A REMARK ON DOUBLE COSETS

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Abstract. If a soluble group $G$ contains two finitely generated abelian subgroups $A, B$ such that the number of double cosets $AgB$ is finite, then $G$ is shown to be virtually polycyclic.

1. Introduction

Suppose $G$ is a group, and $A, B$ are subgroups of $G$ that are, in some sense, ‘small’. If the set $A \backslash G / B$ of double cosets $AgB$ ($g \in G$) of $A$ and $B$ in $G$ is finite, does it necessarily follow that $G$ is also ‘small’?

Questions of this nature have been the subject of some previous investigations in the case where $A = B$ [9, 6, 7]. Under a variety of suitable conditions, it turns out that the hypotheses imply that $A$ has finite index in $G$. This is related to questions of Dunwoody [5] and of Neumann and Rowley [8].

The general question arises in connection with the study of homological finiteness properties of subdirect products of limit groups [1, 3]. For example in [1] it is shown that a certain rational homology group $H_n(G, \mathbb{Q})$ contains a $\mathbb{Q}$-linearly independent set in one-to-one correspondence with a set $A \backslash G / B$ of double cosets, where $A$ is a cyclic subgroup and $B$ a finitely generated abelian group. In the context of [1], the group $G$ is readily shown not to be ‘small’ enough for such a set to be finite, from which one deduces that it is not of type $FP_n(\mathbb{Q})$.

The purpose of this short note is to prove the following.

**Theorem A.** Let $G$ be a virtually soluble group, and let $A, B < G$ be finitely generated virtually abelian subgroups of $G$. If there are only finitely many double cosets $AgB$ ($g \in G$), then $G$ is virtually polycyclic.

I first proved this result in the case where $G$ is (virtually) nilpotent-by-abelian, and $A$ is cyclic, with the intention of using it in the proof of the main theorem of [3] (that a subdirect product of $n$ limit groups has type $FP_n(\mathbb{Q})$ only if it is virtually a direct product of $n$ or fewer limit groups). In the event, an alternative more direct approach was developed for [3] which avoided the use of double cosets. Nevertheless the result seems of some interest in relation to the various references cited above.

The special case $A = B$ is connected to the question of orbits of groups of automorphisms: if $G = N \rtimes A$, then $A \backslash G / A$ is in one-to-one correspondence with the set of orbits of the action of $A$ on $N$. A
partial analogue of this correspondence is a useful tool for the study of
the more general case. Suppose that \( G = N \rtimes A = N \rtimes B \) — in particular
\( A \cong B \) via a preferred isomorphism \( \theta : A \to B \) such that \( a\theta(a)^{-1} \in N \)
for all \( a \in A \). Restricting to the situation where \( N \) is abelian, the map
\( \delta : A \to N, \delta(a) := a\theta(a)^{-1} \) is a derivation, so gives rise to a \( ZA \)
module homomorphism from \( I \) to \( N \) via the rule \( 1 - a \mapsto \delta(a) \) (where
\( I \) denotes the augmentation ideal in the group ring \( ZA \)). We can also
associate to this set-up an action of \( A \) on \( N \) by affine transformations
\[
\begin{align*}
    a \ast \nu := a\nu\theta(a)^{-1} = a \cdot \nu + \delta(a),
\end{align*}
\]
where \( a \cdot \nu = a\nu a^{-1} \) is the standard conjugation action of \( A \) on \( N \). It
is not difficult to check that the set \( A \setminus G/B \) is in one-to-one correspon-
dence with the set of orbits of this affine action of \( A \) on \( N \).

Thus a special case of Theorem A says that, if a finitely generated
virtually abelian group \( A \) acts affinely on an abelian group \( N \) with
finitely many orbits, then \( N \) is finitely generated.

Theorem B. Let \( A \) be a finitely generated virtually abelian group, \( N \) a
\( ZA \)-module, and \( \delta : A \to N \) a derivation. If the resulting affine action
of \( A \) on \( N \) has only finitely many orbits, then \( N \) is finitely generated
as an abelian group.

This should be compared with a theorem of Jabara [7], who proves
that if the action of \( A \) by automorphisms on a (not necessarily abelian)

Theorem [A] is a key step in our proof of Theorem [A]. We prove this in
Section 2 below, and then complete the proof of Theorem [A] in Section
3.

I am grateful to my co-authors of [3], Martin Bridson, Chuck Miller
and Hamish Short, for a fruitful and enjoyable collaboration, and for
their encouragement to publish these double-coset results separately. I
was greatly encouraged while working on this problem by the interest in
it shown by Karl Gruenberg, in the course of a number of conversations
and letters (which culminated in his independent solution of a special
case). Karl was a great source of inspiration to me, as he was to many
mathematicians. This paper is dedicated to his memory.

2. Affine actions

Let \( A \) be a group, and let \( N \) be a (left) \( ZA \)-module. Recall that a
map \( \delta : \to N \) is a derivation if it satisfies the Leibnitz rule
\[
\delta(a_1a_2) = \delta(a_1) + a_1 \cdot \delta(a_2)
\]
for all \( a_1, a_2 \in A \). Here we use \( a \cdot \nu, a \in A, \nu \in N \), to denote the
module action of \( A \) on \( N \).

Given a derivation \( \delta : A \to N \), we define the corresponding affine
action of \( A \) on \( N \) by
\[
\begin{align*}
    a \ast \nu := a \cdot \nu + \delta(a).
\end{align*}
\]
Note that
\[ a_1 * (a_2 * \nu) = a_1 \cdot (a_2 \cdot \nu + \delta(a_2)) + \delta(a_1) \]
\[ = a_1 \cdot (a_2 \cdot \nu) + (\delta(a_1) + a_1 \cdot \delta(a_2)) = (a_1a_2) \cdot \nu + \delta(a_1a_2), \]
so this is a genuine action of \(A\) on the underlying set of \(N\), although it does not in general respect the group structure of \(N\). In the special case where \(\delta\) is the zero derivation, the affine action is just the standard action of \(A\) by group automorphisms on \(N\).

It is well-known (see for example [4, IV.2, Exercise 2]) that the abelian group \(\text{Der}(A, N)\) of derivations \(A \to N\) (where the binary operation is pointwise addition in \(N\)) is isomorphic to \(\text{Hom}_{\mathbb{Z}A}(I, N)\), where \(I\) is the augmentation ideal in \(\mathbb{Z}A\). Here \(\phi : I \to N\) corresponds to \(\delta : A \to N\) defined by \(\delta(a) = \phi(1 - a)\). If \(J\) is any left ideal in \(\mathbb{Z}A\), there is a canonical \(\mathbb{Z}A\)-homomorphism \(I \to \mathbb{Z}A/J\) given by \(x \mapsto x + J\). We call the corresponding derivation \(a \mapsto 1 - a + J\) and affine action \(a \cdot (x + J) := (ax + 1 - a) + J\) the canonical derivation and canonical affine action respectively.

In all the above constructions, one can replace \(\mathbb{Z}\) by another commutative ring. In practice, we will require only the finite fields \(\mathbb{Z}_p\) and the field \(\mathbb{Q}\) of rationals for this purpose.

**Lemma 1.** Let \(R\) be either \(\mathbb{Z}\) or \(\mathbb{Z}_p\) for some prime \(p\), let \(A\) be a finitely generated abelian group, \(I\) the augmentation ideal in \(RA\), and \(J \subset I\) another ideal in \(RA\). Suppose that \(I/J\) is the union of finitely many \(A\)-orbits under the canonical affine action of \(A\) defined by
\[ a \cdot (x + J) := (ax + 1 - a) + J, \quad (a \in A, \ x \in I). \]

Then for each \(a \in A\) there exists a nonzero polynomial \(f(X) \in R[X]\) such that \(f(a) \in J\).

**Proof.** Let \(K\) be an algebraically closed field containing \(R\). For each prime \(q\), let \(f_q(t) = (t-1)^q\). In particular, \(f_q(a) \in I\) for all \(a \in A\). Let \(\Pi\) denote the infinite set of primes (if \(R = \mathbb{Z}\), or the set of primes \(\neq p\) (if \(R = \mathbb{Z}_p\)). Since \(I/J\) is the union of finitely many orbits under the affine action, there are two distinct primes \(q_1, r_1 \in \Pi\) and an element \(a_1 \in A\) such that \(f_{q_1}(a) - a_1 \cdot f_{r_1}(a) \in J\). Applying the same argument to the infinite set \(\Pi \setminus \{q_1, r_1\}\), there are two more primes \(q_2, r_2\) and \(a_2 \in A\) such that \(f_{q_2}(a) - a_2 \cdot f_{r_2}(a) \in J\). Iterating this argument yields an infinite sequence \(q_1, r_1, q_2, r_2, q_3, r_3, \ldots\) of pairwise distinct primes \(\neq p\) in the case \(R = \mathbb{Z}_p\) and a sequence \(a_1, a_2, \ldots\) of elements of \(A\), such that \(f_{q_j}(a) - a_j \cdot f_{r_j}(a) \in J\) for each \(j\).

Since \(A\) is finitely generated, there is a natural number \(n\) and a nonzero \(n\)-tuple \((t_1, \ldots, t_n)\) of integers such that
\[ a_1^{t_1} \cdot a_2^{t_2} \cdots a_n^{t_n} = 1 \text{ in } A. \]
Note that \( f_{r_j}(a) - a_j^{-1} f_{q_j}(a) \in J \) for each \( j \), so by interchanging \( q_j \) and \( r_j \) for each \( j \) in suitable subset of \( \{1, \ldots, n\} \), we can arrange that each \( t_j \) is non-negative in (\( I/J \)).

Now the equations
\[
(a - 1)^{q_j} - 1 = f_{q_j}(a) - 1 = a_j * f_{r_j}(a) - 1
\]
\[
= a_j (f_{r_j}(a) - 1) = a_j ((a - 1)^{r_j} - 1)
\]
in \( I/J \) for \( j = 1, \ldots, n \) give rise to an equation
\[
\prod_{j=1}^{n} ((a - 1)^{q_j} - 1)^{t_j} = \prod_{j=1}^{n} a_j^{t_j} \prod_{j=1}^{n} ((a - 1)^{r_j} - 1)^{t_j} = \prod_{j=1}^{n} ((a - 1)^{r_j} - 1)^{t_j}
\]
in \( I/J \). If \( \alpha \in K \) is a primitive \( q_1 \)-th root of unity, then \( \alpha + 1 \) is a root of \( g_1(X) := \prod_{j=1}^{n} ((X - 1)^{q_j} - 1)^{t_j} \) but not of \( g_2(X) := \prod_{j=1}^{n} ((X - 1)^{r_j} - 1)^{t_j} \). Hence \( f(X) := g_1(X) - g_2(X) \neq 0 \) in \( R[X] \), while \( f(a) \in J \) as required.

\[\square\]

**Lemma 2.** Let \( A \) be a group, \( M \) a \( \mathbb{Z}A \)-module, \( M_0 \) a submodule of \( M \), and \( \delta : A \to M \) a derivation. Then

1. \( A_0 := \delta^{-1}(M_0) \) is a subgroup of \( A \);
2. the affine action \( a * x = ax + \delta(a) \) of \( A \) on \( M \) restricts to an affine action of \( A_0 \) on \( M_0 \), and induces an affine action of \( A \) on the quotient module \( M/M_0 \);
3. the orbits of the affine action of \( A_0 \) on \( M_0 \) are the nonempty intersections of \( M_0 \) with the orbits of the affine action of \( A \) on \( M \); and
4. the orbits of the affine action of \( A \) on \( M/M_0 \) are the images under the natural epimorphism \( M \to M/M_0 \) of the orbits of the affine action of \( A \) on \( M \).

**Proof.**

1. It follows immediately from the Leibnitz rule for derivations that \( A_0 \) is a subgroup of \( A \).
2. By definition, if \( a \in A_0 \) and \( x \in M_0 \) then \( a * x \in M_0 \), so the affine action of \( A \) on \( M \) restricts to an affine action of \( A_0 \) on \( M_0 \). If \( a \in A, x \in M \) and \( y \in M_0 \), then \( a * (x + y) - a * (x) = a * y \in M_0 \), so there is a well-defined induced action of \( A \) on \( M/M_0 \).
3. Two elements \( x, y \in M_0 \) are in the same orbit of the affine \( A \)-action on \( M \) if and only if \( y = a * x = a * x + \delta(a) \) for some \( a \in A \). But \( \delta(a) = y - a * x \in M_0 \) means that \( a \in A_0 \).
4. Let \( x, y \in M \) and \( a \in A \). If \( y + M_0 = a * (x + M_0) = (a * x) + M_0 \) then \( \exists y' \in y + M_0 \) with \( a * x = y' \). It follows that the orbits of the affine \( A \)-action on \( M/M_0 \) are precisely the images of the orbits of the affine \( A \)-action on \( M \).
The next few lemmas take care of special cases of Theorem B.

**Lemma 3.** Let $A$ be a finitely generated virtually abelian group, $N$ a $ZA$-module, and $\delta : A \to N$ a derivation. If the resulting affine action of $A$ on $N$ has only finitely many orbits, then the $\mathbb{Z}$-torsion subgroup $N_0$ of $N$ is finite.

*Proof.* Passing to a subgroup of finite index if necessary, we may assume that $A$ is free abelian of finite rank. Now $N_0$ is a characteristic subgroup of $N$, so is a $ZA$-submodule. By Lemma 2, $N_0$ is the union of finitely many $A_0$-orbits under the restricted affine action of $A_0 = \delta^{-1}(N_0)$ on $N_0$. In particular, $N_0$ is finitely generated as a $ZA_0$-module. Since it is also a $\mathbb{Z}$-torsion module, it has finite exponent. Arguing by induction on the number of prime factors of this exponent, we will assume that $pN_0$ is finite for some prime $p$, and show that $N_1 = N_0/pN_0$ is also finite.

But $N_1$ is a finitely generated $\mathbb{Z}_pA_0$ module, which is the union of finitely many $A_0$-orbits under the induced affine action. If $I_0$ is the augmentation ideal in $\mathbb{Z}_pA_0$ and $\phi_0 : I_0 \to N_1$ the homomorphism corresponding to the derivation $\delta : A_0 \to N_1$, then the kernel of $\phi_0$ is an ideal $J_0$ of $\mathbb{Z}_pA_0$ with $J_0 \subset I_0$.

Let $\{a_1, \ldots, a_k\}$ be a basis for $A_0$ as a free abelian group. Then by Lemma 1 there are nonzero polynomials $f_1(X), \ldots, f_k(X) \in \mathbb{Z}_p[X]$ such that $f_1(a_1), \ldots, f_k(a_k) \in J_0$. It follows that $I_0/J_0$ is finite-dimensional as a $\mathbb{Z}_p$-vector space, and hence finite. Thus $\phi_0(I_0)$ is finite.

On the other hand, $N_1/\phi_0(I_0)$ is the union of finitely many orbits under the induced affine action of $A_0$, by Lemma 2. But this action is just the action by scalar multiplication, since $\delta(A_0) \subset \phi_0(I_0)$. It then follows from [7, Theorem 1] that $N_1/\phi_0(I_0)$ – and hence also $N_1$ – is finite, as required. \hfill $\square$

**Lemma 4.** Let $A$ be a finitely generated virtually abelian group, $N$ a $ZA$-module, and $\delta : A \to N$ a derivation. If the resulting affine action of $A$ on $N$ has only finitely many orbits, then the quotient of $N$ by the submodule generated by $\delta(A)$ is finite.

*Proof.* Let $N_0$ denote the $ZA$-submodule generated by $\delta(A)$. Then the induced affine action of $A$ on $N/N_0$ is just the action by scalar multiplication (since $\delta(A) \subset N_0$). By Lemma 2, $N/N_0$ is the union of finitely many $A$-orbits under this action, so by [7, Theorem 1] it follows that $N/N_0$ is finite. \hfill $\square$

**Lemma 5.** Let $A$ be a finitely generated virtually abelian group, $N$ a $ZA$-module, and $\delta : A \to N$ a derivation. If the resulting affine action of $A$ on $N$ has only finitely many orbits, then $N/IN$ is finitely generated as a $\mathbb{Z}$-module, where $I$ is the augmentation ideal in $ZA$. 

Proof. Since the module $A$-action on $N/IN$ is trivial, the resulting $A$-action is by translations: $a \ast x = a \cdot x + \delta(a) = x + \delta(a)$. The hypotheses then mean that $\delta : A \to N/IN$ is a group homomorphism whose image has finite index in the additive group $N/IN$. The result follows from the fact that $A$ is finitely generated. \qed

**Theorem 6 (= Theorem B).** Let $A$ be a finitely generated virtually abelian group, $N$ a $ZA$-module, and $\delta : A \to N$ a derivation. If the resulting affine action of $A$ on $N$ has only finitely many orbits, then $N$ is finitely generated as a $Z$-module.

**Proof.** Clearly, we may replace $A$ by a finite-index subgroup without changing the hypotheses or the conclusion of the theorem. Hence we may assume without loss of generality that $A$ is free abelian of finite rank.

Now let $N_0$ be the $Z$-torsion subgroup of $N$. Then $N_0$ is finite, by Lemma 3. Lemma 2 thus reduces us to the case where $N$ is $Z$-torsion-free.

By Lemma 4 the submodule of $N$ generated by $\delta(A)$ has finite index in $N$, so there is no loss of generality in assuming that $N$ is generated by $\delta(A)$.

Let $\phi : I \to N$ denote the $ZA$-module homomorphism corresponding to $\delta$. Then $N \cong I/J$ for some ideal $J$ of $ZA$ with $J \subset I$.

By Lemma 5 the quotient $N/IN \cong I/(I^2 + J)$ is finitely generated as a $Z$-module. Moreover, by Lemma 2 $IN$ is the union of finitely many $A_0$-orbits under the restriction of the affine action to $A_0 = \delta^{-1}(IN)$, the kernel of the group homomorphism $A \to N/IN$. If $A_0$ has strictly smaller rank than $A$, then the inductive hypothesis asserts that $IN$ is finitely generated as a $Z$-module, and the result follows. Thus we are reduced to the case where, $A/A_0$ is finite – and hence $N/IN$ is finite.

If $\{b_1, \ldots, b_t\}$ is a basis for $A$ as an abelian group, then Lemma 3 says there are nonzero polynomials $g_1, \ldots, g_t \in \mathbb{Z}[X]$ such that $g_i(b_i) \in J$ for $i = 1, \ldots, t$. It follows that $Q \otimes_Z N$ has finite dimension as a $Q$-vector space.

Now $Q \otimes_Z N = QI/QJ$, where $QI = Q \otimes_Z I$ is the augmentation ideal in $QA$, and $QJ$ is the ideal of $QA$ generated by $J$. Clearly $m := QI/QJ$ is a maximal ideal in the ring $\Lambda := QA/QJ$. Moreover, $m/m^2 = 0$, since $I/(I^2 + J) = N/IN$ is finite.

If $\Lambda$ is a local ring, then $m$ is the unique maximal ideal in $\Lambda$, and $m/m^2 = 0$, so Nakayama’s Lemma (see for example [10, §2.8]) implies that $m = 0$. But $m = Q \otimes_Z N$ with $N$ torsion-free as a $Z$-module, so it follows that $N = 0$.

Hence we may assume that $\Lambda$ is not local. In other words, there is a maximal ideal other than $m$ in $\Lambda$. Thus there is a $Q$-algebra epimorphism $\psi$ from $\Lambda$ onto a field $K$, such that $\psi(I) \neq \{0\}$. Note
that \( QA/QI \cong Q \) and \( QI/QJ \cong Q \otimes_{\mathbb{Z}} N \) are both finite-dimensional over \( Q \), so \( \Lambda \cong QA/QJ \) is finite-dimensional over \( Q \). It follows that \( K \) is finite-dimensional over \( Q \), and so \( K \) is an algebraic number field.

Let \( d \) be the degree of \( K \) over \( Q \). Since \( K \) is spanned by \( \psi(\mathbb{Z}A) = \mathbb{Z} + \psi(I) \) as a vector space over \( Q \), we can choose \( x \in I \) such that \( K = Q[\psi(x)] \). Indeed, multiplying \( x \) by a suitable positive integer, we may assume in addition that \( \psi(x) \) is an algebraic integer in \( K \).

By hypothesis, there exists a finite subset \( U \subset I \) with the property that \( I = \{a * u + y, a \in A, u \in U, y \in J\} \), where \( a * u \) denotes the canonical affine action \( a * u := au + 1 - a \). Note that \( a * u - 1 = a(u - 1) \).

Now define \( \nu : I \to \mathbb{Q} \) by \( \nu(z) = N_{K|\mathbb{Q}}(\psi(z) - 1) \) for all \( z \in I \), where \( N_{K|\mathbb{Q}} \) denotes the norm of the algebraic extension \( K|\mathbb{Q} \).

If \( \xi_1, \ldots, \xi_d \) denote the conjugates of \( \psi(x) \) in \( \mathbb{C} \), then \( \xi_j \neq 0 \) for each \( j \). Moreover, for any positive integer \( n \), the conjugates of \( \psi(nx) \) are \( n\xi_1, \ldots, n\xi_d \). Thus, replacing \( x \) by a sufficiently large integer multiple, we may assume that \( |\xi_j| > 2 \) for each \( j \).

Hence we have, for each integer \( n \geq 1 \),

\[
\nu(nx) = N_{K|\mathbb{Q}}(\psi(nx) - 1) = (n\xi_1 - 1) \cdots (n\xi_d - 1),
\]

an integer of absolute value greater than 1. This expression also shows that

\[
\nu(nx) = (-n)^d \mu(1/n),
\]

where

\[
\mu(X) = (X - \xi_1) \cdots (X - \xi_d) \in \mathbb{Z}[X]
\]

is the minimal polynomial of the algebraic integer \( \psi(x) \). In particular, \( \nu(nx) \) is congruent to \((-1)^d \mod n \), so it is coprime to \( n \). Since \( \nu(nx) \neq \pm 1 \), it has at least one prime divisor, and none of its prime divisors divide \( n \).

Since we can do this for arbitrary integers \( n \), we deduce that there are infinitely many primes \( p \) with the property \( (\exists z \in I) \), \( \nu(z) \) is a nonzero integer divisible by \( p \).

On the other hand, every element of \( I \) has the form \( a * u + y \) for some \( a \in A, u \in U \) and \( y \in J \), and

\[
\nu(a * u + y) = N_{K|\mathbb{Q}}(\psi(au - a) = N_{K|\mathbb{Q}}(\psi(a)(\psi(u) - 1))
\]

\[= N_{K|\mathbb{Q}}(\psi(a))N_{K|\mathbb{Q}}(\psi(u) - 1) = N_{K|\mathbb{Q}}(\psi(a))\nu(u).\]

It follows that, if \( \nu(a * u + y) \) is a nonzero integer, then its prime divisors belong to a finite set – namely those primes that divide the numerator or denominator of \( N_{K|\mathbb{Q}}(\psi(a)) \) for some \( a \) in a fixed basis of \( A \), together with those primes that divide the numerator of \( \nu(u) \) for some \( u \in U \) with \( \psi(u) \neq 1 \).

This gives a contradiction, completing the proof. \( \square \)
3. Double cosets

In this section we use Theorem B to prove Theorem A.

The group \( G \) in Theorem A is virtually soluble. Indeed, one may immediately reduce to the case where \( G \) is soluble, since the hypotheses and the conclusion of Theorem A are stable under the passage to finite index subgroups.

Similarly, we may replace the finitely generated virtually abelian subgroups \( A, B \) of Theorem A by any finite-index subgroups of \( A, B \) respectively. In particular we may assume that \( A, B \) are free abelian of finite rank.

We argue by double induction, on the derived length of the soluble group \( G \) and on the sum of the ranks of the free abelian groups \( A, B \).

If \( G \) is abelian, then the hypotheses imply that \( G \) is finitely generated (for example, by the union of a basis for \( A \), a basis for \( B \), and a set of double-coset representatives). Otherwise, \( G \) has an abelian normal subgroup \( N \) such that the derived length of \( G/N \) is less than that of \( G \). Since \( |AN\backslash G/BN| < \infty \), it follows by inductive hypothesis that \( G/N \) is virtually polycyclic, and it suffices to show that \( N \) is finitely generated as an abelian group.

Now \( |A\backslash AN/(B \cap AN)| < \infty \), so there is no loss of generality in supposing that \( G = AN \). Then \( A \cap N \) is central, and hence normal in \( G \). If \( A \cap N \neq \{1\} \), then we may apply the inductive hypothesis together with the fact that

\[
\frac{A}{A \cap N} \quad \frac{G/(A \cap N)}{A \cap N} \quad \frac{B \cdot (A \cap N)}{A \cap N} < \infty
\]

to deduce that \( G/(A \cap N) \) is virtually polycyclic, and hence that \( G \) is virtually polycyclic. This reduces us to the case where \( A \cap N = \{1\} \).

In a similar way, we may assume that \( BN = G \) and that \( B \cap N = \{1\} \). Hence \( G = N \rtimes A = N \rtimes B \). In particular, \( A \cong B \) via an isomorphism \( \theta : A \to B \) uniquely determined by the property that \( a\theta(a)^{-1} \in N \) for all \( a \in A \). Moreover, \( N \) is a \( ZA \)-module, and the map \( \delta : A \to N \) defined by \( \delta(a) = a\theta(a)^{-1} \) is a derivation. For \( g \in G \), the double coset \( AgB \) intersects \( N \): indeed if \( g \in N \) then \( AgB \cap N \) is precisely the orbit of \( A \) under the affine action

\[
a \ast g = a \cdot g + \delta(a) = aga^{-1}a\theta(a)^{-1} = ag\theta(a)^{-1}.
\]

In particular, \( N \) is the union of finitely many \( A \)-orbits under this action, so \( N \) is finitely generated as an abelian group, by Theorem B.

This concludes the proof of Theorem A.

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