Strengthening the notion of Petrov type I spacetimes

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Abstract. The Petrov classification of spacetime curvature tensors is an algebraic characterization: distinct types have differing multiplicities for their distinct principal null directions. However, a geometrical characterization of a set of vectors requires examining their linear independence, further quantified by the dimension of the spacetime volume they span. We further classify Petrov type I spacetimes as strong type I when the distinct principal null directions are linearly independent and span a 4-space, and weak type I otherwise. A similar distinction applies to Petrov type II spacetimes, whose principal null directions may span a space of dimension 3 or less.

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1 Introduction

When studying the properties of a given spacetime, useful geometrical and physical information is associated with the principal null directions (PNDs) of its Weyl tensor. The Petrov classification [1] categorizes spacetimes by
the generic number of distinct PNDs (algebraic not differential information),
which in turn translates into possible multiplicities of the roots of an eigenvalue problem involving bivectors which can have either simple (unrepeated) eigenvalues or repeated eigenvalues. The multiplicity types for the number of PNDs are

Type I: four simple (four distinct),
Type II: one double and two simple (three distinct),
Type D: two double (two distinct),
Type III: one triple and one simple (two distinct),
Type N: one quadruple (one distinct),
Type O: none (vanishing Weyl tensor).

Type I is the algebraically general case, while the remaining types are referred to as algebraically special.

Why do the PNDs play such an important role? A “rough” argument is the following. The PNDs locate on the light cone at each spacetime point the pillars on which the spacetime itself can stand alone as a solution of the vacuum Einstein equations. If the spacetime then hosts other fields (either test fields or by generalization through perturbation fields which modify the background geometry through back-reaction), it is expected that the characteristic directions of these new fields will coincide, at least in a first approximation, with those of the background. This is true for the Petrov type D Kerr-Newman rotating and charged black hole spacetime, sourced by the electromagnetic field generated by a single massive electric charge. This spacetime generalizes the electrically neutral rotating Kerr black hole: the eigenvectors of the electromagnetic field 2-form are aligned with those of the spacetime curvature. This agreement is captured by the vanishing of the generalized Simon tensor \[2\].

A set of 4 distinct null vectors in a 4-dimensional Lorentzian spacetime may span a subspace of the tangent space of dimension 4 or less, distinguished by the nonvanishing or vanishing of their wedge product, respectively. Similarly a set of 3 distinct null vectors can span a subspace of dimension 3 or less. Little attention has been given previously to this distinction of spanning dimension apart from a general spinor discussion pioneered by Rindler and Penrose \[3\] and later works by McIntosh et al \[4,5,6\]. Indeed it is difficult to find many explicit examples which illustrate all the possible cases. Focusing on this refinement of the Petrov classification by the dimension of the spanning space of the PNDs, we introduce the notions of strong and weak Petrov type I (with four distinct principal null directions spanning either a 4- or 3-dimensional spacetime volume respectively), and illustrate them by discussing several examples. These considerations also extend to type II spacetimes (with three distinct principal null directions, spanning a 3 or 2-dimensional spacetime volume respectively). Obviously, two distinct principal null directions (as in the case of type D spacetimes) are automatically independent, so this sub-classification does not extend lower in the Petrov scheme.

Analytically computing the PNDs of a given spacetime is always possible, but the actual computation can be quite difficult since it involves the roots of a fourth degree polynomial and their use in the subsequent bivector
manipulations. The usual approach starts with a null frame which is then conveniently rotated until one of the frame vectors becomes a PND. In this case, spacetime symmetries may help, in the sense that a null vector $k_{\pm}$ is proportional to the sum or difference of a unit timelike vector $u$ and a unit spacelike one $\hat{\nu}$ orthogonal to $u$, $k_{\pm} \propto u \pm \hat{\nu}$, where either of these might be suggested by any Killing symmetries of the spacetime which might exist.

If one is interested only in characterizing the Petrov type of a given spacetime, it is enough to study the multiplicity of the PNDs without explicitly determining them. However, 1) a dynamical spacetime (including perturbed black hole spacetimes and numerically generated spacetimes), during its evolution, may pass through different spectral types, and it is interesting to study the motivations for this transition; 2) a general family of spacetimes, with a metric depending on several parameters, can also be of different spectral types corresponding to various regions of the parameter space. Since different spectral types are associated with distinct physical properties, it is interesting to study situations in which such changes happen. In particular distinguishing algebraically special spacetimes from the general type I case can be done by evaluating the “speciality index” $S^{[7,8]}$, a particular combination of the Weyl curvature scalars. It has the value $S = 1$ only for algebraically special spacetimes, while $S \neq 1$ characterizes the general type I case, thus identifying the algebraically special cases among a family of spacetimes which is generically of type I.

The main result of the present investigation regards Petrov type I spacetimes for which the 4 distinct PNDs, say $k_i$, $i = 1, \ldots, 4$ span a 4-dimensional hyperplane, i.e., the 4-dimensional volume associated with their wedge product $\Omega_{1234} = k_1 \wedge k_2 \wedge k_3 \wedge k_4$, is nonzero. The actual value has no meaning since the null vectors can be arbitrarily rescaled and only their equivalence classes under rescaling matter (algebraically). We call this the a strong type I case, as opposed to a weak type I case for which $\Omega_{1234} = 0$. The same terminology applies to Petrov type II spacetimes, where there exist three distinct PNDs and one can study the 3-dimensional volume $\Omega_{123} = k_1 \wedge k_2 \wedge k_3$, whose vanishing or nonvanishing defines a strong or weak type II spacetime. Since a set of two or fewer distinct (nonproportional) null vectors is automatically linearly independent, this terminology does not extend lower in the Petrov classification hierarchy.

Apart from the clear geometrical meaning underlying the definition of a strong or weak Petrov type I and II spacetime, its physical meaning is not yet apparent and will require further investigation.

Our conventions and notation will follow the standard ones for the Newman-Penrose (NP) formalism [9,10] (see also Ref. [1]). Furthermore, units are chosen such that $c = 1 = G$.

2 Petrov classification and scalar invariants: a short review

Consider the Weyl tensor $C_{\alpha\beta\gamma\delta}$ of a given spacetime with metric $g$. Define the complex tensor $\bar{C}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} - iC_{\alpha\beta\gamma\delta}$ and introduce in both tensor...
and NP notation the two complex curvature invariants \[ I = \frac{1}{32} \tilde{C}_{\alpha\beta\gamma\delta} \tilde{C}^{\alpha\beta\gamma\delta} = \psi_0 \psi_4 - 4 \psi_1 \psi_3 + 3 \psi_2^2, \] (1)

and

\[ J = \frac{1}{384} \tilde{C}_{\alpha\beta\gamma\delta} \tilde{C}^{\gamma\delta}_{\mu\nu} \tilde{C}^{\mu\nu\alpha\beta} = \psi_0 \psi_2 \psi_4 - \psi_1^2 \psi_4 - \psi_0 \psi_3^2 + 2 \psi_1 \psi_2 \psi_3 - \psi_2^3, \] (2)

where the Weyl scalars refer to a choice of NP frame \( \{ l, n, m, \bar{m} \} \) related to an associated orthonormal frame \( \{ e_\alpha \} = \{ e_0, e_a \} \) by the standard relations

\[ l = \frac{1}{\sqrt{2}} (e_0 + e_1), \quad n = \frac{1}{\sqrt{2}} (e_0 - e_1), \quad m = \frac{1}{\sqrt{2}} (e_2 + i e_3). \] (3)

An observer with 4-velocity \( U \) measures the following electric and magnetic parts of the Weyl tensor

\[ E(U)_{\alpha\beta} = C_{\alpha\mu\beta\nu} U^\mu U^\nu, \]

\[ H(U)_{\alpha\beta} = -^* C_{\alpha\mu\beta\nu} U^\mu U^\nu, \] (4)

respectively, which can be combined into the symmetric tracefree complex tensor

\[ Q(U)_{\alpha\beta} = \tilde{C}_{\alpha\beta\gamma\delta} U^\gamma U^\delta = E(U)_{\alpha\beta} + i H(U)_{\alpha\beta}, \] (5)

in terms of which the scalars \( I \) and \( J \) take the form

\[ I = \frac{1}{32} Q(U)^\alpha_{\beta} Q(U)^\beta_{\alpha}, \quad J = \frac{1}{384} Q(U)^\alpha_{\beta} Q(U)^\beta_{\delta} Q(U)^\delta_{\alpha}. \] (6)

Let \( e_0 = U \), so that the orthonormal frame \( \{ e_\alpha \} \) is adapted to the observer \( U \).

The nonzero components of the tensor \( Q \) with respect to it can be represented by the following 3x3 complex matrix

\[ (Q^a_b) = \begin{pmatrix}
\psi_2 - \frac{i}{2} (\psi_0 + \psi_4) & \frac{i}{2} (\psi_4 - \psi_0) & \psi_1 - \psi_3 \\
\frac{i}{2} (\psi_4 - \psi_0) & \psi_2 + \frac{i}{2} (\psi_0 + \psi_4) & i (\psi_1 + \psi_3) \\
\psi_1 - \psi_3 & i (\psi_1 + \psi_3) & -2 \psi_2
\end{pmatrix}, \] (7)

where \( a, b = 1, 2, 3 \).

The scalars \( I \) and \( J \) are used to define the speciality index \( S \) of the spacetime when \( I \neq 0 \)

\[ S = \frac{27 J^2}{I^3}, \] (8)

characterizing the transition from general Petrov type \( I (S \neq 1) \) to algebraically special behavior \( (S = 1) \). Because of their tensor expressions as scalars, it is clear that both \( I \) and \( J \) (and hence \( S \)) are frame-invariant objects, i.e., they do not change under any allowed transformation of the chosen orthonormal or null frame.

The standard algorithm used to determine the spectral type of a given spacetime involves the evaluation of other scalar objects. One first evaluates the scalars \( I \) and \( J \) and the difference \( I^3 - 27 J^2 \). If the latter quantity is nonzero then the spacetime is of type \( I \). If instead \( I^3 - 27 J^2 = 0 \) one should
distinguish the case of \( I \) and \( J \) both nonvanishing or not, and construct three new scalars [1],

\[
K = \psi_1 \psi_2^2 - 3\psi_1 \psi_3 \psi_2 + 2\psi_3^3,
\]

\[
L = \psi_2 \psi_4 - \psi_3^2,
\]

\[
N = 12L^2 - \psi_4^2 I,
\]

which are related to the discriminants of the quartic equation (13) defining the PNDs, and are not frame-invariant (see Appendix A). The algebraically special types correspond to the following conditions:

- Type II: \( I \neq 0, J \neq 0, K \neq 0 \) or \( N \neq 0 \),
- Type D: \( I \neq 0, J \neq 0, K = 0, N = 0 \),
- Type III: \( I = 0, J = 0, L \neq 0 \) or \( K \neq 0 \),
- Type N: \( I = 0, J = 0, L = 0, K = 0 \).

(10)

Details of this algorithm as well as its representation as a flow chart can be found in Fig. 9.1 of Ref. [1], recalling the underlying assumption \( \psi_4 \neq 0 \) (or \( \psi_0 \neq 0 \)).

An equivalent approach to classifying the Weyl tensor instead solves the eigenvalue problem associated with the matrix \( Q_{ab} \). The matrix criteria for the various Petrov types and the normal forms of the the matrix \( Q_{ab} \) in each case (with corresponding eigenvalues and eigenvectors) are listed in Tables 4.1 and 4.2 of Ref. [1], respectively. The orthonormal frame \( \{e_\alpha\} \) with respect to which the matrix \( Q_{ab} \) has a normal form is uniquely determined (modulo the choice of numbering of the three spatial vectors \( \{e_a\} \)) for the non-degenerate Petrov types I, II and III, and is called a Weyl principal (or canonical) tetrad. The eigenvalues satisfy the equation

\[
\sigma^3 - I\sigma - 2J = 0,
\]

so that

\[
I = \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2), \quad J = \frac{1}{6}(\sigma_1^3 + \sigma_2^3 + \sigma_3^3) = \frac{1}{2}\sigma_1\sigma_2\sigma_3.
\]

(11)

(12)

For Petrov type I spacetimes the Weyl scalars with respect to the principal tetrad are given by \( \psi_0 = \psi_3 = (\sigma_2 - \sigma_1)/2, \psi_1 = \psi_3 = 0, \) and \( \psi_2 = -\sigma_3/2 \), with \( \sigma_3 = -\sigma_1 - \sigma_2 \). For type D we have in addition \( \psi_0 = 0 = \psi_4 \) as \( \sigma_1 = \sigma_2 \).

For type II we have \( \psi_0 = \psi_1 = \psi_3 = 0, \psi_4 = -2, \) and \( \psi_2 = -\sigma_3/2 \).

2.1 PNDs for Petrov types I and II spacetimes

Next we review the explicit determination of the PNDs. Following the notation of Ref. [1], if \( \psi_4 \neq 0 \) for the Petrov type of a given spacetime we have to find the roots \( \lambda \) (with the corresponding multiplicity) of the following algebraic equation

\[
\lambda^4 \psi_4 - 4\lambda^3 \psi_3 + 6\lambda^2 \psi_2 - 4\lambda \psi_1 + \psi_0 = 0,
\]

(13)
whose solutions define the explicit expressions for the four PNDs
\[ k_i = l + \lambda_i^* m + \lambda_i \tilde{m} + |\lambda_i|^2 a, \quad i = 1 \ldots 4. \] (14)

These roots are computed as follows. First divide Eq. (13) through by its leading coefficient
\[ \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 = 0, \] (15)
defining the new coefficients
\[ a_1 = -\frac{4\psi_3}{\psi_4}, \quad a_2 = \frac{6\psi_2}{\psi_4}, \quad a_3 = -\frac{4\psi_1}{\psi_4}, \quad a_4 = \frac{\psi_0}{\psi_4}. \] (16)
This equation can be directly solved by using the standard, rather involved, formulas available in the literature, leading to the four roots \( \lambda_i (i = 1, \ldots, 4) \).

However, one can conveniently rotate the NP frame to put it into its transverse form, i.e., with \( \psi_1 = 0 = \psi_3 \), so that Eq. (15) reduces to a bi-quadratic equation
\[ \lambda^4 + a_2 \lambda^2 + a_4 = 0, \] (17)
with solutions
\[ \lambda_{1,2} = A_\pm, \quad \lambda_{3,4} = -A_\pm, \] (18)
where
\[ A_\pm = \sqrt{-a_2 \pm \sqrt{a_2^2 - 4a_4}}. \] (19)

If the transverse frame is also canonical (\( \psi_0 = \psi_4, a_4 = 1 \)), additional simplifications in the solutions (18) occur, namely
\[ A_\pm = \frac{1}{2} \left[ \sqrt{-a_2 + 2} \pm \sqrt{-a_2 - 2} \right]. \] (20)

For example, for Petrov type I, inserting the value of \( a_2 = 6\psi_2/\psi_0 \) leads to the following (explicit) solutions [6]
\[ \lambda_1, \quad \lambda_2 = -\lambda_1, \quad \lambda_3 = \frac{1}{\lambda_1}, \quad \lambda_4 = -\frac{1}{\lambda_1}, \] (21)
where
\[ \lambda_1 = \left[ -\frac{\psi_2}{\psi_0} - \sqrt{\frac{9 (\psi_2/\psi_0)^2}{2}} \right]^{1/2}. \] (22)

The latter can be also written as
\[ \lambda_1 = \sqrt{\sigma_2 + 2\sigma_1 + \sqrt{\sigma_1 + 2\sigma_2}} \sqrt{\sigma_1 - \sigma_2}^{-1}, \] (23)
in terms of the eigenvalues \( \sigma_i \) of the matrix \( Q_{ab} \) (see Table 4.3 of Ref. [1]).

On the one hand, directly solving the fourth-degree algebraic equation (13) is in general a difficult task (because of the large expressions involved), but which is facilitated if one uses a principal NP frame. In fact, in that case
this equation becomes bi-quadratic with obvious advantages in writing its solutions.

On the other hand, transforming a general NP frame into a principal one is not an easy task, since one generally must use type I, II and III null tetrad rotations in succession to accomplish this, a fact which in most cases works against the advantage of solving a simpler equation at the end.

In the case of Petrov type II spacetimes the canonical tetrad corresponds to \( \psi_0 = 0 = \psi_1 = \psi_3 \) and \( \psi_4 = -2 \), so that Eq. (13) becomes

\[
-2\lambda^2(\lambda^2 - 3\psi_2) = 0,
\]

with solutions

\[
\lambda_1 = 0 = \lambda_2, \quad \lambda_3 = \sqrt{3\psi_2}, \quad \lambda_4 = -\lambda_3 \quad (\psi_2 \neq 0).
\]

Therefore, \( k_1 = l = k_2 \) is a repeated PND with multiplicity 2, while \( k_3, k_4 \) are given by Eq. (14). On the other hand the complex matrix \( Q_{ab} \) has eigenvalues \( \sigma_1 = \sigma_2 = -\sigma/2 \) and \( \sigma_3 = \sigma = -2\psi_2 \), so that \( \lambda_3 = \sqrt{-\frac{3}{2}\sigma} \).

2.2 PND degeneracy

The four PNDs (14) may be either linearly independent or not. In the former case they span a 4-dimensional vector space at each spacetime point, otherwise only a 3-dimensional subspace.

Arianrhod, McIntosh and coworkers [4,5,6] classified the PND degeneracies depending on the nature and value of the scalar invariant

\[
\tilde{M} = \frac{I^3}{J^2} - 27 = \frac{27}{S}(1 - S),
\]

with \( \tilde{M} \) generally complex and possibly infinite. They proved the following theorem [4]: “The four distinct PNDs associated with a metric whose Weyl tensor is of Petrov type I span, at each point, either a 3-dimensional vector space, in which case \( \tilde{M} \) is real and either positive or infinite, or a 4-dimensional vector space for other \( \tilde{M} \).” Furthermore, they showed that if there exists an observer with 4-velocity \( U \) who sees the Weyl tensor as purely electric or purely magnetic, then the PNDs are linearly dependent, and span the 3-dimensional vector space orthogonal to the eigenvector of \( Q_{ab} \) corresponding to the eigenvalue of smallest absolute value [6].

Here we will adopt a different criterion to distinguish between the two cases. Let

\[
\Omega_{1234} = k_1 \wedge k_2 \wedge k_3 \wedge k_4
\]

be the 4-dimensional volume associated with the \( k_i \). When \( \Omega_{1234} \neq 0 \) the four PNDs are linearly independent, our “strong” type I case. The “weak”

\footnote{We denote here such an invariant by \( \tilde{M} \) instead of \( M = \frac{I^3}{J^2} - 6 \) since we are using the definitions of Ref. [11] for \( I \) and \( J \), which slightly differ from those of Penrose and Rindler [3].}
type I case instead corresponds to $\Omega_{1234} = 0$, implying that the PNDs are linearly dependent.

In general, the 4-dimensional volume can be written as

$$\Omega_{1234} = V l \wedge n \wedge m \wedge \bar{m},$$

with

$$V = (\bar{\lambda}_3 \lambda_2 + \bar{\lambda}_2 \lambda_4 - \lambda_3 \bar{\lambda}_2 - \bar{\lambda}_3 \lambda_4 + \lambda_3 \bar{\lambda}_4)|\lambda_1|^2$$

+ $(\lambda_1 \lambda_4 - \bar{\lambda}_1 \lambda_3 + \bar{\lambda}_1 \lambda_3 - \lambda_3 \lambda_4 + \lambda_3 \bar{\lambda}_4 - \lambda_1 \lambda_4)|\lambda_2|^2$

+ $(\lambda_1 \lambda_2 - \bar{\lambda}_1 \lambda_4 + \bar{\lambda}_1 \lambda_4 + \lambda_2 \bar{\lambda}_4 - \bar{\lambda}_1 \lambda_4)|\lambda_3|^2$

+ $(\lambda_3 \lambda_2 + \lambda_1 \lambda_3 - \lambda_1 \lambda_3 - \lambda_1 \lambda_2 - \lambda_3 \lambda_2)|\lambda_4|^2.$

(29)

The factor $V$ vanishes identically when all the $\lambda_i$ are either real or purely imaginary. It may be different from zero if the (distinct) $\lambda_i$ are complex, which is a necessary but not sufficient condition for linear independence.

For Petrov type I spacetimes, substituting into Eq. (29) the solutions (21) corresponding to a canonical tetrad gives

$$V = -\frac{16}{|\lambda_1|^4} \text{Re}(\lambda_1) \text{Im}(\lambda_1)(|\lambda_1|^2 + 1)(|\lambda_1|^2 - 1),$$

(30)

implying that the PNDs are linearly dependent only if one of the following conditions holds: $\text{Re}(\lambda_1) = 0$, $\text{Im}(\lambda_1) = 0$, or $|\lambda_1|^2 = 1$.

For Petrov type II spacetimes, the 3-dimensional volume associated with the canonical tetrad reads

$$\Omega_{123} = k_1 \wedge k_2 \wedge k_3$$

$$= -2|\lambda_3|^2 l \wedge n \wedge (\bar{\lambda}_3 m + \lambda_3 \bar{m})$$

$$= 2\sqrt{2}|\lambda_3|^2 \left[\text{Re}(\lambda_3) \omega^{012} + \text{Im}(\lambda_3) \omega^{013}\right],$$

(31)

with $\omega^{012} = \omega^0 \wedge \omega^1 \wedge \omega^2$ and $\omega^{013} = \omega^0 \wedge \omega^1 \wedge \omega^3$, where $\{\omega^\alpha\}$ is the dual frame of $\{e_\alpha\}$, related to the NP frame in the usual way by Eq. (4), while $\lambda_3 = \sqrt{\frac{3}{\bar{\psi}_2}} \neq 0$ cannot vanish and remain of type II. Therefore, it is always nonzero, implying that there cannot exist spacetimes of weak type II. The nonexistence of type II spacetimes with linearly dependent PNDs has not been pointed out before, and is a novel and unexpected result of the present analysis.

Below we consider explicit examples which prove helpful by illustrating the previous discussion concretely.

### 3 Type I spacetimes: examples

#### 3.1 Kasner spacetime

The simplest Petrov type I spacetime allowing for analytical computations is the vacuum Kasner [11] metric

$$ds^2 = dt^2 - t^{2p_1} dx^2 - t^{2p_2} dy^2 - t^{2p_3} dz^2,$$  

(32)
where the so-called Kasner indices \( p_i \) satisfy
\[
p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1,
\] (33)
and assume values in the closed interval \([-\frac{1}{3}, 1]\). The spatial Cartesian coordinates and these indices are adapted to the eigenvectors of the extrinsic curvature of the intrinsically flat time slices, but the algebraic properties of the spacetime curvature tensor are quite different.

Introduce the following NP frame adapted to the first spatial coordinate
\[
l = \frac{1}{\sqrt{2}}[\partial_t + t^{-p_1}\partial_x],
\]
\[
n = \frac{1}{\sqrt{2}}[\partial_t - t^{-p_1}\partial_x],
\]
\[
m = \frac{1}{\sqrt{2}}[i t^{-p_2}\partial_y + it^{-p_3}\partial_z],
\] (34)
which has the following nonzero Weyl scalars
\[
\psi_0 = \psi_4 = \frac{p_1(p_2 - p_3)}{2t^2}, \quad \psi_2 = -\frac{p_2p_3}{2t^2},
\] (35)
so that the frame is a canonical one (clearly true starting from the other two coordinate directions as well). The associated orthonormal frame
\[
e_0 = \partial_t, \quad e_1 = t^{-p_1}\partial_x, \quad e_2 = t^{-p_2}\partial_y, \quad e_3 = t^{-p_3}\partial_z,
\] (36)
is adapted to the static observers with 4-velocity \( U = e_0 \) whose spatial axes are aligned with the Killing vectors \( \partial_x, \partial_y, \partial_z \), and therefore directly observe the homogeneity of the spacetime. They also see a purely electric Weyl tensor whose electric part is
\[
E(U) = \frac{1}{4t^2} [p_1p_3 e_1 \otimes e_1 + p_1p_2 e_2 \otimes e_2 + p_2p_3 e_3 \otimes e_3],
\] (37)
while its magnetic part \( H(U) \) vanishes identically.

The 1-parameter family of spacetimes (32) is efficiently parametrized by expressing the Kasner indices in terms of the Lifshitz-Khalatnikov (LK) parameter
\[
p_1 = \frac{u}{(1 + u + u^2)}, \quad p_2 = \frac{(1 + u)}{(1 + u + u^2)}, \quad p_3 = \frac{u(1 + u)}{(1 + u + u^2)},
\] (38)
with limiting cases \( u \rightarrow \pm \infty \) capturing the remaining triplet \((0, 0, 1)\), while \( u = 0 \leftrightarrow (0, 1, 0) \) and \( u = -1 \leftrightarrow (1, 0, 0) \), all of which correspond to a flat spacetime. On the other hand the three cases \( u = -2, -\frac{1}{2}, 1 \) correspond to the three triplets which are permutations of \((-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})\) for which the spacetime is a locally rotationally symmetric type \( D \) spacetime, with a spindle-like cosmological singularity [112], expanding in one direction while collapsing in the two orthogonal directions.
Using the LK parametrization, the various scalars turn out to be

\[ I = \frac{u^2(1+u)^2}{(1+u+u^2)^3} \frac{1}{t^4}, \quad J = \frac{1}{2} \frac{u^4(1+u)^4}{(1+u+u^2)^6} \frac{1}{t^6}, \]

\[ K = 0, \quad L = -\frac{1}{4} \frac{u^2(u-1)(1+u)^3}{(1+u+u^2)^4} \frac{1}{t^4}, \]

\[ N = \frac{u^4(1+2)(2u+1)(u-1)^2(1+u)^4}{4(1+u+u^2)^8} \frac{1}{t^8}. \]

Note that both \( I \) and \( J \) vanish for the flat cases \( u = 0, -1, \pm \infty \) but the spectral index is nevertheless always defined and has the constant value

\[ S = -\frac{27}{4} p_1 p_2 p_3 = \frac{27}{4} \frac{u^2(1+u)^2}{(1+u+u^2)^3}, \]

where these expressions here (and their permutations) are equivalent due to Eq. (33). Apart from \( K \) which vanishes identically, other zero values of the remaining NP scalars do occur. All of these scalars vanish for the trivial flat spacetime case for which \( u = 0, -1, \pm \infty \), while for the three type D cases where \( u = -2, -\frac{1}{2}, 1 \) one has \( I = \frac{1}{27} \), \( J = \frac{8}{27} \) and \( u = 1 \): \( L = N = 0 \), \( u = -2, -\frac{1}{2}, N = 0 \). Thus within the Kasner family transitions of Petrov type only occur among types I, D, and O at these particular parameter values.

Consider now the PNDs in the type I case. The Arianrhod-McIntosh invariant (26) reads

\[ \tilde{M} = \frac{(u+2)^2(2u+1)^2(u-1)^2}{u^2(1+u)^2}, \]

and it is always positive for every value of \( u \neq 1 \) and therefore real, implying that the four PNDs must be linearly dependent for all finite values of \( u \) except the trivial case \( u = 1 \) of flat spacetime.

On the other hand we can derive this result directly. The frame (36) is a canonical frame, so that the PNDs are given by Eq. (14) with \( \lambda_i \) specified by Eq. (21). The eigenvalues of the matrix \( Q_{ab} \) are

\[ \sigma_1 = u \sigma_2 = -\frac{u^2(1+u)}{(1+u+u^2)^2} \frac{1}{t^2}, \]

so that by Eq. (23)

\[ \lambda_1 = \frac{u+2}{u-1} + \sqrt{\frac{2u+1}{u-1}}, \]

which is real for \( u > 1 \) and \( u < -2 \), purely imaginary for \(-\frac{1}{2} < u < 1 \), and complex for \(-2 < u < -\frac{1}{2} \) with \( |\lambda_1|^2 = 1 \), all conditions for which by Eq. (30) lead to \( V = 0 \) implying linear dependence of the PNDs for all values of \( u \neq 1 \).
3.2 Petrov spacetime

The Petrov spacetime \[13\] is a homogeneous vacuum solution with line element given by

\[ k^2 ds^2 = e^x[\cos(\sqrt{3}x)(dt^2 - dz^2) + 2 \sin(\sqrt{3}x) dtdz] - dx^2 - e^{-2x} dy^2, \tag{44} \]

where \( k > 0 \) is a constant parameter and \( 0 < \sqrt{3}x < \pi/2 \). The orthonormal frame associated with the principal NP frame is given by

\[
\begin{align*}
    e_0 &= ke^{-x/2} \left[ \cos \left( \frac{\sqrt{3}x}{2} \right) \partial_t + \sin \left( \frac{\sqrt{3}x}{2} \right) \partial_z \right], \\
    e_1 &= \frac{k}{\sqrt{2}} (\partial_x - e^x \partial_y), \\
    e_2 &= \frac{k}{\sqrt{2}} (\partial_x + e^x \partial_y), \\
    e_3 &= ke^{-x/2} \left[ - \sin \left( \frac{\sqrt{3}x}{2} \right) \partial_t + \cos \left( \frac{\sqrt{3}x}{2} \right) \partial_z \right],
\end{align*}
\tag{45} \]

leading to the following nonvanishing Weyl scalars

\[
\begin{align*}
    \psi_0 &= \psi_4 = -\frac{k^2 \sqrt{3}}{2} e^{i\pi/6}, \\
    \psi_2 &= -\frac{k^2}{2} e^{-i\pi/3} = -k^2 - \psi_4.
\end{align*}
\tag{46} \]

The various NP scalars are explicitly

\[
\begin{align*}
    I &= 0, \quad J = -\frac{k^6}{2}, \quad K = 0, \quad L = \frac{\sqrt{3} k^4}{4} e^{-i\pi/6}, \quad N = \frac{9 k^8}{4} e^{-i\pi/3},
\end{align*}
\tag{47} \]

so that the Arianrhod-McIntosh invariant \[26\] is negative (\( \tilde{M} = -27 \)). Both the electric and magnetic parts of the Weyl tensor measured by \( U = e_0 \) are nonzero

\[
\begin{align*}
    E(U) &= \frac{k^2}{2} (e_1 \otimes e_1 + e_2 \otimes e_2 - 2 e_3 \otimes e_3), \\
    H(U) &= \frac{\sqrt{3}k^2}{2} (e_1 \otimes e_1 - e_2 \otimes e_2).
\end{align*}
\tag{48} \]

The complex matrix \( Q_{ab} \) has eigenvalues

\[
\begin{align*}
    \sigma_1 &= -k^2 e^{i\pi/3}, \quad \sigma_2 = -k^2 = e^{i\pi}, \quad \sigma_3 = k^2 e^{-i\pi/3},
\end{align*}
\tag{49} \]

so that from Eq. \[23\]

\[
\begin{align*}
    \lambda_1 &= e^{-i\pi/3} + e^{-i\pi/6} = \frac{1}{2} (1 - i)(1 + \sqrt{3}),
\end{align*}
\tag{50} \]

and \( V = 16\sqrt{3} \), implying linear independence of the PNDs.
4 Type II spacetimes: the Bonnor-Davidson solution example

Consider the Petrov type II Bonnor-Davidson solution describing a stationary nonvacuum spacetime filled with a perfect fluid having nonzero vorticity and obeying the equation of state $\rho + 3p = \text{constant}$ \[14\]. The line element is given by

$$ds^2 = 2Hdu^2 + 2dudr - A(dx^2 + dy^2),$$ (51)

where $H$ and $A$ are (positive) functions of $r$, $x$, $y$ given by

$$H = -3x + n[1 - (kr + m) \cot kr], \quad A = \frac{\sin^2 kr}{4kx^3},$$ (52)

with 3 arbitrary parameters ($m,k,n$). The fluid 4-velocity $U = 1/(\sqrt{2H})\partial_u$ is tangent to the Killing vector $\partial_u$. The energy density $\rho$ and pressure $p$ satisfy the relations

$$2\pi(\rho + 3p) = nk^2, \quad 2\pi(\rho + p) = k^2H.$$ (53)

The positivity of the metric functions implies that for fixed values of the parameters, the allowed ranges for the coordinates $r$ and $x$ are determined by the conditions $0 < x < x_{\text{max}}(r)$ and $0 < r < r_{\text{max}}$, with

$$x_{\text{max}} = \frac{n}{3}[1 - (kr + m) \cot kr].$$ (54)

For example, choosing $n = 1 = k$ and $m = 0$, Eq. (54) gives $x_{\text{max}} = \frac{1}{3}(1 - r \cot r)$, so that $x_{\text{max}} \to 0$ for $r \to 0$ and $x_{\text{max}} \to \infty$ for $r \to \pi = r_{\text{max}}$.

Choosing the NP frame

$$l = \partial_r, \quad n = \partial_u - H\partial_r, \quad m = \frac{1}{\sqrt{2A}}(\partial_x - i\partial_y),$$ (55)

leads to the following nonvanishing Weyl scalars

$$\psi_2 = -nk^2\left[\frac{1}{3} - \frac{1}{\sin^2 kr}(kr + m) \cot kr\right],$$

$$\psi_3 = \frac{3\sqrt{2}k^2x^{3/2} \cos kr}{\sin^2 kr},$$

$$\psi_4 = -\frac{18k^2x^2}{\sin^2 kr}.$$ (56)

The PNDs are $l$ (with multiplicity 2) and $l + \lambda_{\pm}n + \lambda_{\pm}(m + \bar{m})$, with

$$\lambda_{\pm} = \frac{\sqrt{2} \cos kr}{3\sqrt{x}} \pm W,$$

$$W = \frac{1}{3x} \left[(2x + n) \cos^2 kr - 3n(kr + m) \cot kr + 2n\right]^{1/2},$$ (57)

implying

$$\Omega_{123} = -\sqrt{2}\lambda_+ \lambda_- (\lambda_+ - \lambda_-)\omega^{012}. $$ (58)
One can equivalently pass to a canonical frame \( \{ e_\alpha^{\text{can}} \} \) (with dual \( \{ \omega^\alpha_{\text{can}} \} \)) by successively applying to the NP frame (55) a type I null rotation (to eliminate \( \psi_3 \)) and a type III null rotation (to set \( \psi_4 = -2 \)) with (real) respective parameters (see Appendix A)

\[
a = -\frac{\sqrt{2} k^2 x^{3/2} \cos kr}{\psi_2 \sin^2 kr}, \quad A = \frac{\sqrt{3} \psi_2 \sin^2 kr}{9k^2 x^2 W}, \quad \theta = 0, \quad (59)
\]

leaving \( \psi_2 \) unchanged. Using this tetrad the various NP scalars have the simple form

\[
I = 3\psi_2^2, \quad J = -\psi_3^2, \quad K = 0, \quad L = -2\psi_2, \quad N = 36\psi_2^2. \quad (60)
\]

The equation for the PNDs then reduces to Eq. (24) with solutions (25). Finally, the 3-volume (61) associated with the canonical tetrad reads

\[
\Omega_{123} = 2\sqrt{3} \lambda_3 \omega^{012}_{\text{can}}, \quad (61)
\]

where \( \lambda_3 = \sqrt{3} \psi_2 \).

5 Concluding remarks

We have further characterized Petrov type I spacetimes as strong or weak type I according to the geometrical criterion of the nonvanishing or vanishing of the wedge product of their four distinct PNDs, corresponding to spanning a 4- or 3-dimensional subspace of the tangent space at each spacetime point. This completes the Arianrhod-McIntosh classification of PND degeneracies based on the value of the scalar invariant \( \tilde{M} \) defined in Eq. (26). These ideas have been illustrated concretely in two simple type I spacetimes, the Kasner and the Petrov spacetimes, which allow a relatively straightforward computation of the distinct PNDs and their associated wedge product.

A similar criterion could have been applied to Petrov type II spacetimes characterized by three distinct PNDs, but direct computation shows that the wedge product between the three distinct PNDs is always nonzero, so that type II spacetimes are always of strong type. This is illustrated by using the simple Bonnor-Davidson spacetime as an example. The physical content of this further geometrical characterization of Petrov type I spacetimes will be examined in future work.

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A Transformation properties of the NP curvature scalars

A Lorentz transformation of the orthonormal frame associated with a null tetrad transforms that null frame by a so called “null rotation”, which in turn transforms all of the various NP quantities. The curvature scalars $I$ and $J$ are invariants under all the null rotations but the scalars $K$, $N$ and $L$ are only invariant under null rotations of class II. To understand their transformation under null rotations, we review how null rotations affect the NP curvature quantities.

Any null rotation of the basis vectors $l, n, m$ can be achieved by a succession of null rotations of the following types:

1. null rotations of class I, leaving $l$ unchanged;
2. null rotations of class II, leaving $n$ unchanged;
3. null rotations of class III, leaving the directions of $l$ and $n$ unchanged and rotating $m$ by an angle $\theta$ in the $m - \bar{m}$ plane.

The explicit transformations (see Eq. 53 of Ref. [10]) depend on the following six real parameters: $a$ (complex), $b$ (complex) and $\theta$ (real) and $A$ (real), such that

1. class I:
   \[ l \to l, \quad m \to m + a\bar{m}, \quad \bar{m} \to \bar{m} + a\bar{l}, \quad n \to n + \bar{a}m + a\bar{m} + a\bar{a}l. \quad (62) \]

2. class II:
   \[ n \to n, \quad m \to m + b\bar{m}, \quad \bar{m} \to \bar{m} + \bar{b}l, \quad l \to l + \bar{b}m + \bar{b}\bar{m} + \bar{b}a. \quad (63) \]

3. class III:
   \[ l \to A^{-1}l, \quad n \to An, \quad m \to e^{i\theta}m, \quad \bar{m} \to e^{-i\theta}\bar{m}. \quad (64) \]

The resulting transformation laws for the Weyl scalars are listed in many textbooks, for example [10]. They are

1. class I:
   \[
   \begin{align*}
   \psi_0 & \to \psi_0, \quad \psi_1 \to \psi_1 + \bar{a}\psi_0, \\
   \psi_2 & \to \psi_2 + 2\bar{a}\psi_1 + \bar{a}^2\psi_0, \\
   \psi_3 & \to \psi_3 + 3\bar{a}\psi_2 + 3\bar{a}^2\psi_1 + \bar{a}^3\psi_0, \\
   \psi_4 & \to \psi_4 + 4\bar{a}\psi_3 + 6\bar{a}^2\psi_2 + 4\bar{a}^3\psi_1 + \bar{a}^4\psi_0.
   \end{align*}
   \]
   \[ (65) \]

2. class II:
   \[
   \begin{align*}
   \psi_0 & \leftrightarrow \bar{\psi}_4, \quad \psi_1 \leftrightarrow \bar{\psi}_3, \quad \psi_2 \leftrightarrow \bar{\psi}_2, \\
   \psi_3 & \to \psi_3 + b\psi_4, \\
   \psi_4 & \to \psi_4 + 2b\psi_3 + b^2\psi_4, \\
   \psi_1 & \to \psi_1 + 3b\psi_2 + 3b^2\psi_3 + b^3\psi_4, \\
   \psi_0 & \to \psi_0 + 4b\psi_1 + 6b^2\psi_2 + 4b^3\psi_3 + b^4\psi_4.
   \end{align*}
   \]
   \[ (67) \]

3. class III:
   \[
   \begin{align*}
   \psi_0 & \to A^{-2}e^{2i\theta}\psi_0, \quad \psi_1 \to A^{-1}e^{i\theta}\psi_1, \quad \psi_2 \to \psi_2, \\
   \psi_3 & \to Ae^{-i\theta}\psi_3, \quad \psi_4 \to A^2e^{-2i\theta}\psi_4.
   \end{align*}
   \]
   \[ (68) \]
The NP scalars $I, J, K, L, N$ given in Eqs. (1), (2) and (9), respectively, are related to the discriminants of the quartic equation (13) defining the PNDs. Let us start with Eq. (15), i.e.,

$$\lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 = 0,$$

with rescaled coefficients $a_i$ given in Eq. (16). The general solutions can be written as

$$\lambda_{1,2} = \frac{-a_1}{4} - \frac{1}{2} \left[ -\sqrt{y} \pm \sqrt{\frac{y - 2 \left( p + y + \frac{q}{\sqrt{y}} \right)}{2}} \right],$$

$$\lambda_{3,4} = \frac{-a_1}{4} - \frac{1}{2} \left[ \sqrt{y} \pm \sqrt{\frac{y - 2 \left( p + y - \frac{q}{\sqrt{y}} \right)}{2}} \right],$$

where

$$p = a_2 - \frac{3}{8} a_1^2 = \frac{6}{\psi_4^2} L, \quad q = a_3 - \frac{1}{2} a_1 a_2 + \frac{1}{8} a_1^3 = -\frac{4}{\psi_4} K,$$

and $y$ is a solution of the auxiliary cubic equation

$$y^3 + 2py^2 + (p^2 - 4r)y - q^2 = 0,$$

with

$$r = a_4 - \frac{1}{4} a_1 a_3 + \frac{1}{16} a_1^2 a_2 - \frac{3}{256} a_1^4,$$

so that

$$p^2 - 4r = \frac{4}{\psi_4} N.$$

Writing the cubic equation (72) as

$$y^3 + b_1 y^2 + b_2 y + b_3 = 0,$$

with coefficients

$$b_1 = 2p, \quad b_2 = p^2 - 4r, \quad b_3 = -q^2,$$

a solution is given by

$$y = \frac{b_1}{3} + \left[ \frac{Q}{2} + \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}} \right]^{1/3} + \left[ \frac{Q}{2} - \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}} \right]^{1/3},$$

where

$$P = b_2 - \frac{1}{3} b_1^2 = -\frac{4}{\psi_4} I, \quad Q = b_3 - \frac{1}{3} b_1 b_2 + \frac{2}{27} b_1^3 = \frac{16}{\psi_4} J.$$

The scalars $K, N$ and $L$ are invariant under null rotations of class II, but transform under null rotations of class I and III, respectively, as follows:

$$K \rightarrow K + (2\psi_1 \psi_4 \psi_3 - 9\psi_1 \psi_2^2 + 6\psi_1^2 \psi_2 + \psi_1 \psi_4^2)\tilde{a}$$

$$+ 5(-3\psi_1 \psi_2 \psi_4 + 2\psi_1^2 \psi_3 + \psi_1 \psi_2 \psi_4)\tilde{a}^2$$

$$+ 10(\psi_1 \psi_2 \psi_3 - \psi_4^2)\tilde{a}^3 - 5(2\psi_1 \psi_2 \psi_3 - 3\psi_1 \psi_2 \psi_4 + \psi_1 \psi_4)\tilde{a}^4$$

$$- (\psi_4^2 \psi_4 + 6\psi_2^2 \psi_2 + 2\psi_1 \psi_3 \psi_4 - 9\psi_2 \psi_2 \psi_4)\tilde{a}^5 - (3\psi_1 \psi_2 \psi_4 + \psi_0 \psi_4 + 2\psi_1 \psi_4)\tilde{a}^6,$$

$$L \rightarrow L + (-2\psi_2 \psi_3 + 2\psi_1 \psi_4)\tilde{a} + (\psi_0 \psi_4 - 3\psi_2^2 + 2\psi_1 \psi_3)\tilde{a}^2$$

$$+ 2(-\psi_1 \psi_2 + \psi_0 \psi_3)\tilde{a}^3 + (-\psi_1^2 + \psi_2 \psi_4)\tilde{a}^4,$$

$$N \rightarrow 12L^2 - \psi_4^2 I,$$

and

$$K \rightarrow A^3 e^{-3i\theta} K, \quad L \rightarrow A^2 e^{-2i\theta} L, \quad N \rightarrow A^4 e^{-4i\theta} N.$$
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