Intrinsic angular momentum in general relativity

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Abstract

There are several definitions of the notion of angular momentum in general relativity. However none of them can be said to capture the physical notion of intrinsic angular momentum of the sources in the presence of gravitational radiation. We present a definition which is appropriate for the description of intrinsic angular momentum in radiative spacetimes. This notion is required in calculations involving radiation of angular momentum, as for example is expected in binary coalescence of black holes.

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1 Introduction

There is a lot of effort invested in the development and construction of large interferometric gravitational wave detectors such as LIGO, VIRGO, GEO, etc. These observatories are expected to measure the gravitational waves emitted in relativistic astrophysical systems, for example in the coalescence of two compact objects. At the late stages of such systems one can consider the situation of having two black holes or neutron stars in nearly circular orbits which are shrinking due to the loss, by the emission of gravitational waves, of energy and angular momentum.

In order to estimate how important is the radiation mechanism of angular momentum in these kind of processes, let us consider briefly the situation of two Newtonian point masses, with mass $m_0$ each, in circular motion at a distance $r$. Then one can calculate the ratio between total angular momentum $J$ and total mass square $M = 2m_0$, obtaining $\frac{J}{M} = \frac{\sqrt{15}}{16r_s}$; where $r_s$ is the Schwarzschild radius of masses $m_0$, and we are using geometric units for which the speed of light and the gravitational constant have the unit value. This means that for orbits for which $r > 16r_s$, the angular momentum exceeds the Kerr limit to have a final black hole, as opposed to a naked singularity. In other words, some how, one has to account for the radiation of angular momentum before the final collapse of the system, if one is to expect a black hole at the end of the coalescence. It is probably interesting to mention that according to Newtonian dynamics, the magnitude of the velocity of these particles when they are in the $r = 16r_s$ orbit is 25% of the speed of light; which indicates that a relativistic description of these systems is required.

Since the angular momentum loss is crucial for the coalescence mechanism, it is important to have an accurate description of this process. This leads to the first question which is: what is the definition of angular momentum in a relativistic system?

There are several difficulties associated to this question. To begin with, the spacetime appropriate for the description of a relativistic isolated system of compact objects is curved and asymptotically flat. Therefore, since the asymptotic symmetry isolated system of compact objects is curved and asymptotically flat. Therefore, since the asymptotic symmetry group is the infinite dimensional BMS group\cite{1, 14}, the notion of angular momentum in general relativity will have to deal with the so called problem of supertranslations. Roughly speaking, among the generators of the BMS group, one can distinguish a set of 6 rotations, and an infinite set of supertranslations. A normal 4-dimensional subgroup of the BMS group exists which allows for the unambiguous definition of total momentum; the so called Bondi momentum.

To give perspective to this problem, let us recall that in special relativity, the angular momentum $J_{ab}$, the intrinsic angular momentum $S_{ab}$ and the linear momentum $P^a$ are related by

$$J_{ab} = S_{ab} + R^a P^b - P^a R^b,$$

where $a, b$ are numeric spacetime indices and $R^a$ represents the translational freedom. A rest frame is one in which the momentum $P$ has only timelike components. Given a rest reference frame in Minkowski spacetime one needs to use the spacelike translation freedom appearing in expression \(2\) in order to single out the center of mass reference frame. In the center of mass frame one has $J_{ab} = S_{ab}$; that is the total momentum coincide with the intrinsic angular momentum.

In general relativity, instead of having just a momentum, we have an infinite component supermomentum vector (associated with the infinite generators of supertranslations). Therefore the analog of equation \(2\) nec-
We have solved the problem of rest frames in the past with the construction of the so called nice sections [8]. Given an asymptotically flat spacetime, these sections provide a well defined notion of rest frames at future null infinity. We have also proved that they have the expected physical properties [3, 10] (See comments in the next section).

In this article we use this construction to give a definition of intrinsic angular momentum in general relativity, solving in this way the deficiencies of previous works [2, 4, 5, 7, 11, 12, 13, 15, 16, 17, 18].

2 Rest frame systems

In order to define rest frames, we need to fix the notion of supermomentum. We will use the supermomentum ‘ψ’ that was used in the nice section construction [8].

Let the section \( S \) of future null infinity be characterized by the condition \( u = 0 \), of the Bondi coordinate system \((u, \zeta, \bar{\zeta})\). Then the supermomentum is given by

\[
P_{lm}(S) = -\frac{1}{\sqrt{4\pi}} \int_S Y_{lm}(\zeta, \bar{\zeta}) \Psi(u = 0, \zeta, \bar{\zeta}) dS^2, \tag{2}\]

where \( dS^2 \) is the surface element of the unit sphere on \( S \), \( Y_{lm} \) are the spherical harmonics, the scalar \( \Psi \) is given by

\[
\Psi = \Psi^0 + \sigma_0 \delta_0 + \partial^2 \delta_0, \tag{3}\]

where we are using the GHP [5] notation and where \( \Psi^0 \) is the leading order asymptotic behavior of the second Weyl tensor component, \( \sigma_0 \) is the leading order of the Bondi shear, \( \delta \) is the edth operator of the unit sphere, and a dot means partial derivative with respect to the retarded time \( u \).

The condition for a section to be of the nice type is that all the spacelike components of the supermomentum vanish. That is, if \( \tilde{S} \) is nice then \( P_{lm}(\tilde{S}) = 0 \) for \( l \neq 0 \).

Since any section \( S \) can be obtained from an arbitrary reference section \( \hat{S} \) by a supertranslation \( \gamma(\zeta, \bar{\zeta}) \), it is useful to know what is the equivalent condition on \( \gamma \). We have shown that the condition for the section determined by \( \gamma \) to be of the nice type is [8, 10]

\[
\partial^2 \bar{\partial}^2 \gamma = \Psi(\gamma, \zeta, \bar{\zeta}) + K^3(\gamma, \zeta, \bar{\zeta}) M(\gamma), \tag{4}\]

where the conformal factor \( K \) can be related to the Bondi momentum by \( K = \frac{m}{\mathcal{P}^3} \), with

\[
(\mathcal{P}^3) = \left( 1, \frac{\zeta + \bar{\zeta}}{1 + \zeta \bar{\zeta}}, \frac{\zeta - \bar{\zeta}}{i(1 + \zeta \bar{\zeta})}, \frac{\zeta \bar{\zeta} - 1}{1 + \zeta \bar{\zeta}} \right) \tag{5}\]

and \( \mathcal{P}^3 \) is evaluated at the section \( u = \gamma \). Also, the rest mass \( M \) at the same section is given by \( M = \sqrt{\mathcal{P}^3 \mathcal{P}^3} \) [8, 10].

3 Definition of intrinsic angular momentum

We will define intrinsic angular momentum by using the so called ‘Charge integrals of the Riemann tensor’ [7].

A charge integral of the Riemann tensor is a quantity ascribed to a 2-sphere \( S \) by the integration of the 2-form \( C_{ab} \), namely:

\[
Q_S = \int_S C \tag{6}\]

where \( C_{ab} \) is expressed in terms of the curvature by

\[
C_{ab} = R^c_{\ ab} w_{cd} \tag{7}\]

and where a right star means right dual of the Riemann tensor. We have solved the problem of rest frames in the past with the construction of the so called nice sections [8]. An important property is that: if \( S_1 \) is a nice section and \( S_2 \) is another nice section which is generated from \( S_1 \) by a timelike translation, then \( S_2 \) is the future of \( S_1 \). This is exactly what happens in Minkowski space when \( S_1 \) is the intersection of the future null cone emanating from an interior point, let us say \( x_1 \), with future null infinity; and if \( S_2 \) is the intersection of the future null cone corresponding to another point \( x_2 \) which is in the future of \( x_1 \). There is then an analogy between rest frames at future null infinity centered on a nice section \( S \) and rest frames of Minkowski spacetime centered on a point \( x \).
Using the potential $\delta$ of the shear satisfying $\sigma_0 = \partial^2 \delta$, the component $w_1$ can be expressed by

$$ w_1 = b + \frac{1}{3} \partial \delta \partial \bar{a} + \frac{1}{6} (u - \delta) \partial \partial \bar{a}; $$

where the spin weight 0 quantity $b$ satisfies $\bar{b} = 0$ and $\partial^2 b = 0$.

Having a timelike one-parameter family of nice sections at future null infinity, this procedure provides with a 2-form $w$ at future null infinity with the functional dependence

$$ w_{AB} = w_{AB} (\sigma_0(u, \zeta, \bar{\zeta}), a, b; u, \zeta, \bar{\zeta}). $$

Then the charge integral becomes

$$ Q_S(w) = \int_S C $$

$$ = 4 \int_S \left[ -w_2 \left( \Psi_1^0 + 2 \sigma_0 \partial \sigma_0 + \partial (\sigma_0 \sigma_0) \right) 
+ 2 w_1 \left( \Psi_2^0 + \sigma_0 \delta \sigma_0 + \partial^2 \sigma_0 \right) \right] dS^2 + \text{c.c.} $$

In order to pick up the intrinsic angular momentum we need to impose the center of mass condition

$$ Q_S(a) = 0 \quad \text{for all} \quad a = \bar{a}. $$

This is the analog of the fixing of the translation freedom in the Minkowskian case by requiring that three components of the angular momentum tensor vanish, namely $J^{01} = J^{02} = J^{03} = 0$. In this way, there is left a unique one parameter family of nice sections, which determines the center of mass frames.

Let us denote these sections with $S_{cm}$; then the intrinsic angular momentum $j$ is defined by

$$ j(w) = Q_{S_{cm}}(w); $$

where in order to pick up the intrinsic angular momentum one must take $a = -\bar{a}$ and $b = 0$.

### 4 Comments

Several advantages exist when one uses the charge integral approach for the definition of physical quantities. For example, although we have defined the tensor $C$ at future null infinity, let us consider the possibility to extend its definition to the interior of the spacetime. Assume $V$ is a spacelike hypersurface in the interior of the spacetime which extends up to future null infinity, and which has as boundary the section $S$; as shown in figure

![Figure 1: The spacelike hypersurface $V$ has as boundary the section $S$ at future null infinity.](image)

Then, using Stokes’ theorem, one can express the charge integral $Q$ as an integral on $V$, namely

$$ Q_S = \int_S C = \int_V dC. $$

One can prove that the exterior derivative of $C$ can be expressed by

$$ dC_{abc} = 2 \varepsilon_{abcd} R^{*defg} \left( T_{efg} + \frac{1}{3} \varepsilon_{efg} \frac{1}{3} \varepsilon_{efg} v_f \right) $$

$$ = \frac{1}{3} \varepsilon_{abcd} \left( -2 G^{df} v_d + * \varepsilon^{defg} T_{efg} \right), $$

where $G_{ab}$ is the Einstein tensor, $T_{abc}$ is the traceless part of $\nabla_a w_{bc}$ and $\nabla_a w^{ab} = v^a$, its trace; in other words

$$ \nabla_a w_{bc} = T_{abc} + \frac{1}{3} \varepsilon_{ab} v_c - \frac{1}{3} \varepsilon_{ac} v_b. $$

Let us look at equation in the linearized gravity case. Then, it is noted that if the vector $v^a$ were a Killing vector of the flat background metric, and $T_{abc}$ were $O(1)$, then equation will give the conserved quantities in the context of linearized gravity. This is telling us that the charge integrals have the appropriate meaning in the weak field limit.

Also one can use Stokes’ theorem to calculate the flux of angular momentum. Let $S_2$ be a section to the future of the section $S_1$ of future null infinity, and let now $\Sigma$ be the region which has as boundaries $S_1$ and $S_2$; see figure. Then the flux law for angular momentum is given by

$$ Q_{S_2} - Q_{S_1} = \int_{S_2} C - \int_{S_1} C = \int_{\Sigma} dC. $$

When the spacetime is stationary, it can be seen that our definition does give the intrinsic angular momentum of the spacetime; and also that the center of mass, defined in terms of nice sections, is what one expects to be.
of the first and third stages are related in general by a supertranslation.

It should be stressed that given a point at future null infinity, which could be considered as the event of the detection of gravitational waves, then the center of mass sections just defined provides with a unique section where all physical quantities should be defined. A couple of previous works \([7, 13]\) also provide with a prescription for the selection of unique sections at future null infinity, however they are either non-local \([7]\) or they do not mention the subject of supermomentum \([13]\) and therefore can not be considered as either rest frames nor center of mass.

It is probably worth mentioning that several estimates on the gravitational radiation of different astrophysical systems are done with the quadrupole radiation formula. If one needs these estimates at two different times, such that back reaction has occurred due to the emission of gravitational waves between the first and second time, then the multipoles should be calculated at the corresponding center of mass for each time. In other words, even in the use of the quadrupole radiation formula one needs to have a well defined notion of center of mass. Our work provides both notions ‘center of mass’ and ‘intrinsic angular momentum’ in one construction.

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