Domination game on forests

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Abstract

In the domination game studied here, Dominator and Staller alternately choose a vertex of a graph $G$ and take it into a set $D$. The number of vertices dominated by the set $D$ must increase in each single turn and the game ends when $D$ becomes a dominating set of $G$. Dominator aims to minimize whilst Staller aims to maximize the number of turns (or equivalently, the size of the dominating set $D$ obtained at the end). Assuming that Dominator starts and both players play optimally, the number of turns is called the game domination number $\gamma_g(G)$ of $G$.

Kinnersley, West and Zamani verified that $\gamma_g(G) \leq 7n/11$ holds for every isolate-free $n$-vertex forest $G$ and they conjectured that the sharp upper bound is only $3n/5$. Here, we prove the $3/5$-conjecture for forests in which no two leaves are at distance 4 apart. Further, we establish an upper bound $\gamma_g(G) \leq 5n/8$, which is valid for every isolate-free forest $G$.

Keywords: domination game, game domination number, $3/5$-conjecture.

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1 Introduction

1.1 Domination game

The domination game considered here was introduced in 2010 by Brešar, Klavžar and Rall [3], where the original idea is attributed to Henning (2003, personal communication). For this domination game, a graph $G$ is given and two players, called Dominator and Staller, take turns choosing a vertex and taking it into a set $D$. Each vertex chosen dominates itself and its neighbors. The rule of the game prescribes that the set of vertices dominated by $D$ must be enlarged in each single turn. The game ends when no more legal moves can be made; that is, when $D$ becomes a dominating set of $G$. The goal of Dominator is to minimize, while that of Staller is to maximize the length of the game. Equivalently, Dominator wants a small dominating set $D$ and Staller wants $D$ to be as large as possible. The game domination number $\gamma_g(G)$ of $G$ is the number of turns in the game (equals the cardinality of the dominating set $D$ obtained at the end) when Dominator starts the game and each of the two players applies an optimal strategy. Analogously, the Staller-start game domination number $\gamma'_g(G)$ is the number of turns when Staller begins and the players play optimally.

1.2 Standard definitions

For a vertex $v \in V$ of a graph $G = (V, E)$, its open neighborhood is defined as $N(v) = \{u : uv \in E\}$, whilst its closed neighborhood is $N[v] = N(v) \cup \{v\}$. Then the degree $d(v)$ (or $d_G(v)$) of $v$ is just $|N(v)|$. Each vertex dominates itself and its neighbors, moreover a set $S \subseteq V$ dominates all vertices contained in $N[S] = \bigcup_{v \in S} N[v]$. A vertex set $D \subseteq V$ is called dominating set if $D$ dominates all vertices of $G$. The smallest cardinality of a dominating set $D$ is the domination number $\gamma(G)$ of $G$. One can prove that $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$ and $\gamma(G) \leq \gamma'_g(G) \leq 2\gamma(G)$ hold.

In a tree, as usual, a leaf is a vertex of degree 1, while a vertex having a leaf-neighbor is called stem.

1.3 Results on the domination game

The earlier papers discuss several aspects of the domination game, for example, connections between $\gamma_g(G)$ and $\gamma'_g(G)$ [3, 7, 8], the game domination number of Cartesian products [3] moreover the difference between $\gamma_g(G)$ and $\gamma_g(H)$ when $H$ is a spanning subgraph of $G$ [4]. The recent manuscript [1] discusses the possible changes of the game domination number when a vertex or an edge is deleted from the graph.

From our point of view, the following “3/5-conjecture” and the related results are the most important ones.
Conjecture 1 (Kinnersley, West and Zamani, [7]) If $G$ is an isolate-free for-
est of order $n$, then

$$
\gamma_g(G) \leq \frac{3n}{5} \quad \text{and} \quad \gamma'_g(G) \leq \frac{3n + 2}{5}.
$$

Conjecture 1 is proved to be true for graphs each of whose components is a

caterpillar\footnote{A caterpillar is a tree whose non-leaf vertices induce a path.} [7]. Additionally, the authors of the recent paper [2] identify all trees
attaining this bound up to 20 vertices by computer search moreover construct in-
finity many trees satisfying $\gamma_g = 3n/5$. As it follows, the bound $3n/5$ (if true) is sharp.

One of our contributions is the proof of Conjecture 1 for the class of forests in
which no two leaves are connected by a path of length 4. For this class of forests
our upper bound $(3n + 1)/5$ on $\gamma'_g$ is slightly better than the bound conjectured in
[7] for forests in general.

Theorem 1 If $G$ is an isolate-free forest of order $n$ in which no two leaves have
distance 4, then

$$
\gamma_g(G) \leq \frac{3n}{5} \quad \text{and} \quad \gamma'_g(G) \leq \frac{3n + 1}{5}
$$

hold.

Our proof, presented in Section 3, is based on a value-assignment to the vertices,
where the value of a vertex $v$ depends on the current status of $v$ in the game. Then,
we describe a greedy-like strategy for Dominator which ensures that the game ends
within $3n/5$ turns. We introduced this approach in the conference paper [5], where
also Theorem 1 was stated without a completely detailed proof. Now, the strategy
described there is fine-tuned and the proof is extended by a a more detailed analysis
to obtain a further result. This new general upper bound $5n/8$ concerns all isolate-
free forests and improves the earlier bound $\gamma_g(G) \leq 7n/11$, which was recently
proved by Kinnersley, West and Zamani [7].

Theorem 2 If $G$ is an isolate-free forest of order $n$, then

$$
\gamma_g(G) \leq \frac{5n}{8} \quad \text{and} \quad \gamma'_g(G) \leq \frac{5n + 2}{8}.
$$

The paper is organized as follows. In Section 2, the basic value-assignment is
introduced and some general lemmas are obtained. Then, in Section 3, we describe
the strategy and analyze the structure of the residual graph at some crucial points.
In the last subsection of this part, we verify Theorems 1 and 2 based on the previous
lemmas. In Section 4, we make some concluding notes.
2 Preliminaries

At any moment of the game we have three different types of the vertices. We assign them to different colors and to different numbers of points. The letter $D$ always denotes the set of vertices selected by the players up to the considered moment of the game. A vertex $v$ is dominated if $v \in N[D]$, otherwise $v$ is called undominated.

- A vertex is white and its value is 3 points if it is undominated.
- A vertex is blue and its value is 2 points if it is dominated but has at least one undominated neighbor.
- A vertex is red and its value is 0 point if it and all of its neighbors are dominated.

Clearly, selecting a red vertex would not enlarge the set of dominated vertices, hence this choice is not legal in the game. Also, selecting any vertex, the status of a red vertex will not change. Hence, red vertices can be ignored in the continuation of the game. On the other hand, blue vertices can be chosen later by any players as they have white neighbors, but edges connecting two blue vertices can be deleted. Therefore, at any moment of the game, graph $G$ will be meant without red vertices moreover without edges joining two blue vertices. This graph $G$ will be called residual graph as it was introduced already in [7]. Due to our definition, in a residual graph each blue vertex has only white neighbors and definitely has at least one. As relates white vertices, none of their neighbors and none of the edges incident with at least one white vertex were deleted. This implies the following statements.

**Lemma 1**

(i) If $v$ is a white vertex in a residual graph $G$, then $v$ has the same neighborhood in $G$ as it had at the beginning of the game. Particularly, if $v$ is a white leaf in $G$, then it was white leaf in each of the earlier residual graphs.

(ii) If $G$ contains no isolated vertices at the beginning of the game, this property remains valid for each residual graph throughout the game.

When a vertex $v$ is played, it becomes red, each white vertex from $N(v)$ becomes either blue or red and additionally, each blue leaf contained in $N[N(v)]$ turns red. Further, if the game is played on a tree, these are the only possible changes in colors.

The value $p(G)$ of a residual graph $G$ is defined to be the sum of the values associated with its vertices. When a player selects a vertex, $p(G)$ necessarily decreases. We say that the player gets (or seizes) $q$ points in a turn if his move causes decrease $q$ in the value of $G$.

Observe that in each turn the player either selects a white vertex which turns red (this means 3 points by itself, even without additional gain); or selects a blue vertex $v$ which turns red (2 points) moreover $v$ must have at least one white neighbor which becomes blue or red (at least 1 additional point). Hence, we have
**Lemma 2** In each turn, the value of $G$ decreases by at least 3 points.

As a preparation for proving our main theorems, we introduce some further notations and terminology.

- In general, at any moment of the game, $G$ denotes the current residual graph. However, if preciseness requires, we also use the notation $G_i$ for the residual graph obtained after the $i$th turn of the game, moreover the graph given at the beginning is referred to as $G_0$. Similarly, the number of white, blue and red vertices after the $i$th turn are denoted by $w_i$, $b_i$ and $r_i$, respectively, and we set $w_0 = n$, $b_0 = r_0 = 0$. Thus, $p(G_i) = 3w_i + 2b_i$ and the number of points the player got in the $i$th turn is just the difference $p(G_{i-1}) - p(G_i)$. Note that in the Dominator-start version, the $i$th turn belongs to Dominator, if $i$ is odd; otherwise it is Staller’s turn.

- The subgraph of $G$ (or that of $G_i$) induced by the set of its white vertices is $G(W)$ (or $G_i(W)$, respectively).

- As relates colors, we use the abbreviations W, B and R. Hence, an R-vertex is a red vertex, a W-neighbor is a neighbor which is white and a B-leaf is a leaf of $G$ which is blue. Similarly, the notation $v : B \rightarrow R$ means that in the turn considered the color of $v$ changed from blue to red. Also, for a path subgraph of $G$, its type is denoted by the order of colors, for example BWB means a path on three vertices with the color-order indicated.

- A critical $P_5$ is a path on five vertices whose both ends are W-leaves and which is of type WWBWW. The unique blue vertex in a critical $P_5$ is called critical center.

At the end of this section we prove a further useful lemma.

**Lemma 3** If the $i$th turn belongs to Dominator and the residual graph $G_{i-1}$ contains a B-leaf in a component of order at least 3, then Dominator can seize at least 7 points in the $i$th turn.

**Proof** Assume that $G_{i-1}$ contains the B-leaf $v$ and let $u$ be its unique neighbor, which is definitely white. If $u$ has at least two W-neighbors, then Dominator gets at least $7 = 2 + 3 + 1 + 1$ points by playing $u$. If $u$ has exactly one W-neighbor, say $u'$, then choosing $u'$, all the vertices $v$, $u$ and $u'$ become red, hence Dominator can seize at least $8 = 2 + 3 + 3$ points. If $u$ has no W-neighbor but the component consists of at least 3 vertices, then $u$ has a B-neighbor $v'$ which is different from $v$. In this case, if Dominator chooses $v'$, the value of $G_{i-1}$ decreases by at least $7 = 2 + 3 + 2$ points.  

□
3 Proof of the theorems

Here we prove our main results, Theorems 1 and 2. The two proofs are not separated, as they apply the same strategy for Dominator, they proceed by the same structural analysis and use the same lemmas. The special condition in Theorem 1, namely the absence of leaves at distance four apart, will be used only in the final part of the proof.

First, we consider the Dominator-start game on an isolate-free \( n \)-vertex forest \( G \), and describe a strategy for Dominator which ensures the game to end within a limited number of turns whatever strategy is applied by Staller. In our presentation, the game is divided into four phases, some of which might be missing. For each Phase \( i \) (for \( i = 1, 2, 3, 4 \)) we give a strategy prescribed for Dominator. Then, Phase \( i \) itself will be defined due to the applicability of the given strategies.

- **Strategy-Phase(1)** In his turn, Dominator gets at least 7 points, moreover at least two vertices become red in this turn.
- **Strategy-Phase(2)** In his turn, Dominator gets at least 7 points.
- **Strategy-Phase(3)** In his turn, Dominator gets at least 6 points. In Phase 3 we have two additional rules Dominator must apply:
  - \((R.3.1)\) Dominator plays a vertex which results in the possible maximum gain achievable in that turn.
  - \((R.3.2)\) Under the rule \((R.3.1)\) Dominator prefers to play a W-stem having a W-leaf neighbor.
- **Strategy-Phase(4)** In his turn, Dominator gets at least 3 points.

Phase \( i \) may start only with the first turn of Dominator when there is no applicable Strategy-Phase\((j)\) for any integers \( 1 \leq j \leq i - 1 \). But it really starts only if Strategy-Phase\((i)\) can be applied in this turn, otherwise this phase is skipped. If Phase \( i \) was not skipped, then it ends just before the first turn of Dominator when Strategy-Phase\((i)\) is not applicable. Phase 1 is skipped only if all components of \( G \) are of order 2. Let us emphasize that we never go back to an earlier phase (no matter whether it was ended or skipped). For example, at a point of Phase 3, the changes in the structure of the residual graph might cause that Dominator can get 7 points, but then the game is continued in Phase 3. We remark that, by Lemma 2, Dominator always is able to get at least 3 points if the game is not over yet.

In general, we observe that each non-skipped phase begins with a turn of Dominator and ends with a turn of Staller with the only exception when Dominator ends the game and hence the current phase as well.

To prove the theorems, we will keep track of the decrease in \( p(G) \) from phase to phase, moreover analyze the structural properties of the residual graph at some points of the game.
3.1 Phase 1

Due to Strategy-Phase(1) and Lemma 2, Dominator seizes at least 7 points and Staller gets at least 3 points in each of their Phase-1-turns. The extra points seized above these limits are counted separately and will be put to use in Phase 3 when critical $P_5$-subgraphs are treated. Formally, for every $i \geq 1$ if the $i$th turn belongs to Phase 1, we define

$$e_i = \begin{cases} 
(p(G_{i-1}) - p(G_i) - 7 & \text{if } i \text{ is odd} \\
(p(G_{i-1}) - p(G_i) - 3 & \text{if } i \text{ is even}
\end{cases}$$

Moreover, let $e^* = \sum_{i=1}^{k} e_i$, where $k$ is the number of turns belonging to Phase 1.

As Dominator begins the phase, we have

**Lemma 4** If Phase 1 consists of $k$ turns ($k \geq 0$), then the value of $G$ decreases by $5k + e^*$ in this phase, where $e^* \geq 0$.

Further, we estimate the number of critical centers.

**Lemma 5** Let Phase 1 consist of $k$ turns and let $r_k$ denote the number of red vertices at the end of Phase 1. Then, the number of vertices which are critical centers in at least one later residual graph $G_i$ ($i \geq k$) is at most $(r_k/3) + e^*$.

**Proof** Consider the color-changes in the $i$th turn of Phase 1 and denote the number of vertices with changes W→R, B→R, and W→B by $x_1$, $x_2$ and $x_3$ respectively. Then, the change in the number of blue vertices is $b_i - b_{i-1} = x_3 - x_2$, and $p(G_{i-1}) - p(G_i) = 3x_1 + 2x_2 + x_3$.

First, assume that this is Dominator’s turn. Then, Strategy-Phase(1) ensures that $r_i - r_{i-1} = x_1 + x_2 \geq 2$ and hence

$$e_i + 1 = 3x_1 + 2x_2 + x_3 - 6 = 3(x_1 + x_2) - 6 + x_3 - x_2 \geq b_i - b_{i-1}.$$

In the other case, when Staller moves, the vertex selected definitely becomes red and thus, $r_i - r_{i-1} = x_1 + x_2 \geq 1$ holds. This implies

$$e_i = 3x_1 + 2x_2 + x_3 - 3 = 3(x_1 + x_2) - 3 + x_3 - x_2 \geq b_i - b_{i-1}.$$

Consequently, for any two consecutive moves in the phase

$$e_i + e_{i+1} + 1 \geq b_{i+1} - b_{i-1} \quad \text{and} \quad r_{i+1} - r_{i-1} \geq 3$$

hold.

Note that if the game is finished in Phase 1, the lemma clearly holds. Otherwise $k$ is even, and for the number of blue and red vertices

$$\frac{k}{2} + e^* \geq b_k \quad \text{and} \quad r_k \geq \frac{3k}{2} \quad (1)$$

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are valid, which yield
\[ \frac{r_k}{3} + e^* \geq b_k. \] (2)

What remains to prove is that each vertex which occurs as a critical center in any later residual graph is already blue in \( G_k \). Consider a critical \( P_5 \) subgraph \( v_1v_2v_3v_4v_5 \) of a \( G_i \) \( (i \geq k) \). As \( v_1 \) is a W-leaf in \( G_i \), Lemma 1 implies that it is also a W-leaf in \( G_k \) at the end of Phase 1. Clearly, vertex \( v_2 \) is white and \( v_3 \) is not red in \( G_k \). Moreover, if \( v_3 \) was white in \( G_k \), then Strategy-Phase(1) could be applied in the \((k + 1)\)st turn, as Dominator could select \( v_2 \), which would cause the color-changes \( v_1, v_2 : W \rightarrow R \) and \( v_3 : W \rightarrow B \). This cannot be the case as the \( k \)th turn finishes Phase 1. Therefore, each later critical center \( v_3 \) must be blue in \( G_k \), and the lemma follows.

\[ \square \]

3.2 Phase 2

Our first statement is a direct consequence of the definition of Phase 2, of Lemma 2 and of the fact that Dominator starts the phase.

**Lemma 6** If Phase 2 consists of \( k \) turns \( (k \geq 0) \), then the value of \( G \) is decreased by at least 5\( k \) in this phase.

Our main observation concerning this phase is that the structure of the residual graph is quite restricted at the end of Phase 2. In a residual graph \( G \), \( v \) is a **single white vertex** (single-W) if it has only blue neighbors, that is, \( N_{G(W)}(v) = \emptyset \); and a **white pair** (W-pair) consists of two W-vertices \( u \) and \( v \) for which \( N_{G(W)}(u) = \{v\} \) and \( N_{G(W)}(v) = \{u\} \) hold.

**Lemma 7** At the end Phase 2 the residual graph \( G \) has the following properties:

(i) For each white vertex \( v \),

- either \( v \) is a single white vertex,
- or \( v \) is in a white pair.

(ii) If a leaf is contained in a component of order at least 3, then it is white.

(iii) If a blue vertex \( v \) belongs to a component of order at least 3 and \( v \) has a single white neighbor, then \( v \) has exactly one further neighbor, which is necessarily from a white pair.

(iv) Each blue vertex is of degree at most 4.

Moreover, the above statements (i) – (iv) are valid for every residual graph \( G \) from which we have no possibility of choosing a vertex and attaining a gain of at least 7 points.
Proof If the game is finished in Phase 2, then the residual graph in question contains no vertex, and there is nothing to prove. Otherwise, $G$ is a residual graph in which Strategy-Phase(2) cannot be applied; that is, no choice of Dominator can cause a decrease of at least 7 points in the value of $G$.

(i) Assume that $G(W)$ has a component of order at least 3. Then, let $v$ be one of the leaves of this component, and let $u$ be the only neighbor of $v$ in $G(W)$. As the component contains at least one further vertex, there exists a vertex $z$ for which $z \in N_{G(W)}(u)$ and $z \neq v$. Therefore, if Dominator plays vertex $u$, then $u$ and $v$ turn red and additionally $z$ becomes either blue or red. Consequently, Dominator could seize at least $7 = 3 + 3 + 1$ points. This cannot be the case, hence each component of $G(W)$ contains either one or two vertices corresponding to the single-W vertices and to the W-pairs in $G$.

(ii) Due to Lemma 3, otherwise (when the leaf is blue) Dominator could get at least 7 points.

(iii) Assume that $v$ is a B-vertex and has a single-W neighbor $u$. Then, choosing $v$, vertex $u$ becomes dominated and has no undominated neighbor. This already gives $2 + 3 = 5$ points gain for Dominator. Due to (ii), $v$ is not a leaf in $G$ and hence has a further W-neighbor $z$. If $z$ was a single-W vertex, the choice of $v$ would result in at least $5 + 3 = 8$ points, which contradicts our condition. Thus, $z$ is from a W-pair. Moreover, if $v$ had three different W-neighbors, namely $u$, $z$ and $z'$, then Dominator could seize at least $7 = 5 + 1 + 1$ points by choosing $v$. As this is not the case, $v$ has exactly two neighbors $u$ and $z$ and the statement follows.

(iv) If a blue vertex $v$ had five different neighbors, then all of them would be white and the choice of $v$ would give a gain of at least $7 = 2 + 5 \cdot 1$ points.

Finally, we observe that the same arguments are valid for any moment of the game, when Dominator has no possibility to get more than 6 points. □

We remark that properties (i) – (iii) were already satisfied by the residual graph at the end of Phase 1. This could be verified analogously to the above proof, but we do not do so, as we will not use this fact in the present paper.

3.3 Structural lemmas for later phases

Here, we prove some properties which remain valid throughout Phases 3 and 4 (even if during Phase 3 Dominator has the possibility of seizing 7 or more points).

Lemma 8 Throughout Phases 3 and 4, each residual graph $G$ has the following properties:
(i) Each white vertex is either single white or it is in a white pair.

(ii) If a blue vertex $v$ has a single white neighbor $u$, and $u$ has no blue-leaf neighbor, then $v$ has exactly one further neighbor $z$, which is either in a white pair or it is a single white vertex having a blue-leaf neighbor.

(iii) Each blue vertex is of degree at most 4.

Proof

(i) As new white vertices do not arise during the game, this follows from Lemma 7(i).

(ii) If $u, u'$ formed a white pair at the end of Phase 2, then either it remains a white pair, or both $u$ and $u'$ turn red and are deleted, or one of them remains white and the other one becomes a B-leaf. By our conditions, $u$ is a single-W vertex without blue leaf in $G$, hence $u$ also was single-W at the end of Phase 2. Then, our statement follows from Lemma 7(iii).

(iii) Consider a B-vertex $v$ of $G$. If $v$ was already blue at the end of Phase 2, then by Lemma 7(iv), it had at most four W-neighbors and hence, in any later residual graph $G$ its degree is at most 4. In the other case, when $v$ was a W-vertex at the end of Phase 2, it was either single-W, but then it could not be blue in $G$; or $v$ was in the W-pair $uv$ and now it is a B-leaf with the only neighbor $u$. □

Applying Lemma 8, we prove a further lemma, which says that if $G$ has no component of type BWB, then the maximum achievable gain cannot be exactly 7 points.

Lemma 9 In Phases 3 and 4, for any residual graph $G$ the following statements hold.

(i) If $G$ has a component which is not of the type BWB, but it is of order at least 3, moreover this component contains a blue leaf, then Dominator can seize at least 8 points.

(ii) If there is no blue leaf in a component $C$ of order at least 4, then Dominator cannot seize more than 6 points by playing a vertex from $C$.

Proof

(i) Consider a component satisfying the conditions of the lemma, a B-leaf $v$ from it, and the only W-neighbor $u$ of $v$. If $u$ is in the W-pair $uu'$, Dominator can choose $u$ and then all the three vertices $v$, $u$ and $u'$ become red, and Dominator gets at least $8 = 2 + 3 + 3$ points. If $u$ is a single-W vertex and has at least three B-leaf neighbors (including vertex $v$), then the choice of $u$ results in a
gain of at least $9 = 3 + 3 \cdot 2$ points. If none of the previous cases holds, and as it is assumed, the component is not of the type BWB, then $u$ is a single-W vertex and has a B-neighbor $z$ which is not a leaf. If Dominator selects vertex $z$, then $v$, $u$ and $z$ become red, moreover at least one further white neighbor of $z$ turns blue or red. Thus, Dominator seizes at least $8 = 2 + 3 + 2 + 1$ points.

$(ii)$ As there is no blue leaf in the component, selecting any vertex $v \in V(C)$ in a turn, only $v$ and its W-neighbors change their colors. We have the following cases due to the type of the vertex $v$ chosen.

- If $v$ is a B-vertex which has no single white neighbor, then by Lemma $8(iii)$, $v$ has at most four W-neighbors each of which turns blue. Therefore, Dominator may seize at most $6 = 2 + 4 \cdot 1$ points, if he selects $v$.
- If $v$ is a B-vertex with a single-W neighbor $u$, then by Lemma $8(ii)$ and by the absence of B-leaves, $v$ has exactly one further neighbor $z$ which is in a W-pair. Hence, selecting $v$ Dominator gets exactly $6 = 2 + 3 + 1$ points.
- If $v$ is single-W, then no vertex from $N(v)$ changes its color and therefore the gain is exactly 3 points.
- If $v$ is in the W-pair $vu$, the only color-changes are $v, u : W \rightarrow R$ and hence, Dominator gets exactly 6 points.

\[ \square \]

### 3.4 Phase 3: the crucial point

It was easy to see that the average decrease in the value $p(G)$ of the residual graph was at least 5 points per turn in the first two phases. We will see that this average holds in the last phase. Also, if Staller gets at least 4 points in the $i$th turn of Phase 3, then together with the next turn of Dominator, when he seizes at least 6 points the desired average is attained locally. Hence, we focus on the turns when Staller gets only 3 points.

Recall that Strategy-Phase(3) prescribes greedy selection for Dominator. Further, if he cannot get more than 6 points, the preferred choice is to dominate a (W-leaf,W-stem) pair.

**Lemma 10** If Staller gets 3 points in the $i$th turn in Phase 3, then at least one of the following statements is true.

1. Dominator gets at least 8 points in the $(i-1)$st turn.
2. Dominator gets at least 7 points in the $(i+1)$st turn.
3. Dominator chooses a white stem $v_2$ of a critical $P_5$ $v_1v_2v_3v_4v_5$ in the $(i-1)$st turn, and Staller selects the center $v_3$ in the $i$th turn.
Proof  Assuming that Staller gets 3 points in the $i$th turn, we have two cases to consider.

Case 1  Staller selects a single-W vertex $v$.
As this choice results in only 3 points, $v$ is not from a component of order two, moreover $v$ has no B-leaf neighbor in $G_{i-1}$. Therefore, by Lemma 8(ii), each B-neighbor $u$ of $v$ has exactly one further neighbor. Thus, after the selection of $v$ (that is, in $G_i$) $u$ is a B-leaf. Also, it follows from Lemma 8(ii) that the component containing $u$ in $G_i$ is of order at least 3. Then, Lemma 3 and the greedy strategy of Dominator imply that (b) holds.

Case 2  Staller selects a B-vertex $v$.
As he gets only 3 points, $N(v) = \{u\}$, where $u$ is white but not a single-W vertex, moreover, $u$ has no B-leaf neighbor. Then, by Lemma 8(i), $u$ must be from a W-pair $uu'$, and after the move of Staller, $u$ becomes a B-leaf in $G_i$.

If $u'$ is not a leaf in $G_i$, then the component of the B-leaf $u$ is of order at least 3, hence by Lemma 8(Dominator gets at least 7 points in the $(i+1)$st turn and (b) holds.

Suppose thus that $u'$ is a W-leaf in $G_i$ and hence, in $G_{i-1}$ and $G_{i-2}$, too. We also assume that (a) is not valid, that is, Dominator could not get 8 or more points in the $(i-1)$st turn. Our goal is to prove that under these conditions (c) is necessarily true.

First, observe that $v$ was not a B-leaf in $G_{i-2}$ (otherwise choosing $u$ Dominator could seize at least 8 points). Similarly, $v$ was not a W-vertex in $G_{i-2}$, as this would mean a ‘W-triplet’ in Phase 3. Consequently, $v$ was a non-leaf B-vertex in $G_{i-2}$ and had a further W-neighbor, say $z$. This component $C_{i-2}$ of $G_{i-2}$ contains $v$, $u$, $u'$ and $z$, hence its order is at least 4. As we assume (a) not to be valid, by Lemma 9(i) we can conclude that $C_{i-2}$ contains no B-leaf. Next, we apply Lemma 9(ii) and obtain that Dominator can seize at most 6 points in the $(i-1)$st turn, and further, as Phase 3 is not finished at this time, he surely gets exactly 6 points. Due to the rule (R.3.2) given in Strategy-Phase(3), he selects a W-stem of a W-leaf if there exists such a pair. Actually, there does exist one, as the pair $uu'$ is of this type. Since Dominator did not choose $u$ in the $(i-1)$st turn, he played another W-stem with a W-leaf. This caused change in the color of $z$, so the only possibility is that Dominator selected the W-stem $z$ which had a W-leaf $z'$.

Therefore, $z'zvu'u$ was a critical $P_5$ in $G_{i-2}$ and Dominator chose the W-stem $z$ in the $(i-1)$st turn and then Staller played the center in the $i$th turn. This satisfies (c).

When case (c) of Lemma 10 is realized in the game and neither (a) nor (b) holds, the $i$th turn (when the center is selected) is called critical turn. Note that all of such turns belong to Phase 3. In the following lemma we estimate the number $c^*$ of critical turns.
Lemma 11  Let $n_\ell$ denote the number of non-red vertices at the beginning of Phase 3, and let $c^*$ be the number of critical turns. Then, $5c^* \leq n_\ell$ holds.

Proof  It is clear by definition that the $i$th turn might be critical only if Dominator’s choice in the $(i-1)$st turn and Staller’s choice in the next turn together change three vertices to be red in a component of order at least 5, moreover a new component of order 2 (of type BW) arises. These five vertices are associated with the $i$th critical turn. As they were non-red vertices at the beginning of the phase and no vertex is associated with more than one critical turn, the inequality follows. □

Lemma 12  If Phase 3 consists of $k$ turns ($k \geq 0$) and $c^*$ denotes the number of critical turns, then the value of $G$ has been decreased by at least $5k-c^*$ in this phase.

Proof  For the sake of simplicity, let the turns of Phase 3 be indexed from 1 to $k$, and $d_i$ (for $i$ odd) and $s_i$ (for $i$ even) denote the number of points Dominator or Staller seized in the $i$th turn, respectively. Hence, the value of the residual graph $G$ was decreased by

$$P = \sum_{1 \leq i \leq k, \ i \ odd} d_i + \sum_{1 \leq i \leq k, \ i \ even} s_i.$$ 

First, if $s_i = 3$ and $d_{i-1} \geq 8$, we redefine $s_i = 4$ and $d_{i-1} = 7$. Then, if the $i$th turn of Phase 3 is critical, we increase $s_i$ from 3 to 4. For the sum $P'$ of the current values, the inequality $P' \leq P + c^*$ holds.

Now, consider the pairs $s_i + d_{i+1}$ where $i$ is even and $2 \leq i < k$. If $s_i = 3$, neither (a) nor (c) from Lemma 10 is true for this turn, hence (b) must be valid and $s_i + d_{i+1} \geq 10$ follows. If $s_i \geq 4$ and $i < k$, then $d_{i+1} \geq 6$, and we have $s_i + d_{i+1} \geq 10$ again.

If $k$ is even and the last move of the phase is made by Staller, then (b) from Lemma 10 cannot be true. Thus, $s_k \geq 4$ and $d_1 + s_k \geq 10$, from which $P' \geq 5k$. Similarly, if $k$ is odd, $P' \geq d_1 + 10(k-1)/2 > 5k$ holds, and the lemma follows. □

3.5 Phase 4

We show that the structure of the residual graph is very simple throughout this phase.

Lemma 13  If Phase 3 consists of $k$ turns, then the value of $G$ has been decreased by exactly $5k$ in this phase.

Proof  Consider the residual graph $G$ which we have at the beginning of this phase. As Dominator cannot seize 6 or more points, there are no W-pairs, hence each W-vertex is single-W. Now, if a B-vertex $v$ had at least two neighbors, then selecting vertex $v$, all of its neighbors and also $v$ itself would turn red, and Dominator would
seize at least 8 points. Therefore, each blue vertex is a leaf. It is also easy to see that each white vertex has no more than one B-leaf neighbor, and definitely has at least one, as there are no isolated vertices by Lemma 1(ii).

Consequently, each component of $G$ is a $K_2$ with one white and one blue vertex. Therefore, no matter which vertex is selected, in each turn the value of the residual graph is decreased by exactly 5.

\[\square\]

### 3.6 Finalizing the proofs

Here we present the proofs of our theorems, based on the lemmas verified in the previous subsections.

**Proof of Theorem 1**  Consider an isolate-free forest $G$ in which no two leaves are at distance 4 apart. By Lemma 1(i), no new white leaves arise. Thus, we have no critical $P_5$ subgraphs at any moment of the game, and there occur no critical turns in Phase 3.

At the beginning, we have $p(G) = 3n$ and this is decreased to zero during the game. By Lemmas 11, 12, 13 and by $e^* = 0$, the average decrease in the value of the residual graph is at least 5 points per turn for the Dominator-start game. Then, the desired upper bound immediately follows:

$$\gamma_g(G) \leq \frac{p(G)}{5} = \frac{3n}{5}.$$ 

For the Staller-start version, we may define a Phase 0 consisting of just the starting turn indexed by 0. Recall that $G$ contains no isolated vertices and every vertex is white, which implies that in this turn Staller gets at least 4 points. Then, our lemmas on the later phases remain valid and we have

$$\gamma'_g(G) \leq \frac{3n + 1}{5}$$

as stated. \[\square\]

**Proof of Theorem 2**  In this general case, we consider an isolate-free forest $G$ with $p(G) = 3n$. If the described strategy yields a game with $t$ turns, $e^*$ extra points in Phase 1 and $e^*$ critical turns in Phase 3, our Lemmas 4, 6, 12 and 13 imply

$$t \leq \frac{3n - e^* + c^*}{5}.$$ 

The number $c^*$ of critical turns in Phase 3 cannot be greater than the number of critical centers at the beginning of this phase. Moreover, by Lemma 5 the latter
parameter is not greater than \((r_k/3) + e^*\), where \(r_k\) is the number of the vertices turned red in Phase 1. Therefore,

\[ r_k \geq 3(c^* - e^*). \]

On the other hand, by Lemma 11 for the number \(n_\ell\) of vertices which are non-red at the beginning of Phase 3,

\[ n_\ell \geq 5c^* \geq 5(c^* - e^*). \]

Thus, we obtain

\[ n \geq r_k + n_\ell \geq 8(c^* - e^*) \]

and then,

\[ \gamma_g(G) \leq t \leq \frac{3n + (n/8)}{5} = \frac{5n}{8} \]

as stated in Theorem 2.

Similarly to the proof of Theorem 1, the Staller-start game is treated by introducing Phase 0. As \(G\) is isolate-free, Staller gets at least 4 points in the turn indexed by 0 and then, for the number of red and blue vertices \(r_0 \geq 1\) and \(r_0 + b_0 \geq 2\) hold. Lemma 5 for this Staller-start version must be modified, as in the inequalities (1) and (2), parameters \(r_k\) and \(b_k\) must be replaced by \(r_k - r_0\) and by \(b_k - b_0\) respectively. Otherwise, the proof proceeds in the same way. Thus, here we obtain

**Lemma 5’**  
Let Phase 1 consist of \(k\) turns and let \(r_k\) denote the number of red vertices at the end of Phase 1. Then, the number of vertices which are critical centers in at least one later residual graph \(G_i\) \((i \geq k)\) is at most \(((r_k - r_0)/3) + e^* + b_0\).

Observe that the same upper bound holds for \(c^*\).

Let us introduce the notation \(e_0^* = 3r_0 + b_0 - 5\), which is the number of extra points achieved above the desired average 5 points in the starting turn. Note that \(e_0^*\) might equal \(-1\), but otherwise it is non-negative.

For the number \(t'\) of turns in this game,

\[ t' \leq \frac{3n - e^* - e_0^* + c^*}{5} = \frac{3n + 1 - [c^* - e^* - (e_0^* + 1)]}{5}. \]

Applying Lemma 5’,

\[ 3[c^* - e^* - (e_0^* + 1)] \leq r_k - 10r_0 + 12 \leq r_k + 2, \]

and by Lemma 11

\[ 5[c^* - e^* - (e_0^* + 1)] \leq 5c^* \leq n_\ell \]

is obtained. Then, we conclude the inequality

\[ \gamma'_g \leq t' \leq \frac{3n + 1 + \frac{n+2}{8}}{5} = \frac{5n + 2}{8} \]

which proves the theorem. \(\square\)
4 Concluding remarks

Although Conjecture 1 is a challenging open problem in itself, we close this paper with the following more general version of the conjecture.

Conjecture 2 (Kinnersley, West and Zamani, [7]) If $G$ is an isolate-free graph of order $n$, then

$$
\gamma_g(G) \leq \frac{3n}{5} \quad \text{and} \quad \gamma'_g(G) \leq \frac{3n+2}{5}.
$$

It is worth noting that a graph may have greater game domination number than any of its spanning trees. Hence, even if an upper bound on $\gamma_g$ is verified for forests, there is no trivial way to conclude the same bound for graphs in general.

The relation $\gamma_g(G) \leq \lceil 7n/10 \rceil$ is the best result, which has been published up to now for this general case [7]. In the forthcoming manuscript [6], we will improve this upper bound significantly by using our proof technique, where we consider a greedy-type strategy under some value-assignment to the vertices.

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