AN INVERSE PROBLEM OF DETERMINING THE ORDERS OF SYSTEMS OF FRACTIONAL PSEUDO-DIFFERENTIAL EQUATIONS

RAVSHAN ASHUROV AND SABIR UMAROV

Abstract. As it is known various dynamical processes can be modeled through the systems of time-fractional order pseudo-differential equations. In the modeling process one frequently faces with determining adequate orders of time-fractional derivatives in the sense of Riemann-Liouville or Caputo. This problem is qualified as an inverse problem. The right (vector) order can be found utilizing the available data. In this paper we consider this inverse problem for linear systems of fractional order pseudo-differential equations. We prove that the Fourier transform of the vector-solution $\hat{U}(t, \xi)$ evaluated at a fixed time instance, which becomes possible due to the available data, recovers uniquely the unknown vector-order of the system of governing pseudo-differential equations.

Keywords: system of differential equations, fractional order differential equation, pseudo-differential operator, matrix symbol, inverse problem, determination of the fractional derivative’s order

1. Introduction

In modern science and engineering researchers frequently use fractional order differential equations for modeling of dynamics of various complex stochastic processes arising in different fields; see, for example, [12, 22, 28, 32] in physics, [20, 28] in finance, [6] in hydrology, [21] in cell biology, among others. In the last few decades several books, devoted to fractional order differential and pseudo-differential equations and their various applications, have been published (see e.g. [27, 24, 16, 11, 29, 31]).

In fractional order modeling, in contrast to integer order equations, orders of fractional order governing equations are often unknown, and requires to utilize available data to measure. Therefore, one of the key questions arising in the process of modeling is a proper determination of a fractional order of the governing equation. The problem of determining a correct order of an equation (or orders of equations if the model uses more than one governing equation) is classified as an inverse problem. Inverse problems naturally require additional conditions (or information) for a solution. For subdiffusion equations, in which the order is between zero and one, the inverse problem of determination of the order has been considered by a number of authors; see [1] [2] [3] [7] [14] [17] [18] [19] and references therein. For the survey paper, we refer the reader to [17] by Li et. al. Note that in all the refereed works the subdiffusion equation was considered in a bounded domain $\Omega \subset \mathbb{R}^N$. In addition, it should be noted that in publications [7] [11] [18] [10] the following relation was taken as an additional condition

\begin{equation}
    u(x_0, t) = h(t), \ 0 < t < T,
\end{equation}
at a monitoring point \( x_0 \in \Omega \). But this condition, as a rule (an exception is the work \cite{14} by J. Janno, where both the uniqueness and existence are proved), can ensure only the uniqueness of the solution of the inverse problem \cite{7, 15, 19}. In paper \cite{3} authors obtained the existence and uniqueness result, considering as an additional condition, the value of the projection of the solution onto the first eigenfunction of the elliptic part of the subdiffusion equation. Note that the technique used in \cite{3} is applicable only when the first eigenvalue is zero. In more general case the uniqueness and existence of a solution of the inverse problem of determination of an unknown order of the fractional derivative in the subdiffusion equation was proved in the recent work \cite{1}. In this case, the additional condition is \( \|u(x, t_0)\|^2 = d_0 \), and the boundary condition is not necessarily homogeneous. In paper \cite{2} authors studied the inverse problem for the simultaneous determination of the order of the Riemann-Liouville time fractional derivative and the source function in the subdiffusion equations.

The purpose of this work is to investigate the inverse problem of determining the vector-order of the time-fractional derivatives of systems of pseudo-differential equations. We note that systems (linear and non-linear) of fractional ordinary equations and partial differential equations have rich applications and are used in modeling of various processes arising in modern science and engineering. For example, they are used in modeling of processes in biosystems \cite{8, 26, 10}, ecology \cite{15, 25}, epidemiology \cite{33, 13}, etc.

In this paper we consider the following system of linear homogenous time-fractional order pseudo-differential equations

\[
\begin{aligned}
D_{\beta_1}^{{\hat{}}} u_1(t, x) &= A_{1,1}(D)u_1(t, x) + \ldots A_{1,m}(D)u_m(t, x), \\
D_{\beta_2}^{{\hat{}}} u_2(t, x) &= A_{2,1}(D)u_1(t, x) + \ldots A_{2,m}(D)u_m(t, x), \\
\vdots \\
D_{\beta_m}^{{\hat{}}} u_m(t, x) &= A_{m,1}(D)u_1(t, x) + \ldots A_{m,m}(D)u_m(t, x),
\end{aligned}
\]

(1.2)

where \( \mathcal{B} = \langle \beta_1, \ldots, \beta_m \rangle \), \( 0 < \beta_j \leq 1 \), \( j = 1, \ldots, m \), is an unknown vector-order to be determined, the operator \( D \) on the left hand side expresses either the Riemann-Liouville derivative \( D_+ \) or the Caputo derivative \( D_* \), and \( A_{j,k}(D) \) are pseudo-differential operators with (possibly singular) symbols depending only on dual variables (for simplicity) and described later. The initial conditions depend on the form of fractional derivatives.

As it follows from our main result, a predetermined value of the Fourier transform \( \hat{U}(t, \xi) = \langle \hat{u}_1(t, \xi), \ldots, \hat{u}_m(t, \xi) \rangle \) of the solution \( U(t, x) = \langle u_1(t, x), \ldots, u_m(t, x) \rangle \) of the initial value problem for system (1.2) at an appropriate fixed point \( \xi_0 \in \mathbb{R}^m \) satisfying some condition (see Eq. \ref{eq:2.11}) , that is

\[
\hat{u}_j(t_0, \xi_0) = d_j, \quad j = 1, 2, \ldots, m,
\]

(1.3)

where \( t_0 \geq 1 \) is an observation time, uniquely recovers the vector-order \( \mathcal{B} = \langle \beta_1, \ldots, \beta_m \rangle \) of the fractional derivatives.

In the particular case, the determining of a scalar order for one equation was considered in \cite{3} and the forward problem for systems of pseudo-differential equations in \cite{30}. From this point of view the current paper is a logical continuation of these two papers.

We note that in \cite{3} the additional condition is represented in the form

\[
\int_{\Omega} u(t_0, x)v_1(x)dx = d \neq 0,
\]
that is, in the form of a projection of the solution \( u(t_0, x) \) onto the first eigenfunction \( v_1(x) \) of the elliptic part of the equation considered in an arbitrary bounded domain \( \Omega \subset \mathbb{R}^n \). Condition (1.3) can be considered as the projection of the solution \( u(t_0, x) \) onto "the eigenfunction" \( e^{-ix\xi_0} \):

\[
\int_{\mathbb{R}^n} u(t_0, x) e^{-ix\xi_0} dx = d.
\]

To our best knowledge, the inverse problem for the system of equations (1.2) with the vector-order fractional derivatives under the additional condition (1.3) is considered for the first time.

2. Main results

2.1. Notations. We follow the notions and notations introduced in [30]. For the reader’s convenience, below we introduce the main notations used in the current paper; for details see [29, 30]. Let \( G \subseteq \mathbb{R}^n \) be an open set and \( p \geq 1 \). The space \( \Psi_{G,p}(\mathbb{R}^n) \) comprises of functions \( \psi \in L^p(\mathbb{R}^n) \), such that \( \text{supp} \hat{\psi} \subseteq G \), i.e. the Fourier transform

\[
\hat{\psi}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx
\]

of \( \psi \) has a compact support in \( G \). This is a topological-vector space with respect to the following convergence: a sequence \( \psi_n \to \psi \) if \( \text{supp} \hat{\psi}_n \subseteq G \), and \( \psi_n \to \psi \) in \( L^p(\mathbb{R}^n) \). For relations of the spaces \( \Psi_{G,p}(\mathbb{R}^n) \) to Sobolev spaces and Schwartz distributions see [29].

Let \( A(\xi) \) be a continuous function in \( G \). Outside of \( G \) or on its boundary \( A(\xi) \) may have singularities of arbitrary type. For a function \( \varphi \in \Psi_{G,p}(\mathbb{R}^n) \) the pseudo-differential operator \( A(D) \) corresponding to the symbol \( A(\xi) \) is defined by the formula

\[
A(D) \varphi(x) = \frac{1}{(2\pi)^n} \int_{G} A(\xi) \hat{\varphi}(\xi) e^{ix\xi} d\xi \quad x \in \mathbb{R}^n.
\]

For the systematic presentation of the theory of pseudo-differential operators being considered in this paper we refer the reader to [29].

Let

\[
A(D) = \begin{bmatrix} A_{1,1}(D) & \cdots & A_{1,m}(D) \\ \cdots & \cdots & \cdots \\ A_{m,1}(D) & \cdots & A_{m,m}(D) \end{bmatrix}
\]

be the matrix pseudo-differential operator with constant (that is not depending on the variable \( x \)) matrix-symbol

\[
A(\xi) = \begin{bmatrix} A_{1,1}(\xi) & \cdots & A_{1,m}(\xi) \\ \cdots & \cdots & \cdots \\ A_{m,1}(\xi) & \cdots & A_{m,m}(\xi) \end{bmatrix},
\]

defined and continuous in \( G \) in the sense of the matrix norm.

With the matrix form of the pseudo-differential operator we can represent system (1.2) in the vector form:

\[
\mathcal{D}^\beta U(t, x) = \mathcal{A}(D) U(t, x),
\]

where \( \mathcal{D}^\beta U(t, x) = (\mathcal{D}^{\beta_1} u_1(t, x), \ldots, \mathcal{D}^{\beta_m} u_m(t, x)) \).
Below we will use two main forms of fractional derivatives, namely the Riemann-Liouville form and the Caputo form. Let $k$ be a natural number and $k - 1 \leq \beta < k$. Then the fractional derivative of order $\beta$ of a measurable function $f$ in the sense of Riemann–Liouville is defined as

$$D_+^{\beta} f(t) = \frac{1}{\Gamma(k - \beta)} \frac{d^k}{dt^k} \int_0^t \frac{f(\tau)d\tau}{(t - \tau)^{\beta+1}},$$

provided the expression on the right exists. Here $\Gamma(t)$ is Euler’s gamma function. If we replace differentiation and fractional integration in this definition, then we get the definition of a regularized derivative, that is, the definition of a fractional derivative in the sense of Caputo:

$$D^{\beta}_* f(t) = \frac{1}{\Gamma(k - \beta)} \int_0^t \frac{f^{(k)}(\tau)d\tau}{(t - \tau)^{\beta+1}},$$

provided the integral on the right exists.

We assume that the matrix-symbol is symmetric, $A_{k,j}(\xi) = A_{j,k}(\xi)$ for all $k,j = 1,\ldots,m$, and $\xi \in G$, and diagonalizable. Namely, there exists an invertible $(m \times m)$-matrix-function $M(\xi)$, such that

$$(2.4) \quad A(\xi) = M^{-1}(\xi) \Lambda(\xi) M(\xi), \quad \xi \in G,$$

with a diagonal matrix

$$(2.5) \quad \Lambda(\xi) = \begin{bmatrix} \lambda_1(\xi) & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & \lambda_m(\xi) \end{bmatrix}.$$ 

We denote entries of matrices $M(\xi)$ and $M^{-1}(\xi)$ by $\mu_{j,k}(\xi)$, $j,k = 1,\ldots,m$, and $\nu_{j,k}(\xi)$, respectively.

Since initial conditions depend on the form of the fractional derivative on the left hand side of equation (2.3), we will consider the cases with the Caputo and Riemann-Liouville derivatives separately. We first formulate our main result in the case of Caputo fractional derivative. The case of Riemann-Liouville fractional derivative can be treated similarly.

### 2.2. Forward problem

Let $B$ be a known vector-order with $0 < \beta_j \leq 1$, $j = 1, \ldots, m$. Consider the following Cauchy problem

$$D^B_+ U(t, x) = A(D)U(t, x), \quad t > 0, \quad x \in \mathbb{R}^n,$$

$$U(0, x) = \Phi(x), \quad x \in \mathbb{R}^n,$$

where $\Phi(x) = \langle \varphi_1(x), \ldots, \varphi_m(x) \rangle \in \Psi_{G,p}^p(\mathbb{R}^n)$ and the fractional derivatives on the left are in the sense of Caputo.

We call the Cauchy problem (2.6) the forward problem.

A representation formula for the solution of the forward problem was obtained in [30] and it has the form

$$u_j(t, x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \sum_{k=1}^m s_{j,k}(t, \xi) \hat{\varphi}_k(\xi) e^{ix\xi} d\xi, \quad j = 1, \ldots, m,$$
where
\[ s_{j,k}(t,\xi) = \sum_{l=1}^{m} \mu_{j,l}(\xi) \nu_{l,k}(\xi) E_{\beta_l}(\lambda_l(t) t^{\beta_l}) . \]

Here we denoted by \( E_{\beta_j}(z) \), \( j = 1, \ldots, m \), the Mittag-Leffler functions of indices \( \beta_1, \ldots, \beta_m \), respectively.

We rewrite function \( u_j \) as
\[
\begin{aligned}
\hat{u}_j(t,\xi) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \sum_{k=1}^{m} \sum_{l=1}^{m} \mu_{j,l}(\xi) \nu_{l,k}(\xi) E_{\beta_l}(\lambda_l(t) t^{\beta_l}) \hat{\varphi}_k(\xi) e^{ix\xi} d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \sum_{l=1}^{m} E_{\beta_l}(\lambda_l(t) t^{\beta_l}) \left[ \mu_{j,l}(\xi) \sum_{k=1}^{m} \nu_{l,k}(\xi) \hat{\varphi}_k(\xi) \right] e^{ix\xi} d\xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \sum_{l=1}^{m} E_{\beta_l}(\lambda_l(t) t^{\beta_l}) \hat{K}_{j,l}(\xi, \hat{\Phi}(\xi)) e^{ix\xi} d\xi ,
\end{aligned}
\]
where
\[
\hat{K}_{j,l}(\xi, \hat{\Phi}(\xi)) = \mu_{j,l}(\xi) \sum_{k=1}^{m} \nu_{l,k}(\xi) \hat{\varphi}_k(\xi),
\]
and
\[
\hat{\Phi}(\xi) = \langle \hat{\varphi}_1(\xi), \hat{\varphi}_2(\xi), \ldots, \hat{\varphi}_m(\xi) \rangle .
\]

For the Fourier transform of the solution we have
\[
(2.8) \quad \hat{u}_j(t,\xi) = \sum_{l=1}^{m} E_{\beta_l}(\lambda_l(t) t^{\beta_l}) \hat{K}_{j,l}(\xi, \hat{\Phi}(\xi)) .
\]

Note that under the above conditions on the matrix-symbol \( A_{j,k}(\xi) \) and on the function \( \Phi(x) \), this Fourier transform exists at each point \( \xi \in \mathbb{R}^n \).

2.3. Inverse problem. Now let the parameter \( B \) be an unknown vector-order of the time derivative with \( \beta_0 \leq \beta_j < 1 \), \( j = 1, \ldots, m \), \( \beta_0 \in (0,1) \). The main purpose of this paper is to investigate the inverse problem of identifying of these parameters \( \beta_j \). Since there are \( m \) unknown parameters, we need to set \( m \) conditions. We pass on to the determining of these additional conditions.

In what follows, we will assume that
\[
(2.9) \quad |\arg \lambda_j(\xi)| > \frac{\pi}{2}, \quad \xi \in G, \quad j = 1, \ldots, m .
\]

Let \( \xi^0 = (\xi_1^0, \xi_2^0, \ldots, \xi_m^0) \in G \subset \mathbb{R}^n \) be a vector such that the determinant of the matrix
\[
(2.10) \quad \mathcal{K}(\xi^0) \equiv \{ K_{j,l}(\xi^0, \hat{\Phi}(\xi^0)) \}, \quad j, l = 1, \ldots, m ,
\]
satisfies the condition
\[
(2.11) \quad |K_{j,l}(\xi^0, \hat{\Phi}(\xi^0))| \neq 0 .
\]
To find the unknown parameters \( \beta_j, l = 1, \ldots, m \), we consider the following additional conditions
\[
(2.12) \quad f_j(B, t_0, \xi^0) \equiv \hat{u}_j(t_0, \xi^0) = d_j, \quad j = 1, \ldots, m ,
\]
where \( d_j \) are given numbers and \( t_0 \) is defined later (see Lemmas 2.2 and 2.4). We call problem (2.6)–(2.7) together with the additional condition (2.12) the inverse problem.
It follows from (2.8) and (2.12) that for all \( j = 1, 2, ..., m \)

\[
(2.13) \quad \sum_{l=1}^{m} E_{\beta_l}(\lambda_l(\xi^0) t_0^{\beta_l}) K_{j,l}(\xi^0, \hat{\Phi}(\xi^0)) = d_j.
\]

These are in fact the system of equations to define the orders \( \beta_l, l = 1, 2, ..., m \).

Due to condition (2.11) one can solve system (2.13) with respect to the Mittag-Leffler functions \( E_{\beta_l} \), i.e.

\[
(2.14) \quad E_{\beta_l}(\lambda_l(\xi^0) t_0^{\beta_l}) = b_l, \quad l = 1, 2, ..., m,
\]

where \( b_l, l = 1, 2, ..., m \), are components of the vector \( \mathbf{d} = \langle d_1, \ldots, d_m \rangle \) and \( \mathbf{d} = \mathbf{K}(\xi^0) \mathbf{d} \). Thus to define each unknown parameter \( \beta_l \) we obtained a separate equation (2.14).

Let \( R_{C,l} \) be the range of values of the function \( e_{1,\lambda_l}(\beta_l) \equiv E_{\beta_l}(\lambda_l(\xi^0) t_0^{\beta_l}) \) when \( \beta_l \) runs over the half-interval \( [\beta_0, 1) \), i.e.

\[
e_{1,\lambda_l} : [\beta_0, 1) \to R_{C,l} \subset C,
\]

where \( C \) is a complex plane, and the index \( C \) emphasizes that we are considering the case of the Caputo derivatives.

Obviously, for equations (2.14), in order to have solutions with respect to \( \beta_l \), the right-hand sides of these equations must lie within the values of the functions on the left-hand sides of these equations, i.e.

\[
(2.15) \quad b_l \in R_{C,l}, \quad l = 1, 2, ..., m.
\]

On the other hand, by virtue of Rolle’s theorem, the strict monotonicity of either the real part or the imaginary part of the function \( E_{\beta_l}(\lambda_l(\xi^0) t_0^{\beta_l}) \) in the variable \( \beta_l \) is sufficient for the uniqueness of the solution to equation (2.14).

Let us introduce the following notation

\[
R_C(\beta_l) = \Re(E_{\beta_l}(\lambda_l(\xi^0) t_0^{\beta_l})),
\]

where \( \Re(z) \) is the real part of \( z \) and the index \( C \) again emphasizes that we are considering the case of the Caputo derivatives. The necessity of condition (2.9) is that its fulfillment guarantees the strict monotonicity of the function \( R_C(\beta_l) \) it the variable \( \beta_l \) for each fixed \( l \).

2.4. Main results. The main results of this paper are stated in Theorems 2.1 and 2.5.

**Theorem 2.1.** Let \( \xi^0 \) satisfy condition (2.11) and \( t_0 > T_0 \), where \( T_0 \) is identified in Lemma 2.2. Let the numbers \( d_j \) on the right hand side of equation (2.12) be such that the components \( b_l, l = 1, \ldots, m \), of the vector \( \mathbf{K}(\xi^0) \mathbf{d} \) satisfy the conditions (2.15). Then for each \( l \) there exists the unique number \( \beta^*_l \in [\beta_0, 1] \) such that the Fourier transform of the solution \( u_j(t,x) \), \( j = 1, \ldots, m \), of the forward problem with \( \beta_j = \beta^*_j, j = 1, \ldots, m \), satisfies equation (2.14).

The proof of this theorem follows from the existence and uniqueness theorem for the forward problem proved in [30] (see Theorem 3.1) and Lemma 2.2 below. Therefore, in order to prove the theorem we need to prove only this lemma. The proof of Lemma 2.2 is given in Section 3.
Lemma 2.2. Given $\beta_0$ in the interval $0 < \beta_0 < 1$ and $\xi_0$ satisfying condition (2.77), there exists a number $T_0 = T_0(\xi_0, \beta_0) > 1$, such that for all $t_0 \geq T_0$ the function $R_C(\beta_i)$ is positive and strictly monotonically decreasing with respect to $\beta_i \in [\beta_0, 1]$ and
\begin{equation}
R_C(1) \leq R_C(\beta_i) \leq R_C(\beta_0), \quad l = 1, \ldots, m.
\end{equation}

Remark 2.3. Theorem 2.1 defines the vector-order $B^* = (\beta_1^*, \beta_2^*, \ldots, \beta_m^*)$ uniquely from conditions (2.72). Hence, if we define $f_j(B^*, \cdot)$ at another time instant $t_1$ and point $\xi^1$ and get a new $B^{**}$, i.e. $f_j(B^{**}, t_1, \xi^1) = d_j^1$, then from the equality $f_j(B^{**}, t_0, \xi^0) = d_j$, by virtue of the theorem, we obtain $B^{**} = B^*$.

Now consider the following initial-value problem
\begin{align}
D^n_B U(t, x) &= \lambda(D)U(t, x), \quad t > 0, \quad x \in \mathbb{R}^n, \\
J^{1-n}B U(0, x) &= \Phi(x), \quad x \in \mathbb{R}^n,
\end{align}
where $\Phi(x) = \langle \varphi_1(x), \ldots, \varphi_m(x) \rangle \in \Psi_{G,p}(\mathbb{R}^n)$ and the fractional derivatives on the left hand side of equation (2.17) are in the sense of Riemann-Liouville.

We call the Cauchy problem (2.17)-(2.18) the second forward problem.

A representation formula for the solution of the second forward problem was also obtained in [30] and it has the form
\begin{equation}
\hat{u}_j(t, x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \sum_{k=1}^{m} s_{j,k}^+(t, \xi) \hat{\varphi}_k(\xi) e^{ix\xi} d\xi, \quad j = 1, \ldots, m,
\end{equation}
where
\begin{equation}
s_{j,k}^+(t, \xi) = \sum_{l=1}^{m} \mu_{j,l}(\xi) \nu_{j,k}(\xi) t^{\beta_l-1} E_{\beta_l, \beta_l}(\lambda_l(\xi)t^{\beta_l}).
\end{equation}
Here we denoted by $E_{\beta_l, \beta_l}(z), j = 1, \ldots, m$, the two-parametric Mittag-Leffler functions.

For the Fourier transform of the solution we have
\begin{equation}
\hat{u}_j(t, \xi) = \sum_{l=1}^{m} t^{\beta_l-1} E_{\beta_l, \beta_l}(\lambda_l(\xi)t^{\beta_l}) K_{j,l}(\xi, \hat{\Phi}(\xi)).
\end{equation}

Suppose that condition (2.40) is fulfilled and choose $\xi^0 \in G$ so that inequality (2.41) holds.

We call problem (2.17)-(2.18) together with the additional condition (2.12) the second inverse problem.

Note that additional condition (2.12) is in fact the equation to determine the unknown parameters $\beta_i$. Performing similar calculations as above, by virtue of condition (2.11), we rewrite (2.12) as
\begin{equation}
t^{\beta_l-1} E_{\beta_l, \beta_l}(\lambda_l(\xi)t^{\beta_l}) = b_l, \quad l = 1, 2, \ldots, m,
\end{equation}
where $b_l$, $l = 1, 2, \ldots, m$, are the same numbers as above.

Let $R_{RL,l}$ be the range of values of the left-hand side of these equations when $\beta_l$ runs over the half-interval $[\beta_0, 1)$. Here the index $RL$ emphasizes that we are considering the case of the Riemann-Liouville derivatives.

Again, as in case of equations (2.14), a necessary condition for the existence of solutions to equations (2.20) is the inclusion
\begin{equation}
b_l \in R_{RL,l}, \quad l = 1, 2, \ldots, m.
\end{equation}
On the other hand, by virtue of Rolle’s theorem, the strict monotonicity of the function \( \Re(t^{\beta_l-1}E_{\beta_l,\beta_l}(\lambda_l t^{\beta_l})) \) in the variable \( \beta_l \) is sufficient for the uniqueness of the solution to equation (2.20).

However, if \( |\Re(\lambda_l(\xi^0))| = |\Im(\lambda_l(\xi^0))| \) (note, under the condition (2.9) one has \( \Re(\lambda_l(\xi^0)) < 0 \)), then the principal part of \( \Re(t^{\beta_l-1}E_{\beta_l,\beta_l}(\lambda_l t^{\beta_l})) \) vanishes, and in this case it is necessary to go to its next term in the asymptotic. Therefore, to simplify the presentation, we further assume that

\[
(2.22) \quad |\Re(\lambda_l(\xi^0))| \neq |\Im(\lambda_l(\xi^0))|
\]

and therefore \( \lambda_l t \) is positive and strictly monotonically decreasing with respect to \( \beta_l \). Thus, to simplify the presentation, we further simplify the presentation, we further assume that

\[
(2.23) \quad R_{RL}(\beta_l) = \text{sign}(|\Re(\lambda_l(\xi^0))| - |\Im(\lambda_l(\xi^0))|)\Re(t^{\beta_l-1}E_{\beta_l,\beta_l}(\lambda_l t^{\beta_l})).
\]

Here the index \( RL \) again emphasizes that we are considering the case of the Riemann-Liouville derivatives.

**Lemma 2.4.** Given \( \xi_0 \) in the interval \( 0 < \xi_0 < 1 \) and \( \xi_0 \) satisfying condition (2.11), there exists a number \( T_1 = T_1(\xi_0, \beta_0) > 1 \), such that for all \( t_0 \geq T_1 \) the function \( R_{RL}(\beta_l) \) is positive and strictly monotonically decreasing with respect to \( \beta_l \in [\beta_0, 1] \) and

\[
(2.23) \quad R_{RL}(1) \leq R_{RL}(\beta_l) \leq R_{RL}(\beta_0), \ l = 1, \ldots, m.
\]

This lemma, similar to the Caputo derivative case, immediately implies the following main result of this paper in the case of the Riemann-Liouville derivatives. The existence and uniqueness theorem of the corresponding forward problem is proved in [30] (see Theorem 3.4). The proof of Lemma 2.4 is presented in the next section.

**Theorem 2.5.** Let \( \xi_0 \) satisfy condition (2.11) and \( t_0 > T_1 \), where \( T_1 \) is identified in Lemma 2.4. Let the numbers \( \mu_i \) from condition (2.12) be such that the corresponding numbers \( \beta_i \) satisfy the conditions (2.23). Then for each \( l \) there exists the unique number \( \beta_l \in [\beta_0, 1] \) such that the Fourier transform of the solution \( u_j(t, x) \) of the second forward problem with \( \beta_j = \beta_j^* \) satisfies the equation (2.13).

Similar to the Caputo derivative case, Theorem 2.5 defines the vector-order \( B^\ast = (\beta_1^*, \beta_2^*, \ldots, \beta_m^*) \) uniquely from conditions (2.12); see Remark 2.3.

3. Proofs of Lemmata 2.2 and 2.3

Let us denote by \( \delta(1; \theta) \) a contour oriented by non-decreasing arg \( \zeta \) consisting of the following parts: the ray arg \( \zeta = -\theta \) with \( |\zeta| \geq 1 \), the arc \( -\theta \leq \arg \zeta \leq \theta \), \( |\zeta| = 1 \), and the ray arg \( \zeta = \theta \), \( |\zeta| \geq 1 \). If \( 0 < \theta < \pi \), then the contour \( \delta(1; \theta) \) divides the complex \( \zeta \)-plane into two unbounded parts, namely \( G(\ast) \) to the left of \( \delta(1; \theta) \) by orientation, and \( G(\ast) \) to the right of it. The contour \( \delta(1; \theta) \) is called the Hankel path.

In what follows, we fix \( l \) out of \( 1, \ldots, m \) and denote \( \lambda = \lambda_l(\xi^0) \) and \( \rho = \beta_l \). Let \( \lambda = -\lambda_1 + i\lambda_2 \) and by virtue of condition (2.19) one has \( \lambda_1 > 0 \). Further let \( \theta = (\frac{\pi}{2} + \frac{\pi}{2})\rho \), \( \alpha = (\frac{\pi}{2} + \frac{\pi}{2})\rho \), \( \rho \in [\beta_0, 1] \) and \( \varepsilon > 0 \) be such that \( \varepsilon \equiv \varepsilon(\xi^0) < \frac{\pi}{2} \min\{|\arg \lambda(\xi^0)| - \pi/2, \pi/2\} \).

Then

\[
\frac{\pi}{2} \rho < \theta < \alpha < \pi \rho, \quad \alpha < |\arg \lambda|,
\]

and therefore \( \lambda_l(\xi^0) \in G(\ast) \).
3.1. Proof of Lemma 2.2. First we prove Lemma 2.2. By the definition of contour \( \delta(1; \theta) \), we have (see [9], formula (2.29), p. 135)

\[
(3.1) \quad e_{1,\lambda}(\rho) \equiv E_\mu(\lambda t_0^\rho) = -\frac{1}{\lambda t_0^\rho \Gamma(1 - \rho)} + \frac{1}{2\pi i \rho \lambda t_0^\rho} \int_{\delta(1; \theta)} \frac{e^{\xi/\rho}}{\xi + \lambda t_0^\rho} d\xi = f_1(\rho) + f_2(\rho).
\]

To prove the lemma, we need to determine the sign of the real part of the derivative \( \frac{d}{d\rho} e_{1,\lambda}(\rho) \). It is not hard to estimate the derivative \( f_1'(\rho) \). Indeed, let \( \Psi(\rho) \) be the logarithmic derivative of the gamma function \( \Gamma(\rho) \) (for the definition and properties of \( \Psi \) see [5]). Then \( \Gamma'(\rho) = \Gamma(\rho) \Psi(\rho) \), and therefore,

\[
f_1'(\rho) = \frac{\ln t_0 - \Psi(1 - \rho)}{\lambda t_0^\rho \Gamma(1 - \rho)}.
\]

Since

\[
\frac{1}{\Gamma(1 - \rho)} = \frac{1 - \rho}{\Gamma(2 - \rho)}, \quad \Psi(1 - \rho) = \Psi(2 - \rho) - \frac{1}{1 - \rho},
\]

the function \( f_1'(\rho) \) can be represented as follows

\[
f_1'(\rho) = \frac{1}{\lambda t_0^\rho} \left( (1 - \rho)[\ln t_0 - \Psi(2 - \rho)] + 1 \right). \tag{3.2}
\]

If \( \gamma \approx 0,57722 \) is the Euler-Mascheroni constant, then \(-\gamma < \Psi(2 - \rho) < 1 - \gamma\). By virtue of this estimate we may write

\[
-\Re(f_1'(\rho)) \geq \frac{\lambda_1}{|\lambda|^2} \frac{(1 - \rho)[\ln t_0 - (1 - \gamma)] + 1}{\Gamma(2 - \rho)t_0^\rho} \geq \frac{\lambda_1}{|\lambda|^2t_0^\rho}, \tag{3.3}
\]

provided

\[
\ln t_0 > 1 - \gamma \quad \text{or} \quad t_0 > t_0 = e^{1-\gamma} > 1.
\]

To estimate the derivative \( f_2'(\rho) \), we denote the integrand in (3.1) by \( F(\zeta, \rho) \):

\[
F(\zeta, \rho) = \frac{1}{2\pi i \rho \lambda t_0^\rho} \frac{e^{\xi/\rho}}{\xi + \lambda t_0^\rho}.
\]

Note, that the domain of integration \( \delta(1; \theta) \) also depends on \( \rho \). To take this circumstance into account when differentiating the function \( f_2'(\rho) \), we rewrite the integral (3.1) in the form:

\[
f_2'(\rho) = f_{2+}(\rho) + f_{2-}(\rho) + f_{21}(\rho),
\]

where

\[
f_{2\pm}(\rho) = e^{\pm i\theta} \int_1^\infty F(s e^{\pm i\theta}, \rho) ds,
\]

\[
f_{21}(\rho) = i \int_{-\theta}^\theta F(e^{iy}, \rho) e^{iy} dy = i\theta \int_{-1}^1 F(e^{i\theta s}, \rho) e^{i\theta s} ds.
\]

Let us consider the function \( f_{2+}(\rho) \). Since \( \theta = (\frac{\pi}{2} + \varepsilon) \rho \) and \( \zeta = s e^{i\theta} \), then

\[
e^{\xi/\rho} = e^{-s\frac{\pi}{2}(\varepsilon_1 - i\varepsilon_2)}, \quad \cos(\frac{\pi}{2} + \varepsilon) = -\varepsilon_1 < 0. \quad \sin(\frac{\pi}{2} + \varepsilon) = \varepsilon_2 > 0.
\]
The derivative of the function $f_{2+}(\rho)$ has the form

$$f'_{2+}(\rho) = \frac{1}{2\pi i \rho \lambda t_0} \int_1^\infty e^{-s \frac{1}{\rho} (\varepsilon_1 - i \varepsilon_2)} s e^{2ia \rho} \mathcal{M}(s) \frac{ds}{s e^{i \rho} + \lambda t_0^\rho},$$

where $a = \frac{\pi}{2} + \varepsilon$, and

$$\mathcal{M}(s) = \frac{\varepsilon_1 - i \varepsilon_2 s^{1/\rho} \ln s + 2ia - 1}{\rho} - \ln t_0 - \frac{ias e^{i \rho} + \lambda t_0^\rho \ln t_0}{s e^{i \rho} + \lambda t_0^\rho}.$$ 

It is not hard to verify, that

$$|s e^{i \rho} + \lambda t_0^\rho| \geq |\lambda| t_0^\rho \sin(\alpha - \theta) \geq \frac{2}{\pi} |\lambda| t_0^\rho \varepsilon.$$

Therefore we arrive at

$$|f'_{2+}(\rho)| \leq C \frac{\ln t_0}{|\lambda| t_0^\rho}.$$

Now we show that the real part of the derivative $\frac{d}{d \rho} e_{1,\lambda}(\rho)$ is negative. Taking into account estimate (3.2) and the estimates for $f'_{2+}$ and $f'_{21}$, we have

$$\Re \left( \frac{d}{d \rho} e_{1,\lambda}(\rho) \right) < -\frac{\lambda_1}{|\lambda|^2 t_0^\rho} + C \frac{1/\rho + \ln t_0}{(|\lambda| t_0^\rho)^2}.$$

In other words, this derivative is negative if

$$t_0^\rho > C \frac{1/\rho + \ln t_0}{\lambda_1}$$

for all $\rho \in [\beta_0, 1)$. Hence

$$t_0^\beta_0 > C \frac{1/\beta_0 + \ln t_0}{\lambda_1}.$$

Thus, there exists a number $T_0 = T_0(\xi^0, \beta_0) > 1$ (see (3.3)) such, that for all $t_0 \geq T_0$ we have the estimate

$$\Re \left( \frac{d}{d \rho} e_{1,\lambda}(\rho) \right) < 0 \text{ for all } \rho \in [\beta_0, 1].$$

The positivity of $R_C(\beta_1)$ follows from the explicit form of the function $f_1(\rho)$. Lemma 2.2, and therefore Theorem 2.1 are completely proved.
3.2. Proof of Lemma [2.4] We now turn to the proof of Lemma [2.4]. By the definition of contour \( \delta(1; \theta) \), we have for \( e_{2,\lambda}(\rho) \equiv \Gamma(t_0^{-1} E_{\rho,\lambda}(t_0^\rho)) \) the following equation (see [9], formula (2.29), p. 135)

\[
e_{2,\lambda}(\rho) = \frac{1}{\lambda^2 t_0^{\rho+1} \Gamma(-\rho)} + \frac{1}{2\pi i \rho \lambda^2 t_0^{\rho+1}} \int_{\delta(1; \theta)} \frac{e^{\sqrt{\gamma} \sqrt{\lambda}}}{\zeta + \lambda t_0^\rho} d\zeta = g_1(\rho) + g_2(\rho).
\]

Since the positivity of \( R_{RL}(1) \) is obvious, then in order to prove Lemma [2.4] it suffices to show that the derivatives of \( R_{RL}(\rho) \) is negative for all \( \rho \in [\beta_0, 1) \).

For the derivative \( g_1'(\rho) \) we have

\[
g_1'(\rho) = \ln t_0 - \Psi(-\rho) \frac{\rho(1-\rho)}{\lambda^2 t_0^{\rho+1} \Gamma(-\rho)}.
\]

To get rid of the singularity in the denominators, we use the equalities

\[
\frac{1}{\Gamma(-\rho)} = \frac{\rho}{\Gamma(1-\rho)} = -\frac{\rho(1-\rho)}{1(2-\rho)},
\]

\[
\Psi(-\rho) = \Psi(1-\rho) + \frac{1}{\rho} = \Psi(2-\rho) + \frac{1}{\rho} - \frac{1}{1-\rho}.
\]

Then the function \( g_1'(\rho) \) can be represented as follows

\[
g_1'(\rho) = \frac{\rho(1-\rho)\Psi(2-\rho) - \ln t_0 + 1 - 2\rho}{\lambda^2 t_0^{\rho+1} \Gamma(2-\rho)} = -\frac{g_{11}(\rho)}{\lambda^2 t_0^{\rho+1} \Gamma(2-\rho)}.
\]

Since \( \Psi(2-\rho) < 1 - \gamma \), then

\[
g_{11}(\rho) > \rho(1-\rho)[\ln t_0 - (1-\gamma)] + 2\rho - 1.
\]

For \( t_0 = e^{1-\gamma} \), one has \( \rho(1-\rho)[\ln t_0 - (1-\gamma)] + 2\rho - 1 = 1 \). Hence, \( g_{11}(\rho) \geq 1 \), provided \( t_0 \geq T_1 \), where

\[
T_1 = e^{1-\gamma} e^{2/\beta_0} > e^{3-\gamma} > 1.
\]

Thus, by virtue of (3.7), for all such \( t_0 \) we arrive at

\[
\text{sign}(\lambda_2^2 - \lambda_1^2) \Re(g_1'(\rho)) \leq -\frac{|\lambda_2^2 - \lambda_1^2|}{|\lambda_1^4 t_0^{\rho+1}|}.
\]

To estimate the derivative \( g_2'(\rho) \), we denote the integrand in (3.6) by \( G(\zeta, \rho) \):

\[
G(\zeta, \rho) = \frac{1}{2\pi i \rho \lambda^2 t_0^{\rho+1}} \cdot \frac{e^{\sqrt{\gamma} \sqrt{\lambda}}}{\zeta + \lambda t_0^\rho},
\]

and rewrite the integral (3.6) in the form:

\[
g_2(\rho) = g_{2+}(\rho) + g_{2-}(\rho) + g_{21}(\rho),
\]

where

\[
g_{2\pm}(\rho) = e^{\pm i \theta} \int_1^\infty G(s e^{\pm i \theta}, \rho) ds,
\]

and

\[
\text{sign}(\lambda_2^2 - \lambda_1^2) \Re(g_2'(\rho)) \leq -\frac{|\lambda_2^2 - \lambda_1^2|}{|\lambda_1^4 t_0^{\rho+1}|}.
\]
The derivative of the function $g_{2+}(\rho)$ has the form

$$g'_{2+}(\rho) = I \cdot \int_{1}^{\infty} e^{-s^{1/\rho}((\varepsilon_1 + i\varepsilon_2)s^{1/\rho} + 1)} \ln s + 2ia - \frac{1}{\rho} - \ln t_0 - \frac{ias e^{i\rho s + \lambda t_0^0 \ln t_0}}{e^{i\rho s + \lambda t_0^0}}.$$

By virtue of the inequality $|se^{i\rho s + \lambda t_0^0}| \geq \frac{2}{\pi} |\lambda| t_0^0 s^\varepsilon$ we arrive at

$$|g'_{2+}(\rho)| \leq \frac{C}{|\lambda| t_0^{2t_0^0 + 1}} \left[ 1 + \ln t_0 \right],$$

where the constant $C$ depends only on $\xi_0$.

The function $g'_{2-}(\rho)$ has exactly the same estimate.

Now consider the function $g_{21}(\rho)$. For its derivative we have

$$g'_{21}(\rho) = \frac{a}{2\pi i \lambda t_0^0 + 1} \cdot e^{e^{i\rho s} e^{i\rho s + \lambda t_0^0}} \left[ 2ias - \ln t_0 - \frac{ias e^{i\rho s + \lambda t_0^0 \ln t_0}}{e^{i\rho s + \lambda t_0^0}} \right] ds.$$

Therefore,

$$|g'_{21}(\rho)| \leq \frac{C}{|\lambda| t_0^{2t_0^0 + 1}} \left[ 1 + \ln t_0 \right].$$

Taking into account estimate (3.9) and the estimates for $g'_{2+}$ and $g'_{21}$, we have

$$\text{sign}(\lambda_1^2 - \lambda_2^2) \Re(e_{2,1}(\rho)) \leq -\frac{\lambda_1^2 - \lambda_2^2}{|\lambda| t_0^{2t_0^0 + 1}} + C \frac{1/\rho + \ln t_0}{|\lambda| t_0^{2t_0^0 + 1}}.$$

In other words, the left hand side is negative if

$$t_0^0 > \frac{C|\lambda|}{|\lambda_1^2 - \lambda_2^2|} \left( 1/\rho_0 + \ln t_0 \right).$$

Hence, there exists a number $T_1 = T_1(\xi_0, \beta_0) > 1$ (see (3.8)), such, that for all $t_0 \geq T_1$ the function $\text{sign}(\lambda_1^2 - \lambda_2^2) \Re(e_{2,1}(\rho))$ is negative.

Lemma 2.4 and therefore Theorem 2.5 are proved.
4. An example

To illustrate the theorems proved above consider the following Cauchy problem (see [30])

\begin{align}
(4.1) \quad D_\xi^\beta u_1(t, x) &= -D^2 u_1(t, x) - D u_2(t, x), \quad t > 0, \quad -\infty < x < \infty, \\
(4.2) \quad D_\xi^\beta u_2(t, x) &= -D u_1(t, x) - D^2 u_2(t, x), \quad t > 0, \quad -\infty < x < \infty, \\
(4.3) \quad u_1(0, x) &= \varphi_1(x), \quad u_2(0, x) = \varphi_2(x), \quad -\infty < x < \infty.
\end{align}

It is not hard to see that the symbol of the operator on the right hand side of \((4.1)-(4.2)\) is symmetric and has the representation

\begin{equation}
A(\xi) = \begin{bmatrix}
-\xi^2 & -\xi \\
-\xi & -\xi^2
\end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} -\xi^2 + \xi & 0 \\ 0 & -\xi^2 - \xi \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.
\end{equation}

As is seen from \((4.4)\) that \(\lambda_1(\xi) = -\xi^2 + \xi\) and \(\lambda_2(\xi) = -\xi^2 - \xi\). The solution \(U(t, x) = (u_1(t, x), u_2(t, x))\) to Cauchy problem \((4.1)-(4.3)\) has the representation:

\begin{align}
u_1(t, x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{1}{2} E_{\beta_1}((-\xi^2 + \xi) t^{\beta_1}) + \frac{1}{2} E_{\beta_2}((-\xi^2 - \xi) t^{\beta_2}) \right] \hat{\varphi}_1(\xi) d\xi \\
&\quad + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{1}{2} E_{\beta_1}((-\xi^2 + \xi) t^{\beta_1}) - \frac{1}{2} E_{\beta_2}((-\xi^2 - \xi) t^{\beta_2}) \right] \hat{\varphi}_2(\xi) d\xi;
\end{align}

\begin{align}
u_2(t, x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{1}{2} E_{\beta_1}((-\xi^2 + \xi) t^{\beta_1}) - \frac{1}{2} E_{\beta_2}((-\xi^2 - \xi) t^{\beta_2}) \right] \hat{\varphi}_1(\xi) d\xi \\
&\quad + \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{1}{2} E_{\beta_1}((-\xi^2 + \xi) t^{\beta_1}) + \frac{1}{2} E_{\beta_2}((-\xi^2 - \xi) t^{\beta_2}) \right] \hat{\varphi}_2(\xi) d\xi.
\end{align}

Moreover, obviously, \(\lambda_k(\xi) \leq 0, \ k = 1, 2,\) for all \(\xi\) satisfying the inequality \(|\xi| \geq 1\). It is not hard to verify, that

\begin{align}
K_{1,1}(\xi, \hat{\Phi}(\xi)) &= K_{2,1}(\xi, \hat{\Phi}(\xi)) = \frac{1}{2} \hat{\varphi}_1(\xi) + \frac{1}{2} \hat{\varphi}_2(\xi) \\
\end{align}

and

\begin{align}
K_{1,2}(\xi, \hat{\Phi}(\xi)) &= \frac{1}{2} \hat{\varphi}_1(\xi) - \frac{1}{2} \hat{\varphi}_2(\xi), \quad K_{2,2}(\xi, \hat{\Phi}(\xi)) = -\frac{1}{2} \hat{\varphi}_1(\xi) + \frac{1}{2} \hat{\varphi}_2(\xi).
\end{align}

Therefore for the corresponding determinant one has

\begin{equation}
|K_{j,l}(\xi^0, \hat{\Phi}(\xi^0))| = \frac{1}{2} \left( \hat{\varphi}_2^2(\xi^0) - \hat{\varphi}_1^2(\xi^0) \right)
\end{equation}

and condition \((2.11)\) has the form

\begin{equation}
\hat{\varphi}_2^2(\xi^0) \neq \hat{\varphi}_1^2(\xi^0), \quad \text{or} \quad |\hat{\varphi}_2(\xi^0)| \neq |\hat{\varphi}_1(\xi^0)|.
\end{equation}
In this case the unknown orders $\beta_1$ and $\beta_2$ are the unique roots of the following equations

\[ E_{\beta_1}(\lambda_1(\xi^0)t^{\beta_1}_0) = -\frac{d_1 + d_2}{\varphi_1(\xi^0) + \varphi_2(\xi^0)}, \]

\[ E_{\beta_2}(\lambda_2(\xi^0)t^{\beta_2}_0) = \frac{d_1 - d_2}{\varphi_2(\xi^0) - \varphi_1(\xi^0)}, \]

respectively.

**REFERENCES**

1. Sh. Alimov, R. Ashurov, Inverse problem of determining an order of the Caputo time-fractional derivative for a subdiffusion equation, J. Inverse Ill-Posed Probl. 28 (2020), Issue 5, pp. 651-658.
2. R. Ashurov, Yu. Fayziev, Determination of fractional order and source term in a fractional subdiffusion equation, arXiv:submit/3264960[math. AP]8 Jul 2020.
3. R. Ashurov, S. Umarov, Determination of the order of fractional derivative for subdiffusion equations, Fractional Calculus and Applied Analysis, 2020, 12, pp. 1-17.
4. R.R. Ashurov, R. Zunnunov, Initial-boundary value and inverse problems for subdiffusion equation in $\mathbb{R}^N$, Fractional Differential Calculus, 2020, 10 (2), 291–306.
5. H. Bateman, Higher transcendental functions, McGraw-Hill, 1953.
6. D. Benson, M. Meerschaert, J. Revielle, Fractional calculus in hydrology modeling: A numerical perspective. Advances in water resources, 51 479–497, 2013.
7. J. Cheng, J. Nakagawa, M. Yamamoto, T. Yamazaki, Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation, Inverse Prob., 4 (2009), 1–25.
8. S. Das, P.K. Gupta, A mathematical model on fractional Lotka-Volterra equations. Journal of theoretical biology, 277 (1), 1-6, 2011.
9. M. M. Dzherbashian, Integral Transforms and Representation of Functions in the Complex Domain (in Russian), M. NAUKA, 1966.
10. Ch. Guo, Sh. Fang, Stability and approximate analytic solutions of the fractional Lotka-Volterra equations for three competitors. Advanced difference equations, 219, 1-14, 2016.
11. Handbook of Fractional Calculus with Applications. Volume 2: Fractional Differential Equations. Editors: Kochubey A., Luchko Yu. De Gruyter, 2019.
12. R. Hilfer, Applications of Fractional Calculus in Physics. World Scientific, 2000.
13. R. Islam, A. Pease, D. Medina, T. Oraby, Integer Versus Fractional Order SEIR Deterministic and Stochastic Models of Measles. International Journal of Environmental Research and Public Health, 17 (6), 1-19, 2020.
14. J. Janno, Determination of the order of fractional derivative and a kernel in an inverse problem for a generalized time-fractional diffusion equation, Electronic J. Differential Equations. 216 (2016), 1-28.
15. N.A. Khan, O. Razzaq, S.P. Mondal, Q. Rubbab Fractional order ecological system for complexities of interacting species with harvesting threshold in imprecise environment. Advances in Difference Equations, 405, 1-34, 2019.
16. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations. Elsevier Science, 2006.
17. Z. Li, Y. Liu, M. Yamamoto, Inverse problems of determining parameters of the fractional partial differential equations, Handbook of fractional calculus with applications, DeGruyter, 2 (2019), 431–442.
18. Z. Li, Y. Luchko, M. Yamamoto, Analyticity of solutions to a distributed order time-fractional diffusion equation and its application to an inverse problem, Comput. Math. Appl. 73 (2017), 1041-1052.
19. Z. Li, M. Yamamoto, Uniqueness for inverse problems of determining orders of multi-term time-fractional derivatives of distribution equation, Appl. Anal., 94 (2015), 570–579.
20. T. Machado, A. Lopes, Relative fractional dynamics of stock markets. Nonlinear dynamics, 86 (3), 1613–1619, 2016.
21. R. Magin, Fractional Calculus in bioengineering. Critical reviews in biomedical engineering, 32 (1), 1-104, 2004.
22. F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models. Imperial College Press. 2010.
23. R. Metzler, J. Klafter, The random walk’s guide to anomalous diffusion: a fractional dynamics approach. Phys. Rep. 339, no. 1 1–77, 2000.
24. I. Podlubny, Fractional Differential Equations. Academic Press, 1998.
25. Rana S., Bhattacharya S., Pal J., Guerekata G., Chattopadhyay, Paradox of enrichment: A fractional differential approach with memory. Physica A: Statistical Mechanics and its Applications, 392 (17), 3610–3621, 2013.
26. F. Rihan, Numerical Modeling of Fractional-Order Biological Systems. Abstract and Applied Analysis, 2013, 1–13, 2013.
27. S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach Science Publishers, 1993.
28. E. Scalas, R. Gorenflo, F. Mainardi, Fractional calculus and continuous-time finance. Physica A: Statistical Mechanics and its Applications, 284 (1-4), 376–384, 2000.
29. S. Umarov, Introduction to Fractional and Pseudo-Differential Equations with Singular Symbols. Springer, 2015.
30. S. Umarov, R. Ashurov, Y. Chen, On a method of solution of systems of fractional pseudo-differential equations. Fractional Calculus and Applied Analysis, to appear.
31. S. Umarov, M. Hahn, K. Kobayashi, Beyond the Triangle: Brownian Motion, Ito Calculus and Fokker-Planck equations - Fractional Generalizations. World Scientific, 2018.
32. B. West, Physics of Fractal Operators. Springer, 2003.
33. A. Zeb, G. Zaman, M.I. Chohan, Sh. Momani, V.S. Erturk, Analytic numeric solution for SIRC epidemic model in fractional order. Asian J. of Math and Appl. 2013, 1-19, 2013.

1Institute of Mathematics of Academy of Sciences of Republic of Uzbekistan
Email address: ashurovr@gmail.com

2University of New Haven, Department of Mathematics, 300 Boston Post Road, West Haven, CT 06516, USA
Email address: sumarov@newhaven.edu