Nonsingular bouncing cosmology from general relativity

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Abstract

We investigate a particular classical nonsingular bouncing cosmology, which results from general relativity if we allow for degenerate metrics. The simplest model has a constant equation of state and we calculate the modified Hubble diagram for both the luminosity distance and the angular diameter distance. A possibly more realistic model has an equation of state which is different before and after the bounce.

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I. INTRODUCTION

There have been many discussions from various physics perspectives of the possible existence of a pre-big-bang phase, with or without a bounce-type behavior of the cosmic scale factor. See, e.g., Refs. [1–3] and references therein.

Recently, we have obtained a surprising hint for the actual existence of a pre-big-bang phase [4], where we worked with the established theory of general relativity in four spacetime dimensions but allowed for degenerate metrics. (See the last two paragraphs of Sec. I in Ref. [4] for a brief comparison of this extended version of general relativity and the standard version, which considers only nondegenerate metrics.) With an appropriate differential structure and trivial spacetime topology, a nonsingular spatially-flat Friedmann-type solution has been obtained, where the curvature and the matter energy density remain finite (these quantities diverge for the standard Friedmann solution). Most interestingly, this nonsingular Friedmann-type solution suggests the existence of a pre-big-bang phase with a bounce-type behavior of the cosmic scale factor.

The aim of the present article is to review this nonsingular bounce, which stays within the realm of general relativity, and to obtain a better understanding of the nonsingular bouncing cosmology by performing exploratory calculations of certain cosmological observables. In App. A we also give an explicit realization of a particular classical nonsingular bouncing cosmology that was discussed in Ref. [3].

II. NONSINGULAR BOUNCE WITH A CONSTANT EQUATION OF STATE

We start from the classical spacetime manifold of Ref. [4], but use a simplified version of the cosmic time coordinate $T$ and consider only the $T$-even solution for the cosmic scale factor $a(T)$. In this way, we obtain a modified spatially-flat Friedmann–Lemaître–Robertson–Walker (FLRW) universe with a bounce-type behavior of $a(T)$. We can be relatively brief in this section, as further details can be found in Refs. [4–7]. Throughout, we use natural units with $c = 1$ and $\hbar = 1$.

With a cosmic time coordinate $T$ and comoving spatial coordinates $\{x^1, x^2, x^3\}$, an appropriate Ansatz for the metric is given by [4]

$$
\left. ds^2 \right|_{\text{mod. FLRW}} \equiv g_{\mu\nu}(x) dx^\mu dx^\nu \bigg|_{\text{mod. FLRW}} = -\frac{T^2}{b^2 + T^2} dT^2 + a^2(T) \delta_{kl} dx^k dx^l, \quad (2.1a)
$$

$$
b > 0, \quad (2.1b)
$$
The parameter $b$ in the metric (2.1a) corresponds to the characteristic length scale of the spacetime defect localized at $T = 0$ (see Refs. [4–7] and references therein). For the moment, $b$ is simply a model parameter and we remain agnostic as to its physical origin. It may be that $b$ is related to the Planck length, but it is also possible that $b$ is related to a new fundamental length scale of quantum spacetime [8].

Observe that the metric (2.1a) is degenerate: $\det g_{\mu\nu} = 0$ at $T = 0$. The corresponding $T = 0$ spacetime slice may be interpreted as a 3-dimensional “defect” of spacetime with topology $\mathbb{R}^3$. The standard elementary-flatness condition does not hold at the location of this spacetime defect; see App. D in Ref. [5] and Sec. 2 D in Ref. [6] for further discussion. As will be seen shortly, the metric (2.1a) removes the big bang curvature singularity, but does so at the price of introducing a spacetime defect. Remark also that a degenerate metric evades certain singularity theorems; cf. Sec. 3.1.5 in Ref. [7].

Later on, we will simplify the calculations away from the spacetime defect by use of the auxiliary coordinate $\tau$ instead of $T$. These coordinates are related as follows (see also Fig. II):

$$T(\tau) = \begin{cases} +\sqrt{\tau^2 - b^2}, & \text{for } \tau \geq b, \\ -\sqrt{\tau^2 - b^2}, & \text{for } \tau \leq -b, \end{cases}$$

$$\tau \in (-\infty, -b] \cup [b, \infty).$$

The inverse relation reads

$$\tau(T) = \begin{cases} +\sqrt{b^2 + T^2}, & \text{for } T \geq 0, \\ -\sqrt{b^2 + T^2}, & \text{for } T \leq 0. \end{cases}$$

which is multi-valued at $T = 0$. The advantage of using the auxiliary coordinate $\tau$ is that the metric (2.1a) takes the standard spatially-flat FLRW form, $ds^2 = -d\tau^2 + a^2(\tau) \delta_{kl} dx^k dx^l$, and that the reduced field equations are nonsingular. But it is important to realize that the coordinate transformation from $T$ to $\tau$ is not a diffeomorphism (a $C^\infty$ function): the function (2.3) is discontinuous between $T = 0^-$ and $T = 0^+$, as is the (suitably defined) second derivative. In short, the differential structure of the metric (2.1a) is different from the one of the standard spatially-flat FLRW metric; see Ref. [6] for a related discussion.
FIG. 1. Surgery on the real line with coordinate $\tau \in \mathbb{R}$ gives the cosmic time axis $\tau \in (-\infty, -b] \cup [b, \infty)$, where the points $\tau = -b$ and $\tau = b$ are identified (as indicated by the dots). A suitable cosmic time coordinate is given by $T \in \mathbb{R}$ from (2.2a). Each point of the cosmic time axis corresponds to a unique value of the coordinate $T$.

Taking the metric (2.1a) with spacetime coordinates $\{T, x^1, x^2, x^3\}$ and the energy-momentum tensor of a homogeneous perfect fluid of relativistic matter, the Einstein equation gives the following modified spatially-flat Friedmann equation, energy-conservation equation, and equation-of-state parameter $w$:

\[
\left(1 + \frac{b^2}{T^2}\right) \left(\frac{1}{a(T)} \frac{da(T)}{dT}\right)^2 = \frac{8\pi}{3} G_N \rho(T), \tag{2.4a}
\]

\[
\frac{d}{da} \left[a^3 \rho(a)\right] + 3 a^2 P(a) = 0, \tag{2.4b}
\]

\[
w(T) \equiv \frac{P(T)}{\rho(T)} = \frac{1}{3}, \tag{2.4c}
\]

where the last equation corresponds to a constant equation-of-state parameter. As mentioned in Ref. [4], the only new ingredient in (2.4) is the singular factor $(1 + b^2/T^2)$ on the left-hand side of the modified Friedmann equation (2.4a).

The $T$-even bounce-type solution $a(T)$ from (2.4) with normalization $a(T_0) = 1$ at $T_0 > 0$ is given by

\[
a(T) \bigg|_{\text{mod. FLRW}}^{(T\text{-even rel-mat. sol.)}} = \sqrt[4]{\frac{(b^2 + T^2)}{(b^2 + T_0^2)}}, \tag{2.5}
\]

which is perfectly smooth at $T = 0$ as long as $b \neq 0$ (see Fig. 2 for a comparison with the singular solution). The corresponding Kretschmann curvature scalar $K \equiv R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$ and
FIG. 2. Cosmic scale factor (full curve) of the modified spatially-flat FLRW universe with relativistic matter, as given by (2.5) with \( b = 1 \) and \( T_0 = 4 \sqrt{5} \). Also shown is the cosmic scale factor (dashed curve) of the standard FLRW universe with an extended cosmic time coordinate \( T \), as given by (2.5) with \( b = 0 \) and \( T_0 = 4 \sqrt{5} \).

matter energy density \( \rho \) are then finite at \( T = 0 \),

\[
K(T) \propto (b^2 + T^2)^{-2},
\]

\[
\rho(T) \propto (b^2 + T^2)^{-1}.
\]

In terms of the auxiliary coordinate \( \tau \) from (2.3), the bounce solution reads

\[
a(\tau)_{\text{mod. FLRW}}^{(\tau\text{-even rel-mat. sol.)}} = 4 \sqrt{\tau^2/\tau_0^2},
\]

with \( \tau_0^2 \equiv b^2 + T_0^2 \).

We emphasize that the new input for this particular nonsingular bouncing cosmology is the metric Ansatz (2.1a). The other two inputs are standard, the Einstein equation and the energy-momentum tensor of the matter (here, relativistic matter). The resulting modified Friedmann equation (2.4a), together with the standard equations (2.4b) and (2.4c), then gives the bounce-type scale factor (2.5). In the next section, we calculate some cosmological observables for this bounce-type FLRW universe.

III. GEODESICS, HUBBLE DIAGRAM, AND PARTICLE HORIZON

A. Null geodesics

The background metric is given by (2.1a). Particles travel on straight lines in the coordinate system \( \{T, x^1, x^2, x^3\} \). So, we can consider geodesics of light that start at \( T = T_1 < 0 \)
and end at \( T = T_0 > 0 \), while moving in the \( x^1 \equiv X \) direction. Then, the reduced metric is

\[
0 = ds^2 \bigg|_{\text{mod. FLRW}}^{\text{(light)}} = -\frac{T^2}{b^2 + T^2}dT^2 + a^2(T) \, dX^2, \tag{3.1}
\]

where \( c \) has been set to unity. For relativistic matter, the cosmic scale factor \( a(T) \) is given by (2.5).

With boundary condition \( X(0) = 0 \), we now have the following solution \( X = X(T) \) from the reduced metric (3.1) and the cosmic scale factor (2.5):

\[
X(T) = \begin{cases} 
+2 \sqrt{b^2 + T_0^2} \left[ \sqrt{T^2 + b^2} - \sqrt{b} \right], & \text{for } T > 0, \\
-2 \sqrt{b^2 + T_0^2} \left[ \sqrt{T^2 + b^2} - \sqrt{b} \right], & \text{for } T \leq 0.
\end{cases} \tag{3.2}
\]

A plot of this null geodesic is given in Fig. 3.

**B. Modified Hubble diagram**

It is a straightforward exercise to calculate the luminosity distance \( d_L \) as a function of the redshift \( z \), provided we distinguish two cases:

1. the light is emitted by a co-moving galaxy in the expanding phase of the universe \((T_1 > 0)\);

2. the light is emitted by a co-moving galaxy in the contracting phase of the universe \((T_1 \leq 0)\).

In both cases, the light is detected by a co-moving observer in the expanding phase at cosmic time \( T_0 > 0 \) with \( T_0 > T_1 \).

Using the auxiliary time coordinate \( \tau \) from (2.3) with scale factor (2.7) and adapting the relevant formulae in Secs. 14.4 and 14.6 of Ref. [9], we obtain

\[
d_L(z) \bigg|_{\text{case 1}} = \frac{a^2(\tau_0)}{a(\tau_1)} \int_{\tau_1}^{\tau_0} \frac{\, d\tau'}{a(\tau')}, \tag{3.3a}
\]

\[
d_L(z) \bigg|_{\text{case 2}} = d_L^{(\text{pre})}(\tau_1) + d_L^{(\text{post})}(\tau_0) = a^2(-b) \int_{\tau_1}^{-b} \frac{\, d\tau''}{a(\tau'')} + a^2(\tau_0) \int_b^{\tau_0} \frac{\, d\tau'}{a(\tau')}, \tag{3.3b}
\]

where light is emitted at cosmic time \( \tau = \tau_1 \) (with \( \tau_1 > b \) for case 1 and \( \tau_1 \leq -b \) for case 2) and observed at \( \tau = \tau_0 > b > 0 \) with \( \tau_0 > \tau_1 \). Taking the positive function \( a(\tau) \) from (2.7)
and introducing the redshift \( z \equiv \sqrt{a^2(\tau_0)/a^2(\tau_1)} - 1 = a(\tau_0)/a(\tau_1) - 1 \), the integrals in (3.3) give
\[
\begin{align*}
  d_L(z) \bigg| \text{case } 1 & = 2 \tau_0 z, \quad \text{for } z \in [0, z_{\text{max}}), \quad (3.4a) \\
  d_L(z) \bigg| \text{case } 2 & = 2 b \frac{z_{\text{max}} - z}{1 + z_{\text{max}}} + 2 \tau_0 z_{\text{max}}, \quad \text{for } z \in (-1, z_{\text{max}}], \quad (3.4b)
\end{align*}
\]
with definition
\[
z_{\text{max}} \equiv a(\tau_0)/a(b) - 1 = \sqrt{\tau_0/b} - 1. \quad (3.4c)
\]
The length scale \( 2 \tau_0 \) entering (3.4a) and (3.4b) is determined by (2.7),
\[
2 \tau_0 = \left[ \frac{1}{a(\tau)} \left( \frac{da}{d\tau} \right) \right]_{\tau=\tau_0}^{-1} \equiv \left[ H_{0(\tau-\text{def.})} \right]^{-1}, \quad (3.5)
\]
where the Hubble constant \( H_{0(\tau-\text{def.})} \) differs from \( H_{0(T-\text{def.})} \equiv [da(T)/dT]/a(T) \bigg|_{T=T_0} \) by a factor close to unity, as long as \( \tau_0 \gg b \).

The corresponding expressions for the angular diameter distance \( d_A \) read
\[
\begin{align*}
  d_A(z) \bigg| \text{case } 1 & = \frac{a^2(\tau_1)}{a^2(\tau_0)} d_L(z) \bigg| \text{case } 1, \quad (3.6a) \\
  d_A(z) \bigg| \text{case } 2 & = \frac{a^2(\tau_1)}{a^2(-b)} d_L^{\text{(pre)}}(\tau_1) + \frac{a^2(b)}{a^2(\tau_0)} d_L^{\text{(post)}}(\tau_0). \quad (3.6b)
\end{align*}
\]
With the definitions in (3.3) and \( a(\tau) \) from (2.7), the integrals give
\[
\begin{align*}
  d_A(z) \bigg| \text{case } 1 & = 2 \tau_0 \frac{z}{(1 + z)^2}, \quad \text{for } z \in [0, z_{\text{max}}), \quad (3.7a) \\
  d_A(z) \bigg| \text{case } 2 & = 2 \tau_0 \left( \frac{1}{(1 + z)^2} - \frac{1}{(1 + z_{\text{max}})^2} + \frac{1}{1 + z_{\text{max}}} \frac{z}{1 + z} \right), \quad \text{for } z \in (-1, z_{\text{max}}]. \quad (3.7b)
\end{align*}
\]
The modified Hubble diagram with the luminosity distance \( d_L(z) \) is plotted in Fig. 4 and the one with the angular diameter distance \( d_A(z) \) in Fig. 5. The nonsmooth behavior at \( z = z_{\text{max}} \) in Figs. 4 and 5 is a direct manifestation of the spacetime defect. This last point is of interest in that it shows that the spacetime defect at \( T = 0 \) (or \( \tau = \pm b \)) is not just a coordinate artifact, as it leads to observable effects. The discontinuity of the derivative \( d'_L(z) \) at \( z_{\text{max}} \) from (3.4a) and (3.4b) traces back to the nontrivial \( \tau_1 \) behavior in (3.3), due to the sharp change in slope of \( a(\tau_1) \) between \( \tau_1 \leq -b \) and \( \tau_1 \geq b \).
FIG. 3. Null geodesic (3.2) with $b = 1$ and $T_0 = 4 \sqrt{5}$.  

FIG. 4. Hubble diagram with the luminosity distance $d_L$ from (3.4) for $b/\tau_0 = 1/9$ and $z_{\text{max}} = 2$. With an observer in the expanding phase, the full curve corresponds to case 1 (light emitted by a co-moving galaxy in the expanding phase of the universe) and the dashed curve to case 2 (light emitted by a co-moving galaxy in the contracting phase). It may be more realistic to have model parameters $\tau_0 \gg b > 0$ and $z_{\text{max}} \gg 1$, so that the dashed line is nearly horizontal for $z \in (-1, z_{\text{max}}]$.  

FIG. 5. Same as Fig. 4 but now with the angular diameter distance $d_A$ from (3.7).
After we completed our calculation of the luminosity distance, we became aware of Ref. [10], which discusses certain phenomenological aspects of a nonsingular bouncing cosmology but not the dynamics of the bounce. The behavior of the \( n = 1/2 \) curve in Fig. 1 of Ref. [10] agrees with the more or less constant behavior of the dashed curve in our Fig. [1].

To our knowledge, the possible cusp-type behavior of the luminosity distance \( d_L(z) \) has not been obtained before in other bouncing models. In Ref. [10], the authors did calculate the luminosity distances for different contracting phases but not the complete description, from contraction to expansion. Needless to say, a complete description of the luminosity distance is far from trivial for most of the bouncing models in the literature, as it depends on the details of the bouncing models (especially the details at the bounce moment). As we have shown in this subsection, our bouncing model not only gives a complete description of the luminosity distance (or the angular diameter distance) but also displays a nontrivial effect such as the cusp-type behavior, which may be regarded as an important characteristic of our bouncing model.

C. Past particle horizon

At cosmic time \( T > 0 \), the past particle horizon is infinite, as the universe extends back in time indefinitely. Explicitly, the particle horizon at \( T > 0 \) reads

\[
    d_{\text{hor}}(T) = a(T) \lim_{\tau_1 \to -\infty} \left[ \int_{\tau_1}^{-b} \frac{d\tau''}{a(\tau'')} + \int_{b}^{\tau(T)} \frac{d\tau'}{a(\tau')} \right],
\]  

(3.8)

where \( \tau(T) \) is given by (2.3) and \( a(\tau) \) by (2.7). For positive and finite values of \( b, \tau, \) and \( \tau_0 \), we get

\[
    d_{\text{hor}}(T) = 2a(T) \lim_{\tau_1 \to -\infty} \left( \sqrt{-\tau_1 \tau_0} - 2 \sqrt{b \tau_0} + \sqrt{\tau \tau_0} \right)
\]

\[= 2a(T) \lim_{\tau_1 \to -\infty} \sqrt{-\tau_1 \tau_0}, \]  

(3.9)

which goes to \(+\infty\). In other words, the past particle horizon at a finite positive cosmic time \( T \) diverges for this particular bounce-type universe.

IV. DISCUSSION

The construction of the spacetime manifold in Ref. [4] is entirely classical. But it could very well be that the classical length parameter \( b \) appearing in the metric (2.1a) has its
origin in the (unknown) theory of “quantum spacetime,” with a fundamental length scale related to the Planck length or not. It is then possible to imagine that this quantum theory removes the classical times \( \tau \in (-b, b) \) in Fig. 1 and ties together \( \tau = -b \) and \( \tau = b \), so that the resulting manifold of the emerging classical time coordinate \( T = T(\tau) \) has no boundary. In that case, there must be a classical pre-big-bang phase \( T < 0 \) and, in this article, we have studied some cosmological consequences.

Up till now, we have investigated only the simplest possible model with a constant equation-of-state parameter \( \gamma \). But we may also consider a possibly more realistic equation of state, as discussed in Ref. [3]. Details are presented in App. A.

Assuming the relevance of the nonsingular bounce model of Sec. II (and of the model of App. A), the modified Hubble relation in Fig. 4 delivers the following message to astronomers of future generations:

if a large part of the recent history of the universe [in our model, cosmic time coordinate \( T \in (-T_0, T_0) \), with \( T_0 > 0 \) corresponding to the present epoch] is sprinkled with “standard candles,” then it may be worthwhile to look for extremely faint images with reduced redshifts (or even blueshifts).

Obviously, this is a big “if.” Moreover, it is known that the present universe after the big bang (or bounce, for that matter) contained a hot plasma, which would strongly scatter the photons of the assumed standard candles in the pre-bounce phase. But it may be that these standard candles emit gravitational waves instead of electromagnetic waves (light). Indeed, it is possible to consider a gravitational standard candle from a binary-black-hole merger with definite masses (giving a recognizable “chirp”); see Ref. [10] for further discussion. Incidentally, the above restriction on the cosmic time interval with standard candles, \( T \in (-T_0, T_0) \), is to avoid the Olbers paradox [9, 10].

Based on the modified Hubble relation in Fig. 5, a similar argument holds for objects of a known physical size and we have a second message to astronomers of future generations:

if a large part of the recent history of the universe [in our model, cosmic time coordinate \( T \in (-T_0, T_0) \), with \( T_0 > 0 \) corresponding to the present epoch] is sprinkled with “standard-size objects,” then it may be worthwhile to look for extremely small images with reduced redshifts (or even blueshifts).

Again, these objects may emit gravitational waves instead of electromagnetic waves.

In order to quantify the qualification “extremely” in the two italicized messages, note
that the numerical values $cT_0 \approx 10^{10}\text{lyr} \approx 10^{26}\text{m}$ and $b \approx \hbar c/\text{TeV} \approx 2 \times 10^{-19}\text{m}$ imply, according to (3.4c), the huge value $z_{\text{max}} \approx 2 \times 10^{22}$. For the moment, the two italicized messages are entirely academic, but the general idea remains valid and may perhaps be adapted to other circumstances.

In fact, we have been talking primarily about direct images of pre-bounce structures (e.g., binary-black-hole mergers). But, as mentioned in Fig. 4 of Ref. 3, the currently observed “super-horizon” patterns in the cosmic microwave background may also be due to a pre-bounce phase, assuming that there has been such a phase. The crucial question is whether or not a cosmic bounce has actually occurred and, if so, what physics is responsible. The intriguing result from general relativity, extended to allow for degenerate metrics, is that a particular “regularization” of the standard big bang singularity indeed suggests the occurrence of a cosmic bounce.

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Appendix A: Nonsingular bounce with a variable equation of state

In this appendix, we give some results for a modified spatially-flat FLRW universe with a nonconstant equation of state. In fact, we take our cue from the general discussion of a classical nonsingular bouncing cosmology in Ref. 3. With the definition $\epsilon \equiv (3/2)(1 + P/\rho)$, Sec. 4 of that paper states: “According to the bouncing scenario, at some point during or shortly after the bounce, the kinetic energy stored in scalar fields is converted to the matter and radiation we observe, with $\epsilon \leq 2$. The irreversible reheating process accounts for the asymmetry in $\epsilon$ about the bounce point.” The main characteristics of that nonsingular bouncing cosmology are summarized in Fig. 3 of Ref. 3 and the goal of the present appendix is to present a “fully-computable bounce model,” as requested in Sec. 6 of Ref. 3.

With reduced-Planckian units ($8\pi G_N = c = \hbar = 1$), the modified spatially-flat Friedmann equation, the energy-conservation equation, and the assumed equation of state are given by

(A1a) \[
\left(1 + \frac{b^2}{T^2}\right) \left(\frac{1}{a(T)} \frac{da(T)}{dT}\right)^2 = \frac{1}{3} \rho(T),
\]

(A1b) \[
\frac{d}{da} \left[a^4 \rho(a)\right] + 3 a^2 P(a) = 0,
\]
\[ P(T) = w(T) \rho(T), \]  
\[ w(T) = \begin{cases} \frac{1}{3} + \frac{2}{3} \exp\left[ -\left( \sqrt{b^2 + T^2} - b \right)^2 / b^2 \right], & \text{for } T > 0, \\ 1, & \text{for } T \leq 0, \end{cases} \]

(A1c)
(A1d)

where (A1c) and (A1d) provide an explicit realization of the required equation-of-state behavior of the nonsingular bouncing cosmology as plotted in Fig. 3 of Ref. [3]. The particular function \( w(T) \) from (A1d) is shown, for model parameter \( b = 1 \), in the top-left panel of Fig. 6.

By reverting temporarily to the auxiliary coordinate \( \tau \) from (2.3) and by focussing on the Hubble parameter \( h(\tau) \equiv a^{-1}(\tau) [da(\tau)/d\tau] \) it is possible to get an explicit analytic result:

\[ H(T) = \sqrt{\frac{T^2}{b^2 + T^2}} h(T), \]  
\[ \rho(T) = 3 \bar{h}^2(T), \]  
\[ \bar{h}(T) = \begin{cases} \left( b + 2 \sqrt{b^2 + T^2} + \frac{1}{2} b \sqrt{\pi} \text{ erf} \left[ \left( \sqrt{b^2 + T^2} - b \right) / b \right] \right)^{-1}, & \text{for } T > 0, \\ \left( -3 \sqrt{b^2 + T^2} \right)^{-1}, & \text{for } T \leq 0, \end{cases} \]

(A2a)
(A2b)
(A2c)

in terms of the error function

\[ \text{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z dt \exp \left[ -t^2 \right]. \]  

(A3)

From (A2a) and (A2c), we have \( H(T) \sim (1/3) T^{-1} < 0 \) for \( T \ll -b \) and \( H(T) \sim (1/2) T^{-1} > 0 \) for \( T \gg b \). From (A2b), (A2c) and (A1d), we also see that the maximum values of the energy density and pressure (which occur at \( T = 0 \)) remain finite,

\[ \rho(0) = P(0) = \frac{1}{3} E_{\text{planck}}^2 b^{-2}, \]

(A4)

in terms of the reduced Planck energy \( E_{\text{planck}} \equiv \sqrt{\hbar c^5 / (8\pi G_N)} \approx 2.44 \times 10^{18} \text{ GeV} \).

It does not appear possible to get \( a(T) \) in an explicit analytic form, but the ordinary differential equation from (A2a) can be solved numerically for \( a(T) \). Figure 6 shows the cosmological functions for a particular choice of model parameters. The corresponding luminosity distance \( d_L \) and angular diameter distance \( d_A \) (Fig. 7) are found to be qualitatively the same as those from Sec. III (Figs. 4 and 5).
FIG. 6. Bounce-type universe from a modified Friedmann equation (A1a) with a post-bounce change of the equation of state (A1c). Top-left panel: equation-of-state parameter \( w(T) \) from (A1d). Top-right panel: Hubble parameter \( H(T) \) from (A2a) and (A2c). Bottom-left panel: energy density \( \rho(T) \) from (A2b) and (A2c). Bottom-right panel: numerical solution for the cosmic scale factor \( a(T) \) from the ordinary differential equation (A2a) with boundary condition \( a(-T_0) = 1 \). The time-asymmetric behavior of \( a(T) \) in the bottom-right panel is manifest [having, for example, \( a(10) \neq a(-10) \)] and differs from the symmetric behavior in Fig. 2. The model parameters are \( \{b, \tau_0, T_0\} = \{1, 9, 4 \sqrt{5}\} \) in reduced-Planckian units with \( 8\pi G_N = c = \hbar = 1 \).

FIG. 7. Numerical results for the luminosity distance \( d_L \) from (3.3) and the angular diameter distance \( d_A \) from (3.6) for the bounce-type universe of Fig. 6. For the model parameters chosen, the numerical value of the maximum redshift is \( z_{\text{max}} \approx 1.32425 \).
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