THE GLUING FORMULA OF THE REFINED ANALYTIC TORSION FOR AN ACYCLIC HERMITIAN CONNECTION

RUNG-TZUNG HUANG AND YOONWEON LEE

Abstract. In the previous work ([14]) we introduced the well-posed boundary conditions $\mathcal{P}^-,\mathcal{L}_0$ and $\mathcal{P}^+,\mathcal{L}_1$ for the odd signature operator to define the refined analytic torsion on a compact manifold with boundary. In this paper we discuss the gluing formula of the refined analytic torsion for an acyclic Hermitian connection with respect to the boundary conditions $\mathcal{P}^-,\mathcal{L}_0$ and $\mathcal{P}^+,\mathcal{L}_1$. In this case the refined analytic torsion consists of the Ray-Singer analytic torsion, the eta invariant and the values of the zeta functions at zero. We first compare the Ray-Singer analytic torsion and eta invariant subject to the boundary condition $\mathcal{P}^-,\mathcal{L}_0$ or $\mathcal{P}^+,\mathcal{L}_1$ with the Ray-Singer analytic torsion subject to the relative (or absolute) boundary condition and eta invariant subject to the APS boundary condition on a compact manifold with boundary. Using these results together with the well known gluing formula of the Ray-Singer analytic torsion subject to the relative and absolute boundary conditions and eta invariant subject to the APS boundary condition, we obtain the main result.

1. Introduction

The refined analytic torsion was introduced by M. Braverman and T. Kappeler ([4], [5]) on an odd dimensional closed Riemannian manifold with a flat bundle as an analytic analogue of the refined combinatorial torsion introduced by M. Farber and V. Turaev ([10], [11], [25], [26]). Even though these two objects do not coincide exactly, they are closely related. The refined analytic torsion is defined by using the graded zeta-determinant of the odd signature operator and is described as an element of the determinant line of the cohomologies. Specially, when the odd signature operator is defined by an acyclic Hermitian connection on a closed manifold, the refined analytic torsion is a complex number, whose modulus is the Ray-Singer analytic torsion and the phase part is the $\rho$-invariant determined by the given odd signature operator and the trivial odd signature operator acting on the trivial line bundle.

In the previous work ([14]) we introduced the well-posed boundary conditions $\mathcal{P}^-,\mathcal{L}_0$ and $\mathcal{P}^+,\mathcal{L}_1$ for the odd signature operator, which are complementary to each other and have similar properties as the relative and absolute boundary conditions. We showed that the refined analytic torsion is well-defined under these boundary conditions on a compact oriented Riemannian manifold with boundary. In this paper we discuss the gluing formula of the refined analytic torsion with respect to the boundary conditions $\mathcal{P}^-,\mathcal{L}_0$ and $\mathcal{P}^+,\mathcal{L}_1$ when the odd signature operator is given by an acyclic Hermitian connection. In this case the refined analytic torsion consists of the Ray-Singer analytic torsion, the eta invariant and the values of the zeta functions at zero. The gluing formula of the Ray-Singer analytic torsion with respect to the relative and absolute boundary conditions has been obtained by W. Lück ([21]), D. Burghelea, L. Friedlander and T. Kappeler in [9] (cf. [29]). The gluing formula of the eta invariant with respect

2000 Mathematics Subject Classification. Primary: 58J52; Secondary: 58J28, 58J50.

Key words and phrases. refined analytic torsion, zeta-determinant, eta-invariant, odd signature operator, well-posed boundary condition.

The second author was supported by the NRF with the grant number 2010-0008726.
to the Atiyah-Patodi-Singer (APS) boundary condition has been studied by many authors, for instance, K. Wojciechowski ([32], [33]), U. Bunke ([7]), J. Brüning, M. Lesch ([6]), P. Kirk and M. Lesch ([17]). To use these results we first compare the Ray-Singer analytic torsion subject to the boundary condition $\mathcal{P}_-,\mathcal{L}_0$ or $\mathcal{P}_+,\mathcal{L}_1$ with the Ray-Singer analytic torsion subject to the relative or the absolute boundary condition. We next compare the eta invariant associated to the odd signature operator subject to $\mathcal{P}_-,\mathcal{L}_0$ or $\mathcal{P}_+,\mathcal{L}_1$ with the eta invariant subject to the APS boundary condition. To compare the Ray-Singer analytic torsions we are going to use the BFK-gluing formula for zeta-determinants ([8], [18], [19]) and the adiabatic limit method. To compare the eta invariants we are going to follow the method given in [6]. These comparison results together with the well known gluing formulas lead to our main result. The boundary value problem and the gluing formula of the refined analytic torsion have been already studied by B. Vertman ([27], [28]) but our method is completely different from what he presented.

Suppose that $\rho : \pi_1(M) \to GL(n, \mathbb{C})$ is a representation of the fundamental group and $E = \tilde{M} \times_\rho \mathbb{C}^n$ is the associated flat bundle, where $\tilde{M}$ is a universal covering space of $M$. We choose a flat connection $\nabla$ and extend it to a covariant differential $\nabla : \Omega^*(M,E) \to \Omega^{*+1}(M,E)$.

Using the Hodge star operator $*_\rho$, we define the involution $\Gamma = \Gamma(\rho^M) : \Omega^*(M,E) \to \Omega^{m-*}(M,E)$ by

$$\Gamma \omega := i^r(-1)^{\frac{r(r+1)}{2}}*_\rho \omega, \quad \omega \in \Omega^p(M,E), \quad (1.1)$$

where $r$ is given as above by $r = \frac{m+1}{2}$. It is straightforward to see that $\Gamma^2 = \text{Id}$. We define the odd signature operator $\mathcal{B}$ by

$$\mathcal{B} = \mathcal{B}(\nabla, g^M) := \Gamma \nabla + \nabla \Gamma : \Omega^*(M,E) \to \Omega^*(M,E). \quad (1.2)$$

Then $\mathcal{B}$ is an elliptic differential operator of order 1. Let $N$ be a collar neighborhood of $Y$ which is isometric to $[0, 1) \times Y$. Any $q$-form $\omega$ can be written, on $N$, by

$$\omega = \omega_{\text{tan}} + du \wedge \omega_{\text{nor}},$$

where $\omega_{\text{tan}}$ and $\omega_{\text{nor}}$ are the tangential and normal parts of $\omega$ and $du$ is the dual of the inward unit normal vector field $\partial u$ to the boundary $Y$ on $N$. Then we have a natural isomorphism

$$\Psi : \Omega^p(N,E|_N) \to C^\infty([0, 1), \Omega^p(Y,E|_Y) \oplus \Omega^{p-1}(Y,E|_Y)), \quad \Psi(\omega_{\text{tan}} + du \wedge \omega_{\text{nor}}) = (\omega_{\text{tan}}, \omega_{\text{nor}}). \quad (1.3)$$

Using the product structure we can induce a flat connection $\nabla^Y : \Omega^*(Y,E|_Y) \to \Omega^{*+1}(Y,E|_Y)$ from $\nabla$ and a Hodge star operator $*_Y : \Omega^*(Y,E|_Y) \to \Omega^{m-1-*}(Y,E|_Y)$ from $*_\rho$. We define two maps $\beta, \Gamma^Y$ by

$$\beta : \Omega^p(Y,E|_Y) \to \Omega^p(Y,E|_Y), \quad \beta(\omega) = (-1)^p \omega,$$

$$\Gamma^Y : \Omega^p(Y,E|_Y) \to \Omega^{m-1-p}(Y,E|_Y), \quad \Gamma^Y(\omega) = i^{r-1}(-1)^{\frac{r(r+1)}{2}}*_Y \omega. \quad (1.4)$$

It is straightforward that

$$\beta^2 = \text{Id}, \quad \Gamma^Y \Gamma^Y = \text{Id}. \quad (1.5)$$
Simple computation shows that

\[
\Gamma = i\beta \Gamma^Y \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \nabla = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \nabla_{\partial u} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \nabla^Y.
\]

(1.6)

Hence the odd signature operator \(B\) is expressed, under the isomorphism (1.3), by

\[
B = -i\beta \Gamma^Y \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \nabla_{\partial u} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} (\nabla^Y + \Gamma^Y \nabla^Y \Gamma^Y) \right\}.
\]

(1.7)

We denote

\[
\gamma := -i\beta \Gamma^Y \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{A} := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} (\nabla^Y + \Gamma^Y \nabla^Y \Gamma^Y)
\]

(1.8)

so that \(B\) has the form of

\[
\mathcal{B} = \gamma (\partial_u + \mathcal{A}) \quad \text{with} \quad \gamma^2 = -\text{Id}, \quad \gamma \mathcal{A} = -\mathcal{A} \gamma.
\]

(1.9)

Since \(\nabla_{\partial u} \nabla^Y = \nabla^Y \nabla_{\partial u}\), we have

\[
\mathcal{B}^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \nabla^2_{\partial u} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\nabla^Y + \Gamma^Y \nabla^Y \Gamma^Y)^2 = (-\nabla^2_{\partial u} + \mathcal{B}^2_Y) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

(1.10)

where

\[
\mathcal{B}_Y = \Gamma^Y \nabla^Y + \nabla^Y \Gamma^Y.
\]

We next choose a Hermitian inner product \(h^E\). All through this paper we assume that \(\nabla\) is a Hermitian connection with respect to \(h^E\), which means that \(\nabla\) is compatible with \(h^E\), i.e. for any \(\phi, \psi \in C^\infty(E)\),

\[
dh^E(\phi, \psi) = h^E(\nabla \phi, \psi) + h^E(\phi, \nabla \psi).
\]

The Green formula for \(B\) is given as follows (cf. [14]).

**Lemma 1.1.** (1) For \(\phi \in \Omega^q(M, E), \psi \in \Omega^{m-q}(M, E)\), \(\langle \Gamma \phi, \psi \rangle_M = \langle \phi, \Gamma \psi \rangle_M\).

(2) For \(\phi \in \Omega^q(M, E), \psi \in \Omega^{q+1}(M, E)\),

\[
\langle \nabla \phi, \psi \rangle_M = \langle \phi, \Gamma \nabla \psi \rangle_M - \langle \phi_{\text{tan}}|_Y, \psi_{\text{nor}}|_Y \rangle_Y.
\]

(3) For \(\phi, \psi \in \Omega^{\text{even}}(M, E)\) or \(\Omega^{\text{odd}}(M, E)\),

\[
\langle B \phi, \psi \rangle_M - \langle \phi, B \psi \rangle_M = -\langle \phi_{\text{tan}}|_Y, i \beta \Gamma^Y (\psi_{\text{tan}}|_Y) \rangle_Y - \langle \phi_{\text{nor}}|_Y, i \beta \Gamma^Y (\psi_{\text{nor}}|_Y) \rangle_Y = \langle \phi|_Y, (\gamma \psi)|_Y \rangle_Y.
\]

**Remark:** In the assertions (2) and (3) the signs on the inner products on \(Y\) are different from those in [14] because in [14] \(\partial u\) is an outward unit normal vector field.

We note that \(\mathcal{B}_Y\) is a self-adjoint elliptic operator on \(Y\). Putting \(\mathcal{H}^\bullet(Y, E|_Y) := \ker \mathcal{B}_Y^2, \mathcal{H}^\bullet(Y, E|_Y)\) is a finite dimensional vector space and we have

\[
\Omega^\bullet(Y, E|_Y) = \text{Im} \nabla^Y \oplus \text{Im} \Gamma^Y \nabla^Y \Gamma^Y \oplus \mathcal{H}^\bullet(Y, E|_Y).
\]

If \(\nabla \phi = \Gamma \nabla \Gamma \phi = 0\) for \(\phi \in \Omega^\bullet(M, E)\), simple computation shows that \(\phi\) is expressed, near the boundary \(Y\), by

\[
\phi = \nabla^Y_{\text{tan}} \phi_{\text{tan}, h} + du \wedge (\Gamma^Y \nabla^Y \Gamma^Y \phi_{\text{nor}, h} + \phi_{\text{nor}, h}), \quad \phi_{\text{tan}, h}, \phi_{\text{nor}, h} \in \mathcal{H}^\bullet(Y, E|_Y).
\]

(1.11)
We define \( \mathcal{K} \) by
\[
\mathcal{K} := \{ \phi_{\text{tan}, h} \in \mathcal{H}^\bullet(Y, E|_Y) \mid \nabla \phi = \Gamma \nabla \Gamma \phi = 0 \},
\]
where \( \phi \) has the form (1.11). If \( \phi \) satisfies \( \nabla \phi = \Gamma \nabla \Gamma \phi = 0 \), so is \( \Gamma \phi \) and hence
\[
\Gamma^Y \mathcal{K} = \{ \phi_{\text{nor}, h} \in \mathcal{H}^\bullet(Y, E|_Y) \mid \nabla \phi = \Gamma \nabla \Gamma \phi = 0 \},
\]
where \( \phi \) has the form (1.11). The second assertion in Lemma 1.1 shows that \( \mathcal{K} \) is perpendicular to \( \Gamma^Y \mathcal{K} \). We then have the following decomposition (cf. Corollary 8.4 in [17], Lemma 2.4 in [14]).
\[
\mathcal{K} \oplus \Gamma^Y \mathcal{K} = \mathcal{H}^\bullet(Y, E|_Y),
\]
which shows that \( (\mathcal{H}^\bullet(Y, E|_Y), \langle \cdot, \cdot \rangle_Y, -i \beta \Gamma^Y) \) is a symplectic vector space with Lagrangian subspaces \( \mathcal{K} \) and \( \Gamma^Y \mathcal{K} \). We denote by
\[
\mathcal{L}_0 = \left( \begin{array}{c} \mathcal{K} \\ \mathcal{K} \end{array} \right), \quad \mathcal{L}_1 = \left( \begin{array}{c} \Gamma^Y \mathcal{K} \\ \Gamma^Y \mathcal{K} \end{array} \right).
\]

We next define the orthogonal projections \( \mathcal{P}_{-, \mathcal{L}_0}, \mathcal{P}_{+, \mathcal{L}_1} : \Omega^\bullet(Y, E|_Y) \rightarrow \Omega^\bullet(Y, E|_Y) \oplus \Omega^\bullet(Y, E|_Y) \) by
\[
\text{Im } \mathcal{P}_{-, \mathcal{L}_0} = \left( \begin{array}{c} \text{Im } \nabla^Y \oplus \mathcal{K} \\ \text{Im } \nabla^Y \oplus \mathcal{K} \end{array} \right), \quad \text{Im } \mathcal{P}_{+, \mathcal{L}_1} = \left( \begin{array}{c} \text{Im } \Gamma^Y \nabla^Y \Gamma^Y \oplus \Gamma^Y \mathcal{K} \\ \text{Im } \Gamma^Y \nabla^Y \Gamma^Y \oplus \Gamma^Y \mathcal{K} \end{array} \right).
\]
Then \( \mathcal{P}_{-, \mathcal{L}_0}, \mathcal{P}_{+, \mathcal{L}_1} \) are pseudodifferential operators and give well-posed boundary conditions for \( B \) and the refined analytic torsion. We denote by \( B_{\mathcal{P}_{-, \mathcal{L}_0}} \) and \( B_{q,\mathcal{P}_{-, \mathcal{L}_0}} \) the realizations of \( B \) and \( B_q \) with respect to \( \mathcal{P}_{-, \mathcal{L}_0} \), i.e.
\[
\text{Dom } (B_{\mathcal{P}_{-, \mathcal{L}_0}}) = \{ \psi \in \Omega^\bullet(M, E) \mid \mathcal{P}_{-, \mathcal{L}_0} (\psi|_Y) = 0 \},
\]
\[
\text{Dom } (B_{q,\mathcal{P}_{-, \mathcal{L}_0}}) = \{ \psi \in \Omega^q(M, E) \mid \mathcal{P}_{-, \mathcal{L}_0} (\psi|_Y) = 0, \mathcal{P}_{-, \mathcal{L}_0} ((B\psi)|_Y) = 0 \}.
\]

We define \( B_{\mathcal{P}_{+, \mathcal{L}_1}} \), \( B_{q,\mathcal{P}_{+, \mathcal{L}_1}} \), \( B_{q,\text{abs}}, B_{q,\text{rel}} \) and \( B_{\Pi, \mathcal{K}} \), \( B_{\Pi, \mathcal{C}} \) (see Section 3) in the similar way. The following result is straightforward (Lemma 2.11 in [14]).

**Lemma 1.2.**
\[
\ker B_{q,\mathcal{P}_{-, \mathcal{L}_0}}^2 = \ker B_{q,\text{rel}}^2 = H^q(M, Y; E), \quad \ker B_{q,\mathcal{P}_{+, \mathcal{L}_1}}^2 = \ker B_{q,\text{abs}}^2 = H^q(M; E).
\]

We choose an Agmon angle \( \theta \) by \(-\frac{\pi}{2} < \theta < 0\). For \( \mathcal{D} = \mathcal{P}_{-, \mathcal{L}_0} \) or \( \mathcal{P}_{+, \mathcal{L}_1} \) we define the zeta function \( \zeta_{B_{q,\mathcal{D}}}^2(s) \) and eta function \( \eta_{B_{\text{even}, \mathcal{D}}}^2(s) \) by
\[
\zeta_{B_{q,\mathcal{D}}}^2(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left( \text{Tr } e^{-tB_{q,\mathcal{D}}} - \dim \ker B_{q,\mathcal{D}}^2 \right) dt,
\]
\[
\eta_{B_{\text{even}, \mathcal{D}}}^2(s) = \frac{1}{\Gamma(s - 1)} \int_0^\infty t^{s-1} \text{Tr } (Be^{-tB_{\text{even}, \mathcal{D}}}) dt.
\]
It was shown in [14] that \( \zeta_{B_{q,\mathcal{D}}}^2(s) \) and \( \eta_{B_{\text{even}, \mathcal{D}}}^2(s) \) have regular values at \( s = 0 \). We define the zeta-determinant and eta-invariant by
We denote
\[
\log \text{Det}_{2\theta} \mathcal{B}^2_{q, \partial} := -\zeta^\partial_{\mathcal{B}^2_{q, \partial}}(0),
\]
\[
\eta(\mathcal{B}_{\text{even}, \partial}) := \frac{1}{2} (\eta_{\mathcal{B}_{\text{even}, \partial}}(0) + \dim \ker \mathcal{B}_{\text{even}, \partial}).
\]

We denote
\[
\Omega^q_\pm(M, E) = \text{Im} \nabla \cap \Omega^q(M, E), \quad \Omega^\pm_\partial(M, E) = \text{Im} \Gamma \nabla \cap \partial_\partial \Omega^q(M, E),
\]
\[
\Omega^\pm_{\text{even}}(M, E) = \sum_{q=\text{even}} \Omega^q_\pm(M, E),
\]
and denote by \(\mathcal{B}^\pm_{\text{even}}\) the restriction of \(\mathcal{B}_{\text{even}}\) to \(\Omega^\pm_{\text{even}}(M, E)\). The graded zeta-determinant \(\text{Det}_{gr, \theta}(\mathcal{B}_{\text{even}, \partial})\) of \(\mathcal{B}_{\text{even}}\) with respect to the boundary condition \(\partial\) is defined by
\[
\text{Det}_{gr, \theta}(\mathcal{B}_{\text{even}, \partial}) = \frac{\text{Det}_\theta \mathcal{B}^\pm_{\text{even}, \partial}}{\text{Det}_\theta \mathcal{B}^-_{\text{even}, \partial}}.
\]

We next define the projections \(\tilde{\mathcal{P}}_0, \tilde{\mathcal{P}}_1: \Omega^\mp(Y, E|_Y) \oplus \Omega^\mp(Y, E|_Y) \to \Omega^\mp(Y, E|_Y) \oplus \Omega^\mp(Y, E|_Y)\) as follows. For \(\phi \in \Omega^q(M, E)\)
\[
\tilde{\mathcal{P}}_0(\phi|_Y) = \begin{cases} \mathcal{P}_{-\mathcal{L}_0}(\phi|_Y) & \text{if } q \text{ is even} \\ \mathcal{P}_{+\mathcal{L}_0}(\phi|_Y) & \text{if } q \text{ is odd} \end{cases},
\]
\[
\tilde{\mathcal{P}}_1(\phi|_Y) = \begin{cases} \mathcal{P}_{-\mathcal{L}_1}(\phi|_Y) & \text{if } q \text{ is even} \\ \mathcal{P}_{+\mathcal{L}_1}(\phi|_Y) & \text{if } q \text{ is odd} \end{cases}.
\]

We denote by
\[
l_q := \dim \ker \mathcal{B}^2_{Y, q}, \quad l^+_q := \dim \mathcal{K} \cap \ker \mathcal{B}^2_{Y, q}, \quad \text{and} \quad l^-_q := \dim \Gamma Y \cap \ker \mathcal{B}^2_{Y, q},
\]
so that \(l_q = l^+_q + l^-_q \) and \(l^-_q = l^+_{q-1} \). Simple computation shows that \(\log \text{Det}_{gr, \theta}(\mathcal{B}_{\text{even}, -\mathcal{L}_0})\) and \(\log \text{Det}_{gr, \theta}(\mathcal{B}_{\text{even}, +\mathcal{L}_1})\) are described as follows ([14]).

\[
\begin{align*}
(1) \quad \log \text{Det}_{gr, \theta}(\mathcal{B}_{\text{even}, -\mathcal{L}_0}) &= \frac{1}{2} \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \log \text{Det}_{2\theta} \mathcal{B}^2_{q, \mathcal{P}_0} - i\pi \eta(\mathcal{B}_{\text{even}, -\mathcal{L}_0}) \\
&\quad + \frac{\pi i}{2} \left( \frac{1}{4} \sum_{q=0}^{m-1} \zeta_{\mathcal{B}^2_{q, \partial}}(0) + \sum_{q=0}^{r-2} (r - 1 - q)(l^+_q - l^-_q) \right). \\
(2) \quad \log \text{Det}_{gr, \theta}(\mathcal{B}_{\text{even}, +\mathcal{L}_1}) &= \frac{1}{2} \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \log \text{Det}_{2\theta} \mathcal{B}^2_{q, \mathcal{P}_1} - i\pi \eta(\mathcal{B}_{\text{even}, +\mathcal{L}_1}) \\
&\quad - \frac{\pi i}{2} \left( \frac{1}{4} \sum_{q=0}^{m-1} \zeta_{\mathcal{B}^2_{q, \partial}}(0) + \sum_{q=0}^{r-2} (r - 1 - q)(l^+_q - l^-_q) \right).
\end{align*}
\]

To define the refined analytic torsion we introduce the trivial connection \(\nabla^{\text{trivial}}\) acting on the trivial bundle \(M \times \mathbb{C}\) and define the trivial odd signature operator \(\mathcal{B}^{\text{trivial}}_{\text{even}}: \Omega^\text{even}(M, \mathbb{C}) \to \Omega^\text{even}(M, \mathbb{C})\) in the
same way as \([12]\). The eta invariant \(\eta(B_{\text{trivial}})|_{P_{-\epsilon_0}/P_{+\epsilon_1}}\) associated to \(B_{\text{trivial}}\) and subject to the boundary condition \(P_{-\epsilon_0}/P_{+\epsilon_1}\) is defined in the same way as in \([19]\) by simply replacing \(B_{\text{even},P_{-\epsilon_0}/P_{+\epsilon_1}}\) by \(B_{\text{trivial},P_{-\epsilon_0}/P_{+\epsilon_1}}\). When \(\nabla\) is acyclic in the de Rham complex, the refined analytic torsion subject to the boundary condition \(P_{-\epsilon_0}/P_{+\epsilon_1}\) is defined by

\[
\log TP_{-\epsilon_0}(g^M, \nabla) = \log \det_{gr}(B_{\text{even},P_{-\epsilon_0}}) + \frac{\pi i}{2} \langle \text{rank } E \rangle \eta(B_{\text{trivial}})|_{P_{-\epsilon_0}}(0) \tag{1.24}
\]

\[
\log TP_{+\epsilon_1}(g^M, \nabla) = \log \det_{gr}(B_{\text{even},P_{+\epsilon_1}}) + \frac{\pi i}{2} \langle \text{rank } E \rangle \eta(B_{\text{trivial}})|_{P_{+\epsilon_1}}(0) \tag{1.25}
\]

The refined analytic torsion on a closed manifold is defined similarly.

In this paper we are going to discuss the gluing formula of the refined analytic torsion with respect to the boundary conditions \(P_{-\epsilon_0}\) and \(P_{+\epsilon_1}\). For this purpose in the next two sections we are going to compare the Ray-Singer analytic torsion and eta invariant subject to the boundary condition \(P_{-\epsilon_0}\) (or \(P_{+\epsilon_1}\)) with those subject to the relative and APS boundary conditions, respectively.

2. COMPARISON OF THE Ray-SINGER ANALYTIC TORSIONS

In this section we are going to compare the Ray-Singer analytic torsion subject to the boundary condition \(P_{-\epsilon_0}\) with the Ray-Singer analytic torsion subject to the relative boundary condition. For this purpose we are going to use the BFK-gluing formula and the method of the adiabatic limit for stretching the cylinder part. We recall that \((M, g^M)\) is a compact oriented Riemannian manifold with boundary \(Y\) with a collar neighborhood \(N = [0, 1] \times Y\) and \(g^M\) is assumed to be a product metric on \(N\). We denote by \(M_{1,1} = [0, 1] \times Y\) and \(M_2 = M - N\). To use the adiabatic limit we stretch the cylinder part \(M_{1,1}\) to the cylinder of length \(r\) and denote \(M_{1,r} = [0, r] \times Y\) with the product metric and

\[M_r = M_{1,r} \cup_{Y_r} M_2\quad \text{with } Y_r = \{r\} \times Y.\]

Then we can extend the bundle \(E\) and the odd signature operator \(\mathcal{B}\) on \(M\) to \(M_r\) in the natural way and we denote these extensions by \(E_r\) and \(B(r) (B = B(1))\). We denote the restriction of \(B(r)\) to \(M_{1,r}\) and \(M_2\) by \(B_{M_{1,r}}, B_{M_2}\). It is well known (cf. [16], [2]) that the Dirichlet boundary value problem for \(B_2^q\) on \(M_2\) has a unique solution, i.e. for \(f + du \wedge g \in \Omega^q(M_2, E|_{M_2})\), there exists a unique \(\psi \in \Omega^q(M_2, E|_{M_2})\) such that

\[\mathcal{B}_2^q \psi = 0, \quad \psi|_{Y_r} = f + du \wedge g.\]

Let \(\mathcal{D}\) be one of the following boundary conditions: \(\mathcal{P}_{-\epsilon_0}, \mathcal{P}_{+\epsilon_1}\), the absolute boundary condition, the relative boundary condition or the Dirichlet boundary condition. We define the Neumann jump operators

\[
Q_{q,1,\mathcal{D}}(r), \quad Q_{q,2} : \Omega^q(Y_r, E|_{Y_r}) \oplus \Omega^{q-1}(Y_r, E|_{Y_r}) \to \Omega^q(Y_r, E|_{Y_r}) \oplus \Omega^{q-1}(Y_r, E|_{Y_r})
\]

as follows. For \(f + du \wedge g \in \Omega^q(Y_r, E|_{Y_r}) \oplus \Omega^{q-1}(Y_r, E|_{Y_r})\), we choose \(\phi \in \Omega^q(M_{1,r}, E|_{M_{1,r}})\) and \(\psi \in \Omega^q(M_2, E|_{M_2})\) such that

\[
\mathcal{B}_2^q \phi = 0, \quad \mathcal{B}_2^q \psi = 0, \quad \phi|_{Y_r} = \psi|_{Y_r} = f + du \wedge g, \quad \mathcal{D}(\phi|_{Y_0}) = 0. \tag{2.1}
\]
Then we define
\[ Q_{q,1,\mathcal{D}}(f) = (\nabla_{\partial_u} \phi) |_{Y_r}, \quad Q_{q,2}(f) = - (\nabla_{\partial_u} \psi) |_{Y_r}, \]
where \( \partial u \) is the inward unit normal vector field on \( N \subset M \). We next define the Dirichlet-to-Neumann operator \( R_{q,\mathcal{D}}(r) \) as follows.

\[
R_{q,\mathcal{D}}(r): \Omega^2(Y_r, E|_{Y_r}) \oplus \Omega^{g-1}(Y_r, E|_{Y_r}) \rightarrow \Omega^g(Y_r, E|_{Y_r}) \oplus \Omega^{g-1}(Y_r, E|_{Y_r})
\]

\[
R_{q,\mathcal{D}}(r) = Q_{q,1,\mathcal{D}}(r) + Q_{q,2}. \quad (2.2)
\]

The following lemma is well known (cf. [18]).

**Lemma 2.1.** (1) \( R_{q,\mathcal{D}}(r) \) is a non-negative elliptic pseudodifferential operator of order 1 and has the form of

\[
R_{q,\mathcal{D}}(r) = \begin{pmatrix}
2 \sqrt{B_{Y,q}} & 0 \\
0 & 2 \sqrt{B_{Y,q-1}}
\end{pmatrix} + \text{a smoothing operator.} \quad (2.3)
\]

(2) \( \ker R_{q,\mathcal{D}} = \{ \phi|_{Y_r} \mid \phi \in \ker B_{q,\mathcal{D}}^2 \} \).

We denote by \( B_{q,M_1,r,\mathcal{D},D}^2 \) (\( B_{q,M_2,D}^2 \)) the restriction of \( B_{q}^2(r) \) to \( M_1, r \) (\( M_2 \)) subject to the boundary condition \( \mathcal{D} \) on \( Y_0 \) and the Dirichlet boundary condition on \( Y_r \) (the Dirichlet boundary condition on \( Y_r \)). We denote by \( B_{q,\mathcal{D}}^2(r) \) the operator \( B_{q}^2(r) \) on \( M_r \) subject to the boundary condition \( \mathcal{D} \) on \( Y_0 \). Then Lemma [12] shows that \( \dim \ker B_{q,\mathcal{D}}^2(r) \) is a topological invariant. Let \( \dim \ker B_{q,\mathcal{D}}^2(r) = k \) and \( \{ \phi_1, \cdots, \phi_k \} \) be an orthonormal basis of \( \ker B_{q,\mathcal{D}}^2(r) \). We define a positive definite \( k \times k \) Hermitian matrix \( A_{q,\mathcal{D}}(r) \) by

\[
A_{q,\mathcal{D}}(r) = (a_{ij}), \quad a_{ij} = \langle \phi_i|_{Y_0}, \phi_j|_{Y_0} \rangle_{Y_0}.
\]

Then the BFK-gluing formula ([8], [18], [19]) is described as follows. Setting \( l_q = \dim \ker B_{Y,q}^2 \), we have

\[
\log \det_{\theta} B_{q,\mathcal{D}}^2(r) = \log \det_{2\theta} B_{q,M_1,r,\mathcal{D},D}^2 + \log \det_{2\theta} B_{q,M_2,D}^2 + \log \det_{2\theta} R_{q,\mathcal{D}}(r) - \log 2 \cdot (\zeta_{B_{Y,q}^2}(0) + \zeta_{B_{Y,q-1}^2}(0) + l_q + l_{q-1}) - \log \det A_{q,\mathcal{D}}(r). \quad (2.4)
\]

The above equality can be rewritten as follows.

**Corollary 2.2.**

(1) \[
\log \det_{2\theta} B_{q,\mathcal{D}}^2(r) = \log \det_{2\theta} B_{q,M_1,r,\mathcal{D},D}^2 + \log \det_{2\theta} B_{q,M_2,D}^2 + \log \det_{2\theta} R_{q,\mathcal{D}}(r) - \log 2 \cdot (\zeta_{B_{Y,q}^2}(0) + \zeta_{B_{Y,q-1}^2}(0) + l_q + l_{q-1}) - \log \det A_{q,\mathcal{D}}(r).
\]

(2) \[
\log \det_{2\theta} B_{q,p,+,c_1}^2(r) = \log \det_{2\theta} B_{q,M_1,r,+,D}^2 + \log \det_{2\theta} B_{q,M_2,D}^2 + \log \det_{2\theta} R_{q,+,c_1}(r) - \log 2 \cdot (\zeta_{B_{Y,q}^2}(0) + \zeta_{B_{Y,q-1}^2}(0) + l_q + l_{q-1}) - \log \det A_{q,+,c_1}(r).
\]

(3) \[
\log \det_{2\theta} B_{q,rel}^2(r) = \log \det_{2\theta} B_{q,M_1,rel,D}^2 + \log \det_{2\theta} B_{q,M_2,D}^2 + \log \det_{2\theta} R_{q,rel}(r) - \log 2 \cdot (\zeta_{B_{Y,q}^2}(0) + \zeta_{B_{Y,q-1}^2}(0) + l_q + l_{q-1}) - \log \det A_{q,rel}(r)
\]

(4) \[
\log \det_{2\theta} B_{q,abs}^2(r) = \log \det_{2\theta} B_{q,M_1,abs,D}^2 + \log \det_{2\theta} B_{q,M_2,D}^2 + \log \det_{2\theta} R_{q,abs}(r) - \log 2 \cdot (\zeta_{B_{Y,q}^2}(0) + \zeta_{B_{Y,q-1}^2}(0) + l_q + l_{q-1}) - \log \det A_{q,abs}(r).
\]
Remark: The BFK-gluing formula was proved originally on a closed manifold in [8]. But it can be extended to a compact manifold with boundary with only minor modification when a cutting hypersurface does not intersect the boundary.

We define $\Omega^q_{\pm}(Y,E|_V)$ similarly to [1.20] and denote $B^q_{Y;q} := B^q_{Y;q}|_{\Omega^q_{\pm}(Y,E|_V)}$. Simple computation leads to the following results.

**Lemma 2.3.** The spectra of $B^2_{q,M_{1,r},P_{-},\zeta_0,D}$, $B^2_{q,M_{1,r},P_{+},\zeta_1,D}$, $B^2_{q,M_{1,r},rel,D}$ and $B^2_{q,M_{1,r},abs,D}$ are given as follows. Let $k = 1, 2, 3, \cdots$.

1. $\text{Spec} \left( B^2_{q,M_{1,r},P_{-},\zeta_0,D} \right) = \left\{ \lambda_{q-1,j} + \left( \frac{k\pi}{r} \right)^2 \right\} \cup \left\{ \lambda_{q-2,j} + \left( \frac{k\pi}{r} \right)^2 \right\} \cup \left\{ \lambda_{q,j} + \left( \frac{(k - \frac{1}{2})\pi}{r} \right)^2 \right\}$

2. $\text{Spec} \left( B^2_{q,M_{1,r},P_{+},\zeta_1,D} \right) = \left\{ \lambda_{q-1,j} + \left( \frac{(k - \frac{1}{2})\pi}{r} \right)^2 \right\} \cup \left\{ \lambda_{q-2,j} + \left( \frac{(k - \frac{1}{2})\pi}{r} \right)^2 \right\} \cup \left\{ \lambda_{q,j} + \left( \frac{k\pi}{r} \right)^2 \right\}$

3. $\text{Spec} \left( B^2_{q,M_{1,r},rel,D} \right) = \left\{ \lambda_{q-1,j} + \left( \frac{k\pi}{r} \right)^2 \right\} \cup \left\{ \lambda_{q,j} + \left( \frac{k\pi}{r} \right)^2 \right\} \cup \left\{ \lambda_{q-2,j} + \left( \frac{(k - \frac{1}{2})\pi}{r} \right)^2 \right\}$

4. $\text{Spec} \left( B^2_{q,M_{1,r},abs,D} \right) = \left\{ \lambda_{q-1,j} + \left( \frac{(k - \frac{1}{2})\pi}{r} \right)^2 \right\} \cup \left\{ \lambda_{q,j} + \left( \frac{(k - \frac{1}{2})\pi}{r} \right)^2 \right\} \cup \left\{ \lambda_{q-2,j} + \left( \frac{k\pi}{r} \right)^2 \right\}$

where each $\lambda_{q,j}$ runs on $\text{Spec} \left( B^2_{Y;q} \right)$ and $\left\{ \left( \frac{k\pi}{r} \right)^2 \right\}_{1q}$ means that the multiplicity of each $\left( \frac{k\pi}{r} \right)^2$ is $t_q^+$. For each $q$ we define

\[
\zeta_{\Delta,q,N}(s) = \sum_{\lambda_{q,j} \in \text{Spec} \left( B^2_{Y;q} \right)} \sum_{k=1}^{\infty} \left( \lambda_{q,j} + \left( \frac{(k - \frac{1}{2})\pi}{r} \right)^2 \right)^{-s},
\]

\[
\zeta_{\Delta,q,D}(s) = \sum_{\lambda_{q,j} \in \text{Spec} \left( B^2_{Y;q} \right)} \sum_{k=1}^{\infty} \left( \lambda_{q,j} + \left( \frac{k\pi}{r} \right)^2 \right)^{-s}.
\]

The following result is well known (cf. [22]).
Lemma 2.4. We put  
\[ \zeta_{Y,q}(s) = \zeta_{B_{2,q},(s - \frac{1}{2})}. \]
Then:

1. \(-\zeta'_{\Delta_{q,N}}(0) = -r\zeta_{Y,q}(0) + \sum_{\lambda_{q,j} \in \text{Spec}(B_{2,q}^+)} \log(1 + e^{-2r\sqrt{\lambda_{q,j}}}), \]
2. \(-\zeta'_{\Delta_{q,D}}(0) = -r\zeta_{Y,q}(0) - \frac{1}{2} \log \det(B_{2,q}^+) + \sum_{\lambda_{q,j} \in \text{Spec}(B_{2,q}^+)} \log(1 - e^{-2r\sqrt{\lambda_{q,j}}}). \]

Proof. The computation of \(-\zeta'_{\Delta_{q,D}}(0)\) was done in Proposition 5.1 of [22]. Using the Poisson summation formula, we have the following identity
\[ \sum_{k=1}^{\infty} e^{-\pi t^2 (k-\frac{1}{2})^2} = \frac{1}{\sqrt{\pi t}} \left( \frac{1}{2} + 2 \sum_{k=1}^{\infty} e^{-\frac{4k^2}{t}} - \sum_{k=1}^{\infty} e^{-\frac{4k^2}{t}} \right), \]
from which we can compute \(-\zeta'_{\Delta_{q,N}}(0). \)

Corollary 2.5. Putting \(C_q^+(r) = \prod_{\lambda_{q,j} \in \text{Spec}(B_{2,q}^+)} \left( 1 + \frac{2e^{-r\sqrt{\lambda_{q,j}}}}{e^{\sqrt{\lambda_{q,j}}} - e^{-r\sqrt{\lambda_{q,j}}}} \right), \) we have
\[ \left( -\zeta'_{\Delta_{q,N}}(0) \right) - \left( -\zeta'_{\Delta_{q,D}}(0) \right) = \frac{1}{2} \log \det(B_{2,q}^+) + \log C_q^+(r). \]

If we denote the Riemann zeta function by \(\zeta_R(s)\), it is well known that \(\zeta_R(0) = -\frac{1}{2}\) and \(\zeta'_R(0) = -\frac{1}{2} \log 2\pi\), which leads to the following result.

Lemma 2.6. Setting \(\zeta_1(s) = \sum_{k=1}^{\infty} \left( \frac{k^2}{r^2} \right)^{-2s} \) and \(\zeta_2(s) = \sum_{k=1}^{\infty} \left( \frac{(k^2 + 1/4)^2}{r^2} \right)^{-2s} \), we have \(\zeta_1(0) = -\log 2\) and \(\zeta_2(0) = -\log 2\).

Lemma 2.3 together with Corollary 2.5 and Lemma 2.6 yields the following result.

Lemma 2.7.

1. \(\log \det(B_{2,q,1,r,+}^+, -\zeta_{1,D}) = \frac{1}{2} \left( \log \det(B_{2,q,1,r,rel,D}^+) - \log \det(B_{2,q,1,r,rel,D}^+) \right) \]
\[ + \log C_q^+(r) - \log C_{q-2}^+(r) + (l_{q-1}^- - l_q^+) \log r \]
2. \(\log \det(B_{2,q,1,r,+,1,D}) = (l_{q-1}^- - l_q^+) \log r \)
3. \(\sum_{q=0}^{m} (-1)^{q+1} q \left( \log \det(B_{2,q,1,r}, -\zeta_{0,D}) - \log \det(B_{2,q,1,r,rel,D}) \right) = \sum_{q=\text{even}} \log \det(B_{2,q,1,r,rel,D}^+) + 2 \sum_{q=\text{even}} \log C_q^+(r) \]
\[ + \left( \sum_{q=\text{even}} (2q + 1)l_q^- - \sum_{q=\text{odd}} (2q + 1)l_q^+ \right) \log r \]
4. \(\sum_{q=0}^{m} (-1)^{q+1} q \left( \log \det(B_{2,q,1,r}, -\zeta_{0,D}) - \log \det(B_{2,q,1,r,rel,D}) \right) = - \sum_{q=\text{odd}} \log \det(B_{2,q,1,r,rel,D}^+) - 2 \sum_{q=\text{odd}} \log C_q^+(r) \]
\[ + \left( \sum_{q=\text{even}} (2q + 1)l_q^+ - \sum_{q=\text{odd}} (2q + 1)l_q^- \right) \log r \]
\( (5) \sum_{q=0}^{m} (-1)^{q+1} q \left( \log \det B_{q,M_1,r}^2, \mathcal{P}_{-\mathcal{L}_0,D} - \log \det B_{q,M_1,r,\text{rel},D}^2 \right) = \frac{m}{2} \chi(Y, E|Y) \log r, \)

where \( \chi(Y, E|Y) := \sum_{q=0}^{m-1} (-1)^q \cdot l_q \) the Euler characteristic of \( Y \) with respect to \( H^*(Y, E|Y) \).

We finally discuss the Dirichlet-to-Neumann operator \( R_{q,\mathcal{D}}(r) \) defined by \( R_{q,\mathcal{D}}(r) = Q_{q,1,2}(r) + Q_{q,2} \), where \( \mathcal{D} \) is one of \( \mathcal{P}_{-\mathcal{L}}, \mathcal{P}_{+\mathcal{L}} \), the absolute or the relative boundary condition. The following lemma is straightforward.

**Lemma 2.8.** \( R_{q,\mathcal{P}_{-\mathcal{L}_0}}(r), R_{q,\mathcal{P}_{+\mathcal{L}_1}}(r) \) and \( R_{q,\text{rel}}(r) \) are described as follows.

\[
R_{q,\mathcal{P}_{-\mathcal{L}_0}}(r) = Q_{q,2} + \left( \begin{array}{cc} \sqrt{B^2_{Y,q}} & 0 \\ \frac{1}{\sqrt{B^2_{Y,q-1}} - e^{-\sqrt{B^2_{Y,q-1}}} \mathcal{P}_{-\mathcal{L}_0}} \end{array} \right) + \begin{cases} 2 \frac{\sqrt{B^2_{Y,q}} e^{-\sqrt{B^2_{Y,q-1}}} \mathcal{P}_{-\mathcal{L}_0}}{e^{\sqrt{B^2_{Y,q-1}} - e^{-\sqrt{B^2_{Y,q-1}}}} \mathcal{P}_{-\mathcal{L}_0}} & \text{on } \Im \mathcal{P}_{-\mathcal{L}_0} \cap (\ker B^2_{Y})^\perp \\
-2 \frac{\sqrt{B^2_{Y,q}} e^{-\sqrt{B^2_{Y,q-1}}} \mathcal{P}_{-\mathcal{L}_0}}{e^{\sqrt{B^2_{Y,q-1}} + e^{-\sqrt{B^2_{Y,q-1}}}} \mathcal{P}_{-\mathcal{L}_0}} & \text{on } \Im \mathcal{P}_{-\mathcal{L}_0} \cap (\ker B^2_{Y}) \end{cases}
\]

\[
R_{q,\mathcal{P}_{+\mathcal{L}_1}}(r) = Q_{q,2} + \left( \begin{array}{cc} \sqrt{B^2_{Y,q}} & 0 \\ \frac{1}{\sqrt{B^2_{Y,q-1}} + e^{-\sqrt{B^2_{Y,q-1}}} \mathcal{P}_{+\mathcal{L}_1}} \end{array} \right) + \begin{cases} 2 \frac{\sqrt{B^2_{Y,q}} e^{-\sqrt{B^2_{Y,q-1}}} \mathcal{P}_{+\mathcal{L}_1}}{e^{\sqrt{B^2_{Y,q-1}} - e^{-\sqrt{B^2_{Y,q-1}}}} \mathcal{P}_{+\mathcal{L}_1}} & \text{on } \Im \mathcal{P}_{+\mathcal{L}_1} \cap (\ker B^2_{Y})^\perp \\
-2 \frac{\sqrt{B^2_{Y,q}} e^{-\sqrt{B^2_{Y,q-1}}} \mathcal{P}_{+\mathcal{L}_1}}{e^{\sqrt{B^2_{Y,q-1}} + e^{-\sqrt{B^2_{Y,q-1}}}} \mathcal{P}_{+\mathcal{L}_1}} & \text{on } \Im \mathcal{P}_{+\mathcal{L}_1} \cap (\ker B^2_{Y}) \end{cases}
\]

\[
R_{q,\text{rel}}(r) = Q_{q,2} + \left( \begin{array}{cc} \sqrt{B^2_{Y,q}} & 0 \\ \frac{1}{\sqrt{B^2_{Y,q-1}} - e^{-\sqrt{B^2_{Y,q-1}}} \mathcal{P}_{\text{rel}}} \end{array} \right) + \begin{cases} 2 \frac{\sqrt{B^2_{Y,q}} e^{-\sqrt{B^2_{Y,q-1}}} \mathcal{P}_{\text{rel}}} {e^{\sqrt{B^2_{Y,q-1}} - e^{-\sqrt{B^2_{Y,q-1}}}} \mathcal{P}_{\text{rel}}} & \text{on } \Im \mathcal{P}_{\text{rel}} \cap (\ker B^2_{Y})^\perp \\
-2 \frac{\sqrt{B^2_{Y,q}} e^{-\sqrt{B^2_{Y,q-1}}} \mathcal{P}_{\text{rel}}} {e^{\sqrt{B^2_{Y,q-1}} + e^{-\sqrt{B^2_{Y,q-1}}}} \mathcal{P}_{\text{rel}}} & \text{on } \Im \mathcal{P}_{\text{rel}} \cap (\ker B^2_{Y}) \end{cases}
\]

We next discuss \( \lim_{r \to \infty} \left( \log \det R_{q,\mathcal{P}_{-\mathcal{L}_0}}(r) \right) - \log \det R_{q,\text{rel}}(r) \) when \( H^q(M; E) = \{0\} \) for each \( 0 \leq q \leq m \). The Poincaré duality and long exact sequence imply that \( H^q(M; E) = H^q(Y, E|Y) = 0 \) for each \( 0 \leq q \leq m \). Then Lemma 1.2 and Lemma 2.1 show that \( R_{q,\mathcal{P}_{-\mathcal{L}_0}}(r), R_{q,\mathcal{P}_{+\mathcal{L}_1}}(r) \) and \( R_{q,\text{rel}}(r) \) are invertible operators and

\[
\lim_{r \to \infty} R_{q,\mathcal{P}_{-\mathcal{L}_0}}(r) = \lim_{r \to \infty} R_{q,\text{rel}}(r) = Q_{q,2} + \left( \begin{array}{cc} \sqrt{B^2_{Y,q}} & 0 \\ \frac{1}{\sqrt{B^2_{Y,q-1}} - e^{-\sqrt{B^2_{Y,q-1}}} \mathcal{A}} \end{array} \right) = Q_{q,2} + |\mathcal{A}|.
\]

The kernel of \( Q_{q,2} + |\mathcal{A}| \) is described as follows. For \( f \in \Omega^q(M_2, E)|Y \), choose \( \psi \in \Omega^q(M_2, E) \) such that \( B^2 \psi = 0 \) and \( \psi|Y = f \). Then,
We assume that for each Corollary 2.10. Corollary 2.2 and Lemma 2.7 together with Lemma 2.9 lead to the following result.

It is a well known fact (Proposition 4.9 in [1]) that the space of \( L^\infty \) is injective and hence invertible, which leads to the following result.

\[
\text{Let } M_\infty := ((-\infty, 0] \times Y) \cup Y \cdot M_2. \text{ We can extend } E \text{ and } B \text{ canonically to } M_\infty, \text{ which we denote by } E_\infty \text{ and } B_\infty. \text{ Then } \psi \text{ in (2.7) can be extended to } M_\infty \text{ as an } L^2\text{-solution of } B_\infty. \text{ Hence,}
\]

\[
\ker(Q_{q,2} + |A|) = \{ \psi \in Y | \psi \text{ is an } L^2\text{-solution of } B_\infty \text{ in } \Omega^q(M_\infty, E_\infty) \}.
\]

It is a well known fact (Proposition 4.9 in [1]) that the space of \( L^2 \)-solutions of \( B_\infty \) is isomorphic to the image of \( H^\bullet(M, Y; E) \to H^\bullet(M; E) \), which is zero under our assumption. This shows that \( (Q_{q,2} + |A|) \) is injective and hence invertible, which leads to the following result.

**Lemma 2.9.** We assume that for each \( 0 \leq q \leq m, H^q(M; E) = H^q(M, Y; E) = \{ 0 \}. \text{ Then}
\]

\[
\lim_{r \to \infty} \log \det R_{q, p^- \cdot e_0 / p^+ \cdot e_1}(r) = \log \det \lim_{r \to \infty} R_{q, rel}(r) = \log \det (Q_{q,2} + |A|).
\]

Corollary 2.2 and Lemma 2.7 together with Lemma 2.9 lead to the following result.

**Corollary 2.10.** We assume that for each \( 0 \leq q \leq m, H^q(M; E) = H^q(M, Y; E) = \{ 0 \}. \text{ Then :}
\]

1. \[
\lim_{r \to \infty} \sum_{q=0}^{m} (-1)^{q+1} \cdot \left( \log \det B_{q, p^- \cdot e_0 / p^+ \cdot e_1}(r) - \log \det B_{q, rel}(r) \right) = \frac{1}{4} \sum_{q=0}^{m-1} \log \det B_{Y, q}.
\]

2. \[
\lim_{r \to \infty} \sum_{q=0}^{m} (-1)^{q+1} \cdot \left( \log \det B_{q, p_1}(r) - \log \det B_{q, rel}(r) \right) = -\frac{1}{4} \sum_{q=0}^{m-1} \log \det B_{Y, q}.
\]

3. \[
\lim_{r \to \infty} \sum_{q=0}^{m} (-1)^{q+1} \cdot \left( \log \det B_{q, p^- \cdot e_0}(r) - \log \det B_{q, rel}(r) \right) = \lim_{r \to \infty} \sum_{q=0}^{m} (-1)^{q+1} \cdot \left( \log \det B_{q, p_1}(r) - \log \det B_{q, rel}(r) \right) = 0.
\]

The following lemma is well known (cf. [4], [20]).
Lemma 2.11. Let \( M \) be a compact manifold with boundary \( Y \) and \( N \) be a collar neighborhood of \( Y \). We suppose that \( \{ g^M_t | -\delta_0 < t < \delta_0 \} \) is a family of metrics such that each \( g^M_t \) is a product metric and does not vary on \( N \). Let \( \mathcal{D} \) be one of the following boundary conditions: \( \mathcal{P}_0 \), \( \mathcal{P}_1 \), the absolute or the relative boundary condition. We denote by \( B^2_{q,\mathcal{D}}(t) \) the square of the odd signature operator acting on \( q \)-forms subject to \( \mathcal{D} \) with respect to the metric \( g^M_t \). If \( H^q(M;E) = H^q(M,Y;E) = \{0\} \) for each \( 0 \leq q \leq m \), then we have

\[
\frac{d}{dt} \left( \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \log \text{Det} B^2_{q,\mathcal{D}}(t) \right) = 0.
\]

We fix \( \delta_0 > 0 \) sufficiently small and choose a smooth function \( f(r,u) : [0,\infty) \times [0,1] \to [0,\infty) \), \( (r \geq 1) \) such that

\[
\text{supp}_{u} f(r,u) \subset [\delta_0, 1 - \delta_0], \quad \int_{0}^{1} f(r,u) du = r - 1, \quad \text{and} \quad f(1,u) \equiv 0.
\]

Setting \( F(r,u) = u + \int_{0}^{u} f(r,t) dt \), \( F_r := F(r,\cdot) : [0,1] \to [0,r] \) is a diffeomorphism satisfying

\[
F_r(u) = \begin{cases} 
 u & \text{for } 0 \leq u \leq \delta_0 \\
 u + r - 1 & \text{for } 1 - \delta_0 \leq u \leq 1.
\end{cases}
\]

Let \( g^M_t \) be a metric on \( M_r := ([0,r] \times Y) \cup \{r\} \times Y \) which is a product one on \([0,r] \times Y\). Then \( F_r^* g^M_t \) is a metric on \( M \), which is equal to \( \begin{pmatrix} F'(u)^2 & 0 \\ 0 & g_Y \end{pmatrix} \) on \([0,1] \times Y\). Hence, \( F_r^* g^M_t \) is a metric on \( M \) which is a product one near \( Y \). Furthermore, \((M,F_r^* g^M_t)\) and \((M_r, g^M_t)\) are isometric. Let \( \tilde{\mathcal{B}}(r) \) and \( \mathcal{B}(r) \) be the odd signature operators defined on \( M \) and \( M_r \) associated to the metrics \( F_r^* g^M_t \) and \( g^M_t \), respectively. We now assume that for each \( 0 \leq q \leq m \), \( H^q(M;E) = H^q(M,Y;E) = \{0\} \). Then \( \tilde{\mathcal{B}}^2_{q,\mathcal{D}}(r) \) and \( \mathcal{B}^2_{q,\mathcal{D}}(r) \) are invertible operators. Lemma 2.11 leads to the following equalities.

\[
\sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \left( \log \text{Det}_{2q} \tilde{\mathcal{B}}_{q,\mathcal{D}} - \log \text{Det}_{2q} \mathcal{B}_{q,\mathcal{D}} \right)
= \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \left( \log \text{Det}_{2q} \tilde{\mathcal{B}}(r)^2_{q,\mathcal{D}} - \log \text{Det}_{2q} \mathcal{B}(r)^2_{q,\mathcal{D}} \right)
= \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \left( \log \text{Det}_{2q} \tilde{\mathcal{B}}_{q,\mathcal{D}}(r) - \log \text{Det}_{2q} \mathcal{B}_{q,\mathcal{D}}(r) \right)
= \lim_{r \to \infty} \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \left( \log \text{Det}_{2q} \tilde{\mathcal{B}}_{q,\mathcal{D}}(r) - \log \text{Det}_{2q} \mathcal{B}_{q,\mathcal{D}}(r) \right)
= \frac{1}{4} \sum_{q=0}^{m-1} \log \text{Det}_{2q} \mathcal{B}^2_{Y,q}.
\]

Similarly, we have

\[
\sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \left( \log \text{Det}_{2q} \tilde{\mathcal{B}}_{q,\mathcal{D}} - \log \text{Det}_{2q} \mathcal{B}_{q,\mathcal{D}} \right)
= -\frac{1}{4} \sum_{q=0}^{m-1} \log \text{Det}_{2q} \mathcal{B}^2_{Y,q}.
\]

Corollary 2.2 Corollary 2.10 the Poincaré duality and the above equality lead to the following theorem, which is the main result of this section.
Theorem 2.12. Let \((M, g^M)\) be a compact Riemannian manifold with boundary \(Y\) and \(g^M\) be a product metric near \(Y\). We assume that for each \(0 \leq q \leq m\), \(H^q(M; E) = H^q(M, Y; E) = \{0\}\). Then:

\[
\begin{align*}
(1) \quad & \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \log \det_{2q} B^2_{q, \bar{\rho}_0} = \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \log \det_{2q} B^2_{q, \bar{\rho}_1} + \frac{1}{4} \sum_{q=0}^{m-1} \log \det_{2q} B^2_{Y,q} \\
(2) \quad & \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \log \det_{2q} B^2_{q, \bar{\rho}_1} = \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \log \det_{2q} B^2_{q, \bar{\rho}_1} - \frac{1}{4} \sum_{q=0}^{m-1} \log \det_{2q} B^2_{Y,q} \\
(3) \quad & \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \log \det_{2q} B^2_{q, \bar{\rho}_0} = \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \log \det_{2q} B^2_{q, \bar{\rho}_0} = \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \log \det_{2q} B^2_{q, \text{rel}}
\end{align*}
\]

3. Comparison of the eta invariants

In this section we are going to compare the eta-invariant \(\eta(B_{\text{even}}(\mathcal{P}_{-}, \mathcal{L}_0))\) with \(\eta(B_{\text{even}}(\Pi_{-}, \mathcal{L}_0))\), the eta-invariant of \(B_{\text{even}}\) subject to \(\mathcal{P}_{-}, \mathcal{L}_0\) and the generalized APS boundary condition \(\Pi_{-}, \mathcal{L}_0\), where \(\Pi_{-} : \Omega^{\text{even}}(M, E)|_Y \to \Omega^{\text{even}}(M, E)|_Y\) is the orthogonal projection onto the space spanned by the positive eigenspaces of \(A\) (cf. (1.8)). For this purpose we are going to follow the arguments in [6] strongly. Throughout this section we write the odd signature operator acting on \(\Omega^{\text{even}}(M, E)\) by \(B\) rather than \(B_{\text{even}}\) for simplicity. We begin with the descriptions of \(\text{Im} \Pi_{-}\) and \(\text{Im} \mathcal{P}_{-}\) as graphs of some unitary operators.

We denote by \((\Omega^{\text{even}}(M, E)|_Y)^*\) the orthogonal complement of \(
\begin{pmatrix}
\mathcal{H}^{\text{even}}(Y, E|_Y) \\
\mathcal{H}^{\text{odd}}(Y, E|_Y)
\end{pmatrix}
\) in \((\Omega^{\text{even}}(M, E)|_Y)\). Then the action of the unitary operator \(\gamma\) splits according to the following decomposition.

\[
\gamma : (\Omega^{\text{even}}(M, E)|_Y)^* \oplus \begin{pmatrix}
\mathcal{H}^{\text{even}}(Y, E|_Y) \\
\mathcal{H}^{\text{odd}}(Y, E|_Y)
\end{pmatrix} \to (\Omega^{\text{even}}(M, E)|_Y)^* \oplus \begin{pmatrix}
\mathcal{H}^{\text{even}}(Y, E|_Y) \\
\mathcal{H}^{\text{odd}}(Y, E|_Y)
\end{pmatrix}
\]

Since \(\gamma^2 = -\text{Id}\), we denote the \(\pm i\)-eigenspace of \(\gamma\) in \((\Omega^{\text{even}}(M, E)|_Y)^*\) by \((\Omega^{\text{even}}(M, E)|_Y)^*_\pm i\), which are

\[
(\Omega^{\text{even}}(M, E)|_Y)^*_\pm i = \frac{I \mp i\gamma}{2} (\Omega^{\text{even}}(M, E)|_Y)^*.
\] (3.1)

It is a well known fact that \(\text{Im} \Pi_{-}\) and \(\text{Im} \mathcal{P}_{-}\) are expressed by the graphs of some unitary operators from \((\Omega^{\text{even}}(M, E)|_Y)^*_\pm i\) to \((\Omega^{\text{even}}(M, E)|_Y)^*_{\pm i}\). When restricted to \((\Omega^{\text{even}}(M, E)|_Y)^*_{\pm i}\), \(B^2_{\gamma}\) is an invertible operator and we denote its inverse by \((B^2_{\gamma})^{-1}\). In view of (3.3) we define \(U_{\Pi_{-}}, U_{\mathcal{P}_{-}}\) as follows.

\[
U_{\Pi_{-}}, U_{\mathcal{P}_{-}} : (\Omega^{\text{even}}(M, E)|_Y)^*_{\pm i} \to (\Omega^{\text{even}}(M, E)|_Y)^*_{\pm i}
\] (3.2)
\[
U_{\Pi_>} = (B^2_Y - \frac{1}{2})(\nabla Y + \Gamma Y \nabla Y) \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}
\]
\[
U_{\mathcal{P}_-} = (B^2_Y - \frac{1}{2})(B^2_Y - (B^2_Y)^+) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\] (3.3)

where \((B^2_Y)^- := \nabla Y \Gamma Y \nabla Y : \Omega^\bullet(Y, E|Y) \to \Omega^\bullet(Y, E|Y)\) and \((B^2_Y)^+ := \nabla Y \Gamma Y \nabla Y : \Omega^\bullet(Y, E|Y) \to \Omega^\bullet(Y, E|Y)\). Then \(U_{\Pi_>}\) and \(U_{\mathcal{P}_-}\) are well defined \(\Psi\DO\)'s and their adjoints are given by

\[
U_{\Pi_>}^*, \quad U_{\mathcal{P}_-}^*(\Omega_{\text{even}}(M, E)|Y)_{Y, \delta}^* \to (\Omega_{\text{even}}(M, E)|Y)_{Y, \delta}^*
\] (3.4)

\[
U_{\Pi_>}^* = (B^2_Y - \frac{1}{2})(\nabla Y + \Gamma Y \nabla Y) \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}
\]
\[
U_{\mathcal{P}_-}^* = (B^2_Y - \frac{1}{2})(B^2_Y - (B^2_Y)^+) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\] (3.5)

The following lemma is straightforward.

**Lemma 3.1.** (1) Both \(U_{\Pi_>}\) and \(U_{\mathcal{P}_-}\) are unitary operators satisfying

\[
U_{\Pi_>}^* U_{\Pi_>} = U_{\mathcal{P}_-}^* U_{\mathcal{P}_-} = \text{Id}, \quad \gamma U_{\Pi_>} = -U_{\Pi_>} \gamma, \quad \gamma U_{\mathcal{P}_-} = -U_{\mathcal{P}_-} \gamma.
\]

(2) \(\text{Im} \Pi_\rightarrow (\text{Im} \Pi_-)\) and \(\text{Im} \mathcal{P}_- (\text{Im} \mathcal{P}_+)\) are graphs of \(U_{\Pi_>} (-U_{\Pi_>} \) and \(U_{\mathcal{P}_-} (-U_{\mathcal{P}_-} \), respectively, i.e.

\[
\text{Im} \Pi_\rightarrow = \{x + U_{\Pi_>} x \mid x \in (\Omega_{\text{even}}(M, E)|Y)_{Y, \delta}^* \}, \quad \text{Im} \Pi_- = \{x - U_{\Pi_>} x \mid x \in (\Omega_{\text{even}}(M, E)|Y)_{Y, \delta}^* \},
\]
\[
\text{Im} \mathcal{P}_- = \{x + U_{\mathcal{P}_-} x \mid x \in (\Omega_{\text{even}}(M, E)|Y)_{Y, \delta}^* \}, \quad \text{Im} \mathcal{P}_+ = \{x - U_{\mathcal{P}_-} x \mid x \in (\Omega_{\text{even}}(M, E)|Y)_{Y, \delta}^* \}.
\]

(3) \(U_{\Pi_>}\) anticommutes with \(U_{\mathcal{P}_-}\) in the following sense, i.e.

\[
U_{\Pi_>}^* U_{\mathcal{P}_-} = -U_{\mathcal{P}_-}^* U_{\Pi_>} , \quad U_{\Pi_>} U_{\mathcal{P}_-} = -U_{\mathcal{P}_-} U_{\Pi_>}.
\]

We define \(P(\theta) : (\Omega_{\text{even}}(M, E)|Y)_{Y, \delta}^* \to (\Omega_{\text{even}}(M, E)|Y)_{Y, \delta}^*\) by

\[
P(\theta) := U_{\Pi_>} \cos \theta + U_{\mathcal{P}_-} \sin \theta, \quad (0 \leq \theta \leq \frac{\pi}{2}).
\] (3.6)

Then \(P(\theta)\) is a unitary operator satisfying the property (1) in Lemma 3.1 and a smooth path connecting \(U_{\Pi_>}\) and \(U_{\mathcal{P}_-}\). We here note that the orthogonal projections \(\Pi_\rightarrow, \mathcal{P}_- : (\Omega_{\text{even}}(M, E)|Y)_{Y, \delta}^* \oplus (\Omega_{\text{even}}(M, E)|Y)_{Y, \delta}^* \rightarrow (\Omega_{\text{even}}(M, E)|Y)_{Y, \delta}^* \oplus (\Omega_{\text{even}}(M, E)|Y)_{Y, \delta}^*\) are expressed as follows.

\[
\Pi_\rightarrow = \frac{1}{2} \begin{pmatrix} \text{Id} & U_{\Pi_>}^* \\ U_{\Pi_>} & \text{Id} \end{pmatrix} \mathcal{P}_+, \quad \mathcal{P}_- = \frac{1}{2} \begin{pmatrix} \text{Id} & U_{\mathcal{P}_-}^* \\ U_{\mathcal{P}_-} & \text{Id} \end{pmatrix} \mathcal{P}_+.
\]
where $\mathcal{P}$ is the orthogonal projection onto $(\Omega^{\text{even}}(M,E)|_Y)^*$. Let $\mathcal{L}_0 = \left( \begin{array}{c} \mathcal{K} \\ \mathcal{K} \end{array} \right) \cap (\Omega^{\text{even}}(M,E)|_Y)$ and $\mathcal{L}_1 = \left( \begin{array}{c} \Gamma Y \mathcal{K} \\ \Gamma Y \mathcal{K} \end{array} \right) \cap (\Omega^{\text{even}}(M,E)|_Y)$ so that $\mathcal{L}_0 \oplus \mathcal{L}_1 = \left( \begin{array}{c} \mathcal{H}^{\text{even}}(Y,E|_Y) \\ \mathcal{H}^{\text{odd}}(Y,E|_Y) \end{array} \right)$. We denote by $\mathcal{P}_{\mathcal{L}_0}$ and $\mathcal{P}_{\mathcal{L}_1}$ the orthogonal projections onto $\mathcal{L}_0$ and $\mathcal{L}_1$. We define the orthogonal projections $\mathcal{P}_{-\mathcal{L}_0}$ and $\mathcal{P}_{\mathcal{L}_0}$ on $\Omega^{\text{even}}(M,E)|_Y$ as follows.

$$\mathcal{P}_{-\mathcal{L}_0} := \mathcal{P}_{-} + \mathcal{P}_{\mathcal{L}_0}, \quad \Pi_{\mathcal{L}_0} := \Pi_{\mathcal{L}_0}$$

(3.7)

We define $\mathcal{P}_{+\mathcal{L}_1}$ and $\Pi_{\mathcal{L}_1}$ in the same way. Similarly, we define the orthogonal projection $\overline{\mathcal{P}}(\theta)$ by

$$\overline{\mathcal{P}}(\theta) := \frac{1}{2} \left( \begin{array}{cc} \text{Id} & \mathcal{P}(\theta)^* \\ \mathcal{P}(\theta) & \text{Id} \end{array} \right) \mathcal{P} + \mathcal{P}_{\mathcal{L}_0} = \Pi_{\mathcal{L}_0} \cos \theta + \Pi_{\mathcal{L}_0} \sin \theta + \frac{1}{2}(1 - \cos \theta - \sin \theta) \mathcal{P} + \mathcal{P}_{\mathcal{L}_1}. \quad (3.8)$$

$\overline{\mathcal{P}}(\theta)$ satisfies the following properties.

**Lemma 3.2.** (1) $\gamma \overline{\mathcal{P}}(\theta) = (I - \overline{\mathcal{P}}(\theta)) \gamma$, and $\overline{\mathcal{P}}(\theta) \mathcal{B}_Y^2 = \mathcal{B}_Y^2 \overline{\mathcal{P}}(\theta)$.

(2) $\overline{\mathcal{P}}(\theta) \mathcal{A} \overline{\mathcal{P}}(\theta) = \cos \theta |\mathcal{A}| \overline{\mathcal{P}}(\theta) = \cos \theta \sqrt{(\mathcal{B}_Y^2)} \overline{\mathcal{P}}(\theta)$.

**Proof.** : The proofs are straightforward. For the second statement we may need the following identities.

$$\Pi_{\mathcal{L}_0} \Pi_{\mathcal{L}_0} = \frac{1}{2} \Pi_{\mathcal{L}_0}, \quad \mathcal{P}_{-\mathcal{L}_0} + \Pi_{\mathcal{L}_0} \mathcal{P}_- = \left( \mathcal{P}_- + \Pi_{\mathcal{L}_0} - \frac{1}{2} \text{Id} \right) \mathcal{P}_-.$$

$\square$

**Lemma 3.3.** Let $\mathcal{B}_{\overline{\mathcal{P}}(\theta)}$ be the realization of $\mathcal{B}$ with respect to $\overline{\mathcal{P}}(\theta)$, i.e.

$$\text{Dom} \left( \mathcal{B}_{\overline{\mathcal{P}}(\theta)} \right) = \{ \phi \in H^1(\Omega^{\text{even}}(M,E)) \mid \overline{\mathcal{P}}(\theta)(\phi|_Y) = 0 \}. \text{ Then } \mathcal{B}_{\overline{\mathcal{P}}(\theta)} \text{ is essentially self-adjoint.}$$

**Proof.** : It was shown in [24] (cf. [12]) that the adjoint $\left( \mathcal{B}_{\overline{\mathcal{P}}(\theta)} \right)^* \text{ is the realization of } \mathcal{B}^* = \mathcal{B} \text{ with respect to the boundary condition } (I - \overline{\mathcal{P}}(\theta)) \gamma^*, \text{ i.e.}$$

$$\text{Dom} \left( \mathcal{B}_{\overline{\mathcal{P}}(\theta)} \right)^* = \{ \phi \in H^1(\Omega^{\text{even}}(M,E)) \mid (I - \overline{\mathcal{P}}(\theta)) \gamma^*(\phi|_Y) = 0 \} = \text{Dom} \left( \mathcal{B}_{\overline{\mathcal{P}}(\theta)} \right).$$

Hence, it’s enough to show that $\mathcal{B}_{\overline{\mathcal{P}}(\theta)}$ is a symmetric operator. For $\phi, \psi \in \text{Dom} \left( \mathcal{B}_{\overline{\mathcal{P}}(\theta)} \right)$,

$$\langle \mathcal{B}\phi, \psi \rangle_M - \langle \phi, \mathcal{B}\psi \rangle_M = \langle \phi|_Y, \gamma (\psi|_Y) \rangle_Y$$

$$= \langle \overline{\mathcal{P}}(\theta)(I - \overline{\mathcal{P}}(\theta))\phi|_Y, \gamma (\psi|_Y) \rangle_Y = \langle \overline{\mathcal{P}}(\theta)(I - \overline{\mathcal{P}}(\theta))\phi|_Y, \overline{\mathcal{P}}(\theta)\gamma (\psi|_Y) \rangle_Y$$

$$= \langle \overline{\mathcal{P}}(\theta)(I - \overline{\mathcal{P}}(\theta))\phi|_Y, \gamma (\psi|_Y) \rangle_Y = 0,$$

which completes the proof of the lemma. $\square$
Setting
\[ U(\theta) = \begin{pmatrix} P(\theta)^* U_{\Pi_+} & 0 \\ 0 & Id \end{pmatrix} P_* + (Id - P_*) \]
\[ = \begin{pmatrix} \cos \theta + U_{\Pi_+}^* U_{\Pi_+} \sin \theta & 0 \\ 0 & Id \end{pmatrix} P_* + (Id - P_*) , \]
it is straightforward that
\[ U(\theta) \tilde{P}(0) U(\theta)^* = \tilde{P}(\theta). \] (3.9)

Moreover, setting
\[ T(\theta) = -i \theta \begin{pmatrix} U_{\Pi_+}^* U_{\Pi_+} & 0 \\ 0 & 0 \end{pmatrix} P_* , \] (3.10)
\[ T(\theta) \] is a self-adjoint operator and we have
\[ \exp\{iT(\theta)\} = U(\theta). \] (3.11)

Lemma 3.4. \( T(\theta) \) commutes with \( \gamma \) and \( B_2^Y \), i.e.,
\[ \gamma T(\theta) = T(\theta) \gamma, \quad B_2^Y T(\theta) = T(\theta) B_2^Y. \] (3.12)

Remark: Contrary to the case of [6], \( T(\theta) \) does not anticommute with \( A \).

Let \( \phi : [0, 1] \to [0, 1] \) be a decreasing smooth function such that \( \phi = 1 \) on a small neighborhood of 0 and \( \phi = 0 \) on a small neighborhood of 1. We use this cut-off function to extend \( T(\theta) \) defined on \( \Omega^{e\text{ven}}(M, E) \) to an operator defined on \( \Omega^{e\text{ven}}(M, E) \). We define \( \Psi_\theta : \Omega^{e\text{ven}}(M, E) \to \Omega^{e\text{ven}}(M, E) \) by
\[ \Psi_\theta(\omega)(x) = e^{i\phi(x)T(\theta)} \omega(x), \] (3.13)
where the support of \( \phi(x)T(\theta) \) is contained in \( N \), the collar neighborhood of \( Y \).

Lemma 3.5. \( \Psi_\theta \) is a unitary operator mapping from \( \text{Dom}(B_{\tilde{P}(0)}) \) onto \( \text{Dom}(B_{\tilde{P}(\theta)}) \).

Proof. : Clearly \( \Psi_\theta \) is a unitary operator. Let \( \tilde{P}(0)\omega(0) = 0 \). Then
\[ \tilde{P}(\theta)(\Psi_\theta \omega)(0) = U(\theta) \tilde{P}(0) U(\theta)^* e^{i\phi(x)T(\theta)} \omega \big|_{x=0} \]
\[ = U(\theta) \tilde{P}(0) e^{-iT(\theta)} e^{i\phi(x)T(\theta)} \omega \big|_{x=0} = U(\theta) \tilde{P}(0)\omega(0) = 0, \]
which completes the proof of the lemma. \( \square \)

We now consider the following diagram.
Setting $B(\theta) := \Psi_d^* B_{\tilde{P}(\theta)} \Psi_\theta$, 

$$B(\theta) : \text{Dom} \left( B_{\tilde{P}(0)} \right) \to \Omega^{\text{even}}(M, E)$$

is an elliptic $\Psi$DO of order 1 with a fixed domain $\text{Dom} \left( B_{\tilde{P}(0)} \right)$ and have the same spectrum as $B_{\tilde{P}(\theta)}$.

We next discuss one parameter family of eta functions $\eta_{B(\theta)}(s)$ defined by

$$\eta_{B(\theta)}(s) = \frac{1}{\Gamma\left(\frac{3s+1}{2}\right)} \int_0^\infty t^{\frac{s+1}{2}} \text{Tr} \left( B(\theta)e^{-tB(\theta)^2} \right) dt. \quad (3.14)$$

If $\eta_{B(\theta)}(s)$ has a regular value at $s = 0$, we define the eta invariant $\eta(B(\theta))$ by

$$\eta(B(\theta)) = \frac{1}{2} \left( \eta_{B(\theta)}(0) + \dim \ker B(\theta) \right). \quad (3.15)$$

For $0 \leq \theta_0 \leq \frac{\pi}{2}$, there exist $c(\theta_0) > 0$ and $\delta > 0$ such that $c(\theta_0) \notin \text{Spec} (B_\theta)$ for $\theta_0 - \delta < \theta < \theta_0 + \delta$. We denote by $Q(\theta)$ the orthogonal projection onto the space spanned by eigensections of $B(\theta)$ whose eigenvalues are less than $c(\theta)$ for $\theta_0 - \delta < \theta < \theta_0 + \delta$. We define

$$\eta_{B(\theta)}(s ; c(\theta)) = \sum_{|\lambda| < c(\theta)} \text{sign}(\lambda)|\lambda|^{-s} = \frac{1}{\Gamma\left(\frac{3s+1}{2}\right)} \int_0^\infty t^{\frac{s+1}{2}} \text{Tr} \left\{ (I - Q(\theta)) B(\theta)e^{-tB(\theta)^2} \right\} dt. \quad (3.16)$$

Then $\eta_{B(\theta)}(s) - \eta_{B(\theta)}(s ; c(\theta))$ is an entire function and

$$\left\{ \frac{1}{2} (\eta_{B(\theta)}(s) + \dim \ker B(\theta)) - \frac{1}{2} \eta_{B(\theta)}(s ; c(\theta)) \right\}_{s=0}$$

does not depend on $\theta$ for $\theta_0 - \delta < \theta < \theta_0 + \delta$ up to mod $\mathbb{Z}$. Simple computation shows that

$$\frac{\partial}{\partial \theta} \eta_{B(\theta)}(s ; c(\theta)) \quad (3.17)$$

$$= \frac{1}{\Gamma\left(\frac{3s+1}{2}\right)} \int_0^\infty t^{\frac{s+1}{2}} \text{Tr} \left( -\dot{Q}(\theta) B(\theta)e^{-tB(\theta)^2} + (I - Q(\theta)) \frac{\partial}{\partial \theta} \left( B(\theta)e^{-tB(\theta)^2} \right) \right) dt$$

$$= \frac{1}{\Gamma\left(\frac{3s+1}{2}\right)} \int_0^\infty t^{\frac{s+1}{2}} \text{Tr} \left( -\dot{Q}(\theta) B(\theta)e^{-tB(\theta)^2} \right) dt - \frac{s}{\Gamma\left(\frac{3s+1}{2}\right)} \int_0^\infty t^{\frac{s+1}{2}} \text{Tr} \left\{ (I - Q(\theta)) \left( \dot{B}(\theta)e^{-tB(\theta)^2} \right) \right\} dt,$$

where $\dot{Q}(\theta)$ and $\dot{B}(\theta)$ are derivatives of $Q(\theta)$ and $B(\theta)$ with respect to $\theta$. Furthermore, we have (cf. [14])

$$\text{Tr} \left( -\dot{Q}(\theta) B(\theta)e^{-tB(\theta)^2} \right) = 0, \quad \left\{ \frac{s}{\Gamma\left(\frac{3s+1}{2}\right)} \int_0^\infty t^{\frac{s+1}{2}} \text{Tr} \left( Q(\theta) \dot{B}(\theta)e^{-tB(\theta)^2} \right) dt \right\}_{s=0} = 0.$$

These equalities imply that

$$\frac{\partial}{\partial \theta} \eta_{B(\theta)}(s ; c(\theta)) = - \frac{s}{\Gamma\left(\frac{3s+1}{2}\right)} \int_0^\infty t^{\frac{s+1}{2}} \text{Tr} \left( \dot{B}(\theta)e^{-tB(\theta)^2} \right) dt + F(s), \quad (3.18)$$

where $F(s)$ is an analytic function at least for $\text{Re } s > -1$ with $F(0) = 0$. 


Then, we have

\[ \mathcal{B}(\theta) = \Psi^* \mathcal{B}_{\mathcal{P}(\theta)} \Psi_\theta = e^{-i \phi(x)T(\theta)} \gamma(\partial_x + \mathcal{A}) e^{i \phi(x)T(\theta)}. \]  

(3.19)

Using the fact that \( T(\theta)T'(\theta) = T'(\theta)T(\theta) \) and Lemma 3.4, we have

\[ \hat{\mathcal{B}}(\theta) = e^{-i \phi(x)T(\theta)} (i \phi'(x) \gamma T'(\theta) - i \phi(x) \gamma T'(\theta) \mathcal{A} + i \phi(x) \gamma \mathcal{A} T'(\theta)) e^{i \phi(x)T(\theta)}, \]

which leads to

\[ \text{Tr} \left( \hat{\mathcal{B}}(\theta) e^{-t \mathcal{B}(\theta)^2} \right) = \text{Tr} \left\{ (i \phi'(x) \gamma T'(\theta) - i \phi(x) \gamma [T'(\theta), \mathcal{A}] ) e^{-t \mathcal{B}(\theta)^2} \right\}. \]

(3.21)

Since the support of \( \phi \) is in \([0, 1]\), the support of \( \hat{\mathcal{B}}(\theta) \) is in \([0, 1] \times Y\). Let \( \mathcal{B}^{\gamma_1} \) be the odd signature operator defined by (1.24) on \([0, \infty) \times Y\). The heat kernel of \( \left( \mathcal{B}^{\gamma_1}_{\mathcal{P}(\theta)} \right)^2 \) was computed in [6] as follows.

\[ e^{-t \left( \mathcal{B}^{\gamma_1}_{\mathcal{P}(\theta)} \right)^2}(x, y) = \left( 4\pi t \right)^{-\frac{1}{2}} \left( e^{\frac{-|x-y|^2}{4t}} + (I - 2 \tilde{P}(\theta) e^{-\frac{|x-y|^2}{4t}} \right) e^{-t \mathcal{A}^2}

+ (\pi t)^{-\frac{1}{2}} \left( I - \tilde{P}(\theta) \right) \int_0^\infty e^{-\frac{(x+y)^2}{4t}} \tilde{A}(\theta)e^{\tilde{A}(\theta) z - t \mathcal{A}^2} dz \right), \]

(3.22)

where \( \tilde{A}(\theta) := (I - \tilde{P}(\theta)) \mathcal{A}(I - \tilde{P}(\theta)) \). The standard theory for heat kernel \( \mathcal{A}(I - \tilde{P}(\theta)) \) implies that the asymptotic expansions of \( \text{Tr} \left( \hat{\mathcal{B}}(\theta) e^{-t \mathcal{B}(\theta)^2} \right) \) is equal to that of \( \text{Tr} \left( \mathcal{B}^{\gamma_1}(\theta) e^{-t \mathcal{B}(\theta)^2} \right) \) up to \( (e^{-\frac{t}{\tau}}) \) for some \( c > 0 \). With a little abuse of notation we write \( \mathcal{B}^{\gamma_1} \) by \( \mathcal{B} \) again. Equation (3.24) leads to the following equality.

\[ \text{Tr} \left( i \phi'(x) \gamma T'(\theta) e^{-t \mathcal{B}^{\gamma_1}_{\mathcal{P}(\theta)}} \right) \]

(3.23)

\[ = \frac{i}{\sqrt{4 \pi t}} \int_0^\infty \phi'(x) dx \text{ Tr} \left( \gamma T'(\theta) e^{-t \mathcal{A}^2} \right) + \frac{i}{\sqrt{4 \pi t}} \int_0^\infty \phi'(x) e^{-\frac{x^2}{4t}} dx \text{ Tr} \left( \gamma T'(\theta)(I - 2 \tilde{P}(\theta)) e^{-t \mathcal{A}^2} \right)

+ \frac{i}{\sqrt{4 \pi t}} \int_0^\infty \int_0^\infty \phi'(x) e^{-\frac{(x+y)^2}{4t}} \text{ Tr} \left( \gamma T'(\theta)(I - \tilde{P}(\theta)) \tilde{A}(\theta) e^{\tilde{A}(\theta) z - t \mathcal{A}^2} \right) dz dx. \]

Lemma 3.2 and Lemma 3.4 imply that \( \text{Tr} \left( \gamma T'(\theta)(I - 2 \tilde{P}(\theta)) e^{-t \mathcal{A}^2} \right) = 0 \). Since \( \phi(x) = 1 \) near \( x = 0 \), the third integral decays exponentially as \( t \to 0^+ \). Hence,

\[ \text{Tr} \left( i \phi'(x) \gamma T'(\theta) e^{-t \mathcal{B}^{\gamma_1}_{\mathcal{P}(\theta)}} \right) = \frac{-i}{\sqrt{4 \pi t}} \text{ Tr} \left( \gamma T'(\theta) e^{-t \mathcal{A}^2} \right) + O(e^{-\frac{t}{\tau}}). \]

(3.24)

We refer to p.456 in [6] for the proof of the following equality.

\[ \tilde{A}(\theta) e^{\tilde{A}(\theta) z - t \mathcal{A}^2} = (-\cos \theta) |\mathcal{A}|(I - \tilde{P}(\theta)) e^{-\cos \theta} |\mathcal{A}| z - t \mathcal{A}^2. \]

(3.25)

Then, we have
\[
\text{Tr} \left( -i \phi(x) \gamma [T'(\theta), A] e^{-t\bar{P}(\theta)} \right)
\]
\[
= \frac{-i}{\sqrt{4\pi t}} \int_0^\infty \phi(x) dx \text{ Tr} \left( \gamma [T'(\theta), A] e^{-tA^2} \right)
\]
\[
+ \frac{-i}{\sqrt{4\pi t}} \int_0^\infty \phi(x) e^{-\frac{x^2}{4t}} dx \text{ Tr} \left( \gamma [T'(\theta), A] (I - 2\bar{P}(\theta)) e^{-tA^2} \right)
\]
\[
+ \frac{-i}{\sqrt{\pi t}} \int_0^\infty \int_0^\infty \phi(x) e^{-\frac{(x+z)^2}{4t}} \text{ Tr} \left\{ \gamma [T'(\theta), A] (I - \bar{P}(\theta)) (-\cos\theta) |A| (I - \bar{P}(\theta)) e^{-\cos\theta |A| z - tA^2} \right\} \, dz \, dx
\]
\[
=: (I) + (II) + (III).
\]
Lemma 3.4 shows that \( \text{Tr} \left( \gamma [T'(\theta), A] e^{-tA^2} \right) = 0 \), which yields
\[
(I) = 0, \quad (II) = \frac{i}{2} \text{ Tr} \left( \gamma [T'(\theta), A] \bar{P}(\theta) e^{-tA^2} \right) + O(e^{-\frac{t}{2}}).
\]
Change of variables, Lemma 5.2 and Lemma 3.4 show that
\[
(III) = \frac{-2i\cos\theta}{\sqrt{\pi}} \sqrt{t} \int_0^\infty \phi(\sqrt{t}x)e^{-(x+z)^2} \text{ Tr} \left[ \gamma [T'(\theta), A] \bar{P}(\theta) |A| e^{-2\sqrt{t} \cos\theta |A| z - tA^2} \right] \, dx \, dz.
\]
Since \( |A| \) commutes with \( A, T'(\theta) \) and \( \bar{P}(\theta) \), we denote
\[
d(\lambda) := \text{Tr}_{\ker(|A| - \lambda)} \left( \gamma [T'(\theta), A] \bar{P}(\theta) \right).
\]
\[
(III) = -i\cos\theta \sum_{0 \neq \lambda \in \text{Spec}(|A|)} d(\lambda) \int_0^\infty \int_0^\infty \phi(\sqrt{t}x) \frac{2}{\sqrt{\pi}} \sqrt{t} \lambda e^{-(x+z)^2} e^{-2\cos\theta \sqrt{t} \lambda z - t\lambda^2} \, dx \, dz
\]
\[
= -i\cos\theta \sum_{0 \neq \lambda \in \text{Spec}(|A|)} d(\lambda) \int_0^\infty \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-(x+z)^2} \, dx \sqrt{t} \lambda e^{-2\cos\theta \sqrt{t} \lambda z - t\lambda^2} \, dz + O(e^{-\frac{t}{2}})
\]
\[
= -i\cos\theta \sum_{0 \neq \lambda \in \text{Spec}(|A|)} d(\lambda) \int_0^\infty \text{erfc}(z) \sqrt{t} \lambda e^{-2\cos\theta \sqrt{t} \lambda z - t\lambda^2} \, dz + O(e^{-\frac{t}{2}}),
\]
where \( \text{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-y^2} \, dy \). To compute (III) more precisely we introduce the following concepts.

Let \( A, B \) be classical pseudodifferential operators and \( A \) be an elliptic operator of positive order on a compact manifold. Then \( \text{Tr} \left( B e^{-tA^2} \right) \) has an asymptotic expansion of the following type for \( t \to 0^+ \).
\[
\text{Tr} \left( B e^{-tA^2} \right) \sim \sum_{\Re \alpha \to \infty} a_{\alpha,k}(A,B) t^\alpha (\log t)^k.
\]
When \( B \) commutes with \( A^2 \) and vanishes on \( \ker A^2 \), we define the eta function \( \eta(A, B ; s) \) by
\[
\eta(A, B ; s) := \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{\frac{s-1}{2}} \text{ Tr} \left( B e^{-tA^2} \right) \, dt
\]
\[
= \sum_{|\lambda| \in \text{Spec}(|A|) - \{0\}} (\text{Tr}_{\ker(|A| - |\lambda|)} B) |\lambda|^{-s-1}.
\]
Then the noncommutative residue \( \text{res} \) is defined as follows ([30], [31], [15]).

\[
\text{res}(B) := -2 \ord(A) a_{0,1}(A, B) = \ord(A) \Res_{s=-1} \eta(A, B; s).
\]

(3.33)

The following is well known ([30], [6]).

**Lemma 3.6.** If \( B \) is a classical pseudodifferential operator on a compact manifold with \( B^2 = B \), then \( \text{res}(B) = 0 \).

We now go back to (3.30). We define a function \( F_\theta(x) \) and its Mellin transform \( \mathcal{M}F_\theta(s) \) (see [6] for details) by

\[
F_\theta(x) = x \int_0^\infty \text{erfc}(z) e^{-2\cos\theta x z - x^2} dz, \quad \mathcal{M}F_\theta(s) = \int_0^\infty x^{s-1} F_\theta(x) dx.
\]

(3.34)

Using the inverse Mellin transform, we have

\[
\begin{align*}
(III) & = -\cos\theta \sum_{0 \neq \lambda \in \text{Spec}(|A|)} d(\lambda) F_\theta(\sqrt{t}\lambda) + O(e^{-\frac{\pi}{t}}) \\
& = -\cos\theta \sum_{0 \neq \lambda \in \text{Spec}(|A|)} d(\lambda) \frac{1}{2\pi i} \int_{\Re w = c > 0} (\sqrt{t}\lambda)^{-w} \mathcal{M}F_\theta(w) \, dw + O(e^{-\frac{\pi}{t}}) \\
& = -\cos\theta \frac{d}{d\lambda} \sum_{0 \neq \lambda \in \text{Spec}(|A|)} \lambda^{-w} \mathcal{M}F_\theta(w) \, dw + O(e^{-\frac{\pi}{t}}) \\
& = -\cos\theta \cdot \text{Res}_{w = c} \left(t^{-\frac{\pi}{t}} \eta \left(A, \gamma[T'(|\theta|, \mathcal{A} P(\theta); w - 1) \mathcal{M}F_\theta(w)\right) + O(e^{-\frac{\pi}{t}}).\right.
\end{align*}
\]

Equations (3.21), (3.24) and (3.35) show that

\[
\begin{align*}
\text{Tr} \left(-i\phi(x)\gamma[T'(|\theta|, \mathcal{A})e^{-t\mathcal{B}^2(\theta)}\right) & = \frac{i}{2} \text{Tr} \left(\gamma[T'(|\theta|, \mathcal{A})\mathcal{P}(\theta)e^{-t\mathcal{A}^2}\right) \\
- i\cos\theta \cdot \text{Res}_{w = c} \left(t^{-\frac{\pi}{t}} \eta \left(A, \gamma[T'(|\theta|, \mathcal{A})\mathcal{P}(\theta); w - 1\right) \mathcal{M}F_\theta(w)\right) + O(e^{-\frac{\pi}{t}}).
\end{align*}
\]

Equations (3.21), (3.24) and (3.36) lead to the following lemma.

**Lemma 3.7.**

\[
\begin{align*}
\text{Tr} \left(\mathcal{B}(\theta)e^{-t\mathcal{B}^2(\theta)}\right) & = \frac{-i}{\sqrt{4\pi t}} \text{Tr} \left(\gamma[T'(|\theta|)e^{-t\mathcal{A}^2}\right) + \frac{i}{2} \text{Tr} \left(\gamma[T'(|\theta|, \mathcal{A})\mathcal{P}(\theta)e^{-t\mathcal{A}^2}\right) \\
- i\cos\theta \cdot \text{Res}_{w = c} \left(t^{-\frac{\pi}{t}} \eta \left(A, \gamma[T'(|\theta|, \mathcal{A})\mathcal{P}(\theta); w - 1\right) \mathcal{M}F_\theta(w)\right) + O(e^{-\frac{\pi}{t}}).
\end{align*}
\]

It is known that (3.14) has at most a simple pole at \( s = 0 \) (Theorem 3.4 in [6]) and has regular values at \( s = 0 \) for \( \theta = 0 \) and \( \frac{\pi}{2} \) (for the case of \( \theta = \frac{\pi}{2} \), see [14]). Moreover, \( \mathcal{M}F_\theta(w) \) has only simple poles at negative integers (Lemma 3.3 in [6]). The following lemma is due to [13] (cf. [6]).
Lemma 3.8. Let $A$ and $B$ be classical pseudodifferential operators of order $a$ and $b$, respectively, on a compact manifold $M$ with $\dim M = m$. If $A$ is a self-adjoint elliptic operator of positive order, then for $t \to 0^+$,
\[
\text{Tr} \left( B e^{-t A^2} \right) \sim \sum_{j=0}^\infty a_j(A, B) t^{-\frac{m-1}{2}} + \sum_{j=0}^\infty (b_j(A, B) \log t + c_j(A, B)) t^j.
\]

The equation (3.18) with Lemma 3.7 and Lemma 3.8 (cf. Theorem 3.4 and 3.5 in [6]) implies that

\[
\frac{d}{d\theta} \text{Res}_{s=0} \eta_{\mathcal{B}(\theta)}(s) = \text{Res}_{s=0} \left( \frac{d}{d\theta} \eta_{\mathcal{B}(\theta)}(s) \right) = \frac{4}{\sqrt{\pi}} a_{-\frac{1}{2}, 1}(\mathcal{B}(\theta), \tilde{\mathcal{B}}(\theta)) = \frac{1}{\pi} \text{res} (i \gamma T'(\theta)),
\]

where $a_{-\frac{1}{2}, 1}(\mathcal{B}(\theta), \tilde{\mathcal{B}}(\theta))$ is the coefficient of $t^{-\frac{1}{2}} \log t$ in the asymptotic expansion of $\text{Tr} \left( \tilde{\mathcal{B}}(\theta) e^{-t \mathcal{B}(\theta)^2} \right)$ for $t \to 0^+$.

Lemma 3.9.

\[
\text{Tr} \left( i \gamma T'(\theta) e^{-t A^2} \right) = 0.
\]

Hence, $\text{res} (i \gamma T'(\theta)) = 0$ and $\eta_{\mathcal{B}(\theta)}(s)$ has a regular value at $s = 0$ for each $0 \leq \theta \leq \frac{\pi}{2}$.

Proof. We recall that $T'(\theta) = -i \left( U^*_\mathcal{P} \mathcal{U}_{\Pi, >} 0 0 \mathcal{P}_+ \right)$. Using (3) in Lemma 3.1, we have

\[
\text{Tr} \left( i \gamma T'(\theta) e^{-t A^2} \right) = \text{Tr} \left( \gamma \left( U^*_\mathcal{P} \mathcal{U}_{\Pi, >} 0 0 \mathcal{P}_+ e^{-t A^2} \right) \right) = \text{Tr} \left( \gamma \left( 0 0 U^*_\mathcal{P} 0 \right) \left( 0 0 \mathcal{P}_+ e^{-t A^2} \right) \right) = \text{Tr} \left( \gamma \left( 0 0 0 0 \mathcal{P}_+ e^{-t A^2} \right) \right).
\]

Since $\Gamma^Y$ anticommutes with $\left( (B^2_Y)^- - (B^2_Y)^+ \right)$, we have, by (1.8) and (3.20),

\[
\text{Tr} \left( i \gamma T'(\theta) e^{-t A^2} \right) = \frac{1}{2} \text{Tr} \left( \gamma \left( U^*_\mathcal{P} \mathcal{U}_{\Pi, >} 0 0 \mathcal{P}_+ e^{-t A^2} \right) \right) = \frac{1}{2} \text{Tr} \left( i \beta \left( (B^2_Y)^- - (B^2_Y)^+ \right) (B^2_Y)^{-\frac{1}{2}} (\nabla^Y \Gamma^Y + \Gamma^Y \nabla^Y) \left( 0 -1 0 \right) \mathcal{P}_+ e^{-t A^2} \right) = 0,
\]

which completes the proof of the lemma.

Since $\mathcal{M} \Phi(\vartheta)$ has a regular value at $\vartheta = 1$, Lemma 3.7 and (3.18) imply that

\[
\frac{d}{d\theta} \eta_{\mathcal{B}(\theta)}(0, c(\theta)) = -\frac{2}{\sqrt{\pi}} a_{-\frac{1}{2}, 0}(\mathcal{B}(\theta), \tilde{\mathcal{B}}(\theta))
\]

\[
= -\frac{1}{\sqrt{\pi}} a_{-\frac{1}{2}, 0} \left( \mathcal{A}, i \gamma [T'(\theta), \mathcal{A}] \tilde{\mathcal{P}}(\theta) \right) + \frac{2}{\sqrt{\pi}} \cos \theta \cdot \mathcal{M} \Phi(1) \text{Res}_{w=1} \left( \eta \left( \mathcal{A}, i \gamma [T'(\theta), \mathcal{A}] \tilde{\mathcal{P}}(\theta) ; w - 1 \right) \right).
\]

We note that
Lemma 3.10. The following lemma is straightforward.

Lemma 3.11. Equations (3.38), (3.39), (3.40) and Lemma 3.10 lead to the following result.

\[ a_{\pm 0} \left( A, i\gamma |T'(\theta), A| \bar{P}(\theta) \right) = \frac{\sqrt{\pi}}{2} \text{Res}_{s=0} \left( \eta \left( A, i\gamma |T'(\theta), A| \bar{P}(\theta) : s \right) \right) \]
\[ = \frac{\sqrt{\pi}}{2} \text{Res}_{s=0} \left( \eta \left( A, i\gamma |T'(\theta), (\text{sign}A)| \bar{P}(\theta) : s - 1 \right) \right) \]
\[ = \frac{\sqrt{\pi}}{2} \text{Res}_{s=0} \left( i\gamma |T'(\theta), (\text{sign}A)| \bar{P}(\theta) \right). \]  

Similarly,

\[ \text{Res}_{w=1} \left( \eta(A, i\gamma |T'(\theta), A| \bar{P}(\theta) : w - 1 \right) = \text{Res}_{w=1} \left( \eta \left( A, i\gamma |T'(\theta), (\text{sign}A)| \bar{P}(\theta) : w - 2 \right) \right) \]
\[ = \text{Res}_{w=1} \left( \eta \left( A, i\gamma |T'(\theta), (\text{sign}A)| \bar{P}(\theta) : w \right) \right) \]
\[ = \text{res} \left( i\gamma |T'(\theta), (\text{sign}A)| \bar{P}(\theta) \right). \]  

The following lemma is straightforward.

Lemma 3.10.

\[ T'(\theta) \bar{P}(\theta) = \left( I - \bar{P}(\theta) \right) T'(\theta) - \frac{i}{2} \left( \begin{array}{cc} 0 & W^* \\ -W & 0 \end{array} \right) \mathcal{P}_*, \]

where \( W := -U_{1} \sin \theta + U_{\gamma} \cos \theta. \)

Equations (3.38), (3.39), (3.40) and Lemma 3.10 lead to the following result.

Lemma 3.11.

\[ \frac{d}{d\theta} \eta_{B(\theta)}(0 : e(\theta)) = \left( -\frac{1}{2} + \frac{2 \cos \theta}{\sqrt{\pi}} \cdot \mathcal{M} \mathcal{F}_0 (1) \right) \text{res} \left( i\gamma |T'(\theta), (\text{sign}A)| \bar{P}(\theta) \right) = 0. \]

Proof. We note that

\[ \text{res} \left( i\gamma |T'(\theta), (\text{sign}A)| \bar{P}(\theta) \right) = \text{res} \left( i\gamma T'(\theta)(\text{sign}A) \bar{P}(\theta) \right) - \text{res} \left( i\gamma (\text{sign}A)T'(\theta) \bar{P}(\theta) \right). \]

We are going to show that \( \text{res} \left( i\gamma T'(\theta)(\text{sign}A) \bar{P}(\theta) \right) = 0 \) and \( \text{res} \left( i\gamma (\text{sign}A)T'(\theta) \bar{P}(\theta) \right) = 0 \) can be shown in the same way. Since \( \text{res} \) is a trace,

\[ \text{res} \left( i\gamma T'(\theta)(\text{sign}A) \bar{P}(\theta) \right) = \text{res} \left( i \bar{P}(\theta) \gamma T'(\theta)(\text{sign}A) \bar{P}(\theta) \right) \]
\[ = \text{res} \left( i \gamma (I - \bar{P}(\theta)) T'(\theta)(\text{sign}A) \bar{P}(\theta) \right) \]
\[ = \text{res} \left( i \gamma \left( T'(\theta) \bar{P}(\theta) + \frac{i}{2} \left( \begin{array}{cc} 0 & W^* \\ -W & 0 \end{array} \right) \mathcal{P}_* \right)(\text{sign}A) \bar{P}(\theta) \right) \]
\[ = \text{res} \left( i\gamma T'(\theta) \bar{P}(\theta)(\text{sign}A) \bar{P}(\theta) \right) - \frac{1}{2} \text{res} \left( \gamma \left( \begin{array}{cc} 0 & W^* \\ -W & 0 \end{array} \right) \mathcal{P}_*(\text{sign}A) \bar{P}(\theta) \right) \]
\[ = \cos \theta \text{res} \left( i\gamma T'(\theta) \bar{P}(\theta) \right) - \frac{1}{2} \text{res} \left( \gamma \left( \begin{array}{cc} 0 & W^* \\ -W & 0 \end{array} \right) \mathcal{P}_*(\text{sign}A) \bar{P}(\theta) \right). \]
We note that

\[ \text{res} \left( i\gamma T'(\theta) \tilde{P}(\theta) \right) = \text{res} \left( i\gamma T'(\theta)(I - \tilde{P}(\theta)) \right) = \text{res} \left( i\gamma T'(\theta) \right) - \text{res} \left( i\gamma T'(\theta) \tilde{P}(\theta) \right), \]

which together with Lemma 3.9 shows that

\[ \text{res} \left( i\gamma T'(\theta) \tilde{P}(\theta) \right) = \frac{1}{2} \text{res} \left( i\gamma T'(\theta) \right) = 0. \tag{3.41} \]

We note that \((\text{sign} A) = \begin{pmatrix} 0 & U_{\Pi_>^*} \\ U_{\Pi_>} & 0 \end{pmatrix} \) and \(\gamma\) anticommutes with \((\text{sign} A)\) and \(\begin{pmatrix} 0 & W^* \\ -W^* & 0 \end{pmatrix}\).

Hence, we have

\[ \text{res} \left( \gamma \left( \begin{array}{cc} 0 & \ W^* \\ -W & 0 \end{array} \right) \right) P_*(\text{sign} A) \tilde{P}(\theta) = \text{res} \left( \gamma \left( \begin{array}{cc} 0 & \ W^* \\ -W & 0 \end{array} \right) \right) P_*(\text{sign} A)(I - \tilde{P}(\theta)), \]

which shows that

\[ \text{res} \left( \gamma \left( \begin{array}{cc} 0 & \ W^* \\ -W & 0 \end{array} \right) \right) P_*(\text{sign} A) \tilde{P}(\theta) = \frac{1}{2} \text{res} \left( \gamma \left( \begin{array}{cc} 0 & \ W^* \\ -W & 0 \end{array} \right) \right) P_* \left( \begin{array}{cc} 0 & \ W^* \\ -W & 0 \end{array} \right) \right) P_*(\text{sign} A)(I - \tilde{P}(\theta)), \tag{3.42} \]

The above equality with Lemma 3.1 and Lemma 3.6 shows that

\[ \text{res} \left( \gamma \left( \begin{array}{cc} 0 & \ W^* \\ -W & 0 \end{array} \right) \right) P_*(\text{sign} A) \tilde{P}(\theta) = \frac{i}{4} \text{res} \left( \begin{array}{cc} 0 & \ W^* U_{\Pi_>^*} + U_{\Pi_>^*} W \\ 0 \end{array} \right) \right) P_* \left( \begin{array}{cc} 0 & \ W^* U_{\Pi_>^*} + U_{\Pi_>^*} W^* \\ 0 \end{array} \right) \right) P_*(\text{sign} A)(I - \tilde{P}(\theta)), \]

which completes the proof of the lemma. \(\square\)

For one parameter family of essentially self-adjoint Dirac operators \(\mathcal{B}_{\tilde{P}(\theta)} (0 \leq \theta \leq \frac{\pi}{2})\) we define the spectral flow \(\text{SF}(\mathcal{B}_{\tilde{P}(\theta)})_{\theta \in [0, \frac{\pi}{2}]}\) by

\[ \text{SF}(\mathcal{B}_{\tilde{P}(\theta)})_{\theta \in [0, \frac{\pi}{2}]} := m^+ - m^-, \]

where \(m^+ (m^-)\) is the number of eigenvalues which start negative (non-negative) and end non-negative (negative). The following formula is well known (cf. Lemma 3.4 in [17]).

\[ \eta(\mathcal{B}_{\Pi_>^*}) - \eta(\mathcal{B}_{\Pi_>}) = \text{SF}(\mathcal{B}_{\tilde{P}(\theta)})_{\theta \in [0, \frac{\pi}{2}]} + \int_0^{\frac{\pi}{2}} \frac{d}{d\theta} \eta_\theta(\mathcal{B}_{\tilde{P}(\theta)}(0)) \ d\theta. \tag{3.44} \]

Lemma 3.11 and the result of Nicolaescu (Theorem 7.5 in [17], [23]) show that
show that to product metric near \( Y \)

\[
\eta(B_{\Pi^+ \cal L_0}) - \eta(B_{\Pi^- \cal L_0}) = \text{SF}(B_{\bar{\cal P}(\theta)})_{\theta \in [0, \frac{\pi}{2}]} = \text{Mas}(\bar{\cal P}(\theta), \cal C_M)_{\theta \in [0, \frac{\pi}{2}]},
\]

(3.45)

where \( \cal C_M \) is the Calderón projector for \( \cal B \) on \( M \) and \( \text{Mas}(\bar{\cal P}(\theta), \cal C_M)_{\theta \in [0, \frac{\pi}{2}]} \) is the Maslov index for the path \( \bar{\cal P}(\theta) \) and the constant path \( \cal C_M \). We refer to [17] and [23] for the definitions of the Maslov index and Calderón projector.

The unitary operators corresponding to the projection \( \cal P_+ \) is \( -U_{\cal P_-} \), which shows that for \( 0 \leq \theta \leq \frac{\pi}{2} \)

\[
\bar{\cal P}(-\theta) = \frac{1}{2} \begin{pmatrix} \text{Id} & P(-\theta)^* \\ P(-\theta) & \text{Id} \end{pmatrix} \cal P_+ + \cal P_{\cal L_0}
\]

is a smooth path connecting \( \Pi_{\Pi^+ \cal L_0} \) and \( \cal P_+ \cal L_0 \). Similar computation shows that

\[
\eta(B_{\Pi^+ \cal L_0}) - \eta(B_{\Pi^- \cal L_0}) = \text{SF}(B_{\bar{\cal P}(\theta)})_{\theta \in [0, \frac{\pi}{2}]} = \text{Mas}(\bar{\cal P}(\theta), \cal C_M)_{\theta \in [0, \frac{\pi}{2}]},
\]

(3.46)

Summarizing the above arguments we have the following theorem, which is the main result of this section.

**Theorem 3.12.** Let \((M, g^M)\) be a compact Riemannian manifold with boundary \( Y \) and \( g^M \) be a product metric near \( Y \). Then:

1. \( \eta(B_{\cal P^+ \cal L_0}) - \eta(B_{\cal P^- \cal L_0}) = \text{SF}(B_{\bar{\cal P}(\theta)})_{\theta \in [0, \frac{\pi}{2}]} = \text{Mas}(\bar{\cal P}(\theta), \cal C_M)_{\theta \in [0, \frac{\pi}{2}]} \).
2. \( \eta(B_{\cal P_+ \cal L_0}) - \eta(B_{\cal P_- \cal L_0}) = \text{SF}(B_{\bar{\cal P}(\theta)})_{\theta \in [0, \frac{\pi}{2}]} = \text{Mas}(\bar{\cal P}(\theta), \cal C_M)_{\theta \in [0, \frac{\pi}{2}]} \).

4. GLUING FORMULA OF THE REFINED ANALYTIC TORSION

The gluing formula of the analytic torsion with respect to the relative and absolute boundary conditions ([9], [21], [29]) and the gluing formula of the eta invariant with respect to the APS boundary condition ([6], [7], [17], [32], [33]) are well known. In this section we are going to use Theorem 2.12 and Theorem 3.12 together with results in [9], [6] and [17] to obtain the gluing formula of the refined analytic torsion when \( \nabla \) is an ayclic Hermitian connection.

Let \((\widehat{M}, g^{\widehat{M}})\) be a closed Riemannian manifold of dimension \( m = 2r - 1 \) and \( \hat{\cal E} \to \widehat{M} \) be a flat vector bundle with a flat connection \( \nabla \). We denote by \( Y \) a hypersurface of \( \widehat{M} \) such that \( \widehat{M} - Y \) has two components whose closures are denoted by \( M_1 \) and \( M_2 \), i.e. \( \widehat{M} = M_1 \cup_Y M_2 \). We assume that \( g^{\widehat{M}} \) is a product metric near \( Y \) and that \( \nabla \) is a Hermitian connection. Let \( \partial u \) be the unit normal vector field on a collar neighborhood of \( Y \) such that \( \partial u \) is outward on \( M_1 \) and inward on \( M_2 \). We denote by \( \cal B^{\widehat{M}} \) the odd signature operator on \( \widehat{M} \) and denote by \( \cal B^{M_1, M_2} (E_1, E_2, g^{M_1}, g^{M_2}) \) the restriction of \( \cal B^{\widehat{M}} (\hat{\cal E}, g^{\widehat{M}}) \) to \( M_1, M_2 \). We impose the boundary condition \( \cal P_+ \cal L_1 \) on \( M_1 \) and \( \cal P_- \cal L_0 \) on \( M_2 \). Then [12] and [12] show that

\[
\log \text{Det}_{gr, \theta}(B^{M_1}_{even, \cal P_+ \cal L_1}) + \log \text{Det}_{gr, \theta}(B^{M_2}_{even, \cal P_- \cal L_0})
\]

(4.1)

\[
= \frac{1}{2} \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \left( \log \text{Det}_{2q}(B^{M_1}_{q, \cal P_1})^2 + \log \text{Det}_{2q}(B^{M_2}_{q, \cal P_0})^2 \right) - i\pi \left( \eta(B^{M_1}_{even, \cal P_+ \cal L_1}) + \eta(B^{M_2}_{even, \cal P_- \cal L_0}) \right)
\]
Theorem 2.12 together with Theorem 4.3 in [9] (p.36 in [9], cf. [21], [29]) leads to the following result.

**Lemma 4.1.** We assume that for each $0 \leq q \leq m$, $i = 1, 2$, $H^q (\tilde{M}, \tilde{E}) = H^q (M_i, Y; E_i) = H^q (M_i; E_i) = 0$. Then,

$$
\frac{1}{2} \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \left( \log \text{Det}_{2\theta} (B_{q,\tilde{P}_i}^{M_1})^2 + \log \text{Det}_{2\theta} (B_{q,\tilde{P}_i}^{M_2})^2 \right)
$$

$$
= \frac{1}{2} \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \left( \log \text{Det}_{2\theta} (B_{q,\text{abs}}^{M_1})^2 + \log \text{Det}_{2\theta} (B_{q,\text{rel}}^{M_2})^2 \right) = \frac{1}{2} \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \log \text{Det}_{2\theta} (B_{q}^{\tilde{M}})^2.
$$

Under the assumption in Lemma 4.1 Theorem 3.12 shows that

$$
\eta (B_{P_+}^{M_1}) - \eta (B_{P_+}^{M_2}) = \text{Mas} (\tilde{P} (\theta), C_{M_2})_{\theta \in [0, \pi / 2]}.
$$

(4.2)

We next consider $\eta (B_{P_+}^{M_1}) - \eta (B_{P_+}^{M_1})$. Since $\partial u$ is the outward normal derivative on a collar neighborhood of $Y$ on $M_1$, to use Theorem 3.12 we rewrite the odd signature operator $B^{M_1}$ near the boundary by

$$
B^{M_1} = \gamma (\partial u + A) = -\gamma (-\partial u - A).
$$

Here $-\partial u$ is the inward normal derivative to $Y$ on $M_1$. Since $\Pi_{\leq} (A) = \Pi_{\geq} (-A)$ and $I - \tilde{P} (\theta)$ is a path connecting $\Pi_{\leq} (A)$ and $P_+$, Theorem 3.12 shows that

$$
\eta (B_{P_+}^{M_1}) - \eta (B_{P_+}^{M_1}) = \eta (B_{P_+}^{M_1}) - \eta (B_{P_+}^{M_1} - \eta (B_{P_+}^{M_1} - \Pi_{\geq} (-A)) = \text{Mas} (I - \tilde{P} (\theta), C_{M_1})_{\theta \in [0, \pi / 2]}.
$$

(4.3)

Equations (4.2) and (4.3) together with Theorem 8.8 in [17] show that

$$
\eta (B_{P_+}^{M_1}) + \eta (B_{P_+}^{M_1}) = \eta (B_{P_+}^{M_1}) + \eta (B_{P_+}^{M_1} + \text{Mas} (\tilde{P} (\theta), C_{M_2})_{\theta \in [0, \pi / 2]} + \text{Mas} (I - \tilde{P} (\theta), C_{M_1})_{\theta \in [0, \pi / 2]})
$$

$$
= \eta (B^{\tilde{M}}) + \text{Mas} (\tilde{P} (\theta), C_{M_2})_{\theta \in [0, \pi / 2]} + \text{Mas} (I - \tilde{P} (\theta), C_{M_1})_{\theta \in [0, \pi / 2]}.
$$

(4.4)

**Lemma 4.2.** Under the assumption of Lemma 4.1 we have :

$$
\text{Mas} (\tilde{P} (\theta), C_{M_2})_{\theta \in [0, \pi / 2]} = \text{Mas} (I - \tilde{P} (\theta), C_{M_1})_{\theta \in [0, \pi / 2]}.
$$

In particular, $\eta (B_{P_+}^{M_1}) + \eta (B_{P_+}^{M_2}) \equiv \eta (B^{\tilde{M}}) \pmod{2 \mathbb{Z}}$.

**Proof.** We put $M_{1,r} = M_1 \cup_Y ([0, r] \times Y)$, $M_{2,r} = M_2 \cup_Y ([0, r] \times Y)$ and $M_{1,\infty} = M_1 \cup_Y ([0, \infty) \times Y)$, $M_{2,\infty} = M_2 \cup_Y ([0, \infty) \times Y)$. We denote the extensions of $\mathcal{B}$ to $M_{i,r}$, $M_{i,\infty}$ by $\mathcal{B}_{M_{i,r}}$, $\mathcal{B}_{M_{i,\infty}}$ and denote the corresponding Calderón projectors by $\mathcal{C}_{M_{i,r}}$, and $\text{Im} \mathcal{C}_{M_{i,r}} := L_{M_{i,r}}$, and $\lim_{r \to \infty} L_{M_{i,r}} := L_{M_{i,\infty}}$ for $i = 1, 2$. We also denote the orthogonal projection to $L_{M_{i,\infty}}$ by $\mathcal{C}_{M_{i,\infty}}$. Under the assumption of Lemma 4.1 it is shown in [17] (p.610 in [17]) that $L_{M_{1,\infty}}$ and $L_{M_{2,\infty}}$ are Lagrangian subspaces and $L_{M_{2,\infty}} = \gamma L_{M_{1,\infty}}$. Hence $\mathcal{C}_{M_{2,\infty}} = -\gamma \mathcal{C}_{M_{1,\infty}}$. We define a homotopy $(F(\theta, s), G(\theta, s))$ on $M_2$ as follows.

$$
F(\theta, s) = \tilde{P} (\theta), \quad G(\theta, s) = C_{M_{2,r}}, \quad (0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq s \leq \infty).
$$
Then, \((F(\theta, 0), G(\theta, 0)) = (\tilde{P}(\theta), C_{M_2})\) and \((F(\theta, \infty), G(\theta, \infty)) = (\tilde{P}(\theta), C_{M_2, \infty})\). Since \(\ker B_{\Pi_+}\) and \(\ker B_{P_-}\) are topological invariants (cf. Lemma 12 and Proposition 4.9 in [1]), the assumption implies that

\[
\dim (\ker F(0, s) \cap \text{Im} G(0, s)) = \dim (\ker \Pi_+ (A) \cap \text{Im} C_{M_2, s}) = 0,
\]

\[
\dim \left(\ker F\left(\frac{\pi}{2}, s\right) \cap \text{Im} G\left(\frac{\pi}{2}, s\right)\right) = \dim (\ker P_- \cap \text{Im} C_{M_2, s}) = 0,
\]

which shows (cf. p.587 in [17]) that

\[
\text{Mas}(\tilde{P}(\theta), C_{M_2})_{\theta \in [0, \frac{\pi}{2}]} = \text{Mas}(\tilde{P}(\theta), C_{M_2, \infty})_{\theta \in [0, \frac{\pi}{2}]}.
\]

Similarly, we have

\[
\text{Mas}(I - \tilde{P}(\theta), C_{M_1})_{\theta \in [0, \frac{\pi}{2}]} = \text{Mas}(I - \tilde{P}(\theta), C_{M_1, \infty})_{\theta \in [0, \frac{\pi}{2}]} = \text{Mas}(-\gamma \tilde{P}(\theta) \gamma, -\gamma C_{M_2, \infty} \gamma)_{\theta \in [0, \frac{\pi}{2}]}.
\]

Hence, we have (cf. p.586 in [17])

\[
\text{Mas}(\tilde{P}(\theta), C_{M_2})_{\theta \in [0, \frac{\pi}{2}]} = \text{Mas}(\tilde{P}(\theta), C_{M_2, \infty})_{\theta \in [0, \frac{\pi}{2}]} = \text{Mas}(-\gamma \tilde{P}(\theta) \gamma, -\gamma C_{M_2, \infty} \gamma)_{\theta \in [0, \frac{\pi}{2}]},
\]

which completes the proof of the lemma. □

Under the assumption of Lemma 14, the refined analytic torsion \(T_{\tilde{M}}(g^{\tilde{M}}, \nabla)\) on \(\tilde{M}\) is defined by (Definition 10.1 in [4])

\[
\log T_{\tilde{M}}(g^{\tilde{M}}, \nabla) = \log \text{Det}_{\text{gr}, \theta}(B^{\tilde{M}}_{\text{even}}) + \pi i \left(\text{rank}(\tilde{E})\right) \eta(B^{\tilde{M}}_{\text{even}, \text{trivial}}).
\]

The refined analytic torsion \(T_{M_1, P_+}(g^{M_1}, \nabla)\) and \(T_{M_2, P_-}(g^{M_2}, \nabla)\) on \(M_1, M_2\) with respect to the boundary conditions \(P_+\) and \(P_-\) are defined similarly (Definition 4.9 in [14]). Lemma 11 and Lemma 12 lead to the following theorem, which is the main result of this paper.

**Theorem 4.3.** Let \((\tilde{M}, g^{\tilde{M}})\) be a closed Riemannian manifold of dimension \(m = 2r - 1\) and \(Y\) be a hypersurface so that \(\tilde{M} = M_1 \cup_Y M_2\). We assume that \(g^{\tilde{M}}\) is a product metric near \(Y\) and for each \(0 \leq q \leq m, i = 1, 2, H^q(\tilde{M}, \tilde{E}) = H^q(M_i, Y; E_i) = H^q(M_i; E_i) = 0\). Then,

\[
\log T_{\tilde{M}}(g^{\tilde{M}}, \nabla) = \frac{1}{2} \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \log \text{Det}_{2q}(B^{M_{2q}}_{\text{even}})^2 - i\pi \eta(B^{\tilde{M}}_{\text{even}}) + i\pi (\text{rank}(\tilde{E})) \eta(B^{\tilde{M}}_{\text{even}, \text{trivial}})
\]

\[
= \frac{1}{2} \sum_{q=0}^{m} (-1)^{q+1} \cdot q \cdot \left(\log \text{Det}_{2q}(B_{M_1}^{q})^2 + \log \text{Det}_{2q}(B_{M_2}^{q})^2\right)
\]

\[
- i\pi \left(\eta(B_{\text{even}, P_+}^{M_1}) + \eta(B_{\text{even}, P_-}^{M_2})\right) + i\pi (\text{rank}(\tilde{E})) \left(\eta(B_{\text{even}, P_+}^{M_1, \text{trivial}}) + \eta(B_{\text{even}, P_-}^{M_2, \text{trivial}})\right)
\]

\[
= \log T_{M_1, P_+}(g^{M_1}, \nabla) + \log T_{M_2, P_-}(g^{M_2}, \nabla) \pmod{2\pi i \mathbb{Z}}.
\]

Equivalently, we have
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\[ T_M^r (g^\nabla, \nabla) = T_{M_1, P_+} (g^{M_1}, \nabla) \cdot T_{M_2, P_-} (g^{M_2}, \nabla). \]

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Institute of Mathematics, Academia Sinica, 6th floor, Astronomy-Mathematics Building, No. 1, Section 4, Roosevelt Road, Taipei, 106-17, Taiwan

E-mail address: rthuang@math.sinica.edu.tw

Department of Mathematics, Inha University, Incheon, 402-751, Korea

E-mail address: yoonweon@inha.ac.kr