D-branes on Nonabelian Threefold Quotient Singularities

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ABSTRACT

We investigate the classical moduli space of $D$-branes on a nonabelian Calabi-Yau threefold singularity and find that it admits topology-changing transitions. We construct a general formalism of worldvolume field theories in the language of quivers and give a procedure for computing the enlarged Kähler cone of the moduli space. The topology changing transitions achieved by varying the Fayet-Iliopoulos parameters correspond to changes of linearization of a geometric invariant theory quotient and can be studied by methods of algebraic geometry. Quite surprisingly, the structure of the enlarged Kähler cone can be computed by toric methods. By using this approach, we give a detailed discussion of two low-rank examples.

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**Introduction**

The purpose of the present paper is to explore some new aspects of *D-geometry*, the geometry of ultrashort distances in string theory as seen by D-branes. As first pointed out in [1] and discussed in more detail in [3] from a D-brane point of view [9], the presence of non-perturbative objects implies the ability of string theory to probe distances below the string scale. Given the fascinating geometric phenomena known to occur in the moduli space of perturbative string theory (see, for example, [4]), which have changed our understanding of the way in which this theory modifies the general-relativistic concepts of space and time, it is natural to ask whether such processes survive, and in what form, when nonperturbative string physics is included. A first step towards a satisfactory answer to this question is to consider the behaviour of D-branes located at an orbifold singularity of a Calabi-Yau space, and to ask how D-brane physics reflects the underlying geometry of space-time. This approach is similar in spirit to studies of perturbative string theory on orbifold singularities undertaken during the past decade [5] (see also the review [6]), which gave the first indications of how string theory modifies classical geometric concepts.

Various steps in the direction mentioned above have been taken in a series of papers [7, 8], where it was shown that well-known phenomena of perturbative string theory on orbifolds (such as resolution of singularities) survive when D-branes are included, although the underlying physical mechanisms are somewhat different. In the case of quotient singularities in two complex dimensions, i.e. orbifold singularities locally of the form $\mathbb{C}^2/\Gamma$, with $\Gamma$ a finite subgroup of $SU(2)$, an almost complete study has been performed [7] by using the powerful mathematical results of [9, 10, 11]. These investigations lead to a very beautiful and natural realization of minimal resolutions of canonical surface singularities via physical processes in string theory.

In the case of D-brane quotient singularities of Calabi-Yau threefolds (which have the local form $\mathbb{C}^3/\Gamma$ with $\Gamma$ a finite subgroup of $SU(3)$), the focus until now has been exclusively on the case when the group $\Gamma$ is abelian. In this case, the problem can be reduced [8, 12] to one in toric geometry, for which powerful computational tools are available [1]. On the other hand, progress in the nonabelian case has been stifled by the lack of a comparably versatile mathematical tool. One of the main aims of this paper is to show that the problem is tractable (albeit difficult), and that, surprisingly enough, one can extract important information by using abelian (i.e. toric) techniques. This will be achieved by carefully identifying the information we consider most relevant, and by separating it from the context, which allows us to avoid the

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Toric geometry is a machinery for studying varieties $X$ which admit a densely-embedded complex torus, by systematically reducing all geometric questions about $X$ to questions in integral convex geometry. The computational power of this approach is belied by the existence of a rich variety of algorithms in computational convex geometry, a field which has attracted considerable interest during the last few decades due to its widespread applications to various areas of mathematics. While referring the interested reader to the physics-oriented introductions in [4, 13] or to the mathematical account of [14], we hurry to assure him or her that the present paper can be read without having any previous knowledge of this topic.
hard computational task of giving a complete characterization of the geometry. As an example, by following this approach, we will be able to establish the existence of topology-changing transitions in the nonabelian 3-dimensional context. In fact, we will show that some crucial features of the abelian case carry over to nonabelian situations, and that the phenomenon of topology change can be understood in these generalized settings. A more general purpose of the present work is to develop the formalism appropriate for describing the nonabelian case. As we will argue in this paper, the best formalism seems to be that of quivers, a subject which has received considerable attention recently in the mathematical literature [15, 16] concerned with singularity theory and the representation theory of algebras. We explain the relevant concepts of this subject in a form adapted to our problem and in accessible language. As we will discover, quiver technology can be used to considerably simplify the analysis of D-brane effective field theories placed at quotient singularities.

Another motivation for this paper is the recent effort aimed at generalizing the AdS/CFT conjecture of [17] to the case of D-branes on conical singularities of Calabi-Yau manifolds [18]. Such theories can be obtained [19] by turning on Fayet-Iliopoulos parameters in the effective field theory of a large number of branes placed at a quotient singularity, and then flowing to a conformal fixed point of the resulting field theory. The number of examples one can consider as candidates for applying this method has been so far limited to abelian quotient singularities, due the lack of appropriate methods for treating the effective field theory in the nonabelian case. The present paper is a first step toward removing that constraint, thus preparing the way for more extensive studies of such conformal fixed points. In a sense, the situation is similar to that of supersymmetric field theories in 4 dimensions, where a thorough understanding of the classical moduli space is a necessary pre-requisite for a quantum-mechanical study along the lines of [20].

The plan of this paper is as follows. In Section 1 we discuss the effective field theory describing the low energy dynamics of D-branes placed at a general Calabi-Yau threefold quotient singularity and we describe its classical moduli space of vacua. In Section 2, we use quiver techniques in order to simplify the problem of computing the moduli space. We show that the moduli space of vacua can be identified with the moduli space of representations of an associated quiver. This enables us to apply recent mathematical results to the problem of understanding topology-changing transitions, a subject we discuss in Section 3. In Section 4, we explain how one can extract information about such transitions by using toric methods. Finally, in Section 5 we apply these methods to two examples (obtained by taking $\Gamma$ to be a low rank finite subgroup of $SU(3)$), giving a detailed account of the arguments involved. Certain technical details are discussed in the appendices.
1 The projection conditions and the moduli space

In this section, as indicated above, we discuss the field theory degrees of freedom necessary to describe D-branes at a nonabelian quotient singularity in three complex dimensions. To do so, let’s consider a finite subgroup $\Gamma$ of order $N$ of $SU(3)$, acting on $\mathbb{C}^3$ by its defining representation (i.e. via its embedding as a subgroup of $SU(3)$). The model of one $D_p$-brane on a $\mathbb{C}^3/\Gamma$ orbifold, which meets the $D$-brane at a point, can be formulated by the methods of [3]. This approach starts with a $U(N)$ gauge theory with adjoint scalars $\chi_a$ ($a = 1..6$) ($\chi_a^+ = \chi_a$), changing to complex coordinates:

$$X_m := \chi_m + i\chi_{m+3}$$
$$X_{m+3} := \chi_m - i\chi_{m+3},$$

with $m = 1..3$, and imposing the projection conditions:

$$D^{(R)}(\gamma)X_mD^{(R)}(\gamma)^{-1} = D^{(Q)}_{nm}(\gamma)X_n$$
$$D^{(R)}(\gamma)UD^{(R)}(\gamma)^{-1} = U,$$

on the complex fields $X_m$ and the gauge group elements $U$. Here $D^{(R)}$ is the regular $N \times N$ representation of $\Gamma$, while $D^{(Q)}(\gamma) = (D^{(Q)}_{nm}(\gamma))_{n,m=1..3}$ is the defining representation. The well-known decomposition $R = \bigoplus_{\mu=0..r} n_\mu R_\mu$ (where $R_\mu$ ($\mu = 0..r$) are the irreducible representations of $\Gamma$, with $R_0$ the trivial representation and $n_\mu := \text{dim}R_\mu$) shows that we can choose the basis of Chan-Paton factors such that the matrices $D^{(R)}(\gamma)$ have the form:

$$D^{(R)}(\gamma) = \begin{bmatrix}
D^{(0)}(\gamma) \otimes 1_{n_0} & 0 & 0 & \ldots & 0 \\
0 & D^{(1)}(\gamma) \otimes 1_{n_1} & 0 & \ldots & 0 \\
0 & 0 & D^{(2)}(\gamma) \otimes 1_{n_2} & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & D^{(r)}(\gamma) \otimes 1_{n_r}
\end{bmatrix},$$

where $D^{(\mu)}(\gamma) \otimes 1_{n_\mu}$ represents the $n_\mu^2$ by $n_\mu^2$ block-diagonal matrix which contains $n_\mu$ copies of $D^{(\mu)}(\gamma)$:

$$D^{(\mu)}(\gamma) \otimes 1_{n_\mu} = \begin{bmatrix}
D^{(\mu)}(\gamma) & 0 & \ldots & 0 \\
0 & D^{(\mu)}(\gamma) & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 0 & D^{(\mu)}(\gamma)
\end{bmatrix}. $$

More general representations can be chosen, with appropriate physical interpretations [21], but in this paper we consider the regular representation only.

Throughout this paper, the symbol $1_m$ denotes the $m$ by $m$ identity matrix, or the identity map of an $m$-dimensional vector space.
The gauge group $G_0$ of the projected theory is the subgroup of $U(N)$ formed by those elements $U$ which satisfy the projection conditions. Applying Schur’s lemma shows that such an element has the form:

$$U = \begin{bmatrix} 1_{n_0} \otimes U_0 & 0 & \cdots & 0 \\ 0 & 1_{n_1} \otimes U_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1_{n_r} \otimes U_r \end{bmatrix},$$  

(5)

with $U_\mu$ arbitrary $n_\mu$ by $n_\mu$ unitary matrices. Therefore, $G_0$ is isomorphic to $\Pi_{\mu=0..r} U(n_\mu)$.

The scalar potential of the theory is given by:

$$V = - \sum_{a,b=1..6} \text{Tr}([X_a, X_b]^2),$$  

(6)

which in complex variables becomes:

$$V = \frac{1}{4} \text{Tr} \sum_{m,n=1..3} ([X_m, X_n][X_m, X_n]^+ + [X_m, X_m\pi][X_m, X_m\pi]^+) .$$  

(7)

Using the identity:

$$\text{Tr} \sum_{m,n=1..3} [X_m, X_m\pi][X_m, X_m\pi]^+ = \text{Tr} \sum_{m,n=1..3} [X_m, X_n][X_m, X_n]^+ + \text{Tr}([X_m, X_m\pi])(\sum_{m=1..3} [X_m, X_m\pi])^+ ,$$  

(8)

this can be rewritten as:

$$V = \frac{1}{2}[V_f + V_d] ,$$  

(9)

where:

$$V_f := \text{Tr} \sum_{m,n=1..3} [X_m, X_n][X_m, X_n]^+$$

$$V_d := \frac{1}{2} \text{Tr}([X_m, X_m\pi])(\sum_{m=1..3} [X_m, X_m\pi])^+ .$$  

(10)

Introducing the moment map for the gauge group action:

$$\rho = \sum_{m=1..3} [X_m, X_m\pi] ,$$  

(11)

we can write:

$$V_d = \frac{1}{2} \text{Tr}(\rho^2) .$$  

(12)

In the presence of Fayet-Iliopoulos terms, this relation is modified to:

$$V_d = \frac{1}{2} \text{Tr}[(\rho - \zeta)^2] ,$$  

(13)
where
\[
\zeta = \begin{bmatrix}
\zeta_0 1_{n_0^2} & 0 & \ldots & 0 \\
0 & \zeta_1 1_{n_1^2} & \ldots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \zeta_r 1_{n_r^2}
\end{bmatrix},
\] (14)
is a matrix in the centre of the surviving gauge group
(note that \(\zeta_\mu 1_{n_\mu} = \begin{bmatrix}
\zeta_\mu 1_{n_\mu} & 0 & \ldots & 0 \\
0 & \zeta_\mu 1_{n_\mu} & \ldots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \zeta_\mu 1_{n_\mu}
\end{bmatrix}\)). In this situation, the supersymmetric vacuum constraints are:
\[
[X_m, X_n] = 0 \\
\rho(X_1, X_2, X_3) = \zeta.
\] (15)
Equation (11) enforces the constraint \(\text{Tr}(\rho) = 0\), i.e. \(\sum_{\mu=0..r} n_\mu^2 \zeta_\mu = 0\). Dividing out the diagonal \(U(1)\) subgroup of \(G_0\) (which acts trivially on \(X_m, X_n\)) gives an effective action of \(G = G_0/U(1)_{\text{diag.}}\). Define the variety of commuting matrices \(\mathcal{Z}\) to be the set of matrices satisfying the projection conditions, and the equations \([X_m, X_n] = 0\) for all \(m, n = 1..3\). Then the desired moduli space is the Kähler quotient:
\[
\mathcal{M}_\zeta := \{ X \in \mathcal{Z} | \rho(X) = \zeta \}/G.
\] (16)
Notice that the variety of commuting matrices coincides with the set of extremal points of the function:
\[
W = \epsilon_{mnl} \text{Tr}(X_m X_n X_l) .
\] (17)
That is, the conditions \([X_m, X_n] = 0\) for all \(m, n = 1..3\) are equivalent to the constraints \(\partial_{X_m^{ij}} W = 0, \forall i, j = 1..N, \forall m = 1..3\). If one is dealing with D3-branes, then the projected theory has \(N = 1\) supersymmetry in 4 dimensions and \(W\) is its superpotential. In this case, the condition \(V_f = 0\), i.e. \([X_m, X_n] = 0\) for all \(m, n = 1..3\) is simply the F-flatness constraint, while the condition \(V_d = 0\) is the D-flatness constraint in the Wess-Zumino gauge. By analogy with that situation, we will in general call \(V_d = 0\) the D-flatness constraint and \(V_f = 0\) the F-flatness constraint.

2 Quiver formalism

The above formulation is inconvenient because of the presence of linear constraints (imposed by the projection conditions) among the components of the matrices \(X_m\). One can avoid this complication by performing a linear change of variables which

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\footnote{This constraint is a feature of working with the regular representation. If one replaces \(R\) by a more general representation, then the expression of the moment map is different, and the diagonal \(U(1)\) subgroup acts nontrivially on the fields present in that case. In particular, the tracelessness condition ceases to hold.}
solves the projection conditions. This amounts to parametrizing the space of projected matrices \( X_m \) by a set of unconstrained variables \( \phi \) which are free to fill some linear space (further constraints on \( \phi \) will be later imposed by the F-flatness conditions). In essence, the so-called quiver formalism is a procedure which accomplishes this change of variables in a particularly systematic and transparent manner. To see this, view the triple \( X = (X_1, X_2, X_3) \) as a vector-valued matrix \( X = \sum_{m=1..3} X_m \otimes e_m \), where \( (e_m)_{m=1..3} \) is the canonical basis of \( \mathbb{C}^3 \). Let \( R \approx \mathbb{C}^N \) and \( Q \approx \mathbb{C}^3 \) be the vector spaces carrying the regular and defining representations of \( \Gamma \) respectively. One can think of \( Q \) as the covering space of our orbifold and of \( R \) as the space of Chan-Paton factors. Since each of the matrices \( X_m \) can be identified with a linear operator in \( \text{Hom}(R, R) \), we can view \( X \) as an element of \( \text{Hom}(R, Q \otimes R) \). To make everything basis-independent, we let \( \rho_R \) denote the regular representation of \( \Gamma \) and \( \rho_Q \) denote the defining representation. Then the more concrete formulation of the previous section is obtained by picking orthonormal bases \( e_m, (m = 1..3) \) of \( Q \) and \( e_\gamma \) \(( \gamma \in \Gamma \) \) of \( R \) and identifying the linear operators \( \rho_Q(\gamma), \rho_R(\gamma) \) with their matrices \( D^{(Q)}(\gamma), D^{(R)}(\gamma) \) in these bases.

### 2.1 Quiver data

In this abstract language, the projection conditions require \( X \) to be \( \Gamma \)-invariant in the following sense:

\[
X \rho_R(\gamma) = \rho_Q \otimes R(\gamma) X .
\]  

(18)

As before, let \( R_\mu, (\mu = 0..r) \) be the irreducible representations of \( \Gamma \) (with \( R_0 \) the trivial representation) and let \( n_\mu := \text{dim} R_\mu \). One has the standard decomposition \( R = \oplus_{\mu=0..r} V_\mu \otimes R_\mu \), where the \( n_\mu \)-dimensional vector spaces \( V_\mu \) encode the multiplicities of \( R_\mu \) as factors of \( R \).

Consider the decompositions \( Q \otimes R_\nu = \oplus_{\lambda=0..r} A_{\lambda\nu} \otimes R_\lambda \) of the tensor products \( Q \otimes R_\nu \) into irreducible representations of \( \Gamma \), and let \( A_{\lambda\mu} = \text{dim} A_{\lambda\mu} \) be the associated multiplicities. From Schur’s lemma we know that the subspace \( \text{Hom}(R_\mu, R_\lambda)^\Gamma \) of \( \Gamma \)-invariant linear maps from \( R_\mu \) to \( R_\lambda \) is zero unless \( \lambda = \mu \), while the invariant maps from \( R_\mu \) to \( R_\mu \) are the constant multiples of the identity map. Combined with the decompositions discussed above, this shows the existence of an isomorphism:

\[
\text{Hom}(R, Q \otimes R)^\Gamma \approx \oplus_{\mu, \nu=0..r} A_{\mu\nu} \otimes \text{Hom}(V_\mu; V_\nu) ,
\]  

(19)

where \( \text{Hom}(R, Q \otimes R)^\Gamma \) denotes the subspace of \( \Gamma \)-invariant elements of \( \text{Hom}(R, Q \otimes R) \). Modulo this isomorphism, we can identify an element \( X \in \text{Hom}(R, Q \otimes R) \) which satisfies the projection conditions with a set of objects \( \phi^{(\nu \mu)}(\nu \mu; \alpha)(\alpha = 1..a_{\mu\nu}) \) of the vector spaces \( A_{\mu\nu} \), we can write, more specifically:

\[
\phi^{(\nu \mu)} = \sum_{\alpha=1..a_{\mu\nu}} \phi^{(\nu \mu)}_{\alpha} \otimes |\nu \mu; \alpha\rangle \otimes 1_{n_\mu} ,
\]  

(20)
with \( \phi^{(\nu \mu)}_\alpha \in \text{Hom}(V_\mu, V_\nu) \). The linear maps \( \phi^{(\nu \mu)}_\alpha : V_\mu \to V_\nu \) are the desired unconstrained variables, which we will call \textit{quiver data}. Their combinatorial structure can be described in the language of graph theory as follows.

The \textit{McKay quiver} is the graph with \( r + 1 \) nodes, indexed by \( \mu = 0..r \), which is obtained by drawing \( a_{\mu \nu} \) arrows (i.e. oriented edges) starting from the node \( \mu \) and ending at the node \( \nu \), for each ordered pair of distinct indices \((\mu, \nu)\) satisfying \( a_{\mu \nu} \neq 0 \) (if both \( a_{\mu \nu} \) and \( a_{\nu \mu} \) are nonzero, then there will be \( a_{\mu \nu} \) arrows from \( \mu \) to \( \nu \) and \( a_{\nu \mu} \) arrows from \( \nu \) to \( \mu \)), and by drawing \( a_{\mu \mu} \) loops (i.e. edges connecting a node with itself) at the node \( \mu \) for each \( a_{\mu \mu} \neq 0 \). Note that the loops of the quiver do not carry any natural orientation. By indexing the arrows from \( \mu \) to \( \nu \) by an integer \( \alpha = 1..a_{\mu \nu} \), one can think of each arrow as being a pictorial representation of the quiver datum \( \phi^{(\nu \mu)}_\alpha : V_\mu \to V_\nu \) (the same applies to the loops). The McKay quiver encodes the branching rules for the decompositions of the tensor product representations \( Q \otimes R_\nu \) into irreducible representations of \( \Gamma \).

\[ \text{2.2 The projected gauge group and the moment map} \]

The structure of the surviving gauge group \( G_0 := U(R)^\Gamma \) can be found more abstractly as follows. The decomposition of the regular representation shows that:

\[ U(R)^\Gamma \approx \prod_{\mu=0..r} U(V_\mu) \approx \prod_{\mu=0..r} U(n_\mu) . \tag{21} \]

This isomorphism is realized by sending an \((r + 1)-\)tuple of matrices \( (g_0...g_r) \in \prod_{\mu=0..r} U(n_\mu) \) into the block-diagonal \( N \times N \) matrix \( U \in U(R)^\Gamma \) which contains \( n_\mu \) copies of \( g_\mu \) for each \( \mu = 0..r \):

\[
\begin{bmatrix}
U_0 \otimes 1_{n_0} & 0 & \ldots & 0 \\
0 & U_1 \otimes 1_{n_1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & U_r \otimes 1_{n_r}
\end{bmatrix}
\leftrightarrow
\begin{bmatrix}
U_0 & 0 & \ldots & 0 \\
0 & U_1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & U_r
\end{bmatrix} \tag{22}
\]

The action of \( U(R)^\Gamma \) on the matrices \( X \) translates into the natural action of \( \prod_{\mu=0..r} U(V_\mu) \) on the quiver data:

\[ \phi^{(\nu \mu)}_\alpha \rightarrow g_\nu \phi^{(\nu \mu)}_\alpha (g_\mu)^{-1} , \tag{23} \]

for all \( g = (g_0...g_r) \in \prod_{\mu=0..r} U(V_\mu) \). Since the diagonal \( U(1) \) subgroup of \( G_0 \) acts trivially on all such data, the group which acts effectively on their space is:

\[ G = \prod_{\mu=0..r} U(V_\mu)/U(1)_{\text{diag}} \tag{24} \]

which we will call \textit{the effective gauge group}.
The moment map for the action of $G$ on the quiver data is given by general theory as:

$$M = \oplus_{\mu=0..r} M_\mu,$$

where:

$$M_\mu = \sum_{\nu, a_{\mu,\nu} \neq 0} \sum_{\alpha=1..a_{\mu,\nu}} \phi^{(\mu\nu)}_\alpha (\phi^{(\mu\nu)}_\alpha) + \sum_{\nu, a_{\mu,\nu} \neq 0} \sum_{\alpha=1..a_{\mu,\nu}} (\phi^{(\nu\mu)}_\alpha) + (\phi^{(\nu\mu)}_\alpha) \in u(n_\mu).$$

The map $M_\mu$ is related to the original moment map $\rho$ of equation (14) through the isomorphism of Lie algebras induced by (23). In fact, it is not hard to see that:

$$\rho = \oplus_{\mu=0..r} \frac{1}{n_\mu} (M_\mu)_{\oplus n_\mu}.$$

The ‘quiver moment map’ (26) has a simple graphical interpretation (see Figure 1), obtained by noting that the two terms of (26) correspond to summing over all arrows leaving, respectively entering a given node $\mu$ (if there is a loop at that node, then it is considered in both sums, i.e. it is viewed as both leaving and entering the node).

The central levels of $M$ correspond to $(r + 1)$-tuples $(\xi_0...\xi_r) \in \mathbb{R}^{r+1}$ satisfying $\sum_{\mu=0..r} n_\mu \xi_\mu = 0$, which amounts to writing the original matrix $\zeta$ of equation (13) in the form $\zeta = \text{diag}(\frac{\xi_0}{n_0}1_{n_0}^{\alpha=1..a_{\mu,\nu}} \frac{\xi_r}{n_r}1_{n_r})$. The moment map equations become $M = \hat{\zeta}$, where

$$M_\mu = \xi_\mu 1_{n_\mu} \quad (\mu = 0..r).$$

It is convenient to identify the group $G$ with:

$$G \approx S[\Pi_{\mu=0..r} U(n_\mu)]/K,$$

space $Q$ of all quiver data, which is preserved by the action of $G$. It is well-known that the moment map $M : Q \rightarrow g$ for such an action is given by the relation $Tr(M(\Phi) \theta) = \langle \Phi, \theta \Psi \rangle$, where $\theta \in g$ is arbitrary and $\Phi, \Psi$ is any pair of quiver data (here $g$ is the Lie algebra of $G$). Using the presentation (21) of $G$ gives a realization of $g$ as the ‘traceless’ subalgebra of $\oplus_{\alpha=0..r} U(n_\mu)$, so we can write $\theta = \oplus_{\mu=0..r} \theta_\mu$, with $\theta_\mu \in u(n_\mu)$ and $M(\Phi) = \oplus_{\mu=0..r} M_\mu(\Phi)$, where $M_\mu$ are maps from $Q$ to $u(n_\mu)$. Using arbitrariness of $\theta_\mu$, this immediately leads to the expression (26). Note that the original moment map $\rho$ is associated to the hermitian product $\langle X, Y \rangle = Tr(X^c Y_m)$, which differs from the product we used above. Indeed, if $\Phi = (\phi^{(\nu\mu)}_\alpha), \Psi := (\psi^{(\nu\mu)}_\alpha)$ are the quiver data associated to $X, Y$, then $Tr(X^c Y_m) = \sum_{\mu, \nu=0..r} n_\mu (\phi^{(\nu\mu)}_\alpha + \psi^{(\nu\mu)}_\alpha)$. The factors of $n_\mu$ occur because of the presence of the identity map $1_{n_\mu}$ in the decomposition (21).

6This follows easily by noting that $\rho$ has the form: $\rho(X) = \oplus_{\mu=0..r} \rho_\mu(X)^{\oplus n_\mu}$, with $\rho_\mu$ a map from $Q$ to $u(n_\mu)$. The relation between $\rho$ and $M$ is given by the condition $Tr(M(\Phi_X) \theta) = Tr(\rho(X)q(\theta))$ for all $\theta = \oplus_{\mu=0..r} \theta_\mu \in \oplus_{\mu=0..r} u(n_\mu)$, where $q(\theta) = \oplus_{\mu=0..r} (\theta_\mu)^{\oplus n_\mu}$ is the isomorphism induced by (24) and $\Phi_X = (\phi^{(\nu\mu)}_\alpha)$ is the set of quiver data associated to $X$. The factors $n_\mu$ in the relation between $\rho$ and $M$ appear when one evaluates the traces, which gives $\rho_\mu(X) = \frac{1}{n_\mu} M_\mu(\Phi_X)$. 

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where $S[\Pi_{\mu=0..r}U(n_{\mu})]$ is the group of $(r+1)$-tuples $(g_0...g_r) \in \Pi_{\mu=0..r}U(n_{\mu})$ satisfying $\Pi_{\mu=0..r}\det(g_{\mu}) = 1$ and $K \approx \mathbb{Z}_n$ ($n := \sum_{\mu=0..r} n_{\mu}$) is the finite central subgroup given by:

$$K = \{(\eta n_0 ... \eta n_r) \mid \eta^n = 1\}.$$  

(30)

From now on, we will always use this presentation of $G$. Due to the tracelessness condition $\sum_{\mu=0..r} n_{\mu}\xi_{\mu} = 0$, the values of the Fayet-Iliopoulos parameters can be described by the vector $\xi = (\xi_1...\xi_r) \in \mathbb{R}^r$. We will denote the space of such vectors by $\mathbb{R}^r(\xi)$.

### 2.3 Quiver Feynman rules for the superpotential

The parametrization of the variety of commuting matrices in terms of quiver data can be obtained by first expressing the superpotential in the variables $\phi_{\alpha}^{(\mu\nu)}$. Then $Z$ can be obtained as the critical set of $W$.

In order to express the superpotential in terms of $\phi_{\alpha}^{(\mu\nu)}$, we need to know the explicit form of the equivalence (19). This can be obtained as follows. For each irreducible representation $R_{\mu}$ of $\Gamma$, pick a specific realization of $R_{\mu}$ via unitary matrices $D^{(\mu)}(\gamma) \in U(n_{\mu})$. Given a hermitian vector space $S$ carrying a unitary representation $\rho$ of $\Gamma$ which is equivalent to $R_{\mu}$, pick an orthonormal basis $(e_{i}^{(\mu\nu)})_{i=1..n_{\mu}}$ of $S$ will be called fiducial if $\rho(\gamma)e_{i}^{(\mu\nu)} = D^{(\mu)}(\gamma)e_{j}^{(\mu\nu)}$, for all $\gamma \in \Gamma$. By Schur’s lemma, such a basis is determined up to a global phase factor. If the representation $\rho$ carried by $S$ is equivalent to $aR_{\mu} = R_{\mu}^{a}$, then an orthonormal basis $(e_{i}^{(\mu\alpha)})_{\alpha=1..a; i=1..n_{\mu}}$ of $S$ is called fiducial if $\rho(\gamma)e_{i}^{(\mu\alpha)} = D^{(\mu)}(\gamma)e_{j}^{(\mu\alpha)}$ for all $\alpha = 1..a$ and all $\gamma \in \Gamma$. A fiducial basis in this generalized sense is determined up to a transformation of the form $e_{i}^{(\mu\alpha)} \rightarrow U_{\alpha\beta}e_{i}^{(\mu\beta)}$, with $U = (U_{\alpha\beta})_{\alpha,\beta=1..a}$ a unitary $a \times a$ matrix. (The notion of fiducial bases is necessary in order to fix our conventions on the Clebsch-Gordan coefficients below.)

Figure 1: Three types of quiver data $\phi$ related to a node and their contribution to the quiver moment map.
Consider the decomposition:

$$Q \otimes R_\nu = \bigoplus_{\mu=0..r} A_{\mu\nu} \otimes R_\mu \ .$$

(31)

Picking fiducial bases $e_j^{(Q)}$, $e_k^{(\nu)}$ ($j = 1..n_Q$, $k = 1..n_\nu$) of $Q$, respectively $R_\nu$ and $e_i^{(\mu,\alpha)}$ ($i = 1..n_\mu$, $\alpha = 1..a_{\mu\nu}$) of $A_{\mu\nu} \otimes R_\mu$, we can write:

$$e_i^{(\mu,\alpha)} = \sum_{j=1..n_Q}^{k=1..n_\nu} C_{Qj,\nu k}^{i,\mu,\alpha} e_j^{(Q)} \otimes e_k^{(\nu)} \ .$$

(32)

where $C_{Qj,\nu k}^{i,\mu,\alpha}$ are the Clebsch-Gordan coefficients relating these bases. Then it is easy to see that any $X$ satisfying (18) has the form:

$$X = \bigoplus_{\mu,\nu=0..r} \sum_{i=1..n_\mu}^{\alpha=1..a_{\mu\nu}} \sum_{j=1..n_Q}^{k=1..n_\nu} C_{Qj,\nu k}^{i,\mu,\alpha} \phi_{\alpha}^{(\nu \mu)} \langle e_i^{(\mu,\alpha)} | e_j^{(\nu)} \rangle \ ,$$

(33)

while its components are given by:

$$X_m := \bigoplus_{\mu,\nu=0..r} \sum_{i=1..n_\mu}^{j=1..n_Q} \sum_{\alpha=1..a_{\mu\nu}}^{\beta=1..a_{\nu\lambda}} \sum_{k=1..n_\lambda}^{\gamma=1..a_{\lambda\mu}} C_{3m,\nu j}^{i,\mu,\alpha} C_{3n,\nu j}^{i,\nu,\beta} C_{3l,\mu i}^{i,\lambda,\gamma} \langle \phi_{\alpha}^{(\nu \mu)} | \phi_{\beta}^{(\lambda \nu)} | \phi_{\gamma}^{(\lambda \mu)} \rangle \ .$$

(34)

This relation allows us to rewrite the superpotential (17) as:

$$W = - \sum_{\mu, \nu, \lambda=0..r}^{(a_{\mu\nu} a_{\nu\lambda} a_{\lambda\mu} \neq 0)} \sum_{i=1..n_\mu}^{j=1..n_Q} \sum_{\alpha=1..a_{\mu\nu}}^{\beta=1..a_{\nu\lambda}} \sum_{k=1..n_\lambda}^{\gamma=1..a_{\lambda\mu}} \epsilon_{mn\ell} C_{3m,\nu j}^{i,\mu,\alpha} C_{3n,\nu j}^{i,\nu,\beta} C_{3l,\mu i}^{i,\lambda,\gamma} \langle \phi_{\alpha}^{(\nu \mu)} | \phi_{\beta}^{(\lambda \nu)} | \phi_{\gamma}^{(\lambda \mu)} \rangle \ .$$

(35)

This expression admits a graphic description which we now explain. Given a quiver, one can construct its reduction, which is the quiver obtained by keeping only one edge out of each set of edges connecting any two given nodes (the two nodes under consideration need not be distinct, so that reduction is also applied to any loops which may be present in the quiver). This amounts to leaving the nodes $\mu$ unchanged and replacing $a_{\mu\nu}$ by $\tilde{a}_{\mu\nu} = \{ 1, \text{ if } a_{\mu\nu} \neq 0 \}

0, \text{ if } a_{\mu\nu} = 0 , \text{ for all } \mu, \nu = 0..r$. The reduced quiver contains at most one edge between any two nodes.

A triple circuit of a quiver is an ordered triplet of edges $(f, g, h)$ (not necessarily distinct and considered only up to a circular permutation) such that $tail(g) = head(f)$, \footnote{Since our use of the word circuit may be slightly unfamiliar, let us give a formal definition of the objects involved. In precise set-theoretic language, a quiver is a quadruplet $(S, V, h, t)$ where $S$ is a finite set whose elements are called edges, $V$ is a finite set whose elements are called nodes and $h, t$ (also denoted by head, tail) are maps from $S$ to $V$. If $s \in S$ is an edge, then the nodes $h(s), t(s) \in V$ are called the head, respectively the tail of $s$. An edge is called an arrow if $t(s) \neq h(s)$ (one represents this graphically by drawing an arrow from $t(s)$ to $h(s)$). Otherwise, $s$ is called a loop (the graphical representation of a loop does not}
tail$(h) = head(g)$, tail$(f) = head(h)$ (For a loop $l$ we define head$(l) = tail(l)$ to be the associated node.) The nodes of the circuit are the (not necessarily distinct) points tail$(g) = head(f)$, tail$(h) = head(g)$, tail$(f) = head(h)$ which the circuit meets.

A set of ‘Feynman rules’ for computing the superpotential $W$ can now be formulated in terms of the reduced quiver:

(1) To obtain the total superpotential, one must add the contribution of all different triple circuits of the reduced quiver.

(2) For any circuit of the reduced quiver, pick any of the nodes of the circuit and let $\mu$ denote its index. Then follow the circuit in the sense given by its orientation and let $\nu, \lambda$ be the indices of the first and second nodes thus encountered (note that $\mu, \nu, \lambda$ need not be distinct). Then there is a contribution to the superpotential given by (see Figure 2):

\[
-w \sum_{i=1..n_\mu} \sum_{\alpha=1..a_{\mu\nu}} \varepsilon_{\mu\nu\lambda} \sum_{j=1..n_\nu} \sum_{\beta=1..a_{\nu\lambda}} \varepsilon_{\alpha\beta\gamma} \sum_{k=1..n_\lambda} \sum_{\gamma=1..a_{\lambda\mu}} C^{\mu i,\alpha}_{3n_\mu,j} C^{\nu j,\beta}_{3n_\nu,k} C^{\lambda k,\gamma}_{3n_\lambda,i} T_r \left[ \phi^{(\mu\lambda)}_\gamma \phi^{(\lambda\nu)}_\alpha \phi^{(\nu\mu)}_\beta \right],
\]

(36)

where $w$ is a multiplicity equal to:

(a) $w=3$, unless all 3 nodes coincide

(b) $w=1$, if all 3 nodes coincide.

(This multiplicity is a consequence of the invariance of the sum in (36) under cyclic permutations of the triplet $(\mu, \nu, \lambda)$). The types of circuits with 3 nodes which can occur in a reduced quiver are drawn in Figure 3. The third of these consists of a loop which is traversed 3 times and is not orientable.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The graphic structure of a contribution to the superpotential.}
\end{figure}

A marked circuit is a finite ordered set $(s_0...s_k)$ of edges such that $h(s_{j-1}) = t(s_j)$ for all $j = 1..k$ and $h(s_k) = t(s_0)$. Two marked circuits $(s_0...s_k)$ and $(s_0'...s_k')$ are called equivalent if there exists a cyclic permutation $\sigma$ of the set $\{0..k\}$ such that $s_{\sigma(i)} = s'_i$ for all $i = 0..k$ (this defines an equivalence relation). A circuit is an equivalence class of marked circuits modulo this equivalence relation. Intuitively, a marking of a circuit is given by choosing a ‘starting point’ for traversing the circuit, while the circuit itself is obtained by ‘forgetting’ the marking.
2.4 Comparison with the case when $\Gamma$ is abelian

Let us pause for a moment to discuss how the case of an abelian orbifold group fits into the above formalism. If $\Gamma$ is abelian, then all of its irreducible representations are one-dimensional. The number $r + 1$ of irreducible representations is equal to the number $N$ of elements of $\Gamma$ and $n_\mu = 1$ for all $\mu = 0..N - 1$. The spaces $V_\mu$ sitting at the nodes of the quiver are therefore one-dimensional. Choosing a nonzero vector $u_\mu \in V_\mu$ for each $\mu$ allows us to identify $V_\mu \approx \mathbb{C}$. The quiver data $\phi^{(\nu\mu)}_\alpha : V_\mu \to V_\nu$ can be identified with the complex numbers $x^{(\nu\mu)}_\alpha$ given by: $\phi^{(\nu\mu)}_\alpha(u_\mu) = x^{(\nu\mu)}_\alpha u_\nu$. In fact, $x^{(\nu\mu)}_\alpha$ are nothing else than those components of the matrices $X_m$ which survive the projection conditions. Since all $n_\mu = 1$, the projected gauge group is abelian, and it coincides with the compact torus $G_0 = U(1)^N$, while the effective gauge group is given by $G = U(1)^N / U(1)_{\text{diag}} \approx U(1)^{N-1}$. Moreover, equation (27) shows that the original moment map $\rho$ coincides with the quiver moment map $M$.

The moduli space is given by a Kähler reduction of the variety of commuting matrices via the action of $U(1)^{N-1}$, which is equivalent (by results of [23]) with a holomorphic quotient of $\mathcal{Z}$ by the complex torus $(\mathbb{C}^*)^N$. The important fact which simplifies the analysis in this case (and which was pointed out in [8, 12]) is that the variety of commuting matrices is a toric variety. (This can be established, for example, by writing down the commutation relations $[X_m, X_n] = 0$ in terms of those components of $X_m$ which survive the projection conditions and noticing that the resulting F-flatness constraints are given by monomial relations, which assures that $\mathcal{Z}$ is an affine toric variety. A similar argument can be made at the level of the quiver data $x^{(\nu\mu)}_\alpha$.)

Using the alternative presentation [28] of $\mathcal{Z}$ as a holomorphic quotient of a space $\mathbb{C}^d$ by a complex...
torus \((\mathbb{C}^*)^k\) allows one to present the moduli space \(\mathcal{M}_\xi\) itself as a toric variety. The variation of \(\mathcal{M}_\xi\) with \(\xi\) can then be studied by the well-established methods of toric geometry [14].

By contrast with the above, nonabelian groups \(\Gamma\) lead to varieties \(Z\) which are in general not toric. This is due to the fact that, in the general case, some of the irreducible representations of \(\Gamma\) are not one-dimensional (some \(n_\mu\) is different from 1), so that the projected fields \(X_m\) have a complicated block structure. This leads to polynomial (as opposed to monomial) constraints among the surviving components of \(X_m\), when one imposes the commutation conditions \([X_m, X_n] = 0\) which define the variety of commuting matrices. Alternatively, this can be seen at the level of quiver data, if one notices that the traces \(\text{Tr}[\phi^{(\mu\lambda)}_\gamma \phi^{(\nu\mu)}_\beta \phi^{(\lambda\nu)}_\alpha]\) appearing in the quiver Feynman rules (36) will produce polynomials in the components of the maps \(\phi\) if one of the vector spaces \(V_\mu, V_\nu, V_\lambda\) has dimension greater than one.

On the other hand, the effective gauge group \(G\) of equation (29) is in general nonabelian, which means that, even if one has obtained an explicit description of \(Z\), the moduli space is given by a Kähler reduction of \(Z\) via a nonabelian group. While one can still use results of [26] to reduce this to a geometric invariant theory quotient of \(Z\) by the complexified gauge group \(G^C\), the computation of such a quotient is a highly nontrivial problem in algorithmic invariant theory [23]. In the next section we will investigate the variation of the moduli space as a function of the Fayet-Iliopoulos parameters by using methods of symplectic and algebraic geometry.

### 3 The variation of the moduli space

The study of the moduli space of D-branes at nonabelian quotient singularities of threefolds is complicated by the lack of efficient computational methods for treating the resulting symplectic quotient problem. This is in contrast with the case of abelian groups \(\Gamma\), where the problem can reduced to toric geometry, as was sketched above following the detailed discussion of [8, 12]. The nonabelian case is considerably more complicated, since in this situation the moduli space \(\mathcal{M}_\xi\) is not a toric variety, so one cannot resort to the powerful machinery of [14] to reduce the problem to convex geometry. (Apparently, then, the only tools available in such a situation are the general methods of geometric invariant theory [22], which lead to important qualitative results but to hard computational problems. The determination of the moduli space could be achieved, at least in principle, by a computation of ‘relative invariants’ in the spirit of classical invariant theory [23]. Unfortunately, this problem is extremely involved in practice and severely hampers the hope for progress).

However, one can extract considerable information about the variation of the moduli space without performing explicit calculations of invariants, and, rather surprisingly, one can reduce some basic questions about this variation to problems which can be handled by toric methods. While the methods we discuss below are not powerful enough to determine the moduli space explicitly, or to compute its geometric proper-
ties, they suffice to locate the possible topology-changing transitions in the space of Fayet-Iliopoulos parameters.

In order to explain this, let us first outline the general picture for the dependence of $M_\xi$ on $\xi$, which follows from the results of [24, 25, 26]. The allowed levels $\xi \in \mathbb{R}^r$ can be divided into ‘noncritical’ and ‘critical’ values. The critical values occur on walls of codimension at least one, which separate the space of Fayet-Iliopoulos parameters into disjoint chambers. In our case, these chambers will be cones adjoining each other along common faces, which determine the walls. The points lying in the interior of each chamber are the noncritical values of $\xi$. The main results of [24, 25, 26] state that the quotient $M_\xi$ depends only on the chamber to which $\xi$ belongs, and can undergo a topology-changing transition as $\xi$ crosses a wall. Furthermore, the arguments of [24, 25] imply that, if $M_\xi$ is smooth for $\xi$ inside any of the chambers, then this transition is given by a flip. If, moreover, the desingularizations $M_\xi \to \mathbb{C}^3/\Gamma$ are crepant (i.e. if $M_\xi$ is a Calabi-Yau manifold for generic values of $\xi$), then one can invoke results of [27] in order to deduce that such transitions are in fact flops. Hence the situation is similar to the more familiar cases of topology change [4] which are realized in the moduli space of two-dimensional conformal field theories.

In view of the general expectations above, the most important information to extract is the phase structure of the space of Fayet-Iliopoulos parameters, i.e. the location of the system of walls and chambers in the space $\mathbb{R}^r(\xi)$. As we discuss below, and illustrate explicitly with two examples in the next section, this information can be obtained by methods of toric geometry, thus bypassing the computational difficulties involved in determining the precise structure of the moduli space. This follows from the observation that one can perform the Kähler quotient $M_\xi = (Z \cap M^{-1}(\xi))/G$ in two steps, if one writes $G = (T \times H)$ with $T$ a central Lie subgroup of $G$ and $H \approx G/T$ (where we neglect an unimportant finite central subgroup). Then the moduli space $M_\xi$ turns out to be isomorphic with the zero level Kähler quotient of $P_\xi := (Z \cap M_{T^{-1}}(\xi))/T$ by the induced action of $H$. (Here $M_T$ is the projection of the moment map $M$ onto the center of the Lie algebra of $G$). The salient point is that only the first step of this double reduction depends on $\xi$, which suggests that the $\xi$-dependence of the moduli space is essentially encoded by the quotient $P_\xi := (Z \cap M_{T^{-1}}(\xi))/T$. This can be studied by methods of toric geometry, since $T$ will be a compact torus $U(1)^r$ (up to some finite identifications). Indeed, since $Z$ is an affine variety inside of $Q$, this quotient presents $P_\xi$ as a subvariety (given by homogeneous polynomial equations) of the space $E_\xi = \{ \Phi \in Q | M_T(\phi) = \xi \}/T$, which is a toric variety. The wall structure of $\mathbb{R}^r(\xi)$ associated to the quotient $E$ can be easily determined by toric methods, and the walls associated to $P_\xi$ (and thus to $M_\xi$) turn out to form a ‘coarsening’ of the set of walls associated to $E_\xi$.

The plan of this section is as follows. In the first subsection, we discuss the basic features of the variation of the moduli space which follow from general results in symplectic geometry and geometric invariant theory. (The reader unfamiliar with al-

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8A flip is a generalization of a flop, as we explain in more detail in the next subsection.
gebraic geometry and invariant theory can safely skip this section, which is not strictly needed for understanding the rest of the paper.) Subsection 3.2 outlines the double quotient procedure for the computation of $\mathcal{M}_\xi$, while in subsection 3.3 we argue that the variation of the moduli space with the Fayet-Iliopoulos parameters $\xi$ is essentially encoded by the toric part of the quotient. Finally, subsections 3.4 and 3.5 discuss the toric part of the quotient in detail.

### 3.1 Flips and flops

In order to substantiate the above discussion, it is most convenient to translate the problem to one in algebraic geometry, since this will allow us to apply the results of [24, 25].

For this, first note that $Z$ is an affine algebraic variety in the ambient vector space $Q$, since it is given by a finite set of polynomial equations among the quiver variables $\phi_{\alpha}(\nu_{\mu}) \in Q$. Moreover, the universal cover $\tilde{G} = S[\Pi_{\mu=0,r} U(n_{\mu})]$ of the effective gauge group $G$ is a product of a semisimple classical group and a torus. It is well-known that the complexification $\tilde{G}_C = S[\Pi_{\mu=0,r} SL(n_{\mu})]$ of $\tilde{G}$ is a reductive [9] algebraic group, whose induced action on $Z$ is rational. The moduli space $\mathcal{M}_\xi$ will be a (quasiprojective) algebraic variety only if the Kähler form induced by the Kähler quotient procedure is integral. It is not hard to see that this will be the case if one restricts to integral levels of the moment map, i.e. Fayet-Iliopoulos terms $\xi = \sum_{\mu=0,r} \xi_{\mu} n_{\mu}$ whose components $\xi_{\mu}$ are integers. In this case, one can apply results of [26] in order to deduce that the Kähler reduction $\mathcal{M}_\xi$ of $Z$ by $G$ at level $\xi$ is isomorphic (as a complex variety) with the geometric invariant theory quotient $Z//\chi_{\xi} G_C$ of $Z$ by $G_C$ linearized [10] by the rational character associated to $\xi$:

$$\chi_{\xi}(g_0..g_r) = (\det g_0)^{\xi_0}..(\det g_r)^{\xi_r} \quad (g_{\mu} \in SL(n_{\mu})).$$

(37)

In fact, one can allow $\xi$ to be rational [24, 25] by considering ‘fractional linearizations’, so the integrality restriction on $\xi$ is of little consequence.

Once the problem has been reduced to one in algebraic geometry, one can establish the following results. First, the space of moment map levels is divided into a finite number of chambers, which adjoin along a set of walls. These chambers and walls can be shown to be integral convex polytopes with respect to the sublattice determined by the integral moment map levels. The Kähler reduction $\mathcal{M}_\xi$ is unchanged (up to an isomorphism of complex varieties) as long as $\xi$ varies inside of a given chamber, and it suffers a transition known as a flip when $\xi$ crosses a generic wall.

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9A complex algebraic group $U$ is reductive if any finite-dimensional rational representation of $U$ is completely reducible. A rational representation of $U$ is simply a matrix representation such that all entries of the representation matrix $A(u)$ are rational functions of the group element $u$.

10In our case, a linearization of the action of $G_C$ on $X$ is a lift to an action on the total space of the trivial holomorphic line bundle $O_X$, which agrees with the action on $X$. This is the same as a choice of a rational character of $G_C$. 

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Intuitively, a flip is a birational transformation during which a subvariety of $M^{\xi}$ of codimension at least 2 shrinks to a point and is then blown up to another such subvariety. More technically, a flip is defined as a birational transformation $f : X_+ \to X_-$ between two algebraic varieties which can be decomposed into two birational maps $X_+ \to X_0$ and (the inverse of) $X_- \to X_0$, such that $X_\pm \to X_0$ are small contractions. A small contraction $X_\pm \to X_0$ is a proper birational morphism from $X_\pm$ to $X_0$ whose exceptional set (the subset of $X_\pm$ over which the map is not one to one) has codimension at least two in $X_\pm$. For a flip, one also makes the technical requirement that there exist a divisor $D$ on $X_+$ such that $\mathcal{O}(-D)$ and the inverse of its pushforward $f_*(\mathcal{O}(D))$ on $X_-$ are both relatively ample over $X_0$.

According to the discussion above, the moduli space $M^{\xi}$ of our D-brane effective field theories will undergo a flip when $\xi$ crosses a generic wall. On the other hand, general results of [27] assure us that two crepant partial resolutions of a Calabi-Yau threefold singularity can always be related by a sequence of flops. This allows us to conclude that $M^{\xi}$ will in general undergo a flop when it crosses a wall in the space of Fayet-Iliopoulos parameters.

3.2 The double Kähler quotient

Consider the Kähler reduction of $Z$ at level $\xi$ modulo $T$:

$$P^{\xi} := \{ \Phi \in Z | M_T(\Phi) = \xi \}/T \quad (38)$$

Since $T$ is a central subgroup of $G$, one has an induced action of $H := G/T$ on the quotient $P^{\xi}$. This action turns out to be Kählerian with respect to the Kähler form induced on $P^{\xi}$ by the Kähler reduction procedure. Denoting the associated moment map by $M_H$, we can consider the Kähler quotient of $P^{\xi}$ by the action of $H$ at level zero:

$$N^{\xi} := \{ x \in P^{\xi} | M_H(x) = 0 \}/H \quad (39)$$

Then we claim that:

$$M^{\xi} \approx N^{\xi} \quad (40)$$

as Kähler manifolds. An outline of the arguments involved in establishing this fact can be found in Appendix 1.

3.3 Identifying the walls of the extended Kähler cone

3.3.1 Variation of the toric part of the quotient

The first stage (38) in our procedure is a quotient of $Z$ by a hamiltonian torus action. Such a situation has been studied extensively in the literature and can be approached

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11 The reader familiar with Mori theory should note that we do not require that $D$ is the canonical divisor; therefore, a flip in the above sense is apriori more general than a Mori flip.

12 The action is hamiltonian with respect to the symplectic form given by the imaginary part of the Kähler form.
either from a symplectic [2] or a toric [14] perspective. In fact, the quotient (38) realizes \( P_\xi \) as a submanifold of the ambient space \( E_\xi = \{ \Phi \in Q | M(\Phi) = \xi \} / T \). As a complex manifold, this ambient space is a toric variety of the form \( (\mathbb{C}^k - Z_\xi) / (\mathbb{C}^*)^r \) (where \( k = \dim Q \)) and \( Z_\xi \) a subset of \( \mathbb{C}^n \) (a union of intersections of coordinate hyperplanes), called the ‘exceptional set’. It is well-known that the chambers of \( \mathbb{R}^r(\xi) \) for \( E_\xi \) consist of a finite set of polyhedral cones \( \sigma_i \), and that \( Z_\xi \) is constant when \( \xi \) varies inside of a given chamber. Hence choosing \( \xi \) in the interior of the cone \( \sigma_i \) gives a toric variety \( E_\xi = E_i = (Q - Z_i) / (\mathbb{C}^*)^r \approx (Q - Z_i) / (\mathbb{C}^*)^r \), which depends only on the cone \( \sigma_i \). The spaces \( E_i \) are related by toric flips which occur when \( \xi \) crosses one of the walls, a fact which underlies the presence of topology changing-transitions in the moduli space of (2,2) conformal field theories considered in [4]. Indeed, when \( \xi \) crosses a wall, the \( (\mathbb{C}^*)^r \) quotients of the exceptional set \( Z \) on the two sides of the wall will flip. More precisely, if \( E_+ = (Q - Z_+) / (\mathbb{C}^*)^r \) and \( E_- = (Q - Z_-) / (\mathbb{C}^*)^r \) are the toric varieties obtained on the two sides of a generic wall, then \( E_+ \) contains the subvariety \( (Z_- - Z_+ \cap Z_-) / (\mathbb{C}^*)^r \), while \( E_- \) contains the subvariety \( (Z_+ - Z_+ \cap Z_-) / (\mathbb{C}^*)^r \), and these two subvarieties are exchanged during the flip. In fact, the walls for \( E_\xi \) can be identified by methods of toric geometry or with the help of standard results of [29] which characterize the regular levels of the moment map for torus actions.

The underlying complex variety of \( P_\xi \) is given by the quotient \( (Z - (Z_i \cap Z)) / (\mathbb{C}^*)^r \), and will undergo similar transformations as \( \xi \) crosses a wall, since \( Z \) changes for those values of \( \xi \). In fact, the set of walls for \( P_\xi \) will in general be a ‘coarsening’ of the set of walls for \( E_\xi \), since it is possible that the intersection of \( Z \) with \( Z \) stays unchanged even though \( Z \) changes during a flip of \( E \). Hence the only remaining issue is to check whether some of the ambient walls are ‘projected out’ or identified with other walls when restricting to \( Z \), and this can be achieved in each case by a study of the intersection of the variety of commuting matrices with the various exceptional sets \( Z_i \).

### 3.3.2 Variation of the moduli space

Having understood the variation of \( P_\xi \) with \( \xi \), we have to consider the effect of the quotient by \( H \) which produces the moduli space \( \mathcal{M}_\xi \). We claim that this quotient does not modify the wall structure any further, i.e. that the walls in the space of moment map levels are the same for \( \mathcal{M}_\xi \) and \( P_\xi \).

This can be seen as follows. If \( \xi \) and \( \xi' \) both belong to a given chamber for the quotient (38), then there is a canonical diffeomorphism \( \Psi \) between \( P_\xi \) and \( P_{\xi'} \), which can be obtained by considering the orbits of the complexification \( T^C \approx (\mathbb{C}^*)^r \) of \( T \) (the precise statement [4] is as follows. Given an orbit \( O \) of \( T \) inside of \( Z \cap M_T^{-1}(\xi) \),

\[ \text{In the toric case of [3, 12], such a projection often occurs and can be formalized elegantly in the quiver language [2]. The situation is more involved in our case, and we were not able to find a comparably powerful description.} \]

\[ \text{This is one of the main results of [20] in the case } \xi = 0. \text{ The generalization to } \xi \neq 0 \text{ is straightforward and is discussed, for example, in [10].} \]
there exists a unique closed orbit $O$ of $T^C$ which passes through $O$, and this intersects $Z \cap M_T^{-1}(\xi')$ along an orbit $O'$ of $T$ (see Figure 4). Then the isomorphism in question is obtained by sending $O$ to $O'$.

Figure 4. The isomorphism between $\mathcal{M}_\xi$ and $\mathcal{M}_{\xi'}$. The right side of the figure depicts two $G$-orbits (represented as two ‘horizontal’ sheets) inside of two level sets $Z \cap M^{-1}(\xi)$ and $Z \cap M^{-1}(\xi')$ (not shown) for the action of $G$ on the variety of commuting matrices. Two $T$-orbits $O, O'$ (represented as horizontal lines) inside of these $G$-orbits are identified by the map $\Psi$ if they belong to the same orbit (the ruled vertical sheet) of the complexified torus $T^C$. The quotient by the action of $T$ (represented by the dotted arrow on the right) collapses all $T$-orbits to points, thus taking the $G$-orbits into two $H$-orbits inside of the varieties $P_\xi, P_{\xi'}$. These are identified by the map $\Psi$. Further quotienting by $H$ (the dotted arrow on the left) gives an isomorphism (induced from $\Psi$) between $\mathcal{M}_\xi$ and $\mathcal{M}_{\xi'}$.

Since $T$ is central subgroup of $G$, it follows that $\Psi$ commutes with the actions of $H$ on $P_\xi$ and $P_{\xi'}$. This assures us that $\Psi$ induces a diffeomorphism between $\mathcal{M}_\xi$ and $\mathcal{M}_{\xi'}$ as we take the quotient by $H$. It follows that (the differential type of) $\mathcal{M}_\xi$ does not change as $\xi$ varies in a fixed chamber associated to the action of $T$ on $Z$. On the other hand, if $P_\xi$ does become singular as $\xi$ crosses a wall, then $\mathcal{M}_\xi$ will be singular as well. Therefore, the chamber structure for the action of $G$ on $Z$ is the same as that associated to the action of $T$ on $Z$.

In conclusion, the wall structure of the space of Fayet-Iliopoulos parameters can be determined by examining the first stage of the quotient, which is essentially a problem in toric geometry. In the next section we apply this method to two low-rank examples, which allows us to locate the flop transitions in the moduli space of D-branes placed at nonabelian quotient singularities of Calabi-Yau threefolds.
3.4 The toric subgroup of the effective gauge group and its action

In this subsection, we discuss the first step of the double quotient in detail, since, as we argued above, it contains the information we wish to extract.

3.4.1 The action of $T$ on the quiver data

Consider a level $\xi$ of the moment map $M$ and the central subgroup $T = S[\Pi_{\mu=0..r}U(1)]/K$ of $G$ consisting of the elements of the form $(\gamma_0 1_{n_0} ... \gamma_r 1_{n_r}) \in \Pi_{\mu=0..r}U(1)$ (considered up to the identifications given by $K$) which satisfy the constraint $\Pi_{\mu=0..r}\gamma_\mu = 1$. Since $n_0 = 1$, we can solve this constraint for $\gamma_0$ in order to present $T$ as:

$$T \approx U(1)^r/C,$$

i.e. the set of $r$-tuples $(\gamma_1 ... \gamma_r) \in U(1)^r$ modulo the central subgroup $C \approx \mathbb{Z}_n$ given by:

$$C = \{(\eta ... \eta) \in U(1)^r | \eta^n = 1\}. \quad (42)$$

This group acts on the quiver data $\phi_{(\nu\mu)}$ through the action induced from that of $G$:

$$\phi_{(\nu\mu)}^{(\nu\mu)} \rightarrow e^{i(s_{\nu} - s_{\mu})}\phi_{(\nu\mu)}, \quad (43)$$

where we wrote $\gamma_\mu = e^{is_\mu}$ for all $\mu = 0..r$. Using $s_0 = -\sum_{\mu=1..r} n_\mu s_\mu \mod 2\pi$ we obtain the action of $U(1)^r/C$:

$$\phi_{(\nu\mu)}^{(\nu\mu)} \rightarrow e^{il_{\nu\mu}(\lambda)}\phi_{(\nu\mu)}^{(\nu\mu)} \quad (\lambda = 1..r), \quad (44)$$

where

$$l_{\nu\mu}(\lambda) = (1 - \delta_\nu^0)(1 - \delta_\mu^0)(\delta_\nu^\lambda - \delta_\mu^\lambda) - \delta_\nu^0(1 - \delta_\mu^0)[n_\lambda + \delta_\mu^\lambda] + \delta_\mu^0(1 - \delta_\nu^0)[n_\lambda + \delta_\nu^\lambda].$$

3.4.2 The moment map for the toric action

If $Q$ denotes the vector space of all quiver data $\{\phi_{(\nu\mu)}\}$, then the variety $\mathcal{Z}$ of commuting matrices is realized as a subvariety of $Q$ given by the polynomial equations which characterize the critical points of the superpotential $W$. Since the superpotential is gauge-invariant, $\mathcal{Z}$ is stable under the action of $G$ and this gives a Kählerian action of $G$ on $\mathcal{Z}$ induced by the action on $Q$. The moment map for this action is simply the restriction of $M$ to $\mathcal{Z}$.

The Lie algebra $t = u(1)^\oplus r$ of $T = U(1)^r/C$ is embedded as a central subalgebra of the Lie algebra $g$ of $G$ via the map:

$$j(s_1...s_r) = (s_0 1_{n_0}, s_1 1_{n_1}, ..., s_r 1_{n_r}), \quad (45)$$

with $s_0 := -\sum_{\mu=1..r} n_\mu s_\mu$. The moment map $M_T : Q \rightarrow t$ for the action of $T$ on $Q$ is related to $M$ via:

$$Tr(M_T(\Phi)s) = Tr(M(\Phi)j(s)), \quad \text{for all } s = (s_1...s_r) \in u(1)^\oplus r = \mathbb{R}^r. \quad (46)$$
Writing $M(\Phi) = \oplus_{\mu=0..r} M_\mu(\Phi)$ as in equation (25) and $M_T(\Phi) = \oplus_{\lambda=1..r}(M_T)_\lambda(\Phi)$, with $(M_T)_\lambda(\Phi) \in u(1)$, condition (26) gives:

$$(M_T)_\lambda(\Phi) = Tr(M_\lambda(\Phi)) - n_\lambda Tr(M_0(\Phi)) \quad .$$

In this presentation, the central levels $\xi'_\lambda$ of $M_T$ are related to $\xi_\mu$ by:

$${\xi'}_\lambda = n_\lambda (\xi_\lambda - \xi_0) \quad .$$

We will use the presentation (48) when discussing the examples of Section 4. Clearly this trivial reparametrization does not affect any of the considerations of the previous subsection: the space $\mathbb{R}'(\xi')$ has a phase structure obtained from that of $\mathbb{R}'(\xi)$ by the linear transformation (48).

One can simplify the expression (47) by noticing that:

$$tr M_\mu = \sum_{\nu, \alpha} \sum_{a_{\mu\nu} \neq 0} ||\phi^{(\mu\nu)}_\alpha||^2 - \sum_{\nu, \alpha} \sum_{a_{\mu\nu} \neq 0} ||\phi^{(\nu\mu)}_\alpha||^2 \quad ,$$

where $||\phi^{(\nu\mu)}_\alpha||^2 = \sum_{i=1..n_\mu} ||(\phi^{(\nu\mu)}_\alpha)_{ji}||^2$ (in some orthonormal bases of $V_\mu$, $V_\nu$) is the usual operator norm of $\phi^{(\nu\mu)}_\alpha$. Then a simple computation shows that (47) is equivalent with:

$$(M_T)_\lambda = \sum_{\nu, \mu = 0..r} \sum_{\alpha=1..a_{\mu\nu}} t^{(\lambda)}_{\nu\mu} ||\phi^{(\nu\mu)}_\alpha||^2 \quad (\lambda = 1..r),$$

which is the standard form [29] of the moment map for the action (14) of the torus $U(1)^r$ on the vector space $Q$.

4 Examples

4.1 Finite subgroups of $SU(3)$

The finite subgroups of $SU(3)$ fall into 3 series [30]:

(a) A finite series consisting of 6 ‘crystal-like’ groups $\Sigma_1...\Sigma_6$, of which the minimal order is $|\Sigma_1| = 60$.

(b) Two infinite series of ‘dihedral-like’ groups, denoted by:

(a) $\Delta_1(3n^2)$ (of order $3n^2$), with $n$ any positive integer

(b) $\Delta_2(6n^2)$ (of order $6n^2$), with $n$ any positive even integer.

For computational reasons, we are interested in subgroups of low rank. The lowest ranks are attained by $\Delta_1(3)$ ($n = 1$), $\Delta_1(12)$ ($n = 2$) and $\Delta_2(24)$ ($n = 2$). However, $\Delta_1(3)$ is isomorphic to $A_3$ (the alternating group on 3 letters), which is abelian, so we will only consider the subgroups $\Delta_1(12)$ and $\Delta_2(24)$. It should be noted that all of our
statements below apply at the level of classical field theory only. In this paper, we do not consider quantum-mechanical modifications of the moduli space.

4.2 The case \( \Gamma = \Delta_1(12) \)

The subgroup \( \Delta_1(12) \) is given by the following 12 elements:

\[
A(p, q) = \begin{bmatrix}
(-1)^p & 0 & 0 \\
0 & (-1)^q & 0 \\
0 & 0 & (-1)^{p+q}
\end{bmatrix}
\]

\[
C(p, q) = \begin{bmatrix}
0 & 0 & (-1)^p \\
(-1)^q & 0 & 0 \\
0 & (-1)^{p+q} & 0
\end{bmatrix}
\]

\[
E(p, q) = \begin{bmatrix}
0 & (-1)^p & 0 \\
0 & 0 & (-1)^q \\
(-1)^{p+q} & 0 & 0
\end{bmatrix}
\]

Since \( \Delta_1(12) \) has 12 elements, the unprojected D-brane theory has a \( U(12) \) gauge group and \( 3 \times 12 \times 12 = 432 \) complex fields. The presence of such a large field content even in the case of the lowest order nonabelian orbifold group shows that it is difficult to analyze the projection conditions and the variety of commuting matrices without making use of the systematic approach we developed in section 2. Following that approach, we first note that our orbifold group has 4 irreducible representations, which we denote by \( R_0, R_1, R_2 \) and \( R_3 \). The representations \( R_0, R_1, R_2 \) are 1-dimensional (with \( R_0 \) the trivial representation) while \( R_3 \) is the 3-dimensional defining representation induced by the embedding of \( \Delta_1(12) \) in \( SU(3) \). We will chose the action of \( \Delta \) on \( \mathbb{C}^3 \) to be given by \( Q = R_3 \).

The characters of the 4 irreducible representations are given by:

| irrep | \( R_0 \) | \( R_1 \) | \( R_2 \) | \( R_3 \) |
|-------|------|------|------|------|
| \( A(p, q) \) | 1    | 1    | 1    | 1    |
| \( C(p, q) \) | \( \sigma \) | \( \sigma^2 \) | \( \sigma \) |
| \( E(p, q) \) | \( \phi(p, q) \) | 0    | 0 |

where \( \phi(p, q) := (-1)^p + (-1)^q + (-1)^{p+q} \) and \( \sigma := e^{2\pi i/3} \) is the primitive cubic root of unity.

\(^{15}\)In particular, we neglect quantum anomalies, which were recently shown \(^{19}\) to be sometimes present in the field theory of D-branes placed at orbifold singularities.

\(^{16}\)It is not hard to see that this group is isomorphic with the point group of a regular tetrahedron, i.e. the subgroup of \( SO(3) \) which leaves such a tetrahedron invariant.
4.2.1 Branching rules and the quiver

Use of the characters given above establishes the branching rules:

\[ Q \otimes R_0 \approx Q \]
\[ Q \otimes R_1 \approx Q \]
\[ Q \otimes R_2 \approx Q \]
\[ Q \otimes R_3 \approx Q \otimes Q \approx R_0 \oplus R_1 \oplus R_2 \oplus (R_3 \otimes R_3) , \] (52)

which lead to the McKay quiver depicted below. Note that the McKay coefficients are symmetric: \( a_{\mu \nu} = a_{\nu \mu} \), a reflection of the fact that the defining representation \( Q \) is self-dual, \( Q^* \approx Q \). Since each arrow, respectively loop of the quiver corresponds to 3, respectively 9 complex fields, only 36 out of the initial 432 fields survive the projection conditions.

![McKay quiver](image)

Figure 5. The quiver for \( \Delta_i(12) \).

4.2.2 Clebsch-Gordan coefficients and the superpotential

We choose the characters and matrices of the previous subsection as our fiducial matrix form of the irreducible representations \( R_0, R_3 \). With these conventions, the Clebsch-Gordan coefficients can be computed from the information given above by the method
of projectors (a brief review of this method is given in Appendix 2). For the decompositions \( Q \otimes R_{\mu} \approx Q (\mu = 0..2) \), they are given by:

\[
C_{3j,\mu1}^{3i} = a_{i}^{(\mu)} \delta_{ij} ,
\]

(53)

where \( a_{i}^{(\mu)} (\mu = 0..2, i = 1..3) \) are the elements of the matrix:

\[
a = \begin{bmatrix}
1 & 1 & 1 \\
1 & \sigma & \sigma^2 \\
\sigma^2 & \sigma & 1
\end{bmatrix} ,
\]

(54)

(\( \mu \) is the line index and \( i \) is the column index). The Clebsch-Gordan coefficients for the decomposition \( Q \otimes Q \approx Q \otimes R_0 \oplus R_1 \oplus R_2 \oplus R_3 \otimes R_3 \) fall into two types: those associated to the \( R_{\mu} (\mu = 0..2) \) terms, which we denote by \( C_{3i,3j}^{3k} = U_{ij}^{(\mu)} \); and those associated to the \( R_3 \otimes R_3 \) term, which we denote by \( C_{3i,3j} = V_{ij}^{(k\alpha)} \). Here \( i, j = 1..3 \) index the elements \( e_{i}^{(3)} \otimes e_{j}^{(3)} \in Q \otimes Q \), with \( e_{i}^{(\mu)} \) the fiducial basis of \( Q = R_3 \), while \( \alpha, k = 1..3 \) index a fiducial basis \( e_{i}^{(3,\alpha)} \) of \( R_3 \otimes R_3 \), with \( \alpha \) the multiplicity index. The \( R_{\mu} \) terms \( (\mu = 0..2) \) have fiducial bases given by one element \( e_{i}^{(\mu)} \), since they are one-dimensional. With these notations, the Clebsch-Gordan coefficients for the decomposition of \( Q \otimes Q \) are:

\[
U^{(1)} = \frac{1}{\sqrt{3}} I_3 \quad U^{(2)} = \frac{1}{\sqrt{3}} \begin{bmatrix}
\sigma & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \sigma^2
\end{bmatrix} \quad U^{(3)} = \frac{1}{\sqrt{3}} \begin{bmatrix}
1 & 0 & 0 \\
0 & \sigma & 0 \\
0 & 0 & \sigma^2
\end{bmatrix}
\]

(55)

\[
V^{(1,1)} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix} \quad V^{(2,1)} = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \quad V^{(3,1)} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
V^{(1,2)} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \quad V^{(2,2)} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix} \quad V^{(3,2)} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

where we used matrix notation: \( U^{(\mu)} := (U_{ij}^{(\mu)})_{i,j=1..3} \), \( V^{(k\alpha)} := (V_{ij}^{(k\alpha)})_{i,j=1..3} \).

Applying the quiver Feynman rules gives the superpotential:

\[
W = -3K_{\alpha}^{(\mu)} Tr[\phi_{\alpha}^{(33)} \phi_{(3\mu)}^{(3\mu)}] - L_{\alpha\beta\gamma} Tr[\phi_{\alpha}^{(33)} \phi_{\gamma}^{(33)} \phi_{\beta}^{(33)}] 
\]

(56)

where the cubic couplings \( K_{\alpha}^{(\mu)} \) and \( L_{\alpha\beta\gamma} (\mu = 0..2, \alpha, \beta, \gamma = 1,2) \) are given by:

\[
K_{\alpha}^{(\mu)} = \epsilon_{mnl} U_{mk}^{(\mu)} V_{ln}^{(k\alpha)} 
\]

(57)

\[
L_{\alpha\beta\gamma} = \epsilon_{mnl} V_{mj}^{(\alpha)} V_{nk}^{(j\beta)} V_{li}^{(k\gamma)} 
\]

(58)
Direct computation shows that $K_{\alpha}^{(\mu)}$ are the components of the 3 by 2 matrix:

$$
K = \begin{bmatrix} -1 & 1 \\ -\sigma^2 & 1 \\ -\sigma & 1 \end{bmatrix},
$$

(59)

while the only nonzero components of $L_{\alpha\beta\gamma}$ are:

$$
L_{111} = 3 
$$

(60)

$$
L_{222} = -3.
$$

(61)

4.2.3 The variety of commuting matrices

It is convenient to denote the quiver data by:

$$
\begin{align*}
  x &= \phi^{(03)}, \\
  y &= \phi^{(13)}, \\
  z &= \phi^{(23)}, \\
  X &= \phi^{(30)}, \\
  Y &= \phi^{(31)}, \\
  Z &= \phi^{(32)}, \\
  u &= \phi^{(33)}, \\
  v &= \phi^{(33)}.
\end{align*}
$$

(62)

With respect to the $GL(3)$ action, the $x, y, z$ are $3 \times 1$ matrices transforming as covectors, the $X, Y, Z$ are $1 \times 3$ matrices transforming as vectors, and the $u, v$ are $3 \times 3$ matrices transforming as tensors of type $(1,1)$. With this notation, the superpotential becomes:

$$
W = -3[x(v - u)X + y(v - \sigma^2 u)Y + z(v - \sigma u)Z + Tr(u^3) - Tr(v^3)].
$$

(63)

Differentiating $W$ leads to the F-flatness constraints:

$$
\begin{align*}
  (v - u)X &= 0, & (v - \sigma^2 u)Y &= 0, & (v - \sigma u)Z &= 0 \\
  x(v - u) &= 0, & y(v - \sigma^2 u) &= 0, & z(v - \sigma u) &= 0 \\
  3u^2 &= Xx + \sigma^2 Yy + \sigma Zz \\
  3v^2 &= Xx + Yy + Zz,
\end{align*}
$$

(64)

(65)

(66)

(67)

whose solution set is the variety of commuting matrices $\mathcal{Z}$.

4.2.4 The toric part of the quotient and the enlarged Kähler cone

The effective gauge group is given by:

$$
G = U(1)^3 \times U(3)/U(1)_{\text{diag}} \approx S[U(1)^3 \times U(3)]/K,
$$

(68)

where $S[U(1)^3 \times U(3)]$ is the group of quadruples $(g_0, g_1, g_2, g_3) \in U(1)^3 \times U(3)$ such that $g_0 g_1 g_2 det g_3 = 1$ and $K \approx \mathbb{Z}_6$ is its central subgroup:

$$
K = \{(\eta, \eta, \eta, \eta l_3)|\eta^6 = 1\}.
$$

(69)
$G$ has a central subgroup $T = T_0/K$, where:

$$T_0 = \{(g_0, g_1, g_2, \gamma) \in U(1)^4 | g_0 g_1 g_2 \gamma^3 = 1 \}$$

which corresponds to setting $g_3 = \gamma_1 3$. We have an isomorphism $T \approx U(1)^3/C$, induced by:

$$(g_1, g_2, \gamma) \rightarrow (g_1^{-1} g_2^{-1} \gamma^{-3}, g_1, g_2, \gamma)$$

which corresponds to solving for $g_0$ in the constraint $g_0 g_1 g_2 \gamma^3 = 1$. $C \approx \mathbb{Z}_6$ is a central subgroup of $U(1)^3$ given by:

$$C = \{(\eta, \eta, \eta) | \eta^6 = 1 \}$$

Writing $g_1 := e^{i \phi_1}, g_2 := e^{i \phi_2}, \gamma = e^{i \phi_3}$, the subgroup $T \approx U(1)^3/\mathbb{Z}_6$ of $G$ acts on the quiver data as $\phi^{(\mu 3)} \rightarrow e^{i l^{(\lambda)}_{\mu} s_{\lambda} \phi^{(\mu 3)}}, \phi^{(3 \mu)} \rightarrow e^{-i l^{(\lambda)}_{\mu} s_{\lambda} \phi^{(3 \mu)}}$, while leaving $\phi^{(3 1)}$ and $\phi^{(3 2)}$ unchanged, with the charge matrix $L := (l^{(\lambda)}_{\mu})_{\lambda=1..3, \mu=0..2}$ given by:

$$L = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -4 & -1 & -1 \end{bmatrix}$$

(73)

(74)

The moment map for this (effective) $U(1)^3/\mathbb{Z}_6$ action is given by:

$$(M_T)_\lambda = \sum_{\mu=0..2} l^{(\lambda)}_{\mu} [||\phi^{(\mu 3)}||^2 - ||\phi^{(3 \mu)}||^2].$$

(75)

Picking a level $\xi' = (\xi'_\lambda)_{\lambda=1..3} \in \mathbb{R}^3$, we can solve the moment map equations $(M_T)_\lambda = \xi'_\lambda$ as:

$$||\phi^{(\mu 3)}||^2 - ||\phi^{(3 \mu)}||^2 = t_\mu \ (\mu = 0..2)$$

(76)

where $t_\mu$ are the components of the real vector $t = L^{-1} \xi'$. It is well-known that $\xi'$ is a regular level of the moment map if and only if the stabilizer of any point in $M^{-1}(\xi')$ is finite. The points $x \in M^{-1}(\xi')$ which do not satisfy this condition are stabilized by a $U(1)$ subgroup of $G$ and correspond (via the Higgs mechanism) to classical vacua admitting some unbroken gauge symmetry. Therefore, the nonregular values of $\xi'$ are precisely those values for which the classical moduli space contains points of enhanced
gauge symmetry. In our case, the above criterion immediately shows\[\] that $\xi'$ is a regular level of $M$ if and only if $\Pi_{\mu=0.2}\xi' \neq 0$. The set $\Pi_{\mu=0.2}\xi' = 0$ of singular levels coincides with the union of the coordinate planes in the space $\mathbb{R}^3(t)$ of all values of $t$. This divides $\mathbb{R}^3(t)$ into the octants $\Sigma_\epsilon$ ($\epsilon := (\epsilon_1, \epsilon_2, \epsilon_3)$, with $\epsilon_i = -1$ or $1$), which are just the cones $\Sigma_\epsilon = \langle \epsilon_1 e_1, \epsilon_2 e_2, \epsilon_3 e_3 \rangle_{\mathbb{R}^3}$, with $(\epsilon_i)_{i=1..3}$ the canonical basis of $\mathbb{R}^3$. Since $\xi' = Lt$, the space $\mathbb{R}^3(\xi')$ of values of $\xi'$ is similarly divided into the 8 cones $\sigma_\epsilon = \langle \epsilon_1 u_1, \epsilon_2 u_2, \epsilon_3 u_3 \rangle_{\mathbb{R}^3}$ ($\epsilon_i = \pm 1$), where $u_i = Le_i$ are the column vectors of $L$:

$$
\begin{align*}
  u_1 := \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}, & \quad u_2 := \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, & \quad u_3 := \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.
\end{align*}
$$

(77)

These cones are the GIT chambers of the ambient space in our situation. The critical values of the moment map are the points of the walls:

$$
W_1 = \langle u_1, u_2 \rangle_{\mathbb{R}}, \quad W_2 = \langle u_2, u_3 \rangle_{\mathbb{R}}, \quad W_3 = \langle u_3, u_1 \rangle_{\mathbb{R}},
$$

(78)

which give a system of 3 planes in $\mathbb{R}^3(\xi')$, intersecting at the origin and along the vectors $u_1, u_2, u_3$ (see Figure 6).

---

17 Introduce the notation $\phi(\mu^3) := \phi(\mu^+), \phi(3\mu) := \phi(\mu^-)$ and assume first that $t_1, t_2, t_3$ are all nonzero. Then let $\epsilon_\mu := \text{sign}(t_\mu) \in \{-1, 1\}$. If $\phi(\mu^+), \phi(\mu^-)$ satisfy \[\text{(76)},\] then we necessarily have $\|\phi(\mu^\ast)\| \neq 0$ for all $\mu = 0..2$. Such a point is fixed by an element $(\sigma_1..\sigma_3) \in U(1)^3$ if and only if $\Pi_{\lambda=1..3}\sigma_\lambda^{\mu\ast}(\lambda) = 1$ for all $\mu = 0..2$. Let $L^{-1} := \frac{1}{6}A$, where $A$ is the integral matrix in equation \[\text{(74)}.\] Then $LA = 6$, and taking products of the above equations for $\sigma$ shows that $\sigma_\lambda^6 = 1$ for all $\lambda = 1..3$. Hence the stabilizer of any point in the level set considered is a subset of $(C_6)^3$ (with $C_6$ the group of roots of unity of order 6), and therefore is finite. On the other hand, if some $t_\mu$ is zero, then one can easily construct a continuous subgroup of $T$ which fixes some point of the associated level set.
If $\xi'$ belongs to the cone $\sigma_\epsilon$ ($\epsilon = (\epsilon_0, \epsilon_1, \epsilon_2) \in \{-1,1\} \times \{-1,1\} \times \{-1,1\}$) then the exceptional set $Z_\xi'$ is given by:

$$Z_\xi' = \cup_{\mu=0,2} Z_{\epsilon\mu}^{(\mu)} = \{ \phi \in \mathcal{Q} | \phi(0,\epsilon_0) = \phi(1,\epsilon_1) = \phi(2,\epsilon_2) = 0 \} ,$$

(79)

where:

$$Z_{\epsilon\mu}^{(\mu)} = \{ \phi \in \mathcal{Q} | \phi(\mu,\epsilon_\mu) = 0 \} ,$$

(80)

(we use the notation $\phi(\mu) := \phi(\mu^+), \phi(3\mu) := \phi(\mu^-)$). It is not hard to see that the exceptional sets $Z_\epsilon$ intersect the variety of commuting matrices $Z$ along different loci, so that none of the walls is ‘projected out’ (or identified with another wall) upon restricting to $Z$. The defining equations (64) of the variety $Z$ can be solved (on a dense open subset) $^{18}$ which allows one to show that the quotient $Z/G_C$ is complex 3-dimensional, as expected.

\[18\] This can be achieved, for example, by using linear algebra arguments related to completeness relations for the left and right eigenvectors of the 3 by 3 matrices $v - u$, $v - \sigma^2 u$ and $v - \sigma u = 0$. 

---

Figure 6. The enlarged Kähler cone for $\mathbb{C}^3/\Delta_1(12)$.
4.3 The case $\Gamma := \Delta_2(24)$

The subgroup $\Delta_2(24)$ is given by the 12 matrices $A(p,q), C(p,q), E(p,q)$ of (51) together with the following 12 elements:

$$B(p,q) = \begin{bmatrix} (-1)^p & 0 & 0 \\ 0 & (-1)^q & 0 \\ 0 & 0 & (-1)^{p+q} \end{bmatrix}$$

$$D(p,q) = \begin{bmatrix} 0 & (-1)^p & 0 \\ (-1)^q & 0 & 0 \\ 0 & 0 & (-1)^{p+q} \end{bmatrix}$$

$$F(p,q) = \begin{bmatrix} 0 & 0 & (-1)^p \\ 0 & (-1)^q & 0 \\ (-1)^{p+q} & 0 & 0 \end{bmatrix},$$

(81)

where $p, q = 0, 1$.

This group is isomorphic to the more familiar symmetric group on four letters. The unprojected D-brane theory has a $U(24)$ gauge group and $3 \times 24 \times 24 = 1,728$ complex fields. Proceeding as before, we note that our orbifold group has five irreducible representations, which we denote by $R_0, R_1, R_2, R_3$ and $R_4$. The representations $R_0, R_1, R_2$ are induced from representations of the symmetric group on three letters via the isomorphism $\Delta_2(24)/\{A(p,q)\} \approx S_3$. $R_0$ is the trivial representation, $R_1$ is the one dimensional sign representation, and $R_2$ is the two dimensional triangle representation. $R_3$ is the three dimensional defining representation induced by the embedding of $\Delta_2(24)$ in $SU(3)$ via the matrices given above. $R_4$ is the three dimensional representation whose matrices are identical to those of $R_3$ except that $B(p,q), D(p,q)$, and $F(p,q)$ are multiplied by $-1$. $R_4$ is not special unitary, so we only consider $Q = R_3$ as the action of $\Delta_2(24)$ on $\mathbb{C}^3$.

$\Delta_2(24)$ has five conjugacy classes:

$$\begin{align*}
1C1 & = A(0,0) \\
3C2 & = A(0,1), A(1,0), A(1,1) \\
8C3 & = C(p,q), E(p,q), p, q \in 0, 1 \\
6C4 & = B(0,0), B(0,1), D(0,1), D(1,0), F(0,0), F(1,0) \\
6C5 & = B(1,0), B(1,1), D(0,0), D(1,1), F(0,1), F(1,1) ,
\end{align*}$$

(83)

where we use the standard group-theoretic notation $iCj$ for conjugacy classes, with $i
the number of elements of the class and \( j \) its label. The character table is:

| irrep | 1C1 | 3C2 | 8C3 | 6C4 | 6C5 |
|-------|-----|-----|-----|-----|-----|
| \( R_0 \) | 1   | 1   | 1   | 1   | 1   |
| \( R_1 \) | 1   | 1   | 1   | -1  | -1  |
| \( R_2 \) | 2   | 2   | -1  | 0   | 0   |
| \( R_3 \) | 3   | -1  | 0   | 1   | -1  |
| \( R_4 \) | 3   | -1  | 0   | -1  | 1   |

### 4.3.1 Branching rules and the quiver

Use of the characters above establishes the following branching rules:

\[
\begin{align*}
Q \otimes R_0 & \approx Q \\
Q \otimes R_1 & \approx R_4 \\
Q \otimes R_2 & \approx Q \oplus R_4 \\
Q \otimes R_3 & \approx R_0 \oplus R_2 \oplus Q \oplus R_4 \\
Q \otimes R_4 & \approx R_1 \oplus R_2 \oplus Q \oplus R_4 ,
\end{align*}
\]

(84)

which lead to the McKay quiver depicted below. The McKay coefficients are symmetric: \( a_{\mu \nu} = a_{\nu \mu} \), a property due to self-duality of the defining representation \( Q \). The quiver is depicted in the figure below. There are 72 complex fields surviving the projection conditions: \( 3 + 3 = 6 \) from the maps \( \phi^{(30)} \) and \( \phi^{(03)} \), \( 3 + 3 = 6 \) from the maps \( \phi^{(41)} \) and \( \phi^{(14)} \), 9 from \( \phi^{(33)} \), 9 from \( \phi^{(44)} \), \( 9 + 9 = 18 \) from \( \phi^{(43)} \) and \( \phi^{(34)} \), \( 6 + 6 = 12 \) from \( \phi^{(23)} \) and \( \phi^{(32)} \) and \( 6 + 6 = 12 \) from \( \phi^{(24)} \) and \( \phi^{(42)} \).
4.3.2 Clebsch-Gordan coefficients and the superpotential

The Clebsch-Gordan coefficients can be computed from the information given above by the method of projectors. In the case of $\Delta_2(24)$ there are no multiplicities $\alpha$. Our decompositions reduce to the simpler form:

$$Q \otimes R_\nu = \bigoplus_{\mu} R_\mu,$$

where $\mu$ runs over a subrange of 0..r. Again we chose the characters and matrices given above as the fiducial forms of our irreducible representations. Let $n_\mu$ be the dimension of $R_\mu$. Then for each $R_\mu$ appearing in (86), we define $n_\mu$ matrices $C_\mu^{(\nu \mu)}$ (of type $3 \times n_\nu$) by: $(C_\mu^{(\nu \mu)})_{jk} := C_{Q_\nu,\nu k}^{\mu i}$. These matrices can be computed by the method of projectors, with the result:

$$C_0^{(31)} = [1, 0, 0] \quad C_0^{(32)} = [0, 1, 0] \quad C_0^{(33)} = [0, 0, 1]$$

$$C_1^{(41)} = [1, 0, 0] \quad C_1^{(42)} = [0, 1, 0] \quad C_1^{(43)} = [0, 0, 1]$$

Figure 7. The quiver for $\Delta_2(24)$. 

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Applying the quiver Feynman rules gives the superpotential:

\[
W = \frac{3}{\sqrt{2}} \left[ Tr(\phi^{(33)} \phi^{(33)} \phi^{(33)}) + Tr(\phi^{(44)} \phi^{(44)} \phi^{(44)}) \right] \\
-3\sqrt{6} \left[ Tr(\phi^{(30)} \phi^{(33)} \phi^{(03)}) + Tr(\phi^{(41)} \phi^{(44)} \phi^{(14)}) \right] \\
+3\sqrt{3} \left[ Tr(\phi^{(42)} \phi^{(44)} \phi^{(24)}) - Tr(\phi^{(32)} \phi^{(33)} \phi^{(23)}) \right] \\
+9 \left[ Tr(\phi^{(32)} \phi^{(33)} \phi^{(24)}) + Tr(\phi^{(42)} \phi^{(34)} \phi^{(33)}) \right] \\
+ \frac{9}{\sqrt{2}} \left[ Tr(\phi^{(43)} \phi^{(44)} \phi^{(34)}) + Tr(\phi^{(34)} \phi^{(33)} \phi^{(43)}) \right].
\]

(92)
4.3.3 The variety of commuting matrices

Introducing the following notation:

\[ x = \phi^{(30)}, \quad y = \phi^{(41)}, \quad s = \phi^{(43)}, \]
\[ X = \phi^{(03)}, \quad Y = \phi^{(14)}, \quad S = \phi^{(34)}, \]
\[ a = \phi^{(32)}, \quad b = \phi^{(42)}, \quad U = \phi^{(33)}, \]
\[ A = \phi^{(23)}, \quad B = \phi^{(24)}, \quad V = \phi^{(44)}, \]

the superpotential becomes:

\[
W = \frac{3}{\sqrt{2}} \left[ \text{Tr}(U^3) + \text{Tr}(V^3) \right] - 3\sqrt{6} \left[ xUX + yVY \right] + 3\sqrt{3} \left[ \text{Tr}(bVB) - \text{Tr}(aUA) \right] + 9 \left[ \text{Tr}(asB) + \text{Tr}(bSA) \right] + \frac{9}{\sqrt{2}} \left[ \text{Tr}(sVS) + \text{Tr}(SUs) \right].
\]

Differentiating \( W \) leads to the F-flatness constraints:

\[
\frac{9}{\sqrt{2}} U^2 - 3\sqrt{6}X x - 3\sqrt{3}Aa + \frac{9}{\sqrt{2}} Ss = 0
\]
\[
\frac{9}{\sqrt{2}} V^2 - 3\sqrt{6}Y y - 3\sqrt{3}Bb + \frac{9}{\sqrt{2}} sS = 0
\]
\[
9Ba + \frac{9}{\sqrt{2}} S(V + U) = 0
\]
\[
9Ab + \frac{9}{\sqrt{2}} s(V + U) = 0
\]
\[
-3\sqrt{3}\sqrt{2}UX = 0 \quad -3\sqrt{3}\sqrt{2}VY = 0
\]
\[
-3\sqrt{3}\sqrt{2}xU = 0 \quad -3\sqrt{3}\sqrt{2}yV = 0
\]
\[
-3\sqrt{3}UA + 9sB = 0 \quad 3\sqrt{3}VB + 9SA = 0
\]
\[
-3\sqrt{3}aU + 9bS = 0 \quad 3\sqrt{3}bV + 9as = 0,
\]

whose solution set is the variety of commuting matrices \( \mathcal{Z} \).

4.3.4 The toric part of the quotient and the enlarged Kahler cone

The effective gauge group is given by:

\[ G = U(1)^2 \times U(2) \times U(3)^2 / U(1)_{\text{diag}} \approx S[U(1)^2 \times U(2) \times U(3)^2] / K, \]
where $S[U(1)^2 \times U(2) \times U(3)^2]$ is the group of quintuples

$$(g_0, g_1, g_2, g_3, g_4) \in U(1)^2 \times U(2) \times U(3)^2$$

such that

$$g_0 g_1 \det g_2 \det g_3 \det g_4 = 1$$

and $K \approx \mathbb{Z}_{10}$ is its central subgroup:

$$K = \{(\eta, \eta, \eta, \eta, \eta) | \eta^{10} = 1\}.$$ (101)

$G$ has a central subgroup $T = T_0/K$, where:

$$T_0 = \{(g_0, g_1, \delta, \epsilon, \gamma) \in U(1)^5 | g_0 g_1 \delta^2 \epsilon^3 \gamma^3 = 1\},$$ (102)

which corresponds to setting $g_2 = \delta 1_2$, $g_3 = \epsilon 1_3$, $g_3 = \gamma 1_3$. We have an isomorphism $T \approx U(1)^4/C$, given by:

$$(g_1, \delta, \epsilon, \gamma) \rightarrow (g_1^{-1} \delta^{-2} \epsilon^{-3} \gamma^{-3}, g_1, \delta, \epsilon, \gamma),$$ (103)

which corresponds to solving for $g_0$ in the constraint

$$g_0 g_1 \delta^2 \epsilon^3 \gamma^3 = 1.$$ (104)

$C \approx \mathbb{Z}_{10}$ is a central subgroup of $U(1)^4$ given by:

$$C = \{(\eta, \eta, \eta, \eta) | \eta^{10} = 1\}.$$ (105)

Writing $g_\lambda := e^{i \lambda}$, for $\lambda = 1, 4$, the subgroup $T \approx U(1)^4/\mathbb{Z}_{10}$ of $G$ acts on the quiver data as

$$\phi^{(\mu \nu)} \rightarrow e^{i \nu_{\lambda}} \phi^{(\mu \nu)}$$

$$\phi^{(\nu \mu)} \rightarrow e^{-i \mu_{\lambda}} \phi^{(\nu \mu)}$$

for $\mu \neq \nu$ while leaving $\phi^{(33)}$ and $\phi^{(44)}$ unchanged. If we relabel the maps $\phi^{(\nu \mu)}$ for $\mu \neq \nu$ as follows:

$$\phi^{(30)} = \phi^{(0)} \quad \phi^{(41)} = \phi^{(1)} \quad \phi^{(32)} = \phi^{(2)}$$

$$\phi^{(42)} = \phi^{(3)} \quad \phi^{(43)} = \phi^{(4)}$$

(107)
we can re-express the group action as
\[
\phi^{(\rho)} \rightarrow e^{i l^{(\lambda)}_{\rho} s_{\lambda}} \phi^{(\rho)}
\]
(108)
and obtain the charge matrix
\[
L := (l^{(\lambda)}_{\rho})_{\lambda=1..4, \rho=0..4}:
\]
\[
L = \begin{bmatrix}
-1 & 0 & 1 & 1 & 0 \\
-2 & 1 & 0 & 0 & 0 \\
-4 & 0 & -1 & 0 & 1 \\
-3 & -1 & 0 & -1 & -1
\end{bmatrix}.
\]
(109)
(Here \(\lambda\) is the row index and \(\rho\) is the column index.)

Define the vector
\[
e^{\rho} = [||\phi^{(\rho)}||^2 - ||\phi^{(\bar{\rho})}||^2],
\]
where \(\phi^{(\bar{\rho})}\) represents the arrow going in the opposite direction of \(\phi^{(\rho)}\):
\[
\phi^{(\rho)} := \phi^{(\mu\nu)} \Rightarrow \phi^{(\bar{\rho})} := \phi^{(\nu\mu)}
\]
(111)
The moment map for our (effective) \(U(1)^4/\mathbb{Z}_{10}\) action is given by:
\[
M_{\lambda} = \sum_{\rho=0..4} l^{(\lambda)}_{\rho} e^{\rho}.
\]
(112)
The charge matrix \(L\) does not have a left inverse, so we cannot immediately apply the methods of the earlier example. Instead first pick a level \(\xi' = (\xi^{(\lambda)}_{\rho})_{\lambda=1..4} \in \mathbb{R}^4\) and bring the terms containing \(e^4\) in each sum to the left hand side of the equation. Then the moment map equations become:
\[
\xi'_{\lambda} - l^{(\lambda)}_{4} e^4 = \sum_{\rho=0..3} l^{(\lambda)}_{\rho} e^{\rho}.
\]
(113)
The square part of \(L\), for \(\rho = 0..3\), denoted \(\bar{L}\), does possess an inverse:
\[
\bar{L} = \begin{bmatrix}
-1 & 0 & 1 & 1 \\
-2 & 1 & 0 & 0 \\
-4 & 0 & -1 & 0 \\
-3 & -1 & 0 & -1
\end{bmatrix}
\]
(114)
\[
\bar{L}^{-1} = \frac{1}{10} \begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & -8 & 2 & 2 \\
-4 & -4 & 6 & -4 \\
-5 & 5 & -5 & 5
\end{bmatrix}.
\]
(115)
We then obtain the following equations:

\[
\frac{1}{10}(\xi'_1 + \xi'_2 + \xi'_3 + \xi'_4) = \epsilon^0 \\
\frac{1}{10}(2\xi'_1 - 8\xi'_2 + 2\xi'_3 + 2\xi'_4) = \epsilon^1 \\
\frac{1}{10}(-4\xi'_1 - 4\xi'_2 + 6\xi'_3 - 4\xi'_4) = \epsilon^2 - \epsilon^4 \\
\frac{1}{10}(-5\xi'_1 + 5\xi'_2 - 5\xi'_3 + 5\xi'_4) = \epsilon^3 + \epsilon^4.
\]

(116)

In particular, there are no constraints on \(\epsilon^4\). We will therefore have a conical singularity at the point in the moduli space corresponding to \(\phi^{(4)} = \phi^{(\bar{4})} = 0\), which is a fixed point for a single \(U(1) \subset U(1)^4\), independent of the value of \(\xi'^4\).

A chamber \(\Sigma \subset \mathbb{R}^4(\xi')\) is such that for all \(\xi' \in \Sigma\), a maximal number of \(\epsilon^i \neq 0\). In our case, we can only enforce such a restriction on \(\epsilon^0\) and \(\epsilon^1\) via the first two equations of (116). Our chambers will correspond to four cases:

\[
(I) \quad \epsilon^0 > 0 \quad \epsilon^1 > 0 \\
(II) \quad \epsilon^0 > 0 \quad \epsilon^1 < 0 \\
(III) \quad \epsilon^0 < 0 \quad \epsilon^1 > 0 \\
(IV) \quad \epsilon^0 < 0 \quad \epsilon^1 < 0
\]

The chamber walls are determined by the union of the two hyperplanes obtained by setting \(\epsilon^0\) and \(\epsilon^1\) to zero in the first two equations of (116):

\[
\xi'_0 + \xi'_1 + \xi'_2 + \xi'_3 = 0 \quad (117) \\
2\xi'_0 - 8\xi'_1 + 2\xi'_2 + 2\xi'_3 = 0. \quad (118)
\]

The normal vectors of these hyperplanes can be written as \(a_0 = (-1,-1,-1,-1)\) and \(a_1 = (-1,4,-1,-1)\). The hyperplanes intersect along a two-plane spanned by the vectors (1,0,0,−1) and (0,0,1,−1). The four regions of \(\mathbb{R}^4(\xi')\) bounded by these hyperplanes correspond to the GIT chambers in this example.

We can read the chamber conditions in terms of the maps \(\phi^p\). In chamber (I), both \(\epsilon^0\) and \(\epsilon^1\) are strictly positive. This implies that \(||\phi^{(0)}||^2\) and \(||\phi^{(1)}||^2\) are nonzero. Other choices for the signs of \(\epsilon^0\) and \(\epsilon^1\) result in the remaining chambers. Since \(\epsilon^4\) cannot be fixed in such a manner, we are free to have either or both of \(||\phi^{(4)}||^2\) and \(||\phi^{(\bar{4})}||^2\) equal to zero. At any point in the moduli space corresponding both of these 3 × 3 matrices being zero, 18 of our 72 complex field variables vanish and the space develops a conical singularity.

\(19\) This \(U(1)\) subgroup is given by setting \(s_1 = s_3 = -s_2 = -s_4\) in equation (108).
5 Conclusions and directions for further research

The moduli space of D-branes placed at quotient singularities is an important and fascinating subject, with numerous physical and mathematical ramifications. In this paper, we have focused on only a few aspects of this topic; clearly much remains to be resolved.

One of the most obvious problems to address is that of explicitly computing the moduli spaces of our theories. This is related to similar problems in the context of arbitrary supersymmetric field theories in a given dimension [31] and to recent efforts [32] to systematize the computation of classical moduli spaces of supersymmetric field theories with 4 supercharges. This is a prerequisite, for example, for understanding the phenomena of [20] in a more general context. In fact, the phenomenon of topology change is ubiquitous for the moduli space of supersymmetric field theories with unbroken $U(1)$ factors of the gauge group, and deserves a much better understanding. The determination of the classical moduli space can be handled by methods of algebraic geometry, and can be reduced to the problem of computing invariants in the sense of constructive invariant theory [23]. This is essentially an algorithmic problem, albeit it can often be prohibitively intensive from a computational point of view.

Another topic we have not discussed is the more general subject of quiver field theories. By this we mean supersymmetric field theories with a matter content chosen to transform in an arbitrary representation of a quiver (in the sense of [15, 16]). Particular classes of such theories have been considered in [33], and they exhibit rather unusual properties. The mathematical theory of representations of quivers has been considered in [16] and in the more recent mathematical literature [34], and has deep connections with the representation theory of algebras. It would be interesting to understand the relevance of this relation from a physical point of view. Another subject worthy of consideration is the connection between such representations and noncommutative geometry, along the lines of [35], as part of the more general philosophy according to which turning on Fayet-Iliopoulos terms is equivalent to making a noncommutative deformation of the base space. The relevance of noncommutative geometry to matrix theory has been pointed out in a series of recent papers [36].

On a more abstract level, it would be interesting to have a better understanding of the ‘location’ of our brane-theoretic resolutions of singularities among the crepant resolutions guaranteed (for Gorenstein singularities of 3-folds) by the results of [37]. That is, we would like to know which resolutions are realized by the D-brane mechanism out of the multitude of crepant resolutions which usually can be performed on a Gorenstein quotient singularity. This would give us a clearer picture of the way in which the D-brane effective field theory ‘projects out’ certain phases which are otherwise permitted by classical geometry.

Finally, one issue of major physical relevance is to what extent one can use our results and methods in order to extract information on the Maldacena limit of D-brane effective theories placed at various conical singularities which can be obtained from a quotient singularity by performing partial resolutions. A satisfactory answer to this
question may provide a way of generalizing the work of \[19\] to the case of nonabelian quotient singularities, thus adding new strands to the web of connections between supersymmetric field theories, supergravities and string theory which is becoming increasingly apparent.

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A The double symplectic quotient

In order to justify our two-step procedure for taking the Kähler quotient, it suffices to consider the symplectic part of the problem (the fact that the complex structures agree is immediate).

Suppose then that one has a symplectic manifold \((X, \omega)\) and a hamiltonian Lie group action on \((M, \omega)\) with (equivariant) moment map \(M : X \to g\), where \(g\) is the Lie algebra of \(G\). Picking a central Lie subgroup \(T\) of \(G\), its Lie algebra \(t\) is a subalgebra of \(g\). By using the Killing form of \(g\), the Lie algebra \(h = g/t\) of the quotient group \(H := G/T\) can be identified with the orthogonal complement of \(t\). Let \(P\) and \(Q\) be the projectors on the orthogonal subspaces \(t\) and \(h\) of \(g\) (these are orthoprojectors with respect to the Killing form of \(g\)). The action of \(T\) on \(X\) induced from the action of \(G\) is hamiltonian with moment map \(M_T\) given by the projection of \(M\) on \(t\):

\[
M_T = P \circ M : X \to t \quad . \tag{119}
\]

Choosing a level \(\xi \in t\), one can consider the symplectic reduction \(X_T(\xi) := X_T(\xi)/T\) (with \(X_T(\xi) := M_T^{-1}(\xi)\)) at level \(\xi\), endowed with the induced symplectic form \(\omega_T\). If \(\pi_T : X_T(\xi) \to X_T(\xi)\) is the natural projection, then \(\omega_T\) is uniquely determined by the condition \(\pi_T^*(\omega_T) = \omega|_{X_T(\xi)}\). Since \(T\) is a central subgroup of \(G\), we have an induced action of \(H\) on \(X_T(\xi)\), which is easily seen to be hamiltonian with respect to \(\omega_T\). Standard arguments show that its moment map is given by:

\[
M_H \circ \pi_T = Q \circ M : M_T(\xi) \to h \quad . \tag{120}
\]

Further reduction with respect to \(H\) at level zero gives a manifold \(\overline{X}_T(\xi)_H(0) := \overline{X}_T(\xi)_H(0)/H\), where \(\overline{X}_T(\xi)_H(0) := M_H^{-1}(0)\). Denote the natural projection by \(\pi_H : \overline{X}_T(\xi)_H(0) \to \overline{X}_T(\xi)_H(0)\) and the induced symplectic form by \(\omega_T)_H\).

On the other hand, one can consider the symplectic reduction \(\overline{M}_G(\xi) = M_G(\xi)/G\) \((M_G(\xi) := M^{-1}(\xi))\) of \(M\) by \(G\), with projection \(\pi_G : M_G(\xi) \to \overline{M}_G(\xi)\) and induced

\[\text{We always use the Killing form } \langle , \rangle \text{ of } g \text{ to identify it with } g^*.\]
symplectic form $\omega_G$. One clearly has $\pi_T(M_G(\xi)) = \overline{\pi T}(\xi)_H(0)$, which upon projecting by $\pi_H$ gives a diffeomorphism $\phi : M_G(\xi) \to \overline{X_T(\xi)}_H(0)$, which is easily seen to map $\omega_G$ into $(\omega_T)_H$. Hence we have a symplectic isomorphism:

\[ (\overline{M_G(\xi)}, \omega_G) \approx (\overline{X_T(\xi)}_H(0), (\omega_T)_H) \]  

for any level $\xi \in \mathfrak{t}$.

B The method of projectors

Consider a hermitian vector space $(W, <,>)$ carrying a unitary representation $\rho : \Gamma \to U(W, <,>)$ of a finite group $\Gamma$. Let $\rho_\mu (\mu = 0..r)$ be the irreducible representations of $\Gamma$, and let $n_\mu$ be their dimensions. Pick a set of preferred realizations of these abstract irreducible representations by unitary matrices $D^{(\mu)}(\gamma) \in U(n_\mu)$.

Consider the decomposition of $\rho$ into irreducible representations of $\Gamma$, realized by the orthogonal direct sum decomposition:

\[ W = \bigoplus_\mu W_\mu \]  

of $W$ into $\Gamma$-invariant subspaces, where the restriction of $\rho$ to each $W_\mu$ is equivalent to $a_\mu$ copies of $\rho_\mu$.

Assuming that one knows the multiplicities $a_\mu$, and given an orthonormal basis $(e_s)_{s=1..n}$ of $W$, the method of projectors allows one to determine fiducial orthonormal bases $(e^{(\mu, \alpha)}_i)_{i=1..a_\mu}$ of $W_\mu$, i.e. orthonormal bases satisfying the condition: $\rho(\gamma)e^{(\mu, \alpha)}_i = D^{(\mu)}(\gamma)e^{(\mu, \alpha)}_i$. In practice, the method produces the explicit expressions:

\[ e^{(\mu, \alpha)}_i = C^{(\mu\alpha)}_{\nu\lambda} e^{(\nu)}_j \otimes e^{(\lambda)}_k, \]  

of such basis elements in terms of the given basis $(e_s)_{s=1..n}$ of $W$. In particular, if $(W, \rho)$ is a tensor product of two irreducible representations $(R_\nu, \rho_\nu)$ and $(R_\lambda, \rho_\lambda)$ of $\Gamma$, and if one choses the basis of $W$ to be given by $e_{jk} := e^{(\nu)}_j \otimes e^{(\lambda)}_k$, with $e^{(\nu)}_j, e^{(\lambda)}_k$ fiducial bases for $R_\nu, R_\lambda$, then the above expression becomes:

\[ e^{(\mu, \alpha)}_i = C^{(\mu_i\lambda\alpha)}_{\nu j, \lambda k} e^{(\nu)}_j \otimes e^{(\lambda)}_k, \]  

so that the method can be used to compute the Clebsch-Gordan coefficients $C^{(\mu\alpha)}_{\nu\lambda}$.

For the general case of an arbitrary representation $(W, \rho)$, a set of fiducial basis vectors $e^{(\mu, \alpha)}_i$ can be determined as follows:

**Step 1:** Define the linear operators:

\[ P^{(\mu)}_{ji} := \frac{n_\mu}{|\Gamma|} \sum_{\gamma \in \Gamma} D^{(\mu)}_{ji}(\gamma)^* \rho(\gamma) \]  

(125)
where $\ast$ denotes complex conjugation and $|\Gamma|$ is the order of $\Gamma$. The orthogonality relations for the matrices $D^{(\mu)}(\gamma)$ show that $P^{(\mu)}_{ji}$ satisfy the operator relations:

$$
P^{(\mu)}_{kl} P^{(\nu)}_{ji} = \delta_{\mu\nu} \delta_{lj} P^{(\mu)}_{ki},
\sum_i P^{(\mu)}_{ii} = \text{id}_W,
$$

(126)

while unitarity of $D^{(\mu)}(\gamma)$ and $\rho(\gamma)$ imply:

$$
(P^{(\mu)})^+ = P^{(\mu)}.
$$

(127)

In particular, the operators $P^{(\mu)}_{ii}$ form a complete set of orthogonal projectors of $(W, <, >)$.

**Step 2:** For each $\mu$ appearing in the decomposition of $W$ into irreducible representations, consider the space $W^{(\mu)}_1 := P^{(\mu)}_1 (W) = \ker P^{(\mu)}_1$ onto which $P^{(\mu)}_1$ projects. The dimensionality of this space coincides with the multiplicity $a_\mu$ of the irreducible representation $R^{(\mu)}$ in $W$. Compute an arbitrary orthonormal basis $(e^{(\mu,\alpha)}_1)_{\alpha=1}^{a_\mu}$ of $W^{(\mu)}_1$.

**Step 3:** Compute $e^{(\mu,\alpha)}_i := P^{(\mu,\alpha)}_{i1} e^{(\mu,\alpha)}_1$, for each $\alpha = 1..a_\mu$ and $i = 2..n_\mu$. Then the vectors $(e^{(\mu,\alpha)}_i)_{\alpha=1}^{a_\mu,i=1}^{n_\mu}$ form a fiducial basis of the subspace $W^{(\mu)}_\mu$ (in particular, each subspace $W^{(\mu)}_\mu$ is determined as the span of these vectors).

The proof of the above statements consists of direct verifications and can be found for example in [38]. In practice, the choice of a basis $(e_s)_{s=1}^{n}$ presents $W$ as the vector space $\mathbb{C}^n$ and identifies $e_s$ with the canonical basis vectors of $\mathbb{C}^n$. In this case, the representation $\rho$ is given by a set of $n$ by $n$ unitary matrices $\rho(\gamma)$ and the operators $P^{(\mu)}_{ji}$ given by [125] are identified with $n$ by $n$ matrices. Then the fiducial basis elements $e^{(\mu,\alpha)}_i$ produced by the above algorithm are realized as column vectors in $\mathbb{C}^n$. Since their expression in the canonical basis is simply given by their entries, it follows that $C^{(\mu,\alpha)}_s$ is given by the entry number $s$ of the column vector $e^{(\mu,\alpha)}_i$. 

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