Suppose that we have a sequence of quantum states, each drawn from an ensemble with known density matrix $\rho$. Schumacher compression then allows the sequence to be efficiently encoded so that $S(\rho) = -\text{tr} \log_2 \rho$ qubits are required to encode each state in the limit that the length of the sequence goes to infinity. This resembles classical source coding, in which a source can be compressed to a rate asymptotically approaching its Shannon entropy. However, classical compression can be performed by algorithms that are universal (do not depend on a description of the source) and efficient (have running time polynomial in the length of the input). In contrast, most existing quantum compression algorithms either rely on knowing the basis in which $\rho$ is diagonal or have no known polynomial time implementations.

This paper presents an efficient, universal, quantum data compression algorithm; that is, it can compress an unknown i.i.d. quantum source $\rho^\otimes n$ in $\text{poly}(n)$ time to a rate converging to its von Neumann entropy $S(\rho)$ and with error approaching zero as the number of copies, $n$, increases. Another efficient universal quantum data compression algorithm was presented in [3], but our algorithm has the advantages of simplicity and a better rate-disturbance tradeoff.

Our algorithm consists of two parts: a weak measurement of $\rho^\otimes n$ that estimates $\rho$ accurately without causing very much damage to the state, followed by compressing $\rho^\otimes n$ based on this estimate. Conceptually, this resembles classical methods of compression which determine the empirical distribution of their input in their first pass over the data and perform the compression in the second pass. The only new difficulties we will encounter in the quantum case come from the need to perform state tomography on $\rho$ without causing very much damage and to compress $\rho$ based on an imperfect estimate.

The problem of weakly measuring states of the form $\rho^\otimes n$ was introduced in [8] and further developed in [4, 5]. While it is impossible to measure a single state $\rho$ without causing disturbance, we expect ordinary classical logic to apply to $\rho^\otimes n$ when $n$ is large, so that it is possible to measure even non-commuting observables precisely with little disturbance. For example, in Nuclear Magnetic Resonance, the total $x$-magnetization of $n = \mathcal{O}(10^20)$ nuclear spins is continuously measured without causing decoherence by a probe consisting of a coil of wire around the sample. This is possible because the measurement does not precisely determine the number of nuclear spins pointing in the $x$ direction, but only gives a crude estimate of the quantity. In this section, we will introduce a procedure for state tomography on $\rho^\otimes n$ and then show how to modify it so its disturbance vanishes for large $n$ while at the same time it yields an asymptotically accurate estimate of $\rho$.

Let $\{\sigma_k\}_{k=1}^{d^2-1}$ is an orthonormal $(\text{tr} \sigma_j \sigma_k = \delta_{jk})$ basis of traceless Hermitian $d \times d$ matrices, and write the density matrix $\rho$ as $\rho = I/d + \sum_k \rho \sigma_k$. Estimating $\rho$ reduces to estimating the $d^2 - 1$ quantities $\text{tr} \rho \sigma_k$. If we now diagonalize $\sigma_k$ as $\sigma_k = \sum_{i=1}^d \lambda_i |v_i\rangle \langle v_i|$, then $\text{tr} \rho \sigma_k = \sum_i \lambda_i |v_i\rangle \langle v_i|$, so state tomography reduces to estimating $d(d^2 - 1)$ quantites of the form $\langle \phi | \rho | \phi \rangle$ and then performing a classical computation.

If we didn’t mind damaging the state, then one method of estimating $\alpha := \langle \phi | \rho | \phi \rangle$ would be to apply the projective measurement $\{ |\phi\rangle \langle \phi|, I - |\phi\rangle \langle \phi| \}$ to each copy of $\rho$. The number of occurrences of $|\phi\rangle \langle \phi|$ would be binomially distributed with mean $n \alpha$ and variance $n \alpha (1 - \alpha) \leq n/4$, so we could reliably estimate $\alpha$ to an accuracy of $\mathcal{O}(n^{-1/2})$. Of course, this measurement would drastically damage some states, such as $\frac{1}{\sqrt{2}}(|\phi\rangle + |\phi^\perp\rangle)$.

1 Modifying our techniques to only estimate the $d^2 - 1$ quantities $\text{tr} \rho \sigma_k$ would cause the state estimate to converge more quickly, but this would make our exposition slightly more complicated. Unfortunately, there is no known polynomial time implementation of quantum state tomography that has the probability of large deviations vanish at the asymptotically optimal rate.
Instead of measuring each state individually, we can also express this measurement as a collective operation on all \( n \) states simultaneously. It is given by the operators

\[
M_k = \sum_{x \in \{0,1\}^n} x_k |\phi\rangle \langle \phi| + (1 - x_k) (I - |\phi\rangle \langle \phi|).
\] (1)

where \( k \) ranges from 0 to \( n \) and \(|x| \) denotes the number of 1’s in the \( n \)-bit string \( x \). Clearly, measuring \( \{M_k\} \) yields the same statistics as measuring each state individually and counting the \(|\phi\rangle \langle \phi|\) outcomes. The measurement can also be constructed efficiently: we unitarily count the \( n \)-bit string \( x \) and then measure the ancilla (see Fig. 1).

Unfortunately, even the collective measurement in Eq. (1) causes substantial damage to the state. For example, if the measurement \( \{M_k\} \) is repeated, then the distribution of \( k \) will have a variance of \( O(n) \) the first time and 0 on subsequent measurements.\(^2\)

In [4] this problem was solved by initializing the ancilla in Fig. 1 to the state \( \sum_k e^{-k^2/2\Delta^2} |k\rangle \) instead of \(|0\rangle\). The measurement of \( k \) then has variance \( \Delta^2 + O(n) \) and it can be shown that the damage to \( \rho^{\otimes n} \) is \( O(n/(\Delta^2 + n)) \). Ref. [3] proposed a method which causes more damage to the state, but is easier to analyze for our purposes.

To implement the gentle measurement of [3], we will divide up the range from 0 \( \ldots \) \( n \) into \( m \) bins, with boundaries \( 0 = b_0 \leq b_1 \leq \cdots \leq b_m = n + 1 \). Then we will modify the collective measurement of Eq. (1) to measure only the bin that the state lies in instead of determining the exact value of \( k \). The new measurement \( \{M_j\} \) is given in terms of the \( M_k \) of Eq. (1) by

\[
M_j = \sum_{b_{j-1} \leq k < b_j} M_k
\] (2)

where \( j \) ranges from 1 to \( m \).

If the bin size, \( n/m \), is much larger than the \( O(\sqrt{n}) \) width of \( \rho^{\otimes n} \) then we expect to project onto a measurement outcome that contains almost all of the support of \( \rho^{\otimes n} \), thereby causing little disturbance. Since we want to avoid having a bin boundary within \( O(\sqrt{n}) \) of the state, for any choice of \( \rho \), we will choose the \( b_i \) uniformly at random from between 0 and \( n \).

The choice of \( m \) now defines a trade-off between disturbance caused to \( \rho^{\otimes n} \) and information gained about \( \rho \). Choosing a smaller \( m \) means that each bin is larger, so that a measurement outcome lets us infer less about \( \rho \), but we have a smaller probability of damaging \( \rho^{\otimes n} \) by projecting onto only part of its support.

**Proposition 1** The measurement \( \{M_j\} \) described above can be implemented in \( O(n) \) gates. If we choose \( m = n^s \) for \( 0 < s < 1/2 \), then the measurement will fail with probability \( O(n^{s-1/2} \ln n) \). Upon success, the measurement outcome is within \( O(n^{1-s} \ln n) \) of \( \rho \) and the disturbance (in the sense of entanglement fidelity) is less than \( \exp(-O(n^2 \ln n)) \leq O(n^{-p}) \) for any constant \( p \).

**Proof of proposition 1**

We begin by describing how to implement \( \{M_j\} \). First we count the number of times \(|\phi\rangle\) occurs in \( \rho^{\otimes n} \) and store the result \( k \in \{0, \ldots, n\} \) in an ancilla register. Then we perform a classical computation to determine which bin \( j \) contains the result \( k \). We measure \( j \), thus implementing the projective measurement \( M_j \) and then uncompute \( j \) and finally uncompute \( k \). This is demonstrated in Fig 2.

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2 This damage is sometimes useful. In [3], it is used as the first step of entanglement concentration. In fact, our compression protocol may be thought of as the gentle analogue of [3] in the same way that the compression scheme in [3] is the gentle analogue of the entanglement concentration procedure of [2].
of iii) is less than \( \exp(-O(n^2 \ln n)) \). Thus, the possibility of failure is dominated by the probability of i), which is \( O(n^{n+1/2} \ln n) \).

We say that the gentle measurement is successful if none of i), ii) or iii) occur. In this case, we can take as our estimate for \( \alpha \) an arbitrary value within the bin we have measured and by ii) will err by no more than \( 2n^{-s} \ln n \). Finally, let \( M_f \) be the measurement outcome we obtain, let \( |\varphi\rangle_{AB} \) be a purification of \( \rho_A^\otimes n \) and define \( \pi := M_f \otimes I_B \). Then the post-measurement state is \( |\varphi\rangle = \sqrt{\langle \varphi | \pi | \varphi \rangle} \) and the entanglement fidelity is \( F_e = \langle \varphi | \varphi \rangle = \frac{\langle \varphi | \pi | \varphi \rangle}{\sqrt{\langle \varphi | \varphi \rangle \langle \varphi | \varphi \rangle}} \). From iii) we have \( \langle \varphi | \pi | \varphi \rangle \geq 1 - \epsilon \) where \( \epsilon = \exp(-cO(n^2 \ln n)) \), so \( F_e \geq \sqrt{1 - \epsilon} = 1 - \exp(-cO(n^2 \ln n)) \).}

To perform gentle tomography we simply divide the \( n \) states into \( d(d^2 - 1) \) blocks of length \( l = [d(n^2 - 1)] \) and gently measure each block. If \( \{ |v_i^{(k)}\rangle \}_{k=1}^d \) is the basis for \( \sigma_k \), then we can index the blocks by \( i = 1, \ldots, d \) and \( k = 1, \ldots, d^2 - 1 \) and measure \( |v_i^{(k)}\rangle \) on block \((i,k)\).

**Proposition 2 (Gentle tomography)** For any \( 0 < s < 1/2 \) and fixed Hilbert space dimension, applying the procedure described above to \( \rho_A^\otimes n \) requires \( \text{poly}(n) \) time and fails with probability \( O(n^{-1/2} \ln n) \). Upon success, the disturbance is less than \( O(n^{-2}) \) and the estimate \( \hat{\rho} \) satisfies \( \| \rho - \hat{\rho} \|_1 \leq O(n^{-s} \ln n) \).

**Proof:** We say that tomography succeeds when each of the \( d(d^2 - 1) \) measurements succeed individually. Since the dimension \( d \) is a constant, we can use Proposition II to bound the failure probability by \( O(d^{3(\frac{1}{2} - s)} n^{-1/2} \ln n) \sim O(n^{-1/2} \ln n) \) and the state disturbance by \( O(d^2 n^{-2}) \sim O(n^{-2}) \).

We still need to describe how to form an accurate estimate \( \hat{\rho} \). Assume that each gentle measurement has succeeded. Then the \( d(d^2 - 1) \) gentle measurements output not state estimates, but bins, \( (b_1, b_2, |\phi\rangle) \), guaranteeing only that \( b_1 \leq \langle \phi | \rho | \phi \rangle \leq b_2 \). We will try to find a state \( \hat{\rho} \) that is consistent with each bin. Since \( \rho \) is consistent with each bin, we know that some such \( \hat{\rho} \) exists. We can find it efficiently by solving a semi-definite program for \( \hat{\rho} \) given by the constraints: \( \rho \geq 0 \), \( \text{tr} \hat{\rho} = 1 \) and \( b_1 \leq \langle \phi | \rho | \phi \rangle \leq b_2 \) for each bin \( (b_1, b_2, |\phi\rangle) \).

Given such a \( \hat{\rho} \), we have for each gentle measurement that \( |\langle \phi | (\rho - \hat{\rho}) | \phi \rangle| < \epsilon \), where \( \epsilon = O(n^{-1/2} \ln n) \). Then if \( \sigma_k = \sum_i \lambda_i |v_i^{(k)}\rangle \langle v_i^{(k)}| \), \( |\text{tr}(\rho - \hat{\rho}) \sigma_k| = \sum_i |\lambda_i (\rho - \hat{\rho}) | v_i^{(k)}\rangle \langle v_i^{(k)}| \| < \epsilon \sum_i |\lambda_i| \leq \sqrt{d} \epsilon \). Thus, by the Cauchy-Schwartz inequality,

\[ \| \rho - \hat{\rho} \|_1 \leq d \sqrt{\sum_k (\text{tr}(\rho - \hat{\rho}) \sigma_k)^2} \leq d^{1/2} \epsilon \]

This extends our trade-off curve for gentle measurements to full gentle state tomography. It is an interesting question whether the tradeoff we have found between accuracy and probability of failure is optimal up to logarithmic factors.

**III. Universal Compression**

Now look more closely at the quantum coding. Schumacher compression works by identifying the eigenvalues and eigenvectors of \( \rho \), then coherently performing classical Shannon compression on sequences of those eigenvectors with probabilities given by the corresponding eigenvalues. However, we are forced to operate with only an estimate \( \hat{\rho} \approx \rho \), so we will need to use a data compression scheme that deals well with small inaccuracies in the state estimate.

This case has been analyzed in [5], which found that compressing \( \rho \) in the basis \( \{|i\rangle\} \) with any classical algorithm gives an asymptotic rate of \( R = \sum_i \langle i | \rho | i \rangle \log |i\rangle | \langle i| \rangle \). This is because compressing \( \rho \) faithfully reduces to compressing the diagonal entries of \( \rho \) in an arbitrary basis \( \{|i\rangle\} \). Due to the nonnegativity of the relative entropy \( S(\rho | \sigma) = \text{tr} \rho (\log \rho - \log \sigma) \geq 0 \), we have \( R \leq - \text{tr} \rho \log \sigma = S(\rho | S(\rho | \sigma) \) for any density matrix \( \sigma \) that can be diagonalized as \( \sigma = \sum_i \rho_i | i \rangle \langle i | \). Thus, for any density matrix \( \sigma \), we can encode \( \rho \) by diagonalizing it in the basis of \( \sigma \) and then using a classical reversible algorithm. This will achieve a rate \( R \leq S(\rho) + S(\rho | \sigma) \).

Unfortunately, there is no simple bound for \( S(\rho | \hat{\rho}) \) in terms of \( \| \rho - \hat{\rho} \|_1 \); in fact, the relative entropy can be infinite if the support of \( \rho \) is not contained within the support of \( \hat{\rho} \). This problem corresponds to the situation when our state estimate has led the encoder to believe that certain vectors will never appear, so that when it encounters them in \( \rho \), it has made no provision to deal with them. The solution to this is simple: assume that any input vector has a small, but non-zero, chance of occurring. This means that instead of encoding according to \( \rho \), we will use \( \hat{\rho}_\delta := (1 - \delta) \hat{\rho} + \delta I / d \) as our state estimate, for some small \( \delta > 0 \).

Suppose that after performing gentle tomography \( \| \rho - \hat{\rho} \|_1 < \epsilon \). Then if we choose \( \epsilon, \delta = O(n^{-s} \log n) \), we can bound the rate by

\[ R \leq - \text{tr} \rho \log \hat{\rho}_\delta \leq S(\hat{\rho}_\delta) + O(n^{-s} \log^2 n) \]

\[ \leq S(\rho) + O(n^{-s} \log^2 n) \]

The second inequality follows from the operator inequality \( \hat{\rho}_\delta \geq \delta I / d \) (implying \( -\log \hat{\rho}_\delta \geq \log(d/\delta)I = \)]
\(O(\log n)I\) and the last inequality is due to Fannes’ inequality. We have neglected the inefficiency of the classical coding, since we can choose it to be \(O(n^{-s})\) and will incur only exponentially small damage for \(s < \frac{1}{2}\).

To analyze the errors, note that since we usually cannot tell when tomography has failed, we ought to consider failure to be another form of disturbance. Thus, the \(O(n^{-s})\) probability of failure dominates the state disturbance and the errors from classical coding. This is consistent with the observation in [7] that universal compression schemes have yet to achieve better than a polynomially vanishing error.

Since our compression algorithm outputs a variable number of qubits, damage to the encoded state is not the only possible form of error. Upon failure, our algorithm risks producing a string length well above the \(n(S(\rho) + n^{-s} \log^2 n)\) qubits we expect; in fact, the only absolute bound we can establish is \(n \log d\) qubits. Fortunately, the probability that \(\rho^\otimes n\) is compressed to \(nR\) qubits for \(R > S(\rho)\) decreases as \(O(\exp(-nK))\) for some constant \(K\) depending only on \(\rho\) and \(R\). Following [3], we define this overflow exponent as

\[
K = \lim_{n \to \infty} \frac{-1}{n} \log [\text{prob. that } \rho^\otimes n \text{ yields } \geq nR \text{ qubits}]
\]  

(3)

The codes described in [3] achieve the optimal value of \(K\): \(\inf_{\sigma, H(\sigma) \geq R} S(\sigma|\rho)\). In contrast, our algorithm\(^5\) achieves

\[
K = \inf_{\sigma, H(\sigma) \geq R} \frac{1}{d(d^2 - 1)} \sum_{k=1}^{d^2-1} S(M_k(\sigma)\|M_k(\rho))
\]  

(4)

where \(M_k\) denotes the operation of measuring in the eigenbasis of \(\sigma_k\) (i.e. \(M_k(\rho) = \sum_i (|v^{(k)}_i\rangle\langle v^{(k)}_i|\rho|v^{(k)}_i\rangle\langle v^{(k)}_i|\)).

To review, our encoding procedure is:

1. Perform gentle tomography on \(\rho^\otimes n\) using \(n^s\) bins, yielding an estimate \(\hat{\rho}\).
2. Construct a modified estimate \(\hat{\rho}_\delta = (1 - \delta)\hat{\rho} + \delta I/d\) for \(\delta = O(n^{-s})\).
3. Encode \(\rho^\otimes n\) with an efficient classical algorithm (such as arithmetic coding\(^6\)) using the basis of \(\hat{\rho}_\delta\) as the computational basis.
4. Attach a classical description of \(\hat{\rho}_\delta\) with \(O(\sqrt{n})\) bits of precision and a \([\log(n \log d)]\) bit register indicating the length of the compressed data.

The decoding procedure is simply to extract the description of \(\hat{\rho}_\delta\) and use it as the basis for a classical decoding algorithm.

IV. CONCLUSION

We have described a polynomial time algorithm for compressing \(\rho^\otimes n\) into \(nS(\rho) + O(n^{-s} \log^2 n)\) qubits with error rate \(O(n^{-s} \frac{1}{2} \log n)\). This matches the error rate and inefficiency of the proof of [3], though not their overflow exponent. The procedure of [3], on the other hand, can only achieve a compression rate of \(S(\rho) + O(n^{-s})\) by incurring an error rate of \(O(n^{-s} \frac{1}{2} + s(1+d^2))\) (possibly up to logarithmic factors) and an overflow exponent of zero. For example, compressing qubits with constant error is only possible at a rate of \(S(\rho) + O(n^{-1/10})\).

More elegant would be a method for ergodic sources analogous to Lempel-Ziv-Walsh coding that adaptively created a quantum dictionary and compressed quantum information on the fly. But the method proposed here still allows the coding of sources with known statistics to attain the quantum transmission limit for sources with known statistics as the message length approaches infinity.

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overflow exponent \(d\) times higher, though still not optimal.

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