Features of the Use of Symbolic Calculations for Constructing Wave Field Local Asymptotics Using Catastrophe Theory

A S Kryukovsky and Yu I Bova
Russian New University, 105005, Radio st. 22, Moscow, Russia

Abstract. Based on symbol calculation a local asymptotic method is developed that describes the diffraction focusing of electromagnetic fields in the case when a family of primary (geometrooptical) and secondary (edge) rays form focusings of the cuspid type $A_2$ (wave catastrophe $F_4$). Mathematical modeling of the unfolding coefficients and the traveling wave phase was performed. Explicit expressions for the parameters of unfolding are obtained.

1. Introduction
In the course of solving problems of electrodynamics and the propagation of radio waves in inhomogeneous media by asymptotic methods, structurally stable singularities arise that correspond to wave catastrophes of various types. The uniform asymptotics of the fields in the vicinity of such singularities in the focal and diffraction regions in the form of reference structures containing special wave catastrophe functions are described [1-6].

An important problem is the determination of the relationship between the parameters of a physical problem and the parameters of reference structures corresponding to catastrophes of one type or another, that is, finding the unfolding coefficients. In the problems of diffraction and propagation of radio pulses, it is urgent to use the theory of edge wave catastrophes, which make it possible to take into account, along with primary, secondary ray families as a single geometrical-optical (GO) structure.

The determination of unfolding coefficients (similarity coefficients) is a difficult mathematical problem. Recently, however, its explicit solution has become available through the use of symbolic computing and computer mathematics.

In [7-9], the problem was considered by us using the example of a unimodal edge catastrophe $K_{4,2}$. In [4,10], the caustic structure of this singularity was considered. In this paper, we describe a method for constructing “similarity” coefficients using the example of the topological singularity $F_4$ (simple zero-modal catastrophe $F_4$), which is structurally stable in three-dimensional space and allows describing the joint cuspid focusing of $A_2$ type as a family of primary GO rays, so and the family of secondary edge rays [2,11-13].

2. Basic identity and one-to-one conversion
The expression for the unfolding of a edge catastrophe $\Sigma=F_4$ has the form:

$$F_\Sigma = v_2 \xi_2^3 + v_1 \xi_1^3 + \lambda_1 \xi_1 + \lambda_2 \xi_2 + \lambda_3 \xi_1 \xi_2,$$

where $v_1=\pm 1, v_2=\pm 1, a \lambda_j$ - the unfolding coefficients.

We consider the phase function $\Phi(\eta, \eta_2, \tilde{\alpha})$ in the vicinity of a singular point with coordinates $(\tilde{\alpha})$, in which the unfolding transforms into a normal form and has the form...
\[ F_\Sigma = v_2 \xi_2^2 + v_1 \xi_1^3. \]  

(2)  

The identity (see, for example, [1,2,11]):  

\[ \Lambda \Phi = F_\Sigma + \theta, \]  

(3)  

in which \( \Lambda \) – is a large parameter of the problem (\( \Lambda \gg 1 \), as an argument is not considered), \( \theta(\tilde{\alpha}) \) – is the phase of the traveling wave is fulfilled. To simplify the calculations, we introduce the function:  

\[ \mu = \Lambda \Phi. \]  

Then the basic identity takes the form:  

\[ C \equiv \mu(\eta_1(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\alpha}), \eta_2(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\alpha}) - F_\Sigma(\tilde{\xi}_1, \tilde{\xi}_2, a(\tilde{\alpha}), \tilde{\lambda}(\tilde{\alpha}))) - \theta(\tilde{\alpha}) = 0. \]  

(4)  

Between the internal variables of the phase function and the internal unfolding variables there is a one-to-one mapping [1,2,11]:  

\[ \begin{cases} 
\eta_1 = g_1(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\alpha}) \\
\eta_2 = \eta_{o2} + \xi_2 b_2(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\alpha}) 
\end{cases} \]  

(5)  

To determine the coefficients \( \lambda_j(\tilde{\alpha}) \) and phase of the traveling wave \( \theta(\tilde{\alpha}) \) we use the method of local asymptotics developed by us [2, 7-9, 14-18]. Let us find the expressions for the coefficients \( \lambda_j(\tilde{\alpha}) \) and phase \( \theta(\tilde{\alpha}) \) by the method of local asymptotics. We introduce the following notation:  

\[ \mu_i = \left. \frac{\partial \mu}{\partial \eta_i} \right|_{\tilde{\alpha}}, \quad \mu_{ik} = \left. \frac{\partial^2 \mu}{\partial \eta_i \partial \eta_k} \right|_{\tilde{\alpha}}, \quad \mu_{ijk} = \left. \frac{\partial^3 \mu}{\partial \eta_i \partial \eta_j \partial \eta_k} \right|_{\tilde{\alpha}}, \ldots \]  

\[ p^i_j = \left. \frac{\partial \eta_i}{\partial \xi_j} \right|_{\tilde{\alpha}}, \quad p^i_{jk} = \left. \frac{\partial^2 \eta_i}{\partial \xi_j \partial \xi_k} \right|_{\tilde{\alpha}}, \quad p^i_{jkl} = \left. \frac{\partial^3 \eta_i}{\partial \xi_j \partial \xi_k \partial \xi_l} \right|_{\tilde{\alpha}}, \quad (i, j, k, l = 1,2). \]  

(6)  

In more complex cases, we will use the notation:  

\[ \mu_{(n,m)} = \left. \frac{\partial^{n+m} \mu}{\partial \eta^n \partial \eta^m} \right|_{\tilde{\alpha}}, \quad C_{(n,m)} = \left. \frac{\partial^{n+m} C}{\partial \eta^n \partial \eta^m} \right|_{\tilde{\alpha}}, \quad \Omega_{(0,0)} = \left. \frac{\partial^0 \Omega}{\partial \eta_0} \right|_{\tilde{\alpha}}, \quad p^i_{(n,m)} = \left. \frac{\partial^{n+m} \eta_i}{\partial \xi^n \partial \xi^m} \right|_{\tilde{\alpha}}. \]  

(7)  

At a singular point \( (\tilde{\alpha}_0) \) (see [2])  

\[ \tilde{\xi}_1 = \tilde{\xi}_2 = 0, \quad \tilde{\eta} = \tilde{\eta}_0. \]  

(8)  

In addition, with \( \eta_2 = \eta_{o2} \), the internal variable \( \tilde{\xi}_2 = 0 \) (see (5)) (see [2,11,12]), and identity (4) becomes the identity of the restriction:  

\[ \Omega = \mu(\eta_1(0; \tilde{\alpha}), \eta_{o2}, \tilde{\alpha}) - v_1 \xi_1^3 - \lambda_1(\tilde{\alpha}) - \theta(\tilde{\alpha}) = 0. \]  

(9)  

This identity corresponds to the singularity \( A_2 \) – “caustic” for edge rays. In [2, 16, 18] it was shown that at a singular point of type \( A_2 \)  

\[ \mu_1 = \mu_{11} = 0, \quad \mu_{11} \neq 0. \]  

(10)  

Taking into account (10), it is easy to establish that in order to obtain \( p^1_{11} \), it is necessary to differentiate identity (9) at a singular point three times in \( \tilde{\xi}_1 \), to determine \( p^1_{11} \) – three times in \( \tilde{\xi}_1 \) and so on. Performing the calculations, we find (see also [2,14,18]):  

\[ d \equiv p^1_{11} = \sqrt{\frac{6}{|\mu_{111}|}}, \quad p^1_{11} = -\frac{1}{6} \left. \frac{\mu_{4,0}}{\mu_{3,0}} \right| d^2, \quad p^1_{111} = \left. \left( \frac{1}{8} \frac{\mu_{4,0}}{\mu_{3,0}} - \frac{1}{10} \frac{\mu_{5,0}}{\mu_{3,0}} \right) \right| d^3, \]  

\[ \nu_1 = \text{sign} \mu_{111}, \quad p^1_{111} = \left. \left( \frac{35}{216} \frac{\mu_{4,0}}{\mu_{3,0}} - \frac{1}{15} \frac{\mu_{6,0}}{\mu_{3,0}} \right) \right| d^4. \]  

(11)
Thus, formulas (11) are obtained sequentially from the analysis derivatives of \( C_{(n,0)} \) or \( \Omega_{(n,0)} \) with 
\( n=3,4,5,6 \), calculated at a singular point. To obtain the first approximation we need \( p_1^a = \frac{\partial \eta_a}{\partial \alpha_j} \) and 
\( p_{1a}^1 = \frac{\partial^2 \eta_1}{\partial \xi_1 \partial \alpha_j} \). The value \( p_1^a \) is obtained from the analysis \( \Omega_{(3,0)a} \) at a singular point: 
\[
p_1^a = -\frac{\mu_a(2,0)}{\mu(3,0)} + \frac{1}{6} \frac{\mu(4,0)\mu_a^2(1,0)}{\mu^2(3,0)}.
\] (12)
The value \( p_{1a}^1 \) is calculated from the analysis \( \Omega_{(3,0)a} \) at a singular point: 
\[
p_{1a}^1 = -\frac{p_1^a}{3\mu(3,0)} \left( \mu_a'(3,0) + \frac{5}{24} \frac{\mu(4,0)\mu_a^2(2,0)}{\mu^2(3,0)} - \frac{1}{10} \frac{\mu(5,0)\mu_a^3(1,0)}{\mu^3(3,0)} \right).
\] (13)

3. First approximation for coefficients and phase of a traveling wave

We will look for approximate expressions for \( \lambda_j(\vec{a}) \) and \( \theta(\vec{a}) \) in the form:
\[
\lambda_j(\vec{a}) \equiv \sum_{k=1}^{M} \lambda_{ja_k} \Delta \alpha_k, \quad \theta(\vec{a}) \equiv \theta_0 + \sum_{k=1}^{M} \theta_{a_k} \Delta \alpha_k + \sum_{k=1}^{M} \sum_{j=1}^{M} \theta_{a_k a_j} \Delta \alpha_k \Delta \alpha_j,
\] (14)
where \( \Delta \alpha_k = \alpha_k - \alpha_{a_k} \), and \( M \) is the dimension of the configuration space. To reduce the notation we will omit the index \( k \) at \( \alpha_k \), as was done in expressions (12) and (13).

In order to find \( \lambda_{1a} \) we differentiate (9) once in \( \xi_1 \) once in \( \alpha \) (that is, calculate \( \Omega_{(1,0)a} \)) and put:
\[
\vec{a} = \vec{a}_0, \quad \vec{\xi}_1 = 0.
\] (15)
Then we find that:
\[
\lambda_{1a} = \mu_{1a} p_{1\alpha}.
\] (16)

Now we consider the calculation the phase of the traveling wave \( \theta(\vec{a}) \), up to terms of the second order, inclusive. The value \( \theta(\vec{a}_0) \) is found from identity (9):
\[
\theta_0 = \theta(\vec{a}_0) = \mu(\eta(0,0;\vec{a}_0),\eta_{a2},\vec{a}_0),
\] (17)
where \( \eta(0,0;\vec{a}_0) = \eta_{a1} \) – is the value of the first internal parameter of the problem at a singular point.

To define \( \theta_a \) we differentiate identity (9), once in \( \alpha \) (\( \Omega_{(0,0)a} \)) and take into account (15). Then:
\[
\theta_0 = \mu_0.
\] (18)

To calculate the coefficients \( \theta_{a\beta} \), \( \alpha=\alpha_a, \beta=\beta_j \) we differentiate identity (9) with respect to \( \beta \) too. Analyzing \( \Omega_{(0,0)a\beta} \) at a singular point, we find:
\[
\theta_{a\beta} = \mu_{a\beta} + \mu_{a\alpha} p_{1\alpha}^1 + \mu_{a\beta}^1 p_{1\alpha}^1.
\] (19)
Now all quantities in (19) are known (see (12)).

Thus, the restriction allowed us to determine the linear approximation for \( \lambda_1 \) and \( \theta \). To find \( \lambda_2 \) and \( \lambda_3 \) it is necessary to consider the full expression for the unfolding (1) of the singularity \( \Sigma=F_4 \).

It should be noted here that, firstly, all derivatives of \( \eta_2 \) with respect to \( \xi_1 \) and \( \alpha \) are equal to zero:
\[
p_{2(n,0)}^1 = 0, \quad p_{2(n,0)a}^1 = 0,
\] (20)
which clearly follows from (5), and secondly, at a singular point:
\[
\mu_{12} = 0,
\] (21)
which follows from the equality $C_{(1,1)}$ to zero at a singular point.

Find a linear approximation for the coefficient $\lambda_2$. To do this, we differentiate identity (4) once in $\xi_2$, once in $\alpha$ (that is, calculate $C_{(0,1)a}$) and take into account (15), (20), (21). Then:

$$\lambda_{2a} = \mu_{1a} p_2^1 + \mu_{2a} p_2^2.$$  \hspace{1cm} (22)

To calculate $p_2^2$ we differentiate $\xi_2$ identity (4) twice with respect to at a singular point, that is, we calculate $C_{(2,0)}$ and find that

$$p_2^2 = \frac{2}{\sqrt{\mu_{22}}}, \quad \nu_2 = \text{sign} \mu_{22}.$$  \hspace{1cm} (23)

The derivative is more difficult to determine $p_2^1$. To do this, we calculate the derivatives of identities (4) at a singular point, equate to zero and find:

$$p_2^1 = -\frac{p_2^3}{\mu_{11}} \mu_{112}.$$  \hspace{1cm} (24)

Now we turn to the definition of a linear approximation for a coefficient $\lambda_3$. We differentiate identity (4) with respect to $\xi_3$, along $\xi_2$ and along $\alpha$. Then we find:

$$\lambda_{3a} = \left( \mu_{11} p_1^1 p_3^1 + \mu_{112} p_1^1 p_2^1 + \mu_{11a} p_1^1 + \mu_{12a} p_2^1 \right) p_1^1 + \mu_{1a} p_1^1 + \mu_{2a} p_1^1.$$  \hspace{1cm} (26)

In formula (26), we already know all the expressions except $p_{12}^1$ and $p_{12}^2$. To determine $p_{12}^2$ we calculate the identity $C_{(1,2)}$ and equate it to zero:

$$p_{12}^2 = \frac{p_{12}^2 p_1^1}{2 \mu_{111} \mu_{22}} \left( \mu_{112} - \mu_{111} \mu_{122} \right).$$  \hspace{1cm} (27)

To determine the derivative $p_{12}^1$ at the singular point we calculate the derivatives of the identity $C_{(3,1)}$ and equate to zero. Then: $p_{12}^1$:

$$p_{12}^1 = -\frac{p_{12}^2 p_1^1}{6 \mu_{111} \mu_{22}} \left( 3 \mu_{112}^3 - 3 \mu_{11a} \mu_{112} \mu_{122} + 2 \mu_{11} \mu_{111} \mu_{122} - 2 \mu_{1111} \mu_{112} \mu_{22} \right).$$  \hspace{1cm} (28)

4. Conclusion

Thus, in the work by the method of symbolic calculations, formulas that make it possible to calculate to a first approximation of the unfolding parameters of an edge wave catastrophe of the $F_4$ type are obtained. The catastrophe $F_4$ is a single structurally stable diffraction focusing of both the edge rays forming the caustic $A_2$, and the GO rays forming the caustic $A_2$ with the edge. The coefficients forming the vector $\lambda(\vec{\alpha})$, are calculated in the linear approximation, the phase of the traveling wave $\theta(\vec{\alpha})$ is found in the second quadratic approximation.

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