A CONSTRUCTION OF PSEUDO-ANOSOV BRAIDS WITH SMALL NORMALIZED ENTROPIES

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Abstract. Let $b$ be a pseudo-Anosov braid whose permutation has a fixed point and let $M_b$ be the mapping torus by the pseudo-Anosov homeomorphism defined on the genus 0 fiber $F_b$ associated with $b$. We prove that there is a 2-dimensional subcone $C_0$ contained in the fibered cone $C$ of $F_b$ such that the fiber $F_a$ for each primitive integral class $a \in C_0$ has genus 0. We also give a constructive description of the monodromy $\phi_a : F_a \to F_a$ of the fibration on $M_b$ over the circle, and consequently provide a construction of many sequences of pseudo-Anosov braids with small normalized entropies. As an application we prove that the smallest entropy among skew-palindromic braids with $n$ strands is comparable to $1/n$, and the smallest entropy among elements of the odd/even spin mapping class groups of genus $g$ is comparable to $1/g$.

1. Introduction

Let $\Sigma = \Sigma_{g,n}$ be an orientable surface of genus $g$ with $n$ punctures for $n \geq 0$. We set $\Sigma_g = \Sigma_{g,0}$. By mapping class group $\text{Mod}(\Sigma_{g,n})$, we mean the group of isotopy classes of orientation preserving self-homeomorphisms on $\Sigma_{g,n}$ preserving punctures setwise. By Nielsen-Thurston classification, elements in $\text{Mod}(\Sigma)$ are classified into three types: periodic, reducible, pseudo-Anosov [30, 9]. For $\phi \in \text{Mod}(\Sigma)$ we choose a representative $\Phi \in \phi$ and consider the mapping torus $M_{\phi} = \Sigma \times \mathbb{R}/\sim$, where $\sim$ identifies $(x,t+1)$ with $(\Phi(x),t)$ for $x \in \Sigma$ and $t \in \mathbb{R}$. Then $\Sigma$ is a fiber of a

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{(1) $\sigma_i$, (2) $\sigma_1^{-1}\sigma_2$ with the permutation $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$. (3) $\sigma_1^2\sigma_2^{-1}$ whose permutation has a fixed point.}
\end{figure}

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fibration on $M_{\phi}$ over the circle $S^1$ and $\phi$ is called the monodromy. A theorem by Thurston [31] asserts that $M_{\phi}$ admits a hyperbolic structure of finite volume if and only if $\phi$ is pseudo-Anosov.

For a pseudo-Anosov element $\phi \in \text{Mod}(\Sigma)$ there is a representative $\Phi : \Sigma \to \Sigma$ of $\phi$ called a pseudo-Anosov homeomorphism with the following property: $\Phi$ admits a pair of transverse measured foliations $(F^u, \mu^u)$ and $(F^s, \mu^s)$ and a constant $\lambda = \lambda(\phi) > 1$ depending on $\phi$ such that $F^u$ and $F^s$ are invariant under $\Phi$, and $\mu^u$ and $\mu^s$ are uniformly multiplied by $\lambda$ and $\lambda^{-1}$ under $\Phi$. The constant $\lambda(\phi)$ is called the dilatation and $F^u$ and $F^s$ are called the unstable and stable foliation. We call the logarithm $\log(\lambda(\phi))$ the entropy, and call

$$\text{Ent}(\phi) = |\chi(\Sigma)| \log(\lambda(\phi))$$

the normalized entropy of $\phi$, where $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$. Such normalization of the entropy is suited for the context of 3-manifolds [8, 21].

Penner [27] proved that if $\phi \in \text{Mod}(\Sigma_{g,n})$ is pseudo-Anosov, then

$$(1.1) \quad \frac{\log 2}{12g - 12 + 4n} \leq \log(\lambda(\phi)).$$

See also [21, Corollary 2]. For a fixed surface $\Sigma$, the set

$$\{ \log \lambda(\phi) \mid \phi \in \text{Mod}(\Sigma) \text{ is pseudo-Anosov} \}$$

is a closed, discrete subset of $\mathbb{R}$ ([1]). For any subgroup or subset $G \subset \text{Mod}(\Sigma)$ let $\delta(G)$ denote the minimum of $\lambda(\phi)$ over all pseudo-Anosov elements $\phi \in G$. Then $\delta(G) \geq \delta(\text{Mod}(\Sigma))$. We write $f \asymp h$ if there is a universal constant $P > 0$ such that $1/P \leq f/h \leq P$. It is proved by Penner [27] that the minimal entropy among pseudo-Anosov elements in $\text{Mod}(\Sigma_{g,n})$ on the closed surface of genus $g$ satisfies

$$\log \delta(\text{Mod}(\Sigma_g)) \approx \frac{1}{g}.$$

See also [16, 32, 33] for other sequences of mapping class groups.

For any $P > 0$, consider the set $\Psi_P$ consisting of all pseudo-Anosov homeomorphisms $\Phi : \Sigma \to \Sigma$ defined on any surface $\Sigma$ with the normalized entropy $|\chi(\Sigma)| \log \lambda(\Phi) \leq P$. This is an infinite set in general (take $P > 2 \log(2 + \sqrt{3})$ for example) and is well-understood in the context of hyperbolic fibered 3-manifolds. The universal finiteness theorem by Farb-Leininger-Margalit [8] states that the set of homeomorphism classes of mapping tori of pseudo-Anosov homeomorphisms $\Phi^\circ : \Sigma^\circ \to \Sigma^\circ$ is finite, where $\Phi^\circ : \Sigma^\circ \to \Sigma^\circ$ is the fully punctured pseudo-Anosov
homeomorphism obtained from $\Phi \in \Psi_P$. (Clearly $\lambda(\Phi^\circ) = \lambda(\Phi).$) In other words such $\Phi^\circ : \Sigma^\circ \to \Sigma^\circ$ is a monodromy of a fiber in some fibered cone for a hyperbolic fibered 3-manifold in the finite list determined by $P$. Thus 3-manifolds in the finite list govern all pseudo-Anosov elements in $\Psi_P$. It is natural to ask the dynamics and a constructive description of elements in $\Psi_P$. There are some results about this question by several authors [4, 15, 20, 22, 33], but it is not completely understood. In this paper we restrict our attention to the pseudo-Anosov elements in $\Psi_P$ defined on the genus 0 surfaces, and provide an approach for a concrete description of those elements.

Let $B_n$ be the braid group with $n$ strands. The group $B_n$ is generated by the braids $\sigma_1, \ldots, \sigma_{n-1}$ as in Figure 1. Let $S_n$ be the symmetric group, the group of bijections of $\{1, \ldots, n\}$ to itself. A permutation $P \in S_n$ has a fixed point if $P(i) = i$ for some $i$. We have a surjective homomorphism $\pi : B_n \to S_n$ which sends each $\sigma_j$ to the transposition $(j, j + 1)$.

The closure $\cl(b)$ of a braid $b \in B_n$ is a knot or link in the 3-sphere $S^3$. The braided link

$$\br(b) = \cl(b) \cup A$$

is a link in $S^3$ obtained from $\cl(b)$ with its braid axis $A$ (Figure 2). Let $M_b$ denote the exterior of $\br(b)$ which is a 3-manifold with boundary. It is easy to find an $(n + 1)$-holed sphere $F_b$ in $M_b$ (Figure 2(3)). Clearly $F_b$ is a fiber of a fibration on $M_b \to S^1$ and its monodromy $\phi_b : F_b \to F_b$ is determined by $b$. We call $F_b$ the $F$-surface for $b$.

A braid $b \in B_n$ is periodic (resp. reducible, pseudo-Anosov) if the associated mapping class $f_b \in \Mod(\Sigma_{0,n+1})$ is of the corresponding type (Section 2.3). If $b$ is pseudo-Anosov, then the dilatation $\lambda(b)$ is defined by $\lambda(f_b)$ and the normalized entropy $\Ent(b)$ is defined by $\Ent(f_b)$. The following theorem is due to Hironaka-Kin [16, Proposition 3.36] together with the observation by Kin-Takasawa [22, Section 4.1].

**Theorem 1.1.** There is a sequence of pseudo-Anosov braids $z_n \in B_n$ such that $\Ent(z_n) \neq 2\log(2 + \sqrt{3})$, $M_{z_n} \simeq M_{\sigma_1^2\sigma_2^{-1}}$ for each $n \geq 3$ and $\Ent(z_n) \to 2\log(2 + \sqrt{3})$ as $n \to \infty$.

Here $\simeq$ means they are homeomorphic to each other. The limit point $2\log(2 + \sqrt{3})$ is equal to $\Ent(\sigma_1^2\sigma_2^{-1})$. By the lower bound (1.1), Theorem 1.1 implies that

$$\log \delta(\Mod(\Sigma_{0,n})) \asymp \frac{1}{n}.$$

In particular, the hyperbolic fibered 3-manifold $M_{\sigma_1^2\sigma_2^{-1}}$ admits an infinitely family of genus 0 fibers of fibrations over $S^1$.

Let $z_n$ be a pseudo-Anosov braided with $d_n$ strands. We say that a sequence $\{z_n\}$ has a small normalized entropy if $d_n \asymp n$ and there is a constant $P > 0$ which does not depend on $n$ such that $\Ent(z_n) \leq P$. By (1.1) a sequence $\{z_n\}$ having a small normalized entropy means $\log(\lambda(z_n)) \asymp 1/n$. One of the aims in this paper is to give a construction of many sequences of pseudo-Anosov braids with small normalized entropies. The following result generalizes Theorem 1.1.

**Theorem A.** Suppose that $b$ is a pseudo-Anosov braid whose permutation has a fixed point. There is a sequence of pseudo-Anosov braids $\{z_n\}$ with small normalized entropy such that $\Ent(z_n) \to \Ent(b)$ as $n \to \infty$ and $M_{z_n} \simeq M_b$ for $n \geq 1$. 
The proof of Theorem A is constructive. In fact one can describe braids $z_n$ explicitly. For a more general result see Theorems 5.1, 5.2. Let $C \subset H_2(M_0, \partial M_0)$ be the fibered cone containing $[F_0]$. A theorem by Thurston [29] states that for each primitive integral class $a \in C$ there is a connected fiber $F_a$ with the pseudo-Anosov monodromy $\phi_a : F_a \to F_a$ of a fibration on the hyperbolic 3-manifold $M_0$ over $S^1$. The following theorem states a structure of $C$.

**Theorem B.** Suppose that $b$ is a pseudo-Anosov braid whose permutation has a fixed point. Then there are a 2-dimensional subcone $C_0 \subset C$ and an integer $u \geq 1$ with the following properties.

1. The fiber $F_a$ for each primitive integral class $a \in C_0$ has genus 0.
2. The monodromy $\phi_a : F_a \to F_a$ for each primitive integral class $a \in C_0$ is conjugate to

$$((\omega_1 \psi) \cdots (\omega_{u-1} \psi)(\omega_u \psi))^{m-1} : F_a \to F_a,$$

where $m \geq 1$ depends on the class $a$, $\psi$ is periodic and each $\omega_j$ is reducible. Moreover there are homeomorphisms $\tilde{\omega}_j : S_0 \to S_0$ on a surface $S_0$ for $j = 1, \ldots, u$ determined by $b$ and an embedding $h : S_0 \hookrightarrow F_a$ such that $h(S_0)$ is the support of each $w_j$ and

$$w_j|_{h(S_0)} = h \circ \tilde{\omega}_j \circ h^{-1}.$$ 

Theorem B gives a constructive description of $\phi_a$. Also it states that each $w_j : F_a \to F_a$ is reducible supported on a uniformly bounded subsurface $h(S_0) \subset F_a$. It turns out from the proof that the type of the periodic homeomorphism $\psi : F_a \to F_a$ does not depend on $a \in C_0$ (Remark 3.3), see Figure 3(1). Theorem B reminds us of the symmetry conjecture in [23] by Farb-Leininger-Margalit.

Clearly the permutation of each pure braid has a fixed point. For any pseudo-Anosov braid $b$, a suitable power $b^k$ becomes a pure braid and one can apply Theorems A, B for $b^k$.

We have a remark about Theorem A. Theorem 10.2 in [25] by McMullen also tells us the existence of a sequence $(F_n, \phi_n)$ of fibers and monodromies in $C$ such that $\text{Ent}(\phi_n) \to \text{Ent}(b)$ as $n \to \infty$ and $|\chi(F_n)| \asymp n$. However one can not appeal his theorem for the genera of fibers $F_n$. Theorem A says that $F_n$ has genus 0 in fact.

As an application we will determine asymptotic behaviors of the minimal dilatations of a subset of $B_n$ consisting of braids with a symmetry. A braid $b \in B_n$ is *palindromic* if $\text{rev}(b) = b$, where $\text{rev} : B_n \to B_n$ is a map such that if $w$ is a word of letters $\sigma_j^{\pm 1}$ representing $b$, then $\text{rev}(b)$ is the braid obtained from $b$ reversing the order of letters in $w$. A braid $b \in B_n$ is *skew-palindromic* if $\text{skew}(b) = b$, where $\text{skew}(b) = \Delta \text{rev}(b) \Delta^{-1}$ and $\Delta$ is a half twist (Section 2.2). See Figure 4. We will prove that dilatations of palindromic braids have the following lower bound.

**Theorem C.** If $b \in B_n$ is palindromic and pseudo-Anosov for $n \geq 3$, then

$$\lambda(b) \geq \sqrt{2 + \sqrt{5}}.$$

In contrast with palindromic braids we have the following result.
Theorem D. Let $PA_n$ be the set of skew-palindromic elements in $B_n$. We have

$$\log \delta(PA_n) \asymp \frac{1}{n}.$$

The hyperelliptic mapping class group $H(\Sigma_g)$ is the subgroup of $\text{Mod}(\Sigma_g)$ consisting of elements with representative homeomorphisms that commute with some fixed hyperelliptic involution $I : \Sigma_g \to \Sigma_g$ as in Figure 5(1). It is shown in [16] that $\log \delta(H(\Sigma_g)) \asymp 1/g$. See also [7, 15, 19] for other subgroups of $\text{Mod}(\Sigma_g)$. As an application we will determine the asymptotic behavior of the minimal dilatations of the odd/even spin mapping class groups of genus $g$. To define these subgroups let $(\cdot, \cdot)_2$ be the mod-2 intersection form on $H_1(\Sigma_g; \mathbb{Z}_2)$. A map $q : H_1(\Sigma_g; \mathbb{Z}_2) \to \mathbb{Z}_2$ is a quadratic form if $q(v + w) = q(v) + q(w) + (v, w)_2$ for $v, w \in H_1(\Sigma_g; \mathbb{Z}_2)$. For a quadratic form $q$, the spin mapping class group $\text{Mod}_g[q]$ is the subgroup of $\text{Mod}(\Sigma_g)$
We have seen Berrick-Gebhardt-Paris [2].

A result of Dye [5] tells us that $\text{Mod}_{g}[q]$ and $\text{Mod}_{g}[q_1]$ in $\text{Mod}(\Sigma_g)$. We call $\text{Mod}_{g}[q_0]$ and $\text{Mod}_{g}[q_1]$ the even spin and odd spin mapping class group respectively. It is known that $\text{Mod}_{g}[q_1]$ attains the minimum index for a proper subgroup of $\text{Mod}(\Sigma_g)$ and $\text{Mod}_{g}[q_0]$ attains the secondary minimum, see Berrick-Gebhardt-Paris [2].

**Theorem E.** We have

1. $\log(\delta(\text{Mod}_{g}[q_1] \cap \mathcal{H}(\Sigma_g))) \geq \frac{1}{g}$ and
2. $\log(\delta(\text{Mod}_{g}[q_0] \cap \mathcal{H}(\Sigma_g))) \geq \frac{1}{g}$.

In particular $\log(\delta(\text{Mod}_{g}[q]) \geq 1/g$ for each quadratic form $q$.

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2. Preliminaries

2.1. **Links.** Let $L$ be a link in the 3-sphere $S^3$. Let $\mathcal{N}(L)$ denote a tubular neighborhood of $L$ and let $\mathcal{E}(L)$ denote the exterior of $L$, i.e. $\mathcal{E}(L) = S^3 \setminus \text{int}(\mathcal{N}(L))$.

Oriented links $L$ and $L'$ in $S^3$ are equivalent, denoted by $L \sim L'$ if there is an orientation preserving homeomorphism $f : S^3 \rightarrow S^3$ such that $f(L) = L'$ with respect to the orientations of the links. Furthermore for components $K_i$ of $L$ and $K'_i$ of $L'$ with $i = 1, \ldots, m$ if $f$ satisfies $f(K_i) = K'_i$ for each $i$, then $(L, K_1, \ldots, K_m)$ and $(L', K'_1, \ldots, K'_m)$ are equivalent and we write

$$(L, K_1, \ldots, K_m) \sim (L', K'_1, \ldots, K'_m).$$

2.2. **Braid groups $B_n$ and spherical braid groups $SB_n$.** Let $\delta_j = \sigma_1 \sigma_2 \cdots \sigma_{j-1}$ and $\rho_j = \sigma_1 \sigma_2 \cdots \sigma_{j-2} \sigma_{j-1}^2$. The half twist $\Delta_j$ is given by $\Delta_j = \delta_j \delta_{j-1} \cdots \delta_2$. We often omit the subscript $n$ in $\Delta_n$, $\delta_n$, and $\rho_n$ when they are precisely $n$-braids.

We put indices $1, 2, \ldots, n$ from left to right on the bottoms of strands, and give an orientation of strands from the bottom to the top (Figure 1). The closure $\text{cl}(b)$ is oriented by the strands. We think of $\text{br}(b) = \text{cl}(b) \cup A$ as an oriented link in $S^3$ choosing an orientation of $A = A_b$ arbitrarily. (In Section 3 we assign an orientation of the braid axis for $i$-monotonic braids).

If two braids are conjugate to each other, then their braided links are equivalent. Morton proved that the converse holds if their axises are preserved.

**Theorem 2.1.** (Morton [26]). If $(\text{br}(b), A_b)$ is equivalent to $(\text{br}(c), A_c)$ for braids $b, c \in B_n$, then $b$ and $c$ are conjugate in $B_n$.

Let us turn to the spherical braid group $SB_n$ with $n$ strands. We also denote by $\sigma_i$, the element of $SB_n$ as shown in Figure 1(1). The group $SB_n$ is generated
by $\sigma_1, \ldots, \sigma_{n-1}$. For a braid $b \in B_n$ represented by a word of letters $\sigma_j^{\pm 1}$, let $S(b)$ denote the element in $SB_n$ represented by the same word as $b$.

For a braid $b$ in $B_n$ or $SB_n$ the *degree* of $b$ means the number $n$ of the strands, denoted by $d(b)$.

2.3. **Mapping classes and mapping tori from braids.** Let $D_n$ be the $n$-punctured disk. Consider the mapping class group $\text{Mod}(D_n)$, the group of isotopy classes of orientation preserving self-homeomorphisms on $D_n$ preserving the boundary $\partial D$ of the disk setwise. We have a surjective homomorphism

$$\Gamma : B_n \to \text{Mod}(D_n)$$

which sends each generator $\sigma_i$ to the right-handed half twist $t_i$ between the $i$th and $(i+1)$st punctures. The kernel of $\Gamma$ is an infinite cyclic group generated by the full twist $\Delta^2$.

Collapsing $\partial D$ to a puncture in the sphere we have a homomorphism

$$\varsigma : \text{Mod}(D_n) \to \text{Mod}(\Sigma_{0,n+1})$$

We say that $b \in B_n$ is *periodic* (resp. *reducible, pseudo-Anosov*) if $f_b := \varsigma(\Gamma(b))$ is of the corresponding Nielsen-Thurston type. The braids $\delta, \rho \in B_n$ are periodic since some power of each braid is the full twist: $\Delta^2 = \delta^n = \rho^{n-1} \in B_n$.

We also have a surjective homomorphism

$$\hat{\Gamma} : SB_n \to \text{Mod}(\Sigma_{0,n})$$

sending each generator $\sigma_i$ to the right-handed half twist $t_i$. We say that $\eta \in SB_n$ is *pseudo-Anosov* if $\hat{\Gamma}(\eta) \in \text{Mod}(\Sigma_{0,n})$ is pseudo-Anosov. In this case $\lambda(\eta)$ is defined by the dilatation of $\hat{\Gamma}(\eta)$.

2.4. **Stable foliations $\mathcal{F}_b$ for pseudo-Anosov braids $b$.**

Recall the surjective homomorphism $\pi : B_n \to S_n$. We write $\pi_b = \pi(b)$ for $b \in B_n$. Consider a pseudo-Anosov braid $b \in B_n$ with $\pi_b(i) = i$. Removing the $i$th strand $b(i)$ from $b$, we get a braid $b - b(i) \in B_{n-1}$. Taking its spherical element, we have $S(b - b(i)) \in SB_{n-1}$. Note that $b - b(i)$ and $S(b - b(i))$ are not necessarily pseudo-Anosov.

A well-known criterion uses the stable foliation $\mathcal{F}_b$ for the monodromy $\phi_b : F_b \to F_b$ of a fibration on $M_b \to S^1$ as we recall now. Such a fibration on $M_b$ extends naturally to a fibration on the manifold obtained from $M_b$ by Dehn filling a cusp along the boundary slope of the fiber $F_b$ which lies on the torus $\partial N(\text{cl}(b(i)))$. Also $\phi_b$ extends to the monodromy defined on $F_b^*$ of the extended fibration, where $F_b^*$ is obtained from $F_b$ by filling in the boundary component of $F_b$ which lies on $\partial N(\text{cl}(b(i)))$ with
a disk. Then $b - b(i)$ is the corresponding braid for the extended monodromy defined on $F^\bullet$. Suppose that $F_b$ is not 1-pronged at the boundary component in question. (See Figure 6 in the case where $F_b$ is 1-pronged at a boundary component.) Then $F_b$ extends to the stable foliation for $b - b(i)$, and hence $b - b(i)$ is pseudo-Anosov with the same dilatation as $b$. Furthermore if $F_b$ is not 1-pronged at the boundary component of $F_b$ which lies on $\partial N(A)$, then $S(b - b(i))$ is still pseudo-Anosov with the same dilatation as $b$.

2.5. Thurston norm. Let $M$ be a 3-manifold with boundary (possibly $\partial M = \emptyset$). If $M$ is hyperbolic, i.e. the interior of $M$ possess a complete hyperbolic structure of finite volume, then there is a norm $\| \cdot \|$ on $H_2(M, \partial M; \mathbb{R})$, now called the Thurston norm [29]. The norm $\| \cdot \|$ has the property such that for any integral class $a \in H_2(M, \partial M; \mathbb{R})$, $\|a\| = \min_S \{ -\chi(S) \}$, where the minimum is taken over all oriented surface $S$ embedded in $M$ with $a = [S]$ and with no components of non-negative Euler characteristic. The surface $S$ realizing this minimum is called a norm-minimizing surface of $a$.

**Theorem 2.2** (Thurston [29]). The norm $\| \cdot \|$ on $H_2(M, \partial M; \mathbb{R})$ has the following properties.

1. There are a set of maximal open cones $C_1, \ldots, C_k$ in $H_2(M, \partial M; \mathbb{R})$ and a bijection between the set of isotopy classes of connected fibers of fibrations $M \to S^1$ and the set of primitive integral classes in the union $C_1 \cup \cdots \cup C_k$.

2. The restriction of $\| \cdot \|$ to $C_j$ is linear for each $j$.

3. If we let $F_a$ be a fiber of a fibration $M \to S^1$ associated with a primitive integral class $a$ in each $C_j$, then $\|a\| = -\chi(F_a)$.

We call the open cones $C_j$ fibered cones and call integral classes in $C_j$ fibered classes.

**Theorem 2.3** (Fried [11]). For a fibered cone $C$ of a hyperbolic 3-manifold $M$, there is a continuous function $\text{Ent} : C \to \mathbb{R}$ with the following properties.

1. For the monodromy $\phi_a : F_a \to F_a$ of a fibration $M \to S^1$ associated with a primitive integral class $a \in C$, we have $\text{Ent}(a) = \log(\lambda(a))$.

2. $\text{Ent} = \| \cdot \| : C \to \mathbb{R}$ is a continuous function which becomes constant on each ray through the origin.

3. If a sequence $\{a_n\} \subset C$ tends to a point $\neq 0$ in the boundary $\partial C$ as $n$ tends to $\infty$, then $\text{Ent}(a_n) \to \infty$. In particular $\text{Ent}(a_n) = \|a_n\| \text{Ent}(a_n) \to \infty$.

We call $\text{Ent}(a)$ and $\text{Ent}(a)$ the entropy and normalized entropy of the class $a \in C$.

For a pseudo-Anosov element $\phi \in \text{Mod}(\Sigma)$ we consider the mapping torus $M_\phi$. The vector field $\partial / \partial t$ on $\Sigma \times \mathbb{R}$ induces a flow $\phi^t$ on $M_\phi$ called the suspension flow.

**Theorem 2.4** (Fried [10]). Let $\phi$ be a pseudo-Anosov mapping class defined on $\Sigma$ with stable and unstable foliations $F^s$ and $F^u$. Let $F^s$ and $F^u$ denote the suspensions of $F^s$ and $F^u$ by $\phi$. If $C$ is a fibered cone containing the fibered class $[\Sigma]$, then we can modify a norm-minimizing surface $F_a$ associated with each primitive integral class $a \in C$ by an isotopy on $M_\phi$ with the following properties.

1. $F_a$ is transverse to the suspension flow $\phi^t$, and the first return map $\phi_a : F_a \to F_a$ is precisely the pseudo-Anosov monodromy of the fibration on $M_\phi \to S^1$ associated with $a$. Moreover $F_a$ is unique up to isotopy along flow lines.
2.6. Disk twist. Let $L$ be a link in $S^3$. Suppose an unknot $K$ is a component of $L$. Then the exterior $\mathcal{E}(K)$ (resp. $\partial \mathcal{E}(K)$) is a solid torus (resp. torus). We take a disk $D$ bounded by the longitude of a tubular neighborhood $N(K)$ of $K$. We define a mapping class $T_D$ defined on $\mathcal{E}(K)$ as follows. We cut $\mathcal{E}(K)$ along $D$. We have resulting two sides obtained from $D$, and reglue two sides by twisting either of the sides 360 degrees so that the mapping class defined on $\partial \mathcal{E}(K)$ is the right-handed Dehn twist about $\partial D$. Such a mapping class on $\mathcal{E}(K)$ is called the disk twist about $D$. For simplicity we also call a self-homeomorphism representing the mapping class $T_D$ the disk twist about $D$, and denote it by the same notation $T_D : \mathcal{E}(K) \to \mathcal{E}(K)$.

Clearly $T_D$ equals the identity map outside a neighborhood of $D$ in $\mathcal{E}(K)$. We observe that if $u + 1$ segments of $L - K$ pass through $D$ for $u \geq 1$, then $T_D(L - K)$ is obtained from $L - K$ by adding the full twist near $D$. In the case $u = 1$, see Figure 7. We may assume that $T_D$ fixes one of these segments, since any point in $D$ becomes the center of the twisting about $D$.

For any integer $\ell$, consider a homeomorphism

$$T_D^\ell : \mathcal{E}(K) \to \mathcal{E}(K).$$

Observe that $T_D^\ell$ converts $L$ into a link $K \cup T_D^\ell(L - K)$ such that $S^3 \setminus L$ is homeomorphic to $S^3 \setminus (K \cup T_D^\ell(L - K))$. Then $T_D^\ell$ induces a homeomorphism between the exteriors of links

$$(2.1) \quad h_{D,\ell} : \mathcal{E}(L) \to \mathcal{E}(K \cup T_D^\ell(L - K)).$$

We use the homeomorphism in (2.1) in later section.

3. $i$-increasing braids and Theorem 3.2

Definitions of $i$-increasing braids, signs and intersection numbers. Let $L$ be an oriented link in $S^3$ with a trivial component $K$. We take an oriented disk $D$ bounded by the longitude of $N(K)$ so that the orientation of $D$ agrees with the orientation of $K$. For each component $K'$ of $L - K$ such that $D$ and $K'$ intersect transversally with $D \cap K' \neq \emptyset$, we assign each point of intersection $+1$ or $-1$ as shown in Figure 8.
Let $b$ be a braid with $\pi_b(i) = i$. We consider an oriented disk $D = D_{(b,i)}$ bounded by the longitude $\ell_i$ of $N(\text{cl}(b(i)))$. Such a disk $D$ is unique up to isotopy on $E(\text{cl}(b(i)))$. We say that a braid $b \in B_n$ with $\pi_b(i) = i$ is $i$-increasing (resp. $i$-decreasing) if there is a disk $D = D_{(b,i)}$ as above with the following conditions.

(D1) There is at least one component $K'$ of $\text{cl}(b - b(i))$ such that $D \cap K' \neq \emptyset$.

(D2) Each component of $\text{cl}(b - b(i))$ and $D$ intersect with each other transversally, and every point of intersection has the sign +1 (resp. -1).

We set $\epsilon(b,i) = 1$ (resp. $\epsilon(b,i) = -1$), and call it the sign of the pair $(b,i)$. We also call $D$ the associated disk of the pair $(b,i)$. We say that $b$ is $i$-monotonic if $b$ is $i$-increasing or $i$-decreasing. Then we set

$$I(b,i) = D \cap \text{cl}(b - b(i))$$

and let $u(b,i) \geq 1$ be the cardinality of $I(b,i)$. We call $u(b,i)$ the intersection number of the pair $(b,i)$. If the pair $(b,i)$ is specified, then we simply denote $\epsilon(b,i)$ and $u(b,i)$ by $\epsilon$ and $u$ respectively. For example, $\sigma_1^2 \sigma_2^{-1}$ is 1-increasing with $u(\sigma_1^2 \sigma_2^{-1}, 1) = 1$.

A braid $b$ is positive if $b$ is represented by a word in letters $\sigma_j$, but not $\sigma_j^{-1}$. A braid $b$ is irreducible if the Nielsen-Thurston type of $b$ is not reducible.

**Lemma 3.1.** Let $b$ be a positive braid with $\pi_b(i) = i$. Then $b$ is $i$-increasing if $b$ is irreducible.
Proof. Suppose that a positive braid \( b \) with \( \pi_b(i) = i \) is irreducible. Since \( b \) is positive, there is a disk \( D = D(b, i) \) with the condition (D2). Assume that \( D \) fails in (D1). Let \( \partial D_n \) be the boundary of the disk \( D_n \) containing \( n \) punctures. Consider a neighborhood of \( \partial D_n \cup (D_n \cap D) \) in \( D_n \) which is an annulus. One of the boundary components of this annulus is an essential simple closed curve in \( \Gamma(b) \in \text{Mod}(D_n) \). This means that \( b \) is reducible, a contradiction. Thus \( D \) satisfies (D1), and \( b \) is \( i \)-increasing.

Orientation of the axis \( A \). Let \( b \) be \( i \)-monotonic with \( \epsilon(b, i) = \epsilon \) and \( u(b, i) = u \). Consider the braided link \( \text{br}(b) = \text{cl}(b) \cup A \). The associated disk \( D \) has a unique point of intersection with \( A \), and the cardinality of \( I(b, i) \cup (D \cap A) \) is \( u(b, i) + 1 \). To deal with \( \text{br}(b) = \text{cl}(b) \cup A \) as an oriented link, we consider an orientation of \( \text{cl}(b) \) as we described before, and assign an orientation of \( A \) so that the sign of the intersection between \( D \) and \( A \) coincides with \( \epsilon(b, i) \). See Figure 2(2).

Recall that \( M_b = \mathcal{E} (\text{br}(b)) \) is the exterior of \( \text{br}(b) \) which is a surface bundle over \( S^1 \). We consider an orientation of the \( F \)-surface \( F_b \) which agrees with the orientation of \( A \).

E-surface. We now define an oriented surface \( E_{(b,i)} \) of genus 0 embedded in \( M_b \). Consider small \( u(b, i) + 1 \) disks in the oriented disk \( D = D_b(i) \) whose centers are points of \( I(b, i) \cup (D \cap A) \). Then \( E_{(b,i)} \) is a sphere with \( u(b, i) + 2 \) boundary components obtained from \( D \) by removing the interiors of those small disks. We choose the orientation of \( E_{(b,i)} \) so that it agrees with the orientation of \( D \). We call \( E_{(b,i)} \) the E-surface for \( b \). For example, the 1-increasing braid \( \sigma_1 \sigma_2^{-1} \) has the E-surface \( E_{(\sigma_1 \sigma_2^{-1}, i)} \) homeomorphic to a 3-holed sphere.

Subcone \( C_{(b,i)} \). Let us consider the 2-dimensional subcone \( C_{(b,i)} \) of \( H_2(M_b, \partial M_b; \mathbb{R}) \) spanned by \([F_0]\) and \([E_{(b,i)}]\) (Figure 9):

\[
C_{(b,i)} = \{ x[F_0] + y[E_{(b,i)}] \mid x > 0, \ y > 0 \}.
\]

Let \( \overline{C_{(b,i)}} \) denote the closure of \( C_{(b,i)} \). We write \( (x, y) = x[F_0] + y[E_{(b,i)}] \). We prove the following theorem in Section 4.

**Theorem 3.2.** For a pseudo-Anosov, \( i \)-increasing braid \( b \) with \( u(b, i) = u \), let \( C \) be the fibered cone containing \([F_0]\). We have the following.

1. \( C_{(b,i)} \subset C \).
2. The fiber \( F_{(x,y)} \) for each primitive integral class \((x, y) \in C_{(b,i)} \) has genus 0.
3. The monodromy \( \phi(x,y) : F_{(x,y)} \to F_{(x,y)} \) for each primitive integral class \((x, y) \in C_{(b,i)} \) is conjugate to

\[
(\omega_1 \psi) \cdots (\omega_{u-1} \psi)(\omega_u \psi)^{m-1} : F_{(x,y)} \to F_{(x,y)},
\]

where \( m \geq 1 \) depends on \((x, y) \), \( \psi \) is periodic and each \( \omega_j \) is reducible. Moreover there are homeomorphisms \( \widehat{\omega}_j : S_0 \to S_0 \) for \( j = 1, \ldots, u \) on a surface \( S_0 \) determined by \( b \) and an embedding \( h : S_0 \hookrightarrow F_{(x,y)} \) such that the subsurface \( h(S_0) \) of \( F_{(x,y)} \) is the support of each \( w_j \) and

\[
w_j|_{h(S_0)} = h \circ \widehat{\omega}_j \circ h^{-1}.
\]

The conclusion of Theorem 3.2 holds for \( i \)-decreasing braids as well. We now claim that Theorem 3.2 implies Theorem B.
Proof of Theorem B. Suppose that Theorem 3.2 holds. Let \( b \in B_n \) be a pseudo-Anosov braid such that \( \tau_b(i) = i \). We consider the braid \( b\Delta^{2k} \in B_n \) for \( k \geq 1 \). The full twist \( \Delta^2 \) is an element in the center \( Z(B_n) \) and \( \Delta^2 = \sigma_j P_j \) holds for each \( 1 \leq j \leq n - 1 \), where \( P_j \) is positive. Such properties imply that \( b\Delta^{2k} \) is positive for \( k \) large. We fix such large \( k \). Since \( \Gamma(b) = \Gamma(b\Delta^{2k}) \) in \( \text{Mod}(D_n) \), the braid \( b\Delta^{2k} \) is certainly pseudo-Anosov. Hence it is \( i \)-increasing by Lemma 3.1. One can apply Theorem 3.2 for this braid, and obtains the subcone \( C(b\Delta^{2k};i) \).

Remark 3.3. If \( F(x,y) \) is a \((d+1)\)-holed sphere, then the periodic homeomorphism \( \psi : F(x,y) \rightarrow F(x,y) \) in Theorem 3.2 is determined by the periodic braid \( \rho = \sigma_1 \sigma_2 \ldots \sigma_{d-2} \sigma_{d-1}^2 \in B_d \). See the proof of Theorem 3.2(3) in Section 4.3.

4. Proof of Theorem 3.2

We fix integers \( n \geq 3 \) and \( 1 \leq i \leq n \). Throughout Section 4, we assume that \( b \in B_n \) is pseudo-Anosov and \( i \)-increasing with \( u(b,i) = u \). We now choose an associated disk about the pair \((b,i)\) suitably. Let \( D \) denote the unit disk with the center \((0,0)\) in the plane \( \mathbb{R}^2 \). Let \( J = (-1, 1) \times \{0\} \subset \mathbb{D} \) be the interval and let \( A_0 = (-2, 0) \) be a point in \( \mathbb{R}^2 \). We denote by \( \mathbb{D}_n \), the disk \( \mathbb{D} \) with equally spaced \( n \) points in \( J \). Let us denote these \( n \) points by \( A_1, \ldots, A_n \) from left to right. We take a point \( Q_i \neq A_i \in J \) between \( A_{i-1} \) and \( A_i \) so that the Euclidean distance
Figure 11. Case: $b$ is $i$-increasing. (1) Associated disk $D$ with conditions $\diamondsuit 1,2,3$. (2) $\text{br}(b_1)$. Circles $\circ$ indicate points of intersection between $D$ and components of $\text{br}(b - b(i))$. See also Figure 12.

d($Q_i, A_i$) is sufficiently small (e.g. $d(Q_i, A_i) < \frac{1}{n + 1}$). Let $r_i$ denote the closed interval in $[-2,1] \times \{0\}$ with endpoints $A_0$ and $Q_i$. (Figure 10(1).) We regard $b$ as a braid contained in the cylinder $\mathbb{D}^2 \times [0,1] \subset \mathbb{R}^3$ and $b$ is based at $n$ points $A_1 \times \{0\}, \ldots, A_n \times \{0\}$. Since $\pi_b(t) = i$, one can take a representative of $b$ such that $b(i)$ is an interval in the cylinder.

$\diamondsuit 1. \ b(i) = \bigcup_{0 \leq t \leq 1} A_i \times \{t\}.$

Furthermore we may assume that $\partial D(= \ell_i)$ of an associated disk $D$ of $(b,i)$ is a union of the following four segments as a set (Figure 10):

$\diamondsuit 2. \ (\bigcup_{-1 \leq t \leq 2} A_0 \times \{t\}) \cup (r_i \times \{-1\}) \cup (\bigcup_{-1 \leq t \leq 2} Q_i \times \{t\}) \cup (r_i \times \{2\}).$

Preserving $\diamondsuit 1, 2$ we may further assume the following (Figures 10(2), 11(1)):

$\diamondsuit 3. \ For \ a \ regular \ neighborhood \ U_i \ of \ \ell_i \ in \ D, \ we \ have \ I(b(i), U_i.$

This is because every point $x \in D \cap K'$, where $K'$ is a component of $\text{cl}(b - b(i))$, one can slide $x$ along $K'$ so that the resulting point on $K'$ is in $U_i$. Said differently, preserving $\partial D$ pointwise, we can modify a small neighborhood of $D$ near $K'$ so that the resulting associated disk satisfies $\diamondsuit 3$.

Under the conditions $\diamondsuit 1, 2, 3$ we have the following. For each $x \in D \cap K' \subset U_i$, there is a segment $s' \subset K'$ through $x$ such that $s'$ passes over $b(i)$ since $b$ is $i$-increasing. See Figure 11(1). Such a local picture of $\text{cl}(b)$ is used in the next section. Hereafter we assume that associated disks possess conditions $\diamondsuit 1, 2, 3$.

4.1. Proof of Theorem 3.2(1). Let $s$ be the open segment in $H_2(M_b, \partial M_b; \mathbb{R})$ with the endpoints $\frac{n - 1}{u}[E_{(b,i)}] = (0, \frac{n - 1}{u})$ and $[F_0] = (1,0)$:

$$(4.1) \quad s = \{(x,y) \in C_{(b,i)} \mid y = -\frac{n - 1}{u}x + \frac{n - 1}{u}, \ 0 < x < 1\}.$$ 

The ray of each point in $C_{(b,i)}$ through the origin intersects with $s$. Thus for the proof of (1), it suffices to prove that $s \subset C$.
We now introduce a sequence of braided links \{\text{br}(b_p)\}_{p=1}^{\infty} from an \(i\)-increasing braid \(b \in B_n\) such that \(M_{b_p} \simeq M_b\) for each \(p \geq 1\). (We use the 1-increasing braid \(\sigma_1^2\sigma_2^{-1} \in B_3\) to illustrate the idea.) Let \(D\) be an associated disk of the pair \((b,i)\). We take a disk twist

\[ T_D : \mathcal{E}(\text{cl}(b(i))) \to \mathcal{E}(\text{cl}(b(i))) \]

so that the point of intersection \(D \cap A\) becomes the center of the twisting about \(D\), i.e. \(T_D(D \cap A) = D \cap A\). We may assume that \(T_D(A) = A\) as a set. Figure 11 illustrates the image of the segment \(s'\) under \(T_D\). The condition \(\triangle 3\) ensures that \(T_D\) equals the identity map outside a neighborhood of \(U_i\) in \(\mathcal{E}(\text{cl}(b(i)))\). Then by \(\triangle 1, 2\), it follows that

\[ T_D(\text{br}(b - b(i))) \cup \text{cl}(b(i)) \]

is a braided link of some \((i + u)\)-increasing braid with \((n + u)\) strands. We define \(b_1 \in B_{n+u}\) to be such a braid. The trivial knot \(T_D(A)(= A)\) becomes a braid axis of \(b_1\). By definition of the disk twist, we have \(M_{b_1} \simeq M_b\). See Figure 12 for \(\text{br}(\sigma_1^2\sigma_2^{-1})_1\).

As discussed below, there is some ambiguity in defining \(b_1\). As we will see, the ambiguity is irrelevant for the study of pseudo-Anosov monodromies defined on fibers of fibrations on the mapping torus. Suppose that both \(D\) and \(D'\) are the associated disks of the pair \((b,i)\) with conditions \(\triangle 1, 2, 3\). We consider the disk twists \(T_D\) and \(T_{D'}\) with the above condition, i.e. both \(D \cap A\) and \(D' \cap A\) become the center of the twisting about \(D\) and \(D'\) respectively. Observe that the resulting two links obtained from \(D\) and \(D'\) are equivalent:

\[ T_D(\text{br}(b - b(i))) \cup \text{cl}(b(i)) \sim T_{D'}(\text{br}(b - b(i))) \cup \text{cl}(b(i)). \]

They are braided links, say \(\text{br}(b_1)\) and \(\text{br}(b'_1)\) of some braids \(b_1, b'_1 \in B_{n+u}\) respectively with the same axis \(T_D(A) = A = T_{D'}(A)\). This means that a more stronger claim holds:

\[ (\text{br}(b_1), A) \sim (\text{br}(b'_1), A). \]

Thus \(b_1\) and \(b'_1\) are conjugate in \(B_{n+u}\) by Theorem 2.1. In particular both \(b_1\) and \(b'_1\) are pseudo-Anosov (since the initial braid \(b\) is pseudo-Anosov and \(M_b\) is hyperbolic) and they have the same dilatation.
To define \( b_p \) for \( p \geq 1 \), we consider the \( p \)th power

\[
T_p^b : \mathcal{E} \text{cl}(b(i)) \to \mathcal{E} \text{cl}(b(i))
\]

using the above \( T_D \). As in the case of \( p = 1 \),

\[
T_p^b(\text{br}(b - b(i))) \cup \text{cl}(b(i))
\]

is a braided link of some \((i + pu)\)-increasing braid with \((n + pu)\) strands. We define \( b_p \in B_{n+pu} \) to be such a braid. Then \( M_{b_p} \simeq M_b \). As in the case of \( p = 1 \), such a braid \( b_p \) is well-defined up to conjugate. We say that \( b_p \) is obtained from \( b \) by the disk twist. Clearly \( u(b_p, i + pu) = u(b, i) \) for \( p \geq 1 \). See Figure 12.

Let us set

\[
g_p : = h_{D,p} : M_b \to M_{b_p},
\]

where \( h_{D,p} \) is the homeomorphism in (2.1). We consider the isomorphism

\[
g_{p*} : H_2(M_b, \partial M_b) \to H_2(M_{b_p}, \partial M_{b_p}).
\]

**Lemma 4.1.** For each integer \( p \geq 1 \), \( g_{p*} \) sends \((0, 1) \in \mathcal{C}_{(b, i)}^{(b, i)} \) to \((0, 1) \in \mathcal{C}_{(b_p, i + pu)}^{(b_p, i + pu)} \), and sends \((1, p) \in \mathcal{C}_{(b, i)}^{(b, i)} \) to \((1, 0) \in \mathcal{C}_{(b_p, i + pu)}^{(b_p, i + pu)} \). In particular for integers \( x,y \geq 1 \) with \( y = xp + r \) for \( 0 \leq r < p \), \( g_{p*} \) sends \((x,y) \in \mathcal{C}_{(b, i)}^{(b, i)} \) to \((x,r) \in \mathcal{C}_{(b_p, i + pu)}^{(b_p, i + pu)} \).

**Proof.** We consider the oriented sum \( F_{x,y} := xF_b + yE_{(b,i)} \). This is an oriented surface embedded in \( M_b \), and is obtained from the cut and paste construction of parallel \( x \) copies of \( F_b \) and parallel \( y \) copies of \( E_{(b,i)} \). The orientation of \( F_{x,y} \) agrees with those of \( F_b \) and \( E_{(b,i)} \). We have \([F_{x,y}] = (x,y) \in \mathcal{C}_{(b, i)}^{(b, i)} \). Then \( g_p \) sends \( E_{(b,i)} \) to \( E_{(b_p, i + pu)} \), and sends \( F_{1,p} \) to \( F_{b_p} \). Thus \( g_{p*} \) sends \((0, 1) \) to \((0, 1) \), and sends \((1, p) \) to \((1, 0) \). This completes the proof. \( \square \)

By the proof of Lemma 4.1, \( g_1 \) sends \( F_{(1,1)} = F_b + E_{(b,i)} \) to the fiber \( F_{b_1} \) of a fibration on \( M_b \) associated with \((1, 1) \in \mathcal{C}_{(b, i)}^{(b, i)} \). Since the fibers \( F_{(1,1)} \) and \( F_b \) are norm-minimizing, \( E_{(b,i)} \) is also norm-minimizing.

**Proof of Theorem 3.2(1).** We have \([F_b] = n - 1 \) and \([F_{b_p}] = n + pu - 1 \) since \( F_b \) and \( F_{b_p} \) are fibers, and \([E_{(b,i)}] = u \) since \( E_{(b,i)} \) is norm-minimizing. By Lemma 4.1, \([F_{b_p}] = (1, p) \in \mathcal{C}_{(b, i)}^{(b, i)} \). Consider the rational class

\[
c_p : = \frac{n - 1}{n + pu - 1} [F_{b_p}] = \left( \frac{n - 1}{n + pu - 1}, \frac{p(n - 1)}{n + pu - 1} \right).
\]

Then \( \|c_p\| = n - 1 \) for \( p \geq 1 \). The ray of \([F_{b_p}] \) through the origin is contained in some fibered cone for each \( p \geq 1 \). We easily check that \( c_p \) lies on \( s \) in (4.1). This means that three classes \([F_b], c_p \) and \( c_{p+1} \) with the same Thurston norm are contained in \( \mathcal{C} \). Observe that the small segment \( s' \) in \( s \) connecting \([F_b] \) and \( c_{p+1} \) contains \( c_p \), and \( s' \subset \mathcal{C} \) since \( \| \cdot \| \) is linear on each fibered cone. Moreover \( c_p \to (0, \frac{n-1}{n}) \in \partial s \subset \partial \mathcal{C}_{(b, i)}^{(b, i)} \) as \( p \to \infty \). Putting all things together, we conclude that \( s \subset \mathcal{C} \). This completes the proof. \( \square \)

**Remark 4.2.** From the proof of Theorem 3.2(1), one sees the following: If \([E_{(b,i)}] \in \mathcal{C}_{(b,i)}^{(b, i)} \) is a fibered class, then \([E_{(b,i)}] \in \mathcal{C} \). Otherwise \([E_{(b,i)}] \in \partial \mathcal{C} \). See Figure 9(2)(3).
4.2. Proof of Theorem 3.2(2). We start with a simple observation: $\Delta^2 \in B_n$ is $j$-increasing for each $1 \leq j \leq n$, and $u(\Delta^2, j) = n - 1$ holds. The following lemma is immediate.

Lemma 4.3. If $b \in B_n$ is $i$-increasing, then $b\Delta^2 \in B_n$ is $i$-increasing with $u(b\Delta^2, i) = u(b, i) + n - 1$.

We explain the idea of Theorem 3.2(2). Let $D$ be the associated disk of the pair $(b, i)$. We have two types of the disk twist. One is $T^b_{D_i} : \mathcal{E}(A) \to \mathcal{E}(A)$ which appears in the proof of Theorem B in Section 3 and the other is $T^p_{D_i} : \mathcal{E}(\mathrm{cl}(b(i))) \to \mathcal{E}(\mathrm{cl}(b(i)))$. If $k$ and $p$ are positive, then we obtain the $i$-increasing $b\Delta^{2k}$ from the former type $T^b_{D_i}$, and another increasing braid $b_p$ from the latter type $T^p_{D_i}$. Since both resulting braids are increasing, we can further apply two types of the disk twist for the resulting braid. This is a key of the proof. Choosing two types of the disk twist alternatively, we get a sequence of increasing and pseudo-Anosov braids (since the initial braid $b$ is pseudo-Anosov). We shall see that the desired monodromies associated with primitive classes in $C_{(b,i)}$ are given by these braids.

Let $p_1, \ldots, p_j$ be integers such that $p_1 \geq 0$ and $p_2, \ldots, p_j \geq 1$. Given an $i$-increasing braid $b \in B_n$ with $u(b, i) = u$, we define an integer $i[p_1, \ldots, p_j] \geq 1$ and an $i[p_1, \ldots, p_j]$-increasing braid $b[p_1, \ldots, p_j]$ inductively as follows.

- If $j = 1$ and $p_1 = 0$, then $i[0] = i$ and $b[0] = b$. If $j = 1$ and $p_1 = p \geq 1$, then $i[p] = i + pu$ and $b[p] = b_p$.
- If $j > 1$ is even, then
  
  
  \[
  i[p_1, \ldots, p_{j-1}, p_j] = i[p_1, \ldots, p_{j-1}], \\
  b[p_1, \ldots, p_{j-1}, p_j] = (b[p_1, \ldots, p_{j-1}])^{\Delta^{2p_j}}.
  \]

  The right-hand side is $i[p_1, \ldots, p_{j-1}]$-increasing by Lemma 4.3.

- If $j > 1$ is odd, then
  
  \[
  i[p_1, \ldots, p_{j-1}, p_j] = i[p_1, \ldots, p_{j-1}] + p_j u(b[p_1, \ldots, p_{j-1}], i[p_1, \ldots, p_{j-1}]), \\
  b[p_1, \ldots, p_{j-1}, p_j] = (b[p_1, \ldots, p_{j-1}])^{p_j}.
  \]

We say that $b[p_1, \ldots, p_j]$ has length $j$.

Example 4.4.

1. $b[p] = b_p$ by definition.
2. Let $\beta = b\Delta^2$. Then $b[0, 1] = \beta$ and $b[0, 1, p] = \beta_p$.
3. We have $b[0, p] = b\Delta^{2p}$ and $b[0, p, 1] = (b\Delta^{2p})_1$, where $(b\Delta^{2p})_1$ is obtained from $i$-increasing $b\Delta^{2p}$ by the disk twist.

For each $k \geq 1$, let $f_k : M_b \to M_{b\Delta^{2k}}$ be the homeomorphism which in the proof of Theorem B. Consider the isomorphism $f_{k*} : H_2(M_b, \partial M_b) \to H_2(M_{b\Delta^{2k}}, \partial M_{b\Delta^{2k}})$. We have the following property.

Lemma 4.5. For each integer $k \geq 1$, $f_{k*}$ sends $(1, 0) \in C_{(b,i)}$ to $(1, 0) \in C_{(b\Delta^{2k},i)}$, and sends $(k, 1) \in C_{(b,i)}$ to $(0, 1) \in C_{(b\Delta^{2k},i)}$. In particular for integers $x, y \geq 1$ with $x = yk + r$ for $0 \leq r < k$, then $f_{k*}$ sends $(x, y) \in C_{(b,i)}$ to $(r, y) \in C_{(b\Delta^{2k},i)}$.

Proof. The homeomorphism $f_k$ sends $F_0$ to $F_{b\Delta^{2k}}$, and sends $F_{(k,1)} = kF_b + E_{(b,i)}$ to $E_{(b\Delta^{2k},i)}$. This implies that the claim holds. \(\square\)
Proof of Theorem 3.2(2). Let \((x, y) \in C_{(b,i)}\) be a primitive integral class. (Hence \(x, y\) are positive integers with \(\gcd(x, y) = 1\).) We consider the continued fraction of \(y/x\) by the Euclidean algorithm
\[
\frac{y}{x} = p_1 + \frac{1}{p_2 + \frac{1}{p_3 + \cdots + \frac{1}{p_j}}} := p_1 + \frac{1}{p_2 + p_3 + \cdots + p_{j-1} + p_j}
\]
with length \(j\) and \(p_j \geq 2\) and \(p_1 = 0\) if \(0 < y < x\). There is another expression
\[
\frac{y}{x} = p_1 + \frac{1}{p_2 + p_3 + \cdots + p_{j-1} + (p_j - 1) + 1}
\]
with length \(j + 1\). We choose one of the two expressions with odd length \(\ell\):
\[
\frac{y}{x} = p_1 + \frac{1}{p_2 + p_3 + \cdots + p_{\ell-1} + p_{\ell}}.
\]
This encodes the fiber \(F(x,y)\) and its monodromy \(\phi_{(x,y)}\). In fact Lemmas 4.1, 4.5 ensure that
\[
(g_{p_1}^{p_1} g_{p_2-1} \cdots g_{p_\ell} g_{p_\ell})_* : H_2(M_b, \partial M_b) \to H_2(M_b[p_1, \ldots, p_\ell], \partial M_b[p_1, \ldots, p_\ell])
\]
sends \((x, y) = [xF_b + yE_{(b,i)}] \to (1, 0)\) which is the integral class of the \(F\)-surface of \(b[p_1, \ldots, p_\ell]\). \((g_{p_1} = \text{id} : M_b \to M_b\) if \(p_1 = 0\). Thus \(F(x,y)\) has genus 0. Moreover this means that one can take \(F_{b[p_1, \ldots, p_\ell]}\) as a representative of \((x, y) \in C_{(b,i)}\) and the monodromy \(\phi_{(x,y)} : F_{(x,y)} \to F_{(x,y)}\) is determined by \(b[p_1, \ldots, p_\ell]\). This completes the proof.

We denote by \(b_{(x,y)}\) the braid \(b[p_1, \ldots, p_\ell]\) which determines \(\phi_{(x,y)}\). Here is an example: If \((x, y) = (5, 14)\), then \(\frac{14}{5} = 2 + \frac{4}{1 + \frac{1}{2}}\) and \(\phi_{(5,14)}\) is determined by \(b_{(5,14)} = b[2, 1, 4]\). If \((x, y) = (14, 5)\), then \(\frac{5}{14} = 0 + \frac{1}{2 + \frac{1}{3 + \frac{1}{3 + 1}}}\) and \(\phi_{(14,5)}\) is determined by \(b_{(14,5)} = b[0, 2, 1, 3, 1]\).

4.3. Proof of Theorem 3.2(3). We begin with the following lemma.

Lemma 4.6 (Standard form). If \(b \in B_n\) is \(i\)-increasing with \(u(b, i) = u\), then \(b\) is conjugate to an \(n\)-increasing braid \(b'\) of the form
\[
b' = (w_1 \sigma_{n-1} \sigma_{n-2}) \cdots (w_n \sigma_{n-1}),
\]
where each \(w_k\) is a word of \(\sigma_1^{\pm1}, \ldots, \sigma_{n-2}^{\pm1}\), but not \(\sigma_{n-1}^{\pm1}\), possibly \(w_k = \emptyset\) for some \(k\).

Figure 13(1) shows the form of \(b'\) in Lemma 4.6 in case \(u = 2\).

Proof. We regard \(b\) as a braid in \(\mathbb{D} \times [0, 1]\). By \(\Diamond 1\), \(b(i)\) is an interval in \(\mathbb{D} \times [0, 1]\). If \(i = n\), then \(b\) is \(n\)-increasing and it is not hard to see that a representative of \(b\) is of the desired form in Lemma 4.6. Suppose that \(b\) is \(i\)-increasing for \(1 \leq i < n\). We set \(\sigma = \sigma_{n-1} \sigma_{n-2} \cdots \sigma_i\) if \(1 \leq i < n-1\) and \(\sigma = \sigma_{n-1}\) if \(i = n-1\). We consider the \(n\)-braid \(b' = \sigma b \sigma^{-1}\) which is \(n\)-increasing with \(u(b', n) = u\). We pull \(b'(n)\) tight in \(\mathbb{D} \times [0, 1]\) and make it straight. Then a representative of \(b'\) is of the desired form. \(\Box\)
Since each $b$ generates \{b\}. (1) $b = w_1\sigma_2^2w_2\sigma_2^2 = (\nu_1\rho)(\nu_2\rho) \in B_4$, where $\nu_j = w_j(\sigma_1\sigma_2)^{-1}$.
(2) $b_1 = (\nu_1\rho)(\nu_2\rho) \in B_6$. (3) $b_2 = (\nu_1\rho)(\nu_2\rho) \in B_3$.

Proof of Theorem 3.2(3). Since each $i$-increasing braid is conjugate to an $n$-increasing braid of a standard form in Lemma 4.6, we may assume that $b \in B_n$ is an $i$-increasing braid of the form $b = (w_1\sigma_2^2\cdots w_\ell\sigma_2^2)$. Since $\rho \in B_n$ is the periodic braid such that $\rho = \sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}$ we have $\sigma_{n-1} = (\sigma_1\cdots\sigma_{n-2})^{-1}\rho$. Then $b$ is expressed as follows.

$$b = (\nu_1\rho)\cdots(\nu_u\rho),$$

where $\nu_i = w_i(\sigma_1\cdots\sigma_{n-2})^{-1}$ is written by a word of $\sigma_1^{\pm 1}, \cdots, \sigma_{n-2}^{\pm 1}$, but not $\sigma_{n-1}^{\pm 1}$. Each $\nu_j$ in $b$ is a reducible braid and $\rho$ in $b$ is the periodic braid. Let $\omega_j : F_b \to F_b$ denote a reducible representative whose mapping class is determined by $\nu_j$, and let $\psi : F_b \to F_b$ denote a periodic representative whose mapping class determined by $\rho$. The monodromy $\phi_b$ defined on $F_b$ is written by $\phi_b = (\omega_1\psi)\cdots(\omega_u\psi)$.

Recall that $\mathbb{D}_{n-1}$ is the disk $\mathbb{D}$ with marked points $A_1, \cdots, A_{n-1}$. Let $S_0$ be an $n$-holed sphere obtained from $\mathbb{D}_{n-1}$ by removing the interiors of small $(n-1)$ disks with centers $A_1, \cdots, A_{n-1}$. Each $\nu_j$ as an $(n-1)$-braid determines a homeomorphism $\hat{\omega}_j : S_0 \to S_0$. We may assume that $\hat{\omega}_j$ fixes one of the boundary components corresponding to $\partial\mathbb{D}$ pointwise. It is clear that we have an embedding $h : S_0 \to F_b$ such that each $\omega_j$ in $\phi_b$ is reducible supported on the subsurface $h(S_0)$ and the restriction of $\omega_j$ to $h(S_0)$ is given by $h \circ \hat{\omega}_j \circ h^{-1}$.

By the proof of Theorem 3.2(2), $\phi_{(x,y)} : F_{(x,y)} \to F_{(x,y)}$ associated with each primitive class $(x,y) \in C_{(b,i)}$ is determined by the braids of the form $b[p_1, \ldots, p_\ell]$. We now prove by the induction on length $\ell$ that

$$b[p_1, \ldots, p_\ell] = (\nu_1\rho)\cdots(\nu_{u-1}\rho)(\nu_u\rho)\rho^{m-1} = (\nu_1\rho)\cdots(\nu_{u-1}\rho)(\nu_u\rho^m)$$

for some $m \geq 1$ depending on $(x,y)$. Here each $\nu_j$ in $b[p_1, \ldots, p_\ell]$ is a reducible braid which is an extension of $\nu_j$ in $b$ and $\rho$ is the periodic braid with the degree of $b[p_1, \ldots, p_\ell]$. If this holds, then $\phi_{(x,y)}$ has a desired property as in Theorem 3.2(3). Suppose that $\ell = 1$. If $p_1 = 0$, then $b[0] = b$ and we are done. If $p_1 \geq 1$, then $b[p_1] = b_{p_1}$. Using the above expression of $b$ we observe that $b_{p_1}$ is written by

$$b_{p_1} = (\nu_1\rho)\cdots(\nu_u\rho) \in B_{n+p_{1u}}$$

(see Figure 13). We are done.
For \( \ell \geq 2 \), suppose that \( b[p_1, \ldots, p_{\ell-1}] = (v_1 \rho_d) \cdots (v_{\ell-1} \rho_d)(v_\ell \rho_d^m) \) for some \( m \), where \( d \) is the degree of \( b[p_1, \ldots, p_{\ell-1}] \). Consider \( b[p_1, \ldots, p_{\ell-1}] \) with length \( \ell \). If \( \ell \) is even, then by induction hypothesis
\[
b[p_1, \ldots, p_{\ell-1}] = (b[p_1, \ldots, p_{\ell-1}]) \Delta_{d}^{2p_{\ell}} = (v_1 \rho_d) \cdots (v_{\ell-1} \rho_d)(v_\ell \rho_d^m) \Delta_{d}^{2p_{\ell}}.
\]
Since \( \Delta_{d}^2 = \rho_{d-1}^d \) we have \( (v_\ell \rho_d^m) \Delta_{d}^{2p_{\ell}} = v_\ell \rho_d^{m+p(d-1)} \). Thus \( b[p_1, \ldots, p_{\ell-1}] \) has a desired expression and we are done. If \( \ell \) is odd, then by induction hypothesis again
\[
b[p_1, \ldots, p_{\ell-1}] = (b[p_1, \ldots, p_{\ell-1}])_{p_{\ell}} = (v_1 \rho_d) \cdots (v_{\ell-1} \rho_d)(v_\ell \rho_d^m)_{p_{\ell}}.
\]
As in the case of \( \ell = 1 \), the braid in the right-hand side is expressed as
\[
((v_1 \rho_d) \cdots (v_{\ell-1} \rho_d)(v_\ell \rho_d^m))_{p_{\ell}} = (v_1 \rho_1) \cdots (v_{\ell-1} \rho_1)(v_\ell \rho_1^m),
\]
where \( \dagger \) is the degree of \( b[p_1, \ldots, p_{\ell-1}] \). This completes the proof. \( \square \)

5. Sequences of pseudo-Anosov braids with small normalized entropies

In this section we prove Theorem A. We begin with an observation. Let \( \Omega \subset \{ a \in \mathbb{C} \mid |a| = 1 \} \) be a compact set in \( H_2(M_b, \partial M_b; \mathbb{R}) \) and let \( C_\Omega \subset \mathbb{C} \) denote the cone over \( \Omega \) through the origin. By Theorem 2.3(2) there is a constant \( P = P(\Omega) > 0 \) depending on \( \Omega \) such that \( \text{Ent}(a) < P \) for any \( a \in C_\Omega \). This observation provides us many sequences of pseudo-Anosov braids with small normalized entropies from a single pseudo-Anosov braid \( b \).

Theorem 5.1. Suppose that \( b \) is a pseudo-Anosov braid whose permutation has a fixed point. We fix any \( 0 < \ell < \infty \). Let \( \{(x_p, y_p)\} \) be a sequence of primitive integral classes in \( C_{(b, i)} \) such that \( y_p/x_p < \ell \) and \( \|(x_p, y_p)\| \propto p \). Then the sequence of pseudo-Anosov braids \( \{b(x_p, y_p)\} \) has a small normalized entropy.

Proof. If \( \{(x_p, y_p)\} \) is the sequence under the assumption, then we have \( d(b(x_p, y_p)) / \|(x_p, y_p)\| \propto p \). Since \( (1, 0) \in C_{(b, i)} \subset C \) and the slope of \( y_p/x_p \) is bounded by \( \ell \) from above, the set of projective classes \( (x_p, y_p) \) is contained in some compact set in \( \{ a \in \mathbb{C} \mid |a| = 1 \} \) (Figure 9). Thus there is a constant \( P = P(\ell) > 1 \) such that \( \text{Ent}(b(x_p, y_p)) < P \) for any \( p \). This completes the proof. \( \square \)

Let us discuss three sequences coming from Example 4.4. They are \( \{b_p\} \), \( \{\beta_p\} \) and \( \{(b\Delta^{2p})_1\} \) varying \( p \). It is not hard to see that \( d(b_p), d(\beta_p), d((b\Delta^{2p})_1) \propto p \).

Theorem 5.2. For an \( i \)-increasing and pseudo-Anosov \( b \in B_n \), we have the following on the sequences of pseudo-Anosov braids.

1. \( \{b_p\} \) has a small normalized entropy if and only if \( [E_{(b, i)}] \) is a fibered class.
2. For \( \beta = b\Delta^2 \in B_n \), \( \{\beta_p\} \) has a small normalized entropy and \( \text{Ent}(\beta_p) \to \text{Ent}((1, 1)) \) as \( p \to \infty \).
3. \( \{(b\Delta^{2p})_1\} \) has a small normalized entropy and \( \text{Ent}((b\Delta^{2p})_1) \to \text{Ent}(b) \) as \( p \to \infty \).

Proof of Theorem 5.2. For \( a = (x, y) \in \overline{C}_{(b, i)} \), let \( a = (x, y) \) denote its projective class. We have \( [F_{b_p}] = (1, p) \to [E_{(b, i)}] = (0, 1) \) as \( p \to \infty \). If \( [E_{(b, i)}] \) is a fibered class, then \( [E_{(b, i)}] \in C \) by Remark 4.2 and \( \text{Ent}(b_p) \to \text{Ent}([E_{(b, i)}]) \) as \( p \to \infty \) by Theorem 2.3(2). If \( [E_{(b, i)}] \) is a non-fibered class, then \( [E_{(b, i)}] \in \partial C \) by Remark 4.2,
Figure 14. Case: \(b\) is \(i\)-increasing. (1) Meridian and longitude basis. (2) Two boundary slopes \(\partial_{(b,A)}F_{(1,1)}\) (in green) on \(T_{(b,A)}\) and \(\partial_{(b,i)}F_{(1,1)}\) (in red) on \(T_{(b,i)}\) when \((x,y) = (1,1)\).

and \(\text{Ent}(b) \to \infty\) as \(p \to \infty\) by Theorem 2.3(3). We finish the proof of (1). We turn to (2). Since \([F_\beta] = (p + 1, p) \in C_{(b,i)}\), its projective class goes to \((1,1)\) as \(p \to \infty\). Since \((1,1) \in C_{(b,i)} \subset C\) by Theorem 3.2(1), \(\text{Ent}(\beta) \to \text{Ent}((1,1))\) as \(p \to \infty\) by Theorem 2.3(2). This completes the proof of (2). Finally we prove (3). The fibered class of \(F\)-surface of \((\Delta^2 \Delta)^b\) is given by \((p + 1, 1)\) in \(C_{(b,\Delta)}\). Its projective class goes to \([F_\beta] = (1, 0)\) as \(p \to \infty\). Thus \(\text{Ent}((\Delta^2 \Delta)^b) \to \text{Ent}(b)\) as \(p \to \infty\). This completes the proof. \[\square\]

We use Theorem 5.2(1)(2) in Section 8. For an application using (3), see [19].

Proof of Theorem A. Suppose that \(b \in B_n\) is pseudo-Anosov with \(\pi_b(i) = i\). Let \(\beta(k)\) denote \(b\Delta^2k \in B_n\) for \(k \geq 1\). Clearly \(\beta(k)\) is pseudo-Anosov with the same dilatation as \(b\) (for any \(k\)) and \(\beta(k)\) is positive for \(k\) large. We fix such large \(k\). By Lemma 3.1 \(\beta(k)\) is \(i\)-increasing. If we let \(z_p = (\beta(k)\Delta^2)^b\), then \(M_{z_p} \simeq M_{\beta(k)} \simeq M_b\) holds for \(p \geq 1\). By Theorem 5.2(3), \(\{z_p\}\) has a small normalized entropy and \(\text{Ent}(z_p) \to \text{Ent}((\beta(k)) = \text{Ent}(b)\) as \(p \to \infty\). \[\square\]

Let \(b_p^*\) denote the braid obtained from \((i + pu)\)-increasing \(b_p\) by removing the strand of the index \(i + pu\). Taking its spherical element we have \(S(b_p^*)\). A mild generalization of the sequence \(\{b_p\}\) is the ones \(\{b_p^*\}\) and \(\{S(b_p^*)\}\) varyng \(p\). Although \(b_p^*, S(b_p^*)\) may not be pseudo-Anosov, they are frequently pseudo-Anosov. To be more precise, we need to consider the number of prongs of singularities in the stable foliation \(F_{b_p}\) for \(b_p\), as we explained in Section 2.3. This is the motivation of the study in Section 6.

6. Stable foliation for the monodromy

Let \(b\) be pseudo-Anosov and \(i\)-monotonic with the sign \(\epsilon(b, i) = \epsilon\). For any primitive integral class \((x,y) \in C_{(b,i)}\), the oriented sum \(F_{(x,y)} = xF_b + yE_{(b,i)}\) is connected. Let \(T_{(b,A)}\) and \(T_{(b,i)}\) denote the tori \(\partial\mathcal{N}(A)\) and \(\partial\mathcal{N}(\text{cl}(b(i)))\) respectively.
Let us set
\[ \partial_{(b,A)} F(x,y) = \partial F(x,y) \cap T_{(b,A)} \quad \text{and} \quad \partial_{(b,i)} F(x,y) = \partial F(x,y) \cap T_{(b,i)}, \]
each of which is a single simple closed curve on the torus (since \( \gcd(x,y) = 1 \)). Recall that we chose the orientation of the axis for the \( i \)-monotonic \( b \) in Section 3. We use the meridian and longitude basis \( \{m_A, \ell_A\} \) for \( T_{(b,A)} \) to represent a homology class of a disjoint union of simple closed curves on \( T_{(b,A)} \). We also use the meridian and the longitude basis \( \{m_i, \ell_i\} \) for \( T_{(b,i)} \). Observe that the homology classes \( \partial_{(b,A)} F(x,y) \) and \( \partial_{(b,i)} F(x,y) \) are given by the pairs of integers
\[ (6.1) \quad [\partial_{(b,A)} F(x,y)] = (-ey, x) \quad \text{and} \quad [\partial_{(b,i)} F(x,y)] = (-ex, y). \]
They are called boundary slopes of \( F(x,y) \). See Figure 14.

Let \( \phi_b : F_b \to F_b \) be the pseudo-Anosov monodromy of a fiber \( F_b \) of the fibration on \( M_b \to S^1 \). The stable foliation \( F_b \) has singularities on each boundary component of \( F_b \). Now we consider the suspension flow \( \phi_t^b \) (\( t \in \mathbb{R} \)) on the mapping torus \( M_b \). We obtain a disjoint union of simple closed curves \( c_A = c_{(b,A)} \) on \( T_{(b,A)} \) (possibly a single simple closed curve) which is a union of closed orbits for singularities in \( \partial_{(b,A)} F_b \) under the flow. Similarly we have a disjoint union of simple closed curves \( c_i = c_{(b,i)} \) on \( T_{(b,i)} \) (possibly a single simple closed curve again) which is a union of closed orbits for singularities in \( \partial_{(b,i)} F_b \). (Figure 17 depicts these closed curves for some pseudo-Anosov 3-braids.) A useful tool is train track maps which encode those data \( \phi_b, F_b \). They also enable us to compute homology classes \([c_A]\) and \([c_i]\).

The following lemma is a consequence of Theorem 2.4(2) by Fried.

**Lemma 6.1.** Let \( \phi_{(x,y)} : F(x,y) \to F(x,y) \) be the monodromy of a fibration on \( M_b \to S^1 \) associated with a primitive integral class \((x,y) \in C_{(b,i)}\). Then the stable foliation \( F_{(x,y)} \) for \( \phi_{(x,y)} \) is \( i([c_A], [\partial_{(b,A)} F(x,y)]) \)-pronged at \( \partial_{(b,A)} F(x,y) \), and is \( i([c_i], [\partial_{(b,i)} F(x,y)]) \)-pronged at \( \partial_{(b,i)} F(x,y) \), where \( i(\cdot, \cdot) \) means the geometric intersection number between homology classes of closed curves.

**Remark 6.2.** Every closed orbit of the suspension flow \( \phi_t^b \) on the mapping torus \( M_b \) travels around \( S^1 \) direction at least once. This implies that \([c_A]\) has a non-zero first coordinate of the meridian and longitude basis for \( T_{(b,A)} \), i.e., we have \([c_A] = (k, \ell) \in \mathbb{Z}^2 \) with \( k \neq 0 \), since the meridian for \( T_{(b,A)} \) corresponds to the flow direction. Similarly, \([c_i]\) has a non-zero second coordinate of the meridian and longitude basis for \( T_{(b,i)} \), that is we have \([c_i] = (k', \ell') \in \mathbb{Z}^2 \) with \( \ell' \neq 0 \), since the longitude for \( T_{(b,i)} \) corresponds to the flow direction in this case.

Recall that given a braid \( b \in B_n \), we denote by \( S(b) \in SB_n \), the spherical \( n \)-braid with the same word as \( b \). For an \( i \)-increasing braid \( b \) of pseudo-Anosov type, consider the braid \((b\Delta^{2p})_1 = b[0,p,1]\) in Example 4.4(3). This is an \( i[0,p,1]\)-increasing braid. Then we have its spherical braid \( S((b\Delta^{2p})_1) \). We now define other braids obtained from \( (b\Delta^{2p})_1 \). Let \((b\Delta^{2p})_1^* \) denote the braid obtained from \( (b\Delta^{2p})_1 \) by removing the strand of the index \( i[0,p,1] \). Let \( S((b\Delta^{2p})_1^*) \) be the spherical braids corresponding to \((b\Delta^{2p})_1^* \) and \( (b\Delta^{2p})_1^* \) respectively. Then we have the following result.

**Lemma 6.3.** Suppose that \( b \) is an \( i \)-increasing braid of pseudo-Anosov type. For \( p \) large, the braid \((b\Delta^{2p})_1^* \) and the spherical braids \( S((b\Delta^{2p})_1), S((b\Delta^{2p})_1^*) \) are all pseudo-Anosov with the same dilatation as \((b\Delta^{2p})_1 \).
Before proving Lemma 6.3, we recall a formula of the geometric intersection number \( i([c], [c']) \) between two homology classes of simple closed curves \( c, c' \) on a torus. Let \( (p, q) \) and \( (p', q') \) be primitive elements of \( \mathbb{Z}^2 \) which represent \( [c] \) and \( [c'] \) respectively. Then

\[
i([c], [c']) = |pq' - p'q|.
\]

**Proof of Lemma 6.3.** The fibered class of \( F \)-surface of \( (b \Delta^2) \) is \( (p + 1, 1) \in C_{(b,i)} \). We have \( [\partial_{(b,A)} F_{(p+1,1)}] = (-1, p + 1) \) and \( [\partial_{(b,i)} F_{(p+1,1)}] = (-p + 1, 1) \), see (6.1). By Remark 6.2, one can write \([c_A] = (k, \ell) \) with \( k \neq 0 \) and \([c_i] = (k', \ell') \) with \( \ell' \neq 0 \). Then \( i([c_A], [\partial_{(b,A)} F_{(p+1,1)}]) = |k(p+1)+\ell| \) and \( i([c_i], [\partial_{(b,i)} F_{(p+1,1)}]) = |k'+\ell'(p+1)| \). Since \( k \neq 0 \) and \( \ell' \neq 0 \), these intersection numbers are increasing with respective to \( p \) and they are clearly greater than 1 when \( p \) is large. Then Lemma 6.1 says that when \( p \) is large, the stable foliation \( F_{(p+1,1)} \) for the monodromy \( \phi_{(p+1,1)} \) is not 1-pronged at each component of \( \partial_{(b,A)} F_{(p+1,1)} \cup \partial_{(b,i)} F_{(p+1,1)} \). By the discussion in Section 2.4, we are done. \( \square \)

### 7. Properties of \( F \)-surfaces and \( E \)-surfaces

The aim of this section is to study properties of \( E_\tau \), \( F \)-surfaces and to present the technique used in the last section.

**Lemma 7.1.** For an \( i \)-increasing braid \( b \in B_n \) with \( u(b, i) = u \), we set \( \beta = b \Delta^2 \in B_n \). Then there is an \( n \)-increasing braid \( \gamma \in B_{n+u} \) such that

\[
(\text{br}(\beta), \text{cl}(\beta(i)), A_\beta) \sim (\text{br}(\gamma), A_\gamma, \text{cl}(\gamma(n))).
\]

In particular \( M_b \simeq M_{\beta} \simeq M_\tau \) and \( E_{(\beta,1)} = F_\gamma, F_\beta = E_{(\gamma, n)} \) up to isotopy in \( M_\beta \). Moreover if \( b \) is pseudo-Anosov, then \( \gamma \) is also pseudo-Anosov.

A similar claim holds for \( i \)-decreasing braids.

**Proof.** By Lemma 4.6 we may assume that \( b \in B_n \) is an \( n \)-increasing braid of a standard form \( b = (w_1 \sigma_{n-1}^2) \cdots (w_u \sigma_{n-1}^2) \) containing \( u \) subwords \( \sigma_{n-1}^2 \). Using the identity

\[
\Delta^2 = \Delta_{n-1}^2 \sigma_{n-1} \cdots \sigma_2 \sigma_1 \sigma_2 \cdots \sigma_{n-1} \in B_n,
\]

we have (Figure 15(1))

\[
\text{br}(\beta) = \text{br}(\Delta^2) = \text{br}(w_1 \sigma_{n-1}^2 \cdots w_u \sigma_{n-1}^2 \Delta_{n-1}^2 \sigma_{n-1} \cdots \sigma_2 \sigma_1 \sigma_2 \cdots \sigma_{n-1})
\]

We first deform \( \text{br}(\beta) \) into a link as in Figure 15(3). The same figure(1)(2)(3) tells us the process to get the desired link in (3). Then we perform the local moves in the shaded regions containing \( u \) subwords \( \sigma_{n-1}^2 \) in \( b \) so that the link in question is a union of the closure of some \( n \)-increasing braid \( \gamma \in B_{n+u} \) and its braided axis, namely a braided link, see Figure 15(3)(4)(5). As a result,

\[
(\text{br}(\beta), \text{cl}(\beta(n)), A_\beta) \sim (\text{br}(\gamma), A_\gamma, \text{cl}(\gamma(n))).
\]

This expression says that \( M_\beta \simeq M_\tau \) and the \( E_\tau \), \( F \)-surfaces for \( \beta \) are equal to the \( F_\tau \), \( E \)-surfaces for \( \gamma \). Since \( M_b \simeq M_\beta \) we are done. \( \square \)
Here we introduce a simple representative of $\gamma \in B_{n+u}$ in Lemma 7.1. By the deformation as in (5)(6) of Figure 15, we can take the following representative of $\gamma$.

\[
\gamma = \kappa_0 \kappa_1 \cdots \kappa_{u+1} \Delta_{n-1}^2, \quad \text{where}
\]

\[
\kappa_0 = \sigma_{n-1} \sigma_{n-2} \cdots \sigma_1 \sigma_2 \cdots \sigma_{n+u-1},
\]

\[
\kappa_j = w_j \sigma_{n-1} \sigma_n \cdots \sigma_{n+u-j-1} \sigma_{n+u-j-2} \cdots \sigma_{n-1}^{\text{if } 1 \leq j \leq u-1},
\]

\[
\kappa_u = w_u \sigma_{n-1},
\]

\[
\kappa_{u+1} = \sigma_n^{-1} \quad \text{if } u = 1,
\]

\[
\kappa_{u+1} = \sigma_{n+u-1}^{-1} \sigma_{n+u-2}^{-1} \cdots \sigma_n^{-1} \quad \text{if } u \geq 2.
\]
For example if \((n, u) = (3, 2)\), then
\[(7.1) \quad \gamma = \kappa_0 \kappa_1 \kappa_2 \kappa_3 \Delta_2^2 = \sigma_2 \sigma_2 \sigma_3 \sigma_4 \sigma_1 \sigma_2 \sigma_2 \sigma_1^{-1} \sigma_3^{-1} \sigma_1^2.\]

If \((n, u) = (3, 3)\), then \(\gamma = \kappa_0 \kappa_1 \kappa_2 \kappa_3 \kappa_4 \Delta_2^2\), that is
\[(7.2) \quad \gamma = \sigma_2 \sigma_2 \sigma_3 \sigma_5 \sigma_1 \sigma_2 \sigma_3 \sigma_3 \sigma_1 \sigma_2 \sigma_2 \sigma_1^{-1} \sigma_3^{-1} \sigma_2^{-1} \sigma_2 \sigma_2 \sigma_1^{-1} \sigma_3^{-1} \sigma_1^2.\]

Lemma 7.1 is used in the following situation. Suppose that \(\alpha \in B_{n+u}\) is a \(j\)-increasing braid and our task is to prove that \(\alpha\) is pseudo-Anosov and its \(E\)-surface \(E_{(\alpha, j)}\) is a fiber of a fibration on \(M_\alpha \to S^1\). (The conditions are needed to apply Theorem 5.2(1) for \(\alpha\).) To do this, we need to find an \(i\)-increasing and pseudo-Anosov braid \(b \in B_n\) with \(u = u(b, i)\) and need to check the resulting \(n\)-increasing braid \(\gamma \in B_{n+u}\) in Lemma 7.1 satisfies the property
\[(\text{br}(\gamma), A_\gamma, \text{cl}(\gamma(u))) \sim (\text{br}(\alpha), A_\alpha, \text{cl}(\alpha(j))),\]
i.e. \(\gamma\) is conjugate to \(\alpha\) preserving the corresponding strand. If this equivalence holds, then by Lemma 7.1 together with the above equivalence \(\sim\), our task is done. As a result \(\{\alpha_p\}\) has a small normalized entropy by Theorem 5.2(1).

8. Application

In the last section we prove Theorems C, D and E. We first recall a study of pseudo-Anosov 3-braids [14, 24]. Let \(w\) be a word in \(\sigma_1^{-1} \text{ and } \sigma_2\). If both \(\sigma_1^{-1}\) and \(\sigma_2\) occur at least once in \(w\), then we say that \(w\) is a \(pA\) word. It is known that the 3-braid represented by a \(pA\) word is pseudo-Anosov. Conversely a 3-braid \(b\) is pseudo-Anosov, then there is a \(pA\) word \(w\) such that the braid represented by \(w\) is conjugate to \(b\) up to a power of the full twist.

The stable foliation \(\mathcal{F}_b\) is 1-pronged at each boundary component of \(F_b\) for each pseudo-Anosov 3-braid \(b\). Figure 17(3) exhibits a train track automaton. A train track map for the 3-braid represented by a \(pA\) word \(w\) is obtained from the closed loop corresponding to \(w\) in the automaton. For more details, see Ham-Song [13].

8.1. Palindromic/Skew-palindromic braids. We define an anti-homomorphism

\[
\text{rev}: B_n \to B_n
\]
\[
\sigma_{i_1}^{\mu_1} \sigma_{i_2}^{\mu_2} \cdots \sigma_{i_k}^{\mu_k} \mapsto \sigma_{i_k}^{\mu_k} \cdots \sigma_{i_2}^{\mu_2} \sigma_{i_1}^{\mu_1}, \quad \mu_j = \pm 1.
\]

A braid \(b \in B_n\) is palindromic if \(\text{rev}(b) = b\). Clearly \(b \cdot \text{rev}(b)\) is palindromic for any \(b \in B_n\). Let us consider another anti-homomorphism

\[
\text{skew}: B_n \to B_n
\]
\[
\sigma_{i_1}^{\mu_1} \sigma_{i_2}^{\mu_2} \cdots \sigma_{i_k}^{\mu_k} \mapsto \sigma_{n-i_k}^{\mu_k} \cdots \sigma_{n-i_2}^{\mu_2} \sigma_{n-i_1}^{\mu_1}, \quad \mu_j = \pm 1.
\]

A braid \(b \in B_n\) is skew-palindromic if \(\text{skew}(b) = b\). Clearly \(b \cdot \text{skew}(b)\) is skew-palindromic for any \(b \in B_n\).

We now prove Theorems C and D which indicate the asymptotic behaviors of minimal entropies among these subsets are quite distinct.

Proof of Theorem C. For the surjective homomorphism \(\pi: B_n \to S_n\) we write \(\pi_j = \pi(\sigma_j)\). Suppose that an \(n\)-braid \(b = \sigma_{i_1}^{\mu_1} \sigma_{i_2}^{\mu_2} \cdots \sigma_{i_k}^{\mu_k}\) is palindromic. Since \(\text{rev}(b) = b\) we have
\[(\pi_{\text{rev}(b)} = \pi_{i_k} \cdots \pi_{i_2} \pi_{i_1} = \pi_{i_1} \pi_{i_2} \cdots \pi_{i_k} (= \pi_b)).\]
Multiply the both side by $i_1^1 i_2^2 \cdots i_k^k$ from the left:

$$(i_1^1 i_2^2 \cdots i_k^k) (i_k^k \cdots i_2^2 i_1^1) = (i_1^1 i_2^2 \cdots i_k^k) (i_1^1 i_2^2 \cdots i_k^k) = \pi_b^2.$$  

Since $\pi_j^2 = id$ the left-hand side equals $id$. Hence $id = \pi_b^2$ which means that the square $b^2$ is pure. A theorem by Song [28] states that for a pseudo-Anosov pure element $b' \in B_n$, its dilatation has a uniform lower bound $2 + \sqrt{5} \leq \lambda(b')$. In particular if $b' = b^2$, then $2 + \sqrt{5} \leq \lambda(b^2) = (\lambda(b))^2$. This completes the proof. □

**Proof of Theorem D.** We separate the proof into two cases, depending on the parity of the braid degree. We first prove $\log \delta(PA_{2n}) \propto 1/n$. Let us take $\xi = \sigma_1^{-1} \sigma_2^2 \sigma_3^2 \sigma_4 \in B_5$ (Figure 16). The braid $\xi$ is 3-increasing with $u(\xi, 3) = 2$. We consider the disk twist about $D(\xi, 3)$. We obtain the braid $\xi_p$ which is $(3 + 2p)$-increasing for each $p \geq 1$. Observe that $\xi_p^\bullet$ is a skew-palindromic braid with even degree for each $p \geq 1$:

$$\xi_p^\bullet = (\sigma_1 \cdots \sigma_{1+2p})(\sigma_3 \cdots \sigma_{3+2p}) \in B_{4+2p}.$$
(For the definition of \(\xi_p\), see Section 5.) By the lower bound of dilatations by Penner, it is enough to prove that the sequence \(\{\xi_p\}\) has a small normalized entropy. We prove this in the following two steps. In Step 1 we prove that \(\{\xi_p\}\) has a small normalized entropy. In Step 2 we prove that the stable foliation \(F_{\xi_p}\) is not 1-pronged at \(\partial(\xi_p,3+2p)F_{\xi_p}\) for \(p \geq 1\). This tells us that \(\xi_p\) is pseudo-Anosov with the same dilatation as \(\xi_p\). By Step 1 it follows that \(\{\xi_p\}\) has a small normalized entropy.

**Step 1.** The sequence \(\{\xi_p\}\) has a small normalized entropy.

By Theorem 5.2(1) it suffices to prove that \(\xi\) is pseudo-Anosov and \([E_{(\xi,3)}]\) is a fibered class. Consider a pseudo-Anosov braid \(b = \sigma_1^{-1}\sigma_2\sigma_1\sigma_2^{-1} \in B_3\). It is 3-increasing with \(u(b,3) = 2\). For \(\beta = b\Delta^2\) we have \(M_\beta \simeq M_\beta\). By Lemma 7.1 \((\text{br}(\beta),\text{cl}(\beta(3)),A_3) \sim (\text{br}(\gamma),A_\gamma,\text{cl}(\gamma(3)))\), where \(\gamma \in B_5\) is the braid in (7.1) substituting \(\sigma_1^{-1}\) for \(w_1\) and \(\sigma_1^{-1}\) for \(w_2\). It is not hard to check that \(\gamma\) is conjugate to \(\xi\) in \(B_5\) and their permutations have a common fixed point 3. Hence

\[
\text{(8.1)} \quad \text{br}(\beta),\text{cl}(\beta(3)),A_3) \sim (\text{br}(\xi),A_\xi,\text{cl}(\xi(3))).
\]

In particular \(E_{(\xi,3)} = F_\beta\) which means that \(E_{(\xi,3)}\) is a fiber of a fibration on the hyperbolic mapping torus \(M_\beta \simeq M_\xi\) over \(S^1\). Thus \(\xi\) is pseudo-Anosov.

**Step 2.** \(F_{\xi_p}\) is \((p+1)\)-pronged at \(\partial(\xi_p,3+2p)F_{\xi_p}\) for \(p \geq 1\).

We read the singularity data of \(F_{\xi_p}\) from the monodromy \(\phi_3 : F_\beta \to F_\beta\) of the fibration on \(M_\beta \to S^1\). First consider the suspension flow \(\phi_3^t\) on the mapping torus \(M_\beta\). Since \(F_{\xi_p}\) is 1-pronged at each component of \(F_{\xi_p}\), we have simple closed curves \(c_A \subset T_{(\beta,A)}\) and \(c_3 \subset T_{(\beta,3)}\) such that \([c_A] = (1,0),\ [c_3] = (2,1) \in \mathbb{Z}^2\) (Figure 17(1)(2)).

Next we turn to \(\beta = b\Delta^2 \in B_3\) and the suspension flow \(\phi_3^t\) on \(M_\beta \simeq M_\beta\). We have simple closed curves \(c_{(\beta,A)} \subset T_{(\beta,A)}\) and \(c_{(\beta,3)} \subset T_{(\beta,3)}\). Since \(\beta\) is the product of \(b\) and \(\Delta^2\), we get \([c_{(\beta,A)}] = (1,0) + (0,1) = (1,1)\). The first term \((1,0)\) comes from \([c_A]\) and the second one \((0,1)\) comes from \(\Delta^2\). Similarly we have \([c_{(\beta,3)}] = (2,1) + (1,0) = (3,1)\). By (8.1) we have \(F_\beta = E_{(\xi,3)}\) and \(E_{(\beta,3)} = F_\xi\). We also have \(T_{(\beta,A)} = T_{(\xi,3)}\) and \(T_{(\beta,3)} = T_{(\xi,A)}\). Since

\[
p[F_\beta] + [E_{(\beta,3)}] = [F_\xi] + p[E_{(\xi,3)}] = [F_\xi + pE_{(\xi,3)}] = (1,p) \in C_{(\xi,3)},
\]

the stable foliation \(F_{(1,p)}\) associated with an integral class \((1,p) \in C_{(\xi,3)}\) is the stable foliation associated with \((p,1) \in C_{(\beta,3)}\). By (6.1) for \((x,y) = (p,1)\)

\[
[\partial_{(\beta,A)}(F_\xi + pE_{(\xi,3)})] = (-1,p), \quad [\partial_{(\beta,3)}(F_\xi + pE_{(\xi,3)})] = (-p,1) \in \mathbb{Z}^2.
\]

From \(i([c_{(\beta,A)}],[\partial_{(\beta,A)}(F_\xi + pE_{(\xi,3)})]) = p+1\) and \(i([c_{(\beta,3)}],[\partial_{(\beta,3)}(F_\xi + pE_{(\xi,3)})]) = p+3\) together with Lemma 6.1, one sees that \(F_{(1,p)}\) associated with \((1,p) \in C_{(\xi,3)}\) is \((p+1)\)-pronged at \(\partial_{(\beta,A)}F_{(1,p)} = \partial_{(\xi,3)}F_{(1,p)}\), and is \((p+3)\)-pronged at \(\partial_{(\beta,3)}F_{(1,p)} = \partial_{(\xi,A)}F_{(1,p)}\).

Since \(g_p : M_\xi \to M_\xi\) sends \(F_{(1,p)}\) to \(F_{\xi_p}\) the stable foliation \(F_{(1,p)}\) associated with \((1,p) \in C_{(\xi,3)}\) is identified with \(F_{\xi_p}\) via \(g_p\). The boundary components \(\partial_{(\xi,A)}F_{(1,p)}\)

---

1There is a solution for the conjugacy problem on \(B_n\) [6]. The software Braiding [12] can be used to determine whether two braids are conjugate.
and \( \partial(\xi_3)F_{(1,p)} \) correspond to \( \partial(\xi_p,A)F_{\xi_p} \) and \( \partial(\xi_p,3+2p)F_{\xi_p} \) respectively via \( g_p \). Thus \( F_{\xi_p} \) is \((p+1)\)-pronged at \( \partial(\xi_p,3+2p)F_{\xi_p} \). This completes the proof of Step 2.

Next we prove \( \log \delta(PA_{2n+1}) \propto 1/n \) following the above arguments in Steps 1,2.

Take an initial braid

\[ \eta = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7 \in B_8. \]

It is 4-increasing with \( u(\eta,4) = 2 \). Consider \( \eta_p \in B_{8+2p} \) obtained from \( \eta \) by the disk twist. Then \( \eta_p^* \) is a skew-palindromic braid with odd degree for each \( p \geq 1 \):

\[ \eta_p^* = (\sigma_1\sigma_2\cdots\sigma_{4+2p})(\sigma_3\sigma_4\cdots\sigma_{6+2p}) \in B_{7+2p}. \]

For our purpose it suffices to prove that \( \{\eta_p^*\} \) has a small normalized entropy. Following Step 1 we first prove that \( \eta \) is pseudo-Anosov and \( [E_{(\eta,4)}] \) is a fibered class. Consider a pseudo-Anosov braid \( b = \sigma^{-1}\sigma^4\Delta^2 \in B_3 \) which is 3-increasing with \( u(b,3) = 5 \). For \( \beta = b\Delta^2 \) Lemma 7.1 tells us that \( (\text{br}(\beta),\text{cl}(\beta(3)),A_\beta) \sim (\text{br}(\gamma),A_\gamma,\text{cl}(\gamma(3))), \)

where \( \gamma = \kappa_0\kappa_1\cdots\kappa_g\Delta_2^2 \in B_8 \). One sees that \( \gamma \) is conjugate to \( \eta \) in \( B_8 \). Since the permutation \( \pi_\eta \) has a unique fixed point it follows that \( (\text{br}(\beta),\text{cl}(\beta(3)),A_\beta) \sim (\text{br}(\eta),A_\eta,\text{cl}(\eta(4))). \) This expression says that \( E_{(\eta,4)} = F_\beta \) is a fiber of a fibration on the hyperbolic \( M_8 \simeq M_\eta \) over \( S^1 \). Hence \( \eta \) is pseudo-Anosov. We conclude that \( \{\eta_p\} \) has a small normalized entropy.

Following Step 2 one sees that \( F_{\eta_p} \) is \((p+2)\)-pronged at \( \partial(\eta_p,4+2p)F_{\eta_p} \) for \( p \geq 1 \). Thus \( \eta_p^* \) is pseudo-Anosov with the same dilatation as \( \eta_p \). This completes the proof.

\[ \square \]

8.2. Spin mapping class groups.

In this section we prove Theorem E. We first recall a connection between \( \mathcal{H}(\Sigma_g) \) and \( \text{Mod}(\Sigma_{0,2g+2}) \). Let \( t_j \in \text{Mod}(\Sigma_g) \) for \( 1 \leq j \leq 2g+1 \) be the right-handed Dehn twist about the simple closed curve \( C_j \) as in Figure 18. Birman-Hilden [3] proved that \( \mathcal{H}(\Sigma_g) \) is generated by \( t_1, t_2, \ldots, t_{2g+1} \). In fact they prove that

\[ Q : \mathcal{H}(\Sigma_g) \rightarrow \text{Mod}(\Sigma_{0,2g+2}) \]

\[ t_j \mapsto t_j \]

sending \( t_j \) to the right-handed half twist \( t_j \) (see Section 2.3) is well-defined and it is a surjective homomorphism whose kernel is generated by the involution \( \iota = [I] \) as in Figure 5. Using the relation between \( \text{Mod}(\Sigma_{0,2g+2}) \) and \( SB_{2g+2} \) we have

\[ \mathcal{H}(\Sigma_g)/\langle \iota \rangle \simeq \text{Mod}(\Sigma_{0,2g+2}) \simeq SB_{2g+2}/\langle \Delta^2 \rangle. \]
Suppose that.

We prove the lemma by the induction on $t$. Let $t$ with Lemma 8.1(1) implies that $t$.

Note that braid relations one verifi ces that $t$.

Thus Lemma 8.1 tells us that $t$.

Proof. Lemma 8.2.

Lemma 8.1

It is well-known that $\phi \in \mathcal{H}(\Sigma_g)$ is pseudo-Anosov if and only if $Q(\phi)$ is pseudo-Anosov and in this case $\lambda(\phi) = \lambda(Q(\phi))$ holds. The following lemma is useful to find elements of the odd/even spin mapping class groups.

Lemma 8.1 (Theorem 6.1 in [18] for (1), Theorem 3.1 in [17] for (2)). Suppose that $g \geq 3$.

1) $t_2, t_3, t_{j+1}t_{j+1}^{-1}, t_k^g \in \text{Mod}_g[q_1]$ for $4 \leq j \leq 2g$ and $1 \leq k \leq 2g + 1$.

2) $t_{j+1}t_{j+1}^{-1}, t_k^g \in \text{Mod}_g[q_0]$ for $1 \leq j \leq 2g$ and $1 \leq k \leq 2g + 1$.

By the above result of Birman-Hilden, all mapping classes in Lemma 8.1 are elements of $\mathcal{H}(\Sigma_g)$. Using the braid relations: $t_i t_j = t_j t_i$ if $|i - j| \geq 2$ and $t_i t_{j+1} t_j = t_{j+1} t_j t_{j+1}$ for $1 \leq j \leq 2g$, we have

$t_i t_{j+1} t_j^{-1} = t_{j+1} t_j t_{j+1}^{-1} = t_{j+1}^{-1} (t_{j+1} t_j t_{j+1}) t_{j+1}^{-1}$. Thus Lemma 8.1 tells us that $t_i t_{j+1} t_j^{-1} \in \text{Mod}_g[q_1]$ for $4 \leq j \leq 2g$ and $t_i t_{j+1} t_j^{-1} \in \text{Mod}_g[q_0]$ for $1 \leq j \leq 2g$.

The following spin mapping classes are used in the proof of Theorem E.

Lemma 8.2. Let $p \geq 1$ be an integer.

1) $t_2 t_3 (t_4 t_5 \cdots t_{5+2p})^2 t_{5+2p} \in \text{Mod}_g[q_1]$ for any $g \geq p + 2$.

2) $(t_2 t_3 \cdots t_{5+2p})^2 t_{5+2p} \in \text{Mod}_g[q_0]$ for any $g \geq p + 2$.

Proof. We prove the lemma by the induction on $p$. We first prove (1). When $p = 1$

$t_2 t_3 (t_4 t_5 t_6 t_7)^2 t_7 = t_2 \cdot t_3 \cdot t_4 t_5 t_4^{-1} \cdot t_4^2 \cdot t_6 t_7 t_6^{-1} \cdot t_6 t_5 t_6^{-1} \cdot t_5^2 \cdot t_7^2$

which is an element of $\text{Mod}_g[q_1]$ for $g \geq 3$ by Lemma 8.1(1).

Assume that $t_2 t_3 (t_4 t_5 \cdots t_{5+2(p-1)})^2 t_{5+2(p-1)} \in \text{Mod}_g[q_1]$ for $g \geq p - 1 + 2$. By the braid relations one verifies that

$t_2 t_3 (t_4 t_5 \cdots t_{4+2(p-1)} t_{5+2(p-1)} t_{4+2p} t_{5+2p})^2 t_{5+2p}$

$= t_2 t_3 (t_4 t_5 \cdots t_{5+2(p-1)})^2 t_{5+2(p-1)} \cdot t_{5+2(p-1)} \cdot t_{4+2p} t_{5+2p} t_{5+2(p-1)} t_{4+2p} \cdot t_{5+2p}^2$.

Note that $t_i t_{j+1} t_{j-1} = (t_i t_{j+1} t_{j-1}) (t_i t_{j+1} t_{j-1}) t_{j-1}$. Then the assumption together with Lemma 8.1(1) implies that $t_2 t_3 (t_4 t_5 \cdots t_{5+2p})^2 t_{5+2p} \in \text{Mod}_g[q_1]$ for $g \geq p + 2$.

Figure 19. (1) $o \in B_6$. (2) $o^* \in B_{5+2p}$. (3) $sh(o^*) \in B_{6+2p}$.
Consider twist for each $o$

By Theorem 5.2(1) it suffices to prove that $f_s@o$ and $o$ point 4, it follows that $(br(\_))$ in $\text{Mod}(\_)$ is the $\text{braid}$ in (7.2) substituting $(t_3 \cdot t_5 + 2p) \in \text{Mod}(\_)$ for any $g \geq 1$. By the braid relations again, we have

$$\frac{(t_2 t_3 \cdots t_5 + 2p)}{2} \cdot t_5 + 2p = 2t_5 + 2p \cdot t_5 + 2p$$

By the assumption together with Lemma 8.1(2) we have $(t_2 t_3 \cdots t_5 + 2p) \in \text{Mod}(\_)$ for $g \geq p + 2$. This completes the proof. \hfill \square

The shift map $sh : B_n \to B_{n+1}$ is an injective homomorphism sending $\sigma_j$ to $\sigma_{j+1}$ for $1 \leq j \leq n - 1$. Suppose that $b \in B_n$ is pseudo-Anosov. Then $S(sh(b)) \in \text{SB}_{n+1}$ is pseudo-Anosov with the same dilatation as $b$ since $\hat{\Gamma}(S(sh(b)))$ is conjugate to $f_0 = c(\Gamma(b))$ in $\text{Mod}(\Sigma_0, n+1)$. (See Section 2.3 for definitions $\Gamma, \hat{\Gamma}$.) We finally prove Theorem E.

**Proof of Theorem E(1).** Consider $o = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \in B_6$ which is 4-increasing with $u(o, 4) = 2$ (Figure 19). The braid $o_p$ is obtained from $o$ by disk twist for each $p \geq 1$. Then

$$o_p = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \cdot t_5 + 2p \in \text{Mod}_{p+2}[1] \text{ for } g \geq 1, \text{ and it is pseudo-Anosov if } S(sh(o_p)) \text{ is pseudo-Anosov. In this case they have the same dilatation. Thus by the relation between } o_p \text{ and } S(sh(o_p)) \text{ it is enough to prove that } \{o_p^\bullet\} \text{ has a small normalized entropy. We first claim that } \{o_p\} \text{ has a small normalized entropy. By Theorem 5.2(1) it suffices to prove that } o \text{ is a pseudo-Anosov and } [E_{(o, 4)}] \text{ is a fibered class. Consider a 3-braid } b = \sigma_1^2 \sigma_2^2 \sigma_3^2 \text{ which is 3-increasing with } u(b, 3) = 3. \text{ Let } \beta \text{ denote } b\gamma^2. \text{ By Lemma 7.1 } (br(\beta), cl(\beta(3)), A_3) \sim (br(\gamma), A_3, cl(\gamma(3))), \text{ where } \gamma \in B_6 \text{ is the braid in (7.2) substituting } \sigma_1^2, \emptyset \text{ for } w_1, w_2, w_3 \text{ respectively. In this case } \gamma \text{ is conjugate to } o \text{ in } B_6. \text{ Since the permutation } \pi_0 \text{ has a unique fixed point 4, it follows that } (br(\beta), cl(\beta(3)), A_3) \sim (br(o), A_3, cl(o(4))) . \text{ This tells us that } M_\beta \simeq M_0 \text{ and } [E_{(o, 4)}] = [F_\beta] \text{ is a fibered class. On the other hand } \beta \text{ is conjugate to } \sigma_1^4 \sigma_2^2 \Delta^4 \text{ in } B_3 \text{ which means that } \beta \text{ is pseudo-Anosov. Thus } M_\beta \simeq M_0 \text{ is hyperbolic and } o \text{ is pseudo-Anosov.}

Next we prove that $o_p^\bullet$ is pseudo-Anosov with the same dilatation as $o_p$ for $p \geq 1$. By the same argument as in the proof of Theorem D one sees that $F_{o_p}$ is $(p + 2)$-pronged at $\partial_{o_p + 2p} F_{o_p}$. Thus $o_p^\bullet$ has the desired property for $p \geq 1$. We finish the proof of (1).

We turn to (2). Let us consider $v = (\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5)^2 \sigma_1 \sigma_2 \sigma_3^3 \in B_6$ which is 3-increasing with $u(v, 3) = 2$. Let $v_p \in B_{6+2p}$ be the braid obtained from $v$ by the disk twist. Then $v_p$ is $(3 + 2p)$-increasing and

$$v_p = (\sigma_1 \sigma_2 \cdots \sigma_{4+2p}) \cdot \sigma_3^3 \in B_{5+2p},$$

$$S(sh(v_p^\bullet)) = (\sigma_2 \sigma_3 \cdots \sigma_{5+2p}) \cdot \sigma_3^3 \in SB_{6+2p}.$$
By Lemma 8.2(2) it is enough to prove that \( \{v_p^\bullet\} \) has a small normalized entropy. To do this we first prove that \( \{v_p\} \) has a small normalized entropy. Consider a pseudo-Anosov 3-braid

\[
b = \sigma_1^3 \sigma_2^{-2} \Delta^4 = \sigma_1^3 \sigma_2^2 \sigma_1^2 \Delta^2 = \sigma_1^3 \sigma_2^2 \cdot \sigma_1^2 \sigma_1^2
\]

which is 3-increasing with \( u(b,3) = 3 \). Lemma 7.1 tells us that for \( \beta = b \Delta^2 \) we have \( (br(\beta), cl(\beta(3)), A_\beta) \sim (br(\gamma), A_\gamma, cl(\gamma(3))) \), where \( \gamma \in B_6 \) is the braid in (7.2) substituting \( \sigma_1^3 \) for \( w_1 \), \( \sigma_2^2 \) for \( w_2 \) and \( \sigma_1 \) for \( w_3 \). One sees that \( \gamma \) is conjugate to \( v \) in \( B_6 \). Thus \( (br(\beta), cl(\beta(3)), A_\beta) \sim (br(v), A_v, cl(v(3))) \). This implies that \( E(v,3) = [F_\beta] \) is a fibered class of the hyperbolic \( M_\beta \simeq M_v \), and hence \( v \) is pseudo-Anosov. By Theorem 5.2(1), \( \{v_p\} \) has a small normalized entropy.

One sees that \( F_{v_p} \) is \( (p + 3) \)-pronged at \( \partial(v_p,3+2p)F_{v_p} \). Thus \( v_p^\bullet \) is pseudo-Anosov with the same dilatation as \( v_p \) for \( p \geq 1 \). This completes the proof. \( \square \)

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