TOPOLOGICAL CHARACTERIZATIONS OF RECURRENCE, POISSON STABILITY, AND ISOMETRIC PROPERTY OF FLOWS ON SURFACES

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Abstract. The long-time behavior is one of the most fundamental properties of dynamical systems. Poincaré studied the Poisson stability to capture the property of whether points return arbitrarily near the initial positions after a sufficiently long time. Birkhoff introduced the concepts of recurrent points and non-wandering points by introducing the concepts of \( \omega \)-limit set and \( \alpha \)-limit set of a point. Cherry showed that the set of orbits in the closure of a non-closed recurrent orbit of a flow on a manifold contains uncountably many Poisson stable orbits \[7\]. Athanassopoulos characterized a flow that is either irrational or Denjoy on a closed surface by using non-closed Poisson stable orbits \[8\]. On the other hand, the distal property is introduced by Hilbert \[15\] to characterize a rigid group of motions topologically. Isometric dynamics also appear in rigid motions. In this paper, we show that Poisson stability, recurrence, and distal property of flows on surfaces are topological properties of the orbit spaces. More precisely, to state the main results, we recall the following concepts. A topological space is \( T_{1/2} \) if any singletons are closed or open. A topological space is \( S_{1/2} \) (resp. \( S_1, S_2 \)) if the Kolmogorov quotient is \( T_{1/2} \) (resp.

1. Introduction

The long-time behavior of orbits is one of the most fundamental properties in dynamical systems. In \[21, 22\], Poincaré studied the Poisson stability, to capture the property of whether points return arbitrarily near the initial positions after a sufficiently long time. In \[6\], Birkhoff introduced and studied the concepts of non-wandering points and recurrent points, by introducing the concepts of \( \omega \)-limit set and \( \alpha \)-limit set of a point. Cherry showed that the set of orbits in the closure of a non-closed recurrent orbit of a flow on a manifold contains uncountably many Poisson stable orbits \[7\]. Athanassopoulos characterized a flow that is either irrational or Denjoy on a closed surface by using non-closed Poisson stable orbits \[8\]. On the other hand, the distal property is introduced by Hilbert \[15\] to characterize a rigid group of motions topologically. Isometric dynamics also appear in rigid motions. In this paper, we show that Poisson stability, recurrence, and distal property of flows on surfaces are topological properties of the orbit spaces. In fact, the recurrence, Poisson stability, and distal property of flows on surfaces are characterized using separation axioms. More precisely, to state the main results, we recall the following concepts. A topological space is \( T_{1/2} \) if any singletons are closed or open. A topological space is \( S_{1/2} \) (resp. \( S_1, S_2 \)) if the Kolmogorov quotient is \( T_{1/2} \) (resp.
Isometric (up to topological equivalence) $\supseteq$ Minimal

\[
\begin{array}{cccc}
\text{Equicontinuous} & \text{R-closed} & S_2 & \supseteq T_2 \\
\cap \uparrow & & \cap \uparrow & \\
\text{Distal} & \text{Poisson stable} & S_1 & \supseteq T_1 \\
\cap \uparrow & & \cap \uparrow & \\
\text{Recurrent} & S_{1/2} & & T_{1/2} \\
\cap \uparrow & & \cap \uparrow & \\
\text{Non-wandering} & & \text{Non-minimal} & \\
\end{array}
\]

**Figure 1.** Relations among concepts for topological equivalence classes of flows on compact surfaces.

Then we have the following topological characterizations of recurrence and Poisson stability.

**Theorem A.** A flow on a connected compact surface is recurrent if and only if the orbit space is $S_{1/2}$.

**Theorem B.** The following statements are equivalent for a flow on a connected compact surface:
1. The flow is Poisson stable.
2. The orbit space is $S_1$.
3. The flow is distal.

Note that there is a non-Poisson-stable recurrent flow on a closed surface in §8.2. Using the characterizations, we characterize the isometric property for flows on surfaces as follows.

**Theorem C.** The following conditions are equivalent for a flow on a connected compact surface:
1. The flow is topologically equivalent to a (real-analytic) isometric flow.
2. The flow is equicontinuous.
3. The flow is R-closed.
4. The orbit space is $S_2$.
5. The flow is non-wandering, and the singular point set either is the whole surface or consists of at most two centers.

**Theorem D.** The orbit space of a flow on a connected compact surface is $T_2$ if and only if the flow consists of closed points and the singular point set either is the whole surface or consists of at most two centers.

There is a flow $v$ on a sphere whose orbit space is not $S_2$ but $T_1$ such that the singular point set is a “double of lakes of Wada continuum” (see an example in §8.1 for details). Notice that the orbit spaces of any minimal flows on surfaces are $S_2$ but not $T_1$.

The present paper consists of eight sections. In the next section, as preliminaries, we introduce fundamental concepts. In §3, we topologically characterize recurrence
for flows on surfaces. In §4, Poisson stability for flows on surfaces is characterized topologically. In particular, Poisson stability is equivalent to distal property. In §5, the difference between recurrence and Poisson stability for flows on compact surfaces is characterized. In §6, these characterizations imply the topological characterization of $T_2$ separation axiom for the orbit spaces of flows on compact surfaces. In §7, we demonstrate that $S_2$ separation axiom for the orbit spaces for flows on compact surfaces corresponds to the isometric property up to topological equivalence. In the final section, examples are described to state the difference between recurrence, Poisson stability, and $S_2$ separation axiom.

2. Preliminaries

2.1. Topological notion. A surface is a two-dimensional paracompact manifold with or without boundary. By a continuum, we mean a nonempty compact connected metrizable space. A subset $C$ in a topological space $X$ is separating if the complement $X - C$ is disconnected.

2.1.1. Separation axioms for points and spaces. A point $x$ of a topological space $X$ is $T_0$ (or Kolmogorov) if for any point $y \neq x \in X$ there is an open subset $U$ of $X$ such that $\{|x, y| \cap U| = 1$, where $|A|$ is the cardinality of a subset $A$. A point of a topological space is $T_{1/2}$ if its singleton is either closed or open. A point of a topological space is $T_1$ if its singleton is closed. A point $x$ of a topological space $X$ is $T_2$ or Hausdorff if for any point $y \in X - \{x\}$ there are open neighborhoods $U_x$ and $U_y$ of $x$ and $y$ respectively with $U_x \cap U_y = \emptyset$. A topological space is $T_{1/2}$ (resp. $T_1$, $T_2$) if each point is $T_{1/2}$ (resp. $T_1$, $T_2$).

2.1.2. $T_0$-ification of a topological space. Let $X$ be a topological space with the specialization order. Define the class $\hat{\mathcal{X}} := \{ y \in X \mid \overline{\{x\}} = \overline{\{y\}} \}$ for any point $x$ and a decomposition $\hat{X} := \{ \hat{z} \mid z \in X \}$ of classes. Then the decomposition $\hat{X}$ is a $T_0$ space as a quotient space, which is called the $T_0$-ification (or Kolmogorov quotient) of $X$. A topological space is $S_2$ (resp. $S_1$, $S_{1/2}$) if the $T_0$-ification is $T_2$ (resp. $T_1$, $T_{1/2}$).

2.1.3. Cantor manifolds. The small inductive dimension of the empty set is $-1$. By induction, for any non-negative integer $n \in \mathbb{Z}_{\geq 0}$, the small inductive dimension of a nonempty topological space $X$ has small inductive dimension less than or equal to $n$ if for any point $x \in X$ and any open neighborhood $U$ of $x$, there is an open neighborhood $V$ of $x$ with $\overline{V} \subseteq U$ such that the boundary of $V$ has small inductive dimension less than or equal to $n - 1$.

A topological space has small inductive dimension $n$ if it has small inductive dimension less than or equal to $n$ but does not have small inductive dimension less than or equal to $n + 1$. By dimension, we mean small inductive dimension. By Urysohn’s theorem, the Lebesgue covering dimension, the large inductive dimension, and the small inductive dimension correspond in separable metrizable spaces.

A separable metrizable space $X$ whose small inductive dimension is $n > 0$ is an $n$-dimensional Cantor manifold if the complement $X - L$ for any closed subset $L$ of $X$ whose small inductive dimension is less than $n - 1$ is connected. In [27], Urysohn showed that any $n$-dimensional topological manifold is an $n$-dimensional Cantor manifold.
2.2. Notion of dynamical systems. By a flow, we mean a continuous $\mathbb{R}$-action on a topological space. Let $v: \mathbb{R} \times X \to X$ be a flow on a topological space $X$. Then $v_t := v(t, \cdot)$ is a homeomorphism on $S$. For a point $x$ of $X$, we denote by $O(x)$ the orbit of $x$ (i.e. $O(x) := \{v_t(x) \mid t \in \mathbb{R}\}$), $O^+(x)$ the non-negative orbit (i.e. $O^+(x) := \{v_t(x) \mid t \geq 0\}$), and $O^-(x)$ the non-positive orbit (i.e. $O^-(x) := \{v_t(x) \mid t \leq 0\}$). A subset of $X$ is said to be invariant (or saturated) if it is a union of orbits. An invariant subset is minimal if it has no non-empty proper invariant closed subsets. A point $x$ of $X$ is singular if $x = v_t(x)$ for any $t \in \mathbb{R}$, is periodic if there is a positive number $T > 0$ such that $x = v_T(x)$ and $x \neq v_t(x)$ for any $t \in (0, T)$, and is closed if it is either singular or periodic. Denote by $\text{Sing}(v)$ (resp. $\text{Per}(v)$, $\text{Cl}(v)$) the set of singular (resp. periodic, closed) points.

A point is wandering if there are its neighborhood $U$ and a positive number $N$ such that $U(t) \cap U = \emptyset$ for any $t > N$. Then such a neighborhood is called a wandering domain. A point is non-wandering if it is not wandering (i.e. for any its neighborhood $U$ and for any positive number $N$, there is a number $t \in \mathbb{R}$ with $|t| > N$ such that $U(t) \cap U \neq \emptyset$). For a point $x \in X$, define the $\omega$-limit set $\omega(x)$ and the $\alpha$-limit set $\alpha(x)$ of $x$ as follows: $\omega(x) := \bigcap_{n \in \mathbb{R}} \{v_t(x) \mid t \geq n\}$, $\alpha(x) := \bigcap_{n \in \mathbb{R}} \{v_t(x) \mid t \leq n\}$. A point $x$ of $X$ is Poisson stable (or strongly recurrent) if $x \in \omega(x) \cap \alpha(x)$. A point $x$ of $X$ is recurrent if $x \in \omega(x) \cup \alpha(x)$.

Denote by $R(v)$ the set of non-closed recurrent points. The closure of a non-closed recurrent orbit is called a $Q$-set (or quasi-minimal set). An orbit is singular (resp. periodic, closed, non-wandering, recurrent, Poisson stable) if it consists of singular (resp. periodic, closed, non-wandering, recurrent, Poisson stable) points.

A flow on a surface $X$ is trivial if it is either minimal or identical (i.e. $v_t(x) = x$ for any $t \in \mathbb{R}$ and $x \in X$). A flow is non-wandering (resp. Poisson stable, recurrent) if each point is non-wandering (resp. Poisson stable, recurrent). A flow is pointwise almost periodic if any orbit closures are minimal sets. Notice that a flow is pointwise almost periodic if and only if the orbit space is $S_1$. A flow on a surface $X$ is $R$-closed if the orbit closure relation $\{(x, y) \in X \times X \mid y \in O(x)\}$ is closed with respect to the product topology on $X \times X$.

2.2.1. Concepts on flows on paracompact manifolds. Fix the distance $d$ induced by a Riemannian metric on a paracompact manifold $M$. A flow $v$ on the paracompact manifold $M$ is equicontinuous if, for any $\varepsilon \in \mathbb{R}_{>0}$, there is a positive number $\delta \in \mathbb{R}_{>0}$ with $\sup\{d(v_t(x), v_t(y)) \mid t \in \mathbb{R}, x, y \in M, d(x, y) < \delta\} < \varepsilon$. A flow $v$ on $M$ is isometric if $d(x, y) = d(v_t(x), v_t(y))$ for any points $x, y \in M$ and any $t \in \mathbb{R}_{>0}$. A flow $v$ on $M$ is distal if $x = y$ for any points $x, y, z \in M$ and any sequence $(t_n)_{n \in \mathbb{Z}_{>0}}$ with $z = \lim_{n \to \infty} v(t_n, x) = \lim_{n \to \infty} v(t_n, y)$. Note that the distal property implies pointwise almost periodicity (i.e. $T_1$ separation axiom for the orbit class space) for any flow on a compact manifold (more generally, for any group action on a locally compact Hausdorff space such that any orbit closures are compact), from [12, Theorem 1] and [4, Theorem 7 p.11].

2.2.2. Isolated property from closed invariant subsets. A closed invariant subset is isolated from closed invariant subsets if there is its neighborhood that does not contain any closed invariant set except those contained in it.

2.2.3. Positive and negative asymptotically stability. A compact invariant subset $M$ of a flow on a topological space $X$ is positively asymptotically stable if it satisfies the following two conditions: (1) For any neighborhood $U$ of $M$, there is a
neighborhood $V$ of $M$ with $\bigcup_{x \in V} O^+(x) \subseteq U$; (2) the subset $\{y \in X \mid \omega(y) \subseteq M\}$ is a neighborhood of $M$. Notice that a compact invariant subset satisfying the first (resp. second) condition in the previous definition is also called a (positive) attractor and that one satisfying the second condition in the previous definition is also called a (positive) stable (cf. [10, Definition 2.3 and Definition 2.4]). Similarly, a compact invariant subset $M$ of a flow on a topological space $X$ is negatively asymptotically stable if, for any neighborhood $U$ of $M$, there is a neighborhood $V$ of $M$ with $\bigcup_{x \in V} O^-(x) \subseteq U$, and the subset $\{y \in X \mid \alpha(y) \subseteq M\}$ is a neighborhood of $M$.

2.2.4. Orbit classes and orbit class spaces of flows. For a flow $v$ on a topological space $X$ and for an invariant subset $T \subseteq X$, define an equivalence relation $\sim_v$ on $T$ by $x \sim_v y$ if $O(x) = O(y)$. Then the quotient space $T/\sim_v$ of $T$ is called the orbit space of $T$ and is denoted by $T/v$. Notice that an orbit space $T/v$ is the set $\{O(x) \mid x \in T\}$ as a set. Since any minimal flow on a surface consists of non-closed orbits, the orbit space of any minimal flow on a surface is not $T_1$.

The (orbit) class $\hat{O}$ of an orbit $O$ is the union of orbits each of whose orbit closure corresponds to $\hat{O}$ (i.e. $\hat{O} = \{y \in X \mid \overline{O(y)} = \overline{O}\}$). Moreover, the orbit class space $T/\hat{v}$ is defined as the set $\{\hat{O}(x) \mid x \in T\}$ with the quotient topology. In other words, the orbit class space $T/\hat{v}$ is defined as the quotient space $T/\approx_v$ by $x \approx_v y$ if $\overline{O(x)} = \overline{O(y)}$. Note that the orbit class space is a $T_0$-ification of the orbit space. By [28, Lemma 2.2], a flow on a compact surface is $R$-closed if and only if the orbit class space is $S_2$.

2.2.5. Topological properties of orbits. Let $v$ be a flow on a paracompact manifold $M$. An orbit is proper if it is embedded, locally dense if its closure has a nonempty interior, and exceptional if it is neither proper nor locally dense. A point is proper (resp. locally dense, exceptional) if its orbit is proper (resp. locally dense, exceptional). Denote by $LD(v)$ (resp. $E(v)$, $P(v)$) the union of locally dense orbits (resp. exceptional orbits, non-closed proper orbits). Then $M = \overline{P(v)} \cup \overline{LD(v)} \cup \overline{E(v)}$, where $\cup$ denotes a disjoint union. We have the following observation.

**Lemma 2.1.** The following statements hold for a flow $v$ on a paracompact manifold $M$:

1. The union $P(v)$ of non-closed proper orbits is the set of non-recurrent points.
2. The subset $\overline{P(v)} \cup \overline{LD(v)} \cup \overline{E(v)}$ is the set of recurrent points.
3. The subset $\overline{R(v)} = \overline{LD(v)} \cup \overline{E(v)}$ is the union of non-proper orbits.
4. $M = \overline{P(v)} \cup \overline{LD(v)} \cup \overline{E(v)}$.

**Proof.** By definition of recurrence and properness, any closed orbit is recurrent and proper. From the invariance of flows, either the set difference $\overline{O} - O$ is closed or $\overline{O} = \overline{O} - \overline{O}$ for any orbit $O$. Note that an orbit of a flow on a paracompact manifold is proper if and only if it has a neighborhood in which the orbit is closed [30]. Then any orbit $O$ is proper if and only if the set difference $\overline{O} - O$ is closed. In [7, Theorem VI], Cherry showed that the closure of a non-closed recurrent orbit $O$ of a flow on a paracompact manifold contains uncountably many non-closed Poisson stable orbits whose closures are $\overline{O}$. Then any non-closed orbit is recurrent if and only if $\overline{O} = \overline{O} - O$. Any non-closed proper orbits are not recurrent and any non-closed recurrent orbits are not proper. Since any closed orbits are proper and
recurrent, a non-recurrent point is non-closed proper and a non-proper point is recurrent. This implies that assertion (1) holds.

Since the union $P(v)$ of non-closed proper orbits is the set of non-recurrent points, the complement $S - P(v) = Cl(v) \cup LD(v) \cup E(v)$ is the set of recurrent points. This means that assertion (2) holds. Since the union $Cl(v) \cup P(v)$ is the union of proper orbits, the complement $R(v) = M - (Cl(v) \cup P(v))$ is the union of non-proper orbits. This means that assertion (3) holds. By the decomposition $M = Cl(v) \cup P(v) \cup LD(v) \cup E(v)$, assertion (4) holds.

3. Topological characterization of recurrence

We have the following property.

Lemma 3.1. The following are equivalent for a flow $v$ on a connected compact surface $S$:

(1) The flow $v$ is non-wandering.

(2) There are finitely many orbits $O_1, O_2, \ldots, O_k \subset S$ with $S - Cl(v) \subseteq \bigcup_{i=1}^{k} O_i$.
   In the second case, we can choose $O_1, O_2, \ldots, O_k$ such that the orbit classes $\hat{O}_1, \hat{O}_2, \ldots, \hat{O}_k$ are connected components of the set $R(v)$ of non-closed recurrent points.

Proof. Let $v$ be a flow on a connected compact surface $S$. Suppose that there are finitely many non-closed orbits $O_1, O_2, \ldots, O_k \subset S$ with $S - Cl(v) \subseteq \bigcup_{i=1}^{k} O_i$. Then

$$S = \overline{Cl(v)} \cup \bigcup_{i=1}^{k} O_i.$$  

Since any point in $S - Cl(v)$ is contained in the closure $\overline{O_j}$ for some $j \in \{1, 2, \ldots, k\}$, the flow $v$ is non-wandering.

Conversely, suppose that $v$ is non-wandering. By [29] Lemma 2.4 and Theorem 2.5, since $Sing(v)$ is closed, we obtain $S = Cl(v) \cup \partial P(v) \cup LD(v) = Cl(v) \cup LD(v)$, where $\partial A$ is the boundary of a subset $A$. Then $R(v) = LD(v)$. By the Mayer theorem [18][19] (cf. Remark 2 [1]), the total number of $Q$-sets for $v$ is finite. From [29] Proposition 2.2, there are finitely many locally dense orbits $O_1, O_2, \ldots, O_k \subset S$ with $S - Cl(v) \subseteq LD(v) = \bigcup_{i=1}^{k} \hat{O}_i$ and $LD(v) = \bigcup_{i=1}^{k} \hat{O}_i$ such that $\overline{O_i \cap \hat{O}_j} = \overline{O_i \cap \hat{O}_j} \subseteq Sing(v) \cup \partial P(v)$ for any $i \neq j \in \{1, 2, \ldots, k\}$. Then $\hat{O}_i \cap \hat{O}_j = \emptyset$ for any $i \neq j \in \{1, 2, \ldots, k\}$. This means that the orbit classes $\hat{O}_1, \hat{O}_2, \ldots, \hat{O}_k$ are connected components of the set $R(v)$ of non-closed recurrent points.

We have the following characterization of recurrence for flows on surfaces.

Theorem 3.2. The following are equivalent for a flow on a connected compact surface:

(1) The flow is recurrent.

(2) There are no non-closed proper orbits.

(3) Each orbit is closed or locally dense.

(4) The orbit class space is $T_{1/2}$.

(5) The orbit space is $S_{1/2}$.

Proof. Since the orbit class space is the $T_0$-ification of the orbit space, assertions (4) and (5) are equivalent. From [29] Lemma 2.3, assertions (2) and (3) are equivalent. By definition of recurrence, assertion (3) implies assertion (1). Let $v$ be a flow on a connected compact surface $S$. Suppose that $S/v$ is $T_{1/2}$. This means that each
orbit class is closed or open. Therefore each orbit closure is either a minimal set or a locally dense Q-set. This implies that each orbit is recurrent. Hence assertion (4) implies assertion (1).

Conversely, suppose that \( v \) is recurrent. Since the union \( P(v) \) is the set of non-recurrent points, we have \( P(v) = \emptyset \). By definition of non-wandering property and recurrence, the flow \( v \) is non-wandering. From [29, Lemma 2.3], we obtain \( E(v) = \emptyset \). Then \( S = \overline{\text{Cl}(v)} \cup \text{LD}(v) \). This means that assertion (3) holds. [29, Lemma 2.3] implies that \( \text{LD}(v) \cap \overline{\text{Cl}(v)} = \emptyset \) and so that \( \text{LD}(v) \) is open. By Lemma 3.1, there are finitely many non-closed recurrent orbits \( O_1, O_2, \ldots, O_k \) with \( \text{LD}(v) = \bigcup_{i=1}^{k} \overline{O_i} \) such that the orbit classes \( \overline{O_1}, \overline{O_2}, \ldots, \overline{O_k} \) are connected components of \( \text{LD}(v) \). Since any connected components are closed, the finiteness of \( \text{LD}(v) \) implies that the orbit classes \( \overline{O_1}, \overline{O_2}, \ldots, \overline{O_k} \) are open in the open subset \( \text{LD}(v) \) and so open in \( S \). This implies that the orbit classes of any non-closed points are open. Therefore assertion (4) holds. □

Theorem A follows from Theorem 3.2.

4. Topological characterization of Poisson stability

We will show that Poisson stability and pointwise almost periodicity for flows on compact surfaces are equivalent. To demonstrate this, we show the following statements.

4.1. Topological properties of minimal flows on compact surfaces. We have the following properties of minimal flows on compact surfaces.

**Lemma 4.1.** Every flow \( v \) on a connected compact surface \( S \) is minimal if and only if \( \text{LD}(v) = S \).

**Proof.** If \( v \) is minimal, then \( S = \overline{O} \) for any orbit and so \( \text{LD}(v) = S \). Suppose that \( \text{LD}(v) = S \). By [29, Proposition 2.2], since \( \text{Sing}(v) \cup P(v) \cup E(v) = \emptyset \), the union \( \overline{O} = \overline{O} \) is closed for any orbit \( O \). This implies that any orbit closures are minimal sets. By definition of minimal sets, any two distinct minimal sets do not intersect. By the Mañé theorem [18, 19] (cf. Remark 2 [1]), the total number of minimal sets for \( v \) is finite. This means that there are finitely many locally dense orbits \( O_1, O_2, \ldots, O_k \subseteq S \) with \( S = \bigcup_{i=1}^{k} \overline{O_i} = \bigcup_{i=1}^{k} \overline{O_i} \) and \( \overline{O} = \overline{O_1} \). Then the union \( \overline{O} = \overline{O} = S - \bigcup_{i=2}^{k} \overline{O_i} \) is open. By the connectivity of \( S \), the nonempty open and closed subset \( \overline{O} \) is the whole space \( S \). This means that \( v \) is minimal. □

**Proposition 4.2.** The following statements hold for a minimal flow \( v \) on a compact surface \( S \):

1. The surface \( S \) is a torus, and the flow \( v \) is a suspension flow of an irrational rotation on a circle.
2. The flow \( v \) is topologically equivalent to a (real-analytic) isometric flow on a torus.
3. The flow \( v \) is equicontinuous and distal.

**Proof.** Let \( v \) be a minimal flow on a compact surface \( S \). Then \( S \) is connected. By the Poincaré-Hopf theorem, the Euler characteristic of \( S \) is zero. Then \( S \) is either a sphere, a closed annulus, a torus, a projective plane, a Möbius band, or a Klein bottle. The Mañé and Markley works [17,18] (cf. [1, Remark 2]) imply
that no surfaces which are either spheres, closed annuli, projective planes, Möbius bands, or Klein bottles have no Q-sets and so admit no minimal flows. Therefore S is a torus. By the minimality of v, any ω-limit and α-limit sets are the whole surface. [3 Theorem 1.1] implies that v is a suspension flow of an irrational rotation on a circle. This implies that v is topologically equivalent to the isometric flow \( v_X \) generated by a vector field \( X = (1, \alpha) \) on the torus \( \mathbb{R}/\mathbb{Z}^2 \) for some \( \alpha \in \mathbb{R} - \mathbb{Q} \). By the definition of equicontinuity, the isometric flow \( v_X \) is equicontinuous. Since the equicontinuity is invariant under topological equivalence, so is the flow v. By definition of distal property, any equicontinuous flow on a Hausdorff space is distal (cf. [11, Proposition 15.3]).

### 4.2. Characterization of Poisson stability.

We have the following properties.

**Lemma 4.3.** For a flow v on a connected closed surface S, there is a flow w whose singular point set is totally disconnected on a surface T which is a disjoint union of closed surfaces such that the restriction \( v|_{\text{S} - \text{Sing}(v)} \) is topologically equivalent to the restriction \( w|_{T - \text{Sing}(w)} \).

**Proof.** Fix a Riemannian metric on S. Since the singular point set is closed, the complement \( S_0 := S - \text{Sing}(v) \) is open and so a surface with at most finite genus. Let \( S_{\text{mc}} \) be the metric completion of \( S_0 \). Collapsing each connected component of \( S_{\text{mc}} - S_0 \) into a singleton, let \( S_{\text{mc}} \) be the resulting space. Define a flow \( v_{\text{mc}} \) on \( S_{\text{mc}} \) as follows: \( v_{\text{mc}}|_{S_0} = v|_{S_0} \) and the difference \( S_{\text{mc}} - S_0 \) is the set \( \text{Sing}(v_{\text{mc}}) \) of singular points of \( v_{\text{mc}} \). We show that \( v_{\text{mc}} \) is desired. Indeed, by [23 Theorem 3], there is a surface \( T \) which is a disjoint union of closed surfaces such that the surface \( S_0 \) is homeomorphic to the resulting surface from \( T \) by removing a closed totally disconnected subset. Then the surface \( S_{\text{mc}} \) is homeomorphic to \( T \). By construction, the singular point set \( \text{Sing}(v_{\text{mc}}) \) of \( v_{\text{mc}} \) is totally disconnected and the restriction \( v|_{S_0} \) is topologically equivalent to the restriction \( v_{\text{mc}}|_{S_0} \). □

**Lemma 4.4.** Each non-minimal locally dense Q-set of a flow on a connected compact surface contains orbits that are not Poisson stable.

**Proof.** Let \( v \) be a flow on a connected compact surface S and \( \mathcal{M} \) a non-minimal locally dense Q-set of \( v \). Since Poisson stability is invariant under taking the double \( M \cup M - M \) of the manifold \( M \), taking the double of \( M \) if necessary, we may assume that \( M \) is closed. Here the double \( M \cup M - M \) is defined as \( (M \times \{-1, 1\})/\sim \), where \( (x, 1) \sim (x, -1) \) for any \( x \in \partial M \). Fix any locally dense orbit O with \( \overline{O} = \mathcal{M} \subseteq \text{LD}(v) \). By [28 Proposition 2.2], we have \( \overline{O} \cap \text{Per}(v) = \emptyset \) and \( \hat{O} = \overline{O} \setminus (\text{Sing}(v) \cap \text{P}(v)) \).

[28] Lemma 2.3] implies that \( \text{E}(v) \cap \text{Cl}(v) \cap \text{LD}(v) = \emptyset \) and so that the union \( \text{P}(v) \cup \text{E}(v) \) is a neighborhood of \( \text{E}(v) \). Since \( O \cap (\text{P}(v) \cup \text{E}(v)) = \emptyset \), we have \( \overline{O} \cap \text{E}(v) = \emptyset \). Therefore \( \overline{O} \subseteq \text{Sing}(v) \cup \text{P}(v) \cup \text{LD}(v) \) and \( \hat{O} \subseteq \text{LD}(v) \).

Assume that \( \mathcal{M} = \overline{O} \) consists of Poisson stable orbits. Then \( \overline{O}(y) = \omega(y) \) for any point \( y \in \overline{O} \). Since \( \text{P}(v) \) is the set of non-recurrent points, we have \( \overline{O} \cap \text{P}(v) = \emptyset \). Then \( \overline{O} \subseteq \text{Sing}(v) \cup \text{LD}(v) \). [28] Lemma 2.3] implies that \( \text{LD}(v) \cap \text{Cl}(v) \cup \text{E}(v) = \emptyset \) and so that the union \( \text{P}(v) \cup \text{LD}(v) \) is a neighborhood of \( \text{LD}(v) \). Since both \( \overline{O} \) and \( \text{LD}(v) \cup \text{P}(v) \) are neighborhoods of \( \hat{O} \subseteq \text{LD}(v) \), the intersection \( U := \overline{O} \cap (\text{LD}(v) \cup \text{P}(v)) \subseteq \text{LD}(v) \) is an invariant neighborhood of \( \hat{O} \). Then \( U = \overline{O} \cap \text{LD}(v) = \overline{O} \setminus \text{Sing}(v) = \hat{O} \subseteq \text{LD}(v) \) and so that \( U = \hat{O} \) is open. Since \( \overline{U} \setminus \text{Sing}(v) = U = \mathcal{M} \cap \text{LD}(v) \subseteq \text{LD}(v) \), non-minimality implies that U is an open open
surface such that $\emptyset \neq \partial U = \overline{U} - U \subseteq \text{Sing}(v)$. Replacing $S - U$ with singular points, the subset $\mathcal{M} = \overline{O}$ consists of Poisson stable orbits with respect to the resulting flow $w_0$ such that $U = \text{LD}(w_0)$, $S = \text{Sing}(w_0) \cup \text{LD}(w_0)$, and $v|_U = w_0|_U$. By Lemma 4.3 there is a flow $w$ with a nonempty totally disconnected singular point set on a connected closed surface $T$ with $U = \text{LD}(w)$ and $T = \text{Sing}(w) \cup \text{LD}(w)$ such that the restriction $v|_{S - \text{Sing}(w_0)}$ is topologically equivalent to the restriction $w|_{T - \text{Sing}(w)}$. Therefore $w$ is also Poisson stable. In particular, the restriction $v|_U = w_0|_U$ is topologically equivalent to the restriction $w|_{T - \text{Sing}(w)}$. Since $\text{Sing}(w)$ is totally disconnected, we have $T = \overline{T}^T$, where $\overline{T}^T$ is the closure of $U$ in the compact surface $T$. By $\hat{O} = U$, we obtain that $T = \overline{O_w(y)}^T = \alpha_w(y) = \omega_w(y)$ for any point $y \in U = T - \text{Sing}(w)$, where $\overline{O_w(y)}^T$ is the orbit closure of $y$ under the flow $w$ on the compact surface $T$. Then $\text{Sing}(w)$ is a maximal compact invariant subset in any complement $T - \{y\}$ for any point $y \in U = T - \text{Sing}(w)$. Therefore $\text{Sing}(w)$ is isolated from closed invariant subsets of $w$. By results in [24, 26] (cf. [3, Theorem 1.6]) to the compact invariant subset $\text{Sing}(w)$, since $\text{Sing}(w)$ is isolated from closed invariant subsets but neither positively asymptotically stable nor negative asymptotically stable with respect to $w$, there is a locally dense point $y \in T - \text{Sing}(w) = U$ such that either $\alpha_w(y) \subseteq \text{Sing}(w)$ or $\omega_w(y) \subseteq \text{Sing}(w)$. Since $T = \alpha_w(y) = \omega_w(y)$, we have $y \in U = \text{LD}(w) \subseteq T = \alpha_w(y) = \omega_w(y) = \text{Sing}(w)$, which contradict $\text{LD}(w) \cap \text{Sing}(w) = \emptyset$. Thus $\mathcal{M} = \overline{O}$ does not consist of Poisson stable orbits.

We have the following characterization of Poisson stability using $T_1$ separation axiom for the orbit spaces of flows.

**Theorem 4.5.** The following statements are equivalent for a flow $v$ on a connected compact surface $S$:

1. The flow $v$ is Poisson stable.
2. Either the flow $v$ is minimal, or all orbits are closed.
3. Either $P(v) \cup \text{LD}(v) = \emptyset$ or $S = \text{LD}(v)$.
4. The orbit class space $S/\hat{v}$ is $T_1$.
5. The orbit space $S/v$ is $S_1$.
6. The flow $v$ is pointwise almost periodic.
7. The flow $v$ is distal.

**Proof.** Since the orbit class space is a $T_0$-ification of the orbit space, assertions (4) and (5) are equivalent. By definition of pointwise almost periodicity, assertions (4) and (6) are equivalent. From [12, Theorem 1] and [3, Theorem 7 p.11], assertion (7) implies assertion (6). Let $v$ be a flow on a connected compact surface $S$. By Lemma 4.1 the flow is minimal if and only if $S = \text{LD}(v)$. If $v$ is minimal, then Proposition 4.2 implies that $v$ is Poisson stable and distal, $S = \text{LD}(v)$, and the orbit class space $S/\hat{v}$ is a singleton and so $T_1$. If $v$ is identical, then $v$ is Poisson stable and distal, $S = S/v = S/\hat{v}$ is $T_1$, and $P(v) \cup \text{LD}(v) = \emptyset$. Thus we may assume that $v$ is not trivial (i.e. neither identical nor minimal). By definition of orbit class spaces, assertion (2) implies assertion (5). Since any closed orbits are Poisson stable, assertion (2) implies assertion (1).

Suppose that $P(v) \cup \text{LD}(v) = \emptyset$. By [24, Lemma 2.3], we have $E(v) = \emptyset$ and so $S = \text{Cl}(v)$. This means that assertion (3) implies assertion (2).
Suppose that \( v \) is Poisson stable. Since Poisson stable orbits are recurrent, Lemma 3.2 implies that \( S = \text{Cl}(v) \cup \text{LD}(v) \). [29] Lemma 2.3 implies that \( \text{LD}(v) \cap \text{Cl}(v) = \emptyset \) and so that \( \text{LD}(v) \) is open. By Lemma 4.4, Poisson stability implies that each locally dense Q-set of \( v \) is minimal. This implies that the orbit closure of any non-closed recurrent point consists of non-closed recurrent points, and so that the finite union \( \text{LD}(v) \) of Q-sets is closed. Non-minimality implies \( \text{LD}(v) = \emptyset \) and so \( S = \text{Cl}(v) \). This means that assertion (1) implies assertion (2).

Suppose that the orbit class space \( S/\hat{v} \) is \( T_1 \). Then each orbit closure is a minimal set and so \( v \) is non-wandering. Since the closure of a non-recurrent orbit is not minimal, we have \( \text{P}(v) = \emptyset \). By [29] Lemma 2.4, the periodic point set \( \text{Per}(v) \) is open and \( S = \text{Cl}(v) \cup \text{LD}(v) \). [29] Lemma 2.3 implies that \( \text{LD}(v) \cap \text{Cl}(v) = \emptyset \) and so that \( \text{LD}(v) \) is open. By Lemma 4.4, the union \( \text{LD}(v) \) consists of a finite disjoint union of orbit classes. Since each orbit closure is a minimal set, the union \( \text{LD}(v) \) is a finite disjoint union of locally dense minimal sets and so is closed. By non-minimality of \( v \) and connectivity of \( S \), we obtain \( \text{LD}(v) = \emptyset \) and so \( \text{P}(v) \sqcup \text{LD}(v) = \emptyset \). This means that assertion (4) implies assertion (3).

Suppose that assertion (2) holds. The non-minimality implies that any orbit is closed (i.e. the flow is pointwise periodic). Assume that there are points \( x, y, z \in S \) with \( x \neq y \) and a sequence \((t_n)_{n \in \mathbb{Z}_+} \) with \( z = \lim_{n \to \infty} v(t_n, x) = \lim_{n \to \infty} v(t_n, y) \). Then the pointwise periodicity implies that \( v(t_n, x) \in O(x) = \overline{O(x)} \) and \( v(t_n, y) \in O(y) = \overline{O(y)} \) for any \( n \). This means that \( z \in O(x) = O(y) \). If \( x \in \text{Sing}(v) \), then \( z = x = y \), which contradicts \( x \neq y \). Thus \( x \in \text{Per}(v) \). Let \( T \) be the minimal periodic of \( O(x) = O(y) \). Then there is a positive number \( t_y \in (0, T_x) \) such that \( x = v(t_y, y) \). Since \( v(t_x, y) = v(t + t_y, y) \neq v(t, y) \) for any \( t \in \mathbb{R} \), the compactness and periodicity of \( O(x) \) imply that \( \min_{t \in T_x} d(v(t_x, y), v(t, y)) > 0 \). Therefore \( \lim_{n \to \infty} d(v(t_n, x), v(t_n, y)) \geq \min_{t \in [0, T_x]} d(v(t_x, y), v(t, y)) > 0 \). This means that \( \lim_{n \to \infty} v(t_n, x) \neq \lim_{n \to \infty} v(t_n, y) \), which contradicts \( z = \lim_{n \to \infty} v(t_n, x) = \lim_{n \to \infty} v(t_n, y) \). Thus \( v \) is distal. This means that assertion (2) implies assertion (7).

Theorem 13 follows from Theorem 4.3. Notice that there is a non-wandering non-recurrent flow. Indeed, consider a torus \( \mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2 \) and a vector field \( X = (1, 0) \) on \( \mathbb{T}^2 \). Using a bump function \( \varphi: \mathbb{T}^2 \to [0, 1] \) such that the inverse image \( \varphi^{-1}(0) = [0, 0] \), the flow generated by \( \varphi X \) consists one singular point \([0, 0] \), one non-recurrent orbit \( \{[(x, 0)] \mid x \neq 0 \in \mathbb{R}/\mathbb{Z} \} \), and periodic orbits and so is non-wandering and non-recurrent.

5. Difference between recurrence and Poisson stability

The following statements state the difference between recurrence and Poisson stability for flows on compact surfaces.

**Lemma 5.1.** One of the following statements holds exclusively for any Poisson stable flow \( v \) on a connected compact surface \( S \):

1. The surface \( S \) is a torus, and the flow \( v \) is a suspension flow of an irrational rotation on a circle.
2. \( S = \text{Cl}(v) \) and the union \( \text{Per}(v) \) is a disjoint union of invariant annuli.

**Proof.** If \( v \) is minimal, then assertion (1) follows from Proposition 4.2. By Theorem 4.5 we may assume that all orbits are closed. Then the complement \( \text{Per}(v) = \)}
$S - \text{Sing}(v)$ is open. The flow box theorem (cf. [24, Theorem 1.1, p.45]) for periodic orbits implies that the holonomies for any periodic orbits are trivial, and so any periodic orbits have invariant open annular neighborhoods consisting of periodic orbits. This means that the union $\text{Per}(v)$ is a disjoint union of invariant annuli. □

Lemma 5.2. For any recurrent flow $v$ on a compact connected surface $S$, the union $\text{Per}(v)$ is an open subset which is a disjoint union of invariant annuli, the union $\text{LD}(v)$ is open, and $S = \text{Cl}(v) \sqcup \text{LD}(v)$.

Proof. Theorem 4.2 implies that $S = \text{Cl}(v) \sqcup \text{LD}(v)$. By [24, Lemma 2.3], we obtain $\text{Cl}(v) \cap \text{LD}(v) = \emptyset$. This means that the union $\text{LD}(v)$ is open. [29, Lemma 2.1] implies that $\text{Per}(v) \cap \overline{O(x)} = \emptyset$ for any locally dense orbit $O$. From the Maer and Markley works [17,18] (cf. [1, Remark 2]), the closure $\text{LD}(v)$ is a finite union of orbit closures of points in $\text{LD}(v)$ and so $\text{Per}(v) \cap \text{LD}(v) = \emptyset$. By the closedness of $\text{Sing}(v)$, the union $\text{Per}(v) = S - \text{Sing}(v) \sqcup \text{LD}(v)$ is open. Since the resulting flow replacing $\text{LD}(v)$ by singular points are Poisson stable, Lemma 5.1 implies that the union $\text{Per}(v)$ is an open subset which is a disjoint union of invariant annuli. □

The previous lemmas imply the difference between recurrence and Poisson stability for flows on compact surfaces as follows.

Theorem 5.3. The following statements are equivalent for any flow $v$ on a connected compact surface:

1. The flow $v$ is Poisson stable.
2. The flow $v$ is recurrent and either $\text{LD}(v) = \emptyset$ or $\text{Cl}(v) = \emptyset$.

Proof. Let $v$ be a flow on a connected compact surface $S$. If $v$ is Poisson stable, then Lemma 5.1 implies that either $S = \text{Cl}(v)$ or $S = \text{LD}(v)$. This means that assertion (1) implies assertion (2). Suppose that the flow $v$ is recurrent and either $\text{LD}(v) = \emptyset$ or $\text{Cl}(v) = \emptyset$. If $\text{LD}(v) = \emptyset$, then Lemma 5.2 implies that $S = \text{Cl}(v)$ and so that $v$ is Poisson stable. Thus we may assume that $\text{CL}(v) = \emptyset$. From Lemma 5.2 we have that $S = \text{LD}(v)$. Since $S = \text{LD}(v)$, Theorem 4.5 (1) and (3) imply that $v$ is Poisson stable. □

6. CHARACTERIZATIONS OF $T_1$ AND $T_2$ SEPARATION AXIOMS FOR ORBIT SPACES

We observe the following statement.

Lemma 6.1. Every orbit of a flow on a compact surface is not an open subset.

Proof. By Baire category theorem, since every orbit of a flow on a compact surface is a countable union of closed intervals which are nowhere dense, there are no orbits that are open. □

Note that the previous lemma also holds for higher dimensional compact manifolds. By the previous lemma and Theorem 4.6 imply the following characterization of $T_1$ separation axiom for flows on connected compact surfaces.

Theorem 6.2. The following statements are equivalent for a flow on a connected compact surface:

1. The orbit space is $T_1$.
2. The orbit space is $T_{1/2}$.
3. The flow is not minimal but Poisson stable.
Proof. Let $v$ be a flow on a connected compact surface $S$. Suppose that the orbit space is $T_{1/2}$. Lemma 6.1 implies that every orbit is closed and so that the orbit space is $T_1$. Since every $T_1$ space is $T_{1/2}$, assertions (1) and (2) are equivalent. Suppose that the orbit space is $T_1$. Then every orbit is closed, and so $v$ is not minimal because any minimal flows on surfaces consist of infinitely many non-closed orbits. Theorem 4.5 implies that $v$ is Poisson stable. Suppose that $v$ is not minimal but Poisson stable. By Theorem 4.5 every orbit is closed, and so the orbit space is $T_1$. □

Recall that a singular point $x$ of a flow $v$ on a surface $S$ is a (topological) center if there is an open neighborhood $U$ of $x$ such that the restriction $v|_U$ is topologically equivalent to the flow generated by a vector field $X = (-y, x)$ on an open unit disk $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$. We show the following equivalence to characterize $T_1$ separation axiom for flows on connected compact surfaces.

**Lemma 6.3.** The following statements are equivalent for a non-minimal flow with finitely many singular points on a connected compact surface:

1. The flow is non-wandering, and the singular point set consists of at most two centers.
2. The flow is Poisson stable.
3. The orbit space is $T_1$.
4. Each singular point is a center, and there are neither limit cycles nor exceptional $Q$-sets.
5. The flow consists of periodic orbits and at most two centers.

Proof. Let $v$ be a non-minimal flow with finitely many singular points on a connected compact surface $S$. By non-minimality of $v$, Theorem 6.2 implies that assertions (2) and (3) are equivalent. From the definition of orbit space, assertion (5) implies assertion (3).

We claim that there are no limit cycles in any case. Indeed, recall that each Poisson stable flow is non-wandering. The finiteness of singular points implies that each singular point is isolated. The non-existence of wandering domains implies the non-existence of limit cycles in any case.

Suppose that $v$ is non-wandering and the singular point set $\text{Sing}(v)$ consists of at most two centers. By \cite[Lemma 2.4]{24}, there are no exceptional $Q$-sets. This means that assertion (1) implies assertion (4).

Suppose that each singular point is a center and there are neither limit cycles nor exceptional $Q$-sets. Then there are at most finitely many singular points and $E(v) = \emptyset$. From the Maier and Markley works \cite{17,18} (cf. \cite[Remark 2]{1}), the closure $\text{LD}(v)$ is a finite union of orbit closures of points in $\text{LD}(v)$ and so $\text{Per}(v) \cap \text{LD}(v) = \emptyset$. Therefore $\text{LD}(v) \subseteq S - \text{Cl}(v) = \text{P}(v) \cup \text{LD}(v)$. From the definition of local density, each of $\omega$-limit set and $\alpha$-limit set of a point in $S - \text{LD}(v)$ is not a locally dense $Q$-set. By a generalization of the Poincaré-Bendixson theorem for a flow with finitely many singular points (cf. \cite[Theorem 2.6.1]{20}), since any singular points are centers, each of $\omega$-limit set and $\alpha$-limit set of a non-closed point is a locally dense $Q$-set. This means that any non-closed orbit is locally dense and so $\text{P}(v) = \emptyset$. Therefore $S = \text{Cl}(v) \cup \text{LD}(v)$ and $\text{LD}(v) = \text{LD}(v)$. By \cite[Lemma 2.3]{29}, we have that $\text{Cl}(v) \cap \text{LD}(v) = \emptyset$. Then both $\text{Cl}(v)$ and $\text{LD}(v)$ are closed and open. The connectivity of $S$ implies that either $S = \text{Cl}(v)$ or $S = \text{LD}(v)$. By Lemma 4.1
the non-minimality of \( v \) implies \( S = \text{Cl}(v) \). This shows that assertion (4) implies assertion (3).

Suppose that \( S/v \) is \( T_1 \). This means that \( S = \text{Cl}(v) \) and so that \( v \) is non-wandering. [8] Theorem 3 implies that each singular point is either a center or a multi-saddle. By the non-existence of non-closed orbits, each singular point is a center. By connectivity of \( S \), Poincaré-Hopf theorem implies that there are at most two centers. This shows that assertion (3) implies assertions (1) and (5). \( \square \)

We characterize \( T_2 \) separation property of the orbit (class) spaces for non-trivial flows on connected compact surfaces. The orientable case of the following result was stated in [25, Theorem 6.6].

**Proposition 6.4.** Let \( v \) be a non-trivial flow on a connected compact surface \( S \). The following statements are equivalent:

1. The orbit space \( S/v \) is \( T_2 \).
2. The orbit space \( S/v \) is \( S^2 \) (i.e. \( v \) is \( R \)-closed).
3. The flow \( v \) consists of periodic orbits and at most two centers.
4. The flow \( v \) is non-wandering and the singular point set consists of at most two centers.
5. The flow \( v \) is Poisson stable and each singular point is isolated.

In any case, the Euler characteristic of \( S \) is non-negative and the orbit space \( S/v \) is either a closed interval or a circle.

**Proof.** Lemma 6.3 implies that assertions (3)–(5) are equivalent. Form Theorem 4.5 and Theorem 6.2 by assertion (2), the orbit space \( S/v \) is \( T_1 \) for any cases. Let \( v \) be a non-trivial flow on a connected compact surface \( S \) whose orbit space \( S/v \) is \( T_1 \). This means that \( S = \text{Cl}(v) = \text{Sing}(v) \sqcup \text{Per}(v) \) and so assertions (1) and (2) are equivalent. The closedness of the singular point set implies that the union \( \text{Per}(v) \) is open. Since \( v \) is non-trivial, there is a periodic orbit \( O \). Let \( C \) be the connected component of \( \text{Per}(v) \) that contains \( O \).

Suppose that \( v \) consists of periodic orbits and at most two centers. [25, Corollary 2.9] implies that each connected component of \( \text{Per}(v) \) is either an annulus, a torus, a Möbius band, or a Klein bottle whose orbit space is an interval or a circle and whose boundary consists of singular points and one-sided periodic orbits. By the Poincaré-Hopf theorem, the Euler characteristic of \( S \) is non-negative. Moreover, each connected component of the boundary \( \partial C \) is a center and so the complement \( S - \text{Sing}(v) \) is connected. This implies \( S = C \sqcup \text{Sing}(v) \). Since the restriction \( C/v \) is an interval or a circle, the orbit space \( S/v \) is either a closed interval or a circle, and so is \( T_2 \).

Conversely, suppose that \( S/v \) is \( T_2 \). Then \( S = \text{Sing}(v) \sqcup \text{Per}(v) \) and so \( \partial \text{Sing}(v) = \partial \text{Per}(v) \). Each boundary component of \( \text{Per}(v) \) is a singular point and so is each boundary component of \( \text{Sing}(v) \). By definition of dimension, the dimension of \( \text{Sing}(v) \) is at most one and so \( \text{Sing}(v) = \partial \text{Sing}(v) \). This means that each connected component of \( \text{Sing}(v) \) is a singleton. Since a connected compact surface is a Cantor manifold (cf. [16, Theorem 2.1]), the complement \( \text{Per}(v) = S - \text{Sing}(v) \) is connected. Therefore \( \text{Per}(v) = C \), and it contains no singular points. This implies that \( C \) is a surface whose Euler characteristic is zero and so either an annulus, a torus, a Möbius band, or a Klein bottle. Then the whole surface \( S = \overline{\text{Per}(v)} = \overline{C} = C \sqcup \partial C \) is the union of periodic orbits and at most two centers. \( \square \)
Proposition 6.4 implies the characterizations of $T_2$ separation axiom for orbit spaces and orbit class spaces of flows on connected compact surfaces as follows.

**Proof of Theorem D.** Let $v$ be a flow on a connected compact surface $S$. Suppose that $S/v$ is $T_2$. Then $v$ is not minimal. Proposition 6.4 implies that either $v$ is identical or $v$ consists of closed points and at most two centers. This means that $v$ consists of closed points, and that the singular point set either is the whole surface or consists of at most two centers.

Conversely, suppose that $v$ consists of closed points and that the singular point set either is the whole surface or consists of at most two centers. Then $v$ is not minimal. Since the orbit spaces of the identical flows are the original surfaces and so is $T_2$, we may assume that $v$ is non-trivial. Proposition 6.4 implies $S/v$ is $T_2$. □

7. **Characterization of $T_2$ separation axiom for the orbit class spaces**

In this section, we characterize $T_2$ separation axiom for the orbit class spaces of flows on compact surfaces. We have the following equivalence.

**Lemma 7.1.** The orbit space of a flow on a connected compact surface is $S_2$ if and only if the flow is non-wandering and the singular point set either is the whole surface or consists of at most two centers.

**Proof.** Let $v$ be a flow on a connected compact surface $S$. If $v$ is minimal, then $v$ is non-wandering, there are no singular points, and $S/\hat{v}$ is a singleton and so is $T_2$. If $v$ is identical, then $v$ is non-wandering, the singular point set is the whole surface, and $S/\hat{v}$ is the original surface $S$ and so is $T_2$. Thus we may assume that $v$ is non-trivial. Proposition 6.4 implies the assertion. □

To show Theorem C, we show the following equivalence.

**Lemma 7.2.** The following conditions are equivalent for a flow on a connected compact surface:

(1) The orbit space of a flow on a connected compact surface is $S_2$.

(2) The flow is $R$-closed.

(3) The flow is equicontinuous.

(4) The flow is topologically equivalent to a real-analytic isometric flow.

Theorem C follows from Lemma 7.1 and Lemma 7.2. To demonstrate Lemma 7.2, we recall the following concepts. A flow on a closed disk is a closed center disk if it is topologically equivalent to an isometric flow $v_{D^2}$ on the unit disk $D^2 = \{(r \cos \theta, r \sin \theta) \in \mathbb{R}^2 \mid r^2 \leq 1\}$ defined by $v_{D^2}(t, (r \cos \theta, r \sin \theta)) := (r \cos(\theta + \pi), r \sin(\theta + t))$. A flow on a sphere is a rotating sphere if it is topologically equivalent to an isometric flow $v_{S^2}$ on the unit sphere $S^2 = \{(\rho \cos \theta, \rho \sin \theta, z) \in \mathbb{R}^3 \mid \rho^2 + z^2 = 1\}$ defined by $v_{S^2}(t, (\rho \cos \theta, \rho \sin \theta, z)) := (\rho \cos(\theta + t), \rho \sin(\theta + t), z)$. A flow on a projective plane is a rotating projective plane if it is topologically equivalent to the isometric flow $v_{P^2}$ on the projective plane $P^2 := S^2/(x, y, z) \sim (-x, -y, -z)$ induced by the flow $v_{S^2}$. Notice that a rotating sphere (resp. periodic Möbius band) and a center. We have the following tetrachotomy for $R$-closed flows on compact surfaces.

**Proposition 7.3.** One of the following statements holds exclusively up to topological equivalence for any flow $v$ on a connected compact surface $S$ whose orbit space is $S_2$:

...
The flow $v$ is identical.

The surface $S$ is a torus, and the flow $v$ is a suspension flow of a rotation on a circle.

The surface $S$ is either a closed annulus, a Möbius band, or a Klein bottle, and the flow $v$ is a suspension flow of an isometric involution on either a closed interval or a circle.

The flow $v$ is either a rotating sphere, a rotating projective plane, or a closed center disk.

In any case, the flow $v$ is topologically equivalent to a real-analytic isometric flow.

Proof. Let $v$ be a flow on a connected compact surface $S$ whose orbit space is $S^2$. We may assume that $v$ is not identical. If $v$ is minimal, then Proposition 4.2 implies that $S$ is a torus, and the flow $v$ is a suspension flow of an irrational rotation on a circle. Thus we may assume that $v$ is not minimal. Then $v$ is non-trivial and $S = \text{Cl}(v)$. Proposition 6.4 implies that $S/v$ is either a closed interval or a circle and that the flow $v$ consists of periodic orbits and at most two centers.

Suppose that there are no centers. Then $S = \text{Per}(v)$. If $S/v$ is a circle, then $S$ is a torus and the flow $v$ is a suspension flow of a rational rotation on a circle. Thus we may assume that the orbit space $S/v$ is a closed interval. The non-existence of singular points implies that every point of the boundary of the closed interval $S/v$ is a periodic orbit that either is the boundary component of $S$ or has its small invariant neighborhood which is a Möbius band. This means that $S$ is either a closed annulus, a Möbius band, or a Klein bottle, and that the flow $v$ is a suspension flow of an isometric involution on either a closed interval or a circle up to topological equivalence.

Suppose that there are exactly two centers. Then $v$ is topologically equivalent to the rotating sphere $v_{S^2}$.

Suppose that there is exactly one center. This means that $S$ is either a closed disk or a projective plane and that the flow $v$ consists of one center and periodic orbits. If $S$ is a closed disk, then $S$ can be identified with the quotient space $S^2/\sim$ by $(x, y, z) \sim (x, y, -z)$ and the flow $v$ is a closed center disk. Thus we may assume that $S$ is a projective plane. Then $v$ is a rotating projective plane. □

We demonstrate Lemma 7.2 as follows.

Proof of Lemma 7.2. By [28, Lemma 2.2], assertions (1) and (2) are equivalent. From Proposition 7.3, assertion (1) implies assertion (4). By definition of equicontinuity, assertion (4) implies assertion (3). From [13, Proposition 4.10], assertion (3) implies assertion (1). □

8. Examples of recurrent flows and Poisson stable flows

We state some examples to state the difference between recurrence, Poisson stability, and $R$-closedness.

8.1. Distal non-$R$-closed smooth flows with continua like lakes of Wada whose orbit spaces. There is a distal flow $v$ on a sphere $S$ with $\text{Cl}(v) = S$ such that $\text{Sing}(v)$ is a “double of lakes of Wada continuum” and that the orbit space $S/v$ is not $S^2$ but $T_1$. More precisely, for any natural number $k \in \mathbb{Z}_{>0}$, the union $\text{Per}(v)$ is the disjoint union of $k$ open disks $B_i$ with $\text{Sing}(v) = \partial B_i$ for any $i \in \{1, 2, \ldots, k\}$. 

Indeed, by the construction of lakes of Wada [31], there are \( k \) disks \( D_1, D_2, \ldots, D_k \) on the closed square \( D \) with \( \partial D_i = \partial D_j \) for any \( i, j \in \{1, 2, \ldots, k\} \) such that the double \( D \cup \partial D - D \) is a sphere \( S \) and that the unions \( D_i \cup \partial D_i \) are open disks \( B_i \) with \( \partial B_i = \partial B_j \) for any \( i, j \in \{1, 2, \ldots, k\} \). The boundary \( B := \partial B_1 \) is an invariant closed subset of \( S \). Then \( S = B \cup \bigcup_{i=1}^k B_k \) and \( B = \partial B_i \) for any \( i \in \{1, 2, \ldots, k\} \). Considering \( B \) as the singular point set and \( B_i \) as center disks, we can construct a continuous flow \( v \) on the sphere \( S \) such that the orbit space \( S/v \) is \( T_1 \) and that there are \( k \) centers \( p_i \in B_i \) with \( \text{Sing}(v) = B \cup \{p_1, p_2, \ldots, p_k\} \) and \( \text{Per}(v) = \bigcup_{i=1}^k B_i - \{p_i\} \). By Gutierrez’s smoothing theorem [11], we may assume that \( v \) is a \( C^\infty \) flow.

8.2. Recurrent non-Poisson-stable smooth flows with continua like lakes of Wada whose orbit spaces. For any \( k \in \mathbb{Z}_{\geq 2} \), there is a flow \( w \) on an orientable closed surface \( \Sigma_k \) with genus \( k \) and \( \Sigma_k = \text{Sing}(w) \cup \text{LD}(w) \) such that \( \text{Sing}(w) \) is a “double of lakes of Wada continuum”. More precisely, the singular point set \( \text{Sing}(w) \) is both the boundaries of any connected components of \( \text{LD}(w) \) and those of closures of locally dense orbits. Indeed, fix \( k \in \mathbb{Z}_{\geq 2} \). For any \( i \in \{1, 2, \ldots, k\} \), let \( v_i \) be a flow on a torus \( T^2 = (\mathbb{R}/\mathbb{Z})^2 \) generating by a vector field \( X_i = (1, \alpha_i) \) for some \( \alpha_i \in \mathbb{R} - \mathbb{Q} \). Using a bump function \( \varphi : T^2 \rightarrow [0, 1] \) such that the inverse image \( \varphi^{-1}(0) = [0, 0] \), the flow \( v_i' \) generated by \( \varphi X_i \) consists one singular point and dense orbits. Denote by \( w_i \) the restriction of \( v_i \) on the one-punctured torus \( T_i := T^2 - [0, 0] \). Let \( v \) be the flow in the previous example. Replacing all center disks \( B_i \) by all the one-punctured tori \( T_i \), the resulting surface \( \Sigma_k \) is an orientable closed surface with genus \( k \), and the resulting flow \( w \) induced by \( v \) and \( w_i \) consists of singular points and locally dense orbits such that \( \text{Sing}(w) = \partial T_j = \partial O_w(x) \) for any \( j \in \{1, 2, \ldots, k\} \) and any \( x \in \bigcup_{i=1}^k T_i \). Moreover, the one-punctured tori \( T_i \) are the connected components of \( \text{LD}(w) = \Sigma_k - \text{Sing}(w) \).

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