Bosonization of quantum sine-Gordon field with boundary

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Abstract

Boundary operators and boundary ground states in sine-Gordon model with a fixed boundary condition are studied using bosonization and q-deformed oscillators. We also obtain the form-factors of this model.

1 Introduction

An integrable two-dimensional field theory possesses an infinite set of mutually commutative integrals of motion. For the massive theory, this results in the factorizable scattering both in bulk case (i.e without a boundary) and in the case with an integrable reflecting boundary (e.g. field theories defined upon the half line) [1,2,3,4]. In the boundary case, besides the two particle scattering S-matrix, there exists boundary reflecting R-matrix, which describes the process of particle reflecting at the boundary. The R-matrix must be consent with the S-matrix (namely, R-matrix must satisfy the boundary Yang-Baxter equation[2,29,30]). The calculation of the exact factorized S-matrix and R-matrix may be performed by combining the standard requirements of unitarity and crossing symmetry together with the symmetry properties of the model [1,2].

In the bulk case, the two particle S-matrix uniquely specifies the structure of the space of local operators for an integrable model. Its knowledge can be used to calculate off-shell quantities, like correlation functions of elementary or composite fields of the integrable models under investigation. These quantities can be obtained by considering the form-factors of local fields, which are matrix elements of operators between asymptotic states[5,6,10]. It is shown that the general properties of unitary, analyticity and locality lead to a system of functional equations on form-factors. These equations include S-matrix elements as structure

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constants and are enough for the reconstruction of the form-factors (see also [10][11][12]). Therefore, the S-matrix determines the whole fields in the bulk case. Obtaining the form-factors, one can in principle obtain all the off-shell quantities of the field theory. We next introduce these equations for the form-factors of any local operator in the massive integrable model without boundary. It is shown that the form-factors for any local field \( O \)

\[
f(\beta_1, \ldots, \beta_n) \equiv \langle 0| OZ(\beta_1) \ldots Z(\beta_n) | 0 \rangle
\]

are subject to a series axioms\[^{10,12}\] as the following:

1. Riemann-Hilbert problem

\[
S_{ij}(\beta_i - \beta_j) f(\beta_1, \ldots, \beta_i, \ldots, \beta_j, \ldots, \beta_n) = f(\beta_1, \ldots, \beta_j, \ldots, \beta_i, \ldots, \beta_n)
\]

\[
f(\beta_1, \ldots, \beta_i, \ldots, \beta_{n-1}, \beta_n + 2\pi i) = f(\beta_n, \beta_1, \ldots, \beta_i, \ldots, \beta_{n-1}) \quad ;
\]

(1)

2. Residue condition

\[
2\pi i \text{Res}_{\beta_n = \beta_n - 1 + 2\pi i} f(\beta_1, \ldots, \beta_{n-2}, \beta_n) a_1 \ldots a_n = f(\beta_1, \ldots, \beta_{n-2}) a'_1 \ldots a'_{n-2} C_{a'_{n-2}, a_n}
\]

\[
\{ \delta^{a'_1}_{a_1} \ldots \delta^{a'_{n-2}}_{a_{n-2}} \delta^{a'_{n-1}}_{a_{n-1}} - \exp(2i\pi \omega(O, \Psi)) S^{a'_{n-1}, a'_1}_{c_{n-1}, a_1} (\beta_{n-1} - \beta_1) \ldots
\]

\[
S^{c_{n-3}, a'_{n-3}}_{a_{n-3}, a_{n-3}} (\beta_{n-2} - \beta_{n-3}) S^{c_{n-4}, a'_{n-4}}_{a_{n-4}, a_{n-4}} (\beta_{n-2} - \beta_{n-4}) \}.
\]

(2)

In Eq.s(1),(2), we use the usual conventions\[^{10,12}\]: \( S(\beta) \) is a two particle S-matrix and matrix \( C \) is related to the crossing symmetry of S-matrix; the number of \( \omega(O, \Psi) \) is the mutual locality index for the local operator \( O \) with respect to the “elementary field” \( \Psi \)[12]. Therefore, solving the equations (1),(2) plays a very important role in the massive integrable model without boundary. There are two ways to solve the equations (1),(2). The first one is to directly solve the difference equations of q-deformed Knizhnik-Zamolodchikov (KZ) equations\[^{10}\] (q-deformed KZ equations are equivalent to equations (1) and (2)), which can be considered as a generalization of BPZ’s work in two dimensional conformal field theories (CFT)\[^{15}\]. In this way, Smirnov obtained the form-factors of some local operators for SU(2)-Invariant Thirring model (SU(2)-ITM), sine-Gordon (SG) model and O(3)-Nolinear \( \sigma \)-model\[^{10,35}\]. Smirnov and Reshetikhin discussed some properties of the Restricted SG model\[^{18}\]. The second way is by bosonization. In the second way, following the success of the bosonization in CFT\[^{16,17,18,19}\], much attention has been focused on the vertex operators of some q-deformed affine Lie algebra which satisfy q-deformed KZ equation\[^{21,24,38}\]. Using these vertex operators, one can obtain the bosonization of massive integrable field models and solvable lattice models\[^{23,39,12}\]. This method can be considered as the generalization of bosonization in CFT\[^{16,17,19,20}\] and may be a very powerful and efficient method. Using this bosonization method, one can get the general integral representations for form-factors in SU(2)-IMT, SG and XXZ model\[^{12,25}\].
Although much progress has been obtained in the massive integrable model without boundary (in the bulk case), the progress of massive integrable model with integrable boundary condition is still very few. Recently, Ghoshal and Zamolodchikov [2] proposed boundary crossing relation to determine the scalar factor of the boundary reflecting matrix (R-matrix). In the boundary case, it is S-matrix and R-matrix that uniquely specify the structure of the model under investigation. From the S-matrix and R-matrix, Jimbo et al. obtained the general integral representations of form-factors for XXZ model with an integrable boundary (a fixed boundary condition), and found that form-factors for this model satisfied an analogous of difference equation [26] which has only a subtle difference from the bulk case. Some solution to these equations in the XYZ model with boundary was already obtained [26]. In this paper, we use the method of bosonization for integrable model to study the SG model with a fixed boundary condition, which has been proved to be integrable [2,3,4,30,31]. It is the extension of Lukyanov’s work, which only deals with SG model without boundary (in the bulk case), to the integrable boundary case. However, this method can also be extended to other boundary integrable models [36]. We find that in the boundary case of massive integrable quantum field theories, the form-factors also satisfy some difference equations which are quite similar to the solvable lattice model. These functional equations are enough to reconstruct the form-factors as in the bulk case.

This paper is organized as follows. In sect. 2 we give a brief review of SG model without boundary (bulk case) by Lukyanov which will be used in the following section. Boundary operators with the bosonic version will be studied in sect. 3, from these boundary operators we obtain the boundary bound ground states and find the reflecting matrix for the quantum breathers with these bound states. In order to calculate the form-factors, the boundary operators with the q-deformed oscillators expression is given in sect. 4. Then, the form-factors are represented in the term of general integrals in sect. 5. Thus in principle we can obtain all the quantities of the SG model with an integrable boundary.

### 2 SG model in the bulk

The SG field theory in the bulk case is described by the action as the following

\[
\mathcal{L} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt \frac{1}{2} (\partial_{\mu} \varphi)^2 - \frac{m^2}{\beta^2} \cos \beta \varphi
\]

where \(\beta\) is the bare coupling constant. In what follows we shall only use the renormalized coupling constant \(\xi = \frac{\beta^2}{8\pi^2 - \beta^2}\) and for the simplicity we consider a generic \(\xi\). It is well-known that this theory is massive and its particle spectrum consists of a kink-antikink pair \((Z_-, Z_+)\) (with equal masses) and a number of neutral particles \(B_n\) (“quantum breathers”) whose number is dependent on renormalized coupling constant \(\xi(n < \frac{1}{\xi})\) [1,2,27,28]. The asymptotic particle states are generated by the particle-creation operators.
\[ Z_a(\beta) \quad (a = \pm) \]

\[ |Z_{a_1}(\beta_1)Z_{a_2}(\beta_2)\ldots Z_{a_N}(\beta_N)| = Z_{a_1}(\beta_1)\ldots Z_{a_N}(\beta_N)|0\rangle \quad (4) \]

where \( \beta \) is particle’s rapidity and operators \( Z_a \) satisfy the Zamolodchikov-Faddeev (ZF) algebra:

\[ Z_i(\beta)Z_j(\beta) = S_{ij}(\beta - 2\beta)Z_i(\beta_2)Z_k(\beta_1) \quad (5) \]

Lukyanov \([12]\) constructed two kinds of operators: asymptotic operators \( Z_i(\beta) \) and local operators \( Z'_i(\beta) \) — using free bosons and q-deformed oscillators, which behave quite similar to the type I and type II operators in the exact solvable lattice model XXZ \([23]\). These two kinds of operators satisfy relations (5) and (6)

\[ Z'_i(\beta_1)Z'_j(\beta_2) = S'_{ij}(\beta - 2\beta)Z'_i(\beta_2)Z'_k(\beta_1), \quad Z_i(\beta)Z'_j(\beta_2) = i\tan\left(\frac{\pi}{4} + i\frac{\beta_1 - \beta_2}{2}\right)Z'_j(\beta_2)Z_i(\beta_1) \quad (6) \]

The S-matrix in (5) and (6) are given as

\[
S'_{12}(\beta) = s'(\beta) \begin{pmatrix}
1 & \frac{sh}{\beta} & \frac{sh}{\beta} & \frac{sh}{\beta} \\
-\frac{sh}{\beta} & 1 & \frac{sh}{\beta} & \frac{sh}{\beta} \\
-\frac{sh}{\beta} & \frac{sh}{\beta} & 1 & \frac{sh}{\beta} \\
\frac{sh}{\beta} & \frac{sh}{\beta} & \frac{sh}{\beta} & 1
\end{pmatrix},
\]

\[
s'(\beta) = \frac{\Gamma\left(\frac{1}{\xi} - \frac{i\theta}{\pi} \xi\right)\Gamma\left(\frac{i\theta}{\pi} \xi\right)}{\Gamma\left(-\frac{i\theta}{\pi} \xi\right)\Gamma\left(\frac{1}{\xi} + \frac{i\theta}{\pi} \xi\right) \prod_{p=1}^\infty R_p(\beta)}, \quad R_p(\beta) = \frac{\Gamma\left(\frac{2p}{\xi} \pi + \frac{i\theta}{\pi} \xi\right)\Gamma\left(1 + \frac{2p}{\xi} \pi + \frac{i\theta}{\pi} \xi\right)}{\Gamma\left(2p + 1 + \frac{i\theta}{\pi} \xi\right)\Gamma\left(1 + \frac{2p}{\xi} \pi + \frac{i\theta}{\pi} \xi\right)}.
\]

The operators \( Z_i(\beta), Z'_i(\beta) \) can be realized through the bosonic field \( \phi(\beta) \) and \( \phi'(\beta) \) as

\[
Z_+(\beta) = e^{-\frac{i\pi}{\xi} \xi V(\beta)} \equiv e^{\phi(\beta)}; \quad Z_-(\beta) = e^{\phi(\beta)}[q^{\frac{\xi}{2}}\chi V(\beta) - q^{-\frac{\xi}{2}}V(\beta)\chi],
\]

\[
Z'_+(\beta) = e^{\frac{i\theta}{\pi} \xi V(\beta)} \equiv e^{\phi'(\beta)}; \quad Z'_-(\beta) = e^{\phi'(\beta)}[q^{\frac{\xi}{2}}\chi' V'(\beta) - q^{-\frac{\xi}{2}}V'(\beta)\chi'],
\]

\[
\left\{ q = \exp\{i\pi \xi + 1\}, \quad q' = \exp\{i\pi \xi + 1\} \right\} \quad (7)
\]

where \( \phi(\beta) \) and \( \phi'(\beta) \) satisfy

\[
\langle 0 | \phi(\beta_1)\phi(\beta_2)|0 \rangle = -\ln g(\beta_2 - \beta_1), \quad s(\beta) = \frac{g(-\beta)}{g(\beta)}; \quad g(\beta) = \frac{\Gamma\left(\frac{1}{\xi} + \frac{i\theta}{\pi} \xi\right)\prod_{p=1}^\infty [R_p(i\pi)R_p(o)]^{\frac{1}{2}}}{\Gamma\left(\frac{i\theta}{\pi} \xi\right) R_p(\beta)}.
\]

\[
\langle 0 | \phi'(\beta_1)\phi'(\beta_2)|0 \rangle = -\ln g'(\beta_2 - \beta_1), \quad s'(\beta) = \frac{g'(-\beta)}{g'(\beta)}; \quad g'(\beta) = \frac{\Gamma\left(\frac{1}{\xi} + \frac{i\theta}{\pi(\xi+1)} \xi\right)\prod_{p=1}^\infty [R_p(i\pi)R_p(o)]^{\frac{1}{2}}}{\Gamma\left(\frac{1}{\xi} + \frac{i\theta}{\pi(\xi+1)} \xi\right) R_p(\beta)}.
\]

\[
\langle 0 | \phi(\beta_1)\phi'(\beta_2)|0 \rangle = -\ln h(\beta_2 - \beta_1), \quad tg\left(\frac{\pi}{4} + i\frac{\beta}{2}\right) = \frac{h(\beta)}{h(-\beta)}; \quad h(\beta) = \frac{\Gamma\left(\frac{1}{\xi} + \frac{i\theta}{\pi} \xi\right)\prod_{p=1}^\infty [R_p(i\pi)R_p(o)]^{\frac{1}{2}}}{\Gamma\left(\frac{1}{\xi} + \frac{i\theta}{\pi} \xi\right) R_p(\beta)}.
\]

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In Eq.(7) the screening operators $\chi, \chi'$ are defined through
\[
\langle u | \chi | v \rangle \equiv \eta \langle u | \int_C \frac{d\gamma}{2\pi} \nabla(\gamma) | v \rangle, \quad \nabla(\gamma) \equiv e^{-i\phi(\gamma)} \equiv e^{-i(\phi(\gamma - \frac{i\pi}{2}) + \phi(\gamma - \frac{3i\pi}{2}))} ;
\]
\[
\langle u | \chi' | v \rangle \equiv \eta' \langle u | \int_{C'} \frac{d\gamma}{2\pi} \nabla'(\gamma) | v \rangle, \quad \nabla'(\gamma) \equiv e^{-i\phi'(\gamma)} \equiv e^{-i(\phi'(\gamma - \frac{i\pi}{2}) + \phi'(\gamma - \frac{3i\pi}{2}))} ; \quad (7a)
\]

In Eq.(7a) $|u|$ and $|v|$ are some states in the Fock space and its dual space of the bosonic fields $\phi(\beta)$ and $\phi'(\beta)$, $\eta, \eta'$ are irrelevant constants[12]. The integration contour $C$ ($C'$) is taken such that it encloses only the poles originated from the action of $\nabla$, $\nabla'$ on the right-hand state $|v|$ clockwise [12].

In addition to the exchange relation of (5) and (6), the operator product of particle-creation operators $Z_a(\beta)$ has the following singular properties [12] when $\beta_2$ approaches $\beta_1$:
\[
iZ_a(\beta_2)Z_b(\beta_1) = \frac{C_{a,b}}{\beta_2 - \beta_1 - i\pi} + ... , \quad \beta_2 \rightarrow \beta_1 . \quad (8)
\]

From the bosonic representation (7) and its regularized version (see sect.4), Lukyanov got the form-factors of SG model in the bulk case.

3 Boundary operators in the bosonic version

Now let us consider SG field theory with an integrable boundary at $x=0$, which is described by the action [2,4]
\[
L = \int_0^\infty dx \int_{-\infty}^\infty dt \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{m^2}{\beta^2} \cos\beta \varphi - \int_{-\infty}^\infty dt M \cos(\frac{\beta}{2} (\varphi|_{x=0} - \varphi_0)) . \quad (9)
\]

It has been proved that the boundary SG model (9) is integrable[2,3,4] and the asymptotic particle states are generated by the same set of creation operators $Z_a(\beta)$ as in the bulk case in the following way [3]:
\[
|Z_{a1}(\beta_1)Z_{a2}(\beta_2)\ldots Z_{aN}(\beta_N)\rangle_B = |Z_{a1}(\beta_1)\ldots Z_{aN}(\beta_N)\rangle_B . \quad (10)
\]

But now the vacuum state $|0\rangle$ is replaced by the $|B\rangle$, which is the ground state of SG model with a boundary. The state $|B\rangle$ satisfies the relations
\[
Z_i(\beta)|B\rangle = R_i^\dagger(\beta) Z_j(-\beta)|B\rangle . \quad (11)
\]

If such $|B\rangle$ exists, the consistency of (5) and (11) leads to the boundary Yang-Baxter equation and the unitary relation
\[
R_2(\beta_2)S_{12}(\beta_1 + \beta_2)R_1(\beta_1)S_{21}(\beta_1 - \beta_2) = S_{12}(\beta_1 - \beta_2)R_1(\beta_1)S_{21}(\beta_1 + \beta_2)R_2(\beta_2)
\]
\[
R(\beta)R(-\beta) = 1 \quad (12)
\]
which in turn guarantees the integrability of the model\cite{2,3,4,29,30,31}. Supplied (12) with the boundary crossing relation given by Ghoshal and Zamolodchikov\cite{2}

\[ C^{ac}R^b_{c}(i\pi/2 - \beta) = \sum_{c,d,e} S^{ab}_{cd}(2\beta)C^{de}R^c_{e}(i\pi/2 + \beta), \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]  

(13)

one can determine the exact boundary reflecting matrix \( R(\beta) \) upon some “CDD” ambiguity. This ambiguity can be cancelled by some dynamical requirement\cite{1,2}. Thus the crucial problem is to find the ground state \(|B\rangle\). For this purpose we first obtain a \( R \)-matrix satisfying all the requirements (12) and (13).

In this paper, we only consider the diagonal solution to (12) and (13), which corresponds to the fixed boundary condition \( \varphi|_{x=0} = \varphi_0 \) (or to the case of \( M \rightarrow \infty \) in the action (9))\cite{2,4}. In this special case, minimal solution \( R(\beta) \) (which has the minimum poles in the physical strip) reads

\[ R(\beta) = r(\beta) \begin{pmatrix} 1 & 0 \\ 0 & \frac{\sin\beta - \sin\beta}{\sin\beta + \sin\beta} \end{pmatrix}, \quad r(\beta) = r_0(\beta)r_1(\beta), \]  

(14)

We find that this solution is the same as the diagonal solution \((k=0)\) of Ghoshal and Zamolodchikov\cite{2} if one use the following transformation:

\[ \frac{1}{\xi} \longrightarrow \lambda, -i\beta \longrightarrow \mu, \mu \longrightarrow -i\pi(\xi + \frac{1}{2\lambda}) \] .

The diagonal Solution (14) has one formal parameter \( \mu \) which can be related to the physical parameter \( \phi_0 \) in the Lagrangian\cite{2,37}.

We then assume that the boundary ground state \(|B\rangle\) and its dual state \(\langle B|\) can be expressed by

\[ |B\rangle = e^{\Psi_-}|0\rangle, \quad \langle B| = \langle 0|e^{\Psi_+} \]  

(14a)

where \( e^{\Psi_-} \) is called the boundary operator and \( e^{\Psi_+} \) is the dual operator of \( e^{\Psi_-} \). Similar to the XXZ model with a reflection boundary\cite{25}, we need further

\[ \langle B|Z^\dagger_s(-\beta) = \langle B|Z^\dagger_s(\beta)R^\dagger_s(\beta), \text{where} \ Z^\dagger_s(\beta) = C^a_s Z_a(i\pi + \beta), \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]  

(15)

In order to study the properties of the operators \( \Psi_{\pm} \), we first introduce the following decomposition of the bosonic fields \( \phi(\beta) \) and \( \phi^*(\beta) \)

\[ \phi(\beta) = \phi_+(\beta) + \phi_0(\beta) + \phi_-(\beta), \quad \phi^*(\beta) = \phi_+(\beta) + \phi_0^*(\beta) + \phi_-(\beta) \] ,
where $\phi_{\pm}(\beta)$, $\phi'_{\pm}(\beta)$, $\phi_0$ and $\phi'_0$ satisfy the following relations\cite{12}


\[
\begin{align*}
\phi_{+}(\beta)|0\rangle = |0\rangle\phi_{-}(\beta) = 0, & \quad [\phi_0, \phi_0] = 0, & \quad [\phi_0, \phi_{\pm}(\beta)] = 0, & \quad [\phi_{\pm}(\beta_1), \phi_{\pm}(\beta_2)] = 0, \\
[\phi_{+}(\beta_1), \phi_{-}(\beta_2)] = -ln(\beta_2 - \beta_1), & \quad \phi'_{+}(\beta)|0\rangle = |0\rangle\phi'_{-}(\beta) = 0, & \quad [\phi'_0, \phi'_{\pm}(\beta)] = 0, \\
[\phi'_0, \phi'_0] = 0, & \quad [\phi'_{\pm}(\beta_1), \phi'_{\pm}(\beta_2)] = 0, & \quad [\phi'_{+}(\beta_1), \phi'_{-}(\beta_2)] = -ln\gamma(\beta_2 - \beta_1).
\end{align*}
\]

Actually, $\phi_{\pm}(\beta)$ ($\phi'_{\pm}(\beta)$) are proper limit of regularized bosonic field $\phi_{\pm}$ ($\phi'_{\pm}$ respectively), and $\phi_0$ ($\phi'_0$) is the limit of zero mode of regularized one. The regularized bosonic fields will be discussed in the section 4.

Now let us make the ansatz for $\Psi_{\pm}$:

\[
[\Psi_{\pm}, \phi_{\pm}(\beta)] = 0, \quad [\Psi_{\pm}, \phi_0] = 0, \quad [\Psi_{+}, \phi_{-(i\pi + \beta)}] = \kappa_+\phi_{+(i\pi - \beta)} - \frac{i}{2}\gamma_+(-\beta),
\]

\[
[\Psi_{-}, \phi_{+(\beta)}] = \kappa_-\phi_{-(\beta)} - \frac{i}{2}\gamma_-(\beta),
\]

\[
[\kappa_{\pm}, everthing] = 0, \quad [\gamma_{\pm}(\beta), everthing] = 0. \quad (16)
\]

Substituting the above ansatz into the equations (11) and (15) for i=+ in (11) and i=- in (15), we find that $\kappa_{\pm}$ and $\gamma_{\pm}$ can be determined by

\[
e^{i\phi_{+(\beta)}}|B\rangle = \frac{f(\beta)}{f(-\beta)}e^{i\phi_{-}(\beta)}|B\rangle, \quad \langle B|e^{i\phi_{-(i\pi + \beta)}} = e^{i\phi_{+}(\beta)}h^{-\beta}(\beta)\phi_{+(i\pi - \beta)}|B\rangle e^{i\phi_{+(\beta)}}, \quad (17)
\]

where

\[
f(\beta) = \frac{\Gamma\left(\frac{1}{2} - \frac{2\beta}{\pi\xi}\right)}{\Gamma\left(1 - \frac{2\beta}{\pi\xi}\right)} \prod_{p=1}^{\infty} \frac{\Gamma\left(\frac{4p+1}{2} - \frac{2\beta}{\pi}\right)\Gamma\left(1 + \frac{4p-1}{2} - \frac{2\beta}{\pi}\right)}{\Gamma\left(\frac{4p+1}{2} - \frac{2\beta}{\pi}\right)}
\]

\[
\times \prod_{p=0}^{\infty} \frac{\Gamma\left(\frac{\beta - \mu}{1 + 2p} + 2p\right)\Gamma\left(\frac{\beta + \mu}{1 + 2p} + 2p\right)}{\Gamma\left(\frac{\beta - \mu}{1 + 2p} + 2p + 2\right)}.
\]

From directly calculating, we get the exact $\kappa_{\pm}$ and $\gamma_{\pm}$

\[
\kappa_{+} = 1, \quad \gamma_{+(\beta)} = -\frac{\beta}{\xi} + 2lnf(\beta) + 2lnsh\frac{\mu - \beta}{\xi} - ln(\beta - 2\beta),
\]

\[
\kappa_{-} = 1, \quad \gamma_{-(\beta)} = -\frac{\beta}{\xi} + 2lnf(-\beta) - ln(2\beta).
\]

From direct calculating and using the definition of the $\tilde{\phi}(\beta)$

\[
\tilde{\phi}(\beta) = \phi(\beta + \frac{i\pi}{2}) + \phi(\beta - \frac{i\pi}{2}),
\]

one can get

\[
e^{-\tilde{\phi}_{+(\beta)}}|B\rangle = \frac{g(-2\beta - i\pi)g(-2\beta)g(-2\beta + i\pi)}{f(\beta - i\frac{\pi}{2})f(\beta + i\frac{\pi}{2})}e^{-\frac{i\pi}{2}}e^{-i\tilde{\phi}_{-}(\beta)}|B\rangle, \quad (18)
\]

\[
\langle B|e^{-\tilde{\phi}_{-(i\pi + \beta)}} = \frac{g(2\beta + i\pi)g(2\beta - i\pi)}{f(-\beta - i\frac{\pi}{2})f(-\beta + i\frac{\pi}{2})sh\frac{\mu + \beta - i\pi}{\xi}sh\frac{\mu - \beta + i\pi}{\xi}}|B\rangle e^{-\tilde{\phi}_{+(i\pi - \beta)}}.
\]
Then we must check the consistence of our ansatz for the second equation of (11) and (15) i.e. i=- for (11) and i=+ for (15). Using the explicit form of $Z_-(\beta)$ in (7) and (17),(18), we find the second equation of (11) is equivalent to the following equation

$$
\int_{C} d\eta \frac{\Gamma(\frac{\mu+n}{\pi \xi} + \frac{1}{2\xi}) \Gamma(-\frac{1}{2\xi} - i\frac{\beta-n}{\pi \xi}) \Gamma(-\frac{1}{2\xi} + i\frac{\beta+n}{\pi \xi}) \Gamma(-\frac{1}{2\xi} + i\frac{\beta+n}{\pi \xi})}{\Gamma(\frac{\mu-n}{\pi \xi} + \frac{1}{2\xi}) \Gamma(1 + \frac{1}{2\xi} - i\frac{\beta+n}{\pi \xi})}
sh \mu + \frac{\beta}{\xi} \eta (\eta^2 + \frac{\pi^2}{4}) : e^{-i\sigma_-(\eta) - i\sigma_-(\eta)} : |B\rangle = (\beta \leftrightarrow -\beta)
$$

(19)

where the integration contour $C$ encloses the poles $-i\pi \xi(n+1) - i\frac{\pi}{2} - \mu$, $-i\pi \xi + i\frac{\pi}{2} + \beta$, $-i\pi \xi n + i\frac{\pi}{2} - \beta$ but not the poles $i\pi \xi n - i\frac{\pi}{2} + \beta$ ($n \geq 0$). We change the integration parameter $\eta$ to $-\eta$, and with a corresponding new contour $C'$, we still get the same integral as the LHS of (19). We find that the corresponding new integration contour $C'$ can be deformed to the same as $C$. Adding this integral to LHS of (19) we have

$$
2 \times \text{LHS of (19)} = \int_{C} d\eta \frac{\Gamma(\frac{\mu-n}{\pi \xi} - \frac{1}{2\xi} - i\frac{\beta-n}{\pi \xi}) \Gamma(-\frac{1}{2\xi} + i\frac{\beta-n}{\pi \xi}) \Gamma(-\frac{1}{2\xi} - i\frac{\beta+n}{\pi \xi}) \Gamma(-\frac{1}{2\xi} + i\frac{\beta+n}{\pi \xi})}{\Gamma(\frac{\mu-n}{\pi \xi} - \frac{1}{2\xi}) \Gamma(i\frac{\mu+n}{\pi \xi} - \frac{1}{2\xi})}
\sh \mu - \frac{\beta}{\xi} \eta (\eta^2 + \frac{\pi^2}{4}) \sh \frac{\mu+n}{\xi} \eta (\eta^2 + \frac{\pi^2}{4}) : e^{-i\sigma_-(\eta) - i\sigma_-(\eta)} : |B\rangle
$$

(19a)

which is symmetric under $\beta \leftrightarrow -\beta$. Using the same method, it is easy to check that the second equation of (15) also has the same symmetry under $\beta \leftrightarrow -\beta$. Thus, the boundary operator of fixed case is obtained, if (16) can be satisfied for some $\Psi_\pm$. It will be shown that such $\Psi_\pm$ can be obtained in section 4 when one takes the limit of regularized operators $\Psi_{\epsilon \pm}$ as $\epsilon \to 0$.

To study the boundary bound states introduced by Ghoshal and Zamolodchikov, we now analyze the physical poles (in the physical strip $0 < Im(\beta) < \pi$) of the reflecting matrix $R(\beta)$. Poles of $r_1(\beta)$ term (in eq.(14)) are associated with breathers and do not correspond to the boundary (new) bound states. The other poles of $r_1(\beta)$ term correspond to the boundary bound states. For $-\frac{\pi \xi}{2} \leq Im(\mu) \leq 0$ (which is corresponding to Ghoshal and Zamolodchikov’s $0 \leq \xi \leq \frac{\pi}{2}$ in Ref.[2]), there is no pole in the physical strip. For $0 < Im(\mu) \leq \frac{[\frac{\pi}{2}] \pi \xi}{4}$, two series of poles appear in the physical strip: $\mu - i\pi \xi l$ and $i\pi - \mu + i\pi \xi l$ ($0 \leq l \leq \lambda$, $\lambda = \lfloor \frac{Im(\mu)}{\pi \xi} \rfloor$). The structure of physical poles is the same as the results of Skorik and Saleur et al in Ref.[4], where some semi-classical explanation of boundary bound states was given.

The $i\pi - \mu + i\pi \xi l$ series correspond to the “cross-channel” poles of kink and antikink reflecting at the boundary. Hence, we need only to study the bound state corresponding to the poles $\mu - i\pi \xi l$ and denote them by $|B\rangle^{(l)}$. Using the boundary bootstrap methods, we can obtain the corresponding boundary reflecting matrices $R^{(l)}(\beta)$, which describe the reflection between kink or antikink and the boundary bound
crossing condition (12) and (13), which are equivalent to the equations

\[ \Psi \pm \]

are considered as the bound states of kink and antikink pair and are defined as following

\[ \text{where } 0 < n \leq \lambda . \]

One can check that the reflecting matrices \( R^{(l)}(\beta) \) satisfy the boundary unitary and crossing condition (12) and (13), which are equivalent to the equations

\[ r^{(l)}(\beta) r^{(l)}(-\beta) = 1 , \quad r^{(l)}\left(\frac{i\pi}{2} - \beta\right) = s(2\beta) \frac{sh \frac{i\pi + 2\beta}{\xi} \pm sh \frac{\mu - i\pi - \beta + i\xi}{\xi}}{sh \frac{i\pi - 2\beta}{\xi} \pm sh \frac{\mu - i\pi + \beta + i\xi}{\xi}} r^{(l)}\left(\frac{i\pi}{2} + \beta\right). \]

With these \( R^{(l)}(\beta) \), one can construct the corresponding boundary bound states \( |B\rangle^{(+l)} \) and \( \langle B|^{(+l)} \) by the same method as that we already use to construct \( |B\rangle \) and \( \langle B| \) from \( R(\beta) \). On the other hand, due to \( |B\rangle^{(+l)} \) (\( \langle B|^{(+l)} \)) being the bound states of \( |B\rangle \) (\( \langle B| \)) with an incoming particle (kink) carrying rapidity \( \mu - i\pi \xi l \), they can be obtained through the following way

\[ |B\rangle^{(+l)} = \text{Res}_{\beta=\mu-i\pi \xi l} Z_+^{(\beta)}|B\rangle , \quad \langle B|^{(+l)} = \text{Res}_{\beta=\mu-i\pi \xi l} \langle B| Z_+^{\ast}(\beta) . \]  \tag{20} \]

The boundary bound state has been discussed also by Saleur and Skorik et al in Ref.[4] and Ref.[40].

The \( R(\beta) \) and \( R^{(l)}(\beta) \) (\( 0 \leq l \leq \lambda \)) describe the reflection of kink and antikink with the corresponding boundary. However, there exist other particles — breathers \( B_n \) (\( 0 < n \leq \left[ \frac{\lambda}{2} \right] \)) in SG model. The breathers are considered as the bound states of kink and antikink pair and are defined as following

\[ B_n^{(\beta)} = Z_+^{(\beta - \frac{i\pi}{2} + \frac{i\pi \xi n}{2})} Z_-(\beta + \frac{i\pi}{2} + \frac{i\pi \xi n}{2}) + (-1)^n Z_-^{(\beta - \frac{i\pi}{2} + \frac{i\pi \xi n}{2})} Z_+^{(\beta + \frac{i\pi}{2} - \frac{i\pi \xi n}{2})} \]

where \( 0 < n \leq \left[ \frac{\lambda}{2} \right] \). Using these definitions, one can get the reflecting matrices of breathers with the boundary and its bound states

\[ B_n^{(\beta)}|B\rangle = s(2\beta) r^{(l)}\left(\beta + \frac{i\pi}{2} - \frac{i\pi \xi n}{2}\right) r^{(l)}\left(\beta - \frac{i\pi}{2} + \frac{i\pi \xi n}{2}\right) \frac{sh \frac{i\pi + 2\beta}{\xi} \pm sh \frac{\mu - i\pi - \beta + i\xi}{\xi}}{sh \frac{i\pi - 2\beta}{\xi} \pm sh \frac{\mu - i\pi + \beta + i\xi}{\xi}} B_n^{(-\beta)}|B\rangle \]

\[ B_n^{(\beta)}|B\rangle^{(+l)} = s(2\beta) r^{(l)}\left(\beta + \frac{i\pi}{2} - \frac{i\pi \xi n}{2}\right) r^{(l)}\left(\beta - \frac{i\pi}{2} + \frac{i\pi \xi n}{2}\right) \frac{sh \frac{i\pi + 2\beta}{\xi} \pm sh \frac{\mu - i\pi - \beta + i\xi}{\xi}}{sh \frac{i\pi - 2\beta}{\xi} \pm sh \frac{\mu - i\pi + \beta + i\xi}{\xi}} B_n^{(-\beta)}|B\rangle^{(+l)} . \]

4 Boundary operators versus q-oscillator

In sect.3, we give the relation of the boundary operators \( \Psi_{\pm} \) with the bosonic fields \( \phi_{\pm}(\beta) \) in (16). If such \( \Psi_{\pm} \) exist, we can construct a boundary state \( |B\rangle \) and its dual \( \langle B| \) via (14a) which ensures the boundary relations (11) and (15). However, it is still unknown how the boundary operators \( \Psi_{\pm} \) can be explicitly
expressed in the term of bosonic fields $\phi(\beta)$ and $\phi'(\beta)$. Fortunately, in order to calculate form-factors in the bulk case, Lukyanov introduced some regularized field $\phi_r(\beta)$ and $\phi'_r(\beta)$, which can be expanded into an infinite sum of q-oscillator mode. In this paper, with the help of q-oscillator expansion of $\phi_r(\beta)$, we can obtain the explicit form of regularized boundary operators $\Psi_{+,-}$ in terms of the q-oscillators. This not only proves the existence of $\Psi_{\pm}$ but also permits one to calculate the form-factors in the regularized version. After $\epsilon \to 0$, the form-factors of SG model with fixed boundary are obtained (see sect.5). In the following of this section, we will give boundary operators in terms of q-oscillator (The regularized bosonic parameter introduced by Lukyanov modes as the usual free fields does. However, we can perform an oscillator realization by the ultraviolet regularization.)

The bosonic fields $\phi(\beta)$ and $\phi'(\beta)$ in sect.2 cannot be expanded into an infinite sum of oscillator modes as the usual free fields does. However, we can perform an oscillator realization by the ultraviolet regularization introduced by Lukyanov. To do this, the first thing is to introduce a regularization parameter $\epsilon$ such that $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$ and then consider regularized fields $\phi_r(\beta)$ and $\phi'_r(\beta)$ which can be expanded by the q-oscillators. Then the bosonic fields $\phi(\beta)$ $\phi'(\beta)$ can be viewed as proper limit of $\phi_r(\beta)$ $\phi'_r(\beta)$ as $\epsilon \to 0$. The oscillator expansion of the regularized bosonic fields are

$$\phi_r(\beta) = \sqrt{1 + \frac{1 + \xi}{2\xi}(Q - \epsilon\beta P)} + \sum_{m \neq 0} \frac{a_m}{ish\pi\epsilon m} \exp(i m\epsilon \beta),$$

$$\phi'_r(\beta) = -\sqrt{\frac{\xi}{2(1 + \xi)}}(Q - \epsilon\beta P) - \sum_{m \neq 0} \frac{sh\frac{\pi m\epsilon}{2}}{ish\pi m\epsilon sh\frac{\pi m(1 + \xi)}{2}} a_m \exp(i m\epsilon \beta),$$

$$\overline{\phi}_r(\beta) = \phi_r(\beta + \frac{i\pi}{2})\phi_r(\beta - \frac{i\pi}{2}), \quad \overline{\phi'}_r(\beta) = \phi'_r(\beta + \frac{i\pi}{2})\phi'_r(\beta - \frac{i\pi}{2}).$$

$$[P, Q] = -i, \quad [P, a_m] = 0, \quad [Q, a_m] = 0, \quad [a_m, a_n] = sh\frac{m\pi\epsilon}{2} shm\pi\epsilon \delta_{m + n, 0},$$

$$V_\epsilon(\beta) = e^{i\phi_r(\beta)}; \quad V'_\epsilon(\beta) = e^{i\phi'_r(\beta)}; \quad \overline{V}_\epsilon(\beta) = e^{-i\phi_r(\beta + \frac{i\pi}{2}) - i\phi_r(\beta - \frac{i\pi}{2})}; \quad \overline{V'}_\epsilon(\beta) = e^{-i\phi'_r(\beta + \frac{i\pi}{2}) + i\phi'_r(\beta - \frac{i\pi}{2})}. \quad (21)$$

Similarly, we can introduce a regularized boundary reflecting matrix $R_\epsilon(\beta)$

$$R_\epsilon(\beta) = \frac{f_\epsilon(\beta)}{f_\epsilon(-\beta)} \left( \frac{\Gamma_{\phi}(1 - \frac{m\pi\epsilon}{2})}{\Gamma_{\phi}(1 - \frac{m\pi\epsilon}{2})} \frac{\Gamma_{\phi}(\frac{m\pi\epsilon}{2})}{\Gamma_{\phi}(\frac{m\pi\epsilon}{2})} \right),$$

$$f_\epsilon(\beta) = \frac{(e^{-4\pi\epsilon + 2i\pi\beta}; e^{-\pi\xi}; e^{-4\pi\epsilon})(e^{-\pi\xi - 2i\pi\beta}; e^{-\pi\xi}; e^{-4\pi\epsilon})}{(e^{-\pi\xi - 2i\pi\beta}; e^{-\pi\xi}; e^{-4\pi\epsilon})(e^{-\pi\xi - 3i\pi\epsilon + 2i\pi\beta}; e^{-\pi\xi}; e^{-4\pi\epsilon})} \times \frac{(e^{-\pi\xi + i\epsilon(\beta + \mu)}; e^{-\pi\xi}; e^{-2\pi\epsilon})(e^{-\pi\xi + i\epsilon(\beta + \mu)}; e^{-\pi\xi}; e^{-2\pi\epsilon})}{(e^{-\pi\xi + i\epsilon(\beta + \mu)}; e^{-\pi\xi}; e^{-2\pi\epsilon})(e^{-\pi\xi - 2i\pi\epsilon + i\epsilon(\beta + \mu)}; e^{-\pi\xi}; e^{-2\pi\epsilon}}.$$
\[(z; q) = \prod_{n=0}^{\infty} (1 - zq^n) \quad , \quad (z; q_1, q_2) = \prod_{n_1, n_2=0}^{\infty} (1 - zq_1^{n_1}q_2^{n_2}) .\]

(21a)

The reflection matrix \( R(\beta) \) in (14) can be viewed as the limit of the regularized one in (21a). The operators \( \Psi_{\pm} \) in sect.3 can also be viewed as the limit of the regularized operators \( \Psi_{\pm, \epsilon} \). We assume the forms of the q-oscillator expansion of \( \Psi_{\pm} \) are

\[
\Psi_{\pm} = \sum_{m=1}^{\infty} \left( \xi_{\pm} \right)^m \left( 2 \text{sh} \frac{m \pi \epsilon}{2} \right) \left( 2 \text{sh} \frac{m \pi \epsilon}{2} \right) \frac{\alpha_m}{\text{sh} \frac{m \pi \epsilon}{2}} \left( \text{sh} \frac{m \pi \epsilon}{2} \right)^2 a_m + \left( 2 \text{sh} \frac{m \pi \epsilon}{2} \right) \frac{\lambda_m}{\text{sh} \frac{m \pi \epsilon}{2}} \left( \text{sh} \frac{m \pi \epsilon}{2} \right)^2 a_m - \frac{1}{\epsilon \sqrt{2}(1 + \xi)} ,
\]

\[
\Psi_{\pm} = \sum_{m=1}^{\infty} \left( \xi_{\pm} \right)^m \left( 2 \text{sh} \frac{m \pi \epsilon}{2} \right) \left( 2 \text{sh} \frac{m \pi \epsilon}{2} \right) \frac{\sigma_m}{\text{sh} \frac{m \pi \epsilon}{2}} \left( \text{sh} \frac{m \pi \epsilon}{2} \right)^2 a_m + \left( 2 \text{sh} \frac{m \pi \epsilon}{2} \right) \frac{\rho_m}{\text{sh} \frac{m \pi \epsilon}{2}} \left( \text{sh} \frac{m \pi \epsilon}{2} \right)^2 a_m + \frac{1}{\epsilon \sqrt{2}(1 + \xi)} .
\]

(22)

(The zero mode \( Q \) term of \( \Psi_{\pm} \) in this expansion is to reduce the factor \( e^{\frac{\beta}{\epsilon}} \) appearing in (17)). The constants \( \alpha_m, \lambda_m, \sigma_m, \rho_m \) are determined by relations

\[
e^{i\phi_{\pm} + (\beta)} e^{\Psi_{\pm} - |0\rangle} = f_{\epsilon}(\beta) e^{i\phi_{\pm} - |0\rangle} ,
\]

\[
\langle 0 | e^{\Psi_{\pm}} e^{i\phi_{\pm} - (i\pi + \beta)} = f_{\epsilon}(\beta) (e^{-\pi \xi - i\mu - i\beta}; e^{-\pi \xi}) (e^{i\epsilon \mu - i\beta}; e^{-\pi \xi}) \langle 0 | e^{\Psi_{\pm}} e^{i\phi_{\pm} (i\pi - \beta)} .
\]

Substituting (22) into (23), we find

\[
\lambda_m = - \frac{\text{sh} \frac{m \pi \epsilon}{2} \text{sh} \frac{m \pi \epsilon (1 - \xi + \frac{2m}{\epsilon})}{2}}{2 \text{sh} \frac{m \pi \epsilon}{2} \text{sh} \frac{m \pi \epsilon (1 - \xi + \frac{2m}{\epsilon})}{2}} + \theta_m \left( \frac{\text{sh} \frac{m \pi \epsilon}{2} \text{sh} \frac{m \pi \epsilon (1 + \xi)}{2}}{2 \text{sh} \frac{m \pi \epsilon}{2} \text{sh} \frac{m \pi \epsilon (1 + \xi)}{2}} - \frac{\text{sh} \frac{m \pi \epsilon}{2} \text{sh} \frac{m \pi \epsilon (1 - \xi)}{2}}{2 \text{sh} \frac{m \pi \epsilon}{2} \text{ch} \frac{m \pi \epsilon}{2}} \right) ,
\]

\[
\rho_m = \left( \frac{\text{sh} \frac{m \pi \epsilon}{2} \text{sh} \frac{m \pi \epsilon (1 - \xi + \frac{2m}{\epsilon})}{2}}{2 \text{sh} \frac{m \pi \epsilon}{2} \text{sh} \frac{m \pi \epsilon (1 - \xi + \frac{2m}{\epsilon})}{2}} - \frac{\text{sh} \frac{m \pi \epsilon}{2} \text{sh} \frac{m \pi \epsilon (1 + \xi)}{2}}{2 \text{sh} \frac{m \pi \epsilon}{2} \text{sh} \frac{m \pi \epsilon (1 + \xi)}{2}} \right) e^{-\pi \mu} ,
\]

\[
\alpha_m = -1 , \quad \sigma_m = - e^{-2\pi \mu} , \quad \theta_m = \begin{cases} 0 & m \text{ odd} \\ 1 & m \text{ even} \end{cases} .
\]

(24)

Similarly, the regularized versions of vertex operators \( V_\epsilon(\beta), V'_\epsilon(\beta), \text{screening operators } \chi_{\epsilon}, \chi'_{\epsilon} \) can be constructed and \( V(\beta), V'(\beta), \chi, \chi' \) are the limit of their regularized counterparts when \( \epsilon \rightarrow 0 \). In the regularized version, vertex operators, the screening operators and the boundary operators are all expanded by the q-oscillators. Replacing \( \Psi_{\pm}, \phi_{\pm}(\beta), \phi_0 (\phi_0 \propto Q) \) by \( \Psi_{\pm, \epsilon}, \phi_{\pm}(\beta), Q \) in (16), one finds that (16) is satisfied after taking the limit \( \epsilon \rightarrow 0 \). The \( \kappa_{\pm}, \gamma_{\pm} \) are the same as (17a) because the limit of (23) is equivalent to (17). We thus prove the existence of \( \Psi_{\pm} \) and complete the construction of the boundary states \( |B\rangle \) and \( \langle B| \) .
5 Form-factors in the boundary SG model

Our final goal is to obtain the form-factors of SG model with boundary. In principle, these form-factors enable one to calculate all quantities of this quantum field theory. A form-factor is a matrix element of a local field operator between two asymptotic states

\[ f(\beta_1, \ldots, \beta_n) = \langle B | O Z(\beta_1) \ldots Z(\beta_n) | B \rangle \]

The problem of description of all the local operators can be considered in the framework of Ref.[12] and Ref.[23]. The solution to this problem is “a local operator is everything commuting with the type II operators (or the asymptotic generators)” [23]. For SG model, Lukyanov [12] already got this solution: he got three generator function operators of local operator \( \Lambda_s(\alpha) \) \( s = (-1, 0, 1) \)

\[
\Lambda_1(\alpha) = \tilde{V}'(\alpha), \quad \Lambda_0(\alpha) = -\frac{i}{(2 \cos \frac{\pi}{\xi - 1})^2} \left[ q' \chi' \tilde{V}'(\alpha) - q'^{-1} \tilde{V}'(\alpha) \chi' \right], \\
\Lambda_{-1}(\alpha) = -\frac{1}{(2 \cos \frac{\pi}{\xi - 1})^2} \left[ q' \chi'^2 \tilde{V}'(\alpha) - q'^{-1} \tilde{V}'(\alpha) \chi'^2 - (q' + q'^{-1}) \chi' \tilde{V}'(\alpha) \chi' \right]
\]

(25a)

In this equation \( \tilde{V}'(\alpha) \) is the limit of \( \tilde{V}_\epsilon(\alpha) \) in (21) as \( \epsilon \rightarrow 0 \), and the screening operator \( \chi' \) is defined in sect.2. Using the exchange relation (6), one can find that \( \Lambda_m(\beta) \) satisfy the following relations with the particle-creation operator \( Z_a(\beta) \)

\[ \Lambda_m(\alpha) Z_a(\beta) = (-1)^m Z_a(\beta) \Lambda_m(\alpha) \quad . \]

(25)

Thus, in SG model with a boundary a form-factor can be expressed as by multipoint functions of the generator function operators \( \Lambda_s \) \( s = (-1, 0, 1) \)[12] and the particle-creation operators \( Z_a \) \( a = (+, -) \), between ground state \( |B\rangle \) and its dual \( \langle B | \). We can alternatively define the form-factors as

\[ F_{a_1, \ldots, a_n}^{m_1, \ldots, m_k}(\alpha_1, \ldots, \alpha_k | \beta_1, \ldots, \beta_n) = \frac{\langle B | \Lambda_{m_k}(\alpha_k) \ldots \Lambda_{m_1}(\alpha_1) Z_{a_n}(\beta_n) \ldots Z_{a_1}(\beta_1) | B \rangle}{\langle B | B \rangle} \quad . \]

(26)

Using exchange relations (5), (25), the reflection properties (11), (15) and the singular properties (8), one can find that the form-factors in the boundary case defined above satisfy the following difference equations and the residue condition (cf Eq.(1) and (2)), which are quite similar to the equations appearing in the solvable lattice model with integrable boundary[26]:

1. Riemann-Hilbert problem.

\[ F_{a_1, \ldots, a_n}^{m_1, \ldots, m_k}(\alpha_1, \ldots, \alpha_k | \beta_1, \ldots, \beta_i, \ldots, \beta_j, \ldots, \beta_n) = S_{a_1, a_j}^a(\beta_j - \beta_i) F_{a_1, \ldots, a_j, \ldots, a_n}^{m_1, \ldots, m_k}(\alpha_1, \ldots, \alpha_k | \beta_1, \ldots, \beta_j, \ldots, \beta_i, \ldots, \beta_n) \]
\[ F_{a_1, \ldots, a_n}^{m_1, \ldots, m_k}(\alpha_1, \ldots, \alpha_k|\beta_1, \ldots, \beta_n) = R_{a_1}^{\alpha_1}(\beta_1)F_{a_1, \ldots, a_n}^{m_1, \ldots, m_k}(\alpha_1, \ldots, \alpha_k|\beta_1, \ldots, \beta_n) \]
\[ F_{a_1, \ldots, a_n}^{m_1, \ldots, m_k}(\alpha_1, \ldots, \alpha_k|\beta_1, \ldots, \beta_n) = C_j^{\beta_j}R_j^{(i\pi - \beta_n)}C_{a_n}^{\beta_n}F_{a_1, \ldots, a_n}^{m_1, \ldots, m_k}(\alpha_1, \ldots, \alpha_k|\beta_1, \ldots, 2i\pi - \beta_n) \]  
(27)

(The first equation is the result of exchange relation (5),the second one is the result of reflection property (11), and the last one is the result of Eq.(15).)

2. Residue condition.

\[ 2i\pi \text{Res}_{\beta_n=\beta_n-1+i\pi} F_{a_1, \ldots, a_n}^{m_1, \ldots, m_k}(\alpha_1, \ldots, \alpha_k|\beta_1, \ldots, \beta_n) = F_{a_1, \ldots, a_{n-2}}^{m_1, \ldots, m_k}(\alpha_1, \ldots, \alpha_k|\beta_1, \ldots, \beta_{n-2})C_{a_{n-1}, a_n} \]
\[ -C_j^{\alpha_j}R_j^{(-\beta_{n-1})}C_{a_n}^{\beta_n}S_{a_{n-1}, a_{n-1}}^{b_n-1, -\alpha_{n-1}}(i\pi - 2\beta_{n-1})S_{b_{n-2}, a_{n-1}}^{b_n-1, -\alpha_{n-1}}(i\pi - \beta_{n-1} - \beta_1)R_{b_1}^{\beta_1}(\beta_n - i\pi) \]
\[ S_{a_1, b_1}^{\alpha_1, \beta_1}(i\pi + \beta_1 - \beta_n - 1)S_{a_n-2, b_n-2}^{\alpha_n-2, \beta_n-2}(i\pi + \beta_n - 2 - \beta_n - 1)C_{a_n}^{\beta_n}S_{b_n-2, a_n-2}^{\alpha_n-2, \beta_n-2} \]
\[ F_{a_1, \ldots, a_{n-2}}^{m_1, \ldots, m_k}(\alpha_1, \ldots, \alpha_k|\beta_1, \ldots, \beta_n-2) \]  
(28)

(It is the result of singular property (8) with (5), (11), (15), (25).) There are additional poles corresponding to bound particle (breathers) and the boundary bound states.For the poles corresponding to breathers, the residue relation are the same as in the bulk case\[10.\] However, the poles corresponding to boundary bound states is dependent upon the boundary condition. For the fixed boundary case which we study in this paper, the form-factors between the bound state $|B\rangle^{(+)1}$ and its dual states $\langle B\rangle^{(+)1}$ can be directly obtained by the residue form as follow

\[ \langle B\rangle^{(+)1}\Lambda_{m_k}(\alpha_k)\ldots\Lambda_{m_1}Z_{a_n}(\beta_n)\ldots Z_{a_1}(\beta_1)|B\rangle^{(+)1} \]
\[ = \frac{\text{Res}_{\beta=\mu-i\pi L}(B)Z_{\Lambda_{m_k}(\alpha_k)}\ldots Z_{a_n}(\beta_n)\ldots Z_{a_1}(\beta_1)}{\text{Res}_{\beta=\mu-i\pi L}(B)Z_{\Lambda_{m_k}(\alpha_k)}\ldots Z_{a_1}(\beta_1)}|B\rangle^{(+)1} \]  
(29)

The above analogues of difference equations and residue condition also appear in other integrable field theories with integrable boundary, e.g., in SU(2)-ITM with an integrable boundary\[36.\]

From the equation (27), one can obtain the quantum boundary Knizhnik-Zamolodchikov (KZ) equation

\[ F_{a_1, \ldots, a_n}^{m_1, \ldots, m_k}(\alpha_1, \ldots, \alpha_k|\beta_1, \ldots, \beta_n) = R_{a_1}^{\alpha_1}(\beta_1)S_{a_2, b_2}^{\alpha_2, \beta_2}(\beta_2 + \beta_1)\ldots S_{a_n, b_n}^{\alpha_n, \beta_n}(\beta_n + \beta_1) \]
\[ C_j^{\beta_j}R_j^{(i\pi + \beta_1)}C_{b_1, a_2}^{b_1, -\alpha_2}(2i\pi + \beta_1 - \beta_n)\ldots S_{b_{n-2}, a_{n-1}}^{b_{n-2}, -\alpha_{n-1}}(2i\pi + \beta_1 - \beta_2) \]
\[ F_{a_1, \ldots, a_n}^{m_1, \ldots, m_k}(2i\pi + \alpha_1, \ldots, \alpha_k|\beta_1, \ldots, \beta_n) \]  
(30)

Like the bulk case, the system of functional equations (27),(28),(29) permit the reconstruction of the form-factors of boundary massive integrable model. In other word, the equations (27),(28),(29) can be considered as the starting point to construct the form-factors of massive integrable model with integrable boundary. In this direction, some works have been given\[26.\] But the method of bosonization become more powerful and efficient. We will calculate the form-factors directly from the expressions of $|\Lambda_{m_k}, Z_{a_n}\rangle$ and $\langle B|, |B\rangle$ via q-oscillators. In this way, as we already prove, the main functional equations (27),(28),(29)
are the direct results of the exchange relations (5), (25), reflection properties (11), (15) and the singular properties (8).

The procedure is as follows. We have obtain the bosonic representation of the operators $Z_a(\beta)$, $\Lambda_m(\alpha)$ and $\Psi^{\pm}$, which make (5), (8), (11), (15), (25) satisfied. After substituting the bosonic representation of $\Lambda_m(\alpha), Z_a(\beta)$ in (25a) (7) into (26), we find that the form-factors (26) (or solutions to equations (27) (28) (29)) can be given by the combination of multiply contour integrals as the following

$$E_{a_1,...,a_n}^{m_1,...,m_k}(\alpha_1,\ldots,\alpha_k|\beta_1,\ldots,\beta_n) = \prod_{a \in A} \oint_{C_a} \frac{d\gamma_a}{2\pi i} (q^\frac{1}{2} \Delta(\beta_a - \gamma_a) - q^{-\frac{1}{2}}) \prod_{a>j}^n \Delta(\beta_j - \gamma_a)$$

$$\times \prod_{d \in I_1} \oint_{C_d} \frac{d\delta_d^1 d_d^2}{4\pi^2} [q^\prime \Delta(\alpha_d - \delta_d^1) \Delta'(\alpha_d - \delta_d^2) - (q^\prime + q^{-1}) \Delta'(\alpha_d - \delta_d^1) - q^{-1}] \prod_{d>j}^k \Delta'(\alpha_j - \delta_d^1) \Delta'(\alpha_j - \delta_d^2)$$

$$\times \prod_{b \in I_0} \oint_{C_b} \frac{d\delta_b}{2\pi} (q^\prime \Delta'(\alpha_b - \delta_b) - q^{-1}) \prod_{b>j}^k \Delta'(\alpha_j - \delta_b)$$

$$\times R(\alpha_1,\ldots,\alpha_k|\delta_1,\ldots,\delta_1,\ldots,\delta_2,\delta_1,\ldots,\delta_1,\beta_1,\ldots,\beta_n|\gamma_1,\ldots,\gamma_r)$$

(31)

where integral contour $C_a,C_b,C_d$ are taken by the rule in the sect.2, and

$$\Delta(\beta) = \frac{\Gamma(-\frac{1}{2} + i\frac{\beta}{\pi}) \Gamma(1 + \frac{1}{2} - i\frac{\beta}{\pi})}{\Gamma(-\frac{1}{2} + i\frac{\beta}{\pi}) \Gamma(1 + \frac{1}{2} + i\frac{\beta}{\pi})}$$

$$\Delta'(\beta) = \frac{\Gamma(\frac{1}{2(1+\xi)} + \frac{i\beta}{\pi(1+\xi)}) \Gamma(1 - \frac{1}{2(1+\xi)} - \frac{i\beta}{\pi(1+\xi)})}{\Gamma(\frac{1}{2(1+\xi)} - \frac{i\beta}{\pi(1+\xi)}) \Gamma(1 - \frac{1}{2(1+\xi)} + \frac{i\beta}{\pi(1+\xi)})}$$

$$A = \{ a_j | a_j = - \} \quad I_0 = \{ m_j | m_j = 0 \} \quad I_{-1} = \{ m_j | m_j = -1 \}$$

and the function $R(\alpha_1,\ldots,\alpha_k|\delta_1,\ldots,\delta_1,\beta_1,\ldots,\beta_n|\gamma_1,\ldots,\gamma_r)$ are function like

$$R(\alpha_1,\ldots,\alpha_k|\delta_1,\ldots,\delta_1,\beta_1,\ldots,\beta_n|\gamma_1,\ldots,\gamma_r) = \frac{\langle 0 | e^{\Psi^{+}} \tilde{V}(\alpha_k)\ldots\tilde{V}(\alpha_1)\tilde{V}^\dagger(\delta_k)\ldots\tilde{V}^\dagger(\delta_1)V(\beta_1)\ldots V(\beta_n)\tilde{V}(\gamma_r)\ldots \tilde{V}(\gamma_1)e^{-\Psi^{+}}} | 0 \rangle}{\langle 0 | e^{\Psi^{+}} e^{-\Psi^{+}} | 0 \rangle} .$$

(32)

Since the boundary operators $\Psi^{\pm}$ can not be explicitely expressed in the term of bosonic field $\phi(\beta)$ and $\phi'(\beta)$, it is difficult to get the function $R(\alpha_1,\ldots,\alpha_k|\delta_1,\ldots,\delta_1,\beta_1,\ldots,\beta_n|\gamma_1,\ldots,\gamma_r)$. Fortunately, it can be viewed as the limits of their regularized counterparts $R_\epsilon(\alpha_1,\ldots,\alpha_k|\delta_1,\ldots,\delta_1,\beta_1,\ldots,\beta_n|\gamma_1,\ldots,\gamma_r)$ which can be obtained by changing $\tilde{V}(\alpha), \tilde{V}^\dagger(\delta), V(\beta), V(\gamma), \Psi^{\pm}$ to $\tilde{V}_\epsilon(\alpha), \tilde{V}_\epsilon^\dagger(\delta), V_\epsilon(\beta), V_\epsilon(\gamma), \Psi^{\pm}$ in (26). Because $\tilde{V}_\epsilon(\alpha), \tilde{V}_\epsilon^\dagger(\delta), V_\epsilon(\beta), V_\epsilon(\gamma), \Psi^{\pm}$ can be expanded into the q-oscillator (in sect.4), we first move $e^{\Psi^{+}}$ to the left of all vertex operators and reach the right side of $e^{\Psi^{+}}$. Then we insert a complete set of coherent states (38) between $e^{\Psi^{+}}$ and $e^{\Psi^{-}}$ and we can obtain the exact form of function $R_\epsilon(\alpha_1,\ldots,\alpha_k|\delta_1,\ldots,\delta_1,\beta_1,\ldots,\beta_n|\gamma_1,\ldots,\gamma_r)$ using the Gauss type integration formula (40) [25]. After taking the limit of $\epsilon \rightarrow 0$, we finally obtain the
exact form of function \( R(\alpha_1, \ldots, \alpha_k | \delta_1, \ldots, \delta_p | \beta_1, \ldots, \beta_n | \gamma_1, \ldots, \gamma_r) \) (see Appendix for details):

\[
R(\alpha_1, \ldots, \alpha_k | \delta_1, \ldots, \delta_p | \beta_1, \ldots, \beta_n | \gamma_1, \ldots, \gamma_r) = \delta_{k,p} \delta_{n,2} C_1^T C_2
\]

\[
\times \left\{ \prod_{j=1}^{n} F(\gamma_j) \prod_{j<l} G(\gamma_j - \gamma_l) G(\gamma_l - \gamma_j) \right\} \left\{ \prod_{j=1}^{n} W(\gamma_j - \beta_1) W(\gamma_j - \beta_l) \right\}
\]

\[
\times \left\{ \prod_{j=1}^{r} \prod_{m=1}^{p} H(\gamma_j - \delta_m) H(\gamma_j - \delta_m) \right\} \left\{ \prod_{j=1}^{r} \prod_{l=1}^{k} H^{-1}(\gamma_j - \alpha_n) H^{-1}(\gamma_j - \alpha_n) \right\}
\]

\[
\times \left\{ \prod_{j=1}^{n} F(\beta_j) \prod_{j<l} G(\beta_j - \beta_l) G(\beta_l - \beta_j) \right\} \left\{ \prod_{j=1}^{n} U(\beta_j - \delta_l) U(\beta_j - \delta_l) \right\}
\]

\[
\times \left\{ \prod_{j=1}^{p} \prod_{l=1}^{k} U^{-1}(\beta_j - \alpha_l) U^{-1}(\beta_j - \alpha_l) \right\} \left\{ \prod_{j=1}^{p} F'(\delta_j) \prod_{j<l} G'(\delta_j - \delta_l) G'(\delta_j - \delta_l) \right\}
\]

\[
\times \left\{ \prod_{j=1}^{n} F^{(0)}(\gamma_j) \prod_{j=1}^{n} F^{(0)}(\beta_j) \prod_{j=1}^{n} \tilde{F}^{(0)}(\delta_j) \prod_{j=1}^{k} \tilde{F}^{(0)}(\alpha_j) \right\}
\]

where

\[
T(\alpha_1, \alpha_2, a_3, a_4, a_5 | c_1, c_2, c_3, c_4 | b_1, b_2, b_3) = \exp \left\{ \int_0^\infty \frac{\text{shc}_1 x \text{shc}_2 x \text{shc}_3 x \text{shc}_4 x e^{\frac{\beta}{2} e^{-b_2 x}}}{\text{shc}_1 x \text{shc}_2 x \text{shc}_3 x \text{shc}_4 x} \frac{dx}{x} \right\}
\]

\[
\bar{T}(\beta) = \frac{\text{pg}^\frac{1}{2}}{2} (-2\beta) T^{-1}(\bar{T}(\beta)) \left( \frac{1}{2}, \frac{1}{2}, \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1, 1, 1, 0, \beta, 1 \right)
\]

\[
T^{-1}(\frac{1}{2}, \frac{1}{2}, \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1, 1, 1, 0, \beta, 1) \left( \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1, 1, 1, 0, \beta, 1 \right)
\]

\[
F(\beta) = \frac{\text{pg}^\frac{1}{2}}{2} (-2\beta) T(\frac{1}{2}, \frac{1}{2}, \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1, 1, 1, 0, \beta, 1) \left( \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1, 1, 1, 0, \beta, 1 \right)
\]

\[
\bar{T}^\prime(\beta) = \frac{\text{pg}^\frac{1}{2}}{2} (-2\beta) T^{-1}(\bar{T}^\prime(\beta)) \left( \frac{1}{2}, \frac{1}{2}, \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1, 1, 1, 0, \beta, 1 \right)
\]

\[
T^{-1}(\frac{1}{2}, \frac{1}{2}, \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1, 1, 1, 0, \beta, 1) \left( \frac{1}{2}, \frac{1}{2}, \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1, 1, 1, 0, \beta, 1 \right)
\]

\[
\bar{F}(\beta) = \frac{\text{pg}^\frac{1}{2}}{2} (-2\beta) T^{-1}(\bar{F}(\beta)) \left( \frac{1}{2}, \frac{1}{2}, \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1, 1, 1, 0, \beta, 1 \right)
\]

\[
T^{-1}(\frac{1}{2}, \frac{1}{2}, \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1, 1, 1, 0, \beta, 1) \left( \frac{1}{2}, \frac{1}{2}, \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1, 1, 1, 0, \beta, 1 \right)
\]

\[
\bar{G}(\beta) = \frac{\text{pg}^\frac{1}{2}}{2} (-2\beta) T^{-1}(\bar{G}(\beta)) \left( \frac{1}{2}, \frac{1}{2}, \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1, 1, 1, 0, \beta, 1 \right)
\]

\[
G(\beta) = \frac{\text{pg}^\frac{1}{2}}{2} (-2\beta) T^{-1}(G(\beta)) \left( \frac{1}{2}, \frac{1}{2}, \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1, 1, 1, 0, \beta, 1 \right)
\]

\[
\bar{G'}(\beta) = \frac{\text{pg}^\frac{1}{2}}{2} (-2\beta) T^{-1}(\bar{G'}(\beta)) \left( \frac{1}{2}, \frac{1}{2}, \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1, 1, 1, 0, \beta, 1 \right)
\]

\[
G'(\beta) = \frac{\text{pg}^\frac{1}{2}}{2} (-2\beta) T^{-1}(G'(\beta)) \left( \frac{1}{2}, \frac{1}{2}, \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1, 1, 1, 0, \beta, 1 \right)
\]
$W(\beta) = w(\beta)T(\frac{1 + \xi}{2}, \frac{\beta}{2i\pi}, \frac{\beta}{2i\pi}, 1, 1, 1|0, 0, 1)$

$\Pi(\beta) = R(\beta)T^{-2}(\frac{\beta}{2i\pi}, 1, 1, 1|0, 0, 1), U(\beta) = u(\beta)T^{-2}(\frac{\beta}{2i\pi}, 1, 1, 1|0, 0, 1)$

$w(\beta) = \frac{\Gamma(-\frac{1}{2} + i\frac{\beta}{4\pi})}{\Gamma(1 + \frac{\beta}{2i\pi})}, \mu(\beta) = \frac{i\beta}{2\pi}, R(\beta) = \frac{4\pi^2}{\beta^2 + \frac{4\pi}{\xi}}$

$g(\beta) = \left\{ \frac{1}{\Gamma(\frac{1}{2})} \right\}^\frac{1}{2} \frac{\Gamma(\frac{1}{2} + i\frac{\beta}{2\pi})}{\Gamma(\frac{1}{2})} \prod_{p=1}^\infty \frac{R_p(\pi i)}{R_p(\beta)}$

$\Pi(\beta) = \frac{i\beta \Gamma(1 + \frac{1}{2} + i\frac{\beta}{2\pi})}{\pi \xi \Gamma(-\frac{1}{2})}, \quad \Pi'(\beta) = \frac{i\beta \Gamma(1 + \frac{1}{2} + i\frac{\beta}{2\pi})}{\pi \xi \Gamma(-\frac{1}{2})}$

$\rho^2 = \frac{i}{\pi \xi \Gamma(-\frac{1}{2})} \prod_{p=1}^\infty \frac{R_p(\pi i)}{R_p(0)}$

$T^{(0)}(\beta) = T^{-2}(\frac{2i\mu}{\pi} - 1 - \xi, \frac{i\beta + \pi}{2\xi}, 1, 1, 1|0, 0, 1), T^{-2}(\frac{2i\mu}{\pi} - 1 - \xi, \frac{i\beta + \pi}{2\xi}, 1, 1, 1|0, 0, 1)$

$F^{(0)}(\beta) = T^{-2}(\frac{2i\mu}{\pi} - 1 - \xi, \frac{i\beta + \pi}{2\xi}, 1, 1, 1|0, 0, 1), F^{-2}(\frac{2i\mu}{\pi} - 1 - \xi, \frac{i\beta + \pi}{2\xi}, 1, 1, 1|0, 0, 1)$

$\sim(0) = \left\{ F^{(0)}(\beta) \right\}^{-1}$

$C_1 = \exp \left\{ -\int_0^\infty \frac{sh^{\frac{1 + \xi}{2}}}{sh^{\frac{1}{2}}} \left( \frac{1}{sh^{\frac{1 + \xi}{2}}} + \frac{4}{sh^{\frac{1 + \xi}{2}}} \right) e^{-x} dx \right\}$

$C_2 = \exp \left\{ \int_0^\infty \frac{sh^{\frac{1 + \xi}{2}}}{sh^{\frac{1}{2}}} \left( 1 - \frac{sh^{\frac{1 + \xi}{2}}}{sh^{\frac{1 + \xi}{2}}} \right) e^{-x} dx \right\}$
\[
\frac{1}{2} \int_0^\infty \frac{\text{sh} x (\xi + \frac{2i\nu}{\text{sh} x}) (1 - \frac{2\text{sh} x}{\text{sh} x \xi \text{sh} x})}{x} \, dx.
\] (33)

Therefore, we obtained the form-factors of SG model with a fixed boundary condition, which is expressed in terms of integrals ((31)). This is quite similar to the correlation functions in CFT\(^{19,20}\) and the form-factors in massive integrable model without any boundary\(^{10,35}\), they are also expressed in terms of integrals.

6 Discussion

Let us summarize our results. We have shown that the obtained form factors of semi-infinite line SG theory with a fixed boundary obey the quantum boundary KZ equation (27), (28), (29). The quantum KZ equation is a simple consequence of the exchange relation of ZF algebra and the reflection properties (11), (15) of the boundary ground state and its dual. It is believed that these quantum difference relations should govern the form-factors in the integrable field theories with integrable boundary as the classical KZ equation does in CFT.

In this paper, we only consider SG theory with fixed boundary condition which corresponds to the diagonal solution to the boundary Yang-Baxter equation. To solve SG theories with other generic integrable boundary condition is a challenging work. When the coupling constant \(\xi\) is equal to some special value (\(\xi = \frac{1}{N}\)), which may be related to the Restricted SG model, it is worthy of further studying. We hope to present the full results elsewhere. The connection between the symmetries of SG model and deformed Virasoro algebra is also worth of investigating.

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Appendix

A.1

In this appendix, we first calculate the regularized function \(R_\epsilon(\alpha_1, \ldots, \alpha_k | \delta_1, \ldots, \delta_p | \beta_1, \ldots, \beta_n | \gamma_1, \ldots, \gamma_r)\) using the q-oscillator expansion of vertex operators and the boundary operators in sect.4. Then taking the limit of \(\epsilon \to 0\), we obtain the function \(R(\alpha_1, \ldots, \alpha_k | \delta_1, \ldots, \delta_p | \beta_1, \ldots, \beta_n | \gamma_1, \ldots, \gamma_r)\).

Using the exact form of \(\Psi_\epsilon\) in (23), one can find that \(e^{\Psi_\epsilon}\) has the effect of a Bogoliubov transformation.
\[ e^{-\Psi_r - a_n} e^{\Psi_r} = a_n - a_{-n} + \lambda_m , \quad n > 0 \, . \tag{34} \]

From (21) and (34) one can get the reflecting properties

\[
e^{i\Phi_r(\beta) |B\rangle} = \rho_e \exp \left( - \frac{1}{2} \sum_{m=1}^{\infty} \frac{sh \frac{\pi m \epsilon}{2} \pi m e \frac{\pi m e (1 + \xi)}{2}}{sh \pi m e \frac{\pi m e}{2}} e^{2ime\beta} + \sum_{m=1}^{\infty} \frac{\lambda_m}{sh \pi m e} e^{ime\beta} e^{i\phi_r(-\beta) + i\phi_r(-\beta)} |B\rangle \right) \]

\[
e^{-i\Phi_r(\beta) |B\rangle} = \bar{\rho}_e \exp \left( - \frac{1}{2} \sum_{m=1}^{\infty} \frac{sh \pi m e \frac{\pi m e}{2}}{sh \pi m e \frac{\pi m e}{2}} e^{2ime\beta} - \sum_{m=1}^{\infty} \frac{\lambda_m}{sh \pi m e} e^{ime\beta} e^{-i\phi_r(-\beta) - i\phi_r(-\beta)} |B\rangle \right) \]

\[
e^{-i\Phi_r(\beta) |B\rangle} = \bar{\rho}_e \exp \left( - \frac{1}{2} \sum_{m=1}^{\infty} \frac{sh \pi m e \frac{\pi m e}{2}}{sh \pi m e \frac{\pi m e}{2}} e^{2ime\beta} - \sum_{m=1}^{\infty} \frac{\lambda_m}{sh \pi m e} e^{ime\beta} e^{-i\phi_r(-\beta) - i\phi_r(-\beta)} |B\rangle \right) \]

\[
e^{i\Phi_r(\beta) |B\rangle} = \rho_e \exp \left( - \frac{1}{2} \sum_{m=1}^{\infty} \frac{sh \pi m e \frac{\pi m e}{2}}{sh \pi m e \frac{\pi m e}{2}} e^{2ime\beta} - \sum_{m=1}^{\infty} \frac{\lambda_m}{sh \pi m e} e^{ime\beta} e^{i\phi_r(-\beta) + i\phi_r(-\beta)} |B\rangle \right) \]

Noting that the operators \( \tilde{V}_e (\alpha) , \tilde{V}_e (\beta) , V_e (\gamma) \) have the following normal-ordering\[12]\]

\[
V_e (\beta_2) V_e (\gamma_1) = \rho^2_e g_\epsilon (\beta_1 - \beta_2) : V_e (\beta_2) V_e (\gamma_1) :
\]

\[
V_e (\beta_2) \tilde{V}_e (\gamma_1) = \rho \bar{\rho}_e w_\epsilon (\gamma_1 - \beta_2) : V_e (\beta_2) \tilde{V}_e (\gamma_1) :
\]

\[
\tilde{V}_e (\gamma_2) \tilde{V}_e (\gamma_1) = \bar{\rho}^2 \tilde{g}_\epsilon (\gamma_1 - \gamma_2) : \tilde{V}_e (\gamma_2) \tilde{V}_e (\gamma_1) :
\]

\[
\tilde{V}_e (\delta_2) \tilde{V}_e (\delta_1) = \bar{\rho}^2 \tilde{g}_\epsilon (\delta_1 - \delta_2) : \tilde{V}_e (\delta_2) \tilde{V}_e (\delta_1) :
\]

\[
\tilde{V}_e (\delta_2) \tilde{V}_e (\alpha_1) = \bar{\rho}^2 \tilde{g}_\epsilon (\alpha_1 - \delta_2) : \tilde{V}_e (\delta_2) \tilde{V}_e (\alpha_1) :
\]

where functions \( g_\epsilon (\beta) , \tilde{g}_\epsilon (\beta) , w_\epsilon (\beta) , \tilde{W}_e (\beta) , \mu_\epsilon (\beta) \) are the regularized version of \( g(\beta) , \tilde{g}(\beta) , w(\beta) \)

\( \tilde{g}(\beta) , \tilde{W}_e (\beta) , \mu_\epsilon (\beta) \) and using (35), one can move \( e^{\Psi} \) to the left of all vertex operators and obtain

\[
R_\epsilon (\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \beta_3, \beta_4, \gamma_3, \gamma_4, \gamma_5) = \frac{\langle 0 | e^{\Psi} \tilde{V}_e (\alpha_1) \tilde{V}_e (\delta_2) \tilde{V}_e (\delta_1) V_e (\beta_3) V_e (\beta_4) V_e (\gamma_3) V_e (\gamma_4) V_e (\gamma_5) V_e (\gamma_6) e^{\Psi} | 0 \rangle}{\langle 0 | e^{\Psi} e^{\Psi} | 0 \rangle}
\]
We then insert between

\[
\int e^{i\sum_{l=1}^{\infty} a_{-m} (X_m - Y_m)} |0\rangle,
\]

\[
X_m = \sum_{j=1}^{r} e^{-ima_j} - \sum_{l=1}^{n} e^{-ima_l} - \frac{sh \pi mc}{2} \sum_{l=1}^{p} e^{-ima_l} + \frac{sh \pi mc}{2} \sum_{l=1}^{k} e^{-ima_l},
\]

\[
Y_m = - \sum_{j=1}^{r} e^{ima_j} + \sum_{l=1}^{n} e^{ima_l} + \frac{sh \pi mc}{2} \sum_{l=1}^{p} e^{ima_l} + \frac{sh \pi mc}{2} \sum_{l=1}^{k} e^{ima_l}
\]

We then insert between \( e^{\Psi^+} \) and \( e^{\Psi^-} \) the completeness relation of the coherent states (39) and use the integration formula (40) [25]. As a result, we have

\[
I(\alpha_1...\alpha_k |\delta_1...\delta_p |\beta_1...\beta_n |\gamma_1...\gamma_r)
\]

\[
= \exp \left\{ \frac{1}{2} \sum_{m=1}^{\infty} \left[ - \frac{sh \pi mc}{2} + \frac{sh \pi mc}{2} \right] (X_m^2 + Y_m^2 - 2X_mY_m) + \frac{1}{2} - e^{-2\pi mc} \rho_m (X_m - Y_m) \right\}
\]

Substituting eq.(37) into (36), we find the form of regularized function \( R_\epsilon(\alpha_1...\alpha_k |\delta_1...\delta_p |\beta_1...\beta_n |\gamma_1...\gamma_r) \) is

\[
R_\epsilon(\alpha_1...\alpha_k |\delta_1...\delta_p |\beta_1...\beta_n |\gamma_1...\gamma_r)
\]

\[
= C_{\epsilon_1} C_{\epsilon_2} C_{\epsilon_3} \int e^{i\sum_{l=1}^{\infty} a_{-m} (X_m - Y_m)} |0\rangle,
\]

\[
X_m = \sum_{j=1}^{r} e^{-ima_j} - \sum_{l=1}^{n} e^{-ima_l} - \frac{sh \pi mc}{2} \sum_{l=1}^{p} e^{-ima_l} + \frac{sh \pi mc}{2} \sum_{l=1}^{k} e^{-ima_l},
\]

\[
Y_m = - \sum_{j=1}^{r} e^{ima_j} + \sum_{l=1}^{n} e^{ima_l} + \frac{sh \pi mc}{2} \sum_{l=1}^{p} e^{ima_l} + \frac{sh \pi mc}{2} \sum_{l=1}^{k} e^{ima_l}
\]
\[
\begin{align*}
&\times \left\{ \prod_{j=1}^{r} \prod_{l=1}^{p} H_e(\gamma_j - \delta_m) H_e(-\gamma_j - \delta_m) \right\} \left\{ \prod_{j=1}^{k} H_e^{-1}(\gamma_j - \alpha_n) H_e^{-1}(-\gamma_j - \alpha_n) \right\} \\
&\times \left\{ \prod_{j=1}^{n} f_e(\beta_j) \prod_{j<l} G_e(-\beta_j - \beta_l) G_e(\beta_j - \beta_l) \right\} \left\{ \prod_{j=1}^{n} U_e(\beta_j - \delta_l) U_e(-\beta_j - \delta_l) \right\} \\
&\times \left\{ \prod_{j=1}^{n} \prod_{l=1}^{k} U_e^{-1}(\beta_j - \alpha_l) U_e^{-1}(-\beta_j - \alpha_l) \right\} \left\{ \prod_{j=1}^{n} \tilde{F}_e(\delta_j) \prod_{j<l} \tilde{G}_e(-\delta_j - \delta_l) \tilde{G}_e(\delta_j - \delta_l) \right\} \\
&\times \left\{ \prod_{j=1}^{n} \prod_{l=1}^{k} \tilde{G}_e^{-1}(\delta_j - \alpha_l) \tilde{G}_e^{-1}(\delta_j - \alpha_l) \right\} \left\{ \prod_{j=1}^{n} \tilde{F}_e(\alpha_j) \prod_{j<l} \tilde{G}_e(-\alpha_j - \alpha_l) \tilde{G}_e(\alpha_j - \alpha_l) \right\} \\
&\times \left\{ \prod_{j=1}^{r} \prod_{l=1}^{p} F_e^{(0)}(\gamma_j) \prod_{j=1}^{n} F_e^{(0)}(\beta_j) \prod_{j=1}^{k} F_e^{(0)}(\delta_j) \prod_{j=1}^{k} F_e^{(0)}(\alpha_j) \right\},
\end{align*}
\]

where

\[
T_e(a_1, a_2, a_3, a_4, a_5|c_1, c_2, c_3, c_4|b_1, b_2, b_3)
= \exp \left\{ \sum_{m=1}^{\infty} \frac{\text{sha}_1 \text{mesh}_4 \text{mesh}_3 \text{mesh}_2 \text{mesh}_1 \text{mesh}_0 \text{mesh}_2}{\text{mesh}_1 \text{mesh}_2 \text{mesh}_3 \text{mesh}_4 \text{mesh}_5} e^{-b_3 m c} \right\}
\]

\[
\begin{align*}
F_e(\beta) &= \frac{\rho \gamma^L}{(-2\beta)T_e^{-1}(2, \frac{1 + \xi}{2}, \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1|1, \frac{\xi}{2}, 1, 1|0, \beta, 1)} \\
F_e^{-1}(\beta) &= \frac{\rho \gamma^L}{(-2\beta)T_e^{-1}(2, \frac{1 + \xi}{2}, \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1|1, \frac{\xi}{2}, 1, 1|0, \beta, 1)} \\
G_e(\beta) &= \frac{\rho \gamma^L}{(-2\beta)T_e^{-1}(2, \frac{1 + \xi}{2}, \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1|1, \frac{\xi}{2}, 1, 1|0, \beta, 1)} \\
G_e^{-1}(\beta) &= \frac{\rho \gamma^L}{(-2\beta)T_e^{-1}(2, \frac{1 + \xi}{2}, \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1|1, \frac{\xi}{2}, 1, 1|0, \beta, 1)} \\
H_e(\beta) &= \frac{\rho \gamma^L}{(-2\beta)T_e^{-1}(2, \frac{1 + \xi}{2}, \frac{1 + \xi}{2}, \frac{i\beta + \pi}{\pi}, 1|1, \frac{\xi}{2}, 1, 1|0, \beta, 1)}
\end{align*}
\]
\[ F^{(0)}_\epsilon (\beta) = T^2_\epsilon \left( \frac{2i\mu}{\pi} - 1 - \xi, \frac{i\beta + \pi}{\pi}, 1, 1, 1|\xi, 1, 1, 1|0, \beta, 1 \right) T^2_\epsilon \left( \frac{3}{2}, \frac{1 - \xi}{2}, \frac{i\beta + \pi}{\pi}, 1, 1, 1|0, \beta, 1 \right) \]
\[ T^2_\epsilon \left( \frac{2i\mu}{\pi} - 1 - \xi, \frac{i\beta + \pi}{\pi}, 1, 1, 1|\xi, 1, 1, 1|0, \beta, 1 \right) T^2_\epsilon \left( \frac{3}{2}, \frac{1 - \xi}{2}, \frac{i\beta + \pi}{\pi}, 1, 1, 1|0, \beta, 1 \right) \]

Taking the limit of \( \epsilon \to 0 \), one can find

\[ T_\epsilon(a_1, a_2, a_3, a_4, a_5|c_1, c_2, c_3, c_4|b_1, b_2, b_3) \to T(a_1, a_2, a_3, a_4, a_5|c_1, c_2, c_3, c_4|b_1, b_2, b_3), \]
\[ C_{c_1} \to C_1, \quad C_{c_2} \to C_2, \]

and some results which had been obtained by Lukyanov in Ref.[12]:

\[ \rho_\epsilon \to \rho, \quad \overline{\Omega}_\epsilon \to \overline{\Omega}, \quad \overline{\Omega}'_\epsilon \to \overline{\Omega}' \]
\[ g_\epsilon(\beta) \to g(\beta), \quad \overline{\Omega}_\epsilon(\beta) \to \overline{\Omega}(\beta), \quad w_\epsilon(\beta) \to w(\beta) \]
\[ \overline{\Omega}'_\epsilon(\beta) \to \overline{\Omega}'(\beta), \quad \overline{\Omega}_\epsilon(\gamma) \to \overline{\Omega}(\gamma), \quad \mu_\epsilon(\beta) \to \mu(\beta) \]

Therefore, we obtain function \( R(\alpha_1,...,\alpha_k|\delta_1,...,\delta_p|\beta_1,...,\beta_n|\gamma_1,...,\gamma_r) \) as in (33).

A.2

We here summarize formulas concerning coherent states of q-oscillator.

The coherent states \(|z\rangle\) and \(\langle z|\) in the Fock spaces \(\mathcal{H}\) and its dual \(\overline{\mathcal{H}}\) of the q-oscillator \(\{a_n\}\) in the section 4, are defined by

\[ |z\rangle = \exp\left\{ \sum_{m=1}^{\infty} \frac{\text{msh}_{\frac{\pi}{2}}\text{msh}_{\frac{\pi}{2}}}{\text{msh}_{\frac{\pi}{2}}\text{sh}_{\frac{\pi}{2}}\text{sh}_{\frac{\pi}{2}}\text{sh}_{\frac{\pi}{2}}\text{sh}_{\frac{\pi}{2}}\text{sh}_{\frac{\pi}{2}}} z_{n,a-n} \right\} |0\rangle, \]
\[ \langle z| = \langle 0| \exp\left\{ \sum_{m=1}^{\infty} \frac{\text{msh}_{\frac{\pi}{2}}\text{msh}_{\frac{\pi}{2}}}{\text{msh}_{\frac{\pi}{2}}\text{sh}_{\frac{\pi}{2}}\text{sh}_{\frac{\pi}{2}}\text{sh}_{\frac{\pi}{2}}\text{sh}_{\frac{\pi}{2}}\text{sh}_{\frac{\pi}{2}}} z_{n,a_n} \right\} \] (38)
The coherent states $\{\ket{z}\}$ (resp. $\bra{\overline{z}}$) form a complete basis in the Fock space $H$ (resp. $\overline{H}$). Namely it has the completeness relation

$$\text{id}_H = \int \prod_{m=1}^{\infty} \frac{m \text{sh} \frac{m \pi \epsilon}{2} dz_m d\overline{z}_m}{\text{sh} \frac{m \pi \epsilon}{2} \text{sh} m \pi \epsilon \text{sh} \frac{m \pi \epsilon (1+\xi)}{2}} e^{-\sum_{m=1}^{\infty} \frac{m \text{sh} \frac{m \pi \epsilon}{2}}{\text{sh} \frac{m \pi \epsilon}{2} \text{sh} m \pi \epsilon \text{sh} \frac{m \pi \epsilon (1+\xi)}{2}} |z_m|^2} \ket{z_m} \bra{\overline{z}_m}. \quad (39)$$

Here the integration is taken over the entire complex plane. In the proof, the following integration formula is used:

$$\int \prod_{m=1}^{\infty} \frac{m \text{sh} \frac{m \pi \epsilon}{2} dz_m d\overline{z}_m}{\text{sh} \frac{m \pi \epsilon}{2} \text{sh} m \pi \epsilon \text{sh} \frac{m \pi \epsilon (1+\xi)}{2}} \exp \left\{ -\frac{1}{2} \sum_{m=1}^{\infty} \frac{m \text{sh} \frac{m \pi \epsilon}{2}}{\text{sh} \frac{m \pi \epsilon}{2} \text{sh} m \pi \epsilon \text{sh} \frac{m \pi \epsilon (1+\xi)}{2}} (\overline{z}_m, z_m) A_m (\overline{z}_m, z_m)^t + \sum_{m=1}^{\infty} (\overline{z}_m, z_m) B_n \right\}
$$

$$= \prod_{m=1}^{\infty} (m \text{sh} \frac{m \pi \epsilon}{2} \text{sh} \frac{m \pi \epsilon (1+\xi)}{2})^{1/2} B_m A_m^{-1} B_m, \quad (40)$$

where $A_m$ are invertible constant $2 \times 2$ matrices and $B_m$ are constant $2$ component vectors.

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