Abstract. The general method of the supersymmetrization of the soliton equations with the odd (bi)hamiltonian structure is established. New version of the supersymmetric $N=2,4$ (Modified) Korteweg de Vries equation is given, as an example. The second odd Hamiltonian operator of the SUSY KdV equation generates the odd $N=2,4$ SUSY Virasoro - like algebra.

1 Introduction

The Kadomtsev-Petviashvili (KP) hierarchy of integrable soliton nonlinear evolution equations [1,2,3] is among the most important physically relevant integrable systems. One of the main reasons for the interest in the KP hierarchy in the last few years originates from its deep connection with the matrix models providing non - perturbative formulation of string theory [4]. The relations between integrable models and conformal symmetries have been studied intensely since 1984 [5,6]. Quite recently a new class of integrable systems motivated by the Toda field theory appeared both in the mathematical [7] and in the physical literature [8].

On the other side the applications of the supersymmetry (SUSY) to the soliton theory provide us a possibility of the generalization of the integrable systems. The supersymmetric integrable equations [8-20] have drawn a lot of attention in recent years for a variety of reasons. In order to get a supersymmetric theory we have to add to a system of $k$ bosonic equations $kN$ fermions and $k(N-1)$ boson fields ($k=1,2,..N=1,2,..$) in such a way that the final theory becomes SUSY invariant. Interestingly enough, the supersymmetrizations, leads to new effects (not present in the bosonic soliton’s theory). In particular the roots for the SUSY Lax operator are not uniquely defined [15,20], there is no bosonic reduction to the classical equations [21] and the non-local conservation laws [22] appeare. These

1E-mail: ziemek@ift.uni.wroc.pl
effects rely strongly on the hamiltonian description of the supersymmetrical integrable systems.

In this paper we would like to describe a new method of the \( N = 2 \) supersymmetrization of the Modified Korteweg de Vries (MKdV) and Korteweg de Vries (KdV) equations. As the result our super-equations are connected with the complex version of the soliton equations. For this reason this method could be compared with the supercomplexification. We show that it is possible to construct, for the SUSY MKdV equation, the supersymmetric Lax operator. This operator however does not generate the supersymmetric conserved currents defined in the whole superspace. We introduce new Miura transformations which allow to supersymmetrize the KdV.

The SUSY KdV equation, considered in this paper constitute a bi-hamiltonian equation and is completely integrable. The Poisson tensors and the hamiltonians of the supercomplexified version of the MKdV and KdV are superfermionic (odd) objects. The second Hamiltonian operator of the supercomplexified KdV equation generates some odd \( N = 2 \) SUSY algebra in contrast to the even algebra (SUSY \( N = 2 \) Virasoro algebra) of the supersymmetric version of the KdV equation considered in [12,14]. We use therefore the name the odd SUSY Virasoro - like algebra in order to distinguish it from the even SUSY Virasoro algebra of [12,14].

This supercomplexification is a general method and could be applied to the \( N = 2 \) SUSY case as well. Indeed, in this case, we obtained a new \( N = 4 \) SUSY KdV equation with an odd bihamiltonian structure. The model proposed in this paper, differs from the one considered in [14,17].

The idea of introducing an odd hamiltonian structure is not new. Leites noticed almost 20 years ago [23], that in the superspace one can consider both even and odd sympletic structures, with even and odd Poisson brackets respectively. The odd brackets (also known as antibrackets) have recently drawn some interest in the context of BRST formalism in the Lagrangian framework [24], in the supersymmetrical quantum mechanics [25], and in the classical mechanics [26,27]. The present paper proposes some new examples of the odd sympletic structure with the \( N = 2, 4 \) supersymmetry.

The paper is organized as follows: In Section 2 we give a brief descriptions of the (complex) (M)KdV equations. In Section 3 the basic supersymmetric notation is introduced. In Section 4 we present the supercomplexification of the MKdV equation. In particular the Lax formulation, the odd bihamiltonian structure, and the superfermionic conserved currents are described. Section 5 contains supercomplexifications of the KdV equation. We describe the odd bihamiltonian structure and define the hereditary recursion operator. The realization of the odd SUSY Virasoro - like algebra in terms of the solutions of the supercomlexified KdV equation is given. The last section contains the supercomplexification of the \( N = 2 \) SUSY KdV equation.

## 2 Complex (Modified) Korteweg de Vries Equation

The MKdV equation

\[
h_t = -h_{xxx} + 6h^2h_x,
\]

\[(1)\]
has the following Lax representation

\[ L = \partial^2 + 2h\partial, \quad \frac{\partial}{\partial t} L = 4[L, L^\frac{3}{2}] , \]

where \( \geq 1 \) denotes the projection onto the purely differential part of pseudodifferential element.

The MKdV equation constitute the bihamiltonian system

\[ h_t = P_1 \frac{\delta H_2}{\delta h} = P_2 \frac{\delta H_1}{\delta h} , \]

where

\[ P_1 = \partial, \quad P_2 = -\partial^3 + 4\partial h\partial^{-1} h\partial, \]

and

\[ H_1 = \frac{1}{2} \int dx h^2, \quad H_2 = \frac{1}{2} \int dx (h^4 - h_{xx} h) . \]

The conserved currents for the MKdV equation can be obtained using the trace formula

\[ H_n = Tr L_n^{\frac{n+1}{2}} , \]

where \( Tr \) denotes the coefficients standing in the front of \( \partial^{-1} \) in the pseudodifferential element \( L_n^{\frac{n+1}{2}} \). In the following we shall also use the third conserved current

\[ H_3 = \frac{1}{2} \int dx \left(h_{xxxx} h + 2h^6 - 5h_x^2 h^2 - 5h_{xx} h^3\right) . \]

The complex MKdV equation can be obtained from the MKdV equation assuming that \( h = g + if \), where \( g = g(x, t), f = f(x, t) \) are new functions. Inserting this expansion in the MKdV equation we obtain

\[ g_t = \partial \left(-g_{xx} + 2g^3 - 6f^2 g\right) , \]

\[ f_t = \partial \left(-f_{xx} - 2f^3 + 6g^2 f\right) . \]

Our complex MKdV equations constitute the bi-hamiltonian system. This structure could be extracted from the usual bi-hamiltonian structure of MKdV equation using expansion of \( h = g + if \). We have twice as much currents in the complex case. They are given as the real and the complex part of (5,7). For the first two conserved currents (5) we have

\[ H_{1r} = \int dx (g^2 - f^2), \quad H_{1z} = \int dx (gf) , \]

\[ H_{2r} = \int dx \left(g_{xx}g - f_{xx}f + g^4 - 6g^2 f^2 + f^4\right) , \]

\[ H_{2z} = \int dx \left(2g_{xx}f + 4g^3 f - 4gf^3\right) . \]

The MKdV equation is related to the famous KdV equation

\[ w_t := -w_{xxx} + 6ww_x , \]
by the Miura transformation
\[ w := h_x + h^2. \]  

The bihamiltonian structure of the KdV equation is
\[ w_t = P_1 \frac{\delta H_2}{\delta w} = P_2 \frac{\delta H_1}{\delta w}, \]
where
\[ H_2 := \frac{1}{2} \int dx (w_x^2 + 2w^3), \quad H_1 := \frac{1}{2} \int dx w^2, \]
\[ P_1 := \partial, \quad P_2 := -\partial^3 + 2\partial w + 2w\partial. \]

\( P_2 \) is the so called second Hamiltonian operator which generates the Virasoro algebra [5,6]. It is connected with the first Hamiltonian operator of the MKdV equation via
\[ P_2 := \frac{\delta w}{\delta h} \partial \left( \frac{\delta w}{\delta h} \right)^*, \]
where \( * \) denotes the hermitian conjugation, \( \frac{\delta w}{\delta h} \) is the Frechet derivative, and \( w \) is defined by (13).

The complex KdV equation can be obtained from the KdV equation assuming that \( w = s + iv \) where \( s \) and \( v \) are new functions. One gets the system of two equations
\[ s_t = \partial \left( -s_{xx} + 3s^2 - 3u^2 \right), \]
\[ u_t = \partial \left( -u_{xx} + 6su \right). \]

Using the same expansion of \( w \) it is also possible to obtain the Lax representation and the bi-hamiltonian structure of the complex KdV equation. We have the following bi-hamiltonian structure of the equations (18-19):
\[ \frac{d(s, u)^t}{dt} := \begin{pmatrix} \partial & 0 \\ 0 & -\partial \end{pmatrix} \begin{pmatrix} \delta H_{2x} \\ \delta H_{2x} \end{pmatrix}^t = \begin{pmatrix} -\partial^3 + 2\partial s + 2s\partial & 2u\partial + 2\partial u \\ 2u\partial + 2\partial u & \partial^3 - 2\partial s - 2s\partial \end{pmatrix} \begin{pmatrix} \delta H_{1x} \\ \delta H_{1x} \end{pmatrix}^t, \]
where \( (,)^t \) denotes the transposition, while \( H_{1x} \) and \( H_{2x} \) are the real parts of the hamiltonians (15).

### 3 Supersymmetric Notations

We shall consider an \( N = 2 \) superspace with the space coordinates \( x \) and the Grassman coordinates \( \theta_1, \theta_2, \theta_2\theta_1 = -\theta_1\theta_2, \theta_1^2 = \theta_2^2 = 0 \). The so called extended supersymmetry is assumed for which the superfields are superfermions or superbosons. Their Taylor expansions with respect to \( \theta \) take the form
\[ U(x, \theta_1, \theta_2) = u_0(x) + \theta_1 \xi_1(x) + \theta_2 \xi_2(x) + \theta_2 \theta_1 u_1(x), \]
where the fields $u_o, u_1$ are interpreted as bosons (fermions) for the superboson (superfermion) field, while $\xi_1, \xi_2$ as fermions (bosons) for the superboson (superfermion) field, respectively. The supersymmetric covariant derivatives are defined by

$$\partial = \frac{\partial}{\partial x}, \quad D_1 = \frac{\partial}{\partial \theta_1} + \theta_1 \partial, \quad D_2 = \frac{\partial}{\partial \theta_2} + \theta_2 \partial,$$

with the properties

$$D_1^2 = D_2^2 = \partial, \quad D_1 D_2 + D_2 D_1 = 0.$$

$$D_1^{-1} := D_1 \partial^{-1}, \quad D_2^{-1} := D_2 \partial^{-1},$$

where $\partial^{-1}$ is defined as the formal series

$$\partial^{-1} f = f \partial^{-1} - f_x \partial^{-2} + f_{xx} \partial^{-3} - ...,$$

We shall use below the following notation: $(D_i F)$ denotes the outcome of the action of the superderivative on the superfield $F$, while $D_i F$ denotes the action itself.

We define the integration over the $N = 2$ superspace to be

$$\int dX H(x, \theta_1, \theta_2) = \int dx d\theta_1 d\theta_2 H(x, \theta_1, \theta_2),$$

where Berezin’s convention are assumed

$$\int d\theta_i \theta_j := \delta_{i,j}, \quad \int d\theta_i := 0.$$

We always assume that the components of the superfields and their derivatives vanish rapidly enough. Then

$$\int dX (D_1 U) = \int dX \theta_1 \theta_1 \xi_1 x = \int dx \xi_1 x = 0,$$

and similarly for $(D_2 U)$ where $U$ is an arbitrary superfunction vanishing at $\pm \infty$.

The $N = 2$ pseudo-superdifferential operators

$$P := \sum_{-\infty}^{\infty} \left( B_n^1 + F_n^1 D_1 + F_n^2 D_2 + B_n^2 D_1 D_2 \right),$$

where $B_n^i$ are superbosons, and $F_n^i$ are superfermions and $(i = 1, 2)$, form an associative algebra.

### 4 Supercomplexifications of N=2 MKdV equation

The supersymmetric MKdV obtained from the Lax pair representation

$$L := \partial^2 + 2(D_1 D_2 U) \partial + 2U_x D_1 D_2.$$ 

$$\frac{\partial L}{\partial t} = 4 \left[ L, L_{\geq 1}^{\text{even}} \right],$$
reads

\[ U_t = -U_{xxx} - 2U_x^3 + 6(D_1D_2U)^2U_x. \] (32)

where \( \geq 1 \) denotes the projection onto the purely superdifferential part of the pseudosupersymmetric element.

This equation has several surprising properties. First, let us notice that introducing new superfields

\[ g := (D_1D_2U), \quad f := U_x, \] (33)

the supersymmetric MKdV equation can be rewritten as the complex MKdV equation (8-9). This supersymmetric MKdV equation is however not equivalent with the complex MKdV since we have to take into account the constraint (33).

The bosonic components of the SUSY MKdV equation lead to the following system of equations \((U = u_0 + \theta_1\xi_1 + \theta_2\xi_2 + \theta_2\theta_1u_1)\)

\[ u_{0t} = -u_{0xxx} - 2u_0^3u_{0x} + 6u_1^2u_{0x}, \] (34)

\[ u_{1t} = \partial\left( -u_{1xx} + 2u_1^3 - 6u_0^2u_1 \right). \] (35)

Notice that the boson fields do not interact with the fermion ones because there are no such in (34-35). However, the fermion fields interact with boson fields

\[ \xi_{1t} = -\xi_{1xxx} + 6\xi_{1x}u_1 + 6\xi_2u_{0x}, \] (36)

\[ \xi_{2t} = -\xi_{2xxx} - 6\xi_{1x}u_{0x} + 6\xi_{2x}u_1. \] (37)

This way of the supersymmetrization we shall call in the following the supercomplexification. The method is based on the following trick: we first replace the classical function in the equations of motion by \((D_1D_2F) + iF_x\) where \(F\) is some \(N = 2\) superfield, then we extract the equation on \(F\) and the (bi)hamiltonian structure with its hamiltonians.

The most surprising property of the supercomplexified MKdV equation is that the conserved currents of the complex MKdV equation (10-11) are not conserved currents of the supersymmetric MKdV (32) equation if they are defined in the whole superspace. Indeed if we change the usual integration in (10-11) to superintegration \((dx \rightarrow dX)\) and assume the constraint (33) then the currents (10-11) vanishes.

It should be stressed, that the classical currents are conserved quantities for the supercomplexified MKdV equation if they are defined only in the usual space. In order to solve the problem of conserved currents defined in the whole superspace we have checked that our system does not have any superbosonic conserved currents up to fifth conformal dimension. It was verified using the computer algebra [29,30] and assuming the most general form for currents. The Lax operator (30) does not produce any superbosonic conserved currents also.

This observation may suggest that this system is not (bi)hamiltonian on the superspace. There is still a possibility of a superfermionic Hamiltonian operator and superfermionic conserved currents. It turns out that such situation occure in our case. One can construct two superfermionic currents

\[ H_{14} = \int dX \left((D_1U_{xxx})U - 3(D_1U_x)(D_1D_2U)^2U + 3(D_1U_x)U_x^2U \right) \]
which generates the supercomplexified MKdV equation

\[
U_t = D_1^{-1} \frac{\delta H_{14}}{\delta U} = D_2^{-1} \frac{\delta H_{24}}{\delta U}.
\] (40)

The Hamiltonian operator \(D_1^{-1}\) or \(D_2^{-1}\) defines a closed two-form \(\Omega(D_i^{-1})\), \(i = 1, 2\)

\[
\Omega(D_i^{-1})(a, b) := \int dX a D_i^{-1} b = -\Omega(D_i^{-1})(b, a).
\] (41)

where \(a\) and \(b\) are arbitrary odd vector fields.

Although the Hamiltonian operators \(D_1^{-1}\) and \(D_2^{-1}\) are not \(O(2)\) invariant in the superspace, (as the hamiltonians \(H_{14}\) and \(H_{24}\)), they are superpartners. One can restore the \(O(2)\) invariance considering their linear combinations. The operator \(P_1 := D_1^{-1} - D_2^{-1}\) is invariant under \(O(2)\) transformation and generates the same SUSY MKdV equation with the hamiltonian \(H_{14} - H_{24}\).

There are two different "second" Hamiltonian operators for the supercomplexified MKdV equation:

\[
P_{12} := -D_1 \partial - 4(D_1 D_2 U) \partial^{-1} U_x D_2 - 4U_x \partial^{-1} (D_1 D_2 U) D_2
+ 4(D_1 D_2 U) \partial^{-1} (D_1 D_2 U) D_1 - 4U_x \partial^{-1} U_x D_1,
\] (42)

\[
P_{22} := -D_2 \partial + 4(D_1 D_2 U) \partial^{-1} U_x D_1 + 4U_x \partial^{-1} (D_1 D_2 U) D_1
+ 4(D_1 D_2 U) \partial^{-1} (D_1 D_2 U) D_2 - 4U_x \partial^{-1} U_x D_2.
\] (43)

The proof that these operators define a closed two-form is postponed to the next section.

Let us now explain the derivation of the superfermionic conserved currents (38-39). We assume, in the first step of the calculations, the ansatz (33) and use the special computer program [29,30] in order to compute the usual integral of the conserved currents of the complex MKdV equation. The next step is to compute the integral \((D_1^{-1})\) from the nonintegrable part of the first integration. The integrable part yields the first series of conserved currents. The computation of \((D_2^{-1})\) for the nonintegrable part of the second integration leads to a purely integrable result which gives the second series of conserved currents. Symbolically this procedure could be expressed as follows:

\[
\int dx H \Rightarrow K_0 + \int dx K_1,
\] (44)

\[
(D_1^{-1} K_1) \Rightarrow H_1 + (D_1^{-1} S_1), \quad (D_2^{-1} S_1) \Rightarrow H_2.
\] (45)

If we change the order of supersymmetric integration in (45) then the new \(H_1\) and \(H_2\) are equivalent to the old one, modulo the superintegrable term.

\footnote{A new version of the program described in [30], allowing integrating the superfunctions, will appear in the new edition of Reduce 3.7 April 15 1999.}
The first classical conserved current of the complex MKdV equation yields

\[ H_{12} := \int dX(D_1 U) U_x, \quad H_{22} := \int dX(D_2 U) U_x. \] (46)

Our convention for indices of \( H_{ij} \) is as follows: \( j \) denotes the conformal dimension of the classical current, and \( i = 1, 2 \) denotes the index of the supersymmetric integrations. For the second classical current we got the formulae (38-39) while for the third classical current we got

\[
H_{16} := (D_1 U) \left( - (D_1 D_2 U_{xxxx}) - 5(D_1 D_2 U_{xx}) U_x^2 + 5(D_1 D_2 U_{xx}) (D_1 D_2 U)^2 \\
+ 5(D_1 D_2 U_x)^2 - 10(D_1 D_2 U_x) U_{xx} U_x - 2(D_1 D_2 U)^5 + 20(D_1 D_2 U)^3 U_x^2 \\
- 10(D_1 D_2 U) U_{xxx} U_x - 5(D_1 D_2 U) U_{xx}^2 + 10(D_1 D_2 U) U_x^4 \right). \] (47)

The superpartner \( H_{26} \) can be obtained from (47) using the \( O(2) \) transformation.

5 Supersymmetric N=2 KdV Equation, Odd Bihamiltonian Structure, Odd Virasoro - like algebra

The supersymmetric generalizations of the KdV equation based on the supersymmetric version of the Virasoro algebra were considered in [11-16]. The Hamiltonian operator generating this super equation is

\[ P_2 := D_1 D_2 \partial + 2 \partial \Phi + 2 \Phi \partial - D_1 \Phi D_1 - D_2 \Phi D_2, \] (48)

where \( \Phi = \Phi(x, \theta_1, \theta_2) \) is a superboson function. The SUSY \( N = 2 \) KdV equation is the following one parameter family of super-Hamiltonian evolution equation

\[
\Phi_t = P_2 \frac{\delta}{\delta \Phi} \int dX \left( \Phi(D_1 D_2 \Phi) + \frac{\alpha}{3} \Phi^3 \right) \]

\[ = \partial \left( - \Phi_{xx} + 3 \Phi(D_1 D_2 \Phi) + \frac{1}{2}(\alpha - 1)(D_1 D_2 \Phi)^2 + \alpha \Phi^3 \right), \] (50)

where \( \alpha \) is an arbitrary constant. It was show in [12,13,16] that only for three values of the parameter \( \alpha = -2, 1, 4 \) this system is integrable and possesses the Lax representation. The bihamiltonian structure of this generalization was considered in [15].

We show below that it is possible to construct a new bihamiltonian supersymmetric generalization of the KdV equation using our supercomplexified method. The result is

\[ W_t := -W_{xxx} + 6(D_1 D_2 W) W_x, \] (51)

where \( W \) is the superbosonic function.

The supersymmetric generalization of the MKdV equation considered in the previous section is connected with this SUSY KdV equation by the following Miura transformation

\[ W := U_x + 2 \int dx (D_1 D_2 U) U_x. \] (52)
The bosonic limit of (51) in which \( W = w_0 + \theta_2 \theta_1 w_1 \) yields the classical KdV equation when \( w_0 = 0 \). Moreover introducing

\[
s := (D_1 D_2 W), \quad u := W_x,
\]

the SUSY KdV equation (51) reduces to the complex KdV equation (18-19). However, due to the constraint (53) this super equation and the complex KdV equation are not equivalent to each other.

One may expect that our SUSY KdV equation should share the same properties as the SUSY MKdV equation. Indeed from the knowledge of the first hamiltonian structure of the supercomplexified MKdV and the Miura transformation (52) it is possible to obtain two different ”second” Hamiltonian operators

\[
P_{21} := \frac{\delta W}{\delta U} D_1 \partial^{-1} \left( \frac{\delta W}{\delta U} \right)^* = D_1 \partial - 2 \partial^{-1} (D_1 D_2 W) D_1 - 2(D_1 D_2 W) \partial^{-1} D_1 + 2 \partial^{-1} W_x D_2 + 2 W_x \partial^{-1} D_2, \tag{54}
\]

\[
P_{22} := \frac{\delta W}{\delta U} D_2 \partial^{-1} \left( \frac{\delta W}{\delta U} \right)^* = D_2 \partial - 2 \partial^{-1} (D_1 D_2 W) D_2 - 2(D_1 D_2 W) \partial^{-1} D_2 - 2 \partial^{-1} W_x D_1 - 2 W_x \partial^{-1} D_1. \tag{55}
\]

These operators generate the SUSY KdV equation (51)

\[
W_t := P_{21} \frac{\delta H_{12}}{\delta W} = P_{22} \frac{\delta H_{22}}{\delta W}, \tag{56}
\]

where

\[
H_{12} = -\frac{1}{2} \int dX W(D_1 W_x), \quad H_{22} = -\frac{1}{2} \int dX W(D_2 W_x). \tag{57}
\]

It is also possible to obtain the Hamiltonian operators \( P_{21}, P_{22} \) in a different manner. For this purpose one can supercomplexify the formula (20) and extend the gradient of the hamiltonian to the whole superspace

\[
\frac{\delta H_{1r}}{\delta s} = D_2^{-1} \frac{\delta H_{12}}{\delta W}, \quad \frac{\delta H_{1r}}{\delta u} = D_1^{-1} \frac{\delta H_{12}}{\delta W}. \tag{58}
\]

where

\[
H_{1r} = \frac{1}{2} \int dx \left( s^2 - u^2 \right). \tag{59}
\]

Now it is easy to check that these Hamiltonian operators define the closed two - forms. The same is true for the supercomplexified bi-hamiltonians operators of the MKdV equation.

It is possible to define two ”first” Hamiltonian operators:

\[
P_{11} := D_1^{-1}, \quad P_{12} := D_2^{-1}, \tag{60}
\]

generating the same equation

\[
W_t := P_{11} \frac{\delta H_{14}}{\delta W} = P_{12} \frac{\delta H_{24}}{\delta W}, \tag{61}
\]
where
\[ H_{14} := \frac{1}{2} \int dX \left( - W(D_1 W_{xxx}) + 2(D_1 W)((D_1 D_2 W)^2 - W_x^2) \right), \]
\[ H_{24} := \frac{1}{2} \int dX \left( - W(D_2 W_{xxx}) + 2(D_2 W)((W_x^2 - (D_1 D_2 W)^2) \right). \] (62, 63)

We can construct an \( O(2) \) invariant bihamiltonian structure considering the linear combination of \( P_{11} \pm P_{12}, P_{21} \pm P_{22} \) with \( H_{12} \pm H_{22}, H_{14} \pm H_{24} \). These structures define the same SUSY KdV (51).

Notice that the operators \( P_{21}, P_{22} \) or \( P_{21} \pm P_{22} \) play the same role as the Virasoro algebra in the usual KdV equation. There is a basic difference - our Hamiltonian operators generate the odd Poisson brackets in the odd superspace. In order to obtain the explicit realization of this algebra we connect the Hamiltonian operator \( P_{21} - P_{22} \) with the Poisson bracket
\[ \{ W(x, \theta_1, \theta_2), W(y, \theta'_1, \theta'_2) \} = (P_{21} - P_{22})(\theta_1 - \theta'_1)(\theta_2 - \theta'_2)\delta(x - y), \] (64)
where
\[ W(x, \theta_1, \theta_2) = w_0 + \theta_1 \xi_1 + \theta_2 \xi_2 + \theta_1 \theta_2 w_1. \] (65)

Introducing the Fourier decomposition of \( w_0, \xi_1, \xi_2, w_1 \)
\[ \xi_j := \sum_{s=-\infty}^{\infty} G^j_s e^{isx}, \quad j := 1, 2, \] (66)
\[ w_0 := i \sum_{s=-\infty}^{\infty} L_s e^{isx}, \quad w_1 := \sum_{s=-\infty}^{\infty} T_s e^{isx} - \frac{1}{4}, \] (67)
in (64) we obtain
\[ \{ T_n, T_m \} = \{ L_n, L_m \} = \{ L_n, T_m \} = 0, \] (68)
\[ \{ T_n, G^i_m \} = (n^2 - 1)\delta_{n+m,0} + (-1)^i \frac{2m-n}{m} T_{n+m} - \frac{2n^2 - m^2}{m} L_{n+m}, \] (69)
\[ \{ G^i_n, L_m \} = (n - \frac{1}{n})\delta_{n+m,0} + 2\frac{m-n}{nm} T_{n+m} + (-1)^i \frac{2m^2 - n^2}{nm} L_{n+m}, \] (70)
\[ \{ G^i_n, G^j_m \} = (-1)^i \frac{m^2 - n^2}{nm} \left( G^i_{n+m} + G^j_{n+m} \right), \] (71)
\[ \{ G^1_n, G^2_m \} = -2\frac{m^2 - n^2}{nm} \left( G^1_{n+m} - G^2_{n+m} \right). \] (72)

These formulae define the closed algebra with the graded Jacobi identity [10,26]
\[ \sum_{cycl(a,b,c)} (-1)^{[a][b][c]} \{ a, \{ b, c \} \} = 0, \] (73)
where \([a]\) denotes the parity of \(a\). It is the desired odd Virasoro - like algebra.

Unfortunately we did not find any Lax representation for the superxomplexified KdV equation. In order to obtain higher currents we followed the method used in the soliton
theory. In this theory one can apply the so called recursion operator, constructed out of the bihamiltonian operators \( R := P_2P_1^{-1} \), to the derivation of the conserved quantities. This operator generates the currents if it is hereditary [3,31].

We were able to construct such recursion operator for the supercomplexified version of the KdV equation

\[
R_1 := P_{21}P_{11}^{-1}, \quad R_2 := P_{22}P_{12}^{-1},
\]

getting higher superfermionic currents

\[
H_{16} := \frac{1}{6} \int dX \left( 3W(D_1W_{xxxx}) - 20(D_1W_x)(D_1D_2W_{xx})W - 45(D_1W_x)W_x^2W + 45(D_1W_x)(D_1D_2W)^2W - 20(D_1W_{xx})(D_1D_2W_x)W - 20(D_2W_x)W_{xxx}W + 90(D_2W_x)(D_1D_2W)W_xW - 20(D_1W_{xxx})(D_1D_2W)W - 20(D_2W_{xx})W_xW - 20(D_2W_{xxx})W_xW \right).
\]

The superpartner \( H_{26} \) can be obtained from \( H_{16} \) using the \( O(2) \) transformation.

In general \( R_1 \) and \( R_2 \) are hereditary operators. Clearly \( P_{11} \) and \( P_{21} \) or \( P_{12} \) and \( P_{22} \) are compatible. Indeed the deformation

\[
W \Rightarrow W + \left( \epsilon + \theta_1 \eta_1 + \theta_2 \eta_2 + \theta_2 \theta_1 \varepsilon \right),
\]

where \( \epsilon, \varepsilon \) are arbitrary constants while \( \eta_1, \eta_2 \) are arbitrary grassmanian constants, maps \( P_{21} \) or \( P_{22} \) into the Hamiltonian operator \( P_{21} - 4 \varepsilon P_{11} \) or \( P_{22} - 4 \varepsilon P_{12} \). Hence the resulting recursion operators \( R_1 \) and \( R_2 \) are hereditary [3,31]. In this sense the SUSY KdV equation (51) is completely integrable.

6 Supercomplexified N=4 KdV equation

Let us now apply our supercomplexification method to the \( N = 2 \) SUSY KdV equation. We consider the case \( \alpha = 4 \) only. Other cases can be considered in an analogous way. We assume that the superfield \( \Phi \) satisfying the \( N = 2 \) SUSY KdV equation (48) takes after the supercomplexification the following form

\[
\Phi := (D_3D_4 \Upsilon) + i \Upsilon_x,
\]

where \( D_3 = \frac{\partial}{\partial \theta_3} + \theta_3 \partial, D_4 = \frac{\partial}{\partial \theta_4} + \theta_4 \partial \) and \( \Upsilon \) is some \( N = 4 \) superboson field. Substituting this form in (48) we obtain

\[
\Upsilon_t := - \Upsilon_{xxx} + 3(D_1D_2 (\Upsilon_x(D_3D_4 \Upsilon))) + 3 \left( \Upsilon_x(D^4 \Upsilon) + (D_3D_4 \Upsilon)(D_1D_2 \Upsilon_x) \right) - 4 \Upsilon_x^3 + 12(D_3D_4 \Upsilon)^2 \Upsilon_x,
\]

where \( D^4 = D_1D_2D_3D_4 \). It is the desired generalization of the \( N = 4 \) SUSY KdV equation, which is different from the one considered in [14,17].

The supercomplexification of the bihamiltonian structures yields

\[
\Upsilon_t = P_{14} \frac{\delta H_{24}}{\delta \Upsilon} = P_{24} \frac{\delta H_{14}}{\delta \Upsilon},
\]
where

\[ P_{14} := D_1D_2D_4\partial^{-1} + 2(D_3D_4Y)\partial^{-1}D_4 + 2\partial^{-1}(D_3D_4Y)D_4 \]
\[ -D_1\partial^{-1}(D_3D_4Y)\partial^{-1}D_1D_4 - D_2\partial^{-1}(D_3D_4Y)\partial^{-1}D_2D_4 \]
\[ +2\partial^{-1}Y_xD_3\partial^{-1} + 2\partial^{-1}Y_xD_3 - D_1\partial^{-1}Y_x\partial^{-1}D_1D_3 - D_2\partial^{-1}Y_x\partial^{-1}D_2D_3, \] (80)

\[ H_{24} := \frac{1}{2}(D_1D_2D_4Y_x)Y + \frac{2}{3}(D_3Y)(Y_x^2 - (D_3D_4Y)^2), \] (81)

\[ P_{24} := \partial^{-1}D_4 \]

\[ H_{14} := \frac{1}{2}(D_4Y_{xxx})Y - 6(D_1D_2D_4Y)Y_x(D_3D_4Y) \]
\[ -6(D_1Y)Y_x(D_4Y) - 6(D_4Y)(D_3D_4Y)(D_1D_2Y_x) \]
\[ -6(D_4Y_x)Y_x^2Y + 6(D_4Y_x)(D_3D_4Y)^2Y - 12(D_3Y_x)(D_3D_4Y)Y_xY, \] (83)

This is our bihamiltonian formulation of the equation (78). Using the same methods as in the previous sections it is possible to obtain an explicit realization of the odd SUSY \( N = 4 \) Virasoro-like algebra.

Let us remark that the supercomplexification of the \( N = 1 \) SUSY KdV equation gives the even bihamiltonian structure of the \( N = 3 \) SUSY KdV equation.

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