INFINITY ALGEBRAS, COHOMOLOGY AND CYCLIC COHOMOLOGY, AND INFINITESIMAL DEFORMATIONS

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Abstract. An $A_\infty$ algebra is given by a codifferential on the tensor coalgebra of a (graded) vector space. An associative algebra is a special case of an $A_\infty$ algebra, determined by a quadratic codifferential. The notions of Hochschild and cyclic cohomology generalize from associative to $A_\infty$ algebras, and classify the infinitesimal deformations of the algebra, and those deformations preserving an invariant inner product, respectively. Similarly, an $L_\infty$ algebra is given by a codifferential on the exterior coalgebra of a vector space, with Lie algebras being special cases given by quadratic codifferentials. There are natural definitions of cohomology and cyclic cohomology, generalizing the usual Lie algebra cohomology and cyclic cohomology, which classify deformations of the algebra and those which preserve an invariant inner product. This article explores the definitions of these infinity algebras, their cohomology and cyclic cohomology, and the relation to their infinitesimal deformations.

1. Introduction

In a joint paper with Albert Schwarz [16], we gave definitions of Hochschild cohomology and cyclic cohomology of an $A_\infty$ algebra, and showed that these cohomology theories classified the infinitesimal deformations of the $A_\infty$ structure and those deformations preserving an invariant inner product, respectively. Then we showed that the Hochschild cohomology of an associative algebra classifies the deformations of the algebra into an $A_\infty$ algebra, and the cyclic cohomology of an algebra with an invariant inner product classifies the deformations of the algebra into an $A_\infty$ algebra preserving the inner product. Following some ideas of Maxim Kontsevich from [9], we applied these results to show that cyclic cocycles of an associative algebra determine homology cycles in the complex of metric ribbon graphs. In [14], for sake of simplicity we avoided the more standard description of $A_\infty$ algebras in terms of codifferentials.

In this article, the definitions of infinity algebras will be given in terms of codifferentials on coalgebras, so that the theory of $L_\infty$ algebras is seen to be closely analogous to that of $A_\infty$ algebras. The ideas in this text lead immediately to a simple formulation of the cycle in the complex of metric ribbon graphs associated to an $A_\infty$ algebra with an invariant inner product. This same method can be applied to show that $L_\infty$ algebras with an invariant inner product give rise to a cycle in the homology of the complex of metric ordinary graphs (see [15]). This paper is a revision of [14].

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The notion of an $A_\infty$ algebra, also called a strongly homotopy associative algebra, was introduced by J. Stasheff in \cite{19, 20}, and is a generalization of an associative algebra. From a certain point of view, an associative algebra is simply a special case of a codifferential on the tensor coalgebra of a vector space. An $A_\infty$ algebra is given by taking an arbitrary coderivation; in particular associative algebras and differential graded associative algebras are examples of $A_\infty$ algebras.

$L_\infty$ algebras, also called strongly homotopy Lie algebras, first appeared in \cite{17}, and are generalizations of Lie algebras. A Lie algebra can be viewed as simply a special case of a codifferential on the exterior coalgebra of a vector space, and $L_\infty$ algebras are simply arbitrary codifferentials on this coalgebra.

A bracket structure was introduced on the space of cochains of an associative algebra by Murray Gerstenhaber in \cite{4}. From the coalgebra point of view, the Gerstenhaber bracket turns out to be essentially the commutator of coderivations, as was first pointed out by James Stasheff in \cite{21}. In the presence of an invariant inner product, the cyclic cochains can be identified as a subalgebra of the space of ordinary cochains, so that the cyclic cohomology also is equipped with a natural Lie bracket. These notions generalize to the $A_\infty$ case.

In our considerations, we shall be interested in $\mathbb{Z}_2$-graded spaces, but we should point out that the results presented here hold in the $\mathbb{Z}$-graded case as well, because the signs in the $\mathbb{Z}$-graded case coincide with the signs from the associated $\mathbb{Z}_2$-grading. $A_\infty$ algebras were first defined as $\mathbb{Z}_2$-graded objects, but for the applications we have in mind, the $\mathbb{Z}_2$-grading is more appropriate, and the generalization of the results here to the $\mathbb{Z}$-graded case is straightforward. We shall find it necessary to consider the parity reversion of a $\mathbb{Z}_2$-graded space. This is the same space with the parity of elements reversed. (In the $\mathbb{Z}$-graded case, the corresponding notion is that of suspension.)

There is a natural vector space isomorphism between the tensor coalgebra of a $\mathbb{Z}_2$-graded space and the tensor coalgebra of its parity reversion. But in the case of the exterior coalgebra, the isomorphism is to the symmetric coalgebra of the parity reversion.

A notion that will play a crucial role in what follows is that of a grading group. An abelian group $G$ is said to be a grading group if it possesses a symmetric $\mathbb{Z}_2$-valued bilinear form $\langle \cdot, \cdot \rangle$. Any abelian group with a subgroup of index 2 possesses a natural grading form. An element $g$ of $G$ is called odd if $\langle g, g \rangle = 1$. A grading on a group will be called good if $\langle g, h \rangle = 1$, whenever $g$ and $h$ are both odd. Groups equipped with the natural inner product induced by a subgroup of index 2 are good, and these include both $\mathbb{Z}_2$ and $\mathbb{Z}$. If $G$ and $H$ are grading groups, then $G \times H$ has an induced inner product, given by $\langle (g, g'), (h, h') \rangle = \langle g, g' \rangle + \langle h, h' \rangle$. But the induced grading on $G \times H$ is never good when the gradings on $G$ and $H$ are good.

If $V$ is a $G$-graded vector space, then one can define the symmetric and exterior algebras of the tensor algebra $T(V)$. The symmetric algebra is $G$-graded commutative, but the exterior algebra is not. On the other hand, the tensor, symmetric and exterior algebras also are graded by $G \times \mathbb{Z}$, and with respect to the induced inner
product on $G \times \mathbb{Z}$, the exterior algebra is graded commutative. Thus the natural grading associated to the exterior algebra is not good. The consequences of this fact complicates the definition of $A_\infty$ and $L_\infty$ algebras. In the following, we shall always consider the case $G = \mathbb{Z}_2$, although the case $G = \mathbb{Z}$ is similar.

2. **The Exterior and Symmetric Algebras**

Suppose that $V$ is a $\mathbb{Z}_2$-graded $k$-module. The (reduced) tensor algebra $T(V)$ is given by $T(V) = \bigoplus_{n=1}^\infty V^n$, where $V^n$ is the $n$-th tensor power of $V$. The full tensor algebra is defined to include the term $V^0 = k$ as well, and in a notion of an $A_\infty$ algebra on the full tensor algebra is given, but we do not treat this idea here.

For an element $v = v_1 \otimes \cdots \otimes v_n$ in $T(V)$, define its parity $|v| = |v_1| + \cdots + |v_n|$, and its degree by $\deg(v) = n$. We define the bidegree of $v$ by $\text{bid}(v) = (|v|, \deg(v))$. If $\alpha, \beta \in T(V)$, then $\text{bid}(\alpha \otimes \beta) = \text{bid}(\alpha) + \text{bid}(\beta)$, so that $T(V)$ is naturally $\mathbb{Z}_2 \times \mathbb{Z}$-graded by the bidegree, and $\mathbb{Z}_2$-graded if we consider only the parity. For simplicity, for a $\mathbb{Z}_2$-graded space, we will denote $(-1)^{\text{bid}(\alpha) + \text{bid}(\beta)}$. (In formulas like the above, the elements $\alpha$ and $\beta$ are assumed to be homogeneous.)

The (graded) symmetric algebra $S(V)$ is defined as the quotient of the tensor algebra of $V$ by the bigraded ideal generated by all elements of the form $u \otimes v - (-1)^{|u||v|} v \otimes u$. The resulting algebra has a decomposition $S(V) = \bigoplus_{n=1}^\infty S^n(V)$, where $S^n(V)$ is the image of $V^n$ in $S(V)$. We shall denote the induced product in $S(V)$ by juxtaposition. The symmetric algebra is both $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}$-graded, but is graded commutative only with respect to the $\mathbb{Z}_2$-grading.

If $\sigma \in \Sigma_n$, then

$$v_1 \cdots v_n = \epsilon(\sigma; v_1, \ldots, v_n) v_{\sigma(1)} \cdots v_{\sigma(n)}$$

where $\epsilon(\sigma; v_1, \ldots, v_n)$ is a sign which can be determined by the following. If $\sigma$ interchanges $k$ and $k + 1$, then $\epsilon(\sigma; v_1, \ldots, v_n) = (-1)^{|v_k| |v_{k+1}|}$. In addition, if $\tau$ is another permutation, then

$$\epsilon(\tau \sigma; v_1, \ldots, v_n) = \epsilon(\sigma; v_{\tau(1)}, \ldots, v_{\tau(n)}) \epsilon(\tau; v_1, \ldots, v_n).$$

It is conventional to abbreviate $\epsilon(\sigma; v_1, \ldots, v_n)$ as $\epsilon(\sigma)$.

The (graded) exterior algebra $\wedge V$ is defined as the quotient of $T(V)$ by the bigraded ideal generated by all elements of the form $u \otimes v - (-1)^{|u||v|} v \otimes u$. The resulting algebra has a decomposition $\wedge V = \bigoplus_{n=0}^\infty \wedge^n V$, and the induced product is denoted by $\wedge$. The exterior algebra is both $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}$-graded, but is graded commutative only with respect to the $\mathbb{Z}_2 \times \mathbb{Z}$-grading. Let $\alpha, \beta \in \wedge V$. Furthermore it is easy to see that

$$v_1 \wedge \cdots \wedge v_n = (-1)^\sigma \epsilon(\sigma) v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)},$$

where $(-1)^\sigma$ is the sign of the permutation $\sigma$.

3. **The Tensor, Exterior and Symmetric Coalgebras**

A more formal treatment of the symmetric and exterior coalgebras would introduce the coalgebra structure on the tensor algebra, and then describe these coalgebras in terms of coideals. Instead, we will describe these coalgebra structures directly. Recall that a coalgebra structure on a $k$-module $C$ is given by a diagonal
mapping $\Delta : C \to C \otimes C$. We consider only coassociative coalgebras (we do not consider counits). The axiom of coassociativity is

$$(1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta.$$ 

A grading on $C$ is compatible with the coalgebra structure if for homogeneous $c \in C$, $\Delta(c) = \sum u_i \otimes v_i$, where $|u_i| + |v_i| = |c|$ for all $i$; in other words, $\Delta$ has degree 0. We also mention that a coalgebra is graded cocommutative if $S \circ \Delta = \Delta$, where $S : C \otimes C \to C \otimes C$ is the symmetric mapping given by

$$S(\alpha \otimes \beta) = (-1)^{\alpha \beta} \alpha \beta \otimes \alpha.$$ 

The tensor coalgebra structure is given by defining the (reduced) diagonal $\Delta : T(V) \to T(V)$ by

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{k=1}^{n-1} (v_1 \otimes \cdots \otimes v_k) \otimes (v_{k+1} \otimes \cdots \otimes v_n).$$ 

(We use here the reduced diagonal, because we are not including the zero degree term in the tensor coalgebra. As a consequence, $\Delta$ is not injective; in fact, its kernel is $V$.) The tensor coalgebra is not graded cocommutative under either the $\mathbb{Z}_2$ or the $\mathbb{Z}_2 \times \mathbb{Z}$-grading, but both gradings are compatible with the coalgebra structure.

The symmetric coalgebra structure on $S(V)$ is given by defining

$$\Delta(v_1 \cdots v_n) = \sum_{\sigma \in \text{Sh}(k,n-k)} \epsilon(\sigma)v_{\sigma(1)} \cdots v_{\sigma(k)} \otimes v_{\sigma(k+1)} \cdots v_{\sigma(n)},$$

where $\text{Sh}(p,q)$ is the set of all unshuffles of type $(p,q)$, that is, the permutations of $p + q$ such that $\sigma(k) < \sigma(k+1)$ if $k \neq p$.

With this coalgebra structure, and the $\mathbb{Z}_2$-grading, $S(V)$ is a cocommutative, coassociative coalgebra.

Similarly, we define the exterior coalgebra structure on $\bigwedge V$ by

$$\Delta(v_1 \wedge \cdots \wedge v_n) = \sum_{\sigma \in \text{Sh}(k,n-k)} (-1)^{\epsilon(\sigma)}v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} \otimes v_{\sigma(k+1)} \wedge \cdots \wedge v_{\sigma(n)}.$$ 

The coalgebra structure on $\bigwedge V$ is coassociative, and is cocommutative with respect to the $\mathbb{Z}_2 \times \mathbb{Z}$-grading.

4. **Coderivations**

A coderivation on a graded coalgebra $C$ is a map $d : C \to C$ such that

$$\Delta \circ d = (d \otimes 1 + 1 \otimes d) \circ \Delta.$$ 

Note that this definition depends implicitly on the grading group, because

$$(1 \otimes d)(\alpha \otimes \beta) = (-1)^{\alpha,d} \alpha \otimes d(\beta).$$ 

The $k$-module $\text{Coder}(C)$ of graded coderivations has a natural structure of a graded Lie algebra, with the bracket given by

$$[m, n] = m \circ n - (-1)^{(m,n)} n \circ m.$$ 

A codifferential on a coalgebra $C$ is a coderivation $d$ such that $d \circ d = 0$. We examine the coderivation structure of the tensor, symmetric and exterior coalgebras.
4.1. **Coderivations of the Tensor Coalgebra.** Suppose that we wish to extend $d_k : V^k \to V$ to a coderivation $\hat{d}_k$ of $T(V)$. We are interested in extensions satisfying the property that $\hat{d}_k(v_1, \ldots, v_n) = 0$ for $n < k$. How this extension is made depends on whether we consider the $\mathbb{Z}_2$ or the $\mathbb{Z}_2 \times \mathbb{Z}$-grading. First we consider the $\mathbb{Z}_2$-grading, so that only the parity of $d$ is relevant. Then if we define

$$
\hat{d}_k(v_1, \ldots, v_n) = \sum_{i=0}^{n-k} (-1)^{(v_1+\cdots+v_i)d_k} v_1, \ldots, v_i, d_k(v_{i+1}, \ldots, v_{i+k}), v_{i+k+1}, \ldots, v_n,
$$

one can show that $\hat{d}_k$ is a coderivation on $T(V)$ with respect to the $\mathbb{Z}_2$-grading. More generally, one can show that any coderivation $\hat{d}$ on $T(V)$ is completely determined by the induced mappings $d_k : V^k \to V$, and in fact, one obtains

$$
\hat{d}(v_1, \ldots, v_n) = \sum_{1 \leq k \leq n} (-1)^{(v_1+\cdots+v_i)\hat{d}} v_1, \ldots, v_i, d_k(v_{i+1}, \ldots, v_{i+k}), v_{i+k+1}, \ldots, v_n.
$$

Also, one can show that $\hat{d}$ is a codifferential with respect to the $\mathbb{Z}_2$-grading precisely when $n \geq 1$

$$
\sum_{k+l=n+1} (-1)^{(v_1+\cdots+v_i)\hat{d}} d_l(v_1, \ldots, v_i, d_k(v_{i+1}, \ldots, v_{i+k}), v_{i+k+1}, \ldots, v_n) = 0.
$$

The module $\text{Coder}(T(V))$ of coderivations of $T(V)$ with respect to the $\mathbb{Z}_2$-grading is naturally isomorphic to $\text{Hom}(T(V), V)$, so $\text{Hom}(T(V), V)$ inherits a natural structure of a $\mathbb{Z}_2$-graded Lie algebra.

Let us examine the bracket structure on $\text{Hom}(T(V), V)$ more closely. Suppose that for an arbitrary element $d \in \text{Hom}(T(V), V)$, we denote the restriction of $d$ to $V^k$ by $d_k$, and the extensions of $d$ and $d_k$ as coderivations of $T(V)$ by $\hat{d}$ and $\hat{d}_k$, resp. Also denote by $d_{kl}$ the restriction of $\hat{d}_k$ to $V^{k+l-1}$, so that $d_{kl} \in \text{Hom}(V^{k+l-1}, V^l)$. The precise expression for $d_{kl}$ is given by equation (9) with $n = k + l - 1$. It is easy to see that the bracket of $d_k$ and $\delta_l$ is given by

$$
[d_k, \delta_l] = d_k \circ \delta_l - (-1)^{d_k \delta_l} \delta_l \circ d_k.
$$

Furthermore, we have $[d, \delta] = \sum_{k+l=n+1} [d_k, \delta_l]$. The point here is that $d_{kl}$ and $\delta_{kl}$ are determined in a simple manner by $d_k$ and $\delta_k$, so we have given a description of the bracket on $\text{Hom}(T(V), V)$ in a direct fashion. The fact that the bracket so defined has the appropriate properties follows from the fact that if $\rho = [d, \delta]$, then $\hat{\rho} = [\hat{d}, \hat{\delta}]$.

Now we consider how to extend a mapping $m_k : V^k \to V$ to a coderivation $\hat{m}_k$ with respect to the $\mathbb{Z}_2 \times \mathbb{Z}$-grading. In this case, note that the bidegree of $m_k$ is given by $\text{bid}(m_k) = (|m_k|, k-1)$. The formula for the extension is the same as
before, but with the bidegree in place of the parity. In other words,
\[(11) \hat{m}_k(v_1, \cdots, v_n) = \sum_{i=0}^{n-k} (-1)^{(v_1+\cdots+v_i)m_k + (k-1)i} v_1, \cdots, v_i, m_k(v_i+1, \cdots, v_{i+k}), v_{i+k+1}, \cdots, v_n.\]

Similarly, if we consider an arbitrary coderivation \(\hat{m}\) on \(T(V)\), then it is again determined by the induced mappings \(m_k : V^k \to V\), and we see that
\[(12) \hat{m}(v_1, \cdots, v_n) = \sum_{1 \leq k \leq n, 0 \leq i \leq n-k} (-1)^{(v_1+\cdots+v_i)m_k + (k-1)i} v_1, \cdots, v_i, m_k(v_i+1, \cdots, v_{i+k}), v_{i+k+1}, \cdots, v_n.
\]

Also, one obtains that \(\hat{m}\) is a codifferential with respect to the \(\mathbb{Z}_2 \times \mathbb{Z}\)-grading if \(\hat{m}\) is a coderivation with respect to the \(\mathbb{Z}_2 \times \mathbb{Z}\)-grading.

The module \(\text{Coder}(T(V))\) of coderivations of \(T(V)\) with respect to the \(\mathbb{Z}_2 \times \mathbb{Z}\)-grading is naturally isomorphic to \(\bigoplus_{k=1}^\infty \text{Hom}(V^k, V)\), rather than \(\text{Hom}(T(V), V)\), because the latter module is the direct product of the modules \(C^k(V) = \text{Hom}(V^k, V)\). However, we would like to consider elements of the form \(\hat{m} = \sum_{k=1}^\infty \hat{m}_k\), where \(\hat{m}_k\) has bidegree \((|m_k|, k-1)\). Such an infinite sum is a well defined element of \(\text{Hom}(T(V), T(V))\), so by abuse of notation, we will define \(\text{Coder}(T(V))\) to be the module of such infinite sums of coderivations. With this convention, we now have a natural isomorphism between \(\text{Coder}(T(V))\) and \(\text{Hom}(T(V), V)\). Furthermore, the bracket of coderivations is well defined, and we consider \(\text{Coder}(T(V))\) to be a \(\mathbb{Z}_2 \times \mathbb{Z}\)-graded Lie algebra. The reason that the bracket is well defined is that any homogeneous coderivation has bidegree \((m, n)\) for some \(n \geq 0\), so the grading is given by \(\mathbb{Z}_2 \times \mathbb{N}\) rather than the full group \(\mathbb{Z}_2 \times \mathbb{Z}\). In structures where a \(\mathbb{Z}\)-grading reduces to an \(\mathbb{N}\)-grading, it is often advantageous to replace direct sums with direct products.

Using the same notation convention as in the \(\mathbb{Z}_2\)-graded case, we note that if \(m, \mu \in \text{Hom}(T(V), V)\), then we have
\[(14) [m_k, \mu_l] = m_k \circ \mu_l - (-1)^{(m_k, \mu_l)} \mu_l \circ m_k,\]
and \([m, \mu]_n = \sum_{k+l=n+1} [m_k, \mu_l]\).

4.2. Coderivations of the Symmetric Coalgebra. Suppose that we want to extend \(m_k : S^k(V) \to V\) to a coderivation \(\hat{m}_k\) of \(S(V)\) such that \(\hat{m}_k(v_1 \cdots v_n) = 0\) for \(k < n\). Define
\[(15) \hat{m}_k(v_1 \cdots v_n) = \sum_{\sigma \in \text{Sh}(k, n-k)} \epsilon(\sigma) m_k(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) v_{\sigma(k+1)} \cdots v_{\sigma(n)}.\]

Then \(\hat{m}_k\) is a coderivation with respect to the \(\mathbb{Z}_2\)-grading. In general, suppose that \(\hat{m}\) is a coderivation on the symmetric coalgebra. It is not difficult to see that if
$m_k : S^k(V) \rightarrow V$ is the induced map, then $\hat{m}$ can be recovered from these maps by the relations

$$m(v_1 \cdots v_n) = \sum_{1 \leq k \leq n} \epsilon(\sigma)m_k(v_{\sigma(1)}, \cdots, v_{\sigma(k)})v_{\sigma(k+1)} \cdots v_{\sigma(n)}.$$  

From this, we determine that there is a natural isomorphism between $\text{Coder}(S(V))$, the module of coderivations of $V$, and $\text{Hom}(S(V), V)$. Thus $\text{Hom}(S(V), V)$ inherits the structure of a graded Lie algebra. Also, $\hat{m}$ is a coderivation with respect to the $\mathbb{Z}$-grading.

$$\hat{m}(v_1 \cdots v_n) = \sum_{1 \leq k \leq n} \epsilon(\sigma)m_k(v_{\sigma(1)}, \cdots, v_{\sigma(k)})v_{\sigma(k+1)} \cdots v_{\sigma(n)} = 0.$$  

In general, it is not possible to extend a map $m_k : S^k(V) \rightarrow V$ as a coderivation with respect to the $\mathbb{Z}_2 \times \mathbb{Z}$-grading; the signs won’t work out correctly. Similarly, the $\mathbb{Z}_2$-grading is necessary to have a good theory of coderivations of the symmetric coalgebra.

### 4.3. Coderivations of the Exterior Coalgebra

Suppose that we want to extend $l_k : \bigwedge^k(V) \rightarrow V$ to a coderivation $\hat{l}_k$ of $\bigwedge(V)$ such that $\hat{l}_k(v_1, \cdots, v_n) = 0$ for $k < n$. Define

$$\hat{l}_k(v_1, \cdots, v_n) = \sum_{\sigma \in \text{Sh}(k,n-k)} (-1)^\sigma \epsilon(\sigma)l_k(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \wedge v_{\sigma(k+1)} \wedge \cdots \wedge v_{\sigma(n)}.$$  

Then $\hat{l}_k$ is a coderivation with respect to the $\mathbb{Z}_2 \times \mathbb{Z}$-grading. In general, suppose that $\hat{l}$ is a coderivation on the exterior coalgebra. If $l_k : \bigwedge^k(V) \rightarrow V$ is the induced map, then $\hat{l}$ can be recovered from these maps by the relations

$$\hat{l}(v_1 \cdots v_n) = \sum_{1 \leq k \leq n} \epsilon(\sigma)l_k(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \wedge v_{\sigma(k+1)} \wedge \cdots \wedge v_{\sigma(n)}.$$  

From this, we determine that there is a natural isomorphism between $\text{Coder}(\bigwedge(V))$, the module of coderivations of $V$, and $\text{Hom}(\bigwedge(V), V)$. Thus $\text{Hom}(\bigwedge(V), V)$ inherits the structure of a graded Lie algebra. Also, $\hat{l}$ is a coderivation when for all $n$,

$$\sum_{k+l=n+1} (-1)^\sigma \epsilon(\sigma)l_l(v_{\sigma(1)}, \cdots, v_{\sigma(k)})v_{\sigma(k+1)} \cdots v_{\sigma(n)} = 0.$$  

As the $\mathbb{Z}_2$-grading is necessary for coderivations of the symmetric algebra, so the $\mathbb{Z}_2 \times \mathbb{Z}$-grading is essential for a good theory of coderivations of the exterior coalgebra.

### 5. Cohomology of $A_\infty$ Algebras

In [16], a generalization of an associative algebra, called a strongly homotopy associative algebra, or $A_\infty$ algebra was discussed, and cohomology and cyclic cohomology of this structure was defined. $A_\infty$ algebras were introduced by J. Stasheff in [19, 20]. An $A_\infty$ algebra structure is essentially an odd codifferential on the tensor
coalgebra; an associative algebra is simply an odd coderivation determined by a single map \( m_2 : V^2 \to V \).

If \( V \) is a \( \mathbb{Z}_2 \)-graded \( k \)-module, then the parity reversion \( IV \) is the same module, but with the parity of elements reversed. In other words, \( (IV)_0 = V_1 \) and \( (IV)_1 = V_0 \), where \( V_0 \) and \( V_1 \) are the submodules of even and odd elements of \( V \), resp. The map \( \pi : V \to IV \), which is the identity as a map of sets, is odd. There is a natural isomorphism \( \eta : T(V) \to T(IV) \) given by

\[
\eta(v_1 \otimes \cdots \otimes v_n) = (-1)^{(n-1)v_1 + \cdots + (n-2)v_2 + \cdots + v_{n-1}} \pi v_1 \otimes \cdots \otimes \pi v_n
\]

Denote the restriction of \( \eta \) to \( V^k \) by \( \eta_k \). Note that \( \eta_k \) is odd when \( k \) is odd and even when \( k \) is even, so that \( \eta \) is neither an odd nor an even mapping.

Let \( W = IV \). Define a bijection between \( C(W) = \text{Hom}(T(W), W) \) and \( C(V) = \text{Hom}(T(V), V) \) by setting \( \mu = \eta^{-1} \circ \delta \circ \eta \), for \( \delta \in C(W) \). Then \( \mu_k = \eta_k^{-1} \circ \delta_k \circ \eta_k \) and \( |\mu_k| = |\delta_k| + (k-1) \). In particular, note that if \( \delta_k \) is odd in the \( \mathbb{Z}_2 \)-grading, then \( \text{bid}(\mu_k) = (k, k-1) \), so that \( \mu_k \) is odd in the \( \mathbb{Z}_2 \times \mathbb{Z} \)-grading. Now extend \( \delta_k : W^k \to W \) to a coderivation \( \hat{\delta}_k \) on \( T(W) \) with respect to the \( \mathbb{Z}_2 \) grading, so that

\[
\hat{\delta}_k(w_1, \ldots, w_n) = \sum_{i=0}^{n-k} (-1)^{r} \eta^{-1} \delta(w_1, \cdots, w_n) = \eta^{-1}\left(\sum_{i=0}^{n-k} (-1)^{r+s_i} w_1 \otimes \cdots \otimes w_i \otimes \delta_k(w_{i+1}, \cdots, w_{i+k}) \otimes w_{i+k+1} \otimes \cdots \otimes w_n\right)
\]

Let \( \tilde{\mu}_k : T(V) \to T(V) \) be given by \( \tilde{\mu}_k = \eta^{-1} \circ \hat{\delta}_k \circ \eta \). Let \( \hat{\mu} \) be the extension of \( \mu \) as a \( \mathbb{Z}_2 \times \mathbb{Z} \)-graded coderivation of \( T(V) \). We wish to investigate the relationship between \( \tilde{\mu}_k \) and \( \hat{\mu}_k \). For simplicity, write \( w_i = \pi v_i \). Note that \( \eta_1 = \pi \) is the parity reversion operator. So we have \( \delta_k = \pi \circ \mu_k \circ \eta_k^{-1} \). Thus

\[
\hat{\mu}_k(v_1, \ldots, v_n) = (-1)^r \eta^{-1} \delta(w_1, \cdots, w_n) = \eta^{-1} \left(\sum_{i=0}^{n-k} (-1)^{r+s_i} w_1 \otimes \cdots \otimes w_i \otimes \pi \mu_k(v_{i+1}, \cdots, v_{i+k}) \otimes v_{i+k+1} \otimes \cdots \otimes v_n\right)
\]

where

\[
\begin{align*}
 r &= (n-1)v_1 + \cdots + v_{n-1} \\
 s_i &= (w_1 + \cdots + w_i) \delta_k \\
 &= (v_1 + \cdots + v_i) \mu_k + i(\mu_k + 1 - k) + (1 - k)(v_1 + \cdots + v_i) \\
 t_i &= (k - 1)v_{i+1} + \cdots + v_{i+k-1} \\
 u_i &= (n-k)v_1 + \cdots + (n-k-i+1)v_i + (n-k-i)(v_{i+1} + \cdots + v_{i+k}) + (n-k-i+1)v_{i+k+1} + \cdots + v_{n-1}
\end{align*}
\]

Combining these coefficients we find that

\[
r + s_i + t_i + u_i = (v_1 + \cdots + v_i) \mu_k + i(k-1) + (n-k)\mu_k,
\]
so that

\begin{equation}
\bar{\mu}_k(v_1, \ldots, v_n) = \sum_{i=0}^{n-k} (-1)^{(v_1+\cdots+v_i)\mu_k+i(k-1)+(n-k)\mu_k} \\
\times v_1 \otimes \cdots \otimes v_i \otimes \mu_k(v_{i+1}, \ldots, v_{i+k}) \otimes v_{i+k+1} \otimes \cdots \otimes v_n.
\end{equation}

Thus we see that

\begin{equation}
\bar{\mu}_k(v_1, \ldots, v_n) = (-1)^{(n-k)\mu_k} \bar{\mu}_k(v_1, \ldots, v_n).
\end{equation}

Denote the restriction of \( \bar{\mu}_k \) to \( V^{k+l-1} \) by \( \bar{\mu}_{kl} \). Set \( n = k + l - 1 \). Denote \( \bar{\mu}_{kl} = \eta_l^{-1} \circ \delta_{kl} \circ \eta_k \), so that \( \mu_{kl} \) is the restriction of \( \bar{\mu}_k \) to \( V^n \). Then we can express equation (25) in the form \( \bar{\mu}_{kl} = (-1)^{(l-1)\mu_k} \mu_k \).

More generally, if \( \hat{\delta} \) is an arbitrary derivation on \( T(W) \), induced by the maps \( \delta_k : V^k \to V \), then it determines maps \( \mu_k : V^k \to V \), and \( \bar{\mu} : T(V) \to T(V) \), in a similar manner, and we have

\begin{equation}
\bar{\mu}(v_1, \ldots, v_n) = \sum_{1 \leq k \leq n} (-1)^{(v_1+\cdots+v_i)\mu_k+i(k-1)+(n-k)\mu_k} \\
\times v_1 \otimes \cdots \otimes v_i \otimes \mu_k(v_{i+1}, \ldots, v_{i+k}) \otimes v_{i+k+1} \otimes \cdots \otimes v_n.
\end{equation}

The condition that \( \hat{\delta} \) is a codifferential on \( T(W) \) can be expressed in the form

\begin{equation}
\sum_{k+l=n+1} \delta_l \circ \delta_{kl} = 0
\end{equation}

for all \( n \geq 1 \). This condition is equivalent to the condition

\begin{equation}
\sum_{k+l=n+1} (-1)^{(l-1)\mu_k} \mu_l \circ \mu_{kl} = \sum_{k+l=n+1} \mu_l \circ \bar{\mu}_{kl} = 0.
\end{equation}

We can express this condition in the form

\begin{equation}
\sum_{k+l=n+1} (-1)^{(v_1+\cdots+v_i)\mu_k+i(k-1)+(n-k)\mu_k} \\
\times \mu_l(v_1, \ldots, v_i, \mu_k(v_{i+1}, \ldots, v_{i+k}), v_{i+k+1}, \ldots, v_n) = 0.
\end{equation}

When \( \hat{\delta} \) is an odd codifferential, \( |\mu_k| = k \), so the sign in the expression above is simply \( (v_1+\cdots+v_i)k + i(k-1) + nk - k \). This is the form in which the signs appear in the definition of an \( A_\infty \) algebra in [11, 10].

An \( A_\infty \) algebra structure on \( V \) is nothing more than an odd codifferential on \( T(W) \). This can also be expressed in terms of the bracket on \( C(W) \). An odd element \( \delta \in C(W) \) satisfying \( [\delta, \delta] = 0 \) determines a codifferential \( \hat{\delta} \) on \( T(W) \).

More precisely, the condition \( [\delta, \delta] = 0 \) is equivalent to the condition \( \delta^2 = 0 \). Thus \( \mu \in C(V) \) determines an \( A_\infty \) algebra structure on \( V \) when \( \delta = \eta \circ \mu \circ \eta^{-1} \) satisfies \( [\delta, \delta] = 0 \). This is not the same condition as \( [\mu, \mu] = 0 \), nor even the condition \( \bar{\mu}^2 = 0 \), although we shall have more to say about this later.

If one considers the situation \( V = \Pi W \) instead of \( W = \Pi V \), then an odd codifferential \( d \) on \( T(W) \) would give rise to an a mapping \( m : T(V) \to V \) satisfying the
The signs in the expression above agree with the signs in the definition of a $A_\infty$ algebra as given in [4, 3].

From these observations, we see that the two sign conventions for a homotopy associative algebra originate because there are two natural choices for the relation between the space $W$, which carries the structure of an odd differential with respect to the usual $\mathbb{Z}_2$-grading, and $V$, which is its dual. One may choose either $W = IV$, to get the signs in [4, 3], or $V = IV$, to get the signs in [4, 3]. (Actually, one can vary the definition of $\eta$ to obtain both sets of signs from either one of these models.)

Let us examine the bracket structure on the space $\text{Coder}(T(V))$. In [4], M. Gerstenhaber defined a bracket on the space of cochains of an associative algebra, which we shall call the Gerstenhaber bracket. When $V$ is concentrated in degree zero, in other words, in the non $\mathbb{Z}_2$-graded case, the Gerstenhaber bracket is just the bracket of coderivations, with the $\mathbb{Z}$-grading. Thus the Gerstenhaber bracket is given by

$$\langle [\varphi, \psi]_l \rangle = \varphi_k \psi_l - (-1)^{(k-1)(l-1)} \psi_l \varphi_k,$$

for $\varphi \in C^k(V)$ and $\psi \in C^l(V)$.

One of the main results of [4] is that the differential $D$ of cochains in the cohomology of an associative algebra can be expressed in terms of the bracket. It was shown that $D(\varphi) = [\varphi, m]$ where $m$ is the cochain representing the associative multiplication. This formulation leads to a simple proof that $D^2 = 0$, following from the properties of $\mathbb{Z}$-graded Lie algebras. The associativity of $m$ is simply the condition $[m, m] = 0$. Let us recall the proof that an odd homogeneous element $m$ of a graded Lie algebra satisfying $[m, m] = 0$ gives rise to a differential $D$ on the algebra by defining $D(\varphi) = [\varphi, m]$. In other words, we need to show that $[[\varphi, m], m] = 0$. Recall that $m$ is odd when $\langle |m|, |m| \rangle = 1$. The graded Jacobi bracket gives

$$[[\varphi, m]m] = [\varphi[m, m]] + (-1)^{(m,m)}[[\varphi, m]m] = -[[\varphi, m], m],$$

which shows the desired result, in characteristic zero, when the grading is good. Moreover, we point out that the Jacobi identity also shows that $D[[\varphi, \psi]] = [\varphi, D(\psi)] + (-1)^{\psi D}[D(\varphi), \psi]$, so the differential in the cohomology of an associative algebra acts as a graded derivation of the Lie algebra, equipping $C(V)$ with the structure of a differential graded Lie algebra.

We wish to generalize the Gerstenhaber bracket to the $A_\infty$ algebra case, where we are considering a more general codifferential $m$ on $T(V)$, in such a manner that the bracket with $m$ yields a differential graded Lie algebra structure on $C(V)$. If we consider the bracket of coderivations, then a problem arises when the codifferential is not homogeneous. First of all, $m^2 = 0$ is not equivalent to the condition $[m, m] = 0$. Secondly, if we define $D(\varphi) = [\varphi, m]$, then we do not obtain in general that $D^2 = 0$. To see this, note that $[m, m] = 0$ is equivalent to $\sum_{k+l=n+1} [m_k, m_l] = 0$.
for all $n \geq 1$. Let $\varphi_p \in \text{Hom}(V^p, V)$. Then

$$[[\varphi_p, m], m]_{n+p-1} = \sum_{k+l=n+1} [[\varphi_p, m_k], m_l] =$$

$$\sum_{k+l=n+1} [\varphi_p, [m_k, m_l]] + (-1)^{(m_k, m_l)}[[\varphi_p, m_l], m_k] =$$

$$\sum_{k+l=n+1} (-1)^{k+l+1}[[\varphi_p, m_l], m_k] = (-1)^n[[\varphi_p, m], m]_{n+p-1}$$

Thus we only obtain cancellation of terms when $n$ is odd. However, this is sufficient to show that in the particular case where $m_k = 0$ for all even or all odd $k$, then $D^2 = 0$. In this case we also can show that $\hat{m}^2 = 0$ is equivalent to $[m, m] = 0$ as well. These problems occur because the product grading on $\mathbb{Z}_2 \times \mathbb{Z}$ is not good.

Since these problems do not arise if we are considering $\mathbb{Z}_2$-graded codifferentials (or $\mathbb{Z}$-graded codifferentials, in the $\mathbb{Z}$-graded case), it is natural to consider a codifferential on the parity reversion (or suspension) of $V$.

For the remainder of this section, let us assume for definiteness that $W = IV$ and that $d \in C(W)$ satisfies $d^2 = 0$. Because $W$ is $\mathbb{Z}_2$-graded, $d^2 = 0$ is equivalent to $[d, d] = 0$ for $d$ odd, and moreover, $C(W)$ is a differential graded Lie algebra with differential $D(\varphi) = [\varphi, d]$. Let $m_k = \eta^{-1}_k \circ d_k \circ \eta_k$ and $\mu_l = \eta^{-1}_l \circ \delta_l \circ \eta_l$. Define a new bracket $\{\cdot, \cdot\}$ on $C(V)$ by $\{m_k, \mu_l\} = \eta^{-1}_k \circ [d_k, \delta_l] \circ \eta_{k+l-1}$. It follows easily that

$$\{m_k, \mu_l\} = (-1)^{(k-1)\mu_l}[m_k, \mu_l].$$

Of course this new bracket no longer satisfies the graded antisymmetry or graded Jacobi identity with respect to the inner product we have been using on $\mathbb{Z}_2 \times \mathbb{Z}$. However, the bracket does satisfy these properties with respect to the different inner product on $\mathbb{Z}_2 \times \mathbb{Z}$. We state this result in the form of a lemma, whose proof is straightforward.

**Lemma 1.** Let $V$ be equipped with a $\mathbb{Z}_2 \times \mathbb{Z}$-graded Lie bracket $[\cdot, \cdot]$ with respect to the inner product $(\langle \hat{m}, n \rangle, (\hat{m}', n')) = \hat{m}^t + nn'$ on $\mathbb{Z}_2 \times \mathbb{Z}$. Then the bracket $\{\cdot, \cdot\}$ on $V$ given by $\{u, v\} = (-1)^{\deg(u)\deg(v)}[u, v]$ defines the structure of a $\mathbb{Z}_2 \times \mathbb{Z}$-graded Lie algebra on $V$ with respect to the inner product $(\langle \hat{m}, n \rangle, (\hat{m}', n')) = (\hat{m} + n)(\hat{m}' + n')$ on $\mathbb{Z}_2 \times \mathbb{Z}$.

The bracket $\{\cdot, \cdot\}$ given by $\{u, v\} = (-1)^{\deg(u)(\deg(v) + \deg(v))}[u, v]$ also defines the structure of a $\mathbb{Z}_2 \times \mathbb{Z}$-graded Lie algebra on $V$ with respect to the second inner product.

The bracket $\{\cdot, \cdot\}$ on $C(V)$ essentially coincides with the bracket of coderivations on $C(W)$, so $[d, d] = 0$ implies that $\{m, m\} = 0$. Define a differential on $C(V)$ by $D(\varphi) = \{\varphi, m\}$. The homology of this differential is called the cohomology of the $A_\infty$ algebra. We have been considering the picture $V = IV$. This means we have been considering $A_\infty$ algebras as defined in [11, 10]. If we consider instead the picture $V = IIW$, then the bracket induced on $C(V)$ by this on $C(W)$ will be given by $\{m_k, \mu_l\} = (-1)^{(k-1)(\mu_l + l-1)}[m_k, \mu_l]$, which is the second modified bracket described in the lemma. Thus we have similar results. We shall call either one of these two brackets the modified Gerstenhaber bracket.

The Hochschild cohomology of $A_\infty$ algebras was defined in [10] as the cohomology given by $D(\varphi) = \{\varphi, m\}$, and it was shown that this cohomology classifies the
infinitesimal deformations of an $A_\infty$ algebra. We shall not go into the details here. It is important to note however, that unlike the cohomology theory for an associative algebra, where the $\mathbb{N}$-grading on the tensor product gives rise to an $\mathbb{N}$-grading on the cohomology, for an $A_\infty$ algebra there is only one cohomology group $H(V)$. (However, $H(V)$ does inherit a natural $\mathbb{Z}_2$-grading.) The reason is that the image of $\varphi \in C^n(V)$ under the coboundary operator has a part in all $C^k(V)$ with $k \geq n$. Only in the case of an associative or differential graded associative algebra do we get a grading of the cohomology.

5.1. Cyclic Cohomology of $A_\infty$ Algebras. Now let us suppose that $V$ is equipped with a nondegenerate even graded symmetric inner product $\langle \cdot , \cdot \rangle$. Graded symmetry means that $\langle v, w \rangle = (-1)^{vw}\langle w, v \rangle$. The inner product induces an isomorphism between $C^k(V) = \text{Hom}(V^k, V)$ and $C^k(V, k) = \text{Hom}(V^{k+1}, k)$, given by $\varphi \mapsto \tilde{\varphi}$, where

$$\tilde{\varphi}(v_1, \cdots, v_{k+1}) = \langle \varphi(v_1, \cdots, v_k), v_{k+1} \rangle .$$

Non-degeneracy means that the map $\lambda : V \to V^* = \text{Hom}(V, k)$, given by $\lambda(v)(w) = \langle v, w \rangle$ is an isomorphism. (When $k$ is a field, this is equivalent to the usual definition of a non degenerate bilinear form.)

An element $\varphi \in \text{Hom}(V^k, V)$ is said to be cyclic with respect to the inner product if

$$\langle \varphi(v_1, \cdots, v_k), v_{k+1} \rangle = (-1)^{k+v_k} \varphi \langle v_1, \varphi(v_2, \cdots, v_{k+1}) \rangle .$$

Then $\varphi$ is cyclic if and only if $\tilde{\varphi}$ is cyclic in the sense that

$$\tilde{\varphi}(v_1, \cdots, v_{k+1}) = (-1)^{n+v_{k+1}(v_1+\cdots+v_k)}\tilde{\varphi}(v_{k+1}, v_1, \cdots, v_k).$$

If $m \in C(V)$, then we say that $m$ is cyclic if $m_k$ is cyclic for all $k$. If $m$ determines an $A_\infty$ algebra structure on $V$, then we say that the inner product is invariant if $m$ is cyclic with respect to the inner product. Denote the submodule of cyclic elements in $C(V)$ by $CC(V)$, and the submodule of cyclic elements in $C(V, k)$ by $CC(V, k)$. The following lemma shows how to construct cyclic elements from arbitrary elements of $C^n(V, k)$.

Lemma 2. Suppose that $\tilde{f} \in C^n(V, k)$. Define $C(\tilde{f}) \in C^n(V, k)$ by

$$C(f)(v_1, \cdots, v_{n+1}) = \sum_{0 \leq i \leq n} (-1)^{v_i+\cdots+v_i+v_{i+1}+\cdots+v_{n+1}+ni} f(v_{i+1}, \cdots, v_i)$$

Then $C(\tilde{f})$ is cyclic. Furthermore, $C(\tilde{f}) = (n+1)\tilde{f}$ if $\tilde{f}$ is cyclic.

The lemma above follows from the technical lemma below, which simplifies computations with cyclic elements.

Lemma 3. If $\tilde{f} \in C^n(V, k)$, then $\tilde{f}$ is cyclic iff whenever $\alpha = v_1 \otimes \cdots \otimes v_i$ and $\beta = v_i \otimes \cdots \otimes v_{n+1}$, $\tilde{f}(\alpha \otimes \beta) = (-1)^{\alpha \beta + in} \tilde{f}(\beta \otimes \alpha)$.

The following lemma records the fact that $CC(V)$ is a graded Lie subalgebra of $C(V)$, with respect to the bracket of coderivations.
Lemma 4. Let \( \varphi_k \in C^k(V) \) and \( \psi_l \in C^l(V) \). If \( \varphi \) and \( \psi \) are cyclic, then so is \([\varphi, \psi]\). Moreover, if \( n = k + l - 1 \), then
\[
(41) \quad (\varphi, \psi)(v_1, \ldots, v_{n+1}) = \sum_{0 \leq i \leq n} (-1)^{(v_1 + \cdots + v_i)(v_{i+1} + \cdots + v_{n+1}) + in} \varphi_k(\psi_l(v_{i+1}, \ldots, v_i), v_{i+l+1}, \ldots, v_i),
\]
where in expressions of this type, indices should be interpreted \( \mod n + 1 \).

Since \( \{\varphi, \psi\} = (-1)^{(k-1)}[\varphi, \psi] \), it follows that the modified Gerstenhaber bracket of cyclic elements is also cyclic. Thus finally, we can state the main theorem, which allows us to define cyclic cohomology of an \( A_\infty \) algebra.

Theorem 1. i) Suppose that \( V \) is a \( \mathbb{Z}_2 \)-graded \( k \)-module with an inner product \((\cdot, \cdot)\). Suppose that \( \varphi, \psi \in C(V) \) are cyclic. Then \( \{\varphi, \psi\} \) is cyclic. Furthermore, the formula below holds.

\[
(42) \quad (\varphi, \psi)(v_1, \ldots, v_{n+1}) = \sum_{k+l=n+1} (-1)^{(v_1 + \cdots + v_i)(v_{i+1} + \cdots + v_{n+1}) + in} \varphi_k(\psi_l(v_{i+1}, \ldots, v_i), v_{i+l+1}, \ldots, v_i).
\]

Thus the inner product induces a structure of a \( \mathbb{Z}_2 \times \mathbb{Z} \)-graded Lie algebra in \( CC(V, k) \) given by \( \{\varphi, \psi\} = \{\varphi, \psi\} \).

ii) If \( m \) is an \( A_\infty \) structure on \( V \), then there is a differential in \( CC(V, k) \), given by

\[
(43) \quad D(\widetilde{\varphi})(v_1, \ldots, v_{n+1}) = \sum_{k+l=n+1} (-1)^{(v_1 + \cdots + v_i)(v_{i+1} + \cdots + v_{n+1}) + in} \varphi_k(m_l(v_{i+1}, \ldots, v_i), v_{i+l+1}, \ldots, v_i).
\]

iii) If the inner product is invariant, then \( D(\widetilde{\varphi}) = \{\varphi, \widetilde{m}\} \). Thus \( CC(V, k) \) inherits the structure of a differential graded Lie algebra.

Proof. The first statement follows from lemma 3 and the third follows from the first two. The second assertion follows immediately from the first when the inner product is invariant. The general case is a routine verification, which we omit. \( \square \)

Note that the second statement in the theorem holds even in the absence of an inner product, because the definition of cyclicity in \( CC(V, k) \) does not depend on the inner product. Thus cyclic cohomology of an \( A_\infty \) algebra can be defined independently of the inner product.

We define \( HC(V) \) to be the cohomology associated to the coboundary operator on \( CC(V) \). As in the case of Hochschild cohomology, the cyclic cohomology is \( \mathbb{Z}_2 \)-graded, but does not inherit an \( \mathbb{N} \)-grading, except in the associative algebra case.

6. Cohomology of \( L_\infty \) Algebras

There is a natural isomorphism \( \eta \) between \( \Lambda V \) and \( S(IV) \) which is given by
\[
(44) \quad \eta(v_1 \wedge \cdots \wedge v_n) = (-1)^{\pi(v_1 + n-2)v_2 + \cdots + v_n - 1} \pi v_1 \cdots \pi v_n,
\]
Note that $\eta$ is neither even nor odd. The restriction $\eta_k$ of $\eta$ to $\bigwedge^k V$ has parity $k$. Of course, $\eta$ does preserve the exterior degree. For simplicity in the following, let $W = \Pi V$ and let $w_i = \pi v_i$, and denote $C(W) = \text{Hom}(S(W), W)$, and $C(V) = \text{Hom}(\bigwedge(V), V)$. We will use notational conventions as in section 3, so that for $d \in C(W)$, $d_k$ will denote the restriction of this map to $\bigwedge^k W$, $d_{lk}$ will denote the restriction of the associated coderivation $\hat{d}_l$ to $V^{k+l-1}$ etc.

The following lemma will be useful later on.

**Lemma 5.** Suppose that $\sigma$ is a permutation of $n$ elements. Then

$$(-1)^{(n-1)v_1 + \cdots + v_n - 1}(-1)^{\sigma}(\sigma; v_1, \cdots, v_n) = \quad (45) \quad (-1)^{(n-1)v_{\sigma(1)} + \cdots + v_{\sigma(n)} - 1}(-1)^{\sigma}(\sigma; w_1, \cdots, w_n).$$

**Proof.** From the properties of the graded exterior algebra, we have

$$\eta(v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)}) = (-1)^{\sigma} \epsilon(\sigma; v_1, \cdots, v_n)\eta(v_1, \cdots, v_n)
\quad = (-1)^{\sigma} \epsilon(\sigma; v_1, \cdots, v_n)(-1)^{(n-1)v_1 + \cdots + v_n - 1}w_1 \cdots w_n.$$

On the other hand, by direct substitution, we have

$$\eta(v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)}) = (-1)^{(n-1)v_{\sigma(1)} + \cdots + v_{\sigma(n)} - 1}w_{\sigma(1)} \cdots w_{\sigma(n)}
\quad = (-1)^{(n-1)v_{\sigma(1)} + \cdots + v_{\sigma(n)} - 1}(-1)^{\sigma}\epsilon(\sigma; w_1, \cdots, w_n)w_1 \cdots w_n.$$

Comparing the coefficients of the two expressions yields the desired result. \qed

Let $d \in C(W)$ and define $l_k = \eta_k^{-1} \circ d_k \circ \eta_k$, so that

$$d_k(w_{\sigma(1)}, \cdots, w_{\sigma(k)}) = (-1)^{(k-1)v_{\sigma(1)} + \cdots + v_{\sigma(k)} - 1}\pi m_k(v_{\sigma(1)}, \cdots, v_{\sigma(k-1)}).$$

Let us abbreviate $\epsilon(\sigma; v_1, \cdots, v_n)$ by $\epsilon(\sigma; v)$ and $\epsilon(\sigma; w_1, \cdots, w_n)$ by $\epsilon(\sigma; w)$. Let $n = k + l - 1$. Define $l_k = \eta_k^{-1} \circ d_k \circ \eta_n$, where $d_k$ is given by

$$d_k(w_1, \cdots, w_n) = \sum_{\sigma \in \text{Sh}(k, n-k)} \epsilon(\sigma, w)d_l(w_{\sigma(1)}, \cdots, w_{\sigma(l)})w_{\sigma(l+1)} \cdots w_{\sigma(n)}.$$

We wish to compute $l_k$ in terms of $l_l$. Now

$$l_k(v_1, \cdots, v_n) = \sum_{\sigma \in \text{Sh}(l, n-l)} \epsilon(\sigma, w)(-1)^{(n-1)v_1 + \cdots + v_n - 1}\eta_k^{-1}(d_l(w_{\sigma(1)}, \cdots, w_{\sigma(l)})w_{\sigma(l+1)} \cdots w_{\sigma(n)})
\quad = \sum_{\sigma \in \text{Sh}(l, n-l)} (-1)^{\sigma}\epsilon(\sigma, v)(-1)^{\sigma}\eta_k^{-1}(d_l(w_{\sigma(1)}, \cdots, w_{\sigma(l)})w_{\sigma(l+1)} \cdots w_{\sigma(n)})
\quad = \sum_{\sigma \in \text{Sh}(l, n-l)} (-1)^{\sigma}\epsilon(\sigma, v)(-1)^{\sigma}l_l(v_{\sigma(1)}, \cdots, v_{\sigma(l)})v_{\sigma(l+1)} \cdots v_{\sigma(n)}
\quad = \sum_{\sigma \in \text{Sh}(l, n-l)} (-1)^{\sigma}\epsilon(\sigma, v)(-1)^{(k-1)l}l_l(v_{\sigma(1)}, \cdots, v_{\sigma(l)})v_{\sigma(l+1)} \cdots v_{\sigma(n)}.$$
where

\begin{align*}
(49) \quad r &= (n-1)v_{\sigma(1)} + \cdots + v_{\sigma(n-1)} \\
(50) \quad s &= (l-1)v_{\sigma(1)} + \cdots + v_{\sigma(l-1)} \\
(51) \quad t &= (k-1)(l_1 + v_{\sigma(1)} + \cdots + v_{\sigma(l-1)}(k-2)v_{\sigma(l+1)} + \cdots + v_{\sigma(n-1)}
\end{align*}

Thus we deduce immediately that \( \tilde{l}_k = (-1)^{(k-1)l}l_k \), where \( l_k \) is the restriction of the coderivation \( \tilde{l}_k \) to \( V^{k+l-1} \). This formula is identical to the formula we deduced in section 3 connecting \( \tilde{m}_{lk} \) to \( \tilde{n}_{lk} \).

Suppose that \( d \) is an odd codifferential, so that \( d^2 = 0 \). This is equivalent to \( \sum_{k+l=n+1} d_k \circ d_{lk} = 0 \), which is equivalent to the relations \( \sum_{k+l=n+1} (-1)^{(k-1)l}l_k \circ l_{lk} = 0 \), since \( |l| = l \). This last relation can be put in the form

\begin{equation}
\sum_{k+l=n+1, \sigma \in Sh(l,n-l)} (-1)^{\sigma \epsilon}(\sigma, v)(-1)^{(k-1)l}[l_1(v_{\sigma(1)}, \ldots, v_{\sigma(l)}), v_{\sigma(l+1)}, \ldots, v_{\sigma(n)})] = 0.
\end{equation}

We say that the maps \( l_k \) induce the structure of an \( L_\infty \) algebra, or strongly homotopy Lie algebra on \( V \). In [11], the sign \( k(l-1) \) instead of \( (k-1)l \) appears in the definition, but since \( k(l-1) - (k-1)l = n+1 \), this makes no difference in the relations.

If we define \( [\varphi, \psi] \) to be the bracket of coderivations, then we can define the modified bracket \( \{ \varphi, \psi \} = (-1)^{d\text{ex} \varphi | \psi} [\varphi, \psi] \). The definition of an \( L_\infty \) algebra can be recast in terms of the bracket. In this language, \( t \in C(V) \) determines an \( L_\infty \) structure on \( V \) when \( \{l, l\} = 0 \). The cohomology of an \( L_\infty \) algebra is defined to be the cohomology on \( C(W) \) induced by \( l \), in other words, \( D(\varphi) = [\varphi, l] \). This definition makes \( C(V) \) a differential graded Lie algebra, with respect to the second inner product on \( \mathbb{Z}_2 \times \mathbb{Z} \). These results are completely parallel to the \( A_\infty \) case.

In [10], the relationship between infinitesimal deformations of an \( A_\infty \) algebra and the cohomology of the \( A_\infty \) algebra was explored. The basic result is that the cohomology classifies the infinitesimal deformations. Since we did not explore this matter here for \( A_\infty \) algebras, we shall discuss the parallel result for \( L_\infty \) algebras.

An infinitesimal deformation \( l_u \) of an \( L_\infty \) algebra is given by taking \( l_u = l + u\lambda \), where \( u \) is an infinitesimal parameter whose parity chosen so that \( (l_u)_k \) has parity \( k \). It follows that \( |\lambda_k| = |u| + k \). Since \( u \) must have fixed parity, this determines the parity of \( \lambda_k \).

The situation is more transparent if we switch to the \( W \) picture, so suppose that \( l = \eta^{-1} \circ d \circ \eta \), and \( \lambda = \eta^{-1} \circ \delta \circ \eta \). Let \( d_u = d + u\delta \), and let us suppose that \( d^2 = 0 \), which is equivalent to \( l \) giving an \( L_\infty \) structure on \( V \). Then \( d_u \) is an infinitesimal deformation of \( d \) if \( d_u^2 = 0 \). Since \( u \) is an infinitesimal parameter, \( u^2 = 0 \), but we also want the parity of \( d_u \) to be odd, so that \( |u| = 1 - |\delta| \). (We assume here that \( \delta \) is homogeneous.) Now \( d_u^2 = 0 \) is equivalent to \( d^2 + u\delta \circ d + d\delta = 0 \). Also, \( d_u = (-1)^{ud}du = -(-1)^{\delta u}du \), so this condition reduces to \( \delta, d \) = 0; in other words, infinitesimal deformations are given by cocycles in the cohomology of the \( L_\infty \) algebra.

Trivial deformations are more complicated in the \( L_\infty \) case than for Lie algebras. Consider \( d \) as a codifferential of \( T(W) \). Two codifferentials are said to be equivalent if there is an automorphism of \( T(W) \) which takes one of them to the other. In the Lie algebra case, one only considers automorphisms of \( T(W) \) induced by linear
isomorphism of $W$ into itself. One can show that $d_u$ is a trivial infinitesimal deformation precisely when $\delta$ is a coboundary. Thus the cohomology of $C(W)$ classifies the infinitesimal deformations of the $L_\infty$ algebra. When we transfer this back to the $V$ picture, note that $m + (-1)^u \lambda u$ is the deformed product associated to $d_u$. The condition for $l_u$ to be an $L_\infty$ algebra still is that $\{\lambda, m\} = 0$. We summarize these results in the theorem below.

**Theorem 2.** Let $l$ be an $L_\infty$ algebra structure on $V$. Then the cohomology $H(V)$ of $C(V)$ classifies the infinitesimal deformations of the $L_\infty$ algebra.

Suppose that $l$ is a Lie algebra structure on $V$. Then the Lie algebra coboundary operator on $V$ coincides up to a sign with the $L_\infty$ algebra coboundary operator on $V$. This gives a nice interpretation of the cohomology of a Lie algebra.

**Theorem 3.** Let $l$ be an Lie algebra structure on $V$. Then the Lie algebra cohomology $H(V)$ of $V$ classifies the infinitesimal deformations of the Lie algebra into an $L_\infty$ algebra.

### 6.1. Cyclic Cohomology of $L_\infty$ Algebras.

Suppose $V$ is equipped with a non-degenerate graded even symmetric inner product. An element $\varphi \in C^n(V)$ if the tensor $\hat{\varphi} : V^n \rightarrow k$, given by

$$\hat{\varphi}(v_1, \ldots, v_{n+1}) = \langle \varphi(v_1, \ldots, v_n), v_{n+1} \rangle$$

is (graded) antisymmetric; i.e., $\hat{\varphi} \in CC^n(V,k) = \text{Hom}(\Lambda^{n+1}, k)$. Note that the antisymmetry condition is equivalent to the cyclicity condition given in the definition of cyclicity for $A_\infty$ algebras, given that $\varphi$ is antisymmetric. Since the inner product is non-degenerate, the map $\varphi \mapsto \hat{\varphi}$ is an even isomorphism between the submodule $CC^n(V)$ of $C^n(V)$ consisting of cyclic elements, and $CC^n(V,k)$.

**Theorem 4.** i) Suppose that $V$ is a $\mathbb{Z}_2$-graded $k$-module with an inner product $\langle \cdot, \cdot \rangle$. Suppose that $\varphi, \psi \in C(V)$ are cyclic. Then $\{\varphi, \psi\}$ is cyclic. Furthermre, the formula below holds.

$$\hat{\varphi}(v_1, \ldots, v_{n+1}) = \sum_{k+l=n+1} (-1)^{\sigma} (-1)^{(k-1)l} \hat{\varphi}_k(l_1(v_{\sigma(1)}, \ldots, v_{\sigma(l)}), v_{\sigma(l+1)}, \ldots, v_{\sigma(n+1)}).$$

Thus the inner product induces a structure of a $\mathbb{Z}_2 \times \mathbb{Z}$-graded Lie algebra in the module $CC(V)$ consisting of all cyclic elements in $C(V)$, by defining $\{\hat{\varphi}, \hat{\psi}\} = \hat{\varphi}(\cdot, \cdot, \cdot)$.

ii) If $l$ is an $L_\infty$ structure on $V$, then there is a differential in $CC(V)$, given by

$$D(\hat{\varphi})(v_1, \ldots, v_{n+1}) = \sum_{k+l=n+1} (-1)^{\sigma} (-1)^{(k-1)l} \hat{\varphi}_k(l_1(v_{\sigma(1)}, \ldots, v_{\sigma(l)}), v_{\sigma(l+1)}, \ldots, v_{\sigma(n+1)}).$$

iii) If the inner product is invariant, then $D(\hat{\varphi}) = \{\hat{\varphi}, \hat{l}\}$. Thus $CC(V)$ inherits the structure of a differential graded Lie algebra.
We denote the cohomology given by the cyclic coboundary operator on $CC(V)$ as $HC(V)$. It is interesting to note that $HC^n(V) \equiv H^{n+1}(V, k)$, where $H(V, k)$ is interpreted as the cohomology of $V$ with coefficients in the trivial module $k$. Suppose that $V$ is an $L_\infty$ algebra with an invariant inner product. Then an infinitesimal deformation $l_t = l + t\varphi$ preserves the inner product, that is the inner product remains invariant under $l_t$, precisely when $\varphi$ is cyclic. Thus we see that cyclic cocycles correspond to infinitesimal deformations of the $L_\infty$ structure which preserve the inner product. In a similar manner as before, cyclic coboundaries correspond to trivial deformations preserving the inner product. Thus we have the following classification theorem.

**Theorem 5.** Let $l$ be an $L_\infty$ algebra structure on $V$, with an invariant inner product. Then the cyclic cohomology $HC(V)$ classifies the infinitesimal deformations of the $L_\infty$ algebra preserving the inner product.

When $l$ determines a Lie algebra structure on $V$, the cohomology $HC(V)$ has a natural $\mathbb{Z}$-grading, and the group $HC^2(V)$ controls deformations of the Lie algebra structure.

**Theorem 6.** Let $l$ be a Lie algebra structure on $V$, with an invariant inner product. Then the second cyclic cohomology group $HC^2(V)$ classifies the infinitesimal deformations of the Lie algebra preserving the inner product.

For a Lie algebra, we have $HC^n(V) \equiv H^{n+1}(V, k)$, the Lie algebra cohomology of $V$ with trivial coefficients. Thus the deformations preserving an invariant inner product are classified by the cohomology of the Lie algebra with trivial coefficients. Finally, we give a nice interpretation of the cyclic cohomology of a Lie algebra.

**Theorem 7.** Let $l$ be an Lie algebra structure on $V$, with an invariant inner product. Then the Lie algebra cyclic cohomology $H(V)$ of $V$ classifies the infinitesimal deformations of the Lie algebra into an $L_\infty$ algebra preserving the inner product.

Note that as in the case of $A_\infty$ algebras, $C(V) = \text{Hom}(\wedge V, V)$ is really the direct product of its graded subspaces $C^k(V)$, so it is not a graded Lie algebra in the strict sense of the definition, because it is not the direct sum of the graded subspaces. Moreover, the cohomology $H(V)$ does not inherit a natural $\mathbb{Z}$-grading in general. Similarly, the space $CC(V)$ of cyclic cochains is a direct product, and $HC(V)$ does not have a natural $\mathbb{Z}$-grading. Nevertheless, the good $\mathbb{Z}_2 \times \mathbb{Z}$-grading does equip these space with a natural $\mathbb{Z}_2$-grading, so that the cohomology is $\mathbb{Z}_2$-graded. These issues are addressed in more detail in [3].

It is interesting to note that the cyclic cohomology of $L_\infty$ algebras has more symmetry than that of $A_\infty$ algebras; the full symmetric group acts on the space of cyclic cochains for $L_\infty$ algebras. This difference has some consequences in terms of the actions of these algebras on graph complexes. In [13], it is shown that $L_\infty$ algebras with an invariant inner product act on the ordinary graph complex, while $A_\infty$ algebras with an invariant inner product act on the space of ribbon graphs, which are given by graphs equipped with a cyclic order at each vertex.

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REFERENCES

[1] M. Alexandrov, M. Kontsevich, A. Schwarz, and O. Zaboronsky, The geometry of the master equation and topological quantum field theory, Internat. Journ. Modern Phys. A12 (1997), 1405–1423.

[2] A. Connes, Non-commutative differential geometry, Institute Des Hautes Etudes Scientifiques 62 (1985), 41–93.

[3] A. Fialowski and M. Penkava, Deformation theory of infinity algebras, preprint math.RT/0101097, 2000.

[4] M. Gerstenhaber, The cohomology structure of an associative ring, Annals of Mathematics 78 (1963), 267–288.

[5] E. Getzler and J.D.S. Jones, $A\infty$-algebras and the cyclic bar complex, Illinois Journal of Mathematics 34 (1990), no. 2, 256–283.

[6] Operads, homotopy algebra, and iterated integrals for double loop spaces, Preprint, 1995.

[7] G. Hochschild, On the cohomology groups of an associative algebra, Annals of Mathematics 46 (1945), 58–67.

[8] D. Kastler, Cyclic cohomology within the differential envelope, Travaux En Cours, 1988.

[9] M. Kontsevich, Feynman diagrams and low dimensional topology, First European Congress of Mathematics, Paris, 1992, Birkhauser, Basel, 1994, pp. 97–121.

[10] T. Lada and M. Markl, Strongly homotopy Lie algebras, Comm. in Algebra 23 (1995), 2147–2161.

[11] T. Lada and J. Stasheff, Introduction to sh Lie algebras for physicists, Intern. J. Theor. Phys 32 (1993), 1087–1103, Preprint hep-th 9209099.

[12] J. Loday, Cyclic homology, Springer-Verlag, 1992.

[13] M. Markl, A cohomology theory for $A(m)$-algebras and applications, Journal of Pure and Applied Algebra 83 (1992), no. 6, 141–175.

[14] M. Penkava, $L_{\infty}$ algebras and their cohomology, Preprint [alg/9512014].

[15] M. Penkava, Infinity algebras and the homology of graph complexes, Preprint [alg/9601018].

[16] M. Penkava and A. Schwarz, $A\infty$ algebras and the cohomology of moduli spaces, Dykin Seminar, vol. 169, American Mathematical Society, 1995, pp. 91–107.

[17] M. Schlessinger and J. Stasheff, The Lie algebra structure of tangent cohomology and deformation theory, Journal of Pure and Applied Algebra 38 (1985), 313–322.

[18] P. Solt, Cyclic homology of algebras, World Scientific, 1987.

[19] J.D. Stasheff, On the homotopy associativity of $H$-spaces I, Transactions of the AMS 108 (1963), 275–292.

[20] On the homotopy associativity of $H$-spaces II, Transactions of the AMS 108 (1963), 293–312.

[21] The intrinsic bracket on the deformation complex of an associative algebra, Journal of Pure and Applied Algebra 89 (1993), 231–235.

[22] Closed string field theory, strong homotopy Lie algebras and the operad actions of moduli spaces, Conf. Proc. Lecture Notes Math. Phys., III, Internat. Press, Cambridge, MA, 1994, Perspectives in mathematical physics, pp. 265–288.

[23] R. Umble, The deformation complex for differential graded hopf algebras, Preprint, 1994.

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