LIE ALGEBRAS WITH PRESCRIBED
\textit{sl}_3 DECOMPOSITION

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Abstract. In this work, we consider Lie algebras \( \mathcal{L} \) containing a subalgebra isomorphic to \( \text{sl}_3 \) and such that \( \mathcal{L} \) decomposes as a module for that \( \text{sl}_3 \) subalgebra into copies of the adjoint module, the natural 3-dimensional module and its dual, and the trivial one-dimensional module. We determine the multiplication in \( \mathcal{L} \) and establish connections with structurable algebras by exploiting symmetry relative to the symmetric group \( S_4 \).

1. Introduction

The Lie algebra \( \mathfrak{gl}_{n+k} \) of \((n + k) \times (n + k)\) matrices over a field \( \mathbb{F} \) of characteristic 0 under the commutator product \( [x, y] = xy - yx \), when viewed as a module for the copy of \( \mathfrak{gl}_n \) in its northwest corner, decomposes into \( k \) copies of the natural \( n \)-dimensional \( \mathfrak{gl}_n \)-module \( V = \mathbb{F}^n \), \( k \) copies of the dual module \( V^* = \text{Hom}(V, \mathbb{F}) \), a copy of the Lie algebra \( \mathfrak{gl}_k \) in its southeast corner, and the copy of \( \mathfrak{gl}_n \):\[
\mathfrak{gl}_{n+k} = \mathfrak{gl}_n \oplus (V \otimes B) \oplus (V^* \otimes C) \oplus \mathfrak{gl}_k.
\]

As a result, we may write \( \mathfrak{gl}_{n+k} \cong \mathfrak{sl}_n \oplus \mathfrak{gl}_k \), where \( B = C = \mathbb{F}^k \). This second expression reflects the decomposition of \( \mathfrak{gl}_{n+k} \) as a module for \( \mathfrak{sl}_n \oplus \mathfrak{gl}_k \). When restricted to \( \mathfrak{sl}_n \), the \( \mathfrak{gl}_n \)-modules \( V \) and \( V^* \) remain irreducible, while \( \mathfrak{gl}_n \) decomposes into a copy of the adjoint module and a trivial \( \mathfrak{sl}_n \)-module spanned by the identity matrix: \( \mathfrak{gl}_n = \mathfrak{sl}_n \oplus \mathbb{F}I_n \). Thus, we have the \( \mathfrak{sl}_n \) decomposition of \( \mathfrak{gl}_{n+k} \):
\[
\mathfrak{gl}_{n+k} \cong \mathfrak{sl}_n \oplus (V \otimes B) \oplus (V^* \otimes C) \oplus (\mathfrak{gl}_k \oplus \mathbb{F}I_n),
\]
where $\mathfrak{gl}_k \oplus \mathbb{F}l_n$ is the sum of the trivial $\mathfrak{sl}_n$-modules in $\mathfrak{gl}_{n+k}$. Decompositions such as (1.1) also arise in the study of direct limits of simple Lie algebras and give insight into their structure.

Indeed, suppose we have a chain of homomorphisms,

\[(1.2) \quad \mathfrak{g}^{(1)} \xrightarrow{\varphi_1} \mathfrak{g}^{(2)} \xrightarrow{\varphi_2} \ldots \rightarrow \mathfrak{g}^{(i)} \xrightarrow{\varphi_i} \mathfrak{g}^{(i+1)} \rightarrow \ldots ,\]

where $\mathfrak{g}^{(i)} = \mathfrak{sl}(\mathbb{V}^{(i)})$. Assume that $\mathfrak{sl}(\mathbb{V})$ is a fixed term in the chain for some $\mathbb{V} = \mathbb{V}^{(j)}$, and $\dim \mathbb{V} = n$. We identify $\mathfrak{sl}(\mathbb{V})$ with $\mathfrak{sl}_n$ by choosing a basis for $\mathbb{V}$ and assume that $\mathbb{V}^{(i)} = \mathbb{V} \oplus k_i \oplus \mathbb{F} \oplus z_i$ as a module for $\mathfrak{sl}_n$ for $i \geq j$. Then the limit Lie algebra $L = \lim_{\rightarrow} \mathfrak{g}^{(i)}$ admits a decomposition relative to $\mathfrak{sl}_n$,

\[(1.3) \quad L \cong (\mathfrak{sl}_n \otimes A) \oplus (\mathbb{V} \otimes B) \oplus (\mathbb{V}^* \otimes C) \oplus \mathfrak{s},\]

where $\mathfrak{s}$ is the sum of the trivial $\mathfrak{sl}_n$-modules (see [3, Sec. 5]). Bahturin and Benkart in [3, Sec. 4] study Lie algebras having such a decomposition and describe the multiplication in $L$ and the possibilities for $A, B, C, \mathfrak{s}$ when $\dim \mathbb{V} \geq 4$. When $\dim \mathbb{V} = 2$, then $\mathbb{V}^*$ is isomorphic to $\mathbb{V}$ as a module for $\mathfrak{sl}_2 = \mathfrak{sl}(\mathbb{F}^2)$. In this case, a Lie algebra having a decomposition, $L = (\mathfrak{sl}_2 \otimes A) \oplus (\mathbb{V} \otimes B) \oplus \mathfrak{s}$ is graded by the root system $BC_1$, and its structure has been described in [4].

In this paper, we investigate the missing case when $\dim \mathbb{V} = 3$, which presents very distinctive features. For direct limit Lie algebras of the type considered above, we could, of course, choose a larger space $\mathbb{V}^{(j)}$ having $\dim \mathbb{V}^{(j)} \geq 4$ and apply the results of [3]. However, there are many examples of Lie algebras which admit very interesting decompositions as in (1.3) for $n = 3$. The exceptional simple Lie algebras provide examples of this phenomenon.

**Example 1.1.** Each exceptional simple Lie algebra $L$ over an algebraically closed field of characteristic 0 has an automorphism $\psi$ of order 3 that corresponds to a certain node in the Dynkin diagram of the associated affine Lie algebra. The node is marked with a "3" in [10, TABLE Aff 1]. Removing that node gives the Dynkin diagram of a finite-dimensional semisimple Lie algebra $\mathfrak{sl}_3 \oplus \mathfrak{s}$, which is the subalgebra of fixed points of the automorphism $\psi$. The Lie algebra $\mathfrak{s}$ is the centralizer of $\mathfrak{sl}_3$ in $L$; hence, is the sum of trivial $\mathfrak{sl}_3$-modules under the adjoint action. In this table we display the Lie algebra $\mathfrak{s}$:

\[
\begin{array}{ccccccc}
L & \mathbb{G}_2 & \mathbb{F}_4 & \mathbb{E}_6 & \mathbb{E}_7 & \mathbb{E}_8 \\
\mathfrak{s} & 0 & \mathfrak{sl}_3 & \mathfrak{sl}_3 \oplus \mathfrak{sl}_3 & \mathfrak{sl}_6 & \mathfrak{sl}_6 & \mathfrak{E}_6
\end{array}
\]

For the Lie algebra $\mathbb{G}_2$ we have the well-known decomposition (see [9, Prop. 3])

\[
\mathbb{G}_2 \cong \mathfrak{sl}_3 \oplus \mathbb{V} \oplus \mathbb{V}^*
\]

relative to $\mathfrak{sl}_3$ (where $\mathfrak{sl}_3$ corresponds to the long roots of $\mathbb{G}_2$ and $\mathbb{V} = \mathbb{F}^3$). This decomposition can be viewed as the decomposition into eigenspaces
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relative to $\psi$, where $V$ corresponds to the eigenvalue $\omega$ (a primitive cube root of 1); $V^*$ to the eigenvalue $\omega^2$; and $\mathfrak{sl}_3$ to the eigenvalue 1.

For the other exceptional Lie algebras,

(1.5) $\mathcal{L} \cong \mathfrak{sl}_3 \oplus (V \otimes B) \oplus (V^* \otimes C) \oplus \mathfrak{s}$,

where $B$ and $C$ can be identified with $H_3(C)$, the algebra of $3 \times 3$ hermitian matrices over a composition algebra $C$ under the product $h \circ h' = 1/2(hh' + h'h)$. Thus, elements of $B$ have the form

$h = \begin{bmatrix} \alpha & a & b \\ \bar{a} & \beta & c \\ \bar{b} & \bar{c} & \gamma \end{bmatrix}$

where $\alpha, \beta, \gamma \in \mathbb{F}$, $a, b, c \in C$, and $\bar{\cdot}$ is the standard involution in $C$. The composition algebra $C$ is displayed below,

(1.6)

| $\mathcal{L}$ | $\mathbb{F}$ | $E_6$ | $E_7$ | $E_8$ |
|---------------|-------------|-------|-------|-------|
| $\mathcal{C}$ | $\mathbb{F}$ | $K$    | $Q$   | $O$   |

where $K$ is the algebra $\mathbb{F} \times \mathbb{F}$, $Q$ the algebra of quaternions, and $O$ the algebra of octonions. The algebra $\mathfrak{s}$ can be identified with the structure Lie algebra of $B = H_3(C)$, $\mathfrak{s} = \text{Der}(B) \oplus L_{B_0}$, consisting of the derivations and multiplication maps $L_h(h') = h \circ h'$ for $h \in B_0$ (the matrices in $B$ of trace 0). Here $V \otimes B$ is the $\omega$-eigenspace of $\psi$, $V^* \otimes C$ the $\omega^2$-eigenspace, and $\mathfrak{sl}_3 \oplus \mathfrak{s}$ the 1-eigenspace.

For example, when $C = O$, it is well known that $B = H_3(O)$ is the 27-dimensional exceptional simple Jordan algebra, and its structure algebra $\mathfrak{s}$ is a simple Lie algebra of type $E_6$ (see for example, [11, Chap. IV, Sec. 4]). As a module for $E_6$, $B$ is irreducible, and relative to a certain Cartan subalgebra, it has highest weight the first fundamental weight. The module $C$ is an irreducible $E_6$-module, (the dual module of $B$) which has highest weight the last fundamental weight. Thus,

$E_8 = \mathfrak{sl}_3 \oplus (V \otimes B) \oplus (V^* \otimes C) \oplus E_6$.

Reading right to left, we see the decomposition of $E_8$ as a module for the subalgebra of type $E_6$, and reading left to right, its decomposition as an $\mathfrak{sl}_3$-module.

Recently, Lie algebras with a decomposition (1.5) have been considered by Faulkner [8, Lem. 22] in connection with his classification of structurable superalgebras of classical type. (Structurable algebras, which were introduced and studied in [1], form a certain variety of algebras generalizing associative algebras with involution and Jordan algebras.)

In this work, we examine Lie algebras $\mathcal{L}$ such that $\mathcal{L}$ has a subalgebra $\mathfrak{sl}_3$ and such that $\mathcal{L}$ admits a decomposition as in (1.3) into copies of $\mathfrak{sl}_3$, $V = \mathbb{F}^3$, $V^*$, and trivial modules relative to the action of $\mathfrak{sl}_3$. Applying results in [3] and [5], we determine that $A$ is an alternative algebra, $B$ a left
A-module, and C a right A-module, and we describe s and the multiplication in L.

Using the fact that V can be given the structure of a module for the symmetric group \( S_4 \), we obtain an action of \( S_4 \) by automorphisms on L. The elements \( \tau_1 = (1 \ 2)(3 \ 4) \) and \( \tau_2 = (1 \ 4)(2 \ 3) \) generate a normal subgroup of \( S_4 \) which is a Klein 4-subgroup. Results of Elduque and Okubo [7] enable us to deduce that \( L_0 = \{ X \in L \mid \tau_1 X = X, \ \tau_2 X = -X \} \) is a structurable algebra under a certain multiplication. We identify the structurable algebra \( L_0 \) with the space of \( 2 \times 2 \) matrices

\[
A = \begin{bmatrix} A & C \\ B & A \end{bmatrix}
\]

under a suitable multiplication. When L is the exceptional Lie algebra \( E_8 \), then \( A = \begin{bmatrix} F & C \\ B & F \end{bmatrix} \) where \( B = C = H_3(O) \). This is a simple structurable algebra (see [1, Secs. 8 and 9]).

2. Lie algebras with prescribed \( \mathfrak{sl}_3 \) decomposition

Let \( L \) be a Lie algebra over a field \( \mathbb{F} \) of characteristic \( \neq 2,3 \) (this assumption on the underlying field will be kept throughout), which contains a subalgebra isomorphic to \( \mathfrak{sl}(V) \), for a vector space \( V \) of dimension 3, so that \( L \) decomposes, as a module for \( \mathfrak{sl}(V) \) into a direct sum of copies of the adjoint module, the natural module \( V \), its dual \( V^* \), and the trivial one-dimensional module. Thus, we write as in (1.3):

\[
L = (\mathfrak{sl}(V) \otimes A) \oplus (V \otimes B) \oplus (V^* \otimes C) \oplus s
\]

for suitable vector spaces \( A, B, C \), and for a Lie subalgebra \( s \), which is the subalgebra of elements of \( L \) annihilated by the elements in \( \mathfrak{sl}(V) \). The vector space \( A \) contains a distinguished element \( 1 \in A \) such that \( \mathfrak{sl}(V) \otimes 1 \) is the subalgebra isomorphic to \( \mathfrak{sl}(V) \) we have started with.

Fix a nonzero linear map \( \det : \bigwedge^3 V \to \mathbb{F} \). This determines another such form \( \det : \bigwedge^3 V^* \to \mathbb{F} \) such that \( \det(f_1 \wedge f_2 \wedge f_3) \det(v_1 \wedge v_2 \wedge v_3) = \det(f_i(v_j)) \) for any \( f_1, f_2, f_3 \in V^* \) and \( v_1, v_2, v_3 \in V \). (The symbol “\( \det \)” denotes the usual determinant.)

This allows us to identify \( \bigwedge^2 V \) with \( V^* \): \( u_1 \wedge u_2 \leftrightarrow \det(u_1 \wedge u_2 \wedge \_) \) and, in the same vein, \( \bigwedge^2 V^* \) with \( V \).

The invariance of the bracket in \( L \) relative to the subalgebra \( \mathfrak{sl}(V) \) gives equations as in [3 (19)]:
Theorem 2.1. Let \( \mathcal{L} \) be a vector space as in (2.1) and define an anticommutative bracket in \( \mathcal{L} \) by (2.2) for bilinear maps as in (2.3). Then \( \mathcal{L} \) is a Lie algebra if and only if the following conditions are satisfied:

\[
[x \otimes a, y \otimes a'] = [x, y] \otimes \frac{1}{2}a \circ a' + x \circ y \otimes \frac{1}{2}[a, a'] + (x|y)D_{a,a'},
\]

\[
[x \otimes a, u \otimes b] = xu \otimes ab,
\]

\[
[v^* \otimes c, x \otimes a] = v^* x \otimes ca,
\]

\[
[u \otimes b, v^* \otimes c] = (uv^* - \frac{1}{3}(v^* u)l_3) \otimes T(b,c) + \frac{1}{3}(v^* u)D_{b,c},
\]

(2.2)

\[
[u_1 \otimes b_1, u_2 \otimes b_2] = (u_1 \wedge u_2) \otimes (b_1 \times b_2),
\]

\[
[v_1^* \otimes c_1, v_2^* \otimes c_2] = (v_1^* \wedge v_2^*) \otimes (c_1 \times c_2),
\]

\[
[d, x \otimes a] = x \otimes da,
\]

\[
[d, u \otimes b] = u \otimes db,
\]

\[
[d, v^* \otimes c] = v^* \otimes dc,
\]

for any \( x, y \in \mathfrak{sl}(V) \), \( u, u_1, u_2 \in V \), \( v^*, v_1^*, v_2^* \in V^* \), \( d \in \mathfrak{s} \), \( a, a' \in \mathfrak{A} \), \( b, b_1, b_2 \in \mathfrak{B} \) and \( c, c_1, c_2 \in \mathfrak{C} \), and for bilinear maps:

\[
\begin{align*}
A \times A &\to A : (a, a') \mapsto a \circ a' \text{ commutative,} \\
A \times A &\to A : (a, a') \mapsto [a, a'] \text{ anticommutative,} \\
A \times \mathfrak{A} &\to \mathfrak{A} : (a, a') \mapsto D_{a,a'} \text{ skew-symmetric,} \\
A \times \mathfrak{B} &\to \mathfrak{B} : (a, b) \mapsto ab, \\
\mathfrak{C} \times A &\to \mathfrak{C} : (c, a) \mapsto ca, \\
\mathfrak{B} \times \mathfrak{C} &\to A : (b, c) \mapsto T(b,c), \\
\mathfrak{B} \times \mathfrak{C} &\to \mathfrak{A} : (b, c) \mapsto D_{b,c}, \\
\mathfrak{B} \times \mathfrak{B} &\to \mathfrak{C} : (b_1, b_2) \mapsto b_1 \times b_2 \text{ symmetric,} \\
\mathfrak{C} \times \mathfrak{C} &\to \mathfrak{B} : (c_1, c_2) \mapsto c_1 \times c_2 \text{ symmetric,}
\end{align*}
\]

(2.3)

and representations \( \mathfrak{s} \to \mathfrak{gl}(A), \mathfrak{gl}(B), \mathfrak{gl}(C) \), whose action is denoted by \( da \), \( db \), and \( dc \) for \( d \in \mathfrak{s} \) and \( a \in \mathfrak{A} \), \( b \in \mathfrak{B} \) and \( c \in \mathfrak{C} \); where, as in [3, (17)],

\[
\begin{align*}
x \circ y &= xy + yx - \frac{2}{3} \text{tr}(xy)l_3, \\
(x|y) &= \frac{1}{3} \text{tr}(xy),
\end{align*}
\]

(2.4)

for \( x, y \in \mathfrak{sl}(V) \), and \( l_3 \) denotes the identity map. The difference with [3, (19)] lies in the appearance of the symmetric maps \( b_1 \times b_2 \) and \( c_1 \times c_2 \) when \( V \) has dimension 3. This slight difference has a huge impact.

The distinguished element \( 1 \in \mathfrak{A} \) satisfies \( 1 \circ a = a \), \( [1, a] = 0 \), \( D_{1,a} = 0 \), \( 1b = b \), \( c1 = c \), and \( dl = 0 \) for any \( a \in \mathfrak{A} \), \( b \in \mathfrak{B} \), \( c \in \mathfrak{C} \) and \( d \in \mathfrak{s} \).
(0) $\mathfrak{s}$ is a Lie subalgebra of $\mathcal{L}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are modules for $\mathfrak{s}$ relative to the given actions, and the bilinear maps in (2.3) are $\mathfrak{s}$-invariant.

(1) $\mathbf{A}$ is an alternative algebra relative to the multiplication

$$aa' = \frac{1}{2}a \circ a' + \frac{1}{2}[a,a'],$$

and the map $A \times A \to A : (a,a') \mapsto D_{a,a'}$ satisfies the conditions

$$\sum_{cyclic} D_{a_1,a_2a_3} = 0,$$

$$D_{a_1,a_2a_3} = \left([a_1,a_2],a_3\right) + 3\left((a_1a_3)a_2 - a_1(a_3a_2)\right),$$

for any $a_1, a_2, a_3 \in A$.

(2) For any $a_1, a_2 \in \mathbf{A}$, $b \in \mathbf{B}$ and $c \in \mathbf{C}$,

$$a_1(a_2b) = (a_1a_2)b,$$

$$(ca_1)a_2 = c(a_1a_2),$$

so that $\mathbf{B}$ (respectively $\mathbf{C}$) is a left associative module (resp. right associative module) for $\mathbf{A}$, and

$$D_{a_1,a_2}b = [a_1,a_2]b,$$

$$D_{a_1,a_2}c = c[a_2,a_1].$$

(3) For any $a \in \mathbf{A}$, $b \in \mathbf{B}$ and $c \in \mathbf{C}$,

$$aT(b,c) = T(ab,c), \quad T(b,c)a = T(b,ca),$$

$$D_{a,T(b,c)} = D_{ab,c} - D_{b,ca},$$

$$D_{b,ca} = [T(b,c),a].$$

(4) For any $a \in \mathbf{A}$, $b_1, b_2 \in \mathbf{B}$ and $c_1, c_2 \in \mathbf{C}$,

$$(b_1 \times b_2)a = (ab_1) \times b_2 = b_1 \times (ab_2),$$

$$a(c_1 \times c_2) = (c_1a) \times c_2 = c_1 \times (c_2a).$$

(5) $D_{b,bx,b} = 0$ for any $b \in \mathbf{B}$ and $D_{c\times c,c} = 0$ for any $c \in \mathbf{C}$. In addition, the trilinear maps $\mathbf{B} \times \mathbf{B} \times \mathbf{B} \to \mathbf{B} : (b_1, b_2, b_3) \mapsto T(b_1, b_2 \times b_3)$ and $\mathbf{C} \times \mathbf{C} \times \mathbf{C} \to \mathbf{C} : (c_1, c_2, c_3) \mapsto T(c_1 \times c_2, c_3)$ are symmetric.

(6) For any $b, b_1, b_2 \in \mathbf{B}$ and $c, c_1, c_2 \in \mathbf{C}$,

$$(b_1 \times b_2) \times c = -\frac{1}{3}T(b_1,c)b_2 + \frac{1}{3}D_{b_1,c}b_2 - T(b_2,c)b_1,$$

$$b \times (c_1 \times c_2) = -c_2T(b,c_1) - \frac{1}{3}c_1T(b,c_2) - \frac{1}{3}D_{b,c_2}c_1.$$

Proof. If $\mathcal{L}$ is a Lie algebra under the bracket defined in (2.1), then it is clear that $\mathfrak{s}$ is a Lie subalgebra and all the conditions in item (0) are satisfied. Moreover, $(\mathfrak{sl}(\mathfrak{v}) \otimes \mathbf{A}) \oplus \mathfrak{s}$ is a Lie subalgebra, and the arguments in [3, Sec. 3] show that $\mathbf{A}$ is an alternative algebra, and the conditions in item (1) are satisfied.
The arguments in Propositions 4.3 and 4.4, and Equations (25) and (27) in [3], work here and give the conditions in item (2). (Note that there is a minus sign missing in [3, (27)].) Now, equations (30)–(33) in [3] establish the identities in (3). The Jacobi identity applied to elements \( x \otimes a, u_1 \otimes b_1 \) and \( u_2 \otimes b_2 \), for \( x \in \mathfrak{sl}(V) \), \( u_1, u_2 \in V, a \in A \) and \( b_1, b_2 \in B \), give the first equation in item (4), the second one being similar.

The Jacobi identity for elements \( u_i \otimes b_i, i = 1, 2, 3 \), for \( u_i \in V \) and \( b_i \in B \) gives:

\[
\sum_{\text{cyclic}} D_{b_1, b_2 \times b_3} = 0, \quad T(b_1, b_2 \times b_3) = T(b_2, b_3 \times b_1),
\]

which, in view of the symmetry of the bilinear map \( b_1 \times b_2 \), proves half of the assertions in item (5); the other half being implied by the Jacobi identity for elements \( v_i^* \otimes c_i, i = 1, 2, 3 \), for \( v_i^* \in V^* \) and \( c_i \in C \).

Finally, for elements \( u_1, u_2 \in V, v^* \in V^*, b_1, b_2 \in B, c \in C \):

\[
[u_1 \otimes b_1, u_2 \otimes b_2], v^* \otimes c = [(u_1 \wedge u_2) \otimes (b_1 \times b_2), v^* \otimes c] = (u_1 \wedge u_2) \wedge v^* \otimes (b_1 \times b_2) \times c = ((v^* u_1)u_2 - (v^* u_2)u_1) \otimes (b_1 \times b_2) \times c,
\]

while

\[
[u_1 \otimes b_1, v^* \otimes c], u_2 \otimes b_2 = [(u_1 v^* - \frac{1}{3}(v^* u_1)l_3) \otimes T(b_1, c) + \frac{1}{3}v^* u_1D_{b_1,c}, u_2 \otimes b_2]
\]

\[
= ((v^* u_2)u_1 - \frac{1}{3}(v^* u_1)u_2) \otimes T(b_1, c)b_2 + \frac{1}{3}(v^* u_1)u_2 \otimes D_{b_1,c}b_2,
\]

\[
[u_1 \otimes b_1, [u_2 \otimes b_2, v^* \otimes c]] = [u_1 \otimes b_1, (u_2 v^* - \frac{1}{3}(v^* u_2)l_3) \otimes T(b_2, c) + \frac{1}{3}v^* u_2D_{b_2,c}]
\]

\[
= -((v^* u_1)u_2 - \frac{1}{3}(v^* u_2)u_1) \otimes T(b_2, c)b_1 - \frac{1}{3}(v^* u_2)u_1 \otimes D_{b_2,c}b_1.
\]

Hence the Jacobi identity here is equivalent to the first condition in item (6); the second condition can be proven in a similar way.

The converse follows from straightforward computations. \( \square \)

Given an alternative algebra \( A \), the ideal \( E(A) \) generated by the associators \((a_1, a_2, a_3) = (a_1a_2)a_3 - a_1(a_2a_3)\) is \( E(A) = (A, A, A) + (A, A, A)A = (A, A, A) + A(A, A, A) \). The associative nucleus of \( A \) is \( N(A) := \{a \in A \mid (a, A, A) = 0\} \), while the center is \( Z(A) = \{a \in N(A) \mid aa' = a'a, \ \forall a', a' \in A\} \).

**Corollary 2.2.** Let \( L \) be a Lie algebra which contains a subalgebra isomorphic to \( \mathfrak{sl}(V) \) for a vector space \( V \) of dimension 3, so that \( L \) decomposes, as a module for \( \mathfrak{sl}(V) \), as in (2.1). Then, with the notation used so far, the alternative algebra \( A \) is unital (the distinguished element 1 being its unit element), with 1 acting as the identity on both \( B \) and \( C \), and the following conditions hold:
• E(A)B = 0 = CE(A), so that B (respectively C) is a left (resp. right) module for the associative algebra A/E(A).

• T(B, C) is an ideal of A contained in its associative nucleus N(A), and T(B, B × B) and T(C × C, C) are ideals of A contained in Z(A).

• For any b, b₁, b₂ ∈ B and any c, c₁, c₂ ∈ C, the following conditions hold:
  \[ D_{b₁, c₂}b₂ - D_{b₂, c₁}b₁ = 2(T(b₂, c)b₁ - T(b₁, c)b₂), \]
  \[ D_{b₂, c₁}c₁ - D_{b₁, c₂}c₂ = 2(c₁T(b, c₂) - c₂T(b, c₁)). \]

• If the Lie algebra L is simple, then either the algebra A is associative, or else A = E(A) and B = C = 0. Moreover, if B ≠ 0, then C coincides with B × B, and A coincides with T(B, B × B), and A is a commutative and associative algebra.

Proof. For any a₁, a₂, a₃ ∈ A and b ∈ B,
  \[(a₁, a₂, a₃)b = (a₁a₂)(a₃b) - a₁((a₂a₃)b) = a₁(a₂(a₃b)) - a₁(a₂(a₃b)) = 0,\]
because of Theorem 2.1 item (2). Also, this result shows that ann_A(B) = \{a ∈ A | ab = 0\} is an ideal of A. Hence, E(A)B = 0, as E(A) is the ideal generated by (A, A, A). In a similar manner, one proves CE(A) = 0.

For any b ∈ B and c ∈ C, T(b, c) is an element of A, and \(\text{ad}_{T(b, c)} : a \mapsto [T(b, c), a] = D_{b,c}a\) is a derivation of A by the previous theorem. Since A is alternative, this shows that \(T(b, c)\) is in the associative nucleus N(A). Now for \(b₁, b₂, b₃ ∈ B\), \(aT(b₁, b₂ × b₃) = T(ab₁, b₂ × b₃) = T(b₂, b₃ × (ab₁)) = T(b₂, (b₃ × b₁)a) = T(b₂, b₃ × b₁)a = T(b₁, b₂ × b₃)a\), which proves that \(T(B, B × B)\) is an ideal of A contained in the center Z(A). By similar arguments, \(T(C × C, C)\) is shown to be contained in Z(A) too.

For any \(b₁, b₂ ∈ B\) and \(c ∈ C\), the previous theorem gives,
  \[(b₁ × b₂) × c = -\frac{1}{3}T(b₁, c)b₂ + \frac{1}{3}D_{b₁, c}b₂ - T(b₂, c)b₁.\]
We permute \(b₁\) and \(b₂\) and use the fact that × is symmetric to get
  \[D_{b₁, c₂}b₂ - D_{b₂, c₁}b₁ = 2(T(b₂, c)b₁ - T(b₁, c)b₂).\]

With the same arguments we prove
  \[D_{b₂, c₂}c₁ - D_{b₁, c₁}c₂ = 2(c₁T(b, c₂) - c₂T(b, c₁)),\]
for any \(b ∈ B\), and \(c₁, c₂ ∈ C\).

Finally, since the ideal E(A) of the alternative algebra A is invariant under derivations, the subspace \((\mathfrak{sl}(V) ⊗ E(A)) ⊕ D_{E(A), A}\) is an ideal of the Lie algebra \(L\). In particular, if \(L\) is simple, then either \(A = E(A)\) and \(B = C = 0\), or \(E(A) = 0\) and \(A\) is associative. Moreover, if \(B\) is nonzero, the ideal of \(L\) generated by \(V ⊗ B\) is \((\mathfrak{sl}(V) ⊗ T(B, C)) ⊕ (V ⊗ B) ⊕ (V^* ⊗ (B × B)) ⊕ D_{B,C}\). Hence if \(L\) is simple, we obtain \(C = B × B\) and \(A = T(B, C) = T(B, B × B)\), which is commutative and associative. □
3. Structurable algebras

This section is devoted to establishing a relationship between the Lie algebras with prescribed $\mathfrak{sl}_3$ decomposition considered above with a class of structurable algebras. This will be done by exploiting the action of a subgroup of the group of automorphisms of the Lie algebra isomorphic to the symmetric group $S_4$.

**Theorem 3.1.** Let $\mathcal{L}$ be a Lie algebra which contains a subalgebra isomorphic to $\mathfrak{sl}(V)$ for a vector space $V$ of dimension 3, so that $\mathcal{L}$ decomposes, as a module for $\mathfrak{sl}(V)$, as in (2.1). Then, with the notation used so far, the vector space

$$A = \begin{pmatrix} A & C \\ B & A \end{pmatrix},$$

with the multiplication

$$\begin{pmatrix} a_1 & c \\ b & a_2 \end{pmatrix} \cdot \begin{pmatrix} a_1' & c' \\ b & a_2' \end{pmatrix} = \begin{pmatrix} a_1a_1' - T(b', c) & c'a_1 + ca_2' + b \times b' \\ a_1'b + a_2b' + c \times c' & a_2a_2' - T(b, c') \end{pmatrix}$$

and the involution

$$\begin{pmatrix} a_1 & c \\ b & a_2 \end{pmatrix} \rightarrow \begin{pmatrix} a_2 & c \\ b & a_1 \end{pmatrix}$$

is a structurable algebra.

**Proof.** Take a basis $\{e_1, e_2, e_3\}$ of $V$ with $\det(e_1 \wedge e_2 \wedge e_3) = 1$ and its dual basis $\{e_1^*, e_2^*, e_3^*\}$ in $V^*$.

The symmetric group $S_4$ acts on $V$ as follows \([7, (7.1)]\):

$$\tau_1 = (12)(34) : \quad e_1 \mapsto e_1, \quad e_2 \mapsto -e_2, \quad e_3 \mapsto -e_3,$$

$$\tau_2 = (23)(14) : \quad e_1 \mapsto -e_1, \quad e_2 \mapsto e_2, \quad e_3 \mapsto -e_3,$$

$$\varphi = (123) : \quad e_1 \mapsto e_2 \mapsto e_3 \mapsto e_1,$$

$$\tau = (12) : \quad e_1 \mapsto -e_1, \quad e_2 \mapsto -e_3, \quad e_3 \mapsto -e_2.$$

(Thus $V$ is the tensor product of the sign module and the standard irreducible 3-dimensional module for $S_4$, and in this way, $S_4$ embeds in the special linear group $\text{SL}(V)$.)

The inner product given by $(e_i|e_j) = \delta_{ij}$ for any $i, j \in \{1, 2, 3\}$ is invariant under the action of $S_4$, so $V$ is selfdual as an $S_4$-module, and the action of $S_4$ on $V^*$ (where $\sigma \nu^* = v^* \sigma^{-1}$) is given by the “same formulas”:

$$\tau_1 = (12)(34) : \quad e_1^* \mapsto e_1^*, \quad e_2^* \mapsto -e_2^*, \quad e_3^* \mapsto -e_3^*,$$

$$\tau_2 = (23)(14) : \quad e_1^* \mapsto -e_1^*, \quad e_2^* \mapsto e_2^*, \quad e_3^* \mapsto -e_3^*,$$

$$\varphi = (123) : \quad e_1^* \mapsto e_2^* \mapsto e_3^* \mapsto e_1^*,$$

$$\tau = (12) : \quad e_1^* \mapsto -e_1^*, \quad e_2^* \mapsto -e_3^*, \quad e_3^* \mapsto -e_2^*.$$

Since $S_4$ acts by elements in $\text{SL}(V)$, this action of $S_4$ on $V$ and on $V^*$ extends to an action by automorphisms on the whole algebra $\mathcal{L}$. Then the
subspace
\[ \mathcal{L}_0 = \{ X \in \mathcal{L} \mid \tau_1 X = X, \ \tau_2 X = -X \} \]
becomes a structurable algebra [7, Thm. 7.5] with involution and multiplication given by the following formulas,
\[ \tilde{X} = -\tau X, \]
\[ X \cdot Y = -\tau([\varphi X, \varphi^2 Y]), \]
for any \( X, Y \in \mathcal{L}_0 \).

But we easily deduce that
\[ \mathcal{L}_0 = (e_2 e_3^* \otimes A) \oplus (e_3 e_2^* \otimes A) \oplus (e_1 \otimes B) \oplus (e_1^* \otimes C). \]

Identifying \( \mathcal{L}_0 \) with the \( 2 \times 2 \) matrices \( A = \begin{pmatrix} A & C \\ B & A \end{pmatrix} \) by means of
\[ -e_2 e_3^* \otimes a_1 + e_3 e_2^* \otimes a_2 + e_1 \otimes b + e_1^* \otimes c \leftrightarrow \begin{pmatrix} a_1 & c \\ b & a_2 \end{pmatrix}, \]
we determine that the structurable product and the involution become
\[ \begin{pmatrix} a_1 & c \\ b & a_2 \end{pmatrix} \cdot \begin{pmatrix} a'_1 & c' \\ b' & a'_2 \end{pmatrix} = \begin{pmatrix} a_1 a'_1 - T(b', c) & c' a_1 + c a'_2 + b \times b' \\ a'_1 b + a_2 b' + c \times c' & a'_2 a_2 - T(b, c') \end{pmatrix}, \]
\[ \begin{pmatrix} a_1 & c \\ b & a_2 \end{pmatrix} = \begin{pmatrix} a_2 & c \\ b & a_1 \end{pmatrix}, \]
as required. \( \square \)

Items (5) and (6) of Theorem 2.1 show that for any \( b \in B \) and \( c \in C \),
\[ (b \times b) \times (b \times b) = -\frac{4}{3} T(b, b \times b)b, \]
\[ (c \times c) \times (c \times c) = -\frac{4}{3} cT(c \times c, c). \] (3.4)

Also, using Theorem 2.1 and Corollary 2.2 we compute that
\[ (c \times (b \times b)) \times b = -\frac{4}{3} (T(b, c)b) \times b + \frac{1}{3} (D_{b,c}b) \times b \]
\[ = -\frac{4}{3} (b \times b)T(b, c) + \frac{1}{6} D_{b,c}(b \times b) \quad \text{(as the product} \\
\text{\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad b_1 \times b_2 \text{ from} B \times B \text{ into} C \text{ is} s\text{-invariant})} \]
\[ = -\frac{4}{3} (b \times b)T(b, c) + \frac{1}{3} ((b \times b)T(b, c) - cT(b, b \times b)) \]
\[ = -(b \times b)T(b, c) - \frac{1}{3} cT(b, b \times b), \]
and an analogous result with the roles of $b$ and $c$ interchanged. So we conclude that the equations

\[(c \times (b \times b)) \times b = -(b \times b)T(b, c) - \frac{1}{3}cT(b, b \times b), \tag{3.5}\]
\[(b \times (c \times c)) \times c = -T(b, c)(c \times c) - \frac{1}{3}T(c \times c, c)b, \tag{3.5}\]

hold for any $b \in B$ and $c \in C$.

Equations (3.4) and (3.5) are precisely the ones that appear in [2, Ex. 6.4], and are needed to ensure that the algebra defined there, which coincides with our $A$, but with the added restrictions of $A$ being commutative and associative, is structurable. (Note that the bilinear form $T(\cdot, \cdot)$ considered in [2, Ex. 6.4] equals our $-T(\cdot, \cdot)$.)

However some of the previous arguments show that, if our structurable algebra $A$ is simple and $A \neq 0$, then $A$ is simple, and since $T(B, B \times B)$ and $T(C \times C, C)$ are ideals of $A$ contained in the center $Z(A)$, either $A$ is commutative and associative, or else $T(B, B \times B) = 0 = T(C \times C, C)$. But in this case, the subspace \(\begin{pmatrix} 0 & B \times B \\ C \times C & 0 \end{pmatrix}\) becomes an ideal, so if $A$ is simple either $A$ is commutative and associative, or else $B \times B = 0 = C \times C$.

Therefore, when considering simple algebras, we are dealing exactly with the situation considered by Allison and Faulkner in [2].

Theorem 3.1 shows that the restrictions on the bilinear maps involved are sufficient to ensure that the algebra $A$ in (3.1), with multiplication (3.2) and involution (3.3), is a structurable algebra.

A natural question to ask is whether these conditions are also necessary. More precisely, does any structurable algebra of the form $A$ as in (3.1) with multiplication (3.2) and involution (3.3), constructed from a unital alternative algebra $A$, left and right unital “associative” modules $B$ and $C$, and bilinear maps $T(b, c), b_1 \times b_2$, and $c_1 \times c_2$, coordinatize a Lie algebra $\mathcal{L}$ with a subalgebra isomorphic to $\mathfrak{sl}(V)$ for a vector space $V$ of dimension 3 and with decomposition as in (2.1)? (We do not impose any further conditions on these bilinear maps besides requiring that the resulting algebra $A$ be structurable.)

Our last result answers this question in the affirmative.

**Theorem 3.2.** Let $A$ be a unital alternative algebra; let $B$ (respectively $C$) be a left (respectively right) unital associative module for $A$; and let $B \times C \rightarrow A: (b, c) \mapsto T(b, c), B \times B \rightarrow C: (b_1, b_2) \mapsto b_1 \times b_2$, and $C \times C \rightarrow B: (c_1, c_2) \mapsto c_1 \times c_2$ be bilinear maps which make the vector space $A$ in (3.1) with the multiplication (3.2) and involution (3.3) into a structurable algebra. Then there is a Lie algebra $\mathcal{L}$ containing a subalgebra isomorphic to $\mathfrak{sl}(V)$, for a vector space $V$ of dimension 3, such that $\mathcal{L}$ decomposes as in (2.1) for a suitable vector space $s$, such that the Lie bracket on $\mathcal{L}$ is given by (2.2) for some bilinear maps $A \times A \rightarrow s, (a, a') \mapsto D_{a, a'}, B \times C \rightarrow s: (b, c) \mapsto D_{b,c}$, $s \times A \rightarrow A: (d, a) \mapsto da, s \times B \rightarrow B: (d, b) \mapsto db$ and $s \times C \rightarrow C: (d, c) \mapsto dc$. 
Proof. Consider the Lie algebra \( \mathcal{L} = \mathfrak{K}(\mathcal{A}, -, \gamma, \mathcal{V}) \) in [2] Sec. 4 attached to the structurable algebra \((\mathcal{A}, -)\), the triple \(\gamma = (1, 1, 1)\), and the Lie sub-algebra \(\mathcal{V} = T_\mathcal{I}\). This Lie algebra \(\mathcal{L}\), which coincides with the Lie algebra \(\mathfrak{g}(\mathcal{A}, \cdot, -)\) in [6] Ex. 3.1, is the direct sum

\[
\mathcal{L} = T_\mathcal{I} \oplus \mathcal{A}[12] \oplus \mathcal{A}[23] \oplus \mathcal{A}[31],
\]

where \(T_\mathcal{I}\) is the span of the triples \(T = (T_1, T_2, T_3)\) with

\[
\begin{align*}
T_i &= L_x L_y - L_y L_x, \\
T_j &= R_x R_y - R_y R_x, \\
T_k &= R_{xy - \bar{y}x} + L_y L_x - L_x L_y,
\end{align*}
\]

(3.6)

for \(x, y \in \mathcal{A}\) and \((i, j, k)\) a cyclic permutation of \((1, 2, 3)\). Here \(L_x y = x y = R_y x\). The subspace \(T_\mathcal{I}\) is a Lie algebra with componentwise bracket, and the Lie bracket in \(\mathcal{L}\) is given by extending the bracket in \(T_\mathcal{I}\) by setting \(x[ij] = -x[ij]\) for any \(x \in \mathcal{A}\) and

\[
\begin{align*}
[x[ij], y[jk]] &= -(x[jk], y[ij]) = (xy)[ik], \\
[T, x[ij]] &= -[x[ij], T] = T_k(x)[ij], \\
[x[ij], y[ij]] &= T,
\end{align*}
\]

for \(x, y \in \mathcal{A}\), where \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\), and \(T = (T_1, T_2, T_3)\) is as in [3(6)]. Theorems 4.1 and 5.5 in [2] show that \(\mathcal{L}\) is indeed a Lie algebra. Since we are assuming that the characteristic of the field is \(\neq 2, 3\), Corollary 3.5 of [2] shows that \(T_\mathcal{I} = \{(D, D, D) \mid D \in \text{Der}(\mathcal{A}, \cdot, -)\} \oplus \{(L_{s_2} - R_{s_2}, L_{s_3} - R_{s_3}, L_{s_1} - R_{s_1}) \mid s_i \in \mathcal{A}, \bar{s}_i = -s_i, s_1 + s_2 + s_3 = 0\}\).

Here \(\text{Der}(\mathcal{A}, \cdot, -)\) is the space of derivations relative to the product “\(\cdot\)”, which commute with the involution “\(-\)”.

For any \(a \in \mathcal{A}\), consider the linear span \(\mathfrak{sl}_3[a]\) of the elements \((a 0 0) [ij]\) for \(i \neq j\) and the triples \((L_{\alpha_{2s}} - R_{\alpha_{3s}}, L_{\alpha_{3s}} - R_{\alpha_{1s}}, L_{\alpha_{1s}} - R_{\alpha_{2s}})\) for \(\alpha_i \in \mathbb{F}\) with \(\alpha_1 + \alpha_2 + \alpha_3 = 0\) and \(s = (a_0 0)\). (Note that \((a 0 0) [ij] = -(0 0 a) [ij]\).

Also for any \(b \in \mathcal{B}\), consider the linear span \(\mathcal{V}[b]\) of the elements \((0 b 0) [ij]\), and for any \(c \in \mathbb{C}\) the linear span \(\mathcal{V}^*[c]\) of the elements \((0 0 c) [ij]\).

Straightforward computations using (3.7) imply that

- \(\mathfrak{sl}_3[1]\) is a Lie subalgebra of \(\mathcal{L}\) isomorphic to \(\mathfrak{sl}(\mathcal{V})\) \((\dim \mathcal{V} = 3)\),
- \(\mathfrak{sl}_3[a]\) is an adjoint module for \(\mathfrak{sl}_3[1]\) for any \(a \in \mathcal{A}\),
- \(\mathcal{V}[b]\) is the natural module for \(\mathfrak{sl}_3[1]\) for any \(b \in \mathcal{B}\),
- \(\mathcal{V}^*[c]\) is the dual module for \(\mathfrak{sl}_3[1]\) for any \(c \in \mathbb{C}\), and
- \(\mathfrak{sl} = \{(D, D, D) \mid D \in \text{Der}(\mathcal{A}, \cdot, -)\}\) is a Lie subalgebra which commutes with \(\mathfrak{sl}_3[1]\).

Actually, if we fix a basis \(\{e_1, e_2, e_3\}\) of \(\mathcal{V}\) as before with \(\det(e_1 \wedge e_2 \wedge e_3) = 1\) and the dual basis \(\{e_1^*, e_2^*, e_3^*\}\) in \(\mathcal{V}^*\), we may identify \(\mathfrak{sl}_3[a]\) with \(\mathfrak{sl}(\mathcal{V}) \otimes a\).
for \( a \in A \) by means of
\[
e_i e_j^* \otimes a \leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} [ij], \quad \text{for } i \neq j
\]
\[
\sum_{i=1}^{3} \alpha_i e_i e_i^* \otimes a \leftrightarrow (L_{\alpha_2 s} - R_{\alpha_3 s}, L_{\alpha_3 s} - R_{\alpha_1 s}, L_{\alpha_1 s} - R_{\alpha_2 s}),
\]
for \( \alpha_i \in \mathbb{F} \) with \( \alpha_1 + \alpha_2 + \alpha_3 = 0 \) and \( s = \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix} \). Also for any \( b \in B \) and \( c \in C \), we may identify \( V[b] \) with \( V \otimes b \) and \( V^*[c] \) with \( V^* \otimes c \) via
\[
e_i \otimes b \leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} [jk], \quad e_i^* \otimes c \leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} [jk]
\]
where \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\).

In this way we recover the decomposition in \([13]\) with bracket as in \([22]\) for suitable maps \( D_{\ldots} \), as required. \( \square \)

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