Supporting Information

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Magnonic Goos–Hänchen Effect Induced by 1D Solitons

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Supplemental material to “Magnonic Goos-Hänchen effect induced by one dimensional solitons”

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I. Details on numerical computations

The spectral problems \( \Omega_{2k_x} \Omega_{1k_x} \phi_{k_x} = \omega^2 \phi_{k_x} \) were solved numerically for a large discrete set of \( k_x \), on a box \(-L \leq z \leq L\) with Dirichlet boundary conditions at \( z = \pm L \), that is, \( \phi_{k_x}(\pm L) = 0 \). We do not expect sensitivity to (reasonably) boundary conditions if \( L \) is large. The eigenfunctions of bound states are exponentially small as \( z \to \pm \infty \), hence imposing that either they or their derivative vanish at \( z = \pm L \) (Dirichlet or Neumann boundary conditions, respectively) introduces only a very small distortion. On the other hand, the eigenfunctions of bound states have a sinusoidal behaviour as \( z \pm \infty \), and therefore imposing Dirichlet or Neumann boundary conditions merely selects the waves that it into the box \([-L/2, L]\).

The operators \( \Omega_{1k_x} \) and \( \Omega_{2k_x} \) were discretized in the simplest way, with a symmetric difference scheme for the second derivative, which guarantees hermiticity. The spectrum of the discretized operator \( \Omega_{2k_x} \Omega_{1k_x} \) was obtained using the linear algebra package ARPACK\(^1\). In practice, we found it more efficient to make use of the parity symmetry to restrict the operators to the \((0, +L)\) interval, and obtain the even and odd spectrum separately, using the boundary conditions appropriate for each case: \( \phi_{k_x}^{(e)}(-dz) = \phi_{k_x}^{(e)}(+dz) \) for the even eigenfunctions, where \( dz \) is the discretization step, and \( \phi_{k_x}^{(o)}(0) = 0 \) for the odd eigenfunctions. The computation were repeated for several values of \( L \) and \( dz \) to ensure that the results show no noticeable volume or discretization effects.

The phase shifts are computed as follows. The diagonalization of \( \Omega_{2k_x} \Omega_{1k_x} \) provides the eigenvalue \( \omega^2 \) and the corresponding eigenfunction, \( \psi \). From the eigenvalue \( \omega^2 \) we compute the wave number \( k_z \) using equation (15) of the paper [Eq. (83) below]. Since \( \psi(z) \) vanishes at \( z = L \), due to the imposed boundary condition, its leading term in the asymptotic expansion as \( z \to \infty \) has to vanish at \( z = L \), if \( L \) is large, and since the leading term is given by
FIG. 1: Left: Reflection coefficient as a function of the frequency relative to the gap frequency, $\omega - \omega_G$ for the values of $h$ displayed in the legend. Right: frequency, relative to the gap, at which the reflection coefficient decreases to 1/2 (violet) and 1/10 (green), as a function of $h/h_c$.

Eq. (14) of the paper [Eq. (84) below] we obtain the following equations for the cases of even and odd eigenfunction, respectively:

$$\cos(k_zL + \delta_0) = 0, \quad \sin(k_zL + \delta_1) = 0. \quad (1)$$

Then the following relations have to be satisfied: $k_zL + \delta_0 = (n_0 + 1/2)\pi$, and $k_zL + \delta_1 = n_1\pi$, where $n_0$ and $n_1$ are the integers that make $0 \leq \delta_0, \delta_1 \leq \pi$. Hence we have

$$\delta_0 = (-k_zL) \mod \pi + \frac{\pi}{2}, \quad \delta_1 = (-k_zL) \mod \pi. \quad (2)$$

II. SOME NUMERICAL RESULTS

We show here some results that complement those described in the paper.

The reflection coefficient, given by $R = \sin^2(\delta_0 - \delta_1)$, is displayed in figure 1 (left), for different values of the magnetic field. It tends to zero as the frequency grows, as expected. The range of frequencies, relative to the gap frequency, at which reflection is appreciable depends non monotonically on the external magnetic field. That means there is a field strength at which reflection is maximized, as illustrated in figure 1 (right).

The dependence of the phase shifts on $k_x$ is illustrated in Figs. 2 for $\kappa = -5.0$ and $h = 1.0$, were $\delta_0$ and $\delta_1$ are plotted as a function of $k_z$ for several values of $k_x$. We notice that in all cases we have $\delta_0(0) = \pi/2, \delta_1(0) = 0$, and both $\delta_0$ and $\delta_1$ vanish as $k_z \to \infty$. This means that in all cases the phase shifts satify the thesis of Levinson’s theorem\textsuperscript{2,3}.\textsuperscript{2,3}
III. SOME PROPERTIES OF THE K AND Ω OPERATORS

Let us discuss in more detail some properties of the operators that are only briefly summarized on the body of the paper.

A. Positivity of K

For an explanation of the notation see the main paper. The operator $K$ is semidefinite positive if the magnetic state is metastable, since it is a local minimum of the energy, which, for small deviations, $\tilde{\xi}$, from the metastable state, is given by

$$E = E_0 + 2A(1/2)\langle \tilde{\xi}, K \tilde{\xi} \rangle + O(\xi^3),$$

where $E_0$ is the energy of the metastable state, $\tilde{\xi} = [\xi_1, \xi_2]^T$, and

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{pmatrix}.$$  \hfill (4)

The positivity of $K$ means, by definition, the inequality $\langle \tilde{\xi}, K \tilde{\xi} \rangle \geq 0$. Consider two arbitrary complex functions $\xi_1$ and $\xi_2$. Applying the positivity inequality to $\tilde{\xi} = [\xi_1, \xi_2]^T$, $\tilde{\eta} = [\xi_1, -\xi_2]^T$, and $\tilde{\zeta} = [\xi_1, i\xi_2]^T$, we get the constraints

$$\left(\xi_1, K_{11}\xi_1\right) + \left(\xi_2, K_{22}\xi_2\right) \pm 2\Re(\xi_1, K_{12}\xi_2) \geq 0,$$

$$\left(\xi_1, K_{11}\xi_1\right) + \left(\xi_2, K_{22}\xi_2\right) \pm 2\Im(\xi_1, K_{12}\xi_2) \geq 0,$$

which can be written more compactly as

$$\left|\Re(\xi_1, K_{12}\xi_2)\right| \leq \frac{1}{2} \left[ \left|\xi_1, K_{11}\xi_1\right| + \left|\xi_2, K_{22}\xi_2\right| \right],$$

$$\left|\Im(\xi_1, K_{12}\xi_2)\right| \leq \frac{1}{2} \left[ \left|\xi_1, K_{11}\xi_1\right| + \left|\xi_2, K_{22}\xi_2\right| \right].$$

for any $\xi_1$ and $\xi_2$. From these general relations some more specifics constraints on $k_{ij}$ can be derived. For instance, taking either $\xi_2 = 0$ or $\xi_1 = 0$ we get

$$\left(\xi_1, K_{11}\xi_1\right) \geq 0, \quad \left(\xi_2, K_{22}\xi_2\right) \geq 0,$$

which are valid for all $\xi_1$ and $\xi_2$, what implies that $K_{11}$ and $K_{22}$ are positive.
Exchanging $\xi_1$ and $\xi_2$ in (7) and (8) we get the corresponding constraints for the adjoint

\[
|\text{Re}(\xi_1, K_{12}^\dagger \xi_2)| \leq \frac{1}{2} \left[ (\xi_2, K_{11} \xi_1) + (\xi_1, K_{22} \xi_1) \right],
\]

\[
|\text{Im}(\xi_1, K_{12}^\dagger \xi_2)| \leq \frac{1}{2} \left[ (\xi_2, K_{11} \xi_1) + (\xi_1, K_{22} \xi_1) \right].
\]

(10) (11)

And then we obtain constraints for the hermitian and antihermitian operators $K_{12} + K_{12}^\dagger$ and $K_{12} - K_{12}^\dagger$. First, we use the triangle inequality to get

\[
|\text{Re}(\xi_1, [K_{12} + K_{12}^\dagger] \xi_2)| \leq |\text{Re}(\xi_1, K_{12} \xi_2)| + |\text{Re}(\xi_1, K_{12}^\dagger \xi_2)|,
\]

\[
|\text{Im}(\xi_1, [K_{12} + K_{12}^\dagger] \xi_2)| \leq |\text{Im}(\xi_1, K_{12} \xi_2)| + |\text{Im}(\xi_1, K_{12}^\dagger \xi_2)|,
\]

(12) (13)

and then equations (7), (8) (10), and (11), to obtain

\[
|\text{Re}(\xi_1, [K_{12} + K_{12}^\dagger] \xi_2)| \leq \frac{1}{2} \left[ (\xi_1, K_{11} \xi_1) + (\xi_2, K_{11} \xi_1) + (\xi_1, K_{22} \xi_1) + (\xi_2, K_{22} \xi_1) \right],
\]

\[
|\text{Im}(\xi_1, [K_{12} + K_{12}^\dagger] \xi_2)| \leq \frac{1}{2} \left[ (\xi_1, K_{11} \xi_1) + (\xi_2, K_{11} \xi_1) + (\xi_1, K_{22} \xi_1) + (\xi_2, K_{22} \xi_1) \right].
\]

(14) (15)

Thus, in particular the hermitian operator $K_{12} + K_{12}^\dagger$ is not necessarily positive, but its diagonal elements are bounded by those of $K_{11} + k_{22}$ and

\[
|\langle \xi, [K_{12} + K_{12}^\dagger] \xi \rangle| \leq (\xi, [K_{11} + K_{22}] \xi).
\]

(16)

### B. General properties of $\Omega$

Let us analyze in detail the properties of $\Omega$ which are only sketched in the paper. The linearized LLG equation, in absence of damping, takes the form $\partial_t \vec{\xi} = \Omega \vec{\xi}$, where $\Omega = (\omega_0/q_0^2)JK$, with

\[
J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.
\]

(17)

Notice that $J$ is antihermitian: $J^\dagger = -J$. Since $\Omega$ is not anti-hermitian (not even normal), some properties of the spin waves, like the existence of a complete set of well defined modes with definite frequency, are questionable. However, it was shown in the paper that spin waves are well behaved if the underlying magnetic state is metastable. Here we give more detailed derivations of the results.

#### 1. Spectrum of $\Omega$

The spectral equation is $\Omega \vec{\xi} = \nu \vec{\xi}$, with $\nu$ a complex eigenvalue. For a (meta)stable state the square root of $K$ is a well defined hermitian positive definite operator. Here we restrict ourselves to the case that the spectrum of $K$ is strictly positive, that is, it lies on the positive real axis and is separated from zero by a finite gap. Multiplying both sides of the spectral equation by $K^{1/2} \Omega$ we obtain

\[
(\omega_0^2/q_0^2)Q \eta = \nu^2 \eta,
\]

(18)

where $\eta = K^{1/2} \vec{\xi}$ and

\[
Q = K^{1/2}JKJK^{1/2}.
\]

(19)

Hence, the spectral properties of $\Omega$ are derived from the spectral properties of $Q$, which is hermitian, and negative definite, since For any $\xi$ we have

\[
\langle \xi, Q\xi \rangle = \langle \xi, K^{1/2}JKJK^{1/2} \xi \rangle = \langle J^\dagger K^{1/2} \xi, KJK^{1/2} \xi \rangle = -\langle JK^{1/2} \xi, KJK^{1/2} \xi \rangle = -\langle \chi, K\chi \rangle \leq 0
\]

(20)

where $\chi = JK^{1/2} \xi$, and we used $J^\dagger = -J$, that $K^{1/2}$ is hermitian, and the positivity of $K$. 

Therefore, $Q$ is a hermitian negative definite operator, so that its spectrum lies on the negative real axis: $\nu^2 < 0$. Hence we can write $\nu = i\omega$, with $\omega$ real. Thus, for a (meta)stable state, the spectrum of $\Omega$ lies on the imaginary axis and its eigenstates form a complete set.

Notice the relation

$$Q = K^{1/2}JKJ^{1/2} = K^{1/2}\Omega^2K^{-1/2},$$

which shows that $Q$ and $\Omega^2$ are related by a (non-unitary) similarity transformation, given by $K^{1/2}$, so that they have the same spectrum.

2. Eigenstates of $\Omega$

As we have seen, each eigenstate $\xi$ of $\Omega$ gives an eigenstate, $K^{1/2}\xi$, of $Q$. The reciprocal is not true. If $\eta$ is an eigenstate of $Q$, with eigenvalue $-\omega^2$, the state $K^{-1/2}\eta$ is not necessarily an eigenstate of $\Omega$. But the two dimensional space spanned by the two states $K^{-1/2}\eta$ and $\Omega K^{-1/2}\eta$ is invariant under the action of $\Omega$, since

$$\Omega(\Omega K^{-1/2}\eta) = -\omega^2 K^{-1/2}\eta.$$ 

Any $\zeta$ belonging to this subspace is a linear combination of the form

$$\zeta = a K^{-1/2}\eta + b\Omega K^{-1/2}\eta,$$

where $a$ and $b$ are constants. The action of $\Omega$ on $\zeta$ is

$$\Omega\zeta = a\Omega K^{-1/2}\eta + b\Omega(\Omega K^{-1/2}\eta) = a\Omega K^{-1/2}\eta - b\omega^2 K^{-1/2}\eta = a' K^{-1/2}\eta + b'\Omega K^{-1/2}\eta,$$

where $a' = -\omega^2b$ and $b' = a$. Thus, in the basis $\{K^{-1/2}\eta, \Omega K^{-1/2}\eta\}$ the action of $\Omega$ on the invariant subspace is given by

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & -\omega^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$ 

Diagonalization is straightforward. It gives two eigenvalues, written as $i\sigma\omega$, where $\omega$ is the positive square root of $\omega^2$ and $\sigma = \pm 1$. The corresponding eigenstates are given by $a = i\sigma\omega$ and $b = 1$, that is,

$$\zeta^{(\sigma)} = i\sigma\omega K^{-1/2}\eta + \Omega K^{-1/2}\eta.$$ 

By applying $K^{1/2}$ to the above two eigenstates of $\Omega$ we obtain two degenerate eigenstates of $Q$, called $\eta^{(\sigma)}$, with eigenvalue $-\omega^2$:

$$\eta^{(\sigma)} = K^{1/2}\zeta^{(\sigma)} = i\sigma\omega\eta + K^{1/2}\Omega K^{-1/2}\eta.$$ 

Although it is clear that by construction these are eigenvalues of $Q$, since we proved above that the application of $K^{-1/2}$ to an eigenstate of $\Omega$ gives an eigenstate of $Q$, it can be checked directly. It the degeneracy of the subspace associated to the $-\omega^2$ eigenvalue of $Q$ is two dimensional, the two $\eta^{(\sigma)}$ span this subspace. If it has higher degeneracy, we can select another $\eta$ on this subspace and proceed as before, to obtain two more eigenstates of $Q$ with eigenvalue $-\omega^2$. In this way we obtain a one to one correspondence between eigenstates of $Q$ and eigenstates of $\Omega$. Since the eigenstates of $Q$ form a complete set, because $Q$ is hermitian, the set of eigenstates of $\Omega$ is complete.

3. Normalization of the eigenstates of $\Omega$

If $\zeta^{(i\sigma)}$ and $\zeta^{(j\sigma')}$ are the eigenstates of $\Omega$ with eigenvalues $i\sigma\omega_i$ and $i\sigma'\omega_j$, respectively, with $\omega_i > 0$ and $\omega_j > 0$, we have the normalization condition

$$\langle \zeta^{(i\sigma)}, K \zeta^{(j\sigma')} \rangle = N_{ij}^{\sigma\sigma'} \delta_{ij},$$

since

$$\langle \zeta^{(i\sigma)}, K \zeta^{(j\sigma')} \rangle = \langle K^{1/2}\zeta^{(i\sigma)}, K^{1/2}\zeta^{(j\sigma')} \rangle.$$
vanishes if $\omega_i \neq \omega_j$, because in this case $K^{1/2}\xi^{(i\sigma)}$ is orthogonal to $K^{1/2}\xi^{(j\sigma')}$ since they are eigenstates of $Q$ with eigenvalues $-\omega_i^2$ and $-\omega_j^2$, respectively. The normalization constant $N^\sigma_{\sigma'}$ can be obtained using equation (26):

$$
\langle \xi^{(i\sigma)}, K\xi^{(j\sigma')} \rangle = \sigma\sigma' \omega_i \omega_j \langle \eta^{(i)}, \eta^{(j)} \rangle - i\sigma\omega_i \langle \eta^{(i)}, K^{1/2}QK^{-1/2}\eta^{(j)} \rangle + i\sigma'\omega_j \langle \eta^{(i)}, K^{1/2}Q^{1/2}\eta^{(j)} \rangle + \langle \eta^{(i)}, K^{-1/2}Q^{1/2}K\Omega K^{-1/2}\eta^{(j)} \rangle.
$$

(30)

Now we have, for the different terms

$$
\langle \eta^{(i)}, K^{-1/2}Q^{1/2}K\Omega K^{-1/2}\eta^{(j)} \rangle = \langle \eta^{(i)}, K^{-1/2}JKJK^{-1/2}\eta^{(j)} \rangle = -\langle \eta^{(i)}, K^{1/2}JK^{1/2}\eta^{(j)} \rangle = -\langle \eta^{(i)}, K^{1/2}JK^{1/2}\eta^{(j)} \rangle,
$$

(31)

and

$$
\langle \eta^{(i)}, K^{1/2}\Omega K^{-1/2}\eta^{(j)} \rangle = \langle \eta^{(i)}, K^{1/2}JK^{1/2}\eta^{(j)} \rangle,
$$

(32)

$$
\langle \eta^{(i)}, K^{-1/2}Q^{1/2}K\Omega K^{-1/2}\eta^{(j)} \rangle = \langle \eta^{(i)}, K^{-1/2}JK^{1/2}\eta^{(j)} \rangle = -\langle \eta^{(i)}, K^{1/2}JK^{1/2}\eta^{(j)} \rangle.
$$

(33)

Therefore, equation (30) reads

$$
\langle \xi^{(i\sigma)}, K\xi^{(j\sigma')} \rangle = (\sigma\sigma' \omega_i \omega_j + \omega_j^2) \langle \eta^{(i)}, \eta^{(j)} \rangle - i(\sigma\omega_i + \sigma'\omega_j) \langle \eta^{(i)}, K^{1/2}JK^{1/2}\eta^{(j)} \rangle.
$$

(34)

Now we have,

$$
\langle \eta^{(i)}, K^{1/2}JK^{1/2}\eta^{(j)} \rangle = -\frac{1}{\omega_i^2} \langle \eta^{(i)}, K^{1/2}JK^{1/2}Q\eta^{(j)} \rangle - \frac{1}{\omega_j^2} \langle \eta^{(i)}, K^{1/2}JK^{1/2}Q\eta^{(j)} \rangle = -\frac{1}{\omega_i^2} \langle \eta^{(i)}, K^{1/2}JK^{1/2}Q\eta^{(j)} \rangle - \frac{1}{\omega_j^2} \langle \eta^{(i)}, K^{1/2}JK^{1/2}Q\eta^{(j)} \rangle = \omega_i^2 \omega_j^2 \langle \eta^{(i)}, K^{1/2}JK^{1/2}\eta^{(j)} \rangle.
$$

(35)

But this means that $\langle \eta^{(i)}, K^{1/2}JK^{1/2}\eta^{(j)} \rangle$ vanishes if $\omega_i \neq \omega_j$, and we can write

$$
\langle \eta^{(i)}, K^{1/2}JK^{1/2}\eta^{(j)} \rangle = B_i \delta_{ij},
$$

(36)

where the constant $B_i$ depends on how the $\eta^{(i)}$ are normalized. Also, if $\omega_i \neq \omega_j$ we have

$$
\langle \eta^{(i)}, \eta^{(j)} \rangle = A_i \delta_{ij},
$$

(37)

where $A_i$ again depends on how the $\eta^{(i)}$ are normalized.

Inserting this results into equation (34) we obtain

$$
\langle \xi^{(i\sigma)}, K\xi^{(j\sigma')} \rangle = (1 + \sigma\sigma')\omega_i^2 A_i - i(\sigma + \sigma')\omega_i B_i \delta_{ij}.
$$

(38)

Therefore the $\xi^{(i\sigma)}$ satisfy the orhtogonality relation (28), with the normalization constant given by

$$
N^\sigma_{\sigma'} = (1 + \sigma\sigma')\omega_i^2 A_i - i(\sigma + \sigma')\omega_i B_i,
$$

(39)

with $A_i$ and $B_i$ obtained from the eigenstates of $Q$ through relations (36) and (37), respectively.

As explained in the paper, there may be problems with the above arguments if the spectrum of $K$ does not have a gap (if it has zero eigenvalues), since then $K^{-1/2}$ is not well defined. In this case, $\Omega^2 = JKJK$ and $K^{1/2}JKJK^{1/2}$ are not related by a similarity transformation.

C. Case in which all $K_{\alpha\beta}$ commute

The spectral problem for $\Omega$ is easy if the four operators $K_{\alpha\beta}$ commute, as in the ferromagnetic and helical states of monoaxial helimagnets, and in the domain wall of anisotropic ferromagnets, since in this case the problem is reduced to finding the spectrum of one hermitian operator (e.g. $K_{11}$) and the diagonalization of a $2 \times 2$ matrix. Indeed, let $\psi$
an eigenfunction of the four operators $K_{\alpha \beta}$, so that $K_{\alpha \beta} \psi = \kappa_{\alpha \beta} \psi$, with $\kappa_{21} = \kappa_{12}^*$, since $K_{21} = K_{12}^\dagger$. The positivity of $K$ requires $\kappa_{11} \kappa_{22} - |\kappa_{12}|^2 \geq 0$

We can obtain two eigenstates $\xi_1$ of $\Omega$ by setting $\xi = \nu v$, where $v$ is a constant two component vector to be determined as follows. The spectral equation $\Omega \xi = \nu \xi$ becomes, with $\nu$ in units of $\omega_0/q_0^2$,

$$
\left( \begin{array}{cc} -\kappa_{12} & -\kappa_{22} \\
\kappa_{11} & \kappa_{12} \end{array} \right) \left( \begin{array}{c} v_1 \\
v_2 \end{array} \right) = \nu \left( \begin{array}{c} v_1 \\
v_2 \end{array} \right),
$$

(40)

and the two eigenvalues are given by

$$
\nu_{\pm} = i \left( \text{Im} \kappa_{12} \pm \sqrt{\kappa_{11} \kappa_{22} - (\text{Re} \kappa_{12})^2} \right).
$$

(41)

The eigenvalues are imaginary, since the positivity of the $K$ operator guarantees that the term within the square root is positive.

**D. Case $K_{12} = 0$**

In the particular case where $K_{12} = 0$ and $K_{22}$ has a gap, which is the case of domain walls and of the chiral solitons studied in this work, we show in the paper how to obtain a complete solution to the spin wave problem.

In the paper we defined $\Omega_1 = (\omega_0/q_0^2)K_{11}$ and $\Omega_2 = (\omega_0/q_0^2)K_{22}$, so that

$$
\Omega = \left( \begin{array}{cc} 0 & -\Omega_2 \\
\Omega_1 & 0 \end{array} \right).
$$

(42)

Therefore, the spectral equation for $\Omega$ in components gives $\Omega_2 \xi_2 = -i \omega \xi_1$ and $\Omega_1 \xi_1 = i \omega \xi_2$, with $\omega$ real since we consider a (meta)stable state. Substituting the value of $\xi_2$ given by the second equation into the first we obtain

$$
\Omega_2 \Omega_1 \xi_1 = \omega^2 \xi_1.
$$

(43)

The operator $\Omega_2 \Omega_1$ is not hermitian and it is not guaranteed that it is diagonalizable, so that it may not have a complete set of eigenfunctions.

Assume that $\Omega_2$ has a gap. This is the usual case (domain walls, solitons, chiral soliton lattices, etc.). Then $\Omega_2$ is a hermitian positive definite invertible operator, and so it is its square root. We have the obvious relation

$$
\Omega_2 \Omega_1 = \Omega_2^{1/2} (\Omega_1^{1/2} \Omega_2^{1/2}) \Omega_2^{-1/2},
$$

(44)

which shows that $\Omega_2 \Omega_1$ is related to the hermitian positive (semi)definite operator $\Lambda = \Omega_2^{1/2} \Omega_1 \Omega_2^{1/2}$ by a similarity transformation: $\Omega_2 \Omega_1 = S^{-1} \Lambda S$, with $S = \Omega_2^{-1/2}$. This means that $\Omega_2 \Omega_1$ has the same spectrum as $\Lambda$ and that there is a one-to-one correspondence between the eigenfunctions of both operators. Therefore, $\Omega_2 \Omega_1$ is a diagonalizable operator.

Being hermitian, $\Lambda$ has a complete set of orthonormal eigenfunctions, denoted by $\{\psi_i\}$. Let $\omega_0^2$ the eigenvalue corresponding to $\psi_i$. Then $\psi_i = \sqrt{N_i} \omega_0 \Omega_2^{-1/2} \psi_i$ is an eigenfunction of $\Omega_2 \Omega_1$ with eigenvalue $\omega_0^2$. The constant $\sqrt{\omega_0}$ is just convenient for dimensional reasons and $N_i$ is an appropriate normalization constant. The completeness of the set $\{\psi_i\}$ implies the completeness of $\{\psi_i\}$, since $\Omega_2^{-1/2}$ is invertible. Thus, $\Omega_2 \Omega_1$ has a complete set of eigenfunctions that satisfy the normalization condition

$$
(\psi_i, \omega_0 \Omega_2^{-1} \psi_j) = N_i \delta_{ij}.
$$

(45)

These results can be rapidly obtained by noticing that $\Omega_2 \Omega_1$ is a hermitian positive (semi)definite operator with respect to the scalar product $\langle f, g \rangle = (f, \omega_0 \Omega_2^{-1} g)$. Therefore, the eigenvalues of $\Omega_2 \Omega_1$ are real and non-negative and its eigenfunctions are orthogonal with respect to the $\langle , \rangle$ product, what amounts to Eq. (45).

Each eigenfunction $\psi_i$ of $\Omega_2 \Omega_1$, whose eigenvalue is $\omega_0^2$, gives rise to two eigenstates of $\Omega$, with eigenvalues that correspond to the two square roots of $\omega_0^2$, that, for convenience, we write as $-i \sigma \omega_i$, with $\omega_i > 0$ and $\sigma = \pm 1$. Thus the eigenvalues of $\Omega$ come complex conjugate pairs, as it has to be since it is a real operator. The two eigenstates of $\Omega$ associated to $\psi_i$ are given by

$$
\tilde{\xi}^{(i \sigma)} = \left( \begin{array}{c} \psi_i \\
i \sigma \omega_i \Omega_2^{-1} \psi_i \end{array} \right),
$$

(46)
as can be readily checked. They satisfy the normalization condition

$$\langle \tilde{\xi}^{(i\sigma)}, G \tilde{\xi}^{(j\sigma')} \rangle = \left(1 + \sigma \sigma' \frac{\omega_i^2}{\omega_j^2} \right) \delta_{ij},$$

(47)

where

$$G = \begin{pmatrix} \omega_0 \Omega_2^{-1} & 0 \\ 0 & \omega_0^{-1} \Omega_2 \end{pmatrix},$$

(48)

which is a consequence of the normalization condition of $\psi_i$ given by equation (45).

The completeness of the set $\{\psi_i\}$ implies the completeness of the set $\{\tilde{\xi}^{(i\sigma)}\}$: for any given $\tilde{\xi}$ we have $\tilde{\xi} = \sum_{i\sigma} c_{i\sigma} \tilde{\xi}^{(i\sigma)}$, where

$$c_{i\sigma} = \frac{1}{4N_i} \left[ \left(1 + \frac{\omega_i^2}{\omega_i^2} \right) \langle \tilde{\xi}^{(i\sigma)}, G \tilde{\xi} \rangle - \left(1 - \frac{\omega_i^2}{\omega_i^2} \right) \langle \tilde{\xi}^{(i,-\sigma)}, G \tilde{\xi} \rangle \right].$$

(49)

This last relation is straightforwardly obtained using the normalization condition (47). It is equivalent to

$$c_{i\sigma} = \frac{1}{2N_i} \left[ \langle \psi_i, \omega_0 \Omega_2^{-1} \xi \rangle - i \sigma \frac{\omega_0}{\omega_i} \langle \psi_i, \xi \rangle \right].$$

(50)

which is equation (8) of the paper.

The general solution of the wave equation then has the form

$$\tilde{\xi}(\vec{x}, t) = \sum_{i,\sigma} c_{i\sigma} e^{-i\sigma \omega t} \tilde{\xi}^{(i\sigma)}(\vec{x}) + c.c.$$

(51)

If $\omega_i \neq 0$, the eigenstates of $\Omega$ can be also expressed as

$$\tilde{\xi}^{(i\sigma)} = \left( \begin{array}{c} \psi_i \\ i \sigma \frac{1}{\omega_i} \Omega_1 \psi_i \end{array} \right),$$

(52)

simply using in the lower component of $\tilde{\xi}^{(i\sigma)}$ given by equation (46) the relation $\psi_i = (1/\omega_i^2) \Omega_2 \Omega_1 \psi_i$. The above expression is useful to relate the asymptotic behavior of $\tilde{\xi}^{(i\sigma)}$ as $z \to \pm \infty$.

A big simplification arises in the case that $K_{11}$ and $K_{22}$ commute, what means that $\Omega_1$ and $\Omega_2$ commute. Then there exists an orthonormal basis $\{\psi_i\}$ of eigenfunctions of both $\Omega_1$ and $\Omega_2$, which satisfy

$$\Omega_1 \psi_i = \omega_{1i} \psi_i, \quad \Omega_2 \psi_i = \omega_{2i} \psi_i,$$

(53)

where $\omega_{1i}$ and $\omega_{2i}$ are the eigenvalues corresponding to $\Omega_1$ and $\Omega_2$, respectively. Evidently, the $\psi_i$ are also eigenfunctions of $\Omega_2 \Omega_1$:

$$\Omega_2 \Omega_1 \psi_i = \omega_{2i} \omega_{1i} \psi_i,$$

(54)

with eigenvalue $\omega_i^2 = \omega_{1i} \omega_{2i}$. The completeness of the $\{\psi_i\}$ set implies that $\Omega_2 \Omega_1$ has no other independent eigenfunction. The normalization $\langle \psi_i, \psi_j \rangle = \delta_{ij}$ coincides with (45) if we choose $N_i = \omega_{0i}/\omega_{2i}$. The eigenvalues of $\Omega$ are given by $-i \sigma \omega_i = -i \sigma \sqrt{\omega_{1i} \omega_{2i}}$, and the corresponding eigenstates are given by Eq. (46), which takes the form

$$\tilde{\xi}^{(i\sigma)} = \psi_i \left( \begin{array}{c} 1 \\ i \sigma \sqrt{\frac{\omega_{1i}}{\omega_{2i}}} \end{array} \right),$$

(55)

They also satisfy the normalization condition (47), as well as

$$\langle \tilde{\xi}^{(i\sigma)}, \tilde{\xi}^{(j\sigma')} \rangle = \left(1 + \sigma \sigma' \frac{\omega_{1i}}{\omega_{2j}} \right) \delta_{ij}.$$

(56)

As mentioned several times, the cases in which $K_{11}$ and $K_{22}$ commute include the uniform ferromagnetic states and the well known domain walls of anisotropic ferromagnetic systems without DMI.
IV. EXAMPLE: UNIFORMLY MAGNETIZED STATE IN AN ANISOTROPIC FERROMAGNET

Let give some details of the computation of the $K$ operator in the case of the uniformly magnetized state of a ferromagnet with uniaxial anisotropy, of easy-axis type. We take $z$ as the anisotropy axis. An external magnetic field is applied also along the $z$ axis. The energy is given by

$$\mathcal{E} = \int d^3r \left( A \sum_i \partial_i n \cdot \partial_i n - K_u (z \cdot n)^2 - \mu_0 M_s H z \cdot n - \frac{\mu_0}{2} M_s^2 h_m \cdot n \right),$$

where $A$ is the exchange stiffness constant, $K_u \geq 0$ is the anisotropy constant, $M_s$ is the constant magnetization modulus, $H$ the intensity of the applied field, $\mu_0$ the vacuum permeability, and $h_m$ the magnetostatic field, in units of $M_s$, which is the solution of the equations

$$\nabla \times h_m = 0, \quad \nabla \cdot h_m = -\nabla \cdot n.$$

The effective field is given by

$$B_{\text{eff}} = \frac{2A}{M_s} \left( \nabla^2 n + q_a^2 (z \cdot n) z + q_m^2 h_m \right),$$

where we have introduced the quantities $q_a^2 = K_u / A$, $q_m^2 = \mu_0 M_s H / (2A)$, and $q_m^2 = \mu_0 M_s^2 / (2A)$, which have the dimensions of inverse square length. Notice that $q_m$ is the inverse of the exchange length.

We use the following notation for the spatial coordinates: $x_1 = x$, $x_2 = y$, and $x_3 = z$.

Let us consider a spherical system with a very large radius, which eventually will be sent to infinity. The uniformly magnetized state, with magnetization pointing along the $z$ direction, $n_0 = z$, is an equilibrium state, since it is well known that in this case the magnetostatic field is constant, $h_m = -(1/3)z$. Let us study the variations about this equilibrium state, given by $\xi_1$ and $\xi_2$ as

$$n = \sqrt{1 - \xi^2} z + \xi_1 x + \xi_2 y.$$ 

The $K$ operator is obtained by expanding the energy in powers of $\xi_\alpha$ up to second order. The linear terms cancel out since $n_0$ is an equilibrium state. Therefore we can write

$$\mathcal{E} = VA \left( -q_a^2 - 2q_m^2 + \frac{1}{3} q_m^2 \right) + A \int d^3r \sum_{\alpha, \beta=1}^2 \xi_\alpha(r) (K_{\alpha \beta} \xi_\beta)(r) + O(\xi^3),$$

where $V$ is the volume of the system.

We get readily the contributions to $K$ from the different terms entering the energy, except the magnetostatic interaction:

1. Exchange energy:

$$A \int d^3r \sum_{i=1}^3 \sum_{\alpha=1}^2 \left( \partial_i \xi_\alpha \right)^2 = -A \int d^3r \sum_{\alpha=1}^2 \xi_\alpha \nabla^2 \xi_\alpha,$$

where in the last equality we have integrated by parts and neglected the surface term, since the variations vanish at the boundary (which approach infinity).

2. Anisotropy energy: $Aq_m^2 \int d^3r \xi^2$.

3. Applied field energy: $Aq_m^2 \int d^3r \xi^2$.

To get the contribution of the magnetostatic interaction we write the dipolar field as

$$h_m = -\frac{1}{3} z + \delta h_m,$$

where, to second order in $\xi_\alpha$, $\delta h_m$ satisfy the equations

$$\nabla \times \delta h_m = 0, \quad \nabla \cdot \delta h_m = -\partial_x \xi_1 - \partial_y \xi_2 + \sum_{\alpha=1}^2 \xi_\alpha \partial_z \xi_\alpha.$$ 


Since the variations vanish at the boundary, the solution of the above equations is

\[ \delta h_m(r) = -\frac{1}{4\pi} \int_V d^3r' \left( \partial_{x'} \xi_1(r') + \partial_{y'} \xi_2(r') - \sum_{\alpha=1}^{2} \xi_\alpha(r') \partial_{z'} \xi_\alpha(r') \right) \frac{r - r'}{|r - r'|^3} \]  

(65)

where \( V \) represents the volume occupied by the system, a sphere of very large radius, \( R \).

The magnetostatic energy is proportional to

\[ \int_V d^3r \ n(r) \cdot \left( -\frac{1}{3} z + \delta h_m(r) \right), \]  

(66)

where we have to keep only the terms up to \( \xi^2 \). The zeroth order gives the contribution of the magnetostatic energy to the energy of the uniformly magnetized state. The linear term is proportional to

\[ \sum_{\alpha=1}^{2} \int_V d^3r' \left( \sum_{\beta=1}^{2} \partial_{z'} \xi_\alpha(r') \partial_{z} \xi_\beta(r') \right) \frac{r - r'}{|r - r'|^3} \]  

(67)

where the last integral vanishes after integrating by parts, taking into account that \( \xi_\alpha \) vanishes at the boundary, and we used

\[ \int_V d^3r \left( \frac{z - z'}{|r - r'|^3} \right) = -\frac{4\pi}{3} z'. \]  

(68)

(The integral is the colulomb field inside a uniformly charged sphere, since \( r' \) is inside the sphere.) Thus, the linear term vanishes, as it should be.

The quadratic part of the magnetostatic energy is proportional to

\[- \sum_{\alpha=1}^{2} \int_V d^3r \left( \frac{1}{4\pi} \int_V d^3r' \frac{z - z'}{|r - r'|^3} \xi_\alpha(r') \partial_{z'} \xi_\alpha(r') \right) + \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \frac{1}{4\pi} \int_V d^3r \int_V d^3r' \xi_\alpha(r) \xi_\beta(r') \frac{x_{\alpha} - x_{\alpha}'}{|r - r'|^3} \partial_{z'} \xi_\beta(r'). \]  

(69)

Using (68) to perform the integral over \( d^3r \), the middle term reads

\[- \sum_{\alpha=1}^{2} \int_V d^3r \left( \frac{1}{4\pi} \int_V d^3r' \frac{z - z'}{|r - r'|^3} \xi_\alpha(r') \partial_{z'} \xi_\alpha(r') \right) = \int_V d^3r' \left( \frac{z'}{3} \xi_\alpha(r') \partial_{z'} \xi_\alpha(r') \right). \]  

(70)

Integrating by parts we get the relation

\[ \sum_{\alpha=1}^{2} \frac{1}{3} \int_V d^3r' \left( \frac{z'}{3} \xi_\alpha(r') \partial_{z'} \xi_\alpha(r') \right) = - \sum_{\alpha=1}^{2} \frac{1}{3} \int_V d^3r' \xi_\alpha(r') \partial_{z'} (z' \xi_\alpha(r')) = - \frac{1}{3} \int_V d^3r' \xi_\alpha(r') \partial_{z'} \xi_\alpha(r'), \]  

and therefore

\[ \sum_{\alpha=1}^{2} \frac{1}{3} \int_V d^3r' \left( \frac{z'}{3} \xi_\alpha(r') \partial_{z'} \xi_\alpha(r') \right) = - \frac{1}{6} \int_V d^3r' \xi_\alpha(r') \partial_{z'} \xi_\alpha(r'). \]  

(71)

Thus, inserting the proportionality constants, the quadratic part of the magnetostatic energy is

\[- Aq_m^2 \int_V d^3r \left( \frac{1}{3} \xi_\alpha(r) \right) + Aq_m^2 \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \frac{1}{4\pi} \int_V d^3r \int_V d^3r' \xi_\alpha(r) \xi_\beta(r') \frac{x_{\alpha} - x_{\alpha}'}{|r - r'|^3} \partial_{z'} \xi_\beta(r'). \]  

(72)

Collecting the contribution of all interactions to the quadratic part of the energy, the \( K_{\alpha\beta} \) operator entering equation (61) is

\[ (K_{\alpha\beta} \xi_\beta)(r) = \delta_{\alpha\beta} (-\nabla^2 + q^2) \xi_\beta(r) + \frac{q_m^2}{4\pi} \int_V d^3r' \frac{x_{\alpha} - x_{\alpha}'}{|r - r'|^3} \partial_{z'} \xi_\beta(r'), \]  

(73)

where \( q^2 = q_a^2 + 2q_e^2 - q_m^2/3 \).
V. ASYMPTOTIC SOLUTION OF THE SPIN WAVE SPECTRAL PROBLEM

Much insight on the spectrum of $\Omega$ is obtained by analyzing the spectral equation in the asymptotic regime, $z \to \pm \infty$. Although this is standard matter, it is worthwhile to give some details of the computations that are relevant to the results described in the paper.

Consider the spin waves around an isolated soliton in a monoaxial helimagnet discussed in the paper. The case of a domain wall is completely analogous. For the reader convenience, we shall write here also the equations given in the paper. The operators $\Omega_1$ and $\Omega_2$ are given by equations (10) and (11) of the paper:

$$\Omega_1 = \frac{\omega_0}{q_0} \left[ - \nabla^2 + U_1 + q_0^2 h \right],$$

$$\Omega_2 = \frac{\omega_0}{q_0} \left[ - \nabla^2 + U_2 + q_0^2 (h - \kappa) \right].$$

The “potentials” $U_1 = -(1/2)\varphi'^2$ and $U_2 = -(3/2)\varphi'^2 + 2q_0\varphi'$ are even functions of $z$ and decay exponentially to zero when $z \to \pm \infty$, since $\varphi'(z) = 2/[\Delta \cosh(z/\Delta)]$. These functions are independent of $\kappa$, but depend on $h$ through $\Delta$.

Since $U_1$ and $U_2$ are independent of $x$ and $y$, the operator $\Omega_2 \Omega_1$ is partially diagonalized by a Fourier transformation in the $x$ and $y$ coordinates. To lighten the notation, we consider only the $x$ dependence, since the problem is symmetric in $x$ and $y$. The general equations are obtained by substituting $k_x^2$ by $k_x^2 + k_y^2$ and $k_x x$ by $k_x x + k_y y$ in the expressions given here. Therefore, the eigenfunctions of $\Omega_2 \Omega_1$ have the form

$$\psi_{k_x}(x, z) = \exp(ik_x x) \phi_{k_x}(z),$$

where $\phi_{k_x}(z)$ is an eigenfunction of $\Omega_{2k_x} \Omega_{1k_x}$:

$$\Omega_{2k_x} \Omega_{1k_x} \phi_{k_x} = \omega^2 \phi_{k_x},$$

where $\Omega_{1k_x}$ and $\Omega_{2k_x}$ are given by equation (12) of the paper:

$$\Omega_{1k_x} = \frac{\omega_0}{q_0} \left[ - \partial_x^2 + U_1 + k_x^2 + q_0^2 h \right], \quad \Omega_{2k_x} = \frac{\omega_0}{q_0} \left[ - \partial_x^2 + U_2 + k_x^2 + q_0^2 (h - \kappa) \right].$$

For $z \to \pm \infty$ the “potentials” $U_1$ and $U_2$ vanish exponentially and the spectral equation (77) becomes asymptotically

$$[\partial_x^2 - k_x^2 - q_0^2 (h - \kappa)] [\partial_x^2 - k_x^2 - q_0^2 h] \phi_{k_x} = \frac{q_0^4 \omega^2}{\omega_0^2} \phi_{k_x}. $$

The solutions are exponential functions that can in general be written as $\exp(ik_x z)$, for some $k_x$. Equation (79) imposes a relation between $k_x^2$ and $\omega^2$, which can be written as $\omega^2 = \omega_2 \omega_1$, where

$$\frac{\omega_1}{\omega_0} = \frac{k_x^2 + k_y^2}{q_0^2} + h, \quad \frac{\omega_2}{\omega_0} = \frac{k_x^2 + k_y^2}{q_0^2} + h - \kappa.$$

This relation can be inverted to give

$$\frac{k_x^2}{q_0^2} = - \left( h + \frac{k_x^2}{q_0^2} - \kappa \right) \pm \left( \frac{\omega_2}{\omega_0} + \frac{\kappa^2}{4} \right)^{1/2}.$$ 

The right-hand side of the above equation is a real quantity.

A. Bound states states

Bound states in the $z$ direction require $k_x^2 < 0$ (imaginary $k_x$). There are two possibilities: either the minus sign is taken in Eq. (81), in which case there is no restriction in $\omega$, or the plus sign is taken and $\omega < \omega_0$. In the latter case the bound states are below the gap, while in the former bound states above the gap are possible. The numerical results show that, for fixed $k_x$, there is a single bound state, with even parity, located below the gap. At $k_x = 0$ it is the zero mode associated to the translation invariance of the soliton, and has $\omega = 0$ and $k_x = 1/\Delta$. Thus the bound state branch is gapless.
B. Continuum (scattering) states

Continuum states, unbounded in the z direction, have \( k_z \) real, that is \( k_z^2 \geq 0 \). This condition requires to take the plus sign in equation (81) and sets a lower bound on \( \omega \), written as \( \omega > \omega_G \), where

\[
\omega_G(k_z) = \omega_0 \left[ \left( \frac{k_z^2}{q_0^2} + h \right) \left( \frac{k_z^2}{q_0^2} + h - \kappa \right) \right]^{1/2}
\]

(82)

is the gap reported in equation (13) of the paper. The continuum states are conveniently labeled by the wave number \( \pm k_z \), whose relation with the eigenvalue \( \omega^\sigma \) is obtained from equation (81):

\[
k_z = q_0 \left[ \left( \frac{\omega^2}{\omega_0^2} + \frac{\kappa^2}{4} \right)^{1/2} - \left( \frac{k_z^2}{q_0^2} + h - \kappa \right) \right]^{1/2},
\]

(83)

where \( \omega \geq \omega_G \) and \( k_z \geq 0 \).

The eigenfunctions of \( \Omega_{2k_x} \Omega_{1k_z} \) can be chosen real even or odd functions of \( z \), since it is a real operator and \( U_1 \) and \( U_2 \) are even functions of \( z \). The two degenerate values of \( z \) the wave number, \( \pm k_z \) are combined to make the eigenfunctions with definite parity. Thus, the continuum states are labeled by \( \pm k_z \) and \( \pm z \), where \( k_z \geq 0 \) and the parity, denoted by the symbols \( e \) (even) and \( o \) (odd), so that we write \( \phi_{k_z k_z}^{(e)}(z) \) and \( \phi_{k_z k_z}^{(o)}(z) \) for the eigenfunctions of \( \Omega_{2k_x} \Omega_{1k_z} \).

The asymptotic behavior is given by

\[
\phi_{k_z k_z}^{(e)}(z) \sim \cos(k_z z + \delta_0), \quad \phi_{k_z k_z}^{(o)}(z) \sim \cos(k_z z + \delta_1),
\]

(84)

where the phase shifts \( \delta_0 \) and \( \delta_1 \) depend, in general, of \( k_z \) and \( k_z \).

Continuum states start at \( k_z = 0 \), where \( \omega = \omega_G \), and fill the whole frequency region above the gap. For \( k_z = 0 \) the gap is \( \omega_G(0) = \omega_G \), with

\[
\omega_G = \omega_0 \left[ h(h - \kappa) \right]^{1/2}.
\]

(85)

Since for continuum states \( \omega \neq 0 \), we can use for them the convenient expression (55), with the label \( i \) given by the triplet \( (k_x, k_z, p) \), where \( p = e, o \), and with \( \omega_i = \omega(k_z) = \sqrt{\omega^2(k_z)} \) and \( \psi_i \) given by (76). We find notationally convenient to separate the Fourier phase \( \exp(ik_z x) \) from \( \hat{\xi}^\sigma \) by replacing \( \hat{\xi}^\sigma(x, z) \rightarrow \exp(ik_z x)\hat{\xi}^{(p}\sigma \sigma)(x, z) \), where

\[
\hat{\xi}^{(p}\sigma \sigma)(x, z) = \begin{pmatrix}
\phi^{(e)}_{k_z k_z}(z) \\
-\sigma i \Omega_{1k_z} \phi^{(o)}_{k_z k_z}(z)
\end{pmatrix}.
\]

(86)

Given that \( \Omega_{0k_x} \cos(k_z \pm \delta_0) = \omega_1 \cos(k_z \pm \delta_0) \) and \( \Omega_{0k_x} \sin(k_z \pm \delta_1) = \omega_1 \sin(k_z \pm \delta_0) \), the asymptotic behavior of \( \hat{\xi}^{(p}\sigma \sigma}(x, z) \) as \( z \rightarrow \pm \infty \) is the same as that of \( \phi^{(p)}_{k_z k_z}(z) \):

\[
\hat{\xi}^{(p}\sigma \sigma}(z) \sim \hat{\xi}^{(p}\sigma \sigma}(z),
\]

(87)

where

\[
\hat{\xi}^{(e}\sigma}(z) = \cos(k_z z \pm \delta_0) \chi^{(e)}(z), \quad \hat{\xi}^{(o}\sigma}(z) = \sin(k_z z \pm \delta_1) \chi^{(o)}(z),
\]

(88)

with

\[
\chi^{(e)}(z) = \begin{pmatrix} 1 \\ i\sigma \sqrt{\frac{\omega}{\omega_0}} \end{pmatrix}.
\]

(89)

being a constant polarization vector. Note that \( \hat{\xi}^{*} = \hat{\xi} \). Note also that the asymptotic states \( e^{ik_z x} \hat{\xi}^{(p}\sigma \sigma}(z) \) are the eigenstates of the uniform (ferromagnetic) state, as it should be. To avoid too clumsy notation, we do not write explicitly the dependence on \( k_z \) and \( k_z \) of the asymptotic states \( \hat{\xi}^{(p}\sigma \sigma}(z) \), which has to be implicitly understood.

For the case under study, the general solution of the wave equation involving only continuum states has the form

\[
\hat{\xi}(x, z, t) = \sum_{p=e,o} \sum^{1}_{\sigma=\pm} \int^{+\infty}_{-\infty} dk_z \frac{dk_z}{2\pi} \int^{\infty}_{0} dk_z c_{p, \sigma}(k_x, k_z) e^{i(k_z x - \sigma \omega t)} \hat{\xi}^{(p}\sigma \sigma}(z) + c.c.,
\]

(90)

where \( c_{p, \sigma}(k_x, k_z) \) are arbitrary functions of the wave numbers \( k_x \) and \( k_z \).
VI. SCATTERING THEORY

We give here a detailed exposition of scattering theory, which, although it follows very closely the standard techniques for one dimensional systems, deserves a careful analysis when applied to the magnetic system.

Scattering theory relies only on two basic pillars: the asymptotic properties of the wave operator and the completeness of its eigenstates. For the case of the isolated soliton of monoaxial helimagnets, we have shown before that the spin wave operator $\Omega$ has a complete set of eigenfunctions that reproduce asymptotically, as $z \to \pm \infty$ the eigenstates of the uniform ferromagnetic state. Therefore, we can apply to it the concepts of standard scattering theory.

The scattering problem, schematically depicted in Figure 3, is posed as follows. For $t \to \infty$ a spin wave packet located at $x \to -\infty$ and $z \to -\infty$ moves towards the soliton located at $z = 0$. After scattering takes place, for $t \to +\infty$, one reflected wave packet moves towards the $x \to +\infty$, $z \to -\infty$, region, and one transmitted wave packet moves towards the $x \to +\infty$, $z \to +\infty$ region. The whole process is exactly described by the solution of the wave equation given by (90), with suitable functions $c_{p,\sigma}(k_x, k_z)$.

In the asymptotic region $z \to -\infty$ the scattering condition means that the solution of the wave equation has the form

$$\tilde{\xi}_{-\infty}(x, z, t) = \int_{-\infty}^{+\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{\infty} dk_z \left[ A(k_x, k_z) e^{i(k_x x + k_z z - \omega t)} + A_R(k_x, k_z) e^{i(k_x x - k_z z - \omega t)} \right] \tilde{\xi}_+ + \text{c.c.}, \quad (91)$$

where $A(k_x, k_z)$ and $A_R(k_x, k_z)$ are the amplitudes of the incident and reflected waves, respectively. In the asymptotic region $z \to +\infty$ the scattering condition means that the solution of the wave equation has the form

$$\tilde{\xi}_{+\infty}(x, z, t) = \int_{-\infty}^{+\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{\infty} dk_z A_T(k_x, k_z) e^{i(k_x x + k_z z - \omega t)} \tilde{\xi}_+ + \text{c.c.}, \quad (92)$$

where $A_T(k_x, k_z)$ is the amplitude of the transmitted wave. The scattering condition is just the absence of an incident wave in the $z \to +\infty$ region. We consider the amplitude of the incident wave sharply peaked about the wave numbers $k_{x0} > 0$ and $k_{z0}$. Then the same is true for the amplitudes of the reflected and transmitted waves which, as we shall see, are proportional to the amplitude of the incident wave.

For $t \to \pm \infty$ the phases oscillate very rapidly and constructive interference takes place only at the points where the phase is stationary with respect to the integration variables $k_x$ and $k_z$. Since $|t|$ is large, it is clear that for locating approximately the region of constructive interference we can ignore the phases of the amplitudes $A$, $A_R$, and $A_T$. These phases are however the origin of the time delay and the lateral shifts discussed in the paper and, with more detail, below. Therefore, for the incident wave, these points are given by the equations $x = (\partial \omega / \partial k_x)t$ and $z = (\partial \omega / \partial k_z)t$. We see that, since the incident wave is in the $z \to -\infty$ region, its phase is stationary only if $t < 0$ (so in the $t \to -\infty$ case), but not if $t > 0$ (then not for $t \to +\infty$). For the reflected wave the stationary phase points
are given by \( x = (\partial \omega / \partial k_x) t \) and \( z = -(\partial \omega / \partial k_z) t \). Since it is also in the \( z \to -\infty \) region, the phase of the reflected wave is stationary for \( t \to +\infty \), but not for \( t \to -\infty \). For the transmitted wave, the stationary points of the phase are given by \( x = (\partial \omega / \partial k_x) t \) and \( z = (\partial \omega / \partial k_z) t \). Since the transmitted wave is in the \( z \to +\infty \) region, its phase is stationary only if \( t > 0 \) (so in the \( t \to +\infty \) case), but not if \( t < 0 \) (then not for \( t \to -\infty \)). Thus we have shown that for \( t \to -\infty \) there is only an incident wave in the \( z \to -\infty \) region, propagating towards the soliton, while for \( t \to +\infty \) this incident wave is replaced by a reflected wave in the \( z \to -\infty \) region and a transmitted wave in the \( z \to +\infty \), both moving away from the soliton.

The amplitudes of the scattered (reflected and transmitted) waves are obtained by comparing the scattering conditions of equations (91) and (92) with the asymptotic form of the exact solution (90), in which \( c_{\nu}(k_x, k_z) = 0 \) since only the combination \( k_x x - \omega t \) appears in the scattering problem (the incident and scattered waves move from \( x \to \infty \) to \( x \to +\infty \)). Then, for \( z \to -\infty \) equation (90) is asymptotic to

\[
\xi_{-\infty}(x, z, t) = \int_{-\infty}^{+\infty} \frac{dk_x}{2\pi} \int_{0}^{\infty} dk_z e^{i(k_x x - \omega t)} \left[ c_{e+} \cos(k_z z - \delta_0) + c_{o+} \sin(k_z z - \delta_1) \right] \xi_e + c.c.,
\]

where we do not show explicitly the dependence of \( c_{e+} \) and \( c_{o+} \) on \( k_x \) and \( k_z \). For \( z \to -\infty \) equation (90) is asymptotic to

\[
\xi_{+\infty}(x, z, t) = \int_{-\infty}^{+\infty} \frac{dk_x}{2\pi} \int_{0}^{\infty} dk_z e^{i(k_x x - \omega t)} \left[ c_{e+} \cos(k_z z + \delta_0) + c_{o+} \sin(k_z z + \delta_1) \right] \xi_e + c.c.
\]

The comparison of equations (93) and (91) gives the system of two equations

\[
c_{e+} e^{-i\delta_1} - ic_{o+} e^{-i\delta_1} = A, \quad c_{e+} e^{i\delta_0} + ic_{o+} e^{i\delta_1} = A_R,
\]

whose solution is

\[
c_{e+} = \frac{e^{i\delta_1}}{2 \cos(\delta_0 - \delta_1)} (A + e^{-i\delta_1} A_R), \quad c_{o+} = \frac{ie^{i\delta_0}}{2 \cos(\delta_0 - \delta_1)} (A - e^{-i\delta_0} A_R).
\]

And the comparison of equations (94) and (92) gives another the system of two equations:

\[
c_{e+} e^{i\delta_0} - ic_{o+} e^{i\delta_1} = A_T, \quad c_{e+} e^{-i\delta_0} + ic_{o+} e^{-i\delta_1} = 0,
\]

whose solution is

\[
c_{e+} = \frac{e^{-i\delta_1}}{2 \cos(\delta_0 - \delta_1)} A_T, \quad c_{o+} = \frac{ie^{-i\delta_0}}{2 \cos(\delta_0 - \delta_1)} A_T.
\]

Now, the compatibility of equations (96) and (98) provides a system of two equations that determine the amplitudes of the scattered waves, \( A_R \) and \( A_T \), in terms of the amplitude of the incident wave, \( A \):

\[
e^{i\delta_1} (A + e^{-i\delta_1} A_R) = e^{-i\delta_1} A_T, \quad e^{i\delta_0} (A - e^{-i\delta_0} A_R) = e^{-i\delta_0} A_T.
\]

The solution is written as \( A_R = RA \) and \( A_T = TA \), where

\[
R = \sin(\delta_0 - \delta_1)e^{i(\delta_0 + \delta_1 + \pi/2)}, \quad T = \cos(\delta_0 - \delta_1)e^{i(\delta_0 + \delta_1)},
\]

are called the reflected and transmitted amplitudes. The reflection and transmission coefficients are defined as

\[
R = |R|^2 = \sin^2(\delta_0 - \delta_1), \quad T = |T|^2 = \cos^2(\delta_0 - \delta_1),
\]

and satisfy the relation \( R + T = 1 \). In quantum mechanics, this relation expresses the conservation of probability and is guaranteed by the unitarity of the theory (that is, by the hermiticity of the Hamiltonian). In the spin wave scattering, it expresses the conservation of energy, which is guaranteed by the structure of the LLG equation, despite \( \Omega \) is not antihermitian.

The expressions (100) that give the amplitudes of the scattered waves, \( R \) and \( T \), as a function of the phase shifts are exactly the same as those for the quantum mechanical scattering by a one dimensional potential.
VII. TIME DELAY AND LATERAL SHIFT

Let us analyze here the asymptotic waves. Let us write the amplitude of the incident wave as

\[ A = |A| \exp(i\phi_A), \]

where \( \phi_A \) is the phase. As we discuss in the previous section, for \( t \to -\infty \) the incident wave has only a significant intensity at the points where its the phase is stationary with respect to \( k_x \) and \( k_z \), that is, at the points \((x, y)\) with

\[ x = -\frac{\partial \phi_A}{\partial k_x} + \frac{\partial \omega}{\partial k_x} t, \quad z = -\frac{\partial \phi_A}{\partial k_z} + \frac{\partial \omega}{\partial k_z} t. \]  

(102)

Analogously, the transmitted wave, for \( t \to +\infty \) has an appreciable intensity only at the points where its phase is stationary, given by

\[ x = -\frac{\partial \phi_A}{\partial k_x} - \frac{\partial (\delta_0 + \delta_1)}{\partial k_x} + \frac{\partial \omega}{\partial k_x} t, \quad z = -\frac{\partial \phi_A}{\partial k_z} - \frac{\partial (\delta_0 + \delta_1)}{\partial k_z} + \frac{\partial \omega}{\partial k_z} t, \]

(103)

since the phase of the transmitted wave amplitude is \( \phi_A + \delta_0 + \delta_1 \). We see that there is a shift in the center of the transmitted wave packet with respect to the what would be the center of the incident wave if there were no soliton, given by \( \Delta x = -\partial (\delta_0 + \delta_1)/\partial k_x \) and \( \Delta z = -\partial (\delta_0 + \delta_1)/\partial k_z \). Let us assume that the wave is detected on a plane perpendicular to the \( z \) axis, located at \( z = z_D \), far away from the soliton (that is, with \( z_D \) much larger than the soliton width). The transmitted wave reaches this plane at a time \( t_D \) which is obtained from equation (103):

\[ t_D = \frac{z_D + \partial \phi_A/\partial k_x}{\partial \omega/\partial k_x} + \frac{\partial (\delta_0 + \delta_1)/\partial k_z}{\partial \omega/\partial k_z}. \]

(104)

Therefore, there is a time delay in the arrival of the transmitted wave caused by the scattering by the soliton, given by

\[ \delta t_D = \frac{\partial (\delta_0 + \delta_1)/\partial k_x}{\partial \omega/\partial k_x}. \]

(105)

Since \( \delta_0 \) and \( \delta_1 \) can be considered functions of \( k_x \) and \( \omega \), instead of \( k_x \) and \( k_z \), the time delay can be written as

\[ \delta t_D = \frac{\partial (\delta_0 + \delta_1)}{\partial \omega}, \]

(106)

which is the expression quoted in the paper.

The position of the transmitted wave packet on the detection plane is shifted with respect to the position at which the incident wave would have been detected if there were no soliton. This lateral shift, analogous to the Goos-Hänchen effect, is given by

\[ \delta x_s = -\frac{\partial (\delta_0 + \delta_1)}{\partial k_x}. \]

(107)

The shift is illustrated in figure 3.

The existence of a non-vanishing lateral shift requires that the phase shifts \( \delta_0 \) and \( \delta_1 \) depend on the transverse wave number \( k_z \). That is why in quantum mechanics with a one dimensional potential the lateral shift vanishes, at least if interactions do not depend on spin. Consider a two dimensional quantum system described by the Hamiltonian

\[ \mathcal{H} = -(1/2m)(\partial^2/\partial x^2 + \partial^2/\partial z^2) + V(z), \]

where the potential depends only on the \( z \) coordinate. For simplicity, take \( V(z) \) even in \( z \). The eigenfunctions of \( \mathcal{H} \) are of the form \( \psi = \exp(ik_z\phi(z)) \), where \( \phi(z) \) is independent of \( k_x \), since it obeys the equation \( -\partial^2 \phi/\partial z^2 + 2mV(z)\phi = (2m\epsilon - k_x^2)\phi \), where \( \epsilon \) is the energy eigenvalue. The solution is of the form

\[ \epsilon = \epsilon_0 + k_x^2 \]

and \( \phi = \phi_0 \), with \( \epsilon_0 \) and \( \phi_0 \) satisfying the \( k_x = 0 \) equation:

\[ -\partial^2 \phi_0/\partial z^2 + 2mV(z)\phi_0 = 2m\epsilon_0\phi_0. \]

Thus only the eigenvalues \( \epsilon \) depend (trivially) on \( k_x \). The eigenfunctions \( \phi = \phi_0 \) are independent of \( k_x \). For \( z \to \infty \) we have the asymptotic behavior \( \phi(z) \sim \cos(k_xz + \delta_0) \) if \( \phi(z) \) is even in \( z \). Obviously \( \delta_0 \) is independent of \( k_x \), since \( \phi(z) \) is. The same happens with \( \delta_1 \), and thus the lateral shift vanishes. That is why, to our knowledge, the Goos-Hänchen effect has not been studied in quantum systems.

In the case studied here, the relevant eigenfunctions, \( \phi^{(p)}_{k_x,k_z}(z) \), satisfy the equation

\[ \Omega_{2k_x}\Omega_{1k_z}\phi^{(p)}_{k_x,k_z}(z) = \omega^2 \phi^{(p)}_{k_x,k_z}(z), \]

with

\[ \Omega_{2k_x}\Omega_{1k_z} = \Omega_{20}\Omega_{10} + k_x^2(\Omega_{10} + \Omega_{20}) + k_x^4, \]

(108)

where \( \Omega_{10} \) and \( \Omega_{20} \) are the \( k_x = 0 \) operators given by equation (78). They act only on the \( z \) coordinate. Assume that \( \Omega_{10} \) and \( \Omega_{20} \) commute. This is equivalent to say that \( \Omega_{1k_z} \) and \( \Omega_{2k_z} \) commute, or that \( \Omega_1 \) and \( \Omega_2 \) commute. In this
case there is a complete set of eigenfunctions common to $\Omega_{10}$ and $\Omega_{20}$, denoted by $\phi_i(z)$, so that $\Omega_{10}\phi_i = \omega_{1i}\phi_i$ and $\Omega_{20}\phi_i = \omega_{2i}\phi_i$. Evidently, these are also eigenfunctions of $\Omega_{2k_x}\Omega_{1}\phi_i$, since $\Omega_{2k_x}\Omega_{1}\phi_i = [\epsilon_{1i}\epsilon_{11} + k_{x}^{2}(\epsilon_{11} + \epsilon_{1i}) + k_{x}^{4}][\phi_i].$

Any other eigenfunction of $\Omega_{2k_x}\Omega_{1}\phi_i$ is a linear combination of the $\phi_i$, since they form a complete set. The phase shifts $\delta_0$ and $\delta_1$ associated to the $\phi_i$ are independent of $k_x$, since obviously these functions do not depend of $k_x$. Therefore, the lateral shift vanishes.

Hence, mathematically, the non vanishing lateral shift is caused by the non commutativity of $\Omega_1$ and $\Omega_2$. In domain walls without DMI, these two operators commute and there is no lateral shift. The situation is as in the quantum mechanical system discussed above. However, in the case of the chiral soliton studied here, and in domain walls of systems with DMI, these two operators do not commute, and thus the phase shifts depend on the transverse wave number $k_x$, and the lateral shift does not vanish.

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