Kronecker’s and Newton’s approaches to solving:
A first comparison
(Extended Abstract)

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1 Introduction

Let \( K \) be a number field containing the field of Gaussian rationals \( \mathbb{Q}[i] \subseteq K \). In these pages we are mainly interested in the computation of \( K \)-rational points of zero-dimensional algebraic varieties given by systems of multivariate polynomial equations. Namely, let \( f_1, \ldots, f_s \in \mathbb{Z}[X_1,\ldots,X_n] \) be a sequence of multivariate polynomials with integer coefficients. Let \( V(f_1,\ldots,f_s) \subseteq \mathbb{C}^n \) be the complex algebraic variety of their common zeros, i.e.

\[
V(f_1,\ldots,f_s) := \{ x \in \mathbb{C}^n : f_i(x) = 0, \text{ with } 1 \leq i \leq s \}.
\]

For sake of simplicity, let us assume that \( V(f_1,\ldots,f_s) \) is a finite set (i.e. a zero-dimensional algebraic variety). The set of \( K \)-rational points in \( V(f_1,\ldots,f_s) \) is the set of common zeros of the system \( f_1,\ldots,f_s \) whose coordinates lie in \( K^n \), namely

\[
V_K(f_1,\ldots,f_s) := \{ x \in K^n : f_1(x) = \cdots = f_s(x) = 0 \}.
\]

The goal of these pages will be to discuss several aspects of procedures performing the following task: Assume that the field \( K \) is fixed. Given the sequence \( f_1,\ldots,f_s \), compute all \( K \)-rational points in \( V_K(f_1,\ldots,f_s) \) (or eventually all \( K \)-rational points in \( V_K(f_1,\ldots,f_s) \) of bounded height).

Note that the assumption on the field \( K \) is not very restrictive: For every zero \( \zeta \in V(f_1,\ldots,f_s) \), there exists a minimal number field \( K(\zeta) \) containing all the coordinates of \( \zeta \). The degree of the field extension \( K(\zeta) \) over \( \mathbb{Q} \) can also be denoted by \( \deg(\zeta) \). In the sequel, the degree \([ K : \mathbb{Q} ]\) may be replaced by \( \deg(\zeta) \), and the results will equally hold.

For our study, we consider a precomputation task which prepares the input \( F := (f_1,\ldots,f_s) \), before we study the desired \( K \)-rational points. Procedures performing this precomputation task are usually called \textit{multivariate polynomial}
system solvers applied to the input $F$. The output of such polynomial system solvers is called the solution of the system $F$. Observe that all usual notions of solution of $F$ will yield a description of the variety $V(f_1, \ldots, f_s)$ (cf. also [CGH+99]).

Here, we consider two (conceptually different) notions which define what a solution of the system $F$ should be: coming from different fields, the notions are related to a symbolic/geometric and a numerical analysis/diophantine approximation context: Kronecker’s geometric solution and Newton’s approximate zero solution.

Thus, our study includes a comparative study of both approaches with regards to the basic problem described above. It must be said that our study is not intended to be either complete or definitive. It just tries to point out some similarities and differences between both approaches to solving that yield some statements and some open questions of interest. In this sense, we have tried to write down as many comments as possible to clarify (as much as we can) the relations between both approaches to solving.

Moreover, we have tried to put both approaches under the same hypothesis. This means, that our input system of multivariate polynomials $F := (f_1, \ldots, f_s)$ is well-suited for the application of either Kronecker’s or Newton’s approach to solving. Therefore we will assume the following hypotheses:

i) The number of equations equals the number of variables (i.e. $s = n$ above)

ii) The variety $V(f_1, \ldots, f_n)$ is zero-dimensional and contains exactly $D$ points, i.e. the degree of $V(f_1, \ldots, f_n)$ (in the sense of [Hei83]) is exactly $D$.

iii) The $K$-rational points in $V_K(f_1, \ldots, f_n)$ are smooth with respect to the system $F := (f_1, \ldots, f_n)$, i.e. for every $\zeta \in V_K(f_1, \ldots, f_n)$, the jacobian matrix

$$DF(\zeta) := \left( \frac{\partial f_i}{\partial X_j}(\zeta) \right)_{1 \leq i, j \leq n}$$

is a non-singular matrix $DF(\zeta) \in GL(n, K)$.

iv) The sequence $f_1, \ldots, f_n \in \mathbb{Z}[X_1, \ldots, X_n]$ is a reduced regular sequence, i.e. for every $i$, $1 \leq i \leq n - 1$, the ideals $(f_1, \ldots, f_i)$ are radical ideals of codimension $i$ in $\mathbb{Q}[X_1, \ldots, X_n]$.

v) The degrees of the input polynomials satisfy $\deg(f_i) \leq 2$, for $1 \leq i \leq n$.

It must be said that constraints i) and iv) are not relevant for Kronecker’s approach to solving. Applying the iterative version of Bertini’s Theorem (as described in [Mor97], [Hag98], [HMPS] or [GS99]) we can easily reduce the over-determined input system to a system satisfying properties i) and iv). Anyway, we prefer to keep this hypothesis to simplify exposition, notations, and hopefully – reading.

The rest of the introduction presents the new results, classified into three main categories:

i) Newton’s approach to solving.

Here, we show how to extend the approximate zero theory introduced by S. Smale in [Sma81] (cf. also [Sma83], [Sma86a], [Sma86b]), and deeply developed in collaboration with M. Shub in the series of papers [SS85, SS86, SS93a, SS93b, SS94a, SS94b] and in [Mal93, Mal94, Mal95, Yak93a, Yak93b] (see also [Ded97b, Ded97c], to a diophantine approximation context.)
ii) Kronecker’s approach to solving.

This recalls Kronecker’s approach to solving and shows the main statements which relate both approaches by means of an algorithm based on the $L^3$ (or $LLL$) reduction procedure (as introduced in \cite{LLL82} and used in \cite{LLL84, Len84}).

iii) Application: Computation of splitting field and Lagrange resolvent.

Finally we exhibit an algorithm that combines both approaches to compute efficiently the splitting field of an univariate polynomial equation and also the corresponding Lagrange resolvent.

2 Newton’s approach to solving

Let $M_K$ be a proper class of absolute values on the number field $K$ in the sense of \cite{Lan83}. For every $\nu \in M_K$ we have an absolute value $| \cdot |_{\nu} : K \rightarrow \mathbb{R}$. The class $M_K$ is chosen such that it satisfies Weil’s product formula with respect to well-defined multiplicities. We denote by $S \subseteq M_K$ the set of sub–indices $\nu \in M_K$ such that the absolute value $| \cdot |_{\nu}$ is archimedean and, consequently, by $M_K \setminus S$ the class of sub–indices $\nu \in M_K$ such that $| \cdot |_{\nu}$ is non–archimedean. For every $\nu \in M_K$, we shall denote by $K_\nu$ the completion of $K$ with respect to the absolute value $| \cdot |_{\nu}$. We also denote by $| \cdot |_{\nu} : K_\nu \rightarrow \mathbb{R}$ the corresponding extension of $| \cdot |_{\nu}$ to the completion $K_\nu$.

Let $\zeta \in V_K(f_1, \ldots, f_n)$ be a smooth $K$–rational point of the zero–dimensional complex algebraic variety $V(f_1, \ldots, f_n)$. We are interested in approximating $\zeta$ using iterations of the Newton operator. Therefore, we introduce the Newton operator of system $F$ as the following list of rational mappings:

$$N_F(X_1, \ldots, X_n) := \left( \begin{array}{c} X_1 \\ \vdots \\ X_n \end{array} \right) - Df(X_1, \ldots, X_n)^{-1} \left( \begin{array}{c} f_1(X_1, \ldots, X_n) \\ \vdots \\ f_n(X_1, \ldots, X_n) \end{array} \right).$$

An approximate zero $z$ in $K^n$ for the system $F$ with associate zero $\zeta \in V_K(f_1, \ldots, f_n)$ with respect to the absolute value $| \cdot |_{\nu}$ is a point such that the sequence of iterates of the Newton operator is well–defined and converges quadratically to $\zeta$.

Roughly speaking, an approximate zero $z \in K^n$ with associate zero $\zeta \in K^n$ is a point which lies in the basin of attraction of the actual zero $\zeta$ with respect to the Newton operator $N_F$. Formally, we define approximate zeros as follows:

**Definition 1** Let $F := (f_1, \ldots, f_n)$ be a system of multivariate polynomials with integer coefficients : $f_i \in \mathbb{Z}[X_1, \ldots, X_n]$ for $1 \leq i \leq n$. Let $\nu \in M_K$ define an absolute value $| \cdot |_{\nu} : K \rightarrow \mathbb{R}$. Let $\zeta \in V_K(f_1, \ldots, f_n)$ be a smooth $K$–rational point (i.e. $DF(\zeta) \in GL(n, K)$). Let $z := (z_1, \ldots, z_n) \in K^n$ be an affine point. We say that $z$ is an approximate zero of the system $F$ with associate zero $\zeta \in K^n$ with respect to the absolute value $| \cdot |_{\nu}$, if the following properties hold:

- $DF(z) \in GL(n, K)$ is a non–singular matrix.
- The following sequence is well–defined:
  $$z_1 := N_F(z) \in K^n, \text{ and } z_k := N_F(z_{k-1}) \text{ for } k \leq 2.$$
- For every $k \in \mathbb{N}$, $k \geq 1$, the following inequality holds:
  $$\|z_k - \zeta\|_{\nu} \leq \frac{1}{2^{k-1}} \|z - \zeta\|_{\nu},$$

where $\| \cdot \|_{\nu} : K_\nu \rightarrow \mathbb{R}$ is the corresponding norm associated to the absolute value $| \cdot |_{\nu}$. 

3
From a computational point of view, we want to compute approximate zeros of smooth $K$--rational points and we want to write them over a finite alphabet. In particular for every smooth $K$--rational zero $\zeta \in V_K(f_1, \ldots, f_n)$ and every absolute value $\nu \in \mathcal{M}_K$, we consider a subfield $L$ of $K$, such that the completion $L_\nu$ of $L$ with respect to the absolute value $| \cdot |_\nu$ contains the entries of $\zeta$, namely $\zeta \in L_\nu^n$. Thus, we look for approximate zeros $z \in L^n$ with associate zero $\zeta \in L_\nu^n$. Let us observe that if the absolute value $| \cdot |_\nu$ is archimedean, we may fix $L$ to be $L := \mathbb{Q}[i]$. Moreover, we are interested in the heights of approximate zeros $z \in L^n$ with actual zeros $\zeta \in L_\nu^n$. In the case where $L = \mathbb{Q}[i]$, the height of a point $z \in \mathbb{Q}[i]^n$ essentially equals its bit length (i.e. the number of tape cells in a Turing machine required to write down the number list of digits describing $z$). In the sequel, we shall therefore identify the logarithmic height $ht(z)$ and its bit length.

A first relevant task consists in stating conditions which are sufficient for verifying the property of being an approximate zero. This is achieved by means of a local condition based on a quantity (called $\gamma$), which is essentially yielded by the Lipschitz constant appearing in the inverse mapping Theorem (cf. [Dem89], Ch. 1, for instance). These ideas were introduced by S. Smale in the early eighties (cf. [Sm81]) and deeply developed in the series of papers written by M. Shub and S. Smale [SSS93 to SSS94H below).

With the same notations as above, let $\nu \in \mathcal{M}_K$ be an absolute value on the field $K$. We define the quantity $\gamma$:

$$
\gamma(\nu; \zeta) := \sup_{k \geq 2} \left\| (DF(\zeta))^{-1} (D^k F(\zeta)) \right\|_{\nu}^{1/k},
$$

where the norm is considered as the norm with respect to the absolute value $| \cdot |_\nu$ of the multilinear operator

$$
DF(\zeta)^{-1} D^k F(\zeta) : (K_\nu^n)^k \to K_\nu^n.
$$

This quantity yields a locally sufficient condition for having an approximate zero. This statement is known as the $\gamma$–Theorem and it holds equally true for archimedean and non-archimedean absolute values.

**Theorem 2** ($\gamma$–Theorem) With the same notations and assumptions as before, let $F := (f_1, \ldots, f_n)$ be a sequence of multivariate polynomials with coefficients in $K$. Let $\zeta \in V_K(f_1, \ldots, f_n)$ be a smooth $K$–rational zero (i.e. $DF(\zeta) \in GL(n, K)$ is a non–singular matrix). Let $| \cdot |_\nu : K \to \mathbb{R}_+$ be an absolute value on $K$. For every $z \in K^n$ satisfying the inequality:

$$
\|z - \zeta\|_{\nu} \gamma(\nu; \zeta) \leq \frac{3 - \sqrt{7}}{2}
$$

holds: $z$ is an approximate zero of the system $F$ with associate zero $\zeta$ with respect to the absolute value $| \cdot |_\nu$.

The proof of this statement follows step by step the proof of the usual $\gamma$–Theorems (cf. the compiled version in [BCSS98]).

To establish upper and lower bounds for the bit length of approximate zeros, we have established several technical statements. One of them is an extension to the non-archimedean case of the well–known Eckardt & Young Theorem [EY36] :

Let $\nu \in \mathcal{M}_K$ be an absolute value over $K$ and $K_\nu$ the completion of $K$ with respect to the absolute value $| \cdot |_\nu$. Let us denote by $\Sigma_\nu \subseteq \mathcal{M}_n(K_\nu)$ the variety of singular $n \times n$ matrices with entries in $K_\nu$. Similarly, let $\Sigma$ be the subset of $\Sigma_\nu$ of all singular $n \times n$ matrices with entries in $K$. Finally, let

$$
d_F^{(F)} : \mathcal{M}_n(K_\nu) \times \mathcal{M}_n(K_\nu) \to \mathbb{R}_+
$$

be the Frobenius (also called Hilbert–Weil) metric on $\mathcal{M}_n(K_\nu)$ with respect to the absolute value $| \cdot |_\nu$. Then, the following Theorem holds:
Theorem 3 (Eckardt & Young) Let \( \nu \in M_K \) be an absolute value. For every non-singular \( n \times n \) matrix \( A \in GL(n, K) \), the following equality holds:

\[
\deg_{\nu}(A, \Sigma) = \deg_{\nu}(A, \Sigma_{\nu}) = \inf \{ \deg_{\nu}(A, M) : M \in \Sigma \} = \frac{1}{\|A^{-1}\|_{\nu}}.
\]

For every multivariate polynomial \( f \in \mathbb{Z}[X_1, \ldots, X_n] \) with integer coefficients, we define its logarithmic height \( \operatorname{ht}(f) \) as the logarithm of the maximum of the absolute values of its coefficients. This notion introduced, we have the following statement which shows lower bounds for the bit length of approximate zeros.

Theorem 4 (Lower Bounds) Let \( f_1, \ldots, f_n \in \mathbb{Z}[X_1, \ldots, X_n] \) be a sequence of multivariate polynomials. Let us assume that the following properties hold:

i) \( \max \{ \deg(f_i) : 1 \leq i \leq n \} = 2 \),

ii) \( \operatorname{ht}(f_i) \leq h \) for \( 1 \leq i \leq n \).

Let \( \zeta \in V_K(f_1, \ldots, f_n) \) be a smooth \( K \)-rational point of the system \( F := (f_1, \ldots, f_n) \). Let \( | \cdot |_{\nu} : K \longrightarrow \mathbb{R}_{+} \) be an absolute value defined on \( K \), and let \( L \subseteq K \) be a number field such that \( \zeta \in L_{\nu}^0 \). Then, for every \( z \in L^n \), \( z \neq \zeta \) satisfying:

\[
||z - \zeta||_{\nu} \gamma_{\nu}(F, \zeta) \leq \frac{3 - \sqrt{7}}{2}
\]

the following inequality holds:

\[
\operatorname{ht}(z) \geq \frac{1}{3[L : \mathbb{Q}]} \left( \log \gamma_{\nu}(F, \zeta) - [L : \mathbb{Q}](5 \log n + 2h) - 3 \right).
\]

Using Theorem 3 above, the following inequality also holds:

\[
\operatorname{ht}(z) \geq \frac{1}{3[L : \mathbb{Q}]} \left( \log \deg_{\nu}(DF(\zeta)^{-1}, \Sigma_{\nu}) - [L : \mathbb{Q}](7 \log n + 3h) - 5 \right).
\]

Moreover, in the case where \( L = \mathbb{Q}[i] \) is the field of Gaussian rationals, the two previous lower bounds may be rewritten as:

\[
\operatorname{ht}(z) \geq \frac{1}{6} \left( \log \gamma_{\nu}(F, \zeta) - (10 \log n + 4h + 3) \right), \quad \text{and}
\]

\[
\operatorname{ht}(z) \geq \frac{1}{6} \left( \log \deg_{\nu}(DF(\zeta)^{-1}, \Sigma_{\nu}) - (14 \log n + 6h + 5) \right).
\]

Let us observe that the “negative terms” in the previous lower bounds are linear in the input length (i.e., the bit length of the input system \( F := (f_1, \ldots, f_n) \)) whereas the “positive part” depends semantically on the smooth \( K \)-rational solution \( \zeta \in V_K(f_1, \ldots, f_n) \).

Last, but not least, we may also show a few lower bounds for the average height of approximate zeros associated to a \( \mathbb{Q} \)-definable irreducible component of the solution variety \( V(f_1, \ldots, f_n) \). To this end, we introduce some additional notations. Let \( f_1, \ldots, f_n \in \mathbb{Z}[X_1, \ldots, X_n] \) be a sequence of multivariate polynomials satisfying the hypotheses i) to v) above. Let \( \zeta \in K^n \) be a smooth \( K \)-rational zero of the system \( F := (f_1, \ldots, f_n) \). Let \( V := V(f_1, \ldots, f_n) \subseteq \mathbb{C}^n \) be the algebraic variety given as the common zeros of the polynomials \( f_1, \ldots, f_n \). Let \( V_{\zeta} \subseteq V \) be the \( \mathbb{Q} \)-definable irreducible component of \( V \) that contains \( \zeta \). Let us assume \( D := \deg(V_{\zeta}) \) be the number of points in \( V_{\zeta} \). Let us observe that \( D = \deg(\zeta) \leq [K : \mathbb{Q}] \). Let us assume

\[
V_{\zeta} := \{ \zeta_1, \ldots, \zeta_D \}.
\]
Let $\| \cdot \| : \mathbb{K}^n \rightarrow \mathbb{R}$ be the standard hermitian norm induced in $\mathbb{K}^n$ by the inclusion $i : \mathbb{K} \hookrightarrow \mathbb{C}$. A sequence of points $z := (z_1, \ldots, z_D) \in \mathbb{Q}[i]^nD$ is said to be an approximate zero of the system $F$ with associate variety $V_\zeta$ that satisfies the $\gamma$--Theorem if for every $i$, $1 \leq i \leq D$, the following holds:

$$\| z_i - \zeta_i \| \leq \frac{3 - \sqrt{7}}{2\gamma(F, \zeta)},$$

where $\gamma(F, \zeta)$ is the quantity associate to the hermitian norm $\| \cdot \|$.

For every given approximate zero $z := (z_1, \ldots, z_D) \in \mathbb{Q}[i]^nD$ of the system $F$ with associate variety $V_\zeta$, the average height (also the average bit length) of $z$ is defined in the following terms

$$ht_{av}(z) := \frac{1}{D} \sum_{i=1}^{D} ht(z_i).$$

Finally, let us denote by $\mathbb{Z}_K \subset \mathbb{K}$ the ring of algebraic integers of the number field $\mathbb{K}$. Then, we have the following lower bound for the average bit length of approximate zeros with associate variety $V_\zeta$:

**Proposition 5** With the previous notations, let $\zeta \in V_\mathbb{K}(f_1, \ldots, f_n)$ be a smooth $\mathbb{K}$--rational with entries in $\mathbb{Z}_K$, i.e. $\zeta \in \mathbb{Z}_K^n$. Let us also assume that for every archimedean absolute value $| \cdot |_\nu$ (i. e. $\nu \in S$), the following holds:

$$3\| \zeta \|_{\gamma_\nu(F, \zeta)} \geq 3 - \sqrt{7}.$$  

Then the average height of any approximate zero $z \in \mathbb{Q}[i]^nD$ of the system $F$ with associate variety $V_\zeta$ satisfies the following inequality:

$$ht_{av}(z) \geq \frac{1}{2} \left[ ht(\zeta) - \left( \frac{1}{2} \log n + \log 2 \right) \right].$$

In order to illustrate the meaning of this lower bound, we give here a few Corollaries.

**Corollary 6** With the same notations as in Proposition 5 above, let $\zeta \in \mathbb{Z}_K \cap V_\mathbb{K}(f_1, \ldots, f_n)$ be a smooth $\mathbb{K}$--rational zero of the system $F := (f_1, \ldots, f_n)$ and let us assume that for every archimedean absolute value $| \cdot |_\nu : \mathbb{K} \rightarrow \mathbb{R}$ (i. e. for every $\nu \in S$), the following holds:

$$\gamma_\nu(F, \zeta) \geq 3 - \sqrt{7}.$$  

Then the average height of any approximate zero $z \in \mathbb{Q}[i]^nD$ of the system $F$ with associate variety $V_\zeta$ satisfies the following inequality:

$$ht_{av}(z) \geq \frac{1}{2} \left[ ht(\zeta) - \left( \frac{1}{2} \log n + \log 2 \right) \right].$$

Moreover, the previous techniques show how to deform a given system of multivariate polynomials by means of a single additional equation of low degree in such a way that the average bit length of the new system is essentially greater than the height of the actual zero you want to approximate.

**Corollary 7** Let $F := (f_1, \ldots, f_n)$ be a system of multivariate polynomials with integer coefficients satisfying the conditions i) to vi) given on page 4. Let $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{Z}_K \cap V_\mathbb{K}(f_1, \ldots, f_n)$ be a smooth $\mathbb{K}$--rational zero whose coordinates are algebraic integers. Let us now define the system of polynomial equations in $n + 1$ variables:

$$G := (g_1, \ldots, g_{n+1}) \in (\mathbb{Z}[X_1, \ldots, X_{n+1}])^{n+1},$$

given by the following rules:
• $g_i := f_i \in \mathbb{Z}[X_1, \ldots, X_{n+1}]$ for every $i$, $1 \leq i \leq n$,
• $g_{n+1} := (X_{n+1} - X_n) (X_{n+1} - (X_n + 1))$.

Let $\zeta' \in V_K(g_1, \ldots, g_{n+1}) \cap \mathbb{Z}_K$ be the affine point given by:

$$\zeta' := (\zeta_1, \ldots, \zeta_n, \zeta_n) \in \mathbb{Z}_K^{n+1}.$$

Let $V_{\zeta'} \subseteq V(g_1, \ldots, g_{n+1})$ be the $\mathbb{Q}$-definable irreducible component of $V(g_1, \ldots, g_{n+1})$ containing $\zeta'$. Then, the average height of any approximate zero $z \in \mathbb{Q}[i]^{(n+1)D}$ of the system $F$ with associate variety $V_{\zeta'}$ satisfies the following inequality:

$$ht_{av}(z) \geq \frac{1}{2} \left( \frac{1}{2} \log(n+1) + \log 2 \right).$$

We can construct several examples where all of the previous lower bounds for the bit length of approximate zeros apply. For instance we give here an example inspired by a classical univariate example due to M. Mignotte (cf. [Mig89]):

**Example 1 (Using $\log \gamma$ as in Theorem 4)** Let us consider the system of multivariate polynomials $F := (f_1, \ldots, f_{n+1})$ given by the following rules:

• $f_1 := X_1 - 2$,
• $f_i := X_i - X_{i-1}^2$ for every $i$, $2 \leq i \leq n - 1$,
• $f_n := X_{n+1} - X_n^2$,
• $f_{n+1} := X_{n+1} X_n - 2(X_{n-1} X_n - 1)^2$.

This system $F$ has three solutions in $\mathbb{C}^{n+1}$, where two of them, say $\zeta_1, \zeta_2 \in \mathbb{R}^{n+1}$, satisfy the following inequality:

$$\|\zeta_1 - \zeta_2\| \leq \frac{2}{2^{2n-1}} \leq \frac{2}{2^{2n-1}}.$$

Thus, using the well-known relation between the quantity $\gamma$ and the separation of roots, we may conclude:

$$\frac{2}{2^{2n-1}} \geq ||\zeta_1 - \zeta_2|| \geq \frac{3 - \sqrt{7}}{2\gamma(F, \zeta_i)}.$$

By Theorem 4, we conclude that for all approximate zeros $z_1, z_2 \in \mathbb{Q}[i]^{n+1}$ of the system $F$ associated to $\zeta_1, \zeta_2$ respectively and satisfying the corresponding $\gamma$-Theorem, the following holds:

$$ht(z) \geq \frac{1}{6} (2^{n-1} - 2\log(n+1)) - O(1).$$

Therefore this example (an several others which should be constructed) allows us to conclude the following Corollaries 8 to 10.

**Corollary 8** Computing approximate zeros in $\mathbb{Q}[i]$ for archimedean absolute values, using binary encoding of the output requires exponential running time and exponential output length, and these two lower bounds cannot be improved with this encoding. Namely, computing approximate zeros with binary encoding is in the complexity class EXTIME \setminus P.

**Corollary 9** Floating point encoding of approximate zeros requires exponential number of digits and this lower bound cannot be improved. Namely, floating point encoding does not suffice to compute approximate zeros of systems of multivariate polynomial equations.
As continuous fraction encoding of numbers in $\mathbb{Q}[i]$ is close to the binary encoding, we easily conclude the following:

**Corollary 10** Computing approximate zeros in $\mathbb{Q}[i]$ for archimedean absolute values, using continuous fraction encoding of the output requires exponential running time and exponential output length, and these two lower bounds cannot be improved with this encoding. Namely, computing approximate zeros with continuous fraction encoding is in the complexity class $\text{EXTIME} \setminus \text{P}$.

These lower bounds suggest that a central point of interest should be to study the bit length of approximate zeros satisfying the $\gamma$–Theorem. In order to shed some light in this direction, we prove the following statements:

**Theorem 11 (Upper Bounds)** Let $f_1, \ldots, f_n \in \mathbb{Z}[X_1, \ldots, X_n]$ be polynomials with integer coefficients. Let us assume that the following properties hold:

- $\max\{\deg(f_i) : 1 \leq i \leq n\} \leq 2$, and
- $ht(f_i) \leq h$ for $1 \leq i \leq n$.

Let $\zeta \in V_K(f_1, \ldots, f_n)$ be a smooth $K$–rational point. Let $| \cdot |_\nu : K \rightarrow \mathbb{R}_+$ be an absolute value on $K$. Then, the following inequality holds:

$$\log \gamma_\nu(F, \zeta) \leq 3[K : \mathbb{Q}]n (n^2 + 4 \log n + h + ht(\zeta) + 3).$$

In particular, we show the following estimate for the bit length of approximate zeros in $\mathbb{Q}[i]^n$:

**Corollary 12 (Upper bound on the bit length of approximate zeros)**

With the same assumptions and notations as in Theorem 11 above, let $\zeta \in V_K(f_1, \ldots, f_n)$ be a smooth $K$–rational zero and let $| \cdot |_\nu$ be an absolute value on $K$. Let $L \subseteq K$ be a number field such that $\zeta \in L^n$. Then there exist approximate zeros $z \in L^n$ of the system $F := (f_1, \ldots, f_n)$ with approximate zero $\zeta$ with respect to the absolute value $| \cdot |_\nu$, such that the logarithmic height $ht(z)$ of $z$ is at most linear in the following quantities:

$$\frac{1}{[L : \mathbb{Q}]} \log |\Delta_L| + [K : \mathbb{Q}]n (n^2 + h + nht(\zeta)),$$

where $|\Delta_L|$ is the absolute value of the discriminant of the field $L$.

Moreover, in the case where $L = \mathbb{Q}[i]$ (for instance, if $| \cdot |_\nu$ is archimedean), there exist approximate zeros $z \in \mathbb{Q}[i]^n$ for the system $F$ with associate zero $\zeta$ with respect to $| \cdot |_\nu$, such that their bit length is at most linear in the following quantity:

$$[K : \mathbb{Q}]n (n^2 + h + nht(\zeta)), \text{ in other words:}$$

$$ht(z) \leq O \left( [K : \mathbb{Q}]n (n^2 + h + nht(\zeta)) \right).$$

Let us observe, that these two upper bounds above (i.e. Theorem 11 and Corollary 12) depend mainly on the input length $n^2h + n^2$ and on two parameters which in turn depend on the actual zero to approximate: the degree $[K : \mathbb{Q}]$ of a field containing the coordinates of the zero and the logarithmic height of the zero $ht(\zeta)$. These two quantities are bounded respectively by the geometric Bézout inequality (cf. [Hei83], [Ful84], [Vog84]) and the arithmetic Bézout inequality (cf. [BG89], [Ph971], [Ph972], [Ph98] or [KP94], [KPS96], [Som98], [Hag981], [HMS98], for instance). Moreover, combining these two upper bounds (Theorem 11 and Corollary 12) with the previously shown lower bounds and several examples, we may conclude that the upper bounds shown in Theorem 11 and Corollary 12 are optimal.

On the other hand, the $\gamma$–Theorem above has some aesthetic consequences which we may explain in terms of the existence of a universal radius of convergence independent of the absolute value under consideration. To this end, we recall the well–known Implicit Function Theorem for complete noetherian local rings in the following terms:
Theorem 13 (Non–archimedean Basin of Attraction) Let \( F := (f_1, \ldots, f_n) \in \mathbb{Z}[X_1, \ldots, X_n] \) be a system of multivariate polynomials satisfies the hypotheses of Theorem 11 above. Let \( \nu \in M_K \) define a non–archimedean absolute value \(| \cdot |_\nu\) on \( K \). Let us also assume that the restriction
\[
| \cdot |_\nu : \mathbb{Q} \rightarrow \mathbb{R}_+
\]
defines a p–adic absolute value, where \( p \in \mathbb{N} \) is a prime number. Let \( \zeta \in K^n \) be a smooth \( K \)–rational zero of the system which lies in the closed unit sphere of \( K^n \), i.e.
\[
\zeta \in B_\nu(0, 1) := \{ x \in K^n : \| x \|_\nu \leq 1 \}.
\]
Let us finally assume that \( | \det DF(\zeta)|_\nu = 1 \). Then, for every \( z \in B_\nu(0, 1) \) satisfying
\[
\| z - \zeta \|_\nu \leq \frac{1}{p}
\]
holds : \( z \) is an approximate zero of the system \( F \) with associate zero \( \zeta \) with respect to the absolute value \(| \cdot |_\nu\).

This statement is nothing but the usual Hensel Lemma in local algebra (cf. \[ZS58, Mor97\], for instance). However, this statement has a drawback : The radius of the basin of attraction centered at \( \zeta \) depends on the concrete absolute value \(| \cdot |_\nu\). The \( \gamma \)–Theorem 11 above shows that there exists a universal radius, which depends only on the system \( F \) and the smooth \( K \)–rational zero, but does not depend on the absolute value.

In order to prove this claim, let us introduce the following quantity \( \tilde{\gamma}(F, \zeta) \). With the same notations and assumptions as above, let us define the universal quantity
\[
\tilde{\gamma}(F, \zeta) := \left( \prod_{\nu \in M_K} \max \{1, \gamma_\nu(F, \zeta)\}^n \right)^{1/|K|}.
\]
Let us observe, that this quantity is well–defined and finite according to Theorem 11 above. Moreover, it does not depend on any particular absolute value under consideration. Thus, we may conclude the following Theorem :

**Corollary 14 (Universal \( \gamma \)–Theorem)** With the same notations and assumptions as in Theorem 4, for every \( z \in \mathbb{Q}[i]^n \) and every absolute value \(| \cdot |_\nu\) satisfying the following inequality
\[
\| z - \zeta \|_\nu \tilde{\gamma}(F, \zeta) \leq \frac{3 - \sqrt{7}}{2}
\]
holds : \( z \) is an approximate zero for the system \( F \) with associate zero \( \zeta \) and with respect to the absolute value \( \nu \in M_K \).

Let us point out that the existence of such a universal quantity does not imply the existence of a universal basin of attraction independent of the absolute value under consideration. In fact, we show the following (expected) statement :

**Corollary 15** Let \( F := (f_1, \ldots, f_n) \) be a sequence of multivariate polynomials with integer coefficients satisfying conditions i) to v) of page 4. Let \( \zeta \in V_K(f_1, \ldots, f_n) \) be a smooth \( K \)–rational zero. The only point \( z \in K^n \) that satisfies the universal \( \gamma \)–Theorem near \( \gamma \) for all absolute values in \( M_K \) is \( z = \zeta \). Namely, for every \( z \in K^n \) satisfying the following inequality for every \( \nu \in M_K \)
\[
\| z - \zeta \|_\nu \leq \frac{3 - \sqrt{7}}{2\tilde{\gamma}(F, \zeta)}
\]
holds \( z = \zeta \).
3 Kronecker’s approach to solving

In [Kro82], Kronecker introduced a notion of solution of unmixed complex algebraic varieties, which we are going to reproduce here. Let $f_1, \ldots, f_i \in \mathbb{Z}[X_1, \ldots, X_n]$ be a sequence of polynomials defining a radical ideal $(f_1, \ldots, f_i)$ of codimension $i$ in $\mathbb{C}[X_1, \ldots, X_n]$. Let $V := V(f_1, \ldots, f_i) \subseteq \mathbb{C}^n$ be the complex algebraic variety of codimension $i$ given by the common zeros of the $f_i$. A solution of $V$ is a birational isomorphism of $V$ with some complex algebraic hypersurface in a space of adequate dimension.

Technically, this is expressed as follows. First of all, let us assume that the variables $X_1, \ldots, X_n$ are in Noether position with respect to the variety $V$, i.e. we assume that the following is an integral ring extension:

$$Q[X_1, \ldots, X_{n-i}] \hookrightarrow Q[X_1, \ldots, X_n]/(f_1, \ldots, f_i).$$

Let $u := \lambda_{n-i+1}X_{n-i+1} + \cdots + \lambda_n X_n \in Q[X_1, \ldots, X_n]$ be a linear form in the dependent variables $\{X_{n-i+1}, \ldots, X_n\}$. Thus we have a linear projection

$$U : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-i+1}$$

$$(x_1, \ldots, x_n) \longmapsto (x_1, \ldots, x_{n-i}, u(x_1, \ldots, x_n)).$$

Let us also consider the restriction

$$U \mid V : V \longrightarrow \mathbb{C}^{n-i+1}.$$

The linear form $u$ is called a primitive element if and only if the projection $U \mid V$ defined a birational isomorphism of $V$ with some complex hypersurface $H_u$ in $\mathbb{C}^{n-i+1}$ with minimal equation $\chi_u \in Q[X_1, \ldots, X_{n-i}, T]$. Then, a Kronecker solution of variety $V$ is a description of the primitive element $u$, the hypersurface $H_u$ through the minimal equation $\chi_u$, and a description of the inverse of the birational isomorphism, i.e. $(U \mid V)^{-1}$. Formally, this list of items may be described as follows:

- The list of variables in Noether position $X_1, \ldots, X_n$, which includes a description of the dimension of $V$.
- The primitive element $u := \lambda_{n-i+1}X_{n-i+1} + \cdots + \lambda_n X_n$ given by its coefficients in $\mathbb{Z}$.
- The minimal equation of the hypersurface $H_u$, namely $\chi_u \in \mathbb{Z}[X_1, \ldots, X_{n-i}, T]$.
- A description of $(U \mid V)^{-1}$. This description is given by the following list:
  - A non–zero polynomial $\rho \in \mathbb{Z}[X_1, \ldots, X_{n-i}]$.
  - A list of polynomials $v_j \in \mathbb{Z}[X_1, \ldots, X_{n-i}, T]$, $n-i+1 \leq j \leq n$, such that the degrees with respect to variable $T$ satisfy $\deg_T(v_j) \leq \deg_T(\chi_u)$, for every $j$, $n-i+1 \leq j \leq n$.

such that the following holds

$$(U \mid V)^{-1}(x, t) := (x_1, \ldots, x_{n-i}, \rho^{-1}(x)v_{n-i+1}(x, t), \ldots, \rho^{-1}(x)v_n(x, t)), $$

where $x := (x_1, \ldots, x_{n-i}) \in \mathbb{C}^{n-i}$ and $t \in \mathbb{C}$.

Kronecker conceived an iterative procedure to solve multivariate systems of equations $F := (f_1, \ldots, f_n)$ defining zero–dimensional complex varieties, which can be described in the following terms:

First, you start with system $(f_1)$ and you “solve” the unmixed variety of codimension 1, $V(f_1) \subseteq \mathbb{C}^n$. Then you proceed iteratively: From Kronecker’s solution of the variety $V(f_1, \ldots, f_i)$ you eliminate the new equation $f_{i+1}$ to obtain a Kronecker solution of the “next” variety $V(f_1, \ldots, f_{i+1})$. Proceed until you reach $i = n$. This iterative procedure has two main drawbacks which can be explained in the following terms:
• First of all, the space problem arising with the representation of the intermediate polynomials. The polynomials $\chi_u$, $\rho$ and $v_j$ are polynomials of high degree (eventually of degree $2^i$) involving several variables. Thus, to represent them, one has to handle their coefficients, which amounts to the following quantities

$$\left(\frac{2^i + n - i + 1}{n - i + 1}\right),$$

which for $i := n/2$ amounts to more than $2^{n^2/4}$ coefficients of great bit length.

• Secondly, Kronecker’s iterative procedure introduced a nesting of interpolation procedures required for the iterative process and the linear change of coordinates required by every computation of the Noether normalisation. This nesting of interpolation procedures is difficult to avoid and increases the running time complexity.

Therefore, the procedure was forgotten by contemporary mathematicians and hardly mentioned in the literature of algebraic geometry. Macaulay quotes Kronecker’s procedure in [Mac16] and so does König in [Kon03]. But both of them thought, that this procedure requires too much running time to be efficient, and it was progressively forgotten. Traces of this procedure can be found spread over the algebraic geometry literature without giving to it the required relevance. For example, Kronecker’s notion of solution was used by O. Zariski in [Zar95] to define dimension of algebraic varieties, claiming that it was also used in the same form by Severi and others.

In 1995, two works rediscovered Kronecker’s approach to solving without previous knowledge of it’s existing ancestors. These two works [GHM95] and [Par95] were able to overcome the first drawback (space problem of representation) of the previous methods. The technical trick was the use of a data structure coming from semi–numerical modelling : straight–line programs. This idea of representing polynomials by programs that evaluate them goes back to former works of the same research group (as [GH93, FGS95, KP94] or [KP96]). Moreover, these ideas were the natural continuation of the ideas previously developped in [GH91].

To overcome the second drawback (Nesting), the authors introduced a method based on Newton’s method applied in a non–archimedean context (the approximate zeros in the corresponding non–archimedean basin of attraction were called Lifting Fibers in [HH+97]). This result was obtained in the two papers [GHM+97] [GH+98] The key trick to avoid the nesting of interpolation procedures was based on Hensel’s Lemma (also Implicit Mapping Theorem). Perhaps, the following statement could explain the relations existing between Hensel’s Lemma and Approximate Zero Theory. To this end, let us introduce a few more notations. Let $f_1, \ldots, f_r \in \mathbb{C}[X_1, \ldots, X_n]$ be a sequence of polynomials defining a radical ideal of codimension $r$ in $\mathbb{C}[X_1, \ldots, X_n]$. Let us assume that the variables $X_1, \ldots, X_n$ are in Noether position with respect to the ideal $I := (f_1, \ldots, f_r)$, i.e. assume that the following ring extension is integral

$$\mathbb{C}[X_1, \ldots, X_{n-r}] \hookrightarrow \mathbb{C}[X_1, \ldots, X_n]/I.$$

Let $P := (p_1, \ldots, p_{n-r}) \in \mathbb{C}^{n-r}$ be an affine point, let $\mathcal{O}_P$ be the ring of formal power series at $P$, and let $\mathcal{M}_P$ be the field of fractions of $\mathcal{O}_P$. Then, the following is finite ring extension

$$\mathcal{M}_P \hookrightarrow B := \mathcal{M}_P[X_{n-r+1}, \ldots, X_n]/(f_1, \ldots, f_r),$$

and $B$ is a zero–dimensional $\mathcal{M}_P$–algebra. Thus, it has some sense to look for approximate zeros of the solutions in $\mathcal{M}_P$ of the system of polynomial equations $F := (f_1, \ldots, f_r)$. The following statement about Hensel’s Lemma explains the connections existing between Kronecker’s solving and Approximate Zero Theory.
Theorem 16 (Hensel’s Lemma) With the same assumptions and notations as above, let \( \zeta \in \mathcal{M}_p^r \) be a solution of the system \( F \). Let \( \| \cdot \| : \mathcal{M}_p^r \to \mathbb{R} \) be usual non-archimedian norm in \( \mathcal{M}_p^r \). Let \( \mathbb{C}(X_1, \ldots, X_{n-r}) \) be the field of rational functions. Then, for every \( z \in \mathbb{C}(X_1, \ldots, X_{n-r})^r \), if \( \| z \| \leq 1 \), and \( \| z - \zeta \| < \frac{1}{2} \), then \( z \) is an approximate zero for the system \( F := (f_1, \ldots, f_r) \) with associate zero \( \zeta \in \mathcal{M}_p^r \).

Unfortunately, those two works \([GHH+97, GHM+98]\) introduced (for the Lifting Fibers) runtime requirements which depend on the heights of the intermediate varieties (in the sense of \([BGS93, Phi91, Phi94, Phi95, Som98]\)). This drawback was finally overcome in the paper \([GHMP97]\), where integer numbers were represented by straight-line programs and the following result established:

Theorem 17 \([GHMP97]\) There exists a bounded error probability Turing machine \( M \) which performs the following task: Given a system of multivariate polynomial equations \( F := (f_1, \ldots, f_n) \), satisfying the following properties

- \( \deg(f_i) \leq 2 \) and \( \text{ht}(f_i) \leq h \) for \( 1 \leq i \leq n \),
- the ideals \( (f_1, \ldots, f_i) \) are radical ideals of codimension \( i \) in the ring \( \mathbb{Q}[X_1, \ldots, X_n] \) for \( 1 \leq i \leq n-1 \),
- the variety \( V(f_1, \ldots, f_n) \subseteq \mathbb{C}^n \) is a zero-dimensional complex algebraic variety.

Then, the machine \( M \) outputs a Kronecker solution of the variety \( V(f_1, \ldots, f_n) \). The running time of the machine \( M \) is polynomial in the following quantities

\[
\delta(F)nh,
\]

where \( \delta \) is the maximum of the degrees of the intermediate varieties (in the sense of \([Hei83]\)), namely

\[
\delta(F) := \max\{\deg(V(f_1, \ldots, f_i)) : 1 \leq i \leq n-1\}.
\]

It must be said that the coefficients of the polynomials involved in a Kronecker solution of the variety \( V(f_1, \ldots, f_n) \) are given by straight-line programs that evaluate integer numbers. However, the complexity estimates for the Turing machine \( M \) are independent from the height.

Our attempt in these pages is to compare this approach to solving developed by Kronecker to that of Newton as described in the previous Subsection.

The exposition of new results starts with a small improvement of the Witness Theorem in \([HS80]\) and \([BCSS96]\) (cf. also \([BCSS98]\)). When dealing with straight-line program data structures, some relevant technical methods of comparison must be developed. These methods are known as probabilistic zero tests for polynomials given by straight-line programs. Examples of these tests are those introduced in \([Sch79, Zip79, HS80]\) and the Witness Theorem, introduced in \([HS80]\) for the case of polynomials with integer coefficients, and in \([BCSS96]\) for polynomials with coefficients in a number field.

As we already had to introduce a few technical notions and methods spread over the literature of number theory, numerical analysis, algebraic complexity theory and elimination theory, we can give without introducing additional material for free the following improvement of the estimates for the Witness Theorem:

Theorem 18 (Witness Theorem) Let \( P \in K[X_1, \ldots, X_n] \) be a non-zero polynomial evaluable by a non-scalar straight-line program \( \Gamma \) of size \( L \), non-scalar depth \( \ell \) and parameters in \( F \subseteq K \). Let \( \omega_0 \in K \) be such that the following holds:

\[
\text{ht}(\omega_0) \geq \max\{\log 2, \text{ht}(F)\}.
\]
Let \( N \in \mathbb{N} \) be a non-negative integer such that
\[
\log_2 N > \log_2 (\ell + 1) + (\ell + 2)(\log_2 \log_2 (4L)).
\]

Let us define recursively the following sequence of algebraic numbers (known as Kronecker’s scheme):
\[
\omega_1 := \omega_0^N, \text{ and for } 2 \leq i \leq n, \omega_i := \omega_{i-1}^N.
\]

Then, the point \( \omega := (\omega_1, \ldots, \omega_n) \in K^n \) is a witness for \( P \) (i.e. \( P(\omega) \neq 0 \)).

Moreover, we observe that approximate zeros are succinct encodings of generic points of the variety \( V(f_1, \ldots, f_n) \). This means that for every smooth \( K \)-rational zero \( \zeta \in V_K(f_1, \ldots, f_n) \), the binary encoding of an approximate zero \( z \in \mathbb{Q}[i]^n \) is sufficient information to compute the \( \mathbb{Q} \)-irreducible component of \( V(f_1, \ldots, f_n) \) containing \( \zeta \). In more precise terms we show the following statement:

**Theorem 19 (From Approximate Zeros to Geometric Solution)**  With the same assumptions as in Theorem 17 above, there exists a bounded error probability Turing machine \( M \), such that taking as input the binary encoding of an approximate zero \( z \in \mathbb{Q}[i]^n \) of the system \( F \) with associate zero \( \zeta \in V_K(f_1, \ldots, f_n) \) for an archimedean absolute value \( | \cdot |_\nu \) (where \( \nu \in S \)), \( M \) outputs a Kronecker solution of the \( \mathbb{Q} \)-irreducible component \( W \) of \( V(f_1, \ldots, f_n) \) containing \( \zeta \). Moreover, the running time of this probabilistic Turing machine is polynomial in the following quantities
\[
\deg(W) (n h (\ell + 1)) + (\ell + 2)(\log_2 \log_2 (4L)),
\]
where \( \deg(W) \) is the degree of the \( \mathbb{Q} \)-irreducible component \( W \) containing \( \zeta \).

The key idea for the proof of this Theorem is the use of the \( L^3 \) (or \( LLL \)) reduction algorithm.

Conversely, as approximate zeros may depend on the height of the actual zero they approximate, we could be interested in the computation of approximate zeros for actual zeros with small (bounded) height. This is done in the following statement.

**Theorem 20 (From Kronecker’s solution to Newton’s solution)**  There exists a bounded error probability Turing machine \( M \) which performs the following task: Given a sequence of polynomial equations \( F := (f_1, \ldots, f_n) \) of degree at most 2 and height at most \( h \), and given a positive integer number \( H \in \mathbb{N} \) in binary encoding, the machine \( M \) outputs approximate zeros for the archimedean absolute value \( | \cdot | : K \to \mathbb{R} \) induced on \( K \) by the inclusion \( i : K \hookrightarrow \mathbb{C} \) for all those zeros \( \zeta \in V_K(f_1, \ldots, f_n) \), whose logarithmic height is at most \( H \), i.e.
\[
ht(\zeta) \leq H.
\]

The running time of \( M \) is polynomial in the following quantities:
\[
(n d (\ell + 1)) + (\ell + 2)(\log_2 \log_2 (4L)),
\]
where the notations are the same as in Theorem 17 before.

A proof of this statement is based again on an application of the \( L^3 \) reduction algorithm.
4 Application : Computation of splitting field and Lagrange resolvent

Combining both Kronecker’s and Newton’s approach to solving, we exhibit an efficient procedure for computing the splitting field and the Lagrange resolvent of an irreducible monic univariate polynomial \( f \in \mathbb{Q}[X] \) of degree \( d \). Let us recall that the splitting field of \( f \) is the minimal number field \( K(f) \) containing the field of rational numbers \( \mathbb{Q} \) and all complex roots of \( f \) (i.e. the minimal number field where \( f \) splits completely, also called the normal closure of the equation \( f = 0 \)). This normal closure \( K(f) \) is nothing but the Galois field of \( f \) and it satisfies where \( \text{Gal}_{\mathbb{Q}}(f) \) is the Galois group of the polynomial \( f \). The splitting field of \( f \) can be indentified with an irreducible component of the zero–dimensional algebra (known as the universal decomposition algebra)

\[
A := \mathbb{Q}[X_1, \ldots, X_d]/(\sigma_0 - a_0, \ldots, \sigma_{d-1} - a_{d-1}),
\]

where \( \sigma_0, \ldots, \sigma_{d-1} \) are the elementary symmetric functions and \( f \) is written as \( f(X) := a_0 + a_1 X + \cdots + a_{d-1} X^{d-1} + X^d \). Let us also observe that the Lagrange resolvent is nothing but the Chow (or Cayley) elimination polynomial of the zero–dimensional residue algebra \( A/m \), where \( m \) is a well-chosen maximal ideal of \( A \). Therefore, we can also show the following Theorem as a consequence of the comparison between Newton’s and Kronecker’s approach to solving:

**Theorem 21 (Splitting Field and Lagrange Resolvent)** There exists a probabilistic Turing machine, which for every given univariate polynomial \( f \in \mathbb{Z}[X] \) of degree at most \( d \) and logarithmic height at most \( h \) computes the following items :

i) Approximate zeros in \( \mathbb{Q}[i] \) of all zeros of \( f \),

ii) a geometric description of the splitting field \( K(f) \) of the polynomial \( f \),

iii) and the Lagrange resolvent of the equation \( f = 0 \).

The running time of \( M \) is polynomial in the following quantities:

\[
\sharp(\text{Gal}(f))(dh).
\]

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