CONJUGATE CONNECTIONS AND THEIR APPLICATIONS ON
PURE METALLIC METRIC GEOMETRIES

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Abstract. Let \((M, J, g)\) be a metallic pseudo-Riemannian manifold equipped with a metallic structure \(J\) and a pseudo-Riemannian metric \(g\). The paper deals with interactions of Codazzi couplings formed by conjugate connections and tensor structures. The presence of Tachibana operator and Codazzi couplings presented a new characterization for locally metallic pseudo-Riemannian manifold. Also, a necessary and sufficient condition a non-integrable metallic pseudo-Riemannian manifold is a quasi metallic pseudo Riemannian manifold is derived. Finally, it is introduced metallic-like pseudo-Riemannian manifolds and presented some results concerning them.

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1. Introduction

The concept of tensor structures on differentiable manifolds is one of the central concepts of modern differential geometry. Among tensor structures on a differentiable manifold, one of the best known is almost complex structure, that is, a \((1, 1)\)-tensor field \(J\) whose square, at each point, is minus the identity. The manifold must be even-dimensional, that is, \(\dim = 2n\). Usually, it is equipped with a Hermitian metric \(g\) which is a (Riemannian or pseudo-Riemannian) metric that preserves the almost complex structure, that is, the almost complex structure \(J\) acts as an isometry with respect to the (pseudo-)Riemannian metric. In this case, the \((J, g)\) is called an almost Hermitian structure. The associated \((0, 2)\)-tensor of the Hermitian metric \(g\) is a \(2\)-form \(\omega\) which is defined by \(\omega = g(JX, Y)\) and hence the relationship with symplectic geometry.

The other case is that the almost complex structure \(J\) acts as an anti-isometry with respect to a pseudo-Riemannian metric \(g\). Such a metric is known as an anti-Hermitian metric or a Norden metric (first studied by and named after Norden [17]). The Norden metric is necessary pseudo-Riemannian of neutral signature. In this case, \((J, g)\) is called an almost anti-Hermitian structure or almost Norden structure or almost \(B\)-structure. The associated \((0, 2)\)-tensor of any Norden metric which is defined by \(G = g(JX, Y)\) is also a Norden metric. So, in this case we dispose with a pair of mutually associated Norden metrics, known also as twin Norden metrics.

Besides almost complex structures, other tensor structures on differentiable manifolds which are called almost tangent, almost product structures etc. exist and are important for modern differential geometry. All the tensor structures we have talked about so far are actually a polynomial structure. Polynomial structures can
be considered as \((1,1)\)-tensor field which are roots of the algebraic equation

\[ Q(J) = J^n + a_n J^{n-1} + \ldots + a_2 J + a_1 J = 0, \]

where \(a_1, a_2, \ldots, a_n\) are real numbers and \(I\) is the identity tensor of type \((1,1)\).

Polynomial structures on a manifold were defined in \([9]\). In particular, we can say the followings

- If \(Q(J) = J^2 + I = 0\), its solution \(J\) is called an almost complex structure.
- If \(Q(J) = J^2 - I = 0\), its solution \(J\) is called an almost product structure.
- If \(Q(J) = J^2 = 0\), its solution \(J\) is called an almost tangent structure.
- If \(Q(J) = J^2 - J - I = 0\), its solution \(J\) is called a golden structure. This tensor structure was inspired by the golden ratio, which was described by Johannes Kepler (1571–1630). The number \(\eta = \frac{1 + \sqrt{5}}{2} = 1.618\ldots\), which is a solution of the equation \(x^2 - x - 1 = 0\), is the golden ratio.
- If \(Q(J) = J^2 - pJ - qI = 0\), with \(p\) and \(q\) positive integers, its solution \(J\) is called a metallic structure. The name is motivated by the fact that the metallic means family or metallic proportions introduced by de Spinadel \([6]\) is the positive root of the quadratic equation \(x^2 - px - q = 0\), namely \(\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}\). If \(p = 0\), that is, \(J^2 = qI\), then we call them trivial metallic structures.

We will follow the notations and definitions concerning with metallic pseudo-Riemannian manifolds given in \([3]\).

**Definition 1.** \([3]\) Let \(M\) be a smooth (real) manifold and \(J : TM \to TM\) be a tangent bundle isomorphism. A \((1,1)\)-tensor field \(J\) on \(M\) is called a metallic structure if \(J^2 = pJ + qI\), where \(I\) is the identity operator and \(p, q \in \mathbb{R}\).

The classical definition is used for integrability of a metallic structure.

**Definition 2.** \([3]\) For a metallic structure \(J\), the integrability is equivalent to the vanishing of the Nijenhuis tensor \(N_J :\)

\[ N_J(X, Y) = [JX, JY] - J [JX, Y] - J [X, JY] + J^2 [X, Y]. \]

Here and further, \(X, Y, Z\) will stand for arbitrary differentiable vector fields on the considered manifold (or vectors in its tangent space at an arbitrary point of the manifold).

**Definition 3.** A linear connection \(\nabla\) on a smooth manifold \(M\) is called a \(J\)-connection if \(\nabla J = 0\).

Another equivalent definition for integrability of a polynomial structure is given by Vanzura in \([18]\): in order that a metallic structure \(J\) be integrable, it is necessary and sufficient that we can introduce a torsion-free linear connection \(\nabla\) such that \(\nabla J = 0\), that is, the torsion-free linear connection \(\nabla\) is a \(J\)-connection.

**Definition 4.** Let \((M, g)\) be a pseudo-Riemannian manifold and \(J\) be a \(g\)-symmetric \((1,1)\)-tensor field on \(M\) such that \(J^2 = pJ + qI\), where \(p\) and \(q\) are real numbers. Then \((J, g)\) is called a metallic pseudo-Riemannian structure and \((M, J, g)\) is named a metallic pseudo-Riemannian manifold. A metallic pseudo-Riemannian manifold \((M, J, g)\) such that the Levi-Civita connection \(\nabla^0\) with respect to \(g\) is a \(J\)-connection is called a locally metallic pseudo-Riemannian manifold.
Next, we can give the following proposition for another way of describing locally metallic pseudo-Riemannian manifolds. The following proposition can be proven by following the method used in [19]. It will be presented without proof. So, we will avoid unnecessary repetition.

**Proposition 1.** Let \((M, J, g)\) be a metallic pseudo-Riemannian manifold and \(\nabla^g\) be the Levi-Civita connection of \(g\). Then \(\Phi_J g = 0\) is equivalent to \(\nabla^g J = 0\), where \(\Phi_J\) is the Tachibana operator defined by

\[
(\Phi_J g)(X, Y, Z) = JX g(Y, Z) - X g(JY, Z) + g((L_Y J) X, Z) + g(Y, (L_Z J) X).
\]

Here \((L_X J) Y = [X, JY] - J [X, Y]\).

Now let’s turn our attention to the other important actor in our article, the conjugate connections. The notion of conjugate connections with respect to a metric tensor field was originally introduced by Norden in the context of Weyl geometry [17]. Such linear connections were independently developed by Nagao ka and Amari [15] under the name dual connections and used by Lauritzen in the definition of statistical manifolds [12]. In this direction, we can define two conjugate connection on a metallic pseudo-Riemannian manifold \((M, J, g)\), which are respectively called \(g\)–conjugate connection \((\nabla^\ast)\) and \(G\)–conjugate connection \((\nabla^\dagger)\):

\[
Z g (X, Y) = g (\nabla_Z X, Y) + g (X, \nabla^\ast_Z Y)
\]

and

\[
ZG (X, Y) = G (\nabla_Z X, Y) + G (X, \nabla^\dagger_Z Y),
\]

where \(G = g(JX, Y)\) is the twin metric on \((M, J, g)\) and \(\nabla\) is a linear connection. It is easy to see that \(\nabla^\ast\) and \(\nabla^\dagger\) are indeed connections such that \((\nabla^\ast)^\dagger = (\nabla^\dagger)^\ast = \nabla\). Another type of conjugate connections are those concerning with tensor structures (for details, [1, 2, 4]). In our setting, this connection is defined by

\[
\nabla_X^\dagger Y = J^{-1} (\nabla_X (JY))
\]

which is called \(J\)–conjugate connection \((\nabla^\dagger)\).

2. **Codazzi Coupling of \(\nabla\) with \(J\)**

Let \(\nabla\) be an arbitrary linear connection on a pseudo-Riemannian manifold \((M, g)\). A symmetric \((0, 2)\)–tensor field \(\rho\) is Codazzi if it satisfies the symmetry property

\[
(\nabla_X \rho)(Y, Z) = (\nabla_Z \rho)(X, Y).
\]

Alternatively, a \((1, 1)\)–tensor field \(J\) is Codazzi if it is self-adjoint and

\[
(\nabla_X J) Y = (\nabla_Y J) X.
\]

We call the pairs \((\nabla, \rho)\) and \((\nabla, J)\), respectively, a Codazzi-coupled.

Let \((M, J, g)\) be a metallic pseudo-Riemannian manifold. The inverse of the metallic structure \(J\) is as follow: \(J^{-1} = \frac{1}{q} (J - pI)\) \((q \neq 0)\). The following proposition is analogue to the known result given by Fei and Zhang [7] for Hermitian setting.

**Proposition 2.** Let \(\nabla\) be a linear connection and \(J\) be a metallic structure on \(M\). Then the following situations are equivalent

1) \((\nabla, J)\) is Codazzi-coupled;
2) \(\nabla\) and \(\nabla^J\) have equal torsions;
3) \((\nabla^J, J^{-1})\) is Codazzi-coupled.
Proof. 1) ⇒ 2): Assume that $(\nabla, J)$ is Codazzi-coupled. Then we have

\begin{align*}
(2.1) \quad J^{-1}((\nabla_X J) Y - (\nabla_Y J) X) \\
= \quad J^{-1}(\nabla_X J Y - J\nabla_X Y - \nabla_Y J X + J\nabla_Y X) \\
= \quad J^{-1}(\nabla_X J Y) - J^{-1}(\nabla_Y J X) - (\nabla_X Y - \nabla_Y X) \\
= \quad \nabla_X J Y - \nabla_Y J X - (\nabla_X Y - \nabla_Y X) \\
= \quad T^{\nabla J}(X,Y) - T^{\nabla}(X,Y).
\end{align*}

2) ⇒ 3): Suppose that we have $T^{\nabla J}(X,Y) = T^{\nabla}(X,Y)$. Since

\begin{align*}
(2.2) \quad J ((\nabla_X J^{-1}) Y - (\nabla_Y J^{-1}) X) \\
= \quad J (\nabla_X J^{-1} Y) - \nabla_X J^{-1} Y - J (\nabla_Y J^{-1} X) + \nabla_Y J^{-1} X \\
= \quad \nabla_X Y - \nabla_Y X - (\nabla_X J Y - \nabla_Y J X) \\
= \quad T^{\nabla J}(X,Y) - T^{\nabla}(X,Y),
\end{align*}

from which it is clear that $(\nabla^J, J^{-1})$ is Codazzi-coupled.

3) ⇒ 1): The result immediately follows from (2.1) and (2.2). \qed

Fei and Zhang [7] proved that if $J$ is either an almost complex structure or an almost para-complex structure, then $(\nabla^J)^J = \nabla$. For the metallic structure $J$, we have the following proposition.

**Proposition 3.** Let $J$ be a metallic structure on $M$ and $\nabla$ be a linear connection. If $J$ is a trivial metallic structure ($p = 0$), then we get $(\nabla^J)^J = \nabla$.

**Proof.** For a linear connection $\nabla$ and a metallic structure $J$, we know that

$$\nabla^J_X Y = J^{-1}(\nabla_X (JY)).$$

Using the definition of metallic structure $J$, we can rewrite it as

$$\begin{align*}
(\nabla^J)_X^J Y &= J^{-1}(\nabla_X^J (JY)) \\
&= J^{-2}((\nabla_X J^2) Y + J^2 \nabla_X Y) \\
&= pJ^{-2}((\nabla_X J) Y) + qJ^{-2}((\nabla_X I) Y) + \nabla_X Y \\
&= pJ^{-2}((\nabla_X J) Y) + \nabla_X Y,
\end{align*}$$

Thus, if $p = 0$, then we have $(\nabla^J)_X^J Y = \nabla_X Y$. \qed

**Remark 1.** Let $M$ be an almost (para)complex manifold equipped a $(1,1)-$tensor field $J$ such that $J^2 = -I$ (or $J^2 = I$) where $I$ is the identity operation. Fei and Zhang obtained a nice result stating that $g-$conjugation, $\omega-$conjugation and $J-$conjugation (along with an identity operation) together form a 4-element Klein group on the space of linear connections, where $\omega$ is the fundamental 2-form (see Theorem 2.13 in [7]). Unfortunately, for metallic structure $J$, $J-$conjugation is not involutive, that is, $(\nabla^J)^J \neq \nabla$. So, in our setting, we cannot create a 4-element Klein group with our arguments. But, this shows the difference of our structure from structures used in [7].

**Proposition 4.** Let $J$ be a metallic structure on $M$ and $\nabla$ be a linear connection on $M$. If $(\nabla, J)$ is a Codazzi-coupled, then $(\nabla^J, J)$ is so.
Proof. Assume that \((\nabla, J)\) is a Codazzi-coupled. Then, standard calculations give

\[
(\nabla^I_X Y) - (\nabla^I_Y X) = \nabla^I_X (JY) - J\nabla^I_X Y - \nabla^I_Y (JX) + J\nabla^I_Y X
= J^{-1} ((\nabla_X J^2) Y + J^2 (\nabla_X Y)) - \nabla_X (JY)
- J^{-1} ((\nabla_Y J^2) X + J^2 (\nabla_Y X)) + \nabla_Y (JX)
= pJ^{-1}((\nabla_X J) Y - (\nabla_Y J) X)
+ qJ^{-1}((\nabla_X I) Y - (\nabla_Y I) X)
= 0.
\]

□

Let \(J\) be either an almost complex structure or an almost para-complex structure. Then \((\nabla, J)\) is Codazzi-coupled if and only if \((\nabla^I, J)\) is Codazzi-coupled \([7]\). For the metallic structure \(J\), the reverse of the Proposition 4 is only true for trivial metallic structure \((p = 0)\), which is meaningless.

Lemma 1. Let \(J\) be a metallic structure on \(M\) and \(\nabla\) be a linear connection on \(M\). If \((\nabla, J)\) is a Codazzi-coupled, then

\[
N_J (X, Y) = -J^2 T^\nabla (X, Y) + T^\nabla (X, JY) + JT^\nabla (JX, Y) - T^\nabla (JX, JY),
\]

where \(T^\nabla (X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]\) (also, see \([7]\)).

For the integrability of a metallic structure \(J\), the following proposition immediately follows from Lemma 1.

Proposition 5. Let \(J\) be a metallic structure on \(M\) and \(\nabla\) be a linear connection on \(M\). If \((\nabla, J)\) is Codazzi-coupled and \(T^\nabla (X, Y) = 0\), then the metallic structure \(J\) is integrable.

3. Codazzi Coupling of \(\nabla\) with \(g\) and \(G\)

Given a pseudo-Riemannian manifold \((M, g)\) endowed with a metallic structure \(J\), then the triple \((M, J, g)\) is called a metallic pseudo-Riemannian manifold if \([3]\)

\[
g(JX, Y) = g(X, JY).
\]

Also, the twin metallic pseudo-Riemannian metric \(G\) is defined by

\[
G(X, Y) = g(JX, Y).
\]

From the equalities of the above, it is easy to see the following result.

Proposition 6. Let \((M, J, g)\) be metallic pseudo-Riemannian manifold and \(G\) be the twin pseudo-Riemannian metric. Then the following equalities hold:

1) \(G (X, Y) = G (Y, X)\);
2) \(G (JX, Y) = G (X, JY)\), that is, \((M, J, G)\) is a metallic pseudo-Riemannian manifold;
3) \(g (J^{-1} X, Y) = g (X, J^{-1} Y)\), that is, \((M, J^{-1}, g)\) is a metallic pseudo-Riemannian manifold;
4) \(G (J^{-1} X, Y) = G (X, J^{-1} Y)\), that is, \((M, J^{-1}, G)\) is a metallic pseudo-Riemannian manifold.
Let $C$ and $\Gamma$ be the $(0, 3)$-tensor defined by

\begin{equation}
C(X, Y, Z) \equiv (\nabla_Z g)(X, Y) = Zg(X, Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)
\end{equation}

and

\begin{equation}
\Gamma(X, Y, Z) \equiv (\nabla_Z G)(X, Y) = ZG(X, Y) - G(\nabla_Z X, Y) - G(X, \nabla_Z Y).
\end{equation}

Due to symmetry of $g$ and $G$, it is clear that $C(X, Y, Z) = C(Y, X, Z)$ and $\Gamma(X, Y, Z) = \Gamma(Y, X, Z)$. From (1.1) and (3.1) (resp., (1.2) and (3.2)), we get

\begin{equation}
C(X, Y, Z) = g(X, (\nabla^* - \nabla)_Z Y)
\end{equation}

and

\begin{equation}
\Gamma(X, Y, Z) = G(X, (\nabla^* - \nabla)_Z Y).
\end{equation}

Also, it is easy to see that

\begin{equation}
C^*(X, Y, Z) \equiv (\nabla^*_Z g)(X, Y) = -C(X, Y, Z)
\end{equation}

and

\begin{equation}
\Gamma^*(X, Y, Z) \equiv (\nabla^*_Z G)(X, Y) = -\Gamma(X, Y, Z).
\end{equation}

For the metallic structure $J$, the relationship between $C = \nabla g$ and $\Gamma = \nabla G$ is as follow

\begin{equation}
\Gamma(X, Y, Z) = C(X, JY, Z) + g(X, (\nabla_Z J) Y).
\end{equation}

From (3.5), if $\nabla$ is a $J$-connection, then we get $\Gamma(X, Y, Z) = C(X, JY, Z)$.

Next, we give the following proposition which is analogue to result given in [7] for Hermitian setting.

**Proposition 7.** Let $G$ be the twin metallic pseudo-Riemannian metric, $\nabla$ be a linear connection and $\nabla^\dagger$ be the $G$–conjugate connection of $\nabla$. Then the following conditions are equivalent

1) $(\nabla, G)$ is Codazzi-coupled,
2) $(\nabla^\dagger, G)$ is Codazzi-coupled,
3) $\Gamma$ is totally symmetric,
4) $\Gamma^\dagger$ is totally symmetric,
5) $T\nabla = T\nabla^\dagger$,

where $\Gamma = \nabla G$ and $\Gamma^\dagger = \nabla^\dagger G$.

**Proof.** The assertion can be easily proved by following the method in [7, 8]. □

**Proposition 8.** Let $(M, J, g)$ be a metallic pseudo-Riemannian manifold, $G$ be the twin metallic pseudo-Riemannian metric and $\nabla$ be a linear connection. If both $(\nabla, J)$ and $(\nabla, G)$ are Codazzi-coupled, then

\begin{equation}
G \left( (\nabla^\dagger_X J) Y - (\nabla^\dagger_Y J) X, Z \right) = G \left( Y, (\nabla_Z J) X - (\nabla^\dagger_Z J) X \right),
\end{equation}

where $\nabla^\dagger$ is the $G$–conjugate connection of $\nabla$. 

Proof. If both $(\nabla, J)$ and $(\nabla, G)$ are Codazzi-coupled, then
\[
G \left( (\nabla^\dagger_X J) Y - (\nabla^\dagger_Y J) X, Z \right) = G \left( \nabla^\dagger_X JY - J\nabla^\dagger_X Y, Z \right) - G \left( \nabla^\dagger_J XY - J\nabla^\dagger_J X, Z \right) = XG(Z, JY) - G(\nabla_X Z, JY) - YG(Z, JX) + G(\nabla_Y Z, JX) + G(J\nabla^\dagger_X Y, Z) - G(J\nabla^\dagger_J X, Z)
\]
\[
= \Gamma(Z, JY, X) + G(Z, \nabla_X JY) - \Gamma(Z, JX, Y) - G(\nabla_Y JX, Z) + G(JZ, \nabla^\dagger_X Y - \nabla^\dagger_X Y) = \Gamma(Z, JY, X, Z) - \Gamma(Z, JX, Y, Z) = G(Y, (\nabla_Z J) X - \left( \nabla^\dagger_Z J \right) X),
\]
which completes the proof. \hfill \Box

Proposition 9. Let $(M, J, g)$ be a metallic pseudo-Riemannian manifold, $G$ be the twin metallic pseudo-Riemannian metric and $\nabla$ be a linear connection. If both $(\nabla, g)$ and $(\nabla, G)$ are Codazzi-coupled, then $(\nabla^*, J)$ is Codazzi-coupled, where $\nabla^*$ is the $g$–conjugate connection of $\nabla$.

Proof. From (3.5), we have
\[
C(X, JY, Z) = \Gamma(X, Y, Z) - g(X, (\nabla_Z J) Y)
\]
and
\[
C(Z, JY, X) = \Gamma(Z, Y, X) - g(Z, (\nabla_X J) Y).
\]
Since both $(\nabla, g)$ and $(\nabla, G)$ are Codazzi-coupled, we can write
\[
0 = g(Z, (\nabla_X J) Y) - g(X, (\nabla_Z J) Y) = g(Y, (\nabla^\dagger_X J) Z) - g(Y, (\nabla^\dagger_Z J) X) = g(Y, (\nabla^\dagger_Z J) Z - (\nabla^\dagger_Z J) X),
\]
which gives that $(\nabla^*, J)$ is Codazzi-coupled. \hfill \Box

Proposition 10. Let $(M, J, g)$ be a metallic pseudo-Riemannian manifold and $G$ be the twin metallic pseudo-Riemannian metric. Then

1) If both $(\nabla, J)$ and $(\nabla^*, J)$ are Codazzi-coupled, then we have $\nabla = \nabla^*$, where $\nabla^*$ is the $g$–conjugate connection of $\nabla$;

2) If both $(\nabla, J)$ and $(\nabla^\dagger, J)$ are Codazzi-coupled, then we have $\nabla = \nabla^\dagger$, where $\nabla^\dagger$ is the $G$–conjugate connection of $\nabla$.

Proof. Here we will prove only 1). Similarly, 2) can be proven. Assume that both $(\nabla, J)$ and $(\nabla^*, J)$ are Codazzi-coupled. From (3.1), it is easy to see that
\[
g((\nabla_X J) Y, Z) = g(Y, (\nabla^*_X J) Z).
\]
Thus, we can write
\[
0 = g((\nabla_X J) Y - (\nabla_Y J) X, Z) = g(Y, (\nabla^*_Z J) X) - g(Y, (\nabla_Z J) X) = g(Y, (\nabla^*_Z J) X - (\nabla_Z J) X)
\]
that implies $\nabla^* J = \nabla J$.

\[\square\]

**Proposition 11.** Let $(M, J, g)$ be a metallic pseudo-Riemannian manifold and $G$ be the twin metallic pseudo-Riemannian metric. If $\nabla$ is a $J$–connection, then both $\nabla^*$ and $\nabla^\dagger$ are $J$–connections.

**Proof.** Assume that $\nabla$ is a $J$–connection. Then, we have

\[
g(Y, \nabla_X JZ - J\nabla_X Z) = g(Y, \nabla_X JZ - J\nabla_X Z)
= X g(Y, JZ) - g(\nabla_X Y, JZ) - g(Y, J\nabla_X Z)
= X (g(Y, JZ)) - g(\nabla_X Y, JZ) - X (g(JY, Z)) + g(\nabla_X JY, Z)
= 0
\]

such that $\nabla_X JZ = J\nabla_X Z$. Similarly, using the twin metallic pseudo-Riemannian metric $G$, it can be proven that $\nabla^\dagger$ is a $J$–connection.

\[\square\]

**Proposition 12.** Let $(M, J, g)$ be a metallic pseudo-Riemannian manifold, $G$ be the twin metallic pseudo-Riemannian metric, and $\nabla^*$ and $\nabla^\dagger$ respectively denote the $g$–conjugate and $G$–conjugate connection of a linear connection $\nabla$. If $\nabla$ is a $J$–connection, then

\[
C(X, JY, Z) = -C^*(X, JY, Z) = -C^\dagger(X, JY, Z)
= \Gamma(X, Y, Z) = -\Gamma^\dagger(X, Y, Z) = -\Gamma^*(X, Y, Z),
\]

where $C^\dagger = \nabla^\dagger g$ and $\Gamma^* = \nabla^* G$.

**Proof.** Suppose that $\nabla$ is a $J$–connection. From (3.3), (3.4) and (3.6), we have

\[
C(X, JY, Z) = -C^*(X, JY, Z)
\]

and

\[
C^\dagger(X, JY, Z) = \Gamma(X, Y, Z).
\]

Moreover, since

\[
\Gamma^*(X, Y, Z) \equiv (\nabla^*_Z G)(X, Y) = ZG(X, Y) - G(\nabla^*_Z X, Y) - G(X, \nabla^*_Z Y)
= Z g(JX, Y) - g(\nabla^*_Z X, JY) - g(JX, \nabla^*_Z Y)
= g(J\nabla_Z Y, X) - g(X, J\nabla_Z Y)
= -C(X, JY, Z)
\]

and

\[
C^\dagger(X, JY, Z) \equiv (\nabla^\dagger_Z g)(X, JY) = Zg(X, JY) - g(\nabla^\dagger_Z X, JY) - g(JX, \nabla^\dagger_Z Y)
= ZG(X, Y) - G(\nabla^\dagger_Z X, Y) - G(X, \nabla^\dagger_Z Y)
= \Gamma^\dagger(X, Y, Z),
\]

we obtain (3.6) which ends the proof.

\[\square\]

As a direct result of Proposition 12, we have the following proposition.

**Proposition 13.** Let $(M, J, g)$ be a metallic pseudo-Riemannian manifold, $G$ be the twin metallic pseudo-Riemannian metric, and $\nabla^*$ and $\nabla^\dagger$ respectively denote the $g$–conjugate and $G$–conjugate connection of a linear connection $\nabla$. Under the assumption that $\nabla$ is a $J$–connection, If at least one of the pairs $(\nabla, g)$, $(\nabla^*, g)$,
(∇\textsuperscript{\dag}, g), (∇, G), (∇\textsuperscript{*}, G) and (∇\textsuperscript{\dag}, G) is Codazzi-coupled, then all the remaining pairs are Codazzi-coupled.

**Proposition 14.** Let (M, J, g) be a metallic pseudo-Riemannian manifold, G be the twin metallic pseudo-Riemannian metric, and ∇\textsuperscript{*} and ∇\textsuperscript{\dag} respectively denote the g–conjugate and G–conjugate connection of a linear connection ∇. If ∇ is a J–connection, then

\[
\begin{align*}
C (JX, Y, Z) &= C (X, JY, Z), \\
\Gamma (JX, Y, Z) &= \Gamma (X, JY, Z), \\
C^* (JX, Y, Z) &= C^* (X, JY, Z), \\
\Gamma^* (JX, Y, Z) &= \Gamma^* (X, JY, Z), \\
C^\dag (JX, Y, Z) &= C^\dag (X, JY, Z), \\
\Gamma^\dag (JX, Y, Z) &= \Gamma^\dag (X, JY, Z).
\end{align*}
\]

**Proof.** If ∇ is a J–connection, then

\[
\begin{align*}
C (JX, Y, Z) &= Zg (JX, Y) - g (\nabla_J Z JX, Y) - g (JX, \nabla_J Z Y) \\
&= Zg (X, JY) - g (\nabla_J Z X, JY) - g (X, \nabla_J Z JY) \\
&= C (X, JY, Z)
\end{align*}
\]

and

\[
\begin{align*}
\Gamma (JX, Y, Z) &= ZG (JX, Y) - G (\nabla_J Z JX, Y) - G (JX, \nabla_J Z Y) \\
&= ZG (X, JY) - G (\nabla_J Z X, JY) - G (X, \nabla_J Z JY) \\
&= \Gamma (X, JY, Z).
\end{align*}
\]

With help of Proposition 11, the other equalities can be obtained in a similar way. □

Next, let us introduce a tensor operator which is applied to pure tensor fields. We refer to [20] for details about pure tensor fields and tensor operators. One of the most important classes of tensor operators is the class of Tachibana operators associated with a considering fixed (1, 1)–tensor field. The Tachibana operators were firstly defined and studied by Tachibana in [24]. Lather, the Tachibana operators was applied in Norden geometry and in theory of lifts in [19, 20, 21, 22, 10]. In here, we aims to give a new characterization of locally metallic pseudo-Riemannian manifolds by means of the Tachibana operator.

**Proposition 15.** Let (M, J, g) be a metallic pseudo-Riemannian manifold and ∇ be a torsion-free connection. Then

\[(\Phi_J g) (X, Y, Z) = C (Y, Z, JX) - \Gamma (Y, Z, X) + g ((\nabla_J Y) X, Z) + g ((\nabla_J Z) X, Y),\]

where \(\Phi_J g : T^0_2 (M) \rightarrow T^0_3 (M)\) is the Tachibana operator applied to the pseudo-Riemannian metric g.

**Proof.** Using the definition of the operator \(\Phi_J g\), we have

\[(\Phi_J g) (X, Y, Z) = JXg (Y, Z) - X (g (JY, Z)) + g ((L_J Y) X, Z) + g (Y, (L_J Z) X),\]

where \((L_J X) Y = [X, JY] - J [X, Y]\). Since ∇ is a torsion-free linear connection, we can rewrite it as

\[
\begin{align*}
(\Phi_J g) (X, Y, Z) &= JXg (Y, Z) - X (g (JY, Z)) \\
&+ g ((\nabla_J Y) X, Z) - g (\nabla_J X Y, Z) + g (J\nabla_X Y, Z) \\
&+ g (Y, (\nabla_J Z) X) - g (Y, \nabla_J X Z) + g (Y, J\nabla_X Z) \\
&= C (Y, Z, JX) - \Gamma (Y, Z, X) + g ((\nabla_J Y) X, Z) + g ((\nabla_J Z) X, Y),
\end{align*}
\]

where \([X, Y] = \nabla_X Y - \nabla_Y X\). □
A statistical manifold is a (pseudo-) Riemannian manifold \((M, g, \nabla)\) with a (pseudo-) Riemannian metric \(g\) and a torsion-free linear connection \(\nabla\) for which \(C = \nabla g\) is totally symmetric, that is, the Codazzi equation holds [12]

\[(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z) = (\nabla_Z g)(X, Y).\]

The following theorem gives a new characterization of locally metallic pseudo-Riemannian manifolds by means of the Tachibana operator and Codazzi-coupled.

**Theorem 1.** Let \((M, J, g)\) be a metallic pseudo-Riemannian manifold and \(\nabla\) be a torsion-free connection. If \(\nabla\) is a \(J\)-connection and \((\nabla, g)\) is Codazzi-coupled, then \(\Phi_J g = 0\). Furthermore, the triple \((M, J, g)\) is a locally metallic pseudo-Riemannian manifold.

**Proof.** From Proposition [15] and \(\nabla J = 0\), we have

\[(\Phi_J g)(X, Y, Z) = C(Y, Z, JX) - C(Y, JZ, X).\]

Since \((\nabla, g)\) is Codazzi-coupled, from Proposition [14] we get the following

\[(\Phi_J g)(X, Y, Z) = C(JX, Z, Y) - C(X, JZ, Y) = 0.\]

Because of Proposition [1] we can say that the triple \((M, J, g)\) is a locally metallic pseudo-Riemannian manifold. \(\square\)

In [23], for a non-integrable almost paracomplex manifold \((M, P)\) (that is, \(P^2 = I\), where \(I = \text{identity}\)) with compatible metric \(g\), authors give the following

\[(\Phi_P g)(X, Y, Z) + (\Phi_P g)(Y, Z, X) + (\Phi_P g)(Z, X, Y) = g((\nabla_X P) Y, Z) + g((\nabla_Y P) Z, X) + g((\nabla_Z P) X, Y),\]

where \(\nabla g\) is the Levi-Civita connection of \(g\). If \((\Phi_P g)(X, Y, Z) + (\Phi_P g)(Y, Z, X) + (\Phi_P g)(Z, X, Y) = g((\nabla_X P) Y, Z) + g((\nabla_Y P) Z, X) + g((\nabla_Z P) X, Y) = 0\), they called the triple \((M, P, g)\) a quasi (para-)Kähler manifold which is analogue of quasi Kähler manifold in [13].

Analogously, if a non-integrable metallic pseudo Riemannian manifold \((M, J, g)\) satisfies

\[(\Phi_J g)(X, Y, Z) + (\Phi_J g)(Y, Z, X) + (\Phi_J g)(Z, X, Y) = 0,\]

then we are calling the triple \((M, J, g)\) a quasi metallic pseudo-Riemannian manifold.

**Proposition 16.** Let \((M, J, g)\) be a non-integrable metallic pseudo-Riemannian manifold and \(\nabla\) be a torsion-free connection. If \((\nabla, g)\) is Codazzi-coupled, then

\[(\Phi_J g)(X, Y, Z) + (\Phi_J g)(Y, Z, X) + (\Phi_J g)(Z, X, Y) = g((\nabla_X J) Y, Z) + g((\nabla_Y J) Z, X) + g((\nabla_Z J) X, Y).\]

**Proof.** From Proposition [15], we know that

\[(\Phi_J g)(X, Y, Z) = C(Y, Z, JX) - \Gamma(Y, Z, X) + g((\nabla_X J) Y, Z) + g((\nabla_Y J) X, Z) + g((\nabla_Z J) X, Y).\]
If \((\nabla, g)\) is Codazzi-coupled, then
\[
(\Phi_{Jg}) (X, Y, Z) + (\Phi_{Jg}) (Y, Z, X) + (\Phi_{Jg}) (Z, X, Y)
\]
\[
= C(JX, Z, Y) + C(JY, X, Z) + C(JZ, Y, X)
\]
from which we can write the following Codazzi-coupled, the triple \((\Phi_{Jg}) (X, Y, Z) + (\Phi_{Jg}) (Y, Z, X) + (\Phi_{Jg}) (Z, X, Y)\) for generalized connections.

\[
\begin{align*}
\Gamma (Y, Z, X) + \Gamma (Z, X, Y) + \Gamma (X, Y, Z)
+ g ((\nabla_{JX} J) X, Z) + g ((\nabla_{JY} J) Y, Z) + g ((\nabla_{JZ} J) Z, X)
+ g ((\nabla_{JX} Z, Y) + g ((\nabla_{JY} Z, X) + g ((\nabla_{JZ} X, Y).
\end{align*}
\]

By means of the above proposition, we get the following result.

**Corollary 1.** Let \((M, J, g)\) be a non-integrable metallic pseudo-Riemannian manifold and \(\nabla\) be a torsion-free connection. Under the assumption that \((\nabla, g)\) is Codazzi-coupled, the triple \((M, J, g)\) is a quasi metallic pseudo Riemannian manifold if and only if \(g ((\nabla_{JX} J) Y, Z) + g ((\nabla_{JY} J) Z, X) + g ((\nabla_{JZ} J) X, Y) = 0\).

**4. Generalized conjugate connections and their applications**

Generalizations of conjugate connections are studied in this section. Similar problems studied for conjugate connections in the previous section will be searched for generalized connections.

**Definition 5.** Let \((M, J, g)\) be a metallic pseudo-Riemannian manifold and \(\nabla\) be a linear connection on \(M\). The generalized conjugate connection \(\nabla^\tau\) of \(\nabla\) with respect to \(g\) by a 1–form \(\tau\) is defined by [14, 16, 5]

\[
Xg (Y, Z) = g (\nabla_{X} Y, Z) + g (Y, \nabla^\tau_{X} Z) - \tau (X) g (Y, Z).
\]

It can easily be checked that the generalized conjugate connection \(\nabla^\tau\) of \(\nabla\) with respect to \(g\) is involutive, that is, \((\nabla^\tau)^\tau = \nabla\).

**Proposition 17.** Let \((M, J, g)\) be a metallic pseudo-Riemannian manifold and \(\nabla\) be a linear connection on \(M\). If \(\nabla^\tau\) is a generalized conjugate connection of \(\nabla\) with respect to \(g\) by \(\tau\), then we have

\[
\nabla \text{ is a } J - \text{connection if and only if } \nabla^\tau \text{ is a } J - \text{connection.}
\]

**Proof.** Suppose that \(\nabla\) is a \(J\)-connection. Using the definition of the generalized conjugate connection, it is possible to write the following

\[
g (Y, \nabla^\tau_{X} JZ) = Xg (Y, JZ) - g (\nabla_{X} Y, JZ) + \tau (X) g (Y, JZ)
\]
\[
= g (\nabla_{X} JY, Z) + g (JY, \nabla^\tau_{X} Z) - g (\nabla_{X} JY, JZ)
\]
\[
= g (Y, J\nabla^\tau_{X} Z)
\]
such that \(\nabla^\tau_{X} JZ = J\nabla^\tau_{X} Z\). Conversely, let \(\nabla^\tau\) be a \(J\)-connection. Then

\[
g (Y, \nabla_{X} JZ) = Xg (Y, JZ) - g (JZ, \nabla^\tau_{X} Y) + \tau (X) g (Y, JZ)
\]
\[
= g (\nabla^\tau_{X} JY, Z) + g (JY, \nabla_{X} Z) - g (JZ, \nabla^\tau_{X} Y)
\]
\[
= g (Y, J\nabla_{X} Z)
\]

which implies \(\nabla_{X} JZ = J\nabla_{X} Z\). \(\square\)
Proposition 18. Let \((M, J, g)\) be a metallic pseudo-Riemannian manifold and \(\nabla\) be a linear connection, and \(\nabla^{*}\) and \(\nabla'^{\tau}\) denote generalized conjugate connections of \(\nabla\) with respect to \(g\) and \(G\) by \(\tau\), respectively. Then if \(\nabla\) is a \(J\)-connection, then \(\nabla^{*} = \nabla'^{\tau}\).

Proof. The definition of the generalized conjugate connection of \(\nabla\) with respect to \(G\) by \(\tau\) immediately gives
\[
XG(Y, Z) = G(\nabla_X Y, Z) + G\left(Y, \nabla'^{\tau}_X Z\right) - \tau(X) G(Y, Z)
\]
\[
= g\left(J \nabla_X Y, Z\right) + g\left(JY, \nabla'^{\tau}_X Z\right) - \tau(X) g(JY, Z).
\]

Since
\[
X g(JY, Z) = g(\nabla_X JY, Z) + g(JY, \nabla'^{\tau}_X Z) - \tau(X) g(JY, Z),
\]
we find
\[
g(\nabla_X JY - J \nabla_X Y, Z) + g\left(JY, \nabla'^{\tau}_X Z - \nabla'^{\tau}_X Z\right) = 0
\]
from which we immediately obtain that if \(\nabla J = 0\), then \(\nabla^{*} = \nabla'^{\tau}\). \(\square\)

Proposition 19. Let \((M, J, g)\) be a metallic pseudo-Riemannian manifold and \(\nabla^{*}\) be a generalized conjugate connection of \(\nabla\) with respect to \(g\) by \(\tau\). Then \(\nabla^{*}\) is a generalized conjugate connection of \(\nabla^J\) with respect to \(G\) by \(\tau\).

Proof. We have
\[
G\left(\nabla'^{\tau}_X Y, Z\right) = G\left(J^{-1} \nabla_X JY, Z\right)
\]
\[
= g(\nabla_X JY, Z)
\]
\[
= X g(JY, Z) - g(JY, \nabla'^{\tau}_X Z) + \tau(X) g(JY, Z)
\]
\[
= XG(Y, Z) - G(Y, \nabla'^{\tau}_X Z) + \tau(X) G(Y, Z).
\]
\(\square\)

Now, we consider another generalized connection which is called dual-projectively equivalent connection.

Definition 6. Let \((M, J, g)\) be a metallic pseudo-Riemannian manifold and \(\nabla\) and \(\nabla'\) be linear connections on \(M\). We say that \(\nabla\) and \(\nabla'\) are dual-projectively equivalent if there exists a 1-form \(\tau\) such that
\[
\nabla'^{\tau}_X Y = \nabla_X Y - g(X, Y) \tau^\sharp,
\]
where \(g(X, \tau^\sharp) = \tau(X)\).

The following proposition ends this section.

Proposition 20. If \((\nabla, J)\) is Codazzi-coupled, then each of the pairs \((\nabla', J)\), \((\nabla', J^{-1})\), \(((\nabla^J)'\), J\) and \(((\nabla^J)'\), J^{-1})\) is Codazzi-coupled.
Proposition 21. Let \((\nabla, J)\) be Codazzi-coupled. Then we have
\[
(\nabla_X^J) Y - (\nabla_Y^J) X = \nabla_X^J Y - J\nabla_X^J Y - \nabla_Y^J X + J\nabla_Y^J X
\]
\[
= \nabla_X Y - g(X,Y) + g(Y, JX) \tau^z
\]
\[
- J(\nabla_X Y - g(X,Y) + g(Y, JX) \tau^z)
\]
\[
= (\nabla_X J) Y - (\nabla_Y J) X
\]
\[
= 0.
\]
Similarly, it can be easily shown that the pairs \((\nabla', J^{-1})\), \(((\nabla^J)' J)\) and \(((\nabla^J)' J^{-1})\) are Codazzi-coupled too.

5. Metallic-like pseudo-Riemannian manifolds

In this section we shall present some results concerning with a metallic-like pseudo-Riemannian manifold.

Definition 7. Let \((M, g)\) be a pseudo-Riemannian manifold with a metallic structure \(J\) which has another \((1, 1)\)–tensor field \(J^*\) satisfying
\[
g(JX, Y) = g(X, J^*Y),
\]
where \((J^*)^* = J\). Then \((M, J, g)\) is called a metallic-like pseudo-Riemannian manifold.

Standard calculations immediately give the following proposition.

Proposition 22. Let \((M, J, g)\) be a metallic-like pseudo-Riemannian manifold. Then, the following properties hold
1) \((J^*)^2 = pJ^* + qI\),
2) \((J^*)^{-1} = \frac{1}{q} J^* - \frac{q}{p} I\),
3) \(g(J^{-1}X, Y) = g(X, (J^*)^{-1}Y)\),
4) \(g(JX, J^*Y) = pg(X, J^*Y) + gg(X, Y)\),
5) \(g((\nabla_X J^*) Y, Z) = g(Y, (\nabla_X J) Z)\),
6) \(g((\nabla^*_X J) Y, Z) = g(Y, (\nabla_X J) Z)\).

Proof. 1) Assume that \(\nabla J^* = 0\). From (L1), we can write
\[
g(Y, \nabla^*_X JZ) = Xg(Y, JZ) - g(\nabla_X Y, JZ)
\]
\[
= Xg(J^*Y, Z) - g(\nabla_X J^*Y, Z)
\]
\[
= g(J^*Y, \nabla_X Z)
\]
\[
= g(Y, J^*\nabla_X Z)
\]
such that \(\nabla^* JZ = J\nabla_X Z\). Conversely, if \(\nabla^* J\) is \(J\)-connection, then
\[
g(Y, \nabla_X J^* Z) = Xg(J^*Z, Y) - g(J^*Z, \nabla_X Y)
\]
\[
= Xg(Y, J^*Z) - g(Z, \nabla^* JY)
\]
\[
= g(\nabla_X Z, JY)
\]
\[
= g(Y, J^*\nabla_X Z)\]
which implies $\nabla_X J^*Z = J^*\nabla_X Z$.

2) Similarly, it can be obtained with standard calculations. \hfill \Box

**Proposition 23.** Let $(M, J, g)$ be a metallic-like pseudo-Riemannian manifold.

1) If both $(\nabla, J^*)$ and $(\nabla^*, J)$ are Codazzi-coupled, then $\nabla J^* = \nabla^* J$.

2) If both $(\nabla^*, J^*)$ and $(\nabla, J)$ are Codazzi-coupled, then $\nabla^* J^* = \nabla J$.

**Proof.**

1) From (5) of Proposition 21, we have

\[
\begin{align*}
0 &= g((\nabla_X J^*) Y - (\nabla_Y J^*) X, Z) \\
&= g(Y, (\nabla^*_X J) Z) - g(X, (\nabla^*_Y J) Z) \\
&= g(Y, (\nabla^*_Z J) X - g(X, (\nabla^*_Z J) Y) \\
&= g(X, (\nabla^*_Z J^*) Y) - g(X, (\nabla^*_Z J) Y) \\
&= g(Y, (\nabla^*_Z J^*) Y - (\nabla^*_Z J) Y)
\end{align*}
\]

which implies $\nabla J^* = \nabla^* J$.

2) Similarly, it can be easily obtained. \hfill \Box

**Proposition 24.** On a metallic-like pseudo-Riemannian manifold $(M, J, g)$, the following two identities hold

\[
C(JX, Y, Z) = C(X, J^*Y, Z) + g(Y, (\nabla^*_Z J) X - (\nabla^*_Z J) X),
\]

where $C = \nabla g$, and

\[
C^*(JX, Y, Z) = C^*(X, J^*Y, Z) + g(X, (\nabla^*_Z J^*) Y - (\nabla^*_Z J^*) Y),
\]

where $C^* = \nabla^* g$.

**Proof.** From Definition 7, we get

\[
C(JX, Y, Z) = Zg(JX, Y) - g(\nabla_Z JX, Y) - g(JX, \nabla_Z Y) \\
= Zg(X, J^*Y) - g(\nabla_Z X, J^*Y) - g(X, \nabla_Z J^*Y) \\
+ g(X, (\nabla^*_Z J^*) Y) - g((\nabla^*_Z J) X, Y) \\
= C(X, J^*Y, Z) + g(Y, (\nabla^*_Z J) X - (\nabla^*_Z J) X)
\]

and similarly

\[
C^*(JX, Y, Z) = C^*(X, J^*Y, Z) + g(X, (\nabla^*_Z J^*) Y - (\nabla^*_Z J^*) Y).
\]

Also note that if $\nabla J = 0$ and $\nabla J^* = 0$, then we obtain

\[
C(JX, Y, Z) = C(X, J^*Y, Z) = -C^*(JX, Y, Z) = -C^*(X, J^*Y, Z).
\]

**Theorem 2.** On a metallic-like pseudo-Riemannian manifold $(M, J, g)$, if $\nabla J = 0$, $\nabla J^* = 0$ and $(\nabla, g)$ is Codazzi-coupled, where $\nabla$ is a torsion-free connection, then we get

\[
(\Phi_J g)(X, Y, Z) + (\Phi_J g)(X, Y, Z) = 0.
\]

**Proof.** Using the definition of the operator $\Phi_J g$, we have

\[
(\Phi_J g)(X, Y, Z) = J^* X g(Y, Z) - g(\nabla_J X, Y, Z) - g(Y, \nabla_J X, Z) \\
- (X g(J^*Y, Z) - g(J^* \nabla_X Y, Z) - g(Y, J^* \nabla_X Z)) \\
+ g((\nabla^*_Y J^*) X, Z) + g(Y, (\nabla^*_Z J^*) Y) \\
= C(Y, Z, J^* X) - C(Y, J^* Z, X) - g(Y, \nabla_X JZ) + g(Y, J^* \nabla_X Z).
\]
From the hypothesis and Proposition [23], we get
\[
(\Phi_J^* g)(X, Y, Z) = g(Y, (J^* - J) \nabla_X Z).
\]
Similarly, we can calculate
\[
(\Phi_J g)(X, Y, Z) = g(Y, (J - J^*) \nabla_X Z).
\]
When we add the last two equations, we find
\[
(\Phi_J^* g)(X, Y, Z) + (\Phi_J g)(X, Y, Z) = 0.
\]
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