MOTIVES OF GRAPH HYPERSURFACES WITH TORUS OPERATIONS

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Abstract. We investigate graph hypersurfaces and study conditions under which graph hypersurfaces admit algebraic torus operations. This leads in principle to a computation of graph motives using the theorem of Bialynicki-Birula, provided one knows the fixed point loci in a resolution of singularities.

Introduction

Feynman diagrams and their amplitudes are of fundamental importance in perturbative quantum field theory. Extensive calculations of these amplitudes for graphs of low loop numbers by Broadhurst and Kreimer in [6] and [7] revealed the motivic nature of these amplitudes, showing that in many cases they are expressible as rational linear combinations of multiple zeta values. This brought up the question whether all Feynman amplitudes evaluate to multiple zeta values. By general principles [11, 16], this would mean that Feynman amplitudes are periods of mixed Tate motives. Kontsevich [15] related this to point counting on the hypersurface defined by the singularities of the integrand in the Feynman amplitude. Despite the empirical evidence created by Stembridge in [17], Belkale and Brosnan [2] showed that the point counting function for general graph hypersurfaces is not of polynomial type, in fact, it is of the most general type one can conceive in the world of motivic counting functions. Bloch, Esnault and Kreimer [5] investigated the foundations of Feynman amplitudes and their relations to periods of mixed Hodge structures, and studied the mixed Hodge structure of the middle cohomology for wheel-type graphs. Explicit graphs not of mixed Tate type have first been found by Brown-Schnetz [9, 10] and Doryn [13].

The intention of this paper is to explore torus actions on graph hypersurface $X_\Gamma$ and their non-singular models, and to provide a set-up to compute the resulting motive using the theorem of Bialynicki-Birula [3]. In section §1 we study criteria for the existence of algebraic torus operations. In §2 we focus on a particular class of graphs, obtained by a gluing process, where the torus action is evident. In §3, we use the derived category $DM(k)$ of motives and apply the theorem of Bialynicki-Birula in a motivic context in order to study the motive of $X_\Gamma$. The presence of a torus action reduces the complexity of the motive of $X_\Gamma$ with this method to that of the fixed point loci in some resolution of singularities.

1. Existence of torus actions on graph hypersurfaces

Definition 1.1. Let $\Gamma$ be a finite, connected, not necessarily simple graph. The graph polynomial $P(\Gamma)$ of $\Gamma$ is defined as

$$P(\Gamma) := \sum_T \prod_{e \in T} X_e,$$

where $T$ runs through all spanning trees of $\Gamma$, and $X_e$ is a polynomial variable for each $e \in E(\Gamma)$. The polynomial $P(\Gamma)$ is homogenous of degree $h_1 = h_1(\Gamma)$ [5]. We
define the graph hypersurface
\[ X(\Gamma) := \{ P(\Gamma) = 0 \} \subset \mathbb{P}^{n-1}, \quad n = \sharp E(\Gamma). \]

In [5] this polynomial was rewritten in terms of a determinant of a symmetric \((h_1 \times h_1)\)-matrix \(M(\Gamma)\) with linear entries. Since much of this paper relies on this description we will repeat it here. For \(\Gamma\) we choose an orientation of its edges. Define a map \(\vartheta : Z^E(\Gamma) \to Z^V(\Gamma)\), by \(e \mapsto \sum_{v \in V(\Gamma)} sgn(v, e)v\), where \(sgn(v, e) = 1\) if \(v\) is the source of the edge \(e\), further \(sgn(v, e) = -1\) if \(v\) is the target of \(E\). This gives rise to a simplicial complex \(Z^E(\Gamma) \xrightarrow{\vartheta} Z^V(\Gamma)\) and a corresponding exact sequence
\[ 0 \to H_1(\Gamma, \mathbb{Z}) \xrightarrow{\partial} Z^E(\Gamma) \to Z^V(\Gamma) \to H_0(\Gamma, \mathbb{Z}) \to 0. \]

Let \(l_e(\cdot), e \in E(\Gamma)\) denote the dual basis of \(e \in E(\Gamma) \subseteq Z^E(\Gamma)\). Then we can consider the bilinear forms \(b_e\) of rank 1 given by
\[ b_e := (l_e \circ \iota) \cdot (l_e \circ \iota) : H_1(\Gamma, \mathbb{Z}) \times H_1(\Gamma, \mathbb{Z}) \to \mathbb{Z}. \]

Choose a basis \(B\) of \(H_1(\Gamma, \mathbb{Z})\) and let \(M_e = M_e(B)\) be the Gram-matrix associated to \(b_e\) and set
\[ M(\Gamma) := \sum_{e \in E(\Gamma)} X_eM_e \in \mathbb{Z}[X_e : e \in E(\Gamma)] \otimes \mathbb{Z} \text{End}(\mathbb{Z}^{h_1(\Gamma)}). \]

Here \(\mathbb{Z}[X_e : e \in E(\Gamma)]\) denotes the degree 1 part of the algebra \(\mathbb{Z}[X_e : e \in E(\Gamma)]\). We will usually abuse the notation and write \(M(\Gamma)\) without the subscript.

**Lemma 1.2.** One has \(P(\Gamma) = \det M(\Gamma)\).

**Proof.** See [5, Proposition (2.2)]. \(\square\)

**Remark 1.3.** \(P(\Gamma)\) satisfies the deletion-contraction formula
\[ P(\Gamma) = X_eP(\Gamma\setminus e) + P(\Gamma/e), \]
where \(e \in E(\Gamma)\) and \(\Gamma/e\) is the quotient of graphs. From this identity one can immediately deduce that subdivision of edges gives rise to affine fibre bundles over \(X_\Gamma\): Let \(\Gamma'\) be the graph obtained from \(\Gamma\) by subdividing the edge \(e\) into \(e_1\) and \(e_2\). Then
\[ P(\Gamma') = P(\Gamma)|_{X_e=X_{e_1}+X_{e_2}}. \]

Form now on we choose a field \(k\) and consider \(M(\Gamma)\) as a matrix in \(k[X_e : e \in E(\Gamma)]_1\).

**Lemma 1.4.** Let \(\Gamma\) be a graph and \(M(\Gamma)\) as above. Then the diagonal entries of \(M(\Gamma)\) are linearly independent in \(k[X_e : e \in E(\Gamma)]_1\).

**Proof.** By construction the elements on the diagonal correspond to a basis of \(H_1(\Gamma, \mathbb{F}_2)\) consisting of simple cycles. This means that they are independent over \(\mathbb{F}_2\). Hence they are independent over \(\mathbb{Z}\) which implies independence over \(k\). \(\square\)

**Definition 1.5** (Weight lattice). Let \(k\) be a field, \(s \in \mathbb{N}_0\) and \(k[X_1, \ldots, X_n]_s\) be the vector space of homogeneous polynomials in \(m\) variables of degree \(s\). For any
\[ f = \sum_{|\alpha| = s} c_\alpha X^\alpha \in k[X_1, \ldots, X_n]_s \]
let
\[ \Lambda(f) := \\{ \eta \in \mathbb{Z}^n : \eta \cdot \alpha = 0 \text{ for any } \alpha \text{ with } c_\alpha \neq 0 \} \]
be the weight lattice of the polynomial \(f\). Here, by a lattice we mean a free \(\mathbb{Z}\)-module of finite rank. We will refer to the elements of \(\Lambda(f)\) as weight vectors of \(f\).
Definition 1.6 (Torus operation). We define a (faithful) torus operation on a subvariety \( X \subset \mathbb{P}^{n-1} \) to be a (mono)morphism
\[
G_m^r \to \text{Aut}(X) \cap \text{PGL}_n(k)
\]
from a split \( r \)-dimensional torus \( G_m^r \) into the linear automorphisms of \( X \).

From now on, we assume that \( X \subset \mathbb{P}^{n-1} \) is a hypersurface. If \( k \) is algebraically closed and \( X \) admits a (faithful) linear \( G_m^r \)-action, one can always choose a coordinate system \( Z_1, \ldots, Z_n \) where \( G_m^r \) acts diagonally. This torus operation is encoded in the weight lattice \( \Lambda(f) \) (with respect to the coordinates \( Z_1, \ldots, Z_n \)).

The number \( \text{rank}(\Lambda(f)) - 1 \) is the maximal rank \( r \) of a diagonalized torus operation in these coordinates.

We will be mainly interested in the case where the polynomial \( f \) is the determinant of a symmetric \((h \times h)\)-matrix \( M \) with linear polynomials as entries. In that case let \( m = \binom{h+1}{2} \) and think of the elements in \( \mathbb{Z}^m \) as upper triangle entries of \( M \) and write \( \omega = (\omega_{ij})_{1 \leq i \leq j \leq h} \in \mathbb{Z}^m \) with \( \omega_{ij} = \omega_{ji} \). In this case, we define the lattice
\[
\Lambda_h := \{(\omega_{ij})_{1 \leq i \leq j \leq h} \in \mathbb{Z}^m : 2\omega_{ij} = \omega_{ii} + \omega_{jj}\}.
\]
The diagonal sublattice \( \Delta \hookrightarrow \Lambda_h \), where all entries are equal, is irrelevant since it corresponds to the action of the center of the \( \text{SL}(n, k) \). The lattice \( \Lambda_h \) has rank \( h \), since an element is already determined by the diagonal entries. It is a weight lattice of a certain polynomial for the following prototype example:

Example 1.7. Suppose \( M = (Y_{ij}) \in k[X_1, \ldots, X_n]^{h \times h} \) is a symmetric matrix such that all entries in the upper triangle are non-zero linearly independent linear homogenous polynomials. Then consider the maximal torus
\[
G_m^{h-1} \simeq \{\text{diag}(t_1, \ldots, t_h) \in k^{h \times h} : \prod_{i=1}^h t_i = 1\} \subseteq \text{SL}_h(k)
\]
given by diagonal matrices. Then the variety
\[
X := \{\det(M) = 0\} \subseteq \mathbb{P}^{n-1}
\]
carry a faithful \( G_m^{h-1} \)-action, defined by \( Y_{ij} \mapsto t_i t_j Y_{ij} \), since one has
\[
\det(\text{diag}(t_1, \ldots, t_h) M \text{diag}(t_1, \ldots, t_h)) = \det(M) \prod_{i=1}^h t_i^2 = \det(M).
\]
The weight lattice of this polynomial is obviously given by the lattice \( \Lambda_h \).

The following is a version of the previous example with much weaker assumptions:

Theorem 1.8. Let \( k \) be a field. Let \( M \in k[X_1, \ldots, X_n]^{h \times h} \) be a symmetric matrix such that there are \( \ell(M) \) non-zero entries in the upper triangle which are linearly independent linear homogenous polynomials, and all diagonal entries are non-zero. Then the hypersurface
\[
X := \{\det(M) = 0\} \subseteq \mathbb{P}^{n-1}(k),
\]
admits a linear, faithful \( G_m(k)^r \)-action with \( r \geq h - 1 + n - \ell(M) \).

Remark 1.9. The number \( r = h - 1 + n - \ell(M) \) is maximal with this property for the torus action defined in Example 1.7, but there may be additional actions. We construct a torus operation in such way that the weight lattice is contained in \( \Lambda_h \).

Proof. We proceed via induction on \( h \) to prove that there is short exact sequence of \( \mathbb{Z} \)-modules
\[
0 \to \mathbb{Z}^{n-\ell(M)} \to \Lambda(\det(M)) \to \Lambda_h / \Delta \to 0
\]
where $\Delta$ is the diagonal submodule generated by the vector with all entries equal to 1. The kernel of the surjection is obtained by letting $\omega_i = 0$ for all indices $i$ corresponding to variables in $k[Y_1, \ldots, Y_n]$ not occurring in $M$ if the variables $Y_i$ are chosen such that they contain the non-zero entries in the upper triangular matrix of $M$. It suffices to construct a torus operation in the set of variables $Y_{ij}$, which we identify with the non-zero matrix entries of $M$ by abuse of notation. This amounts to a linear change of variables only.

In case $h = 1$, there is nothing to prove. In case $h = 2$, then $M = \begin{pmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{pmatrix}$ or $M = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12} & Y_{22} \end{pmatrix}$. Then $\omega := (0, 2, 1)$ defines a non-zero weight vector of a determinantal hypersurface $X$ with $\Omega$ to $1$ and column. The matrix satisfies the assumption of Thm. 1.8. By induction, let $\Omega' \in \Lambda_{h-1}$ be a weight vector corresponding to the faithful torus action of rank $h - 2$ on $X' = \{\det(M') = 0\}$. Then using the equations $\omega_{1i} = \frac{2ii - 1}{2}$ we fill up $\Omega'$ to a weight matrix

$$
\Omega = \begin{pmatrix}
\omega_{11} & \frac{211 - 1}{2} & \cdots & \frac{211 + \omega_{hh}}{2} \\
\frac{211 + \omega_{hh}}{2} & \ddots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
\frac{2ii - 1}{2} & \cdots & \cdots & \cdots \\
\end{pmatrix}
$$

where $\omega_{11} \in \omega_{22} + 2\mathbb{Z}$ may be chosen arbitrarily. We know by Example 1.7 that the determinantal hypersurface $X$ is certainly invariant under the action defined by $\Omega$, since $\Omega \in \Lambda_h$.

The assumption on the diagonal guarantees that the first row of $M$ contains a non-zero entry $M_{11}$, hence the action defined by $\Omega$ is faithful on the projective subspace defined by the entries in the upper triangle of $M$. Equivalently this means that the induced map $\Lambda(\det(M)) \to \Lambda_h/\Delta$ is surjective. 

**Example 1.10.** Let $k = \mathbb{C}$. Wheels $WS_h$ with $h$ spokes and $2h$ edges satisfy the Proposition, since

$$
M(WS_h) = \begin{pmatrix}
Y_{11} & -X_2 & 0 & \cdots & -X_{i} \\
-X_2 & Y_{22} & -X_3 & \cdots & 0 \\
0 & -X_3 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & Y_{h-1,h-1} & -X_h & \cdots \\
-X_1 & 0 & \cdots & -X_h & Y_{hh} \\
\end{pmatrix},
$$

with $Y_{ii} = X_i + X_{i+1} + X_{h+i}$. Here $i + 1$ is to be considered mod $h$.

As a consequence, the associated hypersurfaces $X_h$ carries a faithful torus operation of rank $h - 1$. This bound is sharp, e.g., in the case $h = 3$, the hypersurface $X_3 \subseteq \mathbb{P}^5 \simeq P(\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)))$ is the complement of the 5-dimensional homogenous space $PSL_3(\mathbb{C})/SO_3(\mathbb{C})$, which carries a faithful rank 2 torus operation. There is no larger linear torus action since the group $PSL_3(\mathbb{C})$ is the stabilizer of $\mathbb{P}^5 \setminus X_3$ in $PSL_3(\mathbb{C}) = Aut(\mathbb{P}^5)$ and it has rank 2 (see [4]).
In general, the condition of linear independence in Thm. 1.8 is too restrictive. We need to define a new invariant for symmetric matrices $M$ to formulate a more general result. The proof of Thm. 1.8 then implies much more as we will see now. Let $k$ be a field. Let $M \in k[X_1, \ldots, X_n]^{h\times h}$ be a symmetric matrix of linear forms such that all diagonal entries are non-zero and linearly independent. We denote by $\ell(M)$ the dimension of the span of all upper-triangular entries, and by $N$ the number of all non-zero upper-triangular entries. By a linear change of variables, we may assume that each non-zero entry $M_{ij}$ of $M$ is either a variable $X_1, \ldots, X_{\ell(M)}$ or a linear form $L_{ij}(X_1, \ldots, X_{\ell(M)})$ in those variables. All diagonal entries are therefore assumed to be independent variables.

**Definition 1.11.** We define an equivalence relation on indices $(ij)$ ($i \leq j$) of the non-zero entries $M_{ij}$ as the transitive hull of the symmetric relation given by

$$(ij) \sim (kl) \iff \text{there is a variable } X \in \{X_1, \ldots, X_n\} \text{ in } M_{ij} \text{ and } M_{kl}.$$

The equivalence classes are called clusters.

An element $(ij)$ in a cluster $C$ is called excessive, if $X_i$ or $X_j$ do not occur in $L_{ij}(X_1, \ldots, X_{\ell(M)})$. Let

$$\delta(M) := \sum_{\text{clusters } C} (|C| - 1) + \sharp \text{ excessive entries in } M$$

be the excess of $M$.

**Theorem 1.12.** Under these assumptions, the dimension $r$ of a linear, faithful $\mathbb{G}_m(k)^\ell$-action on the hypersurface

$$X := \{\det(M) = 0\} \subset \mathbb{P}^{n-1}(k),$$

which is diagonal in the variables $X_1, \ldots, X_n$ is at least

$$r = \max \{0, h - 1 + n - \ell(M) - \delta(M)\}.$$

**Proof.** Substituting new variables for each linear form $L(X_1, \ldots, X_{\ell(M)})$, we arrive at an inclusion

$$i : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{N+n-\ell(M)-1},$$

where $N - \ell(M)$ is the number of additional variables $Y_{ij}$ with $i > j$. This inclusion maps $X$ to a codim $N - \ell(M) + 1$ subvariety $X' = i(X) = \{\det(M') = 0\} \bigcap \{H_{ij} = 0\}$, where $M'$ is the matrix obtained by the same substitutions, and $H_{ij}$ are the linear hyperplanes

$$H_{ij} : Y_{ij} = L_{ij}(X_1, \ldots, X_{\ell(M)}).$$

Theorem 1.8 implies the existence of a torus $T$ of rank $\geq h - 1 + N + n - \ell(M) - \ell(M') = h - 1 + n - \ell(M)$ acting on $\{\det(M') = 0\}$. Now we count conditions to obtain the codimension of a torus stabilizing $X' = i(X)$. For the variables $X_i$ in each cluster $C$ to have equal weight amounts to $|C| - 1$ conditions. The weights of the new variables $Y_{ij}$ are determined by diagonal entries, hence give no new condition on the torus operation if $L_{ij}$ is not excessive.

In total, this gives $\delta(M)$ conditions, and hence we obtain a torus operation of rank $\geq n - \ell(M) + h - 1 - \delta(M)$. \qed

**Remark 1.13.** The bounds in this theorem are far from being sharp. We provide a counter example below. In general, the proof give an algorithm to compute the correct rank of a torus, by checking whether the weight of the entry $Y_{ij}$ coincides with the weight of the variables occurring in $L_{ij}$.

In other words, in order to find a torus operation of maximal rank, we need to find a coordinate system, where the sum over cluster lengths minus one is minimized.
Example 1.14. Consider the graph which is the wheel with 3 spokes with one additional triangle subdivided. This gives rise to the matrix

\[
M = \begin{pmatrix}
X_2 + X_6 + X_8 & X_2 + X_6 & -X_2 & X_2 \\
X_2 + X_6 & X_1 + X_2 + X_4 + X_6 + X_7 & -X_1 - X_2 - X_4 & X_1 + X_2 \\
-X_2 & -X_1 - X_2 - X_4 & X_1 + X_2 + X_4 + X_5 & -X_1 - X_2 \\
X_2 & X_1 + X_2 & -X_1 - X_2 & X_1 + X_2 + X_3
\end{pmatrix}
\]

Substituting as in Theorem 1.12 we arrive at

\[
M = \begin{pmatrix}
Y_1 & Y_5 & Y_8 & -Y_8 \\
Y_5 & Y_2 & Y_6 & -Y_7 \\
Y_8 & Y_6 & Y_3 & Y_7 \\
-Y_8 & -Y_7 & Y_7 & Y_4
\end{pmatrix}
\]

Obviously we have two clusters of length 2 and 6 clusters of length 1. By the theorem this means we can expect \(X_Γ = \{\det(M) = 0\} \subseteq \mathbb{P}^7\) to have no torus action. However, there is a 1-dimensional action given by the weight vector \(ω := (3, -1, -1, -1, 1, -1, -1, 1)\). The algorithm would give the same result, as \(Y_7\) and \(Y_8\) are in excessive positions but impose no extra relation.

2. Examples: \(*\)-graphs

At the beginning of this section we need to introduce a few conventions. We will call a basis \(B \subseteq H_1(Γ)\) a cycle basis if it consists only of simple cycles. Since the matrix \(M(Γ)\) associated to a graph \(Γ\) depends on the chosen basis of \(H_1(Γ)\) we will make this dependence explicit in this section by writing \(M(Γ)_B\).

A class of examples which have linearly independent entries in \(M(Γ)\) and which contains the wheels with \(n\) spokes are the \(*\)-graphs:

Definition 2.1. A polygonal graph \(Γ\) is a connected, not necessarily simple, graph which has a decomposition \(Γ = Δ_1 \cup Δ_2 \cup \cdots \cup Δ_h\) as a successive gluing along non-empty, connected sets of edges inside the polygons \(Δ_i\), and such that no edge is used twice for gluing. Let \(E_0\) be the union of all edges used for the gluing. A \(*\)-graph \(Γ\) is a polygonal graph such that every such decomposition has the property \(h_1(E_0) = 0\).

Example 2.2. The example of a banana graph with 4 edges and 3 loops shows that the condition on \(E_0\) to be simply-connected depends on the gluing order. The matrix \(M(Γ)_B\) (corresponding to the basis \(B\) obtained from the 3 obvious loops) has linearly dependent entries for this graph. This shows that we have to require some strong conditions on the gluings.

Lemma 2.3. For any \(*\)-graph \(Γ\) one has

\(h_1(Γ) = \#\text{ polygons } Δ_i = h\).

Proof. We use the Mayer-Vietoris Theorem and induction on the number of polygons. Assume \(Γ = Γ' \cup Δ\), where \(Δ\) is a polygon. Then the intersection \(Γ' \cap Δ\) is a connected and contractible union of edges, in particular \(h_1(Γ' \cap Δ) = 0\) and \(h_0(Γ' \cap Δ) = 1\). Hence there is an isomorphism \(H_1(Γ') \oplus H_1(Δ) \cong H_1(Γ)\). \(\square\)
While it is natural to define *-graphs as a polygonal graph with an additional property we remark that they form a subclass of a very well known class of graphs:

**Lemma 2.4.** A graph $\Gamma$ is polygonal if and only if it is planar.

**Proof.** $\Gamma$ is polygonal if and only if $\Gamma = \bigcup E_0 \Delta_i$, where all $\Delta_i$ are simple cycles and no edge in $E_0$ is used twice for gluing. This condition means that the $\Delta_i$ contain a 2-basis of the cycle space $H_1(\Gamma) \otimes \mathbb{F}_2$ of $\Gamma$. Hence it is planar by MacLane’s planarity criterion [12]. Conversely consider a plane embedding $\Gamma \to \mathbb{R}^2$. Choose a compact disc $\Gamma \subseteq D \subseteq \mathbb{R}^2$ such that $\partial D \cap \Gamma = \emptyset$ (here $\partial$ means "boundary of").

Define the equivalence relation $\sim \subseteq D \times D$ by $x \sim y$ iff $x$ and $y$ are connected by a path in $D\backslash \Gamma$ or $\Gamma$. This gives a partition $D = \Gamma \cup A \cup \bigcup_{i=1}^{h_1(\Gamma)} \Delta_i^0$, where $A$ is the unique class with $\partial D \subseteq A$. Then $(\partial \Delta_i^0, \ldots, \partial \Delta_i^0)_{\Gamma}$ is a cycle basis of $H_1(\Gamma)$.

Further no edge of $\Gamma$ lies in more than two $\Delta_i$ since a line partitions $\mathbb{R}^2$ into two components (one containing the line). $\square$

**Definition 2.5.** We will call a simple cycle $\Delta \subseteq \Gamma$ an inner cycle of $\Gamma$ if there exist simple cycles $\Delta_2, \ldots, \Delta_{h_1(\Gamma)}$ such that $B := (\Delta_1, \Delta_2, \ldots, \Delta_{h_1(\Gamma)})$ is a cycle basis of $H_1(\Gamma)$ and $\Delta = \sum_{i=2}^{h_1(\Gamma)} \Delta \cap \Delta_i \in H_1(\Gamma, \mathbb{F}_2)$.

**Theorem 2.6.** Let $\Gamma$ be a graph. Then the following conditions are equivalent:

(i) $\Gamma$ is a *-graph.

(ii) There exists a cycle basis $B \subseteq H_1(\Gamma)$ such that the non-zero upper-triangular matrix entries $M_{ij}$ of $M(\Gamma)_B$ are linearly independent polynomials in $k[X_1, \ldots, X_n]_1$.

**Proof.** (ii) $\Rightarrow$ (i): We will first show that $\Gamma$ is planar. To this end we show that

1) $\Gamma$ has no inner cycles.

2) If a graph $\Gamma$ has no inner cycles then it is planar.

Suppose $\Gamma$ has inner cycles. This means there exist cycle basis $B = (\Delta_1, \ldots, \Delta_{h_1(\Gamma)})$ and $B' = (\Delta'_1, \ldots, \Delta'_{h_1(\Gamma)})$ of $H_1(\Gamma, \mathbb{F}_2)$ such that $\Delta'_1 = \sum_{i=2}^{h_1(\Gamma)} \Delta_i \cap \Delta'_1$ and the upper triangular entries $\neq 0$ of $M(\Gamma)_B$ are linearly independent. Since $GL(H_1(\Gamma, \mathbb{F}_2))$ is generated by transvections we can find transvections $t_1, \ldots, t_i$ such that $t_1 \cdots t_i(B) = B'$. We show that we can choose the $t_j$ such that $t_j \cdots t_1(B)$ is a cycle basis for all $i$. In the following I’ll do only one iteration of the reduction since one obtains the full reduction by simply repeating this step. Hence let $\Delta_i = \Delta'_i$ for all $i > 1$ and $\Delta'_1 = \sum_{i=1}^{h_1(\Gamma)} \alpha_i \Delta_i$, with $\alpha_1 = 1$. For $i = 2, \ldots, h_1(\Gamma)$ define $t_i = 1 + \alpha_i E_{1i}$, where $E_{1i}$ is the matrix with 1 at entry $(1, i)$ and 0 else. Then $t(B) = B'$, where $t = \prod_{i=2}^{h_1(\Gamma)} t_i$. Note that the $t_i$ commute pairwise. Suppose (after reordering if necessary) for some $i > 2$ (if $i = 2$ we are done) that $t_j \cdots t_1(\Delta_i)$ is a cycle basis for all $i \leq j$. Then there exists $2 \leq k < i$ such that $t_i \cdots t_1(\Delta_k)$ shares edges with cycle $\Delta_k$, since otherwise $\Delta'_i$ would not be a simple cycle. Now swap the indices of $t_{i-1}$ and $t_k$ and proceed inductively. This reduces us without loss to the situation $\Delta'_i = \Delta_1 + \Delta_2$ and $\Delta_i \cap \Delta_j \neq \emptyset$ for all $i \geq 2$. Now $\Delta'_i \cap \Delta_j = (\Delta_1 \cap \Delta_j) + (\Delta_2 \cap \Delta_j)$ for all $j$. In particular $\Delta'_1 \cap \Delta_j = (\Delta_1 \cap \Delta_j) + (\Delta_2 \cap \Delta_j)$.

Applying this transvection introduces linear relations among the elements of $D_2 := \{\Delta_1', \Delta_2', \ldots, \Delta_{h_1(\Gamma)}, \Delta_i', \Delta_i, \Delta_i \cap \Delta_j: \forall i, j \geq 2\}$, namely if $\Delta_1 \cap \Delta_j = \emptyset$ then $\Delta_1' \cap \Delta_j + \Delta_2 \cap \Delta_j = 0$. 
On the other hand if there is a relation that for some \( j \) with \( \Delta_1 \cap \Delta_j = \emptyset \neq \Delta_2 \cap \Delta_j \) contains exactly one of either \( \Delta'_1 \cap \Delta_j \) or \( \Delta_2 \cap \Delta_j \) then this relation gives rise to a non-trivial relation among the elements of
\[
D_1 := \{ \Delta_1, \ldots, \Delta_{h_1(\Gamma)}, \Delta_i \cap \Delta_j \; \forall i, j \},
\]
since \( \Delta_2 \cap \Delta_j \) does not cancel out. A non-trivial relation among the elements of
\( D_2 \) containing \( \Delta'_i \cap \Delta_j \) for a \( j \) with \( \Delta_1 \cap \Delta_j \neq \emptyset, \Delta_i \cap \Delta_j, \Delta'_1 \) or any \( \Delta_i \) clearly lifts to a non-trivial relation among the elements of \( D_1 \) for similar reasons.

In summary this implies that the relation \( \Delta'_1 = \sum_{i=1}^{h_1(\Gamma)} \Delta'_i \cap \Delta'_j \) from the beginning lifts to a non-trivial relation among the elements of \( \{ \Delta_1, \ldots, \Delta_{h_1(\Gamma)}, \Delta_i \cap \Delta_j \; \forall i, j \} \), a contradiction.

Hence 1 holds. For 2 note that the class of graphs without inner cycles is closed under taking subgraphs and that (all subdivisions of) \( K_3, 3 \) and \( K_3 \) have inner cycles. Thus 2 follows from Kuratowski’s planarity criterion \cite{12}.

Now let \( \Gamma \) be planar. Assume \( \Gamma = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_h \), but \( h_1(E_0) > 0 \). Let \( \Delta_1, \ldots, \Delta_h \) be the natural basis of \( H_1(\Gamma) \) given by the polygons \( \Delta_i \). Given a simple non-zero loop \( \gamma \subset E_0 \), there is a linear relation between the diagonal entries for all \( \Delta_i \) meeting \( \gamma \) and all off-diagonal entries carrying glueing data for these \( \Delta_i \).

(i) \( \Rightarrow \) (ii): Conversely, suppose that \( \Gamma \) is a \( * \)-graph and we have given a linear relation among the entries of \( M(\Gamma) \). By definition of \( * \)-graphs this relation involves a diagonal element, since every edge is only used once for gluing. Hence, we get an equations
\[
\sum_{i=1}^{h} a_i M_{ii} = \sum_{i<j} b_{ij} M_{ij},
\]
with at least one \( a_i \) and one \( b_{ij} \) non-zero by Lemma \cite{1.4}. This is a contradiction, since each \( \Delta_i \) occurring on the left with \( a_i \neq 0 \) has an edge which is not contained in \( E_0 \).

\[ \square \]

**Corollary 2.7.** The \( * \)-graphs admit a faithful \((h_1-1)\)-dimensional torus operation.

**Proof.** By Prop. \cite{160}, the entries of \( M(\Gamma) \) satisfy the assumptions of Prop. \cite{163} \[ \square \]

3. Motivic Bialynicki-Birula decompositions

In this section we discuss how to apply high dimensional torus actions on \( X_\Gamma \) to compute the motive of a graph hypersurface \( X_\Gamma = \{ \det M(\Gamma) = 0 \} \) in specific examples.

The simplest example which is not entirely trivial is \( \Gamma = WS_3 \), the wheel with 3 spokes. The graph hypersurface \( X_\Gamma \) for \( \Gamma = WS_3 \) is isomorphic to \( Sym^2 P^2 \), which has a resolution by blowing up the diagonal and admits a 2-dimensional torus operation. The motive of \( X_\Gamma \) is mixed Tate by \cite{3} Sect. 9. We want to generalize this to a larger class of graphs.

**Lemma 3.1.** Let \( \Gamma \) be a graph such that the non-zero entries in the upper triangular part of the matrix \( M(\Gamma) \) are linearly independent. Then for the action of \( T := \mathbb{G}_m^r \) with \( r = h_1(\Gamma)-1+n-\ell(M) \) on \( X_\Gamma \), as described in Theorem \cite{1.8} and Example \cite{1.7}, the variety \( Fix_{\mathbb{G}_m^r} (\Gamma)-1(T) \) consists of points contained in \( X_\Gamma \).

**Proof.** We may assume that \( n = \ell(M) \), since the action on the \( n-\ell(M) \) extra variables is effective. By Example \cite{1.7} the action on the generic symmetric matrix with independent linear entries is given by \( (t, x) \mapsto (t_1 t_2 x_{ij}) \). Choosing special values for \( t_i \) and \( t_j \) with \( \prod_i t_i = 1 \), one sees that the fixed points in this case are just the points corresponding to the usual standard basis of the underlying space \( \mathbb{P}^{N-1} \) with \( N = \binom{h_1-1}{2} \). In general, the graph hypersurfaces of the type described in the assumption are intersections of the generic zero set of the determinant of this generic
symmetric matrix with ($T$-invariant) linear coordinate subspaces. The induced action is faithful (e.g. by Lemma 1.4). Hence the fixed point set $\text{Fix}_{G_{\Gamma}(r)}(T)$ is given by points in $\mathbb{P}^{N-1}$ with exactly one non-zero entry supported in $E(\Gamma)$. Obviously these points are contained in $X_T$.

Note that all graph hypersurfaces of wheels $WS_n$ with $n$ spokes satisfy this lemma.

In the following we use motives $M(X)$ in the sense of Voevodsky’s triangulated category $\text{DM}(k) = \text{DM}_{gm}(k)$ \cite{18} for any $k$-scheme $X$.

**Definition 3.2.** Let $f : Z \to X$ be a closed immersion of schemes defined over $k$. Then the relative motive corresponding to this morphism is the mapping cone $M(Z) \to M(X)$ to a distinguished triangle in $\text{DM}(k)$:

$$\text{M}(Z) \to M(X) \to M(X, Z).$$

Now we want to give a criterion when the motive of a graph hypersurface $M(X_T) \in \text{DM}(k)$ is mixed Tate. In view of the classical Bialynicki-Birula theorem \cite{3} and its motivic versions \cite{8}, one might expect that the motive should be determined by the fixed point loci. In the presence of singularities, the stratification via affine bundles in the Bialynicki-Birula proof becomes degenerate, so that one cannot apply the same idea. However, we can describe $M(X_T)$ using “smaller” motives in a resolution of singularities. In order to do this, let $X$ be an arbitrary projective $k$-variety with a faithful $T := \mathbb{G}_m$-action, and $\pi = \pi_1 \circ \ldots \circ \pi_{m-1}$ be a resolution of singularities together with a stratification

$$\emptyset \subset X_m \subset \ldots \subset X_1 \subset X_0 = X$$

such that

1. $X_i \setminus X_{i+1}$ is smooth for all $1 \leq i \leq m-1$,
2. $\pi_1 \circ \ldots \circ \pi_{m-1} : X^{(i)} \to X$ (here $X^{(i)}$ denotes the abstract blow-up of $X$ given by $\pi_1 \circ \ldots \circ \pi_{m-1}$) resolves the singularities up to $X_i$, i.e., $(\pi_1 \circ \ldots \circ \pi_{m-1})^{-1}(X_i) \subseteq X^{(i)}$ is a union of smooth components with smooth mutual intersections, and $\pi_i \circ \ldots \circ \pi_{m-1}$ is an isomorphism outside $(\pi_1 \circ \ldots \circ \pi_{m-1})^{-1}(X_{i+1})$.
3. Each $\pi_i$ is equivariant with respect to the $T$-action.

Such a situation can always be obtained in case $k$ is an algebraically closed field with $\text{char}(k) = 0$ by using embedded resolution of singularities \cite{19}. We assume that we are in this situation now.

**Proposition 3.3.** Let

$$\tau := \langle M(F), M(F^{(i)}) : \forall i, F \subseteq \text{Fix}_X(T), F^{(i)} \subseteq \text{Fix}_{X^{(i)}}(T) \text{ irred. components} \rangle$$

be the full triangulated subcategory of $\text{DM}(k)$ obtained by taking the pseudoabelian envelop inside $\text{DM}(k)$ of the full triangulated subcategory, closed under Tate twists, generated by the motives in the brackets. Then $M(X) \in \tau$. In particular if the generating motives lie in $\text{DMT}(k)$ then so does $M(X)$.

**Proof.** Note that for any triangulated category $\tau$ and any sequence $A_r \to A_{r-1} \to \ldots \to A_1 \to A_0$ of arrows in $\tau$, $A_0$ is contained in the full triangulated subcategory $\gamma_0$ of $\tau$ generated by $A_r$ and cone$(A_i \to A_{i-1})$, for $1 \leq i \leq r$. We apply this fact to the sequence $M(X_r) \to M(X_{r-1}) \to \ldots \to M(X)$. Let $Y$ be a $k$-scheme. Let $\pi : \hat{Y} \to Y$ be an abstract blow-up with center $Z \hookrightarrow Y$ in the sense of \cite{18} Prop 4.1.3. In this case there is a distinguished triangle

$$M(\pi^{-1}(Z)) \to M(\hat{Y}) \oplus M(Z) \to M(Y)$$
in DM($k$). The diagram

\[
\begin{array}{ccc}
0 & \rightarrow & M(Z) \\
\downarrow & & \downarrow \\
M(\pi^{-1}(Z)) & \rightarrow & M(\hat{Y}) \oplus M(Z) \\
\downarrow & & \downarrow \\
M(\pi^{-1}(Z)) & \rightarrow & M(\hat{Y}) \\
\end{array}
\]

\[
\begin{array}{ccc}
\rightarrow & & \\
\rightarrow & & \\
\rightarrow & & \\
\end{array}
\]

implies, using Verdier’s lemma [1, Prop. 1.1.11], the Aeppli formula

\[
M(\hat{Y}, \pi^{-1}(Z)) \cong M(Y, Z)
\]

in DM($k$).

Let $A \hookrightarrow W$ be a closed immersion of smooth projective varieties defined over $k$. Let $T = G_m$ be an algebraic torus acting faithfully on $W$ such that $A \hookrightarrow W$ is equivariant. Then

\[
M(W, A) \cong \bigoplus F M(F^+, F^+ \cap A),
\]

where $F$ runs through all components of the fix point locus such that the corresponding cell $F^+$ in the sense of [14] is not contained in $A$. The proof follows from the commutativity of the diagram

\[
\begin{array}{ccc}
M(W) & \rightarrow & \bigoplus F M(F^+) \\
\downarrow & & \downarrow \\
M(W, A) & \rightarrow & \bigoplus F M(F^+, F^+ \cap A),
\end{array}
\]

where the first two isomorphisms follow from the smoothness of the intersections of the components of $W_2$ with $W_1$ and induction and the second isomorphism follows from the usual Bialynicki-Birula and induction. By the functoriality of the Bialynicki-Birula decomposition the second row is just a direct sum of distinguished Mayer-Vietoris triangles. This implies

\[
M(W) = \bigoplus F M(F^+).
\]

Now the assertion of the proposition follows from successively resolving equivariantly the singularities of $X$ and each $X_i$. 
Motives of Graph Hypersurfaces with Torus Operations

\[ M(X_i, X_{i+1}) \cong M(X_i^{(i)}, X_{i+1}^{(i)}) \]
\[ \cong \bigoplus_F M\left((F^{(i)})^+ \cap X_i^{(i)}, (F^{(i)})^+ \cap X_{i+1}^{(i)}(n_\bullet)\right) \]

by the Aeppli formula and the relative Bialynicki-Birula decomposition. Further \( M((F^{(i)})^+ \cap X_i^{(i)}, (F^{(i)})^+ \cap X_{i+1}^{(i)}) \) is the mapping cone of
\[ M((F^{(i)})^+ \cap X_i^{(i)}) \to M((F^{(i)})^+ \cap X_i^{(i)}). \]
Finally \( M((F^{(i)})^+ \cap X_{i+1}^{(i)}) \cong M(F^{(i)} \cap X_{i+1}^{(i)})(n_\bullet) \) and \( M((F^{(i)})^+ \cap X^{(i)}) \cong M(F^{(i)} \cap X^{(i)})(n_\bullet) \), where \( n_\bullet \) indicates an appropriate Tate twist.

\[ \square \]

Remark 3.4. To be more precise, an object \( M \in DM(k) \) is called mixed Tate, if it is in the image of
\[ DMT(k) \to DM(k) \otimes \mathbb{Q}, \]
where \( DMT(k) \) is the \( \mathbb{Q} \)-linear triangulated category defined by Levine [16].

Proposition 3.3 reduces the complexity of the motive of \( X_\Gamma \) with this method to that of the fixed point loci in some resolution of singularities. This method should be successful provided there is some sufficiently high dimensional torus action. However, besides the wheel with 3 spokes, we do not have much evidence yet, as the computational complexity is quite large even in simple examples.

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