HEAT KERNEL ESTIMATES FOR TWO-DIMENSIONAL RELATIVISTIC HAMILTONIANS WITH MAGNETIC FIELD

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To Ari Laptev on the occasion of his 70th birthday.

Abstract. We study semigroups generated by two-dimensional relativistic Hamiltonians with magnetic field. In particular, for compactly supported radial magnetic field we show how the long time behaviour of the associated heat kernel depends on the flux of the field. Similar questions are addressed for Aharonov-Bohm type magnetic field.

1. Introduction

Consider a two-dimensional magnetic Laplacian formally given by
\[ H = P^2, \quad P = i\nabla + A \] (1.1)
in \( L^2(\mathbb{R}^2) \), where \( A : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is a vector potential generating a magnetic field \( B : \mathbb{R}^2 \rightarrow \mathbb{R} \) through the relation \( \text{rot} A = B \). Recall that if \( A \in L^2_{\text{loc}}(\mathbb{R}^2) \), then \( P^2 \) is the unique self-adjoint operator associated with the closed quadratic form
\[ Q[u] = \| (i\nabla + A) u \|_2^2, \quad u \in D(Q), \] (1.2)
with the form domain
\[ D(Q) = \{ u \in L^2(\mathbb{R}^2) : Pu \in L^2(\mathbb{R}^2) \}. \]
The main object of our interest in this paper is the integral kernel of the semigroup generated by the relativistic Hamiltonian
\[ \mathcal{H} = \mathcal{H}(A, m) = \sqrt{P^2 + m^2} - m, \] (1.3)
where \( m \geq 0 \) is the mass of the particle. In particular, we are interested in the long time behaviour of \( e^{-t\mathcal{H}(A,m)}(x,y) \) and in its dependence on the magnetic field. Note that for a massless particle in the absence of magnetic field, we have
\[ e^{-t\mathcal{H}(0,0)}(x,y) = e^{-t\sqrt{-\Delta}}(x,y) = \frac{t}{2\pi(t^2 + |x-y|^2)^{3/2}}, \quad x, y \in \mathbb{R}^2, \quad t > 0, \] (1.4)
see e.g. [12, Sec. 7.11]. Hence
\[ e^{-t\mathcal{H}(0,0)}(x,y) \leq \frac{1}{2\pi t^2} \quad t > 0, \] (1.5)
uniformly in \( x \) and \( y \). By the diamagnetic inequality this upper bound can be extended to \( \mathcal{H}(A, 0) \);
\[ |e^{-t\mathcal{H}(A,0)}(x,y)| \leq \frac{1}{2\pi t^2} \quad t > 0, \] (1.6)
see equation (2.9) below.
However, Laptev and Weidl proved in [11], under mild regularity and decay assumptions on $B$, that the operator $H$ satisfies a Hardy-type inequality

$$
\int_{\mathbb{R}^2} |(i\nabla + A)u|^2 \geq \int_{\mathbb{R}^2} w |u|^2 \quad \forall u \in D(Q),
$$

(1.7)

where $w \geq 0$ is a weight function such that $w(x) = \mathcal{O}(|x|^{-2})$ as $|x| \to \infty$. Hence the presence of a magnetic field removes the singularity of the Green of $-\Delta$ at zero energy. This suggests that it should be possible to improve the decay rate in $t$ of the upper bound (1.6), probably at a cost of spatial weights. One of the results of this paper, Theorem 2.1, confirms this heuristic expectation for radial magnetic fields with compact support.

In the massive case when $m > 0$ the semigroup generated by $H(A, m)$ exhibits different behaviour for $t \to 0$ and for $t \to \infty$. However, for large times one observes again a faster time decay of the associated heat kernel with respect to the heat kernel generated by the non-magnetic operator, see Theorem 2.3. In Section 3 we obtain analogous results for the Aharonov-Bohm type magnetic fields.

**Remark 1.1.** It should be noted that the path integral methods developed in [7] could possibly provide a tool for alternative proofs or even more general results.

### 2. Radial magnetic field

#### 2.1. Preliminaries

In this section we will always assume that $B \in L^1(\mathbb{R}^2)$. Let

$$
\alpha = \frac{1}{2\pi} \int_{\mathbb{R}^2} B
$$

(2.1)

be the total (normalized) magnetic flux, and let

$$
\kappa = \min_{m \in \mathbb{Z}} |m + \alpha| \in [0, 1/2]
$$

(2.2)

be the distance between $\alpha$ and the set of integers. Recall also that for any $A \in L^2_{\text{loc}}(\mathbb{R}^2)$ the semigroup $e^{-tH}$ satisfies the diamagnetic inequality

$$
| e^{-tH}(x, y) | \leq \frac{1}{4\pi t} e^{-|x-y|^2/t^2} \quad \text{a. e. } x, y \in \mathbb{R}^2, \ t > 0,
$$

(2.3)

see [13, 6, 5, 2, 8].

#### 2.2. The case $m = 0$

**Theorem 2.1.** Assume that $B$ is radial, continuous and compactly supported. Then

$$
\text{e}^{-tH} : L^p(\mathbb{R}^2) \to L^q(\mathbb{R}^2), \quad 1 \leq p \leq q \leq \infty,
$$

(2.4)

and its kernel satisfies

$$
| e^{-tH(A, 0)}(x, y) | \lesssim (1 + |x|)^\beta (1 + |y|)^\beta \ t^{-2-2\beta}, \quad \text{if } \kappa > 0,
$$

(2.5)

$$
| e^{-tH(A, 0)}(x, y) | \lesssim \log(2 + |x|)^\theta \log(2 + |y|)^\theta \ t^{-2 \log(2 + t)} - 2\theta \quad \text{if } \kappa = 0,
$$

(2.6)

for any $\beta \in [0, \kappa]$ and any $\theta \in [0, 1]$ respectively.
Proof. We use the Poincaré gauge for the vector potential $A$;

$$A(x) = \frac{(-x_2, x_1)}{|x|^2} \int_{0}^{x} B(r) r \, dr, \quad x \in \mathbb{R}^2. \tag{2.7}$$

Then rot $A = B$ and $A \in C^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. In view of [3, Thm. 6.1] the semigroup

$$e^{-tH} : L^p(\mathbb{R}^2) \to L^q(\mathbb{R}^2), \quad 1 \leq p \leq q \leq \infty,$$

is thus strongly continuous in $t > 0$, and its integral kernel $e^{-tH}(x, y)$ is jointly continuous in $(x, y, t)$. On the other hand, inequality (2.3) shows that $e^{-tH}$ is a contraction on $L^p(\mathbb{R}^2)$ for any $p \in [1, \infty]$. Hence by [9, Thms. 3.9.7, 4.3.1] and [9, Example 3.9.16] we have

$$e^{-t\mathcal{K}(A, 0)} = e^{-t\sqrt{H}} = \frac{t}{\sqrt{4\pi}} \int_{0}^{\infty} s^{-\frac{3}{2}} e^{-\frac{t^2 s}{4}} e^{-sH} \, ds, \tag{2.8}$$

with $e^{-t\mathcal{K}(A, 0)}$ being a strongly continuous contraction semigroup on $L^p(\mathbb{R}^2)$, $p \in [1, \infty]$. Moreover, the above mentioned properties of $e^{-tH}(x, y)$ in combination with (2.8) imply that $e^{-t\mathcal{K}(A, 0)}$ admits an integral kernel $e^{-t\mathcal{K}(A, 0)}(x, y)$ which is jointly continuous in $(x, y, t)$.

Note also that

$$\int_{0}^{\infty} s^{-\frac{5}{2}} e^{-\frac{t^2 s}{4}} ds = t^{-3} \int_{0}^{\infty} s^{-\frac{5}{2}} e^{-\frac{t^2 s}{4}} ds = 8t^{-3} \int_{0}^{\infty} \sqrt{r} e^{-r} \, dr = 8t^{-3} \Gamma(3/2) = 4t^{-3} \sqrt{\pi}.$$ 

This in combination with (2.3) and (2.8) yields

$$\left| e^{-t\mathcal{K}(A, 0)}(x, y) \right| \leq \frac{t}{(4\pi)^{3/2}} \int_{0}^{\infty} s^{-\frac{5}{2}} e^{-\frac{t^2 s}{4}} ds = \frac{1}{2\pi t^2}. \tag{2.9}$$

Hence $e^{-t\mathcal{K}(A, 0)} : L^1(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)$, and (2.4) follows.

Let us now assume that $\kappa > 0$. Then, by [10, Thm.3.3.]

$$\left| e^{-tH}(x, y) \right| \leq C_0 (1 + |x|)^\kappa (1 + |y|)^\kappa t^{-1-\kappa} \quad x, y \in \mathbb{R}^2 \quad t > 0 \tag{2.10}$$

holds for some $C_0$. Hence by (2.8)

$$\left| e^{-t\mathcal{K}(A, 0)}(x, y) \right| \lesssim (1 + |x|)^\kappa (1 + |y|)^\kappa t \int_{0}^{\infty} s^{-\frac{5}{2} - \kappa} e^{-\frac{t^2 s}{4}} ds \lesssim (1 + |x|)^\kappa (1 + |y|)^\kappa t^{-2-2\kappa}.$$ 

In view of (2.9) this proves (2.5). In order to prove (2.6) we note that for $\kappa = 0$,

$$e^{-tH}(x, x) \lesssim \log(2 + |x|) \log(2 + |y|) t^{-1} (\log(2 + t))^{-2}, \tag{2.11}$$

see [10, Thms. 3.1, 3.2]. Thus, proceeding as above we get

$$\left| e^{-t\mathcal{K}(A, 0)}(x, y) \right| \lesssim \log(2 + |x|) \log(2 + |y|) t \int_{0}^{\infty} s^{-\frac{5}{2}} (\log(2 + s))^{-2} e^{-\frac{t^2 s}{4}} ds$$

$$= \log(2 + |x|) \log(2 + |y|) t^{-2} \int_{0}^{\infty} r^{-\frac{5}{2}} (\log(2 + rt^2))^{-2} e^{-\frac{r}{4t}} dr$$

$$\lesssim \log(2 + |x|) \log(2 + |y|) \left( t^{-2} \int_{0}^{t^{-1}} r^{-\frac{5}{2}} e^{-\frac{1}{4r}} dr + (\log(2 + t))^{-2} \int_{t^{-1}}^{\infty} r^{-\frac{5}{2}} e^{-\frac{1}{4r}} dr \right)$$

$$\lesssim \log(2 + |x|) \log(2 + |y|) t^{-2} [\log(2 + t)]^{-2} \quad \forall \, n \geq 1. \tag{2.12}$$

where we have used the fact that

$$(1 + t^n) \int_{0}^{t^{-1}} r^{-\frac{5}{2}} e^{-\frac{1}{4r}} dr = O(1) \quad \forall \, n \geq 1.$$
Inequality (2.6) thus follows from (2.9) and (2.12). □

**Remark 2.2.** The faster time decay of \( e^{-t \mathcal{H}(A,0)}(x,y) \) with respect to \( e^{-t \mathcal{H}(0,0)}(x,y) \) is compensated by the spacial grow of \( x \) and \( y \), as expected.

### 2.3. The case \( m > 0 \)

For particles with positive mass we have

**Theorem 2.3.** Let \( m > 0 \). Under the assumptions of Theorem 2.1 we have

\[
|e^{-t \mathcal{H}(A,m)}(x,y)| \lesssim (1 + |x|)^\beta (1 + |y|)^\beta t^{-1-\beta}, \quad \text{if } \kappa > 0,
\]

\[
|e^{-t \mathcal{H}(A,m)}(x,y)| \lesssim \log(2 + |x|)^\theta \log(2 + |y|)^\theta t^{-1} \left[ \log(2 + t) \right]^{-2\theta} \quad \text{if } \kappa = 0,
\]

for \( t \geq 1 \), and any \( \beta \in [0,\kappa] \) and any \( \theta \in [0,1] \) respectively.

Moreover, if \( t \leq 1 \), then

\[
|e^{-t \mathcal{H}(A,m)}(x,y)| \lesssim t^{-2}
\]

The proof of Theorem 2.3 is based on the following technical result.

**Lemma 2.4.** For any \( a > 0 \) there exist constants \( C_1(a) \) and \( C_2(a) \) such that

\[
\int_0^\infty r^a e^{-t \frac{(r-m)^2}{2}} dr \leq C_1(a) m^{\frac{a}{2}} t^{-\frac{1}{2}} + C_2(a) t^{\frac{1-a}{2}}
\]

holds for all \( t > 0 \).

**Proof.** Note that the function \( s \mapsto s + \sqrt{s^2 + 2m} \) is an increasing bijection which maps \( \mathbb{R} \) onto \((0,\infty)\). Hence we will apply the substitution

\[
r = \frac{s + \sqrt{s^2 + 2m}}{2},
\]

and split the integration in (2.16) in two parts as follows;

\[
\int_0^{\sqrt{m}} r^a e^{-t \frac{(r-m)^2}{2}} dr = 2^{-1-a} \int_0^0 \left( s + \sqrt{s^2 + 2m} \right)^a \left( 1 + \frac{s}{\sqrt{s^2 + 2m}} \right) e^{-ts^2} ds
\]

\[
\leq 2^{-1-a} \int_0^0 \left( s + \sqrt{s^2 + 2m} \right)^a e^{-ts^2} ds
\]

\[
= 2^{-1-a} \int_0^0 \left( \frac{2m}{\sqrt{s^2 + 2m} - s} \right)^a e^{-ts^2} ds
\]

\[
\leq 2^{-1-a} (2m)^{\frac{a}{2}} \int_0^0 e^{-ts^2} ds = 2^{-1-a} (2m)^{\frac{a}{2}} \frac{\sqrt{\pi}}{2} t^{-\frac{1}{2}},
\]

and

\[
\int_{\sqrt{m}}^\infty r^a e^{-t \frac{(r-m)^2}{2}} dr = 2^{-1-a} \int_0^\infty \left( s + \sqrt{s^2 + 2m} \right)^a \left( 1 + \frac{s}{\sqrt{s^2 + 2m}} \right) e^{-ts^2} ds
\]

\[
\leq 2^{-a} \int_0^\infty \left( s + \sqrt{s^2 + 2m} \right)^a e^{-ts^2} ds
\]

\[
\lesssim \int_0^\infty \left( 2s + \sqrt{2m} \right)^a e^{-ts^2} ds \lesssim \int_0^\infty \left( s^a + m^{\frac{a}{2}} \right) e^{-ts^2} ds
\]

\[
\lesssim t^{-\frac{1+a}{2}} + m^{\frac{a}{2}} t^{-\frac{1}{2}}.
\]
Proof of Theorem 2.3. By (2.8)
\[ e^{-t\mathcal{H}(A,m)}(x,y) = e^{mt} \frac{t}{\sqrt{4\pi}} \int_0^\infty s^{-\frac{3}{2}} e^{-\left(\frac{t}{2s} - m^2 s\right)} e^{-sH}(x,y) ds \]
\[ = \frac{t}{\sqrt{4\pi}} \int_0^\infty s^{-\frac{3}{2}} e^{-\left(\frac{t}{2s} - m\sqrt{s}\right)^2} e^{-sH}(x,y) ds. \]
Hence using (2.3) and the substitution
\[ r = \frac{\sqrt{t}}{2\sqrt{s}}, \] (2.18)
we obtain from Lemma 2.4
\[ |e^{-t\mathcal{H}(A,m)}(x,y)| \lesssim t \int_0^\infty s^{-\frac{3}{2}} e^{-\left(\frac{t}{2s} - m\sqrt{s}\right)^2} ds \lesssim t^{-\frac{1}{2}} \int_0^\infty r^2 e^{-t(r-\frac{m}{r})^2} dr \]
\[ \lesssim t^{-1} + t^{-2}. \] (2.19)
This proves (2.15). Similarly, if \( \kappa > 0 \) and \( t \geq 1 \), then using (2.10), (2.18) and Lemma 2.4 we get
\[ |e^{-t\mathcal{H}(A,m)}(x,y)| \lesssim (1 + |x|)^\kappa (1 + |y|)^\kappa \frac{t}{\sqrt{4\pi}} \int_0^\infty s^{-1-\kappa} e^{-\left(\frac{1}{2s} - m\sqrt{s}\right)^2} ds \]
\[ = \frac{2^{3+2\kappa}}{\sqrt{\pi}} (1 + |x|)^\kappa (1 + |y|)^\kappa t^{-\frac{1}{2}-\kappa} \int_0^\infty r^{2+2\kappa} e^{-t(r-\frac{m}{r})^2} dr \]
\[ \lesssim (1 + |x|)^\kappa (1 + |y|)^\kappa t^{-1-\kappa}. \]
The upper bound (2.5) then follows from Lemma 2.4 and (2.19). In the same way we deduce from (2.11) that if \( t \geq 1 \) and \( \kappa = 0 \), then
\[ |e^{-t\mathcal{H}(A,m)}(x,y)| \lesssim \log(2 + |x|) \log(2 + |y|) t^{-\frac{1}{2}} \int_0^\infty r^2 \left[ \log \left(2 + \frac{t}{4r^2}\right) \right]^{-2} e^{-t(r-\frac{m}{r})^2} dr \]
\[ \lesssim \log(2 + |x|) \log(2 + |y|) t^{-\frac{1}{2}} \left( \int_0^{t^\frac{1}{4}} r^2 \left[ \log \left(2 + \frac{\sqrt{t}}{4}\right) \right]^{-2} e^{-t(r-\frac{m}{r})^2} dr + \right. \]
\[ + \left. \int_{t^\frac{1}{4}}^\infty r^2 e^{-t(r-\frac{m}{r})^2} dr \right) \]
\[ \lesssim \log(2 + |x|) \log(2 + |y|) t^{-1} \left[ \log(2 + t) \right]^{-2}. \]
To complete the proof it suffices to use (2.19) once more. \qed

Remark 2.5. Note that in the massive case, contrary to the case \( m = 0 \), the semigroup \( e^{-t\mathcal{H}(A,m)}(x,y) \) exhibits different behaviour for long respectively short times. Indeed, by (2.19) we have
\[ \|e^{-t\mathcal{H}(A,m)}\|_{L^1 \rightarrow L^\infty} = O(t^{-2}) \quad t \to 0, \quad \|e^{-t\mathcal{H}(A,m)}\|_{L^1 \rightarrow L^\infty} = O(t^{-1}) \quad t \to \infty. \] (2.20)
This is caused by the different behaviour of the symbol \( \sqrt{P^2 + m^2} - m \) for \( P \to 0 \) and for \( P \to \infty \) respectively. Analogous effect occurs when \( A = 0 \), see [12, Sec. 7.11].
3. Aharonov-Bohm type magnetic fields

In this section we consider the Aharonov-Bohm magnetic field in $\mathbb{R}^2$. The latter is generated by the vector potential $A$ whose radial and azimuthal components (in the polar coordinates) are given by

$$A(r, \theta) = (a_1(r, \theta), a_2(r, \theta)), \quad a_1 = 0, \quad a_2(r) = \left(0, \frac{\alpha}{r}\right). \quad (3.1)$$

where $\alpha$ is the magnetic. Note that $A \not\in L^2_{loc}(\mathbb{R}^2)$. We will therefore proceed in a different way than in the previous section and define the semigroup $e^{-t\sqrt{H}}$ with the help of the partial wave decomposition. We limit ourselves to the analysis of the massless case, $m = 0$. The main result of section is stated in Theorem 3.2.

We define the Hamiltonian $H_\alpha$ as the Friedrichs extension of $\left(-i\nabla + A\right)^2$ on $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$. In other words, $H_\alpha$ is the self-adjoint operator in $L^2(\mathbb{R}^2)$ generated by the closure, in $L^2(\mathbb{R}^2)$, of the quadratic form

$$Q_\alpha[u] = \int_0^{2\pi} \int_0^\infty \left( |\partial_r u|^2 + r^{-2} |(-i\partial_\theta + \alpha) u|^2 \right) r \, dr \, d\theta, \quad u \in C_0^\infty((0, \infty) \times [0, 2\pi)). \quad (3.2)$$

3.1. Partial wave decomposition. Given a function $f : \mathbb{R}^2 \to \mathbb{R}$ we will often use the polar coordinate representation

$$f(x, y) = f(r, r', \theta, \theta') \iff x = r(\cos \theta, \sin \theta), \quad y = r'(\cos \theta', \sin \theta').$$

By expanding a given function $u \in L^2(\mathbb{R}_+ \times (0, 2\pi))$ into a Fourier series with respect to the basis $\{e^{im\theta}\}_{m \in \mathbb{Z}}$ of $L^2((0, 2\pi))$, we obtain a direct sum decomposition

$$L^2(\mathbb{R}^2) = \bigoplus_{m \in \mathbb{Z}} \mathcal{L}_m, \quad (3.3)$$

where

$$\mathcal{L}_m = \left\{ g \in L^2(\mathbb{R}^2) : g(x) = f(r) e^{im\theta} \ a.e., \ \int_0^\infty |f(r)|^2 \, r \, dr < \infty \right\}. \quad (3.5)$$

Since the vector potential $A$ is radial, the operator $H_\alpha$ can be decomposed accordingly to the direct sum

$$H_\alpha = \bigoplus_{m \in \mathbb{Z}} (h_m \otimes \text{id}) \Pi_m, \quad (3.4)$$

where $h_m$ are operators generated by the closures, in $L^2(\mathbb{R}_+, rdr)$, of the quadratic forms

$$q_m[f] = \int_0^\infty \left( |f'|^2 + \frac{(\alpha + m)^2 |f|^2}{r^2} \right) r \, dr \quad (3.5)$$

defined initially on $C_0^\infty(0, \infty)$, and $\Pi_m : L^2(\mathbb{R}^2) \to \mathcal{L}_m$ is the projector acting as

$$(\Pi_m u)(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} e^{im(\theta - \theta')} u(r, \theta') \, d\theta'.$$

Consider now the operator

$$L_m = U h_m U^{-1} \quad \text{in} \quad L^2(\mathbb{R}_+, dr), \quad (3.6)$$
where $U: L^2(\mathbb{R}_+, r \, dr) \rightarrow L^2(\mathbb{R}_+, dr)$ is the unitary mapping acting as $(U f)(r) = r^{1/2} f(r)$. Note that $L_m$ is subject to Dirichlet boundary condition at zero and that it coincides with the Friedrichs extension of the differential operator

$$\frac{-d^2}{dr^2} + \frac{(m + \alpha)^2 - \frac{1}{4}}{r^2}$$

defined on $C_0^\infty(\mathbb{R}_+)$. From [10, Sect. 5] we know that

$$W_m L_m W_m^{-1} \varphi(p) = p \varphi(p), \quad \varphi \in W_m(D(L_m)),$$

where the mappings $W_m, W_m^{-1}: L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+)$ given by

$$(W_m u)(p) = \int_0^\infty u(r) \sqrt{r} J_{|m+\alpha|}(r \sqrt{p}) \, dr,$$

$$(W_m^{-1} \varphi)(r) = \frac{1}{2} \int_0^\infty \varphi(p) \sqrt{r} J_{|m+\alpha|}(r \sqrt{p}) \, dp$$

extend to unitary operators on $L^2(\mathbb{R}_+)$. 

3.2. The semigroup $e^{-t \sqrt{L_\alpha}}$. By the spectral theorem

$$e^{-t \sqrt{L_\alpha}} = \sum_{m \in \mathbb{Z}} \oplus \left( e^{-t \sqrt{L_m}} \otimes \text{id} \right) \Pi_m.$$ 

Denote by $p_m(r, r', t)$ the integral kernel of $e^{-t \sqrt{L_m}}$ in $L^2(\mathbb{R}_+, r \, dr)$. From (3.6) it follows that

$$p_m(r, r', t) = \frac{1}{\sqrt{rr'}} e^{-t \sqrt{L_m}}(r, r').$$

On the other hand, in view of (3.7) we get

$$(e^{-t \sqrt{L_m}} g)(r) = (W_m^{-1} e^{-t \sqrt{p}} W_m g)(r) = \frac{1}{2} \int_0^\infty \sqrt{rr'} \int_0^\infty e^{-t \sqrt{p}} J_{|m+\alpha|}(r \sqrt{p}) J_{|m+\alpha|}(r' \sqrt{p}) \, dp g(r') \, dr'.$$

Hence

$$p_m(r, r', t) = \frac{1}{2} \int_0^\infty e^{-t \sqrt{p}} J_{|m+\alpha|}(r \sqrt{p}) J_{|m+\alpha|}(r' \sqrt{p}) \, dp$$

$$= \int_0^\infty e^{-t p} J_{|m+\alpha|}(r \, p) J_{|m+\alpha|}(r' \, p) \, dp. \quad (3.11)$$

In order to simplify the notation we will use in the sequel the shorthand

$$z := \frac{r^2}{t^2}, \quad (3.12)$$

From [4, Eq. 4.14.(16)] we then get the explicit expression for $p_m$ on the diagonal:

$$p_m(r, r, t) = \int_0^\infty e^{-t p} J_{|m+\alpha|}(r \, p) \, dp$$

$$= \frac{4^\nu}{\pi} (2\nu + 1) t^{-2} \left( \frac{r}{t} \right)^{2\nu} \frac{\Gamma^2(\nu + \frac{1}{2})}{\Gamma(2\nu + 1)} F \left( \nu + \frac{1}{2}, \nu + \frac{3}{2}; 2\nu + 1; -4z \right), \quad (3.13)$$

where

$$\nu = |m + \alpha|, \quad (3.14)$$
and \( F(a, b, c; w) \) denotes the Gauss hypergeometric series, see e.g. [1, Eq. 15.1.1]. Using its integral representation

\[
F(a, b, c; w) = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c - b)} \int_0^1 s^{b-1} (1 - s)^{c-b-1} (1 - sw)^{-a} \, ds, \quad \Re c > \Re b > 0, \tag{3.15}
\]

see [1, Eq. 15.3.1], in combination with the transformation formula [1, Eq. 15.3.5]:

\[
F(a, b, c; w) = (1 - w)^{-b} F\left(b, c-a, c; \frac{w}{w-1}\right) \tag{3.16}
\]

we find that

\[
p_m(r, r', t) = \frac{2\nu + 1}{\pi t^2} (4z)^\nu \frac{\Gamma^2(\nu + \frac{1}{2})}{\Gamma(2\nu + 1)} (1 + 4z)^{-\nu-\frac{3}{2}} F\left(\nu + \frac{3}{2}, \nu + \frac{1}{2}, 2\nu + 1; \frac{4z}{4z + 1}\right)
\]

\[
= \frac{2\nu + 1}{\pi t^2} (4z)^\nu (1 + 4z)^{-\nu-\frac{3}{2}} \int_0^1 s^{\nu-\frac{1}{2}} (1 - s)^{\nu-\frac{1}{2}} \left(1 - \frac{4zs}{4z + 1}\right)^{-\nu-\frac{3}{2}} \, ds.
\]

\[
= \frac{2\nu + 1}{\pi t^2} (4z)^\nu \int_0^1 s^{\nu-\frac{1}{2}} (1 - s)^{\nu-\frac{1}{2}} (1 + 4z(1-s))^{-\nu-\frac{3}{2}} \, ds
\]

\[
= \frac{2\nu + 1}{\pi t^2} (4z)^\nu \int_0^1 s^{\nu-\frac{1}{2}} (1 - s)^{\nu-\frac{1}{2}} (1 + 4zs)^{-\nu-\frac{3}{2}} \, ds. \tag{3.17}
\]

We have

**Lemma 3.1.** There exists \( \varepsilon_0 > 0 \) such that for every \( \varepsilon \in (0, \varepsilon_0) \) there is \( C_\varepsilon > 0 \) for which the upper bound

\[
(1 + r)^{-\frac{3}{2} - \varepsilon} (1 + r')^{-\frac{3}{2} - \varepsilon} p_m(r, r', t) \leq C_\varepsilon \frac{t^{-2-2\kappa}}{(\lvert m + \alpha \rvert + 1)^{1+\varepsilon}} \tag{3.18}
\]

holds for all \( m \in \mathbb{Z} \) and all \( t \geq 1 \).

**Proof.** Put

\[
\varepsilon_0 := \min\left\{\lvert m + \alpha \rvert - \frac{3}{2} : m \in \mathbb{Z} \land \lvert m + \alpha \rvert > \frac{3}{2}\right\}. \tag{3.19}
\]

Clearly we have \( \varepsilon_0 > 0 \). Let \( \varepsilon < \varepsilon_0 \) and put \( \rho = \frac{3}{2} + \varepsilon \). Keeping in mind the notation (3.14) we will distinguish two cases depending on the value of \( \nu \).

Assume first that \( \nu > 3/2 \). In view of (3.12) and (3.17)

\[
(1 + r^2)^{-\rho} p_m(r, r', t) = \frac{4\nu + 2}{\pi t^2} \frac{r}{t} (1 + r^2)^{-\rho} (4z)^{\nu-\frac{1}{2}} \int_0^1 s^{\nu-\frac{1}{2}} (1 - s)^{\nu-\frac{1}{2}} (1 + 4zs)^{-\nu-\frac{3}{2}} \, ds
\]

\[
\leq t^{-3} \nu (4z)^{\nu-\rho} \int_0^1 s^{\nu-\frac{1}{2}} (1 - s)^{\nu-\frac{1}{2}} (1 + 4zs)^{-\nu-\frac{3}{2}} \, ds
\]

\[
= t^{-3} \nu (4z)^{\nu-\rho} \int_0^1 s^{\nu-\frac{1}{2}} (1 + 4zs)^{\rho-\nu} (1 - s)^{\nu-\frac{1}{2}} (1 + 4zs)^{-\rho-\frac{3}{2}} \, ds,
\]
where we have used the fact that $z \leq r^2$ by assumption. From (3.19) it follows that $\rho < \nu$. Hence
\[
(1 + r^2)^{-\rho} p_m(r, r, t) \lesssim t^{-3} \nu (4z)^{3-\rho} \int_0^1 s^{\nu-\frac{3}{2}} (4zs)^{\rho-\nu} (1-s)^{\frac{3}{2}} ds
\]
\[
= t^{-3} \nu \int_0^1 s^{\rho-\frac{3}{2}} (1-s)^{\nu-\frac{3}{2}} ds = t^{-3} \nu B\left(\rho + \frac{1}{2}, \nu + \frac{1}{2}\right)
\]
where $B(\cdot, \cdot)$ denotes the Euler beta function. Moreover, by the Stirling formula, see e.g. [1, Eq. 6.1.37], we have
\[
\nu \frac{\Gamma \left(\nu + \frac{1}{2}\right)}{\Gamma(\nu + \rho + 1)} \sim \nu^{-\rho+\frac{1}{2}} = \nu^{-1-\varepsilon} \quad \nu \to \infty.
\]
Therefore there exists a constant $C_1$ such that
\[
(1 + r^2)^{-\rho} p_m(r, r, t) \leq C_1 t^{-3} \nu^{-1-\varepsilon} \quad \forall \nu > \frac{3}{2} \quad (3.20)
\]
Now let $0 \leq \nu \leq \frac{3}{2}$. In this case we have $\nu \leq \rho$ and (3.17) thus implies that
\[
(1 + r^2)^{-\rho} p_m(r, r, t) \leq \frac{32}{\pi t^2} (1 + r)^{2-2\rho} \frac{r^2}{t^{2\nu}} \leq \frac{32}{\pi} t^{-2-2\kappa},
\]
since $\nu \geq \kappa$ by definition. This together with (3.14) and (3.20) gives
\[
(1 + r^2)^{-\rho} p_m(r, r, t) \leq C_2 t^{-2-2\kappa} (|m + \alpha| + 1)^{-1-\varepsilon} \quad \forall t \geq 1
\]
holds for all $\nu$ and some $C_2$. To complete the proof it suffices to use the semigroup property of $e^{-t\sqrt{H_m}}$, which implies that
\[
p_m(r, r', t) \leq \sqrt{p_m(r, r, t)p_m(r', r', t)}.
\]
\[\square\]
As a consequence of Lemma 3.1 and equations (3.8), (3.13) we obtain

**Theorem 3.2.** Let $\varepsilon_0$ be given by (3.19). Then for every $\varepsilon \in (0, \varepsilon_0)$ there exists a constant $K_\varepsilon$ such that
\[
\|(1 + |x|)^{-\frac{3}{2} - \varepsilon} e^{-t\sqrt{H_m}} (1 + |x|)^{-\frac{3}{2} + \varepsilon} \|_{L^1(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)} \leq K_\varepsilon t^{-2-2\kappa}. \quad (3.21)
\]
Moreover,
\[
e^{-t\sqrt{H_m}}(x, x) = \frac{t^{-2}}{2\pi^2} \sum_{m \in \mathbb{Z}} 4^\nu (2\nu + 1) \left(\frac{|x|}{t}\right)^{2\nu} \frac{\Gamma^2(\nu + \frac{1}{2})}{\Gamma(2\nu + 1)} F\left(\nu + \frac{1}{2}, \nu + \frac{3}{2}, 2\nu + 1; \frac{4|x|^2}{t^2}\right),
\]
where $\nu = |m + \alpha|$.

**Remark 3.3.** The decay rate $t^{-2-2\kappa}$ in Theorem 3.2 is sharp. This follows from (3.8) and the asymptotic behavior of $p_m(r, r, t)$ as $t \to \infty$. Indeed, let $k \in \mathbb{Z}$ be such that $\kappa = |k + \alpha|$. Equation (3.17) then implies that
\[
\lim_{t \to \infty} t^{2+2\kappa} p_k(r, r, t) = \frac{2\kappa + 1}{\pi} (2r)^{2\kappa} B\left(\frac{1}{2}, \kappa + \frac{1}{2}\right), \quad (3.22)
\]
It should be also noted that if $\alpha \in \mathbb{Z}$, in which case $\kappa = 0$, then the operator $H_\alpha$ is unitarily equivalent to the Laplacian in $L^2(\mathbb{R}^2)$ which satisfies
\[
\| e^{-t\sqrt{-\Delta}} \|_{L^1 \to L^\infty} = \frac{1}{2\pi t^2},
\]
for all $t > 0$, see (1.4).

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