Choosing the best objects by a set of the qualitative features

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Abstract. It is generally accepted that the choice of the best technical solutions in the design of complex technical systems must be carried out on the basis of a multi-criteria approach. However, it is known that at the initial design stages, developers, as a rule, do not have exact values of basic technical indicators. Moreover, many (if not all) indicators are qualitative in nature, and the probability of making a non-optimal decision increases due to the uncertainty of the initial data. Therefore, it is advisable to try to restrict the choice domain, eliminating the deliberately "uncompetitive" objects. In the well-known ELECTRE method, for this purpose, some "preference" relation is introduced, indirectly determined by the choice of "weights" of particular quality indicators and threshold values of the so-called concordance and discordance indices. Unfortunately this approach is characterized by the lack of a clear operational sense of the weights and thresholds introduced, as well as the possible occurrence of oriented cycles in the preference relation graph. This makes it difficult to interpret the results of applying the procedure for excluding "worst" objects by the ELECTRE method. In this regard, the article proposes a new approach to the problem of choice domain restriction. On the set of the objects under consideration, some irreflexive and asymmetric dominance relation is constructed, allowing a solution according to von Neumann-Morgenstern, which simultaneously is also the so-called C-core. At the same time, the considered dominance relation – unlike the preference relation in the ELECTRE method already mentioned – has a clear "physical" meaning and does not require the introduction of any directly immeasurable parameters like "weights" and "thresholds," which always question the adequacy of the model to actual decision maker’s preferences system.

1. Introduction

Formally the task structure of the best objects multiattribute choice (probably there is no need to give examples of this kind of the tasks; as a matter of fact, single attribute tasks of choice are pure abstraction) can be presented in the form of tuple

\[(A; R_1, \ldots, R_k, \ldots, R_n),\]

where \(A\) is the set of an arbitrary nature and \(R_k\) (\(k = 1, 2, \ldots, n\)) is a binary relation on it, meaning "better by \(k\) attribute."

The "classical" approach to the solution of this task is to build some "convolution" of the attributes \(k, k = 1, 2, \ldots, n\), specifying on \(A\) uniform relation of the linear order \(P\) and the choice as the best object "maximum" element from the ordering by \(P\) of set \(A\), i.e., element \(x\), for which there is no \(y \in A: yPx\).

It is well known [1] that order \(P\) can't be satisfied while "coordinating" with relations \(R_k\) (\(k = 1, 2, \ldots, n\)) without using additional information, for example, about relative importance of \(k\).
attributes (subsets). Due to this, it is reasonable to begin with restricting, as much as possible, the choice domain by means of more or less "rough" relation $R$ of the (strictly) domination, majorizing $P$ (They say that relation $R$ majorizes $P$, if for any $x, y, z \in A$ occurs: $xRy \rightarrow xPy, xPy \& yPz \rightarrow yPz$).

In this case natural choice domain restriction is so-called $C$-core of the relation $R$, specified as set $C$ of nondominated elements $x \in A$. Another possible principle of the choice domain restriction relates to a more complicated concept of $R$ relation solution by von Neumann and Morgenstern (threshold, NM-solution of $R$ relation). The latter one is specified as such set $V, V \subset A$, that

a) no two $x, y \in V$ dominate each other and

b) for any $y \not\in V$ can be found some $x \in V$ that dominates $y$.

In a sense, we can say that $C$-core allocates the "best" objects, while the NM-solution only excludes the "worst." Generally, $C \subset V$, and it seems that as an optimality principle, the $C$-core is preferable to the NM-solution (For the first time, the concepts of $C$-core and solution were introduced in game theory [2]; their application to multi-criterion choice problems is discussed in [3].).

Unfortunately, for an arbitrary antireflexive relation $R$ describing the "system of preferences" of the decision-maker (DM), both $C$ and $V$ can be empty, and $V$ can also be defined ambiguously.

In the known method of ELECTRE [4] an attempt is made to circumvent this problem as follows.

Instead of the graph $G$ of the "superiority" relation $R$, we consider the condensation of $G^*$, which defines a new relation $\hat{R}$ of the equivalence classes uniting the objects that correspond to the vertices belonging to the strong components of the graph $G$. Since the relation $\hat{R}$ is acyclic, the existence and uniqueness of the NM-solution $R$ (the core of the graph $G^*$ is according to the author’s terminology) are provided. It is difficult, however, to give a satisfactory interpretation of the cyclic relation $xRyR…RzRx$; in any case, the declaration of objects $x, y, …, z \in A$ in any sense "equivalent" seems, in our opinion, completely unreasonable.

Therefore, it is interesting to construct a rather “substantial” relation of strict domination that would allow the $C$-core or (unique) NM-solution defined directly in $A$. This problem is considered in the framework of the following model below.

2. Model
Consider the set $A$ of objects $x, y, z, …$ each of which is characterized by $k$ dichotomous (further this restriction will be removed) features forming the set $T$. For the sake of simplicity, each object $x \in A$ is identified with the $n$-dimensional Boolean vector of its characteristics

$$x = (x_1, \ldots, x_k, \ldots, x_n),$$

where $x_k (k = 1, 2, \ldots, n)$ is 1 or 0, depending on whether the object $x$ possesses the feature $k \in T$.

Will talk that object $x$ is superior to the object $y$, by the set of attributes $S, S \subset T (xR_3y)$ if, and only if, $x_k > y_k$ for all $k \in S$, which is possible only under the condition that $x_1 = 1, y_k = 0$.

Without limiting generality, it can be assumed that the set $A$ contains only "effective" objects, each of which is superior to any of remaining ones by some (non-empty) set of features of $S$.

A dominance relation on $A$ is defined as

$$R = \bigcup_{S* \subset T} R_{S^*},$$

where $S^*$ is any sufficiently “representative” set of features of $S$, the superiority of which is sufficient – from a decision maker's point of view – to recognize the global superiority of one object over another. All such sets $S^*, S^* \subset T$ are called decisive, and relations – dominance over the (decisive) set $S^*$.

It is clear that the dominance relation $R$ defined in this way is antireflexive for any task of solving sets. However, if we want $R$ to be asymmetric (which should be considered quite reasonable), we need to require that any two decisive sets have at least one common feature (intersect).

Under this condition, the following statements are true:
(1) the dominant object can not dominate any other:
\[ xRy \rightarrow yRz \]

(2) the dominant object cannot be dominated by any other:
\[ xRy \rightarrow zRx \]

(The proof is obvious and given to the reader).

Thus, an asymmetric relation \( R \) must have a very simple structure: it defines a “simple” graph \((V, \overline{V}, \Gamma)\) (In contrast to the traditional definition of a simple graph, the correspondence \( \Gamma \) may not be defined everywhere in the domain \( V \) and, therefore, is not, generally speaking, a (multi-valued) mapping of \( V \) into \( \overline{V} \), where \( V \subseteq \mathcal{A} = \mathcal{A} \setminus V \), and \( \Gamma \) is a correspondence between \( V \) and \( \overline{V} \) such that \( y \in \Gamma x \leftrightarrow xRy \).

It is easy to see that the set \( V \) is simultaneously the C-core and the NM-solution of the relation \( R \), defining, the “multiple maximum” of the latter: no object in \( V \) is “worse than” any of the others, and for each object in \( \overline{V} \) there is at least one "best" object belonging to \( V \).

So, in the case of a nonempty relation \( R \), the subset \( \overline{V} \subseteq \mathcal{A} \) is a natural narrowing of the considered set of objects \( \mathcal{A} \), since no object outside \( V \) can obviously claim to be "optimal." In the future, we will call the \( V \) core of the set \( \mathcal{A} \).

### 3. Algorithm

In accordance with the above, the construction of the relation \( R \) reduces to the enumeration of all the decisive sets \( S^* \). At the same time, of course, it suffices to restrict ourselves to “checking” only those \( S \subseteq \mathcal{T} \) for which \( R_S \neq \emptyset \). (Each check consists in establishing the decision maker of the truth or falsity of the implication \( xR_S y \rightarrow xRy \) for some \( x, y \in \mathcal{A} \); in the first case \( S \) is decisive, in the second case it is not).

We, however, do not aim at a "complete" description of the dominance relation on the initial set of objects \( \mathcal{A} \): the relation \( R \) interests us only insofar as it allows us to exclude some objects from \( \mathcal{A} \) as obviously not the best. What is really important is to minimize the number of checks in the process of core formation.

As will be seen later, not all \( S \) belonging to the set \( \mathcal{U} = \{ S \mid R_S \neq \emptyset \} \) require (immediate) verification; a significant number of them – at least when choosing a “good” decision maker polling strategy – is checked indirectly by checking other \( S \in \mathcal{U} \) or not at all, since this is not required for building the core.

The following remarks will allow to clarify potential possibility of reducing the number of pair comparisons of objects that serve as a heuristic rationale (or rather an excuse) for the enumeration procedure of sets \( S \in \mathcal{U} \), which is an integral part of the core generation algorithm for set \( \mathcal{A} \) considered below.

First, if the object \( y \) is dominated by some object \( x \), then there is no need to check whether it is also dominated by some other object \( z \). The \( xRy \) condition in itself is sufficient to exclude the object \( y \) from \( \mathcal{A} \), so that all relations \( R_S \) can be narrowed to the set \( \mathcal{A} \setminus \{ y \} \) (Recall that, by assertion (1), the excluded object \( y \) could not dominate any other object.).

Secondly, if it is established that \( xRy \), there is no need to check all \( S \) for which \( R_S \ni (z, x) \) since, by virtue of assertion (2), it cannot be decisive.

Thirdly, if \( x \) dominates \( y \) over the (decisive) set \( S^* \), then all sets \( S : S \cup S^* = \emptyset \) cannot be decisive.

Fourthly, from the natural “monotony” of the DM's preferences, it follows that if \( S \) is a decisive set, then each \( S' : S' \supset S \) must also be a decisive set, and vice versa, if \( S \) is not a decisive one, then any \( S' : S' \subset S \) will not be decisive either.

Since, apparently, the establishment of the fact that some \( S \) is a decisive set opens up potentially
richer possibilities of “folding” the task than in the case when $S$ is not decisive, it is reasonable in the verification process to give preference first of all to the sets $S \in U$ that are “more likely” to be decisive. With this in mind, we can propose the following hierarchy of preference rules that determine the choice of $S$ at the next verification step:

1) choose $S$ of maximum power (assuming that all signs are “commensurate in importance”); if there are more than one such set, then

2) choose $S$, for which $R_S$ has the “largest” range of values (when $S$ is a decisive set, a larger number of dominated objects are simultaneously excluded from $A$); if there are more than one such set, then

3) choose $S$ that includes the maximum number of other sets from $U$ (which obviously cannot be decisive when $S$ is not a decisive set); if there are more than one such set, choose any of them.

Rule 1) does not seem so natural when the features vary greatly in importance. In this case, one can, for example, ask the decision maker to evaluate the “weights” $c_k$, characterizing the relative importance of the features of $k \in T$, and choose $S$ from among the sets with the greatest weight $c(S) = \sum_{k \in S} c_k$. Since in this case the checked set $S$ does not have to be maximal in power and, therefore, $U$ can contain not only $S^* : S \subset S$, but also $S^+ : S^+ \supset S$, the procedure for constructing the core is more complicated (we would be inconsistent if, having "improved" the choice of $S$, did not try to use all the benefits provided by this).

Apparently, the situation under discussion in practice will not occur as often and it is difficult to say whether the planned “common” approach will ultimately justify itself (Even if we ignore the purely computational aspects of the problem, the question remains: is the achievable reduction in the number of necessary pair comparisons "worth" the additional decision maker's efforts that will be needed to assess the relative importance of all – possibly numerous – features of $k \in T$).

The following algorithm uses the simple preference rule system proposed above to select the next set $S$.

Description of the algorithm for constructing the core $V$, is given below.

1°. Form the set $U = \{ S \mid A^2 \supset R_S \neq \emptyset \}$, associating with each $S \in U$ the corresponding $R_S$.

2°. If $U = \emptyset$, we assume $V = A$ and stop. Otherwise, go to 3°.

3°. Choose to check $S \in U$ in accordance with the rules 1) – 3). If the set $S$ is decisive, we assume $S^* = S$ and go to 4°. Otherwise, we assume $S^0 = S$ and go to 5°.

4°. Let

$$X(S) = PR_1 R_S,$$

$$Y(S) = PR_2 R_S.$$  

(The projection $PR_i P$ of the relation $P$ to the $i$-th axis is the set of the $i$-components of the pairs $(x, y) \in P$. It is obvious that $PR_1 P$ and $PR_2 P$ represent the domain and the range of values of the relation $P$, respectively.)

Set $A = A^1 \cap H(S^*)$ and restrict all $R_S$ to set $A$, i.e. replace $R_S$ with $R_S \cap A^1$. Remove from $U$ the set $S^*$ and all the sets $S$ for which

a) $R_S = \emptyset$;

b) $R_S \ni (z, x), x \in X(S^*), z \in A$;

c) $S \cap S^* = \emptyset$.

Returning to 2°.

5°. Remove from $U$ the set $S^0$ and all the sets $S$ such that $S \subset S^0$. Returning to 2°.

4. Features measurement

Suppose now that one of the features, say the $k$-th, is evaluated on a point scale characterizing the severity of the corresponding feature:
\[ x_k \in \{0, 1, 2, \ldots, r_k\}. \]

(Without restriction of generality, it can be assumed that large values of \( k \) are preferable). Associate with the considered attribute \( r_k \) of predicates \( \pi_r(k) \) of the form "\( x_k \geq r \)" (\( r = 1, 2, \ldots, r_k \)), setting \( \pi_r(k) \) to 1 or 0, depending on whether the corresponding statement is true or false given feature value \( x_k \). Thus, any rank feature can be represented by a set of dichotomous features. Agree to write \( x_k = r \) as a vector of length \( \sum_{k=1}^{r_k} \), the first \( r_k \) components of which are 1, and the others are 0. Thus, if in the general case \( n \) rank signs are considered, then each object is associated with a vector of length \( \sum_{k=1}^{r_k} \), whose components form \( n \) groups of binary characteristics \( x_{k_1}, x_{k_2}, \ldots, x_{k_n} \) (\( k = 1, 2, \ldots, n; r_k \geq 1 \)).

Let's assume that \( x \) and \( y \) are two objects, differing only in the values of \( x_k = p \) and \( y_k = q \), of the (rank) feature \( k \). If, when comparing them with any third object \( z \), \( xR_{p_k}z \) and \( yR_{q_k}z \) are true, then \( p > q \rightarrow S_p \supset S_q \). As can be seen, in this case decision maker's preferences monotony condition is being formalized the same way as before: when \( S_q \) is a decisive set, then \( S_p \supset S_q \) is a decisive one; when \( S_p \) is not a decisive set, then \( S_q \subset S_p \) is not decisive. Thus, \( x_{k_r} \) characteristics can be treated as "true" dichotomous features values.

Finally, we emphasize that the point scales are considered by us simply as (arbitrary) scales of order \([1]\). It is not required at all, that the scores refer to scales of equidistant intervals (Thurstone scale). We assume that the decision maker bases his judgments solely on the meaningful interpretation of the points attributed to these featured objects.

5. Practical implementation
The algorithm described above was implemented programmatically in the form of a dialogue between a decision maker and a computer. Preliminary experiments showed that the required number of \( S \) sets checks is not great. However, the "informativeness" of \( R \) dominating relation can noticeably change from task to task. As expected, the effective narrowing of the choice area is achieved when objects differ greatly in "quality," and also if in paired comparisons of objects the number of features by which the latter are "equivalent" is relatively small.

The following example, borrowed from [6] is rather typical despite its simplicity.

The problem of choosing the best method for testing computer programs (pp. 187-190 of the cited work) is considered. Each of the six methods is evaluated by seven qualitative features, reflecting various aspects software product ensuring the reliability problem (reliability is considered the main criterion for the quality of a program).

An efficacy check immediately excludes the "modified top-down testing," and the dominance relation (consistent with the weights assigned to the considered features) eliminates the "top-down testing" and "big-bang testing" (the two decisive sets include the first and last four features). Thus, the core is formed by the "bottom-up testing," the "sandwich testing," and the "modified sandwich testing." This result is in complete agreement with the conclusion that "modified sandwich testing and bottom-up testing appear to be the best approaches and big-bang testing appears to be the worst approach" (p. 189). On the other hand, it is quite obvious that the assignment of certain weights to all features should have required much more effort from the decision maker than simply establishing that the first (or last) four features in the aggregate "weigh" more than the others.

6. Conclusion
In conclusion, we emphasize that the dominance relationship we are considering – unlike the superiority relationship in the ELECTRE method already mentioned – has a clear "physical" meaning and does not require the introduction of any directly non-measurable parameters like "weights" and
“thresholds” that always question the adequacy of the model to decision maker's actual preference system.

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