ON THE FEKETE-SZEGÖ INEQUALITY FOR CERTAIN CLASS OF ANALYTIC FUNCTIONS

S. SIVAPRASAD KUMAR* AND VIRENDRA KUMAR

Abstract. In the present investigation, we derive Fekete-Szegő inequality for the class
$S^*_\alpha (\phi)$, introduced here. In addition to that, certain applications of our results are also discussed.

1. Introduction

Let $A$ denote the class of functions of the form
\begin{equation}
    f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\end{equation}
which are analytic in the unit disc $U := \{ z \in \mathbb{C} : |z| < 1 \}$. Further let $S$ denote the subclass of $A$ consisting of univalent functions. Assume that $\phi$ is an analytic function with positive real part in the unit disc $U$ with $\phi(0) = 1$ and $\phi'(0) > 0$, which maps the unit disc $U$ onto a region starlike with respect to 1 and symmetric with respect to the real axis.

For any two analytic functions $f$ and $g$, we say that $f$ is subordinate to $g$ or $g$ is superordinate to $f$, denoted by $f \prec g$, if there exists a Schwarz function $w$ with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$. If $g$ is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$. A function $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ is said to be in the class $P$ if $\text{Re} p(z) > 0$.

Let $S^*(\phi)$ be the class of functions $f \in S$ satisfy
\[ \frac{zf'(z)}{f(z)} < \phi(z) \quad (z \in U) \]
and $C(\phi)$ be the class of functions $f \in S$ satisfy
\[ 1 + \frac{zf''(z)}{f'(z)} < \phi(z) \quad (z \in U), \]
these classes were introduced and studied by Ma and Minda [4]. Note that $S^*(\frac{1+z}{1-z}) =: S^*$ and $C(\frac{1+z}{1-z}) =: C$ are the well known classes of starlike and convex functions respectively.

If $f \in A$ is given by (1) and $g \in A$ is given by

*Corresponding author
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(2) \[ g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \]
then the Hadamard product (or convolution) \( f * g \) of \( f \) and \( g \) is defined by
\[ (f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z). \]

In 1933, M. Fekete and G. Szegö \cite{2} obtained the sharp bounds for \(|a_3 - \mu a_2^2|\) as a function of real parameter \(\mu\) and proved that
\[ |a_3^2 - \mu a_3| \leq 1 + 2 \exp \left( \frac{2\mu}{1 - \mu} \right) \quad (0 \leq \mu \leq 1), \]
for functions belonging to the class \( S \). Later the problem of finding the sharp bounds for the non-linear functional \(|a_3 - \mu a_2^2|\) of any compact family of functions \( f \in A \) is known as the Fekete-Szegö problem or inequality. In the recent years several authors have investigated the Fekete-Szegö inequality for various subclasses of analytic functions \cite{7, 8, 12, 15}.

In the present investigation, we obtain the Fekete-Szegö inequality for functions belonging to the class \( S_{L_g}(\phi) \), defined here. Applications of our main results are also discussed. In fact we generalize many earlier results in this direction \cite{3, 4, 5, 8}.

**Definition 1.1.** Let \( f, g \in A \) are respectively given by (1) and (2). We define the convolution operator \( L_g \) by
\[ L_g(f(z)) := f(z) * g(z). \]

We note that the operator \( L_g \) unifies many earlier linear operators for a suitable choice of the function \( g(z) \). Some are listed below:

1. If \( g(z) = \sum_{n=0}^{\infty} \frac{(a_1)_{n+1}}{(b_1)_{n+1}} \frac{z^{n+1}}{(n+1)!} \), then the operator \( L_g \) coincides with the Dziok-Srivastava \cite{11} linear operator \( H^{\alpha, \beta}[a_1] \).

2. If \( g(z) = \frac{z}{(1-z)^2} \), then \( L_g \) reduces to \( D^n \), where \( D^n \) is the Ruscheweyh derivative operator \cite{3}.

3. For \( g(z) = z + \sum_{n=2}^{\infty} \left( \frac{n+1}{n+1} \right)^r z^n \), the operator \( L_g \) reduces to the operator \( I_1(r, \lambda) \), defined by Sivaprasad et al. \cite{11}.

4. For \( g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} z^n \), the operator \( L_g \) coincides with the fractional derivative operator \( \Omega^\beta \), defined by Owa and Srivastava \cite{11}.

**Definition 1.2.** Let \( \alpha \) be a complex number. A function \( f \in A \) of the form (1) is said to be in the class \( S_{L_g}(\phi) \) if it satisfies
\[ \Psi_g(f)(z) = \frac{f(z)}{L_g(f(z))} < \phi(z), \]
where
\[ \Psi_g(f)(z) := 1 + \frac{z\mathcal{L}_g''(f(z))}{\mathcal{L}_g'(f(z))} + \frac{z\mathcal{L}_g''(f(z))}{\mathcal{L}_g(f(z))} - \frac{(1-\alpha)z^2\mathcal{L}_g''(f(z))}{(1-\alpha)z\mathcal{L}_g'(f(z)) + \alpha\mathcal{L}_g(f(z))} \]
and \( L_g(f(z)) := f(z) * g(z) \).
Remark 1.3. For \( g(z) = \frac{z}{1-z} \), we have \( S^0_{Z_g}(\phi) = S^*(\phi) \) and \( S^1_{Z_g}(\phi) = C(\phi) \).

Remark 1.4. If we take \( g(z) = z + \sum_{n=2}^{\infty} n^m z^n \), then the operator \( Z_g \) reduces to the Sălăgean \( \mathcal{D}^m \) differential operator defined by

\[
\mathcal{D}^m f(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n, \quad m \in \{0, 1, 2, 3, \ldots\}.
\]

Further, if we set \( \phi(z) = \frac{1+z}{1-z} \) and \( g = z + \sum_{n=2}^{\infty} n^m z^n \) in the above Definition [12], then the class \( S^0_{Z_g}(\phi) \) reduces to the class \( H(S^*_m)(\alpha) \), recently introduced by Răducanu [5]. Răducanu in fact investigated the relationship property between the classes \( H(S^*_m)(\alpha) \) and \( S^* \) and obtained the Fekete-Szegő inequality for the class \( H(S^*_m)(\alpha) \).

We need the following Lemmas to prove our main results:

**Lemma 1.5.** [3] If \( p_1(z) = 1 + c_1 z + c_2 z^2 + \ldots \in \mathcal{P} \). Then

\[
|c_2 - vc_1^2| \leq \begin{cases} 
-4v + 2 & \text{if } v \leq 0; \\
2 & \text{if } 0 \leq v \leq 1; \\
4v - 2 & \text{if } v \geq 1.
\end{cases}
\]

When \( v < 0 \) or \( v > 1 \), equality holds if and only if \( p_1(z) \) is \((1+z)/(1-z)\) or one of its rotations. If \( 0 < v < 1 \), then equality holds if and only if \( p_1(z) \) is \((1+z^2)/(1-z^2)\) or one of its rotations. If \( v = 0 \), equality holds if and only if

\[
p_1(z) = \left( \frac{1+\gamma}{2} \right) \frac{1+z}{1-z} + \left( \frac{1-\gamma}{2} \right) \frac{1-z}{1+z} \quad (0 \leq \gamma \leq 1, z \in \mathbb{U})
\]

or one of its rotations. While for \( v = 1 \), equality holds if and only if \( p_1(z) \) is the reciprocal of one of the functions such that equality holds in the case of \( v = 0 \).

Although the above upper bound is sharp, it can be improved as follows when \( 0 < v < 1 \):

\[
|c_2 - vc_1^2| + v|c_1|^2 \leq 2, \quad 0 < v \leq \frac{1}{2}
\]

and

\[
|c_2 - vc_1^2| + (1-v)|c_1|^2 \leq 2, \quad \frac{1}{2} < v \leq 1.
\]

**Lemma 1.6.** [7] If \( p_1(z) = 1 + c_1 z + c_2 z^2 + \ldots \in \mathcal{P} \). Then for any complex number \( v \),

\[
|c_2 - vc_1^2| \leq 2 \max\{1; |2v - 1|\}
\]

and the result is sharp for the functions given by

\[
p_1(z) = \frac{1+z^2}{1-z^2} \quad \text{and} \quad p_1(z) = \frac{1+z}{1-z}.
\]

**Lemma 1.7.** [8] If the function \( p_1(z) = 1 + c_1 z + c_2 z^2 + \ldots \in \mathcal{P} \). Then

1. \(|c_n| \leq 2\) for \( n \geq 1 \),
2. \(|c_2 - \frac{1}{2}c_1^2| \leq 2 - \frac{|c_1|^2}{2} \).
We begin with the following result with a coefficient estimate for the class of functions \( f \in S^0_{\mathcal{L}_a}(\phi) \).

**Theorem 2.1.** Let \( g(z) \) be given by (4) with \( b_2, b_3 \) non zero real numbers. Assume that \( \alpha \geq 0 \) and \( \phi(z) = 1 + B_1z + B_2z^2 + \cdots \). If \( f \in S^0_{\mathcal{L}_a}(\phi) \), then

\[
|a_2| \leq \frac{B_1}{(1 + \alpha)|b_2|} \tag{5}
\]

and for any real number \( \mu \)

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{B_1}{2(2\alpha + 1)|b_1|} \left( \frac{B_2}{B_1} - \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1 + \alpha)^2} - \frac{2\mu(2\alpha + 1)B_1b_3}{(1 + \alpha)^2b_2^2} \right) & \text{if } \mu \leq \sigma_1; \\
\frac{B_1}{2(2\alpha + 1)|b_1|} \left( \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1 + \alpha)^2} + \frac{2\mu(2\alpha + 1)B_1b_3}{(1 + \alpha)^2b_2^2} - \frac{B_3}{B_1} \right) & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\
\frac{B_1}{2(2\alpha + 1)|b_1|} \left( 1 + \frac{B_2}{B_1} \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1 + \alpha)^2} \right) & \text{if } \mu \geq \sigma_2,
\end{cases} \tag{6}
\]

where

\[
\sigma_1 := \frac{(1 + \alpha)^2b_2^2}{(2\alpha + 1)B_1b_3} \left( \frac{B_2}{B_1} - \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1 + \alpha)^2} - 1 \right)
\]

and

\[
\sigma_2 := \frac{(1 + \alpha)^2b_2^2}{(2\alpha + 1)B_1b_3} \left( 1 + \frac{B_2}{B_1} \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1 + \alpha)^2} \right).
\]

The inequality (6) is sharp.

**Proof.** Let \( f \in S^0_{\mathcal{L}_a}(\phi) \) and

\[
(p(z)) = 1 + z\frac{L''_\phi(f(z))}{L'_\phi(f(z))} + z\frac{L''_\phi(f(z))}{L'_\phi(f(z))} - \frac{(1 - \alpha)z^2 L''_\phi(f(z)) + z L'_\phi(f(z))}{(1 - \alpha)z L'_\phi(f(z)) + \alpha L'_\phi(f(z))}
\]

\[
= 1 + d_1z + d_2z^2 + \cdots.
\]

A simple computation shows that

\[
\frac{z L''_\phi(f(z))}{L'_\phi(f(z))} = 1 + a_2b_2z + [2a_3b_3 - a^2_2b_2^2]z^2 + \cdots,
\]

\[
1 + z\frac{L''_\phi(f(z))}{L'_\phi(f(z))} = 1 + 2a_2b_2z + [6a_3b_3 - 4a^2_2b_2^2]z^2 + \cdots
\]

and

\[
\frac{(1 - \alpha)z^2 L''_\phi(f(z)) + z L'_\phi(f(z))}{(1 - \alpha)z L'_\phi(f(z)) + \alpha L'_\phi(f(z))} = 1 + (2 - \alpha)a_2b_2z + [(6 - 4\alpha)a_3b_3 - (\alpha - 2)a^2_2b_2^2]z^2 + \cdots.
\]

Substituting these values in (7), we have

\[
d_1 = (1 + \alpha)a_2b_2 \tag{8}
\]

and

\[
d_2 = 2(2\alpha + 1)a_3b_3 + (\alpha^2 - 4\alpha - 1)a^2_2b_2^2 \tag{9}
\]

Since \( \phi \) is univalent and \( p \prec \phi \), the function \( p_1(z) \) defined by

\[
p_1(z) = \frac{1 + \phi^{-1}(p(z))}{1 - \phi^{-1}(p(z))} = 1 + c_1z + c_2z^2 + \cdots,
\]

where
is analytic with positive real part in the unit disc $U$. Further from (10), we have
\[
p(z) = \phi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) = \phi \left( \frac{c_1 z + c_2 z^2 + \ldots}{2 + c_1 z + c_2 z^2 + \ldots} \right) = 1 + \frac{1}{2} B_1 c_1 z + \left[ \frac{1}{2} B_1 (c_2 - \frac{1}{2} c_1^2) + \frac{1}{4} B_2 c_1^2 \right] z^2 + \ldots.
\]
Thus, we have
\[
(11) \quad d_1 = \frac{1}{2} B_1 c_1
\]
and
\[
(12) \quad d_2 = \frac{1}{2} B_1 \left( c_2 - \frac{1}{2} c_1^2 \right) + \frac{1}{4} B_2 c_1^2.
\]
From (8) and (11), we have
\[
(13) \quad a_2 = \frac{B_1 c_1}{2(1 + \alpha) b_2}.
\]
Similarly from (11) and (12), we obtain
\[
(14) \quad a_3 = \frac{[2B_1 (c_2 - \frac{1}{2} c_1^2) + B_2 c_1^2](1 + \alpha)^2 - (\alpha^2 - 4\alpha - 1)B_2 c_1^2}{8(2\alpha + 1)(1 + \alpha)^2 b_3}.
\]
The inequality (5) now follows from (13) and the first part of Lemma 1.7.
By using (13) and (14), we have
\[
(15) \quad a_3 - \mu a_2 = \frac{B_1}{4(2\alpha + 1) b_3} [c_2 - \mu c_1^2],
\]
where
\[
v := \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} + \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1 + \alpha)^2} + \frac{2\mu(2\alpha + 1) B_1 b_3}{(1 + \alpha)^2 b_2^2} \right].
\]
If $\mu \leq \sigma_1$, then an application of Lemma 1.5 gives
\[
|a_3 - \mu a_2| \leq \frac{B_1}{2(2\alpha + 1) |b_3|} \left( \frac{B_2}{B_1} \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1 + \alpha)^2} - \frac{\mu(2\alpha + 1) B_1 b_3}{(1 + \alpha)^2 b_2^2} \right),
\]
which is the first part of assertion (6).
Next, if $\mu \geq \sigma_2$, then by applying Lemma 1.5, we can write
\[
|a_3 - \mu a_2| \leq \frac{B_1}{2(2\alpha + 1) b_3} \left( \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1 + \alpha)^2} + \frac{\mu(2\alpha + 1) B_1 b_3}{(1 + \alpha)^2 b_2^2} - \frac{B_2}{B_1} \right),
\]
which is the third part of assertion (6).
If $\mu = \sigma_1$, then equality holds if and only if $p_1(z)$ is given by (4) or one of its rotations.
If $\mu = \sigma_2$, then
\[
\frac{1}{2} \left[ 1 - \frac{B_2}{B_1} + \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1 + \alpha)^2} + \frac{2\mu(2\alpha + 1) B_1 b_3}{(1 + \alpha)^2 b_2^2} \right] = 1.
\]
Therefore,
\[
\frac{1}{p_1(z)} = \left(\frac{1 + \gamma}{2}\right) \frac{1 + z}{1 - z} + \left(\frac{1 - \gamma}{2}\right) \frac{1 - z}{1 + z} \quad (0 < \gamma < 1, z \in U).
\]
Finally, we see that
\[
a_3 - \mu a_2^2 = \frac{B_1}{4(2\alpha + 1)b_3} \left[ c_2 - \frac{c_1^2}{2} \left( 1 - \frac{B_2}{B_1} + \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1 + \alpha)^2} + \frac{2\mu(2\alpha + 1)B_1b_3}{(1 + \alpha)^2b_2^2} \right) \right].
\]
Therefore using Lemma 1.5, we get
\[
|a_3 - \mu a_2^2| \leq \frac{B_1}{2(2\alpha + 1)|b_3|} \quad (\sigma_1 \leq \mu \leq \sigma_2).
\]
If \(\sigma_1 < \mu < \sigma_2\), then we have
\[
p_1(z) = \frac{1 + \lambda z^2}{1 - \lambda z^2} \quad (0 \leq \lambda \leq 1).
\]
By an application of Lemma 1.5, we obtain our result. To show that the inequality (0) is sharp, we define the functions \(K^{\phi_n} (n = 2, 3, 4, \ldots)\) by
\[
\Psi_g(K^{\phi_n})(z) = \phi(z^{n-1}) \quad (K^{\phi_n}(0) = 0 = (K^{\phi_n})'(0) - 1)
\]
and the functions \(G^\gamma\) and \(H^\gamma (0 \leq \gamma \leq 1)\) by
\[
\Psi_g(G^\gamma)(z) = \phi \left( \frac{z(z + \gamma)}{1 + \gamma z} \right) \quad (G^\gamma(0) = 0 = (G^\gamma)'(0) - 1)
\]
and
\[
\Psi_g(H^\gamma)(z) = \phi \left( \frac{z(z + \gamma)}{1 + \gamma z} \right) \quad (H^\gamma(0) = 0 = (H^\gamma)'(0) - 1).
\]
It is clear that the functions \(K^{\phi_n} (n = 2, 3, 4, \ldots), G^\gamma\) and \(H^\gamma (0 \leq \gamma \leq 1)\) are in the class \(S_{\mathbb{U}}^{\gamma}(\phi)\). In either cases \(\mu < \sigma_1\) or \(\mu > \sigma_2\), the equality holds if and only if \(f\) is \(K^{\phi_2}\) or one of its rotations. When \(\sigma_1 < \mu < \sigma_2\) the equality occurs if and only if \(f\) is \(K^{\phi_3}\) or one of its rotations. If \(\mu = \sigma_1\), then the equality holds if and only if \(f\) is \(H^\gamma\) or one of its rotations. If \(\mu = \sigma_2\), then the equality holds if and only if \(f\) is \(H^\gamma\) or one of its rotations.

\textbf{Remark 2.2.} Using Lemma 1.5 the result can be improved when \(\sigma_1 \leq \mu \leq \sigma_2\) as follows:

Let
\[
\sigma_3 := \frac{(1 + \alpha)^2b_2^2}{2(2\alpha + 1)B_1b_3} \left( \frac{B_2}{B_1} - \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1 + \alpha)^2} \right).
\]
If \(\sigma_1 \leq \mu \leq \sigma_3\), then
\[
|a_3 - \mu a_2^2| + \frac{(1 + \alpha)^2b_2^2}{2(2\alpha + 1)B_1b_3} \left( 1 - \frac{B_2}{B_1} + \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1 + \alpha)^2} + \frac{2\mu(2\alpha + 1)B_1b_3}{(1 + \alpha)^2b_2^2} \right) |a_2|^2 \leq \frac{B_1}{2(2\alpha + 1)|b_3|}
\]
and if \(\sigma_3 \leq \mu \leq \sigma_2\), then
\[
|a_3 - \mu a_2^2| + \frac{(1 + \alpha)^2b_2^2}{2(2\alpha + 1)B_1b_3} \left( 1 + \frac{B_2}{B_1} - \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1 + \alpha)^2} + \frac{2\mu(2\alpha + 1)B_1b_3}{(1 + \alpha)^2b_2^2} \right) |a_2|^2 \leq \frac{B_1}{2(2\alpha + 1)|b_3|}.
Proof. For the values of \( 0 < \mu \leq \sigma_1 \), we have
\[
|a_3 - \mu a_2^2| + \frac{(1 + \alpha)^2b_2^2}{2(2\alpha + 1)B_1b_3} \left( 1 - \frac{B_2}{B_1} - \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1 + \alpha)^2} - \frac{2\mu(2\alpha + 1)B_1b_3}{(1 + \alpha)^2b_2^2} \right) |a_2|^2
\]
\[
= |a_3 - \mu a_2^2| + (\mu - \sigma_1)|a_2|^2
\]
\[
= \frac{B_1}{4(2\alpha + 1)|b_3|} [(c_2 - v|c_1|^2) + v|c_1|^2]
\]
\[
\leq \frac{B_1}{2(2\alpha + 1)|b_3|}.
\]
Similarly, if \( \sigma_2 \leq \mu \leq \sigma_3 \), then
\[
|a_3 - \mu a_2^2| + \frac{(1 + \alpha)^2b_2^2}{2(2\alpha + 1)B_1b_3} \left( 1 + \frac{B_2}{B_1} - \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1 + \alpha)^2} - \frac{2\mu(2\alpha + 1)B_1b_3}{(1 + \alpha)^2b_2^2} \right) |a_2|^2
\]
\[
= |a_3 - \mu a_2^2| + (\sigma_2 - \mu)|a_2|^2
\]
\[
= \frac{B_1}{4(2\alpha + 1)|b_3|} [(c_2 - v|c_1|^2) + (1 - v)|c_1|^2]
\]
\[
\leq \frac{B_1}{2(2\alpha + 1)|b_3|}.
\]
Thus the proof is complete. \( \square \)

Remark 2.3. If we set \( \alpha = 1 \) and \( g(z) = z/(1 - z) \) in Theorem 2.1, then we have the result \([3, \text{Theorem 3}]\) of Ma and Minda.

Remark 2.4. By setting \( \alpha = 0 \) and \( g(z) = z/(1 - z) \) in Theorem 2.1 we obtain the result of Murugusundaramoorthy et al. \([3, \text{Corollary 2.2}]\).

Using Lemma 1.6 and equation 1.5, we deduce the following:

Theorem 2.5. Let \( g(z) \) be given by (2) with \( b_2, b_3 \) non zero real numbers. Assume that \( \alpha \geq 0 \) and \( \phi(z) = 1 + B_1z + B_2z^2 + \cdots \). If \( f \in S_{2\alpha}'(\phi) \), then for any complex number \( \mu \)
\[
|a_3 - \mu a_2^2| \leq \frac{B_1}{2(2\alpha + 1)|b_3|} \max \left\{ 1; \left| 2\mu(2\alpha + 1)B_1b_3 \right| - \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1 + \alpha)^2} \frac{B_2}{B_1} \right\}.
\]

3. Applications

A few applications of our main results are discussed here.

Definition 3.1. \([13]\) Let \( f(z) \) be an analytic function in a simply connected region of the complex plane containing the origin. The fractional derivative of order \( \delta \) is defined by
\[
D_\delta^\phi f(z) = \frac{1}{\Gamma(1 - \delta)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^\delta} dt \quad (0 \leq \delta < 1),
\]
where the multiplicity of \((z-t)^\delta\) is removed by requiring that \( \log(z-t) \) is real for \((z-t) > 0\).
Using the above Definition 3.1 and its extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [14] introduced the operator $Ω^δ : A → A$ defined by

$$\Omega^δ f(z) = \Gamma(2 - δ) z^δ D^δ f(z), \quad δ ≠ 2, 3, 4 \ldots.$$ 

If we take $g(z) = z + \sum_{n=2}^{∞} \frac{Γ(n+1)Γ(2-δ)}{Γ(n+1-δ)} z^n$ in Theorem 2.1 then we obtain the following:

**Corollary 3.2.** Let $α ≥ 0$ and $φ(z) = 1 + B_1 z + B_2 z^2 + \ldots$. If $f ∈ S^φ_α$, then

$$|a_2| ≤ \frac{(2 - δ)B_1}{2(1 + α)}$$

and for any real number $μ$,

$$|a_3 - μa^2_2| ≤ \begin{cases} \frac{(2 - δ)(3 - δ)B_1}{12(2α + 1)} & B_1 > B_2 - \frac{(α^2 - 4α - 1)B_1}{(1 + α)^2} - \frac{3μ(2α + 1)(2 - δ)B_1}{(1 + α)^2(3 - δ)} \\ \frac{(2 - δ)(3 - δ)B_1}{12(2α + 1)} & B_1 < B_2 - \frac{(α^2 - 4α - 1)B_1}{(1 + α)^2} + \frac{3μ(2α + 1)(2 - δ)B_1}{(1 + α)^2(3 - δ)} - \frac{B_1}{B_1} \end{cases}$$

if $μ ≤ σ_1$;

$$|a_3 - μa^2_2| ≤ \begin{cases} \frac{(1 + α)^2(3 - δ)}{3(2 - δ)(2α + 1)B_1} & B_2 - \frac{(α^2 - 4α - 1)B_1}{(1 + α)^2} - 1 \\ \frac{(1 + α)^2(3 - δ)}{2(2α + 1)B_1} & 1 + B_2 - \frac{(α^2 - 4α - 1)B_1}{(1 + α)^2} \end{cases}$$

$$σ_1 := \frac{(1 + α)^2(3 - δ)}{3(2 - δ)(2α + 1)B_1} \left( B_2 - \frac{(α^2 - 4α - 1)B_1}{(1 + α)^2} - 1 \right)$$

and

$$σ_2 := \frac{(1 + α)^2(3 - δ)}{2(2α + 1)B_1} \left( 1 + B_2 - \frac{(α^2 - 4α - 1)B_1}{(1 + α)^2} \right).$$

The result is sharp.

From Theorem 2.1 and Remark 2.2 we deduce the following:

**Corollary 3.3.** Let $g(z)$ be given by (3) with $b_2, b_3$ non zero real numbers. Assume that $α ≥ 0$ and $-1 ≤ D < C ≤ 1$. If $f ∈ S^g_α$, then

$$|a_2| ≤ \frac{C - D}{(1 + α)|b_2|}$$

and for any real number $μ$,

$$|a_3 - μa^2_2| ≤ \begin{cases} \frac{D - C}{2(2α + 1)|b_3|} & D + \frac{(α^2 - 4α - 1)(C - D)b_3}{(1 + α)^2} + \frac{2μ(2α + 1)(C - D)b_3}{(1 + α)^2b_2^2} \\ \frac{C - D}{2(2α + 1)|b_3|} & C + \frac{(α^2 - 4α - 1)(C - D)b_3}{(1 + α)^2} + \frac{2μ(2α + 1)(C - D)b_3}{(1 + α)^2b_2^2} \end{cases}$$

if $μ ≤ σ_1$;

$$|a_3 - μa^2_2| ≤ \begin{cases} \frac{(1 + α)^2b_2^2}{2(2α + 1)(D - C)b_3} & 1 + D + \frac{(α^2 - 4α - 1)(C - D)}{(1 + α)^2} \\ \frac{(1 + α)^2b_2^2}{2(2α + 1)(C - D)b_3} & 1 - D - \frac{(α^2 - 4α - 1)(C - D)}{(1 + α)^2} \end{cases}$$

$$σ_1 := \frac{(1 + α)^2b_2^2}{2(2α + 1)(D - C)b_3} \left( 1 + D + \frac{(α^2 - 4α - 1)(C - D)}{(1 + α)^2} \right)$$

and

$$σ_2 := \frac{(1 + α)^2b_2^2}{2(2α + 1)(C - D)b_3} \left( 1 - D - \frac{(α^2 - 4α - 1)(C - D)}{(1 + α)^2} \right).$$

The result is sharp.
Remark 3.4. The result can be improved when \( \sigma_1 \leq \mu \leq \sigma_2 \) as follows:

If \( \sigma_1 \leq \mu \leq \sigma_3 \), then

\[
|a_3 - \mu a_2^2| + \frac{(1 + \alpha)^2 b_2^2}{2(2\alpha + 1)(C - D)|b_3|} \left( 1 + D + \frac{(\alpha^2 - 4\alpha - 1)(C - D)}{(1 + \alpha)^2} + \frac{2\mu(2\alpha + 1)(C - D)b_3}{(1 + \alpha)^2 b_2^2} \right) |a_2|^2 \leq \frac{C - D}{2(2\alpha + 1)|b_3|}
\]

and if \( \sigma_3 \leq \mu \leq \sigma_2 \), then

\[
|a_3 - \mu a_2^2| + \frac{(1 + \alpha)^2 b_2^2}{2(2\alpha + 1)(C - D)|b_3|} \left( 1 - D - \frac{(\alpha^2 - 4\alpha - 1)(C - D)}{(1 + \alpha)^2} - \frac{2\mu(2\alpha + 1)(C - D)b_3}{(1 + \alpha)^2 b_2^2} \right) |a_2|^2 \leq \frac{C - D}{2(2\alpha + 1)|b_3|},
\]

where

\[
\sigma_3 := \frac{(1 + \alpha)^2 b_2^2}{2(2\alpha + 1)(D - C)b_3} \left( D + \frac{(\alpha^2 - 4\alpha - 1)(C - D)}{(1 + \alpha)^2} \right).
\]

By taking \( D = -1 \) and \( C = 1 \) in the above Corollary 3.3, we obtain the following:

**Example 3.5.** Let \( \alpha \geq 0 \) and \( g(z) \) be given by (4) with \( b_2, b_3 \) non zero real numbers. If \( f \in S^{\mu}_{\frac{1 + \alpha}{\alpha}} \), then

\[
|a_2| \leq \frac{2}{(1 + \alpha)|b_2|}
\]

and for any real number \( \mu \),

\[
|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{(1 + \alpha)^2 |b_3|} \left( \frac{3 + 10\alpha - \alpha^2}{2\alpha + 1} - \frac{4\mu b_3}{b_2^2} \right) & \text{if } \mu \leq \sigma_1; \\ \frac{1}{(2\alpha + 1)|b_3|} & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{1}{(1 + \alpha)^2 |b_3|} \left( \frac{\alpha^2 - 10\alpha - 3}{2\alpha + 1} + \frac{4\mu b_3}{b_2^2} \right) & \text{if } \mu \geq \sigma_2; \end{cases}
\]

where

\[
\sigma_1 := \frac{(1 + 4\alpha - \alpha^2)b_2^2}{2(2\alpha + 1)b_3} \quad \text{and} \quad \sigma_2 := \frac{(3\alpha + 1)b_2^2}{(2\alpha + 1)b_3}.
\]

The result can be improved when \( \sigma_1 \leq \mu \leq \sigma_2 \) as follows:

If \( \sigma_1 \leq \mu \leq \sigma_3 \), then

\[
|a_3 - \mu a_2^2| + \frac{b_2^2}{2|b_3|} \left( \frac{\alpha^2 - 4\alpha - 1}{2\alpha + 1} + \frac{2\mu b_3}{b_2^2} \right) |a_2|^2 \leq \frac{1}{(2\alpha + 1)|b_3|}
\]

and if \( \sigma_3 \leq \mu \leq \sigma_2 \), then

\[
|a_3 - \mu a_2^2| + \frac{b_2^2}{|b_3|} \left( \frac{3\alpha + 1}{2\alpha + 1} - \frac{\mu b_3}{b_2^2} \right) |a_2|^2 \leq \frac{1}{(2\alpha + 1)|b_3|},
\]

where

\[
\sigma_3 := \frac{(3 + 10\alpha - \alpha^2)b_2^2}{4(2\alpha + 1)b_3}.
\]

The result is sharp.

Remark 3.6. We obtain the result of Răducanu [8, Theorem 2] by taking

\[
g(z) = z + \sum_{n=2}^{\infty} n^m z^n \quad (m \in \{0, 1, 2, 3, \ldots\})
\]

in the above Example 3.5.
Remark 3.7. Setting $\alpha = 0$ and $g(z) = \frac{z}{1-z}$ in the Example 3.5, we obtain the following result [12]:

If $f \in S^*$, then

$$|a_2| \leq 2$$

and for any real number $\mu$

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
3 - 4\mu & \text{if } \mu \leq \frac{1}{2}; \\
1 & \text{if } \frac{1}{2} \leq \mu \leq 1; \\
4\mu - 3 & \text{if } \mu \geq 1.
\end{cases}$$

The result can be improved when $\frac{1}{2} \leq \mu \leq 1$ as follows:

If $\frac{1}{2} \leq \mu \leq \frac{3}{4}$, then

$$|a_3 - \mu a_2^2| + \frac{1}{2}(2\mu - 1)|a_2|^2 \leq 1$$

and if $\frac{3}{4} \leq \mu \leq 1$, then

$$|a_3 - \mu a_2^2| + (1 - \mu)|a_2|^2 \leq 1.$$  

Setting $\alpha = 1$ and $g(z) = \frac{z}{1-z}$ in the Example 3.5, we have the following result:

Let $f \in C$, then

$$|a_2| \leq 1$$

and for any real number $\mu$

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
1 - \mu & \text{if } \mu \leq \frac{2}{3}; \\
\frac{1}{3} & \text{if } \frac{2}{3} \leq \mu \leq \frac{4}{3}; \\
\mu - 1 & \text{if } \mu \geq \frac{4}{3}.
\end{cases}$$

The result can be improved when $\frac{2}{3} \leq \mu \leq \frac{4}{3}$ as follows:

If $\frac{2}{3} \leq \mu \leq 1$, then

$$|a_3 - \mu a_2^2| + \frac{1}{3}(3\mu - 2)|a_2|^2 \leq \frac{1}{3}$$

and if $1 \leq \mu \leq \frac{4}{3}$, then

$$|a_3 - \mu a_2^2| + \frac{1}{3}(4 - 3\mu)|a_2|^2 \leq \frac{1}{3}.$$  

Taking $\phi(z) = \frac{1+Cz}{1+Dz}$, $-1 \leq D < C \leq 1$, in Theorem 2.5, we deduce the following:

**Corollary 3.8.** Let $\alpha \geq 0$ and $g(z)$ be given by (2) with $b_2, b_3$ non zero real numbers. If $f \in S^*_{\frac{\alpha}{2}}(\frac{1+Cz}{1+Dz})$, then for any complex number $\mu$

$$|a_3 - \mu a_2^2| \leq \frac{C-D}{2(2\alpha + 1)|b_3|} \max \left\{ 1; \frac{2\mu(2\alpha + 1)(C-D)b_3}{(1+\alpha)^2 b_2^2} + \frac{\alpha^2 - 4\alpha - 1(C-D)}{(1+\alpha)^2} + D \right\}.$$  

**Remark 3.9.** If we take $g(z) = z + \sum_{n=2}^{\infty} n^m z^n$, $D = -1$ and $C = 1$ in the above Corollary 3.8, we have the following result [8 Theorem 3] of Răducanu:

Let $\alpha \geq 0$. If $f \in H^{S^*_m}(\alpha)$, then for any complex number $\mu$

$$|a_3 - \mu a_2^2| \leq \frac{1}{3^m(1+2\alpha)} \max \left\{ 1; \frac{2^{2m-1}(\alpha^2 - 10\alpha - 3) + 2.3^m(1+2\alpha)\mu}{2^{2m-1}(1+\alpha)^2} \right\}.$$
Remark 3.10. If we set $D = -1, C = 1$ and $g(z) = \frac{z}{1-z}$ in Corollary 3.8, then for $\alpha = 0$, we have the following result [3, Theorem 1](see also [12]):

Let $f \in \mathcal{S}^\ast$. Then for any complex number $\mu$

$$|a_3 - \mu a_2^2| \leq \max \{1; |4\mu - 3|\}.$$ 

Setting $\alpha = 1, D = -1, C = 1$ and $g(z) = \frac{z}{1-z}$ in Corollary 3.8, we obtain the following result [3, Corollary 1] due to Keogh and Merkes:

Let $f \in \mathcal{C}$, then for any complex number $\mu$

$$|a_3 - \mu a_2^2| \leq \max \left\{ \frac{1}{3}; |\mu - 1| \right\}.$$ 

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Department of Applied Mathematics, Delhi Technological University, Delhi-110042, India

E-mail address: spkumar@dce.ac.in

Department of Applied Mathematics, Delhi Technological University, Delhi-110042, India

E-mail address: vktmaths@yahoo.in