Exclusion sets for eigenvalues of matrices

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Abstract

To locate all eigenvalues of a matrix more precisely, we exclude some sets which do not include any eigenvalue of the matrix from the well-known Brauer set to give two new Brauer-type eigenvalue inclusion sets. And it is also shown that the new sets are contained in the Brauer set.

Keywords: Eigenvalue; Exclusion set; Geršgorin set; Brauer set

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1. Introduction

One of the most important problems for eigenvalues of matrices is to locate them \cite{1}, that is, to find regions including all eigenvalue of a given matrix $A$ in the complex plane. The well-known Geršgorin disk theorem \cite{2} stated below provides just such a region, which consists of $n$ disks centered at the diagonal elements of the matrix.

\textbf{Theorem 1.} \cite{2} Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ be a complex matrix, and $\sigma(A)$ the set of all eigenvalues of $A$. Then

$$\sigma(A) \subseteq \Gamma(A) = \bigcup_{i=1}^{n} \Gamma_i(A),$$

where

$$\Gamma_i(A) = \{z \in \mathbb{C} : |z - a_{ii}| \leq r_i(A)\},$$

and $r_i(A) = \sum_{k \neq i} |a_{ik}|$.

Generally, $\Gamma_i(A)$ is called the $i$th disk, and $\Gamma(A)$ is called the Geršgorin set. Although the Geršgorin set is beautiful and simple \cite{2}, it is only a raw result, which inspires researchers to find another sets which are tighter than $\Gamma(A)$. One such well-known set provided by Brauer \cite{3} is described as follows.

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Theorem 2. Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ be a complex matrix. Then

$$\sigma(A) \subseteq \mathcal{K}(A) = \bigcup_{j \neq i, i,j=1}^{n} \mathcal{K}_{ij}(A),$$

where

$$\mathcal{K}_{ij}(A) = \{ z \in \mathbb{C} : |z - a_{ii}| |z - a_{jj}| \leq r_i(A)r_j(A) \}.$$

Note that the Brauer set $\mathcal{K}(A)$ consists of $\frac{n(n-1)}{2}$ Cassini ovals $\mathcal{K}_{ij}(A)$. Hence, $\mathcal{K}(A)$ needs more computations than $\Gamma(A)$ to locate all eigenvalues of $A$, while $\mathcal{K}(A)$ can captures all eigenvalues of $A$ more precisely than $\Gamma(A)$, that is $\mathcal{K}(A) \subseteq \Gamma(A)$.

Besides the Brauer set, there are many sets which are all tighter than the Geršgorin set (see [4, 5, 6, 7, 8, 9, 10, 11]). However, it is worth noting here that one did not consider the problem that whether or not there is some proper subset for these sets in which each eigenvalue of a matrix is not included, until Melman in [12] gave the following Geršgorin-type set.

Theorem 3. Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ be a complex matrix. Then

$$\sigma(A) \subseteq \Omega(A) = \bigcup_{i=1}^{n} \Omega_i(A),$$

where

$$\Omega_i(A) = \Gamma_i(A) \setminus \Delta_i(A), \quad \Delta_i(A) = \bigcup_{j \neq i, j=1}^{n} \Delta_{ij}(A),$$

and

$$\Delta_{ij}(A) = \{ z \in \mathbb{C} : |z - a_{jj}| < 2|a_{ji}| - r_j(A) \}.$$ 

Furthermore, $\Omega(A) \subseteq \Gamma(A)$.

Remark here that there is a typographical error in a single definition, namely, $\geq$ instead of $<$ in the definition of $\Delta_{ij}(A)$ (Theorem 2 in [12]).

Inspired by A. Melman, we in this paper give two new Brauer-type eigenvalue inclusion sets by considering all but the largest modulus component of an eigenvector and its corresponding characteristic polynomial equation of a matrix, and by considering the largest modulus component and the second-largest modulus component. And it is proved that this new Brauer-type sets are better than the Brauer set.

2. Exclusion sets for the Brauer set

In this section, we present two new Brauer-type sets by excluding two kinds of Brauer-type exclusion sets from the Brauer set, and the relations between them and the Brauer set are also given.
**Theorem 4.** Let \( A = [a_{ij}] \in \mathbb{C}^{n \times n} \) be a complex matrix. Then

\[
\sigma(A) \subseteq \Phi(A) = \bigcup_{i,j=1 \atop i \neq j}^{n} \Phi_{ij}(A),
\]

where

\[
\Phi_{ij}(A) = K_{ij}(A) \setminus \mathcal{L}_{i}(A), \quad \mathcal{L}_{i}(A) = \bigcup_{s \neq i, s=1}^{n} \mathcal{L}_{si}(A),
\]

\[
\mathcal{L}_{si}(A) = \{ z \in \mathbb{C} : |z - a_{ss}|(|z - a_{ii}| + r_{s}^{t}(A)) < (|a_{si}| - r_{s}^{i}(A))|a_{is}| \}, \quad (1)
\]

and

\[
r_{k}^{t}(A) = r_{t}(A) - |a_{tk}|, \quad \forall k \neq t.
\]

Furthermore, \( \Phi(A) \subseteq K(A) \).

**Proof.** Suppose that \( \lambda \) is an eigenvalue of \( A \) with a corresponding eigenvector \( x = (x_1, x_2, \ldots, x_n)^{T} \), then

\[
Ax = \lambda x
\]

holds. Let

\[
|x_{p}| \geq |x_{t}| \geq \max_{1 \leq k \leq n} |x_{k}|
\]

hence \( |x_{p}| > 0 \). By the proof of the well-known Brauer theorem in [3], also see Theorem 2.2 in [11], we can easily get that the \( p \)-th equality of (2):

\[
(\lambda - a_{pp})x_{p} = \sum_{k=1 \atop k \neq p}^{n} a_{pk}x_{k}
\]

(3) gives

\[
|\lambda - a_{pp}|x_{p} \leq \sum_{k=1 \atop k \neq p}^{n} |a_{pk}||x_{k}| \leq \sum_{k=1 \atop k \neq p}^{n} |a_{pk}||x_{t}| = r_{p}(A)|x_{t}|,
\]

and the \( t \)-th equality of (2):

\[
(\lambda - a_{tt})x_{t} = \sum_{k=1 \atop k \neq t}^{n} a_{tk}x_{k}
\]

(5) gives

\[
|\lambda - a_{tt}|x_{t} \leq \sum_{k=1 \atop k \neq t}^{n} |a_{tk}||x_{k}| \leq \sum_{k=1 \atop k \neq t}^{n} |a_{tk}||x_{p}| = r_{t}(A)|x_{p}|.
\]

If \( |x_{t}| = 0 \), then from (1), we have \( \lambda = a_{pp} \), which implies \( \lambda \in K_{pt}(A) \). If \( |x_{t}| > 0 \), by (1) and (6), we have

\[
|\lambda - a_{pp}||\lambda - a_{tt}| \leq r_{p}(A)r_{t}(A),
\]
that is,
\[ \lambda \in \mathcal{K}_{pt}(A). \]  

On the other hand, for any \( s \neq p \), and by the \( s \)-th equality of (2), we have
\[ (\lambda - a_{ss})x_s = \sum_{k=1}^{n} a_{sk}x_k + a_{sp}x_p, \]  
then
\[ a_{sp}x_p = (\lambda - a_{ss})x_s - \sum_{k=1 \atop k \neq s,p}^{n} a_{sk}x_k. \]  
Taking absolute values on both sides of (9) and using the triangle inequality gives
\[
|a_{sp}|x_p = |(\lambda - a_{ss})x_s - \sum_{k=1 \atop k \neq s,p}^{n} a_{sk}x_k| 
\leq |\lambda - a_{ss}|x_s + \sum_{k=1 \atop k \neq s,p}^{n} |a_{sk}|x_p 
= |\lambda - a_{ss}|x_s + r_{p}^{s}(A)|x_p|,
\]
then
\[
(|a_{sp}| - r_{p}^{s}(A))|x_p| \leq |\lambda - a_{ss}|x_s. \]  
By the \( p \)-th equation of (2), we have
\[ (\lambda - a_{pp})x_p = \sum_{k=1 \atop k \neq p,s}^{n} a_{pk}x_k + a_{ps}x_s, \]  
then
\[ a_{ps}x_s = (\lambda - a_{pp})x_p - \sum_{k=1 \atop k \neq p,s}^{n} a_{pk}x_k. \]  
Taking absolute values on both sides of (12) and using the triangle inequality yields
\[
|a_{ps}|x_s = |(\lambda - a_{pp})x_p - \sum_{k=1 \atop k \neq p,s}^{n} a_{pk}x_k| 
\leq |\lambda - a_{pp}|x_p + \sum_{k=1 \atop k \neq p,s}^{n} |a_{pk}|x_p 
= |\lambda - a_{pp}|x_p + r_{p}^{p}(A)|x_p|,  
\]
hence

\[ |a_{ps}| |x_s| \leq (|\lambda - a_{pp}| + r_p^p(A)) |x_p|. \]  

(13)

If \(|x_s| > 0\), then by (10) and (13) gives

\[ |\lambda - a_{ss}| (|\lambda - a_{pp}| + r_p^p(A)) \geq (|a_{sp}| - r_s^p(A)) |a_{ps}|, \]  

(14)

that is

\[ \lambda \notin \mathcal{L}_{sp}(A). \]  

(15)

Notice that (15) holds for any \( s \neq p \), then

\[ \lambda \notin \left( \bigcup_{s \neq p, \ s \neq p} \mathcal{L}_{sp}(A) \right) = \mathcal{L}_{p}(A). \]  

(16)

From (7) and (16), we have

\[ \lambda \in (\mathcal{K}_{pt}(A) \setminus \mathcal{L}_p(A)) = \Phi_{pt}(A). \]  

(17)

Since we do not know which \( p \) and \( t \) are appropriate to each eigenvalue \( \lambda \), we can only conclude that

\[ \lambda \in \left( \bigcup_{p,t=1}^{n} \Phi_{pt}(A) \right) = \Phi(A). \]  

(18)

On the other hand, if \(|x_s| = 0\), then from (10), we have \( |a_{sp}| - r_s^p(A) \leq 0 \), which implies (14) holds, and then (18) holds. Hence

\[ \sigma(A) \subseteq \Phi(A). \]

In addition, since

\[ (\mathcal{K}_{pt}(A) \setminus \mathcal{L}_p(A)) \subseteq \mathcal{K}_{pt}(A), \]

then

\[ \Phi(A) \subseteq \mathcal{K}(A). \]

The proof is completed. \( \square \)

Remark 1. (I) Theorem 4 shows that for each eigenvalue \( \lambda \) of \( A \), \( \lambda \notin \mathcal{L}_i(A) \) for \( i \in N \). Note that \( \mathcal{L}_i(A) \) is generated by the union of \( n-1 \) Cassini ovals determined by the elements of \( A \), hence \( \mathcal{L}_i(A) \) is called a Brauer-type exclusion set corresponding to the \((i, j)\)-th Brauer Cassini oval \( \mathcal{K}_{ij}(A) \).

(II) the computation for \( \Phi(A) \) needs \( 3n(n-1) \) Cassini ovals, while the computation for \( \mathcal{K}(A) \) needs \( \frac{n(n-1)}{2} \) Cassini ovals.
Example 3.1 Consider the matrix

$$A = \begin{bmatrix}
14 & 0.01i & 0 & 18 - 2i \\
0 & 9 & 4 + i & 0 \\
0.01 + i & 2 + i & 11 & 0 \\
19 + i & 0 & 0.1 + i & 10
\end{bmatrix}. $$

The sets $\Phi(A)$ in Theorem 4 is drawn in Figure 1. And the exact eigenvalues of $A$ are plotted with asterisks. From Figure 1, we conclude that $\Phi(A)$ locate the eigenvalues of $A$ more precisely than $K(A)$.

Note that Theorem 4 is obtained by considering all but the largest modulus component of an eigenvector and its corresponding characteristic polynomial equation, which needs much computations. To reduce its computations, we next give another Brauer-type set by considering only the largest modulus component and the second-largest modulus component of an eigenvector and its corresponding characteristic polynomial equation.

Theorem 5. Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ be a complex matrix. Then

$$\sigma(A) \subseteq \Theta(A) = \bigcup_{i \neq j, i, j = 1}^{n} \Theta_{ij}(A),$$

where

$$\Theta_{ij}(A) = K_{ij}(A) \setminus \Lambda_{ij}(A),$$

and

$$\Lambda_{ij}(A) = \left\{ z \in \mathbb{C} : (|\lambda - a_{ii}| + r_i(A))(|\lambda - a_{jj}| + r_j(A)) < |a_{ij}||a_{ji}| \right\}. \quad (19)$$

Furthermore, $\Theta(A) \subseteq K(A)$.

Proof. Suppose that $\lambda$ is an eigenvalue of $A$ with a corresponding eigenvector $x = (x_1, x_2, \ldots, x_n)^T$, then (2) holds. Let

$$|x_p| \geq |x_t| = \max_{1 \leq k \leq n} |x_k|,$$

then $|x_p| > 0$. Similar to the proof of Theorem 4,

$$\lambda \in K_{pt}(A) \quad (20)$$

can be easily obtained.

On the other hand, (3) and (5) can be rewritten respectively as

$$\lambda - a_{pp}x_p - \sum_{k=1, k \neq p}^{n} a_{pk}x_k = a_{pt}x_t \quad (21)$$

and

$$\lambda - a_{tt}x_t - \sum_{k=1, k \neq t}^{n} a_{tk}x_k = a_{tp}x_p. \quad (22)$$
Taking absolute values on both sides of (21) and (22), and using the triangle inequality yields

\[ |a_{pt}| |x_t| \leq |\lambda - a_{pp}| |x_p| + \sum_{k=1 \atop k \neq p,t}^{n} |a_{pk}| |x_k| \leq (|\lambda - a_{pp}| + r^p(A)) |x_p| \]

(23)

and

\[ |a_{tp}| |x_p| \leq |\lambda - a_{tt}| |x_t| + \sum_{k=1 \atop k \neq t,p}^{n} |a_{tk}| |x_k| \leq (|\lambda - a_{tt}| + r^t(A)) |x_t|. \]

(24)

If \(|x_t| \neq 0\), then multiplying (23) and (24) gives

\[ |a_{pt}| |a_{tp}| |x_t| |x_p| \leq (|\lambda - a_{pp}| + r^p(A)) (|\lambda - a_{tt}| + r^t(A)) |x_p| |x_t|. \]

that is

\[ (|\lambda - a_{pp}| + r^p(A)) (|\lambda - a_{tt}| + r^t(A)) \geq |a_{pt}| |a_{tp}|, \]

which implies that

\[ \lambda \notin \Lambda_{pt}(A). \]

(26)

If \(|x_t| = 0\), then by (24) we have \(|a_{tp}| = 0\), which also leads to \(\lambda \notin \Lambda_{pt}(A)\).

Furthermore, from (20) and (21), we have

\[ \lambda \in (K_{pt}(A) \setminus \Lambda_{pt}(A)) = \Theta_{pt}(A). \]

(27)

Since we do not know which \(p\) and \(t\) are corresponding to each eigenvalue \(\lambda\), then we can only get that

\[ \lambda \in \left( \bigcup_{t \neq p} \Theta_{pt}(A) \right) = \Theta(A). \]

Hence

\[ \sigma(A) \subseteq \Theta(A). \]

In addition, since

\[ (K_{pt}(A) \setminus \Lambda_{pt}(A)) \subseteq K_{pt}(A), \]

then

\[ \Phi(A) \subseteq K(A) \]

can be easily obtained. The conclusion follows.

\[ \square \]

**Remark 2.** (I) Note that the computation for \(\Theta(A)\) needs \(n(n-1)\) Cassini ovals, which is obviously less than that of \(\Phi(A)\).

(II) By lots of numerical examples we find that \(\Phi(A) \subset \Theta(A)\) in most cases, and the worst case is \(\Phi(A) = \Theta(A)\). We here give a conjecture that

\[ \Phi(A) \subseteq \Theta(A). \]
Example 3.2 Consider again the matrix $A$ in Example 3.1. The set $\Theta(A)$ in Theorem 5 is drawn in Figure 2, and the exact eigenvalues of $A$ are plotted with asterisks. From Figure 2, it is not difficult to see that $\Theta(A)$ can also locate the eigenvalues of $A$ more precisely than $K(A)$, but comparing Figure 1 with Figure 2, we find that $\Phi(A) \subset \Theta(A)$.

Just as the Gershgorin disk theorem leads to the condition of strict diagonal dominance [13], we next give two sufficient criterions for the non-singularity of complex matrices by Theorem 4 and Theorem 5.

**Corollary 1.** Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ be a complex matrix. Then $A$ is non-singular if for each $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n\}$, and $j \neq i$, either

$$|a_{ii}||a_{jj}| > r_i(A)r_j(A)$$

or

$$|a_{ss}|(|a_{ii}| + r_s^i(A)) < (|a_{si}| - r_s^i(A))|a_{is}|$$

for some $s \neq i$ and $s \in \{1, \ldots, n\}$.

**Corollary 2.** Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ be a complex matrix. Then $A$ is non-singular if for each $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n\}$, and $j \neq i$, either

$$|a_{ii}||a_{jj}| > r_i(A)r_j(A)$$

or

$$((|a_{ii}| + r_i^j(A))(|a_{jj}| + r_j^i(A)) < |a_{ij}||a_{ji}|.$$ 

### 3. Conclusion

In this paper, two new Brauer-type sets $\Phi(A)$ and $\Theta(A)$ are given by excluding its corresponding Brauer-type exclusion sets, respectively. To investigate the relations between this two new Brauer-type sets and the Brauer set, we compare them with each other and obtain a novel result. Actually, by the similar method, there are many eigenvalue inclusion sets, such as, the sets in [5, 6, 7, 9, 10], from which we can exclude their corresponding exclusion subsets to provide more precise eigenvalue inclusion sets.

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