Asymptotics of inverse filtration problem in porous media

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Abstract. Filtration problems arise in the design of tunnels and underground structures. A one-dimensional filtration model of a monodisperse suspension in a homogeneous porous medium is considered. For a general nonlinear filtration function, an asymptotic solution is constructed behind the concentrations front of suspended and retained particles. It is shown that the asymptotics is close to the numerical solution. Comparison of the asymptotics with the suspended particles concentration at the outlet of the porous medium allows solving the inverse filtration problem on finding the nonlinear filtration function. The proposed method allows to obtain the filtration function based on the results of standard laboratory experiments.

1. Introduction

Filtration of suspensions and colloids in porous media is actual for many areas of nature and technology [1]. Filtration models are used to describe the processes of strengthening loose soil during construction [2]. Particle transport by a fluid flow, accompanied by the formation of a deposit throughout the whole porous medium, is called deep bed filtration. The reasons for the formation of deposit depend on the physicochemical properties of the particles, the carrier-fluid, and the porous medium frame. If the sizes of particles and pores are close, then the size-exclusion mechanism is predominant: particles freely pass through large pores and get stuck at the inlet of small pores [3].

Macroscopic model of deep bed filtration includes mass balance equation of the suspended and retained particles and the kinetic equation of deposit growth [4-6]. The growth of retained particles concentration is proportional to the suspended particles concentration. The proportionality coefficient is called the filtration function. The simplest model assumes that the filtration function is constant. The most commonly used is linear blocking filtration function, called the Langmuir coefficient. However, experimental studies show that the filtration function is non-linear [5].

The filtration function is determined in laboratory experiments. At the outlet of the porous medium, the suspended particles concentration is measured. Since the concentration of suspended particles is a solution to a mathematical model with a given filtration function, it is necessary to solve the inverse problem.

In the simplest case, under the assumption of constancy, a method for obtaining the filtration coefficient was proposed in [7]. The inverse problem for a monotonically decreasing function is solved in [8, 9]. The inverse problem is reduced to a functional equation, its solvability is proved. However, this method does not allow to represent the filtration function in an explicit form. For a complex filtration model containing several experimental functions, the inverse problem is still not solved.
In [10], the asymptotics of the solution to the filtration problem in a porous medium near the concentrations front of suspended and retained particles is constructed. The parameter of the asymptotic expansions is proportional to the distance to the concentrations front and is zero on the front. In [11], it was shown that the asymptotics approximates well the solution at the porous medium outlet for small and large parameter values. This allows the use of the asymptotic solution over a large time interval to solve the inverse problem.

The filtration function, depending on the retained particles concentration $S$, is expanded in powers of $S$. The asymptotic solution depends on the series terms in explicit form. Equating the asymptotics to the experimentally measured particle’s concentration at different time points, we obtain the asymptotic expansion of the filtration function.

2. Mathematical model
Consider a system of equations for the one-dimensional filtration model [12]
\[
\frac{\partial (C + S)}{\partial t} + \frac{\partial C}{\partial x} = 0, 
\]
\[
\frac{\partial S}{\partial t} = \Lambda(S)C. 
\]
Here, the filtration function $\Lambda(S)$ is non-negative and continuous for $S \geq 0$; $\Lambda(0) > 0$. Let the filtration function have a positive root $S_m$. Assume that $\Lambda(S)$ is smooth on the segment $0 \leq S \leq S_m$. If the filtration function loses smoothness at the point $S_m$, the filtration time is finite [13].

The system of equations (1), (2) is considered in the domain $\Omega = \{0 \leq x \leq 1, \ t \geq 0\}$. The boundary conditions for the system (1), (2) are set at the porous medium inlet $x = 0$ and at the initial moment $t = 0$:
\[
C(x,t)|_{x=0} = 1; 
\]
\[
C(x,t)|_{t=0} = 0; 
\]
\[
S(x,t)|_{t=0} = 0. 
\]

In the domain $\Omega_S = \{0 \leq x \leq 1, \ t > x\}$ the solution is positive $C(x,t) > 0$; $S(x,t) > 0$; in the domain $\Omega_0 = \{0 \leq x \leq 1, \ t < x\}$ the system has a zero solution $C(x,t) = 0$; $S(x,t) = 0$. The boundary and initial conditions do not coincide at the origin and the solution $C(x,t)$ has a gap on the concentrations front $t = x$.

To solve the inverse problem, the asymptotic solution of the direct filtration problem is used [14]. Assume that in a neighborhood of a point $S = 0$ the filtration function $\Lambda(S)$ can be represented as a series in powers of $S$
\[
\Lambda(S) = \lambda_0 + \lambda_1S + \lambda_2S^2 + \lambda_3S^3 + \ldots, \quad \lambda_0 > 0. 
\]

In the domain $\Omega_x$, in the vicinity of the concentrations front $t = x$, the asymptotics of suspended particles concentration has the form [10]
\[
C(x,t) = e^{-\lambda_0 t} + \lambda_1(e^{-\lambda_0 t} - e^{-\lambda_0 x}) (t-x) + \left(\frac{\lambda_1}{2} + \frac{1}{2} \lambda_1 \lambda_0 - \frac{3}{2} \lambda_1^2 e^{-2\lambda_0 t} + \frac{1}{2} \lambda_1^2 \lambda_0 e^{-2\lambda_0 x} \right)(t-x)^2 
\]
\[
+ \left(\frac{\lambda_1^3}{3} + \frac{4}{3} \lambda_1 \lambda_2 \lambda_0 + \frac{1}{3} \lambda_2 \lambda_0^2 \right)e^{-3\lambda_0 t} - \left(2\lambda_1^3 + \lambda_2 \lambda_0^2 \right)e^{-3\lambda_0 x} 
\]
\[
+ \left(\frac{\lambda_1^3}{6} + \frac{4}{3} \lambda_1 \lambda_2 \lambda_0 + \frac{1}{3} \lambda_2 \lambda_0^2 \right)e^{-3\lambda_0 t} + \left(-\frac{1}{6} \lambda_1^3 + \lambda_2 \lambda_1 \lambda_0 - \frac{1}{3} \lambda_2 \lambda_0^2 \right)e^{-3\lambda_0 x} 
\]
\[
(t-x)^3 + \ldots \]
3. Inverse problem

The inverse filtration problem is to obtain the filtration function \( \Lambda(S) \) based on a known concentration of suspended particles at the porous medium outlet \( C(l,t) \). Since at the initial moment \( t = 0 \) the porous medium is empty, and the concentrations front moves in the porous sample with velocity \( v = 1 \), the suspended particles appear at the outlet \( x = 1 \) at the moment \( t = 1 \).

Denote \( \tau = t - 1 \).

The function \( C_e(\tau) = C(l, t - 1) \); \( \tau \geq 0 \) is determined from experiments (possibly \( C_e(\tau) \) is known in a finite number of points \( \tau = \tau_i; 0 = \tau_0 < \tau_1 < \ldots < \tau_\nu \)). The asymptotics of the suspended particles concentration at the porous medium outlet is obtained by substituting \( x = 1 \) in expansion (7)

\[
C(l, \tau) = e^{-\lambda_0} + \lambda_1 (e^{-2\lambda_0} - e^{-\lambda_0})\tau + \left( (\frac{1}{2} \lambda_1 + \frac{1}{2} \lambda_2 \lambda_0) e^{-3\lambda_0} - \frac{3}{2} \lambda_1^2 e^{-2\lambda_0} + \frac{1}{2} (\lambda_1^2 - \lambda_2 \lambda_0) e^{-\lambda_0} \right) \tau^2
\]

\[\]

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\[
+ \left( \frac{1}{3} \lambda_1 \lambda_2 \lambda_0 + \frac{1}{3} \lambda_2^3 \lambda_0^2 e^{-3\lambda_0} - \frac{4}{6} \lambda_1 \lambda_2 \lambda_0 e^{-2\lambda_0} \right) \tau^3 + \ldots \]

(8)

The constant \( \lambda_0 \) is determined by the experimental particle concentration at the breakthrough moment when the suspension first appears at the porous medium outlet.

\( \tau = 0: \quad C(l,0) = e^{-\lambda_0} = C_e(0) \Rightarrow \lambda_0 = -\ln C_e(0) \).

Other equations to determine the remaining parameters are formed for \( \tau = \tau_i > 0 \):

\[ \tau = \tau_i: \quad C(l, \tau_i) = C_e(\tau_i); \]

\( \tau = \tau_2: \quad C(l, \tau_2) = C_e(\tau_2); \]

(10)

\[ \tau = \tau_\nu: \quad C(l, \tau_\nu) = C_e(\tau_\nu). \]

For the solvability of system (10) with respect to the unknown parameters \( \lambda_j; j=1,2,3 \) of the expansion (6) of the filtration function, it is sufficient to prove that the corresponding Jacobian is nonzero.

\[
\frac{DC(l, \tau_i)}{D\lambda_j} \neq 0; \quad i, j = 1,2,3. \quad (11)
\]

Calculation of the derivatives gives

\[
\frac{\partial C(l, \tau_i)}{\partial \lambda_1} = \left( e^{-2\lambda_0} - e^{-\lambda_0} \right) \tau_i + \left( 2\lambda_1 e^{-3\lambda_0} - 3\lambda_1 e^{-2\lambda_0} + \lambda_1 e^{-\lambda_0} \right) \tau_i^2 + \left( 3\lambda_1^2 + \frac{4}{3} \lambda_2 \lambda_0 \right) e^{-4\lambda_0} - (6\lambda_1^2 + \lambda_2 \lambda_0) e^{-3\lambda_0}
\]

\[\]

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\[
+ \left( \frac{7}{2} \lambda_1^2 - \frac{4}{3} \lambda_2 \lambda_0 \right) e^{-2\lambda_0} + \left( \frac{1}{3} \lambda_1 + \lambda_2 \lambda_0 \right) e^{-\lambda_0} \right) \tau_i^3
\]

\[
\frac{\partial C(l, \tau_i)}{\partial \lambda_2} = \left( \frac{1}{2} \lambda_0 e^{-3\lambda_0} - \lambda_0 e^{-\lambda_0} \right) \tau_i^2 + \left( \frac{4}{3} \lambda_1 \lambda_0 e^{-4\lambda_0} - \lambda_1 \lambda_0 e^{-3\lambda_0} - \frac{4}{3} \lambda_1 \lambda_0 e^{-2\lambda_0} + \lambda_1 \lambda_0 e^{-\lambda_0} \right) \tau_i^3
\]

\[
\frac{\partial C(l, \tau_i)}{\partial \lambda_3} = \frac{1}{3} \lambda_0^2 \left( e^{-4\lambda_0} - e^{-3\lambda_0} \right) \tau_i^3
\]

(12)

The order of unknown parameters \( \lambda_j \) can be determined using experimental data for a complex system obtained by Z. You [15, 16]
Table 1. Laboratory filtration function.

| Particle radius, micron | Filtration function |
|------------------------|---------------------|
| \( r_1 = 2.179 \) | \( A(S)=0.51-5.956 \times 10^{-3} S + 2.29 \times 10^{-6} S^2 + 1.35 \times 10^{-8} S^3 \) |
| \( r_2 = 3.168 \) | \( A(S)=1.551-3.467 \times 10^{-3} S - 1.16 \times 10^{-6} S^2 - 1.16 \times 10^{-8} S^3 \) |

Since the experimentally found values of \( \lambda_j \) are small, and the determinant continuously depends on the parameters, it suffices to consider the Jacobian (11) for \( \lambda_j = 0; j = 1,2,3 \). This determinant is proportional to the Vandermond determinant and is calculated in explicit form

\[
\frac{DC(1,\tau_j)}{D\lambda_j} \bigg|_{\lambda_j=0} = \left| \begin{array}{ccc} \left( e^{-2\lambda_j} - e^{-3\lambda_j} \right) \tau_1 & \left( e^{-2\lambda_j} - e^{-\lambda_j} \right) \tau_2 & \left( e^{-2\lambda_j} - e^{-\lambda_j} \right) \tau_3 \\ \left( \frac{1}{2} \lambda_j e^{-3\lambda_j} - \lambda_j e^{-\lambda_j} \right) \tau_1^2 & \left( \frac{1}{2} \lambda_j e^{-\lambda_j} - \lambda_j e^{-3\lambda_j} \right) \tau_2^2 & \left( \frac{1}{2} \lambda_j e^{-3\lambda_j} - \lambda_j e^{-\lambda_j} \right) \tau_3^2 \\ \frac{1}{3} \lambda_j^2 \left( e^{-4\lambda_j} - e^{-\lambda_j} \right) \tau_1^3 & \frac{1}{3} \lambda_j^2 \left( e^{-\lambda_j} - e^{-4\lambda_j} \right) \tau_2^3 & \frac{1}{3} \lambda_j^2 \left( e^{-\lambda_j} - e^{-4\lambda_j} \right) \tau_3^3 \end{array} \right| \frac{1}{\lambda_j^3} e^{-3\lambda_j} \left( e^{-\lambda_j} - 1 \right) \left( e^{-2\lambda_j} - 2 \right) \left( e^{-3\lambda_j} - 1 \right) \tau_1 \tau_2 \tau_3 (\tau_1 - \tau_2)(\tau_2 - \tau_3)(\tau_3 - \tau_1) < 0
\]

(13)

Since Jacobian (11) smoothly depends on parameters \( \lambda_j \), the system of equations (10) is solvable if the modulus of determinant (13) is sufficiently large.

Consider the interval of the asymptotics applicability. In [16], it was shown that the asymptotic solution coincides well with the experiment up to time \( \tau = 80 \). Calculation of the asymptotics \( C(1,\tau) \) for the two filtration functions in table 1 yields

\[
C_2(1,\tau) = 0.600496 \times 0.00142885 \tau - 1.07945 \times 10^{-6} \tau^2 - 4.79506 \times 10^{-9} \tau^3
\]

\[
C_3(1,\tau) = 0.212036 \times 0.000579255 \tau + 7.60479 \times 10^{-7} \tau^2 + 2.0438 \times 10^{-9} \tau^3
\]

For \( \tau = 80 \) the last asymptotic terms are 0.002455 and 0.0104643, respectively, which indicates good convergence of the asymptotic expansions. For \( \tau = 20; \tau = 40; \tau = 60 \) the determinants (13) for the particles \( r_2 \) and \( r_3 \) are \(-1886588\) and \(-6946651\), respectively. Thus, system (10) is solvable with respect to unknowns \( \lambda_j \).

4. Numerical calculations

The numerical solution of the direct problem (1) - (5) was obtained by the finite difference method using an explicit difference scheme [17, 18]. For the convergence of the scheme, the steps in time and coordinate are selected taking into account the Courant convergence condition \( h_t \leq h_x \).

The numerical solution was compared with the asymptotic expansion (8) at the fixed moments \( \tau = 0; 20; 40; 60 \). To account for the deviation of the mathematical model from laboratory experiments, 1% harmonic oscillations were superimposed on the concentration of suspended particles calculated at the output of the porous medium.

The results of numerical calculation of the inverse problem are presented in table 2.
Table 2. Calculated filtration function.

| Particle radius, micron | Filtration function |
|-------------------------|---------------------|
| $r_2 = 2.179$           | $A(S)=0.51-5.958\times10^{-4}S+2.17\times10^{-6}S^2+1.39\times10^{-8}S^3$ |
| $r_3 = 3.168$           | $A(S)=1.551-3.473\times10^{-4}S-1.17\times10^{-6}S^2-1.16\times10^{-7}S^3$ |

A comparison of tables 1 and 2 shows that the relative error in finding the expansion coefficients of the filtration function is 5% for $r_2$-particles and 1% for $r_3$-particles. The maximum relative error of the filtration function for times up to 80 is less than 0.2% for $r_2$-particles and 0.05% for $r_3$-particles.

Figure 1 presents graphs of the suspended particles concentration at a fixed coordinate and at a fixed time.

Since the asymptotics is close to a numerical solution, their graphs in figure 1 match. The plots of high resolution are presented in figure 2.
According to figure 2a), at the porous medium outlet the relative error of the asymptotics and the numerical solution is less than 0.3%.

5. Discussion
To solve the inverse problem, various laboratory data can be used. Methods of [8, 9] need a solution to direct problem with a variable suspended particles concentration at the inlet. In [19], both suspended and retained particles concentrations at the outlet of the porous medium are required. The proposed asymptotic method uses only the suspended particles concentration at the outlet of the porous medium with constant condition at the inlet.

To obtain the filtration function, a numerical solution of the direct filtration problem was used. If the laboratory data is used, the inverse problem error will increase due to the approximation of the filtration model.

The asymptotic method allows to solve the inverse filtration problem for a nonlinear filtration function. Previously, this method was used to obtain the linear filtration function [20]. A sophisticated filtration model with porosity and allowable flow depending on the retained particles concentration contains three unknown non-linear coefficients. An asymptotic solution to a complex model was obtained in [16] giving the chance to use the asymptotic method for solving the inverse problem. This problem will be considered separately.

6. Conclusions
The asymptotic method for solving the inverse filtration problem allows to obtain explicit formulas for the filtration function using the solution of the direct problem at the porous medium outlet.

The asymptotic solution constructed near the concentration front is close to the exact solution on a large time interval. Equating the asymptotics to the suspended particles concentration at the outlet, we obtain a system of algebraic equations for the terms of the model coefficients. It is shown that this system is uniquely solvable and the determined coefficients are close to the model parameters. The expansion terms of the filtration function are obtained up to the high-order using the suspended particles concentration at the outlet of the porous medium at fixed moments.

A small error in comparison with the forced 1% oscillation of the solution to the direct problem indicates the stability of the solution to the inverse problem.

Solving the inverse filtration problem allows to fine-tune the laboratory experiments and predict the formation of deposit in natural rocks [21].

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