The black ring entropy from the Weyl tensor

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Abstract A black ring is an asymptotically flat vacuum solution of the Einstein equations with an event horizon of topology $S^1\times S^2$. A connection between the black ring entropy and its Weyl tensor $C_{\mu\nu\lambda\rho}$ is explored by interpreting the Weyl scalar invariant $C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}$ as the entropy density in 5-dimensional space-time. We calculate the proper volume integral of $C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}$ for a neutral black ring and prove that it is proportional to the entropy of a thin black ring. Similar calculations are extended to more general cases: the black string, the black ring with two angular momenta, and the black ring with a cosmological constant. The proportionality still maintains or is valid at least at the leading order.

1 Introduction

Several decades ago, Penrose conjectured \cite{1} that there could be some latent relation between the Weyl tensor and the entropy of gravitational field. This conjecture was initiated in cosmology, based on the fact that the universe evolves from an almost homogeneous and isotropic space-time to a phase with large-scale structures, during which the Weyl tensor, due to its conformal symmetry, grows monotonically relative to the Ricci tensor. This monotonic increasing behavior reminds us of the second law of thermodynamics, and some scalar invariant constructed from the Weyl tensor could thus be identified with the entropy of the universe. Whereas, to our knowledge, the Penrose conjecture has not been physically interpreted or mathematically formulated in a rigorous way. Therefore, we may infer that the information encoded in the Weyl tensor is associated not to the dynamical, but to the thermodynamical aspects of gravitation. This observation provides us a new and purely geometrical point of view on the thermodynamics of gravitational field.

Even though, it is still very plausible that the Weyl tensor could be related to the thermodynamics of gravitational field, and some entropy measure could be constructed therefrom. This idea can be understood as follows. In mathematics, the curvature of an $n$-dimensional pseudo-Riemann manifold is described by the Riemann tensor $R_{\mu\nu\lambda\rho}$, and it can be decomposed into the Ricci and Weyl sectors,

$$R_{\mu\nu\lambda\rho} = \frac{2}{n-2}(g_{\mu\lambda}R_{\nu\rho} - g_{\nu\lambda}R_{\mu\rho}) - \frac{2}{(n-1)(n-2)}g_{\mu\nu}g_{\lambda\rho}R + C_{\mu\nu\lambda\rho},$$

(1)

where $R_{\mu\nu}$ is the Ricci tensor, $R$ is the Ricci scalar, and $C_{\mu\nu\lambda\rho}$ is the Weyl tensor, respectively. However, in physics, the Einstein equations only link the Ricci tensor to the energy–momentum tensor of gravitational field. As a result, the Weyl tensor is absent in classical general relativity; in other words, it is locally independent of the energy–momentum tensor. Therefore, we may infer that the information encoded in the Weyl tensor is associated not to the dynamical, but to the thermodynamical aspects of gravitation. This observation provides us a new and purely geometrical point of view on the thermodynamics of gravitational field, and can thus serve as a physical realization of the Penrose conjecture. Since the Weyl tensor is traceless and shares the same symmetries as the Riemann tensor, the Weyl scalar invariant, i.e., its complete contraction,

$$W := C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho},$$

is the principal scalar invariant that we are able to construct. Below, we will focus on this scalar invariant and explore its relation to the black ring entropy.
Now, we explain our idea more quantitatively. We first show how the Weyl tensor could be related to the black hole entropy. The metric of the Schwarzschild black hole is \( ds^2 = -(1 - R_S/r) dt^2 + (1 - R_S/r)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \), where the event horizon locates at the Schwarzschild radius \( R_S = 2GM \), and \( M \) is the mass of the black hole. It is direct to obtain the Weyl scalar invariant,

\[
W = \frac{12R_S^2}{r^6}.
\]

On the other hand, the Bekenstein–Hawking formula \([25, 26]\) indicates that the entropy of the Schwarzschild black hole is

\[
S = \frac{A}{4G} = \frac{\pi R_S^2}{G},
\]

where \( A = 4\pi R_S^2 \) is the event horizon area. Comparing these two results, we immediately observe the proportion between \( W \) and \( S \). This proportionality inspires us to go further. Since the Weyl scalar invariant depends on the radial coordinate \( r \) and does not decrease inside the event horizon, if evaluated along the world-lines of the physical observers, we may make an educated interpretation that the Weyl scalar invariant is, or at least proportional to, the entropy density of the gravitational field of the Schwarzschild black hole, and the Bekenstein–Hawking entropy could thus be achieved by its proper volume integral.

Unfortunately, a dimensional analysis easily invalidates this simple attempt. In 4-dimensional space-time, the dimension of the Weyl scalar invariant, \([W]\), is +4 in the natural system of units, but the dimension of the proper volume element, \([dV_4]\), is −3. Therefore, the volume integral \( \int W dV_4 \) has the dimension +1, but entropy itself is dimensionless. Consequently, we are allowed to regard the Weyl scalar invariant as entropy density only in 5-dimensional space-time, in which \([dV_4] = -4\), so the proper volume integral

\[
\int W dV_4
\]

is dimensionless, and could hopefully lead to the correct entropy formula. We may further consider this problem in a general \( n \)-dimensional space-time, in which \([dV_{n-1}] = -(n-1)\), whereas \([W]\) remains +4. Hence, dimension 5 is the unique possibility that \( W \) could be interpreted as the entropy density of gravitational field.

In Ref. [27], this idea was first explored for the 5-dimensional Schwarzschild and Schwarzschild–anti-de Sitter black holes. It was found that the integral in Eq. (2) does lead to the Bekenstein–Hawking formula,

\[
S \propto \int W dV_4,
\]

where the coefficient of proportionality is 1/96 for the Schwarzschild black hole and \( A \)-dependent for the Schwarzschild–anti-de Sitter black hole, with \( A \) being the cosmological constant.

The present paper is a successive work of Ref. [27], as dimension 5 is indeed very special in high-dimensional general relativity and hence deserves careful studies. For instance, it is the unique possibility that allows asymptotically flat solutions for the Einstein equations besides dimension 4. Therefore, in addition to the 5-dimensional black holes with ordinary spherical horizon topology already studied in Ref. [27], we proceed in this paper to consider another important and interesting family of black holes with non-spherical horizon topologies: the 5-dimensional black ring (with one or two angular momenta) and the black string. Detailed calculations show that the proper volume integral of the Weyl scalar invariant in Eq. (2) is again proportional to the entropy of black ring or black string. All these results further confirm our interpretation of the Weyl scalar invariant as the entropy density in 5-dimensional space-time.

This paper is organized as follows. In Sect. 2, a 5-dimensional ring coordinate system is introduced, in which the black ring metric can be expressed more conveniently. Afterward, we explore the possibility to obtain the black ring entropy by integrating its Weyl scalar invariant in Sect. 3. In Sect. 4, we further discuss this procedure for the black string, the black ring with two angular momenta, and the black ring with a cosmological constant, respectively. We conclude in Sect. 5. We work in the natural system of units and set \( c = \hbar = k_B := 1 \), but still keep the 5-dimensional gravitational constant \( G_5 \), and the reason will be seen later.

## 2 The black ring

In this section, we briefly list the relevant properties of the black ring, which is a necessity for the following calculations. A black ring is an asymptotically flat black hole solution of the 5-dimensional Einstein equations with a regular event horizon of topology \( S^1 \times S^2 \) \([28–36]\), and is thus different from other 5-dimensional black holes with spherical event horizon of topology \( S^3 \) \([37]\). Therefore, the black ring solution is of great importance and interest in general relativity, as it serves as an explicit evidence indicating the invalidity of the uniqueness theorems in high-dimensional space-times. (See Refs. \([31, 34–36]\) for excellent reviews.) Compared with a black string or black brane, a black ring is curled, and its total length is finite. This makes the black ring much more stable \([32]\) to small perturbations \([38, 39]\). To avoid gravitational collapse, a black ring must rotate or be charged to balance the contraction. In 5-dimensional space-time, a black ring may rotate in two mutually orthogonal planes and thus have two independent angular momenta. In the following, we focus on the neutral
black ring with one angular momentum in Sect. 3 and with two in Sect. 4.2.

We start from a so-called ring coordinate system [31], which facilitates us to comprehend the geometry of black ring. To see it clearly, let us first consider a 4-dimensional flat space and choose the polar coordinates for two orthogonal planes. By this means, the flat metric is

$$ds^2_4 = dr_1^2 + r_1^2 d\phi^2 + dr_2^2 + r_2^2 d\psi^2,$$

(3)

where $r_1$, $\phi$ and $r_2$, $\psi$ denote the radial and angular coordinates in the two planes. Now, in the ring coordinate system, the coordinate transformation and inverse transformation are

$$x = - \frac{r_1^2 + r_2^2 - R^2}{\Sigma}, \quad y = - \frac{r_1^2 + r_2^2 + R^2}{\Sigma},$$

(4)

and

$$r_1 = \sqrt{\frac{1-x^2}{x-y}} R, \quad r_2 = \sqrt{\frac{1}{x-y}} R,$$

(5)

where $\Sigma = \sqrt{(r_1^2 + r_2^2 + R^2)^2 - 4r_1^2 R^2}$, and $R$ characterizes the scale of the black ring ($\phi$ and $\psi$ are unchanged). From Eq. (5), it is easy to find the coordinate ranges $-1 \leq x \leq 1$ and $-\infty \leq y \leq -1$. Therefore, the coordinates $r_1 = 0$ and $r_2 = R$ correspond to $y = -\infty$, and asymptotic infinity is redefined by $x \rightarrow -1$ and $y \rightarrow -1$. With these transformations, the 4-dimensional flat metric in Eq. (3) can be expressed in terms of the ring coordinates,

$$ds^2 = \frac{R^2}{(x-y)^2} \left[ \frac{dx^2}{1-x^2} + (1-x^2) d\phi^2 - \frac{dy^2}{1-y^2} - (1-y^2) d\psi^2 \right].$$

(6)

We now place a circular neutral black ring at $r_1 = 0$ and $r_2 = R$, and let it rotate only along the $\psi$-direction. The metric of the 5-dimensional black ring preserves most of the properties in Eq. (6) and reads [28, 36]

$$ds^2 = \frac{F(y)}{F(x)} \left( dr - \frac{\lambda (1-y)}{1-\lambda} \frac{1+y}{F(y)} R d\psi \right)^2 + \frac{R^2 F(x)}{(x-y)^2} \left[ \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\phi^2 - \frac{dy^2}{G(y)} - \frac{G(y)}{F(y)} d\psi^2 \right],$$

(7)

where $F(\xi) = 1 + \lambda \xi$ and $G(\xi) = (1 - \xi^2)(1 + v \xi)$. The dimensionless parameters $\lambda$ and $v$ measure the rotation velocity and the shape of the black ring, and smaller values of $v$ correspond to thinner rings. They vary within the range $0 < v \leq \lambda < 1$, and we can recover the 4-dimensional flat metric in Eq. (6) in the limit $\lambda = v = 0$. Moreover, in order to avoid the conical singularities at $x = -1$ and $y = -1$, $\phi$ and $\psi$ must be identified, with the periods

$$\Delta \phi = 2\pi \sqrt{1 - \lambda \over 1 - v}.$$

(8)

Furthermore, to avoid another conical singularity at $x = 1$, we must also set $\Delta \phi = 2\pi \sqrt{1 + \lambda / (1 + v)}$, so the relation between $\lambda$ and $v$ (equilibrium condition) is

$$\lambda = \frac{2v}{1 + v^2}.$$

(9)

With this restriction, Eq. (8) can be finally reduced to

$$\Delta \phi = 2\pi \sqrt{1 + v^2}.$$

 Altogether, there remain only two independent parameters in Eq. (7), $R$ and $v$, the scale and shape of the black ring. The angular momentum must be tuned so as to balance the gravitational self-attraction, so there leaves no free parameter anymore.

3 The black ring entropy from the Weyl scalar invariant

In the following, we calculate in detail the proper volume integral of the Weyl scalar invariant for the black ring, and show that it is proportional to the black ring entropy, up to some coefficient.

From Eq. (1), the explicit form of the Weyl tensor in 5-dimensional space-time reads

$$C_{\mu \nu \lambda \rho} = R_{\mu \nu \lambda \rho} + \frac{1}{3} (g_{\nu \lambda} R_{\mu \rho} + g_{\mu \rho} R_{\nu \lambda} - g_{\mu \lambda} R_{\nu \rho} - g_{\nu \rho} R_{\mu \lambda} + \frac{1}{12} (g_{\nu \rho} g_{\mu \lambda} - g_{\nu \lambda} g_{\mu \rho}) R).$$

(10)

From Eqs. (7) and (10), it is straightforward to obtain the Weyl scalar invariant $W$. The complete expression (a polynomial with respect to the $y$ coordinate) is mathematically rather tedious, but fortunately only its leading order term is physically relevant, because the dominating contribution from $W$ to the integral in Eq. (2) comes from the region around the black ring, where $y$ tends to $-\infty$. Even far from this region, the absolute value of $y$ is still much larger than 1 in the ring coordinate system, so we are always allowed to focus on the leading order term of $W$,

$$W = \frac{12 (1 + v^2)^2 v^2 \lambda^6}{(1 + 2\lambda v + v^2)^2 R^4}.$$

(11)

Furthermore, the leading order term (with respect to $y$) of the 4-dimensional proper volume element is

$$dV_4 = \frac{(1 + v)^{3/2} \sqrt{1 + 2\lambda v + v^2} R^2}{(1 + v^2)^{1/2} (1 - v^2)} d\phi d\psi.$$

(12)
The horizon crossing problem will be discussed in detail at the end of this section.

Altogether, from Eqs. (11) and (12), the proper volume integral of the Weyl scalar invariant is

$$\int W \, dV_4 = \frac{12(1+v^2)(1+v)^{3/2} v^2}{\sqrt{1-v}(1+2iv+v^2)^{3/2}} \, d\phi d\psi. $$

We may first safely perform the integrals for the $x, \phi,$ and $\psi$ coordinates, with the integral intervals $-1 \leq x \leq 1, 0 \leq \phi \leq 2\pi/\sqrt{1+v^2},$ and $0 \leq \psi \leq 2\pi/\sqrt{1+\nu^2},$ so

$$\int W \, dV_4 = \int_{y_0} \frac{96\pi^2 \sqrt{1+v^2}}{(1-v)^{3/2}} \, dy. $$

However, for the $y$ coordinate, we cannot directly set the integral interval to be $-\infty \leq y \leq -1,$ as the lower limit $-\infty$ will blow up the integral. Consequently, our strategy is first to integrate $y$ from $-y_0 (y_0 \gg 1)$ to $-1,$ i.e., from the region just slightly off the black ring to asymptotic infinity, and then to discuss the physical interpretation of $y_0,$ in order to obtain a meaningful result. Therefore,

$$\int W \, dV_4 = \int_{-1}^{y_0} \frac{96\pi^2 \sqrt{1+v^2}}{(1-v)^{3/2}} \, dy \\ \approx \frac{32\pi^2 \sqrt{1+v^2} y_0^3}{(1-v)^{3/2}},$$

where the fact $y_0 \gg 1$ has already been taken into account.

We now discuss the physical meaning of the lower limit $-y_0.$ From Eq. (4), setting $r_1 = 0$ yields

$$y = -\frac{r_2^2 + R^2}{|r_2^2 - R^2|},$$

and $y = -\infty$ corresponds to $r_2 = R.$ However, it is generally acknowledged that classical general relativity is invalid at very small scales (e.g., several Planck lengths), where quantum gravity effects significantly emerge. Hence, we may first choose the lower limit $-y_0$ in such a way that the corresponding ring coordinate $r_2$ satisfies

$$r_2 = R \pm \ell_5,$$

with $\ell_5 = \sqrt{G_5}$ being the 5-dimensional Planck length. Therefore,

$$-y_0 = -\frac{(R \pm \ell_5)^2 + R^2}{|(R \pm \ell_5)^2 - R^2|} \approx -\frac{R}{\ell_5},$$

where we have used the fact $R \gg \ell_5,$ for the scale of a black ring should certainly be much larger than the Planck length.

With this choice, Eq. (13) becomes

$$\int W \, dV_4 = \frac{32\pi^2 \sqrt{1+v^2} R^3}{(1-v)^{3/2} \ell_5^3}. $$

This is the final result for the proper volume integral of the Weyl scalar invariant.

Meanwhile, the Bekenstein–Hawking entropy of the black ring can be obtained by evaluating its event horizon area. From Eq. (7), the black ring has a regular event horizon at $r_h = 1/v,$ so an easy integral over the $x, \phi,$ and $\psi$ coordinates leads to its area,

$$A = \frac{8\sqrt{2}\pi^2 v^2 R^3}{(1-v)(1+v^2)^{3/2}}.$$}

Thus, the black ring entropy is simply written as

$$S = \frac{A}{4G_5} = \frac{A}{4\ell_5^3} = \frac{2\sqrt{2}\pi^2 v^2 R^3}{(1-v)(1+v^2)^{3/2} \ell_5^3},$$

(16)

Till now, from the results in Eqs. (15) and (16), we clearly observe that the black ring entropy is proportional to the proper volume integral of the Weyl scalar invariant (both of them $\approx R^3/\ell_5^3$), albeit the coefficient of proportionality depends on the shape parameter $v$ of the black ring. This complexity may be eliminated to a great extent, if we focus on the limit of thin black rings with $v \ll 1.$ In other words, if we consider the leading order terms (with respect to $v$) in Eqs. (15) and (16),

$$\int W \, dV_4 \approx \frac{32\pi^2 v^2 R^3}{\ell_5^3},$$

and

$$S \approx \frac{2\sqrt{2}\pi^2 v^2 R^3}{\ell_5^3},$$

(18)

we immediately arrive at

$$S = \frac{\sqrt{2}}{16} \int W \, dV_4.$$ (19)

This result eventually confirms our idea that the Weyl scalar invariant can be regarded as the entropy density of gravitational field in 5-dimensional space-time, and its proper volume integral thus leads to the black ring entropy.\(^1\) This idea was first explored in Ref. [27] for the 5-dimensional Schwarzschild and Schwarzschild–anti-de Sitter black holes with simple spherical horizon topology, and is now successfully investigated again for the more complicated 5-dimensional black ring with non-spherical horizon topology. Actually, it will also be applied to other more general black ring solutions in the subsequent section. Altogether, it is reasonable to physically interpret the Weyl scalar invariant as the entropy density in 5-dimensional space-time.

\(^1\)Here, we should point out that the coefficient of proportionality for the thin black rings in Eq. (19), $\sqrt{2}/16,$ seems different from that for the Schwarzschild black hole, $1/96$ [27]. Whereas, this is not a severe problem, as the choice of the ring coordinate $r_2$ in Eq. (14) is just a preliminary estimate. We may of course set $r_2 = R \pm \alpha \ell_5$ for generality, where $\alpha$ is some number of $O(1),$ and adjusting $\alpha$ may make things look better. However, the definite value of $\alpha$ is irrelevant, and only the proportionality between the volume integral and the entropy concerns.
At last, we should give a detailed discussion on the horizon crossing problem when integrating the y coordinate. Strictly speaking, the leading order term of the 4-dimensional proper volume element should be
\[
dV_4 = \frac{(1 + v)^2}{(1 - v^2)^{3/2}} \sqrt{(1 + 2yv + v^2)\psi R^4} \, \mathrm{d}x \mathrm{d}y \mathrm{d}\psi.
\]
Naturally, this result reduces to that in Eq. (12) if \( y \ll -1 \). However, when \( y \) crosses the event horizon at \( y_h = -1/\sqrt{v} \), the term \( 1 + vy \) in the denominator will change sign. Therefore, strictly speaking, the proper volume integral of the Weyl scalar invariant should be
\[
\int W \, dV_4 = \frac{96\pi^2}{(1 - v^2)^{3/2}} \sqrt{(1 + v^2)} \left( \int_{-y_0}^{y_h} \frac{y^5}{1 + vy} \, \mathrm{d}y + \int_{y_h}^{-y_0} \left( -\frac{y^5}{1 + vy} \right) \, \mathrm{d}y \right).
\]
Above, the first integral will give exactly the same result in Eq. (13) in the limit \( y_0 \gg 1 \), meaning that only the region around the black ring dominates the integral. Moreover, the second integral, i.e., the contribution outside the event horizon, is a trivial number independent of \( y_0 \) and can thus be safely disregarded. These observations approve our assessment that there is effectively no horizon crossing problem in the volume integral.

4 Discussions

In this section, we continue to discuss some more complicated cases of the black rings: the black string, the black ring with two angular momenta, and the black ring with a cosmological constant. All the calculations further support our results in the simplest case of the black ring with one angular momentum in Sect. 3.

4.1 Black string

A black string can be viewed as an extreme black ring with an infinite scale \( R \). Therefore, we may obtain the metric of a boosted black string from Eq. (7) in the limit \( R \to \infty \). First, it is convenient to transform the coordinates from \( x, y, \) and \( \psi \) to \( r, \theta, \) and \( z \),
\[
r = -\frac{R}{y} \cos \theta = x, \quad z = \psi R,
\]
and the string is extended along the \( z \) direction (\( \phi \) is unchanged). The parameters \( \lambda \) and \( v \) may also be reparameterized in terms of \( r_0 \) and \( \sigma \),
\[
\lambda = \frac{r_0 \cosh^2 \sigma}{R}, \quad v = \frac{r_0}{R}.
\]
Then, we take the limit \( R \to \infty \), while keeping the coordinates \( r, \theta, z \) and the parameters \( r_0, \sigma \) finite. Now, Eq. (7) turns to
\[
dx^2 = -\hat{f} \left( \frac{dr}{rf} \sinh \sigma \cosh \sigma \, dz \right)^2 + \frac{1}{f} \, dz^2 + \frac{d\theta^2}{f} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2;
\]
where \( f = 1 - r_0/r \) and \( \hat{f} = 1 - r_0 \cosh^2 \sigma/r \). From Eq. (21), it is easy to see that the event horizon of the black string locates at \( r = r_0 \). The equilibrium condition in Eq. (9) is now expressed as \( \cosh^2 \sigma = 2R^2/(R^2 + r_0^2) \), and when \( R \to \infty \),
\[
\cosh^2 \sigma \to 2.
\]
This is the limiting condition for maintaining mechanical equilibrium of a black string of very large scale \( R \).

From Eqs. (10) and (21), we obtain the Weyl scalar invariant for the boosted black string,
\[
W = \frac{12r_0^2}{r^6},
\]
and the 4-dimensional proper volume element is
\[
dV_4 = r^2 \sin \theta \sqrt{\frac{2r - r_0 + r_0 \cosh 2\sigma}{2r - 2r_0}} \, \mathrm{d}\theta \mathrm{d}\phi \mathrm{d}r \mathrm{d}z.
\]
Before integration, we should clarify the integral intervals for the four coordinates. First, from \( \cos \theta = x \in [-1, 1] \), we have \( 0 \leq \theta \leq \pi \). Second, in the limit \( R \to \infty \), the period of \( \phi \) is simplified to \( 0 \leq \phi \leq 2\pi \). Third, as \( r = -R/y \), \( -R/l_5 \leq y \leq -1 \) corresponds to \( l_5 \leq r \leq R/(R \to \infty) \). Last, we should set \( 0 \leq z \leq \Delta z \), meaning that we only integrate in a section of the black string. Following the same procedure in Sect. 3, we arrive at the proper volume integral of the Weyl scalar invariant for the boosted black string,
\[
\int W \, dV_4 = \frac{16\pi r_0^2 \Delta z}{l_5^2},
\]
where we have taken into account the equilibrium condition in Eq. (22) and the fact that only the region \( r \ll r_0 \) dominates the integral, which largely simplifies the calculation.

Furthermore, the event horizon area of the section of the black string is obtained by integrating the area element at the horizon where \( r = r_0 \). From Eqs. (21) and (22), we have
\[
A = \int \sqrt{2r_0^2 \sin \theta} \, \mathrm{d}\theta \mathrm{d}z = 4\sqrt{2} \pi r_0^2 \Delta z.
\]
Therefore, the black string entropy is
\[
S = \frac{A}{4G_5} = \frac{\sqrt{2} \pi r_0^2 \Delta z}{l_5^3}.
\]

From Eqs. (23) and (24), we again obtain the proportionality between the proper volume integral of the Weyl
scalar invariant and the black string entropy, with the same coefficient of proportionality as the black ring,

\[ S = \frac{\sqrt{\pi}}{16} \int W \, dV_4. \]

This result is expected. Since the boosted black string can be viewed as the limiting case of the black ring, they should share the same result. Moreover, if we take the length of the section of the black string as \( \Delta z = 2\pi R_z \), with the coordinate transformation \( v = r_0/R \) in Eq. (20), we find that the black string entropy in Eq. (24) recovers the black ring entropy in Eq. (18). It is noticed that the black string considered here is thin by its nature, as we calculate in the limit \( R \to \infty \), i.e., \( v = r_0/R \to 0 \).

\[ d\tau^2 = \frac{F(y,x)}{F(x,y)} (dt + \omega)^2 + \frac{R^2 F(x,y)}{(x-y)^2(1-\alpha)^2} \left[ \frac{dx^2}{G(x)} - \frac{dy^2}{G(y)} \right] + H(y,x) \, d\tilde{\phi}^2 - H(x,y) \, \tilde{\psi}^2 - 2 J(x,y) \, d\tilde{\phi} d\tilde{\psi}, \quad (25) \]

where the coordinates \( x \) and \( y \) maintain essentially the same meanings as in Eq. (7), the canonical periods of the angles \( \tilde{\phi} \) and \( \tilde{\psi} \) have been rescaled to \( 2\pi \), and the parameter \( \tilde{R} \) is identified to \( R \) in Eq. (7) as \( \tilde{R}^2 = R^2/(1+v^2) \). If the black ring is in equilibrium (without the conical singularities), the 1-form \( \omega \) describing the rotations is

\[ \omega = -\frac{\sqrt{2}\tilde{R}v\sqrt{(1+\alpha)^2-v^2}}{F(x,y)} \left\{ (1-x^2)y\sqrt{\alpha} \, d\tilde{\phi} + \frac{1+y}{1-v/\alpha} \left[ 1+v-\alpha+x^2 y/2(1+\alpha) \right] d\tilde{\psi} \right\}, \]

and

\[ G(x) = (1-x^2)(1+v x+\alpha^2), \]

\[ F(x,y) = 1+v^2-\alpha^2+2v\alpha(1-x^2)y+2xy(1-y^2)\alpha\alpha+xy\alpha(1-v^2-\alpha^2), \]

\[ J(x,y) = \frac{\tilde{R}^2}{(x-y)^2(1-\alpha)^2} \left[ 1+v^2-\alpha^2+2(x+y)\alpha\alpha-xy\alpha(1-v^2-\alpha^2) \right], \]

\[ H(x,y) = \frac{\tilde{R}^2}{(x-y)^2(1-\alpha)^2} \left( G(x)(1-y^2) \left\{ [(1-\alpha)^2-v^2]((1+\alpha)+v(1-v^2+2\alpha-3\alpha^2)) \right\} + G(y) \left\{ 2v^2+3v(1-\alpha)^2 \right\} - 3v(1-\alpha)^2 \right\}. \]

The second angular momentum is characterized by a new parameter \( \alpha \), with the range restricted to \( 0 \leq \alpha < 1 \) and \( 2\sqrt{\alpha} \leq v < 1+\alpha \). Obviously, by setting \( \alpha = 0 \), we recover the metric in Eq. (7). The location of the event horizon is the solution for vanishing \( G(y) \) in Eq. (25).

\[ y_h = -v + \sqrt{v^2 - 4\alpha} \quad (26) \]

Requiring that the root should be real yields the upper bound for \( \alpha \), \( \alpha \leq v^2/4 \). Furthermore, the event horizon area is

\[ A = \frac{8\sqrt{2}\pi^2(1+v+\alpha)\sqrt{\tilde{R}^3}}{(y_h^{-1}-y_h)(1-\alpha)^2}. \quad (27) \]

4.2 Black ring with two angular momenta

We move on to the entropy of the black ring with two independent angular momenta, i.e., the black ring with rotations not only along the \( \phi \)-direction of topology \( S^1 \) but also the \( \phi \)-direction of topology \( S^1 \). The metric of such a black ring is a generalization of Eq. (7) in physics, but much more complicated in mathematics. It was first discovered in Ref. [40] by using the complete integrability of the system via the inverse scattering method. The following expression is adopted from Ref. [36].

The following calculations are totally parallel to those in Sect. 3. However, the complete results are extremely tedious and irrelevant, so we focus on two special cases: (1) \( \alpha \ll v^2/4 \), i.e., the influence from the second angular momentum is much smaller than the first one; (2) \( \alpha = v^2/4 \), i.e., the two angular momenta are comparable, and the event horizon is thus degenerate.

In the first case, the contribution from \( \alpha \) is considered as a small perturbation to the black ring with one angular momentum. Since \( \alpha \ll v^2 \), we set \( \alpha \sim \mathcal{O}(v^2) \), and all the perturbative calculations below are performed up to the orders of \( \alpha \) and \( v^3 \).
Under these conditions, the leading order term of the proper volume integral of the Weyl scalar invariant remains the same as Eq. (17),

\[ \int W \, dV_4 = \frac{32\pi^2 \nu^2 R^3}{l_s^3}. \]

Above, we do not show the next-to-leading order correction, as detailed calculations indicate that the contribution from the second angular momentum is at least at the order of \( \alpha^2 \) (i.e., \( \nu^6 \)), meaning that the influence from \( \alpha \) is absent in \( \int W \, dV_4 \), not only at the leading, but also the next-to-leading order.

From Eq. (27), the event horizon area of the black ring (up to the next-to-leading order) is

\[ A = 8\sqrt{2}\pi^2 R^3 (\nu^2 + \nu^3 - \alpha), \]

and the black ring entropy becomes

\[ S = \frac{A}{4G_5} = \frac{2\sqrt{2}\pi^2 R^3}{l_s^3} (\nu^2 + \nu^3 - \alpha). \]

Here, we find that the entropy picks up a small correction from the second angular momentum at the next-to-leading order. This is a little bit different from the proper volume integral of the Weyl scalar invariant, which is not affected by \( \alpha \). Therefore, the proportionality between \( S \) and \( \int W \, dV_4 \) is slightly violated by the the second angular momentum. Of course, we still have the relation in Eq. (19) at the leading order, but we should admit that it is now a trivial result, since at the leading order, the contribution from \( \alpha \) naturally vanishes in the perturbative approach.

Now, we discuss the second case of the black ring with degenerate event horizon. In this special circumstance, all the corresponding calculations are greatly simplified. At the leading order, we obtain

\[ \int W \, dV_4 = 8 \left( 2\sqrt{3} + \sqrt{2} \tanh^{-1} \sqrt{\frac{\nu}{\sqrt{3}}} \right) \frac{\pi^2 \nu^2 R^3}{l_s^3} \approx \frac{40\pi^2 \nu^2 R^3}{l_s^3}. \]  

(28)

Moreover, from Eq. (26) and the degeneracy condition \( \alpha = \nu^2/4 \), we have \( \gamma_h = -2/\nu \), and the event horizon area in Eq. (27) reduces to

\[ A = \frac{64\sqrt{2}\pi^2 \nu^2 R^3}{(2 + \nu)(2 - \nu)^3(1 + \nu^2)^{3/2}} \approx 4\sqrt{2}\pi^2 \nu^2 R^3. \]

Hence, the leading order term of the black ring entropy is

\[ S = \frac{A}{4G_5} = \frac{\sqrt{2}\pi^2 \nu^2 R^3}{l_s^3}. \]  

(29)

Finally, from Eqs. (28) and (29), we have

\[ S \approx \frac{\sqrt{2}}{40} \int W \, dV_4. \]

One may wonder that the coefficient of proportionality, \( \sqrt{2}/40 \), is not the same as that of the black ring or black string, \( \sqrt{2}/16 \). This is comprehensible, as it is the shape parameter \( \nu \) that reflects physical significance, and the coefficient of proportionality does not encode important information. In this way, we again confirm our interpretation of the Weyl scalar invariant as entropy density. However, we should keep in mind that what we study are still thin black rings.

4.3 Black ring with a cosmological constant

Last, we briefly discuss the black ring with a cosmological constant. Unfortunately, the existence of such a black ring solution is still an unsolved problem. It is acknowledged that the asymptotically flat black ring solutions are always obtained from the integrability of the Einstein equations. However, it is still unclear whether the cosmological constant destroys the integrability [41]. If it does, the black ring solution with a cosmological constant is not supposed to exist. As a result, there are only some numerical black ring solutions in the literature [42]. It should be noticed that the black ring studied in Ref. [42] by approximate methods is also thin, and is thus consistent with our limit. It was shown [42] that in 5-dimensional anti-de Sitter space-time with a negative \( \Lambda \), the event horizon area of the black ring is

\[ A = 8\pi^2 \nu^2 R^3 \sqrt{2 + \frac{3R^2}{L^2}}, \]

where \( L \propto 1/\sqrt{-\Lambda} \) is the characteristic cosmological radius. Thus, the black ring entropy becomes

\[ S = \frac{A}{4G_5} = \frac{2\pi^2 \nu^2 R^3}{L_s^3} \sqrt{2 + \frac{3R^2}{L^2}}. \]  

(30)

This result is identical to the leading order term of the entropy in Eq. (18) in the limit of \( \Lambda \to 0 \) or \( L \to \infty \).

Since the exact black ring metric with a cosmological constant is unavailable, we are not able to calculate the Weyl scalar invariant directly. However, as shown in Ref. [27], the volume integral \( \int W \, dV_4 \) is not likely to be influenced by \( \Lambda \) significantly, because its influence on the integral merely appears at large distances, where it is greatly suppressed by the Weyl scalar invariant, which decays much faster. Therefore, comparing Eqs. (17) and (30), we have

\[ S \approx \frac{\sqrt{2} + \frac{3R^2}{L^2}}{16} \int W \, dV_4. \]
In this way, the proper volume integral of the Weyl scalar invariant is still proportional to the black ring entropy, only with the coefficient of proportionality being $\Lambda$-dependent.

Last, we should also state that the thermodynamics of black ring with a positive cosmological constant, i.e., in the de Sitter space-time, is still a controversial and not well-defined issue [43–56], and we omit the corresponding discussion here.

5 Conclusion

The latent relation between gravitation and thermodynamics is a hot issue in modern theoretical physics. Penrose’s Weyl curvature conjecture is one of the explorations on the role of the Weyl tensor $C_{\mu\nu\lambda\rho}$ in black hole thermodynamics and cosmology. It was preliminarily investigated in Ref. [27] by looking at the relation between the Weyl scalar invariant $C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}$ and the entropy of 5-dimensional black holes. It was found that the Weyl scalar invariant can be interpreted as entropy density, and its proper volume integrals can correctly lead to the Bekenstein–Hawking entropies of the Schwarzschild and Schwarzschild–anti-de Sitter black holes.

In this paper, we generalize the basic idea in Ref. [27] to another important and interesting asymptotically flat vacuum solution of the 5-dimensional Einstein equations: the black ring solution. We perform the proper volume integral of the Weyl scalar invariant for the neutral black ring in the ring coordinate system and show that it is proportional to the black ring entropy. The coefficient of proportionality depends on the shape parameter $\nu$ of the black ring, and reduces to a simple number in the limit of the thin black ring. Furthermore, we extend our calculation to three more complicated cases. (1) For the black string, the proportionality maintains the same as the black ring, as it can be viewed as a limiting case of the black ring with an infinite scale. (2) For the black ring with two angular momenta, two special cases are studied: i) the second angular momentum is much smaller than the first one, and ii) the two angular momenta are comparable. It is found that the proportionality still holds at the leading order in the perturbative approach, and the coefficient of proportionality is modified by the second angular momentum. (3) The black ring with a cosmological constant is not heavily discussed, as the black ring solution is not hoped to exist due to the lack of integrability condition. However, the proportionality is still expected, as the calculation in Ref. [27] indicated that the cosmological constant would not influence the proper volume integral significantly. In conclusion, these various calculations further confirm our interpretation of the Weyl scalar invariant as the entropy density of gravitational field.

Last, we briefly outlook the possible extension of our present work. Till now, all our calculations for black holes and black rings only apply in 5-dimensional space-time, but not the ordinary 4-dimensional one. The next step is naturally to achieve the Bekenstein–Hawking formula for the 4-dimensional black holes via the Weyl tensor. This can be realized by imagining that the mass of the 5-dimensional black ring is distributed along a compact extra dimension, and if this dimension is wrapped to an extremely small scale, the black ring may be effectively viewed as a black hole from 4-dimensional perspective. Some of our related calculations have supported this attempt, and it will be the topic of our next research.

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