CONSTRUCTIONS AND ISOTOPIES OF HIGH-DIMENSIONAL LEGENDRIAN SPHERES

AGNIVA ROY

ABSTRACT. We explore the construction of Legendrian spheres in contact manifolds of any dimension. Two constructions involving open books work in any contact manifold, while one introduced by Ekholm works only in $\mathbb{R}^{2n+1}$. We show that these three constructions are isotopic (whenever defined). We also recover and generalise a result of Courte and Ekholm, that shows Ekholm’s doubling procedure produces the standard Legendrian unknot.

1. Introduction

A contact manifold $(M, \xi)$ is a smooth manifold $M$ equipped with a nowhere integrable hyperplane field $\xi$. The construction and investigation of contact manifolds has historically been aided by studying distinguished submanifolds that interact suitably with the contact structure. In dimension 3, these submanifolds are either convex hypersurfaces, or Legendrian knots, which are 1-dimensional submanifolds. These in conjunction have helped achieve the classification of tight contact structures on several classes of 3-manifolds e.g. [Hon00a, Hon00b, Gir01, Gir00]. A general theme in most of these results can be seen as follows: first understand Legendrian isotopy classes of a family of Legendrian knots in $(S^3, \xi_{st})$, then understand the contact structures that appear by performing contact surgery on these Legendrians.

In higher dimensions, the first hindrance to carrying out this plan is the shortage of examples of Legendrian spheres. Some constructions of high dimensional Legendrian spheres are explored in [EES05, Ekh16, DR11, BST15], in $\mathbb{R}^{2n+1}$ with the standard contact structure. In this article, we describe some general constructions of Legendrian spheres in $(2n + 1)$-dimensional contact manifolds from Lagrangian disks in pages of supporting open books.

Construction 1.1. A natural way to construct Legendrian submanifolds in a contact manifold is via open book decompositions. Suppose $(M, \xi)$ is supported by the open book $(B, \nu)$, where $\nu : M - B \to S^1$ is a fibration, and each page is symplectomorphic to $(W, \omega)$. Consider a properly embedded Lagrangian $n$-disk $L$ on the page. Then consider two pages $W, W'$ and two copies of the same Lagrangian, called $L, L'$ on them. The disks can be individually perturbed to give Legendrian disks in $M$. Then, by Lemma 3.1 the open book can be perturbed so that Legendrian disks actually lie on the page of the open book as Lagrangian disks. We will consider a thickened binding $B$. We can further perturb $L$ and $L'$ so they can be smoothly joined to give the union, a Legendrian sphere $L \cup L'$. This now gives a closed Legendrian in $(M, \xi)$. We will call this construction $S_{join}(L)$. Since $L'$ is an isotopic copy of $L$, the notation suppresses $L'$. This will be described in more detail in Section 4.

Construction 1.2. Consider $(M, \xi)$ and $L$ similarly as above. Then, the open book can be stabilised, by modifying the page by attaching a Weinstein $n$-handle along $\partial L$, and then performing a positive Dehn twist along the resulting Lagrangian $n$-sphere, obtained by taking the union of
and the core of the handle. This resulting manifold is contactomorphic to \((M, \xi)\). Also, the Lagrangian sphere in the new page can be perturbed to a Legendrian sphere. We will call this Legendrian \(S_{\text{stab}}(L)\).

Our main result is that these constructions give Legendrian isotopic spheres and are isotopic to the standard Legendrian unknot. The standard Legendrian unknot is defined to be the Legendrian realisation of the \(n\)-dimensional unknotted sphere in a Darboux neighbourhood of a contact manifold. Its front projection can be inductively constructed by starting from the unknot with maximal Thurston-Bennequin number in \((\mathbb{R}^3, \xi_{\text{st}})\) and successively spinning half of the front projection.

**Theorem 1.3.** Consider a supporting open book decomposition of a contact manifold \((M, \xi)\). Consider a Lagrangian disk \(L\) in the page. Then \(S_{\text{join}}(L)\) is Legendrian isotopic to \(S_{\text{stab}}(L)\), and they are both isotopic to the standard Legendrian unknot.

A technical issue in the above proof is the following: \(S_{\text{join}}\) is defined with respect to a certain open book decomposition of \((M, \xi)\), whereas \(S_{\text{stab}}\) is defined with respect to the stabilisation of the open book decomposition. While the open books support contactomorphic contact manifolds, to prove Legendrian isotopy, one needs to see stabilisation of the open book as an embedded operation. We will show the following in Section 3:

**Theorem 1.4.** Assume \((M^{2n+1}, \xi)\) admits the supporting open book \((B, \nu)\), whose stabilisation along the Legendrian disk \(L\), which is a Lagrangian in the page, is \((B', \nu')\). Then \(\nu\) and \(\nu'\) are obtained from each other by surgering out a \((2n + 1)\)-disk neighbourhood of \(L\), and replacing it by another \((2n + 1)\)-disk.

**Remark 1.5.** Although the theorems are stated for all dimensions, there is a caveat with ambient dimension 3. As observed in Remark 4.3 of [vK17], the proof of Lemma 3.1 does not work similarly in this dimension. However, the theorems hold if we start with Lagrangian arcs in pages that are also Legendrian in the manifold, and that can always be arranged for some open book decomposition.

**Construction 1.6.** This construction of Legendrian spheres in \((\mathbb{R}^{2n+1}, \xi_{\text{st}})\) was introduced by Ekholm in [Ekh16]. Start with a Lagrangian disk \(L\) which is cylindrical near its boundary, in the symplectisation of \(\mathbb{R}^{2n-1}\). Embed it in a hypersurface in \(\mathbb{R}^{2n+1}\) transverse to the Reeb flow. Then join the Legendrian lifts of this disk and a reflection of the disk in the same hypersurface to obtain a Legendrian sphere \(\Lambda(L, L)\). An example of this construction is given in Figure 1.

In [CE17], Courte-Ekholm show that \(\Lambda(L, L)\) is isotopic to the standard Legendrian unknot.

The contact manifold \((S^{2n+1}, \xi_{\text{st}})\) is supported by the open book where the pages are symplectomorphic to \((D^{2n}, \omega_{\text{st}})\), and the monodromy is the identity. The binding is contactomorphic to \((S^{2n-1}, \xi_{\text{st}})\). Given a Lagrangian disk \(L\) in \((B^{2n}, \omega_{\text{st}})\), one can construct all the three Legendrians as mentioned above. We can show that Courte-Ekholm’s result is a particular case of Theorem 1.3.

**Corollary 1.7.** (Originally proven in [CE17]) Given a Lagrangian disk \(L\) in \((B^{2n}, \omega_{\text{st}})\), \(\Lambda(L, L)\) is isotopic to the standard Legendrian unknot.

The paper is organised as follows: In Section 2, we introduce relevant background and give references for further exposition. In Section 3, we prove Theorem 1.4. Then in Section 4, we prove the first half of Theorem 1.3, namely that the join and stabilisation constructions give isotopic spheres. Finally in Section 5, we prove that the constructions give the standard Legendrian
un knot, completing the proof of Theorem 1.3 and also show how to recover Courte-Ekholm’s result.

1.1. Acknowledgements. I would like to thank my advisor John Etnyre for the discussions that led to asking the questions that this paper eventually answers, and also for his patience and guidance in helping me come up with the proof strategies. I am further indebted to his careful reading and comments on various drafts of this paper, and for his assistance in organising the exposition. I thank Sylvain Courte and Tobias Ekholm for answering my questions about their result over email. I am also grateful to Hyunki Min and James Conway for insightful conversations at the beginning of the project. This work was partially supported by NSF grant DMS-1906414.

2. Background

In this section we will give the necessary background to set up the outline for the proofs of the main theorems.

2.1. Legendrian submanifolds, front projections, and Legendrian surgery.

Definition 2.1. Given a contact manifold \((M^{2n+1}, \xi)\), an \(n\)-dimensional submanifold \(L\) is called a Legendrian if \(T_x(L) \subset \xi\) for every \(x \in L\).

It is well-known that a Legendrian sphere in a contact manifold always has a standard neighbourhood.

Lemma 2.2. If \(S\) is a Legendrian \(n\)-sphere in \((M^{2n+1}, \xi)\), then in any open set containing \(S\) there is a neighborhood \(N\) with boundary \(\partial N = S^n \times S^n\) contactomorphic to an \(\epsilon\)-neighbourhood \(N_\epsilon\) of the zero section \(Z\) in the 1-jet space of \(S^n\), denoted \(J^1(S^n)\). We call \(N\) a standard neighbourhood.

The model we will use to describe Legendrian surgery is understood as what is happening on the boundary when a Weinstein handle is attached along the Legendrian sphere. This is called the flat Weinstein model and is described in Section 3 of [vK17] in more generality, for isotropic surgery along \(S^k\) for \(k \leq n\). Our description follows the exposition there.
Notation: To make the notation less cluttered when we talk about $\mathbb{R}^{2n+2}$, we will write the coordinates $(z_1, w_1, \cdots, z_{n+1}, w_{n+1})$ as $(z, w)$. The symplectic form $\omega_0 = \sum_{i=1}^{n+1} dz_i \wedge dw_i$ will be referred to as $dz \wedge dw$. Similar liberties will be taken with 1-jet space coordinates where $(z, p_1, q_1, \cdots, p_n, q_n)$ will be truncated to $(z, p, q)$, and the contact structure there is $\ker (dz + pdq)$. Products between vectors should be thought of as dot products.

Consider the symplectic manifold $(\mathbb{R}^{2n+2}, \omega_0)$, where the coordinates are $(n + 1)$ pairs of $(z, w)$ coordinates, and $\omega_0 = dz \wedge dw$. The vector field $X = 2z \partial_z - w \partial_w$ is Liouville. The set $S_{-1} := \{(z, w) \mid |w|^2 = 1\}$ is transverse to $X$ and inherits the contact form $\alpha = 2zdw + wdz$.

In $S_{-1}$, the sphere $\{z = 0, |w|^2 = 1\}$ describes a Legendrian sphere. Using $\psi_W : J^1(S^n) \to S_{-1}$ given by $(z, q, p) \mapsto (zq + p, q)$, we get a strict contactomorphism between $S_{-1}$ and the standard neighbourhood described in Lemma 2.2. Thus $S_{-1}$ can be regarded as the standard neighbourhood of a Legendrian sphere.

Now, Legendrian surgery along a Legendrian $S$ will involve removing a neighbourhood of $S$ identified with $S_{-1}$ and gluing in another contact hypersurface of $(\mathbb{R}^{2n+2}, \omega_0)$. The contact hypersurface involved in that is called $S_1$ and we describe it here. Define functions $f$ and $g$, described in Figure 2, to satisfy the following:

- $f$ is increasing on $[1 - \epsilon, \infty)$
- $f(w) = 1$ for $w \in [0, 1 - \epsilon)$, $f(w) = w + \epsilon$ for $w > 1 - \frac{\epsilon}{2}$
- $g$ is increasing on $(0, 1 + \epsilon)$
- $g(z) = z$ for $z < 1$, $g(w) = 1 + \epsilon$ for $w > 1 + \epsilon$

Then, define the hypersurface $S_1 := \{(z, w) \mid f(w^2) - g(z^2) = 0\}$. As $X$ is transverse to $S_1$, it inherits a contact structure. Then, Legendrian surgery along $S$ is removing $\nu(S) \cong S_{-1}$ and gluing $S_1$ in its place. If $S \subset (M, \xi)$, and there is a symplectic manifold $W$ obtained by attaching a Weinstein handle to part of the symplectisation $(M \times [0, 1], d(e^\nu \alpha))$, along $S$ in $M \times \{1\}$, the Legendrian surgery along $S$ can be understood as the upper boundary of $W$.

2.2. Weinstein handle attachment. We briefly review the notion of Weinstein handle attachments here, which will need to define stabilisation of open books. For more detailed exposition the reader is encouraged to consult [CE12].

A Weinstein domain is the symplectic analogue of a smooth handlebody. For a $2n$-dimensional domain, Weinstein $k$-handles can have index at most $n$, and are attached along isotropic $(k - 1)$-spheres in the convex boundary. Recall that a submanifold $S$ of a contact manifold is called isotropic if $T_xS \subset \xi_x$ for all $x \in S$.

Definition 2.3. A Weinstein handle of index $k$ is $h^k = D^k \times D^{2n-k}$ with a symplectic structure so that $\partial_- h^k = (\partial D^k) \times D^{2n-k}$ is concave, and $\partial_+ h^k = D^k \times (\partial D^{2n-k})$ is convex. Moreover, $D^k \times \{0\}$ is isotropic and its intersection with $\partial_- h^k$ is an isotropic $S^{k-1}$ in the contact structure induced on $\partial_- h^k$. Thus, the attaching sphere of a Weinstein $k$-handle is an isotropic $S^{k-1}$. Given an isotropic sphere $S^{k-1}$ in the convex boundary of a symplectic manifold with a choice of trivialization of its conformal symplectic normal bundle, one can attach a Weinstein $k$-handle by identifying a neighborhood of the isotropic sphere with $\partial_- h^k$.

A Weinstein handle of index $n$ is called a critical Weinstein handle, and is attached along a Legendrian sphere. It will be useful for us to understand the local model for attaching a critical Weinstein handle. Consider $\mathbb{R}^{2n}$ with the symplectic structure $\sum_{i=1}^{n} dx_i \wedge dy_i$. Now consider $H_{a, b} := D_a \times D_b$, where $D_a$ is the disk of radius $a$ in the $x_i$ subspace and $D_b$ the disk of radius $b$
in the $y_i$ subspace. Then, $H_{a,b}$ is a model for the Weinstein $n$-handle $h^n$. The expanding vector field $v = \sum_{i=1}^n -y_i dy_i + 2x_i dx_i$ induces contact structures on $\partial_-h^n = (\partial D_a) \times D_b$ and $\partial_+h^n = D_a \times (\partial D_b)$.

2.3. Contact open books. The background on contact open books is taken from the lecture notes by Van Koert [vK17]. The reader is referred to the same for more details.

**Definition 2.4.** An (abstract) contact open book $(\Sigma, \lambda, \phi)$, or $\text{Open}(\Sigma, \phi)$ if we suppress the Liouville form from the notation, consists of a compact exact symplectic manifold $(\Sigma, \lambda)$ and a symplectomorphism $\phi : \Sigma \to \Sigma$ with compact support, i.e., it is identity near $\partial \Sigma$.

**Definition 2.5.** An (embedded) supporting open book for a contact manifold $(M, \xi)$ is a pair $(\nu, B)$, where $B$ is a codimension-2 submanifold of $M$ with trivial normal bundle, such that

- $\nu : (M - B) \to S^1$ is a fiber bundle, such that $\nu$ gives the angular coordinate of the $D^2$-factor of a neighbourhood $B \times D^2$ of $B$, and
- if $\alpha$ is a contact form for $\xi$, it induces a positive contact structure on $B$ and $d\alpha$ induces a positive symplectic structure on each fiber of $\nu$.

![Figure 2. The functions $f$ and $g$ used to describe Legendrian surgery.](image)
The embedded open book constructed from Definition 2.4 is the manifold \( \Sigma \times [0, 1]/\sim \), where the equivalence relation \( \sim \) identifies all points \((x, t)\) and \((x, t')\) where \( x \in \partial \Sigma \), and identifies points \((x, 0)\) with \((\phi(x), 1)\). For our purpose, we will employ another (but equivalent, up to contact isotopy) way of building a manifold from an abstract open book, where we will have something called the \textit{thickened binding}. This construction will work as follows:

**Definition 2.6.** A manifold constructed from the abstract open book \( \text{Open}(\Sigma, \phi) \) with \textit{thickened binding} is the quotient of the disjoint union of the mapping torus \( \Sigma \times [0, 1]/((x, 0) \sim (\phi(x), 1)) \) and the thickened binding \( \partial \Sigma \times D^2 \), under the identification \((x, t) \sim (x, 1, t)\), where \( x \in \partial \Sigma \), and \( \{(x, r, \theta) \mid r \in [0, 1], t \in \mathbb{R}/\mathbb{Z}\} \) are the coordinates on \( \partial \Sigma \times D^2 \).

An open book with thickened binding can be given a compatible contact structure, as shown in Section 2.2 of [vK17]. Every contact manifold has a supporting open book decomposition, by work of Giroux. Further, the contact structure supported by an open book is unique up to isotopy, as said by the next theorem, due to Giroux.

**Theorem 2.7** (Giroux). If an open book \((\Sigma, \lambda, \phi)\) supports a contact structure \((M, \xi_1)\), and \(\xi_2\) is another contact structure on \(M\) supported by an open book whose pages are symplectomorphic to \(\Sigma\) and the monodromy is isotopic through symplectomorphisms to \(\phi\), then \(\xi_1\) and \(\xi_2\) are contactomorphic.

An abstract open book defines a supporting open book for the corresponding contact manifold. This follows from work of Thurston-Winkelnkemper [TW75] and Giroux. The reader can refer to [vK17] for a proof (originally by Giroux), and more details. For open books, by a \textit{page} we refer to \(\Sigma\) for abstract open books, and to the closure of a fiber of \(\nu\) for embedded ones. In the manifold built from the abstract open book, the equivalence class \([[(x, t)]\) for \(x \in \partial \Sigma\) is the binding. In the embedded case, \(B\) is the binding. In the thickened binding case, \(\partial \Sigma \times D^2\) is the binding. In the embedded case, as \(M - B\) has the structure of a fibration over \(S^1\), it makes sense to talk about the \textit{monodromy} of an open book. In the abstract setting, \(\phi\) is called the monodromy.

### 2.3.1. Relative Open Books

We will need the idea of open book decompositions of manifolds with boundary, and how to glue such decompositions together. The difference from Definitions 2.4 and 2.5 in this case is that the page topology can change. This definition is closely linked to the notions of \textit{partial open books} [HKM09] and \textit{foliated open books} [LV20] in 3-dimensional contact geometry.

**Definition 2.8.** An (embedded) supporting relative open book for a contact manifold \((M, \partial M, \xi)\) is a pair \((\nu, B)\), where \(B\) is a codimension-2 submanifold of \(M\) with trivial normal bundle, such that

- \(\nu : (M - B) \rightarrow S^1\) is a circle-valued Morse function. Further, \(\nu\) gives the angular coordinate of the \(D^2\)-factor of a neighbourhood \(B \times D^2\) of \(B\), and
- if \(\alpha\) is a contact form for \(\xi\), it induces a positive contact structure on \(B\) and \(d\alpha\) induces a positive symplectic structure on each fiber of \(\nu\)

In our case, the change in page topology will only be effected by adding or removing a Weinstein \(n\)-handle. To help the reader’s intuition, we will give the notion of an abstract relative open book, along the lines of abstract foliated open books [LV20] in 3 dimensions.

**Definition 2.9.** An (abstract) supporting relative open book for a contact manifold is a tuple \(\{(S_i)_{i=0}^{2k}, h\}\) where:

- \(\nu : (M - B) \rightarrow S^1\) is a circle-valued Morse function. Further, \(\nu\) gives the angular coordinate of the \(D^2\)-factor of a neighbourhood \(B \times D^2\) of \(B\), and
- if \(\alpha\) is a contact form for \(\xi\), it induces a positive contact structure on \(B\) and \(d\alpha\) induces a positive symplectic structure on each fiber of \(\nu\)
• $S_i$'s are Liouville domains for odd (or even) $i$ such that $S_{i+1}$ is obtained from $S_i$ by either attaching a standard $2n$-ball or removing one, as a neighbourhood of a properly embedded Lagrangian $n$-disk, which happens alternately.

• The boundary of $S_i$ is a smoothing of $B \cup \alpha_i$, where before smoothing, $B$ is a convex boundary component with an outward pointing Liouville vector field, $\alpha_i$ is diffeomorphic to $(D^n \times S^{n-1})$, and for alternate $i$ it is a convex or concave boundary component. The $2n$-ball can be regarded as $D^n \times D^n$ with the smoothed boundary having two convex parts, the attaching part $\partial_-$ which is $D^n \times S^{n-1}$, and the upper part $S^{n-1} \times D^n$. At every stage, to go from $S_i$ to $S_{i+1}$ the ball is either attached along a concave $\alpha_i$, or the convex $\alpha_i$ is the upper part of the boundary of the ball that gets removed.

• $h$ is a symplectomorphism between $S_{2k}$ and $S_0$.

In this case, we construct the manifold similarly as before by gluing together $S_i \times I$ pieces, then using $h$ to create a ”mapping torus”, and then filling in the binding with $B \times D^2$. There is a natural correspondence between the embedded and abstract descriptions. The schematic of the pages changing is shown in Figure 3. We should point that we expect the full generalisation of partial and foliated open books in high dimensions should involve more ways of changing the page topology, and also the boundary can possibly be enriched to be a convex hypersurface in the sense of Honda-Huang [HH19]. However, for the purpose of this paper, and understanding stabilisation, the simple notion described here suffices.

Example 2.10. We may obtain a relative open book decomposition of a contact ball $B^{2n+1}$, obtained as the neighbourhood of a properly embedded Lagrangian disk in the page of an open book, as follows. A schematic of this decomposition is Figure 4. We derive this in detail in Section 5.

Here $k = 4$. The manifolds $S_0, S_2$, and $S_4$ are the disk cotangent bundles of the $n$-disk $D^n$, while the manifolds $S_1$ and $S_3$ are the disk cotangent bundles of the $n$-dimensional annulus $S^{n-1} \times I$. The map $h$ is the identity. The Lagrangian disk in the page shows up in this neighbourhood as the core $n$-disk of $S_0, S_2$, and $S_4$.

Our main purpose to define these relative open books is to write down how to glue two such decompositions of manifolds together to obtain a closed manifold with an open book decomposition.

Lemma 2.11. Consider two relative open book decompositions on $(M_1, \partial M_1)$ denoted $\{S_{i,1}\}_{i=0}^{2k}, h_1$, and on $M_2, \partial M_2$ denoted $\{S_{i,2}\}_{i=0}^{2k}, h_2$. Suppose the page boundaries are denoted $B_1 \cup \alpha_{i,1}$ and $B_2 \cup \alpha_{i,2}$. These induce relative open book decompositions on the thickened boundary of the manifolds $\partial M_1 \times I$ and $\partial M_2 \times I$. Suppose there is a contactomorphism $\phi : \partial M_1 \times I \to \partial M_2 \times I$, which restricts to each page boundary $\phi_{(\alpha_{i,1} \times I)}$ as a local symplectomorphism that identifies the concave boundary at $\alpha_{i,1}$ with the convex boundary at $\alpha_{i,2}$. This extends the symplectic structures on $S_{i,1}$ and $S_{i,2}$ to a Weinstein structure on $S_{i,1} \cup S_{i,2}$. Then, $M_1 \cup M_2$ is a contact manifold with supporting open book decomposition where the pages are symplectomorphic to $S_{i,1} \cup S_{i,2}$, and the monodromy is $h_1 \cup h_2$.

Proof. The fact that $M_1 \cup_\phi M_2$ is a contact manifold follows because the contact structures on each of them are identified by $\phi$ over the gluing region. The relative open book fibrations, by the given conditions, glue to give a fibration over the glued fibers. □

2.3.2. Generalised Dehn Twist. Suppose $(W, \omega)$ is a symplectic manifold with an embedded Lagrangian sphere $L \subset W$. A neighbourhood $\nu_W(L)$ is symplectomorphic to a neighbourhood of
the zero section of the canonical symplectic structure on $(T^*S^n, d\lambda_{can})$, by the Weinstein neighbourhood theorem. The cotangent bundle of the $n$-sphere $T^*S^n$ can be regarded as a submanifold of $\mathbb{R}^{2n+2}$ as the set $\{(p,q) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} | q \cdot q = 1, q \cdot p = 0\}$. In these coordinates, $\lambda_{can} = pdq$.

Define an auxiliary map describing the normalised geodesic flow

$$\sigma_t(q, p) = \begin{pmatrix} \cos t & |p|^{-1} \sin t \\ -|p| \sin t & \cos t \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

Then define

$$\tau(q, p) = \begin{cases} \sigma_{g_1(|p|)} & p \neq 0 \\ -Id & p = 0 \end{cases}$$

$g_1$ is a smooth map as graphed in Figure 4. Since $\tau$ is identity outside a neighbourhood of the Lagrangian $\{p = 0\}$, it can be extended to all of $(W, \omega)$ by the identity and defines a symplectomorphism. The map $\tau$ is called the generalised Dehn twist about the Lagrangian sphere $L$. 

**Figure 3.** A schematic for how the page topology changes in a relative open book. The darkened part of the boundary is the region $\alpha$. 

[Image of a schematic showing the topology changes in a relative open book with a properly embedded Lagrangian disk and cuts and additions indicated.]
2.3.3. **Stabilisation of open books.** Given a contact open book \( M = \text{Open}(\Sigma^{2n}, \phi) \), suppose \( L \) is an embedded Lagrangian \( n \)-disk in the page \( \Sigma \) whose boundary \( \partial L \) is a Legendrian sphere in the binding. Consider \( \tilde{\Sigma} \) to be the manifold obtained by attaching a Weinstein \( n \)-handle to \( \Sigma \) along \( \partial L \). Then, call \( L_S \) the Lagrangian sphere in \( \tilde{\Sigma} \) defined by the union of \( L \) and the core of the \( n \)-handle.

**Definition 2.12.** The contact open book \( \tilde{M} := \text{Open}(\tilde{\Sigma}, \phi \circ \tau_{L_S}) \), where \( \tau_{L_S} \) is the Dehn twist along \( L_S \), is called the **stabilisation** of \( \text{Open}(\Sigma, \phi) \) along \( L \).

The following is a well-known statement due to Giroux. A proof can be found in [vK17].

**Proposition 2.13.** The stabilisation of a contact open book \( \text{Open}(\Sigma, \phi) \) along a Lagrangian disk \( L \) bounding a Legendrian sphere in \( \partial \Sigma \) is contactomorphic to the contact manifold \( \text{Open}(\Sigma, \phi) \).

In Section 4, we use the following folklore theorem (refer [vK17] for details), that doing Legendrian surgery on a Legendrian sphere that lives on a page is the same as changing the monodromy by a Dehn twist about that sphere.

**Theorem 2.14.** Let \( \text{Open}(\Sigma, \phi) \) be a contact open book with a Legendrian sphere \( L_S \), which is also a Lagrangian sphere in \( \Sigma \). Denote the contact manifold obtained from \( \text{Open}(\Sigma, \phi) \) by Legendrian surgery along \( L_S \) by \( \text{Open}(\Sigma, \phi)_{L_S} \). Then, the contact manifolds

\[
\text{Open}(\Sigma, \phi \circ \tau_{L_S}) \simeq \text{Open}(\Sigma, \phi)_{L_S}
\]

are contactomorphic.

**Example 2.15.** This is the higher dimensional analogue of Example 6.4 in [LV20]. The standard contact \((2n + 1)\)-sphere is supported by the open book \( \text{Open}(D(T^*S^n), \tau_S) \). The page \( D(T^*S^n) \) can be parametrised as \( \{(p, q) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |q|^2 = 1, p \cdot q = 0, |p|^2 \leq 1\} \), and \( \tau_S \) represents a positive Dehn twist about the sphere \( S = \{p_i = 0\} \) in the page. This open book decomposition can be obtained by gluing together two relative open book decompositions on \((2n + 1)\) balls. The first one is the relative decomposition in Example 2.10 but shift the indices on the pages by 1 (we can
do that since the monodromy is identity). So the page $S_{i,1}$ is really the page $S_{i-1}$ in Example 2.10. Here the Lagrangian is $S$. The second one is the complement of the neighbourhood of $S_0$. It can be denoted $(\{S_{i,2}\}_{i=0}^4, \tau_{S_0})$, where $S_{1,2}$ and $S_{3,2}$ are the disk cotangent bundles of $n$-disks, and $S_{0,2}$, $S_{2,2}$, and $S_{4,2}$ are disk cotangent bundles of the $n$-sphere. The schematic of the gluing is given in Figure 5. This example is understood in more detail in Section 3.

Figure 5. The schematic for gluing two relative open books to obtain $S^{2n+1}$ as in Example 2.15. The concave and convex parts of the boundaries, that are relevant for gluing the open books, are labeled. The unlabeled boundaries constitute the binding and are convex.

3. Understanding Stabilisation as a Surgery

In this section, we will prove Theorem 1.4. We will need a lemma, proved in [vK17] that will allow us to assume that if $L$ is a Lagrangian disk on the page of an open book, and $L'$ is the associated Legendrian, the open book pages can be perturbed, without changing the page topology or the monodromy, so that $L'$ is a Lagrangian on a page as well as a Legendrian in the manifold.
Lemma 3.1 (Lemma 4.2 in [VK17]). Suppose $(M, \xi)$ is a contact manifold of dimension greater than $3$ with a supporting open book where the pages are $\Sigma$ and the monodromy is $\psi$, i.e. $M = \text{Open}(\Sigma, \psi)$. If $L$ is a Lagrangian sphere in the page, then we can isotope the contact structure on $M$ and find a supporting open book with symplectomorphic page and isotopic monodromy such $L$ becomes Legendrian in $M$.

We will now discuss the proof of Theorem 1.4.

3.1. Outline of the idea: We will describe stabilisation of an open book supporting $(M, \xi)$ by removing and regluing a $(2n + 1)$-ball in the manifold. The balls will have two different relative open book decompositions. Extending the induced relative open book fibration map of the complement of the first ball, over the new ball will show that after regluing, we get an open book decomposition of the manifold that has the same abstract open book that describes the stabilised manifold. We will glue the relative open books using Lemma 2.11. By Proposition 2.13, we will conclude that the supported contact structure on $M$ by this open book is contactomorphic to $\xi$.

Now we discuss how we obtain the $(2n + 1)$-ball to surger on, as described in the above paragraph. The manifold $M$ has the open book given by $\nu : M - B \rightarrow S^1$, where $\nu^{-1}(pt) \simeq \Sigma$. $L$ is a Lagrangian disk in a page which is also a Legendrian in $M$. We can also assume that $B$ here is a tubular neighbourhood of the binding.

Also, consider the abstract open book $\text{Open}(D(T^*(S^n)), \tau_{S_0})$ which is contactomorphic to $(S^{2n+1}, \xi_{st})$. Here $D(T^*(S^n))$ refers to the disk cotangent bundle of the sphere, which can be parametrised as $\{(p, q) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |q|^2 = 1, p \cdot q = 0, |p|^2 \leq 1\}$, and $\tau_{S_0}$ represents a positive Dehn twist about the sphere $S_0 = \{p_i = 0\}$ in the page. The symplectic form on the page is given by $\omega = dp \wedge dq$. The Lagrangian disk $S := \{(p, q) \mid q = (0, \ldots, 0, 1)\}$ is a Legendrian in the manifold. Here also, we can assume that the binding is thickened.

By the Weinstein neighbourhood theorems, given two Legendrian disks in different contact manifolds, they have contactomorphic neighbourhoods. We will take a $(2n + 1)$-ball neighbourhood of $S$ and identify that with a neighbourhood of $L$. We will call the neighbourhood of $S$ as $D_S$, and that of $L$ as $D_L$. Assume the pages in the open books are indexed by $S^1 = \mathbb{R}/\mathbb{Z}$. This neighbourhood $D$ will be built by taking thickened neighbourhoods of $S$ (or neighbourhoods of $L$) in pages $(0, 0.5)$, and neighbourhoods of $\partial S$ in pages $(0.5, 1)$. This gives $D$ the structure of a relative open book, and its intersection with the bindings of the ambient manifolds creates the binding. Hence, it makes sense to talk about pages of $D$, and other than two singular ones, the pages of $D$ will have two kinds of topology. They will either be $T^*(D^n) = D^n \times D^n$, or $T^*(S^{n-1} \times I) = D^{n+1} \times S^{n-1}$. Thus $D_L = N_L \cup N_{\partial L} \cup B_L$, where $N_L$ is the union of the neighbourhoods of $L$, $N_{\partial L}$ is the union of the neighbourhoods of $\partial L$, and $B_L$ is the portion of the binding that comes within $D_L$. Similarly, $D_S = N_S \cup N_{\partial S} \cup B_S$. This is the same relative open book decomposition that was mentioned in Example 2.10.

Here we describe what we meant by twisting the pages. From the pages of $M$ where neighbourhood of $L$ were removed, we glue in the complements of $\partial S$ neighbourhood, and the remaining pages, we glue in the complement of neighbourhood of $S$. Which is to say, $\partial N_L$ glues to $\partial N_{\partial S}$, $\partial N_{\partial L}$ glues to $\partial N_S$.

This gluing is done so that for the complement of $\partial S$ neighbourhood, which is essentially still $D(T^*S^n)$, a portion is identified with $D(T^*D^n)$, such that it seems as if $L$ is identified with the core of this piece, and the remaining $D(T^*D^n) \sim D^n \times D^n$ is glued on like attaching a Weinstein $n$-handle along the isotropic sphere $\partial L$. In the sense of Section 2.3.1, the initial open book $(B, \nu)$ on $M$ is split as the union of two relative open books on $M \setminus D_L$ and $D_L$. The abstract relative open
Figure 6. This is a schematic of how the pages are being modified under the stabilisation operation. On the left are pages of $M$, while the annuli represent $D(T^*S^n)$, their core being $S_0$. The blue arc in the annuli is $S$. In the page of $M$, the blue arc represents $L$. The shaded portion in the top row left represents a page in $N_L$, while that on the top row middle is a page in $N_{\partial S}$. For the gluing in the top row, the complement of the shaded region in the small rectangular portion of the annulus is first identified with the neighbourhood of $L$, then the remaining region glues on as a critical Weinstein handle. The gluing in the second row is similar, without the identification step.

The rest of the section is devoted to working out the technical details of this gluing, and proving that this operation does indeed yield the stabilised open book as defined in Definition 2.12. For $n = 1$, this can be done using convex surfaces and foliated open books, and is mentioned in [LV20] by Licata-Vertesi. To help the reader’s intuition, we describe that argument first.

Proof of Thm 1.4 in the case $n = 1$ (Example 6.6, [LV20]): The Legendrian $L$ is a properly embedded arc living on the page of an open book supporting $(M, \xi)$, while $S$ is a Legendrian arc in $(S^3, \xi_{st})$.
Figure 7. The smooth ball $D_L$ that gets removed. The foliation is from its intersection with the pages, and the points where the arcs meet are the binding. On the right, is a schematic of the neighbourhoods whose intersection with $D$ is shown in the left figure. The top three are the neighbourhoods of $L$, the leftmost corresponding to two blue arcs near the equator, the one to its right two blue arcs nearer the poles, and the rightmost being one of the neighbourhoods contributing the black arcs. Similarly, the lower three represent, from left to right, the neighbourhoods of $\partial L$ contributing green arcs near the equator, ones that go higher (or lower), and the black arcs.

is $S$, which is the Legendrian corresponding to a fiber in the $D(T^*S^1)$ page of the open book supporting $(S^3, \xi_{st})$. This is exactly the annulus open book for $(S^3, \xi_{st})$, and $S$ is the co-core arc.

As discussed above, a neighbourhood of $L$ in $(M, \xi)$ is $D_L$ is contactomorphic to $D_S$ which is a neighbourhood of $S$. The balls $D_S$ and $D_L$ can be chosen so that their boundaries are convex, and the leaves of the characteristic foliation correspond to their intersection with the pages. This is pictured in Figure 7. The blue represent the intersection of $\partial D_S$ with pages that have a neighbourhood of $S$ inside $D$, i.e. pages in $N_S$, while the green represent the intersection with pages that have a neighbourhood of $\partial S$ inside $D$, i.e., the pages in $N_{\partial S}$. The foliation on $\partial D_L = \partial(M - D_L)$ looks the same. Now glue $M - D_L$ to $S^3 - D_S$ by a rotation that maps the green leaves to the blue leaves. As the leaves of the characteristic foliation are matched up, the contact structure extends over the gluing and the new open book on $M$ is modified exactly as described above. □

3.2. Coordinates on the pages, and neighbourhoods of $L$ and $\partial L$. The pages $D(T^*D^n)$ initially are parametrised with coordinates from $\mathbb{R}^{2n+2}$, as $\{(p_1, p_2, \cdots, p_{n+1}, q_1, q_2, \cdots, q_{n+1}) \mid \sum p_i^2 \leq 1, \sum q_i^2 = 1, \sum p_i q_i = 0\}$. The canonical primitive of the symplectic form is $\lambda = \sum p_i dq_i$. The disk $S$, with which the Lagrangian $L$ in the page in $M$ gets identified, is parametrised as: $S = \{q = (0, \cdots, 0, 1)\}$.

Consider $\psi : \mathbb{R} \times (T^*S^n \cap \{q_{n+1} > 0\}) \to \mathbb{R}^{2n+1}$, given by

$$\psi(z, p_1, p_2, \cdots, p_{n+1}, q_1, q_2, \cdots, q_{n+1}) = (z, x_1, \cdots, x_n, y_1, \cdots, y_n)$$

where $x_i = q_{n+1} p_i, y_i = \frac{q_i}{q_{n+1}}$
This allows us to put coordinates \((x_1, \ldots, x_n, y_1, \ldots, y_n)\) on a neighbourhood of \(S\) on the page, such that \(D(T^*S^n) \cap \{q_{n+1} > 0\}\) is identified with

\[
\{(x, y) \in \mathbb{R}^{2n} \mid \left(\sum x_i^2 + \left(\sum x_iy_i\right)^2\right)\left(\sum y_i^2 + 1\right) \leq 1\}
\]

and \(S\) becomes \(\{y_i = 0\}\). The following calculation verifies this is a symplectomorphism:

\[
\psi^*\left(\sum_{i=1}^{n} x_i dy_i\right) = \sum_{i=1}^{n} (q_{n+1} p_i) d\left(\frac{q_i}{q_{n+1}}\right)
\]

\[
= \sum_{i=1}^{n} (q_{n+1} p_i) \frac{q_{n+1} dq_i - q_i dq_{n+1}}{q_{n+1}^2}
\]

\[
= \sum_{i=1}^{n} p_i dq_i - dq_{n+1} \sum_{i=1}^{n} p_i q_i
\]

\[
= \sum_{i=1}^{n} p_i dq_i + p_{n+1} dq_{n+1}
\]

\[
= \sum_{i=1}^{n+1} p_i dq_i
\]

The above computation uses that \(\sum_{i=1}^{n+1} p_i q_i = 0\) implies \(p_{n+1} = -\frac{1}{q_{n+1}} \sum_{i=1}^{n} p_i q_i\).

The boundary of \(D(T^*S^n)\), which is given by \(|p|^2 = 1\), now becomes

\[
\{(x, y) \mid b(x, y) = 1\} \text{ where } b(x, y) = \left(\sum x_i^2 + \left(\sum x_iy_i\right)^2\right)\left(\sum y_i^2 + 1\right)
\]

Figure 8 illustrates how this looks for \(n = 1\) under the coordinate transformation by \(\psi\). The red is the binding.

Figure 8. Coordinates on a piece of \(D(T^*(S^n))\), which is inside the red
The blue in Figure 8 is the binding "rotated", i.e., \{(x, y) \mid b(y, x) = 1\}. The intersection of the blue and red is transverse. The yellow is the set of points "inside the page" that satisfy \{b(x, y) = b(y, x)\}. Consider the sets \{b(x, y) - b(y, x) = t\}, as \(t\) runs from 0 to \(\frac{1}{2}\). Also, consider a smooth function \(\rho(x, y)\) which is 0 at \{b(x, y) = b(y, x) = 1\}, and 1 on a large sphere inside the page away from the binding. Now, consider the sets

\[ B_t := \{b(y, x) = \rho(x, y)(b(x, y) + t) + (1 - \rho(x, y))\} \]

For \(t\) nonzero, they are indicated by the green in the schematic figure. Define a 1-parameter family of neighbourhoods of \(L\) by

\[ \nu_t(L) := \{(x, y) \mid b(x, y) \leq 1, b(y, x) \leq \rho(x, y)(b(x, y) + t) + (1 - \rho(x, y))b(y, x)\} \]

Similarly, define a 1-parameter family of neighbourhoods of \(\partial L\) by

\[ \nu_t(\partial L) := \{(x, y) \mid b(x, y) \leq 1, b(y, x) \geq \rho(x, y)(b(y, x) + t) + (1 - \rho(x, y))b(y, x)\} \]

Here, \(t\) belongs to the interval \((0, 0.5)\). An indicative picture for the 3 dimensional case is Figure 8.

**Remark 3.2.** The neighbourhoods \(\nu_t(L)\) and \(\nu_t(\partial L)\) are precisely the "pages" that build up \(N_L\) and \(N_{\partial L}\), respectively, from above.

### 3.3. Coordinates on \(D\)

Now, we are in a position to exactly describe the parametrised smooth neighbourhood of the Lagrangian \(L\) that will be removed for stabilisation. Note that the above coordinates parametrise a neighbourhood of \(L\), a Lagrangian disk, sitting on a page in an open book decomposition of \(M\).

In \(M\), suppose the open book is given by \(\nu : (M - B) \to S^1\), while suppose the open book on \(S^{2n+1}\) is given by \(\nu_1 : (S^{2n+1} - B') \to S^1\). Let us first focus on \(D\) as a neighbourhood of \(S\) on a page in \(S^{2n+1}\), and obtain a parametrisation of it. For the Legendrian \(L\) which is Lagrangian in the page, a neighbourhood in \(M\), call it \(D\), can be given standard coordinates from \(I \times \mathbb{R}^{2n}\) modulo some identifications and a standard binding piece, (we assume that the monodromy effects are localised in a place away from this neighbourhood). More precisely, if we assume that the page coordinate or \(S^1\) factor of the open book on \(S^{2n+1}\) is given by \(t \in \mathbb{R}/\mathbb{Z}\), then the monodromy affects the pages in the \(t \in [0.7, 0.8]\) interval, and even in those places, it leaves substantial (as required by the next paragraph) neighbourhoods of \(\partial L\) unaffected, in the sense that they can be parametrised directly as standard pieces of \(I \times \mathbb{R}^{2n}\). What we want is to ensure that the pieces \(D_L\) or \(D_S\) can be described without the monodromy of the open book affecting them.

Suppose in \(M\), the pages of the open book decomposition are parametrised in the \(S^1\) direction by \(t \in \mathbb{R}/\mathbb{Z}\). Consider an interval’s worth of \(L\)-neighbourhoods parametrised by \(t\), given by:

\[ N_L := \{(x, y, t) \mid (x, y) \in \nu_{\beta(t)}(L), t \in [0, 0.5]\}, \]

and an interval’s worth of \((\partial L)\)-neighbourhoods parametrised by \(t\), given by:

\[ N_{\partial L} := \{(x, y, t) \mid (x, y) \in \nu_{\beta(t)}(\partial L), t \in [0.5, 1]\} \]

Here \(\beta(t) = (0.5)\sin(2\pi t)\). Also, we can assume that in this local scenario, the binding \(B\) is thickened, which allows us to choose the standard primitive of the symplectic form, \(\sum x_idy_i\) on the page, giving the contact form \(dt + \sum x_idy_i\) on \(N_L \cup N_{\partial L}\) and match it up with an appropriate choice of contact form on the binding, for example as done in Section 2.2, page 4, of [vK17], to get an explicit description of the contact form restricted to \(N_L \cup N_{\partial L} \cup B_L \subseteq M\). The thickened
binding inside $D_L$ can be parametrised as a quotient set $B = \{(x, y, r, t) \mid (x, y) \in \mathbb{R}^{2n}, b(x, y) = 1, r \in [0, 1], t \in \mathbb{R}/\mathbb{Z}, (x, y, 0, t) \sim (x, y, 0, t')\}$, and in $D_L$, the points in $N_L$ (or $N_{\partial L}$) and $\partial B$ overlap as: $\{(x, y) \in N_L \mid b(x, y) = 1\}$ are matched with $\{(x, y, t) \in B \mid b(y, x) = 1\}$ (similarly for $N_{\partial L}$).

We claim that $D_L = N_L \cup N_{\partial L} \cup B_L$ is a smooth $(2n + 1)$-ball. For this, it suffices to check that the boundary $2n$-sphere $\partial D_L$ is smooth. The boundary $\partial D_L$, in these coordinates, can be described as: $\partial N_L \cup \partial N_{\partial L} \cup \partial B_L$, modulo equivalences that go into the interior of $B_L$. Note that $\nu_{\beta(0)}(L) = \nu_{\beta(0)}(\partial L)$, and $\nu_{\beta(0.5)}(L) = \nu_{\beta(0.5)}(\partial L)$. $\partial D_L$ can be described as:

\[
\bigcup_{t \in [0, 0.5]} \{(x, y, t) \mid b(y, x) = \rho(x, y)(b(x, y) + \beta(t)) + (1 - \rho(x, y))\} \\
\cup \bigcup_{t \in [0.5, 1]} \{(x, y, t) \mid b(x, y) = \rho(x, y)(b(x, y) + \beta(t - 0.5)) + (1 - \rho(x, y))\} \cup \{(x, y, t) \in B \mid b(y, x) = 1\}
\]

Let us focus on the region where $\rho = 1$ and check it is smooth. In that region $\partial D_L$ can be described as:

\[
\bigcup_{t \in [0, 0.5]} \{(x, y, t) \mid b(y, x) = b(x, y) + \beta(t)\} \\
\cup \bigcup_{t \in [0.5, 1]} \{(x, y, t) \in B \mid b(x, y) = b(y, x) + \beta(t - 0.5)\} \cup \{(x, y, r, t) \in B \mid b(y, x) = 1\}
\]

modulo the identifications mentioned above, and $||\{(x, y)||$ everywhere sufficiently small. Other than the binding, the set can be described as a quotient of $\bigcup_{t \in [0, 1]} \{(x, y, t) \mid b(y, x) - b(x, y) = (0.5) \sin(2\pi t)\}$. Thus this is smooth. The smoothness of the whole can be easily checked as it is the quotient under smooth equivalences of smooth pieces. The same holds for $D_S$.

3.4. The surgery step. The above establishes coordinates on the boundary of $D_L$, and they work for both $D_L$ and $D_S$. What we will do now is remove $D_L$ from $M$ and $D_S$ from $S^{2n+1}$, then glue the complements after shifting indices.

We will surger $D_L$ out and glue in $S^{2n+1} \setminus D_S$ by a map $F$ which will give a contactomorphism between a neighbourhoods of $\partial (M \setminus D_L)$ and $(S^{2n+1} \setminus \partial D_S)$. The details follow. Notation: Since both $D_L$ and $D_S$ are being given the same coordinates, we will drop the subscript from $D$ and construct $F$ from $\partial D \times I_1$ to $\partial D \times I_2$, for two intervals $I_1$ and $I_2$.

Coordinates on $\partial D \times I$. As mentioned above, $\partial D$ can be described as a quotient of:

\[
\bigcup_{t \in \mathbb{R}/\mathbb{Z}} \{(x, y, t) \in N_L \cup N_{\partial L} \mid b(y, x) = b(x, y) + \rho(x, y)\beta(t)\} \\
\bigcup \{(x, y, r, t) \in B \mid b(y, x) = 1, r \in [0, 1]\}
\]

where the points $(x, y, t)$, when $b(x, y) = b(y, x) = 1$, are identified with $(x, y, 1, t)$, and $(x, y, 0, t) \sim (x, y, 0, t')$.

On a collar neighbourhood of the boundary in $S^{2n+1} \setminus D$, call it $\partial D \times [0, 1]$, where $s$ represents the $I$ direction, a contact form is given by $dt + e^s \sum x_idy_i$ on the page part, i.e., $(\partial D \times [0, 1] \setminus \partial D \times \{0, 1\}) = \partial M \setminus \partial D$.

\[
\bigcup_{t \in [0.5, 1]} \{(x, y, t) \mid b(x, y) = b(y, x) + \beta(t - 0.5)\} \cup \{(x, y, r, t) \in B \mid b(y, x) = 1\}
\]
([0, 1] \cap (N_L \cup N_{\partial L}), which is extended to the interior of the binding part, i.e., \((\partial D \times [0, 1]) \cap B\), as 
\(e^s h_1(r) \sum x_i dy_i + h_2(r) dt\), where \(h_1\) and \(h_2\) are described in the following figure. The \(s = 1\) end represents the boundary of the manifold.

\[\begin{align*}
F : \partial D \times [0, 1] &\to \partial D \times [-1, 0] \\
(x, y, t, s) &\mapsto (-y, x, t + 0.5, -s)
\end{align*}\]

Using this, we will show in the following section that we can apply Lemma 2.11 to glue \(S^{2n+1} - D\) to \(M - D\), to get back \(M\). We will see that this gives an open book decomposition of \(M\).

3.6. An open book decomposition of \(M\). We would like to show that the above operation induces an honest open book decomposition of \(M\) that supports the contact structure on \(M\). The fact that the operation smoothly does not change \(M\) is because we removed a ball and glued it back in by a map on the boundary (rotation) which is smoothly isotopic to the identity. We need to ensure that the conditions for Lemma 2.11 are met.

The contact manifold \(M\) originally has an open book fibration given by \(\nu : (M - B) \to S^1\). In the neighbourhood of \(L\) we called \(D\), the map \(\nu\) was given on \(D \setminus B\) by \((x, y, t) \mapsto t\). The standard \((2n + 1)\)-sphere \(S^{2n+1}\) originally has an open book fibration given by \(\nu_1 : S^{2n+1} - B' \to S^1\), which in the neighbourhood \((D - B')\) was again given by \((x, y, t) \mapsto t\).
Call the fibers of $\nu$ to be all $2n$-dimensional Liouville domains symplectomorphic to $X$. Then, $L$ is a Lagrangian $n$-disk in $X$, and $\partial L$ is a Legendrian $(n-1)$-sphere in $\partial X$. To index the different pages, call $\nu^{-1}(t) = X_t$. For $t \in (0, 0.5)$, $X_t - D$ is a smooth manifold from which a neighbourhood of $L$ has been removed. For $t \in (0.5, 1)$, $X_t - D$ is a smooth manifold from which a neighbourhood of the $(n-1)$-sphere $\partial L$ has been removed. A neighbourhood of $L$ that contains $X_t \cap D$ is parametrised as $\{(x, y) \in \mathbb{R}^{2n} \mid b(x, y) \leq 1\}$, where locally the symplectic form is $dx \wedge dy$ and $L$ is given by $\{y = 0\}$.

The fibers of $\nu_1$ are disk cotangent bundles of the $n$-sphere, call them $Y$ and index them as $Y_t$ as above. Recall that they can be described as subsets of $\mathbb{R}^{2n+2}$ in the following way: $Y = \{(p, q) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|q\|^2 = 1, p \cdot q = 0, \|p\|^2 \leq 1\}$ with the symplectic form $dp \wedge dq$. As in Section 3.2, we can parametrise a part of $Y$, i.e., where $q_{n+1} > 0$, as subsets of $\mathbb{R}^{2n}$ with the symplectic form $dx \wedge dy$, as $S = Y_{\{q_{n+1} > 0\}} = \{(x, y) \in \mathbb{R}^{2n} \mid b(x, y) \leq 1\}$. This region contains the Lagrangian disk $S = \{q = (0, \ldots, 0, 1)\}$ which becomes $\{y = 0\}$ in $S$. For $t \in (0, 0.5)$, $Y_t \setminus D_S = Y_t \setminus \nu_{\beta(t)}(S)$ has a neighbourhood of $S$ removed, thus is the unit cotangent bundle over the disk. For $t \in (0.5, 1)$, $Y_t \setminus D_S = Y_t \setminus \nu_{\beta(t)}(\partial L)$ is obtained from $Y_t$ by removing a neighbourhood of the $(n-1)$-knot $\partial S$ from $\partial Y_t$, and the core $S^n$ stays. Thus it is still symplectic deformation equivalent to $Y_t$.

After gluing in $S^{2n+1} \setminus D_S$, we want to define an open book fibration on the manifold $(M \setminus D_L) \cup (S^{2n+1} \setminus D_S)$. On the two pieces, we had the restrictions of the fibrations $\nu$ and $\nu_1$, defined on the complements of the respective bindings $B$ and $B'$.

Define the open book fibration on $((M \setminus D_L) \setminus B) \cup ((S^{2n+1} \setminus D_S) \setminus B')$, denoted $\nu_{\text{new}}$, by $\nu$ on $((M \setminus D_L) \setminus B)$ and $(\nu - 0.5)$ on $((S^{2n+1} \setminus D_S) \setminus B')$. The map extends smoothly. In the following we verify that it is a fibration. i.e., $\nu_{\text{new}}^{-1}(t)$ are symplectomorphic for all $t$. For that, we need to check that $X_t \cup_F Y_{t+0.5}$ are symplectomorphic for all $t$.

Consider $U_1 \subset X$ to be the region described by our coordinates as $\{b(x, y) \leq 1\} \cap \{b(y, x) \leq 1\}$. In $Y$ identify a similar neighbourhood of $S$ and call it $U_{1,S}$. Both $U_1$ and $U_{1,S}$ are diffeomorphic to $D^n \times D^n$, with symplectic form $\omega = \sum dx_i \wedge dy_i$, with Liouville forms $\sum x_i dy_i$ and $\sum (-y_i dx_i)$ respectively, and respective Liouville vector fields $\sum x_i \partial_{x_i}$ and $\sum y_i \partial_{y_i}$. The Liouville vector field points out of the boundary of $U_1$.

Now, $Y - U_{1,S}$ can be regarded as a Weinstein n-handle, with the same Liouville vector field, with its attaching region being $Y \cap \{b(y, x) = 1\}$.

Near $\partial(Y - U_{1,S})$ we have the coordinates $(x_1, \ldots, x_m, y_1, \ldots, y_n) \in \mathbb{R}^{2n}$, the attaching sphere being given by $\{x_i = 0, \sum y_i^2 = 1\}$, and the attaching boundary of the handle is given by $\{x_i = 0, b(y, x) = 1\}$. Consider now $G : \partial(Y - U_{1,S}) \to \partial(X \cup U_1)$ given by $G(x, y) = (-y, x)$. Using $G$, one can attach the Weinstein $n$-handle $Y - U_{1,S}$ to $X$ along $\partial L$. Call this manifold $X_L$.

Consider the following: $Y_{t+0.5} - D$ attaching to $X_t - D$. Suppose the gluing of $X_{t+0.5} - D$ to $X_t - D$ happens in two steps. First the gluing under the restriction of $F$ of $(Y_{t+0.5} - D) \cap U_{1,S}$ to $X_t - D$, and then the remainder of $Y_{t+0.5} - D$ is glued on. After the first step, for every $t$, we get back $X$. Then the second step is gluing $Y \cup Y_{1,S}$ to it via $G$. Thus, for every $t$, we get $X$ with a Weinstein handle attached along $\partial L$.

By construction, the contact structure on $M$ post gluing is supported by this open book. The monodromy of the new open book decomposition, by construction, changes by a positive Dehn twist along the sphere $L \cup C$ in the page, where $C$ is the core of the $n$-handle attached to the page. As this open book has the same pages and monodromy as the stabilisation of the open book $\nu$ supporting $(M, \xi)$, the contact structure on $M$ is still contactomorphic to $\xi$, by Theorem 2.7

This finishes the proof of Theorem 1.4.
4. The isotopy between constructions Join and Stab

In this section, we will first rigorously define the constructions $S_{\text{join}}$ and $S_{\text{stab}}$ in open books supporting $(M, \xi)$. Then, we shall establish an isotopy between them, after first ensuring that there is a manifold contactomorphic to $(M, \xi)$ where both the constructions make sense.

At various points in this section, we will do some perturbations by using Legendrians defined by piecewise functions. For that we need to carefully understand how to read the coordinates of such Legendrians in $J^1(S^n)$, which we will address in the following lemma.

**Lemma 4.1.** Consider $J^1(S^n)$ parametrised as $\{(z, q, p) \mid q^2 = 1, p \cdot q = 0\}$. Given a function $f \in C^\infty(S^n)$, $j^1(f)$ defines a Legendrian in $J^1(S^n)$, whose coordinates are given by $(z, q, p)$ such that $z = f(q)$ and $p_i = -\frac{\partial f}{\partial q_i} + (df \cdot q)q_i$, where $df$ is the vector given by $(df)_i = \frac{\partial f}{\partial q_i}$.

**Proof.** The idea here is simply that when a function $f$ is defined on the coordinates $(q_1, \ldots, q_{n+1}) \in \mathbb{R}^{n+1}$, the vector $df|_{\mathbb{R}^{n+1}} := (\frac{\partial f}{\partial q_1}, \ldots, \frac{\partial f}{\partial q_{n+1}})$ belongs to $T^*(\mathbb{R}^{n+1})$. Thus for $f \in C^\infty(S^n)$ but defined on the coordinates $(q_1, \ldots, q_{n+1})$, $df \in T^*(S^n)$ is given by the projection of $df|_{\mathbb{R}^{n+1}}$ from $T^*(\mathbb{R}^{n+1})$ to $T^*(S^n)$. \qed

4.1. Making sense of join Legendrian. The join Legendrian $L_1 \cup L_2$ is described in $M$ (pre-stabilisation) in the following way. Suppose $L_1$ is a Lagrangian $n$-disk on the page, and is also a Legendrian in the manifold. By Lemma 3.1, this can be arranged. Consider two copies $L_1$ and $L_2$ in two pages. They can be glued through the binding. The details follow, and the schematic is given in Figure 10.

**Figure 10.** The above schematic will describe both pre- and post-stabilisation scenarios. Before stabilisation, the left of the vertical represents the thickened binding near $J^1(L) \subset M$. The horizontal lines represent $(z = c)$ slices, and the $q_{n+1}$ coordinate is plotted horizontally. After stabilisation, the page extends to the left, and the picture represents $J^1(S_{\text{Stab}}(L))$. 
The open book on $M$ before stabilisation, by definition, has identity monodromy near the boundaries of the pages. In particular, if we consider all the pages and look near $\partial L_1$, we can parametrise it as $J^1(D^\alpha)$, where \( \{z = \epsilon\} \) represents the pages and the Legendrians \( \{(z, q, 0) \mid z \text{ constant}\} \) represent copies of $L$ near the boundary in different pages. Once two copies of $L$ enter the binding, they can be joined up using the 1-jet of functions defined on $q_{n+1}$ as depicted in Figure 10. Here we are assuming that the contact form on the binding is adjusted so that it still looks like $J^1(D^\alpha)$ near the boundaries of the pages. This amounts to a careful choice of the functions $h_1$ and $h_2$ in Figure 9. This joined up sphere is $S_{\text{join}}(L)$. The portion to the left of the vertical dotted line in Figure 10 is the binding, while to the right are the pages. We will parametrise the $n$-disk as \( \{q \in \mathbb{R}^{n+1} \mid q^2 = 1, q_{n+1} \geq 0\} \).

Now, the way stabilisation is described in Section 2.1, the pages are modified in a small neighbourhood of $\partial L$. Thus, we can interpret in these coordinates as saying that as long as \( \sum p_i^2 \) is large enough, the part of the page is unchanged after stabilisation. In more precise terms, we make the assumption that \( \{(z, q, p) \in T^*S^n \mid q \leq \sqrt{1 + \epsilon}\} \) is the region that is affected. Also, we can assume that this $\epsilon$ is the same for the Legendrian surgery that happens during stabilisation, as described in Section 2.1.

This is the choice of $\epsilon$ that will be used for the remainder of this section.

Thus, we can modify $S_{\text{join}}(L)$ by choosing 1-jets of appropriate functions. At the final stage of this modification, we get $S_{\text{join}}(L)$ obtained by gluing two copies of $L$ “through the pages”. Since by Lemma 4.1, the $p_i$ coordinates depend on this slope, by choosing a high enough slope for the function we can ensure that the $p_i$ coordinates are large enough, and hence this perturbation of $S_{\text{join}}(L)$ lives completely outside the region which is cut out and replaced by stabilisation, and thus is well-defined after stabilisation as well. Also, to make it easier for us to define the isotopy later, we can ensure that $S_{\text{join}}(L)$ does not intersect the interior of the region which is identified with $S_{-1}$ and replaced with $S_1$.

We can choose a Legendrian using a function that depends only on $q_{n+1}$. By Lemma 4.1, for a Legendrian $j^1(H(q_{n+1})) \subset J^1(S^n)$, $p_i = \frac{dH}{dq_{n+1}} q_{n+1} q_i$ for $i = 1, \ldots, n$, and $p_{n+1} = \frac{dH}{dq_{n+1}} (-1 + q_{n+1}^2)$. Thus,

$$z^2 + \sum_{i=1}^{n+1} p_i^2 = H^2 + \left(\frac{dH}{dq_{n+1}}\right)^2 q_{n+1}^2 (1 - q_{n+1}^2) + \left(\frac{dH}{dq_{n+1}}\right)^2 (1 - q_{n+1}^2)^2 = H^2 + \left(\frac{dH}{dq_{n+1}}\right)^2 (q_{n+1}^2 + 1)$$

By choosing $H$ defined on $\{q_{n+1} \geq \epsilon\}$ such that $H^2 + (\frac{dH}{dq_{n+1}})^2 (q_{n+1}^2 + 1) > (1 + \epsilon)$, and such that $H(\epsilon) = H'(\epsilon) = 0$, we can join it with the disk $\{q_{n+1} \geq \epsilon\}$ in the zero section to get a perturbed $S_{\text{join}}$. Such an $H$ would need to satisfy

$$\left(\frac{dH}{dq_{n+1}}\right)^2 \geq \frac{\sqrt{1 + \epsilon} (1 - t)}{1 + q_{n+1}^2}$$

whenever $H = t \sqrt{1 + \epsilon}$.

Such an $H$ can be found, e.g. as shown in Figure 10. Consider a real valued function $H(x)$ on $[0, 1]$ that starts at 0 with value $-\sqrt{1 + \epsilon}$ and slope 0, then increases fast enough so that its slope beats the above inequality, and then reduces to 0 when the value of the function reaches $\sqrt{1 + \epsilon}$ at $x = \frac{1}{2}$.

This perturbed $S_{\text{join}}$ can be seen to be the union of two disks, all described in Figure 10, namely:

- $L_{1,\epsilon}$, the disk corresponding to $j^1(H)$, i.e., the top portion of Figure 10.
• \( L_{2,\epsilon} := \{(c_2, q, 0) \mid q_{n+1} \geq \epsilon\} \) in page that contains \( L_2 \)

The isotopy from \( S_{\text{join}} \) as originally defined to this can be done in \( M \) before stabilisation.

4.2. The stabilisation Legendrian. Our goal is to show that in the stabilised manifold, \( S_{\text{join}}(L) \) is isotopic through smooth Legendrians to \( S_{\text{stab}}(L) \). Recall that when \( M \) is stabilised, each page is modified by adding a Weinstein \( n \)-handle along \( \partial L \), and the core of this handle glues to \( L \) to give a Legendrian sphere, which we call \( S_{\text{stab}}(L) \).

First note that locally, a neighbourhood of \( S_{\text{stab}}(L) \) can be identified with \( J^1(S^n) \) with coordinates \( \{(z, q, p) \mid q^2 = 1\} \), where \( S_{\text{stab}}(L) \) is given by \( (0, q, 0) \). By the description of stabilisation in Section 3, we can assume that the disk \( L \) is exactly the part of \( S_{\text{stab}}(L) \) corresponding to \( q_{n+1} \geq 0 \), and that the page outside a small neighbourhood of the sphere \( \{q_{n+1} = 0\} \) is exactly the page of the open book before stabilisation.

In Figure 10, \( S_{\text{stab}}(L) \) built from the same page that contained \( L_1 \) (above), is seen as the union of the following two disks:

• \( L_\epsilon := \{(c_1, q, 0) \mid q_{n+1} \geq -\epsilon\} \)
• \( C_\epsilon := \{(c_2, q, 0) \mid q_{n+1} \leq -\epsilon\} \)

4.3. Proof of isotopy. In the stabilised manifold, we now have the perturbed \( S_{\text{join}}(L) \) and the \( S_{\text{stab}}(L) \), as described in Sections 4.1 and 4.2 respectively. The perturbed \( S_{\text{join}}(L) \) can be seen as the union of two disks as described in Section 4.1 while the perturbed \( S_{\text{stab}}(L) \) can be seen as the union of two disks as described in Section 4.2. The proof of the isotopy will involve moving \( C_\epsilon \) to \( L_{2,\epsilon} \) and \( L_\epsilon \) to \( L_{1,\epsilon} \), while taking care that their overlaps move consistently. The following subsection is a reading guide for the rest of the proof.

4.4. Proof outline.

1. We will be using different local models at different parts of the argument. As shown in Figure 11, a neighbourhood of \( S_{\text{stab}}(L) \) can be thought of as \( I \times T^*(S^n) \) with a piece cut out and replaced. The orange region is like a portal that represents the “surgery torus” that gets glued in. Recall from Section 2 that Legendrian surgery is done by identifying a neighbourhood of the Legendrian sphere with \( S_{-1} \), then replacing \( S_{-1,c} \) by \( S_{1,c} \). We will use a perturbed region called \( S_1^{st} \) to construct the first half of the isotopy, then move the pieces to \( S_{1,c} \). What will be important is that \( S_{-1} \cap S_1 = S_{-1} \cap S_1^{st} \).

Using the contactomorphisms \( \psi_W : J^1(S^n) \rightarrow S_{-1} \) and \( \psi : S_1^{st} \rightarrow J^1(S^n) \), we will get local coordinates from \( J^1(S^n) \) on the two pieces \( (J^1(S^n) \setminus \psi_W^{-1}(S_{-1,c})) \) and \( \psi(S_1^{st}) \). We carefully constructed \( S_{\text{join}}(L) = L_{1,c} \cup L_{2,c} \) so that \( L_{1,c} \) lives on \( \partial(J^1(S^n) \setminus \psi_W^{-1}(S_{-1,c})) \). The map \( \psi \circ \psi_W \) and its inverse makes sense on \( \psi^{-1}(S_{-1} \cap S_1) \).

2. In Section 4.5.1 we will construct a family of Legendrian disks that live in \( S_1^{st} \), with boundary on \( S_{-1} \cap S_1^{st} \), that define an isotopy between \( \psi_W(C_\epsilon) \) and \( \psi_W(L_{2,\epsilon}) \). We first construct the disks in \( \psi(S_1^{st}) \), and then bring them to \( S_1^{st} \) using \( \psi^{-1} \). A nice observation is that in \( \psi(S_1^{st}) \), these two are disks that appear in the front projection as shown in Figure 12 and the isotopy is essentially twisting their front projections once to a point and then back to the opposite orientation. In some sense, these are a pair of Reidemeister 1-moves.

3. In Section 4.5.2 we show that the above disks can be used to get a family of Legendrian disks in \( S_{1,c} \), with boundary on \( S_{-1} \cap S_1 \), that define an isotopy between \( \psi_W(C_\epsilon) \) and \( \psi_W(L_{2,\epsilon}) \). One can imagine that at this point, in Figure 13 we have a 1-parameter family of disks indexed by \( t \in [-1, 1] \), such that for \( t \in (-1, 1) \), they live inside the orange portal,
but we can see their boundaries at the boundary of the orange region. At \( \{ t = 1 \} \) we get \( C_{t} \), while at \( \{ t = -1 \} \) we get \( L_{2,\epsilon} \).

(4) The last step, done in Section 4.5.3, is to define a family of disks that will give an isotopy between the other halves of \( S_{stab}(L) \) and \( S_{join}(L) \), i.e., \( \hat{L}_{\epsilon} \) and \( L_{1,\epsilon} \). What we want is a family of disks, that in Figure 11 will have their boundaries match up with the boundaries of the disks living inside the orange, and their interiors will be disjoint. The way we do that is ensure we can define disks with the boundary condition of smoothly matching up with the boundaries of the disks found in Section 4.5.2, such that their interiors live in \((J^1(S^n)) \setminus \psi^{-1}_W(S_{-1,\epsilon}))\).

4.5. Modelling the Legendrian surgery: We will assume that in the stabilised manifold the Legendrian surgery corresponding to the Dehn twist about \( S_{stab}(L) \) happens away from the other monodromy of the open book. Stabilisation can be thought of as a two step process happening on the top convex boundary of the symplectisation \((I \times M)\), where the first step is attaching an index \((n)\) Weinstein handle along the isotropic sphere \(\partial L\) living in the binding, and the next step is attaching an \((n + 1)\)-handle along \( L \cup C \). After the first step, on the boundary, we get a manifold \( M' \) presented by an open book \( \nu_{int} \) (as in the open book in the intermediate step) whose pages have been modified by adding an \( n \)-handle to each, while the monodromy has been extended by identity over this handle. Then, the next step modifies the monodromy by adding a Dehn twist. We can call \( L \cup C \) inside \( M' \) as \( S'_{stab}(L) \). After the surgery, we get to \( M \) and the Legendrian sphere \( S'_{stab}(L) \) becomes \( S_{stab}(L) \).

Recall that the identification of \( S_{-1} \) with the standard neighbourhood of a Legendrian sphere, \( \mathbb{R} \times T^*S^n \), in Section 2.1 is done via the following map:

\[
\psi_W : \mathbb{R} \times T^*S^n \to S_{-1}
\]

\[
(z, q, p) \to (zq + p, q) \quad \text{which implies}
\]

\[
\psi_W^{-1}(z_1, w_1) = (z_1 \cdot w_1, w_1, z_1 - (z_1 \cdot w_1)w_1)
\]

Locally, we can think of a neighbourhood \([-\sqrt{1+\epsilon}, \sqrt{1+\epsilon}] \times T^*S^n \) of \( S'_{stab}(L) \subset M' \), across pages indexed by \( t \in [0.4, 0.6] \) which is removed and reglued to achieve the above, i.e., under the contactomorphism between \( \mathbb{R} \times T^*(S^n) \) and \( S_{-1}, [-\sqrt{1+\epsilon}, \sqrt{1+\epsilon}] \times T^*S^n \) is the region that corresponds to \( \{ S_{-1} \mid z^2 \leq 1 + \epsilon \} \), and this corresponds to the neighbourhoods of \( L \) in pages in \([0.4, 0.6]\). As the same indexing of pages outside \( t \in (0.4, 0.6) \) carries over to \( M \), the following sentence makes sense. We will consider \( S_{stab}(L) \) living in \( t = 0.4 \) and \( S_{join}(L) \) formed by joining copies of \( L \) in \( t = 0.4 \) and \( t = 0.6 \).

Now, we will perturb both \( S_{join}(L) \) and \( S_{stab}(L) \) as described in 4.1 and 4.2 respectively, and see them both as unions of two pieces. We will construct the isotopy by constructing isotopies between the two pieces and matching them up along their overlaps. The main idea will be to construct families of functions whose 1-jets will build the isotopies piecewise and ensure they glue together.

4.5.1. Isotopy of \( C_\epsilon \) to \( L_{2,\epsilon} \). Here we first isotope half of \( S_{join}(L) \) to half of \( S_{stab}(L) \). These halves are seen described in Figure 10 \( C_\epsilon \) is the part of \( S_{stab}(L) \) that lives to the left of the picture. It is part of the core of the Weinstein handle that was attached to the page during the stabilisation procedure for the open book.
The idea will be to construct an isotopy between $L_{2,\epsilon}$ and $C_\epsilon$ inside the piece that is glued in during the Legendrian surgery. A schematic is shown in Fig. 11.

We will follow the convention and notation from [2.1] for Legendrian surgery. First, we consider $S'_{\text{stab}}(L)$ lying on a page of $M'$ with the open book $\nu_{\text{int}}$. A neighbourhood of $S'_{\text{stab}}(L)$ is contactomorphic to $S_{-1} = \{(z, w) \in \mathbb{R}^{2n+2} \mid |w|^2 = 1\}$ with the contact form $2zd\bar{w} + wdz$, induced by the symplectic dilation $X = 2z\partial_z - w\partial_w$. This is a contact hypersurface of $(\mathbb{R}^{2n+2}, \omega_{\text{int}})$.

The surgery will be replacing $S_{-1}$ by $S_1$, another contact hypersurface, given by $S_1 = \{(z, w) \in \mathbb{R}^{2n+2} \mid f(|w|^2) - g(|z|^2) = 0\}$, with a similarly induced contact form by $X$. The intersection of $S_1$ with $S_{-1}$ is $\{(z, w) \in \mathbb{R}^{2n+2} \mid |z|^2 \geq 1 + \epsilon, |w|^2 = 1\}$. The surgery replaces $\{S_{-1} \mid |z|^2 \leq 1 + \epsilon\}$ with $\{S_1 \mid |z|^2 \leq 1 + \epsilon\}$. We will call these $S_{-1,c}$ and $S_1,c$ respectively.

The disks we are interested in are described as follows in $S_{-1}$:

$\psi_W(L_{2,\epsilon}) = \{(-\sqrt{1+\epsilon})q, q) \mid q_{n+1} \geq \epsilon\}$

$\psi_W(C_\epsilon) = \{(\sqrt{1+\epsilon})q, q) \mid q_{n+1} \leq -\epsilon\}$
Clearly, they can be seen as living in $S_{-1} \cap S_1 = \{(z, w) \in \mathbb{R}^{2(n+1)} \mid |z|^2 = 1 + \epsilon, |w|^2 = 1\}$.

**Finding isotopy inside $S_1$:** Define $S_{1t}^\text{st} := \{(z, w) \mid |z|^2 = 1 + \epsilon\}$. Clearly, $S_{1t}^\text{st}$ is also transverse to $X$ and hence inherits a contact structure. Now, consider the following contactomorphism $\psi : S_{1t}^\text{st} \to \mathbb{R} \times T^*(S^n)$ given by

\[
\psi(z, w) = \left(\frac{2z.w}{\sqrt{1+\epsilon}}, -\frac{z}{\sqrt{1+\epsilon}}, w - \frac{(z.w)z}{1+\epsilon}\right)
\]

Using this, we can see the disks we are interested in inside $J^1(S^n) = \mathbb{R} \times T^*(S^n)$.

\[
\psi \circ \psi_W(L_{2,\epsilon}) = \{(-2, q, 0) \mid q_{n+1} \geq \epsilon\}
\]

\[
\psi \circ \psi_W(C_\epsilon) = \{(2, -q, 0) \mid q_{n+1} \leq -\epsilon\}
\]

**Remark 4.2.** At this point, one could scale the $z$ and $q$ coordinates to get the above two isotopic as sets. However, for our purpose, we need the isotopy to respect the parametrisation and the induced orientation, i.e., under the isotopy, we want the point $(-2, q, 0) \in \psi \circ \psi_W(L_{2,\epsilon})$ to flow to $(2, -q, 0) \in \psi \circ \psi_W(C_\epsilon)$. So some more work needs to be done. The rest of the isotopy is essentially two Reidemeister twists in $J^1(S^n)$. A schematic is shown in Figure 12.

**Figure 12.** This is a schematic of what happens in $\psi(S_{1t}^\text{st})$. The vertical direction is the $z$ direction, and the figure represents what happens in the isotopy, and how it is really a Reidemeister 1-move. The right hand side represents the front projection, by projecting out the $p$ coordinates.

Consider the unit $n$-disk $D_{\text{model}} := \{(x_1, \ldots, x_n) \mid \sum x_i^2 \leq 1\}$.

For notational convenience, we will define the following

\[
\epsilon_t := (\epsilon - 1)|t| + 1
\]

Also, for $t \in (0, 1]$, consider a sequence of smooth functions $F_t$ defined on the 1-parameter family of disks $\{(2t, q, 0) \mid q_{n+1} \geq \epsilon_t\}$ in $[-2, 2] \times T^*S^n$, that satisfy the piecewise properties.
$F_t \equiv 2t - \delta$

when $q_{n+1} > \epsilon_t + \delta$

$$F_t = \frac{\sqrt{1 - t^2}}{\sqrt{1 - \epsilon_t^2 K_c(t)}} \sum_{i=1}^{n} \frac{q_i^2}{2} + R_t$$

near $q_{n+1} = \epsilon_t$

$F_t$ is the constant function 2. $R_t$ is there to ensure that at the boundary, the value of $F_t$ is exactly $2t$. $K_c(t)$ will be defined to satisfy a boundary condition for the disks. $\delta$ is small and arbitrary.

Now, define a 1-parameter family of maps $D_t : D_{model} \to [-2, 2] \times T^n S^n$ by

$$q_i(D_t(x_1, \ldots, x_n)) := x_i \sqrt{1 - \epsilon_t^2}$$

for $i = 1, \ldots, n$ and

$$q_{n+1}(D_t(x_1, \ldots, x_n)) := \sqrt{1 - \sum_{i=1}^{n} q_i^2}$$

$$z(D_t(x_1, \ldots, x_n)) := F_t(q)$$

$$p_i(D_t(x_1, \ldots, x_n)) = -\frac{\partial F_t}{\partial q_i} + (dF_t, q)q_i \quad \text{when } q_{n+1} \geq \epsilon_t$$

$$\implies \quad \text{when } q_{n+1} = \epsilon_t$$

$$p_i = -\frac{\sqrt{1 - t^2}}{\sqrt{1 - \epsilon_t^2 K_c(t)}} q_i + \frac{\sqrt{1 - t^2}}{\sqrt{1 - \epsilon_t^2 K_c(t)}} (1 - \epsilon_t^2) q_i$$

$$= -\frac{\sqrt{1 - t^2}}{K_c(t)} \epsilon_t^2 x_i$$

for $i = 1, \ldots, n$, and

$$p_{n+1} = \frac{\sqrt{1 - t^2}}{\sqrt{1 - \epsilon_t^2 K_c(t)}} (1 - \epsilon_t^2) \epsilon_t$$

From[1] we can see that

$$\psi^{-1}(z, q, p) = \left( (-\sqrt{1 + \epsilon}) q, p - \frac{z q}{2} \right)$$

So $\psi(S_1 \cap S_{-1})$ is exactly the set $\frac{z^2}{4} + \sum_{i=1}^{n+1} p_i^2 = 1$. Setting this condition for the disks at the extremes, i.e. when $q_{n+1}(D_t) = \epsilon_t$, we will derive the value of $K_c(t)$, as follows. (Note that at these points $\sum_{i=1}^{n} x_i^2 = 1$):
\[ \frac{z^2}{4} + \sum_{i=1}^{n+1} p_i^2 = 1 \implies \sum_{i=1}^{n+1} p_i^2 = 1 - t^2 \]

plugging in the values of \( p_i \)'s, we get

\[ \sum_{i=1}^{n} \left( -\frac{\sqrt{1-t^2}}{\sqrt{1-\epsilon_t^2 K_\epsilon(t)}} q_i + \frac{\sqrt{1-t^2}}{\sqrt{1-\epsilon_t^2 K_\epsilon(t)}} (1-\epsilon_t^2) q_i \right)^2 + \left( \frac{\sqrt{1-t^2}}{\sqrt{1-\epsilon_t^2 K_\epsilon(t)}} (1-\epsilon_t^2) \epsilon_t \right)^2 = 1 - t^2 \]

\[ \implies \frac{1 - t^2}{(1-\epsilon_t^2 K_\epsilon(t))^2} (\epsilon_t^4 \sum_{i=1}^{n} q_i^2 + (1-\epsilon_t^2)^2 \epsilon_t^2) = 1 - t^2 \]

\[ \implies \frac{1}{K_\epsilon(t)^2} (\epsilon_t^4 \sum_{i=1}^{n} x_i^2 + (1-\epsilon_t^2) \epsilon_t) = 1 \implies K_\epsilon(t)^2 = \epsilon_t^2 \]

By Lemma 4.1, the \( D_t \)'s define a 1-parameter family of smooth Legendrian disks in \([-2, 2] \times T^*S^n \). Further, if we define \( D_0 : D_{\text{model}} \to [-2, 2] \times T^*S^n \) by

\[ z(D_0) = q_i(D_0) = 0 \text{ for } i = 1, \ldots, n, q_{n+1}(D_0) = 1, p_i(D_0) = -x_i \text{ for } i = 1, \ldots, n, p_{n+1}(D_0) = 0 \]

then we have a smooth family of disks \( D_t \) for \( t \in [0, 1] \). \( D_1 \) is exactly the disk \( \psi \circ \psi_W(C_\epsilon) \).

The above thus defines an isotopy from \( \psi \circ \psi_W(C_\epsilon) \) to \( D_0 \), which one can see in Figure 12 as being an isotopy from the top disk which is part of the core of \( T^*S^n \), to the disk ''transverse'' to the core. One can think of this as one Reidemeister-1 move, as suggested in Figure 12. Now extend by an isotopy from \( D_0 \) to \( D_{-1} \) where \( D_{-1} \) is exactly \( \psi \circ \psi_W(L_{2,\epsilon}) \), defined as follows:

For \( t \in (0, -1] \), define the functions as follows:

\[ F_t = 2t + \delta \]

\[ = -\frac{\sqrt{1-t^2}}{\sqrt{1-\epsilon_t^2 K_\epsilon(t)}} \sum_{i=1}^{n} \frac{q_i^2}{2} + R_t \]

when \( q_{n+1} > \epsilon_t + \delta \)

when \( q_{n+1} = \epsilon_t \)

Now, analogous to above, for \( t \in (0, -1] \) define a 1-parameter family of maps \( D_t : D_{\text{model}} \to [-2, 2] \times T^*S^n \) by

\[ q_i(D_t(x_1, \ldots, x_n)) := -x_i \sqrt{1-\epsilon_t^2} \]

for \( i = 1, \ldots, n \) and

\[ q_{n+1}(D_t(x_1, \ldots, x_n)) := \sqrt{1 - \sum_{i=1}^{n} q_i^2} \]

\[ z(D_t(x_1, \ldots, x_n)) := F_t(q) \]

\[ p_i(D_t(x_1, \ldots, x_n)) = -\frac{\partial F_t}{\partial q_i} + (dF_t,q)q_i \]

when \( q_{n+1} \geq \epsilon_t \)
This will define an isotopy from $D_0$ to $\psi \circ \psi_W(L_{2,\epsilon})$, and putting everything together, we have defined a smooth 1-parameter family of Legendrians that give an isotopy between $\psi \circ \psi_W(C_t)$ and $\psi \circ \psi_W(L_{2,\epsilon})$, with their boundaries on $\psi(S_1 \cap S_{-1})$. This family can be pulled back via $\psi^{-1}$ to an isotopy in $S_{1}^{std}$ between $\psi_W(C_t)$ and $\psi_W(L_{2,\epsilon})$, with their boundaries on $S_1 \cap S_{-1}$. We will denote this family of disks in $S_{1}^{st}$ by $D_{1}^{st}$, where $D_{1}^{st} = \psi_W(C_t)$ and $D_{-1}^{st} = \psi_W(L_{2,\epsilon})$.

4.5.2. Contact isotopy from $S_{1}^{std}$ to $S_{1}$. Consider a sequence of subsets of $\mathbb{R}^{2n+2}$ defined by $S_{1,t} = \{(z, w) \in \mathbb{R}^{2n+2} | f_1(|w|^2) - g_t(|z|^2) = 0, |z|^2 \leq 1 + \epsilon\}$, where $f_1 = f$ and $g_t = g$ are the functions pictured in Fig 9. $f_0(x) \equiv 1 + \epsilon, g_0(x) = x$, and $f_t$ and $g_t$ are endpoint preserving smooth homotopies between $f_0$ and $f_1$, and $g_0$ and $g_1$, respectively, in the region $\{|z|^2 \leq 1 + \epsilon\}$. Thus, $S_{1,0} = \{(z, w) \in \mathbb{R}^{2n+2} | |z|^2 = 1 + \epsilon\} = S_{1}^{st}$, and $S_{1,1} = S_{1}$. As $f_t$ and $g_t$ have the same values when their arguments take the extremal values 1 and $1 + \epsilon$ respectively, so all the $S_{1,t}$ have the same boundary at $S_1 \cap S_{-1}$. All the $S_{1,t}$ are transverse to the symplectic dilation $X$, thus are a smooth family of contact embeddings of $I \times D(T^*S^n)$ in $\mathbb{R}^{2n+2}$, that agree on the boundary $S^n \times S^n$. Pulling the contact form on $S_{1,t}$ back via the embedding to $S^n \times D^{n+1}$, we get a 1-parameter family of contact forms that agree on the boundary, call them $\alpha_t$. Now, Moser’s method constructs on $S^n \times D^{n+1}$ a vector field that is 0 where the forms agree, and flowing along this vector field gives a contactomorphism between $(S^n \times D^{n+1}, ker(\alpha_0))$ and $(S^n \times D^{n+1}, ker(\alpha_1))$, since the support of the vector field is compact. This gives a contact isotopy taking $S_{1}^{st}$ to $S_{1}$, and we can use that to get a family of disks that live in $S_{1}$.

This means that the isotopy in $S_{1}^{st}$ can be identified with an isotopy in $S_{1}$. Thus, we have a family of disks $D_{1}^{1}$ in $S_{1}$ such that $\partial D_{1}^{t} = \partial D_{t}^{st}$ for all $t$. Moreover, $D_{1}^{1} = D_{1}^{st} = \psi_W(C_t)$ and $D_{-1}^{1} = D_{-1}^{st} = \psi_W(L_{2,\epsilon})$.

Our next step will be to build a family of disks to isotope $L_{\epsilon}$ to $L_{2,\epsilon}$, but ensuring that the boundaries of these disks match up with the boundaries of the $\psi_W^{-1}(D_{1}^{t})$ we have found till now.

4.5.3. Isotopy of $L_{t}$ to $L_{1,\epsilon}$. In this subsection we will construct an isotopy between the remaining halves of $S_{join}(L)$ and $S_{stab}(L)$. We will have to be extra careful, as the family of Legendrian disks we will construct will have to join up with the family of Legendrian disks found above in subsections 4.5.1 and 4.5.2 to give a family of spheres between $S_{join}(L)$ and $S_{stab}(L)$.

We will first identify the boundaries of the disks $D_t$ for $t \in [-1, 1]$. By the description above, we can see that for $t \in [0, 1]$, 

$$\partial(D_t) = \{(z_t, q_t, p_t)\} \quad \text{where}$$

$$z_t = 2t$$

$$q_{i,t} = q_i(S_t(x_1, \ldots, x_n)) = x_i \sqrt{1 - \epsilon_t^2}, \text{ for } i = 1, \ldots, n$$

$$q_{n+1,t} = \epsilon_t$$

$$p_{i,t} = p_i(S_t(x_1, \ldots, x_n)) = -\epsilon_t(\sqrt{1 - t^2})x_i \text{ where } \sum x_i^2 = 1, \text{ for } i = 1, \ldots, n,$$

$$p_{n+1,t} = \sqrt{(1 - t^2)(1 - \epsilon_t^2)}$$
while for \( t \in [-1, 0] \)

\[
\partial(D_t) = \{(z_t, q_t, p_t)\} \quad \text{where} \\
z_t = 2t \\
q_{i,t} = q_i(S_t(x_1, \ldots, x_n)) = -x_i \sqrt{1 - \epsilon_t^2}, \text{ for } i = 1, \ldots, n \\
q_{n+1,t} = \epsilon_t \\
p_{i,t} = p_i(S_t(x_1, \ldots, x_n)) = -\epsilon_t(\sqrt{1 - t^2})x_i \text{ where } \sum x_i^2 = 1, \text{ for } i = 1, \ldots, n, \\
p_{n+1,t} = -\sqrt{(1 - t^2)(1 - \epsilon_t^2)}
\]

We will want to build a family \((D^n)_t\) for \( t \in [-1, 1] \) living in \( S_{-1} \), such that \((\partial D^n)_t = \partial D_t\) as described above, \((D^n)_1 = \psi_W(L_{\epsilon}), (D^n)_{-1} = \psi_W(L_{2, \epsilon})\). We will first see these boundaries in \( \psi_W^{-1}(S_{-1}) \), and for that we will see the images of the above via \( \psi_W^{-1} \circ \psi^{-1} : \mathbb{R} \times T^*S^n \mid \sum x^2 = 1 \to \mathbb{R} \times T^*S^n \), which is given by:

\[
\psi_W^{-1} \circ \psi^{-1}(z, q, p) = \left(\frac{z\sqrt{1 + \epsilon}}{2}, p - \frac{zq}{2}, -q\sqrt{1 + \epsilon} - \frac{z\sqrt{1 + \epsilon}}{2}(p - \frac{zq}{2})\right)
\]

The coordinates of \( \psi_W^{-1}(\partial D_t) \) are as follows:
for \( t \in [0, 1] \),

\[
\psi_W^{-1} \circ \psi^{-1}(\partial D_t) = \{(z_t, q_t, p_t)\} \quad \text{where} \\
z_t = t\sqrt{1 + \epsilon} \\
q_{i,t} = -\epsilon_t(\sqrt{1 - t^2})x_i - t(\sqrt{1 - \epsilon_t^2})x_i \\
q_{n+1,t} = \sqrt{(1 - t^2)(1 - \epsilon_t^2)} - t\epsilon_t \\
p_{i,t} = (\sqrt{1 + \epsilon})x_i \left( -\sqrt{1 - \epsilon_t^2} - t(\sqrt{1 - \epsilon_t^2}) - t(\sqrt{1 - \epsilon_t^2}) \right) \\
p_{n+1,t} = (\sqrt{1 + \epsilon}) \left( -\epsilon_t - t(\sqrt{1 - t^2})(1 - \epsilon_t^2) - t\epsilon_t \right)
\]

for \( t \in [-1, 0] \),
\[ \psi^{-1}_W \circ \psi^{-1}(\partial D_t) = \{(z_t, q_t, p_t)\} \quad \text{where} \]
\[
    z_t = t\sqrt{1 + \epsilon} \\
    q_{i,t} = -\epsilon_t(\sqrt{1 - t^2})x_i + t(\sqrt{1 - \epsilon_t^2})x_i \\
    q_{n+1,t} = -\sqrt{(1 - t^2)(1 - \epsilon_t^2)} - t\epsilon_t \\
    p_{i,t} = (\sqrt{1 + \epsilon})x_i\left(\sqrt{1 - \epsilon_t^2} - t(-\epsilon_t(\sqrt{1 - t^2}) + t(\sqrt{1 - \epsilon_t^2}))\right) \\
    p_{n+1,t} = (\sqrt{1 + \epsilon})(-\epsilon_t - t(-\sqrt{(1 - t^2)(1 - \epsilon_t^2)} - t\epsilon_t))
\]

By choosing functions appropriately we can define an isotopy from \(L_\epsilon\) to \(L_{1,\epsilon}\), by constructing a family of disks whose boundaries match up with the above. First, note that we can extend the disks by a function that depends only on \(q_{n+1}\). Now, note that \(L_\epsilon\) lives on \(\psi^{-1}_W\{(S_1 \mid |z|^2 = (1 + \epsilon))\}\), while \(L_{1,\epsilon}\) meets \(\psi^{-1}_W\{(S_1 \mid |z|^2 \leq (1 + \epsilon))\}\) only at \(\partial L_{1,\epsilon}\). We will choose the family of disks going between them such that their interiors are disjoint from the interior of \(\psi^{-1}_W(S_1)\). This will automatically ensure that when these disks are joined with \(D_t\), we get a family of embedded Legendrians.

The boundary conditions above will ensure that the boundaries live on \(\psi^{-1}_W(S_1)\), so in order that the interior of the disks do not, we need \(\{z^2 + p^2 \geq 1 + \epsilon\}\) for \((z, q, p)\) coordinates on the interiors of the disks.

By Lemma 4.1 and the discussion in 4.1, the required family of Legendrians can be described by \(H_t(q_{n+1})\) defined on \(\{q_{n+1} \geq q_{n+1,t}\}\) which satisfy \(H_t(q_{n+1,t}) = t\sqrt{1 + \epsilon}, H'_t(q_{n+1,t}) = \frac{\sqrt{1 - t^2}}{\epsilon_t\sqrt{1 - t^2} + \sqrt{1 - \epsilon_t^2}}, H_t^2 + \left(\frac{dH_t}{dq_{n+1}}\right)^2(q_{n+1}^2 + 1) \geq (1 + \epsilon)\). Also, \(H_{-1} = H\) from 4.1 while \(H_1 \equiv 0\), so as to give \(L_{2,\epsilon}\) and \(L_{\epsilon}\), respectively.

These initial points are plotted in Figure 13 for \(\epsilon = 0.1\), while the initial slopes are plotted in Figure 14.

---

**Figure 13.** Plot of \(q_{n+1,t}\) vs \(t\) when \(\epsilon = 0.1\).

\[ q_{n+1,t} = \pm \sqrt{(1 - t^2)(1 - \epsilon_t^2)} - t\epsilon_t \]
The choice of generating functions \( H_t \) that will now work, i.e. \( j^1(H_t) \) will describe an isotopy between \( L_\epsilon \) and \( L_{2,\epsilon} \), can be described by the following method.

Similar to the function \( H \) that was chosen at the end of 4.1, consider a smoothly varying family \( H_t(X) \) of functions defined on \([q_{n+1}, t, 1]\) that start with the specified slopes, that increase fast enough, beating the required inequality, and then reduce to 0 when the function value reaches \( \sqrt{1+\epsilon} \).

A schematic of the graphs of \( H_t \) can be seen in Figure 15.

5. THE CONSTRUCTIONS GIVE THE STANDARD UNKNOT

In this section we show that \( S_{\text{join}}(L) \) is isotopic to the standard Legendrian unknot, thus completing the proof of Theorem 1.3. Then we connect our constructions and results to Courte-Ekholm’s work in [CE17] and prove Corollary 1.7. We will show that via a sequence of contactomorphisms and isotopies, \( S_{\text{join}}(L) \) in \( (S^{2n+1}, \xi_{st}) \) can be identified with \( \Lambda(L, L) \).

5.1. Isotoping \( S_{\text{join}} \) to the unknot.

**Proof of Theorem 1.3.** In Section 4 we established that \( S_{\text{join}}(L) \) and \( S_{\text{stab}}(L) \) are isotopic. All that remains now is to show that \( S_{\text{join}} \) is the unknot. Using generating functions as described in Figure 10, \( S_{\text{join}}(L) \) can be isotoped to \( L_{2,\epsilon} \) joined with a pushoff arbitrarily close to it. In the figure, that amounts to bringing the top strand labelled \( L_{1,\epsilon} \) arbitrarily close to the lower one. Thus \( S_{\text{join}}(L) \) is described in a standard neighbourhood of the Legendrian disk \( L \), and hence the isotopy class of the sphere depends on the isotopy class of the disk. By Gromov’s h-principle for Legendrian immersions, any two Legendrian disks are isotopic. Thus we can consider the same construction in a Darboux neighbourhood for the standard Legendrian disk, for which this construction gives the standard Legendrian unknot. This proves that \( S_{\text{join}}(L) \) is isotopic to the unknot. \( \square \)
Here we can see the family $H_t$ that will be used to describe the family of disks isotoping $L_\epsilon$ to $L_{1,\epsilon}$, drawn near their boundary so the picture does not get cluttered. Plotted are the initial points and slopes, following how $q_{n+1,t}$ and $p_{n+1,t}$ change as per the plotted graphs. Now, to describe the full disks, we can pick any family of homotopic functions that start at $H_1$, which is the constant 0 function at the top, then plot functions such that their slopes are large enough to ensure $z^2 + p^2 \geq 1$, and then $H_{-1}$ is the purple graph plotted.

5.2. Identifying with $\Lambda(L, L)$ in the $(S^{2n+1}, \xi_{st})$ case. We will quickly review Ekholm’s [Ekh16] construction and Courte-Ekholm’s proof strategy [CE17]. To start with, one considers a codimension 1 space $W_\rho = \{z = (\rho(x_n)y_n) \cap \{0 < x_n < 1\} \in (\mathbb{R}^{2n+1}, \xi_{st})\}$ which is transverse to the Reeb flow. The function $\rho$ can be considered a smoothing of the function $(1 - |x|)$. A Lagrangian disk $L$ with a cylindrical end can be embedded in $W_\rho$ with its cylindrical end approaching $x_n = 0$. Reflecting the $x_n, y_n$ coordinates, another copy of $L, L^-$, can be similarly embedded with its cylindrical end approaching that of $L$. Taking the Legendrian lift of these and joining along the ends gives the Legendrian sphere $\Lambda(L, L)$. Deforming the hypersurface $W_\rho$ to $\{z = 0\}$, while staying transverse to $\partial_z$, recovers the construction of $\Lambda(L, L)$ as originally described in [Ekh16].

To show that this is the unknot, they describe the construction in $(\mathbb{R}^{2n+1}, \ker dz + \sum_{i=1}^{n-1} y_i dx_i + r_i^2 d\theta_i)$. Then, they modify the contact structure so that two halves of the sphere can be brought close to each other by flowing along $\partial_\theta_n$, sketching out a pre-Lagrangian $(n + 1)$-disk foliated by Legendrian disks in the process.

Proof of Corollary 1.7. By a contactomorphism (refer Example 2.1.10 in [Gei08]), we can identify a hemisphere of $(S^{2n+1}, \xi_{st})$ with $(\mathbb{R}^{2n+1}, \ker (dz + \sum r_i^2 d\theta_i))$. Under this, the modified $S_{\text{join}}(L)$, as in the proof above, which is the Legendrian lift of $L$ in a page, joined with a pushoff, is identified with the Legendrian lift of a disk in $\{z = 0\} \cap \{0 < x_n < 1\}$, joined with a pushoff. (Note that in the proof above we have modified the open book pages so that the Lagrangian in the page is a Legendrian, but for the contactomorphism we want to respect the open book structure given by $\theta_n$, and hence have to use Legendrian lifts.)
Then, by a further sequence of contactomorphisms, we can get to \((\mathbb{R}^{2n+1}, \text{ker}(dz - \sum_{i=1}^{n-1} y_i dx_i + r_0^2 d\theta_n))\), where the Legendrian sphere is still given by the Legendrian lift of a disk in \(\{z = 0\}\) joined with its pushoff. Now deforming the hypersurface \(\{z = 0\}\cap\{0 < x_n < 1\}\) to \(W_\rho\), and tracing back through Courte-Ekholm’s proof, it is clear that this sphere is in fact isotopic to \(\Lambda(L, L)\). Since it is the image under a contactomorphism of the unknot, \(\Lambda(L, L)\) is in fact the unknot. This completes the proof of Corollary 1.7. \(\square\)

Remark 5.1. Courte and Ekholm also use the h-principle for their proof. Hence, the proof here does not use a radically different argument. The point is to see their result as a part of a more general picture.

REFERENCES

[BST15] Frédéric Bourgeois, Joshua M Sabloff, and Lisa Traynor, *Lagrangian cobordisms via generating families: construction and geography*, Algebraic & Geometric Topology 15 (2015), no. 4, 2439–2477.

[CE12] Kai Cieliebak and Yakov Eliashberg, *From stein to weinstein and back: symplectic geometry of affine complex manifolds*, vol. 59, American Mathematical Soc., 2012.

[CE17] Sylvain Courte and Tobias Ekholm, *Lagrangian fillings and complicated legendrian unknots*, arXiv preprint arXiv:1712.07849 (2017).

[DR11] Georgios Dimitroglou Rizell, *Knotted legendrian surfaces with trivial contact homology dga*, Algebraic & Geometric Topology 11 (2011), no. 5, 2903–2936.

[EES05] Tobias Ekholm, John Etnyre, and Michael Sullivan, *Non-isotopic Legendrian submanifolds in \(\mathbb{R}^{2n+1}\)*, J. Differential Geom. 71 (2005), no. 1, 85–128. MR MR2191769

[Ekh16] Tobias Ekholm, *Non-loose legendrian spheres with trivial contact homology dga*, Journal of Topology 9 (2016), no. 3, 826–848.

[Gei08] Hansjörg Geiges, *An introduction to contact topology*, Cambridge Studies in Advanced Mathematics, vol. 109, Cambridge University Press, Cambridge, 2008. MR MR2397738 (2008m:57064)

[Gir00] Emmanuel Giroux, *Structures de contact en dimension trois et bifurcations des feuilletages de surfaces*, Invent. Math. 141 (2000), no. 3, 615–689. MR MR1779622 (2001i:53147)

[Gir01] ———, *Structures de contact sur les variétés fibrées en cercles audessus d’une surface*, Comment. Math. Helv. 76 (2001), no. 2, 218–262. MR MR1839346 (2002c:53138)

[HH19] Ko Honda and Yang Huang, *Convex hypersurface theory in contact topology*, arXiv preprint arXiv:1907.06025 (2019).

[HKM09] Ko Honda, William H. Kazez, and Gordana Matić, *The contact invariant in sutured Floer homology*, Invent. Math. 176 (2009), no. 3, 637–676. MR 2501299 (2010g:57037)

[Hon00a] Ko Honda, *On the classification of tight contact structures. I*, Geom. Topol. 4 (2000), 309–368 (electronic). MR MR1786111 (2001i:53148)

[Hon00b] ———, *On the classification of tight contact structures. II*, J. Differential Geom. 55 (2000), no. 1, 83–143. MR MR1849027 (2002g:53155)

[LV20] Joan E Licata and Vera Vertesi, *Foliated open books*, arXiv preprint arXiv:2002.01752 (2020).

[TW75] W. P. Thurston and H. E. Winkelnkemper, *On the existence of contact forms*, Proc. Amer. Math. Soc. 52 (1975), 345–347. MR MR0375366 (51 #11561)

[vK17] Otto van Koert, *Lecture notes on stabilization of contact open books*, Münster J. Math. 10 (2017), no. 2, 425–455. MR 3725503

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia
Email address: aroy86@gatech.edu