A Fundamental Theorem of Powerful Set-Valued for F-Rough Ring

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Abstract: In this paper, we introduce the upper and lower approximations on the inverse set-valued mapping and the approximations an established on a powerful set valued homomorphism from a ring $R_1$ to power sets of a ring $R_2$. Moreover, the properties of lower and upper approximations of a powerful set valued are studies. In addition, we will give a proof of the theorem of isomorphism over approximations F-rough ring as new result. However, we will prove the kernel of the powerful set-valued homomorphism is a subring of $R_1$. Our result is introduce the fundamental isomorphism theorem of ring as generalized the concept of the set valued mappings.

Keywords: Upper approximations; Powerful Set Valued Mapping; T-Rough Set.

I. INTRODUCTION

The rough set theory has been introduced by Pawlak 1982[1], as the new tool to incomplete information system. Many researchers develop this theory in many areas. Substantively, the rough set an established on two concepts of approximations (lower and upper) [2]. R. Biswas and S. Nanda show rough groups and subgroups. N. Kuroki [3] consider the rough notion of ideal in a semi-group. B. Davvaz have studied roughness in ring [4]. The concepts of rough prime ideals and rough primary ideals in a ring has introduced by O. Kazanci and B. Davvaz [5]. V. Selvan and G. Senthil Kumar [6] consider the rough notion of ideal in a semi-group. B. Davvaz and others has generalized the concept of upper and lower approximations established on a ring by the set valued homomorphism of rings. The properties of T-rough sets in commutative rings has studied by S. B. Hosseini, N. Jafarzadeh, and A. Gholami[9]. However, other researchers have been interested of the Set-valued maps [10],[11]. Set-valued maps have used in many areas such as Economics [12]. In [13], G. SenthilKumarthis gives the proof the fundamental set-valued homomorphism group theorem. Our work, we study the powerful set-valued homomorphism established on ring and some their properties. Moreover, we will show that the kernel of the powerful set-valued homomorphism is a subring of $R_1$. Our result is introduce the fundamental isomorphism theorem of ring as generalized the concept of the set valued mappings.

II. PRELIMINARIES

We will recall the concept of the approximation (lower, upper) based on a set valued mapping for more information and proofs we can see [7]. In addition, we introduce the set valued homomorphism.

Definition 2.1: Suppose $U \neq \emptyset$. Let ~ be an equivalence relation on $U$. Let $R: 2^U \rightarrow 2^U \times 2^U$ where $2^U$ is the set of all non-empty subsets of $U$. A pair $(U, ~)$ is called an approximation space and the upper rough approximation of $X$ is $R(X) = \{x \in U: [x] \subseteq X\}$ and $\overline{R(X)} = \{x \in U: [x] \cap X \neq \emptyset\}$ is the lower rough approximation of $X$.

Definition 2.2: Suppose that $X, Y$ are any two nonempty sets. A set-valued map or multivalued map $F: X \rightarrow 2^Y$ from $X$ to $Y$ is a map that related to any $x \in X$ a subset of $F(x)$ of $Y$. The set called the image of $x$ under $F$ by $F(x)$ and we define the domain of $F$ by $D_F = \{x \in X: F(x) \neq \emptyset\}$. The image of $F$ is a subset of $Y$ defined by $\text{Im}(F) = \bigcup_{x \in X} F(x) = \bigcup_{x \in D_F} F(x)$.

Definition 2.3: Suppose that $X, Y$ are any non-empty sets. Let $M \subseteq Y$ and $F: X \rightarrow 2^Y$ be a set-valued mapping. We called the $\overline{F(M)} = \{x \in X: F(x) \subseteq M\}$ and $\overline{F(M)} = \{x \in X: F(x) \cap M \neq \emptyset\}$ respectively and $\overline{F(M)}$ is called $F$-rough set of $Y$.

Example 2.1: Let $X = \{1, 2, 3, 4, 5, 6\}$ and let $F: X \rightarrow 2^X$ where $\forall x \in X, F(1) = \{1\}, F(2) = \{1, 3\}, F(3) = \{3, 4\}, F(4) = \{4\}, F(5) = \{1, 6\}, F(6) = \{1, 5, 6\}$. Let $A = \{1, 3, 5\}$, then $\overline{F(M)} = \{1, 2\}$ and $\overline{F(A)} = \{1, 2, 3, 5, 6\}$, $B(A) \neq \emptyset$, is rough. $\text{Im}(F) = \bigcup_{x \in X} F(x) = \{1, 3, 4\}$. Now, Let $B = \{2, 4, 6\}$, then $\overline{F(B)} = \{4\}$, and $\overline{F(B)} = \{3, 4, 5, 6\}$, $B(B) \neq \emptyset$, is rough.

Preposition 2.1: Suppose that $X, Y$ are non-empty sets. Let $A, B \subseteq Y$. If $F: X \rightarrow 2^Y$ be a set-valued mapping, then:

1) $\overline{F(A) \cap F(B)} = \overline{F(A)} \cap \overline{F(B)}$;
2) $\overline{F(\emptyset)} = \emptyset$;
3) $\overline{F(A) \cup F(B)} = \overline{F(A)} \cup \overline{F(B)}$.
4)- \( F(A) \cup F(B) = F(A) \cap F(B); \)
5)- \( F(X) = F(A) \cup F(B) \)

**Proof:** By using the definitions upper and lower approximations

**Example 2-2:** let \( X = \{ a, b, c, d, e, f \} \) and let \( F : X \rightarrow 2^X \). If \( F_1(a) = \{ a \}, F_1(b) = \{ c, d \}, F_1(c) = \{ e, f \}, F_1(d) = \{ a, d \}, F_1(e) = \{ a, f \}, F_1(f) = \{ d, e, f \} \). And \( F_2(a) = \{ a, b, f \}, F_2(b) = \{ a, f \}, F_2(c) = \{ c, d \}, F_2(d) = \{ e, f \}, F_2(e) = \{ a, f \} \) and \( A = \{ a, b, c, e \}, \) then \( F_1(A) = \{ a, b \}, \) and \( F(A) = \{ a, b, c, e, f \}, F_2(A) = \{ a, c \}, F_2(A) = \{ a, b, c, e, d, f \} \).

**Theorem 2.1** Let \( F_1; F_2 : X \rightarrow 2^X \) be set-valued map such that \( F_1 \cap F_2 \neq \emptyset ; \forall x \in X \) and \( A \subseteq X \). Then
1- \( F_1(A) \cup F_2(A) = F_1(A) \cup F_2(A); \)
2- \( F_1(A) \cap F_2(A) = F_1(A) \cap F_2(A); \)
3- \( F_1(A) \cap F_2(A) = F_1(A) \cap F_2(A); \)
4- \( F_1(A) \cup F_2(A) = F_1(A) \cup F_2(A). \)

**Proof:** By using the definitions 2-3.

**Definition 2.4:** Let \( 2^X \) be the set of all subsets of a non-empty \( X \). If \( S_1, S_2 \subseteq 2^X \), then we define \( S_1 \cap S_2 = \{ x \in S_1; \text{either} S_1 \cap S_2 \text{or} \in S_2 \text{but not both} \} \) and \( S_1 \cup S_2 = S_1 \cup S_2 \) are called sum and product of \( S_1 \) and \( S_2 \) respectively.

**Proposition 2-2** Let \( 2^X \) be the power of all subsets of a non-empty \( X \) with sum and product of \( S_1 \) and \( S_2 \). Then \( (2^X, +) \) is a commutative ring.

Note that, the empty set \( \emptyset \) is the identity of + and the set \( X \) is the identity of \( - \). Therefore, we called \( 2^X \) is Ring of Subsets of \( X \).

**III. MAIN RESULTS**

We introduce the concepts of the invers set-valued map. Suppose \( X \) and \( Y \) are two nonempty sets.

**Definition 3-1:** Suposethat \( F \) is a set-valued map from \( X \) to \( Y \), we call \( F^I \) the inverse of \( F \) and we write as: \( F^I(y) = \{ x \in X; y \in F(x) \}, \forall y \in Y \). If \( B \subseteq Y \), then the upper inverse image is \( \{ x \in X; F(x) \cap B \neq \emptyset \} \) and lower inverse image \( \{ x \in X; F(x) \subseteq B \} \) and the boundary is \( B(F) = \{ F^{-1}(B); F^{-1}(B) \} \).

**Example 3-1:** Consider the example 1-1, let \( X = \{ a, b, c, d, e, f \} \). Suppose \( F : X \rightarrow 2^X \) is set value map where \( \forall x \in X \), \( F(a) = \{ a \}, F(b) = \{ a, c \}, F(c) = \{ c, d \}, F(d) = \{ d \}, F(e) = \{ a, f \} \), \( F(f) = \{ a, e, f \} \). Let \( B_1 = \{ a, d, e \}, \) then \( F^{-1}(B_1) = \{ a, d \}, \) and \( F^{-1}(B_1) = \{ a, b, c, d, e, f \}, B(B_1) \neq \emptyset, \) is rough. Let \( B_2 = \{ b, e, f \} = \{ b, e, f \}, \) and \( F^{-1}(B_2) = \{ c, d, e, f \} \), is rough.

**Proposition 3-1:** Let \( X, Y \) be non-empty sets and \( B_1, B_2 \subseteq Y \). If \( F : X \rightarrow 2^X \) be a set-valued mapping, then:
1- \( F^{-1}(B_1) \) and \( F^{-1}(B_2) \),\( F^{-1}(B_1) \) and \( F^{-1}(B_2) \);\n2- \( F^{-1}(B_1) \) and \( F^{-1}(B_2) \),\( F^{-1}(B_1) \) and \( F^{-1}(B_2) \);\n3- \( F^{-1}(B_1) \) and \( F^{-1}(B_2) \),\( F^{-1}(B_1) \) and \( F^{-1}(B_2) \).\n
**Proof:** it is explicit.

**Proposition 3-2:** Let \( F_1; F_2 : X \rightarrow 2^X \) be set-valued map such that \( F_1 \cap F_2 \neq \emptyset ; \forall x \in X \) and \( B \subseteq Y \). Then
1- \( \left( F_1 \cup F_2 \right)^{-1}(B) = F_1^{-1}(B) \cup F_2^{-1}(B) \);\n2- \( \left( F_1 \cap F_2 \right)^{-1}(B) = F_1^{-1}(B) \cap F_2^{-1}(B) \);\n3- \( \left( F_1 \cup F_2 \right)^{-1}(B) = F_1^{-1}(B) \cup F_2^{-1}(B) \);\n4- \( \left( F_1 \cap F_2 \right)^{-1}(B) = F_1^{-1}(B) \cap F_2^{-1}(B) \).\n
**Proof:** it is explicit.

**Example 3-2:** Suppose \( U = \{ 1, 2, 3, 4, 5, 6 \} \). Let \( F : U \rightarrow 2^U \) where \( \forall x \in X \), \( F(1) = \{ 1 \}, F(2) = \{ 1, 3 \}, F(3) = \{ 3, 4 \}, F(4) = \{ 4 \}, F(5) = \{ 1, 6 \}, F(6) = \{ 1, 5, 6 \} \). \( (1) \) \( F \) is a \( \emptyset \) ring. Then \( F(A) = \{ 1, 2, 3 \} \) and \( F(A) = \{ 1, 2, 3, 4, 5, 6 \} \). \( B(A) \neq \emptyset \), is rough.

**IV. ISOMORPHISM THEOREM FOR R-ROUGH RINGS.**

**Definition 4.1:** suppose that \( R \) is a ring. Let \( \sim \) be a conformity of \( R \) that is, \( \sim \) is an equivalence relation on \( R \) such that \( (x, y) \sim \) \( \in \) \( \emptyset \) implies \( (x, y) \sim \) \( \in \) \( \sim \) \( \forall x \in X \) and \( \emptyset \) \( \forall a, b \in R \). We denote by \( [x] \). the \( \sim \) conformity class containing the element \( x \in R \).

**Remark 4-1:** Let \( \sim \) be a conformity on a ring \( R \). Define \( F: R \rightarrow 2^R \) by \( F(x) = [x] \), \( \forall x \in X \). Then \( F \) is a set-valued homomorphism.

**Definition 4.2:** Suppose that \( R_1 \) and \( R_2 \) are two rings. Let \( F: R_1 \rightarrow 2^{R_2} \) be a set-valued homomorphism and \( S \) be subring of \( R_2 \). If \( \bar{F} \) is subring of \( R_1 \), then \( S \) is called the lower \( F \)-rough subring of \( R_2 \), \( \bar{F} \) is subring of \( R_1 \) if \( \bar{F} \) is subring of \( R_1 \) and \( S \) is called the upper \( F \)-rough subring of \( R_2 \). If \( \bar{F} \) and \( F \) are subring of \( R_1 \), then we call \( \bar{F} \) \( F \)-rough subring.

**Definition 4.3:** Suppose that \( R_1 \) and \( R_2 \) are rings. A powerful set-valued set-valued homomorphism is a mapping from \( R_1 \) into \( 2^{R_2} \) that preserves the ring operations \( (+, \sim) \), that is, \( \forall x \) and \( y \in R_1 \) such that:
1- \( F(x+y) = [x+y] \neq [x] \neq [y] \); and
2- \( F(x+y) = F(x)+F(y) \); and
3- \( F(x+y) = [a+b] \neq F(x+y) \neq F(x+y) \); and
4- \( F(x+y) = F(x)F(y) \).
**Remark 4.2:** If the powerful set-valued homomorphism is one to one, then we called an epimorphism. If it is onto and one-to-one, then we called an isomorphism.

**Proposition 4.2:** Let ~ be a conformity on a ring \( R \). If \( x, y \in R \) then:
1) \( [x \cdot y] = [x] \cdot [y] \);
2) \( [x + y] = [x] + [y] \);
3) \( (x^-1) \cdot a \in [x^-1] \cdot a \in [x] \).

**Proof:** from definition 4-1

**Proposition 4.2:** Suppose that \( R_1 \) and \( R_2 \) are two rings. Let \( F: R_1 \rightarrow R_2 \) be a subring of \( R_2 \). Then \( F(x) \in R_2 \) and \( F(x) \in \mathbb{Z} \) are rings. Suppose \( F(e(1)) = e(1) \) and \( e(1) \) is the identity element of \( R_2 \).

**Proof:** Suppose \( F(x) = e(1) \); then \( F(x) = x \).

**Theorem 4.1** Suppose that \( R_1 \), \( R_2 \), and \( R_3 \) are rings. Suppose \( F(e(1)) = e(1) \) and \( F(x) \) is a subring of \( R_2 \). Then \( F(x) \) is a subring of \( R_3 \).

**Proof:** Suppose \( S \) is a subring of \( R_2 \). Then there exist elements \( x \), \( y \) such that \( F(x) = x \) and \( F(y) = y \). Since \( S \) is a subset of \( R_2 \), \( x + y \) and \( x \cdot y \) are in \( S \). Therefore, \( F(x + y) = F(x) + F(y) = x + y \) and \( F(x \cdot y) = F(x) \cdot F(y) = x \cdot y \).

Finally, we have \( F(x \cdot y) = F(x) \cdot F(y) = x \cdot y \) for all \( x, y \in S \) from Theorem 4.2. Therefore, \( F(x) \cdot F(y) = x \cdot y \) for all \( x, y \in S \). So, \( F(x) \cdot F(y) = x \cdot y \) for all \( x, y \in S \).

**Definition 4-4:** Suppose that \( R_1 \), \( R_2 \), and \( R_3 \) are rings. Let \( F: R_1 \rightarrow R_2 \) be a subring of \( R_2 \). Then \( F(x) \in R_2 \) and \( F(x) \in \mathbb{Z} \) are rings.

**Proof:** assume that \( x, y \in Ker(F) \). From definition of \( ker(F) \), \( F(x) = F(e_1) \) and \( F(y) = F(e_1) \). We have \( F(x+y)=F(x)F(y)\).
Since $F$ is a powerful set valued homomorphism.

Therefore, $x \in \ker(F)$ and $F(x^0)= F(x^1) = F(e^1) = (F(e^1))^0 = (F(e)) = F(e) = x^0 \in \ker(F)$.

Now, suppose $x+y \in \ker(F)$. Then $F(x+y) = F(x) + F(y) = F(e) + F(e) \in \ker(F)$. Therefore, $x+y \in \ker(F) \Rightarrow \ker(F)$ is a subring of $R$ as we required.

V. THE FUNDAMENTAL THEOREM OF RING HOMOMORPHISM.

Here if we have a ring $R$ and the ideal $I$ and $S$ subring of $R$, then the quotient of $R$ by $I$ is the set $R/I$ of equivalence classes $a+I=\{a+i\mid i \in I\}$ with the two operations($+,\cdot$). We define $(a+I),(b+I) \in R/I$ by: $(a+I) + (b+I) = (a+b)+I$; and $(a+I) \cdot (b+I) = (a \cdot b)+I \forall (a+I), (b+I) \in R/I$. If $F: R \rightarrow 2^S$ is a powerful set-valued homomorphism from a ring to $2^S$, then $\ker(F)$ is a subring of $R$ (Theorem 4.4). We would ultimately like $ker(F)$ to be an ideal of $R$ to define the quotient $R/\ker(F)$.

Now, suppose that $e \in R$ with $a-e=a$ and $a-e=a \forall a \in R$, we need to find identity of operation $+$. By definition of $\ker(F)$ in general cannot be a subring of $R$ because if $e$ is the identity of $R$, then by definition of a ring homomorphism, $F(e)$ is mapped to the multiplicative identity of $S$ and not to the additive identity of $S$. To establish a fundamental theorem of ring homomorphisms, we make a small exception in not requiring that $\ker(F)$ is an ideal for the quotient $R/\ker(F)$ to be defined.

**Theorem 5.1:** (The Fundamental Theorem of Ring Homomorphisms): Let $R$ and $S$ be two rings with ring. Let $F: R/\rightarrow 2^S$ be a powerful set-valued homomorphism. Then $R/\ker(F) \cong F(R)$.

**Proof:** Let $\ker(F)$ be the kernel of $F$ and $\phi: R/\ker(F) \rightarrow F(R)$ and $\forall (a+\ker(F)) \in R$ is $\phi(a+\ker(F))=F(a)$. We show that $\phi$ is well-defined. For $a+\ker(F)=b+\ker(F)$, suppose $a=b+\ker(F)$ for some $k \in \ker(F)$. So, $\phi(a+\ker(F))=\phi((a+k)+\ker(F))=\phi((b+k)+\ker(F))=\phi(b+\ker(F))$. Now, if $(a+\ker(F))$ and $(b+\ker(F)) \in R/\ker(F)$, then $\phi((a+\ker(F))+(b+\ker(F)))=\phi((a+b)+\ker(F))=F(a+b)=F(a)+F(b)=\phi(a+\ker(F))+\phi(b+\ker(F))$.

Finally, we need to show $\phi$ is bijective. Let $\phi(a+\ker(F)), (b+\ker(F)) \in R/\ker(F)$ and suppose that $\phi(a+\ker(F))=\phi((b+\ker(F))$. Then $F(a)=F(b)$. So $F(a-b)=e$. So $a-b \in \ker(F)$. Hence $\phi$ is injective. Now, $\forall a \in F(R)$ we have that $(a+I) \in R/\ker(F)$ is such that $\phi(a+\ker(F))=a$. So $\phi$ is surjective. That mean $\phi$ is bijective. So, $\phi$ is an isomorphism from $R/\ker(F)$ to $F(R)$, that is: $R/\ker(F) \cong F(R)$ as required.

VI. CONCLUSION

Theoretically, rough set based on the upper and lower approximations as an equivalence relation. In this paper, we introduce the upper and lower approximations on the invers set-valued mapping and the approximations an established on a powerful set valued homomorphism from a ring to the power sets of a ring. Moreover, the properties of lower and upper approximations of a powerful set valued are studies. In addition, we will give a proof of the isomorphism theorem for lower and upper F-rough ring as new result. However, we will prove the kernel of the powerful set-valued homomorphism is a subring of $R$. Our result is introduce the first isomorphism theorem of ring as generalized the concept of the set valued mappings. We hope that it will be useful in some applications in the future.

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