On Mutually Orthogonal Graph-Path Squares

Ramadan El-Shanawany

Department of Physics and Engineering Mathematics, Faculty of Electronic Engineering, Menoufia University, Menouf, Egypt
Email: Ramadan_elshanawany380@yahoo.com

Received 14 January 2015; accepted 14 December 2015; published 17 December 2015

Abstract

A decomposition \( \mathcal{G} = \{G_0, G_1, \ldots, G_s\} \) of a graph \( H \) is a partition of the edge set of \( H \) into edge-disjoint subgraphs \( G_0, G_1, \ldots, G_s \). If \( G_i \cong G \) for all \( i \in \{0,1,\ldots,s-1\} \), then \( \mathcal{G} \) is a decomposition of \( H \) by \( G \). Two decompositions \( \mathcal{G} = \{G_0, G_1, \ldots, G_s\} \) and \( \mathcal{F} = \{F_0, F_1, \ldots, F_n\} \) of the complete bipartite graph \( K_{m,n} \) are orthogonal if, \( E(G_i) \cap (F_j) = \emptyset \) for all \( i, j \in \{0,1,\ldots,n-1\} \). A set of decompositions \( \{\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_s\} \) of \( K_{m,n} \) is a set of \( k \) mutually orthogonal graph squares (MOGS) if \( \mathcal{G}_i \) and \( \mathcal{G}_j \) are orthogonal for all \( i, j \in \{0,1,\ldots,k-1\} \) and \( i \neq j \). For any bipartite graph \( G \) with \( n \) edges, \( N_G(n) \) denotes the maximum number \( k \) in a largest possible set \( \{\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_s\} \) of MOGS of \( K_{m,n} \) by \( G \). Our objective in this paper is to compute \( N_G(n) \) where \( G = \mathbb{P}_{d+1}(F) \) is a path of length \( d \) with \( d+1 \) vertices (i.e. Every edge of this path is one-to-one corresponding to an isomorphic to a certain graph \( F \)).

Keywords

Orthogonal Graph Squares, Orthogonal Double Cover

1. Introduction

In this paper we make use of the usual notation: \( K_{m,n} \) for the complete bipartite graph with partition sets of sizes \( m \) and \( n \), \( P_{n+1} \) for the path on \( n+1 \) vertices, \( D \cup F \) for the disjoint union of \( D \) and \( F \), \( D \cup^c F \) for the union of \( D \) and \( F \) with \( L_e \) (set of vertices) that belong to each other (i.e. union of \( D \) and \( F \) with common vertices of the set \( L_e \) belong to \( F \) and \( D \)), \( K_n \) for the complete graph on \( n \) vertices, \( K \) for an isolated vertex. The other terminologies not defined here can be found in [1].

A decomposition \( \mathcal{G} = \{G_0, G_1, \ldots, G_s\} \) of a graph \( H \) is a partition of the edge set of \( H \) into edge-disjoint sub-
graphs \( G_0, G_1, \ldots, G_{s-1} \). If \( G_i \cong G \) for all \( i \in \{0,1,\ldots,s-1\} \), then \( G \) is a decomposition of \( H \) by \( G \). Two decompositions \( G = \{G_0, G_1, \ldots, G_{s-1}\} \) and \( F = \{F_0, F_1, \ldots, F_{s-1}\} \) of the complete bipartite graph \( K_{n,n} \) are orthogonal if \( E(G_i) \cap \{F_j\} = \{1\} \) for all \( i,j \in \{0,1,\ldots,n-1\} \). Orthogonality requires that \( E(G_i) = n = E(F_j) \) for all \( i \in \{0,1,\ldots,n-1\} \). A set of decompositions \( \{G_0, G_1, \ldots, G_{s-1}\} \) of \( K_{n,n} \) is a set of \( k \) mutually orthogonal graph squares (MOGS) if \( G_i \) and \( G_j \) are orthogonal for all \( i, j \in \{0,1,\ldots,k-1\} \) and \( i \neq j \). We use the notation \( N(n,G) \) for the maximum number \( k \) in a largest possible set \( \{G_0, G_1, \ldots, G_{k-1}\} \) of MOGS of \( K_{n,n} \) by \( G \), where \( G \) is a bipartite graph with \( n \) edges.

If two decompositions \( G \) and \( F \) of \( K_{n,n} \) by \( G \) are orthogonal, then \( G \cup F \) is an orthogonal double cover of \( K_{n,n} \) by \( G \). Orthogonal decompositions of graphs and orthogonal double covers (ODC) of graphs have been studied by several authors; see the survey articles [2][3].

It is well-known that orthogonal Latin squares exist for every \( n \neq 2, 6 \). A family of \( k \)-orthogonal Latin squares of order \( n \) is a set of \( k \) Latin squares any two of which are orthogonal. It is customary to denote \( N(n) = \max \{k : 3k MOLS\} \) be the maximal number of squares in the largest possible set of mutually orthogonal Latin squares MOLS of side \( n \). A decomposition of \( K_{n,n} \) by \( nK_2 \) is equivalent to a Latin square of side \( n \); two decompositions \( G \) and \( F \) of \( K_{n,n} \) by \( nK_2 \) are orthogonal if and only if the corresponding Latin squares of side \( n \) are orthogonal; and thus \( N(n,nK_2) = N(n) \). The computation of \( N(n) \) is one of the most difficult problems in combinatorial designs; see the survey articles by Abel et al. [4] and Colbourn and Dinitz in [5]. Since \( N(n,G) \) is a natural extension of \( N(n) \), the study of \( N(n,G) \) for general graphs is interesting.

El-Shanawany [6] establishes the following: i) \( N(n,K_{n,n}) = 2 \); ii) \( N(2, P^s) = 2, N(3, P^t) = 3, N(5, P_5) = 5 \) and \( N(7, P_7) = 7 \); iii) let \( p \) be a prime number, then \( N(p, K_p + ((p-1)/2)P_2) = p \); iv) let \( p \) be a prime number, then \( N(p, (p-2)K_2 + P_2) \geq p - 1 \). Based on ii), El-Shanawany [6] proposed:

**Conjecturer 1.** Let \( p \) be a prime number. Then \( N(p, P_{p+1}) = p \).

Sampathkumar et al. [7] have proved El-Shanawany conjectured. In the following section, we present another technique to prove this conjecture in Theorem 8.

The two sets \( \{0,1,\ldots,(n-1)\} \) and \( \{0,1,\ldots,(n-1)\} \) denote the vertices of the partite sets of \( K_{n,n} \). The length of the edge \( x_0y_1 \) of \( K_{n,n} \) is defined to be the difference \( y-x \), where \( x, y \in \mathbb{Z}_n \). Note that sums and differences are carried over in \( \mathbb{Z}_n \) (that is, sums and differences are carried modulo \( n \)). Let \( G \) be a subgraph of \( K_{n,n} \) without isolated vertices and let \( a \in \{0,1,\ldots,n-1\} \). The \( a \)-translate of \( G \), denoted by \( G + a \), is the edge-induced subgraph of \( K_{n,n} \) induced by \( \{(x+a)(y+a) : x_0y_1 \in E(G)\} \). A subgraph \( G \) of \( K_{n,n} \) is half-starter if \( E(G) = n \) and the lengths of all edges in \( G \) are mutually different.

**Lemma 2 (see [8]).** If \( G \) is a half-starter, then the union of all translates of \( G \) forms an edge decomposition of \( K_{n,n} \) (i.e. \( E(K_{n,n}) = \bigcup_{a \in \mathbb{Z}_n} E(G + a) \)).

In what follows, we denote a half-starter \( G \) by the vector \( v(G) = (v_0, v_1, \ldots, v_{n-1}) \in \mathbb{Z}_n^n = \mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n \), where \( v_0, v_1, \ldots, v_{n-1} \in \mathbb{Z}_n \) and \( v_i \) can be obtained from the unique edge \( (v_i)_0(v_i + i) \) of length \( i \) in \( G \).

**Theorem 3 (see [8]).** Two half-starters \( v(G) = (v_0, v_1, \ldots, v_{n-1}) \in \mathbb{Z}_n^n \) and \( v(F) = (u_0, u_1, \ldots, u_{n-1}) \in \mathbb{Z}_n^n \) are orthogonal if \( \{v_i - u_i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n \).

If two half-starters \( v(G) \) and \( v(F) \) are orthogonal, then the set of translates of \( G \) and the set of translates of \( F \) are orthogonal.

A set of decompositions \( \{G_0, G_1, \ldots, G_{k-1}\} \) of \( K_{n,n} \) is a set of \( k \) mutually orthogonal graph squares (MOGS) if \( G_i \) and \( G_j \) are orthogonal for all \( i, j \in \{0,1,\ldots,k-1\} \) and \( i \neq j \).

Note that
\[
\bigcup_{i=0}^{k-1} G_i = \bigcup_{i=0}^{k-1} \bigcup_{a \in \mathbb{Z}_n} E(G_i + a) = kE(K_{n,n}).
\]

In the following, we define a \( G \)-square over additive group \( \mathbb{Z}_n \). **Definition 4 (see [6]).** Let \( G \) be a subgraph of \( K_{n,n} \). A square matrix \( L \) of order \( n \) is called an \( G \)-square if every element in \( \mathbb{Z}_n \) occur exactly \( n \) times, and the graphs \( G_i, i \in \mathbb{Z}_n \) with
We have already from Lemma 2 and Definition 4 that every half starter vector \( v(G) \) and its translates are equivalent to \( G \)-square. For more illustration, the first matrix \( \mathcal{L}_0 \) in equation (1) is equivalent to the first row in Figure 3, which represented by the half starter vector \( v(G) = (0,0,2) \) and its translates.

**Definition 5.** Two squares matrices \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \) of order \( n \) are said to be orthogonal if for any ordered pair \( (a,b) \), there is exactly one position \( (x,y) \) for \( \mathcal{L}_0(x,y) = a \) and \( \mathcal{L}_1(x,y) = b \).

Now, we shall derive a class of mutually orthogonal subgraphs of \( K_{n,n} \) by a given graph \( G \) as follow.

**Definition 6.** A set of matrices \( \{ \mathcal{L}_i \} \) of \( K_{n,n} \) is called a set of \( k \) mutually orthogonal graph squares (MOGS) if \( \mathcal{L}_i \) and \( \mathcal{L}_j \) are orthogonal for all \( i,j \in \{0,1,\ldots,k-1\} \) and \( i \neq j \).

**Definition 7 (see [9]).** Let \( F \) be a certain graph, the graph \( F \)-path denoted by \( \mathbb{P}_{d+1}(F) \), is a path of a set of vertices \( \forall \{v_i : 0 \leq i \leq d\} \) and a set of edges \( \exists \{E_i : 0 \leq i \leq d-1\} \) if and only if there exists the following two bijective mappings:

1) \( \Psi : \exists \rightarrow \mathcal{F} \) defined by \( \Psi(E_i) = F_i \), where \( \mathcal{F} = \{F_0,F_1,\ldots,F_{d+1}\} \) is a collection of \( d \) graphs, each one is isomorphic to the graph \( F \).

2) \( \phi : \forall \rightarrow \mathcal{Y} \) defined by \( \phi(\forall_i) = X_i \), where \( \mathcal{Y} = \{X_i : 0 \leq i \leq d; \bigcap_i X_i = \emptyset\} \) is a class of disjoint sets of vertices (i.e., \( \mathcal{Y} \) decomposed into \( d+1 \) disjoint sets such that no two vertices within the same set are adjacent).

As a special case if the given graph \( F \) is isomorphic to \( K_{1,1} \) then \( \mathbb{P}_{d+1}(K_{1,1}) \), is the natural path \( P_{d+1} \) that is, \( \mathbb{P}_{d+1}(K_{1,1}) = P_{d+1} \).

For more illustration, see Figure 1, Figure 2.

---

**Figure 1.** \( \mathbb{P}_4(K_{2,2}) \), the path of 6 sets of vertices (every set has only 2 disjoint vertices) and 5 edges of \( K_{2,2} \).

**Figure 2.** \( \mathbb{P}_4(K_{1,3}) \), the path of 6 sets of vertices and 5 edges of \( K_{1,3} \).
Consider \( s \geq 0 \) paths of length \( k \geq 1 \), all attached to the same vertex (root vertex). This tree will be called \( T(s,k) \). Clearly, \( T(s,1) \) is the star with \( s \) edges and \( T(1,k) \) is the path with \( k \) edges. Define \( L_s \) as a set of all leaves of \( T(s,k) \) i.e., \( L_s = \{v : v\text{ is a leaf in } T(s,k)\} \) and \( |L_s| = s \).

In the following section, we will compute \( N(n,G) \) where \( G = E_{d, t}^s(F) \) such that \( F = K_{t,1} \) as in theorem 8 and \( F = T(3,3) \cup^s K_1 \) as in theorem 11.

2. Mutually Orthogonal Graph-Path Squares

The following result was shown in [7]. Here we present another technique for the proof.

**Theorem 8.** Let \( q \) be a prime number. Then \( N(0, P_{d,t}^q(K_{t,1})) = q \).

**Proof.** Let \( G_j \) be a subgraph of \( K_{d,t}^q \) with \( q \) edges; for fixed \( j \in \mathbb{Z}_q \) and \( 0 \leq i \leq q-1 \), define the \( q \) half-starter vectors as follows, \( \psi(G_j) = (j + i + j - 1, 2i + j - 2^2, \ldots, (-2)^{(i+2)} + j, (i+1) + j) \); our task is to prove the orthogonality of those \( q \) half-starter vectors in mutually. Let us define the half starter vector \( \psi(G_{ij}) \) as \( \psi(G_{ij}) = k(i-k) + j \) for all \( k \in \mathbb{Z}_q \). Then for all two different elements \( k, l \in \mathbb{Z}_q \), we have
\[
\psi(G_{ij}) - \psi(G_{ij}) = k(i-k) - l(i-l) = (k-l)(i-(k+l)),
\]
then \( \psi_k, \psi_l \) are mutually orthogonal half-starter vectors of graphs \( G_{ij} \) and \( G_{ij} \) of \( K_{d,t}^q \) respectively iff \( (i-(k+l), q) = 1 \). It remains to prove the isomorphism of \( \psi(G_{ij}) \) half-starter graphs \( G_{ij} \) of \( K_{d,t}^q \) for all \( k \in \mathbb{Z}_q \), Let
\[
\psi(G_{ij}) = k(i-k) + j = \psi_j(G_{ij}) = k(i-k) + s,
\]
and therefore \( j = s \). Furthermore, if \( \psi_k(G_{ij}) = k(i-k) + j = \psi_j(G_{ij}) = l(i-l) + j \), then \( (k-l)(i-(k+l)) = 0 \), since \( i-(k+l) \neq 0 \) (orthogonality of \( \psi_k \) and \( \psi_j \)), and therefore \( k = l \). Moreover, for any \( i, j \in \mathbb{Z}_q \) the \( j^{th} \) graph isomorphic to \( G_{ij} \) has the edges:
\[
E(G_{ij}) = \left\{ -i^2 + j, (i(1-i) + j) \right\}.
\]
An immediate consequence of the Theorem 8 and Conjecture 1 is the following result.

**Example 9.** The three mutually orthogonal decompositions (MOD) of \( K_{3,3} \) by \( P_4 \) given in Figure 3 are associated with the three mutually orthogonal \( P_4 \)-squares as in Equation (1):
\[
\mathcal{L}_0 = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}, \quad \mathcal{L}_1 = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad \mathcal{L}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix}.
\]

Note that, every row in Figure 3 represents edge decompositions of \( K_{3,3} \) by \( P_4 \).

**Figure 3.** 3MOD of \( K_{3,3} \) by \( P_4 \).
The following result is a generalization of the Theorem 8.

**Theorem 10.** Let \( n \) be a prime power such that \( n = q^x \) with integer power \( x \geq 1 \) of a prime number \( q \) and \( G \) be a subgraph of \( K_{n,n} \). Then \( N(n,G) \geq q \).

**Proof.** For fixed \( j \in \mathbb{Z}_n, 0 \leq i \leq q-1 \) and \( G_j \cong G \), define the \( q \) half-starter vectors as follows,

\[
\nu(G_j) = (j, i+j-1, 2i+j-2, \ldots, (-2)(i+2) + j, (-1)(i+1) + j).
\]

(2)

Our task is to prove the orthogonality of those \( q \) half-starter vectors in mutually. Let us define the half starter vector \( \nu(G_j) \) as \( \nu(G_j) = (k(i-k) + j) \) for all \( k \in \mathbb{Z}_q \). Then for all two different elements \( k, l \in \mathbb{Z}_q \), we have \( \nu_k(G_j) - \nu_l(G_j) = (k(i-k) - l(i-l) = (k-l)(i-(k+l))) \), and then \( \nu_k, \nu_l \) are mutually orthogonal half-starter vectors of graphs \( G_j \) of \( K_{n,n} \) respectively iff \( (i-(k+l)), n) = 1 \). It remains to prove the isomorphism of \( \nu_k(G_j) \) half starter graphs \( G_j \) of \( K_{n,n} \) for all \( k \in \mathbb{Z}_q \). Let \( \nu_k(G_j) = k(i-k) + j = \nu_l(G_j) = k(i-k) + s \), and therefore \( j = s \). Furthermore, if \( \nu_k(G_j) = k(i-k) + j = \nu_l(G_j) = l(i-l) + j \), then \( k-l)(i-(k+l)) = 0 \), since \( i-(k+l) \neq 0 \) (orthogonality of \( \nu_k \) and \( \nu_l \)), and therefore \( k = l \). Moreover, for any \( 0 \leq i \leq q-1, j \in \mathbb{Z}_n \) the \( j^\text{th} \) graph \( G_j \) isomorphic to \( G \) has the edges:

\[
E(G_j) = \{ (n-i^2 + j), (n-i(i-1) + j) \}.
\]

(3)

Note that, in the special case \( x = 1 \) the Theorem 10 proved El-Shanawany conjecture; also, in the case \( q = 2, x \neq 1 \), and \( G \neq 2K_2 \), Theorem 10 constructed an orthogonal double cover of \( K_{n,n} \) by \( G \).

Furthermore, we can construct the following result using Theorem 10 in case \( x > 1 \) and \( q = 3 \).

**Theorem 11.** Let \( x \geq 2 \) be a positive integer such that \( n = 3^x \) and \( \mathbb{P}_{3^{x-2},1} \left( T(3,3) \cup^{3^x} K_1 \right) \) be a subgraph of \( K_{3^x,3^x} \). Then \( N\left( 3^x, \mathbb{P}_{3^{x-2},1} \left( T(3,3) \cup^{3^x} K_1 \right) \right) \geq 3 \).

**Proof.** The result follows from the vector in Equation (2) and its edges in Equation (3) with \( G = \mathbb{P}_{d+1}(F), F = T(3,3) \cup^{3^x} K_1 \) such that \( |E(G)| = 3^x = |E(F)|d = 9d \), imply that \( d = 3^{x-2} \) which define the number of graphs isomorphic to \( F \). As a direct application of Theorem 11; see Figure 4.

**Conjecture 12.** \( N\left( q^s, \mathbb{P}_{q^{s-1}} \left( K_{1,1} \right) \right) = q^k \) if \( q \) is a prime number with an integer power \( x \geq 1 \).

**Conjecture 13.** \( N\left( q^s, \mathbb{P}_{q^{s-1}} \left( T(3,3) \cup^{3^x} K_1 \right) \right) \geq q^k \) if \( q \) is a prime number with an integer power \( x \geq 1 \) and \( s, k \) are positive integers.

**References**

[1] Balakrishnan, R. and Ranganathan, K. (2012) A Textbook of Graph Theory. Springer, Berlin. [http://dx.doi.org/10.1007/978-4614-4529-6](http://dx.doi.org/10.1007/978-4614-4529-6)

[2] Alspach, B., Heinrich, K. and Liu, G. (1992) Orthogonal Factorizations of Graphs. In: Dinitz, J.H. and Stinson, D.R., Eds., Contemporary Design Theory, Chapter 2, Wiley, New York, 13-40.

[3] Gronau, H.-D.O.F., Hartmann, S., Grüttmüller, M., Leck, U. and Leck, V. (2002) On Orthogonal Double Covers of
Graphs. Designs, Codes and Cryptography, 27, 49-91. http://dx.doi.org/10.1023/A:1016546402248

[4] Colbourn, C.J. and Dinitz, J.H. (eds.) (2007) Handbook of Combinatorial Designs. 2nd Edition, Chapman & Hall/CRC, London, Boca Raton.

[5] Colbourn, C.J. and Dinitz, J.H. (2001) Mutually Orthogonal Latin Squares: A Brief Survey of Constructions. Journal of Statistical Planning and Inference, 95, 9-48. http://dx.doi.org/10.1016/S0378-3758(00)00276-7

[6] El-Shanawany, R. (2002) Orthogonal Double Covers of Complete Bipartite Graphs. Ph.D. Thesis, Universitat Rostock, Rostock.

[7] Sampathkumar, R. and Srinivasan, S. (2009) Mutually Orthogonal Graph Squares. Journal of Combinatorial Designs, 17, 369-373. http://dx.doi.org/10.1002/jcd.20216

[8] El-Shanawany, R., Gronau, H.-D.O.F. and Grüttmüller, M. (2004) Orthogonal Double Covers of K_{n,n} by Small Graphs. Discrete Applied Mathematics, 138, 47-63. http://dx.doi.org/10.1016/S0166-218X(03)00269-5

[9] El-Shanawany, R., Shabana, H. and ElMesady, A. (2014) On Orthogonal Double Covers of Graphs by Graph-Path and Graph-Cycle. LAP LAMBERT Academic Publishing. https://www.lappublishing.com/