Balayage of Measures with respect to Classes of Subharmonic and Harmonic Functions

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Abstract

We investigate some properties of balayage, or, sweeping (out), of measures with respect to subclasses of subharmonic functions. The following issues are considered: relationships between balayage of measures with respect to classes of harmonic or subharmonic functions and balayage of measures with respect to significantly smaller classes of specific classes of functions; integration of measures and balayage of measures; sensitivity of balayage of measures to polar sets, etc.

1 Introduction

The origins of the concept of balayage, or, “sweeping (out)” etc., of measures or functions are the studies of Henri Poincaré, de la Vallée Poussin, Henri Cartan, Marcel Brelot and many others. A detailed historical review of potential theory is given in in [5]. In [18], we investigate various general concepts of balayage. In this article we deal with particular cases of such balayage with respect to special classes of subharmonic functions.

The general concept of balayage can be defined as follows. Let \( R \) be a (pre-)ordered set with a (pre-)order relation \( \leq \). Let \( L \) be a set with a subset \( H \subset L \). A function \( \omega: L \to R \) can be called the balayage of a function \( \delta: L \to R \) with respect to \( H \), and we write \( \delta \preceq_H \omega \), if the function \( \omega \) majorizes the function \( \delta \) on \( H \):

\[
\delta(h) \leq \omega(h) \quad \text{for each } h \in H \subset L. \tag{1}
\]

In this article, \( R \) is the extended real line, \( L \) is the class of all upper semicontinuous functions on an open set \( O \) in a finite-dimensional Euclidean space, \( H \) is a subclass of subharmonic functions on \( O \), and \( \delta \) and \( \omega \) is a pair of Radon positive measures on \( O \) with compact supports in \( O \). In this case, relationship (1) turns into inequalities of the form

\[
\delta(h) := \int_O h \, d\delta \leq \int_O h \, d\omega =: \omega(h) \quad \text{for each } h \in H \subset L. \tag{2}
\]
We investigate properties of balayage of measures with respect to classes of harmonic, subharmonic, and special subharmonic functions.

We proceed to precise and detailed definitions and formulations.

2 Definitions, notations and conventions

The reader can skip this Section 2 and return to it only if necessary.

We denote by \( \mathbb{N} := \{1, 2, \ldots \} \), \( \mathbb{R} \), and \( \mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\} \) the sets of natural, of real, and of positive numbers, each endowed with its natural order (\( \leq \), sup / inf), algebraic, geometric and topological structure. We denote singleton sets by a symbol without curly brackets. So, \( \mathbb{N}_0 := \{0\} \cup \mathbb{N} =: 0 \cup \mathbb{N} \), and \( \mathbb{R}^+ \setminus 0 := \mathbb{R}^+ \setminus \{0\} \) is the set of strictly positive numbers, etc.

The extended real line \( \overline{\mathbb{R}} := -\infty \sqcup \mathbb{R} \sqcup +\infty \) is the order completion of \( \mathbb{R} \) by the disjoint union \( \sqcup \) with \( +\infty := \sup \mathbb{R} \) and \( -\infty := \inf \mathbb{R} \) equipped with the order topology with two ends \( \pm \infty \), \( \mathbb{R}^+ := \mathbb{R}^+ \sqcup +\infty \); \( \inf \emptyset := +\infty \), sup \( \emptyset := -\infty \) for the empty set \( \emptyset \) etc. The same symbol 0 is also used, depending on the context, to denote zero vector, zero function, zero measure, etc.

We denote by \( \mathbb{R}^d \) the Euclidean space of \( d \in \mathbb{N} \) dimensions with the Euclidean norm \( |x| := \sqrt{x_1^2 + \cdots + x_d^2} \) of \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \).

We denote by \( \mathbb{R}^d_\infty := \mathbb{R}^d \cup \infty \) the Alexandroff one-point compactification of \( \mathbb{R}^d \) obtained by adding one extra point \( \infty \). For a subset \( S \subset \mathbb{R}^d_\infty \) or a subset \( S \subset \mathbb{R}^d \) we let \( \mathcal{C}S := \mathbb{R}^d_\infty \setminus S \), \( \operatorname{clos} S \), \( \operatorname{int} S := \mathcal{C}(\operatorname{clos} S) \), and \( \partial S := \operatorname{clos} S \setminus \operatorname{int} S \) denote its complement, closure, interior, and boundary always in \( \mathbb{R}^d_\infty \), and \( S \) is equipped with the topology induced from \( \mathbb{R}^d_\infty \). If \( S' \) is a relative compact subset in \( S \), i.e., \( \operatorname{clos} S' \subset S \), then we write \( S' \subset S \). We denote by \( B(x, t) := \{y \in \mathbb{R}^d : |y - x| < t\} \), \( \overline{B}(x, t) := \{y \in \mathbb{R}^d : |y - x| \leq t\} \), \( \partial \overline{B}(x, t) := \overline{B}(x, t) \setminus B(x, t) \) an open ball, closed ball, a circle of radius \( t \in \mathbb{R}^+ \) centered at \( x \in \mathbb{R}^d \), respectively. Besides, we denote by \( B := B(0, 1) \), \( \overline{B} := \overline{B}(0, 1) \) and \( \partial B := \partial \overline{B}(0, 1) \) the open unit ball, the closed unit ball and the unit sphere in \( \mathbb{R}^d \), respectively.

Throughout this paper \( O \neq \emptyset \) will denote an open subset in \( \mathbb{R}^d \), and \( D \neq \emptyset \) is a domain in \( \mathbb{R}^d \), i.e., an open connected subset in \( \mathbb{R}^d \).

For \( S \subset \mathbb{R}^d_\infty \), \( C(S) \) is the vector space over \( \mathbb{R} \) of all continuous functions \( f : S \to \mathbb{R} \) with the sup-norm, \( C_0(S) \subset C(S) \) is the subspace of functions \( f \in C(S) \) with compact support \( \operatorname{supp} f \subset S \), and \( \operatorname{usc}(S) \) is the convex cone over \( \mathbb{R}^+ \) of all upper semicontinuous functions \( f : S \to \mathbb{R} \cup -\infty = \mathbb{R} \cup +\infty \). For \( S \subset \mathbb{R}^d \), \( \operatorname{har}(S) \) and \( \operatorname{sbh}(S) \) are the collections of all functions \( u \) which are harmonic and subharmonic on some open set \( O_u \supset S \), respectively. In addition, \( \operatorname{sbh}_c(S) \subset \operatorname{sbh}(S) \) consists only of functions \( u \in \operatorname{sbh}(S) \) such that \( u \neq -\infty \) on each connected component of \( O_u \).

The convex cone over \( \mathbb{R}^+ \) of all Borel, or Radon, positive measures \( \mu \geq 0 \) on the \( \sigma \)-algebra \( \operatorname{Bor}(S) \) of all Borel subsets of \( S \) is denoted by \( \operatorname{Meas}^+(S) \); \( \operatorname{Meas}^+_{\operatorname{cmp}}(S) \subset \operatorname{Meas}^+(S) \) is the subcone of measures \( \mu \in \operatorname{Meas}^+(S) \) with compact support \( \operatorname{supp} \mu \) in \( S \), \( \operatorname{Meas}^{+1}(S) \) is
the convex set of probability measures on $S$, $\text{Meas}_{\text{cmp}}^{1+}(S) := \text{Meas}^{1+}(S) \cap \text{Meas}_{\text{cmp}}(S)$. So, $\delta_x \in \text{Meas}_{\text{cmp}}^{1+}(S)$ is the Dirac measure at a point $x \in S$, i.e., $\text{supp} \delta_x = \{x\}$, $\delta_x(\{x\}) = 1$.

We denote by $\mu \big|_{S'}$ the restriction of $\mu$ to $S' \in \text{Bor}(S)$. The same notation is used for the restrictions of functions and their classes to sets.

Let $\Delta$ be the Laplace operator acting in the sense of the theory of distributions, $\Gamma$ be the gamma function. For $u \in \text{shb}_+(O)$, the Riesz measure of $u$ is a Borel (or Radon [24, A.3]) positive measure

$$\Delta u := c_d \Delta u \in \text{Meas}^+(O), \quad c_d := \frac{\Gamma(d/2)}{2\pi^{d/2} \max\{1, d-2\}}. \quad (3)$$

### 3 Inward filling of subsets in an open set

Let $O$ be a topological space, and $S \subset O$. We denote by $\text{Conn}_O S$ the set of all connected components of $S$. We write $S \subset O$, if the closure of $S$ in $O$ is a compact subset of $O$.

**Definition 1.** The union of a subset $S \subset O$ with all connected component of $C \in \text{Conn}_O(O \setminus S)$ such that $C \subset O$ will be called the inward filling of $S$ with respect to $O$ and is denoted further as

$$\text{in-fill}_O S := S \bigcup \left( \bigcup \{ C \in \text{Conn}_O(O \setminus S) : C \subset O \} \right).$$

Denote by $O_{\infty}$ the Alexandroff one-point compactification of $O$ with underlying set $O \cup \{\infty\}$.

**Proposition 1 ([9, 6.3], [10]).** Let $S$ be a compact set in an open set $O \subset \mathbb{R}^d$. Then

(i) $\text{in-fill}_O S$ is a compact subset in $O$, and $\text{in-fill}_O (\text{in-fill}_O S) = \text{in-fill}_O S$;

(ii) the set $O_{\infty} \setminus \text{in-fill}_O S$ is connected and locally connected;

(iii) the inward filling of $S$ with respect to $O$ coincides with the complement in $O_{\infty}$ of connected component of $O_{\infty} \setminus S$ containing the point $\infty$;

(iv) if $O' \subset \mathbb{R}^d_{\infty}$ is an open subset and $O \subset O'$, then $\text{in-fill}_O S \subset \text{in-fill}_{O'} S$;

(v) $\mathbb{R}^d \setminus \text{in-fill}_O S$ has only finitely many components, i.e., $\# \text{Conn}_{\mathbb{R}^d_{\infty}}(\mathbb{R}^d \setminus \text{in-fill}_O S) < \infty$.

**Proposition 2 ([9, Theorem 1.7]).** Let $O$ be an open set in $\mathbb{R}^d$, let $S$ be a compact subset in $O$, and suppose that $O_{\infty} \setminus S$ is connected. Then, for each $u \in \text{har}(S)$ and each number $b \in \mathbb{R}^+ \setminus 0$, there is $h \in \text{har}(O)$ such that $|u - h| < b$ on $S$.

**Proposition 3.** Let $O$ be an open set in $\mathbb{R}^d$, and let $S$ be a compact subset in $O$. If $h \in \text{har}(\text{in-fill}_O S)$, then there are harmonic functions $h_j \in \text{har}(O)$ such that the sequence $(h_j)_{j \in \mathbb{N}}$ converges to this harmonic function $h$ in $C(\text{in-fill}_O S)$.

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Proposition 3 is the intersection of Proposition 1 (parts (i)–(ii)) and Proposition 2 if we consider in-fill \( O_S \) instead of \( O \) in Proposition 2.

**Proposition 4** ([9, Theorem 6.1], [11, Theorem 1], [10, Theorem 16]). Let \( O \) be an open set in \( \mathbb{R}^d \), let \( S \) be a closed subset in \( O \), and suppose that \( O \setminus S \) is connected and locally connected. Then, for each \( u \in \text{sbh}(S) \), there exists \( U \in \text{sbh}(O) \) such that \( u = U \) on \( S \).

The intersection of Proposition 1 (parts (i)–(ii)) and Proposition 4 is

**Proposition 5.** Let \( O \) be an open set in \( \mathbb{R}^d \), and let \( S \) be a compact subset in \( O \). Then, for each \( u \in \text{sbh}(\text{in-fill}_O S) \), there exists \( U \in \text{sbh}(O) \) such that \( u = U \) on \( \text{in-fill}_O S \).

### 4 Balayage of measures

In this section 4 we traditional classical balayage that is particular case of (1) (see also [18]).

**Definition 2** ([22], [3], [18, Definition 5.2]). Let \( S \subset \text{Bor}(\mathbb{R}^d) \), \( \delta \in \text{Meas}_{\text{cmp}}^+(S) \), \( \omega \in \text{Meas}_{\text{cmp}}^+(S) \). Let \( H \subset \text{usc}(S) \) be a subclass of upper semicontinuous functions on \( S \). We write \( \delta \preceq_H \omega \) and say that the measure \( \omega \) is a balayage, or, sweeping (out), of the measure \( \delta \) with respect to \( H \), or, briefly, \( \omega \) is \( H \)-balayage of \( \delta \), if

\[
\int h \, d\delta \overset{(2)}{\leq} \int h \, d\omega \quad \text{for each } h \in H.
\]

If \( \delta \preceq_H \omega \) and at the same time \( \omega \preceq_H \delta \), then we write \( \delta \simeq_H \omega \).

**Proposition 6.** Let \( O \subset \mathbb{R}^d \) be an open set, \( \omega \in \text{Meas}(O) \) be a \( H \)-balayage of \( \delta \in \text{Meas}(O) \), \( O' \subset \mathbb{R}^d \) be an open set, and \( H' \subset \mathbb{R}^{O'} \).

(i) The binary relation \( \preceq_H \) (respectively \( \simeq_H \) on \( \text{Meas}_{\text{cmp}}^+(S) \) is a preorder, i.e., a reflexive and transitive relation, (respectively, an equivalence) on \( \text{Meas}_{\text{cmp}}^+(S) \).

(ii) If \( H \) contains a strictly positive (respectively, negative) constant, then \( \delta(S) \leq \omega(S) \) (respectively, \( \delta(S) \geq \omega(S) \)).

(iii) If \( H' \subset H \), then \( \omega \) is \( H' \)-balayage of \( \delta \).

(iv) If \( O' \subset O \) and \( \text{supp} \delta \cup \text{supp} \omega \subset O' \), then \( \omega \big|_{O'} \) is a balayage of \( \delta \big|_{O'} \) for \( H \big|_{O'} \).

(v) If \( H = -H \), then the order \( \preceq_H \) is the equivalence \( \simeq_H \). So, if \( H = \text{har}(S) \), then \( \omega \) is a \( \text{har}(S) \)-balayage of \( \delta \) if and only if \( \delta \simeq_{\text{har}(S)} \omega \), i.e.,

\[
\int_S h \, d\delta = \int_S h \, d\omega \quad \text{for each } h \in \text{har}(S) \quad \text{and} \quad \delta(S) = \omega(S).
\]
(vi) If $\delta \preceq_{\text{sbh}(S)} \omega$, then $\delta \preceq_{\text{har}(S)} \omega$. The converse is not true [20, XIB2], [23, Example].

(vii) If $\omega \in \text{Meas}_{\text{cmp}}^+(O)$ is a $(\text{sbh}(O) \cap C^\infty(O))$-balayage of $\delta \in \text{Meas}_{\text{cmp}}^+(O)$, where $C^\infty(O)$ is the class of all infinitely differentiable functions on $O$, then $\delta \preceq_{\text{sbh}(O)} \omega$, since for each function $u \in \text{sbh}(O)$ there exists a sequence of functions $u_j \in \text{sbh}(O) \cap C^\infty(O)$ decreasing to it [7, Ch. 4, 10, Approximation Theorem].

All statements of Proposition 6 are obvious.

**Example 1** ([8], [6], [25], [12]). Let $x \in O$. If a measure $\omega \in \text{Meas}_{\text{cmp}}^+(O)$ is a balayage of the Dirac measure $\delta_x$ with respect to $\text{sbh}(O)$, then this measure $\omega$ is called a **Jensen measure** on $O$ at $x$. The class of all Jensen measures on $O$ at $x \in O$ will be denoted by $J_x(O)$.

**Example 2** ([8], [8, 3], [16], [17, Definition 8]). Let $x \in O$. If $\omega \in \text{Meas}_{\text{cmp}}^+(O)$ is $\text{har}(O)$-balayage of the Dirac measure $\delta_x$, then the measure $\omega$ is called an **Arens – Singer measure** on $O$ at $x \in O$. The class of all Arens – Singer measures on $O$ at $x$ is denoted by $\text{AS}_x(O) \supset J_x(O)$.

For $s \in \mathbb{R}$, we set

$$k_s(t) := \begin{cases} \ln t & \text{if } s = 0, \\ -\text{sgn}(s)t^{-s} & \text{if } s \in \mathbb{R}\backslash 0, \end{cases} \quad t \in \mathbb{R}^+\backslash 0, \quad (6k)$$

$$K_{d-2}(y, x) := \begin{cases} k_{d-2}(|y - x|) & \text{if } y \neq x, \\ -\infty & \text{if } y = x \text{ and } d \geq 2, \\ 0 & \text{if } y = x \text{ and } d = 1, \end{cases} \quad (y, x) \in \mathbb{R}^d \times \mathbb{R}^d, \quad (6K)$$

$$k_x : y \mapsto K_{d-2}(y, x) \in \text{sbh}(\mathbb{R}^d) \cap \text{har}(\mathbb{R}^d \setminus x), \quad x \in \mathbb{R}^d, \quad (6k_x)$$

$$K(X) := \{k_x : x \in X \} \subset \text{sbh}_*(\mathbb{R}^d), \quad X \subset \mathbb{R}^d. \quad (6K)$$

**Theorem 1.** Let $O \subset \mathbb{R}^d$ be an open set, and $\delta \in \text{Meas}_{\text{cmp}}^+(O), \omega \in \text{Meas}_{\text{cmp}}^+(O)$. The measure $\omega$ is $\text{har}(O)$-balayage (respectively, $\text{sbh}(O)$-balayage) of the measure $\delta$ if and only if there exists a compact subset $S \subset O$ such that this measure $\omega$ is a balayage of $\delta$ with respect to

$$K(O \setminus S) \cup (-K(O \setminus S)), \quad (7h)$$

$$\left(\text{respectively, } K(O) \cup (-K(O \setminus S))\right). \quad (7s)$$

**Proof.** We set

$$S_O := \text{in-fill}_O(\text{supp } \delta \cup \text{supp } \omega). \quad (8)$$
By Proposition 3, for each \( x \notin S_O \) there are functions \( \pm h^x_j \in \operatorname{har}(O) \) such that the sequence \((\pm h^x_j)_{j \in \mathbb{N}}\) converges to \( \pm k_x \subset \operatorname{har}(S_O) \) in \( C(S_O) \). Let

\[
\left( \delta \preceq_{\operatorname{har}(O)} \omega \right) \iff \left( \delta \preceq_{\operatorname{har}(O)} \omega \right) \quad \text{(see Definition 2 and Proposition 6(i),(v)).} \tag{9}
\]

If \( x \notin S_O \), then,

\[
\int S_O \pm K_{d-2}(y, x) \, d\delta(y) = \lim_{j \to \infty} \int S_O \pm h^x_j(y) \, d\delta(y) = \lim_{j \to \infty} \int S_O \pm h^x_j \, d\omega.
\]

Thus, (9) implies that \( \omega \) is a balayage of \( \delta \) with respect to the class \((7h)\) with \( S := S_O \).

If \( \delta \preceq_{\operatorname{har}(O)} \omega \), then, by Proposition 6(vi), \( \delta \preceq_{\operatorname{har}(O)} \omega \), and \( \delta \preceq_{K(\mathbb{R}^d \cup O, -K(\mathbb{R}^d \cup O))} \omega \). Besides, in view of (6k_x), we obtain

\[
\int S_O K_{d-2}(y, x) \, d\delta(y) \overset{(6k_x)}{=} \int O k_x(y) \, d\delta(y) = \int O k_x(y) \, d\omega(y) \leq \int S_O K_{d-2}(y, x) \, d\omega(y)
\]

for each \( x \in \mathbb{R}^d \).

Thus, \( \delta \preceq_{\operatorname{har}(O)} \omega \) implies \( \delta \preceq_{K(\mathbb{R}^d \cup O, -K(\mathbb{R}^d \cup O))} \omega \) and \( \omega \) is a balayage of \( \delta \) with respect to \((7s)\) if \( S := S_O \).

So, the necessary conditions of Theorem 1 are proved.

In the opposite direction, let

\[
\delta \overset{(7h)}{\preceq} K(\mathbb{R}^d \cup O, -K(\mathbb{R}^d \cup O)) \omega, \quad \text{where } S \overset{\text{closed}}{=} \text{clos } S \overset{\text{compact}}{=} O. \tag{10}
\]

Then, by Definition 2 and Proposition 6(v), according to equality (5), we have

\[
\int S K_{d-2}(y, x) \, d\delta(y) = \int S K_{d-2}(y, x) \, d\omega(y) \quad \text{for each } x \in O \setminus S. \tag{11}
\]

Let \( u \in \operatorname{har}(O) \). Without loss of generality, we can assume that

\[
\text{supp } \delta \cup \text{supp } \omega \subset \text{int } S \overset{\text{closed}}{=} \text{clos } S \overset{\text{compact}}{=} O. \tag{12}
\]

There is an open subset \( U \subset O \) such that \( S \subset U \), \( \partial U \) is a \( C^1 \) surface and that each point of \( \partial U \) is a one-sided boundary point of \( U \) [9, 1.6]. In particular \( \partial U \subset O \setminus S \). If we apply Green’s identity to \( U \setminus B(x, r) \) and let \( r \) tend to 0, we obtain [9, 1.6]

\[
u(y) = c_d \int_{\partial U} \left( K_{d-2}(y, x) \frac{\partial u}{\partial n_x}(x) - u(x) \frac{\partial}{\partial n_x} K_{d-2}(y, x) \right) \, d\sigma(x) \quad \text{for each } y \in S, \tag{13}\]
where \( \sigma \) denotes surface area measure on \( \partial U \), \( \vec{n}_x \) denotes the outer unit normal to \( \partial U \) at \( x \in \partial U \) and \( c_d \in \mathbb{R}^+ \setminus 0 \) is defined in (3). Integrating both sides of equality (13) with respect to the measure \( \delta \) and the measure \( \omega \), we obtain, respectively,

\[
\frac{1}{c_d} \int_{\text{supp} \delta} u(y) \, d\delta(y) \overset{(12)}{=} \int_S \left( \int_{\partial U} K_{d-2}(y, x) \frac{\partial u}{\partial \vec{n}_x}(x) \, d\sigma(x) \right) \, d\delta(y) - \int_{\partial U} u(x) \frac{\partial}{\partial \vec{n}_x} K_{d-2}(y, x) \, d\sigma(x) \, d\delta(y),
\]

\[
\frac{1}{c_d} \int_{\text{supp} \delta} u(y) \, d\omega(y) \overset{(12)}{=} \int_S \left( \int_{\partial U} K_{d-2}(y, x) \frac{\partial u}{\partial \vec{n}_x}(x) \, d\sigma(x) \right) \, d\omega(y) - \int_{\partial U} u(x) \frac{\partial}{\partial \vec{n}_x} K_{d-2}(y, x) \, d\sigma(x) \, d\omega(y).
\]

Hence, using Fubini’s theorem and differentiation under the integral sign, we have

\[
\frac{1}{c_d} \int_{\text{supp} \delta} u(y) \, d\delta(y) \overset{(14\delta)}{=} \int_{\partial U} \left( \int_S K_{d-2}(y, x) \, d\delta(y) \right) \frac{\partial u}{\partial \vec{n}_x}(x) \, d\sigma(x)
- \int_{\partial U} u(x) \frac{\partial}{\partial \vec{n}_x} \left( \int_S K_{d-2}(y, x) \, d\delta(y) \right) \, d\sigma(x),
\]

\[
\frac{1}{c_d} \int_{\text{supp} \delta} u(y) \, d\omega(y) \overset{(14\omega)}{=} \int_{\partial U} \left( \int_S K_{d-2}(y, x) \, d\omega(y) \right) \frac{\partial u}{\partial \vec{n}_x}(x) \, d\sigma(x)
- \int_{\partial U} u(x) \frac{\partial}{\partial \vec{n}_x} \left( \int_S K_{d-2}(y, x) \, d\omega(y) \right) \, d\sigma(x).
\]

According to equality (11), for each \( x \in \partial U \subset O \cup \bar{S} \), the internal integrals on the right-hand sides of equalities (15\( \delta \)) and (15\( \omega \)) coincide, and the external integrals on the right-hand sides of equalities (15\( \delta \)) and (15\( \omega \)) are of the same form. Therefore, the integrals on the left-hand sides of equalities (15\( \delta \)) and (15\( \omega \)) also coincide for each harmonic function \( u \in \text{har}(O) \). By Definition 2, formula (4), this means that the measure \( \omega \) is \( \text{har}(O) \)-balayage of the measure \( \delta \), i.e., we have (9).

It remains to consider the case when \( \omega \) is a balayage of \( \delta \) with respect to the class (7s). It has already been shown above that in this case we have (9), i.e., \( \delta \preceq_{\text{har}(O)} \omega \).

Let \( u \in \text{sbh}_s(O) \) with the Riesz measure \( \Delta_u \in \text{Meas}_+(O) \). By the Riesz Decomposition Theorem [24, Theorem 3.7.1], [14, Theorem 3.9], [1, Theorem 4.4.1], [13, Theorem 6.18], there exist an open set \( O' \subset O \) and a harmonic functions \( h \in \text{har}(O') \) such that \( S_O \overset{(8)}{=} \text{in-fill} \, S' \subset O' \) and

\[
u(y) = \int_{\text{clo} \, O'} K_{d-2}(x, y) \, d\Delta_u(x) + h(y) \quad \text{for each } y \in S_O \subset O', \quad (16r)
\]

\[
S := \text{supp} \delta \cup \text{supp} \omega, \quad S_O \overset{(8)}{=} \text{in-fill} \, S' \subset O', \quad (16S)
\]
Integrating the representation (16r) with respect to the measures $\delta$ and $\omega$, we obtain

\[
\int_S u(y) \, d\delta(y) = \int_S \int_{\text{clos } O'} K_{d-2}(x, y) \, d\Delta_u(x) \, d\delta(y) + \int_S h(y) \, d\delta(y) \tag{17\delta},
\]

\[
\int_S u(y) \, d\omega(y) = \int_S \int_{\text{clos } O'} K_{d-2}(x, y) \, d\Delta_u(x) \, d\omega(y) + \int_S h(y) \, d\omega(y). \tag{17\omega}
\]

Hence, by Fubini’s theorem and in view of the symmetry of the kernel $K_{d-2}$ from (6K), we can rewrite (17) in the form

\[
\int_S u(y) \, d\delta(y) = \int_{\text{clos } O'} \left( \int_S K_{d-2}(y, x) \, d\delta(y) \right) \, d\Delta_u(x) + \int_S h(y) \, d\delta(y), \tag{18\delta}
\]

\[
\int_S u(y) \, d\omega(y) = \int_{\text{clos } O'} \left( \int_S K_{d-2}(y, x) \, d\omega(y) \right) \, d\Delta_u(x) + \int_S h(y) \, d\omega(y). \tag{18\omega}
\]

By Proposition 3 there are harmonic functions $h_j \in \text{har}(O)$ such that the sequence $(h_j)_{j \in \mathbb{N}}$ converges to this harmonic function $h \in \text{har}(S_O)$ in $C(S_O)$. Hence,

\[
\int_S h \, d\delta = \int_{S_O} h \, d\delta = \lim_{j \to \infty} \int_{S_O} h_j \, d\delta = \lim_{j \to \infty} \int_{S_O} h_j \, d\delta = \lim_{j \to \infty} \int_O h_j \, d\delta = \int_O h \, d\omega. \tag{19}
\]

By construction of class (7s), we have $\delta \preceq_{K(O)} \omega$. Therefore,

\[
\int_S K_{d-2}(y, x) \, d\delta(y) \leq \int_S K_{d-2}(y, x) \, d\omega(y) \quad \text{for each } y \in O \supset \text{clos } O'. \tag{20}
\]

According to equality (19), the last integrals on the right-hand sides of equalities (18\delta) and (18\omega) also coincide, and, in view of (20), we have

\[
\int_{\text{clos } O'} \left( \int_S K_{d-2}(y, x) \, d\delta(y) \right) \, d\Delta_u(x) \leq \int_{\text{clos } O'} \left( \int_S K_{d-2}(y, x) \, d\omega(y) \right) \, d\Delta_u(x)
\]

Hence, by representations (18\delta) and (18\omega), we obtain

\[
\int_O u(y) \, d\delta(y) \overset{(16S)}{=} \int_S u(y) \, d\delta(y) \overset{(18)}{\leq} \int_S u(y) \, d\omega(y) \overset{(16S)}{=} \int_O u(y) \, d\omega(y).
\]

The latter, by Definition 2, formula (4), means that the measure $\omega$ is $\text{sbh}(O)$-balayage of $\delta$. \qed
5 Integration of measures and balayage

Let $S \in \text{Bor}(O)$. Consider a function $\Theta: S \to \text{Meas}^+(O)$ such that

$$\Theta: S \to \text{Meas}^+(O), \quad \vartheta_x := \Theta(x), \quad \bigcup_{x \in S} \text{supp} \vartheta_x \subseteq O, \quad \sup_{x \in S} \vartheta_x(O) < +\infty,$$  \hspace{1cm} (21)\\

$$x \longmapsto \int_{O} f \, d\vartheta_x \quad \text{is a Borel measurable function for each } f \in C_0(O).$$  \hspace{1cm} (21B)

Let

$$\omega \in \text{Meas}^+(O), \quad \text{supp } \omega \subset S \subseteq O.$$  \hspace{1cm} (22)

Under these conditions, we can define the integral $\int \Theta \, d\omega$ of $\Theta$ with respect to measure $\omega$ as a Borel, or, Radon, positive measure on $O$ [21, Introduction, §1], [4, Ch. V, §3], [19, §5]

$$\int \Theta \, d\omega \quad (21)-(22) := \int_{S} \vartheta_x \, d\omega(x) \in \text{Meas}^+(O),$$  \hspace{1cm} (23I)

$$\left( \int \Theta \, d\omega \right)(B) := \int_{S} \vartheta_x(B) \, d\omega(x) \in \mathbb{R} \quad \text{for each } B \in \text{Bor}(O) \text{ such that } B \subseteq O, \hspace{1cm} (23B)$$

$$\int \Theta \, d\omega: f \longmapsto \int_{S} \left( \int f \, d\vartheta_x \right) \, d\omega(x) \in \mathbb{R} \cup -\infty \quad \text{for each } f \in \text{usc}(O).$$  \hspace{1cm} (23f)

Let $r \in \mathbb{R}^+ \setminus 0$ and $\vartheta \in \text{Meas}^+(rB)$. For $x \in \mathbb{R}^d$, we define the shift $\vartheta_x \in \text{Meas}^+(B(x,r))$ of this measure $\vartheta$ to point $x$ as

$$\vartheta_x(B) := \vartheta(B - x) \quad \text{for any } B \in \text{Bor}(B(x,r)), \hspace{1cm} (24B)$$

$$\int f \, d\vartheta_x := \int_{rB} f(x + y) \, d\vartheta(y) \in \mathbb{R} \cup -\infty \quad \text{for each } f \in \text{usc}(B(x,r)).$$  \hspace{1cm} (24f)

For a measure (22), under the condition

$$S^{jr} := \bigcup_{x \in S} B(x,r) \subseteq O,$$  \hspace{1cm} (25)

we can define the convolution $\omega \ast \vartheta \in \text{Meas}^+(O)$ of measures $\omega$ and $\vartheta$ by the integral $\int \Theta \, d\omega$ of $\Theta: S \to \text{Meas}^+(O)$ with respect to the measure $\omega$ as

$$\omega \ast \vartheta \quad (23H) := \int \Theta \, d\omega \quad (24) \int_{S} \vartheta_x \, d\omega(x) \in \text{Meas}^+(O),$$  \hspace{1cm} (26*)

$$(\omega \ast \vartheta)(B) \quad (24B) := \int_{S} \vartheta(B - x) \, d\omega(x) \in \mathbb{R} \quad \text{for each } B \in \text{Bor}(O) \text{ such that } B \subseteq O, \hspace{1cm} (26B)$$

$$\int f \, d(\omega \ast \vartheta) \quad (24f) := \int_{S} \left( \int_{rB} f(x + y) \, d\vartheta(y) \right) \, d\omega(x) \in \mathbb{R} \cup -\infty \quad \text{for each } f \in \text{usc}(O).$$  \hspace{1cm} (26f)
Very special cases of the following Theorem 2 were essentially used for convolutions in [15, Lemmata 7.1, 7.2], [2, 2.1.1, 1b], [18, 8.1].

**Theorem 2.** Let $\omega \in \text{Meas}_{\text{cmp}}(O)$ be a measure from (22).

If $\emptyset \neq H \subset \text{usc}(O)$ and each measure $\vartheta_x \overset{(21\theta)}{=} \Theta(x)$ in (21) is $H$-balayage of the Dirac measure $\delta_x$ at $x \in S$, then the integral $\int \Theta \, d\omega \overset{(23)}{=} \Meas^+(O)$ is $H$-balayage of $\omega$, i.e.,

$$\omega \overset{(4)}{\preceq}_H \int \Theta \, d\omega \overset{(23)}{=} \int_S \vartheta_x \, d\omega(x) \in \Meas^+(O). \quad (27)$$

If $H = \text{har}(O)$ (respectively, $H = \text{sbh}(O)$), $r \in \mathbb{R}^+ \setminus 0$, and a measure $\vartheta \in \Meas^+(rB)$ is an Arens–Singer (respectively, a Jensen) measure on $rB$ at $0 \in rB$, then, under condition (25), the convolution $\omega * \vartheta \overset{(26)}{=} \Meas^+(O)$ is $\text{har}(O)$ (respectively, $\text{sbh}(O)$)-balayage of $\omega$, i.e.,

$$\omega \overset{\text{har}(O)}{\preceq} \left(\overset{\text{respectively, } \text{sbh}(O)}{\preceq}_O\right) \omega * \vartheta \in \Meas^+(O). \quad (28)$$

**Proof.** Under conditions (21)–(22), by definition (23) and by Definition 2 for $H$-balayage $\delta_x \overset{(21\theta)}{=} \Theta(x)$, for each function $h \in H \subset \text{usc}(O)$, we have

$$\int h \, d\omega = \int \int h \, d\delta_x \, d\omega \leq \int_S \left(\int_{\text{supp} \vartheta_x} h \, d\vartheta_x \right) \, d\omega(x) \overset{(23f)}{=} \int h \, d\int \Theta \, d\omega \quad \text{for each } h \in H. \quad (29)$$

By Definition 2, the latter means (27). By definition (26) of convolution $\omega * \vartheta$, the final part of Theorem 2 with formula (28) is a special case of the proved part (27) of Theorem 2. \qed

6 Polar sets and balayage with an example

Remind that a set $E \subset \mathbb{R}^d$ is **polar** if there is $u \in \text{sbh}_*(\mathbb{R}^d)$ such that $E \subset \{x \in \mathbb{R}^d : u(x) = -\infty\}$, or, in equivalent form, $\Cap^*E = 0$ if we use the **outer capacity**

$$\Cap^*(E) := \inf_{E \subset O^\text{open} \subset \text{int} O \subset \text{cl} O \subset \text{cl} O} \sup_{\nu \in \Meas^+(C)} \int K_{d-2}(x, y) \, d\nu(x) \, d\nu(y). \quad (30)$$

**Theorem 3.** If a measure $\omega \in \Meas^+_\text{cmp}(O)$ is $\text{sbh}(O)$-balayage of a measure $\delta \in \Meas^+_\text{cmp}(O)$, i.e., $\omega \overset{\text{sbh}(O)}{\preceq} \delta$, and $E \subset \mathbb{R}^d$ is polar, i.e., $\Cap^*E = 0$, then $\omega(O \cap E \setminus \text{supp } \delta) = 0$.

**Remark 1.** A special case of this Theorem 3 is noted in [6, Corollary 1.8] for a Jensen measure $\omega \in J_x(O)$ on $O$ at $x \in O$ and the Dirac measure $\delta := \delta_x$. It was used in [16, Lemma 3.1].

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By construction, the measure \( \mu \) is an \( \delta \)-limit point in \( \Bbb R^d \). Easy to see that \( \mu \) is a measure on \( \Bbb R \) (development of one example of T. Lyons [20, XIB2]).

\[ O_k := \bigcup_j B(x_j, 1/k) \subseteq O, \quad \text{supp} \delta \subseteq O_k \supset O_{k+1}, \quad k \in k_0 + \Bbb N_0, \quad \text{supp} \delta = \bigcap_{k \in k_0 + \Bbb N_0} O_k, \]

have complements \( \Bbb R^d \setminus O_k \) in \( \Bbb R^d \) without isolated points. Then every open set \( O_k \subseteq O \) is regular for the Dirichlet problem. It suffices to prove that the equality \( \omega(O_k \cap E) = 0 \) holds for every number \( k \in k_0 + \Bbb N_0 \). By definition of polar sets, there is a function \( u \in \text{sbh}_*(O) \) such that \( u(E) = \{-\infty\} \). Consider the functions

\[ U_k = \begin{cases} u \text{ on } O \setminus O_k, \\ \text{the harmonic extension of } u \text{ from } \partial O_k \text{ into } O_k \text{ on } O_k, \end{cases} \quad k \in k_0 + \Bbb N_0. \tag{31} \]

We have \( U_k \in \text{sbh}_*(O) \), and \( U_k \) is bounded from below in \( \text{supp} \delta \subseteq O_k \). Hence

\[-\infty < \int_O U_k \, d\delta \leq \int_O U_k \, d\omega = \left( \int_{O_k \cap E} + \int_{O_k \setminus E} \right) U_k \, d\omega \]

\[= \int_{O_k \cap E} U_k \, d\omega + (-\infty) \cdot \omega(O_k \cap E) \leq \omega(O) \sup_{\text{supp} \omega} U_k + (-\infty) \cdot \omega(O_k \cap E). \]

Thus, we have \( \omega(O_k \cap E) = 0 \). \( \square \)

Generally speaking, Theorem 3 is not true for \( \text{har}(O) \)-balayage. An implicit example is built in [23, Example]. We get in Example 5 another already constructive way to build such examples.

**Example 3** (development of one example of T. Lyons [20, XIB2]). Let \( \lambda \) be the Lebesgue measure on \( \Bbb R^d \), and let \( b \) be the volume of the unit ball \( \Bbb B \subset \Bbb R^d \). Consider

\[ O := \Bbb B, \quad 0 < t < r < 1, \quad \delta := \frac{1}{br^d} \lambda \big|_{t\Bbb B}, \quad \omega := \frac{1}{br^d} \lambda \big|_{r\Bbb B}. \tag{32} \]

Easy to see that \( \delta \preceq \text{sbh}(\Bbb B) \omega \). Let \( E = (e_j)_{j \in \Bbb N} \subseteq r\Bbb B \setminus t\Bbb B \) be a polar countable set without limit point in \( r\Bbb B \setminus t\Bbb B \). Surround each point \( e_j \in E \) with a ball \( B(e_j, r_j) \) of such a small radius \( r_j > 0 \) that the union of all these balls is contained in \( r\Bbb B \setminus t\Bbb B \). Consider a measure

\[ \mu_E := \omega - \frac{1}{br^d} \sum_{j \in \Bbb N} \lambda \big|_{B(e_j, r_j)} + \frac{1}{br^d} \sum_{j \in \Bbb N} \lambda(e_j, r_j) \delta_{e_j} \]

\[\overset{(32)}{=} \frac{1}{br^d} \lambda \big|_{r\Bbb B} - \frac{1}{br^d} \sum_{j \in \Bbb N} \lambda \big|_{B(e_j, r_j)} + \frac{1}{r^d} \sum_{j \in \Bbb N} r_j^d \delta_{e_j}. \]

By construction, the measure \( \mu_E \) is \( \text{har}(\Bbb B) \)-balayage of measure \( \delta \), but

\[ \mu_E(E) = \frac{1}{r^d} \sum_{j \in \Bbb N} r_j^d > 0 \]

in direct contrast to Theorem 3.
7 Balayage for three measures

Proposition 7. Let \( \omega \in \text{Meas}^{\text{cmp}}(O) \) and \( \delta \in \text{Meas}^{\text{cmp}}(O) \).
If \( \omega \) is \( \text{sbh}(O) \)-balayage of \( \delta \), then
\[
\int u \, d\delta \leq \int u \, d\omega \quad \text{for each } u \in \text{sbh}(S_O), \quad \text{where } S_O = \text{in-fill}_O S, \ S := \text{supp } \omega \cup \text{supp } \delta,
\]
(33)
i.e., \( O' \supset S_O \) is a open subset in \( \mathbb{R}^d \), then \( \omega \) is \( \text{sbh}(O') \)-balayage of \( \delta \).

If \( \omega \) is \( \text{har}(O) \)-balayage of \( \delta \), then
\[
\int h \, d\delta = \int h \, d\omega \quad \text{for each } h \in \text{har}(S_O),
\]
(34)
i.e., \( O' \supset S_O \) is a open subset in \( \mathbb{R}^d \), then \( \omega \) is \( \text{har}(O') \)-balayage of \( \delta \).

Proof. If \( u \in \text{sbh}(S_O) \), then, by Proposition 5, there is a function \( U \in \text{sbh}(O) \) such that
\[
\int_{S_O} u \, d\delta = \int_{S_O} U \, d\delta = \int_O U \, d\delta \leq \int_O U \, d\omega = \int_{S_O} U \, d\omega = \int_{S_O} u \, d\omega,
\]
that gives (33). If \( \delta \preceq_{\text{har}(O)} \omega \), then we can repeat (19) using Proposition 3, and we obtain (34). \( \square \)

Very special cases of the following Theorem 4 were essentially used in [2, Proposition 3] only for special Jensen measures on the complex plane on the complex plane identified with \( \mathbb{R}^2 \).

Theorem 4. Suppose that measures \( \beta, \delta, \omega \in \text{Meas}^{\text{cmp}^+}(O) \) satisfy the conditions
\[
\begin{aligned}
\beta \preceq_{\text{har}(O)} \delta, \\
\beta \preceq_{\text{sbh}(O)} \omega,
\end{aligned}
\]
and in-fill(\( \text{supp } \beta \cup \text{supp } \delta \)) \( \subset O' \),
(35)
where \( O' \Subset O \) is an open subset such that \( O' \cap \text{supp } \omega = \emptyset \). Then \( \delta \preceq_{\text{sbh}(O)} \omega \).

Proof. It suffices to consider the case when \( D := O \) and \( D' := O' \) are domains. There exists a regular (for the Dirichlet problem) domain \( D'' \) such that
\[
\text{in-fill}(\text{supp } \beta \cup \text{supp } \delta) \overset{(35)}{\subset} D'' \Subset D' \subset D
\]
(36)
since in-fill(\( \text{supp } \beta \cup \text{supp } \delta \)) is compact subset in \( D' \) by Proposition 1(i).

Let \( u \in \text{sbh}_*(D) \). Then we can build a new subharmonic function \( \tilde{u} \in \text{sbh}_*(D) \) such that
\[
\tilde{u} \big|_{D''} \in \text{har}(D''), \quad \tilde{u} = u \quad \text{on } D \setminus D'', \quad u \leq \tilde{u} \quad \text{on } D.
\]
(37)
By Proposition 7, in view of the inclusion in (36), we have

\[ \int_D u \, d\delta = \int_{D''} \tilde{u} \, d\delta \leq \int_{D''} \tilde{u} \, d\beta = \int_D \tilde{u} \, d\beta \leq \int_D \tilde{u} \, d\omega. \tag{38} \]

Since supp \( \omega \subset D \setminus D' \), we can continue this chain of (in)equalities (38) as

\[ \int_D u \, d\delta \leq \int_D \tilde{u} \, d\omega = \int_{D,D''} \tilde{u} \, d\omega = \int_{D,D''} u \, d\omega = \int_D u \, d\omega. \]

This completes the proof. \( \square \)

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