UNBOUNDED SUPERSOLUTIONS OF SOME QUASILINEAR PARABOLIC EQUATIONS: A DICHOTOMY

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Abstract. We study unbounded "supersolutions" of the Evolutionary $p$-Laplace equation with slow diffusion. They are the same functions as the viscosity supersolutions. A fascinating dichotomy prevails: either they are locally summable to the power $p - 1 + \frac{2}{p} - 0$ or not summable to the power $p - 2$. There is a void gap between these exponents. Those summable to the power $p - 2$ induce a Radon measure, while those of the other kind do not. We also sketch similar results for the Porous Medium Equation.

1. Introduction

The unbounded supersolutions of the Evolutionary $p$-Laplace Equation
\[ \frac{\partial u}{\partial t} - \mathbf{\nabla} \cdot (|\mathbf{\nabla} u|^{p-2} \mathbf{\nabla} u) = 0, \quad 2 < p < \infty, \]
exhibit a fascinating dichotomy in the slow diffusion case $p > 2$. This phenomenon was discovered and investigated in [14]. The purpose of the present work is to give an alternative proof, directly based on the iterative procedure in [8]. Besides the achieved simplification, our proof can readily be extended to more general quasilinear equations of the form
\[ \frac{\partial u}{\partial t} - \mathbf{\nabla} \cdot \mathbf{A}(x, t, u, \mathbf{\nabla} u) = 0, \]
which are treated in the book [DGV]. The expedient analytic tool is the intrinsic Harnack inequality for positive solutions, see [6]. We can avoid to evoke it for supersolutions. We also mention the books [4] and [24] as general references.

The supersolutions that we consider are called $p$-supercaloric functions. They are pointwise defined lower semicontinuous functions, finite in a dense subset, and are required to satisfy the Comparison Principle with respect to the solutions of the equation; see Definition 2.4 below. The definition is the same as the one in classical potential

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\[1^1\text{They are also called } p\text{-parabolic functions, as in [8].}\]
theory for the Heat Equation\textsuperscript{2}, to which the equation reduces when \( p = 2 \), see [23]. Incidentally, the \( p \)-supercaloric functions are exactly the viscosity supersolutions of the equation, see [7].

There are two disjoint classes of \( p \)-supercaloric functions, called class \( \mathcal{B} \) and \( \mathcal{M} \). We begin with the former one. Throughout the paper we assume that \( \Omega \) is an open subset of \( \mathbb{R}^n \) and we denote \( \Omega_T = \Omega \times (0, T) \) for \( T > 0 \).

**Theorem 1.1** (Class \( \mathcal{B} \)). Let \( p > 2 \). For a \( p \)-supercaloric function \( v : \Omega_T \to (\infty, \infty] \) the following conditions are equivalent:

(i) \( v \in L^{p-2}_{\text{loc}}(\Omega_T) \),

(ii) the Sobolev gradient \( \nabla v \) exists and \( \nabla v \in L^{q'}_{\text{loc}}(\Omega_T) \) whenever \( q' < p - 1 + \frac{1}{n+1} \),

(iii) \( v \in L^q_{\text{loc}}(\Omega_T) \) whenever \( q < p - 1 + \frac{2}{n} \).

In this case there exists a non-negative Radon measure \( \mu \) such that

\[
\int_0^T \int_{\Omega} \left( -v \frac{\partial \varphi}{\partial t} + \langle |\nabla v|^{p-2} \nabla v, \nabla \varphi \rangle \right) \, dx \, dt = \int_{\Omega_T} \varphi \, d\mu
\]

for all test functions \( \varphi \in C^\infty_0(\Omega_T) \). In other words, the equation

\[
\frac{\partial v}{\partial t} - \nabla \cdot (|\nabla v|^{p-2} \nabla v) = \mu
\]

holds in the sense of distributions, cf. [12]. It is of utmost importance that the local summability exponent for the gradient in (ii) is at least \( p - 2 \). Such measure data equations have been much studied and we only refer to [2]. For potential estimates we refer to [15], [16].

As an example of a function belonging to class \( \mathcal{B} \) we mention the celebrated Barenblatt solution

\[
\mathcal{B}(x, t) = \begin{cases} 
  t^{-\frac{n}{\lambda}} \left[ C - \frac{p-2}{p} \lambda^{1-p} \left( \frac{|x|}{t^\frac{1}{\lambda}} \right)^{p-1} \right]^{\frac{p-1}{p-2}} + , & \text{when } t > 0, \\
  0, & \text{when } t \leq 0,
\end{cases}
\]

found in 1951, cf. [1]. Here \( \lambda = n(p - 2) + p \) and \( p > 2 \). It is a solution of the Evolutionary \( p \)-Laplace Equation, except at the origin \( x = 0, t = 0 \). Moreover, it is a \( p \)-supercaloric function in the whole \( \mathbb{R}^n \times \mathbb{R} \), where it satisfies the equation

\[
\frac{\partial \mathcal{B}}{\partial t} - \nabla \cdot (|\nabla \mathcal{B}|^{p-2} \nabla \mathcal{B}) = c \delta
\]

in the sense of distributions (\( \delta = \text{Dirac’s delta} \)). It also shows that the exponents in (i) and (ii) of the previous theorem are sharp.

\textsuperscript{2}Yet, the dichotomy we focus our attention on, is impossible for the Heat Equation.
A very different example is the stationary function

\[ v(x, t) = \sum_j \frac{c_j}{|x - q_j|^{\frac{p-2}{p-1}}}, \quad 2 < p < n, \]

where the \( q_j \)'s are an enumeration of the rationals and the \( c_j \geq 0 \) are convergence factors. Indeed, this is a \( p \)-supercaloric function, it has a Sobolev gradient, and \( v(q_j, t) \equiv \infty \) along every rational line \( x = q_j \), \(-\infty < t < \infty\), see [18].

Then we describe class \( \mathcal{M} \).

**Theorem 1.4 (Class \( \mathcal{M} \)).** Let \( p > 2 \). For a \( p \)-supercaloric function \( v : \Omega_T \to (-\infty, \infty] \) the following conditions are equivalent:

(i) \( v \notin L^{p-2}_{\text{loc}}(\Omega_T) \),

(ii) there is a time \( t_0, 0 < t_0 < T \), such that

\[ \liminf_{(y, t) \to (x, t_0)} v(y, t)(t - t_0)^{\frac{1}{p-2}} > 0 \quad \text{for all} \quad x \in \Omega. \]

Notice that the infinities occupy the whole space at some instant \( t_0 \).

As an example of a function from class \( \mathcal{M} \) we mention

\[ \mathfrak{B}(x, t) = \begin{cases} \frac{\mathfrak{U}(x)}{(t - t_0)^{\frac{1}{p-2}}}, & \text{when } t > t_0, \\ 0, & \text{when } t \leq t_0, \end{cases} \]

where \( \mathfrak{U} \in C(\Omega) \cap W^{1,p}_0(\Omega) \) is a weak solution to the elliptic equation

\[ \nabla \cdot (|\nabla \mathfrak{U}|^{p-2} \nabla \mathfrak{U}) + \frac{1}{p-2} \mathfrak{U} = 0 \]

and \( \mathfrak{U} > 0 \) in \( \Omega \). The function \( \mathfrak{B} \) is \( p \)-supercaloric in \( \Omega \times \mathbb{R} \), see equation (3.3) below. This function can serve as a minorant for all functions \( v \geq 0 \) in \( \mathcal{M} \). No \( \sigma \)-finite measure is induced in this case. As far as we know, these functions have not yet been carefully studied.

A function of class \( \mathcal{M} \) always affects the boundary values. Indeed, at some point on \((\xi_0, t_0)\) on the lateral boundary \( \partial \Omega \times (0, T) \) it is necessary to have

\[ \limsup_{(x, t) \to (\xi_0, t_0)} v(x, t) = \infty. \]

This alone does not yet prove that \( v \) would belong to \( \mathcal{M} \). A convenient sufficient condition for membership in class \( \mathfrak{B} \) emerges: If

\[ \limsup_{(x, \tau) \to (\xi, t)} v(x, \tau) < \infty \quad \text{for every} \quad (\xi, t) \in \partial \Omega \times (0, T), \]

then \( v \in \mathfrak{B} \).

It is no surprise that a parallel theory holds for the celebrated Porous Medium Equation

\[ \frac{\partial u}{\partial t} - \Delta(u^m) = 0, \quad 1 < m < \infty. \]
We refer to the monograph [22] about this much studied equation. We sketch the argument in the last section.

2. Preliminaries

We begin with some standard notation. We consider an open domain \( \Omega \) in \( \mathbb{R}^n \) and denote by \( L^p(t_1, t_2; W^{1,p}(\Omega)) \) the Sobolev space of functions \( v = v(x, t) \) such that for almost every \( t, t_1 \leq t \leq t_2 \), the function \( x \mapsto v(x, t) \) belongs to \( W^{1,p}(\Omega) \) and

\[
\int_{t_1}^{t_2} \int_{\Omega} \left( |v(x, t)|^p + |\nabla v(x, t)|^p \right) \, dx \, dt < \infty,
\]

where \( \nabla v = \left( \frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n} \right) \) is the spatial Sobolev gradient. The definitions of the local spaces \( L^p(t_1, t_2; W^{1,p}_{loc}(\Omega)) \) and \( L^p_{loc}(t_1, t_2; W^{1,p}_{loc}(\Omega)) \) are analogous. We denote \( \Omega_{t_1, t_2} = \Omega \times (t_1, t_2) \) and recall that the parabolic boundary of \( \Omega_{t_1, t_2} \) is the set \( (\bar{\Omega} \times \{t_1\}) \cup (\partial \Omega \times (t_1, t_2)) \).

Definition 2.1. A function \( u \in L^p(t_1, t_2; W^{1,p}(\Omega)) \) is a weak solution of the Evolutionary \( p \)-Laplace Equation in \( \Omega_{t_1, t_2} \), if

\[
\int_{t_1}^{t_2} \int_{\Omega} \left( -u \frac{\partial \varphi}{\partial t} + \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle \right) \, dx \, dt = 0
\]

for every \( \varphi \in C_0^\infty(\Omega_{t_1, t_2}) \). If, in addition, \( u \) is continuous, then it is called a \( p \)-caloric function. Further, we say that \( u \) is a weak supersolution, if the above integral is non-negative for all non-negative \( \varphi \in C_0^\infty(\Omega_{t_1, t_2}) \). If the integral is non-positive instead, we say that \( u \) is a weak subsolution.

By parabolic regularity theory, a weak solution is locally Hölder continuous after a possible redefinition in a set of \( n+1 \)-dimensional Lebesgue measure zero, see [21] and [4]. In addition, a weak supersolution is upper semicontinuous with the same interpretation, cf. [13].

Lemma 2.3 (Comparison Principle). Assume that

\[
u, \, v \in L^p(t_1, t_2; W^{1,p}(\Omega)) \cap C(\bar{\Omega} \times [t_1, t_2]).\]

If \( v \) is a weak supersolution and \( u \) a weak subsolution in \( \Omega_{t_1, t_2} \) such that \( v \geq u \) on the parabolic boundary of \( \Omega_{t_1, t_2} \), then \( v \geq u \) in the whole \( \Omega_{t_1, t_2} \).

The Comparison Principle is used to define the class of \( p \)-supercaloric functions.

Definition 2.4. A function \( v : \Omega_{t_1, t_2} \to (-\infty, \infty] \) is called \( p \)-supercaloric, if

(i) \( v \) is lower semicontinuous,
(ii) \( v \) is finite in a dense subset,
(iii) \( v \) satisfies the comparison principle on each interior cylinder \( D_{t_1, t_2} \subset \Omega_{t_1, t_2} \): If \( h \in C(\overline{D_{t_1, t_2}}) \) is a \( p \)-parabolic function in \( D_{t_1, t_2} \), and if \( h \leq v \) on the parabolic boundary of \( D_{t_1, t_2} \), then \( h \leq v \) in the whole \( D_{t_1, t_2} \).

We recall a fundamental result for \textit{bounded} functions, which is also applicable to more general equations.

**Theorem 2.5.** Let \( p \geq 2 \). If \( v \) is a \( p \)-supercaloric function that is locally bounded from above in \( \Omega_T \), then the Sobolev gradient \( \nabla v \) exists and \( \nabla v \in L^p_{loc}(\Omega_T) \). Moreover, \( v \in L^p_{loc}(0, T; W^{1,p}_{loc}(\Omega)) \) and \( v \) is a weak supersolution.

A proof based on auxiliary obstacle problems was given in [9], Theorem 1.4. A more direct proof with infimal convolutions can be found in [17].

In order to apply the previous theorem, we need \textit{bounded} functions. The truncations

\[
v_j(x, t) = \min\{v(x, t), j\}, \quad j = 1, 2, \ldots ,
\]

are \( p \)-supercaloric, if \( v \) is, and since they are bounded from above, they are also weak supersolutions. Thus \( \nabla v_j \) is at our disposal and estimates derived from the inequality

\[
(2.6) \quad \int_0^T \int_\Omega \left( -v_j \frac{\partial \varphi}{\partial t} + \langle |\nabla v_j|^{p-2} \nabla v_j, \nabla \varphi \rangle \right) \, dx \, dt \geq 0,
\]

where \( \varphi \geq 0 \) and \( \varphi \in C_0^\infty(\Omega_T) \), are available. The starting point for our proof is the following theorem for the truncated functions.

**Theorem 2.7.** Let \( p > 2 \). Suppose that \( v \geq 0 \) is a \( p \)-supercaloric function in \( \Omega_T \) with initial values \( v(x, 0) = 0 \) in \( \Omega \). If \( v_j \in L^p(0, T; W^{1,p}_{loc}(\Omega)) \) for every \( j = 1, 2, \ldots , \) then

(i) \( v \in L^q(\Omega_{T_1}) \) whenever \( q < p - 1 + \frac{p}{n} \) and \( T_1 < T \),

(ii) the Sobolev gradient \( \nabla u \) exists and \( \nabla v \in L^{q'}(\Omega_{T_1}) \) whenever \( q' < p - 1 + \frac{1}{n+1} \) and \( T_1 < T \).

\[ \Box \]

Proof. See [9].

We remark that the summability exponents are sharp. It is decisive that the boundary values are zero. The functions of class \( \mathfrak{M} \) cannot satisfy this requirement. As we shall see, those of class \( \mathfrak{B} \) can be modified so that the theorem above applies.

The standard Caccioppoli estimates are valid. We recall the following simple version, which will suffice for us.
Lemma 2.8 (Caccioppoli). Let \( p > 2 \). If \( u \geq 0 \) is a weak subsolution in \( \Omega_T \), then the estimate

\[
\int_{t_1}^{t_2} \int_{\Omega} \zeta^p |\nabla u| \, dx \, dt + \text{ess sup}_{t_1 < t < t_2} \int_{\Omega} \zeta(x)^p u(x, t)^2 \, dx \leq C(p) \left\{ \int_{t_1}^{t_2} \int_{\Omega} u^p |\nabla \zeta|^p \, dx \, dt + \int_{\Omega} \zeta(x)^p u(x, t)^2 \big|_{t_1}^{t_2} \, dx \right\}
\]

holds for every \( \zeta = \zeta(x) \geq 0 \) in \( C_0^\infty(\Omega) \), \( 0 < t_1 < t_2 < T \).

Proof. A formal calculation with the test function \( \phi = v \zeta^p \) gives the inequality. See [4], [9]. \qed

Infimal Convolutions. The infimal convolutions preserve the \( p \)-supercaloric functions and are Lipschitz continuous. Thus they are convenient approximations. If \( v \geq 0 \) is lower semicontinuous and finite in a dense subset of \( \Omega_T \), then the infimal convolution

\[
v^\varepsilon(x, t) = \inf_{(y, \tau) \in \Omega_T} \left\{ v(y, \tau) + \frac{1}{2\varepsilon} (|x - y|^2 + |t - \tau|^2) \right\}
\]

is well defined. It has the properties

- \( v^\varepsilon(x, t) \nearrow v(x, t) \) as \( \varepsilon \to 0 \),
- \( v^\varepsilon \) is locally Lipschitz continuous in \( \Omega_T \),
- the Sobolev derivatives \( \frac{\partial v^\varepsilon}{\partial t} \) and \( \nabla v^\varepsilon \) exist and belong to \( L^\infty_{\text{loc}}(\Omega_T) \).

Assume now that \( v \) is a \( p \)-supercaloric function in \( \Omega_T \). Given a subdomain \( D \subset \Omega_T \), the above \( v^\varepsilon \) is a \( p \)-supercaloric function in \( D \), provided that \( \varepsilon \) is small enough, see [9].

3. A Separable Minorant

We begin with observations, which will simplify some arguments later.

Extension to the past. If \( v \) is a non-negative \( p \)-supercaloric function in \( \Omega_T \), then the extended function

\[
v(x, t) = \begin{cases} v(x, t), & \text{when } 0 < t < T, \\ 0, & \text{when } t \leq 0, \end{cases}
\]

is \( p \)-supercaloric in \( \Omega \times (-\infty, T) \). We use the same notation for the extended function.

A separable minorant. Separation of variables suggests that there are \( p \)-caloric functions of the type

\[
v(x, t) = (t - t_0)^{-\frac{1}{p-2}} u(x).
\]
Indeed, if $\Omega$ is a domain of finite measure, there exists a $p$-caloric function of the form

$$V(x,t) = \frac{\U(x)}{(t-t_0)^{\frac{1}{p-2}}}, \text{ when } t > t_0,$$

where $\U \in C(\Omega) \cap W^{1,p}_0(\Omega)$ is a weak solution to the elliptic equation

$$\nabla \cdot (|\nabla \U|^{p-2} \nabla \U) + \frac{1}{p-2} \U = 0$$

and $\U > 0$ in $\Omega$. The solution $\U$ is unique. (Actually, $\U \in C^{1,\alpha}_{\text{loc}}(\Omega)$ for some exponent $\alpha = \alpha(n,p) > 0$.) The extended function

$$V(x,t) = \begin{cases} \frac{\U(x)}{(t-t_0)^{\frac{1}{p-2}}}, & \text{when } t > t_0, \\ 0, & \text{when } t \leq t_0. \end{cases}$$

is $p$-supercaloric in $\Omega \times \mathbb{R}$. The existence of $\U$ follows by the direct method in the Calculus of Variations, when the quotient

$$J(w) = \frac{\int_{\Omega} |\nabla w|^p \, dx}{\left(\int_{\Omega} w^2 \, dx\right)^{\frac{p}{2}}}$$

is minimized among all functions $w$ in $W^{1,p}_0(\Omega)$ with $w \not\equiv 0$. Replacing $w$ by its absolute value $|w|$, we may assume that all functions are non-negative. Sobolev’s and Hölder’s inequalities imply

$$J(w) \geq c(p,n)|\Omega|^{\frac{1}{n}} \cdot \frac{|\U(x)|}{(t-t_0)^{\frac{1}{p-2}}},$$

for some $c(p,n) > 0$ and so $J_0 = \inf_w J(w) > 0$. Choose a minimizing sequence of admissible normalized functions $w_j$ with

$$\lim_{j \to \infty} J(w_j) = J_0 \quad \text{and} \quad \|w_j\|_{L^p(\Omega)} = 1.$$

By compactness, we may extract a subsequence such that $\nabla w_{j_k} \rightharpoonup \nabla w$ weakly in $L^p(\Omega)$ and $w_{j_k} \to w$ strongly in $L^p(\Omega)$ for some function $w$. The weak lower semicontinuity of the integral implies that

$$J(w) \leq \liminf_{k \to \infty} J(w_{j_k}) = J_0.$$

Since $w \in W^{1,p}_0(\Omega)$ this means that $w$ is a minimizer. We have $w \geq 0$, and $w \not\equiv 0$ because of the normalization.

It follows that $w$ has to be a weak solution of the Euler–Lagrange equation

$$\nabla \cdot (|\nabla w|^{p-2} \nabla w) + J_0 \|w\|_{L^p(\Omega)}^{p-2} w = 0$$

with $\|w\|_{L^p(\Omega)} = 1$. By elliptic regularity theory $w \in C(\Omega)$, see [20]. Finally, since $\nabla \cdot (|\nabla w|^{p-2} \nabla w) \leq 0$ in the weak sense and $w \geq 0$.

\footnote{Unfortunately, the otherwise reliable paper [J. García-Azorero, I. Peral Alonso: Existence and nonuniqueness for the $p$-Laplacian: Nonlinear eigenvalues, Communications in Partial Differential Equations 12, 1987, pp. 1389–1430], contains a misprint exactly for those parameter values that would yield this function.}
we have that \( w > 0 \) by the Harnack inequality \([20]\). A normalization remains to be done. The function

\[
\Omega = Cu, \quad \text{where} \quad J_0C^{p-2} = \frac{1}{p-2},
\]

will do.

**One dimensional case.** In one dimension the equation is

\[
\frac{d}{dx}\left( |U'|^p - 2U' \right) + \frac{1}{p - 2} U = 0, \quad 0 \leq x \leq L.
\]

It has the first integral

\[
\frac{p-1}{p} |U'|^p + \frac{\Omega^2}{2(p-1)} = C
\]

in the interval \([0, L]\). Now \( \Omega(0) = 0 = \Omega(L) \) and \( \Omega'\left(\frac{L}{2}\right) = 0 \). This determines the constant of integration in terms of \( \Omega'(0) \) or of the maximal value \( M = \max \Omega = \Omega\left(\frac{L}{2}\right) \). Solving for \( \Omega' \), separating the variables, and integrating from 0 to \( \frac{L}{2} \), one easily obtains the parameters

\[
M = C_1(p)L^{\frac{2}{p-2}} \quad \text{and} \quad \Omega'(0) = -\Omega'(L) = C_2(p)L^{\frac{2}{p-2}}.
\]

The constants can be evaluated. In passing, we mention that \( \frac{U(x)}{M} \) has interesting properties as a special function.

4. **Harnack’s Convergence Theorem**

A known phenomenon for an increasing sequence of non-negative \( p \)-caloric functions is described in this section. The analytic tool is an intrinsic version of Harnack’s inequality, see \([4]\), pp. 157–158, \([5]\), and \([6]\)

**Lemma 4.1** (Harnack’s inequality). Let \( p > 2 \). There are constants \( C \) and \( \gamma \), depending only on \( n \) and \( p \), such that if \( u > 0 \) is a lower semicontinuous weak solution in

\[
B(x_0, 4R) \times (t_0 - 4\theta, t_0 + 4\theta), \quad \text{where} \quad \theta = \frac{CR^n}{u(x_0, t_0)^{p-2}},
\]

then the inequality

\[
u(x_0, t_0) \leq \gamma \inf_{B_R(x_0)} u(x, t_0 + \theta)
\]

is valid.

Notice that the waiting time \( \theta \) depends on the solution itself.

**Proposition 4.3.** Suppose that we have an increasing sequence \( 0 \leq h_1 \leq h_2 \leq h_3 \leq \ldots \) of \( p \)-caloric functions in \( \Omega_T \) and denote \( h = \lim_{k \to \infty} h_k \). If there is a sequence \( (x_k, t_k) \to (x_0, t_0) \) such that \( h_k(x_k, t_k) \to +\infty \), where \( x_0 \in \Omega \) and \( 0 < t_0 < T \), then

\[
\liminf_{(y, t) \to (x, t_0)} h(y, t)(t - t_0)^{\frac{1}{p-2}} > 0 \quad \text{for all} \quad x \in \Omega.
\]
Thus, at time $t_0$,

$$
\lim_{(y,t) \to (x_0,t_0)} h(y,t) \equiv \infty \quad \text{in} \quad \Omega.
$$

**Remark 4.4.** The limit function $h$ may be finite at every point, though locally unbounded. Keep the function $\mathcal{W}$ in mind. — The proof will give

$$
h(x, t) \geq \frac{\Omega(x)}{(t-t_0)^{\frac{p-2}{2}}} \quad \text{in} \quad \Omega \times (t_0, T).
$$

**Proof:** Let $B(x_0, 4R) \subseteq \Omega$. Since

$$
\theta_k = \frac{CR \rho}{h_k(x_k, t_k)^{p-2}} \to 0,
$$

Harnack’s Inequality (4.2) implies

$$(4.5) \quad h_k(x_k, t_k) \leq \gamma h_k(x, t_k + \theta_k)$$

when $x \in B(x_k, R)$ provided $B(x_k, 4R) \times (t_k - 4\theta_k, t_k + 4\theta_k) \subseteq \Omega_T$. The center is moving, but since $x_k \to x_0$, equation (4.5) holds for sufficiently large indices. Let $\Lambda > 1$. We want to compare the solutions

$$
\frac{\Omega^{R\Lambda}(x)}{(t-t_k + (\Lambda - 1)\theta_k)^{\frac{p-2}{2}}} \quad \text{and} \quad h_k(x, t)
$$

when $t = t_k + \theta_k$ and $x \in B(x_0, R)$. Here $\Omega^{R\Lambda}$ is the positive solution of the elliptic equation (3.2) in $B(x_0, R)$ with boundary values zero. We get

$$
\frac{\Omega^{R\Lambda}(x)}{(t-t_k + (\Lambda - 1)\theta_k)^{\frac{p-2}{2}}} \bigg|_{t=t_k+\theta_k} \leq \frac{\Omega^{R\Lambda}(x)}{(\LambdaCR)^{\frac{p-2}{2}}} h_k(x_k, t_k)
$$

$$
\leq \frac{\Omega^{R\Lambda}(x)}{(\LambdaCR)^{\frac{p-2}{2}}} \gamma h_k(x, t_k + \theta_k) \leq h_k(x, t_k + \theta_k)
$$

by taking $\Lambda$ so large that

$$
\frac{\gamma \|\Omega^{R\Lambda}\|_{L^\infty(B(x_0, R))}}{(\LambdaCR)^{\frac{p-2}{2}}} \leq 1.
$$

By the Comparison Principle

$$
\frac{\Omega^{R\Lambda}(x)}{(t-t_k + (\Lambda - 1)\theta_k)^{\frac{p-2}{2}}} \leq h_k(x, t) \leq h(x, t)
$$

when $t \geq t_k + \theta_k$ and $x \in B(x_0, R)$. By letting $k \to \infty$, we arrive at

$$
\frac{\Omega^{R\Lambda}(x)}{(t-t_0)^{\frac{p-2}{2}}} \leq h(x, t) \quad \text{when} \quad t_0 < t < T.
$$
Here \( \Omega^j \) depended on the ball \( B(x_0, R) \), but now we have many more infinities, so that we may repeat the procedure in a suitable chain of balls to extend the estimate to the whole domain \( \Omega \).

**Proposition 4.6.** Suppose that we have an increasing sequence \( 0 \leq h_1 \leq h_2 \leq h_3 \leq \ldots \) of \( p \)-caloric functions in \( \Omega_T \) and denote \( h = \lim_{k \to \infty} h_k \). If the sequence \( \{h_k\} \) is locally bounded, then the limit function \( h \) is \( p \)-caloric in \( \Omega_T \).

**Proof.** In a strict subdomain we have the Hölder continuity estimate
\[
|h_k(x_1, t_1) - h_k(x_2, t_2)| \leq C\|h_k\| \left| |x_2 - x_1|^{a} + |t_2 - t_1|^{\frac{2}{p}} \right|
\]
so that the family is locally equicontinuous. Hence the convergence \( h_k \to h \) is locally uniform in \( \Omega_T \). Theorem 24 in [LM] implies that \( \{\nabla h_k\} \) is a Cauchy sequence in \( L^{p-1}_{loc}(\Omega_T) \). Thus we can pass to the limit under the integral sign in the equation
\[
\int_0^T \int_\Omega \left( -\frac{\partial \varphi}{\partial t} + \langle |\nabla h_k|^{p-2} \nabla h_k, \nabla \varphi \rangle \right) \, dx \, dt = 0
\]
as \( k \to \infty \). From the Caccioppoli estimate
\[
\int_{t_1}^{t_2} \int_\Omega |\nabla h_k|^p \, dx \, dt \leq C(p) \int_{t_1}^{t_2} \int_\Omega |\nabla \zeta|^p \, dx \, dt + C(p) \int \int_\Omega \zeta(x)^p h_k(x, t)^2 \, dx \left| t_2 - t_1 \right|
\]
we deduce that \( h \in L^p_{loc}(0, T; W^{1,p}_{loc}(\Omega)) \). \( \square \)

**5. Proof of the Theorem**

For the proof we start with a non-negative \( p \)-supercaloric function \( v \) defined in \( \Omega_T \). By the device in the beginning of Section 3, we fix a small \( \delta > 0 \) and redefine \( v \) so that \( v(x, t) \equiv 0 \) when \( t \leq \delta \). This function is \( p \)-supercaloric. This does not affect the statement of the theorem. The initial condition \( v(x, 0) = 0 \) required in Theorem 2.7 is now in order.

Let \( Q_{2l} \subset \subset \Omega \) be a cube with side length \( 4l \) and consider the concentric cube
\[
Q_l = \{x||x_i - x^0_i| < l, i = 1, 2, \ldots n\}
\]
of side length \( 2l \). The center is at \( x^0 \). The main difficulty is that \( v \) is not zero on the lateral boundary, neither does \( v_j \) obey Theorem 2.7. We aim at correcting \( v \) outside \( Q_l \times (0, T) \) so that also the new function is \( p \)-supercaloric and, in addition, satisfies the requirements of zero boundary values in Theorem 2.7. Thus we study the function
\[
w = \begin{cases} v & \text{in } Q_l \times (0, T), \\ h & \text{in } (Q_{2l} \setminus Q_l) \times (0, T), \end{cases}
\]

where \( h \) is defined as the solution of the following boundary value problem:

\[
\begin{align*}
-\Delta_h & = 0 & & \text{in } \Omega \setminus (Q_{2l} \setminus Q_l), \\
\frac{\partial h}{\partial n} & = v - h & & \text{on } \partial \Omega \setminus (Q_{2l} \setminus Q_l), \\
\frac{\partial h}{\partial n} & = 0 & & \text{on } \partial (Q_{2l} \setminus Q_l).
\end{align*}
\]
where the function $h$ is, in the outer region, the weak solution to the boundary value problem

$$
\begin{cases}
    h = 0 & \text{on } \partial Q_l \times (0, T), \\
    h = v & \text{on } \partial Q_l \times (0, T), \\
    h = 0 & \text{on } (Q_{2l} \setminus Q_l) \times \{0\}.
\end{cases}
$$

(5.2)

An essential observation is that the solution $h$ does not always exist. This counts for the dichotomy. If it exists, the truncations $w_j$ satisfy the assumptions in Theorem 2.7, as we shall see.

For the construction we use the infimal convolutions

$$
v^\varepsilon(x, t) = \inf_{(y, \tau) \in \Omega_T} \left\{ v(y) + \frac{1}{2\varepsilon}(|x - y|^2 + |t - \tau|^2) \right\}.
$$

They are Lipschitz continuous in $Q_{2l} \times [0, T]$ and weak supersolutions when $\varepsilon$ is small enough. Then we define the solution $h^\varepsilon$ as in formula (5.2) above, but with $v^\varepsilon$ in place of $v$. Then we define

$$
w^\varepsilon = \begin{cases}
    v^\varepsilon & \text{in } Q_l \times (0, T), \\
    h^\varepsilon & \text{in } (Q_{2l} \setminus Q_l) \times (0, T),
\end{cases}
$$

and $w^\varepsilon(x, 0) = 0$ in $\Omega$. Now $h^\varepsilon \leq v^\varepsilon$, and when $t \leq \delta$ we have $0 \leq h^\varepsilon \leq v^\varepsilon = 0$ so that $h^\varepsilon(x, t) = 0$ when $t \leq \delta$. The function $w^\varepsilon$ satisfies the comparison principle and is therefore a $p$-supercaloric function. Here it is essential that $h^\varepsilon \leq v^\varepsilon$. The function $w^\varepsilon$ is also (locally) bounded; thus we have arrived at the conclusion that $w^\varepsilon$ is a weak supersolution in $Q_{2l} \times (0, T)$.

There are two possibilities, depending on whether the sequence $\{h^\varepsilon\}$ is bounded or not, when $\varepsilon \searrow 0$ through a sequence of values.

**Bounded case.** Assume that there does not exist any sequence of points $(x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$ such that

$$
\lim_{\varepsilon \to 0} h^\varepsilon(x_\varepsilon, t_\varepsilon) = \infty,
$$

where $x_0 \in Q_{2l} \setminus Q_l$ and $0 < t_0 < T$ (that is an interior limit point). By Proposition 4.6, the limit function $h = \lim_{\varepsilon \to 0} h^\varepsilon$ is $p$-caloric in its domain. The function $w = \lim_{\varepsilon \to 0} w^\varepsilon$ itself is $p$-supercaloric and agrees with formula (5.1).

By Theorem 2.5 the truncated functions $w_j = \min\{w(x, t), j\}$, $j = 1, 2, \ldots$, are weak supersolutions in $Q_{2l} \times (0, T)$. We claim that

$$
w_j \in L^p(0, T'; W^{1,p}_0(Q_{2l})) \quad \text{when } T' < T.
$$

This requires an estimation, where we use

$$
L = \sup\{h(x, t) : (x, t) \in (Q_{2l} \setminus Q_{3l/4}) \times (0, T')\}.
$$

Let $\zeta = \zeta(x)$ be a smooth cutoff function such that $0 \leq \zeta \leq 1$, $\zeta = 1$ in $Q_{2l} \setminus Q_{3l/2}$ and $\zeta = 0$ in $Q_{3l/4}$. Using the test function $\zeta^p h$ when
deriving the Caccioppoli estimate we get
\[
\int_0^{T'} \int_{Q_{2l} \setminus Q_{3l/2}} |\nabla w_j|^p \, dx \, dt \\
\leq \int_0^{T'} \int_{Q_{2l} \setminus Q_{3l/2}} |\nabla h|^p \, dx \, dt \leq \int_0^{T'} \int_{Q_{2l} \setminus Q_{5l/4}} \zeta^p |\nabla h|^p \, dx \, dt \\
\leq C(p) \left\{ \int_0^{T'} \int_{Q_{2l} \setminus Q_{l}} h^p|\nabla \zeta|^p \, dx \, dt + \int_0^{T'} \int_{Q_{2l} \setminus Q_{5l/4}} h(x,T')^2 \, dx \right\} \\
\leq C(n,p) (L^p l^{n-p} T + L^2 l^m),
\]
where we used the fact that $|\nabla w_j| = |\nabla \min\{h,j\}| \leq |\nabla h|$ in the outer region. Thus we have an estimate over the outer region $Q_{3l/2}$. Concerning the inner region $Q_{3l/2}$, we first choose a smooth cutoff function $\eta = \eta(x,t)$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $Q_{3l/2}$ and $\eta = 0$ in $Q_{2l} \setminus Q_{5l/4}$. Then the Caccioppoli estimate for the truncated functions $w_j$, $j = 1, 2, \ldots$, takes the form
\[
\int_0^{T'} \int_{Q_{3l/2}} |\nabla w_j|^p \, dx \, dt \\
\leq C j^p \int_0^{T'} \int_{Q_{2l}} |\nabla \eta|^p \, dx \, dt + C j^p \int_0^{T'} \int_{Q_{2l}} |\eta_t|^p \, dx \, dt.
\]
Thus we have obtained the estimate
\[
\int_0^{T'} \int_{Q_{2l}} |\nabla w_j|^p \, dx \, dt \leq C j^p
\]
over the whole domain $Q_{2l} \times (0, T')$ and it follows that $w_j \in L(0, T'; W_0^{1,p}(Q_{2l}))$. In particular, the crucial estimate
\[
\int_0^{T'} \int_{Q_{2l}} |\nabla w_1|^p \, dx \, dt < \infty,
\]
which was taken for granted in [10], is now established.\(^4\)

From Theorem 2.7 we conclude that $v \in L^q(Q_l)$ and $\nabla v \in L^{q'}(Q_l)$ with the correct summability exponents. Either we can proceed like this for all interior cubes, or the following case occurs.

**Unbounded case.** If there is a sequence $(x_\varepsilon, t_\varepsilon) \rightarrow (x_0, t_0)$ such that
\[
\lim_{\varepsilon \rightarrow 0} h^\varepsilon(x_\varepsilon, t_\varepsilon) = \infty
\]
for some $x_0 \in Q_{2l} \setminus \overline{Q_l}$, $0 < t_0 < T$, then
\[
v(x, t) \geq h(x, t) \geq (t - t_0)^{-\frac{1}{r-1}} \Omega(x),
\]
\(^4\)The class $M$ passed unnoticed in [10].
when \( t > t_0 \), according to Proposition 4.3. Thus \( v(x, t_0+) = \infty \) in \( Q_{2l} \setminus Q_l \). But in this construction we can replace the outer cube with \( \Omega \), that is, a new \( h \) is defined in \( \Omega \setminus Q_l \). Then by comparison

\[ v \geq h^\Omega \geq h^{Q_{2l}} \]

and so \( v(x, t_0+) = \infty \) in the whole boundary zone \( \Omega \setminus Q_l \).

It remains to include the inner cube \( Q_l \) in the argument. This is easy. Reflect \( h = h^{Q_{2l}} \) in the plane \( x_1 = x_0^1 + l \), which contains one side of the small cube by setting

\[ h^*(x_1, x_2, \ldots, x_n) = h(2x_1^0 + 2l - x_1, x_2, \ldots, x_n), \]

so that

\[ \frac{x_1 + (2(x_1^0 + l) - x_1)}{2} = x_1^0 + l \]

as it should. Recall that \( x_0^1 \) was the center of the cube. (The same can be done earlier for all the \( h^\varepsilon \).) The reflected function \( h^* \) is \( p \)-caloric. Clearly, \( v \geq h^* \) by comparison. This forces \( v(x, t_0+) = 0 \) when \( x \in Q_l, x_1 > x_0^1 \). A similar reflexion in the plane \( x_1 = x_0^1 - l \) includes the other half \( x_1 < x_0^1 \). We have achieved that \( v(x, t_0+) = \infty \) also in the inner cube \( Q_l \). This proves that

\[ v(x, t_0+) \equiv \infty \quad \text{in the whole} \quad \Omega. \]

6. The Porous Medium Equation

We consider the Porous Medium Equation

\[ \frac{\partial u}{\partial t} - \Delta (u^m) = 0 \]

in the slow diffusion case \( m > 1 \). The equation is treated in detail in the book [22]. We also mention [24] and [19]. In [11] the so-called\(^5\) viscosity supersolutions of the Porous Medium Equation were defined in an analogous way as the \( p \)-supercaloric functions. Thus they are lower semicontinuous functions \( v : \Omega_T \to [0, \infty] \), finite in a dense subset, obeying the Comparison Principle with respect to the solutions of the equation.

Again we get two totally distinct classes of solutions, called class \( \mathcal{B} \) and \( \mathcal{M} \). Now the discriminating summability exponent is \( m - 1 \). We begin with \( \mathcal{B} \).

**Theorem 6.1** (Class \( \mathcal{B} \)). Let \( m > 1 \). For a viscosity supersolution \( v : \Omega_T \to [0, \infty] \) the following conditions are equivalent:

(i) \( v \in L^{m-1}_{\text{loc}}(\Omega_T) \),

(ii) the Sobolev gradient \( \nabla (v^m) \) exists and \( \nabla (v^m) \in L^q_{\text{loc}}(\Omega_T) \) whenever \( q' < 1 + \frac{1}{1+nm} \).

\(^5\)The label "viscosity" was dubbed in order to distinguish them and has little to do with viscosity. The name "\( m \)-superporous function" would perhaps do instead?
(iii) \( v \in L^q_{\text{loc}}(\Omega_T) \) whenever \( q < m + \frac{2}{n} \).

A typical member of this class is the Barenblatt solution for the Porous Medium Equation. In this case a viscosity supersolution is a solution to a corresponding measure data problem with a Radon measure in a similar fashion as for the Evolutionary \( p \)-Laplace Equation. The other class of viscosity supersolutions is \( \mathcal{M} \). Unfortunately, this class was overlooked in [11].

**Theorem 6.2** (Class \( \mathcal{M} \)). Let \( m > 1 \). For a viscosity supersolution \( v : \Omega_T \to [0, \infty] \) the following conditions are equivalent:

1. \( v \notin L^{m-1}_{\text{loc}}(\Omega_T) \).
2. There is a time \( t_0, 0 < t_0 < T \), such that
   \[
   \liminf_{(y,t) \to (x,t_0)} \frac{v(y,t) - v(x,t_0)}{t - t_0} > 0 \quad \text{for all} \quad x \in \Omega.
   \]

Notice that again the infinities occupy the whole space at some instant \( t_0 \). In [11] Theorem 3.2 it was established that a bounded viscosity supersolution \( v \) is a weak supersolution to the equation:

\[
\int_0^T \int_\Omega \left( -v \frac{\partial \phi}{\partial t} + \langle \nabla v^m, \nabla \varphi \rangle \right) dx \, dt \geq 0
\]

whenever \( \varphi \in C_0^\infty(\Omega_T) \) and \( \varphi \geq 0 \).

We shall deduce the above theorems from the following result.

**Theorem 6.3.** Let \( m > 1 \). Suppose that \( v \geq 0 \) is a viscosity supersolution in \( \Omega_T \) with initial values \( v(x,0) = 0 \) in \( \Omega \). If

\[
\min\{v^m, j\} \in L^2(0,T;W^{1,2}_{\text{loc}}(\Omega)), \quad j = 1, 2, \ldots,
\]

then

1. \( v \in L^q(\Omega_{T_1}) \) whenever \( q < 1 + \frac{2}{n} \) and \( T_1 < T \),
2. the function \( v^m \) has a Sobolev gradient \( \nabla(v^m) \in L^{q'}(\Omega_{T_1}) \) whenever \( q' < 1 + \frac{1}{1+mn} \) and \( T_1 < T \).

The summability exponents are sharp.

**Proof.** See [11], Theorem 4.7 and 4.8. \( \square \)

We start from the intrinsic Harnack inequality given in [3, Theorem 3]. This is the fundamental analytic tool here.

**Lemma 6.4** (Harnack’s inequality). Let \( m > 1 \). There are constants \( C \) and \( \gamma \), depending only on \( n \) and \( m \), such that if \( u > 0 \) is a continuous weak solution in

\[
B(x_0,4R) \times (t_0 - 4\theta, t_0 + 4\theta), \quad \text{where} \quad \theta = \frac{CR^2}{u(x_0,t_0)^{m-1}},
\]

\[
\Omega \cap (B(x_0,4R) \times (t_0 - 4\theta, t_0 + 4\theta))
\]

then
\[
\frac{u(x_0,t_0)}{u(x,t)} \leq C \quad \text{for all} \quad x \in B(x_0,4R) \quad \text{and} \quad t \in (t_0 - 4\theta, t_0 + 4\theta).
\]
then the inequality
\[(6.5) \quad u(x_0, t_0) \leq \gamma \inf_{B_R(x_0)} u(x, t_0 + \theta)\]
is valid.

Again the waiting time \(\theta\) depends on the solution itself. Then we need the separable solution
\[G_m(x) \left( t - t_0 \right)^{\frac{1}{m-1}}\]
where the function \(G_m \in W^{1,2}_0(\Omega)\) is a weak solution of the auxiliary equation
\[\Delta(G_m) + \frac{G_m}{m-1} = 0,\]
which is the Euler-Lagrange Equation of the variational integral
\[\int_{\Omega} |\nabla(u_m)|^2 \, dx \quad \int_{\Omega} |u|^{m+1} \, dx.\]
This function is known as “the Friendly Giant”, see [V, p. 111] and often serves as a minorant. When extended as 0 when \(t < t_0\) it becomes a viscosity supersolution in the whole \(\Omega \times \mathbb{R}\).

**Proposition 6.6.** Suppose that we have an increasing sequence \(0 \leq h_1 \leq h_2 \leq h_3 \leq \ldots\) of viscosity supersolutions in \(\Omega_T\) and denote \(h = \lim_{k \to \infty} h_k\). If there is a sequence \((x_k, t_k) \to (x_0, t_0)\) such that \(h_k(x_k, t_k) \to \infty\), where \(x_0 \in \Omega\) and \(0 < t_0 < T\), then
\[\liminf_{(y, t) \to (x, t_0)} h(y, t)(t - t_0)^{\frac{1}{m-1}} > 0 \quad \text{for all} \quad x \in \Omega.\]
Thus, at time \(t_0\),
\[\lim_{(y, t) \to (x, t_0)} h(y, t) \equiv \infty \quad \text{in} \quad \Omega.\]

**Remark 6.7.** Notice that the limit function is not a solution in the whole domain, but it may, nonetheless, be finite at each point. (This is different from the Heat Equation, see [23].)

If it so happens that the subsequence in the Proposition does not exist, then we have the normal situation with a solution:

**Proposition 6.8.** Suppose that we have an increasing sequence \(0 \leq h_1 \leq h_2 \leq h_3 \leq \ldots\) of viscosity supersolutions in \(\Omega_T\) and denote \(h = \lim_{k \to \infty} h_k\). If the sequence \(\{h_k\}\) is locally bounded, then the limit function \(h\) is a supersolution in \(\Omega_T\).

After this, the proof proceeds along the same lines as for the \(p\)-parabolic equation. A difference is that the infimal convolution should be replaced by the solution to an obstacle problem as in Chapter 5 of [11].
References

[1] G. Barenblatt: *On self-similar motions of a compressible fluid in a porous medium*, Prikladnaja Matematika & Mekhanika 16 1952, pp. 679–698. —In Russian.

[2] L. Boccardo, A. Dall’Aglio, T. Gallouët, L. Orsina: *Nonlinear parabolic equations with measure data*, Journal of Functional Analysis 147, 1997, pp. 237–258.

[3] E. DiBenedetto: *Intrinsic Harnack type inequalities for solutions of certain degenerate parabolic equations*, Archive for Rational Mechanics and Analysis 100, 1988, pp. 129–147.

[4] E. DiBenedetto: Degenerate Parabolic Equations, Springer Verlag, Berlin-Heidelberg-New York 1993.

[5] E. DiBenedetto, U. Gianazza, V. Vespri: *Harnack estimates for quasilinear degenerate parabolic differential equations*, Acta Mathematica 200, 2008, pp. 181–209.

[6] E. DiBenedetto, U. Gianazza, V. Vespri: Harnack’s Inequality for Degenerate and Singular Parabolic Equations, Springer, Berlin-Heidelberg-New York 2012.

[7] P. Juutinen, P. Lindqvist, J. Manfredi: *On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation*, SIAM Journal on Mathematical Analysis 33, 2001, pp. 699–717.

[8] T. Kilpeläinen, P. Lindqvist: *On the Dirichlet boundary value problem for a degenerate parabolic equation*, SIAM Journal on Mathematical Analysis 27, 1996, pp. 661–683.

[9] J. Kinnunen, P. Lindqvist: *Pointwise behaviour of semicontinuous supersolutions to a quasilinear parabolic equation*, Annali di Matematica Pura ed Applicata (4) 185, 2006, pp. 411–435.

[10] J. Kinnunen, P. Lindqvist: *Summability of semicontinuous supersolutions to a quasilinear parabolic equation*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze (Serie V) 4, 2005, pp. 59–78.

[11] J. Kinnunen, P. Lindqvist: *Definition and properties of supersolutions to the porous medium equation*, Journal für die reine und angewandte Mathematik 618, 2008, pp. 135–168.

[12] J. Kinnunen, T. Lukkari, M. Parviainen: *An existence result for superparabolic functions*, Journal of Functional Analysis 258, 2010, pp. 713–728.

[13] T. Kuusi: *Lower semicontinuity of weak supersolutions to non-linear parabolic equations*, Differential and Integral Equations 22, 2009, pp. 1211–1222.

[14] T. Kuusi, P. Lindqvist, M. Parviainen: *Shadows of Infinites*, Manuscript 2014, arXiv:1406:6309.

[15] T. Kuusi, G. Mingione: *Riesz potentials and nonlinear parabolic equations*, Archive for Rational Mechanics Analysis 212, 2014, pp. 727–780.

[16] T. Kuusi, G. Mingione: *The Wolff gradient bound for degenerate parabolic equations*, Journal of European Mathematical Society 16(4), 2014, pp. 835–892.

[17] P. Lindqvist, J. Manfredi: *Viscosity supersolutions of the evolutionary p-Laplace equation*, Differential and Integral Equations 20, 2007, pp. 1303–1319.

[18] P. Lindqvist, J. Manfredi: *Note on a remarkable superposition for a quasi-linear equation*, Proceedings of the American Mathematical Society 136, 2008, pp. 136–140.

[19] A. Samarskii, V. Galaktionov, S. Kurdyumov, A. Mikhailov: Blow-up in Quasilinear Parabolic Equations, Walter de Gruyter & Co., Berlin 1995.
[20] N. Trudinger: *On Harnack type inequalities and their application to quasilinear elliptic equations*, Communications on Pure and Applied Mathematics 20, 1967, pp. 721–747.

[21] N. Trudinger: *Pointwise estimates and quasilinear parabolic equations*, Communications on Pure and Applied Mathematics 21, 1968, pp. 205–226.

[22] J. Vázquez: *The Porous Medium Equation* Mathematical Theory, Oxford Mathematical Monographs, Clarendon Press 2007.

[23] N. Watson: *Introduction to Heat Potential Theory*, Mathematical Surveys and Monographs 182, American Mathematical Society, Providence RI 2012.

[24] Z. Wu, J. Zhao, J. Yin, H. Li: *Nonlinear Diffusion Equations*, World Scientific, Singapore 2001.

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