LATTICE-SUPPORTED SPLINES ON POLYTOPAL COMPLEXES

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Abstract. We study the module $C^r(P)$ of piecewise polynomial functions of smoothness $r$ on a pure $n$-dimensional polytopal complex $P \subset \mathbb{R}^n$, via an analysis of certain subcomplexes $P_W$ obtained from the intersection lattice of the interior codimension one faces of $P$. We obtain two main results: first, we show that the vector space $C^r_d(P)$ of splines of degree $\leq d$ has a basis consisting of splines supported on the $P_W$ for $d \gg 0$. We call such splines lattice-supported. This shows that an analog of the notion of a star-supported basis for $C^r_d(\Delta)$ studied by Alfeld-Schumaker in the simplicial case holds [3]. Second, we provide a pair of conjectures, one involving lattice-supported splines, bounding how large $d$ must be so that $\dim_k C^r_d(P)$ agrees with the McDonald-Schenck formula [14]. A family of examples shows that the latter conjecture is tight. The proposed bounds generalize known and conjectured bounds in the simplicial case.

1. Introduction

Let $P$ be a subdivision of a region in $\mathbb{R}^n$ by convex polytopes. $C^r(P)$ denotes the set of piecewise polynomial functions (splines) on $P$ that are continuously differentiable of order $r$. Study of the spaces $C^r(P)$ is a fundamental topic in approximation theory and numerical analysis (see [8]) while within the past decade geometric connections have been made between $C^0(P)$ and equivariant cohomology rings of toric varieties [16]. Practical applications of splines include computer aided design, surface modeling, and computer graphics [8].

A central problem in spline theory is to determine the dimension of (and a basis for) the vector space $C^r_d(P)$ of splines whose restriction to each facet of $P$ has degree at most $d$. In the bivariate, simplicial case, these questions are studied by Alfeld and Schumaker in [1] and [2] using Bernstein-Bezier methods. A signature result appears in [2], which gives a dimension formula for $C^r_d(P)$ when $d \geq 3r + 1$ and $P$ is a generic simplicial complex. An algebraic approach to the dimension question was pioneered by Billera in [5] using homological and commutative algebra. This method has been refined and extended by Schenck, Stillman, and McDonald ([20] and [14]). The last of these gives a polyhedral version of the Alfeld-Schumaker formula in the planar case, building on work of Rose [17], [18] on dual graphs.

In applications it is often important to find a basis of $C^r_d(P)$ which is “locally supported” in some sense. In the simplicial case the natural thing to require is that the basis elements are supported on stars of vertices. Alfeld and Schumaker [3] call such a basis minimally supported or star-supported, and they show that for $d = 3r + 1$ it is not always possible to construct a star-supported basis of $C^r_d(P)$.
However in the planar simplicial case the bases constructed for $C^r_d(\Delta)$ in [12] and [13] for $d \geq 3r + 2$ are in fact star-supported. Alfeld, Schumaker, and Sirvent show in [4] that in the trivariate case $C^r_d(\Delta)$ has a star-supported basis for $d > 8r$.

In this paper, we first show that there are polyhedral analogs of “star-supported splines”. Our main technical tool is Proposition 3.5 which utilizes polytopal subcomplexes $P_W \subset P$ associated to certain linear subspaces $W \subset \mathbb{R}^n$ to give a precise description of localization of the module $C^r(P)$. We use this description to tie together local characterizations of projective dimension and freeness due to Yuzvinsky [22] and Billera and Rose [7]. A consequence of Proposition 3.5 is that there are generators for $C^r(P)$ as an $R = \mathbb{R}[x_1, \ldots, x_n]$-module which are supported on the complexes $P_W$. From this follows Theorem 4.7 that for $d \gg 0$, $C^r_d(P)$ has an $R$-basis which is supported on subcomplexes of the form $P_W$. We call such a basis a lattice-supported basis, where the lattice (in the sense of a graded poset) of interest is the intersection lattice of the interior codimension one faces of $P$. A lattice-supported basis reduces to a star-supported basis in the simplicial case.

In §5 we use the regularity of a graded module to analyze $\dim R C^r_d(P)$, which is also done in [21]. We propose in Conjecture 5.5 a regularity bound on the module of locally supported splines in the case $P \subset \mathbb{R}^2$. If true this conjecture gives a bound for when the McDonald-Schenck formula [14] for $\dim_R C^r_d(P)$ holds which generalizes the Alfeld-Schumaker $3r + 1$ bound in the planar simplicial case. We also propose (Conjecture 5.6) a stronger regularity bound on $C^r(P)$ for $P \subset \mathbb{R}^2$ and give a family of examples to show that this proposed bound is tight. Conjecture 5.6 generalizes a conjecture of Schenck that the Alfeld-Schumaker dimension formula [2] holds when $d \geq 2r + 1$ in the planar simplicial case [19].

1.1. Example of locally supported splines. Consider the two dimensional polytopal complex $Q$ in Figure 1 with 5 faces, 8 interior edges, and 4 interior vertices.

\[ (-2,2) \]
\[ \begin{array}{ccc}
(2,2) & | & (2,-2) \\
| & | & |
\end{array} \]
\[ \begin{array}{ccc}
(-1,1) & | & (1,1) \\
| & | & |
\end{array} \]
\[ \begin{array}{cc}
(-1,-1) & (1,-1) \\
\end{array} \]

It is readily verifiable that the constant function $1 \in C^0(Q)$ cannot be written as a sum of splines which are supported on the stars of the 4 interior vertices, i.e. splines which restrict to 0 outside of the shaded regions in Figure 2.

\[ \text{Figure 1. } Q \]

\[ \text{Figure 2. Stars of Interior Vertices of } Q \]
However, Macaulay2 \cite{11} computes the following decomposition of \(1 \in C^0_2(Q)\).

\[
\begin{array}{c}
1 & 1 & 1 \\ 1 & 1 & 1 \\
\frac{x+y}{2} & \frac{x+y}{2} & 0 \\
0 & 0 & 0 \\
\end{array}
\quad = \quad
\begin{array}{c}
-x+1 & 0 & -x+1 \\ x+1 & 0 & -x+1 \\
\frac{x+y}{2} & \frac{x+y}{2} & 0 \\
0 & 0 & 0 \\
\end{array}
\quad + \quad
\begin{array}{c}
0 & -y+1 \\ -y+1 & 0 \\
\frac{-x+y}{2} & \frac{-x+y}{2} & 0 \\
0 & 0 \\
\end{array}
\quad + \quad
\begin{array}{c}
0 & 0 \\ 0 & 0 \\
\frac{-x+y}{2} & \frac{-x+y}{2} & 0 \\
0 & 0 \\
\end{array}
\]

**Figure 3.** A ‘local’ decomposition of \(1 \in C^0_2(Q)\)

The support of the first spline in this sum is the annular subcomplex in Figure 4.

According to Theorem 4.7, \(C^r_d(Q)\) has a basis of splines supported on either the star of an interior vertex or one of the shaded complexes in Figure 4, for \(d \gg 0\) (see Example 4.10). We call such complexes *lattice complexes*, explained in §2.

**Figure 4.** Lattice Complexes of \(Q\)

It is the presence of such an annular subcomplex in Figure 4 that contributes to the constant term of the dimension formula \(\dim_R C^r_d(Q)\) for \(d \gg 0\) provided in \cite{14}. So the lattice complexes describe in §2 encode subtle interactions between the geometry and combinatorics of \(P\) that are manifested in \(C^r(P)\).

**Figure 5.** \(Q'\)

If we disturb the symmetry of \(Q\) slightly to get \(Q'\) as in Figure 5, then the affine spans of the four edges connecting the inner and outer squares do not all intersect at the same point. By Theorem 4.7, \(C^r_d(Q')\) has a basis of splines with support in either the star of an interior vertex or one of the shaded complexes in Figure 6 for \(d \gg 0\) (see Example 4.10).
2. LATTICE COMPLEXES

We begin with some preliminary notions. A polytopal complex \( P \subset \mathbb{R}^n \) is a finite set of convex polytopes (called faces of \( P \)) in \( \mathbb{R}^n \) such that

- If \( \gamma \in P \), then all faces of \( \gamma \) are in \( P \).
- If \( \gamma, \tau \in P \) then \( \gamma \cap \tau \) is a face of both \( \gamma \) and \( \tau \) (possibly empty).

The dimension of \( P \) is the greatest dimension of a face of \( P \). The faces of \( P \) are ordered via inclusion; a maximal face of \( P \) is called a facet of \( P \), and \( P \) is said to be pure if all facets are equidimensional. \( |P| \) denotes the underlying space of \( P \).

\( P_i \) and \( P_i^0 \) denote the set of \( i \)-faces and the set of interior \( i \)-faces, respectively. In the case that all facets of \( P \) are simplices, \( P \) is a simplicial complex and will be denoted by \( \Delta \).

Given a complex \( P \) and a face \( \gamma \in P \), the star of \( \gamma \) in \( P \), denoted \( \text{st}_P(\gamma) \), is defined by

\[
\text{st}_P(\gamma) := \{ \psi \in P | \exists \sigma \in P, \psi \in \sigma, \gamma \in \sigma \}.
\]

This is the smallest subcomplex of \( P \) which contains all faces which contain \( \gamma \). If the complex \( P \) is understood we will write \( \text{st}(\gamma) \).

For \( P \subset \mathbb{R}^n \), \( G(P) \) is a graph, with a vertex for every facet (element of \( P_n \)); two vertices are joined by an edge iff the corresponding facets \( \sigma \) and \( \sigma' \) satisfy \( \sigma \cap \sigma' \in P_{n-1} \). \( P \) is said to be hereditary if \( G(\text{st}_P(\gamma)) \) is connected for every nonempty \( \gamma \in P \). Throughout this paper, \( P \subset \mathbb{R}^n \) is assumed to be a pure, \( n \)-dimensional, hereditary polytopal complex.

Let \( R = \mathbb{R}[x_1, \ldots, x_n] \) be the polynomial ring in \( n \) variables. For a subset \( S \subset \mathbb{R}^n \), let \( I(S) \subset R \) denote the ideal of polynomials vanishing on \( S \). If \( \tau \in P_{n-1} \) then \( l_{\tau} \) denotes any linear form generating the principal ideal \( I(\tau) \).

In what follows we use a subgraph \( G_J(P) \) of \( G(P) \) determined by an ideal \( J \subset R \). This is a slight generalization of a graph used by McDonald and Schenck in [14].
Definition 2.1. Let $\mathcal{P} \subset \mathbb{R}^n$ be a polyhedral complex, and $J \subset R$ an ideal. The vertices of the graph $G_J(\mathcal{P})$ correspond to facets of $\mathcal{P}$ having a codimension one face $\tau$ such that $l_\tau \in J$. Two vertices corresponding to facets $\sigma_1$ and $\sigma_2$ which intersect along the edge $\tau$ are connected in $G_J(\mathcal{P})$ if $l_\tau \in J$.

Let $G_J(\mathcal{P})$ be the union of $k$ connected components $G^1_J(\mathcal{P}), \ldots, G^k_J(\mathcal{P})$. There is a unique subcomplex $\mathcal{P}^j_1$ of $\mathcal{P}$ whose dual graph is $G^j_J(\mathcal{P})$.

Definition 2.2. With notation as above, define $\mathcal{P}_J$ to be the disjoint union of $\mathcal{P}^1_1, \ldots, \mathcal{P}^k_J$.

We call the $\mathcal{P}^j_J$ the connected components, or simply components, of $\mathcal{P}_J$, although two components may share codimension one faces within $\mathcal{P}$ as will be apparent in Example 2.5. There are only finitely many distinct complexes $\mathcal{P}_J$ associated to $J \subset R$. They are in bijection with the nontrivial elements of the intersection poset of a certain hyperplane arrangement, which we now describe.

Recall that a hyperplane arrangement $\mathcal{H} \subset \mathbb{R}^n$ is a finite set $\mathcal{H} = \{H_1, \ldots, H_k\}$ of hyperplanes. The intersection poset $L(\mathcal{H})$ of $\mathcal{H}$ includes the whole space, the hyperplanes $H_i$, and nonempty intersections of these hyperplanes (called flats) ordered with respect to reverse inclusion. $L(\mathcal{H})$ is a ranked poset with rank function $\text{rk}(W) = \text{codim}(W)$ for $W \in L(\mathcal{H})$. $L(\mathcal{H})$ is a meet semilattice and is a lattice iff $\mathcal{H}$ is central, that is, iff $\cap_i H_i \neq \emptyset$.

Definition 2.3. Let $\mathcal{P} \subset \mathbb{R}^n$ be a polyhedral complex.

1. For $\tau \in \mathcal{P}$ a face, $\text{aff}(\tau)$ denotes the linear (or affine) span of $\tau$.
2. $\mathcal{A}(\mathcal{P})$ denotes the hyperplane arrangement $\bigcup_{\tau \in \mathcal{P}^0_{n-1}} \text{aff}(\tau)$.
3. $L_\mathcal{P}$ denotes the intersection semi-lattice $L(\mathcal{A}(\mathcal{P}))$ of $\mathcal{A}(\mathcal{P})$.

Lemma 2.4. For every ideal $J \subset R$, there is a unique $W \in L_\mathcal{P}$ so that $\mathcal{P}_J = \mathcal{P}_{I(W)}$. Furthermore, the ideal $I(W)$ is minimal with respect to $\mathcal{P}_{I(W)} = \mathcal{P}_J$.

Proof. Set $W = \cap_{\tau \in (\mathcal{P}_J)^{0}_{n-1}} \text{aff}(\tau)$. Clearly $\mathcal{P}_{I(W)} = \mathcal{P}_J$. To prove minimality, let $Q$ be any ideal satisfying $\mathcal{P}_{Q} = \mathcal{P}_J$. Then all codim 1 faces $\tau \in (\mathcal{P}_Q)^0_{n-1}$ satisfy $l_\tau \in Q$. Since $(\mathcal{P}_Q)^{0}_{n-1} = (\mathcal{P}_J)^{0}_{n-1}$, $I(W) \subset Q$. To show uniqueness, assume $V \in L_\mathcal{P}$ and $\mathcal{P}_{I(V)} = \mathcal{P}_J$. By minimality of $I(W)$, $I(W) \subset I(V)$, implying $V \subset W$. If $V \subset W$, then there is some $\tau \in (\mathcal{P}_J)^0_{n-1}$ so that $V \subset \text{aff}(\tau)$. But then $\tau$ is an interior edge of $\mathcal{P}_{I(V)}$ that is not an interior edge of $\mathcal{P}_J$, a contradiction. So $V = W$. □

For brevity, henceforth we write $G_S(\mathcal{P})$ and $\mathcal{P}_S$ to denote $G_{I(S)}(\mathcal{P})$ and $\mathcal{P}_{I(S)}$ for $S \subset R^n$.

Example 2.5. The planar polytopal complex $Q$ from the introduction is shown in Figure 7 along with its associated line arrangement $\mathcal{A}(Q)$ and a representative sample of the complexes $Q_W$ for $W \neq \emptyset \in L_Q$. We label the interior edges of $Q$ by 1, ..., 8 and denote their affine spans by $L1, \ldots, L8$. For each complex $Q_W$ the facets are shaded and the corresponding flat $W$ is labelled. If $Q_W$ is the disjoint union of several subcomplexes $Q_{W_l}$, we display these subcomplexes separately.

2.1. The central case and homegenization. The case where $\mathcal{P}$ is a central complex, i.e. $\mathcal{A}(\mathcal{P})$ is a central arrangement, is of particular interest for splines since then $\mathcal{C}^e(\mathcal{P})$ is a graded $R$-algebra. All the hyperplanes of $\mathcal{A}(\mathcal{P})$ pass through the origin (perhaps after a coordinate change), so we can remove the origin and consider the projective arrangement $\mathcal{P}\mathcal{A}(\mathcal{P}) \subset \mathbb{P}^{n-1}_R$ obtained by quotienting under
the action of \( \mathbb{R}^* \) by scalar multiplication. The intersection poset \( L(\mathbb{P}, A(\mathcal{P})) \) is identical to \( L_{\mathcal{P}} \) except it may not contain the maximal flat of \( L_{\mathcal{P}} \) (if that flat was the origin).

One of the most important central complexes for splines is the homogenization \( \hat{\mathcal{P}} \subset \mathbb{R}^{n+1} \) of a polytopal complex \( \mathcal{P} \subset \mathbb{R}^n \). \( \hat{\mathcal{P}} \subset \mathbb{R}^{n+1} \) is constructed by taking the join of \( i(\mathcal{P}) \) with the origin in \( \mathbb{R}^{n+1} \), where \( i: \mathbb{R}^n \to \mathbb{R}^{n+1} \) is defined by \( i(a_1, \ldots, a_n) = (1, a_1, \ldots, a_n) \). \( C^r(\hat{\mathcal{P}}) \) is a graded algebra whose \( d \)th graded piece \( C^r(\hat{\mathcal{P}})_d \) is a vector space isomorphic to \( C^r_d(\mathcal{P}) \) (see [6]). If we regard \( x_0, \ldots, x_n \) as

\[ Q \]

\[ A(Q) \]

\[ Q_{L1} \]

\[ Q_{L5} \]

\[ Q_{L5}^1 \]

\[ Q_{L5}^2 \]

\[ Q_w \]

\[ Q_w^1 \]

\[ Q_w^2 \]

\[ Q_\xi \]

Figure 7. Lattice Complexes of Example 2.5


coordinate functions on $\mathbb{R}^{n+1}$, then we obtain the original complex $\mathcal{P}$ from $\hat{\mathcal{P}}$ by setting $x_0 = 1$.

**Remark:** We associate a subcomplex $\mathcal{P}_W \subset \mathcal{P}$ to $W \in L_{\hat{\mathcal{P}}}$ by slicing the complex $\hat{\mathcal{P}}_W$ with the hyperplane $x_0 = 1$. Note that the subcomplex $\hat{\mathcal{P}}_W$ is the cone over the subcomplex $\mathcal{P}_W$. The subcomplexes $\mathcal{P}_W \subset \mathcal{P}$ obtained this way are the same as those obtained by first embedding $\mathcal{P}$ in $\mathbb{P}^n_{\mathbb{R}}$ by adding the hyperplane at infinity, taking the arrangement of hyperplanes $\mathcal{A}(\mathcal{P})$ in $\mathbb{P}^n_{\mathbb{R}}$ (including intersections in the hyperplane at infinity), and forming the complexes $\mathcal{P}_W$ for flats $W$ in this projective arrangement.

In Figure 8 we show the arrangement $\mathbb{P}\mathcal{A}(\hat{\mathcal{Q}})$ for the complex $\mathcal{Q}$ in Example 2.5. The lattice $L_{\hat{\mathcal{Q}}}$ has two rank 2 flats $\alpha$ and $\beta$ which do not appear in $L_{\mathcal{Q}}$, corresponding to the intersections of the two pairs of parallel lines $L_1, L_3$ and $L_2, L_4$ in $\mathbb{P}^2$. The complexes $Q_\alpha, Q_\beta$, also depicted in $\mathbb{P}^2$, are identical up to rotation.

2.2. The simplicial case. We close this section by showing that the complexes $\mathcal{P}_W$ reduce to unions of stars of faces when $\mathcal{P} = \Delta$ is a pure $n$-dimensional hereditary simplicial complex, as Proposition 2.8 shows. We use the following lemma.

**Lemma 2.6.** Let $\Delta \subset \mathbb{R}^n$ be an $n+1$-simplex and $\sigma_1, \sigma_2 \in \Delta$. Then $\text{aff}(\sigma_1) \cap \text{aff}(\sigma_2) = \text{aff}(\sigma_1 \cap \sigma_2)$. This includes the case $\text{aff}(\sigma_1) \cap \text{aff}(\sigma_2) = \emptyset$, assuming $\text{aff}(\emptyset) = \emptyset$.

**Proof.** $\Delta$ is the convex hull of $n+1$ vertices $\{v_0, \ldots, v_n\}$. Let $\Delta_i$ be the convex hull of $\{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$, the codimension one face of $\Delta$ determined by leaving out vertex $v_i$. It has supporting hyperplane $H_{v_i}$, the affine hull of all vertices except $v_i$. A $k$-dimensional face $\tau$ of $\Delta$ determined by $\{v_{i_0}, \ldots, v_{i_k}\}$ has $\text{aff}(\tau) = \bigcap_{v \in \Delta_0 \setminus \{v_i\}} H_v$. Now if $\sigma_1, \sigma_2 \in \Delta$,

$$\text{aff}(\sigma_1) \cap \text{aff}(\sigma_2) = \bigcap_{v \in (\Delta_0 \setminus \{\sigma_1\}) \cup (\Delta_0 \setminus \{\sigma_2\})} H_v = \bigcap_{v \in \Delta_0 \setminus (\sigma_1 \cap \sigma_2)} H_v = \text{aff}(\sigma_1 \cap \sigma_2)$$

□

**Lemma 2.7.** Let $\Delta \subset \mathbb{R}^n$ be a pure $n$-dimensional hereditary simplicial complex, and $W \in \mathcal{A}(\Delta)$. Then each component of $\Delta_W$ has the form $\text{st}(\tau)$ for some $\tau \in \Delta$.
Proof. Let $G_W^1(\Delta)$ be a connected component of $G_W(\Delta)$ and $\Delta_W^1 \subset \Delta$ the corresponding complex. Let $\Delta_W^1$ have facets $\sigma_1, \ldots, \sigma_k$, set $V_i = \bigcap_{\tau \in \{\sigma_{i-1}, \ldots, \tau, \ldots, \sigma_k\}} \text{aff}(\tau)$, $V = \bigcap_{\tau \in \Delta_W^1} \text{aff}(\tau) = \bigcap_{i=1}^k V_i$. By applying Lemma 2.6 iteratively, $V_i = \text{aff}(\gamma_i)$ for some face $\gamma_i \in \sigma_i$. Now let $K$ be a walk of length $m+1$ on the graph $G_W^1$ with the corresponding sequence of facets and codimension 1 faces of $\Delta_W^1$ being $S = \{\sigma_j, \sigma_j \cap \sigma_{j-1}, \ldots, \sigma_{j-m}, \sigma_{j-m}\}$, where $\sigma_j = \sigma_{j-1} \cap \sigma_j$ is a codimension 1 face of $\sigma_{j-1}$ and $\sigma_j$. Set $\beta_j = \gamma_j$, $\beta_{j-1} = \bigcap_{i=0}^{j-1} \gamma_i$. We prove $\bigcap_{i=0}^j V_i = \text{aff}(\beta_j)$, for $c = 0, \ldots, m$ by induction. We already have $V_0 = \text{aff}(\gamma_0) = \text{aff}(\beta_0)$. Assume $\bigcap_{i=0}^c V_i = \text{aff}(\beta_c)$. $\beta_j$ is a face of $\gamma_{j+1}$, which in turn is a face of $\tau_{j+1}$, since $\tau_{j+1}$ is a codimension 1 face of $\sigma_{j+1}$ that contains $W$. So $\beta_{j+1}$ is a face of $\sigma_{j+1}$. By Lemma 2.6, $\text{aff}(\beta_c) \cap \text{aff}(\gamma_{j+1}) = \text{aff}(\beta_c \cap \gamma_{j+1})$. Putting everything together, we have $\bigcap_{i=0}^{c+1} V_i = (\bigcap_{i=0}^c V_i) \cap V_{j+1} = \text{aff}(\beta_{j+1}) \cap \text{aff}(\gamma_{j+1}) = \text{aff}(\beta_{j+1})$.

Setting $\tau = \beta_m$ and noting that $V = \bigcap_{i=1}^k V_i = \bigcap_{i=0}^m V_i$, we have $V = \text{aff}(\tau)$. By construction $\tau$ is a face of $\sigma_i$ for $i = 1, \ldots, k$, hence $\Delta_W^1 \subset \text{st}(\tau)$. On the other hand, since $W \subset \text{aff}(\tau)$, $\text{st}(\tau) \subset \Delta_W$. $\Delta$ is hereditary, so $G(\text{st}(\tau))$ is connected and $\text{st}(\tau)$ is a component of $\Delta_W$. Hence $\Delta_W^1 = \text{st}(\tau)$. \hfill $\square$

We indicate precisely which stars appear in $\Delta_W$, following notation of Billera and Rose [7].

Proposition 2.8. Let $\Delta$ be a pure simplicial complex and $W \in L_\Delta$. Set

$$S(W) = \{\tau \in \Delta | W \subset \text{aff}(\tau) \text{ and } \tau \text{ is minimal with respect to this property}\}.$$  

Then

$$\Delta_W = \bigcup_{\tau \in S(W)} \text{st}(\tau)$$

Proof. Let $\Delta_W^1$ be a component of $\Delta_W$. By Lemma 2.7, $\Delta_W^1 = \text{st}(\tau)$ for some $\tau \in \Delta$. Clearly $W \subset \text{aff}(\tau)$. If $\tau$ is not minimal with respect to this property, then there is some $\gamma$ a proper face of $\tau$ so that $W \subset \text{aff}(\gamma)$. But then $\text{st}(\tau) \subset \gamma \subset \Delta_W$, contradicting that $\text{st}(\tau)$ is a component of $\Delta_W$. So $\tau \in S(W)$. Now suppose $\tau \in S(P)$. Clearly $\text{st}(\tau) \subset \Delta_W$ and $G(\text{st}(\tau))$ is connected since $\Delta$ is hereditary. Hence $\text{st}(\tau)$ is contained in a component $\Delta_W^1$ of $\Delta_W$. By Lemma 2.7, $\Delta_W^1 = \text{st}(\gamma)$ for some $\gamma \in \Delta$. We may assume $\tau$ is the intersection of the simplices contained in $\text{st}(\gamma)$, implying $\gamma$ is a face of $\tau$. But $W \subset \text{aff}(\gamma)$ and $\tau \in S(W)$, so $\tau = \gamma$. \hfill $\square$

3. Localization of $C^r(P)$

Our objective in this section is to give an explicit description of $C^r(P)_P$ for any prime $P \subset R$, using the complexes $P_W$ defined in the previous section. We first recall the definition of $C^r(P)$. For $U \in \mathbb{R}^n$, let $C^r(U)$ denote the set of functions $F : U \to \mathbb{R}$ continuously differentiable of order $r$. For $F : |P| \to \mathbb{R}$ a function and $\sigma \in P$, $F_{|\sigma}$ denotes the restriction of $F$ to $\sigma$. The module $C^r(P)$ of piecewise polynomials continuously differentiable of order $r$ on $P$ is defined by

$$C^r(P) := \{F \in C^r(|P|) | F_{|\sigma} \in R \text{ for every } \sigma \in P_n\}$$

The polynomial ring $R$ includes into $C^r(P)$ as globally polynomial functions (these are the trivial splines); this makes $C^r(P)$ an $R$-algebra via pointwise multiplication.

Given a pure $n$-dimensional polytopal complex $P$, the boundary complex of $P$ is a pure $(n - 1)$-dimensional complex denoted by $\partial P$. For a pure $n$-dimensional subcomplex $Q \subset P$, not necessarily hereditary, we use the following notation.
We will assume this throughout the paper.

Observation: $C^r_Q(P) = C^r(Q)_P$ for any prime $P$ such that $L_{\partial Q} \notin P$. \((*)\)

Let $Q_1, \ldots, Q_k$ be a partition of $\partial Q$ if the facets of $Q_1, \ldots, Q_k$ partition the facets of $\partial P$.

Lemma 3.1. Let $Q$ and $O$ be two polyhedral subcomplexes which partition $P$. Let $P$ be a prime of $R$ such that $L_{\partial Q} \notin P$ and $L_{\partial O} \notin P$. Then

$$C^r(P)_P = C^r(Q)_P + C^r(O)_P$$

as submodules of $\oplus_{\sigma \in P} R_P$. More precisely,

$$(C^r_Q(P) + C^r_O(P))_P = C^r(Q)_P + C^r(O)_P.$$  

Proof. It is clear that $C^r(Q) \cap C^r(O) = 0$ and $C^r_Q(P) \cap C^r_O(P) = 0$ in $\oplus_{\tau \in P} R$ since $Q$ and $O$ have no common facets. So both $C^r_Q(P) + C^r_O(P)$ and $C^r(Q) + C^r(O)$ are internal direct sums. The result follows from \((*)\). \qed

Corollary 3.2. Let $Q_1, \ldots, Q_k$ be a partition of $P$ into polyhedral subcomplexes and $P \subset R$ a prime such that $L_{\partial Q_i} \notin P$ for $i = 1, \ldots, k$.

$$C^r(P)_P = \bigoplus_{i=1}^k C^r(Q_i)_P$$

as submodules of $\oplus_{\sigma \in P_n} R$.

Proof. Apply Corollary 3.1 iteratively. \qed

Corollary 3.3. Let $P$ be a prime of $R$ so that $l_\tau \notin P$ for every edge $\tau \in P_0$. Then

$$C^r(P)_P = \bigoplus_{\sigma \in P_n} R_P$$

Proof. Apply Corollary 3.2 to the partition of $P$ into individual facets. This yields

$$C^r(P)_P = \bigoplus_{\sigma \in P_n} C^r(\sigma)_P$$

Since $C^r(\sigma) = R$, we are done. \qed

Now let $I \subset R$ be an ideal and $P_I \subset P$ be the subcomplex defined in the previous section.

Definition 3.4. Let $P_I$ have components $P_I^1, \ldots, P_I^k$. Define

$$C^r(P_I) := \bigoplus_{i=1}^k C^r(P_I^i).$$
Proposition 3.5. Let $P \subset R$ be a prime ideal, and $\mathcal{P} \subset \mathbb{R}^n$ a polyhedral complex. Then

$$C^r(\mathcal{P})_P = C^r(\mathcal{P}_P)_P \oplus \bigoplus_{\sigma \in \mathcal{P}_n \setminus (\mathcal{P}_P)_n} R_P$$

where $W$ is the unique flat of $L_P$ so that $\mathcal{P}_W = \mathcal{P}_P$ guaranteed by Lemma 2.7.

Proof. Consider the partition of $\mathcal{P}$ determined by $\mathcal{P}_P$ and $\mathcal{Q}$, where $\mathcal{Q}$ is the subcomplex of $\mathcal{P}$ generated by all facets $\sigma \in \mathcal{P}_n \setminus (\mathcal{P}_P)_n$. By the construction of $\mathcal{P}_P$, $\tau \in (\mathcal{P}_P)_{n-1}^{0} \iff \ell_\tau \in P$. Hence if $\ell_\tau \in \partial \mathcal{P}_P \setminus \partial \mathcal{P} = \partial \mathcal{Q} \setminus \partial \mathcal{P}$ then $\ell_\tau \not\in P$. Applying Corollary 3.1 we have

$$C^r(\mathcal{P})_P = C^r(\mathcal{P}_P)_P \oplus C^r(\mathcal{Q})_P$$

Again since all $\tau \in \mathcal{P}_{n-1}^{0}$ such that $\ell_\tau \in P$ are interior codim 1 faces of $\mathcal{P}_P$, there is no $\tau \in \mathcal{Q}_{n-1}^{0}$ such that $\ell_\tau \in P$. Applying Corollary 3.3 to $C^r(\mathcal{Q})_P$ gives the result. \square

We get the following result of Billera and Rose, used in the proof of Theorem 2.3 in [22], as a corollary of Proposition 3.5 and Proposition 2.8.

Corollary 3.6. Let $\Delta$ be a simplicial complex and $W \in L_\Delta$. Define $S(W)$ as in Proposition 2.8. Then

$$C^r(\Delta)_P = \bigoplus_{\tau \in S(W)} C^r(st(\tau))_P,$$

where $W \in L_\Delta$ is the unique flat so that $\Delta_P = \Delta_W$.

Note that if a facet $\sigma$ is in $S(W)$ then it is a not a facet of $\Delta_W$. Since $C^r(\sigma) = R$, the sum $\bigoplus_{\tau \in S(W)} C^r(st(\tau))_P$ appearing in Theorem 3.5 is implicit in the sum above.

3.1. Relation to results of Billera-Rose and Yuzvinsky. As an application of Proposition 3.5 we prove a slight variant of a result of Yuzvinsky [22] which reduces computation of the projective dimension of $C^r(\mathcal{P})$ to the central case. Recall that if $M$ is a module over the polynomial ring $R$, a finite free resolution of $M$ of length $r$ is an exact sequence of free modules

$$F_\bullet : 0 \to F_r \xrightarrow{\phi_r} F_{r-1} \xrightarrow{\phi_{r-1}} \cdots \xrightarrow{\phi_1} F_0$$

such that $\text{coker } \phi_1 = M$. The Hilbert syzygy theorem guarantees that $M$ has a finite free resolution. The projective dimension of $M$, denoted $\text{pd}(M)$, is the minimum length of a finite free resolution. If $M$ is a graded $S$-module with $\text{pd}(M) = \delta$ then $M$ has a minimal free resolution $F_\bullet \to M$ of length $\delta$, unique up to graded isomorphism. This resolution is characterized by the property that the entries of any matrix representing the differentials $\phi_i$ in $F_\bullet$ are contained in the homogeneous maximal ideal $(x_1, \ldots, x_n)$.

In [22], Yuzvinsky introduces a poset $L$, different from $L_\mathcal{P}$, associated to a polytopal complex. He defines subcomplexes associated to each $x \in L$ and uses them to reduce the characterization of projective dimension to the graded case (Proposition 2.4). Proposition 3.5 is the analog for $L_P$ of Lemma 2.3 in [22], and we use it to prove the following statement.
Theorem 3.7. Let $P \subset \mathbb{R}^n$ be a polytopal complex. Then

1. $\text{pd}(C^r(P)) \geq \text{pd}(C^r(P_W))$ for all $W \in L_P$.
2. $\text{pd}(C^r(P)) = \max_{W \in L_P} \text{pd}(C^r(P_W))$. In particular, $C^r(P)$ is free if and only if $C^r(P_W)$ is free for all nonempty $W \in L_P$.

Proof. We use the following two facts about projective dimension. Here $M$ is any $R$-module, not necessarily graded.

(A) For any prime $P \subset R$, $\text{pd}(M) \geq \text{pd}(M_P)$.

(B) $\text{pd}(M) = \max_{P \in \text{Spec } R} \text{pd}(M_P)$.

(A) follows from fact (A) above and the second follows from showing that there are primes which preserve $\text{pd}(M)$ under localization. Set $\text{pd}(M) = r$.

We have $\text{Ext}_P^r(M,R) \neq 0$ and taking any prime $P$ in its support will suffice. For such a prime, we have $\text{Ext}_P^r(M_P,R_P) \cong \text{Ext}_P^r(M,R) \otimes_R R_P \neq 0$. Hence $\text{pd}(M_P) \geq r$. Since we already have $\text{pd}(M) \leq r$, $\text{pd}(M_P) = r$. (1) Observe that $C^r(P_W)$ is graded with respect to $I(\xi)$ for any point $\xi \in W$ since the affine span of any interior codimension 1 face of $P_W$ contains $W$, hence also contains $\xi$. Choose $\xi \in W \setminus \cup_{V \subset W} V$, where $V \in L_P$. Then $P_{\xi} = P_W$. We have

$$\text{pd}(C^r(P)) \geq \text{pd}(C^r(P)_{I(\xi)}) = \text{pd}(C^r(P)_{I(\xi)}) = \text{pd}(C^r(P_W)_{I(\xi)})$$

where the first equality follows from fact (A) above and the second follows from Theorem 3.5, since $C^r(P)_{I(\xi)}$ is the direct sum of a free module and $C^r(P)_{I(\xi)}$. $C^r(P_W)$ is graded with respect to $I(\xi)$, so a minimal resolution of $C^r(P_W)$ has differentials with entries in $I(\xi)$. It follows that this remains a minimal resolution under localization with respect to $I(\xi)$, so $\text{pd}(C^r(P_W)_{I(\xi)}) = \text{pd}(C^r(P_W))$, and the result follows. (2) Set $m = \max_{W \in L_P} \text{pd}(C^r(P_W))$. From (1) $\text{pd}(C^r(P)) \geq m$. For any prime $P \subset R$ we have $P_P = P_W$ for some $W \in L_P$ by Lemma 2.4 so

$$\text{pd}(C^r(P)_P) = \text{pd}(C^r(P_W)_P) \leq \text{pd}(C^r(P_W)) \leq m.$$ 

Hence $\text{pd}(C^r(P)) \leq m$ from fact (B) above, and $\text{pd}(C^r(P)) = m$ as desired. $\square$

Since the $P_W$ are central complexes, this reduces computation of $\text{pd}(C^r(P))$ to the central case. Via Proposition 2.8 we obtain the following theorem of Billera and Rose as a corollary to Theorem 3.7. Recall an $R$-module $M$ is free if $\text{pd}(M) = 0$.

Theorem 3.8. [2.3 of [2]] Let $P \subset \mathbb{R}^n$ be a polytopal complex. Then

1. If $C^r(P)$ is free over $R$ then $C^r(st(\tau))$ is free over $R$ for all $\tau \neq \emptyset \in P$.
2. If $P = \Delta$ is simplicial then the converse is also true: if $C^s(st(\sigma))$ is free for all nonempty $\sigma \in \Delta$, then $C^s(\Delta)$ is free.

4. Lattice-Supported Splines

Our main application of Proposition 3.5 is to construct “locally supported approximations” to $C^r(P)$ which allow us to generalize the notion of a star-supported basis of $C^r_\rho(P)$ in the simplicial case. In particular we show that a basis of $C^r_\rho(P)$ consisting of splines supported on complexes of the form $P_W$ always exists for $d \geq 0$ as long as we allow $W \in L_P$.

Recall for $Q$ a pure $n$-dimensional connected subcomplex of $P \subset \mathbb{R}^n$, we defined $C^r_{\infty}(P)$ to be the set of splines vanishing outside of $Q$. Also recall that $\hat{P} \subset \mathbb{R}^{n+1}$ is
the central complex obtained by coning over $P$ in $\mathbb{R}^{n+1}$. In the remark in subsection 2.1, we defined subcomplexes $P_W \subset P$ for $W \in \hat{L}_P$ by slicing the subcomplex $\hat{P}_W$ with the hyperplane $x_0 = 1$. We make the following definitions.

**Definition 4.1.** For $W \in L_P$ let $P_W$ have components $P^1_W, \ldots, P^m_W$.

1. $C^r_W(P) := \bigoplus_{i=1}^m C^r_{P^i_W}(P)$
2. $L^{s,r,k}(P) := \sum_{W \in L_P} C^r_W(P)$

$C^r_W(P)$ is the submodule of $C^r(P)$ generated by splines which are 0 outside of a component of $P_W$. We say that elements of $C^r_W(P)$ are supported at $W$. If a spline $F$ is supported at some $W \in L_P$ then we call $F$ lattice-supported. By Proposition 2.8 lattice-supported splines generalize the notion of star-supported splines [3] to polytopal complexes.

**Remark:** Results in this section which do not depend on grading (Proposition 4.2, Theorem 4.3, and Corollary 4.4) only require the sum in (2) to run over $W \in L_P$, not $W \in L_P$.

**Proposition 4.2.** Let $P \subset \mathbb{R}^n$ be a pure $n$-dimensional polytopal complex. If $P \subset R$ is a prime such that the unique $W \in L_P$ satisfying $P_W = P$, guaranteed by Lemma 2.4, has $\text{rk}(W) \leq k$, then $L^{s,r,k}(P)_P = C^r(P)_P$.

**Proof.** Let $P$ satisfy the given condition. Since $\text{rk}(W) \leq k$ the module $C^r_W(P)$ appears as a summand in $L^{s,r,k}(P)$. Note that $C^r_n(P) = \sum_{\sigma \in \mathcal{P}_n} C^r_\sigma(P)$. Define the submodule $N(P) \subset L^{s,r,k}(P)$ by $N(P) = C^r_W(P) + \sum_{\sigma \in \mathcal{P}_n \setminus \{P\}_n} C^r_\sigma(P)$. The support of the summands of $N(P)$ is disjoint, so it is clear that this is an internal direct sum. Also, for $\sigma \notin (P_P)_n$, $C^r_\sigma(P)_P = R_P$. We have

$$N(P)_P = C^r_W(P)_P \oplus \bigoplus_{\sigma \in \mathcal{P}_n \setminus (P_P)_n} C^r_\sigma(P)_P = C^r(P)_P \oplus \bigoplus_{\sigma \in \mathcal{P}_n \setminus (P_P)_n} R_P = C^r(P)_P$$

where the second equality follows from applying observation (*) to the summands of $C^r_W(P)$ and the third equality follows from Theorem 3.5. Since $N(P) \subset L^{s,r,k}(P) \subset C^r(P)$ and $N(P)_P = C^r(P)_P$, we have $L^{s,r,k}(P)_P = C^r(P)_P$. □

**Theorem 4.3.** Let $P \subset \mathbb{R}^n$ be a polytopal complex. Then $L^{s,r,k}(P)$ fits into a short exact sequence

$$0 \rightarrow L^{s,r,k}(P) \rightarrow C^r(P) \rightarrow C \rightarrow 0$$

where $C$ has codimension $\geq k + 1$ and the primes in the support of $C$ with codimension $k + 1$ are contained in the set $\{I(W) | W \in L_P \text{ and } \text{rk}(W) = k + 1\}$.

Recall the support of $C$ is the set of primes $P \subset R$ satisfying $C_P \neq 0$.

**Proof.** The first statement follows from Proposition 4.2. Now suppose $C$ is supported at $P$ with codim $P = k + 1$, and let $W \in L_P$ be the unique flat so that $P_W = P$ (Lemma 2.4). If $\text{rk}(W) \leq k$ then $L^{s,r,k}(P)_P = C^r(P)_P$ by Proposition 4.2 and $C_P = 0$. So we must have $\text{rk}(W) = k + 1$ and $P = I(W)$. Hence the primes of codimension $k + 1$ in the support of $C$ are contained in $\{I(W) | W \in L_P \text{ and } \text{rk}(W) = k + 1\}$. □
Corollary 4.4. For $P \subset \mathbb{R}^n$, $\text{LS}^{r,n}(P) = C^r(P)$.

4.1. Graded Results. Let $R_{\leq d}$ and $R_d$ be the set of polynomials $f \in R = \mathbb{R}[x_1, \ldots, x_n]$ of degree $\leq d$ and degree $d$, respectively. For $P \subset \mathbb{R}^n$ we have a filtration of $C^r(P)$ by $\mathbb{R}$-vector spaces

$$C^r_d(P) := \{ F \in C^r(P) | F_\sigma \in R_{\leq d} \text{ for all facets } \sigma \in P_n \}.$$  

If $P$ is a central complex, $C^r(P)$ is graded by the vector spaces

$$C^r(P)_d := \{ F \in C^r(P) | F_\sigma \in R_d \text{ for all facets } \sigma \in P_n \}.$$  

In other words, $C^r(P) = \bigoplus_{d \geq 0} C^r_d(P)$. We have the following lemma due to Billera and Rose.

Lemma 4.5 (Theorem 2.6 of [1]). $C^r_d(P) \cong C^r_d(\hat{P})$ as $\mathbb{R}$-vector spaces.

The map $\phi_d : C^r_d(P) \to C^r_d(\hat{P})$ in Lemma 4.5 is given by homogenizing:

$$f(x_1, \ldots, x_n) \in C^r_d(P) \to x_0^df(x_1/x_0, \ldots, x_n/x_0) \in C^r_d(\hat{P}),$$  

while its inverse $\phi_d^{-1} : C^r_d(\hat{P}) \to C^r_d(P)$ is given by

$$F(x_0, \ldots, x_n) \in C^r_d(\hat{P}) \to F(1, x_1, \ldots, x_n) \in C^r_d(P).$$  

We define parallel filtrations and gradings for $\text{LS}^{r,k}(P)$. For $W \in L_{\hat{P}}$, define

$$C^r_W, d(P) := C^r_W(P) \cap C^r_d(P)$$  

$$C^r_W(P)_d := C^r_W(\hat{P}) \cap C^r(P)_d.$$  

Then $\text{LS}^{r,k}(P)$ has a filtration by vector spaces

$$\text{LS}^{r,k}_d(P) := \sum_{W \in L_{\hat{P}}, \atop 0 \leq \text{rk}(W) \leq k} C^r_W, d(P),$$  

which are subspaces of $C^r_d(P)$. If $P$ is a central complex, $\text{LS}^{r,k}(P)$ is graded by the vector spaces

$$\text{LS}^{r,k}(P)_d := \{ F \in \text{LS}^{r,k}(P) | F_\sigma \in R_d \text{ for all facets } \sigma \in P_n \}$$  

$$= \sum_{W \in L_{\hat{P}}, \atop 0 \leq \text{rk}(W) \leq k} C^r_W, (\hat{P})_d,$$  

which are subspaces of $C^r(\hat{P})_d$. If $P$ is central, the set of subcomplexes $P_W$ for $W \in L_{\hat{P}}$ is the same as the set of subcomplexes $P_W$ for $W \in L_P$ (there is a rank preserving isomorphism between $L_P$ and $L_{\hat{P}}$ in this case). Hence we may write

$$\text{LS}^{r,k}(P) = \sum_{W \in L_P, \atop 0 \leq \text{rk}(W) \leq k} C^r_W(\hat{P})_d$$  

Lemma 4.6. $\text{LS}^{r,k}_d(P)$ and $\text{LS}^{r,k}_d(\hat{P})$ are isomorphic as $\mathbb{R}$-vector spaces.
Proof. We show that the homogenization map \( \phi_d : C^r_d(\mathcal{P}) \to C^r(\hat{\mathcal{P}})_d \) restricts to an isomorphism between \( \mathcal{L}S_{r,k}^r(\mathcal{P}) \) and \( \mathcal{L}S_{r,k}^r(\hat{\mathcal{P}})_d \). We have

\[
\mathcal{L}S_{r,k}^r(\mathcal{P}) := \sum_{W \in L_{\hat{\mathcal{P}}}} C^r_{W,d}(\mathcal{P})
\]

\[
\mathcal{L}S_{r,k}^r(\hat{\mathcal{P}})_d = \sum_{W \in L_{\hat{\mathcal{P}}}} C^r_{W}(\hat{\mathcal{P}})_d
\]

Since \( \phi_d \) is \( \mathbb{R} \)-linear, it suffices to show that, given \( W \in L_{\hat{\mathcal{P}}} \), \( \phi_d \) restricts to an isomorphism \( \phi_d : C^r_{W,d}(\mathcal{P}) \to C^r_{W}(\hat{\mathcal{P}})_d \). We show \( \phi_d(C^r_{W,d}(\mathcal{P})) \subset C^r_{W}(\hat{\mathcal{P}})_d \) and \( \phi_d^{-1}(C^r_{W}(\hat{\mathcal{P}})_d) \subset C^r_{W,d}(\mathcal{P}) \). Suppose \( f \in C^r_{W,d}(\mathcal{P}) \). The support of \( f \) is by definition contained in the subcomplex \( \mathcal{P}_W \), so the support of \( \phi(f) \) is contained in the cone over \( \mathcal{P}_W \), which is precisely \( \hat{\mathcal{P}}_W \). It follows that \( \phi(f) \in C^r_{W}(\hat{\mathcal{P}})_d \). Since \( \phi_d(f) \in C^r(\hat{\mathcal{P}})_d \), \( \phi_d(f) \in (C^r_{W}(\hat{\mathcal{P}})_d \cap C^r(\hat{\mathcal{P}})_d = C^r_{W}(\hat{\mathcal{P}})_d \). Now suppose \( F \in C^r_{W}(\hat{\mathcal{P}})_d \). The support of \( F \) is contained in the subcomplex \( \mathcal{P}_W \), so the support of \( \phi^{-1}(F) \) is contained in the complex obtained by slicing \( \hat{\mathcal{P}}_W \) with the hyperplane \( x_0 = 1 \). But this by definition \( \mathcal{P}_W \), so \( \phi^{-1}(F) \in C^r_{W}(\mathcal{P}) \). Since \( \phi^{-1}(F) \in C^r_{W}(\mathcal{P}) \), \( \phi^{-1}(F) \in C^r_{W}(\mathcal{P}) \cap C^r_{W,d}(\mathcal{P})) = C^r_{W,d}(\mathcal{P}) \). □

For the following theorem, recall that the Hilbert function \( HF(M,d) \) of a finitely generated graded module \( M = \bigoplus_d M_d \) over \( R = \mathbb{R}[x_1, \ldots, x_n] \) is defined by \( HF(M,d) = \dim_{\mathbb{R}} M_d \) and the Hilbert polynomial \( HP(M,d) \) of \( M \) is the polynomial with which \( HF(M,d) \) agrees for \( d \gg 0 \).

**Theorem 4.7.** Let \( \mathcal{P} \subset \mathbb{R}^n \) be a polytopal complex.

1. If \( \mathcal{P} \) is central, the first \( k+1 \) coefficients of \( HP(C^r(\mathcal{P})) \) and \( HP(LS_{r,k}^r(\mathcal{P})) \) agree. In particular, \( LS_{r,n-1}^r(\mathcal{P})_d = C^r(\mathcal{P})_d \) for \( d \gg 0 \).

2. \( LS_{r,n}^r(\mathcal{P}) = C^r_{d}(\mathcal{P}) \) for \( d \gg 0 \). Equivalently, \( C^r_{d}(\mathcal{P}) \) has a basis consisting of lattice-supported splines for \( d \gg 0 \).

Proof. (1) Applying Theorem 4.3 to \( LS_{r,k}^r(\mathcal{P}) \) yields the short exact sequence

\[
0 \to LS_{r,k}^r(\mathcal{P}) \to C^r(\mathcal{P}) \to C \to 0,
\]

where \( code C \leq k+1 \). It follows that \( HP(C,d) \) has degree at most \( (n-1) - (k+1) \). On the other hand, \( HP(C^r(\mathcal{P}),d) \) has degree \( n-1 \). Since \( HP(C^r(\mathcal{P}),d) - HP(LS_{r,k}^r(\mathcal{P}),d) = HP(C,d) \), the first \( k+1 \) coefficients of \( HP(C^r(\mathcal{P}),d) \) and \( HP(LS_{r,k}^r(\mathcal{P}),d) \) agree. Now specialize to \( k = n-1 \). Then \( HP(C,d) = 0 \) so \( HP(C^r(\mathcal{P}),d) = HP(LS_{r,n-1}^r(\mathcal{P}),d) \), implying \( HF(C^r(\mathcal{P}),d) = HF(LS_{r,n-1}^r(\mathcal{P}),d) \) for \( d \gg 0 \). Since \( LS_{r,n-1}^r(\mathcal{P})_d \subset C^r(\mathcal{P})_d \), we have \( LS_{r,n-1}^r(\mathcal{P})_d = C^r(\mathcal{P})_d \) for \( d \gg 0 \). (2) From part (1), \( LS_{r,n}^r(\mathcal{P})_d = C^r(\mathcal{P})_d \) for \( d \gg 0 \). The result follows by applying Lemma 4.6 to the left hand side and Lemma 4.5 to the right hand side. □

**Example 4.8.** We give an example to highlight the difference between Corollary 4.4 and Theorem 4.7 part (2). Take the underlying complex to be the complex \( Q \) from Figure 1. In Figure 3 we show a decomposition for the unit in \( C^0(\mathcal{Q}) \), \( 1 = \sum_{j=1}^5 G_j \), with \( G_j \in C_2(\mathcal{Q}) \). The splines \( G_1, \ldots, G_5 \) have support in the subcomplexes \( \mathcal{Q}_W \) for \( W \in L_{\mathcal{Q}} \). Given any spline \( F \in C^0_d(\mathcal{Q}) \), \( \sum_{j=1}^5 G_j \cdot F \) gives
a decomposition of $F$ using lattice-supported splines in $C^{0}_{d+2}(Q)$. It follows that $C^{0}(Q) = \sum_{W \in L_{Q}} C^{0}_{W}(Q)$, without using the two complexes $Q_{\alpha}, Q_{\beta}$ of Figure 8 which correspond to intersections ‘at infinity.’ This is true in general; the statement of Corollary 4.4 can be changed to

$$\sum_{W \in L_{P}} C^{r}_{W}(P) = C^{r}(P),$$

without altering the proof.

However, if we want to write every spline $F \in C^{0}(Q)$ as a sum of lattice-supported splines of degree at most $d$, we must use the two complexes $Q_{\alpha}$ and $Q_{\beta}$. For example, a computation in Macaulay2 shows that the spline $x^{2} \cdot 1 \in C^{0}_{2}(Q)$ is not in the vector space $\sum_{W \in L_{P}} C^{0}_{W}(Q)$, while a decomposition for $x^{2} \cdot 1$ in $\sum_{W \in L_{P}} C^{0}_{W}(Q)$ is shown in Figure 9. Set $l_{1} = x + 1$, $l_{2} = y - 1$, $l_{3} = x - 1$, $l_{4} = y + 1$, $l_{5} = x - y$, $l_{6} = x + y$ below.

\[\begin{array}{ccc}
\begin{array}{ccc}
-\frac{l_{1}}{2} & 0 & 0 \\
\frac{l_{2}}{4} & l_{2} & 0 \\
2l_{2} & 0 & 0
\end{array}
\end{array}\]

\[\begin{array}{ccc}
\begin{array}{ccc}
0 & l_{4} & 0 \\
\frac{l_{5}}{4} & l_{5} & 0 \\
2l_{5} & 0 & 0
\end{array}
\end{array}\]

\[\begin{array}{ccc}
\begin{array}{ccc}
0 & \frac{l_{6}}{2} & 0 \\
\frac{l_{7}}{2} & \frac{l_{7}}{2} & 0 \\
\frac{l_{8}}{2} & \frac{l_{8}}{2} & 0
\end{array}
\end{array}\]

\[\begin{array}{ccc}
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}
\end{array}\]

Figure 9. Decomposition of $x^{2} \cdot 1 \in C^{0}_{2}(Q)$

In [12, 13] and [4], it is shown that in the bivariate simplicial case there exists a star-supported basis for $C^{r}_{d}(\Delta)$ if $d \geq 3r + 2$ and in the trivariate simplicial case for $d > 8r$. These bases are explicitly constructed using the Bernstein-Bezier method. Finding how large $d$ must be in order to obtain a lattice-supported basis for $C^{r}_{d}(P)$ is a difficult question. The results in the bivariate and trivariate simplicial case suggest that the vector spaces $LS^{r,n}_{2}(\hat{P})$ and $C^{n}_{d}(\hat{P})$ begin to agree at a value of $d$ which is a linear function of $r$. We address this in the planar polytopal case in §5 and relate this question to bounding the value of $d$ for which known dimension formulas for $C^{r}_{d}(P)$ hold. Before progressing to this, we show how the poset $L_{P}$ can be refined to give a cleaner description of $LS^{r,k}_{d}(P)$.

4.2. Refining posets for $LS^{r,k}_{d}(P)$. In this section we seek a minimal set of submodules of the form $C^{r}_{Q}(P)$ which generate $LS^{r,k}_{d}(P)$. Observe that if $Q \subseteq O$ are subcomplexes of $P$ then $C^{r}_{Q}(P) \subseteq C^{r}_{O}(P)$. So if there is containment among subcomplexes which are the support of the summands appearing in $LS^{r,k}_{d}(P)$ then we may discard one of the summands. This suggests that while $L_{P}$ is quite useful for describing localization of $C^{r}(P)$, there is a more useful poset to consider for understanding $LS^{r,k}_{d}(P)$. This is the poset $\Gamma_{P}$ whose elements are subcomplexes $Q$.
of \(P\) which are a component of \(P_W\) for some \(W \in L_{\tilde{\Gamma}}\), ordered by inclusion. As we will see, \(\Gamma_P\) may be quite different from \(L_{\tilde{\Gamma}}\).

Define a function \(f_t\) on \(\Gamma_P\) by

\[
f_t(Q) = \begin{cases} 
\text{codim} \left( \cap_{\tau \in Q^0_{n-1}} \text{aff}(\tau) \right) & \text{if } Q^0_{n-1} \neq \emptyset, \\
0 & \text{if } Q^0_{n-1} = \emptyset
\end{cases}
\]
or equivalently

\[
f_t(Q) = \min\{\text{rk}(V) | V \in L_{\tilde{\Gamma}}, Q \text{ a component of } P_V\}.
\]

\(f_t\) is increasing in the sense that if \(O \subseteq Q\) then \(f_t(O) < f_t(Q)\). We call \(f_t(O)\) the \(\Gamma\)-rank of \(O\). Let \(\Gamma^k_P\) be the poset formed by \(\{Q \in \Gamma | f_t(Q) \leq k\}\) and for a poset \(L\) let \(L^{\max}\) denote the maximal elements of \(L\). Then we have

**Proposition 4.9.** \(LS^{r,k}(P) = \sum_{Q \in \Gamma^r_{\max} P} C^r_Q(P)\).

**Proof.** \(C^r_W(P) = \sum_{W_{i=1}^n} C^r_{P_W}(P)\) by definition, where \(P_W, \ldots, P_W\) are the components of \(P_W\). If \(\text{rk}(W) \leq k\), then \(f_t(P_W) \leq k\) for all components \(P_{W_i}\) of \(P_W\). Hence \(P_W \in \Gamma^k_P\) and \(P_{W_i} \subset Q\) for some \(Q \in \Gamma^k_{\max}\). Since this holds for all components \(P_W\) of \(P_W\), \(C^r_W(P) \subset \sum_{Q \in \Gamma^r_{\max} P} C^r_Q(P)\) and we are done. \(\square\)

**Example 4.10.** Let \(Q\) and \(Q'\) be as in Figures 1 and 5. Label the facets of \(Q\) and \(Q'\) by \(A, B, C, D, E\) as shown in Figure 10. The Hasse diagrams of \(L_{\tilde{\Gamma}}\) and \(\Gamma_{Q}\) are shown in Figure 10 organized according to rank and \(\Gamma\)-rank, respectively. For \(L_{\tilde{\Gamma}}\), we use the labels assigned to the flats in Example 2.5. We label the complexes \(\Gamma_{\tilde{\Gamma}}\) and \(\Gamma_Q\) by listing the their facet labels. Figures 2, 4 show the complexes \(\Gamma_{\tilde{\Gamma}}\) as in Figures 1 and 5. Label the complexes \(\Gamma^{2,\max}_{\tilde{\Gamma}}\). Figure 6 shows the subcomplexes of \(\Gamma^{2,\max}_Q\) which are not stars of vertices.

By Proposition 4.9 and Theorem 4.7, \(C^r(Q)\) and \(C^r_{\max}(Q)\) have a basis of splines which vanish outside of the complexes \(\Gamma^r_{\max} \Gamma^r_{\max}\), respectively, for \(k \gg 0\). This proves the claims made in the Example 1.1.

Now suppose \(P = \Delta\) is simplicial. Proposition 2.8 shows that \(\Gamma_{\Delta}\) is the poset of stars of faces \(\tau\) so that \(\text{aff}(\tau)\) appears in \(L_{\Delta}\). Every star in \(\Gamma^r_{\max}\) is contained in the star of a face \(\tau \in \Delta_{n-k}\), so we obtain the following corollary to Proposition 4.9

**Corollary 4.11.** \(LS^{r,k}(\Delta) := \sum_{\tau \in \Delta_{n-k}} C^r_{\text{st}(\tau)}(\Delta)\).

Setting \(k = n\) and applying Theorem 4.7, we obtain the existence of a star-supported basis for \(C^r_{\Delta}(\Delta)\) in any dimension.

**Corollary 4.12.** Let \(\Delta \subseteq \mathbb{R}^n\) be a pure, \(n\)-dimensional, hereditary simplicial complex. Then \(C^r_{\Delta}(\Delta)\) has a basis consisting of splines supported on the star of a vertex for \(d \gg 0\).

5. Lattice-Supported Splines and the McDonald-Schenck Formula

In this section we address the question of when \(\dim_\mathbb{Q} C^r_{\tilde{\Gamma}}(P)\) becomes polynomial, particularly in the planar case where these polynomials have been computed by Alfeld, Schumaker, McDonald, and Schenck (2 and 14). Rephrased, this is a question about when the Hilbert function \(HF(C^r(P), d)\) of the graded module \(C^r(\tilde{\Gamma})\) agrees with the Hilbert polynomial \(HP(C^r(P), d)\). We give an indication
of how we may use lattice-supported splines to address this problem and also give conjectural bounds on $d$ for when $HF(C'(\hat{P}), d) = HP(C''(\hat{P}), d)$ in the case $P \subset \mathbb{R}^2$.

There is a convenient notion for discussing when the Hilbert function $HF(M, d)$ of a graded module $M$ over $R = \mathbb{R}[x_1, \ldots, x_n]$ agrees with the Hilbert polynomial
$HP(M, d)$, namely the regularity of $M$, denoted $\text{reg}(M)$. Let $F_\bullet \to M$ be a minimal free graded resolution of $M$, with $F_i = \bigoplus_j R(-a_{ij})$. Then

$$\text{reg}(M) := \max_{i, j} \{a_{ij} - i\}$$

The relevant facts about $\text{reg}(M)$ may be found in [10], chapter 4 and appendix A. Regularity relates to the Hilbert function becoming polynomial via Theorem 4.2 of [10]:

**Theorem 5.1.** Let $M$ be a finitely generated graded module over $S = K[x_0, x_1, \ldots, x_n]$ ($K$ a field) of projective dimension $\delta$. Then $HF(M, d) = HP(M, d)$ for $d \geq \text{reg}(M) + \delta - n$.

We apply this theorem to $C^r(\hat{P})$. First, since $\mathcal{P}$ is hereditary, $C^r(\hat{P})$ is the kernel of a matrix and hence a second syzygy (see [6]). $C^r(\hat{P})$ is a module over $S = \mathbb{R}[x_0, \ldots, x_n]$, so since $C^r(\hat{P})$ is a second syzygy, $\text{pd}(C^r(\hat{P})) \leq n - 1$. By Theorem 5.1, $HF(C^r(\hat{P}), d) = HP(C^r(\hat{P}), d)$ for $d \geq \text{reg}(C^r(\hat{P})) - 1$. It turns out that the Alfeld-Schumaker result for generic simplicial $\Delta$ conjectures a tightening of this bound, namely $HF(C^r(\hat{\Delta}), d) = HP(C^r(\hat{\Delta}), d)$ for $d \geq 3r + 1$ [2]. Schenck conjectures a tightening of this bound, namely $HF(C^r(\hat{\Delta}), d) = HP(C^r(\hat{\Delta}), d)$ for $d \geq 2r + 1$ [19]. We will call this the $2r + 1$ conjecture. This is equivalent to $\text{reg}(C^r(\hat{\Delta})) \leq 2r + 2$ (see [21] for the equivalence of these statements). We now relate $\text{reg}(C^r(\hat{P}))$ to $\text{reg}(LS^{r,n}(\hat{P}))$.

**Theorem 4.3** provides the short exact sequence

$$0 \to LS^{r,n}(\hat{P}) \to C^r(\hat{P}) \to C \to 0,$$

where $C$ has finite length. The following proposition identifies $C$ as a local cohomology module of $LS^{r,n}(\hat{P})$.

**Proposition 5.2.** Let $\mathcal{P} \subset \mathbb{R}^n$ be a pure hereditary polytopal complex and $C$ be the cokernel of the inclusion $LS^{r,n}(\hat{P}) \to C^r(\hat{P})$. Then

$$C \cong H^1_m(LS^{r,n}(\hat{P})),$$

where $H^1_m(LS^{r,n}(\hat{P}))$ is the first local cohomology module of $LS^{r,n}(\hat{P})$ with respect to $m = (x_0, \ldots, x_n)$, the homogeneous maximal ideal of $S = \mathbb{R}[x_0, \ldots, x_n]$.

**Proof.** If $M$ is a graded $S$-module, let $H^i_m(M)$ denote the $i$th local cohomology module of $M$ with respect to $m = (x_0, \ldots, x_n)$, $\tilde{M}(i)$ the associated twisted sheaf on $\mathbb{P}^n$, and $H^0(\tilde{M}(i))$ the vector space of global sections of $\tilde{M}(i)$. Define $\Gamma(M) = \bigoplus_i H^0(\tilde{M}(i))$. We have the four term exact sequence (see [10] Corollary A1.12)

$$0 \to H^0_m(M) \to M \to \Gamma(M) \to H^1_m(M) \to 0.$$

The graded modules $LS^{r,n}(\hat{P})$ and $C^r(\hat{P})$ determine the same sheaf since their localizations at nonmaximal primes agree by Theorem 4.3. Hence $\Gamma(\text{LS}^{r,n}(\hat{P})) = \Gamma(C^r(\hat{P}))$. Furthermore $\Gamma(C^r(\hat{P})) = \Gamma(C^r(\hat{P}))$. This is a consequence of the fact that $C^r(\hat{P})$ is a second syzygy. From this it follows that $\text{Ext}_{\mathbb{P}}^i(C^r(\hat{P}), S) = 0$ for $i = n, n + 1$ and hence $H^1_m(C^r(\hat{P})) = 0$ for $i = 0, 1$ by local duality ([10] Theorem 10.6). The four term exact sequence above then yields $C^r(\hat{P}) = \Gamma(C^r(\hat{P}))$. 

Putting this all together and using the fact that $H^0_m(LS^{r,n}(\hat{\mathcal{P}})) = 0$ since $LS^{r,n}(\hat{\mathcal{P}})$ has no submodule of finite length, we arrive at the short exact sequence

$$0 \to LS^{r,n}(\hat{\mathcal{P}}) \to C^r(\hat{\mathcal{P}}) \to H^1_m(LS^{r,n}(\hat{\mathcal{P}})) \to 0$$

So $C$, the cokernel of the inclusion $LS^{r,n}(\hat{\mathcal{P}}) \hookrightarrow C^r(\hat{\mathcal{P}})$, may be identified with $H^1_m(LS^{r,n}(\hat{\mathcal{P}}))$. \hfill \Box

We record a couple of facts (see Chapter 4 or Appendix A of \cite{10}) about regularity and local cohomology in the following lemma.

**Lemma 5.3.** Let $M$ be a graded $S$-module, $m \subset S$ the maximal homogeneous ideal.

1. If $M$ has finite length, then $\text{reg } M = \max_j \{j| M_j \neq 0\}$.
2. $H^i(M)$ has finite length for every $i \geq 0$.
3. $\text{reg}(M) = \max \text{reg } H^i_m(M) + j$

**Corollary 5.4.** Let $\mathcal{P} \subset \mathbb{R}^n$ be a pure hereditary polytopal complex. Set $t = \text{reg}(LS^{r,n}(\mathcal{P}))$.

1. If $d \geq t$ then $HF(LS^{r,n}(\mathcal{P}),d) = HF(C^r(\mathcal{P}),d) = HP(C^r(\mathcal{P}),d)$.
2. $\text{reg}(C^r(\mathcal{P})) \leq t$ and $HF(C^r(\mathcal{P}),d) = HP(C^r(\mathcal{P}),d)$ for $k \geq t - 1$.

**Proof.** From Theorem 4.3 we have the short exact sequence

$$0 \to LS^{r,n}(\hat{\mathcal{P}}) \to C^r(\hat{\mathcal{P}}) \to H^1_m(LS^{r,n}(\hat{\mathcal{P}})) \to 0,$$

(1) If $d \geq t$ then $H^1_m(LS^{r,n}(\mathcal{P}))(d) = 0$ by Lemma 5.3 and $HF(LS^{r,n}(\mathcal{P}),d) = HF(C^r(\mathcal{P}),d)$. We have $\text{pd}(LS^{r,n}(\mathcal{P})) \leq n$ since $LS^{r,n}(\mathcal{P})$ has no submodule of finite length, so Theorem 5.1 yields that $HF(LS^{r,n}(\mathcal{P}),d) = HP(LS^{r,n}(\mathcal{P}),d)$ for $d \geq t + n - n = t$. The result follows since $HP(LS^{r,n}(\mathcal{P}),d) = HP(C^r(\mathcal{P}),d)$.

(2) If $0 \to A \to B \to C \to 0$ is a short exact sequence of graded modules, then $\text{reg}(B) \leq \max\{\text{reg}(A), \text{reg}(C)\}$ (see §20.5 of \cite{9}). This fact coupled with Proposition 5.2 yields $\text{reg}(C^r(\mathcal{P})) \leq t$. The second statement of (2) follows from Theorem 5.1 and the fact that $C^r(\hat{\mathcal{P}})$ is a second syzygy. \hfill \Box

In \cite{12} and \cite{13}, star-supported bases are constructed for $C^r_\mathcal{P}(\Delta), \Delta \subset \mathbb{R}^2$ any triangulation of a disk. According to Proposition 5.2, this implies $H^1_m(LS^{r,2}(\hat{\mathcal{P}}))(3r+2) = 0$, which is compatible with (not necessarily equivalent to) the statement $\text{reg } LS^{r,2}(\hat{\mathcal{P}}) \leq 3r + 2$. The following conjecture is a natural generalization of this observation.

**Conjecture 5.5.** Let $\mathcal{P} \subset \mathbb{R}^2$ be a hereditary polytopal complex with $F$ being the maximum length of the boundary of a polytope of $\mathcal{P}$. Then

$$\text{reg}(LS^{r,2}(\hat{\mathcal{P}})) \leq F(r + 1) - 1.$$
dim$_R C^r(\hat{P})_k$ agrees with the McDonald-Schenck formula for $k \geq F(r+1) - 2$. We also propose the following generalization of Schenck’s $2r+1$ conjecture.

**Conjecture 5.6.** Let $\mathcal{P} \subset \mathbb{R}^2$ be a hereditary polytopal complex with $F$ being the maximum length of the boundary of a polytope of $\mathcal{P}$. Then

$$\operatorname{reg}(C^r(\hat{P})) \leq (F-1)(r+1).$$

This conjecture would imply, via Corollary 5.4, that dim$_R C^r(\hat{P})_k$ agrees with the McDonald-Schenck formula for $k \geq (F-1)(r+1) - 1$. The simplicial case of this, Schenck’s $2r+1$ conjecture, is still open. See [21] for an approach using the cohomology of sheaves on $\mathbb{P}^2$ and [15] for an example showing that this bound is tight. We give a family of examples which show that the regularity bound of Conjecture 5.6 cannot be lowered further.

**Theorem 5.7.** There exists a polytopal complex $\mathcal{P} \subset \mathbb{R}^2$ having one triangular face, $n-1$ quadrilateral faces, and two $(n+1)$-gons, such that $C^r(\hat{P})$ has a minimal generator of degree $n(r+1)$ supported on a single facet. Hence $\operatorname{reg}(C^r(\hat{P})) \geq n(r+1)$ and Conjecture 5.6 cannot be made tighter.

**Proof.** Let $T_n \subset \mathbb{R}^2$ be the polyhedral complex with

- $2n+1$ vertices as follows: $v_0 = w_0 = (0,0)$, $v_i = (i,i(i+1)/2)$ for $i = 1,\ldots,n$, and $w_j = (j,j(j+1)/2)$ for $j = 1,\ldots,n$,
- 1 triangular face $P_0$ with vertices $(0,0), v_1, w_1$,
- $n-1$ quadrilateral faces $P_i$ with vertices $v_i, w_i, v_{i+1}, w_{i+1}$ for $i = 1,\ldots,n-1$,
- Two $(n+1)$-gons $A$ and $B$ with vertices $(0,0), v_1,\ldots,v_n$ and $(0,0), w_1,\ldots,w_n$, respectively. (See Figure 11)

Set $S = \mathbb{R}[x,y,z], R = \mathbb{R}[x,z]$. $u_k = (k+1)x - y - (k+1)^2z, h_k = x - k z, l_k = (k+1)x + y - (k+1)^2z$ are the homogenized forms defining the edges between $v_k$ and $v_{k+1}, v_k$ and $w_k, w_k$ and $w_{k+1}$, respectively. Let $\phi_A : C^r(\hat{P}_n) \to S$ denote the map obtained by restricting splines to the facet $A$ and set $NT^n_A = \ker \phi_A$. 

![Figure 11. T3](image-url)
Suppose we have $G \in NT^*_n$. Then $u_i^r+1|G_{\hat{B}}$ and $l_i^{r+1}|G_{\hat{B}} - G_{\hat{B}}$ for $i = 0, \ldots, n - 1$. So $G_{\hat{B}} \in (u_i^{r+1}, l_i^{r+1})$ for $i = 0, \ldots, n - 1$ and $G_{\hat{B}} \in \cap_{i=0}^{n-1} (u_i^{r+1}, l_i^{r+1}) = J_r$. Define $p_y : S \to R$ by $p_y(f(x, y, z)) = f(x, 0, z)$ for $f(x, y, z) \in S$. $p_y(u_k) = p_y(k) = (k + 1)x - \binom{k+1}{2}$, so $p_y(J_r) = I_r \subset R$ is the principal ideal $\cap_{i=0}^{n-1} (x - (k/2)z)^{r+1} = (\prod_{i=0}^{n-1} (x - (k/2)z)^{r+1})$.

So we have a graded homomorphism of $S$-modules $p_y \circ \phi_B : NT^{r+1}_n \to I_r$, where $\phi_B(G) = G_{\hat{B}}$. Define $F(B) \in C^r(\hat{T}_n)$ by $F(B)_\sigma = 0$ for every facet $\sigma$ of $T_n$ other than $B$ and $F(B)_\hat{B} = L_{\hat{B}}$. $(p_y \circ \phi_B)(F(B)) = (\prod_{k=0}^{n-1} (k + 1)(x - (k/2)z))^{r+1}$, which is a minimal generator of the ideal $I_r$. It follows that $F(B)$ is a minimal generator of $C^r(\hat{T}_n)$. Since $F(B)$ has degree $n(r + 1)$, we are done. 

The following corollary indicates how different the polytopal case is from the simplicial, in which case $C^r(\Delta)$ is generated as a module over $S = \mathbb{R}[x, y, z]$ in degree at most $3r + 2$.

**Corollary 5.8.** If $P$ is planar polyhedral complex, then $C^r(\hat{P})$ may be generated as an $S$-module in arbitrarily high degree.

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