NON-NORMAL APPROXIMATION BY STEIN’S METHOD OF EXCHANGEABLE PAIRS WITH APPLICATION TO THE CURIE-WEISS MODEL

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Let \( (W, W') \) be an exchangeable pair. Assume that

\[
E(W - W'|W) = g(W) + r(W),
\]

where \( g(W) \) is a dominated term and \( r(W) \) is negligible. Let \( G(t) = \int_0^t g(s)ds \) and define \( p(t) = c_1 e^{-c_0 G(t)} \), where \( c_0 \) is a properly chosen constant and \( c_1 = 1/\int_{-\infty}^\infty e^{-c_0 G(t)}dt \). Let \( Y \) be a random variable with the probability density function \( p \). It is proved that \( W \) converges to \( Y \) in distribution when the conditional second moment of \( (W - W') \) given \( W \) satisfies a law of large numbers. A Berry-Esseen type bound is also given. We use this technique to obtain a Berry-Esseen error bound of order \( 1/\sqrt{n} \) in the non-central limit theorem for the magnetization in the Curie-Weiss ferromagnet at the critical temperature. Exponential approximation with application to the spectrum of the Bernoulli-Laplace Markov chain is also discussed.

∗Research supported by NSF grant DMS-0707054 and a Sloan Research Fellowship
†Partially supported by Hong Kong RGC 602206 and RGC 602608

AMS 2000 subject classifications: Primary 60F05; secondary 60G09,

Keywords and phrases: Stein’s method, exchangeable pair, Berry-Esseen bound, Curie-Weiss model
1. Introduction and main results. Let $W$ be the random variable of interest. Typical examples of $W$ include the partial sum of independent random variables and functionals of independent random variables or dependent random variables whose joint distribution is known. Since the exact distribution of $W$ is not available for most cases, it is natural to seek the asymptotic distribution of $W$ with a Berry-Esseen type error. Let $(W, W')$ be an exchangeable pair. Assume that

\begin{equation}
E(W - W' | W) = g(W) + r(W),
\end{equation}

where $g(W)$ is a dominated term while $r(W)$ is a negligible term. When $g(W) = \lambda W$, and $E((W' - W)^2 | W)$ is concentrated around a constant, Stein’s method for normal approximation shows that the limiting distribution of $W$ is normal under certain regularity conditions. We refer to Stein (1986), Rinott and Rotar (1997), Chen and Shao (2005) and references therein for the general theory of Stein’s method. The main aim of this paper is to find the limiting distribution of $W$ as well as the rate of convergence for general $g$. The key step is to identify the limiting density function.

As soon as the limiting density function is determined, we can follow the idea of the Stein’s method of exchangeable pairs for normal approximation. Let

\begin{equation}
G(t) = \int_0^t g(s)ds \quad \text{and} \quad p(t) = c_1 e^{-c_0 G(t)},
\end{equation}

where $c_0 > 0$ is a constant that will be specified later and $c_1 = 1/\int_{-\infty}^{\infty} e^{-c_0 G(t)} dt$ is the normalizing constant. Let $Y$ be a random variable with the probability density function $p$. Set

(H1) $g(t)$ is non-decreasing, and $g(t) \geq 0$ for $t > 0$ and $g(t) \leq 0$ for $t \leq 0$;

(H2) there exists $c_2 < \infty$ such that for all $x$,

\[
\min\left(1/c_1, 1/|c_0 g(x)|\right) (|x| + 3/c_1) \max(1, c_0 |g'(x)|) \leq c_2.
\]
(H3) there exists $c_3 < \infty$ such that for all $x$,

$$\min\left(1/c_1, 1/|c_0 g(x)|\right)(|x| + 3/c_1)c_0|g'(x)| \leq c_3.$$  

Let $\Delta = W - W'$. Our main result shows that $W$ converges to $Y$ in distribution as long as $c_0 E(\Delta^2 |W|$ satisfies a law of large numbers.

**Theorem 1.1.** Let $h$ be absolutely continuous with $\|h'\| = \sup_x |h'(x)| < \infty$.

(i) If (H1) and (H2) are satisfied, then

$$|Eh(W) - Eh(Y)| \leq \|h'\| \left\{ \frac{1 + c_2}{c_1} E|1 - (c_0/2)E(\Delta^2 |W)| + \frac{1}{2} c_0 (1 + c_2) E|\Delta|^3 + c_0 c_2 E|r(W)| \right\}$$

(ii) If (H1) and (H3) are satisfied, then

$$|Eh(W) - Eh(Y)| \leq \|h'\| \left\{ \frac{1 + c_3}{c_1} E|1 - (c_0/2)E(\Delta^2 |W)| + \frac{1}{2} c_0 (1 + c_3) E|\Delta|^3 
+ \frac{c_0}{c_1} E((|W| + 3/c_1)|r(W)|) \right\}$$

When $\Delta$ is bounded, next theorem gives a Berry-Esseen type inequality.

**Theorem 1.2.** Assume that $|W - W'| \leq \delta$, where $\delta$ is a constant. If (H1) and (H3) are satisfied, then

$$|P(W \leq z) - P(Y \leq z)| \leq 3E|1 - (c_0/2)E(\Delta^2 |W)| + c_1 \max(1, c_3) \delta + 2c_0 E|r(W)|/c_1 + \delta^3 c_0 \{ (2 + c_3/2)E|c_0 g(W)| + c_1 c_3/2 \}.$$  

We remark that $c_0$ can be chosen as follows. In order to make the error term on the right hand side of (1.3) small, it is necessary that $E|1 -
\( (c_0/2)E(\Delta^2|W|) \to 0 \) and therefore \( E(1 - (c_0/2)E(\Delta^2|W|)) \) must be small and we should choose \( c_0 \) so that \( c_0 \sim 2/E(\Delta^2) \).

The paper is organized as follows. In section 2, we give a concrete application of our general result to the magnetization of the Curie-Weiss model of ferromagnets at the critical temperature, and show that the rate of convergence achieves \( O(n^{-1/2}) \). In section 3 we focus on approximation by the exponential distribution with an application to the spectrum of the Bernoulli-Laplace Markov chain. We present a general approach of Stein’s method of exchangeable pairs in section 4 and postpone detailed proofs of our main results to section 5.

2. Curie-Weiss Model. Consider the Curie-Weiss model for \( n \) spins at temperature \( T \), i.e. the probability distribution on \( \{-1,1\}^n \) that puts mass

\[
Z_T^{-1} \exp \left( \frac{\sum_{1 \leq i < j \leq n} \sigma_i \sigma_j}{Tn} \right)
\]

at \( \sigma \in \{-1,1\}^n \), where \( Z_T \) is the normalizing constant. Let us fix \( T = 1 \), which is the ‘critical temperature’ for this model. Now let

\[
W = W(\sigma) = n^{-3/4} \sum_{i=1}^{n} \sigma_i.
\]

This is a simple statistical mechanical model of ferromagnetic interaction, sometimes called the Ising model on the complete graph. For a detailed mathematical treatment of this model, we refer to the book by Ellis (1985).

Following ideas in Simon and Griffiths (1973), it was proved by Ellis and Newman (1978a, 1978b) that as \( n \to \infty \), the law of \( W \) converges to the distribution with density proportional to \( e^{-x^4/12} \). For various interesting extensions and refinements of their results, let us refer to Ellis, Newman, and Rosen (1980), and Papangelou (1989).
Below, we present a Berry-Esseen bound for this non-central limit theorem obtained via Theorem 1.2. Incidentally, Theorem 1.2 can also be used to obtain similar error bounds for the other limit theorems in the aforementioned papers (in particular, the Curie-Weiss model at non-critical temperatures), but we prefer to stick to this example only, since it is probably the most interesting and relevant one.

Given a random element $\sigma$, construct $\sigma'$ by choosing a coordinate $I$ at random and replacing $\sigma_I$ by $\sigma'_I$, where $\sigma'_I$ is generated from the conditional distribution of $\sigma_I$ given $(\sigma_j)_{j \neq I}$. In other words, we take one step of the Glauber dynamics. It is easy to see that $(\sigma, \sigma')$ is an exchangeable pair. Let $W' = W(\sigma')$. We shall show that (see Section 5)

\begin{align}
(2.1) \quad & E|E(W - W'|W) - \frac{1}{3} n^{-3/2} W^3| = O(n^{-2}), \\
(2.2) \quad & E|E((W' - W)^2|W) - 2n^{-3/2}| = O(n^{-2}), \\
(2.3) \quad & |W' - W| = O(n^{-3/4}), \\
\text{and} \\
(2.4) \quad & E|W|^3 = O(1).
\end{align}

Let us now explain roughly how we arrive at (2.1), which is the most important step. A simple computation shows that at any temperature,

\[ E(W - W'|W) = n^{-3/4}(m - \tanh(m/T)) + O(n^{-2}), \]

where $m := n^{-1/4}W$ is the magnetization. Since $m \simeq 0$ with high probability when $T \geq 1$, and $\tanh x = x - x^3/3 + O(x^5)$ for $x \simeq 0$, we see that the right hand side in the above equation is like $n^{-3/4}m(1 - 1/T)$ when $T > 1$, while it’s like $n^{-3/4}m^3/3$ when $T = 1$. This is what distinguishes between the high
temperature regime $T > 1$ and the critical temperature $T = 1$, and this is how we arrive at equation (2.1).

Let 
\[ g(w) = \frac{1}{3}n^{-3/2}w^3, \quad c_0 = n^{3/2}, \quad \delta = O(n^{-3/4}) \]
Then
\[ G_1(w) = c_0 \int_0^w g(t)dt = w^4/12. \]
With the above information, it can be easily checked that by Theorem 1.2 we get

**Theorem 2.1.** Let $Y$ be a random variable with density function
\[ p(w) = c_1 e^{-w^4/12}, \quad \text{where } c_1 = \frac{1}{\int_{-\infty}^{\infty} e^{-w^4/12}dw} = \frac{2^{1/2}}{3^{1/4}\Gamma(1/4)}. \]
Then for all $z$,
\[ |P(W \leq z) - P(Y \leq z)| \leq c n^{-1/2}, \]
where $c$ is an absolute constant.

Incidentally, after this manuscript was submitted, it was brought to our attention that an article by Eichelsbacher and L"owe (2009) was in preparation, where the same result (Theorem 2.1) is proved, along the same lines as our proof. Eichelsbacher and L"owe (2009) has generalizations of Theorem 2.1 to some other mean-field models.

3. Exponential limit with application to spectrum of the Bernoulli-Laplace Markov chain. In this section we focus on the exponential limit. Let $(W, W')$ be an exchangeable pair satisfying
\[ E(W - W'|W) = 1/c_0 + r(W), \]
where $c_0 > 0$ is a constant. Let $\Delta = W - W'$. As a special case of Theorems 1.1 and 1.2 with a constant function $g$, we have
Theorem 3.1. Let $Y$ have the exponential distribution with mean 1. Assume (3.1) is satisfied.

(i) Let $h$ be absolutely continuous with $\|h'\| < \infty$. Then

$$|Eh(W) - Eh(Y)| \leq \|h'\| \{E|1 - (c_0/2)E(\Delta^2|W)| + c_0E|\Delta|^3 + 3c_0E|Wr(W)|\}.$$ (3.2)

(ii) If $|\Delta| \leq \delta$ for some constant $\delta$, then

$$|P(W \leq z) - P(Y \leq z)| \leq 3E|1 - (c_0/2)E(\Delta^2|W)| + \delta + 2c_0\delta^3 + 3c_0E|Wr(W)|.$$ (3.3)

We refer to Chatterjee, Fulman and Rölлин (2008) and Peköz and Rölлин (2009) for other general results for the exponential approximation.

We now apply Theorem 3.1 to the spectrum of the Bernoulli-Laplace Markov chain, a simple model of diffusion, following the work of Chatterjee, Fulman and Rölлин (2008). Two urns contain $n$ balls each. Initially the balls in each urn are all of a single color, with urn 1 containing all white balls, and urn 2 all black. At each stage, a ball is picked at random from each urn and the two are switched. Let the state of the chain be the number of white balls in the urn 1. Diaconis and Shahshahani (1987) proved that $(n/4) \log(2n) + cn$ steps suffice for this process to reach equilibrium, in the sense that the total variation distance to the stationary distribution is at most $ae^{-dc}$ for positive universal constants $a$ and $d$. In order to prove this, they used the fact that the spectrum of the Markov chain consists of the numbers

$$\lambda_i = 1 - i(2n - i + 1)/n^2 \quad \text{for} \quad i = 0, 1, \ldots, n$$ (3.4)

occurring with multiplicities

$$m_i = \binom{2n}{i} - \binom{2n}{i - 1} \quad \text{for} \quad i = 0, 1, \ldots, n.$$
Let $I$ have distribution $P(I = i) = \pi_i$, where
$$
\pi_i = \frac{\binom{2n}{i} - \binom{2n}{i-1}}{\binom{2n}{n}}
$$
for $0 \leq i \leq n$. Then $\lambda_I$ is a random eigenvalue chosen from \{\lambda_i, 0 \leq i \leq n\} in proportion to their multiplicities. Hora (1998) proved that $W = n\lambda_I + 1$ converges in distribution to an exponential random variable with mean 1.

Noting that $n\lambda_I + 1 = (n-i)(n+1-i)/n := \mu_i$, we can rewrite $W = \mu_I$. To apply Theorem 3.1, we construct an exchangeable pair $(W, W')$ using a reversible Markov chain on \{0, 1, \cdots, n\} with transition probability matrix $K$ satisfying
$$
\pi(i)K(i,j) = \pi(j)K(j,i) \quad \text{for all } i, j \in \{0, 1, \cdots, n\}.
$$

Given such a $K$, we obtain the pair $(W, W')$ by letting $W = u_I$ where $I$ is chosen from the equilibrium distribution $\pi$, and $W' = \mu_J$ where $J$ is determined by taking one step from state $I$ according to the transition probability $K$. As proved in [1], we have (with $\Delta = W - W'$).

$$
E(\Delta|W) = \frac{1}{2n^2} - \frac{n+1}{2n^2}I_{\{W=0\}},
$$

$$
E(W) = 1, \quad E(\Delta^2|W) = \frac{1}{n^2} \quad \text{and} \quad E|\Delta|^3 \leq 6n^{-5/2}.
$$

Now applying Theorem 3.1 we have

**Theorem 3.2.** Let $Y$ have the exponential distribution with mean 1 and $h$ be absolutely continuous with $\|h'\| < \infty$. Then

$$
|Eh(W) - Eh(Y)| \leq 12n^{-1/2}
$$

As the difference between $W$ and $W'$ is large when $I$ is small, Theorem 3.1 doesn’t provide a useful Berry-Esseen type bound. However, using a completely different approach and some heavy machinery, Chatterjee, Fulman...
and Röllin (2008) are able to show that

$$\sup_z |P(W \leq z) - P(Y \leq z)| \leq Cn^{-1/2},$$

where $C$ is a universal constant.

4. **The Stein method via density approach.** Let $p$ be a strictly positive, absolutely continuous probability density function, supported on $(a, b)$, where $-\infty \leq a < b \leq \infty$. Assume that a right limit $p(a+)$ at $a$ and a left limit $p(b-)$ exist. Let $p'$ be a version of the derivative of $p$ and assume that

$$\int_a^b |p'(t)| dt < \infty.$$ 

Let $Y$ be a random variable with the probability density function $p$. In this section we develop the Stein method via density approach. The approach was developed in Stein, Diaconis, Holmes and Reinert (2004), but the properties presented in section 4.2 are new.

4.1. **The Stein identity and equation.** A key step is to have Stein’s identity and Stein’s equation. Let $\mathcal{D}$ be the set of bounded, absolutely continuous functions $f$ with $f(b-) = f(a+) = 0$. Observe that for any $f \in \mathcal{D}$

$$E\{f'(Y) + f(Y)p'(Y)/p(Y)\}$$

$$= E\{(f(Y)p(Y))'/p(Y)\}$$

$$\int_a^b (f(y)p(y))' dy = 0. \quad (4.1)$$

The Stein identity is

$$Ef'(Y) + Ef(Y)p'(Y)/p(Y) = 0 \quad \text{for } f \in \mathcal{D}. \quad (4.2)$$

For any measurable function $h$ with $E|h(Y)| < \infty$, let $f = f_h$ be the solution to Stein’s equation

$$f'(w) + f(w)p'(w)/p(w) = h(w) - Eh(Y). \quad (4.3)$$
It follows from (4.3) that

\[(f(w)p(w))' = (h(w) - Eh(Y))p(w)\]

and hence

\[f(w) = \frac{1}{p(w)} \int_a^w (h(t) - Eh(Y))p(t)dt\]

(4.4)

\[= -\frac{1}{p(w)} \int_w^b (h(t) - Eh(Y))p(t)dt.\]

Note that \(f_h \in \mathcal{D}\).

Consider two classes of density functions. The first one is the family of exponential distributions. It is easy to see that if \(Y\) has the exponential distribution with parameter \(\lambda\), that is, \(Y\) is a random variable with density function \(p(x) = \lambda e^{-\lambda x}\) for \(x > 0\) and \(p(x) = 0\) for \(x \leq 0\). Then \(p'(x)/p(x) = -\lambda\) and the Stein identity (4.2) becomes

(4.5)

\[Ef'(Y) - \lambda Ef(Y) = 0 \quad \text{for} \quad f \in \mathcal{D}.\]

The second is the family

\[p(x) = \frac{\alpha e^{-|x|^{\alpha/\beta}}}{2\beta^{1/\alpha} \Gamma(1/\alpha)}, \quad -\infty < x < \infty\]

where \(\alpha > 0, \beta > 0\). Then \(p'(x)/p(x) = -\frac{\alpha}{\beta} |x|^{\alpha-1} \text{sign}(x)\) and hence the Stein identity reduces to

\[Ef'(Y) - \frac{\alpha}{\beta} E|Y|^{\alpha-1} \text{sign}(Y)f(Y) = 0 \quad \text{for} \quad f \in \mathcal{D}.\]

4.2. Properties of the Stein solution. In order to determine error bounds for the approximation to \(E(h(Y))\), we need to understand some basic properties of the Stein solution \(f_h\). In the following, we use the notation \(\|g\| := \sup_{x \in \mathbb{R}} |g(x)|\).

**Lemma 4.1.** Let \(h\) be a measurable function and \(f_h\) be the Stein solution and let \(F(x) = \int_a^x p(t)dt\).
(i) Assume that $h$ is bounded and that there exist $d_1 > 0$ and $d_2 > 0$

\begin{equation}
\text{min}(1 - F(x), F(x)) \leq d_1 p(x)
\end{equation}

and

\begin{equation}
|p'(x)| \text{min}(F(x), 1 - F(x)) \leq d_2 p^2(x).
\end{equation}

Then

\begin{equation}
\|f_h\| \leq 2d_1 \|h\|
\end{equation}

\begin{equation}
\|f_h' p'/p\| \leq 2d_2 \|h\|
\end{equation}

and

\begin{equation}
\|f_h''\| \leq (2 + 2d_2) \|h\|.
\end{equation}

(ii) Assume that $h$ is absolutely continuous with bounded $h'$. In addition to (4.6), (4.7), assume that there exist $d_3$ and $d_4$ such that

\begin{equation}
\min\left( E|Y|I_\{Y \leq x\} + E|Y|F(x), E|Y|I_\{Y > x\} + E|Y|(1 - F(x)) \right) |(p'/p)'|
\end{equation}

\begin{equation}
\leq d_3 p(x)
\end{equation}

and

\begin{equation}
\min\left( E|Y|I_\{Y \leq x\} + E|Y|F(x), E|Y|I_\{Y > x\} + E|Y|(1 - F(x)) \right)
\end{equation}

\begin{equation}
\leq d_4 p(x).
\end{equation}

Then if $h$ is absolutely continuous with bounded derivative $h'$,

\begin{equation}
\|f_h''\| \leq (1 + d_2)(1 + d_3) \|h''\|
\end{equation}

\begin{equation}
\|f_h\| \leq d_4 \|h'\|
\end{equation}

and

\begin{equation}
\|f_h'\| \leq (1 + d_3) d_1 \|h'\|.
\end{equation}
Proof. (i) Let $Y^*$ be an independent copy of $Y$. Then we can rewrite $f_h$ in (4.4) as

$$f(w) = (1/p(w))E(h(Y) - h(Y^*))I_{Y \leq w}$$

(4.16)

$$= -(1/p(w))E(h(Y) - h(Y^*))I_{Y > w},$$

which yields

(4.17) \[ |f(w)| \leq 2\|h\| \min(F(w), 1 - F(w))/p(w). \]

Inequality (4.8) now follows from (4.6) and (4.17). Inequalities (4.17) and (4.7) imply $|f_h p'|/p \leq 2d_2\|h\|$, that is (4.9), and now (4.10) follows from (4.3).

(ii) Let $g_1(x) = p'(x)/p(x)$. Recall by (4.3)

(4.18) \[ f'' = h' - f'g_1 - fg_1'. \]

To prove (4.13), it suffices to show that

(4.19) \[ \|fg_1'\| \leq d_3\|h'\| \]

and

(4.20) \[ \|f'g_1\| \leq (1 + d_3)d_2\|h'\|. \]

By (4.16) again, we have

(4.21) \[ |f(w)p(w)| \]

\[ \leq \|h'\| \min \left(E(|Y| + |Y^*|)I_{Y \leq w}, E(|Y| + |Y^*|)I_{Y > w}\right) \]

\[ = \|h'\| \min \left(E|Y|I_{Y \leq w} + E|Y|F(w), E|Y|I_{Y > w} + E|Y|(1 - F(w))\right). \]

This proves (4.19) by assumption (4.7). This also proves (4.14) by (4.12).

It follows from (4.18) that

\[ (h' - fg_1')p = p(f'' + f'g_1) = f''p + f'p' = (f'p'). \]
Thus
\[ f'(w)p(w) = \int_{a}^{w} (h' - f'g')pdx = -\int_{w}^{b} (h' - f'g')pdx \]
and hence
\[ |f'(w)p(w)| \leq \|h'\|(1 + d_3) \min(F(w), 1 - F(w)), \]
which gives (4.20) as well as (4.15) by (4.12) and (4.6) respectively. □

The next lemma shows that (4.6) - (4.12) are satisfied for \( p \) defined in (1.2).

**Lemma 4.2.** Let \( p \) be defined as in (1.2). Assume that (H1) and (H2) are satisfied. Then (4.6) - (4.12) hold with \( d_1 = 1/c_1, \) \( d_2 = 1, \) \( d_3 = c_2 \) and \( d_4 = c_2. \)

**Proof.** Let \( g_2(t) = c_0g(t), \) \( G_1(t) = c_0G(t) \) and \( F(t) = P(Y \leq t) \) be the distribution function of \( Y. \) We first show that (4.6) is satisfied with \( d_1 = 1/c_1. \) It suffices to show that

\[ F(t) \leq F(0)p(t)/c_1 \text{ for } t \leq 0 \] \hspace{1cm} (4.22)

and

\[ 1 - F(t) \leq ((1 - F(0))/c_1)p(t) \text{ for } t \geq 0. \] \hspace{1cm} (4.23)

Let \( H(t) = F(t) - (F(0)/c_1)p(t) \) for \( t \leq 0. \) Noting that
\[
H'(t) = p(t) - (F(0)/c_1)p'(t) = p(t) + (F(0)/c_1)g_2(t)p(t) = p(t)(1 + g_2(t)F(0)/c_1),
\]
Since \( g_2(t) \) is non-decreasing, if \( H'(0) > 0, \) then there is at most one \( t_0 \) such that \( H'(t_0) = 0; \) if \( H'(0) \leq 0, \) then \( H'(t) \leq 0 \) for \( t < 0. \) Hence \( H \) achieves...
maximum either at \( t = 0 \) or \( t = -\infty \). Notice that \( H(0) = H(-\infty) = 0 \), \( H(t) \leq 0 \) for all \( t < 0 \). This proves (4.22). Similarly, (4.23) holds.

Next we prove (4.7). Noting that \( p' = -pg_2 \), we have for \( t < 0 \)
\[
F(t) = \int_{-\infty}^{t} p(s) ds \\
\leq \int_{-\infty}^{t} \frac{g_2(s)p(s)}{g_2(t)} ds \\
= \int_{-\infty}^{t} \frac{-p'(s)}{g_2(t)} ds \\
= \frac{p(t)}{-g_2(t)} = \frac{p(t)}{|g_2(t)|}.
\]
Similarly, we have
\[
1 - F(t) \leq \frac{p(t)}{g_2(t)} \text{ for } t \geq 0.
\]
Hence (4.7) is satisfied with \( d_2 = 1 \).

Note that (4.6) and (4.7) imply that
\[
1 - F(x) \leq p(x) \min \left( \frac{1}{c_1}, \frac{1}{|g_2(x)|} \right) \text{ for } x \geq 0
\]
and
\[
F(x) \leq p(x) \min \left( \frac{1}{c_1}, \frac{1}{|g_2(x)|} \right) \text{ for } x \leq 0
\]
To verify (4.11), with \( x \geq 0 \) write
\[
E|Y|I_{\{Y > x\}} = xP(Y > x) + \int_{x}^{\infty} P(Y \geq t) dt \\
\leq xp(x) \min \left( \frac{1}{c_1}, \frac{1}{|g_2(x)|} \right) + \int_{x}^{\infty} p(t) \min \left( \frac{1}{c_1}, \frac{1}{|g_2(t)|} \right) dt \\
\leq xp(x) \min \left( \frac{1}{c_1}, \frac{1}{|g_2(x)|} \right) + \min \left( \frac{1}{c_1}, \frac{1}{|g_2(x)|} \right) \int_{x}^{\infty} p(t) dt \\
\leq \min \left( \frac{1}{c_1}, \frac{1}{|g_2(x)|} \right) \{xp(x) + (1 - F(x))\} \\
\leq \min \left( \frac{1}{c_1}, \frac{1}{|g_2(x)|} \right) \{xp(x) + p(x)/c_1\} \\
\leq p(x) \min \left( \frac{1}{c_1}, \frac{1}{|g_2(x)|} \right) \{x + 1/c_1\}.
\]
Similarly, for \( x < 0 \)

\[
(4.29) \quad E|Y|I_{\{Y < x\}} \leq p(x) \min \left( \frac{1}{c_1}, \frac{1}{|g_2(x)|} \right) \{|x| + 1/c_1\}.
\]

\[
(4.28) \quad \text{and } (4.29) \text{ with } x = 0 \text{ also give } E|Y| \leq 2/c_1.
\]

Hence, recalling \( (4.26) \)

\[
E|Y|I_{\{Y > x\}} + E|Y|(1 - F(x)) \leq p(x) \min \left( \frac{1}{c_1}, \frac{1}{|g_2(x)|} \right) \{|x| + 3/c_1\} \quad \text{for } x > 0
\]

and

\[
E|Y|I_{\{Y < x\}} + E|Y|F(x) \leq p(x) \min \left( \frac{1}{c_1}, \frac{1}{|g_2(x)|} \right) \{|x| + 3/c_1\} \quad \text{for } x \leq 0.
\]

Thus, \( (4.11) \) holds with \( d_3 = c_2 \) by \( (H2) \).

Equations \( (4.30) \) and \( (4.31) \) also show that \( (4.12) \) is satisfied with \( d_4 = c_2 \).

This completes the proof of Lemma 4.2.

From the proof of Lemma 4.2 one can see the following remark is true.

**Remark 4.1.** Assume that \( (H1) \) and \( (H3) \) are satisfied. Then \( (4.6) - (4.11) \) hold with \( d_1 = 1/c_1, d_2 = 1 \) and \( d_3 = c_3 \) and hence \( (4.13) \) and \( (4.15) \).

5. **Proof of main results.** In this section we prove the general error bounds (Theorems 1.1 and Theorem 1.2), the result for the Curie-Weiss model (Theorem 2.1), and Theorem 3.1.

5.1. **Proof of Theorem 1.1** Let \( f = f_h \) be the solution to Stein’s equation \( (4.3) \). Then

\[
(5.1) \quad Eh(W) - Eh(Y) = Ef'(W) + Ef(W)p'(W)/p(W) = Ef'(W) - c_0 Ef(W)g(W).
\]
Recall $\Delta = W - W'$ and observe that for any absolutely continuous function $f$

\begin{align*}
0 &= E(W - W')(f(W') + f(W)) \\
&= 2Ef(W)(W - W') + E(W - W')(f(W') - f(W)) \\
&= 2E\{f(W)E((W - W')|W)\} - E(W - W') \int_0^0 f'(W + t)dt \\
(5.2) &= 2Ef(W)g(W) + 2Ef(W)r(W) - E \int_0^0 \hat{K}(t)dt \\
&= 2Ef(W)g(W) + 2Ef(W)r(W) - E \int_\infty^{-\infty} f'(W + t) \hat{K}(t)dt
\end{align*}

where

$\hat{K}(t) = E\{\Delta(I{-\Delta \leq t} \leq 0) - I{0 < t \leq -\Delta})|W\}.$

Substituting (5.2) into (5.1) gives

\begin{align*}
Ef'(W) - c_0Ef(W)g(W) \\
&= Ef'(W) - (c_0/2) \Big\{ E \int_\infty^{-\infty} f'(W + t) \hat{K}(t)dt - 2Ef(W)r(W) \Big\} \\
&= E\{f'(W)(1 - (c_0/2)E(\Delta^2|W))\} \\
&\quad + (c_0/2)E \int_\infty^{-\infty} (f'(W) - f'(W + t)) \hat{K}(t)dt \\
(5.3) &= +c_0Ef(W)r(W).
\end{align*}

When $(H1)$ and $H(2)$ are satisfied, by Lemmas 4.1 and 4.2

\begin{align*}
\|f_h\| &\leq c_2 \|h'\|, \quad \|f'_h\| \leq (1 + c_2)\|h'\|/c_1, \quad \|f''_h\| \leq 2(1 + c_2)\|h'\| \\
\end{align*}

and hence

\begin{align*}
|Ef'_h(W) - c_0Ef_h(W)g(W)| &\leq \frac{(1 + c_2)\|h'\|}{c_1} E|1 - (c_0/2)E(\Delta^2|W))| \\
&\quad + (1 + c_2)\|h'\|c_0c_2|\Delta|^{3/2} + c_0c_2\|h'\|E|r(W)|.
\end{align*}

This proves (1.3).
Under \((H1)\) and \((H3)\), by Remark 4.1

\[
(5.5) \quad \|f'_h\| \leq (1 + c_3)\|h'\|/c_1, \quad \|f''_h\| \leq 2(1 + c_3)\|h'\|
\]

From (4.16), (4.30) and (4.31) it follows that

\[
|f(w)| \leq (1/p(w))\|h'\| \min(E|Y - Y*|I_{|Y \leq w}), E|Y - Y*|I_{|Y \geq w})
\]

\[
(5.6) \quad \|h'\| \min(1/c_1, 1/|g_2(w)|)(|w| + 3/c_1) \leq \|h'\|(|w| + 3/c_1)/c_1
\]

This proves (1.4) by (5.3), (5.5) and (5.6). \(\square\)

5.2. **Proof of Theorem 1.2** Since (1.5) is trivial when \(c_1c_3\delta > 1\), we assume

\[
(5.7) \quad c_1c_3\delta \leq 1.
\]

Let \(F\) be the distribution function of \(Y\) and let \(f = f_z\) be the solution to the equation

\[
(5.8) \quad f'(w) - c_0f(w)g(w) = I(w \leq z) - F(z).
\]

By (5.2)

\[
2Ef(W)g(W) + 2Ef(W)r(W)
\]

\[
= E \int_{-\infty}^{\infty} f'(W + t)\hat{K}(t)dt
\]

\[
= E \int_{-\delta}^{\delta} \{c_0f(W + t)g(W + t) + I(W + t \leq z) - F(z)\}\hat{K}(t)dt
\]

\[
\geq E \int_{-\delta}^{\delta} c_0f(W + t)g(W + t)\hat{K}(t)dt + EI(W \leq z - \delta)\Delta^2 - F(z)E\Delta^2
\]
and hence

\[ EI(W \leq z - \delta)\Delta^2 - F(z)E\Delta^2 \]

\[ \leq 2Ef(W)g(W) + 2Ef(W)r(W) + -c_0E \int_{-\delta}^{\delta} f(W + t)g(W + t)\hat{K}(t)dt \]

\[ = 2Ef(W)g(W)(1 - (c_0/2)E(\Delta^2|W)) + 2Ef(W)r(W) + c_0E \int_{-\delta}^{\delta} \{f(W)g(W) - f(W + t)g(W + t)\}\hat{K}(t)dt \]

(5.9) \[ J_1 + J_2 + J_3. \]

From Lemmas 4.1 and 4.2 again, we obtain

(5.10) \[ \|f_z\| \leq 2/c_1, \quad \|f_zg\| \leq 2/c_0 \quad \text{and} \quad \|f'_z\| \leq 4. \]

Therefore

(5.11) \[ |J_1| \leq (4/c_0)E|1 - (c_0/2)E(\Delta^2|W)|. \]

and

(5.12) \[ |J_2| \leq (4/c_1)E|r(W)| \]

To bound \( J_3 \), we first show that

(5.13) \[ \sup_{|t|\leq\delta} |g(w + t) - g(w)| \leq \frac{c_1c_3\delta}{2c_0}(c_1 + c_0|g(w)|). \]

From (H2) it follows that

(5.14) \[ |g'(x)| \leq \frac{c_1c_3}{3c_0\min(1/c_1, 1/|c_0g(x)|)} \]

\[ = \frac{c_1c_3}{3c_0}\max(c_1, |c_0g(x)|) \]

\[ \leq \frac{c_1c_3}{3c_0}(c_1 + |c_0g(x)|). \]
Thus by the mean value theorem

$$\sup_{|t| \leq \delta} |g(w + t) - g(w)|$$

$$\leq \delta \sup_{|t| \leq \delta} |g'(w + t)|$$

$$\leq \frac{c_1 c_3 \delta}{3 c_0} (c_1 + c_0 \sup_{|t| \leq \delta} |g(w + t)|)$$

$$\leq \frac{c_1 c_3 \delta}{3 c_0} (c_1 + c_0 |g(w)| + c_0 \sup_{|t| \leq \delta} |g(w + t) - g(w)|)$$

$$= \frac{c_1 c_3 \delta}{3 c_0} (c_1 + c_0 |g(w)|) + \frac{1}{3} \sup_{|t| \leq \delta} |g(w + t) - g(w)|$$

by (5.7). This proves (5.13).

Now by (5.10) and (5.13), when $|t| \leq \delta$

$$|f(w)g(w) - f(w + t)g(w + t)|$$

$$\leq |g(w)||f(w + t) - f(w)| + |f(w + t)||g(w + t) - g(w)|$$

$$\leq 4|g(w)||t| + \frac{2}{c_1} \frac{c_1 c_3 \delta}{2 c_0} (c_1 + c_0 |g(w)|)$$

$$\leq (4 + c_3)\delta|g(w)| + \delta c_1 c_3 / c_0.$$

Therefore

$$|J_3| \leq c_0 (4 + c_3)\delta E|g(W)|\Delta^2 + \delta c_1 c_3 E\Delta^2$$

(5.15)

$$\leq (4 + c_3)\delta^3 E|g(W)| + c_1 c_3 \delta^3.$$

Combining (5.9), (5.12), (5.11) and (5.15) shows that

$$EI(W \leq z - \delta)\Delta^2 - F(z)E\Delta^2$$

$$\leq (4/c_0)E|1 - (c_0/2)E(\Delta^2|W|) + (4/c_1)E|r(W)|$$

(5.16)$$+(4 + c_3)\delta^3 E|g(W)| + c_1 c_3 \delta^3.$$
On the other hand, using $F'(z) = p(z) \leq c_1$, we have

\[ EI(W \leq z - \delta) \Delta^2 - F(z) E \Delta^2 \]
\[ = \frac{2}{c_0} \left( EI(W \leq z - \delta) - F(z - \delta) \right) \]
\[ - \frac{2}{c_0} E \left\{ I(W \leq z - \delta) - F(z) \right\} \left( 1 - \frac{c_0}{2} E(\Delta^2 | W) \right) \]
\[ + \frac{2}{c_0} (F(z - \delta) - F(z)) \]
\[ \geq \frac{2}{c_0} \left( P(W \leq z - \delta) - F(z - \delta) \right) \]
\[ \frac{2}{c_0} E[1 - \frac{c_0}{2} E(\Delta^2 | W)] - \frac{2c_1 \delta}{c_0}, \]  

which together with (5.16) yields

\[ P(W \leq z - \delta) - F(z - \delta) \]
\[ \leq E[1 - (c_0/2) E(\Delta^2 | W)] + c_1 \delta \]
\[ + \frac{c_0}{2} \left( (4/c_0) E[1 - (c_0/2) E(\Delta^2 | W)] + (4/c_1) E|r(W)| \right) \]
\[ + (4 + c_3) \delta^3 E|c_0 g(W)| + c_1 c_3 \delta^3 \]
\[ = 3E[1 - (c_0/2) E(\Delta^2 | W)] + c_1 \delta + 2c_0 E|r(W)|/c_1 \]
\[ + \delta^3 c_0 \{(2 + c_3/2) E|c_0 g(W)| + c_1 c_3/2\}. \]  

Similarly, we have

\[ F(z + \delta) - P(W \leq z + \delta) \]
\[ \leq 3E[1 - (c_0/2) E(\Delta^2 | W)] + c_1 \delta + 2c_0 E|r(W)|/c_1 \]
\[ + \delta^3 c_0 \{(2 + c_3/2) E|c_0 g(W)| + c_1 c_3/2\}. \]  

This completes the proof of (1.5). \( \square \)

5.3. \textit{Proof of Theorem 2.1} By (2.1)-(2.4)

\[ E|r(W)| = O(n^{-2}), \]
\[ E|1 - (c_0/2)E((W - W')^2|W)| = O(n^{-1/2}), \]
\[ E|W|^3 = O(1) \]

Applying Theorem 1.2 gives Theorem 2.1 \( \square \)

We now show that (2.1) - (2.4) hold.

**Lemma 5.1.** With \( W, W' \) as in section 2, we have

\[ E\left| E(W - W'|W) - \frac{n^{-3/2}}{3}W^3 \right| \leq 15n^{-2}, \quad (5.22) \]
\[ E\left| E((W - W')^2|W) - 2n^{-3/2} \right| \leq 15n^{-2} \quad (5.23) \]

and

\[ E|W|^3 \leq 15. \quad (5.24) \]

Also, obviously, \( |W - W'| \leq 2n^{-3/4} \).

**Proof.** Let \( m = n^{-1} \sum_{i=1}^{n} \sigma_i = n^{-1/4}W \), and for each \( i \), let

\[ m_i = n^{-1} \sum_{j \neq i} \sigma_j. \]

It is easy to see that for \( \tau \in \{-1, 1\} \)

\[ P(\sigma'_i = \tau|\sigma) = \frac{e^{m_i\tau}}{e^{m_i} + e^{-m_i}}, \quad (5.25) \]

and so

\[ E(\sigma'_i|\sigma) = \frac{e^{m_i}}{e^{m_i} + e^{-m_i}} - \frac{e^{-m_i}}{e^{m_i} + e^{-m_i}} = \tanh m_i. \]

Hence

\[ E(W - W'|\sigma) = \frac{1}{n} \sum_{i=1}^{n} n^{-3/4}(\sigma_i - E(\sigma'_i|\sigma)) \]
\[ = n^{-3/4}m - n^{-7/4} \sum_{i=1}^{n} \tanh m_i. \quad (5.26) \]
Now it is easy to verify that the function
\[
\frac{d^2}{dx^2} \tanh x = \frac{-2 \sinh x}{\cosh^3 x} = -2(\tanh x)(1 - \tanh^2 x)
\]
has exactly two extrema ±\(x^*\) on the real line, where \(x^*\) solves the equation \(\tanh^2 x^* = \frac{1}{3}\). It follows that the maximum magnitude of this function is \(4/3^{3/2}\). Thus, for all \(x, y \in \mathbb{R}\),
\[|\tanh x - \tanh y - (x - y)(\cosh y)^{-2}| \leq \frac{2(x - y)^2}{3^{3/2}}.
\]
It follows that
\[
\left| \sum_{i=1}^{n} \tanh m_i - n \tanh m + n^{-1}(\cosh m)^{-2} \sum_{i=1}^{n} \sigma_i \right| \leq \frac{2n^{-1}}{3^{3/2}},
\]
and therefore
\[
\left| \sum_{i=1}^{n} \tanh m_i - n \tanh m \right| \leq |m| + \frac{2n^{-1}}{3^{3/2}}.
\]
Using this in (5.26) and the relation \(m = n^{-1/4}W\), we get
\[
(5.27) \quad |E(W - W'|\sigma) + n^{-3/4} \tanh m - n^{-3/4}m| \leq n^{-2}|W| + \frac{2n^{-11/4}}{3^{3/2}}.
\]
Now consider the function \(f(x) = \tanh x - x + \frac{x^3}{3}\). Note that \(f'(x) = (\cosh x)^{-2} - 1 + x^2 \geq 0\) for all \(x\), and hence \(f\) is an increasing function. Also \(f(0) = 0\). Therefore \(f(x) \geq 0\) for all \(x \geq 0\). Now, it can be easily verified that the first four derivatives of \(f\) vanish at zero, and for all \(x \geq 0\),
\[
\frac{d^5 f}{dx^5} = \frac{16}{\cosh^2 x} - 120 \frac{\sinh^2 x}{\cosh^4 x} + 120 \frac{\sinh^4 x}{\cosh^6 x} \leq \frac{16}{\cosh^2 x} \leq 16.
\]
Thus, for all \(x \geq 0\),
\[0 \leq f(x) \leq \frac{16}{5^1} x^5 = \frac{2x^5}{15}.
\]
Since \(f\) is an odd function, we get that for all \(x\),
\[|\tanh x - x + \frac{1}{3}x^3| \leq \frac{2|x|^5}{15}.
\]
Using this information in (5.27), we get
\[ |E(W - W'|\sigma) - \frac{n^{-3/4}}{3} m^3| \leq \frac{2n^{-3/4}|m|^5}{15} + n^{-2}|W| + \frac{2n^{-11/4}}{3^{3/2}}. \]

Using the relation \( m = n^{-1/4}W \), we get
\[ (5.28) \quad |E(W - W'|\sigma) - \frac{n^{-3/2}}{3} W^3| \leq \frac{2n^{-2}|W|^5}{15} + n^{-2}|W| + \frac{2n^{-11/4}}{3^{3/2}}. \]

This implies, in particular, that
\[ (5.29) \quad |E((W - W')W^3) - \frac{n^{-3/2}}{3} E(W^6)| \leq \frac{2n^{-2}E(W^8)}{15} + n^{-2}E(W^4) + \frac{2n^{-11/4}E|W|^3}{3^{3/2}}. \]

Thus,
\[ (5.30) \quad E(W^6) \leq 3n^{3/2}|E((W' - W)W^3)| + \frac{2n^{-1/2}E(W^8)}{5} + 3n^{-1/2}E(W^4) + \frac{2n^{-5/4}E|W|^3}{3^{1/2}}. \]

Using the crude bound \(|W| \leq n^{1/4}\), we get
\[ (5.31) \quad \frac{2n^{-1/2}E(W^8)}{5} + 3n^{-1/2}E(W^4) + \frac{2n^{-5/4}E|W|^3}{3^{1/2}} \leq \frac{2E(W^6)}{5} + 3E(W^2) + \frac{2n^{-1}E(W^2)}{3^{1/2}}. \]

Next, note that by the exchangeability of \((W,W')\),
\[ E((W' - W)W^3) = \frac{1}{2} E((W' - W)(W^3 - W'^3)) \]
\[ = -\frac{1}{2} E((W' - W)^2(W^2 + WW' + W'^2)). \]

Since \(|W - W'| \leq 2n^{-3/4}\), this gives
\[ (5.32) \quad |E((W' - W)W^3)| \leq 6n^{-3/2}E(W^2). \]

Combining (5.30), (5.31), and (5.32), we get
\[ E(W^6) \leq \left(21 + \frac{2n^{-1}}{3^{1/2}}\right)E(W^2) + \frac{2E(W^6)}{5}, \]
and therefore,
\[ E(W^6) \leq \frac{5}{3} \left( 21 + \frac{2n^{-1}}{3^{1/2}} \right) E(W^2) \leq 36.9245E(W^2). \]

Since \( E(W^2) \leq (E(W^6))^{1/3} \), this gives
\[ (5.33) \quad E(W^6) \leq (36.9245)^{3/2} \leq 224.4. \]

and hence (5.24) holds.

Combined with (5.28), this gives
\[ E \left( (W - W')^2 | W \right) - 2n^{-3/2} \| | W \| | \leq \frac{2(224.4)^{5/6}}{15} + (224.4)^{1/6} + \frac{2n^{-11/4}}{3^{3/2}} \leq 15n^{-2}. \]

By (5.33), we have
\[
E((W - W')^2 | \sigma) = \frac{1}{n} \sum_{i=1}^{n} 4n^{-3/2} \frac{e^{-m_i \sigma_i}}{e^{m_i \sigma_i} + e^{-m_i \sigma_i}}
= 2n^{-5/2} \sum_{i=1}^{n} (1 - \tanh(m_i \sigma_i))
= 2n^{-3/2} - 2n^{-5/2} \sum_{i=1}^{n} \sigma_i \tanh m_i.
\]

Using \( |\tanh m_i - \tanh m| \leq |m_i - m| \leq n^{-1} \), we get
\[
\left| E((W - W')^2 | \sigma) - 2n^{-3/2} \right|
\leq 2n^{-5/2} + 2n^{-3/2} m \tanh m
\leq 2n^{-5/2} + 2n^{-3/2} m^2
= 2n^{-5/2} + 2n^{-2} W^2.
\]

Using (5.33), we get
\[
E \left| E((W - W')^2 | W) - 2n^{-3/2} \right| \leq 2n^{-5/2} + 2n^{-2} (224.4)^{1/3} \leq 15n^{-2}.
\]

This completes the proof of the lemma. \( \square \)
5.4. **Proof of Theorem 3.1**  
With \(p(w) = e^{-w}I_{\{w>0\}}\), for given \(h\), let \(f_h\) be the Stein solution given in (4.4)
\[
f_h(w) = e^w \int_0^w (h(t) - Eh(Y))e^{-t}dt = -e^w \int_0^w (h(t) - Eh(Y))e^{-t}dt
\]
for \(w \geq 0\). Following the proof of Theorems 1.1 and 1.2, it suffices to show that
\[
|f_h(w)| \leq 3 \min(|h|, |h'|)w \quad \text{for } w \geq 0.
\]
By (4.17),
\[
|f_h(w)| \leq 2|h| \min(1 - e^{-w}, e^{-w})e^w = 2|h| \min(1, e^w - 1) \leq 3w|h|
\]
and by (4.21)
\[
|f_h(w)| \leq |h'|e^w \min(-we^{-w} + 2(1 - e^{-w}), (w + 1)e^{-w}) \\
\leq |h'| \min(w + 1, 2(e^w - 1)) \leq 3w|h'|.
\]
This proves (5.35) and hence Theorem 3.1.

**Acknowledgement.** The authors thank Larry Goldstein for helping on the exponential approximation and thank an anonymous referee and an Associate editor for their helpful comments.

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