Inverse Time-Dependent Perfusion Coefficient Identification

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Abstract. The identification of the time-dependent perfusion coefficient in the transient bioheat conduction equation is investigated. In this inverse coefficient identification problem, the additional measurement necessary to render a unique solution may be a heat flux, interior temperature or mass measurement which is taken permanently along the time interval of interest. A numerical approach based on the boundary element and mollification methods is developed. Numerical results are presented and discussed.

1. Introduction
In medical expertise the accurate evaluation of both the temperature and the blood perfusion through a certain region of tissue under investigation have always been an important task either before or during a surgical intervention as well as in other termo-regulatory tests. The difficulty of estimating these two quantities depends on the place where this investigation and eventually intervention is required since, for certain parts of the body, only a restricted number of types of measurements could be carried out. Blood perfusion, which refers to the local, multidirectional flow through the living tissue, is defined as the blood volume exchanged per volume of tissue over time and has the units $ml/ml/s$, [13, 34]. However, at macroscopic level, the blood perfusion is considered to be a directionless quantity due to the very complex nature of the pathways through which it evolves. Provided that it is possible, it is always an advantage to be able to estimate a time limit within which the estimated quantities are accurate, this giving a straight indication over the timing during a possible intervention. Blood perfusion’s physiological importance is underlined by its property of providing the oxygen and nutrients necessary for life processes. The temperature and the blood perfusion have been related in 1948 by Pennes under the bio-heat equation, [31], which, in non-dimensional form, in the absence of sources, is given by:

$$\Delta T - P_f T = \frac{\partial T}{\partial t}, \quad \Omega \times (0, t_f],$$

where $t_f > 0$ is a final time of interest, $T$ is the temperature of the tissue, $\Omega$ is the spatial solution domain and the coefficient

$$P_f = \frac{w_b c_b L^2}{k_i},$$

where $w_b$ is the blood perfusion rate, $c_b$ is the specific heat of the blood, $L$ is the reference length of the biological body and $k_i$ is the thermal conductivity of the tissue.
Since, in the inverse problems under investigation, both the temperature \( T \) and the perfusion coefficient \( P_f \) are unknown, the nonlinear second-order parabolic equation (1) will be solved under the prescription of initial and boundary conditions on the boundary \( \partial \Omega \), when additional information is provided from one of various types of measurements that can be taken. However, the measurements accuracy, their invasiveness character or the extent of practical use as well as the ability to take continuous measurements, create constraints over the range of types of possible additional informations that can be considered. Several types of measurements and experimental work that test the equation (1) are described in [13, 23, 29, 33, 34, 35].

In a first approach, the case where \( P_f \) is assumed constant, has already been discussed both from numerical and analytical stand points, [37]. However, the blood perfusion coefficient need not be constant in time in all the regions of the body. This paper discusses the retrieval of the time-dependent coefficient \( P_f(t) \) along with the temperature \( T(x, t) \) from various types of measured noisy and exact data.

The one-dimensional version of equation (1) is given by

\[
\frac{\partial^2 T}{\partial x^2}(x, t) - P_f(t)T(x, t) = \frac{\partial T}{\partial t}(x, t), \quad (x, t) \in (0, 1) \times (0, t_f].
\] (3)

Since the technique described in this paper can be easily extended to higher dimensions, we focus our discussion on the one-dimensional problem, when Dirichlet or mixed boundary conditions are considered and, as additional information, either the time-dependent internal temperature measurements at fixed or moving point inside the region \((0, 1)\) are taken, or total mass or partial mass measurements are supplied, or alternatively we have heat flux measurements on a part of the boundary \( \Gamma = \{0, 1\} \). All the measurement data are assumed to exhibit both exact and noisy character.

### 2. Mathematical Formulation

In mathematical terms, the time-dependent perfusion coefficient identification problem can be formulated as follows: Find \( T(x, t) \) in \( C^{2,1}([0, 1] \times [0, t_f]) \cap C^{1,0}([0, 1] \times [0, t_f]) \) and the time-dependent perfusion coefficient \( P_f(t) \in C^0([0, t_f]), P_f > 0 \), satisfying the one-dimensional time-dependent bio-heat equation (3) subject to the initial condition

\[
T(x, 0) = T_0(x), \quad x \in [0, 1],
\] (4)

the Dirichlet boundary conditions

\[
\begin{align*}
T(0, t) &= f(t), \quad t \in [0, t_f], \quad f(0) = T_0(0), \\
T(1, t) &= g(t), \quad t \in [0, t_f], \quad g(0) = T_0(1),
\end{align*}
\] (5)

and, for the time being, one of the following types of additional information:

a) a permanent interior temperature measurement at a fixed space point \( x_0 \in (0, 1) \)

\[
T(x_0, t) = u(t), \quad t \in [0, t_f], \quad u(0) = T_0(x_0),
\] (6)

b) a permanent total mass measurement

\[
\int_0^1 T(x, t)dx = \mathcal{E}(t), \quad t \in [0, t_f], \quad \mathcal{E}(0) = \int_0^1 T_0(x)dx,
\] (7)
c) a permanent heat flux measurement
\[ \frac{\partial T}{\partial x}(1, t) = h_1(t), \quad t \in [0, t_f], \quad h_1(0) = T_0'(1). \] (8)

The permanent heat flux measurement (8) at \( x = 1 \) can be replaced, with no modification, by a permanent heat flux measurement at \( x = 0 \).

The non-local mass specification (7) arises in many important applications in heat transfer in which the source control parameter \( P_f(t) \) needs to be determined so that a desired thermal energy (7) can be obtained over the spatial domain.

More general types of measurements will be considered at the end of this paper. Moreover, as we will see starting with Section 7, the Dirichlet boundary conditions may also be replaced by mixed boundary conditions.

Defining
\[ r(t) := \exp \left( \int_0^t P_f(\tau) d\tau \right), \quad t \in [0, t_f], \] (9)

the change of variable
\[ v(x, t) = r(t)T(x, t) \] (10)
transforms the time-dependent problem (3)-(5) into a constant coefficient heat equation problem, namely
\[ \frac{\partial^2 v}{\partial x^2}(x, t) = \frac{\partial v}{\partial t}(x, t), \quad (x, t) \in (0, 1) \times (0, t_f], \] (11)
\[ v(x, 0) = T_0(x), \quad x \in [0, 1], \] (12)
\[ v(0, t) = r(t)f(t), \quad t \in [0, t_f], \quad f(0) = T_0(0), \] (13)
\[ v(1, t) = r(t)g(t), \quad t \in [0, t_f], \quad g(0) = T_0(1). \]

3. The Permanent Internal Temperature Measurement at a Fixed Space Point

In this section we investigate the inverse problem given by equations (3)-(6). In (5) and (6), the conditions
\[ f(0) = T_0(0), \quad g(0) = T_0(1), \quad u(0) = T_0(x_0) \] (14)
are called compatibility conditions of order zero. Further, we need compatibility conditions up to first-order which require condition (14) be satisfied and in addition
\[ f'(0) = T_0''(0) + \frac{T_0(0)(u'(0) - T_0'(x_0))}{T_0(x_0)'}, \] (15)
\[ g'(0) = T_1'(1) + \frac{T_0(1)(u'(0) - T_0'(x_0))}{T_0(x_0)'}, \quad T_0(x_0) > 0. \]

The solvability of the inverse problem (3)-(6) in spaces \( \mathcal{C}^{k+\alpha} \), with \( \alpha \) fixed in \( (0, 1) \) and \( k \in \mathbb{N} \), of continuous functions with Holder continuous derivatives, see [24, 27], has been established in [9, 32], as follows

**Theorem 3.1** If \( T_0 \in \mathcal{C}^{2+\alpha}([0, 1]), \ f, g, u \in \mathcal{C}^{1+\alpha/2}([0, t_f]), \ T_0 \geq 0, \ f \geq 0, \ g \geq 0, \ u > 0, \) and the compatibility conditions up to first order are satisfied, then there exists a unique solution \( (T \in \mathcal{C}^{2+\alpha, 1+\alpha/2}([0, 1] \times [0, t_f]), \ P_f \in \mathcal{C}^{\alpha/2}([0, t_f])) \) of the inverse problem (3)-(6) which is continuously dependent upon data.
Remark that the theorem does not guarantee that the solution for $P_j$ is positive, hence only the uniqueness of the solution $(T, P_j(t) > 0)$ can be concluded.

Prior to this study, numerical methods based on finite differences, [2, 19, 21], and radial basis functions (RBF), [22], have been developed for solving (3)-(6), with extensions to two-dimensional rectangular domains given in [3, 10, 15, 16, 18]. However, the finite difference method (FDM) is not easy to implement in higher dimensional irregular domains, whilst the RBF method is only an approximation meshless method which lacks rigor yet. Therefore, in order to overcome some of these difficulties, in this section we propose the boundary element method (BEM) for solving the inverse problem of finding the solution $(v(x, t), r(t))$ with $v \in \mathcal{C}^{2,1}((0,1) \times (0,t_j)) \cap \mathcal{C}^{1,0}([0,1] \times [0,t_j])$, $r \in \mathcal{C}^1([0,t_j])$, $r'(t) > 0$ for $t \in (0,t_j]$, $r(0) = 1$, satisfying (11)-(13) and the transformed permanent interior measurement

$$v(x_0, t) = r(t)u(t), \quad \text{for } t \in [0,t_j], \quad u(0) = T_0(x_0).$$

Even though both the boundary conditions and the measured data for (12)-(16) are unknown, an essential assistance in our approach comes from the integral representation formula for the heat equation (11), namely

$$\mu(x)v(x, t) = \int_0^t \int_\Omega \left[ G(x, t; \xi, \tau) \frac{\partial v}{\partial \tau} (\xi, \tau) - \frac{\partial G}{\partial \tau} (x,t; \xi, \tau) v(\xi, \tau) \right] d\Gamma_\xi d\tau + \int_\Omega G(x, t; \xi, 0)d\Omega_\xi,$$

for $(x, t) \in [0,1] \times (0,t_j]$, where $\mu(0) = \mu(1) = \frac{1}{4}$ and $\mu(x) = 1$ for $x \in (0,1)$, and

$$G(x, t; \xi, \tau) = \frac{H(t-\tau)}{\sqrt{4\pi(t-\tau)}} \exp \left( -\frac{(x-\xi)^2}{4(t-\tau)} \right)$$

is the fundamental solution of the heat equation, where $H$ is the Heaviside step function.

For the boundary of the region $(0,1) \times (0,t_j]$ we consider a uniform discretization with $N$ time boundary nodes

$$\{0,1\} \times \{0\} = \bigcup_{j=1}^N \{0,1\} \times (t_{j-1}, t_j]$$

and $N_0$ space cells

$$\{0,1\} \times \{0\} = \bigcup_{k=1}^{N_0} [x_{k-1}, x_k] \times \{0\}.$$

Further, for $j = 1, \ldots, N$, $k = 1, \ldots, N_0$, $x \in [0,1]$, $\xi \in [0,1]$, $t \in (0,t_j]$, we denote

$$C_j^k(x, t) = \int_{t_{j-1}}^{t_j} G(x, t; \xi, \tau)d\tau, \quad D_j^k(x, t) = \int_{t_{j-1}}^{t_j} \frac{\partial G}{\partial \xi} (x,t; \xi, \tau)d\tau, \quad E_k^j(x, t) = \int_{x_{k-1}}^{x_k} G(x, t; \xi, 0)d\xi.$$

Note that the integrals in (21) can be evaluated analytically as described in [26]. Denoting \( \hat{t}_j = (t_{j-1} + t_j)/2 \), $j = 1, \ldots, N$, let us now define three nonlinear discretization-dependent maps:

$$C, D : [0,1] \longrightarrow L(\mathbb{R}^{2N}, \mathbb{R}^N), \quad E : [0,1] \longrightarrow L(\mathbb{R}^{N_0}, \mathbb{R}^N),$$

where $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^N$ the $N$-dimensional Euclidean space, and $L(\mathbb{R}^2, \mathbb{R})$ the space of continuous functions from $\mathbb{R}$ into $\mathbb{R}$. The above maps satisfy

$$C \circ D = E.$$
given by:

a) for $C$ and $D$:

\[
C(x)(i, j) := C^0_i(x, \tilde{t}_j), \quad i = \overline{1,N}, \quad j = \overline{1,N},
\]

\[
C(x)(i, j) := C^i_j(x, \tilde{t}_j), \quad i = \overline{1,N}, \quad j = \overline{N+1,2N},
\]

\[
D(x)(i, j) := D^0_i(x, \tilde{t}_j), \quad i = \overline{1,N}, \quad j = \overline{1,N},
\]

\[
D(x)(i, j) := D^i_j(x, \tilde{t}_j), \quad i = \overline{1,N}, \quad j = \overline{N+1,2N},
\]

(23)

b) for $E$:

\[
E(x)(j, k) := E_k(x, \tilde{t}_j), \quad j = \overline{1,N}, \quad k = \overline{1,N_0},
\]

(24)

for all $x \in \overline{\Omega}$, where throughout the this work, for any $m, n \geq 1$, by $L(R^m, R^n)$ we denote the space of linear operators defined on $R^m$ and taking values in $R^n$, which can be represented as matrices of dimension $(n \times m)$. Using a BEM with constant boundary elements, [4], from (17) we obtain that, for any space point $x \in \overline{\Omega}$, the solution vector satisfies

\[
\mu(x) [v(x, \tilde{t}_1), ..., v(x, \tilde{t}_N)]^t = C(x)\tilde{v}_r - D(x)\tilde{v}_r + E(x)\tilde{v}_0,
\]

(25)

where the superscript $^t$ denotes the transpose of a vector, the $2N$–dimensional vector $\tilde{v}_r$ captures the Dirichlet temperature boundary conditions (5), the $N_0$–dimensional vector $\tilde{v}_0$ the discretized initial condition (4), and the $2N$–dimensional vector $\tilde{v}_n$ represents the flux $\frac{\partial v}{\partial n}$ over the boundary $\Gamma = \{0,1\}$. These vectors are configured as follows:

\[
\tilde{v}_r := [r(\tilde{t}_1)\varphi(\tilde{t}_1), ..., r(\tilde{t}_N)\varphi(\tilde{t}_N), r(\tilde{t}_1)g(\tilde{t}_1), ..., r(\tilde{t}_N)g(\tilde{t}_N)]^t,
\]

(26)

\[
\tilde{v}_n := \left[\frac{\partial v}{\partial n_0}(0, \tilde{t}_1), ..., \frac{\partial v}{\partial n_0}(0, \tilde{t}_N), \frac{\partial v}{\partial n_1}(1, \tilde{t}_1), ..., \frac{\partial v}{\partial n_1}(1, \tilde{t}_N)\right]^t,
\]

(27)

\[
\tilde{v}_0 := \left[T_0(\tilde{x}_1), ..., T_0(\tilde{x}_{N_0})\right]^t,
\]

(28)

where $n_\xi, \xi \in \Gamma = \{0,1\}$, are the outward normal directions and $\tilde{x}_k = (x_{k-1} + x_k)/2$, $k = \overline{1,N_0}$. Moreover, the boundary associated linear operators $C_r, D_r \in L(R^{2N}, R^{2N})$ and $E_r \in L(R^{N_0}, R^{2N})$, defined by

\[
C_r = \begin{bmatrix} C(0) \\ C(1) \end{bmatrix}, \quad D_r = \begin{bmatrix} D(0) + \frac{1}{2}I_N & 0_N \\ D(1) + 0_N & \frac{1}{2}I_N \end{bmatrix}, \quad E_r = \begin{bmatrix} E(0) \\ E(1) \end{bmatrix},
\]

(29)

where $I_N \in L(R^N, R^N)$ is the usual identity operator and $0_N \in L(R^N, R^N)$ is the null operator, for $x \in \Gamma$, allow us to write the following $2N \times 2N$ system of equations

\[
C_r\tilde{v}_r' - D_r\tilde{v}_r + E_r\tilde{v}_0 = 0.
\]

(30)

which is derived from equation (25), for $x \in \Gamma$. Equation (30) imply

\[
\tilde{v}_r' = C_r^{-1}D_r\tilde{v}_r - C_r^{-1}E_r\tilde{v}_0.
\]

(31)

Having chosen $x_0 \in (0,1)$, the measurement (16) evaluated at the time nodes $\tilde{t}_j$, $j = \overline{1,N}$, supplies us with the vector

\[
r \cdot u := [r(\tilde{t}_1)u(\tilde{t}_1), ..., r(\tilde{t}_N)u(\tilde{t}_N)]^t,
\]

(32)
which, via the equation (25), can be expressed as

\[ r \cdot u = C(x_0)\tilde{v}_r - D(x_0)\tilde{v}_r + E(x_0)\tilde{v}_0. \]  

(33)

Let us now denote with

\[ \tilde{r} := [r(\tilde{t}_1), ..., r(\tilde{t}_N)]^\text{tr}, \quad \tilde{u} := [u(\tilde{t}_1), ..., u(\tilde{t}_N)]^\text{tr}, \]  

(34)

\[ \tilde{f} := [f(\tilde{t}_1), ..., f(\tilde{t}_N)]^\text{tr}, \quad \tilde{g} := [g(\tilde{t}_1), ..., g(\tilde{t}_N)]^\text{tr}, \]  

(35)

the vectors induced by \( r, u, f \) and \( g \) evaluated at the nodes \( \tilde{t}_j, j = 1, N \). Also, throughout the entire paper, by \( \text{diag}(\text{vec}) \) we will understand the diagonal matrix whose main diagonal is composed from the components of the vector \( \text{vec} \), while preserving their order.

Therefore, we notice that the vector \( r \cdot u \) can formally be written in terms of diagonal matrix as follows

\[ r \cdot u = \text{diag}(\tilde{u})\tilde{r}. \]  

(36)

Then, using (31), equation (33) can be expressed as

\[ \text{diag}(\tilde{u})\tilde{r} = (C(x_0)C_r^{-1}D_r - D(x_0))\tilde{v}_r + (-C(x_0)C_r^{-1}E_r + E(x_0))\tilde{v}_0. \]  

(37)

Using the mapping \( J : (0, 1) \to L(\mathbb{R}^{2N}, \mathbb{R}^{N}) \) defined by

\[ J(x) := C(x)C_r^{-1}D_r - D(x), \quad x \in (0, 1), \]  

(38)

and observing that the vector \( \tilde{v}_r \) can be expressed as

\[ \tilde{v}_r = \begin{bmatrix} \text{diag}(\tilde{f}) \\ \text{diag}(\tilde{g}) \end{bmatrix} \tilde{r}, \]  

(39)

from the equation (37) we obtain the following \( N \times N \) linear system

\[ A\tilde{r} = (-C(x_0)C_r^{-1}E_r + E(x_0))\tilde{v}_0, \]  

(40)

where

\[ A := \text{diag}(\tilde{u}) - J(x_0) \begin{bmatrix} \text{diag}(\tilde{f}) \\ \text{diag}(\tilde{g}) \end{bmatrix}. \]  

(41)

The solution \( \tilde{r} \) of the system of equations (40) is then used, via the equation (9), to obtain the time-dependent coefficient \( P_f(t) \) in its discretized form as

\[ \tilde{P}_f = \text{diag}(\frac{1}{\tilde{r}})\tilde{r}', \]  

(42)

where we denoted

\[ \tilde{r}' := [r'(\tilde{t}_1), ..., r'(\tilde{t}_N)]^\text{tr}. \]  

(43)
4. The Permanent Total Mass Measurement

In this section we investigate the inverse problem given by (3)-(5) and (7). For the data in (4)-(5) and (7) we assume the following compatibility conditions:

\[
\begin{align*}
    f(0) &= T_0(0), \quad g(0) = T_0(1), \quad \mathcal{E}(0) = \int_0^1 T_0(x)dx, \\
    f'(0) &= T''_0(0) + \frac{T_0(0)\mathcal{E}'(0) - T'_0(1) + T'_0(0)}{\mathcal{E}(0)}, \\
    g'(0) &= T''_0(1) + \frac{T_0(1)\mathcal{E}'(0) - T'_0(1) + T'_0(0)}{\mathcal{E}(0)}, \quad \mathcal{E}(0) > 0.
\end{align*}
\]

Then we have the following solvability theorem established in [9] and [28].

**Theorem 4.1** If \( T_0 \in \mathcal{C}^{2+\alpha}([0,1]), \ f, g, \mathcal{E} \in \mathcal{C}^{1+\alpha/2}([0,t_f]), \ T_0 \geq 0, \ f \geq 0, \ g \geq 0, \ \mathcal{E} > 0 \) and the compatibility conditions (44) are satisfied, then there exists a unique solution \( (T \in \mathcal{C}^{2+\alpha,1+\alpha/2}([0,1] \times [0,t_f]), \ P_f \in \mathcal{C}^{\alpha/2}([0,t_f])) \) of the inverse problem (3)-(5) and (7), which is continuously dependent upon the data.

In the case that the Dirichlet conditions (5) are replaced by Neumann or Robin boundary conditions, similar solvability theorems are given in [5, 6, 7] or [8], respectively.

Prior to this study, numerical methods based on finite differences, see [1, 9, 12, 14, 38], have been developed for solving (3)-(5) and (7) with extensions to two and three-dimensional rectangular and cuboidal domains given in [17, 20]. However, as the FDM is not easy to extend to higher-dimensional irregular domains, in this section we adopt the BEM.

First, recalling the definition (9), the change of variable (10) transforms the problem (3)-(5) into (11)-(13) and the permanent total mass measurement (7) become

\[
\begin{align*}
    \int_0^1 v(x,t)dx &= r(t)\mathcal{E}(t), \quad t \in [0,t_f], \quad \mathcal{E}(0) = \int_0^1 T_0(x)dx.
\end{align*}
\]

Our approach for the inverse problem (11)-(13) and (45) follows a path that is similar with the one developed for (3)-(6). The BEM for the heat equation (11)-(13) provides us a space discretization for the mass formula, namely

\[
r(t)\mathcal{E}(t) \approx \sum_{i=0}^{N_0-1} (x_{i+1} - x_i)v(\tilde{x}_{i+1},t) = \frac{1}{N_0} \sum_{i=1}^{N_0} v(\tilde{x}_i,t).
\]

Therefore, the vector

\[
r \cdot \mathcal{E} := [r(\tilde{t}_1)\mathcal{E}(\tilde{t}_1), ..., r(\tilde{t}_N)\mathcal{E}(\tilde{t}_N)]^T
\]

satisfies

\[
r \cdot \mathcal{E} = \frac{1}{N_0} \sum_{i=1}^{N_0} [v(\tilde{x}_i,\tilde{t}_1), ..., v(\tilde{x}_i,\tilde{t}_N)]^T.
\]

Thus, via the equation (25), for \( i = 1, N \), we obtain

\[
r \cdot \mathcal{E} = \frac{1}{N_0} \left( \sum_{i=1}^{N_0} C(\tilde{x}_i)\tilde{v}_r^i - \sum_{i=1}^{N_0} D(\tilde{x}_i)\tilde{v}_r^i + \sum_{i=1}^{N_0} E(\tilde{x}_i)\tilde{v}_0 \right).
\]
Using (31), equation (49) can be expressed as
\[ r \cdot \mathcal{E} = \frac{1}{N_0} \sum_{i=1}^{N_0} \left( C(\tilde{x}_i)C_r^{-1}D_r - D(\tilde{x}_i) \right) \tilde{v}_r + \frac{1}{N_0} \sum_{i=1}^{N_0} \left( -C(\tilde{x}_i)C_r^{-1}E_r + E(\tilde{x}_i) \right) \tilde{v}_0. \] (50)

By denoting
\[ \tilde{\mathcal{E}} := [\mathcal{E}(\tilde{t}_1), ..., \mathcal{E}(\tilde{t}_1)]^T, \] (51)
we notice that the vector \( r \cdot \mathcal{E} \) can formally be expressed as
\[ r \cdot \mathcal{E} = diag(\tilde{\mathcal{E}})\tilde{r}. \] (52)

Using (39), and the mapping \( J \) defined in (38), equation (50) becomes
\[ diag(\tilde{\mathcal{E}})\tilde{r} = \frac{1}{N_0} \sum_{i=1}^{N_0} J(\tilde{x}_i) \left[ \begin{array}{c} diag(\tilde{f}) \\ diag(\tilde{g}) \end{array} \right] \tilde{r} + \frac{1}{N_0} \sum_{i=1}^{N_0} \left( -C(\tilde{x}_i)C_r^{-1}E_r + E(\tilde{x}_i) \right) \tilde{v}_0. \] (53)

Therefore, by denoting
\[ A := diag(\tilde{\mathcal{E}}) - \frac{1}{N_0} \sum_{i=1}^{N_0} J(\tilde{x}_i) \left[ \begin{array}{c} diag(\tilde{f}) \\ diag(\tilde{g}) \end{array} \right] \in L(\mathbb{R}^N, \mathbb{R}^N), \] (54)
we obtain the following \( N \times N \) linear system of equations
\[ A\tilde{r} = \frac{1}{N_0} \sum_{i=1}^{N_0} \left( -C(\tilde{x}_i)C_r^{-1}E_r + E(\tilde{x}_i) \right) \tilde{v}_0. \] (55)

Once the system (55) has been solved, equation (42) is used to obtain the discretized time-dependent coefficient \( P_f(t) \).

5. Numerical Results and Discussion for Internal and Total Mass Measurements for the Dirichlet Inverse Problem
A test solution for our inverse problem (3)-(6) and (3)-(5) and (7), namely
\[ T(x, t) = (x^2 + 2t) \exp \left( -t - \frac{t^2}{2} \right); \quad P_f(t) = 1 + t, \quad t \in [0, t_f = 1], \] (56)
provide us with a validation tool for the computed results. Throughout the paper, a percentage \( \rho \in \{0\% , 1\% \} \) of random noise \( \eta(t) \) is usually involved in our data. The noise \( \eta(t) \) is generated from a uniform distribution on the interval \([-1, 1]\) for each time \( \tilde{t}_j, j = 1, N \), and is supplied by the NAG routine G05DAF. In all the figures we use the solid line for the computed values and the dash-dot line for the analytical ones.

The test solution (56) provide us with the measured data for:

a) permanent internal temperature measurement taken at \( x_0 = 0.5 \), namely
\[ u(t) = T(0.5, t) = \left( \frac{1}{4} + 2t \right) \exp \left( -t - \frac{t^2}{2} \right) \left( 1 + \rho \eta(t) \right), \] (57)
b) permanent total mass measurement:

\[ E(t) = \int_0^1 T(x, t) dx = \left( \frac{1}{3} + 2t \right) \exp \left( -t - \frac{t^2}{2} \right) \left( 1 + \rho \eta(t) \right), \tag{58} \]

where the percentage \( \rho \in \{0\%, 1\%\} \) of noise \( \eta(t) \) is involved.

The numerical results for both internal (57) and mass (58) measurements behave almost the same. Therefore, we focus our discussion on the numerical results for the mass measurements. In all the numerical computations performed in this paper we use \( N = 320 \) constant time boundary elements and \( N_0 = 160 \) space cells for the BEM discretization. Figure 1 illustrates the computed and analytical values of \( r(t) = e^{t + \frac{t^2}{2}} \) and \( P_f(t) = 1 + t \) for the no noise case, \( \rho = 0 \). In Figure 1(a) we can see almost perfect agreement between the analytical and computed values of \( \tilde{r} \). This agreement is further extended for \( P_f \), see Figure 1(b).

In Figure 2 we illustrate the computed and analytical values for \( r, r' \) and \( P_f \), when the data (58) is affected by 1\% noise. From Figure 2(a) it can be seen that the noise \( \eta(t) \) from the additional information \( E(t) \) determines the results for the computed \( \tilde{r} \) to be noisy as well. Although these computed values of \( r \) are localized around the exact values, their instability becomes a major problem for computing the numerical derivative \( r'(t) \) of the noisy function \( r(t) \) shown in Figure 2(a). In order to obtain a stable numerical derivative, one can employ the mollification method with a Gaussian mollifier, see [30]. In short, this scheme consists in two steps:

a) we use the Gaussian kernel

\[ J_\delta(t) = \frac{1}{\delta \sqrt{\pi}} \exp \left[ -\frac{t^2}{\delta^2} \right], \tag{59} \]

where \( \delta > 0 \) is the radius of mollification (or the regularization parameter) acting as an averaging filter and perform the convolution \( J_\delta \ast r \), namely

\[ J_\delta \ast r(t) = \int_{-\infty}^{\infty} J_\delta(\tau) r(t - \tau) d\tau. \tag{60} \]
Figure 2. Computed and analytical values of: (a) $r(t)$, (b) $r'(t)$, and (c) $P_f(t)$, when there is 1% noise in the mass data (58).

b) differentiate $J_\delta * r$ to obtain $J_\delta * r' = J_\delta * r$.

We should notice here that the mollifier $J_\delta$ is always positive and becomes very close to zero outside the interval centred at the origin and of radius $3\delta$. Good results for $r'(t)$ are therefore expected in the interval $[3\delta, 1 - 3\delta]$. In our algorithm, the radius of mollification $\delta$ is computed automatically by Generalized Cross-Validation (GCV). We also remark that although the numerical $r(t)$ in Figure 2(a) is non-smooth, its mollification $J_\delta * r(t)$ is a $C^\infty$ function, hence is differentiable.

The computational program includes the mollification procedure which encompasses a large part of the code supplied by Professor D.A. Murio. After performing the data mollification, the solution obtained is stable. In order to alleviate the top end effect at $t = t_f = 1$ of the interval $[0, t_f = 1]$, we double the time interval and, using $N = 320$ boundary nodes and $N_0 = 160$ space cells, we first solve the system of equations (55) for the inverse problem considered on
the time interval $[0, 2]$. Then we perform the mollification for $t \in [0, 2]$ and retain only the results for the interval $t \in [0, t_f] = [0, 1]$, which corresponds to solving the problem on the interval $[0, t_f]$. The mollified results for the derivative of $r$ on $[0, t_f]$ are obtained in terms of the convolution $J_s * r' = J'_s * r$ and, as shown in Figure 2(b), they agree with the analytical values $r'(t) = (1 + t)e^{t^2/2}$. After this, the computed results shown in Figures 2(a) and 2(b) are used in equation (42) to obtain $P_f(t)$, which, as shown in Figure 2(c), approximates fairly close the solution $P_f(t) = 1 + t$. The (20%) inaccuracies near the lower end $t = 0$ are expected, since we cannot solve the problem for negative times $t < 0$.

6. Analysis for the Permanent Flux Measurement

In this section we investigate the inverse problem given by the equations (3)-(5) and (8). For this problem we could not find solvability results in the literature and therefore, a first investigation is attempted below.

Let $\Phi \in \mathcal{C}^{2,1}((0, 1) \times (0, t_f)) \cap \mathcal{C}^{1,0}([0, 1] \times [0, t_f])$ be the unique solution of the well-posed problem:

$$\begin{align*}
\frac{\partial \Phi}{\partial t}(x, t) &= \frac{\partial^2 \Phi}{\partial x^2}(x, t), \quad (x, t) \in (0, 1) \times (0, t_f], \\
\Phi(x, 0) &= T_o(x), \quad x \in [0, 1], \\
\Phi(0, t) &= f(t), \quad t \in [0, t_f], f(0) = T_o(0), \\
\Phi(1, t) &= g(t), \quad t \in [0, t_f], g(0) = T_o(0).
\end{align*}
$$

The change of variable

$$w(x, t) := r(t)T(x, t) - \Phi(x, t)$$

transforms (3)-(5) into

$$\begin{align*}
\frac{\partial w}{\partial t}(x, t) &= \frac{\partial^2 w}{\partial x^2}(x, t), \quad (x, t) \in (0, 1) \times (0, t_f], \\
w(x, 0) &= 0, \quad x \in [0, 1], \\
w(0, t) &= f(t)(r(t) - 1) =: F_o(t), \quad t \in [0, t_f], F_o(0) = 0, \\
w(1, t) &= g(t)(r(t) - 1) =: F_i(t), \quad t \in [0, t_f], F_i(0) = 0.
\end{align*}
$$

Let us first notice that, with no changes in analysis, the additional information (8) may well be replaced by a permanent heat flux measurement at $x = 0$, namely

$$-\frac{\partial T}{\partial x}(0, t) = h_o(t), \quad \text{for any } t \in (0, t_f].$$

The change of variable (64) transforms (8) or (68) into an additional information for the problem (65)-(67), namely:

i) a left boundary flux measurement:

$$-\frac{\partial w}{\partial x}(0, t) = h_o(t)r(t) + \frac{\partial \Phi}{\partial x}(0, t)$$

or

ii) a right boundary flux measurement:

$$\frac{\partial w}{\partial x}(1, t) = h_i(t)r(t) + \frac{\partial \Phi}{\partial x}(1, t),$$
respectively.
However, the two cases can be treated similarly and in the following we focus our attention on the left boundary permanent flux measurement (68). Let us assume that a solution to the problem (65)-(67) and (69) exists.
Since \( F_0, F_1 \in C^0([0, t_f]) \), we have the representation formula, see [25],
\[
  w(x, t) = -\int_0^t \frac{\partial M}{\partial x}(x, t - \tau) F_0(\tau) d\tau + \int_0^t \frac{\partial M}{\partial x}(x - 1, t - \tau) F_1(\tau) d\tau,
\]
(71)
where
\[
  M(\xi, \sigma) := \frac{H(\sigma)}{\sqrt{\pi \sigma}} \sum_{n=-\infty}^{\infty} \exp \left[ -\frac{(\xi + 2n)^2}{4\sigma} \right].
\]
(72)
By differentiating equation (71) with respect to \( x \), we obtain
\[
  \frac{\partial w}{\partial x} = -\int_0^t \frac{\partial^2 M}{\partial x^2}(x, t - \tau) F_0(\tau) d\tau + \int_0^t \frac{\partial^2 M}{\partial x^2}(x - 1, t - \tau) F_1(\tau) d\tau.
\]
(73)
Since
\[
  \frac{\partial M}{\partial \tau}(x - \xi, t - \tau) = -\frac{\partial^2 M}{\partial x^2}(x - \xi, t - \tau), \quad x \neq \xi, \ t > \tau,
\]
(74)
it follows that
\[
  \frac{\partial w}{\partial x}(x, t) = \int_0^t \frac{\partial M}{\partial \tau}(x, t - \tau) F_0(\tau) d\tau - \int_0^t \frac{\partial M}{\partial \tau}(x - 1, t - \tau) F_1(\tau) d\tau.
\]
(75)
On the other hand, let us first notice, from equation (72), that we obtain
\[
  \lim_{\tau \to t} M(x - \xi, t - \tau) = 0, \quad x \neq \xi,
\]
(76)
and since \( F_0, F_1 \in C^1([0, t_f]) \), with \( F_0(0) = F_1(0) = 0 \), by using integration by parts in (75), we have
\[
  \frac{\partial w}{\partial x}(x, t) = -\int_0^t M(x, t - \tau) F'_0(\tau) d\tau + \int_0^t M(x - 1, t - \tau) F'_1(\tau) d\tau,
\]
(77)
for all \((x, t) \in (0, 1) \times (0, t_f)\). By using now the Lebesgue’s Dominative convergence theorem, see [36], we obtain
\[
  \lim_{x \to 0} \frac{\partial w}{\partial x}(x, t) = -\int_0^t M(0, t - \tau) F'_0(\tau) d\tau + \int_0^t M(1, t - \tau) F'_1(\tau) d\tau
\]
(78)
for all \( t \in (0, t_f) \).
For \( t \in (0, t_f) \) and \( P_f(t) > 0 \), let us define:
\[
  Z(t, P_f(t)) := -\int_0^t M(0, t - \tau) F'_0(\tau) d\tau + \int_0^t M(-1, t - \tau) F'_1(\tau) d\tau,
\]
(79)
and
\[ Q(P_f(t)) := - \frac{Z(t, P_f(t)) + \frac{\partial b}{\partial x}(0, t)}{r(t)}, \]  
(80)

From the analysis (71)-(78) and the equation (69), we obtain that
\[ Q(P_f(t)) = h_0(t) \quad \text{for all } t \in (0, t_f]. \]  
(81)

Thus, we arrive to the following remark:

**Remark 6.1** If the inverse problem (3)-(5) and (68) has a solution then the function \( P_f(t) \) needs to satisfy equation (81).

A direct consequence can be stated as follows:

**Theorem 6.2** If equation (81) has a unique positive solution, then the inverse problem (3)-(5) and (68) has a unique solution \((T(x, t); P_f(t))\).

### 6.1. BEM Treatment for the Permanent Flux Measurement at \( x = 1 \)

Let us start by recalling that equation (31) supply us with a formula for \( \hat{v}_r \), where, from (27),
\[ \hat{v}_r = [r(\tilde{t}_1)h_0(\tilde{t}_1), ..., r(\tilde{t}_N)h_0(\tilde{t}_N), r(\tilde{t}_1)h_1(\tilde{t}_1), ..., r(\tilde{t}_N)h_1(\tilde{t}_N)]^{tr}. \]  
(82)

If we further consider the vector and matrix notations
\[ \tilde{h}_i := [h_i(\tilde{t}_1), ..., h_i(\tilde{t}_N)]^{tr}, \quad i \in \{0, 1\}, \]  
(83)

then equation the (30) can be rewritten as
\[ \begin{bmatrix} \text{diag}(\tilde{h}_0) \\ \text{diag}(\tilde{h}_1) \end{bmatrix} \hat{r} - C^{-1}_r D_r \begin{bmatrix} \text{diag}(\tilde{f}) \\ \text{diag}(\tilde{g}) \end{bmatrix} \hat{r} = -C^{-1}_r E_r \tilde{v}_0 \]  
(84)

which is a \( 2N \times 2N \) nonlinear system with the vectors \( \hat{r} \) and \( \tilde{h}_0 \) as the unknowns. However, we will show that, in order to obtain the unknown vector \( \hat{r} \), we only need \( N \) linear equations of the system (84).

Let us define the orthogonal projection
\[ P_{N+1,2N} : \mathbb{R}^{2N} \to R^N, \]
\[ P_{N+1,2N}(\{x_1, ..., x_N, x_{N+1}, ..., x_{2N}\}^{tr}) = \{x_{N+1}, ..., x_{2N}\}^{tr}, \]  
(85)

for all \( \{x_1, ..., x_N, x_{N+1}, ..., x_{2N}\}^{tr} \in \mathbb{R}^{2N} \). In order to disregard the unknown flux information \( h_0(t) \), by applying \( P_{N+1,2N} \) to the system of equations (84), we obtain
\[ P_{N+1,2N} \begin{bmatrix} \text{diag}(\tilde{h}_0) \\ \text{diag}(\tilde{h}_1) \end{bmatrix} \hat{r} - P_{N+1,2N} C^{-1}_r D_r \begin{bmatrix} \text{diag}(\tilde{f}) \\ \text{diag}(\tilde{g}) \end{bmatrix} \hat{r} = -P_{N+1,2N} C^{-1}_r E_r \tilde{v}_0, \]  
(86)

which is equivalent to the following \( N \times N \) system of linear equations
\[ \text{diag}(\tilde{h}_1)\hat{r} - P_{N+1,2N} C^{-1}_r D_r \begin{bmatrix} \text{diag}(\tilde{f}) \\ \text{diag}(\tilde{g}) \end{bmatrix} \hat{r} = -P_{N+1,2N} C^{-1}_r E_r \tilde{v}_0. \]  
(87)

Thus, if we are only supplied with the flux measurement at \( x = 1 \), \( h_1(t) \) for all \( t \in (0, t_f] \), just
by solving the linear system (87), we are able to obtain $\tilde{r}$, which is the BEM discretized version of $r(t)$.

We would like to remark here that the steps proposed so far are symmetric in the sense that, if the flux information at $x = 0$, $h_0(t)$, is available and the flux information at $x = 1$, $h_1(t)$, is not known, then defining the complementary projection $P_{1N}$ which takes a $2N$-dimensional vector into its first $N$ coordinates, a similar system is obtained and, thus, $r(t)$ can be retrieved again.

The test solution (56) provide us with the following heat flux measurement at $x = 1$

$$h_1(t) = \frac{\partial T}{\partial x}(1, t) = 2e^{\exp \left(-t - \frac{t^2}{2}\right)}(1 + \rho \eta(t)).$$

(88)

For the no noise case, $\rho = 0$, the computed results are in a full agreement with the analytical values and their plots look almost identical with the ones in Figures 1(a) and 1(b), and therefore,
they are not presented.

When 1% of noise is included in the flux data \( h_i(t) \), as in (88), the problem becomes again difficult, since the computed \( \tilde{r} \) becomes noisy, so that the numerical derivative \( \tilde{r}' \) needs special attention. However, by pursuing the same steps as in the case of noisy mass or internal measurements, we solve the system of equations (87) obtained for a discretization with \( N = 320 \) boundary nodes and \( N_0 = 160 \) space cells for \( (0, 1) \times [0, 2] \), and perform the Gaussian mollification of the obtained results. Then we restrict our problem to the domain of interest \( (0, 1) \times [0, t_f = 1] \) and, out of the results obtained in the previous step, we retain the only values corresponding to this restriction. In Figure 3(a), the comparison between the results computed and the analytical values for \( \tilde{r} \) unveil the fact that the noise from the flux measurements have been propagated through the computation and have determined \( \tilde{r} \) to be noisy. However, as shown in Figure 3(b), after mollification, the computed derivative becomes very close to its analytical value. Finally, as shown in Figure 3(c), the perfusion coefficient \( P_f(t) \) is retrieved reasonably stable and fairly close to its true value.

7. Mixed Boundary Conditions

A set of mixed boundary conditions are used in this section to replace the Dirichlet boundary conditions (5). Therefore, in this context, the inverse time-dependent perfusion coefficient identification problem satisfying the bio-heat equation (3), with the initial condition (4), has to be solved subject to the mixed boundary conditions

\[
-\frac{\partial T}{\partial x}(0, t) = h_0(t), \quad t \in [0, t_f],
\]

\[
T(1, t) = g(t), \quad t \in [0, t_f], \quad g(0) = T_0(1),
\]

while preserving the same kind of additional information, equations (6) or (7). The case of the additional information (8), being supplied by the heat flux at \( x = 1 \), is expected to produce qualitatively the same conclusions as those of subsection 6.1 and, in addition, since there is no theory on the solvability of the inverse problem, we do not insist on this investigation. Instead, we concentrate on the analysis of Sections 3 and 4 in which the mixed boundary conditions (89) and (90) replace the boundary conditions (5). Similar solvability theorems to those of Sections 3 and 4 can be established, [8, 32], for the inverse problem (3), (4), (89), (90) and (6) or (7).

The change of variable defined in (9)-(10) transforms the inverse problems (3), (4), (89), (90) and (6) and (3), (4), (89), (90) and (7) into mixed boundary condition problems for the standard heat equation (11) subject to the initial condition (12) and the mixed boundary conditions

\[
-\frac{\partial v}{\partial x}(0, t) = r(t)h_0(t), \quad t \in [0, t_f],
\]

\[
v(1, t) = r(t)g(t), \quad t \in [0, t_f], \quad g(0) = v_0(1),
\]

and the additional information could be taken either as (16) or (45).

7.1. Internal Temperature Measurement

Recalling (26)-(29), (34)-(35), (39) and (83), equations (30) and (33) are recasted as follows:

\[
C_r \begin{bmatrix} \text{diag}(\tilde{h}_0) \\ \text{diag}(\tilde{h}_1) \end{bmatrix} \tilde{r} - D_r \begin{bmatrix} \text{diag}(\tilde{f}) \\ \text{diag}(\tilde{g}) \end{bmatrix} \tilde{r} + E_r \tilde{v}_0 = 0,
\]

\[
\text{diag}(\tilde{u})\tilde{r} = C(x_0) \begin{bmatrix} \text{diag}(\tilde{h}_0) \\ \text{diag}(\tilde{h}_1) \end{bmatrix} \tilde{r} - D(x_0) \begin{bmatrix} \text{diag}(\tilde{f}) \\ \text{diag}(\tilde{g}) \end{bmatrix} \tilde{r} + E(x_0) \tilde{v}_0.
\]
From (93), equation (94) can be expressed as

\[
\text{diag}(\vec{u})\vec{r} = C(x_0) \left[ C^{-1}_\Gamma D_r \left[ \text{diag}(\vec{f}) \text{diag}(\vec{g}) \right] \vec{r} - C^{-1}_\Gamma E_r \vec{v}_0 \right] - D(x_0) \left[ \text{diag}(\vec{f}) \text{diag}(\vec{g}) \right] \vec{r} + E(x_0)\vec{v}_0.
\] (95)

Let us define the linear projection

\[
P_{\Gamma,N}^r : \mathbb{R}^{2N} \rightarrow \mathbb{R}^N,
\]

for all \([x_1, \ldots, x_N, x_{N+1}, \ldots, x_{2N}]^r \in \mathbb{R}^{2N}\). By applying \(P_{\Gamma,N}^{-1}C^{-1}_\Gamma\) to the \(2N\)-dimensional operator equation (93), we manage to discard from our analysis the unknown \(\vec{v}_1\), containing the unknown heat flux at \(x = 1\), and we arrive at the \(N\)-dimensional operator equation

\[
\text{diag}(\vec{h}_0)\vec{r} - P_{\Gamma,N}^{-1}C^{-1}_\Gamma D_r \left[ \text{diag}(\vec{f}) \text{diag}(\vec{g}) \right] \vec{r} + P_{\Gamma,N}^{-1}C^{-1}_\Gamma E_r \vec{v}_0 = 0.
\] (97)

Let us denote by

\[
\vec{r} \cdot \vec{f} := [r(\vec{i}_1)f(\vec{i}_1), \ldots, r(\vec{i}_N)f(\vec{i}_N)]^r,
\]

the BEM-discretized version of \(r(t)f(t)\). We can now remark that equations (95) and (97) form the following \(2N \times 2N\) linear system of equations

\[
\begin{align*}
\text{diag}(\vec{h}_0)\vec{r} - P_{\Gamma,N}^{-1}C^{-1}_\Gamma D_r \left[ \text{diag}(\vec{f}) \text{diag}(\vec{g}) \right] \vec{r} + P_{\Gamma,N}^{-1}C^{-1}_\Gamma E_r \vec{v}_0 &= 0, \\
\text{diag}(\vec{u})\vec{r} &= C(x_0) \left[ C^{-1}_\Gamma D_r \left[ \text{diag}(\vec{f}) \text{diag}(\vec{g}) \right] \vec{r} - C^{-1}_\Gamma E_r \vec{v}_0 \right] - D(x_0) \left[ \text{diag}(\vec{f}) \text{diag}(\vec{g}) \right] \vec{r} + E(x_0)\vec{v}_0,
\end{align*}
\] (99)

in the \(2N\)-dimensional vector of unknowns:

\[
\begin{bmatrix} \vec{r} \\ \vec{r} \cdot \vec{f} \end{bmatrix}.
\] (100)

Throughout the paper, for an arbitrary matrix \(A\), by \(\text{col}_i(A)\) we understand the \(i\)–th column of the matrix \(A\). Defining now the finite dimensional operators:

\[
D_{\Gamma,N} := [\text{col}_1(D_r), \ldots, \text{col}_N(D_r)] \in L(\mathbb{R}^N, \mathbb{R}^{2N}),
\] (101)

\[
D_{\Gamma,N+1,2N} := [\text{col}_{N+1}(D_r), \ldots, \text{col}_{2N}(D_r)] \in L(\mathbb{R}^N, \mathbb{R}^{2N}),
\] (102)

\[
D(x)_{\Gamma,N} := [\text{col}_1(D(x)), \ldots, \text{col}_N(D(x))] \in L(\mathbb{R}^N, \mathbb{R}^N), \quad x \in \Omega,
\] (103)

\[
D(x)_{\Gamma,N+1,2N} := [\text{col}_{N+1}(D(x)), \ldots, \text{col}_{2N}(D(x))] \in L(\mathbb{R}^N, \mathbb{R}^N), \quad x \in \Omega,
\] (104)
the system of equations (99) can be equivalently expressed as

\[
\begin{align*}
&\begin{cases}
\text{diag}(\hat{h}_0) - P_{1,N} C^{-1}_N D_{r_{N+1,2N}} \text{diag}(\tilde{g}) \\
\text{diag}(\tilde{u}) - C(x_0) C^{-1}_N D_{r_{N+1,2N}} \text{diag}(\tilde{g}) + D(x_0) D_{r_{N+1,2N}} \text{diag}(\tilde{g})
\end{cases}
\hat{r} - P_{1,N} C^{-1}_N D_{r_{N+1,2N}} r \cdot f = -P_{1,N} C^{-1}_N E_\Gamma \hat{v}_0 \\
+ \begin{cases}
-C(x_0) C^{-1}_N D_{r_{1,N}} + D(x_0) D_{r_{1,N}}
\end{cases} r \cdot f = E(x_0) \hat{v}_0 - C(x_0) C^{-1}_N E_\Gamma \hat{v}_0
\end{align*}
\]

(105)

Thus, the linear operators \( A_{1,N}, A_{N+1,2N} \in L(R^N, R^{2N}) \), defined by

\[
A_{1,N} := \begin{bmatrix}
\text{diag}(\hat{h}_0) - P_{1,N} C^{-1}_N D_{r_{N+1,2N}} \text{diag}(\tilde{g}) \\
\text{diag}(\tilde{u}) - C(x_0) C^{-1}_N D_{r_{N+1,2N}} \text{diag}(\tilde{g}) + D(x_0) D_{r_{N+1,2N}} \text{diag}(\tilde{g})
\end{bmatrix}
\]

(106)

and

\[
A_{N+1,2N} := \begin{bmatrix}
-P_{1,N} C^{-1}_N D_{r_{N+1,2N}} \\
-C(x_0) C^{-1}_N D_{r_{N+1,2N}} + D(x_0) D_{r_{1,N}}
\end{bmatrix}
\]

(107)

enable us to rewrite the system of equations (105) as

\[
A_{1,N} \hat{r} + A_{N+1,2N} r \cdot f = \begin{bmatrix}
-P_{1,N} C^{-1}_N E_\Gamma \hat{v}_0 \\
E(x_0) \hat{v}_0 - C(x_0) C^{-1}_N E_\Gamma \hat{v}_0
\end{bmatrix}.
\]

(108)

Defining the linear operator

\[
A := \begin{bmatrix}
A_{1,N} & A_{N+1,2N}
\end{bmatrix} \in L(R^{2N}, R^{2N}),
\]

(109)

we can finally write the \( 2N \times 2N \) linear system of equations (108) as

\[
A \begin{bmatrix}
\hat{r} \\
r \cdot f
\end{bmatrix} = \begin{bmatrix}
-P_{1,N} C^{-1}_N E_\Gamma \hat{v}_0 \\
E(x_0) \hat{v}_0 - C(x_0) C^{-1}_N E_\Gamma \hat{v}_0
\end{bmatrix}.
\]

(110)

However, we are only interested in the retrieval of the vector \( \hat{r} \), and therefore, we disregard from our considerations the other half of the solution vector, which is summarized in the components of \( r \cdot f \).

### 7.2. Total Mass Measurement Case

When the additional information considered is the mass measurement (7), the mixed boundary value inverse problem (3), (4), (89), (90) and (7) focuses our interest. The equivalent inverse problem (11), (12), (91), (92) and (45) allows us to recall and apply here the analysis from (45)-(49). Using the definitions (26)-(29), (34), (35), (51), (83), we note first that equation (93) is valid also in this case. On the other hand, in equation (49), which is rewritten here as

\[
\text{diag}(\hat{E}) \tilde{r} = -\sum_{i=1}^{N_2} C(\vec{x}_i) \begin{bmatrix}
\text{diag}(\hat{h}_0) \\
\text{diag}(\tilde{f})
\end{bmatrix} \tilde{r}
- \sum_{i=1}^{N_2} D(\vec{x}_i) \begin{bmatrix}
\text{diag}(\tilde{g})
\end{bmatrix} \tilde{r} + \sum_{i=1}^{N_0} E(\vec{x}_i) \hat{v}_0
\]

(111)
we use equation (93) to obtain

\[
diag(\hat{E})\tilde{r} = \frac{1}{N_0} \left( \sum_{i=1}^{N_0} C(\tilde{x}_i) \left\{ C_r^{-1} D_r \left[ diag(\tilde{f}) \right] \tilde{r} - C_r^{-1} E_r \tilde{v}_0 \right\} \right. \\
- \left. \sum_{i=1}^{N_0} D(\tilde{x}_i) \left[ diag(\tilde{f}) \right] \tilde{r} + \sum_{i=1}^{N_0} E(\tilde{x}_i) \tilde{v}_0 \right). 
\]  

(112)

Using the projection (96) in (93), we arrive again at the \(N\)-dimensional operator equation (97). Therefore, equations (97) and (112) give us the \(2N \times 2N\)-dimensional system

\[
\begin{align*}
\text{diag}(\hat{h}_0)\tilde{r} &- P_{\frac{1}{N\times \infty}} C_r^{-1} D_r \left[ diag(\tilde{f}) \right] \tilde{r} + P_{\frac{1}{N\times \infty}} C_r^{-1} E_r \tilde{v}_0 = 0, \\
\text{diag}(\hat{E})\tilde{r} &- \frac{1}{N_0} \sum_{i=1}^{N_0} C(\tilde{x}_i) C_r^{-1} D_r \left[ diag(\tilde{f}) \right] \tilde{r} - D(\tilde{x}_i) \left[ diag(\tilde{f}) \right] \tilde{r} \\
&+ \frac{1}{N_0} \sum_{i=1}^{N_0} \left( -C(\tilde{x}_i) C_r^{-1} E_r + E(\tilde{x}_i) \right) \tilde{v}_0,
\end{align*}
\]  

(113)

with the \(2N\)-dimensional vector of unknowns

\[
\begin{bmatrix}
\tilde{r} \\
\tilde{v} \\
\tilde{f}
\end{bmatrix} = \begin{bmatrix}
\tilde{r} \\
\tilde{v} \\
\tilde{f}
\end{bmatrix},
\]  

(114)

according to the definition (98) of the vector \(\tilde{r} \cdot \tilde{f}\). Using the linear operators defined in (101)-(104), the system (113) can equivalently be expressed as:

\[
\begin{align*}
\text{diag}(\hat{h}_0)\tilde{r} &- P_{\frac{1}{N\times \infty}} C_r^{-1} D_r \left[ \frac{1}{N_0} \sum_{i=1}^{N_0} C(\tilde{x}_i) C_r^{-1} D_r \right] \tilde{r} \\
&+ P_{\frac{1}{N\times \infty}} C_r^{-1} E_r \tilde{v}_0 = 0, \\
\text{diag}(\hat{E})\tilde{r} &- \frac{1}{N_0} \sum_{i=1}^{N_0} C(\tilde{x}_i) C_r^{-1} D_r \left[ \frac{1}{N_0} \sum_{i=1}^{N_0} C(\tilde{x}_i) C_r^{-1} D_r \right] \tilde{r} \\
&- \frac{1}{N_0} \sum_{i=1}^{N_0} D(\tilde{x}_i) \left[ \frac{1}{N_0} \sum_{i=1}^{N_0} C(\tilde{x}_i) C_r^{-1} D_r \right] \tilde{r} \\
&+ \frac{1}{N_0} \sum_{i=1}^{N_0} \left( -C(\tilde{x}_i) C_r^{-1} E_r + E(\tilde{x}_i) \right) \tilde{v}_0.
\end{align*}
\]  

(115)

Thus, using again the linear operator notations \(A_{1,N}, A_{N+1,2N} \in L(\mathbb{R}^N, \mathbb{R}^{2N})\) to describe the matrices

\[
A_{1,N} := \begin{bmatrix}
\text{diag}(\hat{h}_0) & - P_{\frac{1}{N\times \infty}} C_r^{-1} D_r \frac{1}{N_0} \sum_{i=1}^{N_0} C(\tilde{x}_i) C_r^{-1} D_r \text{diag}(\tilde{g}) \\
\text{diag}(\hat{E}) & - \frac{1}{N_0} \sum_{i=1}^{N_0} C(\tilde{x}_i) C_r^{-1} D_r \frac{1}{N_0} \sum_{i=1}^{N_0} C(\tilde{x}_i) C_r^{-1} D_r \text{diag}(\tilde{g}) + \frac{1}{N_0} \sum_{i=1}^{N_0} D(\tilde{x}_i) \frac{1}{N_0} \sum_{i=1}^{N_0} C(\tilde{x}_i) C_r^{-1} D_r \text{diag}(\tilde{g})
\end{bmatrix}
\]  

(116)

and

\[
A_{N+1,2N} := \begin{bmatrix}
- P_{\frac{1}{N\times \infty}} C_r^{-1} D_r \frac{1}{N_0} \sum_{i=1}^{N_0} C(\tilde{x}_i) C_r^{-1} D_r \\
\frac{1}{N_0} \sum_{i=1}^{N_0} \left( -C(\tilde{x}_i) C_r^{-1} D_r + D(\tilde{x}_i) \right)
\end{bmatrix},
\]  

(117)
the system of equations (115) recasts as
\[
\begin{align*}
A_{N+2, N} \tilde{r} + A_{N+1, 2N} r \cdot f &= \begin{bmatrix}
-P_{1,N} C_{1}^{-1} E_1 \tilde{v}_0 \\
\frac{1}{N_0} \sum_{i=1}^{N_0} \left( E(\tilde{x}_i) \tilde{v}_0 - C(\tilde{x}_i) C_{N}^{-1} E_N \tilde{v}_0 \right) 
\end{bmatrix}.
\end{align*}
\]
\begin{equation}
\tag{118}
(18)
\end{equation}
Therefore, by defining the matrix \( A \in L(\mathbb{R}^{2N}, \mathbb{R}^{2N}) \) as in (109), the system of equations (118) can be written as
\[
A \begin{bmatrix} \tilde{r} \\ r \cdot f \end{bmatrix} = \begin{bmatrix}
-P_{1,N} C_{1}^{-1} E_1 \tilde{v}_0 \\
\frac{1}{N_0} \sum_{i=1}^{N_0} \left( E(\tilde{x}_i) \tilde{v}_0 - C(\tilde{x}_i) C_{N}^{-1} E_N \tilde{v}_0 \right) 
\end{bmatrix}.
\begin{equation}
\tag{119}
(19)
\end{equation}

The system of equations (119) is to be solved as a whole, however, of interest for us are only the first \( N \) components of the solution vector, namely, \( \tilde{r} \), and we disregard here the last \( N \) components that form the computed vector \( r \cdot f \).

The numerical results obtained were very similar to those for the Dirichlet problem of Section 5, see Figures 1 and 2, and therefore, they are not presented.

8. Internal Temperature Measurements on Arbitrarily Non-Constant Time-Dependent Paths

So far all the internal measurements considered were set permanently at one single point \( x_0 \in (0, 1) \), which in the \( (0,1) \times (0,t_f) \) domain represents the measured value of the temperature \( T(x,t) \) considered along the straight vertical path given by the graph of the constant function \( \gamma_0 : [0,t_f) \rightarrow (0,1) \), \( \gamma_0(t) := x_0 \). However, as we will see in the following, the function \( \gamma_0 \) need not be constant. We devote this section to investigate both the Dirichlet and the mixed boundary conditions problem, when the internal temperature measurement is taken on arbitrarily non constant paths \( \gamma : [0,t_f) \rightarrow (0,1) \). Therefore, the Dirichlet inverse problem (3)-(5), or the mixed boundary value inverse problem (3), (4), (89), and (90), receives in this section the following internal temperature measurement:
\begin{equation}
T(\gamma(t),t) = u(t), \quad t \in [0,t_f), \quad u(0) = T_0(\gamma(0)),
\end{equation}
\begin{equation}
\tag{120}
(20)
\end{equation}
for a fixed arbitrary \( \gamma : [0,t_f) \rightarrow (0,1) \). Thus, after performing the change of variable (9)-(10), the corresponding inverse problems (11)-(13), and (11), (12), (91) and (92), receives the additional information
\begin{equation}
\begin{align*}
\tilde{v}(\gamma(t), t) &= u(t) r(t), \quad t \in [0,t_f), \quad u(0) = T_0(\gamma(0)).
\end{align*}
\end{equation}
\begin{equation}
\tag{121}
(21)
\end{equation}

While the general BEM technique remains the method used for our numerical investigation, specific path-dependent linear operators are defined as follows. On the set of all possible paths \( \{ \gamma : [0,t_f) \rightarrow (0,1) \} \) let us define the following mappings:
\begin{equation}
C, D : \{ \gamma : [0,t_f) \rightarrow [0,1] \} \rightarrow L(\mathbb{R}^{2N}, \mathbb{R}^N),
\end{equation}
\begin{equation}
E : \{ \gamma : [0,t_f) \rightarrow (0,1) \} \rightarrow L(\mathbb{R}^{N_0}, \mathbb{R}^N),
\end{equation}
\begin{equation}
\tag{122}
(22)
\end{equation}
\begin{equation}
\tag{123}
(23)
\end{equation}
given by:

a) for $C(\gamma)$ and $D(\gamma)$

$$C(\gamma)(i,j) := C^0_j(\gamma(\check{t}_i), \check{t}_j), \quad i = \overline{1,N}, \quad j = \overline{1,N},$$

$$C(\gamma)(i,j) := C^1_j(\gamma(\check{t}_i), \check{t}_j), \quad i = \overline{1,N}, \quad j = N+1, 2N,$$

$$D(\gamma)(i,j) := D^0_j(\gamma(\check{t}_i), \check{t}_j), \quad i = \overline{1,N}, \quad j = \overline{1,N},$$

$$D(\gamma)(i,j) := D^1_j(\gamma(\check{t}_i), \check{t}_j), \quad i = \overline{1,N}, \quad j = N+1, 2N. \tag{124}$$

b) for $E(\gamma)$

$$E(\gamma)(j,k) := E_k(\gamma(\check{t}_j), \check{t}_j), \quad j = \overline{1,N}, \quad k = \overline{1,N}. \tag{125}$$

Then the BEM solution vector along $\gamma$ of the heat equation (11) satisfies

$$[v(\gamma(\check{t}_1), \check{t}_1), \ldots, v(\gamma(\check{t}_N), \check{t}_N)]^T = C(\gamma)\check{v}_\gamma - D(\gamma)\check{v}_\gamma + E(\gamma)\check{v}_0. \tag{126}$$

We notice that the operators (124)-(125) are compatible with the ones defined in Section 3 and we use them together in the subsections that follow. It is worthwhile to remark here that the equations (30) and (31) hold valid for the Dirichlet inverse problem (11)-(13) and (121) and for the mixed boundary condition inverse problem (11), (12), (91), (92) and (121).

### 8.1. The Dirichlet Inverse Problem Revisited

Using the notation (32), from (31) and (126) we obtain

$$r \cdot u = (C(\gamma)C^{-1}_r D_r - D(\gamma))\check{v}_\gamma + (-C(\gamma)C^{-1}_r E_r + E(\gamma))\check{v}_0. \tag{127}$$

After defining

$$J(\gamma) := C(\gamma)C^{-1}_r D_r - D(\gamma) \in L(\mathbb{R}^{2N}, \mathbb{R}^N), \tag{128}$$

from equation (127), and proceeding in a similarly manner as in the case of fixed point internal measurement, we obtain the $N \times N$ linear system:

$$A\hat{r} = (-C(\gamma)C^{-1}_r E_r + E(\gamma))\check{v}_0, \tag{129}$$

where

$$A := diag(\check{u}) - J(\gamma) \begin{bmatrix} diag(\check{f}) \\ diag(\check{g}) \end{bmatrix}, \tag{130}$$

which clearly is similar in structure with the system of equations (40). Moreover, if $\gamma$ is chosen to be the constant path $\gamma_0(t) \equiv x_0$, for all $t \in [0, t_f]$, we immediately recognize that the systems of equations (129) and (40) coincide, as was expected, since the two inverse problems become the same.

### 8.2. The Mixed Boundary Conditions Inverse Problem Revisited

Preserving all the notations from Section 7.1, equation (93) holds valid for the mixed boundary conditions inverse problem defined by (11), (12), (91), (92) and (121). However, equation (95) from Section 7.1 is replaced here by

$$
\begin{align*}
\text{diag}(\check{u})\hat{r} &= C(\gamma) \left[ C^{-1}_r D_r \begin{bmatrix} \text{diag}(\check{f}) \\ \text{diag}(\check{g}) \end{bmatrix} \hat{r} - C^{-1}_r E_r \check{v}_0 \right] \\
&\quad - D(\gamma) \begin{bmatrix} \text{diag}(\check{f}) \\ \text{diag}(\check{g}) \end{bmatrix} \hat{r} + E(\gamma)\check{v}_0.
\end{align*}
\tag{131}$$
Since (97) was obtained as a straight implication of equation (93), this equation holds valid also in this case. Therefore, equations (97) and (131) form the $2N \times 2N$ system

\[
\begin{align*}
\text{diag}(\hat{h}_0)\vec{r} - P_{\Gamma, N} C_{\Gamma}^{-1} D_{\Gamma} \left[ r \cdot f \text{ diag}(\vec{g}) \right] + P_{\Gamma, N} C_{\Gamma}^{-1} E_{\Gamma} \vec{v}_0 &= 0, \\
\text{diag}(\hat{u})\vec{r} &= C(\gamma) \left[ C_{\Gamma}^{-1} D_{\Gamma} \left[ r \cdot f \text{ diag}(\vec{g}) \right] - C_{\Gamma}^{-1} E_{\Gamma} \vec{v}_0 \right] \\
-D(\gamma) \left[ r \cdot f \text{ diag}(\vec{g}) \right] + E(\gamma)\vec{v}_0,
\end{align*}
\]

in the $2N$-dimensional vector of unknowns (100). Let us define the finite dimensional operators

\[
D(\gamma)_{\Gamma, N} := [\text{col}_1(D(\gamma)), ..., \text{col}_N(D(\gamma))] \in L(\mathbb{R}^N, \mathbb{R}^N),
\]

\[
D(\gamma)_{N+1, 2N} := [\text{col}_{N+1}(D(\gamma)), ..., \text{col}_{2N}(D(\gamma))] \in L(\mathbb{R}^N, \mathbb{R}^N).
\]

By continuing now with an analysis identical with the one carried out in (101)-(107), we obtain the corresponding sub-matrices $A_{\Gamma, N}$, $A_{N+1, 2N} \in L(\mathbb{R}^N, \mathbb{R}^{2N})$, which are defined here as follows:

\[
A_{\Gamma, N} := \begin{bmatrix}
\text{diag}(\hat{h}_0) - P_{\Gamma, N} C_{\Gamma}^{-1} D_{\Gamma} \text{ diag}(\vec{g}) \\
\text{diag}(\hat{u}) - C(\gamma) C_{\Gamma}^{-1} D_{\Gamma} \text{ diag}(\vec{g}) + D(\gamma)_{\Gamma, N} \text{ diag}(\vec{g})
\end{bmatrix}
\]

\[
A_{N+1, 2N} := \begin{bmatrix}
-P_{\Gamma, N} C_{\Gamma}^{-1} D_{\Gamma} \vec{v}_0 \\
-C(\gamma) C_{\Gamma}^{-1} D_{\Gamma} \vec{v}_0 + D(\gamma)_{\Gamma, N} \vec{v}_0
\end{bmatrix},
\]

as well as the right hand side part of the system

\[
\begin{bmatrix}
-P_{\Gamma, N} C_{\Gamma}^{-1} E_{\Gamma} \vec{v}_0 \\
E(\gamma)\vec{v}_0 - C(\gamma) C_{\Gamma}^{-1} E_{\Gamma} \vec{v}_0
\end{bmatrix}.
\]

Therefore, creating again the linear operator $A := [A_{\Gamma, N} \quad A_{N+1, 2N}] \in L(\mathbb{R}^{2N}, \mathbb{R}^{2N})$, we obtain the desired system

\[
A \begin{bmatrix}
\vec{r} \\
r \cdot f
\end{bmatrix} = \begin{bmatrix}
-P_{\Gamma, N} C_{\Gamma}^{-1} E_{\Gamma} \vec{v}_0 \\
E(\gamma)\vec{v}_0 - C(\gamma) C_{\Gamma}^{-1} E_{\Gamma} \vec{v}_0
\end{bmatrix}.
\]

However, we are only interested in computing $\vec{r}$ and, even though we solve the entire system (138), we disregard $r \cdot f$ which is the second half of the solution vector.

### 8.3. Numerical Results and Discussion for Arbitrary Path Measurements

Let us consider the following path $\gamma : [0, t_j = 1] \rightarrow (0, 1)$

\[
\gamma(t) = \frac{1}{2} + \frac{\sin t}{4}, \quad t \in [0, 1]
\]
Figure 4. Computed and analytical values of: (a) \( r(t) \), (b) for \( r'(t) \), and (c) \( P_f(t) \), when there is 1% noise in the data (140).

Then, the test solution (56) gives us the path internal temperature measurement:

\[
u(t) = T(\gamma(t), t) = \left( \frac{1}{2} + \frac{\sin t}{4} \right)^2 + 2t \exp \left( -t - \frac{t^2}{2} \right) (1 + \rho \eta(t)).\] (140)

From a computational standpoint, the results from both Dirichlet and mixed boundary conditions behave in a similar manner. Therefore, we will discuss and exemplify our computation only for the mixed boundary conditions.

In the no noise case, the computed results agree very well with their analytical values for both \( \tilde{r} \) and \( P_f \) and their plots look almost exactly as in Figures 1(a) and 1(b).

When 1% noise is included in the path measurements (140), the computed solution of the system of equations (138) is heavily affected by noise. However, applying the same mollification approach described in Section 5, we obtain yet again agreement between the computed and analytical
values of $\tilde{r}$, $\tilde{r}'$ and $P_f$, as illustrated in Figures 4(a), 4(b) and 4(c).

9. Permanent Partial Mass Measurements

A natural connection with the previous section is revealed by a sensible relaxation of the conditions (7) giving the total mass measurement for the inverse problem of the Dirichlet type, (3)-(5) and (7), or of the mixed boundary condition type, (3), (4), (89), (90) and (7).

However, as it occurs in most of the real world situations, environment restrictions or technical capabilities prevent the total measurement of the mass, and instead only a partial measurement is possible for a certain part of the mass. Therefore, we devote this section to discussing both the Dirichlet and the mixed boundary type inverse problems when the information (7) is replaced by the partial mass measurement, which is formally defined by

$$\int_{\gamma_1(t)}^{\gamma_2(t)} T(x,t)dx = \mathcal{E}(t), \quad \text{for all } t \in [0,t_f], \quad \mathcal{E}(0) = \int_{\gamma_1(0)}^{\gamma_2(0)} T_0(x)dx,$$

(141)

where the paths $\gamma_1, \gamma_2 : [0,t_f] \rightarrow [0,1]$, $\gamma_1, \gamma_2 \in \mathcal{C}([0,t_f])$, $\gamma_1 < \gamma_2$, are a priori given.

The solvability of the inverse problem (3)-(5) and (141) has been established in [8, 9, 11, 28], as follows.

**Theorem 9.1** If $T_0 \in \mathcal{C}^2([0,1])$, $f, g \in \mathcal{C}([0,t_f])$, $T_0 \geq 0$, $f \geq 0$, $g \geq 0$, $\gamma_2, \mathcal{E} \in \mathcal{C}^1$, $\mathcal{E} > 0$, $\gamma_1 \equiv 0$, then there exists a unique solution $T \in \mathcal{C}^2, 1 \times (0,t_f]) \bigcap \mathcal{C}([0,1] \times [0,t_f])$, $P_f \in \mathcal{C}([0,t_f])$ of the inverse problem (3)-(5) and (141) which is continuously dependent upon data.

After performing the change of variable (9)-(10), the corresponding inverse problems (11)-(13) and (11), (12), (91), (92) receive the additional information

$$\int_{\gamma_1(t)}^{\gamma_2(t)} v(x,t)dx = r(t)\mathcal{E}(t), \quad t \in [0,t_f], \quad \mathcal{E}(0) = \int_{\gamma_1(0)}^{\gamma_2(0)} T_0(x)dx.$$

(142)

Thus, in terms of the BEM, we obtain

$$r(t)\mathcal{E}(t) \simeq \frac{1}{N_0} \sum_{k=1}^{N_0} r(t)T(\tilde{x}_k,t)\chi_{\{1\}}(\gamma_1(t),\gamma_2(t))(\tilde{x}_k),$$

(143)

where, for any nonempty arbitrary set $\Sigma$ and any subset $\Sigma_1 \subset \Sigma$, the function

$$\chi_{\Sigma_1} : \Sigma \rightarrow \{0,1\},$$

$$\chi_{\Sigma_1}(\sigma) = \begin{cases} 1, & \sigma \in \Sigma_1, \\ 0, & \sigma \in \Sigma \setminus \Sigma_1, \end{cases}$$

(144)

is called the characteristic function of $\Sigma_1$ in $\Sigma$. From (143) we obtain the following $N$-dimensional equation

$$\text{diag}(\mathcal{E})\tilde{r} = \frac{1}{N_0} \sum_{k=1}^{N_0} \text{diag} \left( T(\tilde{x}_k,\tilde{t}_k)\chi_{\{1\}}(\gamma_1(t_1),\gamma_2(t_1))(\tilde{x}_k), ..., T(\tilde{x}_k,\tilde{t}_N)\chi_{\{N\}}(\gamma_1(t_N),\gamma_2(t_N))(\tilde{x}_k) \right)^{tr} \tilde{r}.$$

(145)
Let us now define the integer
\[ N_0(\tilde{E}) := \max_{j=1,N} \text{Card} \{ \tilde{x}_k \mid \tilde{x}_k \in [\gamma_1(i_j), \gamma_2(i_j)] \}, \] (146)
where, in general, for any arbitrary set \( \Sigma \), by \( \text{Card}(\Sigma) \) we understand the cardinal (number of elements) of \( \Sigma \). Obviously,
\[ N_0(\tilde{E}) \leq N_0. \] (147)
Denoting \( A(\tilde{t}_j) := [\gamma_1(i_j), \gamma_2(i_j)] \cap \{ \tilde{x}_k \}_{k=1,N_0} \), \( j = 1, N \), for \( 1 \leq l \leq N_0(\tilde{E}) \), for any \( l \in \{0, ..., N_0(\tilde{E})\} \), we consider the linear interpolating paths \( \gamma^l : [0, t_j] \rightarrow [0, 1] \) given by \( \gamma^0(t) \equiv 0 \), and iteratively we define
\[ \gamma^l(i_j) := \begin{cases} \min \{A(\tilde{t}_j) \setminus \bigcup_{i=0}^{l-1} \gamma^i(i_j)\}, & \text{if } A(\tilde{t}_j) \setminus \bigcup_{i=0}^{l-1} \gamma^i(i_j) \neq \emptyset, \\ \max \{A(\tilde{t}_j)\}, & \text{otherwise}, \end{cases} \] (148)
\[ \gamma^l(i_j + (1-\lambda)i_{j-1}) := \gamma^l(i_j)\lambda + (1-\lambda)\gamma^l(i_{j-1}), \quad \lambda \in (0, 1). \]
Therefore, equation (145) becomes
\[ \text{diag}(\tilde{E})\tilde{r} = \frac{1}{N_0} \sum_{i=1}^{N_0(\tilde{E})} \text{diag} \left( \left[ T(\gamma^i(i_j), i_j), ..., T(\gamma^i(i_N), i_N) \right]^n \right) \tilde{r} \\ = \frac{1}{N_0} \sum_{i=1}^{N_0(\tilde{E})} \left[ C(\gamma^i)C^{-1}_\Gamma D_\Gamma - D(\gamma^i) \right] \tilde{v}_\gamma \\ + \frac{1}{N_0} \sum_{i=1}^{N_0(\tilde{E})} \left[ -C(\gamma^i)C^{-1}_\Gamma E_\Gamma + E(\gamma^i) \right] \tilde{v}_0. \] (149)

9.1. Discussion of the Dirichlet Case
When the Dirichlet boundary conditions (5) are supplied, equation (149) becomes
\[ \text{diag}(\tilde{E})\tilde{r} = \frac{1}{N_0} \sum_{i=1}^{N_0(\tilde{E})} \left[ C(\gamma^i)C^{-1}_\Gamma D_\Gamma - D(\gamma^i) \right] \left[ \begin{array}{c} \text{diag}(\tilde{f}) \\ \text{diag}(\tilde{g}) \end{array} \right] \tilde{r} \\ + \frac{1}{N_0} \sum_{i=1}^{N_0(\tilde{E})} \left[ -C(\gamma^i)C^{-1}_\Gamma E_\Gamma + E(\gamma^i) \right] \tilde{v}_0. \] (150)
Thus, by defining the left hand side matrix \( A \in L(\mathbb{R}^N, \mathbb{R}^N) \) as
\[ A := \text{diag}(\tilde{E}) - \frac{1}{N_0} \sum_{i=1}^{N_0(\tilde{E})} \left[ C(\gamma^i)C^{-1}_\Gamma D_\Gamma - D(\gamma^i) \right] \left[ \begin{array}{c} \text{diag}(\tilde{f}) \\ \text{diag}(\tilde{g}) \end{array} \right], \] (151)
we obtain the \( N \)-dimensional linear system
\[ A\tilde{r} = \frac{1}{N_0} \sum_{i=1}^{N_0(\tilde{E})} \left[ -C(\gamma^i)C^{-1}_\Gamma E_\Gamma + E(\gamma^i) \right] \tilde{v}_0. \] (152)
9.2. Discussion for the Mixed Boundary Condition case

When mixed boundary conditions (89) and (90) are supplied, a useful remark is that equation (97) holds valid. Therefore, from (97) and (149) we obtain the $2N \times 2N$-dimensional system

$$
\begin{array}{c}
\text{diag}(\tilde{h}_0)\tilde{r} - P_{1\to N} C_\Gamma^{-1} D_\Gamma \left[ \begin{array}{c}
\text{diag}(\tilde{f})\tilde{r} \\
\text{diag}(\tilde{g})\tilde{r}
\end{array} \right] + P_{1\to N} C_\Gamma^{-1} E_\Gamma \tilde{v}_0 = 0,
\\
d\left(\tilde{E}\right)\tilde{r} = \frac{1}{N_0} \sum_{i=1}^{N_0(\tilde{E})} \left( C(\gamma^i) C_\Gamma^{-1} D_\Gamma \left[ \begin{array}{c}
\text{diag}(\tilde{f})\tilde{r} \\
\text{diag}(\tilde{g})\tilde{r}
\end{array} \right] - D(\gamma^i) \left[ \begin{array}{c}
\text{diag}(\tilde{f})\tilde{r} \\
\text{diag}(\tilde{g})\tilde{r}
\end{array} \right] \right)
\\
+ \frac{1}{N_0} \sum_{i=1}^{N_0(\tilde{E})} \left( -C(\gamma^i) C_\Gamma^{-1} E_\Gamma + E(\gamma^i) \right) \tilde{v}_0,
\end{array}
$$

(153)

with the $2N$-dimensional vector of unknowns (114). However, only the first $N$ components of the solution for the system of equations (153) is of interest to us, the second half of the components, namely $r \cdot f$, being disregarded.

![Figure 5](image-url)

**Figure 5.** Computed and analytical values of: (a) $r(t)$ and (b) $P_f(t)$, when there is no noise in the partial mass data (154).

9.3. Numerical Results and Discussion for the Permanent Partial Mass Measurement

Let us now consider the following paths $\gamma_1, \gamma_2 : [0, t_f = 1] \to [0, 1]$, $\gamma_1(t) := 0.4 + \frac{\sin t}{4}$, $\gamma_2(t) := 0.6 + \frac{\sin t}{4}$.

Then the test solution (56) gives us the partial mass measurement

$$
E(t) = \int_{\gamma_1(t)}^{\gamma_2(t)} T(x,t)dx = \left( \frac{x^3}{3} + 2tx \right) \exp \left( -t - \frac{t^2}{2} \right) \left[ \begin{array}{c}
0.6 + \frac{\sin t}{4} \\
0.4 + \frac{\sin t}{4}
\end{array} \right], \quad t \in [0, 1].
$$

(154)

The numerical results for the Dirichlet and mixed boundary value cases behave in a fairly similar manner, and therefore, we discuss only the mixed boundary conditions case.

By solving the system of equations (153), Figures 5(a) and 5(b) show that even though we have considered exact data and the computation of $\tilde{r}$ behaves satisfactory, the results obtained
for $P_f(t)$ are highly unstable. The instability shown in Figure 5(b) is caused by the direct computation of the derivative $\tilde{r}'$ using central differences. However, if we apply the mollification steps as described in Section 5, the results stabilize and, as illustrated in Figures 6(a) and 6(b), they agree with the analytical values.

10. Conclusions
The inverse problem regarding the identification of the time-dependent perfusion coefficient in the bioheat equation has been investigated. Both exact and noisy measurements were taken into consideration. For the cases of internal and total mass measurements in the presence of Dirichlet or mixed boundary conditions, the hypotheses of the solvability theorems results were satisfied. In the heat flux measurement case, since there were no uniqueness results previously proved, we have stated and proved a uniqueness criterion which translates the uniqueness issue to the existence of a unique zero for a constructed functional. Natural generalizations to path measurements and partial mass measurements were also approached in our investigation both for the Dirichlet and for the mixed boundary conditions. However, our main effort was focused on developing a general numerical method that would allow us to retrieve of the solution $(T(x, t); P_f(t))$, globally, in a unified manner, for all the types of boundary conditions and measurements considered.

The numerical method that we have developed consists of two parts. First, we have constructed a Boundary Element Method for the time-dependent inverse problem. However, since in the noisy measurement cases, derivatives of noisy resulting functions were supposed to be computed, we have mollified the results using a Gaussian kernel.

The test examples considered show, in the cases of exact measurement, that the analytical and the computed values match almost exactly. For the noisy data, the results obtained from our method approximates fairly well the analytical values of the perfusion coefficient. As it can be immediately observed, from the manner in which it has been built, the method proposed in this paper can be straightforward extended and applied to the higher-dimensional versions of this inverse time-dependent perfusion coefficient identification problem.
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