IDENTITIES FOR MATRIX INVARIANTS OF THE SYMPLECTIC GROUP

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Abstract. The general linear group acts on the space of several linear maps on the vector space as the basis change. Similarly, we have the actions of the orthogonal and symplectic groups. Generators and identities for the corresponding polynomial invariants over a characteristic zero field were described by Sibirskii, Procesi and Razmyslov in 1970s. In 1992 Donkin started to transfer these results to the case of infinite fields of arbitrary characteristic. We completed this transference for fields of odd characteristic by establishing identities for the symplectic matrix invariants over infinite fields of odd characteristic.

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1. Introduction

We work over an infinite field $F$ of arbitrary characteristic $p = \text{char } F$. All vector spaces, algebras and modules are over $F$ and all algebras are associative with unity unless otherwise stated.

Consider a group $G$ from the list $\text{GL}(n)$, $O(n) = \{A \in F^{n \times n} \mid AA^T = E\}$, $SO(n) = \{A \in O(n) \mid \det(A) = 1\}$, $\text{Sp}(n) = \{A \in F^{n \times n} \mid AA^* = E\}$, where we assume that $p \neq 2$ in case $G \in \{O(n), SO(n)\}$ and $n$ is even in case $G = \text{Sp}(n)$. Here $F^{n \times n}$ is the space of $n \times n$ matrices over $F$ and $A^* = -J A^T J$ is the symplectic transpose of $A$, where $J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ is the matrix of the skew-symmetric bilinear form. The group $G$ acts on the space $V = (F^{n \times n})^{\oplus d}$ by the diagonal conjugation:

$$g \cdot (A_1, \ldots, A_d) = (gA_1g^{-1}, \ldots, gA_dg^{-1})$$

for $g \in G$ and $A_1, \ldots, A_d$ in $F^{n \times n}$. The coordinate ring of $V$, i.e. the algebra of all polynomial maps $V \to F$, is the polynomial ring

$$R = F[x_{ij}(k) \mid 1 \leq i, j \leq n, 1 \leq k \leq d]$$

in $n^2 d$ variables, where $x_{ij}(k)$ sends $(A_1, \ldots, A_d)$ to the $(i, j)^{th}$ entry of $A_k$. The algebra of matrix $G$-invariants is the set of all polynomial maps $f \in R$ that are constants on $G$-orbits of $V$, i.e., $f(g \cdot v) = f(v)$ for all $g \in G$ and $v \in V$. We denote this algebra by $R^G$.

For $f \in R$ denote by $\deg f$ its degree and by $\text{md} f$ its multidegree, i.e., $\text{md} f = (t_1, \ldots, t_d)$, where $t_k$ is the total degree of the polynomial $f$ in $x_{ij}(k)$, $1 \leq i, j \leq n$, and $\deg f = t_1 + \cdots + t_d$. By the Hilbert–Nagata Theorem on invariants, each of the considered algebras of invariants $I$ is finitely generated algebras. The algebra $I$ also have $\mathbb{N}_0$-gradings by degrees and $\mathbb{N}_0^d$-grading by multidegrees, where $\mathbb{N}_0$ stands...
for non-negative integers. Denote by $D(I)$ the maximal degree of elements of a minimal (by inclusion) $\mathbb{N}^d$-homogeneous set of generators (m.h.s.g.) for $I$.

Generating sets for the considered algebras of invariants are known (see Section 2) as well as relations between generators for $R^{GL(n)}$ and $R^{O(n)}$ (see Section 3). In case $p = 0$ relations between generators for $R^{Sp(n)}$ are also known (see [8]). The key difference between the case of zero and positive characteristic is the following property obtained in [2]:

$$(1) \quad D(R^{GL(n)}) \to \infty \text{ as } d \to \infty \text{ if and only if } 0 < p \leq n.$$ 

In this paper we describe the ideal of relations for $R^{Sp(n)}$ over a field of odd characteristic (see Theorem 3.2 below).

2. Generators

To formulate the result describing generators of the algebra $R^G$, we introduce the following notations. The ring $R$ is generated by the entries of $n \times n$ generic matrices $X_k = (x_{ij}(k))_{1 \leq i,j \leq n}$ ($1 \leq k \leq d$).

Consider an arbitrary $n \times n$ matrix $A = (a_{ij})$ over some commutative ring. Denote coefficients in the characteristic polynomial of $A$ by $\sigma_t(A)$, i.e.,

$$\det(\lambda E - A) = \sum_{t=0}^{n} (-1)^t \lambda^{n-t} \sigma_t(A).$$

So, $\sigma_0(A) = 1$, $\sigma_1(A) = \text{tr}(A)$ and $\sigma_n(A) = \det(A)$.

Part (a) of the following theorem was proven by Donkin [3], parts (b), (c) by Zubkov [11].

**Theorem 2.1.** The algebra of matrix $G$-invariants $R^G$ is generated by the following elements:

(a) $\sigma_t(A)$ ($1 \leq t \leq t$ and $A$ ranges over all monomials in $X_1, \ldots, X_d$), if $G = GL(n)$;

(b) $\sigma_t(B)$ ($1 \leq t \leq t$ and $B$ ranges over all monomials in $X_1, \ldots, X_d$, $X_1^T, \ldots, X_d^T$), if $G = O(n)$;

(c) $\sigma_t(C)$ ($1 \leq t \leq t$ and $C$ ranges over all monomials in $X_1, \ldots, X_d$, $X_1^*, X_d^*$), if $G = Sp(n)$.

We can assume that in the formulation of Theorem 2.1 each of monomials $A, B, C$ is primitive, i.e., is not equal to a power of a shorter monomial. If $p = 0$ or $p > n$, then in Theorem 2.1 it is enough to take traces $\text{tr}(U)$, where $U$ can be non-primitive, instead of $\sigma_t(U)$ in order to generate the algebra $R^G$. The corresponding results were obtained earlier than Theorem 2.1 by Sibirskii [9] and Procesi [8].

**Remark 2.2.** It is not difficult to see that elements from Theorem 2.1 are in fact invariants. Namely, the action of $G$ on $V$ induces the action on $R$ as follows: $g \cdot x_{ij}(k)$ is the $(i, j)$-th entry of $g^{-1}X_k g$. Given $f \in R$, we have $f \in R^G$ if and only if $g \cdot f = f$ for all $g \in G$. Then the following properties give us the required: $\sigma_t(gAg^{-1}) = \sigma_t(A)$. 

3. Identities

The following definitions were given in [5]. Let \( \langle X \rangle \) be the semigroup (without unity) freely generated by letters
- \( x_1, \ldots, x_d, \) if \( G = \text{GL}(n); \)
- \( x_1, \ldots, x_d, x_1^T, \ldots, x_d^T, \) otherwise.

Denote \( \langle X \rangle^\# = \langle X \rangle \cup \{1\}, \) i.e., we endow \( \langle X \rangle \) with the unity. Let \( \mathbb{F}(X) \) and \( \mathbb{F}(X)^\# \) be the free associative algebras (without and with unity, respectively) with the \( \mathbb{F} \)-bases \( \langle X \rangle \) and \( \langle X \rangle^\# \), respectively. Note that elements of \( \mathbb{F}(X) \) and \( \mathbb{F}(X)^\# \) are finite linear combinations of monomials from \( \langle X \rangle \) and \( \langle X \rangle^\# \), respectively. Assume that \( a = a_1 \cdots a_r \) and \( b \) are elements of \( \langle X \rangle \), where \( a_1, \ldots, a_r \) are letters.

- Introduce the involution \( T \) on \( \langle X \rangle \) as follows. If \( G = \text{GL}(n) \), then \( a^T = a \).
  Otherwise, we set \( b^{TT} = b \) for a letter \( b \) and \( a^T = a_1^T \cdots a_r^T \in \langle X \rangle \).

- We say that \( a \) and \( b \) are cyclic equivalent and write \( a \sim b \) if \( a = a_1a_2b \) and \( b = a_2a_1 \) for some \( a_1, a_2 \in \langle X \rangle^\# \). If \( a \sim b \) or \( a \sim b^T \), then we say that \( a \) and \( b \) are equivalent and write \( a \sim b \).

An element from \( \langle X \rangle \) is called primitive if it is not equal to a power of a shorter monomial.

- Let \( \langle \tilde{X} \rangle \subset \langle X \rangle \) be the subset of primitive elements. Note that if \( a \sim b \) for \( a \in \langle \tilde{X} \rangle \), then \( b \in \langle \tilde{X} \rangle \).

- Let \( \sigma(\tilde{X}) \) (\( \sigma(X) \), respectively) be the ring with unity of commutative polynomials over \( \mathbb{F} \) freely generated by “symbolic” elements \( \sigma_t(a) \), where \( t > 0 \) and \( a \in \langle \tilde{X} \rangle \) ranges over \( \sim \)-equivalence classes \( (a \in \mathbb{F}(X), \) respectively\).

We will use the following conventions: \( \sigma_0(a) = 1 \) and \( \sigma_1(a) = \text{tr}(a) \), where \( a \in \langle \tilde{X} \rangle \).

For a letter \( b \in \langle X \rangle \) define

\[
X_b = \begin{cases} 
X_k, & \text{if } b = x_k \\
X_T, & \text{if } b = x_k^T \text{ and } G = O(n) \\
X_K, & \text{if } b = x_k^T \text{ and } G = Sp(n)
\end{cases}
\]

Given \( a = a_1 \cdots a_r \in \langle X \rangle \), where \( a_i \) is a letter, we set \( X_a = X_{a_1} \cdots X_{a_r} \).

Consider the surjective homomorphism

\[
\phi_n : \sigma(\tilde{X}) \rightarrow R^G
\]

defined by \( \sigma_t(a) \rightarrow \sigma_t(X_a) \), if \( t \leq n \), and \( \sigma_t(a) \rightarrow 0 \) otherwise. Note that for all \( n \times n \) matrices \( A, B \) over \( R \) and \( 1 \leq t \leq n \) we have \( \sigma_t(A^\delta) = \sigma_t(A), (A^\delta)^\delta = A \), and \( (AB)^\delta = B^\delta A^\delta \), where \( \delta \) stands for the transposition or symplectic transposition. Hence the map \( \phi_n \) is well defined. Its kernel \( K_n \) is the ideal of relations for \( R^G \).

In [7] it was shown that \( \sigma(\tilde{X}) \) is the ring \( \sigma(X)/L \) with the generators of the ideal \( L \) were given. Therefore, any element of \( \sigma(X) \) can be considered as an element of \( \sigma(\tilde{X}) \).

Assume that \( G = O(n) \). Let us recall the definition of element \( \sigma_{t,r}(a, b, c) \) of \( \sigma(X) \), where \( t, r \geq 0 \) and \( a, b, c \in \mathbb{F}(X) \). For short, we set \( x = x_1, y = x_2, \) and \( z = x_3 \). Consider the quiver (i.e., the oriented graph) \( Q \):

\[
\begin{array}{c}
x \circ \end{array} \xrightarrow{\left\{ \begin{array}{c} y, y^T \\ z, z^T \end{array} \right.} \begin{array}{c} x^T \end{array}
\]

where there are two arrows from vertex 2 to vertex 1 as well as from 1 to 2. By abuse of notation arrows of \( Q \) are denoted by letters from \( \langle X \rangle \). For an arrow \( a \) denote by
There are non-trivial relations of degree \( n \) in the first case the degree of any relation is greater than \( n \). A path \( a \) is closed if \( a' = a'' \). Denote the multidegree of a monomial \( a \) in arrows of \( Q \) by \( \text{mdeg}(a) = (\text{deg}_x(a) + \text{deg}_y(a), \text{deg}_y(a), \text{deg}_x(a)) \). We set

\[
\sigma_{t,r}(x,y,z) = \sum (-1)^k \sigma_{k_1}(e_1) \cdots \sigma_{k_q}(e_q),
\]

where the sum ranges over all closed paths \( e_1, \ldots, e_q \) in \( Q \) that are pairwise different with respect to \( \sim \)-equivalence and \( k_1, \ldots, k_q > 0 \) (\( q > 0 \)) satisfying \( k_1 \text{mdeg}(e_1) + \cdots + k_q \text{mdeg}(e_q) = (t, r, r) \). Here \( \xi = t + \sum_{i=1}^q k_i(\text{deg}_y e_i + \text{deg}_x e_i + 1) \). Given \( a, b, c \in F(X) \) we define \( \sigma_{t,r}(a, b, c) \) as the result of the substitutions \( x \rightarrow a, y \rightarrow b, z \rightarrow c \) in \( \sigma_{t,r}(x, y, z) \).

Part (a) of the following theorem was proven by Zubkov [10] and part (b) by Lopatin [5], [6].

**Theorem 3.1.** The ideal of relations \( K_n \) for \( R^G \simeq \sigma(X)/K_n \) is generated by

(a) \( \sigma_t(a) \) for \( t > n \), if \( G = \text{GL}(n) \);

(b) \( \sigma_{t,r}(a, b, c) \) for \( t + 2r > n \) (\( t, r \geq 0 \)), if \( G = \text{O}(n) \) and \( p \neq 2 \).

Here \( a, b, c \) ranges over \( F(X) \).

Assume \( G = \text{Sp}(n) \). Define the element \( \varrho_{t,r}(x, y, z) \) of \( \sigma(X) \) by

\[
\varrho_{t,r}(x,y,z) = \sum (-1)^{t+k_1+\cdots+k_q} \sigma_{k_1}(e_1) \cdots \sigma_{k_q}(e_q),
\]

where \( e_1, \ldots, e_q, k_1, \ldots, k_q \) are the same as in the definition of \( \sigma_{t,r} \).

**Theorem 3.2.** Assume \( p \neq 2 \). Then the ideal of relations \( K_n \) for \( R^{\text{Sp}(n)} \simeq \sigma(X)/K_n \) is generated by \( \varrho_{t,r}(a, b, c) \) for \( t + 2r > n \) (\( t, r \geq 0 \)), where \( a, b \in F(X) \) and \( c \in F(X)^\# \).

Key difference of the relations in case \( G = \text{O}(n) \) and in case \( G = \text{Sp}(n) \) is that in the first case the degree of any relation is greater than \( n \), but in the second case there are non-trivial relations of degree \( \frac{n}{2} + 1 \). The proof of the theorem is given at the end of Section 5.

**Remark 3.3.** Denote by \( F_p \subset F \) the field of characteristic \( p \), generated by 1. Note that generators of \( R^{\text{Sp}(n)} \) as well as elements from the formulation of Theorem 3.2 are defined over \( F_p \). Hence the standard linear algebra arguments imply that without loss of generality we can assume that \( F \) is algebraically closed.

4. **Isomorphism of algebras**

In this section we assume that \( F \) is an algebraically closed field of characteristic different from two (see Remark 3.3). To define a subalgebra \( I_n \) of \( R \otimes F[y_{ij}] \) \( 1 \leq i < j \leq n \) we denote by \( Y \) the \( n \times n \) skew-symmetric matrix with \( (i, j) \)th entry equal to \( y_{ij} \) for \( i < j \). The algebra \( J_n \) is generated by all \( \sigma_i(A_1Y \cdots A_tY) \) for \( 1 \leq t \leq n, r > 0 \), where \( A_i \in \{ X_1, \ldots, X_d, X_1^2, \ldots, X_d^2 \} \) for all \( i \). Consider the homomorphism

\[
\Psi_n : I_n \rightarrow R^{\text{Sp}(n)}
\]
defined by $X_k \to X_k J$ and $Y \to -J$. (This notation means that $\Psi_n(x_{ij}(k))$ is equal to $(i,j)\text{th}$ entry of $X_k J$ and similarly for $Y$).

**Lemma 4.1.** The map $\Psi_n$ is an isomorphism of algebras $I_n$ and $R^{Sp}(n)$.

This lemma is proven at the end of this section. Let $I'_n$ be the subalgebra of $R \otimes F[z_{ij} \mid 1 \leq i, j \leq n]$ generated by elements $\sigma_i(A_1ZJZ^T \cdots A_rZJZ^T)$ for $1 \leq t \leq n$, $r > 0$, where $A_i \in \{X_1, \ldots, X_d, X_1^T, \ldots, X_d^T\}$ for all $i$ and $Z = (z_{ij})_{1 \leq i,j \leq n}$.

**Lemma 4.2.** The exists a unique homomorphism $\theta_n : I'_n \to I_n$ that sends $\sigma_i(A_1ZJZ^T \cdots A_rZJZ^T)$ to $\sigma_i(A_1Y \cdots A_rY)$.

**Proof.** Given a monomial $a = A_1ZJZ^T \cdots A_rZJZ^T$, we write $\theta_n(a)$ for the monomial $A_1Y \cdots A_rY$. Let $f = \sum \alpha_i \sigma_{t_i}(a_{i_1}) \cdots \sigma_{t_r}(a_{i_r})$ be an element of $I'_n$, where $\alpha_i \in F$. Denote by $h$ the element $\sum \alpha_i \sigma_{t_i}(\theta_n(a_{i_1})) \cdots \sigma_{t_r}(\theta_n(a_{i_r}))$ of $I_n$. To prove the lemma, it is enough to show that if $f = 0$, then $h = 0$.

Assume that $f = 0$. Then the result of substitution $Z \to B$ in $f$ is zero for every $B \in F^{n \times n}$. It is well-known that for any skew-symmetric $n \times n$ matrix $C$ over an algebraically closed field $F$ with $p \neq 2$ there is $B \in F^{n \times n}$ such that $B^T B = C$. Therefore, the result of substitution $Y \to C$ in $h$ is zero for every skew-symmetric matrix $C \in F^{n\times n}$. Since $F$ is infinite, the last condition implies that $h = 0$. \hfill \Box

We define the homomorphism $\mu_n : R^{Sp}(n) \to I'_n$ of algebras by $X_k \to Z^T X_k Z J$.

Straightforward calculations imply that

\begin{align*}
(2) & \quad \Psi_n \circ \theta_n \circ \mu_n \text{ is the identical map on } R^{Sp}(n) \text{ and} \\ 
(3) & \quad \theta_n \circ \mu_n \circ \Psi_n \text{ is the identical map on } R^{Sp}(n),
\end{align*}

where $\theta_n \circ \mu_n(x)$ stands for $\theta_n(\mu_n(x))$ and so on. Thus $\Psi_n^{-1} = \theta_n \circ \mu_n$ and Lemma 4.1 is proven.

## 5. Invariants of Quivers

Consider the following quivers:

\begin{align*}
&\xymatrix{1 \ar[r]^y & 2} & \quad \text{and} & \xymatrix{1 \ar[r]^{z, z^T} & 2}
\end{align*}

Denote the left hand side quiver by $G_y$ and the right hand side quiver by $G_z$. As in Section 3 by abuse of notation some arrows of this quivers are denoted by letters from $(X)$. Let $\langle G_y \rangle$ (resp. $\langle G_z \rangle$) be the set of all closed paths in $G_y$ (resp. $G_z$). By definition, we set $y^T = -y$. Then we define $\tau$-involutive and $\sim$-equivalence on $\langle G_y \rangle$ and $\langle G_z \rangle$ in the natural way. Considering $\langle G_y \rangle$ instead of $\langle X \rangle$, we define $\langle G_y \rangle$, $F(\langle G_y \rangle)$, $\sigma(\langle G_y \rangle)$ and $\sigma(\langle G_y \rangle)$ similarly to $\langle X \rangle$, $F(X)$, $\sigma(X)$ and $\sigma(X)$, respectively (see Section 3). The same notions we also introduce for $G_z$.

We write $I(Q_y, n)$ for the algebra of polynomial invariants of dimension vector $(n, n)$ of the quiver $Q_y$ with involution. By [4], $I(Q_y, n) = I_n$. We can also see that the algebra $I(Q_z, n)$ is generated by elements $\sigma_i(A_1B_1 \cdots A_rB_r)$ for $1 \leq t \leq n$, $r > 0$, where $A_i \in \{X_1, \ldots, X_d, X_1^T, \ldots, X_d^T\}$ and $B_i \in \{Z, Z^T\}$ for all $i$ (see [12]).
We set \(X_y = Y, X_z = Z,\) and \(X_{z^T} = Z^T,\) where matrices \(Y\) and \(Z\) were defined in Section 4. Hence \(X_a\) is determined for every \(a \in \langle G_y \rangle\) as well as for \(a \in \langle G_z \rangle\) (see also Section 3). Define the surjective homomorphisms

\[\pi_{y,n} : \sigma(G_y) \to I_n \quad \text{and} \quad \pi_{z,n} : \sigma(G_z) \to I(G_z, n)\]

defined by \(\sigma_t(a) \to \sigma_t(X_a),\) if \(t \leq n,\) and \(\sigma_t(a) \to 0\) otherwise. Kernels \(T_{y,n}\) and \(T_{z,n},\) respectively, of these maps are ideals of relations for \(I_n\) and \(I(G_z, n),\) respectively.

Assume that \(v\) is a vertex of \(G_y.\) Given an \(a \in F(G_y),\) we have \(a = \sum_i \alpha_i a_i\) for some \(\alpha_i \in F\) and \(a_i \in \langle G_y \rangle.\) If \(a_i = v\) for all \(i,\) then we write \(a' = v.\) Similarly we define \(a''._\]

A triple \((a, b, c)\) of elements from \(F(G_y)\) is called admissible if \(a' = a'' = b' = c'' = 1\) and \(b'' = c' = 2.\)

**Lemma 5.1.** Assume \(p \neq 2.\) Then the ideal of relations \(T_{y,n}\) for \(I_n\) is generated by \(\sigma_{t,r}(a, b, c)\) for \(t + 2r > n (t, r \geq 0),\) where \((a, b, c)\) is an admissible triple of \(G_y.\)

**Proof.** Consider a relation \(f \in \sigma(G_y)\) for \(I_n.\) Since \(F\) is infinite, without loss of generality we can assume that \(f\) is multihomogeneous. In particular, each monomial of \(f\) has one and the same degree \(k \geq 0\) in letter \(y.\) Denote by \(\sigma \in \sigma(G_z)\) the result of substitution \(y \to z - z^T\) in \(f.\) Obviously, \(\sigma\) is a relation for \(I(G_z, n).\) The general result by Zubkov [13] implies that the ideal \(T(G_z, n)\) of relations for \(G_z\) is generated by \(\sigma_{t,r}(a, b, c)\) for \(t + 2r > n\) and admissible triples \((a, b, c)\) of \(G_z.\) Denote by \(l \in \sigma(G_y)\) the result of substitution \(z \to y, z^T \to y^T = -y\) in \(h.\) Then \(l\) belongs to the ideal of \(\sigma(G_y)\) generated by \(\sigma_{t,r}(a, b, c)\) for \(t + 2r > n\) and admissible triples \((a, b, c)\) of \(G_y.\) On the other hand, \(l = 2^kf\) and the proof is completed. \(\square\)

Now we can prove Theorem 3.2

**Proof.** By Lemma 4.1 relations for \(R^{\Psi(n)}\) are images of relations for \(I_n\) with respect to \(\Psi_n.\) Relations for \(I_n\) are described in Lemma 5.1. The straightforward calculations complete the proof. \(\square\)

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**References**

[1] H. Aslaksen, E.-C. Tan, C.-B. Zhao, *Invariant theory of special orthogonal groups*, Pacific J. Math. **168**(1995), No. 2, 207–215.
[2] M. Domokos, S.G. Kuzmin, A.N. Zubkov, *Rings of matrix invariants in positive characteristic*, J. Pure Appl. Algebra **176**(2002), 61–80.
[3] S. Donkin, *Invariants of several matrices*, Invent. Math. **110**(1992), 389–401.
[4] A.A. Lopatin, *Invariants of quivers under the action of classical groups*, J. Algebra **321**(2009), 1079–1106.
[5] A.A. Lopatin, *Relations between $O(n)$-invariants of several matrices*, Algebra Repr. Theory, **15**(2012), 855–882.
[6] A.A. Lopatin, *Free relations for matrix invariants in modular cases*, J. Pure Appl. Algebra, **216**(2012), 427–437.
[7] A.A. Lopatin, *Matrix identities with forms*, to appear in J. Pure Appl. Algebra.
[8] C. Procesi, *The invariant theory of $n \times n$ matrices*, Adv. Math. **19**(1976), 306–381.
[9] K.S. Sibirskii, *Algebraic invariants for a set of matrices*, Sibirsk. Mat. Zh. 9 (1968), No. 1, 152–164 (in Russian); English translation: Siberian Math. J. 9 (1968), 115–124.

[10] A.N. Zubkov, *On a generalization of the Razmyslov–Procesi theorem*, Algebra and Logic 35 (1996), No. 4, 241–254.

[11] A.N. Zubkov, *Invariants of an adjoint action of classical groups*, Algebra and Logic 38 (1999), No. 5, 299–318.

[12] A.N. Zubkov, *Invariants of mixed representations of quivers I*, J. Algebra Appl. 4 (2005), No. 3, 245–285.

[13] A.N. Zubkov, *Invariants of mixed representations of quivers II: Defining relations and applications*, J. Algebra Appl. 4 (2005), No. 3, 287–312.

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