Several Methods for Solving the Extended Lyapunov Matrix Equations

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Abstract. This suggests four methods for solving the extended Lyapunov matrix equation. The first one is based on the Jordan canonical form. The second one considers the Schur reduction method. The third one takes into account the eigenvalue decomposition of the matrix. The last one is the Squared Smith method.

1. Introduction

The Lyapunov equation plays a significant role in the theory of control, communication, and power systems. It is perhaps best known for its applications in the stability analysis of dynamical systems [1]. There are many cases that require the numerical solutions of (possibly coupled) sets of Lyapunov equations. The first of the cases is model-order reduction for linear dynamical systems. During the last twenty years, almost methods for model-order reduction were based on the solutions of their corresponding Lyapunov equations.

The extended Lyapunov matrix equation

\[ A^T P + PA - 2\sigma P + Q = 0 \]  (1.1)

(where A, P and Q are real n×n with P and Q symmetric, \(\sigma\) is a real constant, and the superscript T denotes matrix transposition) arises in estimating transient response [2] and in connection with the order numbers of the linear differential equations [3]. Kalman and Bertram [2] have shown that the real parts of the eigenvalues of A are less than \(\sigma\) if and only if for any positive definite symmetric matrix Q there exists a unique positive definite symmetric matrix P satisfying (1.1).

When \(\sigma = 0\), (1.1) reduces to the standard Lyapunov matrix equation

\[ A^T P + PA + Q = 0 \]  (1.2)

Equation (1.2) has found application in stability theory [2,4], optimal control [2], and other problems. It is well known [4] that if \(\lambda_i + \lambda_j \neq 0\) for all \(i, j = 1, 2, \ldots, n\), where \(\lambda_i\) is the eigenvalue of A, then for each symmetric matrix Q there exists a unique symmetric matrix P satisfying (1.2). From recent years there already exist some methods for the Lyapunov equations (1.2). For example, the GMRES algorithm [5] for the large Lyapunov equations has been proposed. The Bartel-Stewart’s [6] algorithm is a good method, which is not suitable for solution of the associated Lyapunov equations.

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The paper [7] written by TROCH investigates the relation between the discrete Lyapunov equations and the continuous Lyapunov equations. For some special cases there are corresponding methods. Many numerical methods for solving (1.2) can also be applied to (1.1) if (1.1) is written in the form

\[(A - \sigma I)^T P + P(A - \sigma I) + Q = 0\]  

(1.3)

This paper is organized as follows. In section 2 we will introduce the Jordan canonical form method, and give its corresponding algorithm. In section 3 the Schur reduction method is presented, and the corresponding algorithm is given. In section 4 we discuss the eigenvalue decomposition method and its algorithm. In section 5 we will introduce the squared Smith method, and give its convergent proof and error bound, in the following its algorithm is presented. In section 6 we introduce the extended discrete Lyapunov matrix equation simply, the previous four methods can be similarly applied to this matrix equation. Finally we offer some concluding remarks in section 7.

2. The Jordan canonical form method

In this section we will present the Jordan canonical method. This method is based on the transformation of A to its Jordan canonical form. Let

\[
J = \begin{pmatrix}
J_1 & 0 \\
J_2 & \ddots \\
0 & J_m
\end{pmatrix}, m \leq n
\]

be a Jordan canonical representation of A, where the \(J_i\) is the Jordan block of the form

\[
J_i = \begin{pmatrix}
\lambda_i & 1 & 0 \\
0 & \lambda_i & \ddots \\
0 & 0 & \ddots & 1 \\
0 & 0 & \cdots & \lambda_i
\end{pmatrix}
\]

and where it is assumed that \(\lambda_i + \lambda_j \neq 2\sigma\) for all \(i, j\). Let S be a similarity transformation matrix taking A to its Jordan canonical form J, that is \(J = S^{-1} AS\). Defining \(B = S^T PS\) and \(C = S^T QS\). Multiplying on the left by \(S^T\) and on the right by S, and replacing A by \(SJS^{-1}\), one obtains

\[J^T B + BJ - 2\sigma B + C = 0\]  

(2.1)

Now B and C may be partitioned into submatrices of appropriate dimensions so as to be compatible with J:

\[
B = \begin{pmatrix}
B_{11} & B_{12} & \cdots & B_{1m} \\
B_{21} & B_{22} & \cdots & B_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
B_{m1} & B_{m2} & \cdots & B_{mm}
\end{pmatrix}, C = \begin{pmatrix}
C_{11} & C_{12} & \cdots & C_{1m} \\
C_{21} & C_{22} & \cdots & C \\
\vdots & \vdots & \ddots & \vdots \\
C_{m1} & C_{m2} & \cdots & C_{mm}
\end{pmatrix}
\]

where, due to symmetry, \(B_{ji} = B_{ij}^T\) and \(C_{ij} = C_{ji}^T\) for all \(i, j = 1, 2, \cdots, m\). By expanding of(2.1), facilitated by the block diagonal form of J, one obtains the expressions
\[ J_i^T B_j + B_j J_i - 2\sigma B_j + C_j = 0, i, j = 1, 2, \ldots, m \]  
\hfill (2.2)

Let the elements of \( B_j \) and \( C_j \) be denoted by \( b_{r\sigma}^{ij} \) and \( c_{r\sigma}^{ij} \), respectively, where \( r = 1, 2, \ldots, I \) and \( s = 1, 2, \ldots, j \). Due to its relatively simple form, (2.2) can now be solved for the elements of \( B_j \), yielding the formula

\[ b_{r\sigma}^{ij} = \frac{-b_{r\sigma-1}^{ij} - b_{r\sigma-1}^{ij} - c_{r\sigma}^{ij}}{\lambda_i + \lambda_j - 2\sigma} \]  
\hfill (2.3)

By convention, \( b_{r\sigma}^{ij} = 0 \) whenever \( r = 0 \) or \( s = 0 \) or both. Since, by assumption \( \lambda_i + \lambda_j \neq 2\sigma \), \( b_{r\sigma}^{ij} \) is well defined by (2.3). Once \( B \) is determined directly from the elements \( c_{ij} \) of \( C \), one can obtain \( P \) from \( P = S^{-T} BS^{-1} \).

**Algorithm 1:**

**Step 1** Determine \( S \), \( S^T \), \( S^{-1} \), \( S^{-T} \) and \( J \) using available technique.

**Step 2** Compute \( C \) using \( C = S^T QS \).

**Step 3** Determine \( B \) by repeated application of (2.3).

**Step 4** Obtain \( P \) by \( P = S^{-T} BS^{-1} \).

3. The Schur reduction method

In this section we will discuss the Schur reduction method. In this method we will use the QR algorithm to compute the real Schur decomposition

\[ U^T A U = R \]  
\hfill (3.1)

where \( R \) is upper quasi-triangular, and \( U \) is orthogonal. (A quasi-triangular matrix is triangular with possible nonzero \( 2 \times 2 \) blocks along the diagonal.) Substitute (3.1) into (1.1), we can get

\[ URU^T P + PUR^T U^T - 2\sigma P + Q = 0 \]  
\hfill (3.2)

Premultiplying (3.2) by \( U^T \) and postmultiplying by \( U \), then (3.2) can be expressed by

\[ RU^T PU + U^T PUR^T - 2\sigma U^T PU + U^T QU = 0 \]  
\hfill (3.3)

Let \( \bar{P} = U^T PU \) and \( \bar{Q} = U^T QU \), then we have

\[ R\bar{P} + PR^T - 2\sigma P + \bar{Q} = (R - 2\sigma I)\bar{P} + \bar{PR}^T + \bar{Q} \]  
\hfill (3.4)

Assuming \( r_{kk-1} \) is zero, then it follows that

\[ (R + (r_{kk} - 2\sigma I))p_k = -q_k - \sum_{j=k+1}^n r_{kj} p_j \]  
\hfill (3.5)

where \( \bar{P} = (\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n) \) and \( \bar{Q} = (\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_n) \). Thus \( \bar{p}_k \) can be found from \( \bar{p}_{k+1}, \ldots, \bar{p}_n \) by solving an upper quasi-triangular system.

**Algorithm 2:**

**Step 1** Use the QR algorithm to compute the real Schur decomposition (3.1)
step 2 Substitute (3.6) into Eq(1.1) and multiply it on the left by $U^T$ and on the right by $U$, then we can have (3.3)

step 3 Let $\overline{P} = U^T PU$ and $\overline{Q} = U^T QU$, then (3.4)

step 4 Determine $\overline{P}$ by solving (3.4):
for $k = 1$ to $n - 1$

$$(R + (r_{kk} - 2\sigma)I)p_k = -q_k - \sum_{j=k+1}^{n} r_{kj} p_j$$

end

step 5 Thus $P = U\overline{P}U^T$.

4. The eigenvalue decomposition method
In this section we will investigate the eigenvalue decomposition method. Assume that $A$ can be diagonalized, so we can get the following relation:

$$A = VDV^{-1}$$  \hspace{1cm} (4.1)

With the eigenvalues $d_i (i = 1, 2, \ldots, n)$ of $A$ on the diagonal of the diagonal matrix $D$ and the corresponding eigenvectors forming the columns of the matrix $V$. Substitute (4.1) into Eq(1.1), then we can get

$$V^{-T}DV^T P + PVDV^{-1} - 2\sigma P + Q = 0$$  \hspace{1cm} (4.2)

Multiply (4.2) on the left by $V^T$ and on the right by $V$, we obtain

$$DV^TPV + V^TPVDV - 2\sigma V^TPV + V^TQV = 0$$  \hspace{1cm} (4.3)

Let $\hat{P} = V^TPV$ and $\hat{Q} = V^TQV$, then (4.3) can be expressed as

$$D\hat{P} + \hat{PD} - 2\sigma \hat{P} + \hat{Q} = 0$$  \hspace{1cm} (4.4)

We can get the solution of Eq(1.1) by solving (4.4) easily. In the following we will give the corresponding algorithms.

**Algorithm 3:**

step 1 Calculate the eigenvalue decomposition of $A$ from (1.1)

step 2 Substitute (4.5) into Eq(1.1) and Multiply it on the left by $V^T$ and on the right by $V$, then we can have (4.2)

step 3 Let $\hat{P} = V^TPV$ and $\hat{Q} = V^TQV$, then (4.2) can be written as (4.3)

step 4 Determine $\hat{P}$ by solving (4.3):
for $i = 1$ to $n$
for $j = 1$ to $n$

$$\hat{p}_{ij} = -\hat{q}_{ij}/(d_i + d_j - 2\sigma)$$

end

end

step 5 Thus $P = V^{-T}\hat{P}V^{-1}$.

5. The squared Smith method
In this section we will present the squared Smith method. We will solve (1.3) instead of (1.1) by the squared Smith method.
Let \((A - \sigma I) = W\), then (1.3) can be expressed by

\[ W^T P + PW + Q = 0 \]  

(5.1)

Since \(W\) and \(W^T\) have no common eigenvalue, Eq(5.1) has a unique solution. Let \(I\) be the \(n \times n\) identity matrix, and \(q\) be a non zero real number. Eq(5.1) can be written as the following equivalent form

\[
(qI - W^T)P(qI - W) - (qI + W^T)P(qI + W) = 2qQ
\]

(5.2)

Multiplying on the left by \((qI - W^T)^{-1}\) and on the right by \((qI - W)^{-1}\) (assuming, of course, that these matrices are nonsingular), we obtain

\[ P - EPF = G \]  

(5.3)

with

\[ E = (qI - W^T)^{-1}(qI + W^T), F = E^T = (qI + W)(qI - W)^{-1}, G = 2q(qI - W^T)^{-1}Q(qI - W)^{-1}. \]

Thus

\[ P = \sum_{i=0}^{\infty} E^i G F^i \]  

(5.4)

which converges if the special radius of \(E\) and \(F\) are strictly less than 1. In the following we will give its convergent proof.

**Lemma 1** If \(W\) is stable and \(q > 0\), then \(\rho(E) < 1\) and \(\rho(F) < 1\).

**Proof** From the case that the real parts of the eigenvalues of \(A\) are less than \(\sigma\) if and only if for any positive definite symmetric matrix \(Q\) there exists a unique positive definite symmetric matrix \(P\) satisfying (1.1), then \(\Re(\lambda_i) < 0 (i = 1, 2, \ldots, n)\), where \(\lambda_i\) is the eigenvalue of \(W\) since the eigenvalues of \(W\) are \(\lambda_i - \sigma\). The eigenvalues of \(E\) and \(F\) are respectively

\[
\lambda_E = \frac{q + \lambda_{wT}}{q - \lambda_{wT}}, \quad \lambda_F = \frac{q + \lambda_{w}}{q - \lambda_{w}}.
\]

Since \(\Re(\lambda_i) < 0\) and \(q > 0\) then \(\rho(E) < 1\) and \(\rho(F) < 1\).

**Theorem 2** If \(\rho(E) < 1\) and \(\rho(F) < 1\), then the series \(P = \sum_{i=0}^{\infty} E^i G F^i\) is convergent.

**Proof** Since \(\rho(E) < 1\) and \(\rho(F) = \rho(E^T) < 1\), so there exists \(0 < r^2 < 1\) such that \(\rho(E) < \frac{1}{r^2}\) and \(\rho(F) < \frac{1}{r^2}\). Then we can find a consistent norm such that \(\| E \|_r < \frac{1}{r^2}\) [9].

Since all norms on a finite dimensional space are equivalent, one can find a constant \(\gamma\) such that \(\| A \|_r \leq \gamma \| A \|_i\), for all matrices \(A\), where the constant \(\gamma\) depends only on \(A\) and the choice of \(r\).

We can obtain

\[ \| E^i \|_r \leq \gamma \| E^i \|_i \leq \gamma \| E \|_i \leq \gamma r^2 \frac{1}{i} \]  

(5.5)

Then
Let $\gamma^2 \norm{G}_2 = M$, then Eq. (5.6) can be expressed as

$$\norm{E'GF^i}_2 \leq M \gamma^i$$

(5.7)

From (5.7) we can obtain $\norm{E'GF^i}_2 \to 0$ if $i$ is large enough, so the series $P = \sum_{i=0}^{\infty} E'GF^i$ is convergent.

In terms of Lemma 1 and Theorem 2 we obtain easily the following conclusion.

**Theorem 3** If $W$ is stable and $q > 0$, then the series $P = \sum_{i=0}^{\infty} E'GF^i$ is convergent.

In general, the convergence of $P = \sum_{i=0}^{\infty} E'GF^i$ is slow. So we will consider the sequence of matrices $P_k$ defined by

$$P_k = P_{k-1} + E^{2k-1} P_{k-1} F^{2k-1}$$

with $P_0 = G$. It can be written as

$$P = P_0 + EP_0 F$$
$$P_2 = P_0 + E^2 P_0 F^2$$
$$\cdots$$
$$P_{k-1} = P_{k-2} + E^{2k-2} P_{k-2} F^{2k-2}$$
$$P_k = P_{k-1} + E^{2k-1} P_{k-1} F^{2k-1}.$$

From the above equations we can get the following truncation series:

$$P_k = \sum_{i=0}^{2k-1} E'GF^i$$

(5.8)

We will show that $P_k$ is a good approximation to $P$ by their error bound.

**Theorem 4** Given $P$ in (5.1) and $P_k$ in (5.8), then

$$\norm{P - P_k}_2 \leq \sum_{i=2}^{\infty} \norm{E^i}_2 \norm{\gamma^i}_2 \norm{G}_2 \norm{F^i}_2 \leq M \gamma^2/(1 - r).$$

(5.9)

**proof** Notice that

$$\norm{P - P_k}_2 = \sum_{i=2}^{\infty} \norm{E'GF^i}_2$$

(5.10)

From (5.10) and theorem 2 we can get the following inequality:
\[ \| P - P_k \|_2 \leq \sum_{i=2}^{\infty} \| E^{i-1} G \|_2 \| F^{i-1} \|_2 \leq \sum_{i=2}^{\infty} M r^i = M r^2 / (1 - r) \].

**Remark:** From Theorem 4 we can see that \( P_k \) is quadratically convergent.

**Algorithm 4:**

**Step 1** Choose an appropriate \( q > 0 \) and calculate the matrices \( E, F \) and \( G \):

\[ E = (qI - W)^{-1} (qI + W^T), F = E^T \]

\[ G = 2q(qI - W^T)^{-1} Q(qI - W)^{-1} \]

**Step 2** Let \( P_0 = G \), then

For \( i = 1 \) to \( k \)

\( H = I \)

For \( j = 1 \) to \( 2i - 1 \)

\( H = EH \)

end

\( P_i = P_{i-1} + HP_{i-1}H^T \)

end

**Step 3** Given an \( \varepsilon > 0 \).

If \( \| P_k - P_{k-1} \|_2 < \varepsilon \), then stop;

otherwise choose a \( k_i < k \) and let \( k = k_i \), return to step 2.

### 6. The solution of the extended discrete Lyapunov matrix equation

In this section we will discuss the solution of the extended discrete Lyapunov matrix equation simply. The previous four method can be applied to this matrix equation without any further problems, where we only introduce the extended discrete Lyapunov matrix equation simply. The extended discrete Lyapunov matrix equation[8]

\[ \rho^{-2} A^T P A - P + Q = 0 \]  

(6.1)

where \( A, P \) and \( Q \) are real \( n \times n \) matrices with \( P \) and \( Q \) symmetric, \( \rho \) is a real nonzero constant, is the discrete system analog of the extended Lyapunov matrix equation

\[ A^T P + P A - 2\sigma P + Q = 0 \]  

(6.2)

When \( \rho = 1 \), (5.1) reduces to the standard discrete Lyapunov matrix equation

\[ A^T P A - P + Q = 0 \]  

(6.3)

Several methods have been proposed for solving (5.3) for \( P \). These techniques thus apply to (5.1) only for the case \( \rho = 1 \). However, some of them can be applied to (5.1) if (5.1) is written as

\[ (\rho^{-1} A)^T P (\rho^{-1} A) - P + Q = 0 \]  

(6.4)

So we can get similar four methods for solving (5.1), where we will not present them in detail.

### 7. Conclusions

In this paper we discuss the solution of the extended Lyapunov matrix equations. In the following we give four methods and their corresponding algorithms. The first one is based on the Jordan canonical
form of A. The second one considers the Schur reduction method. The third one takes into account the
eigenvalue decomposition of A. The last one is the Squared Smith method, in this method we give its
convergent proof and error bound. At last we investigate the solution of the extended discrete
Lyapunov matrix equation, where we don’t give the methods in detail, but these methods are similar to
the previous four methods for solving the extended Lyapunov matrix equations.

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References
[1] J.LaSalle, S.Lefschetz, 1961 Stability of Lyapunov’s Direct Method (New York: Academic Press)
[2] R.E.Kalman, J.E.Bertram, 1960 J.Basic Eng. 82 371
[3] W.Hahn, 1963 Theory and application of Liapunov’s Direct Method. (Englewood Cliffs N.J.: Prentice-Hall)
[4] J.LaSalle, S.Lefschetz, 1961 Stability by Liapunov’s Direct Method with Applications. (New York: Academic Press)
[5] K.Jbilou, A.Messaoudi, H.Sadok, 1999 Appl. Number. Math. 31 97
[6] R.H.Bartels, G.W.Stewart, 1972 Comm. ACM. 15 820
[7] TROCH, 1988 IEEE Trans. Automat. Contr. 33 944
[8] R.E.Kalman, J.E.Bertram, 1960 Trans. ASME, J. Basic Eng 82 394
[9] G.W.Stewart, J.G.Sun, 1990 Matrix perturbation Theory Academic Press