Necessary and sufficient criterion of steering for two-qubit T states

Xiao-Gang Fan, Huan Yang, Fei Ming, Xue-Ke Song, Dong Wang, and Liu Ye

School of Physics and Material Science, Anhui University, Hefei 230601, China

Einstein-Podolsky-Rosen (EPR) steering is the ability that an observer persuades a distant observer to share entanglement by making local measurements. Determining a quantum state is steerable or unsteerable remains an open problem. Here, we derive a new steering inequality with infinite measurements corresponding to an arbitrary two-qubit T state, from consideration of EPR steering inequalities with $N$ projective measurement settings for each side. In fact, the steering inequality is also a sufficient criterion for guaranteeing that the T state is unsteerable. Hence, the steering inequality can be viewed as a necessary and sufficient criterion to distinguish whether the T state is steerable or unsteerable. In order to reveal the fact that the set composed of steerable states is the strict subset of the set made up of entangled states, we prove theoretically that all separable T states can not violate the steering inequality. Moreover, we put forward a method to estimate the maximum violation from concurrence for arbitrary two-qubit T states, which indicates that the T state is steerable if its concurrence exceeds 1/4.

I. INTRODUCTION

Schrödinger initially introduced the concept of steering in 1935 [1], which is a significant nonclassical phenomenon. And it formalizedwhat Einstein called “spooky action at a distance” [2]. Although, Einstein-Podolsky-Rosen (EPR) paradox [3-5] was explored for a long time, the investigations concerning steering has been received extensive attentions till recently [7]. As one of nonclassical correlations, steering sits between entanglement and Bell nonlocality [6,11], and the intrinsical asymmetry is a nontrivial characteristic of steering, which is different from entanglement and Bell nonlocality [6,7]. The preliminary works indicate that steering has a number of practical applications, such as one-sided device independent quantum key distribution [12,13], subchannel discrimination [14], various protocols in quantum information processing [15], and so on.

With the development of examinations about steering, a variety of sufficient criteria for detecting steering have been derived [16-28]. As long as one of these steering inequalities is violated, it can be used as a criterion for entanglement witness. All unsteerable states are Bell local, since a local hidden state (LHS) model [8,29] is a particular case of a local hidden variable model. Historically, Wiseman et al. demonstrated that Werner state with weak entanglement does not violate the LHS model [30]. Besides, Bowles et al. put forward a sufficient criterion for guaranteeing that a two-qubit state is unsteerable [31]. Nguyen et al. show that quantum steering can be viewed as an inclusion problem in convex geometry [29]. However, most of these results obtain sufficient criteria for steering or unsteering, and many criteria are only applicable to given numbers of measurement settings and outcomes.

In this paper, we put forward a new steering inequality with infinite measurements corresponding to an arbitrary two-qubit T state. The steering inequality is a necessary and sufficient criterion that is used to make sure whether an arbitrary two-qubit T state is steerable or unsteerable. To test the correctness of this steering inequality, we prove theoretically that all separable T states must follow it. In addition, we establish the function relation between the concurrence and maximum violation for some special T states, and put forward a method to estimate the maximum violation from concurrence for any two-qubit T states.

II. PRELIMINARIES

A. Conditional states and LHS models

Consider two distant observers Alice and Bob, who share a bipartite quantum state $\rho$ with reduced states $\rho_A$ and $\rho_B$. It is supposed that Alice can perform different measurements $M_r = r \cdot \sigma$ on her assigned system. Here $r$ serves as an arbitrary measurement setting (or measurement direction), and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is a matrix-vector consisting of three Pauli matrices. To be exact, each such measurement $M_r$ is described by the set of operators $\{M_r|a\}_a$ with the outcome $a$. For each of Alice’s measurement setting $r$ and outcome $a$, Bob retains a unnormalized conditional state $\sigma_{r|a} = \text{Tr}_A[M_r|a \otimes I_2] \rho$ with the probability $p(r|a) = \text{Tr}(\sigma_{r|a})$, where $I_2$ is the unit matrix of rank-2. And the conditional state obey the condition $\sum_{r|a} \sigma_{r|a} = \rho_B$.

However, Bob is sceptical that Alice can remotely steer his state. And he is unsure whether he has received half of an entangled pair or a pure state sent by Alice. In order to eliminate this doubt, Bob tests whether the conditional state $\sigma_{r|a}$ conforms to a LHS model. In other words, if the state $\sigma_{r|a}$ obey the LHS model, there exists a probability density distribution function $p(r|a, v)$, which makes that the state $\sigma_{r|a}$ can be expressed as $[8,29,31]

$$\sigma_{r|a} = \frac{1}{4\pi} \int_S p(r|a, v) \sigma_v dS,$$

where the distribution function $p(r|a, v)$ is parametrized by the unit Bloch vector $v = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$. The term $|a\rangle$ is the pure unit state with the $a$-th coordinate one and the rest coordinates zero. It is supposed that Alice can perform different measurements $M_r = r \cdot \sigma$ on her assigned system. Here $r$ serves as an arbitrary measurement setting (or measurement direction), and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is a matrix-vector consisting of three Pauli matrices. To be exact, each such measurement $M_r$ is described by the set of operators $\{M_r|a\}_a$ with the outcome $a$. For each of Alice’s measurement setting $r$ and outcome $a$, Bob retains a unnormalized conditional state $\sigma_{r|a} = \text{Tr}_A[M_r|a \otimes I_2] \rho$ with the probability $p(r|a) = \text{Tr}(\sigma_{r|a})$, where $I_2$ is the unit matrix of rank-2. And the conditional state obey the condition $\sum_{r|a} \sigma_{r|a} = \rho_B$.

However, Bob is sceptical that Alice can remotely steer his state. And he is unsure whether he has received half of an entangled pair or a pure state sent by Alice. In order to eliminate this doubt, Bob tests whether the conditional state $\sigma_{r|a}$ conforms to a LHS model. In other words, if the state $\sigma_{r|a}$ obey the LHS model, there exists a probability density distribution function $p(r|a, v)$, which makes that the state $\sigma_{r|a}$ can be expressed as $[8,29,31]

$$\sigma_{r|a} = \frac{1}{4\pi} \int_S p(r|a, v) \sigma_v dS,$$

where the distribution function $p(r|a, v)$ is parametrized by the unit Bloch vector $v = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$.
measurement setting \( r \) and outcome \( a \). Here \( dS = \sin \theta d\theta d\varphi \) represents the surface element, and the local hidden state \( \sigma_v \) denotes a normalized state that related to the Bloch vector \( v \). It is obvious that the probability \( p(r|a) \) can be represented by the integral of the probability distribution function \( p(r|a, v) \), i.e.,

\[
p(r|a) = \frac{1}{4\pi} \int_S p(r|a, v) dS.
\] (2)

If a representation as in Eq. (1) exists, Bob does not need to assume any kind of action at a distance to explain the post-measurement states \( \rho_{r|a} = \sigma_v |a\rangle \langle a| \). Consequently, Alice fails to convince Bob that she can steer his system by her measurements, and one also says that the state \( \rho \) is unsteerable from Alice to Bob. If such a model does not exist, Bob is required to believe that Alice can steer the state in his laboratory by some action at a distance. In this case, the state is said to be steerable from Alice to Bob.

B. Steering inequalities with \( N \) measurements

For qubits, Alice’s and Bob’s \( k \)th measurement settings correspond to the measurement \( M_{r_k} = r_k \cdot \sigma \) and \( M_{s_k} = s_k \cdot \sigma \), respectively. Here the measurement settings \( r_k, s_k \) are unit vectors with three-dimension. The steering inequalities [30] can be expressed as

\[
F_{N}(\rho) = \frac{1}{N} \sum_{k=1}^{N} A_k \langle M_{s_k} \rangle \leq C_N,
\]

where the random variable \( A_k \in \{-1, 1\} \) represents Alice’s corresponding declared result for all \( k \), and \( \langle M_{s_k} \rangle \) is the expected value of measurement \( M_{s_k} \) in the normalized conditional state \( \rho_{r_k|a_k} = \sigma_v |a_k\rangle \langle a_k| \). For the sake of description, we call the quantity \( F_{N}(\rho) \) as the steering parameter for \( N \) measurement settings. The bound \( C_N \) is the maximum value \( F_N \) can have if Bob has a pre-existing state known to Alice. And this bound can be denoted as

\[
C_N = \max_{\{A_k\}} \lambda_{\max}(O_N),
\]

where \( \lambda_{\max}(O_N) \) stands for the largest eigenvalue of operator

\[
O_N = \frac{1}{N} \sum_{k=1}^{N} A_k M_{s_k}.
\]

If a two-qubit state \( \rho \) violates the steering inequality in Eq. (3), then the state \( \rho \) must be steerable from Alice to Bob. However, if a two-qubit state \( \rho \) conforms the steering inequality, then the state \( \rho \) may be steerable or unsteerable from Alice to Bob. In fact, Saunders et al. [30] gave some bounds \( C_N \), such as \( C_2 = 1/\sqrt{2} , C_3 = C_4 = 1/\sqrt{3}, C_6 \approx 0.5394, \) \( C_{10} \approx 0.5236 \) and so on. And their result shows that it should be possible to demonstrate steering if \( \alpha > C_N \), for Werner states \( W_\alpha = \alpha |\varphi\rangle \langle \varphi| + \frac{1-\alpha^2}{4} I_4 \). Here the state \( |\varphi\rangle \) is one of Bell states, \( I_4 \) denotes the unit matrix with rank-4, and the parameter \( \alpha \) represents the probability of \( |\varphi_B\rangle \). It indicates that the steering inequality can detect more and more steerable states, when the number \( N \) of measurements increases.

III. RESULTS

A. Steering inequality with infinite measurements

In order to improve the above inequalities in Eq. (3), we consider a limiting case, i.e., \( N \rightarrow \infty \). Clearly, two key issues need to be addressed. The first question is how to acquire this bound \( C = \lim_{N \rightarrow \infty} C_N \). And the second question is how to get the maximum violation of the steering inequality in the case of \( N \rightarrow \infty \), i.e., how to obtain a limit maximum violation

\[
F(\rho) = \lim_{N \rightarrow \infty} \max_{\{r_k\}} F_{N}(\rho)
\]

of the steering inequality in Eq. (3). For short, we call \( F(\rho) \) as the maximum violation. Obviously, the maximum violation \( F(\rho) \) denotes the maximum of the steering parameter \( \lim_{N \rightarrow \infty} F_{N}(\rho) \) with infinite measurements. Just to keep the following derivation simple and easy to understand, we can write the same vector in two different ways: \( u = (u_1, u_2, u_3) \) and \( |u\rangle = (u_1, u_2, u_3)^T \), where the superscript symbol \( T \) represents transpose of a matrix.

On the one hand, we need to calculate this bound \( C = \lim_{N \rightarrow \infty} C_N \). And the bound \( C \) is based on infinite measurement settings. In order to derive the limit bound \( C \), we suppose that \( N \) unit vectors \( |s_k\rangle \), which start at the sphere centre and end at the sphere surface, are uniformly distributed in Bloch sphere. When the number \( N \) tends to infinity, these \( N \) vectors can divide the whole spherical surface into \( N \) surface elements \( dS \) (as shown in Fig. 1). For the sake of description, we replace \( |s_k\rangle \) with a variable unit vector \( |v\rangle = (\sin \theta \cos \varphi \sin \theta \sin \varphi \cos \theta)^T \). And then the surface element is denoted as \( dS = \sin \theta d\theta d\varphi \approx 4\pi/N \).

Based on the bound \( C_N \), we consider the value of random variable \( A_k \) in the following way to maximize the bound \( C_N \). The way is: when the angle between the measurement setting \( s_k \) and the z-axis is acute, we set the random variable \( A_k = 1; \)
when the angle between the measurement setting $s_k$ and the 
z-axis is obtuse, we set the random variable $A_k = -1$. Hence, 
if $N \to \infty$, the operator $O = \lim_{N \to \infty} O_N$ can be rewritten as 
$$
O = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} A_k M_{s_k}
$$
where $M_{s_k} = v \cdot \sigma$ and $v = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$. 
Here $S^+_1$ (or $S^{-}_1$) stands for the upper (or lower) 
half Bloch spherical surface. It is easy to see that the operator $O$ in Eq. 
(5) can be reduced to $O = \sigma_z/2$. Hence, the bound $C$ can be 
reduced to 
$$
C = \lambda_{\text{max}} (O) = \frac{1}{2}.
$$

On the other hand, we need to obtain the maximum violation 
$F(\rho) = \lim_{N \to \infty} \max_{r_k} F_N(\rho)$. However, for a general 
two-qubit state $\rho$, the maximum violation $F(\rho)$ is hard to be 
obtained. Therefore, in the following discussion, we only consider 
the steering criterion of two-qubit T states. In general, based on Pauli operators $\sigma = (\sigma_x, \sigma_y, \sigma_z)$, an arbitrary 
two-qubit T state $\rho$ can be expressed as 
$$
\rho = \frac{1}{4} (1 + \sum_{m,n} T_{mn} \sigma_m \otimes \sigma_n),
$$
where all information of the state $\rho$ is encoded into the 
elements $T_{mn} = \text{Tr} [\rho (\sigma_m \otimes \sigma_n)]$ of correlation matrix $T(\rho)$. 
Here, $m$ and $n \in \{x, y, z\}$. It is obvious that the expected 
value $\langle M_{s_k} \rangle$ can be denoted as 
$$
\langle M_{s_k} \rangle = \text{Tr} (\rho r_k | a_k M_{s_k} = \langle \hat{b}_k | s_k \rangle,
$$
where $\hat{b}_k = \text{Tr} (\rho r_k | a_k \sigma)$ is Bloch vector of the conditional 
state $\rho_{r_k}|a_k$. And the expected value $\langle M_{s_k} \rangle$ can be given by 
$$
\langle M_{s_k} \rangle = a_k \langle r_k | T(\rho) | s_k \rangle.
$$
In order to be simple and not affect the final result, we might 
as well take $A_k = a_k$, which means that Alice declare the 
outcomes of her own measurements. Hence, the steering 
parameter in Eq. (4) can be simplified as 
$$
F_N(\rho) = \frac{1}{N} \sum_{k=1}^{N} \langle r_k | T(\rho) | s_k \rangle.
$$

Obviously, when the unit vector $| r_k \rangle$ is collinear with the 
applied vector $T(\rho) | s_k \rangle$, the value $\langle r_k | T(\rho) | s_k \rangle$ in Eq. (10) 
can be maximized to be $X_k(\rho) = \max_{| r_k \rangle} \langle r_k | T(\rho) | s_k \rangle$. Just 
for the sake of description, we set that the collinear condition 
of two vectors $| r_k \rangle$ and $T(\rho) | s_k \rangle$ is written as $u_k(\rho) | r_k \rangle = 
T(\rho) | s_k \rangle$. Here, $u_k(\rho) > 0$ stands for the collinear coefficient. 
Based on the property $\langle r_k | r_k \rangle = 1$, the collinear coefficient 
$u_k(\rho)$ is denoted as $u_k(\rho) = \sqrt{\langle s_k | T^T(\rho) T(\rho) | s_k \rangle}$. Hence, the maximum expected value $X_k(\rho)$ of $k$th measurement 
outcome for an arbitrary T state $\rho$ can be indicated as 
$X_k(\rho) = u_k(\rho) = \sqrt{\langle s_k | T^T(\rho) T(\rho) | s_k \rangle}$. Based on the 
maximum expected value $X_k(\rho)$, the maximum of the steering 
parameter in Eq. (10) can be rewritten as 
$$
\max_{| r_k \rangle} F_N(\rho) = \frac{1}{N} \sum_{k=1}^{N} \sqrt{\langle s_k | T^T(\rho) T(\rho) | s_k \rangle}.
$$

In the same way, we replace $| s_k \rangle$ with a variable unit vector 
$| v \rangle = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)^T$. Consequently, the 
maximum violation $F(\rho) = \lim_{N \to \infty} \left[ \max_{| r_k \rangle} F_N(\rho) \right]$ can be reduced to 
$$
F(\rho) = \frac{1}{4\pi} \iint_{S} \sqrt{|v|^T T(\rho) T(\rho) |v|} dS,
$$
where $|v| = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ and $dS = \sin \theta d\theta d\varphi$. Combining Eqs. (6) and (12), we can obtain a 
steering inequality that can be described by the following Theorem 1.

Theorem. For the T state $\rho$, the steering inequality with infinite measurements can be expressed as 
$$
\frac{1}{2\pi} \iint_{S} \sqrt{|v|^T T(\rho) T(\rho) |v|} dS \leq 1.
$$

Theorem shows that the T state $\rho$ is steerable if $F(\rho) > 
1/2$. And it is based on infinite measurements corresponding to 
an arbitrary two-qubit T state. Therefore, it is the best optimization of steering inequalities in Eq. (3) with finite number 
of measurements. In other words, it can detect more steerable 
states than Eq. (3). Specially, Werner state $W_{2e}$ is steerable if 
the probability $\alpha$ conforms to the relation $\alpha > 1/2$.

**B. Properties of the maximum violation**

According to the calculation result of the maximum violation $F(\rho)$, we now provide two important properties that are 
scaling and symmetry.

Property 1. Given a two-qubit T state $\rho$, we consider a family 
of states $\rho_\gamma$ by mixing it with a special kind of separable noise, 
$$
\rho_\gamma = \gamma \rho + (1 - \gamma) \frac{1}{4},
$$
where $0 \leq \gamma \leq 1$. For these states $\rho_\gamma$, we can show that 
$$
F(\rho_\gamma) = \gamma F(\rho).
$$
Proof. In combination with the calculation formula \( T_{mn} = \text{Tr}[\rho (\sigma_m \otimes \sigma_n)] \), the relation between the correlation matrices \( T(\rho) \) and \( T(\rho_B) \) can be given by \( T(\rho_B) = \gamma T(\rho) \). Thus, the derivative process can be given by

\[
F(\rho) = \frac{1}{4\pi} \int_S \sqrt{\langle \psi | T(\rho) | \psi \rangle} dS
\]

\[
= \frac{\gamma}{4\pi} \int_S \sqrt{\langle \psi | T(\rho_B) | \psi \rangle} dS
\]

\[
= \gamma F(\rho).
\]

(16)

Property 2. Given a two-qubit T state \( \rho \), we consider a family of states \( \rho' \) which are formed by the unitary operation applied to the original state \( \rho \),

\[
\rho' = (U_A \otimes U_B) \rho (U_A \otimes U_B)^\dagger,
\]

where \( U_A \) and \( U_B \) are the unitary matrices on Alice’s and Bob’s side, respectively. For these states, we can show that

\[
F(\rho') = F(\rho).
\]

(18)

Proof. When a local unitary operation \( U_A \otimes U_B \) is performed on a T state \( \rho \), the final state \( \rho' \) is also a T state. And the relation between the correlation matrices \( T(\rho) \) and \( T(\rho') \) can be given by \( T(\rho') = R_A T(\rho) R_B^T \), where the elements of \( R_A \) and \( R_B \) can be denoted as \( (R_A)_{kk'} = \frac{1}{2} \text{Tr} \left( \sigma_k U_A \sigma_{k'} U_A^\dagger \right) \) and \( (R_B)_{kk'} = \frac{1}{2} \text{Tr} \left( \sigma_k U_B \sigma_{k'} U_B^\dagger \right) \), respectively. And \( R_A \) and \( R_B \) belong to the three-dimensional rotation group \( SO(3) \). We rewrite the maximum violation of the final state as

\[
F(\rho') = \frac{1}{4\pi} \int_S \sqrt{\langle \psi' | T(\rho') | \psi' \rangle} dS
\]

\[
= \frac{1}{4\pi} \int_S \sqrt{\langle \psi' | R_B T(\rho) R_B^T | \psi' \rangle} dS
\]

\[
= \frac{1}{4\pi} \int_S \sqrt{\langle \psi' | T(\rho) | \psi' \rangle} dS,
\]

where \( |\psi'\rangle = R_B^T |\psi\rangle \) denotes a new unit vector. To represent the surface element \( dS' \) corresponding to the new vector \( |\psi'\rangle \), we set \( |\psi'\rangle = (\sin \theta' \cos \varphi' \sin \theta' \sin \varphi' \cos \theta')^T \). Obviously \( dS' = \sin \theta' d\theta' d\varphi' \). Notice that the new surface element \( dS' \) is given by the original surface element \( dS \) by the rotation operation \( R_B^T \), which indicates \( dS' = dS \). Therefore, local unitary operation does not change the maximum violation. The derivative process can be described as follows

\[
F(\rho') = \frac{1}{4\pi} \int_S \sqrt{\langle \psi' | T(\rho) | \psi' \rangle} dS
\]

\[
= \frac{1}{4\pi} \int_S \sqrt{\langle \psi | T(\rho) | \psi \rangle} dS
\]

\[
= F(\rho).
\]

(20)

Here, the derivation takes advantage of the property that the integral is independent of the integral variable.

Inference. For an arbitrary two-qubit T state \( \rho \), the maximum violation \( F(\rho) \) is only related to three singular values \( \{t_1(\rho), t_2(\rho), t_3(\rho)\} \) of the correlation matrix \( T(\rho) \). And the formula can be expressed as follows

\[
F(\rho) = \frac{1}{4\pi} \int_S \sqrt{\langle \psi | \Lambda^2(\rho) | \psi \rangle} dS,
\]

where \( \Lambda(\rho) = \text{diag} \{t_1(\rho), t_2(\rho), t_3(\rho)\} \) represents a diagonal matrix consisting of these singular values.

Proof. For a general two-qubit T state \( \rho \), there is a local unitary operation that transforms the state \( \rho \) into a Bell diagonal state \( \rho_{\text{Bell}} \), whose correlation matrix \( T(\rho_{\text{Bell}}) \) satisfies the relation \( T^2(\rho_{\text{Bell}}) = \Lambda^2(\rho) \). Therefore, combining with Property 2, we obtain

\[
F(\rho) = F(\rho_{\text{Bell}}) = \frac{1}{4\pi} \int_S \sqrt{\langle \psi | \Lambda^2(\rho) | \psi \rangle} dS
\]

\[
= \frac{1}{4\pi} \int_S \sqrt{\langle \psi | T^2(\rho_{\text{Bell}}) | \psi \rangle} dS.
\]

(22)

C. Sufficient criterion for unsteerability

For each of Alice’s measurement setting \( r \) and outcome \( a \), Bob retains a conditional state \( \rho_{r|a} = \sigma_{r|a}/\rho(\rho_{r|a}) \). And the eigenvalues of \( \rho_{r|a} \) can be reduced to

\[
\lambda_1 = 1 + \sqrt{\langle \rho | T(\rho) T^T(\rho) | \rho \rangle},
\]

\[
\lambda_2 = 1 - \sqrt{\langle \rho | T(\rho) T^T(\rho) | \rho \rangle}.
\]

(23)

In order to acquire the sufficient criterion of unsteering for an arbitrary T state, we start with a any Bell diagonal state \( \rho_{\text{Bell}} \).

According to ‘proof of Theorem 1” in Ref. [31], we conclude that if the conditional state of \( \rho_{\text{Bell}} \) conforms to a LHS model, then its eigenvalues \( \lambda_1 \) and \( \lambda_2 \) satisfy the relation \( \lambda_1 \leq 2\sqrt{\lambda_2} - \lambda_2 \) for the measurement setting \( r \), or equivalently

\[
\sqrt{\langle \rho | T^2(\rho_{\text{Bell}}) | \rho \rangle} \leq \frac{1}{2}.
\]

(24)

Therefore, when we consider infinite \( (N \rightarrow \infty) \) measurements, Eq. [24] can be rewritten as

\[
\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{N} \sqrt{\langle \rho_k | T^2(\rho_{\text{Bell}}) | \rho_k \rangle} \leq \frac{1}{2}.
\]

(25)

Similarly, we replace \( |\rho_k\rangle \) with a variable unit vector \( |\psi\rangle = (\sin \theta \cos \varphi \sin \theta \sin \varphi \cos \theta)^T \). And then, Eq. [25] can be reduced to

\[
\frac{1}{4\pi} \int_S \sqrt{\langle \psi | T^2(\rho_{\text{Bell}}) | \psi \rangle} dS \leq \frac{1}{2}.
\]

(26)
Thus, it is unsteerable for Bell diagonal state $\rho_{\text{Bell}}$, if $F(\rho_{\text{Bell}}) \leq 1/2$. In fact, this result is consistent with the criteria given in Ref. [32].

Considering the symmetry (Property 2 and its Inference) of the maximum violation $F(\rho)$, we obtain a sufficient criterion that it is unsteerable for a general T state $\rho$ if $F(\rho) \leq 1/2$. And Theorem shows that the T state $\rho$ is steerable if $F(\rho) > 1/2$. Therefore, the T state $\rho$ is steerable if only if and if the maximum violation meets the relation $F(\rho) > 1/2$.

D. Separable states don’t violate the steering inequality

In general, the steerable states must be entangled. In other words, the separable states must be unsteerable. Obviously, there is a rule that the separable states must obey the steering inequality. Therefore, it is necessary to test the newly derived steering inequality in Eq. (13).

Rule. If the T state $\rho$ is a separable state, then it conforms the steering inequality in Eq. (13).

Proof. In general, a separable T state can be expressed as $\rho = \sum_i p_i \rho_i$ with Bloch vectors $a_i = \text{Tr}[\rho_i (\sigma \otimes 1_2)] = 0$ and $b_i = \text{Tr}[\rho_i (1_2 \otimes \sigma)] = 0$, where $p_i = \rho_i^B \otimes \rho_i^B$ denotes ith uncorrelated state and the probability $p_i$ satisfies the relation $\sum_i p_i = 1$. In other words, these vectors $a_i = \text{Tr}[\rho_i^B \sigma]$ and $b_i = \text{Tr}[\rho_i^B \sigma]$ meet the relations $\sum_i p_i a_i = 0$ and $\sum_i p_i b_i = 0$. And the correlation matrix $T(\rho)$ can be described by these vectors, i.e., $T(\rho) = \sum_i p_i |a_i\rangle \langle b_i|$. Combining with the integrand $f(\rho, v) = \sqrt{|v| T^+(\rho) T(\rho) |v|}$, we obtain that the integrand $f(\rho, v)$ can be rewritten as

$$f(\rho, v) = \sqrt{\sum_{i,j} p_i p_j |\langle v | b_i\rangle \langle a_i | a_j\rangle \langle b_j | v|}.$$  (27)

Considering the relation $|\langle a_i | a_j\rangle| \leq 1$, we rewrite Eq. (27) as an inequality

$$f(\rho, v) \leq \sqrt{\sum_{i,j} p_i p_j |\langle v | b_i\rangle \langle a_i | a_j\rangle \langle b_j | v|} \leq \sqrt{\sum_{i,j} p_i p_j |\langle v | b_i\rangle \langle b_j | v|} \leq \sqrt{\sum_{i} p_i |\langle v | b_i\rangle|} \sum_{j} p_j |\langle v | b_j\rangle| \leq \sum_{i} p_i |\langle v | b_i\rangle|.$$  (28)

It is obvious that the maximum violation $F(\rho)$ for separable T state $\rho$ satisfies the following relation

$$F(\rho) = \frac{1}{4\pi} \int \int f(\rho, v) dS \leq \frac{1}{4\pi} \int \sum_i p_i |\langle v | b_i\rangle| dS = \frac{1}{2} \sum_i p_i \left( \frac{1}{2\pi} \int \int |\langle v | b_i\rangle| dS \right) = \frac{1}{2} \sum_i p_i \left( \frac{1}{2\pi} \int \int |b_i| \cos \theta |dS\right)$$

$$= \frac{1}{2} \sum_i p_i |b_i| \int_0^\pi \cos \theta \sin \theta d\theta \leq \frac{1}{2} \sum_i p_i = \frac{1}{2}.$$  (29)

Therefore, all separable T states don’t violate the steering inequality in Eq. (13).

E. Relation between the concurrence and maximum violation

We now illustrate the boundary problem of the intrinsic relation between steering and entanglement with some special T states, and try to estimating the maximum violation by using entanglement. In order to better understand the relation between two quantum correlations, we introduce a common measure of entanglement for two-qubit states, i.e., concurrence [33]. For an arbitrary pure state $|\psi\rangle$, its concurrence can be defined as

$$E(|\psi\rangle) = \left|\left<\psi\right|\tilde{\psi}\right|,$$  (30)

where $\tilde{\psi} = (\sigma_y \otimes \sigma_y) |\psi\rangle^*$ represents the spin-flipped state of $|\psi\rangle$ and $|\psi^*\rangle$ is the complex conjugate state of $|\psi\rangle$. For a general T state $\rho$, its spin-flipped state $\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$ is same as the state $\rho$, i.e., $\tilde{\rho} = \rho$. Thus, the concurrence of $\rho$ can be expressed as [33 35 36]

$$E(\rho) = \max \left\{ 0, 2\lambda_{\text{max}} \left( \sqrt{\rho^* \rho} \right) - \text{Tr} \left( \sqrt{\rho^* \rho} \right) \right\} = \max \left\{ 0, 2\lambda_{\text{max}}(\rho) - 1 \right\},$$  (31)

where $\lambda_{\text{max}}(X)$ represents the maximum eigenvalue of the matrix $X$.

(1) Evolutionary states of Werner states.— We consider the evolutionary states $W_{PD}$, which are formed by Werner states $W_{\alpha}$ going through the phase damped (PD) channel. And the states $W_{PD}$ can be denoted as

$$W_{PD} = \sum_{i=0}^d K_i W_{\alpha} K_i^\dagger,$$  (32)
where $K_0 = |0\rangle \langle 0| + \sqrt{1-\eta}|1\rangle \langle 1|$ and $K_1 = \sqrt{\eta}|1\rangle \langle 1|$ are the Kraus operators of PD channel. Obviously, the states $W_{\text{PD}}$ belong to Bell diagonal states. Based on Eq. (31), the concurrence can be expressed as $E(W_{\text{PD}}) = \max \{0, \alpha \sqrt{1-\eta} - \frac{1-\eta}{\sqrt{1-\eta}}\}$. And the correlation matrix of $W_{\text{PD}}$ can be written as a diagonal matrix, i.e., $T(W_{\text{PD}}) = \text{diag} \{ \alpha \sqrt{1-\eta}, -\alpha \sqrt{1-\eta}, \alpha \}$. Hence, the maximum violation $F(W_{\text{PD}})$ can be reduced to

$$F(W_{\text{PD}}) = \frac{\alpha}{2} \left( 1 + \frac{1-\eta}{\sqrt{\eta}} \ln \frac{1+\sqrt{\eta}}{1-\eta} \right),$$

where $\ln$ denotes the natural logarithm. Thus, the sufficient and necessary criterion of steering for the states $W_{\text{PD}}$ is the relation $\alpha \left( 1 + \frac{1-\eta}{\sqrt{\eta}} \ln \frac{1+\sqrt{\eta}}{1-\eta} \right) > 1$. In particular, when $\eta = 0$, the states $W_{\text{PD}}$ are reduced to Werner states $W_{\alpha}$. And the maximum violation corresponding to Werner states $W_{\alpha}$ can be given by $F(W_{\alpha}) = \lim_{\eta \to 0} F(W_{\text{PD}}) = \alpha$. It is apparent that Werner states $W_{\alpha}$ are to demonstrate steering if and only if $\alpha > 1/2$ (as shown in Fig. 2).

(2) $T$ states of rank-2.— Based on the lemma that an arbitrary two-qubit state has a decomposition in which each pure state has the same entanglement [33], an arbitrary two-qubit T state $\psi_T$ of rank-2 can be formed by mixing an arbitrary pure state $|\psi\rangle$ and its spin-flipped state $|\tilde{\psi}\rangle$ with equal probability. And the T states $\psi_T$ can be written as

$$\psi_T = \frac{1}{2} \left( |\psi\rangle \langle \psi| + |\tilde{\psi}\rangle \langle \tilde{\psi}| \right).$$

According to Eqs. (30) and (31), we obtain an inequality that concurrence of the constructed T state $\psi_T$ is equal to concurrence of the original state $|\psi\rangle$, i.e., $E(\psi_T) = E(|\psi\rangle)$. For the state $\psi_T$, three singular values of the correlation matrix $T(\psi_T)$ can be given by $\Lambda(\psi_T) = \text{diag} \{ 1, E(\psi_T), E(\psi_T) \}$. Therefore, the maximum violation for the state $\psi_T$ can be reduced to

$$F(\psi_T) = \frac{1}{2} \left[ 1 + \frac{E^2(\psi_T)}{\sqrt{1-E^2(\psi_T)}} \ln \frac{1+\sqrt{1-E^2(\psi_T)}}{E(\psi_T)} \right].$$

In particular, when $E(\psi_T) = 0$, the maximum violation is limit $F(\psi_T) = 1/2$; when $E(\psi_T) = 1$, the maximum violation is limit $F(\psi_T) = 1$. Hence, for an arbitrary T state with rank-2, the state is steerable if only if this state is entangled.

(3) Any T states.— At the front, we have given the function relation between the concurrence and maximum violation for some special states. For any two-qubit T states $\rho$, what relation could we obtain about the concurrence and maximum violation? When we only know the values of concurrence, where is the value-range of the maximum violation? In other words, we try to establish an inequality relation between the concurrence and maximum violation, and use the concurrence to estimate the maximum violation. In order to accomplish this task, we investigate lots of randomly generated two-qubit T states. The result shows that there is an inequality relation between the concurrence and the maximum violation. For the separable T states, $0 \leq F(\rho) \leq 1/2$. When $E(\rho) > 0$, the inequality can be expressed as follows (see Fig. 3)

$$\frac{1 + 2E(\rho)}{3} \leq F(\rho) \leq \frac{1}{2} \left[ 1 + \frac{E^2(\rho)}{\sqrt{1-E^2(\rho)}} \ln \frac{1+\sqrt{1-E^2(\rho)}}{E(\rho)} \right].$$

IV. CONCLUSION AND DISCUSSION

In this paper, we have completed two main tasks about two-qubit T states. On the one hand, we derive a steering inequality with infinite measurements corresponding to an arbitrary
two-qubit T state. And the steering inequality can be viewed as a necessary and sufficient criterion that is used to distinguish that the T state is steerable or unsteerable. And a two-qubit T state is steerable if and only if the maximum violation is beyond 1/2. For an arbitrary two-qubit T state, the maximum violation satisfies the scaling and symmetry in Eqs. (15) and (18). On the other hand, we establish the function relation between the concurrence and maximum violation for some special T states, and put forward a method to estimate the maximum violation from concurrence for any two-qubit T states by using lots of randomly generated two-qubit T states. And it indicates that an arbitrary T state is steerable if its concurrence exceeds 1/4. Specially, for all T states with rank-2, the state is steerable if only and if this state is entangled.

In future work, it would be interesting to investigate the necessary and sufficient criterion that ensures an arbitrary two-qubit state is steerable or unsteerable from Alice to Bob.

ACKNOWLEDGEMENTS

This work was supported by the National Science Foundation of China under Grant No. 11575001.
[23] I. Kogias, A. R. Lee, S. Ragy, and G. Adesso, Quantification of Gaussian quantum steering, Phys. Rev. Lett. 114, 060403 (2015).
[24] A. C. S. Costa, R. M. Angelo, Quantification of Einstein-Podolsky-Rosen steering for two-qubit states, Phys. Rev. A 93, 020103 (2016).
[25] B. C. Yu, Z. A. Jia, Y. C. Wu, G. C. Guo, Geometric steering criterion for two-qubit states, Phys. Rev. A 97, 012130 (2018).
[26] D. Mondal, D. Kaszlikowski, Complementarity relations between quantum steering criteria, Phys. Rev. A 98, 052330 (2018).
[27] D. Das, S. Sasmal, S. Roy, Detecting Einstein-Podolsky-Rosen steering through entanglement detection, Phys. Rev. A 99, 052109 (2019).
[28] H. Yang, Z. Y. Ding, L. Ye, et al., Experimental observation of Einstein-Podolsky-Rosen steering via entanglement detection, Phys. Rev. A 101, 042115 (2020).
[29] H. C. Nguyen, H. V. Nguyen, and O. Gühne, Geometry of Einstein-Podolsky-Rosen correlations, Phys. Rev. Lett. 122, 240401 (2019).
[30] D. J. Saunders, H. M. Wiseman, G. J. Pryde, et al., Experimental EPR-steering using Bell-local states, Nature Phys. vol 6, 11 (2010).
[31] J. Bowles, F. Hirsch, N. Brunner, et al., Sufficient criterion for guaranteeing that a two-qubit state is unsteerable, Phys. Rev. A 93, 022121 (2016).
[32] S. Jevtic, M. J. W. Hall, M. R. Anderson, M. Zwierz, and H. M. Wiseman, Einstein-Podolsky-Rosen steering and the steering ellipsoid, J. Opt. Soc. Am. B 32, A40 (2015).
[33] Z. F. Su, H. S. Tan and X. Y. Li, Entanglement as upper bound for the nonlocality of a general two-qubit system, Phys. Rev. A 101, 042112 (2020).
[34] S. Jevtic, M. Pusey, D. Jennings, and T. Rudolph, Quantum Steering Ellipsoids, Phys. Rev. lett. 113, 020402 (2014).
[35] W. K. Wootters, Entanglement of Formation of an Arbitrary State of Two Qubits, Phys. Rev. Lett. 80, 2245 (1998).
[36] X. G. Fan, W. Y. Sun, L. Ye, et al., Universal complementarity between coherence and intrinsic concurrence for two-qubit states, New J. Phys. 21, 093053 (2019).