Quantal interferometry with dissipative internal motion

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In presence of dissipation, quantal states may acquire complex-valued phase effects. We suggest a notion of dissipative interferometry that accommodates this complex-valued structure and that may serve as a tool for analyzing the effect of certain kinds of external influences on quantal interference. The concept of mixed-state phase and concomitant gauge invariance is extended to dissipative internal motion. The resulting complex-valued mixed-state interference effects lead to well-known results in the unitary limit and in the case of dissipative motion of pure quantal states. Dissipative interferometry is applied to fault-tolerant geometric quantum computation.

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I. INTRODUCTION

Garrison and Wright [1] were first to address the effect of dissipation on quantal interference, with particular emphasis on the geometric phase in cyclic motion. By modeling the dissipative quantal motion with non-Hermitian Hamiltonians, they arrived at a complex-valued phase concept, the geometric interpretation of which was formulated in a biorthonormal description [2]. This result has triggered further work on complex-valued geometric phase effects [3, 4, 5, 6, 7, 8].

In this paper, we revisit, from the point of view of interferometry, the concept of phase in the presence of dissipation. Our point of departure is the concept of gauge invariance applied to interference in dissipative systems. We focus on a proper treatment of gauge invariance of the non-Hermitian description of dissipative interferometry. When the interfering system carries some internal degrees of freedom, consideration of this gauge symmetry is shown to lead to a complex-valued geometric phase effect for arbitrary input states. This effect constitutes the non-Hermitian generalization of the mixed-state geometric phase put forward in Ref. [4] and, in the particular case of pure cyclic internal states, it reduces to the Garrison-Wright phase [1].

Geometric quantum computation, first proposed in Ref. [10] and experimentally demonstrated in Ref. [11], has attracted considerable interest recently due to its promise of being a tool for analyzing the effect of certain kinds of external influences on quantal interference. This attractive feature has been analyzed from different perspectives in the Abelian case, such as random unitary perturbations [12, 13] and decoherence [14] (for similar analyses of non-Abelian geometric quantum computation, see Refs. [15, 16, 17, 18, 19]). We demonstrate fault-tolerance with respect to dissipative decay for an Abelian geometric phase shift gate.

The biorthonormal approach to dissipative quantal motion is described in the next section. In particular, we put forward an extension to the mixed-state case, in order to pave the way for the subsequent analysis in Sec. III of dissipative interferometry with internal degrees of freedom. Section IV contains an analysis of a nonadiabatic one-qubit geometric phase shift gate undergoing dissipative decay. The paper ends with the conclusions.

II. BIORTHONORMAL APPROACH

In presence of dissipation, we expect the norm of state vectors to change in time. To illustrate this, consider a dissipative system modeled by a partial absorber, characterized by the transmission probability \(0 < T \leq 1\). When passing through the absorber, any \(|\psi\rangle\) in Hilbert space \(\mathcal{H}\) transforms as \(|\psi\rangle \rightarrow \sqrt{T}|\psi\rangle\), which has norm reduced by a factor \(T\). On the other hand, the space of pure states is \(P(\mathcal{H}) = \mathcal{H}/(C - \{0\})\) \(\mathbb{C}[20]\), \(C - \{0\}\) being the set of nonzero complex numbers, i.e., \(|\psi\rangle\) and \(\sqrt{T}|\psi\rangle\) should be regarded as the same state.

To deal with the projective structure \(P(\mathcal{H})\), one defines pure states as generalized one dimensional projectors \(\mathbb{K}\) that take the form

\[
P = |\alpha\rangle\langle\alpha|,
\]

such that \(\text{Tr}P = \langle\beta|\alpha\rangle = 1\). Here \(|\alpha\rangle, |\beta\rangle \in \mathcal{H}\) are said to be biorthonormal. Dissipation may be modeled by a time-dependent non-Hermitian Hamiltonian \(H(t)\), which we assume to have a nondegenerate complex-valued discrete spectrum. In terms of the linear operators \(L(t)\) and \(R(t)\), being solutions of \((h = 1\) from now on\)

\[
i\dot{L}(t) = H(t)L(t),
\]

\[
i\dot{R}(t) = H^\dagger(t)R(t)
\]

(2)

with \(L(0) = R(0) = I\), pure states evolve as

\[
P \rightarrow P(t) = L(t)PR^\dagger(t),
\]

(3)

which is trace preserving for arbitrary \(P\) provided that

\[
R^\dagger(t)L(t) = I.
\]

(4)

Indeed, we obtain from Eq. (2) that

\[
\frac{d}{dt}\left(R^\dagger(t)L(t)\right) = 0,
\]

(5)
which together with $L(0) = R(0) = I$ implies Eq. 4. In the example with a partial absorber discussed above, we may put $L = \sqrt{T}$ and $R = 1/\sqrt{T}$ so that $|\psi\rangle\langle\psi| \rightarrow L|\psi\rangle\langle\psi|R^\dagger = |\psi\rangle\langle\psi|$, in concurrence with the projective structure $P(H)$.

In many physical situations pure states do not provide an accurate state description and one has to resort to mixed states. To deal with these cases, we introduce the generalized density operator

$$\rho = \sum_{k=1}^{N} w_k |\alpha_k\rangle \langle\beta_k|,$$

where $w_k \geq 0$ are real-valued and sum up to unity so that $\text{Tr} \rho = 1$ by requiring $\langle \beta_k |\alpha_l \rangle = \delta_{kl}$. The set $\{|\alpha_k\rangle, |\beta_k\rangle; k = 1, \ldots N\}$ with $\sum_{k=1}^{N} |\alpha_k\rangle \langle\beta_k| = I$ is said to be a biorthonormal complete basis of the $N$ dimensional Hilbert space $H$, such that $|\alpha_k\rangle$ and $|\beta_k\rangle$ are eigenvectors of $\rho$ and $\rho^\dagger$, respectively. $\rho$ evolves as

$$\rho \rightarrow \rho(t) = L(t) \rho R(t)^\dagger,$$

which in conjunction with Eq. 4 assures preserved trace. It further follows that $\{|\alpha_k(t)\rangle = L(t)|\alpha_k\rangle\}$ and $\{|\beta_k(t)\rangle = R(t)|\beta_k\rangle\}$ are nonorthogonal sets of eigenvectors of $\rho(t)$ and $\rho(t)$, respectively, both with time-independent semi-positive eigenvalues $w_k$.

### III. DISSIPATIVE INTERFEROMETRY

Consider a single beam of particles incident on the standard Mach-Zehnder interferometer shown in Fig. 1. At each equal-time slice in the interferometer, $|0\rangle, |1\rangle$ span the Hilbert space $H_s$ describing the spatial beam-pair. All horizontal (vertical) beams are denoted $|0\rangle$ (|1\rangle). Let $1 - T$, $0 < T \leq 1$, be the absorption probability of a static partial absorber in the $|1\rangle$ beam added to it a variable $U(1)$ phase $\chi$. What is the output intensity consistent with the biorthonormal description?

To appreciate what follows, we may first recall that the standard analysis [22] yields the output intensity

$$\mathcal{I}_{|0\rangle} \propto 1 + \frac{2\sqrt{T}}{1+T} \cos \chi$$

in the $|0\rangle$ output channel. It is perhaps tempting to interpret this result in terms of an interference pattern characterized by visibility $\bar{v} = 2\sqrt{T}/(1+T)$ and phase shift $\chi$. On the other hand, as noted in the previous section, the state along the $|1\rangle$ beam inside the interferometer is unaffected by the application of the $U(1)$ phase, as well as of the partial absorber. This gauge symmetry suggests that a mere multiplication of $\sqrt{T}e^{i\chi}$ in the $|1\rangle$ beam should only change the phase in the output, while the visibility should remain unity. We now show that the biorthonormal formalism accommodates a natural notion of phase and visibility, adapted to this intuition.

![FIG. 1: Interferometer setup illustrating the effect of partial absorption and phase shift. 1 – T is the absorption probability and $\chi$ a variable U(1) phase.

Let $z = \sqrt{T}e^{i\chi}$ and identify $L = z$ and $R = 1/z^*$ so that $|\psi\rangle\langle\psi| \rightarrow L|\psi\rangle\langle\psi|R^\dagger = |\psi\rangle\langle\psi|$ for any $|\psi\rangle \in H_s$. In other words, $z$ does not affect the local motion in the $|1\rangle$ beam, but it may show up in interference. To verify this latter point, we note that the effect of $z$ may be represented by the operators

$$\bar{L} = |0\rangle \langle 0| + z|1\rangle \langle 1|,$$

$$\bar{R} = |0\rangle \langle 0| + \frac{1}{z^*}|1\rangle \langle 1|.$$

By analyzing the interferometer, we obtain that the input state $|0\rangle |0\rangle$ transforms as

$$|0\rangle |0\rangle \rightarrow H \sigma_x \bar{L} H |0\rangle |0\rangle H^\dagger \bar{R}^\dagger \sigma_x H^\dagger = \frac{1}{4} (2 + \frac{1}{z} + z) |0\rangle |0\rangle + \frac{1}{4} (2 - \frac{1}{z} - z) |1\rangle |1\rangle + \text{interference terms}$$

with $\sigma_x$ and $H$ the standard Pauli-X and Hadamard operator, respectively, acting on $H_s$. In the $|0\rangle$ output channel, we obtain the $z$-dependent complex-valued intensity

$$\mathcal{I}_{|0\rangle} \propto 2 + \frac{1}{z} + z.$$

This intensity displays a singularity at the origin in the complex $z$ plane. Physically, this singular point at $T = 0$ corresponds to vanishing interference or, equivalently, a situation where the path of the particles is perfectly known. The complex-valued phase shift $\phi$ and visibility $\nu$ are defined as

$$\mathcal{I}_{|0\rangle} \propto 1 + \nu \cos \phi,$$

which yields

$$e^{i\phi} = z,$$

$$\nu = 1,$$
as desired. To further analyze the relation between \( \mathcal{I}_{[0]} \) and the experimental parameters \( \chi \) and \( T \), let us focus on the complex-valued interference term \( \cos \phi \equiv \mathcal{J}_{[0]} \). By introducing the polar decomposition \( \mathcal{J}_{[0]} = |\mathcal{J}_{[0]}| e^{-i\phi} \), we obtain

\[
\tan \vartheta = \frac{1 - T}{1 + T} \tan \chi, \\
|\mathcal{J}_{[0]}| = \sqrt{\cos^2 \chi + \frac{(1 - T)^2}{4T}},
\]

which shows that \( \mathcal{J}_{[0]} \) rotates in the complex plane with angular frequency that tends to \( \chi \) in the singular \( T \to 0 \) limit where \( |\mathcal{J}_{[0]}| \) becomes infinite. In the unitary limit \( T \to 1 \), it follows directly from Eq. (12) that \( \mathcal{J}_{[0]} \) oscillates along the real axis according to the expected \( \cos \chi \).

Next, we extend the above setup and assume that the particles carry some internal degrees of freedom prepared in an input state that is described by the generalized density operator \( \rho \), so that the incoming state is characterized by the generalized density operator \( \varrho_{[0]} = |0\rangle\langle 0| \otimes \rho \) acting on the full Hilbert space \( \mathcal{H}_s \otimes \mathcal{H} \). Suppose that

\[
L = |0\rangle\langle 0| \otimes L + z|1\rangle\langle 1| \otimes I, \\
R = |0\rangle\langle 0| \otimes R + \frac{1}{z}|1\rangle\langle 1| \otimes I
\]

with \( L, R \) being solutions of Eq. (2), for \( H(t) \neq H^\dagger(t) \), are applied between the first beam-splitter and the mirror pair, see Fig. 2 and the complex-valued \( z \) must in general contain absorption in order to fully exploit the complex-valued structure of the phase shift resulting from the non-Hermitian transformation of the internal motion. The output state becomes

\[
\varrho_{\text{out}} = U_B U_M L U_B \varrho_{\text{in}} U_B^\dagger R U_M^\dagger U_B^\dagger
\]

with \( U_M = \sigma_x \otimes I \) and \( U_B = H \otimes I \), where \( \sigma_x \) and \( H \) as above acting on \( \mathcal{H}_s \). The operators \( U_M, U_B, L, \) and \( R \) act on the full Hilbert space \( \mathcal{H}_s \otimes \mathcal{H} \). \( L \) and \( R \) correspond to the application of \( L \) and \( R \) along the \( |0\rangle \) path and the \( z \) operation similarly along \( |1\rangle \). Direct evaluation yields the output state

\[
\varrho_{\text{out}} = \frac{1}{4} |0\rangle\langle 0| \otimes \left( L \rho R^\dagger + \rho + \frac{1}{z} L \rho + z \rho R^\dagger \right) \\
+ \frac{1}{4} |1\rangle\langle 1| \otimes \left( L \rho R^\dagger + \rho - \frac{1}{z} L \rho - z \rho R^\dagger \right)
+ \text{interference terms.}
\]

In the \( |0\rangle \) output channel, we thus obtain

\[
\mathcal{I}_{[0]} \propto 2 + \frac{1}{z} \text{Tr}[L \rho] + z \text{Tr}[\rho R^\dagger].
\]

In analogy with Eq. (12), we may now introduce the complex-valued relative phase \( \Phi \) and visibility \( \mathcal{V} \) as

\[
\mathcal{I}_{[0]} \propto 1 + \mathcal{V} \cos [\phi - \Phi],
\]

with \( z = e^{i\phi} \neq 0, \phi \in \mathbb{C} \). Comparing Eqs. (18) and (19), yields

\[
\mathcal{V} = \sqrt{\frac{\text{Tr}[L \rho]}{\text{Tr}[\rho R^\dagger]}}, \\
\mathcal{V} = \sqrt{\frac{\text{Tr}[U \rho]}{\text{Tr}[\rho U^\dagger]}},
\]

which is the desired complex-valued generalization of relative phase and interference visibility put forward in Ref. 3. In the case where \( H(t) \) is Hermitian, we have that \( L = R \equiv U \) is unitary and we obtain

\[
\mathcal{I}_{[0]} \propto 1 + \mathcal{V} \cos [\phi - \Phi],
\]

which is consistent with Ref. 3 in the case where \( \rho \) is Hermitian. Note, however, that \( \Phi \) may be complex-valued in the general unitary case where \( |\beta_k| \neq |\alpha_k| \).

We now introduce a concept of geometric phase associated with the path \( \mathcal{C} : t \in [0, \tau] \to \rho(t) = L(t) \rho R^\dagger(t) \). The basic observation for this purpose is that there is an equivalence set \( \mathcal{A} \) of operator pairs \( \bar{L}(t), \bar{R}(t) \) that all generate \( \mathcal{C} \), namely those of the form

\[
\bar{L}(t) = L(t) \sum_{k=1}^N z_k(t) |\alpha_k\rangle \langle \beta_k|, \\
\bar{R}(t) = R(t) \sum_{k=1}^N \frac{1}{z_k(t)} |\beta_k\rangle \langle \alpha_k|,
\]

where \( z_k(t), t \in [0, \tau] \), are nonvanishing complex numbers such that \( z_k(0) = 1 \). The equivalence set \( \mathcal{A} \) is the proper non-Hermitian generalization of that in the unitary case introduced in Ref. 28. The existence of \( \mathcal{A} \) expresses the
gauge symmetry of dissipative motion of mixed quantal states. We may identify \( \{ L^\parallel(t), R^\parallel(t) \} \in \mathcal{A} \) fulfilling
\[
(\beta_k | R^\parallel(t) L^\parallel(t) | \alpha_k) = 0, \quad k = 1, \ldots, N.
\]
(23)
These constitute parallel transport conditions in a fiber bundle with structure group \((\mathbb{C} - \{0\})^N\). Substituting \( L^\parallel(t), R^\parallel(t) = \tilde{L}(t), \tilde{R}(t) \), with \( \tilde{L}(t), \tilde{R}(t) \) given by Eq. \[24\], into Eq. \[24\], we obtain
\[
z^\parallel_k(\tau) = e^{-\int_0^\tau (\beta_k | R^\parallel(t) L^\parallel(t) | \alpha_k) dt} = e^{-\int_0^\tau (\beta_k | \dot{\alpha}_k(t)) dt}.
\]
(24)
where we have used that \( z^\parallel_k(0) = 1 \). Putting this into the relative phase, we identify the mixed-state generalization of the complex-valued geometric phase factor as
\[
e^{i\Gamma} = \sqrt{\frac{\text{Tr}[ L^\parallel(t) \rho ]}{\text{Tr}[ R^\parallel(t) \rho ]}} = \frac{\sum_{k=1}^N w_k \langle \beta_k | L(\tau) | \alpha_k \rangle e^{i\int_0^\tau (\beta_k(t)|\dot{\alpha}_k(t)) dt}}{\sum_{k=1}^N w_k \langle \beta_k | R^\parallel(\tau) | \alpha_k \rangle e^{i\int_0^\tau (\beta_k(t)|\dot{\alpha}_k(t)) dt}}.
\]
(25)
We may verify that \( e^{i\Gamma} | \tilde{L}(t), \tilde{R}(t) \rangle = e^{i\Gamma} | L(t), R(t) \rangle \) for any \( \tilde{L}(t), \tilde{R}(t) \in \mathcal{A} \). Thus, \( e^{i\Gamma} \) is a property of \( \mathcal{C} \).

Let us now analyse two important special cases. First, assume that \( H(t) \) is Hermitian, so that \( L(t) = R(t) \equiv U(t) \) is a one-parameter family of unitarities. In this case, Eq. \[24\] takes the form
\[
e^{i\Gamma} = \frac{\sum_{k=1}^N w_k \langle \beta_k | U(\tau) | \alpha_k \rangle e^{i\int_0^\tau (\beta_k(t)|\dot{\alpha}_k(t)) dt}}{\sum_{k=1}^N w_k \langle \beta_k | U^\parallel(\tau) | \alpha_k \rangle e^{i\int_0^\tau (\beta_k(t)|\dot{\alpha}_k(t)) dt}}.
\]
(26)
where \( | \alpha_k(t) \rangle = U(t) | \alpha_k \rangle \) and \( | \beta_k(t) \rangle = U(t) | \beta_k \rangle \). Equation \[26\] is consistent with the mixed-state geometric phase in unitary evolution \[1\] when \( \rho \) is Hermitian, but \( \Gamma \) may still be complex-valued in the general unitary case where \( \beta_k \neq \alpha_k \).

Secondly, in the pure cyclic case, defined by \( \rho = | \alpha \rangle \langle \beta | \), \( L(\tau) | \alpha \rangle = e^{i\zeta} | \alpha \rangle \), and \( R(\tau) | \beta \rangle = e^{i\zeta^*} | \beta \rangle \) with \( \zeta \) some complex number \[1\], we obtain
\[
e^{i\Gamma} = e^{i\zeta} e^{-\int_0^\tau (\beta(t)|\dot{\alpha}(t)) dt}.
\]
(27)
In analogy with Ref. \[24\], we may put
\[
| \tilde{\alpha}(t) \rangle = e^{-if(t)} | \alpha(t) \rangle, \quad | \tilde{\beta}(t) \rangle = e^{-if^*(t)} | \beta(t) \rangle
\]
(28)
such that \( f(\tau) - f(0) = \zeta \). This yields
\[
\Gamma = i \int_0^\tau \langle \beta(t)|\tilde{\alpha}(t) \rangle dt,
\]
(29)
which is consistent with Ref. \[1\].

IV. FAULT-TOLERANT GEOMETRIC QUANTUM COMPUTATION

Abelian nonadiabatic geometric quantum computation has been proposed \[24\] in order to achieve high-speed fault-tolerant implementations of quantum gates. Here, we analyze the resilience of a nonadiabatic one-qubit geometric phase shift gate to dissipation.

The physical scenario for the dissipative phase shift gate is an unstable two-level atom interacting with an external electric field that rotates uniformly around the \( z \) axis. By using the rotating wave approximation, the internal quantal motion of the atom may, in the rotating frame, be described by the non-Hermitian Hamiltonian
\[
H = \frac{1}{2}(\eta - i\gamma) \sigma_z + \frac{1}{2} \omega \sigma_z,
\]
(30)
where \( \eta \) is the detuning, \( \gamma \) is the average decay rate, and for simplicity we have neglected small off-diagonal terms in the weak-amplitude limit of the transverse electric field in the \( x - y \) plane. Furthermore, \( \sigma_z = |g\rangle\langle g| - |e\rangle\langle e| \), where \( |g\rangle \) and \( |e\rangle \) are the unperturbed ground and first excited atomic state, respectively. Solving Eq. \[24\] yields
\[
L(t) = e^{-i\omega \sigma_z t/2}, \quad R(t) = e^{-i\omega^* \sigma_z t/2}.
\]
(31)
Assume
\[
\rho = \frac{1 + r}{2} | \alpha_+ \rangle \langle \alpha_+ | + \frac{1 - r}{2} | \alpha_- \rangle \langle \alpha_- |,
\]
(32)
where \( 0 < r \leq 1 \) is the nonzero length of the Bloch vector of the qubit and
\[
| \alpha_+ \rangle = \cos \frac{\omega}{2} | g \rangle + \sin \frac{\omega}{2} | e \rangle,
\]
\[
| \alpha_- \rangle = -\sin \frac{\omega}{2} | g \rangle + \cos \frac{\omega}{2} | e \rangle
\]
(33)
make an angle \( \theta \in \mathbb{R} \) with respect to the \( z \) axis. In absence of decay, i.e., \( \gamma = 0 \), a geometric phase shift gate of the form
\[
U[\theta] = e^{i(1 - \sigma_\theta) 2\pi (1 - \cos \theta)},
\]
(34)
where \( \sigma_\theta = | \alpha_+ \rangle \langle \alpha_+ | - | \alpha_- \rangle \langle \alpha_- | \), is obtained after one period \( \tau = 2\pi/\eta \), by eliminating the dynamical phase, e.g., using refocusing technique. The dependence upon the solid angle \( 2\pi(1 - \cos \theta) \) expresses the geometric nature of \( U[\theta] \).

Now, we may notice that \( | \alpha_+ \rangle \) undergoes noncyclic evolution for all \( t > 0 \) in the presence of decay, except in the trivial case where \( \theta = 0 \). Yet, we may still compute the noncyclic complex-valued phases by using Eqs. \[24\] and \[26\].

First, let \( \varphi = \omega \tau \) be the complex-valued total precession angle. By using Eq. \[24\], we may compute the
relative phase and visibility as
\[ \Phi = - \arctan \left[ r \cos \theta \tan \frac{\varphi}{2} \right], \]
\[ \mathcal{V} = \sqrt{\cos^2 \frac{\varphi}{2} + r^2 \cos^2 \theta \sin^2 \frac{\varphi}{2}}, \]
which are in general complex-valued unless the average decay rate \( \gamma \) vanishes.

Next, by using Eq. (25), we may compute the geometric phase associated with the path \( t \in [0, \tau] \rightarrow L(t)\rho R^\dagger(t) \) as
\[ \Gamma = - \arctan \left[ r \tan \frac{\Omega}{2} \right], \]
where
\[ \Omega = 2 \arctan \left[ \cos \theta \tan \frac{\varphi}{2} \right] - \varphi \cos \theta \]
is a complex-valued analog of the geodesically closed solid angle on the Bloch sphere, appearing in the Hermitian case. For small \( \gamma \tau \), we may Taylor expand \( \Omega \) and obtain
\[ \Omega = 2 \arctan \left[ \cos \theta \tan \frac{\eta \tau}{2} \right] - \eta \tau \cos \theta - i \gamma \tau \cos \theta \sin^2 \frac{\eta \tau}{2} \sin^2 \theta + \mathcal{O}[(\gamma \tau)^2] \]
whose real part is exactly the desired solid angle to second order in \( \gamma \tau \). This expression further entails that the one-qubit gate \( U[\theta] \) in Eq. (33) is fault-tolerant in \( \gamma \), in that \( \Omega \) and, since \( r \) is independent of \( \eta \) and \( \gamma \), thereby also \( \Gamma \) are robust to second order in the decay for \( \tau = 2\pi/\eta \), i.e.,
\[ \Omega = 2\pi(1 - \cos \theta) + \mathcal{O} \left[ \left( \frac{2\pi \gamma}{\eta} \right)^2 \right]. \]
This feature further supports the predicted resilience of geometric quantum computation to unwanted external influences. A related result in the context of adiabatic evolution of open quantum systems has been found in Ref. [1].

Furthermore, for small \( \gamma \tau \) we may also expand the relative phase as
\[ \Phi = - \arctan \left[ r \cos \theta \tan \frac{\eta \tau}{2} \right] - i \gamma \tau \cos \theta \frac{r \cos \theta}{2 - (1 - r^2 \cos^2 \theta) \sin^2 \frac{\eta \tau}{2}} + \mathcal{O}[(\gamma \tau)^2]. \]
Thus, the imaginary part of \( \Phi \) vanishes for \( \theta = \pi/2 \) but is nonzero otherwise for all \( \gamma \tau > 0 \). In other words, the relative phase, which contains both dynamical and geometric contributions, is not fault-tolerant to the dissipative decay. This suggests that the above demonstrated resilience to the present form of dissipative decay is a consequence of the geometric nature of \( \Gamma \), and that a quantum gate based upon \( \Phi \) would be sensitive to this particular kind of error.

V. CONCLUSIONS

Two-beam interferometry with particles carrying internal degrees of freedom has been analyzed by using a biorthonormal dynamical description. It leads to the notion of dissipative interferometry that may serve as a tool for analyzing the effect of certain kinds of external influences on quantal interference. In dissipative interferometry, phases and visibilities become complex-valued. Gauge invariance, adapted to internal mixed input states, is defined in terms of path invariance in state space, and has been shown to lead to a natural concept of geometric phase that generalizes Ref. [3] to dissipative motion as well as Ref. [1] to the mixed-state case. Fault-tolerance of Abelian nonadiabatic geometric quantum computation is demonstrated for a one-qubit phase shift gate, modeled by an unstable two-level atomic system. It would be pertinent to extend the present framework to the non-Abelian case in order to deal with the robustness of universal sets of quantum gates implemented by geometric means.

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A weaker condition for $\text{Tr}\rho = 1$ is $\langle \beta_k | \alpha_k \rangle = 1$, $\forall k$. The stronger condition $\langle \beta_k | \alpha_l \rangle = \delta_{kl}$ plays a natural role in the subsequent definition of parallel transport in Eq. (23) as it is the biorthonormal analog to the spectral decomposition of standard (Hermitian) density operators.