On the admissibility of observation operators in the context of maximal regularity

O. EL MENNAOUI, S. HADD and Y. KHAROU

Abstract. We study admissible observation operators for perturbed evolution equations using the concept of maximal regularity. We first show the invariance of the maximal $L^p$-regularity under non-autonomous Miyadera–Voigt perturbations. Second, we establish the invariance of admissibility of observation operators under such a class of perturbations. Finally, we illustrate our result with two examples, one on a non-autonomous parabolic system, and the other on an evolution equation subject to a Neumann boundary condition and a non-local perturbation.

1. Introduction

In this work, we propose a study of admissibility of observation operators for non-autonomous linear systems within the framework of the maximal $L^p$-regularity. This problem is studied in [22, Section 2] (see also [11,13,14]) in an abstract way using properties of evolution families. We also note that the admissibility for non-autonomous systems is less understood compared with the autonomous case, see e.g., [15,19,24–27].

Before summarizing precisely our results, we first recall some definitions. Denote by $(X, \|\cdot\|), (D, \|\cdot\|_D)$ and $(Y, \|\cdot\|_Y)$ three Banach spaces such that $D$ is continuously and densely embedded into $X$ ($D \hookrightarrow_d X$), and let $-A : D \to X$ be the generator of a strongly continuous semigroup $\mathbb{T} := (\mathbb{T}(t))_{t \geq 0}$ on $X$. For a real number $\theta > 1$, an operator $C \in \mathcal{L}(D, Y)$ is called an $L^\theta$-admissible observation operator for $(A, \theta)$ if for some (hence all) $\alpha > 0$, there exists a constant $\gamma > 0$ such that

$$
\left( \int_0^\alpha \| C\mathbb{T}(t)x \|^\theta_Y dt \right)^{\frac{1}{\theta}} \leq \gamma \| x \|, \quad (x \in D).
$$

This kind of operators appears naturally if one observes the semigroup solution at the boundary of a domain $\Omega \subset \mathbb{R}^n$ (when $X$ is a space of functions defined on $\Omega$), see

Mathematics Subject Classification: 93C20, 93C73, 93C25

Keywords: Non-autonomous systems, Evolution families, Admissible operators, $L^p$-maximal regularity, Perturbation.

Published online: 15 September 2023
We note that the admissibility in the autonomous case is a problem of integrability at 0 of the function $t \mapsto C \tau(t)x$.

In [22, Definition 2.4], Schnaubelt extended the $L^\theta$-admissibility from semigroups to abstract evolution families $(U(t, s))_{t \geq s \geq 0}$ on $X$, where extra assumptions are added to justify the expression $CU(t, s)x$. Recently, the work [16] studied the well-posedness of the following observed evolution system

$$\begin{align*}
\dot{u}(t) + A(t)u(t) &= 0, \quad u(s) = x, \quad t \in [s, \tau], \\
y(t) &= Cu(t), \quad t \in [s, \tau],
\end{align*}$$

for $\tau > 0$ fixed, where $(A(t))_{t \in [0, \tau]}$ is a family of unbounded operators with constant domain $D$ such that $A(\cdot) : [0, \tau] \rightarrow \mathcal{L}(D, X)$ is a bounded strongly measurable application, $C : D \rightarrow Y$ is a linear unbounded observation operator and $s \in [0, \tau)$. The main assumption used in [16] was that, for each $t \in [0, \tau]$, $A(t)$ has the property of maximal regularity (we write $A(t) \in \mathcal{MR}$, see Definition 2) and $A(\cdot) : [0, \tau] \rightarrow \mathcal{L}(D, X)$ satisfies the relative $\nu$-Dini assumption for some $\nu \in (1, \infty)$ (see the condition (H1) in Sect. 2). According to Theorem 1, essentially taken from [16], the evolution equation (2) is solved by a unique bounded evolution family $U := (U(t, s))_{0 \leq s \leq t \leq \tau}$ on $X$ such that $U(t, s)x \in D$ for all $x \in X$ and for almost every $t \in [s, \tau)$. In this case, the $L^\theta$-admissibility of the non-autonomous system (2) (or $(C, A(\cdot))$) is exactly defined as in (1), see Definition 6.

In the setting of the maximal regularity, the third named author (see Theorem 3 below) proved in [16] that under the condition (H1), $(C, A(\cdot))$ is admissible if and only if $(C, A(t))$ is admissible for each $t \in [0, \tau]$ (in the sense of (1), as each $A(t)$ generates an analytic semigroup on $X$). This means that instead of checking the admissibility condition involving the evolution family $U$, it suffices to check this condition individually for each $A(t)$ with respect to their associated semigroups.

In the autonomous case, it was proved in [12] that $C$ is admissible for $A$ if and only if $C$ is admissible for $A + P$ for any admissible perturbation $P \in \mathcal{O}_X^\theta(A)$, i.e., $\mathcal{O}_Y^\theta(A) = \mathcal{O}_Y^\theta(A + P)$ for any $P \in \mathcal{O}_X^\theta(A)$. The main objective of the present work is to extend this result to non-autonomous systems. To this end, consider a strongly measurable and bounded function $P(\cdot) : [0, \tau] \rightarrow \mathcal{L}(D, X)$ satisfying the condition (H2) (see Sect. 4). Then, we denote by $A^P(\cdot)$ the function $A^P(\cdot) : [0, \tau] \rightarrow \mathcal{L}(D, X)$ given by

$$A^P(t) := A(t) + P(t), \quad t \in [0, \tau].$$

In Theorem 2 (Sect. 4), we show that the conditions (H1) and (H2) imply $A^P(\cdot) \in \mathcal{MR}_p(0, \tau)$ for certain $p$. We mention that the same result is obtained in [3], using a quite different proof mainly based on Lebesgue extensions of the operators $P(t)$. Now the fact that $A^P(\cdot) \in \mathcal{MR}_p(0, \tau)$ shows that $A^P(\cdot)$ is associated with an evolution family $V := (V(t, s))_{0 \leq s \leq t \leq \tau}$ on the trace space, due to [4, Proposition 2.3]. In Proposition 1 we prove that the evolution family $V$ has a unique extension to an evolution family on the whole space $X$.  

[26, Chapter 4].
Under the above conditions, we prove in Theorem 4 that \((C, A(\cdot))\) is admissible if and only if \((C, A^p(\cdot))\) is admissible. This is somehow a non-autonomous version of the result in [12] mentioned above.

The rest of the paper is organized as follows. Section 2 gathers notation and required material from the concept of maximal \(L^p\)-regularity. Section 3 discusses the invariance of maximal \(L^p\)-regularity under non-autonomous Miyadera–Voigt perturbations. In Sect. 4, we prove results on admissibility of observation operators for perturbed evolution equations. Section 5 is devoted to examples which illustrate the main results.

2. Preliminaries on the maximal \(L^p\)-regularity

Throughout this section, \(X\) and \(D\) are Banach spaces with \(D \leftrightarrow \) \(X\). Moreover, we take real numbers \(p \in (1, \infty)\) and \(\tau, \tau' > 0\) such that \(\tau' < \tau\).

We recall some background about the concept of maximal \(L^p\)-regularity for non-autonomous linear systems.

**Definition 1.** A family \(U := (U(t, s))_{0 \leq s \leq t \leq \tau} \subset \mathcal{L}(X)\) is an evolution family on \(X\) if

\[
\begin{align*}
(i) & \quad U(t, s) = U(t, r)U(r, s), \quad U(s, s) = I, \quad \text{for any } s, t, r \in [0, \tau] \text{ with } 0 \leq s \leq r \leq t \leq \tau, \\
(ii) & \quad \left\{(t, s) \in [0, \tau]^2 : t \geq s\right\} \ni (t, s) \mapsto U(t, s) \text{ is strongly continuous.}
\end{align*}
\]

The concept of evolution families is used to solve some classes of evolution equations (see, e.g., [5, Chapter 2], [8, Section VI.9] and [21]). That is, the evolution family arises as the solution operator of the well-posed non-autonomous evolution equation

\[
\dot{u}(t) + A(t)u(t) = 0, \quad u(s) = x \in X, \quad 0 \leq s \leq t \leq \tau,
\]

where \(A(t)\) is (in general) an unbounded linear operator for every fixed \(t\). We mention that, in general, the function \(U(t, s)x\), as a function of \(t\), is not differentiable. However, the differentiability of such a function is guaranteed if \(A(t), t \in [0, \tau]\), satisfy additional conditions (see e.g. [1,23]).

In the following, we will introduce a situation in which one can naturally associate an evolution family to a family of unbounded operators. We first define the notion of maximal regularity for single operators.

**Definition 2.** We say that an operator \(A \in \mathcal{L}(D, X)\) has \(L^p\)-maximal regularity \((p \in (1, \infty))\) and we write \(A \in \mathcal{MR}_p\) if for all bounded intervals \((a, b) \subset \mathbb{R} (a < b)\) and all \(f \in L^p(a, b; X)\), there exists a unique \(u \in W^{1,p}(a, b; X) \cap L^p(a, b; D)\) such that

\[
\dot{u}(t) + Au(t) = f(t) \quad t\text{-a.e. on } [a, b], \quad u(a) = 0. \tag{4}
\]

According to [7], the property of maximal \(L^p\)-regularity is independent of the bounded interval \((a, b)\), and if \(A \in \mathcal{MR}_p\) for some \(p \in (1, \infty)\) then \(A \in \mathcal{MR}_q\) for all \(q \in (1, \infty)\). Hence, we can write \(A \in \mathcal{MR}\) for short. Also it is well known that
if \( A \in \mathcal{M}\mathcal{R} \), then \(-A\) generates an analytic semigroup on \( X \) [17]. The converse is true if we work in Hilbert spaces [17, Corollary 1.7]. We also refer to [6,17] for the concept of \( \mathcal{R}\)-Boundedness and Fourier Multipliers applied to maximal regularity.

Let \( p \in (1, \infty) \) and define the following functional space

\[
\text{MR}_p(a, b) := W^{1,p}(a, b; X) \cap L^p(a, b; D),
\]

which we call the space of maximal regularity. It is equipped with the following norm

\[
\|u\|_{\text{MR}_p(a, b)} := \|u\|_{W^{1,p}(a,b;X)} + \|u\|_{L^p(a,b;D)} \quad (u \in \text{MR}_p(a, b)),
\]

or, with the equivalent norm

\[
\left(\|u\|_{W^{1,p}(a,b;X)}^p + \|u\|_{L^p(a,b;D)}^p\right)^{1/p}.
\]

The space \( \text{MR}_p(a, b) \) is a Banach space when equipped with one of the above norms. Moreover, we consider the trace space defined by

\[
\text{Tr}_p := \{u(a) : u \in \text{MR}_p(a, b)\},
\]

and endowed with the norm

\[
\|x\|_{\text{Tr}_p} := \inf \{\|u\|_{\text{MR}_p(a,b)} : x = u(a)\}.
\]

The space \( \text{Tr}_p \) is isomorphic to the real interpolation space \((X, D)_{1-1/p, p}[20, \text{Chapter 1}]. In particular, \( \text{Tr}_p \) does not depend on the choice of the interval \((a, b)\) and \( D \hookrightarrow_{d} \text{Tr}_p \hookrightarrow_{d} X \). We also note that

\[
\text{MR}_p(a, b) \hookrightarrow_{d} C ([a, b]; \text{Tr}_p),
\]

and the constant of the embedding does not depend on the interval \((a, b)\) [2, chap. 3].

Consider now the non-autonomous evolution equation

\[
\dot{u}(t) + A(t)u(t) = 0 \quad t \text{ a.e. on } [s, \tau], \quad u(s) = x
\]

for any \( s \in [0, \tau) \), \( x \in X \) and \( A(\cdot) : [0, \tau] \rightarrow \mathcal{L}(D, X) \) is a strongly measurable and bounded operator-valued function such that \( A(t) \in \mathcal{M}\mathcal{R} \) for all \( t \in [0, \tau] \).

**Definition 3.** Let \( p \in (1, \infty) \). We say that \( A(\cdot) \) has \( L^p \)-maximal regularity on the bounded interval \([0, \tau]\) (and we write \( A(\cdot) \in \mathcal{M}\mathcal{R}_p(0, \tau)\)), if and only if for all \([a, b]\) a sub-interval of \([0, \tau]\) and all \( f \in L^p(a, b; X) \), there exists a unique \( u \in \text{MR}_p(a, b) \) such that

\[
\dot{u}(t) + A(t)u(t) = f(t) \quad t \text{ a.e. on } [a, b], \quad u(a) = 0.
\]
Note that, by compactness of $[0, \tau]$, $A(\cdot) \in \mathcal{MR}_p(0, \tau)$ if and only if there exists $\alpha_0 > 0$ such that for all $[a, b]$ a sub-interval of $[0, \tau]$ with $|b - a| < \alpha_0$, and all $f \in L^p(a, b; X)$, there exists a unique $u \in MR_p(a, b)$ such that

$$\dot{u}(t) + A(t)u(t) = f(t) \quad a.e. \text{ on } [a, b], \quad u(a) = 0.$$ 

Let us now recall how to solve the evolution equation (5). We assume that $A(\cdot) \in \mathcal{MR}_p(0, \tau)$, and we take $[a, b] \subset [0, \tau]$ and $x \in \text{Tr}$. Let then $w \in MR_p(a, b)$ such that $w(a) = x$. Let now $v \in MR_p(a, b)$ the unique solution of

$$\dot{v}(t) + A(t)v(t) = -\dot{w}(t) - A(t)w(t), \quad a.e. \ t \in [a, b], \quad v(a) = 0.$$ 

Then $u := v + w \in MR_p(a, b)$ and $u$ satisfies the equation (5). This shows the existence of the solution of (5). The uniqueness of this solution is a consequence of maximal regularity.

In principle, one can ask if the condition $A(t) \in \mathcal{MR}$ for each $t \in [0, \tau]$ will imply that the evolution equation (5) is solved by an evolution family on $X$. This is true if $A(\cdot)$ satisfies a kind of continuity.

**Definition 4.** We say that $A(\cdot) : [0, \tau] \to \mathcal{L}(D, X)$ is relatively continuous if for all $\varepsilon > 0$ there exist $\delta > 0$ and $\eta \geq 0$ such that for all $s, t \in [0, \tau]$, we have

$$\|A(t)x - A(s)x\| \leq \varepsilon \|x\|_D + \eta \|x\| \quad (x \in D)$$

whenever $|t - s| \leq \delta$.

Now, if $A(t) \in \mathcal{MR}$ for each $t \in [0, \tau]$ and $A(\cdot)$ is relatively continuous, the authors of [4] proved the existence of an evolution family $U := (U(t, s))_{0 \leq s \leq t \leq \tau}$ on the trace space $\text{Tr}_p$. If the operators $A(t)$, $t \in [0, \tau]$ are, in addition, accretive, they further showed that $U$ extends to a contractive evolution family on $X$.

We need the following regularity on the map $A(\cdot)$.

**Definition 5.** Let $v \in (1, \infty)$. The function $A(\cdot) : [0, \tau] \to \mathcal{L}(D, X)$ satisfies the relative $v$-Dini condition if there exist $\eta \geq 0$, $\omega : [0, \tau] \to [0, \infty)$ a continuous function with $\omega(0) = 0$ and

$$\int_0^\tau \left(\frac{\omega(t)}{t}\right)^v \, dt < \infty \quad (7)$$

such that for all $x \in D, s, t \in [0, \tau]$, we have:

$$\|A(t)x - A(s)x\| \leq \omega(|t - s|) \|x\|_D + \eta \|x\|.$$

In the rest of this paper, we need the following condition:

**H1** For any $t \in [0, \tau]$, $A(t) \in \mathcal{MR}$, and $A(\cdot) : [0, \tau] \to \mathcal{L}(D, X)$ satisfies the relative $v$-Dini condition for some $v \in (1, \infty)$.
Remark 1. (i) It is worth noting that if $A(\cdot)$ satisfies the relative $\nu$-Dini condition, then $A(\cdot)$ is relatively continuous, so the result from [4] mentioned above can be applied. In particular, the condition (H1) implies the existence of an evolution family $U$ on $\text{Tr}_p$ for any $p \in (1, \infty)$, see [4, Theorem 2.7 and Proposition 2.3].

(ii) If $A(\cdot)$ is Hölder continuous, that is, 
\[
\|A(t)x - A(s)x\| \leq |t - s|^\alpha \|x\|_D, \quad t, s \in [0, \tau],
\]
for some $\alpha \in (0, 1)$, then $A(\cdot)$ satisfies the relative $\nu$-Dini condition for all $\nu \in (1, 1 - \alpha)$.

(iii) The condition (H1) implies that $A(\cdot) \in \mathcal{MR}_q(0, \tau)$ for any $q \in (1, \infty)$, see [4].

The following result is a slight modification of [16, Theorem 3.3].

Theorem 1. Assume that $A(\cdot) : [0, \tau] \rightarrow \mathcal{L}(D, X)$ satisfies the condition (H1) and let $U := (U(t, s))_{0 \leq s \leq t \leq \tau}$ be the associated evolution family on the trace space $\text{Tr}_p$ for $p \in (1, \infty)$. Then the following assertions hold:

(i) $U$ extends to a bounded evolution family on $X$.

(ii) For every $x \in X$, the function $u$ given by $u(t) := U(t, 0)x$ is the unique solution of the problem
\[
\dot{u}(t) + A(t)u(t) = 0 \quad t \text{ a.e. on } [0, \tau], \quad u(0) = x.
\]

(iii) For every $q \in (1, v)$, and $x \in X$, we have $v : t \mapsto v(t) = tU(t, 0)x \in \mathcal{MR}_q(0, \tau)$ and
\[
\|v\|_{\mathcal{MR}_q(0, \tau)} \leq M\|x\|, 
\tag{8}
\]
for a constant $M \geq 0$, depending on $q$ but independent of $x \in X$. Moreover, the function $u$ given by $u(t) := U(t, 0)x$ belongs to the space
\[
C \left([0, \tau], X\right) \cap L^q_{loc} ((0, \tau], D) \cap W^{1,q}_{loc} ((0, \tau], X).
\]

3. The stability of maximal $L^p$-regularity under non-autonomous Miyadera–Voigt kind of perturbations

In this section we assume that the function $A(\cdot) : [0, \tau] \rightarrow \mathcal{L}(D, X)$ satisfies the condition (H1). Then we discuss the maximal regularity of $A(\cdot) + P(\cdot)$, where $P(\cdot) : [0, \tau] \rightarrow \mathcal{L}(D, X)$ is a strongly measurable and bounded application such that:

(H2) There exists $\mu \in (1, \infty)$ and a constant $c > 0$ such that for $0 \leq s < \tau$, we have
\[
\left(\int_s^\tau \|P(t)U(t, s)x\|^{\mu} dt\right)^{\frac{1}{\mu}} \leq c\|x\|, \quad (x \in D).
\]
The following result shows the maximal regularity of \( A^P(t) : [0, \tau] \to \mathcal{L}(D, X) \) defined in (3).

**Theorem 2.** Assume that \( A(\cdot) \) and \( P(\cdot) \) satisfy the conditions (H1)–(H2). Then \( A^P(\cdot) \in \mathcal{M}_q(0, \tau) \) for every \( q \in (1, \mu] \).

**Proof.** According to Remark 1 we have \( A(\cdot) \in \mathcal{M}_q(0, \tau) \) for any \( q \in (1, \infty) \). Let \([a, b]\) \((a < b)\) be a sub-interval of \([0, \tau]\) and define the operators \( A \) and \( B \) on \( L^q(a, b; X) \) by:

\[
D(B) := \left\{ u \in W^{1,q}(a, b; X) : u(a) = 0 \right\}, \quad Bu := \dot{u},
\]

\[
D(A) := L^q(a, b; D), \quad Au := A(s)u(s) \quad (s \in (a, b)).
\]

On the other hand, define

\[
(\mathcal{P} u)(t) := P(t)u(t), \quad u \in L^q(a, b; D), \quad a.e. t \in [a, b].
\]

Observe that for any \( u \in L^q(a, b; D) \), \( t \mapsto (\mathcal{P} u)(t) \) is measurable and

\[
\int_a^b \| (\mathcal{P} u)(t) \|^q dt \leq \left( \sup_{t \in [0,\tau]} \| P(t) \|_{\mathcal{L}(D, X)} \right)^q \| u \|^q_{L^q(a, b; D)}.
\]

This shows that \( \mathcal{P} : L^q(a, b; D) \to L^q(a, b; X) \), and

\[
\| \mathcal{P} u \|_{L^q(a, b; X)} \leq \| P(\cdot) \|_\infty \| u \|_{L^q(a, b; D)}, \quad (u \in L^q(a, b; D)).
\]

Now let the operator \( A + B + \mathcal{P} \) on \( L^q(a, b; X) \) with domain \( D(A + B + \mathcal{P}) := D(A) \cap D(B) \). The fact that \( A(\cdot) \in \mathcal{M}_q(0, \tau) \) implies that the inverse \( (A + B)^{-1} \)

exists in \( L^q(a, b; X) \). Moreover, we can write

\[
A + B + \mathcal{P} = (I + \mathcal{P}(A + B)^{-1})(A + B).
\]

Let \( q \in (1, \mu] \) and \( q' \in (1, \infty) \) such that \( \frac{1}{q} + \frac{1}{q'} = 1 \). Then \( P(\cdot) \) satisfies also the condition (H2) if we replace \( \mu \) with \( q \). Now for \( f \in L^q(a, b; X) \),

\[
||P(A + B)^{-1}f||^q_{L^q(a,b;X)} = \int_a^b \| [P(A + B)^{-1}f](t) \|^q dt
\]

\[
= \int_a^b \left| \int_a^t U(t, r) f(r) \, dr \right|^q dt
\]

\[
\leq \int_a^b \left( \int_a^t \| P(t) U(t, r) f(r) \| dr \right)^q dt
\]

\[
\leq (b - a)^{q'/q'} \int_a^b \int_a^t \| P(t) U(t, r) f(r) \|^q dr dt
\]

\[
\leq (b - a)^{q'/q'} \int_a^b \int_r^b \| P(t) U(t, r) f(r) \|^q dt dr
\]

\[
\leq (b - a)^{q'/q'} \int_a^b \| f(r) \|^q dr
\]

\[
= (b - a)^{q'/q'} \| f \|_{L^q(a,b;X)},
\]
by Hölder inequality, Fubini theorem and (H2). By choosing a small interval \([a, b]\), 
we can assume that \((b - a)^{q/q'} c < 1\). This ends the proof due to the compactness of 
\([0, \tau]\) (see comments after Definition 3).

Remark 2. A similar result for the maximal regularity of \(A^p(\cdot)\) is obtained in [3], 
where Lebesgue extensions of the operators \(P(t)\) are used to prove the result.

The following proposition shows that to \(A^p(\cdot)\), we can associate an evolution family 
\(V := (V(t, s))_{0 \leq s \leq t \leq \tau}\) on \(X\) (compare with [11, Proposition 4.3]).

**Proposition 1.** Assume that \(A(\cdot)\) and \(P(\cdot)\) satisfy the conditions (H1)–(H2). Then 
\(A^p(\cdot)\) generates an evolution family \(V := (V(t, s))_{0 \leq s \leq t \leq \tau}\) on \(X\) satisfying the following formula

\[
V(t, s)x = U(t, s)x + \int_s^t V(t, \sigma)P(\sigma)U(\sigma, s)x d\sigma
\]

for any \(0 \leq s \leq t \leq \tau\) and \(x \in D\). Furthermore for any \(x \in X\), the function 
\(u : t \mapsto V(t, 0)x\) solves the problem

\[
\dot{u}(t) + A^p(t)u(t) = 0, \quad a.e. \ t \in [0, \tau], \ u(0) = x,
\]

and belongs to the space

\[
C([0, \tau], X) \cap L^q((0, \tau], D) \cap W^{1,q}_{loc}((0, \tau], X),
\]

for all \(q \in (1, \min\{v, \mu\}]\).

**Proof.** According to Theorem 2, \(A^p(\cdot) \in \mathcal{M}R_q(0, \tau)\) for every \(q \in (1, \mu]\). Then by 
[4, Lemma 2.2], to \(A^p(\cdot)\), we associate a unique evolution family \(V := (V(t, s))_{0 \leq s \leq t \leq \tau}\) on \(Tr_q\) for \(q \in (1, \mu]\). In particular, for \(x \in D\), the functions \(t \mapsto V(t, s)x\) and 
\(t \mapsto U(t, s)x\) belong to \(MR_q(s, \tau)\) for \(q \in (1, \mu]\), so that

\[
\frac{d}{dt} (V(t, s)x - U(t, s)x) = -A^p(t)V(t, s)x + A(t)U(t, s)x
\]

\[
= -A^p(t)(V(t, s)x - U(t, s)x) - P(t)U(t, s)x.
\]

The fact that \(A^p(\cdot) \in \mathcal{M}R_q(0, \tau)\) for \(q \in (1, \mu]\) implies that there exists a constant 
\(\kappa > 0\) such that

\[
\|V(\cdot, s)x - U(\cdot, s)x\|_{MR_q(s, \tau)} \leq \kappa \|P(\cdot)U(\cdot, s)x\|_{L^q([s, \tau], X)} \leq \kappa \gamma \|x\| \quad (10)
\]

for any \(x \in X\), due to (H2). By using the estimate (10) and the embedding \(MR_q(s, \tau) \hookrightarrow d\) 
\(C([s, \tau], X)\) (densely and continuously), there exists a constant \(M > 0\), independent 
of \(t\) and \(s\) such that

\[
\|V(t, s)x - U(t, s)x\| \leq M \|x\|
\]
for any $0 \leq s \leq t \leq \tau$ and $x \in \text{Tr}_p$. Now the density of $\text{Tr}_p$ in $X$ implies that $V$ has an extension to an evolution family on $X$. Now we can write

$$V(t, s)x - U(t, s)x = \int_s^t V(t, \sigma)P(\sigma)U(\sigma, s)x\,d\sigma$$

for any $0 \leq s \leq t \leq \tau$ and $x \in X$.

On the other hand, let $q \in (1, \min\{v, \mu\})$ and $x \in \text{Tr}_q$. By the $L^q$-maximal regularity of $A(\cdot)$, the function $v(t) := tV(t, 0)x$ is the unique solution of the non-homogeneous problem

$$\begin{cases}
\dot{v}(t) + A^P(t)v(t) = V(t, 0)x, & \text{a.e. } t \in [0, \tau], \\
u(0) = 0,
\end{cases}$$

and there exists a constant $\kappa > 0$ such that

$$\|v\|_{\text{MR}_q(0, \tau)} \leq \kappa \|V(\cdot, 0)x\|_{L^q(0, \tau; X)} \leq M'\|x\|$$

for a constant $M' > 0$ depending on $\kappa$, $M$ and $q$. By the density of $\text{Tr}_q$ in $X$, the above estimate holds also for every $x \in X$. In addition, since $\text{MR}_q(0, \tau) \hookrightarrow_d C([0, \tau], X)$, we have

$$\|tV(t, 0)x\| \leq M'\|x\|$$

for almost every $t \in [0, \tau]$ and $x \in X$. In particular, for every $x \in X$ and every $q \in (1, \min\{v, \mu\})$, we have

$$V(\cdot, 0)x \in L^q_{\text{loc}}((0, \tau], D) \cap W^{1, q}_{\text{loc}}((0, \tau], X).$$

\[\square\]

4. The stability of admissibility of observation operators under non-autonomous Miyadera–Voigt kind of perturbations

Throughout this section $X$, $Y$ and $D$ are Banach spaces as in Sect. 1. We take real numbers $p \in (1, \infty)$ and $\tau > 0$. Let $A(\cdot) : [0, \tau] \rightarrow \mathcal{L}(D, X)$ be a strongly measurable and bounded function, and $C : D \rightarrow Y$ is a linear (observation) operator. From Sect. 1, if $A(\cdot)$ satisfies the condition (H1), then we can associate to it a bounded evolution family $U = (U(t, s))_{0 \leq s \leq t \leq \tau}$ on the whole space $X$.

**Definition 6.** Let the assumption (H1) be satisfied. The operator $C$ is $L^\theta$-admissible for $A(\cdot)$ or $(C, A(\cdot))$ is $L^\theta$-admissible with $\theta \in (1, \infty)$ if and only if there exists $\gamma > 0$ such that for any $s \in [0, \tau)$, we have

$$\left(\int^\tau_s \|CU(t, s)x\|^\theta_Y \,dt\right)^{\frac{1}{\theta}} \leq \gamma\|x\|, \quad (x \in D). \quad (11)$$
One can see that \((C, A(\cdot))\) is \(L^\theta\)-admissible if and only if \((C, \lambda + A(\cdot))\) is for some/all \(\lambda \in \mathbb{C}\).

The fact that \((C, A(\cdot))\) is admissible is not easy to verify in the examples. In some cases, individual admissibility may suffice. The relation between the admissibility of observation operators for \(A(\cdot)\) and for each single operator \(A(t_0)\) \((t_0 \in [0, \tau))\) is given in the following result which is taken from [16, Theorem 3.8].

**Theorem 3.** Let \(A(\cdot) : [0, \tau] \rightarrow \mathcal{L}(D, X)\) satisfy the condition \((H1)\). Then \((C, A(\cdot))\) is admissible, if and only if \((C, A(t))\) is admissible for all \(t \in [0, \tau)\).

The following proposition shows a condition on \(A(\cdot)\) for which the property “\((C, A(\cdot))\) is \(L^\theta\)-admissible for all \(t \in [0, \tau]\)” is equivalent to “there exists \(t_0 \in [0, \tau]\) such that \((C, A(t_0))\) is \(L^\theta\)-admissible”.

**Proposition 2.** Let \(A(\cdot) : [0, \tau] \rightarrow \mathcal{L}(D, X)\) be a strongly measurable and bounded function. Assume that \(A(t) \in \mathcal{M}_X\) for every \(t \in [0, \tau]\) and that there exist \(M, \eta > 0\) and \(\beta \in (1, \infty)\) such that

\[
\|A(t)x - A(s)x\| \leq M\|x\|_{\mathcal{L}(D, X)} + \eta\|x\|, \quad (t, s \in [0, \tau], x \in D).
\]

Then

\[
\mathcal{G}_Y^\theta(A(t_0)) = \mathcal{G}_Y^\theta(A(t_1))
\]

for every \(\theta < \frac{\beta}{\beta - 1}\) and \(t_0, t_1 \in [0, \tau]\).

**Proof.** Let \(0 < t \leq \tau, x \in D\) and \(\theta \in (1, \infty)\). Consider the function \(v : [0, \tau] \ni r \mapsto e^{-(t-r)A(t_0)}e^{-rA(t_1)x}.\) Then \(v \in W^{1,\theta}(0, \tau; X)\) and

\[
\frac{d}{dr}v(r) = A(t_0)e^{-(t-r)A(t_0)}e^{-rA(t_1)x} - e^{-(t-r)A(t_0)}A(t_1)e^{-rA(t_1)x} = e^{-(t-r)A(t_0)}(A(t_0) - A(t_1))e^{-rA(t_1)x}.
\]

By integrating between 0 and \(t\), we obtain

\[
e^{-tA(t_1)x} = e^{-tA(t_0)x} + \int_0^t e^{-(t-r)A(t_0)}(A(t_0) - A(t_1))e^{-rA(t_1)x}dr.
\]

Thus, according to [10, Prop.3.3], the \(L^\theta\)-admissibility of \(C\) for \(A(t_0)\) implies

\[
\int_0^\tau \|Ce^{-tA(t_1)x} - Ce^{-tA(t_0)x}\|^\theta dt \\
\leq \int_0^\tau \left\| C \int_0^t e^{-(t-r)A(t_0)}(A(t_0) - A(t_1))e^{-rA(t_1)x}dr \right\|^\theta dt \\
\leq k_\tau \int_0^\tau \| (A(t_0) - A(t_1))e^{-rA(t_1)x} \|^\theta dr \\
\leq k_\tau \int_0^\tau \left( M\|e^{-rA(t_1)x}\|_{\mathcal{L}(D, X)} + \eta\|e^{-rA(t_1)x}\|\right)^\theta dr \\
\leq 2^\theta M^\theta k_\tau \int_0^\tau \frac{1}{r^{\frac{\theta}{\theta - 1}}}dr\|x\|^\theta + 2^\theta \eta^\theta c_\tau^\theta \|x\|^\theta.
\] (12)
Remark 3. We mention that the condition satisfied by $A(\cdot)$ in Proposition 2 means that $A(\cdot)$ is obtained by a non-autonomous perturbation of a somehow lower order of a fixed operator.

Remark 4. Let $A(\cdot) : [0, \tau] \to {\mathcal L}(D, X)$ be a strongly measurable and bounded function, $\theta \in (1, \infty)$ and $B \in \mathcal{O}^\theta_Y(A(t))$ for any $t \in [0, \tau]$. Assume that $A(t) \in \mathcal{M}_\mathcal{R}$ for every $t \in [0, \tau]$ and that there exist $M, \eta > 0$ such that
\[
\|A(t)x - A(s)x\| \leq M\|Bx\| + \eta\|x\|, \quad \forall t, s \in [0, \tau], \forall x \in D.
\] (13)
Then $\mathcal{O}^\theta_Y(A(t_0)) = \mathcal{O}^\theta_Y(A(t_1))$ for any $t_0, t_1 \in [0, \tau]$. Indeed, assume that $C \in \mathcal{O}^\theta_Y(A(t_0))$. By combining the arguments in (12) with (13), we obtain
\[
\int_0^\tau \|C e^{-tA(t_1)}x\|^{\theta} dt \leq \gamma^\theta \|x\|^\theta + \kappa \int_0^\tau \left( M\|B e^{-rA(t_1)}x\| + \eta\|e^{-rA(t_1)}x\| \right)^\theta dr 
\leq c\|x\|^\theta,
\]
for a constant $c := c(\theta, \tau) > 0$, due to the $L^\theta$-admissibility of $B$ for $A(t_1)$.

The following is the main result of this paper, which gives the invariance of admissibility of observation under unbounded perturbations.

**Theorem 4.** Assume that $A(\cdot)$ and $P(\cdot)$ satisfy the conditions (H1)–(H2). Then
\[
\mathcal{O}^\theta_Y(A(\cdot)) = \mathcal{O}^\theta_Y(A^P(\cdot)) \quad (1 < \theta \leq \mu),
\]
where $\mu$ is from (H2).

**Proof.** Let $C \in \mathcal{O}^\theta_Y(A(\cdot))$, $x \in D$ and $s \in [0, \tau)$. According to Proposition 1, we have the evolution family $V$ for $A^P(\cdot)$ and we can write
\[
CV(t, s)x = C(V(t, s)x - U(t, s)x) + CU(t, s)x
\]
for $0 \leq s \leq t \leq \tau$. Thus
\[
\int_s^\tau \|CV(t, s)x\|^{\theta} dt \leq 2P\|C\|_{{\mathcal L}(D, Y)}^\theta \int_s^\tau \|V(t, s)x - U(t, s)x\|^{\theta} dt + (2\beta)^\theta\|x\|^\theta 
\leq 2\|C\|_{{\mathcal L}(D, Y)}^\theta \|V(\cdot, s)x - U(\cdot, s)x\|_{{\mathcal L}(D, Y)}^{\theta(\cdot, s, \tau)} + (2\beta)^\theta\|x\|^\theta 
\leq (2\kappa \gamma\|C\|_{{\mathcal L}(D, Y)}^\theta)\|x\|^{\theta} + (2\beta)^\theta\|x\|^\theta := \delta^\theta\|x\|^\theta,
\]
due to (10).

Conversely, assume that $C \in \mathcal{O}^\theta_Y(A^P(\cdot))$, $x \in D$ and $s \in [0, \tau)$. By the same computation if we replace $C$ with $P(t)$ and using the boundedness of the map $t \in [0, \tau] \mapsto \|P(t)\|_{{\mathcal L}(D, Y)}$, we can see that
\[
\left( \int_s^\tau \|P(t)V(t, s)x\|^{\mu} dt \right)^{\frac{1}{\mu}} \leq \tilde{\gamma}\|x\|.
\]
Hence $P(\cdot)$ satisfies (H2) with respect to $A(\cdot) + P(\cdot)$. We can then use the first case to show that $C \in \mathcal{O}^\theta_Y(A(\cdot))$. \(\Box\)
Remark 5. Let the assumptions of Theorem 4 be satisfied. Let $C(\cdot) : [0, \tau] \to \mathcal{L}(D, Y)$ be a strongly measurable and bounded map and let $\theta \in (1, \mu]$. Using the same arguments as in the proof of Theorem 4, we can prove the following result: for $s \in [0, \tau)$, there exists $\gamma > 0$ such that

$$
\int_{s}^{\tau} \|C(t)U(t, s)x\|^{\theta} \, dt \leq \gamma^{\theta} \|x\|^{\theta} \quad (x \in D)
$$

(14)

if and only if for $s \in [0, \tau)$, there exists $c > 0$ such that

$$
\int_{s}^{\tau} \|C(t)V(t, s)x\|^{\theta} \, dt \leq c^{\theta} \|x\|^{\theta} \quad (x \in D).
$$

Remark 6. Suppose that $A(\cdot) : [0, \tau] \to \mathcal{L}(D, X)$ satisfies the condition $(H_1)$ and let $C(\cdot) : [0, \tau] \to \mathcal{L}(D, Y)$ be a strongly measurable, bounded map satisfying the following assumption

$(\tilde{H}_1)$ there exist constants $\tilde{v} \in (1, \infty)$, $\tilde{\eta} > 0$, and a continuous function $\tilde{\omega} : [0, \tau] \to [0, \infty)$ with $\tilde{\omega}(0) = 0$, and

$$
\int_{0}^{\tau} \left( \frac{\tilde{\omega}(t)}{t} \right)^{\tilde{v}} \, dt < \infty
$$

such that for all $x \in D$, $s, t \in [0, \tau]$, we have

$$
\|C(t)x - C(s)x\| \leq \tilde{\omega}(|t - s|) \|x\|_{D} + \tilde{\eta} \|x\|.
$$

In what follows, when the conditions $(H_1)$ and $(\tilde{H}_1)$ are satisfied, we denote $(T^{a}(t))_{t \geq 0}$ the analytic semigroup generated by $(-A(a), D)$ with $a \in [0, \tau)$, and set $\kappa := \sup_{t \in [0, \tau]} \|C(t)\|$, and $M := \sup_{t \in [0, \tau]} \|T^{a}(t)\|$. Moreover, we denote by $(U(t, s))_{0 \leq s \leq t \leq \tau}$ the evolution family on $X$ associated with $A(\cdot)$ (see Theorem 1). We have the following observations

(i) Assume that $(H_1)$ and $(\tilde{H}_1)$ are satisfied and let $\theta \in (1, \tilde{v}]$. Then, for any $x \in D$, and $a \in [0, \tau)$, there exists a constant $\beta > 0$ such that

$$
\left\| C(\cdot)U(\cdot, a)x - C(a)T^{a}(\cdot - a)x \right\|_{L^{\theta}(a, \tau; Y)} \leq \beta \|x\|.
$$

(15)

In fact, according to the proof of [16, Theorem 3.3], we deduce that $U(\cdot, a)x - T^{a}(\cdot - a)x \in MR_{\theta}(a, \tau)$ and

$$
\|U(\cdot, a)x - T^{a}(\cdot - a)x\|_{MR_{\theta}(a, \tau)} \leq M \|x\|
$$

(16)
for a certain constant \( M > 0 \). Moreover,
\[
\| C(\cdot)U(\cdot, a)x - C(a)T_a^{\theta}(\cdot - a)x \|_{L^\theta(\omega, \tau; Y)} \\
\leq \| C(\cdot)U(\cdot, a)x - T_a^{\theta}(\cdot - a)x \|_{L^\theta(\omega, \tau; Y)} \\
+ \| (C(\cdot) - C(a))T_a^{\theta}(\cdot - a)x \|_{L^\theta(\omega, \tau; Y)} \\
\leq \kappa \| U(\cdot, a)x - T_a^{\theta}(\cdot - a)x \|_{M\omega, \omega, \omega} \\
+ 2\| \tilde{\omega}(\cdot - a)T_a^{\theta}(\cdot - a)x \|_{L^\theta(\omega, \tau; D)} \\
+ 2\tilde{\eta} \| T_a^{\theta}(\cdot - a)x \|_{L^\theta(\omega, \tau; X)} \\
\leq (\kappa + 2\tau^\frac{1}{2}\tilde{\eta}\tilde{M})\| x \| + 2\| \tilde{\omega}(\cdot - a)T_a^{\theta}(\cdot - a)x \|_{L^\theta(\omega, \tau; D)}. \quad (17)
\]
On the other hand, since \( \| tT_a^{\theta}(t)x \|_D \leq c\| x \| \) for any \( t \in [0, \tau] \) and a suitable constant \( c := c(\tau) > 0 \), then
\[
\| \tilde{\omega}(\cdot - a)T_a^{\theta}(\cdot - a)x \|_{L^\theta(\omega, \tau; D)} \leq c^\theta \int_a^\tau \left( \frac{\tilde{\omega}(t - a)}{t - a} \right) \| x \|^\theta dt \\
\leq c^\theta \int_0^\tau \left( \frac{\tilde{\omega}(\sigma)}{\sigma} \right) \tilde{v} \| x \|^\theta := \tilde{c} \| x \|^\theta \quad (18)
\]
for a constant \( \tilde{c} := \tilde{c}(\tau, \tilde{v}, \theta) \). Thus, the result now follows by (17) and (18), and by selecting \( \beta := \kappa + 2\tau^\frac{1}{2}\tilde{\eta}\tilde{M} + 2\tilde{c}^\frac{1}{\theta} \).

(ii) Assume that (H1) and (H1) are satisfied and the family \( (C(t))_{t \in [0, \tau]} \) satisfies the condition (14) for an exponent \( \theta \in (1, \tilde{v}) \). Then \( (C(a), A(a)) \) is \( L^\theta \)-admissible for any \( a \in [0, \tau) \). In fact, let \( a, b \in [0, \tau) \) such that \( a < b \) and \( x \in D \). Using a change of variables and the inequalities (14) and (15), we obtain
\[
\| C(a)T_a^{\theta}(\cdot - a)x \|_{L^\theta(0, b-a; Y)} = \| C(a)T_a^{\theta}(\cdot - a)x \|_{L^\theta(\omega, \omega; Y)} \\
\leq \gamma \| x \| + \| C(\cdot)U(\cdot, a)x - C(a)T_a^{\theta}(\cdot - a)x \|_{L^\theta(\omega, \tau; Y)} \\
\leq (\gamma + \beta)\| x \|.
\]
This shows that \( (C(a), A(a)) \) is \( L^\theta \)-admissible.

(iii) Let (H1) and (H1) be satisfied. If \( (C(a), A(a)) \) is \( L^\theta \)-admissible for any \( a \in [0, \tau) \) and some exponent \( \theta \in (1, \tilde{v}) \), then the family \( (C(t))_{t \in [0, \tau]} \) satisfies the condition (14). In fact, by assumption, there exists a constant \( c > 0 \) such that
\[
\int_0^\tau \| C(a)T_a^{\theta}(\sigma)x \|^\theta d\sigma \leq c^\theta \| x \|^\theta
\]
for any \( x \in D \). Now this inequality together with (16) imply that
\[
\left( \int_a^\tau \| C(t)U(t, a)x \|^\theta dt \right)^\frac{1}{\theta} \\
\leq \| C(a)T_a^{\theta}(\cdot - a)x \|_{L^\theta(\omega, \tau; Y)} + \| C(\cdot)U(\cdot, a)x - C(a)T_a^{\theta}(\cdot - a)x \|_{L^\theta(\omega, \tau; Y)} \\
\leq (c + \beta)\| x \|.
\]
This ends the proof.
**Corollary 1.** Let $-A_0, -A_1 : D \to X$ be two generators on $X$ such that $A_0 \in \mathcal{MR}$ and

$$
\| (A_1 - A_0)x \| \leq M \| x \|_{Tr^2} + \eta \| x \| 
$$

(19)

for and $x \in D$ and for some constants $M, \eta > 0$. Then $O^\theta_Y(A_0) = O^\theta_Y(A_1)$ for any $\theta \in (1, 2)$.

**Proof.** We set $A(t) = A_0$ and $P(t) = A_1 - A_0$ for any $t \in [0, \tau]$. Clearly $A(\cdot)$ satisfies the condition (H1). Let us now verify that $P(\cdot)$ satisfies the condition (H2) for $\mu = \theta \in (1, 2)$. Let $x \in D$ and $s \in [0, \tau)$. By using (19), a change of variables and trace space properties, we obtain

$$
\int_s^\tau \| P(t) e^{-(t-s)A_0}x \|^\theta dt \leq 2^\theta \left( M^\theta \int_0^\tau \frac{dt}{t^2} dt + \eta^\theta \right) \| x \|^\theta 
$$

for a constant $\tilde{\gamma} := \tilde{\gamma}(\theta, \tau, M, \eta) > 0$. Now Theorem 4, shows that $O^\theta_Y(A_0) = O^\theta_Y(A(\cdot)) = O^\theta_Y(A(\cdot) + P(\cdot)) = O^\theta_Y(A_1)$. \qed

5. **Examples**

In this section, we give two situations that illustrate our abstract results. We choose the heat equation as model, but the results can also be applied to other physical models.

5.1. A heat equation with Dirichlet conditions and point observation

Let $p, q \in (1, \infty)$ and $n \in \mathbb{N}^*$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $q > \frac{nq}{2}$. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with $C^2$-boundary $\partial \Omega$, and let $a_{kl} : \mathbb{R}^+ \times \overline{\Omega} \to \mathbb{R}$, $k, l = 1, \ldots, n$ be bounded continuous functions such that

$$
\sum_{i,j} a_{ij}(t, x)x_i x_j \geq \beta \| x \|^2
$$

for a constant $\beta > 0$. In addition, we suppose that

$$
\max_{1 \leq i, j \leq n} \sup_{x \in \Omega} | a_{ij}(t, x) - a_{ij}(s, x) | \leq \omega(t - s)
$$

for any $s, t \in [0, \tau]$, where $\omega : [0, \tau] \to \mathbb{R}^+$ is a continuous function satisfying

$$
\int_0^\tau \left( \frac{w(t)}{t} \right)^\nu dt < \infty
$$

for some $\nu \in (1, \infty)$. Select

$$
X = L^q(\Omega) \quad \text{and} \quad D = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega).
$$
Define a family of unbounded operators on $X$, $A(t) : D \to X$ with
\[
A(t)u := -\sum_{ij} a_{ij}(t, \cdot)\partial_i\partial_j u - b_0(t, \cdot)u, \quad t \in [0, \tau], \; u \in D,
\]
where $b_0 \in L^\infty((0, \tau) \times \Omega)$. According to [4,6] the operator $A(t)$ has the maximal $L^p$ regularity for any $t$. Remark that for any $t, s \in [0, \tau]$ and $u \in D$,
\[
\|A(t) - A(s)\|_{L^q(\Omega)} \leq \omega(t - s) \sum_{ij} \|\partial_i\partial_j u\|_{L^q(\Omega)} + 2\|b_0\|_\infty \|u\|_{L^q(\Omega)}
\]
\[
\leq \omega(t - s)\|u\|_D + 2\|b_0\|_\infty \|u\|_{L^q(\Omega)}.
\]
Thus $A(\cdot)$ satisfies the condition (H1). It is well-known that the evolution family $(U(t, s))_{0 \leq s \leq t \leq \tau}$ on $X$ associated with $A(t)$ is given by
\[
U(t, s)\varphi = \int_\Omega k(t, s, \cdot, x)\varphi(x)dx, \quad 0 \leq s \leq t \leq \tau, \; \varphi \in X,
\]
for a continuous kernel $k(t, s, \cdot, x)$, $\tau \geq t > s \geq 0$ and $x \in \Omega$ such that
\[
|k(t, s, \xi, x)| \leq M(t - s)^{-\frac{\alpha}{2}} \exp\left(-\frac{\delta|\xi - x|^2}{t - s} + \tilde{\delta}(t - s)\right)
\]
for $\xi \in \overline{\Omega}$, constants $M, \delta > 0$ and $\tilde{\delta} \in \mathbb{R}$, see [18, Section IV.16].

Also, for $\alpha \in (0, \frac{1}{p})$, we define
\[
P(t) := (-A(t))^{\alpha}_{|D}, \quad t \in [0, \tau].
\]
From [1], we deduce that there exists a constant $\tilde{M} > 0$ such that for $0 \leq s < \tau$, and $\varphi \in D$,
\[
\int_s^\tau |(-A(t))^{\alpha}U(t, s)\varphi|^p dt \leq \tilde{M} \int_s^\tau (t - s)^{-ap} dt \|\varphi\|_{X}^p
\]
\[
\leq \tilde{M} \frac{\tau^{1-ap}}{1 - ap} \|\varphi\|_{X}^p.
\]
This implies that $P(\cdot)$ satisfies the condition (H2) for $\mu = p$. We also define an observation operator
\[
C : D \to \mathbb{C}, \quad C\varphi := \varphi(c_0), \quad \varphi \in D,
\]
for some $c_0 \in \Omega$. As in [22, p.23], by using Hölder’s inequality and the estimate (20), for $\varphi \in D$, there exists a constant $c > 0$ such that
\[
\int_s^\tau |CU(t, s)\varphi|^p dt \leq c \left(\int_s^\tau (t - s)^{-\frac{np}{2q}} dt\right) \|\varphi\|_{L^q(\Omega)}^p
\]
\[
\leq \gamma^p \|\varphi\|_{L^q(\Omega)}^p
\]
for $q > \frac{np}{2}$ and a certain constant $\gamma > 0$. This shows that $(C, A(\cdot))$ is admissible with exponent $p$. Finally, according to Theorem 4, $(C, A(\cdot) + P(\cdot))$ is admissible.
5.2. A heat equation subject to Neumann boundary conditions and a non local perturbation

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with \( n \geq 2 \) with smooth boundary (\( C^2 \)-boundary) \( \partial \Omega \) and outer unit normal \( N(\cdot) \). Consider an operator \( K \in L^\infty(\partial \Omega \times \Omega) \), and select
\[
X := L^2(\Omega), \quad Y := L^2(\partial \Omega),
\]
\[
(\mathcal{K} \varphi)(x) = \int_{\Omega} K(x, y) \varphi(x) \, dy, \quad x \in \partial \Omega, \quad \varphi \in L^2(\Omega),
\]
\[
D := \left\{ \varphi \in H^2(\Omega) : \nabla \varphi |_{\partial \Omega} = \mathcal{K} \varphi \text{ on } \partial \Omega \right\}.
\]
Now we define the linear operator \( \mathbb{A} := \Delta \) with domain \( D(\mathbb{A}) = D \). According to the proof of [9, Theorem 3.4], the operator \( -\mathbb{A} \) with domain \( D(-\mathbb{A}) = D \) generates an analytic semigroup on \( X \), so that \( \mathbb{A} \in \mathcal{M}(\mathbb{A}) \). Now consider operators \( A(t) : D \to X \), \( t \in [0, \tau] \) such that
\[
A(t)u = \mathbb{A}u + \int_{\partial \Omega} \psi(t, \cdot, z) \phi(z) u(\cdot, z) \, dz, \quad u \in D,
\]
where \( \phi \in C_b(\partial \Omega) \) and \( \psi : [0, \tau] \times \Omega \times \partial \Omega \to \mathbb{R} \) is a measurable function such that \( \psi(t, \cdot, \cdot) \in L^2(\Omega \times \partial \Omega) \) for a.e. \( t \in [0, \tau] \) and
\[
\| \psi(t, \cdot, \cdot) - \psi(s, \cdot, \cdot) \|_{L^2(\Omega \times \partial \Omega)} \leq \omega(t - s) \tag{21}
\]
for a.e. \( t, s \in [0, \tau] \), where \( \omega : [0, \tau] \to [0, \infty) \) is a continuous function with \( w(0) = 0 \) and satisfies the condition (7) for \( p = 2 \).

Let \( b \in L^\infty([0, \tau]) \) be a scalar function and \( \alpha \in (0, \frac{1}{2}) \). Select
\[
P(t) := b(t)\mathbb{A}^\alpha : D \to X, \quad t \in [0, \tau].
\]
Moreover, we consider the observation operator
\[
C : D \to Y, \quad (C \varphi)(y) = \phi(y) \varphi(y), \quad y \in \partial \Omega.
\]
According to the proof of [9, Theorem 3.4], \( C \in \mathcal{S}^2(\mathbb{A}) \). On the other hand, for any \( t \in [0, \tau] \), we can write \( A(t) = \mathbb{A} + B(t)C \) with \( B(t) : Y \to X \) defined by
\[
B(t)g = \int_{\partial \Omega} \psi(t, \cdot, z) g(\cdot, z) \, dz, \quad g \in Y.
\]
Using (21), we obtain
\[
\| B(t)g \| \leq \left( \| \omega \|_\infty + \| \psi(0, \cdot, \cdot) \|_{L^2(\Omega \times \partial \Omega)} \right) \| g \|_Y := c \| g \|_Y
\]
for any \( g \in Y \) and \( t \in [0, \tau] \). Using the admissibility of \( C \) for \( \mathbb{A} \), for any \( s \in [0, \tau) \) and \( f \in D \), we have
\[
\int_s^\tau \| B(t)Ce^{-\mathbb{A}(t-s)}f \|^2 \, dt \leq c^2 \int_0^{\tau-s} \| Ce^{-\mathbb{A}t}f \|^2 \, dt \leq (cy)^2 \| f \|^2,
\]
for a constant $\gamma > 0$. Now by using [11], we associate to $A(\cdot)$ an evolution family $U = (U(t, s))_{0 \leq s \leq t \leq \tau}$ on $X$. On the other hand, as $A \in \mathcal{MR}$, then by [3], $A(t) \in \mathcal{MR}$ for any $t \in [0, \tau]$. In addition, for each $t$, $B(t)C$ is admissible for the generator $A$, then by [12], $(C, A(t))$ is admissible for each $t \in [0, \tau]$. Thus by Theorem 3, $(C, A(\cdot))$ is admissible.

Let us now discuss the admissibility of $C$ for $A(\cdot) + P(\cdot)$. For any $f \in D$ and $s \in [0, \tau)$, and using Hölder’s inequality, we have

$$\int_s^\tau \| P(t)e^{-(t-s)A}f \|^2 dt \leq \| b(\cdot) \|_{\infty} \int_s^\tau \| (-A)^{\alpha}e^{-tA}f \|^2 dt \leq \| b(\cdot) \|_{\infty} \int_0^\tau \| f \|^2 dt = \| b(\cdot) \|_{\infty} \frac{(\tau)^{1-\alpha}}{1-\alpha} \| f \|^2$$

for $\alpha \in (0, \frac{1}{2})$. Now by using Remark 5, $P(\cdot)$ satisfies the condition $(H2)$ for $\mu = 2$. On the other hand, for $t, s \in [0, \tau]$ and $f \in D$, we have

$$\| A(t)f - A(s)f \|^2 \leq \int_\Omega \left( \int_{\partial\Omega} |\psi(t, x, z) - \psi(s, x, z)| ||(Cf)(z)||dz \right)^2 dx \leq \int_\Omega \int_{\partial\Omega} |\psi(t, x, z) - \psi(s, x, z)|^2 dzdx ||Cf||_Y^2 \leq \omega(t-s)^2 ||Cf||_Y^2,$$

due to the estimate (21). This implies that $A(\cdot) : [0, \tau] \to \mathcal{L}(D, X)$ satisfies the relative 2-Dini condition, so that the condition $(H1)$ holds. Thus $(C, A(\cdot) + P(\cdot))$ is $L^\theta$-admissible for $1 < \theta \leq 2$, due to Theorem 4.

Acknowledgements

The authors would like to thank the editors and the referee whose detailed comments helped us to improve the organization and the content of the paper.

Data Availability Statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s);
REFERENCES

[1] Acquistapace, A., Terreni, B.: A unified approach to abstract linear nonautonomous parabolic equations. Rendiconti del Seminario Matematico della Università di Padova, 78, 47–107 (1987)
[2] Amann, H.: Linear and Quasilinear Parabolic Problems. V1 Birkhäuser, Basel, (1995)
[3] Amansag, A., Bounit, H., Driouich, A., Hadd, S.: On the maximal regularity for perturbed autonomous and nonautonomous evolution equations. J. Evol. Equ. 20, 165–190 (2020)
[4] Arendt, W., Chill, R., Fornaro, S., Poupaud, C.: $L^p$-maximal regularity for non-autonomous evolution equations. J. Diff. Equa. 237, 1–26 (2007)
[5] Chicone, C., Latushkin, Y.: Evolution Semigroups in Dynamical Systems and Differential Equations. AMS, Providence, RI, (1999)
[6] Denk, R., Hieber, M., Prüss, J.: $\mathcal{R}$-Boundedness, Fourier Multipliers and Problems of Elliptic and Parabolic Type. 166, Memoirs of the American Mathematical Society, (2003)
[7] Dore, G.: Maximal regularity in $L^p$ spaces for an abstract Cauchy problem. Advances Differential Equations. 5, 293-322 (2000)
[8] Engel, K.-J., Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations. Graduate Texts in Mathematics, 194, Springer-Verlag, (2000)
[9] Fkirine, M., Hadd, S.: On nonlinear Miyadera-Voigt perturbations. Archiv der Mathematik 119, 179-188 (2022).
[10] Hadd, S.: Unbounded Perturbations of $C_0$-Semigroups on Banach Spaces and Applications. Semigroup forum. 70, 451–465 (2005)
[11] Hadd, S.: An evolution equation approach to nonautonomous linear systems with state, input and output delays. SIAM J. Control Optim. 45, 246–272 (2006)
[12] Hadd, S., Idrissi, A.: On the admissibility of observation for perturbed $C_0$-semigroups on Banach spaces. Systems & Control Letters. 55, 1–7 (2006)
[13] Hinrichsen, D., Pritchard, A.J.: Robust stability of linear evolution operators on Banach spaces. SIAM J. Control Optim. 32, 1503–1541 (1994)
[14] Jacob, B., Dragan, V., Pritchard, A.J.: Robust stability of infinite dimensional time-varying systems with respect to nonlinear perturbations. Integral Equations Operator Theory. 22, 440–462 (1995)
[15] Jacob, B., Partington, J.R.: Admissibility of control and observation operators for semigroups: a survey. Operator Theory: Advances and Applications. 149, 199–221 (2004)
[16] Kharou, Y.: On the admissibility of observation operators for evolution families. Semigroup Forum 105, 265–281 (2022)
[17] Kunstmann, C., Weis, L.: Maximal $L^p$-regularity for Parabolic Equations, Fourier Multiplier Theorems and $H^\infty$-functional Calculus. Functional Analytic Methods for Evolution Equations (M. Iannelli, R. Nagel, and S. Piazzera, eds.), Lecture Notes in Mathematics, volume 1855, pp. 65-311, Springer, (2004)
[18] Ladyzenskaja, O.A., Solonnikov, V.A., Uralceva, N.N.: Linear and Quasilinear Equations of Parabolic Type. AMS, Providence, RI, (1968)
[19] Le Merdy, C.: The Weiss Conjecture for Bounded Analytic Semigroups. J. London Mathematical Society. 67, 715–738 (2003)
[20] Lunardi, A.: Analytic Semigroups and Optimal Regularity in Parabolic Problems. Birkhäuser, Basel, (1995)
[21] Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Applied Mathematical Sciences, 44, Springer-Verlag, (1983)
[22] Schnaubelt, R.: Feedbacks for non-autonomous regular linear systems. SIAM J. Control Optim. 41, 1141–1165 (2002)
[23] Schnaubelt, R.: Well-posedness and asymptotic behaviour of non-autonomous linear evolution equations. Progress in Nonlinear Differential Equations and Their Applications, 50, 311–338 (2002)
[24] Salamon, D.: Infinite-dimensional linear system with unbounded control and observation: a functional analytic approach. Trans. Amer. Math. Soc. 300, 383–431 (1987)
[25] Staffans, O.J.: Well-Posed Linear Systems. Cambridge Univ. Press, Cambridge, (2005)
[26] Tucsnak, M., Weiss, G.: Observation and Control for Operator Semigroups. Birkhäuser, (2009)
[27] Weiss, G.: Admissible observation operators for linear semigroups. Israel Journal of Mathematics. 65, 17–43 (1989)

O. El Mennaoui, S. Hadd and Y. Kharou
Department of Mathematics, Faculty of Sciences
Ibn Zohr University
Hay Dakhla
BP 8106
80000 Agadir
Morocco
E-mail: o.elmennaoui@uiz.ac.ma

S. Hadd
E-mail: s.hadd@uiz.ac.ma

Y. Kharou
E-mail: yassine.kharou@edu.uiz.ac.ma

Accepted: 15 August 2023