ON THE BETTI NUMBERS OF LEVEL SETS OF SOLUTIONS TO ELLIPTIC EQUATIONS

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Dedicated to Professor Peter Lax on the occasion of his 90th birthday.

Abstract. In this paper we study the topological properties of the level sets, \( \{ x : u(x) = t \} \), of solutions \( u \) of second order elliptic equations with vanishing zeroth order terms. We show that the total Betti number of level sets \( t \) is a uniformly bounded function of the parameter \( t \). The uniform bound can be estimated in terms of the analytic coefficients as well as the generalized degrees of the corresponding solutions. Such an estimate is also valid for the nodal sets of solutions of the same type equations with zeroth order terms. In general, it is possible to derive from our analysis an estimate for the total Betti numbers of level sets, for large measure set of \( t \)’s, when coefficients are sufficiently smooth, and therefore a \( L^p \) bound on Betti numbers as a function of \( t \). These estimates are obtained by a quantitative Stability Lemma and a quantitative Morse Lemma.

1. Introduction. It is with great pleasure that we dedicate this paper to Professor Peter D. Lax on the occasion of his 90th birthday. The main object of this paper is to study the topological complexity of the level sets of solutions to general elliptic equations with vanishing zeroth order terms. We shall, in particular, establish uniform bounds on the Betti numbers of regular level sets of solutions. The existence of such bounds were known for algebraic varieties and certain analytic sets, see for examples [19], [15], [16] and [10] without explicit estimates.

The nodal and level sets of real-valued solutions to the linear elliptic equations have been studied by many authors, see the work of Cheng ([5]), Donnelly-Fefferman ([7], [8]), Garofalo-Lin ([11]), Han ([13]), Han-Hardt-Lin ([14]), Hardt-Simon ([17]), Lin([20], [21]) and more recent ones, see for examples, by A.Naber et al ([4], [26]), Colding-Minicozzi ([6]), Hamid-Sogge ([12]), Sogge-Zelditch ([30]). For discussions on the nodal sets of complex-valued or vector-valued solutions, see for instance the work of Berger-Rubinstein ([3]), Elliott-Matano-Tang ([9]), C. Bär ([1]), Helffer-Hoffmann-Hoffmann-Owen ([18]) and X.B. Pan ([27]). There are also interesting
related results concerning polynomials and almost polynomials, see the earlier work of Oleinik-Petrovskii [28] and Y. Yomdin [33].

The above works were all focused on mainly the following issues: the Hausdorff dimension and the Hausdorff measure estimates and the local smooth structure of nodal and critical point sets. For the topological complexity of the nodal and level sets, say the Betti numbers, only a few results are known to the authors. In ([32], [34]), Y. Yomdin obtained various integral bounds on the total Betti numbers of level sets of a smooth functions in [34]. The proof of such bounds depends crucially on his earlier entropy estimates on sets of near critical values [32]. Since such estimates are valid for general smooth functions, by constructing simple examples, one may conclude immediately that only certain $L^p$ integral estimates such as those in [32], for some positive finite $p'$s, would be possible in general. Here we shall prove that for analytic solutions of certain partial differential equations that the total Betti numbers of their level sets are uniformly bounded (for sufficiently smooth solutions, one would obtain bounds in $L^p$ as functions of the level values) in terms of the so-called Almgren’s frequency or generalized degrees. We note that there were well-known works concerning the topology of the level sets of polynomials by R. Thom [31] and J. Milnor [24]. In fact, in [24], an upper bound for the total Betti number of a real algebraic variety was proved by using the Morse theory.

Lemma 1.1. ([24]) Let $f(x)$ be a degree $N$ polynomial of $x \in \mathbb{R}^n$, then

the total Betti number of $f^{-1}(0) \leq c(n)N^n$.

This result showed that the total Betti number may be bounded by a constant depending on the degree of the polynomial and the number of unknowns. From earlier analysis for level and critical point sets of solutions of elliptic equations, one tends to believe that the analogy may be valid also for bounds on Betti numbers. Indeed, solutions of elliptic equations possess properties similar to that of analytic functions. In order to generalize Milnor’s proof, the first step one naturally needs to do would be to control the so-called “generalized degree” of solutions. This generalized degree (which bounds local vanishing orders) has a natural substitute, the Almgren’s frequency, which has already been applied in the study of the quantitative unique continuation property of solutions, see for example, [11]. The next important step would be to get a quantified version of Morse Theory that J. Milnor had used in the above lemma. Very much like the proof of a quantitative version of Stability Lemma in Han-Hardt-Lin [14] which played a critical role in obtaining the geometric measure estimates for critical point sets of solutions, the compactness of solutions of second order elliptic equations with an uniformly bounded generalized degree would also be crucial. Moreover, in order for a suitable perturbation argument to work which is necessary for understanding solutions of elliptic equations, the latter are in general non-analytic, one also requires coefficients of equations to be uniformly elliptic and to be of certain fixed number of orders of smoothness. Finally, there was, in addition, a topological component in the Milnor’s proof. The Nodal sets of polynomials are algebraic varieties (which may be singular), and one has pretty good understandings of their topologies, say they are homeomorphic to polyhedrons, see for example [10]. In particular, all classical notions in topology: triangulation properties, Mayer-Vietoris sequences and continuity of Čech homology theory were established, see also [24] and reference therein. However, these topological facts are unknown for the nodal sets of smooth solutions of elliptic partial
differential equations in general. Here instead, we shall avoid this issue by establishing a somewhat weak statement that there is an uniform bound for analytic solutions in terms of generalized degrees. Since, by Sard’s theorem [29], [10] almost all values of a smooth function are regular values, we hence obtained an essential uniform bound on the total Betti numbers as a function of the level parameter $t$ when the coefficients are analytic. It is because we would need an uniform Stability (Lemma 2.1 with a uniform lower bound on $r$) when the levels are varying that the hypothesis of analyticity of coefficients are assumed. For a fixed level set, say the nodal set, of a solution, then one has to know a priori the Betti numbers are in fact defined. The latter again was known for analytic varieties, [15], and hence we would need analyticity of coefficients. Let us first introduce some notations. Throughout this paper, we consider the general elliptic equation,

$$\sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u(x) + \sum_{i=1}^{n} b_i(x) D_i u(x) = 0 \quad \text{in} \quad B_1(0) \subset \mathbb{R}^n,$$

where the coefficients satisfy the following assumptions:

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{3} a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^3, \quad x \in B_1(0), \quad (2)$$

and, for suitably large $M$ and a fixed positive $\alpha < 1$,

$$\sum_{i,j=1}^{n} \|a_{ij}\|_{C^{M,\alpha}(B_1(0))} + \sum_{i=1}^{n} \|b_i\|_{C^{M,\alpha}(B_1(0))} \leq K. \quad (3)$$

where $\lambda, \Lambda$ and $K$ are positive constants. When the zeroth order term $c(x)u$ is presented, we assume $c(x)$ satisfies the same assumptions. In addition, we shall assume that all coefficients can be expanded into power series centered at any points in the unit ball with the radius of convergence has a uniform lower bound, say $r_0 > 0$.

The following theorem is the main result of the paper:

**Theorem 1.2.** Let $u$ be the solution of problem (1), and the coefficients $a_{ij}, b_i$ are analytic and satisfy the conditions (2) and (3). Here $M$ may chosen to be $2(2N_0)^n$. Then the following is true:

The total Betti numbers of $N(u) \cap B_{\frac{1}{2}}(0) \leq C(n, K, \alpha, \lambda, \Lambda, N_0, r_0)$.

where the constant $N_0$ is the upper bound of the generalized degree of $u$ at any point of $B_{\frac{1}{2}}(0)$, $r_0$ is the radius of convergence of power series expansions of coefficients.

In Section 3 below, we will give the precise definition of the generalized degree and the existence of an uniformly upper bound for $N_0$. The above result can be applied when the zeroth order term $c(x)u$ is present.

This paper is organized as follows. In Section 2, we prove some stability results, which play a crucial role in the proof of Theorem 1.2. The proofs of this section does not need 0 is a regular value of the solution $u$. In Section 3, after recalling the definition of the generalized degree of $u$, we give a natural definition of the Betti numbers of $S(u) \cap B_{\frac{1}{2}}(0)$ for a regular value 0 of $u$. In case that 0 is not a regular value of $u$, the nodal sets may not be in general a smooth hypersurface, the notation of its Betti numbers may be defined as it is an analytic variety. We end with this paper with the proof of the main theorem.
2. Stability results. In this section we prove some stability results, which will be used in the proof of Theorem 1.2. First, we prove a general statement 2.1 which is somewhat standard in the study of stability of singularities in the algebra geometry. Given some smooth map \( g \), the stability result holds for any smooth map sufficiently close to \( g \) with the closeness measured by several constants all depending on \( g \). If we take the map \( g \) as a polynomial map in a suitable family that comes naturally from the Taylor expansions of solutions to a wide class of elliptic equations with an uniform bound on the vanishing orders, we get a stronger stability statement in Lemma 2.5. Roughly speaking, given \( N_0 \), for all (L-harmonic) polynomial maps \( g \) with degrees less than \( 2N_0 \) and with the Almgren frequencies of \( g \) bounded by \( N_0 \), then the constants can be chosen to depend only on \( N_0 \) and the dimension such that the conclusion of the stability lemma holds.

We start with the following lemma:

**Lemma 2.1.** Let \( G(x) = (g_1(x), \cdots, g_n(x)) \) be a smooth map defined from \( B_1(0) \subset \mathbb{R}^n \) to \( \mathbb{R}^n \). Assume that the extended map \( G(x) \) from the unit ball in \( \mathbb{C}^n \) to \( \mathbb{C}^n \) is holomorphic. If

\[
\mathcal{H}^0 \{ G^{-1}(0) \cap B_1(0) \} = m < \infty,
\]

then there exist positive constants \( \delta_*, M_*, r_* \) and \( C_* \) all depending on the function \( G(x) \), such that for any \( F(x) = (f_1(x), \cdots, f_n(x)) \in C^{M_*} (B_1(0)) \) with

\[
\| F - G \|_{C^{M_*} (B_1(0))} < \delta_*,
\]

we have

\[
\mathcal{H}^0 \{ F^{-1}(0) \cap B_{r_*}(0) \} \leq C_*
\]

with multiplicity.

**Proof.** The proof is given in [14, Theorem 4.1]. For the sake of the completeness, we sketch it here. Assume that the nodal set of \( G = (g_1, g_2, \cdots, g_n) \) contains \( m \) points, letting

\[
\mathcal{N}(G) = \mathcal{H}^0 \{ G^{-1}(0) \cap B_1(0) \} = \{ 0, x_1, \cdots, x_{m-1} \}.
\]

**Step 1.** For \( 0 \in \mathcal{N}(G) \), one can show that there is an integer \( N_j^0 \) and a holomorphic function \( a_{ij}^0(x) \) such that,

\[
x_j^{N_j^0} = \sum_{i=1}^{n} a_{ij}^0(x) g_i(x), \text{ for each } j = 1, 2, \cdots, n,
\]

where \( x \in \mathbb{R}^n \).

Letting \( \Lambda_0 = n \max \{ N_1^0, N_2^0, \cdots, N_n^0 \} \), then any monomial in \( x_1, \cdots, x_n \) of degree \( \geq \Lambda_0 \) belongs to the ideal \( (G) \) near 0. Therefore, we can choose an integer \( \mu_0 \) such that

\[
\dim_{\mathbb{R}} P^{\mu_0} / (G) \leq \mu_0,
\]

where the set \( P^{\mu_0} (\mathbb{R}^n) \) denotes the collection of all polynomials in \( \mathbb{R} \) of degree \( \mu_0 \).

Following the arguments in the proof of Theorem 4.1 ([14]), we obtain that there exist neighborhoods \( U_0, V_0 \) and \( Q_0 \) of the origin with \( V_0 \subset U_0 \) and \( G(V_0) \subset Q_0 \) such that for any \( a \in C^{2\mu_0} (U_0) \) there exist \( \alpha_1^0, \alpha_2^0, \cdots, \alpha_n^0 \in C^1(Q_0) \) such that

\[
a(x) = \sum_{i=1}^{\mu_0} e_i(x) \alpha_i^0 (g(x)), \text{ for } x \in V_0.
\]
Remark 2.2. When coefficients are analytic, then

\[ \text{of rectangular coordinate system only on } r \]

Moreover, there exists a \( r_0 > 0 \), such that for any \( F \in C^{2\mu_0}(U_0) \) with \( \| F - G \|_{C^{2\mu_0}(U_0)} < \delta_0 \), the identity (4) still holds with \( G \) replaced by \( F \).

As in ([14]), there exists \( r_0 > 0 \), such that \( F \) has \( \mu_0 \) zeros at most in \( B_{r_0}(0) \).

**Step 2.** For each \( x_k \in N(G) \), we repeat the step 1 to get corresponding \( \mu_k, U_k, V_k, Q_k, \delta_k \) and \( r_k \), such that for \( F \) with \( \| F - G \|_{C^{2\mu_k}(U_k)} < \delta_k \), \( F \) has \( \mu_k \) zeros at most in \( B_{r_k}(x_k) \).

Therefore, by taking

\[ \delta_\ast = \min\{\delta_0, \ldots, \delta_{m-1}\}, \quad M_\ast = 2 \sum_{k=0}^{m-1} \mu_k, \quad C_\ast = \sum_{k=0}^{m-1} \mu_k, \]

and choosing \( r_\ast \) such that

\[ B_{r_\ast}(0) \subset \bigcup_{k=0}^{m-1} B_{r_k}(x_k) \subset \bigcup_{k=0}^{m-1} V_k, \]

we have,

\[ \mathcal{H}^0 \{ F^{-1}(0) \cap B_{r_\ast}(0) \} \leq C_\ast. \]

for \( F(x) \in C^M(B_1(0)) \) with \( \| F - G \|_{C^{M_\ast}(B_1(0))} < \delta_\ast \). To end our proof, we remark that the constants \( M_\ast, \delta_\ast, r_\ast \) and \( C_\ast \) all depend on \( G(x) \).

**Remark 2.2.** When coefficients are analytic, then \( r_\ast \) may be chosen to depend only on \( r_0 \) and other bounds. See also [15, 16].

In general, the constant \( M_\ast \) may be very large, which means the function \( F(x) \) must have sufficiently higher orders of derivatives. It is the reason why we need the assumption (3) to make sure solutions of such equations would be also sufficiently smooth for the conclusion of Theorem 1.2 to be valid. On the other hand, if \( G \) is a polynomial of degree at most \( N \), then from the above proof of ([14]), \( M_\ast \) may be chosen to be not bigger than \( 2N^n \).

**Remark 2.3.** Let us introduce the set

\[ \mathcal{G}_{N_0} = \{ g(x) : B_1(0) \to \mathbb{R}^1, \text{a polynomial in } B_1(0) \subset \mathbb{R}^n, \text{ with degree } \leq 2N_0, \]

\[ \| g \|_{L^2} = 1, \int_{B_1(0)} |\nabla g|^2 dx \leq N_0 \}. \]

Then \( \mathcal{G}_{N_0} \) is a compact set in the polynomial space of \( C^M_\ast \), see for example, [14].

For given numbers \( \varepsilon \) and \( \theta \) in \([0, 1]\) we also define the set

\[ \mathcal{F}_{N_0} = \{ f : f = g^2 + \varepsilon^2|x|^2 - \theta^2, \quad g \in \mathcal{G}_{N_0} \}. \]

Then \( \mathcal{F}_{N_0} \) is also compact.

**Corollary 2.4.** For a given \( f_0 \in \mathcal{F}_{N_0} \), there exist positive constants \( M_0, r_0, \delta_0 \) and \( C_0 \), such that if \( f \in C^{M_0+1}(B_1(0)) \) with \( \| f - f_0 \|_{C^{M_0+1}} < \delta_0 \), and the hypersurfaces \( \{ x : f(x) = 0 \} \) and \( \{ x : f_0(x) = 0 \} \) are both regular, then

\[ \mathcal{H}^0 \{ F^{-1}(0) \cap B_{r_0}(0) \} \leq C_0, \]

where \( F \) is a vector field defined by \( F(x) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n-1}}, f) \), for a suitable choice of rectangular coordinate system \((x_1, x_2, \ldots, x_n) \) of \( \mathbb{R}^n \).
Proof. First, let us use the following notations for the two hypersurfaces:

\[ W_1 = \{ x \in B_1(0) : f_0(x) = 0 \}, \quad W_2 = \{ x \in B_1(0) : f(x) = 0 \}. \]

We use \( \tilde{n}_1 \) and \( \tilde{n}_2 \) to denote the Gaussian maps of \( W_1 \) and \( W_2 \). Sard’s theorem ensures that the sets

\[ \{ \tilde{n}_1 : \nabla \tilde{n}_1 = 0 \} \text{ and } \{ \tilde{n}_2 : \nabla \tilde{n}_2 = 0 \} \]

both have measure zero in \( S^{n-1} \).

Up to some coordinate rotation, we may choose directions \((0, \cdots, 0, 1) \in \mathbb{R}^n\), such that \( \tilde{n}_1(q) = (0, \cdots, 0, 1) \), and \((0, \cdots, 0, 1) \) are neither the critical values of \( \tilde{n}_1 \) nor the critical values of \( \tilde{n}_2 \).

Therefore, the height function \((x_1, \cdots, x_n) \rightarrow x_n \) of the hypersurfaces \( W_1 \) and \( W_2 \) has no degenerate critical points. And the critical points of the height function on \( W_1 \) and \( W_2 \) can be characterized as the solutions of the equations

\[ F_0(x) = \left( \frac{\partial f_0}{\partial x_1}, \cdots, \frac{\partial f_0}{\partial x_{n-1}}, f_0 \right) = 0, \quad F(x) = 0. \]

Since \( f_0 \) is a polynomial of degree \( \leq 2N_0 \), we can apply the result of [24] to obtain that

\[ \mathcal{H}^0 \left\{ F_0^{-1}(0) \cap B_1(0) \right\} = m_0 < \infty. \]

For any \( f \in C^{M_0+1}(B_1(0)) \), we have \( \| F - F_0 \|_{C^{M_0}(B_1(0))} < \delta_0 \). Applying Lemma 2.1 to the maps \( F \) and \( F_0 \), there exist positive constants \( r_0 \) and \( C_0 \) depending on \( f_0 \), such that

\[ \mathcal{H}^0 \left\{ F^{-1}(0) \cap B_{r_0}(0) \right\} \leq C_0. \]

The next Stability Lemma is the key to our proof of the main theorem.

**Lemma 2.5.** There exist positive constants \( \delta_0, M_0, r_0 \) and \( C_0 \) only depending on \( N_0 \), such that if \( f \in C^{M_0+1}(B_1(0)) \), and if the hypersurface \( \{ x : f(x) = 0 \} \) is regular with \( \| f - g \|_{C^{M_0+1}(B_1(0))} < \delta_0 \) and \( g \in \mathcal{F}_{N_0} \), satisfying that \( \{ x : g(x) = 0 \} \) is also a regular hypersurface, then we have

\[ \mathcal{H}^0 \left\{ F^{-1}(0) \cap B_{r_0}(0) \right\} \leq C_0, \]

where the map \( F(x) = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_{n-1}}, f) \), for a suitable choice of rectangular coordinate system \((x_1, x_2, \cdots, x_n) \) of \( \mathbb{R}^n \).

**Proof.** From Corollary 2.4, we can see that for each \( g \in \mathcal{F}_{N_0} \), there exist positive constants \( \delta_g, M_g, r_g \) and \( C_g \), such that if \( \| g - f \|_{C^{M_g+1}(B_1(0))} < \delta_g \), then

\[ \mathcal{H}^0 \left\{ F^{-1}(0) \cap B_{r_g}(0) \right\} \leq C_g. \]

Let us remark that the constants \( M_g \) and \( C_g \) for any \( g \) can be chosen uniformly depending only on \( N_0 \), say \( 2(2N_0)^n \).

Therefore, \( \{ \mathcal{O}(g, \frac{\delta_g}{2}) : g \in \mathcal{F}_{N_0} \} \), form an open cover of \( \mathcal{F}_{N_0} \). By the compactness of \( \mathcal{F}_{N_0} \), there exists a finite cover, denoted by

\[ \mathcal{F}_{N_0} \subset \bigcup_{j=1}^{k} \mathcal{O}(g_j, \frac{\delta_0}{2}). \]

Let us denote the representatives by \( \mathcal{R}_\mathcal{F} = \{ g_1, \cdots, g_k \} \), and we take

\[ \delta_0 = \frac{1}{2} \min \{ \delta_1, \cdots, \delta_k \}, \quad r_0 = \min \{ r_1, \cdots, r_k \}. \]
We claim that the constants $\delta_0$ and $r_0$ only depend on $N_0$. In other words, for any $g \in F_{N_0}$, the stability results holds for $\delta_0$ and $r_0$.

In fact, for any given $g \in F_{N_0}$, by the above covering of $F_{N_0}$, there exists some $g_s \in R_{F_{N_0}}$, such that $\|g - g_s\|_{C^{M_0+1}} < \frac{\delta_0}{2}$. If $f \in C^{M_0+1}(B_1(0))$ with $\|f - g\|_{C^{M_0+1}(B_1(0))} < \delta_0$, then

$$\|f - g_s\|_{C^{M_0+1}(B_1(0))} < \frac{\delta_0}{2} \leq \frac{\delta_s}{2} + \frac{\delta_s}{2} = \delta_s.$$

Applying Corollary 2.4 to $g_s$, we get

$$H^0 \{ F^{-1}(0) \cap B_{r_0}(0) \} \subset H^0 \{ F^{-1}(0) \cap B_{r_s}(0) \} \leq C_0.$$

Hence the stability result works uniformly for the fixed constants $\delta_0$ and $r_0$, which only depend on $N_0$.

Therefore, if $f \in C^{M_0+1}(B_1(0))$ with $\|f - g\|_{C^{M_0+1}(B_1(0))} < \delta_0$ for some $g \in F_{N_0}$, we may conclude that

$$H^0 \left\{ \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n-1}}, f \right)^{-1}(0) \cap B_{r_0} \right\} \leq C_0,$$

which completes the proof. \qed

3. **Estimates on the total Betti numbers.** We can now give an upper estimate of the total Betti numbers of a nodal set. To begin with, let us recall the definition of vanishing order of $u$.

Assume that $u \in W^{1,2}(B_1(0))$ is a solution of (1) in $B_1(0) \subset \mathbb{R}^n$. To define the vanishing order of the weak solution $u$, we only need the following weaker assumptions on the coefficients than (2) and (3):

$$\sum_{i,j=1}^3 a_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2, \forall \xi \in \mathbb{R}^3, x \in B_1(0),$$

$$\sum_{i,j=1}^n |a_{ij}| + \sum_{i=1}^n |b_i| + |c| \leq \kappa, \forall x \in B_1(0),$$

$$\sum_{i,j=1}^n |a_{ij}(x) - a_{ij}(y)| \leq L|x - y|, \forall x, y \in B_1(0),$$

for some positive constants $\lambda, \kappa$ and $L$.

In [11], the authors define,

$$H(p, r) = \int_{\partial B_r(p)} \mu u^2 dx,$$

$$D(p, r) = \int_{B_r(p)} a^{ij}(x)u_{x_i}u_{x_j} dx,$$

where the positive function $\mu$ is constructed in [11].

Letting

$$N(p, r) = \frac{rD(p, r)}{H(p, r)},$$

a monotonicity property for $N(p, r)$ with respect to $r$ is proved in [11]: there exist positive constants $r_0 = r_0(n, \lambda, \kappa, L)$ and $\theta = \theta(n, \lambda, \kappa, L)$ such that the function $N(p, r)\exp(\theta r) \in (0, r_0)$.
Therefore, one can define at each point \( p \in B_{\frac{1}{2}}(0) \) the quantity,
\[
O_u(p) := \lim_{r \to 0} N(p, r).
\]
We shall call it the vanishing order of \( u \) at point \( p \). The following lemma says that the vanishing order of \( u \) in \( B_{\frac{1}{2}}(0) \) is uniformly bounded. For details see [20]. In our paper, we denote the uniformly upper bound by the constant \( N_0 \).

**Lemma 3.1** ([20]). Set
\[
N = \frac{\int_{B_1(0)} |Du|^2 dx}{\int_{\partial B_1(0)} w^2 dx}.
\]
Then the vanishing order of \( u \) at any point of \( B_{1/2}(0) \) is never exceeded by \( C(n, \lambda, \kappa, L)N \).

The following lemma comes from [13] and [14]. It yields a decomposition result and Schauder estimates of the error term.

**Lemma 3.2** ([13], [14]). Suppose that \( u \) is the solution to the equation (1). Under the assumption (5), for any \( y \in B_{1/2}(0) \), the solution \( u \) has the following decomposition:
\[
u(x) = P_d(x - y) + \phi(x), \quad x \in B_{1/4}(y),\tag{6}
\]
where the nonzero polynomial \( P_d(x - y) \) is homogeneous of degree \( d = d(y) \leq N_0 \) satisfying
\[
\sum_{i,j=1}^n a_{ij}(y)D_{ij}P_d = 0,\tag{7}
\]
and
\[
|P_d(x - y)| \leq C|x - y|^d, \quad x \in B_{1/4}(y).\tag{8}
\]
Moreover \( \phi \) satisfies the following estimate: for some \( \alpha \in (0, 1) \),
\[
|\phi(x)| \leq C|x - y|^{d+\alpha}, \quad x \in B_{1/4}(y).\tag{9}
\]
The constant \( C \) depends on \( n, N_0, \lambda, \kappa, L \).

Furthermore, if we assume that the assumptions (2) and (3) hold, by using the interior Schauder estimates one can get,
\[
|D^i \phi(x)| \leq \tilde{C}|x - y|^{d-i+\alpha}, \quad \text{for } i = 1, \ldots, d,
\]
\[
|D^i \phi(x)| \leq \tilde{C}, \quad \text{for } i = d + 1, \ldots, M + 1,\tag{10}
\]
for all \( x \in B_{1/8}(y) \). The positive constant \( \tilde{C} \) depends on \( n, N_0, \lambda, \Lambda, K \).

Next, let us use the definition of the total Betti number of \( \mathcal{N}(u) \cap B_{1/2}(0) \) in [24]:

**Definition 3.3.** The total Betti numbers of \( \mathcal{N}(u) \cap B_{1/2}(0) \) is defined to be
\[
\limsup_{\text{regularvalues}(\varepsilon, \theta) \to (0^+, 0^+)} \text{the total Betti numbers of } \mathcal{N}(u) \cap B_{1/2}(0),
\]
where \( \mathcal{N}(u) \cap B_{1/2}(0) = \{ x \in \mathbb{R}^n : u^2 + \varepsilon^2|x|^2 = \theta^2 \} \).

We should note that, the above limiting set \( \mathcal{N}(u) \cap B_{1/2}(0) \) is also a regular hypersurface with boundaries (not necessary connected), its topology is well-defined and it can be controlled by the limit of total Betti numbers of \( \mathcal{N}(u) \cap B_{1/2}(0) \). Before we proving Theorem 1.2, let us also recall the weak Morse inequalities (see [25]).
Lemma 3.4 (\[25\]). (Weak Morse Inequalities) Let $b_k$ be the $k$-th Betti number of a compact manifold $M$ and $c_k$ denote the set of the critical points of index $k$ of a Morse function on $M$, then

$$b_k \leq \#c_k, \quad \text{for all } k \in [0, \dim M],$$

and

$$
\sum_{k=0}^{\dim M} (-1)^k \#c_k = \sum_{k=0}^{\dim M} (-1)^k b_k = \chi(M).
$$

By the above lemma, it is therefore sufficient for us to do the following two things. One is to construct suitable Morse functions. The other is to get a suitable estimate of the numbers of critical points of such Morse functions.

The following lemma comes from [24]. It gives an estimate on the number of zeros of polynomial equations.

Lemma 3.5 ([24]). Let $V_0 \subset \mathbb{R}^m$ be a zero-dimensional variety defined by polynomial equations $f_1 = 0, \cdots, f_m = 0$. Suppose that the gradient vectors $df_1, \cdots, df_m$ are linearly independent at each point of $V_0$. Then the number of points in $V_0$ is at most equal to the product $(\deg f_1)(\deg f_2) \cdots (\deg f_m)$.

Now we can proceed with the proof of Theorem 1.2: Taking any fixed point $y_0 \in \mathcal{N}(u) \cap B_{1/2}(0)$, there exists a nonzero homogeneous polynomial $P_d$ of degree $0 \leq d \leq N_0$ such that

$$u(x) = P_d(x - y_0) + \phi(x), \quad x \in B_{1/4}(y_0),$$

(11)

where $P_d$ and $\phi$ satisfy the properties (7)-(10).

From (11), we have the decomposition of $u^2$,

$$u^2(x) = P_{2d}(x - y_0) + \hat{\phi}(x),$$

(12)

where $\hat{\phi}(x) = 2P_d(x - y_0)\phi(x) + \phi^2(x)$ and $P_{2d}(x - y_0) = P_d^2(x - y_0)$. Then $P_{2d}(x - y_0)$ is a nonzero homogenous polynomial of degree $2d$. Note however, the equation (7) may not be satisfied by $u^2$, but it does not effect the compactness of set formed by those $P_{2N_0}(x - y_0)$’s.

Given two positive numbers $\varepsilon$ and $\theta$, let us introduce some notations first:

$$K_{y_0}(\varepsilon, \theta) = \{ x \in \mathbb{R}^n : u^2 + \varepsilon^2 |x - y_0|^2 \leq \theta^2 \},$$

$$W_{y_0}(\varepsilon, \theta) = \{ x \in \mathbb{R}^n : P_{2d}(x - y_0) + \varepsilon^2 |x - y_0|^2 \leq \theta^2 \},$$

$$\partial K_{y_0}(\varepsilon, \theta) = \{ x \in \mathbb{R}^n : u^2 + \varepsilon^2 |x - y_0|^2 = \theta^2 \},$$

$$\partial W_{y_0}(\varepsilon, \theta) = \{ x \in \mathbb{R}^n : P_{2d} + \varepsilon^2 |x - y_0|^2 = \theta^2 \}.$$

Since $K_{y_0}(\varepsilon, \theta)$ is contained in $B_{\theta}(y_0)$, $K_{y_0}(\varepsilon, \theta)$ is a compact set. The hypersurface $\partial K_{y_0}(\varepsilon, \theta)$ is nonsingular if and only if $\theta^2$ is a regular value of the function $u^2 + \varepsilon^2 |x - y_0|^2$. By Sard’s theorem, for any fixed $\varepsilon$, almost all the values of $\theta$ are regular. Similarly, applying Sard’s theorem to the function $P_{2d} + \varepsilon^2 |x - y_0|^2$, we have that almost all the values of $\theta$ are regular as well. Henceforth, we can always assume that $\varepsilon$ and $\theta$ are chosen so that both the hypersurface $\partial K_{y_0}(\varepsilon, \theta)$ and $\partial W_{y_0}(\varepsilon, \theta)$ are nonsingular. Moreover we may also assume that they are contained in $B_{1/2}(0)$.

Let $\vec{n}_W(x)$ and $\vec{n}_K(x)$ be the exterior unit normal vector fields of the hypersurfaces $\partial K_{y_0}(\varepsilon, \theta)$ and $\partial W_{y_0}(\varepsilon, \theta)$ respectively. Sard’s theorem ensures that the
sets
\[ \{ \bar{n}_W(x) : \nabla \bar{n}_W(x) = 0 \} \text{ and } \{ \bar{n}_K(x) : \nabla \bar{n}_K(x) = 0 \} \]
both have measure zero in \( \mathbb{S}^{n-1} \).

Therefore, up to some coordinate rotation, we can choose directions \((0, \cdots, 0, 1) \in \mathbb{R}^n, \) such that \((0, \cdots, 0, 1) \) are neither the critical values of \( \bar{n}_W(x) \) nor the critical values of \( \bar{n}_K(x) \).

Let us denote the height function \((x_1, \cdots, x_n) \to x_n \) of the hypersurfaces \( \partial K_{y_0}(\varepsilon, \theta) \) and \( \partial W_{y_0}(\varepsilon, \theta) \) by \( h_W \) and \( h_K \) respectively. From Morse theory (for details see the proof of [24, Theorem 1]), we have that both \( h_W \) and \( h_K \) have no degenerate critical points.

Note that the critical point of \( h_W \) and \( h_K \) can be characterized as follows:

\[
F(x - y_0) = \begin{pmatrix}
\frac{\partial P_2(x-y_0)}{\partial x_1} + \phi(x) + 2\varepsilon x_1 (x_1 - y_1) \\
\vdots \\
\frac{\partial P_2(x-y_0)}{\partial x_{n-1}} + 2\varepsilon x_{n-1} (x_{n-1} - y_{n-1}) \\
\end{pmatrix} = 0,
\]

\[
G(x - y_0) = \begin{pmatrix}
\frac{\partial P_2(x-y_0)}{\partial x_1} + 2\varepsilon x_1 (x_1 - y_1) \\
\vdots \\
\frac{\partial P_2(x-y_0)}{\partial x_{n-1}} + 2\varepsilon x_{n-1} (x_{n-1} - y_{n-1}) \\
\end{pmatrix} = 0.
\]

Applying Lemma 3.5 to the polynomial \( G(x - y_0) \), we obtain

\[ \mathcal{H}^0 \{ G^{-1}(0) \cap B_R(y_0) \} \leq C(d), \text{ for any } R \in (0, 1). \quad (13) \]

We introduce the \( C^{M+1} \)-norm weighted with the radius \( R \), denoted by

\[ \|w\|_{C^{M+1}(B_R(\alpha))} = \sum_{i=0}^{M+1} R^i \sup_{x \in B_R(\alpha)} |D^i w(x)|. \]

Using the inequalities (9) and (10), we can choose small \( R = R(y_0) < \frac{1}{8} \) such that

\[ \left\| \frac{1}{R^{2d-1}} (F(x - y_0) - G(x - y_0)) \right\|_{C^{M_0+1}(B_R(\alpha))} \]

\[ \leq \frac{1}{R^{2d-1}} \left( C R^{2d-1+\alpha} + R C R^{2d-2+\alpha} + \cdots + R^{2d-1} C R^{\alpha} + R^{2d} C + \cdots + R^{M_0+1} C \right) \]

\[ \leq C R^\alpha + C R + C R^2 + \cdots + C R^{M_0-2d+2} < \delta_0, \]

where \( \delta_0 \) is the positive constant obtained in Lemma 2.5.

By the transformation \( x \to y_0 + R_x \), we compute

\[ \left\| \frac{1}{R^{2d-1}} F(Rx) - \begin{pmatrix}
\frac{\partial P_2(x)}{\partial x_1} + 2\varepsilon x_1 \frac{R^{2d-2}}{R^{2d-2}} \\
\vdots \\
\frac{\partial P_2(x)}{\partial x_{n-1}} + 2\varepsilon x_{n-1} \frac{R^{2d-2}}{R^{2d-2}} \\
\end{pmatrix} \right\|_{C^{M_0+1}(B_1(\alpha))} \]

\[ = \left\| \frac{1}{R^{2d-1}} (F(x - y_0) - G(x - y_0)) \right\|_{C^{M_0+1}(B_R(\alpha))} \]

\[ < \delta_0. \]
Note that (13) is equivalent to
\[
\mathcal{H}^0 \left\{ \left( \frac{\partial P_{2d}(x)}{\partial x_1} + 2\varepsilon^2 \frac{x_1}{R^{2d-2}} \right)^{-1} \cdots \left( \frac{\partial P_{2d}(x)}{\partial x_{n-1}} + 2\varepsilon^2 \frac{x_{n-1}}{R^{2d-2}} \right)^{-1} \right\} (0) \cap B_1(0) \triangleq l(y_0) \leq C(N_0).
\]

Therefore, we take the polynomial \( g = RP_{2d}(x) + \frac{\varepsilon^2 |x|^2}{R^{2d-3}} - \frac{\theta^2}{R^{2d-7}} \) in Lemma 2.5, then there exists a positive constant \( r_0 \) such that
\[
\mathcal{H}^0 \left\{ \left[ \frac{1}{R^{2d-1}} F(Rx) \right]^{-1} (0) \cap B_{r_0}(0) \right\} \leq C_0.
\]

After transforming back to \( B_R(y_0) \), one can get
\[
\mathcal{H}^0 \left\{ [F(x-y_0)]^{-1} (0) \cap B_{r_0R}(y_0) \right\} \leq C_0.
\]

Therefore, for any \( r < r_0R \),
\[
\mathcal{H}^0 \left\{ [F(x-y_0)]^{-1} (0) \cap B_r(y_0) \right\} \leq C_0.
\]

Here we need to remark that the constant \( C_0 \) in the proof above also depends on \( n, K, \lambda, \Lambda, N_0 \) and \( \alpha, M_0 = 2(N_0)^2 \) which come from the assumptions of the coefficients. For simplicity, we omit these constants in the notation \( C_0 \).

Recalling that the critical points of the height function \( h_K \) can be characterized by the zeros of \( F(x-y_0) \). Therefore, from (3) we can see that the height function \( h_K \), i.e. the Morse function of the hypersurface \( \partial K_{y_0}(\varepsilon, \theta) \), only has finite critical points.

By Morse theory, the \( q \)th Betti number of \( \partial K_{y_0}(\varepsilon, \theta) \) means the rank of the \( \check{\text{C}} \)ech cohomology group \( H^q(\partial K_{y_0}(\varepsilon, \theta)) \). Let
\[
H^*(\partial K_{y_0}(\varepsilon, \theta)) = \bigoplus_q H^q(\partial K_{y_0}(\varepsilon, \theta)),
\]
then the total Betti numbers of \( \partial K_{y_0}(\varepsilon, \theta) \) is equals to rank \( H^*(\partial K_{y_0}(\varepsilon, \theta)) \).

The weak Morse inequalities imply
\[
\text{rank } H^*(\partial K_{y_0}(\varepsilon, \theta)) \leq \# \{ \text{all the critical points of } h_K \} \leq \mathcal{H}^0 \left\{ [F(x-y_0)]^{-1} (0) \cap B_r(y_0) \right\} \leq C_0.
\]

On the other hand, the Alexander duality theorem gives
\[
\text{rank } H^*(K_{y_0}(\varepsilon, \theta)) = \frac{1}{2} \text{rank } H^*(\partial K_{y_0}(\varepsilon, \theta)).
\]

Hence,
\[
\text{rank } H^*(K_{y_0}(\varepsilon, \theta)) \leq C_0.
\]

Now we can choose suitable sequences of regular values of \( \{ \varepsilon_i \} \) and \( \{ \theta_i \} \), such that \( \{ \varepsilon_i \} \) decreases monotonically with the limit zero and \( \{ \theta_i/\varepsilon_i \} \) decreases monotonically with limit \( r(y_0) \). Meanwhile each \( \partial K_{y_0}(\varepsilon_i, \theta_i) \) is a nonsingular hypersurface.

It is easy to see that
\[
K(\varepsilon_1, \theta_1) \supset K(\varepsilon_2, \theta_2) \supset \cdots \supset K(\varepsilon_n, \theta_n) \supset \cdots,
\]
and
\[ \mathcal{N}(u) \cap B_{r(y_0)}(y_0) = \bigcap_{i=1}^{+\infty} K(\varepsilon_i, \theta_i). \]

Therefore,
\[ \text{rank } H^*(\mathcal{N}(u) \cap B_{r(y_0)}(y_0)) \leq \limsup \text{rank } H^*(K_{y_0}(\varepsilon_i, \theta_i)) \leq C_0. \]

It is obvious that the collection of \{B_{r(y)}(y) : y \in \mathcal{N}(u) \cap B_{1/2}(0)\} covers \mathcal{N}(u) \cap B_{1/2}(0). By the compactness of \mathcal{N}(u) \cap B_{1/2}(0), there exist finite balls \{B_{r_j}(y_j) : j = 1, \cdots, \nu\} such that
\[ \mathcal{N}(u) \cap B_{1/2}(0) \subset \bigcup_{j=1}^{\nu} B_{r_j}(y_j), \]

and for each \( j \),
\[ \text{rank } H^*(\mathcal{N}(u) \cap B_{r_j}(y_j)) \leq C_0. \]

Since the total number \( \nu \) can be bounded in terms of \( N_0, K, \lambda, \Lambda \) and \( n \), therefore, the total Betti numbers of \( \mathcal{N}(u) \cap B_{1/2}(0) \leq C(N_0, K, \lambda, \Lambda, n). \)

This completes the proof of the main theorem.

\[ \square \]

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