COHOMOLOGY OF MODULI SPACE OF CUBIC FOURFOLDS
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Abstract. In this paper we compute the cohomology of moduli space of cubic fourfolds with ADE type singularities relying on Kirwan’s blowup and Laza’s GIT construction. More precisely, we obtain the Betti numbers of Kirwan’s resolution of the moduli space. Furthermore, by applying decomposition theorem we obtain the Betti numbers of the intersection cohomology of Baily-Borel compactification of the moduli space.

1. Introduction

The study of cubic fourfolds and their moduli space is a classical topic in algebraic geometry and has attracted lots of attentions in various aspects. The purpose of this paper is to investigate the topology of the coarse moduli space \(M\) of cubic fourfolds with ADE singularities at worst and its various compactifications. The celebrated works of Voisin [36] [37], Hassett [12], Laza [23] and Looijenga [27] establish the global Torelli theorem for the cubic fourfolds completely. Thus we can identify the moduli space \(M\) as the complement \(D/\Gamma - H_\infty\) of a Heegner divisor \(H_\infty\) in the Shimura variety \(D/\Gamma\) (see 4.4). This provides many compactifications of \(M\) from the arithmetic side, e.g. Baily-Borel’s compactification [7], Looijenga’s [26] and toriodal compactifications [1]. These compactifications imply that \(M\) is a quasi-projective variety.

The cohomology of moduli space is a basic invariant of particular interest for the moduli space and it is a part of cycle theory on the moduli space. The cohomology rings of moduli spaces \(M_g\) of genus \(g\) curves have been studied decades since the work of Mumford. Mumford defined the tautological cohomological rings of \(M_g\) as subrings of the cohomology rings generated by kappa classes and similarly defined tautological Chow rings of \(M_g\). It is expected the tautological cohomological rings are isomorphic to tautological Chow rings via cycle classes maps (see Question 0.1 in [31]). But so far it is only known for genus less than 23. The cohomology ring of the moduli spaces \(F_g\) of quasi-polarised K3 surface of fixed degree \(2g - 2\) has also many progress recently (see [5], [32], [4]). The tautological ring of moduli space \(F_g\) involves more ingredients than that of \(M_g\). In [5] and [32], it is shown that tautological rings are generated by Noether-Lefschetz cycles. Cubic fourfolds and K3 surfaces share similar Hodge theory and the Torelli thorem holds for the both. So it is natural to ask the similar questions for moduli spaces of cubic fourfolds. This is the main motivation for us to study the topology of \(M\) and its compactifications as the first step to the similar picture for cubic fourfolds.
For a topological space $Y$, we denote by $$P_t(Y) := \sum b_i(X)t^i, \quad IP_t(Y) := \sum \dim IH^i(X)t^i,$$
the Poincaré polynomial of singular cohomology and intersection cohomology (with respect to middle perversity) of $Y$. Let $\tilde{M}$ be the partial desingularization of the GIT compactification $\overline{M}$ in the sense of Kirwan. The first main result in the paper is

**Theorem 1.1.** The Poincaré polynomial of $\tilde{M}$ is given by

$$P_t(\tilde{M}) = 1 + 9t^2 + 26t^4 + 51t^6 + 81t^8 + 115t^{10} + 152t^{12} + 193t^{14}$$
$$+ 236t^{16} + 280t^{18} + 324t^{20} + 280t^{22} + 236t^{24} + 193t^{26}$$
$$+ 152t^{28} + 115t^{30} + 81t^{32} + 51t^{34} + 26t^{36} + 9t^{38} + t^{40}.$$

Using the explicit resolution of the period maps, we also compute the intersection cohomology of the Baily-Borel compactification of the Shimura variety $D/\Gamma$.

**Theorem 1.2.** The intersection cohomology Poincaré polynomial of $\overline{D/\Gamma}_{BB}$ is given by

$$IP_t(\overline{D/\Gamma}_{BB}) = 1 + 2t^2 + 5t^4 + 13t^6 + 24t^8 + 38t^{10} + 54t^{12} + 70t^{14}$$
$$+ 88t^{16} + 107t^{18} + 137t^{20} + 107t^{22} + 88t^{24} + 70t^{26}$$
$$+ 54t^{28} + 38t^{30} + 24t^{32} + 13t^{34} + 4t^{36} + 2t^{38} + t^{40}.$$

**Remark 1.3.** It is interesting to note that by recent results of Liu [24], the GIT moduli space of cubic fourfolds is isomorphic to the K-moduli space of cubic fourfolds, that is, the space of isomorphic classes of cubic fourfolds admitting Kahler-Einstein metrics. So our computations also provide cohomological results on the K-moduli space (see [38] for a nice survey of K-moduli spaces).

**Remark 1.4.** We are most interested in cohomology of the open part $D/\Gamma$ and the complement of Heegener divisor $D/\Gamma - H_\infty$. But at present there are some technical difficulty. The problem will be investigated in the future.

**Remark 1.5.** As a complement of a divisor of Shimura variety, the cohomology of $M$ is closely related to the representation theory (see [7]). Our computation here is based on geometric results. It is expected there is a representation theoretic explanation. After the computational results, it is an interesting topic to study the generators of the intersection cohomology in each degree and ask whether these generators are generated by special cycles (for example, see [34]).

**Remark 1.6.** It is worth to mention that it is still unknown whether the tautological Chow ring of $F_g$ and its tautological cohomology ring are isomorphic.

The strategy of our approach is as follows: first, the equivariant cohomology of GIT quotient space $\overline{M}$ can be computed by the stratifications. This relies on from Kirwan’s general theory on cohomology of quotient space. Then we apply the partial desingularization procedure, that is, take a successive blowups along
GIT strictly semistable loci. And then We keep track of the change of cohomology for each blowup in partial desingularization and thus we can obtain the cohomology of \( \widetilde{\mathcal{M}} \). For the computation of intersection cohomology of \( \overline{D/\Gamma^{BB}} \), we need to make use of the geometry of moduli spaces. The Torelli theorem provides a birational map \( p: \mathcal{M} \rightarrow \overline{D/\Gamma^{BB}} \) between GIT compactification \( \mathcal{M} \) and the Baily-Borel compactification \( \overline{D/\Gamma^{BB}} \). The birational map can be explicitly resolved via Kirwan’s partial desingularization, that is, there is a diagram

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{p} & \overline{D/\Gamma^{BB}} \\
\downarrow & & \downarrow \\
\mathcal{M} & \xrightarrow{p} & \overline{D/\Gamma^{BB}}
\end{array}
\]

where \( \mathcal{M} \) is the intermediate space during the Kirwan’s partial desingularization. One can use blowup formula of the intersection cohomology reversely to get the cohomology of \( \mathcal{M} \) from the cohomology of \( \widetilde{\mathcal{M}} \). Then the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber [6] will provide a way to compute the intersection cohomology of Baily-Borel compactification of the Shimura variety \( D/\Gamma \).

This strategy has been worked out for the moduli space of K3 surfaces of degree 2 (see [17], [18]) and the moduli space of cubic threefolds (see [8]). The main difficulty in our case is that the boundary strata of the GIT compactification \( \mathcal{M} \) are much more complicated than the above two cases, but the our observation is that the GIT moduli space of K3 surfaces of degree 2 will appear as an exceptional divisor in Kirwan’s desingularization of \( \mathcal{M} \), then the computations of Kirwan-Lee in [18] will help us to simplify the computations.

**Outline.** The paper is organised as follows: In section 2, we review the construction of moduli space \( \mathcal{M} \), its GIT compactification and the cohomology theory used in this paper. We also introduce Kirwan’s partial desingularization package. In section 3, we use Kirwan’s methods to compute the cohomology of the partial resolution \( \mathcal{M} \). In section 4, we introduce the global Torelli theorem for cubic fourfolds and use the decomposition theorems to compute the intersection cohomology of the Baily-Borel compactification \( \overline{D/\Gamma^{BB}} \) of the moduli space of cubic 4-folds.

**Notations and Conventions.**

(1) \( \mathcal{M} \) the moduli space of cubic fourfolds with ADE singularities;
(2) \( \overline{\mathcal{M}} \) the GIT compactification of \( \mathcal{M} \);
(3) \( \widetilde{\mathcal{M}} \) the Kirwan’s desingularization space;
(4) \( \mathbb{C}[x_0, x_2, ..., x_n]_d \) means degree \( d \) homogeneous polynomials in \( n+1 \) variables;
(5) \( l(x), q(x), c(x) \) means linear, quadratic and cubic forms in \( x \);
(6) \{polynomial\} means the vector space spanned by the monomials of the polynomial;
(7) \( \alpha, \mu, \gamma, \delta, \cdots \) the strata of GIT boundaries;
(8) \( Z_{\alpha}, Z_{\mu}, Z_{\gamma}, \cdots \) the parametrizing space of strata \( \alpha, \mu, \gamma, \delta, \cdots \);
(9) \( R_{\alpha}, R_{\mu}, R_{\gamma}, \cdots \) the stabilizer subgroup of strata \( \alpha, \mu, \gamma, \delta, \cdots \);
2. Preliminaries

2.1. Moduli space of Cubic fourfolds. We work over \( \mathbb{C} \). A cubic fourfold \( X \) is a hypersurface in \( \mathbb{P}^5 \) defined by a homogeneous polynomial of degree 3.

**Definition 2.1.** We call a cubic fourfold \( X \) has ADE singularities if it has only isolated singularities and each singularity germ is a 4-dimensional hyper-surface singularity in \( (\mathbb{C}^5, 0) \) that can be written as \( x_1^2 + x_2^2 + f(x_1, x_2, x_3) \) where \( f(x_1, x_2, x_3) \) is the equation of the surface singularity of ADE type.

Let \( \mathcal{M} \) be the coarse moduli space parametrizing isomorphic classes of cubic fourfolds with ADE singularities. It can be constructed as follows: The Hilbert scheme of cubic hypersurfaces in \( \mathbb{P}^5 \) is isomorphic to the projective space \( \mathbb{P}(\mathbb{C}[x_0, x_2, ..., x_5]_3) \). The action \( G = SL(6, \mathbb{C}) \) on \( \mathbb{P}^5 \) will induce the action \( G \) on \( \mathbb{C}[x_0, x_2, ..., x_5]_3 \cong H^0(\mathbb{P}^5, \mathcal{O}(3)) \) and so on \( \mathbb{P}(\mathbb{C}[x_0, x_2, ..., x_5]_3) \). Let \( U \subset \mathbb{P}(\mathbb{C}[x_0, x_2, ..., x_5]_3) \) be the locus of cubic fourfolds with ADE singularities and it is an open subset. By choosing any ample \( G \)-linearization \( L \) on \( \mathbb{P}\mathbb{C}[x_0, x_2, ..., x_5]_3 \), we have a GIT stability for the point in \( \mathbb{P}\mathbb{C}[x_0, x_2, ..., x_5]_3 \) in the sense of Mumford [30]. We say a cubic fourfold \( X \) is (semi) stable if its associated point \( [X] \in \mathbb{P}(\mathbb{C}[x_0, x_2, ..., x_5]_3) \) is GIT (semi)stable. By a result of Laza, we know

**Theorem 2.2** ([22], Theorem 5.6). A cubic fourfold with ADE singularities is stable.

As the stabilizer group of a stable point is finite, the theorem implies the quotient stack \( [U/G] \) is a Deligne-Mumford stack and \( \mathcal{M} \cong U/G \). Clearly, \( \mathcal{M} \) is not a compact space. The GIT theory provides a natural compactification

\[
\overline{\mathcal{M}} := \mathbb{P}(\mathbb{C}[x_0, x_2, ..., x_5]_3)/G = \text{Proj}(R(\mathbb{P}(\mathbb{C}[x_0, x_2, ..., x_5]_3), L)^G)
\]

of \( \mathcal{M} \) also by the theorem where \( R(\mathbb{P}(\mathbb{C}[x_0, x_2, ..., x_5]_3), L)^G \) is the \( G \)-invariant section rings. Let \( \mathcal{M}^s \subset \overline{\mathcal{M}} \) be the locus of GIT stable points, we call \( \overline{\mathcal{M}} - \mathcal{M}^s \) the GIT boundary strata, which parametrize the minimal orbits of semistable cubic fourfolds. For the purpose of computation, we need to understand explicit geometry of these boundary strata. Let us recall Laza’s analysis of GIT stability.

**Proposition 2.3** (Prop 2.6 [22]). A strictly semi-stable cubic fourfold with minimal orbit have defining equation of the following type:

- \( \alpha: x_0q_1(x_2, ..., x_5) + x_1q_2(x_2, ..., x_5) = 0 \);
- \( \mu: ax_0x_2^2 + x_0x_3l_1(x_2, x_3) + bx_3^2 + x_4l_2(x_2, x_3) + c(x_2, x_3) = 0 \);
- \( \gamma: x_0q_3(x_3, ..., x_5) + x_2^4l_1(x_3, ..., x_5) - 2x_1x_2l_2(x_3, ..., x_5) + x_3^2l_3(x_3, ..., x_5) = 0 \);
\[ \delta: x_0 q(x_4, x_5) + f(x_1, x_2, x_3) = 0. \]

where \( l, q, f \) means linear, quadratic and cubic equations respectively. Thus, we have the following stratification

\[ \overline{\mathcal{M}} - \mathcal{M}^s = \alpha \cup \mu \cup \gamma \cup \delta. \]

We still use the notation \( \alpha, \delta, \cdots \mu \) to denote the boundary strata corresponding the equation of type \( \alpha, \delta, \cdots \mu \). So we have

\[ \overline{\mathcal{M}} - \mathcal{M} = \alpha \cup \cdots \cup \mu. \]

The incidence relation is given by the figure 1 where \( \tau \) is a curve that parametrizes cubic fourfolds with equation of the form

\[
\begin{vmatrix}
 x_0 & x_1 & ax_2 \\
 x_1 & x_5 & x_3 \\
 x_2 & x_3 & x_4 
\end{vmatrix} = 0
\]

and \( \zeta \) is the point representing \( x_0x_4x_5 + x_1x_2x_3 \), \( \omega \) is the point representing

\[
-\begin{vmatrix}
 x_0 & x_1 & x_2 \\
 x_1 & x_5 & x_3 \\
 x_2 & x_3 & x_4 
\end{vmatrix} = 0.
\]

By summary of results in [22], we also get the stabilizers of the boundary strata.

**Theorem 2.4** ([22]). The stabilizers of general points the GIT boundary strata are one of the following (up to a conjugate)

- \( R_\omega \cong SL(3, \mathbb{C}) \)
- \( R_\zeta \cong (\mathbb{C}^*)^4 \)
- \( R_\chi \cong SL(2, \mathbb{C}) \)
- \( R_\delta \cong SO(2)(\mathbb{C}) \times \{ \text{diag}(t^{-2}, t, t, 1, 1, 1) : t \in \mathbb{C}^* \} \)
Proof. The GIT boundaries $\overline{\mathcal{M}} - \mathcal{M}$ are the following strata:

1. 1-dimensional strata: $\alpha \cong \mathbb{P}^1$, $\delta \cong \mathbb{P}^1$, $\tau \cong \mathbb{P}^1$, $\chi \cong \mathbb{P}^1$.

2. 2-dimensional strata: $\gamma$ is $\mathbb{P}^1 \times \mathbb{C}$, $\phi \cong \mathbb{P}^1 \times \mathbb{P}^1$.

3. 3-dimensional strata: $\mu \cong \mathbb{P}(1, 3, 6, 8)$, $\epsilon \cong \mathbb{P}^1 \times \mathbb{P}(1, 2, 3)$.

Based on the above results, we have explicit results of boundary strata.

**Proposition 2.5.** The GIT boundaries $\overline{\mathcal{M}} - \mathcal{M}$ are the following strata:

1. 1-dimensional strata: $\alpha \cong \mathbb{P}^1$, $\delta \cong \mathbb{P}^1$, $\tau \cong \mathbb{P}^1$, $\chi \cong \mathbb{P}^1$.

2. 2-dimensional strata: $\gamma$ is $\mathbb{P}^1 \times \mathbb{C}$, $\phi \cong \mathbb{P}^1 \times \mathbb{P}^1$.

3. 3-dimensional strata: $\mu \cong \mathbb{P}(1, 3, 6, 8)$, $\epsilon \cong \mathbb{P}^1 \times \mathbb{P}(1, 2, 3)$.

Proof. $\chi \cong \mathbb{P}^1$ is shown in section 4.1 in [23]. By Lemma 4.5 in [22], we can write the defining equation of $\delta$ as

$$bx_5^3 + \det \begin{pmatrix} x_0 & x_1 & x_2 + 2ax_5 \\ x_1 & x_2 - ax_5 & x_3 \\ x_2 + 2ax_5 & x_3 & x_4 \end{pmatrix} = 0.$$

Here $a, b \in \mathbb{C}$ and the inclusion relation is given by the figure 2.1.

\[ \begin{array}{c}
R_{\alpha} \cong \mathbb{P}^1 \\
R_{\delta} \cong T^4 \\
R_{\gamma} \cong \mathbb{P}^1 \times \mathbb{P}^1 \\
R_{\tau} \cong \mathbb{P}(1, 3, 6, 8) \\
R_{\chi} \cong \mathbb{P}^1 \times \mathbb{P}(1, 2, 3) \\
R_{\omega} \cong SL(3) \end{array} \]

**Figure 2.** The stabilizers of general points in the boundary strata.
Cohomology of moduli space of cubic fourfold

on $\mathbb{A}^3(a, b, c)$ where $a, b, c$ are coefficients in the normal form of $l(x_3, x_4, x_5) = ax_3 + bx_4 + cx_5$, then

$$\gamma \cong \mathbb{C} \times \mathbb{P}^1.$$ 

For $\beta$, we first use the action $GL(2)$ on $x_2, x_3$ to reduce the problem to consider the torus action

$$\text{diag}\{ (t^{a_0}, t^{a_1}, t^{a_2}, t^{a_3}, t^{a_4}, t^{a_5}) : a_2 = a_3, \sum a_i = 0 \}$$

on the space

$$\mathbb{P}V = \mathbb{P}^6(y_0, \ldots, y_6)$$

where $V$ is the vector space spanned by the monomials

$$\{x_1x_3x_4, x_0x_4^2, x_5x_1^2, xx_2^3, x_2x_3^2, x_3x_2^2, x_3^2\}.$$ 

then we compute its invariant ring which has the minimal generators

$$y_0, \ y_1y_2y_5, \ (y_1y_2)^2y_4y_6, \ (y_1y_2)^3y_3y_6$$

and thus we obtain $\beta \cong \mathbb{P}(1, 3, 6, 8)$. From [22] we know that the loci $\varepsilon$ parametrizes the cubic 4-folds singular along an irreducible rational normal curve of degree 4. By Proposition 6.6 in [22], the parameter space for such 4-folds is the product of $\text{Sym}^4(\mathbb{P}^1) \cong \mathbb{P}^4$ and affine space $\mathbb{A}(a, b, c)$ with natural action $SL(2)$ on $\mathbb{P}^1$ and $\mathbb{C}^*$ on $x_5$ in the equation above, thus

$$\varepsilon \cong \mathbb{P}^4/SL(2) \times \mathbb{P}(1, 2, 3) \cong \mathbb{P}^1 \times \mathbb{P}(1, 2, 3).$$

\[\square\]

2.2. Cohomology Theory. Assume $X$ is a smooth variety over $\mathbb{C}$ with a reductive group $G$ acting on it. We are most interested in its $G$-equivariant cohomology (here we use singular cohomology) which captures the $G$-group action. The $i$-th equivariant cohomology $H_i^G(X, \mathbb{Q})$ is defined by the ordinary cohomology $H^i(EG \times_G X, \mathbb{Q})$ of the topological space $EG \times_G X$ where $EG \to BG$ is the universal principal homogeneous $G$-space and $EG \times_G X$ is quotient space of $EG \times X$ under action $g(h, x) := (h \cdot g^{-1}, g \cdot x)$. We denote the $G$-equivariant Poincaré polynomials for a variety $G$ by

$$P_i^G(X) := \sum_{i \geq 0} \dim_{\mathbb{Q}} H_i^G(X, \mathbb{Q}) \cdot t^i.$$ 

The properties below will be frequently used in this paper and one may refer to [13] for more details on general equivariant cohomology theory.

**Theorem 2.6.** With notation as above, we have

(1) If the quotient $X/G$ has only quotient singularities, then

$$H_i^G(X, \mathbb{Q}) = H^i(X/G, \mathbb{Q})$$

(2) If the quotient space $X/G$ is contractible, then

$$H_i^G(X) = H^i(BG)$$
Let \( F \to X \to B \) be a \( G \)-equivariant topological fibration on \( X \) over the base space \( B \) and \( F \) is the fiber, then we have the spectral sequence:

\[
H^q_G(B, H^p(F, \mathbb{Q})) \Rightarrow H^{p+q}_G(X, \mathbb{Q}).
\]

In particular, the spectral sequence implies

\[
P^G_t(X) = P^G_t(B) \cdot P_t(F). \tag{2.1}
\]

If \( X \) is a singular space, we are interested in its intersection which behaves better than the usual cohomology. Let \( IC_X \in D_c(X) \) be an intersection complex defined by Goresky-MacPherson \([10]\), then the intersection cohomology of \( X \) is defined to be the hypercohomology

\[
IH^i(X, \mathbb{Q}) := H^i(X, IC_X).
\]

We refer to \([19]\) for the definition of intersection complex and more details on the theory of intersection cohomology. We have the blowup formula that is obtained by Kirwan.

**Proposition 2.7.** \([17, \text{Proposition 6.2}]\) Let \( Z \subset X \) be a smooth \( G \)-subvariety with the reductive stabilizer subgroup \( R \). Let \( \tilde{X} := Bl_Z(X) \to X \) be the blow up of \( X \) along \( Z \), then

\[
\dim IH^i(X/G) = \dim IH^i(\tilde{X}/G) - \sum_{p+q=i} \dim(H^p(Z//N_0)) \otimes H^{\lambda(q)}(\mathbb{P}/R)_{\pi_0(N)} \tag{2.2}
\]

where \( \lambda(q) := \begin{cases} 
q - 2 & \text{if } q \leq \dim \mathbb{P}/R, \\
q & \text{if others.}
\end{cases} \) and \( \mathbb{P} \) is the projection of a normal vector space of any point in \( Z \), \( N_0 \) is the identity component of the normalizer subgroup \( N \) of \( R \) and \( \pi_0(N) := N/N_0 \). The actions of \( \pi_0(N) \) on \( H^p(Z//N_0) \) and \( H^{\lambda(q)}(\mathbb{P}/R) \) are induced by the actions of \( N \) on \( Z^{ss} \) and projective normal bundle \( \mathbb{P}N_{Z/X} \to Z \) respectively.

**2.3. Kirwan’s desingularization package.** Denote by \( Z_R \subset X \) the locus whose stabilizer is \( R \). Suppose there are only finitely many locus

\[
\{ \ Z_{R_1}, \ldots, Z_{R_r} : \dim R_1 \geq \ldots \geq \dim R_r \}\n\]

such that all the stabilizers \( R_i \) are reductive subgroups of \( G \) and all \( Z_{R_i} \) are smooth, then Kirwan took the blowups successively along these locus (see \([20],[15]\))

\[
\tilde{X} = Bl_{\tilde{Z}_{R_r}} \cdots \to Bl_{\tilde{Z}_{R_1}} X \to X
\]

where \( \tilde{Z}_{R_i} \) is the strict transformations of \( Z_{R_i} \) and showed the \( G \)-action can be lifted to \( \tilde{X} \) under suitable polarization. Moreover, it commutes with the GIT quotient

\[
\begin{array}{ccc}
\tilde{X} & \longrightarrow & X \\
\|/G & \longrightarrow & X//G
\end{array} \tag{2.3}
\]
In this way, after finite steps, Kirwan obtained a partial resolution \( \tilde{X} \) of \( X/G \), which has at worst only orbifold points. We call the final blowup space the Kirwan’s desingularization space of \( X/G \). To study the cohomology of \( \tilde{X} \), Kirwan developed several useful cohomological formulas:

(1) Cohomology formula for GIT quotient: let \( L \) be a \( G \)-linearized polarization. We can choose a \( G \)-equivalent embedding \( X \hookrightarrow \mathbb{P}^N \) via \( L \). Let \( T \subset G \) be a maximal torus and \( t \) be its Lie algebra, fix a positive Weyl chamber \( t^+ \subset t \), then define the index set \( \mathcal{B}_0 \) consists of \( \beta \in t^+ \) such that \( \beta \) is the closest point to the origin 0 of the nonempty convex hull \( \text{cone}(\alpha_1, \ldots, \alpha_m) \) generated by some weights \( \alpha_1, \ldots, \alpha_m \). We call a vector in \( \mathcal{B}_0 \) index vector. Fix a norm \( | \cdot | \) on \( t \) (e.g., the one induced by killing form), set

\[
Z_\beta = \{ [x_0, \ldots, x_N] \in X : x_i = 0, \text{if } \alpha_i . \beta \neq |\beta|^2 \}
\]

\[
Y_\beta = \{ [x_0, \ldots, x_N] \in X : x_j = 0, \text{if } \alpha_j . \beta = |\beta|^2 \& \exists \alpha_i . \beta \neq |\beta|^2 \},
\]

then there is a natural retraction map

\[
p_\beta : Y_\beta \rightarrow Z_\beta.
\]

Denote by \( X^{ss} \) the semistable locus of \( X \) with respect to the polarization \( L \) in the sense of Mumford’s GIT and \( Z_\beta^{ss} \) the locus of semistable points in \( Z_\beta \) and let

\[
Y_\beta^{ss} := p_\beta^{-1}(Z_\beta^{ss}), \quad S_\beta := G \cdot Y_\beta^{ss},
\]

then combing the theory of moment maps and relations of symplectic reduction and geometric invariant theory, it is shown in [21] that \( \{S_\beta\}_{\beta \in \mathcal{B}_0} \) gives \( X \) a \( G \)-equivariant perfect Morse stratification. In particular, for \( \beta = 0 \), \( S_0 = X^{ss} \). Using such stratification, Kirwan obtained the following formula of Poincaré’s polynomials,

\[
P^G_t(X^{ss}) = P_t(X)P_t(BG) - \sum_{0 \neq \beta \in \mathcal{B}_0} t^{2\text{codim}(S_\beta)} P^\text{stab}(\beta)(Z^{ss}_\beta).
\]

(2.4)

We will call the term \( \sum_{0 \neq \beta \in \mathcal{B}_0} t^{2\text{codim}(S_\beta)} P^\text{stab}(\beta)(Z^{ss}_\beta) \) the \textbf{removing part} in the formula 2.4. Moreover, there is a natural identification

\[
S_\beta \cong G \times_{P_\beta} Y_\beta^{ss}
\]

where \( P_\beta \leq G \) is the parabolic subgroup associated to \( \beta \). In this way, we have also have a dimension formula

\[
\dim S_\beta = \dim G + \dim Y_\beta^{ss} - \dim P_\beta.
\]

(2.5)

(2) Cohomology formula for blowups: assume \( R \) is a reductive subgroup of \( G \) with respect to the locus \( Z_R \). We take the blow up

\[
\pi : \tilde{X} \rightarrow X^{ss}
\]

along the smooth center \( G \cdot Z_R^{ss} \). Let \( N(R) \) be the normalizer subgroup of \( R \) in \( G \) and \( d_R \) be the complex codimension of \( Z_R \) in \( X \), then Kirwan’s blowup
formula [17] gives

\[ P^G_t(\tilde{X}^{ss}) = P^G_t(X^{ss}) + (t^2 + \ldots + t^{2dR})P^{N(R)}_t(Z^{ss}_R) \]
\[ - \sum_{\beta \in B_{0,\rho}} t^{2 \cdot \text{codim}(S_\beta)} P^{\text{stab}(\beta) \cap N(R)}_t(Z^{ss}_{\beta,\rho}). \]  

(2.6)

Here \( B_{0,\rho} \) is the index set obtained as in (1) with respect to the normal representation

\[ \rho : R \to \text{Aut}(\mathbb{P}N), \]

where \( N \) is the normal vector space of a point in \( Z^{ss}_R \) and \( Z^{ss}_{\beta,\rho} \subset \mathbb{P}N \) is the associated semi-stable strata given by weight \( \beta \).

Suppose there are series of blowups

\[ \cdots \to X_2 \to X_1 \to X_0 = X. \]

We write the correction terms contributed in the formula of \( i \)-th blowup as follows:

\[ A_i(t) := (t^2 + \ldots + t^{2dR})P^{N(R_i)}_t(Z^{ss}_R) \]
\[ - \sum_{\beta \in B_{0,\rho}} t^{2 \cdot \text{codim}(S_\beta)} P^{\text{stab}(\beta) \cap N(R_i)}_t(Z^{ss}_{\beta,\rho}). \]  

(2.7)

3. Cohomology of partial desingularization \( \tilde{M} \)

In this section, we follow the discussion in section 2 to do the computation. Let \( \tilde{M} \) be the Kirwan’s desingularization space of \( \overline{\mathcal{M}} \). Note that \( \tilde{M} \) has at worst orbifold points and thus the equivariant cohomology of \( H^*_G(\tilde{M}) \) equals to the usual cohomology \( H^*(\tilde{M}) \) by Theorem 2.6. By duality of intersection cohomology, we will only consider the term in \( P^G_t(\tilde{M}) \) of degree lower than 20. Throughout the section, \( X = \mathbb{P}^{55} \) and \( G = SL(6, \mathbb{C}) \).

3.1. Computations for blowups.

3.1.1. Computation of \( P^G_t(X^{ss}) \). According to the formula 2.4, we need to describe the index set \( B_0 \). It consists of all closest points \( \beta \) lying in a positive Weyl chamber \( \tau_* \) to origin 0 in convex hull \( \text{con}(\alpha_1, \ldots, \alpha_5) \) generated by some weights \( \alpha_1, \ldots, \alpha_5 \). Let \( T \) be a maximal torus in \( SU(6, \mathbb{C}) \) and its Lie algebra is

\[ t = \mathbb{R}^5 \cong \{ \text{diag}(\sqrt{-1}\theta_0, \sqrt{-1}\theta_1, \ldots, \sqrt{-1}\theta_5) : \sum \theta_i = 0 \}. \]

Each 1-parameter subgroup is of the form

\[ \lambda(t) = \{ \text{diag}(t^{e_0}, t^{e_1}, t^{e_2}, t^{e_3}, t^{e_4}, t^{e_5}) : \sum r_i = 0 \} \]

which can be identified as a vector in \( t \). For each monomial \( x_0^{i_0} \cdots x_5^{i_5} \) of degree 3, its weight with respect to a 1-parameter subgroup is

\[ i_0\theta_0 + i_1\theta_1 + \cdots + i_5\theta_5 = (i_1 + i - 3) \cdot \theta_1 + \cdots (i_5 + i - 3) \cdot \theta_5 \]

where \( i := i_1 + i_2 + i_3 + i_4 + i_5 \) and \( (\theta_0, \ldots, \theta_5) \in t \) is the vector of 1-parameter subgroup. So we need to identify the monomials with weight vector

\[ W = \{(i_1 + i - 3, i_2 + i - 3, i_3 + i - 3, i_4 + i - 3, i_5 + i - 3) : 0 \leq \sum i_j \leq 3 \}. \]
By choosing a positive root system

\[ \Phi_+ = \{(1, -1, 0, 0, 0), (0, 1, -1, 0, 0), (0, 0, 1, -1, 0), (0, 0, 0, 1, -1), (0, 0, 0, 0, 3)\}, \]

we obtain the positive Weyl chamber in \( t \) as follows

\[ t_+ = (\theta_1, \ldots, \theta_5) : \theta_1 \geq \ldots \geq \theta_5 \geq 0. \]

Given weight vectors \( \alpha_1, \cdots, \alpha_5 \in W \), we can find the closest points \( \beta \) from origin 0 to convex hull \( \text{con}(\alpha_1, \ldots, \alpha_5) \). If \( \beta \in t_+ \), we keep the data \( \beta \). If not, we continue to use other weight vectors. With the help of computer, we find the only data with codimension \( < 10 \) is the case \( \beta = (\frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}) \in \tau_+ \). In order to get \( P^G_t(X^*) \), we need to compute the removing part. In this case, we have

\[ \text{stab} (\beta) = \{ \left( \begin{array}{cc} a & 0 \\ 0 & A \end{array} \right) \in SL(6, \mathbb{C}) : a \cdot \det A = 1, A \in GL(5, \mathbb{C}) \} \]

\[ \cong \mathbb{C}^* \times SL(5, \mathbb{C}) \]

where the first factor acts trivially on \( Z_\beta = \mathbb{P}(\mathbb{C}[x_1, x_2, x_3, x_4, x_5]_3) \). Thus,

\[ P^G_t(\text{stab}(\beta) (Z_\beta^*) = \frac{1}{(1 - t^2)} \cdot P^G_t(\mathbb{C}^* \times SL(5, \mathbb{C}) (Z_\beta^*) \]

\[ = \frac{1}{(1 - t^2)} \cdot P^G_t(\mathbb{C}^* \times SL(5, \mathbb{C}) (\mathbb{P}(\mathbb{C}[x_1, x_2, x_3, x_4, x_5]_3)^{ss}) \].

To compute \( P^G_t(\mathbb{C}^* \times SL(5, \mathbb{C}) (\mathbb{P}(\mathbb{C}[x_1, x_2, x_3, x_4, x_5]_3)^{ss}) \), we will use Kirwan’s formula 2.4 once more since these terms can be viewed as the cohomology of GIT quotient space. As before, we have unstable data given in the table 1 with help of computer.

| \( \beta \) | Monomials | stable(\( \beta \)) |
|----------------|-----------------|-------------------|
| 1. (0.25, 0.25, 0.25, 0.25) | \( \mathbb{C}[x_1, x_2, x_3, x_4]_3 \) | \( \left( \begin{array}{cc} a & 0 \\ 0 & A \end{array} \right) \) |
| 2. (0.9, 0.8, 0.7, 0.6) | \{x_1 x_2^2, x_1 x_3^2, x_1 x_2 x_4 \} | \( T^4 \) |
| 3. (1, 0, 0, 0) | \( x_1 \cdot \mathbb{C}[x_2, x_3, x_4]_2 + x_0 x_1^2 \) | \( \left( \begin{array}{ccc} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & A \end{array} \right) \) |
| 4. (1, 0.5, 0.5) | \{x_1^2, x_1 x_2, x_1 x_3 \} | \( \left( \begin{array}{ccc} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{array} \right) \) |
| 5. (1.08, 0.84, 0.72, 0.36) | \{x_2^3, x_1 x_4, x_3^2 \} | \( T^4 \) |
| 6. (1, 1, 0) | \( \mathbb{C}[x_1, x_2, x_3]_3 \) | \( SL(3, \mathbb{C}) \times T^2 \) |
| 7. (1.2, 0.9, 0.6, 0.3) | \{x_2^3, x_1 x_2, x_3 x_4^2 \} | \( T^4 \) |
| 8. (1.25, 0.75, 0.75, 0.25) | \{x_1^2 x_4, x_1 x_2, x_1 x_2, x_1 x_3 \} | \( \left( \begin{array}{cccc} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & A \end{array} \right) \) |

**Table 1.** Unstable stratification.
In the table 1, by choosing a suitable 1-parameter subgroup, we find some locus $Z_{\beta}^{ss} = \emptyset$ and in this case, $P_{t}^{\text{stab}^{(\beta)}}(Z_{\beta}^{ss}) = 0$. Such index vector is fake. For example, 1-parameter subgroup

$$\lambda(t) = \text{diag}(1, t^3, t^{-1}, t^{-1}, t^{-1})$$

will destabilize fourth data. Checking case by case, we have only two nonzero data: 1st and 6th that make contribution.

(1) For 1st data, $v_1 = \frac{1}{4}(1, 1, 1, 1)$, by the dimension formula 2.5, the codimension of unstable stratum equal to $35 - (24 + 20 - 15) = 6$. By the formula 2.4, we have

$$P_{t}^{\text{stab}^{(v_1)}}(Z_{v_1}^{ss}) = \frac{1}{(1 - t^2)} \cdot P_{t}^{SL(4, \mathbb{C})}(\mathbb{P}[x_1, x_2, x_3, x_4]_{3}) \mod t^{20}$$

The same method give only two unstable data for $SL(4, \mathbb{C}) \subset \mathbb{P}[x_1, x_2, x_3, x_4]_{3}$:

(a) $\beta_1 = (1, 1, 1)$. Such stratum has codimension 4 and

$$P_{t}^{\text{stab}^{(\beta_1)}}(Z_{\beta_1}^{ss}) = \frac{1}{(1 - t^2)} \cdot P_{t}^{SL(3, \mathbb{C})}(\mathbb{P}[x_1, x_2, x_3]_{3}) \mod t^{20}$$

(3.1)

(b) $\beta_2 = (1, 0, 0)$. Such stratum has codimension 5 and so we have

$$P_{t}^{\text{stab}^{(\beta_2)}}(Z_{\beta_2}^{ss}) = \frac{1 + t^2 + t^4 + t^6}{(1 - t^2) \cdot (1 - t^4)} - \frac{1 + t^2 \cdot t^2}{(1 - t^2)^2} = \frac{1}{1 - t^2}. \quad (3.2)$$

Combing these data, we have

$$P_{t}^{\text{stab}^{(v_1)}}(Z_{v_1}^{ss}) = \frac{1}{(1 - t^2)} \cdot \left\{ \frac{1 + t^2 + \ldots + t^{38}}{(1 - t^4) \cdot (1 - t^6) \cdot (1 - t^8)} - \frac{t^{12}}{(1 - t^2)} - \frac{t^8}{(1 - t^2)} \cdot \frac{1 + t^2 + t^{10} + t^{12}}{(1 - t^4) \cdot (1 - t^6)} \right\}.$$

(2) For 6th data, the index vector is $v_2 = (1, 1, 1, 0)$ and thus we have $\text{stab}(v_2) = (\mathbb{C}^*)^2 \times SL(3, \mathbb{C})$.

Then the codimension of the removing strata is

$$34 - (24 + 9 - 15) = 16 > 10.$$ 

Thus by the formula 2.4, we have

$$P_{t}^{\text{stab}^{(v_2)}}(Z_{v_2}^{ss}) = \frac{1}{(1 - t^2)^2} \cdot P_{t}^{SL(3, \mathbb{C})}(\mathbb{P}[x_1, x_2, x_3]_{3}) \mod t^{20}$$

$$= \frac{1}{(1 - t^2)^2} \cdot \frac{1 + t^2 + t^{10} + t^{12}}{(1 - t^4) \cdot (1 - t^6)} \mod t^{20}.$$

Observe the codimension of unstable stratification of this data is given by

$$\text{codim} S_{\beta} = 55 - (\dim G + \dim Y_{\beta} - \dim P_{\beta})$$

$$= 55 - (35 + 34 - 20) = 6.$$
Here $P_\beta$ is a parabolic subgroup consisting of all upper-triangle matrix and thus it has \(\dim P_\beta = \frac{(6+1)\cdot 6}{2} - 1 = 20\) and
\[
\dim Y_\beta = \# \{ \alpha \in W : \alpha, \beta \geq \beta, \beta \} = 34.
\]
Thus by putting all the discussions above into the formula 2.4, we have

**Proposition 3.1.**

\[
P_t^G(X_{ss}) \equiv \frac{1 - t^{112}}{\prod_{1 \leq i \leq 6} (1 - t^{2i})} - t^{12} \frac{\Pi_{1 \leq i \leq 5} (1 - t^{2i})}{(1 - t^{2})} \] (3.3)

Next, we will take the blowups successively along the locus discussed in section 2.3. Here we give the list of locus to be blowuped in the following table 2.

### Table 2. List of datas to be blowed up

| Blowup locus | stabilizer group (up to finite index) | codimension |
|--------------|--------------------------------------|-------------|
| $G_\omega$   | $SL(3, \mathbb{C})$                  | 27          |
| $G_\zeta$    | $T^4$                                | 24          |
| $GZ_\chi$    | $SO(2)$                              | 21          |
| $GZ_\tau$    | $T$                                  | 20          |
| $GZ_\delta$  | $\mathbb{C}^* \times \mathbb{C}^*$   | 19          |
| $GZ_\alpha$  | $\mathbb{C}^*$                       | 19          |
| $GZ_\gamma$  | $\mathbb{C}^*$                       | 18          |
| $GZ_\beta$   | $\mathbb{C}^*$                       | 17          |

3.1.2. **Computation of** $P_t^G(\mathcal{X}_{ss})$.

We take the blow up
\[
\pi : X_1 \rightarrow X_{ss}
\]
along $G \cdot Z_{R_\omega}^{ss}$. By dimension counting, we have
\[
d_R + 1 = \text{codim} G \cdot Z_{R_\omega}^{ss} = 55 - (\dim G - \dim N(R_\omega)) = 55 - (35 - 8) = 28.
\]
Moreover, we have
\[
P_t^{N(R_\omega)}(Z_{R_\omega}^{ss}) = P_t(BN(R_\omega)) = \frac{1}{\prod_{1 \leq i \leq 4} (1 - t^i)}
\]
since $Z_{R_\omega}^{ss}$ is just a point. In [23, Corollary 4.2], by using the fact the cubic 4-fold $\omega$ is the secant variety of Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, Laza proved

**Proposition 3.2.** The representation of $R_\omega$ on the normal slice $\mathcal{N}_\omega$ is isomorphic to $\text{Sym}^6(\mathbb{C}^3)$, where $R_\omega \cong SL(3, \mathbb{C})$ has the natural representation on $\mathbb{C}^3$. In particular, the exceptional divisor $\mathbb{P}\mathcal{N}_\omega/R_\omega$ is isomorphic to the GIT quotient space of plane sextic curves.

A very helpful observation is the following
Corollary 3.3. After the 1st blowup, the incidence relations of boundaries on exception divisor \( \mathbb{P}N_{\omega}/\mathbb{P}SL(3, \mathbb{C}) \) coincide with that of GIT moduli space of degree 2 K3 surfaces. See the figure 3. That is, let \( \chi_1, \beta_1, \tau_1, \beta_1 \subset X_1^{ss}/G \) be the strict transformation of the GIT boundary after the 1st blowup along \( G \cdot Z_\omega \), then

\[
E^{ss}_{\omega}/G \cap \chi_1 = pt, \quad E^{ss}_{\omega}/G \cap \tau_1 = pt \\
E^{ss}_{\omega}/G \cap \gamma_1 \cong |O_{\mathbb{P}^1(4)}/\mathbb{P}SL(2)| = \mathbb{P}^1 \\
E^{ss}_{\omega}/G \cap \beta_1 \cong \mathbb{P}^3/\mathbb{C}^* = \mathbb{P}(1, 2, 3).
\]

(3.4)

Proof. Under the isomorphism \( \mathbb{P}N_{\omega}/\mathbb{P}\omega \cong \mathbb{P}Sym^6(\mathbb{C}^3)/\mathbb{P}SL(3) \) in Proposition 3.2, we can identify the stability on both sides. Since after the first blowup, the rest blowups restricting to the divisor \( \mathbb{P}N_{\omega}/\mathbb{P}\omega \) are isomorphic to the partial resolution of \( \mathbb{P}N_{\omega}/\mathbb{P}\omega \) in the sense of Kirwan, then the locus of GIT strictly semistable locus will coincide with that of the GIT moduli space of plane sextics \( |O_{\mathbb{P}^2}(6)|/\mathbb{P}SL(3) \). Such locus have been explicitly described in [18]. Then from the incidence relation in figure 1, we prove the assertion. \( \square \)

Remark 3.4. This corollary shows in the 2nd-8th blowups, the blowup on \( E^{ss}_{\omega}/G \cap \beta_1 \) will be the same as that in GIT moduli space of degree 2 K3 surfaces, then Kirwan-Lee’s results (see [17] [18]) will help us simplify many computations.

Following the computation in [16] for K3 surface and formula 2.7, we have

\[
A_1 = \frac{(t^2 - t^{56})}{(1 - t^2)(1 - t^4)(1 - t^6)} - \frac{t^{50} - t^{56}}{(1 - t^2)(1 - t^4)(1 - t^6)} + \frac{t^{20} - t^{28}}{(1 - t^2)^2} \\
\equiv \frac{t^2}{(1 - t^2)(1 - t^4)(1 - t^6)} - \frac{t^{20}}{(1 - t^2)^2} \mod t^{20}.
\]

(3.5)

3.1.3. Computation of \( P^G(X_2^{ss}) \). Thanks to the disjointness of orbit \( G\omega \) and \( G\zeta \), for the second blowup we do not need to consider the effect of the first blowup. So we take the second blowup

\[
\pi : X_2 \to X_1^{ss}
\]

(3.6)

along \( G \cdot Z^{ss}_{R_\zeta} \). It’s easy to see that the normalizer of \( R_\zeta \) in \( G = SL(6, \mathbb{C}) \) is given by the extension

\[
0 \to T^5 \to N(R_\zeta) \to S_6 \to 0.
\]
Here the symmetric group $S_6$ is the Weyl group of $R_\zeta$. Then

\[ d_{R_\zeta} + 1 = \text{codim}(G \cdot Z_{R_\zeta}^{ss}) = 55 - (\dim G - \dim N(R_\zeta)) = 25. \]

This gives

\[ (t^2 + \ldots + t^{2d_{R_\zeta}})P_t^{N(R_\zeta)}(Z_{R_\zeta}^{ss}) = (t^2 + \ldots + t^{48})P_t(BN(R_\zeta)) \]

\[ = \frac{(t^2 - t^{48})}{(1 - t^2)(1 - t^{12})} \]

since $G \cdot Z_{R_\zeta}^{ss} = G \times N(R_\zeta) Z_{R_\zeta}^{ss}$ and $Z_{R_\zeta}^{ss}$ is just a point. Actually, following a lemma of section 4.2 in [17], we can compute the normal vector space $N_\zeta$ at $\zeta$ as follows

\[ N_\zeta = \mathbb{C}x_0^3 + \ldots + \mathbb{C}x_5^3 + \{x_0^2, x_4, x_5^2\} \mathbb{C}[x_1, x_2, x_3] \]

\[ + \{x_1^2, x_2, x_3^2\} \mathbb{C}[x_4, x_5, x_6] \mathbb{C} \{x_0 x_4 x_5 + x_1 x_2 x_3\}. \]

So the intersections of exceptional divisor $E_2/N(R)$ with the proper transformation of $\alpha, \delta, \tau$ are 3 distinct points. Note that $R_\zeta = \{\text{diag}(a, b, c, d, c^{-1} d^{-1}, a^{-1}b^{-1}) : a, b, c, d \in \mathbb{C}^*\}$ acts trivially on $x_0 x_4 x_5 + x_1 x_2 x_3$. By Kirwan, the unstable data is identified with the unstable data of natural action $R_\zeta$ on $\mathbb{P}N_\zeta$. So we only consider this action. Each 1-parameter subgroup can be written as $\text{diag}(t^{a_0}, \ldots, t^{a_5})$ and the weight is of the form

\[ \{a \cdot I : x^t \in N_\zeta\} \]

where $x^t = x_0^{i_0} \ldots x_5^{i_5}$ with $i_0 + \ldots + i_5 = 3$ and $a \cdot I = a_0 i_0 + \ldots + a_5 i_5$. Note that in the formula 2.7, the codimension is

\[ \text{codim} = 24 - \# \{a \cdot I \in W : a \cdot I > 0\} \geq 11. \]

Thus, we obtain

\[ A_2 \equiv \frac{t^2}{(1 - t^2)(1 - t^{12})} \text{mod } t^{20}. \] (3.7)

3.1.4. Computation of $P_t^G(X_3^{ss})$. We take the third blowup

\[ \pi : X_3 \to X_2^{ss} \]

along $G \cdot \hat{Z}_{R_\zeta}^{ss}$ where $\hat{Z}_{R_\zeta}^{ss}$ is the strict transformation of $Z_{R_\zeta}^{ss}$ under composition of previous blowups, since $\chi$ contains point $\omega$ and $\zeta$, we have

\[ \text{codim}(G \cdot \hat{Z}_{R_\zeta}^{ss}) = 55 - (\dim G + \dim \hat{Z}_{R_\zeta}^{ss} - \dim N(R_\zeta)) = 23. \]

By Proposition 2.6, we have

\[ \hat{Z}_{R_\zeta}^{ss}/N(R_\zeta) \cong \mathbb{P}(1, 3) \cong \mathbb{P}^1. \]

This gives

\[ (t^2 + \ldots + t^{2d_{R_\zeta}}) \cdot P_t^{N(R_\zeta)}(\hat{Z}_{R_\zeta}^{ss}) = \frac{t^2 - t^{46}}{1 - t^2} \cdot P_t(BR_\zeta) \cdot P_t^{N(R_\zeta)/R_\zeta}(\hat{Z}_{R_\zeta}^{ss}) \]

\[ = \left(\frac{t^2 - t^{46}}{1 - t^2}\right) \cdot \frac{1}{1 - t^4} \cdot (1 + t^2). \]
since the action $N(R_{\chi})$ on $\hat{Z}_{R_{\chi}}$ is isomorphic to the action $N(R_{\chi})/R_{\chi}$ on $\hat{Z}_{R_{\chi}}$. In the same paper [23, Lemma 4.3], Laza showed the normal representation $\rho : R_{\chi} \sim \mathcal{N}_{\chi}$ can be identified as

$$SL(2) \sim H^0(\mathcal{O}_{\mathbb{P}^1}(12)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(8)),$$

This gives

$$\sum t^{2 \cdot \text{codim}(S_3)} P_t^{s(\beta) \cap N(R)}(Z_{ss}) = \frac{t^{24} + t^{26} + t^{28} + t^{30}(1 + t^2) + t^{30} + t^{34} + t^{36} + t^{38} + t^{40}(1 + t^2)}{1 - t^2} \equiv \frac{t^{24} - t^{44}}{(1 - t^2)^2}.$$

By putting these together, we obtain

$$A_3 = \left(\frac{t^2 - t^{46}}{1 - t^2}\right) \cdot \frac{(1 + t^2)}{1 - t^2} - \frac{t^{24} - t^{44}}{(1 - t^2)^2} \quad \text{mod}\ t^{20}. \quad (3.8)$$

3.1.5. To compute $P_t^G(X_4^{ss})$. We take the 4th blowup $\pi : X_4 \to X_3^{ss}$ along $G \cdot \hat{Z}_R^{ss}$. Since $N(R_{\gamma}) = R_{\gamma} \leq N(R_{\omega}) = SL(3)$, we have

$$\text{codim}(G \cdot \hat{Z}_R^{ss}) = 55 - (\dim G + \dim Z_{R_{\gamma}} - \dim N(R_{\gamma})) = 21.$$

Observe that we can identify the normal representation $R_{\gamma} \sim \mathcal{N}_{\gamma} \cong R_1 \sim \mathcal{N}_1$ where the normal representation $R_1 \sim \mathcal{N}_1$ the second blowup in [17, Section 4] by corollary 3.3. Then from [17, table 2], we have

$$A_4 = \frac{t^{2} - t^{42}}{1 - t^2} \cdot P_t(BR_{\gamma}) \cdot (1 + t^2) - \frac{t^{18} + t^{20}}{1 - t^2} \mod t^{20} \equiv \frac{t^2(1 - t^2)}{(1 - t^2)^3} - \frac{t^{18} + t^{20}}{1 - t^2} \mod t^{20} \quad (3.9)$$

where the multiplication term $1 + t^2$ is due to the geometry of locus $\hat{Z}_R^{ss} / (N(R_{\gamma})/R_{\gamma}) \cong \mathbb{P}^1$ by proposition 2.6.

3.1.6. To compute $P_t^G(X_5^{ss})$. We take the blowup $\pi : X_5 \to X_4^{ss}$ along $G \cdot \hat{Z}_R^{ss}$. Note that here

$$Z_{R_5} = \mathbb{P}\{x_0q(x_4, x_5) + c(x_1, \ldots, x_3)\}$$

where $\{x_0q(x_4, x_5) + c(x_1, \ldots, x_3)\}$ means the vector space spanned by the monomials in a general polynomial of the form $x_0q(x_4, x_5) + c(x_1, \ldots, x_3)$. And the normalizer subgroup of such locus is

$$N(R_5) = \{\text{diag}(a, A, B) : a^{-1} = |A| \cdot |B|, A \in GL(3), B \in GL(2)\}.$$

Thus, by the dimension counting, we have

$$\text{codim} GZ_{R_5} = 55 - (35 + 12 - 13) = 21.$$
Observe that $Z_{R_5}$ contains the point $\zeta$ which is represented by the equation $x_0x_4x_5 + x_1x_2x_3$, so we need to take the blow up

$$\hat{Z}_{R_5} \rightarrow Z_{R_5}$$

along $G_5 \cdot Z_{R_5}$ to compute $P_t^{N(R_5)}(\hat{Z}_{R_5}^{ss})$. Note from Proposition 2.6, we know the blowup $\hat{Z}_{R_5}^{ss}/N(R) \rightarrow Z_{R_5}/N(R) = \mathbb{P}^1$ does not change cohomology, ie, $P_t(\hat{Z}_{R_5}^{ss}/N(R)) = 1 + t^2$. Thus

$$A_5 = \frac{t^2}{1 - t^2} \frac{1}{(1 - t^2)(1 - t^4)} - \sum \text{ mod } t^{20}$$

where $\sum$ is the terms due to removing the unstable strata of representation of $R_5$ on the normal vector space of some point in $Z_{R_5}$.

In order to find the weight of normal representation, we choose a point in $Z_5$ distinct to $\zeta$ whose equation is $F = x_0x_4x_5 + f$, where $f$ is a generic cubic polynomial in $x_1, x_2, x_3$. For normal representation $R_5$ on $\mathcal{N}_F$, we take weight $(2, 0, 0, 0, -1, -1)$ of the maximal torus of $R_5$. Here we view the weight embedded into the Lie algebra of $G$. By subtracting the weight from $\frac{\partial F}{\partial x_0}, ..., \frac{\partial F}{\partial x_5}$, and the form $x_0f + f$, we have weight of normal space $\mathcal{N}_F$ in following list:

| weight | 6 | 4 | 0 | -3 | -2 | -1 |
|--------|---|---|---|----|----|----|
| mul    | 1 | 3 | 3 | 2  | 6  | 6  |

Thus, by formula 2.7, the removing term $\sum$ is

$$P_t(\hat{Z}_{R_5}^{ss}/N(R)) \cdot (t^{2.8} P_t(\mathbb{P}^5) + t^{5} P_t(\mathbb{P}^1)) \text{ mod } t^{20}$$

and the correction term for 5-th blowup is

$$A_5 = \frac{t^2}{(1 - t^2)^3} - \frac{(t^{12} + ... + t^{20})(1 + t^2)}{1 - t^2} \text{ mod } t^{20}.$$  

3.1.7. Computation of $P_t^G(X_6^{ss})$. We take the blowup

$$\pi : X_6 \rightarrow X_5^{ss}$$

along $G \cdot \hat{Z}_{R_5}$, where $\hat{Z}_{R_5}^{ss}$ is the strict transformation of $Z_{R_5}^{ss}$ under previous blowups, since $\alpha$ contains point $\zeta$. It is easy to see

$$Z_{R_5} = \mathbb{P}\{x_0q_0(x_2, ..., x_5) + x_1q_1(x_2, ..., x_5)\},$$

$$1 + d_\alpha = \text{codim } G\hat{Z}_{R_5} = 55 - (35 + 19 - 19) = 20,$$

$$N(R_\alpha) = \{\text{diag}(A, B) : \det(A) \cdot \det(B) = 1, A \in GL(2, \mathbb{C}), B \in GL(4, \mathbb{C})\}.$$

The blowup

$$\hat{Z}_{R_\alpha} \rightarrow Z_{R_\alpha}^{ss}$$

along $N(R_\alpha)\zeta$ descending to quotients will not change cohomology of quotients as in the case of 5-th blowup and thus we have the formula

$$P_t^{N(R_\alpha)}(\hat{Z}_{R_\alpha}^{ss}) = P_t(N(R_\alpha))(1 + t^2) = \frac{1 + t^2}{1 - t^2}$$

since $Z_{R_\alpha}/R_\alpha \cong \mathbb{P}^1$ where first identity is due to formula 2.1.
To determine the normal representation of $R_\alpha$, we choose a point in $Z^{ss}_R$ represented by $F = x_0(x_2^2 + x_3^2 + x_4^2) + x_1x_5^2$ which is not in the orbit $\zeta$. Then

$$
F_{x_0} = x_2^2 + x_3^2 + x_4^2, \quad F_{x_1} = x_5^2, \quad F_{x_2} = 2x_0x_2,
$$

$$
F_{x_3} = 2x_0x_3, \quad F_{x_4} = 2x_0x_2, \quad F_{x_5} = 2x_1x_5
$$

where $F_{x_i} := \frac{\partial F}{\partial x_i}$. It is known as before that the tangent space at $F$ is spanned by the monomials $F_{x_0}, \ldots, F_{x_5}$ and monomials in $Z^{ss}_R$. Subtracting from $\mathbb{C}[x_0, x_1, \ldots, x_5]_3$, we obtain the normal vector space

$$
\mathcal{N}_F = \mathbb{C}[x_0, x_1]_3 \oplus \text{span}_{\mathbb{C}} \{x_3V_2, x_1^2x_2, x_1^2x_3, x_1^2x_4\} \oplus V_3
$$

where $V_2$ is the set of monomials in $x_2, x_3, x_4$ of degree 2 and $V_3$ is the vector space of monomials in $x_2, x_3, x_4$ of degree 3 without the terms $x_2F_{x_0}, x_3F_{x_0}, x_4F_{x_0}$. Recall the weight of $R_\alpha$ is $(2, 2, -1, -1, -1, -1)$, then the weight of normal representation is given by

| weight | -3 | 0  | 3  | 6  |
|--------|----|----|----|----|
| mul    | 7  | 5  | 4  | 4  |

So the smallest codimension of unstable strata is $19 - 8 = 11$ and thus the removing term vanishes after $mod\ t^{20}$.

In a summary, the correction term in the 6-th blowup contributes

$$A_6 \equiv \frac{t^2}{1-t^2} \cdot \frac{1+t^2}{1-t^2} \mod t^{20}. \quad (3.10)$$

3.1.8. Computation of $P^G_t(X_7^{ss})$. We take the blowup

$$\tau : X_7 \to X_6^{ss}$$

along $G \cdot \hat{Z}^{ss}_R$ where $\hat{Z}^{ss}_R$ is the strict transform of $Z^{ss}_R$ under $X_6^{ss} \to X$. Then the codimension of $GZ_R|$ is given by

$$1 + d_\gamma = 55 - (35 + 14 - 13) = 19.
$$

Also we have the normalizer subgroup

$$N(R_\gamma) = \{\text{diag}(a, A, B) : a \cdot \det(A) \cdot \det(B) = 1, A \in GL(2, \mathbb{C}), B \in GL(3, \mathbb{C})\}.
$$

By Proposition 2.6, we have $Z_R_{\gamma}/G_\gamma \cong \mathbb{P}^1 \times \mathbb{C}$ and blowup at two points in $Z_{R_\gamma}/G_\gamma$ will give $P_t(\overline{Z_{R_{\gamma}}}/G_\gamma) = 1 + 3t^2$.

Now we consider the normal representation $R_\gamma \cap \mathcal{N}_\gamma$. As before, by choosing a suitable element in $\hat{Z}^{ss}_R$, we compute its normal vector space $\mathcal{N}_\gamma$, which is a vector space spanned by monomials in the following form

$$\{x_0^3, \ldots, x_5^3, x_0^2x_1, \ldots, x_0^2x_4, x_0x_1^2, x_0x_2^2, x_1^2x_2, x_2^2x_3, x_3^2x_4, x_4^2x_5, x_4^2x_3\}.
$$

It can be identified as the normal representation in the third blowup in the case of K3 surfaces of degree 2 in [17], then in a summary the 7th blowup contributes

$$A_7 \equiv (t^2 + ... + t^{2d_\gamma})P^N_t(R_\gamma)(\mathcal{Z}_R) - \sum\text{unstable} \mod t^{20}$$

$$\equiv \frac{1 + 3t^2}{1-t^2}(t^2 + t^4 + ... + t^{14}) \mod t^{20}. \quad (3.11)$$
3.1.9. Computation of $P_t^G(X_{ss}^8)$. We take the blowup
\[ \pi : X_8^s \to X_7^s \]
along $G \cdot \tilde{Z}_{R_{\mu}}^{ss}$ where $\tilde{Z}_{R_{\mu}}^{ss}$ is the strict transform of $Z_{R_{\mu}}^{ss}$ under previous blowups. The normalizer subgroup is
\[ N(R_{\mu}) = \{ \text{diag}(a, b, A, c, d) : abcd \cdot \det(A) = 1, a, b, c, d \in \mathbb{C}^*, A \in GL(2, \mathbb{C}) \} \].
The locus $Z_{R_{\mu}}$ is identified as
\[ Z_{R_{\mu}} = \mathbb{P}\{ ax_0x_4^2 + x_0x_5l_1(x_2, x_3) + bx_1^2x_5 + x_1x_4l_2(x_2, x_3) + c(x_2, x_3) \} \].
Here $\{ f \}$ means the vector space spanned by monomials in $f$. Thus the codimension is
\[ 1 + d_{\mu} = \text{codim} GZ_{R_{\beta}} = 55 - (35 + 9 - 7) = 18 \].
The blowup $\tilde{Z}_{R_{\mu}} \to Z_{R_{\mu}}$ along the orbit $N(R_{\mu})\zeta$ and $N(R_{\mu})\omega$ will descent to the blowup along two points in $Z_{R_{\mu}}//N(R_{\mu}) \cong \mathbb{P}(1, 3, 6, 8)$. This gives
\[ P_t^{N(R_{\beta})}(\tilde{Z}_{R_{\beta}}) = P_t(BR_{\beta}) \cdot P_t(\tilde{Z}_{R_{\mu}}//N(R_{\beta})) = \frac{1}{1 - t^2} \cdot (P_t(\mathbb{P}(1, 3, 6, 8)) + (t^2 + t^4) + (t^2 + t^4)) \].
(3.12)
The normal representation for $R_{\mu}$ can be identified in K3 case as done in last blowup of Kirwan-Lee (see 5.3 in [17]), thus the removing term is giving by
\[ \frac{t^{18}}{1 - t^2} \mod t^{20} \].
In a summary, the correction term will be given by
\[ A_8(t) \equiv \frac{t^2}{1 - t^2} \cdot \frac{1 + 3t^2 + 3t^4 + t^6}{1 - t^2} - \frac{t^{18}}{1 - t^2} \cdot (1 + 3t^2 + 3t^4 + t^6) \mod t^{20} \].
(3.13)
3.2. Proof of Theorem 1.1. By previous computations, we have
\[ P_t(\tilde{M}) = P_t^G(X_{ss}^8) + \sum_{i=1}^{8} A_i(t) \]
\[ = 1 + 9t^2 + 26t^4 + 51t^6 + 81t^8 + 115t^{10} + 152t^{12} + 193t^{14} + 236t^{16} + 280t^{18} + 324t^{20} \mod t^{20} \].
Then the duality of intersection cohomology will imply the formula.

4. Intersection cohomology of Baily-Borel compactification
In this section, we will compute the intersection cohomology of Baily-Borel compactification $\overline{D/\Gamma}^{BB}$ based on the computations in previous sections.
4.1. Baily-Borel compactification of moduli space of cubic fourfolds. It is well-known that for a smooth cubic fourfold $X$ its integral middle cohomology $H^4(X, \mathbb{Z})$ is isomorphic to

$$\Lambda := \langle 1 \rangle^2 \oplus \langle -1 \rangle^3 .$$

Let $h := c_1(O_X(1))^2 \in \Lambda$ be the hyperplane class and $\Lambda_0 = E_8^{\perp} \oplus U^2 \oplus A_2 = h^\perp$ be the lattice associated to the smooth cubic fourfold, which is isomorphic to the primitive cohomology $H^4_p(X, \mathbb{Z})$ of $X$. Denote by

$$D := \{ z \in \mathbb{P}(\Lambda_0 \otimes \mathbb{C}) : \langle z, z \rangle = 0, \langle z, \overline{z} \rangle > 0 \}$$

the period domain. It is a symmetric domain of type IV. Let $\Gamma$ be the monodromy group of cubic fourfolds, then it is shown in [3] that $\Gamma = O^*(\Lambda_0)$ is the automorphism group of $\Lambda_0$ whose elements act trivially on the discriminant group of $\Lambda_0$. The quotient space $D/\Gamma$ is known as a locally symmetric space. By the general result of Baily-Borel in [2], there is a compactification of $D/\Gamma$, whose boundaries correspond to Type II, III degenerations of cubic fourfolds (see also [26] for refinements). Such compactification is well-known as Baily-Borel compactification now and we denote by $\overline{D/\Gamma}$. Following Hassett [12], we define

**Definition 4.1.** A cubic fourfold $X$ is called a special cubic fourfold of discriminant $d$ if it contains a surface $T$ which is not homologous to a complete intersection and the classes $h$ and $[T]$ form a saturated rank 2 sublattice of $\Lambda$ with discriminant $d$.

**Example 4.2.** Let $X$ be a nodal cubic fourfold. That is, $X$ contains a point $x$ whose projective tangent cone is a smooth quadratic. Assume the coordinate of $x$ is $[0, 0, 0, 0, 1]$, then $X$ has defining equation of the following form

$$x_5q(x_0, \cdots, x_4) + c(x_0, \cdots, x_4) = 0,$$

where $q$ and $c$ are quadratic and cubic forms in $x_0, \cdots, x_4$. By associating a K3 surface of degree 6, Hassett showed that a nodal cubic fourfold is a special cubic fourfold of discriminant 6.

**Example 4.3.** Let $X$ be a smooth cubic fourfold containing a plane $P$, then it is not hard to check that the class $[P] \in H^4(X, \mathbb{Z})$ with the product of hyperplane class $h^2$ form the following matrix

| \ | $h^2$ | \ | $[P]$ |
|----------|----------|----------|----------|
| $h^2$    | 3        | 1        |
| $[P]$    | 1        | 3        |

Such cubic fourfold is a special cubic fourfold of discriminant 8.

It is also shown by Hassett in [12] that the locus of special cubic fourfolds of discriminant $d$ is nonempty for $d \equiv 0, 2 \mod 6$. Moreover, it is a divisor in the moduli space of cubic fourfolds and is called Hassett divisor now. On $\overline{D/\Gamma}$, we can use the saturated rank 2 sublattices of $\Lambda$ with discriminant $d$ to define a divisor which is called a Heegner divisor of discriminant $d$. Let $\mathcal{H}_\infty$ be such a divisor on $\overline{D/\Gamma}$ of discriminant 2.
Theorem 4.4. (Global Torelli, see \cite{23} \cite{27}) The period map
\[ p : \overline{\mathcal{M}} \rightarrow \overline{\mathcal{D}/\Gamma}^{BB} \]
is a birational map. It is an open immersion over \( \mathcal{M}^0 \) and can be defined over
\( \mathcal{M} \), whose image is the complement of Heegner divisor \( \mathcal{H}_\infty \).

Thanks to the Torelli theorem 4.4, these divisors defined in 4.1 are also called
Heegner divisors if we view \( \mathcal{D}/\Gamma \) as a Shimura variety. We refer the readers to
\cite{28} for the definition and properties of Shimura varieties.

Remark 4.5. Recently, the property of open immersion of period map on \( \mathcal{M}^0 \)
is also proven by Huybrechts and Rennemo in \cite{14} by using Jacobian rings.

4.2. Intersection cohomology. Let \( \widetilde{\mathcal{M}} \) be the blowups of \( X/\sl(6) \) only along
the point \( \omega \) and then the line \( \chi \). Then there is a natural contraction morphism
\[ f : \widetilde{\mathcal{M}} \rightarrow \widehat{\mathcal{M}}. \]
From \cite{23}, it is known that the period map from GIT compatification to Baily-
Borel compatification is resolved by Looijengena’s semi-toric compatification (see
for \cite{26} the general discussions of Looijengena’s semi-toric compatifications):

\[ \begin{array}{ccc}
\widetilde{\mathcal{M}} & \xrightarrow{p_1} & \overline{\mathcal{D}/\Gamma}^{\Sigma(\mathcal{H}_\infty)} \\
p_2 & \downarrow & \rightarrow \\
\overline{\mathcal{M}} & \xrightarrow{f} & \overline{\mathcal{D}/\Gamma}^{BB} \\
\end{array} \]

where \( p_1 \) is the composition of the two successive blowups along the point \( \omega \)
and then the line \( \chi \). Let \( \overline{\mathcal{D}/\Gamma}^{\Sigma(\mathcal{H}_\infty)} \) be the Looijenga’s semi-toric compatification associated to the Heegner divisor \( \mathcal{H}_\infty \). We observe the following explicit
description of birational morphism \( p_2 \)

Proposition 4.6. The morphism \( p_2 \) is the composition of \( f : \widetilde{\mathcal{M}} \rightarrow \overline{\mathcal{D}/\Gamma}^{\Sigma(\mathcal{H}_\infty)} \)
and \( \nu : \overline{\mathcal{D}/\Gamma}^{\Sigma(\mathcal{H}_\infty)} \rightarrow \overline{\mathcal{D}/\Gamma}^{BB} \). Here \( f \) is the morphism contracting the divisor
\( E_\chi \) to
\[ E_\chi \cap \tilde{E}_\omega \cong \mathbb{P}(H^0(C, \mathcal{O}_C(4)) \oplus H^0(C, \mathcal{O}_C(6)))/\sl(3) \subset \overline{\mathcal{D}/\Gamma}^{\Sigma(\mathcal{H}_\infty)} \]
where \( C \) is a smooth plane conic and \( \nu \) is a small modification whose boundaries
maps are described in the following table 3. Here the locus \( \phi_\infty, \cdots, \epsilon_\infty \) in table

| Exceptional locus in \( \overline{\mathcal{D}/\Gamma}^{\Sigma(\mathcal{H}_\infty)} \) | Boundaries in \( \overline{\mathcal{D}/\Gamma}^{BB} \) | fiber |
|--------------------------------|-----------------|-------|
| \( \phi_\infty \)          | \( A_{17} \)    | \( \mathbb{P}^1 \) |
| \( \gamma_\infty \)         | \( E_7 \oplus D_{10} \) | \( \mathbb{P}^1 \) |
| \( \beta_\infty \)          | \( E_8 \oplus A_2 \) | \( \mathbb{P}^2 \) |
| \( \epsilon_\infty \)       | \( A_2 \oplus D_{16} \) | \( \mathbb{P}^2 \) |

Table 3. Contraction locus of \( p_2 \)

3 is described in Lemma 6.9 in \cite{23}.
Proof. From Section 6 in [23], we know that \( p_2 \) is the composition of a small modification in the sense of Looijenga (see [26]) and a blowup of codimension 2 self-intersection of Heegner divisor \( H_\infty \). It remains to prove the codimension 2 self-intersections is isomorphic to \( E_\chi \cap \tilde{E}_\omega \). This also follows from geometric description of the resolution in Section 6 in [23]. Indeed, denote by \( \tilde{H_\infty} \) its strict transformation and then \( \tilde{H_\infty} \) is isomorphic to the blowup of GIT space of plane sextics at a point from Corollary 3.3. So another divisor \( E_\chi \) obtained from morphism \( p_1 \) is isomorphic to the exceptional divisor of blowup the codimension 2 locus (note that such locus is irreducible by Lemma 6.8 in [23]). As the divisor \( E_\chi \to \mathbb{P}^1 \) is fibration, By the construction in Section 7 in [26], we get the center is isomorphic to the general fiber of the fibration \( E_\chi \to \mathbb{P}^1 \), ie, \( E_\chi \cap \tilde{E}_\omega \cong \mathbb{P}(\mathcal{O}_C(4)) \oplus \mathcal{O}_C(6)) \otimes SO(3) \) where the identification is from the first blowup of GIT space of plane sextics obtained by Shah in [35].

The small modification is determined by the self-intersections of the Heegner divisor \( H_\infty \) and its intersection with Baily-Borel boundaries: As \( H_\infty \) has no 1-dimensional self-intersection by Lemma 6.8 in [23], we can apply Proposition 7.2 [26] and thus we know \( \nu : \mathcal{D} / \Gamma^{\Sigma(H_\infty)} \to \mathcal{D} / \Gamma^B \) is a normalised blowup, ie, the blowup along the Baily-Borel boundaries that intersect with the self-intersections of \( H_\infty \). The root lattices of these boundaries and codimension are described in Lemma 6.9 in [23]. Thus, we get the table 3. \( \square \)

**Proposition 4.7.** Let \( f : X \to Y \) be a birational morphism of \( n \) dimensional irreducible varieties over \( \mathbb{C} \) contracting a divisor \( E \to \mathbb{P}^m \) to a lower dimensional locus \( Z \) and the restriction \( f_E \) of \( f \) is a topological \( \mathbb{P}^m \)-bundle, then for \( k \leq n \), the intersection cohomology has a decomposition

\[
\text{IH}^k(X) \cong \text{IH}^k(Y) \oplus H^{k-c+j}(Z, \mathbb{Q}) \tag{4.3}
\]

where \( c \) is the codimension of \( Z \) in \( X \).

**Proof.** Let \( IC_X \) be the intersection complex on \( X \). By BBDG’s decomposition theorem, there is a decomposition (non-canonical in general, see [9] for more general results)

\[
Rf_* IC_X \cong IC_Y \oplus IC_Z(L_j)[-i]
\]

where \( L_j \) are the local systems on \( Z \) and \( i \) is the degree to be shifted. Following [11], we can determine these local system: each \( L_j \) is an irreducible summand of \( R^j f_E^* Q_E \) where \( f_E \) is the morphism restricting on \( E \). \( L_j \) is rank 1 for \( j \) even since each fiber is \( \mathbb{P}^m \), thus \( L_j \cong R^j f_E^* Q_E \) and the shift degree is \( -i = -j + c \). then by taking cohomology of the decomposition, we obtain the formula 4.3:

\[
\text{IH}^k(X) = H^k(Y, IC_Y) \oplus R^j f_E^* Q_E[-j + c])
\]

\[
= \text{IH}^k(Y) \oplus H^k(Z, \mathbb{Q}[-j + c])
\]

\[
= \text{IH}^k(Y) \oplus H^{k-c+j}(Z, \mathbb{Q})
\]

\( \square \)
Recall that an algebraic map $f : X \to Y$ is called semismall if the defect
$$r(f) := \max\{ i \in \mathbb{Z} : \; p^\mathcal{H}^i(Rf_*IC_X[n]) \neq 0 \}$$
is zero.

**Proposition 4.8.** Let $f : X \to Y$ be a semismall birational morphism of $n$-dimensional irreducible varieties over $\mathbb{C}$ such that $Z \subset Y$ is a connected closed subvariety and $f$ is isomorphic outside $Z$ and over $Z$, $f$ is a $\mathbb{P}^m$-bundle, then for $k \leq n$

$$\text{IH}^k(X) = \bigoplus_{0 \leq j \leq m} H^{k+2j-n}(Z, \mathbb{Q})$$

(4.4)

where $H^l(Z, \mathbb{Q}) = 0$ if $l < 0$.

**Proof.** By semi-small property and semi-simplicity of the decomposition theorem, we have

$$Rf_*IC_X[n] = \bigoplus_{-r(f) \leq i \leq r(f)} p^\mathcal{H}^i(Rf_*IC_X[n])[-i]$$

$$= \bigoplus_j IC(\overline{Y}_i, \mathcal{L}_{i,j})$$

(4.5)

where $\mathcal{L}_{i,j}$ is a local system supported on the closure $\overline{Y}_i$ of strata $Y_i$. In our case, there is a natural stratification $Y_0 = Z, Y_1 = Y - Z$, then we have

$$Rf_*IC_X = IC_Y \bigoplus_{j=2}^m \mathbb{Q}[n-2j]$$

(4.6)

By taking the cohomology, we obtain the result. \qed

**Theorem 4.9.** The intersection cohomology of $\widehat{M}$ is given by

$$\text{IP}_t(\widehat{M}) = 1 + 3t^2 + 8t^4 + 17t^6 + 29t^8 + 44t^{10} + 61t^{12} + 78t^{14}$$

$$+ 99t^{16} + 121t^{18} + 151t^{20} + 121t^{22} + 99t^{24} + 78t^{26}$$

$$+ 61t^{28} + 44t^{30} + 29t^{32} + 17t^{34} + 8t^{36} + 3t^{38} + t^{40}$$

(4.7)

**Proof.** We will use the blowup formula 2.2 of intersection cohomology reversely. Then we need to do the calculations step by step:

(1) Blow down $E_\mu$: in this case, $\pi_0(N_\mu)$ acts on the fiber trivially since $N_\mu$ is connected, thus we need to shift the polynomial by degree 2 according to the formula 2.2, then we get

$$B_\mu(t) = (1 + 3t^2 + 3t^4 + t^6) \cdot (t^2 + t^4 + 2t^6$$

$$+ 2t^8 + 3t^{10} + 3t^{12} + 4t^{14} + 4t^{16} + 4t^{18}$$

$$+ 4t^{20} + 3t^{24} + 2t^{26} + 2t^{28} + t^{30} + t^{32}).$$

(4.8)

(2) Blowing down $E_\gamma$: it is similar to the case $E_\mu$. we get

$$B_\gamma(t) = (1 + 3t^2 + t^4) \cdot (t^2 + 2t^4 + 3t^6 + 4t^8$$

$$+ 5t^{10} + 6t^{12} + 7t^{14} + 8t^{16} + 8t^{18} + 8t^{20} + 7t^{22}$$

$$+ 6t^{24} + 5t^{26} + 4t^{28} + 3t^{30} + 2t^{32} + t^{34}).$$

(4.9)
(3) Blowing down $E_\alpha$: it is similar to the case $E_\mu$. we get
\[ B_\alpha(t) = (1 + t^2) \cdot (t^2 + t^4 + 2t^6 + 3t^8 + 4t^{10} \\
+ 5t^{12} + 6t^{14} + 7t^{16} + 8t^{18} + 8t^{20} + 7t^{22} + 6t^{24} \\
+ 5t^{26} + 4t^{28} + 3t^{30} + 2t^{32} + t^{34} + t^{36}). \] (4.10)

(4) Blowing down $E_\delta$: it is similar to the case $E_\mu$. we get
\[ B_\delta(t) = (1 + t^2) \cdot (t^2 + 2t^4 + 4t^6 + 6t^8 + 9t^{10} + 12t^{12} + \\
16t^{14} + 19t^{16} + 24t^{18} + 24t^{20} + 19t^{22} + 16t^{24} \\
+ 12t^{26} + 9t^{28} + 6t^{30} + 4t^{32} + 2t^{34} + t^{36}). \] (4.11)

(5) Blowing down $E_\tau$: it is similar to the case $E_\mu$. we get
\[ B_\tau(t) = (1 + t^2) \cdot (t^2 + t^4 + 2t^6 + 3t^8 + 4t^{10} \\
+ 5t^{12} + 7t^{14} + 8t^{16} + 9t^{20} + 8t^{22} + 7t^{24} \\
+ 5t^{26} + 4t^{28} + 3t^{30} + 2t^{32} + t^{34} + t^{36}). \] (4.12)

(6) Blowing down $E_\xi$: note in this case, $\pi_0(N_\xi) = S_6$ acts on $H^\ast(\mathbb{P}N_\xi/R_\xi)$ by permutation of coordinates of $\mathbb{P}N_\xi$, thus,
\[ \text{IP}_t(H^\ast(\mathbb{P}N_\xi/R_\xi))^{\pi_0(N_\xi)} \equiv P_t(\mathbb{P}N_\xi)P_t((H^\ast(BR_\xi))^{\pi_0(N_\xi)}) \mod t^{19} \]
\[ \equiv \frac{1}{\prod_{1 \leq i < 6} (1 - t^{2i})} \mod t^{19} \] (4.13)

then using formula 2.2 again, we have
\[ B_\xi(t) = t^2 + t^4 + 2t^6 + 3t^8 + 5t^{10} + 7t^{12} + 11t^{14} + 14t^{16} \\
+ 20t^{18} + 26t^{20} + 20t^{22} + 14t^{24} + 11t^{26} \\
+ 7t^{28} + 5t^{30} + 3t^{32} + 2t^{34} + t^{36} + t^{38} \] (4.14)

Put these together, we obtain our formula from
\[ \text{IP}_t(\hat{M}) = P_t(M) - B_\mu(t) - B_\xi(t) - B_\delta(t). \]

Remark 4.10. In [8], the authors doubted whether the Kirwan resolution of moduli spaces of cubic threefolds is isomorphic to certain toroidal compactification of ball quotient $\mathbb{B}/\Gamma$ with respect to some cone decomposition. Their evidence in [8] is that they compute the cohomology of the toroidal compactification and find the Betti numbers of the two compactification match perfectly. It is quite interesting to ask whether it is also true for the moduli spaces of cubic fourfolds.

Corollary 4.11. The intersection Betti numbers of $\overline{D}/\Gamma^{\overline{BB}}$ are given by
\[ \text{IP}_t(\overline{D}/\Gamma^{\overline{BB}}) = 1 + 2t^2 + 5t^4 + 13t^6 + 24t^8 + 38t^{10} + 54t^{12} + 70t^{14} \\
+ 88t^{16} + 107t^{18} + 137t^{20} + 107t^{22} + 88t^{24} + 70t^{26} \\
+ 54t^{28} + 38t^{30} + 24t^{32} + 13t^{34} + 4t^{36} + 2t^{38} + t^{40}. \] (4.15)
**Proof.** First by applying formula (4.3) to morphism $f$ in Proposition 4.6, we have
\[
\text{IP}_t(\overline{\mathcal{D}}/\Gamma^{(\mathcal{H}_\infty)}) = \text{IP}_t(\widetilde{\mathcal{M}}) - (1 + t^2)(P_t(E_\chi \cap \widetilde{E}_\omega) - 1) \mod t^{20}.
\]
Then it remains to compute the cohomology $E_\chi \cap \widetilde{E}_\omega$. Thanks to 3.3, we identify $E_\chi \cap \widetilde{E}_\omega$ as the exceptional divisor in the 1st blow up of GIT moduli of degree 2 K3 surfaces. According to Section 4.2 [17], after blowup a point $\chi \cap \widetilde{E}_\omega$, we get a partial resolution of $E_\omega$ and the exceptional divisor $E_\chi \cap \widetilde{E}_\omega \cong \mathbb{P}(H^0(C, \mathcal{O}_C(4)) \oplus H^0(C, \mathcal{O}_C(6))) / SO(3)$ by Proposition 4.6. Note that $\mathbb{P}(H^0(C, \mathcal{O}_C(4)) \oplus H^0(C, \mathcal{O}_C(6))) / SO(3)$ has only quotient singularities at worst. As the smooth plane conic $C$ is isomorphic to $\mathbb{P}^1$, this will induces isomorphism
\[
\mathbb{P}(H^0(C, \mathcal{O}_C(4)) \oplus H^0(C, \mathcal{O}_C(6))) / SO(3) \cong \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(8)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(12))) / SL(2).
\]
Let $T = \{\text{diag}(t, t^{-1}) : t \in \mathbb{C}^*\}$ be a maximal torus of $SL(2)$ and
\[
[a_0, \cdots, a_8, b_0, \cdots, b_{12}]
\]
be the homogeneous coordinate for $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(8)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(12)))$, then the action $T$ on $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(8)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(12)))$ is given by
\[
t \cdot [a_0, \cdots, a_8, b_0, \cdots, b_{12}] = [t^{-8}a_0, t^{-6}a_1, \cdots, t^8a_8, t^{-12}b_0, t^{-10}b_1, \cdots, t^{12}a_{12}].
\]
so the number of weight $< 4$ is $4 + 6 = 10$ and the maximal dimension of unstable strata for action $SL(2)$ on $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(8)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(12)))$ is 9 which implies
\[
P_t(E_\chi \cap \widetilde{E}_\omega) = P_t^{SL(2)}(\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(8)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(12))) / SL(2)
\]
\[
\equiv P_t(BSL(2)) - \text{unstable terms} \mod t^{18}
\]
\[
\equiv P_t(BSL(2)) P_t(\mathbb{P}^{21}) \equiv \frac{1}{(1 - t^2)(1 - t^4)} \mod t^{18}.
\]
Thus we have
\[
P_t(E_\chi \cap \widetilde{E}_\omega) = 1 + t^2 + 2t^4 + 2t^6 + 3t^8 + 3t^{10} + 4t^{12}
\]
\[
+ 4t^{14} + 5t^{16} + 5t^{18} + 5t^{20} + 4t^{22} + 4t^{24}
\]
\[
+ 3t^{26} + 3t^{28} + 2t^{30} + 2t^{32} + t^{34} + t^{36}.
\]
Last by applying formula 4.4 to morphism $\nu$ and combine the table 3, we only need to remove
\[
2(t^{18} + t^{20}) + 2(t^{16} + t^{18} + t^{20}) \mod t^{20}
\]
from $\text{IP}_t(\overline{\mathcal{D}}/\Gamma^{(\mathcal{H}_\infty)}) \mod t^{20}$ in order to get $\text{IP}_t(\overline{\mathcal{D}}/\Gamma^{BB})$. In this way, we obtain our formula 4.15. \hfill \Box

**Remark 4.12.** Since the Zucker’s conjecture was established in [25] and [33], the $L^2$-cohomology of $\mathcal{D}/\Gamma$ is isomorphic to the intersection cohomology of $\mathcal{D}/\Gamma$. our result also provides most $L^2$-betti numbers of $\mathcal{D}/\Gamma$ as the dimension of Baily-Borel’s boundaries is 1.

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REFERENCES

[1] A. Ash, D. Mumford, M. Rapoport, and Y. Tai. Smooth compactification of locally symmetric varieties. Math. Sci. Press, Brookline, Mass., 1975. Lie Groups: History, Frontiers and Applications, Vol. IV.

[2] Walter L Baily and A. Borel. Compactification of arithmetic quotients of bounded symmetric domains. S. L, 1966.

[3] Arnaud Beauville. Le groupe de monodromie des familles universelles d’hypersurfaces et d’intersections complètes. In Complex analysis and algebraic geometry (Göttingen, 1985), volume 1194 of Lecture Notes in Math., pages 8–18. Springer, Berlin, 1986.

[4] Nicolas Bergeron and Zhiyuan Li. Tautological classes on moduli spaces of hyper-Kähler manifolds. Duke Math. J., 168(7):1179–1230, 2019.

[5] Nicolas Bergeron, Zhiyuan Li, John Millson, and Colette Moeglin. The Noether-Lefschetz conjecture and generalizations. Invent. Math., 208(2):501–552, 2017.

[6] A. A. Be˘ılinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In Analysis and topology on singular spaces, I (Luminy, 1981), volume 100 of Astérisque, pages 5–171. Soc. Math. France, Paris, 1982.

[7] Armand Borel and Nolan R. Wallach. Continuous cohomology, discrete subgroups, and representations of reductive groups, volume 94 of Annals of Mathematics Studies. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1980.

[8] Sebastian Casalaina-Martin, Samuel Grushevsky, Klaus Hulek, and Radu Laza. Cohomology of the moduli space of cubic threefolds and its smooth models. arXiv preprint arXiv:1904.08728, 2019.

[9] Mark de Cataldo and Luca Migliorini. The decomposition theorem, perverse sheaves and the topology of algebraic maps. Bulletin of the American Mathematical Society, 46(4):535–633, 2009.

[10] Mark Goresky and Robert MacPherson. Intersection homology. II. Invent. Math., 72(1):77–129, 1983.

[11] Samuel Grushevsky and Klaus Hulek. The intersection cohomology of the satake compactification of $a_g$ for $g \leq 4$. Mathematische Annalen, 369(3-4):1353–1381, 2017.

[12] Brendan Hassett. Special cubic fourfolds. Compositio Mathematica, 120(1):1–23, 2000.

[13] Wu Yi Hsiang. Cohomology theory of topological transformation groups, volume 85. Springer Science & Business Media, 2012.

[14] Daniel Huybrechts and Jørgen Vold Rennemo. Hochschild cohomology versus the Jacobian ring and the Torelli theorem for cubic fourfolds. Algebr. Geom., 6(1):76–99, 2019.

[15] Frances Kirwan. Rational intersection cohomology of quotient varieties. Inventiones mathematicae, 86(3):471–505, 1986.

[16] Frances Kirwan et al. Moduli spaces of degree d hypersurfaces in $\mathbb{P}^n$. Duke Mathematical Journal, 58(1):39–78, 1989.

[17] Frances Kirwan and Ronnie Lee. The cohomology of moduli spaces of k3 surfaces of degree 2 (i). Topology, 28(4):495–516, 1989.

[18] Frances Kirwan and Ronnie Lee. The cohomology of moduli spaces of k3 surfaces of degree 2 (ii). Proceedings of the London Mathematical Society, 3(3):559–582, 1989.

[19] Frances Kirwan and Jonathan Woolf. An introduction to intersection homology theory. Chapman & Hall/CRC, Boca Raton, FL, second edition, 2006.

[20] Frances Clare Kirwan. Partial desingularisations of quotients of nonsingular varieties and their betti numbers. Annals of mathematics, 122(1):41–85, 1985.

[21] Frances Clare Kirwan, John N Mather, and Phillip Griffiths. Cohomology of quotients in symplectic and algebraic geometry, volume 31. Princeton University Press, 1984.

[22] Radu Laza. The moduli space of cubic fourfolds. Journal of Algebraic Geometry, 18(3):511–545, 2009.
[23] Radu Laza. The moduli space of cubic fourfolds via the period map. *Annals of mathematics*, pages 673–711, 2010.

[24] Yuchen Liu. K-stability of cubic fourfolds. *arXiv e-prints*, page arXiv:2007.14320, July 2020.

[25] Eduard Looijenga. $l^2$-cohomology of locally symmetric varieties. *Compositio Mathematica*, 67(1):3–20, 1988.

[26] Eduard Looijenga. Compactifications defined by arrangements, ii: Locally symmetric varieties of type iv. *Duke Mathematical Journal*, 119(3):527–588, 2003.

[27] Eduard Looijenga. The period map for cubic fourfolds. *Inventiones mathematicae*, 177(1):213–233, 2009.

[28] J. S. Milne. Introduction to Shimura varieties. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 265–378. Amer. Math. Soc., Providence, RI, 2005.

[29] Shigeru Mukai and Mukai Shigeru. *An introduction to invariants and moduli*, volume 81. Cambridge University Press, 2003.

[30] David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory*, volume 34. Springer Science & Business Media, 1994.

[31] Rahul Pandharipande. A calculus for the moduli space of curves. In *Algebraic geometry: Salt Lake City 2015*, volume 97 of *Proc. Sympos. Pure Math.*, pages 459–487. Amer. Math. Soc., Providence, RI, 2018.

[32] Rahul Pandharipande and Qizheng Yin. Relations in the tautological ring of the moduli space of K3 surfaces. *J. Eur. Math. Soc. (JEMS)*, 22(1):213–252, 2020.

[33] Leslie Saper and Mark Stern. $L^2$-cohomology of arithmetic varieties. *Annals of Mathematics*, pages 1–69, 1990.

[34] Joachim Schwermer. Geometric cycles, arithmetic groups and their cohomology. *Bull. Amer. Math. Soc. (N.S.)*, 47(2):187–279, 2010.

[35] Jayant Shah. A complete moduli space for K3 surfaces of degree 2. *Ann. of Math. (2)*, 112(3):485–510, 1980.

[36] Claire Voisin. Théorème de Torelli pour les cubiques de $\mathbf{P}^5$. *Invent. Math.*, 86(3):577–601, 1986.

[37] Claire Voisin. Théorème de torelli pour les cubiques de 5. *Inventiones Mathematicae*, 172(2):455, 2008.

[38] Chenyang Xu. K-stability of Fano varieties: an algebro-geometric approach. *arXiv e-prints*, page arXiv:2011.10477, November 2020.

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