Multidimensional Topological Foam

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Abstract

Multidimensional cosmological model with the topology \( M = \mathbb{R} \times M_1 \times \cdots \times M_n \) where \( M_i \ (i = 1, \ldots, n) \) undergo a chain splitting into arbitrary number of compact spaces is considered. It is shown that equations of motion can be solved exactly because they depend only on the effective curvatures and dimensions and "forget" about inner topological structure. It is proved that effective cosmological action for the model with \( n = 1 \) in the case of infinite splitting of the internal space coincides with the tree-level effective action for a bosonic string.

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1 Introduction

Our observable universe at the present time at large scales is well described by the Friedmann model with 4-dimensional Friedmann-Robertson-Walker (FRW) metric. However, it is possible that spacetime at short (Planck) distances might have a dimensionality more than four and possess complex topology. Quantum fluctuations in metric at Planck distances might result in branching off or joining onto baby universes. Both of these phenomena (multidimensionality and quantum fluctuations) provide a foamlike structure of the universe at Planck length [1, 2]. In the present paper we shall consider a foamy structure of the universe due to former of these phenomena. Namely, we shall consider universe which has complicated topological structure due to extra dimensions and this type of structure we call topological foam.

At Planck length all dimensions (external and internal) have equal rights. It might have place, for example, in the early universe at Planck times. To describe dynamics of this universe it is natural to generalize the FRW model to a multidimensional cosmological model (MCM) with the topology

\[ M = \mathbb{R} \times M_1 \times \ldots \times M_n, \]  

(1.1)

where \( M_i \) (\( i = 1, \ldots, n \)) denote \( d_i \)-dimensional spaces of constant curvature. One of this spaces, say \( M_1 \), describes our 3-dimensional external space but all others are internal spaces. To make the internal dimensions unobservable at the present time the internal spaces have to be reduced to scales near the Planck length. This can be achieved by contraction of the internal spaces during cosmological evolution or by supposing that they are static (or nearly static) and of the Planck scales from the very beginning. Anyway, the internal spaces have to be compact spaces what can be achieved by appropriate periodicity conditions for the coordinates [4, 5, 6, 7]. As a result, internal spaces may have a nontrivial global topology, being compact (i.e. closed and bounded) for any sign of the spatial curvature. The model (1.1) can be generalized to the case of all spaces being Einstein spaces. Dynamics of the factor spaces \( M_i \) is described by scale factors \( a_i(\tau), \) (\( i = 1, \ldots, n \)). Now, we suppose that factor spaces \( M_i \) have structure of russian ”matreshka”: each \( M_i \) is split into a product of Einstein spaces \( M_i \rightarrow \prod_{k=1}^{n_i} M^k_i. \) In one’s turn each \( M^k_i \) can be split also and so on. We demand only that factor spaces corresponding to
the most elementary splitting should be Einstein spaces. In our paper we show that under such conditions equations of motion can be solved exactly in spite of very complex topological structure of the manifold $M$. It takes place because scalar curvatures of the constituent factor spaces are included into these equations in certain combinations only. They can compensate each other provided they have opposite signs.

Of special interest are exact solutions because they can be used for a detailed study of the evolution of our space, of the compactification of the internal spaces and of the behaviour of matter fields. It was shown up to now that MCM can be integrated if they have at the most two non-Ricci-flat spaces $M_i$ \cite[13, 14]. Now, we can generalize all these solutions to the cases with much more complex topological structure. Similar ideas of splitting were proposed also in \cite[13, 14] for particular models.

We can continue the splitting process up to infinity. Infinite splitting of the internal spaces corresponds to infinite number of the internal dimensions. Scale factors in this case should belong to the Banach space \cite[14]. In our paper we show that in the limit of infinite splitting effective action for the model with one internal space coincides with tree-level effective action for a bosonic string in the presence of a background gravitational field and dilaton field.

\section{Model and splitting procedure}

We consider a cosmological model with the metric

$$g = -e^{2\gamma(\tau)}d\tau \otimes d\tau + e^{2\beta_i(\tau)} \sum_{k=1}^{n_i} g^{(i)}_{(k)} + \ldots + e^{2\beta_n(\tau)} \sum_{k=1}^{n_n} g^{(n)}_{(k)},$$

which is defined on the manifold (1.1) where

$$M_i = \prod_{k=1}^{n_i} M^k_i, \ i = 1, \ldots, n.$$  \hspace{1cm} (2.2)

The manifold $M^k_i$ with the metric $g^{(i)}_{(k)}$ is an Einstein space of dimension $d^k_i$, i.e

$$R_{mn} \left[ g^{(i)}_{(k)} \right] = \lambda^i_{k} g^{(i)}_{(k)mn}, \ m, n = 1, \ldots, d^k_i,$$  \hspace{1cm} (2.3)

and

$$R \left[ g^{(i)}_{(k)} \right] = \lambda^i_{k} d^k_i.$$  \hspace{1cm} (2.4)
The scalar curvature corresponding to the metric (2.1) is

\[ R[g] = \sum_{i=1}^{n} e^{-2\beta^i} R \left[ g^{(i)} \right] + e^{-2\gamma} \sum_{i,j=1}^{n} (d_{i}\delta_{ij} + d_{i}d_{j}) \dot{\beta}^{i} \dot{\beta}^{j} \]

\[ + e^{-2\gamma} \sum_{i=1}^{n} d_{i}(2\ddot{\beta}^{i} - 2\dot{\gamma}\dot{\beta}^{i}) , \tag{2.5} \]

where

\[ R \left[ g^{(i)} \right] = \sum_{k=1}^{n_{i}} R \left[ g^{(i)}_{(k)} \right] \tag{2.6} \]

and

\[ d_{i} = \sum_{k=1}^{n_{i}} d^{k}_{i} . \tag{2.7} \]

Here, \( g^{(i)} = \sum_{k=1}^{n_{i}} g^{(i)}_{(k)} \) is the metric on \( d_{i} \)-dimensional manifold \( M_{i} \). The overdot denotes differentiation with respect to time \( \tau \). It follows from the equation (2.5) that the scalar curvatures and dimensions of the constituent factor spaces \( M^{k}_{i} \) are included there in the combination (2.6) and (2.7) respectively. It is not difficult to show that \( M_{i} \) are also Einstein spaces provided \( M^{k}_{i} \) are not Ricci flat \([13, 14]\). We can achieve it by conformal transformation

\[ g^{(i)}_{(k)} \rightarrow \left( \lambda^{i}_{k}/\lambda^{i} \right) g^{(i)}_{(k)} , \tag{2.8} \]

where \( \lambda^{i} \neq 0 \) are arbitrary constants. After this transformation \( g^{(i)} = \sum_{k=1}^{n_{i}} (\lambda^{i}_{k}/\lambda^{i}) g^{(i)}_{(k)} \).

Using the equations (2.3) and (2.4) we get the following equations:

\[ R_{mn} \left[ g^{(i)} \right] = \sum_{k=1}^{n_{i}} R_{mn} \left[ \left( \lambda^{i}_{k}/\lambda^{i} \right) g^{(i)}_{(k)} \right] = \sum_{k=1}^{n_{i}} \lambda^{i}_{k} g^{(i)}_{(k)mn} = \lambda^{i} g^{(i)}_{mn} , \tag{2.9} \]

\[ R \left[ g^{(i)} \right] = \sum_{k=1}^{n_{i}} \left( \lambda^{i}/\lambda^{i}_{k} \right) R \left[ g^{(i)}_{(k)} \right] = \lambda^{i} d_{i} , \tag{2.10} \]

which prove our statement. It is clear that the metric \( g^{(i)} \) is indefinite now if signs of \( \lambda^{i}, \lambda^{1}, \ldots, \lambda^{n_{i}} \) do not coincide at least for two of them.

The action is adopted in the following form

\[ S = S_{g} + S_{m} + S_{GH} = \frac{1}{2\kappa^{2}} \int_{M} d^{p} \sqrt{|g|}(R[g] - 2\Lambda) + S_{m} + S_{GH} , \tag{2.11} \]

where \( S_{GH} \) is the standard Gibbons-Hawking boundary term \([15]\), \( \kappa^{2} \) is \( D = 1 + \sum_{i=1}^{n} d_{i} \)-dimensional gravitational constant, \( \Lambda \) is a cosmological constant, and \( S_{m} \) is action for matter. For the metric (2.1) the action (2.11) reads

\[ S_{g} + S_{GH} = \mu \int d\tau \left\{ e^{-\gamma+\gamma_{0}} G_{\beta^{i} \dot{\beta}^{j}} \right\} - \]
\[
\frac{1}{2} e^{\gamma + \gamma_0} \left( -\sum_{i=1}^{n} R \left[ g^{(i)} \right] e^{-2\beta^i} + 2\Lambda \right),
\]
(2.12)

where \( \gamma_0 = \sum_1^n d_i \beta^i \). The effective curvatures \( R \left[ g^{(i)} \right] \) and dimensions \( d_i \) are defined by the equations (2.6) and (2.7) respectively. \( \mu = \frac{1}{\kappa^2} \prod_1^n \mu_i \) where \( \mu_i = \prod_{k=1}^n \mu_i^k \) and \( \mu_i^k = \int_{M_i^k} d^k y \sqrt{|g_{(k)}^{(i)}|} \) is the volume of \( M_i^k \) (as was mentioned in the Introduction, all \( M_i^k \) are compact). \( G_{ij} = d_i \delta_{ij} - d_i d_j \) are the components of the minisuperspace metric.

In the case of homogeneous minimally coupled scalar field we get
\[
S_m = \kappa^2 \mu \int d\tau \left\{ \frac{1}{2} e^{-\gamma + \gamma_0} \dot{\varphi}^2 - e^{\gamma + \gamma_0} U(\varphi) \right\}.
\]
(2.13)

We can consider also matter in the form of \( m \)-component perfect fluid with the energy-momentum tensor
\[
T_{MN}^{(a)} = \sum_{a=1}^{m} T^{(a)}_{MN},
\]
(2.14)

\[
T^{(a)}_{MN} = \text{diag} \left( -\rho^{(a)}, \frac{P^{(a)}_1}{d_1}, \ldots, \frac{P^{(a)}_n}{d_n} \right).
\]
(2.15)

The conservation equations are imposed on each component separately:
\[
T^{(a)}_{MN;M} = 0 \quad a = 1, \ldots, m
\]
(2.16)

and for the tensor (2.15) and the metric (2.1) they have the form
\[
\dot{\rho}^{(a)} + \sum_{i=1}^{n} d_i \beta^i \left( \rho^{(a)} + P^{(a)}_i \right).
\]
(2.17)

We suppose that the pressure and the energy density are connected by the equations of state
\[
P^{(a)}_i = \left( \alpha_i^{(a)} - 1 \right) \rho^{(a)}
\]
(2.18)

where \( \alpha_i^{(a)} = \text{const} \). Then, the conservation equations (2.17) have the simple integrals
\[
\rho^{(a)}(\tau) = A^{(a)} \prod_{i=1}^{n} a_i^{-d_i \alpha_i^{(a)}} = A^{(a)} \prod_{i=1}^{n} V_i^{-\alpha_i^{(a)}},
\]
(2.19)

where \( A^{(a)} \) are constants of integration, \( a_i = \exp \beta^i \) are scale factors and \( V_i \) are proportional to the volume of \( M_i : V_i = a_i^{d_i} \). It is not difficult to verify that the Einstein equations with the energy momentum (2.14)-(2.19) are equivalent to the Euler-Lagrange equations for the action (2.11) and (2.12) where
\[
S_m = -\kappa^2 \mu \int d\tau e^{\gamma + \gamma_0} \sum_{a=1}^{m} \rho^{(a)}.
\]
(2.20)
It can be easily seen from the equations (2.12), (2.13) and (2.20) that scalar curvatures $R[g_{(ik)}^{(i)}]$ and dimensions $d_i^k$ of the constituent factor-spaces $M_i^k$ are included into action in combinations (2.6) and (2.7) only. The equations of motion (Euler-Lagrange equations) "forget" about topological structure. Thus, all possible combinations of the spaces $M_i^k$ which give the same effective curvatures $R[g^{(i)}]$ and the same effective dimensions $d_i$ will have the same dynamics. We can use this fact to generalize the solutions obtained in [8]-[14]. Let us consider, for example, the case

$$\sum_{k=1}^{n_i} R[g_{(ik)}^{(i)}] = 0, \quad i = 1, \ldots, n.$$  (2.21)

Then, for MCM in the presence of minimally coupled scalar field and $\Lambda = 0$ we obtain two classes of the solutions [12]:

i) generalized Kasner solutions

$$a_i = a_{(0)i} t^{\alpha^i}, \quad i = 1, \ldots, n,$$  (2.22)

$$\varphi = \ln t^{\alpha^{n+1}} + const,$$  (2.23)

where

$$\sum_{i=1}^{n} d_i \alpha^i = 1, \quad \sum_{i=1}^{n} d_i (\alpha^i)^2 = 1 - (\alpha^{n+1})^2.$$  (2.24)

Here, $t$ is synchronous time. The space volume of the universe is proportional to time: $V \sim \prod_{i=1}^{n} a_i^{d_i} \sim t$.

ii) steady state solutions

$$a_i = a_{(0)i} \exp(b^i t), \quad i = 1, \ldots, n,$$  (2.25)

$$\varphi = b^{n+1} t + const,$$  (2.26)

where

$$\sum_{i=1}^{n} d_i b^i = 0, \quad \sum_{i=1}^{n} d_i (b^i)^2 = 0.$$  (2.27)

For this class of solutions the space volume of the universe is constant: $V = const.$
3 Multidimensional cosmology and string theory

Let us slightly generalize the metric (2.1) to the inhomogeneous case supposing that the metric has the form

$$g = g^{(0)} + e^{2\beta^1(x)} \sum_{k=1}^{n_1} g^{(1)}_{(k)} + \ldots + e^{2\beta^n(x)} \sum_{k=1}^{n_n} g^{(n)}_{(k)},$$

(3.1)

where the metric $g^{(0)}$ is defined on the $D_0$-dimensional manifold $M_0$ and $x$ are some coordinates of $M_0$. Full manifold $M$ is:

$$M = M_0 \times \prod_{i=1}^{n} M_i$$

where $M_i$ is defined by the equation (2.2) and all $M_i^k$ are again Einstein spaces (see eqs. (2.3) and (2.4)). The scalar curvature is

$$R[g] = R[g^{(0)}] + \sum_{i=1}^{n} e^{-2\beta^i} R[g^{(i)}]$$

$$- \sum_{i,j=1}^{n} (d_i \delta_{ij} + d_i d_j) g^{(0)\mu\nu} \partial_{\mu} \beta^i \partial_{\nu} \beta^j - 2 \sum_{i=1}^{n} d_i \Delta[g^{(0)}] \beta^i,$$

(3.2)

where $\Delta[g^{(0)}]$ is the Laplace-Beltrami operator corresponding to $g^{(0)}$ and the effective curvatures $R[g^{(i)}]$ and dimensions $d_i$ are defined again by the equations (2.6) and (2.7) respectively.

After dimensional reduction the effective cosmological action is [16]

$$S = S_g + S_{GH} = \frac{1}{2\kappa_0^2} \int_{M_0} d^{D_0} x \sqrt{|g^{(0)}|} \prod_{i=1}^{n} e^{d_i \beta^i}$$

$$\cdot \left\{ R[g^{(0)}] - \sum_{i,j=1}^{n} G_{ij} g^{(0)\mu\nu} \partial_{\mu} \beta^i \partial_{\nu} \beta^j + \sum_{i=1}^{n} R[g^{(i)}] e^{-2\beta^i} - 2\Lambda \right\}.$$  

(3.3)

Here, $\kappa_0^2 = \kappa^2 / \prod_{i=1}^{n} \mu_i$ is $D_0$-dimensional gravitational constant and $G_{ij} = d_i \delta_{ij} - d_i d_j$ are the components of the midisuperspace metric.

Let us consider in more detail the case of one internal space: $n = 1$. Then the action (3.3) reads

$$S = \frac{1}{2\kappa_0^2} \int_{M_0} d^{D_0} x \sqrt{|g^{(0)}|} e^{-2\Phi}$$

$$\cdot \left( R[g^{(0)}] - 4\omega g^{(0)\mu\nu} \partial_{\mu} \Phi \partial_{\nu} \Phi + R[g^{(1)}] \frac{4}{d_1} e^{\frac{4}{d_1} \Phi} - 2\Lambda \right),$$

(3.4)

where the dilaton $\Phi$ is defined via the scale factor $a_1(x) = \exp \beta^1(x)$ as follows $e^{-2\Phi} := a_1^{d_1}$ and $\omega = \frac{1}{d_1} - 1 < 0$ is the Brans-Dicke parameter. This action is
written in the Brans-Dicke frame. After conformal transformation
\[
\hat{g}_{\mu\nu}^{(0)} = e^{-4\Phi/(D_0-2)} g_{\mu\nu}^{(0)},
\]
\[
\varphi = \pm 2 \left[ \Omega + \frac{D_0 - 1}{D_0 - 2} \right]^{1/2} \Phi
\]
the action (3.4) can be written in the Einstein frame as follows
\[
S = \frac{1}{2\kappa^2_0} \int_{M_0} d^{D_0}x \sqrt{|\hat{g}^{(0)}|} \cdot \left\{ \hat{R} \left[ \hat{g}^{(0)} \right] - \hat{g}^{(0)\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + R \left[ g^{(1)} \right] \exp(-2\lambda_{(1)}c\varphi) - 2\Lambda \exp(-2\lambda_{(2)}c\varphi) \right\},
\]
where the cosmological dilatonic coupling constants are
\[
\lambda_{(1)c} = \frac{D - 2}{d_1(D_0 - 2)}, \quad \lambda_{(2)c} = \frac{d_1}{(D - 2)(D_0 - 2)},
\]
and \(D = D_0 + d_1\) is the dimension of the manifold \(M\).

By analogy with the splitting \(M^i \to \prod_{k=1}^{n_i} M^k_i\) we can split spaces \(M^k_i\) into new factor spaces: \(M^k_i \to \prod_{i=1}^{k_i} M^k_i\) where \(M^k_i\) are arbitrary Einstein spaces. In one’s turn spaces \(M^k_i\) can be split also and so on (chain splitting). We demand only that spaces corresponding to the most elementary splitting are Einstein spaces. We can continue the splitting process up to infinity and under this limit the actions (2.12), (2.13), (2.20), (3.3), (3.4), and (3.7) are well defined provided that
\[
\sum_{i=1}^{n} |R \left[ g^{(i)} \right]| < +\infty, \quad \sum_{i=1}^{n} d_i|\beta^i| < +\infty.
\]
In this case \(\beta = (\beta^i)\) belongs to the Banach space with \(l_1\)-norm [8, 14].

In particular case \(n = 1\), infinite splitting of the internal space corresponds to the limit \(d_1 \to \infty\) and the action (3.7) reads
\[
S = \frac{1}{2\kappa^2_0} \int_{M_0} d^{D_0}x \sqrt{|\hat{g}^{(0)}|} \left\{ \hat{R} \left[ g^{(0)} \right] - \hat{g}^{(0)\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + C \exp(-2\lambda_s\varphi) \right\},
\]
where
\[
\lambda_s^2 := \frac{1}{(D_0 - 2)}, \quad C := R \left[ g^{(1)} \right] - 2\Lambda.
\]
This action coincides with tree-level effective action for a bosonic string in the presence of a background metric \(g^{(0)}_{\mu\nu}\) and dilaton \(\Phi\). Here, \(\lambda_s\) is the string dilatonic coupling constant and \(C\) is the central charge deficit [17, 18].

In conclusion I would like to note that infinite chain splitting may result in topological fractal structure of the multidimensional universe. We investigate this interesting problem in our forthcoming paper [19].
Acknowledgements

The work was supported in part by DFG grant 436 RUS 113/7/0. I wish to thank Prof.Kleinert and Free University of Berlin as well as the Projektgruppe Kosmologie at the Potsdam University for their hospitality during preparation of this paper. I am grateful to V.D.Ivashchuk, U.Günther and M.Rainer for useful discussions.

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