GENERIC ABSENCE OF FINITE BLOCKING FOR INTERIOR POINTS OF BIRKHOFF BILLIARDS

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(Communicated by Rafael de la Llave)

ABSTRACT. Let $x$ and $y$ be points in a billiard table $M = M(\sigma)$ in $\mathbb{R}^2$ that is bounded by a curve $\sigma$. We assume $\sigma \in \Sigma_r$ with $r \geq 2$, where $\Sigma_r$ is the set of simple closed $C^r$ curves in $\mathbb{R}^2$ with positive curvature. A subset $B$ of $M \setminus \{x, y\}$ is called a blocking set for the pair $(x, y)$ if every billiard path in $M$ from $x$ to $y$ passes through a point in $B$. If a finite blocking set exists, the pair $(x, y)$ is called secure; if not, it is called insecure. We show that for $\sigma$ in a dense $G_\delta$ subset of $\Sigma_r$ with the $C^r$ topology, there exists a dense $G_\delta$ subset $R = R(\sigma)$ of $M(\sigma) \times M(\sigma)$ such that $(x, y)$ is insecure in $M(\sigma)$ for each $(x, y) \in R$. In this sense, for the generic Birkhoff billiard, the generic pair of interior points is insecure. This is related to a theorem of S. Tabachnikov [24] that $(x, y)$ is insecure for all sufficiently close points $x$ and $y$ on a strictly convex arc on the boundary of a smooth table.

1. Introduction. Consider a compact plane region $M = M(\sigma)$ ("table") bounded by a simple closed curve $\sigma$. A billiard is the dynamical system consisting of this table and a point mass that moves within the table at unit speed along line segments whose endpoints are on the boundary. At the boundary, the direction of motion changes instantaneously, so that the angle of incidence equals the angle of reflection. The trajectory of the point mass during a given time interval is called a billiard path. We say that a set $B \subset M \setminus \{x, y\}$ is a blocking set for a pair of points $(x, y) \in M \times M$ if every billiard path from $x$ to $y$ passes through a point of $B$. If a finite blocking set exists, the pair $(x, y)$ is called secure; if not, it is called insecure. For example, if the boundary of $M$ is an ellipse with foci $x$ and $y$, then $(x, y)$ is insecure. In particular, if the boundary is a circle with center $C$, the pair $(C, C)$ is insecure, but for $z \in M$ with $z \neq C$, $(C, z)$ is secure. A table is called secure if each pair of points in the table is secure; if not, the table is called insecure.

It is an open problem to characterize secure billiard tables, even in the case of polygonal tables (see the survey [12] by E. Gutkin). In 2004, T. Monteil [20] showed that there exists a rational billiard table (i.e., a polygonal billiard table in which all angles are rational multiples of $\pi$) that is insecure, contradicting some earlier work in this area. According to Gutkin [11], for a regular $n$-gon to be secure, it is necessary and sufficient that $n = 3, 4,$ or $6$. The proof of necessity is deep and is based on earlier work on security in translation surfaces [10, 25, 26, 14, 15, 13].

2010 Mathematics Subject Classification. Primary: 37J99, 37E99, 78A05; Secondary: 53C22.

Key words and phrases. Birkhoff billiards, security, finite blocking, geodesics, genericity.

The authors were supported in part by NSF grant DMS-1156515.
Moreover, Gutkin [11] proved the security of polygons that are tiled under reflection by one of the following: a triangle with angles of 30°, 60°, 90°, a triangle with angles of 45°, 45°, 90°, an equilateral triangle, or a rectangle. Gutkin’s conjecture [12] that this is a necessary condition for a polygonal table to be secure is still open.

We consider Birkhoff billiard tables, which are defined to be convex tables with a smooth boundary (at least $C^2$ in the context of this paper). The only prior result regarding security for points in such tables seems to be the one by S. Tabachnikov [24], who showed that for every compact billiard table $M$ bounded by a smooth curve, in particular a Birkhoff billiard, if $x$ and $y$ are sufficiently close distinct points on a strictly convex arc of $\partial M$, then $(x, y)$ is insecure. We consider the case of points $x$ and $y$ in the interior of a Birkhoff billiard table. While we do not obtain information about the security of any specific pair of points, we show that insecurity is generic in the sense of Baire category, that is, it holds on a residual set (which is a set whose complement is meager). More precisely, for $r \geq 2$, let $\Sigma_r$ be the set of simple closed $C^r$ curves in the plane that have positive curvature (as defined in Section 3), with the $C^r$ topology. For any given pair of distinct points $x$ and $y$ in the plane, let $\Sigma_r(x, y)$ be the set of curves in $\Sigma_r$ that enclose a region containing $x$ and $y$ in its interior. We show that there exists a residual subset $A_r(x, y)$ of $\Sigma_r(x, y)$ such that for every billiard table bounded by a curve in $A_r(x, y)$ the pair $(x, y)$ is insecure. The Kuratowski-Ulam Theorem allows us to reformulate this as follows: There is a residual subset $A_r$ of $\Sigma_r$ such that for $\sigma \in A_r$, insecurity holds for $(x, y)$ in a residual subset of $M(\sigma) \times M(\sigma)$. (See Theorem 6.3.) For Birkhoff billiards we should allow boundary curves of nonnegative curvature, but for our result we may assume positive curvature, since $\Sigma_r$ is an open dense subset of the set of simple closed $C^r$ curves of nonnegative curvature with the $C^r$ topology. Thus, our result shows that for the generic Birkhoff billiard, the generic pair of interior points is insecure.

In the context of compact Riemannian manifolds, the security property (defined by replacing “billiard paths” by “geodesics”) has been studied extensively (see section 5.6 of [12] for a summary of these results and references). According to [5], it is expected that most Riemannian manifolds are totally insecure, that is, all pairs $(x, y)$ are insecure. Total insecurity has been established in various settings [16, 19, 5, 1, 8]. M. Gerber and W.-K. Ku [9] proved that for any compact $C^\infty$ manifold, there is a residual set of metrics for which the set of insecure pairs $(x, y)$ is residual. In the case of dimension two, V. Bangert and Gutkin [1] proved that the following stronger result: All Riemannian manifolds of genus greater than one are totally insecure, and for genus one, a residual set of Riemannian metrics is totally insecure. (See also the work of W. Ho [17] for related results.) Even though the result for Birkhoff billiards in the present paper is analogous to the result for Riemannian manifolds in [9], the techniques are quite different, because Riemannian metrics can be modified anywhere within the manifold, while our modifications can only change the boundary of the table, not the geometry within the table.

2. Outline of our approach. Let $r \geq 2$, and let $M = M(\sigma)$ be a billiard table in $\mathbb{R}^2$ bounded by a curve $\sigma \in \Sigma_r$, as in Section 1. We consider distinct points $x$ and $y$ in the interior of $M$. A vertex of a billiard path from $x$ to $y$ is a point on this path that lies on the boundary of $M$. (See Section 3 for a precise definition of billiard paths.) To prove that the pair $(x, y)$ is insecure, it suffices to show that for every positive integer $n$, there exist $n$ billiard paths from $x$ to $y$ that have no
triple intersection points except \(x\) and \(y\). A \textit{triple intersection point} is a point at which three or more distinct segments of the \(n\) paths meet. We include the case in which two or more of these segments come from the same billiard path, even though we would only need to consider the case in which all three segments come from different paths in order to prove insecurity. The absence of triple intersection points is part of the following definition:

\textbf{Definition 2.1.} Suppose there are \(n\) billiard paths from \(x\) to \(y\) in the billiard table \(M\). We say that these paths are in \textit{general position} if the following four conditions hold:

1. No two paths share a vertex.
2. No point occurs as a vertex on the same path more than once.
3. The paths have no triple intersection points except \(x\) and \(y\).
4. The points \(x\) and \(y\) are not interior points of any of the \(n\) paths.

In Theorem 5.1 we prove that for points \(x \neq y\) in the interior of \(M\) and any positive integer \(n\), there exists an arbitrarily small \(C^r\) perturbation \(\sigma_n\) of \(\sigma\) bounding a table \(M_n\) such that there exist \(n\) billiard paths from \(x\) to \(y\) in \(M_n\) that are in general position. Moreover, we can do this in such a way that there exists a \(C^r\) neighborhood of \(\sigma_n\) in \(\Sigma_r\) such that if \(\tilde{\sigma}\) is in this neighborhood, then we still have \(n\) billiard paths from \(x\) to \(y\) in the corresponding table \(\tilde{M}\) that are in general position (see Lemma 3.1). It then follows (see Corollary 6.2) that the set \(\Sigma_{r,(x,y)}\), consisting of curves in \(\Sigma_r\) that bound a region containing \(x\) and \(y\) in its interior, contains a dense \(G_\delta\) subset of curves for which the existence of \(n\) billiard paths from \(x\) to \(y\) that are in general position holds \textit{for all} \(n\). In Theorem 6.3, we apply the Kuratowski-Ulam Theorem (the analog for Baire Category of the Fubini Theorem), to conclude that there is a dense \(G_\delta\) subset \(A\) of \(\Sigma_r\) such that for each \(\sigma \in A\), the set of insecure pairs \((x,y)\) among those pairs \((x,y)\) such that \(x\) and \(y\) are in the interior of the region bounded by \(\sigma\) contains a dense \(G_\delta\) set.

To begin the proof of Theorem 5.1, we need a method for generating a large collection of billiard paths from \(x\) to \(y\). A \textit{polygonal path} \(P_0P_1\cdots P_{k+1}\) in the plane is defined to be the union of oriented line segments from \(P_i\) to \(P_{i+1}\), for \(i = 0, \ldots, k\), where \(P_0, \ldots, P_k\) are points in \(\mathbb{R}^2\). We fix a positive integer \(k\) and points \(x\) and \(y\) in a billiard table \(M\), and consider polygonal paths \(P_0P_1\cdots P_{k+1}\) with \(P_0 = x\), \(P_{k+1} = y\), and \(P_1, \ldots, P_k \in \partial M\). The well-known Lemma 3.1 shows that any maximal-length polygonal path of this type is a billiard path for \(M\). We use this lemma to obtain a billiard path \(\gamma_{n+1}\) from \(x\) to \(y\) if we are already given billiard paths \(\gamma_1, \ldots, \gamma_n\) from \(x\) to \(y\) that are in general position, and we would like to obtain \(n+1\) paths from \(x\) to \(y\) that are in general position. We would like the new path \(\gamma_{n+1}\) to have at least one vertex that is not a vertex of any of \(\gamma_1, \ldots, \gamma_n\). However, even if we take \(k\) to be very large compared to the total number of vertices of \(\gamma_1, \ldots, \gamma_n\), it is possible that the path \(\gamma_{n+1}\) obtained from Lemma 3.1 is part of a periodic billiard path, and the vertices of \(\gamma_{n+1}\) are contained in the set of vertices of \(\gamma_1, \ldots, \gamma_n\). To avoid this problem, we make a small \(C^r\) perturbation \(\tilde{\sigma}\) of \(\sigma\) and small perturbations of \(\gamma_1, \ldots, \gamma_n\) to obtain billiard paths \(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n\) from \(x\) to \(y\) for the table bounded by \(\tilde{\sigma}\) such that \(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n\) are still in general position, and, in addition, no two distinct vertices of \(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n\) are collinear with \(y\). This implies that there is no periodic billiard path passing through \(x\) and \(y\) whose vertices are a subset of the vertices of \(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n\). Then, if we choose \(k\) sufficiently large, the new path \(\gamma_{n+1}\) obtained from Lemma 3.1 must contain at least one vertex \(V\) that
is not a vertex of $\gamma_1, \ldots, \gamma_n$ (see Lemma 5.3). By making a further perturbation of $\sigma$ if necessary, as in Lemma 5.10, we may assume that $\gamma_{n+1}$ passes through $V$ only once. Then we are free to modify the path $\gamma_{n+1}$ to $\gamma_{n+1}$ by changing the initial angle at $x$, the final angle at $y$, and the table near $V$ so the part of $\gamma_{n+1}$ starting at $x$ until it hits the table near $V$ and the part of $\gamma_{n+1}$ from near $V$ to the point $y$ are joined to form a billiard path for the new table (see Proposition 5.7). This procedure does not change the paths $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$, and it can be done in such a way that $\gamma_{n+1}$ does not have any vertices in common with $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$. Then we can make further perturbations of the table near the vertices of $\gamma_{n+1}$ using Corollaries 5.8 and 5.9 to obtain $n+1$ paths in general position.

3. Notation and preliminaries. Throughout this paper we assume $r \geq 2$, and we let $\Sigma_r$ be the set of simple closed $C^r$ curves $\sigma$ in $\mathbb{R}^2$ such that $\sigma$ has positive curvature, that is, its acceleration vector has a positive component in the direction of the inward-pointing normal vector for the region enclosed by $\sigma$. Let $M = M(\sigma)$ be the compact region in $\mathbb{R}^2$ bounded by some $\sigma \in \Sigma_r$. For convenience, we assume $\sigma$ is parametrized by the circle $S^1 = \mathbb{R}/\mathbb{Z}$ and the parametrization is at constant speed. For $s \in S^1$, let $T(s)$ denote the unit tangent vector, $T(s) = \sigma'(s)/|\sigma'(s)|$, and let $N(s)$ be the unit vector perpendicular to $T(s)$ that is inward pointing for $M$ at $\sigma(s)$. We may assume that $\sigma$ is oriented in the counterclockwise direction, or equivalently, the pair of vectors $T(s), N(s)$ has the same orientation as the standard basis $(1, 0), (0, 1)$ in $\mathbb{R}^2$. Note that $T$ and $N$ are $C^{r-1}$ functions of $s$. We let $\kappa(s)$ denote the curvature of $\sigma$ at $\sigma(s)$. Then $T'(s)/|\sigma'(s)| = \kappa(s)N(s)$, with $\kappa(s) > 0$.

A billiard path $\gamma(t)$, defined for $t \in \mathbb{R}$, represents the position at time $t$ of a point mass moving within $M$ at unit speed with elastic collisions at $\partial M$. More precisely, there is a partition $c_{-2} < c_{-1} < c_0 < c_1 < c_2 < \cdots$ of $\mathbb{R}$ with $c_0 \leq 0 < c_1$ such that for each $i \in \mathbb{Z}$, $\gamma([c_{i-1}, c_i])$ is a line segment in $M$ parametrized at unit speed, $\gamma(c_i) \in \partial M$, and

$$
\begin{align*}
\gamma'_-(c_i) &= (\cos \alpha_i)T - (\sin \alpha_i)N \\
\gamma'_+(c_i) &= (\cos \alpha_i)T + (\sin \alpha_i)N,
\end{align*}
$$

where $T = T(s_i)$ and $N = N(s_i)$, with $s_i \in S^1$ chosen so that $\gamma(c_i) = \sigma(s_i); \alpha_i \in (0, \pi)$ is given by $\alpha_i = 3(\gamma'_-(c_i), T)$; and $\gamma'_-[\gamma'_+]$ denotes the derivative of $\gamma$ from the left [right]. It follows that

$$
\gamma'_+(c_i) = \gamma'_-(c_i) - 2\text{Proj}_v\gamma'_-(c_i),
$$

(3.1)

where $\text{Proj}_w v$ denotes the orthogonal projection of $v$ onto span($w$). We also consider billiard paths from a point $x \in \text{Int}(M)$ to a point $y \in \text{Int}(M)$. Such a path is a billiard path $\gamma(t)$ defined as above, except the domain of $\gamma$ is restricted to a compact interval $[a, b]$; $\gamma(a) = x$, and $\gamma(b) = y$. The points $\gamma(c_i) \in \partial M$ with $c_i \in (a, b)$ are called the vertices of the path. We will refer to the segment of the path from $x$ to the first vertex as the initial segment, and the segment of the path from the last vertex to $y$ as the final segment. It is also possible for the billiard path from $x$ to $y$ to simply be the segment $xy$, in which case there are no vertices, and the segment $xy$ is both the initial and the final segment. A segment of the path that joins two consecutive vertices is called a chord of the path.

For $p \in \text{Int}(M)$, let $T_p^1 M$ denote the set of all unit vectors based at $p$; for $p \in \partial M$, let $T_p^1 M$ denote the set of unit vectors based at $p$ that are inward-pointing for $M$ (not including vectors tangent to $\partial M$). Vectors in $T_p^1 M$ will be written in the form $(p, v)$,
Lemma 3.1. For \( m = 1, 2, \ldots \), and given points \( x, y \in \text{Int}(M) \), a maximal-length polygonal path \( L_m = P_0 P_1 \ldots P_m P_{m+1} \), subject to the conditions \( P_0 = x, P_{m+1} = y, \) and \( P_i \in \partial M \) for all \( i \in \{1, 2, \ldots, m\} \). Since \( (P_1, \ldots, P_m) \in \partial M \times \cdots \times \partial M \), which is a compact set, such an \( L_m \) exists. Note that all consecutive vertices of \( L_m \) must be distinct in order for \( L_m \) to have maximal length: If \( P_i = P_{i+1} \), then by the triangle inequality, the path \( L'_m \) formed by moving \( P_{i+1} \) slightly away from \( P_i \) on \( r \) has length longer than \( L_m \). The following well-known lemma will be useful in obtaining a collection of distinct billiard paths from \( x \) to \( y \).

**Lemma 3.1.** For \( m = 1, 2, \ldots \), and given points \( x, y \in \text{Int}(M) \), a maximal-length polygonal path \( L_m = P_0 P_1 \ldots P_m P_{m+1} \), subject to the conditions \( P_0 = x, P_{m+1} = y, \) and \( P_i \in \partial M \) for all \( i \in \{1, 2, \ldots, m\} \), is a billiard trajectory.

**Proof.** It suffices to prove that for any points \( A \) and \( B \) in \( M \), a polygonal path \( AZB \), with \( Z \in \partial M \), of maximal length must be a billiard path. This follows from the Lagrange multiplier principle applied to the function \( f(Z) = |AZ| + |ZB| \). (See pp. 12-13 of [23].)

**Remark 3.2.** For \( x, y \in \text{Int}(M) \), Lemma 3.1 provides the existence of at least one billiard trajectory from \( x \) to \( y \) that makes exactly \( m \) reflections at \( \partial M \). The argument also holds for \( x, y \in \partial M \). Much deeper work by M. Farber [6] gives a lower bound for the number of billiard trajectories making \( m \) reflections at \( \partial M \) that have a prescribed initial point \( x \) and a prescribed final point \( y \), where \( x, y \in \partial M \). Farber assumes, as we do, that \( M \) is strictly convex, but his result also applies in higher dimensions. In further work, [7], Farber also obtains lower bounds for the number of periodic and of closed billiard paths that start and end at a given point \( z \in \partial M \).

In order to study \( D\Phi \) geometrically it is convenient to introduce \( C^k \) families of oriented lines in \( \mathbb{R}^2 \), for \( k \geq 1 \). In our application, we usually take \( k = r - 1 \).

**Definition 3.3.** Let \( I \subset \mathbb{R} \) be a compact interval of positive length, and for each \( u \in I \), suppose that \( \ell(u) \) is an oriented line in \( \mathbb{R}^2 \). For \( k \geq 1 \), we say that \( \ell(u), u \in I \), is a \( C^k \) family of oriented lines if there exist \( C^k \) functions \( \xi : I \to \mathbb{R}^2 \) and \( v : I \to S^1 \) such that for each \( u \in I \), \( \ell(u) = \{\xi(u) + tv(u) : t \in \mathbb{R}\} \), and the orientation on \( \ell(u) \) is given by \( v(u) \). The point \( \xi(u) \) and the vector \( v(u) \) are the base point and the direction vector, respectively, of the line \( \ell(u) \). Such a family \( \ell(u) \) is said to be non-degenerate if the following condition holds: For each \( u \in I \), if \( v'(u) = 0 \), then \( \xi'(u) \) is not a scalar multiple of \( v(u) \). (It follows that \( \xi'(u) \) and \( v'(u) \) are not both 0.) In particular, the non-degeneracy condition prevents the family of oriented lines \( \ell(u) \) from consisting of just a single line.

If \( \ell(u) = \{\xi(u) + tv(u) : t \in \mathbb{R}\}, u \in I \), is a \( C^k \) family of oriented lines, then we may consider parameter translations along each \( \ell(u) \) by letting \( \xi(u) = \xi(u) + a(u)v(u) \), where \( a : I \to \mathbb{R} \) is \( C^k \). Note that the definition of non-degeneracy is satisfied by \( (\xi, v) \) if and only if it is satisfied by \( (\xi, v) \) replaced by \( (\xi, v) \). Therefore the definition of non-degeneracy is independent of parameter translations. We may
also reparametrize the family $\ell(u), u \in I$, as $\ell(\beta(\tilde{u})), \tilde{u} \in \tilde{I}$, where $\tilde{I}$ is a compact interval of positive length and $\beta : \tilde{I} \to I$ is a $C^k$ diffeomorphism. The definition of non-degeneracy is also independent of such a reparametrization. (In fact, if we use a natural identification of the set of lines in $\mathbb{R}^2$ with an open subset of the Grassmannian $\text{Gr}(2, 3)$ of planes through the origin in $\mathbb{R}^3$ and consider the family $\ell(u)$ as a curve in $\text{Gr}(2, 3)$, then non-degeneracy is equivalent to this curve having non-vanishing derivative.)

**Definition 3.4.** If $\ell(u), u \in I$, is a $C^1$ family of oriented lines parametrized by $\ell(u, t) = \xi(u) + tv(u), u \in I, t \in \mathbb{R}$, then the focusing point (in linear approximation) for the family $\ell(u)$ at $u = u_0$, is $F = F(u_0) := \ell(u_0, f(u_0))$, where $f(u) = -\langle \xi'(u), v'(u) \rangle / \langle v'(u), v'(u) \rangle$ whenever $v'(u) \neq 0$. (See Figure 1.) If $v'(u_0) = 0$, we take $f(u_0) = \infty$, and we say the focusing point is at infinity for $u = u_0$. (See, e.g., Section 2 of [28].)

![Figure 1. Local envelope $\ell(u, f(u))$ and focusing point $F(u_0)$ on $\ell(u_0)$.](image)

A straightforward computation shows that the focusing point for the family $\ell(u)$ at $u = u_0$ does not change under parameter translations along each of the $\ell(u)$. Likewise, if the family $\ell(u), u \in I$, is reparametrized as $\ell(\beta(\tilde{u})), \tilde{u} \in \tilde{I}$, where $\beta$ is as above, then the focusing point for the family $\ell(u)$ at $u = u_0 \in I$ is the same as the focusing point for the family $\ell(\beta(\tilde{u}))$ at $\tilde{u} = \beta^{-1}(u_0) \in \tilde{I}$.

**Remark 3.5.** If $\ell(u), u \in I$, is a $C^1$ family of lines and $v'(u_0) \neq 0$ for some $u_0 \in I$, then a simple calculation shows that $F(u_0)$ is the limit as $u$ approaches $u_0$ of the intersection point of $\ell(u)$ and $\ell(u_0)$. (See Figure 1.) In classical terminology $F(u_0)$ was called the “intersection point of consecutive lines.” The curve $u \mapsto \ell(u, f(u))$, defined for $u$ in an interval on which $v'(u) \neq 0$, is said to be the local envelope of
the local envelope at the parameter value \( u \), as illustrated in Figure 1. (See Chapter 5 of [3] for further discussion of envelopes of lines and curves.)

**Lemma 3.6.** Suppose \( \ell(u) \), \( u \in I \), is a non-degenerate \( C^1 \) family of oriented lines parametrized by \( \ell(u,t) = \xi(u) + tv(u) \), \( t \in \mathbb{R} \). For \( (u_0,t_0) \in I \times \mathbb{R} \), \( D\ell(u_0,t_0) \) is invertible if and only if \( \ell(u_0,t_0) \) is not a focusing point at \( u = u_0 \).

**Proof.** The derivative of \( \ell(u,t) \) is given by

\[
D\ell(u_0,t_0) = [\begin{array}{cc} \xi'(u_0) + t_0v'(u_0) & v(u_0) \end{array}].
\]

**Case 1.** Suppose \( v'(u_0) = 0 \). Then \( \ell(u_0,t_0) \) is not a focusing point at \( u = u_0 \), and the columns of \( D\ell(u_0,t_0) \) are linearly independent, since \( v(u_0) \) is a unit vector and the non-degeneracy condition implies that \( \xi'(u_0) \) is not a scalar multiple of \( v(u_0) \).

**Case 2.** Suppose \( v'(u_0) \neq 0 \). Since \( \langle v'(u), v(u) \rangle \equiv 0 \), the columns of \( D\ell(u_0,t_0) \) are linearly dependent if and only if the orthogonal projection of \( \xi'(u_0) + t_0v'(u_0) \) onto \( v'(u_0) \) is \( 0 \). Thus the columns of \( D\ell(u_0,t_0) \) are linearly dependent if and only if

\[
\frac{\langle \xi'(u_0) + t_0v'(u_0), v'(u_0) \rangle}{\langle v'(u_0), v'(u_0) \rangle} = 0,
\]

which is equivalent to \( \ell(u_0,t_0) \) being a focusing point at \( u = u_0 \). \( \square \)

**Definition 3.7.** Let \( \ell(u) \), \( u \in I \), be a \( C^k \) family of oriented lines, where \( 1 \leq k \leq r - 1 \). Assume that the line \( \ell(u) \) intersects \( \text{Int}(M) \) for each \( u \in I \). Suppose the lines are parametrized by \( \ell(u,t) = \xi(u) + tv(u) \), \( t \in \mathbb{R} \), where \( \xi : I \to M \) and \( v : I \to \mathbb{R}^2 \) are \( C^k \) functions and \( (\xi(u), v(u)) \in T^1_{\xi(u)}M \), for all \( u \), which implies that \( v(u) \) is inward pointing for \( M \) at \( \xi(u) \) if \( \xi(u) \in \partial M \). The reflected family \( \ell_1(u) \), obtained from \( \ell(u) \) is defined as follows: For each \( u \in I \), there are exactly two values of \( t \), say \( t_a = t_a(u) \) and \( t_b = t_b(u) \), where \( t_a < t_b \), such that \( \xi(u) + t_a v(u) \) and \( \xi(u) + t_b v(u) \) are on \( \partial M \). Let \( \xi_1(u) = \xi(u) + t_b(u)v(u) \) and let \( v_1(u) = v(u) - 2\text{Proj}_N(\xi_1(u))v(u) \). By (3.1), we see that a billiard trajectory along the line \( \ell(u) \) (going in the same direction as the orientation of \( \ell(u) \)) is reflected at \( \xi_1(u) \) on \( \partial M \), so that it continues along the line \( \ell_1(u) \) after reflection. The reflected family, \( \ell_1(u) \), \( u \in I \), is the family of oriented lines that can be parametrized by \( \ell_1(u,t) = \xi_1(u) + tv_1(u), t \in \mathbb{R} \).

**Remark 3.8.** The first part of the following lemma is the well-known fact that for a strictly convex table bounded by a \( C^r \) curve, the billiard map is \( C^{r-1} \). (See, e.g., Section 9.2 of [18].) We include the proof, so that we may generalize it in Lemma 3.11.

**Lemma 3.9.** Let \( M \) be a billiard table in \( \mathbb{R}^2 \) bounded by a curve \( \sigma \in \Sigma_r \), where \( r \geq 2 \). Suppose \( \ell(u) \), \( u \in I \), is a \( C^k \) family of oriented lines, where \( 1 \leq k \leq r - 1 \), and \( \ell(u) \) intersects \( \text{Int}(M) \) for each \( u \in I \). Then the reflected family, \( \ell_1(u) \), of oriented lines obtained from \( \ell(u) \) is also \( C^k \). Moreover, if the family \( \ell(u) \) is non-degenerate, then so is \( \ell_1(u) \).

**Proof.** Let \( \xi, \xi_1, v, v_1, t_a, t_b \) be as in Definition 3.7. From the inverse function theorem and the compactness of \( S^1 \) it follows that there exists \( \delta > 0 \) such that the function \( \varphi : S^1 \times (-\delta, \delta) \to \mathbb{R}^2 \) given by \( \varphi(s,w) = \sigma(s) + wN(s) \) is injective and defines a \( C^{r-1} \) coordinate system on a neighborhood \( N_\delta := \{ p \in \mathbb{R}^2 : \text{dist}(p, \partial M) < \delta \} \). This is a special case of the tubular neighborhood theorem (see, e.g., [4]). The \( w \)
coordinate gives the signed distance of a point in $N_\delta$ to $\partial M$, with the sign chosen so that $w < 0$ outside $M$ and $w > 0$ in $\text{Int}(M)$. Since $\nabla w = N(\sigma(s))$ is a $C^{r-1}$ function of $s$, $w$ is a $C^r$ function of the usual $(x, y)$ coordinates on $N_\delta$. For $(u, t) \in I \times \mathbb{R}$ such that $\ell(u, t) \in \partial M$, we have $\partial (w \circ \ell) / \partial t = (\nabla w, v(u)) = (N(\ell(u, t)), v(u)) \neq 0$.

Therefore, by the implicit function theorem, the function $t_0(u)$, which satisfies the equation $w \circ \ell(u, t_0(u)) = 0$, is $C^k$. Hence the function $\xi_1 : I \to \partial M$ defined by $\xi_1(u) := \xi(u) + t_0(u)v(u)$ is $C^k$. By (3.1), we have $v_1(u) := v(u) - 2\text{Proj}_{N(\xi_1(u))}v(u) = v(u) - 2\langle N(\xi_1(u)), v(u) \rangle N(\xi_1(u))$, which is $C^k$. Therefore, $\xi_1(u)$ is $C^k$.

Now assume, in addition, that $\ell(u)$ is non-degenerate. Note that $\ell(u)$ can be reparametrized by $\ell(u, t) = \xi_1(u) + tv(u)$. Thus $\xi_1(u)$ and $v(u)$ satisfy the condition for non-degeneracy. We need to prove that $\xi_1(u)$ and $v_1(u)$ satisfy this condition. We have $v'_1(u) = v'(u) - 2\langle N(\xi_1(u)), v'(u) \rangle N(\xi_1(u)) - 2 \langle N(\xi_1(u)), v(u) \rangle N(\xi_1(u)) - 2 \langle N(\xi_1(u)), v(u) \rangle (N'(-\xi_1(u)))(\xi_1(u))$. Since $\xi_1(u) \in \partial M$, we consider the following two cases for each $u_0 \in I$:

1. Suppose $\xi'_1(u_0)$ is a non-zero scalar multiple of $T = T(\xi_1(u_0))$. Then $\xi'_1(u_0)$ is not a multiple of $v_1(u_0)$, since the line $\ell(u_0)$, as well as its reflection $\xi_1(u_0)$, intersects $\text{Int}(M)$, which implies that $v_1(u_0)$ and $T$ are transversal.

2. Suppose $\xi'_1(u_0) = 0$. By the non-degeneracy of $\ell(u)$, $v'(u_0) \neq 0$. The formula for $v'_1(u_0)$ simplifies to

$$v'_1(u_0) = v'(u_0) - 2 \langle N(\xi_1(u_0)), v'(u_0) \rangle N(\xi_1(u_0)) = v'(u_0) - 2\text{Proj}_{N(\xi_1(u_0))}v'(u_0) = \text{Proj}_{T(\xi_1(u_0))}v'(u_0) - \text{Proj}_{N(\xi_1(u_0))}v'(u_0).$$

Thus in the $(T, N)$ coordinates $v'_1(u_0)$ is obtained from $v'(u_0)$ by changing the sign of the $N$ coordinate. Therefore $v'_1(u_0) \neq 0$.

Therefore the family $\xi_1(u)$ is non-degenerate.

We now give explicit formulations of the standard $C^k$ distance between functions, families of oriented lines, and curves. For later use, we also include a definition of distance between oriented polygonal paths.

**Definition 3.10.** If $K$ is a compact subset of $\mathbb{R}^n$ and $f, \tilde{f} : K \to \mathbb{R}^m$ are $C^k$ maps, then the $C^k$ distance between $f$ and $\tilde{f}$ is defined to be

$$\text{dist}_{C^k(K, \mathbb{R}^m)}(f, \tilde{f}) := \sup_{j \in \{0, 1, \ldots, k\}} \sup_{s \in K} |D^{(j)}f(s) - D^{(j)}\tilde{f}(s)|.$$  

If $\ell(u, t) = \xi(u) + tv(u)$ and $\tilde{\ell}(u, t) = \tilde{\xi}(u) + t\tilde{v}(u)$, $u \in I$, $t \in \mathbb{R}$, are two parametrized families of oriented lines where $\xi, \tilde{\xi}, v, \tilde{v}$ are $C^k$ functions, then the $C^k$ distance between $\ell$ and $\tilde{\ell}$ depends on the parametrizations and is defined to be

$$d_k(\ell, \tilde{\ell}) := \max \{ \text{dist}_{C^k(I, \mathbb{R}^2)}(\xi, \tilde{\xi}), \text{dist}_{C^k(I, \mathbb{R}^2)}(v, \tilde{v}) \}.$$  

If $\sigma$ and $\tilde{\sigma}$ are simple closed $C^k$ curves in the plane, we define the $C^k$ distance geometrically as follows. Let $f_p, \tilde{f} : S^1 \to \mathbb{R}^2$ be constant speed parametrizations of $\sigma$ and $\tilde{\sigma}$, respectively, with $f_p(0) = p \in \sigma$. We regard $S^1 = \mathbb{R}/\mathbb{Z}$ to be $[0, 1]$ with the endpoints identified and apply formula (3.2) to $f_p$ and $\tilde{f}$ with $K = [0, 1]$:

$$d_k(\sigma, \tilde{\sigma}) := \inf_{p \in \sigma} d_k(f_p, \tilde{f}).$$

For $0 \leq k \leq r$, we refer to the topology on $\Sigma_r$ induced by the metric $d_k$ as the $C^k$ topology on $\Sigma_r$. 
If \( \gamma = P_0P_1 \cdots P_{m+1} \) and \( \hat{\gamma} = \hat{P}_0\hat{P}_1 \cdots \hat{P}_{m+1} \) are (oriented) polygonal paths in \( \mathbb{R}^2 \), as defined in Section 2, then the distance between \( \gamma \) and \( \hat{\gamma} \) is
\[
d(\gamma, \hat{\gamma}) := \max\{\text{dist}(P_i, \hat{P}_i) : i = 0, \ldots, m + 1\}.
\]

We assume that \( P_i, P_{i+1}, P_{i+2} \) are noncollinear and \( \hat{P}_i, \hat{P}_{i+1}, \hat{P}_{i+2} \) are noncollinear for \( i = 0, \ldots, m - 1 \), so that the points \( P_0, \ldots, P_{m+1} \) and the points \( \hat{P}_0, \ldots, \hat{P}_{m+1} \) are uniquely determined by \( \gamma \) and \( \hat{\gamma} \), respectively.

The following lemma is a “perturbed” version of the first part of Lemma 3.9.

**Lemma 3.11.** Let \( r \geq 2, \sigma \in \Sigma_r \), and \( M = M(\sigma) \) be the billiard table bounded by \( \sigma \). Suppose \( \ell(u, t) = \xi(u) + tv(u), u \in I, t \in \mathbb{R} \) is a \( C^k \) parametrized family of oriented lines with \( 1 \leq k \leq r - 1 \) such that for each \( u \in I \) the line \( \ell(u, \cdot) \) intersects \( \text{Int}(M) \). Then for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( \tilde{\sigma} \in \Sigma_r \) and \( \tilde{\ell}(u, t) = \tilde{\xi}(u) + tv(u), u \in I, t \in \mathbb{R} \) is a \( C^k \) parametrized family of lines, then the family \( \tilde{\ell}(u, t) = \xi(u) + tv(u), u \in I, t \in \mathbb{R} \) obtained by reflecting \( \ell(u, t) \) from \( \sigma \), and the family \( \tilde{\ell}(u, t) = \tilde{\xi}(u) + tv(u), u \in I, t \in \mathbb{R} \) obtained by reflecting \( \ell(u, t) \) from \( \tilde{\sigma} \), satisfy \( d_k(\ell, \tilde{\ell}) < \epsilon \) whenever \( d_k(\sigma, \tilde{\sigma}) < \delta \) and \( d_k(\ell, \tilde{\ell}) < \delta \). Here we require \( \ell \) and \( \tilde{\ell} \) to be chosen, as in the proof of Lemma 3.9, so that \( \xi(u) \) and \( \tilde{\xi}(u) \) lie on the images of \( \sigma \) and \( \tilde{\sigma} \), respectively.

**Proof.** We use the same notation as in the proof of Lemma 3.9, with all objects obtained from \( \sigma \) and \( \ell \) in an analogous way to objects obtained from \( \sigma \) and \( \ell \) being given a tilde over them. Let \( \ell(u, t) = \xi(u) + tv(u) \) and \( \sigma \) be as in the statement of the lemma. By taking \( \delta \) sufficiently small, we may assume that each \( \ell(u) \) intersects \( \text{Int}(\tilde{M}) \). By following the proof of Lemma 3.9, we see that for small \( \delta \), the signed distances \( \bar{w} \) and \( w \) from \( \bar{\sigma} \) and \( \sigma \) are both defined in a neighborhood of \( \sigma \), \( \nabla \bar{w} = N \circ \bar{\sigma} \) is \( C^{r-1} \) close to \( \nabla w = N \circ \sigma \), and \( \bar{w} \) is \( C^r \) close to \( w \). It then follows that the functions \( t_k(u) \) and \( \tilde{t}_k(u) \) obtained from the implicit function theorem such that \( (\bar{w} \circ \bar{\ell})(u, \tilde{t}_k(u)) = 0 = (w \circ \ell)(u, t_k(u)) \) are \( C^r \) close. Using the procedure of Lemma 3.9 to find \( \xi_1, \tilde{\xi}_1, v_1, \tilde{v}_1 \), we see that for sufficiently small \( \delta \), \( d_k(\ell, \tilde{\ell}) < \epsilon \). \( \square \)

4. Nonconjugacy condition for billiard paths. In this section we discuss the conjugacy condition, which has proved to be useful in the theory of billiards. (See [2] for further information on the adaptation of this condition, as well as other related notions from differential geometry, to the study of billiards.) In Corollary 4.3, we present a consequence of nonconjugacy in the form that we need in Section 5.

Let the billiard table \( M \) be as before, with boundary curve \( \sigma \in \Sigma_r \), where \( r \geq 2 \).

Suppose \( \ell(u), u \in I \), is a \( C^k \) family of oriented lines, where \( 1 \leq k \leq r - 1 \), such that each \( \ell(u) \) intersects \( \text{Int}(M) \), and \( \tilde{\ell}(u), u \in I \), is the reflected family of lines as in Definition 3.7. Suppose the parametrizations of \( \ell(u, t) = \xi(u) + tv(u) \) and \( \tilde{\ell}(u, t) = \xi(u) + tv(u) \) are chosen so that for some \( u_0 \in I, \xi(u_0) = \tilde{\xi}(u_0) \), and \( \xi(u_0) \) lies on \( \sigma \). Let \( f, \tilde{f} \in \mathbb{R} \) be such that \( F = \ell(u_0, f) \) and \( \tilde{F} = \tilde{\ell}(u_0, \tilde{f}) \) are the focusing points of the families \( \ell \) and \( \tilde{\ell} \), respectively, at \( u = u_0 \). Let \( \kappa \) be the curvature of \( \sigma \) at \( \xi(u_0) \), and let \( \alpha \) be the angle that \( v(u_0) \) makes with the tangent vector \( T \) to \( \sigma \) at \( \xi(u_0) \). Then according to the mirror equation (see, e.g., Theorem 5.28 in [23] or Lemma 1 in Section 2 of [28]),
\[
-\frac{1}{f} + \frac{1}{\tilde{f}} = \frac{2\kappa}{\sin \alpha}.
\]
(See Figure 2.) Note that \( \sin \alpha \neq 0 \) for a billiard table whose boundary has positive curvature. In case \( f \) or \( \tilde{f} \) is equal to \( \infty \), we interpret \( \frac{1}{\infty} \) as 0. If either of \( f \) or \( \tilde{f} \) is 0, then the other one is also 0.

**Definition 4.1.** Let \( \alpha \) be the oriented line determined by the initial segment of \( \tau \). Note that \( \sin \alpha \neq 0 \).

**Lemma 4.2.** Let \( K = \overline{B_{r_0}(x_0)} \subset \mathbb{R}^2 \), where \( B_{r_0}(x_0) \) denotes the open ball of radius \( \epsilon_0 \) about some point \( x_0 \in \mathbb{R}^2 \). Suppose \( f : K \to \mathbb{R}^2 \) is \( C^1 \) and \( Df(x_0) \) is invertible. Let \( y_0 = f(x_0) \). Then there exist \( \eta, \delta > 0 \) and \( 0 < \epsilon < \epsilon_0 \), such that for any \( C^1 \) function \( g : K \to \mathbb{R}^2 \) with \( \text{dist}_{C^1(K,\mathbb{R}^2)}(f,g) < \eta \), we have \( g \) one-to-one on \( B_{r}(y_0) \) and \( B_{\delta}(y_0) \subset g(B_{r}(x_0)) \).

**Corollary 4.3.** Let \( p \) and \( q \) be points in \( M = M(\sigma) \), where \( \sigma \in \Sigma_2 \). Let \( \tau \) be a billiard path in \( M \) from \( p \) to \( q \) that makes \( m \geq 0 \) reflections between \( p \) and \( q \). Assume that \( p \) and \( q \) are not conjugate along \( \tau \) in \( M \). Let \( \epsilon_1 > 0 \). Then there exist \( \eta, \delta > 0 \) and \( 0 < \epsilon < \epsilon_1 \) such that the following implication holds: If

1. \( \bar{\sigma} \in \Sigma_2 \),
2. \( d_2(\sigma, \bar{\sigma}) < \eta \),
3. \( p, q_1 \in \tilde{M} \), where \( \tilde{M} = M(\bar{\sigma}) \),
(4) \( \text{dist}(q, q_1) < \delta, \)

then there is a unique billiard path for \( \tilde{M} \) starting at \( p \) and making angle less than \( \epsilon \) with \( \tau \) at \( p \) that passes through \( q_1 \) after reflection number \( m \) and before (or at) reflection number \( m+1 \) from \( \partial \tilde{M} \).

**Proof.** If \( m = 0 \), the result clearly holds; so assume \( m > 0 \). Let \( \ell_0, \ell_1, \ldots, \ell_m \) be the oriented lines determined by the segments of \( \tau \). Let \( \ell_0(u) \) be the non-degenerate \( C^1 \) family of lines that pass through \( p \) and whose direction vectors \( v(u) \) are such that the signed angle from the initial tangent vector of \( \tau \) to \( v(u) \) is \( u \). For \( i = 1, \ldots, m \), let \( \ell_i(u) \) be the family of lines obtained by reflecting \( \ell_{i-1}(u) \) from \( \sigma \). Suppose \( \tilde{\sigma} \) is a \( C^2 \) perturbation of \( \sigma \). Let \( \ell_0(u) = \ell_0(u) \), and for \( i = 1, \ldots, m \), let \( \ell_i(u) \) be the family of lines obtained by reflecting \( \ell_{i-1}(u) \) from \( \tilde{\sigma} \). Let \( t_0 > 0 \) be such that \( \ell_m(0, t_0) = q \). Let \( \epsilon_0, \eta_1 > 0 \) be sufficiently small so that \( (p, v(u)) \in T^1 M \cap T^1 \tilde{M} \) if \( |u| < \epsilon_0 \), \( d_2(\sigma, \tilde{\sigma}) < \eta_1 \), and \( p \in \tilde{M} \). (This restriction of \( u \) is only needed if \( p \in \partial \tilde{M} \).

Let \( \epsilon_1 > 0 \). Let \( f(u, t) = \ell_m(u, t) \) and let \( x_0 = (0, t_0) \). We may assume that \( \epsilon_1 < \min(t_0/2, \epsilon_0) \). Let \( K := B_{\epsilon_1}(x_0) \subset (-\epsilon_0, \epsilon_0) \times (t_0/2, 3t_0/2) \), which is contained in the domain of \( f \). Since \( p \) is a focusing point for \( \ell_0(u) \) at \( u = 0 \), it follows from the nonconjugacy assumption and Lemma 3.6 that \( Df(x_0) \) is invertible. Let \( g(u, t) = \tilde{\ell}_m(u, t) \). By Lemma 4.2 there exist \( \epsilon_2 \in (0, \epsilon_1) \) and \( \alpha > 0 \) such that \( \text{dist}_{C^1(K, \mathbb{R}^2)}(f, g) < \alpha \) implies that \( \tilde{\ell}_m \) is one-to-one on \( B_{2\epsilon_2}(x_0) \). By repeated application of Lemma 3.11, there exists \( \eta_2 \in (0, \eta_1) \) such that \( \text{dist}_{C^1(K, \mathbb{R}^2)}(f, g) < \alpha \) whenever \( d_2(\sigma, \tilde{\sigma}) < \eta_2 \).

Note that \( d(\ell_m(0, t), q) \geq \epsilon_2 \) for \( t \notin (t_0 - \epsilon_2, t_0 + \epsilon_2) \). Thus, there exist \( \epsilon \in (0, \epsilon_2) \) and \( \eta_3 \in (0, \eta_2) \) such that for \( |u| < \epsilon \) and \( d_2(\sigma, \tilde{\sigma}) < \eta_3 \), we have \( d(\ell_m(u, t), q) \geq \epsilon_2/2 \) whenever \( t \notin (t_0 - \epsilon_2, t_0 + \epsilon_2) \). By Lemma 4.2, there exist \( \eta \in (0, \eta_3) \) and \( \delta \in (0, \epsilon_2/2) \) such that \( d_2(\sigma, \tilde{\sigma}) < \eta \) and \( q_1 \in \tilde{M} \) with \( \text{dist}(q, q_1) < \delta, \) then there is a point \( (u_1, t_1) \in B_\epsilon(x_0) \) with \( \ell_m(u_1, t_1) = q_1 \). This proves the existence of a billiard path for \( \tilde{M} \) with the required properties.

The point \( (u_1, t_1) \) is the unique point in \( B_{2\epsilon_2}(x_0) \) with \( \tilde{\ell}_m(u_1, t_1) = q_1 \), because \( \tilde{\ell}_m \) is one-to-one on \( B_{2\epsilon_2}(x_0) \). Now suppose \( |u| < \epsilon, t \in \mathbb{R} \), and \( \ell_m(u, t) = q_1 \). Since \( d(\ell_m(u, t), q) \geq \epsilon_2/2 > \delta \) for \( t \notin (t_0 - \epsilon_2, t_0 + \epsilon_2) \), we must have \( t \in (t_0 - \epsilon_2, t_0 + \epsilon_2) \). But then \( (u, t) \in B_{2\epsilon_2}(x_0) \), and therefore \( (u, t) = (u_1, t_1) \), which proves the uniqueness assertion. \( \square \)

5. **Construction of \( n \) paths without triple intersections.** As before, we let \( M \) be the compact region bounded by a curve \( \sigma \in \Sigma_r \), where \( r \geq 2 \). Suppose \( x \) and \( y \) are distinct points in \( \text{Int}(M) \). Recall that for \( n \) billiard paths from \( x \) to \( y \) in the billiard table \( M \), we say that the paths are in general position if the following four conditions hold:

(GP1) No two paths share a vertex.

(GP2) No point occurs as a vertex on the same path more than once.

(GP3) The paths have no triple intersection points except \( x \) and \( y \).

(GP4) The points \( x \) and \( y \) are not interior points of any of the \( n \) paths.

We say that the paths satisfy the **non-collinearity condition** if the following holds:

(NC) For any distinct vertices \( P \) and \( Q \) of the \( n \) paths, the points \( P, Q, \) and \( y \) are not collinear.
Note that condition (GP4) implies none of the $n$ paths is perpendicular to $\sigma$ at any vertex. Moreover, if there exist $n$ paths from $x$ to $y$ in general position for each positive integer $n$, then $(x, y)$ is insecure for the billiard table $M$. Condition (NC) is not essential for our applications, but it is useful to establish this condition in addition to conditions (GP1)–(GP4) in our proof of Theorem 5.1.

We consider the billiard table $M$, as well as billiard tables obtained by perturbing the boundary of $M$. In this section we prove the following.

**Theorem 5.1.** Suppose $M, \sigma, r, x, y$ are as above. Let $n$ be a positive integer and let $\epsilon > 0$. Then there exists a curve $\sigma_n \in \Sigma_r$ with $d_\epsilon(\sigma, \sigma_n) < \epsilon$ that bounds a region $M_n$ still containing $x$ and $y$ in its interior such that for the billiard system on $M_n$ there are $n$ billiard paths from $x$ to $y$ that are in general position.

We proceed toward a proof of Theorem 5.1, as outlined in Section 2. We first observe that the property of paths being in general position is preserved under small perturbations of the paths.

Let $x$ and $y$ be distinct points in $\mathbb{R}^2$. If $\gamma = P_0 P_1 \cdots P_{m+1}$ is a polygonal path from $x$ to $y$, i.e., $P_0 = x$ and $P_{m+1} = y$, then we refer to $P_1, \ldots, P_m$ as the vertices of $\gamma$. (In our application $\gamma$ will be a billiard path, and $P_1, \ldots, P_m$ will also be vertices in the sense of billiard paths, i.e., they will lie on the boundary of the billiard table.) As for billiard paths, we say that polygonal paths $\gamma_1, \ldots, \gamma_n$ from $x$ to $y$ are in general position if conditions (GP1)–(GP4) hold, and we say they satisfy the non-collinearity condition if (NC) holds. The following lemma is easy to verify, and we omit the proof.

**Lemma 5.2.** Suppose $\gamma_1, \ldots, \gamma_n$ are polygonal paths from $x$ to $y$ that are in general position. Then there exists $\epsilon > 0$ such that for any polygonal paths $\hat{\gamma}_1, \ldots, \hat{\gamma}_n$ from $\hat{x}$ to $\hat{y}$ such that $\hat{\gamma}_i$ has the same number of vertices as $\gamma_i$ and $d(\hat{\gamma}_i, \gamma_i) < \epsilon$ for $i = 1, \ldots, n$, it follows that $\hat{\gamma}_1, \ldots, \hat{\gamma}_n$ are in general position. This result remains true if we replace “are in general position” by “satisfy the non-collinearity condition” in both the hypothesis and the conclusion.

In the next lemma, we show how the non-collinearity condition is used in the proof of Theorem 5.1.

**Lemma 5.3.** Let $M, \sigma, r, x, y$ be as above. Suppose that $\gamma_1, \ldots, \gamma_n$ are billiard paths from $x$ to $y$ in $M$ that satisfy the non-collinearity condition (NC). Then there exists a billiard path $\gamma$ from $x$ to $y$ in $M$ which has at least one vertex $V$ that is not a vertex of any of $\gamma_1, \ldots, \gamma_n$.

**Proof.** Let $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ be the vertices of $\gamma_1, \ldots, \gamma_n$, and let $N$ be an integer greater than $k^2 - k + 1$. By Lemma 3.1, there exists a billiard path $\gamma$ from $x$ to $y$ of the form $xQ_1Q_2\cdots Q_N y$ with vertices $Q_1, Q_2, \ldots, Q_N$. If each $Q_i$, $i = 1, 2, \ldots, N$, were in $\mathcal{P}$ then at least two of the ordered pairs $(Q_j, Q_{j+1})$, $j \in \{1, 2, \ldots, N-1\}$ would be the same. But this would imply that the path is contained in a periodic billiard path through $x$ and $y$, which is impossible due to condition (NC). Therefore the set $\{Q_1, Q_2, \ldots, Q_N\}$ must include at least one point not in $\mathcal{P}$. 

Proposition 5.4 and Corollary 5.5 below allow us to avoid unwanted conjugacies between any given pairs of points $(p_i, q_i)$ along billiard paths $\gamma_i$, for $i = 1, \ldots, n$, by making a small $C^\infty$ change to the table, while $\gamma_1, \ldots, \gamma_n$ remain billiard paths for the new table.
Proposition 5.4. Suppose \( r \geq 2 \), \( P \) is a point on a curve \( \sigma \in \Sigma_r \), and \( \sigma \) has curvature \( \kappa \) at \( P \). Given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for any \( \bar{\kappa} \in (\kappa - \delta, \kappa + \delta) \) there exists a curve \( \bar{\sigma} \in \Sigma_r \) such that \( \bar{\sigma} \) passes through \( P \), \( \bar{\sigma} \) is tangent to \( \sigma \) at \( P \), the curvature of \( \bar{\sigma} \) at \( P \) is \( \bar{\kappa} \), \( \bar{\sigma} \) agrees with \( \sigma \) outside an \( \epsilon \)-neighborhood of \( P \), and \( d_r(\bar{\sigma}, \sigma) < \epsilon \).

Proof. Choose \( \nu > 0 \) and a Cartesian coordinate system centered at \( P \) such that the box \( B = \{(x, y) : |x| \leq \nu, |y| \leq \nu \} \) is contained in the \( \epsilon \)-neighborhood of \( P \), and within this box \( \sigma \) is given by the graph of a function \( f : [-\nu, \nu] \to [-\nu/2, \nu/2] \) with \( f(0) = f'(0) = 0 \) and \( f''(0) > 0 \).

Let \( \Psi : \mathbb{R} \to \mathbb{R} \) be a \( C^\infty \) function such that \( \Psi(0) = \Psi'(0) = 0 \), \( \Psi''(0) \neq 0 \), and \( \Psi(x) = 0 \) for \( x \notin (-\nu, \nu) \). For \( |c| \) sufficiently small, the function \( g(x) = f(x) + c\Psi(x) \) maps \( [-\nu, \nu] \) to \( [-\nu, \nu] \). Let \( \tilde{\sigma} \) be the curve obtained by replacing the graph of \( f \) by the graph of \( g \) and keeping \( \tilde{\sigma} = \sigma \) outside \( B \). Then \( \tilde{\sigma} \) passes through \( P \), \( \tilde{\sigma} \) is tangent to \( \sigma \) at \( P \), and the curvature of \( \sigma \) at \( P \) is \( \kappa + c\Psi''(0) \). There exists \( \eta > 0 \) such that for \( |c| < \eta \), we have \( \tilde{\sigma} \in \Sigma_r \) and \( d_r(\tilde{\sigma}, \sigma) < \epsilon \). Then the result holds with \( \delta = \eta|\Psi''(0)| \).

Corollary 5.5. Let \( M, \sigma, r \) be as above. Suppose that \( \gamma = V_0V_1 \cdots V_m \) is a billiard path for \( M \), where \( p = V_0 \in M \), \( q = V_{m+1} \in M \), and \( V_1, \ldots, V_m \in \partial M \). (The points \( p, q \) are allowed to be on \( \partial M \)). Given \( \epsilon > 0 \), there exists \( \tilde{\sigma} \in \Sigma_r \) bounding a table \( \tilde{M} \) such that \( d_r(\tilde{\sigma}, \sigma) < \epsilon \) and the following conditions are satisfied:

1. \( \gamma \) is still a billiard path in \( \tilde{M} \);
2. \( \sigma \) and \( \tilde{\sigma} \) agree outside the \( \epsilon \)-neighborhoods of \( V_1, \ldots, V_m \);
3. \( p \) is not conjugate to \( q \) along \( \gamma \) in \( \tilde{M} \).

Moreover, for any sufficiently small \( C^2 \) perturbation \( \tilde{\sigma} \) of \( \tilde{\sigma} \) that agrees with \( \tilde{\sigma} \) to first order at each of \( V_1, \ldots, V_m \), \( p \) is not conjugate to \( q \) along \( \gamma \) in the table \( \tilde{M} \) bounded by \( \tilde{\sigma} \).

Proof. Assume that \( p \) and \( q \) are conjugate along \( \gamma \). (Otherwise, we may take \( \tilde{\sigma} = \sigma \).)
Let \( \kappa_i \) be the curvature of \( \sigma \) at \( V_i \) for \( i = 1, \ldots, m \), and let \( \rho_i = \text{dist}(V_i, V_{i+1}) \) for \( i = 0, \ldots, m \). Let \( \ell_0, \ldots, \ell_m \) be the oriented lines along the segments of \( \gamma \), that is, \( \ell_i = V_iV_{i+1} \), for \( i = 0, \ldots, m \). Parametrize \( \ell_i \) by \( V_{i+1} + tv_i \), where \( v_i \) is the unit vector in the direction of \( \ell_i \). Let \( \ell_0(u) \), with \( u \in I = [-\delta, \delta] \) for some \( \delta > 0 \), be the family of lines through \( p \) with \( \ell_0(0) = \ell_0 \), and \( \ell_0(u) \) making signed angle \( u \) with \( \ell_0 \). Suppose \( \tilde{\sigma} = \tilde{\sigma}(z) \) is the boundary of a billiard table \( \tilde{M} \) such that \( \sigma \) and \( \tilde{\sigma} \) agree to first order at \( V_1, \ldots, V_m \), and \( \tilde{\sigma}(z) \) has curvature \( \kappa_i z \) at \( V_i \), for \( i = 1, \ldots, m \). Let \( \tilde{\ell}_0(u) = \ell_0(u) \), and for \( i = 1, \ldots, m \), let \( \tilde{\ell}_{i+1}(u) \) be the family of lines obtained by reflecting \( \tilde{\ell}_i(u) \) from \( \partial \tilde{M} \). Let \( f_i = f_i(z) \in \mathbb{R} \cup \{\infty\} \) be such that \( V_{i+1} + f_i v_i \) is the focusing point for \( \tilde{\ell}_i(u) \), \( u \in I \), at \( u = 0 \). Then \( f_0 = -\rho_0 \), and for \( i = 0, \ldots, m - 1 \), it follows from the mirror equation (4.1) that

\[
f_{i+1} = \frac{1}{f_i + \frac{2\kappa_i z}{\sin \alpha_i}} - \rho_{i+1},
\]

where \( \alpha_i \) is the angle that \( \gamma \) makes with the tangent line to \( \sigma \) at the \((i + 1)\)st reflection. The term \(-\rho_{i+1}\) occurs in equation (5.1) because the origin (time \( t = 0 \)) of \( \ell_{i+1} \) is taken at \( V_{i+2} \) while the origin of \( \ell_i \) is at \( V_{i+1} \). By repeated application of (5.1) we see that \( f_m = f_m(z) \) is a linear fractional transformation of \( z \), that is, it is of the form \((az + b)/(cz + d)\). Note that \( p \) and \( q \) are conjugate along \( \gamma \) within \( \tilde{M} \) if
and only if \( f_m(z) = 0 \). We want to show that we may choose \( z \) arbitrarily close to 1 so that \( f_m(z) \neq 0 \). But if we take \( z = 0 \), we obtain \( f_m(0) = -\rho_0 - \rho_1 - \cdots - \rho_m \neq 0 \). Therefore \( f_m(z) \) is not identically 0. Thus there exists \( z \) arbitrarily close to 1 so that \( f_m(z) \neq 0 \). Given \( \epsilon > 0 \), it follows from Proposition 5.4 that for \( z \) sufficiently close to 1, we can find \( \tilde{\sigma} \in \Sigma_r \) such that \( d_r(\sigma, \tilde{\sigma}) < \epsilon \), \( \tilde{\sigma} \) agrees with \( \sigma \) to first order at \( V_1, \ldots, V_m \), and the curvature of \( \tilde{\sigma} \) at \( V_i \) is \( \kappa_i z \) for \( i = 1, \ldots, m \). Thus, for \( z \) close to 1, but not equal to 1, conditions (1)–(3) are satisfied for \( \tilde{\sigma} \).

It follows from (5.1) that nonconjugacy of \( p \) and \( q \) along \( \gamma \) within \( \tilde{M} \) is preserved when \( \tilde{\sigma} \) is replaced by a small \( C^2 \) perturbation \( \tilde{\sigma} \) that agrees with \( \tilde{\sigma} \) to first order at \( V_1, \ldots, V_m \). □

**Remark 5.6.** In the proof of Corollary 5.5, if one of the vertices \( V_{i_0} \) occurs only once on the list \( V_1, \ldots, V_m \), then \( \tilde{\sigma} \) could be obtained by changing the curvature just at \( V_{i_0} \). The argument that we used takes care of the case in which each \( V_{i_0} \) may occur multiple times, and we have to make sure that the effect on \( f_m \) of changing the curvature at the vertices is not canceled out during multiple passages of \( \gamma \) through these vertices.

The following proposition provides a perturbation technique that will be used repeatedly in the proof of Theorem 5.1.

**Proposition 5.7.** Let \( r \geq 2 \) and \( \sigma \in \Sigma_r \). Suppose \( P \) is a point on \( \sigma \) and \( \ell \) is the tangent line to \( \sigma \) at \( P \). Given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( \tilde{P} \in \mathbb{R}^2 \) is such that \( d(P, \tilde{P}) < \delta \) and \( \tilde{\ell} \) is a line through \( \tilde{P} \) that is either parallel to \( \ell \) or makes angle less than \( \delta \) with \( \ell \), then there exists \( \tilde{\sigma} \in \Sigma_r \) such that \( \tilde{\sigma} \) agrees with \( \sigma \) outside an \( \epsilon \)-neighborhood of \( P \), \( \tilde{\sigma} \) passes through \( \tilde{P} \) with tangent line \( \tilde{\ell} \) at \( \tilde{P} \), and \( d_r(\sigma, \tilde{\sigma}) < \epsilon \).

**Proof.** Let \( \epsilon > 0 \). Choose \( \nu > 0 \) and a Cartesian coordinate system centered at \( P \) such that the box \( B = \{(x, y) : |x| \leq \nu, |y| \leq \nu \} \) is contained in the \( \epsilon \)-neighborhood of \( P \), and within this box \( \sigma \) is given by the graph of a function \( f : [-\nu, \nu] \to [-\nu/2, \nu/2] \) with \( f(0) = f'(0) = 0 \) and \( f''(0) > 0 \). Suppose that in this coordinate system \( \tilde{P} = (a, b) \) and \( \tilde{\ell} \) has slope \( m \).

Let \( \Psi_1 : \mathbb{R} \to \mathbb{R} \) be a \( C^\infty \) function such that \( \Psi_1(x) = 0 \) for all \( x \notin (-\nu/2, \nu/2) \), \( \Psi_1(0) = 1 \), and \( \Psi'_1(0) = 0 \). Let \( \psi_2 : \mathbb{R} \to \mathbb{R} \) be a \( C^\infty \) function such that \( \psi_2(0) = 1 \), \( \int_{-\nu/2}^{\nu/2} \psi_2(x) \, dx = \int_{0}^{\nu/2} \psi_2(x) \, dx = 0 \), and \( \psi_2(x) = 0 \) for \( x \notin (-\nu/2, \nu/2) \). Let \( \Psi_2 : \mathbb{R} \to \mathbb{R} \) be defined by \( \Psi_2(x) = \int_{-\infty}^{x} \psi_2(u) \, du \). Then \( \Psi_2(0) = 0 \), \( \Psi'_2(0) = 1 \), and \( \Psi_2(x) = 0 \) for \( x \notin (-\nu/2, \nu/2) \).

Define a function \( \Psi : [-\nu, \nu] \to \mathbb{R} \) by

\[
\Psi(x) = (b - f(a))\Psi_1(x - a) + (m - f'(a))\Psi_2(x - a).
\]

If \( |a| < \nu/2 \), then \( \Psi \) vanishes in a neighborhood of the boundary of \( [-\nu, \nu] \). Since \( f \) is \( C^1 \), for any \( \eta > 0 \), there exists \( \delta = \delta(\eta) > 0 \) so that \( \max(|a|, |b|, |f(a)|, |f'(a)|, |m|) < \eta \) whenever \( d(P, \tilde{P}) < \delta \) and \( \ell \) and \( \tilde{\ell} \) are either parallel or meet at an angle less than \( \delta \).

For \( \eta \) sufficiently small, the function \( g(x) = f(x) + \Psi(x) \) maps \( [-\nu, \nu] \) to \( [-\nu, \nu] \). Let \( \tilde{\sigma} \) be the curve obtained by replacing the graph of \( f \) by the graph of \( g \) and keeping \( \tilde{\sigma} = \sigma \) outside \( B \). Since \( g(a) = b \) and \( g'(a) = m \), we see that \( \tilde{\sigma} \) passes through \( \tilde{P} \) and has tangent line \( \tilde{\ell} \) at \( \tilde{P} \). Moreover, if \( \eta \) is sufficiently small, then \( \tilde{\sigma} \in \Sigma_r \) and \( d_r(\sigma, \tilde{\sigma}) < \epsilon \). □
Corollary 5.8. Consider a billiard path $APA_1$ in the table $M$ with boundary $\sigma \in \Sigma_r$, for $r \geq 2$, where $A, A_1 \in \text{Int}(M)$ and $P \in \partial M$. For any $\epsilon > 0$, if $\tilde{P}$ is a point on either of the segments $AP$ or $PA_1$ and is sufficiently close to $P$, there exists $\tilde{\sigma} \in \Sigma_r$ with $d_r(\sigma, \tilde{\sigma}) < \epsilon$ such that $A\tilde{P}A_1$ is a billiard path for the table with boundary $\tilde{\sigma}$. (See Figure 3.)

Proof. Let $\ell$ be the tangent line to $\sigma$ at $P$. Then $\ell$ is perpendicular to the angle bisector of $\angle APA_1$. If $\tilde{P}$ on $AP$ or $PA_1$ is sufficiently close to $P$, then the line $\tilde{\ell}$ that is perpendicular to the angle bisector of $\angle A\tilde{P}A_1$ makes a small angle with $\ell$. Then we apply Proposition 5.7.

![Figure 3. Changing the direction of a billiard path at $A$.](image)

Corollary 5.9. Let $APQB$ be a billiard path in the table $M$ with boundary $\sigma \in \Sigma_r$, for $r \geq 2$, where $A, B \in \text{Int}(M)$ and $P, Q \in \partial M$. For any $\epsilon > 0$, if $\tilde{P}$ on $AP$ or $PA_1$ is sufficiently close to $P$, then the line $\tilde{\ell}$ that is perpendicular to the angle bisector of $\angle APA_1$ makes a small angle with $\ell$. Then we apply Proposition 5.7 twice.

![Figure 4. Changing the direction of a billiard path at $A$.](image)

Lemma 5.10. Let $r \geq 2$, $\sigma \in \Sigma_r$, and $M = M(\sigma)$ be as above. Let $x$ and $y$ be distinct points in $\text{Int}(M)$. Suppose $\gamma_1, \ldots, \gamma_{n+1}$ are billiard paths in $M$ from $x$ to $y$ such that $\gamma_1, \ldots, \gamma_n$ are in general position, and $\gamma_{n+1}$ has at least one vertex $V$
that is not a vertex of any of \(\gamma_1, \ldots, \gamma_n\). Then given \(\epsilon > 0\) there exists \(\tilde{\sigma} \in \Sigma_r\) with \(d_r(\sigma, \tilde{\sigma}) < \epsilon\) such that for the table \(\tilde{M}\) bounded by \(\tilde{\sigma}\), we have \(x, y \in \text{Int}(\tilde{M})\), and there are billiard paths \(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n, \tilde{\gamma}_{n+1}\) from \(x\) to \(y\) satisfying the following conditions:

(1) \(d(\gamma_i, \tilde{\gamma}_i) < \epsilon\) for \(i = 1, \ldots, n + 1\),
(2) \(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n\) are in general position,
(3) \(\tilde{\gamma}_{n+1}\) has a vertex \(\tilde{V}\) with \(\text{dist}(V, \tilde{V}) < \epsilon\), where \(\tilde{V}\) is not a vertex of any of \(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n\),
(4) \(\tilde{\gamma}_{n+1}\) passes through \(\tilde{V}\) only once, and
(5) \(\tilde{\gamma}_{n+1}\) is not perpendicular to \(\tilde{\sigma}\) at any of its vertices.

**Proof.** If \(\gamma_{n+1}\) passes through \(V\) only once and \(\gamma_{n+1}\) is not perpendicular to \(\sigma\) at \(V\), then conditions (1)–(5) are satisfied by letting \(\tilde{\sigma} = \sigma, \tilde{V} = V, \tilde{\gamma}_i = \gamma_i\), for \(i = 1, \ldots, n\), and letting \(\tilde{\gamma}_{n+1}\) be the shortest path within \(\gamma_{n+1}\) that starts at \(x\), passes through \(V\), and ends at \(y\). Thus we may assume that \(\gamma_{n+1}\) passes through \(V\) more than once or \(\gamma_{n+1}\) is perpendicular to \(\sigma\) at \(V\).

We give an outline of the rest of the proof, as implemented in the paragraphs below. Let \(s \geq 1\) be the number of times that \(\gamma_{n+1}\) passes through \(V\). We will find a perturbation \(\tilde{\sigma}\) of \(\sigma\) and a billiard path \(\tilde{\gamma}_{n+1}\) for the table \(\tilde{M}\) bounded by \(\tilde{\sigma}\) such that \(\tilde{\gamma}_{n+1}\) is close to \(\gamma_{n+1}\), and if \(s > 1\), then \(\tilde{\gamma}_{n+1}\) passes through \(V\) at least once and at most \(s - 1\) times. Simultaneously we will arrange for \(\tilde{\gamma}_{n+1}\) not to be perpendicular to \(\tilde{\sigma}\) at the first passage through \(V\). By repeating this perturbation process, if necessary, we may assume that \(\gamma_{n+1}\) passes through \(V\) only once and is not perpendicular to \(\sigma\) at \(V\). The curve \(\tilde{\sigma}\) is obtained by perturbing \(\sigma\) near \(V\), but away from the vertices of \(\gamma_1, \ldots, \gamma_n\), so that these \(n\) paths are still billiard paths from \(x\) to \(y\) for \(\tilde{M}\). The path \(\tilde{\gamma}_{n+1}\) may end at a different point \(y_1\) near \(y\), instead of \(y\) itself. To fix this problem, we perturb the paths \(\gamma_1, \ldots, \gamma_n\) (but not the table \(\tilde{M}\)) so that the perturbed billiard paths \(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n\) also end at \(y_1\). Finally we apply
a composition of a homothety (uniform expansion or contraction) centered at \( x \) and a rotation of the plane centered at \( x \) such that \( y_1 \) is mapped to \( y \). Under the composition of these two maps, the images \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{n+1} \) of \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n \) are billiard paths for the image \( \tilde{M} \) of the table \( \tilde{M} \), and these paths start at \( x \) and end at \( y \). The path \( \tilde{\gamma}_{n+1} \) passes through the image \( \tilde{V} \) of \( V \) only once and it is not perpendicular to the boundary of \( \tilde{M} \) at \( \tilde{V} \).

Let \( \epsilon > 0 \). By Corollary 5.5, we may assume that \( x \) and \( y \) are not conjugate along any of \( \gamma_1, \ldots, \gamma_n \), and \( V \) is not conjugate to itself along any path contained in \( \gamma_{n+1} \). We will assume \( \epsilon \) is sufficiently small so that the conditions \( d(\gamma_i, \tilde{\gamma}_i) < \epsilon \) for \( i = 1, \ldots, n \) and \( \text{dist}(V, \tilde{V}) < \epsilon \) imply that \( \tilde{V} \) is not a vertex of any of \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n \); and, in addition, by Lemma 5.2, \( d(\gamma_i, \tilde{\gamma}_i) < \epsilon \) for \( i = 1, \ldots, n \) implies that \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n \) are in general position. Thus conditions (2) and (3) will follow from condition (1) for \( i = 1, \ldots, n \), for \( \epsilon \) sufficiently small. We now fix such a choice of \( \epsilon \).

In this paragraph, we set the stage for the final perturbation, that is, the homothety and rotation mentioned in the outline. We need to determine how small this final perturbation has to be in order to achieve the desired estimates. As a preliminary choice, let \( \delta \) be sufficiently small so that \( 0 < \delta < \epsilon/2 \), \( \text{dist}(x, y) > 2\delta \) and \( B_{2\delta}(y) \subset M \). By Corollary 4.3 and Lemma 3.11, we may assume that \( \delta \) is sufficiently small so that for any \( i \in \{1, 2, \ldots, n\} \), any \( \tilde{x} \in \Sigma_x \) with \( d_2(\sigma, \tilde{x}) < \delta \), and any \( y_1 \in \tilde{M} \) with \( d_2(y, y_1) < \delta \), where \( \tilde{M} \) is the region bounded by \( \tilde{x} \), there exists a billiard path \( \tilde{\gamma}_i \) for \( \tilde{M} \) from \( x \) to \( y_1 \) with \( d(\gamma_i, \tilde{\gamma}_i) < \epsilon/2 \). We may also assume that \( \delta \) is sufficiently small so that whenever \( \text{dist}(y, y_1) < \delta \) and \( d_2(\sigma, \tilde{x}) < \delta \), the curve \( \tilde{\gamma}_i \) obtained from \( \tilde{x} \) by applying the composition, \( T \), of a homothety centered at \( x \) and a rotation of the plane centered at \( x \) such that \( y_1 \) gets mapped to \( y \) satisfies \( d_2(\tilde{x}, \tilde{\gamma}_i) < \epsilon/2 \). In addition, we may assume that \( \delta \) is sufficiently small so that for any polygonal path \( \tilde{\gamma} \) within \( \tilde{M} \), the image \( \gamma \) of \( \tilde{\gamma} \) under \( T \) satisfies \( d_2(\gamma, \gamma') < \epsilon/2 \).

To see this, note that on any compact region, a homothety with expansion factor or contraction factor close to 1 and a rotation by a small angle are both \( C^r \) close to the identity. We now fix a choice of \( \delta \) satisfying the conditions of the present paragraph for the rest of the proof.

Given \( 0 < \eta < \delta/s \), there exists \( \alpha_0 > 0 \) such that for \( 0 < |\alpha| < \alpha_0 \), Proposition 5.7 provides a \( C^r \) perturbation \( \sigma_x \) of \( \sigma \) such that: \( d_2(\sigma, \sigma_x) < \eta \); \( \sigma_x \) agrees with \( \sigma \) outside a small neighborhood of \( V \) that is chosen sufficiently small that it contains no vertex of \( \gamma_{n+1} \) other than \( V \) and it contains no vertices of \( \gamma_1, \ldots, \gamma_n \); \( V \) lies on \( \sigma_x \); the tangent line at \( V \) for \( \sigma_x \) is obtained by rotating the tangent line at \( V \) for \( \sigma \) by angle \( \alpha \) in the counterclockwise direction; and the table \( M_\alpha \) bounded by \( \sigma_x \) contains \( x \) and \( y \) in its interior. We may assume that \( \alpha_0 \) is sufficiently small so that for \( 0 < |\alpha| < \alpha_0 \), \( \gamma_{n+1} \) is not perpendicular to \( \sigma_x \) at the first time that \( \gamma_{n+1} \) leaves \( V \). Let \( \gamma_{n+1, \alpha} \) be the billiard path for \( M_\alpha \) obtained by starting with the initial part of \( \gamma_{n+1} \) from \( x \) to the first occurrence of \( V \) along \( \gamma_{n+1} \) and then continuing from \( V \) with the same number of reflections as the number of reflections \( \gamma_{n+1} \) has after the first occurrence of \( V \). Note that the signed angle from the tangent vector of \( \gamma_{n+1} \) to the tangent vector of \( \gamma_{n+1, \alpha} \) at the time that these paths leave \( V \) for the first time is \( 2\alpha \). We end \( \gamma_{n+1, \alpha} \) after the last reflection at a point \( y_1 \) in the interior of \( M_\alpha \) that is as close as possible to \( y \). If \( s = 1 \), then we let \( \tilde{\sigma} = \sigma_x \), \( \tilde{M} = M_\alpha \), and \( \tilde{\gamma}_{n+1} = \gamma_{n+1, \alpha} \), and omit the further perturbations in the rest of this paragraph. Now consider the case \( s > 1 \). Suppose \( \gamma_{n+1} \) passes through \( V \) the first time at reflection number \( k_1 \), the second time at reflection number \( k_2 \), etc.,
up to the $s$th time at reflection number $k_s$, where we count the reflections starting with the first reflection after $x$. By Corollary 4.3, we may assume that $\alpha_0$ and $\eta$ are sufficiently small so that for $0 < |\alpha| < \alpha_0$, $\gamma_{n+1, \alpha}$ does not pass through $V$ at reflection number $k_2$. We may also assume that $\alpha_0$ and $\eta$ are sufficiently small so that for $0 < |\alpha| < \alpha_0$, we have $d(\gamma_{n+1, \alpha}, \gamma_{n+1}) < \delta/s$, and, in addition, $d(\gamma_{n+1, \alpha}, \gamma_{n+1})$ is sufficiently small so that $\gamma_{n+1, \alpha}$ does not pass through $V$ at the $i$th reflection for $i \notin \{k_1, \ldots, k_s\}$. Thus, $\gamma_{n+1, \alpha}$ passes through $V$ at most $s - 1$ times. (See Figure 5 for a sketch of $\gamma_{n+1}$ and $\gamma_{n+1, \alpha}$ in the case $s = 2$.) If $s > 2$, then the construction may have to be carried out as many as $s - 1$ times in order for the final perturbed path to pass through $V$ exactly once, and we require the perturbations of the path and the boundary to be of size less than $\delta/s$ and $\eta$, respectively, at each step. Let $\tilde{\gamma}_{n+1}$ be the path obtained from $\gamma_{n+1}$ in this way and let $\tilde{\sigma}$ be the boundary obtained from $\sigma$. Then $d(\gamma_{n+1}, \gamma_{n+1}) < \delta < \epsilon/2$ and $d_r(\tilde{\sigma}, \sigma) < s\eta < \delta < \epsilon/2$. Since $d(\gamma_{n+1}, \gamma_{n+1}) < \delta$, the final endpoint $y_1$ of $\tilde{\gamma}_{n+1}$ satisfies $\text{dist}(y_1, y) < \delta$.

Let $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$ be paths in general position from $x$ to $y_1$ for the billiard table $\tilde{M}$ bounded by $\tilde{\sigma}$ that we obtain from perturbing $\gamma_1, \ldots, \gamma_n$ as in the fourth paragraph of the proof. Then $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$ do not pass through $V$, $\gamma_{n+1}$ passes through $V$ exactly once, and $\tilde{\gamma}_{n+1}$ is not perpendicular to $\tilde{\sigma}$ at $V$. We may assume that $\tilde{\gamma}_{n+1}$ is not perpendicular to $\tilde{\sigma}$ at any vertex other than $V$, because if it were, we could avoid this by replacing $\tilde{\gamma}_{n+1}$ by the shortest path within $\tilde{\gamma}_{n+1}$ that starts at $x$, contains $V$, and ends at $y_1$. We have now achieved everything required, except that the paths $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{n+1}$ end at $y_1$ instead of $y$. By the choice of $\delta$ in the fourth paragraph of the proof, if we apply the composition of a homothety and a rotation, both centered at $x$, so that $y_1$ is sent to $y$, then $\tilde{\sigma}$ is mapped to a curve $\tilde{\sigma}$ with $d_r(\tilde{\sigma}, \sigma) \leq d_r(\tilde{\sigma}, \tilde{\sigma}) + d_r(\sigma, \sigma) < \epsilon/2 + \epsilon/2 = \epsilon$; the billiard paths $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{n+1}$ for $\tilde{M}$ get

![Figure 5. Billiard path $\gamma_{n+1, \alpha}$, for the table bounded by $\sigma_{\alpha}$, passes through $V$ only once.](image-url)
mapped to billiard paths \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{n+1} \) for the billiard table \( \tilde{M} \) bounded by \( \tilde{\sigma} \), where 
\[
d(\tilde{\gamma}_n, \gamma_i) \leq d(\tilde{\gamma}_i, \tilde{\gamma}_i) + d(\tilde{\gamma}_i, \gamma_i) < \epsilon/2 + \epsilon/2 = \epsilon,
\]
for \( i = 1, \ldots, n \); and \( V \) is sent to a vertex \( V \) of \( \tilde{\gamma}_{n+1} \), with \( \text{dist}(V, V) < \epsilon/2 \). Then \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{n+1} \) and \( \tilde{V} \) satisfy conditions (1)–(5).

We now complete the proof of the main result of this section.

**Proof of Theorem 5.1.** If \( n = 1 \), we can take \( \gamma_1 \) to be the segment \( xy \). Then \( \gamma_1 \) has no vertices, and conditions (GP1)–(GP4) and (NC) are clearly satisfied with \( \sigma_1 = \sigma \).

Let \( \epsilon > 0 \), and let \( (\epsilon_k)_{k=1}^{\infty} \) be a sequence of positive numbers with sum less than \( \epsilon \).

Assume \( n \) is a positive integer. \( \sigma_n \in \Sigma_\epsilon \), \( d_\epsilon(\sigma, \sigma_n) < \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1} \), and we have \( n \) billiard paths \( \gamma_1, \ldots, \gamma_n \) in the table \( M_n \) bounded by \( \sigma_n \) that satisfy conditions (GP1)–(GP4) and (NC). For the inductive argument, we will show that there exists \( \sigma_{n+1} \in \Sigma_\epsilon \) with \( d_\epsilon(\sigma_n, \sigma_{n+1}) < \epsilon_n \) such that there are \( n + 1 \) billiard paths in the table \( M_{n+1} \) bounded by \( \sigma_{n+1} \) that satisfy conditions (GP1)–(GP4) and (NC). Each of the finitely many perturbations that we will make in obtaining \( M_{n+1} \) from \( M_n \) can be made arbitrarily small in the \( d_\epsilon \) metric.

By Lemma 5.3 and Lemma 5.10, there exists a new table \( \hat{M}_n \) on which there are paths \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n, \gamma_{n+1} \) such that \( \gamma_{n+1} \) has a vertex \( V \) that is not a vertex of any of \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n \) and such that \( \gamma_{n+1} \) passes through \( V \) only once and is not perpendicular to the boundary of \( \hat{M}_n \) at any vertex. In applying Lemma 5.10, \( \gamma_1, \ldots, \gamma_n \) can be chosen as close to \( \gamma_1, \ldots, \gamma_n \) as we like, and therefore by Lemma 5.2 we may assume that \( \gamma_1, \ldots, \gamma_n \) still satisfy (GP1)–(GP4) and (NC).

Next, we describe a perturbation of the table in a small neighborhood of \( V \) (that does not contain any vertices of \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n \)) and a perturbation \( \tilde{\gamma}_{n+1} \) of \( \gamma_{n+1} \) such that \( \tilde{\gamma}_{n+1} \) is a billiard path on the perturbed table and such that \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n, \tilde{\gamma}_{n+1} \) satisfy condition (GP1) on the perturbed table. We may assume the neighborhood of \( V \) where this perturbation takes place is small enough that \( \gamma_1, \ldots, \gamma_n \) are unchanged. To obtain (GP1), it suffices to ensure that \( \tilde{\gamma}_{n+1} \) does not pass through any vertices of \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n \). Suppose there exist vertices of \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n \) that occur in \( \gamma_{n+1} \). (If no such vertices exist, we are finished with condition (GP1).) We may label these vertices so that \( P_1, \ldots, P_k \) occur along \( \gamma_{n+1} \) between \( x \) and \( V \), and \( P_{k+1}, \ldots, P_{l} \) occur between \( V \) and \( y \). Now by Corollary 5.5 we may assume that \( \tilde{M}_n \) is constructed so that \( x \) and \( P_i \) are not conjugate along \( \gamma_{n+1} \) for \( i = 1, \ldots, k \), and \( P_i \) and \( y \) are not conjugate along \( \gamma_{n+1} \) for \( i = k+1, \ldots, l \). Therefore, by Corollary 4.3 there exists a billiard path \( \tau_x \) starting at \( x \) with initial direction making a small but nonzero angle with that of \( \gamma_{n+1} \) and ending at the boundary of \( \tilde{M}_n \) near \( V \) that avoids \( P_1, \ldots, P_k \). Similarly, there exists a billiard path \( \tau_y \) starting at \( y \) with initial direction making a small but nonzero angle with that of \( -\gamma_{n+1} \) and ending at the boundary of \( \tilde{M}_n \) near \( V \) that avoids \( P_{k+1}, \ldots, P_l \). Since \( \gamma_{n+1} \) is not perpendicular to the boundary of \( \tilde{M}_n \) at \( V \), the lines containing the segments of \( \gamma_{n+1} \) with endpoint \( V \) intersect transversely at \( V \). Therefore the final segments of the paths \( \tau_x \) and \( \tau_y \), possibly slightly extended beyond \( \tilde{M}_n \), intersect at some point \( \tilde{V} \) near \( V \), and we end these final segments at \( \tilde{V} \). The angle bisector of the final segments of \( \tau_x \) and \( \tau_y \) is arbitrarily close to the angle bisector of the segments of \( \gamma_{n+1} \) that have \( V \) as an endpoint. By Proposition 5.7, we can now make a small \( C^\infty \) perturbation of the table in a small neighborhood of \( V \) so that the paths \( \tau_x \) and \( -\tau_y \) can be joined at \( \tilde{V} \) to form a billiard path \( \tilde{\gamma}_{n+1} \) for the new table, which we call
\[ \tilde{M}_n \text{, and } \tilde{\gamma}_{n+1} \text{ avoids the vertices of } \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n. \] This perturbation does not affect \[ \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n. \] Therefore \[ \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{n+1} \text{ satisfy condition (GP1) for } \tilde{M}_n. \] Moreover, \[ \tilde{\gamma}_{n+1} \] passes through \( \tilde{V} \) only once and \[ \tilde{\gamma}_{n+1} \] is not perpendicular to the boundary of \( \tilde{M}_n \) at any vertex.

Now we want to satisfy condition (GP2). Suppose \( \tilde{\gamma}_{n+1} \) passes through a vertex \( P \neq \tilde{V} \) more than once. By condition (GP1), \( P \) is not a vertex of any of \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n. \) Then we can replace \( V \) by \( P \) in Lemma 5.10 and obtain a small \( C^\alpha \) perturbation of the boundary of the table, and small perturbations of \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{n+1} \) maintaining condition (GP1) for the \( n+1 \) paths, maintaining conditions (GP2)--(GP4) and (NC) for the first \( n \) paths, and having the \((n+1)\)st path go through a vertex \( P \) near \( P \) only once. This procedure can be repeated until the \((n+1)\)st path goes through each vertex only once. Note that the procedure ends after finitely many steps because the number of reflections made by the \((n+1)\)st path is unchanged by these perturbations. Thus we obtain condition (GP2) for paths \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{n+1} \) on a table \( \tilde{M}_n. \) Again, we may assume that \( \tilde{\gamma}_{n+1} \) is not perpendicular to the boundary of \( \tilde{M}_n \) at any vertex.

We now proceed to achieve conditions (GP3)--(GP4). A segment \( \eta \) of \( \tilde{\gamma}_{n+1} \) can cause a violation of condition (GP3) or (GP4) by passing through any of the following (finitely many) points: (i) an intersection point (other than \( x \) or \( y \)) of two segments of \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n; \) (ii) an intersection point (other than \( x \) or \( y \)) of one of \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n \) and a segment of \( \tilde{\gamma}_{n+1} \) different from \( \eta; \) (iii) an intersection point (other than \( x \) or \( y \)) of two distinct segments of \( \tilde{\gamma}_{n+1} \) other than \( \eta; \) (iv) \( x \) if \( \eta \) is not the initial segment of \( \tilde{\gamma}_{n+1} \); and (v) \( y \) if \( \eta \) is not the final segment of \( \tilde{\gamma}_{n+1} \). If such a violation occurs, then we can apply Corollary 5.9 if \( \eta \) is a chord of \( \tilde{\gamma}_{n+1} \) and Corollary 5.8 if \( \eta \) is an initial or final segment of \( \tilde{\gamma}_{n+1} \) in order to avoid these violations of conditions (GP3)--(GP4). This can be done while maintaining conditions (GP1)--(GP2) for \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n \) and the perturbed version of \( \tilde{\gamma}_{n+1} \), and avoiding any new violations of conditions (GP3)--(GP4). This procedure can be done to each segment of \( \tilde{\gamma}_{n+1} \) that causes a violation of conditions (GP3)--(GP4). We can therefore achieve conditions (GP3)--(GP4) for a new table \( \widehat{M}_n, \) a perturbed version \( \widehat{\gamma}_{n+1} \) of the \((n+1)\)st path, and the same paths \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n. \)

Since condition (NC) is already satisfied for \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n, \) any violation of condition (NC) for the \( n+1 \) paths \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n, \tilde{\gamma}_{n+1} \) on \( \widehat{M}_n \) would mean there exist distinct vertices \( p, q \) of \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{n+1} \) that are collinear with \( y, \) where \( p \) is a vertex of \( \tilde{\gamma}_{n+1}. \) Note that the segment \( pq \) cannot be a chord of \( \tilde{\gamma}_{n+1}, \) because then condition (GP4) would be violated. If \( p \) is not an endpoint of the final segment of \( \tilde{\gamma}_{n+1}, \) then we can apply Corollary 5.9 to a chord through \( p \) to obtain a perturbed version of \( \tilde{\gamma}_{n+1} \) so that the new vertex near \( p \) is not collinear with \( y \) and \( q. \) If \( p \) is an endpoint of the final segment of \( \tilde{\gamma}_{n+1}, \) then we can apply Corollary 5.8 to achieve this result. Again, all perturbations can be done while maintaining conditions (GP1)--(GP4) and not introducing any new violations of condition (NC). The paths \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n \) and the perturbed version of \( \tilde{\gamma}_{n+1} \) are then labeled \( \gamma_1, \ldots, \gamma_{n+1} \) and the perturbed version of \( \widehat{M}_n \) is labeled \( M_{n+1} \). Thus conditions (GP1)--(GP4) and (NC) are satisfied for \( \gamma_1, \ldots, \gamma_{n+1} \) on \( M_{n+1}. \)
The boundary $\sigma_{n+1}$ of $M_{n+1}$ is in $\Sigma_r$ and can be chosen to satisfy $d_r(\sigma_n, \sigma_{n+1}) < \epsilon_n$ since we made arbitrarily small $C^r$ perturbations of the boundaries of the tables at each step in the proof.

6. The main result: Generic insecurity. Let $\Sigma_r$ and $d_r$ be as described in Section 3, and assume $r \geq 2$. Then $\Sigma_r$ is a dense open subset of the complete metric space $(\Sigma^D_r, d_r)$, where $\Sigma^D_r$ is the set of simple closed $C^r$ curves in the plane with nonnegative curvature. Thus, by the Baire Category Theorem, in the $C^r$ topology the intersection of countably many dense open subsets of $\Sigma_r$ is dense in $\Sigma_r$. Theorem 6.3 contains the generic insecurity result described in Section 1. In fact, we prove slightly more: the set $A$ in Theorem 6.3 is not just a dense $G_\delta$ subset of $\Sigma_r$ in the $C^r$ topology—it is the intersection of countably many sets that are $C^2$ open and $C^r$ dense in $\Sigma_r$.

**Lemma 6.1.** Let $\sigma_n \in \Sigma_2$ and suppose $x$ and $y$ are distinct points in the interior of the table $M_n$ bounded by $\sigma_n$. Assume there exist $n$ billiard paths $\gamma_1, \ldots, \gamma_n$ for $M_n$ from $x$ to $y$ that are in general position, and $x$ and $y$ are not conjugate along any of these paths. Then there exists an open neighborhood $N$ of $\sigma_n$ in $\Sigma_2$ with the $C^2$ topology such that for every $\tilde{\sigma} \in N$, $x$ and $y$ are still in the interior of the table $\tilde{M}$ bounded by $\tilde{\sigma}$, and there exist $n$ billiard paths $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$ for $\tilde{M}$ from $x$ to $y$ that are in general position.

**Proof.** Let $\epsilon > 0$ be as in Lemma 5.2. Then it suffices to show that there exists a $C^2$ open neighborhood $N$ of $\sigma_n$ in $\Sigma_2$ such that for all $\tilde{\sigma} \in N$, there exist billiard paths $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$ for the table $\tilde{M}$ bounded by $\tilde{\sigma}$ from $x$ to $y$ such that $\tilde{\gamma}_i$ has the same number of vertices as $\gamma_i$ and $d(\gamma_i, \tilde{\gamma}_i) < \epsilon$ for $i = 1, \ldots, n$. Fix a choice of $i \in \{1, \ldots, n\}$, and let $k_i$ be the number of vertices of $\gamma_i$. It follows from Lemma 3.11 that there exists $\alpha_i > 0$ and a $C^2$ open neighborhood $N_i$ of $\sigma_n$ such that for any billiard path $\tilde{\gamma}_i$ for any $\tilde{\sigma} \in N_i$ that starts at $x$ making angle less than $\alpha_i$ with $\gamma_i$ at $x$, the first $k_i$ vertices of $\tilde{\gamma}_i$ are within distance $\epsilon_i$ of the corresponding vertices of $\gamma_i$. Moreover, by Corollary 4.3, if $N_i \subset N$ is a sufficiently small $C^2$ open neighborhood of $\sigma_n$, then for $\tilde{\sigma} \in N_i$ there exists a billiard path $\tilde{\gamma}_i$ starting at $x$ and making angle less than $\alpha_i$ with $\gamma_i$ at $x$ such that $\tilde{\gamma}_i$ ends at $y$ after $k_i$ reflections. Thus $N := \cap_{i=1}^n N_i$ is the desired neighborhood of $\sigma_n$. 

**Corollary 6.2.** Given two distinct points $x$ and $y$ in $\mathbb{R}^2$, let $\Sigma_{(x,y),r}$ be the set of curves $\sigma$ in $\Sigma_r$ such that $x$ and $y$ are contained in the interior of the table bounded by $\sigma$. Then there exists a dense $G_\delta$ subset $A_{(x,y),r}$ of $\Sigma_{(x,y),r}$ in the $C^r$ topology such that for every $\tau \in A_{(x,y),r}$ and every positive integer $n$ there exist $n$ billiard paths from $x$ to $y$ for the table $M(\tau)$ bounded by $\tau$ that are in general position. In particular, for every $\tau \in A_{(x,y),r}$, the pair $(x, y)$ is insecure for $M(\tau)$. The set $A_{(x,y),r}$ may be chosen to be an intersection of countably many $C^2$ open and $C^r$ dense subsets of $\Sigma_{(x,y),r}$.

**Proof.** Let $n$ be a positive integer, $\epsilon > 0$, and $\sigma \in \Sigma_{(x,y),r}$. By Theorem 5.1, there exists $\sigma_n \in \Sigma_{(x,y),r}$ with $d_r(\sigma, \sigma_n) < \epsilon$ such that there exist $n$ billiard paths $\gamma_1, \ldots, \gamma_n$ from $x$ to $y$ for the table bounded by $\sigma_n$ that are in general position. By Corollary 5.5, we may assume that $x$ and $y$ are not conjugate along any of $\gamma_1, \ldots, \gamma_n$ in the billiard table bounded by $\sigma_n$. Now by Lemma 6.1 there exists a $C^2$ neighborhood $N = N(\sigma, n, \epsilon)$ of $\sigma_n$ in $\Sigma_{(x,y),r}$ such that for every $\tilde{\sigma} \in N$ there are $n$ billiard paths for $M(\tilde{\sigma})$ from $x$ to $y$ that are in general position. Then
the conclusion of the Corollary.

For \( k = 2 \) or \( r \), we refer to the product topology on \( \Sigma^0 \times \mathbb{R}^2 \) obtained from the \( C^k \) topology on \( \Sigma^0 \) and the usual topology on \( \mathbb{R}^2 \times \mathbb{R}^2 \) as the \( C^k \) topology. The \( C^r \) topology on \( \Sigma^r \) is the topology of a complete metric space.

**Theorem 6.3.** For \( r \geq 2 \) there exists a dense \( G_\delta \) subset \( A \subseteq \Sigma_r \) in the \( C^r \) topology such that if \( \sigma \in A \) and \( M(\sigma) \) is the billiard table bounded by \( \sigma \), then there is a dense \( G_\delta \) subset \( \mathcal{R}(\sigma) \) of \( M(\sigma) \times M(\sigma) \) with the topology induced from \( \mathbb{R}^2 \times \mathbb{R}^2 \) such that for each \( (x,y) \in \mathcal{R}(\sigma) \) and each positive integer \( n \), there are \( n \) billiard paths for \( M(\sigma) \) from \( x \) to \( y \) that are in general position. In particular, every pair \( (x,y) \in \mathcal{R}(\sigma) \) is insecure for \( M(\sigma) \). The set \( A \) may be chosen to be an intersection of countably many \( C^2 \) open and \( C^r \) dense subsets of \( \Sigma_r \).

**Proof.** Consider \( \Sigma_r \times B \), where \( B = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : x \neq y\} \), with the \( C^r \) topology induced from \( \Sigma^r \times \mathbb{R}^2 \times \mathbb{R}^2 \). Let \( \mathcal{G} = \{(\sigma,(x,y)) \in \Sigma_r \times B : x \) and \( y \) do not lie on \( \sigma \} \) and let \( \mathcal{G}_0 = \{(\sigma,(x,y)) \in \Sigma_r \times B : (x,y) \in \text{Int}(M(\sigma)) \times \text{Int}(M(\sigma))\} \). Then \( \mathcal{G} \) is a \( C^2 \) open and \( C^r \) dense subset of \( \Sigma^r \times \mathbb{R}^2 \times \mathbb{R}^2 \), and \( \mathcal{G} \setminus \mathcal{G}_0 \) is \( C^2 \) open. If \( (\sigma,(x,y)) \in \mathcal{G}_0 \), then as in the proof of Corollary 6.2, if we are given \( \varepsilon > 0 \) and a positive integer \( n \), there exist \( n \) billiard paths from \( x \) to \( y \) for \( M(\tilde{\sigma}) \) that are in general position. Note that if we make a sufficiently small perturbation of \( (x,y) \) to \((\hat{x},\hat{y})\) in addition to a sufficiently small \( C^2 \) perturbation of \( \sigma \) to \( \tilde{\sigma} \), we will still obtain \( n \) billiard paths from \( \hat{x} \) to \( \hat{y} \) in \( M(\tilde{\sigma}) \) that are in general position, because we may apply a transformation \( T \) of \( \mathbb{R}^2 \) such that \( T \) is the composition of a homothety and a rigid motion, both close to the identity, such that \( T \) maps \( \hat{x} \) and \( \hat{y} \) to \( x \) and \( y \), respectively, and maps \( \tilde{\sigma} \) to a curve in \( N \). Thus there is a \( C^2 \) open neighborhood \( \tilde{N} = \tilde{N}(\sigma,n,\varepsilon,(x,y)) \) of \( (\sigma,(x,y)) \) in \( \mathcal{G}_0 \) such that for each \( (\tilde{\sigma},(\hat{x},\hat{y})) \in \tilde{N} \) there are \( n \) billiard paths for \( M(\tilde{\sigma}) \) from \( \hat{x} \) to \( \hat{y} \) that are in general position. Let \( \mathcal{G}_n = (\mathcal{G} \setminus \mathcal{G}_0) \cup \{(\sigma,(x,y)) \in \mathcal{G}_0 : \tilde{N}(\sigma,n,\varepsilon,(x,y)) \} \). Then \( \mathcal{G}_n \) is a \( C^2 \) open and \( C^r \) dense subset of \( \mathcal{G} \). Thus there exists a subset \( A_n \subseteq \Sigma_r \) which is the intersection of countably many \( C^2 \) open and \( C^r \) dense subsets of \( \Sigma_r \) such that for \( \sigma \in A_n \), \( \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : (\sigma,(x,y)) \in \mathcal{G}_n \} \) is a dense \( G_\delta \) subset of \( \mathbb{R}^2 \times \mathbb{R}^2 \). (This follows from the proof of the Kuratowski-Ulam Theorem, as presented in Chapter 15 of [21].) Let \( A = \cap_{n=1}^\infty A_n \). If \( \sigma \in A \), then \( \mathcal{R}(\sigma) = \cap_{n=1}^\infty \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : (\sigma,(x,y)) \in \mathcal{G}_n \setminus \mathcal{G}_0 \} \) is a dense \( G_\delta \) subset of \( \text{Int}(M(\sigma)) \times \text{Int}(M(\sigma)) \) having the required property.

**Acknowledgments.** We thank Sergei Tabachnikov for a helpful conversation concerning this paper. We also thank the anonymous referee for several useful suggestions.

**REFERENCES**

[1] V. Bangert and E. Guttak, "Insecurity for compact surfaces of positive genus," *Geom. Dedicata*, **146** (2010), 165–191.

[2] R. Bishop, "Circular billiard tables, conjugate loci, and a cardioid," *Regul. ChaoticDyn.*, **8** (2003), 83–95.

[3] J. Bruce and P. Giblin, *Curves and Singularities: A Geometrical Introduction to Singularity Theory*, Cambridge University Press, Cambridge, 1984.

[4] K. Burns and M. Gidea, *Differential Geometry and Topology: With a View to Dynamical Systems*, Chapman & Hall/CRC, Boca Raton, FL, 2005.
[5] K. Burns and E. Gutkin, Growth of the number of geodesics between points and insecurity for Riemannian manifolds, *Discrete Contin. Dyn. Syst.*, 21 (2008), 403–413.

[6] M. Farber, Topology of Billiard Problems, I, *Duke Math. J.*, 115 (2002), 559–585.

[7] M. Farber, Topology of Billiard Problems, II, *Duke Math. J.*, 115 (2002), 587–621.

[8] M. Gerber and L. Liu, Real analytic metrics on $S^2$ with total absence of finite blocking, *Geom. Dedicata*, 166 (2013), 99–128.

[9] M. Gerber and W.-K. Ku, A dense G-delta set of Riemannian metrics without the finite blocking property, *Math. Res. Lett.*, 18 (2011), 389–404.

[10] E. Gutkin, Billiards on almost integrable polyhedral surfaces, *Ergodic Theory Dynam. Sys.*, 4 (1984), 569–584.

[11] E. Gutkin, Blocking of billiard orbits and security for polygons and flat surfaces, *Geom. Funct. Anal.*, 15 (2005), 83–105.

[12] E. Gutkin, Billiard dynamics: An updated survey with the emphasis on open problems, *Chaos*, 22 (2012), 026116, 13pp.

[13] E. Gutkin, P. Hubert and T. Schmidt, Affine diffeomorphisms of translation surfaces: Periodic points, Fuchsian groups, and arithmeticity, *Ann. Sci. École Norm. Sup. (4)*, 36 (2003), 847–866.

[14] E. Gutkin and C. Judge, The geometry and arithmetic of translation surfaces with applications to polygonal billiards, *Math. Res. Lett.*, 3 (1996), 391–403.

[15] E. Gutkin and C. Judge, Affine mappings of translation surfaces: Geometry and arithmetic, *Duke Math. J.*, 103 (2000), 191–213.

[16] E. Gutkin and V. Schroeder, Connecting geodesics and security of configurations in compact locally symmetric spaces, *Geom. Dedicata*, 118 (2006), 185–208.

[17] W. Ho, On blocking numbers of surfaces, preprint, *arXiv:0807.2934v3* (2008).

[18] A. Katok and B. Hasselblatt, *Modern Theory of Dynamical Systems*, Cambridge University Press, Cambridge, 1995.

[19] J.-F. Lafont and B. Schmidt, Blocking light in compact Riemannian manifolds, *Geom. Topol.*, 11 (2007), 867–887.

[20] T. Monteil, A counter-example to the theorem of Hiemer and Snurnikov, *J. Statist. Phys.*, 114 (2004), 1619–1623.

[21] J. Oxtoby, *Measure and Category, Second Edition*, Springer-Verlag, New York-Berlin, 1980.

[22] W. Rudin, *Principles of Mathematical Analysis, Third Edition*, McGraw Hill, New York-Auckland-Düsseldorf, 1976.

[23] S. Tabachnikov, *Geometry and Billiards*, American Mathematical Society, Providence, RI, 2005.

[24] S. Tabachnikov, Birkhoff billiards are insecure, *Discrete Contin. Dyn. Syst.*, 23 (2009), 1035–1040.

[25] W. Veech, Teichmüller curves in moduli space, Eisenstein series, and an application to triangular billiards, *Invent. Math.*, 97 (1989), 553–583.

[26] W. Veech, The billiard in a regular polygon, *Geom. Funct. Anal.*, 2 (1992), 341–379.

[27] Ya. Vorobets, On the measure of the set of periodic points of a billiard, *Math. Notes*, 55 (1994), 455–460.

[28] M. Wojtkowski, Principles for the design of billiards with nonvanishing Lyapunov exponents, *Comm. Math. Phys.*, 105 (1986), 391–414.

Received August 2015; revised March 2016.

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