All binary linear codes that are invariant under PSL$_2(n)$

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Abstract
The projective special linear group PSL$_2(n)$ is 2-transitive for all primes $n$ and 3-homogeneous for $n \equiv 3 \pmod{4}$ on the set $\{0, 1, \ldots, n-1, \infty\}$. It is known that the extended odd-like quadratic residue codes are invariant under PSL$_2(n)$. Hence, the extended quadratic residue codes hold an infinite family of 2-designs for primes $n \equiv 1 \pmod{4}$, an infinite family of 3-designs for primes $n \equiv 3 \pmod{4}$. To construct more $t$-designs with $t \in \{2, 3\}$, one would search for other extended cyclic codes over finite fields that are invariant under the action of PSL$_2(n)$. The objective of this paper is to prove that the extended quadratic residue binary codes are the only nontrivial extended binary cyclic codes that are invariant under PSL$_2(n)$.

Keywords: Cyclic code, linear code, quadratic residue code, projective linear group, $t$-design.

2000 MSC: 05B05, 94B05, 94B15

1. Introduction

An $[n, k, d]$ code $C$ over $\mathbb{F}(q)$ is a $k$-dimensional linear subspace of $\mathbb{F}(q)^n$ with minimum Hamming distance $d$. Trivial linear codes of length $n$ over $\mathbb{F}(q)$ are the linear subspace consisting only of the zero vector of $\mathbb{F}(q)^n$ with dimension 0, the whole space $\mathbb{F}(q)^n$ with dimension $n$, the subspace $\{a(1, 1, \ldots, 1) : a \in \mathbb{F}(q)\}$ with dimension 1, and the subspace

$$\left\{ (c_0, c_1, \ldots, c_{n-1}) \in \mathbb{F}(q)^n : \sum_{i=0}^{n-1} c_i = 0 \right\}$$

with dimension $n - 1$.

A linear code $C$ over $\mathbb{F}(q)$ is cyclic if $(c_0, c_1, \ldots, c_{n-1}) \in C$ implies $(c_{n-1}, c_0, c_1, \ldots, c_{n-2}) \in C$. We may identify a vector $(c_0, c_1, \ldots, c_{n-1}) \in \mathbb{F}(q)^n$ with the polynomial

$$c_0 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1} \in \mathbb{F}(q)[x]/(x^n - 1).$$

In this way, a code $C$ of length $n$ over $\mathbb{F}(q)$ always corresponds to a subset of the quotient ring $\mathbb{F}(q)[x]/(x^n - 1)$. A linear code $C$ is cyclic if and only if the corresponding subset in $\mathbb{F}(q)[x]/(x^n - 1)$ is an ideal of the ring $\mathbb{F}(q)[x]/(x^n - 1)$.
It is well-known that every ideal of \( \text{GF}(q)[x]/(x^n - 1) \) is principal. Let \( C = \langle g(x) \rangle \) be a cyclic code, where \( g(x) \) is monic and has the smallest degree among all the generators of \( C \). Then \( g(x) \) is unique and called the generator polynomial, and \( h(x) = (x^n - 1)/g(x) \) is referred to as the check polynomial of \( C \).

Given a linear code \( C \) of length \( n \) over \( \text{GF}(q) \), we can extend \( C \) into another code \( \overline{C} \) of length \( n + 1 \) over \( \text{GF}(q) \) by adding an extended coordinate, denoted by \( \infty \), as follows:

\[
\overline{C} = \{(c_0, c_1, \cdots, c_n, c_{\infty}) : (c_0, c_1, \cdots, c_{n-1}) \in C\},
\]

where

\[
c_{\infty} = -\sum_{i=0}^{n-1} c_i.
\]

By definition, \( C \) and \( \overline{C} \) have the same dimension, but their minimum distances may be the same or may differ by one.

Let \( n \) be a prime and let \( q \) be a prime power such that \( \gcd(q, n) = 1 \) and \( q \) is a quadratic residue modulo \( n \). Let \( m = \text{ord}_q(q) \), and let \( \alpha \) be a generator of \( \text{GF}(q^m)^* \), which is the multiplicative group of \( \text{GF}(q^m) \). Set \( \beta = \alpha^{(q^m-1)/n} \). Then \( \beta \) is a \( n \)-th primitive root of unity in \( \text{GF}(q^m) \). Define

\[
g_0(x) = \sum_{i \in Q} (x - \beta^i) \quad \text{and} \quad g_1(x) = \sum_{i \in N} (x - \beta^i),
\]

where \( Q \) and \( N \) are the set of quadratic residues and nonresidues modulo \( n \), respectively. It is easily seen that \( g_0(x) \) and \( g_1(x) \) are polynomials over \( \text{GF}(q) \) and are divisors of \( x^n - 1 \). The cyclic codes of length \( n \) over \( \text{GF}(q) \) with generator polynomials \( g_0(x) \) and \( g_1(x) \) are called odd-like quadratic residue codes.

The set of coordinate permutations that map a code \( C \) to itself forms a group, which is referred to as the permutation automorphism group of \( C \) and denoted by \( \text{PAut}(C) \). If \( C \) is a code of length \( n \), then \( \text{PAut}(C) \) is a subgroup of the symmetric group \( \text{Sym}_n \).

A monomial matrix over \( \text{GF}(q) \) is a square matrix having exactly one nonzero element of \( \text{GF}(q) \) in each row and column. A monomial matrix \( M \) can be written either in the form \( DP \) or the form \( PD_1 \), where \( D \) and \( D_1 \) are diagonal matrices and \( P \) is a permutation matrix.

The set of monomial matrices that map \( C \) to itself forms the group \( \text{MAut}(C) \), which is called the monomial automorphism group of \( C \). Clearly, we have

\[
\text{PAut}(C) \subseteq \text{MAut}(C).
\]

The automorphism group of \( C \), denoted by \( \text{Aut}(C) \), is the set of maps of the form \( M\gamma \), where \( M \) is a monomial matrix and \( \gamma \) is a field automorphism, that map \( C \) to itself. In the binary case, \( \text{PAut}(C), \text{MAut}(C) \) and \( \text{Aut}(C) \) are the same. If \( q \) is a prime, \( \text{MAut}(C) \) and \( \text{Aut}(C) \) are identical. In general, we have

\[
\text{PAut}(C) \subseteq \text{MAut}(C) \subseteq \text{Aut}(C).
\]

By definition, every element in \( \text{Aut}(C) \) is of the form \( DP\gamma \), where \( D \) is a diagonal matrix, \( P \) is a permutation matrix, and \( \gamma \) is an automorphism of \( \text{GF}(q) \). The automorphism group \( \text{Aut}(C) \) is said to be \( t \)-transitive if for every pair of \( t \)-element ordered sets of coordinates, there is an element \( DP\gamma \) of the automorphism group \( \text{Aut}(C) \) such that its permutation part \( P \) sends the first set to the second set. The automorphism group \( \text{Aut}(C) \) is said to be \( t \)-homogeneous if for every...
pair of \( t \)-subsets of coordinates, there is an element \( D \gamma \) of the automorphism group \( \text{Aut}(C) \) such that its permutation part \( P \) sends the first set to the second set.

Let \( n \) be a prime. The projective special linear group \( \text{PSL}_2(n) \) consists of all permutations of the set \{\( \infty \)\} \( \cup \) \( \text{GF}(n) \) of the following form:

\[
\tau(a,b,c,d)(x) = \frac{ax + c}{bx + d}
\]

with \( ad - bc = 1 \), and the following conventions:

- \( \frac{a}{b} = \infty \) for all \( a \in \text{GF}(n)^* \).
- \( \frac{a + c}{b + d} = \frac{a}{b} \).

The set of all such permutations is a group under the function composition operation. It is known that \( \text{PSL}_2(n) \) is generated by the following two permutations [5, p. 491]:

\[
S: y \mapsto y + 1,
\]
\[
T: y \mapsto -\frac{1}{y}.
\]

The permutation group \( \text{PSL}_2(n) \) has a number of interesting properties, and has applications in both mathematics and engineering.

Let \( C \) be a cyclic code of length \( n \) over \( \text{GF}(q) \), where \( q \) is a prime power with \( \gcd(n,q) = 1 \). Let \( \overline{C} \) denote the extended code of \( C \), where \( \infty \) is used to index the extended coordinate and other coordinates are indexed by the elements of \( \text{GF}(n) \). For any codeword \( \overline{\tau} = (c_0, c_1, \cdots, c_{n-1}, c_\infty) \) in \( \overline{C} \), any permutation \( \tau \) of \{\( \infty \)\} \( \cup \) \( \text{GF}(n) \) acts on \( \overline{\tau} \) as follows:

\[
\tau(\overline{\tau}) = (c_{\tau(0)}, c_{\tau(1)}, \cdots, c_{\tau(n-1)}, c_{\tau(\infty)}).
\]

The extended code \( \overline{C} \) is said to be invariant under \( \text{PSL}_2(n) \) if

\[
\text{PSL}_2(n)(\overline{C}) = \overline{C}.
\]

In other words, the extended code \( \overline{C} \) is invariant under \( \text{PSL}_2(n) \) if the permutation part of the automorphism group \( \text{Aut}(\overline{C}) \) contains \( \text{PSL}_2(n) \).

Linear codes that are invariant under \( \text{PSL}_2(n) \) have interesting properties. It is well known that the extended odd-like quadratic residue codes are invariant under \( \text{PSL}_2(n) \). The objective of this paper is to prove that the only such binary codes are the extended odd-like quadratic residue codes and the trivial codes.

2. Motivations of this paper

Let \( \mathcal{P} \) be a set of \( v \geq 1 \) elements, and let \( \mathcal{B} \) be a set of \( k \)-subsets of \( \mathcal{P} \), where \( k \) is a positive integer with \( 1 \leq k \leq v \). Let \( t \) be a positive integer with \( t \leq k \). The pair \( \mathbb{D} = (\mathcal{P}, \mathcal{B}) \) is called a \( t-(v,k,\lambda) \) design [3], or simply \( t \)-design, if every \( t \)-subset of \( \mathcal{P} \) is contained in exactly \( \lambda \) elements of \( \mathcal{B} \). The elements of \( \mathcal{P} \) are called points, and those of \( \mathcal{B} \) are referred to as blocks. We use \( b \) to denote the number of blocks in \( \mathcal{B} \). A \( t \)-design is called simple if \( \mathcal{B} \) does not contain any repeated blocks. A \( t \)-design is called symmetric if \( v = b \). It is clear that \( t \)-designs with \( k = t \) or \( k = v \)
always exist. Such \( t \)-designs are trivial. A \( t-(v,k,\lambda) \) design is referred to as a Steiner system if \( t \geq 2 \) and \( \lambda = 1 \), and is denoted by \( S(t,k,v) \).

Let \( C \) be a \([v,k,d]\) linear code over \( \text{GF}(q) \). Let \( A_i := A_i(C) \) be the number of codewords with Hamming weight \( i \) in \( C \), where \( 0 \leq i \leq v \). The sequence \((A_0, A_1, \ldots , A_v)\) is called the weight distribution of \( C \), and \( \sum_{i=0}^{v} A_i z^i \) is referred to as the weight enumerator of \( C \). For each \( k \) with \( A_k \neq 0 \), let \( B_k \) denote the set of supports of all codewords of Hamming weight \( k \) in \( C \), where the coordinates of a codeword are indexed by \( \{0, 1, 2, \ldots , v-1\} \). Let \( \mathcal{P} = \{0, 1, 2, \ldots , v-1\} \). The pair \((\mathcal{P}, B_k)\) may be a \( t-(v,k,\lambda) \) design for some positive integer \( \lambda \) and appropriate \( t \geq 0 \), and is called a design supported by the code \( C \). In this case, we say that \( C \) holds a \( t-(v,k,\lambda) \) design. If \( C \) holds a \( t-(v,k,\lambda) \) design, its dual code \( C^\perp \) admits majority-logic decoding up to a certain number of errors determined by the design parameters \([8],[7],[9, Section 8]\).

A classical approach to obtain \( t \)-designs from linear codes is by using the automorphism groups of linear codes \([1],[4],[5],[9],[10]\). A proof of the following theorem can be found in \([4, p. 308]\).

**Theorem 1.** Let \( C \) be a linear code of length \( n \) over \( \text{GF}(q) \) where \( \text{Aut}(C) \) is \( t \)-transitive. Then the codewords of any weight \( i \geq t \) of \( C \) hold a \( t \)-design.

The following theorem can be derived directly from Propositions 4.6 and 4.8 in \([2]\) (see also \([10, 1.27]\)).

**Theorem 2.** Let \( C \) be a linear code of length \( n \) over \( \text{GF}(q) \) where \( \text{Aut}(C) \) is \( t \)-homogeneous. Then the codewords of any weight \( i \geq t \) of \( C \) hold a \( t \)-design.

The two theorems above give a sufficient condition for a linear code to hold \( t \)-designs. To apply Theorems 1 and 2 we need to determine the automorphism group of \( C \) and show that it is \( t \)-transitive or \( t \)-homogeneous.

The projective special linear group \( \text{PSL}_2(n) \) is \( 2 \)-transitive for all primes \( n \) and \( 3 \)-homogeneous for primes \( n \equiv \pm 1 \text{ (mod 4)} \) on the set \( \{0, 1, \ldots , n-1, \infty\} \). It is known that the extended odd-like quadratic residue codes are invariant under \( \text{PSL}_2(n) \). Hence, the extended quadratic residue codes hold an infinite family of 2-designs for \( n \equiv 1 \text{ (mod 4)} \), an infinite family of 3-designs for \( n \equiv 3 \text{ (mod 4)} \). To construct more \( t \)-designs with \( t \in \{2, 3\} \), one would search for other extended cyclic codes over finite fields that are invariant under the action of \( \text{PSL}_2(n) \) (cf. \([4],[5]\)). This is the main motivation of this paper.

### 3. Binary linear codes invariant under \( \text{PSL}_2(n) \)

#### 3.1. All binary linear codes invariant under \( \text{PSL}_2(n) \) are extended cyclic codes

**Theorem 3.** Let \( n \) be an odd prime. Let \( \tilde{C} \) be a binary linear code of length \( n+1 \) and \( \tilde{C} \neq \text{GF}(2)^{n+1} \). If \( \tilde{C} \) is invariant under \( \text{PSL}_2(n) \), then \( \tilde{C} \) is an extended cyclic code.

**Proof.** If \( \tilde{C} \) has dimension 0 or \( n+1 \) the conclusion is obviously true. We now assume that

\[
1 \leq \dim \left( \tilde{C} \right) \leq n-1.
\]

Let \( \tilde{C} \) be the punctured code of \( \tilde{C} \) at coordinate \( \infty \). Since the permutation \( \tau_i(x) = x+i \) in \( \text{PSL}_2(n) \) acts cyclically on the coordinates \( (0, 1, \ldots , n-1) \) when \( i = 1 \), \( \tilde{C} \) must be a cyclic code.
Since \( \dim(\tilde{C}) \geq 1 \), any minimum weight codeword in \( \tilde{C} \) has weight at least one. Suppose \( \tilde{C} \) has a codeword \( c \) with Hamming weight 1. If
\[
eq (0, 0, \cdots, 0, 1) \in \tilde{C},
\]
then the permutation
\[
T(x) = -\frac{1}{x}
\]
will transform \( c \) into the codeword
\[
T(c) = (1, 0, \cdots, 0)
\]
in \( \tilde{C} \). Note that the permutation \( \tau_i(x) = x + i \) in PSL\(_2(n)\) will transform \( T(c) \) into the following codeword
\[
(0, \cdots, 0, 1, 0, \cdots, 0),
\]
where the nonzero bit 1 could be in any coordinate \( i \) with \( 0 \leq i \leq n - 1 \). This means that all codewords of weight 1 are in \( \tilde{C} \). Consequently, \( \tilde{C} = \text{GF}(q)^{n+1} \). This is contrary to the assumption that \( \tilde{C} \neq \text{GF}(2)^{n+1} \). This proves that the minimum distance \( d(\tilde{C}) > 1 \). It then follows from Theorem 1.5.1 in [4] that
\[
\dim(C) = \dim(\tilde{C}).
\]
Next we prove that \( \tilde{C} \) is the extended code of \( C \).

Let \( g(x) = \sum_{i=0}^{n-1} a_i x^i \) be the generator polynomial of \( C \), and let \( \deg(g) = n - 1 - k \), i.e. \( \dim(C) = k \). Then the first \( k \) cyclic shifts of the codeword \( \mathbf{a}_0 = (a_0, a_1, \ldots, a_{n-1}) \) form a basis of \( \tilde{C} \). Denote these first \( k \) cyclic shifts by \( \{a_j \mid 0 \leq j \leq k - 1\} \), and their corresponding codewords in \( \tilde{C} \) as \( \tilde{a}_j, 0 \leq j \leq k - 1 \), which form a basis of \( \tilde{C} \). Then we must have \( \tilde{a}_j = \tau_{j}(\mathbf{a}_0) \) for \( 0 \leq j \leq k - 1 \), where \( \tau_{j}(\infty) = \infty \), which shows that the \( \infty \) coordinate of \( \tilde{a}_j \) is the same for \( 0 \leq j \leq k - 1 \).

Let \( \tilde{a}_0 = (a_0, a_1, \ldots, a_{n-1}, a_{\infty}) \). If \( g(1) \neq 0 \), the extended coordinate of \( \tilde{a}_0 \) should be 1. Assume that \( \tilde{C} \) is not the extended code of \( C \), which is equivalent to \( a_{\infty} = 0 \). Since the \( \infty \) coordinate of \( \tilde{a}_j \) is the same for \( 0 \leq j \leq k - 1 \), the \( \infty \) coordinate of all codewords in \( \tilde{C} \) is 0. However, as PSL\(_2(n)\) is transitive, there exists a permutation and a codeword that transfer a 1 in the codeword to the \( \infty \) coordinate, which gives a contradiction. Thus, in this case, \( \tilde{C} \) is the extended code of \( C \).

If \( g(1) = 0 \), then \( \tilde{C} \) is an even-weight code and all extended coordinates in its extended code should be 0. Assume \( \tilde{C} \) is not the extended code of \( C \), which is equivalent to \( a_{\infty} = 1 \). Since \( n \) is odd and \( (a_0, a_1, \ldots, a_{n-1}) \) has even weight, there exists an integer \( i \) such that \( a_i = 0 \), where \( 0 \leq i \leq n - 1 \). Since PSL\(_2(n)\) acts transitively on \( \{0, 1, \ldots, n-1, \infty\} \), there must be a permutation \( \tau \in \text{PSL}_2(n) \) that exchanges coordinate \( i \) with coordinate \( \infty \). We have then
\[
\tau(a_0, a_1, \ldots, a_{n-1}, a_{\infty}) = (a_0, \cdots, a_{i-1}, 1, a_{i+1}, \cdots, a_{n-1}, 0)
\]
which is another codeword in \( \tilde{C} \). It then follows that
\[
(a_0, \cdots, a_{i-1}, 1, a_{i+1}, \cdots, a_{n-1}) \in C,
\]
which has odd weight. This is contrary to our assumption that \( C \) has only even weights. \( \square \)
3.2. The main theorem

The main result of this paper is the following.

**Theorem 4.** Let $n$ be an odd prime. If $\tilde{C}$ is a binary code of length $n+1$ invariant under $\text{PSL}_2(n)$, then $\tilde{C}$ must be one of the following:

1. the zero code $C(0) = \{(0,0,\cdots,0)\}$; or
2. the whole space $C(n+1)$, which is the dual of $C(0)$; or
3. the code $C(1) = \{(0,0,\cdots,0),(1,1,\cdots,1)\}$ of dimension $1$; or
4. the code $C(1)^\perp$, denoted by $C(n)$, given by
   $$C(n) = \left\{(c_0,c_1,\cdots,c_n) \in \text{GF}(2)^{n+1} : \sum_i c_i = 0 \right\};$$

   or

5. the extended code of one of the two odd-like quadratic residue binary codes of length $n$.

According to Theorem 4, to prove Theorem 4, we need to consider only extended cyclic codes. Before proving Theorem 4, we need to do some preparations. Specifically, we will make use of the defining set of a cyclic code, and the Fourier transform (also called the Mattson-Solomon polynomial) of a codeword.

Note that $\gcd(2,n) = 1$. Let $m$ denote the order of 2 modulo $n$. The 2-cyclotomic coset $C_i$ modulo $n$ containing $i$ is defined by

$$C_i = \{i,2i,2^2i,\cdots,2^{\ell_i}i\} \mod n,$$

where $\ell_i$ is the least positive integer such that $2^{\ell_i} \equiv i \pmod{n}$. The smallest non-negative integer in $C_i$ is called the coset leader of $C_i$. Let $\Gamma_{(2,n)}$ denote the set of all coset leaders of the 2-cyclotomic cosets modulo $n$. Then $\{C_i : i \in \Gamma_{(2,n)}\}$ is a partition of the set $\mathbb{Z}_n = \{0,1,\cdots,n-1\}$. We identify $\mathbb{Z}_n$ with $\text{GF}(n)$.

Let $\alpha$ be a generator of $\text{GF}(2^m)^*$, and let $\beta = \alpha^{(2^m-1)/n}$. Then $\beta$ is a $n$-th primitive root of unity in $\text{GF}(2^m)$. It is straightforward to see that the minimal polynomial $\mathbb{M}_\beta(x)$ over $\text{GF}(2)$ of $\beta^i$ is given by

$$\mathbb{M}_\beta(x) = \prod_{j \in C_i} (x - \beta^j). \tag{3}$$

Clearly,

$$x^n - 1 = \prod_{j \in \Gamma_{(2,n)}} \mathbb{M}_\beta(x).$$

The generator polynomial $g(x)$ of any cyclic code $C$ over $\text{GF}(2)$ of length $n$ must be the product of some of irreducible polynomials $\mathbb{M}_\beta(x)$. The set

$$T = \{0 \leq i \leq n-1 : g(\beta^i) = 0\}$$

is called the defining set of the cyclic code $C$ with respect to $\beta$, and must be the union of some 2-cyclotomic cosets.

The Fourier transform of a vector $c = (c_0,c_1,\cdots,c_{n-1}) \in \text{GF}(2)^n$, denoted by $C = (C_0,C_1,\cdots,C_{n-1}) \in \text{GF}(2^m)^n$, is given by

$$C_j = \sum_{i=0}^{n-1} \beta^{ij} c_i = c(\beta^j),$$

where $\beta$ is a $n$-th primitive root of unity in $\text{GF}(2^m)$.
where \( c(x) = \sum_{i=0}^{n-1} c_i x^i \in \text{GF}(2)[x] \) and \( 0 \leq j \leq n - 1 \).

Let \( C \) be a cyclic code of prime length \( n \) and \( \pi \) be a primitive element of \( \text{GF}(n) \). Then indices in \( \text{GF}(n) \) can be expressed by powers of \( \pi \), and codewords of \( C \) can be reordered accordingly as
\[
c = (c_0, c_{\pi^1}, c_{\pi^2}, \ldots, c_{\pi^{n-2}}).
\]
Note that \( \pi^{-1} \) is another generator of \( \text{GF}(n)^* \). Similarly, the Fourier transform \( C \) of \( c \) can be written in the permuted order,
\[
C = (C_0, C_{\pi^{-1}}, \ldots, C_{\pi^{-(n-2)}}).
\]
Rewrite the Fourier transform as
\[
C_{\pi^s} = c_0 + \sum_{r=0}^{n-2} \beta^{r-s} c_r, \quad s = 0, 1, \ldots, n - 2.
\]
Let \( C'_s = C_{\pi^s} \) and \( c'_r = c_r \) for \( 0 \leq s, r \leq n - 2 \). Then
\[
C'_s = c_0 + \sum_{r=0}^{n-2} \beta^{r-s} c'_r, \quad s = 0, 1, \ldots, n - 2.
\]
This can be rewritten in the language of polynomials. Define
\[
u(x) = \sum_{r=0}^{n-2} \beta^{r-s} x^r, \quad c'(x) = \sum_{r=0}^{n-2} c'_r x^r, \quad C'(x) = \sum_{r=0}^{n-2} c_r x^r.
\]
Then all the equations in (5) can be compactly expressed into
\[
C'(x) = \left( \nu(x)c'(x) + c_0 \sum_{i=0}^{n-2} x^i \right) \mod (x^{n-1} - 1),
\]
which is a polynomial representation of the equation of the Fourier transform.

Let \( h = (n - 1)/m \). Since \( m \) is the order of 2 modulo \( n \), there exists a primitive element \( \pi \) of \( \text{GF}(n) \) for which \( \pi^h = 2 \) in \( \text{GF}(n) \). Then the nonzero elements of \( \text{GF}(n) \) can be presented as \( \{\pi^r : 0 \leq r \leq n - 2\} \). We denote the set of quadratic residues in it by \( Q := \{\pi^{2r} : 0 \leq r \leq (n - 1)/2\} \) and the set of quadratic nonresidues by \( \mathcal{X} \).

Recall that \( \beta = \alpha^{2^{m-1}/n} \), which is a \( n \)-th primitive root of unity in \( \text{GF}(2^m) \), where \( \alpha \) is a generator of \( \text{GF}(2^m)^* \). We now prove the following lemma.

**Lemma 5.** Let notation and symbols be as before. Define \( \beta_l = \beta^l \) for \( 1 \leq l \leq n - 1 \). Then \( \{1, \beta, \ldots, \beta_{n-1}^{-1}\} \) is a basis of \( \text{GF}(2^m) \) over \( \text{GF}(2) \). Consequently, for any element \( a \in \text{GF}(2^m) \), there exists a polynomial \( f(x) \) over \( \text{GF}(2) \) with degree less than \( m \) such that \( f(\beta^l) = a \).

**Proof.** Let \( \mathcal{C}_l = \{l, 2l, \ldots, 2^m l^{-1}\} \mod n \) be the \( 2 \)-cyclotomic coset modulo \( n \) containing \( l \), where \( 1 \leq l \leq n - 1 \). Recall that \( n \) is a prime. By definition, \( m = \text{ord}_n(l) \). Let \( w \) be a positive integer such that \( 2^w \equiv l \pmod{n} \). Then \( l(2^w - 1) \equiv 0 \pmod{n} \). Since \( n \) is a prime and \( 1 \leq l \leq n - 1 \), \( 2^w \equiv 1 \pmod{n} \). It then follows from \( m = \text{ord}_n(2) \) that \( w \geq m \). Consequently, \( |\mathcal{C}_l| = m \). Hence, the polynomial \( \mathcal{M}_{\beta^l}(x) \) of (3) has degree \( m \) and is irreducible. This is the minimal polynomial of \( \beta^l \) over \( \text{GF}(2) \). It then follows that \( \{1, \beta, \ldots, \beta_{m-1}^{-1}\} \) is a basis of \( \text{GF}(2^m) \) over \( \text{GF}(2) \). Consequently, for any element \( a \in \text{GF}(2^m) \), there exists a polynomial \( f(x) \) over \( \text{GF}(2) \) with degree less than \( m \) such that \( f(\beta^l) = a \). \[\square\]
Let $C$ be a cyclic code over $\text{GF}(2)$ with length $n$ and defining set $T$. Denote the extended code of $C$ by $\overline{C}$, where the extended coordinate $c_\infty$ is defined by

$$c_\infty = \sum_{i=0}^{n-1} c_i$$

for any codeword $c = (c_0, c_1, \ldots, c_{n-1}) \in C$.

Consider now the permutation $T : y \mapsto -1/y$ in $\text{PSL}_2(n)$. For any $\bar{c} = (c_0, c_1, \ldots, c_{n-1}, c_\infty) \in \overline{C}$, let $\bar{d} = (d_0, d_1, \ldots, d_{n-1}, d_\infty)$ be the permuted vector of $c$ under $T$. Let $C$ and $D$ be the Fourier transforms of $(c_0, c_1, \ldots, c_{n-1})$ and $(d_0, d_1, \ldots, d_{n-1})$, respectively. Define the polynomials $D'(x) = \sum_{s=0}^{n-1} D'_s x^s$. We have the following relationship between $D'(x)$ and $C'(x)$, which was stated in [3] without a proof. We state it as a general result here and present a proof.

**Lemma 6.** Let $D'(x)$, $C'(x)$, $u(x)$ be defined as above. We have

$$D'\left(\frac{1}{x}\right) = u(x)^2 C'(x) \pmod{x^{n-1} - 1}. \quad (7)$$

**Proof.** The inverse Fourier transform can be written as

$$c_i = C_0 + \sum_{k=1}^{n-1} \beta^{-ik} C_k = c_\infty + \sum_{k=1}^{n-1} \beta^{-ik} C_k.$$

Now we have

$$d_i = c_{-i/\pi} = c_\infty + \sum_{k=1}^{n-1} \beta^{(1/i)k} C_k, \quad i = 1, 2, \ldots, n-1,$$

and $d_0 = c_\infty$. Consequently, for $1 \leq j \leq n-1$,

$$D_j = d_0 + \sum_{i=1}^{n-1} \beta^j d_i$$

$$= c_\infty + \sum_{i=1}^{n-1} \beta^j \left( c_\infty + \sum_{k=1}^{n-1} \beta^{(1/i)k} C_k \right)$$

$$= \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} \beta^{(1/i)k} C_k. \quad (8)$$

Now we change indices again as follows:

$$i = \pi^r, \quad k = \pi^s, \quad j = \pi^{-t},$$

with $r, s, t \in \mathbb{Z}_{n-1}$. Then (8) becomes

$$D_{\pi^{-t}} = \sum_{r=0}^{n-2} \beta^{\pi^{-t}r} \sum_{s=0}^{n-2} \beta^{\pi^s} C_{\pi^r}. \quad (9)$$

Alternatively, (9) can be expressed as

$$D'_s = \sum_{r=0}^{n-2} u_{s-r} \sum_{t=0}^{n-2} u_{r-t} C'_t. \quad (10)$$
This completes the proof.

We are now ready to prove the following.

**Theorem 7.** Let $\mathcal{C}$ be a binary cyclic code of length $n$ with defining set $T$. Assume the extended code $\overline{\mathcal{C}}$ is invariant under $\text{PSL}_2(n)$. If there exists an $l \in \mathbb{Q}$ ($\mathbb{N}$, respectively) that is not in $T$, then $\mathbb{Q} \cap T = \emptyset$ ($\mathbb{N} \cap T = \emptyset$, respectively). Further, if $0 \in T$, then $\mathcal{C}$ must be the zero code.

**Proof.** Let $c = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$ be a codeword of $\mathcal{C}$ and let the corresponding polynomials $\mathcal{C}'(x)$, $D'(x)$ and $u(x)$ be the same as before. Lemma 6 says that

$$D'(1/x) = \sum_{r=0}^{n-2} D'_r x^{n-1-r} = u^2(x) \mathcal{C}'(x) \bmod (x^{n-1} - 1),$$

where $u(x) = \sum_{r=0}^{n-2} \beta^{w-r} x^r$.

We first consider the case that there exists an $l = \pi^2 u \in \mathbb{Q}$ such that $l \notin T$, where $0 \leq u \leq (n-3)/2$. To show that $T \cap \mathbb{Q} = \emptyset$, we need to prove the following statement.
Statement: For any even number $2s$ with $0 \leq 2s \leq n - 2$, there exists a codeword $c(s) \in C$ such that the corresponding term $D'_x x^{n-1-2s}$ in Equation (11) is nonzero, i.e. $D'_x \neq 0$.

Proof of the Statement: Let $\overline{M}_p(x) = (x^n - 1)/M_p(x)$ be a polynomial in $\mathbb{GF}(2)[x]$. For any $0 \leq s \leq (n - 3)/2$, define a codeword $c \in C$ by the following polynomial.

$$c(x) = a(x) \cdot \overline{M}_p(x),$$

where $a(x)$ is a polynomial of degree less than $m$ over $\mathbb{GF}(2)$, which will be figured out later. Since $l \notin T$, clearly $c(x)$ is a codeword in $C$ with $C_i = c(B') = 0$ for any $i \notin C_i$. Besides, for $i = l \cdot 2^w \in C_i$ it is easy to see that $C_{2w,i} = (C_i)^{2w}$ for $0 \leq w \leq m - 1$. Equivalently, $C_{i} = C_{i'} = \begin{cases} (C_{2w})^{2r} = C_i^{2w}, & r = 2u + hw, 0 \leq w \leq m - 1; \\ 0, & \text{otherwise.} \end{cases}$

From Equation (11) we have

$$D'(1/x) = g^2(x)C'(x) \pmod{x^{n-1} - 1}$$

$$= \left( \sum_{r=0}^{n-2} \beta^{2x^r} x^r \right)^2 \cdot \left( \sum_{w=0}^{m-1} C_{2w+h,w}^2 x^{2w+h} \right) \pmod{x^{n-1} - 1}$$

$$= x^{2n} \left( \sum_{r=0}^{n-2} \beta^{2x^r} x^r \right)^2 \cdot \left( \sum_{w=0}^{m-1} C_{2w}^2 x^{2w} \pmod{x^{n-1} - 1} \right)$$

We continue our discussion by distinguishing the following two circumstances. First we consider the case that $2 = \pi^h$ is a quadratic number in $\mathbb{GF}(n)$. Let $h' = h/2$. It then follows from (12) that

$$D'(1/x) = x^{2n} \sum_{r=0}^{n-2} \sum_{w=0}^{m-1} \beta^{2x^r} C_i^{2w} x^{2r+2hw} \pmod{x^{n-1} - 1}$$

$$= x^{2n} \sum_{w=0}^{m-1} C_i^{2w} \sum_{s=0}^{n/2} (\beta^{2x^s})^{x^s} x^{2s} \pmod{x^{n-1} - 1}$$

$$= x^{2n} \sum_{w=0}^{m-1} C_i^{2w} \sum_{s=0}^{n/2} (\beta^{2x^s})^{x^s} x^{2s} \pmod{x^{n-1} - 1}$$

$$= x^{2n} \left( \sum_{s=0}^{n/2} C_i^{2w} \beta^{2x^s} x^{2s} \right) x^{2s} \pmod{x^{n-1} - 1}$$

$$= x^{2n} \left( \sum_{w=0}^{m-1} C_i^{2w} \beta^{2x^s} x^{2s} \right) x^{2s} + x^{2n} \sum_{s=0}^{n/2} \left( \sum_{w=0}^{m-1} C_i^{2w} \beta^{2x^s} x^{2s} \right) x^{2s} \pmod{x^{n-1} - 1}$$

$$= x^{2n} \left( \sum_{s=0}^{n/2} C_i^{2w} x^{2s} \right) x^{2s} + x^{2n} \sum_{w=0}^{m-1} \left( \sum_{s=0}^{n/2} C_i^{2w} \beta^{2x^s} x^{2s} \right) x^{2s} \pmod{x^{n-1} - 1}$$

$$= x^{2n} \left( \sum_{s=0}^{n/2} \beta^{2x^s} x^{2s} + \beta^{-2x^s} x^{2s} \right) C_i^{2w} x^{2s} \pmod{x^{n-1} - 1}.$$

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Thus for any $0 \leq s \leq (n-3)/2$, by comparing terms of both sides we have

$$D_s' = \sum_{w=0}^{m-1} \left( \beta^{2\pi^w u + \pi^w u + w + u} + \beta^{2\pi^w u + \pi^w u + w} \right) C_{2w}' = \sum_{w=0}^{m-1} \left( \beta^{2\pi^w u + u + w + u} + \beta^{-2\pi^w u + u + w} \right) C_{2w}'$$

Define a polynomial $L_s(x)$ over $\text{GF}(2^n)$ to be

$$L_s(x) = \sum_{w=0}^{m-1} \left( \beta^{2\pi^w u + \pi^w u + w + u} + \beta^{2\pi^w u + \pi^w u} \right) x^w.$$

$L_s(x)$ is a linearized function independent of $c$ and $D_s' = L_s(C_t)$. We now prove that $L_s(x)$ is not the zero function for any $s$. Notice that the degree of $L_s(x)$ is at most $2m-1$. It suffices to prove that

$$\beta^{2\pi^w u + \pi^w u + w} + \beta^{-2\pi^w u + \pi^w u} \neq 0$$

for one $w$ with $0 \leq w \leq m-1$. We do this for $w = 0$. On the contrary, suppose that

$$\beta^{2\pi^w u + \pi^w u + w} + \beta^{-2\pi^w u + \pi^w u} = 0$$

for $w = 0$. We have then $\beta^{2\pi^w u + \pi^w u} + \beta^{-2\pi^w u} = 0$, which is the same as $\beta^{4\pi^w u} = 1$. Note that $x \mapsto x^4$ is a permutation on $\text{GF}(2^n)$. We obtain that $\beta^{4\pi^w u} = 1$. Obviously, $\pi^{w_u} \mod n$ is an integer in the set $\{1, 2, \cdots, n-1\}$. Since $\beta$ is an $n$-th primitive root of unity in $\text{GF}(2^n)$, $\beta^{4\pi^w u} \neq 1$. Thus, we have reached a contradiction. Therefore, $L_s(x)$ is a nonzero function. Consequently, there is an element $\gamma$ in $\text{GF}(2^n)$ such that $L_s(\gamma) \neq 0$. By Lemma 5 there exists a polynomial $a_s(x) \in \text{GF}(2)[x]$ of degree less than $m$ such that $a_s(\beta^i) = \gamma^{m_i(\beta^i)(\beta^i)^{-1}}$. Set the $a(x)$ in the definition of $c(x)$ to be $a_s(x)$. Then we have

$$C_{2w} = C_t = c(\beta^i) = a_s(\beta^i) \cdot \gamma^{m_i(\beta^i)} = \gamma.$$  

Thus $D_s' = L_s(C_{2w}) = L_s(\gamma) \neq 0$.

Finally, we consider the case that $2 = \pi^h \in \mathcal{P}$. Then $h$ is odd and $m = (n-1)/h$ is even.
Theorem 8. Let $C$ be a binary cyclic code with prime length $n > 2$ and $0 < \dim(C) < n$. Let $\overline{C}$ be invariant under $\text{PSL}_2(n)$. Let $g(x)$ denote the generator polynomial of $C$.

If $n \equiv \pm 3 \pmod{8}$, then $g(x) = (x^4 - 1)/(x - 1)$.

If $n \equiv \pm 1 \pmod{8}$, then $g(x) = (x^8 - 1)/(x - 1)$, or $g(x)$ is the generator polynomial of one of the odd-like quadratic residue binary codes.

Proof. Let $C$ be a binary cyclic code with prime length $n > 2$ and $0 < \dim(C) < n$. Let $\overline{C}$ be invariant under $\text{PSL}_2(n)$. By Theorem 3, $\dim(C) \in \{1, (n + 1)/2\}$. 

Equation (12) becomes

$$D'(1/x) = x^2 \left( \sum_{r=0}^{m/2-1} \sum_{w=0}^{n-2} \beta^{2\pi r} x^{2r} \left( \sum_{r=0}^{m/2-1} \sum_{w=0}^{n-2} \beta^{2\pi r} x^{2bw} + C_t^{22w} x^{2bw} + C_t^{22w+1} x^{2bw+h} \right) \right) \pmod{x^{n-1} - 1}$$

$$= x^2 \left( \sum_{r=0}^{m/2-1} \sum_{w=0}^{n-2} \beta^{2\pi r} x^{2r} \left( C_t^{22w} x^{2bw} + C_t^{22w+1} x^{2bw+h} \right) \right) \pmod{x^{n-1} - 1}$$

$$= x^2 \left( \sum_{r=0}^{m/2-1} \sum_{w=0}^{n-2} \beta^{2\pi r} x^{2r} \left( C_t^{22w} x^{2bw} + C_t^{22w+1} x^{2bw+h} \right) \right) \pmod{x^{n-1} - 1}$$

Again by comparison, for any $0 \leq s \leq (n - 3)/2$ we have

$$D_{2s}' = \sum_{w=0}^{m/2-1} \left( \beta^{2\pi s} + \beta^{-2\pi s} \right) C_t^{22w} \pmod{x^{n-1} - 1}$$

With similar arguments to the case $2 \notin Q$, we can find a $c_t \in C$ such that $D_{2s}' \neq 0$. We hereby finish the proof of the statement.

By the statement above, for any $\pi^{2s} \in Q$, there exists a codeword $c(s) \in C$ such that $d(\pi^{2s}) = D_{2s}' = D_{2s}' \neq 0$, where $d(x) = \sum_{i=0}^{n-1} d_i x^i$. Since $\overline{C}$ is also a codeword of $\overline{C}$, $d(\pi^{2s}) \neq 0$ leads to $\pi^{2s} \notin T$. Thus we proved that $Q \cap T = \emptyset$.

For the case that there exists an $l = \pi^{2s+1} \in \mathcal{N}$ such that $l \notin T$, the desired conclusion can be similarly proved.

The conclusion of the last part is implied by the proof of Theorem 3. \qed
It is well known that 2 is a quadratic residue modulo \( n \) if and only if \( n \equiv \pm 1 \pmod{8} \). In the case that \( n \equiv \pm 1 \pmod{8} \), 2 is a quadratic residue. By Theorem 4 we have only the following four possibilities:

1. The defining set \( T = Q \cup N \). In this subcase, \( g(x) = (x^n - 1)/(x - 1) \).
2. The defining set \( T = Q \). In this subcase,
   \[
g(x) = \prod_{i \in Q} (x - \beta^i).
   \]
3. The definition set \( T = N \). In this subcase,
   \[
g(x) = \prod_{i \in N} (x - \beta^i).
   \]

Consider now the case \( n \equiv \pm 3 \pmod{8} \). In this case, 2 must be a quadratic nonresidue. Hence, \( (n - 1)/\ord_n(2) \) must be odd. Note that \( (x^n - 1)/(x - 1) \) is the product of \( (n - 1)/\ord_n(2) \) irreducible polynomials of degree \( \ord_n(2) \) over \( \mathbb{F}_2 \). Since \( (n - 1)/\ord_n(2) \) is odd, there is no binary cyclic code of length \( n \) and dimension \( (n + 1)/2 \). It then follows that \( C \) must have generator polynomial \( (x^n - 1)/(x - 1) \). Another way to prove this conclusion goes as follows. If \( T \) contains a quadratic residue \( a \) modulo \( n \), then \( T \) must contain \( 2a \mod n \), which is a quadratic nonresidue. If \( T \) contains a quadratic nonresidue \( b \) modulo \( n \), then \( T \) must contain \( 2b \mod n \), which is a quadratic residue. Hence, \( C \) must have defining set \( T = \{1, 2, \cdots, n - 1\} \), and thus generator polynomial \( (x^n - 1)/(x - 1) \).

**Proof of Theorem 4**

It is known that the two odd-like quadratic residue binary codes are invariant under \( \text{PSL}_2(n) \) (see, for example, [3]). It is easily seen that the four trivial binary codes \( C(0), C(1), C(n + 1) \) and \( C(n) \) of length \( n + 1 \) are invariant under \( \text{PSL}_2(n) \).

Let \( \tilde{C} \) be a binary code of length \( n + 1 \) which is invariant under \( \text{PSL}_2(n) \). By Theorem 3, \( \tilde{C} \) is either the whole space \( \mathbb{F}_2^{n+1} \) or an extended cyclic code. The desired conclusion of the other part then follows from Theorem 3.

**4. Concluding remarks**

The main result of this paper tells us that the only nontrivial binary linear codes of length \( n + 1 \) that are invariant under \( \text{PSL}_2(n) \), where \( n \) is an odd prime, are the extended codes of the two odd-like quadratic residue codes of length \( n \equiv \pm 1 \pmod{8} \). This means that the extended quadratic residue codes are very special. When \( n \equiv -1 \pmod{8} \), the extended quadratic residue codes over \( \mathbb{F}_2 \) are self-dual and hold 3-designs.

A self-dual binary code with parameters \([N, N/2, d]\) is said to be *extremal and of Type II* if

\[
d = 4 \left\lfloor \frac{N}{24} \right\rfloor + 4.
\]

It is known that only finitely many Type II extremal codes could exist. In fact, Type II extremal codes of length \( N \) are known only for the following \( N \):

\[8, 16, 24, 32, 40, 48, 56, 64, 80, 88, 104, 112, 136.\]
Among these Type II extremal codes, those with length in \{8, 24, 32, 48, 80, 104\} are extended quadratic residue codes [6]. This fact shows another specialty of the quadratic residue codes.

Experimental data indicates that the main conclusion of this paper (i.e., Theorem 4) is also true for linear codes over \(\text{GF}(q)\) for any prime power \(q\). However, even if this is indeed true, it may not be easy to prove it. The reader is cordially invited to settle this open problem.

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