On the embedding of Weyl manifolds

R. Avalos, F. Dahia, C. Romero

Departamento de Física, Universidade Federal da Paraíba,
Caixa Postal 5008, 58059-970 João Pessoa, PB, Brazil and
E-mail: ravalos@gmail.com; fdahia@fisica.ufpb.br; cromero@fisica.ufpb.br

Abstract

We discuss the possibility of extending different versions of the Campbell-Magaard theorem, which have already been established in the context of semi-Riemannian geometry, to the context of Weyl’s geometry. We show that some of the known results can be naturally extended to the new geometric scenario, although new difficulties arise. In pursuit of solving the embedding problem we have obtained some no-go theorems. We also highlight some of the difficulties that appear in the embedding problem, which are typical of the Weylian character of the geometry. The establishing of these new results may be viewed as part of a program that highlights the possible significance of embedding theorems of increasing degrees of generality in the context of modern higher-dimensional space-time theories.
I. INTRODUCTION

The unification of the fundamental forces of nature is now recognized to be one of the most important tasks in theoretical physics. Unification, in fact, has been a feature of all great theories of physics. It is a well known fact that Newton, Maxwell and Einstein, they all succeeded in performing some sort of unification. So, not surprisingly in the last two centuries physicists have recurrently pursued this theme. Broadly speaking one can mention two different paths followed by theoreticians to arrive at a unified field theory. First, there are the early attempts of Einstein, Weyl, Cartan, Eddington, Schrödinger and many others, whose aim consisted of unifying gravity and electromagnetism. Their approach consisted basically in resorting to different kinds of non-Riemannian geometries capable of accommodating new geometrical structures with a sufficient number of degrees of freedom to describe the electromagnetic field. In this way, different types of geometries have been invented, such as affine geometry, Weyl’s geometry (where the notion of parallel transport generalizes that of Levi-Civita’s), Riemann-Cartan geometry (in which torsion is introduced), to quote only a few. In fact, it is not easy to track all further developments of these geometries, most of which were clearly motivated by the desire of extending general relativity to accommodate in its scope the electromagnetic field. However, as we now see, the main problem with all these attempts was that they completely ignored the other two fundamental interactions and did not take into account quantum mechanics, dealing with unification only at classical level. Of course, an approach to unification today would necessarily take into account quantum field theory.

The second approach to unification has to do with the rather old idea that our space-time may have more than four dimensions. This program starts with the work of the Finnish physicist Gunnar Nordström, in 1914. Nordström realised that by postulating the existence of a fifth dimension he was able (in the context of his scalar theory of gravitation) to unify gravity and electromagnetism. Although the idea was quite original and interesting, it seems the paper did not attract much attention due to the fact that his theory of gravitation was not accepted at the time. Then, soon after the completion of general relativity, Théodor Kaluza, and later, Oscar Klein, launched again the same idea, now entirely based on Einstein’s theory of gravity. Kaluza-Klein theory starts from five-dimensional vacuum Einstein’s equations and shows that, under certain assumptions, the field equations reduce
to a four-dimensional system of coupled Einstein-Maxwell equations. This seminal idea has given rise to several different theoretical developments, all of them exploring the possibility of achieving unification from extra dimensionality of space-time. Indeed, through the old and modern versions of Kaluza-Klein theory, supergravity, superstrings, and to the more recent braneworld scenario, induced-matter and M-theory, there has been a strong belief among some physicists that unification might finally be achieved if one is willing to accept that space-time has more than four dimensions.

Among all these higher-dimensional theories, one of them, the induced-matter theory (also referred to as space-time-matter theory) has called our attention since it vividly recalls Einstein’s belief that matter and radiation (not only the gravitational field) should ultimately be viewed as manifestations of pure geometry. Kaluza-Klein theory was a first step in this direction. But it was Paul Wesson, from the University of Waterloo, who pursued the matter further. Wesson and collaborators realized that by embedding the ordinary space-time into a five-dimensional vacuum space, it was possible to describe the macroscopic properties of matter in geometrical terms. In a series of interesting papers Wesson and his group showed how to produce standard cosmological models from five-dimensional vacuum space. It looked like as if any energy-momentum tensor could be generated by an embedding mechanism. At the time these facts were discovered, there was no guarantee that any energy-momentum tensor could be obtained in this way. Putting it in mathematical terms, Wesson’s programm would not always work unless one could prove that any solution of Einstein’s field equations could be isometrically embedded in five-dimensional Ricci-flat space. It turns out, however, that this is exactly the content of a beautiful and powerful theorem of differential geometry now known as the Campbell-Magaard theorem. This theorem, little known until recently, was proposed by English mathematician John Campbell in 1926, and was given a complete proof in 1963 by Lorenz Magaard, as many geometers of his time, was interested in geometrical aspects of Einstein’s general relativity and his works were published a few years before the classical Janet-Cartan theorem on embeddings was established. Compared to the Janet-Cartan theorem the nice thing about the Campbell-Magaard’s result is that the codimension of the embedding space is drastically reduced: one needs only one extra-dimension, and that perfectly fits the requirements of the induced-matter theory. Finally, let us note that both theorems refer to local and analytical embeddings (the global version
of Janet-Cartan theorem was worked out by John Nash [20], in 1956, and adapted for semi-Riemannian geometry by R. Greene [21], in 1970, while a discussion of global aspects of Cambell-Magaard has recently appeared in the literature [22].

II. HIGHER-DIMENSIONAL SPACE-TIMES AND RIEMANNIAN EXTENSIONS OF THE CAMPBELL-MAGAARD THEOREM

Besides the induced-matter proposal, there appeared at the turn of the XX century some other physical models of the Universe, which soon attracted the attention of theoreticians. These models have put forward the idea that the space-time of our everyday perception may be viewed as a four-dimensional hypersurface embedded not in a Ricci-flat space, but in a five-dimensional Einstein space (referred to as the bulk) [8, 9]. Spurred by this proposal new research on the geometrical structure of the proposed models started. It was conjectured [23] and later proved that the Campbell-Magaard theorem could be immediately generalized for embedding Einstein spaces [24]. This was the first extension of the Campbell-Magaard theorem and other extensions, still in the context of Riemannian geometry, were to come. More general local isometric embeddings were next investigated, and it was proved that any \( n \)-dimensional semi-Riemannian analytic manifold can be locally embedded in \( (n + 1) \)-dimensional analytic manifold with a non-degenerate Ricci-tensor, which is equal, up to a local analytical diffeomorphism, to the Ricci-tensor of an arbitrary specified space [25]. Further motivation in this direction came from studying embeddings in the context of non-linear sigma models, a theory proposed by J. Schwinger in the fifties to describe strongly interacting massive particles [26]. It was then showed that any \( n \)-dimensional Lorentzian manifold \( (n \geq 3) \) can be harmonically embedded in an \( (n + 1) \)-dimensional semi-Riemannian Ricci-flat manifold [27].

At this point we should remark that most theories that regard our spacetime as a hypersurface embedded in a higher-dimensional manifold [28] make the tacit assumption that this hypersurface has a semi-Riemannian geometrical structure. Surely, this assumption avoids possible conflicts with the well-established theory of general relativity, which operates in a Riemannian geometrical frame. However, recently there has been some attempts to broaden this scenario. For instance, new theoretical schemes have been proposed, where one of the most simple generalizations of non-Riemannian geometry, namely the Weyl geometry [29],
has been taken into consideration as a viable possibility to describe the geometry of the bulk \[30,32\]. In some of these approaches, the induced-matter theory is revisited to show that it is even possible to generate a cosmological constant, or rather, a cosmological function, from the extra dimensions and the Weyl field \[33\]. In a similar context, it has also been shown how the presence of the Weyl field may affect both the confinement and/or stability of particles motion, and how a purely geometrical field, such as the Weyl field, may effectively act both as a classical and quantum scalar field, which in some theoretical-field modes is the responsible for the confinement of matter in the brane \[34,35\].

There is also another very interesting and compelling argument for considering a Weyl structure as a suitable mathematical model for describing space-time. This is based on the well-known axiomatic approach to space-time theory put forward by Ehlers, Pirani and Schild (EPS), which, through an elegant and powerful theoretical construction, shows that by starting from a minimum set of rather plausible and general axioms concerning the motion of light signals and freely falling particles, one is naturally led to a Weyl structure as the proper framework of space-time \[36\]. In order to reduce this more general framework to that of a semi-Riemannian manifold we need an additional axiom to be added to this minimum set. It turns out, however, that this added axiom does not seem as natural as the others, as was pointed out by Perlick \[37\]. We take Perlick’s point of view as one of the motivations for investigating the geometry of Weyl spaces.

In this paper we shall consider the mathematical problem of extending different versions of the Campbell-Magaard theorem from the Riemannian context to Weyl’s geometry. Specifically, we shall first analyze the possibility of locally and analytically embedding an \(n\)-dimensional Weyl manifold in an \((n+1)\)-dimensional Weyl space, the latter being Ricci-flat. We then weaken this condition to investigate the problem of embedding manifolds whose symmetric part of the Ricci tensor vanishes. These problems can be regarded as extensions of the Campbell-Magaard theorems, which hold in Riemannian geometry, to a more general geometrical setting, namely that of Weyl’s geometry. We believe that an investigation of these seemingly purely geometrical problems may also shed some light on the physics of higher-dimensional theories in which there are extra degrees of freedom coming from the geometric structure of space-time, in particular, those in which there are mechanisms for generating matter and fields from extra dimensions in the case of theories of gravitation whose geometrical framework is based on the Weyl theory, and other higher-dimensional
proposals formulated in Weyl manifolds, such as D-dimensional dilaton gravity [38], higher-dimensional WIST theories [30],[31],[33],[39] and others.

Finally, a few words should be said with regard to the Campbell-Magaard theorem and its application to physics. First, let us note that the proof provided by Magaard is based on the Cauchy-Kovalevskaya theorem. Therefore, some properties of relevance to physics, such as the stability of the embedding, cannot be guaranteed to hold [43]. Nevertheless, the problem of embedding space-time into five-dimensional spaces can be considered in the context of the Cauchy problem for general relativity [44]. Specifically, it has recently been shown that the embedded space-time may arise as a result of physical evolution of proper initial data. This new perspective has some advantages in comparison with the original Campbell-Magaard formulation because, by exploring the hyperbolic character of the field equations, it allows to show that the embedding has stability and domain of dependence (causality) properties [45].

III. WEYL GEOMETRY

When working in Riemannian geometry we consider a pair \((M,g)\), where \(M\) is a differentiable manifold and \(g\) a (semi)-Riemannian metric defined on \(M\). The fundamental theorem of Riemannian geometry states that there is a unique torsionless linear connection compatible with \(g\) [46]. By compatibility we mean the following. When we endow any differentiable manifold with a linear connection \(\nabla\) we have an associated notion of parallel transport. It is well known that parallel transport defines isomorphisms between tangent spaces. The compatibility condition is defined as the requirement that this isomorphism is also an isometry. This turns out to be equivalent to the following requirement

\[ \nabla g = 0. \]

It turns out that in Weyl’s geometry we relax the requirement of \(\nabla\) being compatible with \(g\), and this means that parallel transports are not required to define isometries anymore. We first endow \(M\) not only with a semi-Riemannian metric, but also with a one-form field \(\omega\), so that instead of the pair \((M,g)\) we now consider a triple \((M,g,\omega)\). Weyl’s connection is defined by requiring it to be torsionless and that, for any parallel vector field \(V\) along any
smooth curve $\gamma$, the following condition is satisfied:

$$\frac{d}{dt}g(V(t), V(t)) = \omega(\gamma'(t))g(V(t), V(t)).$$  (1)

Before presenting the main existence and uniqueness theorems for such connection, we shall try to get some insight on what this condition means geometrically. First of all, note that because parallel transport is a linear application, if $V, W$ are parallel fields along some curve $\gamma$, then $V + W$ also is a parallel field along $\gamma$. On the other hand, by polarization we get

$$g(V, W) = \frac{1}{2}(g(V + W, V + W) - g(V, V) - g(W, W)),$$

which together with (1) gives

$$\frac{d}{dt}g(V, W) = \omega(\gamma')g(V, W).$$

We thus say that the connection $\nabla$ is Weyl compatible with $(M, g, \omega)$ if for any pair of parallel vectors along any smooth curve $\gamma = \gamma(t)$ the condition below is satisfied

$$\frac{d}{dt}g(V(t), W(t)) = \omega(\gamma'(t))g(V(t), W(t)),$$

where $\gamma'$ denotes the tangent vector of $\gamma$. Integrating the above equation along the curve $\gamma$ leads to

$$g(V(t), W(t)) = g(V(0), W(0))e^{\int_0^t \omega(\gamma'(s))ds}.$$  (2)

In particular, if $V = W$ this last expression gives us precisely how much the parallel transport fails to be an isometry:

$$g(V(t), V(t)) = g(V(0), V(0))e^{\int_0^t \omega(\gamma'(s))ds}.$$  (2)

Note that if the vectors $V(0)$ and $W(0)$ are orthogonal, then (2) implies that they remain orthogonal when parallel transported along the curve, although their respective "norms" may change.

Let us now state some results that hold for a Weyl connection which are analogues to those valid for a Riemannian connection. All these results are proven in very much the same way as in Riemannian geometry.

**Proposition 1.** A connection $\nabla$ is compatible with a Weyl structure $(M, g, \omega)$ iff for any pair of vector fields $V, W$ along any smooth curve $\gamma$ in $M$ the following holds:

$$\frac{d}{dt}g(V, W) = g(\frac{DV}{dt}, W) + g(V, \frac{DW}{dt}) + \omega(\gamma')g(V, W).$$  (3)
Corollary 1. A linear connection $\nabla$ is compatible with a Weyl structure $(M, g, \omega)$ iff $\forall p \in M$ and for every vector fields $X, Y, Z$ on $M$ the condition below holds

$$X_p(g(Y, Z)) = g_p(\nabla_X Y, Z_p) + g_p(Y_p, \nabla_X Z) + \omega_p(X_p)g_p(Y_p, Z_p)$$  \hspace{1cm} (4)$$

In the last proposition we can actually drop $p$ and write

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + \omega(X)g(Y, Z),$$

which, then, can be used to prove the following:

Proposition 2. A linear connection $\nabla$ is compatible with a Weyl structure $(M, g, \omega)$ iff it satisfies

$$\nabla g = \omega \otimes g$$ \hspace{1cm} (5)$$

Now the following result is easily established.

Proposition 3. There is a unique torsionless connection compatible with the Weyl structure $(M, g, \omega)$.

In the proof of this proposition it is found that the Weyl connection, in a particular coordinate system, takes the following form:

$$\Gamma^u_{ac} = \frac{1}{2}g^{bu}(\partial_aj_{bc} + \partial_cj_{ab} - \partial_bg_{ca}) + \frac{1}{2}g^{bu}(\omega_a^c - \omega_b^a)$$ \hspace{1cm} (6)$$

It is important to note that a Weyl manifold defines an equivalence class of such structures all linked by the following group of transformations:

$$\begin{align*}
\bar{g} &= e^{-f}g \\
\bar{\omega} &= \omega - df
\end{align*}$$ \hspace{1cm} (7)$$

where $f$ is an arbitrary smooth function defined on $M$. It is easy to check that these transformations define an equivalence relation between Weyl manifolds, and that if $\nabla$ is compatible with $(M, g, \omega)$, then it is also compatible with $(M, \bar{g}, \bar{\omega})$. In this way every member of the class is compatible with the same connection, hence has the same geodesics, curvature tensor and any other property that depends only on the connection. This is the reason why it is regarded more natural, when dealing with Weyl manifolds, to consider the whole class of equivalence $(M, [g], [\omega])$ rather than working with a particular element.
of this class. In this sense, it is argued that only geometrical quantities that are invariant under (7) are of real significance in the case of Weyl geometry. Following the same line of argument it is assumed that only physical theories and physical quantities presenting this kind of invariance should be considered of interest in this context. To conclude this section, we remark that when the one-form field $\omega$ is an exact form, then the Weyl structure is called integrable.

A. Weyl submanifolds

Definition 1. Let $(\overline{M}, \overline{g}, \overline{\omega})$ be a Weyl manifold and $M \hookrightarrow \overline{M}$ be a submanifold of $\overline{M}$. If the pullback $i^*(g)$ is a metric tensor on $M$ then $(M, i^*(\overline{g}), i^*(\overline{\omega}))$ is a Weyl submanifold of $\overline{M}$. In this case we will use the notation $g = i^*(\overline{g})$ and $\omega = i^*(\overline{\omega})$ for the induced metric and 1-form.

Using the same conventions as in the previous definition, we denote by $\nabla$ the Weyl-compatible connection associated with $(\overline{M}, \overline{g}, \overline{\omega})$. We define the induced connection $\nabla$ on $M$ following the same reasoning as in the Riemannian case. Thus if $X, Y$ are vector fields on $M$, and $\overline{X}, \overline{Y}$ are extensions of these vector fields to $\overline{M}$, then $\nabla_X Y = (\overline{\nabla}_{\overline{X}} \overline{Y})^T$. It is a well-known fact that this definition does not depend on the extensions [46].

It is worth noticing that both the definition of Weyl submanifold and of induced connection make sense in the whole class $(\overline{M}, [\overline{g}], [\overline{\omega}])$. We can see that the definition of Weyl submanifold satisfies this condition since every such structure that can be obtained from an element of $(\overline{M}, [\overline{g}], [\overline{\omega}])$ lies in $(M, [g], [\omega])$ and vice versa, every element of $(M, [g], [\omega])$ can be obtained from some element of $(\overline{M}, [\overline{g}], [\overline{\omega}])$. The fact that the definition of induced connection is invariant in the whole class $(\overline{M}, [\overline{g}], [\overline{\omega}])$, is because two conformal metrics make the same splitting of the tangent spaces: $T_p\overline{M} = T_pM \oplus T_pM^\perp$.

The following results are obtained in the same way as in Riemannian geometry.

Proposition 4. Given a Weyl manifold $(\overline{M}, \overline{g}, \overline{\omega})$ and a Weyl submanifold $M \hookrightarrow \overline{M}$, the induced connection $\nabla$ on $M$ is the Weyl connection compatible with the induced Weyl structure on $(M, g, \omega)$. 
As usual, we define the second fundamental form $\alpha$ on $M$ as

$$\alpha : TM \times TM \mapsto TM^\perp$$

$$\alpha(X, Y) = (\nabla_X Y)^\perp$$

One can easily check that this definition does not depend on how we extend $X$ and $Y$ to $\overline{M}$. Thus, if $X, Y$ are fields on $M$ we write

$$\nabla_X Y = \nabla_X Y + \alpha(X, Y)$$

The next proposition is analogous to its Riemannian counterpart:

**Proposition 5.** The second fundamental form $\alpha$ is symmetric and $\mathfrak{H}(M)$-linear in both arguments.

From now on we shall consider only hypersurfaces. In this case, we can define a unit normal vector field $\eta$, which, at least locally, is unique up to a sign. We define the scalar second fundamental form $l$ as given by

$$l : TM \times TM \mapsto \mathfrak{H}(M)$$

$$(X, Y) \mapsto \mathfrak{l}(\alpha(X, Y), \eta)$$

We note that although the choice of the unit normal field $\eta$ depends on a particular element of $(\overline{M}, [g], [\omega])$, the definition of $l$ does not.

Now from the last proposition it follows that $l$ is symmetric and $\mathfrak{H}(M)$-linear, i.e., $l$ is a symmetric $(0, 2)$-tensor field on $M$. Following a procedure entirely analogous to what is done in Riemannian geometry, we obtain the Gauss-Codazzi equations for hypersurfaces. Thus, if $X, Y$ and $Z$ are vector fields on $M$, Gauss’ equation takes the form

$$\mathfrak{l}(\mathcal{R}(X, Y)Z, W) = \mathfrak{l}(R(X, Y)Z, W) + \mathfrak{l}(\alpha(X, Z), \alpha(Y, W)) - \mathfrak{l}(\alpha(Y, Z), \alpha(X, W)). \quad (8)$$

If $\xi$ is a unit field normal to $M$, then Codazzi’s equation reads

$$\mathfrak{l}(\mathcal{R}(X, Y)Z, \xi) = \epsilon((\nabla_X l)(Y, Z) = (\nabla_Y l)(X, Z) + \frac{1}{2}(\omega(Y)l(X, Z) - \omega(X)l(Y, Z))). \quad (9)$$

Where $\epsilon = \mathfrak{l}(\xi, \xi) = \pm 1$ and the sign depends on whether the restriction of $\mathfrak{l}$ to each $T_pM^\perp$ is positive or negative definite.

Let us now have a look at the Bianchi identities in Weyl geometry, as they will be useful in our investigation of the embedding problem.
B. Bianchi identities

We start by writing down the symmetries of the Riemann curvature tensor $R$ defined on an $n$-dimensional Weyl manifold. First of all, in order to clarify notation, we remark that in this paper we adopt the following convention for the curvature tensor:

$$R^\rho_{\beta\sigma\alpha} = \partial_\alpha \Gamma^\rho_{\beta\sigma} - \partial_\beta \Gamma^\rho_{\alpha\sigma} + \Gamma^\rho_{\gamma\beta} \Gamma^\gamma_{\alpha\sigma} - \Gamma^\rho_{\gamma\alpha} \Gamma^\gamma_{\beta\sigma}.$$

In terms of the components of $R$ in a coordinate basis it is easy to see that for any connection the following identity holds:

$$R^\rho_{\mu\nu\alpha} = -R^\rho_{\mu\alpha\nu}.$$  

Moreover, if the connection is torsionless we also have the Bianchi identities

$$R^\rho_{\mu\nu\alpha} + R^\rho_{\alpha\mu\nu} + R^\rho_{\nu\alpha\mu} = 0,$$

$$R^\rho_{\mu\nu;\lambda} + R^\rho_{\mu\lambda;\nu} + R^\rho_{\nu\lambda;\mu} = 0,$$

where the semicolon denotes covariant differentiation.

We now look for a contracted version of the Bianchi identities. In particular, we want to get a geometric identity for $g^{\alpha\beta} \nabla^S_\alpha G_{\beta\sigma}$, where the upper index $S$ stands for the "symmetric part". Before doing this we need one more identity, which comes from looking at the following expression for the Riemann tensor

$$R^\rho_{\sigma\beta\alpha} = \circ R^\rho_{\sigma\beta\alpha} + g_{[\beta} |^{\sigma} \nabla_\alpha |^{\rho} - \delta^{\rho}_{[\beta} \nabla^{\sigma]}_\alpha |^{\omega} - \delta^{\sigma}_{[\beta} \nabla^{\omega]}_\alpha |^{\rho} + \frac{1}{2} \delta^{\rho}_{[\beta} \omega^{\sigma]} |^{\omega} + \frac{1}{2} \delta^{\sigma}_{[\beta} \omega^{\rho]} |^{\omega} - \frac{1}{2} g_{[\sigma} \omega^{\beta]} |^{\rho} |^{\omega},$$

where $\circ$ denotes quantities computed with the Riemannian connection. From this expression we can prove the identity

$$R_{\lambda\sigma\beta\alpha} + R_{\sigma\lambda\beta\alpha} = 2 g_{\lambda\sigma} F_{\beta\alpha},$$

where $F_{\beta\alpha} = d\omega_{\alpha\beta} = \frac{1}{2} (\nabla^\beta \omega^\alpha - \nabla^\alpha \omega^\beta).$ In order to compute $g^{\alpha\beta} \nabla^S_\alpha G_{\beta\sigma}$ we can first compute both divergences, which will give us the final result. Using all the previous identities, it is not difficult to see that we are led to the following:

$$g^{\mu\lambda} \nabla^S_\lambda G_{\nu\mu} = \frac{n - 2}{2} g^{\mu\lambda} \nabla_\lambda F_{\nu\mu}.$$
IV. THE EMBEDDING PROBLEM

We now turn to the problem of existence of isometric embeddings of Weyl manifolds. In particular, we are interested in studying possible extensions of the Campell-Magaard-like theorems in the context of Weyl geometry. First of all, we shall define what we understand by an isometric embedding in this context.

**Definition 2.** An isometric immersion \( \phi : M \mapsto \tilde{M} \) between two Weyl manifolds \((M, g, \omega)\) and \((\tilde{M}, \tilde{g}, \tilde{\omega})\) is a smooth mapping satisfying:

1) \( d\phi_p \) is injective \( \forall p \in M \)
2) \( \phi^*(\tilde{g}) = g \)
3) \( \phi^*(\tilde{\omega}) = \omega. \)

If, furthermore, \( \phi \) is one-to-one and the induced map \( M \mapsto \phi(M) \) is an homeomorphism, where \( \phi(M) \subset \tilde{M} \) is seen with the induced topology, then we say that \( \phi \) is an isometric embedding. Also, we shall say that \( \phi \) is a local isometric embedding at \( p \in M \) if there is a neighborhood of \( p \) where \( \phi \) is an embedding.

An important result we shall use when studying Campbell-Magaard-like theorems is the following theorem:

**Theorem 1.** Let \((M^n, g, \omega)\) and \((\tilde{M}^{n+1}, \tilde{g}, \tilde{\omega})\) be Weyl manifolds, and \((U, \mu)\) a coordinate system around \( p \in M^n \). Then \((M, g, \omega)\) has a local analytic isometric embedding in \((\tilde{M}^{n+1}, \tilde{g}, \tilde{\omega})\) around \( p \) if there are analytic functions \( \overline{g}_{ik}, \overline{\psi}, \overline{\omega}_k \) and \( \overline{\omega}_{n+1} \), with \( i, k = 1, \ldots, n \), defined on some open set \( D \subset \mu(U) \times \mathbb{R} \) containing \((x^1_p, \ldots, x^n_p, 0)\) satisfying the following conditions

\[
\overline{g}_{ik}(x^1, \ldots, x^n, 0) = g_{ik}(x^1, \ldots, x^n)
\]

\[
\overline{\omega}_k(x^1, \ldots, x^n, 0) = \omega_k(x^1, \ldots, x^n)
\]

on some open set \( A \subset \mu(U) \), and

\[
\overline{g}_{ik} = \overline{g}_{ki} \tag{14}
\]

\[
det(\overline{g}_{ik}) \neq 0 \tag{15}
\]

\[
\overline{\psi} \neq 0 \tag{16}
\]
on $D$, and such that on some open set $V \subset \tilde{M}^{n+1}$, the metric $\tilde{g}$ and the 1–form $\tilde{\omega}$ can be written in coordinates as

$$\tilde{g} = g_{ik} dx^i \otimes dx^k + \epsilon \psi^2 dx^{n+1} \otimes dx^{n+1}$$

$$\tilde{\omega} = \omega_k dx^k + \tilde{\omega}_{n+1} dx^{n+1}$$

where $\epsilon = \pm 1$.

At first sight a natural extension of the Campbell-Magaard theorem [24] in the context of Weyl geometry seems to be to prove the existence of a local analytic isometric embedding of an arbitrary Weyl manifold $(M^n, g, \omega)$ in an $(n+1)$-dimensional Weyl manifold $(\tilde{M}^{n+1}, \tilde{g}, \tilde{\omega})$ satisfying $\tilde{R}_{\alpha\beta} = 0$ around some arbitrary point $p \in M$. This turns out to be a simple extension which can be treated in complete analogy to [24] after making some considerations.

First, note that $\tilde{R}_{\alpha\beta} = 0$ implies that both its symmetric and antisymmetric parts of $\tilde{R}_{\alpha\beta}$ must vanish. However, we already know that for an $n-$dimensional Weyl manifold we have $A \tilde{R}_{\alpha\beta} = \frac{n}{2} F_{\alpha\beta}$. Therefore, this condition implies $F_{\alpha\beta} = 0$, which is locally equivalent to setting $\omega = d\phi$, for some function $\phi$. In other words, in this case $(\tilde{M}, \tilde{g}, \tilde{\omega})$ gives an integrable Weyl structure. From this, we see that if $(M, g, \omega)$ is non-integrable, then it does not exist any isometric embedding of $(M, g, \omega)$ in Ricci-flat manifolds, irrespective of the codimension considered. Thus, let us first consider integrable Weyl manifolds and look for analytic isometric embeddings of an integrable Weyl manifold $(M^n, g, \phi)$ in a Ricci-flat integrable Weyl manifold $(\tilde{M}^{n+1}, \tilde{g}, \tilde{\phi})$. We now proceed to set up the notation that will be used throughout this paper.

Henceforth we shall consider $\tilde{M} = M \times \mathbb{R}$, a local chart in $M$ defined in a neighbourhood of $p$ with coordinates $(x^1, \ldots, x^n)$, while in the product structure we have a coordinate system around $(p, 0)$ with coordinates $(x^1, \ldots, x^n, y)$, where $y$ denotes the coordinate in $\mathbb{R}$. In this coordinate system we write

$$\tilde{g} = g_{ik} dx^i \otimes dx^k + \epsilon \psi^2 dy \otimes dy,$$

$$\tilde{\omega} = \omega_i dx^i + \tilde{\omega}_{n+1} dy,$$

and consider the unit normal field given by

$$\xi = \frac{1}{|\tilde{g}(\partial_{n+1}, \partial_{n+1})|^\frac{1}{2}} \partial_{n+1}.$$
From Gauss’ equation we obtain
\[ \tilde{R}_{likj} = R_{likj} + \epsilon(l_{ij}l_{kl} - l_{ki}l_{jl}), \quad (17) \]
with
\[ l_{ji} = \bar{\psi} \bar{\Gamma}_{ji}^{n+1} = -\frac{1}{2\bar{\psi}} \frac{\partial}{\partial y} g_{ij} + \frac{1}{2\bar{\psi}} g_{ij} \bar{\omega}_{n+1}. \quad (18) \]
Also, from the Gauss-Codazzi equations and some explicit expressions for the components of the connection, we arrive at the following equations for the components of the Ricci tensor:
\[ \tilde{R}_{ij} = R_{ij} + \frac{1}{\bar{\psi}} \nabla_{j} \nabla_{i} - \frac{\epsilon}{\bar{\psi}} \partial_{n+1} l_{ij} - \frac{1}{2} \nabla_{j} \bar{\omega}_{i} \]
\[ + \frac{1}{2\bar{\psi}} (\bar{\omega}_{i} \partial_{j} \bar{\psi} + \bar{\omega}_{j} \partial_{i} \bar{\psi} - \epsilon \bar{\omega}_{n+1} l_{ji}) \quad (19) \]
Our next step is to compute the component \( \tilde{G}_{n+1}^{n+1} \) of the Einstein tensor to obtain
\[ \tilde{G}_{n+1}^{n+1} = -\frac{1}{2} (\tilde{R} + \epsilon g^{ik} g_{kl} (l_{ij} l_{kl} - l_{ki} l_{jl})). \quad (20) \]

**A. The Weyl integrable case**

In this section we will discuss the embedding problem for Weyl integrable manifolds. It is worth noticing that, in this case, there is a stronger analogy with some Riemannian problems already studied in contact with General Relativity. This is because if, for one particular member of the class \((\tilde{M}^{n+1}, [\tilde{g}], [\tilde{\phi}])\), we split the Ricci tensor into its Riemannian part and the extra terms, then the Ricci-flat condition becomes equivalent to the Einstein field equations with a scalar field as a source. In the Riemannian framework, embeddings in such structures have been studied by Ponce de Leon, who constructed explicit embeddings of general vacuum solutions of \(n\)-dimensional general relativity (with a possible presence of the cosmological constant) into \((n+1)\)-Semi-Riemannian manifolds sourced by a scalar field \[47\]. We should also mention that embeddings in such structures where also treated by Anderson
et al., in which they worked out one of the known extensions of the Campell-Magaard theorem [48]. Even though these results are clearly related to the problem we intend to study here, there are important differences, one of them and maybe the main one, is that, since in both [47] and [48] the underlying structure is Riemannian, the results presented there would not guarantee the embedding of a whole Weyl integrable structure \((M^n, [g], [\phi])\) in a Ricci-flat Weyl integrable structure \((\tilde{M}^{n+1}, [\tilde{g}], [\tilde{\phi}])\), as will be shown in this section. Another difference with respect to [47] is that there it is shown that, given a solution of the vacuum Einstein field equations in \(n\)-dimensions, then it is possible to construct embeddings for such a solutions in \((n + 1)\)-dimensional manifolds sourced by scalar fields. In contrast, we will not impose any restriction, besides the regularity assumptions, for the initial data (it does not need to solve any field equations on the original manifold). In this way, we can make an interesting contact with these results known in Riemannian geometry, while having some important differences with them.

In order to start with the discussion of the present embedding problem, note that in the case where \((\tilde{M}, \tilde{g}, \tilde{\omega})\) is integrable, that is, when \(\tilde{\omega} = d\tilde{\phi}\), the expressions in (19) are simplified. In fact, as we have already seen, the Ricci tensor turns out to be symmetric in this case, and from (13) we obtain

\[ \tilde{g}^\alpha\beta \tilde{\nabla}_\alpha \tilde{G}_{\nu\beta} = 0. \] (21)

From the above, we see that the natural approach to the problem is to follow the same procedure adopted in [24], which consists in considering the evolution equations \(\tilde{R}_{ij} = 0\) in a neighborhood of \(0 \in \mathbb{R}^{n+1}\), as well as the constraint equations \(\tilde{R}_{i(n+1)} = 0\) and \(\tilde{G}^{n+1}_{n+1} = 0\) on the hypersurface \(\Sigma_0\) given by \(y = 0\). Then, the evolution equations together with the identity (21), guarantee that we can propagate the constraint equations in a neighborhood of the origin of \(\mathbb{R}^{n+1}\). In this scheme, we just consider \(\tilde{\phi}\) as being some given analytic function in a neighborhood of the origin satisfying \(\tilde{\phi}(x,0) = \phi(x)\). Proceeding in this way, we find that the problem is totally analogous to the one investigated in [24], immediately leading to the following statement:

**Theorem 2.** Any analytic integrable \(n\)-dimensional Weyl manifold \((M^n, g, \phi)\) admits a local analytic isometric embedding around any point \(p \in M\) in an analytic Ricci-flat integrable Weyl manifold \((\tilde{M}^{n+1}, \tilde{g}, \tilde{\phi})\).
It is interesting to note that this result guarantees the existence of isometric embeddings for Weyl manifolds, not for a Weyl structure \((M, [g], [\omega])\). Indeed, in order to take into account the whole Weyl structure we need to show that for every element of \((M, [g], [\omega])\) there is an isometric embedding of this element in an element of some \((n + 1)\)-dimensional Weyl structure \((\tilde{M}, [\tilde{g}], [\tilde{\omega}])\). Since, as already remarked, when working with Weyl manifolds all the relevant geometric (and physical) quantities are to be defined on the whole class, it is of much more interest to look for an embedding for the whole structure. We claim that we can show this from our previous results. To do this, let us consider the following argument.

Suppose that a particular \(n\)-dimensional Weyl manifold \((M, g, \omega)\) admits a local analytic isometric embedding into an \((n + 1)\)-dimensional Weyl manifold \((\tilde{M}, \tilde{g}, \tilde{\omega})\), and that this embedding has been constructed following our previous prescription, namely, that the embedding is just the inclusion. On the other hand, any other element of the class \((M, [g], [\omega])\) can be written as \((M, e^{-h} g, \omega - dh)\) for some analytic function \(h\). The question is whether there is some analytic function \(f\) on \(\tilde{M}\) such that, for this element of the class, there is a local analytic isometric embedding into \((\tilde{M}, e^{-f} \tilde{g}, \tilde{\omega} - df)\). By using the same set up we have developed, we define the function \(f(x, y)\) in a neighborhood of the point \(p \in \tilde{M}\) (where we know the isometric embedding exists) by:

\[
f(x, y) = h(x) + y.
\]

We then get

\[
e^{-f(x,0)} \tilde{g}_{ij}(x, 0) = e^{-h(x)} g_{ij}(x),
\]

\[
\tilde{\omega}_i(x, 0) - \partial_i f(x, 0) = \omega_i(x) - \partial_i h(x),
\]

which gives us the isometry condition. Also, since the Ricci tensor is an invariant of the class of Weyl manifolds, we have shown the following result.

**Theorem 3.** Any analytic \(n\)-dimensional integrable Weyl structure \((M^n, [g], [\phi])\) admits a local analytic isometric embedding in an \((n + 1)\)-dimensional integrable Weyl structure \((\tilde{M}^{n+1}, [\tilde{g}], [\tilde{\phi}])\) with vanishing Ricci tensor.

We now turn our attention to the more general problem of embedding of Weyl manifolds which are not necessarily integrable, dropping the condition of Ricci-flatness. Thus, in the following sections, we shall weaken this latter condition.
V. EMBEDDINGS IN WEYL MANIFOLDS WHOSE RICCI TENSOR HAS VANISHING SYMMETRIC PART

In this section we shall investigate the existence of a local isometric embedding of an arbitrary Weyl manifold \((M^n, g, \omega)\) around some point \(p \in M\) in a Weyl manifold \((\tilde{M}^{n+1}, \tilde{g}, \tilde{\omega})\) which has \(\tilde{S} \tilde{R}_{\alpha\beta} = 0\). This is the same as requiring that \(\tilde{S} \tilde{G}_{\alpha\beta} = 0\). From the identity

\[
\tilde{g}^{\alpha\beta} \nabla^\alpha \tilde{G}_{\nu\beta} = \frac{n-2}{2} \tilde{g}^{\alpha\beta} \nabla_\alpha F_{\nu\beta},
\]

we see that our requirement on \((\tilde{M}, \tilde{g}, \tilde{\omega})\) imposes the condition

\[
\tilde{g}^{\alpha\beta} \nabla_\alpha F_{\nu\beta} = 0,
\]

which must hold in a neighborhood of \(p\). As we shall see, (22) will impose further restrictions on \((\tilde{M}, \tilde{g}, \tilde{\omega})\). To see this we shall need to make use of some geometric identities.

**Proposition 6.** Suppose we have a semi-Riemannian manifold \(M\) endowed with a torsionless connection \(\nabla\). Then, for any \(T \in \mathfrak{X}_0^2(M)\) the following identity holds:

\[
\nabla_\nu \nabla_\mu T_{\alpha\beta} - \nabla_\mu \nabla_\nu T_{\alpha\beta} = -R^\sigma_{\alpha\mu\nu} T_{\sigma\beta} - R^\sigma_{\beta\mu\nu} T_{\alpha\sigma}.
\]

A corollary of this proposition in the context of Weyl geometry is given by the statement below.

**Corollary 2.** Suppose we have a Weyl manifold \((M, g, \omega)\), endowed with its Weyl-compatible connection \(\nabla\), and let \(F = d\omega\). Then, for any 2-form \(T\) on \(M\) we have the identity

\[
g^{\nu\alpha} g^{\mu\beta} \nabla_\nu \nabla_\mu T_{\alpha\beta} = -R^\sigma_{\alpha\mu\nu} T_{\sigma\beta} + 2F^\sigma_{\alpha\beta} T_{\sigma\beta}.
\]

A direct consequence of the above is the following:

**Corollary 3.** Let \((M, g, \omega)\) be a \(n\)-dimensional Weyl manifold whose symmetric part of the Ricci tensor is zero. Then, for \(n \neq 4\) we must have

\[
F^{\mu\nu} F_{\mu\nu} = 0.
\]

**Proof.** Using Weyl’s compatibility condition we get the following:

\[
g^{\mu\nu} \nabla_\mu (g^{\alpha\beta} \nabla_\alpha F_{\nu\beta}) = g^{\mu\nu} g^{\alpha\beta} \nabla_\mu \nabla_\alpha F_{\nu\beta} - \omega^\nu g^{\alpha\beta} \nabla_\alpha F_{\nu\beta}
\]
We know that under our hypotheses (22) is satisfied. Then the second term in the right-hand side of the previous expression vanishes and so does the left-hand side. Also we know that the previous corollary holds for the 2-form $F$. This gives us the following:

$$0 = - R^{\mu\nu} F_{\mu\nu} + 2 F^{\mu\nu} F_{\mu\nu}$$

$$= - A R^{\mu\nu} F_{\mu\nu} + 2 F^{\mu\nu} F_{\mu\nu}$$

Using the fact that for a Weyl manifold of dimension $n$, the antisymmetric part of its Ricci tensor is $A R_{\mu\nu} = \frac{n}{2} F_{\mu\nu}$ we get the following:

$$0 = \frac{4 - n}{2} F^{\mu\nu} F_{\mu\nu}$$

So we get that if $n \neq 4$ then it must hold that:

$$F^{\mu\nu} F_{\mu\nu} = 0$$

It is worth noticing that the above condition will lead to unexpected and interesting no go results. For example, if $\tilde{g}$ is a positive definite metric, then $F^{\mu\nu} F_{\mu\nu} = 0$ implies $F_{\mu\nu} = 0$; hence $(\tilde{M}, \tilde{g}, \tilde{\omega})$ is integrable. Therefore, we have the following result:

**Theorem 4.** Let $(M, g, \omega)$ be an $n$-dimensional non-integrable Weyl manifold, with $n \geq 5$. If $g$ is positive definite, then it is not possible to isometrically immerse $(M, g, \omega)$ into a Weyl manifold $(\tilde{M}, \tilde{g}, \tilde{\omega})$, with a positive definite metric $\tilde{g}$ and a Ricci tensor, whose symmetric part is vanishing, regardless of the codimension of the embedding.

This result shows that the previous corollary imposes a very strong restriction on the existence of embeddings in the case of Weyl manifolds. For example, Theorem 4 implies that, rather surprisingly, for a non-integrable Weyl manifold of dimension greater that 4, there does not exist an isometric immersion in a Riemann-flat space.

We shall now treat the very particular 4-dimensional case for which this restriction does not apply. In doing this we will make use of the restriction on the dimensionality of the embedding manifold only when necessary, so that the difficulties implied for the general dimensional case are made explicit.
A. The 4-dimensional case

The idea is to divide the equations $S\tilde{R}_{\alpha\beta} = 0$ into a set of constraint equations and a set of evolution equations. To do this, we shall impose an additional set of equations coming from the contracted Bianchi identities. Explicitly, we shall impose the equations $g^{\alpha\beta}\tilde{\nabla}_{\alpha}S_{\beta\gamma} = 0$, which, as can be seen from (13), is equivalent to imposing the following set of additional partial differential equations (PDE):

$$\tilde{g}^{\alpha\beta}\tilde{\nabla}_{\alpha}F_{\beta\gamma} = 0.$$  (24)

The above equations will be looked upon as a set of equations imposed on $\tilde{\omega}_{\beta}$. Thus, our complete system consists of (24) together with the following set of equations:

$$S\tilde{R}_{ij} = 0$$  (25)
$$S\tilde{R}_{i(n+1)} = 0$$  (26)
$$S\tilde{G}_{n+1}^{m+1} = 0$$  (27)

As we shall show, by using this scheme we can treat the problem as consisting of a set of evolution equations plus some constraint equations.

**Lemma 1.** Let $\bar{f}_{ik}(x,y), \bar{\psi}(x,y)$ and $\bar{\omega}_{\alpha}(x,y)$ be analytic functions at $0 \in \Sigma_0 \subset \mathbb{R}^{n+1}$. Suppose that $\bar{f}_{ik} = \bar{f}_{ki}, \text{det}(\bar{f}_{ik}) \neq 0$ and $\bar{\psi} \neq 0$ in a neighborhood of $0 \in \mathbb{R}^{n+1}$, that $\bar{f}_{ik}, \bar{\psi}$ and $\bar{\omega}_{\alpha}$ satisfy (24) and (25) in a neighborhood $V$ of $0 \in \mathbb{R}^{n+1}$ and also (26) and (27) in a neighborhood of $0 \in \Sigma_0$. Then, $\bar{f}_{ik}, \bar{\psi}$ and $\bar{\omega}_{\alpha}$ will satisfy (26) and (27) in a neighborhood of $0 \in \mathbb{R}^{n+1}$.

**Proof.** Since equation (24) is equivalent to $\tilde{g}^{\alpha\beta}\tilde{\nabla}_{\alpha}S\tilde{G}_{\beta\gamma} = 0$, then by hypothesis we have that:

$$\tilde{g}^{ij}\tilde{\nabla}_{j}S\tilde{G}_{i\sigma} + \frac{\epsilon}{\bar{\psi}}\tilde{\nabla}_{n+1}S\tilde{G}_{(n+1)\sigma} = 0$$

which is equivalent to the following:

$$\frac{\partial S\tilde{G}_{(n+1)\sigma}}{\partial y} = -\epsilon\bar{\psi}^2\tilde{g}^{ij}\tilde{\nabla}_{j}S\tilde{G}_{i\sigma} + \bar{\Gamma}_{(n+1)(n+1)}^{\gamma}S\tilde{G}_{\gamma\sigma} + \bar{\Gamma}_{(n+1)\sigma}^{\gamma}S\tilde{G}_{(n+1)\gamma} + \epsilon\bar{\psi}^2\tilde{g}^{ij}(\bar{\Gamma}_{ij}^{\gamma}S\tilde{G}_{\gamma\sigma} + \bar{\Gamma}_{j\sigma}^{\gamma}S\tilde{G}_{i\gamma})$$  (28)

To analyze these equations first set $\sigma = k$. We can use the fact that since (25) holds in a neighborhood of $0 \in \mathbb{R}^{n+1}$, then in such a neighborhood we have that the following holds
\( S\tilde{G}_{ik} = -\frac{\epsilon}{\psi} \tilde{g}_{ik} S\tilde{G}_{(n+1)(n+1)}. \) Then we get that:

\[
\frac{\partial}{\partial y} S\tilde{G}_{(n+1)(n+1)} = \partial_k^2 \tilde{G}_{(n+1)(n+1)} + \frac{\psi^2}{\tilde{g}^{ij}} \partial_j (\frac{\tilde{g}_{ik}}{\psi}) S\tilde{G}_{(n+1)(n+1)} - \frac{\epsilon}{\psi^2} \tilde{\Gamma}_{(n+1)(n+1)} \tilde{g}_{ik} S\tilde{G}_{(n+1)(n+1)} \\
+ \tilde{\Gamma}_{(n+1)(n+1)}^{n+1} S\tilde{G}_{(n+1)k} + \tilde{\Gamma}_{(n+1)(n+1)}^{\gamma} S\tilde{G}_{(n+1)\gamma} + \frac{\psi^2}{\tilde{g}^{ij}} (\tilde{\Gamma}_{ij}^{n+1} S\tilde{G}_{(n+1)k} + \tilde{\Gamma}_{ij}^{n+1} S\tilde{G}_{(n+1)l}) \\
- \frac{\epsilon}{\psi^2} \tilde{\Gamma}_{(n+1)(n+1)}^{l} \tilde{g}_{ik} S\tilde{G}_{(n+1)(n+1)} - \frac{\epsilon}{\psi^2} \tilde{\Gamma}_{(n+1)(n+1)}^{l} \tilde{g}_{ik} S\tilde{G}_{(n+1)(n+1)}
\]

(29)

Also setting \( \sigma = n + 1 \) in (28) we get:

\[
\frac{\partial}{\partial y} S\tilde{G}_{(n+1)(n+1)} = -\frac{\psi^2}{\tilde{g}^{ij}} \partial_j S\tilde{G}_{(n+1)} + 2\tilde{\Gamma}_{(n+1)(n+1)}^{\gamma} S\tilde{G}_{(n+1)} + \frac{\psi^2}{\tilde{g}^{ij}} (\tilde{\Gamma}_{ij}^{n+1} S\tilde{G}_{(n+1)k} + \tilde{\Gamma}_{ij}^{n+1} S\tilde{G}_{(n+1)l}) \\
- \frac{\epsilon}{\psi^2} \tilde{\Gamma}_{(n+1)(n+1)}^{l} \tilde{g}_{ik} S\tilde{G}_{(n+1)(n+1)}
\]

(30)

So we get that (28) is equivalent to the system of PDE formed by the equations (29) and (31), which are linear homogeneous equations on \( S\tilde{G}_{(n+1)\sigma} \) which can be written in the following form:

\[
\frac{\partial}{\partial y} S\tilde{G}_{(n+1)\sigma} = U_\sigma (x, y, S\tilde{G}_{(n+1)\beta}, \partial_j S\tilde{G}_{(n+1)\beta})
\]

(31)

and under our hypothesis the functions on the right hand side are analytic functions on some neighborhood of the origin in \( \mathbb{R}^{n+1} \). Also under our hypothesis we have that, not only this set of equations are satisfied, but they also satisfy the following initial data:

\[
S\tilde{G}_{(n+1)\sigma}(x, 0) = 0
\]

(32)

Now we know that the Cauchy-Kovalevskaya theorem asserts that this system admits just one set of analytic solutions satisfying these initial data, and since the system is homogeneous, we know that the trivial solution \( S\tilde{G}_{(n+1)\sigma} = 0 \) is such a solution, then this is the only solution. Hence the functions \( S\tilde{G}_{(n+1)\sigma} \) are actually zero on a neighborhood of the origin in \( \mathbb{R}^{n+1} \) and this finishes the proof.

First, we shall show that (24) and (25) have a solution in a neighborhood of \( 0 \in \mathbb{R}^{n+1} \). In order to do this we need to write down these equations explicitly. From (19) we find that:

\[
S\tilde{R}_{ij} = -\frac{\epsilon}{\psi} \tilde{R}_{ij} + \frac{\epsilon}{\psi} \tilde{g}^{kl} (l_{ij} l_{kl} - 2l_{ki} l_{jl}) + \frac{1}{\psi} \nabla_j \nabla_i \psi - \frac{1}{4} (\nabla_j \tilde{\omega}_i + \nabla_i \tilde{\omega}_j) + \frac{1}{4} \tilde{\omega}_i \tilde{\omega}_j \\
- \frac{1}{2\psi} (\tilde{\omega}_i \partial_j \tilde{\psi} + \tilde{\omega}_j \partial_i \tilde{\psi} - \epsilon \tilde{\omega}_{n+1} l_{ij}).
\]
By using the fact that
\[ l_{ij} = -\frac{1}{2\psi}\partial_{n+1}g_{ij} + \frac{1}{2\psi}\tilde{\omega}_{n+1}g_{ij}, \]
we can write (25) in the form
\[
\epsilon^2\psi^2\partial_y^2 g_{ij} - \frac{1}{\psi}\tilde{\omega}_n\tilde{\omega}_j - g_{kl}(l_{ij}l_{kl} - 2l_{ik}l_{jl}) - \frac{1}{4}\tilde{\omega}_j\tilde{\omega}_j - \frac{1}{2}\epsilon\tilde{\omega}_{n+1}\tilde{\omega}_j - \epsilon g_{kl}(l_{ij}l_{kl} - 2l_{ik}l_{jl}) - \frac{1}{4}\psi\partial_j\psi + \frac{1}{2}\psi\partial_j\tilde{\omega}_n - \frac{1}{2}\psi\partial_y g_{ij}
\] (33)

On the other hand, (24) is equivalent to:
\[
\tilde{\gamma}^{\mu\lambda\nu}\partial_{\lambda}\partial_{\mu}\tilde{\omega}_{\nu} + \tilde{\gamma}^{\mu\lambda\nu}\partial_{\lambda}
\] (34)

Thus, from the compatibility condition we can rewrite the first term as
\[
\tilde{\gamma}^{\mu\lambda\nu}\partial_{\lambda}\partial_{\mu}\tilde{\omega}_{\nu} = \tilde{\gamma}^{\lambda\nu}\tilde{\omega}_{\lambda} - \tilde{\gamma}^{\lambda\nu}\tilde{\omega}_{\lambda} + \tilde{\gamma}^{\mu\lambda\nu}\partial_{\lambda}\tilde{\omega}_{\nu} + \tilde{\gamma}^{\mu\lambda\nu}\partial_{\lambda}\tilde{\omega}_{\nu} + \tilde{\gamma}^{\mu\lambda\nu}\partial_{\lambda}\tilde{\omega}_{\nu}.
\]

From the definition of the curvature tensor we have
\[
\tilde{\gamma}_{\lambda\nu}\tilde{\omega}_\lambda = \tilde{\gamma}_{\lambda\nu}\tilde{\omega}_\lambda + \tilde{\gamma}_{\lambda\nu}\tilde{\omega}_\lambda + \tilde{\gamma}_{\lambda\nu}\tilde{\omega}_\lambda + \tilde{\gamma}_{\lambda\nu}\tilde{\omega}_\lambda - \tilde{\gamma}_{\lambda\nu}\tilde{\omega}_\lambda.
\]

In this way, we get
\[
\tilde{\gamma}^{\mu\lambda\nu}\partial_{\lambda}\partial_{\mu}\tilde{\omega}_{\nu} - \tilde{\gamma}^{\lambda\nu}\tilde{\omega}_{\lambda} - \tilde{\gamma}^{\lambda\nu}\tilde{\omega}_{\lambda} - \tilde{\gamma}^{\mu\lambda\nu}\partial_{\lambda}\tilde{\omega}_{\nu} + \tilde{\gamma}^{\mu\lambda\nu}\partial_{\lambda}\tilde{\omega}_{\nu} + \tilde{\gamma}^{\mu\lambda\nu}\partial_{\lambda}\tilde{\omega}_{\nu} = 0. \]

These equations are equivalent to (34). Unfortunately, they cannot be written in a form where we can apply the Cauchy-Kovalevskaya theorem. However, if we consider these equations in the Lorentz gauge \( \tilde{\gamma}^{\lambda\nu}\tilde{\omega}_\lambda = 0 \), we can show that the resulting set of reduced equations can be cast in the form required by this theorem. Now, writing these equations explicitly
we get
\[ \epsilon \frac{\partial^2 \tilde{\omega}_\nu}{\partial y^2} = - \bar{g}^{ij} \tilde{\nabla}_i \tilde{\nabla}_j \tilde{\omega}_\nu + \tilde{\omega}^\lambda \tilde{\nabla}_\lambda \tilde{\omega}_\nu + \tilde{g}^{\mu \lambda} \tilde{\nabla}_\lambda \tilde{\omega}_\mu + \tilde{\omega}_\mu \tilde{\nabla}_\mu \tilde{\omega}_\nu - \tilde{\omega}_\mu \tilde{\omega}_\nu - \tilde{R}_{\sigma \nu} \tilde{\omega}_\sigma \]
\[ + \epsilon \frac{\partial \tilde{\Gamma}_\sigma (n+1)\nu \tilde{\omega}_\sigma}{\partial y} + \tilde{\Gamma}_\sigma (n+1)\nu \tilde{\omega}_\sigma - \tilde{\Gamma}_\sigma (n+1)\nu \tilde{\omega}_\sigma \]
\[ + \tilde{\Gamma}_\sigma (n+1)(n+1) \partial \tilde{\omega}_\nu - \tilde{\Gamma}_\sigma (n+1)(n+1) \tilde{\omega}_\sigma \] (36)

We shall regard these equations together with (33) as a system of PDEs for \((\tilde{g}, \tilde{\omega})\). It is important to remark that (36) depends on \(\frac{\partial^2 \tilde{\omega}}{\partial y^2}\) through terms such as \(\frac{\partial \tilde{\Gamma}_\sigma (n+1)\nu \tilde{\omega}_\sigma}{\partial y}\) or \(\tilde{R}_{\sigma \nu} \tilde{\omega}_\sigma\).

But, as we are regarding (33) and (36) as a system, we just replace \(\frac{\partial^2 \tilde{\omega}}{\partial y^2}\) in (36) using (33).

Thus, if we consider that \(\bar{\psi}\) is a given analytic function in a neighborhood of the origin of \(\mathbb{R}^{n+1}\) which satisfies \(\bar{\psi} \neq 0\) in this neighborhood, then (33) and (36) yield a system in the form

\[ \frac{\partial^2 \bar{\sigma}_{ij}}{\partial y^2} = F_{ij}(x, y, \bar{g}_{ij}, \bar{\omega}_\alpha, \partial_\alpha \bar{g}_{ij}, \partial_\beta \bar{\omega}_\alpha, \partial_\alpha i \bar{g}_{ij}, \partial_\beta j \bar{\omega}_\alpha) \ 1 \leq i < j \leq n \ ; \ \alpha, \beta = 1, \ldots, n + 1 \]
\[ \frac{\partial^2 \bar{\omega}_\beta}{\partial y^2} = U_\beta(x, y, \bar{g}_{ij}, \bar{\omega}_\alpha, \partial_\alpha \bar{g}_{ij}, \partial_\beta \bar{\omega}_\alpha, \partial_\alpha i \bar{g}_{ij}, \partial_\beta j \bar{\omega}_\alpha) \ 1 \leq i < j \leq n \ ; \ \alpha, \beta = 1, \ldots, n + 1 \] (37)

Therefore, if we choose a specific order for the \(\frac{n(n+1)}{2}\) components of \(\bar{g}_{ij}\) and the \(n + 1\) components of \(\bar{\omega}_\beta\), then (37) may be regarded as a system of \(\frac{(n+1)(n+2)}{2}\) PDEs for the \(\frac{(n+1)(n+2)}{2}\) functions \((\bar{g}_{ij}, \bar{\omega}_\beta)\). For such a system we give the following initial data

\[ \bar{g}_{ik}(x, 0) = g_{ik}(x) \ 1 \leq i < k \leq n \]
\[ \bar{\omega}_\beta(x, 0) = \omega_\beta(x) \ \beta = 1, \ldots, n + 1 \] (38)

\[ \frac{\partial \bar{g}_{ik}}{\partial y}(x, 0) = -2\bar{\psi}(x, 0)\Omega_{ik}(x) + g_{ik}(x)\omega_{n+1}(x) = g_{ik}'(x) \ 1 \leq i < k \leq n \]
\[ \frac{\partial \bar{\omega}_\beta}{\partial y}(x, 0) = \omega_\beta'(x) \ \beta = 1, \ldots, n + 1 \] (39)

where \(\omega_\beta, \omega_\beta', \Omega_{ik}\) and \(g_{ik}\) are all analytic functions at \(0 \in \mathbb{R}^n\), and it is required that the initial data \(g_{ik}\) also satisfy that the condition \(det(g_{ik})(0) \neq 0\). It is important to note that the right-hand side of (37) consists of rational functions of the variables
\(g_{ij}, \bar{\omega}_\alpha, \partial_\alpha g_{ij}, \partial_\beta \bar{\omega}_\alpha, \partial_{aa} g_{ij}, \partial_{a\beta} \bar{\omega}_\alpha,\) and all the denominators are just \(\det(g_{ik})\). On the other hand, it follows from the initial data that \(\det(g_{ik})(0,0) = \det(g_{ik})(0) \neq 0\). Thus, since \(\det(g_{ik})\) is a polynomial of the functions \(g_{ik}\), and we know that for \(g_{ik} = g_{ik}(0,0)\) this polynomial is different from zero, then there is a neighborhood of \((g_{ik})\) where this polynomial does not vanish. Using both this fact and that the functions \(F_{ij}\) and \(U_\beta\) are just these rational functions multiplied by some power of \(\bar{\psi}\), which, in turn, is an analytic function in a neighborhood of \(0 \in \mathbb{R}^{n+1}\) and \(\bar{\psi} \neq 0\) in this neighborhood, we then see that \(F_{ij}\) and \(U_\beta\) are analytic functions at \(P = (0,0, \bar{g}_{ij}(0,0), \bar{\omega}_\alpha(0,0), \partial_\alpha \bar{g}_{ij}(0,0), \partial_\beta \bar{\omega}_\alpha(0,0), \partial_{aa} \bar{g}_{ij}(0,0), \partial_{a\beta} \bar{\omega}_\alpha(0,0))\). We can thus use the Cauchy-Kovalevskaya theorem to guarantee the existence of solutions of the system (37) with the initial data (38)-(39). It is important to remark that, since the Cauchy-Kovalevskaya theorem guarantees the existence of analytic solutions in a neighborhood of \(0 \in \mathbb{R}^{n+1}\), and as \(\det(g_{ij})(0,0) \neq 0\) from the initial data, then by continuity we know that there exists a neighborhood of \(0 \in \mathbb{R}^{n+1}\) where \(\det(g_{ij})(x,y) \neq 0\).

In this way we have constructed a set of analytic solutions of the reduced equations (37). In order for these solutions to satisfy the original set of equations (24)-(25) we need to show that they satisfy the gauge condition \(\bar{\nabla}_\lambda \bar{\omega}^\lambda = 0\). With this in mind we present the following lemma.

**Lemma 2.** Consider \(n+1 = 4\) and suppose that \((\bar{g}, \bar{\omega})\) is an analytic solution of (37) satisfying the initial data (38)-(39), and also assume that

\[
\bar{\nabla}_\lambda \bar{\omega}^\lambda|_{\Sigma_0} = 0 \tag{40}
\]

\[
\partial \bar{\nabla}_\lambda \bar{\omega}^\lambda|_{\Sigma_0} = 0 \tag{41}
\]

Then, \((\bar{g}, \bar{\omega})\) satisfy the complete system of equations (24)-(25).

**Proof.** If \(\dim(\bar{M}) = 4\) then we have seen that the following identity is satisfied on \(\bar{M}:
\]

\[
\bar{g}^{\alpha\beta} \bar{\nabla}_\alpha (\bar{g}^{\mu\nu} \bar{\nabla}_\mu F_{\beta\nu}) + \bar{\omega}^\beta \bar{g}^{\mu\nu} \bar{\nabla}_\mu F_{\beta\nu} = 0.
\]

Also if \((\bar{g}, \bar{\omega})\) satisfy the reduced equations, then from (35) we get that

\[
\bar{g}^{\mu\nu} \bar{\nabla}_\mu F_{\beta\nu} = - \bar{\nabla}_\beta \bar{\nabla}_\lambda \bar{\omega}^\lambda
\]

(42)

Then the previous identity gives us the following:

\[
\bar{g}^{\alpha\beta} \bar{\nabla}_\alpha \bar{\nabla}_\beta (\bar{\nabla}_\lambda \bar{\omega}^\lambda) + \bar{\omega}^\beta \bar{\nabla}_\beta (\bar{\nabla}_\lambda \bar{\omega}^\lambda) = 0.
\]
It is not difficult to show that this is a second order linear and homogeneous equation for the function $\tilde{\nabla}_\lambda \tilde{\omega}^\lambda$ which can be rewritten as follows:

$$\frac{\partial^2 (\tilde{\nabla}_\lambda \tilde{\omega}^\lambda)}{\partial y^2} = F(x, y, \tilde{\nabla}_\lambda \tilde{\omega}^\lambda, \partial_{\alpha}(\tilde{\nabla}_\lambda \tilde{\omega}^\lambda), \partial_{i\alpha}(\tilde{\nabla}_\lambda \tilde{\omega}^\lambda))$$

where the right-hand side is an analytic function at the origin. Then the Cauchy-Kovalevskaya theorem guarantees the existence of a unique solution for this equation satisfying the initial data (40)-(41). Since $\tilde{\nabla}_\lambda \tilde{\omega}^\lambda = 0$ satisfies all these requirements, we get that this is the unique solution. Using this in (42) we see that under these conditions $(\tilde{g}, \tilde{\omega})$ satisfies the full system (24)-(25).

Using this lemma, which only works in the 4-dimensional case, we see that we should look at (40) and (41) as additional constraints. Thus, our system of constraint equations consists of the equations (26)-(27) on the hypersurface $\Sigma_0$, together with the equations (40)-(41). This system is posed for the second fundamental form of $\Sigma_0$, $\Omega_{ij}$, and the initial data $\omega'_\beta$, and will be referred as the Weyl constraints equations. We shall denote the initial data set by $(\Sigma_0, g, \omega, \Omega, \omega')$, where $(\Sigma_0, g, \omega)$ gives the Weyl structure of the hypersurface $\Sigma_0$. With these notations, we can state the following theorem:

**Theorem 5.** Let $(\Sigma_0, g, \omega, \Omega, \omega')$ be an initial data set satisfying the Weyl constraint equations. Then $(\Sigma_0, g, \omega)$ admits a local analytic isometric embedding around $p \in \Sigma_0$ in a Weyl manifold $(\tilde{M}^4, \tilde{g}, \tilde{\omega})$ such that the symmetric part of the Ricci tensor of the embedding manifold vanishes.

Using this theorem, we see that in order to guarantee the existence of an isometric embedding of $(M^3, g, \omega)$ at $p \in M^3$ in a Weyl manifold $(\tilde{M}^4, \tilde{g}, \tilde{\omega})$ having vanishing symmetric part of its Ricci tensor, we just need to show that we can always find an initial data set $(M^3, g, \omega, \Omega, \omega')$ satisfying the Weyl constraint equations in a neighborhood of $p \in M^3$. When dealing with these constraints, we shall make use of the gauge freedom we have in Weyl’s geometry. We have already seen that if we can construct an embedding for some element $(M, g, \omega) \in (M, [g], [\omega])$ in $(\tilde{M}, \tilde{g}, \tilde{\omega}) \in (\tilde{M}, [\tilde{g}], [\tilde{\omega}])$ then we can construct an embedding for each element of $(M, [g], [\omega])$ in some element of $(\tilde{M}, [\tilde{g}], [\tilde{\omega}])$. Thus, we shall select a particular element of $(M, [g], [\omega])$ where (40) is satisfied. Let us show that we can always do this. First, consider that $(\tilde{g}_{ij}, \tilde{\omega}_\beta)$ is a solution of (37) in a neighborhood $U$ of
0 ∈ \mathbb{R}^{n+1} satisfying the initial data (38)-(39). Then, defining
\[ \tilde{g} = g^i_j dx^i \otimes dx^j + \epsilon \tilde{\psi}^2 dy \otimes dy, \]
we have that \((U, \tilde{g}, \tilde{\omega})\) is a well-defined Weyl manifold. Under these assumptions note that
\[ \tilde{\nabla}_\lambda \tilde{\omega}^\lambda = \nabla_k \omega^k + \tilde{\nabla}_{n+1} \omega^{n+1}, \]
hence
\[ \tilde{\nabla}_\lambda \tilde{\omega}^\lambda|_{\Sigma_0} = \nabla_k \omega^k + \tilde{\nabla}_{n+1} \omega^{n+1}(x, 0). \]
This last expression only depends on the initial data \((g, \omega)\) and \(\tilde{\omega}_{n+1}|_{\Sigma_0}, \partial_{n+1} \tilde{\omega}_{n+1}|_{\Sigma_0}\). We now make the Weyl transformation
\[ g \to e^{-f} g, \]
\[ \omega_k \to \omega_k - \partial_k f \]
for some analytic function \(f\). Then, for \((e^{-f} g, \omega - df)\) (40) is equivalent to the equation
\[ g^{k\mu} \nabla_k \nabla_\mu f + g^{k\mu} (\omega_k - \nabla_k f)(\omega_\mu - \nabla_\mu f) + g^{k\mu} \nabla_k \omega_\mu + e^f (\partial_y \tilde{\omega}^4 + \tilde{\Gamma}_4^{4\sigma} \tilde{\omega}^\sigma)|_{y=0} = 0, \]
where \(\tilde{\Gamma}_4^{4\sigma} = \frac{1}{\psi} \partial_\sigma \tilde{\psi} - \frac{1}{2} \tilde{\omega}_\sigma\). Since \(\tilde{\psi}\) is considered as a given analytic function and both \(\tilde{\omega}_4|_{\Sigma_0}\) and \(\partial_y \tilde{\omega}_4|_{\Sigma_0}\) are also arbitrary given analytic functions, then (44) is a second-order PDE for the function \(f\). Thus, from now on, we shall regard \(\tilde{\psi}|_{\Sigma_0} = \psi(x), \tilde{\omega}_4|_{\Sigma_0} = \omega_4(x)\) and \(\partial_y \tilde{\omega}_4|_{\Sigma_0} = \eta(x)\) as given analytic functions, which will be involved in the initial data of the system (37). Then, we can guarantee the existence of an analytic solution for (44). To see this, we can use a coordinate system \((x^i)\) on \(M\) around \(p\), satisfying that \(g_{1k'} = 0\), with \(k' = 2, 3\). In this way, (41) can be cast in the form
\[ g^{11} \nabla_1 \nabla_1 f + g^{k'u'} \nabla_{k'} \nabla_{u'} f + g^{k\mu} (\omega_k - \nabla_k f)(\omega_\mu - \nabla_\mu f) + g^{k\mu} \nabla_k \omega_\mu + e^f \left\{ \frac{\epsilon}{\psi^2} \eta - \frac{2\epsilon}{\psi^3} \frac{\partial_\psi}{\partial y}|_{\Sigma_0} \omega_4 + \left( \frac{1}{\psi} \partial_k \psi - \frac{1}{2} \omega_k \right) \omega_4 + \frac{\epsilon}{\psi} \left( \frac{1}{\psi} \partial_\psi \tilde{\omega}_4 \right) \right\} = 0, \]
where \(g^{11} \neq 0\) in a neighborhood of the origin, \(k', u' = 2, 3\), and all the known quantities involved are analytic in a neighborhood of \(0 \in \mathbb{R}^n\). Therefore, this last equation has the form
\[ \frac{\partial^2 f}{\partial (x^1)^2} = \mathcal{U}(x, \partial_i f, \partial_{k'} f), \]
25
where the right-hand side is analytic at the origin, and hence the Cauchy-Kovalevskaya theorem guarantees the existence of an analytic solution. We thus have shown that, given a Weyl manifold \((M^3, g, \omega)\) we can always find an element of \((M^3, [g], [\omega])\) for which (40) is satisfied. Hence there is no loss of generality in assuming that \((M^3, g, \omega)\) satisfies this condition. In this way, we can reduce the Weyl constraint equations to the following set of equations:

\[
\epsilon \psi g^{kl}(\nabla_k \Omega_{il} - \nabla_l \Omega_{ik}) + \frac{\psi}{2} g^{kl}(\omega_i \Omega_{kl} - \omega_k \Omega_{il}) + \frac{n-1}{4} (\partial_i \eta - \omega_i') = 0 \tag{45}
\]

\[
g^{ij} g^{kl}(R_{kilj} + \epsilon (\Omega_{ij} \Omega_{kl} - \Omega_{ki} \Omega_{jl})) = 0 \tag{46}
\]

\[
\frac{\partial \tilde{\nabla}_\lambda \tilde{\omega}^\lambda}{\partial y}|_{\Sigma_0} = 0. \tag{47}
\]

Equations (45) and (46) are dealt with in the same way it was done in \([24]\), just carrying the extra terms along the same computations. Doing this, from (46) we find an explicit expression for \(\Omega_{11}\) in terms of the other variables and from (45) a set first-order PDEs of the Cauchy-Kovalevskaya-type for the functions \(\Omega_{k'k}\) and \(\Omega_{r'3}\), where \(k' = 2, 3\) and \(r'\) is fixed, having either the value 2 or 3. In this system, the remaining components of \(\Omega_{ij}\) are set as arbitrary analytic functions. In the same way we did when dealing with (40), in this procedure, a coordinate system on \(M^3\) around \(p\) is chosen such that \(g_{1k'} = 0\), and the coordinate \(x^1\) is chosen as the variable with respect to which we pose the constraint equations in the Cauchy-Kovalevskaya form. Now, we shall deal with the remaining equation (47).

First, let us write it down explicitly:

\[
\frac{\partial (\tilde{\nabla}_\lambda \tilde{\omega}^\lambda)}{\partial y} = \frac{\partial (\partial_k \tilde{\omega}^k)}{\partial y} + \frac{\partial^2 \tilde{\omega}^4}{\partial y^2} + \frac{\partial \tilde{\Gamma}^k_{kj} \tilde{\omega}^j}{\partial y} + \tilde{\Gamma}^k_{kj} \frac{\partial \tilde{\omega}^j}{\partial y} + \frac{\partial \tilde{\Gamma}^4_{4\sigma} \tilde{\omega}^\sigma}{\partial y} + \tilde{\Gamma}^4_{4\sigma} \frac{\partial \tilde{\omega}^\sigma}{\partial y}. \tag{48}
\]

Using the following expressions for the connection components involved in the previous expression

\[
\tilde{\Gamma}^4_{4\beta} = \frac{1}{\psi} \partial_\beta \psi - \frac{1}{2} \tilde{\omega}_\beta,
\]

\[
\tilde{\Gamma}^4_{ij} = \Gamma^4_{ij},
\]

\[
\tilde{\Gamma}^l_{ij} = \Gamma^l_{ij},
\]

26
together with the definition of the second fundamental form $\Omega$,

$$\Omega_{ji} = \left\{ -\frac{1}{2\psi} \frac{\partial g_{ij}}{\partial y} + \frac{1}{2\psi} \tilde{g}_{ij} \tilde{\omega}_4 \right\}|_{\Sigma_0},$$

we see that when we restrict (48) to $\Sigma_0$ the last four terms depend on given data $g_{ij}, \omega_k, \eta, \omega'_4, \psi$ and terms up to first-order in $\Omega_{ij}$ and the remaining $\omega'_k$. Also, since $(\tilde{g}, \tilde{\omega})$ satisfy the reduced equations (37), then:

$$\frac{\partial^2 \tilde{\omega}_4}{\partial y^2} = U_{n+1}(x, y, \tilde{g}_{ij}, \tilde{\omega}_i, \partial \tilde{\omega}_i, \partial \tilde{g}_{ij}, \partial \tilde{\omega}_\alpha), \quad 1 \leq i < j \leq 3; \quad a = 1, 2, 3.$$

It follows that

$$\frac{\partial^2 \tilde{\omega}_4}{\partial y^2}|_{\Sigma_0} = U'_{n+1}(x, \Omega_{ij}, \omega'_\alpha, \partial \omega'_\alpha), \quad 1 \leq i < j \leq 3; \quad a = 1, 2, 3.$$

Thus, constraining (48) to $\Sigma_0$ and setting the left-hand side equal to zero, we get

$$g^{ku} \partial_k \omega'_u + \partial_k g^{ku} \omega'_u + \partial_k \tilde{g}^{ku}|_{\Sigma_0} \omega_u + \frac{\partial \tilde{g}^{ku}}{\partial y}|_{\Sigma_0} \partial_k \omega_u + O(x, \Omega_{ij}, \omega'_k, \partial \Omega_{ij}) = 0.$$

Using the same special form of the metric in the coordinate system used to study the constraints, we can rewrite this last equation as

$$\frac{\partial \omega'_1}{\partial x^1} = O'(x, \Omega_{ij}, \omega'_k, \partial \omega'_k, \partial \Omega_{ij}) \quad i, j, k = 1, 2, 3; \quad j', k' = 2, 3.$$

Then, we see that the constraint equations (45)-(47) can be written as a set of first-order PDEs of the following form:

$$\frac{\partial \Omega'_{1'}}{\partial x^1} = H'_{1'}(x, \Omega_{1j'}, \omega'_1, \partial \omega'_1 \Omega_{1j'}), \quad u', j', k' = 2, 3$$

$$\frac{\partial \Omega'_{3'}}{\partial x^1} = H'_{3'}(x, \Omega_{1j'}, \omega'_1, \partial \omega'_1 \Omega_{1j'}), \quad u', j' = 2, 3; \quad r' \ fixed \ with \ r' = 2 \ or \ r' = 3 \quad (49)$$

$$\frac{\partial \omega'_{1'}}{\partial x^1} = O'(x, \Omega_{1j'}, \omega'_1, \partial \omega'_1 \Omega_{1j'}), \quad u', j' = 2, 3$$

together with an explicit algebraic expression for $\Omega_{11}$. In this set up the rest of the $\Omega_{ij}$ and $\omega'_2, \ldots, \omega'_n$ are set as given arbitrary analytic functions. The equations (49) are of the Cauchy-Kovalevskaya type and hence we know that this system admits a solution. We now can state the main result of this section.

**Theorem 6.** Any 3-dimensional Weyl structure $(M^3, [g], [\omega])$ admits a local analytic isometric embedding at any point $p \in M^3$ in a 4-dimensional Weyl structure $(\tilde{M}^4, [\tilde{g}], [\tilde{\omega}])$ having vanishing symmetric part of its Ricci tensor.
VI. FINAL COMMENTS

In the present article, we have considered the embedding problem in the context of Weyl geometry and have proven that some of the Campbell-Magaard-type theorems can be naturally extended from Riemannian to Weyl’s geometry, although some instances appear that are not exactly analogous to their Riemannian counterpart. The investigation of embeddings in Weyl manifolds has led us to discover an interesting and rather unexpected no-go result in this direction, and to establish an important geometrical identity which seems to be essential for studying embeddings in Weyl spaces, in arbitrary dimensions, in which the symmetric part of its Ricci tensor vanishes. We have worked out the embedding problem in the 3-dimensional case and showed that this solution does not hold in other dimensions. We believe that the complete solution of the general problem, still left open, may be regarded as a mathematical motivation for studying other embedding problems in the framework of Weyl’s geometry, which may originate from modern theoretical physics. Finally, we would like to mention that an extension of \[25\] to the context of Weyl geometry is being studied at the moment. The results coming from these further studies should be considered as a completion of the present article.

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