Perturbations and quasi-normal modes of black holes with time-dependent scalar hair in shift-symmetric scalar-tensor theories

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We study odd parity perturbations of spherically symmetric black holes with time-dependent scalar hair in shift-symmetric higher-order scalar-tensor theories. The analysis is performed in a general way without assuming the degeneracy conditions. Nevertheless, we end up with second-order equations for a single master variable, similarly to cosmological tensor modes. We thus identify the general form of the quadratic Lagrangian for the odd parity perturbations, leading to a generalization of the Regge-Wheeler equation. We also investigate the structure of the effective metric for the master variable and refine the stability conditions. As an application of our generalized Regge-Wheeler equation, we compute the quasi-normal modes of a certain nontrivial black hole solution. Finally, our result is extended to include the matter energy-momentum tensor as a source term.

I. INTRODUCTION

The remarkable first direct detection of gravitational waves from a binary black hole merger [1] has opened a new era of astrophysics and gravitational physics. Gravitational waves are becoming more and more important as a probe of strong gravitational fields and as a tool for testing gravity in the strong field regime. To establish general relativity on a firmer basis, it is necessary to study the predictions from alternative theories such as scalar-tensor theories and test them against gravitational wave observations. In modified theories gravity, black holes may have hair, i.e., nontrivial configurations of scalar or extra degrees of freedom around themselves, and the perturbation dynamics may also differ from that in general relativity. Therefore, identifying the general action for perturbations around a hairy black hole will help to achieve the above purpose. The results will be useful for instance for the computation of quasi-normal modes (QNMs) in modified gravity.

In this paper, we determine the general action governing odd parity perturbations around a spherically symmetric black hole dressed with a linearly time-dependent scalar field. To do so, we start from a covariant action for shift-symmetric higher-order scalar-tensor theories admitting such time-dependent scalar hair and second-order field equations at least in the odd parity sector. In contrast to Refs. [2–4], we do not take the effective-field-theory (EFT) approach, because our background scalar-field configuration depends not only on the radial coordinate but also on time, which breaks the usual assumption of the EFT on the symmetry. In such a case, it is probably more convenient to start from a covariant action.

To make the results as general as possible, we work with general shift-symmetric higher-order scalar-tensor theories whose action depends on the curvature tensors and first and second derivatives of the scalar field in such a way that yields healthy second-order field equations for gravitational-wave degrees of freedom. Probably the most well-known example of such theories is the Horndeski theory [5], or, equivalently, the generalized Galileon theory [6, 7]. The perturbation theory for spherically symmetric black holes with static hair has been developed in Refs. [8, 9]. When restricted to the shift-symmetric subclass, the Horndeski theory admits linearly time-dependent scalar hair, as was first demonstrated in Ref. [10] and later generalized in Ref. [11], thus evading the no-hair theorem in shift-symmetric scalar-tensor theories [12]. The black-hole perturbation theory can be extended to the case with linearly time-dependent scalar hair [13, 14] (see Ref. [15] for a recent update). Though the Horndeski theory is the most general scalar-tensor theory having second-order field equations both for the metric and scalar field, later it was noticed that it can further be generalized while maintaining one scalar and two tensorial degrees of freedom [16–19] (see also Ref. [20] for an earlier work seeking theories beyond Horndeski by means of a disformal transformation of the metric). The basic idea behind this generalization is that if some of the field equations are degenerate, then the number of dynamical degrees of freedom is reduced and thus the system can retain one scalar and two tensorial degrees of freedom even if the field equations are apparently of higher order. Theories with such a structure are called degenerate higher-order scalar-tensor (DHOST) theories. See Refs. [21–23] for a review. In DHOST theories in which second derivatives of the scalar field appear quadratically in the Lagrangian (i.e., quadratic DHOST theories), odd parity perturbations of spherically symmetric black holes with linearly time-dependent scalar hair have been studied in Ref. [24], with the results showing that the master variable for the odd parity perturbations obeys a second-order equation having essentially the same structure as that in the case of the Horndeski theory [13, 14]. More recently, it was argued that the dangerous ghost degrees of freedom remain to be absent even if one relaxes the degeneracy conditions so that the system is degenerate only in the unitary gauge, giving rise to the notion of “U-degenerate” theories [25]. If a
theory is treated as a low-energy effective theory rather than a complete one, it is sufficient to require that no ghost degrees of freedom emerge within the regime of validity of the effective theory. This viewpoint allows us to consider the theories in which the degeneracy conditions are detuned slightly [26]. Detuning the degeneracy conditions help to resolve the problem of infinite strong coupling in the even parity sector [27, 28].

In light of these developments, we will study black-hole perturbations in higher-order scalar-tensor theories that are most closely related to cubic DHOST theories [19], but without imposing the degeneracy conditions. Still, at least in the odd parity sector, we will have a second order equation for a single master variable and can thus determine the general form of the action for the master variable, generalizing the previous results [13, 14, 24].

This paper is organized as follows. In the next section, we present the covariant action for scalar-tensor theories which we will work with. In Sec. III we give an example of spherically symmetric background solutions with time-dependent scalar hair. Then, in Sec. IV we determine the general action for odd parity perturbations and derive the generalized Regge-Wheeler equation. We also refine the previous notion of the stability conditions by investigating the structure of the effective metric for gravitons. In Sec. V, we calculate the QNMs of the black hole solution we present in Sec. III. Section VI is devoted to a summary of conclusions. In the appendices we argue that the theory is degenerate at least in the unitary gauge, which can still provide a healthy class of theories called U-degenerate theories [25]. If the action (1) is regarded as a low-energy truncation of some complete theory, detuning the degeneracy conditions is acceptable because a ghost degree of freedom itself is not problematic from the effective-field-theory viewpoint [26].

In this paper, we do not assume any particular relations among the functions in the action. Nevertheless, we can handle the relevant equations and derive the universal form of the quadratic Lagrangian for odd mode perturbations around a spherically symmetric background with time-dependent scalar hair.

II. HIGHER-ORDER SCALAR-TENSOR THEORIES

We consider a system composed of the metric $g_{\mu\nu}$ and the scalar field $\phi$ described by the action [17–19]

$$S_{\text{grav}} = \int d^4x \sqrt{-g} \left[ F_0(X) + F_1(X) \Box \phi + F_2(X) R \right] + \frac{1}{2} \sum_{I=1}^{10} A_I(X) L_I^{(2)} + F_3(X) G_{\mu\nu} \phi^{\mu\nu} + \sum_{I=1}^{10} B_I(X) L_I^{(3)} ,$$

where $X := -\phi_{\mu} \phi^{\mu} / 2$, $\phi_{\mu} := \nabla_{\mu} \phi$, $\phi_{\mu\nu} = \nabla_{\mu} \nabla_{\nu} \phi$, $R$ is the Ricci scalar, and $G_{\mu\nu}$ is the Einstein tensor. Here, $L_I^{(2)}$ are quadratic in the second derivatives of the scalar field and are written explicitly as

$$L_1^{(2)} = \phi_{\mu\nu} \phi^{\mu\nu} , \quad L_2^{(2)} = (\Box \phi)^2 , \quad L_3^{(2)} = (\Box \phi) \phi_{\mu\nu} \phi^{\mu\nu} ,$$

$$L_4^{(2)} = \phi^{\mu} \phi_{\mu\nu} \phi^{\nu} , \quad L_5^{(2)} = (\phi^{\mu} \phi_{\mu\nu} \phi^{\nu})^2 .$$

Similarly, $L_I^{(3)}$ are cubic in the second derivatives of the scalar field and are given by

$$L_1^{(3)} = (\Box \phi)^3 , \quad L_2^{(3)} = (\Box \phi) \phi_{\mu\nu} \phi^{\mu\nu} ,$$

$$L_3^{(3)} = \phi_{\mu\nu} \phi^{\mu\nu} \phi^{\sigma} \phi_{\rho\sigma} , \quad L_4^{(3)} = (\Box \phi)^2 \phi_{\mu\nu} \phi^{\mu\nu} ,$$

$$L_5^{(3)} = \phi_{\mu\nu} \phi_{\rho\sigma} \phi_{\lambda\mu} \phi_{\nu\sigma} , \quad L_6^{(3)} = \phi_{\mu\nu} \phi_{\rho\sigma} \phi_{\lambda\mu} \phi_{\nu\sigma} ,$$

$$L_7^{(3)} = \phi_{\mu\nu} \phi_{\rho\sigma} \phi_{\lambda\mu} \phi_{\nu\sigma} , \quad L_8^{(3)} = \phi_{\mu\nu} \phi_{\rho\sigma} \phi_{\lambda\mu} \phi_{\nu\sigma} ,$$

$$L_9^{(3)} = \phi_{\mu\nu} \phi_{\rho\sigma} \phi_{\lambda\mu} \phi_{\nu\sigma} , \quad L_10^{(3)} = (\phi_{\mu\nu} \phi_{\rho\sigma} \phi_{\lambda\mu} \phi_{\nu\sigma})^3 .$$

These exhaust possible terms built from $\phi_{\mu}$ and $\phi_{\mu\nu}$ and quadratic/cubic in $\phi_{\mu\nu}$. The functions $F_0, F_1, F_2, F_3, A_1$, and $B_I$ depend only on $X$, so that the theory has shift symmetry, $\phi \rightarrow \phi + c$.

In general, the action (1) yields higher-order equations of motion for the metric and the scalar field, resulting in the dangerous Ostrogradsky ghost. One can circumvent this by imposing the degeneracy conditions among the functions $F_2, F_3, A_1$, and $B_I$ [17–19]. In such degenerate theories, one arrives in the end at a set of second-order equations by combining the different components of field equations, and thus can remove the unstable ghost degrees of freedom. One may relax the degeneracy conditions so that the theory is degenerate at least in the unitary gauge, which can still provide a healthy class of theories called U-degenerate theories [25]. If the action (1) is regarded as a low-energy truncation of some complete theory, detuning the degeneracy conditions is acceptable because a ghost degree of freedom itself is not problematic from the effective-field-theory viewpoint [26].

In this paper, we do not assume any particular relations among the functions in the action. Nevertheless, we can handle the relevant equations and derive the universal form of the quadratic Lagrangian for odd mode perturbations around a spherically symmetric background with time-dependent scalar hair.

III. SPHERICALLY SYMMETRIC BACKGROUND

Let us start with a background solution. We consider static and spherically symmetric spacetime whose metric is of the form

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2 C(r) d\sigma^2 ,$$

where $d\sigma^2 := d\theta^2 + \sin^2 \theta d\varphi^2$. At this point we introduce $C(r)$ to reproduce all the relevant field equations from the action principle. After deriving the field equations one may put $C(r) = 1$ by redefining the radial coordinate (see, e.g., [29]).

The scalar field is assumed to be dependent linearly on the time coordinate,

$$\phi(t, r) = \mu t + \psi(r) ,$$

where $\mu := \beta_1 / 2$. Then, the action (1) reduces to

$$S_{\text{grav}} = \int d^4x \sqrt{-g} \left[ F_0(X) + F_1(X) \Box \phi + F_2(X) R \right] + \frac{1}{2} \sum_{I=1}^{10} A_I(X) L_I^{(2)} + F_3(X) G_{\mu\nu} \phi^{\mu\nu} + \sum_{I=1}^{10} B_I(X) L_I^{(3)} ,$$

where $X := -\phi_{\mu} \phi^{\mu} / 2$, $\phi_{\mu} := \nabla_{\mu} \phi$, $\phi_{\mu\nu} = \nabla_{\mu} \nabla_{\nu} \phi$, $R$ is the Ricci scalar, and $G_{\mu\nu}$ is the Einstein tensor. Here, $L_I^{(2)}$ are quadratic in the second derivatives of the scalar field and are written explicitly as

$$L_1^{(2)} = \phi_{\mu\nu} \phi^{\mu\nu} , \quad L_2^{(2)} = (\Box \phi)^2 , \quad L_3^{(2)} = (\Box \phi) \phi_{\mu\nu} \phi^{\mu\nu} ,$$

$$L_4^{(2)} = \phi^{\mu} \phi_{\mu\nu} \phi^{\nu} , \quad L_5^{(2)} = (\phi^{\mu} \phi_{\mu\nu} \phi^{\nu})^2 .$$

These exhaust possible terms built from $\phi_{\mu}$ and $\phi_{\mu\nu}$ and quadratic/cubic in $\phi_{\mu\nu}$. The functions $F_0, F_1, F_2, F_3, A_1$, and $B_I$ depend only on $X$, so that the theory has shift symmetry, $\phi \rightarrow \phi + c$.

In general, the action (1) yields higher-order equations of motion for the metric and the scalar field, resulting in the dangerous Ostrogradsky ghost. One can circumvent this by imposing the degeneracy conditions among the functions $F_2, F_3, A_1$, and $B_I$ [17–19]. In such degenerate theories, one arrives in the end at a set of second-order equations by combining the different components of field equations, and thus can remove the unstable ghost degrees of freedom. One may relax the degeneracy conditions so that the theory is degenerate at least in the unitary gauge, which can still provide a healthy class of theories called U-degenerate theories [25]. If the action (1) is regarded as a low-energy truncation of some complete theory, detuning the degeneracy conditions is acceptable because a ghost degree of freedom itself is not problematic from the effective-field-theory viewpoint [26].

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where $\mu$ is a constant. Without loss of generality we assume that $\mu > 0$. This configuration is consistent with the static metric (4) because the action (1) depends on $\phi$ only through its derivatives.

The crucial points of the ansatz (5) are the following. First, by assuming a linearly time-dependent scalar field one can avoid the postulate in the no-hair theorem of [12], which makes it easier to obtain hairy solutions. Indeed, spherically symmetric black hole solutions with such a scalar field configuration have been found in the context of the Horndeski theory [10, 11, 30–32] and beyond-Horndeski/DHOST theories [33–37]. Second, it has been assumed in the formulation of the EFT of black hole perturbations in the presence of second order or higher.

Substituting the metric (4) and the scalar field ansatz (5) to the action (1) and varying it with respect to $A, B, C,$ and $\psi,$ one is able to derive the background field equations. We write the resultant field equations as $E_A = 0$, $E_B = 0$, $E_C = 0$, and $E_\psi = 0$, whose explicit expressions are not important in the present paper. These equations do not reduce to second-order differential equations in general, because we do not impose any degeneracy conditions. However, as far as the odd mode perturbations are concerned, we do not need to care about the higher-order nature of the background equations. We will just use (some of) these background equations in their original form to simplify the quadratic action for the odd mode perturbations, whether they are of second order or higher.

Before proceeding to the analysis of perturbations, let us present a simple explicit example of background solutions. An interesting class of solutions often studied in the literature is a stealth Schwarzschild black hole with $X = X_0 = \text{const}$. One can see that our field equations admit the solution

$$A = B = 1 - \frac{r_h}{r}, \quad X = X_0 = \frac{\mu^2}{2},$$

provided that the functions in the action (1) satisfy the following equations (cf. [36]):

$$F_0(X_0) = 0, \quad F_{0X}(X_0) = 0, \quad F_1(X_0) = 0,$$
$$A_1(X_0) + A_2(X_0) = 0, \quad A_{1X}(X_0) + A_{2X}(X_0) = 0,$$
$$B_2(X_0) = -\frac{1}{2} B_3(X_0) = 9 B_1(X_0),$$
$$B_4(X_0) + B_6(X_0) - B_1X(X_0) - B_{2X}(X_0) - \frac{5}{9} B_{3X}(X_0) = \frac{6}{X_0} B_1(X_0).$$

Note that these relations are compatible with the degeneracy conditions in the class $2N-I + 3M-I$ degenerate theories in the terminology of [19], and therefore the above solution is admitted even if one concentrates on a degenerate theory. From $2X = \mu^2 = \mu^2/A - B \frac{d}{dr} \left( \frac{d\psi}{dr} \right)^2$ we have}

$$\psi = \pm \mu \left[ 2\sqrt{r_h r} + r_h \ln \left( \frac{\sqrt{r} - \sqrt{r_h}}{\sqrt{r} + \sqrt{r_h}} \right) \right].$$

We choose the “$-$” branch because we have $\phi \approx \mu \left( t \pm r_h \ln (r/r_h - 1) \right) + \text{const}$ near the horizon and it is regular at the horizon only in the “$+$” branch, as is clear by expressing $\phi$ in terms of the ingoing null coordinate $v = t + r + r_h \ln (r/r_h - 1)$ [10].

**IV. ODD PARITY PERTURBATIONS**

A. Derivation of the Quadratic Lagrangian and the Effective Metric

Let us consider the odd mode metric perturbations,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu},$$

where $\bar{g}_{\mu\nu}$ is the background metric (4) with $C(r) = 1$. The scalar field does not have an odd mode perturbation. Among the ten components, $h_{tt}, h_{t\varphi},$ and $h_{\varphi\varphi}$ are concerned with odd parity modes, where $a = \theta, \varphi$. Using the spherical harmonics $Y_{\ell m}(\theta, \varphi)$, we follow the standard procedure and expand the odd mode perturbations as

$$h_{t\theta} = -\frac{1}{\sin \theta} \partial_\varphi \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h_{0}^{(\ell m)}(t, r) Y_{\ell m}(\theta, \varphi),$$

$$h_{t\varphi} = \sin \theta \partial_\theta \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h_{0}^{(\ell m)}(t, r) Y_{\ell m}(\theta, \varphi),$$

$$h_{r\theta} = -\frac{1}{\sin \theta} \partial_\varphi \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h_{1}^{(\ell m)}(t, r) Y_{\ell m}(\theta, \varphi),$$

$$h_{r\varphi} = \sin \theta \partial_\theta \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h_{1}^{(\ell m)}(t, r) Y_{\ell m}(\theta, \varphi).$$

The odd parity part of $h_{ab}$ can also be expressed using a single pseudo-scalar function, say $h_2$, but we adopt the Regge-Wheeler gauge in which $h_2 = 0$ and accordingly $h_{ab} = 0$.

We substitute Eqs. (10)–(13) to the action (1) and expand it to second order in perturbations. In doing so one can remove many terms by using the background equations. Performing the angular integrations, we arrive in the end at the general action

$$S_{\text{grav}} = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \int dt dr L_{\ell m}^{(2)},$$

where, omitting the labels ($\ell m$) from $h_0$ and $h_1$,
with $c_t = (\ell - 1)(\ell + 1)(\ell + 2)$. Here a dot and a prime denote differentiation with respect to $t$ and $r$, respectively. Following Ref. [24], it is convenient to write these coefficients as

$$a_1 = \frac{c_t}{2r^2\sqrt{AB}} \left\{ F_2 + \frac{\mu^2}{A} A_1 + \frac{2B\psi'X'}{2} F_3X + \frac{\mu^2}{A\psi'} B_2 + \frac{2B(\psi')^2}{r} - \frac{(AX)'}{A} \right\},$$

$$a_2 = -\frac{c_t}{2r^2} \left\{ F_2 - B(\psi')^2 A_1 - \frac{2B\psi'X'}{2} F_3X - \frac{B\psi'}{\psi'} B_2 - \frac{3B^2(\psi')^3}{r} B_3 + B^2(\psi')^3 X'B_0 \right\},$$

$$a_3 = \frac{\ell(\ell + 1)}{2} \sqrt{B} \left\{ F_2 + 2X A_1 - \frac{B\psi'X'}{2} F_3X + \frac{2X}{\psi'} B_2 + \frac{3X}{\psi'} \left[ B(\psi')^2 - \frac{(AX)'}{A} \right] B_3 - 2B\psi'XX'B_0 \right\},$$

$$a_4 = -\frac{c_t}{r^2} \sqrt{B} A \mu \left\{ \psi'A_1 + \frac{X'}{2} F_3X + \frac{2B(\psi')^2}{r} - \frac{(AX)'}{A} \right\} B_2 + \frac{3B(\psi')^2}{r} B_3 - B(\psi')^2 X'B_0 \right\},$$

where $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{J}$ have a dimension of (mass)$^2$. For the Schwarzschild solution in general relativity, we simply have $\mathcal{F} = \mathcal{G} = \mathcal{H} = M_B^2 = (8\pi G)^{-1}$ and $\mathcal{J} = 0$.

Before proceeding to further reduction of the Lagrangian (15), let us point out two things, both of which come essentially from the fact that the odd parity modes constitute a part of tensorial metric perturbations. First, note that only $F_2$, $F_3$, $A_1$, $B_2$, $B_3$, and $B_6$ participate in the above result. Contributions from the other terms are dropped from the action upon using the background equations. This is as expected because it is known that only these terms contribute to the tensor modes on a cosmological background [38, 39]. Second, it should be emphasized that the quadratic Lagrangian (15) is derived without using any degeneracy conditions. This is also not surprising because tensorial metric perturbations in the theory (1) obey second-order equations without regard to the degeneracy conditions. Therefore, our result can be used, for example, to U-degenerate theories [25] and detuned (“scordatura”) DHOST theories [26].

Now let us rewrite the Lagrangian (15) in terms of a single master variable. This can be done straightforwardly, following closely Refs. [13, 14, 24]. First, we introduce an auxiliary field $\chi = \chi^{\ell_0}(t, r)$ and rewrite the Lagrangian (15) in an equivalent way as

$$\mathcal{L}_{\ell_0}^{(2)} = \frac{1}{2} \left\{ a_1|\dot{h}_0|^2 + a_2|\dot{h}_1|^2 + a_3|\dot{h}_1 - 2\dot{h}_1|\dot{h}_0 | + a_4 h_1^2| \right\} + c.c. \right.$$
and
\[ V = 2\mathcal{H} \left[ r^2 b_2 \sqrt{b/ A} \left( \frac{\sqrt{A/B}}{r^2 \mathcal{H}} \right) \right]' - \frac{2\mathcal{H}}{r^2}. \quad (28) \]

The equation of motion that follows from this Lagrangian is given by
\[
\begin{align*}
 b_1 \ddot{x} - \frac{\sqrt{A/B}}{r^2} \left( r^2 \sqrt{\frac{b_2}{A}} \dot{x} \right)' &+ \frac{b_3}{2} \ddot{x}' \\
+ \frac{\sqrt{A/B}}{2r^2} \left( r^2 \sqrt{\frac{b_2}{A}} \dot{x} \right)' &+ \left[ \frac{\ell(\ell+1)}{2} \mathcal{H} + \frac{V}{r^2} \right] \chi = 0.
\end{align*}
\] (29)

At this stage, it can be seen from the Lagrangian (26) that we need to impose
\[ \mathcal{H} > 0, \quad (30) \]
as otherwise modes with large \( \ell \) would have large negative energy and make the system unstable quickly.

One notices that Eq. (29) can be written in the form
\[ \mathcal{H} \Omega^2 Z^\mu\nu D_\mu D_\nu \chi - V \chi = 0, \quad (31) \]
where \( Z^\mu\nu \) is the inverse of the effective metric \( Z_{\mu\nu} \) [40],
\[
Z_{\mu\nu} dx^\mu dx^\nu = \Omega^2 \left( -\frac{G}{\mathcal{H}} Adt^2 - \frac{2J}{\mathcal{H}} dt dr + \frac{\mathcal{F} dr^2}{\mathcal{H} B} \right. \\
+ \left. r^2 d\sigma^2 \right), \quad (32)
\]
with
\[ \Omega^2 := \frac{B}{A} \frac{\mathcal{H}^2}{\sqrt{\mathcal{F}G + (B/A) J^2}}, \quad (33) \]
and \( D_\mu \) is the covariant derivative operator defined in terms of a connection compatible with \( Z_{\mu\nu} \). Note here that the metric perturbations have already been expanded in terms of the spherical harmonics and hence the spherical Laplacian in \( Z^\mu\nu D_\mu D_\nu \) must be replaced with its eigenvalue \( -\ell(\ell+1) \). Note also that
\[ \zeta^2(r) := \mathcal{F}G + \frac{B}{A} J^2 > 0 \quad (34) \]

must be imposed in order for the effective metric to be well-defined. It is easy to see that one has \( Z_{\mu\nu} = M_{\mu\nu}^2 \tilde{g}_{\mu\nu} \) in general relativity, where \( \mathcal{F} = \mathcal{G} = \mathcal{H} = M_{\mu\nu}^2 \) and \( J = 0 \). However, \( Z_{\mu\nu} \) may not be proportional to \( \tilde{g}_{\mu\nu} \) in modified gravity. This fact itself has already been known in the context of the Horndeski theory [41–43].

We introduce a new time coordinate \( \tau \) defined by
\[ d\tau = dt + \frac{J}{AG} dr. \quad (35) \]

Using \( \tau \), the effective metric (32) can be written in a diagonal form as
\[ Z_{\mu\nu} dx^\mu dx^\nu = \Omega^2 \left( -\frac{G}{\mathcal{H} B} Ad\tau^2 + \frac{\zeta^2 dr^2}{\sqrt{\mathcal{H} B}} + r^2 d\sigma^2 \right). \quad (36) \]

It is sometimes more convenient to work in the conformally related effective metric \( \tilde{Z}_{\mu\nu} \) defined as
\[
\tilde{Z}_{\mu\nu} = \Omega^{-2} Z_{\mu\nu}.
\] (37)

In the tilded frame, Eq. (31) is written as
\[
\tilde{Z}^{\mu\nu} \tilde{D}_\mu \tilde{D}_\nu \left( \tilde{\chi} \right) = -\frac{V}{\mathcal{H}} - \frac{\tilde{Z}^{\mu\nu} \tilde{D}_\mu \tilde{D}_\nu \Omega}{\Omega} \tilde{\chi} = 0, \quad (38)
\]
where \( \tilde{\chi} := \Omega r \chi \) and \( \tilde{D}_\mu \) is the covariant derivative operator defined in terms of a connection compatible with \( \tilde{Z}_{\mu\nu} \).

Defining the generalized tortoise coordinate by
\[ dr_* = \frac{\zeta}{\sqrt{\mathcal{G}AB}} dr, \quad (39) \]
Eq. (38) can further be rewritten in a more familiar form as
\[
\left( -\partial^2_\tau + \partial^2_\tau - \tilde{V} \right) \tilde{\chi} = 0, \quad (40)
\]
where
\[
\tilde{V} = \frac{GA}{\mathcal{H}} \left\{ \frac{(\ell + 2)(\ell - 1)}{r^2} + \Omega r \left[ \frac{G \sqrt{AB}}{\zeta} \left( \frac{1}{r^{\zeta^{1/2}}} \right)' \right] \right\}. \quad (41)
\]

This generalizes the Regge-Wheeler equation known in general relativity [44] to higher-order scalar-tensor theories. In Appendix B, we extend the main result of this section to include the energy-momentum tensor of matter and derive the generalized Regge-Wheeler equation with a matter source term.

So far we have focused on the modes with \( \ell \geq 2 \). The dipole (\( \ell = 1 \)) mode must be treated separately, but here we only comment that the dipole perturbation corresponds to adding a slow rotation, as has been already discussed in detail in the previous literature [8, 13, 14, 24].

B. Propagation Speed

In theories described by the action (1), the propagation speed of gravitational waves differs in general from the speed of light. In light of the constraint from GW170817 [45–47], let us identify the subclass of scalar-tensor theories that admits a luminal speed of gravitational waves at least at large \( r \). This weak requirement was also employed in Ref. [4] (see, however, Refs. [48, 49]).
We assume that the background is given by
\[ A = 1 + \mathcal{O}(r^{-1}), \quad B = 1 + \mathcal{O}(r^{-1}), \]
\[ \psi' = \psi'_{\infty} + \mathcal{O}(r^{-1}), \quad (42) \]
for large \( r \), where \( \psi'_{\infty} \) is a constant. We then find
\[ \mathcal{F} = 2 [F_2(X_{\infty}) + \mu^2 A_1(X_{\infty})] + \mathcal{O}(r^{-1}), \quad (43) \]
\[ \mathcal{G} = 2 [F_2(X_{\infty}) - (\psi'_{\infty})^2 A_1(X_{\infty})] + \mathcal{O}(r^{-1}), \quad (44) \]
\[ \mathcal{H} = 2 [F_2(X_{\infty}) + 2X_1 A_1(X_{\infty})] + \mathcal{O}(r^{-1}), \quad (45) \]
\[ \mathcal{J} = -2\mu\psi'_{\infty} A_1(X_{\infty}) + \mathcal{O}(r^{-1}), \quad (46) \]
where \( X_{\infty} := [\mu^2 - (\psi'_{\infty})^2]/2 \). Thus, if one has
\[ A_1(X_{\infty}) = 0, \quad (47) \]
Eq. (40) reduces to \([-\partial_t^2 + \partial_r^2 - \ell(\ell + 1)/r^2]\tilde{\chi} \equiv 0 \) for large \( r \), rendering luminal propagation of gravitational waves sufficiently away from a black hole. Note that \( F_2 \) and \( B_1 \) appear only in the \( \mathcal{O}(r^{-1}) \) terms in Eqs. (43)–(46).

**C. Horizons for Photons and Gravitons**

Suppose that \( r_h \) is the location of the horizon in the metric \( \tilde{g}_{\mu\nu} \), and the metric components are expanded as
\[ A(r) = \sum_{n=1}^{\infty} \alpha_n \epsilon^n, \quad B(r) = \sum_{n=1}^{\infty} \beta_n \epsilon^n, \quad (48) \]
where \( \epsilon := r/r_h - 1 > 0 \). We assume that \( X \) is regular at the horizon, so that \( X \) is of the form
\[ X = X_h + \sum_{n=1}^{\infty} X_n \epsilon^n. \quad (49) \]
Accordingly, one has
\[ \psi' = \frac{\mu}{\sqrt{\alpha_1 \beta_1}} \frac{1}{\epsilon} + \sum_{n=0}^{\infty} \gamma_n \epsilon^n. \quad (50) \]
Note that \( \psi' \) diverges as \( r \to r_h \), but this is not problematic. See the comment below Eq. (8). Substituting Eqs. (48)–(50) to Eqs. (16)–(19), we find, in the vicinity of the horizon,
\[ \mathcal{F} = -\frac{d_0}{\epsilon} - d_1 + \mathcal{O}(\epsilon), \quad \mathcal{G} = \frac{d_0}{\epsilon} - d_2 + \mathcal{O}(\epsilon), \]
\[ \mathcal{H} = d_3 + \mathcal{O}(\epsilon), \quad \mathcal{J} = \sqrt{\frac{B}{A}} \frac{d_0}{\epsilon} + \frac{d_1 + d_2}{2} + \mathcal{O}(\epsilon), \quad (51) \]
and hence \( \zeta = \text{const} + \mathcal{O}(\epsilon) \), where
\[ d_0 = -\frac{2\mu^2}{\alpha_1} A_1(X_h) + \frac{2\mu}{r_h} \sqrt{\frac{\beta_1}{\alpha_1}} \left[ (\alpha_1 X_h - 2\mu^2) B_2(X_h) - 3\mu^2 B_3(X_h) + \mu^2 X_1 B_6(X_h) \right], \quad (52) \]
while the explicit expressions for \( d_1, d_2, \) and \( d_3 \) are more involved. Hereafter we will consider the case where \( d_0 \) is nonvanishing. Thus, at \( r \approx r_h \),
\[ \Omega \approx \text{const}, \quad \tilde{Z}_{rr} \approx \text{const}, \quad \tilde{Z}_{\tau\tau} \approx \text{const}, \quad (53) \]
which shows that nothing special happens in the effective metric at the horizon of the metric \( \tilde{g}_{\mu\nu} \). In particular, this fact implies that \( r = r_h \) is not an appropriate place to impose the inner boundary conditions when solving the Regge-Wheeler equation (40). Rather, the form of the effective metric implies that a possible appropriate boundary will be \( r = r_g \), where \( \mathcal{G}(r_g) = 0 \). To see this more explicitly, let us study some concrete examples.

The first example is given by the special case of the solution in Sec. III, with \( A_1(X_0) \neq 0 \) and \( B_1(X_0) = 0 \). Essentially the same solution is also studied in Ref. [24]. This does not satisfy Eq. (47), but is a good illustrative example. We have
\[ \mathcal{G} = 2F_2(X_0) \cdot \frac{1 - r_g/r}{1 - r_h/r}, \quad \mathcal{H} = 2F_2(X_0)(1 + \mathcal{A}), \quad (54) \]
where
\[ r_g := (1 + \mathcal{A})r_h, \quad \mathcal{A} := \frac{2X_0 A_1(X_0)}{F_2(X_0)}, \quad (55) \]
and we assume that \( F_2(X_0) > 0 \) and \( 1 + \mathcal{A} > 0 \). The conformal factor is a nonvanishing constant, \( \Omega^2 = 2F_2(X_0)(1 + \mathcal{A})^{-3/2} \), and the components of the (tilded) effective metric are given by
\[ \tilde{Z}_{\tau\tau} = -\frac{1 - r_g/r}{1 - \mathcal{A}}, \quad \tilde{Z}_{rr} = \frac{1}{1 - r_g/r}, \quad (56) \]
which shows that the horizon of the effective metric is at \( r = r_g \neq r_h \). In this case, the generalized tortoise coordinate is given by \( r_* = (1 + \mathcal{A})^{-1/2}[r + r_g \ln(r/r_g - 1)] \)
and the potential in Eq. (40) reads
\[ \tilde{V} = \frac{1}{1 + \mathcal{A}} \left[ \frac{\ell(\ell + 1)}{r^2} - \frac{3r_g}{r^3} \right]. \quad (57) \]
Aside from the constant factor of \((1 + \mathcal{A})^{-1}\), this coincides with the well-known potential in the Regge-Wheeler equation in general relativity with the horizon at \( r = r_g \).

In this example, \( \mathcal{G} \) is singular at \( r = r_h \). One also notices that \( \mathcal{G} < 0 \) for \( r_g < r < r_h \) if \( \mathcal{A} < 0 \). However, the effective metric and the potential do not depend on \( r_h \) explicitly and are free from any pathologies. In particular, the sign of \( \mathcal{G} \) does not directly related to the stability of the solution. Indeed, it is now clear that the above solution is stable provided that \( F_2(X_0) > 0 \) and \( 1 + \mathcal{A} > 0 \) are satisfied.

The second example is again the special case of the solution in Sec. III, but now with \( A_1(X_0) = 0 \) and \( B_1(X_0) \neq 0 \). In this case, we have
\[ \mathcal{G} = 2F_2(X_0) \cdot \frac{f(r)}{1 - r_h/r}, \quad \mathcal{H} = 2F_2(X_0), \quad (58) \]
\[ f(r) = 1 - \frac{r_h}{r} + B \left( \frac{r_h}{r} \right)^{5/2}, \quad B := \frac{81}{2} \frac{\mu^3}{r_h^3} B_1(X_0). \]

The conformal factor is given by

\[ \Omega^2 = \frac{2F_2(X_0)}{g^{1/2}(r)}, \]

and the (tilded) effective metric reduces to

\[ \tilde{Z}_{tr} = -f(r), \quad \tilde{Z}_{rr} = \frac{g(r)}{f(r)}, \]

where

\[ g(r) = 1 - B \left( \frac{r_h}{r} \right)^{3/2}. \]

We see that the horizon of the effective metric is at \( r = r_g \neq r_h \), where \( f(r_g) = 0 \).

Let us investigate the structure of the effective metric (61) in more detail. For \( B > 6/25(\sqrt{3}/5) \approx 0.186 \), \( f \) has no zeros, while \( g = 0 \) at \( r = B^{2/3} r_h \). We are not interested in this case. For \( 0 < B < 6/25(\sqrt{3}/5) \), we have \( f = 0 \) at \( r = r_g < r_h \). In this case, \( g \) remains positive outside the horizon of the effective metric, but \( g = 0 \) occurs at \( r = B^{2/3} r_h < r_g \). Finally, for \( B < 0 \), we have \( f = 0 \) at \( r = r_g > r_h \) and \( g \) is always positive for \( r > 0 \). Therefore, in the latter two cases the solution has an outer horizon of the effective metric at \( r = r_g \).

It is straightforward to write the potential \( \tilde{V} \), but the expression is messy. The shape of the potential is shown for different values of \( B \) in Fig. 1. One can check that \( r_* \to -\infty \) as \( r \to r_g \).

## V. QUASI-NORMAL MODES

In this section, we compute the QNMs of the second example of the previous section. Quasi-normal modes in the Horndeski theory have been studied in the case of the Schwarzschild background with a constant scalar field [50] and a nearly Schwarzschild background with a nearly constant scalar field [51].

We assume the time dependence of the master variable as \( \tilde{\chi} = Q(r) e^{-i\omega \tau} \) and solve

\[ \frac{d^2 Q}{dr_*^2} + \left[ \omega^2 - \tilde{V}(r) \right] Q = 0, \]

where \( \tilde{V} \) is the potential obtained from the second example of the background solutions in the previous section, which is characterized by the dimensionless parameter \( B \) (Fig. 1). The boundary conditions for \( Q \) are given by

\[ Q \propto \begin{cases} e^{+i\omega r_*} & r_* \to \infty \quad (r \to \infty) \\ e^{-i\omega r_*} & r_* \to -\infty \quad (r \to r_g). \end{cases} \]

Note again that the inner boundary is located at \( r = r_g \) rather than at \( r = r_h \). In order to obtain the QNMs, we employ direct numerical integration.\(^1\) The lowest overtone quasi-normal frequencies for \( \ell = 2 \) are given in Fig. 2, showing how the frequencies depend on the modified gravity parameter \( B \).

## VI. CONCLUSIONS

In this paper, we have studied odd parity perturbations of black holes with linearly time-dependent scalar hair in shift-symmetric scalar-tensor theories. Due to the time dependence of the scalar field background, the

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\(^1\) Taking \( B \) as a small expansion parameter, one may write the potential as \( \tilde{V} = V_{GR} + \delta V \), where \( V_{GR} \) is the Regge-Wheeler potential in general relativity and \( \delta V \) is a small correction. In the present case, \( \delta V \) contains fractional powers of \( r \), which hinders us from using the convenient formalism of Ref. [52].
effective-field-theory (EFT) approach [2–4] can not be
applied straightforwardly to the present case. Therefore,
we have started from a general covariant action that is
most similar to the action of cubic degenerate higher-
order scalar-tensor theories [19] and derived the general
quadratic action for odd parity perturbations without im-
posing the degeneracy conditions. The degeneracy con-
ditions are not essential for retaining the healthy odd
parity perturbations that are not mixed with the per-
turbation of the scalar field. We have thus derived a
second-order equation for a single master variable as a
generalization of the Regge-Wheeler equation in general
relativity. Starting from the more general action, we have
arrived at qualitatively the same results as the previous
ones [13, 14, 24], showing that no new terms appear in
the quadratic action for odd parity perturbations. Our
generalized Regge-Wheeler equation can be used in a
wide class of scalar-tensor theories such as U-degenerate the-
ories [25] and scordatura theories [26].

We have also refined the stability conditions explored
in the previous literature [24]. The previous conditions
were actually sufficient conditions, and we have argued
that one of the conditions is not directly related to the
stability.

As another application of our results, we have com-
puted the quasi-normal modes of a certain nontrivial
black hole solution. In doing so, we have demonstrated
that it is important to identify the correct location of
the inner boundary by inspecting the effective metric for
gravitons.

It would be interesting to extend the present analysis
to the even parity sector, which would be much more
involved due to its higher derivative nature. It would
also be interesting to perform a complementary analysis
based on the EFT approach along the line of [3, 4].

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Appendix A: Generality of the Quadratic Lagrangian

Starting from the action (1), we have shown in the
main text that the quadratic Lagrangian for the odd
parity modes is given by Eq. (15). Actually, one can
show that more general scalar-tensor theories lead to the
quadratic Lagrangian for the odd parity modes having
the same structure as Eq. (15) as long as the equation of
motion for gravitational-wave degrees of freedom remains
of second order.

For example, one may add to the action (1)
\[ \bar{F}_3(X) R \Box \phi, \]  
(A1)
to consider a general derivative coupling of the form
\[ F_3 G_{\mu \nu} \phi^{\mu \nu} + \bar{F}_3 R \Box \phi = F_3 R_{\mu \nu} \phi^{\mu \nu} + (F_3 - F_3/2) R \Box \phi. \]
This only shifts the coefficients as
\[ F_3 G_{\mu \nu} \phi^{\mu \nu} + \bar{F}_3 R \Box \phi \rightarrow \mathcal{F}, \mathcal{G}, \mathcal{H} \rightarrow \mathcal{F}, \mathcal{G}, \mathcal{H} + \left[ \frac{B \psi'}{r} - \frac{(AX)'}{A \psi'} \right] \bar{F}_3, \]
(A2)
\[ \mathcal{J} \rightarrow \mathcal{J}, \]  
(A3)
and does not give rise to any new terms in Eq. (15).

Similarly, one may also add terms quartic in second
derivatives of \( \phi \) such as
\[ C_1(X) \phi_{\mu \nu} \phi^{\rho \lambda} \phi_{\rho \lambda} \phi^{\mu \nu}, C_2(X) (\Box \phi)^4, \ldots. \]  
(A4)

One can verify by direct computation that such quartic
terms merely shift the coefficients without altering the
structure of the Lagrangian (15) or have no contribution
to the odd parity sector.

We thus conclude that the form of the Lagrangian (15)
is generic to scalar-tensor theories in which gravitational-
wave degrees of freedom obey a second-order equation of
motion.

Appendix B: Sourced Regge-Wheeler Equation

In this appendix, we generalize our main result to in-
clude the source term, which has not been considered in
the previous similar studies [8, 13, 14, 24]. Assuming that
matter is minimally coupled to gravity, the source term
can be obtained from
\[ S_{\text{source}} = \frac{1}{2} \int d^4 x \sqrt{-g} h_{\mu \nu} T^{\mu \nu}, \]  
(B1)
where \( T_{\mu \nu} \) is the matter energy-momentum tensor.
Similarly to the metric perturbations, the odd parity part of
the energy momentum tensor can also be expanded as
\[ T_{\theta \theta} = - \frac{1}{\sin \theta} \partial_\phi \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^\ell S_0^{(\ell m)} (t, r) Y_{\ell m}, \]  
(B2)
\[ T_{t \varphi} = \sin \theta \partial_\varphi \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^\ell S_0^{(\ell m)} (t, r) Y_{\ell m}, \]  
(B3)
\[ T_{r \theta} = - \frac{1}{\sin \theta} \partial_\varphi \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^\ell S_1^{(\ell m)} (t, r) Y_{\ell m}, \]  
(B4)
\[ T_{r \varphi} = \sin \theta \partial_\varphi \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^\ell S_1^{(\ell m)} (t, r) Y_{\ell m}, \]  
(B5)
\[ T_{\theta \varphi} = \frac{2}{\sin \theta} (\partial_\varphi \partial_\varphi - \cot \theta \partial_\varphi) \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^\ell S_2^{(\ell m)} (t, r) Y_{\ell m}, \]  
(B6)
\[ T_{\theta \varphi} = \left( \frac{1}{\sin \theta} \partial^2_{\varphi} + \cos \theta \partial_{\theta} - \sin \theta \partial_{\theta}^2 \right) \]
\[ \times \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} S^{(\ell m)}_2(t, r) Y_{\ell m}, \]
\[ T_{\varphi \varphi} = -2 \sin \theta \left( \partial_{\theta} \partial_{\varphi} - \cot \theta \partial_{\varphi} \right) \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} S^{(\ell m)}_2(t, r) Y_{\ell m}. \]
\[ (B7) \]
\[ (B8) \]

The conservation of the matter energy-momentum tensor, \( \nabla_{\nu} T^{\mu \nu} = 0 \), yields
\[ -\frac{S_0^{(\ell m)}}{A} + \frac{\sqrt{B/A}}{r^2} \left( r^2 \sqrt{AB} S_1^{(\ell m)} \right)' + \frac{\ell(\ell + 1)}{r^2} S^{(\ell m)}_2 = 0. \]
\[ (B9) \]

It is straightforward to perform the angular integrations in Eq. (B1) to obtain
\[ S_{\text{source}} = -\sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\ell(\ell + 1)}{2} \int dt dr \left( \frac{h_0 S_0}{\sqrt{AB}} - \sqrt{AB} h_1 S_1 + \text{c.c.} \right), \]
\[ (B10) \]

where we omitted the labels \((\ell m)\) from \( S_0 \) and \( S_1 \). This is the source action for the odd mode perturbations (see also Ref. [4]). We add the above source action to the gravitational part of the action (14). Then, Eqs. (22) and (23) are generalized to
\[ a_1 h_0 + (a_3 \dot{\chi})' + \frac{2a_3}{r} \chi + \frac{1}{2} a_4 h_1 = \frac{\ell(\ell + 1)}{2\sqrt{AB}} S_0, \]
\[ a_2 h_1 - a_3 \ddot{\chi} + \frac{1}{2} a_4 h_0 = -\frac{\ell(\ell + 1)}{2} \sqrt{AB} S_1, \]
\[ (B11) \]
\[ (B12) \]

Solving these equations for \( h_0 \) and \( h_1 \) and removing \( h_0 \) and \( h_1 \) from the quadratic Lagrangian, we see that the Lagrangian (26) is generalized to include the source as
\[ L_{\ell m, \text{total}}^{(2)} = L_{\ell m}^{(2)} - \frac{\ell(\ell + 1)}{4(\ell - 1)(\ell + 2)} \frac{B}{A} \left( \dot{\chi} S_{\text{odd}} + \text{c.c.} \right), \]
\[ (B13) \]

where \( L_{\ell m}^{(2)} \) in the right-hand side is the same Lagrangian as the one defined as Eq. (26) and
\[ S_{\text{odd}}^{(\ell m)}(t, r) := 2\mathcal{H} \left( \frac{G}{\zeta^2} S_0^{(\ell m)} \right)' - \frac{2F\mathcal{H}}{\zeta^2} S_1^{(\ell m)} - \frac{2H\mathcal{J}}{A\zeta^2} \dot{\chi}_0^{(\ell m)} - 2\mathcal{H} \left( \frac{B\mathcal{J}}{\zeta^2} S_1^{(\ell m)} \right)' , \]
\[ (B14) \]

Now Eq. (31) with the source term reads
\[ \mathcal{H} \Omega^2 Z^{\mu \nu} D_{\mu} D_{\nu} \chi - V \chi = S_{\text{odd}}, \]
\[ (B15) \]

and, accordingly, Eq. (40) with the source term is given by
\[ \left( -\partial_t^2 + \partial_r^2 - \ddot{V} \right) \tilde{\chi} = \frac{Gr \sqrt{AB}}{\mathcal{H}^{1/2}} S_{\text{odd}}. \]
\[ (B16) \]

This is the generalization of the sourced Regge-Wheeler equation.

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