Topological States in Strongly Correlated Systems

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Abstract
Topological order in strongly correlated systems, including quantum spin liquids, quantum Hall states in lattices and topological superconductivity, is treated. Various metallic non-Fermi-liquid states are discussed, including fractionalized Fermi-liquid (FL*) and phase string theories. Classification of topological states and differences between quantum and classical topology is considered.

Keywords Spin liquid · Topological superconductivity · Fractionalized Fermi liquid · Quantum Hall effect

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1 Introduction: Topological Order
Since 1980s, the investigation and classification of a new type of order in the condensed matter—of topological order, became a field of intensive researches. Main experimental grounds were the discovery of fractional quantum Hall effect [1], which is an essentially correlation phenomenon (unlike the integer quantum Hall effect), and of high-temperature superconductivity in strongly correlated cuprates [2]. Later, a possible equivalence of both these phenomena was established within resonating-valence-bond (RVB) theory [3].

The term “topological order” comes historically from the low-energy effective theory of chiral spin states in topological quantum field theory (TQFT) [4] which describes many-body states with topological degeneracy at low energies. The term “topological” here means the long-range entanglement and therefore refers to quantum topology [5]. It is necessary to distinguish this from the classical topology, which treats the vortices in the superfluid liquid, the difference between the sphere and torus, the vortex in superfluid, etc. [6]. Quantum fluctuations eliminate the degeneracy of the ground state of strongly frustrated systems by quantum tunneling, leading to non-degenerate ground states (up to global symmetry transformations or topological degeneracies). One distinguishes topological order (elementary excitations with an energy gap) and quantum order (the more general case of gapless excitations).

The emergence of correlations in topological systems is rather uncommon. Here, there are no usual “energetic” correlations (precursor of long-range order), but specific correlations appear, which are caused by the topology of the sample. In this case, the ground state is degenerate owing to the topological characteristics and not because of symmetry. The simplest example: the system feels the influence of a single pricked point (the transformation of a sphere into a torus).

From the macroscopic viewpoint, the topological order is characterized by the strong ground-state degeneracy. Topological degeneracy can be realized in protected two-level states—qubits, which enables one to perform topological quantum computation [7]. On the other hand, from a microscopic viewpoint the topological order is the state of matter with gapped energy spectrum [8], which is not reduced to the product of one-particle eigenstates, if we do not take into account phase transitions with closing the energy gap. This means long-range correlations, i.e., entanglement.

Topological phases are phases of matter at zero temperature. They are disordered liquids that seem to have no simple characterization, but actually can have rich patterns of many-body entanglement representing new kinds of order.

Authors of Ref. [9, 10] propose the following classification of gapped topological entangled phases. For gapped quantum systems without any symmetry, one has two
classes: short-range entangled (SRE) states and long-range entangled (LRE) states. SRE states can be transformed into direct product states via local unitary transformations (LUT). All SRE states can be transformed into each other via LUT, and therefore belong to the same phase. On the other hand, LRE states cannot be transformed into direct product states via LUT. A number of LRE states also cannot be transformed into each other, so that they belong to different classes and represent different quantum phases. These quantum phases are just the topologically ordered phases, including Fractional quantum Hall states, chiral and quantum spin liquids. Such different topological orders can result in a quasiparticle spectrum with fractional statistics and fractional charges.

For gapped quantum systems with symmetry, the structure of phase diagrams is much richer. The corresponding LRE phases are called Symmetry Enriched Topological (SET) phases and are described by projective symmetry group (PSG). Symmetry breaking (SB) and LRE can appear together forming SB-SRE states. An example is given by the topological superconducting states. Long-range-entangled states that do not break any symmetry can also belong to different symmetric phases. Examples are $Z_2$ symmetric spin liquids with spin rotation, translation, and time-reversal symmetries.

Not only LRE, but also SRE states with symmetry can belong to different phases. Here, there are two possibilities:

(i) the usual Landau SB states which belong to different equivalent classes of LUT and have different broken symmetries;

(ii) states with short-range entanglement can belong to different equivalent classes of the symmetric LUT even if they do not break any symmetry of the system, i.e., have the same symmetry. Such phases are beyond Landau symmetry breaking theory and are called symmetry protected topological (SPT) phases. An example is given by the band and topological insulators which have the same symmetry, but belong to two different equivalent classes of symmetric LUT.

In the present paper, we review various topological states in strongly correlated systems including quantum spin liquids, quantum Hall states and topological superconductivity. Some further details can be found in the reviews [11, 12].

2 Quantum Spin Liquids

Quantum spin liquids (QSLs) are phases of matter, which are characterized by emergent dynamic gauge fields and topological order, and fractional excitations interacting with the gauge field. The existence of nontrivial many-particle states is confirmed by the Lieb–Schultz–Mattis theorem [13] and its higher-dimensional generalization by Hastings [14]. This states that in the system with the half-integer spin per cell and global U(1) symmetry the excitation spectrum in the thermodynamic limit cannot satisfy simultaneously two requirements: (a) the ground state is unique; and (b) there is a finite gap for all excitations. This means that the state with the gap and unbroken symmetry should have a degeneracy of a topological nature.

According to [15], spin liquids can be divided into four classes: (i) Rigid spin liquid where spinons (and all other excitations) are fully gapped and may have Bose, Fermi, or fractional statistics; (ii) Fermi spin liquid where spinons are gapless and are described by a Fermi liquid theory. (iii) Algebraic spin liquid where the spectrum is gapless, but excitations are not described as free quasiparticles; (iv) Bose spin liquid where low-lying gapless excitations are described by a free-boson theory.

Being examples of quantum-ordered states, different spin liquids cannot be distinguished by their symmetry properties. The first discovered topological invariants that define topological order were (i) the robust ground state degeneracy on a torus and other closed space manifolds, (ii) the non-abelian geometric phases of the degenerate ground states, and (iii) the chiral central charge $c$ of the edge states. The ground states of gapped spin liquids always have topological degeneracy, which cannot be related to any symmetry. Some stable quantum phases have gapless excitations even without any spontaneous symmetry breaking. In particular, such excitations in algebraic spin liquids interact in the limit of zero energy, but the interaction does not open an energy gap. Thus, the quantum order (and not symmetry) protects the gapless excitations and makes algebraic spin liquids and Fermi spin liquids stable [5, 16].

The spin liquid can demonstrate some strange properties. Sometimes excitations can carry fractional statistics. Generally, topological systems can have exotic spectrum of quasiparticle excitations (neutral spin-1/2 fermions – spinons, charged bosons – holons, see an example in Fig. 1). A description of gapless quantum spin liquids in strongly correlated metals is provided by various theories using slave (auxiliary) representations for such quasiparticles [17–19].

The exotic quasiparticles were first introduced by Anderson [21] in the theory of two-dimensional cuprates as

$$\tilde{c}_{i\sigma} = X_i(0, \sigma) = b_j^\dagger f_{i\sigma}. \quad (1)$$

According to Anderson, spinons $f_{i\sigma}$ are fermions and holons $b_j$ are bosons. This choice of statistics is not unique and can be changed depending on the physical problem, e.g., on the presence or absence of antiferromagnetic ordering, see Ref. [20]. The representation Eq. (1) leads to difficulties
in simple mean-field treatments, since the problem of nonphysical states occurs: the on-site no-double-occupancy condition is violated. Thus, spinons and holons become coupled by a gauge field required to satisfy this constraint.

A more general SU(2) representation [17] exploits two types of bosons,

\[
\tilde{c}_{i\uparrow} = \frac{1}{\sqrt{2}} \left( b_{i\uparrow}^\dagger + b_{i\downarrow}^\dagger \right), \\
\tilde{c}_{i\downarrow} = \frac{1}{\sqrt{2}} \left( b_{i\uparrow}^\dagger - b_{i\downarrow}^\dagger \right). 
\]

Upon the treatment of strongly correlated compounds, e.g., of copper–oxygen high-temperature superconductors, the \( t - J \) model (the Hubbard model with the on-site Coulomb repulsion \( U \rightarrow \infty \) and including the Heisenberg exchange) is widely used. The Hamiltonian of this model in the many-electron representation of Hubbard’s operators

\[ X(\Gamma, \Gamma') = |\Gamma\rangle \langle \Gamma'| \]

(where \( \Gamma = 0, \sigma \)) takes the following form:

\[ H = -\sum_{ij} t_{ij} X_{ij}(0, \sigma) X_{ij}(\sigma, 0) + \sum_{ij} J_{ij} S_i S_j. \]

In the representation of auxiliary bosons \( X_{i}(\sigma 0) = f_{i\sigma}^\dagger b_{i\sigma} \), one can introduce the pairings

\[ \chi_{ij} = \sum_\sigma \langle f_{i\sigma}^\dagger f_{j\sigma} \rangle, \quad \Delta_{ij} = \langle f_{i\downarrow} f_{j\downarrow} - f_{i\uparrow} f_{j\uparrow} \rangle. \]

Anderson [21, 22] used the slave-boson approach to construct a uniform RVB state—a symmetric spin liquid with all the lattice symmetries (SU(2)-gapless state). In the uniform RVB phase (uRVB), \( \chi_{ij} = \chi \) for all bonds and is real, and the gap \( \Delta_{ij} \) is zero, so that the spectrum of \( f \)-fermions has the form \( E_k = 2J\chi (\cos k_x + \cos k_y) \).

Later two more spin-liquid states were constructed using the same U(1) slave-boson approach (for details, see [17]). One is the \( \pi \)-flux (\( \pi \)-FL) phase which is a SU(2)-linear symmetric spin liquid with the spectrum

\[ E_k = \pm \frac{3}{4} J |\chi| \sqrt{\sin^2 k_x + \sin^2 k_y}. \]

The other is staggered-flux/d-wave state (staggered flux liquid, sFL) which is a U(1)-linear symmetric state. This phase demonstrates a linear Dirac spectrum at the points \( \pm \pi / 2, \pm \pi / 2 \).

\[ E_k = \pm \frac{3}{4} J \sqrt{\chi^2 (\cos k_x + \cos k_y)^2 + \Delta^2 (\cos k_x - \cos k_y)^2}. \]

Generally, the U(1) and SU(2) spin liquids are unstable at low energies and cannot be treated as the ground states. However, they can provide a good description in a broad intermediate region of length and time scales close to a quantum phase transitions. Thus, the first known stable spin liquid is the chiral spin liquid where the spinons and holons carry fractional statistics and true spin-charge separation occurs. Such a state violates the time reversal and parity symmetries and is a SU(2)-gapped state. The coupling between the slave fermions and the gauge field in the chiral state is identical to the coupling between the electrons and the electromagnetic field. Therefore, the system has properties similar to the Hall effect [16]. The SU(2) gauge fluctuations in the chiral spin state do not result in an instability since the gauge fluctuations are suppressed and become massive due to the Chern–Simons term [24].

A way to obtain a stable deconfined phase is lowering the U(1) or SU(2) symmetry down to a Z_{2sp} gauge structure [17]. Such a phase is called a Z_{2sp} spin liquid or a short-range RVB state. The corresponding low-energy effective theory deals with massless Dirac fermions and fermions with small Fermi surfaces, coupled to a Z_{2sp} gauge field. Since the Z_{2sp} gauge interaction is irrelevant at low energies, the spinons are free fermions at low energies and we have a true spin-charge separation in the Z_{2sp}-gapped spin liquid.

The state with noncollinear SU(2) flux is a Z_{2sp} state where all the gauge fluctuations possess a gap. Here, the fluctuations mediate short-range interactions between fermions and do not radically change the properties of the mean-field solution, the gauge interactions being irrelevant. This implies the existence of a true physical spin liquid which contains fractionalized spin-1/2 neutral Fermi excitations—spinons. Such a spin liquid also has Z_{2sp} vortex excitations, so-called visons. The bound state of a spinon and a Z_{2sp} vortex gives us a spin-1/2 Bose excitation. The ground state degeneracy of the Z_{2sp} RVB state is a reflection of its long-range entanglement. Although all spin-spin correlation functions decay exponentially in the ground state, there are
Einstein–Podolsky–Rosen-type long-range correlations, so that the quantum state retains information about the global topology [25].

Fluctuations of the gauge field are physically chirality fluctuations or fluctuations of orbital current. The corresponding staggered flux (SF) phase is obtained in the slave-boson mean-field approach [17]. The SF state is competing with $d$-wave superconductivity and antiferromagnetic (AFM) ordering in systems with nodal points (high-$T_c$ cuprates). The SF state provides an example of collinear SU(2) flux state which is invariant only under a U(1) rotation. This is a marginal situation.

Another way to get a deconfined phase is to make the gauge boson massive. As mentioned above, most simple example of such a situation is provided by the chiral spin liquid. The picture is as follows [16]. The excitation in the mean field approximation is obtained by inserting a spinon into the conduction band. However, this excitation itself is not physical, since this insertion violates the constraint $\sum_{\sigma} \langle f^\dagger_{i\sigma} f_{i\sigma} \rangle = 1$. The additional density of spinons can be eliminated by adding a vortex flow of the gauge field

$$\Phi = -\pi \sum_i \left( \sum_{\sigma} \langle f^\dagger_{i\sigma} f_{i\sigma} \rangle - 1 \right).$$

(7)

Thus, the physical quasiparticles are spinons dressed by a $\pi$ vortex which carry spin 1/2. At the same time, spinon, which bears a unit charge of the gauge field, has a fractional (semion) statistics, being a bound state of charge and vortex [16].

In the presence of fluxes, quantization in the gauge field gives rise to Landau-type levels [16, 26] (see Fig. 2). In this case, the zero level has a degeneracy equal to the number of the flux quanta. The addition of a flux quantum yields a zero fermionic mode for each type of Dirac fermions. After inclusion the crystal potential, the Landau levels are transformed into narrow correlated bands. In this sense, the Hubbard bands (which can be described in the simplest Hubbard-I approximation, including for degenerate bands [27, 28], as broadened atomic levels) are spinon bands. It can be hypothesized that the Hubbard broadening of levels (for example, due to scattering and resonant broadening [29]) plays a role similar to that of disorder which is crucial for the quantum Hall effect.

A similar theory was developed by Levin and Wen [30] within the doubled Chern–Simons theories where macroscopic chirality is absent and time reversal is not broken. They treated the mutual statistics of spinon and vison excitations which can be considered as mutual semions (see also recent work [31]). Here, encircling a spinon around a vison yields a Berry phase of $\pi$, the charges moving on the square lattice, while the fluxes on the dual lattice. Some corresponding three-dimensional models can be also formulated [30].
In the undoped case (for the Mott insulator), the fermionic sign structure on non-frustrated lattices is completely reducible (removable), since the problem of interacting electrons is simply the localized-spin problem with distinguishable electrons (however, for a frustrated lattice, this problem can also include the problem of sign). A doped hole remains stable because of the confinement of spinons and holons by the field of the phase shift, in spite of the fact that the background is a spinon–holon sea. The exact deconfinement appears only in the limit of the zero doping, when the hole is decomposed into a holon and a spinon. At doping, there appears an irreducible sign structure, but for low doping this structure is very rarefied in comparison with the equivalent system of free fermions.

A way to treat the Weng statistics is systematically calculating the irreducible signs in terms of world lines [34]. The irreducible signs entering into the partition function of the $t - J$ model can be represented as

$$Z_{t - J} = \sum_c (-1)^{N^c_\uparrow + N^c_\downarrow} Z[c]. \tag{9}$$

The sum is taken over the configurations of the world lines $c$ of spinons and holes. Relative to each other, the holes behave as fermions, and $N^c_{\uparrow/\downarrow}$ numerates the exchanges in the configuration $c$. The spinons are bosons with a solid core, the up spins being considered as the background. The sign structure is determined by the parity of the number of hole–spinon collisions for a concrete configuration of world lines.

The projected annihilation electron operator is written in the slave-fermion representation, the sign coefficient $(-\sigma)^j$ being included explicitly:

$$\tilde{c}_{ja} = (-\sigma)^j f_j^\dagger b^\sigma_a,$$

where $f_j^\dagger$ is the Fermi holon, and the bosons $b^\sigma_a$ are used for the spin system. Because of the sign coefficient, the spin-spin superexchange acquires the general negative sign. Therefore, the ground state wave function for the purely spin system in the half filled case does not contain nodes.

Fractionalized Fermi liquid (FL*) picture [19, 35] presents another scenario for anomalous frustrated metallic systems. This was initially formulated within the framework of the $s - d(f)$ exchange (Kondo lattice) model. This theory starts from a fractionalized spin liquid with exotic excitations and adds conventional carriers in a second band. Low-temperature properties are Fermi-liquid-like or not, depending on that the excitations of the spin-liquid subsystem are gapped or gapless. FL* state may demonstrate a number of instabilities at lowering temperature, including AFM ordering and unconventional superconductivity. In this state, which is a metallic spin liquid of topological nature, charged excitations have conventional quantum numbers (charge $\pm e$ and spin 1/2), but coexist with additional fractionalized degrees of freedom in the second band. The FL* concept was applied to frustrated Kondo lattices [36]. The most important topological feature of FL* is that violates the Luttinger theorem—has the small Fermi surface (FS) which does not include localized $d(f)$-electrons, unlike the ground state of the Kondo lattice—heavy Fermi liquid (FL). In this state, fermionic spinon excitations form a “ghost” Fermi surface.

In the FL* phase, the anomalous Kondo pairing, i.e., the condensation of Higgs boson $b$ is absent, $\langle b_i \rangle = \langle f_j^\dagger c_i \rangle = 0$. At the same time, there is an anomalous RVB-type average at different lattice sites, $\chi_{ij} = \langle f^\dagger c_i f_j \rangle$. Thus, the localized spins do not take part in the formation of the Fermi surface (however, they form a separate spinon FS). Owing to the heritage of the spin liquid, FL* state possesses deconfinement exotic excitations. In the space dimensions $d \geq 2$, a $Z_2$ spin liquid type is stable, and at $d \geq 3$ a U(1) spin liquid can exist. In this phase, electronic specific heat is not linear, but $C/T$ diverges logarithmically [19]. Against the background of this state, there can occur a magnetic instability for the spinon, rather than electron FS, which leads to an itinerant-magnetic state with the spin-density wave, SDW*.

As follows from the Lieb–Schulz–Mattis theorem, a state with a gap and unbroken symmetry should have a ground-state degeneracy which has a topological nature. In the two-dimensional case, the existence of topologically different sectors, being supported by existence of gapped vison excitations on a torus, protects the FL* state. Usually the FL* theory is applied to 2D case, but it can be generalized to 3D case due to Oshikawa’s topological analysis [19, 37]. The FL* state was treated for near both the metal-insulator transition and transition between metallic states with large and small Fermi surfaces. The corresponding phase diagrams were built in both doping case and the situation of interaction-driven transition [19, 38, 39].

In the FL* state, there are low-energy excitations of local moments, which give a topological contribution to a change in the momentum of the crystal. Indeed, the action of a vortex flux is analogous to the Lieb–Schulz–Mattis transformation [13], so that the spin-liquid state in the dimensionality $d = 2$ acquires this momentum change. According to the treatment [37], based on the threading the flux quantum and on the global gauge symmetry $U(1)$ (charge conservation), the existence of a nonmagnetic FL* state with small FS is permitted if we include global topological excitations which naturally appear in the gauge theories. Therefore, the violation of the Luttinger theorem must be accompanied by the appearance of a topological order [19]. Generally, the formation of a small FS, of the Hubbard splitting, and of a state with disordered local moments can be connected with the topological order in a spin-liquid-like state. The Oshikawa theorem [37] holds for both the metal and insulator, so that we can deal with the Mott–Hubbard state too.
An exotic type of quantum order occurs in the string-net condensation theory [5, 30]. Such string states are similar to Bose-condensed ones, but the condensate is formed here from extended objects. The collective excitations in this condensate are not usual Bose particles: the vibrations of closed strings generate gauge bosons. Moreover, at breaking of strings, their ends give fermions. The quantum topology describing such internal degrees of freedom can be reformulated as a field-theory of closed strings, which are constructed from local spins or pseudospins (qubits [7]). In turn, one can to pass from these strings to the concept of “electric” and “magnetic” gauge fields [40].

3 Quantum Hall States

Similar to a quantum chiral spin liquid, in the quantum Hall effect (QHE) situation we deal with a gapped essentially topological matter including topological order. Owing to occurrence of a gap and topological degeneracy, in the effective magnetic (e.g., gauge) field $T$-symmetry becomes violated, at least dynamically, and the Chern–Simons term arises which represents charge-flux attachment. It is important that any system with nonzero Chern number includes gapless edge modes. These modes, being initially discovered in the integer quantum Hall effect, are chiral, i.e., propagate only in one direction [41].

Low-energy effective theories of fractional quantum Hall (FQH) states are U(1) Chern–Simons theories which describe universal properties. The dependence of the ground-state degeneracy on the topology of the space indicates the existence of a long-range order in FQH liquids, despite the absence of long-range correlations for local physical operators. Thus, FQH liquids are characterized by hidden long-range orders [24].

In the QHE situation, electrons in a magnetic field move along circular cyclotron orbits, and also around other electrons. Then, the Landau level number is determined by the number of wavelengths on a given circle. The description of strongly correlated systems requires the formation of flat-band structure, which is similar to system of the Landau levels. Band structures with a gap can be classified topologically by considering the equivalence classes of the Bloch Hamiltonian that can be continuously deformed into one another without gap closing [42]. In this case, the total Berry flux over the Brillouin zone (the sum of the integrals of the Berry curvature over all occupied bands) is a Chern topological invariant $C$ which is the total Berry flux in the Brillouin zone. Thus, we obtain the Chern insulator which has gapless edge states, similar to topological insulators (however, in the latter case the topological case is invertible—there are no topological excitations [5]). These states can be treated as a consequence of the cyclotron orbit bounce off the edge. The electron states responsible for this motion are chiral.

QHE in two-dimensional (2D) systems is usually related to the presence of an external magnetic field, which splits the electron spectrum into Landau levels. However, QHE may also result from breaking time-reversal symmetry (i.e., by a magnetic order) without a net magnetic flux through the unit cell of a periodic 2D system [43].

There is another topological class of insulating band structure where, although the $T$ symmetry is not broken initially, the spin–orbit interaction is present [42]. Such a 2D topological insulator is called a quantum spin Hall insulator. This system is described by the so-called double Haldane model [43] where the Hall conductivity has opposite signs for up and down spins. In an applied electric field, the up and down spins give Hall currents that flow in opposite directions. Thus, the total Hall conductivity is zero, but there occurs a spin current and a spin Hall conductivity which is quantized. Taking into account the complex amplitudes of hopping between next-nearest neighbors, a gap opens at the Dirac points, so that the $T$ symmetry becomes broken. Thus, we have two bands with Chern numbers (of $C = \pm 1$) and, in the case of half filling, the integer QHE. At some partial fillings, the fractional QHE can arise, i.e., topological state of a quantum incompressible liquid [44].

As first stated in Ref. [45], appropriate combination of geometric frustration, ferromagnetism, and spin-orbit interactions can give rise to nearly flat bands with a large band gap and nonzero Chern number. Such bands (which mimic Landau levels) can give rise to fractional QHE even at high temperatures.

The realization of topological Hall phases on a lattice in the absence of an external magnetic field (the “anomalous Hall effect”) requires a number of simultaneous conditions. The first one is the presence of a nearly dispersionless (“flat”) bare energy band with nontrivial topology (nonzero Chern number), which provides the picture similar to that of the Landau level (the quasiparticle kinetic energy is quenched in such a topological band). The second condition is a strong interelectron interaction violating the Fermi liquid picture. Strong correlations are especially important for the fractional Hall effect, when the Landau levels are highly degenerate. Since the Hall conductivity is odd with respect to time reversal, topologically nontrivial states can arise when the $T$ symmetry becomes violated.

In a number of papers, attempts were made to take into account strong interactions in a frustrated system in order to obtain phases with topological order within the framework of simple mean-field-type approximations. In particular, the anomalous QHE can occur dynamically in the generalized Hubbard model on a honeycomb lattice and other systems with a quadratic zone intersection point, including the diamond and kagome lattices. However, later numerical studies
did not confirm the occurrence of exotic topological phases predicted by mean field theories. The matter is that, instead of causing spontaneous $T$-symmetry breaking, strong interactions also tend to stabilize competing long-range orders breaking translational symmetry (see the discussion in [46]). Thus, when describing lattice QHE systems, it would be correct to start immediately from a strongly correlated state.

In the presence of interaction, the system can be described within the framework of the Hubbard model with the band width $W$ and the Coulomb repulsion $U$. In the case of the Chern zone, the charge cannot be localized (“the obstruction of the Wannier states”), so the narrow band regime cannot be treated in the pure spin model [47]. For small $W/U$ ratios, the charge gap is determined by $U$ and is not necessarily removed by the spin disorder. A destruction of quantum Hall ferromagnetism by the kinetic term is possible at integer filling of the system [46, 47]. For small $W/U$ ratios, the charge gap is determined by $U$ and is not necessarily removed by the spin disorder. A destruction of quantum Hall ferromagnetism by the kinetic term is possible at integer filling of the system.

The effective Lagrangian describing QHE for electrons in a magnetic field with inclusion of the Chern–Simons term has the form [16, 48]:

$$\mathcal{L}_{CS} = -\frac{m}{4\pi} e^{\nu a} a_{\mu} \partial_{\nu} a_{\mu} - \frac{e}{2\pi} \epsilon^{\nu \mu \lambda} A_{\mu} \partial_{\nu} A_{\lambda}. \quad (10)$$

Here, $a_{\mu}$ is the internal gauge field, $A_{\mu}$ is the vector potential of the external electromagnetic field, and $e$ is the antisymmetric second-rank tensor. The filling is given by $v = 1/m$, $m$ being the charge of the gauge field, i.e., the number of wavelengths in the situation where one electron encircles another.

Equation (10) describes only the linear response of the ground state to the external field. To obtain a more complete description of a fractional quantum Hall liquid, one has to introduce excitations. Although in the Laughlin fractional Hall ground state the electron is a fermion, the excited states of the system have a Bose type. Then, $m$ is an even integer for Bose states and odd for Fermi states. A fractional quantum Hall liquid contains two types of quasiparticles: a quasi-hole (or vortex) in the original electron condensate and a quasi-hole (or vortex) in the new Bose condensates.

Introducing the gauge field $\tilde{a}_{\mu}$ which describes a bosonic current, we can represent the total Lagrangian in a matrix form which is analogous to (10):

$$\mathcal{L} = -\frac{1}{4\pi} e^{\nu a} K_{IJ} a_{\mu} \partial_{\nu} a_{\mu} - \frac{e}{2\pi} \epsilon^{\nu \mu \lambda} q_{L} A_{\mu} \partial_{\nu} a_{\lambda}. \quad (11)$$

Here, $(a_{1\mu}, a_{2\mu}) = (a_{\mu}, \tilde{a}_{\mu})$, and the matrix $K$ reads

$$K = \begin{pmatrix} p_1 & -1 \\ -1 & p_2 \end{pmatrix}. \quad (12)$$

where $p_1$ is the starting number $m$ for fermion electronic states, $p_2$ is an even number describing the bosonic field, $q$ is the charge vector, and $q^f = (q_1, q_2) = (1, 0)$. The occupation numbers now are $v = q^T K^{-1} q$. This construction corresponds to the doubled Chern–Simons theory for the topological interaction among both quasiparticles and vortices [49].

The topological theory of abelian phases of a general gapped two-dimensional matter is described by the Lagrangian [48, 50]

$$\mathcal{L}_{\text{bulk}} = -\frac{1}{4\pi} e^{\nu a} a_{\mu}^J K_{IJ} \partial_{\nu} a_{\mu}^J - \frac{e}{2\pi} t_{AI} \epsilon^{\nu \mu \lambda} A_{\mu}^I \partial_{\nu} a_{\lambda}^I. \quad (13)$$

Here, $A_I^I (I = 1, 2, \ldots, N)$ is a set of gauge fields, $K_{IJ}$ is now an $N \times N$ symmetric integer-number matrix determining the mutual statistics of excitations, $j_I$ are quasiparticle currents, and $t_{AI}$ is a charge vector determining the occupation numbers. The ground state degeneracy on the torus (which is a characteristic of the topological order) is given by the determinant of the matrix $K$. This determinant also gives the number of independent types of anyons, i.e., fractional charge particles. The last term in Eq. (13) describes the coupling to external source fields $A_{\mu}^I$ ($A = 1, 2, \ldots, M$) which have the global $U(1)_A$ symmetry.

The Hall conductance of the system is quantized if the many-body ground state on a torus is not degenerate and has a finite energy gap. If the many-body ground states have degeneracies, then $K$ and the Hall conductance can be

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**Fig. 3** Two possible phase diagrams for spinful Chern band at integer filling $\nu = 1$ according to [47]. QHFM, QHAF and QHSL stand for quantum Hall ferromagnetism, quantum Hall antiferromagnetism and quantum Hall spin liquid. QHSL passes into QHAF through a continuous transition. In the QHAF state, partial spin polarization coexists with antiferromagnetic order.
Fractional quantum Hall states are incompressible and possess a finite energy gap for all their bulk excitations. At the same time, such liquids in finite-size systems always contain one-dimensional gapless edge excitations with a complex structure that reflects a bulk topological order (the bulk–boundary correspondence). In the case of non-abelian fractional quantum Hall liquids, the edge states produce even more exotic one-dimensional correlated systems [16].

Effective field theory Eq. (13) provides understanding of the edge states physics. In the absence of external sources \( \mathcal{A}^4 \), a chiral Luttinger theory arises [50, 51]:

\[
\mathcal{L}_{\text{edge}} = \frac{1}{4\pi} \left[ K_{IJ} \partial_i \phi^I \partial_i \phi^J - V_{IJ} \partial_i \phi^I \partial_j \phi^J \right].
\]  

Here, \( \phi^I \) are \( N \) chiral bosons, \( V_{IJ} \) is a nonuniversal positive definite real matrix depending on the microscopic properties of the edge. Thus, the above \( K \) matrix determinant gives the number of edge states.

In the case of a narrow strongly correlated band, the representation of many-electron Hubbard operators \( \hat{c}_{i\sigma} = \chi_i(0, \sigma) \) in terms of slave (auxiliary) bosons and fermions can be used again to describe various classes of quantum Hall spin liquids. This construction has two forms: charged fermionic holons and neutral bosonic (Schwinger) spinons (slave-fermion representation) or neutral fermionic (Abrikosov) spinons and bosonic holons (slave-boson representation). In addition, a \( U(1) \) gauge field arises due to the constraint on filling at a site. Such a construction (which is also called particle representation) can be introduced for arbitrary symmetry groups describing different types of anyons [52, 53]. This enables one to establish the correspondence in the physics of Landau levels and Chern bands, continuous fractional quantum Hall phases, and spin liquids for lattice models. In this context, phase transitions at half-filling are described as transitions with a change in the Chern number between insulator phases [54].

The topological interaction can be treated within a quantum field theory (see, e.g., [49]). In 2+1 dimensions, we have to take into consideration both quasiparticles and vortices, so that the situation is similar to the bosonic Chern–Simons QHE theory and flux and charge can be attributed to the particles in such a way that the Berry phases occur. The situation is similar to the Aharonov–Bohm effect, where the Fermi–Bose statistics transmutation can take place. The wavefunction acquires a well-defined Berry phase given by the integral of the Berry connection. This may be expressed as a surface integral of the Berry curvature, the Chern invariant being the total Berry flux in the Brillouin zone [42].

The Chern–Simons Lagrangian [49] describes coupling of a quasiparticle current \( j^\mu \) and a vortex current \( J^\mu \) to electric and magnetic gauge potential components \( a_\mu \) and \( b_\mu \):

\[
\mathcal{L} = \frac{1}{\pi} \epsilon^{\mu\nu\sigma} b_\mu \partial_\nu a_\sigma - a_\mu b^\mu - b_\mu J^\mu.
\]  

In 3+1 dimensions, the situation is similar, but we have strings instead of the vortices, the vector potential \( b \) being an antisymmetric tensor \( b_{\mu\nu} \).

The slave-fermion representation \( \hat{c}_{i\sigma} = b_{i\sigma} f_i \) allows one to describe a \( Z_2 \) spin liquid as singlet pairing of Schwinger bosons with a spin gap and topological order. In the Bose condensate regime, also the antiferromagnetic phase can be described (cf. [55]). In this case, the charge response of the Chern insulator is preserved, which just means the state of the quantum Hall spin liquid. The construction of this state is generalized to the fractional filling (fractional QHE). Thus, the quantum Hall \( Z_2 \) spin liquid gives a realization of eight different abelian topological orders with four anyons (ldet \( K \) = 4). In this sense, they are equivalent to eight abelian topological superconductors of Kitaev’s 16-fold way [41].

The Fermi spinon representation \( \bar{c}_{i\sigma} = b_{i\sigma} f_i \) describes a \( U(1) \) spin liquid having a spinon Fermi surface. This is the parent state for the \( Z_2 \) spin liquid which occurs at lower-lying the gauge symmetry. For even Chern invariants \( C \), the so-called composite Fermi liquid state can be constructed. This is a paramagnetic and metallic analogue of a spin liquid with a purely spinon Fermi surface [47]. Such a state enables one to partially preserve the electronic degrees of freedom, including nonzero residue. The formation of this state is owing to the separation of spin and charge degrees of freedom, being connected with strong correlations.

The bosonic integer QHE (BIQHE) phase presents the simplest Chern band with an even Chern invariant \( C \) [47]. The effective action for this phase reads

\[
\mathcal{L} = \mathcal{L}_{\text{BIQHE}}[A, A + a] + \mathcal{L}_{FS}[f', -a].
\]  

where \( A \) is an external \( U(1) \) gauge field. One has in terms of a differential form

\[
\mathcal{L}_{\text{BIQHE}} = \frac{C}{4\pi} (A + a) d(A + a),
\]  

\[
\mathcal{L}_{FS} = f'_i \left( \partial_\tau - \mu + ia_0 \right) f_i - \frac{\hbar^2}{2m^*} f'_i \left( -i \partial_\tau + a_0 \right) f_i.
\]  

Here, \( \mu \) is the chemical potential, \( m^* \) is the effective mass, and \( a_0 \) and \( a_i \) are the temporal and spatial components of the gauge field [48]. The Fermi spinons \( f_i \) partially fill the band without Chern number and form a Fermi surface.

For odd Chern invariants \( C = 1, 3, 5, 7, \ldots \), exotic states with the Hall conductivity \( \sigma_{xy} = C \) are formed. These are eight types of spin liquids with a half-integer chiral central
charge $e = C - 1/2$, which are analogous to eight non-abelian Kitaev’s superconductors [41].

4 Topological Superconductivity

The experimental discovery of the superconductive order by Kamerlingh Onnes historically resulted in the theory of broken symmetry; however, a later quantum topological investigation complicates this picture. In fact, the superconductivity can be described by the Ginzburg–Landau theory with a dynamic $U(1)$ gauge field [5]. Frequently, the superconductivity is described by the Ginzburg–Landau theory without this field, instead of which a violated symmetry $U(1)$ is treated. According to recent theoretical developments [49, 56], the real superconductors in an electromagnetic gauge field are not states with violated $U(1)$ symmetry, but can be treated as topologically ordered states. The condensation of an Cooper pair with charge $2e$ breaks the $U(1)$ gauge theory, reducing it to the $Z_2$ gauge theory at low energies. The latter is the effective theory of topological order $Z_2$, so that a real superconductor has such a topological order. The superconductor has string excitations, which characterize a topological order—loops of the flux $hc/2e$.

Topological superconductors have a fully gapped spectrum below the critical $T_c$, the topological surface states dominating above $T_c$, similar to topological insulators [57]. Topological superconductivity is characteristic for some doped topological insulators. To realize topological superconductivity, strong spin-orbit coupling, especially for heavy atoms, is favorable. The corresponding examples are some Sn-based superconductors, e.g., Sn$_x$Sb$_{1-x}$ and SnAs [58].

The presence of essential topology in SnAs is confirmed by first principle calculations of $Z_2$ invariant [59].

A topologically non-trivial structure exists only in two dimensions (2D), since General lattice Hamiltonians do not have such a structure in one and three-dimensions. Nevertheless, topological superconductors are possible in 1D and 3D, as well as in 2D, owing to the specific symmetry on superconductors [57]. If no symmetry is assumed, the only possible topological phase is the quantum Hall state with a non-trivial Chern number. Thus, some additional symmetry is required to obtain other topological phases. If the time-reversal $T$ symmetry is present, one can obtain new topological phases, including the quantum spin Hall phase in the 2D case and the topological insulator phase in the 3D case.

For the 2D case, the particle-hole symmetry does not result in a new topological structure. Similar to the quantum Hall or spin Hall states, one can define the Chern number or the 2D $Z_2$ index $(-1)^{\nu_2}$ in the absence or presence of $T$ symmetry, respectively.

In the presence of $T$ symmetry, the 3D $Z_2$ index can be introduced in the same way as in topological insulators, but the combination with particle-hole symmetry enables one to introduce a more sophisticated integer topological number [57]. Thus, the particle-hole symmetry plays an important role in the 3D case.

Due to the Higgs phenomenon, the ground state spectrum of a two-dimensional superconductors with electrically charged paired fermions interacting with a dynamical electromagnetic field has a gap. This means a description in terms of 2D $Z_2$ spin liquid within the framework of effective $Z_2$ gauge theory with a $Z_2$ topological order. According to [5, 49], this construction can be generalized to treat 3D $Z_2$ spin liquid, its $Z_2$ topological order being the same as in an $s$-wave superconductor.

A 2D superconductor possesses two types of point excitations: Bogoliubov quasiparticles and vortices, their braiding being possible. Moroz et al. [50] developed a low-energy description for spin-singlet pairing states by elaborating Chern–Simons field theories for even-wave superconductors: $s$-wave, $d + id$, etc. Combining topological order and symmetries in superconductors, this approach demonstrates that these systems are symmetry-enriched topological (SET) phases discussed in Sect. 1.

The corresponding theories of chiral spin-singlet superconductors are described by Kitaev’s classification of topological superconductors (the “16-fold way”). For chiral spin-singlet superconductors having the chirality parameter $k$ (i.e., the orbital angular momentum of a Cooper pair, which is an even integer ($0$ for $s$-pairing and $2$ for $d$-pairing), one finds $c = k$. According to Kitaev [41], chiral superconductors have a $Z_{16}$ bulk classification. Since each Majorana mode gives a half unit of the central charge, the total chiral central charge is $e = \nu/2$ with $\nu$ the Chern number.

For $s$- and $d$-wave superconductor, one has the four-component theory with the resulting $K$-matrix, respectively

$$K = \begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \quad (19)$$

This $K$-matrix contains the bosonic block $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$. In the case of a $s$-wave superconductor, Bogoliubov particles carry spin and vortices—magnetic flux, their composite being a boson which bears both spin and vortex charge. Their mutual braiding gives the statistical angle $\pi$: the Bogoliubov quasi-particle accumulates a minus sign when encircling a vortex.

In a chiral $d + id$ superconductor, parity and $T$ symmetry are violated spontaneously leading thereby to anyon statistics of excitations. Now, the vortex is a semion, which is demonstrated by the Berry phase gained via exchange of two identical vortices.
For a spin-singlet superconductor, the chiral central charge $c$ is equal to the chirality parameter $k$. The states with an odd $c$ constitute another class of abelian states probably describing some spin-triplet superconducting phases [50]. Thus, spin-triplet superconducting pairing is important from the topological point of view, since it can mean existence of topological states and Majorana fermions. Recently, superconducting compound UTe$_2$ has been supposed to be a candidate for a chiral spin-triplet topological superconductor near a ferromagnetic instability [60]. Inelastic neutron scattering demonstrates that superconductivity in UTe$_2$ near antiferromagnetic order is coupled to a sharp magnetic excitatio (a resonance) at the Brillouin zone boundary. This suggests that antiferromagnetic spin fluctuations may lead to spin-triplet pairing.

A doping-induced transition from Mott insulator to superconductor occurs naturally provided that the parent Mott insulator is a $Z_2$ spin liquid. On the other hand, doping a U(1) spin liquid naturally provides a Fermi liquid phase, and not a superconductor, since there is no pairing of spinons in the parent Mott insulator [39, 61].

According to Ref. [61], in the case of the triangular lattice two different chiral spin liquid phases are possible: CSL1 which is an analogue of fractional quantum Hall state in spin system, and CSL2 is a “gauged” chiral superconductor, the $n = 4$ member of the Kitaev classification. Both CSL1 and CSL2 are described in mean-field approach of Abrikosov fermions, and the doping is treated by introducing charged holons. In CSL1, the Fermi spinons are in a Chern insulator state with $C = 2$. For CSL2, the spinon is in a $d + id$ superconductor state, which is similar to a $Z_2$ spin liquid. Indeed, a $Z_2$ spin liquid can be treated in terms of Fermi spinons in a BCS state. Doped charges are treated in terms of Bose holons, which can condense. Holon condensation transforms the spinon pairing into usual Cooper electron one, which results in a superconducting phase.

At doping CSL2, holons are condensed to form a topological $d + id$ superconductor. At doping CSL1, two scenarios are possible. Holon condensation results in a chiral metal with doubled unit cell and finite Hall conductivity. In the second scenario, the internal magnetic flux is set with doping and holons form a bosonic integer quantum Hall (BIQH) state which is identical to a $d + id$ superconductor.

5 Conclusions

In the present paper, we have considered various aspects of quantum topological states. This issue should be distinguished from the problems of the Fermi surface topology [62]: their phenomenological treatment, as well as physics of topological insulators [42], is close to classical topology.

The traditional approach to the problem of phase transitions was based on the Landau broken-symmetry theory. The new theoretical developments in the condensed matter physics are connected with new kinds of orders and classes of matter which can violate the Landau theory. Moreover, there exist various combinations of symmetry breaking and topological orders (long-range entanglements) which occur together in such states as SB-LRE (see the Introduction). The topological superconducting states provide examples of such phases [5].

The occurrence of exotic states is possible too. For example, in Ref. [63], a Chern–Simons theory was constructed for the doped spin-1/2 kagome system. This system turns put to be an exotic superconductor that violates time-reversal symmetry, carries minimal vortices of flux $hc/4e$ and, in addition to the usual spin-1/2 fermionic Bogoliubov quasiparticles, contains fractional excitations. These are Fermi quasiparticles with spinon mutual statistics and spin-1/2 quasiparticles with Bose self-statistics.

It seems that there is a whole new world before us, which is to be investigated. The new topological paradigm may contribute to our understanding of fundamental questions of nature (see discussion in Refs. [7, 11, 16]). Most interesting results have been up to now obtained for 2D quantum systems, whereas a detailed classification of 3D ones is an open problem [5].

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