Interchanging $A_\infty$ and $E_n$ structures

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Dedicated to the memory of Roland Schwänzl

Abstract

Let $\mathcal{A}$ and $\mathcal{B}$ be operads and let $X$ be an object with an $\mathcal{A}$-algebra and a $\mathcal{B}$-algebra structure. These structures are said to interchange if each operation $\alpha : X^n \to X$ of the $\mathcal{A}$-structure is a homomorphism with respect to the $\mathcal{B}$-structure and vice versa. In this case the combined structure is codified by the tensor product $\mathcal{A} \otimes \mathcal{B}$ of the two operads. There is not much known about $\mathcal{A} \otimes \mathcal{B}$ in general, because the analysis of the tensor product requires the solution of a tricky word problem.

Intuitively one might expect that the tensor product of an $E_k$-operad with an $E_l$-operad (which encode the multiplicative structures of $k$-fold, respectively $l$-fold loop spaces) ought to be an $E_{k+l}$-operad. However, there are easy counterexamples to this naive conjecture. In this paper we essentially solve the word problem for the nullary, unary, and binary operations of the tensor product of arbitrary topological operads and show that the the tensor product of a cofibrant $E_k$-operad with a cofibrant $E_l$-operad is an $E_{k+l}$-operad. It follows that if $\mathcal{A}_i$ are $E_{k_i}$ operads for $i = 1, 2, \ldots, n$ then $\mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n$ is at least an $E_{k_1+\ldots+k_n}$ operad, i.e. there is an $E_{k_1+\ldots+k_n}$-operad $C$ and a map of operads $C \to \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n$.

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1. Introduction

Two algebraic structures are said to interchange if the structure maps of one structure are homomorphisms with respect to the second structure and vice versa. A precise definition will be given below.

Interchange features are abundant in algebra, category theory, algebraic topology and related fields. A well-known exercise in introductory algebra is to show that two interchanging group structures coincide and are abelian group structures. In iterated loop space theory interchanging loop structures provide rich algebraic structures. In the theory of $n$-categories the interchange of the various category structures is of central interest, and one of the main problems in the search for the “best” notion of a weak $n$-category is the determination of the “right” notion of interchange.

In the present paper we address the interchange of $A_{\infty}$ and $E_n$ structures extending a program originally suggested by J.M. Boardman [4] in the context of a recognition principle for $n$-fold loop spaces. This program has experienced a revival of interests for various reasons: In connection with the research on weak $n$-categories, $A_{\infty}$-categories, and Segal categories the question of the uniqueness of $n$-fold delooping machines for $1 \leq n \leq \infty$ has become of interest again. The solution offered in [10] has gaps. The analysis of the delooping problem brings up the question of interchanging $A_{\infty}$ structures.

Kontsevich’s generalization of Deligne’s Hochschild cohomology conjecture to algebras over the little $n$-cubes operad, and the dual problem about the multiplicative structure of (topological) Hochschild homology lead directly into interchange problems of $E_n$ structures and $A_{\infty}$ structures.

A space $X$ has two interchanging structures encoded by operads $\mathcal{B}$ and $\mathcal{C}$ iff it has a structure encoded by the tensor product $\mathcal{B} \otimes \mathcal{C}$. Our main result is

**Theorem 4.3:** If $\mathcal{B}$ is an $E_k$ operad and $\mathcal{C}$ an $E_l$ operad and both are cofibrant, then $\mathcal{B} \otimes \mathcal{C}$ is an $E_{k+l}$ operad.

If we drop the cofibrancy condition, then $\mathcal{B} \otimes \mathcal{C}$ is at least $E_{k+l}$, which means that there is an $E_{k+l}$ operad $\mathcal{D}$ and a map of operads $\mathcal{D} \to \mathcal{B} \otimes \mathcal{C}$. But there might be an $E_r$ operad $\mathcal{D}'$ with $r > k + l$ and a map of operads $\mathcal{D}' \to \mathcal{B} \otimes \mathcal{C}$.

The tensor product of operads is quite elusive, and during our work on this paper we often fell into traps. In Section 3 we will list some surprising examples which will give an indication that we have to solve a non-trivial word problem.

We are convinced that the methods presented in our paper will be useful in the analysis of dendroidal sets introduced by Moerdijk and Weiss [12], [14], [17], and we believe that the same is true for the study of higher order category theory.

The strategy of the proof of Theorem 4.3 is as follows: For a particular choice of universal operads $\mathcal{B} = W|N\mathcal{M}_k|$ and $\mathcal{C} = W|N\mathcal{M}_l|$ we cover the tensor product $\mathcal{B} \otimes \mathcal{C}$ by suitable contractible subsets and relate the nerve of this covering to $\mathcal{M}_{k+l}$. Here $\mathcal{M}_k$ is the $Cat$ operad which was analyzed in [1] and shown to be
an $E_k$ operad, $N$ is the nerve functor, and $W$ is a cofibrant replacement functor.

The paper is organized as follows: In Sections 2 and 3 we recall the definitions of $A_\infty$ and $E_k$ operads, of interchange and the tensor product of operads. As mentioned above, Section 3 also contains some surprising examples of tensor products. Our main results and a recollection of the operad $M_k$ follow in Section 4. In Section 5 we explain the strategy of the proof of Theorem 4.3 in greater detail. Section 6 deals with the unary and binary operations in the tensor product of arbitrary operads.

The forgetful functor $U$ from reduced operads to topological monoids has a right adjoint $R$, and we call a $\Sigma$-free operad $B$ *axial* if the unit $B(n) \rightarrow RU(B)(n)$ is a closed cofibration for each $n$. This property is crucial in our proof. Therefore we study the adjoint pair $U$ and $R$ in Section 7. In Section 8 we recall the $W$-construction and, using the tree category of [12] but allowing stumps, interpret $W|NM_k|$ as the topological realization of a simplicial $Cat$ operad. This part may be of independent interest. We prove the axially of $W|NM_k|$ and related properties used in the proof of the main result. In the remaining sections we define the cover of $(W|NM_k| \otimes W|NM_l|)(m)$ by homotopy cells, set up the diagram and the homotopy equivalences mentioned above.

We dedicate this paper to the memory of Roland Schwänzl [19], our dear friend and collaborator, who started off with us on this project but tragically succumbed to a fatal illness at an early stage of its development.

2. $E_n$ structures and $A_\infty$ structures

$E_n$ structures are closely related to the algebraic structure of an $n$-fold loop space. They are best described using operads.

**2.1 Definition:** Let $S$ be a symmetric monoidal category with multiplication $\times$. An operad $B$ in $S$ is a collection $\{B(k)\}_{k \geq 0}$ of objects equipped with symmetric group actions $B(k) \times \Sigma_k \rightarrow B(k)$, composition maps

$$B(k) \times (B(j_1) \times \ldots \times B(j_k)) \rightarrow B(j_1 + \ldots + j_k),$$

and a unit $id \in B(1)$ satisfying the appropriate equivariance, associativity and unitality conditions - see [13] for details.

**2.2 Remark:** Throughout this paper $S$ will be $Cat$, the category of small categories, $Sets$, the category of sets, $SSets$, the category of simplicial sets, or $Top$, the category of $k$-spaces and continuous maps. For topological operads we require the additional technical condition that $\{id\} \subset B(1)$ is a closed cofibration (this can always be achieved by whiskering $B(1)$).

In all three cases the symmetric monoidal structure is given by the product and the categories are complete and cocomplete. Moreover, they are self-enriched and the product distributes over the coproduct. In particular, we can define the
$S$-endomorphism operad $E_X$ of an object $X$ in $S$ by $E_X(n) = S(X^n, X)$ with the obvious $\Sigma_n$-action and the obvious composition maps and unit.

2.3 While (2.1) is the most common definition of an operad, it is often helpful to think of it in the following equivalent way, which is the original version from [5]. An operad $B$ in a symmetric monoidal category $S$ of Remark 2.2 is an $S$-enriched symmetric monoidal category $(B, \oplus, 0)$ such that

(i) $ob B = \mathbb{N}$ and $m \oplus n = m + n$
(ii) $\oplus$ is a strictly associative $S$-functor with strict unit $0$
(iii) $\coprod_{r_1 + \ldots + r_n = k} B(r_1, 1) \times \ldots \times B(r_n, 1) \times \Sigma_{r_1} \times \ldots \times \Sigma_{r_n} \Sigma_k \to B(k, n)$

is an isomorphism in $S$.

In the topological case the additional requirement translates to the assumption that $\{id\} \subset B(1, 1)$ is a closed cofibration.

Using (iii), each operad in the sense of (2.1) determines a category as in (2.3). Conversely, given a category as in (2.3) we obtain an operad by taking the collection $\{B(k, 1)\}_{k \geq 0}$.

For some inductive arguments we will use the following blown-up version of (2.3):

2.4 Each operad $B$ gives rise to an $S$ enriched symmetric monoidal category $(B, \oplus, 0)$ defined by

(i) $ob B = \{\text{totally ordered finite sets}\}$ and $S \oplus T = S \sqcup T$, the ordered disjoint union.
(ii) $\oplus$ is a strictly associative $S$-functor with strict unit $0 = \emptyset$
(iii) $B(S, \{t\}) = B(|S|)$, where $|S|$ is the cardinality of $S$.
(iv) $\prod_{S_1 \sqcup \ldots \sqcup S_n = U} B(S_1, \{t_1\}) \times \ldots \times B(S_n, \{t_n\}) \times \Sigma(S_1) \times \ldots \times \Sigma(S_n) \Sigma(U) \to B(U, T)$

is an isomorphism in $S$, where $\Sigma(U)$ is the permutation group of (the underlying set of) $U$ and $T = \{t_1 < \ldots < t_n\}$.

We will also find it convenient to use this blown-up version in operad notation; e.g. $B(S)$ stands for $B(|S|)$.

2.5 Definition: Let $B$ and $C$ be $S$-operads.

(1) $B$ is called $\Sigma$-free if the $\Sigma_n$-action on $B(n)$ is free for each $n$ in the cases $S = \text{Cat}$, $\text{Sets}$, or $\text{SSets}$. If $S = \text{Top}$ we require that $B(n) \to B(n)/\Sigma_n$ is a numerable principal $\Sigma_n$-bundle for each $n$. 

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(2) An operad map $B \to C$ is a collection of equivariant maps $B(n) \to C(n)$ in $S$, compatible with the operad structure.

(3) A $B$-structure on an object $X$ in $S$ is an operad map $B \to \mathcal{E}_X$ into the endomorphism operad $\mathcal{E}_X$ of $X$. We say that $B$ acts on $X$, or that $X$ is a $B$-algebra; if $S = Top$ we also call $X$ a $B$-space.

(4) An operad map is called a weak equivalence if each map $B(n) \to C(n)$ is an equivariant homotopy equivalence (in $\mathcal{C}$ or $\mathcal{SSets}$ this means that each map is an equivariant homotopy equivalence after applying the classifying space functor, respectively the topological realization functor).

(5) Two operads are called equivalent if there is a finite chain of weak equivalences connecting them.

(6) A topological operad is called $E_n$ if it is equivalent to the little $n$-cubes operad $C_n$.

(7) An $A_\infty$ operad is another term for an $E_1$ operad.

(8) An operad $B$ is called reduced if $B(0) = \{0\}$. We denote the categories of operads and reduced operads by $\mathcal{O}_{pr}$ and $\mathcal{O}_{pr0}$ respectively. Here we use the same notation for each of our categories $S$; it will be clear from the context which $S$ we mean.

Recall that $C_n(k)$ is the space of $k$-fold configurations of subcubes of the unit cube $I^n$, whose axes are parallel to those of $I^n$ and whose interiors are disjoint. Any $n$-fold loop space $\Omega^n Y$ has a natural action by this operad. Conversely, each connected space with a $C_n$-structure is of the weak homotopy type of an $n$-fold loop space (c.f. [5], [6], and [14] for details).

2.6 Definition: Let $S$ and $T = \{t_1, \ldots, t_n\}$ be totally ordered finite sets and let $B$ be a reduced operad. We define restriction maps

$$ - \cap S : B(T) \to B(S \cap T) $$

by composing with $(\varepsilon_1 \oplus \ldots \oplus \varepsilon_n)$, where $\varepsilon_i = id$ if $t_i \in S$ and is 0 otherwise.

We will also use this notation in related situations like products, sums etc. of $B(T)$’s.

We will also make use of the operads $\mathcal{Ass}$ and $\mathcal{Com}$ which encode the structures of a monoid and a commutative monoid respectively. By definition, $\mathcal{Ass}(n) = \Sigma_n$, where $\sigma \in \Sigma_n$ stands for the operation

$$(x_1, \ldots, x_n) \mapsto x_{\sigma^{-1}(1)} \cdot \ldots \cdot x_{\sigma^{-1}(n)}$$

in a monoid. From this the operad data for $\mathcal{Ass}$ can be deduced. $\mathcal{Com}(n) = \{\lambda_n\}$ is a single point. Here $\lambda_n$ stands for the operation

$$ (x_1, \ldots, x_n) \quad \mapsto \quad x_1 + \ldots + x_n. $$

An $n$-fold loop space $\Omega^n Y$ has $n$ interchanging loop space structures. Since $C_n$ acts naturally on $\Omega^n Y$, these $n$ interchanging structures should somehow be encoded in $C_n$. Before we can make this precise we need to formally define the notion of interchange.
3. Interchange

Since this paper is about interchange of topological operad structures the operads in this section will be topological operads. The interested reader can easily make the necessary adjustments for our other categories $\mathcal{S}$.

Let $X$ be a space with actions of operads $B$ and $C$. Then the coproduct $B \amalg C$ of $B$ and $C$ in the category of operads acts on $X$.

3.1 Definition: We say that the $B$- and $C$-actions on $X$ interchange if each operation $\beta : X^k \to X$, $\beta \in B(k)$, in the $B$-structure is a homomorphism of $C$-spaces, and vice versa. Explicitly, this means that for each $\beta \in B(k)$ and each $\gamma \in C(l)$ the square

\[
\begin{array}{ccc}
(X^k)^l & \xrightarrow{\tau_{k,l}} & (X^l)^k \\
\downarrow{\beta^l} & & \downarrow{\gamma^k} \\
X^l & \xrightarrow{\gamma} & X
\end{array}
\]

commutes, where $\tau_{k,l} \in \Sigma_{k \cdot l}$ is the permutation which reorders the coordinates of $X^{k \cdot l}$ from lexicographical to reserve lexicographical order.

Note that the two composites $\beta \circ \gamma^k \circ \tau_{k,l}$ and $\gamma \circ \beta^l$ are elements in $(B \amalg C)(k \cdot l)$. If $k > 0$ and $l > 0$, we may interpret the interchange condition as follows. Given a $k \times l$ array $\{x_{ij}\}_{i=1, j=1}^{k, l}$ of elements of $X$, we can apply $\beta$ to the columns of the array and then $\gamma$ to the resulting products. Alternatively we can apply $\gamma$ to each row of the array, then $\beta$ to the resulting products. The interchange condition states that we obtain the same final result either way.

3.2 Definition: The tensor product $B \otimes C$ of operads $B$ and $C$ is obtained from the coproduct $B \amalg C$ by factoring out the interchange relation [3.1].

A more detailed description of $B \amalg C$ and $B \otimes C$ will be given in the beginning of Section 5.

3.3 Remark: In the case when $\beta \in B(1)$, the interchange relation implies

$$\beta \circ \gamma = \gamma \circ \beta^l.$$ 

This relation and the dual relation when $\gamma \in C(1)$ are called unary interchanges and will play an important role in our analysis of the tensor product of operads. More generally we shall refer to interchanges involving $\beta \in B(k)$ and $\gamma \in C(l)$ as $(k, l)$-interchanges.
By definition, a space $X$ admits a $B$-structure interchanging with a $C$-structure iff it admits a $(B \otimes C)$-structure.

The following two results about tensor products of operads are extant in the literature:

3.4 Proposition: (Dunn [8]) There is a canonical weak equivalence of operads

$$C_1 \otimes \ldots \otimes C_1 \to C_n$$

(n tensor factors)

So we “recover” the $n$ interchanging loop space structures in $C_n$.

3.5 Proposition: ([8]) The operad $A_{ss} \otimes C_k$ is equivalent to $C_{k+1}$

In view of these two results, we might conjecture that:

3.6 Naive Conjecture: If $B$ is $E_k$ and $C$ is $E_l$, then $B \otimes C$ is $E_{k+l}$.

Unfortunately, the situation is not that simple: The functor $B \otimes -$ does not preserve weak equivalences, and the structure of $B \otimes C$ is anything but clear. In general, one has to solve a substantial word problem. For instance unary operations in $B(1)$ and $C(1)$ may be factored in many different ways and these factors may then be redistributed in a complicated way using the unary interchange relations mentioned in Remark 3.3. The following examples illustrate some rather surprising consequences that these and other interchange relation may imply.

3.7 Proposition: ([6, Lemma 2.23]) Let $B$ and $C$ be operads such that $B(0) \neq \emptyset \neq C(0)$ and let $B'$ and $C'$ be their universal quotients with exactly one nullary operation. Then

(1) $(B \otimes C)(0)$ contains exactly one element.

(2) $B \otimes C \simeq B' \otimes C'$

Proof: Given $\beta \in B(0)$ and $\gamma \in C(0)$, the interchange relation gives

$$\ast = (X^0)^0 \xrightarrow{\gamma_{0,0}=id} (X^0)^0 \xrightarrow{\gamma_{0}=id} X^0 = \ast$$

$$\beta^0=id \quad \gamma \quad \beta$$

Hence $\beta = \gamma$ in $B \otimes C$.

The second part is an immediate consequence of the first part. 

3.8 Proposition: Let $B$ and $C$ be operads such that $B(1) = \{id\}$, $C(1) = \{id\}$ and $B(0)$, $B(2)$, $C(0)$, $C(2)$ are not empty. Then $B \otimes C \simeq Com.$
Proof Since $B(2)$ and $C(2)$ are not empty, so are $B(k)$ and $C(k)$ for $k \geq 2$. Let $\lambda \in (B \otimes C)(0)$ be the unique nullary operation and let $X$ be a $(B \otimes C)$-space. For $\beta \in B(k)$ and $\gamma \in C(k)$ consider the following $(k \times k)$-array of points in $X$,

$$
\begin{array}{cccc}
  x_1 & \ast & \ldots & \ast \\
  \ast & x_2 & \ldots & \ast \\
  \ldots & \ldots & \ldots & \ldots \\
  \ast & \ast & \ldots & x_k
\end{array}
$$

Let $\beta$ act horizontally on this array and $\gamma$ vertically. By the interchange relation, the results should be the same if we compute the action first horizontally, then vertically or vice versa. Now $\beta(\ast, \ldots, \ast, x_i, \ast, \ldots, \ast) = \beta \circ (\lambda, \ldots, \lambda, id, \lambda, \ldots, \lambda)(x_i) = x_i$ for $i = 1, \ldots, k$, since $\beta \circ (\lambda, \ldots, \lambda, id, \lambda, \ldots, \lambda) \in B(1) = \{id\}$. Hence computing the action first horizontally and then vertically gives $\gamma(x_1, \ldots, x_k)$. Similarly computing it the other way we get $\beta(x_1, \ldots, x_k)$, so that $\beta = \gamma$ in $B \otimes C$ for all $\beta$ and $\gamma$.

3.9 Corollary: (i) Suppose $\mu_1, \mu_2 : X^2 \rightarrow X$ are two (not necessarily associative) $H$-space structures on a topological space $X$. Suppose that $\mu_1$ and $\mu_2$ both have the same strict 2-sided unit $\ast$ and that $\mu_1$ and $\mu_2$ satisfy the $(2, 2)$-interchange relation

$$(E-H) \quad \mu_1(\mu_2(x_1, x_2), \mu_2(x_3, x_4)) = \mu_2(\mu_1(x_1, x_3), \mu_1(x_2, x_4))$$

for all $x_1, x_2, x_3, x_4 \in X$. Then $\mu_1 = \mu_2$ and this multiplication is strictly associative and commutative.

(ii) More generally, if $B(2) \times X^2 \rightarrow X$, $C(2) \times X^2 \rightarrow X$, are continuous nonempty families of (not necessarily associative) multiplications on a topological space $X$, which have a common 2-sided unit $\ast$, and which satisfy the $(2, 2)$-interchange relations $(E-H)$ for all $\mu_1 \in B(2)$ and $\mu_2 \in C(2)$, then $B(2) = C(2) = \{\mu\}$ and $\mu$ defines a commutative monoid structure on $X$.

(iii) $\text{Ass} \otimes \text{Ass} \cong \text{Ass} \otimes \text{Com} \cong \text{Com} \otimes \text{Com} \cong \text{Com}$

Corollary 3.9 (i) with the additional hypotheses that $\mu_1$ and $\mu_2$ are both associative is well known to topologists as Eckmann-Hilton interchange [11]. Surprisingly the fact that the associativity hypotheses are superfluous seems to be generally unknown.

Proof We first prove (i). Let $B$ be the operad generated by $\mu_1$ and the 2-sided unit. Specifically $B(0) = \{\ast\}$, $B(1) = \{id\}$, $B(2) = \{\mu_1, \mu_1 \circ (1 2)\}$, and for $k > 2$, $B(k)$ consists of all $k$-fold iterates of $\mu_1$ and their permutations. (For $k \geq 2$, the elements of $B(k)$ are in 1-1 correspondence with planar binary trees with $k$ labelled inputs and no stumps.) Similarly let $C$ be the operad generated
by $\mu_2$ and the 2-sided unit. Then $B$ and $C$ both act on $X$. We must show that these operad actions interchange.

The unary interchanges between $B$ and $C$ hold trivially since $B(1) = \{id\} = C(1)$. Similarly the nullary $(k,0)$- and $(0,l)$-interchanges follow from $B(0) = \{*\} = C(0)$. Finally we obtain that all $(2,2)$-interchanges hold by applying appropriate permutations to the interchange relation (E-H). Thus it remains to show that $(k,l)$-interchanges hold when $k > 2$ or $l > 2$.

Let $\beta_2 \in B(2)$. Then $\beta_2$ defines an $H$-space structure on $X$ and on all products $X^l$ (via coordinatewise multiplication). From the already established $(2,2)$-interchange relations, $\mu_2$ is a homomorphism $X^2 \to X$ with respect to $\beta_2$. Thus iterates of $\mu_2$ are composites of products of $\beta_2$-homomorphisms and thus are also homomorphisms. Clearly permutations of $X^l$ are also homomorphisms. This establishes all $(2,l)$-interchanges for all $l > 2$.

Now fix an element $\gamma_l \in C(l)$. Then $\gamma_l$ defines an $l$-ary operation on $X$ and coordinatewise on all products $X^k$. From the already established $(2,l)$-interchange relations, we obtain that $\mu_1$, its iterates and permutations thereof determine homorphisms of these $\gamma_l$ structures. This establishes all $(k,l)$-interchanges for $k > 2$.

Thus we obtain that the $B$ and $C$ actions on $X$ interchange and thus determine a $B \otimes C$ action. But by $B \otimes C \cong \text{Com}$. Thus $\mu_1 = \mu_2$ is strictly associative and commutative, establishing (i).

Part (ii) is an immediate consequence. For we can pick $\mu_1 \in B(2)$ and $\mu_2 \in C(2)$ and apply (i). We obtain that $\mu_1 = \mu_2$ is associative and commutative. Since $\mu_1$ and $\mu_2$ are arbitrary, (ii) follows.

Part (iii) is an immediate consequence of $B \otimes C \cong \text{Com}$.

The same argument as in the proof of part (i) establishes the following result.

3.10 Proposition: Suppose that two operads $B$ and $C$ act on a space $X$. Suppose that the interchange relations hold between all the generating elements of $B$ and $C$. Then the actions of $B$ and $C$ on $X$ interchange.

3.11 Remark: The tensor product of $\text{Ass}$ with an operad $B$ satisfying $B(0) = *$ has been determined in [7]. We will recall this result in Section 7 and deduce from it the structure of $\text{Com} \otimes B$. In the meantime we point out another surprising consequence of that result.

3.12 Proposition: If the bar construction $B_\bullet M$ on a well-pointed topological monoid $M$ is an $H$-space in the category of simplicial topological spaces, then the geometric realization $|B_\bullet M|$ is homotopy equivalent to a loop space, as an $H$-space.

Proof As is shown in [7], loop space structures are parametrized by an operad $W\text{Ass}$, a blown-up version of $\text{Ass}$ (which we will discuss in Section 8 below).
It is also shown there that $H$-space structures are parametrized by the suboperad $W^{(2)}\mathcal{Ass}$ of $W\mathcal{Ass}$ generated by the nullary, unary and binary operations $W\mathcal{Ass}(0), W\mathcal{Ass}(1), \text{and } W\mathcal{Ass}(2)$. Now a simplicial $H$-space structure on $B\mathcal{B}M$ amounts to the same thing as an $\mathcal{Ass} \otimes W^{(2)}\mathcal{Ass}$ structure on $M$. But according to [7], the induced map

$\mathcal{Ass} \otimes W^{(2)}\mathcal{Ass} \to \mathcal{Ass} \otimes W\mathcal{Ass}$

is an isomorphism of operads. Hence $M$ is an $\mathcal{Ass} \otimes W\mathcal{Ass}$ algebra and it follows that $|B\mathcal{B}M|$ has a $W\mathcal{Ass}$-structure. Since $|B\mathcal{B}M|$ is a path-connected Dold space [18, Cor. 5.2] this structure admits a homotopy inverse (e.g. [18, Prop. 3.16]). It is well-known that a $W\mathcal{Ass}$-space admitting a homotopy inverse is homotopy equivalent through homotopy homomorphisms to a loop space.

Since $\mathcal{Ass}$ is $E_1$ and $\mathcal{Com}$ certainly is not $E_2$, Corollary 3.14 provides a counterexample to the Naive Conjecture 3.6. In fact, each connected abelian topological monoid is of the weak homotopy type of an infinite loop space, indeed equivalent to a product of Eilenberg-MacLane spaces. So we adjust the conjecture in the following way:

3.13 Definition: An operad $\mathcal{B}$ is called at least $E_n$ if there is an $E_n$ operad $\mathcal{C}$ and a map of operads $\mathcal{C} \to \mathcal{B}$. (So any $\mathcal{B}$ space has an $E_n$ structure.)

3.14 Conjecture: If $\mathcal{B}$ is $E_k$ and $\mathcal{C}$ is $E_l$, then $\mathcal{B} \otimes \mathcal{C}$ is at least $E_{k+l}$.

3.15 Remark: Given maps of operads $\mathcal{A} \to \mathcal{B} \to \mathcal{A}$ with $\mathcal{A}$ being $E_n$, then each $\mathcal{B}$-space is an $\mathcal{A}$-space and each $\mathcal{A}$-space is a $\mathcal{B}$-space. $\mathcal{B}$ is at least $E_n$, and one might expect $\mathcal{B}$ to be in fact $E_n$. This need not be true even if the composite $\mathcal{A} \to \mathcal{B} \to \mathcal{A}$ is the identity, as the following example shows:

$\mathcal{C}_n \to \mathcal{C}_n \amalg C_n \to \mathcal{C}_n$

where the first map is the inclusion of the first summand and the second is the folding map. $\mathcal{C}_n \amalg \mathcal{C}_n$ codifies two non-interchanging $\mathcal{C}_n$-structures and is certainly not $E_n$: The operad $\pi_0(C_n)$ of path components of $\mathcal{C}_n$ is isomorphic to $\mathcal{Com}$ for $n \geq 2$. Hence there is a surjection (in fact, a bijection) $\pi_0(\mathcal{C}_n \amalg \mathcal{C}_n) \to \mathcal{Com} \amalg \mathcal{Com}$. Now $(\mathcal{Com} \amalg \mathcal{Com})(2)$ has two elements, while $\mathcal{Com}(2)$ has only one. So $\mathcal{C}_n \amalg \mathcal{C}_n$ has too many path components to be $E_n$. For $n = 1$ the same argument works if one replaces $\mathcal{Com}$ by $\mathcal{Ass}$.

4. Main results

4.1 Convention: In view of Proposition 3.7, we only work with reduced topological operads unless explicitly stated otherwise. So operad will mean reduced topological operad.
4.2 Definition: An operad $B$ in $Opr_0$ is called cofibrant if for any diagram

\[
\begin{array}{ccc}
P & \xrightarrow{p} & C \\
\downarrow & & \downarrow \\
B & \xrightarrow{f} & C
\end{array}
\]

of operad maps with $p$ a weak equivalence and each $p : \mathcal{P}(n) \to C(n)$ an equivariant fibration, there is a lift $g : B \to \mathcal{P}$ such that $p \circ g = f$.

It is shown in [6] that for any $\Sigma$-free operad $B$, there is a cofibrant operad $W B$ and an operad equivalence $W B \to B$. We call $W B \to B$ a cofibrant resolution of $B$.

The main result of our paper is the following.

4.3 Theorem: If $B$ is an $E_k$ operad and $C$ an $E_l$ operad and both are cofibrant, then $B \otimes C$ is an $E_{k+l}$ operad.

This verifies the Naive Conjecture 3.6 in the special case of cofibrant operads. It also verifies the modified Conjecture 3.14. Indeed it is equally easy to prove a strengthened version of this conjecture.

4.4 Corollary: Suppose that $A_i$ are at least $E_{k_i}$ for $i = 1, 2, \ldots, n$. Then the tensor product $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ is at least $E_{k_1+k_2+\cdots+k_n}$.

Proof We proceed by induction on $n$. The result is trivially true for $n = 1$. Assume it holds for $n - 1$. Then there is an $E_{k_1+k_2+\cdots+k_{n-1}}$ operad $B$ which maps into $A_1 \otimes A_2 \otimes \cdots \otimes A_{n-1}$. Let $C$ be a $E_{k_n}$ operad which maps into $A_n$. Let $W B \to B$ and $W C \to C$ be cofibrant resolutions. Then according to Theorem 4.3 $W B \otimes W C$ is $E_{k_1+k_2+\cdots+k_n}$ and we have a chain of operad maps

\[
W B \otimes W C \to B \otimes C \to A_1 \otimes A_2 \otimes \cdots \otimes A_n.
\]

This completes the induction and the proof. □

We will now make use of

4.5 Lemma: (e.g. see [20]): Given a diagram of operads

\[
\begin{array}{ccc}
P & \xrightarrow{p} & C \\
\downarrow & & \downarrow \\
B & \xrightarrow{f} & C
\end{array}
\]

with $B$ cofibrant and $p$ a weak equivalence. Then there is a lift $g : B \to \mathcal{P}$ up to homotopy, uniquely up to homotopy. (Here homotopy means homotopy through operad maps.)
4.6 Corollary:  (i) Any two cofibrant $E_k$ operads $B$ and $C$ are homotopy equivalent in the strong sense; i.e. there are operad maps $f : B \to C$ and $g : C \to B$ such that $g \circ f \simeq id$ and $f \circ g \simeq id$ through operad maps.

(ii) If $A$ is any operad and $B$ and $C$ are cofibrant $E_k$ operads, then $A \otimes B$ and $A \otimes C$ are homotopy equivalent in the strong sense.

(iii) If $B$ and $C$ are $E_k$ and $B$ is cofibrant, there is a weak equivalence $B \to C$.

In view of the corollary it suffices to prove Theorems 4.3 for our favorite cofibrant operads.

Our favorite cofibrant $E_k$ operad will be $W|NM_k|$, a cofibrant resolution of $|NM_k|$, the topological realization of the nerve $\mathcal{N}M_k$ of the $\text{Cat}$-operad $\mathcal{M}_k$ which parametrizes the algebraic structure of a $k$-fold monoidal category [1].

We can briefly describe $\mathcal{M}_k(m)$ as a poset whose objects are words of length $m$ in the alphabet $\{1, 2, \ldots, m\}$ combined together using $k$ binary operations $\Box_1, \Box_2, \ldots, \Box_k$, which are strictly associative and have common unit $0$. We moreover require that the objects of $\mathcal{M}_k(m)$ are precisely those words where each generator $\{1, 2, \ldots, m\}$ occurs exactly once. The morphisms of $\mathcal{M}_k(m)$ are generated by interchanges

$$\eta^i_{j,A,B,C,D} : (A \Box_j B) \Box_i (C \Box_j D) \longrightarrow (A \Box_i C) \Box_j (B \Box_i D),$$

where $1 \leq i < j \leq k$ and $A, B, C, D$ are words in $\{1, 2, \ldots, m\}$ such that $(A \Box_j B) \Box_i (C \Box_j D)$ (and hence also $(A \Box_i C) \Box_j (B \Box_i D)$) represent objects of $\mathcal{M}_k(m)$. These interchange morphisms are then combined using the binary operations $\Box_1, \Box_2, \ldots, \Box_k$ as well as composition of morphisms. The coherence theorem of [1] proves the nonobvious fact that $\mathcal{M}_k(m)$ is a poset, i.e. there is at most one morphism between any two objects, and gives a simple algorithm for determining when there is a morphism between any two objects (c.f. the proof of Lemma 5.4).

4.7 The topological operads $|NM_k|$ and $|NM_l|$ are far from cofibrant. Indeed $|NM_k|(1)$ and $|NM_l|(1)$ are both $\{id\}$ and hence by Proposition 3.8 $|NM_k| \otimes |NM_l| = \text{Com}$. On the other hand we will show that $W|NM_k| \otimes W|NM_l|$ is $E_{k+l}$. In anticipation of this result, it will be convenient to adopt the following convention. We will identify $\mathcal{M}_k$ with the obvious suboperad of $\mathcal{M}_{k+l}$ and we will identify $\mathcal{M}_l$ with the suboperad of $\mathcal{M}_{k+l}$ obtained by shifting the indices on the binary operations from $\Box_1, \Box_2, \ldots, \Box_l$ to $\Box_{k+1}, \Box_{k+2}, \ldots, \Box_{k+l}$.

In the course of proving Theorem 4.3 we obtain more results about the spaces of unary and binary operations in an arbitrary tensor product of operads. As these results may be of separate interest, we state them below.

4.8 Proposition: Let $A$ and $B$ be arbitrary topological operads with $A(0) = \{0\}$ and $B(0) = \{0\}$. Then

- $(A \otimes B)(1) \cong A(1) \times B(1)$
\( (A \otimes B)(2) \) is homeomorphic to the pushout of the following diagram

\[
\begin{array}{c}
A(2) \times B(2) \xrightarrow{f} A(1)^2 \times B(2) \\
\downarrow g \\
A(2) \times B(1)^2
\end{array}
\]

Here \( f(\alpha, \beta) = (\alpha \circ (id \oplus 0), \alpha \circ (0 \oplus id), \beta) \) and \( g(\alpha, \beta) = (\alpha, \beta \circ (id \oplus 0), \beta \circ (0 \oplus id)) \).

To get a handle on the homotopy type of the space of binary operations, we need to impose additional hypotheses (which are satisfied by \( A = W\mathcal{N}\mathcal{M}_k \) and \( B = W\mathcal{N}\mathcal{M}_l \) in particular). We then obtain the following result.

**4.9 Corollary:** Let \( A \) and \( B \) be arbitrary topological operads with \( A(0) = \{0\} \) and \( B(0) = \{0\} \). Suppose also that the spaces \( A(1) \) and \( B(1) \) are contractible and the map \( A(2) \longrightarrow A(1)^2 \) given by \( \alpha \mapsto (\alpha \circ (id \oplus 0), \alpha \circ (0 \oplus id)) \) is a cofibration. Then \( (A \otimes B)(2) \) has the homotopy type of the join \( A(2) \ast B(2) \).

**Proof** Because of the cofibration hypothesis, the pushout diagram for \( (A \otimes B)(2) \) has the same homotopy type as the homotopy pushout. Because \( A(1) \) and \( B(1) \) are contractible, this in turn has the same homotopy type as the homotopy pushout of

\[
\begin{array}{c}
A(2) \times B(2) \xrightarrow{pr_2} B(2) \\
\downarrow pr_1 \\
A(2)
\end{array}
\]

which is by definition the join \( A(2) \ast B(2) \). \( \square \)

If \( A \) is an \( E_k \) operad and satisfies the cofibration hypothesis of Corollary 4.9, while \( B \) is an arbitrary \( E_l \) operad, then \( A(2) \) has the homotopy type of \( S^{k-1} \), and \( B(2) \) has the homotopy type of \( S^{l-1} \). Thus by Corollary 4.9 \( (A \otimes B)(2) \) has the homotopy type of \( S^{k-1} \ast S^{l-1} \cong S^{k+l-1} \). This is consistent with \( A \otimes B \) being an \( E_{k+l} \) operad, and suggests that Theorem 4.3 might hold with a weaker hypothesis. However, we will not pursue this further in this paper.

**5. Outline of the proofs**

Before we discuss the proof of Theorem 4.3, we need a more explicit description of the tensor product of operads. We begin with a description of the coproduct \( B \amalg C \) of \( B \) and \( C \). The elements of \( (A \amalg B)(m) \) are equivalence classes of planar trees with \( m \) labelled inputs and one output, and whose nodes are labelled by elements of the the operads \( A \) and \( B \). It is required that the arity of each node
label correspond to the number of input branches coming into that node. So we have to allow nodes without inputs, which we call stumps labelled by elements in \( \mathcal{A}(0) \) or \( \mathcal{B}(0) \). For a more detailed description see Definition 8.1. The topology imposed is the quotient topology on the evident disjoint union of products of spaces \( \mathcal{A}(i) \) and \( \mathcal{B}(j) \). Operad composition in \( \mathcal{A} \Pi \mathcal{B} \) is given by splicing together trees.

The following equivalence relations are imposed on the trees. First of all if two trees have subtrees of the form shown below but are otherwise identical, then they are identified:

\[
\begin{align*}
\lambda_1 & \quad \lambda_2 & \quad \ldots & \quad \lambda_k \\
& \quad \lambda & \quad \lambda \circ (\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_k)
\end{align*}
\]

Here we are assuming that the node labels \( \lambda, \lambda_1, \lambda_2, \ldots, \lambda_k \) all belong to \( \mathcal{A} \) or all belong to \( \mathcal{B} \), and \( \lambda \circ (\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_k) \) denotes operad composition.

Additionally we also allow changing unary nodes which are labelled by the unit of \( \mathcal{A} \) to unary nodes labelled by the unit of \( \mathcal{B} \) and vice-versa.

5.2 We also need to impose an equivariance relation

\[
\begin{align*}
T_1 & \quad T_2 & \quad \ldots & \quad T_k \\
& \quad \lambda \circ \sigma & \quad \lambda \circ \sigma^{-1}(1) & \quad \lambda \circ \sigma^{-1}(2) & \quad \ldots & \quad \lambda \circ \sigma^{-1}(k)
\end{align*}
\]

Here we are assuming that the node label \( \lambda \) is either in \( \mathcal{A}(k) \) or in \( \mathcal{B}(k) \) and \( \sigma \in \Sigma_k \).

The resulting spaces of equivalence classes of trees form the coproduct operad \( \mathcal{A} \Pi \mathcal{B} \). The two equivalence relations insure that the images of \( \mathcal{A} \) and \( \mathcal{B} \) are suboperads of \( \mathcal{A} \Pi \mathcal{B} \).

5.3 To pass from the coproduct \( \mathcal{A} \Pi \mathcal{B} \) to the tensor product we need to impose another relation corresponding to the interchange. We identify two trees if they
have subtrees of the form shown below, but are otherwise identical:

Here $\alpha \in \mathcal{A}(k)$, $\beta \in \mathcal{B}(\ell)$, and $T_{ij}$ denote the branches of the trees lying above the nodes shown. Note that these branches are permuted on the two sides of the relation. The resulting spaces of equivalence classes of trees is the tensor product $\mathcal{A} \otimes \mathcal{B}$. For future reference, we will define the equivalence relation shown in the picture above as a $(k, l)$-interchange on trees. Trees related by a sequence of $(k, 1)$- or $(1, l)$-interchanges will be said to be related by unary interchanges (cf. Remark 3.3).

In order to determine the homotopy type of a tensor product of operads, we need to describe its spaces of operations as colimits of “nice” diagrams. Since the operad $\mathcal{M}_2$ encodes the algebraic structure of a 2-fold monoidal category, i.e. a category with two multiplications which interchange with each other up to coherent natural transformations, it should not be surprising that $\mathcal{M}_2$ should serve as the basis for constructing such diagrams. However we need to make some adjustments.

First of all, to avoid confusion between the multiplications in $\mathcal{M}_2$ and the internal multiplications in $\mathcal{W}(\mathcal{M}_k \otimes \mathcal{W}(\mathcal{M}_l)$ arising from those in $\mathcal{M}_k$ and $\mathcal{M}_l$, we will denote the multiplications in $\mathcal{M}_2$ by $\boxtimes_1$ and $\boxtimes_2$ instead of $\Box_1$ and $\Box_2$.

5.4 Lemma: The poset operad $\mathcal{M}_2$ has a quotient poset operad $\mathcal{M}_2^{ab}$, obtained from $\mathcal{M}_2$ by making the operations $\boxtimes_1$ and $\boxtimes_2$ commutative.

Proof In order to verify that making $\boxtimes_1$ and $\boxtimes_2$ commutative is compatible with the poset structure of $\mathcal{M}_2(m)$, we recall the criterion for the existence of a morphism $\alpha \to \beta$ in $\mathcal{M}_2(m)$. If $\{a, b\}$ is any two element subset of $\{1, 2, \ldots, m\}$ and $\alpha \cap \{a, b\} = a \boxtimes_1 b$, then either $\beta \cap \{a, b\} = a \boxtimes_2 b$ for $j \geq i$ or $\beta \cap \{a, b\} = b \boxtimes_2 a$ for $j > i$. Using this criterion we see that the number of pairs $\{a, b\}$ for which $\alpha \cap \{a, b\} = a \boxtimes_2 b$ is less than or equal to the number of pairs $\{c, d\}$ for which $\beta \cap \{c, d\} = c \boxtimes_2 d$, and that the counts are equal iff $\alpha = \beta$. Making $\boxtimes_1$ and $\boxtimes_2$ commutative does not affect these counts, so the poset structure on $\mathcal{M}_2(m)$ induces a poset structure on $\mathcal{M}_2^{ab}(m)$. It is straightforward to verify that the operad structure on $\mathcal{M}_2$ passes to an operad structure on $\mathcal{M}_2^{ab}$.

5.5 Definition: We will not need to use the poset structure of $\mathcal{M}_2^{ab}$ until Section 11. However we will need to use the underlying sets of objects of $\mathcal{M}_2^{ab}$.
in various constructions before that. Hence we will use the separate notation \( \mathcal{M}_{2B}^b(m) \) to denote the underlying set of objects of \( \mathcal{M}_{2b}^b(m) \). Obviously the operad structure on \( \mathcal{M}_{2b}^b \) restricts to an operad structure on \( \mathcal{M}_{2B}^b(m) \).

5.6 Definition: We define a collection of simplicial complexes (in the classical sense of the term, rather than simplicial sets) \( \{ \mathcal{R}_*(m) \}_{m \geq 0} \) as follows. The vertex set of \( \mathcal{R}_*(m) \) is \( \mathcal{M}_{2b}^b \). The complexes \( \mathcal{R}_*(0) \) and \( \mathcal{R}_*(1) \) are 0-dimensional consisting of a single vertex while \( \mathcal{R}_*(2) \) is the simplicial complex consisting of the single 1-simplex \( \{ 1 \otimes 2, 1 \otimes 2, 2 \} \) and its subsimplices. The complex \( \mathcal{R}_*(3) \) has the vertex set

\[
(1 \otimes 2) \otimes 1 \quad (2 \otimes 1) \otimes 2
\]

\[
(1 \otimes 2 \otimes 3) \quad (1 \otimes 2) \otimes 1 \quad (1 \otimes 3) \otimes 2 \quad (1 \otimes 2) \otimes 2 \otimes 3
\]

The simplices are those collections of vertices which contain at most one vertex from each column. For \( m \geq 3 \) the simplicial complex \( \mathcal{R}_*(m) \) has as \( r \)-simplices those \( (r+1) \)-tuples \( \{ \alpha_0, \alpha_1, \ldots, \alpha_r \} \) such that for each subset \( \{ a < b < c \} \subset \{ 1, 2, \ldots, m \} \) the restrictions \( \alpha_0 \cap \{ a, b, c \}, \alpha_1 \cap \{ a, b, c \}, \ldots, \alpha_r \cap \{ a, b, c \} \) forms a simplex in \( \mathcal{R}_* \{ a, b, c \} \), which is identified with \( \mathcal{R}_*(3) \) via the isomorphism induced by \( 1 \mapsto a, 2 \mapsto b \) and \( 3 \mapsto c \).

5.7 Lemma: For any topological operads \( \mathcal{A} \) and \( \mathcal{B} \), there is a natural map of operads

\[
\varepsilon : \mathcal{A} \boxplus \mathcal{B} \longrightarrow \mathcal{M}_{2b}^b
\]

Proof To define \( \varepsilon \), take a tree representative \( T \) of an element in \( (\mathcal{A} \boxplus \mathcal{B})(m) \). Replace each stump by 0 and delete each node with exactly one input by combining its input and output. Regard each node in \( T \) with more than 1 input and label coming from \( \mathcal{A} \) as an iterated multiplication \( \otimes_1 \) and label coming from \( \mathcal{B} \) as an iterated multiplication \( \otimes_2 \). Interpret the edges of the resulting tree as compositions in the operad \( \mathcal{M}_{2b}^b \). This evaluated tree gives an element of \( \mathcal{M}_{2b}^b(m) \). It is obvious that this construction is compatible with the equivalence relations on trees which define the elements of \( (\mathcal{A} \boxplus \mathcal{B})(m) \). [Note that this construction does not give a well defined map \( \mathcal{A} \boxplus \mathcal{B} \) to the set of objects of \( \mathcal{M}_{2b}^b \), since that would not be compatible with the equivariance relation on trees.]

5.8 We observe that Lemma gives a trivial colimit decomposition of \( (\mathcal{A} \boxplus \mathcal{B})(m) \), namely as a disjoint union of \( \varepsilon^{-1}(\alpha) \) indexed over all the elements of \( \mathcal{M}_{2B}^b(m) \). For future reference, we will use the notation \( \tilde{\mathcal{G}}_m(\alpha) \) to refer to \( \varepsilon^{-1}(\alpha) \). The first step in obtaining a colimit decomposition for the tensor product is to
observe that trees in \((A \amalg B)(m)\) which are related by unary interchanges have the same image under \(\varepsilon\), since \(M_2(1) = \{1 = id\}\). Thus for any object \(\alpha\) in \(M_2(m)\) we define

\[ G_m(\alpha) = \tilde{G}_m(\alpha)/\text{unary interchanges} \]

From now on we will take \(A = W[\mathcal{N}\mathcal{M}_k]\) and \(B = W[\mathcal{N}\mathcal{M}_l]\).

The proofs of the remaining statements are postponed to Sections 10 and 11.

5.9 Proposition: For each element \(\alpha\) in \(M_2(m)\), the natural map

\[ G_m(\alpha) \longrightarrow (W[\mathcal{N}\mathcal{M}_k] \otimes W[\mathcal{N}\mathcal{M}_l])(m) \]

is a cofibration.

Thus \((W[\mathcal{N}\mathcal{M}_k] \otimes W[\mathcal{N}\mathcal{M}_l])(m)\) is a union of subspaces \(G_m(\alpha)\) over all objects \(\alpha\) in \(M_2(m)\). To determine the homotopy type \((W[\mathcal{N}\mathcal{M}_k] \otimes W[\mathcal{N}\mathcal{M}_l])(m)\), we need to analyze the intersections of these closed subspaces. We would expect that these intersections should correspond to nonunary interchanges, which are encoded by the simplices of \(K^\bullet(m)\).

5.10 Definition: The interchange diagram for \((W[\mathcal{N}\mathcal{M}_k] \otimes W[\mathcal{N}\mathcal{M}_l])(m)\) is the following diagram indexed by \(I(m) = Sd \mathcal{R}_\bullet(m)\), the barycentric subdivision of \(\mathcal{R}_\bullet(m)\) with poset structure opposite to the inclusions of the faces of \(\mathcal{R}_\bullet(m)\):

- To each vertex \(\alpha \in \mathcal{R}_0(m) = M_2(m)\) assign \(G_m(\alpha)\) to \(\alpha\).
- To each barycenter of a simplex in \(\mathcal{R}_\bullet(m)\) assign the intersection \(\cap G_m(\alpha_i)\), where \(\{\alpha_i\}\) are the vertices of that simplex.
- The maps in the diagram are inclusions.

5.11 Proposition: The interchange diagram for \((W[\mathcal{N}\mathcal{M}_k] \otimes W[\mathcal{N}\mathcal{M}_l])(m)\) is a diagram of cofibrations, and \((W[\mathcal{N}\mathcal{M}_k] \otimes W[\mathcal{N}\mathcal{M}_l])(m)\) is the colimit of that diagram. Moreover each space in that diagram has the general form

\[ \prod W[\mathcal{N}\mathcal{M}_k](r_i) \times \prod W[\mathcal{N}\mathcal{M}_l](s_j)/\text{unary interchanges}, \]

and the maps in the diagram are given by operad compositions (including insertion of the unique constant).

The operad spaces \(\mathcal{N}\mathcal{M}_k(r)\) and \(\mathcal{N}\mathcal{M}_l(s)\) can be described as colimits of diagrams of contractible spaces indexed by the posets \(\mathcal{M}_k(r)\) and \(\mathcal{M}_l(s)\), namely by assigning to each object in \(\mathcal{M}_k(r)\) or \(\mathcal{M}_l(s)\) the realization of the nerve of the subposet for which that object is terminal. Pulling back these diagrams along the augmentations \(W[\mathcal{N}\mathcal{M}_k](r) \longrightarrow \mathcal{N}\mathcal{M}_k(r)\) and \(W[\mathcal{N}\mathcal{M}_l](s) \longrightarrow\)
Combining these colimits with the colimit diagram of Proposition 5.11, we obtain a refined iterated colimit decomposition of \((W|\mathcal{N}M_k| \otimes W|\mathcal{N}M_l|)(m)\). By Proposition 5.2 of [8], such an iterated colimit diagram can be reexpressed as a single colimit diagram over an appropriate Grothendieck construction, which we denote \(\mathcal{I}(k,l)(m)\). We thus obtain the following result.

5.12 Proposition: \((W|\mathcal{N}M_k| \otimes W|\mathcal{N}M_l|)(m)\) is the colimit of a diagram \(G'_m\) of cofibrations of contractible spaces indexed by the poset \(\mathcal{I}(k,l)(m)\), obtained by the Grothendieck construction from the interchange diagram of Proposition 5.11 by replacing the space at each node by the product of posets \(\prod \mathcal{M}_k(r_i) \times \prod \mathcal{M}_l(s_j)\) parametrizing the colimit decomposition at that node. The functors in that diagram are given by operad compositions (including insertions of the unique constant).

5.13 Definition: Following the convention of [1] we call a diagram of closed cofibrations of contractible spaces indexed by a poset a cellular decomposition of its colimit. (This definition is inspired but differs slightly from the one of Berger [2]).

5.14 Proposition: There is a functor

\[ L : \mathcal{I}(k,l)(m) \rightarrow \mathcal{M}_{k+l}(m) \]

which satisfies Quillen’s Theorem A, and thus induces an equivalence upon passage to nerves.

We thus have a chain of equivalences

\[
(W|\mathcal{N}M_k| \otimes W|\mathcal{N}M_l|)(m) \cong \colim_{\mathcal{I}(k,l)(m)} G'_m \cong \hocolim_{\mathcal{I}(k,l)(m)} G'_m \cong |\mathcal{N}\mathcal{I}(k,l)(m)| = |\mathcal{N}\mathcal{M}_{k+l}(m)|
\]

Unfortunately this chain of equivalences is not compatible with the operad structure on the two ends. The problem originates from the fact that the operad structure on \(\mathcal{M}^{ab}\), which forms the vertex sets of \(\mathfrak{R}_*(m)\) \(m \geq 0\) does not extend to the barycentric subdivision of \(\mathfrak{R}_*\). Consequently \(\mathcal{N}\mathcal{I}(k,l)\) has no natural operad structure.

Fortunately there is a straightforward way to deal with this problem, which is essentially the same strategy employed in [1] – we coarsen the decomposition so that it is indexed by an operad.

5.15 Definition: For each object \(\gamma \in \mathcal{M}_{k+l}(m)\) define

\[ F_m(\gamma) = \cup G'_m(\bar{\pi}, \beta) \subset (W|\mathcal{N}M_k| \otimes W|\mathcal{N}M_l|)(m), \]
where the union is indexed over all objects $(\alpha, \beta)$ in $\mathcal{T}(k,l)(m)$ such that there is a morphism $L(\alpha, \beta) \to \gamma$.

It is not difficult to check that the resulting diagrams $\{F_m\}_{m \geq 0}$ over $\mathcal{M}_{k+l}$ form a cellular decomposition which is compatible with the operad structures. Thus we obtain a diagram of operad equivalences

$$W|\mathcal{N}M_k| \otimes W|\mathcal{N}M_l| \xrightarrow{\colim M_{k+l}} F_\ast \xleftarrow{\hocolim M_{k+l}} \sim |\mathcal{N}M_{k+l}|$$

This completes the proof of Theorem 4.3.

6. Unary and binary operations

In this section we analyze the spaces of unary and binary operations in a tensor product $\mathcal{A} \otimes \mathcal{B}$ of two arbitrary reduced operads $\mathcal{A}$ and $\mathcal{B}$, and thus prove Proposition 4.8.

First of all observe that in the coproduct operad $\mathcal{A} \coprod \mathcal{B}$, the unary operations are arbitrary compositions of unary operations in $\mathcal{A}$ and $\mathcal{B}$, with the units of $\mathcal{A}(1)$ and $\mathcal{B}(1)$ identified and with the single relation that adjacent factors both in $\mathcal{A}(1)$ or both in $\mathcal{B}(1)$ can be combined into a single factor using the multiplications in $\mathcal{A}(1)$ or $\mathcal{B}(1)$. In other words, $(\mathcal{A} \coprod \mathcal{B})(1)$ is the coproduct (or free product) $\mathcal{A}(1) * \mathcal{B}(1)$ in the category of topological monoids. The interchange relations in $\mathcal{A} \otimes \mathcal{B}$ restrict on the space of unary operations to the relation that factors from $\mathcal{A}(1)$ commute with factors from $\mathcal{B}(1)$. Thus the factors from $\mathcal{A}(1)$ can be commuted to the front of a word, leaving the factors from $\mathcal{B}(1)$ at the back of the word. Then the factors from $\mathcal{A}(1)$ and from $\mathcal{B}(1)$ can be combined into single factors, using the multiplications in $\mathcal{A}(1)$ and $\mathcal{B}(1)$, leaving a pair of factors, the first from $\mathcal{A}(1)$ and the second from $\mathcal{B}(1)$. This establishes that

$$(\mathcal{A} \otimes \mathcal{B})(1) \cong \mathcal{A}(1) \times \mathcal{B}(1)$$

and thus proves the first part of Proposition 4.8.

To analyze the binary operations in $\mathcal{A} \otimes \mathcal{B}$, we begin by noting that according to Proposition 5.7 and the follow up discussion in paragraph 5.8 we have

$$(\mathcal{A} \coprod \mathcal{B})(2) = \tilde{G}_2(1 \boxplus 1) \coprod \tilde{G}_2(1 \boxplus 2).$$

Now by definition the elements of $\tilde{G}_2(1 \boxplus 2)$ are represented by trees with one binary node labelled by an element $a_2$ of $\mathcal{A}(2)$ and arbitrary chains of unary nodes labelled by arbitrary elements of $\mathcal{A}(1)$ and $\mathcal{B}(1)$ on both input branches coming into the binary node and on the output branch from that node. Upon dividing out by the unary interchange relations, i.e. passing to $G_2(1 \boxplus 1)$, we may commute the factors coming from $\mathcal{B}(1)$ past factors coming from $\mathcal{A}(1)$ towards the top of each branch and combine them into single factors, using the multiplication in $\mathcal{B}(1)$. Let $b \in \mathcal{B}(1)$ be the factor remaining on the top of the
output branch of the binary node labelled by $a_2$. We can then use the unary interchange relation

$$b \circ a_2 = a_2 \circ (b \oplus b)$$

to move $b$ from below the binary node to both branches above that node. We can then commute these copies of $b$ past factors of $A(1)$ on these branches and then combine them with the factors from $B(1)$ at the top of these branches. This leaves a tree with one factor of $B(1)$ at the top of each input branch and factors of $A(1)$ below them on the input branches, $a_2 \in A(2)$ on the binary node and additional factors of $A(1)$ on the output branch below that node. These $A$ nodes can all be composed together to produce a single binary node with a label in $A(2)$. Finally we can use the equivariance relation to insure that input 1 is on the left branch of the binary node and input 2 is on the right. Thus each element of $G_2(1 \boxtimes_1 2)$ has a unique tree representative of the form

![Tree Diagram]

which we will refer to as the reduced form of that element. This shows that

$$G_2(1 \boxtimes_1 2) \cong A(2) \times B(1)^2.$$ 

A similar argument, with the roles of $A$ and $B$ interchanged, shows that each element of $G_2(1 \boxtimes_2 2)$ has a unique reduced form representative

![Tree Diagram]

and thus

$$G_2(1 \boxtimes_2 2) \cong A(1)^2 \times B(2).$$

It now remains to analyze the nonunary interchanges in $(A \otimes B)(2)$. It is easy to see that any such interchange reduces to a $(2, 2)$-interchange. A reduced representative in $G_2(1 \boxtimes 1 2)$ can only interchange this way if both $b_1$ and $b_2$ factor through a common element in $b \in B(2)$ by composing with the constant 0. The resulting interchanges are shown below

![Tree Diagram]

$$\approx$$

Finally, we refer to this as the reduced form of that element. This shows that

$$G_2(1 \boxtimes 1 2) \cong A(2) \times B(1)^2.$$ 

A similar argument, with the roles of $A$ and $B$ interchanged, shows that each element of $G_2(1 \boxtimes 2 2)$ has a unique reduced form representative

![Tree Diagram]

and thus

$$G_2(1 \boxtimes 2 2) \cong A(1)^2 \times B(2).$$
Note that the second interchange follows from the first by replacing \( b \) by \( b\tau \), where \( \tau \) is the transposition \((1, 2)\), using the equivariance relation. Similarly the fourth interchange follows from the third. Also note that in the third interchange we can apply the simplification \( a \circ (0 \oplus 0) = 0 \) (since \( A(0) = \{0\} \)). If we then reduce both sides of that interchange (using unary interchanges and composition), we see that both sides become the same. Thus the third (and hence the fourth) interchanges are superfluous. This leaves only the first interchange. Moreover even this interchange can be simplified by noting that \( b'_1 \circ 0 = 0 \) and \( b'_2 \circ 0 = 0 \) (since \( B(0) = \{0\} \)). This converts the first interchange to the following form

Taking \( b' = b \circ (b'_1 \oplus b'_2) \), this in turn simplifies to the interchange
This can be briefly summarized as follows. If an element \((a, b_1, b_2) \in G_2(1 \boxtimes_1 2) \cong \mathcal{A}(2) \times B(1)^2\) is the image of an element \((a, b) \in \mathcal{A}(2) \times B(2)\) under the map \((a, b) \mapsto (a, b \circ (\text{id} \oplus 0), b \circ (0 \oplus \text{id}))\), then it is identified with the element \((a \circ (\text{id} \oplus 0), a \circ (0 \oplus \text{id}), b)\) in \(G_2(1 \boxtimes_2 2) \cong \mathcal{A}(1)^2 \times B(2)\). A similar analysis of interchanges starting from an element \(G_2(1 \boxtimes_2 2)\), shows that the only relation obtained is the reverse of the above relation. It follows that \((\mathcal{A} \times B)(2)\) is a pushout as in the second part of Proposition 4.8.

7. Axial operads

It is technically convenient to prove some of our results in the category \(\text{SSets}\) of simplicial sets. So throughout this section we work in the categories \(\text{Top}\) and \(\text{SSets}\) of \(k\)-spaces and simplicial sets respectively. The formulas given in this section make sense in \(\text{Top}\). If we work in \(\text{SSets}\), they are meant to be applied degreewise.

7.1 Observation: If \(\mathcal{A}\) and \(\mathcal{B}\) are simplicial operads, then

\[|\mathcal{A} \otimes \mathcal{B}| \cong |\mathcal{A}| \otimes |\mathcal{B}|,\]

because \(\mathcal{A} \otimes \mathcal{B}\) is a colimit of a diagram involving finite products of simplicial sets \(\mathcal{A}(k)\) and \(\mathcal{B}(l)\) and the realization functor preserves colimits and finite limits.

7.2 Lemma: (Igusa [13]) For both categories, the forgetful functor

\[U : \text{Opr}_0 \longrightarrow \text{Monoids}, \quad \mathcal{B} \mapsto \mathcal{B}(1)\]

has a right adjoint \(R : \text{Monoids} \longrightarrow \text{Opr}_0\), where \(\text{Monoids}\) is the category of monoids in \(\text{Top}\) respectively \(\text{SSets}\).

**Proof** The right adjoint \(R : \text{Monoids} \longrightarrow \text{Opr}_0\) is defined as follows: \(RM(k) = M^k\) for a monoid \(M\). The symmetric group permutes the factors, and composition is defined by

\[(x_1, \ldots, x_n) \circ ((y_{11}, \ldots, y_{k_1}) \oplus \ldots \oplus (y_{n1}, \ldots, y_{nk_n})) = (x_1 \cdot y_{11}, \ldots, x_1 \cdot y_{k_1}, \ldots, x_n \cdot y_{n1}, \ldots, x_n \cdot y_{nk_n}).\]

The unit of the adjunction

\[\xi : \mathcal{B} \longrightarrow RUB\]

is given by the *axial maps*

\[\mathcal{B}(n) \longrightarrow \mathcal{B}(1)^n\]

whose \(i\)-th coordinates are the compositions

\[\xi_i : \mathcal{B}(n) \cong \mathcal{B}(n) \times (\mathcal{B}(0)^{i-1} \times \{\text{id}\} \times \mathcal{B}(0)^{n-i}) \longrightarrow \mathcal{B}(1)\]

The counit \(URM \longrightarrow M\) is the identity map. \(\square\)

\(RU(\mathcal{B})\) is closely related to \(\text{Ass} \otimes \mathcal{B}\) and \(\text{Com} \otimes \mathcal{B}\). The following result is a reformulation of [7, Thm. 5.5]; the topological proof of [7] also works in \(\text{SSets}\).
7.3 Proposition: In $\text{Top}$ or $\text{SSets}$, if $B \in \mathcal{O}_{pr}$, then

\[ \text{Ass} \otimes B \cong (\text{Ass} \times RU(B))/\sim, \]

where the relation $\sim$ is defined on

\[ (\text{Ass} \times RU(B))(n) = \Sigma_n \times B(1)^n \]

by $(\pi, (b_1, \ldots, b_n)) \sim (\rho, (b_1, \ldots, b_n))$ iff $\pi^{-1}(i) < \pi^{-1}(j)$ and $\rho^{-1}(i) > \rho^{-1}(j)$ imply that there is a $c \in B(2)$ such that $(b_i, b_j) = \xi(c)$.

Note that $(\text{Ass} \otimes B)(1) = B(1)$, so that $RU(\text{Ass} \otimes B) = RU(B)$.

As a simple consequence we get

7.4 Proposition: In $\text{Top}$ or $\text{SSets}$, if $B \in \mathcal{O}_{pr}$, then

\[ \text{Com} \otimes B \cong RU(B) \]

and the adjunction map $\xi : B \to RU(B)$ corresponds to the canonical map $B \to \text{Com} \otimes B$.

Proof Using Corollary 7.3 and Proposition 7.3 we obtain

\[ \text{Com} \otimes B \cong \text{Ass} \otimes \text{Ass} \otimes B \cong (\text{Ass} \times RU(\text{Ass} \otimes B))/\sim \cong (\text{Ass} \times RU(B))/\sim \]

Now $(\text{Ass} \otimes B)(2) = \Sigma_2 \times B(1)^2/\sim$, and a pair $(a, b) \in B(1)^2$ is the image of $(id, (a, b)) \in (\text{Ass} \otimes B)(2)$ under $\xi$. Hence

\[ (\text{Ass} \times RU(B)/\sim)(n) = (\Sigma_n \times B(1)^n)/\Sigma_n = B(1)^n \]

Let $\beta \in B(n)$ and let $\lambda_n$ denote the unique element in $\text{Com}(n)$. Then by the same argument as in the proof of Proposition 7.3 we have $\beta = \lambda_n \circ (\xi_1(\beta) \oplus \ldots \oplus \xi_n(\beta))$. Hence the adjunction map $\xi$ corresponds to the canonical map $B \to \text{Com} \otimes B$. \qed

7.5 Remark: Proposition 7.4 is a special case of the following general result, which is a simple consequence of Proposition 7.3 and Proposition 4.8. Let $A$ and $B$ be reduced operads and let $M$ and $N$ be monoids in $\text{Top}$ or $\text{SSets}$. Then there are natural isomorphisms

1. $R(M) \otimes A \cong R(M \times U(A))$
2. $U(A \otimes B) \cong U(A) \times U(B)$
3. $R(M \times N) \cong R(M) \times R(N) \cong R(M) \otimes R(N)$
4. $RU(A \otimes B) \cong RU(A) \otimes RU(B) \cong RU(A) \times RU(B)$

7.6 Definition: A $\Sigma$-free topological operad $B$ is called axial if $\xi : B \to RU(B)$ is a closed cofibration. A $\Sigma$-free simplicial operad $B$ is called axial if $\xi : B \to RU(B)$ is injective. (In these cases we consider $B(n)$ as a subspace of $B(1)^n$.)
8. The reduced $W$-construction

8.1 Definition: The unreduced operad $T^u B_\bullet$ of trees over a simplicial operad $B_\bullet$ is an operad in the category of simplicial categories. The morphisms of degree $k$ of the simplicial category $T^u B_\bullet(n)$ are equivalence classes of quadruples $(\psi, f, g, h)$ consisting of

- a finite directed rooted planar tree $\psi$. Each vertex $v$ has a finite set $\text{In}(v)$ of incoming edges and exactly one outgoing edge. $\text{In}(v) = \emptyset$ is allowed. Thus $\psi$ has a finite set $\text{In}(\psi)$ of inputs and exactly one output, the root. We require that $|\text{In}(\psi)| = n$, where $|M|$ denotes the cardinality of $M$.
- a function $f$ assigning to each vertex $v$ a label $f(v) \in B_k(|\text{In}(v)|)$.
- a bijection $g : \text{In}(\psi) \to \{1, \ldots, n\}$, where $n = |\text{In}(\psi)|$. We call $g(i)$ an input-label and think of it as label of input $i$.
- a function $h$ assigning to each edge a label 0 or 1, called edge-label. The label 0 is only allowed on inner edges, i.e. edges with vertices on both ends.

We usually suppress $f, g, h$ from the notation and think of $(\psi, f, g, h)$ as a tree $\psi$ with labelled inputs, edges, and vertices. We allow the trivial tree, i.e. a single edge with label 1 (direction is from top to bottom), and call vertices without an input a stump.

The equivalence relation between such trees is generated by the equivariance relation $\text{eq}$ applied to labelled trees and the following:

8.2 Identity relation

\[
\begin{array}{c|c|c}
\epsilon_1 & id & \sim \\
\epsilon_2 & \max(\epsilon_1, \epsilon_2)
\end{array}
\]

The simplicial structure is induced by the simplicial structure of the vertex labels.

The identity morphisms and hence the objects are represented by those trees whose edge-labels are all 1. The source of a morphism is obtained by replacing all edge-labels by 1 and the target by shrinking all edges labelled 0 and composing the labels of the vertices at their ends using the operad structure of $B_\bullet$. There is a morphism $\varphi \rightarrow \psi$ of labelled trees iff $\psi$ is obtained from $\varphi$ by shrinking some internal edges of $\varphi$ and composing the vertex labels at their ends. This also determines the composition. But note that there can be more than one morphism $\varphi \rightarrow \psi$. 

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The operad structure is given by tree composition: The composite tree \( \varphi \circ (\psi_1 \oplus \ldots \oplus \psi_n) \) is obtained by grafting \( \psi_i \) on the input of \( \varphi \) with input-label \( i \). The newly created inner edges obtain the labels 1.

### 8.3 Observation:
The nerve of \( \mathcal{T}^n \mathcal{B}_\bullet \) is a bisimplicial set whose diagonal is the simplicial Boardman-Vogt construction \( W^n \mathcal{B}_\bullet \) as described in [3] - one glues in a 1-simplex for each edge labelled 0. Its topological realization is \( W^n |\mathcal{B}_\bullet| \) as defined in [6] (see also [3, 8.2]).

### 8.4 Remark:
In [3] and [6] the symbol \( W \) is used instead of \( W^n \). We reserve \( W \) for the reduced version defined below, which corresponds to the \( W' \)-construction of [6, p. 159].

### 8.5 Definition:
The **reduced operad** \( \mathcal{T}^r \mathcal{B}_\bullet \) over a reduced simplicial operad \( \mathcal{B}_\bullet \) is obtained from \( \mathcal{T}^n \mathcal{B}_\bullet \) by imposing the

### 8.6 Stump relation:
If a labelled tree \( \psi \) has an edge labelled 1 such that the subtree above that edge has no inputs (i.e. it is a subtree topped by stumps) then this subtree may be replaced by a single stump.

As before \( \mathcal{T} \mathcal{B}_\bullet \) is an operad in the category of simplicial categories.

We denote the diagonal of the nerve of \( \mathcal{T} \mathcal{B}_\bullet \) by \( \mathcal{W} \mathcal{B}_\bullet \) and its realization by \( \mathcal{W} |\mathcal{B}_\bullet| \). The stump relation ensures that \( \mathcal{W} \mathcal{B}_\bullet \) and \( \mathcal{W} |\mathcal{B}_\bullet| \) are reduced simplicial, respectively topological operads.

There are functors of simplicial categories

\[ \varepsilon(n) : \mathcal{T} \mathcal{B}_\bullet(n) \to \mathcal{B}_\bullet(n) \]

from the simplicial category \( \mathcal{T} \mathcal{B}_\bullet(n) \) to the simplicial set \( \mathcal{B}_\bullet(n) \) considered as simplicial discrete category, given by shrinking all tree edges. These functors are compatible with the operad structure and hence define **augmentations** of operads

\[ \mathcal{T} \mathcal{B}_\bullet \to \mathcal{B}_\bullet, \quad \mathcal{W} \mathcal{B}_\bullet \to \mathcal{B}_\bullet, \quad |\mathcal{W} \mathcal{B}_\bullet| \to |\mathcal{B}_\bullet| \]

in the categories of simplicial categories, \( \mathcal{SSets} \), and \( \mathcal{Top} \) respectively. The functor \( \varepsilon(n) \) has a functorial section

\[ \eta(n) : \mathcal{B}_\bullet(n) \to \mathcal{T} \mathcal{B}_\bullet(n) \]

sending \( a \in \mathcal{B}_\bullet(n) \) to the tree with a single vertex labelled \( a \). The \( \eta(n) \) are not compatible with the operad structure but define **canonical sections**

\[ \eta(n) : \mathcal{B}_\bullet(n) \to \mathcal{W} \mathcal{B}_\bullet(n), \quad \eta(n) : |\mathcal{B}_\bullet|(n) \to |\mathcal{W} \mathcal{B}_\bullet|(n) \]

in \( \mathcal{SSets} \) and \( \mathcal{Top} \).
8.7 Remark: Since we are mainly interested in $|\mathcal{M}_k|$ we give an explicit description of the reduced $W$-construction $WB$ for a reduced topological operad $B$. Well-pointed means that the inclusion $\{id\} \subset B(1)$ is a closed cofibration. The elements of $WB(n)$ are represented by labelled trees with $n$ inputs as above with the difference that the edge-labels are replaced by edge-lengths $t \in I = [0,1]$. Only internal edges are allowed to have lengths smaller than 1. The set of trees inherits a topology from the vertex labels and edge-lengths.

We impose the equivariance relation 5.2, the identity relation 8.3 with $\epsilon_i \in I$, the stump relation 8.8 for edges of lengths 1, and the following

8.8 Shrinking relation: Edges of lengths 0 may be shrunk composing the labels of the vertices of their two ends.

The operad structure is defined by tree composition as in $TB_*$. As pointed out in [2,3] we often think of $WB$ as a topologically enriched symmetric monoidal category. Then composition is defined by the operad composition. So composites are represented by trees having inner edges of lengths 1.

8.9 Notation: We call a representing tree of an element in $TB_*$ or $WC$ reduced if it cannot be further reduced by applying the identity-, stump-, or in case of a topological operad $C$ the shrinking relation.

An input path of a tree is a directed edge path from an input to the root.

8.10 Proposition: $WB_*$ is an axial operad.

Proof Since $WB_*$ is the diagonal of the nerve of $TB_*$ it suffices to show that the axial map

$$\xi : TB_*(n) \to TB_*(1)^n$$

is injective. This is clear for the unreduced version $TB_*$. In the reduced version, the stump relation may reduce the trees in the axial image. Nevertheless one can recover each reduced tree from its axial image, because if $v$ is the vertex of the $i$-th input path where it meets another input path the first time, then the subtree above $v$ containing the $i$-th input path is contained in the $i$-th coordinate of the axial image. \qed

8.11 Definition: A monoid $M$ is called left-factorial if any $(x_1, \ldots, x_n) \in M^n$ has a maximal left factorization $x_i = y \cdot x_i', i = 1, \ldots, n$: if $x_i = z \cdot x_i''$ is any other factorization, $i = 1, \ldots, n$, it can be factored further as

$$x_i'' = w \cdot x_i', \quad i = 1, \ldots, n$$

with $y = z \circ w$

We call $y = mlf(x_1, \ldots, x_n)$ the maximal left factor of $(x_1, \ldots, x_n)$.

8.12 Lemma: Let $B$ be a topological operad. Then $WB$ is left factorial.
Proof Since $\text{mlf}(\alpha_1, \alpha_2, \alpha_3) = \text{mlf}(\alpha_1, \alpha_2, \alpha_3)$ it suffices to show the existence of a maximal left factorization for a pair $(\alpha_1, \alpha_2) \in W\mathcal{B}(1)^2$. Let $A$ be a reduced tree representing $\alpha_1$. The left factors of $\alpha_1$ are represented by subtrees $T$ obtained from $A$ by deleting the subtree on top of an edge of length 1 of the input path. The height of $T$ is defined to be the length of its input path. Then $\text{mlf}(\alpha_1, \alpha_2)$ is represented by the subtree $T$ of $A$ of maximal height which also represents a left factor of $\alpha_2$.

8.13 Lemma: Let $\mathcal{B}$ be a topological operad. If $\alpha_1, \ldots, \alpha_n \in W\mathcal{B}(1)$, $n \geq 2$, then there is a pair $i < j$ such that

$$\text{mlf}(\alpha_1, \ldots, \alpha_n) = \text{mlf}(\alpha_i, \alpha_j)$$

Proof This follows from $\text{mlf}(\text{mlf}(\alpha_1, \alpha_2), \alpha_3) = \text{mlf}(\alpha_1, \alpha_2, \alpha_3)$.

8.14 Lemma: Let $\mathcal{B}_\bullet$ be a simplicial operad such that $\mathcal{B}_\bullet(1) = \{id\}$. Given $\alpha_1, \ldots, \alpha_n \in W|\mathcal{B}_\bullet|(1)$ such that $(\alpha_i, \alpha_j) \in W|\mathcal{B}_\bullet|(2)$ for each pair $i < j$, then $(\alpha_1, \ldots, \alpha_n) \in W|\mathcal{B}_\bullet|(n)$.

Proof Let $J$ denote the category of subsets of $\{1, \ldots, n\}$ of cardinality 1 or 2 and inclusions as morphisms. Let $P$ denote the limit of the $J^{op}$ diagram in $\text{Top}$ sending $\{i\}$ to $W|\mathcal{B}_\bullet|(1)$, and $\{i, j\}$ to $W|\mathcal{B}_\bullet|(2)$. The two maps $W|\mathcal{B}_\bullet|(2) \to W|\mathcal{B}_\bullet|(1)$ corresponding to the inclusions of $\{i\}$ and $\{j\}$ into $\{i, j\}$ are given by the components of the axial map. The lemma states that there is a surjection $W|\mathcal{B}_\bullet|(n) \to P$. Since the nerve functor and the realization functor preserve finite limits and surjections it suffices to prove the lemma for $\mathcal{T}\mathcal{B}_\bullet$. Moreover, since the $\mathcal{T}$-construction is defined degreewise and is functorial it suffices to prove the lemma for each simplicial degree $\mathcal{B}_r$ separately.

So let $\mathcal{B}$ denote an operad in the category of sets and let the morphisms $\alpha_i$ in $\mathcal{T}\mathcal{B}(1)$ satisfy the condition of the lemma. We proceed by induction. For $n = 2$ the statement is trivial. Now suppose that $n \geq 3$.

Throughout our proof we will argue with reduced trees representing morphisms in $\mathcal{T}\mathcal{B}$. Let $T_i$ represent the morphism $\alpha_i$. By assumption there are reduced trees $S_{ij}$ with two inputs, mapped by the axial map to $(T_i, T_j)$ (we may have to reduce the image). Let $s_{ij}$ denote the node where the input paths of $S_{ij}$ meet. The height of $s_{ij}$ is the number of edges between $s_{ij}$ and the root along an input path. By renumbering the $\alpha_i$ we may assume that $s_{1n}$ has the minimal height of all the $s_{ij}$.

By induction there is a tree $U \in \mathcal{T}\mathcal{B}(n-1)$ whose axial image is $(T_1, \ldots, T_{n-1})$. Let $v$ be its node of lowest height where two input paths meet. Then $U$ has the form
where the stump stands for a subtree topped by stumps if its outgoing edge is labelled 0. Of course, there may be more stumps. We may assume that the input path 1 passes through $U_1$. Then $T_1$ is obtained from $U$ by replacing $U_1$ by $U_1^\prime$, obtained from $U_1$ by putting stumps on all inputs of $U_1$ except of input 1 and reducing, and by putting stumps on $U_2, \ldots, U_r$ and reducing. In particular, $T_1$ contains the node $u$ and the subtree of $U$ below it.

By assumption, there is a tree $V \in TB(2)$ whose axial image is $(T_1, T_n)$. So $V$ contains the edge path of $U$ from $u$ to the root. By minimality of $s_{1n}$ the two input paths of $V$, which we denote by 1 and $n$, meet in this edge path. If they meet in a vertex $v$ below $u$ we replace the corresponding incoming edge of $v$ in $U$ by the subtree of $V$ consisting of all nodes between input $n$ and $v$, and obtain a tree in $TB(n)$ whose axial image is $(T_1, \ldots, T_n)$. We recover our old $U$ by putting a stump on the input of this new subtree. We can proceed the same way if the two input paths meet in $u$ and input path $n$ of $V$ meets $u$ at an input corresponding to a stump in $U$. This procedure only fails if input path $n$ meets $u$ in an input which is topped by a tree $U_i$ with $1 < i \leq r$. Suppose that the input paths $i_1, \ldots, i_q$ of $U$ pass through $U_i$. Let $(T_{i_1}', \ldots, T_{i_q}')$ be the axial image of $U_i$ and let $V'$ be the subtree of $V$ consisting of the nodes of input path $n$ above $u$. Then $T_j$ has the form

for all $j \in \{i_1, \ldots, i_q, n\}$ where the stumps again stand for subtrees topped by stumps. The condition $(T_k, T_l) \in TB(2)$ for all $k \neq l$ therefore implies that $(T_{k}', T_{l}') \in TB(2)$ for all $k, l \in \{i_1, \ldots, i_q, n\}$, $k \neq l$. By induction there is a tree in $X \in TB(q + 1)$ whose axial image is $(T_{i_1}', \ldots, T_{i_q}', T_n')$. If we replace $U_i$ in $U$ by $X$, we obtain a tree in $TB(n)$ whose axial image is $(T_1, \ldots, T_n)$.

**8.15 Lemma:** Let $B$ be a $\Sigma$-free topological operad. If $\alpha, \beta, \gamma, \delta \in WB(1)$ satisfy that $(\alpha \circ \beta, \gamma)$ and $(\alpha, \gamma \circ \delta)$ are in $WB(2)$, then $(\alpha, \gamma)$ is in $WB(2)$.

**Proof** Let $A, B, G, D$ be reduced trees representing $\alpha, \beta, \gamma, \delta$. By assumption there are trees $S$ and $T$ whose axial image is $(A \circ B, G)$ respectively $(A, G \circ D)$.

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In $S$ the input paths corresponding to $A \circ B$ and $G$ can either meet in the $A$-part or the $B$-part. If they meet in the $A$-part we delete the $B$-part from $S$ and obtain a tree whose axial image is $(A, G)$. If they meet in the $B$-part, then $G$ is of the form $G = A \circ G'$. But since $(\alpha \circ \rho, \alpha) \notin WB(2)$ for all $\rho \in WB(1)$, the condition $(\alpha, \gamma \circ \delta) \in WB(2)$ then cannot hold. 

9. Binodal trees

In this section we will discuss a tree notation which is very convenient in analyzing the colimit decomposition of the tensor product $W_NM_k \otimes W_NM_l$.

9.1 Definition: A binodal tree is a nonplanar rooted tree with labelled inputs, no stumps and no unary nodes. Also all the nodes are colored either black or white and there are no edges connecting two white nodes. [If some construction below gives rise to an edge connecting two white nodes, it is to be understood that such an edge is to be shrunk to a point and the two white nodes merged together.]

When we draw pictures of binodal trees, we will represent black nodes by solid dots and white nodes by hollow dots. We will also use the following convention: if $T$ is a binodal tree, then $|T|$ will denote the set of inputs for $T$.

9.2 Definition: Let $A$ be an operad in one of our categories $S$, let $S$ be a finite nonempty set and let $T$ be a binodal tree with $|T| = S$. We define the object $A(T)$ with a left $A(1)$ action recursively as follows.

- If $|T|$ has cardinality $1$, then $A(T) = A(1)$ with left action given by multiplication.

- If the bottom node of $T$ is colored white and thus $T$ looks like

$$T_1 \quad T_2 \quad \cdots \quad T_r$$




then

$$A(T) = A(T_1) \times A(T_2) \times \cdots \times A(T_r)$$

with the diagonal left action of $A(1)$ on the right hand side.

- If the bottom node of $T$ is colored black and thus $T$ looks like

$$T_1 \quad T_2 \quad \cdots \quad T_r$$





then
\[ \mathcal{A}(T) = \mathcal{A}(r) \times_{\mathcal{A}(1)} \mathcal{A}(T_1) \times \mathcal{A}(T_2) \times \cdots \times \mathcal{A}(T_r) \]

with left \( \mathcal{A}(1) \) action coming from that the left action of \( \mathcal{A}(r) \).

**9.3 Remark:** If we work in \( \mathcal{C} \) or \( \mathcal{S} \) it is helpful to think of an object in \( \mathcal{A}(T) \) in the following way. We extend this point of view to \( \mathcal{S} \) by applying it degreewise.

An element of \( \mathcal{A}(T) \) is a **planar** binodal tree \( T \) in which each black node \( v \) is decorated with an element of \( \mathcal{A}(\text{In}(v)) \) and each edge with an element of \( \mathcal{A}(1) \) subject to the following relations:

1. The equivariance relation 5.2 (here we give each white node a dummy decoration which is invariant under the action of the symmetric group).
2. If an edge below a black node has decoration \( \alpha \cdot \beta \) then \( \beta \) can be moved into the black node by composing with the node’s decoration.
3. If an edge above a black node has decoration \( \alpha \cdot \beta \) then \( \alpha \) can be moved into the black node by composing in the canonical way with the node’s decoration.
4. If an edge below a white node has decoration \( \alpha \cdot \beta \) then \( \beta \) can be moved into all edges above that node simultaneously by composing with the decorations of these edges.
5. The converse relation of (3): If the \( r \) incoming edges of a white node have labels \( \beta \cdot \gamma_1 , \beta \cdot \gamma_2 , \ldots , \beta \cdot \gamma_r \) then \( \beta \) can be moved to the outgoing edge of the node by composing with the decoration of this edge.

**9.4 Remark:** While we are primarily concerned with the above construction for topological operads, we will also need to consider this construction for simplicial operads and for operads in \( \mathcal{C} \). In particular, we will use the poset \( \mathcal{M}_k(T) \) as indexing category in our colimit decomposition of \( W[\mathcal{M}_k] \otimes W[\mathcal{M}_l] \).

**9.5 Proposition:** Let \( \mathcal{A} \) be an axial operad and \( T \) a binodal tree with \( |T| = S \). Then there is a natural imbedding \( \mathcal{A}(T) \subseteq \mathcal{R} \mathcal{U} \mathcal{A}(S) = \mathcal{A}(1)^S \).
**Proof** We proceed inductively on the cardinality of $S$. If $S$ has cardinality 1, then $A(T) = A(1) = A(1)^S$ by definition. Assuming we have established the result for binodal trees whose sets of inputs have cardinality less than $S$, we consider two cases.

If the bottom node of $T$ is colored white and thus $T$ looks like

$$
T_1 \rightarrow T_2 \rightarrow \cdots \rightarrow T_r
$$

then by induction we have

$$
A(T) = A(T_1) \times A(T_2) \times \cdots \times A(T_r) \subseteq \prod_{i=1}^{r} A(1)^{|T_i|} = A(1)^S
$$

If the bottom node of $T$ is colored black and thus $T$ looks like

$$
T_1 \rightarrow T_2 \rightarrow \cdots \rightarrow T_r
$$

then by induction hypothesis and the axiality of $A$

$$
A(T) = A(r) \times A(1)^r \times A(T_1) \times A(T_2) \times \cdots \times A(T_r) \subseteq A(1)^r \times A(1)^r \times \prod_{i=1}^{r} A(1)^{|T_i|} = A(1)^S
$$

where the last map is given by composition in the operad $RU.A$.

**9.6 Corollary:** Suppose $A$ is an axial and left factorial topological or simplicial operad. Then for any binodal tree $T$ with $|T| = S$, $A(T)$ can be identified with the subspace of $RU.A(S) = A(1)^S$ defined recursively as follows:

- If $S$ has cardinality 1 (and thus $T$ consists of a single edge with one input and one output), we define $A(T) = RU.A(S) \cong A(1)$.
- If the bottom node of $T$ is colored white and thus $T$ looks like

$$
T_1 \rightarrow T_2 \rightarrow \cdots \rightarrow T_r
$$

we define

$$
A(T) = A(T_1) \times A(T_2) \times \cdots \times A(T_r) \subseteq \prod_{i=1}^{r} A(1)^{|T_i|} \cong A(1)^S
$$
• If the bottom node of $T$ is colored black and thus $T$ looks like

\[
\begin{array}{c}
T_1 \\
\downarrow \\
T_r \\
\end{array}
\]

we define $A(T)$ to be the subspace of $A(T_1) \times A(T_2) \times \cdots \times A(T_r) \subseteq \prod_{i=1}^r A(1)^{|T_i|} \cong A(1)^S$ consisting of those $r$-tuples $(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_r)$ such that the corresponding $r$-tuple of maximal left factors $(a_1, a_2, \ldots, a_r)$ is in the image of the axial map $A(r) \rightarrow A(1)^r$.

This follows immediately from the proof of Proposition 9.5.

**9.7 Proposition:** Let $\mathcal{E}_\bullet$ be a simplicial operad with $\mathcal{E}_\bullet(1) = \{id\}$. Let $T$ be a binodal tree with $|T| = S$. Then $(W|\mathcal{E}_\bullet|)(T)$ can be identified with the subspace $(W|\mathcal{E}_\bullet|)'(T) \subseteq W|\mathcal{E}_\bullet|'(1)^S$ consisting of all tuples $(\alpha_s)_{s \in S}$ such that, for any subset $\{a, b, c\} \subseteq S$ (of not necessarily distinct elements),

\[(\alpha_s)_{s \in S} \cap \{a, b, c\} \in (W|\mathcal{E}_\bullet|)(T \cap \{a, b, c\}).\]

**Proof** By 8.10 and 8.12 $W|\mathcal{E}_\bullet|$ is axial and left factorial, so Corollary 9.6 applies. The inclusion of $(W|\mathcal{E}_\bullet|)(T) \subseteq (W|\mathcal{E}_\bullet|)'(T)$ follows by naturality. We prove the reverse inclusion by induction on the cardinality of $S$. If the cardinality of $S$ is $\leq 3$, this holds trivially. So assume the assertion is true for binodal trees $T'$ with $|T'|$ having cardinality less than $S$.

If the bottom node of $T$ is colored white and thus $T$ looks like

\[
\begin{array}{c}
T_1 \\
\downarrow \\
T_r \\
\end{array}
\]

then it follows by induction that

\[
(W|\mathcal{E}_\bullet|)'(T) = (W|\mathcal{E}_\bullet|)'(T_1) \times (W|\mathcal{E}_\bullet|)'(T_2) \times \cdots \times (W|\mathcal{E}_\bullet|)'(T_r) = (W|\mathcal{E}_\bullet|)(T_1) \times (W|\mathcal{E}_\bullet|)(T_2) \times \cdots \times (W|\mathcal{E}_\bullet|)(T_r) = (W|\mathcal{E}_\bullet|)(T)
\]

If the bottom node of $T$ is colored black and thus $T$ looks like

\[
\begin{array}{c}
T_1 \\
\downarrow \\
T_r \\
\end{array}
\]

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Hence we can reduce questions about the spaces \( (W|E_\star)\)'(T) \subseteq (W|E_\star)'(T_1) \times (W|E_\star)'(T_2) \times \cdots \times (W|E_\star)'(T_r) \\
= (W|E_\star)(T_1) \times (W|E_\star)(T_2) \times \cdots \times (W|E_\star)(T_r)

Thus it remains to show that if

\[ (\alpha_s)_{s \in S} = (\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_r) \in (W|E_\star)'(T) \]

then \((\beta_1, \beta_2, \ldots, \beta_r)\) is in the image of \(W|E_\star|(r) \rightarrow W|E_\star|(1)^r\), where \(\beta_i\) is the maximal left factor of \(\overline{\alpha}_i\) for \(i = 1, 2, \ldots, r\).

By \(\text{S.14}\) and the induction hypothesis, we can reduce to the case \(r = 2\). If both \(|T_1|\) and \(|T_2|\) have cardinality 1 then \(|T|\) has cardinality 2 and we are done by the induction hypothesis. We can assume wolog that \(|T_1|\) has cardinality \(\geq 2\). By \(\text{S.13}\) we can choose \(\{s_1, s_2\} \subseteq |T_1|\) so that the maximal left factor of \((\alpha_{s_1}, \alpha_{s_2})\) is \(\beta_1\). If \(|T_2| = \{s_3\}\) has cardinality 1, then \(\beta_2 = \alpha_{s_3}\) and we have \((\alpha_{s_1}, \alpha_{s_2}, \alpha_{s_3}) \in (W|E_\star)|(T \cap \{s_1, s_2, s_3\})\) which implies that \((\beta_1, \beta_2)\) is in the image of \(W|E_\star|(2) \rightarrow W|E_\star|(1)^2\). If \(|T_2|\) also has cardinality \(\geq 2\), pick \(\{s_3, s_4\} \subseteq |T_2|\) so that the maximal left factor of \((\alpha_{s_3}, \alpha_{s_4})\) is \(\beta_2\). Then we have factorizations in \(W|E_\star|(1)\) of the following form:

\[
\alpha_{s_1} = \beta_1 \cdot \gamma_1, \quad \alpha_{s_2} = \beta_1 \cdot \gamma_2, \quad \alpha_{s_3} = \beta_2 \cdot \gamma_3, \quad \alpha_{s_4} = \beta_2 \cdot \gamma_4
\]

We then have

\[(\alpha_{s_1}, \alpha_{s_2}, \alpha_{s_3}) = (\beta_1 \cdot \gamma_1, \beta_1 \cdot \gamma_2, \beta_2 \cdot \gamma_3) \in (W|E_\star)|(T \cap \{s_1, s_2, s_3\})\],

which implies that \((\beta_1, \beta_2 \cdot \gamma_3)\) is in the image of \(W|E_\star|(2) \rightarrow W|E_\star|(1)^2\). Similarly we have

\[(\alpha_{s_1}, \alpha_{s_3}, \alpha_{s_4}) = (\beta_1 \cdot \gamma_1, \beta_2 \cdot \gamma_3, \beta_2 \cdot \gamma_4) \in (W|E_\star)|(T \cap \{s_1, s_3, s_4\})\],

which implies that \((\beta_1 \cdot \gamma_1, \beta_2)\) is in the image of \(W|E_\star|(2) \rightarrow W|E_\star|(1)^2\). By \(\text{S.13}\) this implies that \((\beta_1, \beta_2)\) is in the image of \(W|E_\star|(2) \rightarrow W|E_\star|(1)^2\). This concludes the induction and proof.

Hence we can reduce questions about the spaces \((W|E_\star)|(T)\) to the case of binodal trees with three inputs. For future reference we will need to work out the intersections \((W|E_\star)|(T_1) \cap (W|E_\star)|(T_2)\) in this case. First we note that the following is the complete list of all binodal trees with three inputs.

```
1 2 3 1 2 3 1 2 3 2 3 1
```

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9.8 Proposition: Let $B_\bullet$ be a simplicial operad with $B_\bullet(1) = \{id\}$. Let $T_1$ and $T_2$ be binodal trees with three inputs. Then we have $(W|B_\bullet|(T_1)) \cap (W|B_\bullet|(T_2)) = (W|B_\bullet|(T_3))$ or $\emptyset$ as shown in the table below and in the table in the Appendix.

|   | $T_1$ | $T_2$ | $(W|B_\bullet|(T_1)) \cap (W|B_\bullet|(T_2))$ |
|---|-------|-------|----------------------------------|
| 1 | i j k | i k j | ∅                               |
| 2 | i j k | i k j | ∅                               |
| 3 | i j k | i k j | ∅                               |

Proof Let $T_{ij}$ be the tree in row $i$ and column $j$ of our lists. We start with the list in the Appendix.

Row 1 is trivially true because $(W|B_\bullet|(T_{11})) = W|B_\bullet|(1)^3$.

Rows 2 to 4: Since $T_{21}$ is symmetric in 1, 2, 3 we may assume that $(i, j, k) = (1, 2, 3)$. Then

$(W|B_\bullet|(T_{21})) = W|B_\bullet|(3)$

$(W|B_\bullet|(T_{22})) = W|B_\bullet|(2) \times W|B_\bullet|(1)^2 (W|B_\bullet|(2) \times W|B_\bullet|(1)) \subset W|B_\bullet|(3)$

$(W|B_\bullet|(T_{23})) = W|B_\bullet|(2) \times W|B_\bullet|(1) \supset W|B_\bullet|(3)$

$(W|B_\bullet|(T_{24})) = W|B_\bullet|(2) \times W|B_\bullet|(1)^2 (W|B_\bullet|(1)^2 \times W|B_\bullet|(1))$,

where $W|B_\bullet|(1)^2$ acts on $W|B_\bullet|(1)$ by

$\quad (x, y) \cdot (a_1, a_2, b) = (x \circ a_1, x \circ a_2, y \circ b)$

Clearly $(W|B_\bullet|(T_{31})) \subset (W|B_\bullet|(T_{31})) \cap (W|B_\bullet|(T_{32}))$. Conversely, if you stump $k$ (i.e. you graft a stump on input $k$) of an element in the intersection, you obtain an element in $W|B_\bullet|(2)$, i.e. the intersection lies in $(W|B_\bullet|(T_{33}))$.

Row 5: Since $T_{51}$ is symmetric in $i$ and $j$ we have to consider the cases $(p, q, r) = (i, j, k)$ and $(p, q, r) = (i, k, j)$. In the first case $(W|B_\bullet|(T_{51})) \subset (W|B_\bullet|(T_{52}))$.

In the second case we use commutativity to give the inputs of both trees the order $k, i, j$. We find

$$(W|B_\bullet|(T_{51})) = W|B_\bullet|(2) \times W|B_\bullet|(1)^2 (W|B_\bullet|(1) \times W|B_\bullet|(2)) \subset W|B_\bullet|(2) \times W|B_\bullet|(2) = (W|B_\bullet|(T_{52}))$$

The first components of the inclusion is given by stumpying $j$, the second by stumpying $k$ and $i$. 

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Row 6: \((W|E_\bullet|)(T_{61}) \subset (W|E_\bullet|)(T_{62})\).
Row 7: Clearly \((W|E_\bullet|)(T_{73}) \subset (W|E_\bullet|)(T_{71}) \cap (W|E_\bullet|)(T_{72})\). Conversely, by stumping \(k\) we see that the intersection has to lie in \((W|E_\bullet|)(T_{73})\).
Row 8: We use commutativity to give the inputs of both trees the order \(k,i,j\).
Then
\[
(W|E_\bullet|)(T_{81}) \subset W|E_\bullet| (2) \times W|E_\bullet| (1) = (W|E_\bullet|)(T_{82})
\]
The first components of the inclusion is given by stumping \(j\), the second by stumping \(k\) and \(i\).
Row 9: Again we give the inputs of both trees the order \(k,i,j\) and obtain the intersection
\[
W|E_\bullet| (1) \times W|E_\bullet| (2) \cap W|E_\bullet| (2) \times W|E_\bullet| (1)
\]
which contains \(W|E_\bullet| (3)\), but is not of the form \((W|E_\bullet|)(T)\).
Now consider the second list: In all three cases we arrange the trees such that the order of inputs is \(k,i,j\).
Row 1: If \(x\) is in \((W|E_\bullet|)(T_{11})\) then the \(i\)-th and \(j\)-th input paths meet above the \(i\)-th and \(k\)-th ones, while if \(x\) is in \((W|E_\bullet|)(T_{12})\) it is the other way around. So \((W|E_\bullet|)(T_{11}) \cap (W|E_\bullet|)(T_{12}) = \emptyset\).
Rows 2 and 3:
\[
(W|E_\bullet|)(T_{21}) \cap (W|E_\bullet|)(T_{22}) \subset (W|E_\bullet|)(T_{31}) \cap (W|E_\bullet|)(T_{32})
\]
Stumping \(j\) or \(k\) of an element \(y \in (W|E_\bullet|)(T_{31}) \cap (W|E_\bullet|)(T_{32})\) makes it an element in \(W|E_\bullet| (2)\). Hence
\[
(W|E_\bullet|)(T_{31}) \cap (W|E_\bullet|)(T_{32}) \subset (W|E_\bullet|)(T_{11}) \cap (W|E_\bullet|)(T_{12}) = \emptyset
\]
\[\blacksquare\]

9.9 Proposition: Let \(E_\bullet\) be a simplicial operad. Then

- The imbedding \((W|E_\bullet|)(T) \subseteq W|E_\bullet| (1)^S\) of Proposition 9.5 is a cofibration.
- If \(S\) and \(T\) are binodal trees such that \((W|E_\bullet|)(S) \subset (W|E_\bullet|)(T)\), then the inclusion map is a cofibration.

Proof It is obvious that the imbedding \((W|E_\bullet|)(T) \subseteq W|E_\bullet| (1)^S\) is the realization of an injective simplicial map. It follows that whenever we have \((W|E_\bullet|)(S) \subset (W|E_\bullet|)(T)\), the inclusion is also the realization of an injective simplicial map. Since realizations of injective simplicial maps are cofibrations, the result follows. \[\blacksquare\]

9.10 Proposition: Let \(E_\bullet\) be a simplicial operad such that \(E_\bullet (1) = \{id\}\), and let \(\varepsilon : W|E_\bullet| \rightarrow |E_\bullet|\) be the augmentation map. Then
• For any binodal tree $T$, there is an induced map $\varepsilon : (W|B_\bullet|)(T) \to |B_\bullet(T)|$

• For any subcomplex $K_\bullet \subset B_\bullet(T)$, the restriction $\varepsilon : \varepsilon^{-1}(|K_\bullet|) \to |K_\bullet|$ is an equivalence.

**Proof** Since $B_\bullet(1) = \{id\}$, $B_\bullet(T)$ is simply a product of the form $\prod_{i} B_\bullet(k_i)$. Thus $\varepsilon$ induces a map of products $\prod_{i} (W|B_\bullet|(k_i)) \to \prod_{i} |B_\bullet(k_i)| = |B_\bullet(T)|$ which factors through a map $\varepsilon : (W|B_\bullet|)(T) \to |B_\bullet(T)|$. It is moreover clear from our description of the augmentation in Section 8 that this map is the realization of the nerve of a map of simplicial categories $\varepsilon_\bullet : TB_\bullet(T) \to B_\bullet(T)$, where $B_\bullet(T)$ is a simplicial discrete category. This in turn restricts to a map of simplicial categories $\varepsilon_\bullet : \varepsilon^{-1}(K_\bullet) \to K_\bullet$. It suffices to check that in each simplicial degree $r$, $\varepsilon_r : \varepsilon^{-1}(K_r) \to K_r$ satisfies Quillen’s Theorem A. This follows immediately from the observation that for any $x \in K_r$, the object $\eta(x) \in \varepsilon^{-1}(K_r)$ is a terminal object in the overcategory $\varepsilon/x$. Here $\eta(x)$ stands for the image of the product of the canonical sections $\prod_{i} \eta(k_i) : \prod_{i} B_\bullet(k_i) \to \prod_{i} TB_\bullet(k_i)$ followed by the quotient map into $(TB_\bullet)(T)$.

**9.11 Corollary:** For any binodal tree $T$, $|\mathcal{N}\mathcal{M}_k(T)|$ has a cellular decomposition (recall Definition 5.3) over the poset $\mathcal{M}_k(T)$ which lifts to a cellular decomposition of $(W|\mathcal{N}\mathcal{M}_k|)(T)$ over $\mathcal{M}_k(T)$.

**Proof** Since $\mathcal{N}\mathcal{M}(1) = \{id\}$, $|\mathcal{N}\mathcal{M}_k(T)|$ is simply a product of the form $\prod_{i} \mathcal{N}\mathcal{M}(k_i)$, which has a cellular decomposition over $\prod_{i} \mathcal{M}(k_i) = \mathcal{M}_k(T)$. Since this cellular decomposition is the realization of a simplicial cellular decomposition, by the second part of Proposition 9.10 the inverse images of the cells under the augmentation $\varepsilon : (W|\mathcal{N}\mathcal{M}_k|)(T) \to |\mathcal{N}\mathcal{M}(T)|$ provide the required cellular decomposition.

**10. The interchange diagram for a tensor product**

We begin by showing how to associate a pair of binodal trees $(S_\alpha, T_\alpha)$ to any element $\alpha$ in $\mathfrak{M}_2^{ab}(m)$. First of all we can represent $\alpha$ by a nonplanar rooted tree with inputs labelled by $\{1, 2, \ldots, m\}$ with nodes marked by $\mathfrak{E}_1$ and $\mathfrak{E}_2$. Given such a tree, the corresponding object $\alpha$ is obtained by interpreting the edges as compositions in the operad $\mathfrak{M}_2^{ab}$. There is more than one tree representation of $\alpha$, but there is a unique reduced form obtained by removing all univalent edges (which correspond to composing with the identity of $\mathfrak{M}_2^{ab}$) and by shrinking all edges connecting nodes which are both marked by $\mathfrak{E}_1$ and or both marked by $\mathfrak{E}_2$ (which corresponds to the strict associativity of $\mathfrak{E}_1$ and $\mathfrak{E}_2$). Now given the reduced form representation of $\alpha$ we obtain the binodal tree $S_\alpha$ by replacing each node decorated with $\mathfrak{E}_1$ by a black node and each node decorated with $\mathfrak{E}_2$ by a white node. The binodal tree $T_\alpha$ is obtained similarly by reversing the roles of $\mathfrak{E}_1$ and $\mathfrak{E}_2$. Thus we obtain planar representatives of the binodal trees $S_\alpha$. 

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and $T_\alpha$ having exactly the same shape and input labels, but each white node in $S_\alpha$ corresponds to a black node in $T_\alpha$ and vice-versa. Clearly $\alpha$ can be uniquely reconstructed from the pair $(S_\alpha, T_\alpha)$.

As a first step in our analysis of $(W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)(m)$ we prove Proposition 5.9 by proving a stronger result.

10.1 Proposition: Let $\alpha$ be an element of $\mathfrak{W}^{ab}_m(m)$. Then the composite map

$$G_m(\alpha) \longrightarrow (W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)(m) \xrightarrow{id} RU(W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)(m) = (W|\mathcal{N}\mathcal{M}_k|_n(1))^m \times (W|\mathcal{N}\mathcal{M}_l|_n(1))^m$$

is an embedding and its image is $(W|\mathcal{N}\mathcal{M}_k|)(S_\alpha) \times (W|\mathcal{N}\mathcal{M}_l|)(T_\alpha)$.

Proof We use the descriptions of $(W|\mathcal{N}\mathcal{M}_k|)(S_\alpha)$ and $(W|\mathcal{N}\mathcal{M}_l|)(T_\alpha)$ of Remark 9.3. Let $T$ be a tree representative of an element in $G_m(\alpha)$. We may reduce this tree by interpreting any edge connecting nodes both labelled by elements of $W|\mathcal{N}\mathcal{M}_k|$ or both labelled by elements of $W|\mathcal{N}\mathcal{M}_l|$ as compositions within that operad. After reducing, we obtain the $(W|\mathcal{N}\mathcal{M}_k|)(S_\alpha)$ component of the image of $T$, by replacing all multivalent nodes labelled with elements of $W|\mathcal{N}\mathcal{M}_l|$ by unlabelled white nodes and dropping all univalent nodes on any edge and labelling that edge by the product of all the elements of $W|\mathcal{N}\mathcal{M}_k|_n(1)$ which labelled the univalent nodes on that edge (or by $id \in W|\mathcal{N}\mathcal{M}_k|_n(1)$ if there were no such univalent nodes). Similarly we obtain the $(W|\mathcal{N}\mathcal{M}_l|)(T_\alpha)$ component by reversing the roles of $W|\mathcal{N}\mathcal{M}_k|$ and $W|\mathcal{N}\mathcal{M}_l|$.

We obtain the map in the other direction as follows. Choose compatible planar representatives of the binodal trees $S_\alpha$ and $T_\alpha$, and then pick corresponding labelled planar tree representatives $S \in (W|\mathcal{N}\mathcal{M}_k|)(S_\alpha)$ and $T \in (W|\mathcal{N}\mathcal{M}_l|)(T_\alpha)$. Now for any unlabelled white node in $S$ there is a unique matching black node in $T$, labelled by an element of $W|\mathcal{N}\mathcal{M}_l|$. Replace the white node in $S$ by the corresponding labelled black node in $T$. Also combine the edge labels from $S$ and $T$ by putting a univalent node marked with the edge label $(W|\mathcal{N}\mathcal{M}_k|_n(1))$ from $T$ above another univalent node marked with the edge label $(W|\mathcal{N}\mathcal{M}_l|_n(1))$ from $S$. The resulting tree represents an element of $G_m(\alpha)$.

The equivariance relations, relations (2)-(3), and relations (4)-(5) of 9.3 in $(W|\mathcal{N}\mathcal{M}_k|)(S_\alpha) \times (W|\mathcal{N}\mathcal{M}_l|)(T_\alpha)$ correspond to the equivariance, composition and unary relations respectively in $G_m(\alpha)$. This insures that the maps in both directions are well-defined and mutually inverse to each other.

Thus we can think of $G_m(\alpha)$ as a subspace, either of $(W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)(m)$ or of $(W|\mathcal{N}\mathcal{M}_k|_n(1))^m \times (W|\mathcal{N}\mathcal{M}_l|_n(1))^m$, and it makes sense to talk about intersections of the form $\cap_i G_m(\alpha_i)$. However at this point we cannot determine if these intersections are the same in $(W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)(m)$ as they are in $(W|\mathcal{N}\mathcal{M}_k|_n(1))^m \times (W|\mathcal{N}\mathcal{M}_l|_n(1))^m$ because we so far have not proved that $(W|\mathcal{N}\mathcal{M}_k| \otimes W|\mathcal{N}\mathcal{M}_l|)$ is axial. To distinguish between the two types of intersections, we will write $G^a_m(\alpha)$ (i.e. axial image) when we think of $G_m(\alpha)$ as a subset of the latter instead of the former.
Moreover since the map $G_m(\alpha) \to (W|\mathcal{M}_k| \otimes W|\mathcal{M}_l|)(m)$ is the realization of a simplicial map, it follows that this map is cofibration, thus establishing Proposition 5.9.

Our next goal is to show that there are no nonempty intersections in either $(W|\mathcal{M}_k| \otimes W|\mathcal{M}_l|)(m)$ or $W|\mathcal{M}_k|(1)^m \times W|\mathcal{M}_l|(1)^m$ except those encoded by the simplices of $\mathcal{K}(m)$.

**10.2 Proposition:** If $\alpha_0, \ldots, \alpha_r$ are not vertices of a simplex of $\mathcal{K}(m)$, then $\cap_{i=0}^r G_m(\alpha_i) = \cap_{i=0}^r G_m^{\pi}(\alpha_i) = \emptyset$.

**Proof** By assumption there is a triple $\{a, b, c\} \subset \{1, \ldots, m\}$ such that the set of vertices of $\{\alpha_0 \cap \{a, b, c\}, \ldots, \alpha_r \cap \{a, b, c\}\}$ is not a simplex in $\mathcal{K}(\{a, b, c\})$. Wolog $\{a, b, c\} = \{1, 2, 3\}$. Hence this collection of vertices must contain two different objects $\beta_1$ and $\beta_2$ in the second or third column of the table of $5.6$. By 9.8 if $\beta_1$ and $\beta_2$ are in the second column, then $(W|\mathcal{E}_1|)(S_{\beta_1}) \cap (W|\mathcal{E}_1|)(S_{\beta_2}) = \emptyset$, and if $\beta_1$ and $\beta_2$ are in the third column, then $(W|\mathcal{E}_1|)(T_{\beta_1}) \cap (W|\mathcal{E}_1|)(T_{\beta_2}) = \emptyset$. Since the restriction into $\mathcal{K}(3)$ corresponds to looking at the first three components of the axial image, we conclude that $\cap_{i=0}^r G_m^{\pi}(\alpha_i) = \emptyset$. Since the image of a nonempty set must be nonempty, it follows that $\cap_{i=0}^r G_m(\alpha_i) = \emptyset$.

Our next aim is to show that the intersections in $W|\mathcal{M}_k|(1)^m \times W|\mathcal{M}_l|(1)^m$ along a simplex are not empty.

**10.3 Lemma:** If $\sigma = \{\alpha_0, \ldots, \alpha_r\}$ is an $r$-simplex, $r \geq 1$, in $\mathcal{K}(m)$ and all vertices $\alpha_i$ have outermost operation $\boxtimes_1$, then for all $i$

$$\alpha_i = \alpha_{i1} \boxtimes_1 \alpha_{i2}$$

with $|\alpha_{i1}| = |\alpha_{j1}|$ and $|\alpha_{i2}| = |\alpha_{j2}|$ for all $i, j$, where $|\alpha|$ is the underlying set of generators of $\alpha$.

The same holds if we interchange $\boxtimes_1$ and $\boxtimes_2$.

**Proof** Special case: There is a vertex $\alpha = 1 \boxtimes_1 \alpha'$ in $\sigma$ where $\alpha'$ has outermost operation $\boxtimes_2$. We show that any other vertex $\beta$ of $\sigma$ has the form $\beta = 1 \boxtimes_1 \beta'$.

We proceed by induction. For $m = 3$ the statement is true (see table in 5.6). Assume that $m > 3$. We can find a generator (wolog 2) such that replacing 2 by 0 leaves an outermost $\boxtimes_2$ in $\alpha' \cap \{1, 3, \ldots, m\}$. Then by induction, $\beta \cap \{1, 3, \ldots, m\} = 1 \boxtimes_1 \beta''$. If $\beta \neq 1 \boxtimes_2 \beta''$ it must have the form $\beta = (1 \boxtimes_2 2) \boxtimes_1 \beta''$. There is a generator (wolog 3) such that $\alpha' \cap \{2, 3\} = 2 \boxtimes_2 3$. Then $\alpha \cap \{1, 2, 3\} = 1 \boxtimes_1 (2 \boxtimes_2 3)$ and $\beta \cap \{1, 2, 3\} = (1 \boxtimes_2 2) \boxtimes_1 3 = 3 \boxtimes_1 (1 \boxtimes_2 2)$.

But then $\{\alpha, \beta\}$ is not a 1-simplex in $\mathcal{K}(m)$.

General case: Let

$$\alpha_i = \alpha_{i1} \boxtimes_1 \alpha_{i2} \boxtimes_1 \ldots \boxtimes_1 \alpha_{iq_i}$$

with $q_i$ maximal for such a decomposition. Among the $\alpha_{is}, 0 \leq i \leq n, 1 \leq s \leq q_i$, there is a not necessarily unique block with a maximal number of generators.
Wolog we may assume that this block is $\alpha_{01}$ and that $|\alpha_{01}| = \{1, \ldots, p\}$. Then $\alpha_{0} = \alpha_{01} \boxtimes_{1} \alpha_{0}'$ and $|\alpha_{0}'| = \{p + 1, \ldots, m\}$. For each $j \in \{p + 1, \ldots, m\}$ define $\alpha_{i}(j) = \alpha_{i} \cap \{1, 2, \ldots, p, j\}$. Then $\alpha_{i}(j)$ has outermost operation $\boxtimes_{1}$ by maximality of $\alpha_{01}$. Since $\alpha_{0}(j) = j \boxtimes_{1} \alpha_{01}$ and $\alpha_{01}$ has outermost operation $\boxtimes_{2}$ we obtain that each $\alpha_{i}(j)$ has the form $\alpha_{i}(j) = j \boxtimes_{1} \alpha_{i}(j)'$ by the special case above. Since this holds for each $j \in \{p + 1, \ldots, m\}$ we obtain a decomposition

$$\alpha_{i} = (\alpha_{i} \cap \{1, \ldots, p\}) \boxtimes_{1} (\alpha_{i} \cap \{p + 1, \ldots, m\})$$

which proves the lemma.

\[\Box\]

10.4 Remark: Note that the proof of Lemma 10.3 is constructive. That is, we give an explicit algorithm for constructing the decomposition in question.

By iterating Lemma 10.3 we obtain

10.5 Corollary: If $\sigma = \{\alpha_{0}, \ldots, \alpha_{r}\}$ is an $r$-simplex, $r \geq 1$, in $\mathcal{K}_{*}(m)$ and all vertices $\alpha_{i}$ have outermost operation $\boxtimes_{1}$, then there is a maximal decomposition such that for all $i$

$$\alpha_{i} = \alpha_{i1} \boxtimes_{1} \alpha_{i2} \boxtimes_{1} \cdots \boxtimes_{1} \alpha_{ip}$$

and for all $1 \leq j \leq p$

$$|\alpha_{0j}| = |\alpha_{1j}| = \cdots = |\alpha_{rj}|.$$ 

Maximal means that for all $1 \leq j \leq p$, either $|\alpha_{0j}| = |\alpha_{1j}| = \cdots = |\alpha_{rj}|$ consists of a single generator, or $|\alpha_{ij}|$ has outermost operation $\boxtimes_{2}$ for at least one $i$. The same holds if we interchange $\boxtimes_{1}$ and $\boxtimes_{2}$.

We shall refer to the decomposition specified in Corollary 10.5 as the maximal common $\boxtimes_{1}$ decomposition of the vertices of a simplex of the first kind or the maximal common $\boxtimes_{2}$ decomposition of the vertices of a simplex of the second kind.

10.6 Proposition: Suppose that

$$\{\alpha_{0}, \alpha_{1}, \ldots, \alpha_{r}\}$$

is an $r$-simplex in $\mathcal{K}_{*}(m)$ (which we denote $\mathcal{P}$). Then $\cap_{i=0}^{r} G^{ax}_{m}(\alpha_{i})$ is nonempty and we may find a pair of binodal trees $(S_{\mathcal{P}}, T_{\mathcal{P}})$ such that

$$\cap_{i=0}^{r} G^{ax}_{m}(\alpha_{i}) = W|N_{k}\mathcal{M}_{l}|(S_{\mathcal{P}}) \times W|N_{l}\mathcal{M}_{k}|(T_{\mathcal{P}})$$

Proof We will first describe a recursive algorithm for constructing the trees $S_{\mathcal{P}}$ and $T_{\mathcal{P}}$. We defer proving that

$$\cap_{i=0}^{r} G^{ax}_{m}(\alpha_{i}) = (W|N_{k}\mathcal{M}_{l}|)(S_{\mathcal{P}}) \times (W|N_{l}\mathcal{M}_{k}|)(T_{\mathcal{P}})$$

until later.
The first case of a simplex of positive dimension occurs when \( m = 2 \), where we have the 1-simplex \( \tau = \{1 \otimes_1 2, 1 \otimes_2 2\} \). In this case we take

\[
S_\tau = T_\tau = \begin{array}{c}
1 \\
2 \\
\bullet
\end{array}
\]

Now assume we have constructed the trees \( S_\beta, T_\beta \) for all simplices \( \beta \) in \( \mathcal{K}_a(m) \), for \( a < m \). For a simplex \( \tau \) in \( \mathcal{K}_m(m) \), we consider three possible cases.

**Case 1:** Some of the objects \( \alpha_i \) have outermost operation \( \otimes_1 \) while others have outermost operation \( \otimes_2 \).

In this case reindex the vertices so that \( \alpha_0, \alpha_1, \ldots, \alpha_s \) have outermost operation \( \otimes_1 \), while \( \alpha_{s+1}, \ldots, \alpha_r \) have outermost operation \( \otimes_2 \). Let

\[
\alpha_i = \alpha_{i1} \otimes_1 \alpha_{i2} \otimes_1 \cdots \otimes_1 \alpha_{ip} \quad 0 \leq i \leq s
\]
\[
\alpha_i = \alpha_{i1} \otimes_2 \alpha_{i2} \cdots \otimes_2 \alpha_{iq} \quad s + 1 \leq i \leq r
\]

be the maximal common decompositions as in Corollary 10.5. Let

\[
U_j = |\alpha_{0j}| = |\alpha_{1j}| = \cdots = |\alpha_{sj}| \quad 1 \leq j \leq p
\]
\[
V_j = |\alpha_{s+1,j}| = |\alpha_{s+2,j}| = \cdots = |\alpha_{rj}| \quad 1 \leq j \leq q
\]

By recursion, we have already defined pairs of binodal trees \( (S_{\tau \cap U_j}, T_{\tau \cap U_j}) \) for each \( j = 1, 2, \ldots, p \). Similarly for each \( j = 1, 2, \ldots, q \) we have already defined pairs of binodal trees \( (S_{\tau \cap V_j}, T_{\tau \cap V_j}) \). We define \( S_\tau \) to be the binodal tree

\[
S_{\tau \cap U_1} \quad S_{\tau \cap U_2} \quad \cdots \quad S_{\tau \cap U_p}
\]

and we define \( T_\tau \) to be the binodal tree

\[
T_{\tau \cap V_1} \quad T_{\tau \cap V_2} \quad \cdots \quad T_{\tau \cap V_q}
\]

**Case 2:** All of the objects \( \alpha_i \) have outermost operation \( \otimes_1 \).

Let

\[
\alpha_i = \alpha_{i1} \otimes_1 \alpha_{i2} \otimes_1 \cdots \otimes_1 \alpha_{ip} \quad 0 \leq i \leq r
\]

and

\[
U_j = |\alpha_{0j}| = |\alpha_{1j}| = \cdots = |\alpha_{rj}| \quad 1 \leq j \leq p
\]

be as in Case 1.
By recursion, for each \( j = 1, 2, \ldots, p \) we have already defined pairs of binodal trees \((S_{\alpha \cap U_j}, T_{\alpha \cap U_j})\). We define \( S_{\alpha} \) to be the binodal tree
\[
S_{\alpha \cap U_1} \quad S_{\alpha \cap U_2} \quad \cdots \quad S_{\alpha \cap U_p}
\]
and \( T_{\alpha} \) to be the binodal tree
\[
T_{\alpha \cap U_1} \quad T_{\alpha \cap U_2} \quad \cdots \quad T_{\alpha \cap U_p}
\]

Case 3: All of the objects \( \alpha_i \) have outermost operation \( \boxdot_2 \).

Let
\[
\alpha_i = \alpha_{i_1} \boxdot_2 \alpha_{i_2} \boxdot_2 \cdots \boxdot_2 \alpha_{i_q} \quad 0 \leq i \leq r
\]
and
\[
V_j = |\alpha_{0j}| = |\alpha_{1j}| = \cdots = |\alpha_{rj}| \quad 1 \leq j \leq q
\]
be as in Case 1.

By recursion, for each \( j = 1, 2, \ldots, q \) we have already defined pairs of binodal trees \((S_{\alpha \cap V_j}, T_{\alpha \cap V_j})\). We define \( S_{\alpha} \) to be the binodal tree
\[
S_{\alpha \cap V_1} \quad S_{\alpha \cap V_2} \quad \cdots \quad S_{\alpha \cap V_q}
\]
and \( T_{\alpha} \) to be the binodal tree
\[
T_{\alpha \cap V_1} \quad T_{\alpha \cap V_2} \quad \cdots \quad T_{\alpha \cap V_q}
\]

To check that
\[
\cap_{i=1}^r G^m_i (\alpha_i) = (W|\mathcal{N}, \mathcal{M}_k|)(S_{\alpha}) \times (W|\mathcal{N}, \mathcal{M}_l|)(T_{\alpha})
\]
we observe that it is true for the unique 1-simplex in the case \( m = 2 \) (compare the discussion of binary operations in Section 6). If \( m > 2 \), then by 9.7 we
can reduce checking this to the case $m = 3$. Up to permutations, there are two maximal simplices in $\mathcal{M}_{2b}^b(3)$:

\[
\begin{align*}
\{ & 1 \boxtimes_1 2 \boxtimes_1 3, (1 \boxtimes_2 2 \boxtimes_1 3, (2 \boxtimes_1 3 \boxtimes_2 1, 1 \boxtimes_2 2 \boxtimes_2 3) \\
\{ & 1 \boxtimes_1 2 \boxtimes_1 3, (1 \boxtimes_2 2 \boxtimes_1 3, (1 \boxtimes_1 2 \boxtimes_2 3, 1 \boxtimes_2 2 \boxtimes_2 3) \\
\end{align*}
\]

We index the vertices of the first simplex by the indices 0, 1, 2, 3 and we use the notations $\alpha[i]$, $\alpha[i,j]$, $\alpha[i,j,k]$, $\alpha[0,1,2,3]$ to indicate the subsimplices spanned by the vertices with those indices. We list below the pairs $(S_{\alpha}, T_{\alpha})$ assigned to these simplices by our recursive algorithm.

\[
\begin{align*}
(S_{\alpha[0]}, T_{\alpha[0]}) & = \begin{pmatrix}
1 & 2 & 3 \\
\end{pmatrix}, \\
(S_{\alpha[1]}, T_{\alpha[1]}) & = \begin{pmatrix}
1 & 2 & 3 \\
\end{pmatrix}, \\
(S_{\alpha[2]}, T_{\alpha[2]}) & = \begin{pmatrix}
2 & 3 & 1 \\
\end{pmatrix}, \\
(S_{\alpha[3]}, T_{\alpha[3]}) & = \begin{pmatrix}
1 & 2 & 3 \\
\end{pmatrix}, \\
(S_{\alpha[0,1]}, T_{\alpha[0,1]}) & = \begin{pmatrix}
1 & 2 & 3 \\
\end{pmatrix}
\end{align*}
\]
\[
(S_{\alpha_{[0,2]}}, T_{\alpha_{[0,2]}}) = \begin{pmatrix}
1 & 2 & 3 & 2 & 3 & 1 \\
\end{pmatrix},
\]

\[
(S_{\alpha_{[0,3]}}, T_{\alpha_{[0,3]}}) = \begin{pmatrix}
1 & 2 & 3 & 1 & 2 & 3 \\
\end{pmatrix},
\]

\[
(S_{\alpha_{[1,2]}}, T_{\alpha_{[1,2]}}) = \begin{pmatrix}
1 & 2 & 3 & 2 & 3 & 1 \\
\end{pmatrix},
\]

\[
(S_{\alpha_{[1,3]}}, T_{\alpha_{[1,3]}}) = \begin{pmatrix}
1 & 2 & 3 & 1 & 2 & 3 \\
\end{pmatrix},
\]

\[
(S_{\alpha_{[2,3]}}, T_{\alpha_{[2,3]}}) = \begin{pmatrix}
2 & 3 & 1 & 2 & 3 & 1 \\
\end{pmatrix},
\]

\[
(S_{\alpha_{[0,1,2]}}, T_{\alpha_{[0,1,2]}}) = \begin{pmatrix}
1 & 2 & 3 & 2 & 3 & 1 \\
\end{pmatrix},
\]

\[
(S_{\alpha_{[0,1,3]}}, T_{\alpha_{[0,1,3]}}) = \begin{pmatrix}
1 & 2 & 3 & 1 & 2 & 3 \\
\end{pmatrix},
\]

\[
(S_{\alpha_{[0,2,3]}}, T_{\alpha_{[0,2,3]}}) = \begin{pmatrix}
1 & 2 & 3 & 2 & 3 & 1 \\
\end{pmatrix},
\]

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\( (S_{\alpha[1,2,3]}, T_{\alpha[1,2,3]}) = \begin{pmatrix}
1 & 2 & 3 & \rightarrow & \circ & \rightarrow & \rightarrow & \rightarrow & \circ \\
1 & 2 & 3 & \rightarrow & \circ & \rightarrow & \rightarrow & \rightarrow & \circ 
\end{pmatrix} \)

\( (S_{\alpha[0,1,2,3]}, T_{\alpha[0,1,2,3]}) = \begin{pmatrix}
1 & 2 & 3 & \rightarrow & \circ & \rightarrow & \rightarrow & \rightarrow & \circ \\
1 & 2 & 3 & \rightarrow & \circ & \rightarrow & \rightarrow & \rightarrow & \circ 
\end{pmatrix} \)

We can check that
\( \cap_{i=0}^{r} G^m_{\alpha_i}(\alpha_i) = (W|\mathcal{N}M_k)|(S_{\alpha}) \times (W|\mathcal{N}M_i)|(T_{\alpha}) \)
holds for each of these simplices by repeatedly applying Proposition 9.8.

For the second simplex
\( \{1 \natural_1 2 \natural_1 3, (1 \natural_2 2) \natural_1 3, (1 \natural_1 2) \natural_2 3, 1 \natural_2 2 \natural_2 3 \} \)
any subsimplex not containing the edge \( \{(1 \natural_2 2) \natural_1 3, (1 \natural_1 2) \natural_2 3\} \) is, up to permutation, a subsimplex of the first maximal simplex and thus has already been analyzed. For any subsimplex containing this edge, the recursive algorithm assigns
\( (S_{\alpha}, T_{\alpha}) = \begin{pmatrix}
1 & 2 & 3 & \rightarrow & \circ & \rightarrow & \rightarrow & \rightarrow & \circ \\
1 & 2 & 3 & \rightarrow & \circ & \rightarrow & \rightarrow & \rightarrow & \circ 
\end{pmatrix} \)

Again we can check that
\( \cap_{i=0}^{r} G^m_{\alpha_i}(\alpha_i) = (W|\mathcal{N}M_k)|(S_{\alpha}) \times (W|\mathcal{N}M_i)|(T_{\alpha}) \)
holds for each of these simplices by repeatedly applying Proposition 9.8 \( \square \)

**10.7 Proposition:** Let \( \overline{\alpha} = \{\alpha_0, \alpha_1, \ldots, \alpha_r\} \) be an \( r \)-simplex in \( \mathfrak{S}_*(m) \) and let
\( \cap_{i=0}^{r} G^m_{\alpha_i}(\alpha_i) = (W|\mathcal{N}M_k)|(S_{\alpha}) \times (W|\mathcal{N}M_i)|(T_{\alpha}). \)

Then
1. \( \cap_{i=0}^{r} G^m_{\alpha_i}(\alpha_i) \) has a cellular decomposition over the poset \( \mathcal{M}_k(S_{\alpha}) \times \mathcal{M}_k(T_{\alpha}) \)
2. If \( \overline{\alpha} \) is a subsimplex of \( \overline{\beta} \), then the inclusion \( \cap_{i} G^m_{\alpha_i}(\alpha_i) \subset \cap_{i} G^m_{\alpha_i}(\alpha_i) \) is compatible with the cellular decomposition.
The axial map sends $\cap_{i=0} G_m(\alpha_i) \cap_\sigma G_m(\alpha_i)$ homeomorphically onto $\cap_{i=0} G_m^\sigma(\alpha_i)$.

**Proof** Part (1) is an immediate consequence of Corollary 9.11. To establish the second part, we proceed by induction on $m$ and refer back to the proof of Proposition 10.6. If $m = 1$, the diagram is trivial and there is nothing to prove. So let us assume we have established this for all $a < m$. It suffices to check the compatibility with the cellular decompositions for inclusions $G_m'(\overline{\beta}) \subset G_m'(\overline{\alpha})$, where the simplex $\overline{\beta}$ is obtained by adding a single vertex $\lambda$ to the simplex $\overline{\alpha}$.

We refer back to the recursive algorithm of Proposition 10.6 and analyze several different cases. For convenience sake we will call vertices of our simplices as $\boxtimes_1$-vertices if their outermost operation is $\boxtimes_1$, and $\boxtimes_2$-vertices if their outermost operation is $\boxtimes_2$.

**Case (a):** Suppose $\overline{\alpha}$ contains both $\boxtimes_1$-vertices and $\boxtimes_2$-vertices, the new vertex $\lambda$ is a $\boxtimes_1$-vertex and has a $\boxtimes_1$-decomposition compatible with the maximal $\boxtimes_1$-decomposition of the $\boxtimes_1$-vertices of $\overline{\alpha}$.

In this case both simplicies $\overline{\alpha}$ and $\overline{\beta}$ are in Case 1 of the proof of Proposition 10.6. Thus the inclusion $(W|\mathcal{M}_k)(\mathcal{S}_\overline{\beta}) \subset (W|\mathcal{M}_k)(\mathcal{S}_\overline{\alpha})$ takes the form

$$W|\mathcal{M}_k|(p) \times W|\mathcal{M}_k|(1) \prod_{j=1}^p (W|\mathcal{M}_k|(\mathcal{S}_\overline{\beta} \cap U_j))$$

and is thus compatible with the cellular decomposition by the induction hypothesis. Similarly the inclusion $(W|\mathcal{M}_l)|(\mathcal{T}_\overline{\beta}) \subset (W|\mathcal{M}_l)|(\mathcal{T}_\overline{\alpha})$ takes the form

$$W|\mathcal{M}_l|(q) \times W|\mathcal{M}_l|(1) \prod_{j=1}^q (W|\mathcal{M}_l|(\mathcal{T}_\overline{\beta} \cap V_j))$$

which again is compatible with the cellular decompositions for the same reason.

A similar analysis, reducing the proof to the induction hypothesis, applies in Cases (b), (c) and (d) below.

**Case (b):** Suppose $\overline{\alpha}$ contains both $\boxtimes_1$-vertices and $\boxtimes_2$-vertices, the new vertex $\lambda$ is a $\boxtimes_2$-vertex and has a $\boxtimes_2$-decomposition compatible with the maximal common $\boxtimes_2$-decomposition of the $\boxtimes_2$-vertices of $\overline{\alpha}$.

**Case (c):** Suppose $\overline{\beta}$ contains only $\boxtimes_1$-vertices and the new vertex $\lambda$ has a $\boxtimes_1$-decomposition compatible with the maximal common $\boxtimes_1$-decomposition of the vertices of $\overline{\alpha}$.
Case (d): Suppose $\beta$ contains only $\boxplus_2$-vertices and the new vertex $\lambda$ has a $\boxplus_2$ decomposition compatible with the maximal common $\boxplus_2$ decomposition of the vertices of $\alpha$.

Case (e): Suppose $\alpha$ contains both $\boxplus_1$-vertices and $\boxplus_2$-vertices and the new vertex $\lambda$ is a $\boxplus_1$-vertex which does not have a $\boxplus_1$ decomposition compatible with the maximal common $\boxplus_1$ decomposition of the $\boxplus_1$-vertices of $\alpha$.

Again both simplicies $\alpha$ and $\beta$ are in Case 1 of the proof of Proposition 10.6 but now the maximal common $\boxplus_1$-decomposition of the $\boxplus_1$-vertices of $\alpha$ is finer (i.e. has more summands) than the maximal common $\boxplus_1$-decomposition of the $\boxplus_1$-vertices of $\beta$.

In this case the inclusion $W|\mathcal{N}\mathcal{M}_k|(S_\beta) \subset W|\mathcal{N}\mathcal{M}_k|(S_\alpha)$ corresponds to a sequence of intersections of the type

$$W|\mathcal{N}\mathcal{M}_k| \left( \begin{array}{c} \cdots \cdots \\ \vdots \vdots \\ \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \\ \end{array} \right) \cap W|\mathcal{N}\mathcal{M}_k| \left( \begin{array}{c} \cdots \cdots \\ \vdots \vdots \\ \downarrow \downarrow \downarrow \downarrow \downarrow \\ \end{array} \right) = W|\mathcal{N}\mathcal{M}_k| \left( \begin{array}{c} \cdots \cdots \\ \vdots \vdots \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ \end{array} \right)$$

Hence $S_\alpha$ can be obtained from $S_\beta$ by a sequence of the following types of moves:

$$W_1 W_2 \cdots W_{u-1} W_u W_{u+1} \cdots W_s \quad W'_1 W'_2 \cdots W'_{u-1} W_u W_{u+1} \cdots W_s$$

where $W_j$ and $W'_j$ are obtained by restricting $S_\beta$, respectively $S_\alpha$, to the same set of inputs. The induced map on $W|\mathcal{N}\mathcal{M}_k|(-)$ is the product of the composition map:

$$W|\mathcal{N}\mathcal{M}_k|(s-u+1) \times W|\mathcal{N}\mathcal{M}_k|(u-1) \quad \rightarrow \quad W|\mathcal{N}\mathcal{M}_k|(s)$$

$$(a, b) \quad \rightarrow \quad a \circ (b \oplus id \oplus \cdots \oplus id),$$

which is obviously compatible with the cellular decomposition, and the inclusion map:

$$\prod_{j=1}^s W|\mathcal{N}\mathcal{M}_k|(W_j) \subset \prod_{j=1}^s W|\mathcal{N}\mathcal{M}_k|(W'_j)$$

which is compatible with the cellular decomposition by inductions hypothesis.
The induced map $(W|\mathcal{M}_l|)(\overline{T_{\overline{\nu}}}) \subset (W|\mathcal{N}_M l|)(\overline{T_{\overline{\nu}}})$ takes the form

$$W|\mathcal{M}_l|((q) \times_{W|\mathcal{M}_l|((1))} \prod_{j=1}^{p}(W|\mathcal{M}_l|)(\overline{T_{\overline{\nu}}} \cap V_j))$$

$$\subset W|\mathcal{M}_l|((q) \times_{W|\mathcal{M}_l|((1))} \prod_{j=1}^{q}(W|\mathcal{M}_l|)(\overline{T_{\overline{\nu}}} \cap V_j)),$$

for which compatibility with the cellular decomposition follows by induction hypothesis.

**Case (f):** Suppose $\overline{\alpha}$ contains both $\boxtimes_1$-vertices and $\boxtimes_2$-vertices and the new vertex $\lambda$ is a $\boxtimes_2$-vertex which does not have a $\boxtimes_2$ decomposition compatible with the maximal common $\boxtimes_2$ decomposition of the $\boxtimes_2$-vertices of $\overline{\alpha}$.

In this case, a similar analysis as in Case (e) applies, with the roles of the inclusions $(W|\mathcal{N}_M k|)(S_{\overline{\nu}}) \subset (W|\mathcal{N}_M k|)(S_{\overline{\nu}})$ and $(W|\mathcal{M}_l|)(\overline{T_{\overline{\nu}}}) \subset (W|\mathcal{M}_l|)(\overline{T_{\overline{\nu}}})$ reversed.

**Case (g):** Suppose $\overline{\alpha}$ contains only $\boxtimes_1$-vertices and the new vertex $\lambda$ is a $\boxtimes_1$-vertex which does not have a $\boxtimes_1$ decomposition compatible with the maximal common $\boxtimes_1$ decomposition of the vertices of $\overline{\alpha}$.

In this case the argument that the resulting inclusion is compatible with the cellular decomposition is similar to that in Case (e).

**Case (h):** Suppose $\overline{\alpha}$ contains only $\boxtimes_2$-vertices and the new vertex $\lambda$ is a $\boxtimes_2$-vertex which does not have a $\boxtimes_2$ decomposition compatible with the maximal common $\boxtimes_2$ decomposition of the vertices of $\overline{\alpha}$.

In this case the argument that the resulting inclusion is compatible with the cellular decomposition is similar to that in Case (f).

**Case (i):** Suppose $\overline{\alpha}$ contains only $\boxtimes_1$-vertices and the new vertex $\lambda$ is $\boxtimes_2$-vertex.

In this case the induced inclusion $(W|\mathcal{M}_k|)(S_{\overline{\nu}}) \subset (W|\mathcal{M}_k|)(S_{\overline{\nu}})$ takes the form

$$W|\mathcal{M}_k|((p) \times_{W|\mathcal{M}_k|((1))} \prod_{j=1}^{p}(W|\mathcal{M}_k|)(S_{\overline{\nu}} \cap U_j))$$

$$\subset W|\mathcal{M}_k|((p) \times_{W|\mathcal{M}_k|((1))} \prod_{j=1}^{p}(W|\mathcal{M}_k|)(S_{\overline{\nu}} \cap V_j))$$

and is thus compatible with the cellular decomposition by the induction hypothesis. The inclusion $(W|\mathcal{M}_l|)(\overline{T_{\overline{\nu}}}) \subset (W|\mathcal{M}_l|)(\overline{T_{\overline{\nu}}})$ is more complicated.
The trees $T_\alpha$ and $T_\beta$ are related as shown below:

Here $W_{ij}$ and $W'_{ij}$ have the same sets of inputs, some of which may be empty, in which case the trees are understood to be empty as well. The induced inclusion $(W|N\mathcal{M}_i|)(T_\alpha) \subset (W|N\mathcal{M}_i|)(T_\beta)$ is then the product of the maps

\[ N\mathcal{M}_i(q) \longrightarrow \prod_{j=1}^p N\mathcal{M}_i(a_j) \]

and

\[ \prod_{i,j} N\mathcal{M}_i(W_{ij}) \subset \prod_{i,j} N\mathcal{M}_i(W'_{ij}). \]

Here $a_i$ is the cardinality of $\bigcup_{j=1}^q |W'_{ij}|$ and the first map is given by

\[ \phi \mapsto (\phi \cdot (\epsilon_{11} \oplus \cdots \oplus \epsilon_{1q}), \phi \cdot (\epsilon_{21} \oplus \cdots \oplus \epsilon_{2q}), \ldots, \phi \cdot (\epsilon_{p1} \oplus \cdots \oplus \epsilon_{pq})), \]

where $\epsilon_{ij} = \text{id} \in N\mathcal{M}_i(1)$ if $|W'_{ij}|$ is nonempty or $\epsilon_{ij} = 0 \in N\mathcal{M}_i(0)$ otherwise. Clearly this is compatible with the cellular decompositions while the same is true for the second map by the induction hypothesis.

**Case (j):** Suppose $\tau$ contains only $\mathbb{Z}_2$-vertices and the new vertex $\lambda$ is $\mathbb{Z}_1$-vertex.

In this case we argue as in Case (i), except with the roles of the inclusions $(W|N\mathcal{M}_k|)(S_\tau) \subset (W|N\mathcal{M}_k|)(S_\alpha)$ and $(W|N\mathcal{M}_k|)(T_\alpha) \subset (W|N\mathcal{M}_k|)(T_\beta)$ interchanged.

This concludes the proof of part (2).

To prove part (3) we proceed by a double induction on $m$ and $r$. If $m = 1$ or $r = 1$, there is nothing to prove. So assume we have established the result for $m' < m$ and all $r'$ as well as for $m = m$ and $r' \leq r$. Now consider an $(r+1)$-simplex $\overline{\beta}$ in $\mathcal{R}_n$ obtained by adding a new vertex $\lambda$ to an $r$-simplex $\overline{\alpha}$. Suppose we are given an element $x \in \cap_{i=0}^r G_m^n(\alpha_i) \cap G_m^n(\lambda)$, and let $y \in G_m(\lambda)$ be the unique preimage of $x$. It suffices to find a single vertex $\alpha_j$ such that the preimage $z_j \in G_m(\alpha_j)$ of $x$ is the same element of $(W|N\mathcal{M}_k| \otimes W|N\mathcal{M}_i|)(m)$ as $y$. For then we would have $y = z_j \in G_m(\alpha_j) \cap G_m(\lambda)$ and by the second induction hypothesis $z_j \in \cap_{i=0}^r G_m^n(\alpha_i).$ Thus we would have $y = z_j \in \cap_{i=0}^r G_m^n(\alpha_i) \cap G_m(\lambda)$ and we would be done.

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To establish this fact, we do the same case by case analysis as in the proof of part (2). In cases (a)-(d), that analysis establishes that we can find a vertex $\alpha_j$ such that $y$ and $z_j$ have tree representatives whose root nodes are identical, and all the axial images of the corresponding tree branches of $y$ and $z_j$ above the common root node are the same. By the primary induction hypothesis, these tree branches of $y$ and $z_j$ are also the same in $W[\mathcal{N}\mathcal{M}_k] \otimes W[\mathcal{N}\mathcal{M}_l]$. Hence $y = z_j$.

In cases (e)-(h), the analysis of the proof of part (2) shows that $y$ has a tree representative which can be converted by series of compositions, either in $W[\mathcal{N}\mathcal{M}_k]$ or in $W[\mathcal{N}\mathcal{M}_l]$, into a tree which is related to some $z_j \in G_m(\alpha_j)$ in the same way as in cases (a)-(d). By the same argument as above $y = z_j$. In cases (i) and (j), that analysis shows $y$ has a tree representative which can be converted by an interchange into a tree which is related to some $z_j \in G_m(\alpha_j)$ in the same way as in cases (a)-(d). Again we can conclude that $y = z_j$. 

By combining the above results, we obtain the following more precise version of Proposition 5.11.

10.8 Corollary: 

1. $(W[\mathcal{N}\mathcal{M}_k] \otimes W[\mathcal{N}\mathcal{M}_l])(m)$ is the colimit of the diagram $G_m : I(m) \rightarrow \text{Top}$.
2. The maps in this diagram are all cofibrations.
3. Each space $G_m(\overline{\alpha})$ has a cellular decomposition over the poset $\mathcal{M}_k(S_{\overline{\alpha}}) \times \mathcal{M}_l(T_{\overline{\alpha}})$ and the maps in the diagram are compatible with these cellular decompositions.
4. $(W[\mathcal{N}\mathcal{M}_k] \otimes W[\mathcal{N}\mathcal{M}_l])(m)$ is axial and left factorial.

Proof Part (1) is tautological. In view of Proposition 10.7, we have for any object $\overline{\alpha}$ in $I(m)$, or equivalently any simplex $\overline{\alpha} = \{\alpha_0, \alpha_1, \ldots, \alpha_r\}$ in $\mathcal{R}_*(m)$

$$G_m(\overline{\alpha}) = \cap_{i=0}^{r} G_m(\alpha_i) \cong \cap_{i=0}^{r} G^\text{ax}_m(\alpha_i) = (W[\mathcal{N}\mathcal{M}_k](S_{\overline{\alpha}}) \times (W[\mathcal{N}\mathcal{M}_l](T_{\overline{\alpha}}))$$

Thus the maps in the diagram are realizations of simplicial injections and thus are cofibrations. Part (3) follows from parts (1) and (2) of Proposition 10.7. Axiality follows from part (3) of Proposition 10.7 whereas left factoriality immediately follows from the obvious observation that the product of two left factorial monoids is left factorial.

11. The Grothendieck construction for interchanges

According to Corollary 10.8 $(W[\mathcal{N}\mathcal{M}_k] \otimes W[\mathcal{N}\mathcal{M}_l])(m)$ is homeomorphic to an iterated colimit of diagrams of cofibrations. We can now apply Proposition 5.2 of [8] which identifies such an iterated colimit as a simple colimit of the Grothendieck construction of the diagram of categories which to each node $\overline{\alpha}$ of the outer diagram assigns the category indexing the inner colimit at that node.
11.1 **Definition:** Let $F : A \to \mathcal{C}at$ be any functor. Then the Grothendieck construction $\mathcal{A} \downarrow F$ is the category whose objects are pairs $(A, B)$ with $A \in \text{obj}\mathcal{A}$ and with $B \in \text{obj}F(A)$. A morphism $(A_1, B_1) \to (A_2, B_2)$ is a pair $(\alpha, \beta)$ where $\alpha : A_1 \to A_2$ and $\beta : F(\alpha)(B_1) \to B_2$.

11.2 **Definition:** In Corollary 10.8 the appropriate diagram for the Grothendieck construction is the diagram of posets $\mathcal{G}_m' : \mathcal{I}(m) \longrightarrow \mathcal{C}at$ given by $\mathcal{G}_m'((\mathcal{T}_m\), \mathcal{S}_m)) = \mathcal{M}_k(S_m) \times \mathcal{M}_l(T_m)$. We denote the Grothendieck construction $\mathcal{I}(m) \downarrow \mathcal{G}_m'$ by $\mathcal{I}(k, l)(m)$.

Thus we have established that

11.3 **Proposition:** $(W\lceil \mathcal{N}\mathcal{M}_k \rceil \otimes W\lceil \mathcal{N}\mathcal{M}_l \rceil)(m)$ has a cellular decomposition indexed by $\mathcal{I}(k, l)(m)$. This decomposition assigns to an object $(\mathcal{T}_m, \mathcal{S}_m)$ the cell $\mathcal{G}_m'((\mathcal{T}_m, \mathcal{S}_m))$ in $\mathcal{G}_m(\mathcal{T}_m) \cong (W\lceil \mathcal{N}\mathcal{M}_k \rceil(S_m)) \times (W\lceil \mathcal{N}\mathcal{M}_l \rceil(T_m))$ indexed by $\beta \in \mathcal{M}_k(S_m) \times \mathcal{M}_l(T_m)$.

We now prove Proposition 5.14. We first need the following result.

11.4 **Lemma:** Let $\mathcal{T}_m$ be a simplex in $\mathcal{R}_m(m)$. Consider the set $\mathcal{S}_m$ of all objects $\lambda \in M_2^b(m) = \text{Ob}M_2^b(m)$ such that $G_m(\mathcal{T}_m) \subseteq G_m(\lambda)$. Then $\mathcal{S}_m$ contains a unique minimal element $L'(\mathcal{T}_m)$ with respect to the order relation on the poset $M_2^b(m)$. The mapping $\mathcal{T}_m \mapsto L'(\mathcal{T}_m)$ defines a poset map $L' : \mathcal{I}(m) \longrightarrow M_2^b(m)$.

**Proof** By construction and Proposition 10.2 $\mathcal{T}_m$ is also a simplex in $\mathcal{R}_m(m)$ and we have $G_m(\mathcal{T}_m) = G_m(\mathcal{T})$. Hence wolog we may take $\mathcal{T}_m = \mathcal{T}$, i.e. that $G_m(\mathcal{T}_m) \subseteq G_m(\lambda)$ implies $\lambda \in \mathcal{T}_m$.

Let $(S_{\mathcal{T}_m}, T_{\mathcal{T}_m})$ be the pair of binodal trees corresponding to $\mathcal{T}_m$ in Proposition 10.6. We define $L'(\mathcal{T}_m)$ as follows. In the tree $S_{\mathcal{T}_m}$ replace each node label $\bullet$ by node label $\bullet_1$ and each node label $\circ$ by node label $\circ_2$. Let $L'(\mathcal{T}_m)$ be the element of $M_2^b(m)$ corresponding to this relabelled tree.

To check that $G_m(\mathcal{T}_m) \subseteq G_m(L'(\mathcal{T}_m))$, it suffices to check that $G_m(\mathcal{T}_m \cap \{a, b, c\}) \subseteq G_m(L'(\mathcal{T}_m)) \cap \{a, b, c\}$ for all $\{a < b < c\} \subseteq \{1, 2, \ldots, m\}$, which is easily verified. To check that $L'(\mathcal{T}_m) \subseteq a_j$ for all vertices $a_j$ in $\mathcal{T}_m = \mathcal{T}$, it suffices to check that $L'(\mathcal{T}_m) \cap \{a, b\} \subseteq a_j \cap \{a, b\}$ for all $\{a < b\} \subseteq \{1, 2, \ldots, m\}$, which is even easier to verify.

Finally if $\mathcal{T}_m \subseteq \mathcal{T}_n$ is an inclusion of simplices in $\mathcal{R}_n(m)$, then it easily follows that $L'(\mathcal{T}_m) \subseteq L'(\mathcal{T}_n)$ for all $\{a < b\} \subseteq \{1, 2, \ldots, m\}$. This implies $L'(\mathcal{T}_m) \leq L'(\mathcal{T}_n)$, and so $L' : \mathcal{I}(m) \longrightarrow M_2^b(m)$ is a map of posets. (Recall that the poset structure on $\mathcal{I}(m)$ is opposite to inclusion of simplices, cf. Definition 5.10) □

**Proof of Proposition 5.14** The construction of $L : \mathcal{I}(k, l)(m) \longrightarrow M_{k+l}(m)$
goes as follows. An object in $\mathcal{I}(k,l)(m)$ is a pair $(\overline{\alpha}, \beta)$, where $\overline{\alpha}$ is an object in $\mathcal{I}(m) = \text{Sd} \mathcal{R}_*(m)$, or equivalently a simplex $\{\alpha_0, \alpha_1, \ldots, \alpha_r\} \in \mathcal{R}_*(m)$, and $\beta$ is an object in the poset $\mathcal{M}_k(SL) \times \mathcal{M}_l(TL)$. 

There is a unique morphism $\overline{\alpha} \to L'(\overline{\alpha})$ in $\mathcal{I}(m)$. Let $\beta'$ denote the image of $\beta$ under the corresponding functor $G'_m(\overline{\alpha}) \to G'_m(L'(\overline{\alpha}))$. Then $\beta'$ is an object in the poset $\mathcal{M}_k(\overline{\alpha}) \times \mathcal{M}_l(\overline{\alpha})$. But $S_L(\overline{\alpha})$ and $T_L(\overline{\alpha})$ are trees of the same shape with black nodes in $S_L(\overline{\alpha})$ corresponding to white nodes in $T_L(\overline{\alpha})$ and vice-versa. Now an element $\beta'$ in $\mathcal{M}_k(S_L(\overline{\alpha})) \times \mathcal{M}_l(T_L(\overline{\alpha}))$ can be identified with a pair of planar trees $(S', T')$ with $S'$ looking exactly like $S_L(\overline{\alpha})$, and $T'$ looking exactly like $T_L(\overline{\alpha})$, except that every black node is decorated with an object of $\mathcal{M}_k$ of the appropriate arity and with $T'$ looking exactly like $T_L(\overline{\alpha})$, except that every black node is decorated with an object of $\mathcal{M}_l$ of the appropriate arity. Putting together the decorations on $S'$ and $T'$, we obtain a tree all of whose nodes are decorated with objects either of $\mathcal{M}_k$ or of $\mathcal{M}_l$. Regarding $\mathcal{M}_k$ and $\mathcal{M}_l$ as suboperads of $\mathcal{M}_{k+l}$ as in paragraph 11.5, we can interpret the resulting tree as an object $\beta''$ in $\mathcal{M}_{k+l}$, by considering the edges of the tree to be compositions in the operad $\mathcal{M}_{k+l}$. We define $L(\overline{\alpha}, \beta') = \beta''$. This is easily checked to be a map of posets.

To check Quillen’s Theorem A for the functor $L$, we show that the over categories $L/\alpha$ have contractible nerves for any object $\alpha$ in $\mathcal{M}_{k+l}(m)$. We note that we can obtain an object $\alpha'$ in $\mathcal{M}_2(m)$ by replacing each $\Box_i$ in $\alpha$ by $\mathcal{E}_1$ if $i \leq k$ and by $\mathcal{E}_2$ if $i > k$. This resulting object $\alpha'$ can also be regarded as an object in $\mathcal{I}(m)$. It is easy to see that there is a unique object $\beta'$ in $\mathcal{M}_k(S_{\alpha'}) \times \mathcal{M}_l(T_{\alpha'})$ such that $L(\alpha', \beta') = \alpha$ and that id : $L(\alpha', \beta') \to \alpha$ is the terminal object in $L/\alpha$. \hfill \Box

It remains to prove

11.5 Proposition: The decomposition of 6.15 provides a cellular decomposition of $(W|N\mathcal{M}_k \otimes W|N\mathcal{M}_l)(m)$ indexed by the poset $\mathcal{M}_{k+l}(m)$. This decomposition is compatible with the operad structures on $(W|N\mathcal{M}_k \otimes W|N\mathcal{M}_l)(m)$ and $\mathcal{M}_{k+l}$. Thus we obtain a chain of operad equivalences

$$W|N\mathcal{M}_k \otimes W|N\mathcal{M}_l \cong \text{colim}_{\mathcal{M}_{k+l}} F_* \xleftarrow{\sim} \text{hocolim}_{\mathcal{M}_{k+l}} F_* \xrightarrow{\sim} |N\mathcal{M}_{k+l}|$$

(we recall $F_*$ below), proving that $W|N\mathcal{M}_k \otimes W|N\mathcal{M}_l$ is an $E_{k+l}$-operad.

Proof For an object $\gamma$ in $\mathcal{M}_{k+l}(m)$ we define

$$F_m(\gamma) = \bigcup_{L(\overline{\alpha}, \beta) \leq \gamma} G'_m(\overline{\alpha}, \beta)$$

In other words, $F_m(\gamma)$ has a cellular decomposition into finer cells indexed by the poset $L/\gamma$. We have just shown in the proof of Proposition 11.3 that this poset is contractible. It follows that $F_m(\gamma)$ is contractible.
We also have
\[
F_m(\gamma_1) \cap F_m(\gamma_2) = \bigcup_{L(\overline{\alpha}, \beta_1) \leq \gamma_1} \bigcup_{L(\overline{\alpha}, \beta_2) \leq \gamma_2} G'_m(\overline{\alpha}, \beta_1) \cap G'_m(\overline{\alpha}, \beta_2)
\]
\[
= \bigcup_{L(\overline{\alpha}, \beta_1) \leq \gamma_1} \bigcup_{L(\overline{\alpha}, \beta_2) \leq \gamma_2} G'_m(\overline{\alpha}, \beta_1) \cap G'_m(\overline{\alpha}, \beta_2)
\]
\[
\subseteq \bigcup_{L(\overline{\alpha}, \beta_1) \leq \gamma_1} G'_m(\overline{\alpha}, \beta_1) \cap G'_m(\overline{\alpha}, \beta_1)
\]
\[
\subseteq \bigcup_{L(\overline{\alpha}, \beta_1) \leq \gamma_1} \bigcup_{L(\overline{\alpha}, \beta_2) \leq \gamma_2} G'_m(\overline{\alpha}, \beta_2)
\]
\[
\subseteq \bigcup_{L(\overline{\alpha}, \beta_1) \leq \gamma_1} \bigcup_{L(\overline{\alpha}, \beta_2) \leq \gamma_2} \bigcup_{L(\overline{\alpha}, \beta_3) \leq \gamma_3} F_m(\gamma_3)
\]
\[
\gamma_3 \leq \gamma_1
\]
\[
\gamma_3 \leq \gamma_2
\]

Thus \( \{F_m(\gamma)\}_{\gamma \in M_{k+1}} \) is a cellular decomposition.

Before we move on to the last part of the proof, let us note that if \( L(\overline{\alpha}, \beta) \leq \gamma \) and if \( \alpha_r \) is the least vertex of \( \overline{\alpha} \) and \( \beta' \) is the image of \( \beta \) under the map \( \hat{G}_m'(\overline{\alpha}) \rightarrow \hat{G}_m'(\alpha_r) \), then we have \( L(\alpha_r, \beta') = L(\overline{\alpha}, \beta) \leq \gamma \) and \( G'_m(\overline{\alpha}, \beta) \subseteq G'_m(\alpha_r, \beta') \). It follows that

\[
(*) \quad F_m(\gamma) = \bigcup G'_m(\alpha, \beta)
\]

where the union runs over all pairs \((\alpha, \beta)\) where \( \alpha \) is an object of \( \mathcal{M}_2^h(m) \) and \( \beta \) is an object in the poset \( \mathcal{M}_k(S_{\alpha}) \times \mathcal{M}_1(T_{\alpha}) \), such that \( L(\alpha, \beta) \leq \gamma \).

To establish compatibility with the operad structure, we need to establish the following. Let \( \gamma \in M_{k+1}(m) \), \( \phi_i \in M_{k+1}(p_i) \) for \( i = 1, 2, \ldots, m \). Then the operad composition

\[
(W|\mathcal{M}_k| \otimes W|\mathcal{M}_l|)(m) \times \prod_{i=1}^{p} (W|\mathcal{M}_k| \otimes W|\mathcal{M}_l|)(p_i)
\]
\[
\rightarrow \quad (W|\mathcal{M}_k| \otimes W|\mathcal{M}_l|)(p_1 + p_2 + \ldots + p_m)
\]

restricts to

\[
F_m(\gamma) \times \prod_{i=1}^{m} F_{p_i}(\phi_i) \rightarrow F_{p_1 + p_2 + \ldots + p_m} (\gamma \cdot (\phi_1 \oplus \phi_2 \oplus \ldots \oplus \phi_m))
\]

By \( (*) \) it suffices to check that for pairs \((\alpha, \beta), (\kappa_i, \lambda_i), i = 1, 2, \ldots, m \) with \( \alpha \in \mathcal{M}_2^h(m), \kappa_i \in \mathcal{M}_2^h(p_i), \beta \in \mathcal{M}_k(S_{\alpha}) \times \mathcal{M}_1(T_{\alpha}), \) and \( \lambda_i \in \mathcal{M}_k(S_{\kappa_i}) \times \mathcal{M}_1(T_{\kappa_i}) \), satisfying \( L(\alpha, \beta) \leq \gamma \) and \( L(\kappa_i, \lambda_i) \leq \phi_i \), the operad composition restricts to

\[
G'_m(\alpha, \beta) \times \prod_{i=1}^{m} G'_m(\kappa_i, \lambda_i) \rightarrow F_{p_1 + p_2 + \ldots + p_m} (\gamma \cdot (\phi_1 \oplus \phi_2 \oplus \ldots \oplus \phi_m))
\]
This follows from the fact that

\[ L(\alpha, \beta) \cdot (L(\kappa_1, \lambda_1) \oplus L(\kappa_2, \lambda_2) \oplus \cdots \oplus L(\kappa_m, \lambda_m)) \leq \gamma \cdot (\phi_1 \oplus \phi_2 \oplus \cdots \oplus \phi_m). \]
12. Appendix: Intersection table for Proposition 9.8

|   | $T_1$ | $T_2$ | $T_3$ |
|---|---|---|---|
| 1 | 1 2 3 | $T_2$ arbitrary | $T_2$ |
| 2 | 1 2 3 | $i j k$ | $i j k$ |
| 3 | 1 2 3 | $i j k$ | 1 2 3 |
| 4 | 1 2 3 | $i j k$ | $i j k$ |
| 5 | $i j k$ | $p q r$ | $i j k$ |
| 6 | $i j k$ | $i j k$ | $i j k$ |
| 7 | $i j k$ | $i j k$ | $i j k$ |
| 8 | $i j k$ | $i k j$ | $i j k$ |
| 9 | $i j k$ | $i k j$ | nonempty but not of the form $(W\mathcal{E}_*)(T_3)$ |
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