Soliton molecules, rational positons and rogue waves for the extended complex modified KdV equation

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Abstract In this paper, we consider the integrable extended complex modified Korteweg–de Vries equation. Based on Darboux transformation, we obtain soliton molecules, positon solutions, rational positon solutions and rogue waves for integrable extended complex modified Korteweg–de Vries equation. Further, under the standard decomposition, we divide the rogue waves into three patterns: fundamental pattern, triangular pattern and ring pattern. On the basis of fundamental pattern, we define the length and width of rogue waves and discuss the effect of different parameters on rogue waves.

Keywords Darboux transformation · Soliton molecules · Positon solution · Rational positon solution · Rogue waves

1 Introduction

It is well known the integrable partial differential equations (PDEs) are an important part of modern mathematical and theoretical physics with far-reaching implications. Integrable equations have many valuable properties: Lax pairs, Hamiltonian, conservation laws, exact solutions, etc. Integrable equations can describe many phenomena in science. It is interesting to find their exact solutions for an integrable equation. There are many methods to find solutions of the integrable equation, for instance Lie group [4], the Darboux transformation [10,25,29], the bilinear method [11], the Bäcklund transformation [24], the algebraic geometry method [3] and the inverse scattering transformation [1,2].

The Darboux transformation, first proposed by the French mathematician Gaston Darboux in 1882 on study of the linear Sturm–Liouville problem, is a powerful method to construct solutions for integrable PDEs. In particular, the well-known soliton solution that appear in many PDEs like the nonlinear Schrödinger equation, complex modified Korteweg–de Vries (cmKdV) equation can be computed thereby. In 2012, Guo et al. [7] have obtained rogue waves of the nonlinear Schrödinger equation by the generalized Darboux transformation. The concept of rogue waves originated from oceanography [16]. Rogue waves are rare, large amplitude waves whose heights exceed 2.2 times the significant wave height of the background sea. Now rogue waves have been proposed in many fields: nonlinear optics [28,35], finance [34], Bose–Einstein condensates [12], plasmas [26], water waves [35], etc. More research on rogue waves can be found in monograph [9] and its references.

Soliton molecules, also called the bound-state solitons, are an attractive phenomenon caused by the inter-
action between individual solitons. Recently, soliton molecules were obtained in optical experiments and attracted researcher's attention. Scientists have discovered soliton molecules in Bose–Einstein condensates [22], few-cycle mode-locked laser [13], etc. Lou [19] has presented a velocity resonance mechanism and theoretically obtained soliton molecules of integrable systems and asymmetric solitons three-dimensional fluid system.

In this paper, we investigate an extended complex modified Korteweg–de Vries (ecmKdV) equation, which takes the form

\[ q_t + \alpha (q_{xxx} + 6|q|^2 q_x) + \beta (-30q_x|q|^4 - 10q_x|q|^2 - 10q^*(q^*_x)_x - 10q((q^*_x)_{xx} + q^*_x q_{xxx}) - \partial_x^5 q) = 0, \]

where \( \alpha \ll 1 \) and \( \beta \ll 1 \) stand for the third-order and fifth-order dispersion coefficients matching with the relevant nonlinear terms, respectively. The solitons and inverse scattering transform of Eq. (1) have been considered in [1]. If \( \beta \) is replaced by \( (-\beta) \) and \( q(x, t) \) is a real function in ecmKdV Eq. (1), the ecmKdV equation can be reduced to

\[ q_t + \alpha (q_{xxx} + 6q^2 q_x) + \beta (30q^4 q_x) + 10q_x^3 + 20q(q^*_x)_x + 10q^2 q_{xxx} + \partial_x^5 q = 0. \]

Wazwaz and Xu [31] have considered the Painlevé test and multi-soliton solutions via the simplified Hirota direct method for Eq. (2). The conservation laws, Darboux transformation, periodic solutions and soliton molecules of Eq. (2) have been obtained in [32] and [27], respectively. The longtime asymptotics for Eq. (2) with initial data or initial boundary values have been considered in [20,21]. Liu [18] has obtained the explicit solitons and breather solutions for Eq. (2) by the Riemann–Hilbert method. If we take \( \beta = 0 \), Eq. (1) reduce to the classical cmKdV equations

\[ q_t + \alpha (q_{xxx} + 6|q|^2 q_x) = 0, \]

where \( q = q(t, x) \) is a complex function. The cmKdV equation has many applications in science. For example, the cmKdV equation has been proposed as a model for nonlinear evolution of plasma waves [17], and it has been derived to describe the propagation of transverse waves in a molecular chain model [8] and in a generalized elastic solid [5,6]. He et al. [14] have constructed a generalized Darboux transformation for the cmKdV equation which obtained the rogue waves and analyzed the dynamics of rogue waves. The soliton molecules for the cmKdV equation have been considered in [37]. There are many references on the research of Eq. (1); we will not list them one by one.

Solutions for Eq. (1) is obtained through the Darboux transformation in this paper. Our manuscript is organized as follows: In Sect. 2, after recalling the Lax pair of Eq. (1), we introduce the Darboux transformation for Eq. (1). In Sect. 3, we obtain soliton molecules, position solutions of Eq. (1) from seed solution \( q = 0 \). In Sect. 4, we obtain rational positon solutions of Eq. (1) from nonzero seed solution \( q = c \). In Sect. 5, we construct higher order rogue waves solution from a periodic seed solution with constant amplitude and analyze their structures in detail by choosing suitable system parameters. Finally, we give the conclusions in Sect. 6.

2 Lax pair and Darboux transformation

2.1 Lax pair

Introducing \( r = q^* \), Eq. (1) can be rewritten as

\[ q_t + \alpha (q_{xxx} + 6qrq_x) + \beta (-30q^2 r^2 q_x - 10q^2 r_x - 10r (q^*_x)_x - 10q((q^*_r)_{xx} + rq_{xxx}) - \partial_x^5 q) = 0. \]

According to the AKNS method [1], we obtain the Lax pair corresponding to Eq. (4), i.e.,

\[ \psi_x = M \psi, \quad \psi_t = W \psi, \]

where

\[ M = \begin{pmatrix} -\lambda & q \\ -r & \lambda \end{pmatrix}, \]

\[ W = W_5 \lambda^5 + W_4 \lambda^4 + W_3 \lambda^3 + W_2 \lambda^2 + W_1 \lambda + W_0, \]

and

\[ \psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad W_5 = \begin{pmatrix} -16\beta & 0 \\ 0 & 16\beta \end{pmatrix}, \quad W_4 = \begin{pmatrix} 0 & 16\beta q \\ -16\beta & 0 \end{pmatrix}, \quad W_0 = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}, \]

\[ W_3 = \begin{pmatrix} 4\alpha - 8\beta qr & -8\beta q_x \\ -8\beta q_x & -4\alpha + 8\beta qr \end{pmatrix}, \]

\[ W_2 = \begin{pmatrix} (\beta(4rq_{rs} - 4rq_x) - 4aq - 4\beta(-2q^2 r - q_{xx}) \\ 4\alpha r + 4\beta(-2q^2 r - r_{xx}) \end{pmatrix}. \]
\[ W_1 = \begin{pmatrix}
2\alpha qr + \beta(-6q^2r^2 + 2qr_{xx}) \\
-2qr_{xx} - 2qr_{xx} \\
2\alpha quar - 2\beta(6qrq_{xx} + qr_{xxx}) \\
2\alpha r_2 - 2\beta(6qr_{xx} + qr_{xxx}) \\
-2\alpha qr - \beta(-6q^2r^2 + 2qr_{xx}) \\
-2qr_{xx} - 2qr_{xx}
\end{pmatrix}, \]

\[ w_{11} = -\alpha(rq_{xx} - qr_{xx}) + \beta(6\alpha q^2r_{xx} - qr_{xx}), \]

\[ w_{12} = -2\alpha q^2r - \alpha qr_{xx} + \beta(6\alpha q^2r^2 + 6qr_{xx}) \\
+ 4\alpha qr_{xx} + 8\alpha qr_{xx} + 2\alpha qr_{xxx} + qr_{xxx}, \]

\[ w_{21} = 2\alpha q^2r + \alpha qr_{xx} - \beta(6\alpha q^2r^3 + 6qr_{xx}) \\
+ 4\alpha qr_{xx} + 8\alpha qr_{xx} + 2\alpha qr_{xxx} + qr_{xxx}, \]

\[ w_{22} = \alpha(qr_{xx} - qr_{xx}) - \beta(6\alpha q^2r_{xx} - qr_{xx}) \\
+ qr_{xx} + qr_{xxx} - qr_{xxx}. \]

Eq. (4) can be obtained by zero-curvature condition \( M_I - W_x + [M, W] = 0 \), where the commutator \([M, W] := MW - WM\).

2.2 Darboux transformation

In order to obtain solutions of the ecmKdV equation, we will construct the \( n \)-fold Darboux transformation. Starting from the onefold Darboux transformation \( \psi^{[1]} = T_1 \psi \),

\[ \psi^{[1]}_\lambda = M^{[1]}\psi^{[1]}, \quad \psi^{[1]}_t = W^{[1]}\psi^{[1]} \] (7)

The functions \( M^{[1]} \) and \( W^{[1]} \) satisfy the definitions obtained by replacing \( q \) and \( r \) with \( q^{[1]} \) and \( r^{[1]} \), respectively, in (6). We assume the onefold Darboux transformation

\[ T_1 = T_1(\lambda) = \begin{pmatrix} a_1 & b_1 \\ c_1 \end{pmatrix} \lambda + \begin{pmatrix} a_0 & b_0 \\ c_0 \end{pmatrix} \] (8)

where \( a_i, b_i, c_i, d_i, i = 0, 1 \) are functions of \( x, t \). Combining Eqs. (5) and (7), it is easy to get \( T_x + TM = M^{[1]}T \), \( T_t + TW = W^{[1]}T \).

Substituting Eq. (8) into Eq. (9) and comparing the coefficient of \( \lambda_j^2 \), we obtain

\[ q^{[1]} = q - 2 \frac{\Omega}{\Omega} \begin{pmatrix} \phi_{11} & \lambda \phi_{11} \\ \phi_{21} \end{pmatrix}, \quad \Omega = \begin{pmatrix} \phi_{11} \phi_{12} \\ \phi_{21} \phi_{22} \end{pmatrix} \]

\[ T_1(\lambda; \lambda_1) = \left( \begin{array}{cc} \lambda - \frac{1}{\Omega} \frac{\lambda_1 \phi_{11}}{\phi_{21}} & -\frac{1}{\Omega} \phi_{11} \\ \frac{\lambda_1 \phi_{12}}{\phi_{22}} & \lambda - \frac{1}{\Omega} \frac{\lambda_1 \phi_{12}}{\phi_{22}} \end{array} \right) \left( \begin{array}{cc} \phi_{11} \phi_{12} \\ \phi_{21} \phi_{22} \end{array} \right) \]

In order to get the expression for \( n \)-fold Darboux transformation \( T_n \) and new solution \( q^{[n]} \) of Eq. (1), we give Theorem 2.1.

**Theorem 2.1** If the function \( q(x, t) \) is a solution of Eq. (1), then \( q^{[n]} \) is new solution of Eq. (1) which is defined by

\[ q^{[n]} = q + 2 \frac{N_{2n}}{W_{2n}} \]

and \( n \)-fold Darboux transformation

\[ T_n = T_n(\lambda; \lambda_1, \lambda_2, ..., \lambda_{2n}) = \begin{pmatrix} (T_{2n})_{11} & (T_{2n})_{12} \\ (T_{2n})_{21} & (T_{2n})_{22} \end{pmatrix} \]

where
Darboux transformation has the form

\[ T_{n} = \begin{bmatrix}
\lambda_{1}^{n-1} & \lambda_{1}^{n-1} \phi_{11} & \lambda_{1}^{n-2} \phi_{12} & \lambda_{1}^{n-2} \phi_{11} & \cdots & \lambda_{1}^{n-2} \phi_{12} & \phi_{11} & \phi_{12} \\
\lambda_{2}^{n-1} & \lambda_{2}^{n-1} \phi_{21} & \lambda_{2}^{n-2} \phi_{22} & \lambda_{2}^{n-2} \phi_{21} & \cdots & \lambda_{2}^{n-2} \phi_{22} & \phi_{21} & \phi_{22} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\lambda_{2n}^{n-1} & \lambda_{2n}^{n-1} \phi_{2n,1} & \lambda_{2n}^{n-2} \phi_{2n,2} & \lambda_{2n}^{n-2} \phi_{2n,1} & \cdots & \lambda_{2n}^{n-2} \phi_{2n,2} & \phi_{2n,1} & \phi_{2n,2}
\end{bmatrix},
\]

**Proof** According to the form of \( T_{1} \) in Eq. (12), \( n \)-fold Darboux transformation has the form

\[ T_{n} = T_{n}(\lambda; \lambda_{1}, \lambda_{3}, \ldots, \lambda_{2n-1}) = E \lambda^{n} + \sum_{i=0}^{n-1} P_{i} \lambda^{i} \quad (14) \]

with

\[ E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P_{i} = \begin{bmatrix} a_{i} & b_{i} \\ c_{i} & d_{i} \end{bmatrix}. \]

Combining Eq. (14) and properties of Darboux transformation \( T_{n}(\lambda; \lambda_{1}, \lambda_{2}, \ldots, \lambda_{2n})|_{\lambda=\lambda_{k}} = 0, k = 1, 2, \ldots, 2n \), the element of \( P_{i} \) can be solved by Cramer’s rule. Substituting \( P_{i} \) into Eq. (14), \( T_{n} \) in Eq. (13) is obtained after calculation and simplification. Based on properties of \( n \)-fold Darboux transformation \( T_{n} + T M = M^{[n]} T, T_{n} + T W = W^{[n]} T \), the expression for \( q^{[n]} \) in Eq. (13) is generated by comparing the coefficients of \( \lambda^{n+1} \).

\[ \square \]

### 3 Zero seed solution

Taking the seed solution \( q = r^{*} = 0 \), the spectral problem (5) reduces to
\[ \psi_x = M' \psi, \quad \psi_t = W' \psi, \]

where

\[ M' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \lambda, \quad W' = \begin{pmatrix} -16\beta & 0 \\ 0 & 16\beta \end{pmatrix} \lambda^5 \]

\[ + \left( 4\alpha \begin{pmatrix} 0 & 0 \\ 0 & -4\alpha \end{pmatrix} \right) \lambda^3. \]

By a simple calculation, the eigenfunctions corresponding to eigenvalue \( \lambda_{2j-1} \) are given by the following:

\[ (\phi_{2j-1,1}, \phi_{2j-1,2}) = \left( e^{-\lambda_{2j-1} x + (-16\beta \alpha_{2j-1} + 4\alpha \lambda_{2j-1} \lambda) / \beta} \right), \]

\[ j = 1, 2, ..., n, \] (15)

where \( \zeta \) is a real constant.

In what following subsections, we will use the eigenfunctions \( \phi_{2j-1,1}, \phi_{2j-1,2} \) to construct the solution of the ecmKdV equation through Darboux transformation. In particular, soliton molecules and positon solutions can be obtained at \( \zeta \neq 0 \) and \( \zeta = 0 \), respectively.

### 3.1 Soliton molecules

In this subsection, we will consider soliton molecules of ecmKdV equation. Substituting eigenfunctions (15) corresponding to eigenvalue \( \lambda_1 = a_1 + ib_1 \) into Eq. (13), we obtain one-soliton solution \( |q_{1-s}|^2 \) as follows:

\[ |q_{1-s}|^2 = 4a_1^2 \text{sech}^2 \left( 2a_1 \left( H - \frac{\zeta}{a_1} \right) \right), \] (16)

where

\[ H = -16\beta t a_1^4 + 160\beta t a_1^2 b_1^2 \]

\[ -80\beta t b_1^4 + 4\alpha t a_1^2 - 12\alpha t b_1^2 - x. \]

According to the expression of \( |q_{1-s}|^2 \), the solution consisting of \( l \) solitons and \( m \) molecules consisting of two same solitons can be generated if Eq. (13) satisfies the following resonance conditions:

\[ \lambda_1 = -\lambda_3, \lambda_5 = -\lambda_7, ..., \lambda_{4m-3} = -\lambda_{4m-1}, \quad n = 2m + l. \]

Figure 1a shows a soliton molecule consisting of two same solitons. The collision process of two soliton molecules composed of two same solitons is shown in Fig. 1b. A molecule consisting of \( l \) solitons can be obtained if parameters in Eq. (13) satisfy following conditions:

\[ -16\beta a_{2j-1}^4 + 160\beta a_{2j-1}^2 b_{2j-1}^2 \]

\[ + 4a_1^2 a_{2j-1}^2 - 80\beta b_{2j-1}^4 - 12\alpha b_{2j-1}^2 = v_0, \] (17)

\[ \lambda_1 \neq \lambda_3, ..., \neq \lambda_{2j-1}, \quad j = 1, 2, ..., n. \]

On the basis of condition (17), the distance between adjacent solitons is equal if parameters satisfy following conditions:

\[ \frac{\zeta}{a_{2j+1}} - \frac{\zeta}{a_{2j-1}} = d_0, \] (18)

\( v_0 \) and \( d_0 \) are real constants. Figure 1c, d shows a molecule consisting of 3 solitons under condition Eq. (17) and Eq. (18). If we take \( \alpha = 1, \beta = 0 \), then the soliton molecules of Eq. (13) reduced to the case of complex modified KdV equation as shown in [37].

### 3.2 Positon solutions

In this subsection, we will construct the \( n \)-positon solutions of ecmKdV equation. It is trivial to see in Eq. (13) that \( q[l] \) becomes 0 when \( \lambda_{2j-1} \to \lambda_1, \quad j = 2, 3, ..., n. \)

Taking \( \lambda_{2j-1} = \lambda_1 + \epsilon, \epsilon \) is an infinitesimal parameter, the \( n \)-positon solution \( q_{n-p} (n \geq 2) \) can be obtained by we perform the higher-order Taylor expansion (see, for example, ref. [15,33]), where

\[ q_{n-p} = 2 \frac{N'_{n-2n}}{W_{2n}}, \]

and

\[ N'_{2n} = \left( \frac{\partial n_{i-1}}{\partial e} \right)_{\epsilon=0} (N_{2n})_{ij} (\lambda_1 + \epsilon) \]

\[ W'_{2n} = \left( \frac{\partial n_{i-1}}{\partial e} \right)_{\epsilon=0} (W_{2n})_{ij} (\lambda_1 + \epsilon) \]

\[ n_i = \lfloor \frac{i+1}{2} \rfloor. \] \( \lfloor x \rfloor \) denotes the floor function of \( x \). Considering the choice in eq.(3.2) with \( \lambda_1 = a_1 + ib_1 \) and \( n = 2 \), one can get the expression of 2-positon solution

\[ q_{2-p} = -\frac{B_1}{B_2} a_1 \]

\[ e^{-32i\beta b_1 t^3 + 320i\beta t a_1^2 b_1^3 - 8i a b_1^3 - 16i\beta t a_1^4 b_1^4 + 24i a t a_1^2 b_1^2 - 2i x b_1}, \]

where

\[ B_1 = (16 + 10240 i b_1 a_1^2 - 10240 i b_1 a_1^4 b_1^4 + 768 i b_1 a_1^2 a_1) \cosh(2a_1 H) + (2560 \beta a_1^5 - 15360 \beta a_1^3 b_1^2 + 2560 \beta a_1 b_1^4 - 384 a a_1^3 + 384 a t a_1 b_1^2 + 32 a x_1) \sinh(2a_1 H), \]

\[ B_2 = (102400 \beta^2 a_1^8 + 409600 \beta^2 a_1^6 b_1^2) \]
In this section, the rational positon solutions of ecmKdV equation can be obtained through degenerate Darboux transformation from exponential solution $q(x, t) = ce^{i\rho}$ of the ecmKdV equation, where $\rho = ax + bt$, $b = \beta(a^5 - 20ax^3 c^3 + 30ac^4) + \alpha(a^3 - 6ac^2)$ and $a, b, c$ are real constants. By the principle of superposition of the linear differential equations, new eigenfunctions corresponding to $\lambda_j$ is

$$\psi_j = \begin{pmatrix} \phi_{j1} \\ \phi_{j2} \end{pmatrix}$$

Although we did not give expressions of 3-positon and 4-positon solution because of their verbosity and complexity, the 2-positon, 3-positon and 4-positon solutions are plotted in Fig. 2.

### 4 Rational positon solutions

In this section, the rational positon solutions of ecmKdV equation can be obtained through degenerate Darboux transformation from exponential solution $q(x, t) = ce^{i\rho}$ of the ecmKdV equation, where $\rho = ax + bt$, $b = \beta(a^5 - 20ax^3 c^3 + 30ac^4) + \alpha(a^3 - 6ac^2)$ and $a, b, c$ are real constants. By the principle of superposition of the linear differential equations, new eigenfunctions corresponding to $\lambda_j$ is

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$$\psi_j = \begin{pmatrix} \phi_{j1} \\ \phi_{j2} \end{pmatrix}$$

Although we did not give expressions of 3-positon and 4-positon solution because of their verbosity and complexity, the 2-positon, 3-positon and 4-positon solutions are plotted in Fig. 2.
Proposition 4.1 Suppose that $q(x, t)$ is a nonzero seed solution of Eq. (1) and eigenfunctions are degenerate at eigenvalue $\lambda_0$, when $\lambda_{2j-1} \rightarrow \lambda_0$, the degenerate $n$-fold Darboux transformation produces new solution

$$q^n(x, t; \lambda_0) = q + 2 \frac{N'_{2n}}{W_{2n}}, \quad (21)$$

where

$$N'_{2n} = \left( \frac{\partial n_i}{\partial \epsilon} \right)_{\epsilon=0}^{(N_{2n})_{ij}(\lambda_0 + \epsilon)}_{2n \times 2n},$$

$$W_{2n} = \left( \frac{\partial n_i}{\partial \epsilon} \right)_{\epsilon=0}^{(W_{2n})_{ij}(\lambda_0 + \epsilon)}_{2n \times 2n},$$

and $n_i = \left\lfloor \frac{i+1}{2} \right\rfloor$, $\lfloor x \rfloor$ denotes the floor function.

Combining Proposition 4.1 and setting $s_i = 0, i = 0, 1, \ldots, n - 1$, $n$-order rational positon solution $q_{n-r}$ of Eq. (1) can be obtained. The expression of first-order rational positon $q_{1-r}$ (rational line wave) and second-order rational positon solution $q_{2-r}$ are given as

$$q_{1-r} = c - 2c \frac{L_1 - 1}{L_1 + 1},$$

$$q_{2-r} = c + 2 \frac{L_2}{L_3},$$

where

$$L_1 = 3600 \beta^2 e^{10} t^2 - 1440 \alpha \beta e^8 t^2 + 144 \alpha^2 c^6 t^2 + 240 \beta e^6 t X - 48 \alpha c^4 t X + 4 \epsilon^2 e^2 X^2,$$

$L_2$ and $L_3$ are shown in Appendix A. If the values of $\alpha$ and $\beta$ are given, first-order rational positon solution $|q_{1-r}| = |c|$ when $x \rightarrow \infty, t \rightarrow \infty$ and the height of $|q_{1-r}|$ is $|3c|$. Figure 3 shows the second-order rational positon solution $|q_{2-r}|^2$ and its density plot.

5 Rogue waves solution and their dynamics analysis

In this section, to construct the rogue waves solution of ecmKdV equation, let us choose the special solution of the ecmKdV equation as

$$q = ce^{\rho}, \quad \rho = ax + bt,$$

where

$$b = \beta(a^5 - 20a^3 c^2 + 30ac^4) + \alpha(a^3 - 6ac^2), \quad a, b, c \in \mathbb{R}, \quad a, c \neq 0. \quad (22)$$

It is trivial to find the eigenfunctions (20) are degenerate at $\lambda_0 = -\frac{1}{2}a + c$. The expression of $n$-order rogue wave $q_n$ can be obtained by Proposition 4.1. Due to the length and complexity of higher-order rogue waves, we only give expression of the first-order rogue wave as follows:

$$q_1 = ce^{a|x + (a\beta - 20a^2 c^2 \beta + 30c^4 \beta + a^2 - 6ac^2)|t|} - A_1 - A_1 - 160ia^3 \beta c^2 t + 480ia\beta c^4 t - 48iaac^2 t + 3 \frac{1}{A_1 + 1},$$

where $A_1$ is given in Appendix B.

A simple computation gives $|q_1|^2 \leq 9c^2$ and $|q_1|^2 = c^2$ when $x \rightarrow \infty, t \rightarrow \infty$. It is not difficult to find that the selection of parameters $d_1, d_2$ will produce different types of rogue waves. Let $\alpha = \beta = 0.5$ for the convenience of discussion in this section. Next, let us discuss the first-order to fourth-order rogue wave because of the complexity of higher-order rogue waves.

Setting $(s_i = 0)^{n-1}$, the fundamental pattern of rogue waves can be generated. The first-order rogue wave $|q_1|^2$ to fifth-order rogue wave $|q_4|^2$ are shown.
takes the maximum value at $|q_2| = 0.5$, $|q_1| = 0.8$, $|q_3| = 0.2$. It is easy to see that Fig. 4.

Fundamental patterns of the rogue waves. $\alpha = \beta = 0.5$, $c_1 = 1$, $c_2 = 0$, $i = 0, 1, 2, 3, 4$. From (a) to (e) are first-order rogue wave $|q_1|^2$, second-order rogue wave $|q_2|^2$, third-order rogue wave $|q_3|^2$ and fourth-order rogue wave $|q_4|^2$ with $\alpha = 0.65, 0.7, 0.7$.

in Fig. 4. It easy to see that $n$-order rogue wave $|q_n|^2$ takes the maximum value at $(x, t) = (0, 0)$ and has $n$ peaks on each side of $t = 0 (n > 1)$. Next, we analyze the contour line of first-order rogue wave $|q_1|^2$ at different heights. Let us fix the value of $c$ and assume $c = 1$.

1. At height $c^2$, the contour line of first-order rogue wave $|q_1|^2$ is

\[
4x^2 + \left(20a^4t - 228a^2t + 96t\right)x + 25a^8t^2 - 970a^6t^6 + 5649a^4t^4 - 5652a^2t^2 + 576t^2 = 1.
\]

It’s a hyperbola which has two asymptotes,

\[
l_1 : x = \left(-\frac{(5a^4 - 57a^2 + 24)}{2a} + (10a^3 - 27a)\right) t,
\]

\[
l_2 : x = \left(-\frac{(5a^4 - 57a^2 + 24)}{2} - (10a^3 - 27a)\right) t.
\]

2. At height $c^2 + 1$, the contour line of first-order rogue wave $|q_1|^2$ is $A_2 = 0$ which has two end points,

\[
P_1 = \left(\frac{\sqrt{7}}{2a(10a^2 - 27)} - \frac{\sqrt{7}(5a^4 - 57a^2 + 24)}{4a(10a^2 - 27)}\right),
\]

\[
P_2 = \left(\frac{\sqrt{7}}{2a(10a^2 - 27)} - \sqrt{7}(5a^4 - 57a^2 + 24)\right).
\]

3. At height $\frac{c^2}{2}$, the contour line of first-order rogue wave $|q_1|^2$ is $A_3 = 0$ and two centers of valleys $P_2 = (0, \frac{\sqrt{7}}{2c}, P_4 = (0, -\frac{\sqrt{7}}{2c})$.

$A_2, A_3$ are defined in Appendix B. Figure. 5a gives density plot of first-order rogue wave $|q_1|^2$. Figure 5b–d shows the contour line of first-order rogue wave $|q_1|^2$ with $h = c, \frac{c^2}{2}$, respectively.

By the definitions in [14], we define the length and width of first-order rogue wave in similar way. The length $d_L$ of first-order rogue wave is distance between
Fig. 5 $\alpha = \beta = 0.5, c = 1$. (a) The density plot of the first-order rogue wave $|q_1|^2$ with $\alpha = 0.65$. The blue point and green dashed line are the asymptotes and imaginary axis of the contour line of $|q_1|^2$ at height $h = c^2$. From (b) to (d) are the contour line of first-order rogue wave $|q_1|^2$ at height $h = c^2 + 1$, $\alpha = 0.5$ (blue point), $\alpha = 0.65$ (red curve), $\alpha = 0.8$ (green dotted line). (b) Two fixed points $(0,0.5), (0,-0.5)$. (c) Two fixed points $(0,0.41), (0,-0.41)$. (d) Four fixed points $(0,0.58), (0,-0.58), (0,1.78), (0,-1.78)$.

$P_1$ and $P_2$, i.e.,

$$d_L = \frac{\sqrt{7}}{a(10a^2 - 27)} \sqrt{1 + k_i^2} = \frac{\sqrt{7}}{2a(10a^2 - 27)} \sqrt{25a^8 - 570a^6 + 3489a^4 - 2736a^2 + 580}.$$  \hspace{1cm} (23)

The projection of line segment $P_3P_4$ in width direction is width $d_W$ of first-order rogue wave, that is,

$$d_W = \frac{\sqrt{3}}{\sqrt{1 + k_i^2}} = \frac{2\sqrt{3}}{\sqrt{25a^8 - 570a^6 + 3489a^4 - 2736a^2 + 580}}.$$  \hspace{1cm} (24)

Figure 6 shows the length and width of first-order rogue wave in the case of $c = 1$. When $a > 0$, combining Eqs. (23) and (24), we find that the length keeps decreasing and the width keeps increasing when $0 < a < 0.66$ and $1.64 < a < 3.31$. The length keeps increasing and the width keeps decreasing when $0.66 < a < 1.64$ and $a > 3.31$. When $a < 0$, the change of length and width is opposite to that of $a > 0$. This is first effect of $a$. Furthermore, we find that first-order rogue wave rotates counterclockwise with the increase of $a$. This is second effect of $a$.

In the above discussion, we have considered the length and width of first-order rogue wave in the case of $c = 1$. When $c \neq 1$, the length $d_{cL}$ and width $d_{cW}$ of first-order rogue wave can be obtained by the method analogous to that used above, where

$$d_{cL} = \frac{\sqrt{8c^2 - 1}}{2ac^2(10a^2 - 30c^2 + 3)} \sqrt{4 + (5a^4 - 60c^2a^2 + 30c^4 + 3a^2 - 6c^2)^2},$$

$$d_{cW} = \frac{\sqrt{3}}{c\sqrt{1 + k_c^2}} = \frac{2\sqrt{3}}{c \sqrt{4 + (5a^4 - 60c^2a^2 + 30c^4 + 3a^2 - 6c^2)^2}}.$$  \hspace{1cm} (25)

Taking $s_1 \gg 1, n \geq 2, s_i = 0, i = 0, 2, 3... n - 1$, the structure of rogue wave is similar to a triangle which is the so-called triangular pattern. Figure 7 shows...
triangular patterns from second-order to fourth-order rogue wave. Setting $s_{n-1} \gg 1, n \geq 3, s_i = 0, i = 0, 1, 2, 3, \ldots, n - 2$, the structure of rogue wave is similar to a ring, and this is the so-called ring pattern. Figure 8 shows ring patterns of third-order and fourth-order rogue wave.

Rogue waves composed of triangular and ring patterns also can be generated. For example, Fig. 9a shows fourth-order rogue wave $|q_4|^2$ consisting of a triangular and a ring pattern. Rogue waves composed of the fundamental patterns, triangular patterns and ring patterns is called the standard decomposition of rogue waves. Due to the diversity of parameter $\{s_i\}^{n-1}_{i=0}$, we can generate more patterns of rogue waves. Figure 9b shows nonstandard decomposition of fourth-order rogue wave $|q_4|^2$.

**Remark 5.1** In the above discussion, fundamental patterns, triangular patterns and ring patterns of rogue waves are obtained in the case of $s_0 = 0$. If $s_0 \neq 0$, rogue waves similar to the above three patterns can also be generated, as shown in Fig. 10.

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**6 Conclusions**

In this paper, we have presented the soliton molecules, positon solutions, rational positon solutions and rogue waves for the extended complex modified KdV equation (1) which is general complex cmKdV equation. The results obtained in our paper are as follows:

- For soliton molecules of Eq. (1), we have obtained the general soliton molecules and soliton molecules with equal distances between adjacent solitons under condition (17) and (18). It can be seen that there is no energy loss after the collision of the two soliton molecules, but the phase of the molecules has changed.
- By degenerate Darboux transformation, we get the positon solutions and rational positon solutions for Eq. (1).
- In the discussion about rogue waves and their dynamics analysis of Eq. (1), we find that the contour line of first-order rogue wave is a hyperbola in the case of $|q_1|^2 = c^2$. At height of $c^2 + 1$ and $c^2$,
Fig. 9. $\alpha = \beta = 0.5, a = 0.7, c = 1$. (a) Standard decomposition of $|q_4|^2$ with $s_0 = s_2 = 0, s_1 = 50, s_3 = 10^5$. (b) Nonstandard decomposition of $|q_4|^2$ with $s_0 = s_1 = s_3 = 0, s_2 = 10^5$. 

![Figure 9](image1)

Fig. 10. Rogue waves similar to the fundamental pattern, triangular pattern and ring pattern with $s_0 \neq 0$. $\alpha = \beta = 0.5, a = 0.7, c = 1$. (a) The 3D plot of $|q_3|^2$ with $s_0 = 10, s_1 = 0, s_2 = 0$. (b) The density plot of $|q_3|^2$ with $s_0 = 10, s_1 = 200, s_2 = 0$. (c) The density plot of $|q_3|^2$ with $s_0 = 10, s_1 = 0, s_2 = 2000$.

![Figure 10](image2)

the contour line of first-order rogue wave $|q|^2$ is a closed curve. In the case of $c = 1$, the length and width of first-order rogue wave take two extreme values when $a > 0$, and first-order rogue wave rotate counterclockwise as the increase of $a$. The fundamental patterns [Fig. 4], triangular patterns [Fig. 7], ring patterns [Fig. 8], standard decomposition [Fig. 9a] and nonstandard decomposition [Fig. 9b] of rogue waves are shown in paper. We obtain third-order rogue wave similar to fundamental pattern, triangular pattern and ring pattern at $s_0 \neq 0$.

If we consider the special case of the $\alpha = 1, \beta = 0$, our results can be reduce to the case of complex modified KdV equation which consider in [14, 37].

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Data Availability Statement. All data generated or analyzed during this study are included in this article.

Declarations

Conflict of interest. The authors declare that they have no conflict of interest.

Appendix A

\[ L_2 = 34560000 \beta^4 t^4 c^{21} - 27648000 \alpha \beta^3 t^4 c^{19} + 4608000 \beta^3 t^3 c^{17} + 8294400 \alpha^2 \beta^2 t^4 c^{17} - 1105920 \alpha^3 \beta t^4 c^{15} - 27648000 \alpha \beta^2 t^3 c^{15} + 55296 \alpha^4 t^4 c^{13} + 230400 \beta^2 t^2 x^2 c^{13} + 552960 \alpha^2 \beta t^3 c^{13} - 36864 \alpha^3 t^3 c^{11} - 92160 \alpha \beta t^2 x^2 c^{11} + 364800 \beta^2 t^2 c^{11} + 9216 \alpha^2 t^2 x^2 c^9 + 5120 \beta t x^3 c^9 - 8 c \]
\[-115200 \alpha \beta \tau^2 c^9 - 1024 \alpha \tau x^3 c^7 + 8448 \alpha^2 r^2 c^7 + 14080 \beta \tau x c^7 + 128 r^4 x^5 - \frac{1792}{3} \alpha \tau x c^5 + 64 x^2 c^3,\]

\[L_3 = -2073600000 \beta \alpha^6 c^{30} + 2488320000 \alpha \beta^6 c^{28} - 12441600000 \alpha^2 \beta^6 c^{26} - 4147200000 \beta^3 r^5 \tau x c^{26} + 4147200000 \alpha^3 r^5 \tau x c^{24} + 3317760000 \alpha^3 \beta^3 r^5 \tau x c^{24} - 1658800000 \alpha^2 \beta^3 r^5 \tau x c^{22} - 345600000 \beta^4 r^5 \tau x c^{22} - 497664000 \alpha^4 \beta^3 r^5 \tau x c^{22} + 167040000 \beta^4 r^4 \tau x c^{20} + 276480000 \alpha^2 \beta^3 r^4 \tau x c^{20} + 331776000 \alpha^3 \beta^3 r^3 \tau x c^{20} + 39813120 \alpha^5 \beta r^6 c^{20} - 115200000 \alpha \beta^3 r^4 c^{18} - 1327104 \alpha^6 \beta^3 c^{18} - 82944000 \alpha^2 \beta^2 r^4 x^2 c^{18} - 33177600 \alpha^4 \beta \tau^5 x c^{18} - 15360000 \beta^3 \tau^3 x^3 c^{18} + 16128000 \beta^3 \tau^3 x c^{16} + 29030400 \alpha^2 \beta^2 r^4 c^{16} + 9216000 \alpha^2 \beta^2 r^3 x^2 c^{16} + 11059200 \alpha^3 \beta^3 r^3 x^2 c^{16} + 1327104 \alpha^4 \beta \tau^4 x c^{16} - 3133440 \alpha^3 \beta \tau^4 x c^{14} - 7833600 \alpha^2 \beta^2 r^4 x c^{14} - 1843200 \alpha^3 \beta \tau^2 x^2 c^{14} - 384000 \beta^2 r^2 x^4 c^{14} - 552960 \alpha^4 \tau^4 x^2 c^{14} + 119080 \alpha^4 \beta^4 c^{12} + 499200 \beta^2 r^2 x^2 c^{12} + 119080 \alpha^2 \beta^3 r^3 x c^{12} + 153600 \alpha \beta^2 r^5 x^3 c^{12} - 606400 \beta^2 r^2 c^{10} - 55296 \alpha^3 \beta^3 x c^{10} - 138240 \alpha \beta \tau^2 x^2 c^{10} - 15360 \alpha^2 r^2 x^4 c^{10} - 5120 \beta \tau x^5 c^{10} + 145280 \alpha \beta \tau^2 \beta^8 + 7680 \alpha^2 r^2 x^2 c^8 + \frac{12800}{3} \beta \tau x c^8 + 1024 \alpha \tau x^5 c^8 - \frac{256}{9} \alpha \tau x c^6 - 8996 \alpha^2 \tau r^2 c^6 - 8000 \beta \tau x c^6 - \frac{512 \alpha}{3} \tau x^3 c^6 + 1088 \alpha \tau x^4 c^4 - \frac{64}{3} \alpha \tau x^4 c^4 - 48 x^2 c^2 - 4.\]

**Appendix B**

\[A_1 = 100 c^8 \alpha^2 r^2 - 800 c^6 \beta^2 r^2 + 6000 c^2 \alpha^2 r^2 \beta^6 c^{12} + 36000 c^{10} \alpha^2 \beta^2 r^2 + 120 c^8 \beta^2 r^2 \alpha + 720 c^8 \alpha^2 \beta^4 r^2 + 720 c^6 \tau^2 r^2 \beta^6 \alpha + 1440 c^8 \beta^2 r^2 \alpha + 36 c^8 \beta^2 r^2 \tau^2 + 144 c^8 \alpha^2 \tau^2 r^2 + 40 c^8 \alpha^2 \beta \tau x + 40 c^8 \alpha^2 \beta \tau x s_0 - 480 c^8 \alpha^2 \beta \tau x.\]

\[A_2 = 16 x^4 + (160 a \tau^2 - 1824 a \tau^2 + 768 t) x^3 + (600 a \tau^2 - 1048 a t^6 + 6645a t^4 - 42336 a t^2 \tau^4 + 13824 t^2 \tau^4 + 40 a t^2 + (1000 a \tau^2 - 1820 a \tau^4 t^3 + 135480 a \tau^8 t^3 - 631464 a \tau^8 t^3 + 196128 a \tau^8 t^3 + 200 a \tau^4 t - 228096 a \tau^3 t^3 + 2280 a \tau^2 t + 110592 a \tau^3 - 960 a t) x + 625 a \tau^4 t^4 - 800 a t^4 a \tau^4 t^4 + 95350 a \tau^4 t^4 - 143260 a \tau^4 t^4 + 3850 a \tau^4 t^4 + 207360 a \tau^4 t^4 + 44856 a \tau^2 t^2 + 331776 t^4 + 5760 t^2 - 7.\]

\[A_3 = -8 x^4 + (-80 a \tau + 912 a \tau - 384 t) x^3 + [28 + (-300 a^8 + 5240 a^6 - 33228 a^4 + 21168 a^2 - 6912 t^2)] x^2 + [(-500 a^{12} + 9100 a^{10} - 67740 a^8 + 315732 a^6 - 98064 a^4 + 114048 a^2 + 55296 \tau^3 + (140 a^4 - 1596 a^2 + 672) t] x + (-65888 + 4250 a^{14} - 625 a^{16} - 103680 a^2 - 781704 a^4 - 141300 a^6 - 1733841 a^8 + 221430 a^{10} - 47675 a^{12} t^4 + (175 a^8 - 7590 a^6 + 43863 a^4 - 45396 a^2 + 4032) t^2 - \frac{17}{2}.\]

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