Fault-Tolerant Resolvability in Some Classes of Line Graphs

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1. Introduction and Preliminaries

Let \( G \) be a simple connected graph with vertex set \( V(G) \) and edge set \( E(G) \). The distance \( d(s, t) \) between two vertices \( s, t \in V(G) \) is the length of a shortest path between them. The degree of a vertex is the number of edges that are incident to it. Let \( W = \{w_1, w_2, \ldots, w_l\} \subset V(G) \) be an ordered set and \( s \in V(G) \); then, the representation \( r(s | W) \) of \( s \) with respect to \( W \) is the \( l \)-tuple \((d(s, w_1), d(s, w_2), \ldots, d(s, w_l))\). \( W \) is called a resolving set if different vertices of \( G \) have different representations with respect to \( W \). A resolving set with a minimum number of elements is called a basis for \( G \), and the cardinality of the basis is known as the metric dimension of \( G \), represented by \( \beta(G) \). For \( W = \{w_1, w_2, \ldots, w_l\} \subset V(G) \), the \( i \)th component of \( r(s | W) \) is 0 if and only if \( s = w_i \). Hence, to prove that \( W \) is a resolving set, it is enough to show that \( r(s | W) \neq r(t | W) \) for each pair \( s \neq t \in V(G) \). The absolute difference representation of \( s, t \in V(G) \) with respect to \( W \) is \( AD((s, t) | W) = (|d_G(s, w_1) - d_G(t, w_1)|, \ldots, |d_G(s, w_l) - d_G(t, w_l)|) \). So, \( W \) is a resolving set if \( AD((s, t) | W) \) has at least one entry in the l-vector different from zero for \( s \neq t \in V(G) \).

The concept of metric dimension of the general metric space was presented in 1953 (see [1]). After twenty years, the concept of resolving set in graphs was first introduced by Slater [2, 3] in 1975 and also independently by Harary and Melter [4] in 1976. The resolving sets were basically defined to determine the location of the intruder in a network, but later, Chartrand and Zhang used metric bases in the fields of robotics, chemistry, and biology in 2003, see [5, 6].

It will be a difficult task to locate an interrupter if one of the sensors does not function in a proper way. In order to tackle such problems, Hernando et al. [7] gave the idea of the fault-tolerant resolving set. Fault-tolerant resolving set is a resolving set if the removal of any element keeps it resolving. Formally, a resolving set \( W' \) of any graph \( G \) is called a fault-tolerant resolving set if \( (W' \setminus \{w\}) \) for all \( w \in W' \) is also a resolving set of \( G \). The minimum cardinality of the fault-tolerant resolving set is called the fault-tolerant metric dimension, and it is denoted by \( \beta'(G) \). In other words, for all \( s, t \in V(G) \), \( AD((s, t) | W') \) has at least two entries in the l-vector different from zero.

Fault-tolerant metric dimension is an interesting concept and has been studied by many authors. For instance, Hernando et al. in [7] computed the fault-tolerant resolving set for tree graphs and proved that \( \beta'(P_n) = 2 \) for the path graph \( P_n \) on \( n \geq 2 \) vertices. Voronov in [8] computed the...
fault-tolerant metric dimension of the king's graph. Recently, Hussain et al. in [9] computed closed formulas for the fault-tolerant metric dimension of wheel-related graphs. Raza et al. in [10] computed the fault-tolerant metric dimension of some classes of convex polytopes. Raza et al. in [11] showed that the fault-tolerant metric dimension of the complete graph on \( n \) vertices is \( n \). Javaid et al. in [12] proved that \( \beta'(C_n) = 3 \) for the cycle graph \( C_n \) on \( n \geq 3 \) vertices. For more details on the fault-tolerant metric dimension, see [13–15].

The line graph \( L(G) \) of graph \( G \) is the graph whose vertices are the edges of \( G \), and two vertices \( e \) and \( f \) of \( L(G) \) are connected if and only if they have a common end vertex in \( G \). The metric dimension of line graphs is studied in [6, 10, 16–18]. For more details, see [19–21]. Here, we determine the fault-tolerant metric dimension in line graphs. The fault-tolerant metric dimension in line graphs is only known for path and cycle graphs as given in the following theorem.

**Theorem 1.** The fault-tolerant metric dimension of the line graphs of path and cycle graphs of order \( n \) is 2 and 3, respectively.

**Proof.** The results are obvious from the definition of the line graph, and the results are proved in [7, 12], respectively. Since it is difficult to compute the exact values of \( \beta'(G) \) for every graph \( G \), A. Estrado-Moreno et al. gave some of the important bounds on the fault-tolerant metric dimension of graphs as follows.

**Lemma 1** (see [22]). Let \( G \) be any graph; then, \( \beta(G) < \beta'(G) \).

**Lemma 2** (see [22]). If \( G \neq P_n \), then \( \beta'(G) \geq 3 \) for all \( G \).

In [23], Khuller et al. studied an important property of graphs with metric dimension 2 as follows.

**Lemma 3** (see [23]). Let \( G \) be a graph with metric dimension 2, and let \( \{v_1, v_2\} \subset V(G) \) be a resolving set in \( G \). Then, the degree of both \( v_1 \) and \( v_2 \) is at most 3.

Consequently, similar argument works for the graphs with fault-tolerant metric dimension 3 are given in the following lemma.

**Lemma 4.** Let \( G \) be a graph with fault-tolerant metric dimension 3, and let \( \{v_1, v_2, v_3\} \subset V(G) \) be a fault-tolerant resolving set in \( G \). Then, the degree of each vertex \( v_1, v_2, \) and \( v_3 \) is at most 3. The rest of the paper is structured as follows: in Section 2, we will compute the fault-tolerant metric dimension of the line graph of the necklace graph. In Section 3, we will compute the fault-tolerant metric dimension of the line graph of the prism graph.

**2. The Fault-Tolerant Metric Dimension of the Line Graph of the Necklace Graph**

The necklace graph \( N_n' \) for \( n \geq 2 \) consists of the edge set \( E(N_n') = \{f, g, h_i : 1 \leq i \leq n + 1\} \) as shown in Figure 1.

For the fault-tolerant metric dimension of the line graph of the necklace graph, we have to construct a line graph \( L(N_n') \) of \( N_n' \) with \( n \geq 2 \) (see Figure 2).

In the following theorem, the result for the metric dimension of the line graph of the necklace graph is given.

**Theorem 2** (see [17]). The metric dimension of the line graph of the necklace graph \( L(N_n') \) is 3 for \( n \geq 2 \). Now, we will compute the fault-tolerant metric dimension of the line graph of the necklace graph.

**Theorem 3.** The fault-tolerant metric dimension of the line graph of the necklace graph is 4 for \( n \geq 2 \).

**Proof.** To prove this theorem, consider the following cases.

**Case 1** \( (n \) is odd). Let \( n \geq 3 \), and take \( W' = \{f_1, f_{(n+1)/2}, f_{(n+1)/2}, h_{(n+1)/2}\} \subseteq V(L(N_n')). \) Representation of vertices \( f_i \) with respect to \( W' \) is

\[
r(f_i | W') = \begin{cases} 
(i - 1, n + 1/2, -i, i + 1, n + 3/2, -i), & \text{if } 1 \leq i \leq \frac{n-1}{2}, \\
(i - 1, i - \frac{n + 1}{2}, n + 1 - i, 1), & \text{if } \frac{n + 1}{2} \leq i \leq \frac{n + 3}{2}, \\
(n + 3 - i, i - \frac{n + 1}{2}, n + 1 - i, i - \frac{n + 1}{2}), & \text{if } \frac{n + 5}{2} \leq i \leq n + 1.
\end{cases}
\]
Now, representation of vertices $g_i$ with respect to $W'$ is

$$r(g_i | W') = \begin{cases} 
(i, \frac{n+3}{2} - i, i + 1, \frac{n+3}{2} - i), & \text{if } 1 \leq i \leq \frac{n-1}{2}, \\
(i, 2, n + 2 - i, 1), & \text{if } \frac{n+1}{2} \leq i \leq \frac{n+3}{2}, \\
(n + 3 - i, i - \frac{n-1}{2}, n + 2 - i, i - \frac{n+1}{2}), & \text{if } \frac{n+5}{2} \leq i \leq n + 1.
\end{cases}$$

(2)

Representation of vertices $h_i$ with respect to $W'$ is

$$r(h_i | W') = \begin{cases} 
(i, \frac{n+1}{2} - i, i + 2, \frac{n+3}{2} - i), & \text{if } 1 \leq i \leq \frac{n-1}{2}, \\
\left(\frac{n+1}{2}, 1, \frac{n+1}{2}, 0\right), & \text{if } i = \frac{n+1}{2}, \\
\left(n + 3 - i, i - \frac{n-1}{2}, n + 1 - i, i - \frac{n-1}{2}\right), & \text{if } \frac{n+3}{2} \leq i \leq n, \\
\left(1, \frac{n+1}{2}, 1, \frac{n+3}{2}\right), & \text{if } i = n + 1.
\end{cases}$$

(3)
It can be easily seen that AD $\{(s, t) \mid W'\}$ has at least two entries in the 4-vector different from zero for any $s, t \in V(L(N_e))$. Hence, $W'$ is a fault-tolerant resolving set of $L(N_e)$. So, by using Theorem 2 and Lemma 1, $\beta' (L(N_e)) = 4$ for every odd $n$.

Case 2 ($n$ is even). Let $n \geq 2$, and take $W' = \{f_{(\lceil n/2 \rceil)}, h_1, h_{(\lceil n/2 \rceil)}, g_{(\lceil n/2 \rceil)}\} \subseteq V(L(N_e))$. For $n = 2$, it is easy to verify that all the representations are distinct.

Now, representation of vertices $f_i$ for $n \geq 4$ with respect to $W'$ is

$$r(f_i \mid W') = \begin{cases} \left(\frac{n}{2}, \frac{n + 2}{2}, \frac{n + 2}{2}, \frac{n + 2}{2}\right), & \text{if } i = 1, \\ \left(\frac{n + 2}{2} - i, \frac{n + 4}{2} - i, \frac{n + 4}{2} - i, \frac{n + 4}{2} - i\right), & \text{if } 2 \leq i \leq \frac{n}{2}, \\ \left(0, \frac{n}{2}, 1, 2\right), & \text{if } i = \frac{n + 2}{2}, \\ \left(1, \frac{n + 2}{2}, 1, 2\right), & \text{if } i = \frac{n + 4}{2}, \\ \left(i - \frac{n + 2}{2}, n + 4 - i, i - \frac{n + 2}{2}, i - \frac{n}{2}\right), & \text{if } \frac{n + 6}{2} \leq i \leq n + 1. \end{cases}$$

(4)

Representation of vertices $g_i$ for $n \geq 4$ with respect to $W'$ is

$$r(g_i \mid W') = \begin{cases} \left(\frac{n + 2}{2}, \frac{n + 2}{2}, \frac{n}{2}, \frac{n}{2}\right), & \text{if } i = 1, \\ \left(\frac{n + 4}{2} - i, \frac{n + 4}{2} - i, \frac{n + 2}{2} - i, \frac{n + 2}{2} - i\right), & \text{if } 2 \leq i \leq \frac{n}{2}, \\ \left(2, i - 1, 1, i - \frac{n + 2}{2}\right), & \text{if } \frac{n + 2}{2} \leq i \leq \frac{n + 4}{2}, \\ \left(i - \frac{n}{2}, n + 4 - i, i - \frac{n + 2}{2}, i - \frac{n + 2}{2}\right), & \text{if } \frac{n + 6}{2} \leq i \leq n + 1. \end{cases}$$

(5)

Representation of vertices $h_i$ for $n \geq 4$ with respect to $W'$ is

$$r(h_i \mid W') = \begin{cases} \left(\frac{n}{2}, 0, \frac{n + 2}{2}, \frac{n}{2}\right), & \text{if } i = 1, \\ \left(\frac{n + 2}{2} - i, \frac{n + 4}{2} - i, \frac{n + 2}{2} - i, \frac{n + 2}{2} - i\right), & \text{if } 2 \leq i \leq \frac{n}{2}, \\ \left(1, \frac{n + 2}{2}, 0, 1\right), & \text{if } i = \frac{n + 2}{2}, \\ \left(i - \frac{n}{2}, n + 4 - i, i - \frac{n}{2}, i - \frac{n}{2}\right), & \text{if } \frac{n + 4}{2} \leq i \leq n, \\ \left(\frac{n + 2}{2}, 2, \frac{n + 2}{2}, \frac{n + 2}{2}\right), & \text{if } i = n + 1. \end{cases}$$

(6)
It can be easily seen that AD \((s, t) \mid W'\) has at least two entries in the 4-vector different from zero for any \(s, t \in V(L(N_e))\). Hence, \(W'\) is a fault-tolerant resolving set of \(L(N_e)\). So, by using Theorem 2 and Lemma 1, \(\beta'(L(N_e)) = 4\) for every even \(n\).

3. The Fault-Tolerant Metric Dimension of the Line Graph of the Prism Graph

The prism graph \(Y_n\) is Cartesian product graph \(C_n \times P_2\), where \(C_n\) is the cycle graph of order \(n\) and \(P_2\) is a path of order 2. The prism graph \(Y_n\) consists of 4-sided faces and \(n\)-sided faces with edge set \(E(Y_n) = \{e_i, f_i, g_i; 1 \leq i \leq n\}\) as shown in Figure 3. The line graph \(L(Y_n)\) of the prism graph consists of 3-sided faces, 4-sided faces, and \(n\)-sided faces as shown in Figure 4. For our purpose, we label the inner cycle vertices of \(L(Y_n)\) by \(e_i; 1 \leq i \leq n\), middle vertices by \(f_i; 1 \leq i \leq n\), and the outer cycle vertices by \(g_i; 1 \leq i \leq n\).

In the following theorem, the result for the metric dimension of the line graph of the prism graph is presented.

**Theorem 4** (see [24]). Let \(Y_n\) be the prism graph; then, the metric dimension of the line graph of the prism graph is 3 for \(n \geq 3\). Now, in the following theorem, we will compute the fault-tolerant metric dimension of the line graph of the prism graph.

**Theorem 5.** Let \(Y_n\) be the prism graph; then, the fault-tolerant metric dimension of the line graph of the prism graph is 4 for \(n \geq 3\).

**Proof.** To prove this theorem, consider the following cases. □

**Case 1** \((n\text{ is even})\). Let \(n \geq 4\), and take \(W' = \{e_1, e_2, e_{\lceil (n+2)/2 \rceil}, e_{\lceil (n+4)/2 \rceil} \subseteq V(L(Y_n))\}\). Representation of vertices \(e_i\) with respect to \(W'\) is

\[
\begin{align*}
  r(e_i \mid W') &= \begin{cases} 
    (0, 1, \frac{n + 2}{2}, \frac{n - 2}{2}) & \text{if } i = 1, \\
    (i - 1, i - 2, \frac{n + 2}{2} - i, \frac{n + 4}{2} - i) & \text{if } 2 \leq i \leq \frac{n + 2}{2}, \\
    (n + 1 - i, n + 2 - i, \frac{n + 2}{2} - i, \frac{n + 4}{2} - i) & \text{if } \frac{n + 4}{2} \leq i \leq n.
  \end{cases}
\end{align*}
\]

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It can be easily seen that AD \((s, t) | W'\) has at least two entries in the 4-vector different from zero for any \(s, t \in V(L(Y_n))\). Hence, \(W'\) is a fault-tolerant resolving set of \(L(Y_n)\). So, by using Theorem 4 and Lemma 1, \(\beta'(L(Y_n)) = 4\) for every even \(n\).

Case 2 (\(n\) is odd). For \(n = 3\), take \(W' = \{e_1, f_1, f_3, g_1\} \subseteq V(L(Y_3))\). It is easy to check that \(W'\) is a fault-tolerant resolving set. Let \(n \geq 5\), and take \(W' = \{e_1, e_2, e_{(n+1)/2}, g_1\} \subseteq V(L(Y_n))\). Representation of vertices \(e_i\) with respect to \(W'\) is
\[ r(e_i | W') = \begin{cases} 
(i - 1, 2 - i, \frac{n + 1}{2} - i, 2), & \text{if } 1 \leq i \leq 2, \\
(i - 1, i - 2, \frac{n + 1}{2} - i, i - 1), & \text{if } 3 \leq i \leq \frac{n + 1}{2}, \\
\left( \frac{n - 1}{2}, \frac{n - 1}{2}, 1, \frac{n + 1}{2} \right), & \text{if } i = \frac{n + 3}{2}, \\
\left( n + 1 - i, n + 2 - i, i - \frac{n + 1}{2}, n + 3 - i \right), & \text{if } \frac{n + 5}{2} \leq i \leq n. 
\end{cases} \]  

(11)

Representation of vertices \( f_i \) with respect to \( W' \) is

\[ r(f_i | W') = \begin{cases} 
(1, 1, \frac{n - 1}{2}, 1), & \text{if } i = 1, \\
(i, i - 1, \frac{n + 1}{2} - i, i - 1), & \text{if } 2 \leq i \leq \frac{n - 1}{2}, \\
\left( \frac{n + 1}{2}, \frac{n - 1}{2}, 1, \frac{n - 1}{2} \right), & \text{if } i = \frac{n + 1}{2}, \\
\left( n + 1 - i, n + 2 - i, i - \frac{n - 1}{2}, n + 2 - i \right), & \text{if } \frac{n + 3}{2} \leq i \leq n. 
\end{cases} \]  

(12)

Representation of vertices \( g_i \) when \( n = 5 \) with respect to \( W' \) is

\[ r(g_i | W') = \begin{cases} 
(2, 2, 4 - i, 2 - i), & \text{if } 1 \leq i \leq 2, \\
(3, 2, 2, 1), & \text{if } i = 3, \\
(7 - i, 3, i - 2, 2), & \text{if } 4 \leq i \leq 5. 
\end{cases} \]  

(13)

Representation of vertices \( g_i \) when \( n \geq 7 \) with respect to \( W' \) is

\[ r(g_i | W') = \begin{cases} 
\left( 2, 2, \frac{n + 3}{2} - i, 2 - i \right), & \text{if } 1 \leq i \leq 2, \\
\left( i, i - 1, \frac{n + 3}{2} - i, i - 2 \right), & \text{if } 3 \leq i \leq \frac{n - 1}{2}, \\
\left( \frac{n + 1}{2}, i - 1, 2, i - 2 \right), & \text{if } \frac{n + 1}{2} \leq i \leq \frac{n + 3}{2}, \\
\left( n + 2 - i, n + 3 - i, i - \frac{n - 1}{2}, n + 2 - i \right), & \text{if } \frac{n + 5}{2} \leq i \leq n. 
\end{cases} \]  

(14)

4. Conclusion

In this paper, we have studied for the first time the fault-tolerant metric dimension of the line graph of a graph. We have given the exact values of the fault-tolerant metric...
dimension of the line graphs of necklace and prism graphs and found these values are independent of the number of vertices of a graph. In future, we will compute the exact value of the fault-tolerant metric dimension of the line graph of the kayak paddle graph. We will also discuss the fault-tolerant resolvability in graphs by using other graph operations.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors contributed equally to this study.

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