Option Pricing Accuracy for Estimated Heston Models

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Abstract

We consider assets for which price $X_t$ and squared volatility $Y_t$ are jointly driven by Heston joint stochastic differential equations (SDEs). When the parameters of these SDEs are estimated from $N$ sub-sampled data $(X_{nT}, Y_{nT})$, estimation errors do impact the classical option pricing PDEs. We estimate these option pricing errors by combining numerical evaluation of estimation errors for Heston SDEs parameters with the computation of option price partial derivatives with respect to these SDEs parameters. This is achieved by solving six parabolic PDEs with adequate boundary conditions. To implement this approach, we also develop an estimator $\hat{\lambda}$ for the market price of volatility risk, and we study the sensitivity of option pricing to estimation errors affecting $\hat{\lambda}$. We illustrate this approach by fitting Heston SDEs to 252 daily joint observations of the S&P 500 index and of its approximate volatility VIX, and by numerical applications to European options written on the S&P 500 index.

Keywords : Heston SDEs , Option pricing errors, initial boundary value problems, option price sensitivities

1 Introduction

Option based hedging relies on accurate pricing of option contracts, generally computed after modelling the joint dynamics of the underlying asset price $X_t$ and squared volatility $Y_t$. The shortcomings of Black-Scholes models ([15]) have been well identified (see [14], [60], [53]), and have led to studies of a wide range of stochastic volatility models (e.g. [12], [39], [40], [43], [50], [53], [57]). For option pricing, one needs to estimate the parameters of the stochastic dynamics driving $X_t$ and $Y_t$. We study here the option pricing errors due to the parameter

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estimation errors induced by model fitting to actual data, since these errors can indeed be sizeable for small or moderately large market data sets.

This paper focuses on computing the impact of parameter estimation errors on European option pricing, when $X_t$ and $Y_t$ are driven by the classical Heston joint stochastic differential equations (SDEs). The Heston model (39), which has often been applied to concrete market data, does enable the numerical computation of option prices, either by Fourier inversion (21) or by solving directly the well known option pricing PDE (2, 39).

To fit Heston model parameters to data, we use discretized maximum likelihood parameter estimators, as developed and studied in [7] and [8]. These estimators have explicit closed form expressions, combining sub-sampled data $X_{nT}$ and $Y_{nT}$. In practice volatility “data” are not directly available, and are naturally replaced by well established estimates, such as “implied volatility” or “realized volatility”.

Volatility estimation has been intensively studied, often in combination with parameter estimation for stochastic volatility models. We refer for instance to [3] and to papers such as [1, 5, 17, 19, 32, 35, 42, 44, 51]. Parameter estimation based on both asset and option prices data has been explored in [4, 9, 21, 27, 30, 49].

We consider generic European options on assets for which price and volatility are driven by Heston joint SDEs. The parabolic PDEs verified by option prices involve four Heston model parameters as well as the unknown market price of volatility risk. We define option price sensitivities to these five parameters through partial derivatives of the option price with respect to these parameters. We derive the five PDEs and boundary conditions satisfied by the option price sensitivities, and we outline the efficient numerical schemes we have implemented to compute these sensitivities. We present our discretized maximum likelihood estimators for SDEs parameters, as well as an estimation technique for the market price of volatility risk. We then indicate how to quantify and compute the impact of parameter estimation errors on option pricing.

Finally, we illustrate our approach by analyzing market data for options based on the S&P 500 index, using the VIX index as a proxy for the S&P 500 volatility.

2 Heston Stochastic Volatility Model

Let $X_t$ be the asset price at time $t \geq 0$. The squared instantaneous volatility $Y_t$ of the returns process is defined by $Y_t dt = \text{var}(dX_t/X_t)$.

In the classical Heston model (39), the pair $\{X_t, Y_t\}$ is a progressively measurable stochastic process defined on a probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, endowed with an increasing filtration $\mathcal{F}_t$, and is driven by the following system $\mathcal{H}$ of SDEs,

\begin{align*}
    dX_t &= \mu X_t dt + \sqrt{Y_t} X_t dW_t, \\
    dY_t &= \kappa (\theta - Y_t) dt + \gamma \sqrt{Y_t} dB_t.
\end{align*}

Here $W_t$ and $B_t$ are standard Brownian motions on $\mathbb{R}$, adapted to the filtration $\mathcal{F}_t$, and have constant instantaneous correlation $\rho$, so that $E[dW_t dB_t] = \rho dt$. 

The parameter vector $\Theta = (\kappa, \theta, \gamma, \rho)$ is required to verify the well known constraints

$$|\rho| < 1, \quad \kappa > 0, \quad \theta > 0, \quad \gamma > 0, \quad 2\kappa\theta > \gamma^2,$$  \hspace{1cm} (3)

where the last inequality ensures the almost sure positivity of $Y_t$ as soon as $Y_0 > 0$. There are no constraints on the drift parameter $\mu$, which as is well known does not appear at all in option pricing equations. $Y_t$ is the “mean reverting” process studied by Feller ([29]) and originally used in ([24]) to model short-term interest rates. In practice $\Theta$ is unknown and has to be estimated from market data, which are usually sub-sampled at $N + 1$ successive times $nT$ starting with $n = 0$, for some fixed $T > 0$ and positive integer $N$.

### 2.1 Option pricing PDE

A European call option based on an asset $A$ is a contract signed at time $t = 0$, which fixes a strike or exercise price $K$ and a maturity time or exercise date $\tau$. At maturity time, the option holder has the option to buy at price $K$, from the option writer, one share of asset $A$.

Call $X_t$ the price of asset $A$ at time $t$. The pay-off of the option at maturity time is then given by $\Psi(X_\tau)$ where the pay-off function $\Psi(x)$ is defined for $x > 0$ by

$$\Psi(x) = (x - K)^+ = \max(x - K, 0).$$

We systematically assume that the price $X_t$ and the squared volatility $Y_t$ of $A$ are driven by the joint Heston SDEs ([1], [2]) with coefficients verifying ([3]). European call option prices based on asset $A$ verify then a well known parabolic PDE associated to the elliptic second order differential operator $L$ defined for $x > 0, y > 0$ by

$$L = \frac{1}{2}x^2 y \partial_x^2 + \frac{1}{2} \gamma^2 y \partial_y^2 + \rho \gamma x y \partial_x \partial_y + r x \partial_x + \left[\kappa (\theta - y) - \lambda_t \gamma \sqrt{y}\right] \partial_y - r $$ \hspace{1cm} (4)

The coefficients of $L$ involve the known risk free rate of return $r$, and the four unknown parameters $\kappa, \theta, \gamma, \rho$ of the Heston SDEs. Note that the drift parameter $\mu$ of these SDEs does not appear in $L$.

The coefficient of $\partial_y$ in $L$ involves the market price of volatility risk $\lambda_t \geq 0$, which is generally a time dependent random variable, but remains the same for all European call options based on a fixed asset $A$ (see [13]).

The emergence of $\lambda_t$ in $L$ and the associated option pricing PDE is due to the fact that the squared volatility $Y_t$ is typically a non tradable asset (see [13]). Several empirical studies have shown that $\lambda_t$ cannot be neglected ([10], [12], [23], [22], [28], [30], [15]). Since our numerical applications below involve only short term options, we will assume the market price of risk $\lambda_t$ to be constant in time and denote this unknown constant by $\lambda$ as in [20], [29]. In section 9 we will outline how to estimate $\lambda$ from market data.

As is well known (see [29]), the option price $Z_t$ is of the form

$$Z_t = g(X_t, Y_t, \tau - t),$$
where \( g(x, y, t) \) is of class 2 in \((x, y)\) and class 1 in \(t\), and is the unique solution of the parabolic PDE

\[
[\partial_t - \mathcal{L}]g = 0 \quad \text{for } 0 < x, 0 < y, 0 < t < \tau,
\]

verifying on the boundary \( \partial G \) of \( G = \mathbb{R}^+ \times \mathbb{R}^+ \times [0, \tau] \) the following boundary conditions

\[
g(x, y, 0) = \Psi(x) = (x - K)^+ \quad \text{for } 0 < x, 0 < y, \quad (6)
g(0, y, t) = 0 \quad \text{for } 0 < y, 0 < t \leq \tau, \quad (7)
\]

\[
[\partial_t - rx\partial_x - \kappa\theta\partial_y + r] g(x, 0, t) = 0 \quad \text{for } 0 < x, 0 < t \leq \tau, \quad (8)
\]

\[
\lim_{x \to \infty} \partial_x g(x, y, t) = 1 \quad \text{for } 0 < y, 0 < t \leq \tau, \quad (9)
\]

\[
\lim_{y \to \infty} \partial_y g(x, y, t) = 0 \quad \text{for } 0 < x, 0 < t \leq \tau. \quad (10)
\]

The first condition asserts that at maturity time, the option price is equal to the option pay-off. The second condition states that when the asset price \( x = 0 \), the option price must also be 0. The third condition is the formal limit, as \( y \to 0 \), of the PDE (5). Indeed note that, as \( y \to 0 \), the elliptic operator \( \mathcal{L} \) does degenerate into the first order differential operator \( D = rx\partial_x + \kappa\theta\partial_y - r \).

The fourth and fifth boundary conditions require the option price to be approximately linear in the asset price \( x \) for large \( x \), and to be roughly independent of the squared volatility \( y \) for large \( y \).

Four of the boundary conditions are standard, namely the initial value (6), the Dirichlet condition (7), and the two Neumann boundary conditions (9) and (10). But the boundary condition (8) at \( y = 0 \) is of a more general type (see \[36\]).

### 3 Option price sensitivity to parameters estimation errors

#### 3.1 Option price differentiability with respect to parameters

Due to the specificity of the 3rd boundary condition in (6) - (10) and to the degenerescence of the elliptic operator \( \mathcal{L} \) on the boundary \( y = 0 \), classical generic results on parabolic PDEs (see \[52\], \[59\]) do not directly imply the differentiability of \( g(x, y, t) \) with respect to the parameter vector \( \pi = [\kappa, \theta, \gamma, \rho, \lambda] \) which determines the coefficients of the elliptic operator \( \mathcal{L} \).

To directly prove this differentiability one can proceed as follows. When one fixes \( x, y, t, \pi, \tau \), then \( g(x, y, t) \) can be viewed as a function \( \Gamma(K) \) of the strike price \( K \) only. For \( z \in \mathcal{R} \), the one-dimensional Fourier transform \( \hat{\Gamma}(z) = \int_{K > 0} \Gamma(K) dK \) of \( \Gamma(K) \) has a known complicated expression as an explicit function \( \psi(z, x, y, t, \pi, \tau) \) given in \[39\]. The formal partial derivatives of \( \psi \) with respect to \( \kappa, \theta, \gamma, \rho, \lambda \) then provide formal explicit expressions for the Fourier transforms with respect to \( K \) of the partial derivatives of \( g(x, y, t) \) with respect to \( \kappa, \theta, \gamma, \rho, \lambda \). Proving the actual convergence of the integrals naturally
involved in these new Fourier transform formulas requires a cumbersome and fairly extensive amount of mathematical verifications which did not seem to be the shortest path to prove the desired differentiability result. Hence we are not presenting this proof here, and we will simply consider as a known result the existence of the gradient

\[ \partial_\pi g = [\partial_\kappa g, \partial_\theta g, \partial_\gamma g, \partial_\rho g, \partial_\lambda g] \]  

(11)

### 3.2 Option price sensitivity: definition

In practical option pricing, the parameter vector \( \pi \) has to be estimated from price data, approximate volatility data, and observed option trading prices. The unavoidable estimation errors on \( \pi \) necessarily induce option pricing errors. We define the five sensitivities of the option pricing function \( f(x, y, t) = g(x, y, \tau - t) \) with respect to small errors affecting the parameter vector \( \pi \) by the formulas

\[ \text{Sen}_\kappa = |\partial_\kappa g| \quad \text{Sen}_\theta = |\partial_\theta g| \quad \text{Sen}_\gamma = |\partial_\gamma g| \quad \text{Sen}_\rho = |\partial_\rho g| \quad \text{Sen}_\lambda = |\partial_\lambda g| \]  

(12)

where all partial derivatives are computed at the point \((x, y, \tau - t)\). These option price sensitivities are hence functions of \((x, y, t, \pi, \tau, K)\). The numerical computation of option price sensitivities has been explored in \cite{18, 23} and we propose here a new approach.

### 3.3 Sensitivity Equations

Differentiating equation (5) with respect to each one of the coordinates of the parameter vector \( \pi \), one obtains the five independent PDEs verified by the 5 coordinates of the gradient \( \partial_\pi g(x, y, t) \), in the open domain \( 0 < x, 0 < y, 0 < t < \tau \).

\[
\left( \frac{\partial}{\partial t} - L \right) \partial_\kappa g = (\theta - y) \frac{\partial g}{\partial y} \
\]  

(13)

\[
\left( \frac{\partial}{\partial t} - L \right) \partial_\theta g = \kappa \frac{\partial g}{\partial y} \
\]  

(14)

\[
\left( \frac{\partial}{\partial t} - L \right) \partial_\gamma g = \gamma y \frac{\partial^2 g}{\partial y^2} + \rho xy \frac{\partial^2 g}{\partial x \partial y} - \lambda \sqrt{y} \frac{\partial g}{\partial y} \
\]  

(15)

\[
\left( \frac{\partial}{\partial t} - L \right) \partial_\rho g = \gamma xy \frac{\partial^2 g}{\partial x \partial y} \
\]  

(16)

\[
\left( \frac{\partial}{\partial t} - L \right) \partial_\lambda g = -\gamma \sqrt{y} \frac{\partial g}{\partial y} \
\]  

(17)

Each one of these five PDEs is associated to five boundary conditions. The first four of these conditions are given in compact form by the following vector
equations, where $[0]$ is the vector of $\mathbb{R}^5$ with all coordinates equal to 0,

$$
\partial_x g = 0 \text{ for } 0 < x, 0 < y, t = 0 \quad (18)
$$

$$
\partial_y g = 0 \text{ for } x = 0, 0 < y, 0 < t \leq \tau \quad (19)
$$

$$
\lim_{x \to \infty} \partial_x [\partial_y g] = [0] \text{ for } 0 < y, 0 < t \leq \tau \quad (20)
$$

$$
\lim_{y \to \infty} \partial_y [\partial_x g] = 0 \text{ for } 0 < x, 0 < t \leq \tau \quad (21)
$$

The fifth boundary conditions are obtained by differentiating equation (8) with respect to each one of the parameters. This yields the following boundary conditions, valid for $0 < x, y = 0, 0 < t \leq \tau$,

$$
\mathcal{D} \partial_x g = \theta \partial_y g, \quad \mathcal{D} \partial_y g = \kappa \partial_x g, \quad \mathcal{D} \partial_y g = 0, \quad \mathcal{D} \partial_x g = 0, \quad \mathcal{D} \partial_{\lambda} g = 0. \quad (22)
$$

where $\mathcal{D}$ is the 1st order differential operator given by

$$
\mathcal{D} = \frac{\partial}{\partial t} - r_x \frac{\partial}{\partial x} - \kappa \theta \frac{\partial}{\partial y} + r. \quad (23)
$$

We now outline the numerical scheme we have implemented to solve the option price PDE and the five preceding non homogeneous PDEs.

## 4 Numerical computation of option price sensitivities to parametric errors

In concrete numerical option pricing, the asset price $x > 0$ and the squared volatility $y > 0$ have obvious explicit realistic upper bounds $x_B, y_B$, so it is standard computing practice to replace the unbounded domain $G = \mathbb{R}^+ \times \mathbb{R}^+ \times [0, \tau]$ by the bounded domain $U = [0, x_B] \times [0, y_B] \times [0, \tau]$ where $x_B, y_B$ are fixed large positive numbers.

Then $g(x, y, t)$ will be computed as the unique function verifying the parabolic PDE (3) for $(x, y, t)$ in the interior $U^0$ of $U$ and the following five conditions on the boundary $\partial U$ of $U$

$$
g(x, y, 0) = \Psi(x) = (x - K)^+ \text{ for } 0 < x < x_B, 0 < y < y_B \quad (24)
$$

$$
g(0, y, t) = 0 \text{ for } 0 < y < y_B, 0 < t \leq \tau \quad (25)
$$

$$
[\partial_t - r_x \partial_x - \kappa \theta \partial_y + r] g(x, 0, t) = 0 \text{ for } 0 < x < x_B, 0 < t \leq \tau \quad (26)
$$

$$
\lim_{x \to x_B} \partial_x g(x, y, t) = 1 \text{ for } 0 < y < y_B, 0 < t \leq \tau \quad (27)
$$

$$
\lim_{y \to y_B} \partial_y g(x, y, t) = 0 \text{ for } 0 < x < x_B, 0 < t \leq \tau \quad (28)
$$

To solve the PDE (5), we discretize the elliptic operator $\mathcal{L}$ in (4), by standard numerical schemes well known to be stable under discretization refining. Numerical schemes for option pricing are discussed in multiple papers such as [2], [41]. For European options under the Heston model, the papers [38] and [48]
both present explicit specific numerical schemes. We apply a uniform space-time finite difference grid on the domain $U$ using the second order space discretization outlined in [41] and the backward differentiation formula for time discretization given in [47].

Let the number of grid steps be $m, n$ and $s$ in the $x, y$ and $t$ directions respectively. Grid step sizes in each direction are denoted

$$
\Delta x = x_B / m; \quad \Delta y = y_B / n; \quad \Delta t = \tau / s.
$$

At grid points, the values of $g$ are indexed as follows

$$
g_{i,j}^{k} = g(x_i, y_j, t_k) = g(i \Delta x, j \Delta y, k \Delta t) \quad \text{for} \quad i = 0, \ldots, m; j = 0, \ldots, n; k = 0, \ldots, s.
$$

### 4.1 Space discretization

All space partial derivatives in the parabolic PDE (5) have variable coefficients, so that in some parts of the domain the first order derivative terms may dominate the second order terms. To discretize these spatial derivatives we apply a space discretization scheme used in [41] for American options. After discretization, $L$ becomes a matrix $A$ well studied in [41].

Recall that M-matrices which are strictly diagonally dominant with positive diagonal elements and non-positive off diagonal elements have good stability properties (see [55], [58]). In general $A$ is not an M-matrix but as remarked in [41], when the time discretization of (5) has sufficiently small time steps, $A$ becomes diagonally dominant.

We apply the seven point spatial discretization scheme of [41] to solve for the option price. We use a second order accurate finite difference scheme for the space derivatives, namely the classical central difference scheme for first order derivatives and the usual three point scheme for second order derivatives.

The corresponding finite difference operators are then

$$
\delta_x g_{i,j}^k = \frac{g_{i+1,j}^k - g_{i-1,j}^k}{2 \Delta x}, \quad \delta_y g_{i,j}^k = \frac{g_{i,j+1}^k - g_{i,j-1}^k}{2 \Delta y},
$$

$$
\delta_x^2 g_{i,j}^k = \frac{g_{i+1,j}^k - 2g_{i,j}^k + g_{i-1,j}^k}{\Delta x^2}, \quad \delta_y g_{i,j}^k = \frac{g_{i,j+1}^k - 2g_{i,j}^k + g_{i,j-1}^k}{\Delta y^2}.
$$

On the boundary $j = 0$, the derivative in the $y$ direction in (26) is evaluated by the following upwind discretization scheme (see [37]),

$$
\delta_y g_{i,0}^k = \frac{-3g_{i,0}^k + 4g_{i,1}^k - g_{i,2}^k}{2 \Delta y}.
$$

As proposed in [41], the mixed derivatives are discretized using a seven point stencil $\delta_{xy}$, where $2 \Delta x \Delta y \delta_{xy} g_{i,j}^k$ is given by

$$
2g_{i,j}^k + g_{i+1,j+1}^k + g_{i-1,j-1}^k - g_{i+1,j-1}^k - g_{i-1,j+1}^k - g_{i,j+1}^k - g_{i,j-1}^k.
$$
We handle the Neumann boundary conditions (27) and (28) in the same way as in [41]. This space discretization leads to a semi-discrete equation,

\[
\frac{dg}{dt} + Ag = B,
\]

where \( A \) is an \( mn \times mn \) matrix and \( B \) is a column vector of length \( mn \). The vector \( g \) of length \( mn \) gathers all the option price values at grid points. The vector \( B \) gathers terms due to the Neumann boundary condition in the \( x \) direction and does not depend on \( t \).

### 4.2 Time discretization

For time discretization, as in [47], we use the BDF2 scheme (Backward Difference Formula), which is an implicit scheme with second order accuracy [47]. At time \( k\Delta t \) the BDF2 scheme reads,

\[
\frac{3g^{k+1} - 4g^k + g^{k-1}}{2\Delta t} + Ag^{k+1} = B,
\]

for \( k = 1, 2, \ldots, l-1 \). The stability properties of this scheme are studied in [41] and [47]. At each iterate of the BDF2 scheme we require the value of the last two iterates. To obtain the 1st iterate, we use an Implicit Euler scheme [37]: given the initial value \( g^0 \), we compute \( g^1 \) by

\[
\frac{g^1 - g^0}{\Delta t} + Ag^1 = B.
\]

At moderate grid sizes, we have numerically verified that this choice of space-time discretization of our initial-boundary value problem gives us stable solutions for the option price. At each time step, we use an LU decomposition to solve the following system of linear equations,

\[
(I + \frac{2}{3}\Delta t A)g^{k+1} = \frac{4}{3}g^k - \frac{1}{3}g^{k-1} + \Delta tB,
\]

(32)

where \( I \) is the \( mn \times mn \) identity matrix.

Once the function \( g \) has been computed on our discrete grid, we solve the sensitivity equations (13)–(17) by a numerical scheme quite similar to the scheme just described. To evaluate the right-hand side of equations (13), we use a central difference scheme to discretize the first spatial derivatives of \( g \), a 3-point stencil similar to (30) for the second spatial derivatives of \( g \), and a 7 point stencil similar to (31) for the mixed second derivative of \( g \). The space-time discretization of the sensitivity equations is the same as above, and we can then solve these discretized equations on the same grid used to compute the option price.

We have verified empirically, by successive grid refinements, that this numerical scheme generates converging approximations for the solutions of the sensitivity equations.
5 Estimation of Parameters for Heston joint SDEs

Consider $N$ joint observations $(U_n = X_{nT}$ and $V_n = Y_{nT}, \ n = 1, 2, \ldots, N, \ T > 0)$ of the asset price and approximate squared volatility, where $T$ is some fixed sub-sampling time. In practice $T$ is user selected, with standard values such as $T = 1/252$ for daily data.

Asset prices are directly observed, but squared volatilities $V_n = Y_{nT}$ have to be estimated for instance by “realized volatility” (see [4, 11, 26, 33, 34] for realized volatility estimation).

In a companion paper [8], we have introduced and studied an explicit vector $\hat{\Theta} = (\hat{\kappa}, \hat{\theta}, \hat{\gamma}^2, \hat{\rho})$ of discretized maximum likelihood estimators for the unknown vector $\Theta = (\kappa, \theta, \gamma^2, \rho)$ of Heston model coefficients.

The vector $\hat{\Theta}$ is computed in [8] by likelihood maximization after formal Euler discretization of the Heston joint SDEs, which leads to define the five sufficient statistics $a, b, c, d, f$

$$a = \frac{1}{N} \sum_{n=0}^{N-1} \frac{(V_{n+1} - V_n)^2}{V_n}, \quad b = -\frac{2}{N} \sum_{n=0}^{N-1} \frac{V_{n+1} - V_n}{V_n}, \quad c = \frac{2}{N} (V_N - V_0)$$

$$d = \frac{2}{N} \sum_{n=0}^{N-1} \frac{1}{V_n}, \quad f = \frac{2}{N} \sum_{n=0}^{N-1} V_n.$$

(33)

These statistics almost surely verify

$$a > 0, \quad d > 0, \quad f > 0, \quad df - 4 > 0, \quad 2af - c^2 > 0, \quad d + f - 4 > 0.$$  

Our nearly maximum likelihood estimators of $\kappa, \theta, \gamma^2$ are then explicitly given by

$$\hat{\kappa} = -\frac{2b + cd}{T(df - 4)}, \quad \hat{\theta} = \frac{bf + 2c}{2b + cd}, \quad \hat{\gamma}^2 = \frac{a}{T} - \frac{b^2f + 4bc + c^2d}{2T(df - 4)}.$$  

(34)

When $NT \to \infty$ and $T \to 0$, we have shown in [8] that, with probability tending to 1, these three estimators verify all the required natural constraints and converge in probability to the true parameters. We have also shown the same properties when $N \to \infty$ with $T$ fixed, provided $T$ is small enough.

Recall that the Heston SDEs involve two Brownian motions $W_t$ and $B_t$. Once $\kappa, \theta, \gamma$ are estimated, the discretization of the SDEs with time step $T$ provides natural estimates $\hat{\mu}$ for $\mu$ and $DZ_n, DB_n$ for the Brownian increments $W_{(n+1)T} - W_{nT}$ and $B_{(n+1)T} - B_{nT}$. The natural estimator $\hat{\rho}$ of $\rho$ is then the empirical correlation of $DZ_n$ and $DB_n$.

Call $\Delta \Theta$ the vector $(\hat{\Theta} - \Theta)$ of parameter estimation errors. These theoretical errors are not necessarily in $L_2$, because $1/Y_t$ does not always have a finite second moment but any slight truncation of our estimators eliminates this difficulty in numerical applications to concrete data. The covariance matrix $\Sigma$ of
\( \Delta \Theta \) provides then the \( L_2 \)-sizes of estimation errors and their correlations. We have validated in ([8], [31]) that for \( N \) moderately large, one can generate reasonable empirical estimates \( \Sigma \) of the error covariance matrix \( \Sigma \) as follows.

Use the \( N \) observations \( X_{nT}, Y_{nT} \) to compute the associated value \( \Theta_0 \) of our vector of estimators \( \Theta \). Then simulate a moderately large number \( q \) of random diffusion trajectories \( \omega_j \) of duration \( NT \) driven by joint Heston SDEs parametrized by \( \Theta_0 \). This is achieved by a standard Euler time discretization of the Heston SDEs, with very small discretization time step \( \delta << T \). The explicit estimation formulas outlined above then provide one value \( \Theta_j \) of the random vector \( \hat{\Theta} \) for each simulated trajectory \( \omega_j \). The empirical covariance matrix \( \hat{\Sigma} \) of the \( \Theta_j - \Theta_0 \) is then a natural estimator of the covariance matrix \( \Sigma \).

6 Impact of parametric estimation errors on option pricing

To implement option pricing, we start from \( N \) joint observations \((X_{nT}, Y_{nT})\) of the underlying asset price and squared volatility, where \( T \) is a fixed known (or user selected) sub-sampling time step. As just described, we use these data to compute an estimator \( \hat{\Theta} \) for the vector \( \Theta \) of Heston model coefficients.

Consider a European option based on this asset, with given strike price \( K \) and maturity date \( \tau \). In the option pricing PDE ([9], the unknown Heston model coefficients are then replaced by the estimators just computed. The option price \( f(x,y,t) = g(x,y,\tau - t) \) computed by solving ([9] is then affected by a (random) error \( \Delta f(x,y,t) \). Our objective is to compute numerical bounds for this option pricing error. For the moment we consider that the market price \( \lambda \) of volatility risk is known accurately. The impact of estimation errors affecting \( \lambda \) will be studied separately below in section [9]. The option price \( f(x,y,t) = g(x,y,\tau - t) \) depends also on the underlying parameter vector \( \Theta \). To simplify notations we often omit below the variables \((x,y,t,\Theta)\) for \( f \) and the associated variables \((x,y,\tau - t,\Theta)\) for \( g \).

With this shorthand convention, the option pricing error \( \Delta f \) is equal to the error \( \Delta g \) affecting \( g(x,y,\tau - t) \). For \( NT \) large enough and \( T \) small enough, the vector of estimation errors \( \Delta \Theta = \hat{\Theta} - \Theta \) becomes arbitrarily small (see [8]), so that one can legitimately apply a first order Taylor expansion to obtain the approximation

\[
\Delta f = \Delta g \simeq \partial_{\Theta} g. \Delta \Theta. \tag{35}
\]

For each fixed quadruplet \((x,y,t,\Theta)\), parameter estimation errors have a perturbation impact on option pricing, and we quantify this impact by the \( L_2 \)-norm \( \varepsilon \) of \( \Delta f \). Due to equation (35), we have the approximation

\[
\varepsilon^2 \simeq \partial_{\Theta} g^* \cdot \Sigma \cdot \partial_{\Theta} g, \tag{36}
\]

where the gradient \( \partial_{\Theta} g \) should be evaluated at \((x,y,\tau - t,\Theta)\), and * denotes matrix transpose. Since \( \Sigma \) is positive definite, we have \(|\Sigma_{i,j}| < \sqrt{\Sigma_{i,i}} \Sigma_{j,j} \) for all
i, j ∈ \{1, 2, 3, 4\}, which implies, for all v ∈ R^4, the inequality

\[ 0 \leq v^*. \Sigma \cdot v \leq ( \sum_{i=1}^{4} |v_i| \Sigma_{i,i}^{1/2} )^2. \]  \hspace{1cm} (37)

The squared L^2-norms \( s_\kappa^2, s_\theta^2, s_\gamma^2, s_\rho^2 \) of estimation errors on \( \kappa, \theta, \gamma, \rho \) are the diagonal terms \( \Sigma_{i,i} \) of \( \Sigma \), so that equations (37) and (36) yield the upper bound

\[ \varepsilon \leq s_\kappa |\partial_\kappa g| + s_\theta |\partial_\theta g| + s_\gamma |\partial_\gamma g| + s_\rho |\partial_\rho g|. \] \hspace{1cm} (38)

Introducing the option price sensitivities \( Sen_\kappa, Sen_\theta, Sen_\gamma, Sen_\rho \) as defined in section 3.2, the last inequality becomes

\[ \varepsilon \leq s_\kappa Sen_\kappa + s_\theta Sen_\theta + s_\gamma Sen_\gamma + s_\rho Sen_\rho = \varepsilon_\kappa + \varepsilon_\theta + \varepsilon_\gamma + \varepsilon_\rho. \] \hspace{1cm} (39)

where the individual impacts on option pricing of the estimation errors respectively affecting \( \kappa, \theta, \gamma, \rho \) are defined by

\[ \varepsilon_\kappa = s_\kappa Sen_\kappa, \quad \varepsilon_\theta = s_\theta Sen_\theta, \quad \varepsilon_\gamma = s_\gamma Sen_\gamma, \quad \varepsilon_\rho = s_\rho Sen_\rho. \] \hspace{1cm} (40)

### 7 Numerical Study of Option pricing accuracy

#### 7.1 Joint SDEs model fitting for S&P 500 and VIX

To illustrate our approach to quantify the impact of parameter estimation errors on option pricing, we present one case study for two options written on the index S&P 500.

We used a dataset of \( N = 252 \) joint daily observations recorded in 2006 for the index SPX = S&P 500 and its approximate annualized volatility VIX. Recall that VIX, as maintained by CBOE, estimates SPX volatility through the implied volatility of options with a 30 day maturity, and is annualized on the standard basis of 252 trading days per year (see [56]).

As indicated in section 5 above, after fixing a conventional time interval \( T = 1/252 \) between two successive daily observations, we fit joint Heston SDEs to this dataset by the approximate maximum likelihood estimators (34). This yields the vector \( \hat{\Theta} \) of estimated SDEs parameters values

\[ \hat{\kappa} = 16.6 ; \hat{\theta} = 0.017 ; \hat{\gamma} = 0.28 ; \hat{\rho} = -0.54. \] \hspace{1cm} (41)

The estimated drift \( \mu \) is not listed since it has no impact on option pricing.

As outlined in section 5 we then fix \( \Theta \) at the estimated values just obtained to simulate 5000 trajectories of the just fitted Heston SDEs, and this enables the computation of empirical evaluations for the root mean squared errors of estimation \( s_\kappa, s_\theta, s_\gamma, s_\rho \). This easily yields the values

\[ \hat{s}_\kappa \sim 5.7 ; \hat{s}_\theta \sim 0.002 ; \hat{s}_\gamma \sim 0.01 ; \hat{s}_\rho \sim 0.06. \] \hspace{1cm} (42)

The number \( N = 252 \) of joint daily observations is realistic but rather small, so that the relative errors of estimation on SDEs parameters are naturally still
fairly high. For numerical evaluations below, we consider that with reasonably high probability, the parameters \( \kappa, \theta, \gamma, \rho \) belong to four intervals centered around \( \hat{\kappa}, \hat{\theta}, \hat{\gamma}, \hat{\rho} \), and with half-widths \( \hat{s}_\kappa, \hat{s}_\theta, \hat{s}_\gamma, \hat{s}_\rho \). The product \( J \subset \mathbb{R}^4 \) of these four intervals is then a high probability “localization box” for the true vector \( \Theta \) of unknown parameters.

### 7.2 Numerical computing of option price sensitivities

We study two benchmark European options \( \Omega_1 \) and \( \Omega_2 \) written on the SPX index, maturing at 63 days and 126 days, and with identical strike price \( K = 1380 \). These options were actively traded during the 1st quarter of 2007. The time between daily observations for the underlying asset model has been conventionally set at \( T = \frac{1}{252} \), so in option pricing PDEs (5), the maturity \( \tau \) must be \( \tau_1 = \frac{63}{252} = 0.25 \) for \( \Omega_1 \) and \( \tau_2 = \frac{126}{252} = 0.5 \) for \( \Omega_2 \).

The risk free rate of return is set at \( r = 1\% \). As detailed in section 9 below, we derive from other options a constant estimate \( \hat{\lambda} = 2 \) roughly approximating all the unknown market prices of risk \( \lambda_t \).

In the first quarter of 2007, \( X_t = \text{SPX} \) ranged from 1370 to 1460 with median 1426, and \( \sqrt{Y_t} = \text{VIX} \) ranged from 10% to 20% with median 11%. Hence over the lifetime of \( \Omega_1 \) and \( \Omega_2 \), the triple \( X_t, Y_t, t \) remained within the \( \mathbb{R}^3 \)-domains defined by \( 0 < x < 2800, \ 0 < y < 1, \ 0 < t < \tau \), with \( \tau_1 = 0.25 \) for \( \Omega_1 \) and \( \tau_2 = 0.5 \) for \( \Omega_2 \). In these two domains, we need to solve the option pricing PDE (5) for \( g(x, y, t) = f(x, y, \tau - t) \) with boundary conditions (6)-(10).

After checking numerically that for \( 100 < x < 2800 \) option prices are not significantly affected when the initial boundary \( x = 0 \) is shifted to the position \( x = 100 \), we did implement this shift for substantial CPU reduction. The computing domain is then

\[
100 < x < 2800, \quad 0 < y < 1, \quad 0 < t < \tau, \quad \text{with} \quad \tau_1 = 0.25 \quad \text{and} \quad \tau_2 = 0.50.
\]

We discretize this domain by a grid \( G_1 \) of size \( 90 \times 80 \times 63 \) for \( \Omega_1 \), and a grid \( G_2 \) of size \( 90 \times 80 \times 126 \) for \( \Omega_2 \). As indicated in section 3, we then solve the four PDEs (13) – (16), verified by option price partial derivatives with respect to \( \kappa, \theta, \gamma, \rho \), to obtain the vector \( SEN(\Theta) \) of option pricing sensitivities \((SEN_\kappa, SEn_\theta, SEn_\gamma, SEn_\rho)\).

By refining the grids \( G_1 \) and \( G_2 \), we have numerically validated that these discretizations were dense enough to accurately solve all the necessary PDEs.

Since the unknown \( \Theta \) may essentially be any point of the high probability localization box \( J \subset \mathbb{R}^4 \) defined above in (6.2), the individual impacts on option pricing of parameter estimation errors, denoted \( \varepsilon_\kappa, \varepsilon_\theta, \varepsilon_\gamma, \varepsilon_\rho \) in equation (40), are then evaluated by their upper bounds for \( \Theta \) in a large finite subgrid of \( J \).

On a standard desktop PC, solving all the option pricing and sensitivity PDEs as above required for each \( \Theta \) a CPU-time of 1 minute for option \( \Omega_1 \) and 2 minutes for option \( \Omega_2 \). When the grid size \( m \times n \times p \) increases, these CPU times are enhanced by considering a non-uniform grid in \( y \).\footnote{For small values of the squared volatility \( y \) close to zero the numerical scheme can be enhanced by considering a non-uniform grid in \( y \).}
roughly linear in the grid time size $p$ and behave like low degree polynomials in
the grid spatial size $mn$; indeed, a key computational cost is the LU decom-
position and backward substitution for a single $mn \times mn$ matrix.

8 Impact of parameter estimation errors on option pricing

We now present numerical results for the two European options $\Omega_1, \Omega_2$ written on the SPX index, and actually traded in 2007. The 3D-graphs in Fig.
1 display the prices of options $\Omega_1, \Omega_2$ computed at option creation time, as
functions of current SPX price $x = SPX_t \in [1120,1570]$ and VIX value $\sqrt{y} =
VIX_t \in [11\%, 38\%]$. For options $\Omega_1$ and $\Omega_2$, the figures Fig. 2, 3, 4, 5 display
the individual impacts $\varepsilon_\kappa, \varepsilon_\theta, \varepsilon_\gamma, \varepsilon_\rho$ of parameter estimation errors on
option pricing. These separate error impacts on option pricing are computed
by equation (40), and are displayed in our 3D-graphs as functions of the SPX
price $1120 < x < 1570$ and of its volatility $VIX = \sqrt{y}$, covering deep-out-of-the
money cases as well as deep-in-the money cases.

Note that the estimation errors on $\kappa$ and $\theta$ induce option pricing errors which
tend to be decreasing functions of $|x - K|$ where $x = SPX_t$ and $K$ is the option
strike price. Since the individual impacts of estimation errors on option price
are clearly stronger for $\kappa$ and $\theta$ than for $\gamma$ and $\rho$, we see that option pricing
tends to be more sensitive to parameter estimation errors for options close to
the money than for options far from the money.

The bound on the global option pricing error $\varepsilon$ induced by the combined effects
of all parametric estimation errors was computed as the upper bound of the
right-hand-side of equation (39) for $\Theta$ in a large finite subgrid of the localization
box $J$ and displayed in Fig. 6 and 7 as a function of current asset price
$x = SPX_t$ and volatility $\sqrt{y} = VIX_t$.

Relative global option pricing errors are defined by $\varepsilon(x,y,t)/f(x,y,t)$ and are
displayed in Fig. 8 and 9 for $1120 < x < 1270, 22\% < y < 38\%$.

9 An estimator for the market price of volatility risk

To solve the option pricing PDE, one first has to estimate the market price $\lambda_t$
of volatility risk which depends on investors preferences, liquidity concerns, risk
aversion, etc. For various approaches to estimate $\lambda_t$, see for instance [13], [30],
[39], [45].

Here the estimation of $\lambda_t$ was not our main focus, so in the spirit of [30], we
derive a constant estimator of $\lambda_t$ by minimizing a natural criterion.
9.1 A rough constant estimator for $\lambda_t$

Consider $q$ benchmark options $\Omega_j$, $j = 1 \ldots q$, with strikes $K_j$ and maturity $\tau_j$, written on the same underlying asset. The asset price $X_t$ and its volatility $Y_t$ are driven by SDEs (1)-(2) with known parameters. Call $T$ the user selected sub-sampling time. Then for each day $k$, let $X_{kT}$ be the closing asset price and $\Omega_j(kT)$ be the average of closing bid and ask for $\Omega_j$.

We seek a constant $\hat{\lambda}$ reasonably approximating all the unknown $\lambda_{kT}$ for $1 \leq k \leq \tau_j/T$, that is over the life span of all the options $\Omega_j$. Fix an arbitrary constant $\lambda$ and assume temporarily that $\lambda_t$ is identically equal to $\lambda$ for all $t$. Then let $f_{j,\lambda}(x, y, t)$ be the corresponding solution of the option pricing PDE (5) for option $\Omega_j$. The predicted option price $\hat{\Omega}_j(kT)$ for day $k$ becomes $f_{j,\lambda}(X_{kT}, Y_{kT}, kT)$. To compare this option price predictor to the observed option price $\Omega_j(kT)$, we compute the root mean squared prediction error $RMS_j(\lambda)$ defined by

$$RMS_j(\lambda)^2 = \frac{T}{\tau_j} \sum_{k=1}^{\tau_j/T} \left[ (\hat{\Omega}_j(kT) - \Omega_j(kT))^2 \right]$$

Let $p_j$ be the median price of $\Omega_j$ over its lifetime. To combine option pricing accuracy across multiple options, introduce the relative size of pricing prediction errors $RMS_j(\lambda)/p_j$ for each $j$. We quantify the performance of the constant estimator $\lambda$ by the median perf ($\lambda$) of pricing prediction errors over all benchmark options, so that

$$\text{perf (\lambda)} = \text{median}_{j=1 \ldots q} \left[ RMS_j(\lambda)/p_j \right]$$

Our constant estimator $\hat{\lambda}$ is then determined by minimizing the median pricing error perf ($\lambda$) over all $\lambda$,

$$\hat{\lambda} = \arg \min_{\lambda} \text{perf (\lambda)}$$

9.2 Numerical estimate of $\lambda$ for options written on S&P 500

For option pricing sensitivity computations above, we have focused on the S&P 500 index SPX and its approximate annualized volatility VIX in the first quarter of 2007. This process was modeled above by joint Heston SDEs with sub-sampling time $T = 1/252$ and parameters estimated on the basis of 252 daily observations recorded in 2006.

As outlined in paragraph 9.1, we seek a constant estimator $\hat{\lambda}$ approximating all the $\lambda_t$ for options written on SPX between Jan 3rd, 2007 and Feb 2nd 2007. To this end we have selected 16 such benchmark options with the same 22 days trading period. Eight of these options were maturing on Feb 17th 2007, with strike prices 1380, 1400, 1410, 1420, 1425, 1430, 1450, 1460. The other 8 options had the same strike prices but with maturity date Mar 17th 2007. The risk free rate of return is set at $r = 1\%$. As indicated above, for each
benchmark option Ω_\text{j}, and each constant \( \lambda \), we consider the situation where \( \lambda_t \equiv \lambda \) for all \( t \) and we solve the \( \Omega_j \) pricing PDE to evaluate the mean squared difference \( RMS_j(\lambda) \) between real and computed \( \Omega_j \) price over the 22 days option lifetime. We then compute the median perf(\( \lambda \)) of the 16 relative pricing errors \( RMS_j(\lambda)/p_j \) where \( p_j \) is the median price of \( \Omega_j \) over its lifetime. The graph of perf(\( \lambda \)) is displayed in figure 10 as a function of \( \lambda \), which reaches a minimum of 8% for \( \lambda = 2.0 \). This is the constant value by which we have approximated all the \( \lambda_t \) in the computations of option pricing sensitivities presented earlier.

### 9.3 Impact on option price of estimation errors affecting \( \lambda \)

Our estimate \( \hat{\lambda} = 2.0 \) was derived by minimizing the median option pricing accuracy, where the median was computed over a set \( K \) of 16 benchmark options \( \Omega_j \) written on SPX within a short trading period. To evaluate the error of estimation \( s_\lambda \) affecting \( \hat{\lambda} \), we implement a rough “bootstrap” evaluation as follows. For each subset \( Q \) of 12 options arbitrarily selected in \( K \), one can as above compute perf\(_Q\)(\( \lambda \)) as the median pricing accuracy over the 12 options in \( Q \) and then minimize perf\(_Q\)(\( \lambda \)) in \( \lambda \), which yields another estimate \( \hat{\lambda}(Q) \) of \( \lambda \). The average of the \( 16 \choose 4 \times 12 \) shifts \( |\hat{\lambda}(Q) - \hat{\lambda}| \) is a rough evaluation of the error \( s_\lambda \) affecting the estimate \( \hat{\lambda} \). This procedure provides here the value \( s_\lambda \approx 0.5 \).

The sensitivity \( Sen_\lambda = |\partial_\lambda f(x,y,t)| \) of the option price \( f(x,y,t) \) to errors affecting \( \lambda \) is computed by solving the adequate PDE as indicated in section 3.3. Fig. 11 displays the sensitivities of options \( \Omega_1 \) and \( \Omega_2 \) with respect to \( \lambda \), computed at option creation time.

### 10 Conclusion

Our main goal was to quantify the impact on European call option pricing of the parameter estimation errors generated by fitting Heston joint SDEs to market data. We have developed and numerically tested an impact quantification technique combining the computation of consistent estimators for the underlying Heston SDEs parameters, the evaluation of root mean squared accuracy for these estimators, and the numerical solution of six parabolic partial differential equations in \( \mathbb{R}^2 \), namely the option pricing PDE and the PDEs verified by the partial derivatives of option price with respect to model parameters.

We have also derived and implemented an algorithm to estimate the market price of volatility risk by analysis of multiple benchmark options.

We have tested our approach by numerical fitting of Heston joint SDEs to the 252 daily data recorded in 2006 for the S&P 500 and VIX indices, and by studying several European call options written on S&P 500. For two such options, we compute and display the average option pricing shifts induced by errors of estimation on the SDEs coefficients, as well as by approximation errors for the market price of volatility risk.
Since our algorithmic implementations are fairly fast on a standard laptop, our study strongly suggests that quantification of errors induced on option pricing by model parameters estimation errors should not be neglected, and could be systematically computed when option pricing is performed after fitting joint Heston SDEs to daily or intraday market data. Our numerical results indeed show that Heston SDEs fitting to a year of daily data can generate sizeable inaccuracies for European option pricing.

We expect that our methods will perform just as well for American options as for European options. In future work, we also plan to extend our approach to develop fast accuracy monitoring algorithms for portfolio hedging.
Figure 1: Computed price of options $\Omega_1$ (top) and $\Omega_2$ (bottom) written on SPX. Option prices are computed at option creation time; the horizontal coordinates $x$ and $\sqrt{y}$ denote the SPX price and its approximate volatility VIX.
Figure 2: Option pricing errors due to estimation errors on $\kappa$, computed at option creation time.

Figure 3: Option pricing errors due to estimation errors on $\theta$, computed at option creation time.

Figure 4: Option pricing errors due to estimation errors on $\gamma$, computed at option creation time.
Figure 5: Option pricing errors due to estimation errors on $\rho$, computed at option creation time.

Figure 6: Global option pricing errors in dollars for option $\Omega_1$, computed at option creation time.
Figure 7: Global option pricing errors in dollars for option $\Omega_2$, computed at option creation time.

Figure 8: Relative Errors on option prices for option $\Omega_1$, computed at option creation time.
Figure 9: Relative Errors on option prices for option Ω₂, computed at option creation time.

Figure 10: For each potential value λ of the market price of volatility risk, and each one of 16 benchmark option written on SPX, we compute the relative accuracy of option pricing, and display the median perf(λ) of these 16 pricing accuracies. Note that perf(λ) reaches its minimum at $\hat{\lambda} = 2.0$. 
Figure 11: Option price sensitivity with respect to $\lambda$, computed at option creation time.
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