Abstract

We study dimensional reduction of M5 branes on a circle bundle when the supersymmetry parameter is not constant along the circle. When the gauge group is Abelian and the fields appear quadratically in the Lagrangian, we can always obtain a supersymmetric five-dimensional theory by keeping fermionic nonzero modes that match with the corresponding nonzero modes of the supersymmetry parameter, and by keeping the zero modes for the bosonic fields as usual. But a supersymmetric non-Abelian generalization can be found only under special circumstances. One instance where we find a non-Abelian supersymmetric generalization is when we perform dimensional reduction along a null direction.
1 Introduction

There is a supersymmetric $2,0$ Abelian tensor multiplet in $\mathbb{R}^{1,5}$ which has a self-dual three-form, five scalar fields and four real Weyl fermionic fields. We can put this tensor multiplet on any six-manifold for which there exists a nontrivial solution to the six-dimensional conformal Killing spinor equation

$$\nabla_M \varepsilon = \Gamma_M \eta \quad (1.1)$$

Here $M = 0, 1, 2, 3, 4, 5$ is a vector index on the six-manifold that we will take to be Lorentzian, and $\varepsilon$ will then be the supersymmetry parameter. The equation (1.1) can be relaxed by turning on supergravity background fields. But we will not study such a generalization here. So $\nabla_M$ here is denoting a curvature covariant derivative that only involves the spin connection and no $R$-gauge field is turned on.

The classical non-Abelian tensor multiplet is not known and perhaps it does not exist. One approach is then to consider the Abelian tensor multiplet on a circle bundle and perform dimensional reduction along the circle. Then one finds an Abelian 5d Yang-Mills theory for which one can find a non-Abelian generalization. If the supersymmetry parameter is constant along the circle, then it will survive as a supersymmetry under dimensional reduction. Otherwise the supersymmetry will be broken but one may then get a supersymmetric theory by turning on a background R-gauge field that will relax the requirement (1.1). But that changes the problem that one may want to study. So we would like to analyze whether one can avoid turning on the R-gauge field and somehow take advantage of the fact that the 6d theory is supersymmetric.

One example that one may want to study is the M5 brane on $S^6$ that one may conformally map to $S^1 \times H_5$. If one wants to study this problem without any background fields turned on, then one finds that the supersymmetry parameter will have a non-trivial dependence on $S^1$ in $S^1 \times H_5$, and dimensional reduction down to $H_5$ yields a non-supersymmetric Yang-Mills theory that is quite difficult to study. Being a nonrenormalizable theory it has no clear well-defined perturbative expansion and there are not many tools to study this theory and supersymmetric localization can not be used if the Yang-Mills theory one gets on $H_5$ is not supersymmetric.

In this paper we will study the following situation. We assume that the 6d theory is supersymmetric on a circle bundle with fiber coordinate $u$. We also assume that the supersymmetry parameter is not constant along $u$. So under dimensional reduction along $u$ all supersymmetry is gone. That is the case if we consider the bosonic and fermionic zero modes. But what if we consider the bosonic zero modes and some fermionic nonzero modes? Is there a consistent truncation of supersymmetry where bosonic zero modes are
kept such that supersymmetry exists in that 5d truncation?

If the fields appear only quadratically in the Lagrangian so that the gauge group
is Abelian, then there always exists such a consistent truncation. To see this, let us
schematically write the 6d Lagrangian as

$$
\mathcal{L}_{6d} = (\partial \phi)^2 + \psi \partial \bar{\psi}
$$

where $\phi$ denotes bosonic fields and $\psi$ denotes fermionic fields. The supersymmetry varia-
tion is schematically on the form

$$
\delta \phi = \epsilon \psi \\
\delta \psi = \epsilon \partial \phi
$$

Then the supersymmetry variation of the Lagrangian is a sum of terms on the form

$$
0 = \delta \mathcal{L}_{6d} = \sum \partial^2 \phi \epsilon \psi
$$

and the sum vanishes since the 6d Lagrangian is supersymmetric. Now let us make the
truncation where we keep the bosonic zero mode along the $u$ direction,

$$
\phi_0 = \int du \phi
$$

Its supersymmetry variation is

$$
\delta \phi_0 = \int du \epsilon \psi
$$

(1.2)

Now let us assume that the supersymmetry parameter has only two nonzero modes,

$$
\epsilon = e^{iau} \epsilon_{+1} + e^{-iau} \epsilon_{-1}
$$

for some real parameter $a$ that depends on the geometry of the six-manifold. This is the
generic structure for any solution of (1.1) on a circle bundle. Here subscripts denote the
mode number. Then the integral in (1.2) picks up corresponding nonzero modes from $\psi$,

$$
\delta \phi_0 = \epsilon_{+1} \psi_{-1} + \epsilon_{-1} \psi_{+1}
$$

whose supersymmetry variations are

$$
\delta \psi_{\pm 1} = \epsilon_{\pm 1} \partial \phi_0
$$

Now let check if the truncated Lagrangian

$$
\mathcal{L}_{5d} = (\partial \phi_0)^2 + \psi_{+1} \partial \psi_{-1} + \psi_{-1} \partial \psi_{+1}
$$
is supersymmetric. We get

$$\delta L_{5d} = \sum \partial^2 \phi_0 (\varepsilon_{+1} \psi_{-1} + \varepsilon_{-1} \psi_{+1})$$

but this we can also write as

$$\delta L_{5d} = \int du \sum \partial^2 \phi_0 \varepsilon \psi$$

Now let us go back to the 6d Lagrangian. If we expand $\phi$ in its Fourier modes as $\phi = \sum_n \phi_n e^{inu}$, then we get

$$0 = \delta L_{6d} = \sum \sum_n \partial^2 \phi_n e^{inu} \varepsilon \psi$$

and we know that this is zero since the 6d Lagrangian is supersymmetric. Of course, if we integrate zero along the fiber, it is still zero, so we have

$$0 = \int du \delta L_{6d} = \sum \sum_n \partial^2 \phi_n \int du e^{inu} \varepsilon \psi$$

If we then put $\phi_n = 0$ for all $n$ except for the zero mode $\phi_0$, then this reduces to

$$0 = \sum \partial^2 \phi_0 \int du \varepsilon \psi = \delta L_{5d}$$

which means that the truncated Lagrangian where only $\phi_0$ is kept, is supersymmetric under the truncated supersymmetries.

This general argument fails for the non-Abelian generalization where the Lagrangian has higher order terms. For instance if the 6d Lagrangian contains a cubic interaction term of the form $\phi_{+2} \psi_{-1} \psi_{-1}$ and if we have a supersymmetry variation of the form $\delta \phi_{+2} = \varepsilon_{+1} \psi_{+1}$, then the variation of that term will contain a term of the form $\varepsilon_{+1} \psi_{+1} \psi_{-1} \psi_{-1}$ that should survive if the truncation down to the modes $\phi_0$ and $\psi_{\pm 1}$ were a consistent truncation. But we will never get that term if we first truncate the Lagrangian to the modes $\phi_0$ and $\psi_{\pm 1}$ and then make the supersymmetry variation since then we will put the term $\phi_{+2} \psi_{-1} \psi_{-1}$ to zero in that truncated Lagrangian. So the truncation becomes inconsistent in general, when there are higher order terms. However, there can be exceptions where a truncated non-Abelian generalization can be found that is supersymmetric.

This argument also shows that the critical term to analyze in the supersymmetry variation of the non-Abelian Lagrangian will be the terms that are cubic in the fermionic fields. Typically these term are the most difficult ones to analyse since it usually requires a Fierz rearrangement to see whether the sum of these cubic terms is zero or not. But it
is really important to analyze precisely these cubic terms to see whether the non-Abelian Lagrangian is supersymmetric or not. This will become more clear as we proceed with our concrete examples.

In this paper we will study the M5 brane on $\mathbb{R} \times S^5$ where we have the Lorentzian time along $\mathbb{R}$. The supersymmetry parameter depends nontrivially on the time direction. First, in section 2 we perform dimensional reduction along the time direction and obtain a supersymmetric Abelian Lagrangian. We then show that no non-Abelian generalization exists if we insist on keeping all the supersymmetries of the Abelian theory. In section 2.1 we reduce the amount of supersymmetry and consider the smaller tensor multiplet that has just one real scalar field. Here we almost seem to find a supersymmetric non-Abelian Lagrangian in 5d by using our truncation, but it turns out to fail. While most terms cancel out nicely, there are cubic terms in the fermionic fields that arises upon a supersymmetry variation and these have to vanish by using a Fierz rearrangement, but these terms do not vanish in that way. We then make a further Weyl projection that reduces supersymmetry further, and then finally we are able to find a supersymmetric Lagrangian. But then, in section 2.2 we discover that if we make a simple field redefinition, our Lagrangian becomes identical with the Lagrangian that was already found in the literature on $S^5$ [2] and that was derived from the M5 brane in [3] by turning on an R-gauge field along the time direction.

We next consider our second example, in section 3 where we consider a null reduction by following closely [6]. We take our null direction as a combination of the Hopf circle on $S^5$ and the time direction. We first obtain the Abelian truncated theory and show that it is supersymmetric. We next show that the Abelian theory does not immediately generalize to the non-Abelian case, but if we impose further Weyl projections, then we are able to obtain a non-Abelian Lagrangian.

There are five appendices. In particular, in appendix A we review a 6d formulation of non-Abelian 5d SYM where one introduces an auxiliary geometrical vector field [4], [7], [5] and present the closure relations that one gets for these supersymmetry variations and it was this analysis that originally led us to consider the two examples that we are presenting in this paper. Namely these two examples are following from making the two Weyl projections in equations (A.8) and (A.9) respectively. The first Weyl projection leads us to the time reduction and the small tensor multiplet. The second Weyl projection leads us to the null reduction.
2 M5 brane on $\mathbb{R} \times S^5$

The six-manifold $\mathbb{R} \times S^5$ can be conformally mapped to $S^6$ if we assume an Euclidean signature. But here we will assume a Lorentzian signature with time along the $\mathbb{R}$ direction. Our first goal is to see whether we can derive a supersymmetric theory on $S^5$ from an M5 brane on $\mathbb{R} \times S^5$ without turning on an R-gauge field along the time direction. The Abelian M5 brane on $\mathbb{R} \times S^5$ is well-understood. In fact one can generalize to any six-manifold for which (1.1) has at least one solution. In that case we have the following supersymmetry variations

$$\delta \phi^A = i \bar{\epsilon} \Gamma^A \psi$$
$$\delta B_{MN} = i \bar{\epsilon} \Gamma_{MN} \psi$$
$$\delta \psi = \frac{1}{12} \Gamma^{MNP} \bar{\epsilon} H_{MNP} + \Gamma^M \Gamma^A \psi \nabla_M \phi^A - 4 \Gamma^A \eta \phi^A$$

and the supersymmetric Lagrangian may be expressed as

$$\mathcal{L} = \mathcal{L}_B - \frac{1}{2} (\nabla_M \phi^A)^2 + \frac{i}{2} \bar{\psi} \Gamma^M \nabla_M \psi - \frac{R}{10} (\phi^A)^2$$

where $\mathcal{L}_B$ is some Lagrangian for the selfdual tensor field whose precise form will not be very important for us now, since we will shortly reduce this Lagrangian down to five dimensions. Here $R$ is the Ricci curvature scalar on six-manifold. We will now specialize to $\mathbb{R} \times S^5$ and write the metric as

$$ds^2 = g_{MN} dx^M dx^N = -dt^2 + G_{mn} dx^m dx^n$$

To reduce down to $S^5$, we will represent the gamma matrices in terms of five-dimensional gamma matrices $\gamma^m$ and $\tau^A$ as follows,

$$\Gamma^t = i \sigma^2 \otimes 1 \otimes 1$$
$$\Gamma^m = \sigma^1 \otimes \gamma^m \otimes 1$$
$$\Gamma^A = \sigma^3 \otimes 1 \otimes \tau^A$$

The 6d chirality matrix is

$$\Gamma = \sigma^3 \otimes 1 \otimes 1$$

and $\bar{\epsilon}$ and $\psi$ have opposite chiralities

$$\Gamma \bar{\epsilon} = -\bar{\epsilon}$$
$$\Gamma \psi = \psi$$
and they are Majorana spinors in eleven dimensions,
\[
\bar{\epsilon} = \epsilon^T C_{11d} \\
\bar{\psi} = \psi^T C_{11d}
\]
where the Dirac conjugate is defined as \( \bar{\psi} = \psi^\dagger \Gamma^t \). We may solve (1.1) by separating its components as
\[
\partial_t \epsilon = \Gamma_t \eta \\
\nabla_m \epsilon = \Gamma_m \eta
\]
(2.1)
We use the relation
\[
\Gamma^{mn} \nabla_m \nabla_n \epsilon = -\frac{R}{4} \epsilon
\]
where \( R = \frac{20}{r^4} \) is the Ricci scalar on \( S^5 \) with radius \( r \), to find the solution
\[
\epsilon = e^{\frac{t}{2r}} \begin{pmatrix} 0 \\ \mathcal{E} \end{pmatrix} + e^{-\frac{t}{2r}} \begin{pmatrix} 0 \\ \mathcal{F} \end{pmatrix}
\]
We also get
\[
\eta = i \frac{e^{\frac{t}{2r}}}{2r} \begin{pmatrix} \mathcal{E} \\ 0 \end{pmatrix} - i \frac{e^{-\frac{t}{2r}}}{2r} \begin{pmatrix} \mathcal{F} \\ 0 \end{pmatrix}
\]
Here
\[
\nabla_m \mathcal{E} = i \frac{1}{2r} \gamma_m \mathcal{E} \\
\nabla_m \mathcal{F} = -i \frac{1}{2r} \gamma_m \mathcal{F}
\]
Perhaps the best way to see that this solves (1.1) is by simply plugging in this solution into (2.1) to see that these equations are both satisfied. Let us now study the Majorana condition more closely. The eleven-dimensional charge conjugation matrix is antisymmetric,
\[
C^T_{11d} = -C_{11d}
\]
and we will represent it as
\[
C_{11d} = \epsilon \otimes C \otimes \tilde{C}
\]
where $C$ and $\tilde{C}$ are antisymmetric charge conjugation matrices in 5d, and $\varepsilon$ is the antisymmetric tensor. At this point, things get clearer when we write out all the spinor indices explicitly though, so let us do that here,

$$(C_{1d})_{\alpha \beta, \dot{\alpha} \dot{\beta}} = \varepsilon_{ab} C_{\alpha \beta} C_{\dot{\alpha} \dot{\beta}}$$

Then the Majorana condition becomes

$$(\psi^{+\alpha\dot{\alpha}})^* i(\sigma^2)^b_a = \psi^{a\beta\dot{\beta}} \varepsilon_{ab} C_{\beta \alpha} C_{\beta \alpha}$$

We will define the antisymmetric tensor $\varepsilon_{ab}$ such that

$$\varepsilon_{+-} = 1$$

and then we get

$$(\psi^{+\alpha\dot{\alpha}})^* = \psi^{+\beta\dot{\beta}} C_{\beta \alpha} C_{\beta \alpha}$$

$$(\varepsilon^{-\alpha\dot{\alpha}})^* = \varepsilon^{-\beta\dot{\beta}} C_{\beta \alpha} C_{\beta \alpha}$$

From now on we will drop the 6d chirality indices $\pm$ as they play no significant role in 5d. In 5d we do not really have a Majorana condition for the nonzero modes. What we have instead is a relation between $E$ and $F$,

$$(\mathcal{E}^{\alpha\dot{\alpha}})^* = C_{\alpha \beta} C_{\dot{\alpha} \dot{\beta}} F^{\beta \dot{\beta}}$$

$$(\mathcal{F}^{\alpha\dot{\alpha}})^* = C_{\alpha \beta} C_{\dot{\alpha} \dot{\beta}} \mathcal{E}^{\beta \dot{\beta}}$$

These relations follow easily from using the explicit form of our solution, equation (2.2). But now we would also like to derive the second condition from the first one by taking the complex conjugate. Taking the complex conjugate of the first equation, we get

$$\mathcal{E}^{\alpha\dot{\alpha}} = (C_{\alpha \beta})^* (C_{\dot{\alpha} \dot{\beta}})^* (F^{\beta \dot{\beta}})^*$$

We may now multiply by charge conjugation matrices on both sides to get

$$C_{\alpha \beta} C_{\dot{\alpha} \dot{\beta}} \mathcal{E}^{\beta \dot{\beta}} = C_{\alpha \beta} C_{\dot{\alpha} \dot{\beta}} (C_{\beta \gamma})^* (C_{\dot{\beta} \dot{\gamma}})^* (F^{\gamma \dot{\gamma}})^*$$

We shall require that

$$C_{\alpha \beta} (C_{\beta \gamma})^* = -\delta_{\alpha}^{\gamma}$$

The reason why we put the minus sign here will become clear later on. We can now introduce the inverse

$$C^{\gamma \beta} = (C_{\beta \gamma})^*$$
We use \( C_{\alpha\beta} \) and \( C^{\alpha\beta} \) to lower and rise spinor indices by always acting from the left,

\[
\psi_\alpha = C_{\alpha\beta} \psi^\beta \\
\psi^\alpha = C^{\alpha\beta} \psi_\beta
\]

So we define for example

\[
(\gamma^m)^{\alpha\beta} = C^{\gamma\gamma}(\gamma^m)^{\alpha}_\gamma
\]

We may now find the following relations

\[
C^\alpha_\beta = C^{\alpha\gamma} C_{\gamma\beta} = \delta^\alpha_\beta \\
C_{\alpha\beta} = C^{\beta\gamma} C_{\alpha\gamma} = -\delta^\alpha_\beta
\]

We have the Fierz expansion of two anticommuting spinors,

\[
\psi^\alpha \psi^\beta = A C^{\alpha\beta} + B_m (\gamma^m)^{\alpha\beta} + C_{mn} (\gamma^{mn})^{\alpha\beta}
\]

It corresponds to the following expansion of the tensor product of two spinor representations

\[
4 \otimes 4 = 1_a \oplus 5_a \oplus 10_s
\]

The subscripts \( a \) and \( s \) stand for antisymmetric and symmetric representations, so we must have that \( C^{\alpha\beta} \) and \( (\gamma^m)^{\alpha\beta} \) are antisymmetric, whereas \( (\gamma^{mn})^{\alpha\beta} \) is symmetric in \( \alpha \) and \( \beta \). Our 5d spinor notations follow closely the reference [2].

The time direction in Euclidean \( \mathbb{R} \times S^5 \) is noncompact if this shall be related by a conformal map to \( S^6 \). But in Lorentzian signature that we will consider here, the time direction can be taken to be a compact circle with radius \( 2\pi r \). We will refrain from discussing any physical implications of having a compact time direction. From a purely mathematical viewpoint of classical supersymmetric field theory, having a compact time direction simply means that we may expand the fields in Fourier modes in the time direction by assuming that time has a periodicity \( t \sim t + 2\pi r \). For fermions there is as always a possibility of having either periodic or antiperiodic boundary conditions. Since the supersymmetry parameter depends on time through the exponential factors \( e^{\pm \frac{i}{2} t} \) which is antiperiodic as \( t \) goes to \( t + 2\pi r \), we conclude that fermions shall have antiperiodic boundary conditions if we want to have a supersymmetric theory. The bosonic fields must be periodic and therefore only even modes are kept for the bosonic fields, whereas for the fermionic field only the odd modes are kept. And if only the odd modes are kept, it means that there is no fermionic zero mode present.
But we do not think that we will be able to find a non-Abelian theory if we keep infinitely many Kaluza-Klein modes, neither do we think this is really the right thing to do when the gauge group is non-Abelian because then we shall have instanton particles that are expected to fill in missing modes when we truncate the modes to a finite number of modes. Now instead of truncating to the fermionic zero modes as one normally does in usual dimensional reduction, we will truncate to the lowest lying odd Fourier modes 

\[ \psi = e^{i \frac{\chi}{r}} \begin{pmatrix} \chi \\ 0 \end{pmatrix} + e^{-i \frac{\zeta}{r}} \begin{pmatrix} \chi \\ 0 \end{pmatrix} \]

Then the fermionic field has the same type of expansion as the supersymmetry parameter \( \varepsilon \) and there is a chance that this will preserve some supersymmetry. There is no Majorana condition on these modes but instead there is a relation between the two modes,

\[ (\chi^{\alpha \dot{\alpha}})^* = C_{\alpha \beta} C_{\dot{\alpha} \dot{\beta}} \]

The supersymmetry variations can be derived easily by truncating the supersymmetry variations for the Abelian M5 brane. We get

\[ \delta \phi^A = -i \mathcal{E}^\dagger \tau^A \chi - i \mathcal{F}^\dagger \tau^A \zeta \]
\[ \delta A_m = -i \mathcal{E}^\dagger \gamma_m \chi - i \mathcal{F}^\dagger \gamma_m \zeta \]
\[ \delta \chi = \frac{1}{2} \gamma^{mn} \mathcal{E} F_{mn} - \gamma^m \tau^A \mathcal{E} \nabla_m \phi^A - \frac{2i}{r} \tau^A \mathcal{E} \phi^A \]
\[ \delta \zeta = \frac{1}{2} \gamma^{mn} \mathcal{F} F_{mn} - \gamma^m \tau^A \mathcal{F} \nabla_m \phi^A + \frac{2i}{r} \tau^A \mathcal{F} \phi^A \]

The corresponding supersymmetric Lagrangian is given by

\[ \mathcal{L} = \frac{1}{4} F_{mn}^2 - \frac{1}{2} (\nabla_m \phi^A)^2 + \frac{i}{2} \chi^\dagger \gamma^m \nabla_m \chi + \frac{i}{2} \zeta^\dagger \gamma^m \nabla_m \zeta - \frac{2}{r^2} (\phi^A)^2 + \frac{1}{4r} (\chi^\dagger \chi - \zeta^\dagger \zeta) \]

The natural choice is to take \( \varepsilon \) to be an anti-commuting parameter. In that case the variations of the bosonic fields become hermitian, and we may write these variations as

\[ \delta \phi^A = -i \mathcal{E}^\dagger \tau^A \chi + i \chi^\dagger \tau^A \mathcal{E} \]
\[ \delta A_m = -i \mathcal{E}^\dagger \gamma_m \chi + i \chi^\dagger \gamma_m \mathcal{E} \]
\[ \delta \chi = \frac{1}{2} \gamma^{mn} \mathcal{E} F_{mn} - \gamma^m \tau^A \mathcal{E} \partial_m \phi^A - \frac{2i}{r} \tau^A \mathcal{E} \phi^A \]

We may also write the Lagrangian as

\[ \mathcal{L} = \frac{1}{4} F_{mn}^2 - \frac{1}{2} (\nabla_m \phi^A)^2 + i \chi^\dagger \gamma^m \nabla_m \chi \]
One may now easily verify that this Lagrangian is invariant under these supersymmetry variations by just using the Killing spinor equation

\[ \nabla_m \mathcal{E} = \frac{i}{2r} \gamma_m \mathcal{E} \]

This result is encouraging because it provides our first example of a dimensionally reduced theory that has supersymmetry although the 6d theory has a supersymmetry parameter that depends nontrivially on the circle along which we reduce. Having a supersymmetric Lagrangian, we may also expect that these supersymmetry variations close on some symmetry variations of the Lagrangian.

However, we will now see that no non-Abelian generalization of this Abelian Lagrangian can be constructed that is supersymmetric. To show this we will proceed iteratively. First we just replace all the derivatives \( \nabla_m \) with gauge covariant derivatives \( D_m = \nabla_m - i [A_m, \bullet] \) and assume all fields are in the adjoint representation. Then of course the Lagrangian will not be supersymmetric. We then find correction terms such that we cancel the unwanted terms, but such correction terms will also generate new terms that we also need to cancel by adding further correction terms. This can be analysed fairly systematically. In the end, we will find a fully corrected Lagrangian and corresponding supersymmetry variations but still that Lagrangian will not be supersymmetric. Because of the apparent uniqueness of each term we find in each iteration step, we consider this to be a no-go proof.

First, if we just replace \( \nabla_m \) with \( D_m \) everywhere, then we get the following nonvanishing variation of the Lagrangian,

\[
\delta \mathcal{L} = -\frac{1}{2} \chi^\dagger \gamma^m \tau^A \mathcal{E} [F_{mn}, \phi^A] - \chi^\dagger \gamma^m \mathcal{E} [\phi^A, D_m \phi^A]
\]

where we define the gauge covariant derivative so that

\[
[D_m, D_n] \phi = -i [F_{mn}, \phi]
\]

We next cancel both these terms by adding to the Lagrangian the following coupling term

\[
\mathcal{L}_1 = \chi^\dagger \tau^A [\chi, \phi^A]
\]

We can not imagine any other term can do this job. But by adding this term, there will be generated some new terms as well, and so now we get

\[
\delta \mathcal{L} + \delta \mathcal{L}_1 = \frac{1}{2} \chi^\dagger \tau^{AB} \gamma^m \mathcal{E} D_m ( [\phi^A, \phi^B] ) + \frac{2i}{r} \chi^\dagger \tau^{AB} \mathcal{E} [\phi^A, \phi^B]
\]
plus some cubic terms in $\chi$ that we will not need to analyse further here. Now these two terms can be canceled by modifying the supersymmetry variation by adding the term

$$\delta_1 \chi = \frac{i}{2} \tau^{AB} \mathcal{E} [\phi^A, \phi^B]$$

to $\delta \chi$. But that will also generate another term

$$\delta_1 \mathcal{L}_1 = i \chi^+ \tau^C \mathcal{E} [[\phi^A, \phi^C], \phi^C]$$

but that we can easily cancel by adding the term

$$\mathcal{L}_2 = -\frac{1}{4} [\phi^A, \phi^B]^2$$

But even when taking into account all these non-Abelian correction terms, we will still end up with a nonvanishing variation

$$(\delta + \delta_1) (\mathcal{L} + \mathcal{L}_1) = \left( \frac{-5i}{4r} + \frac{i}{4r} + \frac{2i}{r} \right) \chi^+ \tau^{AB} \mathcal{E} [\phi^A, \phi^B] = \frac{i}{r} \chi^+ \tau^{AB} \mathcal{E} [\phi^A, \phi^B]$$

plus those cubic terms in the fermionic fields that we did not analyse here since it is already clear that no non-Abelian Lagrangian can be found. There now is no further terms that we can add that could cancel this nonvanishing variation. This finishes our no-go proof.

### 2.1 The small vector multiplet

We may be more successful with finding a non-Abelian generalization if we make our tensor multiplet smaller. To this end we will impose the Weyl projection

$$\tau^5 \mathcal{E} = \mathcal{E}$$

on the supersymmetry parameter, thus reducing the amount of supersymmetry by half. This will reduce the R-symmetry as $SO(5) \to SU(2)_R$. But of course, by selecting the fifth direction in $SO(5)$, we will just break $SO(5) \to SO(4) = SU(2)_F \times SU(2)_R$ but the $SU(2)_F$ will not rotated the supercharges, it will be a flavor symmetry. The original Abelian tensor multiplet breaks into one smaller tensor multiplet with just one real scalar field $\phi = \phi^5$ and a fermionic field that is also subject to the Weyl projection

$$\tau^5 \psi = \psi$$
Then the remaining fields are four real scalars, and another fermionic field subject to the opposite Weyl projection $\tau^5 \psi = -\psi$. These fields form a hypermultiplet. We will discard this hypermultiplet and only focus on the small tensor multiplet.

Let us now introduce some index notations for the R-symmetry. We denote a spinor as

$$\psi^{\alpha \dot{\alpha}} = \begin{pmatrix} \psi^\alpha_I \\ \psi_{\alpha A} \end{pmatrix}$$

The flavor index $A$ is a two-component spinor index that shall not be confused with the $SO(5)$ vector index $A$. We define the gamma matrices $\tau^A = (\tau^i, \tau^5)$ as

$$\tau^i = \begin{pmatrix} 0 & \sigma^i_{IB} \\ \sigma^{i,AJ} & 0 \end{pmatrix}$$

$$\tau^5 = \begin{pmatrix} \delta^J_I & 0 \\ 0 & -\delta^A_B \end{pmatrix}$$

The supersymmetry parameter that satisfies $\tau^5 \mathcal{E} = \mathcal{E}$ has a nonvanishing component $\mathcal{E}_I$,

$$\mathcal{E} = \begin{pmatrix} \mathcal{E}_I \\ 0 \end{pmatrix}$$

The antisymmetric charge conjugation matrix is represented as

$$C_{\dot{\alpha} \dot{\beta}} = \begin{pmatrix} \epsilon^{IJ} & 0 \\ 0 & \epsilon_{AB} \end{pmatrix}$$

We have

$$(\mathcal{E}^\alpha_I)^* = C_{\alpha \beta} \epsilon^{IJ} \mathcal{F}^\beta_J$$

$$(\chi^\alpha_I)^* = C_{\alpha \beta} \epsilon^{IJ} \zeta^\beta_J$$

(2.3)

The Killing spinor equations are

$$\nabla_m \mathcal{E}_I = \frac{i}{2r} \gamma_m \mathcal{E}_I$$

$$\nabla_m \mathcal{F}_I = -\frac{i}{2r} \gamma_m \mathcal{F}_I$$

The derivation of the second equation from the first by taking the complex conjugate is as follows,

$$\nabla_m \mathcal{F}^\alpha_I = C^{\alpha \beta} \epsilon_{IJ} (\nabla_m \mathcal{F}^\beta_J)^*$$

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\[
\begin{align*}
\delta \phi &= -i(\mathcal{E}_I)^\dagger \chi_I - i(\mathcal{F}_J)^\dagger \zeta_I \\
\delta A_m &= -i(\mathcal{E}_I)^\dagger \gamma_m \chi_I - i(\mathcal{F}_J)^\dagger \gamma_m \zeta_I \\
\delta \chi_I &= \frac{1}{2} \gamma^{mn} \mathcal{E}_I F_{mn} - \gamma^m \mathcal{E}_I D_m \phi - \frac{2i}{r} \mathcal{E}_I \phi
\end{align*}
\]

and the supersymmetric Lagrangian is

\[
\mathcal{L} = \frac{1}{4} F_{mn}^2 - \frac{1}{2} (D_m \phi)^2 - \frac{2}{r^2} \phi^2 + i(\chi_I)^\dagger \gamma^m D_m \phi + \frac{1}{2r} (\chi_I)^\dagger \chi_I
\]

The closure relations for these supersymmetry variations are highly nonstandard,

\[
\begin{align*}
[\delta_2, \delta_1] \phi &= 2i \mathcal{L}_v \phi \\
[\delta_2, \delta_1] A_m &= 2i \mathcal{L}_v A_m + D_m \Lambda \\
[\delta_2, \delta_1] \chi_I &= 8i \mathcal{L}_B \chi_I + \frac{12}{r} A_I^J \chi_J - i \left( 3A_I^J + 3B_{pl}^J \gamma^p - C_{pq}^J \gamma^{pq} \right) \left( \gamma^m \nabla_m \chi_J + \frac{1}{2r} \chi_J \right) \\
&\quad + 8i \mathcal{L}_{\tilde{B}} \chi_I - \frac{16}{r} \tilde{A}_I^J \zeta_J - \frac{4}{r} \tilde{B}_{pl}^J \gamma^p \zeta_J \\
&\quad - i \left( 3\tilde{A}_I^J + 3\tilde{B}_{pl}^J \gamma^p - \tilde{C}_{pq}^J \gamma^{pq} \right) \left( \gamma^m \nabla_m \zeta_J - \frac{1}{2r} \zeta_J \right)
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{L}_B \chi_I &:= B_{mI}^J \nabla_m \chi_J + \frac{1}{4} \nabla_m B_{nl}^J \gamma^{mn} \chi_J = B_{mI}^J \nabla_m \chi_J + \frac{i}{2r} C_{mn}^J \gamma^{mn} \chi_J \\
\mathcal{L}_{\tilde{B}} \zeta_I &:= \tilde{B}_{mI}^J \nabla_m \zeta_J + \frac{1}{4} \nabla_m \tilde{B}_{nl}^J \gamma^{mn} \zeta_J = \tilde{B}_{mI}^J \nabla_m \zeta_J
\end{align*}
\]

Here the various coefficients are defined as

\[
\begin{align*}
\mathcal{E}_I(\mathcal{E}_J)^\dagger &= A_I^J + B_{mI}^J \gamma^m + C_{mn}^J \gamma^{mn} \\
\mathcal{E}_I(\mathcal{F}_J)^\dagger &= \tilde{A}_I^J + \tilde{B}_{mI}^J \gamma^m + \tilde{C}_{mn}^J \gamma^{mn}
\end{align*}
\]
where

\[ A^J_I = -\frac{1}{4}(\mathcal{E}_J)^I \mathcal{E}_I \]
\[ B^J_m I = -\frac{1}{4}(\mathcal{E}_J)^I \gamma_m \mathcal{E}_I \]
\[ C^J_{mn} I = \frac{1}{8}(\mathcal{E}_J)^I \gamma_{mn} \mathcal{E}_I \]

and

\[ \tilde{A}^J_I = -\frac{1}{4}(\mathcal{F}_J)^I \mathcal{E}_I \]
\[ \tilde{B}^J_m I = -\frac{1}{4}(\mathcal{F}_J)^I \gamma_m \mathcal{E}_I \]
\[ \tilde{C}^J_{mn} I = \frac{1}{8}(\mathcal{F}_J)^I \gamma_{mn} \mathcal{E}_I \]

There are the following differential relations between these coefficients that one may derive by using the Killing spinor equations,

\[ \nabla_m A^J_I = 0 \]
\[ \nabla_m B^J_m I = \frac{2i}{r} C^J_{mn} I \]
\[ \nabla_m \tilde{A}^J_I = -\frac{i}{r} \tilde{B}^J_m I \]
\[ \nabla_m \tilde{B}^J_m I = -\frac{i}{r} \tilde{A}^J_I G_{mn} \]

These closure relations reflect the fact that there are many more fermionic degrees of freedom than there are bosonic ones, so closure on the fermion does not give back the same fermion translated or gauge transformed, but instead it maps us back to into a linear combination of \( \chi_I \) and \( \zeta_I \).

Let us now turn our attention to a non-Abelian Lagrangian

\[ \mathcal{L} = \frac{1}{4} F^2_{mn} - \frac{1}{2} (D_m \phi)^2 - \frac{2}{r^2} \phi^2 + i(\chi_I)^I \gamma^m D_m \chi_I + \frac{1}{2r} (\chi_I)^I \chi_I + e(\chi_I)^I [\chi_I, \phi] \]

and first examine whether this Lagrangian is supersymmetric. This is indeed straightforward to show for all terms, except for the cubic terms in the fermionic fields,

\[ T := e(\chi_I)^I [\chi_I, \delta \phi] + i(\chi_I)^I \gamma^m (2ie)[\delta A_m, \chi_I] \]
\[ = -ie(\mathcal{E}_j)^* \left( \chi_j^\gamma (\chi_I)^\gamma \delta \phi - (\gamma^m)^\gamma \delta \chi_j^\beta (\chi_I)^\beta \Gamma^\gamma \right) \chi_I^\gamma \]

We expand

\[ \chi_I^{\alpha a} (\chi_J^{\beta b})^* = \delta^\alpha_\beta A^{ab} I_J + (\gamma^m)^\alpha_\beta B_{m I J}^{ab} + (\gamma^m)^\alpha_\beta C_{mn I J}^{ab} \]
and then

\[ T = 4ie(\mathcal{E}^\gamma J)^* \left[ \delta^\gamma_\beta A^{ca} J - (\gamma^m)^\gamma_\beta B^{ca}_m J \right] \chi \beta I J \]

Here

\[ A^{ca} J = -\frac{1}{4}(\chi^a_J)^* \chi^c_I \]
\[ B^{ca}_m J = -\frac{1}{4}(\chi^a_J)^* (\gamma_m)^a_\beta \chi \beta I J \]

So we have

\[ T = ie(\mathcal{E}^\gamma J)^* \chi^c_I (\chi^a_I)^* \chi^b_J - ie(\mathcal{E}^\gamma J)^* (\gamma^m)^\gamma_\beta (\chi^a_I)^* (\gamma^m)^\alpha_\beta \chi^b_J \]

We now see that we got an expression that looks similar to the expression that we started with, but with some indices I and J permuted and an overall sign changed. The up-shot of this analysis is that we can not deduce that \( T = 0 \) from this result. Now, if we repeat the same steps again, then one may expect we will get a similar expression with the indices I and J in the right order, possibly with a different overall factor from what originally had? Let us now examine this in detail. We start by putting the above expression in the form

\[ T = ie(\mathcal{E}^\gamma J)^* \chi^c_I (\chi^a_I)^* \chi^b_J - ie(\mathcal{E}^\gamma J)^* (\gamma^m)^\gamma_\beta (\chi^a_I)^* (\gamma^m)^\alpha_\beta \chi^b_J \]

Now if we use the Fierz expansion, then we get

\[ T = -4ie(\mathcal{E}^\gamma J)^* \left[ \delta_\beta^\gamma A^{ca} - (\gamma^m)^\gamma_\beta B^{ca}_m \right] \chi^b_J \]

where

\[ A^{ca} = -\frac{1}{4}(\chi^a_J)^* \chi^c_I \]
\[ B^{ca}_m = -\frac{1}{4}(\chi^a_J)^* (\gamma_m)^a_\beta \chi \beta I J \]

So we have

\[ T = -ie(\mathcal{E}^\gamma J)^* \chi^c_I (\chi^a_I)^* \chi^b_J + ie(\mathcal{E}^\gamma J)^* (\gamma^m)^\gamma_\beta \chi \beta I J (\chi^a_I)^* (\gamma^m)^\alpha_\beta \chi^b_J \]

and we got back the same expression as we started with. So these lines were insufficient to show that \( T \) is vanishing, and most probably \( T \) is not vanishing. It may be difficult to actually prove it, but the argument we have presented seems sufficiently convincing to us.

So we conclude that there is no non-Abelian supersymmetric Lagrangian with this amount of supersymmetry. We can reduce the amount of supersymmetry so that the R-symmetry is further reduced from \( SU(2)_R \) down to \( U(1)_R \) by imposing the Weyl condition

\[ (\sigma^3)_I J \mathcal{E}_J = \mathcal{E}_J \]
Then there is just one complex supersymmetry parameter $E = E_1$. With this projection, one finds that the component $\chi_2$ does not enter the supersymmetry multiplet as its supersymmetry variation becomes zero,

$$\delta\chi_2 = 0$$

and so we define $\chi := \chi_1$ for which we find the supersymmetry variations

$$\delta\phi = -iE^\dagger\chi - iF^\dagger\zeta$$
$$\delta A_m = -iE^\dagger\gamma_m\chi - iF^\dagger\gamma_m\zeta$$
$$\delta\chi = \frac{1}{2}\gamma^{mn}EF_{mn} - \gamma^m E D_m \phi - \frac{2i}{r} E \phi$$

The Lagrangian is

$$L = \frac{1}{4}F_{mn}^2 - \frac{1}{2}(D_m \phi)^2 - \frac{2}{r^2}\phi^2$$
$$+ i\chi^\dagger\gamma^m D_m \chi + \frac{1}{2r}\chi^\dagger\chi + e\chi^\dagger[\chi, \phi]$$

The Killing spinor equation is

$$\nabla_m E = \frac{i}{2r}\gamma_m E$$

Originally we had

$$F_2^\alpha = \varepsilon_{21}C^{\alpha\beta}(E_1^\beta)^*$$
$$\zeta_2^\alpha = \varepsilon_{21}C^{\alpha\beta}(\chi_1^\beta)^*$$

Now we define $F^\alpha := F_2^\alpha$ and $\zeta^\alpha := \zeta_2^\alpha$ so with $\varepsilon^{12} = 1$, we get the relations

$$F^\alpha = C^{\alpha\beta}(E^\beta)^*$$
$$\zeta^\alpha = C^{\alpha\beta}(\chi^\beta)^*$$

Let us now again analyze the cubic terms in the fermionic field that arise upon a supersymmetry variation of this Lagrangian. These terms are

$$T := e(\chi)^\dagger[\chi, \delta\phi] + i(\chi)^\dagger\gamma^m(-ie)[\delta A_m, \chi]$$

However, we still have the Lagrangian for $\chi_2$ as well,

$$L_2 = i(\chi_2)^\dagger\gamma^m D_m \chi_2 + \frac{1}{2r}(\chi_2)^\dagger\chi_2 + e(\chi_2)^\dagger[\chi_2, \phi]$$

but this Lagrangian is not supersymmetric since the corresponding cubic term $T$ upon a supersymmetry variation will not be vanishing, but it is now consistent with supersymmetry to truncate to $\chi_2 = 0$ since the supersymmetry variation of $\chi_2$ is vanishing. So then we will simply get $L_2 = 0$ and we retain supersymmetry of $L_2$ trivially by putting $\chi_2 = 0$ as a truncation that is consistent with supersymmetry.
\[ = -ie(\mathcal{E}^\gamma)^* \left[ \chi^{\gamma c} (\chi^{\beta a})^* - (\gamma_m)^\gamma (\gamma_m)^\gamma \chi^{\delta c} (\chi^{\alpha a})^* (\gamma_m)^\alpha \right] \chi^{\beta b} \]

We expand

\[ \chi^{\alpha a}(\chi^{\beta b})^* = \delta_\beta^\alpha A^{ab} + (\gamma^m)^\alpha_\beta B_{m}^{ab} + (\gamma^m)^\alpha_\beta C_{mn}^{ab} \]

and then

\[ T = 4ie(\mathcal{E}^\gamma)^* \left[ \delta_\beta^\alpha A^{ca} - (\gamma^m)^\gamma_\beta B_{m}^{ca} \right] \chi^{\beta b} \]

Here

\[ A^{ca} = -\frac{1}{4} (\chi^{\alpha a})^* \chi^{\alpha c} \]
\[ B_{m}^{ca} = -\frac{1}{4} (\chi^{\alpha a})^* (\gamma_m)^\alpha_\beta \chi^{\beta c} \]

So we have

\[ T = ie(\mathcal{E}^\gamma)^* \chi^{\gamma c} (\chi^{\alpha a})^* \chi^{\alpha b} - ie(\mathcal{E}^\gamma)^* (\gamma^m)^\gamma_\beta \chi^{\beta c} (\chi^{\delta a})^* (\gamma_m)^\delta \chi^{eb} \]

We now see that we got back the same expression as the one we started with, but with an overall minus sign, so \( T = -T \), which clearly shows that \( T = 0 \) and the Lagrangian is supersymmetric.

### 2.2 A dual description with an R-gauge field

By making a few changes of viewpoint we may recover the theory one gets by turning on an R-gauge field and make contact with the results in [2]. We relabel the spinor field and its complex conjugate field as

\[ \chi = \psi_1 \]
\[ \zeta = \psi_2 \]

and similarly

\[ \mathcal{E} = \mathcal{E}_1 \]
\[ \mathcal{F} = \mathcal{E}_2 \]

Then we may state a Majorana condition as

\[ \psi_I^\alpha = \varepsilon_{IJ} C^{\alpha \beta} (\psi_j^\beta)^* \]

that we get from

\[ \zeta^\alpha = C^{\alpha \beta} (\chi^\beta)^* \]
Moreover, the Killing spinor equations for $\chi$ and $\zeta$ can now be grouped together into one Killing spinor equation for the Majorana spinor $\mathcal{E}_I$

$$\nabla_m \mathcal{E}_I = \frac{i}{2r} (\sigma^3)_I^J \gamma_m \mathcal{E}_J$$

So there is an exact isomorphism between the theory we get by turning on an R-gauge field, and the theory we get in this entirely different way by keeping nonzero modes for the fermionic field and not turning on any R-gauge field.

In one viewpoint, $\chi$ and $\zeta$ are nonzero Kaluza-Klein modes who receive an extra mass simply by the fact that they are nonzero modes. In the other viewpoint, $\chi$ and $\zeta$ form two components in an $SU(2)_R$ Majorana spinor which is a zero mode spinor upon dimensional reduction with an R-gauge field turned on and the mass of these fermions is induced from that R-gauge field in the six-dimensional theory. Both ways result in the same 5d theory, but the 6d theories seem to be very different.

Once having realized this kind of dual description, we can proceed and use all knowledge that we already have of this 5d theory from say [2]. We will review that theory below in order to put it in relation to the 6d theory on $\mathbb{R} \times S^5$. We will focus only on the case of Abelian gauge group for simplicity. The non-Abelian generalization will be straightforward and can be found in [2]. We begin by turning on an R-symmetry gauge field to preserve supersymmetry for fermionic zero modes. The Killing spinor equation is modified to

$$D_t \mathcal{E}_I = \frac{i}{2r} (\sigma^3)_I^J \mathcal{E}_J$$

$$\nabla_m \mathcal{E}_I = \frac{i}{2r} \gamma_m (\sigma^3)_I^J \mathcal{E}_J$$

We have the Majorana condition

$$(\mathcal{E}_I^\alpha)^* = C_{\alpha\beta} \mathcal{E}^I \mathcal{E}_J^\beta$$

We also have

$$\eta_I = \frac{i}{2r} (\sigma^3)_I^J \mathcal{E}_J$$

The supersymmetry variations are

$$\delta \phi = -i (\mathcal{E}_I)^\dagger \psi_I$$

$$\delta A_m = -i (\mathcal{E}_I)^\dagger \gamma_m \psi_I$$

$$\delta \psi_I = \frac{1}{2} \gamma^{mn} \mathcal{E}_I F_{mn} - \gamma^m \mathcal{E}_I \partial_m \phi - \frac{2i}{r} (\sigma^3)_I^J \mathcal{E}_J \phi$$
With a commuting supersymmetry parameter, we have the following closure relations.

Closure on $\phi$,

$$\delta^2 \phi = i(\mathcal{E}_I)^\dagger \gamma^m \mathcal{E}_I \partial_m \phi$$

Closure on $A_m$,

$$\delta^2 A_m = i(\mathcal{E}_I)^\dagger \gamma^m \mathcal{E}_I F_{nm} + \partial_m \left(-i(\mathcal{E}_I)^\dagger \mathcal{E}_I \phi\right)$$

Closure on $\psi_I$,

$$\delta^2 \psi_I = -8iB^m \nabla_m \psi_I + \frac{12A}{r} (\sigma^3)_I^J \psi_J$$

$$- (3A + 3B_p \gamma^p) \left(i \gamma^m \nabla_m \psi_I + \frac{1}{2r} (\sigma^3)_I^J \psi_J \right)$$

The supersymmetric Lagrangian is $\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F + \mathcal{L}_F^{II}$ where

$$\mathcal{L}_B = \frac{1}{4} F_{mn}^2 - \frac{1}{2} (\nabla_m \phi)^2 - \frac{2}{r^2} \phi^2$$

$$\mathcal{L}_F^I = \frac{i}{2} (\psi_I)^\dagger \gamma^m \nabla_m \psi_I$$

$$\mathcal{L}_F^{II} = \frac{1}{4r} (\psi_I)^\dagger (\sigma^3)_I^J \psi_J$$

For an anticommuting supersymmetry parameter, we have

$$\delta \phi = i(\psi_I)^\dagger \mathcal{E}_I$$

$$\delta A_m = i(\psi_I)^\dagger \gamma^m \mathcal{E}_I$$

and then we get

$$\delta \mathcal{L}_B = -\nabla_m F_{mn} i(\psi_I)^\dagger \gamma^m \mathcal{E}_I + \nabla^2 \phi i(\psi_I)^\dagger \mathcal{E}_I - \frac{4}{r^2} \phi i(\psi_I)^\dagger \mathcal{E}_I$$

$$\delta \mathcal{L}_F^I = i(\psi_I)^\dagger \gamma^m \nabla_m \left(\frac{1}{2} \gamma^{pq} \mathcal{E}_I F_{pq} - \gamma^p \mathcal{E}_I \partial_p \phi - \frac{2i}{r} (\sigma^3)_I^J \mathcal{E}_J \phi \right)$$

$$= i(\psi_I)^\dagger \gamma^m \nabla_m \mathcal{E}_I F_{pq} - i(\psi_I)^\dagger \gamma^m \nabla_m \mathcal{E}_I + \frac{2i}{r} (\psi_I)^\dagger (\sigma^3)_I^J \mathcal{E}_J \nabla_m \phi$$

$$+ \frac{i}{2} (\psi_I)^\dagger \gamma^m \nabla_m \mathcal{E}_I F_{pq} - i(\psi_I)^\dagger \gamma^m \nabla_m \mathcal{E}_I + \frac{2i}{r} (\psi_I)^\dagger (\sigma^3)_I^J \mathcal{E}_J \phi$$

$$\delta \mathcal{L}_F^{II} = \frac{1}{2r} (\psi_I)^\dagger (\sigma^3)_I^J \left(\frac{1}{2} \gamma^{mn} \mathcal{E}_J F_{mn} - \gamma^m \mathcal{E}_J \nabla_m \phi - \frac{2i}{r} (\sigma^3)_J^K \mathcal{E}_K \phi \right)$$

Using

$$\nabla_m \mathcal{E}_I = \frac{i}{2r} (\sigma^3)_I^J \mathcal{E}_J$$

and

$$\gamma^m \gamma^p \gamma_m = -3 \gamma^p$$
we can show that all terms cancel against each other so that $\delta L = 0$.

We may take the supersymmetry variations off-shell,

\[
\begin{align*}
\delta \phi &= -i(E_I)^{\dagger} \psi_I \\
\delta A_m &= -i(E_I)^{\dagger} \gamma_m \psi_I \\
\delta \psi_I &= \frac{1}{2} \gamma^{mn} E_I F_{mn} - \gamma^m E_I \partial_m \phi - \frac{i}{r} (\sigma^3)_I^J E_J \phi + E_J D^J I \\
\delta D^J I &= 2(E_I)^{\dagger} \left( i \gamma^m \nabla_m \psi_I + \frac{1}{2r} (\sigma^3)_I^L \psi_L \right) - \frac{1}{r} (\sigma^3)_I^J (E_K)^{\dagger} \psi_K \\
&\quad - \delta_I^J (E_K)^{\dagger} \left( i \gamma^m \nabla_m \psi_K + \frac{1}{2r} (\sigma^3)_K^L \psi_L \right)
\end{align*}
\]

where the second line in the variation of $D^J I$ removes the trace part, where we notice that $\sigma^3$ is already traceless. The Lagrangian is

\[
\mathcal{L} = \frac{1}{4} F_{mn}^2 - \frac{1}{2} (\nabla_m \phi)^2 + \frac{1}{4} D^J J D^J I + \frac{i}{2r} (\sigma^3)_I^J \phi - \frac{5}{2r^2} \phi^2 + \frac{i}{2} (\psi_I)^{\dagger} \gamma^m \nabla_m \psi_I + \frac{1}{4r} (\psi_I)^{\dagger} (\sigma^3)_I^L \psi_L
\]

Integrating out $D^J J$ amounts to putting

\[
D^J J = \frac{i}{r} (\sigma^3)_I^J \phi
\]

and then the second line in the Lagrangian becomes

\[
\frac{1}{2r^2} \phi^2 - \frac{5}{2r^2} \phi^2 = -\frac{2}{r^2} \phi^2
\]

which is the right on-shell action, and also the supersymmetry variation becomes

\[
\begin{align*}
\delta \psi_I &= \frac{1}{2} \gamma^{mn} E_I F_{mn} - \gamma^m E_I \partial_m \phi - \frac{2i}{r} (\sigma^3)_I^J E_J \phi \\
\delta D^J I &= -\frac{i}{r} (\sigma^3)_I^J \delta \phi
\end{align*}
\]

which are the right on-shell variation. The on-shell variation of $D^J I$ corresponds to a variation of the on-shell saddle point equation (2.4).

But this does not explain why we shall make this funny shift away from say the saddle point value zero for $D^J J$. To understand why we shall construct the Lagrangian such that we have the shifted saddle point value (2.4), we look at the supersymmetry variation of the fermionic part of the Lagrangian with $\delta \psi_I = E_J D^J I$. We notice that there is no term that involves a derivative of $D^J I$ as this field is an auxiliary non-dynamical field.
Therefore we shall make an integration by parts such that the variation of the fermionic
terms becomes
\[ \delta L_F = (\delta \psi_I)\dagger \left( i\gamma^m \nabla_m \psi_I + \frac{1}{2r} (\sigma^3)_{I}^{K} \psi_K \right) \]
This is opposite the the convention we used before where made integrations by parts so
that no derivatives acted on the fermionic field. But here this new convention makes better
sense because we do not get derivative of the auxiliary field from the bosonic terms by
varying the auxiliary field. Now let us compute this variation with \((\delta \psi_I)\dagger = (D^I_J)\dagger (E_J)\dagger\).
We then notice that
\[ -(D^I_J)^* = \varepsilon_{JK} \varepsilon^{IL} D^K_L = \varepsilon_{JK} D^{KI} = \varepsilon_{JK} D^{JK} = D^I_J \]
where the first equality is a consequence of demanding
\[ (\delta \psi_I)\dagger = \delta (\psi_I)\dagger \]
with \(\delta \psi_I = E_I D^I_J\). Here is the computation. First,
\[ (\delta \psi_I^\alpha)^* = (E_I^\alpha D^I_J)^* = C_{\alpha\beta} \varepsilon^{JK} E_K^\beta (D^I_J)^* \]
and second,
\[ \delta (\psi_I^\alpha)^* = C_{\alpha\beta} \varepsilon^{IJ} \delta \psi_J = C_{\alpha\beta} \varepsilon^{IJ} E_K^\beta D^K_J \]
Then by identifying these two results, we get
\[ \varepsilon^{JK} (D^I_J)^* = \epsilon^{IJ} D^K_J \]
After these preliminaries, we get
\[ \delta L_F = - \frac{1}{2r} D^I_J (E_J)^\dagger (\sigma^3)_{I}^{K} \psi_K \]
We also get
\[ \delta L_B = \frac{1}{2r} D^I_J (E_J)^\dagger (\sigma^3)_{I}^{K} \psi_K \]
and so we see that the sum is zero, \(\delta L_F + \delta L_B = 0\). This shows that the Lagrangian is
supersymmetric.

Offshell closure is slightly modified from onshell closure as follows. We have
\[ \delta^2 \psi_I = 8i B^m \nabla_m \psi_I + \frac{12}{r} (\sigma^3 \psi)^I \]
\[ \delta^2 D'_I = i(\mathcal{E}_K)^{1\gamma^m} \mathcal{E}_K \nabla_m D'_I - \frac{3}{r}(\mathcal{E}_L)^1 \mathcal{E}_L (\sigma^3)_K^J D^K_I \]

Now these results can be recast in the form

\[ \delta^2 \psi_I = \ldots - \frac{3}{2r} (\mathcal{E}_L)^1 \mathcal{E}_L (\sigma^3)_I^J \psi_J \]
\[ \delta^2 D'_I = \ldots - \frac{3}{r} (\mathcal{E}_L)^1 \mathcal{E}_L (\sigma^3)_K^J D^K_I \]

and we see that we got an R-symmetry rotation. Of course the scalar field \( \phi \) is an R-symmetry singlet so it will not be R-symmetry rotated.

The results we have found here all followed from straightforward computations. But it remains a mystery to us why two different kind of dimensional reductions result in the same 5d Lagrangian. In one instance we did not turn on any R-gauge field but instead we kept the modes \( \phi_0 \) and \( \psi_{\pm 1} \). In the other instance we turn on an R-gauge field and keep the zero modes \( \phi_0 \) and \( \psi_0 \). Both ways lead us to the exact same Lagrangian in 5d if we impose the appropriate Weyl projections, but we do not understand why that is so.

3 Null reduction

A general null reduction of the M5 brane was studied in [6]. Here we will stay with our example of \( \mathbb{R} \times S^5 \) with Lorentzian time along \( \mathbb{R} \) for simplicity, although we believe that our results can be generalized to any Lorentzian six-manifold without any new conceptional difficulties, beyond those we will address here. We will perform the dimensional reduction along the null direction that is formed out of the time direction and a circle fiber direction on \( S^5 \) when viewed as a circle fiber over \( \mathbb{C}P^2 \). However, once we specify a circle fiber, there are two null directions, \( x^+ \) and \( x^- \) and we need to make a choice. We will make the choice such that we perform the dimensional reduction along the \( x^- \) direction. This choice of null direction is correlated with some chirality choices for the supersymmetry parameter that we wish to make, as we will now explain.

We start by writing the metric on \( \mathbb{R} \times S^5 \) as a metric over the base-manifold \( \mathbb{R} \times \mathbb{C}P^2 \). The M5 brane on (a Hopf circle bundle over) \( \mathbb{R} \times \mathbb{C}P^2 \) was first studied in [1]. We start by writing the 6d metric in the form

\[ ds^2 = r^2 (dy + \kappa_i dx^i)^2 - dt^2 + G_{ij} dx^i dx^j \]

where the five coordinates \( x^m \) on \( S^5 \) are separated as \( y \sim y + 2\pi \) for the circle fiber, and \( x^i \) for the base manifold \( \mathbb{C}P^2 \), and \( \kappa_i \) is the graviphoton whose nonvanishing curvature components are

\[ w_{12} = w_{34} = \frac{2}{r^2} \]
where the hats on these indices indicate that they are tangent space indices of \( \mathbb{CP}^2 \). Here we use \( G_{ij} \) to denote the 4d metric tensor on \( \mathbb{CP}^2 \) whose inverse is denoted \( G^{ij} \). Further details regarding this Hopf fibration over \( \mathbb{CP}^2 \) can be found in appendix D.

We then also split the indices in the 5d Killing spinor equation

\[
\nabla_m \mathcal{E}^{\alpha \dot{\alpha}} = \frac{i}{2r} (\gamma_m)^\alpha_{\beta} \mathcal{E}^{\beta \dot{\alpha}}
\]

on \( S^5 \) into two equations

\[
\begin{align*}
\nabla_y \mathcal{E} &= \frac{i}{2r} \gamma_y \mathcal{E} \\
\nabla_i \mathcal{E} &= \frac{i}{2r} \gamma^i \mathcal{E}
\end{align*}
\]

(3.1)

associated to the fiber and the base-manifold respectively (and from now, we suppress the spinor indices). To analyse these equations further, we need expressions for these covariant derivatives in terms of spin connections and we need to express the 5d gamma matrices in terms of 4d gamma matrices. To this end, we start by writing down expressions for the vielbein

\[
\begin{align*}
\hat{e}^t &= dt \\
\hat{e}^\theta &= r (dy + \kappa_i dx^i) \\
\hat{e}^i &= E^j_{\ i} dx^j
\end{align*}
\]

and its inverse

\[
\begin{align*}
\hat{e}_t &= \partial_t \\
\hat{e}_\theta &= \frac{1}{r} \partial_y \\
\hat{e}_i &= E^j_{\ i} (\partial_j - \kappa_j \partial_y)
\end{align*}
\]

Using these vielbeins, we may expand the 5d gamma matrices \( \gamma_m \) in terms of 4d gamma matrices \( \hat{\gamma}_i = E^j_{\ i} \gamma_j \) and \( \gamma := \hat{\gamma}^{1234} \) as follows,

\[
\begin{align*}
\gamma_y &= r \gamma \\
\gamma_i &= \hat{\gamma}_i + r \kappa_i \gamma
\end{align*}
\]

and then we use standard circle bundle expressions for the 5d covariant derivative acting on a 5d spinor \( \psi \),

\[
\begin{align*}
\nabla_y \psi &= \partial_y \psi - \frac{r^2}{8} w_{ij} \hat{\gamma}^{ij} \psi \\
\nabla_i \psi &= \hat{\nabla}_i \psi - \frac{r^2}{8} \kappa_i w_{kl} \hat{\gamma}^{kl} \psi + \frac{r}{4} w_{ij} \hat{\gamma}^{ij} \gamma \psi
\end{align*}
\]
where $\tilde{\nabla}_i$ denotes the covariant derivative with respect to the metric on the 4d base space. We are now ready to express (3.1) in 4d quantities,

$$\partial_y E - \frac{r^2}{8} w_{ij} \gamma^{ij} E = \frac{i}{2} \gamma E$$
$$\nabla_i E - \frac{r^2}{8} \kappa_i w_{kl} \gamma^{kl} E + \frac{r}{4} w_{ij} \gamma^j \gamma E = \frac{i}{2r} (\gamma_i + r \kappa_i \gamma) E$$

where now all quantities are 4d quantities, and so we have dropped the tildes for notational simplicity. We may also express the second equation more simply as

$$\mathcal{D}_i E = \frac{i}{2r} \gamma_i E - \frac{r}{4} w_{ij} \gamma^j \gamma E$$

where we have introduced the curly derivative

$$\mathcal{D}_i \psi = \nabla_i \psi - \kappa_i \partial_y \psi$$

But let us first analyze the first equation. Plugging in the explicit form of $w_{ij}$, this equation reads

$$\partial_y E = \frac{1}{2} \left( \gamma^{\bar{1}\bar{2}} + \gamma^{\bar{3}\bar{4}} + i \gamma \right) E$$

Of course the spinor $E^{\alpha\dot{\alpha}}$ has four different indices $\alpha$. To see the meaning of these various indices more clearly, we will introduce a spin notation $\alpha = (s_1, s_2)$ where the spins $s_1$ and $s_2$ are defined by

$$\frac{i}{2} \gamma^{\bar{1}\bar{2}} E = s_1 E$$
$$\frac{i}{2} \gamma^{\bar{3}\bar{4}} E = s_2 E$$

Let us first consider the spinor component $(s_1, s_2) = (+, +)$ where $\pm$ represent spins $\pm \frac{1}{2}$. The Killing spinor equations then reduce to

$$\partial_y E = -\frac{3i}{2} E$$
$$\mathcal{D}_i E = 0$$

Moving up to 6d, we have the conformal Killing spinor solution

$$\epsilon = e^{\frac{i}{2} t - \frac{3i}{2} y} E + e^{-\frac{i}{2} t + \frac{3i}{2} y} \mathcal{F}$$

This is the singlet solution. The other cases are $(s_1, s_2) = \{(-, -), (+, -), (-, +)\}$ that form a triplet. For any of these components, the first Killing spinor equation becomes

$$\partial_y E = \frac{i}{2} E$$
and then the 6d solution becomes

\[ \varepsilon = e^{\frac{i}{2}t + \frac{i}{2}y} \mathcal{E} + e^{-\frac{i}{2}t - \frac{i}{2}y} \mathcal{F} \]

but the Killing spinor equations for \( \mathcal{E} \) and \( \mathcal{F} \) now become more complicated. We introduce light cone coordinates

\[ x^\pm = \frac{1}{\sqrt{2}}(t \pm ry) \]

Expressed in these light-cone coordinates, the singlet solution is

\[ \varepsilon = e^{\frac{i}{\sqrt{2}}(-x^+ + 2x^-)} \mathcal{E} + e^{-\frac{i}{\sqrt{2}}(-x^+ + 2x^-)} \mathcal{F} \]

and the triplet solutions are

\[ \varepsilon = e^{\frac{i}{\sqrt{2}}x^+} \mathcal{E} + e^{-\frac{i}{\sqrt{2}}x^+} \mathcal{F} \]

Since these triplet supersymmetry parameters do not depend on \( x^- \), the corresponding supersymmetry survives upon dimensional reduction along \( x^- \) without any need to turn on an R-gauge field. While this is nice, the price we have to pay is having a more complicated Killing spinor equation.

We will study the singlet solution instead. This has a simpler Killing spinor equation, and it gives us an opportunity to study a situation where the supersymmetry parameter depends nontrivially on the fiber direction along which we dimensionally reduce. But again the question arises, along which direction we shall reduce. Let us start by recalling the 6d Weyl condition \( \Gamma \varepsilon = -\varepsilon \) that we will write as

\[ \Gamma^{i\nu} \Gamma^{1234} \varepsilon = -\varepsilon \quad (3.2) \]

As we mentioned in the Introduction, we also want to impose the Weyl projection

\[ \Gamma_M \varepsilon v^M = 0 \]

where \( v^M \) is now to be either one of the lightcone directions, \( v^M = \delta_M^\pm \). So the above Weyl projection amounts to

\[ \Gamma_{\pm} \varepsilon = 0 \]

where

\[ \Gamma_{\pm} = \frac{1}{\sqrt{2}} \left( \Gamma_t \pm \frac{1}{r} \Gamma_y \right) \]
so we may also express this Weyl projection as

$$\Gamma^{ty} \varepsilon = \mp \varepsilon$$

(3.3)

Now by combining (3.2) and (3.3), we get

$$\Gamma^{1234} \varepsilon = \pm \varepsilon$$

The singlet supersymmetry parameter has $\Gamma^{1234} \varepsilon = -\varepsilon$ and therefore we shall take $v^M = \delta^M$ and perform the dimensional reduction along the $x^-$ direction. Let us write down the singlet solution again as

$$\varepsilon = e^{\frac{\sqrt{2}}{r} x^-} \mathcal{E} + e^{-\frac{\sqrt{2}}{r} x^-} \mathcal{F}$$

Then upon dimensional reduction, we shall expand the fermionic field in the same modes as

$$\psi = e^{\frac{\sqrt{2}}{r} x^-} \chi + e^{-\frac{\sqrt{2}}{r} x^-} \zeta$$

Of course we do not know the non-Abelian supersymmetry variations for the M5 brane. The strategy will therefore be to start with the Abelian supersymmetry variations for the M5 brane, and reduce these along the $x^-$ direction by using the above mode expansion for the fermionic field. We will also find a corresponding Abelian Lagrangian that is supersymmetric. These steps are in parallel with what we have already done when we reduced along the time direction, although the reduction along $x^-$ requires a lot more computations. Once we have obtained these Abelian supersymmetries and Lagrangian, the generalization to the non-Abelian case will be examined. We start by replacing derivatives with gauge covariant derivatives and examine the term in the variation of the Lagrangian that is cubic in the fermionic field. But this term is vanishing, not because of some Fierz rearrangement, but simply because, as we will see, the supersymmetry variation of the following combination of gauge fields is vanishing:

$$\delta \left( A_i - \kappa_i A_y \right) = 0$$

and it is precisely this combination that enters in the kinetic term for the fermionic field

$$i \chi^\dagger \gamma^i D_i \chi$$

So when we vary the gauge potential in this term, there will be no cubic term generated. Let us now show this in more detail. Let us start with the 6d supersymmetry variation

$$\delta A_M = -i \bar{\psi} \Gamma_{MN} \varepsilon v^N$$

\footnote{In 6d we also have the gauge fixing condition $A_M v^M = A_\perp = 0$ that can be seen as a consequence of $A_M = B_{MN} v^N$.}
from which we obtain

\[
\delta A_i = -\frac{ir}{\sqrt{2}}\kappa_i \bar{\psi} \epsilon
\]

\[
\delta A_+ = -i\bar{\psi} \epsilon
\]

Then

\[
\mathcal{D}_i \psi = (D_i - \kappa_i D_y) \psi
\]

\[
= \left( D_i - \frac{r}{\sqrt{2}}\kappa_i D_+ \right) \psi
\]

The important observation is now that

\[
\delta \mathcal{D}_i = -i\epsilon \delta \left( A_i - \frac{r}{\sqrt{2}}\kappa_i A_+ \right) = 0
\]

For this computation we have used

\[
\Gamma_{\pm} = \Gamma_{\pm}
\]

\[
\Gamma_i = \tilde{\Gamma}_i + \frac{r}{\sqrt{2}} (\Gamma_+ - \Gamma_-)
\]

and then

\[
\Gamma_{i \pm} = \tilde{\Gamma}_i \Gamma_{\pm} + \frac{r}{\sqrt{2}} \kappa_i \Gamma_{++}
\]

We have

\[
\Gamma_{\pm} = \frac{1}{\sqrt{2}} \left( \Gamma^i \pm \frac{1}{r} \Gamma_y \right)
\]

\[
\Gamma^{\pm} = \frac{1}{\sqrt{2}} \left( \Gamma^i \pm r \Gamma^y \right)
\]

and then we get

\[
\Gamma_{++} = \Gamma_{\tilde{g}}
\]

We impose the Weyl projection

\[
\Gamma_- \epsilon = 0
\]

and then we get

\[
\Gamma_- \Gamma_+ \epsilon = \left( \{\Gamma_-, \Gamma_+\} - \Gamma_+ \Gamma_- \right) \epsilon
\]

\[
= -2 \epsilon
\]

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where we notice the metric is

\[ ds^2 = -2e^\hat{\gamma}e^\hat{i} + e^\hat{i}e^\hat{i} \]

Now having shown that \( \delta D_i = 0 \) is, as we will see below, just one crucial step among many other steps towards obtaining a supersymmetry non-Abelian Lagrangian.

We begin with assuming the gauge group is Abelian and let us first study the supersymmetry variation of the tensor gauge field in 6d,

\[ \delta H_{MNP} = -3i\partial_M (\bar{\psi}\Gamma_{NP}\epsilon) \]

for an anticommuting supersymmetry parameter, for which we have the relation

\[ \bar{\epsilon}\Gamma_{MN}\psi = (\epsilon^T C^T \Gamma_{MN}\psi)^T = -\psi^T (-C\Gamma_{MN}C^{-1})(-C)\epsilon = -\bar{\psi}\Gamma_{MN}\epsilon \]

where we used the 11d Majorana condition. We would first like to show a correspondence with the fermionic equation of motion and selfduality of \( H_{MNP} \). In 6d, this correspondence is almost trivial to show. Namely, we have

\[ (\delta H_{MNP})^- = -\frac{i}{2}\nabla Q (\bar{\psi}\Gamma^Q\Gamma_{MNP}\epsilon) \]

and by using the identity \( \Gamma^Q\Gamma_{MNP}\Gamma_Q = 0 \) and \( \nabla_M\epsilon = \Gamma_M\eta \), we get

\[ (\delta H_{MNP})^- = -\frac{i}{2}\nabla Q \bar{\psi}\Gamma^Q\Gamma_{MNP}\epsilon \]

and we see that this variation vanishes on the fermionic equation of motion \( \Gamma^M\nabla_M\psi = 0 \).

We would now like to show this correspondence between selfduality and the fermionic equation of motion again, but now in lightcone coordinates, following closely [6]. To this end, we define

\[ G_{ij} = G_{ij} - r\sqrt{2}F_{i+}\kappa_j \]

where

\[ G_{ij} = H_{ij+} \]
\[ F_{i+} = H_{i+-} \]

and we want to show that the selfdual part vanishes, \( (\delta G_{ij})^+ = 0 \), on the fermionic equation of motion. So we first need to obtain the explicit expressions for the supersymmetry variation and for the fermionic equation of motion in lightcone coordinates. We begin with the supersymmetry variation. We have

\[ \delta G_{ij} = -2i\nabla_i (\bar{\psi}\Gamma_{j+}\epsilon) - i\partial_+ (\bar{\psi}\Gamma_{ij}\epsilon) \]
\[
\delta F_{i+} = -i \partial_i (\bar{\psi} \Gamma_{+-} \varepsilon) + i \partial_+ (\bar{\psi} \Gamma_{i-} \varepsilon) - i \partial_- (\bar{\psi} \Gamma_{i+} \varepsilon)
\]

where \( \nabla_i \) are 4d covariant derivatives. We expand

\[
\varepsilon = e^{i \sqrt{2} x^-} \mathcal{E} + e^{-i \sqrt{2} x^-} \mathcal{F}
\]
\[
\psi = e^{i \sqrt{2} x^-} \chi + e^{-i \sqrt{2} x^-} \zeta
\]

where

\[
\nabla_i \mathcal{E} = -\frac{3i}{2} \kappa_i \mathcal{E}
\]
\[
\partial_+ \mathcal{E} = -i \frac{r}{\sqrt{2}} \mathcal{E}
\]

and corresponding relations for \( \mathcal{F} \). We also expand

\[
\Gamma_{i \pm} = \tilde{\Gamma}_{i \pm} + \frac{r}{\sqrt{2}} \kappa_i \Gamma_{+-}
\]
\[
\Gamma_{ij} = \tilde{\Gamma}_{ij} - r \sqrt{2} \kappa_i \Gamma_{j} (\Gamma_{+-} - \Gamma_{--})
\]

Then we get

\[
\delta G_{ij} = -2i \nabla_i \left( \bar{\chi} \tilde{\Gamma}_j \Gamma_{+-} \mathcal{E} \right) - i \sqrt{2} r \nabla_i (\bar{\chi} \Gamma_{+-} \mathcal{E} \kappa_j)
\]
\[
- i \partial_+ \left( \bar{\chi} \tilde{\Gamma}_{ij} \mathcal{E} \right) + i \sqrt{2} r \partial_+ \left( \kappa_i \bar{\chi} \tilde{\Gamma}_j \mathcal{E} \right)
\]

We may now notice the appearance of a curly derivative from

\[
-2i \left( \nabla_i - \frac{r}{\sqrt{2}} \kappa_i \partial_+ \right) \left( \bar{\chi} \tilde{\Gamma}_j \Gamma_{+-} \mathcal{E} \right) = -2i D_i \left( \bar{\chi} \tilde{\Gamma}_j \Gamma_{+-} \mathcal{E} \right)
\]

where we assume that \( \partial_+ \kappa_i = 0 \). So then we have

\[
\delta G_{ij} = -2i D_i \left( \bar{\chi} \tilde{\Gamma}_j \Gamma_{+-} \mathcal{E} \right) - i \sqrt{2} r \nabla_i (\bar{\chi} \Gamma_{+-} \mathcal{E} \kappa_j)
\]
\[
- i \partial_+ \left( \bar{\chi} \tilde{\Gamma}_{ij} \mathcal{E} \right)
\]

We have

\[
\delta F_{i+} = -i \nabla_i (\bar{\chi} \Gamma_{+-} \mathcal{E}) + \frac{ir}{\sqrt{2}} \kappa_i \partial_+ (\bar{\chi} \Gamma_{+-} \mathcal{E})
\]

and then we get

\[
\delta G_{ij} = -2i D_i \left( \bar{\chi} \tilde{\Gamma}_j \Gamma_{+-} \mathcal{E} \right) - i \sqrt{2} r \bar{\chi} \Gamma_{+-} \mathcal{E} \nabla_i \kappa_j
\]
\[
- i \partial_+ \left( \bar{\chi} \tilde{\Gamma}_{ij} \mathcal{E} \right)
\]
or if we define

\[ w_{ij} = \nabla_i \kappa_j - \nabla_j \kappa_i \]

then we can write this as

\[
\delta G_{ij} = -2iD_i \left( \bar{\chi} \tilde{\Gamma}_j \Gamma_+ \mathcal{E} \right) - \frac{ir}{\sqrt{2}} \bar{\chi} \tilde{\Gamma}_j \Gamma_+ \mathcal{E} w_{ij} \\
- i\partial_+ \left( \bar{\chi} \tilde{\Gamma}_j \mathcal{E} \right)
\]

We are now interested in extracting the selfdual part of this variation. To do this, we first recall the Weyl projection

\[ \Gamma_- \mathcal{E} = 0 \]

We have

\[
\Gamma_\pm = \frac{1}{\sqrt{2}} \left( \Gamma_t \pm \frac{1}{r} \Gamma_y \right) \\
\Gamma^\pm = \frac{1}{\sqrt{2}} \left( \Gamma^t \pm r \Gamma^y \right)
\]

The Weyl projection can be written in the following alternative forms

\[
\Gamma^{\hat{t} \hat{y}} \mathcal{E} = \mathcal{E} \\
\Gamma_{+-} \mathcal{E} = \mathcal{E}
\]

Expressed in terms of 4d gamma matrices, we get

\[
\delta G_{ij} = 2\sqrt{2}i \partial_i \bar{\chi}^{*} \gamma_j \mathcal{E} - \frac{ir}{\sqrt{2}} \bar{\chi}^{*} \mathcal{E} w_{ij} - i\partial_+ (\bar{\chi}^{*} \gamma_{ij} \mathcal{E})
\]

We can further write this as

\[
\delta G_{ij} = \frac{i}{\sqrt{2}} D_k \bar{\chi}^{*} [\gamma^k, \gamma_{ij}] \mathcal{E} - i\partial_+ \bar{\chi}^{*} \gamma_{ij} \mathcal{E} - \frac{1}{r\sqrt{2}} \bar{\chi}^{*} \gamma_{ij} \mathcal{E} - \frac{ir}{\sqrt{2}} \bar{\chi}^{*} \mathcal{E} w_{ij}
\]

Here we have rewritten this in terms of 6d Weyl components so that now all that remains of the \( \Gamma_- \mathcal{E} = 0 \) Weyl projection is

\[ \gamma \mathcal{E} = -\mathcal{E} \]

which amounts to that \( \gamma_{ij} \mathcal{E} \) will be selfdual, and also \( \gamma_{ij} \gamma_k \mathcal{E} \) will be antiselfdual simply because \( \gamma_k \mathcal{E} \) is satisfying the opposite Weyl projection

\[ \gamma \gamma_k \mathcal{E} = \gamma_k \mathcal{E} \]

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as \( \{ \gamma_k, \gamma \} = 0 \). Also since \( w_{ij} \) is selfdual, we can now extract the selfdual part of the variation,

\[
(\delta G_{ij})^+ = \frac{i}{\sqrt{2}} D_k \chi^* \gamma^k \gamma_{ij} \mathcal{E} - i \partial_+ \chi^* \gamma_{ij} \mathcal{E} - \frac{1}{r \sqrt{2}} \chi^* \gamma_{ij} \mathcal{E} - \frac{ir}{\sqrt{2}} \chi^* \mathcal{E} w_{ij}
\]

We can also write this in the form

\[
\delta G_{ij} = -2\sqrt{2} i \mathcal{F}^* \gamma_j D_i \zeta + \frac{ir}{\sqrt{2}} \mathcal{F}^* \zeta w_{ij} + i \partial_+ (\mathcal{F}^* \gamma_{ij} \zeta)
\]

and then

\[
(\delta G_{ij})^+ = -\frac{i}{\sqrt{2}} \mathcal{F}^* \gamma_{ij} \gamma^k D_k \zeta + \frac{ir}{\sqrt{2}} \mathcal{F}^* \zeta w_{ij} + i \mathcal{F}^* \gamma_{ij} \partial_+ \zeta + \frac{1}{r \sqrt{2}} \mathcal{F}^* \gamma_{ij} \zeta
\]

that we can write as

\[
= -\frac{i}{\sqrt{2}} \mathcal{F}^* \gamma_{ij} \left( \gamma^k D_k \zeta - \sqrt{2} \partial_+ \zeta + \frac{i}{r} \zeta \right) + \frac{ir}{\sqrt{2}} \mathcal{F}^* \zeta w_{ij}
\]

We now use the identity

\[
\mathcal{E} = \frac{ir^2}{8} \gamma^{ij} \mathcal{E} w_{ij}
\]

to rewrite one term as

\[
-\frac{1}{r \sqrt{2}} \chi^* \gamma_{ij} \mathcal{E} = -\frac{ir}{8 \sqrt{2}} \chi^* \gamma_{ij} \gamma^{kl} \mathcal{E} w_{kl}
\]

and then we decompose

\[
\gamma_{ij} \gamma^{kl} = \{ \gamma_{ij}, \gamma^{kl} \} - \gamma^{kl} \gamma_{ij}
\]

Noting that \( \{ \gamma_{ij}, \gamma^{kl} \} = -8 \delta_{ij}^{kl} \) when acting on selfdual \( w_{kl} \), the first term gives rise to a term

\[
-\frac{ir}{\sqrt{2}} \chi^* \mathcal{E} w_{ij}
\]

that cancels that corresponding term in \( \delta G_{ij} \), and we are left with

\[
\delta G_{ij} = \frac{i}{\sqrt{2}} D_k \chi^* \gamma^k \gamma_{ij} \mathcal{E} - i \partial_+ \chi^* \gamma_{ij} \mathcal{E} + \frac{ir}{8 \sqrt{2}} \chi^* \gamma^{kl} \gamma_{ij} \mathcal{E}
\]

and consequently

\[
(\delta G_{ij})^+ = \frac{i}{\sqrt{2}} D_k \chi^* \gamma^k \gamma_{ij} \mathcal{E} - i \partial_+ \chi^* \gamma_{ij} \mathcal{E} + \frac{ir}{8 \sqrt{2}} \chi^* \gamma^{kl} \gamma_{ij} \mathcal{E}
\]
We can write this as
\[
(\delta G_{ij})^+ = \frac{i}{\sqrt{2}} \left( D_k \chi^* \gamma^k - \sqrt{2} \partial_+ \chi^* + \frac{r}{8} \chi^* \gamma^{kl} w_{kl} \right) \gamma_{ij} \mathcal{E}
\]

We now wish to show that this vanishes when the fermionic equation of motion is satisfied. Taking the complex conjugate of what is inside the parenthesis, we get the requirement
\[
\gamma^i D_i \chi - \sqrt{2} \partial_+ \chi - \frac{r}{8} \gamma^{kl} \chi w_{kl} = 0
\]
and indeed this is (a Weyl component of) the equation of motion.

Let us complete the supersymmetry variations. We have
\[
\delta F_{i+} = -i \left( \nabla_i (\bar{\chi} \mathcal{E}) - \frac{r}{\sqrt{2}} \kappa_i \partial_+ (\bar{\chi} \mathcal{E}) \right)
= -i D_i (\bar{\chi} \mathcal{E})
= -i D_i (\chi^* \mathcal{E})
\]
and, quite interestingly,
\[
\delta F_{ij} = -\frac{ir}{\sqrt{2}} \chi^* \mathcal{E} w_{ij}
\]
This is interesting, because it is zero, up to a term that is proportional to \( w_{ij} \). This is nothing like the usual supersymmetry variation, and in fact \( \delta D_i = 0 \). And trivially
\[
(\delta F_{ij})^- = 0
\]
since \( w_{ij} \) is selfdual. We do not even need to use the fermionic equation of motion here.

We will now derive a 5d Lagrangian from the selfdual tensor field in 6d dimensions, following closely [6]. We start by noting that
\[
H_{ij} = E^i E^j \left( H_{ij} - \frac{3r}{\sqrt{2}} H_{ij+} \kappa_k + \frac{3r}{\sqrt{2}} H_{ij-} \kappa_k \right)
H_{ij} = E^i E^j \left( H_{ij} - \frac{3r}{\sqrt{2}} H_{ij+} \kappa_k + \frac{3r}{\sqrt{2}} H_{ij-} \kappa_k \right)
H_{i+} = E^i H_{i+} -
or if we define
\[
F_{ij} = H_{ij-}
G_{ij} = H_{ij+}
F_{i+} = H_{i+}
\]
then
\[
H_{ij} = E^i E^j \left( G_{ij} - \frac{r}{2} F_{i+} \kappa_j \right)
\]
\[
H_{ij} = E^i E^j \left( F_{ij} - r \sqrt{2} F_{i+} \kappa_j \right)
\]
\[
H_{ijk} = E^i E^j E^k \left( H_{ijk} + \frac{3r}{\sqrt{2}} \left( F_{ij} - G_{ij} \right) \kappa_k \right)
\]
\[
H_{k+} = E^k F_{i+}
\]

We have the Bianchi identity

\[3 \partial_i H_{jk|+} - \partial_+ H_{ijk} = 0 \quad (3.4)\]

and we have the selfduality relation

\[H_{ijk} = \varepsilon_{ijk}^+ H_{i+}^\perp\]

We define

\[\varepsilon_{ijk}^+ = \varepsilon_{ijkl}^\perp \]

so we have

\[H_{ijk} = -\varepsilon_{ijk}^+ H_{i+}^\perp\]

that we can write this as

\[H_{ijk} + \frac{3r}{\sqrt{2}} \left( F_{ij} - G_{ij} \right) \kappa_k + \varepsilon_{ijk}^+ F_{i+} = 0\]

The Bianchi identity (3.4) then becomes

\[3 \partial_i G_{jk} = -\partial_+ \left( \frac{3r}{\sqrt{2}} \left( F_{ij} - G_{ij} \right) \kappa_k + \varepsilon_{ijk}^+ F_{i+} \right) \quad (3.5)\]

We define

\[G_{ij} = G_{ij} - r \sqrt{2} F_{i+} \kappa_j\]
\[F_{ij} = F_{ij} - r \sqrt{2} F_{i+} \kappa_j\]

that enable us to express (3.5) in the following simple form

\[\varepsilon_{ijkl}^+ D_i G_{jk} = -2 \partial_+ F_{i+}^l\]

and from

\[H_{ij}^\perp = \frac{1}{2} \varepsilon_{ijk}^{\perp k} H_{k+}^\perp\]

we get, by noting that \[\varepsilon_{ijk}^{\perp k} = -\varepsilon_{ijk}^{\perp k+} = -\varepsilon_{ijl}^{\perp k+} = -\varepsilon_{ijkl}^\perp,\]

\[G_{ij} = \frac{1}{2} \varepsilon_{ijkl}^\perp G_{kl}\]
\[ F_{ij} = \frac{1}{2} \varepsilon_{ijkl} F_{kl} \]

The next step will therefore be to replace straight capital letters with curly ones,

\[ \partial_i \left( G_{jk} + r \sqrt{2} F_{j+} \kappa_k \right) + \partial_+ \left( \frac{r}{\sqrt{2}} (F_{ij} - G_{ij}) \kappa_k + \frac{1}{3} \varepsilon_{ijk} F_{l+} \right) = 0 \]

because then we can dualize and get

\[ -D_i G^{il} + \frac{r}{\sqrt{2}} \varepsilon^{ijkl} D_i (F_{j+} \kappa_k) + \frac{r}{\sqrt{2}} \kappa_i \partial_+ (F^{il} + G^{il}) + \partial_+ F^{il} = 0 \]

As a consequence of this equation, we have

\[ - (D_i G^{il}) \kappa_l + \partial_+ F^{il} + \kappa_l = 0 \]

that we can also write as

\[ -D_i (G^{ij} \kappa_j) + \frac{1}{2} G^{il} w_{il} + \partial_+ F^{il} + \kappa_l = 0 \]

but the second term is vanishing, as one can see by replacing \( G^{il} \) with \( G^{il} \) which is antiselfdual so contracting with a selfdual \( w_{il} \) gives zero. And moreover \( \kappa^l w_{il} = 0 \). So we have

\[ D_i (G^{ij} \kappa_j) = \kappa_i \partial_+ F^{il} \]

which will be a useful relation that we will use later. We may also write

\[ -D_i G^{il} + \frac{r}{\sqrt{2}} \varepsilon^{ijkl} D_i (F_{j+} \kappa_k) + \frac{r}{\sqrt{2}} \kappa_i \partial_+ F^{il} + \partial_+ F^{il} = 0 \]

We have the Bianchi identity

\[ 3 \partial_i H_{jk} = \partial_- H_{ijk} \]

but if we put \( \partial_- = 0 \) upon dimensional reduction, then this reduces to

\[ \varepsilon^{ijkl} \partial_i F_{jk} = 0 \]

Again replacing straight capital \( F \) with curly \( F \), we first get

\[ \varepsilon^{ijkl} \partial_i \left( F_{jk} + r \sqrt{2} F_{j+} \kappa_k \right) = 0 \]

and then by using selfduality this becomes

\[ D_i F^{il} + \frac{r}{\sqrt{2}} \varepsilon^{ijkl} D_i (F_{j+} \kappa_k) = 0 \]
But the nicest way to express this same equation is as

$$\varepsilon^{ijkl} D_i F_{jk} = 0$$

We have the Bianchi identity

$$2\partial_t F_{j} + \partial_+ F_{ij} = 0$$

Replacing $F$ with $\mathcal{F}$ it becomes

$$2D_i F_{j} + \partial_+ \mathcal{F}_{ij} = 0$$

Finally, we return to

$$H_{ijk} + 3r \sqrt{2} (F_{ij} - G_{ij}) \kappa_k + \varepsilon_{ijk} l F_{l} = 0$$

and apply the Bianchi identity $\varepsilon^{ijkl} \partial_i H_{ijk} = 0$. We then get

$$D_i F_{i} = \frac{r}{2\sqrt{2}} \mathcal{F}_{ij} \omega^{ij}$$

where we define

$$D_i = D_i - \frac{r}{\sqrt{2}} \kappa_i \partial_+$$

We have thus got two types of equations of motion,

$$-D_i G^{il} + \frac{r}{\sqrt{2}} \varepsilon^{ijkl} D_i (F_{j} + \kappa_k) + \frac{r}{\sqrt{2}} \kappa_i \partial_+ F^{il} + \frac{1}{2} \partial_+ F^{l} = 0$$

$$D_i F_{i} - \frac{r}{2\sqrt{2}} \mathcal{F}_{ij} \omega^{ij} = 0$$

and in addition to these, we have the selfduality equations

$$G_{ij} = -\frac{1}{2} \varepsilon^{ijkl} G_{kl}$$

$$\mathcal{F}_{ij} = \frac{1}{2} \varepsilon^{ijkl} \mathcal{F}_{kl}$$

The equation

$$D_i \mathcal{F}^{il} + \frac{r}{\sqrt{2}} \varepsilon^{ijkl} D_i (F_{j} + \kappa_k) = 0$$

surely looks very much like an independent equation of motion, but actually it is not. It is a direct consequence of $\varepsilon^{ijkl} \partial_i F_{jk} = 0$ together with the selfduality equation of motion for $\mathcal{F}_{ij}$. That means we do not need to demand that the equation (3.6) follows from an action upon the variation of a gauge field as one normally would expect. Now one may ask
some questions about number of components. Let us be very brief and just notice that selfdual $H_{MNP}$ has 10 components, just as do selfdual $F_{ij}$ and $F_{i+}$ together, as $6 + 4 = 10$. So we do not expect $G_{ij}$ shall be part of the supermultiplet upon dimensional reduction. Only $F_{ij}, F_{i+}$ should be part of the vector multiplet. It then seems reasonable to assume that the antiselfdual $G_{ij}$ shall be viewed as a Lagrange multiplier field that is imposing selfduality on $F_{ij}$, rather than as a dynamical field that contributes to additional degrees of freedom. We now make the following ansatz for a gauge field Lagrangian,

$$\mathcal{L}_A = b F_{ij} G_{ij} + c F_{i+} + d \varepsilon^{ijkl} F_{ij} F_{k+} \kappa_l + e \varepsilon^{ijkl} G_{ij} F_{k+} \kappa_l$$

and treat $G_{ij}$ (assumed to be antiselfdual from the outset), $A_i$ and $A_{i+}$ as independent fields that we shall vary to derive the classical equations of motion. Then these equations of motion become

$$\mathcal{F}_{ij} - \frac{1}{2} \varepsilon_{ijkl} \mathcal{F}_{kl} = 0$$

$$\left( br \sqrt{2} + 2e \right) D_i \left( G^{ij} \kappa_j \right) - 2c D_i F_{i+} - d \mathcal{F}_{ij} w^{ij} = 0$$

$$-2b D_i g^{im} + \left( br \sqrt{2} + 2e \right) \kappa_i \partial_+ g^{im} + 2d \varepsilon^{mkl} D_i \left( F_{k+} \kappa_l \right) + 2d \partial_+ F^{m+} \kappa_j + 2c \partial_+ F^{m+} = 0$$

We now write the second relation as

$$-2c D_i F_{i+} + \left( br \sqrt{2} + 2e \right) \kappa_i \partial_+ F_{i+} - d \mathcal{F}_{ij} w^{ij} = 0$$

By now requiring the combination $D_i = D_i - \frac{r}{\sqrt{2}} \kappa_i \partial_+$ to appear, we get the following equations

$$\frac{br \sqrt{2} + 2e}{2c} = \frac{r}{\sqrt{2}}$$

$$\frac{br \sqrt{2} + 2e}{2b} = \frac{r}{\sqrt{2}}$$

$$\frac{d}{b} = -\frac{r}{\sqrt{2}}$$

$$\frac{c}{b} = 1$$

These equations have the following unique solution

$$b = 1$$

$$c = 1$$

$$d = -\frac{r}{\sqrt{2}}$$

$$e = 0$$
up to one overall constant. Fixing that overall constant to be $1/4$, the Lagrangian is given by

$$\mathcal{L}_A = \frac{1}{4} \left( \mathcal{F}^{ij} \mathcal{G}_{ij} + F^i_+ F_i^+ - \frac{r}{\sqrt{2}} \varepsilon^{ijkl} \mathcal{F}_{ij} F_{k+l} \right)$$

where we have also replaced $F$ with $\mathcal{F}$ in the graviphoton term, which we can do freely by just noting that $\kappa_j \kappa_l = 0$ upon antisymmetrization in $j$ and $l$. The supersymmetry variation of this Lagrangian is

$$\delta \mathcal{L}_A = -i \sqrt{2} \chi^* \gamma_j \mathcal{E} D_j \mathcal{F}^{ij} + i \frac{\gamma^j \mathcal{E} \partial_+ \mathcal{F}_{ij}}{4} + i \frac{\gamma^i \mathcal{E} D^i F_i^+}{2}$$

$$- \frac{ir}{4} \sqrt{2} \chi^* \gamma^i \mathcal{G}^{ij} w_{ij} - \frac{ir}{2 \sqrt{2}} \chi^* \mathcal{E} F^{ij} w_{ij}$$

The fourth term is identically zero because $\mathcal{G}_{ij}$ is antiselfdual off-shell.

Next, we obtain the supersymmetry variation

$$\delta \psi = \frac{1}{12} \Gamma^{MNP} \varepsilon H_{MNP}$$

in 4d. To this end, it is advantageous to first recast this in flat space indices,

$$\delta \psi = \frac{1}{4} \Gamma^{ij} \varepsilon H_{ij} + \frac{1}{4} \Gamma^{ij} \varepsilon H_{ij} - \frac{1}{4} \Gamma^{ij} \varepsilon H_{ij} + \frac{1}{2} \Gamma^{ij} \varepsilon H_{ij}$$

Then it immediately follows that

$$\delta \psi = \frac{1}{4} \Gamma^{ij} \varepsilon \mathcal{F}_{ij} + \frac{1}{2} \Gamma^{ij} \varepsilon \mathcal{F}_{ij}$$

which in terms of 4d gamma matrices reads

$$\delta \chi = \frac{1}{2 \sqrt{2}} \gamma^{ij} \mathcal{E} F_{ij} - \frac{1}{2} \gamma^i \mathcal{E} F_i^+$$

Then let us look at each term in turn in the fermionic action

$$\mathcal{L}_F = \frac{i}{2} \chi^* \gamma^i D_i \chi - \frac{i}{\sqrt{2}} \chi^* P_- \partial_+ \chi$$

$$+ \frac{1}{r} \chi^* P_+ \chi - \frac{ir}{16} \chi^* \gamma^{ij} P_- \chi w_{ij}$$

The variation of the first two derivative terms becomes after using two types of Bianchi identities

$$\delta \left( \mathcal{L}_F + \mathcal{L}_F^{II} \right) = \frac{i}{\sqrt{2}} \gamma^i \mathcal{E} D^i \mathcal{F}_{ij} - \frac{1}{2} \gamma^i \mathcal{E} D^i F_i^+ - \frac{i}{4} \chi^* \gamma^{ij} \mathcal{E} \partial_+ \mathcal{F}_{ij}$$

$$- \frac{1}{r 2 \sqrt{2}} \chi^* \gamma^{ij} \mathcal{E} F_{ij}$$
The first line is exactly canceling corresponding terms in $\delta L_A$. The variation of the two last mass terms gives

$$\delta \left( L^IV_F + L^IV_F \right) = \frac{-ir}{16\sqrt{2}} \chi^* \gamma^i E F_i + \frac{ir}{16\sqrt{2}} \chi^* \gamma^i \gamma^j \gamma^{kl} E w_{ij} F_{kl}$$

Ideally we had wanted these to cancel against the last term in $\delta L_A$,

$$\delta L^VI_A = -\frac{ir}{2\sqrt{2}} \chi^* w_{ij} F_j$$

We do not seem to get a perfect cancelation, but let us note that we can rewrite the last term in $\delta \left( L^IV_F + L^IV_F \right)$ as

$$-\frac{ir}{16\sqrt{2}} \chi^* \left( \{\gamma^i, \gamma^{kl}\} - \gamma^{kl} \gamma^{ij} \right) E w_{ij} F_{kl} = \frac{ir}{2\sqrt{2}} \chi^* w_{ij} F_{ij} + \frac{ir}{16\sqrt{2}} \chi^* \gamma^i \gamma^{kl} E w_{ij} F_{kl} = \frac{ir}{2\sqrt{2}} \chi^* w_{ij} F_{ij} + \frac{1}{r} \frac{ir}{2\sqrt{2}} \chi^* \gamma^{ij} E F_{ij}$$

The first term cancels against $\delta L^VI_A$ and the second term cancels the last term in $\delta \left( L^IV_F + L^IV_F \right)$. The final result is that we have the following nonzero variation of the Lagrangian,

$$\delta L = -\frac{1}{r} \chi^* \gamma^i E F_i$$ (3.7)

Since the 6d metric inverse $g^{ij}$ is equal to the 4d metric inverse $G^{ij}$ and since the index $i$ in $F_{++} = H_{++}$ can be extended to indices $+$ and $-$ without changing anything since $H_{++}$ and $H_{+-}$ are zero anyway, we can view $i$ as a 6d index contracted by the 6d metric. This means that we can write this result in terms of 6d flat space indices as

$$\delta L = \frac{i}{r} \sqrt{2} \delta B_+ H_{++}$$

and by using the selfduality relation

$$H_+ = \varepsilon_+ \varepsilon_+ F_+$$

we can further write this as

$$\delta L = \frac{i}{r} \sqrt{2} \varepsilon_+ \varepsilon_+ \delta B_+ H_{++}$$

Now we can change to 6d curved space indices and then this becomes

$$\delta L = \frac{i}{r} \sqrt{2} \varepsilon^{ijkl} H_{ijk} \delta B_i$$

where we define $\varepsilon^{ijkl} = -\varepsilon^{ijkl+}$. We now wish to show that this can be expressed as a total variation of some topological term of the form

$$L_{top} = \varepsilon^{ijkl} H_{ijk} B_i$$
up to some constant factor. When we expand its variation, we find two types of terms,

\[ \delta L_{\text{top}} = 3 \epsilon^{ijkl} \partial_i \delta B_{jk} B_{l+} + \epsilon^{ijkl} H_{ijkl} \delta B_{l+} \]

The first term here can be further written as

\[ -3 \epsilon^{ijkl} B_{jk} \partial_i B_{l+} = -\frac{3}{2} \epsilon^{ijkl} H_{ijk} \delta B_{l+} \]

where we dropped a couple of total derivative terms. Now, if we change to flat space indices we see the emergence of an antiselfdual \( H_{\hat{i}\hat{j}} = \hat{G}_{\hat{i}\hat{j}} \) and so what this term becomes is something that is proportional to \( \delta B^{ij} \hat{G}_{ij} \) and this is zero, because \( \delta B^{ij} \sim \chi^i \gamma^j \mathcal{E} \) and we have that \( \gamma^{ij} \mathcal{E} \mathcal{G}_{ij} = 0 \) since \( \mathcal{G}_{ij} \) is antiselfdual and \( \mathcal{E} \) is Weyl. One way to see this is by noting that \( \gamma^{ij} \mathcal{E} w_{ij} \) is nonzero where \( w_{ij} \) is selfdual. This means that we are left with only the second term,

\[ \delta L_{\text{top}} = \epsilon^{ijkl} H_{ijkl} \delta B_{l+} \]

as we wanted to show. So by adding the topological term

\[ L_{\text{top}} = \frac{i}{r \sqrt{2}} \epsilon^{ijkl} H_{ijkl} B_{l+} \]

we find that its variation cancels the variation \( \delta L \) in (3.7) above.

Let us now study the matter part supersymmetry. The Lagrangian is

\[ L_F = \frac{i}{2} \chi^i \gamma^i D_i \chi - \frac{i}{\sqrt{2}} \chi^i P_- \partial_+ \chi + \frac{1}{r} \chi^i P_+ \chi \]

\[ -\frac{i}{2r} \chi \gamma^{12} P_- \chi - \frac{ir}{2 \sqrt{2}} \kappa_i \chi \gamma^i \partial_+ \chi \]

\[ -\frac{1}{2} \kappa_i \chi \gamma^i \chi \]

The supersymmetry variation is

\[ \delta \chi = -\gamma^i \tau^A \mathcal{E} D_i \phi^A - \frac{2i}{r} \tau^A \mathcal{E} \phi^A \]

where we define

\[ D_i = D_i - \kappa_i \partial_y \]

\[ \partial_y = \frac{1}{\sqrt{2}} (\partial_+ - \partial_-) \]

Using this generalized derivative on the fermion, and the expansion where

\[ \partial_- \chi \rightarrow \frac{i \sqrt{2}}{r} \chi \]

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we find that the Lagrangian simplifies to
\[
\mathcal{L}_F = \frac{i}{2} \chi \gamma^i D_i \chi - \frac{i}{\sqrt{2}} \chi^\dagger P_- \partial^+ \chi + \frac{1}{r} \chi^\dagger P_+ \chi - \frac{i}{2r} \chi \gamma^{12} P_- \chi
\]

We thus need to carefully define the operator \( D_i \) acting on bosons and fermions respectively, as
\[
D_i \phi = \partial_i \phi - \frac{r}{\sqrt{2}} \partial^+ \phi \\
D_i \chi = D_i \chi - \frac{\kappa_i \partial^+ \chi + i \kappa_i \chi}{\sqrt{2}}
\]

Similarly then when this generalized derivative acts on the supersymmetry parameter, and then one finds the following Killing spinor equation
\[
D_i \mathcal{E} = 0
\]

We get the supersymmetry variation
\[
\delta \mathcal{L}_F = - \frac{i}{2} \chi \gamma^{ij} \tau^A \mathcal{E} [D_i, D_j] \phi^A - i \chi \gamma^{ij} \tau^A \mathcal{E} D_i^2 \phi^A \\
- \frac{2\sqrt{2}}{r} \chi \gamma^{ij} \tau^A \mathcal{E} \partial^+ \phi^A \\
+ \frac{2i}{r^2} \chi \gamma^{ij} \tau^A \mathcal{E} \phi^A - \frac{2}{r} \chi \gamma^{12} \tau^A \mathcal{E} \phi^A
\]

Two terms cancel by using
\[
[D_i, D_j] \phi = - \frac{r}{\sqrt{2}} w_{ij} \partial^+ \phi
\]

and
\[
\gamma^{12} \mathcal{E} = - i \mathcal{E} \\
\gamma^{34} \mathcal{E} = - i \mathcal{E}
\]

and we get
\[
\delta \mathcal{L}_F = - i \chi \gamma^{ij} \tau^A \mathcal{E} D_i^2 \phi^A \\
+ \frac{4i}{r^2} \chi \gamma^{ij} \tau^A \mathcal{E} \phi^A
\]

Let us now turn to the scalar fields’ Lagrangian
\[
\mathcal{L}_S = - \frac{1}{2} (D_i \phi^A)^2 - \frac{2}{r^2} (\phi^A)^2
\]
Using the variation
\[ \delta \phi^A = i \chi^i \tau^A \mathcal{E} \]
we find that \( \delta \mathcal{L}_S + \delta \mathcal{L}_F = 0 \).

Before turning to the non-Abelian case, let us first summarize the Abelian case. We have the Lagrangian
\[ \mathcal{L} = \mathcal{L}_A + \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{top}} \]
where
\[ \mathcal{L}_A = \frac{1}{4} \left( F_{ij} \mathcal{G}_{ij} + F_i^i F_i^+ - \frac{r}{\sqrt{2}} \varepsilon^{ijkl} F_k^j F_{k+} \right) \]
\[ \mathcal{L}_{\text{matter}} = \frac{i}{2} \chi^* \gamma^i \mathcal{D}_i \chi - \frac{i}{\sqrt{2}} \chi^* P_+ \partial_+ \chi 
+ \frac{1}{r} \chi^* P_+ \chi - \frac{i r}{16} \chi^* \gamma^{ij} P_- \chi w_{ij} 
- \frac{1}{2} (D_i \phi^A)^2 - \frac{2}{r^2} (\phi^A)^2 \]
\[ \mathcal{L}_{\text{top}} = \frac{1}{r} \sqrt{2} \varepsilon^{ijkl} H_{ijk} B_{k+} \]
and the supersymmetry variations
\[ \delta \phi^A = \frac{i}{2} \chi^* \tau^A \mathcal{E} \]
\[ \delta \chi = \frac{i}{2 \sqrt{2}} \gamma^{ij} \mathcal{E} F_{ij} - \frac{1}{2} \gamma^i \mathcal{E} F_i^+ - \gamma^i \tau^A \mathcal{E} D_i \phi^A - \frac{2i}{r} \tau^A \mathcal{E} \phi^A \]
\[ \delta A_i = - \frac{ir}{\sqrt{2}} \kappa_i \chi^* \mathcal{E} \]
\[ \delta A_+ = - i \chi^* \mathcal{E} \]
\[ \delta F_i^+ = - i D_i (\chi^* \mathcal{E}) \]
\[ \delta F_{ij} = - \frac{ir}{\sqrt{2}} \chi^* \mathcal{E} w_{ij} \]
\[ \delta G_{ij} = 2 \sqrt{2} i D_i \chi^* \gamma_j \mathcal{E} - \frac{ir}{\sqrt{2}} \chi^* \mathcal{E} w_{ij} - i \partial_+ (\chi^* \gamma_j \mathcal{E}) \]

To see whether a non-Abelian generalization is possible, let us start by replacing all derivatives with gauge covariant derivatives,
\[ \mathcal{D}_i \phi^A = D_i \phi^A - \kappa_i D_y \phi^A \]
\[ D_i \phi^A = \partial_i \phi^A - ie [A_i, \phi^A] \]
\[ D_y \phi^A = \partial_y \phi^A - ie [A_y, \phi^A] \]
in the supersymmetry variations. Then by noting that
\[ [\mathcal{D}_i, \mathcal{D}_j] \phi^A = -ie [\mathcal{F}_{ij}, \phi^A] \]
we get
\[ \delta \mathcal{L} = -\frac{e}{2} \chi^* \gamma^{ij} \tau^A \mathcal{E} \left[ F_{ij}, \phi^A \right] \]

To cancel this variation, one might be tempted to add the following term to the Lagrangian,
\[ \Delta \mathcal{L} = \frac{e}{\sqrt{2}} \chi^* \tau^A \left[ \chi, \phi^A \right] \]

But if we do that, then that term will upon a supersymmetry variation generate a host of new terms, such as
\[ \chi^* \gamma^i \mathcal{E} \left[ D_i \phi^A, \phi^A \right] \] (3.8)

but we cannot cancel this term by anything. The only candidate term \((D_i \phi^A)^2\) does not work because the supersymmetry variation of the gauge potential \(A_i\) is vanishing, so it cannot give rise to something that is proportional to \(\chi^* \gamma_i \mathcal{E}\). So we cannot cancel the variation (3.8) and therefore we shall not add any extra commutator terms to the Lagrangian.

Instead we shall modify the supersymmetry variation of \(G_{ij}\) by adding a term\(^3\)
\[ \Delta \delta G_{ij} = \frac{e}{2} \left[ \chi^* \gamma_{ij} \tau^A \mathcal{E}, \phi^A \right] \]

If we could make the gauge choice \(A_+ = 0\) and then just forget about \(\delta A_+\) altogether, then since \(\delta A_i = 0\), we would have no cubic term in the fermionic fields that could appear when we vary the gauge potential in the fermionic kinetic term. But imposing the gauge choice \(A_+ = 0\) is unsatisfactory since this gauge choice breaks supersymmetry by itself. We can avoid this problem of gauge fixing by reducing supersymmetry by another half. We then impose the Weyl projection
\[ \tau^5 \mathcal{E} = \mathcal{E} \]

Then we have the supersymmetry variation
\[ \delta \phi^5 = i \chi^* \mathcal{E} \]

\(^3\)This is in accordance with the Lambert-Papageorgakis theory, where
\[ \delta H_{MNP} \sim \ldots + [\phi^A, \bar{\psi}] \Gamma^A \Gamma_{MNPQ} \varepsilon^Q \]

if we notice that the only surviving combination of gamma matrices can be \(\Gamma_{ij+} \), which simply means that the commutator only enters in \(H_{ij+}\), or in other words \(G_{ij}\).
and we see that the combination $A_+ - \phi^5$ is a supersymmetric invariant,

$$\delta (A_+ - \phi^5) = 0$$

We then obtain a supersymmetric Lagrangian by simply adding commutator terms that involve $\phi^5$ for each place where there is a gauge field $A_+$. Such commutator terms are of course gauge invariant by themselves. But we can repackage these terms into a new derivative

$$D_+ = D_+ + ie[\phi^5, \cdot]$$

where $D_+ = \partial_+ - ie[A_+, \cdot]$. One may worry that ordinary derivative acts on a fermionic field, but that is just because of how we have set up our Lagrangian. We have already taken into account all those curvature corrections when we analysed the Abelian case and those curvature corrections will not be affected in any significant way by the non-Abelian generalization. We now obtain a full supersymmetric non-Abelian Lagrangian by replacing every occurrence of $\partial_+$ with $D_+$ as we defined it above (with an ordinary derivative $\partial_+$ rather than a curvature covariant $\nabla_+$). There is now at this stage no need to impose any gauge fixing condition on $A_+$.

The $B \wedge H$ term is straightforwardly generalized to the non-Abelian case as $B^a \wedge H^a$ where $a$ is the adjoint gauge group index. The supersymmetry variation of $\tilde{B}_{i+}$ is similarly generalized by just attaching that adjoint gauge group index as $\delta \tilde{B}_{i+}^a = -i\sqrt{2}(\chi^a)^* \gamma_i \mathcal{E}$. We also assume the duality relation is generalized to the non-Abelian case as $H_{ijk}^a = \varepsilon_{ijkl} F_{l+}^a$.

As we did not put any component of the fermionic field to zero here, as we did for the case of time reduction, we do not expect our 5d Lagrangian will be possible to derive by tuning on an R-gauge field in some dual formulation. In particular, we do not expect the closure relations will be of a standard form.

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A A 6d formulation of 5d SYM

There is a 6d formulation of 5d SYM where one introduces a vector field $v^M$ and requires all fields to have vanishing Lie derivatives along that vector field [4], [7], [5]. We did not make explicit use of this 6d formulation of 5d SYM. But it was this formulation that

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originally motivated us to search for consistent supersymmetric truncations, and the two cases that we have studied in this paper can be at least intuitively quite clearly understood by looking at this formulation of the theory where they emerge as the Weyl projections (A.8) and (A.9) respectively.

The 6d supersymmetry variations look like a non-Abelian generalization of the Abelian M5 brane, but of course there is a catch. Namely we do not have closure relations satisfied for these variations, unless two terms vanish, namely the terms in (A.3) and (A.4). Let us present this in detail. The supersymmetry variations are given by

\[
\begin{align*}
\delta \phi^A &= i \bar{\epsilon} \Gamma^A \psi \\
\delta H_{MNP} &= 3i D_P (\bar{\epsilon} \Gamma_{MN} \psi) + e \bar{\epsilon} \Gamma_{MNPQ} \Gamma^A [\psi, \phi^A] v^Q \\
\delta A_N &= i \bar{\epsilon} \Gamma_{NP} \psi v^P \\
\delta \psi &= \frac{1}{12} \Gamma^{MNP} \bar{\epsilon} H_{MNP} + \Gamma^M \Gamma^A \bar{\epsilon} D_M \phi^A - 4 \Gamma^A \eta \phi^A - \frac{i e}{2} \Gamma_M \Gamma^{AB} \bar{\epsilon} [\phi^A, \phi^B] v^M 
\end{align*}
\]

Here

\[
D_M \phi^A = \partial_M \phi^A - i \epsilon [A_M, \phi^A] + V_M^{AB} \phi^B
\]

where \( V_M \) is an R-gauge field, and

\[
\begin{align*}
D_M \bar{\epsilon} &= \Gamma_M \bar{\eta} - \frac{1}{8} \Gamma^A \Gamma^{RST} \Gamma_M \bar{\epsilon} T^A_{RST} \\
D_M \epsilon &= - \bar{\eta} \Gamma_M - \frac{1}{8} \epsilon \Gamma_M \Gamma^{RST} \Gamma^A \bar{T}^A_{RST} 
\end{align*}
\]

(A.1)

Here \( v^M \) is a Killing vector field and \( \mathcal{L}_v \) denotes the Lie derivative along this Killing vector field. We will impose the gauge condition

\[
A_M v^M = 0 \quad \text{(A.2)}
\]

which is a very natural gauge condition if we think on \( A_M \) as \( B_{MN} v^N \). Now this correspondence is at present unknown to us for the nonabelian case where \( H_{MNP} \) is all that we have. We would like to know how to express the theory in terms of some nonabelian gauge potential \( B_{MN} \) but at present we do not have such a formulation. Nevertheless, the gauge potential will be assumed to satisfy the gauge condition (A.2).

We define the 6d chirality matrix

\[
\Gamma = \Gamma^{012345}
\]

in flat tangent space and we assume that spinor and supersymmetry parameter have opposite chiralities

\[
\Gamma \psi = \psi
\]

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\[\Gamma \varepsilon = -\varepsilon\]

We use 11d gamma matrices where \(\Gamma_M\) denote spacetime gamma matrices for \(M = 0, 1, \ldots, 5\) and \(\Gamma_A\) denote five transverse space gamma matrices for \(A = 1, 2, 3, 4, 5\) and these anticommute, \(\{\Gamma_M, \Gamma_A\} = 0\).

For the closure computation of these supersymmetry variations, we define
\[S^M = \bar{\varepsilon} \Gamma^M \varepsilon\]
and the gauge parameter
\[\Lambda = -i\bar{\varepsilon} \Gamma_Q \Gamma^A \varepsilon^A \phi^Q\]
and we assume that \(\varepsilon\) is a commuting spinor, since that simplifies the closure computation a bit yet without imposing any restrictions.

The superconformal algebra in curved space is
\[\delta_\varepsilon^2 = -i\mathcal{L}_S - 2i\mathcal{W} - 2\bar{\varepsilon} \Gamma^{AB} \eta R^{AB} + \delta_{\text{gauge}}\]

where
\[R^{AB} = \frac{i}{2} \Gamma^{AB}\]
\[R^{CD}_{\ AB} = \frac{1}{2} \delta^{\ AB}_{\ CD}\]
and
\[\mathcal{W}_{\phi^A} = 2\]
\[\mathcal{W}_{\psi} = 5\]
\[\mathcal{W}_{\varphi M N P} = 0\]
are the Weyl weights.

As always with closure relations, we can express these in terms of conventional Lie derivatives \(\mathcal{L}_S\), or in terms of gauge covariant Lie derivatives \(\mathbb{L}_S\). These are related as
\[-i\mathbb{L}_S \phi^A = -i\mathcal{L}_S \phi^A - ie[\phi^A, \Delta \lambda] - iS^M V^{AB}_M \phi^B\]
\[-i\mathbb{L}_S \psi = -i\mathcal{L}_S \psi - ie[\psi, \Delta \lambda] - i\frac{1}{4} S^M V^A_M \Gamma^{AB} \psi\]
\[-i\mathbb{L}_S H_{MNP} = -i\mathcal{L}_S H_{MNP} - ie[H_{MNP}, \Delta \lambda]\]
\[-i\mathbb{L}_S A_M = -iS^N F_{NM}\]
\[= -i\mathcal{L}_S A_M + D_M (\Delta \lambda)\]
where \(\Delta \lambda = iS^M A_M\). We thus see that from \(\mathbb{L}_S\) we get \(\mathcal{L}_S\) plus some extra gauge transformation and R-rotation.
If we assume that $\varepsilon$ is commuting, then we find the following closure relations,

$$
\delta^2 \phi^A = -iL_S \phi^A - 4i\bar{\varepsilon}\eta\phi^A - 4i\bar{\varepsilon} \Gamma^{AB} \eta\phi^B - ie[\phi^A, \Lambda]
$$

$$
\delta^2 H_{MNP} = -iL_S H_{MNP}
$$

$$
+3iD_M \left( S^T (H^A_{NPT} - 6\phi^A T^A_{NPT}) \right)
$$

$$
-4iS^T D_P H_{MNT} - e\varepsilon_{MNPQRS} \varepsilon \left[ D_R \phi^A, \phi^A \right] \nu^Q - \frac{ie}{2} \varepsilon \varepsilon_{MNPQUV} \bar{\psi} \Gamma^U \psi^a \nu^Q
$$

$$
+3i\bar{\varepsilon} \varepsilon_P \Gamma^A \varepsilon \left[ H_{MNQ} \nu^Q - F_{MN} \phi^A \right]
$$

$$
- ie[H_{MNP}, \Lambda]
$$

$$
+ \frac{e}{2} L_v \left( \varepsilon \Gamma^{AB} \Gamma_{MNP} \varepsilon \right) [\phi^A, \phi^B]
$$

$$
\delta^2 \psi = -iL_S \psi - 2i\bar{\varepsilon}\frac{5}{2} \psi - i\bar{\varepsilon} \Gamma^{AB} \bar{\psi} - ie[\psi, \Lambda]
$$

$$
+3i \frac{S^Q \Gamma Q}{8} \left( \Gamma^P D_P \psi + \frac{1}{4} \Gamma^RST \Gamma^A \psi T^A_{RST} - ie \Gamma_M \Gamma^A[\psi, \phi^A] \nu^M \right)
$$

$$
+2ie \bar{\varepsilon} \Gamma^Q \Gamma Q \bar{\psi} \Gamma^B \left( \Gamma^P D_P \psi + \frac{1}{4} \Gamma^RST \Gamma^A \psi T^A_{RST} - ie \Gamma_M \Gamma^A[\psi, \phi^A] \nu^M \right)
$$

$$
\delta^2 A_M = -iL_S A_M + D_M \left( -i\bar{\varepsilon} \Gamma_N \Gamma^A \varepsilon \phi^A \nu^N \right)
$$

$$
+iS^T F_{TM} - iS^T \left( H^A_{MNT} + 6T^A_{MNP} \phi^A \right) \nu^N
$$

$$
+ L_v \left( \varepsilon \Gamma_M \Gamma^A \varepsilon \phi^A \right)
$$

Apart from the term

$$
\frac{e}{2} L_v \left( \varepsilon \Gamma^{AB} \Gamma_{MNP} \varepsilon \right) [\phi^A, \phi^B] \quad (A.3)
$$

in $\delta^2 H_{MNP}$ and the term

$$
iL_v \left( \varepsilon \Gamma_M \Gamma^A \varepsilon \phi^A \right) \quad (A.4)
$$

in $\delta^2 A_M$, we can now obtain closure up to a gauge transformation with gauge parameter $\lambda = -i\bar{\varepsilon} \Gamma_M \Gamma^A \varepsilon \phi^A \nu^M$ if certain equations of motion are satisfied. Closure on $H_{MNP}$ requires the following equations of motion,

$$
H_{NPT} - 6\phi^A T^A_{NPT} = 0 \quad (A.5)
$$

$$
H_{MNQ} \nu^Q - F_{MN} = 0 \quad (A.6)
$$

and closure on $A_M$ requires the equation of motion

$$
F_{TM} - \left( H^A_{MNT} + 6T^A_{MNP} \phi^A \right) \nu^N = 0 \quad (A.7)
$$

By adding $0 = H^A_{NPT} - 6\phi^A T^A_{NPT}$ to $H^A_{MNT} + 6T^A_{MNP} \phi^A$ we get $H_{MNT} = H^A_{MNT} + H_{MNT}^-$ and (A.7) reduces to (A.6). Of course the presence of the terms (A.3) and (A.4) means that these 6d supersymmetry variations do not close, unless both these terms vanish.
One way to make these two terms vanish is by requiring the Lie derivative vanishes on every field and also on the supersymmetry parameter, $L_v \varepsilon = 0$, where $L_v$ denotes the Lie derivative along $v^M$. This is the usual dimensional reduction along the vector field $v^M$.

Could there be some other ways to achieve closure? At least for the first term (A.3), we can make that term disappear without requiring $L_v \varepsilon = 0$. To see this more clearly, let us notice that a corresponding commutator term sits in the supersymmetry variation of the $(2,0)$ tensor multiplet fermion $\psi$ as

$$\delta \psi = \ldots - \frac{i e}{2} \Gamma_M \Gamma^{AB} \varepsilon [\phi^A, \phi^B] v^M$$

and here we can see two ways for this commutator term to vanish.

One is by just keeping one scalar field, say $\phi^5$ and reduce supersymmetry by imposing the R-symmetry Weyl projection

$$\Gamma^{A=5} \varepsilon = \varepsilon$$

and discarding the hypermultiplet. Of course, with just one scalar field, there will be no nontrivial commutator term $[\phi^A, \phi^B]$, but having to discard the hypermultiplet is of course unsatisfactory.

The other way to get rid of this term is by taking $v^M$ to be a null vector and imposing the Weyl projection

$$\Gamma_M \varepsilon v^M = 0$$

and again this commutator term will vanish. The advantage of the null reduction is clearly that we can keep the full tensor multiplet structure with the five scalar fields intact.

**B The Euclidean M5 brane**

So far we have discussed only the Lorentzian M5 brane. But if we eventually would like to study the M5 brane on say $S^6$, then we will need to understand what the Euclidean M5 brane really means in terms of its tensor multiplet structure and its supersymmetry. So here we will clarify this point. First we begin with what is familiar to us though, namely the Lorentzian tensor multiplet and then we seek a way to modify this so that we can allow a Euclidean signature.

**B.1 The Lorentzian $(2,0)$ and $(0,2)$ tensor multiplets**

We begin with Lorentzian $SO(1,5) \times SO(5) \subset SO(1,10)$ where we have the Dirac conjugate $\bar{\varepsilon} = \varepsilon^\dagger \Gamma^0$ and the Majorana condition $\bar{\varepsilon} = \varepsilon^T C$ that in terms of Weyl components
reads $\varepsilon^\dagger \Gamma = \varepsilon^{T} C$ and hence is compatible with Weyl projection $\varepsilon^+ = 0$. We then have the chiral $(2, 0)$ tensor multiplet

$$
\begin{align*}
\delta \phi^+ &= i \varepsilon^A \psi^+ \\
\delta B^+_{MN} &= i \varepsilon_{MN} \psi^+ \\
\delta \psi^+ &= \frac{1}{12} \Gamma^{MNP} \varepsilon H^+_{MNP} + \Gamma^M \Gamma^A \varepsilon D_M \phi^+ A - 4 \Gamma^A \eta \phi^+ A
\end{align*}
$$

We may also consider the anti-chiral $(0, 2)$ tensor multiplet

$$
\begin{align*}
\delta \phi^- &= i \varepsilon^A \psi^- \\
\delta B^-_{MN} &= i \varepsilon_{MN} \psi^- \\
\delta \psi^- &= \frac{1}{12} \Gamma^{MNP} \varepsilon H^-_{MNP} + \Gamma^M \Gamma^A \varepsilon D_M \phi^- A - 4 \Gamma^A \eta \phi^- A
\end{align*}
$$

and if we put them together we can write a Lagrangian

$$
\mathcal{L}_{(2,0)+(0,2)} = -\frac{1}{24} H^2_{MNP} + \mathcal{L}^+ + \mathcal{L}^-
$$

that is invariant under both the $(2, 0)$ and the $(0, 2)$ superconformal symmetries where the corresponding supersymmetry parameters satisfy

$$
D_M \varepsilon^\pm = \Gamma_M \eta^\pm - \frac{1}{8} \Gamma^A \Gamma^{RST} \Gamma_M \varepsilon^\mp T^A_{RST}
$$

These Killing spinor equations are compatible with the Majorana conditions $\varepsilon^\dagger \Gamma^0 = \varepsilon^{T} C$ only if we require that

$$
\begin{align*}
(\eta^+) \Gamma^0 &= (\eta^+)^T C \\
(T^A_{RST})^* &= T^A_{RST}
\end{align*}
$$

To see that we use $(\Gamma^M)^\dagger = \Gamma^0 \Gamma^M \Gamma^0$.

### B.2 The Euclidean $(2, 2)$ tensor multiplet

We change to Euclidean signature $SO(6) \times SO(5) \subset SO(6,5)$ by defining the Dirac conjugate as $\bar{\varepsilon} = \varepsilon^T \Gamma$. We impose the 11d Majorana condition $\bar{\varepsilon} = \varepsilon^{T} C$ with that new Dirac conjugate. In terms of Weyl components, this reads $(\varepsilon^\pm)^\dagger \Gamma = (\varepsilon^\mp)^T C$ and we can not impose the 6d Weyl condition. We have the Euclidean nonchiral $(2,2)$ multiplet

$$
\delta \phi^{\pm A} = i \varepsilon^{\pm 1} \Gamma^A \psi^{\pm}
$$
\[
\delta B_{MN} = \varepsilon^{\dagger} \Gamma_{MN} \psi
\]
\[
\delta \psi^\pm = \frac{i}{12} \Gamma^{MNP} \varepsilon^{\dagger} H_{MNP} + \Gamma^M \Gamma^A \varepsilon^{\dagger} D_M \phi^{\pm A} - 4 \Gamma^A \eta^{\pm} \phi^{\pm A}
\]
where we have removed a factor of \(i\) from the variation \(\delta B_{MN}^\pm\) to make the variation hermitian by using the Majorana condition. We also multiplied \(H_{MNP}\) by a factor of \(i\) in \(\delta \psi\) to make the variation compatible with the Majorana condition with \(H_{MNP}\) real. Because of this \(i\), there is a change of sign in the kinetic term for the tensor field and the Lagrangian is
\[
L_{(2,2)} = \frac{1}{24} H_{MNP}^2 + L^+ + L^-
\]
where the matter part looks identical with that of the Lorentzian \((2,0) + (0,2)\) theory if we write the Dirac conjugates as \(\psi^T C\). But if we use the new Majorana condition then it will look like
\[
L^\pm = -\frac{1}{2} (D_M \phi^{\pm A})^2 - \frac{1}{2} \mu^{AB} \phi^{\pm A} \phi^{\pm B}
\]
\[
+ \frac{i}{2} \psi^{\dagger} \Gamma^M D_M \psi^\pm - \frac{1}{8} \psi^{\dagger} \Gamma^M \Gamma^{MNP} \Gamma^A \psi^\pm T^{\pm A}_{MNP}
\]
where we also multiplied \(T^A_{MNP}\) with a factor of \(i\), which is in line with having the same factor of \(i\) multiplying \(H_{MNP}\). We may notice that the chiral parts \(H_{MNP}^\pm\) will be complex fields, but the sum, \(H_{MNP} = H^+_{MNP} + H^-_{MNP}\) will be real. This observation may be used for holomorphic factorization of the partition function in Euclidean signature. We get back to the \((2,0)\) tensor multiplet by replacing \(\psi^{-\dagger} \Gamma\) with \(\psi^{+T} C\). Once we have done that replacement, we drop the 11d Majorana condition and impose the Weyl projection \(\psi^- = 0\). Then \(L^+\) will become identical with \(L_{(2,0)}\) (although we are now in signature \(SO(6,5)\)). We can do the corresponding replacements for the \((0,2)\) theory. These two supersymmetries do not mix once we formulate the theory in terms of \(\psi^T C\). The supersymmetry parameters satisfy
\[
D_M \varepsilon^{\dagger} = \Gamma_M \eta^{\pm} - \frac{i}{8} \Gamma^A \Gamma^{RST} \Gamma_M \varepsilon^{\dagger} T^{\pm A}_{RST}
\]
where consistency with the Majorana condition implies that
\[
\eta^{\pm\dagger} \Gamma = -\eta^{\pm T} C
\]
\[
(T^{\pm A}_{MNP})^\dagger = T^{\pm A}_{MNP}
\]

C The Majorana condition in various dimensions

The 11d Majorana condition is
\[
\bar{\psi} = \psi^T C
\]
where we define $\tilde{\psi} = \psi^\dagger \Gamma^t$. We will represent the 11d gamma matrices as

$$\Gamma^t = i(\sigma^2)^A B \delta^\alpha_\beta \delta^{\dot{\alpha}}_{\dot{\beta}}$$
$$\Gamma^m = (\sigma^1)^A B (\gamma^m)^\alpha_\beta \delta^{\dot{\alpha}}_{\dot{\beta}}$$
$$\Gamma^A = (\sigma^3)^A B \delta^\alpha_\beta (r^A)^{\dot{\alpha}}_{\dot{\beta}}$$

The charge conjugation matrix is

$$C = \varepsilon_{AB} C_{\alpha\beta} C^{\dot{\alpha}\dot{\beta}}$$

Hence the 11d Majorana condition is

$$(\psi^{A\alpha\dot{\alpha}})^\ast i(\sigma^2)^A_B = \psi^{B\beta\dot{\beta}} \varepsilon_{BAC} C_{\beta\alpha} C^{\dot{\beta}\dot{\alpha}}$$

The 6d chirality matrix is

$$\Gamma = (\sigma^3)^A_B \delta^\alpha_\beta \delta^{\dot{\alpha}}_{\dot{\beta}}$$

So if we define $\varepsilon_{+-} = 1$, then we find

$$(\psi^{+\alpha\dot{\alpha}})^\ast = C_{\alpha\beta} C^{\dot{\alpha}\dot{\beta}} \psi^{\beta\dot{\beta}}$$
$$(\varepsilon^{-\alpha\dot{\alpha}})^\ast = C_{\alpha\beta} C^{\dot{\alpha}\dot{\beta}} \varepsilon^{\beta\dot{\beta}}$$

If we reduce to 5d then we have the spinor zero modes that satisfy the above Majorana condition, but the chirality has lost its significance so we choose to not display it when we work in 5d language, so instead of writing $\psi^{+\alpha\dot{\alpha}}$, we will just write $\psi^{\alpha\dot{\alpha}}$ when this is a 5d spinor.

From

$$D_M \varepsilon = \Gamma_M \eta$$

we get

$$D_M \varepsilon^\dagger \Gamma^t = -\eta^\dagger \Gamma^t \Gamma_M$$
$$D_M \varepsilon^T C = -\eta^T C \Gamma_M$$

Applying the Majorana condition on the left-hand side of the first equation, we get

$$D_M \varepsilon^T C = -\eta^\dagger \Gamma^t \Gamma_M$$

and by identifying this with the right hand side of the second equation, we conclude that

$$\eta^\dagger \Gamma^t = \eta^T C$$
D Metric and Kahler form on \( \mathbb{C}P^2 \)

Here we follow [8], [1] and obtain the explicit form of the metric and of the Kahler form on \( \mathbb{C}P^2 \). We begin by defining \( S^5 \) as a sphere that is embedded in \( \mathbb{C}^3 \)

\[
r^2 = |Z^0|^2 + |Z^1|^2 + |Z^2|^2
\]

with the ambient flat space metric

\[
ds^2 = |dZ^0|^2 + |dZ^1|^2 + |dZ^2|^2
\]

We define inhomogeneous coordinates

\[
\zeta^1 = \frac{Z^1}{Z^0} \\
\zeta^2 = \frac{Z^2}{Z^0}
\]

and put

\[
Z^0 = \rho e^{iy}
\]

where

\[
\rho^2 = \frac{r^2}{1 + \sum_{a=1,2} |\zeta^a|^2}
\]

and

\[
y \sim y + 2\pi
\]

We then get the metric on \( S^5 \) as

\[
ds^2 = r^2 \left( (dy + V)^2 + \frac{d\zeta^a d\bar{\zeta}^a}{1 + \sum_a |\zeta^a|^2} - \frac{\zeta^a \bar{\zeta}^b d\zeta^a d\bar{\zeta}^b}{(1 + \sum_a |\zeta^a|^2)^2} \right)
\]

where

\[
V = \frac{i}{2 (1 + \sum_a |\zeta^a|^2)} (\zeta^a d\bar{\zeta}^a - \bar{\zeta}^a d\zeta^a)
\]

If we parametrize

\[
\zeta^1 = f(\chi, \psi) \cos \frac{\theta}{2} e^{i\frac{\psi}{2}} \\
\zeta^2 = f(\chi, \psi) \sin \frac{\theta}{2} e^{-i\frac{\psi}{2}}
\]
where
\[ f(\chi, \psi) = \tan \chi e^{i\psi} \]
then we get
\[ ds^2 = r^2 (dy + V)^2 + ds_{\mathbb{CP}^2}^2 \]
where
\[ V = \frac{1}{2} \sin^2 \chi \sigma_3 \]
\[ ds_{\mathbb{CP}^2}^2 = r^2 \left( d\chi^2 + \frac{1}{4} \sin^2 \chi \left( \sigma_1^2 + \sigma_2^2 + \cos^2 \chi \sigma_3^2 \right) \right) \]
and
\[ \sigma_1 = \sin \theta \cos \psi d\phi - \sin \psi d\theta \]
\[ \sigma_2 = \sin \theta \sin \psi d\phi + \cos \psi d\theta \]
\[ \sigma_3 = d\psi + \cos \theta d\phi \]
for which we find that
\[ d\sigma_3 = \sigma_1 \wedge \sigma_2 \]
and cyclically related relations. We define \( \tan \chi \geq 0 \) so that \( \chi \in [0, \pi/2] \) and we make the identification
\[ \psi \sim \psi + 4\pi \]
We define the vielbein
\[ e^4 = r d\chi \]
\[ e^1 = \frac{r}{3} \sin \chi \sigma_1 \]
\[ e^2 = \frac{r}{3} \cos \chi \sigma_2 \]
\[ e^3 = \frac{r}{2} \sin \chi \cos \chi \sigma_3 \]
We then find that
\[ F = dV = \frac{2}{r^2} J \]
where
\[ J = e^4 \wedge e^3 + e^1 \wedge e^2 \]
is the Kahler form.

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E The vielbein components in lightcone coordinates

In lightcone coordinates on \( \mathbb{R} \times S^5 \), the vielbein has the components

\[
\begin{pmatrix}
  e^\pm_+ & e^\pm_- & e^\pm_i \\
  e^-_+ & e^-_- & e^-_i \\
  e^i_+ & e^i_- & e^i_i
\end{pmatrix}
= \begin{pmatrix}
  1 & 0 & \frac{r}{\sqrt{2}} \kappa_i \\
  0 & 1 & -\frac{r}{\sqrt{2}} \kappa_i \\
  0 & 0 & E^i_i
\end{pmatrix}
\]

and its inverse is

\[
\begin{pmatrix}
  e^+_+ & e^+_+ & e^+_i \\
  e^-_+ & e^-_- & e^-_i \\
  e^i_+ & e^i_- & e^i_i
\end{pmatrix}
= \begin{pmatrix}
  1 & 0 & -\frac{r}{\sqrt{2}} \kappa_i \\
  0 & 1 & \frac{r}{\sqrt{2}} \kappa_i \\
  0 & 0 & E^i_i
\end{pmatrix}
\]

The metric is

\[
ds^2 = -2e^+ e^- + e^i e^i
\]

and \( E^i \) denotes the vielbein on \( \mathbb{C}P^2 \). Since \( \kappa_i \) is a Killing vector, we have the important identity

\[
\kappa^i w_{ij} = 0
\]

where \( w_{ij} \) is the Kahler form. Here \( \kappa \) was denoted as \( V \) and \( w = d\kappa \) was denoted as \( J \) in appendix D.

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