Spectral gaps, missing faces and minimal degrees

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Abstract

Let \( X \) be a simplicial complex with \( n \) vertices. A missing face of \( X \) is a simplex \( \sigma \not\in X \) such that \( \tau \in X \) for any \( \tau \subseteq \sigma \). For a \( k \)-dimensional simplex \( \sigma \) in \( X \), its degree in \( X \) is the number of \((k+1)\)-dimensional simplices in \( X \) containing it. Let \( \delta_k \) denote the minimal degree of a \( k \)-dimensional simplex in \( X \). Let \( L_k \) denote the \( k \)-Laplacian acting on real \( k \)-cochains of \( X \) and let \( \mu_k(X) \) denote its minimal eigenvalue. We prove the following lower bound on the spectral gaps \( \mu_k(X) \), for complexes \( X \) without missing faces of dimension larger than \( d \):

\[
\mu_k(X) \geq (d + 1)(\delta_k + k + 1) - dn.
\]

As a consequence we obtain a new proof of a vanishing result for the homology of simplicial complexes without large missing faces. We present a family of examples achieving equality at all dimensions, showing that the bound is tight. For \( d = 1 \) we characterize the equality case.

Keywords: High dimensional Laplacian, Simplicial cohomology.

1 Introduction

Let \( X \) be a finite simplicial complex on vertex set \( V \). For \( k \geq -1 \) let \( X(k) \) denote the set of all \( k \)-dimensional simplices of \( X \), let \( C^k(X) \) be the space of real valued \( k \)-cochains of \( X \) and let \( d_k : C^k(X) \to C^{k+1}(X) \) be the coboundary operator. The reduced \( k \)-dimensional Laplacian of \( X \) is defined by

\[
L_k(X) = d_{k-1}d_{k-1}^* + d_k^*d_k.
\]

\( L_k \) is a positive semi-definite operator from \( C^k(X) \) to itself. The \( k \)-th spectral gap of \( X \), denoted by \( \mu_k(X) \), is the smallest eigenvalue of \( L_k \).

A missing face of \( X \) is a subset \( \sigma \subset V \) such that \( \sigma \not\in X \) but \( \tau \in X \) for any \( \tau \subset \sigma \). Let \( h(X) \) denote the maximal dimension of a missing face of \( X \). For example, \( h(X) = 1 \) if and only if \( X \) is the clique complex of a graph \( G \) (the missing faces of \( X \) are the edges of the complement of \( G \)).

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Let $\sigma \in X(k)$. The degree of $\sigma$ in $X$ is defined by
$$\deg_X(\sigma) = |\{\eta \in X(k+1) : \sigma \subset \eta\}|.$$  

Let $\delta_k = \delta_k(X)$ be the minimal degree of a $k$-dimensional simplex, that is,
$$\delta_k = \min_{\sigma \in X(k)} \deg_X(\sigma).$$

Our main result is the following lower bound on the spectral gaps of $X$:

**Theorem 1.1.** Let $X$ be a simplicial complex on vertex set $V$ of size $n$, with $h(X) = d$. Then for $k \geq -1$,
$$\mu_k(X) \geq (d+1)(\delta_k + k + 1) - dn.$$  

As a consequence we obtain a new proof of the following known result (see [1, Prop. 5.4]):

**Theorem 1.2.** Let $X$ be a simplicial complex on vertex set $V$ of size $n$, with $h(X) = d$. Then $\tilde{H}^k(X; \mathbb{R}) = 0$ for all $k > \frac{d}{d+1}n - 1$.

The proof of Theorem 1.1 relies on two main ingredients. The first one is the following theorem of Geršgorin (see [9, Chapter 6]).

**Theorem 1.3 (Geršgorin circle theorem).** Let $A \in \mathbb{C}^{n \times n}$ and $\lambda \in \mathbb{C}$ be an eigenvalue of $A$. Then there is some $i \in [n]$ such that
$$|\lambda - A_{ii}| \leq \sum_{j \neq i} |A_{ij}|.$$  

The second ingredient is the following inequality concerning sums of degrees of simplices in $X$, which generalizes a known result for clique complexes (see [2, Claim 3.4], [3]):

**Lemma 1.4.** Let $X$ be a simplicial complex on vertex set $V$ of size $n$, with $h(X) = d$. Let $k \geq 0$ and $\sigma \in X(k)$. Then
$$\sum_{\tau \in \sigma(k-1)} \deg_X(\tau) - (k - d + 1) \deg_X(\sigma) \leq dn - (d - 1)(k + 1).$$  

Lemma 1.4 is closely related to Lemma 2.8 in [10]. For completeness, we include a full proof in Section 3.

Let $X$ and $Y$ be two simplicial complexes on disjoint vertex sets. The join of $X$ and $Y$ is the complex
$$X \ast Y = \{\sigma \cup \tau : \sigma \in X, \tau \in Y\}.$$  

We will denote by $X \ast X$ the join of $X$ with a disjoint copy of itself. Also, we will denote the complex $X \ast X \ast \cdots \ast X$ ($k$ times) by $X^k$.  

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Let $\Delta_m$ be the complete simplicial complex on $m + 1$ vertices, and let $\Delta_m^{(k)}$ be its $k$-dimensional skeleton, i.e. the complex whose simplices are all the sets $\sigma \subset [m + 1]$ such that $|\sigma| \leq k + 1$. The following example shows that the inequalities in Theorem 1.1 are tight:

Let $Z = (\Delta_1^{(d-1)})^* * \Delta_{r-1}$. Note that all the missing faces of $Z$ are of dimension $d$, and $\dim(Z) = dt + r - 1$. Let $n = (d + 1)t + r$ be the number of vertices of $Z$. We have:

**Proposition 1.5.**

$$\mu_k(Z) = \begin{cases} (d + 1) \left( t - \left\lfloor \frac{k + 1}{d} \right\rfloor \right) + r & \text{if } -1 \leq k \leq dt - 1, \\ r & \text{if } dt \leq k \leq dt + r - 1, \end{cases}$$

and

$$\delta_k(Z) = \begin{cases} n - (k + 1) - \left\lfloor \frac{k + 1}{d} \right\rfloor & \text{if } -1 \leq k \leq dt - 1, \\ n - (k + 1) - t & \text{if } dt \leq k \leq dt + r - 1. \end{cases}$$

In particular,

$$\mu_k(Z) = (d + 1)(\delta_k(Z) + k + 1) - dn$$

for all $-1 \leq k \leq \dim(Z)$.

Now we look at the case $d = 1$. If $X$ is a clique complex with $n$ vertices, then by Theorem 1.1 we have $\mu_k(X) \geq 2(\delta_k + k + 1) - n$ for all $k \geq -1$, and we found a family of examples achieving equality in all dimensions. In particular, at the top dimension $k_t = \dim(Z)$ we obtain $\mu_k(Z) = 2(k_t + 1) - n$. The next proposition shows that these are the only examples achieving such an equality:

**Proposition 1.6.** Let $X$ be a clique complex on vertex set $V$ of size $n$, such that $\mu_k(X) = 2(k + 1) - n$ for some $k$. Then

$$X \cong (\Delta_1^{(0)})^{*n-k-1} * \Delta_{2(k+1)-n-1},$$

(and in particular, $\dim(X) = k$).

The paper is organized as follows: In Section 2 we recall some definitions and results on simplicial cohomology and high dimensional Laplacians that we will use later. In Section 3 we prove Lemma 1.4. Section 4 contains the proofs of Theorems 1.1 and 1.2. In Section 5 we prove Propositions 1.5 and 1.6.

## 2 Preliminaries

Let $X$ be a simplicial complex on vertex set $V$, where $|V| = n$. An ordered simplex is a simplex with a linear order of its vertices. For two ordered
simplices $\sigma$ and $\tau$ denote by $[\sigma, \tau]$ their ordered union. For $v \in V$ denote by $v\sigma$ the ordered union of $\{v\}$ and $\sigma$.

For $\tau \subset \sigma$, both given an order on their vertices, we define $(\sigma : \tau)$ to be the sign of the permutation on the vertices of $\sigma$ that maps the ordered simplex $\sigma$ to the ordered simplex $[\sigma \setminus \tau, \tau]$ (where the order on the vertices of $\sigma \setminus \tau$ is the one induced by the order on $\sigma$).

A simplicial $k$-cochain is a real valued skew-symmetric function on all ordered $k$-dimensional simplices. That is, $\phi$ is a $k$-cochain if for any two ordered $k$-dimensional simplices $\sigma, \tilde{\sigma}$ in $X$ that are equal as sets, it satisfies $\phi(\tilde{\sigma}) = (\tilde{\sigma} : \sigma)\phi(\sigma)$.

For $k \geq -1$ let $X(k)$ be the set of $k$-dimensional simplices of $X$, each given some fixed order on its vertices. Let $C^k(X)$ denote the space of $k$-cochains on $X$. For $k = -1$ we have $X(-1) = \{\emptyset\}$, so we can identify $C^{-1}(X) = \mathbb{R}$.

For $\sigma \in X(k)$, let

$$\text{lk}(X, \sigma) = \{\tau \in X : \tau \cup \sigma \in X, \tau \cap \sigma = \emptyset\}$$

be the link of $\sigma$ in $X$. For $U \subset V$, let $X[U] = \{\sigma \in X : \sigma \subset U\}$ be the subcomplex of $X$ induced by $U$.

The coboundary operator $d_k : C^k(X) \to C^{k+1}(X)$ is the linear operator defined by

$$d_k \phi(\sigma) = \sum_{\tau \in \sigma(k)} (\sigma : \tau)\phi(\tau),$$

where $\sigma(k) \subset X(k)$ is the set of all $k$-dimensional faces of $\sigma$.

For $k = -1$ we have, under the identification $C^{-1}(X) = \mathbb{R}$, $d_{-1}a(v) = a$ for every $a \in \mathbb{R}$, $v \in V$.

Let $\tilde{H}^k(X; \mathbb{R}) = \text{Ker}(d_k)/\text{Im}(d_{k-1})$ be the $k$-th reduced cohomology group of $X$ with real coefficients.

We define an inner product on $C^k(X)$ by

$$\langle \phi, \psi \rangle = \sum_{\sigma \in X(k)} \phi(\sigma)\psi(\sigma).$$

This induces a norm on $C^k(X)$:

$$\|\phi\| = \left( \sum_{\sigma \in X(k)} \phi(\sigma)^2 \right)^{1/2}.$$ 

Let $d^*_k : C^{k+1}(X) \to C^k(X)$ be the adjoint of $d_k$ with respect to this inner product.

Let $k \geq 0$. The reduced $k$-Laplacian of $X$ is the positive semi-definite operator on $C^k(X)$ given by $L_k = d_{k-1}d_{k-1}^* + d_k^*d_k$. 

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For $k = -1$ define $L_{-1} = d_1^* d_{-1} : \mathbb{R} \to \mathbb{R}$. We have $L_{-1}(a) = n$ for all $a \in \mathbb{R}$.

Let $\sigma \in X(k)$. We define the $k$-cochain $1_\sigma$ by

$$1_\sigma(\tau) = \begin{cases} (\sigma : \tau) & \text{if } \sigma = \tau \text{ (as sets)}, \\ 0 & \text{otherwise.} \end{cases}$$

The set $\{1_\sigma\}_{\sigma \in X(k)}$ forms a basis of the space $C_k(X)$. We identify $L_k$ with its matrix representation with respect to the basis $\{1_\sigma\}_{\sigma \in X(k)}$. We denote the matrix element of $L_k$ at index $(1_\sigma, 1_\tau)$ by $L_k(\sigma, \tau)$.

We can write the matrix $L_k$ explicitly (see e.g. [4, 7]):

**Claim 2.1.** For $k \geq 0$

$$L_k(\sigma, \tau) = \begin{cases} \deg_X(\sigma) + k + 1 & \text{if } \sigma = \tau, \\ (\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) & \text{if } |\sigma \cap \tau| = k, \sigma \cup \tau \notin X(k+1), \\ 0 & \text{otherwise.} \end{cases}$$

An important property of the Laplacian operators is their relation to the cohomology of the complex $X$, first observed by Eckmann in [5]:

**Theorem 2.2** (Simplicial Hodge theorem).

$$\tilde{H}^k(X; \mathbb{R}) \cong \ker(L_k).$$

In particular we obtain

**Corollary 2.3.** $\tilde{H}^k(X; \mathbb{R}) = 0$ if and only if $\mu_k(X) > 0$.

Let $\Spec_k(X)$ be the spectrum of $L_k(X)$, i.e. a multiset whose elements are the eigenvalues of the Laplacian. The following theorem allows us to compute the spectrum of the join of simplicial complexes (see [4, Theorem 4.10]).

**Theorem 2.4.** Let $X = X_1 \ast \cdots \ast X_m$. Then

$$\Spec_k(X) = \bigcup_{i_1 + \ldots + i_m = k-m+1, \atop 1 \leq i_j \leq \dim(X_j) \forall j \in [m]} \Spec_{i_1}(X_1) + \cdots + \Spec_{i_m}(X_m),$$

We will need the following well known result on the spectrum of the complex $\Delta^{(k)}_{n-1}$ (see e.g. [8, Lemma 8]):

**Claim 2.5.**

$$\Spec_i(\Delta^{(k)}_{n-1}) = \begin{cases} \{n, n, \ldots, n\} & \text{if } -1 \leq i \leq k-1, \\ \binom{n}{i+1} \text{ times} & \text{if } i = k. \end{cases}$$
3 Sums of degrees

In this section we prove Lemma 1.4.

Claim 3.1. Let $X$ be a simplicial complex on vertex set $V$. Let $k \geq 0$ and $\sigma \in X(k)$. Then

$$\sum_{\tau \in \sigma(k-1)} \deg_X(\tau) = (k + 1)(\deg_X(\sigma) + 1)$$

$$+ \sum_{v \in V \setminus \sigma, \ v \not\in \text{lk}(X, \sigma)} |\{\tau \in \sigma(k-1) : v \in \text{lk}(X, \tau)\}|.$$

Proof.

$$\sum_{\tau \in \sigma(k-1)} \deg_X(\tau) = \sum_{\tau \in \sigma(k-1)} \sum_{v \in \text{lk}(X, \tau)} 1 = \sum_{v \in V} \sum_{\tau \in \sigma(k-1), \ \tau \in \text{lk}(X, v)} 1$$

$$= \sum_{v \in \sigma} \sum_{\tau \in \sigma(k-1), \ \tau \in \text{lk}(X, v)} 1 + \sum_{v \in \text{lk}(X, \sigma)} \sum_{\tau \in \sigma(k-1), \ \tau \in \text{lk}(X, v)} 1 + \sum_{v \in V \setminus \sigma} \sum_{\tau \in \sigma(k-1), \ \tau \in \text{lk}(X, v)} 1.$$  (3.1)

We consider separately the first two summands on the right hand side of (3.1):

1. For $v \in \sigma$, there is only one $\tau \in \sigma(k-1)$ such that $\tau \in \text{lk}(X, v)$, namely $\tau = \sigma \setminus \{v\}$. Thus the first summand is $k + 1$.

2. For $v \in \text{lk}(X, \sigma)$, any $\tau \in \sigma(k-1)$ is in $\text{lk}(X, v)$, therefore the second summand is $(k + 1) \deg_X(\sigma)$.

Thus we obtain

$$\sum_{\tau \in \sigma(k-1)} \deg_X(\tau) = (k + 1)(\deg_X(\sigma) + 1)$$

$$+ \sum_{v \in V \setminus \sigma, \ v \not\in \text{lk}(X, \sigma)} |\{\tau \in \sigma(k-1) : v \in \text{lk}(X, \tau)\}|.$$

Proof of Lemma 1.4. By Claim 3.1 we have

$$\sum_{\tau \in \sigma(k-1)} \deg_X(\tau) = (k + 1)(\deg_X(\sigma) + 1)$$

$$+ \sum_{v \in V \setminus \sigma, \ \tau \in \sigma(k-1), \ v \not\in \text{lk}(X, \sigma)} 1.$$  (3.1)

Let $v \in V \setminus \sigma$ such that $v \not\in \text{lk}(X, \sigma)$. For each $\tau \in \sigma(k-1)$ such that $\tau \in \text{lk}(X, v)$, let $u$ be the unique vertex in $\sigma \setminus \tau$. Since $v \tau \in X$ but $v \sigma \not\in X$,
must belong to every missing face of $X$ contained in $v\sigma$. Also $v$ must belong to every such missing face, since $\sigma \in X$. Therefore, since all the missing faces of $X$ are of size at most $d + 1$, there can be at most $d$ such different vertices $u$, so

$$|\{ \tau \in \sigma(k - 1) : \tau \in \text{lk}(X, v) \}| \leq d.$$ 

Thus we obtain

$$\sum_{\tau \in \sigma(k-1)} \deg_X(\tau) \leq (k + 1)(\deg_X(\sigma) + 1) + \sum_{v \in V \setminus \sigma, v \notin \text{lk}(X, \sigma)} d \leq (k + 1)(\deg_X(\sigma) + 1) + (n - k - 1 - \deg_X(\sigma))d = dn - (d - 1)(k + 1) + (k - d + 1) \deg_X(\sigma).$$

\[4.1\]

\[4\] Main results

In this section we prove our main results, Theorem 1.1 and its corollary Theorem 1.2.

Proof of Theorem 1.1. For $k = -1$ the claim holds since $\delta^{-1}(X) = \mu^{-1}(X) = n$. Assume now $k \geq 0$. By Claim 2.1 we have for $\sigma \in X(k)$

$$L_k(\sigma, \sigma) = \deg_X(\sigma) + k + 1$$

and

$$\sum_{\eta \in X(k), \eta \neq \sigma} |L_k(\sigma, \eta)| = |\{ \eta \in X(k) : |\sigma \cap \eta| = k, \sigma \cup \eta \notin X(k + 1) \}|$$

$$= \sum_{\tau \in \sigma(k-1)} |\{ v \in V \setminus \sigma : v \in \text{lk}(X, \tau), v \notin \text{lk}(X, \sigma) \}|$$

$$= \sum_{\tau \in \sigma(k-1)} (\deg_X(\tau) - 1 - \deg_X(\sigma))$$

$$= \sum_{\tau \in \sigma(k-1)} \deg_X(\tau) - (k + 1)(\deg_X(\sigma) + 1).$$

(4.1)
So by Geršgorin’s theorem (Theorem 1.3) we obtain

\[
\mu_k(X) \geq \min_{\sigma \in X(k)} \left( L_k(\sigma, \sigma) - \sum_{\eta \in X(k), \eta \neq \sigma} |L_k(\sigma, \eta)| \right)
\]

\[
= \min_{\sigma \in X(k)} \left( \deg_X(\sigma) + k + 1 - \sum_{\tau \in \sigma \cap (k-1)} \deg_X(\tau) + (k+1)(\deg_X(\sigma) + 1) \right)
\]

\[
= \min_{\sigma \in X(k)} \left( (k + 2) \deg_X(\sigma) + 2(k + 1) - \sum_{\tau \in \sigma \cap (k-1)} \deg_X(\tau) \right). \quad (4.2)
\]

Recall that by Lemma 1.4 we have

\[
\sum_{\tau \in \sigma \cap (k-1)} \deg_X(\tau) - (d - 1)(k + 1) \leq (d - 1)(k + 1). \quad (4.3)
\]

Combining (4.2) and (4.3) we obtain

\[
\mu_k(X) \geq \min_{\sigma \in X(k)} \left( (d + 1)(\deg_X(\sigma) + k + 1) - dn \right)
\]

\[
= (d + 1)(\delta_k + k + 1) - dn,
\]

as wanted.

\[\square\]

**Remark.** A slightly different approach to the proof of Theorem 1.1 is the following: We build a graph \(G_k\) on vertex set \(V_k = X(k)\), with edge set

\[
E_k = \{ \{\sigma, \tau\} : \sigma, \tau \in X(k), |\sigma \cap \tau| = k, \sigma \cup \tau \notin X(k+1) \}.
\]

We make \(G_k\) into a signed graph (see [11]) by defining the sign function \(\phi : E_k \to \{-1, +1\}\) by

\[
\phi(\{\sigma, \tau\}) = -(\sigma : \sigma \cap \tau)(\tau : \sigma \cap \tau).
\]

The incidence matrix \(H_k\) is the \(V_k \times E_k\) matrix

\[
H_k(\sigma, \{\eta, \tau\}) = \begin{cases} (\sigma : \eta \cap \tau) & \text{if } \sigma \in \{\eta, \tau\}, \\ 0 & \text{otherwise.} \end{cases}
\]

Define the Laplacian of \(G_k\) to be the \(V_k \times V_k\) matrix \(K_k = H_kH_k^T\). So \(K_k\) is
positive semi-definite, and we have

\[ K_k(\sigma, \tau) = \begin{cases} 
\deg_G(\sigma) & \text{if } \sigma = \tau, \\
-\phi(\{\sigma, \tau\}) & \text{if } \{\sigma, \tau\} \in E_k, \\
0 & \text{otherwise.}
\end{cases} \]

By Equation (4.1) we obtain

\[ L_k = D_k + K_k, \]

where \( D_k \) is the diagonal matrix with diagonal elements

\[ D_k(\sigma, \sigma) = 2(k + 1) + (k + 2) \deg_X(\sigma) - \sum_{\tau \in \sigma(k-1)} \deg_X(\tau). \]

This decomposition of the Laplacian first appeared in [6], where the graph Laplacian \( K_k \) is called the Bochner Laplacian of \( X \).

Using the fact that \( K_k \) is positive semi-definite, and applying Lemma 1.4 as before, we obtain Theorem 1.1.

Now we can prove Theorem 1.2:

**Proof of Theorem 1.2.** Let \( k > \frac{d}{d+1}n - 1 \). By Theorem 1.1 we have

\[ \mu_k(X) \geq (d + 1)(k + 1) - dn > (d + 1)\frac{d}{d+1}n - dn = 0. \]

So by the simplicial Hodge theorem (Corollary 2.3), \( \tilde{H}^k(X; \mathbb{R}) = 0 \).

5 Extremal examples

In this section we prove Propositions 1.5 and 1.6 about complexes achieving equality in Theorem 1.1.

**Proof of Proposition 1.5.** Let \( -1 \leq k \leq dt + r - 1 \). By Theorem 2.4 we have

\[ \mu_k(Z) = \min \left\{ \mu_{i_1}(\Delta_d^{(d-1)}) + \cdots + \mu_{i_t}(\Delta_d^{(d-1)}) + \mu_j(\Delta_{r-1}) : -1 \leq i_1, \ldots, i_t \leq d-1, \right. \]

\[ \left. -1 \leq j \leq r - 1, \right\}_{i_1 + \cdots + i_t + j = k-t}. \]

By Claim 2.5 we have

\[ \mu_j(\Delta_d^{(d-1)}) = \begin{cases} 
 d + 1 & \text{if } -1 \leq j \leq d - 2, \\
0 & \text{if } j = d - 1,
\end{cases} \]
Figure 1: A simplex $\sigma \in Z$. The black dots are the vertices in $\sigma$. The white dots are the vertices in $\text{lk}(Z, \sigma)$. The crosses are the vertices in $V \setminus \sigma$ that do not belong to $\text{lk}(Z, \sigma)$ (these are the vertices that, when added to $\sigma$, complete a missing face).

and $\mu_j(\Delta_{r-1}) = r$ for all $-1 \leq j \leq r-1$. Therefore $\mu_k(Z) = (d+1)(t-m)+r$, where $m$ is the maximal number of indices in $i_1, \ldots, i_t$ that can be chosen to be equal to $d-1$. That is, $m$ is the maximal integer between 0 and $t$ such that there exist $-1 \leq i_1, \ldots, i_{t-m} \leq d-2$ and $-1 \leq j \leq r-1$ satisfying

$$m(d - 1) + i_1 + \cdots + i_{t-m} + j = k - t.$$  

We obtain

$$m = \begin{cases} \left\lfloor \frac{k+1}{d} \right\rfloor & \text{if } -1 \leq k \leq dt - 1, \\ t & \text{if } dt \leq k \leq dt + r - 1. \end{cases}$$

So

$$\mu_k(Z) = \begin{cases} (d+1)(t - \left\lfloor \frac{k+1}{d} \right\rfloor) + r & \text{if } -1 \leq k \leq dt - 1, \\ r & \text{if } dt \leq k \leq dt + r - 1. \end{cases}$$

Next we consider the degrees of simplices in $Z$: Let $V$ be the vertex set of $Z$. Recall that $|V| = n = (d+1)t + r$. Let $V_1, \ldots, V_t \subset V$ be the vertex sets of the $t$ copies of $\Delta_{d-1}$.  

Let $\sigma \in Z(k)$. A vertex $v \in V \setminus \sigma$ belongs to $\text{lk}(Z, \sigma)$ unless $V_i \subset \sigma \cup \{v\}$ for some $i \in [t]$ (see Figure 1). Therefore $\deg_Z(\sigma) = n - (k+1) - s(\sigma)$, where

$$s(\sigma) = |\{i \in [t] : |\sigma \cap V_i| = d\}|.$$

So the minimal degree of a simplex in $Z(k)$ is

$$\delta_k(Z) = \begin{cases} n - (k+1) - \left\lfloor \frac{k+1}{d} \right\rfloor & \text{if } -1 \leq k \leq dt - 1, \\ n - (k+1) - t & \text{if } dt \leq k \leq dt + r - 1. \end{cases}$$

Therefore $\delta_k(Z) = n - (k+1) - m$, thus

$$(d+1)(\delta_k(Z) + k + 1) - dn = n - (d+1)m = \mu_k(Z).$$
Proof of Proposition 1.6. Let $\sigma_0 \in X(k)$ such that

$$
\min_{\sigma \in X(k)} \left( (k + 2) \deg_X(\sigma) + 2(k + 1) - \sum_{\tau \in \sigma(k-1)} \deg_X(\tau) \right) = (k + 2) \deg_X(\sigma_0) + 2(k + 1) - \sum_{\tau \in \sigma_0(k-1)} \deg_X(\tau).
$$

By Inequality (4.2) and Lemma (1.4) we have

$$
2(k + 1) - n = \mu_k(X) \geq \min_{\sigma \in X(k)} \left( (k + 2) \deg_X(\sigma) + 2(k + 1) - \sum_{\tau \in \sigma(k-1)} \deg_X(\tau) \right) = (k + 2) \deg_X(\sigma_0) + 2(k + 1) - \sum_{\tau \in \sigma_0(k-1)} \deg_X(\tau).
$$

So all the inequalities are actually equalities, therefore we obtain

$$
\deg_X(\sigma_0) = 0
$$

and

$$
\sum_{\tau \in \sigma_0(k-1)} \deg_X(\tau) = n.
$$

By Claim 3.1 we have

$$
n = \sum_{\tau \in \sigma_0(k-1)} \deg_X(\tau) = k+1 + \sum_{\substack{v \in V \setminus \sigma_0, \\ v \not\in \text{lk}(X,\sigma_0)}} |\{\tau \in \sigma_0(k-1) : v \in \text{lk}(X,\tau)\}|.
$$

(5.1)

Let $v \in V \setminus \sigma_0$ and $\tau \in \sigma_0(k-1)$. Denote $\{u\} = \sigma_0 \setminus \tau$. We have

$$
v \not\in \text{lk}(X,\sigma_0) \quad \text{and} \quad v \in \text{lk}(X,\tau) \iff \text{the only missing face of } X \text{ contained in } v\sigma_0 \text{ is the edge } uv.
$$

Thus, if $v \not\in \text{lk}(X,\sigma_0)$ and $v \in \text{lk}(X,\tau)$, then for any $\tau \neq \tau' \in \sigma_0(k-1)$ we must have $v \not\in \text{lk}(X,\tau')$. Denote by $Q(\sigma_0)$ the set of vertices $v \in V \setminus \sigma_0$ such that $v\sigma_0$ contains only one missing face. We obtain:

$$
\sum_{\substack{v \in V \setminus \sigma_0, \\ v \not\in \text{lk}(X,\sigma_0)}} |\{\tau \in \sigma_0(k-1) : v \in \text{lk}(X,\tau)\}| = |Q(\sigma_0)|,
$$

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therefore by Equation (5.1) we have
\[ n = (k + 1) + |Q(\sigma_0)| = |\sigma_0| + |Q(\sigma_0)|. \]
So \( |Q(\sigma_0)| = n - |\sigma_0| = |V \setminus \sigma_0| \), thus \( Q(\sigma_0) = V \setminus \sigma_0 \). Hence, for every vertex \( v \in V \setminus \sigma_0 \) there is exactly one vertex \( u \in \sigma_0 \) such that \( uv \notin X(1) \).

Denote the vertices in \( V \setminus \sigma_0 \) by \( v_1, \ldots, v_{n-k-1} \). For each \( v_i \) denote by \( u_i \) the unique vertex in \( \sigma_0 \) such that \( u_i v_i \notin X(1) \).

Let \( A = \sigma_0 \setminus \{ u_1, \ldots, u_{n-k-1} \} \) and \( r = |A| \). Each vertex in \( A \) is connected in the graph \( X(1) \) to any other vertex. Therefore, since \( X \) is a clique complex, we have \( X = X[A] * Y \), where \( Y = X[V \setminus A] \).

But \( X[A] \cong \Delta_{r-1} \), therefore \( \mu_\sigma(X[A]) = r \) for all \( -1 \leq i \leq r - 1 \), so by Theorem 2.4,
\[
\mu_k(X) = \min_{i+j=k-1} (\mu_i(X[A]) + \mu_j(Y))
= r + \min \left\{ \mu_j(Y) : \max\{-1, k-r\} \leq j \leq \min\{k, \dim(Y)\} \right\} \geq r. \quad (5.2)
\]
If the vertices \( u_1, \ldots, u_{n-k-1} \) are not all distinct, then we have
\[ r > |\sigma_0| - (n - k - 1) = 2(k + 1) - n = \mu_k(X), \]
a contradiction to (5.2). Therefore \( u_1, \ldots, u_{n-k-1} \) are all different vertices (see Figure 2), so \( r = \mu_k(X) = 2(k+1) - n \). This implies that the inequality in (5.2) is an equality. Hence, there exists some \( j \geq k - r = n - k - 2 \) such that \( \mu_j(Y) = 0 \). Let
\[
Y' = \{u_1, v_1\} * \{u_2, v_2\} * \cdots * \{u_{n-k-1}, v_{n-k-1}\} \cong \left( \Delta_1^{(0)} \right)^{(n-k-1)}.
\]
The geometric realization of \( Y' \) is the boundary of the \((n-k-1)\)-dimensional cross-polytope, so \( Y' \) is a triangulation of the \((n-k-2)\)-dimensional sphere.

We have \( Y \subseteq Y' \), therefore \( \dim(Y') \leq \dim(Y) = n - k - 2 \). Thus we must have \( \mu_{n-k-2}(Y) = 0 \), so by the simplicial Hodge theorem (Corollary 2.3), \( \check{H}^{n-k-2}(Y; \mathbb{R}) \neq 0 \). But any proper subcomplex of \( Y' \) has trivial \((n-k-2)\)-dimensional cohomology, therefore \( Y = Y' \).

Hence,
\[
X \cong \left( \Delta_1^{(0)} \right)^{(n-k-1)} * \Delta_2^{(k+1) - n - 1}.
\]

Proposition 1.6 characterizes, for the case of clique complexes \( h(X) = d = 1 \), the complexes achieving the equality
\[
\mu_k(X) = (d + 1)(k + 1) - dn \]
at some dimension \( k \). It would be interesting to extend this characterization to complexes with higher dimensional missing faces. We expect the situation to be similar to the case \( d = 1 \), that is:
Figure 2: The vertices in $V$. Each vertex $v_i \in V \setminus \sigma_0$ is connected to all the vertices in $\sigma_0$ except $u_i$. Each vertex in $A$ is connected to every other vertex in $V$.

**Conjecture 5.1.** Let $X$ be a simplicial complex on vertex set $V$ of size $n$, with $h(X) = d$, such that $\mu_k(X) = (d+1)(k+1) - dn$ for some $k$. Then

$$X \cong \left(\Delta_{d}^{(d-1)}\right)^{(n-k-1)} \ast \Delta_{(d+1)(k+1)} - dn - 1,$$

(and in particular, $\dim(X) = k$).

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