TOP DEGREE OF JACK CHARACTERS
AND ENUMERATION OF MAPS
WITH APPENDIX BY VALENTIN FÉRAY

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ABSTRACT. Jack characters are (suitably normalized) coefficients in the expansion of Jack symmetric functions in the basis of power-sum symmetric functions. These quantities have been introduced recently by Lassalle who formulated some challenging conjectures about them. We give explicit formulas for the top-degree homogeneous part of these Jack characters in terms of bicolored oriented maps with an arbitrary face structure. We also give an abstract characterization of Jack characters which does not involve Jack polynomials.

0. PROLOGUE

We study enumerative problems related to Jack characters $Ch_\pi$, which form a natural family (indexed by partition $\pi$) of functions on the set $\mathcal{Y}$ of Young diagrams. Jack characters can be viewed as a natural deformation of the classical normalized characters of the symmetric groups; a deformation that is associated with Jack symmetric functions. The structure of Jack characters remains mysterious; our ultimate goal (which is outside of our reach) would be to find some closed formulas for them. In order to motivate the Reader and to give her some flavor of the results to expect, we shall present now some selected highlights before getting involved in somewhat lengthy definitions in the Introduction. We also postpone the bibliographic details.

0.1. Kerov–Lassalle positivity conjecture. Each Jack character $Ch_\pi$ can be uniquely expressed as a polynomial — called Kerov–Lassalle polynomial — in the celebrated free cumulants $R_2, R_3, \ldots$ (which are convenient, explicit functions on the set $\mathcal{Y}$ of Young diagrams). We will concentrate our attention on the simplest case of the Jack characters $Ch_n$ corresponding to
the partitions $\pi = (n)$ with a single part. For example:

\begin{align}
(0.1) & \quad \text{Ch}_1 = \underbrace{R_2}_{\text{Ch}_1^{\text{top}}}, \\
(0.2) & \quad \text{Ch}_2 = \underbrace{R_3 + R_2 \gamma}_{\text{Ch}_2^{\text{top}}}, \\
(0.3) & \quad \text{Ch}_3 = \underbrace{R_4 + 3R_3 \gamma + 2R_2 \gamma^2 + R_2}_{\text{Ch}_3^{\text{top}}}, \\
(0.4) & \quad \text{Ch}_4 = \underbrace{R_5 + 6R_4 \gamma + R_3^2 \gamma + 11R_3 \gamma^2 + 6R_2 \gamma^3 + 5R_3 + 7R_2 \gamma}_{\text{Ch}_4^{\text{top}}}.
\end{align}

Above and in the following $\gamma$ denotes the deformation parameter on which the Jack characters depend implicitly.

Vast amount of numerical examples (such as the ones above) suggest that the coefficients of these polynomials are non-negative integers. This phenomenon — referred to as Kerov–Lassalle positivity conjecture — might be seen as an indication of some hypothetical, mysterious, unexpected combinatorial structure behind Jack characters. The conjecture in its full generality remains open.

0.2. **Top-degree of Jack characters.** The main result of the current paper is a number of explicit descriptions of the top-degree part $\text{Ch}_n^{\text{top}}$ of the Jack character $\text{Ch}_n$ with respect to a certain natural gradation. These descriptions are given in terms of various convenient parametrizations of the set of functions on the set $\mathcal{Y}$ of Young diagrams.

In the case of the parametrization given by the free cumulants considered above, the degrees of the generators are given by:

$$
\text{deg} R_n = n \quad \text{for each } n \geq 2; \\
\text{deg} \gamma = 1.
$$

According to this gradation, the top-degree part $\text{Ch}_n^{\text{top}}$ of the Jack character $\text{Ch}_n$ has been indicated in (0.1)–(0.4) by the curly brackets.

The first main result of the current paper is the following partial result supporting Kerov–Lassalle positivity conjecture.

**Theorem 0.1.** For each $n \geq 1$, the coefficients of Kerov–Lassalle polynomial for the top-degree part $\text{Ch}_n^{\text{top}}$ of Jack character are non-negative integers.

We will also provide an explicit combinatorial interpretation for these coefficients; it turns out that they are equal to the number of certain *oriented*
maps with a fixed number of edges and an arbitrary face structure. We will present the details below. Theorem 0.1 would follow from one of these more explicit results (to be more specific, from Corollary 0.4).

0.3. Kerov–Lassalle polynomials for the top-degree of Jack characters. We denote by $\mathcal{S}(n)$ the symmetric group, consisting of the permutations of the set $[n] := \{1, \ldots, n\}$. For a permutation $\pi \in \mathcal{S}(n)$ we denote by $C(\pi)$ the set of its cycles.

Let $\sigma_1, \sigma_2 \in \mathcal{S}(n)$ be some fixed permutations. We say that “$\langle \sigma_1, \sigma_2 \rangle$ is transitive” if the group generated by $\sigma_1$ and $\sigma_2$ acts transitively on the underlying set $[n]$. Let $q: C(\sigma_2) \to \{2, 3, \ldots\}$ be a function on the set $C(\sigma_2)$ of cycles of the permutation $\sigma_2$; since the definition of the property “$(\sigma_1, \sigma_2, q)$ is an expander” is a bit technical, we postpone it until Section 9.1.

An example of an enumerative result proved in the current paper is provided by the following theorem.

**Theorem 0.2** (Kerov–Lassalle polynomial for $\text{Ch}_n^{\text{top}}$). For each $n \geq 1$

\[(0.5) \quad \text{Ch}_n^{\text{top}}(\lambda) = \frac{1}{(n-1)!} \sum_{\sigma_1, \sigma_2 \in \mathcal{S}(n)} \gamma^{n+1-|C(\sigma_1)|-|C(\sigma_2)|} \sum_{\langle \sigma_1, \sigma_2 \rangle \text{ is transitive}} \prod_{c \in C(\sigma_2)} \mathcal{R}_{q(c)}.\]

The proof is postponed to Section 9.3. In the following we will explore this formula and its consequences.

0.4. Labeled maps. Recall that a map [LZ04] is a graph $G$ (possibly, with multiple edges) drawn on a surface $\Sigma$. We denote the vertex set by $\mathcal{V}$ and the edge set by $\mathcal{E}$. As usual, we assume that $\Sigma \setminus \mathcal{E}$ is homeomorphic to a collection of open discs.

The first sum in (0.5) is taken over the set

\[(0.6) \quad \mathcal{X}_n := \{(\sigma_1, \sigma_2) \in \mathcal{S}(n) \times \mathcal{S}(n) : \langle \sigma_1, \sigma_2 \rangle \text{ is transitive}\}.

To any pair $(\sigma_1, \sigma_2) \in \mathcal{X}_n$ in this set we can canonically associate a map $M$ which is:

- **labeled, with $n$ edges**, i.e., each edge carries some label from the set $[n]$ and each label is used exactly once;
- **bicolored**, i.e., the set of vertices $\mathcal{V} = \mathcal{V}(M)$ is decomposed $\mathcal{V} = \mathcal{V}_o \sqcup \mathcal{V}_w$ into the set $\mathcal{V}_o = \mathcal{V}_o(M)$ of white vertices and the set $\mathcal{V}_w = \mathcal{V}_w(M)$ of black vertices.
$\Upsilon_\bullet(M)$ of black vertices; each edge connects two vertices with the opposite colors;

- **connected**, i.e., the graph $G$ is connected or, equivalently, the surface $\Sigma$ is connected;
- **oriented**, i.e., the surface $\Sigma$ is orientable and has some fixed orientation.

This correspondence follows from the observation that the structure of such a map is uniquely determined by the counterclockwise cyclic order of the edges around the white vertices (which we declare to be encoded by the disjoint cycle decomposition of the permutation $\sigma_1$) and by the counterclockwise cyclic order of the edges around the black vertices (which we declare to be encoded by the disjoint cycle decomposition of the permutation $\sigma_2$).

**Example 0.3.** The map shown in Figure 1 corresponds to the pair

$$
\sigma_1 = (1, 4, 9, 5, 7)(2, 6)(3, 8), \quad \sigma_2 = (1, 9)(2, 3, 5)(4, 7)(6, 8).
$$

Due to this correspondence the first sum in (0.5) can be viewed as a summation over labeled, oriented, connected maps.

**0.5. A formula with integer coefficients.** Due to the division by $(n-1)!$ in (0.5), the Reader may suspect that the coefficients in this formula might be non-integer rational numbers. In fact, this is not the case, as we shall see from the following discussion.

On the set $\mathcal{X}_n$ defined in (0.6) we consider the following equivalence relation:

$$
(\sigma_1, \sigma_2) \sim (\sigma_1', \sigma_2') \iff \exists \pi \in S(n), \pi(n) = n \quad \sigma_i' = \pi \sigma_i \pi^{-1} \text{ for each } i \in \{1, 2\}.
$$

The equivalence classes are nothing else but the orbits of the obvious action of the group

$$
\mathcal{G}(n-1) := \{ \pi \in S(n) : \pi(n) = n \}
$$

on $\mathcal{X}_n$ by coordinate-wise conjugation.

Let $\pi \in \text{Stab}(\sigma_1, \sigma_2) \subseteq \mathcal{G}(n-1)$ belong to the stabilizer of some $(\sigma_1, \sigma_2) \in \mathcal{X}_n$ with respect to the above action of $\mathcal{G}(n-1)$; in other words

$$
\sigma_i = \pi \sigma_i \pi^{-1} \text{ for each } i \in \{1, 2\}.
$$

The set of fixpoints of $\pi$ is non-empty (it contains, for example, $n$). Furthermore, if $x \in [n]$ is a fixpoint of $\pi$, then (0.8) implies that $\sigma_i(x)$ is also a fixpoint. As $(\sigma_1, \sigma_2)$ is transitive, it follows that all elements of $[n]$ are fixpoints, thus $\pi = \text{id}$. In this way we proved that $\text{Stab}(\sigma_1, \sigma_2) = \{\text{id}\}$, thus each equivalence class consists of exactly $\frac{|\mathcal{G}(n-1)|}{|\text{Stab}(\sigma_1, \sigma_2)|} = (n-1)!$ elements.
Figure 1. Example of a labeled map drawn on the torus. The left side of the square should be glued to the right side, as well as bottom to top, as indicated by the arrows.

Figure 2. The unlabeled map corresponding to the labeled map from Figure 1. The root edge is marked by the dashed line.

In this way we have almost shown the following formula which involves only integer numbers and no division (almost, because it remains to check that the expression marked (♣) in (0.5) is invariant under the above action of $\mathfrak{S}(n - 1)$; the Reader is advised to revisit this proof after reading the definition of the expanders in Section 9.1).
**Corollary 0.4** (Formula for Kerov–Lassalle polynomial with integer coefficients). For each \( n \geq 1 \)

\[
\text{Ch}_n^{\text{top}}(\lambda) = \sum_{[(\sigma_1, \sigma_2)] \in \mathcal{X}_n/\sim} \gamma^{n+1-|C(\sigma_1)|-|C(\sigma_2)|} \sum_{q: C(\sigma_2) \rightarrow \{2, 3, \ldots \}} \prod_{c \in C(\sigma_2)} R_q(c),
\]

where the sum runs over representatives of the equivalence classes.

Note that this result implies immediately Theorem 0.1 which was announced before.

**0.6. Unlabeled maps.** Informally speaking, an unlabeled, rooted, oriented map with \( n \) edges is a labeled oriented map, from which all labels have been removed, except for a single edge. This special edge is called the root edge. For an example, see Figure 2.

This concept can be formalized as follows: on the set of labeled oriented maps with \( n \) edges we consider the action of the symmetric group \( \mathbb{S}(n-1) \) (which is viewed as in (0.7)) by the permutation of the labels of the edges. An unlabeled map is defined as an orbit of this action. The root edge is defined as the edge with the label \( n \), which is invariant under the action of \( \mathbb{S}(n-1) \). Such unlabeled maps are in a bijective correspondence with the equivalence classes in \( \mathcal{X}_n/\sim \).

With these notations, Corollary 0.4 has the following geometric interpretation.

**Corollary 0.5.** For each \( n \geq 1 \)

\[
\text{Ch}_n^{\text{top}}(\lambda) = \sum_M \gamma^{n+1-|V|} \sum_{q: V \rightarrow \{2, 3, \ldots \}} \prod_{v \in V_{\bullet}} R_q(c),
\]

where the sum runs over rooted, oriented, bicolored, connected maps \( M \) with \( n \) unlabeled edges and \( V = V(M) \) (respectively, \( V_{\circ} \) and \( V_{\bullet} \)) denotes the set of its vertices (respectively, white vertices and black vertices).

A surprising feature of the above formula is that — contrary to several previously known formulas for the normalized characters of the symmetric groups and for some special cases of Jack characters (see [DFS14, Section 1.7] for an overview) — it involves no restrictions on the face-structure of the map \( M \). We will come back to this issue in Section 1.16.
1. Introduction

The goal of this Introduction is to present the remaining main results of the paper which have not been discussed in Prologue, namely Theorem 1.7 and Theorem 1.21. We will present the bare minimum of the definitions and of the auxiliary results necessary in order to state these results. A minor drawback of this minimalistic approach is that a certain definition (Inconvenient Definition 1.5) although compact, will turn out to be not very useful for some proofs and will have to be reformulated later in an equivalent, less compact but more convenient way.

We shall also sketch the background and present the bibliographical details, both of which were deliberately not given in Prologue.

1.1. Jack polynomials. Jack polynomials \((J^{(\alpha)}_\pi)\) \cite{Jac71} are a family (indexed by an integer partition \(\pi\)) of symmetric functions which depend on an additional parameter \(\alpha\). During the last forty years, many connections of Jack polynomials with various fields of mathematics and physics were established: it turned out that the combinatorial structure of Jack polynomials plays a crucial role in understanding Ewens random permutations model \cite{DH92}, generalized \(\beta\)-ensembles and some statistical mechanics models \cite{OO97, BF97, DE02}, Selberg-type integrals \cite{Kan93, Kad97}, certain random partition models \cite{Ker00b, BO05, Mat08, DF16}, and some problems of the algebraic geometry \cite{Nak96, Oko03}, among many others. Over these years, the beautiful and complicated structure of Jack polynomials became an important source as well as an impenetrable apex for many research investigations that led to some fascinating, still unresolved problems such as understanding the structure of Littlewood–Richardson coefficients for Jack polynomials \cite{Sta89} or \(\beta\)-conjecture and relation to combinatorics of maps \cite{GJ96}. Better understanding of Jack polynomials is also very desirable in the context of generalized \(\beta\)-ensembles and their discrete counterpart model \cite{DF16}. Last, but not least, Jack polynomials are a special case of the celebrated Macdonald polynomials which “have found applications in special function theory, representation theory, algebraic geometry, group theory, statistics and quantum mechanics” \cite{GR05}. Indeed, some surprising features of Jack polynomials \cite{Sta89} have led in the past to the discovery of Macdonald polynomials \cite{Mac95} and Jack polynomials have been regarded as a relatively easy case \cite{LV95} which later allowed understanding of the more difficult case of Macdonald polynomials \cite{LV97}.

For some special values of the parameter \(\alpha\), Jack polynomials coincide with some established families of symmetric functions; namely, up to multiplicative constants, for \(\alpha = 1\) Jack polynomials coincide with the Schur polynomials, for \(\alpha = 2\) with the zonal polynomials, for \(\alpha = 1/2\) with the
symplectic zonal polynomials, for $\alpha = 0$ with the elementary symmetric functions, and in some sense for $\alpha = \infty$ with the monomial symmetric functions. Over the time it has been shown that several results concerning Schur and zonal polynomials can be generalized in a rather natural way to Jack polynomials ([Mac95, Section (VI.10)] gives a few results of this kind), therefore Jack polynomials can be viewed as a natural interpolation between several interesting families of symmetric functions.

1.2. Jack characters — the first definition. Firstly, as there are several of them, we have to fix a normalization of Jack polynomials. In our context it is most convenient to use the functions denoted by $J$ in the book of Macdonald [Mac95, Section VI, Eq. (10.22)].

We expand the Jack polynomial $J_\pi^{(\alpha)}$ in the power-sum symmetric function basis:

$$J_\pi^{(\alpha)} = \sum_\pi \theta_\pi^{(\alpha)}(\lambda) p_\pi.$$  

The coefficient $\theta_\pi^{(\alpha)}(\lambda)$ in this expansion will be called unnormalized Jack character. One of the motivations for studying such quantities comes from the fact that for a special choice of the deformation parameter $\alpha = 1$, Jack characters coincide (up to simple multiplicative constants) with the characters of the symmetric groups (which will be discussed in detail in Section 1.5), while for generic values of $\alpha$ they (conjecturally) fulfill several algebraic–combinatorial properties with a representation–theoretic flavor. We will postpone presentation of some more concrete motivations until Section 1.3.

The usual way of viewing the characters of the symmetric groups is to fix the representation $\lambda$ and to consider the character as a function of the conjugacy class $\pi$. However, there is also another very successful viewpoint due to Kerov and Olshanski [KO94], called dual approach, which suggests to do roughly the opposite. We will mention only one of its success stories, namely Kerov’s Central Limit Theorem and its generalizations [Ker93c, IO02, Sni06]. Lassalle [Las08, Las09a] adapted this dual approach to the framework of Jack characters.

In order for the dual approach to be successful (both with respect to the usual characters of the symmetric groups and for the Jack characters) one has to choose the most convenient normalization constants. In the current paper we will use the normalization introduced by Dolega and Féray [DF16] which offers some advantages over the original normalization of Lassalle. Thus, with the right choice of the multiplicative constant, the unnormalized Jack character $\theta_\pi^{(\alpha)}(\pi)$ becomes the normalized Jack character $\text{Ch}_\pi^{(\alpha)}(\lambda)$, defined as follows.
Definition 1.1. Let $\alpha > 0$ be given and let $\pi$ be a partition. The normalized Jack character $\text{Ch}_\pi(\lambda)$ is given by:

\[
\text{Ch}_\pi(\lambda) := \begin{cases} 
\alpha^{\frac{|\pi|-\ell(\pi)}{2}} \left( \frac{|\lambda|-|\pi|+m_1(\pi)}{m_1(\pi)} \right) z_\pi \theta_{\pi,1|\lambda|-|\pi|}^{(\alpha)}(\lambda) & \text{if } |\lambda| \geq |\pi|; \\
0 & \text{if } |\lambda| < |\pi|,
\end{cases}
\]

where $|\pi|$ denotes the size of the partition $\pi$, where $\ell(\pi)$ denotes its length while $m_i(\pi)$ its number of parts of equal to $i$; furthermore

\[
z_\pi = \prod_i i^{m_i(\pi)} m_i(\pi)!
\]

is the standard numerical factor. Each Jack character depends on the deformation parameter $\alpha$; in order to keep the notation light we make this dependence implicit.

Regretfully, the above definition is quite technical and ‘low-level’ (as in ‘low-level programming language’). Fortunately, there are two bright points.

- Firstly, for the purposes of the current paper this definition is irrelevant and can be completely skipped by the Reader. The reason is that our proofs will be based on some ‘high-level’ (as in ‘high-level programming language’) results of Dolega and Féray [DF16].
- Secondly, one of the main results of this paper is an equivalent but more comprehensible, ‘high-level’, abstract characterization of Jack characters which will be given in Theorem 1.7 and Theorem A.2. For this reason the Reader who wants to gain a heuristic, intuitional meaning of Jack characters is advised to take Theorem 1.7 and Theorem A.2 as a definition of Jack characters; then the first main result of the current paper can be viewed as an equivalence of this more intuitive definition and the one from the literature.

Apart from the results of Dolega and Féray [DF16] we will use the following vanishing property.

Property 1.2 (The vanishing property). If $\pi$ and $\lambda$ are partitions such that $|\lambda| < |\pi|$ then

\[
\text{Ch}_\pi(\lambda) = 0.
\]

Proof. It is a direct consequence of the definition (1.1) of Jack characters.

1.3. Motivations for Jack characters. Our main motivation for studying Jack characters comes from the fact that they encode the information about Jack polynomials, thus a better understanding of the former might shed some light on the latter.
Furthermore, the results of computer calculations (such as the ones shown in (0.1)–(0.4)) support some conjectures (such as Kerov–Lassalle positivity conjecture) that there might be some particularly nice combinatorial structures behind Jack characters. Such combinatorial structures would make Jack characters more convenient for investigation than Jack polynomials themselves. In fact, from the viewpoint of an algebraic combinatorialist, the mysterious results of such computer calculations are a self-standing motivation for the investigation of the topic.

Yet another motivation comes from the fact that Jack characters form a linear basis of an important algebra of \(\alpha\)-polynomial functions, which will be discussed later on.

1.4. The deformation parameters. Laurent polynomials. In order to avoid dealing with the square root of the variable \(\alpha\), we introduce an indeterminate \(A\) such that

\[ A^2 = \alpha. \]

Several quantities in this paper will be viewed as elements of \(\mathbb{Q}[A, A^{-1}]\), i.e., as Laurent polynomials in the variable \(A\).

**Definition 1.3.** For an integer \(d\) we will say that a Laurent polynomial

\[ f = \sum_{k \in \mathbb{Z}} f_k A^k \in \mathbb{Q}[A, A^{-1}] \]

is of degree at most \(d\) if \(f_k = 0\) holds for each integer \(k > d\).

Several quantities and functions which we will introduce in the following depend on the value of \(A\); in order to keep the notation light we will make this dependence implicit.

1.5. Normalized characters of the symmetric groups. In the special case of \(A = \alpha = 1\) the Jack polynomials coincide (up to a simple multiplicative constant) with Schur polynomials. Using this fact one can show that in this special case the Jack character \(\text{Ch}_{A=1}\) coincides with the (suitably normalized) character of the symmetric group:

\[
\text{Ch}_{A=1}(\lambda) = \begin{cases} 
\left|\lambda\right| \cdot \left(|\lambda| - 1\right) \cdots \left(|\lambda| - |\pi| + 1\right) \frac{\text{Tr} \rho^\lambda(\pi, 1^{\left|\lambda\right|-|\pi|})}{\text{Tr} \rho^\lambda(1)} & \text{if } |\lambda| \geq |\pi|, \\
0 & \text{if } |\lambda| < |\pi|,
\end{cases}
\]

where \(\rho^\lambda\) denotes the irreducible representation of the symmetric group \(\mathfrak{S}(\left|\lambda\right|)\) and where \(\text{Tr} \rho^\lambda(\pi, 1^{\left|\lambda\right|-|\pi|})\) denotes the character of this representation evaluated on an arbitrary permutation with the cycle structure given
by the partition \( \pi \), augmented by an appropriate number of additional fix-points. For the details of the proof see the work of Lassalle [Las09a] (who used a different normalization) as well as the work of Dolega and Féray [DF16].

Such normalized characters of the symmetric groups \( \text{Ch}^{A_{=1}}_\pi \) have a much longer history than Jack characters. They have been introduced by Kerov and Olshanski [KO94] and they turned out to be the key tool for studying problems of the asymptotic representation theory of the symmetric groups. Also their structure is understood much better, especially in the context of Kerov polynomials [GR07, DF´S10, PST11, DS12].

Thus, Jack characters can be viewed as a challenging generalization of the usual normalized characters of the symmetric groups, which serves as another motivation for studying this subject.

1.6. \( \alpha \)-content. The set of Young diagrams will be denoted by \( \mathcal{Y} \). For drawing Young diagrams we use the French convention and the usual Cartesian coordinate system; in particular, the box \((x, y) \in \mathbb{N}^2\) is the one in the intersection of the column with the index \(x\) and the row with the index \(y\). We index the rows and the columns by the elements of the set \( \mathbb{N} = \{1, 2, \ldots \} \) of positive integers; in particular the first row as well as the first column correspond to the number 1.

Definition 1.4. For a box \( \Box = (x, y) \) of a Young diagram we define its \( \alpha \)-content by

\[
\alpha\text{-content}(\Box) = \alpha\text{-content}(x, y) := Ax - \frac{1}{A}y \in \mathbb{Q}[A, A^{-1}].
\]

This definition of the \( \alpha \)-content is the one used by Dolega and Féray [DF16] and is different from the one used by Lassalle [Las09a]. Later on we will use this definition also in the more general context when \((x, y) \in \mathbb{R}^2\) is an arbitrary point on the plane.

1.7. The algebra \( \mathcal{P} \) of \( \alpha \)-polynomial functions on the set of Young diagrams. We will present below (in Inconvenient Definition 1.5) the definition of the algebra \( \mathcal{P} \) of \( \alpha \)-polynomial functions and the first definition of the filtration on this algebra. Regretfully, the latter definition is not very convenient and is presented here only in order to state quickly one of the main results, Theorem 1.7. For this reason the Reader is advised to forget this definition of the filtration immediately after Section 1.8.

For the other goals of the current paper it will be more convenient to consider first a certain gradation on the algebra \( \mathcal{P} \); a gradation for which the corresponding filtration is the one from Inconvenient Definition 1.5. This
gradation will be defined and studied in Section 1.10 (on a certain larger algebra \( \mathcal{S} \)) as well as in Section 2.8 (on the algebra \( \mathcal{P} \) itself). The corresponding filtration will be defined and studied in Section 2.9, where we shall also prove that it is equal to the one from Inconvenient Definition 1.5.

**Inconvenient Definition 1.5.** For an integer \( d \geq 0 \) we say that
\[
F: \mathcal{Y} \to \mathbb{Q}[A, A^{-1}]
\]
is an \( \alpha \)-polynomial function of degree at most \( d \) if there exists a sequence \( p_0, p_1, \ldots \) of polynomials which satisfies the following properties:

- for each \( k \geq 0 \) we have that \( p_k \in \mathbb{Q}[\gamma, c_1, \ldots, c_k] \) is a polynomial in the variables \( \gamma, c_1, \ldots, c_k \) of degree at most \( d - 2k \);
- for each Young diagram \( \lambda \),
\[
F(\lambda) = \sum_{k \geq 0} \sum_{\square_1, \ldots, \square_k \in \lambda} p_k(\gamma, c_1, \ldots, c_k) \in \mathbb{Q}[A, A^{-1}],
\]
where the second sum runs over all tuples of boxes of the Young diagram; furthermore
\[
c_1 := \alpha\text{-content}(\square_1), \ldots, c_k := \alpha\text{-content}(\square_k)
\]
are the corresponding \( \alpha \)-contents, and the substitution
\[
\gamma := -A + \frac{1}{A}
\]
is used. Equation (1.4) should be understood as equality between Laurent polynomials in \( \mathbb{Q}[A, A^{-1}] \).

The set of \( \alpha \)-polynomial functions forms a filtered algebra which will be denoted by \( \mathcal{P} \).

Condition (1.4) appeared for the first time in the work of Corteel, Goupil and Schaeffer [CGS04] for the special case \( A = 1 \) of the normalized characters of the symmetric groups \( \text{Ch}_{\pi=1}^{A} \), see Section 1.5.

**Remark 1.6.** The algebra \( \mathcal{P} \) is closely related to another algebra which in the literature carries the same name of the algebra of \( \alpha \)-polynomial functions on the set of Young diagrams. The latter algebra — although viewed slightly differently — coincides with the algebra \( \Lambda^{(\alpha)} \) of \( \alpha \)-shifted symmetric functions [OO97].

Note however a small but important difference between \( \Lambda^{(\alpha)} \) and our algebra \( \mathcal{P} \), namely the definition of \( \Lambda^{(\alpha)} \) makes sense both when we view \( \alpha > 0 \) as a fixed number as well as when we view it as an indeterminate, while the elements of the algebra \( \mathcal{P} \) — and, more importantly, the filtration on \( \mathcal{P} \) — make sense only when we view \( A = \sqrt{\alpha} \) as an indeterminate.
1.8. The first main result: abstract characterization of Jack characters. The following theorem is one of the main results of the current paper. It gives an abstract characterization of Jack characters; a characterization that does not involve Jack polynomials. It was modeled after the characterization given by Okounkov and Olshanski \cite{OO98, Section 1.4.3} of shifted Jack polynomials, quantities which are somewhat related to Jack characters.

\textbf{Theorem 1.7} (The first main result). \textit{Let }\pi\textit{ be a partition. Then the corresponding Jack character }\text{Ch}_\pi\textit{ is the unique }\alpha\textit{-polynomial function }\mathcal{F}\textit{ which fulfills the following properties:}

\text{(K1)} \quad \mathcal{F} \in \mathcal{P} \text{ is of degree at most } |\pi| + \ell(\pi) \text{ (see Inconvenient Definition 1.5);}
\text{(K2)} \quad \text{for each } m \geq 1 \text{ the polynomial function in } m \text{ variables}
\begin{equation}
    \mathcal{Y} \ni (\lambda_1, \ldots, \lambda_m) \mapsto \mathcal{F}(\lambda_1, \ldots, \lambda_m) \in \mathbb{Q}[A, A^{-1}]
\end{equation}
\text{is of degree } |\pi| \text{ and its homogeneous top-degree part is equal to}
\begin{equation}
    A^{|\pi| - \ell(\pi)} \, p_{\pi}(\lambda_1, \ldots, \lambda_m),
\end{equation}

where
\begin{equation}
    p_{\pi}(\lambda_1, \ldots, \lambda_m) = \prod_r \sum_i \lambda_i^{\pi_r}
\end{equation}

\text{is the power-sum symmetric polynomial;}
\text{(K3)} \quad \text{for each } \lambda \in \mathcal{Y} \text{ such that } |\lambda| < |\pi| \text{ we have}
\begin{equation}
    \mathcal{F}(\lambda) = 0;
\end{equation}
\text{(K4)} \quad \text{if } \ell(\pi) \geq 2 \text{ then for each Young diagram } \lambda \in \mathcal{Y} \text{ the evaluation}
\begin{equation}
    \mathcal{F}(\lambda) \in \mathbb{Q}[A, A^{-1}]
\end{equation}
\text{is a Laurent polynomial of degree at most } |\pi| - \ell(\pi) \text{ (see Definition 1.3).}

The sense in which the polynomial function (1.6) exists will be discussed later in Lemma 3.1.

\textbf{Remark 1.8.} When this paper was almost finished, Valentin Féray pointed out that this characterization of \text{Ch}_\pi\text{ can be further improved. We present his findings in Appendix A where also an improved version of Theorem 1.7 is given, see Theorem A.2. Even though an improved version of Theorem 1.7 is available we decided to keep it in the current form because it fits better our goal of finding an abstract characterization of the top-degree of Jack characters (Theorem 7.3). Our original proof of Theorem 1.7 was based on Theorem 7.2, a new proof based on Féray’s characterization will be given in Appendix A.5.}
Remark 1.9. If we view Theorem 1.7 as a definition of the Jack character $Ch_\pi$, it is equivalent to the classical definition (Definition 1.1) in the following strict sense: for any $\alpha > 0$ and any Young diagram $\lambda$ the Jack character $Ch_\pi(\lambda)$ given by Definition 1.1 coincides with the evaluation at $A := \sqrt{\alpha}$ of the Jack character $Ch_\pi(\lambda) \in \mathbb{Q} [A, A^{-1}]$ given by Theorem 1.7. Note, however, that there is an essential philosophical difference between these two definitions of the Jack character: Definition 1.1 makes perfectly sense both when we view $\alpha > 0$ as an indeterminate and when we view it as a fixed number, while Theorem 1.7 makes sense only when we view $A = \sqrt{\alpha}$ as an indeterminate. This kind of phenomenon was already discussed in Remark 1.6. On the other hand, Féray’s characterization of Jack characters from Theorem A.2 makes sense if we view $A = \sqrt{\alpha}$ as an arbitrary fixed number (except for a countable number of exceptions).

Heuristically, conditions (K1), (K2), and (K4) specify the asymptotic behavior of Jack character $Ch_\pi(\lambda)$ in the following three asymptotic regimes:

- condition (K1) in the limit as the shape of the Young diagram $\lambda$ in some sense tends to infinity, together with the deformation parameter $\gamma$ (for details see Section 1.10.3);
- condition (K2) in the limit studied by Vershik and Kerov [VK81] as the lengths $\lambda_1, \lambda_2, \ldots$ of the rows of the Young diagram $\lambda$ tend to infinity;
- condition (K4) in the limit as the deformation parameter $A$ tends to infinity.

On the other hand, condition (K3) is of quite different flavor and it specifies the behavior of Jack character on small Young diagrams.

Example 1.10. By easily checking that the assumptions of Theorem 1.7 are fulfilled, one can verify the following equalities:

\[
Ch_\emptyset(\lambda) = 1,
\]

\[
Ch_1(\lambda) = \sum_{\Box \in \lambda} 1,
\]

\[
Ch_2(\lambda) = \sum_{\Box \in \lambda} 2(c_1 + \gamma),
\]

\[
Ch_3(\lambda) = \sum_{\Box \in \lambda} \left( 3(c_1 + \gamma)(c_1 + 2\gamma) + \frac{3}{2} \right) + \sum_{\Box_1, \Box_2 \in \lambda} \left( -\frac{3}{2} \right),
\]

\[
Ch_{1,1}(\lambda) = \sum_{\Box \in \lambda} (-1) + \sum_{\Box_1, \Box_2 \in \lambda} 1,
\]

where $\gamma$ is given by (1.5).
1.9. **For minimalistic readers.** The readers with a minimalistic approach who would like to have an insight on the results of the current paper without reading too many definitions are advised to skip the following sections and to move fast forward to Section 1.14.

1.10. **The algebra \( \mathcal{S} \) and its gradation.** In the following we will present a certain graded algebra \( \mathcal{S} \) of functions on the set \( \mathcal{Y} \) of Young diagrams. As we shall see later (Corollary 2.11), this class contains the algebra \( \mathcal{P} \) of \( \alpha \)-polynomial functions.

1.10.1. **Multirectangular coordinates.** We start with anisotropic multirectangular coordinates \( P = (p_1, \ldots, p_\ell) \) and \( Q = (q_1, \ldots, q_\ell) \). They give rise to isotropic multirectangular coordinates given by

\[
P' = (p'_1, \ldots, p'_\ell) := (Ap_1, \ldots, Ap_\ell),
\]

\[
Q' = (q'_1, \ldots, q'_\ell) := \left( \frac{1}{A} q_1, \ldots, \frac{1}{A} q_\ell \right).
\]

Note that \( P' \) and \( Q' \) depend implicitly on \( P \) and \( Q \).

Suppose that \( P' = (p'_1, \ldots, p'_\ell) \) and \( Q = (q_1, \ldots, q_\ell) \) are sequences of non-negative integers such that \( q'_1 \geq \cdots \geq q'_\ell \); we consider the multirectangular Young diagram

\[
P' \times Q' = (q'_1, \ldots, q'_1, \ldots, q'_\ell, \ldots, q'_\ell)_{p'_1 \text{ times}}, (q_1, \ldots, q_\ell)_{p'_\ell \text{ times}}.
\]

This concept is illustrated in Figure 3.

**Remark 1.11.** The passage between the isotropic coordinates \( P', Q' \) and the anisotropic coordinates \( P, Q \) corresponds to the passage between the
diagram \( \lambda := P' \times Q' \) and its anisotropic deformation

\[
D_{A,\frac{1}{A}}(\lambda) := P \times Q = \left( \frac{1}{A} P' \right) \times (A Q')
\]

which is the diagram \( \lambda \) stretched horizontally by the factor \( A \) and vertically by the factor \( \frac{1}{A} \). The difficulty with this approach is that (except for some exceptional cases) it is not possible that both \( \lambda \) and \( D_{A,\frac{1}{A}}(\lambda) \) are Young diagrams and one has to use the notions of generalized Young diagrams [Ker00b] or continuous Young diagrams [Ker93a, Ker99]. In the current paper we decided to avoid dealing with such objects.

1.10.2. Stanley polynomials. Let \( St = (St_1, St_2, \ldots) \) be a sequence of polynomials such that for each \( \ell \geq 1 \)

\[
St_\ell = St_\ell(\gamma; p_1, \ldots, p_\ell; q_1, \ldots, q_\ell) = St_\ell(\gamma; P; Q)
\]

is a polynomial in \( 2\ell + 1 \) variables and

\[
St_{\ell+1}(\gamma; p_1, \ldots, p_\ell, 0; q_1, \ldots, q_\ell, 0) = St_\ell(\gamma; p_1, \ldots, p_\ell; q_1, \ldots, q_\ell).
\]

We assume furthermore that the degrees of the polynomials \( St_1, St_2, \ldots \) are uniformly bounded. In other words, \( St \) is an element of the inverse limit in the category of graded algebras; informally it can be viewed as a polynomial

\[
St = St(\gamma; p_1, p_2, \ldots; q_1, q_2, \ldots)
\]

in the deformation parameter \( \gamma \) and two infinite sequences of indeterminates: \( p_1, p_2, \ldots \) and \( q_1, q_2, \ldots \).

**Definition** 1.12. Let \( F : \mathbb{Y} \to \mathbb{Q}[A, A^{-1}] \) be a function on the set \( \mathbb{Y} \) of Young diagrams. Suppose that for each \( \ell \geq 1 \) the equality

\[
F(P' \times Q') = St_\ell(\gamma; P; Q)
\]

— with the usual substitution (1.5) for the variable \( \gamma \) — holds true for all choices of \( P, Q \) and \( A \neq 0 \) for which the multirectangular diagram \( P' \times Q' \) is well-defined. Then we say that \( St \) is the (anisotropic) Stanley polynomial for \( F \). For a given function \( F \), the corresponding Stanley polynomial, if exists, is unique (in order to show this, one can adapt the corresponding part of the proof of [DFS14, Lemma 2.4]). The set of functions \( F \) with the above property forms an algebra which will be denoted by \( \mathcal{S} \).

**Example** 1.13. Using data provided by Lassalle [Las09b] and some methods which are out of scope of the current paper, one can show that the Stanley
polynomials for \( \text{Ch}_1 \), \( \text{Ch}_2 \) and \( \text{Ch}_3 \) are given by

\[
\text{Ch}_1(P' \times Q') = \sum_i p_i q_i,
\]

\[
\text{Ch}_2(P' \times Q') = \sum_i p_i q_i [q_i - p_i + \gamma] - 2 \sum_{i < j} p_i p_j q_j.
\]

(1.9)

\[
\text{Ch}_3(P' \times Q') = \sum_i p_i q_i \left[q_i^2 - 3p_i q_i + p_i^2 + 3\gamma(q_i - p_i) + 2\gamma^2 + 1\right] + 
-3 \sum_{i < j} p_i p_j q_j [(q_i - p_i + \gamma) + (q_j - p_j + \gamma)] + 
+ \sum_{i < j < k} 6p_i p_j p_k q_k.
\]

Formally speaking, in these examples the appropriate Stanley polynomial \( \text{St} = (\text{St}_1, \text{St}_2, \ldots) \) is the sequence in which \( \text{St}_\ell \) is equal to the corresponding right-hand in which the summation over \( i \) (respectively, the double summation over \( i < j, \) etc.) is restricted to \( 1 \leq i \leq \ell \) (respectively, to \( 1 \leq i < j \leq \ell, \) etc.).

In the current paper we treat Stanley polynomials only as a technical tool. We refer the Reader interested in this topic to [Sni14] and the references therein.

### 1.1.0.3. Gradation on \( \mathcal{S} \).

**Definition 1.14.** Let \( F \in \mathcal{S} \) be a function on \( \mathcal{Y} \) for which there exists the corresponding Stanley polynomial \( \text{St} = (\text{St}_1, \text{St}_2, \ldots) \). We say that \( F \) is homogeneous of degree \( d \) if for each \( \ell \geq 1 \) the polynomial \( \text{St}_\ell = \text{St}_\ell(\gamma; p_1, \ldots, p_\ell; q_1, \ldots, q_\ell) \) in \( 2\ell + 1 \) variables is homogeneous of degree \( d \).

With this definition, \( \mathcal{S} \) becomes a graded algebra.

**Example 1.15.** The data from Example [1.13] shows that \( \text{Ch}_1 \in \mathcal{S} \) is homogeneous of degree 2, and \( \text{Ch}_2 \in \mathcal{S} \) is homogeneous of degree 3; furthermore, \( \text{Ch}_3 \in \mathcal{S} \) is of degree 4, but it is not homogeneous.

### 1.11. Top-degree part \( \text{Ch}_n^{\text{top}} \) of Jack character.

In Theorem [2.15] we will show that for each integer \( n \geq 1 \) we have that \( \text{Ch}_n \in \mathcal{S} \) and its degree is equal to \( n + 1 \). This motivates the following definition.

**Definition 1.16.** The top-degree part \( \text{Ch}_n^{\text{top}} \) of Jack character is defined as the homogeneous part of degree \( n + 1 \) of Jack character \( \text{Ch}_n \).

With the above definition, a priori we have only that \( \text{Ch}_n^{\text{top}} \in \mathcal{S} \), but later on (in Corollary [2.16]) we will show that, in fact, \( \text{Ch}_n^{\text{top}} \in \mathcal{P} \). The
The main problem which we investigate in the current paper is to find an explicit formula for this top-degree part $\text{Ch}_n^{\text{top}}$.

**Example 1.17.** Using the data from Example 1.13 one can easily check that Jack characters $\text{Ch}_1$ and $\text{Ch}_2$ are already homogeneous of degree, respectively, $2$ and $3$, thus $\text{Ch}_1^{\text{top}} = \text{Ch}_1$ and $\text{Ch}_2^{\text{top}} = \text{Ch}_2$. On the other hand, the formula (1.9) for $\text{Ch}_3$ involves only one term which is not homogeneous of degree $4$, thus

$$
\text{Ch}_3^{\text{top}}(P' \times Q') = \sum_i p_i q_i \left[ q_i^2 - 3p_i q_i + p_i^2 + 3\gamma(q_i - p_i) + 2\gamma^2 \right] +
- 3 \sum_{i<j} p_i p_j q_j \left[ (q_i - p_i + \gamma) + (q_j - p_j + \gamma) \right] +
+ \sum_{i<j<k} 6p_i p_j p_k q_k.
$$

1.12. **Number of embeddings.** Let $G$ be a bicolor graph, i.e., a bipartite graph together with the choice of the coloring of the vertices. We denote the set of its white (respectively, black) vertices by $V_\circ$ (respectively, $V_\bullet$). We will always assume that $G$ has no isolated vertices. Furthermore, let $\lambda$ be a Young diagram.

**Definition 1.18.** We say that $f = (f_1, f_2)$ is an embedding of $G$ into $\lambda$ if the functions

$$
f_1 : V_\circ \to \mathbb{N}, \quad f_2 : V_\bullet \to \mathbb{N}
$$

are such that the condition

$$(1.10) \quad (f_1(w), f_2(b)) \text{ is one of the boxes of } \lambda$$

holds true for each pair of vertices $w \in V_\circ$, $b \in V_\bullet$ connected by an edge. We denote by $N_G(\lambda)$ the number of embeddings of $G$ into $\lambda$.

The above quantity has been introduced in [FS11a] and it turned out to be a convenient tool of algebraic combinatorics, in particular in the context of Kerov polynomials [DF10].

**Definition 1.19.** We define the normalized number of embeddings which is a Laurent polynomial in $A$:

$$(1.11) \quad \mathcal{N}_G(\lambda) := A^{|V_\circ(G)|} \left( \frac{-1}{A} \right)^{|V_\bullet(G)|} N_G(\lambda) \in \mathbb{Q}[A, A^{-1}].$$

A very similar quantity denoted by $N_G^{(\alpha)}(\lambda)$ — which differs from $\mathcal{N}_G(\lambda)$ only by the choice of the sign — was considered in [DFS14].
Definition 1.20. To a pair \((\sigma_1, \sigma_2) \in \mathfrak{S}(n) \times \mathfrak{S}(n)\) of permutations one can associate a natural bicolored graph \(G(\sigma_1, \sigma_2)\) with the white vertices \(V_\circ := C(\sigma_1)\) corresponding to the cycles of \(\sigma_1\) and the black vertices \(V_\bullet := C(\sigma_2)\) corresponding to the cycles of \(\sigma_2\). A pair of vertices \(w \in C(\sigma_1), b \in C(\sigma_2)\) is connected by an edge if the corresponding cycles are not disjoint. Note that \(G(\sigma_1, \sigma_2)\) coincides (as a bicolored graph) with the map considered in Section 0.4.

We will write

\[
N_{\sigma_1,\sigma_2}(\lambda) := N_{G(\sigma_1,\sigma_2)}(\lambda),
\]

\[
\mathfrak{N}_{\sigma_1,\sigma_2}(\lambda) := \mathfrak{N}_{G(\sigma_1,\sigma_2)}(\lambda).
\]

1.13. **The second main result.** The following explicit formula for the top-degree part \(\text{Ch}_{n}^{\text{top}}\) of Jack polynomial is the second (but the more important one) main result of the paper. This formula is expressed in terms of the (normalized) numbers of embeddings; as we shall see in Section 9.3, analogous formulas in terms of other quantities (such as Theorem 0.2 which we already discussed in Prologue) are corollaries from this result.

**Theorem 1.21** (The second main result). For each \(n \geq 1\) and each Young diagram \(\lambda \in \mathbb{Y}\)

\[
\text{Ch}_{n}^{\text{top}}(\lambda) = (-1)^{n-1} \frac{1}{(n-1)!} \sum_{\substack{\sigma_1,\sigma_2 \in \mathfrak{S}(n) \\
(\sigma_1,\sigma_2) \text{ is transitive} \}} \gamma^{n+1-|C(\sigma_1)|-|C(\sigma_2)|} N_{\sigma_1,\sigma_2}(\lambda).
\]

The proof is postponed until Section 8. Note that the discussion from Sections 0.5 and 0.6 applies also to the above formula, thus it possible to convert the above formula into one which involves only integer coefficients, as well as to write it as a sum over unlabeled, rooted, oriented maps.

1.14. **Minimalistic readers, welcome back.** We welcome back the readers with a minimalistic approach who skipped the definition of the gradation on the algebra \(\mathcal{P}\) and know only the definition of the corresponding filtration (Inconvenient Definition 1.5). With this approach, without the gradation on \(\mathcal{P}\), it is not possible to define the top-degree part \(\text{Ch}_{n}^{\text{top}}\) of the Jack character and thus it is not even possible to state Theorem 1.21. Nevertheless, it is possible to state the following substitute to Theorem 1.21.

**Theorem 1.22** (The second main result, alternative formulation). For each \(n \geq 1\) the function

\[
\lambda \mapsto \text{Ch}_{n}(\lambda) + \frac{1}{(n-1)!} \sum_{\substack{\sigma_1,\sigma_2 \in \mathfrak{S}(n) \\
(\sigma_1,\sigma_2) \text{ is transitive} \}} \gamma^{n+1-|C(\sigma_1)|-|C(\sigma_2)|} N_{\sigma_1,\sigma_2}(\lambda)
\]
is an element of $\mathcal{P}$ of degree at most $n - 1$; the quantity $\mathfrak{m}_{\sigma_1, \sigma_2}$ has been defined in Section 1.12.

The proof is postponed to Section 8.3.

1.15. **How to prove the second main result?** If we managed to solve a much more general (and probably more difficult) problem than finding a formula for the top-degree $\text{Ch}^\text{top}_n$ and we guessed a closed formula for the Jack character $\text{Ch}_n$, a verification of its validity would be rather straightforward; namely it would be enough to check that this candidate expression for $\text{Ch}_n$ fulfills the requirements from Theorem 1.7. Unfortunately, this characterization of Jack characters from Theorem 1.7 is not particularly useful for verifying that a given conjectural formula for the top-degree $\text{Ch}^\text{top}_n$ is correct.

What we need is an abstract characterization of the top-degree $\text{Ch}^\text{top}_n$ of Jack characters; a characterization which would use only intrinsic properties of $\text{Ch}^\text{top}_n$ and which would not refer to the much more complicated Jack character $\text{Ch}_n$. We will provide such an abstract characterization of $\text{Ch}^\text{top}_n$ in Theorem 7.3. We shall discuss it in the following.

Our characterization of $\text{Ch}^\text{top}_n$ is quite analogous to the characterization of Jack characters from Theorem 1.7. The only challenging task was to find a proper replacement for condition (K3) about the vanishing of the characters on small Young diagrams. Since the difference

\begin{equation}
\delta_n := \text{Ch}_n - \text{Ch}^\text{top}_n
\end{equation}

between the Jack character and its top-degree part is usually non-zero, if in (1.8) we mechanically replace the Jack character $\text{Ch}_n$ by its top-degree part $\text{Ch}^\text{top}_n$, we would get a statement which is clearly false.

A solution which we present in Theorem 7.3 (condition (T3)) is to require that certain linear combinations (over $\lambda$) of the values of the top-degree $\text{Ch}^\text{top}_n(\lambda)$ vanish. These linear combinations were chosen in such a way that analogous linear combinations for $\delta_n$ vanish tautologically for any $\delta_n \in \mathcal{P}$ which is of degree smaller than the degree of $\text{Ch}_n$. We shall continue this discussion in Section 4.

1.16. **How to guess the second main result?** **Orientability generating series.** In Section 1.15 above we discussed how to prove the second main result once the closed formula for $\text{Ch}^\text{top}_n$ — such as (1.12) — is guessed correctly. Now it is the time to explain how we managed to overcome the only remaining difficulty: to make the right guess.

Our starting point was a recurrence relation discovered by Lassalle [Las08, formula (6.2)]. This recurrence relates the values of Jack characters $\text{Ch}_\pi(p \times q)$ evaluated on a fixed rectangular Young diagram $p \times q$, corresponding
to various partitions $\pi$. Together with Dołęga and Féray [DFS14] we attempted to reverse-engineer Lassalle’s recurrence relation and to guess the hypothetical combinatorial structure behind it. The next step would be to extrapolate these findings from rectangular to generic Young diagrams.

In this way we have found some heuristic arguments [DFS14] that each Jack character $Ch_{\pi} : \mathcal{Y} \to \mathbb{Q}[A, A^{-1}]$ can be viewed as a hypothetical weighted generating series of (non-oriented) maps with the face structure specified by the partition $\pi$. The weight in this series should be related to some hypothetical measure of (non-)orientability of a given map. This should not be viewed as a surprise because existence of some measures of (non-)orientability was postulated already by Goulden and Jackson [GJ96] in a different, but closely related context of the connection coefficients for Jack polynomials; an extensive bibliography on this subject can be found in [LC09]. In [DFS14] we also conjectured the specific form of this weight. In this way we defined a new quantity

$$\widehat{Ch}_{\pi} : \mathcal{Y} \to \mathbb{Q}[A, A^{-1}]$$

for which we coined the name orientability generating series with the hope that we have guessed the closed formula for Jack characters correctly or, in other words, $Ch_{\pi} = \widehat{Ch}_{\pi}$.

Regrettfully, extensive computer calculations showed that this is not the case and $Ch_{\pi} \neq \widehat{Ch}_{\pi}$ in general. Nevertheless, computer exploration indicated that the discrepancy between the Jack character $Ch_{\pi}$ and the corresponding orientability generating series $\widehat{Ch}_{\pi}$ is surprisingly small in the following sense: if each of these two quantities is expressed as a polynomial in terms of some convenient parameters describing functions on the set of Young diagrams — such as multirectangular coordinates (see Sections 1.10.1 and 1.10.2) — a lot of coefficients of these polynomials coincide. This might be seen as a heuristic hint that while our attempt [DFS14] to find a closed formula for Jack character $Ch_{\pi}$ was a failure, there might be some truth behind it.

In the current paper, as a guess for the closed formula for $Ch_{n}^{\text{top}}$ we attempted the homogeneous part of degree $n + 1$ of the orientability generating series $\widehat{Ch}_{n}$; and this turned out to be a correct guess! In other words, the main result of the current paper, Theorem 1.21, can be reformulated as follows.

**Theorem 1.23.** For each $n \geq 1$ the top-degree part $Ch_{n}^{\text{top}}$ of Jack character is equal to the homogeneous part of degree $n + 1$ of the orientability generating series $\widehat{Ch}_{n}$. 
Equivalently, the difference between the Jack character and the orientability generating series

\[ \text{Ch}_n - \hat{\text{Ch}}_n \]

is an element of \( S \) of degree at most \( n - 1 \).

This result is a strong support for our heuristic claim from [DFS14] that the orientability generating series describes some properties of Jack characters surprisingly well.

A careful Reader may feel surprised by our claim that the right-hand side of the main formula (1.12) (which can be viewed as a summation over oriented maps with an arbitrary face structure) should be related to the orientability generating series (which can be viewed as a summation over non-oriented maps with a very specific face structure). Indeed, this relationship is not immediate. The current paper is already quite long, for this reason — regretfully — we decided to postpone the proof of Theorem 1.23 to a forthcoming paper [CJS16].

1.17. **Beyond the top-degree.** One of the main results of the current paper is a closed formula (Theorem 1.21) for the homogeneous part of the Jack character \( \text{Ch}_n \) of degree \( n + 1 \) (i.e., top-degree). Can we say something about the sub-dominant terms of degree \( n, n - 1 \), etc.?

The homogeneous part of \( \text{Ch}_n \) of degree \( n \) turns out to have a trivial structure and is equal to zero, see the proof of Theorem 1.22 in Section 8.3. As a consequence, the error term (1.13) as well as (1.14) in Theorem 1.23 turn out to be smaller than expected, namely of degree at most \( n - 1 \), see Theorem 1.22. Therefore, the next challenging step would be to find and prove a closed formula for the homogeneous part of \( \text{Ch}_n \) of degree \( n - 1 \).

If we would attempt to solve this problem with the methods presented in the current paper, the first difficulty would be to guess the correct closed formula. Unfortunately, computer calculations show that the orientability generating series \( \hat{\text{Ch}}_n \) gives incorrect predictions for this homogeneous part [DFS14 Section 7] and, in a moment, we have no alternative candidates. We hope however that the results and methods of the current paper might be adapted to this more general setup and will shed some light on this problem in the future.

1.18. **Jack characters corresponding to more complicated partitions.** In the current paper we restricted our attention to the Jack characters \( \text{Ch}_n = \text{Ch}_{(n)} \) which correspond to the partition \( (n) \) which consists of a single part. Can we say something about the Jack character \( \text{Ch}_\pi \) corresponding to a generic partition \( \pi = (\pi_1, \ldots, \pi_\ell) \) with an arbitrary number of parts?
It turns out that such a generic Jack character $\text{Ch}_\pi$ is closely related to the Jack characters $\text{Ch}_{(\pi_i)}$ corresponding to the components of the partition $\pi$. Informally speaking, the following approximate factorization of Jack characters holds true:

\begin{equation}
\text{Ch}_\pi \approx \text{Ch}_{(\pi_1)} \cdot \cdots \cdot \text{Ch}_{(\pi_\ell)} = \text{Ch}_{\pi_1} \cdots \text{Ch}_{\pi_\ell}.
\end{equation}

There are several ways in which this statement can be made rigorous. We shall discuss two of them in the following.

The simplest way is the following one: the top-degree homogeneous part of $\text{Ch}_\pi$ (which is of degree $|\pi| + \ell(\pi)$) is given by the product of the top-degree parts of the Jack characters corresponding to the components of $\pi$:

$$
\text{Ch}_\pi^\text{top} = \text{Ch}_{\pi_1}^\text{top} \cdot \cdots \cdot \text{Ch}_{\pi_\ell}^\text{top}.
$$

Note that in the current paper we have found a closed formula for each of the factors on the right-hand side, thus the above formula solves the problem of calculating the top-degree part of a generic Jack character $\text{Ch}_\pi$.

Another way — more challenging and more powerful — of giving sense to (1.15) is to consider analogues of the notions of covariance and classical cumulants from the classical probability theory in which the role of random variables is played by partitions and the role of the expected value is played by the Jack character.

For example, in the case $\ell = 2$, the covariance of the partitions $(\pi_1)$ and $(\pi_2)$ — each consisting of a single part — is given by

\begin{equation}
\kappa(\pi_1, \pi_2) := \text{Ch}_{(\pi_1, \pi_2)} - \text{Ch}_{\pi_1} \cdot \text{Ch}_{\pi_2} : \mathcal{Y} \to \mathbb{Q}[A, A^{-1}].
\end{equation}

In the case $\ell = 2$ the informal statement (1.15) can be interpreted as a claim that (1.16) is of degree at most $\pi_1 + \pi_2$. An important point here is a drop in degree, since each of the two summands in (1.16) is of degree $\pi_1 + \pi_2 + 2$.

In the general case $\ell \geq 2$ we claim that the cumulant $\kappa(\pi_1, \ldots, \pi_\ell)$ is of degree $\pi_1 + \cdots + \pi_\ell + 2 - \ell$, which is a significant drop in degree since each of the summands involved is of degree $\pi_1 + \cdots + \pi_\ell + \ell$. This kind of approximate factorization of characters appeared before in the more restricted context when $\text{Ch}_\pi := \text{Ch}_{\pi}^{A=1}$ denotes not the Jack character itself but its specialization: the normalized character of the symmetric groups given by (1.2). The Reader may consult our work [Sni06] which was an extension of the results of Biane [Bia01]. Approximate factorization of characters might seem to be a technical curiosity; in fact it turned out to be a powerful tool for investigation of random Young diagrams, as we shall see in the following.

In a typical problem, one prescribes some natural reducible representation $\rho_n$ of the symmetric group $\mathfrak{S}(n)$ and asks about the statistical properties of the corresponding random Young diagram (i.e., the Young diagram
corresponding to the random irreducible representation of $\rho_n$) in the limit $n \to \infty$ as the number of boxes tends to infinity. For a large class of representations Biane [Bia01] proved a form of law of large numbers in which the corresponding random Young diagram concentrates with high probability around some limit shape, while we [´Sni06] proved an extension of Kerov’s Central Limit Theorem [Ker93c] in which the fluctuations of the random Young diagram around the limit shape are Gaussian. The proofs of both results [Bia01, ´Sni06] were based on the fact that the characters of the symmetric groups fulfill the property of approximate factorization.

Using the ideas and methods presented in the current paper one can show the property of approximate factorization for Jack characters. Nevertheless, since the paper is already quite lengthy, we decided to postpone to forthcoming papers [´Sni16, D´S16] both the proof as well as the discussion of the ramifications of this result to Jack deformations of some classical probability measures on the set of Young diagrams.

1.19. Kerov–Lassalle polynomials. We shall revisit Kerov–Lassalle polynomials which were discussed in the Prologue and provide the bibliographical background on this topic.

In the special case $\alpha = A = 1$, the idea of expressing the normalized character of the symmetric groups $\text{Ch}_n^{A=1}$ in terms of the free cumulants originates from Kerov [Ker00a]; the corresponding polynomial carries the name of Kerov polynomial. Kerov also formulated the conjecture (Kerov positivity conjecture) that the coefficients these polynomials are non-negative integers. Despite some difficult open problems, quite much is known about Kerov polynomials [GR07, DF´S10, PS11, D´S12]; in particular it was proved by Féray [Fér09] that Kerov positivity conjecture holds true.

Investigation of the analogous polynomials (Kerov–Lassalle polynomials) for general Jack character $\text{Ch}_n$ was initiated by Lassalle [Las09a]. He also formulated a conjecture (Kerov–Lassalle positivity conjecture) that the coefficients of these polynomials are non-negative integers.

In the current paper we use different normalizations of Jack characters and free cumulants than in the original paper of Lassalle [Las09a], namely the normalizations from [DF16]. For this reason Kerov–Lassalle polynomials in the current paper differ slightly from their counterparts from [Las09a]; the relationship between these two normalizations is explained in detail in [DF16, Appendix A].

1.20. Outlook. The content of Section 2 is purely technical: we investigate the gradation, as well as the corresponding filtration, on the algebra $P$ of
α-polynomial functions. We also investigate some convenient families of generators of this algebra.

The content of Section 3 is also quite technical. Its purpose is twofold. Firstly, we investigate the Jack character $\mathcal{Y} \ni (\lambda_1, \lambda_2, \ldots) \mapsto \text{Ch}_\pi (\lambda_1, \lambda_2, \ldots)$ viewed as a function of the lengths of the rows of a Young diagram. It turns out to be a multivariate polynomial and we give bounds on the top-degree of this polynomial. Secondly, we investigate the evaluation $\text{Ch}_\pi (\lambda) \in \mathbb{Q} [A, A^{-1}]$ on a fixed Young diagram $\lambda$, this time viewed as a Laurent polynomial in the indeterminate $A$.

Section 4 can be mostly skipped by an impatient Reader (except for Definitions 4.2 and 4.3). We present there a rather informal, heuristic discussion of the general direction of our proofs and present the difficulties which we will encounter.

Section 5 is quite independent from the remaining part of the paper and can be skipped by an impatient Reader. We revisit here the considerations from Section 4 and present some partial algebraic–combinatorial results about the simplest summands for the top-degree part $\text{Ch}_{\pi}^{\text{top}}$ of the Jack character.

In Section 6 we introduce the algebra $\mathcal{R}$ of row functions which will be the key tool for investigations both in the current paper as well as in the forthcoming work [Sni16].

In Section 7 we will prove Theorem 7.2, the key tool. We also present some of its applications: a proof of one of the main results of the paper, namely an abstract characterization of Jack characters (Theorem 1.7), as well as an abstract characterization of the top-degree of Jack characters (Theorem 7.3). Theorem 7.2 will be also essential for the forthcoming work [Sni16].

Section 8 is devoted to the proof of one of the main results of the paper, an explicit formula for the top-degree of the Jack character (Theorem 1.21).

Finally, in Section 9 we shall prove a formula for Kerov–Lassalle polynomials for the top-degree part of Jack characters (Theorem 0.2).

2. Gradation and filtration on the algebra $\mathcal{R}$ of α-polynomial functions

This section is purely technical; it provides some tools for the investigations in the further sections. We shall present here three convenient families of functions on the set of Young diagrams:

- the (anisotropic) free cumulants $\mathcal{R}_2, \mathcal{R}_3, \ldots$ in the normalization considered by Dolega and Féray [DFT16];
- the functionals $S_2, S_3, \ldots$ which are anisotropic analogues of their counterparts from [DFS10];
• the functionals $T_2, T_3, \ldots$ which are a discretized version of the functionals $S_2, S_3, \ldots$ above.

We will show that each of these three families generates (with the coefficients in $\mathbb{Q}[\gamma]$) the same algebra: the algebra $\mathcal{P}$ of $\alpha$-polynomial functions.

The fourth family of functions:

• Jack characters $\mathrm{Ch}_\pi$ over partitions $\pi$

is not algebraically independent, but provides a convenient linear basis (with the coefficients in $\mathbb{Q}[\gamma]$) of the algebra $\mathcal{P}$.

We shall also consider a gradation (respectively, the corresponding filtration) on $\mathcal{P}$ and we shall describe it in terms of the above generators.

This section contains no surprises; all proofs will be rather straightforward or will be reformulations of some results of Dołęga and Féray [DF16].

2.1. Discrete functionals of shape. For an integer $n \geq 2$ we define the discrete functional of shape $T_n : \mathbb{Y} \to \mathbb{Q}[A, A^{-1}]$ given by

$$T_n(\lambda) := (n-1) \sum_{\square \in \lambda} (\alpha\text{-content}(\square))^{n-2} \in \mathbb{Q}[A, A^{-1}],$$

where the sum runs over the boxes of the Young diagram $\lambda$, and the $\alpha$-content has been defined in (1.3).

Example 2.1. For the Young diagram $\lambda = (2)$ — which consists of a single row with two boxes, namely $(1, 1)$ and $(2, 1)$ — we have

$$T_n((2)) = (n-1) \left[ (A - A^{-1})^{n-2} + (2A - A^{-1})^{n-2} \right].$$

2.2. Smooth functionals of shape. For an integer $n \geq 2$ we define the (anisotropic) functional of shape $S_n(\lambda) := (n-1) \int \int_{(x,y) \in \lambda} (\alpha\text{-content}(x, y))^{n-2} \, dx \, dy \in \mathbb{Q}[A, A^{-1}]$,

where the integral is taken over the Young diagram $\lambda$ viewed as a subset of $\mathbb{R}^2$; in other words it is an integration over $x$ and $y$ such that

$$y > 0 \quad \text{and} \quad 0 < x \leq \lambda[y].$$

Example 2.2. The Young diagram $\lambda = (2)$ — the same as the one from Example 2.1 — corresponds to the rectangle

$$\{(x,y) : 0 < y \leq 1, \ 0 < x \leq 2\},$$
thus
\[ S_n((2)) = \int_0^2 \left[ \int_0^1 (n-1)(Ax - A^{-1}y)^{n-2} \, dy \right] \, dx = \int_0^2 (-A) \left[ (Ax - A^{-1})^{n-1} - (Ax)^{n-1} \right] \, dx = \frac{-1}{n} [ (2A - A^{-1})^n - (-A^{-1})^n - (2A)^n ]. \]

2.3. **Relationship between the discrete and the smooth functionals of shape.**

2.3.1. \((A \leftrightarrow -A^{-1})\)-invariant Laurent polynomials. We consider an automorphism
\[ S_{A=\; -A^{-1}} : \mathbb{Q}[A, A^{-1}] \to \mathbb{Q}[A, A^{-1}] \]
of the algebra of Laurent polynomials which is given by the substitution
\[ A := -A^{-1}, \]
in other words
\[ S_{A=\; -A^{-1}} : \sum_{k \in \mathbb{Z}} f_k A^k \mapsto \sum_{k \in \mathbb{Z}} f_k (-A)^{-k}. \]

The following lemma will be used only in the proof of Proposition 2.4 below. For the definition of the degree for Laurent polynomials see Definition 1.3.

**Lemma 2.3.** The usual substitution (1.5), i.e.,
\[ \gamma := -A + \frac{1}{A} \]
gives an isomorphism \(S_{\gamma=\; -A+\; A^{-1}}\) between the algebra of polynomials \(\mathbb{Q}[\gamma]\) and the algebra of \(S_{A=\; -A^{-1}}\)-invariant Laurent polynomials in \(\mathbb{Q}[A, A^{-1}]\).

This isomorphism preserves the degree, i.e., the image of a polynomial in \(\mathbb{Q}[\gamma]\) which is exactly of degree \(d\) is a Laurent polynomial in \(\mathbb{Q}[A, A^{-1}]\) which is exactly of the same degree \(d\).

**Proof.** Clearly, the substitution (1.5) gives an algebra homomorphism. In order to prove the first part of the lemma it remains to show that this homomorphism is surjective.

We denote by \(\mathcal{I}\) the linear space of these Laurent polynomials
\[ \sum_{k \in \mathbb{Z}} f_k A^k \in \mathbb{Q}[A, A^{-1}] \]
which are \(S_{A=\; -A^{-1}}\)-invariant, or — in other words — for which
\[ f_{-k} = (-1)^k f_k \quad \text{for each } k \in \mathbb{Z}. \]
We also consider the *forgetting map* given by

\[ \mathcal{F} : \mathbb{Q}[A, A^{-1}] \ni \sum_{k \in \mathbb{Z}} f_k A^k \mapsto \sum_{k \geq 0} f_k A^k \in \mathbb{Q}[A], \]

which to a Laurent polynomial in the variable \( A \) associates the polynomial which involves only the summands with non-negative exponents of the variable \( A \). Condition (2.2) implies that the map \( \mathcal{F} \) gives a linear isomorphism between the linear space \( \mathcal{I} \) and the linear space of \( \alpha \)-polynomials \( \mathbb{Q}[A] \).

Monomials

\[ 1, \gamma, \gamma^2, \ldots \]

form a linear basis of \( \mathbb{Q}[[\gamma]] \). Their images under the map \( \mathcal{F} \circ S_{\gamma := -A + A^{-1}} \) form the sequence

\[ (P_0, P_1, \ldots) := \left( \mathcal{F}[1], \mathcal{F}[-A + A^{-1}], \mathcal{F}((-A + A^{-1})^2), \ldots \right) \]

of polynomials in \( \mathbb{Q}[A] \) with the property that for each \( i \geq 0 \) the polynomial \( P_i \) is exactly of degree \( i \). This has twofold consequences.

- It follows that \( P_0, P_1, \ldots \) form a linear basis of \( \mathbb{Q}[A] \). In this way we proved that \( \mathcal{F} \circ S_{\gamma := -A + A^{-1}} \) is surjective thus \( S_{\gamma := -A + A^{-1}} \) is surjective as well, which completes the proof of the first part of the lemma.
- The map \( S_{\gamma := -A + A^{-1}} \) is degree-preserving, as required in the second part of the lemma.

\[ \square \]
2.3.2. *Relationship between the discrete and the smooth functionals of shape.*

**Proposition 2.4.**

(A) For each integer \( n \geq 2 \) there exists a sequence of polynomials \( P_2, \ldots, P_n \in \mathbb{Q}[\gamma] \) with the property that

\[
S_n = \sum_{2 \leq k \leq n} P_k(\gamma) T_k,
\]

where on the right-hand side the usual substitution (1.5) for \( \gamma \) is used.

Furthermore, \( P_n = 1 \) and, for each \( 2 \leq k \leq n \), the polynomial \( P_k \) is of degree at most \( n - k \).

For example,

\[
S_2 = T_2, \\
S_3 = T_3 + \gamma T_2, \\
S_4 = T_4 + \frac{3}{2} \gamma T_3 + \frac{2\gamma^2 + 1}{2} T_2.
\]

(B) For each integer \( n \geq 2 \) there exists a sequence of polynomials \( Q_2, \ldots, Q_n \in \mathbb{Q}[\gamma] \) with the property that

\[
T_n = \sum_{2 \leq k \leq n} Q_k(\gamma) S_k,
\]

where on the right-hand side the usual substitution (1.5) for \( \gamma \) is used. Furthermore, \( Q_n = 1 \) and for each \( 2 \leq k \leq n \) the polynomial \( Q_k \) is of degree at most \( n - k \).

For example,

\[
T_2 = S_2, \\
T_3 = S_3 - \gamma S_2, \\
T_4 = S_4 - \frac{3}{2} \gamma S_3 + \frac{\gamma^2 - 1}{2} S_2.
\]

**Proof.** A single box \((x_0, y_0) \in \mathbb{N}^2\) of a Young diagram, when viewed as a subset of the plane, becomes

\[
\{(x, y) : x_0 < x \leq x_0 + 1, \ y_0 < y \leq y_0 + 1\} \subset \mathbb{R}^2.
\]
The integral on the right-hand side of (2.1) restricted to this box is given by (for an analogous calculation see Example 2.2):

\begin{equation}
(n - 1) \int_{x_0}^{x_0 + 1} \left[ \int_{y_0}^{y_0 + 1} (Ax - A^{-1}y)^{n-2} \, dy \right] \, dx = \\
\int_{x_0}^{x_0 + 1} (-A) \left[ (Ax - A^{-1}(y_0 + 1))^{n-1} - (Ax - A^{-1}y_0)^{n-1} \right] \, dx = \\
- \frac{1}{n} \left[ (c + A - A^{-1})^n - (c + A)^n - (c - A^{-1})^n + c^n \right],
\end{equation}

where on the right-hand side
\[ c := \alpha\text{-content}(x_0, y_0) = Ax_0 - A^{-1}y_0. \]

We shall view the right-hand side of (2.5) as a polynomial in the variable \( c \) of the following form:
\[
\sum_{2 \leq k \leq n+2} d_k (k - 1) c^{k-2}
\]
with the coefficients \( d_2, \ldots, d_{n+2} \in \mathbb{Q}[A, A^{-1}] \) given by
\[
d_k = \frac{n}{n(k-1)} \left[ - (A - A^{-1})^{n+2-k} + A^{n+2-k} + (A^{-1})^{n+2-k} - 0^{n+2-k} \right].
\]

An elementary calculation based on the binomial formula shows that — due to cancellations — \( d_k \) is a Laurent polynomial of degree at most \( n - k \) for each \( 2 \leq k \leq n \). Furthermore, \( d_{n+1} = d_{n+2} = 0 \) and \( d_n = 1 \). For this reason, Lemma 2.3 applied to \( d_k \in \mathbb{Q}[A, A^{-1}] \) shows existence of a polynomial \( P_k \in \mathbb{Q}[\gamma] \) of degree at most \( n - k \) with the property that
\[
d_k = P_k,
\]
where on the right-hand side the usual substitution (1.5) is applied.

As the integral over a Young diagram \( \lambda \subset \mathbb{R}^2 \) can be written as a sum of the integrals over the individual boxes, it follows immediately that for an arbitrary Young diagram \( \lambda \) the equality (2.3) holds true with the polynomials \( P_2, \ldots, P_n \) given by the above construction. Furthermore, these polynomials fulfill the required degree bounds. This concludes the proof of part (A).

In order to prove part (B) we use induction with respect to the variable \( n \). Part (A) implies that (2.3) can be written in the form
\[
\mathcal{T}_n = \mathcal{S}_n - \sum_{2 \leq k \leq n-1} P_k(\gamma) \mathcal{T}_k.
\]
The inductive assertion can be applied to each of the expressions $T_2, \ldots, T_{n-1}$ on the right-hand side. It follows that $T_n$ can be written, as required, in the form (2.4) with the proper bounds on the degrees of the polynomials $Q_2, \ldots, Q_n$. This concludes the proof of the inductive step.

2.4. Free cumulants.

2.4.1. Labeled oriented maps revisited. We shall continue the discussion of maps from Section 0.4. Recall that we consider an oriented, labeled map which corresponds to a pair of permutations $\sigma_1, \sigma_2 \in S(n)$. It is not very difficult to see that the faces of the map correspond to the cycles of the product $\sigma_1 \sigma_2$ in the following strict sense: if we go clockwise around the boundary of some face, and write down the labels of every second visited edge (to be more precise: we write down only the labels of the edges in which we visit their white endpoint first and then their black endpoint), we shall get the elements of one of the cycles of $\sigma_1 \sigma_2$ in the same cyclic order.

Example 2.5. We continue Example 0.3. The product $\sigma_1 \sigma_2 = (1, 5, 6, 3, 7, 9, 4)(2, 8)$ has two cycles; the first one corresponds to the face with $2 \times 7 = 14$ edges which in Figure 1 was indicated by hatched red lines, while the second one corresponds to the face with $2 \times 2 = 4$ edges which on the same figure was indicated by a solid blue fill.

2.4.2. Free cumulants, the definition. We are now ready to present a formal definition of the notion of free cumulants of a Young diagram; the notion that played an important role in Prologue.

Definition 2.6. For a Young diagram $\lambda$, we define the sequence of its (anisotropic) free cumulants $R_2(\lambda), R_3(\lambda), \ldots \in \mathbb{Q}[A, A^{-1}]$ by the formula

\[ R_k(\lambda) := (-1) \sum_{\sigma_1, \sigma_2} \mathfrak{N}_{\sigma_1, \sigma_2}(\lambda), \]

where the normalized number of embeddings $\mathfrak{N}_{\sigma_1, \sigma_2}$ has been defined in (1.11). The sum in (2.6) runs over pairs of permutations $\sigma_1, \sigma_2 \in S(k - 1)$ with the property that:

(a) their product $\sigma_1 \sigma_2 = (1, 2, \ldots, k - 1)$ is the full cycle, and
(b) their total number of cycles fulfills $|C(\sigma_1)| + |C(\sigma_2)| = k$.

Following the discussion from Sections 0.4 and 2.4.1 the right-hand side of (2.6) can be viewed as a sum over labeled, bicolored, oriented maps with some additional properties which will be discussed in the following.
Condition (a) implies that the map has exactly one face; furthermore the labels around this face are arranged in a special way. This means that if we erase the labels from all edges, except for the single edge with the label \( k - 1 \) (we shall call this the root edge of the map), all the remaining labels can be recovered uniquely. After erasing the labels, the map becomes a rooted, unlabeled, oriented map with a single face and \( k - 1 \) edges.

Condition (b) states that the map has \( k \) vertices. Since the map is connected, and it has \( k - 1 \) edges, this implies that the map is a tree.

The above discussion shows that the right-hand side of (2.6) can be viewed as a sum over rooted, bicolored, oriented trees with \( k - 1 \) edges. Note that in the literature the name plane rooted tree is used more frequently.

2.4.3. Free cumulants, historical comments. The usual way of defining the free cumulants of a Young diagram \( \lambda \) is a two-step procedure [Bia98]: it uses Kerov’s transition measure of \( \lambda \) [Ker93b] and the notion of free cumulants of a probability measure on the real line [NS06] which originates in Voiculescu’s free probability theory. Our definition has the advantage of being more direct. The equivalence between both definitions has been established by Rattan [Rat08], see also [FS11a Theorem 9].

To be very strict: the quantities \( R_2, R_3, \ldots : \mathbb{Y} \to \mathbb{Q}[A, A^{-1}] \) correspond not to the usual Biane’s free cumulants of a Young diagram, but to Lassalle’s anisotropic free cumulants [Las09a]. However, the normalization used here is different and corresponds to the one from [DF16, DFS14].

2.5. Anisotropic and isotropic quantities. The anisotropic Stanley polynomials considered in the current paper were defined (see Definition 1.12) in terms of the anisotropic multirectangular coordinates \( P, Q \). In the special case \( A = 1, \gamma = 0 \) considered in [FS11a, DFS10] there is no difference between the anisotropic and the isotropic coordinates and the definition of Stanley polynomials takes a simpler form. We shall refer to the Stanley polynomials considered in these papers as isotropic Stanley polynomials. Similarly, the free cumulants \( R_2, R_3, \ldots \) and functionals of shape \( S_2, S_3, \ldots \) considered in [DFS10] will be referred to as isotropic free cumulants and isotropic functionals of shape.

Several times our strategy will be to reduce the general anisotropic case to the simpler isotropic one. The following lemma is the tool which relates some anisotropic quantities with their isotropic counterparts.

**Lemma 2.7.**

(1) For each bicolored graph \( G \) the anisotropic Stanley polynomial for the function

\[
\lambda \mapsto (-1)^{|V(G)|} \mathfrak{M}_G(\lambda)
\]
exists and coincides with the isotropic Stanley polynomial for the function \( \lambda \mapsto N_G(\lambda) \). This polynomial is homogeneous of degree \( |V(G)| \). When viewed as a polynomial in the variables \( q_1, q_2, \ldots \) with coefficients in \( \mathbb{Q}[p_1, p_2, \ldots] \), this polynomial is homogeneous of degree \( |V_e(G)| \).

(2) For each integer \( n \geq 2 \), the anisotropic Stanley polynomial for \( \lambda \mapsto S_n(\lambda) \) exists and coincides with the isotropic Stanley polynomial for the isotropic functional \( \lambda \mapsto S_n^\lambda \) from [DFS10].

(3) For each integer \( n \geq 2 \), the anisotropic Stanley polynomial for the (anisotropic) free cumulant \( \lambda \mapsto R_n(\lambda) \) exists and coincides with the isotropic Stanley polynomial for the free cumulant \( \lambda \mapsto R_n^\lambda \) from [DFS10].

Proof. As pointed out in the proof of [DFS14, Lemma 2.4], a slight variation of [FS11b, Lemma 3.9] shows that if \( G \) is an arbitrary bicolored graph, then for the function \( \lambda \mapsto (-1)^{|V_e(G)|} N_G(\lambda) \)
the corresponding anisotropic Stanley polynomial exists and does not involve the variable \( \gamma \). In particular, the anisotropic Stanley polynomial for (2.7) coincides with the isotropic Stanley polynomial for the function \( N_G \) from Definition 1.18 (in [FS11b] this function has been denoted by \( N_G^{(1)} \)); note that this function appeared already — for the special case of the graph \( G \) corresponding to a pair of permutations — in [FS11a], as well as somewhat implicitly in [DFS10]).

We use the notations from Section 1.10.1. An elementary integration, analogous to (2.5), shows that whenever \( P' \times Q' \in \mathbb{Y} \), then

\[
S_n(P' \times Q') = \frac{-1}{n} \sum_{i \geq 1} \left[ \left( - (p_1 + \cdots + p_{i-1}) \right)^n - \left( - (p_1 + \cdots + p_i) \right)^n \right.
\]

\[
- \left( q_i - (p_1 + \cdots + p_{i-1}) \right)^n + \left( q_i - (p_1 + \cdots + p_i) \right)^n \right].
\]

The right-hand side does not involve the variable \( \gamma \), which implies that the anisotropic Stanley polynomial for \( S_n \) coincides with the isotropic Stanley polynomial for the isotropic functional \( \lambda \mapsto S_n^\lambda \) from [DFS10].

The above observation and Definition 2.6 imply that also the anisotropic Stanley polynomial for any free cumulant \( \lambda \mapsto R_n(\lambda) \) coincides with the isotropic Stanley polynomial for the free cumulant \( \lambda \mapsto R_n^\lambda \) from [DFS10].

2.6. The relationship between \((S_n)\) and \((R_n)\). If \( \mu = (\mu_1, \mu_2, \ldots) \) is a partition which does not contain any parts equal to 1, we define the functions
Lemma 2.8.

(a) For each $n \geq 2$ the anisotropic free cumulant $R_n$ can be expressed uniquely as a linear combination of $S_\mu$ over partitions $\mu$ such that $|\mu| = n$, with the coefficients in $\mathbb{Q}$. This linear combination coincides with the polynomial \cite[Eq. (15)]{DFS10} which expresses the isotropic free cumulant $R_n$ in terms of the isotropic functionals $S_2, S_3, \ldots$. In this linear combination, the coefficient standing at $S_n$ is given by $[S_n]R_n = 1$.

(b) For each $n \geq 2$ the anisotropic free cumulant $S_n$ can be expressed uniquely as a linear combination of $R_\mu$ over partitions $\mu$ such that $|\mu| = n$, with the coefficients in $\mathbb{Q}$. This polynomial coincides with the polynomial \cite[Eq. (14)]{DFS10} which expresses the isotropic functional $S_n$ in terms of the isotropic free cumulants $R_2, R_3, \ldots$.

Proof. We start with the proof of part (a). Our ultimate goal is to prove that $R_n$ can be expressed as a polynomial $F(S_2, S_3, \ldots)$ for some multivariate polynomial $F$ and to investigate the latter. In order to do this we shall study the relationship between the anisotropic Stanley polynomials (Section 1.10.2) and their isotropic counterparts (Section 2.5).

Each anisotropic Stanley polynomial determines uniquely the corresponding function on the set $\mathcal{Y}$ of Young diagrams. In other words, the map which to a function from $\mathcal{S}$ associates the corresponding anisotropic Stanley polynomial (see Definition 1.12) is an injective algebra homomorphism. Therefore our ultimate goal can be reformulated as expressing the anisotropic Stanley polynomial for $R_n$ as a polynomial $F$ in terms of the anisotropic Stanley polynomial for $S_2, the anisotropic Stanley polynomial for $S_3, \ldots$.

Lemma 2.7 shows equality between the anisotropic Stanley polynomial for $R_n$ (respectively, $S_n$) and the isotropic Stanley polynomial for $\lambda \mapsto R_\lambda^\lambda$ (respectively, for $\lambda \mapsto S_\lambda^\lambda$). For this reason our ultimate goal is equivalent to proving the existence of a multivariate polynomial $F$ with the property that

$$R_n^\lambda = F(S_2^\lambda, S_3^\lambda, \ldots).$$

The latter polynomial is known to exist and its exact form is known \cite[Eq. (15)]{DFS10}. The latter formula also implies the remaining properties claimed in the part (a).
The proof of the remaining part (b) is analogous with the roles of the quantities $S_n$ and $R_n$ interchanged. In the last step of the proof one should use [D.F.S10, Eq. (14)] instead.

\[\square\]

2.7. Equivalent definitions of the algebra $\mathcal{P}$ of $\alpha$-polynomial functions.

The following proposition can be viewed as a definition of the algebra $\mathcal{P}$, alternative to Inconvenient Definition \[1.5\]

**Proposition 2.9.**

(a) The algebra $\mathcal{P}$ of $\alpha$-polynomial functions is the algebra generated by the functions

\begin{equation}
\gamma, T_2, T_3, T_4, \ldots : \mathcal{Y} \to \mathbb{Q}[A, A^{-1}]
\end{equation}

over the field $\mathbb{Q}$ of rational numbers, where $\gamma : \mathcal{Y} \to \mathbb{Q}[A, A^{-1}]$ is viewed as a constant function

$$\gamma = -A + \frac{1}{A}.$$ 

The multiplication (respectively, addition) in this algebra is just the pointwise multiplication (respectively, addition) of functions on the set of Young diagrams.

(b) The algebra $\mathcal{P}$ of $\alpha$-polynomial functions is the algebra generated by the functions

$$\gamma, S_2, S_3, S_4, \ldots : \mathcal{Y} \to \mathbb{Q}[A, A^{-1}].$$

(c) The algebra $\mathcal{P}$ of $\alpha$-polynomial functions is the algebra generated by the functions

$$\gamma, R_2, R_3, R_4, \ldots : \mathcal{Y} \to \mathbb{Q}[A, A^{-1}].$$

**Proof.** Part (a) is a reformulation of Inconvenient Definition \[1.5\]. Part (b) follows from part (a) and Proposition 2.4. Part (c) follows from part (b) and Lemma 2.8.\[\square\]

2.8. Gradation on the algebra $\mathcal{P}$ of $\alpha$-polynomial functions revisited.

In Definition \[1.14\] we specified a gradation on the algebra $\mathcal{J}$. Proposition \[2.13\] below gives a convenient description of this gradation, when restricted to the algebra $\mathcal{P}$. It can also be viewed as an alternative definition of this gradation.

**Lemma 2.10.** For each integer $n \geq 2$ we have that $S_n \in \mathcal{J}$ is homogeneous of degree $n$ (in the sense given by Definition \[1.14\]).

The same is true for $R_n$.

**Proof.** Equation \[2.8\] implies that the Stanley polynomial for $S_n$ exists and is homogeneous of degree $n$. This concludes the proof of the first part.

For the second part we apply Lemma \[2.8\] and the above observation. \[\square\]
Corollary 2.11. \( \mathcal{P} \subseteq \mathcal{S} \) or, in other words, each \( \alpha \)-polynomial function has a Stanley polynomial.

Proof. It is a direct consequence of Lemma 2.10 and Proposition 2.9(b). \( \square \)

Definition 2.12. We equip the algebra \( \mathcal{P} \) with a gradation defined as a restriction of the gradation on \( \mathcal{S} \). In other words, we say that \( F \in \mathcal{P} \) is homogeneous of degree \( d \) if \( F \), viewed as an element of \( \mathcal{S} \), is homogeneous of degree \( d \).

Proposition 2.13.  
(a) The set of homogeneous elements of \( \mathcal{P} \) of degree \( n \) is spanned (as a linear space over \( \mathbb{Q} \)) by the elements of the form 

\[
\gamma \, S_s^2 \, S_3^3 \cdots 
\]

over \( g, s_2, s_3, \ldots \geq 0 \) such that \( g + \sum s_k k = n \).

(b) The set of homogeneous elements of \( \mathcal{P} \) of degree \( n \) is spanned (as a linear space over \( \mathbb{Q} \)) by the elements of the form 

\[
\gamma \, R_s^2 \, R_3^3 \cdots 
\]

over \( g, s_2, s_3, \ldots \geq 0 \) such that \( g + \sum s_k k = n \).

Proof. It is a direct consequence of Proposition 2.9(b)(c) and Lemma 2.10. \( \square \)

2.9. Filtration on the algebra \( \mathcal{P} \) revisited. In Inconvenient Definition 1.5 we defined a certain filtration on the algebra \( \mathcal{P} \) of \( \alpha \)-polynomial functions. The following proposition gives a relationship between this filtration and the gradation considered in Definition 2.12, namely that the former is the filtration corresponding to the latter.

Proposition 2.14. Let \( n \geq 0 \) be an integer. Then the following hold true.

(a) The following two sets are equal:
(i) the set of \( \alpha \)-polynomial functions which are of degree at most \( n \) (in the sense of Inconvenient Definition 1.5),
(ii) the span of homogeneous \( \alpha \)-polynomial functions of degree \( d \) (in the sense of Definition 2.12) over \( d \leq n \).

(b) The above set of elements of \( \mathcal{P} \) of degree at most \( n \) is spanned (as a linear space over \( \mathbb{Q} \)) by the elements of the form

\[
\gamma \, S_s^2 \, S_3^3 \cdots 
\]

over \( g, s_2, s_3, \ldots \geq 0 \) such that \( g + \sum s_k k \leq n \).
This set is also spanned (as a linear space over $\mathbb{Q}$) by the elements of the form

$$
\gamma^g \mathcal{T}_2^{s_2} \mathcal{T}_3^{s_3} \cdots
$$

over $g, s_2, s_3, \ldots \geq 0$ such that $g + \sum_k s_k k \leq n$.

**Proof.** By Proposition 2.13(a), the set (ii) is equal to the span of (2.10). Proposition 2.4 implies that the latter is equal to the span of (2.11). The latter is clearly equal to the set (i). \qed

2.10. **Degree of the Jack characters.** Further considerations in this paper (in particular, the proof of Theorem 7.3) will be based on the following theorem which might seem innocent at the first sight; nevertheless it is a highly nontrivial result of Dołęga and Féray [DF16].

**Theorem 2.15.** Let $\pi$ be an arbitrary partition.

Then $\text{Ch}_\pi \in \mathcal{P}$ is an $\alpha$-polynomial function of degree at most $|\pi| + \ell(\pi)$.

**Proof.** Dołęga and Féray studied the Kerov–Lassalle polynomial, i.e., the expansion of the Jack character $\text{Ch}_\pi$ in terms of the free cumulants $\mathcal{R}_2, \mathcal{R}_3, \ldots$ with the coefficients which a priori belong to the field of rational functions $\mathbb{Q}(A)$. They proved [DF16, Corollary 3.5] that the coefficients are, in fact, polynomials in the variable $\gamma$; this result together with Proposition 2.9(c) implies that $\text{Ch}_\pi \in \mathcal{P}$, as required.

In [DF16 Proposition 3.7] Dołęga and Féray found an upper bound on the degrees of these polynomials which, by Proposition 2.13(b), implies the required bound on the degree of $\text{Ch}_\pi$. \qed

2.11. **The top-degree part** $\text{Ch}_n^{\text{top}}$. Recall that with Definition 1.16 we have a priori only that $\text{Ch}_n^{\text{top}} \in \mathcal{S}$.

**Corollary 2.16.** For each integer $n \geq 2$ we have that $\text{Ch}_n^{\text{top}} \in \mathcal{P}$.

**Proof.** By Theorem 2.15 and Proposition 2.13, $\text{Ch}_n \in \mathcal{P}$ thus it can be expressed as a polynomial in terms of generators $\gamma, \mathcal{S}_2, \mathcal{S}_3, \ldots \in \mathcal{P}$ which are homogeneous. For this reason, $\text{Ch}_n^{\text{top}} \in \mathcal{P}$ which is the homogeneous part of $\text{Ch}_n \in \mathcal{S}$ also belongs to $\mathcal{P}$. \qed

2.12. **Filtration on the algebra** $\mathcal{P}$ **revisited again.** The filtration on $\mathcal{P}$ allows also the following alternative description which will be used in [Śni16].

**Proposition 2.17.** The family of functions

$$
\gamma^g \text{Ch}_\pi : \mathcal{Y} \to \mathbb{Q} \left[ A, A^{-1} \right]
$$

over integers $g \geq 0$ and partitions $\pi$ is linearly independent over $\mathbb{Q}$. 
The linear space of the elements of $\mathcal{P}$ of degree at most $n$ has a linear basis (over $\mathbb{Q}$) given by the elements (2.12) such that
\[(2.13)\quad g + |\pi| + \ell(\pi) \leq n.\]

**Proof.** Assume that the first part is not true. It would mean that there exists some finite collection of polynomials $P_\pi \in \mathbb{Q}[\gamma]$ with the property that
\[\sum_\pi P_\pi \text{Ch}_\pi = 0,\]
where the above equation should be viewed as an equality between functions $\mathbb{Y} \to \mathbb{Q}[A, A^{-1}]$. Dołęga and Féray [DF16, Proposition 2.9] proved that for each value of $A \in \mathbb{C}$ the functions $\text{Ch}_\pi: \mathbb{Y} \to \mathbb{C}$ obtained by a specialization of the indeterminate $A$ are linearly independent over $\mathbb{C}$. It follows that for each value of $A \in \mathbb{C}$ the specialization of $P_\pi$ is equal to zero. Since the polynomial $P_\pi \in \mathbb{Q}[\gamma]$ is uniquely determined by its values it follows that $P_\pi = 0$ as required. This concludes the proof of the first part.

The above first part and Theorem 2.15 show that the elements (2.12) such as in the second part of the proposition are a linear basis of some linear subspace of the space of the elements of $\mathcal{P}$ of degree at most $n$. An obvious bijective correspondence
\[(g, \pi) \mapsto (\pi_1 + 1, \pi_2 + 1, \ldots, \pi_{\ell(\pi)} + 1, g)\]
between the set of tuples $(g, \pi)$ which contribute to (2.13) and the set $\mathcal{P}_{\leq n}$ of partitions $\lambda$ such that $|\lambda| \leq n$ shows that the dimension of this subspace is equal to $|\mathcal{P}_{\leq n}|$.

On the other hand there is an obvious bijective correspondence
\[(g, s_2, s_3, \ldots) \mapsto (\underbrace{\ldots, 3, \ldots, 3, 2, \ldots, 2, 1, \ldots, 1}_{s_3 \text{ times } s_2 \text{ times } g \text{ times}})\]
between the set of tuples $(g, s_2, s_3, \ldots)$ which contribute to Proposition 2.14(b) and the set $\mathcal{P}_{\leq n}$. Thus Proposition 2.14(b) implies that the dimension of the linear space of the elements of $\mathcal{P}$ of degree at most $n$ is bounded from above by $|\mathcal{P}_{\leq n}|$.

The equality of the dimensions concludes the proof of the second part. 

\[\square\]

3. VERSHIK–KEROV SCALING AND LAURENT POLYNOMIALS

3.1. Vershik–Kerov scaling. In the following we shall investigate the function
\[(3.1)\quad \mathbb{Y} \ni (\lambda_1, \ldots, \lambda_m) \mapsto F(\lambda_1, \ldots, \lambda_m) \in \mathbb{Q}[A, A^{-1}]\]
in the case when $F \in \mathcal{P}$ is an $\alpha$-polynomial function and $m \geq 1$ is a fixed integer. It turns out that (3.1) can be identified with a polynomial in the variables $\lambda_1, \ldots, \lambda_m$; the following lemma explains the precise sense in which this happens. The main result of this section is Proposition 3.4 which gives information about this polynomial in the special case when $F = \text{Ch}_\pi$ is the Jack character.

**Lemma 3.1.** Let $F \in \mathcal{P}$ be given.

Then, for each integer $m \geq 1$, there exists a unique polynomial $W(\lambda_1, \ldots, \lambda_m)$ with the coefficients in $\mathbb{Q}[A, A^{-1}]$ with the property that for each Young diagram $(\lambda_1, \ldots, \lambda_m)$ with at most $m$ rows

$$(3.2) \quad F(\lambda_1, \ldots, \lambda_m) = W(\lambda_1, \ldots, \lambda_m).$$

**Proof.**

**Proof of the existence.** We use the notations from Section 1.10.1. Corollary 2.11 shows that $F$ has a Stanley polynomial and

$$F(P' \times Q') = \text{St}_m(\gamma; P, Q);$$

now it is enough to consider the specialization $P' = (1, \ldots, 1)$, $Q' = (\lambda_1, \ldots, \lambda_m)$, $P = (A^{-1}, \ldots, A^{-1})$, $Q = (A\lambda_1, \ldots, A\lambda_m)$.

**Proof of uniqueness.** By linearity of the problem, it is enough to consider the case when $F = 0$; our goal is to prove that $W = 0$. We shall use induction with respect to the variable $m$. The induction basis, case $m = 1$, follows from the observation that the polynomial $W = W(\lambda_1)$ has infinitely many nullpoints (namely all non-negative integers) thus must be the zero polynomial.

We assume that the result holds true for $m' := m - 1$ and we shall prove that it holds true for $m \geq 2$. We view $W$ as a polynomial in a single variable $\lambda_1$:

$$(3.3) \quad W = \sum_{k \geq 0} W_k(\lambda_2, \ldots, \lambda_m) \lambda_1^k$$

with the coefficients $W_k \in \mathbb{Q}[\lambda_2, \ldots, \lambda_m] \otimes \mathbb{Q}[A, A^{-1}]$. We shall prove that $W_k = 0$ for each value of the index $k$.

Let us fix for the moment the values of non-negative integers $\lambda_2 \geq \cdots \geq \lambda_m$. Equation (3.2) implies that the polynomial (3.3) has infinitely many nullpoints (namely, all integers $\lambda_1$ such that $\lambda_1 \geq \lambda_2$); it follows that each of the coefficients of the polynomial (3.3) vanishes:

$$W_k(\lambda_2, \ldots, \lambda_m) = 0.$$

Since the above equality holds true for all non-negative integers $\lambda_2 \geq \cdots \geq \lambda_m$, the inductive hypothesis can be applied to $W_k$ and thus $W_k = 0$. This concludes the proof of the uniqueness. \qed
Lemma 3.2. For each $n \geq 2$ and $F := S_n$ the polynomial (3.1) is of degree $n - 1$, with the homogeneous part of degree $n - 1$ equal to

$$(\lambda_1, \lambda_2, \ldots) \mapsto A^{n-2} \sum_i \lambda_i^{n-1} = A^{n-2} p_{n-1}(\lambda_1, \lambda_2, \ldots).$$

The same holds true for $F := R_n$.

Proof. The first part of the lemma follows by (2.8) applied to $P' = (1, \ldots, 1)$, $Q' = (\lambda_1, \ldots, \lambda_m)$, $P = (A^{-1}, \ldots, A^{-1})$, $Q = (A\lambda_1, \ldots, A\lambda_m)$.

By multiplicativity it follows that for $F := S_\mu$ the polynomial (3.1) is of degree $|\mu| - \ell(\mu)$. For the second part, we apply Lemma 2.8(a) and the above observation. \qed

Lemma 3.3. Let $n \geq 1$ be an integer. Assume that $F \in \mathcal{P}$ is of degree at most $n$.

Then the polynomial (3.1) is of degree at most $n - 1$.

Proof. It is a direct application of Lemma 3.2 and Proposition 2.14. \qed

Vershik and Kerov [VK81] proved a special case of the following result; namely in the case $A = 1$ which corresponds to the characters $Ch_{\pi}^{A=1}$ of the symmetric groups given by (1.2). In the setup of Jack characters it was proved (in a slightly different formulation) by Lassalle [Las08, Proposition 2]. Nevertheless, we provide an alternative proof below.

Proposition 3.4. Let $\pi$ be a partition and $m \geq 1$ be an integer.

Then

(3.4) \hspace{1cm} Y \ni (\lambda_1, \ldots, \lambda_m) \mapsto Ch_{\pi}(\lambda_1, \ldots, \lambda_m) \in Q[A, A^{-1}]

is a polynomial of degree $|\pi|$; its homogeneous top-degree part is equal to

$$A^{|\pi| - \ell(\pi)} p_\pi(\lambda_1, \ldots, \lambda_m),$$

where $p_\pi$ is the power-sum symmetric polynomial (1.7).

Proof. We follow the investigations of Dołęga and Féray [DF16, Section 3] and we consider the expansion of $Ch_{\pi}$ as a linear combination of the functions $R_\mu$ with the coefficients in $Q[\gamma]$; Dołęga and Féray proved that this expansion is unique. They also proved [DF16, Proposition 3.7, Proposition 3.10] that if $\mu$ is a partition for which the corresponding coefficient in this expansion is non-zero, then

(3.5) \hspace{1cm} |\mu| + \deg \leq |\pi| + \ell(\pi)

(3.6) \hspace{1cm} |\mu| - 2\ell(\mu) + \deg \leq |\pi| - \ell(\pi),
where $\text{deg}$ denotes the degree of the polynomial appearing as the coefficient. By taking the mean of the above inequalities we obtain

\begin{equation}
|\mu| - \ell(\mu) + \text{deg} \leq |\pi|.
\end{equation}

Lemma 3.2 implies by multiplicativity that for $F := \mathcal{R}_\mu$ the corresponding polynomial (3.1) is of degree at most $|\mu| - \ell(\mu)$. Thus in order to prove the degree bound for the polynomial (3.4) it would be enough to show that

\begin{equation}
|\mu| - \ell(\mu) \leq |\pi|.
\end{equation}

However, the latter is a direct consequence of (3.7) which concludes the proof of the first part.

We shall find now the partitions $\mu$ which contribute to the top-degree part of this polynomial. Firstly, the bound (3.8) is saturated only if $\text{deg} = 0$, thus the corresponding coefficient does not depend on $\gamma$. For this reason we can specialize to the case $A := 1$, $\gamma := 0$ of the normalized characters of the symmetric groups $\text{Ch}_A^{A=1}$.

Secondly, the bound (3.8) is saturated only if (3.5) becomes an equality; in other words the corresponding product of free cumulants $\mathcal{R}_\mu$ must be of the degree $|\pi| + \ell(\pi)$. For this reason we will investigate the homogeneous part of $\text{Ch}_A^{A=1} \in \mathcal{P}$ of degree $|\pi| + \ell(\pi)$. It is an old result of Kerov \cite{Ker00a,Bia03} that the latter is equal to $\mathcal{R}_\pi$, which concludes the proof by Lemma 3.2.

\section{Degrees of Laurent polynomials.}

\textbf{Proposition 3.5.} For any partition $\pi$ and any Young diagram $\lambda$ the evaluation $\text{Ch}_\pi(\lambda)$ is a Laurent polynomial of degree at most $|\pi| - \ell(\pi)$.

\textbf{Proof.} From the very definition of free cumulants (Definition 2.6) it follows that for any $n \geq 2$ and any Young diagram $\lambda$, the evaluation $\mathcal{R}_n(\lambda) \in \mathbb{Q}[A, A^{-1}]$ is a Laurent polynomial of degree at most $n - 2$. By multiplicativity it follows that $\mathcal{R}_\mu(\lambda) \in \mathbb{Q}[A, A^{-1}]$ is a Laurent polynomial of degree at most $|\mu| - 2\ell(\mu)$.

We revisit the above proof of Proposition 3.4: inequality (3.6) gives the desired bound. \hfill \Box

\section{The difference operator $\Delta_\lambda$}

The purpose of the current section is to provide some intuitional background for Sections 6 and 7 and in particular to explain some hidden difficulties that must be overcome. The Readers who are not interested in such heuristic considerations may skip this section entirely, except for Definitions 4.2 and 4.3. Or, better, may jump directly to Section 4.10 which presents the summary and a conclusion of the considerations in this section.
4.1. **The problem: functionals on the quotient space.** As we already hinted at the end of Section [1.15](#), the only challenging task is to find a proper replacement for condition (K3) from Theorem [1.7](#) which would hold for the top-degree $\text{Ch}_n^{\text{top}}$ of the Jack character. In order to achieve this goal we need to solve the following more general problem first.

**Problem 4.1.** For a given integer $d \geq 0$ (we are interested in the case $d = n + 1$) find explicitly some convenient family $(\phi_i)$ of linear functionals $\phi_i: \mathcal{P} \to \mathbb{Q}$ with the following properties.

(R1) We require that $\phi_i(F)$ does not depend on the homogeneous parts of $F \in \mathcal{P}$ of degree at most $d - 1$. In other words, we require that $\phi_i$ vanishes on each $F \in \mathcal{P}$ of degree at most $d - 1$ and thus can be viewed as a functional on the appropriate quotient space.

(R2) We require that $\phi_i$ vanishes on each $F \in \mathcal{P}$ which fulfills an analogue of Property [1.2](#) (The vanishing property), namely

$$F(\lambda) = 0 \quad \text{for each } \lambda \in \mathcal{Y} \text{ such that } |\lambda| \leq n - 1.$$  

(R3) It would be desirable if the collection $(\phi_i)$ of the functionals is such that for a given $F \in \mathcal{P}$ which is of degree at most $d$, the collection of numbers $(\phi_i(F))$ provides as much information about $F$ as possible.

Once this problem is solved, the wanted replacement for condition [K3](#) from Theorem [1.7](#) would be a collection of the equalities $\phi_i(F) = 0$ over all the elements of collection $(\phi_i)$. An impatient reader may check that the replacement for condition [K3](#) which we present in Theorem 7.3 as condition [T3](#) indeed has this form.

Let us have a try: any $F \in \mathcal{P}$ which is of degree at most $d$ can be written in the form given by (1.4):

$$F(\lambda) = \underbrace{p_0(\gamma)}_{F_0(\lambda):=} + \sum_{\square_1 \in \lambda} \underbrace{p_1(\gamma, c_1)}_{F_1(\lambda):=} + \sum_{\square_1, \square_2 \in \lambda} \underbrace{p_2(\gamma, c_1, c_2)}_{F_2(\lambda):=} + \cdots ,$$

where for each $k \geq 0$ the polynomial $p_k$ is of degree at most $d - 2k$ and is symmetric with respect to the variables $c_1, \ldots, c_k$. In the above sum $c_i := \alpha$-content($\square_i$) and the usual substitution (1.5) is used.

We denote by $p_k^{\text{top}}$ the homogeneous part of $p_k$ of the degree $d - 2k$. The right-hand side of (4.2) serves as a definition of the functions $F_0, F_1, \ldots \in \mathcal{P}$. We are interested in the information about the homogeneous part of $F_k$ of degree $d$ and the latter information is encoded by $p_k^{\text{top}}$. How to extract
this top-degree part $p_k^{\text{top}}$ from the information about $F$? In other words: how to find explicitly the functionals $(\phi_i)$?

The first step is to notice that both $\gamma \in \mathbb{Q}[A, A^{-1}]$ as well as $\alpha\text{-content}(\Box) \in \mathbb{Q}[A, A^{-1}]$ (for an arbitrary box $\Box \in \mathbb{N}^2$) are Laurent polynomials of degree equal to 1. It follows that each value $F_k(\lambda) \in \mathbb{Q}[A, A^{-1}]$ is a Laurent polynomial of degree at most $d - 2k$. For this reason the coefficient $[A^{d-2k}]F(\lambda)$ depends only on the polynomials $p_0, \ldots, p_k$. This opens the possibility of solving the problem inductively by considering first the case $k = 0$, then using this information in order to tackle the case $k = 1$, etc. We shall do it in the following.

4.2. The case $k = 0$. This case is very simple: the top-degree part $p_0^{\text{top}}$ is some multiple of the monomial $\gamma^d$. The corresponding coefficient is encoded by the Laurent polynomial coefficient $[A^d]F(\emptyset)$:

$$(4.3) \quad [A^d]F(\emptyset) = (-1)^d [\gamma^d]p_0(\gamma) = (-1)^d [\gamma^d]p_0^{\text{top}}(\gamma) = p_0^{\text{top}}(-1).$$

In other words, as one of our functionals $\phi_i$ we may take

$$\phi(F) = [A^d]F(\emptyset).$$

4.3. The case $k = 1$. This case is a bit more challenging because the coefficient

$$(4.4) \quad [A^{d-2}]F(\lambda) = (-1)^{d-2} [\gamma^{d-2}]p_0 + (-1)^{d-1}d [\gamma^d]p_0^{\text{top}} + \sum_{(x,y) \in \lambda} [A^{d-2}]p_1(\gamma, \alpha\text{-content}(x, y)) = (-1)^{d-2} [\gamma^{d-2}]p_0 + (-1)^{d-1}d [\gamma^d]p_0^{\text{top}} + \sum_{(x,y) \in \lambda} p_1^{\text{top}}(-1, x)$$

depends also on the sub-dominant part $[\gamma^{d-2}]p_0$ of the polynomial $p_0$, therefore it does not vanish for each $F \in \mathcal{P}$ of degree at most $d - 1$. The solution is to find some linear combination of the expressions of the form (4.4) which would vanish on any $F_0$, i.e. on an arbitrary constant function. This serves as the motivation for the following operator.

4.4. The difference operator.

Definition 4.2. If $F = F(\lambda_1, \ldots, \lambda_\ell)$ is a function of $\ell$ arguments and $1 \leq j \leq \ell$, we define a new function $\Delta_{\lambda_j}F$ by

$$\Delta_{\lambda_j}F(\lambda_1, \ldots, \lambda_\ell) := F(\lambda_1, \ldots, \lambda_{j-1}, \lambda_j + 1, \lambda_{j+1}, \ldots, \lambda_\ell) - F(\lambda_1, \ldots, \lambda_\ell).$$
We call $\Delta_{\lambda_j}$ a \textit{difference operator}.

4.5. \textbf{The case $k = 1$, continued.} We continue our discussion. Our candidate for the functional $\phi$ is given by

$$\phi(F) = (\Delta_{\lambda_1} [A^{d-2}] F)(\lambda) = [A^{d-2}] [F(\lambda_1 + 1, \lambda_2, \lambda_3, \ldots) - F(\lambda_1, \lambda_2, \lambda_3, \ldots)]$$

for some suitable Young diagram $\lambda$. We shall investigate this quantity in the following. The first step is to notice that $\phi$ is linear, thus

$$\phi(F) = \phi(F_0) + \phi(F_1) + \cdots$$

(the sum involves only finitely many summands) and it is enough to study $\phi(F_l)$ for $l \in \mathbb{N}_0$.

For $l = 0$ the function $F_0$ is constant thus $\Delta_{\lambda_1} F_0 = 0$ and $\phi(F_0) = 0$.

For each $l \geq 2$ the evaluation $F_l(\lambda) \in \mathbb{Q}[A, A^{-1}]$ is a Laurent polynomial of degree at most $d - 4l < d - 2$, thus $\phi(F_l) = 0$.

In this way we proved that

$$(4.5) \quad \phi(F) = \phi(F_1) = [A^{d-2}] p_1(\gamma, \alpha\text{-content}(\lambda_1 + 1)) = p_{1\text{top}}(-1, \lambda_1 + 1).$$

This is quite encouraging because it shows that $\phi$ fulfills the requirement \textbf{(R1)} from Problem 4.1.

Note also that the value of $\Delta_{\lambda_1} F(\lambda)$ depends only on the values of $F$ on the Young diagrams with at most $|\lambda| + 1$ boxes, thus if $F$ fulfills \textbf{(4.1)} then $\phi(F) = 0$ for any Young diagram $\lambda$ with at most $n - 2$ boxes which is the requirement \textbf{(R2)}.

Concluding, we may consider a family of $n - 1 = d - 2$ functionals $(\phi_i)$ of the form

$$(4.6) \quad \phi(F) = (\Delta_{\lambda_1} [A^{d-2}] F)(\lambda)$$

where $\lambda = (\lambda_1)$ consists of one row and $\lambda_1 \in \{0, 1, \ldots, n - 2\}$.

The specialization $p_{1\text{top}}(\gamma, c_1) \mapsto p_{1\text{top}}(-1, c_1)$ gives a linear isomorphism between the space of homogeneous polynomials of degree $d - 2$ in the variables $\gamma$ and $c_1$ and the space of (non-homogeneous) polynomials in a single variable $c_1$, of degree at most $d - 2$. By (4.5), the functionals (4.6) provide information about the values of the polynomial $c_1 \mapsto p_{1\text{top}}(-1, c_1)$ in $d - 2$ points. This would determine this polynomial uniquely if we had additional information about the leading coefficient $[c_1^{d-2}] p_{1\text{top}}(-1, c_1)$. 
4.6. **The case** $k = 2$. Let us mimic the above ideas in the case $k = 2$. It is reasonable to consider the functionals of the form

$$
\phi(F) = [A^{d-4}] \Delta_{\lambda_1} \Delta_{\lambda_2} F(\lambda) =
[A^{d-4}](F(\lambda_1 + 1, \lambda_2 + 1, \lambda_3, \ldots) - F(\lambda_1 + 1, \lambda_2, \lambda_3, \ldots) - F(\lambda_1, \lambda_2 + 1, \lambda_3, \ldots) + F(\lambda_1, \lambda_2, \lambda_3, \ldots)).
$$

We encounter an unexpected difficulty: in the case when $\lambda_1 = \lambda_2$, the third summand on the right-hand side involves evaluation of the function $F$ on $(\lambda_1, \lambda_2 + 1, \lambda_3, \ldots)$ which is **not** a Young diagram, thus the value of $F$ on it is not well-defined. This serves as a motivation for the following definition.

4.7. **Extension of the domain of functions on** $\mathbb{Y}$. Let $F$ be a function on the set of Young diagrams. Such a function can be viewed as a function $F(\lambda_1, \ldots, \lambda_\ell)$ defined for all non-negative integers $\lambda_1 \geq \cdots \geq \lambda_\ell$. We will extend its domain, as follows.

**Definition 4.3.** If $(\xi_1, \ldots, \xi_\ell)$ is an arbitrary sequence of non-negative integers, we denote

$$
F^{\text{sym}}(\xi_1, \ldots, \xi_\ell) := F(\lambda_1, \ldots, \lambda_\ell),
$$

where $(\lambda_1, \ldots, \lambda_\ell) \in \mathbb{Y}$ is the sequence $(\xi_1, \ldots, \xi_\ell)$ sorted in the reverse order $\lambda_1 \geq \cdots \geq \lambda_\ell$. In this way $F^{\text{sym}}(\xi_1, \ldots, \xi_\ell)$ is a symmetric function of its arguments.

A Reader accustomed with the very natural notion of $\alpha$-shifted symmetric functions (which is closely related to the algebra of $\alpha$-polynomial functions) may be surprised that the extended function which we consider above is a symmetric function instead of being $\alpha$-shifted symmetric.

This reservation can be worded differently: in Section 3.1 we proved that each function $F \in \mathcal{P}$ can be identified with a certain polynomial $F(\lambda_1, \ldots, \lambda_\ell)$ in the variables $\lambda_1, \ldots, \lambda_\ell$. Why do not we extend the domain of the function $F$ by the means of this polynomial?

The reason for our choice is that with our definition, if a function $F$ fulfills the vanishing condition (4.1) then

$$
F^{\text{sym}}(\xi_1, \ldots, \xi_\ell) = 0
$$

holds true for all tuples of integers $\xi_1, \ldots, \xi_\ell \geq 0$ such that $\xi_1 + \cdots + \xi_\ell \leq n - 1$. This property will be critical for the applications, and it is not guaranteed to hold true for the alternative ways of extending the domain of the function. Nevertheless, our choice of extending the domain of the functions will turn out soon to be a source of some serious difficulties.
4.8. **The case** \( k = 2 \), continued. Our second attempt is to consider the functionals of the form

\[
\phi(F) = [A^{d-4}] \Delta_{\lambda_1} \Delta_{\lambda_2} F^{\text{sym}}(\lambda)
\]

where \( \lambda = (\lambda_1, \lambda_2) \) has at most two rows and at most \( n - 3 \) boxes. Heuristically, the operator \( \Delta_{\lambda_1} \Delta_{\lambda_2} \) measures how a given function changes when one box is added to the first row and one box is added to the second row. In particular, we expect that

\[
(4.7) \quad \Delta_{\lambda_2} F_1(\lambda) = p_1(\gamma, \alpha-\text{content}(\lambda_2 + 1, 1))
\]

does not depend on the value of \( \lambda_1 \) thus

\[
(4.8) \quad \Delta_{\lambda_1} \Delta_{\lambda_2} F_1(\lambda) = 0.
\]

In the following we shall verify if these heuristic claims hold true in reality.

Indeed, in the case \( \lambda_1 > \lambda_2 \) we get no surprises: due to cancellations \( \phi(F_l) = 0 \) for \( l \in \{0, 1\} \) and due to the bounds on degrees of Laurent polynomials \( \phi(F_l) = 0 \) for \( l \geq 3 \). For this reason

\[
\phi(F) = \phi(F_2) = 2[A^{d-4}] p_2(\gamma, \alpha-\text{content}(1, \lambda_1 + 1), \alpha-\text{content}(2, \lambda_2 + 1)) = 2p_2^{\text{top}}(-1, \lambda_1 + 1, \lambda_2 + 1),
\]

which is encouraging.

However, in the case \( \lambda_1 = \lambda_2 \) (which already once turned out to be troublesome)

\[
\phi(F) = [A^{d-4}] \Delta_{\lambda_1} \Delta_{\lambda_2} F^{\text{sym}}(\lambda) = [A^{d-4}](F(\lambda_1 + 1, \lambda_2 + 1, \lambda_3, \ldots) - 2F(\lambda_1 + 1, \lambda_2, \lambda_3, \ldots) + F(\lambda_1, \lambda_2, \lambda_3, \ldots)).
\]

Due to cancellations, \( \phi(F_0) = 0 \); due to bounds on the degrees of Laurent polynomials polynomials \( \phi(F_l) = 0 \) for \( l \geq 3 \) and, just as in the previous case,

\[
\phi(F_2) = 2p_2^{\text{top}}(-1, \lambda_1 + 1, \lambda_2 + 1).
\]

This time, however, we encounter a sad surprise:

\[
\phi(F_1) = [A^{d-4}] [p_1(\gamma, \alpha-\text{content}(\lambda_1 + 1, 2)) - p_1(\gamma, \alpha-\text{content}(\lambda_1 + 1, 1))] = -\frac{\partial}{\partial \lambda_1} p_1^{\text{top}}(-1, \lambda_1 + 1)
\]
which stands in a strong contradiction to (4.8). The explanation is as follows: even though (4.7) holds true, it is not true anymore if \( F_1 \) is replaced by \( F_1^{\text{sym}} \). For the same reason, (4.8) does not hold true if \( F_1 \) is replaced by \( F_1^{\text{sym}} \). Or, worded differently, (4.8) holds true only if \( \lambda_1 > \lambda_2 \) and the case \( \lambda_1 = \lambda_2 \) is the source of the difficulties.

Concluding, for a general Young diagram \( \lambda \)

\[
\phi(F) = [A^{d-4}]\Delta_{\lambda_1}\Delta_{\lambda_2}F^{\text{sym}}(\lambda) = \\
- [\lambda_1 = \lambda_2] \frac{\partial}{\partial \lambda_1} p_{1}^{\text{top}}(-1, \lambda_1 + 1) + 2p_{2}^{\text{top}}(-1, \lambda_1 + 1, \lambda_2 + 1).
\]

This formula has twofold consequences.

On the bright side, the right hand side of (4.9) depends only on the polynomials \( p_{1}^{\text{top}} \), thus it vanishes if \( F \) is of degree at most \( d - 1 \). This shows that as a collection of functionals \( \phi_i \) we may take (4.9) over all Young diagrams \( \lambda = (\lambda_1, \lambda_2) \) with at most two rows and at most \( n - 3 \) boxes. Since the polynomial \( p_{1}^{\text{top}} \) has been determined previously, the functionals (4.9) provide the information about the values of the polynomial \( p_{2}^{\text{top}}(-1, c_1, c_2) \) on all pairs of integers \( c_1, c_2 \) such that \( c_1, c_2 \geq 1 \) and \( c_1 + c_2 \leq n - 1 = d - 2 \). One can show that since \( p_{2}^{\text{top}}(-1, c_1, c_2) \) is a symmetric polynomial of degree \( d - 4 \), this determines the polynomial \( p_{2}^{\text{top}}(-1, c_1, c_2) \) uniquely (the uniqueness will be shown in Lemma 7.1 the proof of existence we leave as an exercise).

On the dark side, the fact that the right-hand side of (4.9) involves not only the polynomial \( p_{2}^{\text{top}} \) but \( p_{1}^{\text{top}} \) as well will be the source of serious difficulties.

Similarly as in the case \( k = 1 \) considered above, the specialization \( p_{2}^{\text{top}}(\gamma, c_1, c_2) \mapsto p_{2}^{\text{top}}(-1, c_1, c_2) \) gives a linear isomorphism between:

- the space of homogeneous polynomials of degree \( d - 4 \) in the variables \( \gamma, c_1, c_2 \) which are symmetric with respect to the variables \( c_1 \) and \( c_2 \), and
- the space of (non-homogeneous) symmetric polynomials in \( c_1, c_2 \), of degree at most \( d - 4 \).

This shows that the values of the functionals (4.9) together with the information about \( p_{1}^{\text{top}} \) determine \( p_{2}^{\text{top}}(\gamma, c_1, c_2) \) uniquely.

4.9. The general value of \( k \). For a general value of \( k \geq 1 \) as a collection of functionals \( \phi_i \) we may take

\[
\phi(F) = [A^{d-2k}]\Delta_{\lambda_1} \cdots \Delta_{\lambda_k}F^{\text{sym}}(\lambda_1, \ldots, \lambda_k)
\]
over the Young diagrams $\lambda = (\lambda_1, \ldots, \lambda_k)$ which have at most $k$ rows and $|\lambda| \leq n - k - 1$. However, it is not obvious that so defined functionals have the required properties, in particular if they fulfill property (R1) from Problem 4.1. We will prove it later on in Lemma 6.8.

It is worth pointing out that for $k \geq 3$ the above collection of functionals is, in fact, too large and we will show later on (Theorem 7.3, condition (T3)) that it is enough to consider the functionals (4.10) over the Young diagrams $\lambda = (\lambda_1, \ldots, \lambda_k)$ which have at most $k$ rows and $|\lambda| \leq n + 1 - 2k < n - k - 1$. We find this fact rather surprising and, regretfully, we do not have a good intuitive explanation for it.

A good intuitive way of thinking about the quantity
\[(4.11) \quad \Delta_{\lambda_1} \cdots \Delta_{\lambda_k} F^{\text{sym}}(\lambda_1, \ldots, \lambda_k)\]
which contributes to (4.10) is that we consider all ways of adding at most one box to each of the rows of the diagram $\lambda = (\lambda_1, \ldots, \lambda_k)$. We let the boxes fall down in a Tetris-like manner so that the resulting configuration of the boxes corresponds to a Young diagram. The quantity (4.11) is a signed sum of the values of the function $F$ on the resulting Young diagrams.

4.10. The moral lessons. It is time to summarize the discussion from this section.

(M1) Usually the difference operator $\Delta_{\lambda_1} \cdots \Delta_{\lambda_n} F^{\text{sym}}$ answers the question how the function $F$ would change if one box is added to the first row, one box is added to the second row, \ldots, and one box is added to the $n$-th row. Regretfully, this interpretation breaks down at the “diagonals” where some of the coordinates $\lambda_1, \ldots, \lambda_n$ are repeated.\[100\]

(M2) The viewpoint that for $F \in \mathcal{P}$ its evaluation $F(\lambda)$ is a symmetric function applied to the alphabet of $\alpha$-contents of $\lambda$ (see Eq. (4.2)) is not very convenient if we try to evaluate $F^{\text{sym}}(\lambda)$ on $\lambda$ which is not a Young diagram or evaluate $\phi(F)$ for the functional (4.10).\[100\]

(M3) The latter observation motivates searching for some alternative ways of viewing the functions $F \in \mathcal{P}$, so that $F^{\text{sym}}$ and thus $\phi(F)$ would be better tractable. One such a viewpoint will be given in Section 6 where we consider a larger algebra $\mathcal{R} \supseteq \mathcal{P}$ of functions which admit a convenient representation for $F^{\text{sym}}$. Another viewpoint is to consider the elements of $\mathcal{P}$ as linear combinations of the functions $\lambda \mapsto \mathfrak{N}_F(\lambda)$ for some bipartite graphs $F$.\[100\]

5. Enumerative results for the top-degree part of Jack character

This section is quite independent from the remaining part of the paper and may be skipped by an impatient reader. We shall revisit Section 4 for
the special case when \( F = \text{Ch}_n \) is the Jack character in order to find some explicit closed formulas for its top-degree part \( \text{Ch}_n^{\text{top}} \).

Just as in (4.2) we consider the expansion

\[
\text{Ch}_n(\lambda) = p_0(\gamma) + \sum_{\square_1 \in \lambda} p_1(\gamma, c_1) + \sum_{\square_1, \square_2 \in \lambda} p_2(\gamma, c_1, c_2) + \cdots ;
\]

our goal is to find explicitly the top-degree part \( p_k^{\text{top}} \) of the polynomial \( p_k \). The key observation is that, by the very definition (requirement (R2) in Problem 4.1), all functionals \( \phi_i \) which were found in Section 4 vanish on \( \text{Ch}_n \).

It will be convenient to use the following notation: for integer \( k \geq 0 \) we define

\[
x_k = (x + \gamma)(x + 2\gamma) \cdots (x + k\gamma) \quad \text{for } k \text{ factors},
\]

5.1. The case \( k = 0 \). Equation (4.3) shows that

\[
p_0^{\text{top}} = 0
\]

(in fact, it is elementary to show a stronger result that \( p_0 = 0 \)).

5.2. The case \( k = 1 \). Equation (4.5) shows that the polynomial \( c_1 \mapsto p_1^{\text{top}}(-1, c_1) \) of degree \( n - 1 \) vanishes for \( c_1 \in \{1, 2, \ldots, n - 1\} \). It follows that

\[
p_1^{\text{top}}(-1, c_1) = s (c_1 - 1) \cdots (c_1 - (n - 1)) \quad \text{for some constant } s \in \mathbb{Q}.
\]

The exact value of this constant can be found by the investigation of the polynomial

\[\mathbb{N}_0 \ni \lambda_1 \mapsto \text{Ch}_n(\lambda_1, 0, 0, \ldots)\]

when the Jack character is evaluated on a Young diagram with a single row. Polynomials of this kind were considered (in a bigger generality) in Section 3. In particular, Proposition 3.4 shows that

\[ [\lambda_1^n] \text{Ch}_n(\lambda_1, 0, 0, \ldots) = A_n^{n-1}. \]

On the other hand, (5.1) implies that

\[ [\lambda_1^n] \text{Ch}_n(\lambda_1, 0, 0, \ldots) = \frac{A_n^{n-1}}{n} [c_1^{n-1}] p_1^{\text{top}}. \]

We conclude that \( s = n \).
As we already discussed at the very end of Section 4.5, the knowledge of \( p_{1}^{\text{top}}(-1, c_1) \) allows us to recover the polynomial \( p_{1}^{\text{top}} \) itself, thus

\[
(5.4) \quad p_{1}^{\text{top}}(\gamma, c_1) = n c_1^{\gamma - \gamma_1}.
\]

5.3. The case \( k = 2 \). Denote

\[
Q(c_1, c_2) = p_{2}^{\text{top}}(-1, c_1, c_2).
\]

We have to find a symmetric polynomial \( Q \) of degree at most \( d - 4 = n - 3 \) for which the right-hand side of (4.9) would vanish for any Young diagram \( \lambda = (\lambda_1, \lambda_2) \) with at most two rows and at most \( n - 3 \) boxes; in other words we require that

\[
(5.5) \quad Q(c_1, c_2) = [c_1 = c_2] \frac{n}{2} \frac{d}{dc_1} (c_1 - 1) \cdots (c_1 - (n - 1))
\]

holds for all pairs of integers \( c_1, c_2 \geq 1 \) such that \( c_1 + c_2 \leq n - 1 \).

It is not clear if there is some systematic way of finding such a function; our strategy is to make the right guess. The special form of the right-hand side of (5.5) suggests that we may try some kind of divided difference

\[
Q_{\text{guessed}}(c_1, c_2) := \frac{n}{2} \frac{d}{dc_1} (c_1 - 1) \cdots (c_1 - (n - 1)) - (c_2 - 1) \cdots (c_2 - (n - 1)).
\]

Note that despite the division, the right-hand side is a polynomial. The guess given by the function \( Q_{\text{guessed}} \) is not bad: it fulfills the condition (5.5). Unfortunately, \( Q_{\text{guessed}} \) is a polynomial of degree \( n - 2 \) and not, as required, of degree \( n - 3 \).

We shall try to improve our guess by adding to \( Q_{\text{guessed}} \) some counterterms which would cancel the monomials of degree \( n - 2 \). Our second attempt is

\[
Q_{\text{improved}}(c_1, c_2) := Q_{\text{guessed}}(c_1, c_2) - \frac{n}{2} \sum_{i, j \geq 0 \atop i + j = n - 2} \binom{c_1 - 1}{i} \binom{c_2 - 1}{j}.
\]

Indeed, \( Q_{\text{improved}} \) has all required properties. Furthermore, Lemma 7.1 shows that such a polynomial is unique, therefore

\[
p_{2}^{\text{top}}(-1, c_1, c_2) = Q_{\text{improved}}(c_1, c_2).
\]
Just as we discussed at the very end of Section 4.8 this allows us to recover the polynomial $p_{2}^{\text{top}}$ itself:

\[
p_{2}^{\text{top}}(\gamma, c_1, c_2) = \frac{n}{2\gamma} \left[ \left( \sum_{i,j \geq 0, i+j=n-2} c_1^{i} c_2^{j} \right) - \frac{c_1^{n-1} - c_2^{n-1}}{c_1 - c_2} \right].
\]

5.4. The general $k$. One could continue this procedure for $k = 3, 4, \ldots$ and then look for some pattern governing the closed formulas for the polynomials $p_{k}^{\text{top}}$. Regretfully, we failed to find a closed formula already for $k = 3$.

Our positive findings so far (Equations (5.3), (5.4), (5.6)) can be summarized as follows.

**Theorem 5.1.** For each integer $n \geq 1$

\[
Ch_n(\lambda) = n \sum_{\square_1 \in \lambda} c_1^{n-1} + \frac{n}{2\gamma} \sum_{\square_1, \square_2 \in \lambda} \left[ \left( \sum_{i,j \geq 0, i+j=n-2} c_1^{i} c_2^{j} \right) - \frac{c_1^{n-1} - c_2^{n-1}}{c_1 - c_2} \right] + \cdots + (\text{terms of degree at most } n),
\]

where the ellipsis symbol on the right hand side denotes the terms for which the author of the manuscript failed to find a closed formula. The sums run over all the boxes of $\lambda$ and the symbol $c_i = \alpha$-content($\square_i$) denotes the corresponding content. The symbol $x^\lambda$ has been defined in (5.2).

Our failure to find a closed formula for Jack character $Ch_n$ or its top-degree $Ch_n^{\text{top}}$ of the form (5.7) is not very frustrating, since in the remaining part of the paper we shall prove a closed formula for $Ch_n^{\text{top}}$ of a different flavour (see Theorem 1.21).

6. The algebra $R$ of row functions

In this section we shall investigate a certain filtered algebra $R$ (the algebra of row functions) which was invented in order to overcome some difficulties which we discussed in Section 4. In particular, this algebra is very convenient for the investigation of the difference operators of the form $\Delta_{\lambda_1} \cdots \Delta_{\lambda_n} F_{\sym}$. This algebra $R$ will be the key ingredient in the proofs of the main results of the current paper, as well as for the forthcoming paper [Sni16].
Before delving into the technical details, we shall provide here a brief heuristic overview.

The first advantage of the algebra \( \mathcal{R} \) of row functions over the algebra \( \mathcal{P} \) of polynomial functions is that for each \( F \in \mathcal{R} \) the extension of its domain \( F^{\text{sym}} \) is given in a very simple way (see Remark 6.3 below), unlike for the elements of \( \mathcal{P} \) (see (M2) from Section 4.10).

The second advantage of \( \mathcal{R} \) over \( \mathcal{P} \) lies in the fact that the action of the difference operators on the elements of \( \mathcal{R} \) is very simple.

The third convenient feature is that each element of \( \mathcal{P} \) of degree \( d \) is also an element of \( \mathcal{R} \) of the same degree \( d \) (see the results of Section 6.2 below). For this reason it is sometimes convenient to regard the polynomial functions as row functions. An example of this viewpoint is Lemma 6.8 which, roughly speaking, shows that the functionals (4.10) have the required property (R1) from Problem 4.1.

For all these advantages of \( \mathcal{R} \) over \( \mathcal{P} \) there is some price to pay. Namely, the relationship between:

- the information about a function \( F \), viewed as an element of \( \mathcal{P} \) (for example, the information about the polynomials \( p_0, p_1, \ldots \) from (1.4)), and
- the information about \( F \), viewed this time as an element of \( \mathcal{R} \) (for example, the information about the convolution kernel \( f_0, f_1, \ldots \) from (6.1))

is quite complicated. This relationship is provided (somewhat implicitly) by Lemma 6.5 and Lemma 6.4 below. On the bright side, Section 6.3 provides some partial results about this relationship, at least for the top-degree terms.

### 6.1. Row functions.

**Definition 6.1.** Let a sequence (indexed by \( r \geq 0 \)) of symmetric functions \( f_r : \mathbb{N}_0^r \rightarrow \mathbb{Q}[A, A^{-1}] \) be given, where

\[ \mathbb{N}_0 = \{0, 1, 2, \ldots \}. \]

We assume that:

- if \( 0 \in \{x_1, \ldots , x_r\} \) then \( f_r(x_1, \ldots , x_r) = 0 \),
- \( f_r = 0 \) except for finitely many values of \( r \),
- there exists some integer \( d \geq 0 \) with the property that for all \( r \geq 0 \) and all \( x_1, \ldots , x_r \in \mathbb{N}_0 \), the evaluation \( f_r(x_1, \ldots , x_r) \in \mathbb{Q}[A, A^{-1}] \) is a Laurent polynomial of degree at most \( d - 2r \).

We define a function \( F : \mathbb{Y} \rightarrow \mathbb{Q}[A, A^{-1}] \) given by

\[
F(\lambda) := \sum_{r \geq 0} \sum_{i_1 < \cdots < i_r} f_r(\lambda_{i_1}, \ldots , \lambda_{i_r}).
\]
We will say that $F$ is a row function of degree at most $d$ and that $(f_r)$ is the convolution kernel of $F$. The set of such row functions will be denoted by $R$.

**Remark 6.2.** Note that the right-hand side of (6.1) involves only the values of $f_r$ over $r \leq \ell(\lambda)$ thus the collection of equalities (6.1) can be viewed as an upper-triangular system of linear equations. It follows immediately that for a given row function $F$, the corresponding convolution kernel is determined uniquely. This explains why the above definition of the degree of a row function is well-defined, i.e., it does not depend on some arbitrary choice of the convolution kernel.

**Remark 6.3.** In Section 4.7 we explained how to extend the domain of a function $F : \mathbb{Y} \to \mathbb{Q}[A, A^{-1}]$ which was originally defined on the set of Young diagrams. Note that the definition (6.1) of a row function does not require any modifications in order to give rise to such an extension. For this reason, if $F$ is a row function, we will identify it with its extension $F^{\text{sym}}$.

### 6.2. Filtration on $R$.

**Lemma 6.4.** The set $R$ of row functions equipped with the pointwise product and pointwise addition forms a filtered algebra.

**Proof.** A product of two terms appearing on the right-hand side of (6.1) is again of the same form:

\[
(6.2) \sum_{i_1 < \cdots < i_r} f_r(\lambda_{i_1}, \ldots, \lambda_{i_r}) \cdot \sum_{j_1 < \cdots < j_s} g_s(\lambda_{j_1}, \ldots, \lambda_{j_s}) = \sum_{0 \leq t \leq r+s} \sum_{k_1 < \cdots < k_t} f_r(\lambda_{i_1}, \ldots, \lambda_{i_r}) \cdot g_s(\lambda_{j_1}, \ldots, \lambda_{j_s}) = h_t(\lambda_{k_1}, \ldots, \lambda_{k_t}) := \sum_{0 \leq t \leq r+s} \sum_{k_1 < \cdots < k_t} h_t(\lambda_{k_1}, \ldots, \lambda_{k_t})
\]

which shows that $R$ forms an algebra.

Assume that the two factors on the left-hand of (6.2) are row functions of degree at most, respectively, $d$ and $e$. It follows that each value of $f_r$ is a Laurent polynomial of degree at most $d - 2r$ and each value of $g_s$ is a Laurent polynomial of degree at most $e - 2s$. Thus each value of $h_t$, the convolution kernel defined by the curly bracket on the right-hand side of
is a Laurent polynomial of degree at most \( d + e - 2(r + s) \leq d + e - 2t \) which shows that the product is a row function of degree at most \( d + e \), which concludes the proof that \( \mathcal{R} \) is a filtered algebra.

**Lemma 6.5.** Let \( F \in \mathcal{P} \) be an \( \alpha \)-polynomial function of degree at most \( d \). Then \( F \in \mathcal{R} \) is also a row function of degree at most \( d \).

**Proof.** By Lemma 6.4 and the last part of Proposition 2.14 it is enough to prove the claim for the generators (2.9), i.e., to show for each \( n \geq 2 \) that \( T_n \) is a row function of degree at most \( n \) and that \( \gamma \) is a row function of degree 1. We will do it in the following.

For a Young diagram \( \lambda \) we denote by \( \lambda^T = (\lambda^T_1, \lambda^T_2, \ldots) \in \mathbb{Y} \) the transposed diagram. The binomial formula implies that

\[
(6.3) \quad T_n(\lambda) = (n - 1) \sum_{x \geq 1} \sum_{1 \leq y \leq \lambda^T_x} \left( Ax - A^{-1} y \right)^{n-2} =
\]

\[
(n - 1) \sum_{q \geq 0} A^{n-2-2q} \binom{n-2}{q} (-1)^q \sum_{x \geq 1} x^{n-2-q} \sum_{1 \leq y \leq \lambda^T_x} y^q =
\]

\[
(6.3) \quad (n - 1) \sum_{u \geq 1} A^{n-2u} \binom{n-2}{u-1} (-1)^{u-1} \sum_{x \geq 1} x^{n-1-u} \sum_{1 \leq y \leq \lambda^T_x} y^{u-1},
\]

where the last equality follows from the change of variables \( u := q + 1 \).

The expression (♠) marked above by the curly bracket, namely

\[
\mathbb{N}_0 \ni s \mapsto \sum_{1 \leq y \leq s} y^{u-1},
\]

is a polynomial function of degree \( u \), thus it can be written as a linear combination (with rational coefficients) of the family of polynomials \( \mathbb{N}_0 \ni s \mapsto \binom{s}{r} \) indexed by \( r \in \{0, 1, \ldots, u\} \). Notice that \( \lambda^T_x \) is the number of rows of \( \lambda \) which are bigger or equal than \( x \), thus (♠) is a linear combination (with rational coefficients) of

\[
(6.4) \quad \left( \binom{\lambda^T_x}{r} \middle| \sum_{i_1 < \cdots < i_r} [\lambda_{i_1} \geq x] \cdots [\lambda_{i_r} \geq x] \right)
\]

over \( r \in \{0, 1, \ldots, u\} \). This shows that \( T_n \) is a row function.

Let \( f_0, f_1, \ldots \) be the corresponding convolution kernel. Equation (6.3) shows that each value \( f_r(x_1, \ldots, x_r) \) is a linear combination (with rational coefficients) of the expressions

\[
A^{n-2u} \left[ x_1 \geq x \right] \cdots \left[ x_r \geq x \right] \in \mathbb{Q} \left[ A, A^{-1} \right]
\]
over \( u \geq 1 \), over \( x \geq 1 \), and \( r \leq u \). This Laurent polynomial is degree at most \( n - 2u \leq n - 2r \), which shows that \( T_n \) is a row function of degree at most \( n \), as required.

We define

\[
 f_r = \begin{cases} 
 \gamma & \text{if } r = 0, \\
 0 & \text{if } r \geq 1.
\end{cases}
\]

Clearly, the corresponding row function \( F \) fulfills \( F(\lambda) = \gamma \) for any \( \lambda \in \mathcal{Y} \). This shows that \( \gamma \) is a row function of degree at most 1, as required. \( \square \)

6.3. **Top-degree part of row functions.**

**Definition 6.6.** Let \( d \geq 0 \) be an integer and let \( F \) given by (6.1) be a row function of degree at most \( d \). We define its top-degree part as:

\[
 F^\text{top}(\lambda) := \sum_{r \geq 0} A^{d-2r} \sum_{i_1 < \cdots < i_r} [A^{d-2r}] f_r(\lambda_{i_1}, \ldots, \lambda_{i_r}).
\]

We will also say that the summand corresponding to a specified value of \( r \) (i.e., the \( r \)-fold sum over the rows) has rank \( r \).

**Lemma 6.7.** Let \( r \geq 0 \) and \( d \geq 2r \) be integers and let \( p(c_1, \ldots, c_r) \) be a symmetric polynomial in its \( r \) arguments with the coefficients in the polynomial ring \( \mathbb{Q}[\gamma] \). We assume that \( p \), viewed as a polynomial in \( \gamma, c_1, \ldots, c_r \), is a homogeneous polynomial of degree \( d - 2r \). We consider the row function of degree at most \( d \) given by

\[
 F(\lambda) = \sum_{\Box_1, \ldots, \Box_r \in \lambda} p(c_1, \ldots, c_r),
\]

where

\[
 c_1 := \alpha\text{-content}(\Box_1), \ldots, \ c_r := \alpha\text{-content}(\Box_r).
\]

Then each summand for the top-degree part of \( F \) has rank at least \( r \); the summand with the rank equal to \( r \) is given by

\[
\lambda \mapsto A^{d-2r} r! \sum_{i_1 < \cdots < i_r} \sum_{1 \leq x_1 \leq \lambda_{i_1}} \cdots \sum_{1 \leq x_r \leq \lambda_{i_r}} p(x_1, \ldots, x_r) \bigg|_{\gamma = -1},
\]

where on the right-hand side we consider the evaluation of the polynomial \( p \) for \( \gamma = -1 \).

**Proof.** We start with a general investigation of the top-degree part of various row functions. The proof of Lemma 6.5 shows that

\[
 \gamma^\text{top} = -A
\]

consists of a single summand of rank 0.
The extraction of the top-degree part of the row function (6.3) corresponds to the restriction to the summand \( r = u \geq 1 \) (with the notations of (6.4)). For this reason \( T_n^{\text{top}}(\lambda) \) involves only the summands with the rank at least 1. Furthermore, the term of rank 1 is given explicitly in the following expansion:

\[
T_n^{\text{top}}(\lambda) = A^{n-2} \sum_{i} \sum_{1 \leq x \leq \lambda_i} (n-1)x^{n-2} + \text{(summands of rank at least 2)}.
\]

We shall revisit (6.2) in order to investigate the top-degree part of a product of two row functions. The summands on the right-hand side do not contribute to the top-degree part unless \( t = r + s \) and

\[
\{k_1, \ldots, k_t\} = \{i_1, \ldots, i_r\} \sqcup \{j_1, \ldots, j_s\}
\]

is a decomposition into disjoint sets. This shows that the top-degree part of a product involves only the summands with the rank at least the sum of the ranks of the original factors. Furthermore, the top-degree summand of this minimal rank is given very explicitly.

We come back to the proof of the Lemma. Assume for simplicity that the polynomial \( p \) is a monomial. In this case \( F \) is — up to simple numerical factors — a product of some power of \( \gamma \) and of exactly \( r \) factors of the form \( T_n \) over \( n \geq 2 \). The above discussion shows that \( F^{\text{top}} \) involves only the summands of rank at least \( r \), and gives a concrete formula for the sum- 

\[
\Delta_{\lambda_1} \cdots \Delta_{\lambda_k} F^\text{sym}(\lambda_1, \lambda_2, \ldots) = 0.
\]

Proof. We know that \( F \) is a sum of the functions of the form

\[
\sum_{i_1 < \cdots < i_l} f_i(\lambda_{i_1}, \ldots, \lambda_{i_l}),
\]

over \( l \geq 0 \) and each value of \( f_i \) is a Laurent polynomial of degree at most \( d - 1 - 2l \).

6.4. The difference operator vanishes on elements of small degree.

**Lemma 6.8.** Let \( d \geq 1 \) be an integer and assume that \( F \in \mathcal{R} \) is of degree at most \( d - 1 \).

Then for each integer \( k \geq 0 \) and each Young diagram \( \lambda = (\lambda_1, \lambda_2, \ldots) \)

\[
[A^{d-2k}] \Delta_{\lambda_1} \cdots \Delta_{\lambda_k} F^\text{sym}(\lambda_1, \lambda_2, \ldots) = 0.
\]

Proof. We know that \( F \) is a sum of the functions of the form

\[
\sum_{i_1 < \cdots < i_l} f_i(\lambda_{i_1}, \ldots, \lambda_{i_l}),
\]

over \( l \geq 0 \) and each value of \( f_i \) is a Laurent polynomial of degree at most \( d - 1 - 2l \).
Clearly,\[
\Delta_{\lambda_1} \ldots \Delta_{\lambda_k} f_l(\lambda_{i_1}, \ldots, \lambda_{i_l}) = 0 \quad \text{if } \{1, \ldots, k\} \not\subseteq \{i_1, \ldots, i_l\}
\]
thus\[
(6.6) \quad [A^{d-2k}] \Delta_{\lambda_1} \ldots \Delta_{\lambda_k} \sum_{i_1 < \cdots < i_l} f_l(\lambda_{i_1}, \ldots, \lambda_{i_l})
\]
vanishes if \(l < k\). On the other hand, for \(l \geq k\), the expression \(f_l(\lambda)\) is a Laurent polynomial of degree at most \(d - 1 - 2l \leq d - 1 - 2k\) thus (6.6) vanishes as well. □

7. The key theorem 7.2

The main result of the current section is Theorem 7.2 which will be the key tool for the proofs of the main results of the current paper, namely Theorems 1.7 and 1.21, as well as for the results of the forthcoming paper [Sn16].

7.1. Multivariate polynomials having lots of zeros.

**Lemma 7.1.** Let \(k \geq 0\) and \(d \geq 0\) be integers.

- Let \(p(x_1, \ldots, x_k) \in \mathbb{Q}[x_1, \ldots, x_k]\) be a polynomial of degree at most \(d\). Assume that
  \[
  p(x_1, \ldots, x_k) = 0
  \]
  holds true for all integers \(x_1, \ldots, x_k \geq 1\) such that
  \[
  x_1 + \cdots + x_k \leq d + k.
  \]
  Then \(p = 0\).

- Let \(p(x_1, \ldots, x_k) \in \mathbb{Q}[x_1, \ldots, x_k]\) be a symmetric polynomial of degree at most \(d\). Assume that (7.1) holds true for all integers \(x_1 \geq \cdots \geq x_k \geq 1\) such that (7.2) holds true. Then \(p = 0\).

**Proof.** We will show the first part of the claim by induction over \(k\).

The case \(k = 1\). In the case \(k = 1\) if follows that \(p(x_1) \in \mathbb{Q}[x_1]\) is a polynomial of degree at most \(d\) which has at least \(d + 1\) zeros; it follows that \(p = 0\).

We consider the case \(k \geq 2\) and we assume that the first part of the lemma is true for \(k' := k - 1\). The polynomials
\[
1, \ x_k - 1, \ (x_k - 1)(x_k - 2), \ldots, \ (x_k - 1)(x_k - 2) \cdots (x_k - d_k) \in \mathbb{Q}[x_k]
\]
form a linear basis of the space of these polynomials in a single variable which are of degree at most $d_k$. Thus any monomial

$$x_1^{d_1} \cdots x_k^{d_k}$$

(7.3)

can be written as a linear combination of the polynomials of the form

$$x_1^{d_1} \cdots x_{k-1}^{d_{k-1}} (x_k - 1)(x_k - 2) \cdots (x_k - r)$$

over $r \leq d_k$. Note that the degree of each of these polynomials does not exceed the degree of the original monomial (7.3). Thus $p$ can be written in the form

$$p = \sum_{0 \leq r \leq d} p_r(x_1, \ldots, x_k-1) \cdot (x_k - 1)(x_k - 2) \cdots (x_k - r),$$

where $p_r \in \mathbb{Q}[x_1, \ldots, x_{k-1}]$ is a polynomial of degree at most $d - r$.

We will show by a nested induction over the variable $r$ that $p_r = 0$ for each $0 \leq r \leq d$. Assume that $p_l = 0$ for each $l < r$. It follows that

$$p(x_1, \ldots, x_k-1, r+1) = r! \cdot p_r(x_1, \ldots, x_k-1),$$

thus

$$p_r(x_1, \ldots, x_k-1) = 0$$

holds true for all integers $x_1, \ldots, x_{k-1} \geq 1$ such that

$$x_1 + \cdots + x_{k-1} \leq (d - r) + (k - 1).$$

It follows that $p_r$ fulfills the condition (2.1) for $d' := d - r$ and $k' := k - 1$ thus the inductive hypothesis (with respect to the variable $k$) can be applied. It follows that $p_r = 0$. This concludes the proof of the inductive step over the variable $r$.

The second part of the lemma is a direct consequence of the first part. 

7.2. **The key Theorem**

The assumptions of the following theorem might seem complicated at the first sight. They have been modeled after the properties of the Jack characters; in particular $F := \text{Ch}_\pi$ fulfills the assumptions (except for the assumption (Z2)) for $n := |\pi|$ and $r := \ell(\pi)$.

**Theorem 7.2** (The key tool). Let integers $n \geq 0$ and $r \geq 1$ be given. Assume that:

1. **(Z1)** $F \in \mathcal{P}$ is of degree at most $n + r$;
2. **(Z2)** we assume that for each $m \geq 1$ the polynomial in $m$ variables

$$\mathbb{Y} \ni (\lambda_1, \ldots, \lambda_m) \mapsto F(\lambda_1, \ldots, \lambda_m) \in \mathbb{Q}[A, A^{-1}]$$

is of degree at most $n - 1$;
(Z3) the equality

\[ [A^{n+r-2k}] \Delta_{\lambda_1} \cdots \Delta_{\lambda_k} F_{\text{sym}}(\lambda_1, \ldots, \lambda_k) = 0 \]

holds true for the following values of \( k \) and \( \lambda \):

- \( k = r \) and \( \lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Y} \) with at most \( r \) rows is such that \( |\lambda| \leq n + r - 2k - 1 \);
- \( k > r \) and \( \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Y} \) with at most \( k \) rows is such that \( |\lambda| \leq n + r - 2k \);

(Z4) we consider the following cases:

- in the case when \( r = 0 \) we make no assumptions (this might occur only for the alternative formulation, see the end of the theorem);
- in the case when \( r = 1 \) we assume that \( F(\emptyset) \in \mathbb{Q}[A, A^{-1}] \) is a Laurent polynomial of degree at most \( n \);
- in the case when \( r \geq 2 \) we assume that for each \( \lambda \in \mathbb{Y} \), the Laurent polynomial \( F(\lambda) \in \mathbb{Q}[A, A^{-1}] \) is of degree at most \( n - r + 1 \).

Then \( F \in \mathcal{P} \) is of degree at most \( n + r - 1 \).

The result remains valid for all integers \( n \geq 0 \) and \( r \geq 0 \) if the assumption (Z2) is removed and the condition (Z3) is replaced by the following one:

(Z3a) the equality \( (7.4) \) holds true for all \( k \geq r \) and \( \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Y} \) with at most \( k \) rows such that \( |\lambda| \leq n + r - 2k \).

Proof. Recall that in Proposition 2.14 we proved that the degree on \( \mathcal{P} \) specified in Section 2.8 coincides with the degree from Inconvenient Definition 1.5. Thus, by Inconvenient Definition 1.5, it follows that the function \( F \) can be written in the form

\[ F(\lambda) = \sum_{k \geq 0} \sum_{\square_1, \ldots, \square_k \in \lambda} p_k(c_1, \ldots, c_k) \]

where

\[ c_1 := \alpha\text{-content}(\square_1), \quad \ldots, \quad c_k := \alpha\text{-content}(\square_k), \]

and where \( p_k \) is a symmetric polynomial in its \( k \) arguments with the coefficients in the polynomial ring \( \mathbb{Q}[\gamma] \). Furthermore, the condition (Z1) implies that we may assume that \( p_k(c_1, \ldots, c_k) \) — this time viewed as a polynomial in \( k + 1 \) variables: \( \gamma, c_1, \ldots, c_k \) — is a polynomial of degree at most \( n + r - 2k \) (it is a reformulation of Proposition 2.14).
The statement of the theorem would follow if we can show that \( p_k(c_1, \ldots, c_k) \) is, in fact, of degree at most \( n + r - 2k - 1 \). We will show this claim by induction over \( k \geq 0 \); assume that \( p_m(c_1, \ldots, c_m) \) is of degree at most \( n + r - 2m - 1 \) for each \( m < k \).

The curly bracket in (7.5) serves as the definition of the functions \( F_0, F_1, \ldots \) on the set \( \mathbb{Y} \) of Young diagrams. In the following we will investigate the quantity — which is analogous to (7.4) — given by

\[
(7.6) \quad [A^{n+r-2k}] \Delta_{\lambda_1} \cdots \Delta_{\lambda_k} F^\text{sym}_m(\lambda_1, \ldots, \lambda_k)
\]

for \((\lambda_1, \ldots, \lambda_k) \in \mathbb{Y} \) and various choices of the variable \( m \geq 0 \).

- **The case \( m > k \).** Firstly, observe that \( \gamma \) as well as \( \alpha\)-content(\( \square \)) (for any box \( \square \in \mathbb{N}^2 \)), viewed as Laurent polynomials in the variable \( A \), are of degree at most 1, thus \( F_m(\lambda) \) is a Laurent polynomial of degree at most \( n + r - 2m \) which is strictly smaller than \( n + r - 2k \). It follows that for \( m > k \) the quantity (7.6) vanishes.

- **The case \( m < k \).** From the inductive hypothesis, \( F_m \in \mathcal{P} \) is an \( \alpha \)-polynomial function of degree at most \( n + r - 1 \). We apply Lemma 6.8 for \( d := n + r \). In this way we proved that for \( m < k \) the quantity (7.6) vanishes.

- **The case \( m = k \).** Let \( p_k^{\text{top}} \) denote the homogeneous part of degree \( n + r - 2k \) of the polynomial \( p_k(c_1, \ldots, c_k) \) viewed as a polynomial in the variables \( \gamma, c_1, \ldots, c_k \). Our ultimate goal will be to prove that \( p_k^{\text{top}} = 0 \).

  We will revisit the case \( m < k \) considered above and discuss the changes in the reasoning. We study now the function \( F_k \). As the induction hypothesis cannot be applied, the assumption of Lemma 6.8 is not satisfied for \( F := F_k \) and \( d := n + r \) and we have to revisit its proof. We shall do it in the following.

  The upper bound on the degree of the Laurent polynomial \( f_l(\lambda) \) is weaker, given by \( n + r - 2l \). One can easily see that the only case in which (6.6) could possibly be non-zero is for \( l = k = m \), thus

\[
(7.7) \quad (7.6) = \Delta_{\lambda_1} \cdots \Delta_{\lambda_k} [A^{n+r-2k}] \sum_{i_1 < \cdots < i_k} f_k(\lambda_{i_1}, \ldots, \lambda_{i_k}).
\]

Clearly, the expression (\( \diamond \)) is directly related to the top-degree part of \( F_k \) of rank \( k \). The latter can be computed explicitly by Lemma 6.7.
Thus

\[(7.7)\]
\[
\Delta_{\lambda_1} \ldots \Delta_{\lambda_k} k! \sum_{i_1 < \cdots < i_k} \sum_{1 \leq x_1 \leq \lambda_{i_1}} \cdots \sum_{1 \leq x_k \leq \lambda_{i_k}} p_k^{\text{top}}(x_{i_1}, \ldots, x_{i_k})\mid_{\gamma := -1} =
\]
\[
k! \ p_k^{\text{top}}(\lambda_1 + 1, \ldots, \lambda_k + 1)\mid_{\gamma := -1}.
\]

This finishes our discussion of the quantity \((7.6)\) for various choices of the variable \(m\). The conclusion is that

\[(7.8)\]
\[
[a^{n+r-2k}] \Delta_{\lambda_1} \ldots \Delta_{\lambda_k} F_{\text{sym}}(\lambda_1, \ldots, \lambda_k) =
\]
\[
k! \ p_k^{\text{top}}(\lambda_1 + 1, \ldots, \lambda_k + 1)\mid_{\gamma := -1}
\]
holds true for an arbitrary Young diagram \((\lambda_1, \ldots, \lambda_k) \in \mathbb{Y}\) with at most \(k\) rows. We will use this equality to finish the proof.

Notice that the polynomial

\[(7.9)\]
\[
p_k^{\text{top}}(c_1, \ldots, c_k)\mid_{\gamma := -1}
\]
in which we used the substitution \(\gamma := -1\) is an (inhomogeneous) symmetric polynomial of degree at most \(n + r - 2k\). In order to achieve our ultimate goal and show that the homogeneous polynomial \(p_k^{\text{top}}\) is equal to zero it is enough to show that the inhomogeneous polynomial \((7.9)\) is equal to zero. We shall do it in the following.

- **Firstly, consider the case** \(k < r\). Assumption \(\text{(Z4)}\) implies that

\[
[a^{n+r-2k}] F_{\text{sym}}(\lambda_1, \ldots, \lambda_k) = 0
\]

holds true for any Young diagram with at most \(k\) rows. Thus the left-hand side of \((7.8)\) is constantly equal to zero. Lemma \(7.1\) can be applied to the polynomial \((7.9)\); it follows that \((7.9)\) is the zero polynomial as required.

- **Secondly, consider the case** \(k > r\). Assumption \(\text{(Z3)}\) implies that the left-hand side of \((7.8)\) is equal to zero for \(|\lambda| \leq n + r - 2k\), therefore

\[
 p_k^{\text{top}}(c_1, \ldots, c_k)\mid_{\gamma := -1} = 0
\]

for all integers \(c_1, \ldots, c_k\) such that \(c_1 \geq \cdots \geq c_k \geq 1\) and \(c_1 + \cdots + c_k \leq n + r - k\). Thus Lemma \(7.1\) implies that \((7.9)\) is the zero polynomial, as required. The same proof works for the alternative assumption \(\text{(Z3a)}\) in the case \(k \geq r\).

- **Finally, consider the case** \(k = r\). Note that this case for the alternative assumption \(\text{(Z3a)}\) was already considered above and the following discussion is not applicable.
We consider the set of Young diagrams \( \lambda = (\lambda_1, \ldots, \lambda_k) \) with the property that \( \lambda_1 > \cdots > \lambda_k \). For any Young diagram in this set

\[
\Delta_{\lambda_1} \cdots \Delta_{\lambda_k} F_{\text{sym}}(\lambda_1, \ldots, \lambda_k) = \Delta_{\lambda_1} \cdots \Delta_{\lambda_k} F(\lambda_1, \ldots, \lambda_k).
\] (7.10)

By Lemma 3.1 we can view \( F \) as a polynomial in the indeterminates \( \lambda_1, \ldots, \lambda_k \). One can easily show that if two polynomials in the variables \( \lambda_1, \ldots, \lambda_k \) coincide on the above set of Young diagrams then they must be equal; (7.8) and (7.10) imply therefore the following equality between polynomials:

\[
[A^{n+1-2k}] \Delta_{\lambda_1} \cdots \Delta_{\lambda_k} F(\lambda_1, \ldots, \lambda_k) = k! \ p_{k}^{\text{top}}(\lambda_1 + 1, \ldots, \lambda_k + 1) \bigg|_{\gamma_i = -1}.
\]

Each application of a difference operator decreases the degree of a polynomial by one. Together with assumption (Z2) this implies that the left-hand side is as a polynomial in \( \lambda_1, \ldots, \lambda_k \) of degree at most \( n - r - 1 \), so \( p_{k}^{\text{top}} \) must be also of degree at most \( n - r - 1 \).

Assumption (Z3) implies that the left-hand side of (7.8) is equal to zero for \( |\lambda| \leq n - r - 1 \); thus Lemma 7.1 can be applied again to show that (7.9) is the zero polynomial, as required.

This concludes the proof of the inductive step over the variable \( k \). \( \square \)

### 7.3. Characterization of the top-degree of Jack characters.

**Theorem 7.3.** Let \( n \geq 1 \) be an integer.

Then \( \text{Ch}_{n}^{\text{top}} \) is the unique function \( G \) which fulfills the following conditions:

1. \( (T1) \) \( G \in \mathcal{P} \) is homogeneous of degree \( n + 1 \);
2. \( (T2) \) for each \( m \geq 1 \) the polynomial in \( m \) variables

\[
\mathbb{Y} \ni (\lambda_1, \ldots, \lambda_m) \mapsto G(\lambda_1, \ldots, \lambda_m) \in \mathbb{Q} \left[ A, A^{-1} \right]
\]

is of degree \( n \) and its homogeneous top-degree part is equal to

\[
A^{n-1} \sum_i \lambda_i^n;
\]

3. \( (T3) \) the equality

\[
[A^{n+1-2k}] \Delta_{\lambda_1} \cdots \Delta_{\lambda_k} G_{\text{sym}}(\lambda_1, \ldots, \lambda_k) = 0
\]

holds true for the following values of \( k \) and \( \lambda \in \mathbb{Y} \):

- \( k = 0 \) and \( \lambda = \emptyset \);
- \( k = 1 \) and \( \lambda = (\lambda_1) \) with at most one row is such that \( |\lambda| \leq n - 2 \);
- \( k \geq 2 \) and \( \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Y} \) with at most \( k \) rows is such that \( |\lambda| \leq n + 1 - 2k \).
Proof.

Proof that $G := \text{Ch}_n^\text{top}$ indeed fulfills the conditions of the theorem. Condition (T1) is an immediate consequence of Definition 1.16.

In order to prove condition (T2) we write

$$G = \text{Ch}_n^\text{top} = \underbrace{\text{Ch}_n^\text{top}}_{G' :=} + \underbrace{\text{Ch}_n^\text{top} - \text{Ch}_n}_{G'' :}.$$

Proposition 3.4 says that $G' := \text{Ch}_n$ fulfills condition (T2). On the other hand, Lemma 3.3 shows that the polynomial (7.11) corresponding to $G''$ is of degree at most $n - 1$. This concludes the proof that $G := \text{Ch}_n^\text{top}$ fulfills (T2).

With the above notations we will show in the following that both $G'$ and $G''$ fulfill condition (T3). From Property 1.2 it follows that $\text{Ch}_n(\mu) = 0$ if $|\mu| \leq n - 1$. As $\Delta_{\lambda_1} \cdots \Delta_{\lambda_k} \text{Ch}_n^\text{sym}(\lambda)$ is a linear combination of the expressions of the form $\text{Ch}_n(\mu)$ over $|\mu| \leq |\lambda| + k$, it follows that

$$\Delta_{\lambda_1} \cdots \Delta_{\lambda_k} \text{Ch}_n^\text{sym}(\lambda) = 0 \quad \text{whenever } |\lambda| \leq n - k - 1.$$

In this way we proved that $G' := \text{Ch}_n^\text{sym}$ fulfills condition (T3). On the other hand, $G'' \in \mathcal{P}$ is of degree at most $n$, thus Lemma 6.8 implies that $G''$ fulfills (7.12) for all choices of $\lambda$ and $k$.

Proof of the uniqueness. Assume that both $G_1$ as well as $G_2$ fulfill the conditions from the Theorem. We set $F := G_1 - G_2$. Our ultimate goal is to show that $F = 0$. We will verify that $F$ fulfills the assumptions of Theorem 7.2 for the special choice $r = 1$.

Condition (Z1) follows from the assumption (T1).

Assumption (T1) implies that $G(\emptyset) \in \mathbb{Q}[A, A^{-1}]$ is a Laurent polynomial of degree at most $n + 1$. Furthermore, assumption (T3) for $k = 0$ and $\lambda = \emptyset$ implies that $G_i(\emptyset)$ is a Laurent polynomial of degree at most $n$ for $i \in \{1, 2\}$. This shows that $F$ fulfills condition (Z4).

By assumption (T2) the top-degree homogeneous parts of the polynomials (3.1) corresponding to $G_1$ and $G_2$ cancel each other, and the condition (Z2) follows.

Assumption (T3) implies that the condition (Z3) is fulfilled.

Thus, we proved that the assumptions of Theorem 7.2 are fulfilled for $F := G_1 - G_2$ and $r = 1$. Theorem 7.2 implies therefore that $F$ is of degree at most $n$. On the other hand, assumption (T1) implies that $F$ is homogeneous of degree $n + 1$. This is possible only if $F = 0$. This concludes the proof. □
8. Proof of the Second Main Result

8.1. The candidate formula for $\text{Ch}_{n}^{\text{top}}$ is an $\alpha$-polynomial function.

Lemma 8.1. For a permutation $\pi \in \Sigma(n)$ and a Young diagram $\lambda$ we define

$$\mathcal{K}_\pi(\lambda) := (-1)^{|C(\pi)|} \sum_{\sigma_1, \sigma_2 \in \Sigma(n), \sigma_1 \sigma_2 = \pi, \langle \sigma_1, \sigma_2 \rangle \text{ is transitive}} \mathcal{N}_{\sigma_1, \sigma_2}(\lambda).$$

Then $\mathcal{K}_\pi \in \mathcal{P}$.

Proof. If $X$ is an arbitrary set, we denote by $\text{Part}(X)$ the set of all set partitions of $X$. Let $\pi \in \Sigma(n)$ be a fixed permutation. We denote

$$\text{Part}_\pi := \{ P \in \text{Part}([n]) : P \supseteq C(\pi) \}$$

which is the set of the partitions $P$ of the underlying set $[n]$ which have the property that each cycle $c \in C(\pi)$ (we view $c \subseteq [n]$) is contained in one of the blocks of the partition $P$. We define

$$\mathcal{M}_\pi(\lambda) := \sum_{P \in \text{Part}_\pi} \prod_{B \in P} \mathcal{K}_{\pi|_B}(\lambda),$$

where the product runs over the blocks of the partition $P$ and $\pi|_B : B \to B$ denotes the restriction of the permutation $\pi$ to the set $B \subseteq [n]$.

It is not hard to see that

$$\mathcal{M}_\pi(\lambda) = (-1)^{|C(\pi)|} \sum_{\sigma_1, \sigma_2 \in \Sigma(n), \sigma_1 \sigma_2 = \pi} \mathcal{N}_{\sigma_1, \sigma_2}(\lambda)$$

(the difference between the right-hand side of (8.3) and (8.1) lies in the requirement on transitivity); indeed, the summands on the right-hand side of (8.3) can be pigeonholed according to the set of orbits of the group $\langle \sigma_1, \sigma_2 \rangle$ and each such a class of summands corresponds to an appropriate summand on the right-hand side of (8.2).

The function $\mathcal{K}_\pi$ (respectively, the function $\mathcal{M}_\pi$) depends only on the conjugacy class of the permutation $\pi$. Since such conjugacy classes are in a bijective correspondence with the integer partitions (i.e., the Young diagrams), we may index the family $\{\mathcal{K}_\pi\}$ (respectively, the family $\{\mathcal{M}_\pi\}$) by $\pi$ being an integer partition. With this perspective, (8.2) can be viewed
as the following system of equalities:

\[
\begin{align*}
\mathcal{M}_i &= \mathcal{K}_i & \text{for all } i \geq 1, \\
\mathcal{M}_{i,j} &= \mathcal{K}_{i,j} + \mathcal{K}_i \mathcal{K}_j & \text{for all } i \geq j \geq 1, \\
\mathcal{M}_{i,j,k} &= \mathcal{K}_{i,j,k} + \mathcal{K}_i \mathcal{K}_{j,k} + \mathcal{K}_j \mathcal{K}_{i,k} + \mathcal{K}_k \mathcal{K}_{i,j} + \mathcal{K}_i \mathcal{K}_j \mathcal{K}_k & \text{for all } i \geq j \geq k \geq 1,
\end{align*}
\]

(8.4)

If we view \((\mathcal{K}_\pi)\) as variables, (8.4) becomes an upper-triangular system of equations which can be solved. This shows that each \(\mathcal{K}_\sigma\) can be expressed as a polynomial in the variables \((\mathcal{M}_\pi)\). Thus for our purposes it is enough to show that \(\mathcal{M}_\pi\) is an \(\alpha\)-polynomial function. We will do it by showing that \(\mathcal{M}_\pi\) can be expressed as a polynomial in the indeterminates \(S_2, S_3, \ldots\).

We will use the ideas presented in the proof of Lemma 2.8 and we will show that it is possible to express the anisotropic Stanley polynomial for \(\mathcal{M}_\pi\) as a polynomial in terms of the anisotropic Stanley polynomial for \(\mathcal{R}_2\), the anisotropic Stanley polynomial for \(\mathcal{R}_3\), \ldots.

Our first step is to notice that, by Lemma 2.7, the anisotropic Stanley polynomial for \(\mathcal{M}_\pi\) is equal to the isotropic Stanley polynomial for the function

\[
\lambda \mapsto (-1)^{|\mathcal{C}(\pi)|} \sum_{\sigma_1, \sigma_2 \in \mathfrak{S}(n), \sigma_1 \sigma_2 = \pi} (-1)^{|\mathcal{C}(\sigma_2)|} N_{\sigma_1, \sigma_2} (\lambda) = \sum_{\sigma_1, \sigma_2 \in \mathfrak{S}(n), \sigma_1 \sigma_2 = \pi} (-1)^{\sigma_1} N_{\sigma_1, \sigma_2} (\lambda) = \text{Ch}_\alpha^\pi = 1 (\lambda),
\]

where the last equality follows from [FS11a, Theorem 2]. On the other hand, by Lemma 2.7 the anisotropic Stanley polynomial for \(\mathcal{R}_n\) coincides with the isotropic Stanley polynomial for \(\lambda \mapsto R_\lambda\). Thus our problem is equivalent to proving that it is possible to express the isotropic Stanley polynomial for \(\text{Ch}_{\alpha=1}^\pi\) as a polynomial in terms of the isotropic Stanley polynomial for \(R_2^\lambda\), the isotropic Stanley polynomial for \(R_3^\lambda\), \ldots.

The latter is equivalent to proving that \(\text{Ch}_{\alpha=1}^\pi\) itself is a polynomial in \(R_2^\lambda, R_3^\lambda, \ldots\). However, this is the celebrated Kerov polynomial and its existence has been proved by Biane [Bia03].

**Remark 8.2.** It is worth pointing out that \(\mathcal{M}_\pi\) is the anisotropic version of the usual normalized character of the symmetric groups \(\text{Ch}_\alpha^\pi = 1\) while \(\mathcal{K}_\pi\) is the anisotropic version of the quantity \(K_\pi := \kappa^{\text{id}}(\text{Ch}^{\alpha=1}_{{\pi}_1}, \text{Ch}^{\alpha=1}_{{\pi}_2}, \ldots)\) which was introduced by Rattan and Śniady [RS08, Section 5.2] and was further investigated by Dołęga, Féray and Śniady [DFS10, Section 1.6]. The
reader should be warned that the latter paper wrongfully gives a reference to the paper [Sni06], where a somewhat similar, but different quantity has been considered.

**Proposition 8.3.** For each integer \( n \geq 1 \) the function (which gives the candidate formula for \( \text{Ch}_{\text{top}}^n \) from (1.12))

\[
\lambda \mapsto (-1) \frac{1}{(n-1)!} \sum_{\sigma_1,\sigma_2 \in S(n)} \langle \sigma_1, \sigma_2 \rangle \mathfrak{m}_{\sigma_1,\sigma_2}(\lambda),
\]

is an \( \alpha \)-polynomial function.

**Proof.** The function (8.5) is a linear combination with coefficients in \( \mathbb{Q}[\gamma] \) (over permutations \( \pi \in S(n) \) and integers \( l \)) of the functions

\[
K_{\pi,l}(\lambda) := (-1)^{|C(\pi)|} \sum_{\sigma_1,\sigma_2 \in S(n), \sigma_1 \sigma_2 = \pi, |C(\sigma_1)| + |C(\sigma_2)| = l, \langle \sigma_1, \sigma_2 \rangle \text{ is transitive}} \mathfrak{m}_{\sigma_1,\sigma_2}(\lambda).
\]

For this reason it is enough to show that for each fixed \( \pi \in S(n) \) and each integer \( l \) the function \( K_{\pi,l} \in \mathcal{P} \) is an \( \alpha \)-polynomial function. We shall do it in the following.

By Lemma 2.7 the anisotropic Stanley polynomial for \( K_{\pi,l} \) exists and is homogeneous of degree \( l \); in other words \( K_{\pi,l} \in \mathcal{P} \) is homogeneous of degree \( l \) and

\[
K_{\pi} = \sum_l K_{\pi,l}
\]

is a decomposition of \( K_{\pi} \), viewed as an element of \( \mathcal{P} \), into homogeneous components.

Note that \( K_{\pi} \), this time viewed as an element of \( \mathcal{P} \), has another decomposition into homogeneous components:

\[
K_{\pi} = \sum_l K'_{\pi,l},
\]

where \( K'_{\pi,l} \in \mathcal{P} \) is homogeneous with degree \( l \) (with respect to the gradation on \( \mathcal{P} \)).

Since the restriction of the gradation on \( \mathcal{P} \) coincides with the gradation on \( \mathcal{P} \), the decomposition (8.6) can be also viewed as a decomposition of \( K_{\pi} \), viewed as an element of \( \mathcal{P} \), into the homogeneous components. Since such a decomposition must be unique, it follows that \( K_{\pi,l} = K'_{\pi,l} \) and thus \( K_{\pi,l} \in \mathcal{P} \). This concludes the proof. \( \square \)
Remark 8.4. The reader acquainted with the notion of generalized Young diagrams (see Remark [1.11]) may see that (1.12) can be written (for $\gamma > 0$ and $0 < \alpha < 1$) in the very appealing form

\begin{equation}
\text{Ch}_{\text{top}}^n(\lambda) = (-1)^n \frac{1}{(n-1)!} \sum_{\sigma_1, \sigma_2 \in \mathcal{S}(n)} \mathcal{N}_{\sigma_1, \sigma_2}(D_{\frac{1}{\gamma}} \lambda) =
\end{equation}

\begin{equation}
(-1)^n \frac{1}{(n-1)!} \sum_{\pi \in \mathcal{S}(n)} (\lambda_{\sigma_1, \sigma_2}(\pi)) |\mathcal{C}(\pi)| K_{\pi}(D_{\frac{1}{\gamma}} \lambda) =
\end{equation}

\begin{equation}
(-1)^n \frac{1}{(n-1)!} \sum_{\pi \in \mathcal{S}(n)} (\lambda_{\sigma_1, \sigma_2}(\pi)) |\mathcal{C}(\pi)| K_{\pi}(D_{\frac{1}{\gamma}} \lambda),
\end{equation}

where $D_{c}(\lambda)$ denotes the isotropic dilation (scaling) of the Young diagram $\lambda$ by the factor $c > 0$ and $D_{c_1, c_2}(\lambda)$ denotes the anisotropic dilation of $\lambda$, where the first coordinate is stretched by the factor $c_1 > 0$ and the second factor by the factor $c_2 > 0$; furthermore $K_{\pi}$ is the quantity which was discussed in Remark [8.2]. An interesting feature of the formula (8.7) is that the right-hand side is a quantity which is expressed purely in terms of the normalized characters of the symmetric group (which corresponds to the deformation parameter $\alpha = 1$). This is rather surprising because its left-hand side is related to the general anisotropic case.

8.2. Proof of the second main result: top-degree of Jack characters.

We are now ready to prove the second main result of this paper, which was stated in Section [1.13].

Theorem 1.21 (The second main result). For each $n \geq 1$ and each Young diagram $\lambda \in \mathcal{Y}$

\begin{equation}
(1.12)
\text{Ch}_{\text{top}}^n(\lambda) = (-1)^n \frac{1}{(n-1)!} \sum_{\sigma_1, \sigma_2 \in \mathcal{S}(n)} \mathcal{N}_{\sigma_1, \sigma_2}(\lambda).
\end{equation}

Proof. In the following we will use the right-hand side of (1.12) as a definition of $F(\lambda)$ and we will show that it fulfills the assumptions of Theorem [7.3].

Condition [T1] Proposition 8.3 states that $F \in \mathcal{P}$. Stanley polynomial for $\mathcal{N}_{\sigma_1, \sigma_2}$ is homogeneous, of degree $|C(\sigma_1)| + |C(\sigma_2)|$. Thus $F$ is homogeneous of degree at $n + 1$ and the condition [T1] is satisfied.

Condition [T2] We revisit the beginning of the proof of Lemma 3.1. In order to investigate the homogeneous part of the polynomial (7.11) of some
high degree \( d \) it is enough to study the part of Stanley polynomial for \( F \) which is homogeneous, of degree \( d \) with respect to the variables \( q_1, q_2, \ldots \). Such summands correspond to the terms \( \mathfrak{N}_{\sigma_1, \sigma_2} \) for which \( |C(\sigma_1)| = d \). This has twofold consequences. Firstly, \( d \leq n \), thus (7.11) is of degree at most \( n \). Secondly, the summands for which \( d = |C(\sigma_1)| = n \) are exactly those for which \( \sigma_1 = \text{id} \). The transitivity requirement implies that in this case \( \sigma_2 \) must have exactly one cycle; there are \((n-1)!\) such permutations. This completes the proof of condition (T2).

**Condition (T3)** The definition of an embedding, as well as the definitions of \( N(\lambda) \) and \( \mathfrak{N}(\lambda) \) from Section 1.12 can be naturally extended to an arbitrary tuple \( \lambda = (\lambda_1, \ldots, \lambda_k) \) of non-negative integers; condition (1.10) should be simply replaced by
\[
1 \leq f_1(w) \leq \lambda f_2(b)
\]
for each pair of vertices \( w \in V_0, b \in V_* \) connected by an edge. It is easy to check that so defined \( N(\lambda) \) is a symmetric function of its \( k \) arguments. Thus \( F(\lambda_1, \ldots, \lambda_k) \) defined as the right-hand side of (1.12) is a symmetric function of its \( k \) arguments; in other words \( F_{\text{sym}} = F \).

For \( \sigma_1, \sigma_2 \in \mathfrak{S}(n) \), any embedding \( (f_1, f_2) \) of the bicolored graph \( G_{\sigma_1, \sigma_2} \) into \( \lambda = (\lambda_1, \ldots, \lambda_k) \) can be alternatively viewed as a pair of functions
\[
f_1 : [n] \to \mathbb{N}, \quad f_2 : [n] \to [k]
\]
with the property that \( f_s \) is constant on each cycle of \( \sigma_s \) for \( s \in \{1, 2\} \) and such that
\[
(8.8) \quad 1 \leq f_1(m) \leq \lambda f_2(m) \quad \text{holds true for any } m \in [n].
\]
It follows that the sum on the right-hand-side of (1.12) can be alternatively written as
\[
(8.9) \quad \sum_{\substack{f_1 : [n] \to \mathbb{N}, \\ f_2 : [n] \to [k], \\ \text{condition } (8.8) \text{ holds true}}} \sum_{\sigma_2 \in \mathfrak{S}(n), \\ f_2 \text{ is constant on each cycle of } \sigma_2} \left\{ \sum_{\sigma_1 \in \mathfrak{S}(n), \\ f_1 \text{ is constant on each cycle of } \sigma_1, \\ \langle \sigma_1, \sigma_2 \rangle \text{ is transitive}} \gamma^{n+1-|C(\sigma_1)|-|C(\sigma_2)|} A^{|C(\sigma_1)|} \left( \frac{-1}{A} \right)^{|C(\sigma_2)|} \right\}.
\]

Let us fix the values of \( f_1, f_2 \) and \( \sigma_2 \); we denote by \( H(\lambda_1, \ldots, \lambda_k) \) the value of the curly bracket in the above expression (8.9). We will investigate in the following the contribution of \( H \) to the left-hand-side of (7.12).
Firstly, notice that if $i \in [k]$ is such that $i \notin \text{Im} f_2$ then $H(\lambda_1, \ldots, \lambda_k)$ does not depend on the variable $\lambda_i$ thus $\Delta_\lambda H(\lambda_1, \ldots, \lambda_k) = 0$ and thus the contribution of $H$ to the left-hand-side of (7.12) vanishes. Thus it is enough to consider only surjective functions $f_2 : [n] \to [k]$.

Secondly, $H(\lambda_1, \ldots, \lambda_k)$ is a Laurent polynomial of degree at most $n + 1 - 2|\mathcal{C}(\sigma_2)|$, thus in order for the coefficient of $A^{n+1-2k}$ to be non-zero, we must have $|\mathcal{C}(\sigma_2)| \leq k$.

The above two observations imply that in order to have a nontrivial contribution we must have $|\mathcal{C}(\sigma_2)| = k$ and $f_2 : \mathcal{C}(\sigma_2) \to [k]$ must be a bijection; we will assume this in the following.

Assume that $f : [n] \to \mathbb{N}^2$ given by $f(i) = (f_1(i), f_2(i))$ is not injective. It follows that there exist $i \neq j$ with $i, j \in [n]$ such that $f(i) = f(j)$. For a given $\sigma_1 \in \mathcal{S}(n)$ we denote $\sigma_1' := (i, j) \sigma_1$, where $(i, j) \in \mathcal{S}(n)$ denotes the transposition interchanging $i$ and $j$.

Note that $f = f \circ (i, j)$ thus
\[
(f_1 \text{ is constant on each cycle of } \sigma_1) \iff f_1 = f_1 \circ \sigma_1 \iff f_1 = f_1 \circ (i, j) \sigma_1 \iff (f_1 \text{ is constant on each cycle of } \sigma_1').
\]

We will show now that
\[
\langle \sigma_1, \sigma_2 \rangle \text{ is transitive } \iff \langle \sigma_1', \sigma_2 \rangle \text{ is transitive.}
\]

We will show only that the left-hand side implies the right-hand side; the opposite implication will follow by interchanging the values of $\sigma_1$ and $\sigma_1'$.

Consider the case when $i$ and $j$ belong to different cycles of $\sigma_1$. Then $\mathcal{C}(\sigma_1') = \mathcal{C}(\sigma_1) \cup \{\{i, j\}\}$ is the set-partition obtained from the set-partition $\mathcal{C}(\sigma_1)$ by merging the two blocks containing $i$ and $j$. The left-hand side of (8.10) implies that $\mathcal{C}(\sigma_1) \cup \mathcal{C}(\sigma_2) = 1_n$ is the maximal partition, thus $\mathcal{C}(\sigma_1') \cup \mathcal{C}(\sigma_2) = \mathcal{C}(\sigma_1) \cup \mathcal{C}(\sigma_2) \cup \{\{i, j\}\} = 1_n$ as well. This implies the right-hand side of (8.10).

Consider the case when $i$ and $j$ belong to the same cycle of $\sigma_1$. Then $\mathcal{C}(\sigma_1) = \mathcal{C}(\sigma_1') \cup \{\{i, j\}\}$. Since $f_2 : \mathcal{C}(\sigma_2) \to [k]$ is a bijection, the equality $f_2(i) = f_2(j)$ implies that $i$ and $j$ belong to the same cycle of $\sigma_2$. It follows that
\[
1_n = \mathcal{C}(\sigma_1) \cup \mathcal{C}(\sigma_2) = (\mathcal{C}(\sigma_1') \cup \{\{i, j\}\}) \cup \mathcal{C}(\sigma_2) = \mathcal{C}(\sigma_1') \cup (\{\{i, j\} \cup \mathcal{C}(\sigma_2)) = \mathcal{C}(\sigma_1') \cup \mathcal{C}(\sigma_2).
\]

The latter is equivalent to the right-hand side of (8.10).
The equivalence (8.10) implies that $\sigma_1$ contributes to the sum within $H$ in (8.9) if and only if $\sigma'_1$ contributes to this sum. The map $\sigma_1 \mapsto \sigma'_1$ is an involution without fixpoints. It is easy to check that the contributions of $\sigma_1$ and $\sigma'_1$ to $[A^{n+1-2k}]H$ cancel. In this way we proved that it is enough to consider only the functions $f$ which are injective. The injectivity requirement implies that if $|\lambda| < n$ then the left-hand-side of (7.12) vanishes. This shows that the condition (T3) holds true.

In this way we completed the proof that the right-hand side of (1.12) fulfills all conditions from Theorem 7.3. This concludes the proof. \hfill \Box

### 8.3. Proof of the second main result in the alternative formulation

We are now ready to prove the following result which was stated in Section 1.14.

**Theorem [1.22]** (The second main result, alternative formulation). *For each $n \geq 1$ the function*

\[
\lambda \mapsto \text{Ch}_n(\lambda) + \frac{1}{(n - 1)!} \sum_{\sigma_1, \sigma_2 \in S(n)} \gamma^{n+1-|C(\sigma_1)|-|C(\sigma_2)|} n_{\sigma_1, \sigma_2}(\lambda)
\]

*is an element of $\mathcal{P}$ of degree at most $n - 1$.*

**Proof.** By Theorem 2.15 and Proposition 8.3 the right-hand side of (8.11) belongs to $\mathcal{P}$.

By Theorem 2.15 and Lemma 2.7 it is of degree at most $n + 1$. Theorem 1.21 implies that its homogeneous part of degree $n + 1$ is equal to zero, thus (8.11) is, in fact, of degree at most $n$.

Furthermore, the homogeneous part of $\text{Ch}_n$ of degree $n$ is equal to zero by a result of Dołęga and Féray [DF16, Proposition 3.7]; the same is true for the right-hand side of (8.11). This completes the proof. \hfill \Box

### 9. Proof of Theorem 0.2

#### 9.1. Expanders

The intuitive meaning of Definition 9.1 is the following: we require that each nontrivial set of black vertices should have a sufficiently big neighborhood of white vertices; the size of this neighborhood is determined by the weight $q$.

**Definition 9.1.** We say that $(G, q)$ is an expander if the following conditions are fulfilled:

(a) $G$ is a bicolored graph with the set of black vertices $\mathcal{V}_\bullet$ and the set of white vertices $\mathcal{V}_\circ$;
(b) $q \colon \mathcal{V}_\bullet \to \{2, 3, \ldots\}$ is a function on the set of the black vertices;
(c) \(|V_0| = \sum_{v \in V_0} (q(v) - 1)\),
(d) for every set \(A \subset V_0\) such that \(A \neq \emptyset\) and \(A \neq V_0\) we require that
\[\# \{ v \in V_0 : v \text{ is connected to at least one vertex in } A \} > \sum_{i \in A} (q(i) - 1).\]

We will apply this definition mostly to the special case when \(G = G(\sigma_1, \sigma_2)\) is the bicolored graph corresponding to a pair of permutations, see Definition 1.20. In this special case the above definition takes the following form.

**Definition 9.2.** Let a positive integer \(n\) be fixed. We say that \((\sigma_1, \sigma_2, q)\) is an expander if the following conditions are fulfilled:

(a) \(\sigma_1, \sigma_2 \in S(n)\) are permutations;
(b) \(q : C(\sigma_2) \to \{2, 3, \ldots\}\) is a function on the set of cycles of \(\sigma_2\);
(c) \(|C(\sigma_1)| = \sum_{c \in C(\sigma_2)} (q(c) - 1)\),
(d) for every set \(A \subset C(\sigma_2)\) such that \(A \neq \emptyset\) and \(A \neq C(\sigma_2)\) we require that
\[\# \{ c \in C(\sigma_1) : c \text{ intersects at least one of the cycles in } A \} > \sum_{i \in A} (q(i) - 1).\]

9.2. Kerov polynomials and expanders.

**Proposition 9.3.** Let \(F \in \mathcal{P}\), let \(\mathcal{G}\) be a finite collection of connected bicolored graphs and let \(\mathcal{G} \ni G \mapsto m_G \in \mathbb{Q}[\gamma]\) be a function on it. Assume that for each \(\lambda \in \mathcal{Y}\)
\[F(\lambda) = \sum_{G \in \mathcal{G}} m_G \mathcal{N}_G(\lambda).\]

Then the Kerov’s polynomial for \(F\) is explicitly given by
\[F = \sum_{G \in \mathcal{G}} \sum_q (-m_G) \prod_{v \in V_0(G)} R_{q(v)},\]
where the sums run over \(G\) and \(q\) for which \((G, q)\) is an expander.

**Proof.** This kind of result was proved in the special case \(A = 1, \gamma = 0\) in our joint work with Dolega and Féray [DFS10]. In the following we will explain how to extend that result to our more general setup.

Our goal is to find a multivariate polynomial \(K\) (with coefficients in \(\mathbb{Q}[\gamma]\)) with the property that
\[F = K(\mathcal{R}_2, \mathcal{R}_3, \ldots).\]
We shall reuse the ideas presented in the proof of Lemma 2.8. Our current goal can be reformulated as expressing the anisotropic Stanley polynomial for $F$ as the polynomial $K$ in terms of the anisotropic Stanley polynomial for $R_2$, the anisotropic Stanley polynomial for $R_3$, . . . with the coefficients in $\mathbb{Q}[\gamma]$.

Lemma 2.7 shows equalities between the Stanley polynomials in the isotropic setup and in its anisotropic counterpart, thus the original problem is equivalent to the following one: we define

$$
\bar{F}(\lambda) := \sum_{G \in \mathcal{G}} (-1)^{|V(G)|} m_G N_G(\lambda)
$$

and we ask how to express the function $\bar{F}$ in terms of the isotropic free cumulants:

$$
\bar{F} = K(R_2, R_3, \ldots)
$$

This problem has been explicitly solved in [DF̆S10] for the special case when $\bar{F} = \text{Ch}_{n^{A=1}}$ is the character of the symmetric groups and (9.1) takes a specific form of the Stanley’s character formula. However, as we explained in a joint work with Féray [F̆S11b, Lemma 4.2], the argument holds for any polynomial function $\bar{F}$ (note that the sign in [F̆S11b, Lemma 4.2] is incorrect).

9.3. **Proof of Theorem 0.2.** We are now ready to prove a theorem which was stated in Section 0.3.

**Theorem 0.2** (Kerov–Lassalle polynomial for $\text{Ch}_{n^{\top}}$). For each $n \geq 1$

$$(0.5) \quad \text{Ch}_{n^{\top}}(\lambda) = \frac{1}{(n-1)!} \sum_{\sigma_1, \sigma_2 \in \mathfrak{S}(n)} \gamma^{n+1-|C(\sigma_1)|-|C(\sigma_2)|} \sum_{q : C(\sigma_2) \to \{2, 3, \ldots\}} \prod_{c \in C(\sigma_2)} R_{q(c)} \cdot _{\bullet_{\sigma_1, \sigma_2, q \text{ is an expander}}}$$

**Proof.** It is a direct consequence of Theorem 1.21 and Proposition 9.3. Q.E.D.

**Appendix A. Abstract characterization of Jack characters.**

**Written by Valentin Féray**

**A.1. Shifted symmetric functions.** In the following we shall assume that $A \in \mathbb{C} \setminus \{0\}$ is a fixed complex number and $\alpha = A^2$.

**Definition A.1.** We say that a polynomial $F(\lambda_1, \ldots, \lambda_r)$ is shifted-symmetric if the polynomial

$$
F\left(m_1, m_2 + \frac{1}{\alpha}, m_3 + \frac{2}{\alpha}, \ldots, m_r + \frac{r-1}{\alpha}\right)
$$
is a symmetric polynomial in the indeterminates \( m_1, \ldots, m_r \).

With this definition one can define \textit{shifted-symmetric functions} in the same way as symmetric polynomials give rise to symmetric functions, see the references from [Las08, Section 2]. The degree of a shifted-symmetric function \( F \) is defined as the maximum of the degrees of the corresponding polynomials \( F(\lambda_1, \ldots, \lambda_r) \).

\section*{A.2. Abstract characterization of Jack characters.}

\begin{theorem} \label{thm:abstract_characterization}
Let \( \pi \) be a partition and let \( A \neq 0 \) be a complex number such that \( -\frac{1}{\alpha} = -\frac{1}{A^2} \) is not a positive integer number.

Then there exist a unique shifted-symmetric function \( F \) such that:

\begin{enumerate}[(J1)]
\item \( F \) is a shifted-symmetric function of degree \( |\pi| \) and its top-degree homogeneous part is equal to \( A^{|\pi| - \ell(\pi)} p_\pi \), where \( p_\pi \) is the power-sum symmetric function;
\item \( F(\lambda) = 0 \) holds true for each Young diagram \( \lambda \) such that \( |\lambda| < |\pi| \).
\end{enumerate}

Moreover, if \( \alpha \) is a positive real number, \( F \) has the property that
\[
\text{Ch}_\pi(\lambda) = F(\lambda_1, \ldots, \lambda_r)
\]
holds true for each Young diagram \( \lambda = (\lambda_1, \ldots, \lambda_r) \).
\end{theorem}

Existence and uniqueness of \( F \) come from earlier work of Knop and Sahi [KS96] and are presented in Appendix \[A.3\] The link with \( \text{Ch}_\pi \) follows from the work of Lassalle and is explained in Appendix \[A.4\].

\section*{A.3. Existence and uniqueness of \( F \).}

Let \( \mathcal{P}(r, d) \) be the set of partitions of size at most \( d \) and length at most \( r \). We identify such partitions with lists \( \lambda_1 \geq \cdots \geq \lambda_r \geq 0 \) so that \( \mathcal{P}(r, d) \) is a subset of \( \mathbb{Z}^r_+ \). A special case of a result of Knop and Sahi [KS96, Theorem 2.1] is the following interpolation theorem.

\begin{theorem} \label{thm:interpolation}
Fix \( r, d \geq 1 \). Assume that \(-1/\alpha\) is not a positive integer. Then, for every function \( \bar{f} : \mathcal{P}(r, d) \rightarrow \mathbb{Q} \), there exists a unique shifted-symmetric polynomial \( F \) in \( r \) variables, of degree at most \( d \) such that \( F(\lambda) = \bar{f}(\lambda) \) for all \( \lambda \) in \( \mathcal{P}(r, d) \).
\end{theorem}

\begin{proof}
In fact, Knop and Sahi have proved a slightly different statement: if some list of shifts \( \rho = (\rho_1, \ldots, \rho_r) \) is such that, for each \( i < j \), \( \rho_i - \rho_j \) is not a negative integer (they called such sequences \textit{dominant}), then there exists a unique symmetric polynomial \( f \) such that: for all \( \lambda \) in \( \mathcal{P}(r, d) \), \( f(\lambda + \rho) = \bar{f}(\lambda) \). By definition \( f \) is symmetric if and only if \( F(\lambda) := f(\lambda + \rho) \) is shifted-symmetric, where \( \rho = (-1/\alpha, -2/\alpha, \ldots, -r/\alpha) \). Since
\((-1/\alpha, -2/\alpha, \ldots, -r/\alpha)\) is dominant when \(-1/\alpha\) is not a positive integer, Theorem [A.3] follows from Knop and Sahi’s result. □

Proof of the existence and uniqueness of \(F\) in Theorem [A.2]. We start by proving the existence. First observe that it is easy to construct a function satisfying only (J1) set
\[
F_k(x_1, x_2, \ldots) := A^{k-1} \sum_{i \geq 1} \left( (x_i - i-1/\alpha)^k - (-i-1/\alpha)^k \right),
\]
and
\[
F_\pi := \prod_{j=1}^{\ell(\pi)} F_{\pi_j}.
\]

Using Theorem [A.3] for each integer \(r \geq |\pi|\) there exists a unique shifted-symmetric polynomial \(G_\pi\) in \(r\) variables, of degree at most \(|\pi| - 1\), such that \(G_\pi(\lambda) = F_\pi(\lambda)\) for every diagram \(\lambda\) of size smaller than \(|\pi|\). Like symmetric functions, shifted-symmetric functions are determined by their restrictions to a finite number of variables, thus \(G_\pi\) can be extended to a unique shifted-symmetric function that we abusively also denote \(G_\pi\). Then \(F_\pi - G_\pi\) satisfies (J1) and (J2).

We now prove uniqueness. Let \(F\) and \(G\) be two shifted-symmetric functions satisfying conditions (J1) and (J2). Then \(H := F - G\) is a shifted-symmetric function of degree at most \(|\pi| - 1\) that vanishes on each Young diagram of size smaller than \(|\pi|\).

Consider its restriction \(H'\) to \(r\) variables, where \(r \geq |\pi|\) is an integer. Using Theorem [A.3] we have that \(H' = 0\) is the zero polynomial: both \(H'\) and 0 are shifted-symmetric polynomials of degree at most \(|\pi| - 1\) that vanish on all partitions \(\lambda\) in \(P(|\pi| - 1, r)\) (i.e. all partitions of size smaller than \(|\pi|\)). Like symmetric functions, shifted-symmetric functions are determined by their restrictions to a finite number of variables, so that \(H = 0\). □

A.4. Link with \(\text{Ch}_\pi\). We now prove the second part of the theorem: when \(\alpha\) is a positive integer, the function \(F\) defined by (J1) and (J2) coincides with \(\text{Ch}_\pi\) on Young diagrams.

We follow the work of Lassalle [Las08 Section 3]. He constructed a linear isomorphism \(f \mapsto f^\#\) between the space of symmetric functions and the space of shifted-symmetric functions with the following properties.

(P1) If \(f\) is homogeneous, the highest degree homogeneous part of \(f^\#\) is equal to \(f\).
(P2) For all Young diagram \(\lambda\),
\[
\text{Ch}_\pi(\lambda) = A^{|\pi| - \ell(\pi)} p^\#_\pi(\lambda).
\]
Proof of \([P2]\) Lassalle \([Las08, \text{Proposition 2}]\) proved that both sides are equal when \(|\lambda| \geq |\pi|\). It remains to show equality in the case when \(|\lambda| < |\pi|\).

When \(|\lambda| < |\pi|\), we have that \(\text{Ch}_\pi(\lambda) = 0\) by definition. Let us recall from Lassalle’s paper \([Las08, \text{Equation (3.1)}]\) that the family \((p^{\#}_\pi(\nu))\) (where \(\nu\) runs over all partitions) is defined implicitly by the following equation:

\[
\exp(p_1) p_\pi = \sum_{\nu} p^{\#}_\pi(\nu) \frac{J_\nu}{j_\nu},
\]

where \(j_\nu\) is a combinatorial factor that is not relevant here. The Reader should be advised that this equation is an equality between the usual (i.e. non-shifted) symmetric functions. Observe that, after expanding the exponential, the left-hand side involves homogeneous symmetric functions of degree at least \(|\pi|\). Therefore, its expansion on the Jack basis given in the right-hand side involves only Jack functions indexed by partition \(\nu\) of size at least \(|\pi|\). In particular, since \(|\lambda| < |\pi|\), one has \(p^{\#}_\pi(\lambda) = 0\) and the equality is proved.

\[\square\]

It is now clear that \(F = A|\pi| - \ell(\pi) p^{\#}_\pi\) satisfies \([J1]\) and \([J2]\) of Theorem \(A.2\). Thus \(\text{Ch}_\pi(\lambda) = F(\lambda)\) for all Young diagram \(\lambda\) as claimed.

A.5. \textbf{Proof of Theorem 1.7}

\textit{Proof of Theorem 1.7} We first check that \(F := \text{Ch}_\pi\) indeed fulfills the required conditions. Indeed, condition \([K1]\) coincides with Theorem 2.15; condition \([K2]\) coincides with Proposition 3.4; condition \([K3]\) coincides with Property 1.2; condition \([K4]\) coincides with Proposition 3.5 as required.

In order to prove the uniqueness part we assume that there are two functions \(F_1, F_2\) which fulfill the conditions \([K1]-[K3]\). There are infinitely many values of the deformation parameter \(A\) for which Theorem \(A.2\) can be applied and conditions \([K2],[J2]\) imply that the specializations of \(F_1\) and \(F_2\) for this particular value of \(A\) are equal. On the other hand, the assumption that \(F_1, F_2 \in \mathcal{P}\) implies that each value \(F_1(\lambda), F_2(\lambda) \in \mathbb{Q}[A, A^{-1}]\) is a Laurent polynomial in \(A\). Equality between the values of two Laurent polynomials for infinitely many values of \(A\) implies their equality for all \(A \in \mathbb{C}\). In particular, \(F_1 = F_2\), as desired.

\[\square\]

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