Quantum Discord of Certain Two-Qubit States

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Abstract. Quantum discord is an effective measure of quantum correlation introduced by Olliver and Zurek. We evaluate analytically the quantum discord for a large family of non-X-states. Exact solutions of the quantum discord are obtained of the four parametric space for non-X-states. Dynamic behavior of the quantum discord is also explored under the action of the Kraus operator.

1. Introduction

Quantum correlations are one of the fundamental features in quantum computation and quantum information. Among various measurements of quantum correlations, quantum discord has been studied as a particularly important problem [1, 2, 3].

The notion of quantum discord was introduced by Olliver and Zurek [7] to measure the difference of two natural quantum extensions of the classical mutual information [8, 9, 10, 11, 12, 13, 14]. The quantum classical mutual information is usually used to quantify the total correlations with well documented basic physical significance [15, 16, 17, 18]. Let’s consider the bipartite quantum state $\rho$, the quantum mutual information is defined as

$$I(\rho) := S(\rho^a) + S(\rho^b) - S(\rho),$$

where $S(\rho) := -\text{Tr} \rho \log_2(\rho)$ is the von Neumann entropy of the quantum state.

In order to reveal the essence of quantum correlation, measurement entropy based conditional density operators are used to study the classical correlation [7]. The von Neumann measurement is an entire set of projectors $\{B_k\}$ such that $\sum_k B_k = I$ and $B_j B_k = \delta_{jk} B_k$. If the measurement $\{B_k\}$ is executed locally on one side of the bipartite quantum state $\rho$, the

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The quantum state is turned into
\[ \rho_k = \frac{1}{p_k} \text{Tr}_b(I \otimes B_k) \rho(I \otimes B_k) \]
with the probability \( p_k = \text{Tr}(I \otimes B_k) \rho(I \otimes B_k) \), and here \( I \) is the identity operator for the party \( a \). The quantum conditional entropy with respect to the measurement \( \{B_k\} \) is defined as
\[ S(\rho|\{B_k\}) = \sum_k p_k S(\rho_k), \]
then the quantum mutual information is defined as
\[ \mathcal{I}(\rho|\{B_k\}) = S(\rho_a) - S(\rho|\{B_k\}) \]
and the classical correlation is measured in terms of the quantity
\[ \mathcal{C}(\rho) := \sup_{\{B_k\}} \mathcal{I}(\rho|\{B_k\}). \]
To compare the two quantum analogs of the classical quantum information the so-called quantum discord is defined as the difference of these two quantities:
\[ \mathcal{Q}(\rho) := \mathcal{I}(\rho) - \mathcal{C}(\rho). \]

There are considerable studies on quantum discords for various types of quantum states [2]. In [20] an exact formula was obtained for the Bell state. There are several well-known methods to compute the quantum discord for general X-states [9], and in [21] exact and analytic formulas for the general X type states were given, and in this latter work some of the confusions in previous computations of the quantum discord were clarified. In [22] it was shown that the quantum discord of 2-qubit is more robust than entanglement. In [23] the quantum discord dynamics of 2-qubit states in independent and common non-Markovian environments are evaluated. Ref. [24] provided an approach to compute one way quantum deficit of 2-qubit states. In [25] an analytical formula of quantum discord was presented for the two-qubit quantum state of rank-2 by studying its classical correlation (see also [21]). Despite all these progresses, it is still a difficult problem to find exact formula of the quantum discord for the general bipartite state, for instance, the quantum discord of the non-X-type of two-qubit states with rank more than 2 is unknown. In this paper, we study the quantum discord for the two-qubit states of all rank and derive exact formulas for several nontrivial cases.

We also study the dynamics of the quantum discord in this important case. We use the Kraus operators \( K_i \) to discuss the behavior of the 2-qubit non-X-state \( \rho \) through the phase damping channels, where \( \sum_i K_i^\dagger K_i = 1 \). Under the phase damping \( \rho \) changes into
\[ \tilde{\rho} = \sum_{i,j=1,2} K_i^A \otimes K_j^B \cdot \rho \cdot (K_i^A \otimes K_j^B)^\dagger, \]
where the Kraus operators can be defined as $K_1^{A(B)} = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|$ and $K_2^{A(B)} = \sqrt{\gamma}|1\rangle\langle 1|$ with the decoherence rate $\gamma \in [0,1]$.

This paper is organized into two parts. First we derive a formula of the quantum discord for a general quantum bipartite state in non-X-type and then give the exact quantum discord for several nontrivial regions. In the last part we study how the quantum discord behaves under the action of Kraus operators. Through this study one hopes to understand better the quantum discord for the general quantum state.

2. Quantum discord for non-X-states

The quantum discord for the Bell diagonal state was completely calculated by Luo [20], and the analytical expression of the general X-state quantum discord is obtained in [21]. Here we consider a certain non-X states and compute its exact quantum discord.

Let $\rho$ be the following quantum state

$$\rho = \frac{1}{4}(I \otimes I + r \cdot \vec{\sigma} \otimes I + I \otimes s \cdot \vec{\sigma} + \sum_{i=1}^{3} c_i \sigma_i \otimes \sigma_i),$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli matrices, $r = (r_1, r_2, r_3), s = (s_1, s_2, s_3), c = (c_1, c_2, c_3) \in \mathbb{R}^3$. Here $\sigma_i$ are normalized as $\text{tr}(\sigma_i \sigma_j) = 2\delta_{ij}$. It is clear that the coefficients $r_i, s_i$ can be confined within the interval $[-1,1]$. The two marginal states of $\rho$ are given by

$$\rho^a = \text{Tr}_b \rho = \frac{1}{2}(I + r \cdot \vec{\sigma}), \quad \rho^b = \text{Tr}_a \rho = \frac{1}{2}(I + s \cdot \vec{\sigma}).$$

Then the von Neumann entropy of the quantum marginal states are given by

$$S(\rho^a) = 1 - \frac{1 + |r|}{2} \log_2(1 + |r|) - \frac{1 - |r|}{2} \log_2(1 - |r|),$$
$$S(\rho^b) = 1 - \frac{1 + |s|}{2} \log_2(1 + |s|) - \frac{1 - |s|}{2} \log_2(1 - |s|).$$

Let us introduce the entropic function

$$H_\epsilon(x) = \frac{1}{2}(1 + \epsilon + x) \log_2(1 + \epsilon + x) + \frac{1}{2}(1 + \epsilon - x) \log_2(1 + \epsilon - x).$$

It is easy to see that $H_\epsilon(x)$ is an even function. Also $H_\epsilon(x) = H_\epsilon(0) = (1 + \epsilon)\log_2(1 + \epsilon)$ and $\max H_\epsilon(x) = H_\epsilon(\max |x|)$. Thus the quantum mutual information of $\rho$ is obtained as

$$I(\rho) = 2 - H_{\epsilon=0}(|r|) - H_{\epsilon=0}(|s|) + \sum_{i=1}^{4} \lambda_i \log_2(\lambda_i),$$

where $\lambda_i (i = 1, \cdots, 4)$ are the eigenvalues of $\rho$.

Now let’s turn to the second mutual information—the classical correlation $C(\rho)$, which is defined with help of the von Neumann measurements.
As it is well known that \( \{B_k = V|k\rangle\langle k|V^\dagger, k = 0, 1\} \), where \( V \in \text{SU}(2) \), parameterize the von Neumann measures.

Note that \( \text{SU}(2) \) is homeomorphic to the unit sphere, so any unitary matrix \( V = tI + \sqrt{-1}\sum_{i=1}^{3} y_i\sigma_i \) where \( t, y_i \in \mathbb{R} \) \((i = 1, 2, 3)\) are on the unit sphere:

\[
t^2 + \sum_{i=1}^{3} y_i^2 = 1
\]

Under the unitary transformation the two marginal states of \( \rho \) in (2.2) are changed to

\[
\rho_0 = \frac{1}{2(1 + sz)}[(1 + sz)I + (r + cz) \cdot \vec{\sigma}],
\]

\[
\rho_1 = \frac{1}{2(1 - sz)}[(1 - sz)I + (r - cz) \cdot \vec{\sigma}]
\]

with \( p_0 = \frac{1+sz}{2}, p_1 = \frac{1-sz}{2} \) and the unit vector \( z = (z_1, z_2, z_3) \) is given by

\[
z_1 = 2(-ty_2 + y_1y_3), z_2 = 2(ty_1 + y_2y_3), z_3 = t^2 + y_3^2 - y_1^2 - y_2^2.
\]

The eigenvalues of \( \rho_0 \) and \( \rho_1 \) are seen to be

\[
\lambda_{\rho_0}^\pm = \frac{1}{2(1 + sz)}(1 + sz \pm |r + cz|)
\]

\[
\lambda_{\rho_1}^\pm = \frac{1}{2(1 - sz)}(1 - sz \pm |r - cz|)
\]

Let

\[
G(z) = -H_{\epsilon=0}(sz) + \frac{1}{2}H_{\epsilon=sz}(|r + cz|) + \frac{1}{2}H_{\epsilon=-sz}(|r - cz|),
\]

then the classical correlations can be given by

\[
C(\rho) = \sup_{\{B_k\}} I(\rho|\{B_k\}) = S(\rho^a) - \sup \{ \sum_{k=0,1} p_k S(\rho_k) \}
\]

\[
= S(\rho^a) - \sup \{ \sum_{k=0,1} p_k (\lambda_{\rho_k}^+ \log_2 \lambda_{\rho_k}^+ + \lambda_{\rho_k}^- \log_2 \lambda_{\rho_k}^-) \}
\]

\[
= -H_{\epsilon=0}(|r|) + \max \{ G(z) \}.
\]

The following result computes the quantum discord \([133] \) for some non-X states.

**Theorem 2.1.** When \( s = 0 \) and \( c_1 = c_2 = c_3 = c \), the quantum discord is

\[
Q(\rho) = \frac{1}{2}H_{\epsilon=c}(|r|) + \frac{1}{2}H_{\epsilon=-c}(|r| + \sqrt{4c^2 + |r|^2})
\]

\[
- \frac{1}{2}H_{\epsilon=0}(|r| + |c|) + H_{\epsilon=0}(|r| - |c|).
\]
In particular, when $c = |r| \neq 0$, the quantum discord is

$$Q(\rho) = \frac{1}{4} (1 - c + \sqrt{5c}) \log_2 (1 - c + \sqrt{5c}) \quad (2.14)$$

$$+ \frac{1}{4} (1 - c - \sqrt{5c}) \log_2 (1 - c - \sqrt{5c})$$

$$- \frac{1}{4} (1 - 2c) \log_2 (1 - 2c);$$

When $|r| = |s| = 0$, the quantum discord is

$$Q(\rho) = \frac{1}{4} [1 - 3c] \log_2 (1 - 3c) - 2(1 - c) \log_2 (1 - c) + (1 + c) \log_2 (1 + c)] \quad (2.15)$$

We first prove the following lemma.

**Lemma 2.2.** Let $\theta = |r + cz|^2$, then $\min \theta = (|r| - |c|)^2$ and $\max \theta = (|r| + |c|)^2$.

**Proof.** Since $\theta = r^2 + c^2 + 2c(r_1 z_1 + r_2 z_2 + r_3 z_3)$ and $z_1^2 + z_2^2 + z_3^2 = 1$, we consider

$$F(z_1, z_2, z_3, \mu) = 2c(r_1 z_1 + r_2 z_2 + r_3 z_3) + \mu(1 - z_1^2 - z_2^2 - z_3^2),$$

where $\mu$ is a parameter. Then $\frac{\partial F}{\partial z_i} = 2c r_i - 2 \mu z_i = 0, (i = 1, 2, 3)$ implies that

$$\mu = \pm \sqrt{c^2 (r_1^2 + r_2^2 + r_3^2)} = \pm |c||r|.$$ 

When $\mu = |c||r|$, then $z_i = \frac{r_i}{|r|}$ and the minimal value $\theta_{\min} = (|r| - |c|)^2$. Similarly when $\mu = -|c||r|$, the maximal value $\theta_{\max} = (|r| + |c|)^2$. \qed

When $s = 0, c_1 = c_2 = c_3 = c$, the function $G(z)$ in (2.11) becomes

$$G(\theta) = \frac{1}{2} H_{c=0}(\sqrt{\theta}) + \frac{1}{2} H_{c=0}(\sqrt{2(|r|^2 + c^2) - \theta}). \quad (2.16)$$

Meanwhile, the eigenvalues of $\rho$ in this case are

$$\lambda_{1,2} = \frac{1}{4} (1 + c \pm |r|); \quad \lambda_{3,4} = \frac{1}{4} (1 - c \pm \sqrt{4c^2 + |r|^2}). \quad (2.17)$$

As $\rho$ is nonnegative, $(1 + c)^2 \geq r^2$ and $(1 - c)^2 \geq 4c^2 + |r|^2$. Subsequently $|r|^2 + c^2 \leq 1$, therefore both $|r|, |c| \leq 1$. Moreover, $|r| - c \leq 1$ and $|r| + c \leq 1$. Now we can prove the theorem.

**Proof.** It is obvious that

$$G(|r| + |c|) = G(|r| - |c|) = \frac{1}{2} H_{c=0}(|r| + |c|) + \frac{1}{2} H_{c=0}(|r| - |c|) \quad (2.18)$$

The derivative of $G(\theta)$ is equal to

$$\frac{\partial G(\theta)}{\partial \theta} = \frac{1}{8} \left[ \frac{1}{\sqrt{\theta}} \log_2 \frac{1 + \sqrt{\theta}}{1 - \sqrt{\theta}} \right]$$

$$- \frac{1}{\sqrt{2(|r|^2 + c^2) - \theta}} \log_2 \frac{1 + \sqrt{2(|r|^2 + c^2) - \theta}}{1 - \sqrt{2(|r|^2 + c^2) - \theta}}. \quad (2.19)$$
Let \( g(x) = \frac{1}{x} \log_2 \frac{1 + x}{1 - x} \), \( x \in (0, 1) \). The function \( g(x) \) is strictly increasing as

\[
\frac{\partial g(x)}{\partial x} = \frac{2}{x \ln 2} \left( \sum_{n=0}^{\infty} \frac{-x^{2n}}{2n+1} + \sum_{n=0}^{\infty} x^{2n} \right) > 0.
\]

So when \( \theta > |r|^2 + c^2 \), (2.19) implies that \( \frac{\partial G(\theta)}{\partial \theta} > 0 \), so \( G(\theta) \) is an increasing function. Similarly, when \( \theta < |r|^2 + c^2 \), \( G(\theta) \) is a decreasing function. Hence \( G(\theta) \) has the minimal value at \( \theta = |r|^2 + c^2 \), and

\[
\max G(\theta) = G((|r| + |c|)^2) = G((|r| - |c|)^2).
\]

In particular, if \( |r| = |c| \neq 0 \), then \( \theta \in [0, 2(|r|^2 + c^2)] \), we have

\[
\max G(\theta) = G(0) = G(2(|r|^2 + c^2)) = \frac{1}{2} H(2(|r|^2 + c^2)) = \frac{1}{4}[(1 + 2c)\log_2(1 + 2c) + (1 - 2c)\log_2(1 - 2c)]
\]

If \( |r| = 0, c_1 = c_2 = c_3 = c \), the state \( \rho \) in (2.1) degenerate to the Werner state. We have \( \theta = c^2 \), then

\[
(2.21) \max G(\theta) = G(c^2) = H(c) = \frac{1}{2} [(1+c)\log_2(1+c) + (1-c)\log_2(1-c)].
\]

\[ \square \]

Similarly, when \( |r| = 0 \) and \( c_1 = c_2 = c_3 = c \), the quantum discord is

\[
(2.22) Q(\rho) = \frac{1}{2} H_{\epsilon=-c}(\sqrt{4c^2 + |s|^2}) - \frac{1}{2} H_{\epsilon=-c}(|s|).
\]

**Theorem 2.3.** When \( |s| = 0, c_1 = c_2 = 0 \), the quantum discord \( Q(\rho) = 0 \); When \( |r| = 0, c_1 = c_2 = 0 \), the quantum discord is

\[
(2.23) Q(\rho) = H_{\epsilon=0}(\frac{|s|}{\sqrt{s_1^2 + s_2^2 + (c_3 + s_3)^2}}).
\]

**Theorem 2.4.** When \( s = 0, c_3 = 0, c_1 = c_2 = c \), the quantum discord is

\[
(2.24) Q(\rho) = \frac{1}{2} [H_{\epsilon=0}(\alpha_+) + H_{\epsilon=0}(\alpha_-) - H_{\epsilon=0}(\beta_+ - H_{\epsilon=0}(\beta_-)],
\]

where

\[
\alpha_+ = \sqrt{2c^2 + r_1^2 + r_2^2 + r_3^2 + 2\sqrt{c^2r_1^2 + c^2r_2^2}},
\]

\[
\beta_+ = \sqrt{2c^2 + r_1^2 + r_2^2 + r_3^2 + 2\sqrt{c^2r_1^2 + c^2r_2^2}},
\]

Example 1. Consider \( \rho \) with \( s_1 = 0.1, s_2 = 0.2, s_3 = 0.2, c = 0.3 \). \( \rho \) can be written in the form

\[
(2.25) \rho = \begin{pmatrix}
0.375 & 0.025 - 0.05i & 0 & 0 \\
0.025 + 0.05i & 0.125 & 0.15 & 0 \\
0 & 0.15 & 0.225 & 0.025 - 0.05i \\
0 & 0 & 0.025 + 0.05i & 0.275
\end{pmatrix}
\]
The behavior of $G(\theta)$ for $\theta \in [0, 0.0697]$ with parameters $s_1 = 0.1, s_2 = 0.2, s_3 = 0.2, c = 0.3$

The eigenvalues of $\rho$ are $\lambda_1 = 0.0073, \lambda_2 = 0.25, \lambda_3 = 0.3427, \lambda_4 = 0.4$ and the behavior of $G(\theta)$ is described in Fig.1. The quantum discord $Q(\rho) = 0.1058844$.

Example 2. Let $s_1 = s_2 = s_3 = 0, r_1 = 0.1, r_2 = 0.2, r_3 = 0.25, c = 0.3$, then $\rho$ is given by

\[
\rho = \begin{pmatrix}
0.25 & 0 & 0.025 - 0.05i & 0 \\
0 & 0.25 & 0.15 & 0.025 - 0.05i \\
0.025 + 0.05i & 0.15 & 0.25 & 0 \\
0 & 0.025 + 0.05i & 0 & 0.25
\end{pmatrix}
\]

The eigenvalues of $\rho$ are $\lambda_1 = 0.0815, \lambda_2 = 0.2315, \lambda_3 = 0.2685, \lambda_4 = 0.4185$, the quantum discord $Q(\rho) = 0.0271$. The behavior of $G(\theta)$ is depicted in Fig.2. We can observe that the max of $G(\theta)$ is 0.2321.
3. Dynamics of quantum discord under phase damping channel

In this section, we use the Kraus operators $K_i$ to discuss the behavior of the 2-qubit non-X-state $\rho$ through the phase damping channels [20], where $\sum_i K_i^\dagger K_i = 1$. Under the phase damping $\rho$ is changed into

$$\tilde{\rho} = \sum_{i,j=1,2} K_i^A \otimes K_j^B \cdot \rho \cdot (K_i^A \otimes K_j^B)^\dagger$$

where the Kraus operators can be defined as $K_1^{A(B)} = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|$ and $K_2^{A(B)} = \sqrt{\gamma}|1\rangle\langle 1|$ with the decoherence rate $\gamma \in [0,1]$. Therefore, under the phase damping $\rho$ in (2.1) becomes

$$\tilde{\rho} = \frac{1}{4}[I \otimes I + \sum_{i=1,2} r_i \sqrt{1-\gamma}\sigma_i \otimes I + r_3\sigma_3 \otimes I + I \otimes \sum_{i=1,2} s_i \sqrt{1-\gamma}\sigma_i$$

$$+ I \otimes s_3\sigma_3 + c_3\sigma_3 \otimes \sigma_3 + \sum_{i=1,2} (1-\gamma)c_i\sigma_i \otimes \sigma_i].$$

The two marginal states of $\tilde{\rho}$ are

$$\tilde{\rho}^a = \frac{1}{2}(I + \sum_{i=1,2} r_i \sqrt{1-\gamma}\sigma_i + r_3\sigma_3);$$

$$\tilde{\rho}^b = \frac{1}{2}(I + \sum_{i=1,2} s_i \sqrt{1-\gamma}\sigma_i + s_3\sigma_3).$$

Thus the quantum mutual information of $\tilde{\rho}$ can be written as

$$\mathcal{I}(\tilde{\rho}) = S(\tilde{\rho}^a) + S(\tilde{\rho}^b) - S(\tilde{\rho})$$

$$= 2 - H_{\epsilon=0}(\sqrt{|s|^2 - \gamma s_1^2 - \gamma s_2^2}) - H_{\epsilon=0}(\sqrt{|s|^2 - \gamma s_1^2 - \gamma s_2^2})$$

$$+ \sum_i \lambda_i \log_2 \lambda_i,$$

where $\lambda_i (i = 1, \cdots, 4)$ are eigenvalues of $\tilde{\rho}$. Under the unitary transformation, the two marginal states of $\tilde{\rho}$ becomes

$$\tilde{\rho}_k = \frac{1}{2}[(1 + (-1)^k) \sqrt{1 - \gamma}(s_1 z_1 + s_2 z_2) + (-1)^k s_3 z_3)I$$

$$+ \sum_i r_i \sqrt{1-\gamma}\sigma_i + r_3\sigma_3 + (-1)^k c_3 z_3\sigma_3 + (-1)^k \sum_i c_i (1 - \gamma)\sigma_1 z_1];$$

with $p_k = \frac{1}{2}(1 + (-1)^k) \sqrt{1 - \gamma}(s_1 z_1 + s_2 z_2) + (-1)^k s_3 z_3)$ and $k = 0, 1$. The eigenvalues of $\tilde{\rho}_0$ and $\tilde{\rho}_1$ are given by

$$\lambda_{\tilde{\rho}_k}^{\pm} = \frac{1}{2(1 + \epsilon_{+})}(1 + \epsilon_{+} \pm \sqrt{\zeta_{+}}); \quad \lambda_{\tilde{\rho}_1}^{\pm} = \frac{1}{2(1 + \epsilon_{-})}(1 + \epsilon_{-} \pm \sqrt{\zeta_{-}}),$$
where \( \epsilon_\pm = \pm \sqrt{1 - \gamma}(s_1z_1 + s_2z_2) \pm s_3z_3 \), \( \zeta_\pm = (1 - \gamma)[(r_1 \pm \sqrt{1 - \gamma}c_1z_1)^2 + (r_2 \pm \sqrt{1 - \gamma}c_2z_2)^2 + (r_3 \pm c_3z_3)^2 \) The classical correlation \( C(\rho) \) can be given by

\[
C(\rho) = -H_{\epsilon=0}(\sqrt{|r|^2 - \gamma r_1^2 - \gamma r_2^2}) + \max \tilde{G}(z),
\]

where

\[
\tilde{G}(z) = -H_{\epsilon=0}(\epsilon_+) + \frac{1}{2}(H_{\epsilon=\epsilon_+}(\delta_+) + H_{\epsilon=\epsilon_-}(\delta_-))
\]

with \( \delta_\pm = \sqrt{(1 - \gamma)[(r_1 \pm \sqrt{1 - \gamma}c_1z_1)^2 + (r_2 \pm \sqrt{1 - \gamma}c_2z_2)^2 + (r_3 \pm c_3z_3)^2}. \)

Then the quantum discord of \( \tilde{\rho} \) is

\[
Q(\tilde{\rho}) = 2 + \sum_1^4 \tilde{\lambda}_i \log_2 \tilde{\lambda}_i - \max \tilde{G}(z).
\]

Under the phase damping channel, the Werner state \( \rho \) becomes

\[
\tilde{\rho} = \frac{1}{4}[I \otimes I - c\sigma_3 \otimes \sigma_3 - c \sum_{i=1,2}^4 (1 - \gamma)\sigma_i \otimes \sigma_i].
\]

The eigenvalues of \( \tilde{\rho} \) are \( \tilde{\lambda}_1 = \frac{1-c}{4}, \tilde{\lambda}_2 = \frac{1-c}{4}, \tilde{\lambda}_3 = 1+3c - 2c\gamma, \tilde{\lambda}_4 = 1-c+2c\gamma. \)

The maximal value of \( \tilde{G}(z) \) is written as

\[
\max \tilde{G}(z) = \frac{1+c}{2} \log_2(1+c) + \frac{1-c}{2} \log_2(1-c).
\]

Then the quantum discord of \( \tilde{\rho} \) is given by

\[
Q(\tilde{\rho}) = \frac{1+3c - 2c\gamma}{4} \log_2(1+3c - 2c\gamma)
+ \frac{1-c + 2c\gamma}{4} \log_2(1-c + 2c\gamma)
- \frac{(1+c)}{2} \log_2(1+c).
\]

Thus,

\[
Q(\rho) - Q(\tilde{\rho}) = \frac{1-c}{4} \log_2(1-c) + \frac{1+3c}{4} \log_2(1+3c)
- \frac{1+3c - 2c\gamma}{4} \log_2(1+3c - 2c\gamma)
- \frac{1-c + 2c\gamma}{4} \log_2(1-c + 2c\gamma).
\]

Let \( T(c, \gamma) = Q(\rho) - Q(\tilde{\rho}) \), the derivative of \( T(c, \gamma) \) is equal to

\[
\frac{\partial T}{\partial \gamma} = \frac{c}{2} \log_2 \frac{1+3c - 2c\gamma}{1-c + 2c\gamma}.
\]

This is a strictly increasing function of \( \gamma \). Thus, for fixed \( c \in [0, 1] \), the minimum of \( T(c, \gamma) \) is at \( \gamma = 0 \). Therefore, when \( \gamma \neq 0 \), \( Q(\rho) > Q(\tilde{\rho}) \). This shows that the quantum discord of Werner state decreases under the phase damping channel.
When $c_1 = c_2 = 0$ and $s = 0$, the $\rho$ under the phase damping channel is given by

$$\hat{\rho} = \frac{1}{4}[I \otimes I + \sum_{i=1,2} r_i \sqrt{1 - \gamma \sigma_i} \otimes I + r_3 \sigma_3 \otimes I + c_3 \sigma_3 \otimes \sigma_3],$$

(3.16) 

the eigenvalues of $\hat{\rho}$ are

$$\hat{\lambda}_{1,2} = \frac{1}{4} \pm \sqrt{(r_1^2 + r_2^2)(1 - \gamma) + (c_3 - r_3)^2},$$

$$\hat{\lambda}_{3,4} = \frac{1}{4} \pm \sqrt{(r_1^2 + r_2^2)(1 - \gamma) + (c_3 + r_3)^2}.$$

The maximal value of $\hat{G}(z)$ in (3.9) is given by

$$\max \hat{G}(z) = \frac{1}{2}(H_{c=0}(\varrho_+) + H_{c=0}(\varrho_-)),$$

(3.17) 

where

$$\varrho_{\pm} = [1 + \sqrt{(r_1^2 + r_2^2)(1 - \gamma) + (r_3 \pm c_3)^2}].$$

Then $Q(\hat{\rho}) = Q(\rho)$.

When $|s| = 0, c_3 = 0$ and $c_1 = c_2 = c$, the $\rho$ under the phase damping is described by

$$\hat{\rho} = \frac{1}{4}[I \otimes I + \sum_{i=1,2} r_i \sqrt{1 - \gamma \sigma_i} \otimes I + r_3 \sigma_3 \otimes I + \sum_{i=1,2} (1 - \gamma) c \sigma_i \otimes \sigma_i]$$

(3.18)

We can also get that the eigenvalues of $\hat{\rho}$ are

$$\hat{\lambda}_{1,2} = \frac{1}{4} \pm \sqrt{(1 - \gamma)^2 2c^2 + (1 - \gamma)(r_1^2 + r_2^2) + r_3^2 + 2\varsigma},$$

$$\hat{\lambda}_{3,4} = \frac{1}{4} \pm \sqrt{(1 - \gamma)^2 2c^2 + (1 - \gamma)(r_1^2 + r_2^2) + r_3^2 - 2\varsigma},$$

where $\varsigma = \sqrt{c^4(1 - \gamma)^4 + (c^2 r_1^2 + c^2 r_2^2)(1 - \gamma)^3}$. Thus the maximal value of $\hat{G}(z)$ in (3.9) is given by

$$\max \hat{G}(z) = \frac{1}{2}(H_{c=0}(\xi_3) + H_{c=0}(\xi_4)).$$

(3.19) 

The difference between quantum discord of the $\hat{\rho}$ of under the phase damping channel and quantum discord of $\rho$ is given by

$$Q(\rho) - Q(\hat{\rho}) = \frac{1}{2}\{H_{c=0}(\alpha_+) + H_{c=0}(\alpha_-) - (H_{c=0}(\beta_+) + H_{c=0}(\beta_-))$$

$$- \{H_{c=0}(\mu_+) + H_{c=0}(\mu_-) - (H_{c=0}(\sigma_+) + H_{c=0}(\sigma_-))\}$$

(3.20)

where

$$\alpha_{\pm} = \sqrt{2c^2 + r_1^2 + r_2^2 + r_3^2 \pm 2\sqrt{c^4 + c^2 r_1^2 + c^2 r_2^2}},$$

$$\beta_{\pm} = \sqrt{(r_1 \pm \frac{r_1 d}{\sqrt{r_1^2 + r_2^2}})^2 + (r_2 \pm \frac{r_2 d}{\sqrt{r_1^2 + r_2^2}})^2 + r_3^2},$$

$$\mu_{\pm} = \sqrt{(r_1 \pm \frac{r_1 d}{\sqrt{r_1^2 + r_2^2}})^2 + (r_2 \pm \frac{r_2 d}{\sqrt{r_1^2 + r_2^2}})^2 + r_3^2},$$

$$\sigma_{\pm} = \sqrt{2c^2 + r_1^2 + r_2^2 + r_3^2 \pm 2\sqrt{c^4 + c^2 r_1^2 + c^2 r_2^2}}.$$
\[\mu_\pm = \sqrt{2c^2(1 - \gamma) + (r_1^2 + r_3^2)(1 - \gamma) + r_2^2 \pm 2\varsigma},\]
\[\sigma_\pm = \sqrt{(1 - \gamma)((r_1 \pm \sqrt{1 - \gamma}\sqrt{r_1^2 + r_2^2})^2 + (r_2 \pm \sqrt{1 - \gamma}\sqrt{r_1^2 + r_2^2})^2) + r_3^2}.\]

Now, we consider the state \(\rho\) in Example 2. The state \(\rho\) under phase damping channel is given by

\[
\tilde{\rho} = \frac{1}{4} [I \otimes I + 0.1\sqrt{1 - \gamma}\sigma_1 \otimes I + 0.2\sqrt{1 - \gamma}\sigma_2 \otimes I + 0.3\sigma_3 \otimes I
+ 0.25\sigma_3 \otimes \sigma_3 + 0.25(1 - \gamma)\sigma_1 \otimes \sigma_1 + 0.25(1 - \gamma)\sigma_2 \otimes \sigma_2].
\]

For \(\gamma = 0.2\), the eigenvalues of \(\tilde{\rho}\) are \(\tilde{\lambda}_1 = 0.399691, \tilde{\lambda}_2 = 0.328613, \tilde{\lambda}_3 = 0.217934, \tilde{\lambda}_4 = 0.0537617\) and the difference \(Q(\rho) - Q(\tilde{\rho}) = 0.0583\). For \(\gamma = 0.7\), the eigenvalues of \(\tilde{\rho}\) are \(\tilde{\lambda}_1 = 0.391011, \tilde{\lambda}_2 = 0.288475, \tilde{\lambda}_3 = 0.220322, \tilde{\lambda}_4 = 0.100192\) and the difference \(Q(\rho) - Q(\tilde{\rho}) = 0.1426\). It is easily seen that \(Q(\rho) - Q(\tilde{\rho})\) is different when \(\gamma\) is different.

### 4. Conclusions

The quantum discord is an important quantum correlation with interesting applications. It measures the difference between two natural quantum analogs of the classical mutual information. Its computation is usually hard and exact formulas are difficult to derive. For the general non-X-type quantum state, we have given an analytical solution of the quantum discord in terms of the maximum of a one variable function.

We have shown that the quantum discord essentially follows the similar pattern as the other types of quantum states. As an example, we have shown that the quantum discord in the non-X-type case can also be computed exactly in several interesting regions. Using an example, our method is demonstrated to be able to solve general non-X-type quantum states. We also studied the dynamics of the quantum discord under the Kraus operators, and we have explained that there are cases the quantum discord is invariant under the process, while there are also examples the quantum discord is changed. This is basically similar to the other situations.

In summary, the problem of the quantum discord for the general bipartite states follows the similar pattern either in the X-type or non-X-type.

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