An Integro-Differential Conservation Law 

arising in a Model of Granular Flow

Debora Amadori* and Wen Shen**

(*): Dipartimento di Matematica Pura & Applicata, University of L’Aquila, Italy. 
E-mail: amadori@univaq.it

(**): Department of Mathematics, Penn State University, U.S.A.. 
E-mail: shen_w@math.psu.edu

Abstract

We study a scalar integro-differential conservation law. The equation was first derived in [2] as the slow erosion limit of granular flow. Considering a set of more general erosion functions, we study the initial boundary value problem for which one can not adapt the standard theory of conservation laws. We construct approximate solutions with a fractional step method, by recomputing the integral term at each time step. A-priori $L^\infty$ bounds and BV estimates yield convergence and global existence of BV solutions. Furthermore, we present a well-posedness analysis, showing that the solutions are stable in $L^1$ with respect to the initial data.

1 Introduction

We consider the initial boundary value problem for the scalar integro-differential equation

$$q_t + \left( \exp \left\{ \int_x^0 f(q(t, \xi)) d\xi \right\} f(q) \right)_x = 0, \quad t \geq 0, \quad x \leq 0,$$

(1.1)

with initial condition

$$q(0, x) = \bar{q}(x), \quad x \leq 0.$$ 

(1.2)

Note that the flux includes a non-local integral term. For notational convenience, we introduce

$$K(q(t, \cdot))(x) \doteq \exp \left\{ \int_x^0 f(q(t, \xi)) d\xi \right\}.$$ 

(1.3)

The function $f : (-1, +\infty) \rightarrow \mathbb{R} \in C^2(\mathbb{R})$ is called the erosion function. The following assumptions apply to $f$:

$$f(0) = 0, \quad f' > 0, \quad f'' < 0, \quad \lim_{q \rightarrow -1} f(q) = -\infty, \quad \lim_{q \rightarrow +\infty} \frac{f(q)}{q} = 0.$$ 

(1.4)
We remark that the characteristic speed of (1.1) is
\[ \dot{x} = f'(q)K. \]
By (1.3) and (1.4), the characteristic speed is always positive, therefore no boundary condition is assigned at \( x = 0 \) for (1.1).

The equation (1.1) arises as the slow erosion limit in a model of granular flow, studied in [2], with a specific erosion function
\[ f(q) = \frac{q}{q + 1}. \tag{1.5} \]
Note that this function satisfies all the assumptions in (1.4). In more details, let \( h \) be the height of the moving layer, and \( p \) be the slope of the standing profile. Assuming \( p > 0 \), the following \( 2 \times 2 \) system of balance laws was proposed in [12]
\[
\begin{cases}
    h_t - (hp)_x &= (p - 1)h, \\
p_t + ((p - 1)h)_x &= 0.
\end{cases} \tag{1.6}
\]
This model describes the following phenomenon. The material is divided in two parts: a moving layer with height \( h \) on top and a standing layer with slope \( p > 0 \) at the bottom. The moving layer slides downhill with speed \( p \). If the slope \( p = 1 \) (the critical slope), the moving layer passes through without interaction with the standing layer. If the slope \( p > 1 \), then grains initially at rest are hit by rolling grains of the moving layer and start moving as well. Hence the moving layer gets bigger. On the other hand, if \( p < 1 \), grains which are rolling can be deposited on the bed. Hence the moving layer becomes smaller.

In the slow erosion limit as \( ||h||_{L^\infty} \to 0 \), we proved in [2] that the solution for the slope \( p \) in (1.6) provides the weak solution of the following scalar integro-differential equation
\[ p_\mu + \left( \frac{p - 1}{p} \right) \exp \int_0^x \frac{1}{p(\mu, y)} - \frac{1}{p(\mu, y)} \, dy \right)_x = 0. \]
Here, the new time variable \( \mu \) accounts for the total mass of granular material being poured downhill. Introducing \( q = p - 1 \) and writing \( t \) for \( \mu \), we obtain the equation (1.1) with (1.5).

The result in [2] provides the existence of entropy weak solutions to the initial boundary value problem (1.1) with \( f \) given in (1.5) for finite “time” (which is actually finite total mass). However, well-posedness property was left open due to the technical difficulties caused by the non-local term in the flux. Furthermore, due to the discontinuities in \( q \), the function \( k(t, x) = K(q(t, \cdot))(x) \) is only Lipschitz continuous in its variables, therefore one can not apply directly previous results. Indeed, classical results as [15] require more smoothness on the coefficients; see also [9]. Some closer results can be found in [14, 16] where the coefficient \( k = k(x) \) does not depend on time.

In this paper we consider a class of more general erosion functions \( f \) that satisfy the assumptions in (1.4), and we study existence and well-posedness of BV solutions of (1.1). Assuming that the slope is always positive, i.e., \( q > -1 \), we seek BV solutions with bounded total mass. Therefore, we define \( \mathcal{D} = \mathcal{D}_{C_0, \kappa_0} \) as the set of functions that satisfy
\[ \mathcal{D}_{C_0, \kappa_0} = \left\{ q(x) : \inf_{x < 0} q(x) \geq \kappa_0 > -1, \quad TV \{q\} \leq C_0, \quad ||q||_{L^1(\mathbb{R}_{-})} \leq C_0 \right\}. \tag{1.7} \]
Assume that the initial data satisfies $\bar{q} \in D_{C_0, \kappa_0}$ for some constants $C_0 > 0, \kappa_0 > -1$. A natural definition of entropy weak solution is given below.

**Definition 1** Let $T > 0$. A function $q$ is an entropy weak solution to $(1.1)$ on $[0, T] \times \mathbb{R}_-$ with initial condition $(1.2)$, if the following holds.

(H1) $q : [0, T] \to L^1(\mathbb{R}_-\cap BV(\mathbb{R}_-))$, $\inf_x q(t, x) > -1$, and the map $[0, T] \ni t \mapsto q(t)$ is Lipschitz in $L^1(\mathbb{R}_-)$;

(H2) $q$ is a weak entropy solution of the scalar conservation law

\[
\begin{align*}
&\begin{cases}
q_t + (k(t, x)f(q))_x = 0, \\
q(0, x) = \bar{q}(x)
\end{cases} \\
\text{with } k \text{ defined by }
\end{align*}
\]

\[ k(t, x) = K(q(t, \cdot))(x) = \exp \left\{ \int_x^0 f(q(t, \xi)) d\xi \right\}. \tag{1.9} \]

Notice that, thanks to (H1), the coefficient $k(t, x)$ in $(1.9)$ is Lipschitz continuous on $[0, T] \times \mathbb{R}_-$. Now we state the main result of this paper.

**Theorem 1** Assume $(1.4)$ and let $C_0 > 0, \kappa_0 > -1$ be given constants. Then for any initial data $\bar{q} \in D_{C_0, \kappa_0}$ there exists an entropy weak solution $q(t, x)$ to the initial-boundary value problem $(1.1)$–$(1.2)$ for all $t \geq 0$. Moreover, consider two solutions $q_1(t, \cdot), q_2(t, \cdot)$ of the integro-differential equation $(1.1)$, corresponding to the initial data

$q_1(0, x) = \bar{q}_1(x), \quad q_2(0, x) = \bar{q}_2(x), \quad x < 0,$

with $\bar{q}_1, \bar{q}_2 \in D_{C_0, \kappa_0}$. Then for any $T > 0$ there exists $L = L(T, C_0, \kappa_0) > 0$ such that

\[ \|q_1(t, \cdot) - q_2(t, \cdot)\|_{L^1(\mathbb{R}_-)} \leq e^{Lt} \|\bar{q}_1 - \bar{q}_2\|_{L^1(\mathbb{R}_-)}, \quad t \in [0, T]. \tag{1.10} \]

Recalling that $q = p - 1 = u_x - 1$, the solution $q$ established by Theorem 1 allows us to recover the profile $u$ of the standing layer:

\[ u(t, x) - x = \int_{-\infty}^x q(t, y) dy. \tag{1.11} \]

Moreover, since $K_x = -Kf(q(t, x))$, the equation $(1.1)$ can be rewritten as

\[ q_t - K_{xx} = 0. \]

Integrating in space on $(-\infty, x)$, using $(1.11)$ and that $K_x(q(t, \cdot)) \in L^1(\mathbb{R}_-)$, we arrive at

\[ u_t - K_x = u_t + Kf(u_x - 1) = 0. \]
This nonlocal Hamilton-Jacobi equation is studied in [17], with a different class of erosion functions \( f \). Assuming more erosion for large slope, i.e., \( \lim_{q \to +\infty} f'(q) = \eta_0 > 0 \), the slope \( u_x \) of the standing layer would blowup, leading to jumps in the standing profile \( u \). Notice that, in our case, only upward jumps in \( u_x \) can occur as singularities, which corresponds to convex kinks in the profile \( u \).

About the continuous dependence notice that, when \( k \) is a prescribed coefficient, the \( L^1 \) stability estimate (1.10) holds with \( L = 0 \), see (2.3). On the other hand, for the integral equation (1.1), one cannot expect \( L = 0 \) in general. Indeed, a small variation in the \( L^1 \) norm of the initial data may cause a variation in the global term and then in the overall solution. However, a special case in which (1.10) holds with \( L = 0 \) is when \( q_2 \equiv 0 \), which indeed is a solution of (1.1).

Other problems involving a nonlocal term in the flux have been considered in [10, 7, 8]. Well-known integro-differential equations which lead to blow up of the gradients include the Camassa-Holm equation [6] and the variational wave equation [5]. The Cauchy problem for (1.1) with initial data with bounded support is studied in [3] where we use piecewise constant approximation generated by front tracing and obtain similar results.

The rest of the paper is structured in the following way. As a step toward the final result, in Section 2 we study the existence and well-posedness of the scalar equation (1.8) for a given coefficient \( k(t, x) \). Here \( k(t, x) \) is a local term, and preserves the properties of the global integral term. Such equation does not fall directly within the classical framework of [15], where more regularity on the coefficients is required \((C^1)\). In particular, BV estimates for solutions of (1.8) are needed to obtain the continuous dependence on the initial data, see (2.23). We employ a fractional step argument to deal with the time dependence of \( k \), and then follow an approach similar to [4] (see also [11]), where the authors deal with the case of \( k = k(x) \in L^\infty \). We further refer to [9] on total variation estimates for general scalar balance laws: their result, in our context, would require more regularity \((C^1)\) on the coefficient \( k \).

The properties of the integral operator \( K \), defined at (1.3), are summarized in the last Appendix.

## 2 Local well-posedness of solutions with a given coefficient \( k \)

In this section we study the well-posedness of the scalar equation (1.8) for a given coefficient \( k(t, x) \), by reviewing some related results and completing the arguments where needed.

Throughout this section, we will use \( u \) as the unknown variable. Consider

\[
\begin{align*}
  u_t + \left( k(t, x)f(u) \right)_x &= 0, & x \leq 0, \ t \geq 0 \\
  u(0, x) &= \bar{u}(x), & x < 0
\end{align*}
\]

where \( k(t, x) \) satisfies the following assumptions, for some \( T > 0 \):

- \( k(t, x) \in L^\infty ([0, T] \times \mathbb{R}_-) \), it is Lipschitz continuous and \( \inf_{t, x} k > 0 \);
- \( (K) \) \( TV \{k(t, \cdot)\}, TV \{k_x(t, \cdot)\} \) are bounded uniformly in time;
- \( [0, T] \ni t \mapsto k_x(t, \cdot) \in L^1(\mathbb{R}_-) \) is Lipschitz continuous.

4
The above assumptions on $k$ are motivated by the properties of the integral operator $K$, see Proposition 2 in the Appendix.

**Theorem 2** Assume $f$ satisfies (1.4) and $k(t,x)$ satisfies (K). Let $C_0 > 0$, $\kappa_0 > -1$ be given constants. Then there exist two constants $C_1$ and $\kappa_1$, with possibly $C_1 \geq C_0$ and $-1 < \kappa_1 \leq \kappa_0$, and an operator $P : [0,T] \times D_{C_0,\kappa_0} \to D_{C_1,\kappa_1}$ such that:

1) the function $u(t,x) = P_t(\bar{u})$ is a weak entropy solution of (2.1) with initial data $u(0,\cdot) = \bar{u} \in D_{C_0,\kappa_0}$;

2) for any $\bar{u}_1$, $\bar{u}_2 \in D_{C_0,\kappa_0}$ one has

$$
\|P_t(\bar{u}_1) - P_t(\bar{u}_2)\|_{L^1(\mathbb{R}_-)} \leq \|\bar{u}_1 - \bar{u}_2\|_{L^1(\mathbb{R}_-)}.
$$

**(Proof.** Let $\bar{u} \in D_{C_0,\kappa_0}$. We introduce the parameter $\Delta t > 0$ and define $t_n = n\Delta t$ for any integer $n \geq 0$. We approximate the coefficient $k$ by

$$
k_{\Delta t}(t,x) = k(t_n,x) \quad (t,x) \in [t_n,t_{n+1}) \times \mathbb{R}_-, \quad n \geq 0
$$

which is constant in time on each interval $[t_n,t_{n+1})$, and consider the equation

$$u_t + \left(k_{\Delta t}(t,x)f(u)\right)_x = 0.
$$

By adapting the analysis in [4], on each interval $[t_n,t_{n+1})$ the entropy solution for (2.5), call it $u_{\Delta t}$, exists and the corresponding operator $(t,\bar{u}) \mapsto u_{\Delta t}(t,\cdot)$ is contractive in $L^1(\mathbb{R}_-)$, provided that $u_{\Delta t}$ is bounded from both below and above. Furthermore, the complete flux

$$F(t,x) = k_{\Delta t}(t,x)f(u_{\Delta t}(t,x))$$

has the following properties: its sup norm does not increase in time,

$$|F(t,x)| \leq \sup |F(t_n,\cdot)|, \quad t \in (t_n,t_{n+1}),
$$

as well as its total variation:

$$TV \{F(t,\cdot)\} \leq TV \{F(t_n,\cdot)\}, \quad t \in (t_n,t_{n+1}).
$$

We now establish the lower and upper bounds for $u_{\Delta t}$. For notation simplicity, in the following we denote by $k(t,x)$ and $u(t,x)$ the approximate coefficient and solution respectively, without causing confusion. We define the constants $k_0$, $L$, $L_1$ such that, recalling (K), one has:

$$
k_0 = \inf_{t,x} k > 0; \quad (2.8)
$$

$$|k(t_1,x_1) - k(t_2,x_2)| \leq L \left(|t_1 - t_2| + |x_1 - x_2|\right) \quad \text{for all } t_i, x_i, \ i = 1, 2; \quad (2.9)
$$

$$TV \{k(t_1,\cdot) - k(t_2,\cdot)\} = \|k_x(t_1,\cdot) - k_x(t_2,\cdot)\|_{L^1(\mathbb{R}_-)} \leq L_1|t_1 - t_2|\quad (2.10)
$$
and set \( L_2 = L/k_0 \). We first give some formal arguments. The evolution of the complete flux \( F = kf(u) \) along the characteristic \( x(t) \) with \( \dot{x} = f'(u)k \) follows the equation

\[
\frac{d}{dt}F(t, x(t)) = (kf)_t + f'k(kf)_x = k_tF = \frac{k_t}{k}F. \tag{2.11}
\]

By our assumptions (K), the term \( k_t/k \) is uniformly bounded. Therefore, \(|F|\) grows at most at an exponential rate, and remains bounded for finite time \( t \leq T \). Therefore \(|f(u)|\) remains bounded as well. By the 4th assumption in (1.4), \( u \) never reaches \(-1\) in finite time, leading to a lower bound on \( u \).

The same argument leads to an upper bound for \( f(u) \), if \( f(u) \to +\infty \) as \( u \to +\infty \). However, if \( f(u) \to f_0 > 0 \) as \( u \to +\infty \), we need a different argument. We observe that, along a characteristic \( x(t) \), one has

\[
\frac{d}{dt}u(t, x(t)) = -k_x(t, x)f(u). \tag{2.12}
\]

By the lower bound on \( u \), the growth of \( u \) remains uniformly bounded, yielding an upper bound.

We now make these arguments rigorous for the approximate solutions. At time \( t = 0 \) one has

\[
|k(0, x)f(\bar{u}(x))| \leq C_1 \tag{2.13}
\]

for some \( C_1 \geq 0 \) that depends on the bounds for \( k \) and \( \bar{u} \). We claim that, as long as the approximate solution exists, we have

\[
|k(t, x)f(u(t, x))| \leq C_1e^{L_2t}. \tag{2.14}
\]

Indeed, by (2.13) and (2.6), the inequality (2.13) is valid on \([0, t_1)\). Assume now that (2.14) is valid on \([0, t_{n+1}), n \geq 0 \), i.e.,

\[
|F(t, x)| = |k(t_n, x)f(u(t_n, x))| \leq C_1 e^{L_2t_n}, \quad t \in [t_n, t_{n+1}). \tag{2.15}
\]

At time \( t = t_{n+1} \) one has

\[
|k(t_{n+1}, x)f(u(t_{n+1}, x))| = \frac{k(t_{n+1}, x)}{k(t_n, x)}|k(t_n, x)f(u(t_{n+1}, x))| \leq \left(1 + \frac{L}{k_0} \Delta t \right) \sup_x |k(t_n, x)f(u(t_{n+1}, x))| \leq e^{L_2\Delta t}C_1 e^{L_2t_n} = C_1 e^{L_2t_{n+1}}
\]

By induction, this proves (2.14), which in turn gives the lower bound \( \kappa_1 \) for \( u \). The upper bound also follows if \( f(u) \to +\infty \) as \( u \to +\infty \).

Finally, we consider the case that \( f(u) \to f_0 > 0 \) as \( u \to +\infty \). At any given point \((\bar{t}, \bar{x})\) one can trace back along an extremal backward generalized characteristic, which is classical on each \((t_n, t_{n+1})\) and continuous up to \( t = 0 \). Since now the r.h.s. of (2.12) is bounded, then \( u \) grows at a linear rate, and therefore remains bounded.

We remark that the lower bound yields an a-priori bound on the wave speed. Indeed, since \( f' \) is a decreasing function, the characteristic speed is bounded,

\[
\lambda = kf'(u) \leq \|k\|_\infty f'(\kappa_1).
\]
Bound on total variation. We estimate the total variation of $F(t, x) = k(t, x)f(u(t, x))$. On the interval $(t_n, t_{n+1})$ the coefficient $k$ is constant in time and we use (2.7). On the other hand, the total variation might increase at $t_n$ when $k$ is updated. Then we observe that

$$F(t_n, x) = \left[ 1 + \frac{k(t_n, x) - k(t_{n-1}, x)}{k(t_{n-1}, x)} \right] F(t_{n-1}, x), \quad (2.16)$$

therefore

$$TV \{F(t_n, \cdot)\} \leq \left( 1 + \frac{\|k(t_n, \cdot) - k(t_{n-1}, \cdot)\|_{\infty}}{\inf k(t_{n-1}, \cdot)} \right) TV \{F(t_{n-1}, \cdot)\} + \sup |F| \cdot TV \left\{ \frac{k(t_n, \cdot) - k(t_{n-1}, \cdot)}{k(t_{n-1}, \cdot)} \right\}. \quad (2.17)$$

Thanks to (2.8)–(2.10), we have

$$\frac{\|k(t_n, \cdot) - k(t_{n-1}, \cdot)\|_{\infty}}{\inf k(t_{n-1}, \cdot)} \leq L_2 \Delta t, \quad TV \left\{ \frac{k(t_n, \cdot) - k(t_{n-1}, \cdot)}{k(t_{n-1}, \cdot)} \right\} \leq L_3 \Delta t,$$

for a suitable constant $L_3$ independent on $\Delta t$. Moreover $F = kf$ is uniformly bounded thanks to (K) and the bounds on $u$. Hence we conclude that

$$TV \{F(t_n, \cdot)\} \leq (1 + L_2 \Delta t) TV \{F(t_{n-1}, \cdot)\} + L_4 \Delta t$$

for a suitable $L_4 > 0$. By induction it follows that

$$TV \{F(t, \cdot)\} \leq e^{L_2 t} TV \{F(0+, \cdot)\} + \frac{L_4}{L_2} (e^{L_2 t} - 1).$$

Recalling that $f(u) = F/k$, one obtains the $BV$ bound for $f(u(t))$,

$$(\inf f') TV \{u(t, \cdot)\} \leq TV \{f(u(t, \cdot))\} \leq \frac{1}{\inf k} TV \{F(t, \cdot)\} + \frac{\|F\|_{\infty}}{(\inf k)^2} TV \{k(t, \cdot)\}.$$  

This gives a bound on the total variation for $u(t)$:

$$TV \{u(t)\} \leq C [TV \{F(t, \cdot)\} + TV \{k(t, \cdot)\}] \leq C_1(t) \quad (2.18)$$

where the constant $C$ depends on $\inf_x u$, $\sup_x u$, $\inf_x k$, $\sup_x k$. Hence the total variation of $u$ may increase in time but it remains bounded as long as $u$ remains bounded.

Taking the limit $\Delta t \to 0$, the coefficient $k_{\Delta t}$ converges uniformly to $k$. Correspondingly, the family $u_{\Delta t}$ converges (up to a subsequence) to a weak solution $u$ of the original equation, satisfying the same upper and lower bounds and (2.18).

Moreover, in the limit as $\Delta t \to 0$, the Kružkov entropy inequalities for equation (2.1)

$$\partial_t |u - \alpha| + \partial_x [k(x, t)|f(u) - f(\alpha)|| + \text{sign}(u - \alpha) k_x(x, t) f(\alpha) \leq 0 \quad (2.19)$$

for all $\alpha \in \mathbb{R}$, hold in the sense of distributions. \hfill \Box
Next we establish the continuous dependence on the coefficient function. We rely on a result in [14] (Corollary 3.2) that applies to Cauchy problems and to the case of $k = k(x)$, that is, the coefficient does not depend on time.

For convenience of the reader we report that statement of [14] adapted to our situation.

Consider the two equations
\begin{align}
  u_t + \left( k f(u) \right)_x &= 0, \quad t \geq 0, \quad (2.20) \\
  u_t + \left( \tilde{k} f(u) \right)_x &= 0, \quad t \geq 0. \quad (2.21)
\end{align}

**Proposition 1** For $x \in \mathbb{R}$, let $k(x), \tilde{k}(x) \in BV(\mathbb{R})$ satisfy
\[ k_x, \tilde{k}_x \in BV(\mathbb{R}); \quad \inf k, \inf \tilde{k} \geq \alpha > 0 \]
for some positive $\alpha$. Consider the initial data $u_0, \tilde{u}_0 \in BV(\mathbb{R})$ for the two equations $(2.20), (2.21)$ respectively and let $u(t, \cdot), \tilde{u}(t, \cdot)$ be the corresponding solutions, assuming that they are bounded from above and bounded away from $-1$. Let $C_1$ be a bound on $|f|$ over the range of the solutions. Then
\[ \| u(t, \cdot) - \tilde{u}(t, \cdot) \|_{L^1(\mathbb{R})} \leq \| u_0 - \tilde{u}_0 \|_{L^1(\mathbb{R})} + t \left( C_1 TV \{ k - \tilde{k} \} + C_2 (1 + TV u_0 + TV \tilde{u}_0) \| k - \tilde{k} \|_{L^\infty([0, t] \times \mathbb{R}^{-})} \right) \] 
(2.22)

where $C_2$ depends on the bounds on $u, k$, $TV \{ k \}$ and on $\tilde{u}, \tilde{k}, TV \{ \tilde{k} \}$.

The continuous dependence property for our problem follows from Proposition 1 by properly extending the IBVP into Cauchy problems.

**Theorem 3** For $x < 0$, let $k(t, x), \tilde{k}(t, x)$ satisfy the assumption (K), and assume that the initial data $\tilde{u}$ belongs to $D_{C_0, \kappa_0}$ (defined at (1.7)). Let $u(t, \cdot), \tilde{u}(t, \cdot)$ be the solutions of the conservation laws $(2.20), (2.21)$ respectively, with the same initial data $\tilde{u}$, for some time interval $[0, T]$ ($T > 0$).

Then, the following estimate holds
\[ 1 \| u(t, \cdot) - \tilde{u}(t, \cdot) \|_{L^1(\mathbb{R}^{-})} \leq \hat{C}_1 \sup_{t \in [0, T]} TV \left\{ k(t, \cdot) - \tilde{k}(t, \cdot) \right\} \]
\[ + \hat{C}_2 \left( 1 + \sup_{\tau} TV u(\tau, \cdot) + \sup_{\tau} TV \tilde{u}(\tau, \cdot) \right) \| k - \tilde{k} \|_{L^\infty([0, t] \times \mathbb{R}^{-})}, \] 
(2.23)
where $\hat{C}_1$ is a bound on $|f|$ over the range of the solutions and $\hat{C}_2$ depends on the bounds on the solutions, the coefficients and their total variation $TV \{ k(t, \cdot) \}, TV \{ \tilde{k}(t, \cdot) \}$.

**Proof.** The IBVP $(2.21) - (2.22)$ can be extended to the following Cauchy problem
\[ u_t + \left( k(t, x) f(u) \right)_x = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \] 
(2.24)
with extended initial data

\[ u(0, x) = \begin{cases} \bar{u}(x) & \text{for } x \leq 0, \\ \bar{u}(0-) & \text{for } x > 0 \end{cases} \quad (2.25) \]

and the extended coefficient function \( k(t, x) \)

\[ k(t, x) = \lim_{y \to 0-} k(t, y) \quad \text{for } x > 0. \]

Due to the fact that the characteristic speed is positive, the solution for the Cauchy problem \( (2.24) - (2.25) \) restricted on \( x \leq 0 \) will match the solution for the IBVP \( (2.1) \).

In a same way, the IBVP \( (2.21) \) is extended to the Cauchy problem for

\[ u_t + \left( \tilde{k}(t, x)f(u) \right)_x = 0, \quad x \in \mathbb{R}, \quad t \geq 0 \quad (2.26) \]

with data \( (2.25) \). Without causing confusion, let’s still denote \( u(t, x) \) and \( \bar{u}(t, x) \) the solutions for \( (2.24) \) and \( (2.26) \), respectively, and let \( u_\Delta(t, x) \) and \( \bar{u}_\Delta(t, x) \) be the corresponding approximate solutions, constructed in the same way as in the proof of Theorem \( 2 \) with approximate coefficients \( k_\Delta \) and \( \tilde{k}_\Delta \) as in \( (2.4) \).

Denote the distance between these two solutions by

\[ e_\Delta(t) \doteq \| u_\Delta(t, \cdot) - \bar{u}_\Delta(t, \cdot) \|_{L^1(\mathbb{R})}. \]

Notice that \( e_\Delta(0) = 0 \) and that \( e_\Delta(t) \geq \| u_\Delta(t, \cdot) - \bar{u}_\Delta(t, \cdot) \|_{L^1(\mathbb{R})} \).

On each time interval \([t_n, t_{n+1})\) the coefficient is constant in time and the assumptions of Proposition \( 1 \) are satisfied. Hence, from \( (2.22) \), we have the following estimate

\[ e_\Delta(t_{n+1}) - e_\Delta(t_n) \leq \Delta t \tilde{C}_1 TV_{\mathbb{R}} \left\{ k_\Delta(t_n, \cdot) - \tilde{k}_\Delta(t_n, \cdot) \right\} + \Delta t \tilde{C}_2 \left( 1 + TV_{\mathbb{R}_-} u(t_n, \cdot) + TV_{\mathbb{R}_-} \bar{u}(t_n, \cdot) \right) \| k_\Delta(t_n, \cdot) - \tilde{k}_\Delta(t_n, \cdot) \|_{L^\infty(\mathbb{R})} \quad (2.27) \]

for some constants \( \tilde{C}_1 \) and \( \tilde{C}_2 \) that are uniform on \([0, T]\). Notice that, in the above lines, \( TV_{\mathbb{R}} \left\{ k_\Delta - \tilde{k}_\Delta \right\} \) coincides with \( TV_{\mathbb{R}_-} \) of the same quantity and, similarly, the \( L^\infty \)-norm on \( \mathbb{R} \) coincides with the \( L^\infty \)-norm on \( \mathbb{R}_- \). Concerning \( TV_{\mathbb{R}} u \) (similarly for \( TV_{\mathbb{R}_-} \bar{u} \)), we replaced it with \( TV_{\mathbb{R}_-} u \) with an error that is bounded and possibly depending on \( T \).

Summing up \( (2.27) \) in \( n \), we get

\[ e_\Delta(t_N) - e_\Delta(0) = \sum_{n=0}^{N-1} e_\Delta(t_{n+1}) - e_\Delta(t_n) \leq t_N \tilde{C}_1 \sup_{t \in [0, t_N]} TV_{\mathbb{R}_-} \left\{ k_\Delta - \tilde{k}_\Delta \right\} + t_N \tilde{C}_2 \left( 1 + \sup_{t} TV_{\mathbb{R}_-} u(t, \cdot) + \sup_{t} TV_{\mathbb{R}_-} \bar{u}(t, \cdot) \right) \| k_\Delta - \tilde{k}_\Delta \|_{L^\infty([0, t_N] \times \mathbb{R}_-)} \]

Now taking the limit \( \Delta t \to 0 \), we get \( (2.23) \), completing the proof. \( \square \)
3 Well-posedness of the integro-differential equation

In this section we prove the main Theorem 1. In Subsection 3.1 we define a family of approximate solutions to (1.1)–(1.2) and show their compactness, locally in time. Then in Subsection 3.2 we show that the limit solution can be prolonged beyond the existence time, by improving the estimates on upper and lower bound for the exact solution of (1.1)–(1.2). Finally, in Subsection 3.3 we show that the flow generated by the integro-differential equation (1.1) is Lipschitz continuous, restricted to any domain \( D \) given at (1.7).

3.1 Local in time existence of BV solutions

In this Subsection we prove the following existence theorem.

**Theorem 4** Let \( C_0, \kappa_0 \) be given constants and let \( \bar{q}(x) \in L^1(\mathbb{R}^-) \cap BV(\mathbb{R}^-) \) such that

(a) \( \inf_{x < 0} \bar{q}(x) \geq \kappa_0 > -1 \);

(b) \( TV\{\bar{q}(\cdot)\} \leq C_0 \);

(c) \( \|\bar{q}\|_{L^1(\mathbb{R}^-)} \leq C_0 \).

Then there exist \( T > 0, \kappa_1 > -1 \) and \( C_1 > 0 \) such that

\[
\begin{cases}
q_t + \left( \exp \left\{ \int_0^t f(q(t, \xi)) \, d\xi \right\} \right)_x = 0, \\
q(0, x) = \bar{q}(x),
\end{cases}
\]

admits an entropy weak solution \( q(t, x) \) on \([0, T] \times \mathbb{R}^-\) that satisfies

(a)' \( \inf_{x < 0} q(t, x) \geq \kappa_1 > -1 \);

(b)' \( TV\{q(t, \cdot)\} \leq C_1 \);

(c)' \( \|q(t, \cdot)\|_{L^1(\mathbb{R}^-)} \leq \|\bar{q}\|_{L^1(\mathbb{R}^-)} \).

**Proof.** We define a sequence of approximate solution to the scalar equation (1.1)–(1.3). We fix \( \Delta t > 0 \) and set \( t_n = n\Delta t, \, n \in \mathbb{N} \). The approximation is generated recursively, as \( n \) starts from 0 and increases by 1 after each step. For each step with \( n \geq 0 \), let \( q(t, x) \) be defined on \([0, t_n) \times \mathbb{R}^-\) and set

\[
k_n(x) = \exp \left\{ \int_x^0 f(q(t_n, \xi)) \, d\xi \right\}.
\]

Then we define \( q \) on \([t_n, t_{n+1}) \times \mathbb{R}^-\) as the solution of the problem

\[
\begin{cases}
q_t + (k_n(x) f(q))_x = 0, & t \in [t_n, t_{n+1}) \\
q(t_n, x) = q(t_n-, x).
\end{cases}
\]
This procedure leads to a solution operator \( t \mapsto S_t^{\Delta t} \bar{q} = q^{\Delta t}(t, \cdot) \), defined up to a certain time \( T = T(\Delta t, \bar{q}) > 0 \), of the problem

\[
\begin{aligned}
q_t + (k^{\Delta t}(t, x) f(q))_x &= 0, & t > 0 \\
q(0, x) &= \bar{q}(x),
\end{aligned}
\]  

(3.2)

where \( k = k^{\Delta t} \) is defined by

\[
k^{\Delta t}(t, x) = \sum_{n \geq 0} \chi_{[t_n, t_{n+1}]}(t) \cdot k_n(x).
\]  

(3.3)

Notice that the operator \( S_t^{\Delta t} \) has the semigroup property \( S_{t_1+t_2}^{\Delta t} = S_{t_1}^{\Delta t} S_{t_2}^{\Delta t} \) for \( t_1, t_2 \in (\Delta t) \mathbb{N} \).

Now we prove uniform bounds, independent of \( \Delta t \), on the family of approximate solutions.

**The \( \mathcal{L}^1 \) bound.** This follows by the application of (2.3) in Theorem 2 at each time step \([t_n, t_{n+1})\), and the fact that \( t \mapsto q(t, \cdot) \) is continuous in \( \mathcal{L}^1 \). Until the solution is defined, we have

\[
\|q(t, \cdot)\|_{\mathcal{L}^1} \leq \|q(0, \cdot)\|_{\mathcal{L}^1}.
\]  

(3.4)

**Lower and upper bound on \( q \).** Define

\[
z(t) = \inf_x q(t, x), \quad w(t) = \sup_x q(t, x).
\]

We observe that, by comparison with the equilibrium solution \( u \equiv 0 \), (i) if \( z(0) \geq 0 \) then \( z(t) \geq 0 \); and (ii) if \( w(0) \leq 0 \) then \( w(t) \leq 0 \) for all \( t > 0 \).

Now consider \(-1 < z(0) < 0 \) and \( w(t) > 0 \). Choose \( \delta \) and \( M \) such that \( z(0) \geq -1 + 2\delta \) and \( w(0) \leq M/2 \). For example, one can take \( \delta = (\kappa_0 + 1)/2 \) and \( M = 2w(0) \). Let \( T = T(\delta, M) > 0 \) be the first time that one of the following bounds fails,

\[
z(t) \geq -1 + \delta, \quad w(t) \leq M.
\]  

(3.5)

Then, for \( t \leq T \), from the analysis of equation (3.2) (see (2.12)), we find that \( z \) and \( w \) are continuous and satisfy

\[
z(t) \geq z(0) + \sup_x |k_x^{\Delta t}(t, x)| \int_0^t f(z(\tau)) d\tau, \quad z < 0,
\]  

(3.6)

\[
w(t) \leq w(0) + \sup_x |k_x^{\Delta t}(t, x)| \int_0^t f(w(\tau)) d\tau, \quad w > 0.
\]  

(3.7)

Note that in (3.6) we have \( f(z) \leq 0 \), and in (3.7) we have \( f(w) \geq 0 \). For \( |k_x^{\Delta t}| \), we have the estimate

\[
|k_x^{\Delta t}(t, x)| = |k^{\Delta t}(x) f(q(t_n, x))| \leq \exp \left\{ \int_x^0 |f(q(t_n, \xi))| d\xi \right\} f(M) \\
\leq f(M) \exp \{ f'(1 - \delta) \|q\|_{\mathcal{L}^1} \} \leq C(\delta, M).
\]
This gives us
\[
\begin{align*}
z(t) & \geq z(0) + C(\delta, M) \int_0^t f(z(\tau)) \, d\tau \geq z(0) - C(\delta, M) t \left| f'(-1 + \delta) \right|, \\
w(t) & \leq w(0) + C(\delta, M) \int_0^t f(w(\tau)) \, d\tau \leq w(0) + C(\delta, M) t f(M).
\end{align*}
\]
We conclude that the bounds in (3.5) hold for \( t \leq T \) with
\[
T(\delta, M) = \min\{T_1, T_2\},
\]
where
\[
T_1(\delta, M) = \frac{\delta}{C(\delta, M) \left| f'(-1 + \delta) \right|}, \quad T_2(\delta, M) = \frac{M/2}{C(\delta, M) f(M)},
\]
yielding the lower and upper bounds.

Finally, if \( z(0) \geq 0 \) and \( w(0) > 0 \), or if \( z(0) < 0 \) and \( w(0) \leq 0 \), then we would only need to establish one of the bounds in (3.5), and the result follows.

**Bounds on \( f, f', k \).** Once we have a lower, upper bound on \( q \) and the bound on \( \|q\|_{L^1} \), we immediately find that
\[
\begin{align*}
f(q(t,x)), \quad f'(q(t,x)), \quad \int_x^0 f(q(t,\xi)) \, d\xi & \in L^\infty([0,T] \times \mathbb{R}_-) \quad (3.8)
\end{align*}
\]
uniformly w.r.t. \( \Delta t \). By definition (3.3) of \( k \), we can easily verify that the following properties hold uniformly w.r.t. \( \Delta t \):

(i) \( k \in L^\infty([0,T] \times \mathbb{R}_-) \), \( \inf_{t,x} k > 0 \);

(ii) \( k_x \in L^\infty([0,T] \times \mathbb{R}_-) \);

(iii) \( TV k(t,\cdot) \) is bounded uniformly in time.

Indeed, (i) follows from the definition of \( k \) and (3.8). About (ii), at each time \( t \) we have \( k(t,\cdot) = k_n(\cdot) \) for some \( n \), and \( k_x = -k_n f(q(t_n,\cdot)) \). Then \( k_x \in L^\infty \) because of (i) and (3.8). Finally
\[
TV k(t,\cdot) = \|k_x\|_{L^1} = \|k_n f(q(t_n,\cdot))\|_{L^1} \leq M \|k\|_{\infty} \|q(t_n,\cdot)\|_{L^1} \leq M \|k\|_{\infty} \|\bar{q}\|_{L^1},
\]
where \( M = \sup f' \), that depends on the lower bound on \( q \).

Lastly, from (i) and (3.8) one obtains a uniform bound on the characteristic speed \( kf'(q) \).

**Bound on the total variation of \( q \).** By definition of the total variation
\[
TV \{q(t,\cdot)\} \doteq \lim_{h \to 0+} \frac{1}{h} \int_{-\infty}^0 |q(t,x) - q(t,x - h)| \, dx,
\]

12
In conclusion, using also (3.9), we obtain

\[
\frac{1}{h} \int_{-\infty}^{0} |q(t,x) - q(t,x-h)| \, dx \leq \text{TV} \{q(t,\cdot)\}. \tag{3.9}
\]

The total variation of \(q\) does not change at time \(t_n\) when \(k\) is updated. Now consider a time interval \(t \in [t_n, t_{n+1})\), and we estimate the change of the total variation of \(q\) in this time interval. We have

\[
\int_{-\infty}^{0} |q(t_{n+1},x) - q(t_{n+1},x-h)| \, dx \leq \int_{-\infty}^{0} |q(t_n,x) - q(t_n,x-h)| \, dx + \int_{t_n}^{t_{n+1}} E(\tau) \, d\tau \tag{3.10}
\]

where

\[
E(\tau) = \limsup_{\theta \to 0^+} \frac{\int_{-\infty}^{0} |q(\tau + \theta, x-h) - \hat{q}(\tau + \theta, x)| \, dx}{\theta}.
\]

Here \(\hat{q}\) is the entropy solution to

\[
\begin{cases}
  u_t + (k_n(x)f(u))_x = 0, & t \geq \tau, \; x < 0 \\
  u(\tau,x) = q(\tau,x-h).
\end{cases}
\]

On the other hand, \(q(t,x-h)\) is a solution of

\[
\begin{cases}
  u_t + (k_n(x-h)f(u))_x = 0, & t \geq \tau, \; x < 0 \\
  u(\tau,x) = q(\tau,x-h).
\end{cases}
\]

Using the estimate (2.22) we find

\[
E(\tau) \leq \|f\|_\infty \text{TV} \{k_n(\cdot-h) - k_n(\cdot)\} + C(1 + \text{TV} \{q(\tau,\cdot)\}) \|k_n(\cdot-h) - k_n(\cdot)\|_\infty
\]

for a suitable constant \(C\). Notice that

\[
|k_n(x-h) - k_n(x)| = \left| \int_{x-h}^{x} (k_n)_x(\tau,y) \, dy \right| \leq h \|k_n f\|_\infty
\]

and that

\[
\text{TV} \{k_n(\cdot-h) - k_n(\cdot)\} = \| (k_n)_x(\cdot-h) - (k_n)_x(\cdot) \|_{L^1}
\]

\[
\leq \| (k_n(\cdot-h) - k_n(\cdot)) \cdot f(q(\tau,\cdot)) \|_{L^1} + \|k_n(\cdot-h) \cdot (f(q(\tau,\cdot-h)) - f(q(\tau,\cdot))) \|_{L^1}
\]

\[
\leq h \|k_n f\|_\infty \cdot \|f(q(\tau,\cdot))\|_{L^1} + \|k_n\|_{L^\infty} \|f\|_\infty \|q(\tau,\cdot) - q(\tau,\cdot-h)\|_{L^1}.
\]

In conclusion, using also (3.9), we obtain

\[
E(\tau) \leq h \left\{ M_1 + M_2 \text{TV} \{q(\tau,\cdot)\} + M_3 \frac{1}{h} \|q(\tau,\cdot) - q(\tau,\cdot-h)\|_{L^1} \right\}
\]

\[
\leq h \left\{ M_1 + (M_2 + M_3) \text{TV} \{q(\tau,\cdot)\} \right\}
\]

where \(M_i\) depend only on a-priori bounded quantities. Now from (3.10) we obtain

\[
\text{TV} \{q(t_{n+1},\cdot)\} \leq \text{TV} \{q(t_n,\cdot)\} + \int_{t_n}^{t_{n+1}} \left[ M_1 + (M_2 + M_3) \text{TV} \{q(\tau,\cdot)\} \right] d\tau. \tag{3.11}
\]

We conclude that the total variation of \(q\) may grow exponentially in \(t\) on each interval \((t_n,t_{n+1})\), but it remains bounded for any bounded time \(t\).
Convergence to weak solutions; Existence of BV solutions. Now, without causing confusion, we will use $q^\Delta(t,x)$ for the approximate solution, where $\Delta = \Delta t$ is the step size. Let $k^\Delta$ be the approximated coefficient of the equation, defined in (3.3).

By compactness, there exists a subsequence of $\{q^\Delta(t,x)\}$, as $\Delta \to 0$, that converges to a limit function $q(t,x)$ in $L^1_{loc}$. Let $k(t,x)$ be the integral term, (1.9), corresponding to $q$, which is uniformly bounded as well as the $k^\Delta$. We have

$$k^\Delta(t,x) - k(t,x) = \mathcal{O}(1) \left\{ \int_x^0 f(q^\Delta(t_n,\xi)) \, d\xi - \int_x^0 f(q(t,\xi)) \, d\xi \right\}$$

$$= \mathcal{O}(1) \left\{ \sup_{\tau} \text{TV} \{ f(q^\Delta(\tau,\cdot)) \} \sup_\Delta + \int_x^0 [f(q^\Delta(t,\xi)) - f(q(t,\xi))] \, d\xi \right\}$$

that vanishes as $\Delta \to 0$. Therefore we can pass to the limit in the weak formulation. On the interval $[t_n,t_{n+1}]$, $q^\Delta(t,x)$ satisfies

$$\int_{t_n}^{t_{n+1}} \int_{-\infty}^0 (q^\Delta \phi_t + k^\Delta f(q^\Delta) \phi_x) \, dx \, dt = \int_{-\infty}^0 [q^\Delta \phi(t_{n+1},x) - q^\Delta \phi(t_n,x)] \, dx$$

for some test function $\phi$ with compact support inside $[0,T] \times \mathbb{R}_-$. Summing this up over $n$, we get

$$\int_0^T \int_{-\infty}^0 (q^\Delta \phi_t + k^\Delta f(q^\Delta) \phi_x) \, dx \, dt = \int_{-\infty}^0 [q^\Delta \phi(T,x) - q^\Delta \phi(0,x)] \, dx. \quad (3.12)$$

Since $q^\Delta \to q$ in $L^1_{loc}$, $f(q^\Delta) \to f(q)$ in $L^1_{loc}$, $k^\Delta \to k$ pointwise and $k^\Delta$, $k$ are uniformly bounded, by dominated convergence we can take the limit as $\Delta \to 0$ and have the convergence of (3.12) to

$$\int_0^T \int_{-\infty}^0 [q(t,x) \phi_t(t,x) + k(t,x) f(q(t,x)) \phi_x(t,x)] \, dx \, dt = \int_{-\infty}^0 [q(\phi(T,x) - q^\phi(0,x)] \, dx.$$

This completes the proof of existence of BV solutions for (1.1). \hfill \Box

3.2 Global existence of BV solutions

Once the BV solutions exist locally in time, we can further show that they enjoy better properties than the ones deduced from the approximate solutions. In particular we show that the lower and upper bounds on $q$ do not depend on time $t$, leading to global in time existence of BV solutions.

Let $q$ be an entropy weak solution of (3.1) on $[0,T] \times \mathbb{R}_-$. We will now improve the needed bounds.

Lower bound on $q$. Given any point $(\bar{t}, \bar{x}) \in (0,T) \times \mathbb{R}_-$, let $t \to x(t)$ be the minimal backward characteristic (which is classical), defined for $t \in [0,\bar{t}]$. By setting $q(t) = q(t, x(t))$, we have

$$\begin{align*}
x'(t) &= k(t,x) f'(q(t)), \\
q'(t) &= -k_x(t,x) f(q) = kf(q)^2 \geq 0, \quad q(\bar{t}) = q(\bar{t}, \bar{x}-).
\end{align*} \quad (3.13)$$

We see that the solution $q$ is non-decreasing along any characteristics. Therefore, we have $\inf_x q(t,x) \geq \inf \bar{q}(x) \geq \kappa_0 > -1$ for all $t \geq 0$. 

14
Upper bound on \( q \). Again, consider a point \((\bar{t}, \bar{x})\) and let \( t \to x(t) \) be the minimal backward characteristic through it. From the second equation in (3.13) we see that if \( q(0, t(0)) \leq 0 \), then \( q \to 0 \) as \( t \to +\infty \). Now consider \( q(0, x(0)) > 0 \), and we have \( q(t, x(t)) \geq 0 \) for all \( t \). Define

\[
W(t, x) = \int_{-\infty}^{x} |q(t, y)| \, dy, \quad x < 0, 
\]

that satisfies

\[
0 \leq W(t, x) \leq \|q(0, \cdot)\|_{L^1(\mathbb{R}_-)}. 
\]

Using (2.19) with \( \alpha = 1 \), we have

\[
W_t = \int_{-\infty}^{x} |q(t, y)|_t \, dy \leq -\int_{-\infty}^{x} \left( k(t, x) |f(q)| \right)_x \, dy = -k|f(q)|. 
\]

The variation of \( W \) along the characteristic is

\[
\frac{d}{dt}W(t, x(t)) = W_t + x'W_x \leq -k|f| + |q|k|f'| = k\left(-|f| + |q|f'(q)\right) 
\]

\[
= k\left(-f + qf'(q)\right) = -f^2k\left(\frac{f - qf'(q)}{f^2}\right) = -\frac{d}{dt}\left(\frac{q(t, x(t))}{f(q(t, x(t)))}\right). \tag{3.15} 
\]

Here we remove the absolute value signs because \( q > 0 \). Then, (3.15) implies that

\[
W(t, x(t)) + \frac{q(t, x(t))}{f(q(t, x(t)))} \equiv C 
\]

along characteristics. This gives the bound

\[
\frac{q(t, x(t))}{f(q(t, x(t)))} = \frac{q(0, x(0))}{f(q(0, x(0)))} + W(0, x(0)) - W(t, x(t)) \leq C_1, \tag{3.16} 
\]

where \( C_1 \) can be chosen independently of \((\bar{t}, \bar{x})\). Recalling (1.4), we have

\[
\lim_{q \to +\infty} \frac{q}{f(q)} = +\infty. 
\]

Therefore, (3.16) implies an upper bound for \( q \) for all \( t \). The uniform bound on the total variation follows because the constants \( M_i \) in (3.11) are now bounded uniformly in time.

### 3.3 Continuous dependence from the data for the integro-differential equation

In this section we prove the last part of Theorem 1, showing that the flow generated by the integro-differential equation (1.1) is Lipschitz continuous, restricted to any domain \( D \subset L^1(\mathbb{R}_-) \) of functions \( q(\cdot) \) satisfying the following uniform bounds in (1.7), for some constants \( C_0, \kappa_0 \).

Consider two solutions \( q_1(t, \cdot), q_2(t, \cdot) \) of the integro-differential equation (1.1), say with initial data

\[
q_1(0, x) = \bar{q}_1(x), \quad q_2(0, x) = \bar{q}_2(x) \quad x < 0, 
\]
and satisfying the conditions in (1.7) for $t \in [0, T]$. We are going to prove that

$$
\|q_1(t, \cdot) - q_2(t, \cdot)\|_{L^1(\mathbb{R}^-)} \leq \|\bar{q}_1 - \bar{q}_2\|_{L^1(\mathbb{R}^-)} + L \int_0^t \|q_1(s, \cdot) - q_2(s, \cdot)\|_{L^1(\mathbb{R}^-)} ds,
$$

(3.17)

for a suitable constant $L$. By Gronwall lemma, this yields (1.10), hence the Lipschitz continuous dependence of solutions of (1.1) on the initial data.

Define the functions $k_1(t, x), k_2(t, x)$ as in (1.9), corresponding to $q_1(t, x), q_2(t, x)$ respectively. Now set

$$
k^\theta(t, x) = \begin{cases} 
k_1(t, x) & \text{if } t \in [0, \theta], \\
k_2(t, x) & \text{if } t > \theta.
\end{cases}
$$

Finally, for any given $\theta \in [0, T]$, let $q^\theta = q^\theta(t, x)$ be the solution of the conservation law

$$
q_t + (k^\theta(t, x) f(q))_x = 0, \quad q^\theta(0, x) = \bar{q}_2(x).
$$

(3.18)

Observe that, for each fixed $\theta$, the distance between any two entropy-admissible solutions of the conservation law (3.18) is non-increasing in time. In particular, for $\theta = T$, call $\hat{q}$ the solution of

$$
q_t + (k_1(t, x) f(q))_x = 0,
$$

with initial data $\hat{q}(0, x) = \bar{q}_2(x)$ (see Figure 1). We have

$$
\|q_1(t, \cdot) - \hat{q}(t, \cdot)\|_{L^1(\mathbb{R}^-)} \leq \|\bar{q}_1 - \bar{q}_2\|_{L^1(\mathbb{R}^-)} \quad \text{for all } t \in [0, T].
$$

(3.19)

Figure 1: The flow of solutions $q_1, \hat{q}, q^\theta, q_2$ for the integro-differential equation.

Moreover we can use the Lipschitz property of the solution operator for (1.1) with $k = k_2$ fixed, and get the distance estimate

$$
\|\hat{q}(T, \cdot) - q_2(T, \cdot)\|_{L^1(\mathbb{R}^-)} \leq \int_0^T E(\tau) \, d\tau,
$$

(3.20)
where

\[ E(\tau) \doteq \lim_{h \to 0+} \sup_{h} \frac{\|q^h(\tau + h, \cdot) - \hat{q}(\tau + h, \cdot)\|_{L^1}}{h}. \]

Indeed, observe that \( \hat{q}(\tau, \cdot) = q^0(\tau, \cdot) \) whenever \( \tau \leq \theta \), for any \( \tau \in [0,T] \).

To compute the integrand in (3.20), observe that the functions \( h \mapsto q^h(\tau + h, \cdot) \) and \( h \mapsto \hat{q}(\tau + h, \cdot) \) take the same value \( \hat{q}(\tau, \cdot) \) when \( h = 0 \), and \( h \mapsto q^h(\tau + h, x) \) satisfies the conservation law

\[ q_h + (k_2(\tau + h, x) f(q))_x = 0, \quad (3.21) \]

while \( h \mapsto \hat{q}(\tau + h, x) \) solves

\[ q_h + (k_1(\tau + h, x) f(q))_x = 0, \quad (3.22) \]

for \( h \geq 0 \). By using (2.23) in Theorem 3, we can measure the error term \( E(\tau) \). By the facts that \( \|q^h(\tau, \cdot)\|_{L^\infty}, \|\hat{q}(\tau, \cdot)\|_{L^\infty}, \text{TV} \{q^h(\tau, \cdot)\}, \text{TV} \{\hat{q}(\tau, \cdot)\}, \text{TV} \{k_1(\tau, \cdot)\} \) and \( \text{TV} \{k_2(\tau, \cdot)\} \) are all bounded, the coefficients \( C_1 \) and \( C_2 \) in (2.23) are all bounded constants. Let \( M \) be a generic bounded constant, we get

\[ \|q^h(\tau + h, \cdot) - \hat{q}(\tau + h, \cdot)\|_{L^1} \leq M h \left[ \sup_{\tau \leq t \leq \tau + h} \text{TV} (k_1(t, \cdot) - k_2(t, \cdot)) + \|k_1 - k_2\|_{L^\infty([\tau, \tau + h] \times \mathbb{R} \_\_)} \right]. \]

Therefore, we have

\[ E(\tau) = M \cdot \text{TV} \{k_1(\tau, \cdot) - k_2(\tau, \cdot)\} + M \cdot \|k_1(\tau, \cdot) - k_2(\tau, \cdot)\|_{L^\infty} \quad (3.23) \]

Recalling the definitions of \( k_1, k_2 \) we deduce that

\[ \|k_1(\tau, \cdot) - k_2(\tau, \cdot)\|_{L^\infty} = M \cdot \sup_{x < 0} \left| \int_{x}^{0} f(q_1(\tau, y)) \ dy - \int_{x}^{0} f(q_2(\tau, y)) \ dy \right| \]

\[ = M \cdot \|q_1(\tau, \cdot) - q_2(\tau, \cdot)\|_{L^1}, \quad (3.24) \]

and, using also (3.24),

\[ \text{TV} \{k_1(\tau, \cdot) - k_2(\tau, \cdot)\} = \|(k_1)_x(\tau, \cdot) - (k_2)_x(\tau, \cdot)\|_{L^1} \]

\[ = \|(k_1(\tau, \cdot) f(q_1(\tau, \cdot)) - k_2(\tau, \cdot) f(q_2(\tau, \cdot))\|_{L^1} \]

\[ \leq \|(k_1(\tau, \cdot) - k_2(\tau, \cdot)) f(q_1(\tau, \cdot))\|_{L^1} + \|k_2(\tau, \cdot) \cdot (f(q_1(\tau, \cdot)) - f(q_2(\tau, \cdot)))\|_{L^1} \]

\[ = \|k_1(\tau, \cdot) - k_2(\tau, \cdot)\|_{L^\infty} \cdot \|f(q_1(\tau, \cdot))\|_{L^1} + \|k_2(\tau, \cdot)\|_{L^\infty} \|q_1(\tau, \cdot) - q_2(\tau, \cdot)\|_{L^1} \]

\[ = M \|q_1(\tau, \cdot) - q_2(\tau, \cdot)\|_{L^1}. \quad (3.25) \]

Putting the estimates (3.24) and (3.25) into (3.23), we get

\[ E(\tau) \leq L \cdot \|q_1(\tau, \cdot) - q_2(\tau, \cdot)\|_{L^1} \]

for a suitable constant \( L \). Inserting this estimate in (3.20) and using (3.19) one finally obtains (3.17).
A Properties of the integral operator

In this Appendix we prove some properties of the integral term \( k \) in terms of a Lipschitz flow \( t \mapsto q(t, \cdot) \). The operator \( K \), see (1.3), is defined on the set

\[
\left\{ q \in L^1(\mathbb{R}_-) \cap BV(\mathbb{R}_-) ; \quad \inf_{x<0} q(x) > -1 \right\}
\]

and valued in \( Lip(\mathbb{R}_-) \). Its properties are summarized in the following Proposition.

**Proposition 2** Let \( C_0, \kappa_0, T \) be given positive constants. Assume that the map \( q : [0, T] \rightarrow D_{C_0,\kappa_0} \) is Lipschitz continuous as a function in \( L^1(\mathbb{R}_-) \).

Define \( k \) as in (1.9). Then

\[
\begin{align*}
(\text{K}) & \quad k(t, x) : [0, T] \times \mathbb{R}_- \rightarrow \mathbb{R}_+ \text{ is bounded and Lipschitz continuous, with } \inf_{t,x} k > 0; \\
& \quad \text{TV} k(t, \cdot) , \text{ TV} k_x(t, \cdot) \text{ are bounded uniformly in time; } \\
& \quad [0, T] \ni t \mapsto k_x(t, \cdot) \in L^1(\mathbb{R}_-) \text{ is Lipschitz continuous.}
\end{align*}
\]

**Proof.** To begin, notice that the quantity \( k \) is well-defined and is Lipschitz continuous on \([0, T] \times \mathbb{R}_-\).

Let \( L \) be a Lipschitz constant of the map \([0, T] \ni t \mapsto q(t) \in L^1(\mathbb{R}_-)\). From the bounds (1.7) one easily deduces that

\[
\begin{align*}
\|q(t, \cdot)\|_{L^\infty(\mathbb{R}_-)} & \leq C_0, \quad \text{(A.1)} \\
\|f(q(t, \cdot))\|_{L^\infty(\mathbb{R}_-)} & \leq \max \{|f(C_0)|, |f(\kappa_0)|\}, \quad \text{(A.2)} \\
\|f(q(t, \cdot))\|_{L^1(\mathbb{R}_-)} & \leq |f'(\kappa_0)| \cdot \|q(t, \cdot)\|_{L^1(\mathbb{R}_-)} \leq C_0|f'(\kappa_0)|, \quad \text{(A.3)} \\
\|f(q(t_1, \cdot)) - f(q(t_2, \cdot))\|_{L^1(\mathbb{R}_-)} & \leq L|f'(\kappa_0)| \cdot |t_1 - t_2|. \quad \text{(A.4)}
\end{align*}
\]

By the assumptions on \( q \) we find that

\[
\left| \int_x^0 f(q(t, \xi)) d\xi \right| \leq \|f(q(t, \cdot))\|_{L^1(\mathbb{R}_-)} \leq C_0|f'(\kappa_0)|.
\]

Hence the integral term \( k \) is bounded and satisfies

\[
0 < \exp \left(-C_0|f'(\kappa_0)|\right) \leq k(t, x) \leq \exp \left(C_0|f'(\kappa_0)|\right).
\]

Moreover, for all \( 0 \leq t_1 < t_2 \) we have

\[
\left| \int_x^0 [f(q(t_1, \xi)) - f(q(t_2, \xi))] d\xi \right| \leq \|f(q(t_1, \cdot)) - f(q(t_2, \cdot))\|_{L^1(\mathbb{R}_-)} \leq L|f'(\kappa_0)| \cdot |t_1 - t_2|.
\]

This leads to the Lipschitz continuity in \( t \) for \( k(t, x) \). Namely, for all \( x \) we have

\[
|k(t_1, x) - k(t_2, x)| = O(1) \left| \int_x^0 [f(q(t_1, \xi)) - f(q(t_2, \xi))] d\xi \right| \leq \hat{L} |t_1 - t_2|. \quad \text{(A.5)}
\]

Here the Lipschitz constant \( \hat{L} \) depends on the parameters \( L, C_0, \kappa_0 \).
From the definition of $k$, the derivative function $k_x$ satisfies
\[ k_x = -kf(q) \in L^1 \cap L^\infty. \quad (A.6) \]
This immediately shows three facts: (i) $k(t,x)$ is Lipschitz in space variable $x$, (ii) $k(t,\cdot) \in BV(\mathbb{R}_-)$ where the BV bounds are uniform in $t$, and (iii) $k_x(t,\cdot) \in BV(\mathbb{R}_-)$. From (A.6) we get the estimate on the total variation of $k_x$
\[ TV(k_x) \leq TV(k) \cdot \|f(q(t,\cdot))\|_{L^\infty(\mathbb{R}_-)} + \|k\|_{L^\infty(\mathbb{R}_-)} TV(f(q(t,\cdot))) \leq M TV(q), \]
with $M$ depending on the parameters $L, C_0, \kappa_0$.

Finally, we show that $[0,T] \ni t \mapsto k_x(t,\cdot) \in L^1(\mathbb{R}_-)$ is Lipschitz continuous. By using (A.6), (A.3) and (A.4), one has
\[
\|k_x(t_1,\cdot) - k_x(t_2,\cdot)\|_{L^1(\mathbb{R}_-)} = TV\{k(t_1,\cdot) - k(t_2,\cdot)\} = \|k(t_1,\cdot)f(q(t_1,\cdot)) - k(t_2,\cdot)f(q(t_2,\cdot))\|_{L^1(\mathbb{R}_-)} \\
\leq \|k(t_1,\cdot) - k(t_2,\cdot)\|_{L^\infty(\mathbb{R}_-)} \|f(q(t_1,\cdot))\|_{L^1(\mathbb{R}_-)} + \|k(t_2,\cdot)\|_{L^\infty(\mathbb{R}_-)} \|f(q(t_1,\cdot)) - f(q(t_2,\cdot))\|_{L^1(\mathbb{R}_-)} \\
\leq \hat{M}|t_1 - t_2|
\]
with $\hat{M}$ depending on the parameters $L, C_0, \kappa_0$. \hfill \Box

Acknowledgement. This paper was started as part of the international research program on Nonlinear Partial Differential Equations at the Centre for Advanced Study at the Norwegian Academy of Science and Letters in Oslo during the academic year 2008–09. The first author would like to acknowledge also the kind hospitality of the Department of Mathematics, University of Ferrara. The work of the second author is partially supported by NSF grant DMS-0908047.

References

[1] Amadori, D., Gosse, L. and Guerra, G.; Godunov-type approximation for a general resonant balance law with large data. *J. Differential Equations* **198** (2004), 233–274

[2] Amadori, D. and Shen, W.; The Slow Erosion Limit in a Model of Granular Flow, *Arch. Ration. Mech. Anal.*, **199** (2011), 1–31

[3] Amadori, D. and Shen, W.; Front Tracing Approximations for Slow Erosion in Granular Flow. Preprint 2010

[4] Baiti, P. and Jenssen, H. K.; Well-posedness for a class of $2 \times 2$ conservation laws with $L^\infty$ data. *J. Differential Equations* **140** (1997), 161–185

[5] Bressan, A. and Zhang, P. and Zheng, Y.; *Asymptotic variational wave equations*. Arch. Ration. Mech. Anal. **183** (2007), 163–185
[6] Camassa, R. and Holm, D.; An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. 71 (1993), 1661–1664

[7] Chen, G.-Q. and Christoforou, C.: Solutions for a nonlocal conservation law with fading memory. Proc. Amer. Math. Soc. 135 (2007), 3905–3915

[8] Colombo, R.M., Herty, M. and Mercier, M.; Control of the Continuity Equation with a Non Local Flow. To appear on ESAIM COCV

[9] Colombo, R.M., Mercier, M. and Rosini M.; Stability and total variation estimates on general scalar balance laws. Commun. Math. Sci. 7 (2009), 37–65

[10] Dafermos, C.M.; Solutions in $L^\infty$ for a conservation law with memory. Analyse mathematique et applications, 117–128, Gauthier-Villars, Montrouge, 1988

[11] Guerra, G.; Well-posedness for a scalar conservation law with singular nonconservative source. J. Differential Equations 206 (2004), 438–469

[12] Hadeler, K.P. and Kuttler, C.; Dynamical models for granular matter. Granular Matter 2 (1999), 9–18

[13] Karlsen, K.-H. and Risebro, N.-H.; On the uniqueness and stability of entropy solutions of nonlinear degenerate parabolic equations with rough coefficients. Discrete Contin. Dyn. Syst. 9 (2003), 1081–1104

[14] Klausen, R. A. and Risebro, N. H.; Stability of conservation laws with discontinuous coefficients. J. Differential Equations 157 (1999), 41–60

[15] Kružkov, S.N.; First order quasilinear equations with several independent variables. Mat. Sb. (N.S.) 81 (123) (1970), 228–255

[16] Lin, L. and Temple, J. B. and Wang, J.; Suppression of oscillations in Godunov’s method of a resonant non-strictly hyperbolic system, SIAM J. Numer. Anal. 32(3) (1995), 841–864

[17] Shen, W. and Zhang, T.; Erosion Profile by a Global Model for Granular Flow. Preprint 2010, accepted for publication on Arch. Ration. Mech. Anal.