A note on submanifolds and mappings in
generalized complex geometry

by

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ABSTRACT. In generalized complex geometry, we revisit linear subspaces and submanifolds
that have an induced generalized complex structure. We give an expression of the induced
structure that allows us to deduce a smoothness criteria, we dualize the results to submer-
sions and we make a few comments on generalized complex mappings. Then, we discuss
submanifolds of generalized Kähler manifolds that have an induced generalized Kähler struc-
ture. These turn out to be the common invariant submanifolds of the two classical complex
structures of the generalized Kähler manifold.

1 Introduction

Induced generalized complex structures of submanifolds were introduced and studied in [3] and the study was continued in [2] [6] [12], etc. In Section 2 of this note, we recall some of the results, we give a general expression of the
induced structure and derive a smoothness criteria. The results are dualized to submersions and a few remarks on more general mappings are made, including a proposed, new definition of generalized complex mappings. In Section 3, we
consider the generalized Hermitian and generalized Kähler case, which is known to be given by a 1-1 correspondence with quadruples \((\gamma, \psi, J_\pm)\), where \(\gamma\) is a
metric, \(\psi\) is a 2-form and \(J_\pm\) are classical \(\gamma\)-compatible complex structures.
Then, we define the notion of induced structure in a way that is compatible
with the generalized complex case and is such that existence of the induced
structure is equivalent to \(J_\pm\)-invariance. We work in the \(C^\infty\)-category and use
the classical notation of Differential Geometry [8].

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2 Submanifolds with induced generalized complex structure

Let $M$ be an almost generalized complex manifold \cite{7} with the structure $J \in \text{End}(TM)$, where $TM = TM \oplus T^*M$, $J^2 = -\text{Id}$ and $J$ is skew-symmetric with respect to the zero-signature metric

$$g((X, \alpha), (Y, \mu)) = \frac{1}{2}(\alpha(Y) + \mu(X)), \quad X, Y \in V, \alpha, \mu \in V^*.$$  

We will use the representation \cite{7,12}

$$J \begin{pmatrix} X \\ \alpha \end{pmatrix} = \begin{pmatrix} A & \sharp_\pi \\ \flat_\sigma & -A^* \end{pmatrix} \begin{pmatrix} X \\ \alpha \end{pmatrix},$$  \hspace{1cm} (2.1)

where $A \in \text{End}(TM), \pi \in \wedge^2 TM, \sigma \in \wedge^2 T^*M$ and $\ast$ denotes transposition. The conditions satisfied by $J$ are equivalent to

$$A^2 + \sharp_\pi \circ \flat_\sigma = -\text{Id}, \quad \sharp_\pi \circ A^* = A \circ \sharp_\pi, \quad \flat_\sigma \circ A = A^* \circ \flat_\sigma. \hspace{1cm} (2.2)$$

The structure $J$ is equivalent with its $i$-eigenbundle $L$, provided that the latter is maximally $g$-isotropic (Dirac) and $L \cap \bar{L} = 0$ (bar denotes complex conjugation). With (2.1), we have

$$L = \{(X - i(AX + \sharp_\pi \alpha), \alpha - i(\flat_\sigma X - \alpha \circ A))\}. \hspace{1cm} (2.3)$$

The generalized almost complex manifold $M$ is integrable, equivalently, it is a generalized complex manifold, if the space of cross sections $\Gamma L$ is closed under the Dirac bracket \cite{7}.

Below, we will use the same notation in the linear case, where the bundle $TM$ is replaced by a finite-dimensional vector space $W$ and $W = W \oplus W^*$. Then, $(W, J)$ is a linear generalized complex space.

**Definition 2.1.** \cite{3} A subspace $\iota_* : V \subseteq W$ of a linear generalized complex space is a **subspace with induced structure** or **generalized complex subspace** if the pullback space $\bar{L} = \iota^* L$ defines an induced linear generalized complex structure $\bar{J}$ on $V$. A submanifold $\iota : N \hookrightarrow M$ of an almost generalized complex manifold $(M, J)$ is a **submanifold with induced structure** $\bar{J}$ (generalized complex submanifold) if, $\forall x \in N$, $\iota_{*x} : T_x N \subseteq (T_{\iota(x)}M, J_x)$ is a subspace with induced structure $\bar{J}_x$ and the field of induced structures is a smooth bundle endomorphism $\bar{J}$ of $TN$.

We recall the definition

$$\iota^* L = \{(X, \iota^* \eta) \mid (\iota_* X, \eta) \in L, \ X \in V, \eta \in W^*\},$$

where upper star denotes the transposed of lower star. It applies to any mapping and subspace, it sends Dirac subspaces to Dirac subspaces (e.g., \cite{3}) and we have added an arrow to the notation to avoid confusion with set-theoretic image. The
existence of the induced structure is equivalent to $\tilde{\nu}^*L \cap \tilde{\nu}^*\bar{L} = 0$ and in the case of submanifolds one must add the smoothness condition. The definition and formula \eqref{2.3} yield

$$
\tilde{\nu}^*L = \{(X + iX', \iota^*\eta - it^*(\tilde{\nu}_sX - \eta \circ A)) / X, X' \in V, \eta \in W^*, \iota_sX' = -A\iota_sX - \tilde{\nu}_\pi\eta\}.
$$

(2.4)

For the subspace $\iota_s : V \subseteq W$ we identify $V \approx im\iota_s$, $V^* \approx W^*/ker\iota^*$, ker $\iota^*$ $\approx$ ann $V$, $V \approx B/ann V$,

where $B = V \oplus W^*$. The last quotient exists since $ker\iota^* = B^\perp \subseteq B$ and we will use the projection $s : B \rightarrow B/ker\iota^*$ \cite{2}.

In \cite{12} it was shown that the subspace $V \subseteq (W, J)$ has the induced structure iff

$$
V \cap \tilde{\pi}^*(ann V) = 0, \text{ } A(V) \subseteq V + im\tilde{\pi}^*
$$

and, then, one also has

$$
V + im\tilde{\pi}^* = V \oplus \tilde{\pi}^*(ann V).
$$

The meaning of the first condition \eqref{2.5} is that $V$ is a Poisson-Dirac subspace of $W$, i.e., $\tilde{\pi}^*(graph\tilde{\pi}^*)$ is the graph of a bivector $\tilde{\pi} \in \wedge^2V$, which may be calculated as follows \cite{5}. The first condition is equivalent to

$$
ann V + [ann\tilde{\pi}^*(ann V)] = W^* \quad (ann\tilde{\pi}^*(ann V) = (ann V)^\perp).
$$

(2.7)

By \eqref{2.7}, ann $V$ has a $\pi$-orthogonal complement $\Pi$ in $W^*$ and $V^* \approx \Pi$, $\tilde{\pi} = \pi|_\Pi$.

The classical tensors of the induced structure are $\tilde{\pi}, \tilde{A} = pr_V \circ A, \tilde{\nu}_\beta = pr_\Pi \circ \nu_\sigma$ \cite{12}.

In \cite{2} it was shown that $V$ is a generalized complex subspace of $(W, J)$ iff one of the following equivalent conditions holds: (a) $B = B \cap JB + ann V$, (b) $JB \subseteq B + J(ann V)$, (c) $(JB)^\perp \cap B \subseteq ann V$.

**Example 2.1.** Let $V$ be a subspace such that $JB \subseteq B$, equivalently, since $ann V = B^\perp$, $J(ann V) \subseteq ann V$ ($V$ is a totally invariant subspace or has an invariant conormal space) \cite{5, 6}. Then, $AV \subseteq V, im\tilde{\pi}^* \subseteq V$ and \eqref{2.5} holds. For a second example, notice that if $\sigma$ is non degenerate, only the first condition \eqref{2.5} is needed. In particular, if $(W, \varpi)$ is a symplectic space and $W = V \oplus V^\perp$, then, $V$ has a generalized complex structure induced by that of $W$, where $A = 0, \sigma = \varpi, \pi = -\varpi^{-1}$, as well as by any $B$-field transform of the former \cite{3, 12}. After the transformation, we get $A = -\varpi^{-1} \circ \nu_B$ and, since it is easy to construct a 2-form $B$ such that $\varpi^{-1} \circ \nu_B(V) \not\subseteq V$, we have an example where a subspace with induced structure is not $A$-invariant. Finally, assume that the generalized complex space $(W, J)$ is endowed with a $\pi$-preserving linear involution $\phi_s (\phi_s^2 = Id)$ such that $A(\phi_sX) - \phi_s(AX) \in im\tilde{\pi}^*, \forall X \in W$. Then, it is easy to check \eqref{2.4} for the subspace $V = \{X + \phi_sX / X \in W\}$ of the fixed vectors of $\phi_s$, which, therefore, has the induced generalized complex structure.
Furthermore, since \( \text{ann} \) means that, if we apply \( \tilde{\mathcal{J}} \) field on \( N \times X \), cross sections again. We use formula (2.9), where \( X \) is a subbundle of \( \text{ann} TN \). This isomorphism tells us that the smooth local vector field \( \tilde{\mathcal{J}} \) may be defined with a smooth form \( \nu \in \text{ann} TN \) and, then, (2.9) shows the smoothness of \( \tilde{\mathcal{J}} \).

**Proposition 2.1.** Let \( N \subseteq (M, \mathcal{J}) \) be a submanifold of a generalized almost complex manifold such that, \( \forall x \in M \), the subspace \( T_x N \subseteq T_x M \) has the induced structure \( \mathcal{J}_x \). Then, if \( TN + \text{im} \pi_\pi \) is a subbundle of \( TN \), the field of induced structures is smooth.

**Proof.** For convenience, we shall denote pairs like \((X, \alpha)\) by calligraphic letters \( \mathcal{X}, \mathcal{Y}, \) etc. We come back to linear subspaces \( V \subseteq (W, \mathcal{J}) \) and look for a general expression of the induced structure \( \mathcal{J} \). Property (a) mentioned above yields \( V = s(B \cap \mathcal{J}B) \), therefore, the elements of \( V \) may be written as \( s\mathcal{X}, \) where \( \mathcal{X} \in \mathcal{W} \) is not unique and \( \exists Y \in B \cap (\mathcal{J}B) \), equivalently, \( Y \in B \) and \( Y \in \mathcal{J}B \), such that \( s\mathcal{X} = sY \). Accordingly, we get

\[
\tilde{\mathcal{J}}(s\mathcal{X}) = \tilde{\mathcal{J}}(sY) = \frac{1}{2}\tilde{\mathcal{J}}[s(Y - i\mathcal{J}Y) + s(Y + i\mathcal{J}Y)] = \frac{1}{2}[s(Y - i\mathcal{J}Y) - s(Y + i\mathcal{J}Y)] = s(\mathcal{J}Y),
\]

(2.8)

where the third equality holds because the terms belong to the \( \pm i \)-eigenspaces of \( \tilde{\mathcal{J}} \) (check that \( \mathcal{L}^*L = s(L \cap B) \) and use \( s\mathcal{Y}, \mathcal{J}Y \in B \)).

For \( \mathcal{X} = (X, \alpha) \in B \), a corresponding \( \mathcal{Y} \) may be obtained as follows. In view of the second relation (2.5) and of (2.6), there exists a decomposition \( AX = X' + \pi_\pi \nu, \) where \( X' \in V, \nu \in \text{ann} V \). On the other hand, (2.7) yields a decomposition \( \alpha = \eta + \theta, \) where \( \eta \in \text{ann} V, \theta \in \text{ann} \pi_\pi (\text{ann} V) \), hence, \( \pi_\pi \theta \in V \).

Using these decompositions and the expression (2.1), we see that

\[
\text{pr}_V[\mathcal{J}(X, \theta - \nu)] = AX - \pi_\pi \nu + \pi_\pi \theta = X' + \pi_\pi \theta \in V,
\]

therefore, \( \mathcal{J}(X, \theta - \nu) \in B \) and \( (X, \theta - \nu) \in B \cap (\mathcal{J}B) \). Since \( \eta, \nu \in \text{ann} V \), we have \( s(X, \alpha) = s(X, \theta - \nu) \) and it follows that we may take \( \mathcal{Y} = (X, \theta - \nu) \) and the induced structure is given by

\[
\tilde{\mathcal{J}}(s(X, \alpha)) = s(\mathcal{J}(X, \theta - \nu)).
\]

(2.9)

Now, we go to the manifolds \( N \subseteq M \) of the proposition. Smoothness of \( \tilde{\mathcal{J}} \) means that, if we apply \( \tilde{\mathcal{J}} \) to smooth, local, cross sections of \( TN \) we get smooth cross sections again. We use formula (2.9), where \( X \) is a smooth, local vector field on \( N \) and \( \alpha \) is a local 1-form on \( M \). Then, we may assume that the terms \( X', \pi_\pi \nu, \eta, \theta \) of the corresponding decompositions used to get (2.9) are smooth. Furthermore, since \( \text{ann}(\text{im} \pi_\pi) = \ker \pi_\pi \), if \( TN + \text{im} \pi_\pi \) is a subbundle of \( TN \),

\[
\text{ann}(TN + \text{im} \pi_\pi) = (\text{ann} TN) \cap (\ker \pi_\pi)
\]

is a subbundle of \( \text{ann} TN \) and \( \pi_\pi(\text{ann} TN) \approx \text{ann} TN/[(\text{ann} TN) \cap (\ker \pi_\pi)] \). This isomorphism tells us that the smooth local vector field \( \pi_\pi \nu \) may be defined with a smooth form \( \nu \in \text{ann} TN \) and, then, (2.9) shows the smoothness of \( \tilde{\mathcal{J}} \). \( \square \)
**Remark 2.1.** Another expression of \( \tilde{\mathcal{J}} \) follows from property (b) which yields decompositions

\[
\mathcal{J}X = X' + \mathcal{J}\eta_0, \ X \in B, X' \in B, \eta_0 \in \text{ann} V.
\]

Then, \( \mathcal{V} = X - \eta_0 \in B \cap (\mathcal{J}B) \) and \( s\mathcal{V} = sX \). Therefore, by (2.8),

\[
\tilde{\mathcal{J}}(sX) = s(\mathcal{J}(X - \eta_0)).
\]

In the remaining part of this section we will discuss other types of mappings.

**Definition 2.2.** The linear mapping \( f^*: V \rightarrow (W, \mathcal{J}) \) is **backward regular** if \( \tilde{\mathcal{J}}^*L \cap \tilde{\mathcal{J}}^*\overline{L} = (\ker f^*)^c \) (which is the smallest possible intersection; the upper index \( c \) denotes complexification). The smooth mapping \( f : N \rightarrow (M, \mathcal{J}) \) is **backward regular** if \( f_*X \) is backward regular for all \( x \in N \) and the field \( \tilde{\mathcal{J}}^*L \) is a smooth vector bundle.

**Proposition 2.2.** The mapping \( f_* \) is backward regular iff it satisfies the conditions

\[
im f_* \cap (\sharp_\pi (\ker f^*)) = 0, \ A(\im f_*) \subseteq \im f_* + \im \sharp_\pi.
\] (2.10)

**Proof.** Using (2.4) with \( \iota \) replaced by \( f \), we get

\[
\tilde{\mathcal{J}}^*L \cap \tilde{\mathcal{J}}^*\overline{L} = \{(X + iX', f^*\eta - i f^*(\flat_\sigma f_*X - \eta \circ A)) \\
/ f^*(\flat_\sigma f_*X - \eta \circ A) = f^*(\nu \circ A), \\
f_*X' = -Af_*X - \sharp_\pi \eta, \\
f_*X' = Af_*X + \sharp_\pi \eta + 2\sharp_\pi \nu, \nu \in \ker f^* \}
\] (2.11)

The last two conditions are equivalent to

\[
f_*X' = \sharp_\pi \nu, \ Af_*X + \sharp_\pi \eta = -\sharp_\pi \nu.
\] (2.12)

Backward regularity occurs iff the triple of conditions included in (2.11) hold only for \( f_*X' = f_*X = 0, f^*\eta = 0, f^*(\eta \circ A) = 0 \). For \( X' \), the only condition is that given by the first equality (2.12) and it implies \( f_*X' = 0 \) iff the first condition (2.10) holds. Then, if we apply \( A \) to the second condition (2.12) and use (2.2) and the first condition in (2.11), we see that the first condition (2.10) also implies \( f_*X = 0 \). Therefore, after imposing the first condition (2.10), what remains from (2.12) is \( \sharp_\pi \eta = 0 \), which, in view of (2.2), also implies \( \sharp_\pi (\eta \circ A) = 0 \). Now, we see that \( \eta \) satisfies the requirements \( f^*\eta = 0, f^*(\eta \circ A) = 0 \) iff

\[
\ker \sharp_\pi \subseteq \ker f^* \cap \ker(f^* \circ A).
\]

Taking above the \( g \)-orthogonal spaces we get an equivalent condition that exactly is the second condition (2.10). \( \square \)
Now, we shall refer to the dual case of a surjection. For the linear surjection \( f_\ast : V \to W \) (also considered in [3]), we have \( \ker f_\ast = 0 \) and we identify

\[
W = \text{im} \ f_\ast \approx V/\ker f_\ast, \ W^* \approx \text{ann}(\ker f_\ast) \subseteq V^*, \ W = B/\ker f_\ast;
\]

we still have \( B = V \oplus W^* \), but, \( \ker f_\ast \subseteq B \) and there exists a projection \( q : B \to B/\ker f_\ast \). We shall also recall the general definition of the push-forward operation

\[
\tilde{f}_\ast \mathcal{L} = \{ (f_\ast X, \eta) / (X, f^\ast \eta) \in \mathcal{L} \subseteq V, X \in V, \eta \in W^* \},
\]

which also makes sense for general linear mappings and always sends Dirac spaces to Dirac spaces (e.g., [3]).

**Definition 2.3.** The surjection \( f_\ast : (V, J) \to W \) has image with projected structure if \( \tilde{f}_\ast \mathcal{L} \cap (\tilde{f}_\ast \mathcal{L}) = 0 \), where \( \mathcal{L} \) is the \( i \)-eigenspace of \( J \).

From definitions, it follows that the surjection \( f_\ast : V \to W \) has the image with projected structure iff the injection \( f^\ast : W^* \to (V^*, J^*) \) induces a generalized complex structure on the subspace \( W^* \subseteq V^* \). This transposition means that we look at the pairs as \( (\alpha, X) \) instead of \( (X, \alpha) \), which has the effect of interchanging \( \pi \) with \( \sigma \) and making the push-forward \( \tilde{f}_\ast \mathcal{L} \subseteq W \) become the pullback \( \tilde{f}^\ast \mathcal{L} \subseteq W \). If we transpose the existence conditions of the induced structure, we get the following results.

**Proposition 2.3.** 1. The surjection \( f_\ast : (V, J) \to W \) produces a projected generalized complex structure \( \tilde{J} \) on \( W \) iff

\[
W^* \cap \beta_\sigma(\ker f_\ast) = 0, \ A^*(W^*) \subseteq W^* + \text{im} \beta_\sigma,
\]

equivalently,

\[
V = \ker f_\ast + \text{ann}(\beta_\sigma(\ker f_\ast)), \ A^*(W^*) \subseteq W^* \oplus \beta_\sigma(\ker f_\ast).
\]

2. The existence of the projected structure is also characterized by each of the following equivalent conditions (a*) \( B = B \cap (J^* B) + \ker f_\ast \), (b*) \( J^* B \subseteq B + J(\ker f_\ast) \), (c*) \( (J^* B)^\perp \cap B \subseteq \ker f_\ast \).

**Proposition 2.4.** Assume the surjection \( f_\ast : (V, J) \to W \) produces the projected generalized complex structure \( \tilde{J} \). For any pair \( \chi = (X, \alpha) \in B \), use the following decompositions given by Proposition 2.3 and (b*)

\[
A^* \chi = \beta_\sigma Z + \nu, \ X = Z' + U, \ J\chi = X' + J\chi_0,
\]

where \( Z, Z' \in \ker f_\ast, U \in \text{ann}(\beta_\sigma(\ker f_\ast)), \nu \in W^* \) and \( X' \in B, \chi_0 \in \ker f_\ast \). Then,

\[
\tilde{J}(q(X, \alpha)) = q(J(U + Z, \alpha)) (X, \alpha) \in B
\]

and also \( \tilde{J}(qX) = q(X') = q(J(X - \chi_0)) \).
The first condition (2.13) is equivalent to \( f^*(\text{graph} \, \sigma) = \text{graph} \, \bar{\sigma} \) for some 2-form \( \bar{\sigma} \in \wedge^2 W^* \). Then, formula (2.14) implies a decomposition \( V = Q \oplus (\ker f) \) with \( Q \perp_{\sigma} (\ker f) \), whence, we deduce \( f^* \bar{\sigma} = \sigma|_Q \). Furthermore, using (2.15) we obtain the tensors of the projected structure \( \tilde{J} \), namely \( \bar{A} \) deduced above and, with the identification \( W \approx Q, \tilde{A} = pr_Q \circ A, \tilde{\sigma} = pr_Q \circ \bar{\sigma} \).

The conditions for a linear mapping \( f_* : (V, \mathcal{J}) \rightarrow W \) to be forward regular in the sense that \( \tilde{J} \circ L \cap (\tilde{J} \circ L) = (\ker f)^c \) are those ensuring that \( f^*: W^* \rightarrow (V^*, \mathcal{J}^*) \) is backward regular. Thus, they will be obtained by transferring (2.10) to \( f^* \). The result is

\[
im f^* \cap \nu_{\sigma}(\ker f_*) = 0, \quad A^*(\im f^*) \subseteq \im f^* + \im \nu_{\sigma}.
\]

The case of a smooth manifold submersion \( f : (N, \mathcal{J}) \rightarrow M \) cannot be handled in the same way as the case of an immersion since the inverse image of \( f^{-1}(x) \) \((x \in M)\) is not just one point of \( N \). Instead, we may proceed as follows. We will say that the structure \( \mathcal{J} \) is projectable if, \( \forall (X, f^* \eta) \in \Gamma TN, \) where \( \eta \in \Omega^1(M) \) and \( X \in \Gamma_{pr} \), where \( \Gamma_{pr} \) denotes the space of projectable vector fields, one has \( \mathcal{J}(X, f^* \eta) = (X', f^* \eta') \) where \( \eta' \in \Omega^1(M) \) and \( X' \in \Gamma_{pr} N \). It is easy to see that this condition is equivalent to the projectability of the tensors \( A, \sigma, \pi \), i.e., the existence of \( f \)-related tensors \( A, \bar{\sigma}, \bar{\pi} \) on \( M \), and there exists a projected structure \( \tilde{\mathcal{J}} \) such that, \( \forall y \in M, \bar{L}_{\eta} = \tilde{J} |_{s_x L_x} \) whenever \( y = f(x) \).

Now, we shall consider a different issue. In the classical case, the induced structure is characterized by the fact that the immersion is a complex (holomorphic) mapping. The literature on generalized complex geometry contains several attempts to extend this notion \([3, 9, 11]\), which lead to rather restrictive situations and a really good notion of generalized complex mapping may not exist.

We shall refer to the linear case. Then, the case of smooth mappings \( f : N \rightarrow M \) may be handled by applying the linear case definitions to the differential of \( f \).

The liner mapping \( f_* : (V, \mathcal{J}_V) \rightarrow (W, \mathcal{J}_W) \) is generalized complex in the sense of \([3]\) iff the three tensors \( A_V, \pi_V, \sigma_V \) are \( f_* \)-related to \( A_W, \pi_W, \sigma_W \), respectively \([12]\). This condition may not hold for induced structures.

The liner mapping \( f_* : (V, \mathcal{J}_V) \rightarrow (W, \mathcal{J}_W) \) is generalized holomorphic in the sense of \([11]\) if \( f_* (\rho_{pr} L_V) \subseteq \rho_{pr} L_W \) and \( \pi_V, \pi_W \) are \( f \)-related. Calculations show that \( \pi \) exactly is the Poisson structure considered in \([11]\) and that the conditions of \([11]\) are equivalent to

\[
\begin{align*}
\sharp_{\pi_W} &= f_* \circ \sharp_{\pi_V} \circ f^*, \quad f_* (\im \sharp_{\pi_V}) \subseteq \im \sharp_{\pi_W}, \\
\im (f_* \circ A_V - A_W \circ f_*) &\subseteq \im \sharp_{\pi_W}.
\end{align*}
\]  

(2.16)

Again, these conditions may not hold for induced structures.

We leave the study of the definitions given in \([9]\) to the reader. On the other hand, we propose one more definition.
\textbf{Definition 2.4.} Let \( f_* : (V, \mathcal{J}_V) \to (W, \mathcal{J}_W) \) be a linear mapping. Then, \( f_* \) is \textit{generalized complex} if
\begin{equation}
\overline{f_*}^*L_V \subseteq L_W + (\ker f^*)^e, \quad f_*^*L_W \subseteq L_V + (\ker f_*)^e. \tag{2.17}
\end{equation}

By carefully looking at the expression of the bundles included in (2.17), using formula (2.18), it follows that \( f_* \) is generalized complex in the new sense iff, \( \forall X \in V, \eta \in W^* \), the following equivalence holds
\begin{align}
(f^*A_W^* - A_V^*f^*)\eta &= (f^*\sigma_\eta f_* - \sigma_{\eta_1})X \\
\iff (A_W f_* - f_* A_V)X &= (f_* \pi_V f^* - \pi_W \eta). \tag{2.18}
\end{align}

If \( \sigma_1 = 0, \sigma_2 = 0, \pi_1 = 0, \pi_2 = 0 \), condition (2.18) becomes \((f^*A_W^* - A_V^*f^*)\theta = 0 \iff (A_W f_* - f_* A_V)X = 0\). Since the first part of the equivalence holds for \( \theta = 0 \), the second part must hold for any \( X \), hence, we are exactly in the case of a classical complex mapping. By a similar argument, if \( f_* \) is generalized complex and if the condition \( A_W f_* - f_* A_V = 0 \) holds, then, we must also have \( f^*\sigma_W = \sigma_V, f_* \pi_V = \pi_W \) and we are in the case of [4]. In particular, the case \( A_* = A_W = 0 \) shows that the generalized complex mappings between symplectic spaces (seen as generalized complex) are just the symplectic mappings.

If the inclusion \( \iota_* : V \subseteq (W, \mathcal{J}) \) is generalized complex in the new sense, the second relation (2.17) shows that \( V \) is a generalized complex subspace. Conversely, if the inclusion \( \iota_* : V \subseteq (W, \mathcal{J}) \) induces a generalized complex structure \( \mathcal{J} \), then, \( \iota_* \) is a generalized complex mapping. Indeed, the second relation (2.17) obviously holds and we have to check that \( \overline{\iota_*}^*\mathcal{J} \subseteq L + (\ker f^*)^e \). Since \( \iota_* \) is injective, \( (\iota_*X, \eta) \in \overline{\iota_*}^*\overline{\mathcal{J}} \) (\( X \in V, \eta \in W^* \)) iff \( (X, \iota^*\eta) \in \overline{\mathcal{J}}^*L \). The latter condition is equivalent to \( \exists \eta^0 \in (\ker f^*)^e \) such that \( (\iota_*X, \eta + \eta^0) = (Y, \zeta) \in L \). Accordingly, \( (\iota_*X, \eta) = (Y, \zeta) + (0, -\eta^0) \) and we are done.

### 3 Submanifolds with induced generalized Kähler structure

In this section we discuss induced generalized Kähler structures.

A Euclidean metric \( G \) of a space \( W = W \oplus W^* \) is a \textit{generalized metric} of \( W \) if there exists a decomposition \( W = W_+ \oplus W_- \), where \( W_\pm \) are maximal \( g \)-positive and \( g \)-negative subspaces (thus, \( \dim W_\pm = \dim W \)), such that \( W_+ \perp_g W_-, W_+ \perp_G W_-, G|_{W_\pm} = \pm 2g|_{W_\pm} \). Such a metric is equivalent with a \( g \)-symmetric product structure \( H \in \text{End}(W) \), \( H^2 = \text{Id} \). The subspaces \( W_\pm \) are the \pm 1-eigenspaces of \( H \) (thus, \( H \) is a paracomplex structure) and
\[ G((X_1, \alpha_1), (X_2, \alpha_2)) = 2g(H(X_1, \alpha_1), (X_2, \alpha_2)). \]

Since the elements of \( W_\pm \) are not \( g \)-isotropic, we have \( W_\pm \cap W = 0 \), hence \( pr_W : W_\pm \rightarrow W \) are isomorphisms whose inverse will be denoted by \( \tau_\pm : W \rightarrow W_\pm \) and \( \tau_\pm X = (X, \theta_\pm X) \) (\( X \in W \)) where \( \theta_\pm \) are non degenerate, 2-covariant.
tensors of $W$. Furthermore, $g(\tau_+ X, \tau_- X') = 0$ implies $\theta_+(X, X') = -\theta_-(X', X)$ and the two tensors must have the same skew-symmetric part and opposite symmetric parts. I.e., $\theta_\pm = \psi \pm \gamma$ where $\psi \in \wedge^2 W^*$ and

$$
\gamma(X, X') = g(\tau_\pm X, \tau_\pm X') = (1/2) G(\tau_\pm X, \tau_\pm X')
$$

is a positive definite metric on $W$. This shows that generalized metrics $G$ bijectively correspond to pairs $(\gamma, \psi)$ \cite{7}.

The formula

$$(X, \alpha) = (X_+, b_{\psi+\gamma} X_+) + (X_-, b_{\psi-\gamma} X_-), \quad X_\pm = \frac{1}{2} ((Id \pm \varphi)X \pm \sharp \gamma \alpha),$$

where $\varphi = -\sharp \gamma \circ (Id - \varphi^2)$ and $\beta$ is also symmetric and positive definite.

A generalized Hermitian structure of $(W, G)$ is a generalized complex structure $J$ that is $G$-skew-symmetric, equivalently, commutes with $H$. If this happens, $J' = H \circ J$ is a second generalized Hermitian structure of $(W, G)$, which commutes with $J$ and $H = -J \circ J'$. If $J$ is given, $W_\pm^\pm = L_\pm \oplus \bar{L}_\pm$, where the terms are $\pm i$-eigenspaces of the complex structures $J_\pm$ induced by $J$ in $W_\pm$ \cite{7}. It follows that the generalized Hermitian structure is equivalent with a decomposition (see also \cite{2})

$$W^c = L_+ \oplus L_- \oplus \bar{L}_+ \oplus \bar{L}_-$$

such that: (1) $L_+ \oplus L_-, L_+ \oplus \bar{L}_-$ are Dirac subspaces, (2) $L_+ \oplus \bar{L}_+, L_- \oplus \bar{L}_-$ are the complexification of the $\pm 1$-eigenspaces of a product structure $H$ associated to a generalized Euclidean metric $G$. From (1), (2) it follows that the terms of the decomposition are $g$-isotropic subspaces of equal dimensions. The spaces in condition (1) define the generalized complex structures $J \pm, J'$. A space $W$ endowed with a generalized Hermitian structure $(G, J)$ is a generalized Hermitian space.

Here are a few other important facts \cite{7}. The projections of the structures $J_\pm$ to $W$ are complex structures $J_\pm$ such that $(\gamma, J_\pm)$, where $G$ corresponds to the pair $(\gamma, \psi)$, are classical Hermitian structures. Thus, the generalized Hermitian structures are in a bijective correspondence with quadruples $(\gamma, \psi, J_+, -J_-)$. The structure $(G, J')$ corresponds to the quadruple $(\gamma, \psi, J_+, -J_-)$.

Furthermore \cite{13}, if $J$ is given by (2.1), we get

$$J_\pm = pr_{W_\pm} \circ J_\pm \circ \tau_\pm = A + \sharp \gamma \circ b_{\psi+\gamma}, \quad (3.3)$$
which yields the classical tensors of a generalized Hermitian structure:

\[
\begin{align*}
\hat{x}_\pi &= \frac{1}{2} (J_+ - J_-) \circ \hat{x}_\pi, \\
\hat{x}'_\pi &= \frac{1}{2} (J_+ + J_-) \circ \hat{x}_\pi, \\
A &= \frac{1}{2} [J_+ \circ (Id + \varphi) + J_- \circ (Id - \varphi)], \\
A' &= \frac{1}{2} [J_+ \circ (Id + \varphi) - J_- \circ (Id - \varphi)], \\
b_\gamma &= b_\gamma \circ (A \circ \varphi - \varphi \circ A + \hat{x}_\pi \circ b_\beta), \\
b_{\gamma'} &= b_\gamma \circ (A' \circ \varphi - \varphi \circ A' + \hat{x}'_\pi \circ b_\beta).
\end{align*}
\] (3.4)

We may give a classical, Hermitian structure \((\gamma, J_+)\) on a space \(W^{2n}\) by an adapted frame in \(W^c\), i.e., line matrices of complex vectors \(e = (e_1, \ldots, e_n), \bar{e} = (\bar{e}_1, \ldots, \bar{e}_n)\) such that

\[
J_+ e = ie, \quad \gamma(e_i, e_j) = \gamma(\bar{e}_i, \bar{e}_j) = 0, \quad \gamma(e_i, \bar{e}_j) = \delta_{ij},
\] (3.5)

where \(J_+, \gamma\) are extended to \(W^c\) by complex linearity. A corresponding, real, \(\gamma\)-orthonormal basis of \(W\) is given by

\[
w_i = \frac{1}{\sqrt{2}} (e_i + \bar{e}_i), \quad J_+ w_i = \frac{i}{\sqrt{2}} (e_i - \bar{e}_i), \quad i = 1, \ldots, 2n.
\] (3.6)

Then, a second \(\gamma\)-Hermitian structure \(J_-\) will be determined by a frame \((f = e\Theta + \bar{e}\Psi, \bar{f} = e\Theta + \bar{e}\Phi)\), where \(\Theta, \Psi\) are complex \((n, n)\)-matrices such that the following conditions hold (1) \(\text{rank } \begin{pmatrix} \Theta & \bar{\Phi} \\ \Psi & \bar{\Theta} \end{pmatrix} = 2n\), (2) \(\gamma(f_i, f_j) = 0\). Condition (1) shows linear independence and the equality \(\text{span}\{f\} \cap \text{span}\{\bar{f}\} = 0\), therefore, \(J_- f = if\) defines a complex structure on \(W\). Condition (2) is equivalent to \(\Theta^* \Psi + \Psi^* \Theta = 0\) and ensures the compatibility property \(\gamma(J_- X, J_- Y) = \gamma(X, Y)\), \(\forall X, Y \in W\).

Now, we shall refer to the question of induced structures.

**Proposition 3.1.** Any subspace \(V \subseteq (W, G)\), where \(G\) is a generalized Euclidean metric, inherits a naturally induced generalized Euclidean metric.

**Proof.** Let \((\gamma, \psi)\) be the corresponding pair of \(G\). For any linear mapping \(f_* : V \rightarrow W\), we get

\[
\begin{align*}
\overrightarrow{f}(W_\pm) &= \{ (X, f^* \psi_\pm f_* X) / X \in V \}, \\
\overrightarrow{f}(W_+) \cap \overrightarrow{f}(W_-) &= \{ (X, f^* \psi_\pm f_* X) / \gamma(f_* X, f_* X) = 0 \}
\end{align*}
\]

and, since \(\gamma\) is positive definite, we get

\[
\overrightarrow{f}(W_+) \cap \overrightarrow{f}(W_-) = \ker f_*.
\] (3.7)

Thus, if \(f_*\) is an injection, we get \(V = V_+ \oplus V_-\), where \(V_+ = \overrightarrow{f}(W_+), V_- = \overrightarrow{f}(W_-)\), and the decomposition defines the induced metric. We will denote it by \(\overrightarrow{f}^* G\) and it corresponds to the pair \((f^* \gamma, f^* \psi)\). \(\square\)
Accordingly, since the kernel is included in any pullback, we get

\[ \gamma V \text{ characterize the decomposition (3.2) of a generalized Hermitian space iff } \gamma \\text{ show that the spaces } \gamma \]
This result implies the conclusion of the proposition.

Furthermore, for $X \in V$, (3.3) gives

$$2AX = (J_+ + J_-)X + (J_+ - J_-)\varphi X$$

$$= (J_+ + J_-)X + (J_+ - J_-)(pr_V \varphi X) + (J_+ - J_-)(pr_{V^\perp} \varphi X)$$

$$= (J_+ + J_-)X + (J_+ - J_-)(pr_V \varphi X) + \frac{1}{2} \varphi \lambda (pr_{V^\perp} \varphi X).$$

In the case of a bicomplex subspace the first two terms belong to $V$ and we see that the second condition of the first line of (3.11) holds. The second condition in the second line of (3.11) is justified similarly.

Now, for a moment, we denote by $\tilde{\mathcal{J}}, \tilde{\mathcal{J}}'$ the structures induced by $\mathcal{J}, \mathcal{J}'$. These have the $i$-eigenspaces $\tilde{L} = \tilde{f}^*(L_+ \oplus L_-), \tilde{L}' = \tilde{f}^*(L_+ \oplus L_-)$. On the other hand, the generalized complex structures included in the induced generalized Hermitian structure have the $i$-eigenspaces $(\tilde{f}^*L_+) \oplus (\tilde{f}^*L_-), (\tilde{f}^*L'_+) \oplus (\tilde{f}^*L'_-)$. The latter are included in $\tilde{L}, \tilde{L}'$ and have the same dimension. Hence, $\tilde{\mathcal{J}} = \tilde{\mathcal{J}}, \tilde{\mathcal{J}}' = \tilde{\mathcal{J}}'$. \hfill \qed

**Proposition 3.4.** $V$ is a generalized Hermitian subspace of $W$ iff $V$ is a generalized complex subspace of both $(W, \mathcal{J}), (W, \mathcal{J}')$ and $s(B \cap (\mathcal{J}B) \cap (\mathcal{J}'B) \cap (HB)) = V$. 

**Proof.** Recall that $B = V \oplus W^\ast \subseteq W$ and $s : B \to V$ is the projection onto the quotient of $B$ by $\text{ann} V$. Denote $\mathcal{B} = B \cap (\mathcal{J}B) \cap (\mathcal{J}'B) \cap (HB)$. The last condition of the proposition is equivalent to $B = B + \text{ann} V$ and $\mathcal{X} \subseteq \mathcal{B}$ iff $\mathcal{X}, \mathcal{J}\mathcal{X}, \mathcal{J}'\mathcal{X}, H\mathcal{X} \in B$. Accordingly, and since $L_\pm = W_\pm \cap L$ are intersections of eigenspaces, for $\mathcal{X} \subseteq \mathcal{B}$, we get

$$B \cap L_+ \subseteq \mathcal{B}, B \cap L_- \subseteq \mathcal{B}, B \cap \bar{L}_+ \subseteq \mathcal{B}, B \cap \bar{L}_- \subseteq \mathcal{B}$$

and also

$$pr_{L_+} \mathcal{X} = \frac{1}{4} (Id + H)(Id - i\mathcal{J}) \mathcal{X} = \frac{1}{4} (\mathcal{X} - i\mathcal{J}\mathcal{X} - i\mathcal{J}'\mathcal{X} + H\mathcal{X}) \in B \cap L_+$$

as well as similar results for projections of $\mathcal{X}$ on $L_-, \bar{L}_+, \bar{L}_-$. Using these facts, we see that

$$\mathcal{B} = (B \cap L_+) \oplus (B \cap L_-) \oplus (B \cap \bar{L}_+) \oplus (B \cap \bar{L}_-). \quad (3.12)$$

For any subset $U \subseteq W$ we may identify $\tilde{f}^*U = s(B \cap U)$. Accordingly, if we apply $s$ to (3.12) and use $(\tilde{f}^*L) \cap (\tilde{f}^*\bar{L}) = 0, (\tilde{f}^*L') \cap (\tilde{f}^*\bar{L}') = 0$, which follow from the fact that $W$ is a generalized complex subspace for both $\mathcal{J}, \mathcal{J}'$, we get

$$s(\mathcal{B}) = (\tilde{f}^*L_+) \oplus (\tilde{f}^*L_-) \oplus (\tilde{f}^*\bar{L}_+) \oplus (\tilde{f}^*\bar{L}_-). \quad (3.13)$$

This result implies the conclusion of the proposition. \hfill \qed

---

\[\text{Theorem 8.1 of [2] asserts that the condition that follows is ensured by the first condition, but, it seems the proof has an error that we could not overcome.}\]
Remark 3.1. By a similar procedure, one can prove that, if $H$ is the paracomplex structure that corresponds to an arbitrary generalized Euclidean metric $G$, then, $B = B \cap (HB) + \text{ann } V$.

Example 3.1. For any space $(W, G, J, J')$, put $S_+ = (T_+ \cap T_-) + (\bar{T}_+ \cap \bar{T}_-)$, $S_- = (T_+ \cap \bar{T}_-) + (\bar{T}_+ \cap T_-)$, $U = (S_+ + S_-)^{\perp \gamma}$ (obviously, $S_+ \cap S_- = 0$).

By decomposing vectors as sums of eigenvectors of $J_{\pm}$, it follows that $S_{\pm}$ are bicomplex subspaces and the $\gamma$-compatibility of $J_{\pm}$ implies that the same holds for $U$.

Let $W$ be a generalized Hermitian space such that $J_{\pm}$ commute. Then, if $S$ is a $J_+$-invariant subspace of $W$ the subspace $V = S \cap (J_- S)$ is bicomplex. If $V \subseteq (W, G, J, J')$ is $J$-totally invariant, $JB = B$ and $V$ has the induced structure (Example 2.1). Moreover, $V$ is a bicomplex subspace [6]. Indeed, we have, $J'B = H(JB) = HB$ and using Remark 3.1 we get $B = B \cap (HB) + \text{ann } V = B \cap (J'B) + \text{ann } V$, which shows that $V$ is also a $J'$-generalized complex subspace. The last condition of Proposition 3.5 holds too, because of Remark 3.1 and since $B \cap (J'B) \subseteq (J'B) \cap (HB) = B \cap (HB)$.

Finally, it is easy to check that the $B$-field transformation of the generalized Hermitian structure defined by the quadruple $(\gamma, \psi, J_\pm)$ yields the generalized Hermitian structure defined by the quadruple $(\gamma, \psi - B, J_{\pm})$. Since $J_{\pm}$ remain unchanged, we see that a bicomplex subspace remains bicomplex after a $B$-field transformation.

Proposition 3.5. A subspace $V$ of a generalized Hermitian space $W$ is generalized complex with respect to the two structures $J, J'$ of $W$ iff it satisfies the conditions

$$V \cap [(J_+ - J_-)(V^{\perp \gamma})] = 0, \quad V \cap [(J_+ + J_-)(V^{\perp \gamma})] = 0$$

and one of each pair of conditions

$$J_\pm(V) \subseteq V + (J_+ - J_-)(W), \quad J_\pm(V) \subseteq V + (J_+ + J_-)(W).$$

Proof. The expression (3.4) of $\pi, \pi'$ yields

$$\sharp_{\pi}(\text{ann } V) = (J_+ - J_-)(V^{\perp \gamma}), \quad \sharp_{\pi'}(\text{ann } V) = (J_+ + J_-)(V^{\perp \gamma}).$$

Accordingly, the first conditions in the two lines of (3.11) translate into (3.14) and their meaning is that $V$ is a Poisson-Dirac subspace for the bivectors $\pi, \pi'$. If this happens, formula (3.3) allows us to check the equivalence of the second and fourth condition (3.11) with either choice in the pairs (3.15).

Remark 3.2. Formulas (3.4) shows that $\pi, \pi'$ are non degenerate iff $J_+ \pm J_-$ are non degenerate. Then, (3.11) holds iff $V$ is a symplectic subspace for both structures $\pi^{-1}, \pi'^{-1}$. Moreover, in this case, all the conditions (3.15) necessarily hold and $V$ is a generalized subcomplex space of both $(W, J)$ and $(W, J')$. Notice that

$$J_+ - J_- = 2i(pr_{T_+} - pr_{T_-}), \quad J_+ + J_- = 2i(pr_{T_+} - pr_{T_-}).$$
where the projections are defined by the decompositions $W^c = T_+ \oplus \bar{T}_+$, $W^c = T_- \oplus \bar{T}_-$. Hence, $J_\pm \pm J_-$ have kernel zero and image $W$ (are isomorphisms) iff $T_+ \cap T_- = 0, T_+ \cap \bar{T}_- = 0$. Notice also that, if $\psi = 0$, (3.4) imply $A = (J_+ + J_-)/2, A' = (J_+ - J_-)/2$.

Now, let $f_\ast : (V, G, J) \rightarrow W$ be a surjection defined on a generalized Hermitian space with the generalized complex structure $J$ and the metric $G$ equivalent to the paracomplex structure $H$. We will establish the conditions for the pushforward spaces $f_\ast L_\pm$ to define a projected generalized Hermitian structure on $W$.

**Proposition 3.6.** The projected Hermitian structure exists iff $\ker f_\ast$ is a bicomplex subspace of $(V, J_\pm)$.

**Proof.** Obviously, the projected structure exists iff the injection $f^* : W^* \subseteq (V^*, G^*, J^*)$, where the star always denotes the dual object, makes $W^*$ into a generalized Hermitian subspace of $V^*$. By Proposition 3.2, $W^*$ has an induced generalized Hermitian structure iff it is invariant by the complex structures $J_\pm$ and this condition is equivalent to $J_\pm(\ker f_\ast) = \ker f_\ast$.

We give a few more details about the projected structure. Assume that the metric $G$ of $V$ corresponds to the quadruple $(\gamma, \psi, J_\pm)$ and put $Q = (\ker f_\ast)^\perp \gamma$. We may identify $Q$ with $W$ by the isomorphism $f_\ast|_Q : Q \rightarrow W$. By Proposition 3.2, $W^*$ has an induced generalized Hermitian structure iff it is invariant by the complex structures $J_\pm$ and this condition is equivalent to $J_\pm(\ker f_\ast) = \ker f_\ast$.

On the other hand, the projected structure of $W$ is the dual of the structure defined on $W^*$ by the quadruple $(\gamma^*|_Q, \psi^*|_Q, J^*_\pm|_Q)$. The structures of $Q, W$ coincide under the identification $Q \approx W$.

Now, we shall refer to submanifolds $\iota : N \hookrightarrow (M, G, J, J')$, where $M$ has a smooth generalized almost Hermitian structure.

**Definition 3.2.** $N$ is a generalized almost Hermitian submanifold if it has the induced structure at every point $x \in N$ and the differentials $\iota^*|_x$ pull back the subbundles $L_\pm$ of $T^c M$ to subbundles of $T^c N$.

If the two structures $J, J'$ are integrable, $M$ is a generalized Kähler manifold and this integrability condition is equivalent to the closure of the spaces $\Gamma L_\pm$ under the Dirac bracket [7]. Accordingly, the argument used in the generalized complex case [3] may be used again and it follows that a generalized almost Hermitian submanifold of a generalized Kähler manifold is generalized Kähler too.

From the results obtained in the linear case, it follows that $N$ is a generalized almost Hermitian submanifold iff it is a bicomplex submanifold of $M$, i.e., $N$ is invariant by the two structures $J_\pm$. Particularly, formula (3.8) shows that $f^* L_\pm$ are smooth subbundles of $T^c N$. 
Let us also recall that $M$ is integrable iff the classical structures $J_{\pm}$ are integrable and

$$(\nabla X J_{\pm})Y = \pm \frac{1}{2} \iota^*_\gamma [i(X)i(J_{\pm}Y)d\psi - (i(Y)i(X)d\psi) \circ J_{\pm}],$$

(3.16)

where $\nabla$ is the Levi-Civita connection of the metric $\gamma$ of the quadruple $(\gamma, \psi, J_{\pm})$ of $M$. This leads to the following result.

**Proposition 3.7.** A submanifold $N$ of a generalized Kähler manifold $M$ is a generalized Kähler submanifold iff $N$ is a complex submanifold of the two complex manifolds $(M, J_{\pm})$.

**Proof.** We already know that $N$ must be $J_{\pm}$-invariant and, of course, if $J_{\pm}$ are integrable, the induced structures are integrable too. The required supplementary condition (3.10) for $N$ follows by taking the $\gamma$-orthogonal projection of the equality (3.16) for $M$ onto the tangent spaces of $N$. The projection sends the left hand side of (3.16) to the covariant derivative with respect to Levi-Civita connection of the induced metric $\iota^* \gamma$ (see the Gauss-Weingarten equations [8]) and the right hand side to the similar expression for the induced form $\iota^* \psi$.

In order to get a manifold version of Proposition 3.6 we proceed as in the generalized complex case. We consider a submersion $f : (N, J, J', G) \to M$ where the structures $J, J'$ define a generalized almost Hermitian structure of $N$ and are projectable. Then, $H = -J \circ J'$ is projectable too, i.e., it sends smooth cross sections to smooth cross sections. In this case, we obviously get a projection of the generalized almost Hermitian structure of $N$ to $M$. Formulas (3.3) and (3.4) show that the projectability of $J, J', H$ is equivalent to the projectability of the corresponding classical structures $(\gamma, \psi, J_{\pm})$. The latter condition means that $\gamma|_{(\ker f)^*} = \gamma|_{(\ker f)^*}$ are pullbacks of a metric and a form on $M$ and $J_{\pm}|_{(\ker f)^*}$ sends projectable vector fields to projectable vector fields. Then, Proposition 3.6 tells us that a submersion projects an almost generalized Hermitian structure iff $J_{\pm}$ are projectable and preserve the kernel $\ker f_*$. Moreover, if the structure of $N$ is integrable, i.e., $J_{\pm}$ are integrable and (3.10) holds, the projected structure has the same properties. Indeed, the Lie bracket of projectable vector fields is projectable to the corresponding Lie bracket and the Levi-Civita connection of $G$ projects to that of the projected metric (for details, a text on foliations and metrics, e.g., [10], may be consulted). The conclusion is that a submersion sends a projectable generalized Kähler structure to a generalized Kähler structure.

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