TAYLOR’S SERIES EXPANSIONS FOR REAL POWERS OF FUNCTIONS
CONTAINING SQUARES OF INVERSE (HYPERBOLIC) COSINE FUNCTIONS,
EXPLICIT FORMULAS FOR SPECIAL PARTIAL BELL POLYNOMIALS, AND
SERIES REPRESENTATIONS FOR POWERS OF CIRCULAR CONSTANT

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Dedicated to people facing and battling COVID-19

Abstract. In the paper, by virtue of expansions of two finite products of finitely many square sums, with
the aid of series expansions of composite functions of (hyperbolic) sine and cosine functions with inverse
sine and cosine functions, and in the light of properties of partial Bell polynomials, the author establishes
Taylor’s series expansions of real powers of two functions containing squares of inverse (hyperbolic) cosine
functions in terms of the Stirling numbers of the first kind, presents an explicit formula of specific partial
Bell polynomials at a sequence of derivatives of a function containing the square of inverse cosine function,
derives several combinatorial identities involving the Stirling numbers of the first kind, demonstrates
several series representations of the circular constant Pi and its real powers, recovers series expansions
of positive integer powers of inverse (hyperbolic) sine functions in terms of the Stirling numbers of the
first kind, and also deduces other useful, meaningful, and significant conclusions.

CONTENTS

1. Simple preliminaries 1
2. Motivations 2
3. Important lemmas 4
4. Taylor’s series expansions of \[ \left(\arccos x\right)^2 \] and \[ \left(\arccosh x\right)^2 \] 12
5. Taylor’s series expansions of \[ \left(\arccos x\right)^{2\alpha} \] and \[ \left(\arccosh x\right)^{2\alpha} \] 17
6. Recovering Maclaurin’s series expansion of \( (\arcsin x)^{\alpha} \) 19
7. Conclusions 20
8. Declarations 20
References 21

1. Simple preliminaries

In this paper, we use the notation
\[ \mathbb{N} = \{1, 2, \ldots\}, \quad \mathbb{N}_0 = \{0, 1, 2, \ldots\}, \quad \mathbb{N}_{-} = \{-1, -2, \ldots\}, \quad \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}. \]
The classical Euler gamma function \( \Gamma(z) \) can be defined [37, Chapter 3] by
\[ \Gamma(z) = \lim_{n \to \infty} \frac{n!n^z}{\prod_{k=0}^{n}(z + k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}. \]
The rising factorial, or say, the Pochhammer symbol, of \( \beta \in \mathbb{C} \) is defined [11, p. 7497] by
\[ (\beta)_n = \prod_{k=0}^{n-1}(\beta + k) = \begin{cases} \beta(\beta + 1)\cdots(\beta + n - 1), & n \in \mathbb{N}; \\ 1, & n = 0. \end{cases} \] (1.1)

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inverse sine function; inverse hyperbolic sine function; Stirling number of the first kind; combinatorial identity; composite;
series representation; circular constant; partial Bell polynomial; explicit formula.

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For \( \alpha, \beta \in \mathbb{C} \) with \( \alpha + \beta \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} \), extended Pochhammer symbol \((\beta)_\alpha\) is defined [12] by

\[
(\beta)_\alpha = \frac{\Gamma(\alpha + \beta)}{\Gamma(\beta)}.
\]

The falling factorial for \( \beta \in \mathbb{C} \) and \( n \in \mathbb{N} \) is defined by

\[
(\beta)_n = \prod_{k=0}^{n-1} (\beta - k) = \begin{cases} 
\beta(\beta - 1) \cdots (\beta - n + 1), & n \in \mathbb{N}; \\
1, & n = 0.
\end{cases}
\]

The extended binomial coefficient \( \binom{z}{w} \) is defined [39] by

\[
\binom{z}{w} = \begin{cases} 
\frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}, & z \notin \mathbb{N}_+, \ w, z-w \notin \mathbb{N}_-; \\
0, & z \notin \mathbb{N}_- \text{ or } z-w \notin \mathbb{N}_-; \\
\frac{(z)_w}{w!}, & z \in \mathbb{N}_-, \ w \in \mathbb{N}_0; \\
\frac{(z)_{z-w}}{(z-w)!}, & z, w \in \mathbb{N}_-, \ z-w \in \mathbb{N}_0; \\
0, & z \in \mathbb{N}_-, \ w \in \mathbb{N}_0; \\
\infty, & z \in \mathbb{N}_-, \ w \notin \mathbb{Z}.
\end{cases}
\]

The Stirling numbers of the first kind \( s(n, k) \) for \( n \geq k \geq 0 \) can be generated [37, p. 20, (1.30)] by

\[
\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n, k) \frac{x^n}{m^n}, \quad |x| < 1
\]

and satisfy diagonal recursive relations

\[
s(n+k, k) = \sum_{\ell=0}^{n} (-1)^\ell \binom{k}{\ell} \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} s(n+m, m) \binom{n}{n+m}
\]

and

\[
s(n, k) = (-1)^k \sum_{m=1}^{n} (-1)^m \sum_{\ell=k-m}^{k-1} (-1)^\ell \binom{n}{\ell} \binom{\ell}{k-m} s(n-\ell, k-\ell)
\]

\[
= (-1)^n-k-1 \sum_{\ell=0}^{k-1} (-1)^\ell \binom{n}{\ell} \binom{\ell-1}{k-n-1} s(n-\ell, k-\ell)
\]

in [24, p. 23, Theorem 1.1] and [28, p. 156, Theorem 4].

The partial Bell polynomials, or say, the Bell polynomials of the second kind, can be denoted and defined by

\[
B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{\ell_1+\cdots+\ell_n = k} \frac{n!}{\ell_1! \cdots \ell_n!} \prod_{i=1}^{n-k+1} \frac{x_i^{\ell_i}}{\ell_i!},
\]

in [7, p. 412, Definition 11.2] and [8, p. 134, Theorem A].

2. Motivations

Let \( f(z) \) and \( h(z) \) be infinitely differentiable functions such that the function \( f(z) \) has the formal series expansion \( f(z) = \sum_{k=0}^{\infty} c_k z^k \) and the composite function \( h(f(z)) \) is defined on a non-empty open interval. A natural problem is to find the series expansion of the composite function \( h(f(z)) \). This problem can be regarded as how to compute derivatives of the composite function \( h(f(z)) \). There have been a long history and a number of literature in textbooks, handbooks, monographs, and research articles on this problem. See the references [1, 12, 31, 32, 37], for example.

In the above general theory, the cases \( h(z) = z^r \) for \( r \in \mathbb{C} \setminus \{1\} \) and \( f(z) \) being concrete elementary functions are of special interest and attract some mathematicians. We recall some results as follows.
In [38, p. 377, (3.5)] and [40, pp. 109–110, Lemma 1], it was obtained that

\[ I_\mu(x)I_\nu(x) = \frac{1}{\Gamma(\mu + 1)\Gamma(\nu + 1)} \sum_{n=0}^{\infty} \frac{(\mu + \nu + n + 1)_n}{n!} (\frac{x}{2})^{2n+\mu+\nu}, \]

where the first kind modified Bessel function \( I_\nu(z) \) can be represented [1, p. 375, 9.6.10] by

\[ I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(\nu + n + 1)} \left( \frac{z}{2} \right)^{2n+\nu}, \quad z \in \mathbb{C}. \]

In [4, p. 310], the power series expansion

\[ [I_\nu(z)]^2 = \sum_{k=0}^{\infty} \frac{1}{k!(\nu + k + 1)^2} \left( \frac{2k + 2\nu}{k} \right) \left( \frac{z}{2} \right)^{2k+2\nu} \]

was listed. As for the series expansion of the function \( [I_\nu(z)]^r \) for \( \nu \in \mathbb{C} \setminus \{-1, -2, \ldots\} \) and \( r, z \in \mathbb{C} \), please refer to [3, 4, 13, 14, 19]. One of the reasons why one investigated the series expansions of the functions \( [I_\nu(z)]^r \) is that the products of the (modified) Bessel functions of the first kind appear frequently in problems of statistical mechanics and plasma physics, see [2, 20, 21].

In the papers [5, 6, 10, 11, 17, 23, 27, 31], Maclaurin’s series expansions of the powers

\[
\begin{align*}
\frac{\arcsin z}{x}^m, & \quad \frac{\sinh z}{x}^m, \\
\frac{\arcsinh z}{\sqrt{1-z^2}}^m, & \quad \frac{\sinh z}{\sqrt{1+z^2}}^m, \\
\frac{\arctan z}{x}^m, & \quad \frac{\tanh z}{x}^m, \\
\frac{\tan z}{x}^m, & \quad \frac{\sec z}{x}^m, \\
\frac{\cot z}{x}^m, & \quad \frac{\csc z}{x}^m
\end{align*}
\]

for \( m \geq 2 \) and their history were reviewed, surveyed, established, discussed, and applied. Here now we recite the following two series expansions.

**Theorem 2.1** ([10, Theorem 2.1]). For \( k \in \mathbb{N} \) and \( |x| < 1 \), the function \( \frac{\arcsin x}{x}^k \), whose value at \( x = 0 \) is defined to be 1, has Maclaurin’s series expansion

\[
\left( \frac{\arcsin x}{x} \right)^k = 1 + \sum_{m=1}^{\infty} (-1)^m Q(k, 2m) \frac{(2x)^{2m}}{(k+2m)!},
\]

where

\[
Q(k, m) = \sum_{\ell=0}^{m} \left( \frac{k + \ell - 1}{k - 1} \right) s(k + m - 1, k + \ell - 1) \left( \frac{k + m - 2}{2} \right)^\ell
\]

for \( k \in \mathbb{N} \) and \( m, \ell \geq 2 \).

**Theorem 2.2** ([10, Theorem 5.1]). For \( k \in \mathbb{N} \) and \( |x| < \infty \), the function \( \frac{\arcsinh x}{x}^k \), whose value at \( x = 0 \) is defined to be 1, has Maclaurin’s series expansion

\[
\left( \frac{\arcsinh x}{x} \right)^k = 1 + \sum_{m=1}^{\infty} Q(k, 2m) \frac{(2x)^{2m}}{(k+2m)!},
\]

where \( Q(k, 2m) \) is given by (2.2).

In the papers [10, 11], the series expansion (2.1) has been applied to derive closed-form formulas for specific partial Bell polynomials and to establish series representations of the generalized logsine function. These results are needed and considered in [9, 15, 22] respectively.

In the community of mathematics, the circular constant \( \pi \) has attracted a number of mathematicians spending long time and utilizing many methods to calculate it. The setting-up of the international Pi Day is the best demonstration of the importance of the circular constant \( \pi \). Taking \( x = \frac{\pi}{2\sqrt{2}} \) in (2.1) produces the series representation

\[
\left( \frac{\pi}{2\sqrt{2}} \right)^k = 1 + k! \sum_{m=1}^{\infty} (-1)^m 2^m Q(k, 2m) \frac{1}{(k+2m)!}.
\]

In this paper, by virtue of expansions of two finite products of finitely many square sums

\[
\prod_{\ell=1}^{k} (\ell^2 + \alpha^2) \quad \text{and} \quad \prod_{\ell=1}^{k} \left( (2\ell - 1)^2 + \alpha^2 \right)
\]
for $k \in \mathbb{N}$ in Lemmas 3.1 and 3.2 below, with the aid of Taylor’s series expansions around $x = 1^-$ of the functions $\cosh(\alpha \arccos x)$ and $\cos(\alpha \arccos x)$ in Lemma 3.3, and in the light of properties of partial Bell polynomials selected in Lemma 3.4, we will

1. establish Taylor’s series expansions around $x = 1^-$ of the functions

$$\left[ \frac{(\arccos x)^2}{2(1-x)} \right]^\alpha$$

and

$$\left[ \frac{(\arccosh x)^2}{2(1-x)} \right]^k$$

for $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$ in terms of $Q(k, m)$ in Theorems 4.1 and 5.2;

2. present an explicit formula of the specific partial Bell polynomials

$$B_{m,k}\left( -\frac{1}{12}, \frac{2}{45}, -\frac{3}{70}, \frac{32}{525}, -\frac{80}{693}, \ldots, \frac{(2m-2k+2)!}{(2m-2k+4)!}Q(2,2m-2k+2) \right)$$

for $m \geq k \in \mathbb{N}$ in Theorem 5.1;

3. derive several combinatorial identities involving the Stirling numbers of the first kind $s(n, k)$ in Lemmas 3.1 and 3.2, in Corollaries 4.7 and 5.1, and in the proof of Theorem 5.1;

4. demonstrate several series representations of the circular constant $\pi$ and its powers $\pi^\alpha$ for $\alpha \in \mathbb{R}$ in Corollaries 4.2 and 5.2;

5. recover Maclaurin’s series expansions (2.1) and (2.3) in Theorem 2.1 and 2.2; and

6. also deduce other useful, meaningful, and significant conclusions in Corollaries 4.1, 4.3, 4.4, 4.5, and 4.6, in Remarks 3.3 and 4.1, and elsewhere in this paper.

3. Important Lemmas

For attaining our aims mentioned just now, we need the following four important lemmas.

**Lemma 3.1.** For $k \in \mathbb{N}$ and $\alpha \in \mathbb{C} \setminus \{0\}$, we have

$$\prod_{\ell=1}^{k} (\ell^2 + \alpha^2) = (-1)^k \prod_{j=0}^{k-1} (-1)^j \left[ \sum_{\ell=2j+1}^{2k+1} \left( \frac{\ell}{2j+1} \right) s(2k+1, \ell) k^{\ell-2j-1} \right] \alpha^{2j} \quad (3.1)$$

and

$$\sum_{\ell=0}^{2k} (\ell + 1) s(2k+1, \ell + 1) k^\ell = (-1)^k (k!)^2. \quad (3.2)$$

For $0 \leq j \leq k - 1$, we have

$$\sum_{\ell=2j+1}^{2k-1} \left( \frac{\ell}{2j} \right) s(2k-1, \ell)(k-1)^{\ell-2j} = -s(2k-1, 2j). \quad (3.3)$$

**Proof.** In [35, p. 165, (12.1)], there exists the formula

$$n! \frac{z^n}{n!} = \sum_{k=0}^{n} s(n, k) z^k, \quad z \in \mathbb{C}, \quad n \geq 0. \quad (3.4)$$

It is not difficult to verify that

$$\prod_{\ell=1}^{k} [(\ell - 1)^2 + \alpha^2] = \prod_{\ell=0}^{k-1} (\ell^2 + \alpha^2)$$

$$= (-1)^k \frac{(i \alpha)_k}{(i \alpha + 1)_{-k}}$$

$$= (-1)^k i \alpha \frac{\Gamma(i \alpha + k)}{\Gamma(i \alpha - k + 1)}$$

$$= (-1)^k i \alpha (2k-1)! \left( \frac{i \alpha + k - 1}{2k-1} \right)$$

for $\alpha \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{N}$, where $i = \sqrt{-1}$ is the imaginary unit and $\alpha \in \mathbb{C}$. Substituting the formula (3.4) into (3.5) results in

$$\prod_{\ell=0}^{k-1} (\ell^2 + \alpha^2) = (-1)^k i \alpha \sum_{\ell=0}^{2k-1} s(2k-1, \ell)(i \alpha + k - 1)^\ell$$
Further regarding $\alpha$ which is equivalent to the combinatorial identity (3.3). The proof of Lemma 3.1 is complete.

**Lemma 3.2.** For $\alpha \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{N}$, we have

$$\prod_{\ell=1}^{k} (\ell^2 + \alpha^2) = (-1)^k i^k \frac{2^{k-1}}{\alpha} \sum_{\ell=0}^{2k-1} \sum_{j=0}^{\ell} \binom{\ell}{j} (i\alpha)^j (k-1)^{\ell-j}$$

$$= (-1)^k \frac{2^{k-1}}{\alpha} \sum_{j=0}^{\ell} \sum_{\ell=j}^{2k-1} \binom{\ell}{j} s(2k-1, \ell) (k-1)^{\ell-j} (i\alpha)^j$$

$$= (-1)^k \frac{2^{k-1}}{\alpha} \sum_{j=0}^{\ell} \sum_{\ell=j}^{2k-1} \binom{\ell}{j} s(2k-1, \ell) (k-1)^{\ell-j} \alpha^{j+1} \cos \left( \frac{(j+1)\pi}{2} \right)$$

$$+ i(-1)^k \frac{2^{k-1}}{\alpha} \sum_{j=0}^{\ell} \sum_{\ell=j}^{2k-1} \binom{\ell}{j} s(2k-1, \ell) (k-1)^{\ell-j} \alpha^{j+1} \sin \left( \frac{(j+1)\pi}{2} \right)$$

for $\alpha \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{N}$, where we used the identity

$$i^k = \cos \frac{k\pi}{2} + i \sin \frac{k\pi}{2}, \quad k \geq 0. \quad (3.6)$$

As a result, equating the real and imaginary parts, we obtain

$$\prod_{\ell=0}^{k-1} (\ell^2 + \alpha^2) = (-1)^k \frac{2^{k-1}}{\alpha} \sum_{j=0}^{2k-1} \cos \left( \frac{(j+1)\pi}{2} \right) s(2k-1, j) + \sum_{\ell=j+1}^{2k-1} \binom{\ell}{j} s(2k-1, \ell) (k-1)^{\ell-j} \alpha^{j+1} \quad (3.7)$$

and

$$\sum_{j=0}^{2k-1} \sin \left( \frac{(j+1)\pi}{2} \right) s(2k-1, j) + \sum_{\ell=j+1}^{2k-1} \binom{\ell}{j} s(2k-1, \ell) (k-1)^{\ell-j} \alpha^{j+1} = 0. \quad (3.8)$$

Since $\cos(j\pi) = (-1)^j$ and $\cos \left( \frac{(2j+1)\pi}{2} \right) = 0$ for $j \in \mathbb{Z}$, the equality (3.7) can be simplified as (3.1). Taking $\alpha \to 0$ in (3.1) reduces to (3.2).

Since $\sin(j\pi) = 0$ and $\sin \left( \frac{(2j+1)\pi}{2} \right) = (-1)^j$ for $j \in \mathbb{Z}$, the equality (3.8) becomes

$$\sum_{j=0}^{2k-1} (-1)^j s(2k-1, 2j) + \sum_{\ell=2j+1}^{2k-1} \binom{\ell}{2j} s(2k-1, \ell) (k-1)^{\ell-2j} \alpha^{2j+1} = 0.$$

Further regarding $\alpha$ as a variable leads to

$$(-1)^j s(2k-1, 2j) + \sum_{\ell=2j+1}^{2k-1} \binom{\ell}{2j} s(2k-1, \ell) (k-1)^{\ell-2j} = 0$$

which is equivalent to the combinatorial identity (3.3). The proof of Lemma 3.1 is complete. \hfill \Box

**Lemma 3.2.** For $k \in \mathbb{N}$ and $\alpha \in \mathbb{C} \setminus \{0\}$, we have

$$\prod_{\ell=1}^{k} ((2\ell - 1)^2 + \alpha^2) = (-1)^k \frac{2^{k-1}}{\alpha^2} \sum_{j=0}^{2k-1} \frac{1}{2\ell} \binom{2\ell}{2j} s(2k, \ell) (2k-1)^{\ell-2j} \alpha^{2j} \quad (3.9)$$

and

$$\sum_{\ell=2j+1}^{2k} s(2k, \ell) \left( k - \frac{1}{2} \right)^\ell = (-1)^k \left[ \frac{(2k-1)!!}{2k} \right]^2. \quad (3.10)$$

For $0 \leq j < k \in \mathbb{N}$, we have

$$\sum_{\ell=2j+1}^{2k} \binom{\ell}{2j+1} s(2k, \ell) \left( k - \frac{1}{2} \right)^\ell = 0. \quad (3.11)$$

**Proof.** The identities in (3.5) can be rearranged as

$$\prod_{\ell=1}^{k} (\ell^2 + \alpha^2) = (-1)^{k+1} \frac{1}{\alpha^2} \frac{(i\alpha)_{k+1}}{\alpha^2 (i\alpha + 1)_{-(k+1)}}$$

$$= (-1)^{k+1} \Gamma(i\alpha + k + 1) \alpha^k \Gamma(i\alpha - k)$$
for $\alpha \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{N}$. By this, we acquire
\[
\prod_{\ell=1}^{k} [(2\ell - 1)^2 + \alpha^2] = \frac{\prod_{\ell=1}^{2k}(\ell^2 + \alpha^2)}{\prod_{\ell=1}^{2k}[\alpha^2 + (2\ell)^2]} = \frac{\prod_{\ell=1}^{2k}(\ell^2 + \alpha^2)}{4^k \prod_{\ell=1}^{k}[(\alpha/2)^2 + \ell^2]}
\]
\[= \frac{1}{2^{2k+1}} \frac{\Gamma(i\alpha + 2k + 1)}{\alpha \Gamma(i\alpha/2 - k)} \frac{1}{\alpha \Gamma(i\alpha/2 - k)} (-1)^{k+1} \frac{\Gamma(2i\alpha/2 + 1)}{2 \Gamma(i\alpha/2 + k + 1)}
\]
for $\alpha \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{N}$. Making use of the Gauss multiplication formula
\[\Gamma(nz) = \frac{n^{nz-1/2}}{(2\pi)^{(n-1)/2}} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right)
\]
in [1, p. 256, 6.1.20] leads to
\[
\frac{\Gamma(i\alpha - 2k)}{\Gamma(i\alpha/2 - k)} = \frac{1}{\Gamma(i\alpha/2 - k)} \frac{2^{i\alpha - 2k - 1/2}}{(2\pi)^{1/2}} \Gamma\left(\frac{i\alpha}{2} - k\right) \Gamma\left(\frac{i\alpha}{2} - k + \frac{1}{2}\right)
\]
and
\[
\frac{\Gamma(i\alpha + 2k + 1)}{\Gamma(i\alpha/2 + k + 1)} = \frac{\Gamma(2i\alpha/2 + k)}{(2\pi)^{1/2} \Gamma(i\alpha/2 + k + 1)}
\]
for $\alpha \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{N}$. Substituting these two equalities into (3.12), utilizing the formula (3.4), interchanging the order of double sums, and employing the identity (3.6) yield
\[
\prod_{\ell=1}^{k} [(2\ell - 1)^2 + \alpha^2] = (-1)^{k+1} \frac{(2k + 1)!}{\alpha} \left(\frac{i\alpha + k}{2k + 1}\right)
\]
for $\alpha \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{N}$. By this, we acquire
\[
\prod_{\ell=1}^{k} [(2\ell - 1)^2 + \alpha^2] = \frac{\prod_{\ell=1}^{2k}(\ell^2 + \alpha^2)}{\prod_{\ell=1}^{2k}[\alpha^2 + (2\ell)^2]} = \frac{\prod_{\ell=1}^{2k}(\ell^2 + \alpha^2)}{4^k \prod_{\ell=1}^{k}[(\alpha/2)^2 + \ell^2]}
\]
\[= \frac{1}{2^{2k+1}} \frac{\Gamma(i\alpha + 2k + 1)}{\alpha \Gamma(i\alpha/2 - k)} \frac{1}{\alpha \Gamma(i\alpha/2 - k)} (-1)^{k+1} \frac{\Gamma(2i\alpha/2 + 1)}{2 \Gamma(i\alpha/2 + k + 1)}
\]
for $\alpha \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{N}$. Making use of the Gauss multiplication formula
\[\Gamma(nz) = \frac{n^{nz-1/2}}{(2\pi)^{(n-1)/2}} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right)
\]
in [1, p. 256, 6.1.20] leads to
\[
\frac{\Gamma(i\alpha - 2k)}{\Gamma(i\alpha/2 - k)} = \frac{1}{\Gamma(i\alpha/2 - k)} \frac{2^{i\alpha - 2k - 1/2}}{(2\pi)^{1/2}} \Gamma\left(\frac{i\alpha}{2} - k\right) \Gamma\left(\frac{i\alpha}{2} - k + \frac{1}{2}\right)
\]
and
\[
\frac{\Gamma(i\alpha + 2k + 1)}{\Gamma(i\alpha/2 + k + 1)} = \frac{\Gamma(2i\alpha/2 + k)}{(2\pi)^{1/2} \Gamma(i\alpha/2 + k + 1)}
\]
for $\alpha \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{N}$. Substituting these two equalities into (3.12), utilizing the formula (3.4), interchanging the order of double sums, and employing the identity (3.6) yield
\[
\prod_{\ell=1}^{k} [(2\ell - 1)^2 + \alpha^2] = (-1)^{k+1} \frac{(2k + 1)!}{\alpha} \left(\frac{i\alpha + k}{2k + 1}\right)
\]
For Lemma 3.3.

Comparing this with the definition (1.1), we can regard (3.1) and (3.9) in Lemmas 3.1 and 3.2 as generalizations of the formula (3.13).

Regarding (2, 2k) can be reformulated as
\[(z)_n = \sum_{k=0}^{n} (-1)^{n-k} s(n, k) z^k. \tag{3.13} \]

Comparing this with the definition (1.1), we can regard (3.1) and (3.9) in Lemmas 3.1 and 3.2 as generalizations of the formula (3.13).

Remark 3.1. The combinatorial identities (3.2) and (3.10) can be rearranged as
\[ Q(2, 2k) = (-1)^k (k!)^2, \quad k \in \mathbb{N} \tag{3.14} \]
and
\[ Q(1, 2k) = (-1)^k \left( \frac{(2k - 1)!!}{2^k} \right)^2, \quad k \in \mathbb{N}. \tag{3.15} \]

Due to the trivial result \( s(n, n) = 1 \) for \( n \in \mathbb{N}_0 \), the combinatorial identities (3.3) and (3.11) can be rearranged as
\[ s(2j + 1, 2j) = -j(2j + 1), \quad Q(2j + 1, 2m - 1) = 0 \]
for \( m \geq 2 \) and \( j \in \mathbb{N}_0 \), and
\[ \sum_{\ell=1}^{2m} \binom{2j + \ell}{2j + 1} s(2j + 2m, 2j + \ell) \left( j + m - \frac{1}{2} \right)^\ell = 0 \]
for \( j \in \mathbb{N}_0 \) and \( m \in \mathbb{N} \). On the other hand, we have
\[ Q(2j, 2m) = s(2j + 2m - 1, 2j - 1) + 2j(j + m - 1)s(2j + 2m - 1, 2j) + 2j(j + m - 1) \sum_{\ell=1}^{2m-1} \frac{1}{\ell + 1} \binom{2j + \ell}{2j} s(2j + 2m - 1, 2j + \ell)(j + m - 1)^\ell \]
for \( j, m \in \mathbb{N} \). Can one give a simple form for the quantity
\[ \sum_{\ell=1}^{2m-1} \frac{1}{\ell + 1} \binom{2j + \ell}{2j} s(2j + 2m - 1, 2j + \ell)(j + m - 1)^\ell \]
for \( j, m \in \mathbb{N} \)? Can one discover more simple forms, similar to \( Q(1, 2k) \) and \( Q(2, 2k) \) in (3.14) and (3.15), of \( Q(k, m) \) for some \( k \in \mathbb{N} \) and \( m \geq 2 \)?

Lemma 3.3. For \( \alpha \in \mathbb{C} \setminus \{0\} \) and \( |x| < 1 \), we have
\begin{align*}
cosh(\alpha \arcsin x) &= \sum_{k=0}^{\infty} \left( \prod_{\ell=1}^{k} \left[ 4(\ell - 1)^2 + \alpha^2 \right] \right) \frac{x^{2k}}{(2k)!}, \tag{3.16} \\
sinh(\alpha \arcsin x) &= \alpha \sum_{k=0}^{\infty} \left( \prod_{\ell=1}^{k} \left[ (2\ell - 1)^2 + \alpha^2 \right] \right) \frac{x^{2k+1}}{(2k+1)!}. \tag{3.17} 
\end{align*}
\[
cosh(\alpha \arccos x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k-1)!!} \left( \prod_{\ell=1}^{k} [4(\ell - 1)^2 - \alpha^2] \right) \frac{(x-1)^k}{k!} \tag{3.18}
\]
\[
= \left( \cosh \frac{\alpha \pi}{2} \right) \sum_{k=0}^{\infty} \left( \prod_{\ell=1}^{k} [4(\ell - 1)^2 + \alpha^2] \right) \frac{x^{2k}}{(2k)!},
\tag{3.19}
\]
\[
sinh(\alpha \arccos x) = \left( \sinh \frac{\alpha \pi}{2} \right) \sum_{k=0}^{\infty} \left( \prod_{\ell=1}^{k} [(2\ell - 1)^2 + \alpha^2] \right) \frac{x^{2k}}{(2k)!},
\tag{3.20}
\]
\[
cos(\alpha \arcsin x) = \sum_{k=0}^{\infty} \prod_{\ell=1}^{k} [4(\ell - 1)^2 - \alpha^2] \frac{x^{2k}}{(2k)!},
\tag{3.21}
\]
\[
sin(\alpha \arcsin x) = \alpha \sum_{k=0}^{\infty} \prod_{\ell=1}^{k} [(2\ell - 1)^2 - \alpha^2] \frac{x^{2k+1}}{(2k+1)!},
\tag{3.22}
\]
\[
cosh(\alpha \arcsin x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k-1)!!} \left( \prod_{\ell=1}^{k-1} [(\ell - 1)^2 - \alpha^2] \right) \frac{(x-1)^k}{k!} \tag{3.23}
\]
\[
= \left( \cos \frac{\alpha \pi}{2} \right) \sum_{k=0}^{\infty} \left( \prod_{\ell=1}^{k} [4(\ell - 1)^2 - \alpha^2] \right) \frac{x^{2k}}{(2k)!}
\tag{3.24}
\]
\[
sinh(\alpha \arcsin x) = \left( \sin \frac{\alpha \pi}{2} \right) \sum_{k=0}^{\infty} \left( \prod_{\ell=1}^{k} [(2\ell - 1)^2 - \alpha^2] \right) \frac{x^{2k}}{(2k)!},
\tag{3.25}
\]

where \((-1)!! = 1\) and any empty product is understood to be 1.

**Proof.** Let \( f_\alpha(x) = \cosh(\alpha \arccos x) \). Then consecutive differentiations and simplifications give

\[
f'_\alpha(x) = \frac{\alpha}{\sqrt{1-x^2}} \sinh(\alpha \arcsin x) = \frac{\alpha}{\sqrt{1-x^2} \sqrt{f'^2_\alpha(x) - 1}},
\]

\[
(1-x^2)f'^2_\alpha(x) - \alpha^2 f'^2_\alpha(x) + \alpha^2 = 0,
\]

\[
(1-x^2)f''_\alpha(x) - \alpha f''_\alpha(x) - \alpha^2 f_\alpha(x) = 0,
\]

\[
(1-x^2) f^{(3)}_\alpha(x) - 3x f'^{\prime}_{\alpha}(x) - (1 + \alpha^2) f''_\alpha(x) = 0.
\]

Accordingly, the differential equation

\[
(1-x^2) f^{(k+2)}_\alpha(x) - (2k+1)x f^{(k+1)}_\alpha(x) - (k^2 + \alpha^2) f^{(k)}_\alpha(x) = 0 \tag{3.26}
\]

is valid for \( k = 0, 1 \) respectively. Differentiating on both sides of (3.26) results in

\[
(1-x^2) f^{(k+3)}_\alpha(x) - (2k+3)x f^{(k+2)}_\alpha(x) - [(k+1)^2 + \alpha^2] f^{(k+1)}_\alpha(x) = 0.
\]

By induction, the equation (3.26) is valid for all \( k \geq 0 \). Taking \( x \to 0 \) in (3.26) gives

\[
f^{(k+2)}_\alpha(0) - (k^2 + \alpha^2) f^{(k)}_\alpha(0) = 0, \quad k \geq 0. \tag{3.27}
\]
It is clear that $f_\alpha(0) = 1$ and $f_\alpha'(0) = 0$. Substituting these two initial values into the recursive relation (3.27) and consecutively recursing reveal $f_\alpha^{(2k-1)}(0) = 0$ and

$$f_\alpha^{(2k)}(0) = \prod_{\ell=1}^{k} [4(\ell - 1)^2 + \alpha^2], \quad k \in \mathbb{N}.$$  

Consequently, the series expansion (3.16) follows.

Let $f_\alpha(x) = \sinh(\alpha \arcsin x)$. Then consecutive differentiations and simplifications give

$$f_\alpha'(x) = \frac{\alpha}{\sqrt{1 - x^2}} \cosh(\alpha \arcsin x) = \frac{\alpha}{\sqrt{1 - x^2}} \sqrt{f_\alpha^2(x) + 1},$$

$$\left(1 - x^2\right) [f_\alpha'(x)]^2 - \alpha^2 f_\alpha^2(x) - \alpha^2 = 0,$$

$$\left(1 - x^2\right) f_\alpha''(x) - xf_\alpha'(x) - \alpha^2 f_\alpha(x) = 0,$$

$$\left(1 - x^2\right) f_\alpha^{(3)}(x) - 3xf_\alpha''(x) - (1 + \alpha^2) f_\alpha'(x) = 0.$$

By the same argument as above, the derivative $f_\alpha^{(k)}(0)$ for $n \geq 0$ satisfy the recursive relation (3.27). Furthermore, from the facts that $f_\alpha(0) = 0$ and $f_\alpha'(0) = \alpha$, we conclude $f_\alpha^{(2k)}(0) = 0$ and

$$f_\alpha^{(2k+1)}(0) = \alpha \prod_{\ell=1}^{k} [4(2\ell - 1)^2 + \alpha^2], \quad k \geq 0.$$  

Consequently, the series expansion (3.17) follows.

Let $f_\alpha(x) = \cosh(\alpha \arccos x)$. Then successively differentiating yields

$$f_\alpha'(x) = -\frac{\alpha}{\sqrt{1 - x^2}} \sinh(\alpha \arccos x) = -\frac{\alpha}{\sqrt{1 - x^2}} \sqrt{f_\alpha^2(x) - 1},$$

$$\left(1 - x^2\right) [f_\alpha'(x)]^2 - \alpha^2 [f_\alpha^2(x) - 1] = 0,$$

$$\left(1 - x^2\right) f_\alpha''(x) - xf_\alpha'(x) - \alpha^2 f_\alpha(x) = 0,$$

$$\left(1 - x^2\right) f_\alpha^{(3)}(x) - 3xf_\alpha''(x) - (1 + \alpha^2) f_\alpha'(x) = 0.$$

As argued above, we conclude that the derivatives $f_\alpha^{(k)}(x)$ for $k \geq 0$ satisfy the equation (3.26). Letting $x \to 1^{-}$ in (3.26) gives

$$(2k + 1) f_\alpha^{(k+1)}(1) + (k^2 + \alpha^2) f_\alpha^{(k)}(1) = 0. \tag{3.28}$$

Setting $x \to 0$ in (3.26) leads to (3.27) It is easy to see that

$$f_\alpha(1) = 1, \quad f_\alpha'(1) = -\alpha^2, \quad f_\alpha(0) = \cosh \frac{\alpha \pi}{2}, \quad f_\alpha'(0) = -\alpha \sinh \frac{\alpha \pi}{2}.$$  

Substituting these four initial values into (3.28) and inductively recursing reveal

$$f_\alpha^{(k)}(1) = (-1)^k \prod_{\ell=1}^{k} \frac{\left(\ell^2 - 1\right)^2 + \alpha^2}{(2k - 1)!!},$$

$$f_\alpha^{(2k)}(0) = \left(\cosh \frac{\alpha \pi}{2}\right) \prod_{\ell=1}^{k} [4(\ell - 1)^2 + \alpha^2],$$

and

$$f_\alpha^{(2k+1)}(0) = -\alpha \left(\sinh \frac{\alpha \pi}{2}\right) \prod_{\ell=1}^{k} [2(2\ell - 1)^2 + \alpha^2]$$

for $k \geq 0$. Consequently, the series expansions (3.18) and (3.19) follow.

Let $f_\alpha(x) = \sinh(\alpha \arccos x)$. Then

$$f_\alpha'(x) = -\frac{\alpha}{\sqrt{1 - x^2}} \cosh(\alpha \arccos x) = -\frac{\alpha}{\sqrt{1 - x^2}} \sqrt{f_\alpha^2(x) + 1},$$

$$\left(1 - x^2\right) [f_\alpha'(x)]^2 - \alpha^2 [f_\alpha^2(x) + 1] = 0,$$

$$\left(1 - x^2\right) f_\alpha''(x) - xf_\alpha'(x) - \alpha^2 f_\alpha(x) = 0,$$
and, inductively, the derivatives $f^{(k)}_\alpha(x)$ for $k \geq 0$ satisfy the equations (3.26) and (3.27). Since $f_\alpha(0) = \sinh \frac{\alpha \pi}{2}$ and $f'_\alpha(0) = -\alpha \cosh \frac{\alpha \pi}{2}$, we obtain

$$f^{(2k)}_\alpha(0) = \left( \prod_{\ell=1}^{k} [4(\ell - 1)^2 + \alpha^2] \right) \sinh \frac{\alpha \pi}{2}$$

and

$$f^{(2k+1)}_\alpha(0) = -\alpha \left( \prod_{\ell=1}^{k} [(2\ell - 1)^2 + \alpha^2] \right) \cosh \frac{\alpha \pi}{2}$$

for $k \geq 0$. Consequently, the series expansion (3.20) follows.

Let $f_\alpha(x) = \cos(\alpha \arcsin x)$. Then

$$f'_\alpha(x) = -\frac{\alpha}{\sqrt{1-x^2}} \sin(\alpha \arcsin x) = -\frac{\alpha}{\sqrt{1-x^2}} \sqrt{1-f^2_\alpha(x)}$$

and

$$(1-x^2) \left[ f'_\alpha(x) \right]^2 - \alpha^2 [1-f^2_\alpha(x)] = 0,$$

$$(1-x^2) f''_\alpha(x) - x f'_\alpha(x) + \alpha^2 f_\alpha(x) = 0,$$

$$(1-x^2) f^{(3)}_\alpha(x) - 3x f''_\alpha(x) + (\alpha^2 - 1) f'_\alpha(x) = 0,$$

$$(1-x^2) f^{(4)}_\alpha(x) - 5x f^{(3)}_\alpha(x) + (\alpha^2 - 4) f''_\alpha(x) = 0,$$

and, inductively,

$$(1-x^2) f^{(k+2)}_\alpha(x) - (2k+1)x f^{(k+1)}_\alpha(x) + (\alpha^2 - k^2) f^{(k)}_\alpha(x) = 0, \quad k \geq 0. \quad (3.29)$$

Letting $x \to 0$ in (3.29) results in

$$f^{(k+2)}_\alpha(0) + (\alpha^2 - k^2) f^{(k)}_\alpha(0) = 0, \quad k \geq 0. \quad (3.30)$$

Recusing the relation (3.30) and considering $f_\alpha(0) = 1$ and $f'_\alpha(0) = 0$ arrive at

$$f^{(2k)}(0) = \left( \prod_{\ell=1}^{k} [4(\ell - 1)^2 - \alpha^2] \right) \quad \text{and} \quad f^{(2k+1)}(0) = 0$$

for $k \geq 0$. Consequently, the series expansion (3.21) is valid.

The series expansion (3.22) can be derived similarly.

Let $f_\alpha(x) = \cos(\alpha \arccos x)$. Then the derivatives $f^{(k)}_\alpha(x)$ and $f^{(k)}_\alpha(0)$ satisfy

$$f'_\alpha(x) = \frac{\alpha}{\sqrt{1-x^2}} \sin(\alpha \arccos x) = \frac{\alpha}{\sqrt{1-x^2}} \sqrt{1-f^2_\alpha(x)},$$

and

$$(1-x^2) \left[ f'_\alpha(x) \right]^2 - \alpha^2 [1-f^2_\alpha(x)] = 0,$$

and, inductively, the differential equation (3.29). Setting $x \to 1^-$ in (3.29) acquires

$$(2k+1) f^{(k+1)}_\alpha(1) = (\alpha^2 - k^2) f^{(k)}_\alpha(1), \quad k \geq 0. \quad (3.31)$$

Letting $x \to 0$ in (3.29) leads to (3.30). Since $f_\alpha(1) = 1$, $f_\alpha(0) = \cos \frac{\alpha \pi}{2}$, and $f'_\alpha(0) = \alpha \sin \frac{\alpha \pi}{2}$, from (3.31) and (3.30), we obtain

$$f^{(k)}_\alpha(1) = \frac{\alpha^2 - k^2}{2k+1} \quad \text{and} \quad f^{(k)}_\alpha(0) = \left( \cos \frac{\alpha \pi}{2} \right) \left( \prod_{\ell=1}^{k} [4(\ell - 1)^2 - \alpha^2] \right),$$

and

$$f^{(2k+1)}_\alpha(0) = \alpha \left( \sin \frac{\alpha \pi}{2} \right) \left( \prod_{\ell=1}^{k} [(2\ell - 1)^2 - \alpha^2] \right)$$

for $k \geq 0$. As a result, we acquire the series expansions (3.23) and (3.24).

Let $f_\alpha(x) = \sin(\alpha \arcsin x)$. Then

$$f'_\alpha(x) = -\frac{\alpha}{\sqrt{1-x^2}} \cos(\alpha \arcsin x) = -\frac{\alpha}{\sqrt{1-x^2}} \sqrt{1-f^2_\alpha(x)},$$

$$(1-x^2) \left[ f'_\alpha(x) \right]^2 - \alpha^2 [1-f^2_\alpha(x)] = 0,$$

and

$$(1-x^2) f''_\alpha(x) - x f'_\alpha(x) + \alpha^2 f_\alpha(x) = 0,$$
and, inductively, the differential equation (3.29) and the recursive relation (3.30) are valid. Employing the recursive relation (3.30) and using

\[ f_\alpha(0) = \sin \frac{\alpha \pi}{2} \quad \text{and} \quad f'_\alpha(0) = -\alpha \cos \frac{\alpha \pi}{2} \]

results in

\[ f^{(2k)}_\alpha(0) = \left( \sin \frac{\alpha \pi}{2} \right) \prod_{\ell=1}^{k} [4(\ell - 1)^2 - \alpha^2] \]

and

\[ f^{(2k+1)}_\alpha(0) = -\alpha \left( \cos \frac{\alpha \pi}{2} \right) \prod_{\ell=1}^{k} [(2\ell - 1)^2 - \alpha^2] \]

for \( k \geq 0 \). Accordingly, the series expansion (3.25) follows. The proof of Lemma 3.3 is complete.

**Remark 3.4.** In the paper [36], among other things, three authors established series expansions at \( x = 0 \) or \( x = 1 \) of the functions

\[
\begin{align*}
\exp(\alpha \arccos x), & \quad \exp(\alpha \arccos x) \frac{\alpha \arccos x}{\alpha \arccos x}, \\
\sin(\alpha \arccos x) \frac{\alpha \arccos x}{\sqrt{1-x^2}}, & \quad \sin(\alpha \arccosh x) \frac{\alpha \arccosh x}{\sqrt{x^2-1}}, \\
\sinh(\alpha \arccosh x) \frac{\alpha \arccosh x}{\sqrt{x^2-1}}, & \quad \cos(\alpha \arccos x), \\
\cos(\alpha \arccosh x) \frac{\alpha \arccosh x}{\sqrt{x^2-1}}, & \quad \sin(\alpha \arccos x), \\
\sin(\alpha \arccosh x) \frac{\alpha \arccosh x}{\sqrt{x^2-1}}, & \quad \cosh(\alpha \arccos x), \\
\cos(\alpha \arccosh x) \frac{\alpha \arccosh x}{\sqrt{x^2-1}}, & \quad \cosh(\alpha \arccosh x), \\
\end{align*}
\]

for \( \alpha \in \mathbb{C} \setminus \{0\} \) in terms of the Gauss hypergeometric function \( _2F_1(\alpha, \beta; \gamma; z) \) which can be defined [37, Section 5.9] by

\[
_2F_1(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} z^k, \quad |z| < 1
\]

for complex numbers \( \alpha, \beta \in \mathbb{C} \setminus \{0\} \) and \( \gamma \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} \), where \( (\alpha)_k, (\beta)_k, \) and \( (\gamma)_k \) are defined by (1.1) or (1.2).

By the way, we point out that the series expansions (2.1), (4.1), (4.3), (4.4), (4.7), and (4.9) in the paper [36] should be wrong.

**Lemma 3.4.** Let \( n \geq k \geq 0 \) and \( \alpha, \beta \in \mathbb{C} \). Then

1. The Faà di Bruno formula can be described in terms of partial Bell polynomials \( B_{n,k} \) by

\[
\frac{d^n}{dt^n} f \circ h(t) = \sum_{k=0}^{n} f^{(k)}(h(t)) B_{n,k}(h'(t), h''(t), \ldots, h^{(n-k+1)}(t)), \quad n \in \mathbb{N}_0. \tag{3.32}
\]

2. Partial Bell polynomials \( B_{n,k} \) satisfy the identities

\[
B_{n,k}(\alpha x_1, \alpha^2 x_2, \ldots, \alpha^k x_{n-k+1}) = \alpha^k B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}), \tag{3.33}
\]

\[
B_{n,k}(\alpha, 1, 0, \ldots, 0) = \frac{(n-k)!}{2^{n-k}} \binom{n}{k} \binom{k}{n-k} \alpha^{2k-n}, \tag{3.34}
\]

\[
B_{n,k}((-1)!!, 1!!), 3!!, \ldots, 2(n-k) - 1]!!) = [2(n-k) - 1]!! \frac{2^{n-k-1}}{2(n-k)}, \tag{3.35}
\]

and

\[
\frac{1}{k!} \left( \sum_{m=1}^{\infty} \frac{x^m}{m!} \right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \frac{t^n}{m!}. \tag{3.36}
\]
These formulas in Lemma 3.4 can be found in [7, p. 412], [8, pp. 134–135 and p. 139], [29, Theorem 4.1], [30, p. 169, (3.6)], and [33, Theorem 1.2], respectively. These identities in Lemma 3.4 can also be found in the survey and review article [31].

4. Taylor’s series expansions of \( \frac{\arccos x}{2(1-x)} \) and \( \frac{\arccosh x}{2(1-x)} \)

In this section, by virtue of some conclusions in Lemmas 3.1, 3.2, and 3.3, we establish Taylor’s series expansions around \( x = 1 \) of the functions \( \frac{\arccos x}{2(1-x)} \) and \( \frac{\arccosh x}{2(1-x)} \) in terms of \( Q(k, m) \) defined by the formula (2.2).

**Theorem 4.1.** For \( k \in \mathbb{N} \) and \( |x| < 1 \), we have

\[
\left( \frac{\arccos x}{2(1-x)} \right)^k = 1 + (2k)! \sum_{n=1}^{\infty} \frac{Q(2k, 2n)}{(2k + 2n)!} (2(x - 1))^n
\]

and

\[
\left( \frac{\arccosh x}{2(1-x)} \right)^k = 1 + (2k)! \sum_{n=1}^{\infty} \frac{Q(2k, 2n)}{(2k + 2n)!} (2(x - 1))^n,
\]

where \( Q(2k, 2n) \) is defined by (2.2).

**Proof.** Replacing \( \alpha \) by \( i \alpha \) in (3.1) leads to

\[
\prod_{\ell=1}^{k} (\ell^2 - \alpha^2) = (-1)^k \sum_{j=0}^{k} \sum_{\ell=2j+1} (\ell - 1) s(2k + 1, \ell) \kappa^{2j-1} \alpha^{2j}.
\]

From Taylor’s series expansion (3.23) and the identity (4.3), it follows that

\[
\sum_{k=0}^{\infty} (-1)^k \frac{(\alpha \arccos x)^{2k}}{(2k)!} = 1 + (x - 1) \alpha^2 - \alpha^2 \sum_{k=2}^{\infty} \frac{(-1)^k}{(2k - 1)!!} \prod_{\ell=1}^{k-1} (\ell^2 - \alpha^2) \frac{(x - 1)^k}{k!}
\]

\[
= 1 + (x - 1) \alpha^2 + \alpha^2 \sum_{k=2}^{\infty} \frac{1}{(2k - 1)!!} \sum_{j=0}^{k-1} \sum_{\ell=2j+1} (\ell - 1) s(2k - 1, \ell) \kappa^{2j-1} \alpha^{2j} \frac{(x - 1)^k}{k!}
\]

\[
= 1 + (x - 1) \alpha^2 + \alpha^2 \sum_{k=2}^{\infty} \frac{1}{(2k - 1)!!} \sum_{j=2}^{k-2} \sum_{\ell=2j-1}^{2k-1} (\ell - 1) s(2k - 1, \ell) \kappa^{2j-1} \alpha^{2j-2} \frac{(x - 1)^k}{k!}
\]

\[
= 1 + (x - 1) \alpha^2 + \alpha^2 \sum_{k=2}^{\infty} \frac{1}{(2k - 1)!!} \sum_{j=2}^{k-2} \sum_{\ell=2j-1}^{2k-1} (\ell - 1) s(2k - 1, \ell) \kappa^{2j-1} \alpha^{2j-2} \frac{(x - 1)^k}{k!}
\]

This means that

\[
- \frac{(\arccos x)^2}{2!} = x - 1 + \sum_{k=2}^{\infty} \frac{1}{(2k - 1)!!} \sum_{\ell=1}^{2k-1} (\ell - 1) s(2k - 1, \ell) \kappa^{2j-1} \alpha^{2j-2} \frac{(x - 1)^k}{k!}
\]

where we used the identity (3.2) in Lemma 3.1 or the identity (3.14), and that

\[
\frac{(-1)^k (\arccos x)^{2k}}{(2k)!} = \sum_{m=0}^{\infty} \frac{Q(2k, 2m)}{(2k + 2m)!} \frac{2(x - 1)^{k+m}}{(2m)!}.
\]
for \( k \geq 2 \). Consequently, the series expansions

\[
\frac{(\arccos x)^2}{2!} = \sum_{m=0}^{\infty} \frac{m!}{(2m + 1)!} \frac{(1 - x)^{m+1}}{m+1}, \quad |x| < 1
\]

(4.4)

and

\[
\frac{(\arccos x)^{2k}}{(2k)!} = \sum_{m=0}^{\infty} (-1)^m Q(2k, 2m) \frac{[2(1 - x)]^{m+k}}{(2k + 2m)!}
\]

(4.5)

for \( k \geq 2 \) and \(|x| < 1\) are valid.

By similar arguments as done above, from the series expansion (3.18), we can also recover the series expansion (4.4) and (4.5).

The series expansions (4.4) and (4.5) can be reformulated as

\[
\frac{(\arccos x)^2}{2(1 - x)} = 1 + \sum_{m=1}^{\infty} \frac{(m!)^2}{(2m + 1)!!(m + 1)!} \frac{(1 - x)^m}{m!}
\]

and

\[
\left[ \frac{(\arccos x)^2}{2(1 - x)} \right]^k = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m + 1)!!(2m + 1)} Q(2k, 2m) \frac{(1 - x)^m}{m!}
\]

for \( k \geq 2 \). Making use of the identity (3.2) or (3.14), we obtain

\[
\frac{(-1)^m}{(2m - 1)!!(2m + 1)} Q(2k, 2m) = \frac{(m!)^2}{(2m + 1)!!(m + 1)!}, \quad m \in \mathbb{N}.
\]

Consequently, the series expansions (4.4) and (4.5) can be unified as the series expansion (4.1).

By the relation

\[
\arccos x = -i \arccosh x,
\]

(4.6)

from (4.1), we deduce (4.2). The proof of Theorem 4.1 is complete.

**Corollary 4.1.** For \( k \in \mathbb{N} \) and \(|x| < 1\), we have

\[
\left[ \frac{(\pi - \arccos x)^2}{2(1 + x)} \right]^k = 1 + (2k)! \sum_{m=1}^{\infty} (-1)^m Q(2k, 2m) \frac{[2(x + 1)]^m}{(2k + 2m)!}
\]

(4.7)

and

\[
(-1)^k \left[ \frac{(\pi + i \arccosh x)^2}{2(1 + x)} \right]^k = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{(2k)!(2m)!}{(2k + 2m)!} Q(2k, 2m) \frac{(x + 1)^m}{m!}
\]

(4.8)

where \( Q(2k, 2m) \) is defined by (2.2).

**Proof.** This follows from replacing \( x \) by \(-x\) in (4.1) and (4.2) and utilizing the relations

\[
\arccos x + \arccos(-x) = \pi
\]

and (4.6).

**Corollary 4.2.** For \( k \in \mathbb{N} \), we have

\[
\left( \frac{\pi^2}{8} \right)^k = 1 + (2k)! \sum_{m=1}^{\infty} (-1)^m Q(2k, 2m) \frac{[2(x + 1)]^m}{(2k + 2m)!}
\]

(4.9)

In particular, we have

\[
\frac{\pi^2}{8} = \sum_{m=1}^{\infty} \frac{2^m}{m^2 \left( \frac{2m}{m} \right)^m}
\]

(4.10)

**Proof.** The series representation (4.9) of \( \pi^{2k} \) follows from letting \( x = 0 \) in either (4.1), (4.2), (4.7), or (4.8).

The series representation (4.10) follows from taking \( k = 1 \) in (4.9), or setting \( k = 2 \) in (2.4), and then making use of the identity (3.2) or (3.14).
Corollary 4.4. The series representation (4.10) recovers a conclusion in [25, Theorem 5.1].
As for series representations of $\pi^2$, in [16, p. 453, (14)] and the paper [18], among other things, we find out

$$
\frac{\pi^2}{6} = \sum_{m=0}^{\infty} \frac{1}{(m+1)^2}, \quad \frac{\pi^2}{8} = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}, \quad \frac{\pi^2}{12} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)^2}, \quad \frac{\pi^2}{18} = \sum_{m=1}^{\infty} \frac{1}{m^2 (2m)}
$$

(4.11)

which are different from (4.10). The last series representation in (4.11) is derived from letting $k = 2$ and $x = \frac{1}{2}$ in (2.1).

Because

$$
\lim_{m \to \infty} \left[ \frac{2^m}{m^2 (2m)} \right]^{1/m} = \frac{1}{2}, \quad \lim_{m \to \infty} \left[ \frac{6}{(m+1)^2} \right]^{1/m} = 1, \quad \lim_{m \to \infty} \left[ \frac{8}{(2m+1)^2} \right]^{1/m} = 1,
$$

(4.12)

we regard that the last series representation in (4.11) in [16, p. 453, (14)] converges to $\pi^2$ quicker than (4.10) and other three in (4.11). The first unsolved problem posed on December 13, 2010 by Herbert S. Wilf (1931–2012) is about the convergent speed of rational approximations of the circular constant $\pi$. For more details on this unsolved problem, see [25, Remark 7.7].

For $k \in \mathbb{N}$, let

$$
L(k) = \lim_{m \to \infty} \left[ \frac{(2k)!(-1)^m Q(2k, 2m)}{(2k + m)!} \right]^{1/m} = 2 \lim_{m \to \infty} \left[ \frac{(-1)^m Q(2k, 2m)}{(2k + m)!} \right]^{1/m}.
$$

The first limit in (4.12) means that $L(1) = \frac{1}{2}$. What is the convergent speed of the hypergeometric term in (4.9) for $k \geq 2$? Equivalently speaking, what is the limit $L(k)$ for $k \geq 2$? Is the limit $L(k)$ a decreasing sequence in $k \geq 2$? What is the limit $\lim_{k \to \infty} [L(k)]^{1/k}$?

Corollary 4.3. For $k, m \in \mathbb{N}$, we have

$$
\left[ \frac{\arccos x}{2(1-x)} \right]^k \bigg|_{x=1^-} = \frac{(2k)!}{(2k+2m)!} Q(2k, 2m),
$$

$$
\left[ \frac{\text{arccosh} x}{2(1-x)} \right]^k \bigg|_{x=1^-} = (-1)^k \frac{(2k)!}{(2k+2m)!} Q(2k, 2m),
$$

$$
\left[ \frac{(\pi - \arccos x)^2}{2(1+x)} \right]^k \bigg|_{x=1^-} = (-1)^m \frac{(2k)!}{(2k+2m)!} Q(2k, 2m),
$$

and

$$
\left[ \frac{(\pi + \text{arccosh} x)^2}{2(1+x)} \right]^k \bigg|_{x=1^-} = (-1)^{k+m} \frac{(2k)!}{(2k+2m)!} Q(2k, 2m),
$$

where $Q(2k, 2m)$ is defined by (2.2).

Proof. These left-hand and right-hand derivatives follow from Taylor’s series expansions (4.1), (4.2), (4.7), and (4.8). \qed

Corollary 4.4. For $k, n \in \mathbb{N}$, we have

$$
\left[ (\arccos x)^2 \right]^k \bigg|_{x=1^-} = \begin{cases} 
0, & n < k; \\
(-1)^k (2k)!!, & n = k; \\
(-1)^k \frac{2^k (2k)!}{(2n-1)!!} Q(2k, 2n-2k), & n > k
\end{cases}
$$

(4.13)
and

\[
[(\arccosh x)^{2k}]^{(n)} |_{x=1^-} = \begin{cases} 
0, & n < k; \\
(2k)! n, & n = k; \\
(2k)! Q(2k, 2m - 2k), & n > k.
\end{cases}
\] (4.14)

**Proof.** For \( k \in \mathbb{N} \) and \( |x| < 1 \), the series expansions (4.1) and (4.2) can be reformulated as

\[
(\arccosh x)^{2k} = (-1)^k (2k)!! \frac{(x - 1)^k}{k!} + (-1)^k (2k)!! \sum_{m=1}^{\infty} \frac{Q(2k, 2m)}{(2k + 2m - 1)!!} \frac{(x - 1)^{k+m}}{(k+m)!}
\] (4.15)

and

\[
(\arccosh x)^{2k} = (2k)!! \frac{(x - 1)^k}{k!} + (2k)!! \sum_{m=1}^{\infty} \frac{Q(2k, 2m - 2k)}{(2k + 2m - 1)!!} \frac{(x - 1)^{k+m}}{(k+m)!}.
\] (4.16)

These forms of series expansions (4.15) and (4.16) imply the formulas in (4.13) and (4.14). \( \square \)

**Corollary 4.5.** For \( k \in \mathbb{N} \) and \( |x| < 1 \), we have

\[
(\arccos x)^{2k} = \sum_{j=0}^{k} (-1)^j \left[ 2^k \binom{k}{j} + (2k)!! \sum_{m=1}^{\infty} \frac{(-1)^m Q(2k, 2m)}{(k+m)!! (2k + 2m - 1)!!} \binom{k+m}{j} \right] x^j
\] (4.17)

and

\[
(\arccos x)^{2k} = (-1)^k \sum_{j=0}^{k} (-1)^j \left[ 2^k \binom{k}{j} + (2k)!! \sum_{m=1}^{\infty} \frac{(-1)^m Q(2k, 2m - 2k)}{(j+m)!! (2j - 1)!!} \binom{j+m}{j} \right] x^j,
\] (4.18)

where \( Q(2k, 2m) \) is given by (2.2).

**Proof.** For \( k \in \mathbb{N} \) and \( |x| < 1 \), by the binomial theorem, the series expansion (4.15) can be rewritten as

\[
(\arccos x)^{2k} = 2^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^j + (2k)!! \sum_{m=1}^{\infty} \binom{k+m}{0} \frac{(-1)^m Q(2k, 2m)}{(k+m)!! (2k + 2m - 1)!!} \binom{k+m}{j} x^j
\]

\[
= 2^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^j + (2k)!! \left( \sum_{j=0}^{k} \sum_{m=0}^{k-j} \frac{(-1)^{m-j} Q(2k, 2m)}{(k+m)!! (2k + 2m - 1)!!} \binom{k+m}{j} \right) x^j
\]

\[
= \sum_{j=0}^{k} (-1)^j \left[ 2^k \binom{k}{j} + (2k)!! \sum_{m=1}^{\infty} \frac{(-1)^m Q(2k, 2m)}{(k+m)!! (2k + 2m - 1)!!} \binom{k+m}{j} \right] x^j
\]

\[
+ (2k)!! \sum_{j=k+1}^{\infty} (-1)^j \left[ \sum_{m=j-k}^{\infty} \frac{(-1)^m Q(2k, 2m)}{(k+m)!! (2k + 2m - 1)!!} \binom{k+m}{j} \right] x^j.
\]

The series expansion (4.17) is thus proved.

Similarly, from (4.16), we conclude (4.18). Theorem 4.5 is thus proved. \( \square \)

**Remark 4.2.** What are closed-form expressions of coefficients in Maclearin’s series expansion around the point \( x = 0 \) of the real power \((\arccos x)^n\) for \( n \in \mathbb{R} \) and \( |x| < 1 \)? For answers to this question, please refer to [25, Section 3].

**Corollary 4.6.** For \( m, n \in \mathbb{N} \), we have

\[
[(\arccos x)^{2n-1}]^{(m)} |_{x=1^-} = \begin{cases} 
0, & m < n; \\
\infty, & m \geq n.
\end{cases}
\] (4.19)
and
\[
[(\arccosh x)^{2n-1}]^{(m)}\Big|_{x=1} = \begin{cases} 0, & m < n; \\ \infty, & m \geq n. \end{cases}
\]

Consequently, the functions \((\arccos x)^{2n-1}\) and \((\arccosh x)^{2n-1}\) for \(n \in \mathbb{N}\) cannot be expanded into Taylor’s series expansions at the point \(x = 1^−\).

**Proof.** It is easy to verify that
\[ (\arccos x)^2 \sim 2(1-x), \quad x \to 1^−. \]  
By the Faà di Bruno formula (3.32) and in the light of the identities (3.33) and (3.34), we obtain
\[
\left( \frac{1}{\sqrt{1-x^2}} \right)^{(m)} = \sum_{j=0}^{m} \left( -\frac{1}{2} \right)^j (1-x^2)^{-1/2-j} B_{m,j} (-2x, -2, 0, \ldots, 0) = \sum_{j=0}^{m} \left( -\frac{1}{2} \right)^j \frac{(2j-1)!!}{2^j} \frac{(-2)^j}{(1-x^2)^{j+1/2}} B_{m,j}(x, 1, 0, \ldots, 0)
\]
\[
= \sum_{j=0}^{m} \frac{(2j-1)!!}{(1-x^2)^{j+1/2}} \frac{(m-j)!}{2^{m-j}} \binom{m}{j} (m-j) \binom{j}{m-j} x^{2j-m} = \sum_{j=0}^{m} \frac{(2j-1)!!}{(1+x)^{j+1/2}} \frac{(m-j)!}{2^{m-j}} \binom{m}{j} \binom{j}{m-j} x^{2j-m}
\]
for \(m \in \mathbb{N}_0\). This implies that
\[
\left( \frac{1}{\sqrt{1-x^2}} \right)^{(m)} \sim \frac{(2m-1)!!}{[2(1-x)]^{m+1/2}}, \quad x \to 1^−, \quad m \in \mathbb{N}_0.
\]
Utilizing the Faà di Bruno formula (3.32), employing the identities (3.33) and (3.35), and making use of (4.21) and (4.22), we acquire
\[
[(\arccos x)^{2n-1}]^{(m)} = \sum_{j=0}^{m} (2n-1)_j (\arccos x)^{2n-j-1}
\times B_{m,j} \left( -\frac{1}{\sqrt{1-x^2}}, \left( -\frac{1}{\sqrt{1-x^2}} \right), \ldots, \left( -\frac{1}{\sqrt{1-x^2}} \right) \right)^{(m-j)} \sim \sum_{j=0}^{m} (2n-1)_j [2(1-x)]^{n-(j+1)/2} (-1)^j
\times B_{m,j} \left( \frac{(-1)!!}{[2(1-x)]^{1/2}}, \frac{1!!}{[2(1-x)]^{3/2}}, \frac{3!!}{[2(1-x)]^{5/2}}, \ldots, \frac{[2(m-j)-1]!!}{[2(1-x)]^{m-j+1/2}} \right)
\]
\[
= \sum_{j=0}^{m} (-1)^j (2n-1)_j [2(1-x)]^{n-(j+1)/2} \left[ \frac{2(1-x)]^{j/2}}{[2(1-x)]^{1/2}} \right] B_{m,j} \left( (-1)!!, 1!!, 3!!, \ldots, [2(m-j)-1]!! \right)
\]
\[
= [2(1-x)]^{n-m-1/2} \sum_{j=0}^{m} (-1)^j (2n-1)_j [2(m-j)-1]!! \left( \frac{2m-j-1}{2(m-j)} \right) \to \begin{cases} 0, & n > m \\ \infty, & n \leq m \end{cases}
\]
as \(x \to 1^−\) for \(m, n \in \mathbb{N}\). The results in (4.19) are thus proved.

Substituting (4.6) into (4.19) leads to (4.20). Theorem 4.6 is therefore proved. □

**Corollary 4.7.** For \(k \in \mathbb{N}_0\), we have
\[
\sum_{j=0}^{k} (-1)^j (2k)_j [2(k-j)-1]!! \left( \frac{2k-j-1}{2(k-j)} \right) = (-1)^k (2k)!!.
\]
Proof. As done in the proof of Theorem 4.6, we arrive at
\[
[(\arccos x)^2]^n = \sum_{j=0}^{n} (2k)_j (\arccos x)^{2k-j} B_{n,j} \left( -\frac{1}{\sqrt{1-x^2}}, -\frac{1}{\sqrt{1-x^2}}, \ldots, -\frac{1}{\sqrt{1-x^2}} \right)^{(n-j)}
\]
\[
\sim \sum_{j=0}^{n} (2k)_j [2(1-x)]^{k-j/2-1} B_{n,j} \left( \frac{(-1)!!}{[2(1-x)]^{1/2}}, \frac{1!!}{[2(1-x)]^{3/2}}, \ldots, \frac{[2(n-j)-1]!!}{[2(1-x)]^{n-j+1/2}} \right)
\]
\[
= \sum_{j=0}^{n} (-1)^j (2k)_j [2(1-x)]^{k-j/2} \frac{[2(1-x)]^{1/2}}{[2(1-x)]} B_{n,j} ((-1)!!, 1!!, 3!!, \ldots, [2(n-j)-1]!!)
\]
\[
= [2(1-x)]^{k-n} \sum_{j=0}^{n} (-1)^j (2k)_j [2(n-j)-1]!! \left( \frac{2n-j-1}{2(n-j)} \right)^{k+n} \rightarrow \begin{cases} 0, & k > n \\ \sum_{j=0}^{k} (-1)^j (2k)_j [2(k-j)-1]!! \left( \frac{2k-j-1}{2(k-j)} \right), & k = n \end{cases}
\]
as \( x \to 1^- \) for \( k, n \in \mathbb{N} \). Comparing this result with (4.13) gives (4.23). \qed

Remark 4.3. The identity (4.23) is similar to
\[
\sum_{k=0}^{n} k!! [2(n-k) - 1]!! \left( \frac{2n-k-1}{2(n-k)} \right) = (2n-1)!!, \quad n \in \mathbb{N}_0,
\]
which was respectively established in [34, p.10, (3.12)] and in an unpublished paper titled “Partial Bell polynomials, falling and rising factorials, Stirling numbers, and combinatorial identities”.

5. Taylor’s Series Expansions of \( (\arccos x^2)^\alpha \)

In this section, via establishing a closed-form expression for the specific partial Bell polynomials at a sequence of the derivatives at \( x = 1^- \) of the function \( (\arccos x^2)^\alpha \), we present Taylor’s series expansion at \( x = 1^- \) of the function \( (\arccos x^2)^\alpha \) for \( \alpha \in \mathbb{R} \).

Theorem 5.1. For \( m \geq k \in \mathbb{N} \), we have
\[
B_{m,k} \left( \left[ \frac{\arccos x^2}{2(1-x)} \right]^{\left( m \right)} \right)_{x=1^-} = 2^k B_{m,k} \left( \frac{1}{12}, \frac{2}{45}, -\frac{3}{70}, \frac{32}{525}, -\frac{68}{693}, \ldots, \frac{(2m-2k+2)!!}{(2m-2k+4)!!}, \frac{Q(2, 2m-2k+2)}{Q(2, 2m-2k+2)} \right)
\]
\[(5.1)\]
where \( Q(2j, 2m) \) is defined by (2.2).

Proof. Let
\[
x_m = \left[ \frac{\arccos x^2}{2(1-x)} \right]^{\left( m \right)}\bigg|_{x=1^-}, \quad m \in \mathbb{N}.
\]
Then, from (3.36) and (4.1), it follows that
\[
B_{n+k}(x_1, x_2, \ldots, x_{n+1}) = \left( \begin{array}{c} n+k \\ k \end{array} \right) \lim_{t \to 0} \frac{d^n}{dt^n} \left[ \sum_{m=0}^{\infty} \frac{x_{m+1}}{(m+1)!} t^m \right]^{k}
\]
\[
= \left( \begin{array}{c} n+k \\ k \end{array} \right) \lim_{t \to 0} \frac{d^n}{dt^n} \left[ \frac{1}{x-1} \sum_{m=1}^{\infty} \frac{(\arccos x)^2}{2(1-x)} (x-1)^m \right]^{k}
\]
\[
= \left( \begin{array}{c} n+k \\ k \end{array} \right) \lim_{x \to 1^-} \frac{d^n}{dx^n} \left[ \frac{1}{x-1} \sum_{m=1}^{\infty} \frac{(\arccos x)^2}{2(1-x)} (x-1)^m \right]^{k}.
\]
Accordingly, we derive
\[
\begin{align*}
\frac{\partial^n}{\partial x^n} \left( \frac{1}{x-1} \left[ \frac{(\arccos x)^2}{2(1-x)} - 1 \right] \right)^k \\
= \left( \frac{n+k}{k} \right) \lim_{x \to 1^-} \frac{d^n}{dx^n} \left( \frac{1}{x-1} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left( \frac{(\arccos x)^2}{2(1-x)} \right)^j \right) \\
= \left( \frac{n+k}{k} \right) \lim_{x \to 1^-} \frac{d^n}{dx^n} \left( \frac{(-1)^k}{x-1} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \left( 1 + (2j)! \sum_{m=1}^{\infty} \frac{Q(2j,2m)}{(2j+2m)!} [2(x-1)]^m \right) \right) \\
= (-1)^k \left( \frac{n+k}{k} \right) \lim_{x \to 1^-} \frac{d^n}{dx^n} \left( \sum_{m=1}^{\infty} 2^m \sum_{j=1}^{k} (-1)^j (2j)! \binom{k}{j} \frac{Q(2j,2m)}{(2j+2m)!} (x-1)^{m-k} \right) \\
\end{align*}
\]
for \( k \in \mathbb{N} \). This implies that
\[
\sum_{j=1}^{k} (-1)^j (2j)! \binom{k}{j} \frac{Q(2j,2m)}{(2j+2m)!} = 0, \quad 1 \leq m < k. \tag{5.2}
\]
Accordingly, we derive
\[
B_{n+k,k}(x_1, x_2, \ldots, x_{n+1}) = (-1)^k \binom{n+k}{k} \\
\times \lim_{x \to 1^-} \frac{d^n}{dx^n} \left( \sum_{m=k}^{\infty} 2^m \sum_{j=1}^{k} (-1)^j (2j)! \binom{k}{j} \frac{Q(2j,2m)}{(2j+2m)!} (x-1)^{m-k} \right) \\
= (-2)^k \binom{n+k}{k} \lim_{x \to 1^-} \frac{d^n}{dx^n} \left( \sum_{m=k}^{\infty} 2^m \sum_{j=1}^{k} (-1)^j (2j)! \binom{k}{j} \frac{Q(2j,2m+2k)}{(2j+2m+2k)!} (x-1)^{m-n} \right) \\
= (-2)^k (2n)!! \binom{n+k}{k} \sum_{m=0}^{\infty} (2n)! (2m)! \binom{k}{m} \frac{Q(2j,2m+2k)}{(2j+2m+2k)!} \\
= (-2)^k (2n)!! \binom{n+k}{k} \sum_{m=0}^{\infty} (2n)! (2m)! \binom{k}{m} \frac{Q(2j,2m+2k)}{(2j+2m+2k)!} \\
\]
for \( n \geq k \in \mathbb{N} \). Replacing \( n+k \) by \( m \) results in
\[
B_{m,k}(x_1, x_2, \ldots, x_{m-k+1}) = (-2)^k (2m-k)!! \binom{m}{k} \sum_{j=1}^{k} (-1)^j (2j)! \binom{k}{j} \frac{Q(2j,2m)}{(2j+2m)!} \\
\]
for \( m \geq k \in \mathbb{N} \). The required result is thus proved. \( \square \)

**Theorem 5.2.** For \( \alpha \in \mathbb{R} \), we have
\[
\left[ \frac{(\arccos x)^2}{2(1-x)} \right]^n = 1 + \sum_{j=1}^{\infty} \sum_{l=0}^{j} \binom{j}{l} (2l)! \binom{n}{j} \frac{Q(2l,2n)}{(2l+2n)!} [2(x-1)]^n. \tag{5.3}
\]

**Proof.** By virtue of the Faà di Bruno formula (3.32) and the formula (5.1) in Theorem 5.1, we obtain
\[
\left[ \frac{(\arccos x)^2}{2(1-x)} \right]^n = \sum_{j=1}^{n} (\alpha_j)^j B_{n,j} \left[ \frac{(\arccos x)^2}{2(1-x)} \right]^j, \quad n \in \mathbb{N}, \tag{5.1}
\]
for \( n \in \mathbb{N} \). Taking the limit \( x \to 1^- \) and employing (5.1) in Theorem 5.1 lead to
\[
\lim_{x \to 1^-} \left[ \frac{(\arccos x)^2}{2(1-x)} \right]^n = \sum_{j=1}^{n} (\alpha_j)^j B_{n,j} \left[ \lim_{x \to 1^-} \frac{(\arccos x)^2}{2(1-x)} \right]^j, \quad n \in \mathbb{N}, \tag{5.2}
\]
for \( n \in \mathbb{N} \). Consequently, the required result (5.3) is proved. \( \square \)
Corollary 5.1. For \( k, n \in \mathbb{N} \), we have

\[
\sum_{j=1}^{n} (-1)^j \frac{k}{j!} \sum_{\ell=1}^{j} (-1)^\ell \frac{Q(2\ell, 2n)}{(2\ell + 2n)!} = (2k)! \frac{Q(2k, 2n)}{(2k + 2n)!}.
\] (5.4)

Proof. This follows from letting \( \alpha = k \in \mathbb{N} \) in (5.3) and equating coefficients of factors \((x - 1)^n\) in (4.1). \( \square \)

Corollary 5.2. For \( \alpha \in \mathbb{R} \), we have

\[
\left( \frac{x^2}{9} \right)^\alpha = 1 + \sum_{n=1}^{\infty} \sum_{j=1}^{n} (-1)^n \frac{\alpha}{j!} \sum_{\ell=1}^{j} (-1)^\ell \frac{Q(2\ell, 2n)}{(2\ell + 2n)!}.
\] (5.5)

Proof. This follows from setting \( x = \frac{t}{2} \) in (5.3). \( \square \)

Remark 5.1. The formula (5.1) in Theorem 5.1 can be used to compute Taylor’s series expansions of functions like \( f\left(\arccos\left(\frac{x}{2(1-x)}\right)\right) \) around the point \( x = 1^- \), only if all the derivatives of \( f(x) \) at \( x = 1^- \) are explicitly computable.

6. Recovering Maclaurin’s series expansion of \( \left( \frac{\arcsin x}{x} \right)^k \)

In this section, by virtue of some conclusions in Lemmas 3.1, 3.2, and 3.3, we recover Maclaurin’s series expansions (2.1) and (2.3) in Theorems 2.1 and 2.2.

It is easy to see that

\[
cosh t = \frac{e^t + e^{-t}}{2} = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}.
\]

Then, making use of the identity (3.1) in Lemma 3.1, the series expansion (3.16) can be reformulated as

\[
\sum_{k=0}^{\infty} \frac{(\arcsin x)^{2k}}{(2k)!} \alpha^{2k} = 1 + \frac{x^2}{2} \alpha^2 + \sum_{k=2}^{\infty} \frac{x^{2k}}{(2k)!} \sum_{j=1}^{k-1} \frac{(-1)^{j-1}}{j!} \left( \sum_{\ell=1}^{j} \ell^2 \left( \alpha + \frac{\alpha}{2} \right)^{2j} \right) \left( \frac{2k-1}{2j-1} \right) \alpha^{2j}.
\]

\[
= 1 + \frac{x^2}{2} \alpha^2 + \sum_{k=2}^{\infty} \frac{x^{2k}}{(2k)!} \sum_{j=1}^{k-1} \frac{(-1)^{j-1}}{j!} \left( \sum_{\ell=1}^{j} \ell^2 \left( \alpha + \frac{\alpha}{2} \right)^{2j} \right) \left( \frac{2k-1}{2j-1} \right) \alpha^{2j}.
\]

\[
= 1 + \frac{x^2}{2} \alpha^2 + \sum_{k=2}^{\infty} \frac{x^{2k}}{(2k)!} \sum_{j=1}^{k-1} \frac{(-1)^{j-1}}{j!} \left( \sum_{\ell=1}^{j} \ell^2 \left( \alpha + \frac{\alpha}{2} \right)^{2j} \right) \left( \frac{2k-1}{2j-1} \right) \alpha^{2j}.
\]

where we used the identity (3.2) or (3.14). Regarding \( \alpha \) as a variable and equating coefficients of \( \alpha^{2k} \) arrive at

\[
\frac{\arcsin x}{x}^2 = \frac{x^2}{2} + \sum_{k=2}^{\infty} \frac{[(2k - 2)!]^2 x^{2k}}{(2k)!} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(2k - 2)! x^{2k}}{(2k - 1)! k}. \] (6.1)
\[
\frac{\arcsin x}{(2k)!} = \sum_{m=k}^{\infty} \frac{(-1)^{m-k}}{2^m m!} \left( \sum_{\ell=2k+1}^{2m-1} \frac{\ell}{2^\ell} \frac{s(2m-1, \ell)(m-1)^{\ell-2k+1}}{(2k-1)!} \right) x^{2m} \tag{6.2}
\]

for \( k \geq 2 \).

Making use of the series expansion (3.17) and the identity (3.9) in Lemma 3.2, we obtain

\[
\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \alpha^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} x^{2k+1}}{(2k+1)!} \sum_{j=0}^{2k} \left[ \sum_{\ell=2j}^{2k} \frac{s(2k, \ell)}{2^\ell} \frac{\ell}{2j} \right] (2k-1)^{\ell-2j} \alpha^{2j} \tag{6.3}
\]

where we used the identity (3.10) or (3.15) and \( \lceil x \rceil \) stands for the ceiling function which gives the smallest integer not less than \( x \). Regarding \( \alpha \) as a variable and equating coefficients of \( \alpha^{2k+1} \) reduce to

\[
\frac{\arcsin x}{(2k+1)!} = \frac{(-1)^k}{(2k+1)!} \sum_{m=[k/2]}^{\infty} \frac{(-1)^m}{m!} \left( \sum_{\ell=2k}^{2m} \frac{s(2m, \ell)}{2^\ell} \frac{\ell}{2k} \right) (2k-1)^{\ell-2k} \frac{(2\ell^{2m+1})}{(2m+1)!} \alpha^{2m+1} \tag{6.3}
\]

Combining the series expansions (6.1), (6.2), and (6.3) leads to the series expansion (2.1).

By similar arguments as above, from the series expansions (3.21) and (3.22), we can recover series expansion (2.1) once again.

Utilizing the relation \( \arcsinh t = -i \arcsin(it) \) or, equivalently, the relation \( \arcsin t = -i \arcsinh(it) \), the series expansions (2.1) and (2.3) can be derived from each other.

7. Conclusions

In this paper, by virtue of Lemmas 3.1 and 3.2, with the aid of Taylor’s series expansion (3.23) or (3.18) in Lemma 3.3, and in the light of properties recited in Lemma 3.4 of partial Bell polynomials, the author establishes Taylor’s series expansions (4.1), (4.2), and (5.3) in Theorems 4.1 and 5.2, presents an explicit formula (5.1), derives several combinatorial identities (3.2), (3.3), (3.11), (4.23), (5.2), and (5.4), demonstrates several series representations (4.9), (4.10), and (5.5) in Corollaries 4.2 and 5.2 of the circular constant \( \pi \) and its real powers, and recovers Maclaurin’s series expansions (2.1) and (2.3) in Section 6.

Those conclusions stated in Corollaries 4.1, 4.3, 4.4, 4.5, and 4.6 are useful, meaningful, and significant.

Several Maclaurin’s series expansions of the functions \((\arccos x)^m\) and \((\arccosh x)^m\) for \( m \in \mathbb{N} \) have been surveyed, reviewed, collected, and mentioned in [10, Section 7], but comparatively their forms or formulations are not more beautiful, not more satisfactory, not simpler, not more concise, or not nicer than these newly-established ones in this paper.

This paper is an extended version of the preprint [26] and a companion of the papers [10, 11, 25, 34].

8. Declarations

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SERIES EXPANSIONS FOR POWERS OF INVERSE COSINE

REFERENCES

[1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series 55, 10th printing, Washington, 1972.

[2] M. Bakker and N. M. Temme, *Sum rule for products of Bessel functions: Comments on a paper by Newberger*, J. Math. Phys. 25 (1984), no. 5, 1266–1267; available online at https://doi.org/10.1063/1.526282.

[3] A. Baricz, *Powers of modified Bessel functions of the first kind*, Appl. Math. Lett. 23 (2010), no. 6, 722–724; available online at https://doi.org/10.1016/j.aml.2010.02.015.

[4] C. M. Bender, D. C. Brody, and B. K. Meister, *On powers of Bessel functions*, J. Math. Phys. 44 (2003), no. 1, 309–314; available online at https://doi.org/10.1063/1.1526940.

[5] J. M. Borwein and M. Chamberland, *Integer powers of arcsin*, Int. J. Math. Math. Sci. 2007, Art. ID 19381, 10 pages; available online at https://doi.org/10.1155/2007/19381.

[6] Y. Hong, B.-N. Guo, and F. Qi, *New sum rule for products of Bessel functions with application to plasma physics*, Contrib. Discrete Math. 11 (2016), no. 1, 22–30; available online at https://doi.org/10.11575/cdm.v11i1.62389.

[7] F. Qi, *Taylor’s series expansions for even powers of inverse cosine function and series representations for powers of Pi*, arXiv (2021), available online at https://arxiv.org/abs/2102.02749v1.

[8] F. Qi, C.-P. Chen, and D. Lim, *Several identities containing central binomial coefficients and derived from series expansions of powers of the arcsine function*, Results Nonlinear Anal. 4 (2021), no. 1, 57–64; available online at https://doi.org/10.5306/rna.867047.

[9] F. Qi and B.-N. Guo, *A diagonal recurrence relation for the Stirling numbers of the first kind*, Appl. Anal. Discrete Math. 12 (2018), no. 1, 153–165; available online at https://doi.org/10.2298/ADMM170405004Q.

[10] F. Qi and B.-N. Guo, *Explicit formulas for special values of the Bell polynomials of the second kind and for the Euler numbers and polynomials*, Mediterr. J. Math. 14 (2017), no. 3, Art. 140, 14 pages; available online at https://doi.org/10.1007/s00009-017-0939-1.
[30] F. Qi, D.-W. Niu, D. Lim, and B.-N. Guo, Closed formulas and identities for the Bell polynomials and falling factorials, Contrib. Discrete Math. 15 (2020), no. 1, 163–174; available online at https://doi.org/10.11575/cdm.v15i1.68111.

[31] F. Qi, D.-W. Niu, D. Lim, and Y.-H. Yao, Special values of the Bell polynomials of the second kind for some sequences and functions, J. Math. Anal. Appl. 491 (2020), no. 2, Article 124382, 31 pages; available online at https://doi.org/10.1016/j.jmaa.2020.124382.

[32] F. Qi, X.-T. Shi, and F.-F. Liu, Expansions of the exponential and the logarithm of power series and applications, Arab. J. Math. (Springer) 6 (2017), no. 2, 95–108; available online at https://doi.org/10.1007/s40065-017-0166-4.

[33] F. Qi, X.-T. Shi, F.-F. Liu, and D. V. Kruchinin, Several formulas for special values of the Bell polynomials of the second kind and applications, J. Appl. Anal. Comput. 7 (2017), no. 3, 857–871; available online at https://doi.org/10.11948/2017054.

[34] F. Qi and M. D. Ward, Closed-form formulas and properties of coefficients in Maclaurin’s series expansion of Wilf’s function, arXiv (2021), available online at https://arxiv.org/abs/2110.08576v1.

[35] J. Quaintance and H. W. Gould, Combinatorial Identities for Stirling Numbers. The unpublished notes of H. W. Gould. With a foreword by George E. Andrews. World Scientific Publishing Co. Pte. Ltd., Singapore, 2016.

[36] M. I. Qureshi, J. Majid, and A. H. Bhat, Hypergeometric forms of some composite functions containing $\arccos(x)$ using Maclaurin’s expansion, South East Asian J. Math. Math. Sci. 16 (2020), no. 3, 83–95.

[37] N. M. Temme, Special Functions: An Introduction to Classical Functions of Mathematical Physics, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1996; available online at http://dx.doi.org/10.1002/9781118032572.

[38] V. R. Thiruvenkatachar and T. S. Nanjundiah, Inequalities concerning Bessel functions and orthogonal polynomials, Proc. Ind. Acad. Sci. Sect. A 33 (1951), 373–384.

[39] C.-F. Wei, Integral representations and inequalities of extended central binomial coefficients, Authorea (2021), available online at https://doi.org/10.22541/au.163355864.99215800/v1.

[40] Z.-H. Yang and S.-Z. Zheng, Monotonicity and convexity of the ratios of the first kind modified Bessel functions and applications, Math. Inequal. Appl. 21 (2018), no. 1, 107–125; available online at https://doi.org/10.7153/mia-2018-21-09.

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