ON THE EFFECTS OF THE EXTERIOR MATRIX HOSTILITY AND A U-SHAPED DENSITY DEPENDENT DISPERSAL ON A DIFFUSIVE LOGISTIC GROWTH MODEL

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Abstract. We study positive solutions to a steady state reaction diffusion equation arising in population dynamics, namely,

\[
\begin{aligned}
-\Delta u &= \lambda u(1-u); \quad x \in \Omega \\
\frac{\partial u}{\partial \eta} + \gamma \sqrt{\lambda} [(A-u)^2 + \epsilon] u &= 0; \quad x \in \partial \Omega
\end{aligned}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N; \) \( N > 1 \) with smooth boundary \( \partial \Omega \) or \( \Omega = (0,1) \), \( \frac{\partial u}{\partial \eta} \) is the outward normal derivative of \( u \) on \( \partial \Omega \), \( \lambda \) is a domain scaling parameter, \( \gamma \) is a measure of the exterior matrix (\( \Omega^c \)) hostility, and \( A \in (0,1) \) and \( \epsilon > 0 \) are constants. The boundary condition here represents a case when the dispersal at the boundary is U-shaped. In particular, the dispersal is decreasing for \( u < A \) and increasing for \( u > A \). We will establish non-existence, existence, multiplicity and uniqueness results. In particular, we will discuss the occurrence of an Allee effect for certain range of \( \lambda \). When \( \Omega = (0,1) \) we will provide more detailed bifurcation diagrams for positive solutions and their evolution as the hostility parameter \( \gamma \) varies. Our results indicate that when \( \gamma \) is large there is no Allee effect for any \( \lambda \). We employ a method of sub-supersolutions to obtain existence and multiplicity results when \( N > 1 \), and the quadrature method to study the case \( N = 1 \).

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1. **Introduction.** We study the steady state reaction diffusion model governed by a logistic growth and given by:

\[
\begin{cases}
-\Delta u = \lambda u(1-u); & x \in \Omega \\
\alpha(u) \frac{\partial u}{\partial n} + \gamma \sqrt{\lambda} (1 - \alpha(u)) u = 0; & x \in \partial \Omega
\end{cases}
\]  

(1.1)

where \( \Omega \) is a bounded region in \( \mathbb{R}^N; N > 1 \) with smooth boundary \( \partial \Omega \) or \( \Omega = (0, 1) \), \( \frac{\partial u}{\partial n} \) is the outward normal derivative of \( u \) on \( \partial \Omega \), \( \lambda \) is a domain scaling parameter, \( \gamma \) is a measure of the exterior matrix (\( \Omega^c \)) hostility, and \( \alpha(u) \) is the probability of the population staying in the habitat \( \Omega \) when it reaches the boundary. See [3], [6], [11] and [14] for the discussion on the derivation of the model.

In this paper, we will focus on a case when the dispersal \( 1 - \alpha(u) \) is U-shaped, that is when it is decreasing for lower densities and increasing for higher densities. In particular, we will discuss the case when \( \alpha \) is of the form:

\[
\alpha(s) = \frac{1}{1 + (A - s)^2 + \epsilon}; \quad s \in [0, 1]
\]  

(1.2)

where \( A \in (0, 1) \) and \( \epsilon \geq 0 \) are constants. Then the graph of

\[
1 - \alpha(s) = \frac{(A - s)^2 + \epsilon}{1 + (A - s)^2 + \epsilon}; \quad s \in [0, 1]
\]  

(1.3)

is as shown in Figure 2.

Note that the minimum dispersal is \( \epsilon/[1 + \epsilon] \). In [6] the authors studied the case when \( \epsilon = 0 \). However, ecologists have noted that in many cases the minimum dispersal on the boundary never becomes zero (\( \epsilon > 0 \)). In this paper we focus on the case when \( \epsilon > 0 \), and establish non-existence, existence, uniqueness and multiplicity results for the model:

\[
\begin{cases}
-\Delta u = \lambda u(1-u); & x \in \Omega \\
\frac{\partial u}{\partial n} + \gamma \sqrt{\lambda} (A - u)^2 + \epsilon) u = 0; & x \in \partial \Omega
\end{cases}
\]  

(1.4)

It turns out that the analysis in [6] for the \( \epsilon = 0 \) does not extend to this case and we require new ideas to study (1.4).

Let \( D > 0 \) be a constant and \( E_1(\gamma, D) \) be the principal eigenvalue of

\[
\begin{cases}
-\Delta v = Ev; & x \in \Omega \\
\frac{\partial v}{\partial n} + \gamma \sqrt{E} (A - v)^2 + \epsilon) v = 0; & x \in \partial \Omega
\end{cases}
\]  

(1.5)
Figure 2. An example that illustrates U-shaped density dependent dispersal \((1 - \alpha(u))\) on the boundary.

Figure 3. Eigencurve \(B(\kappa)\) and principal eigenvalue of (1.5).

Note that the existence of \(E_1(\gamma, D) > 0\) follows from \([13]\) where the authors study the eigenvalue problem:

\[
\begin{align*}
-\Delta \phi &= B\phi; \quad x \in \Omega \\
\frac{\partial \phi}{\partial \eta} &= \kappa \phi; \quad x \in \partial \Omega
\end{align*}
\]

for any \(\kappa \in \mathbb{R}\). They prove for each \(\kappa\), the principal eigenvalue \(B(\kappa)\) exists, and the eigencurve \(B(\kappa)\) is Lipschitz continuous, strictly decreasing, and concave. Further, \(B(0) = 0\) and \(\lim_{\kappa \to -\infty} B(\kappa) = B_D\), where \(B_D\) is the principal eigenvalue of:

\[
\begin{align*}
-\Delta \phi &= B\phi; \quad x \in \Omega \\
\phi &= 0; \quad x \in \partial \Omega
\end{align*}
\]

In the case of (1.5), treating \(\kappa = -\gamma D\sqrt{E}\) (or \(E = \frac{\kappa^2}{D^2\gamma^2}\)), we see that the principal eigenvalue \(E_1(\gamma, D)\) of (1.5) is given by \(E_1(\gamma, D) = C\) where \((-\gamma D\sqrt{C}, C)\) with \(C > 0\) is the point of intersection of the curves \(B(\kappa)\) and \(\frac{\kappa^2}{D^2\gamma^2}\) as shown in Figure 3.

We establish the following results:

**Theorem 1.1.** Let \(\gamma > 0\) and \(\epsilon > 0\). There is no positive solution of (1.4) for \(\lambda \in (0, E_1(\gamma, \epsilon)]\).
that the trivial solution of (1.4) is asymptotically stable for

1.3 is asymptotically stable (see Theorem 6.7 of Chapter 5 in [12]). We also note

particular, when

λ →∞

and unstable for

∥ u ∥_

solution of (1.8). In addition, if there exists ˜ v that

∥ v (∈) − u ∥_

model.

the following multiplicity result which ensures a prediction of an Allee effect in the

to

λ

∈

and

E

γ,A

< E

γ,A

+ γ < E

γ,A

2 + 2A

2

Let

Theorem 1.2. Let γ > 0 and ε > 0. Then (1.4) has a positive solution for

λ > E

γ,A

+ ε.

Next we recall that for γ > 0 fixed, the boundary value problem:

\[\begin{align*}
-Δw &= λw(1-w); \quad x ∈ Ω \\
\frac{∂w}{∂n} + 2γ\sqrt{λ}(A-w)^2w &= 0; \quad x ∈ ∂Ω
\end{align*}\]  

(1.7)

has a positive solution w_λ for λ > 0 such that A < w_λ(x) ≤ 1 for x ∈ ∂Ω, and

this solution is unique (see [6]). We also note that w_λ is continuous with respect

to λ and E_1(γ,A^2) < E_1(γ,A^2 + ε) < E_1(γ,2A^2) for ε ∈ (0,A^2) (see [7]). Let

w^*_λ := min_{x ∈ ∂Ω} w_λ(x) and δ_γ := \min_{λ ∈ [E_1(γ,A^2),E_1(γ,2A^2)]} (w^*_λ - A)^2. Then we establish

the following multiplicity result which ensures a prediction of an Allee effect in the

model.

Theorem 1.3. Let γ > 0, ε^*_γ := min{δ_γ, A^2} and Γ := \{u ∈ C^2(Ω) ∩ C^1(Ω) | u(x) ∈ [A,1] for x ∈ ∂Ω\}. For each ε ∈ (0, ε^*_γ), there exists λ_* > 0 such that if λ ∈ (λ_*,E_1(γ,A^2 + ε)) then (1.4) has at least two positive solutions u_* and u^* such that u_* ∈ Γ and u^* ∉ Γ. In particular, in Γ, (1.4) has a unique solution and this solution

is u_*.

Remark 1. Note that the time dependent problem related to (1.4) is of the form:

\[\begin{align*}
\frac{∂u}{∂t} &= \frac{1}{λ}Δu + u(1-u); \quad x ∈ Ω, \quad t > 0 \\
\frac{∂u}{∂n} + γ\sqrt{λ}(A-u)^2 + εv &= 0; \quad x ∈ ∂Ω, \quad t > 0 \\
u(0,x) &= u_0(x); \quad x ∈ Ω.
\end{align*}\]  

(1.8)

A solution u of (1.4) is called stable if for every ε > 0 there exists δ > 0 such that

∥ v(t,.) − u ∥∞ < ε for t > 0 whenever ∥ u_0 − u ∥∞ < δ where v(t,x) is the

solution of (1.8). In addition, if there exists ˜ δ > 0 such that when ∥ u_0 − u ∥∞ < ˜ δ,

∥ v(t,.) − u ∥∞ → 0 as t →∞, then u is called asymptotically stable. The solution

u is called unstable if it is not stable. We note that the solution u_* ∈ Γ in Theorem

1.3 is asymptotically stable (see Theorem 6.7 of Chapter 5 in [12]). We also note

that the trivial solution of (1.4) is asymptotically stable for λ < E_1(γ,A^2 + ε)

and unstable for λ > E_1(γ,A^2 + ε) following the proof of Theorem 1.1 in [9]. In

particular, when λ ∈ (λ_*,E_1(γ,A^2 + ε)), if ∥ u_0 ∥∞ ≈ 0 then ∥ v(t,.) ∥∞ → 0 as

t →∞, while if ∥ u_0 − u_* ∥∞ → 0 then ∥ v(t,.) − u_* ∥∞ → 0 as t →∞. Hence

there is an Allee effect for λ ∈ (λ_*,E_1(γ,A^2 + ε)). See also [2] where the authors show

Figure 4. Bifurcation diagrams for (1.4).
existence of an Allee effect in a logistic model but with negative density dependent emigration. Note that with Dirichlet boundary condition, an Allee effect does not occur for a logistic growth model. For more details on the discussion of an Allee effect, see [1] and [15].

Next we consider the case when \( \Omega = (0,1) \). In this case (1.4) reduces to the two-point boundary value problem:

\[
\begin{align*}
-u'' &= \lambda u(1-u); \ (0,1) \\
-u'(0) + \gamma \sqrt{A}([A-u(0)]^2 + \epsilon)u(0) &= 0 \\
u'(1) + \gamma \sqrt{A}([A-u(1)]^2 + \epsilon)u(1) &= 0.
\end{align*}
\] (1.9)

We note that if \( u \) is a positive solution of (1.9) then \( u \) has a unique interior maximum, say at \( t_0 \), and the solution is symmetric about \( t_0 \) (see [8]).

We establish:

**Theorem 1.4.** For \( \lambda > 0 \), (1.9) has a positive solution \( u \) such that \( u(t_0) = \|u\|_\infty = \rho \), \( u(0) = q_1 \), \( u(1) = q_2 \), with \( 0 < q_1, q_2 < \rho \) if and only if \( \lambda, \rho, q_1 \) and \( q_2 \) satisfy:

\[
\lambda = \frac{1}{2} \left( \int_{q_1}^{\rho} \frac{ds}{\sqrt{F(s)-F(\rho)}} + \int_{q_2}^{\rho} \frac{ds}{\sqrt{F(s)-F(\rho)}} \right)^2 \]

(1.10)

\[
\begin{align*}
2[F(\rho) - F(q_1)] &= \gamma^2 q_1^2 ([A-q_1]^2 + \epsilon)^2 \\
2[F(\rho) - F(q_2)] &= \gamma^2 q_2^2 ([A-q_2]^2 + \epsilon)^2
\end{align*}
\]

(1.11)

where \( F(s) = \int_0^s r(1-r)dr \). Further, \( t_0 \) is given by

\[
t_0 = \frac{\int_{q_1}^{\rho} \frac{ds}{\sqrt{F(s)-F(\rho)}}}{\int_{q_1}^{\rho} \sqrt{F(s)-F(\rho)} + \int_{q_2}^{\rho} \sqrt{F(s)-F(\rho)}}.
\]

We will use Theorem 1.4, namely, equations (1.10) and (1.11) to obtain numerical bifurcation diagrams via Mathematica.

Next we establish conditions that ensures the symmetry of positive solutions of (1.9).

**Theorem 1.5.** If \( \epsilon > \frac{A^2}{3} \) then all positive solutions of (1.9) are symmetric.
Theorem 1.6. If \( \gamma \gg 1 \) or \( \gamma \approx 0 \) then all positive solutions of (1.9) are symmetric.

In Section 2, we state some preliminaries. In Section 3, we present the proofs of Theorems 1.1 - 1.3. The proof of Theorem 1.4 will be discussed in Section 4. Finally, the proofs of Theorems 1.5 - 1.6 and the numerical bifurcation results will be presented in Section 5.

2. Preliminaries. In this section, we present definitions of a subsolution and a supersolution of (1.4). We also provide a sub-supersolution theorem and a three solution theorem that we will use to prove our existence and multiplicity results.

A function \( Z \in C^2(\Omega) \cap C^1(\Omega) \) is called a supersolution of (1.4) if \( Z \) satisfies

\[
\begin{cases}
-\Delta u \geq \lambda Z(1-Z); & x \in \Omega \\
\frac{\partial u}{\partial \eta} + \gamma \sqrt{\lambda} (A-\psi)^2 + \epsilon \psi \leq 0; & x \in \partial \Omega.
\end{cases}
\]

A function \( Z \in C^2(\Omega) \cap C^1(\Omega) \) is called a subsolution of (1.4) if \( Z \) satisfies

\[
\begin{cases}
-\Delta Z \leq \lambda Z(1-Z); & x \in \Omega \\
\frac{\partial Z}{\partial \eta} + \gamma \sqrt{\lambda} (A-Z)^2 + \epsilon Z \geq 0; & x \in \partial \Omega.
\end{cases}
\]

A strict subsolution of (1.4) is a subsolution which is not a solution. A strict supersolution of (1.4) is a supersolution which is not a solution.

Then the following results hold (see [4], [5], [10] and [16]).

Lemma 2.1. Let \( \psi \) and \( Z \) be a subsolution and a supersolution of (1.4) respectively such that \( \psi \leq Z \). Then (1.4) has a solution \( u \in C^2(\Omega) \cap C^1(\Omega) \) such that \( u \in [\psi, Z] \).

Lemma 2.2. Let \( u_1 \) and \( \pi_2 \) be a subsolution and a supersolution of (1.4) respectively such that \( u_1 \leq \pi_2 \). Let \( u_2 \) and \( \pi_1 \) be a strict subsolution and a strict supersolution of (1.4) respectively such that \( u_2, \pi_1 \in [u_1, \pi_2] \) and \( u_2, \pi_1 \leq \pi_2 \). Then (1.4) has at least three solutions \( u_1, u_2 \) and \( u_3 \) where \( u_i \in [u_1, \pi_1] \); \( i = 1, 2 \) and \( u_3 \in [u_1, \pi_2] \setminus ([u_1, \pi_1] \cup [u_2, \pi_2]) \).

3. Proofs of Theorems 1.1 - 1.3. In this section, we will provide proofs of Theorems 1.1 - 1.3. First, we present the proof of Theorem 1.1.

Proof of Theorem 1.1. Let \( \lambda \leq E_1(\gamma, \epsilon) \). Assume to the contrary that (1.4) has a positive solution \( u \). Then there exist a unique \( \epsilon_\lambda \leq \epsilon \) such that \( \lambda \) is the principal eigenvalue of the boundary value problem:

\[
\begin{cases}
-\Delta u = \epsilon \lambda u; & x \in \Omega \\
\frac{\partial u}{\partial \eta} + \gamma \sqrt{\lambda} e = 0; & x \in \partial \Omega
\end{cases}
\]

and equality holds if and only if \( \lambda = E_1(\gamma, \epsilon) \). This easily follows from the behavior of \( \frac{\lambda^2}{\gamma^2} \) as \( \epsilon \) varies (see Figure 6). See also [7].

Let \( \epsilon > 0 \) be the corresponding normalized eigenfunction for the principal eigenvalue \( \lambda \) in (3.1). Then we have

\[
\int_{\Omega} [(-\Delta u)e + (\Delta e)u] \, dx = \int_{\Omega} \lambda u (1-u)e - \lambda \epsilon u \, dx = -\int_{\Omega} \lambda e u^2 \, dx < 0.
\]

However, by the Green’s second identity we have

\[
\int_{\Omega} [(-\Delta u)e + (\Delta e)u] \, dx = \int_{\partial \Omega} \left[ -\frac{\partial u}{\partial \eta} e + \frac{\partial e}{\partial \eta} u \right] \, ds
\]
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Figure 6. Plot that illustrates the existence of $\epsilon_\lambda$.

$$\int_{\partial\Omega} \left[ \gamma \sqrt{\lambda} (A - u)^2 + \epsilon |ue - \gamma \epsilon \lambda e u| \right] ds$$

$$\geq \int_{\partial\Omega} \gamma (\epsilon - \epsilon_\lambda) \sqrt{\lambda} e u ds$$

$$\geq 0.$$ This is a contradiction since $\epsilon_\lambda \leq \epsilon$. Hence the proof is complete.

Next, we provide a proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\lambda > E_1(\gamma, A^2 + \epsilon)$. It is easy to see that $\phi \equiv 1$ is a supersolution for (1.4). Next we construct a subsolution for (1.4). Let $\mu_\lambda$ be the principal eigenvalue and $z > 0$ be the corresponding normalized eigenfunction for the boundary value problem:

$$\begin{cases}
-\Delta z = (\lambda + \mu) z; & x \in \Omega \\
\frac{\partial z}{\partial \eta} + [\gamma \sqrt{\lambda} (A^2 + \epsilon) - \mu] z = 0; & x \in \partial\Omega.
\end{cases}$$

We note that $\mu_\lambda < 0$ for $\lambda > E_1(\gamma, A^2 + \epsilon)$ (see Lemma 6.2 in Appendix) and $\min_{x \in \Omega} z(x) > 0$. Let $\psi := \alpha_\lambda z$ where $\alpha_\lambda > 0$ will be chosen later. Then $\psi$ satisfies

$$-\Delta \psi = \alpha_\lambda (\lambda + \mu) z \leq \lambda \alpha_\lambda z (1 - \alpha_\lambda z) = \lambda \psi (1 - \psi)$$

for $x \in \Omega$ provided $\mu_\lambda + \lambda \alpha_\lambda z \leq 0$. Further, $\psi$ satisfies

$$\frac{\partial \psi}{\partial \eta} = \alpha_\lambda [-\gamma \sqrt{\lambda} (A^2 + \epsilon) + \mu_\lambda] z \leq -\alpha_\lambda \gamma \sqrt{\lambda} [(A - \alpha_\lambda z)^2 + \epsilon] z = -\gamma \sqrt{\lambda} [(A - \psi)^2 + \epsilon] \psi$$

for all $x \in \partial\Omega$ provided $\gamma \sqrt{\lambda} [(A - \alpha_\lambda z)^2 - A^2] + \mu_\lambda \leq 0$. Since $\mu_\lambda < 0$, choosing $\alpha_\lambda \approx 0$, it follows that $\psi$ is a subsolution for (1.4) and $\psi \leq \phi$ in $\Omega$. Hence for $\lambda > E_1(\gamma, A^2 + \epsilon)$, (1.4) has a positive solution $u$ such that $\psi \leq u \leq \phi$. □

Finally, a proof of Theorem 1.3 is provided.

Proof of Theorem 1.3. Let $\lambda < E_1(\gamma, A^2 + \epsilon)$. It is easy to see that $\phi_1 \equiv 1$ is a supersolution and $\psi_1 \equiv 0$ is a subsolution for (1.4). We now construct a strict supersolution for (1.4). Let $\mu_\lambda$ be the principal eigenvalue and $z > 0$ be
the corresponding normalized eigenfunction for (3.2). We note that $\mu_\lambda > 0$ for $
abla < E_1(\gamma, A^2 + \varepsilon)$ (see Lemma 6.2 in Appendix). Let $\phi_2 := \beta_\lambda z$ for $\beta_\lambda \in (0, A)$. Then $\phi_2$ satisfies

$$-\Delta \phi_2 = \beta_\lambda (\lambda + \mu_\lambda)z \geq \lambda \beta_\lambda z(1 - \beta_\lambda z) = \lambda \phi_2(1 - \phi_2)$$

for $x \in \Omega$. Further, $\phi_2$ satisfies

$$\frac{\partial \phi_2}{\partial \eta} = \beta_\lambda [\gamma \sqrt{A}(A^2 + \varepsilon) + \mu_\lambda] z > -\beta_\lambda \gamma \sqrt{A}[(A - \beta_\lambda z)^2 + \varepsilon] z = -\gamma \sqrt{A}[(A - \phi_2)^2 + \epsilon] \phi_2$$

for $x \in \partial \Omega$ provided $\gamma \sqrt{A}[(A - \beta_\lambda z)^2 - A^2] + \mu_\lambda > 0$. Since $\mu_\lambda > 0$, choosing $\beta_\lambda \approx 0$, it follows that $\phi_2$ is a strict supersolution for (1.4). We next construct a strict subsolution for (1.4). For each $\epsilon \in (0, \epsilon^*)$, there exists $\lambda_* < E_1(\gamma, A^2)$ such that $(w_1^*)^2 - A^2 > \epsilon$ for $\lambda \in (\lambda_*, E_1(\gamma, 2A^2))$. We note that $E_1(\gamma, A^2 + \epsilon) < E_1(\gamma, 2A^2)$ since $\epsilon < \epsilon^* \leq A^2$. Let $\psi_2 := w_\lambda$ for $\lambda \in (\lambda_*, E_1(\gamma, A^2 + \epsilon))$. Then $\frac{\partial \psi_2}{\partial \eta} = -2\gamma \sqrt{A}(A - w_\lambda)\lambda w_\lambda < -\gamma \sqrt{A}(A - w_\lambda)^2 + \epsilon \lambda w_\lambda$ on $\partial \Omega$ and hence $\psi_2$ is a strict subsolution. We note that $\|\psi_2\|_\infty > A$ and $\|\phi_2\|_\infty < A$. By Lemma 2.2, we obtain solutions $u_*$ and $u^*$ such that $u \in [\psi_1, \phi_2], u_* \in [\psi_1, \phi_1]$ and $u^* \in [\psi_1, \phi_1] \setminus [\psi_1, \phi_2]$. Clearly $u_*$ and $u^*$ are positive solutions. Further, $u_* \in \Gamma$ since $u_* \geq \psi_2 > A$ on $\overline{\Omega}$.

Next in $\Gamma$, we show that (1.4) has a unique positive solution. Assume to the contrary that in $\Gamma$ there exist two distinct positive solutions $u$ and $v$. Without loss of generality, we assume $u \leq v$ since $\phi_1 \equiv 1$ is a global supersolution. Therefore we have

$$\int_\Omega [(\Delta u)u + (\Delta u)v] \, dx = \int_\Omega \lambda uv(u - v) \, dx < 0.$$

However, by the Green’s second identity we have

$$\int_\Omega [(\Delta u)u + (\Delta u)v] \, dx = \int_{\partial \Omega} [\frac{\partial u}{\partial \eta} u + \frac{\partial v}{\partial \eta} v] \, ds$$

$$= \int_{\partial \Omega} \gamma \sqrt{A}w_\lambda [(A - v)^2 - (A - u)^2] \, ds$$

$$= \int_{\partial \Omega} \gamma \sqrt{A}w_\lambda (u - v)(2A - u - v) \, ds \geq 0$$

which is a contradiction. Hence in $\Gamma$, there exists a unique positive solution, which is $u_*$, and $u^*$ is a positive solution which does not belong to $\Gamma$. Hence the proof is complete.

4. Proof of Theorem 1.4. In this section, we provide a proof of Theorem 1.4.

Proof of Theorem 1.4. Let $f(s) = s(1 - s)$. Multiplying both sides of (1.9) by $u'$ and integrating we obtain (using the fact that $u(t_0) = \rho$)

$$u'(t) = \left\{ \begin{array}{ll} \sqrt{2\lambda |F(\rho) - F(u(t))|}; & (0, t_0) \\ -\sqrt{2\lambda |F(\rho) - F(u(t))|}; & [t_0, 1). \end{array} \right. \tag{4.1}$$

Next integrating (4.1) yields

$$\int_{q_1}^{u(t)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda t}; \ (0, t_0) \tag{4.2}$$

and

$$\int_{q_2}^{u(t)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda (1 - t)}; \ (t_0, 1). \tag{4.3}$$
Now taking $t \to t_0$, $\lambda$, $\rho$, $q_1$, $q_2$ and $t_0$ must satisfy:

$$
\int_{q_1}^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda} t_0
$$

(4.4)

$$
\int_{q_2}^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda} (1 - t_0).
$$

(4.5)

Now from (4.4) and (4.5), we have

$$
\lambda = \frac{1}{2} \left( \int_{q_1}^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_{q_2}^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} \right)^2
$$

(4.6)

and from (1.9) and (4.1) we obtain:

$$
\begin{align*}
2[F(\rho) - F(q_1)] &= \gamma^2 q_1^2 [(A - q_1)^2 + \epsilon]^2 \\
2[F(\rho) - F(q_2)] &= \gamma^2 q_2^2 [(A - q_2)^2 + \epsilon]^2.
\end{align*}
$$

By (4.4) and (4.6), we also obtain

$$
t_0 = \frac{\int_{q_1}^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}}}{\int_{q_1}^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_{q_2}^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}}}.
$$

Next let $\lambda$, $q_1$, $q_2$ and $\rho$ satisfy (1.10) and (1.11). Define $u : [0, 1] \to [0, \rho]$ such that $u$ satisfies (4.2) for $t \in (0, t_0)$ and (4.3) for $t \in (t_0, 1)$. Note that $u$ is well-defined for $t \in (0, t_0)$ since both

$$
\int_{q_1}^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}}
$$

and $\sqrt{2\lambda} t$ increase from 0 to

$$
\int_{q_1}^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}}
$$

as $u$ increases from $q_1$ to $\rho$ and $t$ increases from 0 to $t_0$. Similarly we can see that $u$ is well defined for $t \in (t_0, 1)$. Define $H : (0, t_0) \times (q_1, \rho) \to R$ by

$$
H(t, u(t)) = \int_{q_1}^{u(t)} \frac{ds}{\sqrt{F(\rho) - F(s)}} - \sqrt{2\lambda} t.
$$

Clearly $H$ is $C^1$, $H(t, u(t)) = 0$ for $t \in (0, t_0)$, and

$$
H_u|_{(t, u(t))} = \frac{1}{\sqrt{F(\rho) - F(u(t))}} \neq 0.
$$

By the Implicit Function Theorem, $u$ is $C^1$ on $(0, t_0)$. Similarly we can show that $u$ is $C^1$ on $(t_0, 1)$. Differentiating (4.2) and (4.3), we obtain

$$
u'(t) = \begin{cases} 
\sqrt{2\lambda} [F(\rho) - F(u(t))]; & (0, t_0) \\
-\sqrt{2\lambda} [F(\rho) - F(u(t))]; & (t_0, 1).
\end{cases}
$$

(4.7)

Now it is easy to see that $u \in C^2(0, 1)$ and is a solution of $-u''(t) = \lambda f(u(t)); (0, 1)$. Further, $u \in C^1[0, 1]$ and from (1.11) and (4.7), we obtain

$$
u'(0) = \sqrt{2\lambda} [F(\rho) - F(q_1)] = \sqrt{\lambda}\gamma [(A - q_1)^2 + \epsilon] q_1 = \sqrt{\lambda}\gamma [(A - u(0))^2 + \epsilon] u(0).
$$

Similarly, we can show that $u'(1) = -\sqrt{\lambda}\gamma [(A - u(1))^2 + \epsilon] u(1)$. Hence the proof is complete. \qed
5. Proofs of Theorems 1.5 - 1.6 and numerical bifurcation results. In this section, we will provide proofs of Theorems 1.5 - 1.6 and we will present our numerical bifurcation results. First, we present the proof of Theorem 1.5.

Proof of Theorem 1.5. Let $u$ be a positive solution such that $u(0) = q_1$ and $u(1) = q_2$. Assume $t_0 < \frac{1}{2}$. Since $u$ is symmetric about $t_0$ and $u$ is concave, $q_1 > q_2$ and hence $|u'(0)| < |u'(1)|$. By the boundary conditions we have $\gamma \sqrt{\lambda}[(A - q_1)^2 + \epsilon] q_1 < \gamma \sqrt{\lambda}[(A - q_2)^2 + \epsilon] q_2$. Let $G(q) = \gamma \sqrt{\lambda}[(A - q)^2 + \epsilon] q$. It is easy to show that if $\epsilon > \frac{A^2}{3}$ then $G'(q) > 0$. This implies $\gamma \sqrt{\lambda}[(A - q_1)^2 + \epsilon] q_1 > \gamma \sqrt{\lambda}[(A - q_2)^2 + \epsilon] q_2$. This is a contradiction. Similar contradiction can be obtained when $t_0 > \frac{1}{2}$. Hence the solution is symmetric if $\epsilon > \frac{A^2}{3}$.

Next, we provide the proof of Theorem 1.6.

Proof of Theorem 1.6. Let $u$ be a positive solution such that $u(0) = q_1$ and $u(1) = q_2$. To show that the solution is symmetric, we need to show $q_1 = q_2$. By Theorem 1.4, this follows by showing that for any fixed $\rho \in (0, 1)$,

$$
\frac{\sqrt{F(\rho) - F(q)}}{q[(A - q)^2 + \epsilon]} = \frac{\gamma}{\sqrt{2}}
$$

has only one solution $q \in (0, \rho)$. Let $H(q) = \frac{\sqrt{F(\rho) - F(q)}}{q[(A - q)^2 + \epsilon]}$. It is easy to see that $\lim_{q \to 0^+} H(q) = \infty$ and $H(\rho) = 0$. Further, we have

$$
H'(q) = \frac{-q f(q)(A - q)^2 + \epsilon - 2[F(\rho) - F(q)][(A - q)(A - 3q) + \epsilon]}{2q^2[(A - q)^2 + \epsilon]^2 \sqrt{F(\rho) - F(q)}}
$$

Thus we obtain $\lim_{q \to 0^+} H'(q) = \lim_{q \to \rho} H'(q) = -\infty$. This implies $\frac{\gamma}{\sqrt{2}}$ has only one solution $q \in (0, \rho)$ for $\gamma \gg 1$ or $\gamma \approx 0$ (see Figure 7 for an illustration). Hence the proof is complete.
Finally, we present some bifurcation curves for a couple of parameter selections. Here we briefly explain how we obtain numerical bifurcation diagrams. Let \( \gamma > 0 \) be fixed and let \( x_i = \frac{i}{n+1}; \ i = 1, \ldots, n \) for some \( n \geq 1 \). Letting \( \rho = x_1 \), we numerically solve the equation (1.11) for \( q_1 \) and \( q_2 \) using the FindRoot command in Mathematica. The values of \( q_1, q_2 \) and \( \rho \) are substituted into (1.10) to find the corresponding value of \( \lambda \). Repeating this procedure for \( \rho = x_i, \ i = 2, \ldots, n \), we obtain \((\lambda, \rho)\) points for the bifurcation diagram.

**Example 1.** Let \( \epsilon = 0.1 \) and \( A = 0.5 \). We note that by Theorem 1.5, every positive solutions of (1.9) is symmetric. Here we provide bifurcation curves numerically generated via Mathematica for various \( \gamma \) values. See Figure 8 consisting of 6 bifurcation curves, the first five are in the ascending order of \( \gamma \) from left to right and the last one is the bifurcation curve with Dirichlet boundary condition. We note that as \( \gamma \) increases the bifurcation diagrams shift to right. In particular, the Allee effect is lost when \( \gamma > 67 \).

**Example 2.** Here we present an example where we get both symmetric and non-symmetric solutions of (1.9) for certain values of \( \gamma \), when \( \epsilon = 0.01 \) and \( A = 0.8 \). We observe that solutions are symmetric for \( \gamma = 1, \gamma = 23 \) and \( \gamma = 25 \) (see (a), (f) and (g) in Figure 9). We also find that for some \( \gamma \) values, (5.1) has three distinct \( q \)-values, say \( q_1, q_2 \) and \( q_3 \), for a certain range of \( \rho \) values. This implies that there exist three symmetric solutions such that \( \|u\|_\infty = \rho \) and \( u(0) = u(1) = q_i \) for \( i = 1, 2, 3 \) and six non-symmetric solutions such that \( \|u\|_\infty = \rho \), \( u(0) = q_i \) and \( u(1) = q_j \) for \( i, j = 1, 2, 3 \) and \( i \neq j \) (Note: In general, if (5.1) has \( n \) \( q \)-value solutions then there are \( n^2 \) total solutions). See (c), (d) and (e) in Figure 9 for bifurcation diagrams when \( \gamma = 6, \gamma = 10 \) and \( \gamma = 16 \), respectively. Here the bifurcation curves for symmetric solutions are in red and the bifurcation curves for non-symmetric solutions are in green (Note: green points represent two solutions each while red represent only one solution each). Note that (h) in Figure 9 is the bifurcation curve with Dirichlet boundary condition i.e., the boundary condition.
is $u(0) = 0 = u(1)$. We observe that bifurcation curves of (1.9) approaches the bifurcation curve with Dirichlet boundary condition when $\gamma \to \infty$. However, for a fixed $\gamma > 3$, we observe that there always exists a range of $\lambda$ in which there exists at least three solutions.

6. Appendix. In this section, we recall important results from [7], namely, Lemmas 4.3 and 4.4 in the Appendix. For the convenience of the reader we also provide the proofs of these results here.

Lemma 6.1. For a given $D > 0$, $\gamma > 0$ and $\lambda > 0$, let $\sigma_1(\lambda, \gamma, D)$ be the principal eigenvalue of the problem:

$$
\begin{cases}
-\Delta \theta = (\sigma + \lambda)\theta; & x \in \Omega \\
\frac{\partial \theta}{\partial n} + \gamma D\sqrt{\lambda} \theta = 0; & x \in \partial \Omega.
\end{cases}
$$

(6.1)

If $\lambda \leq E_1(\gamma, D)$ then $\sigma_1(\lambda, \gamma, D) \geq 0$ and if $\lambda > E_1(\gamma, D)$ then $\sigma_1(\lambda, \gamma, D) < 0$.

Proof. Let $\lambda \leq E_1(\gamma, D)$. Clearly $\sigma_1(\lambda, \gamma, D) + \lambda = B(-\gamma D\sqrt{\lambda})$ by the definition of $B$ (see (1.6)). But it is easy to see from Figure 10, if $\lambda \leq E_1(\gamma, D)$ then $B(-\gamma D\sqrt{\lambda}) \geq \lambda$. Hence $\sigma_1(\lambda, \gamma, D) \geq 0$. Similarly if $\lambda > E_1(\gamma, D)$ then we have $\sigma_1(\lambda, \gamma, D) < 0$. \hfill \Box

Next consider the eigenvalue problem:

$$
\begin{cases}
-\Delta \phi = (\mu + \lambda)\phi; & x \in \Omega \\
\frac{\partial \phi}{\partial n} + \gamma D\sqrt{\lambda} \phi = \mu \phi; & x \in \partial \Omega.
\end{cases}
$$

(6.2)

Notice that by letting $\kappa = \mu - \gamma D\sqrt{\lambda}$ implies (6.2) becomes

$$
\begin{cases}
-\Delta \phi = (\kappa + \lambda + \gamma D\sqrt{\lambda})\phi; & x \in \Omega \\
\frac{\partial \phi}{\partial n} = \kappa \phi; & x \in \partial \Omega
\end{cases}
$$

(6.3)

and the principal eigenvalue $\kappa_1(\lambda, \gamma, D)$ is the $x$-coordinate of the intersection of the curves $B(\kappa)$ and $\kappa + \lambda + \gamma D\sqrt{\lambda}$ (see Figure 11).

Hence the principal eigenvalue $\mu_1(\lambda, \gamma, D)$ of (6.2) exists and is given by:

$$
\mu_1(\lambda, \gamma, D) = \kappa_1(\lambda, \gamma, D) + \gamma D\sqrt{\lambda}.
$$

Lemma 6.2. If $\lambda \leq E_1(\gamma, D)$ then $\mu_1(\lambda, \gamma, D) \geq 0$ and if $\lambda > E_1(\gamma, D)$ then $\mu_1(\lambda, \gamma, D) < 0$.

Proof. We will first show that $\text{sign}(\mu_1(\lambda, \gamma, D)) = \text{sign}(\sigma_1(\lambda, \gamma, D))$. Let $\phi_1$ and $\phi_2$ be the corresponding positive eigenfunctions of (6.1) and (6.2) respectively. Then by the Green’s second identity, we have that

$$
\int_{\Omega} [\phi_2 \Delta \phi_1 - \phi_1 \Delta \phi_2] dx = \int_{\partial \Omega} \left[ \phi_2 \frac{\partial \phi_1}{\partial n} - \phi_1 \frac{\partial \phi_2}{\partial n} \right] ds.
$$

This implies that

$$
[\mu_1(\lambda, \gamma, D) - \sigma_1(\lambda, \gamma, D)] \int_{\Omega} \phi_1 \phi_2 dx = -\mu_1(\lambda, \gamma, D) \int_{\partial \Omega} \phi_1 \phi_2 ds.
$$

It follows that $\sigma_1(\lambda, \gamma, D) = 0$ if and only if $\mu_1(\lambda, \gamma, D) = 0$, and if $\mu_1(\lambda, \gamma, D) \neq 0$ then we have

$$
\sigma_1(\lambda, \gamma, D) = \frac{\mu_1(\lambda, \gamma, D)}{\mu_1(\lambda, \gamma, D)} > 0.
$$
Figure 9. Bifurcation diagrams for (1.9) for several values of $\gamma$, when $\epsilon = 0.01$ and $A = 0.8$. 

(a) $\gamma = 1$

(b) $\gamma = 3$

(c) $\gamma = 6$

(d) $\gamma = 10$

(e) $\gamma = 16$

(f) $\gamma = 23$

(g) $\gamma = 25$

(h) $\gamma = \infty$
This implies that if \( \mu_1(\lambda, \gamma, D) > 0 \) then \( \sigma_1(\lambda, \gamma, D) > \mu_1(\lambda, \gamma, D) > 0 \) and if \( \mu_1(\lambda, \gamma, D) < 0 \) then \( \sigma_1(\lambda, \gamma, D) < \mu_1(\lambda, \gamma, D) < 0 \). Hence \( \sign(\mu_1(\lambda, \gamma, D)) = \sign(\sigma_1(\lambda, \gamma, D)) \). This together with Lemma 6.1 implies that Lemma 6.2 holds. □

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