Iterative methods for approximate solution of the Ornstein-Uhlenbeck process with normalised brownian motion

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Abstract. This work considers the concept of the Normalised Brownian motion for the solutions of the Ornstein-Uhlenbeck process using the Daftardar-Jafari Method (DJM) and Picard Iterative Method (PIM) as the approximate-analytical methods of solutions. The results obtained from DJM are compared with those of the PIM. The obtained results, therefore, show the effectiveness of the proposed methods.

1. Introduction
Stochastic differential equations (SDEs) are particular types of differential equation possessing a term or more that is a stochastic process, whereby the resultant solution gives a random process. In general, it has its application in applied sciences and Engineering. In most cases, obtaining their analytical solutions appears intricate and challenging [1-4]. In this work, a particular class of SDE known as the Ornstein-Uhlenbeck process (O-U process), will be considered. The O-U process is a stationary Gaussian (stochastic) process with its applications in financial mathematics and other physical sciences [5-6].

The general Stochastic differential equation can be expressed as:

$$dS(t) = a(S(t), t)dt + b(S(t), t)dB(t), t \in [0, 1],$$

(1.1)

where $a$, and $b$ denote the drift and volatility coefficients respectively, and $S(t)$ represents the Stock price in the financial mathematics domain. $B(t)$, is the standard Brownian motion with its differential equivalence as $dB(t)$, which is the noise term.

In integral form, (1.1) can be re-written as:

$$S(t) = S_0 + \int_0^t a(S(\tau), \tau) d\tau + \int_0^t b(S(\tau), \tau)dB_{\tau}.$$  

(1.2)

The first integral in (1.2) is known as the Riemann-Stieltjes, and the second is a stochastic integral driven by the Brownian motion $B(t)$. Researchers have used different numerical methods to investigate the approximate solutions of related models [7-16].

Here, two approximate methods: Daftardar-Gejji Jafari method (DJM) and Picards Iterative method (PIM) are employed for the approximate solutions using the Normalised induced Brownian motion transform.

2. Methods of Solution
Here we consider the notions of DJM, PIM, and the normalized Induced Brownian Motion.
2.1 DJM Structure
To analyse the DJM, we consider the general functional equation [17, 18]:

\[ h = g + L(h) + N[h], \]  
(2.1)

where \( g \) is a given or a known function, \( N[\cdot] \) and \( L[\cdot] \) are the nonlinear and linear operators, respectively. Suppose we define \( M[h] \) as:

\[ M[h] = L[h] + N[h], \]  
(2.2)

so (2.1) becomes:

\[ h = g + M[h]. \]  
(2.3)

By considering the solution, \( h \) of (2.2) with the series form:

\[ h = \sum_{i=0}^{\infty} h_i, \]

\[ M[h] = \tilde{N} \left( \sum_{i=0}^{\infty} h_i \right), \]  
(2.4)

the nonlinear operator \( M \) can easily be decomposed as

\[ M \left( \sum_{i=0}^{\infty} h_i \right) = M \left[ h_0 \right] + \sum_{i=1}^{\infty} \left[ M \left( \sum_{i=0}^{n} h_i \right) - M \left( \sum_{i=0}^{n-1} h_i \right) \right], \quad n = 1, 2, \ldots \]  
(2.5)

Therefore, putting (2.4) and (2.5) into (2.3), we obtain

\[ \sum_{i=0}^{\infty} h_i = g + M \left[ s_0 \right] + \sum_{i=1}^{\infty} \left[ M \left( \sum_{i=0}^{n} h_i \right) - M \left( \sum_{i=0}^{n-1} h_i \right) \right], \quad n = 1, 2, \ldots , \]  
(2.6)

So, the recurrence relation is:

\[ \begin{aligned}
    h_0 &= h \\
    s_i &= M(h_i) \\
    h_{n+1} &= M \left( \sum_{i=0}^{n} h_i \right) - M \left( \sum_{i=0}^{n-1} h_i \right), \quad n = 1, 2, \ldots 
\end{aligned} \]  
(2.7)

such that:

\[ h = g + \sum_{i=1}^{\infty} h_i \]

\[ = \sum_{i=0}^{\infty} h_i. \]  
(2.8)

The series converges absolutely and uniformly to the solution of (2.1).

2.2 Picard Iterative Method [19, 20]
Consider the differential equation of this form:

\[ \begin{aligned}
    s' &= f(t, s), \\
    s(0) &= s_0.
\end{aligned} \]  
(2.9)

First order differential equations fall under this type of equation, and PIM is one of the suitable method for handling this type of differential equation. Now, by integrating both sides of equation (2.8), we get
\[ \int_0^t s'(\tau) d\tau = \int_0^t f(\tau, s(\tau)) d\tau. \]  
(2.10)

Therefore, following the basic concept of calculus (2.9) becomes:

\[ s(t) - s(0) = \int_0^t f(\tau, s(\tau)) d\tau, \]

\[ s(t) = s(0) + \int_0^t f(\tau, s(\tau)) d\tau. \]  
(2.11)

Since \( s(t) \) is appearing on both sides of equation (2.10) for arbitrary \( t \), we therefore adopt this iterative process by choosing an initial condition

\[ s(0) = s_0, \quad \text{for } n \geq 1, \quad n \in \mathbb{Z}^+: \]

\[ s_{n+1} = s_0 + \int_0^t f(\tau, s_n(\tau)) d\tau. \]  
(2.12)

Therefore, the approximation of (2.9) yields:

\[ s(t) = \lim_{n \to \infty} s_{n+1}(t), \quad \text{as } n \to \infty. \]

We refer the readers to see references [21-24], for other dimensions of applications, numerical or exact solution methods for functional differential equations, including SDEs.

### 2.3 Normalized Brownian Motion [25]

We will take \( Z_0, Z_1, \ldots \) to be mutually and independent random variables, such that the distribution are Gaussian and identically independent with \( N(0,1) \). The random process,

\[ B(t) = \frac{Z_0}{\sqrt{2\pi}} t + \sum_{k=1}^{\infty} \frac{2}{\sqrt{\pi}} \frac{Z_k}{k} \sin(kt), \quad \text{for } t \in [0,\pi], \]  
(2.13)

is therefore referred to as normalized Brownian Motion on the interval \([0,\pi]\).

Then by differentiating (2.12), it gives:

\[ dB(t) = \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \frac{2}{\sqrt{\pi}} Z_k \cos(kt). \]  
(2.14)

In finite form, (2.13) is expressed as:

\[ dB(t) = \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{5} \frac{2}{\sqrt{\pi}} Z_k \cos(kt), \]  
(2.15)

then replacing the normalized Brownian Motion in (1.1) with (2.14), gives:

\[ \begin{cases} dS(t) = a(S,t)dt + b(S,t) \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{5} \frac{2}{\sqrt{\pi}} Z_k \cos(kt), \\ S(0) = S_0, \end{cases} \]  
(2.16)

where \( L = 5 \) and \( Z_k \) are randomly generated random variables.

### 3. Numerical Example

Consider the Stochastic Differential Equation:
\[
\begin{align*}
\begin{cases}
dS_t = -aS_t dt + b dW_t, \\
S(0) = 1,
\end{cases}
\end{align*}
\tag{3.1}
\]

where \(a,b > 0\), then choose \( a = 1, b = 1\).

The linear SDE (3.1) can be expressed in the integral form as:

\[
S_t = S_0 - \int_0^t S_u du + \int_0^t dW_u
\tag{3.2}
\]

here, the second integral is the Ito integral. This type of system is called the Ornstein-Uhlenbeck (O-U) process [26-29].

**Solution (DJM):**

In integral form, (4.1) becomes:

\[
S_t = 1 - a \int_0^t S_u du + b \int_0^t dW_u.
\tag{3.3}
\]

Therefore, with the following definitions,

\[
M(S_t) = - \int_0^t S_u du + \int_0^t dW_u.
\tag{3.4}
\]

\[
dW_u = \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \cos(kt),
\tag{3.5}
\]

and

\[
M(S_t) = - \int_0^t S_u du + \int_0^t \left( \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \cos(kt) \right) du,
\tag{3.6}
\]

we obtain the following iteration via DJM:

\[
S_0 = 1,
\]

\[
S_1 = M(S_0)
\]

\[
= - \int_0^t S_u du + \int_0^t (dW_u) du
\]

\[
= - \int_0^t (S_0) du + \int_0^t \left( \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \cos(kt) \right) du,
\]

\[
S_2 = M(S_0 + S_1) - M(S_0)
\]

\[
= \left\{ \int_0^t (S_0 + S_1) du - \int_0^t (dW_u) du \right\} - \left\{ \int_0^t (S_0) du + \int_0^t (dW_u) du \right\}
\]

\[
= \left\{ \int_0^t (S_0 + S_1) du - \int_0^t \left( \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \cos(kt) \right) du \right\} - \{S_1\}.
\]
\[ S_3 = M (S_0 + S_1 + S_2) - M (S_0 + S_1) \]
\[ \quad = -\left\{ \int_0^t (S_0 + S_1 + S_2) \, du - \int_0^t (dW_u) \, du \right\} + \left\{ \int_0^t (S_0 + S_1) \, du - \int_0^t (dW_u) \, du \right\} \]
\[ \quad = -\left\{ \int_0^t (S_0 + S_1 + S_2) \, du - \int_0^t \left( \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \cos(kt) \right) \, du \right\} \]
\[ \quad \quad + \left\{ \int_0^t (S_0 + Z_1) \, du - \int_0^t \left( \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \cos(kt) \right) \, du \right\} \].

\[ S_6 = \mathcal{N} (S_0 + S_1 + S_2 + S_3 + S_4 + S_5) - \mathcal{N} (S_0 + S_1 + S_2 + S_3 + S_4 + S_5) \]
\[ \quad = -\left\{ \int_0^t (S_0 + S_1 + S_2 + S_3 + S_4 + S_5) \, du - \int_0^t (dW_u) \, du \right\} \]
\[ \quad \quad + \left\{ \int_0^t (S_0 + S_1 + S_2 + S_3 + S_4 + S_5) \, du - \int_0^t (dW_u) \, du \right\} \]
\[ \quad = -\left\{ \int_0^t (S_0 + S_1 + S_2 + S_3 + S_4 + S_5) \, du - \int_0^t \left( \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \cos(kt) \right) \, du \right\} \]
\[ \quad \quad + \left\{ \int_0^t (S_0 + S_1 + S_2 + S_3 + S_4 + S_5) \, du - \int_0^t \left( \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \cos(kt) \right) \, du \right\} . \]

\[ S = \sum_{i=0}^{6} S_i . \]

**Solution (PIM):**

Next, we re-express (3.1) in an integral form as in (2.11):

\[ S_{n+1} = S_0 + \int_0^t f \left( u, S_n(u) \right) \, du \]  

(3.7)

\[ S_0 = 1 \]

\[ S_1 = 1 - \int_0^t S_0 \, du + \int_0^t (dW_u) \, du, \]

\[ = 1 - \int_0^t S_0 \, du + \int_0^t \left( \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \cos(kt) \right) \, du. \]

\[ S_2 = 1 - \int_0^t S_1 \, du + \int_0^t (dW_u) \, du, \]

\[ = 1 - \int_0^t S_1 \, du + \int_0^t \left( \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \cos(kt) \right) \, du. \]

\[ S_3 = 1 - \int_0^t (S_2) \, du + \int_0^t (dW_u) \, du, \]

\[ = 1 - \int_0^t (S_2) \, du + \int_0^t \left( \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} Z_k \cos(kt) \right) \, du. \]
\[ S_4 = 1 - \frac{1}{2} \int_0^t (S_3') du + \int_0^t (dW_4) du, \]

\[ = 1 - \int_0^t (S_3) du + \int_0^t \left( \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \frac{Z_k}{\sqrt{2\pi}} \cos(kt) \right) du. \]

\[ S_5 = 1 - \int_0^t (S_4) du + \int_0^t (dW_5) du, \]

\[ = 1 - \int_0^t (S_5) du + \int_0^t \left( \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \frac{Z_k}{\sqrt{2\pi}} \cos(kt) \right) du. \]

\[ S_6 = 1 - \int_0^t (S_5) du + \int_0^t (dW_6) du, \]

\[ = 1 - \int_0^t (S_6) du + \int_0^t \left( \frac{Z_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \frac{Z_k}{\sqrt{2\pi}} \cos(kt) \right) du. \]

\[ \vdots \]

\[ S_{n+1} = S_0 + \int_0^t ((S_n) dW_n) du. \]

**Table 1:** Error analysis for the O-U Process

| t   | \( Z_6^{DIM} (t) \)       | \( Z_6^{PIM} (t) \)       | \(| Z_6^{DIM} - Z_6^{PIM} | \) |
|-----|--------------------------|--------------------------|---------------------------------|
| 0.0 | 1.000000000000000000    | 1.000000000000000000    | 0.000E+00                       |
| 0.1 | 1.094638480760513       | 1.094638563724062       | 8.296E-08                       |
| 0.2 | 1.179085738282314       | 1.179088391665922       | 2.653E-06                       |
| 0.3 | 1.250846853088319       | 1.25086976984742        | 2.012E-05                       |
| 0.4 | 1.303968602180530       | 1.304053197396319       | 8.460E-05                       |
| 0.5 | 1.329414276295206       | 1.329671348303800       | 2.571E-04                       |
| 0.6 | 1.317032674675577       | 1.31766970452804       | 6.354E-04                       |
| 0.7 | 1.258651282423057       | 1.260011018982764       | 1.360E-03                       |
| 0.8 | 1.151443473767513       | 1.154058021873473       | 2.612E-03                       |
| 0.9 | 1.000562230460032       | 1.005187751214804       | 4.626e-03                       |
| 1.0 | 8.201461236883663       | 8.277972780380032       | 7.651E-03                       |
4. Conclusion
The induced normalized Brownian motion approach for approximate solution of the Ornstein-Uhlenbeck process has been successfully considered in the present work. The solutions were obtained easily by the proposed methods: DJM and PIM, even with less computing time. Therefore, it is noted for effectiveness and consequently suggested for other well-known stochastic models.

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Conflict of Interests
The authors declare that there is no conflict of interest.

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