AFFINE GRASSMANNIANS IN $\mathbb{A}^1$-ALGEBRAIC TOPOLOGY

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Abstract. Let $k$ be a field. Denote by $\text{Spc}(k)_*$ the unstable, pointed motivic homotopy category and by $R^A\Omega_{G_m} : \text{Spc}(k)_* \to \text{Spc}(k)_*$ the $(\mathbb{A}^1$-derived) $G_m$-loops functor. For a $k$-group $G$, denote by $\text{Gr}_G$ the affine Grassmannian of $G$. If $G$ is isotropic reductive, we provide a canonical motivic equivalence $R^A\Omega_{G_m}G \simeq \text{Gr}_G$. We use this to compute the motive $M(R^A\Omega_{G_m}G) \in \mathcal{D}(k, \mathbb{Z}[1/e])$.

1. Introduction

This note deals with the subject of $\mathbb{A}^1$-algebraic topology. In other words it deals with with the $\infty$-category $\text{Spc}(k)$ of motivic spaces over a base field $k$, together with the canonical functor $\text{Sm}_k \to \text{Spc}(k)$ and, importantly, convenient models for $\text{Spc}(k)$. Since our results depend crucially on the seminal papers [1, 2], we shall use their definition of $\text{Spc}(k)$ (which is of course equivalent to the other definitions in the literature): start with the category $\text{Sm}_k$ of smooth (separated, finite type) $k$-schemes, form the universal homotopy theory on $\text{Sm}_k$ (i.e. pass to the $\infty$-category $\mathcal{P}(\text{Sm}_k)$ of space-valued presheaves on $k$), and then impose the relations of Nisnevich descent and contractibility of the affine line $\mathbb{A}^1_k$ (i.e. localise $\mathcal{P}(\text{Sm}_k)$ at an appropriate family of maps).

The $\infty$-category $\text{Spc}(k)$ is presentable, so in particular has finite products, and the functor $\text{Sm}_k \to \text{Spc}(k)$ preserves finite products. Let $* \in \text{Spc}(k)$ denote the final object (corresponding to the final $k$-scheme $k$); then we can form the pointed unstable motivic homotopy category $\text{Spc}(k)_* := \text{Spc}(k)/_*$. It carries a symmetric monoidal structure coming from the smash product. Thus, for any $P \in \text{Spc}(k)_*$ we have the functor $P \wedge : \text{Spc}(k)_* \to \text{Spc}(k)_*$. By abstract nonsense, this functor has a right adjoint $\Omega_P : \text{Spc}(k)_* \to \text{Spc}(k)_*$, called the $(\mathbb{A}^1$-derived) $P$-loops functor.

For us, the most important instance of this is when $P = \mathbb{G}_m$ corresponds to the pointed scheme $\mathbb{G}_m := (\mathbb{A}^1 \setminus \{0, 1\}) \in \text{Sm}_k$. Indeed studying the functor $\Omega_{\mathbb{G}_m}$ is one of the central open problems of unstable motivic homotopy theory, since it is crucial in the passage from unstable to stable motivic homotopy theory. (The functor $\Omega_{\mathbb{G}_1}$ is similarly important but much better understood.) The main contribution of this note is the computation of $\Omega_{\mathbb{G}_m}G$, where $G$ is (the image in $\text{Spc}(k)_*$ of) an appropriate group scheme, as corresponding via the functor $(\text{Sm}_k)_* \to \text{Spc}(k)_*$ to a certain ind-variety known as the affine Grassmannian $\text{Gr}_G$:

$$\Omega_{\mathbb{G}_m}G \simeq \text{Gr}_G.$$

For a definition of $\text{Gr}_G$, see [18] or Section 3. For us, the main points are as follows: there exists a pointed presheaf of sets $\text{Gr}_G \in \text{Pre}(\text{Aff}_k)$ (where $\text{Aff}_k$ is the category of all affine $k$-schemes) which is in fact an fpqc sheaf. Moreover, in the category $\text{Pre}(\text{Aff}_k)$, the sheaf $\text{Gr}_G$ is a filtered colimit

$$X_1 \to X_2 \to \cdots \to \text{Gr}_G,$$

where each $X_i$ is (the presheaf represented by) a finite type (but in general highly singular) $k$-scheme.

Classical analog. Our result (yet to be stated precisely) has the following classical analog. Suppose that $k = \mathbb{C}$. Then the complex points $\text{Gr}_G(\mathbb{C})$ can be given the structure of a topological space, namely the colimit of the spaces $X_i(\mathbb{C})$ (with their strong topology). Then $\text{Gr}_G(\mathbb{C})$ is homeomorphic to the so-called polynomial loop-Grassmannian $\text{Gr}_0^{\mathbb{C}}$ of the Lie group $G(\mathbb{C})$ [14, 7.2(i)]. This space is homotopy equivalent to the space of smooth loops $\Omega^m(G(\mathbb{C})')$, where $G(\mathbb{C})'$ is the compact form of $G(\mathbb{C})$ [14, Proposition 8.6.6, Theorem 8.6.2], which itself is well-known to be homotopy equivalent to the usual loop space $\Omega(G(\mathbb{C})')$. Finally since $G(\mathbb{C}) \simeq G(\mathbb{C})$ (by the Iwasawa decomposition) we have $\Omega(G(\mathbb{C})') \simeq \Omega(G(\mathbb{C}))$. Putting everything together, we have found that

$$\text{Gr}_G(\mathbb{C}) \cong \text{Gr}_0^{\mathbb{C}} \simeq \Omega^m(G(\mathbb{C})') \simeq \Omega(G(\mathbb{C})') \simeq \Omega(G(\mathbb{C})).$$

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Main result. In order to state our result precisely, we need to make sense of the “image of \( \text{Gr}_G \) in \( \text{Spc}(S)_m \)”. For this we use that the functor \( \text{Sm}^\text{aff}_k \subset \text{Sm}_k \rightarrow \text{Spc}(k) \) extends, by construction, to a functor \( \mathcal{P}(\text{Sm}^\text{aff}_k) \rightarrow \text{Spc}(k) \), and that we have a fully faithful inclusion \( \text{Pre}(\text{Sm}^\text{aff}_k) \subset \mathcal{P}(\text{Sm}^\text{aff}_k) \). Here \( \text{Sm}^\text{aff}_k = \text{Sm}_k \cap \text{Aff}_k \). We thus obtain a functor
\[
\rho : \text{Pre}(\text{Aff}_k) \rightarrow \text{Pre}(\text{Sm}^\text{aff}_k) \rightarrow \mathcal{P}(\text{Sm}^\text{aff}_k) \rightarrow \text{Spc}(k),
\]
and we also denote by \( \rho \) the pointed version \( \text{Pre}(\text{Aff}_k)_* \rightarrow \text{Spc}(k)_* \). This finally allows us to state our main result. For the somewhat technical notion of isotropic groups, see [2, Definition 3.3.5]. This includes in particular all split groups.

Theorem (See Theorem 20). Let \( k \) be an infinite* field and \( G \) an isotropic reductive \( k \)-group. Then we have a canonical equivalence
\[
\Omega_{G_m} G \simeq \rho(\text{Gr}_G)
\]
in \( \text{Spc}(k)_* \).

Organisation and further results. In Section 2 we study the interaction of \( \text{Sing}_a \) and various models of \( \Omega_{G_m} \). Combining this with results of [2], we obtain a preliminary form of our main computation (see Proposition 11): \( \Omega_{G_m} G \) is motivically equivalent to the presheaf
\[
X \mapsto G(X[t, t^{-1}])/G(X).
\]

In Section 3, we review affine Grassmannians. We make no claims to originality here. The main point is this: \( \text{Gr}_G \) is usually defined as the fpqc sheafification of the presheaf \( X \mapsto G(X((t)))/G(X[t]) \). We show that at least over an infinite field, and assuming that \( G \) is split, this is isomorphic to the Zariski sheafification of (2); see Proposition 13. We also prove that this is an isomorphism on sections over smooth affine schemes, for any field \( k \), and only assuming that \( G \) is isotropic; see Proposition 19. This is enough for our eventual application.

In Section 4, we first deduce the main theorem. This is trivial by now, since Zariski sheafification is a motivic equivalence. After that we explore some consequences. We show in Corollary 24 that if \( k \) is perfect, then the \( \mathbb{Z}[1/e] \)-linear motive of \( \rho(\text{Gr}_G) \simeq \Omega_{G_m} G \) is in fact the filtered colimits of the motives of the singular varieties \( X_t \) from (1). Since the geometry of the \( X_t \) is well-understood, this allows us in Corollary 25 to determine the motive of \( \Omega_{G_m} G \).

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Language and models. Throughout this note, we frequently switch between various models for motivic homotopy theory. Section 2 is written in the language of simplicial presheaves and model categories. This is because our manipulations here are essentially on a point-set level. In particular we employ an appropriate localisation of the injective local model structure on \( \text{sPre}(\text{Sm}^\text{aff}_k) \) as our model for \( \text{Spc}(k) \). Section 3 deals exclusively with presheaves of sets, this time on \( \text{Aff}_k \), reflecting its essentially geometric nature. Finally Section 4 is written in the language of \( \infty \)-categories, since we find our manipulations there are most easily understood in this abstract, model-independent framework.

Notation. If \( C \) is a small 1-category, we write \( \text{Pre}(C) \) for the 1-category of presheaves of sets on \( C \), we write \( \text{sPre}(C) \) for the 1-category of presheaves of simplicial sets on \( C \), and we write \( \mathcal{P}(C) \) for the \( \infty \)-category of presheaves of spaces on \( C \).

2. \( \mathbb{G}_m \)-Loops of Groups

Let \( C \) be an essentially small 1-category with finite products. We write \( * \in C \) for the final object. Throughout we fix \( G \in C_* := C_{*/} \). We write \( \text{sPre}(C) \) for the 1-category of simplicial presheaves on \( C \) and \( \text{sPre}(C)_* := \text{sPre}(C)_{*/} \) for the pointed version. This admits an injective model structure where the weak equivalences are given objectwise, and the cofibrations are the monomorphisms [8, Theorem II.5.8]. We note that the canonical map \( * \rightarrow G \in C \) has a section, so is a monomorphism; in particular \( G \in \text{sPre}(C)_* \) is a cofibrant object.

We fix a further object \( A \in C \) together with a map \( G \rightarrow A \). We call \( X \in \text{sPre}(C) \) \( A \)-invariant if for all \( c \in C \), the canonical map \( X^c \rightarrow X(A \times c) \) is a weak equivalence.

*Throughout this note, we make frequent reference to [2]. The main results there are stated only for infinite fields. However they also apply to finite fields [personal communication], and an update will appear soon. In this note, whenever we assume that a field is infinite only because of this reason, we denote this as “infinite”.*
Example 1. The case we have in mind is, of course, where \( C = \operatorname{Sm}_S, G = \mathbb{G}_m \), and \( A = \mathbb{A}^1 \).

Let us recall that the functor \( \operatorname{sPre}(C)_\ast \to \operatorname{sPre}(C)_\ast \), \( X \to G \land X \) has a right adjoint \( \Omega_G^{\text{naive}} : \operatorname{sPre}(C)_\ast \to \operatorname{sPre}(C)_\ast \). It is specified in formulas by asserting that the following square is cartesian (which in general need not imply that it is homotopy cartesian)

\[
\begin{array}{ccc}
\Omega_G^{\text{naive}}(X)(c) & \longrightarrow & X(G \times c) \\
\downarrow & & \downarrow i^\ast \\
* & \longrightarrow & X(c).
\end{array}
\]

Here \( i : * \to G \) is the canonical pointing, as is \( j : * \to X \). Since \( G \) is cofibrant, the functor \( \Omega_G^{\text{naive}} \) is right Quillen (in the injective model structure we are using), and hence admits a total derived functor \( R\Omega_G \) which can be computed as \( R\Omega_G X \simeq \Omega_G^{\text{naive}} R_f X \), where \( R_f \) is a fibrant replacement functor.

Remark 2. We denote the underived functor by \( \Omega_G^{\text{naive}} \) instead of just \( \Omega_G \) in order to make its point set level nature notationally explicit.

Remark 3. Even if \( X \) is objectwise fibrant (i.e. projective fibrant), it need not be injective fibrant. Indeed a further condition for injective fibrancy is that for any monomorphism \( c \to d \in C \), the induced map \( X(d) \to X(c) \) must be a fibration. In particular \( X(G \times c) \to X(c) \) is a fibration, and we deduce from right properness of the model structure on simplicial sets [6, Corollaries II.8.6 and II.8.13] that for any \( X \in \operatorname{sPre}(C)_\ast \), the following diagram is homotopy cartesian:

\[
\begin{array}{ccc}
R\Omega_G(X)(c) & \longrightarrow & X(G \times c) \\
\downarrow & & \downarrow i^\ast \\
* & \longrightarrow & X(c).
\end{array}
\]

Since \( G \) is not projective cofibrant (in general), the functor \( \Omega_G^{\text{naive}} \) is not right Quillen in the projective model structure. In order to derive it in the projective setting, we first have to cofibrantly replace \( G \), for example by the cone \( G \) on \( * \to G \). Of course then \( \Omega_G^{R^{proj}} X \simeq R\Omega_G X \).

Now suppose that \( G \in \operatorname{sPre}(C) \) is a presheaf of simplicial groups. Then \( G \) has a canonical pointing, given by the identity section. Thus \( G \in \operatorname{sPre}(C)_\ast \), in a canonical way.

Definition 4. We denote by \( \Omega_G^{gr}(G \in \operatorname{sPre}(C) \) the simplicial presheaf

\[
\Omega_G^{gr}(G)(c) = G(G \times c)/p^\ast G(c),
\]

where \( p : G \times c \to c \) denotes the projection. We define a further variant

\[
\Omega_G^{gr,A}(G)(c) = G(G \times c)/i^\ast G(A \times G),
\]

where \( i : G \to A \) is the canonical map.

Since \( p \) has a section, \( p^\ast \) is injective and identifies \( G(c) \) with a subgroup of \( G(G \times c) \), so we will drop \( p^\ast \) from the notation. Clearly \( \Omega_G^{gr}(G), \Omega_G^{gr,A}(G) \) are functorial in the presheaf of simplicial groups \( G \). Note that unless \( G \) is abelian, \( \Omega_G^{gr}(G), \Omega_G^{gr,A}(G) \) are not a priori a presheaves of groups. Note also that \( G(c) \subset G(A \times c) \), and hence there is a canonical surjection \( \Omega_G^{gr}(G) \to \Omega_G^{gr,A}(G) \).

Definition 5. We call \( G \in \operatorname{sPre}(C) \) a presheaf of simplicial groups, \( (G,A) \)-injective if for each \( c \in C \), the restriction \( G(A \times c) \to G(G \times c) \) is injective.

Proposition 6. Let \( G \in \operatorname{sPre}(C)_\ast \) be a presheaf of simplicial groups, canonically pointed by the identity.

1. There is a canonical isomorphism \( \Omega_G^{\text{naive}}(G) \to \Omega_G^{gr}(G) \).
2. The canonical map \( \Omega_G^{\text{naive}}(G) \to R\Omega_G(X) \) is an objectwise weak equivalence.
3. Suppose that \( G \) is \( \ast \)-invariant and \( (G,A) \)-injective. Then the canonical map \( \alpha : \Omega_G^{gr}(G) \to \Omega_G^{gr,A}(G) \) is an objectwise weak equivalence.

Proof. (1) We have for \( c \in C \) the canonical map

\[
\alpha_c : \Omega_G^{\text{naive}}(G)(c) = \ker (G(G \times c) \to G(c)) \to G(G \times c) \to G(G \times c)/G(c) = \Omega_G^{gr}(G)(c).
\]

These fit together to form a canonical map \( \Omega_G^{\text{naive}}(G) \to \Omega_G^{gr}(G) \), which we claim is an isomorphism. Write \( j^\ast : G(G \times c) \to G(c) \) for pullback along \( * \to G \). Define a map of sets \( \beta : G(G \times c) \to G(G \times c) \) via \( \beta(g) = (p^\ast j^\ast g)^{-1} g \). If \( a \in G(c) \), then \( \beta(aq) = (p^\ast j^\ast (aq))^{-1} aq = (ap^\ast j^\ast g)^{-1} aq = \beta(g) \). Furthermore \( p^\ast \beta(g) \) is the identity element of \( G(c) \), by construction. It follows that \( \beta \) takes values in \( \Omega_G^{\text{naive}}(G)(c) \) and
factors through the surjection $G \times c \to \Omega^\text{gr}_G(G)(c)$ to define $\beta : \Omega^\text{gr}_G(G)(c) \to \Omega^\text{naive}_G(G)(c)$. We check immediately that $\beta$ is inverse to $\alpha$.

(2) Since $j : * \to G$ has a section (* being final), the induced map $j^* : G \times c \to G(c)$ is a surjection of simplicial groups, and hence a fibration [6, Corollary V.2.7]. It follows from right properness of the model structure on simplicial sets [6, Corollary II.8.6] that

$$\Omega^\text{naive}_G(G)(c) = \text{fib}(G \times c \to G(c)) \simeq \text{hofib}(G \times c \to G(c)) \simeq R\Omega_G(G)(c);$$

see Remark 3 for the last weak equivalence. Thus the canonical map is indeed an objectwise weak equivalence.

(3) is an immediate consequence of Lemma 7 below (applied with $G_* = G'_* = G(G \times c)$, $H_* = G(c), H'_* = G(A \times c)$).

\begin{lemma}
Let $\theta : G_* \to G'_*$ be a homomorphism of simplicial groups and $H_* \subset G_*, H'_* \subset G'_*$ simplicial subgroups such that $\theta(H_*) \subset H'_*$. If each of the maps $\theta : G_* \to G'_*$ and $\theta : H_* \to H'_*$ are weak equivalences, then so is the induced map $G/H_* \to G'/H'_*$.
\end{lemma}

\begin{proof}
We have $G/H_* \simeq \text{hocolim}_{B,H} G_*$, since the action is free. Since the right hand side only depends on $G_*$ and $H_*$ up to weak equivalence, the result follows.

We can make the above slightly sketchy argument precise as follows. Write $G_*$ for $G_*$ viewed as a bisimplicial set constant in the second variable, i.e. $G_n = G_*$ for all $n$. Define $H_*$ similarly. Let $B(H,G)_n$ be the bisimplicial set $(E)H_\times G_*$, where $(EH)_n = H_{n+1}$. We let $H_*$ act on $B(H,G)$, diagonally. Then $[B(H,G)/H_*]_n = (H_\times H^\times_n \times G_*)/H_* \cong H^\times_n \times G_*$, and so the canonical map $B(H,G)/H_* \to B'(H',G')/H'_*$ is a levelwise weak equivalence. By [6, Proposition IV.1.7], hence so is the induced map on diagonals $d(B(H,G)/H_*) \to d(B'(H',G')/H'_*)$. It is thus enough to prove that $d(B(H,G)/H_*) \simeq G_*/H_*$. The unique map $(EH)_n \to *$ induces $B(H,G) \to G_*$ and then $B(H,G)/H_* \to G/H_*$. Since $d(G/H)_* = G/H_*$, it is enough to show that $B(H,G)/H_*$ acts weakly equivalent levelwise in the other variable (since taking diagonals is manifestly symmetric in the two variables). This map is $B(H_n,G_n)/H_n \to G_n/H_n$. It is well-known that the left hand side is the homotopy orbits of the action of the discrete group $H_n$ on $G_n$, and the right hand side is the ordinary quotient. They are weakly equivalent because the action is free.
\end{proof}

To go further, we need to assume that $A$ is given the structure of a \textit{representable interval object} [1, Definition 4.1.1]. In this case there is a functor

$$\text{Sing}_* : s\text{Pre}(\mathcal{C})_* \to s\text{Pre}(\mathcal{C})_*$$

with $\text{Sing}_*(\mathcal{X})(c) = \mathcal{X}_*(A^n \times c)$. The functor $\text{Sing}_*$ preserves objectwise weak equivalences and is in fact a functorial “$A$-localization”; in particular it produces $A$-invariant objects. All of these properties are mentioned in [1], right after Definition 4.1.4.

\begin{lemma}
Let $\mathcal{X} \in s\text{Pre}(\mathcal{C})_*$. Then there is a canonical isomorphism

$$\Omega^\text{naive}_G(\mathcal{X}) \cong \text{Sing}_* \big( \Omega^\text{naive}_G(\mathcal{X}) \big).$$

If $\mathcal{G}$ is a presheaf of simplicial groups, then moreover

$$\Omega^\text{gr}_G(\mathcal{G}) \cong \text{Sing}_* \big( \Omega^\text{gr}_G(\mathcal{G}) \big)$$

and

$$\Omega^\text{gr}_{G,A}(\mathcal{G}) \cong \text{Sing}_* \big( \Omega^\text{gr}_{G,A}(\mathcal{G}) \big).$$

\end{lemma}

\begin{proof}
Clear from the defining formulas.
\end{proof}

\begin{corollary}
Let $\mathcal{G} \in s\text{Pre}(\mathcal{C})_*$ be a presheaf of simplicial groups. Then $R\Omega_G \text{Sing}_* \mathcal{G} \simeq \text{Sing}_* \Omega^\text{gr}_G \mathcal{G}$. If furthermore $\mathcal{G}$ is $(G,A)$-injective, then $R\Omega_G \text{Sing}_* \mathcal{G} \simeq \text{Sing}_* \Omega^\text{gr}_{G,A} \mathcal{G}$.
\end{corollary}

\begin{proof}
We have

$$R\Omega_G \text{Sing}_* \mathcal{G} \simeq \Omega^\text{gr}_G \text{Sing}_* \mathcal{G} \cong \text{Sing}_* \Omega^\text{gr}_G \mathcal{G},$$

where the first step is by Proposition 6(1,2) and the second step is by Lemma 8. This proves the first claim. If $\mathcal{G}$ is $(G,A)$-injective, we have furthermore

$$\Omega^\text{gr}_{G,A} \text{Sing}_* \mathcal{G} \cong \text{Sing}_* \Omega^\text{gr}_{G,A} \mathcal{G},$$

where the first step is by Proposition 6(3), using that $\text{Sing}_*$ produces $A$-invariant objects and preserves $(G,A)$-injective objects, and the second step is by Lemma 8 again. This proves the second claim.
\end{proof}
Specialisation to $\mathbb{A}^1$-algebraic topology. We now consider the situation where $C = Sm^\text{aff}_S$, $G = G_m$, $A = \mathbb{A}$. Here $S$ is a Noetherian scheme of finite Krull dimension (in all our applications it will be the spectrum of a field), and $Sm^\text{aff}_S$ denotes the category of smooth, finite type, (relative) affine $S$-schemes.

We write $L_{\text{mot}}sPre(Sm^\text{aff}_S)$ for the motivic localization of the model category $sPre(Sm^\text{aff}_S)$ (with its injective global model structure); in other words the localization inverting $\mathbb{A}^1$-homotopy equivalences and the distinguished Nisnevich squares (equivalently, the Nisnevich-local weak equivalences [11, Lemma 3.1.18]). It is well-known that $L_{\text{mot}}sPre(Sm^\text{aff}_S)$ is Quillen equivalent to $L_{\text{mot}}sPre(Sm_S)$, the usual model for the pointed, unstable motivic homotopy category.\(^1\) We write $L_{\text{mot}} : sPre(Sm^\text{aff}_S) \to sPre(Sm^\text{aff}_S)$ for a fibrant replacement functor for the motivic model structure.

Let us note right away that $(G, A) = (G_m, \mathbb{A})$-injectivity is common in our situation.

**Lemma 10.** If $X \in sPre(Sm^\text{aff}_S)$ is represented by a separated $S$-scheme, then $X$ is $(G_m, \mathbb{A})$-injective.

**Proof.** By definition the diagonal $X \to X \times_S X$ is a closed immersion, whence any two maps $f, g : \mathbb{A} \times U \to X$ over $S$ which agree on $G_m \times U$ must agree on its closure, which is all of $\mathbb{A} \times U$. In other words, the restriction is injective. This was to be shown. \(\square\)

We can now state the main result of this section.

**Proposition 11.** Let $k$ be an infinite* field and $G$ an isotropic reductive $k$-group. Then

$$\Omega^1_{\text{triv}}G \simeq \text{Sing} \ast \Omega^1 G \simeq \text{Sing} \ast \Omega^1 G,$$

$$\Omega^1_{\text{triv}}G \simeq \text{Sing} \ast \Omega^1 G \simeq \text{Sing} \ast \Omega^1 G,$$

where $\simeq$ means (global) weak equivalence in $sPre(Sm^\text{aff}_S)$.

**Proof.** The main point is that under our assumptions, $L_{\text{mot}}G \simeq \text{Sing} \ast G$ [2, Theorem 4.3.1 and Definition 2.1.1]. Also $G$ is affine [2, Definition 3.1.1], so separated, whence $(G_m, \mathbb{A})$-injective by Lemma 10. The result now follows from Lemma 8 and Corollary 9. \(\square\)

3. Affine Grassmannians

We review some basic results about affine Grassmannians. Surely they are all well-known to workers in the field (i.e., not the author). Our main reference is [18]. Throughout, we fix a field $k$ and write $\text{Aff}_k$ for the category of all affine $k$-schemes (not necessarily of finite type, not necessarily smooth). We extensively work in the category $\text{Pre}($Aff$k)$ of presheaves on affine schemes; as is well-known we have the Yoneda embedding $\text{Sch}_k \to \text{Pre}(\text{Aff}_k)$. On $\text{Pre}(\text{Aff}_k)$ we have many topologies, the most relevant for us are the $\text{fpqc}$ topology [15, Tag 03NV] and the Zariski topology; we denote the relevant sheafification functors by $\text{a}_{\text{fpqc}}$ (which may not always exist!) and $\text{a}_{\text{zar}}$. For elements $F \in \text{Pre}(\text{Aff}_k)$ and $A$ any $k$-algebra, we put $F(A) := F(\text{Spec}(A))$.

**Definition 12.** Let $X \in \text{Pre}(\text{Aff}_k)$ be a presheaf. We have the presheaves $L^+X, LX \in \text{Pre}(\text{Aff}_k)$ defined by

$$L^+X(A) = X(A)[t]\]$$

and

$$LX(A) = X(A[t]).$$

Note that there is a canonical morphism $L^+X \to LX$ induced by $A[t] \to A[t]$.

Let $G \in \text{Pre}(\text{Aff}_k)$ be a presheaf of groups. Then $L^+G, LG$ are presheaves of groups and we define the affine Grassmannian as $Gr_G = a_{\text{fpqc}}LG/L^+G$.

We note right away that at least if $G$ is represented by a group scheme, then $Gr_G = a_{\text{fpqc}}LG/L^+G$ exists and is given by $a_{\text{fpqc}}LG/L^+G$ [18, Proposition 1.3.6, Lemma 1.3.7]. Let us further put $L_0G(A) = G(A[t,t^{-1}])$ and $L_+G(A) = G(A[t])$. Then we have a commutative square

$$\begin{array}{ccc}
L_0G & \longrightarrow & LG \\
\uparrow & & \uparrow \\
L_+G & \longrightarrow & L^+G.
\end{array}$$

The main result of this section is the following. See also Proposition 19 at the end of this section for a related and sometimes stronger result.

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\(^1\)One way of seeing this is as follows. The model category $L_{\text{mot}}sPre(Sm_S)$ is Quillen equivalent to $L_{\text{mot}}sShv(Sm_S)$, where $s\text{Shv}$ denotes the category of simplicial Nisnevich-sheaves [8, Theorem II.5.9], and similarly for $L_{\text{mot}}sPre(Sm^\text{aff}_S)$. But the categories $s\text{Shv}(Sm_S)$ and $s\text{Shv}(Sm^\text{aff}_S)$ are equivalent, because $Sm_S$ and $Sm^\text{aff}_S$ define the same site.
Proposition 13. Let $G$ be a split reductive group over an infinite field $k$. Then the canonical map

$$a_{Zar} L_0 G / L_0^+ G \to \text{Gr}_G$$

induced by (3) is an isomorphism (of objects in $\text{Pre}(\text{Aff}_k)$).

Before giving the proof, we need some background material. If $\tau$ is a topology, we call a morphism of presheaves $f : \mathcal{X} \to \mathcal{Y}$ a $\tau$-epimorphism if $a_{\tau} f$ is an epimorphism in the topos of $\tau$-sheaves.

Definition 14. Let $\mathcal{G} \in \text{Pre}(\text{Aff}_k)$ be a presheaf of groups acting on $\mathcal{X} \in \text{Pre}(\text{Aff}_k)$. Suppose given a $\mathcal{G}$-equivariant map $f : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{Y} \in \text{Pre}(\text{Aff}_k)$ has the trivial $\mathcal{G}$-action. Let $\tau$ be a topology on $\text{Aff}_k$. We call $f$ a $\tau$-locally trivial $\mathcal{G}$-torsor if:

1. $f$ is a $\tau$-epimorphism,
2. the canonical map $\mathcal{G} \times \mathcal{X} \to \mathcal{X} \times \mathcal{Y}$ is an isomorphism.

Let us note that condition (1) implies that $\mathcal{G} \times \mathcal{X}$ and $\mathcal{X} \times \mathcal{Y}$ are $\tau$-sheaves, so condition (3) is $\tau$-local. We call a $\mathcal{G}$-torsor trivial if there is a $\mathcal{G}$-equivariant isomorphism $\mathcal{X} \cong \mathcal{G} \times \mathcal{Y}$.

Lemma 15. Suppose that $\mathcal{G}$ is a presheaf of groups acting on $\mathcal{X}$, and $f : \mathcal{X} \to \mathcal{Y}$ is a $\mathcal{G}$-equivariant map, where $\mathcal{G}$ acts trivially on $\mathcal{Y}$. Suppose that $\mathcal{G}, \mathcal{X}, \mathcal{Y}$ are $\tau$-sheaves. The following are equivalent.

1. $f$ is a $\tau$-locally trivial $\mathcal{G}$-torsor.
2. For every affine scheme $S$ and every morphism $S \to \mathcal{Y}$, there exists a $\tau$-cover $\{S_i \to S\}_i$ such that $\mathcal{X}_{S_i}$ is a trivial $\mathcal{G}$-torsor (for every $i$).
3. There exists a $\tau$-epimorphism $\mathcal{U} \to \mathcal{Y}$ such that $\mathcal{X}_\mathcal{U} \to \mathcal{X}$ is a trivial $\mathcal{G}$-torsor.

Proof. We will work in the topos of $\tau$-sheaves, so suppress any mention of $\tau$-sheafification, and also say “epimorphism” instead of “$\tau$-epimorphisms”, and so on.

1. $\Rightarrow$ (2): Let $S \to \mathcal{Y}$ be any map. Since epimorphisms in a topos are stable under base change (e.g., by universality of colimits), $\alpha : \mathcal{X}_S \to S$ is also a $\mathcal{G}$-torsor, and in particular an epimorphism. There exists then a cover $\{S_i \to S\}_i$ over which $\alpha$ has a section, being an epimorphism. In other words, $\mathcal{X}_{S_i}$ is trivial, as required.

2. $\Rightarrow$ (3): Taking the coproduct $\coprod_{S \to \mathcal{Y}} \prod_i S_i \to \mathcal{Y}$ over a sufficiently large collection of affine schemes $S$ mapping to $\mathcal{Y}$, we obtain a trivializing epimorphism as required.

3. $\Rightarrow$ (1): We need to prove that $\mathcal{X} \to \mathcal{Y}$ is an epimorphism and that $\mathcal{G} \times \mathcal{X} \to \mathcal{X} \times \mathcal{Y}$ is an isomorphism. Both statements may be checked after pullback along the (effective) epimorphism $\mathcal{U} \to \mathcal{Y}$. We may thus assume that $\mathcal{X} \to \mathcal{Y}$ is trivial, in which case the result is clear.

Lemma 16. Let $\mathcal{X} \to \mathcal{Y}$ be a $\tau$-locally trivial $\mathcal{G}$-torsor. Then $\mathcal{Y} \cong a_{\tau} \mathcal{X} / \mathcal{G}$.

Proof. We again work in the topos of $\tau$-sheaves. By definition $\mathcal{X} \to \mathcal{Y}$ is an epimorphism. Since every epimorphism in a topos is effective [15, Tag 086K], we have a coequalizer $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \rightrightarrows \mathcal{X} \to \mathcal{Y}$ in $\tau$-sheaves. By condition (3) of Definition 14, this is the action coequalizer. The result follows.

Proof of Proposition 13. By Lemma 16, it suffices to prove that $L_0 G \to \text{Gr}_G$ is a Zariski-locally trivial $L_0^+ G$-torsor. All presheaves involved are fpqc-, and hence Zariski-sheaves.

We shall make use of results from [18, Section 2]. There $k$ is assumed to be algebraically closed. This will not matter in each case we cite this reference, because the property we are checking will be fpqc local.

We introduce some additional notation. We denote by $L^- G$ the presheaf $A \mapsto G(A[t^{-1}])$. We have $L^- G \cong L_0^+ G$, but the canonical embedding $L^- G \to L_0 G$ is different. Evaluation at $t^{-1} = 0$ induces $L^- G \to G$, and we let $L^\leq G = \text{ker}(L^- G \to G)$. I claim that the following square is a pullback, where $i$ is the multiplication map

$$L^\leq G \times L_0^+ G \xrightarrow{i} L_0 G \xrightarrow{p_{\tau}} L^\leq G \xrightarrow{j} \text{Gr}_G.$$
Now let $A \in L_0G(k)$. We obtain a map $j_A : L^{<0}G \to \text{Gr}_G, x \mapsto A \cdot j(x)$. Define similarly $i_A : L^{<0}G \times L^+_0G \to L_0G, (x,y) \mapsto Ayx$. Since $A$ is invertible, the following square is also a pullback
\[
\begin{array}{c}
L^{<0}G \times L^+_0G \\
\downarrow pr_1 \quad \\
\downarrow \quad \\
L^{<0}G \quad j_A \quad \text{Gr}_G.
\end{array}
\]
By Lemma 15, it is thus enough to show that $j' = \prod_{A \in L_0G(k)} j_A$ is a Zariski-epimorphism. Note first that $j_A$ is a morphism of ind-schemes [18, Theorem 1.2.2], and an open immersion [7, Lemma 3.1]. Consequently each $j_A$ identifies an open ind-subscheme. In order to check that $j'$ is a Zariski-epimorphism, it suffices to check that the $j_A$ form a covering. Let $k$ denote an algebraic closure of $k$. Since $\text{Gr}_G$ is of ind-finite type, it suffices to check this on $k$-points. The result thus follows from Lemma 17 below.

The above proof is complete if $k = \bar{k}$. In the general case, we need the following result, which is probably well-known to experts. A proof was kindly communicated by Timo Richarz.

**Lemma 17.** Let $k$ be an infinite field, $\bar{k}$ an algebraic closure, and $G$ a split reductive group over $k$. Then $\text{Gr}_G(\bar{k})$ is covered by the translates $AL^{<0}G(\bar{k}) \subset \text{Gr}_G(\bar{k})$, for $A \in L_0G(k)$.

**Proof.** We will make use of the Bruhat decomposition of $\text{Gr}_G$. Namely, there exists a set $X$, together with for each $\mu \in X$ an element $t^\mu \in L_0G(k)$ and a $k$-scheme $U_\mu \subset L_0G$ such that

1. The canonical map $U_\mu \to \text{Gr}_G, A \mapsto At^\mu \cdot e$ is a locally closed embedding. Denote the image by $Y_\mu$.

2. There is an isomorphism $U_\mu \cong A^{l(\mu)}$ for some non-negative integer $l(\mu)$.

3. The schemes $Y_\mu \to \text{Gr}_G$ form a locally closed cover.

We do not know a good reference for the statement in this generality, but see for example [14, Theorem 8.6.3].

It is clear that $L_0G \to \text{Gr}_G$ is trivial over $Y_\mu$. We deduce that (1) $L_0G \to \text{Gr}_G$ is surjective on $k$-points. Put $G = G_{\bar{k}}$. We claim that (2) any non-empty open $L^+_0G$-orbit in $L_0G$ contains (the image of) a $k$-point (of $L_0G$). Using surjectivity on $k$-points, for this it suffices to prove that any non-empty open $U \subset \text{Gr}_G$ contains a $k$-rational point. Being non-empty, $U$ meets $Y_\mu := (Y_\mu)_k$ for some $\mu$. Then $V := Y_\mu \cap U$ is a non-empty open subset of $Y_\mu \cong A^{n_\mu}$ for some $n$. Its image $V$ in $A^{n_\mu}$ is open [15, Tag 083] and non-empty. Since $k$ is infinite, $V$ has a rational point.\footnote{This result is widely known and easy to prove, yet we could not locate a reference. A proof is recorded on MathOverflow at [13].} This establishes the claim.

Finally let $A \in \text{Gr}_G(\bar{k})$. By surjectivity on $k$-points (1), we find $A \in L_0G(\bar{k})$ mapping to $A$. Consider the $L^+_0G$-orbit $O = AL^+_0G \subset L_0G$. I claim that $O$ contains a $k$-point. The automorphism rev : $L_0G \to L_0G$ induced by $t \mapsto t^{-1}$ interchanges $L^+$ and $L^-$, and hence converts $L^+$-orbits into $L_0^+$-orbits. Since it is defined over $k$ it preserves $k$-points. It is hence enough to show that rev($O$) has a $k$-point, and by the claim (2) it is enough to show that rev($O$) is open. But rev($O$) $=$ rev($A$) $L^+G \text{Gr}_G$ which is open, being the preimage of rev($A$) $L^+G \subset \text{Gr}_G$.

We thus find $B \in L^+_0G(k), C \in L^-(k)$ such that $ABC \in L_0G(k)$. Then
\[
\bar{A} = A \cdot e = (ABC)C^{-1}B^{-1} \cdot e = (ABC)C^{-1} \cdot e
\in (ABC)L^-G(\bar{k}) \cdot e = (ABC)L^{<0}G(\bar{k}) \cdot e \subset \text{Gr}_G(\bar{k}).
\]

This was to be shown.

**Remark 18.** There is an alternative proof of Proposition 13, using a recent result of Fedorov [4]. Moreover this proof does not require $k$ to be infinite, or a field. It was also kindly communicated by Timo Richarz.

**Alternative proof of Proposition 13.** It follows from the Beauville-Laszlo gluing lemma [3] that $\text{Gr}_G \cong \alpha_{\text{spec}} \cdot L_0G/L_0^+G \cong T$, where $T$ is the functor sending $\text{Spec}(A)$ to the set of isomorphism classes of tuples $(F, \alpha)$ with $F$ a $G$-torsor on $A^1$ and $\alpha$ a trivialization of $F$ over $A^1 \setminus \{0\}$. The map $L_0G \to T$ sends $M \in L_0G(A)$ to the pair $(F^0, \alpha_M)$, where $F^0$ is the trivial $G$-torsor and $\alpha_M$ is the trivialization induced by $M$. By Lemmas 15 and 16, what we need to show is that this map $L_0G \to T$ admits sections Zariski-locally on $T$. In other words if $\text{Spec}(A) \in \text{Aff}_k$ and $(F, \alpha) \in T(A)$, then the $G$-torsor $F$ over $A^1$ is Zariski-locally on $A$ trivial.
If $A$ is Noetherian local, this is [4, Theorem 2]. We need to extend this to more general $A$, so let $\text{Spec}(A) \in \text{Aff}_k$ and $(\mathcal{F}, \alpha) \in T(A)$ be arbitrary. We may write $A$ as a filtering colimit of Noetherian rings $A_i$. Since $\text{Gr}_{G}$ is of ind-finite type, we find $(\mathcal{F}', \alpha') \in T(A_i)$ for some $i$ inducing $(\mathcal{F}, \alpha)$. Thus we may assume that $A$ is Noetherian. Fedorov’s result assures us that $\mathcal{F}$ is trivial over any local ring of $A$. Thus what remains to show is that triviality of $\mathcal{F}$ on $A^\text{h}_\mathbb{A}$ (or equivalently $\mathcal{F}'_A$) is an open condition on $\text{Spec}(A)$. This is proved in [7, proof of Lemma 3.1].

We can also prove the following related result, tailored to our narrower applications.

**Proposition 19.** Let $G$ be an isotropic reductive group over an infinite field $k$. Then the canonical map

$$a_{Zar} L_0 G/L_0^+ G \to \text{Gr}_{G}$$

induced by (3) is an isomorphism on sections over smooth affine varieties.

**Proof.** By arguing as in the alternative proof of Proposition 13, what we need to show is the following: if $X$ is a smooth affine variety and $\mathcal{F}$ is a $G$-torsor on $A^\text{h}_X$ which is trivial over $A^\text{h}_X \setminus \{0\}$, then $\mathcal{F}$ is Zariski-locally on $X$ trivial. By definition $\mathcal{F}$ is generically trivial, and hence by the resolution of the Grothendieck-Serre conjecture over fields [5, 12], $\mathcal{F}$ is Zariski-locally trivial (on $A^\text{h}_X$). By homotopy invariance for $G$-torsors over smooth affine schemes [2, Theorem 3.3.7], we find that $\mathcal{F} \cong (h_{\text{Zar}}^A \to X)^* G$, for some Nisnevich-locally trivial $G$-torsor $\mathcal{G}$ on $X$. Now $\mathcal{G}$ is generically trivial, so by Grothendieck-Serre again $\mathcal{G}$ is Zariski-locally trivial. This concludes the proof.

**4. MAIN RESULT**

We now come to our main result. Let $\mathcal{Spc}(k)_*$ denote the $\infty$-category of pointed motivic spaces. As usual we have a canonical functor $(\text{Sm}_k)_* \to \mathcal{Spc}(k)_*$. We also have the functor $\rho : \text{Pre}(\text{Aff}_k)_* \to \mathcal{Spc}(k)_*$. It is obtained as the composite

$$\text{Pre}(\text{Aff}_k)_* \xrightarrow{j^*} \text{Pre}(\text{Sm}_k^\text{aff})_* \xrightarrow{L} \mathcal{Spc}(k)_*,$$

where the $j^*$ is restriction along $j : \text{Sm}_k^\text{aff} \to \text{Aff}_k$ and $L$ is the motivic localization functor. Recall also the $G_m$-loops functor $R^A \Omega_{G_m} : \mathcal{Spc}(k)_* \to \mathcal{Spc}(k)_*$. We also recall its definition below in the proof of Proposition 22. For $X \in (\text{Sm}_k)_*$ we have $M X \simeq MX$, where on the right hand side we view $X$ as an element of $(\text{Ft}_k)_* \subset \text{Pre}(\text{Ft}_k)_*$. In other words, the functor $M$ allows us to make sense the motive of (among other things) singular varieties.

**Motives of singular varieties.** The presheaves $\text{Gr}_{G}$ are well-understood: they are filtered colimits of projective varieties. Unfortunately, these projective varieties are highly singular. Thus we need to incorporate motives of singular varieties in order to make the best use of Theorem 20.

Let $\text{Ft}_k$ denote the category of finite type $k$-schemes, and suppose that $k$ has exponential characteristic $e$ (i.e. $e = 1$ if char($k$) = 0 and $e = p$ if char($k$) = $p > 0$). Recall the $\infty$-category $\mathcal{D}M(k, Z[1/e])$ of $Z[1/e]$-linear motives [16] and the functor $M : \mathcal{Spc}(k)_* \to \mathcal{D}M(k, Z) \to \mathcal{D}M(k, Z[1/e])$ sending a pointed motivic space to its motive. There also is a more complicated functor

$$M : \text{Pre}(\text{Ft}_k)_* \to \mathcal{D}M(k, Z[1/e]);$$

we recall its definition below in the proof of Proposition 22. For $X \in (\text{Sm}_k)_*$ we have $M X \simeq MX$, where on the right hand side we view $X$ as an element of $(\text{Ft}_k)_* \subset \text{Pre}(\text{Ft}_k)_*$. In other words, the functor $M$ allows us to make sense the motive of (among other things) singular varieties.

Denote by $e^* : \text{Pre}(\text{Ft}_k)_* \to \text{Pre}(\text{Sm}_k^\text{aff})_*$ the functor of restriction along the canonical inclusion $\text{Sm}_k^\text{aff} \to \text{Ft}_k$.

**Proposition 22.** Let $k$ be a perfect field and $X \in \text{Pre}(\text{Ft}_k)_*$. Then $M e^* X \simeq MX$.
Corollary 25.

Let \( \mathcal{C} \) be any category. If \( f : \mathcal{C} \to \mathcal{D} \) is a functor, we get an equivalence \( f^* : \mathcal{P}(\mathcal{D}) \cong \mathcal{P}(\mathcal{C}) \). We have a full inclusion \( \text{Pre}(\mathcal{C}) \subset \mathcal{P}(\mathcal{C}) \) and similarly for \( \mathcal{D} \), and the following diagram commutes:

\[
\begin{array}{ccc}
\text{Pre}(\mathcal{C}) & \longrightarrow & \mathcal{P}(\mathcal{C}) \\
\text{Pre}(\mathcal{D}) & \longrightarrow & \mathcal{P}(\mathcal{D}).
\end{array}
\]

The functor \( \mu \) is constructed via the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{P}(\text{Ft}_k) & \longrightarrow & \mathcal{DM}(k, \mathbb{Z}[1/e]) \\
\epsilon & \longrightarrow & \epsilon^* \\
\mathcal{P}(\text{Sm}^{\text{eff}}) & \longrightarrow & \mathcal{DM}(k, \mathbb{Z}[1/e]).
\end{array}
\]

The category \( \mathcal{DM}(k, \mathbb{Z}[1/e]) \) can be defined as the \( T \)-stabilisation of \( L_{\text{cdh,} k} \mathcal{P}_2(\text{Cor}(k)) \), where \( \text{Cor}(k) \) is the category of finite correspondences \( \{9, \text{Theorem 2.5.9}\} \) and \( \mathcal{P}_2 \) denotes the nonabelian derived category \( \{10, \text{Section 5.5.8}\} \). The functor \( \epsilon^* \) is the stabilisation of the derived left Kan extension functor \( \epsilon : \mathcal{P}_2(\text{SmCor}(k)) \to \mathcal{P}_2(\text{Cor}(k)) \). The important result is that \( \epsilon^* \) is an equivalence \( \{9, \text{Corollary 5.3.9}\} \): one puts \( \mu = (\epsilon^*)^{-1} \circ \mu \).

In order to prove our result, it is thus enough to show that the co-unit map \( \eta : \epsilon \circ \epsilon^* \to \epsilon \) is inverted by the functor \( \mu \). For this it suffices to show that the image \( \mu_\eta \in \mathcal{DM}(k, \mathbb{Z}(1)) \) of \( \mu \) is an equivalence for all primes \( l \neq p \). The functor \( \mu_\eta \) inverts local equivalences for the so-called \( \text{Idh-topology} \{9, \text{Corollary 5.3.9}\} \), and all finite type \( k \)-schemes are \( \text{Idh-locally smooth} \{9, \text{Corollary 3.2.13}\} \). It is hence enough to show that \( \epsilon^*(\eta) : \epsilon^* \epsilon \circ \epsilon^* \to \epsilon^* \) is an equivalence. This follows from the fact that \( \epsilon^* \epsilon \simeq \text{id}_{\mathcal{DM}} \), which is a consequence of fully faithfulness of \( \epsilon : \mathcal{DM}^{\text{eff}} \to \text{Ft}_k \).

Remark 23. If \( k \) has characteristic 0, then using \( \{17\} \) for \( \mathcal{X} \in \text{Pre}(\text{Ft}_k) \), one may define the \( S^1 \)-stable homotopy type \( \Sigma^\infty \mathcal{X} \in \mathcal{SH}^{S^1}(k) \). Essentially the same proof as above shows that \( \Sigma^\infty \epsilon^* \mathcal{X} \simeq \Sigma^\infty \mathcal{X} \in \mathcal{SH}^{S^1}(k) \). In positive characteristic, there does not seem to be an equally accessible \( S^1 \)-stable homotopy type of singular varieties.

Corollary 24. Let \( X_1 \to X_2 \to \cdots \in (\text{Ft}_k)_* \) be a directed system of pointed finite type \( k \)-schemes. View each \( X_i \) as an element of \( \text{Pre}(\text{Aff}_k)_* \) and put \( \mathcal{X} = \text{colim}_i X_i \in \text{Pre}(\text{Aff}_k)_* \). Then we have \( M(\rho(\mathcal{X})) \simeq \text{colim}_i M(X_i) \).

We note that a filtered colimit of fpqc-sheaves (computed in \( \text{Pre}(\text{Aff}_k) \)) is an fpqc-sheaf \( \{15, \text{Tags 0738 (4) and 022E}\} \), and so the colimit in the corollary can be computed in the category of fpqc-sheaves.

Proof. Let \( \mathcal{X}' \in \text{Pre}(\text{Ft}_k)_* \) be the colimit viewed as a presheaf on finite type schemes. Then \( \epsilon^* \mathcal{X}' = j^*(\mathcal{X}) \) and the result follows from Proposition 22, using that all functors in sight commute with filtered colimits.

Corollary 25. Let \( G \) be an isotropic reductive group over an infinite perfect field \( k \) of exponential characteristic \( e \). Then we have

\[
M(\rho(\mathcal{G})) \simeq \bigoplus_{\mu \in \mathcal{X}(G)} \mathbb{Z}[1/e](l(\mu))[2l(\mu)] / \mathcal{D}M(k, \mathbb{Z}[1/e]).
\]

Here \( \mathcal{X}(G) \) is the set of cocharacters of \( G \), and \( l(\mu) \) is the (non-negative) integer from the proof of Lemma 17.

Proof. The Bruhat decomposition provides a filtration of \( \text{Gr}_G \) by closed subschemes \( 0 = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset \text{Gr}_G \). Here \( X_i = \bigcup_{\mu \leq l \leq Y_i} \mu_i \), in the notation of the proof of Lemma 17. Then \( X_{i-1} \to X_i \) is a closed immersion with open complement isomorphic to \( \prod_{l(\mu) = i} k^\times \). It is well-known that this implies our result. For the convenience of the reader, we review the standard argument. We have \( M(\rho(\text{Gr}_G)) = \text{colim}_i M(X_i) \) (by Corollary 24), and \( M(X_{-1}) = 0 \), so it suffices to prove that \( M(X_i) = M(X_{i-1}) \oplus \mathbb{Z}[\{1\}][2] \). Since each \( X_i \) is projective, we have \( M(X_i) = M^c(X_i) \), where \( M^c \) denotes the motive with compact support \( \{16, \text{p. 9}\} \). Thus we can use the Gysin triangle with compact support [16, Proposition 4.1.5]

\[
M^c(X_{i-1}) \to M^c(X_i) \to M^c(X_i \setminus X_{i-1}) \to M^c(X_{i-1})[1].
\]
Since $M^c(A^i) = Z(i)[2i]$, the boundary map vanishes for weight reasons (by induction, $M^c(X_{i-1})$ is a sum of Tate motives of weight $< i$), giving the desired splitting. □

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