SPINORIALITY OF ORTHOGONAL REPRESENTATIONS OF REDUCTIVE GROUPS

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Abstract. Let $G$ be a connected reductive group over a field of characteristic 0, and $\varphi : G \to \text{SO}(V)$ an orthogonal representation. We give a simple criterion for whether $\varphi$ lifts to Spin$(V)$, in terms of the highest weights of the irreducible constituents of $\varphi$.

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1. Introduction

Let $G$ be a connected reductive algebraic group over a field $F$ of characteristic 0. Let $(\varphi, V)$ be a representation of $G$, which in this paper always means a finite-dimensional $F$-representation of $G$. Suppose
that $V$ is orthogonal, i.e., carries a symmetric nondegenerate bilinear form preserved by $\varphi$. Thus $\varphi$ is a morphism from $G$ to $\text{SO}(V)$. Write $\rho : \text{Spin}(V) \to \text{SO}(V)$ for the usual isogeny ([SV00]). Following ([Bou05], Chapter IX, Section 7, Exercise 7), we say that $\varphi$ is spinorial if it lifts to $\text{Spin}(V)$, i.e., if there exists a morphism $\hat{\varphi} : G \to \text{Spin}(V)$ so that $\varphi = \rho \circ \hat{\varphi}$. We call $\varphi$ aspinorial otherwise.

By an argument in Section 13, we may assume that $F$ is algebraically closed, which we do for the rest of this introduction. Let $T$ be a maximal torus of $G$. Write $\pi_1(G)$ for the fundamental group of $G$ (the cocharacter group of $T$ modulo the subgroup $Q(T)$ generated by coroots), and $T_V$ for a maximal torus of $\text{SO}(V)$ containing $\varphi(T)$. Then $\varphi$ induces a homomorphism $\varphi_* : \pi_1(G) \to \pi_1(\text{SO}(V)) \cong \mathbb{Z}/2\mathbb{Z}$, and $\varphi$ is spinorial iff $\varphi_*$ is trivial. If we take cocharacters $\nu_1, \ldots, \nu_r$ whose images generate $\pi_1(G)$, then $\varphi$ is spinorial iff each cocharacter $\varphi \circ \nu_i$ of $T_V$ lifts to $\text{Spin}(V)$. (See Section 3.) The cocharacters of $T_V$ which lift are precisely those whose pairing with a certain weight $\omega_{\Sigma}$ of $T_V$ is even.

This leads to a lifting criterion for the representation $(\varphi, V)$ in terms of the multiplicities $m_{\varphi}(\mu)$ of its weights $\mu$. This much was essentially done in [Bou05] and [PR95]. However, the determination of these multiplicities is difficult, and it is better to have a criterion in terms of the highest weights of the irreducible constituents of $\varphi$. This paper offers such a criterion, as requested in [PR95]; it is in the spirit of the Weyl Dimension Formula.

Write $g$ for the Lie algebra of $G$, and $X^*(T)$ for the character group of $T$. Suppose $\varphi = \varphi_\lambda$ is irreducible with highest weight $\lambda \in X^*(T)$. Write $\chi_\lambda(C)$ for the value of the central character of $\varphi_\lambda$ at the Casimir element. Given a cocharacter $\nu$ of $T$, put

$$|\nu|^2 = \sum_{\alpha \in R} \langle \alpha, \nu \rangle^2 \in 2\mathbb{Z}.$$  

(In Section 2 we review the usual pairing $\langle \alpha, \nu \rangle$, fix norms on $t$ and $t^*$ associated to the Killing form, and recall the formula for $\chi_\lambda(C)$.) Pick cocharacters $\nu_1, \ldots, \nu_r$ whose images generate $\pi_1(G)$, and consider the integer

$$p(G) = p(\nu_1, \ldots, \nu_r) = \frac{1}{2} \gcd (|\nu_1|^2, \ldots, |\nu_r|^2).$$

**Theorem 1.** Suppose that $g$ is simple. The quantity

$$q_\lambda = p(G) \cdot \frac{\dim V_\lambda \cdot \chi_\lambda(C)}{\dim g}$$

is an integer, and $\varphi_\lambda$ is spinorial iff $q_\lambda$ is even.
For example, if $G = \text{PGL}_n$ with $n$ even, then $p(G) = n - 1$ and $\dim g$ are odd. Therefore $\varphi_\lambda$ is spinorial iff the product $\dim V_\lambda \cdot \chi_\lambda(C)$ is even. (Throughout this paper we will regard a rational number as even if, when written in lowest terms, its numerator is even.)

The rational number $\frac{\dim V_\lambda \cdot \chi_\lambda(C)}{\dim g}$ is closely related to the “Dynkin index” $\text{dyn}(\varphi) \in \mathbb{N}$ of the representation (see [GOV97]).

**Theorem 2.** Suppose $\mathfrak{g}$ is simple and let $\alpha$ be a long root of $\mathfrak{g}$. Then

$$q_\lambda = p(G) \cdot |\alpha|^2 \cdot \frac{1}{2} \text{dyn}(\varphi).$$

Turning to reducible representations, note that for any representation $\varphi$, one can form an orthogonal representation $S(\varphi) = \varphi \oplus \varphi^\vee$. When $G$ is semisimple, $S(\varphi)$ is always spinorial. For the general case we have:

**Theorem 3.** $S(\varphi_\lambda)$ is spinorial iff the integers

$$\langle \lambda, \nu^z \rangle \cdot \dim V_\lambda$$

are even for all cocharacters $\nu$ of $T$.

In this formula, $\nu^z$ is the $z$-component of $\nu$ corresponding to the decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and $\mathfrak{g}'$ is the derived algebra of $\mathfrak{g}$. It is enough to check this parity condition for $\nu_1, \ldots, \nu_r$.

An arbitrary finite-dimensional orthogonal representation of a connected reductive group $G$ can be decomposed as

$$\varphi = S(\sigma) \oplus \bigoplus_j \varphi_j,$$

where each $\varphi_j$ is irreducible orthogonal and $\sigma$ is arbitrary. Theorem 8 in Section 9 gives a general lifting criterion for $\varphi$ in terms of the highest weights of the irreducible constituents. It is particularly straightforward when $\mathfrak{g}$ is simple:

**Theorem 4.** Suppose that $\mathfrak{g}$ is simple. Then $\varphi$ as above is spinorial iff an even number of the $\varphi_i$ are aspinorial.

Equivalently, if $\varphi_i = \varphi_{\lambda_i}$, then $\varphi$ is spinorial iff $\sum_i q_{\lambda_i}$ is even.

Returning to the irreducible case, consider $q_\lambda$ as a function of $\lambda$. Since (1) is a polynomial in $\lambda$, the “spinorial weights” form a periodic subset of the highest weight lattice. To be more precise, let
$X^\text{orth} \subset X^\text{adj}(T)$ be the set of highest weights of irreducible orthogonal representations.

**Theorem 5.** There is a $k \in \mathbb{N}$ so that for all $\lambda_0, \lambda \in X^\text{orth}$, the representation $\varphi_{\lambda_0}$ is spinorial iff $\varphi_{\lambda_0+2^k\lambda}$ is spinorial.

This paper is organized as follows. We establish notation in Section 2. In Section 3 we give a criterion for spinoriality in terms of the weights of $\varphi$. This approach is along the lines of [PR95] and the exercise in [Bou05]. This much settles the case of adjoint representations.

We advance the theory in Section 4 by employing an algebraic trick involving palindromic Laurent polynomials; this gives a lifting condition in terms of the integers

$$q_\varphi(\nu) = \frac{1}{2} \cdot \frac{d^2}{dt^2} \Theta_\varphi(\nu(t)) |_{t=1}$$

for cocharacters $\nu$. Here $\Theta_\varphi$ denotes the character of $\varphi$.

In Section 5 we essentially take two derivatives of Weyl’s Character Formula to give a preliminary expression for $q_\varphi(\nu)$. This expression involves the interesting quantity

$$(2) \sum_{w \in W} \text{sgn}(w) \langle w(\lambda), \nu \rangle^{N+2},$$

where $N$ is the number of positive roots of $T$ in $G$. Realizing this quantity as an anti $W$-invariant polynomial allows for simplification, which we perform in Section 6 to arrive at Proposition 7 a satisfactory expression for $q_\varphi(\nu)$. In Section 7 we explain the connection with the Dynkin Index. We treat $S(\varphi)$ in Section 8 where the main result is Theorem 9.

Section 9 works out the case of reducible orthogonal representations; Theorem 10 and the general lifting criterion are established here. Spinoriality for tensor products is understood in Section 10. In Section 11 we prove Theorem 11 the periodicity of the spinorial weights. We illustrate our theory with several examples in Section 12, in particular we work out the periodicity lattice for $\text{PGL}_2$, $\text{SO}_4$, and discuss the cases of type $A_n$ and $\text{SO}_n$.

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2. Preliminaries

2.1. Notation. Throughout this paper $G$ is a connected reductive algebraic group over $F$ with Lie algebra $\mathfrak{g}$. Until the final section, $F$ is algebraically closed. Write $\mathfrak{g}'$ for the derived algebra of $\mathfrak{g}$. We write $T$ for a maximal torus of $G$, with Lie algebra $\mathfrak{t}$ and Weyl group $W$. Let $t' = t \cap \mathfrak{g}'$. Let $\text{sgn}: W \to \{\pm 1\}$ be the usual sign character of $W$. Let $(X^*, R, X_*, R^\vee)$ be the root datum associated to $G$, as in [Spr98].

The groups $X^* = X^*(T) = \text{Hom}(T, \mathbb{G}_m)$ and $X_* = X_*(T) = \text{Hom}(\mathbb{G}_m, T)$ are the character and cocharacter lattices of $T$. One has injections $X^* \hookrightarrow t^*$ and $X_* \hookrightarrow t$ given by differentiation for the former, and $\nu \mapsto d\nu(1)$ for the latter. We will often identify $X^*$, $R$, $X_*$, and $R^\vee$ with their images under these injections. Let $Q(T) \subseteq X^*(T)$ be the group generated by the coroots of $T$ in $G$. Write $R^+$ for a set of positive roots of $T$ in $G$, and $\delta \in t^*$ for the half-sum of these positive roots. Let $w_0 \in W$ denote the longest Weyl group element.

In this paper all representations $V$ of $G$ are finite-dimensional $F$-representations, equivalently morphisms $\varphi: G \to \text{GL}(V)$ of algebraic groups. For $\mu \in X^*(T)$, write $V^\mu$ for the $\mu$-eigenspace of $V$, and put $m_\varphi(\mu) = \dim V^\mu$, the multiplicity of $\mu$ as a weight of $V$.

If $H$ is an algebraic group, write $H^0$ for the connected component of the identity.

2.2. Pairings. Write $\langle , \rangle_T: X^*(T) \times X_*(T) \to \mathbb{Z}$ for the pairing

$$\langle \mu, \nu \rangle_T = n \Leftrightarrow \mu(\nu(t)) = t^n$$

for $t \in F^\times$, and $\langle , \rangle_t: t^* \times t$ for the natural pairing. Note that for $\mu \in X^*(T)$ and $\nu \in X_*(T)$, we have

$$\langle d\mu, d\nu(1) \rangle_t = \langle \mu, \nu \rangle_T.$$

So we may drop the subscripts and simply write $\langle \mu, \nu \rangle$.

Write $(\, , \,)$ for the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{t}$; it may be computed by

$$(x, y) = \sum_{\alpha \in R} \alpha(x)\alpha(y),$$

for $x, y \in \mathfrak{t}$. Also set $|x|^2 = (x, x)$. In particular, for $\nu \in X_*(T)$ we have $|\nu|^2 = \sum_{\alpha \in R}(\alpha, \nu)^2$. The Killing form restricted further to $t'$ induces an isomorphism $\sigma: (t')^* \cong t'$. We use the same notation $(\, , \,)'$ to denote the inverse form on $(t')^*$ defined for $\mu_1, \mu_2 \in t'$ by

$$(\mu_1, \mu_2) = (\sigma(\mu_1), \sigma(\mu_2)).$$
In [Bou02] this form on \((t')^*\) is called the “canonical bilinear form” \(\Phi_R\). Write \(|y|^2 = (y, y)\) for \(y \in (t')^*\). For roots \(\alpha \in R\) we have
\[ |\alpha|^2 \cdot |\alpha^\vee|^2 = 4. \]

For \(\lambda \in X^*(T)\), the quantity
\[ |\lambda + \delta|^2 - |\delta|^2 = (\lambda, \lambda + 2\delta) \]
is equal to \(\chi_\lambda(C)\), the value of the central character of the irreducible representation \(\varphi_\lambda\) at the Casimir element \(C\). (See [Kir09].)

Let \(X^*(T)^+\) be the set of characters \(\lambda\) so that \(\langle \lambda, \alpha^\vee \rangle \geq 0\) for all \(\alpha \in R^+\).

### 3. Lifting Cocharacters

Following the approaches of [Bou05] and [PR95], we reformulate the lifting problem for an orthogonal representation in terms of its weights. Throughout this section \(G\) is a connected reductive group over an algebraically closed field \(F\), and \(T\) is a maximal torus of \(G\).

**Proposition 1.** Let \(\rho : \tilde{H} \to H\) be an isogeny of connected semisimple groups over \(F\), and \(\varphi : G \to H\) a morphism. Pick a maximal torus \(T_H \leq H\) containing \(\varphi(T)\), and write \(\varphi_* : X_*(T) \to X_*(T_H)\) for the induced map. Let \(\tilde{T}_H = \rho^{-1}(T_H) \leq \tilde{H}\), and write \(\rho_* : X_*(\tilde{T}_H) \to X_*(T_H)\) for the induced map. Then there exists a morphism \(\tilde{\varphi} : G \to \tilde{H}\) such that \(\rho \circ \tilde{\varphi} = \varphi\), iff \(\text{im } \varphi_* \subseteq \text{im } \rho_*\).

**Proof.** Let \(G' = (G \times_H \tilde{H})^0\), with projection maps \(\rho_G : G' \to G\) and \(\tilde{\varphi} : G' \to \tilde{H}\). We have the diagram:

\[
\begin{array}{ccc}
G' & \xrightarrow{\tilde{\varphi}} & \tilde{H} \\
\downarrow{\rho_G} & \nearrow{\varphi} & \downarrow{\rho} \\
G & \xrightarrow{\varphi} & H
\end{array}
\]

Put \(\tilde{T}_H = \rho^{-1}(T_H)\) and \(T' = \rho^{-1}_G(T)\). It is routine to check that the following are equivalent:

1. \(\varphi\) lifts to \(\tilde{\varphi} : G \to \tilde{H}\)
2. \(\tilde{\varphi}\) factors through \(\rho_G\)
3. \(\ker \rho_G \subseteq \ker \tilde{\varphi}\)
4. \(\ker \rho_G \subseteq \ker \tilde{\varphi}|_{T'}\)
5. \(\tilde{\varphi}|_{T'}\) factors through \(T\)
6. \(\varphi|_T\) lifts to \(\tilde{T}_H\)
(7) \( \varphi_* \) lifts in the diagram:

\[
\begin{array}{ccc}
X_*(\tilde{T}_H) & \xrightarrow{\rho_*} & X_*(T_H) \\
\varphi_* & \downarrow & \\
X_*(T) & \rightarrow & X_*(T_H)
\end{array}
\]

(8) \( \text{im } \varphi_* \subseteq \text{im } \rho_* \).

Let \( \pi_1(G) = X_*(T)/Q(T) \). A morphism between reductive groups maps one coroot lattice into the other, so \( \varphi_* \) descends to \( \varphi_* : \pi_1(G) \rightarrow \pi_1(H) \). Since \( \rho \) is an isogeny, moreover \( \rho_*(Q(\tilde{T}_H)) = Q(T_H) \). Therefore a lift \( \hat{\varphi} \) in the diagram (3) exists iff \( \varphi_* \) lifts in the diagram

\[
\begin{array}{ccc}
\pi_1(\tilde{H}) & \xrightarrow{\rho_*} & \pi_1(H) \\
\varphi_* & \downarrow & \\
\pi_1(G) & \rightarrow & \pi_1(H)
\end{array}
\]

**Corollary 1.** Let \( \nu_1, \ldots, \nu_r \) generate \( \pi_1(G) \). Then a lift \( \hat{\varphi} \) as in the above proposition exists iff \( \varphi_*(\nu_i) \in \text{im } \rho_* \) for each \( i \).

Let \( V \) be an orthogonal vector space over \( F \), and \( T_V \leq \text{SO}(V) \) a maximal torus. Write \( \rho : \text{Spin}(V) \rightarrow \text{SO}(V) \) for the usual spin double cover.

Let \( P_V \subset X^*(T_V) \) be the set of weights of \( V \) as a \( T_V \)-module. Then \( P_V = -P_V \), and for each \( \omega \in P_V \), the eigenspace \( V^\omega \) is one-dimensional. Suppose we have picked one representative from each nonzero pair \( \{\omega, -\omega\} \) to form a subset \( \Sigma \subset P_V \). Thus we have

\[
V = V^0 \oplus \sum_{\omega \in \Sigma} (V^\omega \oplus V^{-\omega}),
\]

with \( V^0 \), the fixed points under \( T_V \), being one-dimensional if \( \dim V \) is odd and trivial if \( \dim V \) is even. Moreover \( \Sigma \) is a basis of \( X^*(T_V) \) as a \( \mathbb{Z} \)-module. Let \( \tilde{T}_V = \rho^{-1}(T_V) \), a maximal torus in \( \text{Spin}(V) \). Put \( \omega_\Sigma = \sum_{\omega \in \Sigma} \omega \).

**Lemma 1.** A cocharacter \( \theta \in X_*(T_V) \) lies in \( \text{im } \rho_* \) iff

\[
\langle \omega_\Sigma, \theta \rangle = \sum_{\omega \in \Sigma} \langle \omega, \theta \rangle
\]

is even.

**Proof.** This condition is satisfied exactly by members of \( Q(T_V) \). \( \square \)
Definition 1. Let \((\varphi, V)\) be a representation of \(G\). For \(\nu \in X_*(T)\), put
\[
L_{\varphi}(\nu) = \sum_{\{\mu \in X^*(T) | \langle \mu, \nu \rangle > 0\}} m_{\varphi}(\mu) \langle \mu, \nu \rangle \in \mathbb{Z}.
\]

Proposition 2. Let \(\varphi : G \to SO(V)\) be an orthogonal representation. For \(\nu \in X_*(T)\), the cocharacter \(\varphi_*(\nu) \in \text{im}\rho_*\) iff \(L_{\varphi}(\nu)\) is even. If \(\nu_1, \ldots, \nu_r\) generate \(\pi_1(G)\), then \(\varphi\) is spinorial iff all the integers \(L_{\varphi}(\nu_i)\) are even.

(Compare Exercise 7 in Section 8, Chapter IX of [Bou05] and Lemma 3 in [PR95].)

Proof. We may assume that \(\varphi(T) \leq T_V\). Put \(\theta = \varphi \circ \nu \in X_*(T_V)\). Let \(\mu\) be a weight of \(V\) relative to \(T\). Then \(V^\mu\) is \(T_V\)-invariant, and therefore a direct sum of weight spaces for \(T_V\). Write \(\omega \in \mu\) whenever \(0 \neq V^\omega \subseteq V^\mu\). If \(\omega \in \mu\), then
\[
\langle \varphi_\ast \nu, \omega \rangle_{T_V} = \langle \nu, \mu \rangle_{T}.
\]
Note that if \(\langle \mu, \nu \rangle > 0\), then \(\omega \in \mu \Rightarrow -\omega \notin \mu\). We may therefore pick \(\Sigma \subset P_V\) so that if \(\langle \mu, \nu \rangle > 0\), and \(\omega \in \mu\), then \(\omega \in \Sigma\). If \(\omega \in \Sigma\), then \(\omega \in \mu\) for some \(\mu\) with \(\langle \mu, \nu \rangle \geq 0\).

It follows that
\[
\langle \omega_{\Sigma}, \theta \rangle = \sum_{\omega \in \Sigma} \langle \omega, \theta \rangle
\]
\[
= \sum_{\mu} \sum_{\omega \in \mu, \omega \in \Sigma} \langle \mu, \nu \rangle
\]
\[
= \sum_{\mu \mid \langle \nu, \mu \rangle > 0} \langle \mu, \nu \rangle \cdot \dim V^\mu
\]
\[
= L_{\varphi}(\nu).
\]

By Lemma 1 we deduce that \(\varphi \circ \nu \in \text{im}\rho_*\) iff \(L_{\varphi}(\nu)\) is even. From Proposition 1 we conclude that \(\varphi\) lifts iff \(L_{\varphi}(\nu)\) is even for all \(\nu \in X_*(T)\). \(\square\)

Since \(\varphi(Q(T)) \subseteq Q(T_V)\) we note:

Corollary 2. If \(\nu \in Q(T)\), then \(L_{\varphi}(\nu)\) is even.

For two representations \(\varphi_1, \varphi_2\), we have
\[
L_{\varphi_1 \oplus \varphi_2}(\nu) = L_{\varphi_1}(\nu) + L_{\varphi_2}(\nu),
\]
since \(m_{\varphi_1 \oplus \varphi_2}(\mu) = m_{\varphi_1}(\mu) + m_{\varphi_2}(\mu)\).
Corollary 3. The adjoint representation of $G$ on $\mathfrak{g}$ is spinorial iff $\delta \in X^*(T)$.

**Proof.** If $\varphi$ is the adjoint representation, then

$$L_{\varphi}(\nu) = \sum_{\{\alpha \in R | \langle \alpha, \nu \rangle > 0\}} \langle \alpha, \nu \rangle \equiv \sum_{\alpha \in R^+} \langle \alpha, \nu \rangle \mod 2 = 2 \langle \delta, \nu \rangle.$$

The corollary follows since the pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$ is perfect. \qed

Remark: For $G$ a compact connected Lie group, this corollary can be found in [Bou05].

4. Palindromy

This section is the cornerstone of our paper. The difficulty with determining the parity $L_{\varphi}(\nu)$ is in somehow getting ahold of “half” of the weights of $V$, one for each positive/negative pair. This amounts to knowledge of the polynomial part of a certain palindromic Laurent polynomial, and this we accomplish with a derivative trick.

**Definition 2.** For $(\varphi, V)$ a representation of $G$ and $\nu \in X_*(T)$, consider the function $Q_{(\varphi, \nu)} : F^* \to F$ defined by

$$Q_{(\varphi, \nu)}(t) = \Theta_{\varphi}(\nu(t)) = \text{tr}(\varphi(\nu(t))).$$

If $\varphi$ is understood we may simply write ‘$Q_{\nu}(t)$’. For $\gamma \in T$, we have

$$\Theta_{\varphi}(\gamma) = \sum_{\mu \in X^*} m_{\varphi}(\mu) \mu(\gamma),$$

so in particular

$$Q_{\nu}(t) = \sum_{\mu \in X^*} m_{\varphi}(\mu) t^{\langle \mu, \nu \rangle} \in \mathbb{Z}[t, t^{-1}].$$

We note

- $Q_{\nu}(1) = \dim V$,
- $Q'_{\nu}(1) = \sum_{\mu} m_{\varphi}(\mu) \langle \mu, \nu \rangle$,
- $Q''_{\nu}(1) = \sum_{\mu} (m_{\varphi}(\mu) \langle \mu, \nu \rangle)^2 - m_{\varphi}(\mu) \langle \mu, \nu \rangle$.
**Definition 3.** For \((\varphi, V)\) a representation of \(G\) and \(\nu \in X_*(T)\), we set
\[
q_\varphi(\nu) = \frac{1}{2}Q''_\nu(1).
\]

When \(\varphi\) is self-dual, \(m_\varphi(-\mu) = m_\varphi(\mu)\) for all \(\mu \in X^*\), so in this case:
- \(Q_\nu(t) = Q_\nu(t^{-1})\), i.e., \(Q_\nu\) is “palindromic”.
- \(Q'_\nu(1) = 0\).
- \(Q''_\nu(1) = \sum_\mu m_\varphi(\mu, \nu)^2 \in 2\mathbb{Z}\).

**Lemma 2.** For \(\varphi\) self-dual and \(\nu_1, \nu_2 \in X_*,\) we have \(q_\varphi(\nu) \in \mathbb{Z}\) and
\[
q_\varphi(\nu_1 + \nu_2) \equiv q_\varphi(\nu_1) + q_\varphi(\nu_2) \mod 2.
\]

**Proof.** Breaking the sum over \(\mu\) into a sum over nonzero pairs \(\{\mu, -\mu\}\) gives
\[
q_\varphi = \frac{1}{2} \sum_{\mu \in X^*} m_\varphi(\mu) \langle \mu, \nu \rangle^2
= \sum_{\{\mu, -\mu\}} m_\varphi(\mu) \langle \mu, \nu \rangle^2 \in \mathbb{Z}.
\]

Therefore
\[
q_\varphi(\nu_1 + \nu_2) = \sum_{\{\mu, -\mu\}} m_\varphi(\mu) \left( \langle \mu, \nu_1 \rangle^2 + 2\langle \mu, \nu_1 \rangle \langle \mu, \nu_2 \rangle + \langle \mu, \nu_2 \rangle^2 \right)
\equiv q_\varphi(\nu_1) + q_\varphi(\nu_2) \mod 2
\]

Thus \(q_\varphi\) may be viewed as a homomorphism from \(X_*(T)\) to \(\mathbb{Z}/2\mathbb{Z}\). Since \(Q_\nu\) is palindromic, it may be expressed in the form
\[
Q_\nu(t) = H_\nu(t) + H_\nu(t^{-1})
\]
for a unique polynomial \(H_\nu \in \mathbb{Z}[t] + \frac{1}{2}\mathbb{Z}\). Thus \(H_\nu\) has integer coefficients, except its constant term may be half-integral. More precisely,
\[
H_\nu(t) = \sum_{\langle \mu, \nu \rangle > 0} m_\varphi(\mu) t^{\langle \mu, \nu \rangle} + \frac{1}{2} \sum_{\langle \mu, \nu \rangle = 0} m_\varphi(\mu).
\]

What we want, at least mod 2, is the integer
\[
H'_\nu(1) = \sum_{\langle \mu, \nu \rangle > 0} m_\varphi(\mu, \nu) = L_\varphi(\nu).
\]

By calculus we compute
\[
Q''_\nu(1) = 2(H'_\nu(1) + H''_\nu(1)).
\]

But \(H''_\nu(1)\) is even! This gives the crucial result:
Proposition 3. If \( \varphi \) is self-dual, then
\[
L_{\varphi}(\nu) \equiv q_{\varphi}(\nu) \mod 2.
\]

Corollary 4. Let \( \varphi \) be an orthogonal representation of \( G \). Then \( \varphi \) is spinorial iff \( q_{\varphi}(\nu) \) is even for every \( \nu \) in a generating set for \( \pi_1(G) \).

Proof. This follows from Corollary 1, Proposition 2, and the above equation. \( \square \)

5. Two Derivatives of Weyl’s Character Formula

In this section we compute \( q_{\varphi}(\nu) \) when \( \varphi \) is irreducible. Our method follows the proof of Weyl’s Character Formula in [GW09]. For \( \lambda \in X^*(T)^+ \), write \( (\varphi_{\lambda}, V_{\lambda}) \) for the irreducible representation of \( G \) with highest weight \( \lambda \). For simplicity, we use the notation \( q_{\varphi_{\lambda}}, m_{\varphi_{\lambda}}(\mu) \), etc.

For \( \nu \in t \), put
\[
d_{\nu} = \prod_{\alpha \in R^+} \langle \alpha, \nu \rangle,
\]
and for \( \mu \in t^* \), put
\[
d_{\mu} = \prod_{\alpha \in R^+} \langle \alpha^\vee, \mu \rangle.
\]

Definition 4. Put
\[
t_{\text{reg}} = \{ \nu \in t \mid d_{\nu} \neq 0 \}.
\]

Extend \( q_{\lambda} \), originally with domain \( X_* \), to the polynomial function \( q_{\lambda} : t \to F \) defined by the formula
\[
q_{\lambda}(\nu) = \frac{1}{2} \sum_{\mu \in X^*} \langle \mu, \nu \rangle^2 m_{\lambda}(\mu).
\]

We let \( \mathbb{Z}[t^*] \) denote the usual algebra of the monoid \( t^* \) with basis \( e^\mu \) for \( \mu \in t^* \). It contains the elements
\[
J(e^\mu) = \sum_{w \in W} \text{sgn}(w) e^{w\mu} \quad \text{and} \quad \text{ch}(V_{\lambda}) = \sum_{\mu \in X^*} m_{\lambda}(\mu) e^\mu.
\]

Recall the Weyl Character Formula (Prop. 5.10 in [Jan03]):
\[
\text{ch}(V_{\lambda}) J(e^\delta) = J(e^{\lambda+\delta}).
\]

Write \( \varepsilon : \mathbb{Z}[t^*] \to \mathbb{Z} \) for the \( \mathbb{Z} \)-linear map so that \( \varepsilon(e^\mu) = 1 \) for all \( \mu \in t^* \) (i.e. the augmentation); it is a ring homomorphism. Given \( \nu \in t \), write \( \frac{\partial}{\partial \nu} : \mathbb{Z}[t^*] \to \mathbb{Z}[t^*] \) for the \( \mathbb{Z} \)-linear map so that \( \frac{\partial}{\partial \nu}(e^\mu) = \langle \mu, \nu \rangle e^\mu \); it is a \( \mathbb{Z} \)-derivation. Note that \( \varepsilon(\text{ch}(V_{\lambda})) = \dim V_{\lambda} \), and
\[
(7) \quad \left( \varepsilon \circ \frac{\partial^2}{\partial \nu^2} \right) \text{ch}(V_{\lambda}) = Q'_{\nu}(1).
\]
Proposition 4. For $\nu \in t_{\text{reg}}$, we have

$$q_\lambda(\nu) = \sum_{w \in W} \text{sgn}(w) \langle w(\lambda + \delta), \nu \rangle ^{N+2} (N+2)! \dim V_\lambda |\nu|^2,$$

where $N = |R^+|$.

Proof. We apply $\varepsilon \circ \frac{\partial^{N+2}}{\partial \nu^{N+2}}$ to both sides of $J(e^{\lambda+\delta}) = \text{ch}(V_\lambda)J(e^\delta)$. On the left we have

$$\left( \varepsilon \circ \frac{\partial^{N+2}}{\partial \nu^{N+2}} \right) J(e^{\lambda+\delta}) = \sum_{w \in W} \text{sgn}(w) \langle w(\lambda + \delta), \nu \rangle ^{N+2}. \tag{8}$$

The right hand side requires more preparation. For $\alpha \in R^+$, let $r_\alpha = e^{\alpha/2} - e^{-\alpha/2}$. Then

- $\varepsilon(r_\alpha) = 0$,
- $\varepsilon \circ \frac{\partial}{\partial \nu}(r_\alpha) = \langle \alpha, \nu \rangle$,
- $\frac{\partial^2}{\partial \nu^2} r_\alpha = \frac{1}{4} \langle \alpha, \nu \rangle^2 r_\alpha$,
- $J(e^\delta) = \prod_{\alpha \in R^+} r_\alpha$.

The last equality is a familiar identity from [Bou02]. We may now apply the following lemma:

Lemma 3. Let $R$ be a commutative ring, $D : R \to R$ a derivation, and $\varepsilon : R \to R'$ a ring homomorphism. Suppose that $r_1, \ldots, r_N \in \ker \varepsilon$. Then

1. $\varepsilon(D^n(r_1 \cdots r_N)) = 0$ for $0 \leq n < N$.
2. $\varepsilon(D^N(r_1 \cdots r_N)) = \frac{N!}{\prod_{i=1}^N \varepsilon(D(r_i))}$.
3. If also $D^2(r_i) \in \ker \varepsilon$ for all $i$ then $\varepsilon(D^{N+1}(r_1 \cdots r_N)) = 0$.
4. Suppose further that there are $c_i \in R$ so that $D^2(r_i) = c_i r_i$. Then

$$\varepsilon(D^{N+2}(r_1 \cdots r_N)) = \frac{(N+2)!}{6} \left( \prod_i \varepsilon(D(r_i)) \right) \left( \sum_i c_i \right).$$

Proof. This follows from the Leibniz rule for derivations:

$$D^n(r_1 \cdots r_k) = \sum_{i_1 + \cdots + i_k = n} \binom{n}{i_1, \ldots, i_k} D^{i_1}(r_1) \cdots D^{i_k}(r_k).$$

Thus in our case,

1. $(\varepsilon \circ \frac{\partial}{\partial \nu^n})J(e^\delta) = 0$ for $0 \leq n < N$,
2. $(\varepsilon \circ \frac{\partial}{\partial \nu^N})J(e^\delta) = N!d\nu$,
3. $(\varepsilon \circ \frac{\partial}{\partial \nu^{N+1}})J(e^\delta) = 0$,

$\square$
\[ (4) \quad (\varepsilon \circ \frac{\partial^{N+2}}{\partial \nu^{N+2}}) J(e^\delta) = \frac{(N+2)!}{24} d_\nu \sum_{\alpha > 0} \langle \alpha, \nu \rangle^2. \]

Now we are ready to consider
\[ (\varepsilon \circ \frac{\partial^{N+2}}{\partial \nu^{N+2}}) (\text{ch}(V_\lambda J(e^\delta))). \]

Applying the Leibniz rule to the above gives
\[ \binom{N+2}{2} Q''(1) N! d_\nu + \dim V_\lambda \frac{(N+2)!}{24} d_\nu \sum_{\alpha > 0} \langle \alpha, \nu \rangle^2. \]

Equating this with (8) yields the identity
\[ (9) \quad \sum_{w \in W} \text{sgn}(w) \langle w(\lambda+\delta), \nu \rangle^{N+2} = (N+2)! d_\nu \left( q_\lambda(\nu) + \frac{\dim V_\lambda}{24} \sum_{\alpha > 0} \langle \alpha, \nu \rangle^2 \right), \]

whence the proposition. \[\square\]

6. Anti-\(W\)-invariant Polynomials

The expression “\(\sum_{w \in W} \text{sgn}(w) \langle w(\lambda+\delta), \nu \rangle^{N+2}\)” in our formula demands simplification. This can be done by applying the theory of Anti-\(W\)-invariant polynomials.

Let \(f : t \to F\) be a polynomial function. We say that \(f\) is anti-\(W\)-invariant, provided that for all \(w \in W\) and \(\nu \in t\) we have
\[ f(w(\nu)) = \text{sgn}(w) f(\nu). \]

The polynomial \(\nu \mapsto d_\nu\) is a homogeneous anti-\(W\)-invariant polynomial of degree \(N\). According to [Bou02], page 118, if \(f\) is a homogeneous anti-\(W\)-invariant polynomial of degree \(d\), then there exists a homogeneous \(W\)-invariant polynomial \(p : t \to F\) so that \(f(\nu) = p(\nu)d_\nu\). Necessarily \(d = \deg f \geq N\) and \(p\) has degree \(d - N\). Similarly, if \(g : t^* \to F\) is a homogeneous anti-\(W\)-invariant polynomial, then \(g(\mu) = p(\mu)d_\mu\) for a \(W\)-invariant polynomial \(p\) on \(t^*\).

**Definition 5.** Let \(k\) be a nonnegative integer. Put
\[ F_k(\mu, \nu) = \sum_{w \in W} \text{sgn}(w) \langle w(\mu), \nu \rangle^k, \]
for \(\mu \in t^*\) and \(\nu \in t\).
Proposition 5. Let \( g \) be simple. Then

\[
F_k(\mu, \nu) = \begin{cases} 
0 & \text{if } 0 \leq k < N \text{ or } k = N + 1, \\
N! \cdot \frac{d_\mu d_\nu}{d_\delta} & \text{if } k = N, \\
\frac{(N + 2)!}{48|\delta|^2} \cdot \frac{d_\mu d_\nu}{d_\delta} |\mu|^2 |\nu|^2 & \text{if } k = N + 2.
\end{cases}
\]

Proof. Each \( F_k \) may be viewed as a polynomial in two ways: as a function of \( \mu \) and as a function of \( \nu \). It is either identically 0, or homogeneous of degree \( k \). Both the functions \( \mu \mapsto F_k(\mu, \nu) \) and \( \nu \mapsto F_k(\mu, \nu) \) are anti-\( W \)-invariant. Therefore \( F_k(\mu, \nu) \) either vanishes, or is the product of \( d_\mu d_\nu \) and a homogeneous \( W \)-invariant polynomial of degree \( k - N \) in both \( \nu \) and \( \mu \). By degree considerations, \( F_k \) must vanish for \( 0 \leq k < N \).

Case \( k = N \): Here \( F_N(\mu, \nu) = c d_\mu d_\nu \) for some constant \( c \in F \), independent of \( \mu \) and \( \nu \). To determine \( c \), we apply \( \varepsilon \circ \frac{\partial^N}{\partial \nu^N} \) to both sides of \( J(e^{\lambda + \delta}) = \text{ch}(V_\lambda) J(e^\delta) \). On the left we have

\[
(10) \quad \left( \varepsilon \circ \frac{\partial^N}{\partial \nu^N} \right) J(e^{\lambda + \delta}) = F_N(\lambda + \delta, \nu).
\]

On the right we proceed as in the proof of Proposition 4 to obtain

\[
N! \cdot \text{dim } V_\lambda \cdot d_\nu.
\]

Therefore

\[
c \cdot d_{\lambda + \delta} d_\nu = N! \cdot \text{dim } V_\lambda \cdot d_\nu,
\]

so that \( c = \frac{N!}{d_\delta} \).

Case \( k = N + 1 \): Since \( g \) is simple, both \( t \) and \( t^* \) are irreducible representations of \( W \). If \( \text{dim } t > 1 \), there is no 1-dimensional invariant subspace. When \( \text{dim } t = 1 \), \( W \) acts by a nontrivial reflection. Therefore there is no \( W \)-invariant vector, i.e., no \( W \)-invariant polynomial of degree 1. Thus in all cases \( F_{N+1} \) vanishes.

Case \( k = N + 2 \): Let us write \( F_{N+2}(\mu, \nu) = Q_\mu(\nu) d_\nu \) with \( Q_\mu \) a \( W \)-invariant quadratic form on \( t \). The corresponding bilinear form on \( t \) is \( W \)-invariant; as \( t \) is an irreducible \( W \)-representation, this bilinear form must be a scalar multiple of the Killing form. Thus we may write

\[
F_{N+2}(\mu, \nu) = c_R d_\mu d_\nu |\mu|^2 |\nu|^2;
\]

it remains to determine \( c_R \).
Employing \[ \text{Bou05}, \text{Ch. VIII}, \text{Section 9, Exercise 7} \], we obtain the value at \( \nu = \sigma(\delta) \in \mathfrak{t} \):

\[
Q''_{\sigma(\delta)}(1) = \sum_{\mu} \langle \mu, \sigma(\delta) \rangle^2 m_{\mu}(\mu)
= \frac{\text{dim} V_{\lambda}}{24} \cdot (\lambda, \lambda + 2\delta).
\]

Substituting this into (9) gives

\[
F_{N+2}(\lambda + \delta, \sigma(\delta)) = \frac{1}{2} d_{\sigma(\delta)}(N + 2)! \left( Q''_{\sigma(\delta)}(1) + \frac{\text{dim} V_{\lambda}}{24} |\delta|^2 \right)
= d_{\sigma(\delta)}(N + 2)! \cdot \frac{\text{dim} V_{\lambda}}{48} |\lambda + \delta|^2.
\]

On the other hand, from (11) we have

\[
F_{N+2}(\lambda + \delta, \sigma(\delta)) = c_R d_{\lambda + \delta} d_{\sigma(\delta)} |\lambda + \delta|^2 |\delta|^2
= c_R \text{dim} V_{\lambda} d_{\delta} d_{\sigma(\delta)} |\lambda + \delta|^2 |\delta|^2.
\]

We deduce that

\[
c_R = \frac{(N + 2)!}{48 d_{\delta} |\delta|^2}.
\]

The proposition follows from this.

\[ \square \]

For the general case, say \( \mathfrak{g} = \mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^\ell \oplus \mathfrak{z} \) with each \( \mathfrak{g}^i \) simple, and \( \mathfrak{z} \) abelian. A Cartan subalgebra \( \mathfrak{t} \subset \mathfrak{g} \) is the direct sum of the center \( \mathfrak{z} \) and Cartan subalgebras \( \mathfrak{t}^i \subset \mathfrak{g}^i \), and the Weyl group \( W = W(\mathfrak{g}, \mathfrak{t}) \) is the direct product of the Weyl groups \( W^i = W(\mathfrak{g}^i, \mathfrak{t}^i) \). Any \( \mu \in \mathfrak{t}^* \) is equal to \( \mu^z + \sum_i \mu^i \) with \( \mu^i \in (\mathfrak{t}^i)^* \) and \( \mu^z \in \mathfrak{z}^* \); similarly for \( \nu \in \mathfrak{t} \). Let \( N_i \) (resp. \( N \)) be the number of positive roots in \( \mathfrak{g}^i \) (resp. \( \mathfrak{g} \)).

**Proposition 6.** Let \( \mu \in \mathfrak{t}^* \) and \( \nu \in \mathfrak{t} \), with notation as above. Then

\[
F_k(\mu, \nu) = \begin{cases} 
0 & \text{if } 0 \leq k < N, \\
N! \cdot \frac{d_\mu d_\nu}{d_\delta} & \text{if } k = N, \\
(N + 1)! \cdot \frac{d_\mu d_\nu}{d_\delta} (\mu^z, \nu^z) & \text{if } k = N + 1, \\
\frac{(N + 2)!}{48} \cdot \frac{d_\mu d_\nu}{d_\delta} \sum_i |\mu^i|^2 |\nu^i|^2 + \frac{(N + 2)!}{2} \cdot \frac{d_\mu d_\nu}{d_\delta} (\mu^z, \nu^z)^2 & \text{if } k = N + 2.
\end{cases}
\]
Proof. If \( z = 0 \), we have

\[
F_k(\mu, \nu) = \sum_{w \in W} \text{sgn}(w) \langle w(\mu^1 + \cdots + \mu^\ell), \nu^1 + \cdots + \nu^\ell \rangle^k
\]

\[
= \sum_{w=(w_1, \ldots, w_\ell) \in W} \text{sgn}(w) \left( \sum_{i=1}^\ell \langle w_i(\mu^i), \nu^i \rangle \right)^k
\]

\[
= \sum_{w=(w_1, \ldots, w_\ell) \in W} \text{sgn}(w) \left( \sum_{k_1 + \cdots + k_\ell = k} \prod_{i=1}^\ell \langle w_i(\mu^i), \nu^i \rangle^{k_i} \right)
\]

\[
= \sum_{k_1 + \cdots + k_\ell = k} \left( \prod_{i=1}^\ell F_{k_i}(\mu^i, \nu^i) \right).
\]

The product \( \prod_i F_{k_i}(\mu^i, \nu^i) \) vanishes unless \( k_i \geq N_i \) for all \( i \). So \( F_k(\mu, \nu) \) vanishes for \( k < N \).

Now put \( k = N + 2 \). Since \( k_1 + \cdots + k_\ell = N + 2 \), we see by Proposition 5 that this product is only nonzero when some \( k_i = N_i + 2 \) and the other \( k_i \) equal \( N_i \). Therefore

\[
F_{N+2}(\mu, \nu) = \sum_{i=1}^\ell \binom{N+2}{N_1, \ldots, N_i+2, \ldots, N_\ell} F_{N_1}(\mu^1, \nu^1) \cdots F_{N_i+2}(\mu^i, \nu^i) \cdots F_{N_\ell}(\mu^\ell, \nu^\ell)
\]

\[
= \frac{(N+2)!}{48} \cdot \frac{d_\mu d_\nu}{d_\delta} \sum_{i=1}^\ell |\mu^i|^2 |\nu^i|^2 |\delta_i|^2.
\]

If \( z \neq 0 \), there is an extra term \( \frac{(N+2)!}{2} \cdot \frac{d_\mu d_\nu}{d_\delta} (\mu^z, \nu^z)^2 \). The other cases are similar. \( \square \)

Proposition 7. Let \( \mathfrak{g} \) be simple and \( \varphi = \varphi_\lambda \) irreducible self-dual. Then for all \( \nu \in \mathfrak{t} \), we have

\[
q_\lambda(\nu) = \frac{\dim V_\lambda \cdot \chi_\lambda(C)}{\dim \mathfrak{g}} \cdot \frac{|\nu|^2}{2}.
\]
Proof. Let $\nu \in \mathfrak{t}_{\text{reg}}$. By Proposition 4,

$$q_\lambda(\nu) = \frac{F_{N+2}(\lambda + \delta, \nu)}{(N + 2)!d_\nu} - \frac{1}{48} \dim V_\lambda |\nu|^2$$

$$= \frac{1}{48|\delta|^2} \cdot \frac{d_{\lambda + \delta}}{d_\delta} |\lambda + \delta|^2 |\nu|^2 - \frac{1}{48} \dim V_\lambda |\nu|^2$$

$$= \frac{1}{48|\delta|^2} \dim V_\lambda |\nu|^2 (|\lambda + \delta|^2 - |\delta|^2).$$

Recall that $\chi_\lambda(C) = |\lambda + \delta|^2 - |\delta|^2$. Moreover, by [Bou05], Exercise 7, page 256, we have $|\delta|^2 = \dim g/24$. These substitutions give the proposition for the case $\nu \in \mathfrak{t}_{\text{reg}}$; by continuity it holds for $\nu \in \mathfrak{t}$. $\square$

The general case is similar:

**Proposition 8.** With notation as before, and $\varphi_\lambda$ irreducible orthogonal, we have

$$q_\lambda(\nu) = \frac{1}{2} \dim V_\lambda \cdot \sum_i |\nu_i|^2 \chi_\lambda(C_i) \dim g_i.$$

Proof. For $\mathfrak{z} = 0$, we have

$$q_\lambda(\nu) = \frac{F_{N+2}(\lambda + \delta, \nu)}{(N + 2)!d_\nu} - \frac{1}{48} \dim V_\lambda |\nu|^2$$

$$= \frac{1}{48} \dim V_\lambda \sum_{i=1}^\ell |\lambda^i + \delta^i|^2 |\nu_i|^2 - \frac{1}{48} \dim V_\lambda \sum_i |\nu_i|^2$$

$$= \frac{1}{48} \dim V_\lambda \sum_{i=1}^\ell |\nu_i|^2 \left( \frac{|\lambda^i + \delta^i|^2 - |\delta^i|^2}{|\delta|^2} \right).$$

The substitution $|\delta|^2 = \dim g'/24$, gives the proposition in the semisimple case. If $\mathfrak{z} \neq 0$, one must add $\frac{1}{2} (\lambda, \nu^z)^2 \dim V_\lambda$. However for $\varphi_\lambda$ irreducible orthogonal, necessarily $\lambda$ annihilates the center. $\square$

**Corollary 5.** An irreducible orthogonal representation $\varphi_\lambda$ of $G$ is spinorial iff

$$q_\lambda(\nu) = \frac{1}{2} \dim V_\lambda \sum_i |\nu_i|^2 \chi_\lambda(C_i) \dim g_i$$

is even for all cocharacters $\nu$ in a generating set for $\pi_1(G)$.

Proof. This follows from Proposition 8 and Corollary 4. $\square$
Remark: By the following, spinoriality for irreducible orthogonal representations of connected reductive groups reduces to the semisimple case:

**Proposition 9.** Let $G$ be a connected reductive group and $\varphi : G \to \text{SO}(V)$ an irreducible orthogonal representation. Then $\varphi$ factors through the quotient $p : G \to G/Z(G)^\circ$, so that $\varphi = \varphi' \circ p$ with $\varphi' : G/Z(G)^\circ \to \text{SO}(V)$. Moreover $\varphi$ is spinorial iff $\varphi'$ is spinorial.

**Proof.** By Schur’s Lemma, $\varphi(Z(G))$ is a subgroup of the scalars in $\text{SO}(V)$, namely $\{\pm \text{id}_V\}$. Therefore $\varphi(Z(G)^\circ)$ is trivial. This gives the first part, and the second part is similar. □

### 7. Dynkin Index

Let $\mathfrak{g}$ be a simple Lie algebra with a long root $\alpha$. Following Dynkin, [[Dyn52]] we define a bilinear form on $\mathfrak{g}$ by

$$(x, y)_d = \frac{2}{|\alpha|^2}(x, y),$$

for $x, y \in \mathfrak{g}$. In other words, we renormalize the Killing form so that $(\alpha, \alpha)_d = 2$.

**Definition 6.** Let $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a homomorphism of simple Lie algebras. Then there exists an integer $\text{dyn}(\phi)$, called the Dynkin index of $\phi$, so that for $x, y \in \mathfrak{g}$, we have

$$(\phi(x), \phi(y))_d = \text{dyn}(\phi) \cdot (x, y)_d.$$

If $\phi \neq 0$, then $\text{dyn}(\phi) \neq 0$. Also, if $f' : \mathfrak{g}_2 \to \mathfrak{g}_3$ is another homomorphism of simple Lie algebras, then $\text{dyn}(f' \circ f) = \text{dyn}(f') \cdot \text{dyn}(f)$. We refer the reader to [[Dyn00]], page 195, Theorem 2.2, and (2.4).

We assume for the rest of this section that $\mathfrak{so}_V$ is simple, equivalently $\dim V \neq 1, 2, 4$. Note that there are no nontrivial irreducible orthogonal representations of $\mathfrak{g}$ with those degrees. The following is an easy calculation:

**Lemma 4.** If $\iota_V : \mathfrak{so}_V \hookrightarrow \mathfrak{sl}_V$ is the standard inclusion, then $\text{dyn}(\iota_V) = 2$.

□

Now let $\varphi : \mathfrak{g} \to \mathfrak{sl}_V$ be a nontrivial orthogonal Lie algebra representation. Then we may write $\varphi = \iota_V \circ \varphi'$, where $\varphi' : \mathfrak{g} \to \mathfrak{so}_V$. We define $\text{dyn}^o(\varphi) = \text{dyn}(\varphi') \in \mathbb{N}$; thus $\text{dyn}(\varphi) = 2 \text{dyn}^o(\varphi)$. 
Theorem 6. (Theorem 2.5 of [Dyn00], page 197) For $\varphi: g \to \mathfrak{sl}_V$ an irreducible representation of highest weight $\lambda$, we have

$$\text{dyn}(\varphi) = \frac{2 \dim V_\lambda \cdot \chi_\lambda(C)}{\dim g \cdot |\alpha|^2}.$$ 

Corollary 6. For $g$ simple and $\varphi_\lambda$ irreducible, we have

$$q_\lambda(\nu) = \frac{|\nu|^2}{2} \cdot \text{dyn}^o(\varphi_\lambda) \cdot |\alpha|^2.$$ 

8. Spinoriality of $\varphi \oplus \varphi^\vee$

For a representation $(\varphi, V)$ of our connected reductive group $G$, consider the orthogonal representation $(S(\varphi), V \oplus V^\vee)$ defined as follows. We give $V \oplus V^\vee$ the quadratic form

$$Q((v, v^*)) = \langle v^*, v \rangle,$$

and write $S(\varphi)$ for the representation of $G$ on $V \oplus V^\vee$ given by

$$g(v, v^*) = (\varphi(g)v, \varphi^\vee(g)v^*).$$

For $\nu \in X_*(T)$, we have

$$\det \varphi(\nu(t)) = t^{s_\varphi(\nu)},$$

where

$$s_\varphi(\nu) = \sum_{\mu \in X^*(T)} m_\varphi(\mu) \langle \mu, \nu \rangle.$$

Proposition 10. $L_S(\varphi)(\nu) \equiv s_\varphi(\nu) \mod 2$. Therefore $S(\varphi)$ is spinorial iff $s_\varphi(\nu)$ is even for all $\nu \in X_*(T)$.

Proof.

$$L_S(\varphi)(\nu) = \sum_{\{\mu|\mu, \nu\rangle > 0\}} (m_\varphi(\mu) + m_\varphi(-\mu)) \langle \mu, \nu \rangle$$

$$\equiv \sum_{\{\mu|\mu, \nu\rangle > 0\}} (m_\varphi(\mu) - m_\varphi(-\mu)) \langle \mu, \nu \rangle \mod 2$$

$$= s_\varphi(\nu).$$

Note that when $G$ is semisimple, the image of $\varphi$ lies in SL$(V)$, and therefore $s_\varphi(\nu) = 0$. Therefore by the proposition, $L_S(\varphi)(\nu)$ is even and so $S(\varphi)$ is spinorial in the semisimple case.

Now, assume $\varphi = \varphi_\lambda$ is irreducible. Let $\nu = \nu^z + \nu'$ correspond to the decompositions $t = t^z \oplus t'$. 
Theorem 7. $S(\varphi_\lambda)$ is spinorial iff the integers
$$\langle \lambda, \nu^z \rangle \cdot \dim V_\lambda$$
are even for all $\nu \in X_*(T)$.

Proof. Differentiating both sides of (12) at $t = 1$ gives
$$s_\varphi(\nu) = \tr d\varphi(d\nu(1)),$$
where $\tr : t_V \to F$ is the trace.

Write $\mathfrak{z}_V$ for the center of the Lie algebra of $GL(V)$. Write $t'_V$ for the Lie algebra of the maximal torus in $SL(V)$. We have a direct sum decomposition $t_V = t'_V \oplus \mathfrak{z}_V$, and similarly for $t$. Write $pr_V : t_V \to \mathfrak{z}_V$ and $pr : t \to \mathfrak{z}$ for the projections.

Note that the diagram
$$\begin{array}{ccc}
F & \xrightarrow{d\varphi} & t_V \xrightarrow{\tr} F \\
\downarrow{pr} & & \downarrow{pr_V} \\
\mathfrak{z} & \xrightarrow{d\varphi} & \mathfrak{z}_V
\end{array}$$
is commutative. Moreover $\tr(d\varphi(z)) = d\lambda(z) \cdot \dim V_\lambda$ for $z \in \mathfrak{z}$, by Schur’s Lemma. It follows that
$$s_\varphi(\nu) = d\lambda(\nu^z) \cdot \dim V_\lambda,$$
so the theorem follows from the previous proposition.

For example, let $G = GL_2$. We may identify $X^*(T)^+$ with integers $(m, n)$ with $0 \leq n \leq m$ via:
$$\lambda_{m,n} \left( \begin{array}{c} t_1 \\ t_2 \end{array} \right) = t_1^m t_2^n.$$

Let $\nu_0(t) = \left( \begin{array}{cc} t & 0 \\ 0 & 1 \end{array} \right)$, so that $(\nu_0)^z = \frac{1}{2}(1, 1)$. Then $\dim V_{\lambda_{m,n}} = m - n + 1$ and $\langle \lambda, \nu_0^z \rangle = \frac{1}{2}(m + n)$, so $s_{\lambda_{m,n}}(\nu_0) = \frac{1}{2}(m + n)(m - n + 1)$. From Theorem 7 we deduce that the representation $S(\varphi_{\lambda_{m,n}})$ of $GL_2$ is spinorial iff the integer $\frac{1}{2}(m + n)(m - n + 1)$ is even.

9. Reducible Orthogonal Representations

We begin this section by gathering our results to give a general lifting condition for reducible orthogonal representations.
9.1. General Lifting Condition. Recall we have \( g = g^1 \oplus \cdots \oplus g^t \oplus \mathfrak{z} \) with each \( g_i \) simple, and \( \mathfrak{z} \) abelian. Thus our \( \nu \in \mathfrak{t} \) decomposes into \( \nu^z + \sum_i \nu^i \) with \( \nu^i \in \mathfrak{t}^i \) and \( \nu^z \in \mathfrak{z} \).

**Proposition 11.** If \( \varphi \) is an orthogonal representation of \( G \), then \( \varphi \) is a direct sum of representations of the following type:

- Irreducible orthogonal representations.
- The representations \( S(\sigma) \), with \( \sigma \) irreducible.

**Proof.** This follows from Lemma C in Section 3.11 of [Sam90]. \( \square \)

**Theorem 8.** Let \( \varphi = S(\sigma) \oplus \bigoplus_j \varphi_j \), with each \( \varphi_j \) irreducible orthogonal with highest weight \( \lambda_j \), and \( \sigma = \bigoplus_k \sigma_k \), with each \( \sigma_k \) irreducible with highest weight \( \gamma_k \). Then \( \varphi \) is spinorial iff for all \( \nu \in X_*(T) \), the integer

\[
q_{\varphi}(\nu) = \sum_k \langle \gamma_k, \nu^z \rangle \cdot \dim V_{\gamma_k} + \sum_i \frac{|\nu^i|^2}{2} \sum_j \frac{\dim V_{\lambda_j} \cdot \chi_{\lambda_j}(C^i)}{\dim g^i} \cdot \dim g^i
\]

is even.

Again by Corollary 11 it suffices to take \( \nu_1, \ldots, \nu_r \) which generate \( \pi_1(G) \).

**Proof.** We have

\[
L_{\varphi}(\nu) = \sum_k L_{S(\sigma_k)}(\nu) + \sum_j L_{\varphi_j}(\nu)
\]

\[
\equiv \sum_k s_{\sigma_k}(\nu) + \sum_j q_{\varphi_j}(\nu) \mod 2.
\]

The first equality is by (4), and the congruence is by Proposition 10 and Proposition 3. The conclusion then follows from Theorem 7 and Corollary 5. \( \square \)

Note that when \( G \) is semisimple, the sum over \( k \) vanishes.

9.2. Case of \( g \) simple. When \( g \) is simple, the above simplifies to

\[
q_{\varphi}(\nu) = \frac{|\nu|^2}{2} \sum_j \frac{\dim V_{\lambda_j} \cdot \chi_{\lambda_j}(C)}{\dim g}.
\]

As in the introduction, pick cocharacters \( \nu_1, \ldots, \nu_r \) whose images generate \( \pi_1(G) \), and consider the integer

\[
p(G) = p(\nu_1, \ldots, \nu_r) = \frac{1}{2} \gcd \(|\nu_1|^2, \ldots, |\nu_r|^2|).
\]
Theorem 9. The representation $\varphi$ is spinorial iff the integer 
$$p(G) \cdot \sum_j \frac{\dim V_{\lambda_j} \cdot \chi_{\lambda_j}(C)}{\dim g}$$

is even.

Proof. By Theorem 8, it is enough to check that each 
$$\frac{|\nu_m|^2}{2} \sum_j \frac{\dim V_{\lambda_j} \cdot \chi_{\lambda_j}(C)}{\dim g}$$

is even for $1 \leq m \leq r$, thus we need only check $p(G)$ times this sum. 

□

Theorem 2 in the Introduction now follows from Corollary 6.

Note: In general, $p(G)$ and also $\text{ord}_2(p(G))$ depend on the choice of cocharacters. This leads to the curious consequence that if, for a given group $G$, the integer $\text{ord}_2(p(G))$ is not well-defined, then every orthogonal representation of $G$ is spinorial.

Corollary 7. Let $\mathfrak{g}$ be simple. The representation $\varphi$ is spinorial iff an even number of the $\varphi_i$ are aspinorial.

9.3. General $G$. Corollary 7 is not true for general $G$. We give here a counterexample, and the weaker statement Proposition 12 below.

Example: Let $G_1$ and $G_2$ be connected semisimple groups, with orthogonal representations $(\varphi_1, V_1)$ and $(\varphi_2, V_2)$ respectively. Let $G = G_1 \times G_2$, and write $\Phi_i : G_1 \times G_2 \to \text{SO}(V_i)$ for the inflations of $\varphi_1, \varphi_2$ to $G$ via the two projections. Put $\Phi = \Phi_1 \oplus \Phi_2$. For $\nu_1, \nu_2$ cocharacters of tori of $G_1, G_2$, put $\nu = \nu_1 \times \nu_2$. It is easy to see that 
$$L_{\Phi}(\nu) = L_{\varphi_1}(\nu_1) + L_{\varphi_2}(\nu_2).$$

Therefore in this situation, 
$$\Phi \text{ is spinorial } \iff \varphi_1 \text{ and } \varphi_2 \text{ are spinorial} \iff \Phi_1 \text{ and } \Phi_2 \text{ are spinorial}.$$

For example, if $G = \text{SO}(3) \times \text{SO}(3)$, and $\varphi_1, \varphi_2$ are aspinorial (e.g. the defining representation of $\text{SO}(3)$), then each of $\Phi_1, \Phi_2$, and $\Phi_1 \oplus \Phi_2$ is aspinorial.

Proposition 12. Let $\varphi_1, \varphi_2$ be orthogonal representations of $G$, and $\varphi = \varphi_1 \oplus \varphi_2$ their direct sum. If $\varphi_1$ is spinorial, then $\varphi$ is spinorial iff $\varphi_2$ is spinorial.
Proof. This follows from the equality $L_\varphi(\nu) = L_{\varphi_1}(\nu) + L_{\varphi_2}(\nu)$. □

10. Tensor Products

In this section, we explain how the spinoriality of a tensor product of two orthogonal representations is related to the spinoriality of the factors. Both internal and external tensor products are considered.

First, let $(\varphi_1, V_1), (\varphi_2, V_2)$ be orthogonal representations of connected reductive groups $G_1, G_2$, respectively. Write $(\varphi, V) = (\varphi_1 \boxtimes \varphi_2, V_1 \otimes V_2)$ for the external tensor product representation of $G = G_1 \times G_2$. If $T_1, T_2$ are maximal tori for $G_1, G_2$, then $T = T_1 \times T_2$ is a maximal torus of $G$.

Proposition 13. For $\nu = (\nu_1, \nu_2) \in X_*(T) = X_*(T_1) \oplus X_*(T_2)$, we have

$$q_\varphi(\nu) = \dim V_1 \cdot q_{\varphi_2}(\nu_2) + \dim V_2 \cdot q_{\varphi_1}(\nu_1).$$

Proof. For $t \in F^\times$, we have

$$\Theta_\varphi(\nu(t)) = \Theta_{\varphi_1}(\nu_1(t)) \Theta_{\varphi_2}(\nu_2(t)).$$

Therefore

$$Q''_{(\varphi, \nu)}(t) = Q_{(\varphi_1, \nu_1)}(t)Q''_{(\varphi_2, \nu_2)}(t) + 2Q'_{(\varphi_1, \nu_1)}(t)Q'_{(\varphi_2, \nu_2)}(t) + Q''_{(\varphi_1, \nu_1)}(t)Q_{(\varphi_2, \nu_2)}(t),$$

and so

$$Q''_{(\varphi, \nu)}(1) = \dim V_1 \cdot Q''_{(\varphi_2, \nu_2)}(1) + \dim V_2 \cdot Q''_{(\varphi_1, \nu_1)}(1).$$

The proposition follows. □

Corollary 8. With notation as above:

1. If $V_1, V_2$ are even-dimensional, then $\varphi$ is spinorial.
2. If $V_1$ is even-dimensional and $V_2$ is odd-dimensional, then $\varphi$ is spinorial iff $\varphi_1$ is spinorial.
3. If $V_1, V_2$ are odd-dimensional, then $\varphi$ is spinorial iff both $\varphi_1, \varphi_2$ are spinorial.

Next, let $G$ be a connected reductive group and $(\varphi_1, V_1), (\varphi_2, V_2)$ orthogonal representations of $G$. Write $(\varphi, V) = (\varphi_1 \otimes \varphi_2, V_1 \otimes V_2)$ for the (internal) tensor product representation of $G$. As in the previous proposition, we have

$$q_\varphi(\nu) = \dim V_1 \cdot q_{\varphi_2}(\nu) + \dim V_2 \cdot q_{\varphi_1}(\nu).$$

Therefore the first two statements of Corollary hold in this situation. As for the third statement, we offer:

Corollary 9. Suppose that $g$ is simple. If $V_1, V_2$ are odd-dimensional representations of $G$, then $\varphi$ is spinorial iff either $\varphi_1, \varphi_2$ are both spinorial, or $\varphi_1, \varphi_2$ are both aspinorial.
Proof. Since \( \dim V_1 \) and \( \dim V_2 \) are odd, we have
\[
q_{\varphi}(\nu) \equiv q_{\varphi_2}(\nu) + q_{\varphi_1}(\nu) \mod 2
\]
for all \( \nu \in X_\ast \). By Bézout’s Identity, there are integers \( b_i \) so that
\[
p(G) = \sum_i b_i |\nu_i|^2.
\]
Now
\[
\sum_i b_i q_{\varphi}(\nu_i) \equiv \sum_i b_i q_{\varphi_2}(\nu_i) + \sum_i b_i q_{\varphi_1}(\nu_i) \mod 2.
\]
If we let ‘\( \sum(\varphi) \)’ denote the sum in (13), then
\[
\sum_i b_i q_{\varphi}(\nu_i) = p(G) \sum(\varphi),
\]
and similarly for the right-hand side. Thus
\[
p(G) \sum(\varphi) \equiv p(G) \sum(\varphi_1) + p(G) \sum(\varphi_2) \mod 2.
\]
The conclusion then follows from Theorem 9. \( \square \)

11. Periodicity

For the representations \( \varphi_\lambda \), our lifting criterion amounts to determining the parity of one or more \( q_\lambda(\nu) \), each an integer-valued polynomial function of \( \lambda \). As we explain in this section, this entails a certain periodicity of the spinorial highest weights in the character lattice.

11.1. Polynomials with Integer Values. Let \( V \) be a finite-dimensional rational vector space, \( V^\ast \) its dual, \( L \) a lattice in \( V \), and \( L^\vee \) the dual lattice in \( V^\ast \). Recall that \( L^\vee \) is the \( \mathbb{Z} \)-module of \( \mathbb{Q} \)-linear maps \( f : V \to \mathbb{Q} \) so that \( f(L) \subseteq \mathbb{Z} \). Denote by \( \left( \frac{L^\vee}{\mathbb{Z}} \right) \) the \( \mathbb{Z} \)-algebra of polynomial functions on \( V \) which take integer values on \( L \). Given \( f \in L^\vee \), and \( n \in \mathbb{N} \), define \( \left( \frac{f}{n} \right) \in \left( \frac{L^\vee}{\mathbb{Z}} \right) \) by the prescription
\[
\left( \frac{f}{n} \right) : x \mapsto \left( \frac{f(x)}{n} \right) = \frac{f(x)(f(x) - 1) \cdots (f(x) - n + 1)}{n!}
\]
for \( x \in L \).

Proposition 14. The \( \mathbb{Z} \)-algebra \( \left( \frac{L^\vee}{\mathbb{Z}} \right) \) is generated by the \( \left( \frac{f}{n} \right) \) for \( f \in L^\vee \) and \( n \in \mathbb{N} \). If \( \{ f_1, \ldots, f_r \} \) is a \( \mathbb{Z} \)-basis of \( L^\vee \), then the products
\[
\left( \frac{f_1}{n_1} \right) \cdots \left( \frac{f_r}{n_r} \right),
\]
where \( n_1, \ldots, n_r \in \mathbb{N} \), form a basis of the \( \mathbb{Z} \)-module \((L^\vee)_\mathbb{Z}\).

**Proof.** See Proposition 2 in [Bou05], Chapter 8, Section 12, no. 4. □

Given a basis of \( V \), we can form the set \( C \) of its nonnegative linear combinations. Call \( C \) a “full polyhedral cone” if it arises in this way, and write \( L^+ = L \cap C \).

**Proposition 15.** Suppose \( f \) is a polynomial map from \( V \) to \( \mathbb{Q} \) that take integer values on \( L^+ \). Then \( f \in (L^\vee)_\mathbb{Z} \).

**Proof.** We omit the elementary proof (see [Jos18]) of the following lemma:

**Lemma 5.** Suppose that \( V \) is a finite-dimensional rational vector space, that \( C \) is a full polyhedral cone in \( V \), and that \( L \subset V \) is a lattice. Let \( p \in L \). Then

1. \( C \cap (p + C) \) is a translation of \( C \).
2. The intersection \( L \cap C \cap (p + C) \) is nonempty.
3. Suppose \( p' \) is in the above intersection, and write \( v = p' - p \). Then \( p + nv \in L \cap C \cap (p + C) \) for all positive integers \( n \).

Continuing with the proof of the proposition, let \( \ell \in L \); we must show that \( f(\ell) \in \mathbb{Z} \). By the lemma there is a \( v \in L \) so that \( \ell + nv \in L^+ \) for all positive integers \( n \). For \( x \in \mathbb{Z} \), put \( g(x) = f(\ell + xv) \). Then \( g \in \mathbb{Q}[x] \), and by hypothesis it takes integer values on positive integers. It is elementary to see that such a polynomial takes integer values at all integers, and in particular \( g(0) = f(\ell) \in \mathbb{Z} \).

□

**Lemma 6.** Fix an integer \( n \geq 1 \) and put \( k = \lfloor \log_2 n \rfloor + 1 \). Then \( \left( \frac{a + 2k}{n} \right) \equiv \left( \frac{a}{n} \right) \mod 2 \) for every integer \( a \geq 1 \).

**Proof.** This follows from the Lucas Congruence (see e.g., [Sta12]). □

**Proposition 16.** Let \( f \in \left( \frac{L^\vee}{\mathbb{Z}} \right) \). Then there is a \( k \in \mathbb{N} \) so that for all \( x, y \in L \) we have

\[
f(x + 2^k y) \equiv f(x) \mod 2.
\]

**Proof.** By Proposition 13 there are \( f_1, \ldots, f_r \in L^\vee \), integers \( n_1, \ldots, n_r \), and a polynomial \( g \in \mathbb{Z}[x_1, \ldots, x_r] \) so that

\[
f = g \left( \left( \frac{f_1}{n_1} \right), \ldots, \left( \frac{f_r}{n_r} \right) \right).
\]
Let \( k_i = \lceil \log_2 n_i \rceil + 1 \); by Lemma 6 we have

\[
\left( \frac{f_i}{n_i} \right)(x + 2^{k_i} y) \equiv \left( \frac{f_i}{n_i} \right)(x) \mod 2
\]

for all \( x, y \in L \). If we put \( k = \max(k_1, \ldots, k_r) \) we obtain the proposition. \( \square \)

11.2. **Example: Parity of Dimensions.** To illustrate the above, let \( G \) be connected reductive with notation as before. Take \( L = X^*(T) \subseteq V = X^*(T) \otimes \mathbb{Q} \rightarrow t^* \). Define \( f : t^* \rightarrow F \) by

\[
f(\lambda) = \frac{d_{\lambda+\delta}}{d_{\delta}}.
\]

The Weyl Dimension Formula says that for \( \lambda \in X^*(T)^+ \), we have \( f(\lambda) = \dim V_\lambda \in \mathbb{N} \). From Propositions 15 and 16 we deduce:

**Corollary 10.** With notation as above:

1. \( f(\lambda) \in \mathbb{Z} \) for all \( \lambda \in X^*(T)^+ \); equivalently \( f \in \left( \frac{X^*(T)}{Z} \right) \).

2. There is a \( k \in \mathbb{N} \) so that \( f(\lambda_0 + 2^k \lambda) \equiv f(\lambda_0) \mod 2 \) for all \( \lambda_0, \lambda \in X^*(T) \).

11.3. **Proof of Theorem 5.** We continue with \( G \) connected reductive. Now put

\[
X_{sd} = \{ \lambda \in X^*(T) \mid w_0 \lambda = -\lambda \}
\]

and

\[
X_{\text{orth}} = \{ \lambda \in X_{sd} \mid \langle \lambda, 2\delta^\vee \rangle \text{ is even} \}.
\]

According to [Bou05], \( X_{sd}^+ \) is the set of highest weights of irreducible self-dual representations, and \( X_{\text{orth}}^+ \) is the set of highest weights of irreducible orthogonal representations.

If \( g \) is simple put

\[
H(\lambda) = p(G) \cdot \frac{\dim V_\lambda \cdot \chi(\lambda)(C)}{\dim g}.
\]

Then:

1. \( H \) is a polynomial in \( \lambda \),

2. \( H(\lambda) \in \mathbb{Z} \) for \( \lambda \in X_{sd}^+ \), and

3. \( \varphi_\lambda \) is spinorial iff \( H(\lambda) \) is even.

If \( g \) is not necessarily simple, let \( \nu_1, \ldots, \nu_r \) generate \( \pi_1(G) \) as before. We may instead put

\[
H(\lambda) = 1 + \prod_{i=1}^r (q_\lambda(\nu_i) - 1),
\]
and the same three properties hold. From Propositions 15 and 16 we deduce:

**Corollary 11.** With notation as above,

1. \( H(\lambda) \in \mathbb{Z} \) for all \( \lambda \in X_{\text{orth}} \); equivalently \( H \in \left( \frac{X_{\text{orth}}^\vee}{\mathbb{Z}} \right) \).

2. There is a \( k \in \mathbb{N} \) so that \( H(\lambda_0 + 2^k \lambda) \equiv H(\lambda_0) \mod 2 \) for all \( \lambda_0, \lambda \in X_{\text{orth}} \).

Theorem 5 in the introduction follows from this. If we put \( L^+ = 2^k X_{\text{orth}}^+ \), then the theorem says that the set of spinorial highest weights is stable under addition from \( L^+ \). Since the index \( [X_{\text{orth}} : 2^k X_{\text{orth}}^+] \) is finite, the determination of the full set of spinorial weights amounts to a finite computation.

The problem of finding the exact largest lattice \( L \subseteq X_{\text{orth}} \) so that the spinorialities of \( \varphi_{\lambda_0} \) and \( \varphi_{\lambda_0 + \ell} \) agree for all \( \lambda \in X_{\text{orth}}^+ \) and \( \ell \in L^+ \) seems interesting, as does the problem of determining the proportion of spinorial irreducible representations. We do not settle these questions here, but see the next section for \( \text{PGL}_2 \) and \( \text{SO}_4 \), and [Jos18] for more examples.

### 12. Examples

**12.1. Type \( A_n \).** For \( G = \text{PGL}_n \), the fundamental group \( \pi_1(G) \) is cyclic of order \( n \), generated by \( \nu_0(t) = \text{diag}(t, 1, \ldots, 1) \) (mod center). If \( n \) is odd, then all orthogonal representations are spinorial, since there are no nontrivial homorphisms from \( \pi_1(G) \) to \( \mathbb{Z}/2\mathbb{Z} \). So assume \( n \) is even. We have \( |\nu_0|^2 = 2(n - 1) \) and so \( p(G) = n - 1 \). By Theorem 9, \( \varphi_\lambda \) is spinorial iff

\[
\frac{\dim V_\lambda \cdot \chi_\lambda(C)}{n + 1}
\]

is even. Since \( n \) is even, this expression is congruent to \( \dim V_\lambda \cdot \chi_\lambda(C) \) modulo 2. This proves:

**Proposition 17.** For \( G = \text{PGL}_n \), the representation \( \varphi_\lambda \) is spinorial iff \( \dim V_\lambda \cdot \chi_\lambda(C) \) is even.

Note that irreducible orthogonal representations of \( \text{GL}_n \) factor through \( \text{PGL}_n \) by Proposition 9, so this settles the case of irreducible orthogonal representations of \( \text{GL}_n \).

For \( d|n \), write \( G_d \) for the cover of \( \text{PGL}_n \) with fundamental group \( \mathbb{Z}/d\mathbb{Z} \). These comprise all groups of type \( A_{n-1} \). We may take

\[
p(G_d) = \left( \frac{n}{d} \right)^2 (n - 1).
\]
Therefore when \( n \) is even, \( \varphi_\lambda \) is spinorial iff

\[
\left( \frac{n}{d} \right)^2 \dim V_\lambda \cdot \chi_\lambda(C)
\]

is even. (Again when \( n \) is odd, \( \varphi_\lambda \) is always spinorial.)

Let us examine \( G = \text{PGL}_2 \) more closely. We have \( X^*(T) = X_{sd} = X_{\text{orth}} \). For integers \( j \geq 0 \) define \( \lambda_j \in X^*(T) \) by

\[
\lambda_j \left( \begin{pmatrix} a \\ b \end{pmatrix} \right) = (ab^{-1})^j.
\]

Then \( \dim V_{\lambda_j} = 2j + 1 \) and \( \chi_{\lambda_j}(C) = \frac{1}{2}(j^2 + j) \), so \( \varphi_{\lambda_j} \) is spinorial iff

\[
\frac{j(j + 1)(2j + 1)}{2}
\]

is even. Equivalently, \( j \equiv 0, 3 \mod 4 \). We may therefore take \( k = 2 \) in Theorem 5.

12.2. Orthogonal Groups.

Consider \( G = \text{SO}_n \) with \( n \geq 5 \). We compute that \( p(G) = n - 2 \). Applying Theorem 5 again, it follows that \( \varphi_\lambda \) is spinorial iff the integer

\[
\frac{2(n - 2)}{n^2 - n} \dim V_\lambda \cdot \chi_\lambda(C)
\]

is even.

For \( G = \text{SO}_4 \), the Lie algebra \( \mathfrak{g} \) is not simple. Again \( X^*(T) = X_{sd} = X_{\text{orth}} \). Now \( \text{Spin}_4 \) is \( \text{SL}_2 \times \text{SL}_2 \), whose irreducible representations are the external tensor products \( V_{a,b} = \text{Sym}^a V_0 \otimes \text{Sym}^b V_0 \), where \( V_0 \) is the standard 2-dimensional representation of \( \text{SL}_2 \). Here \( a, b \) are nonnegative integers; the representation \( V_{a,b} \) descends to a representation of \( G \) when \( a \equiv b \mod 2 \). Let \( \nu_s = \text{diag}(s, -s) \in \mathfrak{t} \); then \( \nu_{s,t} = (\nu_s, \nu_t) \) corresponds to a cocharacter of \( G \) iff either \( s, t \in \mathbb{Z} \), or \( 2s \) and \( 2t \) are both odd integers. Proposition 8 gives

\[
q_{a,b}(\nu_{s,t}) = s^2(b + 1) \binom{a + 2}{3} + t^2(a + 1) \binom{b + 2}{3}.
\]

The representation \( V_{a,b} \) of \( G \) is spinorial iff for all \( s, t \) as above, this integer is even. Equivalently, the integer

\[
F(a, b) = \frac{1}{4} \left( (b + 1) \binom{a + 2}{3} + (a + 1) \binom{b + 2}{3} \right)
\]

is even. It is elementary to see that \( F(a + 8i, b + 8j) \equiv F(a, b) \mod 2 \) for integers \( i, j \). In particular we may take \( k = 3 \) in Theorem 5.
13. Reduction to the Case of an Algebraically Closed Field

For this section, $G$ is a connected reductive group defined over a field $F$ of characteristic 0, not necessarily algebraically closed. Let $V$ be a quadratic vector space over $F$, and $\varphi : G \to \text{SO}(V)$ a homomorphism defined over $F$. The isogeny $\rho : \text{Spin}(V) \to \text{SO}(V)$ is also defined over $F$. By extending scalars to the algebraic closure $\overline{F}$ of $F$, we may use the rest of this paper to determine whether there exists a lift $\hat{\varphi} : G \to \text{Spin}(V)$ of $\varphi$ defined over $\overline{F}$.

Lemma 7. If $\hat{\varphi} : G \to \text{Spin}(V)$ is a lift defined over $\overline{F}$, then it arises from a lift defined over $F$.

Proof. The Galois group acts by Zariski-continuous automorphisms on the $\overline{F}$-points of $G$ and Spin$(V)$. We must show that for every $\sigma \in \text{Gal}(F)$ and $x \in G(\overline{F})$, we have $\sigma \hat{\varphi}(x) = \hat{\varphi}(\sigma x)$. Since $\rho$ and $\varphi$ are defined over $F$, the identity $\rho(\hat{\varphi}(x)) = \varphi(x)$ implies that

$$\rho(\hat{\varphi}(x)^{-1} \cdot \sigma^{-1} \hat{\varphi}(\sigma x)) = 1.$$ 

Thus the argument of $\rho$ above gives a Zariski-continuous map $G(\overline{F}) \to \ker \rho$. Since $G$ is connected and $\ker \rho$ is discrete, it must be that $\hat{\varphi}(x) = \sigma^{-1} \hat{\varphi}(\sigma x)$, and the lemma follows.

Therefore: The $F$-representation $\varphi$ is spinorial iff its extension to $\overline{F}$-points is spinorial.

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