HOMOLOGICAL MIRROR SYMMETRY FOR THE UNIVERSAL CENTRALIZERS II: THE GENERAL CASE AND INDUCTION PATTERN

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Abstract. We prove homological mirror symmetry for the universal centralizer \( J_G \), associated with any complex reductive Lie group \( G \). This is a continuation of the author’s previous work [Jin1] on proving the result for \( G \) of adjoint form.

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1. Introduction

This is the second part of the study of homological mirror symmetry (HMS) of the universal centralizer \( J_G \) for a complex reductive Lie group \( G \). In the first part [Jin1], we have proved HMS for \( G \) of adjoint form. In this part, we prove the general case. The main result is stated in Theorem 2.9. We give two proofs of the main theorem: one is using microlocal sheaf category and the other is using wrapped Fukaya category. Although arguing in different languages, the underlying principles of the proofs are essentially the same: one combines the HMS for adjoint \( G \) and monadicity properties of functors associated with finite central quotients for a reductive group. Lastly, we describe the diagram of co-restriction functors among wrapped Fukaya categories associated with the sector covering of \( J_G \) by \( J_{L_S} \) of the standard Levi subgroups \( L_S \). Each co-restriction functor serves as an induction functor from a smaller Levi to a bigger Levi.

2. Homological mirror symmetry of \( J_G \) for reductive groups \( G \)

2.1. Basic set-up.
2.1.1. Regular coverings. Let $G$ be any connected complex reductive Lie group, with $G^{\text{der}} := [G, G]$, and let $Z(G)$ (resp. $Z(G)_0$, $Z(G^{\text{der}})_0$) denote for its center (resp. the identity component, $Z(G^{\text{der}}) \cap Z(G)_0$). Then

$$G = G^{\text{der}} \times Z(G)_0, \quad Z(G) / Z(G^{\text{der}}) = Z(G)_0 / Z(G^{\text{der}})_0,$$

and

$$q : J_G = (J_G^{\text{der}} \times T^*Z(G)_0) / Z(G^{\text{der}})_0 \longrightarrow J_G^{\text{ad}} \times T^*(Z(G) / Z(G^{\text{der}}))$$

is a regular covering with covering group $Z(G^{\text{der}})$. By [Jin1, Section 3], $J_G^{\text{der}}$ can be partially compactified to be a Liouville (or Weinstein) sector, which is $Z(G^{\text{der}})$-equivariant. Then $J_G^{\text{der}} \times T^*Z(G)_0$ can be equipped with the product sector structure that is $Z(G^{\text{der}}) \times Z(G^{\text{der}})_0$-equivariant, and the quotient by the diagonal $Z(G^{\text{der}})_0$ gives a Liouville sector structure on a partial compactification of $J_G$.

Let $\Lambda_G$ denote for the Lagrangian skeleton of $J_G$ in the Liouville completion. Then we have

$$q|_{\Lambda_G} : \Lambda_G = (\Lambda_G^{\text{der}} \times Z(G)_0) / Z(G^{\text{der}})_0 \longrightarrow \tilde{\Lambda}_G^{\text{ad}} := \Lambda_G^{\text{ad}} \times (Z(G)_0 / Z(G^{\text{der}})_0)$$

a regular covering as well. Here and after, for notations regarding Lie groups, their Langlands dual, maximal tori and their different forms (e.g. adjoint, simply connected etc.), we follow the notations in [Jin1, Section 1.1].

2.1.2. Sector inclusions. Assume first that $G$ is semisimple. Using the same argument as in [Jin1, Subsection 5.4.1], we have natural sector inclusions, for each $S \subseteq \Pi$,

$$J_{LS} \hookrightarrow J,$$

compatible with $q$, which gives a covering of $\Lambda_G$ by $\Lambda_{LS}$. In more details, let us stratify $H^{sm} \cong \mathfrak{c}^{n-1}$ from loc. cit. as in Figure 1, with each open stratum $\mathcal{S}_S$ corresponding to a barycenter $c_S$ (cf. [Jin1, 3.2.1, 3.2.2] for more details), indexed by $S \subseteq \Pi$, and let $\mathcal{W}_S$ be an open neighborhood of $\mathcal{G}_{S^*} := \bigcup_{S^* \subseteq S} \mathcal{G}_{S^*}$ in $H^{sm}$. Recall the map $\pi|_{\mathcal{W}} : J_G \rightarrow \mathbb{R}_n^{n_1 \lambda^{1/\lambda}(b_0)}$ [Jin1, (3.2.8)] and the homogeneous function $\tilde{N} : \mathbb{R}_n^{n_1 \lambda^{1/\lambda}(b_0)} \rightarrow \mathbb{R}$ of weight $-\frac{1}{2}$ with respect to the Liouville flow (i.e. the contracting radial flow on $\mathbb{R}_n^{n_1 \lambda^{1/\lambda}(b_0)}$). Then by definition in loc. cit., $H^{sm} = \tilde{N}^{-1}(1)$, and $\pi|_{\mathcal{W}}^{-1}(H^{sm})$ is a contact hypersurface which is isomorphic to the product $\mathfrak{g} \times \mathbb{R}$ for a Liouville hypersurface $\mathfrak{g}$ that is a section of the Reeb flow, i.e. the isomorphism $\mathfrak{g} \times \mathbb{R} \rightarrow \pi|_{\mathcal{W}}^{-1}(H^{sm})$ sends $(x, t)$ to $\varphi(t)$ of the Reeb flow $\varphi(t)$ (cf. Proposition 3.9 in loc. cit. for more details). Now consider $\pi|_{\mathcal{W}} : \mathfrak{g} \rightarrow H^{sm} \cong \mathfrak{c}^{n-1}$. Let

$$C_S = \{b_{\lambda^\vee}^{1/\lambda^\vee}(b_0) = 0 \iff \beta \in S\}$$

be the coordinate plane (as defined in [Jin1, (3.2.3)])). Following the notation in [GPS1], for a Liouville sector $\mathcal{X}$, let $\tilde{X}$ denote for its Liouville completion.

**Lemma 2.1.** Let $G$ be semisimple. We have $\tilde{\mathfrak{g}}_S := (\pi|_{\mathcal{W}})^{-1}(\mathcal{W}_S)$ a Weinstein subdomain of $\mathfrak{g}$ whose Liouville completion is (Liouville) homotopic to $\tilde{J}_{\mathfrak{g}^{\text{der}}} \times T^*Z(\mathcal{L}_S)_{0,\text{cpt}} \times T^*\Omega_{S^\perp}$ for the contractible open $\Omega_{S^\perp} := \mathcal{S}_S \cap C_S$ in $H^{sm} \cap C_S$. 
Proof. First, by construction, \((\pi|_\mathfrak{g})^{-1}\mathcal{S}_S\) is isomorphic to \((J_{L^\text{der}})_{|\beta}\mathcal{S}_S\times T^*\Omega_S/\lambda_{\beta}\mathcal{S})_\text{cpt}\times T^*\Omega_S/\lambda_{\beta}\mathcal{S}\) as a symplectic manifold, where \(\|\beta\|=\sum_{\beta\in S}\|b\|_{\beta}\leq \delta\). Using the Liouville completion of \(\mathcal{J}_{L^\text{der}}\) in [Jin1, Section 3.2.3], denoted by \(\mathcal{J}_{L^\text{der}}\) as above, we can equip \(\mathfrak{g}' := (\pi|_\mathfrak{g})^{-1}\mathcal{S}_S\) with a Liouville domain structure (which amounts to modifying the restriction of the Liouville 1-form from \(\mathfrak{g}\) to it) such that the Liouville completion is isomorphic to \(\mathcal{J}_{L^\text{der}}\times T^*\Omega_S/\lambda_{\beta}\mathcal{S}\times T^*\Omega_S/\lambda_{\beta}\mathcal{S}\).

Second, consider the product hypersurface in \(\mathbb{R}^{n}_{\|\beta\|_{\beta}(\mathfrak{g})}\) defined by \(\mathcal{S} := (H^{sm}\cap C_S)\times \mathbb{R}_{\geq 0}\), where \(\mathbb{R}_{\geq 0}\) is (the nonnegative part of) the orthogonal complement to \(C_S\). Let \(\mathcal{S}^0_\mathcal{S} \subset \mathcal{S}_S\) be the open part \((H^{sm}\cap C_S)\times \mathbb{R}_{\geq 0}\), i.e. the part that does not intersect other coordinate planes except for the interior of \(C_S\). Let \(\mathfrak{U}_S\) be the preimage of \(\mathfrak{U}_S\) in \(\mathcal{S}^0_\mathcal{S}\) under the radial projection. Let \(\mathfrak{g}_S\) be the Hamiltonian reduction given by \(\pi\|\beta\|_{\beta}(\mathfrak{g})/(\text{char. foliations})\), which can be identified with a smooth (incomplete) symplectic hypersurface in \(\pi\|\beta\|_{\beta}(\mathfrak{g})\). Since both \(\mathfrak{U}_S\) and \(\mathfrak{g}_S\) are transverse to the radial vector field, the Liouville flow gives a diffeomorphism \(\varphi : \mathfrak{g}_S \sim \mathfrak{g}_S\), such that \(\varphi^*(\vartheta|_\mathfrak{g}_S) = f\cdot\vartheta|_\mathfrak{g}_S\) for some \(f > 0\) on \(\mathfrak{g}_S\), where \(\vartheta\) is the standard Liouville 1-form on \(J_G\). In particular, we have \(\omega_s = (1-s)\omega|_\mathfrak{g}_S + s\cdot\varphi^*(\omega|_\mathfrak{g}_S), s \in [0, 1]\) an isotopy of symplectic forms on \(\mathfrak{g}_S\).

Now identify \(\pi\|\beta\|_{\beta}(\mathcal{S}^0_\mathcal{S})\) with \(\mathcal{J}_{L^\text{der}}\times T^*\Omega_S/\lambda_{\beta}\mathcal{S}\times T^*\Omega_S/\lambda_{\beta}\mathcal{S}\), and equip it with the Liouville manifold structure(s) \(\mathcal{J}_{L^\text{der}}\times T^*\Omega_S/\lambda_{\beta}\mathcal{S}\times T^*\Omega_S/\lambda_{\beta}\mathcal{S}\). By appropriate choices of such structures (up to Liouville homotopies), we can make the Hamiltonian reduction \(\mathfrak{g}_S\) into a Liouville domain, whose Liouville completion is \(\mathcal{J}_{L^\text{der}}\times T^*\Omega_S/\lambda_{\beta}\mathcal{S}\times T^*\Omega_S/\lambda_{\beta}\mathcal{S}\).

Then there is an obvious isotopy between the pullback Liouville domain structure on \(\mathfrak{g}_S\) through \(\varphi\) and that on \(\mathfrak{g}'_S\). This finishes the proof of the lemma.

\[
\square
\]
It is clear from the proof of Lemma 2.1 that we can equip \( \overline{\mathcal{C}}_S := (\pi|_S)^{-1}(\mathcal{E}_C|_S) \) with a Weinstein sector structure that is isotopic to \( \overline{\mathcal{C}}_{S_{\text{der}}} \times T^*\mathcal{Z}_S \). Hence, we have the following:

**Proposition 2.2.** There is a natural sector inclusion \( \overline{\mathcal{C}}_{S_{\text{der}}} \hookrightarrow \overline{\mathcal{C}}_S \), for any \( S \subseteq \Pi \).

**Proof.** Using the identification \( \overline{\mathcal{C}}_S - \mathcal{B}_1 \cong \mathcal{F} \times \mathbb{C}_{-R_2 \leq 0} \) (cf. [Jin1, Subsection 3.2.3]), we have \( \mathcal{F} \times \mathbb{C}_{-R_2 \leq 0} \) a subsector of \( \overline{\mathcal{C}}_S \), which is Liouville isotopic to \( \mathcal{B}_1 \).

For a reductive group \( G \), we have the regular covering

\[
\overline{\mathcal{C}}_S = J_{G_{\text{der}}} \times T^*\mathcal{Z}_S \to J_{G/\mathcal{Z}_S} \cong J_{G_{\text{ad}}} \times T^*(\mathcal{Z}_S).
\]

Then for any \( S \subset \Pi \), the sector inclusion \( (2.1.3) \) for \( G_{\text{ad}} \) induces sector inclusion \( \overline{\mathcal{C}}_{S_{\text{der}}} \hookrightarrow \overline{\mathcal{C}}_{S_{\text{ad}}} \). Taking the preimages in \( \overline{\mathcal{C}}_{S} \), we get the sector inclusion \( J_{S_{\text{der}}} \hookrightarrow J_{S_{\text{ad}}} \). It is clear from the proof of Lemma 2.1 that for any sequence \( S_1 \subset S_2 \subset \cdots \subset S_k \subset \Pi \), we have compatible sector inclusions \( J_{S_1} \subset J_{S_2} \subset \cdots \subset J_{S_k} \).

### 2.2. Proof of HMS using microlocal sheaf categories.

By [Jin1, Lemma 5.28], we have

\[
\pi_1(\tilde{\Lambda}_{G_{\text{ad}}}) = \pi_1(\tilde{\Lambda}_{G_{\text{ad}}}) \times \pi_1(\mathcal{Z}_S/\mathcal{Z}_S) \cong \pi_1(\mathcal{Z}_S/\mathcal{Z}_S).
\]

The covering \( q|_{\Lambda_{G_{\text{ad}}}} \) corresponds to

\[
(2.2.1) \quad \pi_1(\tilde{\Lambda}_{G_{\text{ad}}}) \times \pi_1(\mathcal{Z}_S/\mathcal{Z}_S) \cong \mathcal{Z}_S/\mathcal{Z}_S \to \mathcal{Z}_S/\mathcal{Z}_S,
\]

which is the multiplication of the two natural projections from the two factors on the LHS. Let \( L_{\tilde{\Lambda}_{G_{\text{ad}}}} \) be the local system \( q|_{\Lambda_{G_{\text{ad}}}} \) on \( \tilde{\Lambda}_{G_{\text{ad}}} \), which is corresponding to the representation of \( \pi_1(\tilde{\Lambda}_{G_{\text{ad}}}) \) on the space of functions \( \mathbb{C}[\mathcal{Z}_S/\mathcal{Z}_S] \) through \( (2.2.1) \). Let

\[
T_\mathcal{B} = T_{\mathcal{B}_{\text{ad}}} \times (\mathcal{Z}_S/\mathcal{Z}_S).
\]

In the following, for a Lagrangian skeleton \( \Lambda \) of a Weinstein sector \( X \), let \( \mu_{\text{Shv}_A} \) be the sheaf of microlocal categories (over \( k = \mathbb{C} \)) on \( \Lambda \), in the sense of [NaSh]. The definition of \( \mu_{\text{Shv}_A} \) requires the same data for defining the wrapped Fukaya category, and they are equivalent for a fixed choice of data (as remarked in loc. cit.). For \( X = \overline{\mathcal{C}}_S \), \( \mu_{\text{Shv}_{\Lambda_{G_{\text{ad}}}}} \) can be defined over \( \mathbb{Z} \), for which the data amount to a trivialization of \( \overline{\mathcal{C}}_S \to B^2\mathbb{Z} \times B^3(\mathbb{Z}/2\mathbb{Z}) \). Since \( J_G \) is hyperKahler, there is a canonical trivialization of \( \overline{\mathcal{C}}_S \to B^2\mathbb{Z} \) (cf. [Jin1, beginning of Section 4]). The map \( \overline{\mathcal{C}}_S \to B^3(\mathbb{Z}/2\mathbb{Z}) \) is trivial (this is always the case), and different choices of trivializations give equivalent categories.

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1 More precisely, one should modify \( \mathbb{C}_{-R \leq 0} \) to be \( \mathcal{P} \) as in [Jin1, (5.4.4)].

2 More explicitly and more generally, on the microlocal sheaf side, the data amount to a trivialization of the composition \( X \to BU \to B^3\text{Pic}(k) \), where the first map is the classifying map for the stable \( U \)-bundle on \( X \) (from the symplectic structure of \( X \)) and the second map factors through the delooping of the (complex) J-homomorphism, for the coefficient commutative ring (more generally \( E_3 \)-ring spectrum) \( k \), cf. [Gui, Jin2] (the former for discrete rings) and loc. cit. for the proof of this. The case for \( k \) a discrete ring is much simpler.
Consider the Cartesian diagram

$$\mu \text{Shv}_{\Lambda_G}(A_G) \xrightarrow{q_* \simeq q^*_T} \mu \text{Shv}_{\tilde{\Lambda}_{G,ad}}(\tilde{\Lambda}_{G,ad}) \xrightarrow{\sim} \text{IndCoh}((T_\flat)^\vee \sslash W)$$

satisfying natural commutativity and identities:

$$\text{res} \circ q_* \simeq (q_T)^* \circ \text{res} : \mu \text{Shv}_{\Lambda_G}(\tilde{\Lambda}_G) \to \text{Loc}(T_\flat),$$

$$q_* \circ \text{co-res} \simeq \text{co-res} \circ (q_T)_* : \text{Loc}(T_\flat) \to \mu \text{Shv}_{\Lambda_G}(\tilde{\Lambda}_G),$$

$$(q_T)_* \circ \text{res} \simeq \text{res} \circ q_* : \mu \text{Shv}_{\Lambda_G}(A_G) \to \text{Loc}(T_\flat)$$

$$q^* \circ \text{co-res} \simeq \text{co-res} \circ q^*_T : \text{Loc}(T_\flat) \to \mu \text{Shv}_{\Lambda_G}(A_G)$$

$$q_*q^* \simeq (-) \otimes \mathcal{L}_{\tilde{\Lambda}_{G,ad}}, \quad q^*q_* \simeq (-) \otimes \mathbb{C}[\mathbb{Z}(G^\text{der})].$$

**Proof.** The top right equivalence follows from the main theorem in [Jin1] and the Kunneth formula for microlocal sheaves or wrapped Fukaya category (cf. [GPS2]). The bottom two equivalences are well known. The first and the third identification of functors (involving $q_*$) are obvious, and the second and the fourth follow from taking left adjoint of theformer. The last claims on the two compositions of $q^*$ and $q_*$ are obvious.

**Theorem 2.4.** For any connected complex reductive Lie group $G$, we have

$$\mu \text{Shv}_{\Lambda_G}(A_G) \simeq \text{IndCoh}((T/\mathbb{Z}(G^\text{der}))^\vee \sslash W)^{\mathbb{Z}(G^\text{der})^*},$$

where $\mathbb{Z}(G^\text{der})^*$ is the Pontryagin dual of $\mathbb{Z}(G^\text{der})$. If $\mathbb{Z}(G)$ is connected, then $\mathbb{Z}(G^\text{der})^*$ acts freely on $(T/\mathbb{Z}(G^\text{der}))^\vee \sslash W$, so we have

$$\mu \text{Shv}_{\Lambda_G}(A_G) \simeq \text{IndCoh}(T^\vee \sslash W).$$

**Proof.** By (2.2.2) and the Barr–Beck–Lurie theorem [Lur, Theorem 4.7.4.5], we have

$$\mu \text{Shv}_{\Lambda_G}(A_G) \simeq q_*q^* - \text{Mod}(\mu \text{Shv}_{\tilde{\Lambda}_{G,ad}}(\tilde{\Lambda}_{G,ad}))$$

$$\simeq \text{res} \circ q_*q^* - \text{Mod}(\text{Loc}(T_\flat))$$

$$\simeq (q_T)_* \circ \text{res} \circ \text{co-res} \circ q^*_T - \text{Mod}(\text{Loc}(T_\flat))$$

$$\simeq (\text{co-res} \circ \text{res}) \circ (q_T)_*q^*_T - \text{Mod}(\text{Loc}(T_\flat)).$$

Consider the Cartesian diagram

$$\begin{array}{ccc}
T^\vee_\flat & \xrightarrow{\pi_u} & T^\vee_\flat/\mathbb{Z}(G^\text{der})^* \simeq T^\vee \\
\downarrow{p_T} & & \downarrow{p_T} \\
T^\vee_\flat \sslash W & \xrightarrow{\pi_d} & (T^\vee_\flat \sslash W)/\mathbb{Z}(G^\text{der})^* \\
\end{array}$$

3 To simplify notations, we denote $q|_{\Lambda_G}$ by $q$ as well.
Then we have
\[(\text{co-res} \circ \text{res}) \circ (q_T)_*, q_T^* \simeq (\pi_d p_\ell)^* (\pi_d p_\ell)_*: \text{IndCoh}(T^\vee_0) \to \text{IndCoh}(T_0^\vee).\]
Therefore,
\[
\mu_{\text{Shv}_{\mathcal{A}_G}}(\Lambda_G) \simeq (\pi_d p_\ell)^* (\pi_d p_\ell)_* - \text{Mod}(\text{IndCoh}(T^\vee_0)) \\
\simeq \text{IndCoh}(T^\vee_0/\mathbb{Z}(G^\text{der})^*).
\]
Since
\[
T^\vee_0/\mathbb{Z} = (T^\vee_0/\mathbb{Z})/\mathbb{Z}(G^\text{der})^*,
\]
if \(\mathbb{Z}(G)\) is connected, then \(\mathbb{Z}(G^\text{der})^*\) acts freely on it.

We remark that throughout the paper, we can replace all IndCoh by QCoh, because they are only taken on smooth (Deligne-Mumford) stacks.

2.3. Proof of HMS using wrapped Fukaya categories. First, we assume that \(G\) is semisimple. Choose a wrapping Hamiltonian \(H\) on \(J_G\) as in Section [Jin1, Section 4.1], then \(q^*H\) is a well defined \(\mathbb{Z}(G)\)-invariant wrapping Hamiltonian. For any cylindrical Lagrangian \(L \subset J_G\), we have
\[(2.3.1) \quad q^{-1} \varphi^t_H(L) = \varphi^{t*}_{q^*H} q^{-1}(L),\]
and \(\varphi^t_H(L), t \in \mathbb{R}_{\geq 0}\) is cofinal in the wrapping category \((L \to -)^+\) if and only if (every connected component of) \(\varphi_{q^*H}^{t*} q^{-1}(L)\) is cofinal in the the wrapping category \((q^{-1}(L) \to -)^+\).

Lemma 2.5. The quotient map \(q\) (2.1.1) induces an adjoint pair of functors (with \(F^L\) viewed as the left adjoint)\(^4\) on wrapped Fukaya categories:
\[(2.3.2) \quad \mathcal{W}(J_G) \xrightarrow{F_R} \mathcal{W}(J_G^\text{ad}) \xleftarrow{F^L} \mathcal{W}(J_G^\text{ad}).\]

Proof. First, the functor \(F^L\) can be defined as follows. Since \(q\) is a regular covering map, for any cylindrical Lagrangian \(L_1, L_2 \subset \mathcal{W}(J_G^\text{ad})\), define \(F^L\) as an \(A_\infty\)-functor following the notations in [Sei, Chapter I. 1 (1b)]
\[(F^L)^1: CF_{J_G^\text{ad}}(\varphi^t_H(L_1), L_2) \to CF_{J_G}(\varphi^{t*}_{q^*H} q^{-1}(L_1)), q^{-1}(L_2))\]
\[(2.3.3) \quad x \mapsto \sum_{y \in q^{-1}(x)} y\]
for every generator \(x \in \varphi^t_H(L_1) \cap L_2, t \geq 0\). Let \((F^L)^d = 0, d > 1\). This is clearly compatible with continuation maps. For any collection of cylindrical Lagrangians \(L_1, \ldots, L_k \subset\)
\[\mathcal{W}(J_G)\]
\(^4\)We use the same convention of notation as in [Jin1] in that \(\mathcal{W}(J_G)\) means \(\mathcal{W}(J_G)\).
and \( t_1 \geq t_2 \geq \cdots \geq t_{k-1} \geq 0 \), we have a canonical commutative diagram

\[
\begin{array}{ccc}
CF(L_{k-1}^{t_{k-1}}, L_k) \otimes \cdots \otimes CF(L_1^{t_1}, L_2^{t_2}) & \xrightarrow{\mu^{k-1}} & CF(L_1^{t_1}, L_k) \\
\downarrow & & \downarrow \\
CF(q^{-1}(L_{k-1}^{t_{k-1}}), q^{-1}(L_k)) \otimes \cdots \otimes CF(q^{-1}(L_1^{t_1}), q^{-1}(L_2^{t_2})) & \xrightarrow{\mu^{k-1}} & CF(q^{-1}(L_1^{t_1}), q^{-1}(L_k))
\end{array}
\]

where we use the notation \( L_j^{t_j} \) for \( \varphi_H^j(L_j) \) and \( q^{-1}(L_j)^{t_j} \) for \( \varphi_{q,H}^j(q^{-1}(L_j)) \). This follows from the unique lifting property of a \( J \)-holomorphic disc with (the only) incoming vertex at \( y \in L_{t_1}^{k-1} \cap L_k \) to a \( q^* \eta \)-holomorphic\(^5\) disc with incoming vertex at each point in \( q^{-1}(y) \).

Second, we give a definition of \( F_R \) on the full subcategory of (connected and) simply connected Lagrangian branes \( L \subset J_G \) satisfying \( q(L) \) is embedded. Equivalently, this means \( q : L \to q(L) \) is an isomorphism. Such Lagrangians include the Kostant sections \( \Sigma_z, z \in \mathcal{Z}(G) \), which are generators of \( \mathcal{W}(J_G) \), so the definition of \( F_R \) extends to \( \mathcal{W}(J_G) \) in a unique way (up to equivalences). On the object level, \( F_R(L) = q(L) \).

\[
(F_R)^1 : CF_{J_G}(L_1^{t_1}, L_2) \to CF_{J_G_{ad}}(q(L_1^{t_1}), q(L_2^{t_2}))
\]

for every generator \( x \in L_1^{t_1} \cap L_2 \). Let \((F_R)^d = 0, d > 1 \). For any collection \( L_1, \cdots, L_k \subset J_G \) and \( t_1 \geq t_2 \geq \cdots \geq t_{k-1} \geq 0 \), we have an obvious commutative diagram

\[
\begin{array}{ccc}
CF(L_{k-1}^{t_{k-1}}, L_k) \otimes \cdots \otimes CF(L_1^{t_1}, L_2^{t_2}) & \xrightarrow{\mu^{k-1}} & CF(L_1^{t_1}, L_k) \\
\downarrow & & \downarrow \\
CF(q(L_{k-1}^{t_{k-1}}), q(L_k)) \otimes \cdots \otimes CF(q(L_1^{t_1}), q(L_2^{t_2})) & \xrightarrow{\mu^{k-1}} & CF(q(L_1^{t_1}), q(L_k))
\end{array}
\]

due to the isomorphism \( q : L \cong q(L) \).

Lastly, we verify the adjunction property about \( F^L, F_R \). The co-unit map on each generator \( \Sigma_{z_j} \in \mathcal{W}(J_G) \), which is the Kostant section corresponding to \( z_j \in \mathcal{Z}(G) \),

\[
F^L F_R(\Sigma_{z_j}) = \bigoplus_{z \in \mathcal{Z}(G)} \Sigma_z \to \Sigma_{z_j}
\]

is given by the projection \( \text{proj}_{z_j} \) to the \( z_j \)-component. For each \( x \in \Sigma_{z_j} \cap \Sigma_{z_k} \) corresponding to a generator of \( CF(\Sigma_{z_j}^t, \Sigma_{z_k}^t) \) (recall the cochain complex is concentrated in degree 0 for a sequence of \( t \to \infty \); cf. \cite[Proposition 4.5]{Jin1}), we have a strictly commutative diagram

\[
\begin{array}{ccc}
F^L F_R(\Sigma_{z_j}) = \bigoplus_{z \in \mathcal{Z}(G)} \Sigma_z^{\text{proj}_{z_j}} \to \Sigma_{z_j} \\
\downarrow \quad \sum_{u \in \mathcal{Z}(G)} x_{u}^{x+u} & & \downarrow \quad x_x^{x} \\
F^L F_R(\Sigma_{z_k}) \cong \bigoplus_{z \in \mathcal{Z}(G)} \Sigma_z^{\text{proj}_{z_k}} \to \Sigma_{z_k}
\end{array}
\]

\(^5\)Here \( q^* \eta \) is well defined for \( q \) is a local isomorphism.
where \( u \cdot x \) means the image of \( x \) under the center action by \( u \). Such data completely determine the co-unit map. The unit map on the generator \( \Sigma_I \in \mathcal{W}(J_{G_{ad}}) \)

\[
\Sigma_I \rightarrow F_R F^L(\Sigma_I) = \Sigma_I^{\oplus \mathcal{Z}(G)}
\]
is given by the diagonal embedding. For any \( x \in \Sigma_I' \cap \Sigma_I \) corresponding to a generator of \( CF(\Sigma_I', \Sigma_I) \), we have a strictly commutative diagram

\[
\begin{array}{ccc}
\Sigma_I & \xrightarrow{\Sigma_I} & F_R F^L(\Sigma_I) = \Sigma_I^{\oplus \mathcal{Z}(G)} \\
x & \downarrow & \downarrow \left((x^{\oplus \mathcal{Z}(G)}) \circ \sigma_x\right) \\
\Sigma_I & \xrightarrow{\Sigma_I} & F_R F^L(\Sigma_I) = \Sigma_I^{\oplus \mathcal{Z}(G)}
\end{array}
\]

where \( \sigma_x \) is the permutation on the indexed set \( \mathcal{Z}(G) \) given by multiplying the the inverse of the element in \( \mathcal{Z}(G) \) corresponding to \( x \in X_+(T_{ad})^+ \) (cf. the same proposition in loc. cit.), and \( x^{\oplus \mathcal{Z}(G)} \) means the morphism given by a \( \mathcal{Z}(G) \times \mathcal{Z}(G) \)-matrix with diagonal entries all equal to \( x \).

The identities

\[
(F_R \xrightarrow{\text{unit}} F_R) F_R F^L(\Sigma_I) \xrightarrow{F_R \circ \text{co-unit}} F_R \simeq id_{F_R}
\]

\[
(F^L \xrightarrow{\text{unit}} F^L) F^L F^L(\Sigma_I) \xrightarrow{(\text{co-unit}) \circ F^L} F^L \simeq id_{F^L}
\]
can be directly checked on the generators. We leave the details to the interested reader. \( \square \)

We can also easily deduce that

**Lemma 2.6.** There is another adjoint pair

\[
\mathcal{W}(J_{G_{ad}}) \xrightarrow{F^L} \xleftarrow{F_R} \mathcal{W}(J_G),
\]

where \( F_R \) serves as the left adjoint.

**Proof.** The co-unit map on \( \Sigma_I \in \mathcal{W}(J_{G_{ad}}) \)

\[
\Sigma_I^{\oplus \mathcal{Z}(G)} \simeq F_R F^L(\Sigma_I) \rightarrow \Sigma_I
\]
is given by \( (id_{\Sigma_I}, \cdots, id_{\Sigma_I}) \). For any \( x \in \Sigma_I' \cap \Sigma_I \) corresponding to a generator of \( CF(\Sigma_I', \Sigma_I) \), we have a strictly commutative diagram

\[
\begin{array}{ccc}
F_R F^L(\Sigma_I) & \equiv \Sigma_I^{\oplus \mathcal{Z}(G)} & \Sigma_I \\
(x^{\oplus \mathcal{Z}(G)}) \circ \sigma_x & \downarrow x \\
F_R F^L(\Sigma_I) & \equiv \Sigma_I^{\oplus \mathcal{Z}(G)} & \Sigma_I
\end{array}
\]

where \( \sigma_x \) and \( x^{\oplus \mathcal{Z}(G)} \) are as in the proof of Lemma 2.5.
The unit map on $\Sigma_{z_j} \in W(J_G)$

$$\Sigma_{z_j} \longrightarrow F^LF_R(\Sigma_{z_j}) \cong \bigoplus_{z \in \mathbb{Z}(G)} \Sigma_z$$

is the embedding into the component of $\Sigma_{z_j}$. For any generator $x \in CF(\Sigma_{z_j}, \Sigma_{z_k})$ indexed by an element in $X_*(T_{ad})^+$, we have a strictly commutative diagram

$$\begin{array}{ccc}
\Sigma_{z_j} & \xrightarrow{F^LF_R} & \Sigma_{z_j} \\
x & \downarrow & \sum_{u \in \mathbb{Z}(G)} u \cdot x \\
\Sigma_{z_k} & \xrightarrow{F^LF_R} & \Sigma_{z_k}
\end{array}$$

The identities

$$(F^L)^{(\text{unit})} \circ F^L \cong F^L \circ (F^L)^{(\text{co-unit})} \cong id_{F^L}$$

$$(F_R \circ \text{unit}) \cong (\text{co-unit}) \circ F_R \cong id_{F_R}$$

can be directly checked on the generators. We leave the details to the interested reader. \(\square\)

Consider the diagram

$$\begin{array}{ccc}
W(J_G) & \xleftarrow{res_{\text{co-res}}} & W(B_{w_0}) \cong W(T^*T) \\
& \xrightarrow{F^L} & \ \\
& \xrightarrow{F_R} & \ \\
W(J_{G_{ad}}) & \xleftarrow{res_{\text{co-res}_{ad}}} & W(B_{w_0}^{ad}) \cong W(T^*T_{ad})
\end{array}$$

(2.3.6)

where $F^L_{w_0}$ and $F_{R,w_0}$ are defined in the same way as $F^L$ and $F_R$ (here $B_{w_0}$ and $B_{w_0}^{ad}$ are the open Bruhat “cells” in $J_G$ and $J_{G_{ad}}$, respectively, viewed as subsectors).

**Lemma 2.7.** We have canonical isomorphisms of functors

(i) $$(2.3.7) \quad \text{co-res} \circ F^L_{w_0} \cong F^L \circ \text{co-res}_{\text{ad}}$$

(ii) $$(2.3.8) \quad F_{R,w_0} \circ \text{res} \cong \text{res}_{\text{ad}} \circ F_R.$$

(iii) $$(2.3.9) \quad F_R \circ \text{co-res} \cong \text{co-res}_{\text{ad}} \circ F_{R,w_0}$$

(iv) $$(2.3.10) \quad \text{res} \circ F^L \cong F^L_{w_0} \circ \text{res}_{\text{ad}}.$$
Proof. (i) Here the first one (2.3.7) follows from (1) the compatibility of cofinality of wrapping in \( J_G \) and \( J_{G_{ad}} \) through \( q^{-1} \); (2) the definition of \( co-res \) and \( co-res_{ad} \) as “wrapping more”. To be more precise, following the notations in [GPS1, Section 3.5, 3.6], let \( \mathcal{J} = \{ T_I T_{ad} \} \) and \( \mathcal{J}' = \{ \Sigma_I, T_I T_{ad} \} \) in \( J_{G_{ad}} \). Let \( \mathcal{C}' \) (resp. \( \mathcal{C}'' \)) be the \( A_{\infty} \)-category of Lagrangians for \( \mathcal{B}_{w_0} \) (resp. \( J_{G_{ad}} \)) associated to \( \mathcal{J} \) (resp. \( \mathcal{J}' \)). Then \( \mathcal{C}' \hookrightarrow \mathcal{C}'' \) is an inclusion of a full subcategory. Let \( q^{-1}(\mathcal{C}') \) (resp. \( q^{-1}(\mathcal{C}'') \)) be the \( A_{\infty} \)-category from taking the inverse image of every element (together with each of their connected components) in \( \mathcal{C}' \) (resp. \( \mathcal{C}'' \)) through \( q \). Let \( C \) (resp. \( C'' \)) be the set of all continuation elements in \( \mathcal{C}' \) (resp. \( \mathcal{C}'' \)), then \( q^{-1}(C) \) and \( q^{-1}(C'') \) give the set of all continuation elements in \( q^{-1}(\mathcal{C}') \) and \( q^{-1}(\mathcal{C}'') \), respectively. There are the natural inclusions \( C \hookrightarrow C' \) and \( q^{-1}(C) \hookrightarrow q^{-1}(C'') \). The commutative diagram of \( A_{\infty} \)-categories

\[
\begin{array}{ccc}
q^{-1}(\mathcal{C}') & \longrightarrow & q^{-1}(\mathcal{C}) \\
F_L \downarrow & & \downarrow F_{L_0} = F_L|_{\mathcal{C}'} \\
\mathcal{C}' & \longrightarrow & \mathcal{C}
\end{array}
\]

induces the commutative diagram on localizations

\[
\begin{array}{ccc}
q^{-1}(\mathcal{C}'(q^{-1}(C')))^{-1} & \longleftarrow & q^{-1}(\mathcal{C})(q^{-1}(C))^{-1} \\
F_L \downarrow & & \downarrow F_{L_0} \\
\mathcal{C}'(C')^{-1} & \longleftarrow & \mathcal{C}(C^{-1})
\end{array}
\]

which gives (2.3.7).

The second isomorphism of functors (2.3.8) is from taking the right adjoint on both sides of the first one (2.3.7).

(ii) From (2.3.7) and adjunction, we get a morphism of functors

\[
\begin{align*}
\text{co-res}_{ad} \circ F_{R,w_0} & \longrightarrow F_R \circ F_L \circ \text{co-res}_{ad} \circ F_{R,w_0} \\
& \simeq F_R \circ \text{co-res} \circ F_{L_0} \circ F_{R,w_0} \\
& \longrightarrow F_R \circ \text{co-res},
\end{align*}
\]

where the first map is from the unit map for \((F_L, F_R)\) and the last map is from the co-unit map for \((F_{L_0}, F_{R,w_0})\). To confirm (2.3.9), we just need to show that (2.3.11) on the generator \( T_I T \in \mathcal{W}(\mathcal{B}_{w_0}) \) is an isomorphism. This is straightforward, and we leave the details for the interested reader.

The last isomorphism of functors (2.3.10) follows from taking the right adjoint on both sides of (2.3.9), using Lemma 2.6.

\[\Box\]

**Theorem 2.8.** For any semisimple complex Lie group \( G \), we have an equivalence of categories

\[
\mathcal{W}(J_G) \simeq \text{Coh}(T_{sc}^{\vee} \parallel W)^{2(G)^*}.
\]

**Proof.** We use the Barr–Beck–Lurie theorem [Lur, Theorem 4.7.4.5]. For this, we take the cocompletion of each wrapped Fukaya category \( \mathcal{W}(M) \), and denote them by \( \text{Ind}\mathcal{W}(M) \) (using standard notation). The functors \( F_L, F_R \) (resp. \( F_{L_0}, F_{R,w_0} \)) between \( \mathcal{W}(J_G) \) and
\(\mathcal{W}(J_{G_{\mathrm{ad}}})\) (resp. \(\mathcal{W}(\mathcal{B}_{w_0})\) and \(\mathcal{W}(\mathcal{B}_{w_0,\mathrm{ad}})\)) uniquely extend, and Lemma 2.5, Lemma 2.6 and Lemma 2.7 remain unchanged after the extension.

First, we check that \(F_R\) is conservative. Suppose an object \(L \in \text{Ind}\mathcal{W}(J_G)\) is sent to the zero object in \(\text{Ind}\mathcal{W}(J_{G_{\mathrm{ad}}})\) through \(F_R\), then by adjunction
\[
0 \simeq \text{Hom}_{\text{Ind}\mathcal{W}(J_{G_{\mathrm{ad}}})}(\Sigma_I, F_R(L)) \simeq \text{Hom}_{\text{Ind}\mathcal{W}(J_G)}(F^L(\Sigma_I), L)
\]
so \(L \simeq 0\). Similarly, one gets that \(F_{R,w_0}, \text{res}_{ad}\) and \(\text{res}\) are also conservative. Second, by definition, all functors mentioned above preserve filtered colimits, so in particular preserve geometric realizations.

Using Lemma 2.7, we get a commuting pair of monads \(\text{res}_{ad}\text{-co-res}_{ad}\) and \(F_{R,w_0}F^L_{w_0}\) on \(\mathcal{W}(\mathcal{B}_{w_0})\). Now we apply the Barr–Beck–Lurie theorem and get an equivalence
\[
(\text{2.3.13}) \quad \text{res}_{ad} \circ F_R : \text{Ind}\mathcal{W}(J_G) \xrightarrow{\sim} (\text{res}_{ad}F_RF^L\text{-co-res}_{ad}) - \text{Mod}(\text{Ind}\mathcal{W}(\mathcal{B}_{w_0}))
\]
where \(\text{res}_{ad}F_RF^L\text{-co-res}_{ad} \simeq F_{R,w_0}F^L_{w_0}\text{res}_{ad}\text{-co-res}_{ad}\) are isomorphic monads on \(\text{Ind}\mathcal{W}(\mathcal{B}_{w_0})\).

By [Jin1, Theorem 5.1] and its proof, the monad \(\text{res}_{ad}\text{-co-res}_{ad}\) is isomorphic to \(f^*f_*\) on \(\text{Ind}\mathcal{Coh}(T^{\vee}_{\text{sc}})\) for
\[
f : T^{\vee}_{\text{sc}} \longrightarrow T^{\vee}_{\text{sc}} \parallel W.
\]
It is clear that the monad \(F_{R,w_0}F^L_{w_0}\) is isomorphic to \(\varpi^*\varpi_*\) on \(\text{Ind}\mathcal{Coh}(T^{\vee}_{\text{sc}})\) for
\[
\varpi : T^{\vee}_{\text{sc}} \longrightarrow T^{\vee}_{\text{sc}} \parallel W = \text{Ind}\mathcal{Coh}(T^{\vee}_{\text{sc}} \parallel W)/Z(G)^{\star}.
\]
Therefore, the last line of (2.3.13) is equivalent to \(\text{Ind}\mathcal{Coh}((T^{\vee}_{\text{sc}} \parallel W)/Z(G)^{\star}) = \text{Ind}\mathcal{Coh}(T^{\vee}_{\text{sc}} \parallel W)/Z(G)^{\star}\) through the Cartesian diagram
\[
\begin{array}{ccc}
T^{\vee}_{\text{sc}} \parallel W & \xrightarrow{\varpi} & T^{\vee}_{\text{sc}}/Z(G)^{\star} \\
\downarrow f & & \downarrow \\
T^{\vee}_{\text{sc}}/Z(G)^{\star} & \longrightarrow & (T^{\vee}_{\text{sc}} \parallel W)/Z(G)^{\star}
\end{array}
\]
Lastly, taking compact objects, we get (2.3.12) as desired. \(\square\)

**Theorem 2.9.** For any reductive \(G = G^{\text{der}}_\times Z(G)^{\star}\), we have
\[
\mathcal{W}(J_G) \simeq \text{Coh}((T/Z(G)^{\text{der}})^{\vee}/W)/Z(G)^{\star}\text{Coh}(T^{\vee}/W).
\]
If \(Z(G)^{\star}\) is connected, then \(Z(G)^{\text{der}})\) acts freely on \((T/Z(G)^{\text{der}})^{\vee}/W\), and we have equivalently
\[
\mathcal{W}(J_G) \simeq \text{Coh}(T^{\vee} \parallel W).
\]

**Proof.** The proof goes the same way as that for Theorem 2.8, once we use the Kunneth formula for wrapped Fukaya categories as in the proof of Theorem 2.4. \(\square\)
2.4. Induction pattern. For any $S \subseteq \Pi$, Theorem 2.9 tells us that
\[ W(J_L) \simeq \text{Coh}(T/\mathcal{Z}(L_L^\text{der}))^\vee \parallel W_S)^{\mathcal{Z}(L_S^\text{der})^*}. \]
In the following, let $T_{p,S} := T/\mathcal{Z}(L_S^\text{der})$. The restriction and co-restriction functors for the inclusion of Liouville sectors $\mathcal{J}_{L_S} \subseteq \mathcal{J}_S$ (cf. Subsection 2.1.2) correspond to
\[ \text{Coh}(T^\vee \parallel W_{S})^{\mathcal{Z}(L_S^\text{der})^*} \simeq \text{Coh}(T^\vee \parallel W_S)^{\mathcal{Z}(L_S^\text{der})^*}, \]
where
\[ p_{\emptyset,S} : T^\vee = T_{p,S}^\vee/\mathcal{Z}(L_S^\text{der})^* \to (T^\vee \parallel W_S)/\mathcal{Z}(L_S^\text{der})^* \]
is the natural projection.

**Proposition 2.10.** Let $G$ be any reductive Lie group. For any $S \subseteq S' \subseteq \Pi$, we have the restriction and co-restriction functors between $W(J_{L_S'})$ and $W(J_L)$ given by
\[ W(J_{L_S'}) \simeq \text{Coh}(T^\vee \parallel W_{S'}^{\mathcal{Z}(L_S')^*})^{\mathcal{Z}(L_S')^*} \]
where
\[ p_{S,S'} : (T^\vee \parallel W_{S'})/\mathcal{Z}(L_S')^* \to (T^\vee \parallel W_S)/\mathcal{Z}(L_S^\text{der})^* \]
is the natural projection.

**Proof.** First, we assume $G$ is of adjoint type. For any $S \subseteq S'$, we have the diagram
\[ \begin{array}{ccc}
W(J_{L_S}) & \simeq & \text{Coh}(T^\vee) \\
\downarrow^{\text{res} \simeq p_{\emptyset,S}^*} & & \downarrow^{\text{co-res} \simeq (p_{\emptyset,S})_*} \\
W(J_{L_S'}) & \simeq & \text{Coh}(T^\vee \parallel W_{S'}) \\
\downarrow^{\text{res} \simeq p_{S,S'}^*} & & \downarrow^{\text{co-res} \simeq (p_{S,S'})_*} \\
W(J_L) & \simeq & \text{Coh}(T^\vee \parallel W_S) \\
\end{array} \]
with the co-restriction functors forming a commutative triangle\(^6\), and the restriction functors forming another commutative triangle from adjunction. It implies that $\text{co-res}_{S,S'}$ takes any skyscraper sheaf on $T^\vee \parallel W_S$ to a skyscraper sheaf on $T^\vee \parallel W_{S'}$. In particular, it can be identified with the pushforward functor for a morphism of schemes
\[ f_{S,S'} : T^\vee \parallel W_S \to T^\vee \parallel W_{S'}. \]

On the other hand, we have
\[ \text{Hom}_{\text{Coh}(T^\vee \parallel W_{S'})} (\mathcal{O}, \mathcal{O}) \xrightarrow{p_{\emptyset,S}^*} \text{Hom}_{\text{Coh}(T^\vee)} (\mathcal{O}, \mathcal{O}) \]
\(^6\)The triangle is in fact strictly commutative using the definition of wrapped Fukaya categories of (sub)sectors and co-restriction functors, cf. [GPS2].
factors through
\[ \text{Hom}_{\text{Coh}(T^\vee \sslash W_S)}(\mathcal{O}, \mathcal{O}) \xrightarrow{p^*_S} \text{Hom}_{\text{Coh}(T^\vee)}(\mathcal{O}, \mathcal{O}). \]
This implies that \( f_{S,S'} = p_{S,S'} \).

For general reductive \( G \), we can use the functoriality as in Lemma 2.7 to get the result. \( \square \)

By Proposition 2.2, we have a sector covering \( J_{L_S}, S \subset \Pi \), of \( J_G \) (note that the cover includes \( J_G \) itself). Taking the wrapped Fukaya category of each subsector and using co-restriction functors for sector inclusions, we get a well defined functor
\[
\left( \{ S \subset \Pi \}, \subset \right) \rightarrow 1\text{-Cat}_{\mathbb{C}}
\]
\[
S \mapsto \mathcal{W}(J_{L_S}),
\]
where 1-Cat (resp. 1-Cat^R) is the infinity-category of presentable stable \( \mathbb{C} \)-linear categories with right adjointable (resp. left adjointable) functors, i.e. those admit a right adjoint (resp. left adjoint). Let DMStk^{prop}_{\mathbb{C}} be the ordinary (2,1)-category of Deligne-Mumford stacks over \( \mathbb{C} \) with proper morphisms. Then we have the functor \( \text{Coh}_* : \text{DMStk}^{prop}_{\mathbb{C}} \rightarrow 1\text{-Cat}_{\mathbb{C}} \) that takes each stack \( X \) to \( \text{Coh}(X) \) and a proper morphism \( X \rightarrow Y \) to the pushforward functor on coherent sheaves.

**Corollary 2.11.** Under the canonical equivalences \( \mathcal{W}(J_{L_S}) \simeq \text{Coh}(T_{p,S}^\vee \sslash W_S)_{\mathcal{Z}(L^\text{der}_S)^*} \), the functor (2.4.4) is canonically equivalent to the composition of the functor
\[
\left( \{ S \subset \Pi \}, \subset \right) \rightarrow \text{DMStk}^{prop}_{\mathbb{C}}
\]
\[
S \mapsto (T_{p,S}^\vee \sslash W_S)_{\mathcal{Z}(L^\text{der}_S)^*},
\]
that sends each inclusion to the natural projection on stacks, and \( \text{Coh}_* \).

**Proof.** The canonical equivalences follow from

- the proof of [Jin1, Theorem 5.1] in which the morphism \( \hat{f} \) is uniquely determined;
- the maximal torus \( T \) is canonically identified with the abstract maximal torus, through the inclusion \( T \subset B \) to the fixed Borel \( B \).

The rest is an immediate consequence of Proposition 2.10. \( \square \)

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