On the uniqueness and non-uniqueness of the steady planar Navier-Stokes equations in an exterior domain

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Abstract: In this paper we investigate the uniqueness of solutions of the steady planar Navier-Stokes equations with different boundary conditions in the exterior domain. For a class of incompressible flow with constant vorticity, we prove the uniqueness of the solution under the enhanced Navier boundary conditions. At the same time, some counterexamples are given to show that the uniqueness of the solution fails under the Navier boundary conditions. For the general incompressible flow with Dirichlet boundary condition, we prove various sufficient conditions for the uniqueness of the solution.

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1 Introduction and main results

The motion of viscous incompressible fluid past an obstacle can be described by the Navier-Stokes equations. More precisely, the velocity $u$ of fluid and the pressure $\pi$ satisfy the following stationary incompressible Navier-Stokes problem:

\[
\begin{aligned}
&\{ 
  u \cdot \nabla u + \nabla \pi = \Delta u, \quad \text{in } \Omega, \\
&\text{div } u = 0, \quad \text{in } \Omega,
\end{aligned}
\] (1)

where the flow domain $\Omega$ is a two-dimensional exterior domain, i.e., the complement of a bounded domain which represents the obstacle. Without loss of generality, one assumes that $\Omega = \mathbb{R}^2 \setminus \overline{B_1}$ and $B_1$ is the disk of radius 1 centered at the origin. There are several possibilities of boundary conditions. According to the idea that the fluid

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cannot slip on the boundary due to its viscosity, the widely used are the following Dirichlet or no-slip boundary conditions:

$$u = 0 \quad \text{on} \quad \partial B_1,$$

(2)

However, in the case where the obstacles have an approximate limit, the Dirichlet boundary conditions are no longer valid (see for example [31]). Due to the roughness of the boundary and the viscosity of the fluid, it is usually assumed that there is a stagnant fluid layer near the boundary, which allows the fluid to slip. This situation seems to match the reality. Then, it is really important to introduce another boundary conditions to describe the behavior of fluid on the boundary. In 1827, C. Navier [27] was the first mathematician who considered the slip phenomena and proposed the Navier-slip boundary conditions:

$$\begin{cases} 
    u \cdot n = 0, \\
    2[D(u) \cdot n]_\tau + \alpha(x)u_\tau = 0,
\end{cases}$$

(3)

where $D(u)$ is the stress tensor of fluid, $n$ and $\tau$ are the unit outer normal vector and tangential vector of the boundary, $\alpha(x)$ is a physical parameter, which can be a positive constant or a $L^\infty$ function on the boundary. For the far field of the fluid, one usually assumes that

$$u(x_1, x_2) \to \tilde{u}_\infty \quad \text{as} \quad |(x_1, x_2)| \to \infty.$$  

(4)

So, it is interesting to find solutions to (1) with different boundary conditions. An arbitrary solution $u$ to the Navier-Stokes equations (1) having the finite Dirichlet integral

$$\int_\Omega |\nabla u|^2 \, dx < \infty,$$

(5)

is usually called $D$-solution [15], and as is well known (see [24]), such solutions are real analytic in $\Omega$.

The study of (1) with conditions (2)(4) and (5) began with Leray [25] who sought the solutions as the limit of certain approximate solutions, but the behavior of Leray solution at infinity was not found. Indeed, it was not even apparent that Leray solution was non-trivial. The Navier-Stokes equations have been shown to have a solution by Finn and Smith [13] with some smallness assumptions on $|\tilde{u}_\infty|$ by using the contraction mapping principle. Then, Amick [1] proved the existence of solutions for given external forces when the exterior domain is invariant under the transformation $(x_1, x_2) \mapsto (-x_1, x_2)$, this work was generalized by Pileckas-Russo [28]. Hillairet-Wittwer [19] proved the existence of vanishing at infinity solutions to (1) with non-zero Dirichlet boundary conditions by perturbation around the radial and rotational flow $\mu x^\perp/|x|^2$, while the flow $\mu x^{\perp}/|x|^2$ is the exact solution decaying in the scale-critical order $O(|x|^{-1})$ under the zero flux condition. The problem of the asymptotic behavior at infinity of an arbitrary $D$-solution $(u, \pi)$ to (1) was initiated
by Gilbarg-Weinberger \[17\] and Amick \[2\]. In \[17\], the authors have shown that the pressure \(\pi\) has a finite limit at infinity, and
\[
\begin{align*}
  u(x) &= o(\log^{1/2} r), \\
  \nabla u(x) &= o(r^{-3/4} \log^{9/8} r),
\end{align*}
\]
and
\[
\begin{align*}
  w(x) &= o(r^{-3/4} \log^{1/8} r),
\end{align*}
\]
where \(r = |x| = \sqrt{x_1^2 + x_2^2}\) and \(w(x)\) is the vorticity \(w = \partial_2 u_1 - \partial_1 u_2\). In the elegant paper \[2\], it was shown that if \((u, \pi)\) is a solution of (1)-(2) and (5), then \(u \in L^\infty(\Omega)\) (see Theorem 12 of \[2\]). The assumption (2) was recently removed by Korobkov-Pileckas-Russo in \[21\]. Moreover, there exists a constant vector \(u_\infty\) such that
\[
\lim_{r \to \infty} \int_0^{2\pi} |u(r, \theta) - u_\infty|^2 d\theta = 0,
\]
and if \(u_\infty = 0\), then \(u(x) \to 0\), uniformly as \(|x| \to \infty\), where \(x = (x_1, x_2)\). Some decay properties on vorticity \(w(x)\) and \(\nabla u(x)\) were also obtained. Particularly, it was proved in \[2\] that the following uniform limit at infinity holds
\[
|u(x)| \to |u_\infty| \quad \text{as} \quad |x| \to \infty.
\]
Furthermore, for symmetric flow, there holds the following uniform convergence of the velocity
\[
|u(x) - u_\infty| \to 0 \quad \text{as} \quad |x| \to \infty. \tag{6}
\]
However, the uniform convergence (6) was also proved very recently without Amick’s symmetric condition or zero boundary condition on \(\partial \Omega\) by Korobkov-Pileckas-Russo in \[22\], this ensures that the solution behaves at infinity as that of the linear Oseen equations (see, for example, \[15\]). For asymptotic behaviour of steady solutions to the Navier-Stokes equations at infinity, one can refer to \[3, 11, 12, 29\]. Note that the problem of the coincidence with \(u_\infty\) and the prescribed data \(\tilde{u}_\infty\) is still open. When \(\tilde{u}_\infty = 0\), one always has at least the trivial solution \(u \equiv 0\), but it is not sure that:

**Whether \(u = 0\) is the unique solution of (11) with the conditions of (2), (4) and (5)?**

which is exactly the conjecture raised by Amick in \[2\], and usually called the Liouville problem. There are few studies on the Liouville problem in two dimensional exterior domains for the Navier-Stokes equations and we refer to the recent result by Korobkov-Ren in \[23\] for \(\tilde{u}_\infty \neq 0\).

While in three dimensional case, Galdi \[15\] proved the Liouville type theorems by assuming that \(u \in L^{9/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)\). Chae in \[8\] showed that the condition \(\Delta u \in L^{6/5}(\mathbb{R}^3)\) is enough to guarantee the triviality of \(u\). For related discussion, we refer to \[6, 7, 9, 10, 20, 30, 32\] and references therein. To our knowledge, the following Liouville problem is still open: Is a \(D\)-solution to (11) in \(\mathbb{R}^3\), vanishing at infinity, identically zero? As a matter of fact, this problem is also related to
the very hard problem of uniqueness of solutions to nonhomogeneous problem for the Navier-Stokes equations. However, the case in \( \mathbb{R}^2 \) is different, it was proved by Gilbarg-Weinberger in [17] by using the maximum principle for the vorticity equation

\[
\Delta w - u \cdot \nabla w = 0. \tag{7}
\]

In this paper, we focus on the Amick’s conjecture for \( \tilde{u}_\infty = 0 \) in [2], and consider the uniqueness problem of steady solutions to (1) with some prescribed boundary conditions. The first result is concerned with the uniqueness of the constant vorticity flow of \( u_0 = \frac{a}{2}(x_2, -x_1) \) under the following boundary condition:

\[
u|_{\partial B_1} = \frac{a}{2}(x_2, -x_1)|_{\partial B_1}, \tag{8}
\]

where \( a \) is a constant, which is stated as following for a perturbation of \( L^q \) energy norm.

**Theorem 1.1** Let \( (u, \pi) \) be a smooth solution of the 2D Navier-Stokes equations (1) defined on \( \Omega \) and \( u \in C^1(\bar{\Omega}) \) satisfies the boundary conditions (8). Moreover, let \( v = u - u_0 \) with \( u_0 = \frac{a}{2}(x_2, -x_1) \) and

\[
v \in L^q(\Omega), \quad 1 < q \leq 2. \tag{9}
\]

Then \( u \equiv u_0 \).

The second result is concerned with the uniqueness (up to some constant) of the constant vorticity flow of \( u_0 = \frac{a}{2}(x_2, -x_1) \) under the enhanced Navier slip boundary conditions, which is stated as following for a perturbation of \( L^q \) energy norm. Recall the Navier slip boundary conditions is as follows:

\[
u|_{\partial B_1} = 0, \quad w|_{\partial B_1} = a, \tag{10}
\]

and here we added an additional condition:

\[
\int_{\partial B_1} \frac{\partial v}{\partial n} \cdot v dS = 0, \text{ or } \frac{\partial v}{\partial n}|_{\partial B_1} = 0, \text{ or } |v|_{\partial B_1} \equiv C. \tag{11}
\]

**Theorem 1.2** Let \( (u, \pi) \) be a smooth solution of the 2D Navier-Stokes equations (1) defined in \( \Omega \) and \( u \in C^2(\bar{\Omega}) \) satisfies the boundary conditions (10) and (11). Moreover, let \( v = u - u_0 \) with \( u_0 = \frac{a}{2}(x_2, -x_1) \) and

\[
\nabla v \in L^q(\Omega), \quad 1 < q < \infty. \tag{12}
\]

Then \( u - u_0 \equiv C \).

Moreover, by assuming a perturbation of \( L^p \) norm under the enhanced Navier slip boundary conditions, we have the following conclusions:
Theorem 1.3 Let \((u, \pi)\) be a smooth solution of the 2D Navier-Stokes equations (1) defined in \(\Omega\) and \(u \in C^2(\Omega)\) satisfies the boundary conditions (10) and (11). Moreover, let \(v = u - u_0\) with \(u_0 = \frac{a^2}{8}(x_2, -x_1)\) and
\[
v \in L^p(\Omega), \quad 1 < p < \infty.
\] Then \(u - u_0 \equiv C\).

Remark 1.4 (Non-uniqueness for the Navier slip boundary condition) The boundary conditions in Theorem 1.2 and 1.3 seem to be sharp in a sense. Consider the usual slip condition as follows
\[
|n \cdot u|_{\partial B_1} = 0, \quad w|_{\partial B_1} = a,
\] which is just (14). At this time, one can find another non-trivial solution of (1), which is different from the known solution \(u_0 = \frac{a^2}{8}(x_2, -x_1)\) and \(\pi_0 = \frac{a^2}{8}(x_1^2 + x_2^2)\). For example,
\[
u = u_0 + C_1 \frac{1}{x_1^2 + x_2^2}(x_2, -x_1)
\]
solves (1) with the boundary conditions (11), and \(\nabla v \in L^p\) with \(1 < p < \infty\). However, the condition (11) does not hold.

As an immediate corollary of Theorem 1.2 and Remark 1.4 one has the following result.

Corollary 1.5 Let \((u, \pi)\) be a smooth solution of the 2D Navier-Stokes equations (1) defined in \(\Omega\) and \(u_0 = \frac{a^2}{8}(x_2, -x_1)\) with \(a = 0\) when the solution \(u\) satisfies the no-slip boundary condition (2) and is vanishing at infinity, i.e.,
\[
u(x_1, x_2) \to 0 \quad \text{as} \quad |(x_1, x_2)| \to \infty.
\]
On the other hand, we consider the special constant vorticity flow of \(u_0 = \frac{a^2}{8}(x_2, -x_1)\) with \(a = 0\) when the solution \(u\) satisfies the no-slip boundary condition (2) and is vanishing at infinity, i.e.,
\[
u(x_1, x_2) \to 0 \quad \text{as} \quad |(x_1, x_2)| \to \infty.
\]
Then, some sufficient conditions which guarantee the triviality of \(q\)-generalized solutions to the Navier-Stokes equations (1) are established in the following theorem. First let us recall the definition of \(q\)-generalized solutions.

Definition 1.6 A vector field \(u : \Omega \to \mathbb{R}^2\) is called a \(q\)-generalized solution to (1), (2) and (15) if for some \(q \in (1, \infty)\) the following properties are met:

(i) \(u \in D^{1,q}_0(\Omega)\);
(ii) $u$ is (weakly) divergence-free in $\Omega$;

(iii) $u$ verifies the identity

$$(\nabla u, \nabla \psi) = -(u \cdot \nabla u, \psi), \text{ for all } \psi \in \mathcal{D}(\Omega).$$

If $q = 2$, $u$ is usually called a generalized solution (or $D$-solution).

Moreover, for the 2D Navier-Stokes equations, all $q$-generalized solutions with $q > 1$ are smooth (see, for example, Chapter IX in [15]).

Our result is stated as follows:

**Theorem 1.7** Let $(u, \pi)$ be a $q$-generalized solution to the Navier-Stokes equations (7) with boundary conditions (2), (15) in the exterior domain $\Omega$. Then, $u(x) \equiv 0$ in $\Omega$ under one of the following conditions

1. $u(x)$ is a $q$-generalized solution for $1 < q \leq 3/2$.
2. $u(x) \in BMO^{-1}(\Omega)$ for $3/2 < q < 2$.
3. $u(x) \in L^4(\Omega)$ for a $D$-solution.

The rest of this paper is organized as follows. Some elementary results on functions with finite Dirichlet energy or $q$-generalized integrals, and the Giaquinta’s iteration lemma are collected in Section 2, which are important for the analysis in the rest of this paper. The proofs of Theorems 1.1, 1.2, 1.3 and 1.7 are presented in Section 3 - Section 6, respectively.

## 2 Preliminaries

Before going to the detailed proofs of the theorems, for reader’s convenience, we would like to collect some basic lemmas, which will be used in the proof.

First, the space $BMO$ in $\Omega$ is defined as follows, which is similar to the case in $\mathbb{R}^2$ as defined in [4].

**Definition 2.1** The space $BMO(\Omega)$ of bounded mean oscillations is the set of locally integrable functions $f$ such that

$$\|f\|_{BMO} \overset{\text{def}}{=} \sup_B \frac{1}{|B|} \int_B |f - f_B|dx < \infty \quad \text{with} \quad f_B \overset{\text{def}}{=} \frac{1}{|B|} \int_B f dx. \quad (16)$$

The above supremum is taken over the set of Euclidean balls.

It is clear that this quantity $\| \cdot \|_{BMO}$ is in general a seminorm, unless one argues modulo constant functions, and for $f \in BMO$, the following inequality holds true for all balls $B$

$$\frac{1}{|B|} \int_B |f - f_B|^p dx \leq C_p \|f\|_{BMO}^p, \quad (17)$$
where $1 \leq p < \infty$. In the following, a space that will be used is provided by the set of functions which are derivatives of functions in $BMO$. More precisely, we are talking about the space introduced by Koch and Tataru in [18], which is denoted by $BMO^{-1}$ (or by $\nabla BMO$) and is defined as the space of tempered distributions $f$ such that there exists a vector function $g = (g_1, g_2, g_3)$ belonging to $BMO$ such that $f = \nabla \cdot g$. The norm in $BMO^{-1}$ is defined by

$$
\|f\|_{BMO^{-1}} = \inf_{g \in BMO} \sum_{j=1}^{3} \|g_j\|_{BMO}.
$$

Second, for $1 \leq p \leq \infty$, let $L^p$ denote the usual scalar-valued and vector-valued $L^p$ space over $\Omega$. Let

$$W^{m,p}(\Omega) = \{u \in L^p : D^\alpha u \in L^q(\Omega), |\alpha| \leq m, m \in \mathbb{N}\}.$$

When $p = 2$, one abbreviates $H^m(\Omega) = W^{m,2}(\Omega)$. If $q \in [1, n)$, the space $D^{1,q}_0(\Omega)$ is the following:

$$D^{1,q}_0(\Omega) = \{u \in D^{1,q}(\Omega) : \|u\|_{nq/(n-q)} < \infty, \varphi u \in W^{1,q}_0(\Omega), \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^2)\},$$

if $q \geq n$, and complementary set $\Omega^c \supset B_a$, for some $a > 0$:

$$D^{1,q}_0(\Omega) = \{u \in D^{1,q}(\Omega) : \varphi u \in W^{1,q}_0(\Omega), \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^2)\},$$

where $W^{1,q}_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the classical Sobolev space $W^{1,q}(\Omega)$, and

$$D^{1,q}(\Omega) = \{u \in L^1_{loc}(\Omega) : \nabla u \in L^q(\Omega)\},$$

$$D(\Omega) = \{\psi \in C_0^\infty(\Omega) : \text{div} u = 0 \text{ in } \Omega\}.$$

Finally, let us recall some necessary lemmas. The first one is a lemma of Gilbarg-Weinberger in [17] about the decay of functions with finite Dirichlet integrals.

**Lemma 2.2 (Lemma 2.1, [17])** Let a $C^1$ vector-valued function $f(x) = (f_1, f_2)(x) = f(r, \theta)$ with $r = |x|$ and $x_1 = r \cos \theta$. There holds finite Dirichlet integral in the range $r > r_0$, that is

$$\int_{r>r_0} |\nabla f|^2 \, dx < \infty.$$

Then, we have

$$\lim_{r \to \infty} \frac{1}{\ln r} \int_0^{2\pi} |f(r, \theta)|^2 \, d\theta = 0.$$

For general energy integrals, we have the following:
Lemma 2.3 (Theorem II.9.1, [15]) Let $\Omega \subset \mathbb{R}^2$ be an exterior domain.

(i) Let $\nabla f \in L^r \cap L^p(\Omega)$, for some $1 \leq r < 2 < p < \infty$. Then there exists $f_0 \in \mathbb{R}$ such that
$$\lim_{|x| \to \infty} |f(x) - f_0| = 0,$$
uniformly.

(ii) Let $\nabla f \in L^2 \cap L^p(\Omega)$, for some $2 < p < \infty$. Then
$$\lim_{|x| \to \infty} \frac{|f(x)|}{\sqrt{\ln(|x|)}} = 0,$$
uniformly.

(iii) Let $\nabla f \in L^p(\Omega)$, for some $2 < p < \infty$. Then
$$\lim_{|x| \to \infty} \frac{|f(x)|}{|x|^{\frac{2}{p}}} = 0,$$
uniformly.

Moreover, we recall a lemma from [33].

Lemma 2.4 Let a $C^1$ vector-valued function $f(x) = (f_1, f_2)(x) = f(r, \theta)$ with $r = |x|$ and $x_1 = r \cos \theta$. There holds
$$\int_{r > r_0} |\nabla f|^q dx < \infty, \quad 1 < q < 2.$$

Then, we have
$$\limsup_{r \to \infty} \int_0^{2\pi} |f(r, \theta)|^q d\theta < \infty.$$

The following one is the Giaquinta’s iteration lemma, which gives the estimates of the $L^2$ norm of $\nabla u$ in our proof.

Lemma 2.5 (Lemma 3.1, Page 161, [16]) Let $f(t)$ be a nonnegative bounded function defined in $[r_0, r_1]$, $r_0 \geq 0$. Suppose that for $r_0 \leq t < s \leq r_1$ we have
$$f(t) \leq [A(s - t)^{-\alpha} + B] + \theta f(s),$$

8
where $A, B, \alpha, \theta$ are nonnegative constants with $0 \leq \theta < 1$. Then for all $r_0 \leq \rho < R \leq r_1$ we have
\[
f(\rho) \leq c[A(R - \rho)^{-\alpha} + B],
\]
where $c$ is a constant depending on $\alpha$ and $\theta$.

Next, the following Gagliardo-Nirenberg inequality (see [14, Theorem 10.1, Page 27]) will be frequently used.

**Lemma 2.6** Let $\Omega_0 \subset \mathbb{R}^2$ be a bounded smooth domain. Assume that $1 \leq q, r \leq \infty$, and $j, m$ are arbitrary integers satisfying $0 \leq j < m$. If $v \in W^{m,r}(\Omega_0) \cap L^q(\Omega_0)$, then we have
\[
\|D^j v\|_{L^p} \leq C \|v\|_{L^q}^{1-a} \|v\|_{W^{m,r}}^a,
\]
where
\[
-j + \frac{2}{p} = (1 - a)\frac{2}{q} + a \left(-m + \frac{2}{r}\right),
\]
and
\[
a \in \left\{ \left[ \frac{j}{m}, 1 \right], \text{ if } m - j - \frac{2}{r} \text{ is a nonnegative integer,} \right\}
\]
\[
\left[ \frac{j}{m}, 1 \right], \text{ otherwise,}
\]
the constant $C$ depends only on $m, j, q, r, a$, and $\Omega_0$.

### 3 Proof of Theorem 1.1

We perturb the Navier-Stokes equations around $(u_0, \pi_0) = \left(\frac{a_2}{2}(x_2, -x_1), \frac{a_2^2}{4}(x_1^2 + x_2^2)\right)$, then try to show the triviality of the perturbed system. Since the constant vorticity flow of $(u_0, \pi_0)$ solves the system [1], then $v = u - u_0$ and $\pi_1 = \pi - \pi_0$ satisfy the following system
\[
\begin{aligned}
-\Delta v + u \cdot \nabla v + v \cdot \nabla u_0 + \nabla \pi_1 &= 0, \\
\text{div } v &= 0,
\end{aligned}
\]
with the boundary condition
\[
v|_{\partial B_1} = 0,
\]
due to [5]. Next we show that $v \equiv 0$ under the assumptions of Theorem 1.1.

First, we introduce a cut-off function $\phi(x) \in C_0^{\infty}(B_R)$ with $0 \leq \phi \leq 1$ satisfying the following two properties:

i). $\phi$ is radially decreasing and satisfies
\[
\phi(x) = \phi(|x|) = \begin{cases} 1, & |x| \leq \rho, \\
0, & |x| \geq \tau,
\end{cases}
\]
where $1 < \frac{R}{2} \leq \frac{2}{3}\tau \leq \rho < \tau \leq R$;
ii). \(|\nabla \phi(x)| \leq \frac{C}{\tau - \rho}, |\nabla^2 \phi(x)| \leq \frac{C}{(\tau - \rho)^2}\) for all \(x \in \mathbb{R}^2\).

Second, due to the choosing of \(\phi\) and (19), one has
\[
\int_{B_{\tau} \setminus B_{\frac{1}{2}\tau}} \nabla \cdot (\phi v) dx = - \int_{\partial B_1} n \cdot v \phi dS = -\phi(1) \int_{\partial B_1} n \cdot v dS = 0.
\]

We recall now the Bogovskiî problem:
\[
\nabla \cdot \hat{w} = \nabla \cdot [\phi v].
\] (21)

where a vector-valued function \(\hat{w} : B_{\tau} \setminus B_{\frac{1}{2}\tau} \rightarrow \mathbb{R}^2\). Due to Bogovskiî’s result in [5] (see also, Theorem III 3.1 in [15]), there exists a constant \(C(s)\) and a vector-valued function \(\hat{w}\) such that \(\hat{w} \in W^{1,s}_0(B_{\tau} \setminus B_{\frac{1}{2}\tau})\) and (21) holds. Furthermore, we obtain
\[
\int_{B_{\tau} \setminus B_{\frac{1}{2}\tau}} |\nabla \hat{w}|^s dx \leq C(s) \int_{B_{\tau}} |\nabla \phi \cdot v|^s dx. \tag{22}
\]

Making the inner products \((\phi v - \hat{w})\) on both sides of the equation (18), by \(\nabla \cdot \hat{w} = \nabla \cdot [\phi v]\) we have
\[
\int_{B_{\tau} \setminus B_{\frac{1}{2}\tau}} \phi |\nabla v|^2 dx = - \int_{B_{\tau} \setminus B_{\frac{1}{2}\tau}} \nabla \phi \cdot \nabla v \cdot v dx + \int_{B_{\tau} \setminus B_{\frac{1}{2}\tau}} \nabla \hat{w} : \nabla v dx - \int_{B_{\tau} \setminus B_{\frac{1}{2}\tau}} u \cdot \nabla v \cdot \phi v dx + \int_{B_{\tau} \setminus B_{\frac{1}{2}\tau}} u \cdot \nabla u_0 \cdot \phi v dx + \int_{B_{\tau} \setminus B_{\frac{1}{2}\tau}} v \cdot \nabla u_0 \cdot \hat{w} dx
\]
\[
= I_1 + \cdots + I_6,
\]

For the term \(I_1\), it follows from H"{o}lder’s inequality that
\[
|I_1| \leq \frac{C}{\tau - \rho} \left( \int_{B_{\tau} \setminus B_{\frac{1}{2}\tau}} |\nabla v|^2 dx \right)^\frac{1}{2} \left( \int_{B_{\tau} \setminus B_{\frac{1}{2}\tau}} |v|^2 dx \right)^\frac{1}{2}.
\]

For the term \(I_2\), H"{o}lder’s inequality and (22) imply that
\[
|I_2| \leq C \left( \int_{B_{\tau} \setminus B_{\frac{1}{2}\tau}} |\nabla v|^2 dx \right)^\frac{1}{2} \|\nabla \hat{w}\|_{L^2(B_{\tau} \setminus B_{\frac{1}{2}\tau})}
\]
\[
\leq \frac{C}{\tau - \rho} \|\nabla v\|_{L^2(B_{\tau} \setminus B_{\frac{1}{2}\tau})} \|v\|_{L^2(B_{\tau} \setminus B_{\frac{1}{2}\tau})}.
\]

By integration by parts and (22), we find that
\[
|I_3| = \left| \int_{B_{\tau} \setminus B_{\frac{1}{2}\tau}} u \cdot \nabla v \cdot \phi v dx \right| = \left| \int_{B_{\tau} \setminus B_{\frac{1}{2}\tau}} v \cdot \nabla v \cdot \phi v dx \right|
\]
\[
\leq \frac{C}{\tau - \rho} \|v\|_{L^3(B_{\tau} \setminus B_{\frac{1}{2}\tau})}^3.
\]
Moreover, \( I_5 = 0 \) due to the antisymmetric matrix \( \nabla u_0 \), and

\[
|I_6| \leq \left| \int_{B_r \setminus B_{\frac{3}{4}}} v \cdot \nabla \hat{w} \cdot u_0 \, dx \right| \leq C \frac{R}{\tau - \rho} \left( \int_{B_r \setminus B_{\frac{3}{4}}} |v|^2 \, dx \right).
\]

Combining the estimates of \( I_1 - I_6 \), we get

\[
\int_{B_r \setminus B_{\frac{3}{4}}} \phi |\nabla v|^2 \, dx \leq \frac{1}{4} \| \nabla v \|_{L^2(B_r \setminus B_{\frac{3}{4}})}^2 + \frac{C R^2}{(\tau - \rho)^2} \| v \|_{L^2(B_r \setminus B_{\frac{3}{4}})}^2 + C \frac{R}{\tau - \rho} \| v \|_{L^3(B_r \setminus B_{\frac{3}{4}})}^3.
\]

Finally, we deal with the case of \( v \in L^p \) with \( 1 < p \leq 2 \). Recall that the following Poincaré-Sobolev inequality holds (see, for example, Lemma 2.6 or Theorem 8.11 and 8.12 [26])

\[
\| f \|_{L^2(B_r \setminus B_{\frac{3}{4}})} \leq C \| \nabla f \|_{L^2(B_r \setminus B_{\frac{3}{4}})} \| f \|_{L^p(B_r \setminus B_{\frac{3}{4}})} + C \tau^{1 - \frac{2}{p}} \| f \|_{L^p(B_r \setminus B_{\frac{3}{4}})},
\]

and

\[
\| f \|_{L^3(B_r \setminus B_{\frac{3}{4}})} \leq C \| \nabla f \|_{L^2(B_r \setminus B_{\frac{3}{4}})} \| f \|_{L^p(B_r \setminus B_{\frac{3}{4}})} + C \tau^{\frac{3}{2} - \frac{5}{p}} \| f \|_{L^p(B_r \setminus B_{\frac{3}{4}})},
\]

which imply that

\[
\int_{B_r \setminus B_{\frac{3}{4}}} \phi |\nabla v|^2 \, dx \leq \frac{1}{2} \| \nabla v \|_{L^2(B_r \setminus B_{\frac{3}{4}})}^2 + \frac{C R^2}{(\tau - \rho)^2} \| v \|_{L^2(B_r \setminus B_{\frac{3}{4}})}^2 + C \frac{R}{\tau - \rho} \| v \|_{L^3(B_r \setminus B_{\frac{3}{4}})}^3 \leq \frac{1}{2} \| \nabla v \|_{L^2(B_r \setminus B_{\frac{3}{4}})}^2 + C \frac{R^2}{(\tau - \rho)^2} \left( \| v \|_{L^2(B_r \setminus B_{\frac{3}{4}})}^2 + \| v \|_{L^3(B_r \setminus B_{\frac{3}{4}})}^3 \right) + C \frac{R}{\tau - \rho} \left( \| v \|_{L^2(B_r \setminus B_{\frac{3}{4}})}^2 + \tau^{\frac{3}{2} - \frac{5}{p}} \| v \|_{L^p(B_r \setminus B_{\frac{3}{4}})}^3 \right) + C \frac{R^2}{(\tau - \rho)^2} \left( \| v \|_{L^2(B_r \setminus B_{\frac{3}{4}})}^2 \right)^{\frac{3}{2} - \frac{5}{p}} + C \frac{R^2}{(\tau - \rho)^2} \left( \| v \|_{L^3(B_r \setminus B_{\frac{3}{4}})}^3 \right)^{\frac{3}{2} - \frac{5}{p}} + C \frac{R^2}{(\tau - \rho)^2} \left( \| v \|_{L^p(B_r \setminus B_{\frac{3}{4}})}^p \right)^{\frac{3}{2} - \frac{5}{p}}.
\]

It follows from Young’s inequality and \( v \in L^p \) for \( 1 < p \leq 2 \) that

\[
\int_{B_r \setminus B_{\frac{3}{4}}} \phi |\nabla v|^2 \, dx \leq \frac{3}{4} \| \nabla v \|_{L^2(B_r \setminus B_{\frac{3}{4}})}^2 + C \left( \frac{R^2}{(\tau - \rho)^2} \right)^{\frac{3}{2} - \frac{5}{p}} + C \frac{R^2}{(\tau - \rho)^2} + C \frac{R}{(\tau - \rho)^2} + C \frac{R^2}{(\tau - \rho)^2} \left( \| v \|_{L^p(B_r \setminus B_{\frac{3}{4}})}^p \right)^{\frac{3}{2} - \frac{5}{p}}.
\]
Applying Lemma 2.5, we have
\[ \int_{B_{R/2} \setminus B_1} |\nabla v|^2 \, dx \leq C, \quad \text{for any } R > 2. \]

Using this and (24) again, by taking \( \tau = 2\rho = R \to \infty \) we have
\[ \int_{\mathbb{R}^2 \setminus B_1} |\nabla v|^2 \, dx = 0, \quad (25) \]
which implies \( v \equiv 0 \) due to (8). Thus the proof is complete.

4 Proof of Theorem 1.2

Recall that \( v = u - u_0 \) and \( \pi_1 = \pi - \pi_0 \) satisfy (18). Define the vorticity \( \tilde{w} = \partial_2 v_1 - \partial_1 v_2 = w - a \). Then the equation of the vorticity \( \tilde{w} \) is as follows:
\[ -\Delta \tilde{w} + v \cdot \nabla \tilde{w} + u_0 \cdot \nabla \tilde{w} = 0. \quad (26) \]
Furthermore, let \( v' = \psi v \), where \( \psi \) a smooth cut-off function with \( 0 \leq \psi \leq 1 \) satisfying
\[ \psi(x) = \psi(|x|) = \begin{cases} 1, & |x| \leq 2, \\ 0, & |x| > 2. \end{cases} \quad (27) \]
Then \( v' \in C^\infty(\mathbb{R}^2) \) and \( v'(x) \equiv v(x) \) for \( |x| \geq 3 \). Similarly, define the vorticity \( w' = \partial_2 v'_1 - \partial_1 v'_2 \), then \( w'(x) \equiv \tilde{w}(x) \) for \( |x| \geq 3 \).

Step 1. Case of \( 2 < q < \infty \).

Let \( \eta(x_1, x_2) \in C^\infty_0(\mathbb{R}^2) \) be a cut-off function with \( 0 \leq \eta \leq 1 \) satisfying \( \eta(x) = \eta_1(|x|) \), where
\[ \eta_1(t) = \begin{cases} 1, & |t| \leq 1, \\ 0, & |t| > 2. \end{cases} \quad (28) \]
Multiply \( q \eta|w - a|^q|w - a| \) on both sides of (26), then we have
\[
\frac{4(q-1)}{q} \int_{\Omega} |\nabla (|w - a|^{\frac{q}{2}})|^2 \eta dx \\
\leq \int_{\Omega} |w - a|^q \Delta \eta dx + \int_{\Omega} |w - a|^q v \cdot \nabla \eta dx \\
+ \int_{\Omega} |w - a|^q u_0 \cdot \nabla \eta dx \triangleq K_1 + K_2 + K_3. \quad (29)
\]
Since \( \tilde{w} = w - a \in L^q \) by (12), obviously \( K_1 \to 0 \) as \( R \to \infty \). For the term \( K_2 \), due to (iii) in Lemma 2.3 and (12), for large \( R > 0 \) we have
\[ |v(x_1, x_2)| \leq |(x_1, x_2)|^{1 - \frac{2}{q}}. \]
Thus we have
\[ K_2 \leq CR^{(1-\frac{2}{q})-1} \to 0, \]
as \( R \to \infty \). It is worth noting that the third term is vanishing, since \( u_0 = \frac{2}{7}(x_2, -x_1) \) belongs to the tangent vector and \( \nabla \eta \) is the radial vector. Consequently, we get \( \nabla (|\tilde{w}|^\frac{2}{q}) \equiv 0 \), which implies that \( \tilde{w} \equiv 0 \) by (10). Due to \( \text{div} \ v = 0 \), it follows that
\[ \Delta v \equiv 0, \quad \text{in} \ \Omega. \]

**Claim that:**
\[ v \equiv C, \quad \text{in} \ \Omega. \quad (30) \]

Firstly, due to \( \Delta v' = \nabla^\perp w' \), there holds
\[ \|\nabla v'\|_{L^2(R^2)} + \|\nabla^2 v'\|_{L^p(R^2)} \leq C(\|w'\|_{L^2(R^2)} + \|\nabla w'\|_{L^p(R^2)}) < \infty, \quad (31) \]
by the help of Calderón-Zygmund estimates, since \( w' = \tilde{w} \equiv 0 \) for \( |x| \geq 3 \). Due to Lemma 2.3, there exists a constant vector \( v_0 \) such that
\[ \lim_{|x| \to \infty} |v' - v_0| = 0, \]
uniformly, which implies that
\[ \|v\|_{L^\infty(\Omega)} \leq C_0. \quad (32) \]

Secondly, for any \( r > 1 \), by (11) we have
\[
0 = \int_{B_r \setminus B_1} \Delta v \cdot v dx \\
= -\int_{B_r \setminus B_1} |\nabla v|^2 dx + \int_{\partial B_r} \frac{\partial v}{\partial n} \cdot v dS - \int_{\partial B_1} \frac{\partial v}{\partial n} \cdot v dS \\
= -\int_{B_r \setminus B_1} |\nabla v|^2 dx + r \int_{\partial B_1} \frac{\partial}{\partial r} [v(rz)]^2 dS_z,
\]
which yields that
\[ rG'(r) = 2 \int_{B_r \setminus B_1} |\nabla v|^2 dx, \]
provided that
\[ \int_{\partial B_1} [v(rz)]^2 dS_z = G(r). \]
Then by solving the ODE equation we have
\[
G(r) \geq G(r_0) + \left( 2 \int_{B_{r_0} \setminus B_1} |\nabla v|^2 dxdy \right) \ln \frac{r}{r_0}
\]

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for any $r > r_0 > 1$. Note that (32) implies that $G(r) \leq C_0^2|\partial B_1|$ for any $r > 1$, then

$$
\int_{B_r \setminus B_1} |\nabla v|^2 dxdy = 0.
$$

That is to say

$$
\nabla v \equiv 0, \quad \text{in } \Omega,
$$

and $v \equiv v_0$. Thus we have $v \equiv C$. The proof of (30) is complete.

**Step 2. Case of $1 < q \leq 2$.**

We take a cut-off function $\phi$ as in (20). Multiplying both sides of (26) by $\phi(w - a)$ and then applying integration by parts, we arrive at

$$
\int_{B_1} \phi |\nabla w|^2 dx = - \int_{B_1} \nabla \phi (w - a) \cdot \nabla w dx + \frac{1}{2} \int_{B_1} w \cdot \nabla \phi (w - a)^2 dx + \frac{1}{2} \int_{B_1} u_0 \cdot \nabla \phi (w - a)^2 dx
$$

$$
\equiv I_1' + I_2' + I_3'.
$$

In what follows we shall estimate $I_j'$ for $j = 1, 2, 3$ one by one. As in Step 1, $I_3' = 0$.

For the term $I_1'$, by Hölder’s inequality we have

$$
I_1' \leq \frac{C}{\tau - \rho} \| \nabla w \|_{L^2(B_\tau \setminus B_1)} \| w - a \|_{L^2(B_\tau \setminus B_{1/4})},
$$

Using the following multiplicative Gagliardo-Nirenberg inequality again

$$
\| w - a \|_{L^2(B_\tau \setminus B_{1/4})} \leq C \| \nabla w \|_{L^2(B_\tau \setminus B_{1/4})} \| w - a \|_{L^2(B_\tau \setminus B_{1/4})}^{1 - \frac{2}{q}} + C \tau^{1 - \frac{2}{q}} \| w - a \|_{L^q(B_\tau \setminus B_{1/4})},
$$

which yields that

$$
I_1' \leq \frac{1}{8} \int_{B_\tau \setminus B_1} |\nabla w|^2 dx + \frac{C}{(\tau - \rho)^{\frac{2}{q}}} + \frac{C \tau^{2 - \frac{4}{q}}}{(\tau - \rho)^2},
$$

by noting that $\| w - a \|_{L^q(B_\tau \setminus B_{1/4})} < \infty$.

For the terms $I_2'$, let

$$
\tilde{f}(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r, \theta) d\theta,
$$

then by Writinger’s inequality (for example, for $p = 2$ see Chapter II.5 [15]) we have

$$
\int_0^{2\pi} |f - \tilde{f}|^p d\theta \leq C(p) \int_0^{2\pi} |\partial_\theta f|^p d\theta,
$$
for $1 \leq p < \infty$.

Then by using (36), Lemma 2.2 and Lemma 2.4 we have

$$I_2' \leq \left| \int_{\mathbb{R}^2} (w - a)^2 (v - \bar{v}) \cdot \nabla \phi \, dx \right| + \left| \int_{\mathbb{R}^2} (w - a)^2 \bar{v} \cdot \nabla \phi \, dx \right|$$

$$\leq \frac{C}{\tau - \rho} \left( \int_{B_r \setminus B_{\frac{3}{4}\tau}} (w - a)^{2q'} \right)^{\frac{1}{q'}} \left( \int_{\frac{3}{4}\tau < r < \tau} \int_0^{2\pi} |v(r, \theta)|^q \, d\theta \, dr \right)^{\frac{1}{q}}$$

$$+ \frac{C}{\tau - \rho} \int_{B_{r} \setminus B_{\frac{3}{4}\tau}} (w - a)^2 \left( \int_0^{2\pi} |v(r, \theta)|^q \, d\theta \right)^{\frac{1}{q}} \, dx$$

$$\leq \frac{CR}{\tau - \rho} \left( \int_{B_r \setminus B_{\frac{3}{4}\tau}} (w - a)^{2q'} \right)^{\frac{1}{q'}} \left( \int_{\frac{3}{4}\tau < r < \tau} \int_0^{2\pi} |\partial_{\theta} v|^q \, d\theta \, dr \right)^{\frac{1}{q}}$$

$$+ C \left( \ln R \right)^{\frac{1}{2}} \int_{B_{r} \setminus B_{\frac{3}{4}\tau}} (w - a)^2 \, dx.$$

Using Gagliardo-Nirenberg inequality again, one has

$$\|w - a\|_{L^{2q'}(B_{r} \setminus B_{\frac{3}{4}\tau})} \leq C \|\nabla w\|_{L^2(B_{r} \setminus B_{\frac{3}{4}\tau})}^{1 - \frac{2q'}{2p}} \|w - a\|_{L^p(B_{r} \setminus B_{\frac{3}{4}\tau})}^{\frac{2q'}{2p}} + C \tau^{1 - \frac{2}{q}} \|w - a\|_{L^q(B_{r} \setminus B_{\frac{3}{4}\tau})}. \quad \text{(37)}$$

It follows from (34) and (37) that

$$I_2' \leq \frac{1}{8} \left( \int_{B_{r} \setminus B_1} |\nabla w|^2 \right) + C \left( \frac{R}{\tau - \rho} \right)^{\frac{2q'}{q}} \left( \|\nabla v\|_{L^q(B_{r} \setminus B_{\frac{3}{4}\tau})} \right)^{\frac{2q'}{q} + 2} + CR^{3 - \frac{3}{q}} (\tau - \rho)^{-1}$$

$$+ C \left( \frac{\sqrt{\ln R}}{\tau - \rho} \right)^{\frac{2}{q}} + C \left( \frac{\sqrt{\ln R}}{\tau - \rho} \right)^{2 - \frac{2}{q}}, \quad \text{(38)}$$

where we used the boundedness of $\|\nabla v\|_{L^q(B_{r} \setminus B_{\frac{3}{4}\tau})}$.

Collecting the estimates of $I_1', I_2'$, by (35) and (38) we have

$$\int_{B_{r} \setminus B_1} |\nabla w|^2 \, dx$$

$$\leq \frac{1}{2} \int_{B_{r} \setminus B_1} |\nabla w|^2 + \frac{C}{(\tau - \rho)^2} + C \tau^{2 - \frac{4}{q}} + CR^{3 - \frac{3}{q}} (\tau - \rho)^{-1}$$

$$+ C \left( \frac{\sqrt{\ln R}}{\tau - \rho} \right)^{\frac{2}{q}} + C \left( \frac{\sqrt{\ln R}}{\tau - \rho} \right)^{2 - \frac{2}{q}} + C \left( \frac{R}{\tau - \rho} \right)^{\frac{2q'}{q} + 2} \left( \|\nabla v\|_{L^q(B_{r} \setminus B_{\frac{3}{4}\tau})} \right)^{\frac{2q'}{q} + 2}.$$
Then an application of Lemma 2.5 yields
\[\int_{B_{R/2}} |\nabla w|^2 dx \leq CR^{-\frac{4}{q}} + C \left(\frac{\sqrt{\ln R}}{R}\right) + C \left(\|\nabla v\|_{L^q(B_R \setminus B_{R/2})}\right)^{\frac{2q'}{q}}.\]

Letting \( R \to \infty \), by noting that (12) we have
\[\nabla w \equiv 0, \quad \tilde{w} \equiv 0.\]

Similar arguments as in Step 1, we complete the proof.

5 Proof of Theorem 1.3

In this case, we want to prove that \( \nabla v \in L^2(\Omega) \), then the proof is complete by Theorem 1.2.

Case of \( v \in L^p \) with \( 1 < p \leq 2 \).

It’s similar to Theorem 1.1. In fact, we let \( \phi(x) \in C^\infty_0(B_R) \) with \( 0 \leq \phi \leq 1 \) as in (20). Since
\[\int_{B_r \setminus B_1} \nabla (\phi v) dx = -\int_{\partial B_1} n \cdot v \phi dS = -\phi(1) \int_{\partial B_1} n \cdot v dS = 0,\]
due to (11), one could take a vector-valued function \( \hat{w} : B_r \setminus B_{\frac{3}{2}} \to \mathbb{R}^2 \) such that \( \hat{w} \in W^{1,s}_0(B_r \setminus B_{\frac{3}{2}}) \) and \( \nabla \cdot \hat{w} = \nabla \cdot [\phi v] \) as in (22). Then making the inner products \( (\phi v - \hat{w}) \) on both sides of the equation (18), by \( \nabla \cdot \hat{w} = \nabla \cdot [\phi v] \) we have
\[\int_{B_r \setminus B_1} |\nabla v|^2 dx \\ \leq -\int_{B_r \setminus B_1} \nabla \phi \cdot \nabla v \cdot v dx + \int_{B_r \setminus B_1} \nabla \hat{w} : \nabla v dx - \int_{B_r \setminus B_1} u \cdot \nabla v \cdot \phi v dx \\ + \int_{B_r \setminus B_1} u \cdot \nabla v \cdot \hat{w} dx - \int_{B_r \setminus B_1} v \cdot \nabla u_0 \cdot \phi v dx + \int_{B_r \setminus B_1} v \cdot \nabla u_0 \cdot \hat{w} dx \\ - \int_{\partial B_1} \frac{\partial v}{\partial n} \cdot v \phi dx \equiv I_1 + \cdots + I_7,\]
where \( I_7 \leq C \), the boundary value of the velocity at \( \partial B_1 \) is bounded. Similarly,
\[I_3 \leq \frac{1}{2} \int_{B_r \setminus B_1} u \cdot \nabla \phi |v|^2 dx \leq C(\tau - \rho)^{-1} \int_{B_r \setminus B_{\frac{3}{2}}} |v|^3 dx + C\]
and other terms are similar to Theorem 1.1. The same arguments yield that
\[\int_{B_{R/2} \setminus B_1} |\nabla v|^2 dx \leq C. \tag{39}\]
Applying Theorem 1.2, we complete the proof.

**Case of** \(v \in L^p\) **with** \(2 < p < \infty\).

Let \(\phi(x) \in C_0^\infty(\mathbb{R}^d)\) be a cut-off function defined as in (20). Write \(\tilde{w}^{2q} = (\tilde{w}^2)^q\). For \(q \geq 1\), by Hölder and Young inequalities we have

\[
\int_\Omega \tilde{w}^{2q} \phi^{2q} dx = \int_\Omega (v_2, -v_1) \cdot \nabla [\tilde{w}^{2q-2} \tilde{w} \phi^{2q}] dx
\]

\[
\leq (2q - 1) \int_\Omega |v||\nabla \tilde{w}| \tilde{w}^{2q-2} \phi^{2q} dx + 2q \int_\Omega |v||\nabla \phi||\tilde{w}^{2q-1} \phi^{2q-1} dx
\]

\[
\leq \frac{1}{2} \int_\Omega \tilde{w}^{2q} \phi^{2q} dx + C(q) \|v\|^{2q}_{L^{\frac{2q}{2q-1}}} \left( \int_\Omega |\nabla \tilde{w}|^2 \tilde{w}^{2q-2} \phi^{2q} dx \right)^{\frac{q}{q+1}}
\]

\[
+ C(q) \|v\|^{2q} (\tau - \rho)^{-2q}.
\]

(40)

On the other hand, multiply \(\phi^{2q} \tilde{w}^{2q-2} \tilde{w}\) on both sides of (20), and we have

\[
(2q - 1) \int_\Omega |\nabla \tilde{w}|^2 \tilde{w}^{2q-2} \phi^{2q} dx
\]

\[
\leq \frac{1}{2q} \int_\Omega \tilde{w}^{2q} \Delta (\phi^{2q}) dx + \frac{1}{2q} \int_\Omega \tilde{w}^{2q} v \cdot \nabla (\phi^{2q}) dx
\]

\[
+ \frac{1}{2q} \int_\Omega \tilde{w}^{2q} u_0 \cdot \nabla (\phi^{2q}) dx
\]

\[
= II_1 + \cdots + II_3,
\]

(41)

and the last term vanishes. For the first two terms, there hold

\[
II_1 \leq C(\tau - \rho)^{-2} \int_{B_r \setminus B_1} |\tilde{w}|^{2q} dx,
\]

and

\[
II_2 \leq C(\tau - \rho)^{-1} \|v\|_{2q} \|\tilde{w}\|^{2q}_{L^\frac{2q}{2q-1}(B_r \setminus B_\frac{1}{2}^+)}.
\]

Noting that

\[
\|\tilde{w}^q\|_{L^\frac{4q}{2q-1}(B_r \setminus B_\frac{1}{2}^+)} \leq C \|\nabla (\tilde{w}^q)\|_{L^2(B_r \setminus B_\frac{1}{2}^+)}^{\frac{1}{2}} \|\tilde{w}^q\|_{L^2(B_r \setminus B_\frac{1}{2}^+)}^{1 - \frac{1}{2q}} + C \tau^{-\frac{1}{2q}} \|\tilde{w}^q\|_{L^2(B_r \setminus B_\frac{1}{2}^+)}^{\frac{2q}{2q-1}},
\]

which implies

\[
II_2 \leq C(\tau - \rho)^{-1} \|v\|_{2q} \|\nabla (\tilde{w}^q)\|_{L^2(B_r \setminus B_\frac{1}{2}^+)}^{\frac{1}{2}} \|\tilde{w}^q\|_{L^2(B_r \setminus B_\frac{1}{2}^+)}^{\frac{2q}{2q-1}}
\]

\[
+ C(\tau - \rho)^{-1} \tau^{-\frac{1}{2q}} \|v\|_{2q} \|\tilde{w}^q\|_{L^2(B_r \setminus B_\frac{1}{2}^+)}^2.
\]
Then
\[
\int_{\Omega} |\nabla \tilde{w}|^2 \tilde{w}^{2q-2} \phi^{2q} \, dx \\
\leq C(\tau - \rho)^{-2} \int_{B_r \setminus B_1} |\tilde{w}|^{2q} \, dx \\
+ C(\tau - \rho)^{-1} \|v\|_{2q} \|\nabla (\tilde{w} q^{1/2})\|_{L^2(B_r \setminus B_{2r})}^2 \|\tilde{w} q^{1/2}\|_{L^2(B_r \setminus B_{2r})}^{2-\frac{1}{2}} \\
+ C(\tau - \rho)^{-1} \tau^{-\frac{1}{4}} \|v\|_{2q} \|\tilde{w} q^{1/2}\|_{L^2(B_r \setminus B_{2r})}^2. 
\]  
(42)

Substituting these estimates into the previous estimate of (40), we get
\[
\int_{\Omega} \tilde{w}^{2q} \phi^{2q} \, dx \leq C(q) \|v\|_{2q}^{\frac{2q}{2q+4}} \left( \int_{B_r \setminus B_1} |\tilde{w}|^{2q} \, dx \right)^{\frac{1}{2q+4}} \\
+ C(q) \|v\|_{2q}^{\frac{2q}{2q+4}} \left( \int_{B_r \setminus B_1} |\nabla (\tilde{w} q^{1/2})|^2 \, dx \right)^{\frac{1}{2q+4}} \\
+ C(q) \|v\|_{2q}^{\frac{2q}{2q+4}} \left( \int_{B_r \setminus B_1} |\tilde{w} q^{1/2}|^2 \, dx \right)^{\frac{1}{2q+4}} \\
+ C(q) \|v\|_{2q}^{\frac{2q}{2q+4}} (\tau - \rho)^{-2q}. 
\]  
(43)

Combining (42) and (43) let \(2q = p\) and the known condition \(\|v\|_{2q} \leq C\) implies that
\[
\int_{B_r \setminus B_1} |\tilde{w}|^{2q} \, dx + \left( \int_{B_r \setminus B_1} |\nabla (\tilde{w} q^{1/2})|^2 \, dx \right)^{\frac{1}{2}} \leq \frac{1}{2} \int_{B_r \setminus B_1} |\tilde{w}|^{2q} \, dx + \frac{1}{2} \left( \int_{B_r \setminus B_1} |\nabla (\tilde{w} q^{1/2})|^2 \, dx \right)^{\frac{1}{2}} \\
+ C(q) (\tau - \rho)^{-2q} + C(q) (\tau - \rho)^{-q} \\
+ C(q) (\tau - \rho)^{-4q} + C(q) (\tau - \rho)^{-2q(\frac{2q+1}{4q+1})} + C(q) (\tau - \rho)^{-2q(\frac{2q+1}{4q+1})},
\]
where we used \(\tau > 1\) and Young’s inequality with the index
\[
\frac{4q + 1}{2(q + 1)(2q + 1)} + \frac{2q - 1}{2q + 1} + \frac{1}{2q + 1} = 1,
\]
and
\[
\frac{2q - 1}{4q + 1} + \frac{1}{2q} + \frac{(2q - 1)(2q + 1)}{4q(2q + 1)} = 1.
\]
Applying Lemma 2.5 again, we have
\[
\tilde{w} = 0.
\]
Recall \(v'\) and \(w'\) and estimate it as in (31), then it follows that \(\nabla v \in L^p\) for any \(p > 1\) due to \(\Delta v' = \nabla^+ w'\). Applying Theorem 1.2, the proof is complete.
6 Proof of Theorem 1.7

Case 1: \( u(x) \) is a \( q \)-generalized solution for \( 1 < q \leq \frac{3}{2} \).

It is obvious to see that the pressure \( \pi \) is almost silent in the definition of \( q \)-generalized solutions, which conceals some information about pressure. Note that in exterior domains, the Calderón-Zygmund inequality does not work, and the estimate for pressure via velocity is not available. One can apply the techniques used in Theorem 1.1 to prove the triviality of the \( q \)-generalized solutions. Similar as the local energy estimate (44), one can get

\[
\int_{B_\tau \setminus B_1} \phi |\nabla u|^2 \, dx \\
\leq \frac{1}{4} \| \nabla u \|_{L^2(B_\tau \setminus B_1)}^2 + \frac{C}{(\tau - \rho)^2} \| u \|_{L^2(B_\tau \setminus B_{3\tau} \setminus B_{\frac{3}{2} \tau})}^2 + \frac{C}{\tau - \rho} \| u \|_{L^3(B_\tau \setminus B_{\frac{3}{2} \tau})}^3,
\]

(44)

which implies the required result by using the same arguments, since \( u \in L^p \) with \( 2 < p \leq 6 \). So, for concision, we skip the details.

Case 2: \( u(x) \in BMO^{-1}(\Omega) \) for \( \frac{3}{2} < q < 2 \).

We construct a cut-off radially nonincreasing function \( \zeta_R(x) \in C_0^\infty(\mathbb{R}^2) \) for \( R \gg 1 \) by \( 0 \leq \zeta_R(x) \leq 1 \) which satisfies the followings

\[
\zeta_R(x) = \begin{cases} 
1, & x \in B_\rho \\
0, & x \in B_\rho^c
\end{cases}
\]

with

\[
\frac{R}{2} < \frac{\tau}{2} \leq R < \rho < \tau < 2R,
\]

moreover,

\[
\| \nabla \zeta_R(x) \|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{\tau - \rho}, \quad \| \nabla^2 \zeta_R(x) \|_{L^\infty(\mathbb{R}^2)} \leq \frac{C}{(\tau - \rho)^2},
\]

where \( C \) is independent of \( x \) and \( R \). It is easy to know that \( \nabla \zeta_R(x) \) is supported in \( A_R = B_\rho \setminus \overline{B}_\rho \). Now, multiplying both sides of (11) with \( \zeta_R u - \varphi \), and noticing that \( \zeta_R u - \varphi \) is divergence-free, one has

\[
\int_{B_\tau \setminus B_1} |\nabla u|^2 \zeta_R \, dx = - \int_{B_\tau \setminus B_{\pi/4}} \nabla u \cdot \nabla \zeta_R \cdot u \, dx + \int_{B_\tau \setminus B_{\pi/4}} \nabla u : \nabla \varphi \, dx \\
- \int_{B_\tau \setminus B_{\pi/4}} (u \cdot \nabla) u \cdot \zeta_R \, dx + \int_{B_\tau \setminus B_{\pi/4}} (u \cdot \nabla) u \cdot \varphi \, dx \\
= - \frac{1}{2} \int_{B_\tau \setminus B_{\pi/4}} |u|^2 \Delta \zeta_R \, dx + \int_{B_\tau \setminus B_{\pi/4}} \nabla u : \nabla \varphi \, dx \\
+ \frac{1}{2} \int_{B_\tau \setminus B_{\pi/4}} |u|^2 (u \cdot \nabla \zeta_R) \, dx + \int_{B_\tau \setminus B_{\pi/4}} (u \cdot \nabla) u \cdot \varphi \, dx \\
:= J_1 + J_2 + J_3 + J_4.
\]
We aim to prove that each $J_j$ ($j = 1, 2, 3, 4$) tends to zero as $R$ goes to infinity, which implies that $\|\nabla u\|_{L^2(\Omega)} = 0$, then it implies that $u \equiv 0$. Firstly, it follows by Hölder’s inequality that

$$J_1 \leq C \|u\|^2_{L^{\frac{2q}{n}}(B_r \setminus B_{r/4})} \|\nabla \zeta R\|_{L^{\frac{n}{q-1}}(B_r \setminus B_{r/4})}$$

$$\leq \frac{C}{(\tau - \rho)^{2}}\|u\|^2_{L^{\frac{2q}{n}}(B_r \setminus B_{r/4})} (\tau - \rho)^{\left(\frac{4}{3} - \frac{3}{q}\right)} \leq \frac{C}{R^{\left(\frac{4}{3} - 2\right)}}\|u\|^2_{L^{\frac{2q}{n}}(B_r \setminus B_{r/4})},$$

and

$$J_2 \leq \|\nabla u\|_{L^q(B_r \setminus B_{r/4})} \|\nabla \varphi\|_{L^{\frac{2q}{q-1}}(B_r \setminus B_{r/4})} \leq \|\nabla u\|_{L^q(B_r \setminus B_{r/4})} \|\nabla \zeta_R \cdot u\|_{L^{\frac{2q}{q-1}}(B_r \setminus B_{r/4})}$$

$$\leq \frac{C}{(\tau - \rho)^{2}}\|\nabla u\|_{L^q(B_r \setminus B_{r/4})} \|u\|_{L^{\frac{2q}{q-1}}(B_r \setminus B_{r/4})} (\tau - \rho)^{\left(\frac{3}{2} - \frac{1}{q}\right)}$$

$$\leq \frac{C}{R^{\left(\frac{3}{2} - 2\right)}}\|\nabla u\|_{L^q(B_r \setminus B_{r/4})} \|u\|_{L^{\frac{2q}{q-1}}(B_r \setminus B_{r/4})}.$$

The estimate of $J_3$ is slightly different, we proceed with the assumption that $u \in BMO^{-1}(\Omega)$. Since $u \in BMO^{-1}(\Omega)$, then each component of $u$ can be represented by

$$u_i = \sum_{j=1}^{2} \partial_j g_i^j, \quad i = 1, 2,$$

for some suitable functions $g_i^j \in BMO(\Omega)$. The estimate of $J_3$ is given as follows.

$$J_3 = \frac{1}{2} \int_{B_r \setminus B_{r/4}} |u|^2(u \cdot \nabla \zeta_R)dx = \frac{1}{2} \sum_{i=1}^{2} \int_{B_r \setminus B_{r/4}} |u|^2(u_i \partial_i \zeta_R)dx$$

$$= \frac{1}{2} \sum_{1 \leq i, j \leq 2} \int_{B_r \setminus B_{r/4}} |u|^2 \partial_j (g_i^j - [g_i^j]_\tau) \partial_i \zeta_R dx$$

$$= -\frac{1}{2} \sum_{1 \leq i, j \leq 2} \int_{B_r \setminus B_{r/4}} \partial_j \left( |u|^2 \partial_i \zeta_R \right) (g_i^j - [g_i^j]_\tau) dx$$

$$= -\frac{1}{2} \sum_{1 \leq i, j \leq 2} \int_{B_r \setminus B_{r/4}} |u|^2 \partial_i \zeta_R (g_i^j - [g_i^j]_\tau) dx$$

$$- \sum_{1 \leq i, j \leq 2} \int_{B_r \setminus B_{r/4}} (u \cdot \partial_j u) \partial_i \zeta_R (g_i^j - [g_i^j]_\tau) dx$$

$$:= J_{31} + J_{32},$$

where $[g_i^j]_\tau$ is the mean value of $\int_{B_r \setminus B_{r/4}} g_i^j dx$ on $B_r \setminus B_{r/4}$. Then

$$|J_{31}| \leq \sup_{i,j} \|g_i^j - [g_i^j]_\tau\|_{L^{\frac{2q}{n}}(B_r \setminus B_{r/4})} \|u\|^2_{L^{\frac{2q}{q-1}}(B_r \setminus B_{r/4})} \|\nabla \zeta_R\|_{L^{\frac{n}{q-1}}(B_r \setminus B_{r/4})}$$

$$\leq CR^{\left(\frac{4}{3} - 2\right)} \sup_{i,j} \|g_i^j\|_{BMO(B_r \setminus B_{r/4})} \|u\|^2_{L^{\frac{2q}{q-1}}(B_r \setminus B_{r/4})},$$



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where we used the inequality (17), and similarly

\[ |J_{32}| \leq \sup_{i,j} \left\| g^i_j - [g^i_j]_{r} \right\|_{L^{2R} \setminus (B_r \setminus B_{r/4})} \| u \|_{L^{2R} \setminus (B_r \setminus B_{r/4})} \left\| \nabla u \right\|_{L^q(B_r \setminus B_{3r})} \| \nabla \zeta_R \|_{L^{2R} \setminus (B_r \setminus B_{r/4})} \]

\[ \leq CR^{2 - \frac{d}{q}} \sup_{i,j} \left\| g^i_j \right\|_{BMO(B_r \setminus B_{r/4})} \| u \|_{L^{2R} \setminus (B_r \setminus B_{r/4})} \left\| \nabla u \right\|_{L^q(B_r \setminus B_{r/4})} . \]

Note that \( 3/2 < q < 2 \), then

\[ 2q - 3 > 0, \quad 2 - q > 0, \quad 2 - \frac{4}{q} \leq 0. \]

Finally, we estimate \( J_4 \).

\[ J_4 = \int_{B_r \setminus B_{r/4}} \left( u \cdot \nabla \right) u \cdot \varphi dx = \sum_{1 \leq i,j \leq 2} \int_{B_r \setminus B_{r/4}} u_i \partial_i u_j \varphi_j dx \]

\[ = - \sum_{1 \leq i,j \leq 2} \int_{B_r \setminus B_{r/4}} \partial_i (g^i_j - [g^i_j]_r) u_j \partial_i \varphi_j dx \]

\[ = \sum_{1 \leq i,j \leq 2} \int_{B_r \setminus B_{r/4}} (g^i_j - [g^i_j]_r) (\partial_i u_j \partial_i \varphi_j + u_j \partial_i^2 \varphi_j) dx \]

\[ := J_{41} + J_{42} . \]

In a similar way of \( J_3 \), one has

\[ J_{41} \leq \frac{C}{R} \sup_{i,j} \left\| g^i_j - [g^i_j]_R \right\|_{L^{2R} \setminus (B_{2R})} \| \nabla u \|_{L^q(B_r \setminus B_{3r})} \| u \|_{L^{2R} \setminus (B_r \setminus B_{3r})} . \]

\[ \leq CR^{2 - \frac{d}{q}} \sup_{i,j} \left\| g^i_j \right\|_{BMO(B_{2R})} \| u \|_{L^{2R} \setminus (B_r \setminus B_{r/4})} \left\| \nabla u \right\|_{L^q(B_r \setminus B_{r/4})} ; \]

and

\[ J_{42} \leq \frac{C}{R^2} \sup_{i,j} \left\| g^i_j - [g^i_j]_{r} \right\|_{L^{2R} \setminus (B_r \setminus B_{r/4})} \| u \|_{L^{2R} \setminus (B_r \setminus B_{r/4})} \| \nabla \zeta_R \cdot \nabla u \|_{L^q(B_r \setminus B_{3r})} \]

\[ \leq CR^{2 - \frac{d}{q}} \sup_{i,j} \left\| g^i_j \right\|_{BMO(B_r \setminus B_{r/4})} \| u \|_{L^{2R} \setminus (B_r \setminus B_{r/4})} \left\| \nabla u \right\|_{L^q(B_r \setminus B_{r/4})} ; \]

It is obvious to see that \( J_1, J_2, J_3 \) and \( J_4 \) tend to zero as \( R \) goes to \( \infty \), so we conclude that \( u \equiv 0 \).
Case 3: $u(x) \in L^4(\Omega)$ for a $D$-solution

As discussed in Case 1, the estimate of pressure is important to obtain suitable estimates in this situation. To this end, multiplying the Navier-Stokes equations by \( \psi \in C_0^\infty(\Omega) \) (not necessarily solenoidal), then integrating by parts yields

$$
(\nabla u, \nabla \hat{\psi}) = -(u \cdot \nabla u, \hat{\psi}) + (\pi, \nabla \cdot \hat{\psi}).
$$

(46)

If the convective term $u \cdot \nabla u$ has a mild degree of regularity, to every $q$-generalized solution we are able to associate a pressure $\pi$ such that (46) holds. Therefore, for a locally Lipschitz exterior domain of $\mathbb{R}^2$, if $u \cdot \nabla u \in D_{-1,q}^{-1}(\Omega)$, there exists a unique $\pi \in L^q(\Omega)$ satisfying (46) for all $\hat{\psi} \in C_0^\infty(\Omega)$. Furthermore, the following inequality holds

$$
\|\pi\|_{L^q} \leq C \left( |u \cdot \nabla u|_{-1,q} + \|\nabla u\|_{L^q} \right).
$$

(47)

The proof of this argument can be found in [15, Lemma V.1.1 in Page 305]. Now let $\psi(\xi)$ be a nonincreasing smooth function defined in $\mathbb{R}^2$ with $\psi(\xi) = 1$ if $|\xi| \leq 1/2$ and $\psi(\xi) = 0$ if $|\xi| \geq 1$, and set, for $R$ large enough,

$$
\psi_R(x) = \psi \left( \frac{\ln \ln |x|}{\ln \ln R} \right), \quad x \in \Omega.
$$

Note that, for a suitable constant $c$ independent of $R$, there holds

$$
|\nabla \psi_R(x)| \leq \frac{c}{\ln \ln R |x| \ln |x|},
$$

and $\nabla \psi_R(x) \neq 0$, only if $x \in \tilde{\Omega}_R$, where

$$
\tilde{\Omega}_R = \{x \in \Omega : \exp \sqrt{\ln R} < |x| < R\}.
$$

Multiplying equation (1) by $\psi_R u$, integrating by parts over $\Omega$ and taking the divergence free condition into account yield

$$
\int_{\Omega} \psi_R u \cdot (u \cdot \nabla) u = -\frac{1}{2} \int_{\Omega} |u|^2 u \cdot \nabla \psi_R dx,
$$

$$
\int_{\Omega} \psi_R u \cdot \nabla \pi = -\int_{\Omega} \pi \nabla \psi_R \cdot u dx,
$$

and

$$
\int_{\Omega} \psi_R u \cdot \Delta u dx = -\int_{\Omega} \nabla \psi_R \cdot \nabla u \cdot u dx - \int_{\Omega} \psi_R \nabla u : \nabla u dx,
$$

where we have used the zero boundary condition (2). Then it follows that

$$
\int_{\Omega} \psi_R \nabla u : \nabla u = -\int_{\Omega} \nabla \psi_R \cdot \nabla u \cdot u dx + \int_{\Omega} \pi \nabla \psi_R \cdot u dx + \frac{1}{2} \int_{\Omega} |u|^2 u \cdot \nabla \psi_R dx.
$$

(48)
By Hölder’s inequality, one has
\[
\int_{B_R \setminus \bar{B}_1} \psi_R |\nabla u|^2 \, dx = \int_\Omega \psi_R |\nabla u|^2 \, dx \leq \|\nabla \psi_R u\|_{L^2(\Omega_R)} \|\nabla u\|_{L^2(\Omega)} + \|\pi\|_{L^2(\Omega)} \|\nabla \psi_R u\|_{L^2(\Omega_R)} + C \|u\|_{L^4(\Omega)}^2 \|\nabla \psi_R u\|_{L^2(\Omega_R)};
\]  
(49)

We found by definition that
\[
\|u \cdot \nabla u\|_{-1,2} = \sup_{w \in D_{0}^{-1,2}(\Omega); \|\nabla w\|_{L^2} = 1} \int_\Omega u \cdot \nabla u \cdot w \leq \|u\|_{L^4} \|\nabla w\|_{L^2}.
\]  
(50)

Since we assume that \(u \in L^4\), (50) implies that \(u \cdot \nabla u \in D_{0}^{-1,2}(\Omega)\). Thus, by (47) one has
\[
\pi \in L^2(\Omega).
\]  
(51)

On the other hand, it follows that
\[
\|\nabla \psi_R u\|_{L^2(\Omega_R)}^2 \leq \frac{c_1}{(\ln \ln R)^2} \int_{\exp \sqrt{\ln R}}^R \frac{|u|^2}{|z| \ln |z|}^2 \, dz = \frac{c_1}{(\ln \ln R)^2} \int_{\exp \sqrt{\ln R}}^R \int_0^{2\pi} \frac{|u(r, \theta)|^2 r^{-1}}{\ln |r|} \, dr \, d\theta.
\]  
(52)

It remains to estimate the right-hand side of (52). Note that
\[
u(r, \theta) = u(r_0, \theta) + \int_{r_0}^r \frac{\partial u}{\partial \xi} d\xi, \quad \text{for } r \geq r_0 > 1.
\]

Then
\[
|u(r, \theta)|^2 = \left( u(r_0, \theta) + \int_{r_0}^r \frac{\partial u}{\partial \xi} d\xi \right)^2 \leq 2 \left( |u(r_0, \theta)|^2 + \left| \int_{r_0}^r \frac{\partial u}{\partial \xi} d\xi \right|^2 \right).
\]

It follows that
\[
\int_0^{2\pi} |u(r, \theta)|^2 \, d\theta \leq 2 \left( \int_0^{2\pi} |u(r_0, \theta)|^2 \, d\theta + \int_0^{2\pi} \left| \int_{r_0}^r \frac{\partial u}{\partial \xi} d\xi \right|^2 \, d\theta \right) \leq 2 \left( \int_0^{2\pi} |u(r_0, \theta)|^2 \, d\theta + \int_0^{2\pi} \left| \int_{r_0}^r \frac{\partial u}{\partial \xi} d\xi \right|^2 \, d\theta \right).
\]

By Hölder’s inequality
\[
\left| \int_{r_0}^r \frac{\partial u}{\partial \xi} \, d\xi \right|^2 = \left| \int_{r_0}^r \frac{\partial u}{\partial \xi} \xi^{\pm \frac{1}{2}} \, d\xi \right|^2 \leq \left( \int_{r_0}^r \left( \frac{\partial u}{\partial \xi} \right)^2 |\xi| \, d\xi \right) \ln r;
\]
Therefore, we obtain
\[
\int_0^{2\pi} \left| \int_{r_0}^r \frac{\partial u}{\partial \xi} d\xi \right|^2 d\theta \leq \int_0^{2\pi} G(r, \theta) d\theta \leq C_1 \ln r \left\| \nabla u \right\|_{L^2(B_r \setminus B_{r_0})}^2.
\]

Then from (52), one has
\[
\left\| \nabla \psi_R u \right\|_{L^2(\Omega_R)}^2 \leq C_2 \left( \frac{\ln \ln R}{(\ln R)^2} \right) \int_{\exp \sqrt{\ln R}}^R r^{-1}(\ln r + C)(\ln r)^2 dr 
\leq C_2 \ln \ln R,
\]
which implies that
\[
\lim_{R \to \infty} \left\| \nabla \psi_R u \right\|_{L^2(\Omega_R)} = 0,
\]
and from inequality (49), we conclude that
\[
\lim_{R \to \infty} \int_{B_R \setminus B_1} \psi_R |\nabla u|^2 dx = 0. \tag{53}
\]
Relations (53) and (48) imply, by the monotone convergence theorem, $\nabla u \equiv 0$. It follows that $u$ must identically be 0 by our assumption. Therefore, the proof of Theorem 1.7 is complete.

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References

[1] C.J. Amick, Existence of solutions to the nonhomogeneous steady Navier-Stokes equations, Indiana Univ. Math. J., 33 (1984) 817-830.

[2] C.J. Amick, On Leray’s problem of steady Navier-Stokes flow past a body in the plane, Acta. Math., 161 (1988) 71-130.

[3] K.I. Babenko, On stationary solutions of the problem of flow past a body, Mat. Sb. 91 (1973) 3-27. English translation: Math. SSSR Sbornik, 20 (1973) 1-25.

[4] H. Bahouri, J. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), Vol. 343, Springer, Herdelberg, 2011.
[5] M.E. Bogovski˘ı, Solution of some vector analysis problems connected with operators Div and Grad, Trudy Seminar S.L. Sobolev, #1, 80, Akademia Nauk SSSR, Sibirskoe Otdelnie Matematiki, Nowosibirsk, (1980) 5-40.

[6] B. Carrilloa, X. Pan, Q. Zhang, Decay and vanishing of some axially symmetric D-solutions of the Navier-Stokes equations, J. Funct. Anal., 279 (2020) 108504.

[7] B. Carrilloa, X. Pan, Q. Zhang, N. Zhao, Decay and vanishing of some D-Solutions of the Navier-Stokes equations, Arch. Rational Mech. Anal., 237 (2020) 1383-1419.

[8] D. Chae, Liouville-type theorem for the forced Euler equations and the Navier-Stokes equations, Commun. Math. Phys., 326 (2014) 37-48.

[9] D. Chae, S. Weng, Liouville type theorems for the steady axially symmetric Navier-Stokes and Magnetohydrodynamic equations, Discrete Contin. Dyn. Syst., 36 (2016) 5267-5285.

[10] D. Chae, T. Yoneda, On the Liouville theorem for the stationary Navier-Stokes equations in a critical case, J. Math. Anal. Appl., 405 (2013) 706-710.

[11] H.J. Choe, B.J. Jin, Asymptotic properties of axis-symmetric D-solutions of the Navier Stokes equations, J. Math. Fluid Mech., 11 (2009) 208-232.

[12] R. Farwig, The stationary Navier-Stokes equations in a 3D-exterior domain, recent topics on mathematical theory of viscous incompressible fluid, Lect. Notes Numer. App. Anal., 16 (1998) 53-1961.

[13] R. Finn, D.R. Smith, on the stationary solution of the Navier-Stokes equations in two dimensions, Arch. Rational Mech. Anal., 25 (1967) 26-39.

[14] A. Friedman, Partial differential equations, Dover Books on Mathematics, New York, 2008.

[15] G.P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations, Steady-state Problems, Second Edition, Springer, 2011.

[16] M. Giaquinta, Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems, Princeton University Press, Princeton, New Jersey, (1983).

[17] D. Gilbarg, H.F. Weinberger, Asymptotic properties of steady plane solutions of the Navier-Stokes equations with bounded Dirichlet integral, Ann. Scuola Norm. Sup. Pisa Cl. Sci., (4) 5 (1978) 381-404.
[18] H. Koch, D. Tataru, Well-posedness for the Navier-Stokes equations, Adv. Math., 157 (2001) 22-35.

[19] H. Hillairet, P. Wittwer, On the existence of solutions to the planar exterior Navier-Stokes system, J. Differential Equations, 255 (2013) 2996-3019.

[20] M. Korobkov, K. Pileckas, R. Russo, The Liouville theorem for the steady-state Navier-Stokes problem for axially symmetric 3D solutions in absence of swirl, J. Math. Fluid Mech., 17 (2015) 287-293.

[21] M. Korobkov, K. Pileckas, R. Russo, On the steady Navier-Stokes equations in 2D exterior domains, J. Differential Equations, 269 (2020) 1796-1828.

[22] M. Korobkov, K. Pileckas, R. Russo, On convergence of arbitrary D-solution of steady Navier-Stokes system in 2D exterior domains, Arch. Rational Mech. Anal., 233 (2019) 385-407.

[23] M. Korobkov, X. Ren, Uniqueness of plane stationary Navier-Stokes flow past an obstacle, Arch. Rational Mech. Anal., 240 (2021), no. 3, 1487-1519.

[24] O.A. Ladyzhenskaya, The mathematical theory of viscous incompressible fluid, Gordon and Breach (1969).

[25] J. Leray, Étude de diverses équations intégrales non linéaire et de quelques problèmes que pose l’hydrodynamique, J. Math. Pures Appl., 12 (1933) 1-82.

[26] E. H. Lieb and M. Loss, Analysis, second edition, Amer. Math. Soc., Providence, RI (2001).

[27] C. L. M. H. Navier, Mémoire sur les Lois du Mouvement des fluides, Mem. Acad. Sci. Inst. de France, 6 (1827) 389-440.

[28] K. Pileckas, R. Russo, On the existence of vanishing at infinity symmetric solutions to the plane stationary exterior Navier-Stokes problem, Math. Ann., 352 (2012) 643-658.

[29] V. Šverák, T-P. Tsai, On the spatial decay of 3-D steady-state Navier-Stokes flows, Commun. Partial Differ. Equ., 25 (2000) 2107-2117.

[30] G. Seregin, Liouville type theorem for stationary Navier-Stokes equations, Nonlinearity, 29 (2016), 2191-2195.

[31] J. Serrin, Mathematical principles of classical fluid mechanics. In: Handbuch der Physik (herausgegeben von S. Flügge), Bd. 8/1, Strömungsmechanik I (Mitherausgeber C. Truesdell), Springer-Verlag, Berlin-Göttingen-Heidelberg (1959) 125-263.
[32] W. Wang, Remarks on Liouville type theorems for the 3D steady axially symmetric Navier-Stokes equations, J. Differential Equations, 266 (2019) 6507-6524.

[33] W. Wang, Stability of the Couette flow under the 2D steady Navier-Stokes flow, Math. Nach., to appear.