PARTITION’S SENSITIVITY FOR MEASURABLE MAPS

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Abstract. We study countable partitions for measurable maps on measure spaces such that for all point \( x \) the set of points with the same itinerary of \( x \) is negligible. We prove that in nonatomic probability spaces every strong generator (Parry, W., *Aperiodic transformations and generators*, J. London Math. Soc. 43 (1968), 191–194) satisfies this property but not conversely. In addition, measurable maps carrying partitions with this property are aperiodic and their corresponding spaces are nonatomic. From this we obtain a characterization of nonsingular countable to one mappings with these partitions on nonatomic Lebesgue probability spaces as those having strong generators. Furthermore, maps carrying these partitions include the ergodic measure-preserving ones with positive entropy on probability spaces (thus extending a result in Cadre, B., Jacob, P., *On pairwise sensitivity*, J. Math. Anal. Appl. 309 (2005), no. 1, 375–382). Some applications are given.

1. Introduction

In this paper we study countable partitions \( P \) for measurable maps \( f : X \to X \) on measure spaces \( (X, \mathcal{B}, \mu) \) such that for all \( x \in X \) the set of points with the same itinerary of \( x \) is negligible. In other words,

\[
\mu\left\{ y \in X : f^n(y) \in P(f^n(x)), \ \forall n \in \mathbb{N} \right\} = 0, \ \forall x \in X,
\]

where \( P(x) \) stands for the element of \( P \) containing \( x \in X \). For simplicity, we call these partitions *measure-sensitive partitions*.

We prove that in a nonatomic probability space every *strong generator* is a measure-sensitive partition but not conversely (results about strong generators can be found in [5], [6], [7], [11], [12] and [13]). We also exhibit examples of measurable maps in nonatomic probability spaces carrying measure-sensitive partitions which are not strong generators. Motivated by these examples we shall study measurable maps on measure spaces carrying measure-sensitive partitions (called *measure-expansive maps* for short). Indeed, we prove that every measure-expansive map is aperiodic and also, in the probabilistic case, that its corresponding space is nonatomic. From this we obtain a characterization of nonsingular countable to one mappings on nonatomic Lebesgue probability spaces as those having strong generators. Furthermore, we prove that every ergodic measure-preserving map with positive entropy is a probability space is measure-expansive (thus extending a result in [4]). As an application we obtain some properties for ergodic measure-preserving maps with positive entropy (c.f. corollaries [2.1] and [2.4]).
2. Statements and proofs

Hereafter the term countable will mean either finite or countably infinite.

A measure space is a triple \((X, \mathcal{B}, \mu)\) where \(X\) is a set, \(\mathcal{B}\) is a \(\sigma\)-algebra of subsets of \(X\) and \(\mu\) is a positive measure in \(\mathcal{B}\). A probability space is one for which \(\mu(X) = 1\).

A measure space is nonatomic if it has no atoms, i.e., measurable sets \(A\) of positive measure satisfying \(\mu(B) \in \{0, \mu(A)\}\) for every measurable set \(B \subset A\). A partition is a disjoint collection \(P\) of nonempty measurable sets whose union is \(X\). We allow \(\mu(\xi) = 0\) for some \(\xi \in P\). If \(f : X \to X\) is measurable and \(k \in \mathbb{N}\) we define \(f^{-k}(P) = \{f^{-k}(\xi) : \xi \in P\}\) which is a (countable) partition if \(P\) is. A strong generator of \(f\) is a countable partition \(P\) for which the smallest \(\sigma\)-algebra of \(\mathcal{B}\) containing \(\bigcup_{k \in \mathbb{N}} f^{-k}(P)\) equals \(\mathcal{B}\) (mod 0) (see [11]).

The result below is the central motivation of this paper.

**Theorem 2.1.** Every strong generator of a measurable map in a nonatomic probability space is a measure-sensitive partition.

**Proof.** Recall that the join of finitely many partitions \(P_0, \cdots, P_n\) is the partition defined by

\[
\bigvee_{k=0}^{n} P_k = \left\{ \bigcap_{k=0}^{n} \xi_k : \xi_k \in P_k, \forall 0 \leq k \leq n \right\}.
\]

Given partitions \(P\) and \(Q\) we write \(P \leq Q\) to mean that each member of \(Q\) is contained in some member of \(P\) (mod 0). A sequence of partitions \(\{P_n : n \in \mathbb{N}\}\) (or simply \(P_n\)) is increasing if \(P_0 \leq P_1 \leq \cdots \leq P_n \leq \cdots\). Certainly

\[
(2.1) \quad P_n = \bigvee_{k=0}^{n} f^{-k}(P), \quad n \in \mathbb{N},
\]

defines an increasing sequence of countable partitions satisfying

\[
P_n(x) = \bigcap_{k=0}^{n} f^{-k}(P(f^k(x))), \quad \forall x \in X.
\]

Since for all \(x \in X\) one has

\[
\{y \in X : f^n(y) \in P(f^n(x)), \quad \forall n \in \mathbb{N}\} = \bigcap_{n=0}^{\infty} f^{-n}(P(f^n(x))) = \bigcap_{n=0}^{\infty} P_n(x),
\]

we obtain that the identity below

\[
(2.2) \quad \lim_{n \to \infty} \sup_{\xi \in P_n} \mu(\xi) = 0
\]

implies (1.1).

Now suppose that \(P\) is a strong generator of a measurable map \(f : X \to X\) in a nonatomic probability space \((X, \mathcal{B}, \mu)\). Then, the sequence \((2.1)\) generates \(\mathcal{B}\) (mod 0). From this and Lemma 5.2 p. 8 in [8] we obtain that the set of all finite unions of elements of these partitions is everywhere dense in the measure algebra associated to \((X, \mathcal{B}, \mu)\). Consequently, Lemma 9.3.3 p. 278 in [3] implies that the sequence \((2.1)\) satisfies \((2.2)\) and then \((1.1)\) holds.

We shall see later in Example 2.4 that the converse of this theorem is false, i.e., there are certain measurable maps in nonatomic probability spaces carrying
measure-sensitive partitions which are not strong generators. These examples motivates the study of measure-sensitive partitions for measurable maps in measure spaces. For this we use the following auxiliary concept motivated by the notion of Lebesgue sequence of partitions (c.f. p. 81 in [8]).

**Definition 2.1.** A measure-sensitive sequence of partitions of \((X, \mathcal{B}, \mu)\) is an increasing sequence of countable partitions \(P_n\) such that \(\mu (\bigcap_{n \in \mathbb{N}} \xi_n) = 0\) for all sequence of measurable sets \(\xi_n\) satisfying \(\xi_n \in P_n, \forall n \in \mathbb{N}\). A measure-sensitive space is a measure space carrying measure-sensitive sequences of partitions.

At first glance we observe that (2.2) is sufficient condition for an increasing sequence \(P_n\) of countable partitions to be measure-sensitive (it is also necessary in probability spaces). On the other hand, the class of measure-sensitive spaces is broad enough to include all nonatomic standard probability spaces. Recall that a standard probability space is a probability space \((X, \mathcal{B}, \mu)\) whose underlying measurable space \((X, \mathcal{B})\) is isomorphic to a Polish space equipped with its Borel \(\sigma\)-algebra (e.g. [1]). Precisely we have the following proposition.

**Proposition 2.1.** All nonatomic standard probability spaces are measure-sensitive.

**Proof.** As is well-known, for every nonatomic standard probability space \((X, \mathcal{B}, \mu)\) there are a measurable subset \(X_0 \subset X\) with \(\mu(X \setminus X_0) = 0\) and a sequence of countable partitions \(Q_n\) of \(X_0\) such that \(\bigcap_{n \in \mathbb{N}} \xi_n\) contains at most one point for every sequence of measurable sets \(\zeta_n\) in \(X_0\) satisfying \(\zeta_n \in Q_n, \forall n \in \mathbb{N}\) (c.f. [8] p. 81). Defining \(P_n = \{X \setminus X_0\} \cup Q_n\) we obtain an increasing sequence of countable partitions of \((X, \mathcal{B}, \mu)\). It suffices to prove that this sequence is measure-sensitive. For this take a fixed (but arbitrary) sequence of measurable sets \(\xi_n\) of \(X\) with \(\xi_n \in P_n\) for all \(n \in \mathbb{N}\). It follows from the definition of \(P_n\) that either \(\xi_n = X \setminus X_0\) for some \(n \in \mathbb{N}\), or, \(\xi_n \in Q_n\) for all \(n \in \mathbb{N}\). Then, the intersection \(\bigcap_{n \in \mathbb{N}} \xi_n\) either is contained in \(X \setminus X_0\) or reduces to a single measurable point. Since both \(X \setminus X_0\) and the measurable points have measure zero (for nonatomic spaces are diffuse [3]) we obtain \(\mu (\bigcap_{n \in \mathbb{N}} \xi_n) = 0\). As \(\xi_n\) is arbitrary we are done. \(\square\)

Although measure-sensitive probability spaces need not be standard we have that all of them are nonatomic. Indeed, we have the following result of later usage.

**Proposition 2.2.** All measure-sensitive probability spaces are nonatomic.

**Proof.** Suppose by contradiction that there is a measure-sensitive probability space \((X, \mathcal{B}, \mu)\) with an atom \(A\). Take a measure-sensitive sequence of partitions \(P_n\). Since \(A\) is an atom one has that \(\forall n \in \mathbb{N} \exists \xi_n \in P_n\) such that \(\mu(A \cap \xi_n) > 0\) (and so \(\mu(A \cap \xi_n) = \mu(A)\)). Notice that \(\mu(\xi_n \cap \xi_{n+1}) > 0\) for, otherwise, \(\mu(A) = \mu(A \cap (\xi_n \cup \xi_{n+1})) = \mu(A \cap \xi_n) + \mu(A \cap \xi_{n+1}) = 2\mu(A)\) which is impossible in probability spaces. Now observe that \(\xi_n \in P_n\) and \(P_n \leq P_{n+1}\), so, there is \(L \subset P_{n+1}\) such that

\[
\mu \left( \xi_n \triangle \bigcup_{\zeta \in L} \zeta \right) = 0.
\]

If \(\xi_{n+1} \cap \left( \bigcup_{\zeta \in L} \zeta \right) = \emptyset\) we would have \(\xi_n \cap \xi_{n+1} = \xi_n \cap \xi_{n+1} \setminus \bigcup_{\zeta \in L} \zeta\) yielding

\[
\mu(\xi_n \cap \xi_{n+1}) = \mu \left( \xi_n \cap \xi_{n+1} \setminus \bigcup_{\zeta \in L} \zeta \right) \leq \mu \left( \xi_n \setminus \bigcup_{\zeta \in L} \zeta \right) = 0.
\]
which is absurd. Hence \( \xi_{n+1} \cap \left( \bigcup_{i \in I} \xi_i \right) \neq \emptyset \) and then \( \xi_{n+1} \in L \) for \( P_{n+1} \) is a partition and \( \xi_{n+1} \in P_{n+1} \). Using (2.3) we obtain \( \xi_{n+1} \subset \xi_n \) (mod 0) so \( A \cap \xi_{n+1} \subset A \cap \xi_n \) (mod 0) for all \( n \in \mathbb{N}^+ \). From this and well-known properties of probability spaces we obtain
\[
\mu \left( A \cap \bigcap_{n \in \mathbb{N}} \xi_n \right) = \mu \left( \bigcap_{n \in \mathbb{N}} (A \cap \xi_n) \right) = \lim_{n \to \infty} \mu(A \cap \xi_n) = \mu(A) > 0.
\]
But \( P_n \) is measure-sensitive and \( \xi_n \in P_n, \forall n \in \mathbb{N} \), so \( \mu \left( \bigcap_{n \in \mathbb{N}} \xi_n \right) = 0 \) yielding \( \mu \left( A \cap \bigcap_{n \in \mathbb{N}} \xi_n \right) = 0 \) which contradicts the above expression. This contradiction yields the proof. \( \square \)

The following equivalence relates both measure-sensitive partitions and measure-sensitive sequences of partitions.

**Lemma 2.1.** The following properties are equivalent for measurable maps \( f : X \to X \) and countable partitions \( P \) on measure spaces \((X, \mathcal{B}, \mu)\):

(i) The sequence \( P_n \) in (2.1) is measure-sensitive.
(ii) The partition \( P \) is measure-sensitive.
(iii) The partition \( P \) satisfies
\[
\mu \{ y \in X : f^n(y) \in P(f^n(x)), \forall n \in \mathbb{N} \} = 0, \quad \forall \mu\text{-a.e. } x \in X.
\]

**Proof.** Previously we state some notation.

Given a partition \( P \) and \( f : X \to X \) measurable we define
\[
P_\infty(x) = \{ y \in X : f^n(y) \in P(f^n(x)), \forall n \in \mathbb{N} \}, \quad \forall x \in X.
\]
Notice that
\[
P_\infty(x) = \bigcap_{n \in \mathbb{N}^+} P_n(x) \tag{2.4}
\]
and
\[
P_n(x) = \bigcap_{i=0}^n f^{-i}(P(f^i(x))) \tag{2.5}
\]
so each \( P_\infty(x) \) is a measurable set. For later use we keep the following identity
\[
\left( \bigcap_{i=0}^n f^{-i}(P) \right)(x) = P_n(x), \quad \forall x \in X. \tag{2.6}
\]
Clearly (i) (resp. (iii)) is equivalent to \( \mu(P_\infty(x)) = 0 \) for every \( x \in X \) (resp. for \( \mu\text{-a.e. } x \in X \)).

First we prove that (i) implies (ii). Suppose that the sequence (2.1) is measure-sensitive and fix \( x \in X \). By (2.4) and (2.5) we have \( P_\infty(x) = \bigcap_{n \in \mathbb{N}} \xi_n \) where \( \xi_n = P_n(x) \in P_n \). As the sequence \( P_n \) is measure-sensitive we obtain
\[
\mu(P_\infty(x)) = \mu \left( \bigcap_{n \in \mathbb{N}} \xi_n \right) = 0 \text{ proving (ii)}.
\]
Conversely, suppose that (ii) holds and let \( \xi_n \) be a sequence of measurable sets with \( \xi_n \in P_n \) for all \( n \). Take \( y \in \bigcap_{n \in \mathbb{N}} \xi_n \). It follows that \( y \in P_n(x) \) for all \( n \in \mathbb{N} \) whence \( y \in P_\infty(x) \) by (2.4). We conclude that \( \bigcap_{n \in \mathbb{N}} \xi_n \subset P_\infty(x) \) therefore \( \mu \left( \bigcap_{n \in \mathbb{N}} \xi_n \right) \leq \mu(P_\infty(x)) = 0 \) proving (i).

To prove that (ii) and (iii) are equivalent we only have to prove that (iii) implies (i). Assume by contradiction that \( P \) satisfies (iii) but not (ii). Since \( \mu \) is a probability and (3) holds the set \( X' = \{ x \in X : \mu(P_\infty(x)) = 0 \} \) has measure one. Since (ii)
does not hold there is \( x \in X \) such that \( \mu(P_\infty(x)) > 0 \). Since \( \mu \) is a probability and \( X' \) has measure one we would have \( P_\infty(x) \cap X' \neq \emptyset \) so there is \( y \in P_\infty(x) \) such that \( \mu(P_\infty(y)) = 0 \). But clearly the collection \( \{P_\infty(x) : x \in X\} \) is a partition (for \( P \) is) so \( P_\infty(x) = P_\infty(y) \) whence \( \mu(P_\infty(x)) = \mu(P_\infty(y)) = 0 \) which is a contradiction. This ends the proof.

Recall that a measurable map \( f : X \to X \) is measure-preserving if 
\[
\mu \circ f^{-1} = \mu.
\]
Moreover, it is ergodic if every measurable invariant set \( A \) (i.e. \( A = f^{-1}(A) \) (mod 0)) satisfies either \( \mu(A) = 0 \) or \( \mu(X \setminus A) = 0 \); and totally ergodic if \( f^n \) is ergodic for all \( n \in \mathbb{N}^+ \).

**Example 2.1.** If \( f \) is a totally ergodic measure-preserving map of a probability space, then every countable partition \( P \) with \( 0 < \mu(\xi) < 1 \) for some \( \xi \in P \) is measure-sensitive with respect to \( f \) (this follows from the equivalence (iii) in Lemma 2.1 and Lemma 1.1 p. 208 in [8]).

Hereafter we fix a measure space \((X, B, \mu)\) and a measurable map \( f : X \to X \). We shall not assume that \( f \) is measure-preserving unless otherwise stated.

Using the Kolmogorov-Sinai’s entropy we obtain sufficient conditions for the measure-sensitivity of a given partition. Recall that the entropy of a finite partition \( P \) is defined by
\[
H(P) = - \sum_{\xi \in P} \mu(\xi) \log \mu(\xi).
\]

The entropy of a finite partition \( P \) with respect to a measure-preserving map \( f \) is defined by
\[
h(f, P) = \lim_{n \to \infty} \frac{1}{n} H(P_{n-1}).
\]

Then, we have the following lemma.

**Lemma 2.2.** A finite partition with finite positive entropy of an ergodic measure-preserving map in a probability space is measure-sensitive.

**Proof.** Since the map \( f \) (say) is ergodic, the Shannon-McMillan-Breiman Theorem (c.f. [8] p. 209) implies that the partition \( P \) (say) satisfies
\[
\lim_{n \to \infty} \frac{1}{n} \log(\mu(P_n(x))) = h(f, P), \quad \mu\text{-a.e. } x \in X,
\]
where \( P_n(x) \) is as in (2.5). On the other hand, \( P_{n+1}(x) \subset P_n(x) \) for all \( n \) so (2.4) implies
\[
\mu(P_\infty(x)) = \lim_{n \to \infty} \mu(P_n(x)), \quad \forall x \in X.
\]

But \( h(f, P) > 0 \) so (2.7) implies that \( \mu(P_n(x)) \) goes to zero for \( \mu\text{-a.e. } x \in X \). This together with (2.8) implies that \( P \) satisfy the equivalence (iii) in Lemma 2.1 so \( P \) is measure-sensitive.

In the sequel we study measurable maps carrying measure-sensitive partitions (we call them measure-expansive maps for short). It follows at once from Lemma 2.1 that these maps only exist on measure-sensitive spaces. Consequently we obtain the following result from Proposition 2.2.

**Theorem 2.2.** A probability space carrying measure-expansive maps is nonatomic.

A simple but useful example is as follows.
Example 2.2. The irrational rotations in the circle are measure-expansive maps with respect to the Lebesgue measure. This follows from Example 2.1 since all such maps are measure-preserving and totally ergodic.

On the other hand, it is not difficult to find examples of measure-expansive measure-preserving maps which are not ergodic. These examples together with Example 2.2 suggest the question whether an ergodic measure-preserving map is measure-expansive. However, the answer is negative by the following example.

Example 2.3. A measure-sensitive partition has necessarily more than one element. Consequently, if \( B = \{ X, \emptyset \} \) then no map is measure-expansive although they are all ergodic measure-preserving.

Despite of this we still can give conditions for the measure-expansivity of ergodic measure-preserving maps as follows.

Recall that the entropy (c.f. [8], [15]) of \( f \) is defined by

\[
 h(f) = \sup \{ h(f, Q) : Q \text{ is a finite partition of } X \}.
\]

We obtain a natural generalization of Theorem 3.1 in [4].

Theorem 2.3. Ergodic measure-preserving maps with positive entropy in probability spaces are measure-expansive.

Proof. Let \( f \) be one of such a map with entropy \( h(f) > 0 \). We can assume that \( h(f) < \infty \). It follows that there is a finite partition \( Q \) with \( 0 < h(f, Q) < \infty \). Taking \( P = \bigvee_{i=0}^{n-1} f^{-i}(Q) \) with \( n \) large we obtain a finite partition with finite positive entropy since \( h(f, P) = h(f, Q) > 0 \). It follows that \( P \) is measure-sensitive by Lemma 2.2 whence \( f \) is measure-expansive by definition. \( \square \)

A first consequence of the above result is that the converse of Theorem 2.1 is false.

Example 2.4. Let \( f : X \to X \) be a homeomorphism with positive topological entropy of a compact metric space \( X \). By the variational principle [15] there is a Borel probability measures \( \mu \) with respect to which \( f \) is an ergodic measure-preserving map with positive entropy. Then, by Theorem 2.3, \( f \) carries a measure-sensitive partition which, by Corollary 4.18.1 in [15], cannot be a strong generator. Consequently, there are measurable maps in certain nonatomic probability spaces carrying measure-sensitive partitions which are not strong generators.

On the other hand, it is also false that ergodic measure-expansive measure-preserving maps on probability spaces have positive entropy. The counterexamples are precisely the irrational circle rotations (c.f. Example 2.2). Theorems 2.2 and 2.3 imply the probably well-known result below.

Corollary 2.1. Probability spaces carrying ergodic measure-preserving maps with positive entropy are nonatomic.

In the sequel we analyse the aperiodicity of measure-expansive maps. According to [11] a measurable map \( f \) is aperiodic whenever for all \( n \in \mathbb{N}^+ \) if \( n \in \mathbb{N}^+ \) and \( f^n(x) = x \) on a measurable set \( A \), then \( \mu(A) = 0 \). Let us extend this definition in the following way.
Definition 2.2. We say that \( f \) is eventually aperiodic whenever the following property holds for every \((n, k) \in \mathbb{N}^+ \times \mathbb{N}\): If \( A \) is a measurable set such that for every \( x \in A \) there is \( 0 \leq i \leq k \) such that \( f^{n+i}(x) = f^i(x) \), then \( \mu(A) = 0 \).

It follows easily from the definition that an eventually periodic map is aperiodic. The converse is true for invertible maps but not in general (e.g. the constant map \( f(x) = c \) where \( c \) is a measurable point of zero mass).

With this definition we can state the following result.

Theorem 2.4. Every measure-expansive map is eventually aperiodic (and so aperiodic).

Proof. Let \( f \) be a measure-expansive map of \( X \). Take \((n, k) \in \mathbb{N}^+ \times \mathbb{N}\) and a measurable set \( A \) such that for every \( x \in A \) there is \( 0 \leq i \leq k \) such that \( f^{n+i}(x) = f^i(x) \). Then,

\[
A \subset \bigcup_{i=0}^{k} f^{-i}(\text{Fix}(f^n)),
\]

where \( \text{Fix}(g) = \{x \in X : g(x) = x\} \) denotes the set of fixed points of a map \( g \).

Let \( P \) be a measure-sensitive partition of \( f \). Then, \( \bigvee_{m=0}^{k+n} f^{-m}(P) \) is a countable partition. Fix \( x, y \in A \cap \xi \). In particular \( \xi = \left( \bigvee_{m=0}^{k+n} f^{-m}(P) \right)(x) \) whence \( y \in \left( \bigvee_{m=0}^{k+n} f^{-m}(P) \right)(x) \). This together with (2.5) and (2.6) yields

\[
f^m(y) \in P(f^m(x)), \quad \forall 0 \leq m \leq k + n.
\]

But \( x, y \in A \) so (2.4) implies \( f^i(x), f^j(y) \in \text{Fix}(f^n) \) for some \( i, j \in \{0, \cdots, k\} \). We can assume that \( j \geq i \) (otherwise we interchange the roles of \( x \) and \( y \) in the argument below).

Now take \( m > k + n \). Then, \( m > j + n \) so \( m - j = pn + r \) for some \( p \in \mathbb{N}^+ \) and some integer \( 0 \leq r < n \). Since \( 0 \leq j + r < k + n \) (for \( 0 \leq j \leq k \) and \( 0 \leq r < n \)) one gets

\[
f^m(y) = f^{m-j}(f^j(y)) = f^{pn+r}(f^j(y)) = f^r(f^{pn}(f^j(y))) = f^{i+r}(y) \tag{2.10}
\]

But

\[
P(f^{i+r}(x)) = P(f^{i+r-i}(f^i(x))) = P(f^{i+r-i}(f^m(f^i(x)))) = P(f^{m-i}(f^i(x))) = P(f^m(x))
\]

so

\[
f^m(y) \in P(f^m(x)), \quad \forall m > k + n.
\]

This together with (2.10) implies that \( f^m(y) \in P(f^m(x)) \) for all \( m \in \mathbb{N} \) whence \( y \in P_{\infty}(x) \). Consequently \( A \cap \xi \subset P_{\infty}(x) \). As \( P \) is measure-sensitive Lemma 2.1 implies

\[
\mu(A \cap \xi) = 0, \quad \forall \xi \in \bigvee_{i=0}^{k+n} f^{-i}(P).
\]
On the other hand, $\bigvee_{i=0}^{k+n} f^{-i}(P)$ is a partition so

$$A = \bigcup_{\xi \in \bigvee_{i=0}^{k+n} f^{-i}(P)} (A \cap \xi)$$

and then $\mu(A) = 0$ since $\bigvee_{i=0}^{k+n} f^{-i}(P)$ is countable. This ends the proof. □

It follows from Lemma 2.1 that, in nonatomic probability spaces, every measurable map carrying strong generators is measure-expansive. This motivates the question as to whether every measure-expansive map has a strong generator. We give a partial positive answer for certain maps defined as follows. We say that $f$ is countable to one (mod 0) if $f^{-1}(x)$ is countable for $\mu$-a.e. $x \in X$. We say that $f$ is nonsingular if a measurable set $A$ has measure zero if and only if $f^{-1}(A)$ also does. All measure-preserving maps are nonsingular. A Lebesgue probability space is a complete measure space which is isomorphic to the completion of a standard probability space (c.f. [1], [3]).

Corollary 2.2. The following properties are equivalent for nonsingular countable to one (mod 0) maps $f$ on nonatomic Lebesgue probability spaces:

1. $f$ is measure-expansive.
2. $f$ is eventually aperiodic.
3. $f$ is aperiodic.
4. $f$ has a strong generator.

Proof. Notice that (1) $\Rightarrow$ (2) by Theorem 2.4 and (2) $\Rightarrow$ (3) follows from the definitions. On the other hand, (3) $\Rightarrow$ (4) by a Parry’s theorem (c.f. [11], [13], [12]) while (4) $\Rightarrow$ (1) by Lemma 2.1. □

Denote by $\text{Fix}(g) = \{ x \in X : g(x) = x \}$ the set of fixed points of a mapping $g$.

Corollary 2.3. If $f^k = f$ for some integer $k \geq 2$, then $f$ is not measure-expansive.

Proof. Suppose by contradiction that it does. Then, $f$ is eventually aperiodic by Theorem 2.4. On the other hand, if $x \in X$ then $f^k(x) = f(x)$ so $f^{k-1}(f^k(x)) = f^{k-1}(f(x)) = f^k(x)$ therefore $f^k(x) \in \text{Fix}(f^{k-1})$ whence $X \subset f^{-k}(\text{Fix}(f^{k-1}))$. But since $f$ is eventually aperiodic, $n = k - 1 \in \mathbb{N}^+$ and $X$ measurable we obtain from the definition that $\mu(X) = 0$ which is absurd. This ends the proof. □

Example 2.5. By Corollary 2.3 neither the identity $f(x) = x$ nor the constant map $f(x) = c$ are measure-expansive (for they satisfy $f^2 = f$). In particular, the converse of Theorem 2.4 is false for the constant maps are eventually aperiodic but not measure-expansive.

It is not difficult to prove that an ergodic measure-preserving map of a nonatomic probability space is aperiodic. Then, Corollary 2.1 implies the well-known fact that all ergodic measure-preserving maps with positive entropy on probability spaces are aperiodic. However, using theorems 2.3 and 2.4 we obtain the following stronger result.

Corollary 2.4. All ergodic measure-preserving maps with positive entropy on probability spaces are eventually aperiodic.

Now we study the following variant of aperiodicity introduced in [5] p. 180.
Definition 2.3. We say that \( f \) is aperiodic* whenever for every measurable set of positive measure \( A \) and \( n \in \mathbb{N}^+ \) there is a measurable subset \( B \subset A \) such that \( \mu(B \setminus f^{-n}(B)) > 0 \).

Notice that aperiodicity* implies the aperiodicity used in [6] or [14] (for further comparisons see p. 88 in [7]).

On the other hand, a measurable map \( f \) is negative nonsingular if \( \mu(f^{-1}(A)) = 0 \) whenever \( A \) is a measurable set with \( \mu(A) = 0 \). Some consequences of the aperiodicity* on negative nonsingular maps in probability spaces are given in [7]. Observe that every measure-preserving map is negatively nonsingular.

Let us present two technical (but simple) results for later usage. We call a measurable set \( A \) satisfying \( A \subset f^{-1}(A) \) (mod 0) a positively invariant set (mod 0). For completeness we prove the following property of these sets.

Lemma 2.3. If \( A \) is a positively invariant set (mod 0) of finite measure of a negative nonsingular map \( f \), then

\[
(2.11) \quad \mu \left( \bigcap_{n=0}^{\infty} f^{-n}(A) \right) = \mu(A).
\]

Proof. Since \( \mu(A) = \mu(A \setminus f^{-1}(A)) + \mu(A \cap f^{-1}(A)) \) and \( A \) is positively invariant (mod 0) one has \( \mu(A) = \mu(A \cap f^{-1}(A)) \), i.e.,

\[
\mu \left( \bigcap_{n=0}^{1} f^{-n}(A) \right) = \mu(A).
\]

Now suppose that \( m \in \mathbb{N}^+ \) satisfies

\[
\mu \left( \bigcap_{n=0}^{m} f^{-n}(A) \right) = \mu(A).
\]

Since

\[
\mu \left( \bigcap_{n=0}^{m+1} f^{-n}(A) \right) = \mu \left( \bigcap_{n=0}^{m} f^{-n}(A) \right) - \mu \left( \left( \bigcap_{n=0}^{m} f^{-n}(A) \right) \setminus f^{-m-1}(A) \right)
\]

and

\[
\mu \left( \left( \bigcap_{n=0}^{m} f^{-n}(A) \right) \setminus f^{-m-1}(A) \right) \leq \mu(f^{-m}(A \setminus f^{-1}(A))) = \mu(f^{-m}(A \setminus f^{-1}(A))) = 0
\]

because \( f \) is negative nonsingular and \( A \) is positively invariant (mod 0), one has

\[
\mu \left( \bigcap_{n=0}^{m+1} f^{-n}(A) \right) = \mu(A).
\]

Therefore

\[
(2.12) \quad \mu \left( \bigcap_{n=0}^{m} f^{-n}(A) \right) = \mu(A), \quad \forall m \in \mathbb{N},
\]

\(^1\text{called aperiodic in [5].}\)
Then, on the other hand,

$$\bigcap_{n=0}^{\infty} f^{-n}(A) = \bigcap_{m=0}^{\infty} \bigcap_{n=0}^{m} f^{-n}(A)$$

and $$\bigcap_{n=0}^{m+1} f^{-n}(A) \subset \bigcap_{n=0}^{m} f^{-n}(A)$$. As $$\mu(A) < \infty$$ we conclude that

$$\mu \left( \bigcap_{n=0}^{\infty} f^{-n}(A) \right) = \lim_{m \to \infty} \mu \left( \bigcap_{n=0}^{m} f^{-n}(A) \right) = \lim_{m \to \infty} \mu(A) = \mu(A)$$

proving (2.11).

We use the above lemma only in the proof of the proposition below.

**Proposition 2.3.** Let $$P$$ be a measure-sensitive partition of a negative nonsingular map $$f$$. Then, no $$\xi \in P$$ with positive finite measure is positively invariant (mod 0).

**Proof.** Suppose by contradiction that there is $$\xi \in P$$ with $$0 < \mu(\xi) < \infty$$ which is positively invariant (mod 0). Taking $$A = \xi$$ in Lemma 2.3 we obtain

(2.13)

$$\mu \left( \bigcap_{n=0}^{\infty} f^{-n}(\xi) \right) = \mu(\xi).$$

As $$\mu(\xi) > 0$$ we conclude that $$\bigcap_{n=0}^{\infty} f^{-n}(\xi) \neq \emptyset$$, and so, there is $$x \in \xi$$ such that $$f^n(x) \in \xi$$ for all $$n \in \mathbb{N}$$. As $$\xi \in P$$ we obtain $$P(f^n(x)) = \xi$$ and so $$f^{-n}(P(f^n(x))) = f^{-n}(\xi)$$ for all $$n \in \mathbb{N}$$. Using (2.3) we get

$$P_m(x) = \bigcap_{n=0}^{m} f^{-n}(\xi).$$

Then, (2.4) yields

$$P_\infty(x) = \bigcap_{m=0}^{\infty} P_m(x) = \bigcap_{m=0}^{\infty} \bigcap_{n=0}^{m} f^{-n}(\xi) = \bigcap_{n=0}^{\infty} f^{-n}(\xi)$$

and so $$\mu(P_\infty(x)) = \mu(\xi)$$ by (2.13). Then, $$\mu(\xi) = 0$$ by Lemma 2.1 since $$P$$ is measure-sensitive which is absurd. This contradiction proves the result. \(\square\)

We also need the following lemma resembling a well-known property of the expansive maps.

**Lemma 2.4.** If $$k \in \mathbb{N}^+$$, then $$f$$ is measure-expansive if and only if $$f^k$$ is.

**Proof.** The notation $$P^f_k(x)$$ will indicate the dependence of $$P_\infty(x)$$ on $$f$$.

First suppose that $$f^k$$ is an measure-expansive with measure-sensitive partition $$P$$. Then, $$\mu(P^f_k(x)) = 0$$ for all $$x \in X$$ by Lemma 2.1. But by definition one has $$P^f_k(x) \subset P^f_\infty(x)$$ so $$\mu(P^f_\infty(x)) = 0$$ for all $$x \in X$$. Therefore, $$f$$ is measure-expansive with measure-sensitive partition $$P$$. Conversely, suppose that $$f$$ is measure-expansive with expansivity constant $$P$$. Consider $$Q = \bigvee_{i=0}^{k} f^{-i}(P)$$ which is a countable partition satisfying $$Q(x) = \bigcap_{i=0}^{k} f^{-i}(P(f^i(x)))$$ by (2.10).

Now, take $$y \in Q^f_k(x)$$ in particular, $$y \in Q(x)$$ hence $$f^i(y) \in P(f^i(x))$$ for every $$0 \leq i \leq k$$. Take $$n > k$$ so $$n = pk + r$$ for some nonnegative integers $$p$$ and $$0 \leq r < k$$. As $$y \in Q^f_k(x)$$ one has $$f^{pk+1}(y) \in Q(f^{pk}(x))$$ and then $$f^n(y) = f^{pk+1}(y) = f^i(f^{pk}(y)) \in P(f^i(f^{pk}(x)) = P(f^n(x))$$ proving $$f^n(y) \in P(f^n(x))$$ for all $$n \in \mathbb{N}$$. Then, $$y \in P_\infty(x)$$ yielding $$Q^f_k(x) \subset P_\infty(x)$$. Thus $$\mu(Q^f_k(x)) = 0$$ for all $$x \in X$$.\(\square\)
the equivalence (ii) in Lemma 2.1 since $P$ is measure-sensitive. It follows that $f^k$ is measure-expansive with measure-sensitive partition $Q$. □

With these definitions and preliminary results we obtain the following.

**Theorem 2.5.** Every measure-expansive negative nonsingular map in a probability space is aperiodic*.

**Proof.** Suppose by contradiction that there is a measure-expansive map $f$ which is negative nonsingular but not aperiodic*. Then, there are a measurable set of positive measure $A$ and $n \in \mathbb{N}^+$ such that $\mu(B \setminus f^{-n}(B)) = 0$ for every measurable subset $B \subset A$. It follows that every measurable subset $B \subset A$ is positively invariant (mod 0) with respect to $f^n$. By Lemma 2.4 we can assume $n = 1$.

Now, let $P$ be a measure-sensitive partition of $f$. Clearly, since $\mu(A) > 0$ there is $\xi \in P$ such that $\mu(A \cap \xi) > 0$. Taking $\eta = A \cap \xi$ we obtain that $\eta$ is positively invariant (mod 0) with positive measure. In addition, consider the new partition $Q = (P \setminus \{\xi\}) \cup \{\eta, \xi \setminus A\}$ which is clearly measure-sensitive (for $P$ is). Since this partition also carries a positively invariant (mod 0) member of positive measure (say $\eta$) we obtain a contradiction by Proposition 2.5. The proof follows. □

To finish we compare the measure-expansivity with the notion of pairwise sensitivity in metric measure spaces introduced in p. 376 of [4].

By a **metric measure space** we mean a triple $(X, d, \mu)$ where $(X, d)$ is a metric space and $\mu$ is a measure in the corresponding Borel $\sigma$-algebra. Hereafter the term **measurable** will mean **Borel measurable**. The product measure in $X \times X$ will be denoted by $\mu^\otimes 2$.

**Definition 2.4.** A measurable map $f : X \to X$ of a metric measure space $(X, d, \mu)$ is pairwise sensitive if there is $\delta > 0$ such that

$$
\mu^\otimes 2 \left( \{(x, y) \in X \times X : \exists n \in \mathbb{N} \text{ such that } d(f^n(x), f^n(y)) \geq \delta \} \right) = 1.
$$

The following is a characterization of pairwise sensitivity which is similar to one in [9]. Since this reference is not available yet we include its proof here for the sake of completeness. By a **metric probability space** we mean a metric measure space of total mass one. Given a map $f : X \to X$ and $\delta > 0$ we define the dynamical $\delta$-balls

$$
\Phi_\delta(x) = \{y \in X : d(f^n(x), f^n(y)) \leq \delta, \forall n \in \mathbb{N}\}, \quad \forall x \in X.
$$

**Lemma 2.5.** The following properties are equivalent for measurable maps $f$ on metric probability spaces $(X, d, \mu)$:

1. $f$ is pairwise sensitive.
2. There is $\delta > 0$ such that

$$
\mu(\Phi_\delta(x)) = 0, \quad \forall x \in X.
$$

3. There is $\delta > 0$ such that

$$
\mu(\Phi_\delta(x)) = 0, \quad \forall \mu\text{-a.e. } x \in X.
$$

**Proof.** First we prove (2) and (3) are equivalent. Indeed, we only have to prove that (3) implies (2). Fix $\delta > 0$ satisfying (2.15) and suppose by contradiction that (2) fails. Then, there is $x_0 \in X$ such that $\mu(\Phi_\delta/2(x_0)) > 0$. Denote $X_\delta = \{x \in X : \mu(\Phi_\delta(x)) = 0\}$ so $\mu(X_\delta) = 1$. Since $\mu$ is a probability we obtain $X_\delta \cap \Phi_\delta(x_0) \neq \emptyset$ so there is $y_0 \in \Phi_\delta(x_0)$ such that $\mu(\Phi_\delta(y_0)) = 0$. Now if
Theorem 2.6. All pairwise sensitive maps on separable probability spaces are measure-expansive.

Proof. Let \( f \) be a pairwise sensitive map of a separable probability space \((X, d, \mu)\). By Lemma 2.5 there is \( \delta > 0 \) satisfying (2.14). Since \((X, d)\) is separable we can select a countable covering \( \{ B_k : k \in I \} \) of \( X \) consisting of balls of radius \( \delta \), where \( I \) is either \( \mathbb{N} \) or \( \{0, 1, \ldots, s\} \) for some \( s \in \mathbb{N} \). As usual we can transform this covering into a countable partition \( P = \{ \xi_k : k \in I \} \) by taking \( \xi_0 = B_0 \) and \( \xi_k = B_k \setminus \bigcup_{i=0}^{k-1} B_i \) for \( k \geq 1 \). Clearly this partition satisfies \( P(\infty) \cap P \delta(x) \neq \emptyset \). The following example shows that converse of Theorem 2.6 is false.
Example 2.6. An irrational circle rotation is measure-expansive with respect to the Lebesgue measure (c.f. Example 2.2 or Corollary 2.2) but not pairwise sensitive with respect to that measure (c.f. [4] p. 378).

Recall that a map $f : X \to X$ of a metric space $(X, d)$ is expansive if there is $\delta > 0$ such that $x = y$ whenever $x, y \in X$ and $d(f^n(x), f^n(y)) \leq \delta$ for all $n \in \mathbb{N}$.

Corollary 2.5. Every measurable expansive map in a nonatomic separable probability space is measure-expansive.

Proof. Notice that a map $f$ is expansive if and only if there is $\delta > 0$ such that $\Phi_\delta(x) = \{x\}$ for every $x \in X$. Then, Lemma 2.5 implies that every expansive measurable map of a nonatomic metric measure space is pairwise sensitive. Now apply Theorem 2.6. □

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