Fractional Order Integrals for the Sustainable Development Model

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Abstract. Sustainable development is now engaged in many leading countries. The notion is connected with the activity of countries, their economies, social and political institutions. One of the criteria of sustainability is the economic sphere, the study of which will allow you to control and predict its activity. It is known that the description of continuous processes that arise in economic problems, using definite integrals. In the case when the studied continuous development is carried out in small jumps, when it is the description of the use of fractional integral and differential equations. However, their exact solution is possible only in special cases. Therefore, the construction of approximate methods for solving fractional integral and differential equations is of great importance. In this paper, we give an approximate solution of fractional integral equations of the first kind, where the fractional integral is constructed using the Weyl fractional integral. It turns out that the constructed operator of fractional integration is symmetrical and positively definite. These properties are built by the operator enables to introduce the scalar product on the basis of the specified operator and the corresponding energy space. Next, using the same methodology that is used in the finite element method, approximate solution is constructed, constructed a rough scheme of the method and rationale of the proposed method to construct the energy space. The integral equation of the standard form is given as a specific example, as well as an example solution to illustrate the effectiveness of the proposed method.

Keywords: sustainable development, fractional order integrals, approximate methods, direct methods, projection methods, integral equations, generalized solution.

1 Introduction
Sustainable development is currently on the agenda of all the leading countries of the world. So in 2015, at the UN assembly, world leaders discussed the issue of sustainable development of all countries, agreed on goals and objectives which cover such indicators of development of countries as economic, humanitarian and social. Their solution takes into account both external and internal conditions, both private and global. As a result of this work, a document was prepared that defined the goals and objectives of sustainable development for the next fifteen years, up to 2030 [1]. The main global tasks are highlighted, such as studying climate change on the planet, fighting poverty. The goals and objectives of sustainable development lie in the plane of social, economic, technological, political and partnerships [2].

The development of each area is subject to its own laws and is determined by the fulfillment of certain conditions and criteria. For the study of which models are set that take into account the influence of factors and indicators affecting the result Evaluation of the relevance of indicators is of paramount importance for determining their quality [3]. For each sphere of development, there are laws and rules, so creating a universal model is very difficult and not relevant, we will focus on the economic component of sustainable development.

Economic development is determined by socio-economic processes, which most often change stepwise, the jumps of which are close to zero. The summation of such changes is possible through integral sums. Thus, the definition of the integral amount allows you to use the concept of a certain integral in the socio-economic sphere. In addition, the spasmodic nature of changes with small-order jumps can lead to the use of fractional order integrals, as well as the use of fractional differentiation operators in diffusion problems, biology and medicine, and economics [4].
In this regard, the study of equations with fractional-integral operators is currently being carried out very actively. There are a number of theoretical and applied problems that lead to the need to solve equations with fractional integration operators. Such tasks include diffusion problems, electrochemical, economic processes, as well as the issues mentioned above. These problems, as a rule, are not exactly solved, therefore the development and application of approximate methods of solution with their subsequent theoretical justification for these equations are very acute. We also note that recently in the scientific literature there are works in which numerical methods for some classes of equations are proposed (see, for example, [5-12]). Estimates are also obtained for special cases of fractional integral operators [13]. The solutions of some generalized Riemann-Liouville boundary value problems are theoretically substantiated, the existence and uniqueness of their solutions are determined [14]. In [15], some new generalized fractional integral inequalities of the midpoint and trapezoid type were obtained for doubly differentiable convex functions. Theoretical substantiation of the existence of solutions to a number of generalized problems using fractional-order integrals has been the subject of many works, especially over the past year [16].

The authors also propose methods for calculating fractional order integrals, for example, using the Conhauser matrix polynomials [17], using suitable modifications in the classical process of monotone iterative methods [18].

Very promising and convenient methods for solving fractional-integral and fractional-differential equations are direct methods, methods that ultimately come down to solving systems of linear algebraic equations. A special case of such methods are projection methods, the substantiation of which is the work of [19, 20]. The convergence of direct methods for the fractional-integral equation with the fractional Riemann-Liouville integral was substantiated in [21].

However, despite the success achieved in this direction, the question of theoretical justification for the application of approximate methods for a more general class of similar problems remains open.

The paper proposes a generalized Bubnov-Galerkin method for finding an approximate solution of fractional-integral equations. Estimates are obtained for the convergence of the approximate solution to the exact solution by the metric of the energy space generated by the fractional-integral operator. The constructed computational method is illustrated by a particular example and the method is estimated.

2 Materials and methods

Consider the equation:

\[ f^{(\alpha)}u + Tu = f, \quad u, f \in L_2[0,2\pi], \]  
(1)

where \( f^{(\alpha)} = \frac{f^{(\alpha)}}{\alpha} \) expressed using fractional integration operators \( f^{(\alpha)}_{\pm} \), defined for fractional Weyl integrals for functions \( f(x) \in L_1(a,b) \), according to the formulas:

\[ f_{\pm}^{(\alpha)} \varphi(x) = \begin{cases} \frac{1}{(\alpha)} \int_{a}^{x} \varphi(t)dt \frac{1}{(x-t)^{1-\alpha}}, & x > a, \\ \frac{1}{(\alpha)} \int_{x}^{b} \varphi(t)dt \frac{1}{(x-t)^{1-\alpha}}, & x < b, \end{cases} 
(2)

Here \( \alpha > 0 \). Integrals (2) are also called fractional Riemann-Liouville integrals of order \( \alpha \), left-handed and right-handed, respectively. \( u \) is unknown, and \( f \) is a given function from the space \( L_2[0,2\pi] \). \( T \) is some operator for which \( (f^{(\alpha)} + T) \) is a linear operator, and, in the general case, unbounded and not positive definite.

We note that the usual form \( f^{(\alpha)}_+, f^{(\alpha)}_- \) (see (2)) of Riemann-Liouville fractional integration is inconvenient in the theory of trigonometric series dealing with periodic functions, since fractional integration does not have the property of translating periodic functions to periodic ones. Therefore, it is customary to use another definition of fractional integration proposed by G. Weil, according to which the operator \( f^{(\alpha)} \) acts according to the rule:
Here \( u_k \) are the Fourier coefficients for the function \( u \).

The following lemmas are valid for operator (3).

**Lemma 1.** \( I^{(\alpha)} \) is a positive definite operator.

**Proof.** It is known [22] that \( I^{(\alpha)} \) is a positive definite operator if the condition:

\[
\left( I^{(\alpha)} u, u \right) > 0
\]

is satisfied. In addition, the system \( e^{ikx} \) is complete in the space \( L_2(0,2\pi) \), then

\[
\left( I^{(\alpha)} u, u \right) = \sum_{k=-\infty}^{\infty} \left( I^{(\alpha)} u, e^{ikx} \right) (u, e^{ikx}) = \sum_{k=-\infty}^{\infty} u_k^2 > 0.
\]

Q.E.D.

**Lemma 2.** \( I^{(\alpha)} \) is a symmetric operator.

**Proof.** Consider the scalar product \( \langle I^{(\alpha)} u, v \rangle \) for any \( u, v \in L_2(0,2\pi) \):

\[
\langle I^{(\alpha)} u, v \rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2\pi} \int_0^{2\pi} u(x-t) \sum_{k=-\infty}^{\infty} \frac{e^{ikx}}{(ik)^\alpha} dt \ v(x) dx =
\]

\[
= -\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2\pi} \int_x^{x-2\pi} u(t) \sum_{k=-\infty}^{\infty} \frac{e^{iku(x-t)}}{(ik)^\alpha} dt \ v(x) dx =
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2\pi} \int_0^{2\pi} v(x) \sum_{k=-\infty}^{\infty} \frac{e^{-iku(x-t)}}{(ik)^\alpha} dx \ u(t) dt = (u, I^{(\alpha)} v).
\]

Similarly for \( I^{(\alpha)} \) we have: \( \langle I^{(\alpha)} u, v \rangle = \langle u, I^{(\alpha)} v \rangle \).

Since the operator \( I^{(\alpha)} \) is the sum of the symmetric operators \( I^{(\alpha)}_1 \) and \( I^{(\alpha)}_2 \), it is also a symmetric operator.

We introduce the scalar product and the norm: \( [u, v] = (I^{(\alpha)} u, v), \quad [u] = (I^{(\alpha)} u, u)^{1/2} \).

The scalar product will have the form:

\[
[u, v] = (I^{(\alpha)} u, v) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=-\infty}^{\infty} \frac{u_k e^{ikx}}{|k|^{\alpha/2}} v(x) dx = \sum_{k=-\infty}^{\infty} u_k^2 v_k.
\]

Replenishing \( D(I^{(\alpha)}) \) according to the introduced norm, we obtain the energy space, which, according to [22], is denoted by \( H_1 \).

Multiplying the original equation (1) by an arbitrary function \( v \in D(I^{(\alpha)}) \), we obtain the following equation:

\[
[u, v] + (Tu, v) = (f, v).
\]

Equation (4) admits a generalized statement of the problem. A generalized solution of equation (1) is a function \( u \in H_1 \), satisfying equation (4) for any function \( v \in H_1 \).

**Bubnov-Galerkin Method**

According to the Bubnov-Galerkin method, in the energy space \( H_1 \), the system of basis functions \( \phi_j, j = 1..N \) is selected. An approximate solution is sought in the form of a polynomial in the selected system of basis functions in the form:

\[
u_N = \sum_{j=1}^{N} a_j \phi_j.
\]

Unknown coefficients \( a_j \) are determined from a system of equations of the form

\[
[u_N, c_k] + (Tu_N, c_k) = (f, c_k), \quad k = 1...N.
\]

Given the representation (5) and the linearity of the introduced and ordinary scalar products, we obtain the following system of linear algebraic equations:

\[
\sum_{j=1}^{N} a_j [c_j, c_k] + \sum_{j=1}^{N} a_j (Tc_j, c_k) = (f, c_k), \quad k = 1...N.
\]
Theorem 1.

Let

1) equation (1) have a unique solution for a given right-hand side.
2) The form
   \[ L(u, v) = [u, v] + (Tu, v) \]
   is defined and \( \mathcal{R} \) limited, i.e. conditions are met:
   \[ L(u, u) \geq \epsilon_0^2 [u]^2, \quad L(u, v) \leq \epsilon_1^2 [u][v], \quad \epsilon_0, \epsilon_1 \equiv \text{const.} \]
3) Subspace sequence \( \mathcal{R}_N \) - linear shell functions \( \mathcal{S}_i, j = 1..N \) - is extremely dense in \( \mathcal{R} \).

Then, for any finite \( N \), system (6) is uniquely solvable and the approximate solution \( u_N \) converges to the exact solution \( u \) as \( N \to \infty \) by the metric \([\cdot]\) and the error estimate is valid:

\[ [u - u_N] \leq \epsilon(u, N), \]

where \( \epsilon(u, N) \) - given function of \( N \) (estimate of the approximation error) satisfying the inequality:

\[ \min_{j} \left| f^{(6)} \left( u - \sum_{j=1}^{N} c_j u_j \right) \right| \leq \epsilon(u, N). \]

Special case

Consider the special case of equation (1). As the operator \( T \), we take the following integral operator:

\[ (Tu)(x) = \frac{1}{2\pi} \int_{0}^{2\pi} h(t, x) u(t) dt. \]

We assume that it has the necessary properties.

As the basis functions, we choose \( e^{ikx}, k = -N..N \). We will look for an approximate solution in the form:

\[ u_N = \sum_{j=-N}^{N} a_j e^{ijx}. \]

Then the terms in system (7) are transformed in the form:

\[ \sum_{j=-N}^{N} a_j [c_j, c_k] = \frac{a_k}{|k|^6}, \]

\[ \sum_{j=-N}^{N} a_j (Tc_j, c_k) = \sum_{j=-N}^{N} a_j (h_j, c_k) = \sum_{j=-N}^{N} a_j h_{-j,k}. \]

Therefore, system (7) can be written as:

\[ \frac{a_k}{|k|^6} + \sum_{j=-N}^{N} a_j h_{-j,k} = f_k, \quad k = 1..N. \]

What will match the matrix equation:
\[
\begin{pmatrix}
\frac{1}{|N|^6} + h_{N,-N} \ldots & h_{j,-N} & \ldots & h_{-N,-N} \\
\vdots & \ddots & \ddots & \vdots \\
h_{k,-N} & \vdots & \ddots & h_{k,-N} \\
h_{N,N} \ldots & h_{j,N} & \ldots & \frac{1}{|N|^6} + h_{-N,N}
\end{pmatrix} \cdot \begin{pmatrix}
(a_{-N}) \\
\vdots \\
(a_N)
\end{pmatrix} = \begin{pmatrix}
f_{-N} \\
\vdots \\
f_N
\end{pmatrix}.
\]

The error estimate in this case will take the form:

\[
\min_{c_j} \left\|\mathcal{I}^{(5)} \left( u - \sum_{j=1}^{N} c_j e^{i j x} \right) \right\| \leq e(u, N).
\]

### 3 Results and Discussion

As indicated above, sustainable development issues are dealt with worldwide. Due to the fact that this concept includes many areas of human life and activity, it is impossible to propose any universal model of sustainable development. However, in some areas of socio-economic activity, the processes in which change stepwise, with small jumps close to zero, it is possible to use the integral calculus to interpret the process under study. Since the jumps are small, the summation of such changes is possible through integral sums, which can lead to the use of fractional order integrals. Consequently, the question of using such integrals depends on the possibility of finding their solutions. To do this, it is necessary to offer an apparatus for their calculation, in this case, approximative. The article presents an approximate method for solving such integrals based on the finite element method. An accuracy assessment of the proposed method is also given. The presented work is theoretical in nature, the results presented in it can be used later for other models in which fractional integrals can be applied that have the same properties as those presented in the article.

### 4 Conclusions

The construction of approximate methods for solving fractional integral equations is a very urgent task. It is also important to justify such methods, applications, and rates of convergence. In this paper, this method can be proposed for a wide class of fractional integral equations; a concrete example of the application of this method.

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