COMMON PERIODIC POINTS FOR A CLASS OF CONTINUOUS COMMUTING MAPPINGS ON AN INTERVAL

SHIN MIN KANG and WEILI WANG

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The existence of common periodic points for a family of continuous commuting self-mappings on an interval is proved and two illustrative examples are given in support of our theorem and definition.

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1. Introduction and preliminaries. All mappings considered here are assumed to be continuous from the interval $I = [u, v]$ to itself. Let $F(f)$ and $P(f)$ be the set of fixed and periodic points of $f$, respectively, and let $\overline{P(f)}$ be the closure of $P(f)$. Denote $L(x, f)$ by the set of limit points of the sequence $\{f^n(x)\}_{n=0}^{\infty}$. By Schwartz’s theorem [4], it is easy to show that $L(x, f) \cap \overline{P(f)} \neq \emptyset$ for each $x$ in $I$. Obviously, $F(f)$ is a closed set and $\emptyset \neq F(f) \subset P(f)$. Define the classes of mappings

$$
A = \{f : I \to I \mid F(f) = [a_f, b_f], \ a_f \leq b_f \}, \\
B = \{f : I \to I \mid P(f) = F(f) \}, \\
D = \{f : I \to I \mid P(f) = P(f) \}.
$$

(1.1)

The following definition was introduced by Cano [2].

DEFINITION 1.1. A class of mappings $T$ is said to be an $H$-class if $T = T' \cup \{h\}$, where $T'$ is any subset of $A \cup B$ composed of commuting mappings and $h$ is any mapping which commutes with the elements of $T'$.

Boyce [1] and Huneke [3] showed that if $f$ and $g$ are two commuting self-mappings of $I$, then $f$ and $g$ need not have a common fixed point in $I$. Cano [2] proved the following theorem.

THEOREM 1.2. There is a common fixed point for every $H$-class in $I$.

In this note, we consider a larger class of mappings which has the common periodic point property and properly contains the class $H$ considered by Cano. Two illustrative examples are given in support of our theorem and definition.

We first introduce the following definition.
**Definition 1.3.** A class of mappings $T$ is said to be a $C$-class if $T = T' \cup \{ h \}$ and $T$ is a commuting family of mappings, where $T'$ is any subset of $A \cup D$ and $h$ is any mapping.

Obviously, $B \subset D$. The following example proves that $B$ is a proper subset of $D$.

**Example 1.4.** Let $I = [0,1]$ and $f(x) = 1 - x$. It is easy to show that $F(f) = \{1/2\} \neq [0,1] = P(f) = P(f)$, that is, $f \in D$ and $f \not\in B$.

**Remark 1.5.** Clearly, $H$-class is $C$-class, but the converse is not true.

2. Main results. Our main result is as follows.

**Theorem 2.1.** There is a common periodic point for every $C$-class in $I$.

**Proof.** Let $T$ be a $C$-class and $T_1$ a finite subset of $T$. We can write $T_1$ as

$$T_1 = \{ f_1, f_2, \ldots, f_n \} \cup \{ h \} \cup \{ g_1, g_2, \ldots, g_m \},$$

where $f_i \in A$, $i = 1, 2, \ldots, n$, and $h$ is a possible arbitrary mapping that commutes with the elements of $T$, $g_j \in D$, $j = 1, 2, \ldots, m$. Suppose that there are different $i, k \in \{1, 2, \ldots, n\}$ such that $F(f_i) \cap F(f_k)$ is not an interval, that is, $F(f_i) \cap F(f_k) = \emptyset$. Let $F(f_i) = [a_i, b_i]$ and $F(f_k) = [a_k, b_k]$. Clearly, $\max\{a_i, a_k\} > \min\{a_i, a_k\}$. Without loss of generality, we can assume $a_k > a_i$. Since $f_i$ and $f_k$ commute and $a_i, b_i \in F(f_i)$, then $f_i(f_k(a_i)) = f_k(f_i(a_i)) = f_k(a_i)$, that is, $f_k(a_i) \in F(f_i)$. Hence, $f_k(a_i) > a_i$. Similarly, we can show that $f_k(b_i) < b_i$. Let $w(x) = f(x) - x$ for $x \in F(f_i)$. Since $w(a_i) > 0$ and $w(b_i) < 0$, there is $c \in (a_i, b_i)$ such that $w(c) = 0$, that is, $f_k(c) = c$. Therefore,

$$c \in (a_i, b_i) \cap F(f_k) \subset F(f_i) \cap F(f_k) \neq \emptyset,$$

a contradiction. Thus, $F(f_i) \cap F(f_k)$ is an interval for any two distinct $i, k \in \{1, 2, \ldots, n\}$. It is easy to show that $\cap_{i=1}^{n} F(f_i)$ is an interval. Let $\cap_{i=1}^{n} F(f_i) = [a, b]$. By the commutativity of $h$ with the $f_i$'s, $h$ takes $[a, b]$ into $[a, b]$, and so, it must have a fixed point $z \in [a, b]$. Now, $\{g_1^n(z)\}_{n=0}^{\infty}$ has a limit point $z_1 \in P(g_1)$ because $P(g_1)$ is a closed set. Clearly, there exists a subsequence $\{g_1^{n_k}(z)\}_{k=1}^{\infty}$ of $\{g_1^n(z)\}_{n=1}^{\infty}$ such that

$$\lim_{k \to \infty} g_1^{n_k}(z) = z_1 = g_1'(z_1) \in P(g_1).$$

Since $z \in (\cap_{i=1}^{n} F(f_i)) \cap F(h)$, by (2.3), we have

$$f_i(g_1^{n_k}(z)) = g_1^{n_k}(f_i(z)) = g_1^{n_k}(z) \to z_1, \quad k \to \infty,$$

$$f_i(g_1^{n_k}(z)) \to f_i(z_1), \quad k \to \infty.$$
From (2.4), we have $f_i(z_1) \in F(f_i)$. Using the same method, we can show that $z_1 \in F(h)$. So,

$$z_1 \in (\cap_{i=1}^n F(f_i)) \cap F(h) \cap P(g_1). \quad (2.5)$$

Similarly, $\{g_j^n(z_{j-1})\}_{n=0}^\infty, j = 2, 3, \ldots, m$, has a limit point

$$z_j \in (\cap_{i=1}^n F(f_i)) \cap F(h) \cap (\cap_{i=1}^j P(g_i)). \quad (2.6)$$

Thus,

$$\emptyset \neq (\cap_{i=1}^n F(f_i)) \cap F(h) \cap (\cap_{j=1}^m P(g_j)) \quad (2.7)$$

which implies that

$$\emptyset \neq (\cap_{f \in T \cap A} F(f)) \cap F(h) \cap (\cap_{f \in T \cap D} P(f)) \subset \cap_{f \in T} P(f) \quad (2.8)$$

by the compactness of $I$. When $T$ contains no such $h$, $T \cap A = \emptyset$, or $T \cap D = \emptyset$, we have the same result from the above proof. This completes the proof. □

We at last give an example in which Theorem 2.1 holds but Theorem 1.2 is not applicable.

**Example 2.2.** Let $I = [-1, 1]$,

$$f(x) = \begin{cases} 1 + x & \text{if } x \in [-1, 0], \\ 1 - x & \text{if } x \in (0, 1), \end{cases} \quad g(x) = \begin{cases} -x & \text{if } x \in [-1, 0], \\ x & \text{if } x \in (0, 1). \end{cases} \quad (2.9)$$

Let $h$ be a continuous mapping and commute with $f$ and $g$. It is easy to see that

$$F(f) = \left\{ \frac{1}{2} \right\}, \quad P(f) = \overline{P(f)} = [0, 1], \quad F(g) = [0, 1]; \quad (2.10)$$

that is, $f \in D$, $f \in B$, and $g \in A$. Clearly, $f$ and $g$ are continuous and

$$f(g(x)) = g(f(x)) = \begin{cases} 1 + x & \text{if } x \in [-1, 0], \\ 1 - x & \text{if } x \in (0, 1). \end{cases} \quad (2.11)$$

Thus, $\{f, g, h\}$ is a $C$-class but $\{f, g, h\}$ is not an $H$-class. Hence, Theorem 2.1 holds, that is, $f$, $g$, and $h$ have a common periodic point. But Theorem 1.2 is not applicable.

**Remark 2.3.** Example 2.2 and Remark 1.5 prove the greater generality of Theorem 2.1 over Theorem 1.2.
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Shin Min Kang: Department of Mathematics, Gyeongsang National University, Chinju 660-701, Korea
E-mail address: smkang@nongae.gsnu.ac.kr

Weili Wang: Basis Courses Teaching Department, Dalian Institute of Light Industry, Dalian, Liaoning 116034, China
