LEFT INVERTIBILITY OF I/O QUANTIZED LINEAR SYSTEMS IN
DIMENSION 1: A NUMBER THEORETIC APPROACH

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ABSTRACT. This paper studies left invertibility of discrete-time linear I/O quantized linear
systems of dimension 1. Quantized outputs are generated according to a given partition
of the state-space, while inputs are sequences on a finite alphabet. Left invertibility, i.e. in-
jectivity of I/O map, is reduced to left D-invertibility, under suitable conditions. While left
invertibility takes into account membership in sets of a given partition, left D-invertibility
considers only distances, and is very easy to detect. Considering the system \( x' = ax + u \),
our main result states that left invertibility and left D-invertibility are equivalent, for all but
a (computable) set of \( a \)’s, discrete except for the possible presence of two accumulation
point. In other words, from a practical point of view left invertibility and left D-invertibility
are equivalent except for a finite number of cases. The proof of this equivalence involves
some number theoretic techniques that have revealed a mathematical problem important
in itself. Finally, some examples are presented to show the application of the proposed
method.

1. INTRODUCTION

Left invertibility is an important problem of systems theory, which corresponds to in-
jectivity of I/O map. It deals with the possibility of recovering unknown inputs applied to
the system from the knowledge of the outputs.

We investigate left invertibility of discrete-time linear I/O quantized systems in a con-
tinuous state-space of dimension 1. In particular, inputs are arbitrary sequences of symbols
in a finite alphabet: each symbol is associated to an action on the system. Information avail-
able on the system is represented by sequences of output values, generated by the system
evolution according to a given partition of the state-space (uniform quantization).

In recent years there has been a considerable amount of work on quantized control
systems (see for instance [9], [22], [26] and references therein), stimulated also by the
growing number of applications involving “networked” control systems, interconnected
through channels of limited capacity (see e.g. [3, 6, 27]). The quantization and the finite
cardinality of the input set occur in many communication and control systems. Finite inputs
arise because of the intrinsic nature of the actuator, or in presence of a logical supervisor,
while output quantization may occur because of the digital nature of the sensor, or if data
need a digital transmission.

Applications of left invertibility include fault detection in Supervisory Control and Data
Acquisition (SCADA) systems, system identification, and cryptography ([14, 18]). In-
vertibility of linear systems is a well understood problem, first handled in [5], and then
considered with algebraic approaches (see e.g. [24]), frequency domain techniques ([19],
[20]), and geometric tools (cf. [21]). Invertibility of nonlinear systems is discussed in
([23]). More recent work has addressed the left invertibility for switched systems ([28]),
and for I/O quantized contractive systems([10]).

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time.
The main intent of the paper is to show that the analysis of left invertibility can be substituted, under suitable conditions, by an analysis of a stronger notion, called left D-invertibility. While left invertibility takes in account whether two states are in the same element of a given partition, left D-invertibility considers only the distance between the two states. For this reason left D-invertibility is very easy to detect. For the system \( x^+ = ax + u \), the condition under which left invertibility and left D-invertibility are equivalent has to do with the existence of an infinite (periodic) orbit inside a certain set and the contemporary occurrence of an algebraic condition satisfied by \( a \). This two conditions are as a matter of fact not restrictive, and indeed the main theorem (Theorem 5) states that the set of \( a \) such that left D-invertibility and left invertibility are not equivalent is discrete but possibly 2 accumulation points. In other words from a practical point of view ULI and ULDI are equivalent except for a finite number of cases (see Theorem 5).

The main tools used in the paper are a generalization of a classical density theorem of Kronecker, and some geometry of numbers. The Kronecker’s theorem has to do with density in the unit cube of the fractional part of real numbers. By means of a particular construction the problem of “turning” left D-invertibility into left invertibility can be handled with a Kronecker-type density theorem. Geometry of numbers helps us to show that, even if the Kronecker’s theorem has not a straightforward application (we do not have density) we can obtain our result anyway (we have \( \varepsilon \)-density, with \( \varepsilon \) small enough).

The paper is organized as follows: section 2 contains a precise statement of the problem under study, while section 3 concerns the number theoretic background needed. Section 4 shows the procedure to prove the equivalence between left D-invertibility and left invertibility: the rational case is treated first, to show in a more direct way ideas involved. This section contains also the main result of the paper (Theorem 5). In section 5 explicit calculations are done in a comprehensive example. Conclusions and future work are explained in section 6. Finally, there is a “special” section, the 7-th, in which we collect the notations used in the paper.

2. STATEMENT OF THE PROBLEM

**Definition 1.** The uniform partition of rate \( \delta \) of \( \mathbb{R} \) is

\[
\mathcal{P} = \{ \mathcal{P}_i \}_{i \in \mathbb{Z}} = \{ [i\delta, (i+1)\delta) | i \in \mathbb{Z} \}.
\]

In this paper we consider discrete-time, time-invariant, I/O quantized linear systems of the form

\[
\begin{aligned}
x(k+1) &= ax(k) + bu(k) \\
y(k) &= q_\mathcal{P}(cx(k))
\end{aligned}
\]  

(1)

where \( x(k) \in \mathbb{R} \) is the state, \( y(k) \in \mathbb{Z} \) is the output, \( u(k) \in \mathcal{U} \subset \mathbb{R} \) is the input, and \( a, b, c \in \mathbb{R} \). The map \( q_\mathcal{P} : \mathbb{R} \rightarrow \mathbb{Z} \) is induced by the uniform partition \( \mathcal{P} = \{ \mathcal{P}_i \}_{i \in \mathbb{Z}} \) of \( \mathbb{R} \) of rate \( \delta \) through \( q_\mathcal{P} : (x \in \mathcal{P}_i) \mapsto i \) and will be referred to as the output quantizer. We assume that \( \mathcal{U} \) is a finite set of cardinality \( n \).

**Remark 1.** Without loss of generality in the system (2) we can suppose \( \delta = 1, b = 1, c = 1 \).

**Proof:** Operate the substitutions \( x(k) = \frac{\delta}{\varepsilon}x(k) \) and \( u(k) = \frac{\delta}{\varepsilon}u(k) \).

So we consider only systems of the form

\[
\begin{aligned}
x(k+1) &= ax(k) + u(k) = f_u(x(k)) \\
y(k) &= \lfloor x(k) \rfloor
\end{aligned}
\]  

(2)

where \( \lfloor \cdot \rfloor \) denotes the integer part. Indicate with \( f_u^k(x_0, u_1, \ldots, u_{k_2}) \) the sequence of outputs \( (y_1, \ldots, y_{k_2}) \) generated by the system (2) with initial condition \( x_0 \) and input string \( (u_1, \ldots, u_{k_2}) \).
Definition 2. A pair of input strings \( \{u_i\}_{i \in \mathbb{N}}, \{u'_i\}_{i \in \mathbb{N}} \) is uniformly distinguishable in \( k \) steps, \( k \in \mathbb{N} \), (or with distinguishability time \( k \)) if there exists \( l \in \mathbb{N} \) such that \( \forall (x_0, x'_0) \in \mathbb{R}^2 \) and \( \forall m > l \) the following holds:

\[
u_m \neq u'_m \Rightarrow f^{m+k}_m(x_0, u_1, \ldots, u_{m+k}) \neq f^{m+k}_m(x'_0, u'_1, \ldots, u'_{m+k}).\]

In this case, we say that the strings are uniformly distinguishable with waiting time \( l \).

Definition 3. A system of type \( \{3\} \) is uniformly left invertible (ULI) in \( k \) steps if every pair of distinct input sequences is uniformly distinguishable in \( k \) steps after a finite time \( l \), where \( k \) and \( l \) are constant.

For a ULI system, it is possible to recover the input string until instant \( m \) observing the output string until instant \( m + k \). For applications, however, it is important to obtain an algorithm to reconstruct the input symbol used at time \( m > l \) by processing the output symbols from time \( m \) to \( m + k \).

Definition 4. Define

\[
Q = \bigcup_{y \in \mathbb{Z}} \{q^{-1}(y) \times q^{-1}(y)\} = \bigcup_{y \in \mathbb{Z}} \{[y, y+1] \times [y, y+1]\} \subset \mathbb{R}^2
\]

i.e. the union of the preimages of two identical output symbols. In other words, \( Q \) contains all pairs of states that are in the same element of the partition \( \mathcal{P} \).

To address invertibility, we are interested in studying the following system on \( \mathbb{R}^2 \):

\[
X(k+1) = F_{U(k)}(X(k)) = \begin{bmatrix} f(x(k), u(k)) \\ f(x'(k), u'(k)) \end{bmatrix}
\]

where \( X(k) = \begin{bmatrix} x(k) \\ x'(k) \end{bmatrix}, \quad U(k) = (u(k), u'(k)). \) If it is possible to find an initial state in \( Q \) and an appropriate choice of the strings \( \{u_i\}, \{u'_i\} \) such that the orbit of \( \{3\} \) remains in \( Q \), it means that the two strings of inputs give rise to the same output for the system \( \{3\} \). Therefore conditions ensuring that the state is outside \( Q \) for some \( k \) will be sought to guarantee left invertibility. We will need another notion of left invertibility, stronger but very easy to check, that we define in the following. It will be central in our discussion.

Definition 5. The difference system associated with the system \( \{3\} \) is

\[
z(k+1) = az(k) + v(k) = f_{U(k)}(z(k))
\]

where \( z(k) \in \mathbb{R}, v(k) \in \mathcal{Y} = \mathcal{W} - \mathcal{F} \).

Remark 2. The difference system represents at any instant the difference between the two states \( z(k) = x(k) - x'(k) \) when the input symbols \( u(k) - u'(k) = v(k) \) are performed. So we are interested in understanding the conditions under which

\[
\{z(k)\} \cap [-1, 1] = \emptyset.
\]

Indeed, this implies that \( y(k) \neq y'(k) \). The converse is obviously not true.

Indicate with \( D^k_{m_0}([z_0, v_1, \ldots, v_{k_0}) \) the sequence \( (z(k_1), \ldots, z(k_2)) \) generated by the difference system with initial condition \( z_0 \) and input string \( (v_1, \ldots, v_{k_0}) \).

Definition 6. A pair of input strings \( \{u_i\}_{i \in \mathbb{N}}, \{u'_i\}_{i \in \mathbb{N}} \) is uniformly \( D \)-distinguishable in \( k \) steps, \( k \in \mathbb{N} \) (or with distinguishability time \( k \)), if there exists \( l \in \mathbb{N} \) such that \( \forall (z_0) \in \mathbb{R} \) and \( \forall m > l \) the following holds:

\[
v_m \neq u'_m \Rightarrow D^m_{m+k}(z_0, v_1, \ldots, v_{m+k}) \not\subseteq [-1, 1], \quad \forall m > l,
\]

where \( v_i = u_i - u'_i \). In this case, we say that the strings are uniformly \( D \)-distinguishable with waiting time \( l \).
D-invertible. Indeed in this case the system of the form (1).

Remark 3. Thanks to Remark 2 uniform left D-invertibility implies uniform left invertibility.

Proposition 1. The system is either ULDI in time 1, or not ULDI at all, depending on the following condition is satisfied:

\[
\min_{0 \neq v \in \mathcal{Y}} |v| \geq |a| + 1.
\]

Proof: A sufficient condition for uniform left D-invertibility in one step is

\[
\forall v \in \mathcal{Y}, v \neq 0 : |v| \geq |a| + 1
\]

indeed in this hypothesis \(\forall v \in \mathcal{Y}, v \neq 0 \):

\[
|v| = |v| - 1, 1 \cap \{a \cdot [1 - 1, 1] + v\} = |v| - 1, 1 \cap |v| - a + v, a + |v| = \emptyset
\]

We now prove that if \(3v \in \mathcal{Y}, v \neq 0 : |v| < |a| + 1\), then the system is not uniformly left D-invertible. Indeed in this case the system

\[
\begin{cases}
ax_1 + v = x_2 \\
ax_2 - v = x_1
\end{cases}
\]

has the solution \(x_1 = \frac{-v}{a - 1}, x_2 = \frac{-v}{a + 1}\). Since \(|x_1|, |x_2| < 1\) the difference system has the infinite orbit \(\{x_1, x_2, x_3, x_4, \ldots\} \subset [1, 1]\). Therefore system (2) is not left D-invertible.

Proposition 1 shows a trivial way to check ULDI for systems (2). The problem under study is the following:

Problem 1. State mathematical conditions for the equivalence between ULDI and ULI of a uniformly quantized linear system of the form (1). ◯

3. MATHEMATICAL BACKGROUND

We will mainly need results from number theory: our proofs are essentially based on the application of a density Theorem of Kronecker (see [17]), sufficient in the case in which \(a\) is transcendental. For the algebraic case we need further computations involving the Mahler measure of polynomials.

Definition 8. The numbers \(\theta_1, \ldots, \theta_M \in \mathbb{R}\) are linearly independent over \(\mathbb{Z}\) if the following holds:

\[
k_1, \ldots, k_M \in \mathbb{Z} : k_1 \theta_1 + \ldots + k_M \theta_M = 0 \Rightarrow k_1 = \ldots = k_M = 0.
\]

Theorem 1 (Kronecker). [17] If \(a_1, \ldots, a_M, l \in \mathbb{R}\) are linearly independent over \(\mathbb{Z}\), then, for every \(\theta_1, \ldots, \theta_M \in \mathbb{R}\) the set of points

\[
\{\frac{1}{l} \langle a_1 + \theta_1 \rangle, \ldots, \frac{1}{l} \langle a_M + \theta_M \rangle : l \in \mathbb{R}\}
\]

is dense in the unit cube of \(\mathbb{R}^M\).

Definition 9. A set of independent linear relations among \(a_1, \ldots, a_M \in \mathbb{R}\) is said to be maximal if no other independent linear relation can be found among these numbers.

Remark 4. A corollary of the Kronecker’s Theorem (clear from the proof) is that, if the numbers \(a_1, \ldots, a_M \in \mathbb{R}\) satisfy a maximal set of nontrivial linear equations \(L = \{L^j(a_1, \ldots, a_M) = 0 \ for \ j = 1, \ldots, J\}\), then the set of points

\[
\{\frac{1}{l} \langle a_1 + l\theta_1 \rangle, \ldots, \frac{1}{l} \langle a_M + l\theta_M \rangle : l \in \mathbb{R}\}
\]

is dense in

\[
C_L = \frac{1}{l}(\{x_1, \ldots, x_M : L^j(x_1, \ldots, x_M) = 0, \ j = 1, \ldots, J\}).
\]
Definition 10. We define the set of linear relations $L$ to be integer-maximal for the numbers $\alpha_1, \ldots, \alpha_M$ if

- The set of points \( \left\{ \left[ \frac{\alpha_1 + l \vartheta_1}{}, \ldots, \frac{\alpha_M + l \vartheta_M}{}, \ldots \right] : l \in \mathbb{R} \right\} \) is dense in $C_L$;
- The linear relations $L_j$ are formed with integer coefficients;
- $C_L = \{ x \in \mathbb{R}^M : L_j x \in \mathbb{Z} J \} \cap [0, 1]^M$. ◦

Definition 11. A number $\rho \in \mathbb{C}$ is called algebraic if there exists a polynomial $R(x) \in \mathbb{Z}[x]$ such that $R(\rho) = 0$. In this case there exists a unique monic polynomial $R(x) \in \mathbb{Z}[x]$ with minimal degree $q$. $R(x)$ is called the minimal polynomial of $\rho$ and $q$ its degree. A number $\rho \in \mathbb{C}$ is called trascendental if it is not algebraic. ◦

Note that $1, \rho, \rho^2, \ldots, \rho^M$ are linearly independent if and only if the degree of $\rho$ is at least $M + 1$.

Definition 12. The $i$–th symmetric polynomial in $q$ variables is

\[
e_i(x_1, \ldots, x_q) = \sum_{1 \leq j_1 \leq \ldots \leq j_q} x_{j_1} \cdots x_{j_q}.
\]

Definition 13. If $R(x)$ is the polynomial

\[
R(x) = \sum_{i=0}^{q} r_i x^i = r_q \prod_{j=1}^{q} (x - \rho_j),
\]

where the $\rho_j$’s are the roots of the polynomial, its Mahler measure is defined as

\[
\mathfrak{M}(R) = r_q \prod_{j=1}^{q} \max \left\{ 1, |\rho_j| \right\}.
\]

Mahler measure has many interesting properties. For instance, since $r_i$ is equal to $r_q$ multiplied by the $i$-th symmetric polynomial of the $\rho_j$, which is made of precisely $\binom{q}{i}$ monomials in the $\rho_j$ where each $\rho_j$ appears with degree at most 1, we have that $r_i$ is sum of $\binom{q}{i}$ terms each $\leq \mathfrak{M}(A)$ in absolute value, and consequently

\[
|r_i| \leq \binom{q}{i} \cdot \mathfrak{M}(R), \quad \text{for } 0 \leq i \leq q. \tag{5}
\]

If $\|R\|_{\infty}$ is the norm

\[
\|R\|_{\infty} = \max_{0 \leq i \leq q} |r_i|, \tag{6}
\]

we obtain from (5)

\[
\|R\|_{\infty} \leq \binom{q}{\lfloor q/2 \rfloor} \cdot \mathfrak{M}(R). \tag{7}
\]

In the following we will also have to consider the quantity

\[
\mathfrak{M}(R(x/2)) = r_q \prod_{j=1}^{q} \max \left\{ \frac{1}{2}, |\rho_j| \right\},
\]

i.e. the Mahler measure of the polynomial $R(x/2)$. The last equality is easily proved since

\[
\mathfrak{M}(R(x/2)) = \frac{r_q}{2^q} \prod_{j=1}^{q} \max \left\{ 1, |2\rho_j| \right\} = r_q \prod_{j=1}^{q} \max \left\{ \frac{1}{2}, |\rho_j| \right\}. \tag{8}
\]

In particular, note that

\[
\mathfrak{M}(R) \leq 2^q \cdot \mathfrak{M}(R(x/2)). \tag{9}
\]
Figure 1. Here, for \( i = 1, 2, 3 \), the point \( X_t(i) \) has a distance \( k_i \) from the union of positive coordinate axes along the line \( r_i \) (drawn with a dashed line), and “velocity” \( a_i \) (with respect to \( t \)).

4. ULI: The Number Theoretic Approach

Our strategy is the following: for transcendental \( a \) in the system (2), we prove that ULDI is equivalent to ULI. Moreover, for \( a \) algebraic (and rational), we will show that these two notions are very close, in a sense precisely specified later.

Notations: Consider the system of dimension 2 given by (3), and suppose that there exists at least one proper orbit included in the set

\[
Q' = \left\{ \left( \begin{array}{c} t \\ s \end{array} \right) : s \in [-1, 1], t \in \mathbb{R} \right\}. \tag{10}
\]

(such an orbit exists if and only if system (2) is not ULDI). Take as initial condition \( X_t(0) = \left( \begin{array}{c} t \\ s \end{array} \right) \in \mathbb{R}^2 \), with \( t \) considered as a parameter, varying in \( \mathbb{R} \) and \( s \in [-1, 1] \) fixed. Then, for fixed input string

\[
X_t(k) = \left( \begin{array}{c} \left( a_i^k t + a_i^{k-1} u_1 + \ldots + u_k \right) \\ \left( a_i^k t + a_i^{k-1} u'_1 + \ldots + u'_k \right) \end{array} \right) = a_i^k \left( \begin{array}{c} t \\ s \end{array} \right) + \left( \begin{array}{c} c_i^k \\ c_i^k' \end{array} \right). \tag{11}
\]

Suppose that an orbit \( \{X_t(i)\}_{i=1}^\infty \) is included in \( Q' \). We can see the points \( X_t(i) \), when \( t \) varies in \( \mathbb{R} \), as points moving along the line

\[
\rho_i = \left\{ a_i^k \left( \begin{array}{c} t \\ c_i \end{array} \right) + \left( \begin{array}{c} c_i \\ c_i' \end{array} \right) : t \in \mathbb{R} \right\} \tag{12}
\]

with initial condition \( \left( \begin{array}{c} c_i \\ c_i' \end{array} \right) \) and velocity \( a_i \). Call \( k_i \) the distance between the point \( \left( \begin{array}{c} c_i \\ c_i' \end{array} \right) \) and the union of positive coordinate axes along the line \( \rho_i \) (refer to the figure 1).
4.1. Trascendental $a$.

The following technical lemma gives a necessary condition for uniform left invertibility, a basilar ingredient in the proof of Theorem 2.

**Lemma 1.** Consider the $2$-dimensional system (3) with the notations just introduced. Suppose that $\forall \varepsilon > 0, \forall J \in \mathbb{N}, \forall s \in [-1, 1], \forall \{U(j)\}_{j \in \mathbb{N}}$ there exists $t \in \mathbb{R}$ such that, if $\{X(j)\}_{j=0}^J \subset Q'$ is the orbit by $X(0) = \begin{pmatrix} t+s \\ t \end{pmatrix}$ and input sequence $U(j)$, the following holds for every $j = 1, \ldots, J$:

$$\frac{k_j + ta_j}{\alpha_j} < \varepsilon.$$ (13)

Then the system is not ULI.

**Proof:** Suppose that an orbit $\{X_i(j)\}_{j=1}^\infty$ is included in $Q'$. Observe that $\frac{k_j + ta_j}{\alpha_j} = 0$ if and only if $X_i(j)$ belongs to some translation of

$$\Omega = [0, 1] \times \{0\} \cup \{0\} \times [0, 1]$$ (14)

along the diagonal of $\mathbb{R}^2$, that is entirely included in $Q$, i.e., a translation that takes $\Omega$ to the “bottom-left boundary” of a square of $Q$. It’s now easy to see that, for every $X_i(j)$ there exists $\varepsilon > 0$ such that, if $\frac{k_j + ta_j}{\alpha_j} < \varepsilon$ then $X_i(j) \in Q$. Therefore, if the relations (13) are satisfied, then there exists an arbitrary long orbit included in $Q$. ♦

**Proposition 2.** Suppose that the system (2) is not ULDI. If $a$ is an algebraic number of degree $K$ then the system is not ULI in $K - 1$ steps.

**Proof:** Since the system is not ULDI there exist arbitrary long orbits included in $Q'$. Fix one of these orbits of length greater than $K - 1$.

If, for every $\varepsilon > 0$, and every $k_1, \ldots, k_K \in \mathbb{R}$ there exists a $t \in \mathbb{R}$ such that

$$\frac{k_i + at}{\alpha_i} < \varepsilon \text{ for } i = 0, \ldots, K - 1$$ (15)

then the system (2) is not ULI in $K - 1$ steps by Lemma 1. Equation (15) is equivalent to find integers $N_0, \ldots, N_K$ such that for every $i = 0, \ldots, K$

$$N_i \leq k_i + at < N_i + \varepsilon$$

But, if $a$ is algebraic of degree $K$, then numbers $a^i, i = 0, \ldots, K - 1$ are linearly independent over $\mathbb{Z}$, and by Theorem ?? there always exists a $t$ such that equation (15) holds, and so the system is not uniformly left invertible in $K - 1$ steps. ♦

The following Theorem can be deduced immediately from Proposition 2.

**Theorem 2.** Suppose that $a$ is trascendental. Then the system (2) is ULI if and only if it is ULDI.

**Proof:** Suppose that system (2) is not ULDI. Proposition 2 states that, if the system is ULI in $K$ steps then $a$ cannot be algebraic of degree greater than $K + 1$. The result follows easily since a trascendental number is not algebraic of any degree. ♦

**Corollary 1.** Consider the unidimensional system (2), with trascendental $a$. Then it is either ULI in one step, or it is not ULI. ◇
4.2. Algebraic a.

Suppose now that \( a \) is algebraic of degree \( K \), and that the minimum polynomial of \( a \) is
\[
\alpha_K t^K + \ldots + \alpha_0 \in \mathbb{Z}[t].
\]
We are interested in finding an \( \varepsilon \) (the minimum \( \varepsilon \)) such that for every \( J \in \mathbb{N} \) there exists an \( i \in \mathbb{N} \) and a point in
\[
[0, \varepsilon]^{i+J+1}
\]
in the sequence
\[
\{ \frac{1}{i+1}, \ldots, \frac{1}{i+J+1} \}
\]
for every orbit \( \{ X_k \}_{k \in \mathbb{N}} \subset Q \). Considering the \( J \)-dimensional torus \( T^J \), the linear manifold (of dimension \( K \)) associated with \( a \), i.e. the linear manifold whose image mod 1 is what we called \( C_L \) in Remark 4 is given by the following equations:
\[
P^j_k = (t_0, \ldots, t_{i+K}) \in \mathbb{R}^{J+K+1}: \begin{cases}
\alpha_0 t_0 + \alpha_1 t_1 + \ldots + \alpha_K t_K = 0 \\
\alpha_0 t_1 + \alpha_1 t_2 + \ldots + \alpha_K t_{K+1} = 0 \\
\vdots \\
\alpha_0 t_J + \alpha_1 t_{J+1} + \ldots + \alpha_K t_{K+J} = 0
\end{cases}
\]
\[
= \text{Ker}(\Psi^j_k) = \text{Ker} \left\{ \begin{pmatrix}
\alpha_0 & \alpha_1 & \ldots & \alpha_K & 0 & \ldots & 0 \\
0 & \alpha_0 & \alpha_1 & \ldots & \alpha_K & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & \alpha_0 & \ldots & \alpha_{K-1} & \alpha_K
\end{pmatrix} \right\}.
\] (17)

In other words, \( \text{frac}(P^j_k) \) is the set in which the sequence \( \{ \frac{1}{i+1}, \ldots, \frac{1}{i+J+1} \} \)
is dense (by Remark 4).

**Definition 14.** Denote with \( b_j \) the vector \( (a, \ldots, a^i) \), and define
\[
\varepsilon(a) = \sup_{\zeta \in \mathbb{R}^J} \sup_{J \in \mathbb{N}} \inf_{t \in \mathbb{R}} \max_{i \in [i+J]} \left| \text{frac}(\zeta + t b_j) \right| = 
\]
\[
= \inf \left\{ \varepsilon \in \mathbb{R} : [0, \varepsilon]^{K+J+1} \cap (P^j_k + v + \mathbb{Z}^{K+J+1}) \neq \emptyset, \text{ for each } v \in \mathbb{R}^{K+J+1}, J \in \mathbb{N} \right\}
\] (19)

Let us explain the meaning of \( \varepsilon(a) \). Suppose we are given any trajectory of the 2-dimensional system \( \langle \rangle \) included in \( Q \). Then, letting \( t \) vary as a parameter, it has the form \( \langle \rangle \), and we can investigate ULI looking at fractional parts of \( k_i + ta_i \), for \( i = 1, \ldots, J \), for every \( J \in \mathbb{N} \). Now, *modulo* the \( k_i \)'s (i.e. modulo the inputs), that is taking the sup on \( \zeta \) in the definition, \( \varepsilon(a) \) is the smallest \( \varepsilon \) such that for every \( J \in \mathbb{N} \) there exists \( t \in \mathbb{R} \)
\[
\text{frac}(k_i + ta_i) < \varepsilon \text{ for } i = 1 \ldots J.
\]
It’s now easy to see, looking at Lemma 1 that \( \varepsilon(a) \) is useful to put in relation ULDI with ULI. Moreover, by Remark 4 the set
\[
\{ \text{frac}(t a_i) : t \in \mathbb{R}, \ i = 1, \ldots, J \}
\]
is dense in \( \text{frac}(P^j_k) \). So \( \varepsilon(a) \) equals the following quantity:
\[
\varepsilon(a) = \sup_{\zeta \in \mathbb{R}^J} \sup_{J \in \mathbb{N}} \max_{i \in [i+J]} \left| \text{frac}(\zeta + P^j_k) \right|.
\]
This implies the second equivalent definition of \( \varepsilon(a) \) in the definition 14.

**Proposition 3.** The map defined by the matrix \( \Psi^j_k : \mathbb{Z}^{J+K+1} \rightarrow \mathbb{Z}^{J+1} \) is surjective.
Proof: This is an immediate consequence of [?, Lemma 2, Chap. 1], which says that a rectangular integer \( m \times l \) matrix, for \( m > l \), can be completed to a square invertible integer \( m \times m \) matrix with determinant 1 if and only if the greatest common divisor of the \( l \times l \) minors is 1. Now, if a rectangular integer \( m \times l \) matrix, for \( m > l \), can be completed to an invertible integer \( m \times m \) matrix, then the original matrix must be clearly surjective from \( \mathbb{Z}^m \to \mathbb{Z}^l \).

All we have to do to apply the lemma is checking that the greatest common divisor of the \( k \times k \) minors of \( \Psi^t_k \) is 1, but this is easy since for each prime \( p \) we can consider the first coefficient \( \alpha_1 \) of our polynomial such that \( p \) does not divide \( \alpha_1 \), and take the \( k \times k \) minor made of the columns \( K - i + 1, K - i + 2, \ldots, K - i + k \). Since this minor is lower triangular when reduced modulo \( p \) with all the elements on the diagonal equal to \( \alpha_1 \mod p \), its determinant does not vanish modulo \( p \), and we are done. \( \diamondsuit \)

Proposition 4.

\[ \varepsilon(a) = \inf \{ \varepsilon \in \mathbb{R} : (\Psi^t_k \cdot [0, \varepsilon]^{K+J+1} + w) \cap \mathbb{Z}^{J+1} \neq \emptyset, \text{ for each } w \in \mathbb{R}^{J+1} \} \]

Proof: First note that, for the set of vectors such that \( \Psi^t_k \cdot v \) has integer components, it holds

\[ S = \{ v \in \mathbb{R}^{K+J+1} : \Psi^t_k \cdot v \in \mathbb{Z}^{J+1} \} = \text{Ker}(\Psi^t_k) + \mathbb{Z}^{K+J+1}. \]

Indeed, \( \text{Ker}(\Psi^t_k) + \mathbb{Z}^{K+J+1} \subseteq S \) clearly, and for each vector \( w \in S \) there exist a vector \( z \in \mathbb{Z}^{K+J+1} \) such that \( \Psi^t_k \cdot w = \Psi^t_k \cdot z \), and consequently the difference \( v = w - z \) is in \( \text{Ker}(\Psi^t_k) \), and we have that \( w = v + z \in \text{Ker}(\Psi^t_k) + \mathbb{Z}^{K+J+1} \). Now, since in the (19) we are quantifying over all vectors \( v \in \mathbb{R}^{K+J+1} \), we can equivalently say that

\[ \varepsilon(a) = \inf \{ \varepsilon \in \mathbb{R} : ([0, \varepsilon]^{K+J+1} + v) \cap S \neq \emptyset, \text{ for each } v \in \mathbb{R}^{K+J+1} \} \]

(20)

applying the matrix \( \Psi^t_k \) to the expression, and where we denoted \( \Psi^t_k \cdot [0, \varepsilon]^{K+J+1} \) the image of \([0, \varepsilon]^{K+J+1}\) under the map \( \Psi^t_k \). This passage must be justified because the matrix \( \Psi^t_k \) clearly does not have rank \( K + J + 1 \), but since \( S \) contains all the vectors that are mapped to \( \mathbb{Z}^{J+1} \) the first intersection will be non-empty whenever the second one is (the other direction being trivial). \( \diamondsuit \)

Following Proposition 4, we are investigating how big must be \( \varepsilon \) to ensure that each set obtained translating \( \Psi^t_k \cdot [0, \varepsilon]^{K+J+1} \) contains an integer vector. This will be true if and only if

\[ \Psi^t_k \cdot [0, \varepsilon]^{K+J+1} + \mathbb{Z}^{J+1} = \mathbb{R}^{J+1}, \]

and equivalently if and only if \( \Psi^t_k \cdot [0, \varepsilon]^{K+J+1} \) contains a representative for each class in \( \mathbb{R}^{J+1}/\mathbb{Z}^{J+1} \).

Theorem 3. Suppose that in the system (2) there exists an infinite orbit of the difference system in \([−1 + \varepsilon(a), 1 − \varepsilon(a)]\). Then the system is not uniformly left invertible.

Proof: In the hypotheses of the Theorem we can find, for every \( J \in \mathbb{N} \), an orbit of the 2-dimensional system (3) \( X(0), \ldots, X(J) \), such that for every \( i \in 0, \ldots, J \)

\[ X(i) \in \left\{ \begin{pmatrix} t \\ t \end{pmatrix} + \begin{pmatrix} s \\ 0 \end{pmatrix} : t \in \mathbb{R}, \ s \in [-1 + \varepsilon(a), 1 - \varepsilon(a)] \right\}, \]

\[ \text{frac}(k_i) < \varepsilon(a). \]

This implies clearly that the system is not ULI (see figure 2). \( \diamondsuit \)
In the hypotheses of Theorem 3 we can find a trajectory (we represent here only $X(0), X(1), X(2)$) inside the “strip” $\{[-1 + \varepsilon(a), 1 - \varepsilon(a)] + (t, t) : t \in \mathbb{R}\}$, drawn inside the dashed-dotted line.

4.2.1. $a \in \mathbb{Q}$.

We investigate first the rational case, because the estimates are easier, and the results are straightforward. For the algebraic case we need an harder work. Suppose that $a = \frac{p}{q} \in \mathbb{Q}$, with $\gcd(p, q) = 1$. Then the minimal polynomial of $a$ is $P_a(x) = qx - p$. So:

$$
\begin{align*}
pt_0 + qt_1 &= 0 \\
pt_1 + qt_2 &= 0 \\
&\vdots \\
pt_{i-2} + qt_{i-1} &= 0 \\
pt_{i-1} + qt_i &= 0
\end{align*}
$$

(21)

**Proposition 5.** Suppose that in the system (2) $a = \frac{p}{q} \in \mathbb{Q}$. Then $\varepsilon(a) \leq \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

**Proof:** We show that the image of cube $[0, 1/p)^{J+2}$ under $P_a^I$ assumes each value modulo $\mathbb{Z}^{J+1}$, and this can easily be done inductively in the following way. Let $w = (w_1, \ldots, w_{J+2}) \in \mathbb{R}^{J+2}$, we will build a vector $v = (v_1, \ldots, v_{J+1})$ such that $\Psi^I_K \cdot w - v \in \mathbb{Z}^{J+1}$. Suppose that $v$ is such that the first $i > 0$ components of $\Psi^I_K \cdot w - v$ are in $\mathbb{Z}$, and observe that while $w_{i+1}$ varies in the interval $[0, 1/p]$ the $i + 1$-th component of $P_a^I \cdot w$ varies in an interval large 1, while the first $i$ components of $P_a^I \cdot w$ stay fixed. Consequently we can change $w_{i+1}$ to ensure that the first $i + 1$ components of $P_a^I \cdot w - v$ are in $\mathbb{Z}$, and continuing in this way we prove our assertion. If $q \geq p$ we can clearly proceed similarly but downwards, starting from the last component. $\diamond$

**Corollary 2.** Suppose that in the system (2) there exists an infinite orbit of the difference system in

$$
[-1 + \min\left\{\frac{1}{p}, \frac{1}{q}\right\}, 1 - \min\left\{\frac{1}{p}, \frac{1}{q}\right\}]
$$

Then the system is not uniformly left invertible. $\diamond$
Suppose that \( a \) is algebraic, with minimum polynomial \( P_a(x) = \alpha_0 + \ldots + \alpha_K x^K \) of degree \( K > 2 \). Then denote with
\[
P^I_K = \left\{ (t_0, \ldots, t_{I+K}) \in \mathbb{R}^{I+K+1} : \begin{cases} \alpha_0 t_0 + \alpha_1 t_1 + \ldots + \alpha_K t_K = 0 \\ \alpha_0 t_1 + \alpha_1 t_2 + \ldots + \alpha_K t_{K+1} = 0 \\ \vdots \\ \alpha_0 t_I + \alpha_1 t_{I+1} + \ldots + \alpha_K t_{I+K} = 0 \end{cases} \right\} = \text{Ker}(\Psi^I_K).
\]

**Proposition 6.** Indicating with \( P_a(x) \) the minimal polynomial of an algebraic number \( a \), the following estimate holds:
\[
\varepsilon(a) \leq \min \left\{ \frac{1}{\mathfrak{M}(P_a(x/2))}, \frac{1}{\mathfrak{M}(2^{-K}P_a(2x))} \right\}.
\]

**Proof:** See Theorem 1 of [13]. \( \diamond \)

**Theorem 4.** Indicating with \( P_a(x) = \alpha_K t^K + \ldots + \alpha_0 \) the minimal polynomial of an algebraic number \( a \), the following estimate holds:
\[
\varepsilon(a) \leq \text{const} \cdot \min \left\{ \frac{1}{|\alpha_i|} : i = 1, \ldots, k \right\},
\]
with the constant depending only on the degree of \( a \). Moreover, the constant is less than or equal to
\[
\min_i \left\{ \frac{k}{i} \right\} \frac{\min \{|i-k-i|\}}{\left| \alpha_i \right|}.
\]

**Proof:** By Proposition 5 it holds the estimate (22). The two terms \( 2^{-K} \mathfrak{M}(P_a(2x)) \) and \( \mathfrak{M}(P_a(x/2)) \) are the Mahler measures of respectively the polynomial with coefficients \( 2^{-K} \alpha_i, \ldots, 2^{-1} \alpha_i, \alpha_0 \), and the polynomial with coefficients \( \alpha_i, \ldots, 2^{-k+1} \alpha_i, 2^{-k} \alpha_0 \). Moreover by (7) it holds
\[
2^{-i} \alpha_i \leq \left( \frac{k}{i} \right) M(P(x/2)),
\]
\[
2^{-k+i} \alpha_i \leq \left( \frac{k}{i} \right) M(2^{-k}P(2x))
\]
and we are done. \( \diamond \)

**Corollary 3.** Suppose that in the system (3) the degree of \( a \) is at least 2. Suppose that there exists a proper path of the attractor of the difference system in
\[
-1 + \min_i \left\{ \left( \frac{K}{|K/2|} \right) \frac{2^{K/2}}{\alpha_i} \right\}, \quad 1 - \min_i \left\{ \left( \frac{K}{|K/2|} \right) \frac{2^{K/2}}{\alpha_i} \right\}.
\]
Then the system is not uniformly left invertible. \( \diamond \)

**Theorem 5.** Fix \( \mathcal{U} \subset \mathbb{R} \). Then the set of \( a \in \mathbb{R} \) of degree at most \( K \) for which ULDI is not equivalent to ULI (in the system (3)) is discrete except for possibly 2 accumulation points given by
\[
|a| = \min_{0 \neq v \in \mathcal{U}} |v| - 1.
\]
Therefore, for any fixed \( \delta > 0 \), the set of \( a \) belonging to
\[
\left\{ a \in \mathbb{R} : \text{a algebraic of degree at most K, } |a| - \left( \min_{0 \neq v \in \mathcal{U}} |v| - 1 \right) > \delta \right\}
\]
is finite.
exists a measure of $u$ where

\begin{align*}
\text{of order 2 (see the proof of Proposition 1) in the difference system given by}
\end{align*}

\begin{align*}
ax_1 + v &= x_2 \\
ax_2 - v &= x_1
\end{align*}

has the solution $x_1 = \frac{-v}{a+1}, x_2 = \frac{v}{a+1}$. As soon as

\begin{align*}
\min_{0 \neq v \in \mathcal{V}} |v| = |a| + 1 \iff |a| = \min_{0 \neq v \in \mathcal{V}} |v| - 1
\end{align*}

this periodic point of order 2 lies on $]-1,1[$ and the system is not ULDI. Moreover,

\begin{align*}
\lim_{|a| \to \infty} \frac{|v|}{|a| + 1} = 0.
\end{align*}

This fact, together with Theorem 3 implies that there exists $M : |a| > M$ implies that the system is not ULI.

Suppose now that for a particular $a \in \mathbb{R}$ the system (2) is ULI but not ULDI. Then it must be

\begin{align*}
a - \varepsilon(a) < \min_{0 \neq v \in \mathcal{V}} |v| - 1 < a.
\end{align*}

Moreover a fixed $\delta$ such that

\begin{align*}
|a - (\min_{0 \neq v \in \mathcal{V}} |v| - 1)| > \delta
\end{align*}

can be supposed to exist (because an accumulation point in $a = \min_{0 \neq v \in \mathcal{V}} |v| - 1$ is not excluded). It’s now easy to see that, once $\delta$ is fixed, there exists a $\delta' = \delta'(a)$ such that

\begin{align*}
|a' - a| < \delta' \Rightarrow a' \text{ does not satisfy (24)}.
\end{align*}

This is simply because, thanks to Theorem 4 the set of algebraic $a'$ of degree at most $K$ such that $\varepsilon(a) < \delta$ is finite. Therefore $\delta'(a)$ can be indeed taken independently of $a$, and Theorem is thus proved. ◊

**Remark 5.** The condition given by equation (23) is not important from a practical point of view, since an infinitesimal change in the quantity $\delta$, the rate of the uniform partition $\mathcal{P}$, is enough to satisfy it. ◊

For $|a| > 2$, even if Theorem 5 doesn’t work, we have the following Theorem, that inductively construct two initial states and two sequences of inputs that give rise to the same output, if a particular inequality (a bit stronger than ULDI) is satisfied.

**Theorem 6.** Suppose that in the system (2) $|a| > 2$. If there exist $u_1, u_2 \in \mathcal{V}, u_1 \neq u_2$ such that $|u_1 - u_2| < |a|$, or equivalently if

\begin{align*}
\min_{0 \neq v \in \mathcal{V}} |v| < |a|,
\end{align*}

then the system is not ULI.

**Proof:** We will consider sequences of sets of type

\begin{align*}
\begin{cases}
S_{i+1} = \{a(S_i) + u(i)\} \cap \{a(S_i) + u'(i)\} \cap \mathcal{P}(i+1) \\
S_0 = [0,1],
\end{cases}
\end{align*}

where $u(i), u'(i) \in \{u_1, u_2\}$ and $\mathcal{P}(i+1) \in \mathcal{P}$ is chosen at each step to maximize the measure of $S_{i+1}$.

In the sequence (25) take $u(1) = u_1, u'(1) = u_2$ and $\mathcal{P}(1)$. Since $|u_1 - u_2| < |a|$, there exists a $\mathcal{P}(1) \in \mathcal{P}$ such that $\mu(S_1) > 0$. Then, for $i > 1$ define

\begin{align*}
u(i) = u'(i) = u_1.
\end{align*}
Since $|a| > 2$ there exists an $i_0$ such that $\mu(S_{i_0}) = 1$, therefore, applying again $u(i_0 + 1) = u_1$ and $u'(i_0 + 1) = u_2$
\[ \mu \{ A(S_{i_0}) + Bu_1 \cap A(S_{i_0}) + Bu_2 \} > 0. \]
So there exists $x_0, x'_0 \in \mathbb{R}$ and $(u(1), \ldots, u(i_0 + 1)), (u'(1), \ldots, u'(i_0 + 1))$, with $u(1) \neq u'(1)$ and $u(i_0 + 1) \neq u'(i_0 + 1)$, such that for the corresponding outputs it holds
\[ (y(0), \ldots, y(i_0 + 1)) = (y'(0), \ldots, y'(i_0 + 1)) \]
It is then enough to point out that, since we can achieve every pair of states $x, x' \in S_{i_0}$ in the above described way, we can again go on in the same way and find a new instant $i_1$, a pair of initial states $x_{1,0}, x'_{1,0}$, and control sequences $(u(1), \ldots, u(i_1)), (u'(1), \ldots, u'(i_1))$, with $u(i_1) \neq u'(i_1)$, such that for the corresponding output it holds
\[ (y(0), \ldots, y(i_1)) = (y'(0), \ldots, y'(i_1)). \]
This contradicts the uniform left invertibility property. $\diamond$

Before giving some examples we observe that our original aim, to show the equivalence between ULDI and ULI, has been reached, modulo cases described in theorem 5. This equivalence is actually stronger than what we showed: indeed we didn’t take into account any influence of input sequences in proofs!

5. Examples

Example 1. Consider the system
\[
\begin{align*}
    x(k+1) &= ax(k) + u(k) \\
    y(k) &= |x(k)| \\
    \mathcal{U} &= \{-M\delta, -(M-1)\delta, \ldots, 0, \ldots, (M-1)\delta, M\delta\},
\end{align*}
\]
where $a, x(k), u(k), \delta > 0 \in \mathbb{R}$, $y(k) \in \mathbb{Z}$, $M \in \mathbb{N}$. Straightforward calculations show that $\mathcal{Y} = \{-2M\delta, -(2M-1)\delta, \ldots, 0, \ldots, (2M-1)\delta, 2M\delta\}$. For any fixed $\delta$, following the proof of Theorem 5, the solutions $a$ of the equation (24) should be studied:
\[ a - \varepsilon(a) < \delta - 1 < a. \]

With regard this example $a$ is supposed to be rational,
\[ a = \frac{p}{q} \quad p > q > 0, \]
because it is possible to exclude the case $|a| < 1$ (that can be solved with methods described in [10]) and because the cases $p < 0, q > 0$ or $p > 0, q < 0$ can be obtained in a similar way. So, calling $\tau = \delta - 1$, suppose $a = \frac{p}{q}$ with $p$ of the form
\[ \lfloor \frac{aq}{q} \rfloor + k, \quad k \geq 1. \]
So equation (24) becomes
\[ \frac{\lfloor \frac{aq}{q} \rfloor + k}{q} - \tau < \frac{1}{\lfloor \frac{aq}{q} \rfloor + k} \Leftrightarrow \]
\[ \frac{k - \text{frac}(\frac{aq}{q})}{q} < \frac{1}{\lfloor \frac{aq}{q} \rfloor + k}. \]
This last equation implies that $k = 1$ (otherwise the first member would be greater than $\frac{1}{q}$ and the second smaller than $\frac{1}{q\tau}$). So it must be

$$1 - \text{frac}(\tau q) < \frac{q}{[\tau q] + 1}.$$  \hspace{1cm} (27)

It’s obvious now that, if $\delta \notin \mathbb{Q}$, since the fractional parts

$$\{\text{frac}(\tau q) : q \in \mathbb{N}\}$$

are dense in $[0, 1]$, there is an infinite set of $q \in \mathbb{N}$ such that (27) is satisfied, and so there exists an infinite set of rational $a$ such that (24) is satisfied, i.e. an infinite set of a such that ULDI is not equivalent to ULI. Therefore the two possible accumulation points given by $|a| = \delta - 1$ are effectively present.

Considering instead only the $a$’s belonging to the set

$$\{a \in \mathbb{Q} : |a - (\delta - 1)| > \theta\},$$

the following is obtained

$$\begin{cases} \tau + \theta < a \\ a - \varepsilon(a) < \tau \end{cases} \Rightarrow \begin{cases} a > \tau + \theta \\ \varepsilon(a) > \delta \end{cases} \Rightarrow \begin{cases} \frac{p}{q} > \tau + \theta \\ \frac{p}{q} > \delta \end{cases}.$$

In this case the set of $a$’s for which ULDI is not equivalent to ULI must be found among the solutions of the latter system, and is clearly finite: this is the set of rationals with numerator $p < \frac{1}{\theta}$ and denominator $q < \frac{\tau + \theta}{\delta}$.

Suppose instead $\tau = \frac{1}{m} \in \mathbb{Q}$. Then (27) becomes

$$1 - \text{frac} \left( \frac{1}{m} q \right) < \frac{q}{[\frac{1}{m} q] + 1}.$$  \hspace{1cm} (27)

In this case note that the left-hand side can assume $m$ possible values $h_i$ for $q$ varying in $\mathbb{N}$, and that the right-hand side tends to $\frac{1}{\tau + 1}$ when $q$ tends to infinity. So, if one of the $h_i$ is $< \frac{1}{\tau + 1}$ there is an infinite set of $a \in \mathbb{Q}$ such that ULDI and ULI are equivalent (there are the two accumulation points), otherwise there is a finite set of $a \in \mathbb{Q}$ (possibly empty) such that ULDI and ULI are equivalent (no accumulation points). ♦

6. Conclusions

In this paper we studied left invertibility of I/O quantized linear systems of dimension 1, and we proved that it is equivalent, except for a finite number of cases (but there is the possibility of having two accumulation points), to left D-invertibility, very easy to detect (Proposition 1). Notice that algebraic conditions play a central role in investigation of left invertibility of quantized systems as well in other fields when a quantization is introduced (see for instance [3, 8]).

Future research will include further investigation on the equivalence between left invertibility and left D-invertibility to higher dimensions.

7. Notations

In this “special” section we collect all the notations used in this paper, ordered as they appear.

1. $\text{frac}(\cdot) : \mathbb{R} \rightarrow \mathbb{Z}$: the function that associates to each real number its fractional part: $\text{frac}(r) = r - \lfloor r \rfloor$;
2. $\mathcal{P}$: uniform partition, Definition 1;
3. $f_{k_2}^{k_1}(x_0, u_1, \ldots, u_{k_1})$: the sequence of outputs $(y_{k_1}, \ldots, y_{k_2})$ generated by the system (2) with initial condition $x_0$ and input string $(u_1, \ldots, u_{k_1})$;
(4) $Q$: the set $\subset \mathbb{R}^2$ containing all pairs of states that are in the same element of the uniform partition $\mathcal{P}$, Definition 4.

(5) $F_{\nu(k)}(X(k))$: the updating map of the 2-dimensional system 3;

(6) $z(k)$: state of the difference system, Definition 5.

(7) $\nu$: $\mathcal{P} \rightarrow \mathcal{P}$;

(8) $D(k)_{\nu}(\{z_0, v_1, \ldots, v_k\})$: the sequence $(\pi_{\nu}(k_1), \ldots, \pi_{\nu}(k_2))$ generated by the difference system with initial condition $z_0$ and input string $(v_1, \ldots, v_k)$;

(9) $C_1$: the image mod. 1 of the linear manifold given by linear relations $L = \{L^1_i\}_i$ (Remark 4);

(10) $R(x)$: generic polynomial, whose roots are $\rho_i$ and degree is $q$ (Definition 13).

(11) $\mathfrak{M}$: Mahler measure (Definition 13).

(12) $\|R\|_{\infty}$: the norm given by the maximum modulus of the coefficients of a polynomial (eq. 6);

(13) $\partial$: topological boundary of a set;

(14) $\mu$: Lebesgue measure of a set;

(15) $Q'$: the "strip" $(x_1, x_2) \in \mathbb{R}^2$ such that $|x_2 - x_1| < 1$, defined in eq. (10);

(16) $\rho_i$: the line defined in equation (12);

(17) $k$: the distance between the point $\left(\frac{c_i}{c_i}, \frac{c_i}{c_i}\right)$, defined in (12), and the union of positive coordinate axes, along the line $\rho_i$ (refer to the Fig. 1);

(18) $\Omega$: the set defined in (14);

(19) $P_a(x) = a_0 + \ldots + a_K x^K$: minimal polynomial of $a$, with coefficients $a_i$ and degree $K$;

(20) $P_a^j\Psi_{j}^K$: the linear manifold and the matrix defined by equations (16) and (17);

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