NP-hardness of sortedness constraints

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Abstract

In Constraint Programming, global constraints allow to model and solve many combinatorial problems. Among these constraints, several sortedness constraints have been defined, for which propagation algorithms are available, but for which the tractability is not settled. We show that the $\text{sort}(U, V)$ constraint (Older et. al, 1995) is intractable (assuming $P \neq NP$) for integer variables whose domains are not limited to intervals. As a consequence, the similar result holds for the $\text{sort}(U, V, P)$ constraint (Zhou, 1996). Moreover, the intractability holds even under the stability condition present in the recently introduced $\text{keysorting}(U, V, Keys, P)$ constraint (Carlsson et al., 2014), and requiring that the order of the variables with the same value in the list $U$ be preserved in the list $V$. Therefore, $\text{keysorting}(U, V, Keys, P)$ is intractable as well.

Keywords: sortedness constraints; NP-hardness; graph matching

1 Introduction

Constraint programming systems support an increasing number of global constraints, i.e. constraints for which the number of variables is arbitrary. Such constraints define an important search space, that may be pruned using constraint propagation algorithms. Implementing a certain notion of consistency, a propagation algorithm removes infeasible values from the domains of the variables, and its efficiency is evaluated both with regard to its ability to limit the search space, and with regard to its running time. Dealing with global constraints in general, that is, without fixing a constraint (or a set of constraints), is intractable [1]. However, each constraint has its own complexity, which may range from tractability at all levels of consistency, as for the $\text{alldifferent}$ constraint [15], to intractability at relatively low levels of consistency, as for linear equations [11].

The tractability of a particular constraint is not always settled when the constraint is defined, and this is the case for the sortedness constraints $\text{sort}(U, V)$ [9], $\text{sort}(U, V, P)$ [16] and $\text{keysorting}(U, V, Keys, P)$ [2]. Although one or several of these constraints are implemented in well-known systems like SICStus Prolog [2], Gecode [12] and Choco [13], as well as in the constraint modelling language MiniZinc [8], their hardness is unknown.

In this paper we show that the intuitively simplest of these sortedness constraints, namely $\text{sort}(U, V)$, is intractable (unless $P = NP$) even in the case where the domains of the variables in $U$ are disjoint, and this leads to the intractability of $\text{sort}(U, V, P)$ and of $\text{keysorting}(U, V, Keys, P)$.

The organisation of the paper is as follows. In Section 2, we give the terminology and notations used in the paper. In Section 3 we transform, using ideas from [7], the search for a support of $\text{sort}(U, V)$ into a graph matching problem that we call SORTSUPPORT, and we show how to associate with each instance of the NP-complete problem NOT-ALL-EQUAL 3SAT an instance of SORTSUPPORT. The next section is devoted to the proof that our construction is a polynomial transformation [4], implying the NP-completeness of SORTSUPPORT. In Section 5 we deduce hardness results about the three sortedness problems. Section 6 is the conclusion.
2 Constraints and consistency

In this paper, we deal with constraints over integer domains. Given a variable \( w \), we denote \( \text{Dom}(w) \) its domain, which is assumed to be a finite set of integers. When \( \text{Dom}(w) \) is written as an interval \([l..r]\) (with integer \( l, r \) such that \( l \leq r \)) or a union of intervals, we understand that it contains only the integers in the (union of) interval(s), i.e. \([l..r]\) is defined as \([l..r] := \{ d \in \mathbb{Z} \mid l \leq d \leq r \} \).

A constraint \( C \) is a couple \((W, R)\), where \( W = \{w_1, w_2, \ldots, w_t\} \) is a set of variables with associated domains \( \text{Dom}(w_i) \), and \( R \) is a \( t \)-ary relation over \( \mathbb{Z} \) (equivalently, a subset of \( \mathbb{Z}^t \)). The constraint \( C = (W, R) \) is satisfied by a \( t \)-tuple \( \delta = (\delta_1, \delta_2, \ldots, \delta_t) \) assigning the value \( \delta_i \) to variable \( w_i \), \( 1 \leq i \leq t \), if \( \delta \in R \). Denote \( D := \text{Dom}(w_1) \times \text{Dom}(w_2) \times \cdots \times \text{Dom}(w_t) \). If \( C \) is satisfied by a \( t \)-tuple \( \delta \in D \), then \( \delta \) is a support of \( C \).

A constraint satisfaction problem (or CSP) is defined as a set of variables with their associated domains, and a set of constraints defined on subsets of the variable set. A solution of a CSP is an assignment of values from the associated domains to the variables that satisfies all the constraints. In order to solve a CSP, constraints are successively used to prune the search space, with the help of propagation algorithms that often seek to enforce various consistency properties, defined below (following \([3]\)). A domain \( D \) is said domain consistent for the constraint \( C = (W, R) \) if, for each variable \( w_i \), \( 1 \leq i \leq t \), and for each value \( \delta_i \in \text{Dom}(w_i) \), there is a support of \( C \) assigning the value \( \delta_i \) to \( w_i \). Domain consistency is a strong requirement, for which the following variants of bounds consistency are progressively weaker, but often very useful, alternatives.

Denote \( \text{inf}_D(w_i) \) and \( \text{sup}_D(w_i) \) respectively the minimum and maximum value in \( \text{Dom}(w_i) \). We say that a domain \( D \) is bounds(\( \mathbb{Z} \)) consistent for \( C \) if for each variable \( w_i \) and for each value \( \delta_i \in \{ \text{inf}_D(w_i), \text{sup}_D(w_i) \} \) there exist integers \( \delta_j \) with \( \delta_j \in \text{Dom}(w_j) \), \( 1 \leq j \leq t \) and \( j \neq i \), such that \((\delta_1, \delta_2, \ldots, \delta_t)\) satisfies \( C \). We say that a domain \( D \) is bounds(\( \mathbb{R} \)) consistent for \( C \) if for each variable \( w_i \) and for each value \( \delta_i \in \{ \text{inf}_D(w_i), \text{sup}_D(w_i) \} \) there exist real numbers \( \delta_j \) with \( \text{inf}_D(w_j) \leq \delta_j \leq \text{inf}_D(w_j) \), \( 1 \leq j \leq t \) and \( j \neq i \), such that \((\delta_1, \delta_2, \ldots, \delta_t)\) satisfies \( C \). Finally, we say that a domain \( D \) is bounds(\( \mathbb{Z} \)) consistent for \( C \) if for each variable \( w_i \) and for each value \( \delta_i \in \{ \text{inf}_D(w_i), \text{sup}_D(w_i) \} \) there exist real numbers \( \delta_j \) with \( \text{inf}_D(w_j) \leq \delta_j \leq \text{inf}_D(w_j) \), \( 1 \leq j \leq t \) and \( j \neq i \), such that \((\delta_1, \delta_2, \ldots, \delta_t)\) satisfies \( C \).

We now define the sortedness constraints:

- the sort\((U, V)\) constraint, defined in \([9]\), has variable set \( U \cup V \), where \( U = \{u_1, u_2, \ldots, u_n\} \) and \( V = \{v_1, v_2, \ldots, v_n\} \), and is satisfied by a \( 2n \)-tuple of values assigned to the variables if and only if the variables in \( V \) are the sorted list of the variables in \( U \). The correspondence between the variables in \( U \) and those in \( V \) is therefore a permutation. Propagation algorithms achieving bounds(\( \mathbb{Z} \))-consistency have been proposed in \([5, 7]\).

- the sort\((U, V, P)\) constraint, defined in \([16]\), generalizes the sort\((U, V)\) constraint by adding a set \( P \) of \( n \) variables with domains included in \( \{1, 2, \ldots, n\} \) in order to bring the permutation into the variable set of the constraint. This constraint thus has variable set \( U \cup V \cup P \) and is satisfied by a \( 3n \)-tuple of values if and only if (a) the variables in \( V \) are the sorted list of the variables in \( U \), (b) the variables in \( P \) are all distinct, and (c) the permutation associating the variables from \( U \) and \( V \) is the one defined by the variables in \( P \). The propagation algorithms for the sort\((U, V)\) constraint are able to reduce the domains of the variables in \( U \cup V \) similarly, but achieve bound(\( \mathbb{Z} \))-consistency only on the \( V \)-domains, and not on the \( U \) and \( P \)-domains \([14]\).

- the key-sorting\((U, V, Keys, P)\) constraint (where \( Keys \) is a positive integer), defined in \([2]\), allows to add two features with respect to sort\((U, V, P)\) : (a) each variable is a \( h \)-tuple \((h \geq 1\) and integer, common to all variables), whose first \( Keys \) elements form the sorting key of the variable, using lexicographic ordering; and (b) the sorting has to be stable, i.e.
any pair of variables with the same key value must have the same order in \( U \) and in \( V \).
The domain \( \text{Dom}(z) \) of any variable \( z \) from \( U \cup V \) is thus a \( h \)-tuple of domains. When \( \text{Keys} = 1 \), the lexicographic order of the keys is the classical order between integers, and thus key-sorting(\( U, V, 1, P \)) is similar to sort(\( U, V, P \)), except that it requires the stability of the sorting.

Given two non-empty sets of integers \( D \) and \( E \), we write \( D \leq \text{lex} E \) whenever there exist values \( d \in D \) and \( e \in E \) such that \( d \leq e \). This is not (and is not intended to be) an order on sets, but allows to compare the domains of the variables with respect to the possibility to have a given order between the assigned values.

## 3 Links between sortedness, graph matching and 3SAT

Consider two sets of variables \( U = \{ u_i | 1 \leq i \leq n \} \) and \( V = \{ v_i | 1 \leq i \leq n \} \), with finite integer domains \( \text{Dom}(u_i) \) and \( \text{Dom}(v_j) \), for all \( i, 1 \leq i \leq n \). Following [7], we define the intersection graph \( \Gamma(U, V) \) of \( U \) and \( V \) as the bipartite graph with vertex set \( U \cup V \) and edge set \( \{ (u_i, v_j) | \text{Dom}(u_i) \cap \text{Dom}(v_j) \neq \emptyset \} \). A matching of \( \Gamma(U, V) \) is an injective function \( \sigma : V' \subseteq V \to U \) such that \( (\sigma(v), v) \) is an edge of \( \Gamma \) for each \( v \in V' \). We also use the notation \( M = \{ (\sigma(v), v), v \in V' \} \) to designate the same matching. A matching \( M \) saturates a vertex \( x \) if there exists an edge in \( M \) with endpoint \( x \). We say that \( M \) is a perfect matching if it saturates all the vertices in \( \Gamma(U, V) \).

Denote \( Q_i = \text{Dom}(\sigma(v_i)) \cap \text{Dom}(v_i) \), for each \( v_i \) for which \( \sigma(v_i) \) is defined. Then testing whether sort(\( U, V \)) has a support is equivalent to solving the following problem:

**SortSupport**

**Instance:** Two sets of variables \( U = \{ u_i | 1 \leq i \leq n \} \) and \( V = \{ v_i | 1 \leq i \leq n \} \), with finite integer domains \( \text{Dom}(u_i) \) and \( \text{Dom}(v_i) \), for all \( i, 1 \leq i \leq n \).

**Question:** Is there a perfect matching \( \sigma : V \to U \) of \( \Gamma(U, V) \) such that \( Q_i \neq \emptyset \) for all \( i \) with \( 1 \leq i \leq n \) and \( Q_1 \leq \text{lex} Q_2 \leq \text{lex} \ldots \leq \text{lex} Q_n \)?

We show that SortSupport is NP-complete, and this even in the case where the domains \( \text{Dom}(u_i) \), \( 1 \leq i \leq n \), are disjoint. To this end, given \( \Gamma(U, V) \), a matching fulfilling the required conditions is called a sort-matching. Note that the order \( v_1, v_2, \ldots, v_n \) of the elements in \( V \) is important, since it defines the sort-matching.

We adapt the graph construction in [10], and therefore use the same notations. The reduction is from the NP-complete problem Not-All-Equal 3SAT [4], the variant of 3SAT in which each clause is required to have at least one true and at least one false literal.

Let \( H = H_1 \land H_2 \land \ldots \land H_k \) be an instance of Not-All-Equal 3SAT, where each clause \( H_i \), \( i = 1, 2, \ldots, k \), contains three literals from the set \( x_1, \overline{x_1}, x_2, \overline{x_2}, \ldots, x_p, \overline{x_p} \). We assume that, for each \( j = 1, 2, \ldots, p \), the literal \( x_j \) occurs in the instance \( H \) as many times as \( \overline{x_j} \) (otherwise, add to \( H \) an appropriate number of clauses \( (x_j \lor \overline{x_j} \lor \overline{x_j}) \) or \( (x_j \lor \overline{x_j} \lor \overline{x_j}) \)). We note \( \text{occ}(j) \) the total number of occurrences of \( x_j \) in a clause, either as a positive occurrence (i.e. as \( x_j \)) or as a negative occurrence (i.e. as \( \overline{x_j} \)).

We wish to build an instance \( U, V \) of SortSupport such that \( \Gamma(U, V) \) consists of:

- a unit graph \( G^j_i \) (see Figure 1), for each positive or negative occurrence of a literal \( x_j \) in a clause \( H_i \). The vertices of \( G^j_i \) are the variables \( a^j_i, b^j_i, c^j_i \) from \( U \) and \( (a^j_i)', (b^j_i)', (c^j_i)' \) from \( V \) joined by the four edges \( (a^j_i, (b^j_i)'), (b^j_i, (c^j_i)'), (c^j_i, (b^j_i)'), (a^j_i, (c^j_i)') \) (called down-edges). Unit graphs are positive or negative depending on the occurrence of \( x_j \) they represent.
Figure 1: The unit graph $G^i_j$ for a positive or negative occurrence of $x_j$ in $H_i$. Note that variables $a^i_j, b^i_j, c^i_j$ belong to $U$ and variables $(a^i'_j), (b^i'_j), (c^i'_j)$ belong to $V$.

Figure 2: The consistency component $CC_j$ assuming that the literal $x_j$ occurs in clauses $H_s, H_r$ and literal $x_j$ occurs in clauses $H_i, H_g$. The horizontal edges are respectively the up-linking edges and the down-linking edges, according to their position.

- a consistency component $CC_j$ (see Figure 2) for each literal $x_j$, connecting all unit graphs associated with positive and negative occurrences of $x_j$. The unit graphs are arbitrarily ordered such that they correspond alternately to a positive and to a negative occurrence, the first unit graph being associated with a positive occurrence of $x_j$. Up-linking edges join the $a^i_j$ vertex of a negative unit graph to the $(c^i'_j)$ vertex of the positive unit graph following it, in a circular way. Down-linking edges join the $c^i_j$ vertex of a positive unit graph to the $(a^i'_j)$ vertex of the negative unit graph following it.

- a truth component $D_i$ (see Figure 3) for each clause $H_i$, connecting the three unit graphs associated with the literals in $H_i$. Note that the negative unit graphs are drawn with up-edges down, and vice versa. Four vertices $d_i, e_i$ (defined to be in $U$) and $d^r_i, e^r_i$ (defined to be in $V$) are added, the former ones joined to the $(c^i'_j)$ vertex of every negative unit graph and to the $(a^i'_j)$ vertex of every positive unit graph, whereas the latter ones are joined to the $c^i_j$ vertex of every negative unit graph and to the $a^i_j$ vertex of every positive unit graph. These edges are called lateral edges.

- a completion component $E$ providing, for each $i$ with $1 \leq i \leq k$, an edge between $e_i$ and each $e'_j$ such that $i \neq j$.

The truth components allow to locally give truth values to the literals, whereas the consistency components guarantee that the locally given truth values are globally correct, that is, each literal is either true or false but not both. The completion component ensures the existence of a perfect matching in the graph. Notice that, if a clause $H_i$ contains two (or more) occurrences
Figure 3: The truth component $D_i$ assuming that the associated clause $H_i$ is $(\neg x_i \lor x_i \lor x_j)$. The dashed lines outgoing from vertices belonging to unit graphs are the up- and down-linking edges inside the consistency component containing the vertex. The dashed lines outgoing from $e_i$ and $e'_i$ symbolize the set of edges in the completion component $E$. Note that the negative unit graphs are drawn with up-edges down, and vice versa.

of the same literal $x_j$, then notations $G_{ij}^1$ and $G_{ij}^2$ should be used to identify the unit graph associated with each occurrence. We do not enter into such details in our presentation, in order to keep it as simple as possible.

To define the required order on the elements in $V$, we assume that in each unit graph the elements $(a_j)^i$, $(b_j)^i$, $(c_j)^i$ are ranged in this order, and in each consistency component $CC_j$ unit graphs are ordered according to the alternate arbitrary order chosen to build $CC_j$ (recall that the first unit graph in $CC_j$ is positive). Then, the global order on $V$ is built by considering the $CC_1$, $CC_2$, $CC_3$, $CC_p$ consistency components in this order, and by adding vertices $d'_1, d'_2, \ldots, d'_k$, $e'_1, e'_2, \ldots, e'_k$ in this order at the end.

**Example.** Let $H = H_1 \land H_2$ with $H_1 : x_1 \lor x_2 \lor x_3$ and $H_2 : x_1 \lor x_2 \lor x_3$. Then the consistency component $CC_1$ contains the unit graphs $G_{ij}^1$ (positive) and $G_{ij}^2$ (negative), in this order; $CC_2$ contains $G_{ij}^1$ (positive) and $G_{ij}^2$ (negative), in this order; and $CC_3$ contains $G_{ij}^3$ (positive) and $G_{ij}^3$ (negative), in this order. There are two truth components $D_1$ and $D_2$ corresponding respectively to $H_1$ and $H_2$. By definition, $D_1$ includes the unit graphs $G_{ij}^1$, $G_{ij}^2$ and $G_{ij}^3$, whereas $D_2$ contains the unit graphs $G_{ij}^1$, $G_{ij}^2$ and $G_{ij}^3$. The completion component has only two edges, $(e_1, e'_1)$ and $(e_1, e'_2)$. The order of the variables in $V$ is $(a_j)^i$, $(b_j)^i$, $(c_j)^i$, $(a_j)^i$, $(b_j)^i$, $(c_j)^i$, $(a_j)^i$, $(b_j)^i$, $(c_j)^i$, $(a_j)^i$, $(b_j)^i$, $(c_j)^i$, $d'_1, d'_2, e'_1, e'_2$.

To finish our construction, we have to define the domains of the vertices in $U$ and in $V$ that exactly define the sought intersection graph $\Gamma(U, V)$. To simplify the notations, the domain of a vertex is denoted similarly to the vertex, but with an upper case instead of a lower case, (e.g. $Dom(a_j)$ is denoted $A_j^i$), except for the vertices $d_i, d'_i, e_i, e'_i$.

**Remark 1** Note that in the sequel we do not seek to minimize the sizes of the domains we define, as this is not important for the proof of NP-completeness. In particular, we avoid domains that are singletons, in order to allow a better illustration of the domains and their intersections in Figure 4.
The clause $H_i$ is assumed to contain occurrences of literals $x_i, x_1, x_j$. The notations $A_i^j$ and $(A_i^j)'$ refer respectively to the domains of the variables $a_i^j$ and $(a_i^j)'$, from the unit graphs $G_i^j$ and $G_i^j$ that precede and respectively follow $G_i^j$ in the circular order of $CC_j$. The main domains of the variables are drawn with plain lines. Their secondary domains are drawn with dashed lines for a positive $G_i^j$, and with dotted lines for a negative $G_i^j$ (and this convention is extended to $G_i^0$ and $G_i^0$). We assumed that we are not in the third case of the definition of $X_j^i$.

Each unit graph (see Figure 4) is defined by domains included into an interval of $\mathbb{Z}$, of $t = 24$ consecutive integers (that we call a block), so that each consistency component $CC_j$ is defined on $t * \text{occ}(j)$ consecutive integers, and all the consistency components are represented on the interval $[1..t * \sum_{j=1}^{p} \text{occ}(j)]$. Noticing that $\sum_{j=1}^{p} \text{occ}(j) = 3k$, where $k$ is the number of clauses, and defining $m = 3kt$, we define first the domains for the variables $d_i^j$, $1 \leq i \leq k$, used in the truth components, as follows:

$$\text{Dom}(d_i^j) = [m + 2i - 1..m + 2i]$$

This interval is devoted to creating the lateral edges in $D_i$. Note that the last integer integer by an interval in $\text{Dom}(d_i^j)$ is $m + 2k$, that we denote $q$. For each $i$, we define:

$$\text{Dom}(e_i^j) = [q + 2i - 1..q + 2i].$$

This interval is dedicated both to the lateral edges in $D_i$ and to the edges in $E$.

Now, each unit graph $G_i^j$ is defined by domains inside the block $[ht + 1..(h + 1)t]$ (with little exceptions), where $h$ is the number of unit graphs before $G_i^j$ in the global order, $i.e.$

Figure 4: Domains of the variables in $G_i^j$ and $D_i$, for fixed $i$ and $j$, and their overlaps with other domains.
\[ h = \sum_{q \subset j} \text{occ}(q) + r - 1, \]  
where \( r \) is the position of \( G^i_j \) in the alternate order of \( CC_j \). The domains of \( a^i_j, b^i_j, c^i_j, (a^i_j)' , (b^i_j)' , (c^i_j)' \) are respectively defined as (see Figure 1): 

\begin{align*}
A^i_j &= [ht + 3.\ldots ht + 6] \cup X^i_j, \\
B^i_j &= [ht + 9.\ldots ht + 12], \\
C^i_j &= [ht + 15.\ldots ht + 18] \cup T^i_j, \\
(A^i_j)' &= [ht + 7.\ldots ht + 10] \cup Y^i_j, \\
(B^i_j)' &= [ht + 5.\ldots ht + 8] \cup [ht + 13.\ldots ht + 16], \\
(C^i_j)' &= [ht + 11.\ldots ht + 14] \cup Z^i_j
\end{align*}

where

\[
X^i_j = \begin{cases} 
\text{Dom}(d^i_j) \cup \text{Dom}(e^i_j), & \text{if } G^i_j \text{ is a positive unit graph} \\
[(h + 1)t + 21..(h + 1)t + 22], & \text{if } G^i_j \text{ is a negative unit graph and, moreover,} \\
[(h + 1 - \text{occ}(j))t + 21..(h + 1 - \text{occ}(j))t + 22], & \text{if } G^i_j \text{ is not the last unit graph in } CC_j
\end{cases}
\]

\[
T^i_j = \begin{cases} 
\emptyset, & \text{if } G^i_j \text{ is a positive unit graph} \\
\text{Dom}(d^i_j) \cup \text{Dom}(e^i_j), & \text{if } G^i_j \text{ is a negative unit graph}
\end{cases}
\]

\[
Y^i_j = \begin{cases} 
[ht + 1..ht + 2], & \text{if } G^i_j \text{ is a positive unit graph} \\
[ht - 7..ht - 4], & \text{if } G^i_j \text{ is a negative unit graph}
\end{cases}
\]

\[
Z^i_j = \begin{cases} 
[ht + 21..ht + 22], & \text{if } G^i_j \text{ is a positive unit graph} \\
[ht + 23..ht + 24], & \text{if } G^i_j \text{ is a negative unit graph}
\end{cases}
\]

The part of each domain defined by \( X^i_j, T^i_j, Y^i_j \) or \( Z^i_j \) before is called the secondary domain of respectively \( a^i_j, b^i_j, c^i_j, (a^i_j)' , (b^i_j)' , (c^i_j)' \). The remaining part of the domain, i.e. the one explicitly stated in the definitions of \( A^i_j, B^i_j, C^i_j, (A^i_j)' , (B^i_j)' , (C^i_j)' \) above, is called the main domain of the corresponding variables. Intuitively, the main domains allow to build the unit graphs and the down-linking edges of the consistency components. For a unit graph \( G^i_j \), the main domains of its variables belong to its block \([ht + 1..(h + 1)t]\). The secondary domains are devoted to the connections with other unit graphs or with the other vertices of the truth components.

It remains to give the domains of \( d_i \) and \( e_i \) for each \( i \). Recalling that a unit graph \( G^i_j \) exists if and only if \( x_j \) has a positive or egative occurrence in \( H_i \), we define:

\[
\text{Dom}(d_i) = \cup_{G_j^i \text{ is positive}} Y_j^i \cup \cup_{G_j^i \text{ is negative}} Z_j^i \\
\text{Dom}(e_i) = \cup_{G_j^i \text{ is positive}} Y_j^i \cup \cup_{G_j^i \text{ is negative}} Z_j^i \cup \cup_{s \neq i} \text{Dom}(e^i_s)
\]

Then, \( \text{Dom}(d_i) \) contains intervals from the domains of the V-variables in \( D_i \), whereas \( \text{Dom}(e_i) \) contains in addition the domains of all \( e_s^i \) with \( s \neq i \).

The construction before, obviously polynomial, yields a bipartite graph with \( n = 11k \) vertices in each part of the bipartition. Among these vertices \( 3k \ast 6 \) (\( 3k \ast 3 \) in each part) are in some consistency component and \( 4k \) (\( 2k \) in each part) are in some truth component (but not in a consistency component). We show now that the intersection graph of the variables we defined is indeed the graph we wished to build.
Claim 1 The edges built using the domains defined above for the variables in $U$ and $V$ are exactly those of the unit graphs $G^i_j$, consistency components $CC_j$, truth components $D_i$, and completion component $E$, for all $i$ and $j$ such that $x_j$ has a positive or negative occurrence in $D_i$.

Proof. We consider each variable and the edges it belongs to. It is easy to notice that the main domain of a variable among $a^i_j, b^i_j, (a^i_j)', (b^i_j)'$ or $(c^i_j)'$ can have non-empty intersections only with the main domains of the same set of variables. The only exception to this rule is $c^i_j$ (see below and Figure 3). Moreover, the secondary domains of $a^i_j, c^i_j, (a^i_j)', (c^i_j)'$ are specially defined to guarantee intersections with domains of other variables (according to the desired edges in $CC_j$ and $D_i$), and thus have only empty intersections with the other domains in the block of $G^i_j$.

Variable $a^i_j$. The main domain of $a^i_j$ has non-empty intersection only with $(B^i_j)'$ resulting into the edge $(a^i_j', (b^i_j)')$. The secondary domain of $a^i_j$ contains the domains of $d^i_j$ and $e^i_j$, if $G^i_j$ is a positive unit graph, meaning that $A^i_j \cap Dom(d^i_j) \neq \emptyset$ and $A^i_j \cap Dom(e^i_j) \neq \emptyset$ and thus that the edges $(a^i_j, d^i_j), (a^i_j, e^i_j)$ exist. It is easy to see that no other edge with endpoint $a^i_j$ exists in this case. If $G^i_j$ is a negative unit graph, then we have two sub-cases. If it is not the last unit graph in $CC_j$, then $X^j_i = [(h+1)t+21..(h+1)t+22]$ and it has non-empty intersection only with the secondary domain $Z^j_i = [(h+1)t+21..(h+1)t+22]$ of the variable $(c^i_j)'$ in the positive unit graph $G^i_j$ immediately following it in $CC_j$. Then we have the up-linking edge between $G^i_j$ and $G^j_i$. If $G^i_j$ is the last unit graph in $CC_j$, then $X^j_i = [(h+1−occ(j))t+21..(h+1−occ(j))t+22]$ and $X^j_i$ is included in the block of the first positive unit graph of $CC_j$, denoted $G^1_i$. Then $X^j_i$ has non-empty intersection only with the interval $Z^j_i$ in the domain of the variable $(c^i_j)'$ of the first unit graph, and thus we have again the up-linking edge between the two unit graphs.

Variable $b^i_j$. Obviously, $B^i_j$ has non-empty intersections with the main domains of $(A^i_j)'$ and $(C^i_j)'$, defining the edges $(b^i_j, (a^i_j)')$ and $(b^i_j, (c^i_j)')$ of the unit graph. No other non-empty intersections exist with $B^i_j$.

Variable $c^i_j$. Again, we have an obvious intersection of $C^i_j$ with $(B^i_j)'$, implying the edge $(c^i_j, (b^i_j)')$. and no other intersection with main domains. However, the main domain $[ht+15..ht+18]$ of $c^i_j$ also has non-empty intersection with the secondary domain $Y^j_i = [(h+1)t−7..(h+1)t−4]$ of the variable $(a^i_j)'$ in the unit graph following the current one, if this latter graph is negative. We are therefore in the case where the two unit graphs are joined by the down-linking edge $(c^i_j, (a^i_j)')$. Analyzing now the intersections of the secondary domain of $c^i_j$, we have again two sub-cases. In the case $G^i_j$ is negative, then its intersections with $Dom(d^i_j)$ and $Dom(e^i_j)$ are non-empty and we have the desired lateral edges in the truth component for the clause $H$. In the case $G^i_j$ is positive, there is no secondary domain for $c^i_j$.

Variable $(a^i_j)'$. On the main domain, $(A^i_j)'$ has non-empty intersection with $B^i_j$, yielding the edge $(b^i_j, (a^i_j)')$, and also with $(B^i_j)'$, yielding no edge since both variables are in $V$. Consider now the secondary domain of $(a^i_j)'$. When $G^i_j$ is positive, $Y^j_i$ is included into $Dom(d^i_j)$ and $Dom(e^i_j)$ and we have the sought lateral edges in $D_i$. When $G^i_j$ is negative, we have $Y^j_i = [ht−7..ht−4]$ which is the same as $[(h−1)t+17..(h−1)t+20]$, and thus has non-empty intersection with the main domain of the vertex $c^i_j$ in the unit graph immediately preceding $G^i_j$ in $CC_j$. This confirms the down-linking edge already found for the variable $c^i_j$.

Variable $(b^i_j)'$. Non-empty intersections with domains of variables from $U$ are found only for variables $a^i_j$ and $c^i_j$, confirming the edges already found above.

Variable $(c^i_j)'$. The edge $(b^i_j, (c^i_j)')$ is confirmed by the non-empty intersection of $(C^i_j)'$ with $B^i_j$. No other intersection is found with the main domain of $(c^i_j)'$. For its second domain, we have the intersection with $Dom(d^i_j)$ and $Dom(e^i_j)$ yielding the corresponding lateral edges, in the case where $G^i_j$ is negative. In the contrary case, $Z^j_i = [ht+21..ht+22]$ and the only
possible intersection is with some $X_j^s$ defined for another unit graph $G_j^s$, which must be negative according to the definition of $X_j^s$. With $X_j^s = (h^r+1)t+21, (h^r+1)t+22]$, we must have $h^r = h-1$, meaning that $G_j^s$ is the negative unit graph immediately preceding $G_j^s$ in $CC_j$, and thus we have the up-linking edge $(a_j^s, (c_j^s)'')$. With $X_j^s = [(h^r+1-occ(j))t+21..(h^r+1-occ(j))t+22]$, we have that $X_j^s$ is the secondary domain of the $a_j^s$ vertex in the last unit graph $G_j^s$ in $CC_j$. We also have that $X_j^s$ is included in the block of the first unit graph $G_j^t$ in $CC_j$. Thus $X_j^s$ has non-empty intersection with $Z_j^s$ if and only if $Z_j^s$ also concerns $G_j^t$, that is if $(c_j^s)' = (c_j^t)'$. The resulting edge is then $(a_j^s, (c_j^s)''')$, which is the up-linking edge closing the circuit of unit graphs in $CC_j$.

**Variable $d_i$.** By definition, $Dom(d_i)$ is made of the three secondary domains of variables of type $(a_i^r)'$ and $(c_i^r)'$, according to the positive or negative occurrence of $x_j$ in $D_i$, for which the edges have also been confirmed above. We notice that all the other intersections with domains of variables from $V$ are empty.

**Variable $d_i'$.** By definition, $Dom(d_i')$ has non-empty intersection only with $X_j^s$ (when $G_j^s$ is positive) and with $T_j^s$ (when $G_j^s$ is negative) and this yields the expected lateral edges.

**Variable $e_i$.** Again by definition, $Dom(e_i)$ is, as $Dom(d_i')$, made of three secondary domains of variables of type $(a_i^r)'$ and $(c_i^r)'$, allowing to build the expected lateral edges, but also of $k-1$ domains of variables $e_i'$ allowing to build the edges from $E$ incident with $e_i$.

**Variable $e_i'$.** As all the edges and non-edges with vertices from $U$ have already been verified, there is nothing more to check here.

All the edges in the unit graphs, consistency components, truth components and completion component are correctly built, and no undesirable edge is added. The claim is proved.

### 4 The proposed construction is a polynomial transformation

In this section, we show that there is a truth assignment satisfying $H$ with at least one true and one false literal in each clause if and only if $\Gamma(U,V)$ has a sort-matching. To this end, we first prove that:

**Claim 2** Let $G_j^s$ be an arbitrary unit graph. Then no sort-matching $M$ can contain simultaneously the edges $(a_j^s, (b_j^s)')$ and $(b_j^s, (a_j^s)')$, nor simultaneously the edges $(c_j', (b_j^s)')$ and $(b_j^s, (c_j)')$.

**Proof.** We have that $A_j^s \cap (B_j^s)' = [ht + 5..ht + 6]$, whereas $B_j^s \cap (A_j^s)' = [ht + 9..ht + 10]$. Since $(a_j^s)'$ precedes $(b_j^s)'$ in the order on $V$, if a sort-matching contained both edges $(a_j^s, (b_j^s)')$ and $(b_j^s, (a_j^s)')$, then we should have $B_j^s \cap (A_j^s)' \leq lex A_j^s \cap (B_j^s)'$, and this is obviously false.

Similarly, $C_j^s \cap (B_j^s)' = [ht + 15..ht + 16]$, whereas $B_j^s \cap (C_j^s)' = [ht + 11..ht + 12]$, which does not satisfy $C_j^s \cap (B_j^s)' \leq lex B_j^s \cap (C_j^s)'$.

**Claim 3** Let $CC_j$ be an arbitrary consistency component. Then any sort-matching $M$ satisfies the following property:

a) either all the up-edges and all the down-linking edges in $CC_j$ belong to $M$;

b) or all the down-edges and all the up-linking edges in $CC_j$ belong to $M$.

**Proof.** Given a unit graph $G_j^s$ from $CC_j$, $M$ must contain exactly one edge among $(a_j^s, (b_j^s)')$ and $(c_j, (b_j^s)')$, so that $(b_j^s)'$ is saturated, and similarly exactly one edge among $(b_j^s, (a_j^s)')$ and $(b_j^s, (c_j)')$. By Claim 2 it results that $M$ contains either $(a_j^s, (b_j^s)')$ and $(b_j^s, (c_j)')$ (that is, the up-edges in $G_j^s$) or $(c_j, (b_j^s)')$ and $(b_j^s, (a_j^s)')$ (that is, the down-edges in $G_j^s$).
In the former case, \( e_j \) (if \( G_j^1 \) is positive) or \((a_j^1)'\) (if \( G_j^2 \) is negative) can only be saturated by the down-linking edge with one endpoint in \( G_j^i \) (see Figure 2), implying that the next (previous, respectively) unit graph \( G_j^3 \) in \( CC_j \), in a circular way, also has its up-edges in \( M \). The same deduction may be done for the unit graph \( G_j^3 \) following (respectively preceding) \( G_j^3 \), as follows. Since the up-linking edge is not used by \( M \) (the up-edges are already in \( M \)), the vertex of \( G_j^3 \) incident with this edge must be saturated locally, and this can only be done by an up-edge of \( G_j^3 \). Thus, the property of a unit graph to have its up-edges and its incident down-linking edge in \( M \) is propagated to all the consistency component \( CC_j \).

The reasoning is similar in the latter case.

Now we are ready to prove the main result. For a truth component \( D_i \) and a unit graph \( G_j^1 \) in it, we call \( d_i^1 \)-close the up-edges of \( G_j^1 \) if \( G_j^1 \) is positive, and the down-edges of \( G_j^1 \) if \( G_j^1 \) is negative. The other edges in \( G_j^3 \) are called \( d_i^3 \)-close. In other words, given that in the definition of \( D_i \) the negative unit graphs are drawn upside down (see Figure 3), the \( d_i^3 \)-close edges of \( G_j^1 \) are the pairs of up- or down-edges one of whose endpoints is joined to \( d_i \) (and similarly for \( d_i \)).

Claim 4 There is a truth assignment satisfying \( H \) with at least one true and one false literal in each clause if and only if \( \Gamma(U,V) \) admits a sort-matching.

Proof. To prove the Only if part, assume that \( x_1, x_2, \ldots, x_p \) have been assigned boolean values satisfying \( H \) as required. Build \( M \), initially empty, as follows. For each clause \( H_i \), assume it contains the literals \( x_j, x_i, x_f \), with either a positive or a negative occurrence each. Assume without loss of generality, that the (positive or negative) occurrence of \( x_j \) is true and that the (positive or negative) occurrence of \( x_i \) is false.

Then add to \( M \): the edges of \( G_j^1 \) close to \( d_i^1 \); the edges of \( G_j^1 \) close to \( d_i^3 \) or respectively to \( d_i \) depending whether the occurrence of \( x_f \) is true or respectively false; the lateral edge joining \( d_i^1 \) to \( G_j^1 \) and the lateral edge joining \( d_i^3 \) to \( G_j^1 \). Moreover, add to \( M \) the up- or down-linking edges that are needed to saturate three of the four remaining unsaturated vertices of \( G_j^1, G_j^3 \), and \( G_j^3 \). The remaining unsaturated vertex belonging to a unit graph of \( D_i \) is a vertex \( y \) of \( G_j^3 \). If the occurrence of \( x_f \) in \( H_i \) is false, then this vertex is either \( a_j^1 \) (when the occurrence is positive) or \( c_j^1 \) (when the occurrence is negative), and is always adjacent to \( c_j^1 \). If, on the contrary, the occurrence of \( x_f \) in \( H_i \) is true, then this vertex is either \((a_j^1)'\) (when the occurrence is positive) or \((c_j^1)'\) (when the occurrence is negative), and is always adjacent to \( e_j \). Then add \((y, e_j)\) to \( M \) if \((y, e_j)\) is an edge, and add \((e_j, y)\) to \( M \) in the contrary case, leaving thus \( e_j \) unsaturated when the occurrence of \( x_f \) is false in \( H_i \), and \( e_j \) unsaturated when the occurrence of \( x_f \) is true in \( H_i \). Equivalently, \( e_j \) (respectively \( e_j^i \)) remains unsaturated when \( H_i \) is oversupplied of (respectively true) literals.

Now, as each consistency component has the same number of positive and negative occurrences of its corresponding literal, it results that there are \( 3k/2 \) true literals and \( 3k/2 \) false literals in \( H \). Therefore, the number of clauses that are oversupplied of true literals is the same as the number of clauses that are oversupplied of positive literals, namely \( k/2 \) clauses in each case. Consequently, in the completion component \( E \) the unsaturated vertices induce a \( k/2 \)-regular bipartite graph. By Hall’s theorem [6], this graph has a perfect matching \( M' \), that we add to \( M \). The construction of the matching is now complete.

We first have to show that \( M \) is correctly built. The construction implies that in every unit graph either both up-edges or both down-edges are in \( M \). Inside any consistency component, all unit graphs are in the same case among these two cases. To see this, let \( G_j^1 \) and \( G_j^3 \) be two neighboring unit graphs and assume without loss of generality that \( G_j^1 \) is positive and \( G_j^3 \) is negative, and also that \( x_j \) is true, implying that \( \overline{x_j} \) is false (the other cases are similar). Then in \( D_i \) the up-edges of \( G_j^1 \) are put into \( M \) since they are \( d_i^1 \)-close, whereas in \( D_i \) the up-edges of \( G_j^3 \) are put into \( M \) since they are \( d_i^3 \)-close. Thus in \( CC_j \) either all the up-edges or all the
down-edges are in $M$, together with the down-, respectively up-, linking edges by the definition of $M$. Moreover, in each $D_i$ and in $E$ the matching is correctly built by definition. Thus $M$ is correctly built.

Obviously, $M$ is a perfect matching. Recall that we associate with it a bijective function $\sigma : V \rightarrow U$ such that $\sigma(u) = v$ if and only if $(u, v) \in M$. In order to show the inclusion property between sets $Q(v) := Dom(\sigma(v)) \cap Dom(v)$ required by a sort-matching, we show that for each pair $v, w \in V$ such that $v$ precedes $w$ in the order on $V$ we have $Q(v) \subseteq Q(w)$.

This deduction is based on the following seven affirmations:

**A1.** for each unit graph $G_j$, $Q((a_j^t)') \subseteq Q((b_j^t)') \subseteq Q((c_j^t)')$.

Indeed, as shown above, we have two cases. In the case where the up-edges of $G_j^t$ belong to $M$, we have that $\sigma((b_j^t)) = a_j^t$, $\sigma((c_j^t)) = b_j^t$. Moreover, $\sigma((a_j^t)) \in \{d_i, e_i\}$ if $G_j^t$ is positive, and $\sigma((a_j^t)) = c_j^t$ where $G_j^t$ precedes $G_j^i$ in $CC_j$, if $G_j^i$ is negative. Then, $Q((a_j^t)') = Y_j^t = [ht + 1..ht + 2]$ and respectively $Q((a_j^t)') = [ht + 1..ht + 2]$ (where $h$ defines the block of $G_j^i$).

**A2.** for each pair of consecutive unit graphs $G_j^t, G_j^r$ (in this order) in the same consistency component $CC_j$, $Q((c_j^r)') \subseteq Q((a_j^t)')$.

As before, consider first the case where the up-edges of $G_j^t$ belong to $M$ and deduce in the same way that $Q((c_j^r)') = [ht + 11..ht + 12]$ (where $h$ defines the block of $G_j^r$). Then, the up-edges of $G_j^r$ also belong to $M$, therefore $Q((a_j^r)') = Y_j^r = [ht + 1..ht + 12]$ (where $h$ defines the block of $G_j^r$). In this case the down-edges of $G_j^r$ belong to $M$ and deduce as before that $Q((a_j^r)') = [ht + 11..ht + 12]$ and respectively $Q((a_j^r)') = [ht + 11..ht + 12]$, proving the affirmation.

**A3.** for each pair of consecutive unit graphs $G_j^t, G_{j+1}^r$ (in this order) in two consecutive consistency components, $Q((c_j^r)') \subseteq Q((a_{j+1}^t)')$.

Notice that in this case $G_j^t$ is negative and $G_{j+1}^r$ is positive, therefore as before we have either $Q((c_j^r)') = [ht + 11..ht + 12]$ or $Q((c_j^r)') = [ht + 23..ht + 24]$, depending whether the up- or the down-edges of $G_j^t$ belong to $M$. Similarly, we also have $Q((a_{j+1}^r)') = Y_j^r = [ht + 1..ht + 12]$ (where $h$ defines the block of $G_j^r$). In all four possible cases, the affirmation holds.

**A4.** for the last unit graph $G_p^t$ in the last consistency component $CC_p$, we have $Q((c_p^t)') \subseteq Q((d_1^t))$.

With the same deductions as above and since $G_p^t$ is negative, we have that either $Q(c_p^t) = [ht + 11..ht + 12]$ or $Q(c_p^t) = [ht + 23..ht + 24]$, where $h$ defines the block of $G_p^t$, i.e. $ht + 24 = m = 3kt$ (with $t = 24$) since we have $3k$ literals in the $k$ clauses and thus $3k$ blocks, of which $G_p^t$ uses the last one. Now, $Q(d_1^t) = [m + 1..m + 2]$ since $Dom(d_1^t) = [m + 1..m + 2]$ and it is included in the domains of the three variables yielding vertices adjacent to $d_1^t$ in $D_1$, and the affirmation follows.

**A5.** for each $i$ with $1 \leq i < k$, we have $Q(d_i^t) \subseteq Q(d_{i+1}^t)$.
This is obvious by the definition of the domains and the observation that \( Q(d'_s) = \text{Dom}(d'_s) \) for all \( s \) with \( 1 \leq s \leq k \).

**A6.** we have \( Q(d'_k) \leq_{lex} Q(e'_1) \).

Again, \( Q(d'_k) = \text{Dom}(d'_k) = [m + 2k - 1, m + 2k] \) and similarly \( Q(e'_1) = \text{Dom}(e'_1) = [q + 1, q + 2] \) where \( q = m + 2k \). The affirmation is proved.

**A7.** for each \( i \) with \( 1 \leq i < k \), we have \( Q(e'_i) \leq_{lex} Q(e'_{i+1}) \).

This is obvious by the definition of the domains and the observation that \( Q(e'_s) = \text{Dom}(e'_s) \) for all \( s \) with \( 1 \leq s \leq k \).

Affirmations **A1-A7** allow to deduce that \( M \) is a sort-matching.

We consider now the If part of the theorem. Assume therefore that a sort-matching exists in \( \Gamma(U,V) \). By Claim 3 such a matching must contain, for each unit graph, either both its low-edges or both its up-edges. Given that the matching is perfect, we deduce that all vertices \( d_i \) and \( d'_i \) are saturated, and therefore that in each \( D_i \) there is a unit graph \( G^i_j \) whose \( d'_i \)-close edges are in \( M \) (the unit graph containing the vertex \( y \) such that \( (d_i, y) \in M \)) and a unit graph \( G^i_1 \) whose \( d_i \)-close edges are in \( M \) (the unit graph containing the vertex \( z \) such that \( (z, d'_i) \in M \)). Call \( G^i_2 \) the third unit graph in \( D_i \).

Define, locally to \( D_i \), the (positive or negative) occurrence of \( x_j \) in \( D_i \) to be true, that of \( x_l \) to be false, and that of \( x_f \) to be true if \( M \) contains the edges of \( G^i_j \) close to \( d'_i \), respectively false if \( M \) contains the edges of \( G^i_1 \) close to \( d_i \). Deduce a local truth assignment for each literal \( x_j, x_l \) and \( x_f \). Notice that:

**A8.** The local truth assignment of the literal \( x_r \), for each \( r \in \{ j, l, f \} \), is true if and only if the up-edges of \( G^i_r \) belong to \( M \).

Indeed, we have that \( x_r \) is true if and only if exactly one of the two following cases occurs: either \( x_r \) has a positive occurrence in \( D_i \), in which case this occurrence has been assigned the true value, so the \( d'_i \)-close edges of \( G^i_r \) belong to \( M \); or \( x_r \) has a negative occurrence in \( D_i \), in which case \( x_r \) has been assigned a true value if and only if its negative occurrence has been assigned a false value, and this happens only if \( M \) contains the \( d_i \)-close edges of \( G^i_r \). In the former case, the \( d'_i \)-close edges are the up-edges of \( G^i_r \). In the latter case, due to the upside-down position of negative unit graphs in \( D_i \), the \( d_i \)-close edges are also the up-edges of \( G^i_r \). And we are done.

Due to the affirmato **A8** and to Claim 3 we deduce that all the local truth assignments are coherent. Moreover, each clause \( H_i \) has at least one true literal and one false literal, namely the occurrences of \( x_j \) and of \( x_l \) (with the notations above).

## 5 Hardness of sortedness constraints

The construction in the preceding section allows us to deduce the NP-completeness of \textsc{SortSupport} in general, but also in a particular case of it, as follows.

**Claim 5** \textsc{SortSupport} is NP-complete.

**Proof.** Obviously \textsc{SortSupport} belongs to NP. To show it is NP-complete, apply Claim 4.

With a slight modification of the domains we defined for the variables, we also have that:

**Claim 6** \textsc{SortSupport} is NP-complete, even in the case where the domains \( \text{Dom}(u_i), 1 \leq i \leq n \), are pairwise disjoint.
When a port is NP-complete, even in the case where the variables in $Z$ that are used by several sets $Dom(u)$ with $u \in U$ (thus shifting to right all the other intervals so as to avoid unwished overlaps); (b) to cut them into a sufficiently large number of sub-intervals; and (c) to use a specific sub-interval for each $Dom(u)$, thus insuring the disjointness without modifying the relative positions on the real line of the intervals defining the sets $Q(v)$.

The shared intervals are as follows: $Y_j^i = [ht + 1..ht + 2]$ (shared by $Dom(d_i)$ and $Dom(e_i)$, when $G_j^i$ is positive), $Z_j^i = [ht + 23..ht + 24]$ (shared by $Dom(d_i)$ and $Dom(e_i)$, when $G_j^i$ is negative), $Dom(d_i') = [m + 2i - 1..m + 2i]$ (shared by $X_j^i$ when $G_j^i$ is positive, and by $T_j^i$ when $G_j^i$ is negative) and $Dom(e_i') = [q + 2i - 1..q + 2i]$ (shared by $X_j^i$ when $G_j^i$ is positive, by $T_j^i$ when $G_j^i$ is negative and by $Dom(e_s)$ with $s \neq i$). As an example, consider $Y_j^i$ in the case of a positive unit graph $G_j^i$. As $Y_j^i$ is shared by $Dom(d_i)$ and $Dom(e_i)$, it should be extended to an interval of length three (e.g. $[ht + 1..ht + 4]$ instead of $[ht + 1..ht + 2]$ as it is now), in which case $Dom(d_i)$ and $Dom(e_i)$ would be affected a sub-interval each (e.g. $[ht + 1..ht + 2]$ and $[ht + 3..ht + 4]$ respectively). Of course, in this case, the domain $A_j^i$ of $a_j^i$ should start at $ht + 5$ instead of $ht + 3$ (and similarly for the other domains), so as to avoid overlaps. This would result into an augmentation of the size $t$ of each block.

The proof of the correctness is very similar to the one above. The main difference is that some intersections between domains are slightly shifted.

**Theorem 1** Testing whether $\text{sort}(U, V)$, $\text{sort}(U, V, P)$ or $\text{keysorting}(U, V, Keys, P)$ has a support is NP-complete, even in the case where the variables in $U$ have pairwise disjoint domains.

**Proof.** For $\text{sort}(U, V)$, the affirmation follows immediately by the equivalence to $\text{SortSupport}$ noticed in Section 3 and by Claim 6. Furthermore, $\text{sort}(U, V)$ is the variant of $\text{sort}(U, V, P)$ where each variable in $P$ has the domain $\{1, 2, \ldots, n\}$. As testing whether $\text{sort}(U, V, P)$ has a support is obviously in NP, the previous remark allows to deduce the NP-completeness of the problem. Finally, $\text{sort}(U, V, P)$ and $\text{keysorting}(U, V, 1, P)$ are equivalent when the domains of the variables in $U$ are pairwise disjoint, since the stability of the sorting is trivially satisfied by any assignment of values to the variables.

Given a constraint $C$ defined as in Section 2, enforcing domain consistency requires to test whether for a given variable $w_i$ and a given value $\delta_i \in Dom(w_i)$, a support of $C$ exists assigning the value $\delta_i$ to $w_i$. We can easily deduce that:

**Corollary 1** Enforcing domain consistency for each of the constraints $\text{sort}(U, V)$, $\text{sort}(U, V, P)$ and $\text{keysorting}(U, V, Keys, P)$ is intractable, even in the case where the variables in $U$ have pairwise disjoint domains.

**Proof.** By contradiction and for each of the three constraints, assume a polynomial algorithm $\mathcal{A}$ exists for testing the existence of a support with a given value for a given variable. Recall that, by Claim 4, an instance $H$ of $\text{Not-All-Equal \ 3SAT}$ is satisfiable if and only if $\Gamma(U, V)$ admits a sort-matching, and this latter affirmation holds if and only if $\text{sort}(U, V)$ has a support. By applying $\mathcal{A}$ to all the four values in $B_j^i$ for an arbitrarily chosen variable $b_j^i$, we test in polynomial time whether $\text{sort}(U, V)$ has a support. Then we have a polynomial algorithm for solving $\text{Not-All-Equal \ 3SAT}$, a contradiction. The results for $\text{sort}(U, V, P)$ and $\text{keysorting}(U, V, Keys, P)$ easily follow.

Focusing now on enforcing bounds consistency, we need to test whether for a given variable $w_i$ and a given value $\delta_i \in \{\inf(w_i), \sup_{\text{Dom}}(w_i)\}$, a $t$-tuple $(\delta_1, \delta_2, \ldots, \delta_t)$ satisfying $C$ exists whose values $\delta_j$, $j \neq i$ are more or less constrained. More precisely, $\delta_j$ must belong
to $Dom(w_j)$, respectively to $[\inf_D(w_j), \sup_D(w_j)]$, and respectively to $[\inf_D(w_i), \sup_D(w_i)]$ to allow bounds($\mathbb{D}$), respectively bounds($\mathbb{Z}$) and respectively bounds($\mathbb{R}$) consistency. We are able to show that:

**Theorem 2** Enforcing bounds($\mathbb{D}$) consistency for each of the constraints $\text{sort}(U, V)$, $\text{sort}(U, V, P)$ and $\text{keysorting}(U, V, \text{Keys}, P)$ is NP-complete.

**Proof.** It is easy to notice that these problems are in NP. We show that the reduction from NOT-ALL-EQUAL 3SAT to SORTSUPPORT in Section 3 allows to deduce the result for $\text{sort}(U, V)$. Then the other results follow.

In the instance of SORTSUPPORT built in Section 3, the variable $d'_i$ has the domain $Dom(d'_i) = [m + 2i - 1, m + 2i]$, so that $\inf_D(d'_i) = m + 2i - 1$ and $\sup_D(d'_i) = m + 2i$. Let $y = m + 2i - 1$ and let us notice that $\Gamma(U, V)$ has a sort-matching if and only if $y \in Q(d'_i)$. This is an easy consequence of the observation that $\inf_D(d'_i)$ has non-empty intersection with another domain if and only if it is included in it, i.e. if and only if $y$ belongs to the intersection. Then by Claim 4, we deduce the NP-completeness of testing whether there is a support of $\text{sort}(U, V)$ assigning to $d'_i$ the value $y$.

Notice that the previous result is not proved for pairwise disjoint domains of variables in $U$. The reason is that in this variant the domain of $d'_i$ strictly overlaps the domains of other variables and the proof of Theorem 2 is no longer valid.

### 6 Conclusion

In this paper we have shown that the three sortedness constraints defined up to now are intractable, even in the particular case where the variables to sort have pairwise disjoint domains, and even if we do not seek domain consistency but only enforcing bounds($\mathbb{D}$) consistency. The tractability of the lower levels of bounds consistency, i.e. bounds($\mathbb{Z}$) and bounds($\mathbb{R}$) consistency, is shown for $\text{sort}(U, V)$ [5, 7], but is still open for $\text{sort}(U, V, P)$ and $\text{keysorting}(U, V, \text{Keys}, P)$.

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