Wonderful models for generalized Dowling arrangements

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Abstract

For any triple given by a positive integer $n$, a finite group $G$, and a faithful representation $V$ of $G$, one can describe a subspace arrangement whose intersection lattice is a generalized Dowling lattice in the sense of Hanlon [10]. In this paper we construct the minimal De Concini-Procesi wonderful model associated to this subspace arrangement and give a description of its boundary. Our aim is to point out the nice poset provided by the intersections of the irreducible components in the boundary, which provides a geometric realization of the nested set poset of this generalized Dowling lattice. It can be represented by a family of forests with leaves and labelings that depend on the triple $(n, G, V)$ and we will study it from the enumerative point of view.

1 Introduction

The De Concini-Procesi wonderful models of subspace arrangements were introduced in [5] and [4] and play since then a crucial role in the study of configuration spaces and in various other various fields of mathematics. The relevance of their combinatorial properties and their relation with discrete geometry were pointed out for instance in [8], [9], [18], [19], [13]; their relations with Bergman fans, toric and tropical geometry were enlightened in [16] and [6]; the connections between the geometry of these models and the Chow rings of matroids were pointed out first in [11] and then in [1], where they also played a crucial role in the study of some log-concavity problems.

Let us recall their definition: given a subspace arrangement $\mathcal{G}$ in $(\mathbb{C}^n)^*$, we consider for each $D \in \mathcal{G}$ its annihilator $D^\perp$ in $\mathbb{C}^n$ and the projective space $\mathbb{P}(\mathbb{C}^n/D^\perp)$. Let $\mathcal{A}(\mathcal{G})$ be the complement of $\bigcup_{D \in \mathcal{G}} D^\perp$ in $\mathbb{C}^n$; we can then define the following embedding:

$$i : \mathcal{A}(\mathcal{G}) \to \mathbb{C}^n \times \prod_{D \in \mathcal{G}} \mathbb{P}(\mathbb{C}^n/D^\perp).$$

The wonderful model $Y_{\mathcal{G}}$ is defined as the closure of the image of $i$. If $\mathcal{G}$ is a building set (this is a combinatorial property that will be discussed in Section 2), $Y_{\mathcal{G}}$ turns out to be a smooth variety such that $Y_{\mathcal{G}} - i(\mathcal{A}(\mathcal{G}))$ is a divisor with normal crossings.

The geometric and topological properties of a wonderful model are deeply connected with its initial combinatorial data. For instance, the poset of the intersections of the irreducible components in the boundary $Y_{\mathcal{G}} - i(\mathcal{A}(\mathcal{G}))$ is the nested set poset associated to $\mathcal{G}$ (see Sections 2 and 3). Moreover, the integer cohomology ring of $Y_{\mathcal{G}}$ can be described using some functions with integer values called 'admissible functions' defined on $\mathcal{G}$ (see [5], [20]).
The hyperplane arrangements associated to real and complex reflection groups give rise to particularly interesting wonderful models that have been widely studied in the literature. In the case of the arrangements of type $A_{n-1}$, for example, the minimal building set associated to its poset of intersections produces a wonderful model which is isomorphic to the moduli space $M_{0,n+1}$ of genus 0 stable $n+1$-pointed curves.

In the case of complex reflection groups of type $G(r,p,n)$ (according to the classification in [19]), the lattice of intersections associated to the arrangement has the property of being a Dowling lattice. This is a combinatorial object defined in the early 70's by Dowling in [7] through the action of a finite group $G$ on the set $G \times \{0,1,\ldots,n\}$. The corresponding minimal wonderful models have been studied for instance in [17], [9], [14].

In the 90’s, Hanlon introduced in [16] a generalized version of the Dowling lattices. These new objects are defined given a finite group, a family of its subgroups with a particular property (‘closed subgroups’) and a positive integer $n$. In particular, given any finite group $G$, any faithful representation $V$ of $G$ and any positive integer $n$, one can construct as in Section 3 of [16] a subspace arrangement $H(n,G,V)$ in $V^n$ whose intersection lattice is a generalized Dowling lattice. We will call the arrangements $H(n,G,V)$ generalized Dowling arrangements.

The question that motivated this paper is whether it is possible to give a concrete description of the minimal wonderful model associated to $H(n,G,V)$, generalizing the case of complex reflection groups. In particular our aim is to point out the nice poset provided by the intersections of the irreducible components in the boundary, which provides a geometric realization of the nested set poset of the generalized Dowling lattice. We observe that it can be represented by a family of forests with leaves and labelings that depend on the triple $(n,G,V)$ and we will study it from the enumerative point of view. We will provide formulas for the associated exponential generating series in Theorems 6.1 and 6.2.

This work is divided into five sections. In Section 2 we recall the definition and the main properties of the building sets associated to a collection of subspaces, and of the nested sets associated to a building set.

In Section 3 we deal with the De Concini-Procesi’s wonderful model $Y_G$ associated to a subspace arrangement $G$; in particular the section ends with a description of the boundary of $Y_G$.

In Section 4, following Hanlon, we define the subspace arrangement $H(n,G,V)$ associated to a faithful representation $V$ of a finite group $G$ and to a positive integer $n$; as we mentioned above, the lattice of its intersections, ordered by reverse inclusion, turns out to be a particular example of generalized Dowling lattice. We then study in Section 5 the minimal building set and the nested set poset associated to $H(n,G,V)$.

Finally in Section 6 we define a bijection between the nested set poset and a family of labeled forest. By this bijection we will obtain some exponential formulas (see Theorems 6.1 and 6.2) that enumerate the nested sets and therefore the boundary components of the associated wonderful models. Part of the computation of these exponential formulas is completed in Section 7.

## 2 Building and nested sets

Given a complex space $V$ and its dual space $V^*$, we consider a finite set $H$ of subspaces in $V^*$. We denote by $C_H$ the set of subspaces in $V^*$ that can be written as sum of elements in $H$.

**Definition 2.1.** The set $G$ of subspaces in $V^*$ is building if each $C \in C_G$ can be written as $C = G_1 \oplus \ldots \oplus G_s$ where the $G_i$s are the maximal elements in $G$ among the ones that are contained in $C$.

The nested sets associated to a given building set are defined as follows:
Definition 2.2. Given a building set $G$, a subset $S \subset G$ is said to be $(G)$-nested if for any integer $p \geq 2$ and for any choice of $S_1, \ldots, S_p \in S$ not comparable with respect to inclusion, then $S_1, \ldots, S_p$ are in direct sum and $S_1 \oplus \ldots \oplus S_p \notin G$.

Given any finite set $\mathcal{H}$ of subspaces in $V^*$, the set $C_\mathcal{H}$ defined before is building. The following definition allows us to find another building set naturally associated to $\mathcal{H}$.

Definition 2.3. Given a set $\mathcal{H}$ of subspaces in $V^*$ and a subspace $U \in C_\mathcal{H}$, a decomposition of $U$ is a list of non zero subspaces $U_1, U_2, \ldots, U_k \in C_\mathcal{H}$ with $U = U_1 \oplus \ldots \oplus U_k$ such that for every subspace $A \subset U$ in $\mathcal{H}$, also $A \cap U_1, A \cap U_2, \ldots, A \cap U_k$ lie in $\mathcal{H}$ and $A = (A \cap U_1) \oplus (A \cap U_2) \oplus \ldots \oplus (A \cap U_k)$.

If a subspace does not admit a decomposition it is called irreducible. The set of irreducible subspaces of $C_\mathcal{H}$ is denoted $F_\mathcal{H}$ and it turns out to be a building set. It is the minimal building set that contains $\mathcal{H}$ and is contained in $C_\mathcal{H}$.

3 Wonderful models

Wonderful models have been introduced by De Concini and Procesi in their papers [5] and [4]. Here we recall briefly how such varieties are defined.

Let $V$ be a finite dimensional vector space over $\mathbb{C}$. Let us consider a finite family $G$ of subspaces of $V^*$, and for every $A \in G$ consider its annihilator $A^\perp \subset V$. We also denote by $P_A$ the projective space of lines in $V/A^\perp$.

Let $V_G := \bigcup_{A \in G} A^\perp$ be the union of all the subspaces $A^\perp$ and $\mathcal{A}_G$ be the complement of $V_G$ in $V$. The rational map
\[
\pi_A : V \rightarrow V/A^\perp \rightarrow P_A
\]
is defined outside $A^\perp$ and thus we have a regular morphism $\mathcal{A}_G \rightarrow \prod_{A \in G} P_A$. The graph of this morphism is a closed subset of $\mathcal{A}_G \times \prod_{A \in G} P_A$ which embeds as an open set into $V \times \prod_{A \in G} P_A$.

Finally we have an embedding
\[
\rho : \mathcal{A}_G \rightarrow V \times \prod_{A \in G} P_A
\]
as a locally closed subset.

Definition 3.1. The wonderful model $Y_G$ associated to $G$ is the closure of the image of $\rho$.

In section 3 of [5] De Concini and Procesi prove that if $G$ is a building set then $Y_G$ is a smooth variety. They also provide the following description of the boundary of $Y_G$:

Theorem 3.2. (1) The complement $D$ of $\mathcal{A}_G$ in $Y_G$ is the union of some smooth irreducible divisors $D_G$ indexed by the elements $G \in G$. In particular $D_G$ is the unique irreducible component in $D$ such that $\delta(D_G) = G^\perp$ (where $\delta$ is the projection map $\delta : Y_G \rightarrow V$).

(2) The divisors $D_A_1, \ldots, D_A_n$ have nonempty intersection if and only if the set $\{A_1, \ldots, A_n\}$ is $G$-nested. In this case their intersection is transversal and irreducible.

As a consequence of this theorem, we notice that the poset provided by the intersections of the irreducible components in the boundary ordered by reverse inclusion is isomorphic to the poset of the $G$-nested sets ordered by inclusion.
4 Generalized Dowling arrangements

We start considering a finite group $G$, a positive integer $n$, and a finite dimensional complex faithful representation $\rho : G \to GL(V)$ which does not contain the trivial representation. For each subgroup $H$ of $G$ we denote by $\text{Fix}(H) = \{v \in V : \rho(h)v = v \text{ for all } h \in H\}$ the set of his fixed points in $V$. Notice that since $V$ does not contain the trivial representation we have $\text{Fix}(G) = \{0\}$.

Let us now denote by $\mathcal{P}(G)$ the poset (with respect to inclusion) of all the subgroups of $G$. We define a function $\phi : \mathcal{P}(G) \to \mathcal{P}(G)$ requesting that, for every subgroup $H$, $\phi(H)$ is the maximal subgroup, with respect to inclusion, such that $\text{Fix}(\phi(H)) = \text{Fix}(H)$. We denote by $K$ the set of subgroups fixed by $\phi$, which we will call closed subgroups following Hanlon’s terminology in [16]. We consider $K$ as a poset with respect to inclusion: we notice that $G$ is a closed subgroup since it is maximal with respect to inclusion and $\{e\}$ is a closed subgroup since the representation $V$ is faithful.

Let us then consider the subspace arrangement $\mathcal{H}(n, G, V)$ given by the following subspaces in $V^n$:

$$H(i, j, g) = \{(v_1, \ldots, v_n)|v_j = \rho(g)v_i\}$$

for $1 \leq i < j \leq n$ and $g \in G$, and

$$H(i, i, g) = \{(v_1, \ldots, v_n)|v_i = \rho(g)v_i\}$$

for $1 \leq i \leq n$ and $g \in G - \{e\}$.

We notice that for $i \neq j$ the subspace $H(i, j, g)$ has codimension $\dim V$ in $V^n$ while the subspace $H(i, i, g)$ has codimension $\dim V - \dim \text{Fix}(<g>)$, where $<g>$ is the cyclic subgroup generated by $g$.

We call $\mathcal{H}(n, G, V)$ the generalized Dowling arrangement associated to the triple $(n, G, V)$.

The motivation for this name comes from the following remark: let $L(n, G, V)$ be its intersection lattice, ordered by reverse inclusion. It provides an example of the generalized Dowling lattices introduced by Hanlon, as it was shown in Section 3 of [16]. More precisely, using Hanlon’s terminology, the function $\phi$ is a closure operator and $L(n, G, V)$ is isomorphic to $D_n(G, K(\phi))$.

In particular, when $G = \mathbb{Z}_r$ and $V$ is an irreducible (one dimensional) representation, the lattice $L(n, G, V)$ is isomorphic to the Dowling lattice $Q_n(\mathbb{Z}_r)$. It can be seen as the intersection lattice of the hyperplane arrangement associated to the complex reflection group $G(r, 1, n)$.

5 Nested sets and labeled forests

After fixing the triple $(n, G, V)$, let us denote for brevity by $\mathcal{L}'$ the intersection lattice $L(n, G, V)$ described in the preceding section.

We define $\mathcal{G}'$ to be the subset of $\mathcal{L}'$ whose elements are the subspaces:

- $H^K(i_1^{g_1K}, \ldots, i_k^{g_kK}) := \{(v_1, \ldots, v_n) \in V^n|v_{i_1} = g_1v, \ldots, v_{i_k} = g_kv, v \in \text{Fix}(K)\}$,

where $1 < k \leq n$, $1 \leq i_1 < \ldots < i_k \leq n$, $K$ ranges in the set $K$ of closed subgroups of $G$ and the $g_jK$’s are cosets;

- $H^K(i^{gK}) := \{(v_1, \ldots, v_n) \in V^n|v_i \in \text{Fix}(K)\}$,

where $1 \leq i \leq n$ and $K$ ranges in the set $K$ of closed subgroups of $G$ different from $\{e\}$. 
We notice that here (and from now on) we omitted \( p \) for brevity, using the notation \( gv \) instead of \( p(g)(v) \). It is easy to check that the intersection lattice of \( G' \) is equal to \( L' \); more precisely, every element \( A \) of \( L' \) can be expressed as the transversal intersection of the minimal elements of \( G' \) containing \( A \).

Therefore, if we denote by \( G \) the set of the annihilators in \((V^\ast)^\ast\) of the elements in \( G' \), we have that \( C_G \) is the set of the annihilators of the elements in \( L' \) and \( G \) is the building set of irreducibles of \( C_G \).

Let us now study the nested sets associated to the building set \( G \). We begin by stating two simple lemmas.

Lemma 5.1. Given a closed subgroup \( H < G \), all its conjugate subgroups are closed as well.

Proof. Let us suppose that \( H' = gHg^{-1} \) is not a closed subgroup. We know that there is a \( K \supset H' \) such that \( K \) is closed and \( \text{Fix}(K) = \text{Fix}(H') \). Hence we have \( g^{-1}Kg \supset g^{-1}H'g = H \) and \( \text{Fix}(H) = g^{-1}\text{Fix}(H') = g^{-1}\text{Fix}(g^{-1}Kg) \) and this is absurd. \( \square \)

We now have the main property for the elements in \( G \):

Lemma 5.2. Given \( K \subset G \) a subgroup and \( K' = gKg^{-1} \),

\[
H^K(i^g_1, i^g_2, \ldots, i^g_k) = H^{K'}(i^g_1, i^g_2, \ldots, i^g_k)
\]

holds.

Proof. Let \( w \in H^K(i^g_1, i^g_2, \ldots, i^g_k) \), \( w = (v_1, \ldots, v_n) \), this implies \( v_{i_1} = gv, v \in \text{Fix}(K) \) and \( v_j = g_jv \) for \( j \in i_2, \ldots, i_n \). So \( v_{i_1} \in \text{Fix}(K') \) and \( v_j = g_jg^{-1}v_{i_1} \), these are the conditions for \( v \in H^{K'}(i^g_1, i^g_2, \ldots, i^g_k) \).

Thanks to this lemma we can write

\[
G' = \{ H^K(i^g_1, i^g_2, \ldots, i^g_k) \mid k = 1 \ K \in K - \{ e \}, \text{ if } k > 1 \ K \in K \}.
\]

Given two elements in \( G' \), \( H^1 = H^K(i_1^g, \ldots, i_k^g) \) and \( H^2 = H^{K'}(j_1^g, \ldots, j_l^g) \), we want to determine when there exists a nested set containing both their duals.

Proposition 5.3. Given \( H^1 \) and \( H^2 \) as above, then \( H^1 \supset H^2 \) if and only if the following facts hold:

1. \( \{i_1, \ldots, i_k\} \subseteq \{j_1, \ldots, j_l\} \).
2. For all the \( p \in \{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_l\} \) then \( \text{Fix}(K) \supset g_p^{-1}h_p\text{Fix}(K') \).
3. For all \( p, s \in \{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_l\}, h_s^{-1}g_p^{-1}h_p \in K' \).

Proof. We start supposing \( H^2 \subset H^1 \)

1. Obvious.
2. Let us take \( p \in \{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_l\} \), then \( v \in H_2 \Rightarrow v \in H_1 \) so we must have \( v_1 = g_1w' \), \( v \in \text{Fix}(K') \Rightarrow v_1 = h_1w, w \in \text{Fix}(K) \), this implies \( \text{Fix}(K) \supset g_p^{-1}h_p\text{Fix}(K') \).
3. Given \( p, s \in \{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_l\} \), we must have \( g_pg_s^{-1}v_s = v_p \) and \( h_ph_s^{-1}v_s = v_p \), so \( (g_pg_s^{-1})^{-1}h_ph_s^{-1}v_s = v_s \). Since \( v_s \) can be any element in \( h_s\text{Fix}(K') = \text{Fix}(h_sK'h_s^{-1}) \) we deduce that \( (g_pg_s^{-1})^{-1}h_ph_s^{-1} \in h_sK'h_s^{-1} \) and so \( h_s^{-1}g_pg_p \in K' \).
Suppose now that the three conditions are true. Taken \( v \in H^2 \) we want to show that \( v \in H^1 \). We have \( v = (v_1, \ldots, v_n) \) and for all \( j \in \{j_1, \ldots, j_t\} \), \( v_j = h_j w' \), with \( w' \in \text{Fix}(K') \). Thanks to conditions (1) and (2) we have that for all \( i \in \{i_1, \ldots, i_k\} \), \( q_i^{-1}v_i = w \) with \( w \in \text{Fix}(K) \). Now we have to check that for all \( p, s \in \{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_t\} \) we have \( g_{p}^{-1}v_p = g_{s}^{-1}v_s \), which is true because of (3).

So if the annihilators of \( H^1 \) and \( H^2 \) belong to the same \( G \)-nested set then one of the following cases holds:

- \( H^1 \) and \( H^2 \) are one included into the other, and the three conditions of Proposition 5.3 hold.
- \( \{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_t\} = \emptyset \) and \( K \neq G \) or \( K' \neq G \). In this case the annihilator of \( H^1 \cap H^2 \) is not an element of the building set \( G \).

In the case none of the two conditions above holds we have that \( H^1 \cap H^2 \in G' \) so its annihilator belongs to \( G \). We now want to show that the collection of all the \( G \)-nested sets is in bijection with a family of forests. We start considering \( n \) vertices labeled with the elements in \( M = \{1, \ldots, n\} \) and a partition of \( M \) into some sets \( M_1, \ldots, M_k \). We now consider for each \( M_i \) a rooted oriented tree \( T_i \) whose leaves are labeled by the elements of \( M_i \) (so in the end we will have a forest \( F \) with \( k \) trees). Then we label each internal vertex (i.e. each vertex different from a leaf) and each edge of \( T_i \) according to the following rules:

1. Each internal vertex is labeled with a closed subgroup of \( G \). The label of each vertex (weakly) contains the labels of the internal vertices that are its descendants.

2. If an internal vertex \( w \) is the only direct descendent of a vertex \( v \) the label of \( v \) strictly contains the label of \( w \). If a leaf is the only direct descendent of a vertex \( v \), the label of \( v \) is different from \( \{e\} \).

3. The label \( G \) appears in at most one tree of \( F \), and a vertex labeled with \( G \) has at most one direct descendant labeled with \( G \).

4. If an edge stems from a vertex, it is labeled with a coset of the subgroup labeling the vertex.

5. Let us consider an internal vertex labeled with \( K \); then at least one of its outgoing edges is labeled with the coset \( eK \). In particular the edge that connects the vertex with the subtree containing the smallest leaf is labeled with \( eK \).

The following picture is an example of a forest associated to a nested set:

![Forest Diagram](image-url)
where $H_1 \supseteq H_2 \neq \{e\}$, $H'_1 \supseteq H_2$ and $a, b, c, d \in G$.

We will denote by $\mathcal{F}(n, G, V)$ the set of all the forests (with at least a non-leaf vertex) constructed as above. From the observations of this section it follows that this set is in bijection with the family of nested sets. In the next section we will compute a series that enumerates such forests.

6 Enumeration of the nested sets

Given an oriented forest $F$ in $\mathcal{F}(n, G, V)$, we associate to it a set of subforests, selected according to the labels of its vertices, that are closed subgroups of $G$. Let us consider a closed subgroup $H$ that appears among the labels of the vertices of $F$. The subforest associated to $H$ is provided by all the vertices of $F$ labeled by $H$ and by all the labeled edges that go out (according to the orientation) from these vertices. If the other end of one of these edges is (in $F$) a vertex $w$ which is labeled by a subgroup different from $H$ or is a leaf, then $w$ will be an unlabeled leaf of our subforest. We also consider as a subforest the set of the leaves that are disconnected from the rest of the forest (we will call them fallen leaves). For instance in the Figure 6.1 the leaves 1 and 8 are fallen leaves and the decomposition into subforests is the following:

![Figure 6.1](image)

Given a closed subgroup $H \neq G$, let us now focus on the $H$-forests, i.e. the forests whose internal vertices have the same label $H$; in particular we are interested in counting all the forests of this type whose $n$ leaves are labeled by a fixed set of $n$ numbers.

We will call $H$-tree a rooted oriented tree whose internal vertices are labeled with the subgroup $H$, while the edges are labeled with cosets of $H$ and the leaves are labeled by a fixed set of numbers.

Let us denote by $\lambda_H(t_H)$ the exponential series that counts the number of possible $H$-trees with labeled leaves:

$$\lambda_H(t_H) = \sum_{i \geq 1} \frac{\lambda_{H,i} t_H^i}{i!}$$  \hspace{1cm} (6.1)

where $\lambda_{H,i}$ is the number of $H$-trees with $i$ leaves. A formula for this series will be shown in Section 7.

Suppose that we want to compute the number of forest with three trees, having respectively $l_1$, $l_2$ and $l_3$ leaves, with $l_1 > l_2 > l_3$. This is

$$(l_1 + l_2 + l_3) \lambda_{H,l_1} \binom{l_2 + l_3}{l_2} \lambda_{H,l_2} \lambda_{H,l_3}.$$

The binomials are needed to assign the $l_1 + l_2 + l_3$ labels to the three trees. We then have that

$$\lambda_{H,l_1} \frac{l_1}{l_1} \lambda_{H,l_2} \frac{l_2}{l_2} \lambda_{H,l_3} \frac{l_3}{l_3} = \lambda_{H,l_1} \lambda_{H,l_2} \lambda_{H,l_3} \binom{l_1 + l_2 + l_3}{l_1} \binom{l_2 + l_3}{l_2} \frac{l_1 + l_2 + l_3}{(l_1 + l_2 + l_3)}.$$
This elementary observation can be easily generalized to show that the $H$-forests with three $H$-trees are counted by the series $\frac{\lambda_H(t_H)}{s!}$.

Therefore the contribution given by the forests whose internal vertices are labeled with $H$ is

$$\sum_{n \geq 0} \frac{(s\lambda_H(t_H))^n}{n!} = e^{s\lambda_H(t_H)}.$$

Here we also added the new variable $s$ that counts the number of connected components (trees) of the forest: the coefficient of the monomial $\frac{s^j}{j!}$ is the number of forests with $k$ trees and $j$ leaves.

Let us now focus on the following question: how many are the forests in $F(n, G, V)$ whose decomposition into subforests is given by an $H$-tree with $j$ leaves and a $K$-tree with $i$ leaves, where $K \supseteq H$?

We know that there are $\lambda_{K,i}$ different $K$-trees with $i$ leaves and $\lambda_{H,j}$ $H$-trees with $j$ leaves, then we have the following cases:

- the forest has two trees; the number of such forests is $\lambda_{K,i} \lambda_{H,j} \binom{i+j}{j}$;
- the forest is given by a single tree. We can think of this forest as if it was obtained by gluing the root of an $H$-tree with $j$ leaves with one of the leaves of a $K$-tree with $i$ leaves. In this case the final number of leaves of the forests is $i+j−1$ and the number of all these forests is $\binom{i+j−1}{j} \lambda_{K,i} \lambda_{H,j}$.

Now we observe that

$$(s+\frac{\partial}{\partial s_H}) \frac{\lambda_{K,i} t_H^i}{i!} s \lambda_{H,j} t_H^j = s^2 \lambda_{K,i} \lambda_{H,j} \binom{i+j}{j} \frac{t_H^i}{(i+j)!} + s \lambda_{K,i} \lambda_{H,j} \binom{i+j−1}{j} \frac{t_H^{i+j−1}}{(i+j−1)!}.$$

Therefore the exponential series that computes the number of forests whose decomposition is given by one $K$-tree and one $H$-tree is:

$$\lambda_{H,K} = s_H \lambda_H(t_H) s \lambda_K(t_K) \quad (6.2)$$

where $s_H = s + \frac{\partial}{\partial s_K}$ and $s$ again counts the number of connected components.

Let $F'(n, G, V)$ be the subset of $F(n, G, V)$ given by the forests whose decomposition does not produce $G$-trees and let $F''(n, G, V)$ be the subset of $F'(n, G, V)$ given by the forests whose decomposition does not contain fallen leaves. Putting $H = K - \{G\}$ we denote by

1. $\gamma''(j, (a_H)_{H \in H})$ the number of forests in $F''(n, G, V)$ that have $j$ connected components and $k = \sum_{H \in H} a_H$ leaves such that, for every $H \in H$ there are $a_H$ leaves attached to $H$-trees;
2. $\gamma'(h, j, (a_H)_{H \in H})$ the number of forests in $F'(n, G, V)$ that have $h \geq 0$ fallen leaves, $j$ (with $j > h$) connected components and $k = \sum_{H \in H} a_H$ leaves such that, for every $H \in H$ there are $a_H$ leaves attached to $H$-trees.

We will consider the following series:

$$\tilde{\Gamma} = \tilde{\Gamma}(s, (t_H)_{H \in H}) = 1 + \sum_{j, (a_H)_{H \in H}} \gamma''(j, (a_H)_{H \in H}) s^j \prod_{H \in H} \frac{t_H^{a_H}}{(\sum_{H \in H} a_H)!} \text{ s.t. } \sum_{H \in H} a_H \geq 1.$$
\[ \Gamma = \Gamma(s, t, (t_H)_{H \in \mathcal{H}}) = \sum_{h, j, (a_H)_H \in \mathcal{H}} \gamma'(h, j, (a_H)_H \in \mathcal{H}) s^j \prod_{h \in \mathcal{H}} \frac{t_H^{a_H}}{(h + \sum_{h \in \mathcal{H}} a_H)!}. \]

Theorem 6.1. We have

\[ \tilde{\Gamma}(s, (t_H)_{H \in \mathcal{H}}) = \prod_{H \in \mathcal{H}} e^{s_H \lambda_H(t_H)} \]  
(6.3)

\[ \Gamma(s, t, (t_H)_{H \in \mathcal{H}}) = e^{st}(\tilde{\Gamma} - 1) \]  
(6.4)

where for every \( H \in \mathcal{H} \) we put \( s_H = (s + \sum_{K \in \mathcal{H}} \frac{\partial}{\partial t_K}) \), where every formal differential \( \frac{\partial}{\partial t_K} \) takes into account the attachments of the \( H \)-tree to a leaf of a \( K \)-tree, with \( H \subseteq K \).

In order to prove formula (6.4) we need to take into account the fallen leaves. These are labeled leaves disconnected from the rest of the graph, so we have \( \Gamma(s, t, (t_H)_{H \in \mathcal{H}}) = e^{st}\tilde{\Gamma} - e^{st} = e^{st}(\tilde{\Gamma} - 1) \). We removed from the computation (by subtracting \( e^{st} \)) the contribution of the forests whose only components are fallen leaves, since they do not belong to \( \mathcal{F}'(n, G, V) \) (they do not represent a nested set).

Now we want to take into account the forests \( F \in \mathcal{F}(n, G, V) \) such that \( G \in \mathcal{H}_F \). Let \( \gamma(j, n) \) be the number of forests in \( \mathcal{F}(n, G, V) \) that have \( j \) (with \( j > 0 \)) connected components and \( n > 0 \) leaves. We want to give a formula for the series

\[ \mathcal{G}(s, t) = \sum_{j>0, n>0} \gamma(j, n) s^j t^n \frac{n!}{n!}. \]

We start recalling that given a forest \( F \in \mathcal{F}(n, G, V) \), no more than one tree of \( F \) can have \( G \)-labeled internal vertices and that the \( G \)-labeled vertices are totally ordered by inclusion.

Then we denote by \( \tilde{\Gamma}(s, t) \) (resp. \( \Gamma(s, t) \)) the series \( \tilde{\Gamma}(s, (t_H)_{H \in \mathcal{H}}) \) (resp. \( \Gamma(s, (t_H)_{H \in \mathcal{H}}) \)) evaluated in \( t_K = t \), for all \( K \in \mathcal{H} \). We also put \( \Gamma(s, t) = e^{st}\tilde{\Gamma}(s, t) \).

Let us now count for instance the number of trees in \( \mathcal{F}(n, G, V) \) with two \( G \)-vertices \( \alpha \) and \( \beta \) (see Figure 6.2).
As a first step we imagine to delete from the graph all the edges stemming from $\alpha$ or $\beta$. Then let us consider the following two subforests (with no $G$-vertices) which we suppose to be nonempty: the one made by the trees whose roots were connected by an edge to $\alpha$ and the one made by the trees whose roots were connected by an edge to $\beta$. We will call these respectively the higher forest and the lower forest and we suppose that they have respectively $k > 0$ and $p > 0$ leaves.

The labels for the leaves in the higher forest can be chosen in \(\binom{k+p}{k}\) ways.

The number of possible higher forests is the coefficient $a$ of $t^k$ in $\Gamma(1, t)$ and the number of possible lower forests is the coefficient $b$ of $t^p$. Notice that we are evaluating $\Gamma(s, t)$ in $s = 1$ since in this computation the number of connected components is not relevant (we are subdividing a tree). The number of trees described in the example is therefore

\[
\binom{k+p}{k}ab.
\]

An immediate generalization of the same reasoning shows that the exponential generating series for the cardinality of the set of trees with $l$ $G$-vertices is computed by the series $(\Gamma(1, t) - 1)^l$. We are now ready to compute the exponential series that enumerates the trees with at least one $G$-vertex. This is:

\[
\Phi(t) = (\Gamma(1, t) - 1) + (\Gamma(1, t) - 1)^2 + \ldots + (\Gamma(1, t) - 1)^n \ldots = \frac{1}{2 - \Gamma(1, t)} - 1.
\]

Now, since a forest has at most one connected component that contains $G$-vertices, we obtain:

**Theorem 6.2.** The following formula for $G(s, t)$ holds:

\[
G(s, t) = s\frac{1}{2 - \Gamma(1, t)} - 1)\Gamma(s, t) + \Gamma(s, t).
\]

The number of nested sets in $L_{G,n}$ is the coefficient of $\frac{t^n}{n!}$ in $G(1, t)$.

**Remark 6.3.** A reader who is familiar with the theory of species of structures may have noticed that, given a finite group representation and a set of cardinality $m$, the family of forests associated to the family of nested sets is a species of structure. From this point of view our computation is in the same spirit of the results in Chapter 3 of [2].
We conclude our computation of $G(s, t)$ by providing formulas for the series $\lambda_H(t_H)$, for every closed subgroup $H \neq G$. For brevity of notation in this section we will write $t$ instead than $t_H$.

We start with the series $\lambda(e)(t) = \lambda(t)$. The $\{e\}$-trees are rooted, oriented trees with labeled leaves and labeled edges. Every internal vertex different from the root is connected to at least three other vertices, while the root of the tree must have at least two outgoing edges. If there are $n$ leaves, their labels are the elements of $\{1, ..., n\}$.

The labels of the edges are cosets of the subgroup $\{e\}$, i.e. elements of $G$. According to Lemma 5.2, for every internal vertex, we can consider one of its outgoing edges to be labeled with $e$. In particular we put the edge that connects the vertex with the subtree containing the smallest leaf to be always labeled with $e$. We will denote the set of trees described above as $T_{\{e\}}$.

Our first goal is to put $T_{\{e\}}$ in bijection with a family of weighted partitions. This can be done using a slight modification of the bijection in Theorem 2.1 of [12]. In the following example we show how a labeled partition is associated to a tree in $T_{\{e\}}$.

**Example 7.1.** Let us consider the tree in Figure 7.1 (a). We label its internal vertices with the elements of the set of integers $s = \{7, 8, 9, 10\}$, as in picture Figure 7.1 (b) (see the green labels). The criterion we used is the following one: we associate to every internal vertex $v$ the set (of the labels) of the leaves $L_v$ of the subgraph stemming out of it. Then the label of a vertex $v$ is less than the label of a vertex $w$ if and only if either $L_v$ is included into $L_w$ or the two sets are disjoint and $\min L_v < \min L_w$.

We are now ready to associate to the tree the labeled partition obtained considering, for every internal vertex, the labels of the vertices connected to it by an outgoing edge:

$$\begin{align*}
\{1^e, 2^e, 3^e\} \{4^e, 6^e\} \{5^e, 8^e\} \{7^e, 9^e\}.
\end{align*}$$

Notice that the elements in the partition are 9, which is equal to the number of leaves of the tree plus the number of its internal vertices minus 1.

Therefore we have to count partitions instead than trees. Mimicking the strategy used in Theorem 5.1 of [14], we find

$$\lambda(t) = \left[ \prod_{i \geq 2} e^{\frac{r^i - 1}{i}} \right] - 1$$

where $r = |G|$.

![Figure 7.1](image-url)
More in detail, the contribution of a partition of the type described above with \( j \) parts of the same cardinality \( i \geq 2 \) is provided by \( e^{\frac{z}{i}} \left( \frac{r_i}{r} \right)^j \), where the exponent of \( z \) counts the number of parts in the partition while the exponent of \( t \) counts the sum of the cardinality of all the parts. So all the partitions whose parts have the same cardinality \( i \) are counted by \( e^{\frac{z \cdot i}{i!}} - 1 \). Thus the formula

\[
\left( \prod_{i \geq 2} e^{\frac{z \cdot i}{i!}} \right) - 1
\]

takes into account all these contributions. Then we put \( z = \frac{\partial}{\partial t} \): since in the exponential series \( \lambda(t) \) a tree with \( n \) leaves contributes to the coefficient of \( \frac{t^n}{n!} \), a final integration provides the formula above.

Now we can compute the series \( \lambda_H(t) \) for all the closed subgroups \( H \) different from \( \{e\} \) and \( \{G\} \). Unlike the case \( H = \{e\} \) there may be internal vertices with only one outgoing edge. In particular the trees must satisfy:

- If an internal vertex is connected to only two vertices, then one of these two vertices must be a leaf. We will call these vertices one leaf vertices. We can also have the case of the one leaf tree, with only the root and one leaf.
- All the edges are labeled with cosets of \( H \). Among all the edges stemming from a node, the one connecting the vertex to the subtree with the smallest leaf is labeled with \( eH \).

First we consider the set \( T_H \) of \( H \)-trees with no one leaf vertices and we compute its exponential series \( \overline{\lambda}_H(t) \). As in the case \( H = \{e\} \) we can put \( T_H \) in bijection with a set of partitions with the only difference that the labels of the elements in the partitions are now taken in \( G/H \). This allows us to write

\[
\overline{\lambda}_H(t) = \int \left[ \left( \prod_{i \geq 2} e^{\frac{\partial}{\partial t} \left( \frac{r_i}{r} \right)^j} \right) - 1 \right]
\]

where now \( r = |G/H| \). To count the number of \( H \)-trees we observe that each \( H \)-tree can be seen as a tree in \( T_H \) with possibly some one leaf vertices glued to some of the leaves. So a tree in \( T_H \) with \( i \) leaves produces \( \sum_{k=0}^{i} (\frac{i}{k}) = 2^i \) different \( H \)-trees. Assuming that \( \overline{\lambda}_H(t) = \sum_{i \geq 2} \frac{a_i t^i}{i!} \), we have that

\[
\lambda_H(t) = t + \sum_{i \geq 2} \frac{a_i t^i}{i!} 2^i = t + \overline{\lambda}_H(2t).
\]

In this case the addendum \( t \) appears to take into account the one leaf tree.

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