Stability of a modified Jordan-Brans-Dicke theory in the dilatonic frame

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ABSTRACT: We investigate the Jordan-Brans-Dicke action in the cosmological scenario of FLRW spacetime with zero spatially curvature and with an extra scalar field minimally coupled to gravity as matter source. The field equations are studied with two ways. The method of group invariant transformations, i.e., symmetries of differential equations, applied in order to constraint the free functions of the theory and determine conservation laws for the gravitational field equations. The second method that we apply for the study of the evolution of the field equations is that of the stability analysis of equilibrium points. Particularly, we find solutions with $w_{\text{tot}} = -1$, and we study their stability by means of the Center Manifold Theorem. We show this solution is an attractor in the dilatonic frame but it is an intermediate accelerated solution $a \simeq e^{At^p}$, $p := \frac{2}{2+t} \frac{32}{37+6\omega_0} < p < \frac{2}{3}$, as $t \to \infty$, and not a de Sitter solution. The exponent $p$ is reduced, in a particular case, to the exponent already found for the Jordan’s and Einstein’s frames by A. Cid, G. Leon and Y. Leyva, JCAP 1602, no. 02, 027 (2016). We obtain some equilibrium points that represent stiff solutions. Additionally we find solutions that can be a phantom solution, a solution with $w_{\text{tot}} = -1$ or a quintessence solution. Other equilibrium points mimics a standard dark matter source ($0 < w_{\text{tot}} < 1$), radiation ($w_{\text{tot}} = \frac{1}{3}$), among other interesting features. For the dynamical system analysis we develop an extension of the method of $F$-devisers. The new approach relies upon two arbitrary functions $h(\lambda, s)$ and $F(s)$. The main advantage of this procedure is that it allows us to perform a phase-space analysis of the cosmological model, without the need for specifying the potentials, revealing the full capabilities of the model.

KEYWORDS: Modified Gravity, Jordan-Brans-Dicke, Dark Energy, Asymptotic Structure, Symmetries
1 Introduction

Various models have been proposed for the explanation of the results which followed from the detailed analysis of the recent cosmological data [1–5]. The observable late time acceleration have been attributed to the so-called cosmological fluid Dark energy. The nature of dark energy it is unknown and the theoretical approaches to the problem can be classified in two categories. In the first category in the context of General Relativity an “exotic” matter source is introduced which provides the late time acceleration of the universe [6–10]. On the other hand in the second category the expansion of the universe it is attribute to terms which follows from the modification of General Relativity (GR), see for instance [11–17] and references therein. In the latter theories the new terms which follow from the modification of the Einstein-Hilbert action provide a geometric explanation for the acceleration of the universe.

In the context of this work we are interested on the Brans-Dicke gravitational action in cosmological studies. Brans and Dicke in 1961 proposed a gravitational action which satisfies Mach’s principle [18]. In that theory a new degree of freedom is introduced which is attribute to a scalar field which is nonminimally coupled to gravity. The importance of that theory is that it is equivalent under conformal transformation with GR which includes minimally coupled scalar field. Furthermore, other higher-order theories can be written
in terms of Brans-Dicke field by using Lagrangian multipliers [19]. In the cosmological scenario of a spatially flat Friedmann-Lemaître-Robertson-Walker geometry we assume the existence of a second perfect fluid which is described by a scalar-field minimally coupled to gravity. In this consideration and in the Einstein frame the gravitational field equation is that of GR in the so-called \( \sigma \)-models. That is, two scalar fields with interactions in the kinetic and in the dynamical parts of the Lagrangian. In [20] was recently presented exact solutions in the context of multi-scalar field cosmologies. Two-scalar cosmology was discussed, with interesting results, in the seminal works [21, 22]. Integrable cosmological models with non-minimal coupling have been studied, e.g., in [23]. In [24] it was shown that sometimes it is more easy to prove the integrability of the model with non-minimal coupling then the corresponding model in the Einstein frame. Bianchi I model with non-minimal coupling has a general solution in the analytic form, but in the case of zero potential [25].

In this paper we propose a modified Brans-Dicke theory where the Brans-Dicke field \( \Phi \) is driven by a potential \( U(\Phi) \) and the matter content is modeled by a second scalar field \( \psi \) with potential \( W(\psi) \). The potentials are not specified from the starting point. So, in order to specify the unknown potentials, we first express the action in the dilatonic frame by introducing the dilaton field \( \varphi \) with potential \( V(\varphi) \). The potentials can be derived by applying the method of group invariant transformations. The existence of a symmetry vector is important since the latter can be used in order a invariant surface to be defined in the phase-space of the dynamical system. More details on the application of group invariant transformations in cosmological studies can be found in [27–30] and references therein. In the other hand one can consider the potentials to be free functions and then find the generic features of the dynamical system, under the assumption that the system can be written in closed form. In this regard, we propose a general method for the construction of the phase space that relies in the specification of two arbitrary functions \( F(s) \) and \( h(s, \lambda) \). The equilibrium points with \( s \) constant such that \( h \) is only a function of \( \lambda \) (depending on the choice of \( W \)), and with \( F \) identically zero, are easily found due to the problem can be reduced in one dimension. When \( F(s) \) is not trivial, we discuss a general classification that can be implemented straightforwardly, as for any of the specific choices of \( F \) for the scalar field potentials commonly used in the literature. The search of the equilibrium points with \( \lambda \neq 0 \), on the other hand, is not an easy task, and the success on it depends crucially on the choice of \( h(s, \lambda) \).

The main advantage of this procedure is that it allows us to perform a phase-space analysis of the cosmological model, without the need for specifying the potentials. This phase-space and stability examination let us to bypass the non-linearities and complications of the cosmological equations, which prevent complete analytical treatments by obtaining a qualitative description of the global dynamics of these scenarios, which is independent of the initial conditions and the specific evolution of the universe. Furthermore, in these asymptotic solutions we are able to calculate various observable quantities, such as the dark-energy and total equation-of-state parameters, the deceleration parameter, the various density parameters, etc. However, in order to remain general, we extend beyond the usual procedure [31–47]. As far as we know this methodology has not introduced yet in the literature, although it is inspired in the method of the \( F \)- devisers extensively used in the
relativistic setting in [45–50] and that has been formalized in [51–53].

For illustrating the advantages of the method we consider some specific forms of the potentials $V(\varphi)$ and $W(\psi)$ which leads to specific forms on the functions $F(s)$ and $h(s, \lambda)$. For the Brans-Dicke field $\Phi$ we consider a power-law potential where in terms of the field $\varphi$ has the exponential form $V(\varphi) = V_0 e^{l \varphi}$. As far as concerns the second scalar field we study the cases where the potential is (a) exponential and (b) power-law. Finally, we comment about general features of the equilibrium points of the dynamical system.

Comparing with the Jordan-Brans-Dicke theory introduced and studied in [26] in the Jordan’s and Einstein’s frames we have the following. In the Jordan frame the potentials of [26] are (we have renamed the original constants as $\lambda_U$ and $\lambda_W$):

$$U(\Phi) = U_0 \Phi^{2 - \lambda_U \sqrt{\omega_0 + \frac{3}{2}}}, \quad W(\psi) = W_0 e^{-\lambda_W \psi}.$$ 

Therefore, the fields of this theory in the dilatonic action will be the dilaton $\varphi$ with potential

$$V(\varphi) = U_0 e^{(1 - \lambda_U \sqrt{\omega_0 + \frac{3}{2}}) \varphi},$$

and a second scalar field $\psi$ with potential

$$W(\psi) = W_0 e^{-\lambda_W \psi}.$$ 

Hence, the model studied in [26] can be considered as an special case of the model studied in section 4.1 Case: $W(\psi) = W_0 e^{k \psi}$ and $V(\varphi) = V_0 e^{l \varphi}$ with $k = -\lambda_W, l = 1 - \lambda_U \sqrt{\omega_0 + \frac{3}{2}}, \omega_0 > -\frac{3}{2}$.

The paper is organized as follows. Our model is defined in Section 2. The point-like Lagrangian and some exact solutions by using group invariant transformations are presented in Section 3. In Section 4 we rewrite the field equations in dimensionless variables and we end up with a five first-order differential-algebraic system with two unknown functions which are related with the potentials of the two scalar fields. For some explicitly forms of the potentials, we study the evolution of the field equations by using dynamical systems tools. In particular we consider the cases where the Brans-Dicke scalar field is power law while the minimally coupled field has an exponential potential or a power law potential. The case: $W(\psi) = W_0 e^{k \psi}$ and $V(\varphi) = V_0 e^{l \varphi}$ is studied in Section 4.1, whereas, the case: $W(\psi) = W_0 \psi^k$ and $V(\varphi) = V_0 e^{l \varphi}$ is studied in Section 4.2. Going to the general set up, we find generic features of the dynamics without specifying the potentials in Section 5. This allows to find generic results that are independent of the model choice. The cosmological implications of the model at hand are discussed in Section 6. Finally our conclusions and discussions are given in Section 7.

## 2 Gravitational model

Let us consider the gravitational Action integral to be

$$S = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{\Phi R}{2} - \frac{\omega_0}{2\Phi} \nabla^\mu \Phi \nabla_\mu \Phi - U(\Phi) - \frac{1}{2} \nabla^\mu \psi \nabla_\mu \psi - W(\psi) \right\}, \quad (2.1)$$
where $\Phi$ is the Brans-Dicke field and $\psi$ represents a quintessence field. $U(\Phi)$ and $W(\psi)$ are the corresponding potentials for the scalar fields. For the sake of simplicity and without loss of generality we rescale the Brans-Dicke field $\Phi$ and the associated potential $U(\Phi)$ as,

$$\Phi = e^\varphi \quad \text{and} \quad U(\Phi) = e^\varphi V(\varphi). \quad (2.2)$$

Consequently, under a conformal transformation the action (2.1) is transformed into the dilatonic action:

$$S = \int d^4x \sqrt{-g} e^\varphi \left\{ \frac{R}{2} - \frac{\omega_0}{2} \nabla^\mu \varphi \nabla_\mu \varphi - V(\varphi) - \frac{1}{2} e^{-\varphi} \nabla^\mu \psi \nabla_\mu \psi - e^{-\varphi} W(\psi) \right\}. \quad (2.3)$$

The field equations associated to action (2.3) are given by:

$$G_{\mu\nu} = (1 + \omega_0) \left( \nabla_\mu \varphi \nabla_\nu \varphi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \varphi \nabla^\alpha \varphi \right) - g_{\mu\nu} \left( \frac{1}{2} \nabla_\alpha \varphi \nabla^\alpha \varphi + V(\varphi) \right) + \nabla_\mu \nabla_\nu \varphi - g_{\mu\nu} \nabla^2 \varphi + e^{-\varphi} T^{(\psi)}_{\mu\nu} \right. \quad (2.4a)$$

$$\nabla^2 \varphi + \nabla_\mu \varphi \nabla^\mu \varphi = \frac{2}{3 + 2\omega_0} (V'(\varphi) - V(\varphi)) + \frac{e^{-\varphi}}{3 + 2\omega_0} T^{(\psi)}, \quad (2.4b)$$

$$\nabla^2 \psi = W'(\psi), \quad (2.4c)$$

where $\nabla^2 \equiv \nabla^\mu \nabla_\mu$ and $T^{(\psi)} = -\nabla^\mu \psi \nabla_\mu \psi - 4W(\psi)$ is the trace of the energy-momentum tensor $T^{(\psi)}_{\mu\nu} = \nabla_\mu \psi \nabla_\nu \psi - g_{\mu\nu} \left( \frac{4}{3} \nabla^\alpha \psi \nabla_\alpha \psi + W(\psi) \right)$.

We assume that the geometry which describes the universe is that of spatially flat Friedmann-Lemaître-Robertson-Walker spacetime

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j. \quad (2.5)$$

For the latter line element and for the comoving observer $(u^a = \delta_i^a, \ u^a u_a = -1)$ we calculate the field equations to be

$$3H^2 = \frac{\omega_0}{2} \dot{\varphi}^2 - 3H \dot{\varphi} + V(\varphi) + e^{-\varphi} \left( \frac{1}{2} \dot{\psi}^2 + W(\psi) \right), \quad (2.6a)$$

$$\dot{H} = -\frac{1}{2} \omega_0 \dot{\varphi}^2 + 2H \dot{\varphi} + \frac{V'(\varphi) - V(\varphi)}{3 + 2\omega_0} - \frac{e^{-\varphi} (1 + \omega_0) \dot{\psi}^2}{3 + 2\omega_0} - \frac{2e^{-\varphi} W(\psi)}{3 + 2\omega_0}, \quad (2.6b)$$

$$\ddot{\varphi} + 3H \dot{\varphi} + \varphi^2 = 2 \frac{V(\varphi) - V'(\varphi)}{3 + 2\omega_0} - e^{-\varphi} \frac{\dot{\psi}^2 - 4W(\psi)}{3 + 2\omega_0}, \quad (2.6c)$$

$$\ddot{\psi} + 3H \dot{\psi} + W'(\psi) = 0. \quad (2.6d)$$

where (2.6a) is the modified first Friedmann’s equation, equation (2.6b) is the Raychaudhuri (acceleration) equation and equations (2.6c), (2.6d) are the “Klein-Gordon” equations in which the two scalar fields should satisfy.

In the following section we determine the point-like Lagrangian for the field equations as also we search for solutions by using the method of group invariant transformations.
3 Minisuperspace approach and exact solutions

From the Action integral (2.3) and for the FRW spacetime with line element

$$ds^2 = -N^2(t)dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right),$$

the following Lagrangian density can be defined by

$$\mathcal{L} = \frac{1}{N} \left(-3e^\varphi a\ddot{a}^2 - 3e^\varphi a^2\dot{a}\dot{\varphi} + \frac{\dot{\psi}_0}{2}a^3 e^{2\varphi}\dot{\psi}^2 + \frac{1}{2}a^3 \dot{\psi}^2\right) - Na^3(e^\varphi V(\varphi) + W(\psi)),$$

where the field equations (2.6a)-(2.6d) follow from the Euler-Lagrange with respect to the variables \{N, a, \varphi, \psi\}. Lagrangian (3.8) describes a singular system of second-order differential equations, because the determinant of the Hessian matrix is zero, i.e. \[\frac{\partial^2 \mathcal{L}}{\partial \dot{\varphi}^2}\] = 0. Specifically the field equations form a constraint dynamical system [54], with constraint equation \[\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = 0\] = 0.

Without loss of generality we can consider that \(N(t) = N(a(t), \varphi(t))\), where now Lagrangian (3.8) is autonomous and admits the symmetry vector field \(\partial_t\), where the corresponding conservation law is the Hamiltonian function \(\mathcal{H} = \text{const}\). However from the first modified Friedmann’s equation we have that \(\mathcal{H} = 0\).

We consider that \(N = \bar{N}(t) e^{-\varphi/2}\) and \(a = A e^{-\varphi/2}\), where now the line element (3.7) becomes

$$ds^2 = e^{-\varphi}(\bar{N}^2(t)dt^2 + A^2(t)(dx^2 + dy^2 + dz^2)),$$

while the Lagrangian of the field equations is written as follows

$$\mathcal{L} = \frac{1}{\bar{N}} \left[-3A\bar{\varphi}^2 + \frac{1}{2}A^3 \left(\Omega_0 \bar{\psi}^2 + e^{-\varphi}\dot{\psi}^2\right)\right] - \bar{N}A^3(e^{-\varphi}V(\varphi) + e^{-2\varphi}W(\psi)),$$

in which \(\Omega_0 = \frac{3+2\omega_0}{2}\).

Lagrangian (3.10) is nothing else that the cosmological model of two scalar fields minimally coupled in gravity but with interaction in the kinetic and dynamic terms. Specifically Lagrangian (3.10) describes the field equations for the action integral

$$S = \int d^4x \sqrt{-g} \left(\bar{R}(\bar{g}_{\mu\nu}) - \frac{\Omega_0}{2} \bar{g}^{\mu\nu} \varphi;\mu \varphi;\nu - \frac{1}{2}e^{-\varphi} \bar{g}^{\mu\nu} \psi;\mu \psi;\nu - e^{-\varphi}V(\varphi) - e^{-2\varphi}W(\psi)\right).$$

where \(\bar{g}_{\mu\nu} = e^\varphi g_{\mu\nu}\).

The last action belong to the action of the so-called nonlinear \(\sigma\)-models [55]. On the other hand the action integral (3.11) can be seen like that of complex scalar field where the norm of the complex plane is not defined by the unitary matrix but from a space of constant curvature \(E_B^A = \text{diag}(\Omega_0, e^{-\varphi})\), with Ricciscalar \(R_{(2)} = -\frac{A^2}{\Omega_0}\). Finally because of the constraint equation any solution of the dynamical system with Lagrangian (3.10) will be also a solution for the system (3.8) (for a discussion see [27]). Some exact solutions for cosmological models of the form of (3.11) can be found in [28, 56] and reference therein. In the following without loss of generality in (3.10) we select \(\bar{N} = 1\).
In order to specify the unknown potentials \( V(\varphi) \) and \( W(\psi) \) we apply the method of group invariant transformations. We find that for

\[
V(\varphi) = V_0 e^{(\beta-1)\varphi}, \quad W(\psi) = W_0 \psi^{2\beta}
\]  

(3.12)

Lagrangian (3.10) admits the Noether point symmetry vector

\[
X = (2-\beta) t \partial_t + \frac{2-\beta}{3} a \partial_a + 2 \partial_\varphi + \psi \partial_\psi,
\]  

(3.13)

where the corresponding conservation law is\(^1\)

\[
I_X = (\beta-2) a^2 \dot{a} + \Omega_0 a^3 \dot{\varphi} + \frac{1}{2} e^{-\varphi} a^3 \dot{\psi}.
\]  

(3.14)

Consider now that \( \beta = 2 \), and that the value of the conservation law is zero, that is, \( I_X = 0 \), then from (3.14) follows

\[
\dot{\varphi} = -\frac{1}{2\Omega_0} e^{-\varphi} \dot{\psi} \rightarrow e^{\varphi} = -\frac{1}{4\Omega_0} \psi^2 + c.
\]  

(3.15)

or

\[
\varphi = \ln \left(-\frac{1}{4\Omega_0} \psi^2 + c\right)
\]  

By replacing in the Hamiltonian function we have

\[
\mathcal{H} = -3AA^2 + \frac{1}{2} A^3 \left( \frac{1}{4\Omega_0} \left(1 - \frac{1}{4\Omega_0}\right) \psi^2 + c\right) \dot{\psi}^2 + A^3 \left( V_0 + W_0 \frac{\psi^4}{\left(-\frac{1}{4\Omega_0} \psi^2 + c\right)^2}\right) = 0
\]  

(3.16)

where in the limit \( c = 0 \), the field equations corresponds to that of GR with a cosmological constant and a stiff matter, the latter follows from the kinetic part of the scalar field \( \dot{\Psi} \), where

\[
d\Psi = \sqrt{\left(1 - \frac{1}{4\Omega_0}\right) \psi^2 + c}\sqrt{\left(-\frac{1}{4\Omega_0} \psi^2 + c\right)^2} d\psi.
\]  

(3.17)

For a nonzero constant \( c \), (3.16) corresponds to the first Friedmann’s equation of GR with a minimally coupled scalar field, where the general solution is given in \( [57] \). In the limit where \( \Omega_0 = -\frac{1}{4} \), i.e. \( \omega_0 = -\frac{5}{4} \), from (3.17) we have the closed-form expression \( \psi = \sqrt{c} \tanh \Psi \), where (3.16) becomes

\[
\mathcal{H} = -3AA^2 + \frac{1}{2} A^3 \dot{\psi}^2 + A^3 \sinh^4(\Psi) = 0
\]  

(3.18)

that is, of a quintessence field with the hyperbolic potential \( W(\Psi) = \sinh^4(\Psi) \).

In general, for \( \beta \neq 2 \) and from the symmetry vector (3.13) we define the Lagrange system

\[
\frac{dt}{(2-\beta) t} = \frac{da}{a} = \frac{d\varphi}{\psi} = \frac{d\psi}{\psi}
\]  

(3.19)

\(^1\)The constraint equation \( \frac{\partial L}{\partial \dot{\Psi}} = 0 \), have been applied.
from where we define the invariants \( u = At^{\frac{1}{3}}, \ v = e^{\varphi}t^{\frac{2}{3}}, \ w = \psi t^{\frac{1}{3}} \). Recall that a Noether symmetry is also a Lie point symmetry for the field equations.

The invariants can be used to reduce the order of the differential equations or to determine a special solution. Consider that the invariants are constants, i.e. \( (u,v,w) \rightarrow (A_0, e^{\varphi_0}, \psi_0) \), then we observe that

\[
A(t) = A_0 t^{-\frac{1}{3}}, \quad e^{\varphi} = e^{\varphi_0} t^{-\frac{2}{3}}, \quad \psi = \psi_0 t^{-\frac{1}{3}}
\]

(3.20)
solve the field equations for the gravitational field equations with Lagrangian (3.10) and \( \bar{N}(t) = 1 \), for the potentials (3.12) when the constants \( W_0, V_0, \Omega_0 \) and \( \beta \) are related as follows

\[
W_0 = \frac{2\beta - 5}{2\beta (\beta - 2)} e^{\varphi_0} (\psi_0)^{2(1-\beta)}, \quad V_0 = e^{(1-\beta)\varphi_0} \left( \frac{5 (\psi_0)^2 + 8e^{\varphi_0} \beta \omega_0}{2\beta (\beta - 2)^2} \right),
\]

(3.21)
and

\[
\Omega_0 = -\frac{e^{-\varphi_0}}{12} \left( 2e^{2\varphi_0} \beta^2 - 8e^{\varphi_0} \beta + 3\psi_0^2 - 8e^{\varphi_0} \right)
\]

(3.22)

Solution (3.20) is a special solution of the field equations in the Einstein frame. By going back now in the Jordan frame, where

\[
a(t) = \sigma_1 (\tau) e^{-\varphi(t)/2}, \quad e^{-\varphi/2} dt = d\tau
\]

(3.23)
we have \( t = \sigma_1 (\varphi_0, \beta) t^{\frac{\beta - 2}{5 - \beta}}, \ \beta \neq 1, 2 \), and \( t = \sigma_2 (\varphi_0) e^t \) for \( \beta = 1 \), hence for the scale-factor holds \( a(t) \simeq \tau^{\frac{3 - \beta}{5 - \beta}} \), \( \beta \neq 1, 2 \), and \( a(t) \simeq e^{\sigma_2 \tau} \). The latter is a de Sitter solution while the first one is a perfect fluid solution in which

\[
w_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = -\frac{3\beta - 7}{\beta - 5},
\]

where there exists acceleration, i.e. \( w_{\text{eff}} < -\frac{1}{3} \), for \( \beta \in (-\infty, 2) \cup (5, +\infty) \), while for \( \beta = \frac{13}{3} \), we have a radiation solution and for \( \beta = \frac{7}{3} \) the solution is that of a pressureless fluid.

We continue our analysis with the equilibrium point analysis for the gravitational field equations, but we keep now the potentials unspecified.

### 4 The dynamical system

In order to express the above equations as an autonomous closed dynamical system we define the normalized variables

\[
x = \frac{\dot{\varphi}}{\sqrt{6H}}, \quad y = \frac{\sqrt{V(\varphi)}}{\sqrt{3H}}, \quad z = \frac{e^{-\frac{2}{3} \psi}}{\sqrt{6H}},
\]

(4.1)
and the auxiliary variables

\[
s = -\frac{V'(\varphi)}{V(\varphi)}, \quad \lambda = -e^{\frac{2}{3} \psi} \frac{W'(\psi)}{W(\psi)}, \quad \Gamma_\varphi = \frac{V(\varphi)V''(\varphi)}{V'(\varphi)^2}, \quad \Gamma_\psi = \frac{W(\psi)W''(\psi)}{W'(\psi)^2}.
\]

(4.2)
which are related by

\[- \sqrt{6}x + x^2\omega_0 + y^2 + z^2 + \frac{e^{-\varphi}W(\psi)}{3H^2} = 1. \quad (4.3)\]

Since by definition \(\Gamma_{\varphi} := \frac{V'(\varphi)}{V(\varphi)}\) depends only on \(\varphi\) and simultaneously \(\varphi\) is an implicit function of \(s\) through \(s = -\frac{V'(\varphi)}{V(\varphi)}\), it follows \(\Gamma_{\varphi} = f(s)\). Furthermore, since by definition \(\lambda := \frac{2W'(\psi)}{W(\psi)}\), i.e., it depends on both \(\varphi\) and \(\psi\), thus, using the implicit relation between \(\varphi\) and \(s\) through \(s = -\frac{V'(\varphi)}{V(\varphi)}\) and between \(\psi\) and \(\Gamma_{\psi}\) through \(\Gamma_{\psi} = \frac{W'(\psi)}{W(\psi)}\), we obtain \(\lambda = g(s, \Gamma_{\psi})\). Assume that \(\lambda = g(s, \Gamma_{\psi})\) can be explicitly solved for \(\Gamma_{\psi}\), say \(\Gamma_{\psi} = h(s, \lambda)\).

Then, the evolution equations are

\[
x' = x \left[ \frac{3(s-1)y^2}{2\omega_0 + 3} + \frac{18}{2\omega_0 + 3} + \frac{6\omega_0 z^2}{2\omega_0 + 3} - 3 \right] + \frac{\sqrt{6}(s-1)y^2}{2\omega_0 + 3} \]

\[
+ \frac{3x^3\omega_0(2\omega_0 + 1)}{2\omega_0 + 3} - \frac{\sqrt{6}x^2(8\omega_0 + 3)}{2\omega_0 + 3} + \frac{2\sqrt{6}}{3\omega_0 + 3} \quad (4.4a)
\]

\[
y' = \frac{-\sqrt{6}xy(2(s + 4)\omega_0 + 3s)}{4\omega_0 + 6} + \frac{3(s-1)y^3}{2\omega_0 + 3} + \frac{3x^2y\omega_0(2\omega_0 + 1)}{2\omega_0 + 3} \]

\[
+ y \left[ \frac{6}{2\omega_0 + 3} + \frac{6\omega_0 z^2}{2\omega_0 + 3} \right], \quad (4.4b)
\]

\[
z' = \sqrt{\frac{3}{2}}\lambda + y^2 \left[ \frac{3(s-1)z}{2\omega_0 + 3} - \sqrt{\frac{3}{2}}\lambda \right] + x \left[ \frac{3\omega_0(2\omega_0 + 1)z}{2\omega_0 + 3} - \sqrt{\frac{3}{2}}\lambda \omega_0 \right] +
\]

\[
+ x \left[ \frac{3\lambda - \sqrt{6}(10\omega_0 + 3)z}{4\omega_0 + 6} \right] + \frac{6\omega_0 z^3}{2\omega_0 + 3} - \sqrt{\frac{3}{2}}\lambda z^2 + \left[ \frac{6}{2\omega_0 + 3} - 3 \right] z, \quad (4.4c)
\]

\[
\lambda' = \frac{\sqrt{3}}{2} \lambda [x - 2(h(s, \lambda) - 1)\lambda z], \quad (4.4d)
\]

\[
s' = -\sqrt{6}xF(s), \quad (4.4e)
\]

where \(F(s) := s^2(f(s) - 1)\).

| Potential | References | \(F(s)\) |
|-----------|------------|--------|
| \(V(\varphi) = V_0 e^{-k\varphi} + V_1\) | [58–60] | \(-s(s-k)\) |
| \(V(\varphi) = V_0 [e^{\alpha\varphi} + e^{\beta\varphi}]\) | [61–63] | \(-(s+\alpha)(s+\beta)\) |
| \(V(\varphi) = V_0 [\cosh(\xi\varphi) - 1]\) | [31, 45, 46, 52, 59, 64–69] | \(-\frac{1}{2}(s^2 - \xi^2)\) |
| \(V(\varphi) = V_0 \sinh^{-\alpha}(\beta\varphi)\) | [31, 46, 59, 65, 68–70] | \(\frac{\alpha^2}{\alpha} - \alpha\beta^2\) |

**Table 1:** The function \(F(s)\) for the most common quintessence potentials [51].
We have a dynamical system for the state vector \((x, y, z, \lambda, s)\) defined in the phase space
\[
\left\{(x, y, z, \lambda, s) : -\sqrt{6}x + x^2 \omega_0 + y^2 + z^2 \leq 1, \lambda \in \mathbb{R}, s \in \mathbb{R}\right\},
\] (4.5)
whose evolution is given respectively by (4.4).

Defining the function
\[
C(x, y, z, s, \lambda) = -x^2 \omega_0 + \sqrt{6}x - y^2 - z^2 + 1 \geq 0,
\] (4.6)
and calculating the total derivative we have
\[
C' = C\left[\frac{6 ((s - 1)y^2 + 3x^2 + 2\sqrt{6}x - 3z^2 + 2)}{2\omega_0 + 3} + 6x^2 \omega_0 - 6x^2 - 5\sqrt{6}x + 6z^2 - \sqrt{6}\lambda z\right].
\] (4.7)

From this it follows that if we take the initial conditions over the surface \(C = 0\), the solutions remain on this surface all the time. And if we take the initial conditions on the half-space \(C > 0\), the solutions remain on this region for all the time. By estimating
\[
\frac{6 ((s - 1)y^2 + 3x^2 + 2\sqrt{6}x - 3z^2 + 2)}{2\omega_0 + 3} + 6x^2 \omega_0 - 6x^2 - 5\sqrt{6}x + 6z^2 - \sqrt{6}\lambda z
\]
we can see how the errors propagate if we take the initial conditions on the surface \(C(x, y, z, s, \lambda) = C_0\), with \(C_0\) arbitrarily small.

To explicitly obtain an autonomous dynamical system, first, it is necessary to determine a specific potential form \(V(\varphi)\) and \(W(\psi)\). However, one could alternatively handle the potential differentiations when \(F\) can be expressed as an explicit one-valued function of \(s\), that is \(F = F(s)\), as well as it can be defined an explicit function \(h = h(s, \lambda)\) for some examples. Therefore we result to a closed dynamical system for \(s, \lambda\), and a set of normalized-variables. A similar approach has been applied in isotropic (FRW) scenarios [45–50], however for the purpose of the present work we will improve it. Such a procedure is possible for general physical potentials, and for the usual ansätze of the cosmological literature it results to very simple forms for \(F(s)\), as can be seen in Table 1.

In order to continue we consider some specific forms of the potentials \(V(\varphi)\) and \(W(\psi)\) which leads to specific forms on the functions \(F(s)\) and \(h(s, \lambda)\). For the Brans-Dicke field \(\Phi\) we consider a power-law potential where in terms of the field \(\varphi\) has the exponential form \(V(\varphi) = V_0 e^{\ell \varphi}\). As far as concerns the second scalar field we study the cases where the potential is (a) exponential and (b) power-law. Finally, we comment about general features of the equilibrium points of (4.4) for arbitrary \(h(\lambda, s)\) and \(F(s)\) functions.
4.1 Case: \( W(\psi) = W_0 e^{k \psi} \) and \( V(\varphi) = V_0 e^{l \varphi} \)

In this example we have \( \Gamma_\psi = 1, \Gamma_\varphi = 1 \), thus \( f(s) = 1, F(s) = 0 \) and \( h(s, \lambda) = 1 \). Furthermore, \( s = -l = \text{const.} \). In this particular the system (4.4) simplifies to

\[
x' = x \left[ -\frac{3(l + 1)y^2}{2\omega_0 + 3} + \frac{18}{2\omega_0 + 3} + \frac{6\omega_0z^2}{2\omega_0 + 3} - 3 \right] - \frac{\sqrt{6}(l + 1)y^2}{2\omega_0 + 3} + \frac{3x^3\omega_0(2\omega_0 + 1)}{2\omega_0 + 3} - \frac{\sqrt{6}x^2(8\omega_0 + 3)}{2\omega_0 + 3} + \frac{2\sqrt{6}}{2\omega_0 + 3} - \frac{3\sqrt{6}z^2}{2\omega_0 + 3} \tag{4.8a}
\]

\[
y' = \frac{\sqrt{6}xy(2(l - 4)\omega_0 + 3)}{4\omega_0 + 6} - \frac{3(l + 1)y^3}{2\omega_0 + 3} + \frac{3x^2y\omega_0(2\omega_0 + 1)}{2\omega_0 + 3} + y \left[ \frac{6}{2\omega_0 + 3} + \frac{6\omega_0z^2}{2\omega_0 + 3} \right], \tag{4.8b}
\]

\[
z' = \sqrt{\frac{3}{2}} \lambda - y^2 \left[ \frac{3(l + 1)}{2\omega_0 + 3} + \sqrt{\frac{3}{2}} \lambda \right] + x \left[ \frac{3\omega_0(2\omega_0 + 1)}{2\omega_0 + 3} - \sqrt{\frac{3}{2}} \lambda \omega_0 \right] + x \left[ \frac{6\omega_0z^3}{2\omega_0 + 3} - \frac{\sqrt{3}}{2} \lambda z^2 + \frac{6}{2\omega_0 + 3} - 3 \right], \tag{4.8c}
\]

\[
\lambda' = \sqrt{\frac{3}{2}} \lambda x. \tag{4.8d}
\]

The system is form-invariant under the change \((y, z, \lambda) \rightarrow (y, -z, -\lambda)\). Therefore, without losing generality we can investigate just the sector \(y \geq 0, z \geq 0, \lambda \geq 0\). Henceforth, we will focus on the stability properties of the system (4.8) for the state vector \((x, y, z, \lambda)\) defined in the phase space

\[
\left\{ (x, y, z, \lambda) : -\sqrt{6}x + x^2\omega_0 + y^2 + z^2 \leq 1, y \geq 0, z \geq 0, \lambda \geq 0 \right\}, \tag{4.9}
\]

whose evolution is given by (4.8).

The equilibrium points of the system (4.8) are the following:

\[P_1: (x, y, z, \lambda) = \left( 0, \frac{\sqrt{2}}{\sqrt{l+1}}, 0, 0 \right).\]

Exists for \(2\omega_0 + 3 \neq 0, l \geq 1\).

The eigenvalues are
\[
\left\{ 0, -3, -\frac{1}{2} \left( 3 + \sqrt{\frac{48 + 18\omega_0 + 7\omega_0^2}{2\omega_0 + 3}} \right), -\frac{1}{2} \left( 3 - \sqrt{\frac{48 + 18\omega_0 + 7\omega_0^2}{2\omega_0 + 3}} \right) \right\}.
\]

It is nonhyperbolic with a three dimensional stable manifold provided \(\omega_0 > -\frac{3}{2}, 1 < l \leq \frac{1}{16} (6\omega_0 + 25)\).

\[P_2: (x, y, z, \lambda) = \left( -\frac{\sqrt{2}}{\sqrt{l+1}}, 0, \sqrt{\frac{2\omega_0^3}{3} - \frac{2\omega_0}{3}} - 1, 0 \right).\]

Exists for \(\omega_0 < -\frac{3}{2}\).

The eigenvalues are \(\{ -1, 1, 6, 2 - l \}\).

It is always a saddle with a three dimensional unstable manifold if \(l < 2\).

\[P_3: (x, y, z, \lambda) = \left( -\frac{\sqrt{2}}{\sqrt{l+1}}, -\frac{\sqrt{2}}{\sqrt{l+1}}, -\sqrt{\frac{l(l-3)\omega_0 - 7}{l+1}}, 0 \right).\]
(a) \( l < -1, \omega_0 < -\frac{3}{2} \), or
(b) \( l < -1, -\frac{3}{2} < \omega_0 \leq \frac{1}{6} (l^2 - 3l - 7) \).

The eigenvalues are
\[
\begin{align*}
\left\{ -\frac{3}{l+1}, \frac{3}{2(l+1)} - \frac{\sqrt{6(17-8l)\omega_0 + 8l(l-5)(l+1)+12l}}{2(l+1)\sqrt{2\omega_0+1}}, \frac{3}{2(l+1)} + \frac{\sqrt{6(17-8l)\omega_0 + 8l(l-5)(l+1)+12l}}{2(l+1)\sqrt{2\omega_0+1}} \right\}.
\end{align*}
\]
It is a saddle.

\( P_4: (x, y, z, \lambda) = \left( -\frac{\sqrt{2}}{2\omega_0+1}, \frac{\sqrt{2}}{2\omega_0+1}, \frac{\sqrt{l-3l-7}}{l+1} \right) \). Exists for

(a) \(-1 < l \leq 1, \omega_0 < -\frac{\sqrt{2}}{2} \) or
(b) \(-1 < l \leq 1, -\frac{\sqrt{2}}{2} < \omega_0 \leq \frac{1}{6} (l^2 - 3l - 7) \),
(c) \( 1 < l < 2, \omega_0 \leq \frac{1}{6} (l^2 - 3l - 7) \) or
(d) \( l = 2, \omega_0 < -\frac{3}{2} \).

The eigenvalues are
\[
\begin{align*}
\left\{ -\frac{3}{l+1}, \frac{3}{2(l+1)} - \frac{\sqrt{6(17-8l)\omega_0 + 8l(l-5)(l+1)+12l}}{2(l+1)\sqrt{2\omega_0+1}}, \frac{3}{2(l+1)} + \frac{\sqrt{6(17-8l)\omega_0 + 8l(l-5)(l+1)+12l}}{2(l+1)\sqrt{2\omega_0+1}} \right\}.
\end{align*}
\]
It is a saddle.

\( P_5: (x, y, z, \lambda) = \left( \frac{2}{2\omega_0+1}, 0, 0, 0 \right) \). Exists for

(a) \( \omega_0 < -\frac{\sqrt{2}}{2} \) or
(b) \( -\frac{\sqrt{2}}{2} \leq \omega_0 < -\frac{1}{2} \) or
(c) \( \omega_0 > -\frac{1}{2} \).

The eigenvalues are \( \left\{ \frac{2}{2\omega_0+1}, \frac{2(l+1)}{2\omega_0+1}, -3, -\frac{2}{2\omega_0+1} - 3 \right\} \).
It is a sink for \( l > -1, \omega_0 < -\frac{3}{2} \). It is a saddle otherwise.

\( P_6: (x, y, z, \lambda) = \left( \frac{\sqrt{l(l-1)}}{l+2\omega_0+2}, -\frac{\sqrt{(2\omega_0+3)(2\omega_0+4(l-2))}}{l+2\omega_0+2} \right) \).

Exists for

(a) \( l < -2, \omega_0 = \frac{1}{6} (l-4)(l+2) \) or
(b) \( l > 1, \omega_0 = \frac{1}{6} (l-4)(l+2) \) or
(c) \( -2 < l < 1, \frac{1}{6} (l-4)(l+2) \leq \omega_0 < \frac{1}{2} (-l-2) \) or
(d) \( l \leq 1, \omega_0 < -\frac{3}{2} \) or
(e) \( l > 1, \omega_0 < \frac{1}{2} (-l-2) \).

The eigenvalues are \( \left\{ -\frac{3}{l+2\omega_0+2}, \frac{2(l+1)}{l+2\omega_0+2}, \frac{P-2-6\omega_0-7}{l+2\omega_0+2}, \frac{P-2-6\omega_0-8}{l+2\omega_0+2} \right\} \).
It is a sink for \( l < -1, \omega_0 < -\frac{3}{2} \). It is a saddle otherwise.
\[P_7: (x, y, z, \lambda) = \left( \frac{\sqrt{\frac{1}{4}(l-1)}}{l+2\omega_0+2}, \frac{\sqrt{(2\omega_0+3)(2\omega_0+\frac{l}{4}(l+2))}}{l+2\omega_0+2}, 0, 0 \right).\]

Exists for

(a) \( l > 1, \omega_0 \geq \frac{1}{6}(l-4)(l+2) \) or
(b) \( l < -2, \omega_0 \geq \frac{1}{6}(l-4)(l+2) \) or
(c) \( l = -2, \omega_0 > 0 \) or
(d) \( l = 1, \omega_0 > -\frac{3}{2} \) or
(e) \(-2 < l < 1, \omega_0 > \frac{1}{2}(-l-2) \) or
(f) \( l > 1, \frac{1}{6}(-l-2) < \omega_0 < -\frac{3}{2} \) or
(g) \(-2 < l < 1, \omega_0 = \frac{1}{6}(l-4)(l+2) \).

The eigenvalues are \( \left\{ -\frac{l-1}{l+2\omega_0+2}, \frac{2(l-1)(l+1)}{l+2\omega_0+2}, \frac{l^2-3l-6\omega_0-7}{l+2\omega_0+2}, \frac{l^2-2l-6\omega_0-8}{l+2\omega_0+2} \right\} \).

It is a saddle.

\[P_8: (x, y, z, \lambda) = \left( \frac{\sqrt{3-\sqrt{2\omega_0+3}}}{\sqrt{2\omega_0}}, 0, 0, 0 \right).\]

Exists for \( \omega_0 > -\frac{3}{2}, \omega_0 \neq 0 \).

The eigenvalues are \( \left\{ -\frac{\sqrt{6\omega_0+3}}{2\omega_0}, \frac{\sqrt{6\omega_0+3}}{2\omega_0}, \frac{6\omega_0-\sqrt{6\omega_0+3}}{\omega_0}, \frac{6\omega_0-\omega_0(l+2)(\sqrt{6\omega_0+3})}{2\omega_0} \right\} \).

It is a saddle.

\[P_9: (x, y, z, \lambda) = \left( \frac{\sqrt{3+\sqrt{2\omega_0+3}}}{\sqrt{2\omega_0}}, 0, 0, 0 \right).\]

Exists for \( \omega_0 > -\frac{3}{2}, \omega_0 \neq 0 \).

The eigenvalues are \( \left\{ \frac{\sqrt{6\omega_0+3}}{2\omega_0}, \frac{\sqrt{6\omega_0+3}}{2\omega_0}, \frac{6\omega_0+\sqrt{6\omega_0+3}}{\omega_0}, \frac{6\omega_0+(l+2)(\sqrt{6\omega_0+3})+6\omega_0}{2\omega_0} \right\} \).

It is a sink for

(a) \(-2 < l \leq -1, \frac{1}{6}(l-4)(l+2) < \omega_0 < 0 \), or
(b) \( l > -1, -\frac{5}{6} < \omega_0 < 0 \).

It is a source for

(a) \( l \leq -2, \omega_0 > \frac{1}{6}(l-4)(l+2) \), or
(b) \( l > -2, \omega_0 > 0 \).

It is a saddle otherwise.

\[P_{10}: (x, y, z, \lambda) = \left( 0, 0, \frac{\sqrt{2^2}}{3^2} \right). \text{ Always exists. The eigenvalues are} \]
\[\left\{ -1, 2, \frac{1}{3} \left( -\frac{4l}{2\omega_0+3} - 1 \right), \frac{1}{3} \left( 1 - \frac{4l}{2\omega_0+3} - 1 \right) \right\}. \]

It is a saddle with a three dimensional stable manifold provided \( \omega_0 \geq \frac{45}{2} \).
4.1.1 Center manifold of $P_1$.

From the previous linear analysis we have found that the equilibrium point $P_1$ is nonhyperbolic with a three dimensional stable manifold provided $\omega_0 > -\frac{3}{2}, 1 < l \leq \frac{1}{10} (6\omega_0 + 25)$. In this subsection we use the Center Manifold Theorem to show that the solution corresponding to $P_1$ is indeed locally asymptotically stable under the above conditions.

Introducing the new variables

\[
\begin{align*}
\lambda &= \lambda, \\
v_1 &= z - \frac{\lambda(l-1)}{\sqrt{6}(l+1)}, \\
v_2 &= \frac{-x(4(l-4)\omega_0 + 6)}{\sqrt{l+1}} + \left(y - \frac{\sqrt{2}}{\sqrt{l+1}}\right) \left(\sqrt{(2\omega_0 + 3)(-16l + 6\omega_0 + 25)} - 2\sqrt{3}\omega_0 + 5\sqrt{3}\right), \\
v_3 &= \frac{x(4(l-4)\omega_0 + 6)}{\sqrt{l+1}} - \left(y - \frac{\sqrt{2}}{\sqrt{l+1}}\right) \left(-\sqrt{(2\omega_0 + 3)(-16l + 6\omega_0 + 25)} - 2\sqrt{3}\omega_0 + 5\sqrt{3}\right),
\end{align*}
\]

which are real, the point $P_1$ is shifted to the origin and the linear part of the vector field is transformed to its real Jordan canonical form. Therefore, the evolution equations becomes

\[
\begin{pmatrix}
\frac{d}{dt} u' \\
\frac{d}{dt} v_1' \\
\frac{d}{dt} v_2' \\
\frac{d}{dt} v_3'
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 \\
0 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & \lambda_3
\end{pmatrix}
\begin{pmatrix}
u' \\
v_1' \\
v_2' \\
v_3'
\end{pmatrix}
+ \begin{pmatrix}f(u,v) \\
g_1(u,v) \\
g_2(u,v) \\
g_3(u,v)
\end{pmatrix}
\]

(4.11)

where

\[
\begin{align*}
\lambda_2 &= -\frac{\sqrt{3}\sqrt{(2\omega_0 + 3)(-16l + 6\omega_0 + 25)} + 6\omega_0 + 9}{4\omega_0 + 6},
\lambda_3 &= -\frac{6\omega_0 + 9 - \sqrt{3}\sqrt{(2\omega_0 + 3)(-16l + 6\omega_0 + 25)}}{4\omega_0 + 6},
\end{align*}
\]

(4.12a)

(4.12b)

\[
f(u,v) = \frac{\sqrt{l+1}u \left(\sqrt{3}\sqrt{(2\omega_0 + 3)(-16l + 6\omega_0 + 25)}(v_3 - v_2) - 6\omega_0(v_2 + v_3) + 15(v_2 + v_3)\right)}{\sqrt{2}(4(l-4)\omega_0 + 6)}
\]

(4.12c)

and the $g_1(u,v), g_2(u,v), g_3(u,v)$ are more complicated expressions.

The system (4.11) is written in diagonal form

\[
\begin{align*}
u' &= Cu + f(u,v) \\
v' &= Pv + g(u,v)
\end{align*}
\]

(4.13)

where $(u,v) \in \mathbb{R} \times \mathbb{R}^3$, $C$ is the zero $1 \times 1$ matrix, $P$ is a $3 \times 3$ matrix with negative eigenvalues and $f, g$ vanish at $0$ and have vanishing derivatives at $0$. The center manifold
Theorem asserts that there exists a 1-dimensional invariant local center manifold $W^c(0)$ of (4.13) tangent to the center subspace (the $v = 0$ space) at 0. Moreover, $W^c(0)$ can be represented as

$$ W^c(0) = \{(u, v) \in \mathbb{R} \times \mathbb{R}^3 : v = h(u), h(0) = 0, Dh(0) = 0, |u| < \delta \} , $$

The restriction of (4.13) to the center manifold is

$$ u' = f(u, h(u)). $$

(4.14)

If the origin of (4.14) is stable (asymptotically stable) (unstable) then the origin of (4.13) is also stable (asymptotically stable) (unstable). Therefore, we have to find the local center manifold, i.e., the problem reduces to the computation of $h(u)$.

Substituting $v = h(u)$ in the second component of (4.13) and using the chain rule, $v' = Dh(u)u'$, one can show that the function $h(u)$ that defines the local center manifold satisfies

$$ Dh(u)[f(u, h(u)) - Ph(u) - g(u, h(u))] = 0. $$

(4.15)

The equation (4.15) can be solved approximately by expanding $h(u)$ in Taylor series at $u = 0$. Since $h(0) = 0$ and $Dh(0) = 0$, it is obvious that $h(u)$ commences with quadratic terms. We substitute

$$ h(x) := \begin{bmatrix} h_1(u) \\ h_2(u) \\ h_3(u) \end{bmatrix} = \begin{bmatrix} a_1 u^2 + O(u^3) \\ a_2 u^2 + O(u^3) \\ a_3 u^2 + O(u^3) \end{bmatrix} $$

into (4.15) and set the coefficients of like powers of $u$ equal to zero to find the non-zero coefficients are

$$ a_2 = -\frac{\sqrt{2}(l - 1)^2 \left( \omega_0 \sqrt{(2\omega_0 + 3)(-16l + 6\omega_0 + 25)} + \sqrt{3l}(2\omega_0 + 3) - \sqrt{3}(2\omega_0 + 3) \omega_0 \right)}{(l + 1)^{5/2}(2\omega_0 + 3) \left( -3\sqrt{(2\omega_0 + 3)(-16l + 6\omega_0 + 25)} + 16\sqrt{3l} - \sqrt{3}(6\omega_0 + 25) \right)}, $$

$$ a_3 = -\frac{\sqrt{2}(l - 1)^2 \left( -\omega_0 \sqrt{(2\omega_0 + 3)(-16l + 6\omega_0 + 25)} + \sqrt{3l}(2\omega_0 + 3) - \sqrt{3}(2\omega_0 + 3) \omega_0 \right)}{(l + 1)^{5/2}(2\omega_0 + 3) \left( 3\sqrt{(2\omega_0 + 3)(-16l + 6\omega_0 + 25)} + 16\sqrt{3l} - \sqrt{3}(6\omega_0 + 25) \right)}. $$

Therefore, the local center manifold of the origin can be expressed

$$ \{ (u, v_1, v_2, v_3) \in \mathbb{R}^4 : v_1 = 0, \}

$$

$$ v_2 = -\frac{\sqrt{2}(l - 1)^2 \left( -\sqrt{3}(2\omega_0 + 3) \omega_0 + \sqrt{(2\omega_0 + 3)(-16l + 6\omega_0 + 25)} \omega_0 + \sqrt{3l}(2\omega_0 + 3) \right) u^2}{(l + 1)^{5/2}(2\omega_0 + 3) \left( 16\sqrt{3l} - \sqrt{3}(6\omega_0 + 25) - 3\sqrt{(2\omega_0 + 3)(-16l + 6\omega_0 + 25)} \right)}, $$

$$ v_3 = -\frac{\sqrt{2}(l - 1)^2 \left( -\sqrt{3}(2\omega_0 + 3) \omega_0 - \sqrt{(2\omega_0 + 3)(-16l + 6\omega_0 + 25)} \omega_0 + \sqrt{3l}(2\omega_0 + 3) \right) u^2}{(l + 1)^{5/2}(2\omega_0 + 3) \left( 16\sqrt{3l} - \sqrt{3}(6\omega_0 + 25) + 3\sqrt{(2\omega_0 + 3)(-16l + 6\omega_0 + 25)} \right)}. $$

The dynamics on the center manifold is given by a gradient like equation $u' = -\nabla \Pi(u)$, where $\Pi(u) = \frac{(l - 1)u^4}{8(l + 1)^2}$, for which the origin is degenerate local minimum whenever $l > 1$. 

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\[ \text{Page 14} \]
Figure 1: (Color online) Case: $W(\psi) = W_0 e^{k\psi}$ and $V(\varphi) = V_0 e^{l\varphi}$. Evolution of some orbits of the dynamical system (4.8) projected on space $(x, y, z)$ for $\omega_0 = 50, l = 8$. The initial conditions are chosen randomly to show that, irrespectively of the initial conditions, the orbits are attracted by the center manifold of the equilibrium point $P_1$.

Figure 2: (Color online) Case: $W(\psi) = W_0 e^{k\psi}$ and $V(\varphi) = V_0 e^{l\varphi}$. Evolution of some orbits of the dynamical system (4.8) projected on space $(x, y, z)$ for $\omega_0 = -2, l = -3$. The initial conditions are chosen randomly to show that, irrespectively of the initial conditions, the orbits are attracted by the the equilibrium point $P_6$.

(recall the existence conditions for $P_1$ are $2\omega_0 + 3 \neq 0, l \geq 1$). This implies that the center manifold of $P_1$ is stable when $1 < l \leq \frac{1}{16}(6\omega_0 + 25)$. For $l > \frac{1}{16}(6\omega_0 + 25)$ the unstable
manifold is not empty. Neglecting the order terms \( O(\lambda^3) \) the center can be given in the original variables by the graph:

\[
\begin{align*}
    x &= -\frac{(l-1)\lambda^2}{\sqrt{6(l+1)^2}}, \\
    y &= \frac{\sqrt{2}}{\sqrt{l+1}} - \frac{(l-1)(l-2\omega_0)) \lambda^2}{4\left(\sqrt{2(l+1)^{5/2}}\right)}, \\
    z &= \frac{(l-1)\lambda}{\sqrt{6(l+1)}}.
\end{align*}
\]

(4.16a)

(4.16b)

(4.16c)

In the figure 1 we present some orbits of the dynamical system (4.8) projected on the space \((x,y,z)\) for \( W(\psi) = W_0 e^{k\psi} \) and \( V(\varphi) = V_0 e^{l\varphi} \) with \( \omega_0 = 50 \) and \( l = 8 \). The initial conditions were chosen randomly to show that, irrespectively of the initial conditions, the orbits are attracted by the center manifold of the equilibrium point \( P_1 \). Latter on, in Section 5.1 it will be shown that the cosmological solutions represented by these orbits, tends to the solution associated to \( P_1 \). Furthermore, as shown in Figure 3, the cosmological parameters behaves in accordance with the current cosmological paradigm. This feature makes the model very interesting from the cosmological point of view.

In the Fig. 2 are displayed some orbits of the dynamical system (4.8) projected on space \((x,y,z)\) for \( \omega_0 = -2, l = -3 \). The initial conditions are chosen randomly to show that, irrespectively of the initial conditions, the orbits are attracted by the equilibrium point \( P_6 \). In this example, the phase space is the interior of an hyperboloid that corresponds to the boundary of the phase space and it is represented by a gray mesh. The late-time attractor is a phantom dominated solution.

As we have commented before, the model studied in [26] can be considered as a special case of the model \( W(\psi) = W_0 e^{k\psi} \) and \( V(\varphi) = V_0 e^{l\varphi} \) with \( k = -\lambda W, l = 1 - \lambda U \sqrt{\omega_0 + \frac{3}{2}}, \omega_0 > -\frac{3}{2} \). In this section we have investigated the stability of the equilibrium solutions in the dilatonic frame. In the reference [26] it was studied the stability of the equilibrium points in both the Jordan and the Einstein frames, so our results complements those found in [26]. In particular, notice that the equilibrium point (in the Jordan frame), named \( J_4 \) in [26] corresponds to \( P_1 \) investigated in this section with the identification \( \lambda_U = (1-l)\gamma, \gamma^{-1} = \sqrt{\omega_0 + \frac{3}{2}} \), due to it satisfies

\[
J_4 : \left( e^{-\frac{\sqrt{2}}{\sqrt{6H}}} \frac{\dot{\psi}}{H}, \frac{\dot{\varphi}}{H}, \frac{\sqrt{V(\varphi)}}{\sqrt{3H}} \right) = \left( 0, 0, \frac{\sqrt{2\gamma}}{\sqrt{2\gamma - \lambda_U}} \right) = \left( 0, 0, \frac{\sqrt{2}}{\sqrt{1 + l}} \right).
\]

(4.17)

The stability conditions deduced in [26] in the Jordan’s frame formulation and also in the Einstein’s frame formulation are \( \lambda_U < 0, \gamma > 0 \). That is, \( \omega_0 > -\frac{3}{2}, l > 1 \). The stability in the dilatonic frame formulation is \( 1 < l \leq \frac{1}{16} (6\omega_0 + 25) \). Which are the equivalent conditions with the identifications \( \lambda_U = (1-l)\gamma, \gamma^{-1} = \sqrt{\omega_0 + \frac{3}{2}} \). The equilibrium points \( J_4 \) and \( E_4 \) (the representations of \( P_1 \) in the Jordan’s frame and in the Einstein’s frame, respectively) corresponds to an intermediate accelerated solution instead of a de Sitter solution (see derivation in [26]). That is, attractor in the Jordan frame corresponds to
the solution of the form $a(t) \simeq e^{\alpha_1 t^{p_1}}$, as $t \to \infty$ where $\alpha_1 > 0$ and $0 < p_1 < 1$ for a wide range of parameters. Furthermore, when we work in the Einstein frame we get that the attractor is also the solution of the form $\bar{a}(\bar{t}) \simeq e^{\alpha_2 t^{p_2}}$, as $\bar{t} \to \infty$ where $\alpha_2 > 0$ and $0 < p_2 < 1$, for the same conditions on the parameter space as in the Jordan frame. An equivalent result can be deduced straightforwardly for the dilatonic frame. We proceed as follows. According to the center manifold calculation, we have from (4.16a), the definition $\lambda := k e^{-\varphi/2}$, and the definition (4.1) that (as $\varphi \to \infty$):

$$\frac{d\varphi}{d\ln a} = \frac{(l-1)k^2 e^{-\varphi}}{(l+1)^2},$$

(4.18a)

$$\frac{d\psi}{d\ln a} = \frac{(l-1)k}{(l+1)},$$

(4.18b)

$$\frac{dt}{d\ln a} = \frac{8(l+1)^2 e^{\varphi} - k^2(l-1)(l-2\omega_0)}{4\sqrt{\frac{2}{3}(l+1)^{5/2}\sqrt{V_0} e^{-\varphi}}},$$

(4.18c)

With general solution

$$\varphi(a) = \ln \left| c_1 - \frac{k^2(l-1)\ln(a)}{(l+1)^2} \right|,$$

(4.19a)

$$\psi(a) = c_3 + \frac{k(l-1)\ln(a)}{l+1},$$

(4.19b)

$$t(a) = \sqrt[3]{\frac{2}{3}} c_1 - \frac{k^2(l-1)\ln(a)}{(l+1)^2} \left[ l/2 \left( k^2(l-1)(l(l+8\ln(a)+2) - 2(l+2)\omega_0) - 8c_1 l(l+1)^2 \right) \right] + c_2,$$

(4.19c)

where $c_1, c_2, c_3$ are integration constants.

For large $a$, the leading terms are

$$t(a) \simeq \frac{2\sqrt{6} \ln(a) \left| c_1 - \frac{k^2(l-1)\ln(a)}{(l+1)^2} \right|^{l/2}}{\sqrt{l+1}(l+2)\sqrt{V_0}} \quad \Rightarrow \quad \ln a \simeq t^{2/p} \quad \Rightarrow \quad a \simeq e^{At^p},$$

$$p := \frac{2}{2+l}, \quad \frac{32}{57 + 6\omega_0} < p < \frac{2}{3}, \quad \text{as} \quad t \to \infty.$$ (4.20)

With the identifications $\lambda_U = (1-l)\gamma, \gamma^{-1} = \sqrt{\omega_0 + \frac{3}{2}}$ we obtain the same exponent $p = p_1 = p_2 = \frac{2\gamma}{3\gamma - \lambda_U}$. Since $p < \frac{2}{3}$, $P_1$ is not a de Sitter solution (that requires $p = 1$).
4.2 Case: \( W(\psi) = W_0 \psi^k \) and \( V(\varphi) = V_0 e^{i\varphi} \)

In this example we have \( s = -l = \text{const.}, \lambda = \frac{k \psi^2}{\psi}, \Gamma_\varphi = 1, \Gamma_\psi = \frac{k}{k-1} \). Thus \( h(s, \lambda) = \frac{k-1}{k} \). Assuming \( k \neq 0 \) and introducing \( x_1 = \frac{2 \lambda z}{k} + x \) the system (4.4) becomes

\[
x_1' = \frac{2 \sqrt{6}}{2 \omega_0 + 3} + \left[ -3(l+1) y^2 + \frac{6 \lambda^2}{k} + z \left( \frac{4 \sqrt{6} \kappa \omega_0 \lambda^3}{k^2} + \frac{12 \sqrt{6} (2 \omega_0 + 1) \lambda}{k (2 \omega_0 + 3)} \right) \right] + z^2 \left[ \frac{12 \omega_0 (2 \omega_0 + 1) \lambda^2}{k^2 (2 \omega_0 + 3)} + \frac{6 \omega_0}{2 \omega_0 + 3} \right] + \frac{18}{2 \omega_0 + 3} x_1 \]

\[
+ \left[ -\frac{2 \sqrt{6} \kappa \omega_0 \lambda^2}{k} - \frac{12 \omega_0 (2 \omega_0 + 1) \lambda}{k (2 \omega_0 + 3)} - \frac{\sqrt{6} (8 \omega_0 + 3)}{2 \omega_0 + 3} \right] x_1^2 \]

\[
+ \left[ 3 \omega_0 + \frac{9}{2 \omega_0 + 3} - 3 \right] x_1^3 + y^2 \left[ -\frac{\sqrt{6} \lambda^2}{k} - \frac{\sqrt{6} (l+1)}{2 \omega_0 + 3} \right] + \frac{\sqrt{6} \lambda^2}{k} \]

\[
+ z \left[ -\frac{12 \lambda^3}{k^2} - \frac{24 \lambda}{2 \omega_0 k + 3k} \right] \]

\[
y' = \left[ \frac{6}{2 \omega_0 + 3} \right] y + \left[ \frac{\sqrt{6} (3l + 2 (l-4) \omega_0)}{4 \omega_0 + 6} - \frac{12 \omega_0 (2 \omega_0 + 1)}{k (2 \omega_0 + 3)} \right] x_1 y \]

\[
+ \left( \frac{12 \omega_0 (2 \omega_0 + 1) \lambda^2}{k^2 (2 \omega_0 + 3)} + \frac{6 \omega_0}{2 \omega_0 + 3} \right) z + \frac{\sqrt{6} \lambda (3l - 2 (l-4) \omega_0)}{k (2 \omega_0 + 3)} \right] z y \]

\[
- \frac{3 (l+1) y^3}{2 \omega_0 + 3} + \left[ 3 \omega_0 + \frac{9}{2 \omega_0 + 3} - 3 \right] x_1^2 y, \quad (4.21a) \]

\[
z' = \sqrt{\frac{3}{2}} \lambda + \left[ \frac{12 \omega_0 (2 \omega_0 + 1) \lambda^2}{k^2 (2 \omega_0 + 3)} + \frac{6 \omega_0}{2 \omega_0 + 3} \right] z^3 \]

\[
+ \left[ \frac{\sqrt{6} \lambda (-3k - 2 (k-10) \omega_0 + 6)}{k (2 \omega_0 + 3)} - \frac{2 \sqrt{6} \lambda^3 \omega_0}{k^2} \right] z^2 \]

\[
+ \left[ \frac{6 \lambda^2}{k} + \frac{6}{2 \omega_0 + 3} - 3 \right] z + \left[ \frac{3 (l+1) z}{2 \omega_0 + 3} - \frac{\sqrt{3}}{2} \lambda \right] \]

\[
+ x_1 \left[ \frac{z}{2} \left( 3 \omega_0 + \frac{9}{2 \omega_0 + 3} - 3 \right) - \frac{\sqrt{3}}{2} \lambda \omega_0 \right] \]

\[
+ x_1 \left( -\frac{12 \lambda \omega_0 (2 \omega_0 + 1) z^2}{k (2 \omega_0 + 3)} + \frac{\sqrt{6} \lambda^2 \omega_0}{k} - \frac{\sqrt{6} (10 \omega_0 + 3)}{4 \omega_0 + 6} \right) z + 3 \lambda \right], \quad (4.21b) \]

\[
\lambda' = \sqrt{\frac{3}{2}} x_1 \lambda. \quad (4.21d) \]

The system is form-invariant under the change \((y, z, \lambda) \rightarrow (-y, -z, -\lambda)\). Therefore, without losing generality we can investigate just the sector \( y \geq 0, z \geq 0, \lambda \geq 0 \). Henceforth,
we will focus on the stability properties of the system (4.21) for the state vector \((x,y,z,\lambda)\) defined in the phase space

\[
\left\{(x, y, z, \lambda) : y \geq 0, z \geq 0, \lambda \geq 0, 2\omega_0 + 3 \neq 0, \right. \\
\left. k \left( \sqrt{6}x_1 + y^2 + z^2 - 1 \right) + 2\sqrt{6}\lambda z + \omega_0(kx_1 - 2\lambda z)^2 \leq 0 \right\}.
\] (4.22)

We will focus on the study the particular choice of potentials (3.12):

\[
V(\varphi) = V_0 e^{(\beta-1)x}, \quad W(\psi) = W_0 \psi^{2\beta},
\] (4.23)

that lead to Noether pointlike symmetries, corresponding to the choice \(l = \beta - 1\), and \(k = 2\beta\). The equilibrium points are the following

\[P_1 := (x_1, y, z, \lambda) = \left(0, \frac{\sqrt{3}}{\sqrt{2}}, 0, 0\right)\].

The eigenvalues are

\[
\left\{0, -3, -\frac{\sqrt{3}}{\sqrt{2}} \left(\frac{2\omega_0 + 3}{2\omega_0 + 3} - 1\right) - \frac{3}{2}, \frac{1}{2} \left(\frac{\sqrt{3}}{\sqrt{2}} \left(\frac{2\omega_0 + 3}{2\omega_0 + 3} - 3\right)\right)\right\}.
\]

The stable manifold is three dimensional for \(\beta > 2, \omega_0 \geq \frac{1}{6}(16\beta - 41)\).

\[P_2 := (x, y, z, \lambda) = \left(-\frac{\sqrt{2}}{3}, 0, \sqrt{\frac{2\omega_0}{3} - 1}, 0\right)\].

The eigenvalues are

\[
\{-1, 1, 6, 3 - \beta\}.
\]

It is a saddle point with a three dimensional unstable manifold for \(\beta < 3\).

\[P_3 := (x, y, z, \lambda) = \left(-\frac{\sqrt{3}}{\sqrt{2}}, -\sqrt{\frac{3}{\beta}}, -\sqrt{\frac{(\beta - 5)\beta - 6\omega_0 - 3}{\beta}}, 0\right)\].

The eigenvalues are

\[
\left\{-\frac{3}{\beta}, 6, \frac{3}{2\beta} - \frac{\sqrt{6(25 - 8\beta)\omega_0 + 8(\beta - 6)(\beta - 2)\beta + 81)}{2\beta \sqrt{\frac{2\omega_0}{3} + 1}}, \frac{3}{2\beta} + \frac{\sqrt{6(25 - 8\beta)\omega_0 + 8(\beta - 6)(\beta - 2)\beta + 81)}{2\beta \sqrt{\frac{2\omega_0}{3} + 1}}\right\}.
\]

It is a saddle.

\[P_4 := (x, y, z, \lambda) = \left(-\frac{\sqrt{3}}{\sqrt{2}}, \sqrt{\frac{3}{\beta}}, \sqrt{\frac{(\beta - 5)\beta - 6\omega_0 - 3}{\beta}}, 0\right)\].

The eigenvalues are

\[
\left\{-\frac{3}{\beta}, 6, \frac{3}{2\beta} - \frac{\sqrt{6(25 - 8\beta)\omega_0 + 8(\beta - 6)(\beta - 2)\beta + 81)}{2\beta \sqrt{\frac{2\omega_0}{3} + 1}}, \frac{3}{2\beta} + \frac{\sqrt{6(25 - 8\beta)\omega_0 + 8(\beta - 6)(\beta - 2)\beta + 81)}{2\beta \sqrt{\frac{2\omega_0}{3} + 1}}\right\}.
\]

It is a saddle.

\[P_5 := (x, y, z, \lambda) = \left(2\sqrt{\frac{2}{2\omega_0 + 1}}, 0, 0, 0\right)\].

The eigenvalues are

\[
\left\{\frac{2}{2\omega_0 + 1}, \frac{2\beta}{2\omega_0 + 1}, -3, -\frac{6\omega_0 + 5}{2\omega_0 + 1}\right\}.
\]

It is a sink for \(\beta > 0, \omega_0 < -\frac{3}{2}\).

\[P_6 := (x, y, z, \lambda) = \left(-\sqrt{\frac{2}{\beta + 2\omega_0 + 1}}, -\sqrt{\frac{(2\omega_0 + 3)(2\omega_0 - \frac{1}{3}(\beta - 5)(\beta + 1))}{\beta + 2\omega_0 + 1}}, 0, 0\right)\].

The eigenvalues are
\[ \{ \frac{-\beta - 2}{\beta + 2\omega_0 + 1}, \frac{2(\beta - 2)\beta}{\beta^2 + 2\omega_0 + 1}, \frac{\beta^2 - 5\beta - 6\omega_0 - 3}{\beta + 2\omega_0 + 1}, \frac{\beta^2 - 4\beta - 6\omega_0 - 5}{\beta + 2\omega_0 + 1} \}. \]

It is a sink for \( \beta < 0, \omega_0 < -\frac{2}{3} \). It is a saddle otherwise.

\[ P_7 := (x_1, y, z, \lambda) = \left( -\frac{\sqrt{2}}{\beta + 2\omega_0 + 1}, \frac{\sqrt{2(\beta - 2)\beta}}{\beta + 2\omega_0 + 1}, \frac{\sqrt{(2\omega_0 + 3)(2\omega_0 - \frac{1}{2}(\beta - 5)(\beta + 1))}}{\beta + 2\omega_0 + 1}, 0, 0 \right). \]

The eigenvalues are
\[ \{ \frac{-\beta - 2}{\beta + 2\omega_0 + 1}, \frac{2(\beta - 2)\beta}{\beta^2 + 2\omega_0 + 1}, \frac{\beta^2 - 5\beta - 6\omega_0 - 3}{\beta + 2\omega_0 + 1}, \frac{\beta^2 - 4\beta - 6\omega_0 - 5}{\beta + 2\omega_0 + 1} \}. \]

It is a saddle.

\[ P_8 := (x_1, y, z, \lambda) = \left( 0, 0, 0 \right). \]

The eigenvalues are
\[ \{ \frac{-\sqrt{2(\beta - 2)\beta}}{\beta + 2\omega_0 + 1}, \frac{\sqrt{2(\beta - 2)\beta}}{\beta + 2\omega_0 + 1}, \frac{\sqrt{2(\beta - 2)\beta}}{\beta + 2\omega_0 + 1}, \frac{\sqrt{2(\beta - 2)\beta}}{\beta + 2\omega_0 + 1} \}. \]

It is a saddle.

\[ P_9 := (x_1, y, z, \lambda) = \left( 0, 0, 0 \right). \]

The eigenvalues are
\[ \{ \frac{\sqrt{2(\beta - 2)\beta}}{\beta + 2\omega_0 + 1}, \frac{\sqrt{2(\beta - 2)\beta}}{\beta + 2\omega_0 + 1}, \frac{\sqrt{2(\beta - 2)\beta}}{\beta + 2\omega_0 + 1}, \frac{\sqrt{2(\beta - 2)\beta}}{\beta + 2\omega_0 + 1} \}. \]

It is a saddle for
(a) \( -1 < \beta < 0, \frac{1}{3}(\beta - 5)(\beta + 1) < \omega_0 < 0 \), or
(b) \( \beta > 0, -\frac{5}{6} < \omega_0 < 0 \).

It is a source for
(a) \( \beta < -1, \omega_0 > \frac{1}{3}(\beta - 5)(\beta + 1) \), or
(b) \( \beta > -1, \omega_0 > 0 \).

It is a saddle otherwise.

\[ P_{11}(\lambda) := (x_1, y, z, \lambda) = \left( 0, -\sqrt{2\lambda^2\omega_0(14 - (\beta - 4)\beta)\lambda^2 + 6\beta(\beta + 2) + 6\lambda^2\omega_0) - 3((\beta - 5)\lambda^2 - 4\beta)(2\beta^2 + (\beta + 1)\lambda^2), \frac{\sqrt{2}(\beta - 2)\beta\lambda}{2\beta^2 + (\beta + 1)\lambda^2 + 2\lambda^2\omega_0}, \lambda \right). \]

The eigenvalues and the nature of the equilibrium points has to be handled for specific choices of the parameters in the region of existence.

\[ P_{12}(\lambda) := (x_1, y, z, \lambda) = \left( 0, \sqrt{2\lambda^2\omega_0(14 - (\beta - 4)\beta)\lambda^2 + 6\beta(\beta + 2) + 6\lambda^2\omega_0) - 3((\beta - 5)\lambda^2 - 4\beta)(2\beta^2 + (\beta + 1)\lambda^2), \frac{-\sqrt{2}(\beta - 2)\beta\lambda}{2\beta^2 + (\beta + 1)\lambda^2 + 2\lambda^2\omega_0}, \lambda \right). \]

The eigenvalues and the nature of the equilibrium points has to be handled for specific choices of the parameters in the region of existence.

\[ P_{13} := (x_1, y, z, \lambda) = \left( 0, 0, \sqrt{-\frac{2\omega_0}{3} - 1}, \frac{\beta}{\sqrt{-\omega_0 - \frac{2}{3}}} \right). \]

The eigenvalues are
\{3 - \beta, 1, -2(\beta - 3), 1\}.

It is a source for

(a) \(\omega_0 \in \mathbb{R}, \beta = 0\), or

(b) \(0 < \beta < 3, \omega_0 < -\frac{3}{2}\).

\[P_{14} := (x_1, y, z, \lambda) = \left(0, 0, \frac{1}{96\beta^2 \sqrt{(-10 + 2\beta)(2 + 2\beta) - 24\omega_0 \omega_0 \sqrt[3]{3 + 2\omega_0}}}
\begin{pmatrix}
-48\sqrt{3}\omega_0^2 + 3(2 + 2\beta) \\
4\omega_0 \\
\sqrt{3(-6 + 2\beta)(2 + 2\beta)} + 3\sqrt{3(-2 + 2\beta)^2 + 8\omega_0 (6 + 2\beta + 6\omega_0)}
\end{pmatrix}
\]

\[\begin{pmatrix}
-2\sqrt{3} + 2\sqrt{3\beta} + \sqrt{3(-2 + 2\beta)^2 + 8\omega_0 (6 + 2\beta + 6\omega_0)}
\end{pmatrix}
\]

\[\begin{pmatrix}
-6\beta(-8 + 2\beta(-14 + 2\beta)) + \\
2\beta \\
(8\sqrt{3}\beta^{3/2}) / (\sqrt{(6\beta(-8 + 2\beta(-14 + 2\beta)))}
\end{pmatrix}
\]

\[\begin{pmatrix}
2\beta \\
(-4 + 2\beta)\sqrt{9(-2 + 2\beta)^2 + 24\omega_0 (6 + 2\beta + 6\omega_0)}
\end{pmatrix}
\]

The eigenvalues and the nature of the equilibrium points has to be handled for specific choices of the parameters in the region of existence.

\[P_{15} := (x_1, y, z, \lambda) = \left(0, 0, \frac{1}{96\beta^2 \sqrt{(-10 + 2\beta)(2 + 2\beta) - 24\omega_0 \omega_0 \sqrt[3]{3 + 2\omega_0}}}
\begin{pmatrix}
-48\sqrt{3}\omega_0^2 + 4\omega_0 \\
3(2 + 2\beta) \\
\sqrt{3(-6 + 2\beta)(2 + 2\beta) - 3\sqrt{3(-2 + 2\beta)^2 + 8\omega_0 (6 + 2\beta + 6\omega_0)}
\end{pmatrix}
\]

\[\begin{pmatrix}
-2\sqrt{3} + 2\sqrt{3\beta} - \sqrt{3(-2 + 2\beta)^2 + 8\omega_0 (6 + 2\beta + 6\omega_0)}
\end{pmatrix}
\]

\[\begin{pmatrix}
-6\beta(-8 + 2\beta(-14 + 2\beta)) + \\
2\beta \\
(8\sqrt{3}\beta^{3/2}) / (\sqrt{(6\beta(-8 + 2\beta(-14 + 2\beta)))}
\end{pmatrix}
\]

\[\begin{pmatrix}
2\beta(-12 + 2\beta)\omega_0 + (-4 + 2\beta)\sqrt{9(-2 + 2\beta)^2 + 24\omega_0 (6 + 2\beta + 6\omega_0)}
\end{pmatrix}
\]

The eigenvalues and the nature of the equilibrium points has to be handled for specific choices of the parameters in the region of existence.

4.2.1 Center manifold of \(P_1\).

From the previous linear analysis we found that the equilibrium point \(P_1\) is nonhyperbolic with a three dimensional stable manifold provided \(\beta > 2, \omega_0 \geq \frac{1}{6}(16\beta - 41)\).

Introducing the new variables

\[u = \lambda,\]

\[v_1 = z - \frac{(\beta - 2)\lambda}{\sqrt{3}\beta},\]

\[v_2 = \frac{x(4(\beta - 5)\omega_0 + 6(\beta - 1))}{2\sqrt{3}\sqrt{(2\omega_0 + 3)(-16\beta + 6\omega_0 + 41)}} + \frac{(\sqrt{2\omega_0 + 3}) (-16\beta + 6\omega_0 + 41) - 2\sqrt{3}\omega_0 + 5\sqrt{3}}{2\sqrt{2\omega_0 + 3} (16\beta + 6\omega_0 + 41)} \left(y - \frac{\sqrt{2}}{\sqrt{3}}\right),\]

\[v_3 = \frac{x(4(\beta - 5)\omega_0 + 6(\beta - 1))}{2\sqrt{3}\sqrt{(2\omega_0 + 3)(-16\beta + 6\omega_0 + 41)}} + \frac{(\sqrt{2\omega_0 + 3}) (-16\beta + 6\omega_0 + 41) + 2\sqrt{3}\omega_0 - 5\sqrt{3}}{2\sqrt{2\omega_0 + 3} (16\beta + 6\omega_0 + 41)} \left(y - \frac{\sqrt{2}}{\sqrt{3}}\right),\]
such that
\[
\begin{align*}
u' &= -\sqrt{3}uv_2 \left( \sqrt{3}(2\omega_0 + 3)(-16\beta + 6\omega_0 + 41) + 6\omega_0 - 15 \right) + \\
&\quad \frac{\sqrt{3}uv_3 \left( \sqrt{3}(2\omega_0 + 3)(-16\beta + 6\omega_0 + 41) - 6\omega_0 + 15 \right)}{2\sqrt{2}(2\beta - 5)\omega_0 + 3(\beta - 1)}, \tag{4.24a}
\end{align*}
\]
\[
\begin{align*}
v'_1 &= -3v_1 + O(2), \tag{4.24b}
\end{align*}
\]
\[
\begin{align*}
v'_2 &= - \frac{v_2 \left( \sqrt{3}(2\omega_0 + 3)(-16\beta + 6\omega_0 + 41) + 6\omega_0 + 9 \right)}{4\omega_0 + 6} + O(2), \tag{4.24c}
\end{align*}
\]
\[
\begin{align*}
v'_3 &= \frac{v_3 \left( \sqrt{3}(2\omega_0 + 3)(-16\beta + 6\omega_0 + 41) - 6\omega_0 + 9 \right)}{4\omega_0 + 6} + O(2). \tag{4.24d}
\end{align*}
\]

The system (4.24) is written in diagonal form
\[
\begin{align*}
u' &= Cu + f(u, v) \\
v' &= Pv + g(u, v), \tag{4.25}
\end{align*}
\]
where \((u, v) \in \mathbb{R} \times \mathbb{R}^3\), \(C\) is the zero \(1 \times 1\) matrix, \(P\) is a \(3 \times 3\) matrix with negative eigenvalues and \(f, g\) vanish at \(0\) and have vanishing derivatives at \(0\). The center manifold theorem asserts that there exists a 1-dimensional invariant local center manifold \(W^c(0)\) of (4.13) tangent to the center subspace (the \(v = 0\) space) at \(0\). Moreover, \(W^c(0)\) can be represented as
\[
W^c(0) = \left\{ (u, v) \in \mathbb{R} \times \mathbb{R}^3 : v = h(u), h(0) = 0, \ Dh(0) = 0, \ |u| < \delta \right\},
\]
for \(\delta\) sufficiently small. The restriction of the dynamics to the center manifold is
\[
u' = f(u, h(u)), \tag{4.26}
\]
where the function \(h(u)\) that defines the local center manifold satisfies
\[
\begin{align*}
Dh(u) [f(u, h(u))] - Ph(u) - g(u, h(u)) = 0.
\end{align*}
\]

Following the same procedure implemented in section 4.1.1 we obtain
\[
h(u) := \begin{bmatrix} h_1(u) \\ h_2(u) \\ h_3(u) \end{bmatrix} = \begin{bmatrix} a_1 u^2 + O(u^3) \\ a_2 u^2 + O(u^3) \\ a_3 u^2 + O(u^3) \end{bmatrix},
\]
where \(a_1 = 0,\)
\[
a_2 = \frac{\sqrt{2}(\beta - 2)^2 \left( \omega_0 \sqrt{(2\omega_0 + 3)(-16\beta + 6\omega_0 + 41)} + \sqrt{3}\beta(2\omega_0 + 3) - \sqrt{3}(\omega_0 + 1)(2\omega_0 + 3) \right)}{\beta^{5/2}(2\omega_0 + 3) \left( 3\sqrt{(2\omega_0 + 3)(-16\beta - 6\omega_0 - 41)} - 16\sqrt{3}\beta + \sqrt{3}(6\omega_0 + 41) \right)}.
\]
\[ a_3 = \frac{\sqrt{2}(\beta - 2)^2 (\omega_0 \left( -\sqrt{(2\omega_0 + 3)(-16\beta + 6\omega_0 + 41)} + \sqrt{3}(2\omega_0 + 3) - \sqrt{3}(\omega_0 + 1)(2\omega_0 + 3) \right) \sqrt{\beta^5/(2\omega_0 + 3)} \left( -3\sqrt{(2\omega_0 + 3)(-16\beta - 6\omega_0 - 41)} - 16\sqrt{3}\beta + \sqrt{3}(6\omega_0 + 41) \right)^2}}{\beta^5/(2\omega_0 + 3) \left( -3\sqrt{(2\omega_0 + 3)(-16\beta - 6\omega_0 - 41)} - 16\sqrt{3}\beta + \sqrt{3}(6\omega_0 + 41) \right)^2} \]

Therefore, the dynamics on the center manifold is given by the gradient-like equation

\[ u' = -\frac{(\beta - 2)u^3}{2\beta^2} = -\frac{d}{du} \left[ \frac{(\beta - 2)u^4}{8\beta^2} \right], \]

under the potential \( \Pi(u) = \frac{(\beta - 2)u^4}{8\beta^2} \), for which the origin is a degenerate local minimum whenever \( \beta > 2 \) (recall the existence conditions for \( P_1 \) are \( \beta > 2, \omega_0 \geq \frac{1}{6}(16\beta - 41) \)), and under these conditions, the center manifold of \( P_1 \) is stable. In the original variables (4.1) the center manifold can be locally expressed as the graph

\[ \begin{align*}
  x &= -\frac{(\beta - 2)u^2}{\sqrt{6}\beta^2}, \\
y &= -\frac{(\beta - 2)u^2(\beta - 2\omega_0 - 1) - 8\beta^2}{4\sqrt{2}\beta^5/2}, \\
z &= \frac{(\beta - 2)u}{\sqrt{6}\beta}, \\
\lambda &= u.
\end{align*} \]

According to the center manifold calculation, we have from (4.16a), the definition \( \lambda := -\frac{2\beta}{\omega_0} e^{-\varphi/2} \), and the definition (4.1), and introducing the time rescaling \( \frac{df}{d\tau} = \psi^2 \frac{df}{d\ln a} \) we have that (as \( \lambda \to 0 \)):

\[ \begin{align*}
  \frac{d\varphi}{d\tau} &= 4(2 - \beta) e^{-\varphi}, \\
  \frac{d\psi}{d\tau} &= 2(2 - \beta) \psi, \\
  \frac{dt}{d\tau} &= \frac{2}{\sqrt{2} \beta} e^{-\frac{1}{2}(\beta + 1)\psi} \left( 2\psi^2 e^{\beta} - (\beta - 2)(\beta - 2\omega_0 - 1) \right) \frac{1}{\sqrt{\beta} \sqrt{\psi_0}}.
\end{align*} \]

Integrating the equations, and using the first integral \( \ln \left[ \frac{a}{a_0} \right] = \int \psi^2 d\tau \) we have the general solution

\[ \begin{align*}
  \varphi(\tau) &= \ln |c_1 - 4(\beta - 2)\tau|, \\
  \psi(\tau) &= c_2 e^{-2(\beta-2)\tau}, \\
  a &= a_0 \exp \left[ -\frac{c_2 e^{-4(\beta-2)\tau}}{4(\beta - 2)} \right], \\
  t - t_0 &= \frac{\sqrt{2}}{\beta} \int |c_1 - 4(\beta - 2)\tau| \frac{1}{\psi} \left( (2 - \beta)(\beta - 2\omega_0 - 1) + 2c_2 e^{-4(\beta-2)\tau} |c_1 - 4(\beta - 2)\tau| \right) d\tau, \\
  t - t_0 &\approx \frac{\sqrt{2}}{2(\beta - 1)^{1/2} \sqrt{\psi_0}} \frac{1}{\beta} |c_1 + 8\tau| \frac{1}{\beta - 2} \frac{1}{\sqrt{\psi_0}} \quad \text{for large } \tau.
\end{align*} \]

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5 Cosmological consequences

By considering now the equations written in the dilatonic frame: (2.6a), (2.6b), (2.6c), and (2.6d), we can define the following observable quantities:

\[
\Omega_1 \equiv e^{-\varphi} \frac{\rho_1}{3H^2}, \quad \text{where } \rho_1 := e^\varphi \left( \frac{\omega_0 \dot{\varphi}^2}{2} - 3H \dot{\varphi} + V(\varphi) \right),
\]

\[
\Omega_2 \equiv e^{-\varphi} \frac{\rho_2}{3H^2}, \quad \text{where } \rho_2 \equiv \frac{1}{2} \dot{\psi}^2 + W(\psi),
\]

\[
w_1 \equiv \frac{p_1}{\rho_1}, \quad \text{where}
\]

\[
p_1 := \frac{4 (3 - 2\omega_0) H e^\varphi \dot{\varphi}}{4\omega_0 + 6} + \frac{12H e^\varphi}{2\omega_0 + 3} + \frac{e^\varphi (4V(\varphi) + 2(2\omega_0 + 5) V(\varphi) + (1 - 2\omega_0) \omega_0 \dot{\varphi}^2)}{4\omega_0 + 6},
\]

\[
w_2 = w_\psi \equiv \frac{p_2}{\rho_2}, \quad \text{where } p_2 \equiv \frac{1}{2} \dot{\psi}^2 - W(\psi), \quad \text{and}
\]

\[
w_{\text{tot}} := \frac{p_1 + p_2}{\rho_1 + \rho_2}.
\]

These cosmological parameters can be written in terms of the phase space variables expressed as

\[
\Omega_1 = x^2 \omega_0 - \sqrt{6} x + y^2,
\]

\[
\Omega_2 = -x^2 \omega_0 + \sqrt{6} x - y^2 + 1,
\]

\[
w_1 = \frac{3 \left( (2s - 5)y^2 - 2\omega_0 y^2 - 6z^2 + 4 \right) + 3x^2 \omega_0 (2\omega_0 - 1) + 2\sqrt{6} x (3 - 2\omega_0)}{3(2\omega_0 + 3) \left( x^2 \omega_0 - \sqrt{6} x + y^2 \right)},
\]

\[
w_2 = \frac{x^2 \omega_0 - \sqrt{6} x + y^2 + 2z^2 - 1}{x^2 (-\omega_0) + \sqrt{6} x - y^2 + 1},
\]

\[
w_{\text{tot}} = \frac{6(s - 1)y^2 + 2\omega_0 (6x^2 \omega_0 + 3z^2 - 5\sqrt{6} x + 6z^2 - 3) - 3\sqrt{6} x + 3}{6\omega_0 + 9}.
\]

\(w_{\text{tot}}\) is related to the deceleration for isotropic metrics by \(q = \frac{1}{2}(1 + 3w_{\text{tot}})\).

We continue with the discussion for the interpretations of the model for the choices studied in sections: 4.1 and 4.2. We finish the section with a discussion of the generic features of the models.

5.1 Case: \(W(\psi) = W_0 e^{k\psi}\) and \(V(\varphi) = V_0 e^{l\varphi}\)

In Table 2 we present the cosmological parameters corresponding to the formulation in the dilatonic frame as given by Eqs. (2.6) for \(W(\psi) = W_0 e^{k\psi}\) and \(V(\varphi) = V_0 e^{l\varphi}\). We have the following results:

- \(P_1\) satisfies \(\Omega_2 = 1 - \Omega_1 = \frac{(l-1)}{4+1}\), with \(w_1 = -1, w_2 = -1\) and \(w_{\text{tot}} = -1\). Both energy densities \(\Omega_1, \Omega_2\) are of the same order of magnitude, that is, it is a scaling solution.

We have proved that its center manifold is stable for \(1 < l \leq \frac{1}{16} (6\omega_0 + 25)\). Hence, this point is a late-time attractor.
The equilibrium points $P_2, P_3$ and $P_4$ satisfy $w_1 = 1$, $w_2 = 1$, $w_{\text{tot}} = 1$. That is, they represent stiff solutions. The three solutions are saddle therefore they are not relevant for the late-time cosmology, neither for the early-time cosmology.

The equilibrium point $P_5$, which exists $\omega_0 < -\frac{3}{2}$ or $-\frac{5}{6} \leq \omega_0 < -\frac{1}{2}$ or $\omega_0 > -\frac{1}{2}$, satisfies $\Omega_2 = \frac{(2\omega_0+3)(6\omega_0+5)}{3(2\omega_0+1)^2}$, $w_1 = -1$, $w_2 = -1$ and $w_{\text{tot}} = -1$. Both energy densities $\Omega_1, \Omega_2$ are of the same order of magnitude, that is, it is scaling solution. It is a sink for $l > -1, \omega_0 < -\frac{3}{2}$ or a saddle otherwise.

The equilibrium point $P_6$, which exists for $l < -2, \omega_0 = \frac{1}{6}(l - 4)(l + 2)$ or $l > 1, \omega_0 = \frac{1}{6}(l - 4)(l + 2)$ or $-2 < l < 1, \frac{1}{6}(l - 4)(l + 2) \leq \omega_0 < \frac{1}{2}(-l - 2)$ or $l < 1, \omega_0 < -\frac{5}{6}$ or $l > 1, \omega_0 < -\frac{1}{2}(-l - 2)$, satisfies $\Omega_2 = 0$, that is, the energy density of the dilatonic field is dominant and the energy density of the quintessence field is negligible. Furthermore, $w_1 = \frac{2(l^2-1)}{3(l^2+2\omega_0+2)} - 1$ and $w_{\text{tot}} = \frac{2(l^2-1)}{3(l^2+2\omega_0+2)} - 1$. That is, according to whether $w_{\text{tot}} < -1, w_{\text{tot}} = -1$ or $-1 < w_{\text{tot}} < -\frac{1}{3}$ it represent a phantom solution, a solution with $w_{\text{tot}} = -1$ or a quintessence solution. It is a sink for $l < -1, \omega_0 < -\frac{5}{6}$. It is a saddle otherwise.

The equilibrium point $P_7$ exists for $l > 1, \omega_0 \geq \frac{1}{6}(l - 4)(l + 2)$ or $l < -2, \omega_0 \geq \frac{1}{6}(l - 4)(l + 2)$ or $l = -2, \omega_0 > 0$ or $l = 1, \omega_0 > -\frac{3}{2}$ or $-2 < l < 1, \omega_0 > \frac{1}{2}(-l - 2)$ or $l > 1, \frac{1}{6}(-l - 2) < \omega_0 < -\frac{3}{2}$ or $-2 < l < 1, \omega_0 = \frac{1}{6}(l - 4)(l + 2)$. The cosmological observables are $\Omega_2 = 0$, $w_1 = \frac{2(l^2-1)}{3(l^2+2\omega_0+2)} - 1$ and $w_{\text{tot}} = \frac{2(l^2-1)}{3(l^2+2\omega_0+2)} - 1$. It is a saddle, therefore they are not relevant for the late-time cosmology, neither for the early-time cosmology.

The equilibrium point $P_8$, which exists for $\omega_0 > -\frac{3}{2}, \omega_0 \neq 0$, satisfy $\Omega_2 = 0$ that is, the energy density of the dilatonic field is dominant and the energy density of

| Label | $\Omega_2$ | $w_1$ | $w_2$ | $w_{\text{tot}}$ |
|-------|-----------|--------|--------|------------------|
| $P_1$ | $\frac{(l-1)}{l+1}$ | $1$ | $-1$ | $-1$ |
| $P_2$ | $-\frac{2\omega_0}{l+1} - 1$ | $1$ | $1$ | $1$ |
| $P_3$ | $\frac{(l-3)(l-2\omega_0-7)}{(l+1)^2}$ | $1$ | $1$ | $1$ |
| $P_4$ | $\frac{(l-3)(l-2\omega_0-7)}{(l+1)^2}$ | $1$ | $1$ | $1$ |
| $P_5$ | $\frac{(2\omega_0+3)(6\omega_0+5)}{3(2\omega_0+1)^2}$ | $-1$ | $-1$ | $-1$ |
| $P_6$ | $0$ | $\frac{2(l^2-1)}{3(l^2+2\omega_0+2)} - 1$ | Indeterminate | $2(l^2-1)$ |
| $P_7$ | $0$ | $\frac{2(l^2-1)}{3(l^2+2\omega_0+2)} - 1$ | Indeterminate | $2(l^2-1)$ |
| $P_8$ | $0$ | $\frac{3\omega_0+\sqrt{6\omega_0+9+3}}{3\omega_0}$ | $-1$ | $\frac{3\omega_0-\sqrt{6\omega_0+9+3}}{3\omega_0}$ |
| $P_9$ | $0$ | $\frac{3\omega_0+\sqrt{6\omega_0+9+3}}{3\omega_0}$ | $-1$ | $\frac{3\omega_0+\sqrt{6\omega_0+9+3}}{3\omega_0}$ |
| $P_{10}$ | $1$ | Indeterminate | $\frac{1}{7}$ | $\frac{1}{7}$ |

Table 2: Cosmological parameters corresponding to the formulation in the dilatonic frame as given by Eqs. (2.6) for $W(\psi) = W_0 e^{l\psi}$ and $V(\varphi) = V_0 e^{l\varphi}$. 
the quintessence field is negligible. Furthermore, \( w_1 = \frac{3\omega_0 - \sqrt{6\omega_0 + 7} + 3}{3\omega_0} \), \( w_2 = -1 \), and \( w_{\text{tot}} = \frac{3\omega_0 + \sqrt{6\omega_0 + 7} + 3}{3\omega_0} \). The total energy density represents a standard matter source with \( 0 < w_{\text{tot}} < 1 \). It is a saddle. Therefore they are not relevant for the late-time cosmology.

- The equilibrium point \( P_9 \), which exists for \( \omega_0 > -\frac{3}{2}, \omega_0 \neq 0 \), satisfy \( \Omega_2 = 0 \) that is, the energy density of the dilatonic field is dominant and the energy density of the quintessence field is negligible. Furthermore, \( w_1 = \frac{3\omega_0 + \sqrt{6\omega_0 + 7} + 3}{3\omega_0} \), \( w_2 = -1 \), and \( w_{\text{tot}} = \frac{3\omega_0 + \sqrt{6\omega_0 + 7} + 3}{3\omega_0} \). That is, the second fluid behaves as a cosmological constant whereas the effective equation of state (of the total cosmic budget) is that of quintessence field for \( -\frac{9}{8} < \omega_0 < -\frac{5}{6} \), a cosmological constant for \( \omega_0 = -\frac{5}{6} \) and the phantom field for \( -\frac{5}{6} < \omega_0 < 0 \). It is a sink for \( -2 < l \leq -1, \frac{1}{6}(l-4)(l+2) < \omega_0 < 0 \), or \( l > -1, -\frac{5}{6} < \omega_0 < 0 \) (in both cases it is a phantom attractor). It is a source for \( l \leq -2, \omega_0 > \frac{1}{6}(l-4)(l+2) \), or \( l > -2, \omega_0 > 0 \) (and it behaves as an standard matter source then). It is a saddle otherwise.

- The equilibrium point \( P_{10} \) satisfies \( \Omega_2 = 1 \). That is, it is dominated by the quintessence field and the contribution of the dilatonic field to the total energy density is negligible. It satisfies \( w_2 = \frac{1}{6} \) and \( w_{\text{tot}} = \frac{1}{6} \). This means that the corresponding cosmological solution mimics radiation. Interestingly, it is a saddle with a three dimensional stable manifold provided \( \omega_0 \geq \frac{45}{4} \).

In the Fig. 3 is presented evolution of the dimensionless energy densities \( \Omega_1, \Omega_2 \) and the observables \( w_2, w_{\text{tot}}, q \) vs \( \ln(a) \) for \( W(\psi) = W_0 e^{k\psi} \) and \( V(\varphi) = V_0 e^{l\varphi} \) with \( \omega_0 = 50, l = 8 \).

In the Fig. 4 is presented evolution of the dimensionless energy densities \( \Omega_1, \Omega_2 \) and the observables \( w_2, w_{\text{tot}}, q \) vs \( \ln(a) \) for \( W(\psi) = W_0 e^{k\psi} \) and \( V(\varphi) = V_0 e^{l\varphi} \) with \( \omega_0 = -2, l = -3 \).

### 5.2 Case: \( W(\psi) = W_0 e^{2\beta \psi} \) and \( V(\varphi) = V_0 e^{(\beta-1)\varphi} \)

For this model we have the same results for the first nine equilibrium points in Table 2, in section 5.1 replacing \( l = \beta - 1, k = 2\beta \), and the additional equilibrium points \( P_{11} - P_{15} \).

The recent early- and late-time attractors are the following:

- The stable manifold of \( P_1 \) is three dimensional for \( \beta > 2, \omega_0 \geq \frac{1}{6}(16\beta - 41) \).
- The equilibrium point \( P_5 \) is a sink for \( \beta > 0, \omega_0 < -\frac{5}{6} \).
- The equilibrium point \( P_6 \) is a sink for \( \beta < 0, \omega_0 < -\frac{5}{6} \).
- The equilibrium point \( P_8 \) is a sink for \( -1 < \beta \leq 0, \frac{1}{6}(\beta - 5)(\beta + 1) < \omega_0 < 0 \), or \( \beta > 0, -\frac{5}{6} < \omega_0 < 0 \). It is a source for \( \beta \leq -1, \omega_0 > \frac{1}{6}(\beta - 5)(\beta + 1) \), or \( \beta > -1, \omega_0 > 0 \).
- \( P_{13} \) is a source for \( \omega_0 \in \mathbb{R}, \beta = 0, \) or \( 0 < \beta < 3, \omega_0 < -\frac{2}{3} \).
- The observables for \( P_{11,12} \) and \( P_{14,15} \) have to be evaluated for specific choices of the parameters.
Figure 3: Qualitative evolution of the dimensionless energy densities $\Omega_1, \Omega_2$ and the observables $w_2, w_{\text{tot}}, q$ vs $\ln(a)$ for $W(\psi) = W_0 e^{k\psi}$ and $V(\varphi) = V_0 e^{l\varphi}$ with $\omega_0 = 50, l = 8$.

6 Some results for arbitrary potentials

In the sections 4.1 and 4.2 we have investigated an exponential potential $V(\varphi)$ for which $s$ is a constant such that $h$ is only a function of $\lambda$ that depends on the choice of $W$, and $F$ is identically zero. For complement these results, in this section we comment about the generic features of the equilibrium points of (4.4) for arbitrary $h(\lambda, s)$ and $F(s)$ functions.

Since the system is form-invariant under the change $(y, z, \lambda) \rightarrow (-y, -z, -\lambda)$. Thus, without losing generality we can investigate just the sector $y \geq 0, z \geq 0, \lambda \geq 0$. Henceforth, we will focus on the stability properties of the system (4.4) for the state vector $(x, y, z, \lambda)$.
Figure 4: Qualitative evolution of the dimensionless energy densities $\Omega_1, \Omega_2$ and the observables $w_2, w_{\text{tot}}, q$ vs $\ln(a)$ for $W(\psi) = W_0 e^{k\psi}$ and $V(\varphi) = V_0 e^{l\varphi}$ with $\omega_0 = -2, l = -3$.

defined in the phase space
\[
\left\{ (x, y, z, s, \lambda) : -\sqrt{6}x + x^2\omega_0 + y^2 + z^2 \leq 1, y \geq 0, z \geq 0, \lambda \geq 0, s \in \mathbb{R} \right\}.
\] (6.1)

The equilibrium points of (4.4) that are independent of $h(s, \lambda)$ are summarized below. In table 3 are shown the cosmological parameters corresponding to the formulation in the dilatonic frame as given by Eqs. (2.6) for the equilibrium points in the invariant set $\lambda = 0$ for arbitrary potentials.

$P_1: (x, y, z, \lambda, s) = \left( 0, \frac{\sqrt{2}}{\sqrt{1-s_c}}, 0, 0, s_c \right)$. 

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Exists for $s_c \leq -1$.
The eigenvalues are
\[
\left\{ 0, 0, -3, \frac{1}{2} \left( -\frac{3\sqrt{(s_c-1)(16F(s_c)+s_c-1)(16s_c+6\omega_0+25)}}{2\omega_0+3(s_c-1)} - 3 \right) \right\},
\]
\[
\frac{1}{2} \left( \frac{\sqrt{\sqrt{(s_c-1)(16F(s_c)+s_c-1)(16s_c+6\omega_0+25)}}}{2\omega_0+3(s_c-1)} - 3 \right).
\]
This line of equilibrium points contains the cases $s_c = \hat{s} : F(\hat{s}) = 0$, for which the eigenvalues simplify to
\[
\left\{ 0, 0, -3, \frac{1}{2} \left( -\frac{3\sqrt{16s_c+6\omega_0+25}}{2\omega_0+3} - 3 \right) \right\}.
\]
It is non-hyperbolic. The stable manifold of $P_1$ is three dimensional provided $s_c \leq -1, 1-s_c^2 < F(s_c) \leq -\frac{1}{16} (s_c-1)(16s_c+6\omega_0+25)$, in other case the stable manifold is lower dimensional.

$P_2(\hat{s}): (x, y, z, \lambda, s) = \left( -\sqrt{\frac{2}{3}}, 0, \sqrt{\frac{2\omega_0}{3}} - 1, 0, \hat{s} \right)$, where $F(\hat{s}) = 0$.

Exists for $\omega_0 < -\frac{3}{2}$.
The eigenvalues are $\{6, -1, 1, \hat{s} + 2, 2F'(\hat{s})\}$.
The equilibrium point is a saddle and has a four dimensional unstable manifold provided $\hat{s} > -2, F'(\hat{s}) > 0$. In other case the unstable manifold is lower dimensional.

$P_3(\hat{s}): (x, y, z, \lambda, s) = \left( \frac{\sqrt{6}}{s-1}, \frac{\sqrt{s+2}}{s-1}, \frac{\sqrt{s(s+3)-6\omega_0}-7}{s-1}, 0, \hat{s} \right)$.

Exists for
(a) $\hat{s} > 1, \omega_0 < -\frac{3}{2}$, or
(b) $\hat{s} > 1, -\frac{3}{2} < \omega_0 \leq \frac{1}{6} (s^2 + 3\hat{s} - 7)$.

The eigenvalues are
\[
\left\{ \frac{3}{s-1}, 6, -\frac{3}{2(\hat{s}-1)} - \frac{8\hat{s}\omega_0-1+102\omega_0+121}{2(\hat{s}-1)\sqrt{2\omega_0+1}}, -\frac{3}{2(\hat{s}-1)} + \frac{8\hat{s}\omega_0-1+102\omega_0+121}{2(\hat{s}-1)\sqrt{2\omega_0+1}}, -\frac{6\omega_0}{s-1} \right\}.
\]
It is a saddle.

$P_4(\hat{s}): (x, y, z, \lambda, s) = \left( \frac{\sqrt{6}}{s-1}, \frac{\sqrt{s+2}}{1-s}, \frac{\sqrt{s(s+3)-6\omega_0}-7}{1-s}, 0, \hat{s} \right)$. Exists for
(a) $\hat{s} = -2, \omega_0 < -\frac{3}{2}$, or
(b) $-2 < \hat{s} < -1, \omega_0 \leq \frac{1}{6} (s^2 + 3\hat{s} - 7)$, or
(c) $-1 < \hat{s} < 1, -\frac{3}{2} < \omega_0 \leq \frac{1}{6} (s^2 + 3\hat{s} - 7)$, or
(d) $-1 < \hat{s} < 1, \omega_0 < -\frac{3}{7}$.

The eigenvalues are
\[
\left\{ \frac{3}{s-1}, 6, -\frac{3}{2(\hat{s}-1)} - \frac{8\hat{s}\omega_0-1+102\omega_0+121}{2(\hat{s}-1)\sqrt{2\omega_0+1}}, -\frac{3}{2(\hat{s}-1)} + \frac{8\hat{s}\omega_0-1+102\omega_0+121}{2(\hat{s}-1)\sqrt{2\omega_0+1}}, -\frac{6\omega_0}{s-1} \right\}.
\]
It is a saddle.
\[ P_5(\hat{s}) : (x, y, z, \lambda, s) = \left( \frac{2\sqrt{\pi}}{2\omega_0 + 1}, 0, 0, \hat{s} \right), \text{ where } F(\hat{s}) = 0. \]

Exists for

\( (a) \quad \omega_0 < -\frac{3}{2}, \text{ or} \)
\( (b) \quad -\frac{3}{6} \leq \omega_0 < -\frac{1}{2}, \text{ or} \)
\( (c) \quad \omega_0 > -\frac{1}{2}. \)

The eigenvalues are \( \left\{ \frac{2}{2\omega_0 + 1}, \frac{2(\hat{s}-1)}{2\omega_0 + 1}, -3, -\frac{6\omega_0 + 5}{2\omega_0 + 1}, -\frac{4F'(\hat{s})}{2\omega_0 + 1} \right\} \).

It is a sink for \( \omega_0 < -\frac{3}{2}, \hat{s} < 1, F'(\hat{s}) < 0 \). It is a saddle otherwise.

\[ P_6(\hat{s}) : (x, y, z, \lambda, s) = \left( -\frac{\sqrt{\pi}(\hat{s}+1)}{s-2(\omega_0+1)}, \frac{\sqrt{(2\omega_0+3)(-\hat{s}(\hat{s}+2)+6\omega_0+8)}}{\sqrt{3(\hat{s}-2(\omega_0+1))}}, 0, 0, \hat{s} \right), \text{ where } F(\hat{s}) = 0. \]

Exists for

\( (a) \quad \hat{s} < -1, \omega_0 = \frac{1}{6} (\hat{s} - 2) (\hat{s} + 4), \text{ or} \)
\( (b) \quad \hat{s} < -1, \omega_0 < \frac{1}{2} (\hat{s} - 2), \text{ or} \)
\( (c) \quad -1 \leq \hat{s} \leq 2, \omega_0 < -\frac{3}{2}, \text{ or} \)
\( (d) \quad -1 < \hat{s} < 2, \frac{1}{6} (\hat{s} - 2) (\hat{s} + 4) \leq \omega_0 < \frac{1}{2} (\hat{s} - 2), \text{ or} \)
\( (e) \quad \hat{s} > 2, \omega_0 = \frac{1}{6} (\hat{s} - 2) (\hat{s} + 4), \text{ or} \)
\( (f) \quad \hat{s} > 2, \omega_0 < -\frac{3}{2}. \)

The eigenvalues are \( \left\{ -\frac{\sqrt{\pi}(\hat{s}+1)}{s-2(\omega_0+1)}, \frac{\sqrt{(2\omega_0+3)(-\hat{s}(\hat{s}+2)+6\omega_0+8)}}{\sqrt{3(\hat{s}-2(\omega_0+1))}}, -\frac{2(\hat{s}+1)F'(\hat{s})}{s-2(\omega_0+1)} \right\} \).

It is a sink for \( \omega_0 < -\frac{3}{2}, \hat{s} > 1, F'(\hat{s}) < 0 \), or a saddle otherwise.

\[ P_7(\hat{s}) : (x, y, z, \lambda, s) = \left( -\frac{\sqrt{\pi}(\hat{s}+1)}{s-2(\omega_0+1)}, \frac{\sqrt{(2\omega_0+3)(-\hat{s}(\hat{s}+2)+6\omega_0+8)}}{\sqrt{3(\hat{s}-2(\omega_0+1))}}, 0, 0, \hat{s} \right), \text{ where } F(\hat{s}) = 0. \]

Exists for

\( (a) \quad \hat{s} < -1, \frac{1}{2} (\hat{s} - 2) < \omega_0 < -\frac{3}{2}, \text{ or} \)
\( (b) \quad \hat{s} < -1, \omega_0 \geq \frac{1}{6} (\hat{s} - 2) (\hat{s} + 4), \text{ or} \)
\( (c) \quad \hat{s} = -1, \omega_0 > -\frac{3}{2}, \text{ or} \)
\( (d) \quad -1 < \hat{s} < 2, \omega_0 = \frac{1}{6} (\hat{s} - 2) (\hat{s} + 4), \text{ or} \)
\( (e) \quad -1 < \hat{s} < 2, \omega_0 > \frac{1}{2} (\hat{s} - 2), \text{ or} \)
\( (f) \quad \hat{s} = 2, \omega_0 > 0, \text{ or} \)
\( (g) \quad \hat{s} > 2, \omega_0 \geq \frac{1}{6} (\hat{s} - 2) (\hat{s} + 4). \)

The eigenvalues are \( \left\{ -\frac{\sqrt{\pi}(\hat{s}+1)}{s-2(\omega_0+1)}, \frac{\sqrt{(2\omega_0+3)(-\hat{s}(\hat{s}+2)+6\omega_0+8)}}{\sqrt{3(\hat{s}-2(\omega_0+1))}}, -\frac{2(\hat{s}+1)F'(\hat{s})}{s-2(\omega_0+1)} \right\} \).

It is always a saddle point.
Table 3: Cosmological parameters corresponding to the formulation in the dilatonic frame as given by Eqs. (2.6) for the equilibrium points in the invariant set \( \lambda = 0 \) for arbitrary potentials.

\[
\begin{align*}
\text{Label} & \quad \Omega_2 & \quad w_1 & \quad w_2 & \quad w_{\text{tot}} \\
P_1 & \frac{\omega_{-1}}{\omega_{-1}} & -1 & -1 & -1 \\
P_2(\hat{s}) & -\frac{\omega_{0}^2}{\omega_{-1}} - 1 & 1 & 1 & 1 \\
P_3(\hat{s}) & \frac{\omega_{0}(\omega_{-1} - \omega_{0}) + \gamma}{(\omega_{-1})^2} & 1 & 1 & 1 \\
P_4(\hat{s}) & \frac{\omega_{0}(\omega_{-1} - \omega_{0}) + \gamma}{(\omega_{-1})^2} & 1 & 1 & 1 \\
P_5(\hat{s}) & \frac{2(\hat{s} - 1)}{3(\omega_{-1} + 1)} & -1 & -1 & -1 \\
P_6(\hat{s}) & \frac{2(\hat{s} - 1)}{3(\omega_{-1} + 1)} & \text{Indeterminate} & \frac{2(\hat{s} - 1)}{3(\omega_{-1} + 1)} & -1 \\
P_7(\hat{s}) & \frac{2(\hat{s} - 1)}{3(\omega_{-1} + 1)} & \text{Indeterminate} & \frac{2(\hat{s} - 1)}{3(\omega_{-1} + 1)} & -1 \\
P_8(\hat{s}) & \frac{2\omega_{0} - \sqrt{6\omega_{0} + 9} + 3}{2\omega_{0}} & -1 & \frac{2\omega_{0} - \sqrt{6\omega_{0} + 9} + 3}{2\omega_{0}} \\
P_9(\hat{s}) & \frac{2\omega_{0} + \sqrt{6\omega_{0} + 9} + 3}{2\omega_{0}} & -1 & \frac{2\omega_{0} + \sqrt{6\omega_{0} + 9} + 3}{2\omega_{0}} \\
\end{align*}
\]

\( P_8(\hat{s}) \): \((x, y, z, \lambda, s) = \left( \sqrt{3 - \sqrt{6\omega_{0} + 9}} \omega_{0}, 0, 0, 0, \hat{s} \right) \), where \( F(\hat{s}) = 0 \).

- Exists for \( \omega_{0} > \frac{3}{2}, \hat{s} \neq 0 \).
- The eigenvalues are
  \[\left\{ -\frac{\sqrt{6\omega_{0} + 9} - 3}{2\omega_{0}}, -\frac{\sqrt{6\omega_{0} + 9} - 3}{2\omega_{0}}, \frac{6\omega_{0} - \sqrt{6\omega_{0} + 9} + 3}{\omega_{0}}, \frac{6\omega_{0} + \sqrt{6\omega_{0} + 9} + 3}{\omega_{0}}, \frac{\sqrt{6\omega_{0} + 9} - 3}{2\omega_{0}} \right\} \].
- It is a saddle.

\( P_9(\hat{s}) \): \((x, y, z, \lambda, s) = \left( \sqrt{3 + \sqrt{6\omega_{0} + 9}} \omega_{0}, 0, 0, 0, \hat{s} \right) \), where \( F(\hat{s}) = 0 \).

- Exists for \( \omega_{0} > \frac{3}{2}, \hat{s} \neq 0 \).
- The eigenvalues are
  \[\left\{ \frac{\sqrt{6\omega_{0} + 9} + 3}{2\omega_{0}}, \frac{\sqrt{6\omega_{0} + 9} + 3}{2\omega_{0}}, \frac{\sqrt{6\omega_{0} + 9} - 3}{2\omega_{0}}, \frac{\sqrt{6\omega_{0} + 9} - 3}{2\omega_{0}}, -\frac{\sqrt{6\omega_{0} + 9} + 3}{\omega_{0}} \right\} \].
- It is a source for
  (a) \( \hat{s} \leq 2, \omega_{0} > 0, F'(\hat{s}) < 0 \), or
  (b) \( \hat{s} > 2, \omega_{0} > \frac{1}{6} (\hat{s} - 2) (\hat{s} + 4), F'(\hat{s}) < 0 \). It is a sink for
  (a) \( \hat{s} \leq 1, -\frac{\hat{s}}{6} < \omega_{0} < 0, F'(\hat{s}) < 0 \), or
  (b) \( 1 < \hat{s} < 2, \frac{1}{6} (\hat{s} - 2) (\hat{s} + 4) < \omega_{0} < 0, F'(\hat{s}) < 0 \).

As we see, the stability of these points depends on the character of the zeros \( \hat{s} \) of the function \( F(\hat{s}) \), and their first order derivative evaluated at \( \hat{s} \). The function \( F(s) \) for the most common quintessence potentials [51] are displayed in table 1.
The points $P_1$- $P_9$ were found in the previous three examples, for which $s$ is a constant such that $h$ is only a function of $\lambda$ that depends on the choice of $W$, and $F$ is identically zero (that is, the problem can be reduced in one dimension). When $F(s)$ is not trivial, the above classification can be implemented straightforwardly, as for the specific choices of $F$ in table 1. The search of the equilibrium points with $\lambda \neq 0$ is not an easy task, and the success on it depends crucially on the choice of $h(s, \lambda)$. Indeed, for a given $h$, there are equilibrium points on the surface $x - 2\lambda z(h(s, \lambda) - 1) = 0$. On this surface the existence conditions of an equilibrium point are $\left\{(x, y, z, \lambda, s) : y \geq 0, z \geq 0, \lambda \geq 0, 2\omega \lambda + 3 \neq 0, z^2 (4\lambda^2 \omega_0 (h(s, \lambda) - 1)^2 + 1) + y^2 \leq 2\sqrt{6} \lambda z(h(s, \lambda) - 1) + 1 \right\}$.

For example, given $h(s, \lambda) \equiv 1$, we have the additional equilibrium points $(x, y, z, \lambda, s) = (0, 0, \sqrt{\frac{3}{5}}, 2, s_c)$, where $h(s_c, 2) = 1$. For $h(s, \lambda) = 1 - \frac{1}{s}$, we have the additional points $P_{11}$- $P_{15}$ investigated in section 4.2.

7 Discussion and Conclusions

In this work the Brans-Dicke action have been considered in the cosmological scenario of FLRW spacetime with spatially flat curvature; while a minimally coupled scalar field was considered as a matter source. We show that this action in the Einstein frame provides the dilatonic action integral and it is equivalent with the $\sigma$-models. The method of group invariant transformations, i.e., symmetries of differential equations, was applied in order to constraint the free functions of the theory and determine conservation laws for the gravitational field equations. We found that for a family of potentials there exists a Noetherian conservation law. From the admitted symmetries we derived the zero-order invariants and we derived specific solutions for the field equations which correspond to matter-like dominant eras. Additionally, we have studied the stability of the equilibrium points of the dynamical system for to specific and for arbitrary potentials.

For the model 1, corresponding to the formulation in the dilatonic frame as given by Eqs. (2.6) for $W(\psi) = W_0 e^{k\psi}$ and $V(\varphi) = V_0 e^{l\varphi}$, we have obtained the following main results. The equilibrium por $P_1$ corresponds to a solution with $w_{tot} = -1$. We have proved that its center manifold is stable for $1 < l \leq \frac{1}{16} (6\omega_0 + 25)$. We show this solution is an attractor in the dilatonic frame but it is an intermediate accelerated solution $a \simeq e^{At}$, $p := \frac{2}{3l}$, $\frac{3l^2}{4(3l^2 + 4\omega_0)} \lt p < \frac{2}{3}$, as $t \to \infty$, and not a de Sitter solution. The exponent $p$ is reduced, in a particular case, to the exponent already found for the Jordan’s and Einstein’s frames by [26]. We have obtained some equilibrium points, $P_2, P_3$ and $P_4$, that represent stiff solutions which are saddle. The equilibrium point $P_5$, satisfies $w_{tot} = -1$. It is a sink for $l > -1, \omega_0 < -\frac{3}{2}$ or a saddle otherwise. The equilibrium point $P_6$, corresponds to a solution where the energy density of the dilatonic field is dominant and the energy density of the quintessence field is negligible. According to whether $w_{tot} := \frac{2(l^2 - 1)}{3(l^2 + 2\omega_0 + s)} - 1$ satisfies $w_{tot} < -1, w_{tot} = -1$ or $-1 < w_{tot} < -\frac{1}{3}$, we have found it represents a phantom solution, a solution with $w_{tot} = -1$ or a quintessence solution. It is a sink for $l < -1, \omega_0 < -\frac{3}{2}$. It is a saddle otherwise. Other equilibrium points as $P_8$, mimics a standard dark matter source with $0 < w_{tot} < 1$. It is a saddle. The equilibrium point $P_9$, corresponds to
a solution where the energy density of the dilatonic field is dominant and the energy density of the quintessence field is negligible. Furthermore, \( w_1 = \frac{\omega_0 + \sqrt{\omega_0^2 + 9}}{3\omega_0} \), \( w_2 = -1 \), and \( \omega_{\text{tot}} = \frac{3\omega_0 + \sqrt{36\omega_0^2 + 9}}{3\omega_0} \). That is, the second fluid behaves as a cosmological constant whereas the effective equation of state (of the total cosmic budget) is that of quintessence field for \(-\frac{5}{6} < \omega_0 < -\frac{1}{3}\), a cosmological constant for \( \omega_0 = -\frac{1}{3} \), and the phantom field for \(-\frac{5}{6} < \omega_0 < 0\). It is a sink for \(-2 < l \leq -1, \frac{1}{6} (l-4)(l+2) < \omega_0 < 0\), or \(l > -1, -\frac{5}{6} < \omega_0 < 0\) (in both cases it is a phantom attractor). It is a source for \( l \leq -2, \omega_0 > \frac{1}{6} (l-4)(l+2)\), or \(l > -2, \omega_0 > 0\) (and it behaves as an standard matter source then). It is a saddle otherwise. Finally, the equilibrium point \( P_{10} \) is dominated by the quintessence field and the contribution of the dilatonic field to the total energy density is negligible. It satisfies \( w_2 = \frac{1}{3} \) and \( \omega_{\text{tot}} = \frac{1}{3} \). This means that the corresponding cosmological solution mimics radiation. Interestingly, it is a saddle with a three dimensional stable manifold provided \( \omega_0 \geq \frac{45}{7} \). These results illustrates the capabilities of the model.

For the second model, we have \( V(\varphi) = V_0 e^{(\beta-1)\varphi}, W(\psi) = W_0 \psi^{2\beta} \). The particular parameters where chosen to lead to Noether pointlike symmetries. For this model we have the same results for the first nine equilibrium points in Table 2, in section 5.1, and discussed before, by replacing \( l = \beta - 1, k = 2\beta \). We have found the additional equilibrium points \( P_{11} - P_{15} \), whose stability and cosmological observables has to be evaluated numerically.

We recall that the points \( P_{1-3}, P_9 \) were found in the previous examples, under the assumption \( s \) is a constant such that \( h \) is only a function of \( \lambda \) that depends on the choice of \( W \), and \( F \) is identically zero (that is, the problem can be reduced in one dimension). When \( F(s) \) is not trivial, the above classification can be implemented straightforwardly, as for the specific choices of \( F \) in table 1. The search of the equilibrium points with \( \lambda \neq 0 \) is not an easy task, and the success on it depends crucially on the choice of \( h(s, \lambda) \). For example, given \( h(s, \lambda) = 1 \), we have the additional equilibrium points \( (x, y, z, \lambda, s) = \left( 0, 0, \sqrt{\frac{3}{2}}, 2, s_e \right) \), where \( h(s_e, 2) = 1 \). For \( h(s, \lambda) = 1 - \frac{1}{s} \), we have the additional points \( P_{11} - P_{15} \) investigated in section 4.2. A more complete study requires the specification of the free functions and this is far the scope of the present research.

A possible generalization on the context of scalar-tensor theories will be of interested. In this respect, after dealing with two simple examples, we made the first steps to provide a complete dynamical system analysis of dilatonic JBD cosmology keeping the potentials arbitrary, which is a major improvement since it allows for the extraction of information that is related to the foundations of the cosmological model and not to the specific potentials forms. In particular, we apply an extended version of the method of \( f \)-devisers [51–53] - in the sense that is was developed for two free functions such that additionally to the \( f \)-deviser we have an \( h \)-deviser. Using this approach one first performs the analysis without the need of an a priori specification of the potentials forms, and in the end one just substitutes the specific potential forms in the results, instead of having to repeat the whole dynamical elaboration from the start. Therefore, the results are richer and more general, revealing the full capabilities of dilatonic JBD cosmology.
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