CRITICAL THRESHOLDS IN 1D EULER EQUATIONS WITH
NONLOCAL FORCES

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Abstract. We study the critical thresholds for the compressible pressureless Euler equations with pairwise attractive or repulsive interaction forces and non-local alignment forces in velocity in one dimension. We provide a complete description for the critical threshold to the system without interaction forces leading to a sharp dichotomy condition between global in time existence or finite-time blow-up of strong solutions. When the interaction forces are considered, we also give a classification of the critical thresholds according to the different type of interaction forces. We also analyze conditions for global in time existence when the repulsion is modeled by the isothermal pressure law.

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1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

We are concerned with the following 1D system of pressureless Euler equations with non-local interaction and alignment forces

\begin{align}
\partial_t \rho + \partial_x (\rho u) &= 0, \quad x \in \mathbb{R}, \quad t > 0, \\
\partial_t u + u \partial_x u &= \int_{\mathbb{R}} \psi(x-y)(u(y,t) - u(x,t))\rho(y,t)dy - \partial_x K \ast \rho,
\end{align}

subject to initial density and velocity

\[(\rho(\cdot,t), u(\cdot,t))|_{t=0} = (\rho_0, u_0)\].

Since the total mass is conserved in time, we may assume, without loss of generality, that \(\rho\) is a probability density function, i.e., \(\|\rho(\cdot,t)\|_{L^1} = 1\).
The system involves two types of non-local forces arising in many different fields such as collective behavior patterns in mathematical biology, opinion dynamics, granular media and others. In the particular context of multi-agents interactions, the system \((1.1)\) arises as macroscopic descriptions for individual based models (IBMs) of the form

\[
\dot{x}_i = v_i, \quad \dot{v}_i = \frac{1}{N} \sum_{j=1}^{N} \psi(x_i - x_j)(v_j - v_i) - \frac{1}{N} \sum_{j=1}^{N} \partial x_i K(x_i - x_j),
\]

where the force consists of attractive-alignment-repulsive interactions, under a “three-zone” framework proposed in \([1, 17, 11]\). Starting from the basic system of IBMs, one can derive a kinetic description by BBGKY hierarchies or mean field limits, see \([10, 2, 4]\) and the references therein. The hydrodynamic equations of the form \((1.1)\) are obtained by taking moments on the kinetic equations and assuming a closure based on a monokinetic distribution, see \([7, 10, 3, 4]\) for details. These hydrodynamic equations lead to numerical solutions which share common features with the original IBMs systems such as flocks and mills patterns as demonstrated in \([5]\).

The first term on the right of \((1.1b)\) represents a non-local alignment, where \(\psi \in \mathcal{W}^{1,\infty}(\mathbb{R})\) is the influence function which is assumed symmetric and uniformly bounded

\[
0 \leq \psi_m \leq \psi(x) = \psi(-x) \leq \psi_M.
\]

The second term on the right of \((1.1b)\) represents attractive and/or repulsive forces, through a symmetric smooth enough interaction potential. We will start by assuming the regularity \(K \in \dot{\mathcal{W}}^{2,1}(\mathbb{R})\). If the potential is convex (resp. concave) in \(x\), the forces are attractive (resp. repulsive).

We begin our discussion with the case in which the particles are only driven by alignment. Setting the attraction/repulsion force \(K \equiv 0\), we arrive at a system of 1D mass and momentum Euler equations coupled with the alignment force,

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0, \\
\partial_t u + u \partial_x u &= \int_{\mathbb{R}} \psi(x - y)(u(y,t) - u(x,t))\rho(y,t)dy.
\end{align*}
\]

We refer \((1.2)\) as the Euler-Alignment system for short. It is realized as the hydrodynamic system of the Cucker-Smale flocking model \([8, 10]\).

The authors in \([8, 10]\) have recently shown that global regularity of Euler-Alignment system is determined by the initial configurations. They show that there are only two possible scenarios, depending on upper- and lower-thresholds \(\sigma_+ > \sigma_-\). If the initial data lie above the upper threshold in the sense that \(\partial_x u_0(x) > \sigma_+\) for all \(x \in \mathbb{R}\), then such initial data lead to global smooth solutions which must flock; on the other hand, if the initial data lie below the lower threshold \(\sigma_-\), namely, if there exists \(x \in \mathbb{R}\) such that \(\partial_x u_0(x) < \sigma_-\), then such supercritical initial data lead to finite time blowup of solutions. Our first result, investigated in Section 2, refines this critical threshold phenomenon and quantifies the precise threshold in this case of Euler-Alignment dynamics.

**Theorem 1.1.** Consider Euler-alignment system \((1.2)\).

- **[Subcritical region].** If \(\partial_x u_0(x) \geq -\psi \star \rho_0(x)\) for all \(x \in \mathbb{R}\), then the system has a global classical solution, \((\rho, u) \in C(\mathbb{R}^+; L^\infty(\mathbb{R})) \times C(\mathbb{R}^+; \dot{W}^{1,\infty}(\mathbb{R}))\).

- **[Supercritical region].** If there exists an \(x\) such that \(\partial_x u_0(x) < -\psi \star \rho_0(x)\), then the solution blows up in a finite time.
Theorem 1.2. Consider Euler-Poisson-Alignment system (1.4). If \( \sigma_- < \psi \ast \rho \ast \sigma_+ \), where \( \sigma_\pm \) are the thresholds function obtained in [15]. Hence, Theorem 1.1 provides larger subcritical and supercritical regions, and the result is sharper.

Next, we deal with the case of \( K \neq 0 \), i.e., there is an additional force through the attractive-repulsive potential \( K \). We begin by analyzing the behavior of the solutions for the particular potential \( K(x) = \frac{k}{2}|x|, k \in \mathbb{R} \). This special potential is the 1D Newtonian potential, where \( \partial^2_x K = k\delta_0 \) is the Dirac delta function. When there is no alignment force, \( \psi \equiv 0 \), the system coincides with the 1D pressureless Euler-Poisson equations

\[
\partial_t \rho + \partial_x (\rho u) = 0, \\
\partial_t u + u \partial_x u = -k \partial_x \phi, \\
\phi = \rho.
\] (1.3)

Critical thresholds of the system (1.3) were studied in [9], followed by a series of extensions on multi-dimensional systems [14, 15, 16]. The result in [9] shows that the system has finite time blow-up in the attractive case, \( k > 0 \), while in the repulsive force, \( k < 0 \), there exists a critical threshold.

With the alignment force (\( \psi \neq 0 \)), we can naturally expect that the solution tends to be smoother than for Euler-Poisson. In Section 3.1, we investigate the critical threshold phenomenon for the following Euler-Poisson-Alignment system:

\[
\partial_t \rho + \partial_x (\rho u) = 0, \\
\partial_t u + u \partial_x u = -k \partial_x \phi + \int_{\mathbb{R}} \psi(x-y)(u(y,t) - u(x,t))\rho(y,t)dy, \\
\phi = \rho.
\] (1.4)

**Theorem 1.2.** Consider Euler-Poisson-Alignment system (1.4).

1. **Attractive Poisson forcing** \( k > 0 \). An unconditional finite-time blow up of the solution for all initial configurations.
2. **Repulsive Poisson forcing** \( k < 0 \). We distinguish between two cases:
   - **[Subcritical region]**. If \( \partial_x u_0(x) > -\psi \ast \rho_0(x) + \sigma_+(x) \) for all \( x \in \mathbb{R} \), then the system has a global classical solution. Here, \( \sigma_+(x) = 0 \) whenever \( \rho_0(x) = 0 \) and elsewhere \( \sigma_+(x) \) is the (unique) negative root of the equation
     \[
     \rho_0^{-1}(x) - \frac{1}{\psi^2_M} \left( k + \psi_M \sigma_+(x)/\rho_0(x) - ke^{\psi_M \sigma_+(x)/k \rho_0(x)} \right) = 0, \quad \rho_0(x) > 0.
     \]
   - **[Supercritical region]**. If there exists an \( x \) such that
     \[
     \partial_x u_0(x) < -\psi \ast \rho_0(x) + \sigma_-(x), \quad \sigma_-(x) := -\sqrt{-2k\rho_0(x)},
     \]
     then the solution blows up in a finite time.

**Remarks.**

1. In the attractive case, the blowup is “unconditional”, independent of the choice of initial configuration. It indicates that Poisson force dominates the alignment force.
2. In the repulsive case, alignment force enhances regularity. Indeed, we have a larger subcritical region than the case of \( K \equiv 0 \), as \( \sigma_+(x) < 0 \).
3. If \( \psi \) has a positive lower bound \( \psi_m > 0 \), we can obtain a better supercritical region for the repulsive case. In particular, the threshold condition is sharp when \( \psi \) is a constant. Consult Remark 3.1 below for details.
In Section 3.2 we consider general potentials with enough smoothness, $K \in \dot{W}^{2,\infty}(\mathbb{R})$. We show the following threshold conditions

**Theorem 1.3.** Consider the system (1.1) with alignment and attractive-repulsive forces.

1. **Attractive case**, $\partial_x^2 K < 0$. If there exists an $x$ such that $\partial_x u_0(x) < -\psi \ast \rho_0(x)$, then the solution blows up in a finite time.
2. **Repulsive case**, $\partial_x^2 K > 0$. We distinguish between two cases.
   - **[Subcritical region]**. If $\partial_x u_0(x) \geq -\psi \ast \rho_0(x)$ for all $x \in \mathbb{R}$, then the system has a global classical solution.
   - **[Supercritical region]**. If there exists an $x$ such that $\partial_x u_0(x) < -\psi \ast \rho_0(x) - \sqrt{\|\partial_x^2 K\|_{L^\infty}}$

then the solution blows up in finite time.

**Remark.** If $\psi$ has a positive lower bound $\psi_m > 0$, then we have the following refined critical thresholds irrespective of the sign of interaction force. Consult Remark 3.2.

Finally, we consider Euler-Alignment system with pressure $p(\rho) = A\rho$ instead of the interaction force $\partial_x K \ast \rho$ as the ingredient to model repulsion in the system:

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0, \quad x \in \mathbb{R}, \quad t > 0, \\
\partial_t u + u\partial_x u + \rho^{-1}\partial_x p(\rho) &= \int_{\mathbb{R}} \psi(x-y)(u(y,t) - u(x,t))\rho(y,t) \, dy.
\end{align*}
\] (1.5)

The presence of the pressure destroys the original characteristic structure, which brings additional difficulties in obtaining critical thresholds. Without the alignment force $\psi \equiv 0$, it is well known that the solutions of (1.5) generically develop singularities in finite time no matter how smooth initial data are. We refer to [6] and the references therein for a general survey of the compressible Euler equations. In Section 4 we are concerned with the system (1.5), and we find a subcritical region of initial conditions leading to global solvability of the system (1.5).

**Theorem 1.4.** Consider (1.5) with isothermal pressure $p(\rho) = A\rho$. Suppose $0 < \psi_M < 2\psi_m$, and $A$ is small enough (see condition (4.5) for detailed description). Then there exists $M_1 < M_2$ (specified in (4.6) below) such that if initially,

$\partial_x u_0 \pm \sqrt{A}\partial_x \ln \rho_0 + \psi \ast \rho_0 \in [M_1, M_2]$,

then the system has a global classical solution such that $(\ln \rho, u) \in C(\mathbb{R}^+; \dot{W}^{1,\infty}(\mathbb{R}))$.

**Remark.** System (1.5) with isothermal pressure is rigorously derived in [12] as the hydrodynamic limit of the kinetic Cucker-Smale system, with the assumptions that $\partial_x u$ and $\partial_x \ln \rho$ are bounded. Therefore, a direct consequence of Theorem 1.4 under subcritical initial data, the solution $(\rho, u)$ of system (1.5) is the hydrodynamic limit of the kinetic Cucker-Smale system in all time.

We postpone to the Appendix A the proof of global regularity for the system (1.1) and (1.5) with smooth initial data and suitable assumptions on the influence function $\psi$ and the potential function $K$. These results render fully rigorous the statements of global in time existence of strong solutions in the subcritical cases for all the critical thresholds found in this work.
2. Euler-Alignment system

In this section, we study the one dimensional Euler-Alignment system \((1.2)\), without taking into account the interaction potential \(K\).

Differentiate the momentum equation in \((1.2)\) with respect to \(x\), then \(\psi := \partial_x u\) satisfies

\[
\partial_t \rho + u \partial_x \rho = -\rho v,
\]

\[
\partial_t v + u \partial_x v + v^2 = -u \int_R \partial_x \psi(x-y)\rho(y)dy - \int_R \psi(x-y)\partial_t \rho(y)dy - v \int_R \psi(x-y)\rho(y)dy,
\]

where we used

\[
\int_R \partial_x \psi(x-y)(u(y) - u(x))\rho(y)dy = -u(x)\int_R \partial_x \psi(x-y)\rho(y)dy - \int_R \psi(x-y)\partial_t \rho(y)dy.
\]

Here the symmetry assumption of the influence function \(\psi\) is essential. Consider the characteristic flow \(x(a, t)\) associated to the velocity field \(u\) defined by

\[
\frac{d}{dt} x(a, t) = u(x(a, t), t), \quad \text{with} \quad x(a, 0) = a.
\]

Then along this characteristic flow we find

\[
\partial_t \rho(x(a, t), t) = -\rho(x(a, t), t)v(x(a, t), t),
\]

\[
\partial_t (v(x(a, t), t) + (\psi \ast \rho)(x(a, t), t)) = -v^2(x(a, t), t) - v(x(a, t), t)(\psi \ast \rho)(x(a, t), t).
\]

Set \(d = v + \psi \ast \rho\). Then we again rewrite the above system:

\[
\rho' = -\rho v = -\rho(d - \psi \ast \rho), \quad (2.1a)
\]

\[
d' = -v(v + \psi \ast \rho) = -d(d - \psi \ast \rho), \quad (2.1b)
\]

where \(^'\) denotes the time derivative along the characteristic flow \(x(a, t)\).

**Proposition 2.1.** Consider the equation \((2.1)\). Then we have

- If \(d_0 < 0\), then \(d \to -\infty\) in finite time.
- If \(d_0 = 0\), then \(d(t) = 0\) for all \(t \geq 0\).
- If \(d_0 > 0\), then \(d(t)\) remains bounded for all time, and \(d(t) \to \psi \ast \rho(t)\) as \(t \to \infty\).

**Proof.** First case \(d_0 < 0\). It is clear to have that if \(d_0 < 0\), then \(d(t) \leq 0\) for all \(t \geq 0\). Then it follows from \((2.1b)\) that \(d' \leq -d^2\) which in turn yields \(d(t) \leq \frac{d_0}{t + d_0}\). Hence, \(d(t)\) blows up at \(t^* \leq -d_0\).

The second case \(d_0 = 0\) is trivial.

Finally the third case \(d_0 > 0\). Note that if \(d(t) \in (0, \psi \ast \rho(t))\), then \(d'(t) > 0\) thus \(d(t)\) is increasing up to \(\psi \ast \rho(t)\). On the other hand, if \(d(t) > \psi \ast \rho(t)\), then \(d(t)\) is decreasing up to \(\psi \ast \rho(t)\). Note that

\[
\|\psi \ast \rho\|_{L^\infty} \leq \|\psi\|_{L^\infty} \|\rho\|_{L^1} = \psi_M < \infty.
\]

Thus \(\psi \ast \rho\) is bounded, and we conclude with the third statement of the proposition. \(\square\)

We can also trace the dynamics of \(\rho\) along the characteristic flow from \((2.1a)\).

- If \(\rho_0 = 0\), clearly \(\rho(t) = 0\) for all \(t \geq 0\).

---

\(^1\)We suppress the time dependence whenever it is clear from the context.
the value of $\beta$ and its dynamics along particle path is easily found to be

$$\beta' = \frac{d'\rho - dp'\rho}{\rho^2} = \frac{1}{\rho^2} \left(-d(d - \psi \ast \rho)\rho + dp(d - \psi \ast \rho)\right) = 0.$$ 

Thus $\beta(t) = \beta_0$ for all $t \geq 0$ and $\rho$ remains proportional to $d$ along each path, $\rho(t)\beta_0 = d(t)$,

As a conclusion we have the following complete description of the critical threshold for system (1.2), and Theorem 1.1 then follows as a direct consequence.

**Corollary 2.1.** Consider the Euler-Alignment system (1.2). Then we have

- If $\partial_x u_0(a) < -\psi \ast \rho_0(a)$, then $\partial_x u(x(a), t) \to -\infty$. Moreover, if $\rho_0(a) > 0$, $\rho(x(a), t) \to +\infty$ in a finite time.
- If $\partial_x u(a) = -\psi \ast \rho(a)$, then $\partial_x u(x(a), t) = -\psi \ast \rho(x(a), t)$ for all time $t \geq 0$.
- If $\partial_x u_0(a) > -\psi \ast \rho_0(a)$, then $\partial_x u(x(a), t)$ and $\rho(x(a), t)$ remains uniformly bounded for all $t \geq 0$, and furthermore $\partial_x u(x(a), t) \to 0$ as $t \to +\infty$.

3. **Euler-Alignment system with attractive-repulsive potentials.**

In this section, we study the main system (1.1) of Euler equations with alignment, and attractive-repulsive forces.

3.1. A special potential: Euler-Poisson-Alignment system. We first consider Euler-Alignment system (1.1) with a special Newtonian potential:

$$K(x) = \frac{k|x|^2}{2}.$$ 

As $\partial_x^2 K = k\delta_0$, $k(\partial_x K \ast \rho)$ is a Poisson force which is attractive if $k > 0$, or repulsive if $k < 0$. We recall the corresponding Poisson-Alignment system (1.4):

$$\partial_t \rho + \partial_x (\rho u) = 0,$$

$$\partial_t u + u\partial_x u = -k\partial_x \phi + \int_{\mathbb{R}} (\psi(x-y) u(y, t) - u(x, t)) \rho(y, t) dy, \quad \partial_x^2 \phi = \rho.$$

By using similar arguments to Section 2 we find

$$\rho' = -\rho(d - \psi \ast \rho), \quad (3.1a)$$

$$d' = -d(d - \psi \ast \rho) - kp. \quad (3.1b)$$

In the case of vacuum $\rho_0 = 0$, the dynamics of $d$ (3.1b) are the same as for the Euler-Alignment case (2.1b). Thus Proposition 2.1 holds.

We now focus on the case $\rho_0 > 0$. Set $\beta = d/\rho$, then we can find

$$\beta' = -k, \quad \text{i.e.,} \quad \beta(t) = \beta_0 - kt. \quad (3.2)$$

This again yields that

$$\rho' = -\rho(d - \psi \ast \rho) = -\rho(\beta_0 - kt) - \psi \ast \rho = -\beta_0 \rho^2 + k t \rho^2 + \rho(\psi \ast \rho). \quad (3.3)$$

Then we obtain the explicit form of solution $\rho$ from (3.3) that

$$\rho^{-1}(t) = e^{-\int_0^t (\psi \ast \rho) ds} \left(\rho_0^{-1} + \int_0^t (\beta_0 - ks)e^{\int_0^s (\psi \ast \rho) dr} ds\right). \quad (3.4)$$

For the attractive case $k > 0$, $\beta_0 - ks$ becomes negative in finite time, irrespective of the value of $\beta_0$. The right hand side decreases to 0 in finite time, resulting a blowup of
\(\rho\). Hence, we have the following lemma, and this directly implies the result in part (1) of Theorem 1.2.

**Lemma 3.1. [Blowup with attractive potential]**. If \(k > 0\), then \(\rho(t) \to +\infty\) in finite time.

For the repulsive case \(k < 0\), critical thresholds are expected since a similar phenomenon is proved for both Euler-Poisson [9] and Euler-Alignment (Section 2) systems. We start with the following rough estimate.

**Lemma 3.2. [Rough subcritical region with repulsive potential]**. If \(k < 0\) and \(\beta_0 \geq 0\), then \(\rho(t)\) remains bounded for all time \(t \geq 0\).

**Proof.** By our assumption on \(k\) and \(\beta_0\), we have \(\beta_0 - kt \geq 0\) for all \(t \geq 0\). Also,

\[
0 = \psi_m t \leq \int_0^t (\psi * \rho) ds \leq \psi_M t. \tag{3.5}
\]

Hence, the following lower bound,

\[
\rho - \frac{1}{\psi_M} > 0,
\]

follows directly from (3.4). \(\square\)

We notice that lemma 3.2 provides the same subcritical region as in the Euler-Alignment system, namely, \(\partial_x u_0 + \psi * \rho_0 > 0\). With the additional repulsive force, however, the subcritical region is expected to be larger. Indeed, we turn to derive a refined estimate which yields the larger subcritical region stated in Theorem 1.2.

Consider the case when \(\beta_0 < 0\). To bound \(\rho\), it is enough check when the following value is zero or not:

\[
\rho_0^{-1} + \int_0^t (\beta_0 - ks)e^{\int_0^s (\psi * \rho) ds} ds > 0.
\]

Since \(\beta_0 - ks \leq 0\) for \(s \leq \frac{\beta_0}{k}\), we rewrite (3.6) as

\[
\rho_0^{-1} + \int_0^t (\beta_0 - ks)e^{\int_0^s (\psi * \rho) ds} ds + \int_0^t (\beta_0 - ks)e^{\psi_M s} ds > 0. \tag{3.7}
\]

Thus if there is no blow-up of solutions until \(t \leq \frac{\beta_0}{k}\) then it holds for all times, due to the positivity of the last term in (3.7).

**Proposition 3.1.** Consider the dynamics (3.3) with \(k < 0\) and \(\beta_0 < 0\). Then \(\rho(\cdot, t)\) remains bounded if and only if

\[
\rho_0^{-1} + \int_0^t (\beta_0 - ks)e^{\int_0^s (\psi * \rho) ds} ds > 0. \tag{3.8}
\]

In order to derive a sufficient condition for (3.8) determined by the initial conditions, we use (3.5) to get

\[
\int_0^t (\beta_0 - ks)e^{\int_0^s (\psi * \rho) ds} ds \geq \int_0^t (\beta_0 - ks)e^{\psi_M s} ds \geq -\frac{\beta_0}{\psi_M} + k + \frac{k}{\psi_M} \left( e^{\psi_M \beta_0 / k} - 1 \right) = -\frac{1}{\psi_M^2} \left( k + \psi_M \beta_0 - k e^{\psi_M \beta_0 / k} \right). \tag{3.10}
\]

Thus, we deduce that if initially

\[
\rho_0^{-1} - \frac{1}{\psi_M^2} \left( k + \psi_M \beta_0 - k e^{\psi_M \beta_0 / k} \right) > 0,
\]

then there is no finite-time blow-up of classical solutions. Note that the left hand side is increasing with respect to \(\beta_0\) if \(\beta_0 < 0\). This together with Lemma 3.2 implies the subcritical region in part (2) of Theorem 1.2.
Next, we estimate the blow-up criterion of solutions. According to Proposition 3.1, we shall find a sufficient condition of $d_0$ that makes

$$\rho_0^{-1} + \int_0^{\rho_0} (\beta_0 - ks) e^{\int_0^t (\psi \ast \rho) ds} dt \leq 0.$$  

Since

$$\int_0^{\rho_0} (\beta_0 - ks) e^{\int_0^t (\psi \ast \rho) ds} dt \leq \int_0^{\rho_0} (\beta_0 - ks) dt = \frac{1}{2} \beta_0^2,$$

we obtain $\rho_0^{-1} + \beta_0^2 / 2k \leq 0$, and this holds if $d_0 \leq -\sqrt{-2k\rho_0}$. It concludes that if $d_0 \leq -\sqrt{-2k\rho_0}$ then there exists $t_*$ such that $\rho(t) \to +\infty$ until $t \leq t_*$. 

**Remark 3.1.** If $\psi$ has a positive lower bound, i.e., $\psi_m > 0$, a better bound can be obtained as follows:

$$\int_0^{\rho_0} (\beta_0 - ks) e^{\int_0^t (\psi \ast \rho) ds} dt \leq \int_0^{\rho_0} (\beta_0 - ks) e^{\psi_m s} ds = \frac{1}{\psi_m^2} \left( k + \psi_m \beta_0 - ke^{\psi_m \beta_0 / k} \right).$$

Therefore, we arrive at a refined supercritical region, where $\sigma_-$ in Theorem 1.2 can be redefined as

$$\rho_0^{-1}(x) = \frac{1}{\psi_m^2} \left( k + \psi_m \sigma_-(x) / \rho_0(x) - ke^{\psi_m \sigma_-(x) / \rho_0(x)} \right) = 0,$$

for $\rho_0(x) > 0$, and $\sigma_-(x) = 0$ for $\rho_0(x) = 0$. In particular when $\psi$ is a constant, $\sigma_+ = \sigma_-$, the two thresholds matches and the results are sharp.

It follows from (3.2) that if $\rho(t)$ blows up in finite time, then $d(t)$ is also blowing up in finite-time. Similarly, if $\rho(t)$ remains bounded, then $d(t)$ remains bounded as well. As $|\psi \ast \rho| \leq \psi_M$, $\rho(t)$ and $\partial_x u(t)$ blow up simultaneously, concluding the proof of Theorem 1.2.

### 3.2. Euler-Alignment with general attractive-repulsive potentials

In this part, we consider Euler-Alignment system with general attractive-repulsive potentials:

$$\partial_t \rho + \partial_x (\rho u) = 0,$$

$$\partial_t u + u \partial_x u = \int_{\mathbb{R}} \psi(x - y)(u(y, t) - u(x, t)) \rho(y, t) dy + \partial_x K \ast \rho.$$

By using the similar arguments in Section 2, we find

$$\rho' = -\rho(d - \psi \ast \rho),$$

$$d' = -d(d - \psi \ast \rho) + \partial_x^2 K \ast \rho. \quad (3.9)$$

For this system, we can classify the initial configurations that leading to the global regularity or the finite-time breakdown, when $\partial_x^2 K$ is bounded.

**Proposition 3.2.** Consider the system (3.9). Then the following holds.

- **[Attractive case $\partial_x^2 K > 0$.]** If $d_0 < 0$, then $d(t) \to -\infty$ in finite time.
- **[Repulsive case $\partial_x^2 K < 0$.]** If $d_0 \geq 0$, then $d(t)$ remains uniformly bounded for all time $t \geq 0$. On the other hand, if $d_0 < -\sqrt{\|\partial_x^2 K\|_{L^\infty}}$, then $d(t) \to -\infty$ in finite time.

**Proof.** We begin with the attractive case ($\partial_x^2 K \geq 0$). In this case, we find from (1.1) that

$$\rho' = -\rho(d - \psi \ast \rho),$$

$$d' \leq -d(d - \psi \ast \rho). \quad (3.10)$$
Then one can use the comparison principle for the above system \((3.10)\) with the system \((2.1)\) to obtain
\[
d(t) \to -\infty \quad \text{in finite time} \quad \text{if} \quad d_0 < 0.
\]
We turn to the repulsive case, \((\partial_x^2 K < 0)\). We have
\[
d' = -d^2 + (\psi \ast \rho)d + \partial_x^2 K \ast \rho.
\]
To obtain a global bound on \(d\), we estimate when \(d \geq 0\),
\[
d' \geq -d^2 + \psi \ast \rho d.
\]
Thus \(d(t) \geq 0\) if \(d_0 \geq 0\) due to the comparison principle with the system \((2.1)\). Moreover, we can also obtain the upper bound when \(d > 0\).
\[
d' \leq -d^2 + \psi_M d + B = -(d - d_1^*)(d - d_2^*), \quad B := \|\partial_x^2 K\|_{L^\infty},
\]
where
\[
d_1^* := \frac{\psi_M - \sqrt{\psi_M^2 + 4B}}{2} \quad \text{and} \quad d_2^* := \frac{\psi_M + \sqrt{\psi_M^2 + 4B}}{2} > 0.
\]
It implies \(d' < 0\) for \(d > d_2^*\), and this deduces \(d\) has an upper bound. Hence we have the global boundedness of \(d\).

On the other hand, for \(d < 0\), the upper bound is given as
\[
d' \leq -d^2 + B = -(d - \sqrt{B}) (d + \sqrt{B}),
\]
If \(d_0 < -\sqrt{B}\), through the comparison principle, it is easy to see that \(d(t) \to -\infty\) in finite time. \(\square\)

Collecting all characteristic flows together, we deduce Theorem 1.3.

We can also have more refined estimates by using the same argument as in Proposition 3.2 when the influence function \(\psi\) is bounded from below by \(\psi_m > 0\). In this case, we can treat the case of combined attractive and repulsive forces.

**Corollary 3.1.** Consider the system \((1.1)\) with the nonlocal interaction force \(K \in \dot{W}^{2,\infty}(\mathbb{R})\). Suppose that the influence function \(\psi\) satisfies \(\psi(x) \geq \psi_m > 0\). If the initial slope \(u_0'\) is not “too negative” in the sense that
\[
\psi_m^2 \geq 4B \quad \text{and} \quad d_0 \geq -\frac{\psi_m + \sqrt{\psi_m^2 - 4B}}{2},
\]
then \(d(t)\) is bounded for all time \(t \geq 0\). On the other hand, if \(u_0'\) is “too negative” in the sense that
\[
d_0 < -\frac{\psi_m - \sqrt{\psi_m^2 + 4B}}{2},
\]
then \(d(t) \to -\infty\) in a finite time.

**Remark 3.2.** Corollary 3.1 implies that if the influence function \(\psi\) is bounded from below and it is sufficiently large, then we have better threshold conditions:

- **[Subcritical region].** If \(\psi_m^2 \geq 4B\) and
\[
\partial_x u_0(x) \geq -\psi \ast \rho_0(x) - \frac{1}{2} \left(\psi_m + \sqrt{\psi_m^2 - 4B}\right)
\]
for all \(x \in \mathbb{R}\), then the system has a global classical solution.
• [Supercritical region]. If there exists an \( x \) such that
\[
\partial_x u_0(x) < -\psi \ast \rho_0(x) + \frac{1}{2} \left( \psi_m - \sqrt{\psi_M^2 + 4B} \right),
\]
then the solution blows up in finite time.

4. Critical thresholds in Euler-Alignment with pressure

In this section, we discuss the Euler-Alignment system with pressure \([1.5]\). Here, we recall the system
\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) &= 0, \quad x \in \mathbb{R}, \quad t > 0, \\
\partial_t u + u \partial_x u + \rho^{-1} \partial_x p(\rho) &= \int_{\mathbb{R}} \psi(x-y)(u(y,t) - u(x,t)) \rho(y,t) dy.
\end{align*}
\]
(4.1)
The pressure is usually modeled through a power law \( p(\rho) = A \rho^\gamma \), with \( A \geq 0 \) and \( \gamma \geq 1 \).

In particular, if \( A = 0 \), we recover the pressure-less system \([1.2]\).

Our goal is to obtain a subcritical region of initial conditions such that the solution of \((4.1)\) is smooth. To this end, we follow the argument of \([19]\), where Euler-Poisson equations with pressure is discussed.

Rewrite \((4.1)\) as the following system
\[
\begin{align*}
\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u \\ A \gamma \rho^{\gamma-2} \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_x &= \begin{pmatrix} 0 \\ -\int_{\mathbb{R}} \psi(x-y)(u(y,t) - u(x,t)) \rho(y,t) dy \end{pmatrix}.
\end{align*}
\]
(4.2)
We decouple the system by diagonalizing the matrix in \((4.2)\). It yields
\[
\begin{align*}
R_t + \lambda R_x &= \int_{\mathbb{R}} \psi(x-y)(u(y,t) - u(x,t)) \rho(y,t) dy, \\
S_t + \mu S_x &= \int_{\mathbb{R}} \psi(x-y)(u(y,t) - u(x,t)) \rho(y,t) dy,
\end{align*}
\]
where \( \lambda, \mu \) are eigenvalues of the matrix
\[
\lambda := u - \sqrt{A \gamma \rho^{(\gamma-1)/2}}, \quad \mu := u + \sqrt{A \gamma \rho^{(\gamma-1)/2}},
\]
and \( R, S \) are the corresponding Riemann invariants
\[
R = \begin{cases} u - \frac{2\sqrt{A \gamma}}{\gamma-1} \rho^{(\gamma-1)/2} & \gamma \neq 1 \\
\frac{1}{u - \sqrt{A \gamma}} \rho^{(\gamma-1)/2} & \gamma = 1 \end{cases}, \quad S = \begin{cases} u + \frac{2\sqrt{A \gamma}}{\gamma-1} \rho^{(\gamma-1)/2} & \gamma \neq 1 \\
\frac{1}{u + \sqrt{A \gamma}} \rho^{(\gamma-1)/2} & \gamma = 1 \end{cases}.
\]
Let us denote material derivatives \( ' = \partial_t + \lambda \partial_x \) and \( ' = \partial_t + \mu \partial_x \) along two particle paths. We derive the dynamics of \( R_x \) and \( S_x \) using the same procedure as in \([19]\):
\[
\begin{align*}
R_x' + \frac{1 + \theta}{2} R_x^2 + \frac{1 - \theta}{2} R_x S_x &= \partial_x \int_{\mathbb{R}} \psi(x-y)(u(y,t) - u(x,t)) \rho(y,t) dy, \\
S_x' + \frac{1 + \theta}{2} S_x^2 + \frac{1 - \theta}{2} R_x S_x &= \partial_x \int_{\mathbb{R}} \psi(x-y)(u(y,t) - u(x,t)) \rho(y,t) dy.
\end{align*}
\]
(4.3)
where \( \theta = \frac{\gamma-1}{2} \).

We treat the right hand side of the system similarly as the pressure-less system. Define
\[
\begin{align*}
r &= R_x + \psi \ast \rho \quad \text{and} \quad s = S_x + \psi \ast \rho.
\end{align*}
\]
The paired equations (4.3) can be written in terms of \((r, s)\).

\[
\begin{align*}
  r' + \frac{1 + \theta}{2} r^2 + \frac{1 - \theta}{2} rs &= \left( \frac{1 + \theta}{2} r - \frac{\theta}{2} s \right) (\psi \ast \rho) - \sqrt{A \gamma} \rho^\theta (\psi_x \ast \rho), \\
  s' + \frac{1 + \theta}{2} s^2 + \frac{1 - \theta}{2} rs &= \left( \frac{1 + \theta}{2} s - \frac{\theta}{2} r \right) (\psi \ast \rho) + \sqrt{A \gamma} \rho^\theta (\psi_x \ast \rho).
\end{align*}
\]

In particular, we study the isothermal case, where \(\gamma = 1\) and \(\theta = 0\). In this case, the dynamics is relatively easier.

\[
\begin{align*}
  r' &= -\frac{1}{2} r^2 - \frac{1}{2} rs + (\psi \ast \rho) r - \sqrt{A} (\psi_x \ast \rho), \\
  s' &= -\frac{1}{2} s^2 - \frac{1}{2} rs + (\psi \ast \rho) s + \sqrt{A} (\psi_x \ast \rho).
\end{align*}
\]

(4.4)

The following bounds are easily obtained for the nonlocal terms

\[
\psi_m \leq \psi \ast \rho \leq \psi_M, \quad \text{and} \quad |\sqrt{A} (\psi_x \ast \rho)| \leq \sqrt{\psi}_{\text{Lip}} =: C.
\]

Proposition 4.1. [Invariant region]. Suppose the influence function and the pressure satisfy

\[
\psi_M < 2 \psi_m - \left( 2 \sqrt{2} C + \frac{C}{2 \psi_m - 2 \sqrt{2} C} \right). \tag{4.5}
\]

Then, there exists an invariant region \([M_1, M_2]\), given by

\[
\begin{align*}
  M_1 &= 2 (\psi_M - \psi_m) + 2 \sqrt{2} C + \frac{2 C}{2 \psi_m - 2 \sqrt{2} C}, \\
  M_2 &= 2 \psi_m - 2 \sqrt{2} C,
\end{align*}
\]

(4.6)

such that if \(r_0, s_0 \in [M_1, M_2]\), then along their own characteristics, \(r(t), s(t) \in [M_1, M_2]\) for all \(t \geq 0\).

Remark. The condition \((4.5)\) requires \(\psi_m > 0\). For any \(\psi\) such that \(\psi_M < 2 \psi_m\), we can always find small \(C\) where the condition is satisfied. In particular, if \(\psi \equiv 1\), then \(\psi_m = \psi_M\) and \(\sqrt{\psi}_{\text{Lip}} = 0\). Therefore, condition \((4.5)\) is satisfied, and the invariant region is \([0, 2]\).

Proof. It suffices to prove that \(r\) can not exit the invariant region, namely, \(r' \geq 0\) when \(r = M_1\), and \(r' \leq 0\) when \(r = M_2\). The dynamic of \(s\) can be treated with the same argument.

We start with \(r = M_1 \geq 0\). From \((4.4a)\), we have the following bound on \(r'\):

\[
r' \geq -\frac{1}{2} r^2 + \left( \psi_m - \frac{1}{2} M_2 \right) r - C.
\]

The right hand side is a quadratic form of \(r\) which is non-negative under conditions

\[
\psi_m - \frac{1}{2} M_2 \geq \sqrt{2} C, \tag{4.7}
\]

and

\[
M_1 \geq \psi_m - \frac{1}{2} M_2 - \sqrt{\left( \psi_m - \frac{1}{2} M_2 \right)^2 - 2C}. \tag{4.8}
\]

Similarly, for \(r = M_2\), we use the upper bound

\[
r' \leq -\frac{1}{2} r^2 + \left( \psi_M - \frac{1}{2} M_1 \right) r + C.
\]
Hence, \( r^3 \leq 0 \) if

\[
M_2 \geq \psi_M - \frac{1}{2} M_1 - \sqrt{\left( \psi_M - \frac{1}{2} M_1 \right)^2 + 2C}. \tag{4.9}
\]

To summarize, \( r \) stays in the invariant region \([M_1, M_2]\) for all time if (4.7)-(4.9) are satisfied. In particular, set (4.7) and (4.9) to be equal, we obtain the bounds (4.6). It is easy to check that (4.8) holds as well with the choice of \( M_1, M_2 \).

Since the boundedness of \( r \) and \( s \) imply the boundedness of \( \partial_x u \) and \( \partial_x \ln \rho \), we conclude with Theorem A.1.

**Appendix A. Global regularity**

In this part, we consider smoother subcritical initial data, and prove that initial regularity persists globally in time, under suitable assumptions on the influence function \( \psi \) and the interaction potential \( K \).

**A.1. Pressureless Euler equations with nonlocal forces.** We start with our main system (1.1).

**Theorem A.1.** Let \( s \geq 0 \) be an integer. Consider system (1.1) with smooth influence function \( \psi \) satisfying

\( \psi \in L^1(\mathbb{R}) + \text{const}, \) \quad and \quad \( x\partial_x \psi \in L^1(\mathbb{R}), \)

and potential \( K \) such that

\( \partial_x^2 K \in L^1(\mathbb{R}). \) \tag{A.1}

Suppose the initial data \( (\rho_0, u_0) \) lie in the subcritical region, and satisfy \( \rho_0, \partial_x u_0 \in H^s(\mathbb{R}) \). Then, there exists a unique global solution \( (\rho, u) \) such that

\( \rho \in C([0, T]; H^s(\mathbb{R})) \) \quad and \quad \( \partial_x u \in C([0, T]; H^s(\mathbb{R})) \)

for any time \( T \).

**Remark.** The condition (A.1) on \( \psi \) is valid for constant influence function \( \psi \equiv 1 \), as well as the typical Cucker-Smale weight \( \psi(x) = (1 + x^2)^{-\gamma} \), with \( \gamma > 1/2 \). The condition on \( K \) is valid for Newtonian potential \( K = \frac{k}{2|x|} \).

As subcritical initial data imply global in time bounds on \( \|\rho\|_{L^\infty} \) and \( \|\partial_x u\|_{L^\infty} \), it suffices to prove the following estimate.

**Theorem A.2.** Let \( s \geq 0 \) be an integer. Define \( Y(t) := \|\partial_x u(\cdot, t)\|_{H^s}^2 + \|\rho(\cdot, t)\|_{H^s}^2 \). If the influence function \( \psi \) and the potential \( K \) satisfy (A.1) and (A.2), respectively, then

\[
Y(T) \lesssim Y(0) \exp \left[ \int_0^T \left( 1 + \|\rho(\cdot, t)\|_{L^\infty} + \|\partial_x u(\cdot, t)\|_{L^\infty} \right) dt \right]. \tag{A.3}
\]

**Proof.** We start with acting operator \( \Lambda^s \) on equation (1.1) and integrate by parts against \( \Lambda^s \rho \). Here \( \Lambda := (I - \Delta)^{1/2} \) is the pseudo-differential operator. We also denote \( (\cdot, \cdot) \) as \( L^2 \) inner product in \( \mathbb{R} \).

The evolution of the \( H^s \) norm reads

\[
\frac{1}{2} \frac{d}{dt} \|\rho(\cdot, t)\|_{H^s}^2 = - (\Lambda^s \partial_x u, \rho, \Lambda^s \rho) + \frac{1}{2} (\Lambda^s \rho, (\partial_x u)\Lambda^s \rho).
\]

We postpone the proof of the following commutator estimate to Lemma A.1

\[
\|\Lambda^s \partial_x u, \rho\|_{L^2} \lesssim \|\partial_x u\|_{L^\infty} \|\rho\|_{H^s} + \|\partial_x u\|_{H^s} \|\rho\|_{L^\infty}.
\]
With this, we deduce the following estimate
\[ \frac{d}{dt}\|\rho(\cdot,t)\|_{H^s}^2 \lesssim \|\rho\|_{L^\infty} + \|\partial_x u\|_{L^\infty} \left(\|\rho\|_{H^s}^2 + \|\partial_x u\|_{H^s}^2\right). \]

Similarly, for equation (1.11), we have
\[ \frac{1}{2} \frac{d}{dt}\|\partial_x u(\cdot,t)\|_{H^s}^2 = -\left(\{\Lambda^s \partial_x, u\} \partial_x u, \Lambda^s \partial_x u\right) + \frac{1}{2} \left(\Lambda^s \partial_x u, (\partial_x u)\Lambda^s \partial_x u\right) \]
\[ + \left(\Lambda^s \partial_x u, \Lambda^s \partial_x \int \psi(x-y)(u(y) - u(x))\rho(y)dy\right) \]
\[ + \left(\Lambda^s \partial_x u, \Lambda^s \partial_x \int \partial_x K(x-y)\rho(y)dy\right). \]

For the commutator, we obtain the same estimate from Lemma A.1
\[ \|\{\Lambda^s \partial_x, u\} \partial_x u\|_{L^2} \lesssim \|\partial_x u\|_{L^\infty} \|\partial_x u\|_{H^s}. \]

For the alignment term, we claim and will prove in Lemma A.2 that
\[ \left(\Lambda^s \partial_x u, \Lambda^s \partial_x \int \psi(x-y)(u(y) - u(x))\rho(y)dy\right) \lesssim \|\partial_x u\|_{H^s} \left[\|\rho\|_{H^s} \|\partial_x u\|_{L^\infty} + (1 + \|\rho\|_{L^\infty}) \|\partial_x u\|_{H^s}\right]. \]

For the attraction-repulsion term, as \(\partial_x^2 K \in L^1(\mathbb{R})\),
\[ \left(\Lambda^s \partial_x u, \Lambda^s \partial_x \int \partial_x K(x-y)\rho(y)dy\right) \lesssim \|\partial_x u\|_{H^s} \|\partial_x^2 K\|_{L^1} \|\Lambda^s \rho\|_{L^2} \lesssim \|\partial_x u\|_{H^s} \|\rho\|_{H^s}. \]

Putting everything together, we obtain
\[ \frac{d}{dt}\|\partial_x u(\cdot,t)\|_{H^s}^2 \lesssim \left[1 + \|\rho\|_{L^\infty} + \|\partial_x u\|_{L^\infty}\right] \left(\|\rho\|_{H^s}^2 + \|\partial_x u\|_{H^s}^2\right). \]

A Gronwall’s inequality implies (A.3). □

Next, we provide a short proof of the Kato-Ponce type commutator estimate [13] which is used in the regularity estimates.

**Lemma A.1** (Commutator estimate). Let \(f, \partial_x u \in (L^\infty \cap H^s)(\mathbb{R})\). Take \(s\) to be an non-negative integer. Then,
\[ \|\partial_x^{s+1} u f\|_{L^2} \lesssim \|f\|_{L^\infty} \|\partial_x u\|_{H^s} + \|\partial_x u\|_{L^\infty} \|f\|_{H^s}. \]

**Remark.** Take \(f = \rho\), we get
\[ \|\partial_x^{s+1} u \rho\|_{L^2} \lesssim \|\rho\|_{L^\infty} \|\partial_x u\|_{H^s} + \|\partial_x u\|_{L^\infty} \|\rho\|_{H^s}. \]

Take \(f = \partial_x u\), we get
\[ \|\partial_x^{s+1} u \partial_x u\|_{L^2} \lesssim \|\partial_x u\|_{L^\infty} \|\partial_x u\|_{H^s}. \]

These imply the two commutator inequalities in Theorem A.2

**Proof of Lemma A.1.** We first rewrite the commutator and use appropriate Hölder inequality to get
\[ \|\partial_x^{s+1} u f\|_{L^2} = \|\partial_x^{s+1} (uf) - u \cdot \partial_x^{s+1} f\|_{L^2} \leq \sum_{i=0}^{s} \binom{s+1}{i} \|\partial_x^{s+1-i} u \cdot \partial_x^i f\|_{L^2} \]
\[ \leq \sum_{i=0}^{s} \binom{s+1}{i} \|\partial_x^{s-i} (\partial_x u)\|_{L^2} \|\partial_x^i f\|_{L^2}. \]

This completes the proof. □
We control the first term in the similar way as Lemma A.1, along with the assumption that
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Taking \((g, j) = (\partial_x u, s - i)\) and \((g, j) = (f, i)\), we continue the estimate
\[
\left\| \partial_x^{s+1} f \right\|_{L^2} \lesssim \sum_{i=0}^{s} \left( s + 1 \right) \left( \| \partial_x^{s-i} \partial_x u \|_{L^2} \right) \left( \| f \|_{L^\infty} \right) \left( \| \partial_x^s f \|_{L^2} \| \partial_x u \|_{L^\infty} \right)^2 \lesssim \| \partial_x^s (\partial_x u) \|_{L^2} \| f \|_{L^\infty} + \| \partial_x^s f \|_{L^2} \| \partial_x u \|_{L^\infty}.
\]
The last estimate is due to Young’s inequality. This ends the proof. \( \square \)

We are left with the final estimate (A.3).

**Lemma A.2.** If the influence function \( \psi \) satisfies (A.1), then
\[
\left\| \partial_x^{s+1} \int_\mathbb{R} \psi(x - y)(u(y) - u(x))\rho(y)dy \right\|_{L^2} \lesssim (\| \rho \|_{L^\infty} + 1)\| \partial_x u \|_{H^s} + \| \partial_x u \|_{L^\infty} \| \rho \|_{H^s}.
\]

**Proof.** We first assume \( \psi \in L^1(\mathbb{R}) \). Estimate the left hand side as follows.

\[
\text{LHS} \leq \sum_{i=0}^{s} \left( s + 1 \right) \left( \| \partial_x^{s-i} \partial_x u \|_{L^2} \right) \left( \| \partial_x^i \rho \|_{L^\infty} \right) \| \psi \|_{L^1} \lesssim \| \partial_x^s (\partial_x u) \|_{L^2} \| f \|_{L^\infty} + \| \partial_x^s f \|_{L^2} \| \partial_x u \|_{L^\infty}.
\]

For the second term, we again break it into two terms (again, suppressing the \( t \)-dependence),

\[
\text{II} = \left\| \int_\mathbb{R} \partial_y^{s+1} \psi(x - y)(u(y) - u(x))\rho(y)dy \right\|_{L^2} = \left\| \int_\mathbb{R} \psi(x - y)\partial_y^{s+1} [(u(y) - u(x))\rho(y)] dy \right\|_{L^2} \leq \left\| \int_\mathbb{R} \psi(x - y) \sum_{i=0}^{s} \left( s + 1 \right) \left( \partial_y^{s-i} \partial_y u \right) \partial_y^i \rho(y) dy \right\|_{L^2} + \left\| \int_\mathbb{R} \psi(x - y)(u(y) - u(x))\partial_y^{s+1} \rho(y)dy \right\|_{L^2} = \text{III} + \text{IV}.
\]

The third term can be controlled by Lemma A.1 after applying Young’s inequality
\[
\text{III} \leq \| \psi \|_{L^1} \sum_{i=0}^{s} \left( s + 1 \right) \| \partial_x^{s+1-i} \partial_x u \|_{L^2} \lesssim \| \partial_x^s (\partial_x u) \|_{L^2} \| \rho \|_{L^\infty} + \| \partial_x^s \rho \|_{L^2} \| \partial_x u \|_{L^\infty}.
\]

Finally, for the last term, we have
\[
\text{IV} = \left\| \int_\mathbb{R} \partial_y \left[ \psi(x - y)(u(y) - u(x)) \right] \partial_y^i \rho(y,t) dy \right\|_{L^2} \leq \left\| \int_\mathbb{R} \partial_y \psi(x - y)(u(y) - u(x))\partial_y^i \rho(y) dy \right\|_{L^2} + \left\| \int_\mathbb{R} \psi(x - y)\partial_y u(y)\partial_y^i \rho(y) dy \right\|_{L^2} \leq \| x \partial_x \psi \|_{L^1} \| \partial_x u \|_{L^\infty} \| \partial_x^s \rho \|_{L^2} + \| \psi \|_{L^1} \| \partial_x u \|_{L^\infty} \| \partial_x^s \rho \|_{L^2}.
\]

For \( \psi \in L^1 + \text{const} \), it is easy to check that for constant \( c \),
\[
\partial_x^{s+1} \int_\mathbb{R} c(u(y,t) - u(x,t))\rho(y,t)dy = -c\partial_x^{s+1} u(x).
\]
Thus, we conclude with the desired estimate.

\[ \square \]

A.2. Isothermal Euler equations with nonlocal dissipation. In this part, we study the global regularity for the system (1.5) with the pressure law \( p(\rho) = \rho \). The system can be rewritten in the following form.

\[
\begin{align*}
\partial_t \eta + u \partial_x \eta &= -\partial_x u, \\
\partial_t u + u \partial_x u + \partial_x \eta &= \int_{\mathbb{R}} \psi(x - y)(u(y, t) - u(x, t))\rho(y, t)dy.
\end{align*}
\] (A.5)

**Theorem A.3.** Let \( s \geq 0 \) be an integer. Consider the system (A.5) with the influence function \( \psi \) satisfying

\[
\partial_x \psi \in W^{s-1,\infty}(\mathbb{R}), \quad |x|^{1/2}\partial_x^s \psi \in L^2(\mathbb{R}) \quad \text{for } 1 \leq \alpha \leq s + 1.
\] (A.6)

Suppose the initial data \( (\eta_0 := \ln \rho_0, u_0) \) lies in the subcritical region described in Theorem 1.4 and satisfy \( \partial_x \eta_0 \in H^s(\mathbb{R}) \), and \( \partial_x u_0 \in H^s(\mathbb{R}) \). Then there exists a unique global solution \( (\eta, u) \) such that

\[
\partial_x \eta \in C([0, T]; H^s(\mathbb{R})) \quad \text{and} \quad \partial_x u \in C([0, T]; H^s(\mathbb{R})),
\]

for any time \( T \).

As subcritical initial data imply global in time bounds on \( \|\partial_x \eta\|_{L^\infty} \) and \( \|\partial_x u\|_{L^\infty} \), it suffices to prove the following estimate.

**Theorem A.4.** Let \( s \geq 0 \) be an integer. Define \( Y(t) := \|\partial_x \eta(\cdot, t)\|_{H^s} + \|\partial_x u(\cdot, t)\|_{H^s}^2 \). If the influence function \( \psi \) satisfies (A.6), then

\[
Y(T) \lesssim Y(0) \exp \left( \int_0^T \left( 1 + \|\partial_x \eta(\cdot, t)\|_{L^\infty} \right)^2 + \|\partial_x u(\cdot, t)\|_{L^\infty} \right) dt .
\]

**Proof.** The proof is similar as in Theorem A.2.

We start with acting operator \( \Lambda^s \partial_x \) on equation (A.5) and integrate by parts against \( \Lambda^s \partial_x \eta : \nabla \frac{d}{dt} \|\partial_x \eta\|_{H^s} = -\left( [\Lambda^s \partial_x, u] \partial_x \eta, \Lambda^s \partial_x \eta \right) + \frac{1}{2} \left( [\Lambda^s \partial_x, (\partial_x u)\Lambda^s \partial_x \eta] \right) - \left( \Lambda^s \partial_x \eta, \Lambda^s \partial_x^2 \eta \right).

The commutator estimate in Lemma A.1 implies

\[
\| [\Lambda^s \partial_x, u] \partial_x \eta \|_{L^2} \lesssim \|\partial_x u\|_{L^\infty} \|\partial_x \eta\|_{H^s} + \|\partial_x u\|_{H^s} \|\partial_x \eta\|_{L^\infty} .
\]

With this, we deduce the following estimate

\[
\frac{1}{2} \frac{d}{dt} \|\partial_x \eta(\cdot, t)\|_{H^s}^2 + (\Lambda^s \partial_x \eta, \Lambda^s \partial_x^2 \eta) \lesssim \left( \|\partial_x \eta\|_{L^\infty} \right)^2 + \|\partial_x u\|_{L^\infty}\left( \|\partial_x \eta\|_{H^s}^2 + \|\partial_x u\|_{H^s}^2 \right). \tag{A.7}
\]

Similarly, for equation (A.5), we have

\[
\frac{1}{2} \frac{d}{dt} \|\partial_x u(\cdot, t)\|_{H^s}^2 = -\left( [\Lambda^s \partial_x, u] \partial_x u, \Lambda^s \partial_x u \right) + \frac{1}{2} \left( [\Lambda^s \partial_x, (\partial_x u)\Lambda^s \partial_x u] \right) - \left( \Lambda^s \partial_x u, \partial_x \Lambda^s \partial_x^2 u \right)
\]

\[
+ \left( \Lambda^s \partial_x u, \Lambda^s \partial_x \int_{\mathbb{R}} \psi(x - y)(u(y) - u(x))\rho(y)dy \right) .
\]

With the same argument as in Theorem A.2, we get

\[
\frac{1}{2} \frac{d}{dt} \|\partial_x u(\cdot, t)\|_{H^s}^2 + (\Lambda^s \partial_x u, \partial_x \Lambda^s \partial_x^2 u) \lesssim \left( 1 + \|\rho\|_{L^\infty} \right) \left( \|\partial_x \eta\|_{H^s}^2 + \|\partial_x u\|_{H^s}^2 \right) , \tag{A.8}
\]
provided the following estimate for the alignment term is true.

\[
\left( \Lambda^s \partial_x u, \Lambda^s \partial_x \int_{\mathbb{R}} \psi(x-y)(u(y) - u(x))\rho(y)dy \right) \lesssim \|\partial_x u\|^2_{H^s}. \tag{A.9}
\]

See Lemma A.3 for the proof of this inequality.

Finally, we add \eqref{eq:A.7}-\eqref{eq:A.8} and use Gronwall’s inequality to end the proof. \hfill \Box

We end this section by proving the estimate \eqref{eq:A.9}. A stronger regularity on \(\psi\) is required (compared with Lemma A.2) as we do not have regularity for \(\rho\) any more.

**Lemma A.3.** If \(\psi\) satisfies \eqref{eq:A.6}, then the estimate \eqref{eq:A.9} is satisfied.

**Proof.** It suffices to prove for all \(1 \leq \alpha \leq s + 1\),

\[
\left( \partial_x^\alpha u, \partial_x^\alpha \int_{\mathbb{R}} \psi(x-y)(u(y) - u(x))\rho(y)dy \right) \lesssim \|\partial_x u\|^2_{H^s}.
\]

In fact, we get

\[
\text{LHS} = \left( \partial_x^\alpha u, \sum_{i=0}^{\alpha} \left( \frac{\alpha}{i} \right) \int_{\mathbb{R}} \partial_x^i \psi(x-y)\partial_x^{\alpha-i}(u(y) - u(x))\rho(y)dy \right)
\]

\[
\leq - (\psi \ast \rho)\|\partial_x^\alpha u\|^2_{L^2} + \|\partial_x^\alpha u\|_{L^2} \sum_{i=1}^{\alpha-1} \left( \frac{\alpha}{i} \right) \|\partial_x^{\alpha-i} u \cdot \partial_x^i (\psi \ast \rho)\|_{L^2}
\]

\[
+ \|\partial_x^\alpha u\|_{L^2} \left\| \int_{\mathbb{R}} \partial_x^\alpha \psi(x-y)(u(y,t) - u(x,t))\rho(y,t)dy \right\|_{L^2}
\]

\[
\leq 0 + I + II.
\]

Next, we estimate I and II term by term.

\[
I \leq \|\partial_x^\alpha u\|_{L^2} \sum_{i=1}^{\alpha-1} \left( \frac{\alpha}{i} \right) \|\partial_x^{\alpha-i} u\|_{L^2} \|\partial_x^i \psi\|_{L^\infty} \|\rho\|_{L^1} \lesssim \|\partial_x \psi\|_{W^{s-1, \infty}} \|\partial_x u\|^2_{H^s} \lesssim \|\partial_x u\|^2_{H^s},
\]

\[
II \leq \|\partial_x^\alpha u\|_{L^2} \|u\|_{C^{1/2}L^1} \|x|^{1/2} \partial_x^\alpha \psi(x) \ast \rho\|_{L^2}
\]

\[
\lesssim \|\partial_x^\alpha u\|_{L^2} \|\partial_x u\|_{L^2} \|x|^{1/2} \partial_x^\alpha \psi(x)\|_{L^2} \|\rho\|_{L^1} \lesssim \|\partial_x u\|^2_{H^s}.
\]

This concludes the proof. \hfill \Box

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**References**

[1] I. Aoki, *A Simulation Study on the Schooling Mechanism in Fish*, Bull. Jap. Soc. Sci. Fisheries, 48, (1982): 1081–1088.
[2] J. A. Cañizo, J. A. Carrillo, and J. Rosado, *A well-posedness theory in measures for some kinetic models of collective motion*, Math. Mod. Meth. in Appl. Sci., 21, (2011): 515–539.
[3] J. A. Carrillo, M. R. D’Orsogna, and V. Panferov, *Double milling in self-propelled swarms from kinetic theory*, Kinetic and Related Models, 2, (2009): 363–378.
[4] J.A. Carrillo, M. Fornasier, G. Toscani, and F. Vecil, *Particle, Kinetic, and Hydrodynamic Models of Swarming*, Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences, Series: Modelling and Simulation in Science and Technology, Birkhauser, (2010): 297–336.

[5] J.A. Carrillo, A. Klar, S. Martin, and S. Tiwari, *Self-propelled interacting particle systems with roosting force*, Math. Mod. Meth. Appl. Sci., 20, (2010): 1533–1552.

[6] G.-Q. Chen, *Euler equations and related hyperbolic conservation laws*, In: Handbook of differential equations, 2, edited by C. M. Dafermos, E. Feireisl, Amsterdam: Elsevier Science, (2005): 1–104.

[7] Y.-L. Chuang, M. R. D’Orsogna, D. Marthaler, A. L. Bertozzi and L. Chayes, *State transitions and the continuum limit for a 2D interacting, self-propelled particle system*, Physica D, 232, (2007): 33–47.

[8] F. Cucker and S. Smale, *Emergent behavior in flocks*, IEEE Trans. Autom. Control, 52, no. 5, (2007): 852.

[9] S. Engelberg, H. Liu and E. Tadmor, *Critical threshold in Euler-Poisson equations*, Indiana Univ. Math. J., 50, (2001): 109–157.

[10] S.-Y. Ha and E. Tadmor, *From particle to kinetic and hydrodynamic descriptions of flocking*, Kinetic and Related Models, 1, no. 3, (2008): 415–435.

[11] A. Huth and C. Wissel, *The Simulation of the Movement of Fish Schools*, J. Theo. Bio., 156, (1992): 365–385.

[12] T. Karper, A. Mellet and K. Trivisa, *Hydrodynamic limit of the kinetic Cucker-Smale flocking model*, Math. Mod. Meth. in Appl. Sci., DOI: 10.1142/S0218202515500050.

[13] T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Communications on Pure and Applied Mathematics, 41, no. 7, (1988): 891–907.

[14] Y. Lee and H. Liu, *Thresholds in three-dimensional restricted Euler-Poisson equations*, Physica D, 262, (2013): 59–70.

[15] H. Liu and E. Tadmor, *Critical thresholds in 2-D restricted Euler-Poisson equations*, SIAM J. Appl. Math., 63, no. 6, (2003): 1889–1910.

[16] H. Liu, E. Tadmor, and D. Wei, *Global regularity of the 4D restricted Euler equations*, Physica D, 239, (2010): 1225–1231.

[17] CW. Reynolds, *Flocks, herds and schools: A distributed behavioral model*, ACM SIGGRAPH Computer Graphics, 21, no. 4, (1987): 25–34.

[18] E. Tadmor and C. Tan, *Critical thresholds in flocking hydrodynamics with nonlocal alignment*, Proc. Royal Soc. A, 372, (2014): 20130401.

[19] E. Tadmor and D. Wei, *On the global regularity of sub-critical Euler-Poisson equations with pressure*, J. European Math. Society, 10, (2008): 757–769.

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