LIFTING PSEUDO-HOLOMORPHIC POLYGONS TO THE SYMPLECTISATION OF $P \times \mathbb{R}$ AND APPLICATIONS

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ABSTRACT. Let $\mathbb{R} \times (P \times \mathbb{R})$ be the symplectisation of the contactisation of an exact symplectic manifold $P$, and let $\mathbb{R} \times \Lambda$ be a cylinder over a Legendrian submanifold in the contactisation. We show that a pseudo-holomorphic polygon in $P$ having boundary on the projection of $\Lambda$ can be lifted to a pseudo-holomorphic disc in the symplectisation having boundary on $\mathbb{R} \times \Lambda$. It follows that Legendrian contact homology may be equivalently defined by counting either of these discs. Using this result, we give a proof of the isomorphism of the linearised Legendrian contact homology induced by an exact Lagrangian filling and the singular homology of the filling, which is a result first observed by Seidel.

1. INTRODUCTION

Legendrian contact homology is a Legendrian isotopy invariant which was introduced in [EGH] by Eliashberg, Hofer and Givental and in [Che] by Chekanov. It associates a non-commutative free differential graded algebra (DGA for short) to a Legendrian submanifold $\Lambda$ of a contact manifold $Y$. The algebra is generated by the set of Reeb chords on $\Lambda$. Roughly speaking, the differential of the associated DGA counts rigid pseudo-holomorphic discs in a certain symplectic manifold associated to $Y$ endowed with a compatible almost complex structure, where the discs are determined by $\Lambda$ and its Reeb chords. See Section 3 for definitions of the above geometric objects and Section 4 for an introduction to Legendrian contact homology.

The Legendrian contact homology DGA depends on the choice of representative in the Legendrian isotopy class as well as the almost complex structure but its homotopy type has in many cases been shown to be invariant under these choices. It should be pointed out that there are some transversality issues which makes it a non-trivial task to show invariance in the case of a Legendrian submanifold of a general contact manifold.

The version in [Che] was the first rigorous construction of Legendrian contact homology, where it was defined for Legendrian submanifolds of standard contact 3-space

$$(\mathbb{C} \times \mathbb{R}, dz - ydx).$$

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Here $z$ is a coordinate on the $\mathbb{R}$-factor. Legendrian contact homology was later defined for standard contact $(2n + 1)$-space
\[(\mathbb{C}^n \times \mathbb{R}, dz - y_idx^i)\]
in [EES1] by Ekholm, Etnyre and Sullivan, as well as for a general contactisation
\[(P \times \mathbb{R}, dz + \theta)\]
of an exact symplectic manifold $(P, d\theta)$ in [EES3] by the same authors, were $P$ is assumed to satisfy certain technical conditions.

The canonical projection
\[\Pi_{\text{Lag}} : P \times \mathbb{R} \to P\]
of a contactisation is called the \textit{Lagrangian projection}. Observe that the Reeb chords on a Legendrian submanifold $\Lambda \subset (P \times \mathbb{R}, dz + \theta)$ coincide with the double-points of $\Pi_{\text{Lag}}(\Lambda)$, and that we may assume the latter to be a finite set of transverse double-points.

Fix a compatible almost complex structure $J_P$ on $(P, d\theta)$, let $a, b_1, \ldots, b_m$ be double-points of $\Pi_{\text{Lag}}(\Lambda)$ and write $b = b_1 \cdots b_m$. We are interested in the moduli spaces
\[\mathcal{M}_{a:b}(\Pi_{\text{Lag}}(\Lambda); J_P)\]
of $J_P$-holomorphic polygons $u : (D^2, \partial D^2) \to (P, \Pi_{\text{Lag}}(\Lambda))$ having a positive boundary-puncture mapping to the double-point $a$ and negative boundary-punctures mapping to the double-points $b_1, \ldots, b_m$, where the boundary-punctures moreover appear in this cyclic order with respect to the orientation of the boundary. See Section 4.2.1 for more details.

The differential in the above versions of Legendrian contact homology is defined by counting rigid $J_P$-holomorphic polygons, that is, solutions inside a moduli space as above which moreover is zero-dimensional.

The above constructions of Legendrian contact homology obviously depend heavily on the existence of the projection $\Pi_{\text{Lag}}$ and does not work for a general contact manifold, or even for a different choice of contact form for the above contact structure on $P \times \mathbb{R}$.

Following the philosophy of symplectic field theory [EGH], the construction in [Ekh2] can be used to define a version of Legendrian contact homology defined for more general contact manifolds $(Y, \lambda)$, where $\lambda$ satisfies certain technical assumptions. In particular, this construction works when $(Y, \lambda) = (P \times \mathbb{R}, dz + \theta)$.

Fix a cylindrical almost complex structure $J$ on the symplectisation $(\mathbb{R} \times Y, d(e^t \lambda))$ of a contact manifold $(Y, \lambda)$, where $t$ is a coordinate on the $\mathbb{R}$-factor. Let $a, b_1, \ldots, b_m$ be Reeb chords on $\Lambda$ and write $b = b_1 \cdots b_m$. We are interested in the moduli spaces
\[\mathcal{M}_{a:b}(\Lambda; J)\]
of $J$-holomorphic discs $\bar{u} : (D^2, \partial D^2) \to (\mathbb{R} \times (P \times \mathbb{R}), \mathbb{R} \times \Lambda)$ having a positive boundary-puncture asymptotic to the Reeb chord $a$ at $t = +\infty$ and
negative boundary-punctures asymptotic to the Reeb chords $b_1, \ldots, b_m$ at $t = -\infty$, where the boundary punctures moreover appear in this cyclic order with respect to the orientation of the boundary. Observe that, since $J$ is cylindrical, this moduli-spaces has a natural $\mathbb{R}$-action induced by translations of the $t$-coordinate. See Section 1.2.2 for more details.

The latter definition of Legendrian contact homology is equipped with a differential defined by counting non-trivial (that is, not coinciding with a strip $\mathbb{R} \times c$ over a Reeb chord) $J$-holomorphic discs as defined above which moreover are rigid up to translation.

This version of Legendrian contact homology has the advantage that it fits more directly into the algebraic framework of symplectic field theory, which was introduced in [EGH]. Indeed, it was shown in [Ekh2] that an exact Lagrangian cobordism in $\mathbb{R} \times Y$ between Legendrian submanifolds induce DGA-morphisms between the respective DGAs.

Remark 1.1. It should be pointed out that it is possible to obtain results about the DGA-morphisms induced by exact Lagrangian cobordisms by using the version of Legendrian contact homology in [EES3] as well. In [EHK], using the technique of gradient flow trees from [Ekh1], such results where obtained for Legendrian contact homology in the standard contact 3-space.

It is a natural question whether the two versions of Legendrian contact homology defined for a contactisation are equivalent. This has been expected and, indeed, was shown to be true for the standard contact 3-manifold $\mathbb{C} \times \mathbb{R}$ in [ENS, Theorem 7.7]. Also, we refer to [EES2, Section 2.7] for a discussion.

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2. Results

The main result is that, given some not too restrictive assumptions on a compatible almost complex structure $J_P$ on $(P, d\theta)$ and a Legendrian submanifold $\Lambda \subset (P \times \mathbb{R}, dz + \theta)$, rigid $J_P$-holomorphic polygons in $P$ with boundary on $\Pi_{Lag}(\Lambda)$ lift to pseudo-holomorphic discs in the symplectisation with boundary on $\mathbb{R} \times \Lambda$ under the canonical projection to $P$. As a direct consequence, it will follow that the two versions of Legendrian contact homology discussed above are equivalent.

An exact Lagrangian filling (inside the symplectisation) of a Legendrian submanifold $\Lambda \subset \mathbb{R} \times P$ is an exact Lagrangian submanifold $L \subset (\mathbb{R} \times (P \times \mathbb{R}), d(e^t(dz + \theta)))$ which coincides with the cylinder $[N, +\infty) \times \Lambda$ outside of a compact set.
We use the above result for computing the wrapped Floer homology of the pair consisting of an exact Lagrangian filling and a small push-off of itself which, in turn, is used for deducing properties linearised Legendrian contact homology of Λ. In particular, it proves that the linearised Legendrian contact cohomology induced by the filling is isomorphic to the singular homology of the filling, a result first observed by Seidel and proven in [Ekh3] up to some technical details.

2.1. Lifting pseudo-holomorphic polygons to the symplectisation.
A compatible almost complex structure $J_P$ on $P$ lifts to a unique cylindrical (see Section 3.4) almost complex structure $J$ on the symplectisation
$$(\mathbb{R} \times (P \times \mathbb{R}), d(e^t(dz + \theta)))$$
determined by the property that the canonical projection
$$\pi_P: \mathbb{R} \times (P \times \mathbb{R}) \to P,$$
is $(J, J_P)$-holomorphic, that is, $(D\pi_P)J = J_P(D\pi_P)$.

Let $\Lambda \subset P \times \mathbb{R}$ be a fixed chord-generic closed, not necessarily connected, Legendrian submanifold and $J_P$ be a compatible almost complex structure on $P$ which is integrable in a neighbourhood of the double-points of $\Pi_{\text{Lag}}(\Lambda)$. We require $J_P$ to be regular for the moduli spaces $\mathcal{M}_{a:b}(\Pi_{\text{Lag}}(\Lambda); J_P)$ of pseudo-holomorphic discs with boundary on $\Pi_{\text{Lag}}(\Lambda)$ and one positive puncture, that is, chosen so that the latter moduli spaces are transversely cut out.

By [EES3, Proposition 2.3, Lemma 4.5], after approximating $\Lambda$ by a Legendrian submanifold in the same Legendrian isotopy class which moreover is real-analytic in the above holomorphic charts around the double-points (alternatively, after choosing $J_P$ with some care in a neighbourhood of the double-points), $J_P$ can be $C^\infty$-approximated by a regular compatible almost complex structure which still is integrable in some neighbourhood of the double-points.

**Theorem 2.1.** Let $J_P$ be a regular compatible almost complex structure on $(P, d\theta)$ satisfying the above and let $J$ be the cylindrical lift of $J_P$ to the symplectisation of $(P \times \mathbb{R}, dz + \theta)$. It follows that $J$ is regular for the corresponding moduli spaces of $J$-holomorphic discs and that $\pi_P$ induces a diffeomorphism
$$\mathcal{M}_{a:b}(\mathbb{R} \times \Lambda; J)/\mathbb{R} \to \mathcal{M}_{a:b}(\Pi_{\text{Lag}}(\Lambda); J_P),$$
$$\tilde{u} \mapsto \pi_P \circ \tilde{u}.$$ We refer to Section 7 for the proof.

**Remark 2.2.** In the case when
$$(P = \mathbb{C}, \theta = xdy = -(1/2)d(x^2)(J_P), J_P = i),$$
the fact that the above map is a bijection follows from [ENS] Theorem 7.7]. The proof can be generalised to an arbitrary almost Stein manifold, that is
\((P, \theta = -d\alpha(J_P^*, J_P))\) where \(\alpha: P \to \mathbb{R}\) is smooth and strictly \(J_P\)-convex. See Lemma \[7.1\] together with Remark \[7.2\] below.

**Remark 2.3.** We say that an almost complex structure \(J_P\) is *adjusted* to \(\Lambda\) if it is induced by a metric on \(\Lambda\) in a neighbourhood of \(\Pi_{\text{Lag}}(\Lambda) \subset P\), see Section \[3.4\] below for more details. This property is be used to control certain limits of pseudo-holomorphic discs (see \[EES4\], \[Ekh1\]). Observe that we may assume that \(J_P\) in the above theorem is adjusted to \(\Pi_{\text{Lag}}(\Lambda)\).

By the above theorem, the \(J_P\)-holomorphic polygons in the definition of the Legendrian contact homology differential given in \[EES3\] correspond bijectively to the \(J\)-holomorphic discs in the definition of the differential given in \[Ekh3\]. The following corollary thus immediately follows.

**Corollary 2.4.** Let \(\Lambda \subset P \times \mathbb{R}\) be a Legendrian submanifold of a contactisation. There are choices of regular compatible almost complex structures for which the differentials as defined in \[EES3\] and \[EGH\] are equal.

2.2. **Applications.** An *augmentation* of a freely generated DGA \((A^\bullet, \partial)\) over \(\mathbb{Z}_2\) is a unital DGA-morphism

\[\epsilon: (A^\bullet, \partial) \to (\mathbb{Z}_2, 0).\]

Augmentations can be used to define the so called linearisation of the DGA, which is a complex spanned by the generators over \(\mathbb{Z}_2\). See Section \[4.4\] for more details. If the Legendrian contact homology DGA has an augmentation, we use \((CL^\bullet(\Lambda) = \mathbb{Z}_2(\mathcal{Q}), d_\epsilon)\) to denote the corresponding linearised co-complex, where \(\mathcal{Q}(\Lambda)\) denotes the set of Reeb chords on \(\Lambda\), and use \(HCL^\bullet(\Lambda; \epsilon)\) to denote the cohomology.

Observe that the cohomology depends on the choice of augmentation, but that it still can be used as a Legendrian isotopy invariant. The invariance result in Theorem \[4.2\] shows that Legendrian isotopies act on the sets of augmentations while preserving the homotopy class of the corresponding linearisations.

It is shown in \[Ekh3\] that, according with the principles of symplectic field theory, an exact Lagrangian filling of \(\Lambda\) induces an augmentation of its Legendrian contact homology DGA. For augmentations arising this way, there are some strong consequences for the corresponding linearised Legendrian contact cohomology. Namely, by using Theorem \[2.1\] we prove Theorem \[6.2\] of which the following is a direct consequence.

**Theorem 2.5** (Seidel, Conjecture 1.2 in \[Ekh3\]). Let \(\Lambda \subset P \times \mathbb{R}\) be a closed Legendrian \(n\)-dimensional submanifold which has an exact Lagrangian filling \(L \subset \mathbb{R} \times (P \times \mathbb{R})\) inside the symplectisation. It follows that there is an isomorphism

\[\delta: \text{H}_{n-\bullet}(L) \to HCL^\bullet(\Lambda; \epsilon),\]

where \(\epsilon\) is the augmentation induced by the filling.
The proof, the idea of which goes back to Seidel, was given in [Ekh2] but there depends on the conjectural [Ekh3, Lemma 4.11]. The idea is to use a relation between the wrapped Floer homology of an exact Lagrangian filling and the linearised Legendrian contact cohomology of its end (also, see [Abo]), together with the observation that wrapped Floer homology vanishes inside symplectisations of contactisations (see Proposition 5.12).

Wrapped Floer homology is a version of Lagrangian intersection Floer homology, originally defined in [Flo] by Floer, generalised to non-compact exact Lagrangian submanifold inside a Liouville domain. It first appeared in the literature in [AS1]. Different versions have later been developed in [AS2], [Abo], [FSS] and [Ekh3]. Also, see Section 5.

The following consequence of [Ekh3, Conjecture 1.2] was also shown in [Ekh3], where the proof again depends on the the conjectural [Ekh3, Lemma 4.11]. However, it can be seen that the needed results follow from Theorem 6.2 below, from which the algebraic consequences of the conjectural lemma follow. See Remark 6.3 for a further discussion. We thus get.

**Corollary 2.6** (Corollary 1.3 in [Ekh3]). Let $\Lambda \subset P \times \mathbb{R}$ be a closed Legendrian $n$-dimensional submanifold which has an exact Lagrangian filling $L \subset \mathbb{R} \times (P \times \mathbb{R})$. The below diagram is commutative, where the horizontal sequences are exact, the upper sequence is the standard long exact sequence for the homology of a pair, and where the vertical arrows are isomorphism.

$$
\begin{array}{ccc}
H_{k+1}(\Lambda) & \longrightarrow & H_{k+1}(L) & \longrightarrow & H_{k+1}(L, \Lambda) & \longrightarrow & H_k(\Lambda) \\
\downarrow \text{id} & & \downarrow \delta & & \downarrow H^{-1}\delta' & & \downarrow \text{id} \\
H_{k+1}(\Lambda) & \longrightarrow & HCL_{n-k-1}(\Lambda; \epsilon) & \longrightarrow & HCL_k(\Lambda; \epsilon) & \longrightarrow & H_k(\Lambda)
\end{array}
$$

Here $H := (\eta \sigma)$, $\delta = (g \sigma)$, $\delta' := (\gamma g)^t$ where we refer to Section 6.1 for the definition of $\rho$, $\eta$, and $\sigma$, and to Theorem 6.2 for the definition of $g$ and $\gamma$.

Here the long exact sequence at the bottom is as constructed in [EES4], where Proposition 6.13 shows that these results apply under the given assumptions. Theorem 6.1 has then been used to translate these results to the formulation of Legendrian contact homology using the symplectisation.

Finally we note that result analogous to Theorem 2.5 and Corollary 2.6 for the generating family homology have been obtained in [ST] under the assumption that the Legendrian submanifold and its filling posses generating families.

### 3. General definitions

**3.1. Symplectic and contact manifolds.** A contact manifold $(Y, \xi)$ is a smooth $(2n + 1)$-dimensional manifold $Y$ together with a maximally non-integrable field of tangent hyperplanes $\xi$. We will consider the case when
\( \xi = \ker \lambda \) for a fixed one-form \( \lambda \), the so-called contact form. Maximal non-integrability in this case means that the equation 
\[ \lambda \wedge (d\lambda)^n \neq 0 \]
is satisfied pointwise.

An \( n \)-dimensional submanifold \( \Lambda \subset (Y, \xi) \) of a \( (2n+1) \)-dimensional contact manifold is called Legendrian if it is tangent to \( \xi \). Two Legendrian submanifolds are Legendrian isotopic if they are smoothly isotopic through Legendrian submanifolds. Determining Legendrian isotopy classes is an important, but subtle, question in contact geometry.

A choice of a contact form \( \lambda \) on \( Y \) determines the so-called Reeb vector field \( R \) by
\[ \lambda(R) = 1, \quad \iota_R d\lambda = 0. \]

An integral-curve of \( R \) starting and ending on two different sheets of \( \Lambda \) is called a Reeb chord.

A symplectic manifold \( (X, \omega) \) is an even-dimensional manifold together with a closed non-degenerate two-form. An \( n \)-dimensional submanifold \( L \subset (X, \omega) \) of a \( 2n \)-dimensional symplectic manifold is called Lagrangian if \( \omega|_{TL} = 0 \).

We say that the symplectic manifold is exact if \( \omega \) is exact. Observe that an exact symplectic manifold never is closed. An immersion of an \( n \)-dimensional manifold \( L \) into an exact symplectic \( 2n \)-manifold \( (X, d\theta) \) is called exact Lagrangian if the pull-back of \( \theta \) to \( L \) is an exact one-form.

The contactisation of an exact symplectic manifold \( (P, d\theta) \) is the contact manifold
\[ (P \times \mathbb{R}, dz + \theta), \]
where \( z \) is a coordinate on the \( \mathbb{R} \)-factor. For this choice of contact form, the Reeb vector-field is given by \( R = \partial_z \).

The canonical projection
\[ \Pi_{\text{Lag}} : P \times \mathbb{R} \to P \]
is called the Lagrangian projection and it is easily checked that if \( \Lambda \subset P \times \mathbb{R} \) is Legendrian, then \( \Pi_{\text{Lag}}(\Lambda) \subset (P, d\theta) \) is an exact Lagrangian immersion. Observe that the double-points of \( \Pi_{\text{Lag}}(\Lambda) \) correspond bijectively to the Reeb chords on \( \Lambda \).

**Example 3.1.** The archetypical example of a contactisation the 1-jet space
\[ (J^1(M) = T^*M \times \mathbb{R}, dz + \theta_M), \]
where \( \theta_M \) is minus the canonical one-form (also often called the Liouville form). The canonical one-form is given by \( y_idx^i \) in local canonical coordinates on \( T^*M \), i.e. in coordinates on the form \( (x_i, y_idx_i) \) for some choice of local coordinates \( x_i \) on \( M \). Specializing to the case \( M = \mathbb{R}^n \) we obtain the standard contact \( (2n+1) \)-space.
3.2. Hamiltonian isotopies. A smooth time-dependent Hamiltonian $H_s : X \to \mathbb{R}$ on a symplectic manifold $(X, \omega)$ gives rise to the corresponding Hamiltonian vector-field $X_{H_s}$ on $X$, which is determined by

$$\iota_{X_{H_s}} \omega = dH_s.$$ 

We denote the induced one-parameter flow by $\mathbb{R} \times X \to X$, $(s, x) \mapsto \varphi^s_H(x)$, which can be seen to preserve the symplectic form. A Hamiltonian isotopy is an isotopy of a symplectic manifold induced by a Hamiltonian as above. It follows by Weinstein’s Lagrangian neighbourhood theorem that any one-parameter family of exact Lagrangian embeddings can be realised by a time-dependent Hamiltonian isotopy of the ambient symplectic manifold.

3.3. Exact Lagrangian cobordisms and fillings. The symplectisation of a contact manifold $(Y, \lambda)$ is the exact symplectic manifold $(\mathbb{R} \times Y, d(e^t \lambda))$, where $t$ is a coordinate on the $\mathbb{R}$-factor. It is easily checked that $\Lambda \subset (Y, \lambda)$ is Legendrian if and only if the cylinder $\mathbb{R} \times \Lambda \subset \mathbb{R} \times Y$ is exact Lagrangian.

An exact Lagrangian cobordism $L$ (inside the symplectisation) from the Legendrian submanifold $\Lambda_-$ to $\Lambda_+$ is an exact Lagrangian submanifold in $\mathbb{R} \times Y$ which, for some $N \gg 0$, coincides with

$$((-\infty, -N) \times \Lambda_-) \cup ((N, +\infty) \times \Lambda_+)$$

outside of a compact set. In the case $\Lambda_- = \emptyset$, we say that $L$ is an exact Lagrangian filling of $\Lambda_+$.

Suppose that $V$ is an exact Lagrangian cobordism from $\Lambda_-$ to $\Lambda_0$ and that $W$ is an exact Lagrangian cobordism from $\Lambda_0$ to $\Lambda_+$. We also assume that $V$ and $W$ have been translated appropriately in the $t$-direction, in order for

$$\{-1 \leq t\} \cap V = [-1, +\infty) \times \Lambda_0,$$

$$\{t \leq 1\} \cap W = (-\infty, 1] \times \Lambda_0,$$

to hold. We may then define the concatenation of $V$ and $W$ to be

$$V \odot W := (\{t \leq 0\} \cap V) \cup (\{0 \leq t\} \cap W),$$

which is an Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$. In the case when $\Lambda_0$ is connected, it follows that the concatenation is exact as well.

Consider the translation

$$\tau_s : \mathbb{R} \times Y \to \mathbb{R} \times Y,$$

$$\tau_s(t, y) = (s + t, y)$$
For each $s \geq 0$ we also define the concatenation

$$V \circ_s W := (\{t \leq 0\} \cap V) \cup (\{0 \leq t\} \cap \tau_s(W)),$$

which by construction is cylindrical in the set $\{-1 \leq t \leq 1 + s\}$.

Observe that all the concatenations $V \circ_s W$ are Hamiltonian isotopic.

3.4. Almost complex structures. An almost complex structure $J$ on $X$ is a bundle-endomorphism of $TX$ satisfying $J^2 = -\text{id}$. We say that an almost complex structure $J$ on a symplectic manifold $(X,\omega)$ is compatible with the symplectic form if $\omega(\cdot,J\cdot)$ is a Riemannian metric on $X$. It is well-known that the space of compatible almost complex structures is a non-empty and contractible space.

An almost complex structure on the symplectisation $\mathbb{R} \times Y$ of $(Y,\lambda)$ is said to be cylindrical if it is compatible with the symplectic form, invariant under translations of the $t$-coordinate, satisfies $J\partial_t = R$, and preserves the contact-planes $\ker \lambda \subset TY$.

Let $\iota: L \to (X,\omega)$ be a Lagrangian immersion with transverse double-points. A compatible almost complex structure on $(X,\omega)$ is said to be adjusted to $\iota(L)$ if it can be obtained by the following construction. Using the Weinstein Lagrangian neighbourhood theorem, the immersion $\iota$ can be extended to a symplectic immersion $\tilde{\iota}: (D^*L,d\theta) \to (X,\omega)$ of the co-disc bundle $(D^*L,d\theta) \subset (T^*L,d\theta)$ having fibres of sufficiently small radius such that moreover $\tilde{\iota}|_L = \iota$.

Again, using the Weinstein Lagrangian neighbourhood theorem, it can be shown that each double-point $q \in \Pi_{\text{Log}}(\Lambda)$ has a neighbourhood which can be symplectically identified with $(D^n \times D^n, \epsilon \sum_{i=1}^n dx_i \wedge dy_i)$ by a symplectic embedding $\phi(x,y)$, such that moreover the two intersecting sheets are given by $\phi(x,0) \cup \phi(0,y)$. Here $D^n$ denotes the unit disc and $x_i$ and $y_i$ are the standard coordinates on the first and second factor, respectively.

We may furthermore assume that there are neighbourhoods $U_{p_i} \subset L$, $i = 1, 2$, of the pre-image points $p_1, p_2 \in L$ of $q$ for which the restrictions of $\tilde{\iota}$ to $(D^*U_{p_i},\theta_{U_{p_i}}) \subset (D^*L,\theta_L)$, $i = 1, 2$, are given by $\phi(q_1,p_1)$ and $\phi(-p_2,q_2)$, respectively, relative suitable canonical coordinates $(q_i,p_i)$ on $D^*U_{p_i}$ induced by the coordinates $q_i$ on $U_{p_i}$.

Fix a metric $g$ on $L$ which is required to be the flat metric with respect to the above local coordinates $q_i$ on $U_{p_i}$, $i = 1, 2$. The metric $g$ and the symplectic immersion $\tilde{\iota}$ constructed above determine a compatible almost complex structure $J$ defined in a neighbourhood of $\iota(L)$ as follows. The metric $g$ induces the Levi-Civita connection on the cotangent bundle $\pi: T^*L \to L$ which, in turn, determines the horizontal subbundle $H_x \subset T_x T^*L$ for every $x \in L$. The almost complex structure $J$ is then defined by $J_x = \pi_*^{-1}(J \circ \pi)$ on $H_x$. This construction is compatible with the symplectic form and invariant under translations of the $t$-coordinate, as required.
The horizontal subbundle is a complement to the corresponding vertical subbundle \( V_x := \ker(T\pi)_x \). There are canonical identifications \( H_x \cong T_{\pi(x)}L \) and \( V_x \cong T^*_{\pi(x)}L \).

We require that \( JH_x = V_x \) and that, for the horizontal vector identified with \( h \in T_{\pi(x)}L \), the vertical vector \( Jh \) is identified with the covector \( g(h, \cdot) \in T^*_{\pi(x)}L \). We refer to [EES4, Remark 6.1] for an expression of this almost complex structure in local coordinates. Finally, observe that in a neighbourhood of the double-points, the almost complex structure is standard with respect to the above coordinates \((x, y)\).

4. Background on Legendrian contact homology

In the following, we assume that \( \Lambda \subset P \times \mathbb{R} \) is a closed Legendrian submanifold which is chord generic, that is, \( \Pi_{\text{Lag}}(\Lambda) \) is a generic immersion whose self-intersections thus consist of transverse double-points. Observe that this always can be assumed to hold after an arbitrarily \( C^\infty \)-small Legendrian isotopy. In particular, it follows that the set \( Q(\Lambda) \) of Reeb chords on \( \Lambda \) is finite.

We refer to [EES3] for a discussion about the conditions on \((P, d\theta)\), as well as on the behaviour of the involved almost complex structures outside of a compact set, in order for Legendrian contact homology to be well-defined and invariant.

4.1. The grading. To each Reeb chord \( c \) on \( \Lambda \) we associate a grading

\[
|c| = CZ(\Gamma_c) - 1,
\]

where \( \Gamma_c \) is a path of Lagrangian tangent-planes in \( \mathbb{C}^n \) associated to \( c \) and where \( CZ \) denote the Conley-Zehnder index as defined in [EES1]. The path \( \Gamma_c \) is obtained as follows.

One says that a Reeb chord is pure if its endpoints lie on the same component of \( \Lambda \) and, otherwise, one says that it is mixed.

In the case when \( c \) is a pure Reeb chord we let \( \Gamma_c \) be the tangent-planes of \( \Pi_{\text{Lag}}(\Lambda) \) along the choice of a capping path \( \gamma_c \) of \( c \), which is a continuous path on \( \Lambda \) with starting point (respectively end point) at the end point (respectively starting point) of \( c \). Furthermore, we assume that \( \Pi_{\text{Lag}}(\gamma_c) \) is null-homologous in \( P \) and we choose a symplectic trivialization of \( TP \) along \( \Pi_{\text{Lag}}(\gamma_c) \) induced by a chain bounding \( \Pi_{\text{Lag}}(\gamma_c) \).

This construction provides a well-defined grading of the pure Reeb chords modulo the Maslov number of \( \Lambda \) and twice the Chern number of \( P \). For simplicity, we will in the following assume both to be zero.

For a mixed Reeb chord there is obviously no capping path in the above sense. Instead, we proceed as follows. Fix points \( p, q \) on two different components of \( \Lambda \), together with a path \( \gamma \) in \( P \) connecting \( p \) and \( q \). Also fix a choice of a path of Lagrangian planes in \( TP \) along \( \gamma \) starting at \( T_p \Pi_{\text{Lag}}(\Lambda) \) and ending at \( T_q \Pi_{\text{Lag}}(\Lambda) \).
For each mixed Reeb chord $c$ from the component containing $p$ to the component containing $q$, we construct a capping path by choosing a path on $\Lambda$ from the end-point of the Reeb chord to $p$, followed by $\gamma$, and finally followed by a choice of path on $\Lambda$ from $q$ to the starting point of the Reeb chord. Joining the corresponding paths of Lagrangian tangent-planes of $\Pi_{\text{Lag}}(\Lambda)$ with the above choices of Lagrangian planes along $\gamma$, we get the desired path $\Gamma_c$.

It should be pointed out that the grading of the mixed Reeb chords starting and ending at two fixed components of $\Lambda$ depend on the choice of path of Lagrangian planes along the curve $\gamma$ in the above construction, but that the difference in grading of two mixed Reeb chords between the same components is independent of this choice.

4.2. The relevant moduli spaces. The differential in Legendrian contact homology is defined by a count of certain pseudo-holomorphic discs. We begin with the definitions of the moduli spaces that contain these discs.

4.2.1. The moduli spaces of pseudo-holomorphic polygons. Fix a compatible almost complex structure $J_P$ on $P$. Given double-points $a, b_1, \ldots, b_m$ of $\Pi_{\text{Lag}}(\Lambda)$, writing $b := b_1 \cdot \ldots \cdot b_m$, we let

$$\mathcal{M}_{a:b}(\Pi_{\text{Lag}}(\Lambda); J_P),$$

denote the moduli space of continuous maps

$$u: (D^2, \partial D^2) \to (P, \Pi_{\text{Lag}}(\Lambda)),$$

which are smooth in the interior where they moreover satisfy

$$\overline{\partial}_{J_P}(u) := du + J_P \circ du \circ i = 0.$$

Furthermore, we require there to be $m+1$ distinct boundary points $p_0, \ldots, p_m$, appearing in this cyclic order relative the boundary orientation, such that the following holds.

- The map $u|_{\partial D^2 \setminus \{p_i\}}$ has a continuous lift to $\Lambda$ under $\Pi_{\text{Lag}}$.
- $u$ maps $p_0$ to $a$ and the $z$-coordinate of the above lift of $u|_{\partial D^2 \setminus \{p_i\}}$ is required to make a positive jump when traversing $p_0$ in positive direction according to the boundary orientation.
- $u$ maps $p_i$ to $b_i$ for $i > 0$ and the $z$-coordinate of the above lift of $u|_{\partial D^2 \setminus \{p_i\}}$ is required to make a negative jump when traversing $p_i$ in positive direction according to the boundary orientation.

Finally, two solutions are identified if they differ by a biholomorphism of the domain. We refer to $p_0$ as a positive boundary puncture, and $p_i, i > 0$, as negative boundary punctures. We will call the above discs pseudo-holomorphic polygons.
4.2.2. The moduli spaces of pseudo-holomorphic discs with strip-like ends in the symplectisation. Fix a compatible almost complex structure $J$ on 
$$(\mathbb{R} \times (P \times \mathbb{R}), e^\ell (dz + \theta))$$
which we moreover required to be cylindrical. Given Reeb chords $a, b_1, \ldots, b_m$ on $\Lambda$, writing $b := b_1 \cdot \ldots \cdot b_m$, we let $\mathcal{M}_{a; b}(\mathbb{R} \times \Lambda; J)$, denote the moduli space of continuous maps
$$\tilde{u}: (\tilde{D}^2, \partial \tilde{D}^2) \to (\mathbb{R} \times (P \times \mathbb{R}), \mathbb{R} \times \Lambda)$$
which are smooth in the interior where they moreover satisfy
$$\partial J(\tilde{u}) := d\tilde{u} + J \circ d\tilde{u} \circ i = 0.$$
Here $\tilde{D}^2 = D^2 \setminus \{p_0, \ldots, p_m\}$ for some choice of $m + 1$ distinct points $p_0, \ldots, p_m \in \partial D^2$, appearing in this cyclic order relative the boundary orientation. We moreover require the following. Let $\gamma_c(t): [0, 1] \rightarrow P \times \mathbb{R}$ be a parametrization of a Reeb chord $c$ on $\Lambda$ for which $\gamma'_c(t) = C \partial_z$ is constant and $C > 0$. Fix a metric $g$ on $P \times \mathbb{R}$ and consider the induced product-metric $dt \otimes dt + g$ on the symplectisation.

- Let $s + it$ be coordinates on $D^2$ induced by a biholomorphic identification of $D^2$ with the strip $\{s + it; 0 \leq t \leq 1\} \subset \mathbb{C}$ such that $p_0$ corresponds to $s = +\infty$. We require that there are $s_0 \in \mathbb{R}$ and $\theta_0 > 0$ for which
  $$\|\tilde{u}(s + it) - (s + s_0, \gamma_a(t))\| \leq e^{-\theta_0 s}$$
  relative the above metric for $s \gg 0$ large enough.

- Let $s + it$ be coordinates on $D^2$ induced by a biholomorphic identification of $D^2$ with the strip $\{s + it; 0 \leq t \leq 1\} \subset \mathbb{C}$ such that $p_i$ corresponds to $s = -\infty$. We require that there are $s_0 \in \mathbb{R}$ and $\theta_0 > 0$ for which
  $$\|\tilde{u}(s + it) - (s + s_0, \gamma_b(t))\| \leq e^{-\theta_0 s}$$
  relative the above metric for $s \ll 0$ small enough.

Finally, two solutions are identified if they differ by a biholomorphism of the domain. We refer to $p_0$ as a positive boundary puncture, and $p_i, i > 0$, as negative boundary punctures.

Observe that, since $J$ is cylindrical, these moduli-spaces have natural $\mathbb{R}$-actions induced by translation of the $t$-coordinate.

4.2.3. Energies. We will use the following notions of energies for the pseudo-holomorphic discs involved. For a disc $u: D^2 \rightarrow P$, we define its symplectic area, also called $d\theta$-energy, by
$$E_{d\theta}(u) := \int_u d\theta.$$
For a $J$-holomorphic disc $\tilde{u}: D^2 \to \mathbb{R} \times (P \times \mathbb{R})$, where $J$ is cylindrical, we define its $d\theta$ and $\lambda$-energy by

$$E_{d\theta}(\tilde{u}) := \int_{\tilde{u}} d\theta,$$

$$E_\lambda(\tilde{u}) := \sup_{\rho \in \mathcal{C}} \int_{\tilde{u}} \rho(t)dt \wedge (dz + \theta),$$

respectively, where $\mathcal{C}$ is the set of smooth functions $\rho: \mathbb{R} \to \mathbb{R}_{\geq 0}$ having compact support and satisfying $\int_{\mathbb{R}} \rho(t)dt = 1$.

The above energies are finite for the solutions in the above moduli-spaces. If one assigns the action

$$\ell(a) := \int_a (dz + \theta)$$

to a Reeb chord $a$, pseudo-holomorphic discs $u$ and $\tilde{u}$ as above having a positive puncture at $a$ and negative punctures at $b_1, \ldots, b_m$ can be seen to have energies given by

$$0 \leq E_{d\theta}(u) = E_{d\theta}(\tilde{u}) = \ell(a) - (\ell(b_1) + \ldots + \ell(b_m)),$$

$$0 < E_\lambda(\tilde{u}) = \ell(a).$$

Observe that the $d\theta$-energy of a pseudo-holomorphic disc in $P$ vanishes if and only if the disc is constant, while the $d\theta$-energy of a pseudo-holomorphic disc in the symplectisation vanishes if and only if it is contained entirely in a cylinder $\mathbb{R} \times c$ over a Reeb chord $c$.

4.2.4. Dimension formulae. We call an almost complex structure regular if the appropriate moduli spaces are transversely cut out, and hence are smooth finite-dimensional manifolds. Since the pseudo-holomorphic discs in the above moduli spaces have only one positive puncture, the regular almost complex structures form a Baire set according to [EES3] and [Ekh2].

When the almost complex structure is regular, the dimensions of the above moduli spaces satisfy

$$|a| - |b_1| - \ldots - |b_m| = \dim \mathcal{M}_{a;b}(\mathbb{R} \times \Lambda; J)$$

$$= \dim \mathcal{M}_{a;b}(\Pi_{Log}(\Lambda); J_P) + 1.$$ 

In the case when $J$ is cylindrical, the extra degree of freedom in the moduli space of $J$-holomorphic discs in the symplectisation should be thought of as coming from the action by translation of the $t$-coordinate.

4.3. The Legendrian contact homology DGA. Consider the unital graded $\mathbb{Z}_2$ algebra $A_\bullet(\Lambda)$ freely generated by the Reeb chord on $\Lambda$ with grading determined by the Conley-Zehnder index as above. The Legendrian contact homology DGA of $\Lambda$ is $(A_\bullet(\Lambda), \partial)$, where the differential $\partial$ is defined as follows.
Let $a$ be a Reeb chord generator of the algebra. The differential given in \[EES3\] is defined by

$$\partial(a) := \sum_{|a| - |b| = 1} |\mathcal{M}_{a;b}(\Pi_{\text{Lag}}(\Lambda); J_P)| b$$

for some choice of regular compatible almost complex structure $J_P$ on $P$ while the differential given in \[Ekh2\] is defined by

$$\partial(a) := \sum_{|a| - |b| = 1} |\mathcal{M}_{a;b}(\mathbb{R} \times \Lambda; J)| b$$

for some choice of regular cylindrical almost complex structure $J$ on $\mathbb{R} \times (P \times \mathbb{R})$. Observe that the dimension formula implies that the sum is taken over zero-dimensional moduli spaces. Together with the Gromov-Hofer compactness in \[EES3\] and \[BEH^+\], it follows that the above counts make sense.

In both of the above cases the differential is extended to the whole algebra via the Leibniz rule

$$\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b),$$

and it follows that $\partial$ is of degree $-1$. Moreover, the above formula for the $d\theta$-energy implies that the differential is action-decreasing, and hence that there is an induced filtration of the complex $\mathcal{A}^\bullet$ by the action $\ell$.

We now refer to the invariance results for the above two versions of Legendrian contact homology.

**Theorem 4.1** (Theorem 1.1 in \[EES3\], \[Ekh2\]). Consider a closed Legendrian submanifold $\Lambda \subset P \times \mathbb{R}$ of the contactisation of a Liouville domain. The following holds for either of the above two definitions of the boundary operator $\partial$.

- $\partial^2 = 0$.
- The homotopy type of $(\mathcal{A}^\bullet(\Lambda), \partial)$ is independent of the choice of a regular compactible almost complex structure and Legendrian isotopy class.

**4.4. Linearised Legendrian contact homology.** The results in \[Ekh2\] Lemma 3.15 and \[Ekh2\] Section 4 show that an exact Lagrangian cobordism $V$ from $\Lambda_-$ to $\Lambda_+$ induces a unital DGA-morphism

$$\Phi_V : (\mathcal{A}(\Lambda_+), \partial_+) \to (\mathcal{A}(\Lambda_-), \partial_-)$$

defined by counting rigid $J$-holomorphic discs in $\mathbb{R} \times (P \times \mathbb{R})$ having boundary on $L$ and boundary-punctures asymptotic to Reeb chords. Here we require $J$ to coincide with the cylindrical almost complex structures $J_{\pm\infty}$ used in the definition of the above DGAs on sets $\{N \leq t\}$ and $\{t \leq N\}$, respectively, for $N \gg 0$ sufficiently large.

In particular, as also is shown in \[Ekh3\], an exact Lagrangian filling $L$ of $\Lambda$ together with an appropriate choice of almost complex structure induces
a unital DGA-morphism
\[ \epsilon_L : (A(\Lambda), \partial) \to (\mathbb{Z}_2, 0), \]
where the right-hand side is considered as the trivial DGA. In general, a unital DGA-morphism to \((\mathbb{Z}_2, 0)\) is called an augmentation.

Given an augmentation \(\epsilon\) of a semi-free DGA \((A_\bullet, \partial)\), one can construct the following chain complex. Define an algebra automorphism \(\Psi^\epsilon\) of \(A_\bullet\) by prescribing \(\Psi^\epsilon(a) = a + \epsilon(a)\) for each generator \(a\). It follows that the constant part of \(\Psi^\epsilon \circ \partial\) vanishes and, consequently, its linear part
\[ \partial_\epsilon := (\Psi^\epsilon \circ \partial)^1 \]
is itself a differential of the graded \(\mathbb{Z}_2\)-vector space \(A^1_\bullet\) spanned by the generators of \(A_\bullet\). We call the chain complex \((A^1_\bullet, \partial_\epsilon)\) the linearisation of the DGA induced by \(\epsilon\).

Any augmentation of \((A, \partial)\) pulls back under a unital DGA-morphism \(\Phi : (A', \partial') \to (A, \partial)\) to an augmentation \(\epsilon' := \epsilon \circ \Phi\) of \((A', \partial')\). Moreover, the map \(\Psi^\epsilon \circ \Phi \circ (\Psi^{\epsilon'})^{-1}\) has vanishing constant part. We denote its linear part by
\[ \Phi_\epsilon := (\Psi^\epsilon \circ \Phi \circ (\Psi^{\epsilon'})^{-1})^1 : (A'^1, \partial'_\epsilon) \to (A^1, \partial_\epsilon), \]
which can be seen to be a chain-map between the corresponding linearisations.

If the DGA of a Legendrian \(\Lambda\) has an augmentation \(\epsilon\), we denote the corresponding linearisation by
\[ (CL_\bullet(\Lambda), \partial_\epsilon) := (A^1_\bullet(\Lambda) = \mathbb{Z}_2\langle Q(\Lambda) \rangle, \partial_\epsilon), \]
the so called linearised Legendrian contact homology complex, and we use \(HCL_\bullet(\Lambda; \epsilon)\) to denote its homology. We will also use \((CL^\bullet(\Lambda), d_\epsilon)\) to denote the associated co-complex, and \(HCL^\bullet(\Lambda; \epsilon)\) to denote the corresponding cohomology.

The homotopy type of this complex does indeed depend on the choice of augmentation. However, \cite[Theorem 5.2]{Che} shows that the set of all linearised homologies is a Legendrian isotopy invariant, the proof of which uses the invariance result established for the version of Legendrian contact homology defined in terms of pseudo-holomorphic polygons in \(P\).

For the version of Legendrian contact homology defined in terms of pseudo-holomorphic discs in the symplectisation one proceeds as follows. Suppose \(\Lambda'\) is Legendrian isotopic to \(\Lambda\). A choice of Legendrian isotopy induces an exact Lagrangian cobordism \(V\) from \(\Lambda\) to \(\Lambda'\) which is diffeomorphic to a cylinder, as is shown in \cite[Theorem 3.4]{Cha}. Suppose that the DGA of \(\Lambda\) has an augmentation \(\epsilon\). Let
\[ \Phi_V : (A_\bullet(\Lambda'), \partial') \to (A_\bullet(\Lambda), \partial) \]
be the unital DGA-morphism induced by \(V\), where the almost complex structure has been appropriately chosen. In this setting, we have the following invariance result for the linearised Legendrian contact homology.
Theorem 4.2 ([Ekh2], Theorem 1.1 in [Ekh3]). Let $\Lambda \subset P \times \mathbb{R}$ be a closed Legendrian submanifold whose DGA has an augmentation $\epsilon$. If $\Lambda'$ is Legendrian isotopic to $\Lambda$ and if the isotopy induces the exact Lagrangian cylinder $V$, it follows that

$$\Phi_V: (CL_\bullet(\Lambda'), \partial'_{\epsilon \circ \Phi_V}) \to (CL_\bullet(\Lambda), \partial_\epsilon)$$

is a homotopy equivalence. Furthermore, if $\epsilon = \epsilon_L$ is induced by an exact Lagrangian filling $L$, then

- Up to an DGA-automorphism, the augmentation $\epsilon_L$ is invariant under compactly supported Hamiltonian isotopy as well as of compactly supported deformations of the almost complex structure. In particular, the isomorphism class of the complex $(CL_\bullet(\Lambda), \partial_\epsilon)$ is invariant under these choices.
- For appropriate choices of almost complex structures, $\epsilon \circ \Phi_V = \epsilon_L \circ V$ is induced by the filling $L \circ V$.

The last result above follows by a standard stretching-of-the-neck argument, see [BEH+]. The other results are algebraic consequences of the abstract perturbation constructed in [Ekh2, Appendix B] and can be proved similarly to [Ekh3, Lemma 4.4]. More precisely, they are algebraic consequences of more general results proved for the relative SFT introduced in [Ekh2, in particular see [Ekh2, Lemma B.15]]. Also, we refer to [Bon, Theorem 2.8] for similar calculations in the non-relative case. For completeness, we here outline the main steps.

Proposition 4.3. Let $V$ and $W$ be exact Lagrangian cobordisms from $\Lambda$ to $\Lambda'$ which are related by a compactly supported Hamiltonian isotopy and let $\epsilon$ be an augmentation of the DGA of $\Lambda$. Consider the two chain-maps

$$\Phi_V: (CL_\bullet(\Lambda'), \partial'_{\epsilon \circ \Phi_V}) \to (CL_\bullet(\Lambda), \partial_\epsilon),$$

$$\Phi_W: (CL_\bullet(\Lambda'), \partial'_{\epsilon \circ \Phi_W}) \to (CL_\bullet(\Lambda), \partial_\epsilon),$$

for two choices almost complex structures coinciding on the ends. There is an isomorphism of DGAs

$$\Psi: (A_\bullet(\Lambda'), \partial') \to (A_\bullet(\Lambda'), \partial')$$

satisfying

- $(\epsilon \circ \Phi_V) \circ \Psi = \epsilon \circ \Phi_W$.
- $(\Phi_V)_{\epsilon} \simeq (\Phi_W)_{\epsilon} \circ \Psi_{\epsilon \circ \Phi_W}$.

Sketch of proof. Let $V_s$ be the 1-parameter family of exact Lagrangian cobordisms obtained by the Hamiltonian isotopy and let $J_s$ be an 1-parameter family interpolating between the almost complex structures which is fixed outside of a compact set.

The proof relies on counting $J_s$-holomorphic discs having boundary on $V_s$ and being of formal dimension $-1$ appearing for $s \in [0,1]$. Generically such solutions exist and are rigid. To achieve a situation which can be handled
algebraically, it is necessary to use the abstract perturbation scheme outlined in [Ekh2, Appendix B].

In the following we consider the case when there is only one disc of formal dimension \(-1\) in the unperturbed family. Similarly to the proof of [Ekh3, Lemma 4.4], the abstract perturbation yields

\begin{equation}
\Phi_V = \Phi_W + \partial \circ K + K \circ \partial',
\end{equation}

where

\[ K : A_\bullet(\Lambda') \rightarrow A_{\bullet+1}(\Lambda) \]

is an operator defined by

\[ a_1 \cdot \ldots \cdot a_m \mapsto \sum_{i=1}^{m} \Phi_W(a_1 \cdot \ldots \cdot a_{i-1})K(a_i)\Phi_V(a_{i+1} \cdot \ldots \cdot a_m) \]

\[ \equiv \sum_{i=1}^{m} \Phi_W(a_1 \cdot \ldots \cdot a_{i-1})K(a_i)(\Phi_W + \partial \circ K + K \circ \partial')(a_{i+1} \cdot \ldots \cdot a_m), \]

where

\[ K : A_\bullet(\Lambda') \rightarrow A_{\bullet+1}(\Lambda) \]

is defined by a count of discs having formal dimension \(-1\) appearing after the abstract perturbation.

To show (4.1) one proceeds as follows. First, by the nature of the abstract perturbation, the left-hand side and the right-hand side agree on generators. Second, using the definition of \(K\) together with the fact that \(\Phi_V\) and \(\Phi_W\) are chain maps, it is readily seen that the right-hand side is an algebra-map as well.

We now define the map

\[ \Psi := (\text{id} + \partial' \circ K^\epsilon + K^\epsilon \circ \partial') : A_\bullet(\Lambda') \rightarrow A_\bullet(\Lambda') \]

where

\[ K^\epsilon : A_\bullet(\Lambda') \rightarrow A_{\bullet+1}(\Lambda') \]

is a linear map defined by induction on action via the formula

\[ a_1 \cdot \ldots \cdot a_m \mapsto \sum_{i=1}^{m} a_1 \cdot \ldots \cdot a_{i-1}(\epsilon \circ K(a_i))(\text{id} + \partial' \circ K^\epsilon + K^\epsilon \circ \partial')(a_{i+1} \cdot \ldots \cdot a_m) \]

\[ = \sum_{i=1}^{m} a_1 \cdot \ldots \cdot a_{i-1}(\epsilon \circ K(a_i))\Psi(a_{i+1} \cdot \ldots \cdot a_m). \]

By definition \(\Psi\) is a linear chain-map. Moreover, by the inductive definition of \(K^\epsilon\) together with the chain-map property of \(\Psi\), it follows that \(\Psi - \text{id}\) is action-decreasing, in fact is an isomorphism.

For the first statement, using (4.1), one can check that

\[ (\epsilon \circ \Phi_W) \circ \Psi = \epsilon \circ (\Phi_W + \partial \circ K + K \circ \partial') = \epsilon \circ \Phi_V \]
holds on generators and thus on the whole algebra.

For the last statement, we define the chain-homotopy

$$K_\varepsilon : A_\ast(\Lambda')^1 \to A_{\ast+1}(\Lambda)^1$$

to be the linear part of $K$, and one readily computes

$$(\Phi_V)_\varepsilon = ((\Phi_W)_\varepsilon + \partial_\varepsilon \circ K_\varepsilon + K_\varepsilon \circ \partial_\varepsilon \circ \Phi_W) \circ \Psi_\varepsilon \circ \Phi_W.$$ 

\[ \square \]

5. Background on wrapped Floer homology

We will now give an outline of wrapped Floer homology as defined in \cite{Ekh3}. Wrapped Floer homology is a Hamiltonian isotopy-invariant of pairs of exact Lagrangian fillings inside an exact symplectic manifold. Here we will only consider the when the ambient symplectic manifold is the symplectisation of a contactisation, even though the theory is defined in more generality.

In the following, we thus let $L$ and $L'$ be exact Lagrangian fillings of $\Lambda$ and $\Lambda'$, respectively, inside the symplectisation of a contactisation

$$\left( \mathbb{R} \times (\mathbb{R} \times P), d(\epsilon^t \lambda) \right), \quad \lambda := dz + \theta,$$

where $P$ is $2n$-dimensional. To simplify the situation, we assume that both $L$ and $L'$ are connected.

5.1. Definition of the wrapped Floer homology complex. After a generic Hamiltonian perturbation of $L'$ we may assume that $L$ and $L'$ intersect transversely in finitely many double-points $x_1, \ldots, x_N$, and that the Legendrian link $\Lambda \cup \Lambda'$ is embedded and chord-generic.

We use $c_1, \ldots, c_M$ to denote the Reeb chords starting on $\Lambda$ and ending on $\Lambda'$, and $x_1, \ldots, x_N$ to denote the double-points of $L \cup L'$.

5.1.1. The graded vector space. We define the vector spaces

$$CF^0(L, L') := \mathbb{Z}_2\langle x_1, \ldots, x_N \rangle,$$

$$CF^\infty(L, L') := \mathbb{Z}_2\langle c_1, \ldots, c_M \rangle,$$

$$CF(L, L') := CF^0(L, L') \oplus CF^\infty(L, L'),$$

which we endow with the following grading. For simplicity, we again assume $L$ and $L'$ to have zero Maslov number, and $P$ to have zero Chern number.

Assume that both $L$ and $L'$ are cylindrical in the set $\{ t \geq N \}$ and consider a function $\rho(t) : \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfying $\rho(t) = 0$ for $t \leq N$, $\rho(t) = 1$ for $t \gg 0$, and $\rho'(t) \geq 0$. The Hamiltonian

$$H := \rho(t)e^t : \mathbb{R} \times (P \times \mathbb{R}) \to \mathbb{R}$$

on the symplectisation has the property that, for generic $\rho(t)$ and $s \gg 0$ large enough, the double-points of $L \cup \varphi^s_H(L')$ appearing in $\{ t \geq N \}$ are transverse double-points which naturally are in bijective correspondence with the Reeb
chords starting on \( \Lambda \) and ending on \( \Lambda' \). To that end, observe that the Hamiltonian flow is given by
\[-(\rho(t) + \rho'(t))\partial z.\]
In other words, the Hamiltonian flow wraps \( L' \) in the negative Reeb-direction.

We fix a Legendrian lift of the exact Lagrangian immersion \( L \cup \varphi_{t}^{*}(L') \) to the contactisation of the symplectisation, which moreover has the property that all Reeb chords start on the lift of \( L \). Using the constructions in Section 4.1, we can associate paths \( \Gamma \) of Lagrangian tangent-planes in the symplectisation to each generator (considered as a mixed Reeb chord on the lift to the contactisation of the symplectisation). The gradings of the generators are then defined to be
\[|x_i| := CZ(\Gamma_{x_i}), \quad |c_i| := CZ(\Gamma_{c_i}).\]

**Remark 5.1.** Observe that for a Reeb-chord generator \( c_i \), this grading differs from the grading \(|c_i|_{LCH}\) in Section 4.1 obtained when considering \( c_i \) as a generator of the Legendrian contact homology DGA for \( \Lambda \cup \Lambda' \). More precisely, the capping paths in \( P \) used for the grading of the latter lift to capping paths in the symplectisation, and it can be seen that the corresponding gradings satisfy \(|c_i| = |c_i|_{LCH} + 2\).

**Remark 5.2.** Recall that the above grading is not canonical; the different choices of paths of Lagrangian planes along the curve \( \gamma \) in the construction given in Section 4.1 may induce a global shift in grading of \( CF_\bullet \). However, in the case when \( L' \) is a sufficiently \( C^1 \)-small perturbation of \( L \), one can choose \( \gamma \) as well as the path of Lagrangian planes along \( \gamma \) to be sufficiently small, and in this way get a canonical grading.

### 5.1.2. The differential

Under the decomposition
\[CF_\bullet(L, L') = CF_0^\bullet(L, L') \oplus CF_\infty^\bullet(L, L')\]
the differential will be given on the form
\[\partial := \begin{pmatrix} \partial_0 & 0 \\ \delta & \partial_\infty \end{pmatrix},\]
where the entries in the matrix are to be defined below. In other words, we will construct the wrapped Floer homology complex as the mapping cone
\[(CF(L, L'), \partial) := Cone(\delta)\]
of a chain-map \( \delta \).

We fix a compatible almost complex structure \( J \) on \( \mathbb{R} \times (P \times \mathbb{R}) \) which coincides with the cylindrical almost complex structure \( J_\infty \) in the set \( \{t \geq N\} \) for some \( N \gg 0 \). In the following we assume that \( J \) is regular for the below spaces of pseudo-holomorphic discs.

We will consider pseudo-holomorphic discs with boundary on \( L \cup L' \) and boundary-punctures of which some are asymptotic to Reeb chords of the Legendrian end and some are mapped to double-points of \( L \cup L' \). We will
require that these pseudo-holomorphic discs satisfy the conditions in Section 4.2.2 outside of some compact set, and that they satisfy the conditions in Section 4.2.1 in a neighbourhood of the punctures mapped to double-points.

Recall that we have fixed a Legendrian lift of $L \cup L'$ with the property that every Reeb chord starts on $L$ and ends on $L'$ and that this choice induces a notion of positivity and negativity for the latter punctures.

5.1.3. The sub-complex $CF^\infty$. The two fillings $L$ and $L'$, together with the almost complex structure $J$, induce augmentations $\epsilon_1$ and $\epsilon_2$ of the Legendrian contact homology DGAs of $\Lambda$ and $\Lambda'$, respectively, where the DGA is defined using the cylindrical almost complex structure $J_\infty$.

There is an induced augmentation $\epsilon$ on the Legendrian contact homology DGA of the Legendrian link $\Lambda \cup \Lambda'$ which vanishes on generators corresponding to mixed chords and which takes the value $\epsilon_1$ and $\epsilon_2$ on generators corresponding to the chords on $\Lambda$ and $\Lambda'$, respectively.

The mixed Reeb chords on $\Lambda \cup \Lambda'$ with starting point on $\Lambda$ span a sub-complex

$$(CL_\bullet(\Lambda, \Lambda'), \partial_\epsilon) \subset (CL_\bullet(\Lambda \cup \Lambda'), \partial_\epsilon)$$

of the linearised Legendrian contact homology complex of the link $\Lambda \cup \Lambda'$. We use $(CL^\bullet(\Lambda, \Lambda'), d_\epsilon)$ to denote the corresponding co-complex.

By the above grading conventions, we can identify

$$(CF^\infty(L, L'), \partial_\infty) := (CL^{\bullet-2}(\Lambda, \Lambda'), d_\epsilon).$$

5.1.4. The quotient-complex $CF^0$. The complex

$$(CF^0(L, L'), \partial_0) := (\mathbb{Z}_2\langle x_1, \ldots, x_N \rangle, \partial_0)$$

is defined by

$$\partial_0(x_i) = \sum_{|x_j| - |x_i| = 1} |M_{x_j;x_i}(L \cup L'; J)| x_j$$

where $M_{x_j;x_i}(L \cup L')$ is the moduli space of $J$-holomorphic polygons defined in Section 4.2.1. Recall that $|x_j| - |x_i| - 1$ is the dimension of this moduli space in the case when $J$ is regular. Since these polygons have only two punctures, we will sometimes refer to them as strips.

Remark 5.3. This complex is nothing else than the linearised Legendrian contact cohomology complex of the above Legendrian lift of $L \cup L'$ to the contactisation of the symplectisation, for which all Reeb chords start on $L$ and end on $L'$.

One can associate an action $\ell_{d(e^t\lambda)}$ to every generator in $(CF^0(L, L'), \partial_0)$ by associating to it the action of the corresponding Reeb chord in the above choice of Legendrian lift of $L \cup L'$. It follows that a pseudo-holomorphic strip $u \in M_{x_j;x_i}(L \cup L')$ has $d(e^t\lambda)$-area given by

$$0 < E_{d(e^t\lambda)}(u) = \ell_{d(e^t\lambda)}(x_j) - \ell_{d(e^t\lambda)}(x_i),$$

and that hence $\partial_0$ is action-increasing with respect to the action $\ell_{d(e^t\lambda)}$. 
5.1.5. The mapping-cone. There is a chain map

\[ \delta : (CF_\bullet^0(L, L'), \partial_0) \to (CL_{\bullet-1}(\Lambda, \Lambda'), d_\varepsilon) \]

defined by

\[ \delta(x_i) := \sum_{|c_j| - |x_i| = 1} |M_{c_j ; x_i} (L \cup L'; J)| c_j, \]

where the moduli-space

\[ M_{c_j ; x_i} (L \cup L'; J) \]

consists of \( J \)-holomorphic discs inside \( \mathbb{R} \times (P \times \mathbb{R}) \) having boundary on \( L \cup L' \), a positive puncture asymptotic to the Reeb chord \( c_j \) from \( \Lambda \) to \( \Lambda' \), and a negative puncture mapping to a double-point \( x_i \in L \cap L' \).

Finally, observe that \(|c_j| - |x_i| - 1\) is the dimension of the above moduli space in the case when \( J \) is regular.

5.2. Transfer-maps induced by exact Lagrangian cobordisms. Let \( L \) and \( L' \) be exact Lagrangian fillings of \( \Lambda \) and \( \Lambda' \) as above, and let \( V \subset \mathbb{R} \times (P \times \mathbb{R}) \) be an exact Lagrangian cobordism from \( \Lambda' \) to \( \Lambda'' \). We moreover assume that \( L \) and \( L' \) are cylindrical in the set \( \{ t \geq -N \} \) for some \( N \gg 0 \).

Let \( J \) denote the regular almost complex structure on \( \mathbb{R} \times (P \times \mathbb{R}) \) defining \( (CF(L, L'), \partial) \), which we assume coincides with the cylindrical almost complex structure \( J_\infty \) in the set \( \{ t \geq -1 \} \).

Assuming that \( V \) is cylindrical on \( \{ t \leq 1 \} \), recall that the concatenation

\[ L''_s := L' \circ_s V, \ s \geq 0 \]

is again an exact Lagrangian filling of \( \Lambda'' \). We assume that \( L \) and \( L'' \) intersect transversely. The double-points of \( L \cup L''_s \) satisfy

\[ L \cap L''_s = (L \cap L') \cup ((\mathbb{R} \times \Lambda) \cap V). \]

Fix an almost complex structure \( J_V \) which coincides with \( J_\infty \) on \( \{ t \leq 1 \} \) and with the cylindrical almost complex structure \( J''_\infty \) on \( \{ t \geq 0 \} \). For each \( s \), we consider the almost complex structure \( J''_s \) which coincides with \( J \) in the set \( \{ t \leq s + 1 \} \) and with \( J_V (t - s, p, z) \) in the set \( \{ t \geq s + 1 \} \). We let \( (CF(L, L''_s), \partial''_s) \) be the induced wrapped Floer homology complex.

If we by

\[ CF^0_\bullet (\mathbb{R} \times \Lambda, V) \subset CF^0_\bullet (L, L' \circ_s V) \]

denote the subspace spanned by the double-points \( (\mathbb{R} \times \Lambda) \cap V \), a monotonicity argument (see Proposition 4.7.2(ii) and Proposition 7.3.1(ii) in [Sik]) for the \( d(e^\lambda) \)-area of pseudo-holomorphic discs shows that for \( s \gg 0 \) sufficiently large, the above wrapped Floer homology complex is on the form

\[ CF_\bullet (L, L' \circ_s V) = CF^0_\bullet (L, L') \oplus CF^0_\bullet (\mathbb{R} \times \Lambda, V) \oplus CF^\infty_\bullet (L, L''_s), \]

\[ \partial''_s = \begin{pmatrix} \partial_0 & \delta''_1 & 0 \\ \delta''_1 & \partial_V & 0 \\ \delta''_2 & \delta''_3 & \partial''_\infty \end{pmatrix}, \]

where \( \partial_0 \) is the differential of \( CF^0_\bullet (L, L') \).
Remark 5.4. In the case when all generators of $\text{CF}_\bullet^0(\mathbb{R} \times \Lambda, V)$ have $\ell_{\epsilon^i\lambda}$-action greater than $\text{CF}_\bullet^0(L, L')$, it immediately follows that $\delta'' = 0$.

In [Ekh3, Section 4.2.2] the so called transfer map is constructed which, for $s \gg 0$ sufficiently large, is a chain map on the form

$$\Phi_V: (\text{CF}_\bullet(L, L'), \partial) \to (\text{CF}_\bullet(L, L' \circ_s V), \partial'_s),$$

relative the above decomposition.

To describe its components, we proceed as follows. Let $d_i \in \text{CF}^\infty(L, L' \circ V)$, $x_i, x_j \in \text{CF}^\bullet_0(\mathbb{R} \times \Lambda, V)$, and $a$ and $b$ define words of Reeb chords on $\Lambda'$ and $\Lambda$, respectively.

We use $\mathcal{M}_{d_i; a, c_i, b}(\mathbb{R} \times \Lambda) \cup V; J_V)$ to denote the moduli space of $J_V$-holomorphic discs as defined in Section 4.2.2 having boundary on $(\mathbb{R} \times \Lambda) \cup V$ and boundary punctures asymptotic to the prescribed Reeb chords.

Also, we define the moduli space $\mathcal{M}_{x_j; a, c_i, b}(\mathbb{R} \times \Lambda \cup V; J_V)$ having boundary on $(\mathbb{R} \times \Lambda) \cup V$, a positive puncture mapping to $x_j$ and its negative punctures asymptotic to the prescribed Reeb chords. Here the Legendrian lift of of $(\mathbb{R} \times \Lambda) \cup V$ has been chosen so that all Reeb chords start on the lift of $\mathbb{R} \times \Lambda$, which induces a notion of positivity and negativity for a puncture at a double-point.

Remark 5.5. Observe that with this notion of positivity and negativity of boundary punctures, it is not necessary for a $J_V$-holomorphic disc as above to posses a positive puncture. However, in the case when there are no positive punctures, it must have a puncture asymptotic to a Reeb chord at $-\infty$ starting on $\Lambda'$ and ending on $\Lambda$.

For a generator $c_i \in \text{CF}^\infty(L, L')$, the components of $\Phi_V$ are given by the following counts of rigid $J_V$-holomorphic discs in the above moduli-spaces:

$$\phi_0(c_i) := \sum_{|x_j| - |c_i| = 0 \atop |a = b| = 0} |\mathcal{M}_{x_j; a, c_i, b}(\mathbb{R} \times \Lambda) \cup V; J_V)|\epsilon_L(a)\epsilon_{L'}(b)x_j,$$

$$\phi_\infty(c_i) := \sum_{|d_j| - |c_i| = 0 \atop |a = b| = 0} |\mathcal{M}_{d_j; a, c_i, b}(\mathbb{R} \times \Lambda, V; J_V)|\epsilon_L(a)\epsilon_{L'}(b)x_j.$$

In the case when $J_V$ is regular, $|x_j| - |c_i| - |a| - |b|$ and $|d_j| - |c_i| - |a| - |b|$ denote the dimensions of the above respective moduli spaces.

Again, a stretching-of-the-neck argument can be used to show the following.

Proposition 5.6. Let $L$ and $L'$ be exact Lagrangian fillings. Given exact Lagrangian cobordisms $V$ and $W$, where the positive end of $V$ equals the negative end of $W$, it follows that

$$\Phi_V \circ W = \Phi_W \circ \Phi_V: (\text{CF}_\bullet(L, L'), \partial) \to (\text{CF}_\bullet(L, L' \circ V \circ W), \partial''),$$
where all almost complex structures have been appropriately chosen.

5.3. Invariance under Hamiltonian isotopy. Wrapped Floer homology satisfies the following invariance.

Theorem 5.7. Let $L, L', L'' \subset \mathbb{R} \times (P \times \mathbb{R})$ be exact Lagrangian fillings, where $L'' = \varphi_{H_s}^1(L')$ is Hamiltonian isotopic to $L'$ for a Hamiltonian $H_s$ having support in $\mathbb{R} \times K$ for some compact set $K$. There is a homotopy equivalence

$$\Phi: (CF_\bullet(L, L'), \partial) \to (CF_\bullet(L, L''), \partial'').$$

Since the proof of Theorem 6.2 is based on some constructions in the proof of this invariance theorem, we formulate its main ingredients below. The core of the argument is as follows. A standard fact (see Lemma A.1 in the appendix) implies that there are exact Lagrangian cobordisms $V$ and $W$ satisfying the following.

- $L' \circ V$ and $L'' = \varphi_{H_s}^1(L')$ differ by a compactly supported Hamiltonian isotopy.
- $V \circ W$ is isotopic to $\mathbb{R} \times \Lambda$ by a compactly supported Hamiltonian isotopy.

The idea is now to use transfer-maps induced by $V$ and $W$, together with invariance under compactly supported Hamiltonian isotopies. To that end, the following propositions are used.

Proposition 5.8. Let $L$ and $L'$ be exact Lagrangian fillings. Given a path $\varphi_{H_s}^1(L')$ of fillings, where $H_s$ has compact support, together with a path $J_s$ of almost complex structures fixed outside of some compact set, there is an induced homotopy-equivalence

$$\Phi_{H_s, J_s}: (CF_\bullet(L, L'), \partial) \to (CF_\bullet(L, \varphi_{H_s}^1(L')), \partial''),$$

where the former complex is defined using $J_0$ and the latter is defined using $J_1$, and where $\Phi_{H_s, J_s}|_{CF_\bullet}$ is an isomorphism of complexes. Moreover, if there are no births or deaths of double-points during the Hamiltonian isotopy, then $\Phi_{H_s, J_s}$ is an isomorphism of complexes as well.

Proposition 5.9. Let $L$ and $L'$ be exact Lagrangian fillings and $V$ an exact Lagrangian cobordism with negative end given by $\Lambda'$. If $W := \varphi_{H_s}^1(V)$ is obtained by a compactly supported Hamiltonian isotopy, then the diagram

$$
\begin{array}{ccc}
(CF_\bullet(L, L'), \partial) & \xrightarrow{\Phi_V} & (CF_\bullet(L, L' \circ V), \partial'') \\
\downarrow \Phi_{H_s, J_s} & & \downarrow \Phi_{H_s, J_s} \\
(CF_\bullet(L, L'), \partial) & \xrightarrow{\Phi_W} & (CF_\bullet(L, L' \circ W), \partial'')
\end{array}
$$

commutes up to homotopy.
5.4. The transfer map induced by the negative Reeb-flow. Let $L$ and $L'$ be fillings of $\Lambda$ and $\Lambda'$ as above, which we assume are cylindrical in the set $\{t \geq -1\}$. In this section we obtain a refined invariance result in the spacial case when the Hamiltonian
\[ H := \rho(t)e^t : \mathbb{R} \times (P \times \mathbb{R}) \to \mathbb{R}_{\geq 0} \]
only depends on the $t$-coordinate and moreover satisfies the following.

- $\rho(t)$ has support in $\{t \geq -1\}$.
- $\rho(t), \rho'(t) \geq 0$.
- $\rho(t)$ is constant on some set $\{t \geq A\}$.

Since the corresponding Hamiltonian vector field is of the form
\[ X_H = -(H(t) + H'(t)) \partial_z, \]
the Hamiltonian isotopy fixes the hypersurfaces $\{t\} \times (P \times \mathbb{R})$, where it acts by some reparametrisation of the negative Reeb flow.

Consider the exact Lagrangian cylinder
\[ V := \varphi_H^1(\mathbb{R} \times \Lambda') \]
and, for each $s \geq 0$, the filling
\[ L''_s := L' \circ_s V \]
of $\Lambda''$. It follows that that $L''_0 = L''$, and that $L''_s$ is isotopic to $L''$ by a compactly supported Hamiltonian isotopy.

Observe that $V$ satisfies
\[ \pi_P(V) = \Pi_{\text{Lag}}(\Lambda'') = \Pi_{\text{Lag}}(\Lambda') \]
and that, for generic functions $\rho(t)$ as above, the self-intersections of $(\mathbb{R} \times \Lambda) \cup V$ are transverse double-points corresponding to a subset of the mixed Reeb chords on $\Lambda \cup \Lambda'$ starting on $\Lambda$ and ending on $\Lambda'$. Moreover, it can be checked that there is a natural identification
\[ CF_*(L, L') = CF_*(L, L''_s) \]
of graded vector spaces.

We now show that this identification may be assumed to hold on the level of complexes as well. Let $J$ be a compatible almost complex structure coinciding with the cylindrical almost complex structure $J_\infty$ in the set $\{t \geq s/2\}$. We moreover assume that $J_\infty$ is the cylindrical lift of the compatible almost complex structure $J_P$ on $P$, or in other words, that the canonical projection
\[ \pi_P : \mathbb{R} \times (P \times \mathbb{R}) \to P \]
is $(J_\infty, J_P)$-holomorphic. The following proposition is a key step in the proof of Theorem 5.2 below.

**Proposition 5.10.** For generic choices of a compatible almost complex structure $J_P$ on $P$ and $s \gg 0$ sufficiently large, the almost complex structure $J$ defined above may be assumed to be regular for the moduli spaces in the
definition of \((CF(L, L'), \partial)\) and \((CF(L, L''_s), \partial'')\). Furthermore, the transfer map
\[
\Phi_V: (CF_\bullet(L, L'), \partial) \to (CF_\bullet(L, L''_s), \partial''),
\]
corresponds to the identity map under the natural identifications.

Proof. Using the transversality result \([\text{EES}3, \text{Proposition 2.3}]\), which can be generalised to the discs in the symplectisation under consideration, every \(J\)-holomorphic disc having boundary on \(L \cup L'\) and exactly one positive puncture (either at a double-point or asymptotic to a Reeb chord) is transversely cut out after a perturbation of \(J\) inside a compact set which, without loss of generality, may be assumed to be contained in \(\{t \leq s/2\}\).

Choosing a regular \(J_P\) for the moduli-spaces of \(J_P\)-holomorphic discs with boundary on \(\Pi_{Lag}(\Lambda \cup \Lambda')\) and one positive puncture, it follows that the moduli spaces of \(J\)-holomorphic discs in the definition of \(\partial_\infty\) may be assumed to be regular by Lemma 7.3. In particular, this choice of \(J\) defines the complex
\[
\left(\begin{array}{c}
CF_\bullet(L, L'), \partial = \left(\begin{array}{cc}
\partial_0 & 0 \\
\delta & \partial_\infty
\end{array}\right)
\end{array}\right).
\]

Since \(\pi_P(V) = \Pi_{Lag}(\Lambda')\), the \((J_\infty, J_P)\)-holomorphic projection \(\pi_P\) induces maps
\[
M_{f, a, e, b}(\mathbb{R} \times \Lambda \cup V; J_\infty) \to M_{f, a, e, b}(\Pi_{Lag}(\Lambda \cup \Lambda'); J_P)
\]

between moduli spaces, where \(e\) and \(f\) are either double-points of \((\mathbb{R} \times \Lambda) \cup V\) or mixed Reeb chords on the Legendrian ends, and where \(a = a_1 \cdot \ldots \cdot a_{m_1}\) and \(b = b_1 \cdot \ldots \cdot b_{m_1}\) are words of Reeb chords on \(\Lambda'\) and \(\Lambda\), respectively.

By considering the \(d\theta\)-area of the projection, it immediately follows that \(J_\infty\)-holomorphic discs as above must have a positive puncture (see Remark 5.6).

Writing \(I^\infty_{f, a, e, b}\) and \(I^P_{f, a, e, b}\) for the expected dimensions of the above moduli spaces of \(J_\infty\) and \(J_P\)-holomorphic discs, respectively, Lemma 7.4 implies that
\[
I^\infty_{f, a, e, b} = I^P_{f, a, e, b} + 1, \quad e = c
\]
in the case when \(e = c\) is a Reeb chord, while
\[
I^\infty_{f, a, e, b} = I^P_{f, a, e, b}, \quad e = x
\]
in the case when \(e = x\) is a double-point.

From these index formulas, it follows that the rigid discs in the moduli spaces \(M_{f, a, e, b}(\mathbb{R} \times \Lambda \cup V; J_\infty)\) contributing to \(\Phi_V\), where \(f\) is either a double-point or a mixed Reeb chord, project to \(J_P\)-holomorphic discs of negative index. The regularity of \(J_P\) hence implies that such a disc is a strip contained entirely in a \((t, z)\)-plane. Finally, Lemma 7.4 shows that these discs are transversely cut-out, from which it follows that \(\Phi_V\) is on the form as claimed.
It remains to show that the moduli spaces of $J$-holomorphic discs in the definition of $\partial''$ can be assumed to be transversely cut out. To that end, we stretch the neck by considering the limit of $J$-holomorphic discs, for fixed $J$ as above, having boundary on $L \cup L''$ as $s \to \infty$. The compactness result in [BEH+] shows that a sequence of $J$-holomorphic discs contributing to the differential $\partial''$ and having at least one puncture at a generator corresponding to $CF^0(L, L'')$ converges to one of the following objects.

1. A rigid $J$-holomorphic strip having compact image and boundary on $L \cup L'$, that is, a strip contributing to $\partial_0$.
2. An index zero $J_\infty$-holomorphic strip in $\mathcal{M}_{f;x}((\mathbb{R} \times \Lambda) \cup V; J_\infty)$ whose positive puncture is either asymptotic to a Reeb chord or mapped to a double-point.
3. A two-level holomorphic building whose top level consists of an index zero disc in $\mathcal{M}_{f;a,x,b}((\mathbb{R} \times \Lambda) \cup V; J_\infty)$ whose positive puncture is either asymptotic to a Reeb chord or mapped to a double-point, and whose bottom level consists of rigid discs in the moduli spaces $\mathcal{M}_{a;\emptyset}(L'; J)$ and $\mathcal{M}_{b;\emptyset}(L; J)$.
4. A two-level holomorphic building whose top level consists of an index zero disc in $\mathcal{M}_{f;a,x,b}((\mathbb{R} \times \Lambda) \cup V; J_\infty)$ whose positive puncture is either asymptotic to a Reeb chord or mapped to a double-point, and whose bottom level consists of rigid discs in the moduli spaces $\mathcal{M}_{a;\emptyset}(L'; J)$, $\mathcal{M}_{b;\emptyset}(L; J)$ and $\mathcal{M}_{c;\emptyset}(L \cup L'; J)$.

First, observe that the above index formulas show that, for a regular almost complex structure $J_P$, each $J_\infty$-holomorphic disc with one positive puncture and boundary on $(\mathbb{R} \times \Lambda) \cup V$ has non-negative index. Moreover, the top-level in configuration (4) consists of a strip contained inside a $(t, z)$-plane as also follows by the above index formulas.

The $J$-holomorphic discs in (1) as well as in the bottom levels of (3) and (4) are transversely cut-out by assumption. By using Lemma 7.4 it follows that the $J_\infty$-holomorphic discs appearing in (2) as well as in the top-layers of configuration (3) and (4) can be assumed to be transversely cut out. By gluing these broken configurations it thus follows that, for $s \gg 0$ is big enough, the corresponding $J$-holomorphic discs are transversely cut out as well.

A similar argument shows that the $J$-holomorphic discs having boundary on $L' \odot V$ in the definition of the augmentation $\epsilon_{L''}$ are transversely cut out and moreover coincide bijectively with the $J$-holomorphic discs having boundary on $L'$ in the definition of the augmentation $\epsilon_{L'}$. □

**Remark 5.11.** Suppose $H = \rho(t)e^t$ is a non-vanishing Hamiltonian as above. For $s \gg 0$ sufficiently large every mixed Reeb chord on the positive end of $L \cup \varphi_H^s(L')$ starts on the positive end of $\varphi_H^s(L')$. Consequently,

$$CF_\bullet(L, L') = CF_\bullet(L, \varphi_H^s(L')) = CF^0(L, \varphi_H^s(L')).$$
The co-complex associated to \((CF^0_H(L, \varphi^s_H(L'), \partial_0))\) is related to the version of the wrapped Floer cohomology complex as defined in \([Ab0], [FSS]\). However, one technical difference is that the latter versions use moduli-spaces of solutions to a \(\overline{\partial}\)-equations with a perturbation-term depending on a Hamiltonian vector field.

5.5. Consequences of the invariance. An immediate consequence of the invariance theorem is the following vanishing result.

**Proposition 5.12.** Let \(L, L'\) be exact Lagrangian fillings inside the symplectisation of a contactisation. It follows that \(HF_\bullet(L, L') = 0\).

**Proof.** Using the negative Reeb flow \(-\partial_z\) one can isotope \(L'\) to an exact Lagrangian filling \(L''\) for which \(CF(L, L'') = 0\). Since this is a Hamiltonian isotopy, the claim follows by the invariance theorem. \(\Box\)

A similar argument shows an analogous property for the linearised Legendrian contact cohomology of Legendrian links in certain positions.

**Proposition 5.13.** Let \(L, L' \subset \mathbb{R} \times (P \times \mathbb{R})\) be exact Lagrangian fillings of \(\Lambda\) and \(\Lambda'\), respectively, which induce an augmentation \(\epsilon\) of the DGA of the link \(\Lambda \cup \Lambda'\). If all mixed Reeb-chords on \(\Lambda \cup \Lambda'\) start on \(\Lambda\), it follows that

\[HCL_\bullet(\Lambda, \Lambda'; \epsilon) = 0,\]

or equivalently,

\[HCL_\bullet(\Lambda \cup \Lambda'; \epsilon) = HCL_\bullet(\Lambda; \epsilon_L) \oplus HCL_\bullet(\Lambda'; \epsilon_{L'}).\]

**Proof.** There is a Lagrangian filling \(L''\) which is isotopic to \(L'\) by a compactly supported Hamiltonian isotopy for which \(CF^0_H(L, L'') = 0\). It immediately follows that

\[(CF_\bullet(L, L''), \partial'') = (CF_\bullet^\infty(L, L''), \partial_\infty) = (CL^{*-2}_\bullet(\Lambda, \Lambda'), d_\epsilon)\]

and the invariance theorem finally implies that the above complexes are acyclic. \(\Box\)

In particular, the assumptions of the previous proposition are fulfilled for a link consisting a Legendrian submanifold \(\Lambda \subset P \times \mathbb{R}\) admitting a filling inside the symplectisation, together with a copy of \(\Lambda\) translated sufficiently far away in the positive \(z\)-direction. This shows that \(\Lambda\) together with the augmentation induced by such a filling satisfies the requirements for the existence of the duality long exact sequence in \([EES4]\).

**Remark 5.14.** The fact that the augmentation is induced by a filling *inside the symplectisation* is crucial. For instance, let \(\Lambda\) be the zero-section inside \(J^1(\partial M)\), and let \(\Lambda'\) be a copy of \(\Lambda\) shifted far away in the positive \(z\)-direction. \(\Lambda\) has a filling consisting of the zero-section inside the symplectic manifold \(T^*(M)\), where the latter is considered as an exact symplectic manifold having a convex cylindrical end over the contact manifold \(J^1(\partial M)\).
However, using the theory of gradient flow trees in \cite{Ekh1}, it can be checked that the corresponding linearisation of the DGA of $\Lambda \cup \Lambda'$ satisfies

$$HCL_\ast(\Lambda \cup \Lambda'; \epsilon) = H_{\ast+1}(\partial M; \mathbb{Z}_2),$$

which clearly is non-zero.

6. Applications of Theorem \ref{thm:main}

6.1. The Legendrian contact cohomology of the Legendrian two-copy link. Suppose that $\Lambda, \tilde{\Lambda} \subset P \times \mathbb{R}$ are two Legendrian embeddings of the same $n$-dimensional manifold whose Lagrangian projections $\Pi_{Lag}(\tilde{\Lambda})$ and $\Pi_{Lag}(\Lambda)$ are sufficiently $C^0$-close.

In \cite{EES4}, in the case when $J$ is adjusted to $\Pi_{Lag}(\Lambda)$, the rigid $J$-holomorphic polygons on $\Pi_{Lag}(\Lambda \cup \tilde{\Lambda})$ with one positive puncture are shown to correspond to (generalised) $J$-holomorphic polygons on $\Pi_{Lag}(\Lambda)$ together with gradient flow-lines on $\Lambda$. In this section we recall these results.

Observe that, in the case when $J$ is the cylindrical lift of $J_\ast$, Theorem \ref{thm:main} immediately gives that the result also applies to the corresponding $J$-holomorphic discs inside the symplectisation having boundary on $(\mathbb{R} \times \Lambda) \cup (\mathbb{R} \times \Lambda')$.

For a Legendrian submanifold $\Lambda \subset P \times \mathbb{R}$ we define the following Legendrian two-copy links. Fix a positive Morse function $f: \Lambda \to [0, 1/2]$ whose critical points are disjoint form the end-points of the Reeb chords on $\Lambda$. We assume that there is a metric $g$ on $\Lambda$ such that the pair $(f, g)$ is Morse-Smale, where $g$ is moreover assumed to be flat in neighbourhoods of the end-points of the Reeb chords on $\Lambda$.

There is a contact-form preserving identification of a neighbourhood of $\Lambda$ with a neighbourhood of the zero-section in $(J^1(\Lambda), dz + \theta_\Lambda)$. We consider the following Legendrian submanifolds, where some constants have been chosen to agree with choices needed in Section 6.2 below.

- $\Lambda_+$ denotes the push-offs of $\Lambda$ corresponding to $(\eta^2 df, \epsilon^2 + \eta^2 f)$, for some sufficiently small $\epsilon > \eta > 0$, where we assume that $\epsilon^2 + \eta^2$ is smaller than the shortest Reeb chord on $\Lambda$.
- $\Lambda_-$ denotes the push-offs of $\Lambda$ corresponding to $(\eta^2 df, -\epsilon^{1/2} + \eta^2 f)$, for some sufficiently small $\epsilon > \eta > 0$, where we assume that $\epsilon^{1/2} - \eta^2 > 0$ is smaller than the shortest Reeb chord on $\Lambda$.
- $\Lambda_+\infty$, which is a copy of $\Lambda_+$ translated sufficiently far in the positive $z$-direction, so that all mixed Reeb chords on $\Lambda \cup \Lambda_+\infty$ start on $\Lambda$.

6.1.1. Generalised pseudo-holomorphic discs. We define a generalised pseudo-holomorphic disc in $P$ to consist of the following data. Let $c$ be a critical point of $f$ and let $u: (D^2, \partial D^2) \to (P, \Pi_{Lag}(\Lambda))$
be a pseudo-holomorphic polygon having boundary on $\Pi_{\text{Lag}}(\Lambda)$ and boundary punctures mapping to double-points, together with an additional marked point $p_f \in \partial D^2$. We require $u(p_f)$ to be connected to $c$ via a flow-line of $\nabla f$ (we allow $c = u(p_f)$). In the case when $c$ is connected to $u(p_f)$ by the positive (respectively negative) gradient flow, we say that $p_f$ is a positive (respectively negative) puncture.

We refer to [EES4] for the expected dimension and transversality results for these spaces.

6.1.2. Computations of the complexes for the different two-copies. We assume that the DGA of $\Lambda$ has an augmentation $\epsilon$. We may assume that $\Lambda_{\pm}$ are sufficiently $C^1$-close to $\Lambda$ and hence that the DGA of $\Lambda_i$ for $i = +, -, +\infty$ coincide with the DGA of $\Lambda$. In particular, $\epsilon$ induces an augmentation of each component of the link $\Lambda \cup \Lambda_i$, and thus of the link itself.

We use $(CL^\bullet(\Lambda, \Lambda_i), d_i)$ to denote the linearised Legendrian contact cohomology complex generated by Reeb chords starting on $\Lambda$ and ending on $\Lambda_i$, where the complex is linearised with respect to an augmentation $\epsilon$.

Choosing canonically defined capping paths, it can be shown that

$$CL^\bullet(\Lambda, \Lambda_-) = CL^\bullet(\Lambda),$$
$$CL^\bullet(\Lambda, \Lambda_+) = C_{\text{Morse}}^{\bullet + 1}(f) \oplus CL^\bullet(\Lambda),$$
$$CL^\bullet(\Lambda, \Lambda_{+\infty}) = CL_{n-2-\bullet}(\Lambda) \oplus C_{\text{Morse}}^{\bullet + 1}(f) \oplus CL^\bullet(\Lambda).$$

To that end, observe that the mixed Reeb chords either correspond to a pure Reeb chord on $\Lambda$ or to a critical point of the Morse function $f$. We refer to [EES4, Section 3.1] for more details.

Since the co-differential is action-increasing, by comparing the length of the Reeb chords in the different summands above, one concludes that the they are on the following form with respect to the above decompositions.

$$d_- = dq,$$
$$d_+ = \begin{pmatrix} df & 0 \\ \rho & dq \end{pmatrix},$$
$$d_{+\infty} = \begin{pmatrix} dp & 0 & 0 \\ \rho & df & 0 \\ \eta & \sigma & dq \end{pmatrix}.$$

Remark 6.1. That the exact Lagrangian immersion $\Xi_{\text{Lag}}(\Lambda \cup \Lambda_i)$ is the same for $i = -, +, +\infty$ and, hence, the set of pseudo-holomorphic polygons in $P$ with boundary on $\Pi_{\text{Lag}}(\Lambda \cup \Lambda_i)$ is independent of $i$. However, the notion of being a positive or a negative puncture does depend on $i$.

According to [EES4, Theorem 3.6], for an almost complex structure $J_P$ adjusted to $\Pi_{\text{Lag}}(\Lambda)$, and for $\eta > 0$ small enough, the above co-differentials can be defined by counting the following (generalised) $J_P$-holomorphic polygons.
\[d_p = \partial_\epsilon\] is the differential of the linearised Legendrian contact homology complex of \(\Lambda\) with respect to the augmentation \(\epsilon\).

\[d_q = d_\epsilon\] is the differential on the linearised Legendrian contact cohomology complex with respect to the augmentation \(\epsilon\).

The differential \(d_f\) is the differential on the Morse co-complex (i.e. counting positive gradient flow lines).

The map \(\sigma\) counts generalised pseudo-holomorphic discs on \((\Lambda, f)\) with one positive puncture and one negative Morse-puncture.

The map \(\eta\) counts rigid pseudo-holomorphic discs on \(\Lambda\) with two positive punctures after a domain-dependant perturbation of the boundary condition.

The map \(\rho\) counts generalised pseudo-holomorphic discs on \((\Lambda, f)\) with two positive punctures, of which one is a Morse-puncture.

Finally, we remark that, in the case when \(J\) is a cylindrical lift of \(J_P\), Theorem 2.1 implies that the above description of the complexes holds for the version of Legendrian contact homology defined in terms of \(J\)-holomorphic discs in the symplectisation as well.

### 6.2. The wrapped Floer homology of the two-copy of a filling.

Let \(L \subset \mathbb{R} \times (P \times \mathbb{R})\) be an exact Lagrangian filling of the Legendrian submanifold \(\Lambda\). We assume that \(L\) is cylindrical in the set \(\{t \geq -2\}\).

#### 6.2.1. Construction of the Morse functions \(F_\pm\) on \(L\).

Start by fixing a positive Morse function \(f: \Lambda \rightarrow [0, 1/2]\) and a smooth cut-off function \(\rho: \mathbb{R} \rightarrow [0, 1]\) satisfying \(\rho'(t) \geq 0, \rho|_{(-\infty, -1]} = 0,\) and \(\rho|_{[-1/2, +\infty)} = 1\). For each numbers \(0 < \eta < \epsilon < 1\) and some fixed \(B > 0\) we define the smooth function

\[f_\eta: \mathbb{R} \times \Lambda \rightarrow \mathbb{R}\]

\[(t, p) \mapsto f_\eta(t, p) := \eta^2 \rho(t - 1/2)f(p).\]

We also define the smooth cut-off functions

\[\sigma_\epsilon(t) := \epsilon^5 + (\epsilon^2 - \epsilon^5)\rho(t),\]

\[\tilde{\sigma}_\epsilon(t) := \epsilon^5 + (\epsilon^2 - \epsilon^5)\rho(t) + (\epsilon^{1/2} - \epsilon^2)\rho(t - (B + 3/2)).\]

It follows that \(\sigma_\epsilon|_{(-\infty, -1]} = \epsilon^5, \sigma_\epsilon|_{[-1/2, +\infty]} = \epsilon^2,\) while \(\tilde{\sigma}_\epsilon|_{(-\infty, B+1/2]} = \sigma_\epsilon,\) and \(\tilde{\sigma}_\epsilon|_{[B+1, +\infty)} = \epsilon^{1/2}.\)

Now consider a fixed Morse function \(G\) on \(L \cap \{t \leq -1\}\) which is given by \(t|_L\) in \(\{-2 \leq t \leq -1\}\). We define

\[F_{\pm}^{\eta, \epsilon}: L \rightarrow \mathbb{R}\]

to be the Morse function which coincides with \(\epsilon^5G\) on \(L \cap \{t \leq -1\}\) and with \(\sigma_\epsilon(t) + tf_\eta\) on \(\{t \geq -1\}.\)

Let \(0 < A < B\) be fixed. We also consider the Morse function

\[F_{-}^{\eta, \epsilon}: L \rightarrow \mathbb{R}\]
which coincides with $F_{\pm}^{\eta,\epsilon}$ on 
$$\overline{L} := L \cap \{ t \leq A - 1 \}$$
and which is given by $\tilde{\sigma}_\epsilon(t)\alpha(t) + tf_\eta$ in the set $\{ t \geq A - 1 \}$, where $0 < A < B$ have been chosen sufficiently large, and $\alpha : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions.

- $\alpha(t) = t$ for $t \leq A - 1$ and $\alpha(t) > 0$ for $A \leq t \leq B$.
- $\alpha'(A) = 0$, $\alpha'(B) < -1/2$, and $\alpha'(t) = -1$ for $t \geq B + 1$.
- $\alpha''(t) \leq 0$ for all $t$ and $\alpha''(t) < 0$ for $A \leq t \leq B$.

We will sometimes use $F_{\pm}$ to denote $F_{\eta,\epsilon}^{\eta,\epsilon}$.

The critical points of $F_+$ correspond to the critical points of $G$ and are all contained in $\overline{L}$.

The critical points of $F_-$ consist of critical points inside $L$, which hence correspond to the critical points of $F_+$, together with critical points inside the set $L \cap \{ A < t < B \}$ which are determined by the equation

$$\left( 2\alpha'(t) + \eta^2 f(p) \right)dt + t\eta^2 df(p) = 0.$$ 

Since $\alpha''(t) < 0$ in this set, it follows that these critical points are non-degenerate and correspond bijectively to critical points of $f$. Moreover, the Morse index of such a critical point is one greater than the Morse index of the corresponding critical point of $f$.

We let $g$ be the choice of a metric on $\Lambda$ for which $(f, g)$ is Morse-Smale and let $(C_{\text{Morse}}^\bullet(f), df)$ be the induced Morse co-complex. Consider a metric on $L$ coinciding with a perturbation of the product metric $dt \otimes dt + g$ in the set $\{ t \geq 0 \}$ and which is Morse-Smale together with a small perturbation of $F_-$. By Lemma A.2 we may assume that the corresponding Morse co-complex 
$$\left( C_{\text{Morse}}^\bullet(F_-), df_- \right) = \left( C_{\text{Morse}}^\bullet(F_+) \oplus C_{\text{Morse}}^{\bullet-1}(f), df_- \right),$$

has differential given by

$$d_{F_-} = \begin{pmatrix} d_{F_+} & 0 \\ \Gamma & df \end{pmatrix}.$$ 

Finally, the Morse cohomology groups associated to $F_\pm$ are as follows.

$$H_{\text{Morse}}(F_+; \mathbb{Z}_2) = H^\bullet(L; \mathbb{Z}_2) = H_{(n+1)-\bullet}(\overline{L}, \partial \overline{L}; \mathbb{Z}_2),$$

$$H_{\text{Morse}}(F_-; \mathbb{Z}_2) = H^\bullet(\overline{L}, \partial \overline{L}; \mathbb{Z}_2) = H_{(n+1)-\bullet}(L; \mathbb{Z}_2).$$

The equalities on the left follow from the fact the differential in Morse cohomology counts the positive gradient flow-lines, while the negative gradients of $F_+$ and $F_-$ points inwards and outwards of $L \cap \{ t \leq B \}$, respectively. The equalities on the right follow by relative Poincaré duality.

6.2.2. Construction of the push-offs $L_\pm$ of $L$. We will use the above Morse functions $F$ and $\tilde{F}$ to construct certain exact Lagrangian fillings corresponding to push-offs of $L$. We start by fixing a contact-form preserving identification

$$\varphi : (V, dz + \Theta) \to (U, dz + \Theta_\Lambda)$$
of a neighbourhood $V \subset P \times \mathbb{R}$ of $\Lambda$ with a neighbourhood $U \subset J^1(\Lambda)$ of the zero-section which moreover identifies $\Lambda$ with the zero-section.

It immediately follows that the map

$$(\text{id}, \varphi) : \left(\begin{array}{cc} (-2, +\infty) \times V, d(e^t (dz + \theta)) \end{array}\right) \to \left(\begin{array}{cc} (-2, +\infty) \times U, d(e^t (dz + \theta_\Lambda)) \end{array}\right)$$

is an exact symplectomorphism identifying a neighbourhood of $L \cap \{ t \geq -2 \}$ with a neighbourhood of the cylinder over the zero-section in the symplectisation of $(J^1(\Lambda), dz + \theta_\Lambda)$.

Furthermore, the symplectisation of $(U, dz + \theta_\Lambda)$ is symplectomorphic to a neighbourhood of the zero-section of the cotangent bundle of $\mathbb{R} \times \Lambda$ via the (non-exact) symplectomorphism

$$\psi : (\mathbb{R} \times J^1(\Lambda), d(e^t (dz + \theta_\Lambda))) \to (T^*(\mathbb{R} \times \Lambda), d\theta_{\mathbb{R} \times \Lambda}),$$

$$(t, (\mathbf{q}, \mathbf{p}, z)) \mapsto ((e^t, \mathbf{q}), (z, e^t \mathbf{p})).$$

Using the Weinstein Lagrangian neighbourhood theorem together with the above symplectomorphism we can construct a symplectetic identification of a neighbourhood of the zero-section in $(T^* L, d\theta_L)$ with a neighbourhood of $L$ which coincides with $(\psi \circ (\text{id}, \varphi))^{-1}$ on a neighbourhood of the zero-section of

$$(T^*([-2, +\infty) \times \Lambda), d\theta_{[-2, +\infty) \times \Lambda}) \subset (T^* L, \theta_L).$$

For $0 < \eta < \epsilon < 1$ sufficiently small, we use the above identification to construct the exact Lagrangian fillings $L^\eta_+$ and $L^\eta_-$ inside $\mathbb{R} \times (P \times \mathbb{R})$ by letting them correspond to the graphs of $dF^\eta_+$ and $dF^\eta_-$ inside of $T^* L$, respectively.

By construction, we have

$$L^\eta_+ \cap \{ t \geq -1/2 \} = [-1/2, +\infty) \times \Lambda_+,$$

$$L^\eta_- \cap \{ t \geq B + 1 \} = [B + 1, +\infty) \times \Lambda_-,$$

where the Legendrian submanifold $\Lambda_+$ corresponds to the 1-jet

$$(d(e^2 + \eta^2 f), (e^2 + \eta^2 f)) \subset (T^* \Lambda \times \mathbb{R} = J^1(\Lambda), dz + \theta_\Lambda),$$

and $\Lambda_-$ corresponds to the 1-jet

$$(d(-e^{1/2} + \eta^2 f), (-e^{1/2} + \eta^2 f)) \subset (T^* \Lambda \times \mathbb{R} = J^1(\Lambda), dz + \theta_\Lambda),$$

under the above identification. In other words, the Legendrian ends correspond to the push-offs constructed in Section 6.1 above.

Finally, observe that the double-points of $L \cup L_\pm$ are in bijective correspondence with critical points of $F_\pm$.

6.2.3. **Computation of** $(CF(L_+, \partial))$. We are now ready to state and prove the main result of this section.

**Theorem 6.2.** For $0 < \eta < \epsilon < 1$ sufficiently small, there is an almost complex structure on $\mathbb{R} \times (P \times \mathbb{R})$ for which the wrapped Floer homology complex

$$(CF_*(L, L^\eta_+), \partial) = (CF^0_*(L, L_+) \oplus CF^\infty_*(L, L_+), \partial),$$
given by

$$CF^0(L, L_+) = C^\bullet_{Morse}(F_+),$$
$$CF^\infty(L, L_+) = C^\bullet_{Morse}(f) \oplus CL^{\bullet-2}(\Lambda),$$

has differential on the form

$$\partial = \begin{pmatrix} d_{F_+} & 0 & 0 \\ \gamma & d_f & 0 \\ g & \sigma & d_q \end{pmatrix},$$

where moreover \( \gamma \) is homotopic to \( \Gamma \) defined above, \((CL^{\bullet-2}(\Lambda), d_q)\) is the linearised Legendrian contact cohomology complex for \( \Lambda \) induced by the filling \( L \), and \( \sigma \) is the map described in Section 6.1. In particular, Cone(\( \gamma \)) is homotopy equivalent to \((C^\bullet_{Morse}(F_-), d_{F_-})\).

Remark 6.3. The proof of Corollary 2.6 given in [Ekh3] depends on the conjectural analytical results (1)-(5) stated in [Ekh2, Conjectural Lemma 4.11]. More precisely, the following statements are needed. First, the proof uses statements (1)-(3) together with a translation of the results in [EES4] to the version of Legendrian contact homology defined via the symplectisation. These results follow from Theorem 2.1, as shown in Section 6.1 above. Second, the proof depends on a consequence of statement (5). More precisely, it relies on the fact that \( \gamma \) is homotopic to \( \Gamma \) as shown in in Theorem 6.2. It should be noted that part (5) of the conjecture is stronger than the results obtained here. Namely, it claims that there exists an almost complex structure for which the discs in the definition of \( \gamma \) are in bijection with the flow-lines in the definition of \( \Gamma \).

Proof. By construction \( L_+ \) is cylindrical in the set \( \{ t \geq -1/2 \} \), \( L_- \) is cylindrical in the set \( \{ t \geq B+1 \} \), and \( L_- \) coincides with \( L_+ \) in the set \( \{ t \leq A-1 \} \).

It also follows that \( L_- = L_+ \circ V \), where \( V \) is the exact Lagrangian cylinder

$$\phi^1_H(\mathbb{R} \times \Lambda)$$

where

$$H = \rho(t)e^t: \mathbb{R} \times (P \times \mathbb{R}) \to \mathbb{R}_{\geq 0}$$

for a function \( \rho(t) \) satisfying \( \rho(t), \rho'(t) \geq 0, \rho|\{t \leq A-1\} = 0, \) and \( \rho|\{t \geq B+1\} = \epsilon^2 + \epsilon^{1/2} \).

Let \( J_P \) be a regular compatible almost complex structure on \( P \) which is adjusted to \( \Pi_{Log}(\Lambda) \) and moreover satisfies the assumptions of Theorem 2.1. We use \( J_\infty \) to denote its cylindrical lift. We fix a regular compatible almost complex structure \( J \) on \( \mathbb{R} \times (P \times \mathbb{R}) \) which is adjusted to \( L \cap \{ t \leq 0 \} \) and which coincides with \( J_\infty \) on \( \{ t \geq A-1 \} \).

Observe that the results in Section 6.1 may be assumed to hold for the linearised Legendrian cohomology of \( \Lambda \cup \Lambda_\pm \) defined in terms of \( J_\infty \) and the augmentation \( \epsilon \), which can be assumed to be induced by the filling \( L \) and almost complex structure \( J \). Here we may suppose that the augmentations induced by \( L \) and \( L_\pm \) can be identified, since \( L_\pm \) can be assumed to be arbitrarily \( C^1 \)-close to \( L \).
We let \((CF_\ast(L, L_\pm), \partial)\) be the complex determined by \(J\). First, the results in Section 6.1 proves that \(\partial_\infty\) is as claimed. Second, since \(J\) is adjusted to \(L \cap \{t \leq 0\}\), using the monotonicity result in Lemma 6.4 together with the theory in [Ekh1] shows that \(\partial_0 = d_{F_+}\) for \(\epsilon > 0\) small enough. We have thus concluded that

\[
\partial = \begin{pmatrix}
d_{F_+} & 0 & 0 \\
g & d_f & 0 \\
* & \sigma & d_q \\
\end{pmatrix},
\]

and it remains to show that \(\gamma \simeq \Gamma\).

By Proposition 5.10 we may assume that \(J\) also is regular for the \(J\)-holomorphic discs in the definition of the complex \((CF(L, L_-), \partial_-)\) and that the obvious identification of generators induces an isomorphism of complexes. In particular, \(\partial_- = \partial\) under the above identification.

We now consider a compatible almost complex structure \(J'\) which coincides with \(J\) in the set \(\{t \leq 0\} \cup \{t \geq B + 3\}\), which is adjusted to \(L \cap \{t \leq B + 2\}\) with respect to some (possibly different) taming symplectic form defined in a neighbourhood of \(L\), and which still is a (not necessarily cylindrical) lift of \(J_P\) in the set \(\{t \geq A - 1\}\).

To see the existence of such an adjusted almost complex structure we argue as follows. Let \(\tilde{\iota}\) be the symplectic immersion of a neighbourhood of the zero-section of \(T^*\Lambda\) into \(P\) which extends the immersion \(\iota: \Lambda \hookrightarrow \Pi_{Lag}(\Lambda) \subset P\) and which is used in the definition of the adjusted almost complex structure \(J_P\). Consider the (non-symplectic) embedding

\[
T^*\mathbb{R} \times T^*\Lambda \simeq T^*(\mathbb{R} \times \Lambda) \to \mathbb{R} \times (P \times \mathbb{R}),
\]

\[
((t, z), (q, p)) \mapsto (t, (\tilde{\iota}(q, p), h(q) + z)),
\]

defined in a neighbourhood of the zero-section of the domain, where \(h(q)\) denotes the \(z\)-coordinate of the lift of \(\iota\) to \(\Lambda \subset P \times \mathbb{R}\). Using the construction in Section 3.4, the product metric \(dt \otimes dt + g\) induces an almost complex structure in the domain which is pushed forward to a compatible almost complex structure satisfying the sought properties in a neighbourhood of \(L \cap \{A - 1 \leq t \leq B + 1\}\). It then suffices to perform an appropriate interpolation of this almost complex structure with \(J\).

After a perturbation \(J''\) of \(J'\) inside a compact set, we may assume that \(J''\) is regular for the pseudo-holomorphic discs in the definition of the complex \((CF(L, L_-), \partial')\). The assumptions on \(J'\) together with the monotonicity result in Lemma 6.4 shows that, for \(\epsilon > 0\) small enough, the results in [Ekh1] applies, giving

\[
\partial_0' = d_{F_-}.
\]

Together with Lemma A.2, it follows that \(\partial'\) is on the form

\[
\partial' = \begin{pmatrix}
d_{F_+} & 0 & 0 \\
\Gamma & d_f & 0 \\
* & \sigma & d_q \\
\end{pmatrix}.
\]
We now consider a path \( \{ J_s \} \) of compatible almost complex structures coinciding with \( J \) in the set \( \{ t \leq 0 \} \cup \{ t \geq B + 3 \} \), where \( J_0 = J \), \( J_1 = J' \), and such that the almost complex structures \( J_s \) all are lifts of \( J_P \) in the set \( \{ t \geq A - 1 \} \).

Using the monotonicity result in the below Lemma 6.4, for \( \epsilon > \eta > 0 \) sufficiently small, we may assume that every \( J_s \)-holomorphic strip with both punctures at generators corresponding to \( C_{Morse}(f) \) are contained inside \( \{ A \leq t \leq B \} \). Since \( \pi_F \) is \( (J_s, J_P) \)-holomorphic when restricted to this set, it follows that any such strip projects to a \( J_P \)-holomorphic strip in \( (P, \Pi_{Lag}(\Lambda \cup \Lambda')) \). Consequently, there can be no such \( J_s \)-holomorphic strips of negative index, since such a strip necessarily would project to a non-trivial \( J_P \)-holomorphic strip of negative index, contradicting the regularity of \( J_P \).

Furthermore, Lemma 6.4 implies that any \( J_s \)-holomorphic strip having punctures at double-points corresponding to \( C_{Morse}(F +) \) are contained inside \( \{ t \leq 0 \} \). Since \( J_s = J \) on this set, the regularity of \( J \) implies that there are no such strips of negative index.

The non-existence of the pseudo-holomorphic strips of negative index having both punctures at double-points corresponding to \( C_{Morse}(f) \) and \( C_{Morse}(F +) \), respectively, still holds if we consider a sufficiently small perturbation \( J'_s \) of the path \( J_s \). Without loss of generality, we may thus assume this property to hold for a path starting at \( J'_0 = J \) and ending at \( J''_1 = J'' \).

The above behaviour of the \( J'_s \)-holomorphic discs of index \(-1\) implies that the isomorphism of chain-complexes induced by Proposition 5.8 applied to the path \( J'_s \) is on the form

\[
\Psi : (CF(L, L_-), \partial_-) \to CF(L, L_-), \partial'
\]

\[
\Psi = \begin{pmatrix}
\text{id}_{C(F_+)} & 0 & 0 \\
\psi & \text{id}_{C(f)} & 0 \\
* & * & *
\end{pmatrix}.
\]

Finally, the chain-map property

\[\Psi \circ \partial_- = \partial' \circ \Psi,\]

can be written as

\[
\begin{pmatrix}
d_{F_+} \\
\psi \circ d_{F_+} + \gamma \\
* 
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & d_f \\
* & *
\end{pmatrix}
\]

\[
= \begin{pmatrix}
d_{F_+} \\
\Gamma + d_f \circ \psi \\
* 
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & d_f \\
* & *
\end{pmatrix},
\]

from which it immediately follows that \( \Gamma \simeq \gamma \).

\[\square\]

**Lemma 6.4.** Let \( L \) and \( L_{\eta_\pm} \) be the exact Lagrangian fillings as constructed in Section 6.2 and let \( J_s, s \in [0, 1] \), be a family of compatible almost complex structures on \( \mathbb{R} \times (P \times \mathbb{R}) \). Let \( U_0, U_{B+2} \), and \( U_{[A, B]} \) denote fixed compact neighbourhoods of \( L \cap \{ t \leq 0 \} \), \( L \cap \{ t \leq B + 2 \} \), and \( L \cap \{ A \leq t \leq B \} \), respectively. Choosing \( 0 < \eta < \epsilon < 1 \) sufficiently small we may suppose the following.
(1) Any $J_s$-holomorphic strip having boundary on $L \cup L_{+}^{\eta, \epsilon}$ and both punctures at double-points corresponding to $C_{\text{Morse}}^{\bullet}(F_{+}^{\eta, \epsilon})$ is contained inside $U_0$.

(2) Any $J_s$-holomorphic strip having boundary on $L \cup L_{-}^{\eta, \epsilon}$ and both punctures at double-points corresponding to $C_{\text{Morse}}^{\bullet}(F_{-}^{\eta, \epsilon})$ is contained inside $U_{B+2}$.

(3) Any $J_s$-holomorphic strip having boundary on $L \cup L_{+}^{\eta, \epsilon}$ and both punctures at double-points corresponding to $C_{\text{Morse}}^{\bullet}(f)$ is contained inside $U_{[A, B]}$.

Proof. Consider the symplectic form $\omega := d(e^{\epsilon}(dz + \theta))$ on $\mathbb{R} \times (P \times \mathbb{R})$. The monotonicity property for the $\omega$-area of $J_s$-holomorphic curves with boundary applies (see Proposition 4.7.2(ii) and Proposition 7.3.1(ii) in [Sik]), giving a lower bound on the $\omega$-area of a strip passing through a fixed point.

More precisely, there is a fixed metric on $\mathbb{R} \times (P \times \mathbb{R})$ and a constant $C > 0$ independent of $\epsilon$ and $\eta$ for which the following holds. The two components $(L \cup L_{+}^{\eta, \epsilon}) \cap \{ t = t_0 \}$, for $t_0 \in \{0, A, B\}$, are at a distance at least $C\epsilon^2$ while two components $(L \cup L_{-}^{\eta, \epsilon}) \cap \{ t = B+2 \}$ are at a distance at least $C(\epsilon^{1/2} - \eta^2)$.

By the monotonicity properties there exists a constant $D > 0$ independent of small enough choices of $\epsilon > \eta > 0$ such that any $J_s$-holomorphic strip as above passing through the sets $\partial U_0$ or $\partial U_{[A, B]}$ (with either a boundary point or an interior point) has $\omega$-area bounded from below by $D\epsilon^4$, while any such strip passing through $\partial U_{B+2}$ has $\omega$-area bounded from below by $D(\epsilon^{1/2} - \eta^2)^2$.

Using the exactness of $L$ and $L_{\pm}$, we compute the $\omega$-area of the above discs to be as follows. There is a constant $E > 0$ independent of $\epsilon$ and $\eta$ such that any $J_s$-holomorphic strip in (1), (2) and (3) have $\omega$-area bounded from above by $E\epsilon^5$, $E\epsilon^2$, and $E\eta^2$, respectively.

For $0 < \eta < \epsilon < 1$ sufficiently small, by comparing the two area bounds, we deduce that $J_s$-holomorphic strip in (1), (2), and (3) are disjoint from $\partial U_0$, $\partial U_{B+2}$, and $\partial U_{[A, B]}$, respectively. \qed

7. Proof of Theorem 2.1

In the following, we fix a front-generic Legendrian submanifold $\Lambda \subset P \times \mathbb{R}$. We begin with the below lemma, whose proof is similar to the analysis made in the proof of [ENS Theorem 7.7]. Here we make the identification $P \times \mathbb{R} = P \times i\mathbb{R} \subset P \times \mathbb{C}$ and define

$L := \{(p, x + iy) \in P \times \mathbb{C}; \ (p, iy) \in \Lambda \} \simeq \Lambda \times \mathbb{R}$.

Lemma 7.1. Given a $J_P$-holomorphic polygon $u$: $(D^2, \partial D^2) \to (P, \Pi_{\text{Log}}(\Lambda))$ with punctures mapping to double-points, there is a $(J_P \oplus i)$-holomorphic lift $(u, a): (D^2, \partial D^2) \to (P \times \mathbb{C}, L)$.

Moreover, if $p \in \partial D^2$ is a puncture which is mapped by $u$ to a double-point corresponding to the Reeb chord $\{ u(p) \} \times [A, A + \ell]$, it follows that there are
holomorphic coordinates identifying $D^2$ with \{s + it; 0 ≤ t ≤ 1\} ⊂ $\mathbb{C}$ such that $p$ corresponds to $s = +\infty$ and for which the bound

$$\|a(s + it) - C \pm \ell(s + it)\| \leq e^{-\lambda s}$$

holds for some $C \in \mathbb{C}$, $\lambda > 0$, and $s \gg 0$ large enough, where the norm is induced by some choice of metric near the double-points of $\Pi_{\text{Lag}}(\Lambda)$.

**Proof.** Away from the boundary-punctures, there is a lift

$$(u|_{\partial D^2}, h) : \partial D^2 \to \Lambda \subset \mathbb{P} \times \mathbb{R}.$$  

By abuse of notation, we let $h : D^2 \to \mathbb{R}$ denote the harmonic extension its $\mathbb{R}$-factor. Observe that $h$ is bounded and $C^\infty$ away from the boundary punctures, where it has jump discontinuities.

Let $-g$ be the Harmonic conjugate of $h$, which is smooth in the interior of $D^2$. We define

$$a(s + it) := g(s, t) + ih(s, t),$$

which is holomorphic in the interior of $D^2$ and satisfies the correct boundary condition. It remains to show that $a$ has the above asymptotic behaviour at its boundary punctures.

We fix a boundary puncture $p \in \partial D^2$ of $u$. Choose a conformal identification of the domain $D^2$ with the strip \{s + it; 0 ≤ t ≤ 1\} ⊂ $\mathbb{C}$ such that the boundary-puncture $p$ corresponds to $s = +\infty$. Also fix some coordinates in a neighbourhood of $u(p)$. An application of [RS, Theorem B] yields that there are $C^\infty$ functions $v, w : \{s + it; 0 ≤ t ≤ 1\} \to \mathbb{C}^n$ for which

$$u(s + it) = -\frac{1}{\lambda} e^{-\lambda s} v(t) + w(s, t)$$

for $s \gg 0$ sufficiently large and some $\lambda > 0$. Moreover, $\|w(s, t)\|_{C^k[s, +\infty)} = O(e^{-(\lambda+\delta)s})$ for $s \gg 0$ and some $\delta > 0$.

Let $h_+(s)$ and $h_-(s)$ denote $h$ above restricted to the boundary component \{t = 1\} and \{t = 0\}, respectively. Recall that the contact form is given by $dz + \theta$, where $\theta$ is an 1-form on $\mathbb{P}$. Considering the expressions

$$h_+(s_1) - h_+(s_0) = \int_{s_0}^{s_1} u|_{\{t=1\}} \ast (-\theta) ds,$$

$$h_-(s_1) - h_-(s_0) = \int_{s_0}^{s_1} u|_{\{t=0\}} \ast (-\theta) ds,$$

and using the above asymptotic expansion, it follows that

$$h_\pm(s) = C_\pm + O(e^{-\lambda s}),$$

$$\left(\frac{d}{ds}\right)^k h_\pm(s) = O(e^{-\lambda s}), \quad k > 0,$$

for $s \gg 0$.

The Poisson kernel on the above strip \{s + it; 0 ≤ t ≤ 1\} is given by

$$P(s, t) = \frac{\sin \pi t}{\cosh s - \cos \pi t}$$
as computed in [Wid]. The harmonic extension to this strip of the function given by \( h_{\pm}(s) \) on the respective boundary components is given by

\[
h(s, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} P(\sigma - s, t)h_{\pm}(\sigma) d\sigma \\
+ \frac{1}{2\pi} \int_{-\infty}^{+\infty} P(\sigma - s, 1 - t)h_{\pm}(\sigma) d\sigma.
\]

Observe that \( h(s, t) \) is \( C^\infty \) up to the boundary for \( s \gg 0 \) as follows by standard properties of the Poisson kernel. An explicit comparison with the harmonic function

\[
C_- + (C_+ - C_-)t + e^{-\lambda s}(D \cos \lambda t + E \sin \lambda t)
\]
on the strip, where \( C_{\pm}, D, E \in \mathbb{R} \), we conclude that

\[
h(s, t) = C_- + (C_+ - C_-)t + O(e^{-\lambda s})
\]
for \( s \gg 0 \). Here we have used the fact that all of the above functions are obtained by convolutions of their smooth restrictions to the boundary with the above Poisson kernel.

Similarly, since convolving with the Poisson kernel commutes with \( \partial_s \), it also follows that

\[
(\partial_s)^k h(s, t) = O(e^{-\lambda s}), \quad k > 0.
\]

In order to find a bound for \( \partial_t h(s, t) \) we proceed as follows. First, write \( \tilde{h}(s, t) := h(s, t) - (C_- + (C_+ - C_-)t). \) Observe that

\[
\partial_t \tilde{h}(s, 1/2) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\partial_t P)(\sigma - s, 1/2)(h_{-}(\sigma) - C_-) d\sigma \\
- \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\partial_t P)(\sigma - s, 1/2)(h_{+}(\sigma) - C_+) d\sigma,
\]

where

\[
\partial_t P(s, 1/2) = \pi \left. \frac{\cos \pi t \cosh s - \frac{1}{2}}{(\cosh s - \cos \pi t)^2} \right|_{t=1/2} = O(e^{-2|s|})
\]
for \( |s| \gg 0 \). From this one can deduce the estimate

\[
(7.2) \quad \partial_t \tilde{h}(s, 1/2) = O(e^{-\lambda s})
\]
for \( s \gg 0 \), assuming that \( 0 < \lambda < 2 \). The harmonicity of \( \tilde{h}(s, t) \) together with \((7.1)\) implies that

\[
(\partial_t)^2 \tilde{h}(s, t) = -(\partial_s)^2 \tilde{h}(s, t) = O(e^{-\lambda s}),
\]
and which together with \((7.2)\) can be integrated to

\[
\partial_t \tilde{h}(s, t) = O(e^{-\lambda s})
\]
for \( s \gg 0 \).
In particular, the Harmonic conjugate $-g(s,t)$ of $h(s,t)$ on the strip satisfies
\[ g(s,t) = B + (C_+ - C_-)s + O(e^{-\lambda s}) , \]
for some $B \in \mathbb{R}$, as follows by studying the Cauchy-Riemann equations
\[ \partial_s g = \partial_t h, \quad \partial_t g = -\partial_s h \]
and using the above asymptotic bounds $\partial_t h = (C_+ - C_-) + O(e^{-\lambda s})$ and $\partial_s h = O(e^{-\lambda s})$.

From this the sought asymptotic bound on $g(s,t) + ih(s,t)$ follows. □

Remark 7.2. In the case when $\theta = -d^{J_P} \alpha = -d\alpha(J_P \cdot)$ for some smooth function $\alpha : P \to \mathbb{R}$, the map
\[ (\mathbb{R} \times (P \times P), \mathbb{R} \times \Lambda) \to (P \times \mathbb{C}, L) \]
\[ (t, (p,z)) \mapsto (p, t - \alpha(p) + iz) \]
can be seen to pull back $J_P \oplus i$ to the cylindrical lift of $J_P$.

In the following we fix a regular compatible almost complex structure $J_P$ on $P$ which moreover is integrable in some neighbourhood of the double-points of $\Pi_{\text{Lag}}(\Lambda)$.

Proof of Theorem 2.1. Since the projection $\pi_P$ is $(J, J_P)$-holomorphic, and since $J_P$ is regular, Lemma 7.3 below implies that $J$ is regular as well and that the smooth map
\[ \mathcal{M}_{a;b}(\mathbb{R} \times \Lambda; J)/\mathbb{R} \to \mathcal{M}_{a;b}(\Pi_{\text{Lag}}(\Lambda); J_P), \]
\[ \tilde{u} \mapsto \pi_P \circ \tilde{u} \]
is a local diffeomorphism.

Since two pseudo-holomorphic discs which agree on the boundary coincide by [Laz, Theorem 3.5], this map is injective as well. To show that the map is a diffeomorphism, it thus suffices to check that the above map takes values in every connected component of $\mathcal{M}_{a;b}(\Pi_{\text{Lag}}(\Lambda); J_P)$.

Observe that in the almost Stein case, that is when $\theta = -d^{J_P} \alpha$, Lemma 7.1 together with Remark 7.2 shows that an explicit lift may be constructed. However, in the general we proceed as follows.

The above moduli spaces can be compactified by using Gromov-Hofer compactness. See [BES3] for the case of the moduli space of $J_P$-holomorphic polygons in $P$ and [BEH+] for the case of the moduli space of $J$-holomorphic discs in the symplectisation. The compactified moduli spaces are manifolds having boundary with corners.

Moreover, points in the boundary strata correspond to so called broken configurations. Roughly speaking, a broken configuration is a tree whose vertices consist of pseudo-holomorphic discs and whose edges denote a matching positive and negative puncture of the respective discs at the vertices. Such a broken configuration can be glued to give a pseudo-holomorphic solution. Also, it is easily shown that the sum of $d\theta$-areas of the discs in a broken
configuration equals the $d\theta$-area of corresponding glued pseudo-holomorphic disc.

We let

$$\mathcal{P} : \overline{M}_{a:b}(\mathbb{R} \times \Lambda; J)/\mathbb{R} \to \overline{M}_{a:b}(\Pi_{\text{Lag}}(\Lambda); J_P)$$

be the above projection extended continuously to the compactified moduli spaces.

By the formula for the $d\theta$-energy in terms of the action of Reeb chords, together with the fact that there are only finitely many Reeb chords on $\Lambda$, it follows that there are finitely many non-empty moduli spaces as above in the case under consideration. Moreover, the above compactness theorems imply that each moduli space has finitely many components.

We prove the theorem by induction on the $d\theta$-energy. Assume that the statement has been shown for every moduli space consisting of discs having $d\theta$-energy strictly less than $A$. We now show that $\mathcal{P}$ maps into each component of the moduli space $\overline{M}_{a:b}(\Pi_{\text{Lag}}(\Lambda); J_P)$ consisting of $J_P$-holomorphic discs having $d\theta$-energy equal to $A$.

We begin with the case of a component of $\overline{M}_{a:b}(\Pi_{\text{Lag}}(\Lambda); J_P)$ having non-empty boundary. By the induction hypothesis, the above map $\mathcal{P}$ maps into its boundary. Consider a broken configuration which is the pre-image under $\mathcal{P}$ of a boundary point of the above component. Gluing this broken configuration, one obtains a solution which is contained in the interior of $\overline{M}_{a:b}(\mathbb{R} \times \Lambda; J)/\mathbb{R}$ and which is mapped by $\mathcal{P}$ to the corresponding component of $\overline{M}_{a:b}(\Pi_{\text{Lag}}(\Lambda); J_P)$.

The remainder of the proof concerns the case when a component of $\overline{M}_{a:b}(\Pi_{\text{Lag}}(\Lambda); J_P)$ has empty boundary. In this case we proceed as follows to construct a lift under the map $\mathcal{P}$.

The integrability of $J_P$ in a neighbourhood of the double-points of $\Pi_{\text{Lag}}(\Lambda)$ together with the compatibility of $J_P$ implies that $d\theta$ is a Kähler form there. It thus follows that $d\theta = -dd^cJ_P g$ in this neighbourhood for some real-valued function $g$. Since $\theta = -d^cJ_P g + df$ for some smooth real-valued function $f$ defined in a possibly even small neighbourhood, writing

$$\alpha := g - f(-J_P)$$

we conclude that $\theta = -d^cJ_P \alpha$ in some open neighbourhood $U$ of the double-points.

We choose a cut-off function $\rho : P \to \mathbb{R}$ which has support inside $U$ and which has the property that $\rho = 1$ in a neighbourhood $V \subset U$ of the double-points. We let $\beta : P \times \mathbb{R} \to \mathbb{R}$ be the smooth extension of $\rho \cdot \alpha$ by zero to all of $P$.

Consider the one-parameter family of smooth one-forms $\lambda_s$ on $P \times \mathbb{R}$ defined by

$$\lambda_s = dz + s\theta$$

in $(P \setminus U) \times \mathbb{R}$, and by

$$\lambda_s = dz - d^cJ_P((1-s)\beta + s\alpha)$$
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in the neighbourhood $U \times \mathbb{R}$. Observe that $\lambda_1 = dz + \theta$ coincides with the original contact form on $P \times \mathbb{R}$ and that each $\lambda_s$ coincides with the original contact form in the neighbourhood $V \times \mathbb{R}$ containing the Reeb chords.

We consider the 1-parameter family $\xi_s := \ker \lambda_s \subset T(P \times \mathbb{R})$ of tangent hyper-plane fields. For each $s \in [0, 1]$, $\xi_s$ satisfies the following properties.

- $\xi_s$ is transverse to $\partial_z$.
- $\xi_s$ is invariant with respect to translations of the $z$-coordinate.
- $\xi_s$ coincides with the contact-distribution $\ker(dz + \theta)$ in $V \times \mathbb{R}$.
- $\xi_s$ is symplectic with respect to $d\theta$.

For every $s \in [0, 1]$ we lift $J_P$ to an almost complex structure $J_s$ on the symplectisation $\mathbb{R} \times (P \times \mathbb{R})$ which is uniquely defined by the following properties.

- $J_s$ is invariant with respect to translations in the $(t,z)$-planes.
- $J_s \partial_t = \partial_z$.
- $J_s \xi_s = \xi_s$.
- $\pi_P$ is $(J_s,J_P)$-holomorphic.

It follows that each $J_s$ is tamed by the symplectic form on the symplectisation.

First observe that, since $\xi_1 = \ker(dz + \theta)$ is the contact distribution, it follows that $J_1 = J$. For the same reasons, since $\xi_s$ agrees with the contact distribution in $V \times \mathbb{R}$, it follows that $J_s = J$ in this neighbourhood for each $s \in [0, 1]$.

Second, it can be checked that $J_0$ coincides with the pull-back of the almost complex structure $J_P \oplus i$ on $P \times \mathbb{C}$ under the diffeomorphism

$$\iota : \mathbb{R} \times (P \times \mathbb{R}) \to P \times \mathbb{C},$$

$$\iota(t, (p, z)) \mapsto (p, t - \beta(p) + iz).$$

To that end, observe that the hyper-surfaces $\{t\} \times P \times \mathbb{R}$ are mapped by $\iota$ to level-sets of the smooth function

$$\tau : P \times \mathbb{C} \to \mathbb{R},$$

$$(p, x + iy) \mapsto x + \beta(p),$$

and that, since

$$\iota^*(-d^{J_P \oplus i} \tau) = \iota^*(dy - d^{J_P} \beta) = dz - d^{J_P} \beta = \lambda_0,$$

the $(J_P \oplus i)$-complex tangencies to the latter hyper-surfaces correspond to $J_0$-complex tangencies of the former.

Recall that we want to lift a $J_P$-holomorphic disc lying in a transversely cut-out component $\mathcal{M} \subset \mathcal{M}_{a;b}(\Pi_{\text{Lag}}(\Lambda); J_P)$ without boundary. By using Lemma 7.1 together with the above $(J_0, J_P \oplus i)$-holomorphic diffeomorphism $\iota$ we can lift each $u \in \mathcal{M}$ to a finite-energy $J_0$-holomorphic disc in the symplectisation. We let $\widetilde{\mathcal{M}} \subset \mathcal{M}_{a;b}(\mathbb{R} \times \Lambda; J_0)$ denote the component of the moduli space containing these lifts, which is transversely cut out by Lemma 7.3.
Consider the connected component
\[ \mathcal{W} \subset \bigcup_{s \in [0,1]} \mathcal{M}_{a,b}(\mathbb{R} \times \Lambda; J_s) \]
containing \( \tilde{\mathcal{M}} \). Lemma 7.3 implies that the above moduli-spaces are transversely cut out for each \( s \in [0,1] \) and consequently \( \mathcal{W} \) is a smooth trivial cobordism from \( \tilde{\mathcal{M}} \) to \( \mathcal{W} \cap \{ s = 1 \} \). This shows that there is a \( J_1 \)-holomorphic lift of \( \mathcal{M} \).

To that end, the compactness result in [BEH+] has been used. This applies in our setting since \( (d\theta, \lambda_s) \) is a stable Hamiltonian structure on \( P \times \mathbb{R} \), as follows from
\[ R \partial z = \ker d\theta \subset \ker d\lambda_s, \ i_\partial z \lambda_s = 1, \]
together with the fact that the almost complex structure \( J_s \) is cylindrical with respect to this stable Hamiltonian structure.

Finally, the computation
\[ -dJ_s^*(t - (1 - s)\beta) = \lambda_s + (1 - s)dJ_s^*\beta = \lambda_s + (1 - s)d^P \beta = dz + s\theta \]
shows that the function \( t - (1 - s)\beta \) on the symplectisation is weakly \( J_s \)-convex. This implies that the maximum-principle applies to this function pulled back to a \( J_s \)-holomorphic disc. \( \square \)

7.1. **Functional-analytic set-up.** Here we sketch the construction of the appropriate functional-analytical spaces in order to prove the needed transversality results. We refer to [EES3] for more details.

The moduli spaces of the above \( J_P \)-holomorphic discs are constructed as the vanishing-locus of the section
\[ u \mapsto \overline{\partial}_{J_P} u = du + J_P du \circ i \]
of a bundle \( \mathcal{E}^{0,1} \to \mathcal{B} \). Here \( \mathcal{B} \) denotes a Banach manifold consisting of maps \( u: (D^2, \partial D^2) \to (P, \Pi_{\text{Log}}(\Lambda)) \) in an appropriate Sobolev class. We refer to [EES3] for more details.

By the linearisation of \( \overline{\partial}_{J_P} \) at a \( J_P \)-holomorphic disc \( u \in \mathcal{B} \) we denote the linear operator
\[ D_u := \pi_F \circ T\overline{\partial}_{J_P} : T_u \mathcal{B} \to \Omega^{0,1}(u^* TP), \]
where \( \pi_F \) is the projection to the fibre of the bundle \( \mathcal{E}^{0,1} \).

By a choice of metric, one can make the identification
\[ T_u \mathcal{B} \simeq \Gamma(u^* TP, u^* T_P(\Pi_{\text{Log}}(\Lambda))), \]
where the latter denotes the space of sections in \( u^* TP \) which take value in the sub-bundle \( u^* T_P(\Pi_{\text{Log}}(\Lambda)) \) along the boundary. Observe that the above spaces of sections are the completions of the smooth sections with respect to an appropriate Sobolev norm, but we have suppressed this from the notation.
The above constructions can be extended to the case of a symplectisation \( \mathbb{R} \times (P \times \mathbb{R}) \) as well, where \( \mathcal{B} \) becomes a section in the bundle \( \tilde{E}^{0,1} \rightarrow \tilde{B} \) over the Banach manifold \( \tilde{B} \) consisting of maps \( \tilde{u}: (D^2, \partial D^2) \rightarrow (\mathbb{R} \times (P \times \mathbb{R}), \mathbb{R} \times \Lambda) \) in an appropriate Sobolev class. We refer to \([\text{Abb}]\) and \([\text{Dra}]\) for examples of this, although the analytical set-up there is a bit different from that in \([\text{EES3}]\).

In any case, when \( B \) and \( \tilde{B} \) consists of maps in suitable Sobolev spaces with positive weights, the linearisation \( D \) of the respective non-linear Cauchy-Riemann operator (which itself is a first order operator of Cauchy-Riemann type) is elliptic. We use

\[
\text{index} \ D = \dim \ker \ D - \dim \coker \ D.
\]

to denote its Fredholm index. In the case when the linearisation \( D \) at a solution is surjective, that is \( \ker D = 0 \), it follows that a neighbourhood of the solution in its moduli space is transversely cut out, and hence a smooth finite-dimensional manifold.

Let \( J \) be an almost complex structure on \( \mathbb{R} \times (P \times \mathbb{R}) \) which is invariant under translation in the \((t, z)\)-planes, satisfies \( J \partial_t = \partial_z \), and which has the property that \( \pi_P: (\mathbb{R} \times (P \times \mathbb{R}), J) \rightarrow (P, J_P) \) is \((J, J_P)\)-holomorphic. Furthermore, we assume that \( J \) preserves the contact-planes above neighbourhoods of the double-points of \( \Pi_{\text{Lag}}(\Lambda) \subset P \). Observe that it follows that \( J \) is tamed by the symplectic form.

**Lemma 7.3.** Let \( J \) be as above, and consider a finite-energy \( J \)-holomorphic disc

\[
\tilde{u}: (D^2, \partial D^2) \rightarrow (\mathbb{R} \times (P \times \mathbb{R}), \mathbb{R} \times \Lambda)
\]

with punctures asymptotic to Reeb chords. It follows that

\[
\text{index} \ D_{\tilde{u}} = \text{index} \ D_{\pi_P \circ \tilde{u}} + 1.
\]

If \( D_{\pi_P \circ \tilde{u}} \) is surjective it follows that \( D_{\tilde{u}} \) is surjective as well and, furthermore, the tangent-map

\[
T_{\tilde{u}}(\mathcal{M}_{a,b}(\mathbb{R} \times \Lambda; J)/\mathbb{R}) \rightarrow T_{\pi_P \circ \tilde{u}}(\mathcal{M}_{a,b}(\Pi_{\text{Lag}}(\Lambda); J_P))
\]

induced by the projection \( \tilde{u} \mapsto \pi_P \circ \tilde{u} \) is an isomorphism.

**Proof.** Observe that a positive (respectively negative) puncture of \( \tilde{u} \) projects to a positive (respectively negative) puncture of the \( J_P \)-holomorphic polygon

\[
u := \pi_P \circ \tilde{u}
\]

under \( \pi_P \).

The relation between the Fredholm indices follows by the computations in \([\text{EES3}]\) and \([\text{Ekh2}]\) which express the indices in terms of Conley-Zehnder indices of the involved Reeb chords. Also, see Section 4.2.3.

We fix the metric

\[
\tilde{g} := dt \otimes dt + g + dz \otimes dz, \quad g(v_1, v_2) := d\theta(v_1, J_P v_2)
\]
on \( \mathbb{R} \times (P \times \mathbb{R}) \), where \( g \) is the metric on \( P \) induced by the symplectic form and choice of compatible almost complex structure. Observe that under the splitting
\[
T_{(t_0, (p_0, z_0))}(\mathbb{R} \times (P \times \mathbb{R})) = \mathbb{R} \oplus T_{p_0}P \oplus \mathbb{R}
\]
the exponential map induced by \( \tilde{g} \) takes the form
\[
\tilde{\exp}_{(t_0, (p_0, z_0))}((t, v, z)) = (t_0 + t, (\exp_{p_0}(v), z_0 + z)),
\]
where \( \exp \) denotes the exponential map on \( P \) induced by the metric \( g \).

We use \( \tilde{\exp} \) to make the identification
\[
T_u\tilde{\mathcal{B}} \simeq \Gamma\left(\tilde{u}^*T(\mathbb{R} \times (P \times \mathbb{R})), \tilde{u}|_{\partial D^2_\Lambda}(T(\mathbb{R} \times \Lambda))\right)
\]
and \( \exp \) to make the identification
\[
T_u\mathcal{B} \simeq \Gamma\left(u^*TP, u|_{\partial D^2_\Lambda}(TP_{\Pi_{Log}}(\Lambda))\right).
\]
The tangent-map
\[
(T\pi_P): u^*T(\mathbb{R} \times (P \times \mathbb{R})) \to u^*TP
\]
induces a canonical map
\[
\Phi: \Omega^{0,1}(\tilde{u}^*T(\mathbb{R} \times (P \times \mathbb{R}))) \to \Omega^{0,1}(u^*TP).
\]
Using the above identification, it readily follows that \( \Phi \circ D_{\tilde{u}} = D_u \circ T\pi_P \), and hence that \( T\pi_P \) restricts to a map
\[
T: \ker D_{\tilde{u}} \to \ker D_u.
\]
We will now investigate \( \ker T = \ker D_{\tilde{u}} \cap E \) where
\[
E \subset \Gamma\left(\tilde{u}^*T(\mathbb{R} \times (P \times \mathbb{R})), \tilde{u}|_{\partial D^2_\Lambda}(T(\mathbb{R} \times \Lambda))\right)
\]
consists of sections in the trivial complex sub-bundle spanned by \( \partial_t \) and \( \partial_z \). Using the fact that \( J \) is invariant under translations in the \((t, z)\)-planes, it follows that \( D_{\tilde{u}}|_E = \overline{\partial} \), i.e. the linearisation restricts to the standard Cauchy-Riemann operator on \( E \).

Since \( \ker T \subset E \) thus consists of holomorphic sections and since the linearised boundary condition implies that the sections of \( E \) are on the form \( f\partial_t \) along \( \partial D^2 \) for some function \( f: \partial D^2 \to \mathbb{R} \), we conclude that \( \ker T = \mathbb{R}\partial_t \) is one-dimensional.

In conclusion, we have shown that
\[
1 + \dim \ker D_u \geq \dim \ker D_{\tilde{u}}.
\]
Moreover, in the case when \( \operatorname{coker} D_u = 0 \) it follows that
\[
\operatorname{index} D_{\tilde{u}} = 1 + \operatorname{index} D_u = 1 + \dim \ker D_u,
\]
and the above inequality yields \( \operatorname{index} D_{\tilde{u}} \geq \dim \ker D_{\tilde{u}} \) and hence \( \operatorname{coker} D_{\tilde{u}} = 0 \). Finally, this also shows that \( T \) is surjective. \( \square \)
The following Lemma is needed in the proof of Proposition 5.1.10. Since it follows by an analysis similar to the previous lemma, we show it here.

Let $V$ be an exact Lagrangian cobordism arising as $\varphi^H_1(\mathbb{R} \times \Lambda')$, where $H = \rho(t)e^t$ and $\rho(t), \rho'(t) \geq 0$. Furthermore, we assume that $(\mathbb{R} \times \Lambda) \cup V$ has transverse self-intersections and that the Legendrian ends are chord-generic.

Finally, we assume that $\mathcal{J}$ is integrable in a neighbourhood of the double-points of $\Pi_{\text{Lag}}(\Lambda \cup \Lambda')$ and that $\mathcal{J}$ is the cylindrical lift of $\mathcal{J}_P$. Observe that the projection induces a map

$$M_{f,a,e,b}((\mathbb{R} \times \Lambda) \cup V; J_\infty) \to M_{f,a,e,b}(\Pi_{\text{Lag}}(\Lambda \cup \Lambda'); J_P)$$

$$\tilde{u} \mapsto \pi_P \circ \tilde{u}$$

of moduli spaces in this setting as well, where $f$ and $e$ denote either Reeb-chords or double-points.

Lemma 7.4. Let $J$ be the cylindrical lift of $J_P$ as above and consider a $J$-holomorphic disc

$$\tilde{u}: (D^2, \partial D^2) \to (\mathbb{R} \times (P \times \mathbb{R}), (\mathbb{R} \times \Lambda) \cup V)$$

in the moduli space $M_{f,a,e,b}((\mathbb{R} \times \Lambda) \cup V; J)$ as above. It follows that

$$\text{index} D_{\tilde{u}} = \text{index} D_{\pi_P \circ \tilde{u}}$$

in the case when $e$ is a double-point and

$$\text{index} D_{\tilde{u}} = \text{index} D_{\pi_P \circ \tilde{u}} + 1$$

in the case when $e$ is a Reeb chord. If $\Pi_P \circ \tilde{u}$ is constant, i.e. $\tilde{u}$ is a strip contained in the $(t,z)$-plane, it follows that $D_{\tilde{u}}$ is surjective. After perturbing $J_P$, one may assume that $D_{\tilde{u}}$ is surjective in the case when $\pi_P \circ \tilde{u}$ is non-constant as well.

Proof. The statement concerning the indices follow from their expressions in terms of the Conley-Zehnder indices of the Reeb chords. See Sections 4.2.3 and 5.2.

We begin with the case of an $J$-holomorphic strip $\tilde{u}$ projecting to a constant $J_P$-holomorphic disc under $\pi_P$. More precisely, we assume that the domain of $\tilde{u}$ is $D^2 \setminus \{p_1, p_2\}$ and that its image is contained in the $(z,t)$-plane over a double-point $q \in \Pi_{\text{Lag}}(\Lambda \cup \Lambda') \subset P$. We let $L_{\pm} \subset T_qP$ denote the tangent-planes of the two branches of $\Pi_{\text{Lag}}(\Lambda \cup \Lambda')$ at $q$.

As in the proof of Theorem 2.1 since $J_P$ is integrable in a neighbourhood of $q$, there is a neighbourhood $U \subset P$ of $q$ for which $(\mathbb{R} \times (U \times \mathbb{R}), J)$ is biholomorphic to $(U \times \mathbb{C}, J_P \oplus i)$ by a map on the form

$$(t, (p,z)) \mapsto (p, t - \alpha(p) + iz),$$

where $\alpha$ is a smooth real-valued function. We use the Euclidean metric in a holomorphic chart $U \times \mathbb{C}$ to make the identification

$$T_q\mathcal{S} \simeq \Gamma (\tilde{u}^*T(\mathbb{R} \times (P \times \mathbb{R})), \tilde{u}|_{\partial D^2}(T(\mathbb{R} \times \Lambda)))$$,
which in this case is the space of sections of the trivial complex bundle 

\[(T_q P, J_P) \oplus (\mathbb{R}\partial_t \oplus i\mathbb{R}\partial_z \simeq \mathbb{C}) \times D^2 \to D^2\]

satisfying the appropriate boundary condition. Moreover, the linearised \(\overline{\mathcal{D}}_J\)-operator \(\overline{\mathcal{D}}_u\) is the standard \(\partial\)-operator acting on these sections.

Let \(s = (s_{T_q P}, s_C) \in \ker \overline{\mathcal{D}}_u\) be a holomorphic section satisfying the linearised boundary condition, which implies that the component \(s_{T_q P}\) is contained in \(L_\pm \subset T_q P\) along the respective boundary-arcs of \(\partial D^2 \setminus \{p_0, p_1\}\). Without loss of generality we may assume that the subspace \(L_-\) is identified with the real part \(\mathbb{R}^n \subset \mathbb{C}^n\) while \(L_+\) is the real span of \(e^{i\theta_i}e_i\) for the standard basis \(e_i\) of \(\mathbb{C}^n\), where \(0 < \theta_i < \pi\).

Consider a holomorphic identification of \(D^2 \setminus \{p_1, p_2\}\) with the strip \(\{s + it; 0 \leq t \leq 1\} \subset \mathbb{C}\) for which \(p_1\) and \(p_2\) correspond to \(s = -\infty\) and \(s = +\infty\), respectively. It follows that the \(j\):th component of the holomorphic map \(s_{T_q P} : \{0 \leq t \leq 1\} \to T_q P\) has a power-series expansion

\[s_{T_q P,j}(s + it) = \sum_{k \in \mathbb{Z}} c_{j,k} e^{(\theta_j + k\pi)(s + it)}, \quad c_{j,k} \in \mathbb{R}.\]

Since this section is in a weighted Sobolev space with positive weights on its two ends, and since \(\theta_j \neq 0\), the component \(s_{T_q P}\) vanishes.

From this we conclude the elements of \(\ker \overline{\mathcal{D}}_u\) are on the form \((0, s_C)\). Using the above coordinates on the disc, these sections are identified with holomorphic tangent vector-fields on the strip \(\{s + it; 0 \leq t \leq 1\}\) which are real along the boundary. A power-series expansion as above shows that \(\ker \overline{\mathcal{D}}_u\) is one-dimensional and generated by the constant real vector-fields.

In the remaining cases, the transversality result \(\text{[EES3, Proposition 2.3, Lemma 4.5]}\) adapted to the symplectisation shows that perturbations of \(J\) within the space of cylindrical lifts of almost complex structures \(J_P\) on \(P\) suffices to achieve transversality, since the projection of the disc to \(P\) is non-trivial. Also, see the proof of \(\text{[Dra, Theorem 4.2]}\). \(\square\)

Appendix A.

**Lemma A.1.** Let \(V\) be an exact Lagrangian cobordism in the symplectisation \(\mathbb{R} \times Y\) from \(\Lambda_-\) to \(\Lambda\) and let \(V' = \varphi_{H_s}^1(V)\) be an exact Lagrangian cobordism from \(\Lambda_-\) to \(\Lambda'\) which is Hamiltonian isotopic to \(V\) (we allow \(\Lambda_- = \emptyset\)).

If \(H_s\) has support in \([-N, +\infty) \times K\) for some compact set \(K \subset Y\) and \(N > 0\), there is an exact Lagrangian cobordism \(W\) from \(\Lambda\) to \(\Lambda'\) satisfying the following.

- \(V \circ W\) is isotopic to \(V'\) via a compactly supported Hamiltonian isotopy.
- \(W\) is Hamiltonian isotopic to \(\mathbb{R} \times \Lambda\) by an Hamiltonian isotopy as above.
Proof. For subsets $A \subset \mathbb{R} \times Y$ and $I \subset \mathbb{R}$ we write

$$A_I := A \cap (I \times Y).$$

Take $a \gg 0$ such that $V_{[a, +\infty)}$ and $V'_{[a, +\infty)}$ both are cylindrical. Consider a smooth cut-off function $\rho: \mathbb{R} \to \mathbb{R}$ satisfying $0 \leq \rho(t) \leq 1$, $\rho|_{(-\infty, a]} = 1$, and whose support is contained inside $(-\infty, b]$ for some $b > a$.

We define

$$W_s := ((-\infty, a) \times \Lambda) \cup (\varphi^s_{(1-\rho)H}(V))_{[a, +\infty)},$$

and observe that $W_0 = \mathbb{R} \times \Lambda$, while $W := W_1$ is an exact Lagrangian cobordism from $\Lambda$ to $\Lambda'$.

To conclude that $W_1$ has a cylindrical end we have used the assumption that $H_s$ has support inside $\mathbb{R} \times K$ where $K$ is compact, which implies that the $t$-coordinate of $\varphi^s_{H}([p, q] \times Y)$ is bounded for every $s$ and $p < q$.

We can make the identification

$$V \circ W = \varphi^1_{(1-\rho)H}(V'),$$

where the right-hand side obviously is exact Lagrangian isotopic to $V'$ via the compactly supported isotopy

$$V^\lambda := \varphi^1_{(1-\lambda\rho)H}(V)$$

parametrised by $\lambda \in [0, 1]$, for which $V_0 = V'$ and $V_1 = V \circ W$.

Finally, $W$ is Hamiltonian isotopic to $\mathbb{R} \times \Lambda$ via a Hamiltonian isotopy having support in $[a, +\infty) \times K$ by construction. □

For the following lemma, we assume that $g$ is a metric on $\Lambda$ for which $(f, g)$ constitutes a Morse-Smale pair. We let $C^\bullet_{\text{Morse}}(f, d_f)$ denote the induced Morse co-complex. Recall that $L$ is a manifold with a cylindrical end $[-2, +\infty) \times \Lambda$ and that $F_\pm$ are Morse functions on $L$ as defined in Section 6.2.1.

Lemma A.2. Consider a metric on $L$ which coincides with $dt \otimes dt + g$ on \{t \geq 0\}. After a perturbation of this metric and the function $F_-$, we may assume that the induced Morse co-complex satisfies

$$(C^\bullet_{\text{Morse}}(F_-), d_{F_-}) = (C^\bullet_{\text{Morse}}(F_+), d_{F_+}) \oplus C^{-1}_{\text{Morse}}(f), d_{F_-}),$$

where

$$d_{F_-} = \begin{pmatrix} d_{F_+} & 0 \\ \Gamma & d_f \end{pmatrix}.$$ 

In other words, we may view this complex as Cone($\Gamma$) for a chain-map

$$\Gamma: (C^\bullet_{\text{Morse}}(F_-), d_{F_-}) \to (C^\bullet_{\text{Morse}}(f), d_f).$$

Proof. By considering the action-filtration of this Morse co-complex it follows that

$$C^{-1}_{\text{Morse}}(f) \subset C^\bullet_{\text{Morse}}(F_-).$$
is a sub-complex. The only non-trivial part of the statement is thus that the differential $d_{F_-}$ restricted to this sub-complex may be assumed to coincide with $d_f$.

We restrict ourselves to the domain $U := [0, B] \times \Lambda \subset L$, on which we construct a 1-parameter family

$$F_s(t, p) := \epsilon^2 \alpha(t) + ((1 - s)t + s\alpha(t))\eta^2 f(p)$$

of smooth functions, where $\alpha(t)$ and $f$ are as constructed in Section 6.2.1.

Recall that $\alpha(t) = t$ for $t \leq A - 1$ while $\alpha'(B) < 0$. One can check that $F_s$ is a 1-parameter family of functions starting at $F_0 = F_-|_U$ and whose negative gradient points outwards of $U$. The critical points of $F_s$ satisfy

$$(\epsilon^2 \alpha'(t) + ((1 - s) + \eta^2 f(p))dt + ((1 - s)t + s\alpha(t))\eta^2 df(p) = 0.$$  

For $0 < \eta < \epsilon$ sufficiently small, it follows that critical points $(t, p)$ of $F_s$ are in bijection with critical points $p$ of $f$, and have $t$-coordinate satisfying $A \leq t < B$. Moreover, since $\alpha''(t) < 0$ for $t \in [A, B]$, the critical points of $F_s$ are non-degenerate.

Recall that $A$ is the unique critical point of $\alpha(t)$, and that $\alpha(A) > 0$. All critical points of $F_1$ are thus contained in the hypersurface $\{t = A\}$. Since $F_1$ is on the form $\alpha(t)((\epsilon^2 + \eta^2)f)$ in a neighbourhood of this hypersurface, we may assume that $(F_1, dt \otimes dt + g)$ is a Morse-Smale pair for which

$$(\text{C}_{\text{Morse}}^{-1}(F_1), d_{F_1}) = (\text{C}_{\text{Morse}}^{-1}(f), d_f)$$

under the obvious identification of critical points.

Consider the canonical projection

$$\pi: U = [0, B] \times \Lambda \to \Lambda,$$

which maps gradient flow lines of $(F_s, dt \otimes dt + g)$ to gradient flow lines of $(f, g)$.

Any gradient flow line of $F_s$ which is tangent to the $t$-direction in $U$ must be contained entirely in a set on the form $\mathbb{R} \times \{p\}$, where $p$ is a critical point of $f$. Consequently, such a gradient flow line cannot connect two critical points. A non-trivial gradient flow line of negative expected dimension occurring in the family $(F_s, dt \otimes dt + g)$ which connects two critical points hence projects under $\pi$ to a non-trivial gradient flow line of $(f, g)$ in $\Lambda$. The projected gradient flow line moreover connects the corresponding critical points, and an index computation implies that they both have the same expected dimension.

Since $(f, g)$ is a Morse-Smale pair by assumption, the above argument shows that there cannot be any non-trivial gradient flow lines of negative expected dimension in the family $(F_s, dt \otimes dt + g)$. It follows that this family induces a trivial cobordism of the rigid gradient flow lines.

After a generic perturbation of this family, we conclude that

$$(\text{C}_{\text{Morse}}^{-1}(F_0), d_{F_0}) = (\text{C}_{\text{Morse}}^{-1}(F_1), d_{F_1}),$$
where the complex on the left is obtained by some generic perturbation of the pair \((F_0, dt \otimes dt + g)\).

Finally, this shows that we may assume that
\[
(C_{\bullet-1}^{\bullet-1}(f), df) \subset (C_{\bullet}^{\bullet}(F^-), dF^-)
\]
has differential
\[
d_{F^-}|_{C(f)} = d_{F_0} = d_{F_1} = df.
\]

\[
\square
\]

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