Beta Function Quintessence Cosmological Parameters and Fundamental Constants I: Power and Inverse Power Law Dark Energy Potentials

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ABSTRACT

This investigation explores using the beta function formalism to calculate analytic solutions for the observable parameters in rolling scalar field cosmologies. The beta function in this case is the derivative of the scalar \( \phi \) with respect to the natural log of the scale factor \( a \), \( \beta(\phi) = \frac{d\phi}{d\ln(a)} \). Once the beta function is specified, modulo a boundary condition, the evolution of the scalar \( \phi \) as a function of the scale factor is completely determined. A rolling scalar field cosmology is defined by its action which can contain a range of physically motivated dark energy potentials. The beta function is chosen so that the associated "beta potential" is an accurate, but not exact, representation of the appropriate dark energy model potential. The basic concept is that the action with the beta potential is so similar to the action with the model potential that solutions using the beta action are accurate representations of solutions using the model action. The beta function provides an extra equation to calculate analytic functions of the cosmologies parameters as a function of the scale factor that are not calculable using only the model action. As an example this investigation uses a quintessence cosmology to demonstrate the method for power and inverse power law dark energy potentials. An interesting result of the investigation is that the Hubble parameter \( H \) is almost completely insensitive to the power of the potentials and that \( \Lambda \)CDM is part of the family of quintessence cosmology power law potentials with a power of zero.

Key words: (cosmology:) cosmological parameters – dark energy – theory – early universe.

1 INTRODUCTION

The nature of dark energy is one of the key cosmological questions of our time. A basic component of the question is whether dark energy is static as predicted by the cosmological constant \( \Lambda \) or dynamical as predicted by rolling scalar field cosmologies. The proper test is to determine which theory best fits the observations. The predictions of the cosmological constant are well known and appear to be consistent with current observations. Ideally the predictions of scalar field cosmologies should start with the action of the cosmology which can accommodate various physically motivated model dark energy potentials \( V(\phi) \) where \( \phi \) is the scalar field. Unfortunately it is often mathematically difficult or impossible to make calculations based on the resulting action even for simple dark energy models such as power law potentials (Narain 2011). This work investigates the use of the beta formalism to provide accurate analytic equations for the evolution of cosmological parameters as a function of the observable scale factor \( a \) as opposed to the generally unobservable scalar \( \phi \).

The beta function is defined as the derivative of the scalar with respect to the natural log of the scale factor

\[
\beta(\phi) \equiv \frac{d\phi}{d\ln(a)} = \phi'
\]

where the second equality notes the common cosmological practice of denoting the derivative with respect to \( \ln(a) \) with a prime. As described in section 3 the beta function is chosen so that the resultant "beta potential" is an accurate representation of the model dark energy potential in the model action. For most cases the action with the beta potential is so similar to the action with the model potential that solutions using the beta action are accurate representations of solutions using the model action. Once the form of the beta function is defined analytic solutions of the evolution
of the cosmological parameters can be found as a function of the scalar $\phi$. The beta function also provides the means to express the solutions in terms of the scale factor $a$ rather than the scalar $\phi$. This investigation explores the bounds of the parameter space where the beta function formalism produces solutions that deviate from the exact solution by only on the order of 1% or less. The primary purpose of the investigation is the provision of accurate, analytic functions of the evolution of the cosmological parameters to determine which cosmologies and potentials are consistent with the observed universe and which must be discarded as untenable in the face of the data. The functions also serve as excellent starting points for more exact numerical calculations.

The beta function formalism has its roots in a perceived correspondence between cosmological inflation and the Quantum Field Theory renormalization group flow equations (Binetruy et al. 2015; Kohri and Matsui 2014). In that context it is valid as the solution for the slow evolution of a system approaching or leaving a critical (fixed) point (Binetruy et al. 2015). Both have considered the formalism for the late time dark energy inflation where the critical point is in the infinite future. The descriptions here follow these references with particular dependence on (Cicciarella and Pieroni 2017) who have incorporated matter as well as dark energy in order to describe a real universe.

The beta function formalism is often associated with the term universality (Binetruy et al. 2013; Cicciarella and Pieroni 2017; Kohri and Matsui 2017) referring to a commonality among seemingly disparate cosmologies revealed by the beta function formalism. The example used in this work is too limited to fully show this but section 4 hints at this where a common analytic function is found for the Hubble parameter $H = \frac{\dot{a}}{a}$ which is shared by ΛCDM.

This work concentrates on the "late time" evolution of the universe which is taken to be the time between a scale factor of 0.1 and 1.0 corresponding to redshifts between zero and nine. As a demonstration of the method a quintessence cosmology is considered with power and inverse power law dark energy potentials. Natural units with $\frac{8\pi G}{c^4}$ and the Planck mass equal to one are used. A flat universe is assumed with $H_0 = 70$ km/sec per megaparsec. The current ratio of the dark energy density to the critical density as a function of the scalar $\phi$. The analytic functions have $H_0$ and $\Omega_{\phi_0}$ as parameters therefore results for other choices are easily obtained. Integer powers of $\phi$ are taken to be $\pm (1, 2, 3, 4, 5)$ as examples but the derived functions are valid for fractional powers as well. The current values of the dark energy equation of state $w = \frac{p_\phi}{\rho_\phi}$ are taken to be $w_0 = (-0.98, -0.96, -0.94, -0.92, -0.90)$ where $p_\phi$ is the dark energy pressure and $\rho_\phi$ is the dark energy density. The last two values of $w_0$ are unlikely but are included to determine the limits of the formalism.

2 QUINTESSENCE

Quintessence is of the most studied rolling scalar field cosmologies still standing after the observation of gravity waves from merging neutron stars (Ezquiaga & Zumalacarregui 2017, Durrive et al. 2018). It is characterized by an action of the form

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - V(\phi) \right] + S_m$$

where $R$ is the Ricci scalar, $g$ is the determinant of the metric $g^{\mu\nu}$, $V(\phi)$ is the dark energy potential, and, $S_m$ is the action of the matter fluid. Different types of quintessence are defined by different forms of the dark energy potential.

The dark energy density, $\rho_\phi$, and pressure, $p_\phi$, are derived from the energy momentum tensor which again involves $V(\phi)$.

$$\rho_\phi \equiv \frac{i^2}{2} + V(\phi), \quad p_\phi \equiv \frac{i^2}{2} - V(\phi)$$

An essential observable cosmological parameter is the dark energy equation of state $w = \frac{p_\phi}{\rho_\phi}$. Note that if $\phi$ is zero then $w = -1$ for all time as in ΛCDM. For a quintessence cosmology (Nunes & Liddle 2004) give the dark energy equation of state as

$$w + 1 = \frac{\phi'^2}{3\Omega_\phi} = \frac{\beta^2(\phi)}{3\Omega_\phi}$$

where $\Omega_\phi$ is the ratio of the dark energy density to the critical density. The factor $\Omega_\phi$ recognizes that there can be matter as well as dark energy in the universe so that for a flat universe matter $\Omega_\phi$ is not 1 but rather $1 - \Omega_m$ where $\Omega_m$ is the ratio of the matter density to the critical density. The current value of the equation of state $w_0$ is therefore a possible boundary condition in the solution for the scalar $\phi$.

3 THE BETA FUNCTION

The beta function is defined in eqn. 1 as the derivative of the scalar with respect to the natural log of the scale factor. Analytic solutions for the cosmological parameters are possible because the beta function provides an additional equation that determines the evolution of the scalar $\phi$ as a function of the scale factor. The beta function is not an arbitrarily chosen relation of $\phi$. It is directly tied to the physically relevant model dark energy potential $V(\phi)$ in the action.

For a given model potential $V(\phi)$, the beta function $\beta(\phi)$ is chosen so that

$$V_m(\phi) = \exp \{ - \int \beta(\phi) d\phi \}$$

where $V_m(\phi)$ is the model potential rather than the full potential given in eqn. 12. With the proper choice of $\beta(\phi)$ any function for $V(\phi)$ can be represented, not just the functions considered in this investigation. From eqn. 15 $\beta(\phi)$ is chosen such that the integral of $\beta(\phi)$ equals the logarithmic derivative of $V$. The power and inverse power law potential beta functions are then

$$\beta(\phi) = -\frac{\beta_p}{\phi}, \quad \beta(\phi) = \frac{\beta_i}{\phi}.$$
The evolution of the beta function \( \beta(\phi) \) as a function of the scalar \( \phi \) with \( \beta_{p,i} = 3 \) and the five different values of \( w_0 \).

The power law \( \beta(\phi) \) (solid line) is negative and the inverse power law \( \beta(\phi) \) (dashed lined) is positive.

**Figure 1.** The evolution of the beta function \( \beta(\phi) \) as a function of the scalar \( \phi \) with \( w_0 = -0.94 \) for the five values of \( \beta_{p,i} \).

Except where otherwise noted in subsequent figures power law functions are denoted with a solid line and inverse power law functions with a dashed line. In figure 2 \( w_0 = -0.94 \) for all five \( \beta_{p,i} \) values for the power and inverse power law potentials. Note that the values of \( \beta(\phi) \) for a given value of \( \beta_{p,i} \) are sensitive to the value of \( w_0 \) but for a given value of \( w_0 \) the values are relatively insensitive to \( \beta_{p,i} \). This is a pattern that occurs for many of the functions and parameters considered here.

4 Evolution of the Scalar

From the definition of the beta function a simple integration of eqns. [6] gives

\[
\phi_p(a) = \sqrt{-2\beta_p \ln(a) + \phi_0^2}, \quad \phi_i(a) = \sqrt{2\beta_i \ln(a) + \phi_0^2}
\]

where \( \phi_0 \) is the present day value of \( \phi \). As is evident when \( \beta(\phi) \) is used in eqn. [4] to replace \( \phi' \) the value of \( \phi_0 \) is related to the current dark energy equation of state \( w_0 \) by

\[
\phi_0 = \frac{\beta_{p,i}}{\sqrt{3\Omega_\phi (1 + w_0)}}
\]

for a quintessence cosmology where \( \Omega_\phi \) is the current value of \( \Omega_\phi \). Note that \( \phi_0 \) is the same for both the power and inverse power law beta functions with the same values of \( \beta_{p,i} \).

4.1 Limitations on the Inverse Power Law Beta Function

At all times in the past the value of \( \ln(a) \) is negative, therefore, the term in the square root for the inverse power law in eqn. [7] becomes negative at some time in the past limiting the range of the scale factor. This is one justification for considering the power law and inverse power law beta functions as two separate cases. For the inverse power law case \( \phi_0^2 \) must be larger than \( |2\beta_i \ln(a)| \) to avoid a negative argument. Using eqn. [8] this sets a requirement that

\[
2 \ln(a) + \frac{\beta_i}{3\Omega_\phi (w_0 + 1)} > 0
\]

to insure that \( \phi \) is a real number. For the scale factors between 0.1 and 1 considered in this work the constraint in eqn. [9] is satisfied for all values of \( \beta_i \) and \( w_0 \) utilized in the investigation. For \( \beta_i = 1 \) and \( w_0 = -0.9 \), however, it is not satisfied at scale factors less than 0.0925, very close to the smallest scale factor of 0.1. As \( 2\beta_i \ln(a) \approx -\phi_0^2 \) the beta function evolves rapidly to large numbers making the solutions in this region unreliable. The increased deviation of the \( \beta_i = 1 \) track in fig. 2 is an indicator of the problem. A restriction that only scale factors that are at least some number larger than the scale factor where the argument of eqn. [9] becomes zero are considered reliable could be adopted. Instead in section 4.2 a more physically motivated restrictions are imposed on the scale factors based on the accuracy of the beta potentials match to the model potential. These restrictions are applied to both the power law and inverse power law potentials.

4.2 The Scalar as a Function of the Scale Factor

Figure 3 shows an example of the evolution of \( \phi \) for both the power and inverse power law cases for \( w_0 = 0.94 \). The power law scalar decreases as the scale factor increases while the inverse power law scalar increases with increasing \( a \). Both converge to the same value \( (\phi_0) \) at \( a = 1 \). Even though \( \phi_0 \) changes significantly with the value of \( \beta_{p,i} \), the scalar \( \phi \) evolves relatively little over \( a \) between 0.1 and 1. Figure 4 shows the evolution of the scalar with \( \beta_{p,i} = 3.0 \) and the five different values of \( w_0 \). Figures 3 and 4 quantify the small variation of \( \phi \) by plotting the ratio of \( \phi \) to \( \phi_0 \) with \( w_0 = -0.94 \).
the absolute change in the scalar $\Delta \phi$ in $\phi/\phi_0$ for the five different values of $w_0$. The power law scalar (solid line) decreases to $\phi_0$ and the inverse power law scalar (dashed line) increases to $\phi_0$.

Figure 5. The evolution of the ratio of $\phi$ to $\phi_0$ with $w_0 = 0.96$ for the five different values of $\beta_{p,i}$.

$-0.94$ in fig. 4 and for the five values of $w_0$ with $\beta_{p,i} = 3$ in fig. 3. The figures show that the scalar varies by relatively little over the look back time of 13 gigayears considered in this study. They also show that smaller values of $\beta_{p,i}$ and larger deviations of $w_0$ from minus one result larger changes in $\phi/\phi_0$.

In some cases the evolution of a parameter depends on the absolute change in the scalar $\Delta \phi = \phi - \phi_0$ rather than the relative change in $\phi$. Figure 7 shows the values of $\Delta \phi$ for the five values of $\beta_{p,i}$ for $w_0 = -0.94$. The value of $\Delta \phi$ is essentially independent of the value of $\beta_{p,i}$ for a given value of $w_0$. This is a primary factor in the later conclusions that several parameters appear insensitive to the power, $\beta_{p,i}$, of the power laws considered in this work.

5 THE POTENTIALS

In the beta function formalism two potentials play a prominent role. The first is the dark energy potential in the action $V(\phi)$ that does not depend on matter. The second, in analogy with particle physics, is termed the super potential $W$ given by

$$W(\phi) = -2H(\phi) = -2 \frac{\dot{\phi}}{a}$$

(10)

Even though the Hubble parameter $H$ is the parameter of interest the development of the method utilizes $W$ to be consistent with the literature on beta functions. Both the potential $V(\phi)$ and the super potential $W(\phi)$ can be expressed in terms of $\beta(\phi)$ by Cicciarella and Pieroni (2017) by

$$W(\phi) = W_0 \exp \left\{ \frac{1}{2} \int_{\phi_0}^{\phi} \beta(x) dx \right\}$$

(11)

and

$$V(\phi) = \frac{3}{4} W_0^2 \exp \left\{ - \int_{\phi_0}^{\phi} \beta(x) dx \right\} \left( 1 - \frac{\beta^2(\phi)}{6} \right)$$

(12)

where $W_0$ is the current value of $W$ equal to $-2H_0$. Note that the super potential is always denoted as a capital $W$ and the dark energy equation of state by a lower case $w$.

The power law beta function results in simple forms of the two potentials

$$W(\phi) = W_0 \left( \frac{\phi}{\phi_0} \right)^{\beta_p}$$

(13)

and

$$V(\phi) = \frac{3}{4} W_0^2 \left( \frac{\phi}{\phi_0} \right)^{\beta_p} \left( 1 - \frac{\beta^2(\phi)}{6} \right)$$

(14)

The inverse power law also has simple forms for the potentials.

$$W(\phi) = W_0 \left( \frac{\phi}{\phi_0} \right)^{-\beta_i}$$

(15)
It is clear that the beta dark energy potentials have the desired power and inverse power law potentials multiplied by \((1 - \frac{\beta_n^2}{6\phi^2})\) which produces both an offset and a deviation from the model potentials. The deviation is expected to be small since \(\frac{\beta_n^2}{6\phi^2}\) is much less than one in most cases. The offset can be corrected by a simple normalization \((1 - \frac{\beta_n^2}{6\phi^2})^{-1}\) where \(a_n\) is the scale factor where the normalization occurs. The average deviation can be minimized by choosing a midway point such as \(a_n = 0.5\), however, in this work the normalization point is \(a_n = 1\), the current epoch since that is where the boundary condition is set such that \(H(a = 1) = H_0\). Numerical accuracy could be increased by normalizing piecewise at several scale factors. A goal of this work is to create analytic solutions, rather than numerical tables, therefore only one normalization point is utilized.

\[
V(\phi) = \frac{3}{4}V_0^2 \left( \frac{\phi}{\phi_0} \right)^{\beta_1} \left( 1 - \frac{\beta_n^2}{6\phi^2} \right) \tag{16}
\]

### 5.2 Accuracy of Fit

The cosmological parameters derived by the beta function formalism are only useful if the beta potentials accurately represent the model potentials. Figures 8 and 9 show the evolution of the power and inverse power law potentials respectively. In contrast to previous figures the solid lines are the model potentials and the dashed lines are the beta potentials. The value of \(\beta_{p,i}\) is set to 3.0. The beta potentials are an excellent match to the model potentials and the dashed lines are the beta potentials for comparison. The quality of the fits makes it difficult to resolve the solid lines from the dashed. The average deviation can be minimized by choosing a midway point such as \(a_n = 0.5\), however, in this work the normalization point is \(a_n = 1\), the current epoch since that is where the boundary condition is set such that \(H(a = 1) = H_0\). Numerical accuracy could be increased by normalizing piecewise at several scale factors. A goal of this work is to create analytic solutions, rather than numerical tables, therefore only one normalization point is utilized. The fractional deviation of the beta power law potentials from the model potentials with \(\beta_p = 1.0\), dashed lines, \(\beta_p = 3.0\), solid lines, and \(\beta_p = 5.0\), dot dashed lines. For each \(\beta_p\) the tracks with the minimum deviation are for \(w_0 = -0.98\) and the tracks with the maximum deviation are for \(w_0 = -0.90\) the extremes without excessive overlap of tracks in the figures. The power law beta potentials are quantitatively good matches to model potentials with the fit improving as \(\beta_p\) increases and as \(w_0\) decreases toward minus one. Only the \(\beta_p = 1\) with \(w_0 = -0.90\) case exceeds a fractional deviation of 1% and then only at scale factors less than 0.4. The inverse power law beta potentials show the same trends but are less well behaved. It is clear that for low \(\beta_i\) values and large deviations of \(w_0\) from minus one some of the beta potentials deviate from the model potentials by much more than 1%.

In this investigation the conservative limit of no more than 1% deviation of the beta potential from the model po-
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| \( \beta_p \) | \(-0.98\) | \(-0.96\) | \(-0.94\) | \(-0.92\) | \(-0.90\) |
|-------------|--------|--------|--------|--------|--------|
| 1           | v      | v      | v      | 0.2    | 0.4    |
| 2           | v      | v      | v      | v      | 0.15   |
| 3           | v      | v      | v      | v      | v      |
| 4           | v      | v      | v      | v      | v      |
| 5           | v      | v      | v      | v      | v      |

Table 1. Valid values of the scale factor for the power law beta potentials. The scale factor must be greater than the entered value for the given value of \( \beta_p \) and \( w_0 \). An entry of v indicates that all scale factors between 0.1 and 1 are valid.

| \( \beta_p \) | \(-0.98\) | \(-0.96\) | \(-0.94\) | \(-0.92\) | \(-0.90\) |
|-------------|--------|--------|--------|--------|--------|
| 1           | v      | v      | 0.28   | 0.47   | 0.6    |
| 2           | v      | v      | 0.21   | 0.37   |        |
| 3           | v      | v      | v      | 0.20   |        |
| 4           | v      | v      | v      | v      |        |
| 5           | v      | v      | v      | v      |        |

Table 2. The same as table 1 for the inverse power law potentials.

The range of \( w_0 \) values for this investigation was extended past -0.94 to test the limits of the validity of the method. For the power law beta functions the only cases that are not valid over all scale factors are for \( w_0 \) values of -0.98 and -0.9 with \( \beta_p \) values of 1 and 2. The minimum \( a \) value of \( \beta_p = 2 \) and \( w_0 = -0.9 \) is 0.15, therefore, most of the range of the scale factor is valid. For the inverse power law case only the \( \beta = 1 \) with \( w_0 = -0.94 \) has a limitation on the scale factor for the three values of \( w_0 \) nearest minus one. This leads to the conclusion the beta function formalism is a useful method for power and inverse power dark energy potentials within the expected values of \( w_0 \). Caution, however, must be exercised for \( w_0 \) values further from minus one than -0.94 as is shown in the tables. It is clear from tables 1 and 2 that as the value of \( \beta_{p,i} \) approaches one the solutions for the beta potentials deviate from the model potentials by more than 1% over a larger fraction of the scale factors under consideration. Except for the special case of \( \beta_{p,i} = 0 \), LambdaCDM values of \( \beta_{p,i} < 1 \) are considered unreliable and are not considered in the investigation.

6 ADDING MATTER TO THE UNIVERSE

A real universe includes matter as well as dark energy. The explicit inclusion of matter is discussed in Cicciarella and Pieroni (2017) and is the basis for this work. As before there is no attempt to rederive the work presented there except where it is useful for clarity. The purpose of this work is useful analytic models for comparison with observational rather than a theoretical extension of previous work. Matter is represented by the \( S_m \) term the action, eqn. 2.

6.1 The Matter Density

The matter density \( \rho_m \) follows the mass continuity equation

\[
\dot{\rho}_m + 3H\rho_m = 0
\]

In keeping with the notation of Binetruy et al. (2013) and Cicciarella and Pieroni (2017) the subscript \( \phi \) indicates the derivative with respect to \( \phi \). This leads to the equations

\[
\frac{\rho_{m,\phi}}{\rho_m} = -\frac{3H}{\phi} = -\frac{3}{\beta(\phi)}
\]

Integrating the logarithmic derivative in eqn. 18 yields the equation for \( \rho_m(\phi) \)

\[
\rho_m(\phi) = \rho_{m0} \exp(-3 \int_{\phi_0}^{\phi} \frac{d\phi}{\beta(\phi)})
\]

Different beta functions produce different functions for \( \rho_m \) as a function of \( \phi \). The emphasis in this work, however, is expressing the cosmological parameters as a function of the observable scale factor \( a \) rather than the unobservable scalar \( \phi \). From the definition of \( \beta(\phi) \) in eqn. 16 eqn. 19 becomes

\[
\rho_m(a) = \rho_{m0} \exp(-3 \int_{1}^{a} d\ln(a)) = \rho_{m0} a^{-3}
\]

as expected, independent of \( \beta(\phi) \).

6.2 The Super Potential \( W \) with Mass

The Einstein equations with mass become

\[
H^2 = \frac{\rho_m + 3\phi}{3}
\]

\[
-2\dot{H} = \rho_m + \rho_\phi + p_\phi
\]

Cicciarella and Pieroni (2017) show that the inclusion of matter results in differential equation for \( W \) of the form

\[
WW_\phi + \frac{1}{2} \beta W^2 = -2\frac{\rho_m}{\beta}
\]

For the power law beta function \( \beta(\phi) = -\frac{\phi}{\phi} \) eqn. 23 becomes

\[
WW_\phi + \frac{1}{2} \beta_\phi W^2 = -2\rho_m \frac{\phi}{\beta_p}
\]

Equation 23 is solved by multiplying it by an integrating factor that makes the left hand side an exact differential and the right hand side an integral that can be solved preferably analytically or by numerical integration. The integrating factor for the power law beta function is \( \phi^{-\beta_p} \). The equation then reads

\[
\frac{d}{d\phi} \left( \frac{1}{2} W^2 \phi^{-\beta_p} \right) = 2\rho_m(\phi) \frac{\phi^{1-\beta_p}}{\beta_p}
\]

which is a general equation for any positive value of \( \beta_p \).

The derivation of the super potential deviates from the discussion of Cicciarella and Pieroni (2017) at this point to
derive \( W(a) \) rather than \( W(\phi) \) since the goal is observable predictions. Substituting eqn. 20 into eqn. 25 results in

\[
|\phi| W^2 a^{-\beta p} = 4 \rho_{m0} \beta_p \int_0^a \phi^{1-\beta_p} a^{-3} d\phi
\]  

(26)

Using \( d\phi = -\beta_p (-2 \beta_p \ln(a) + \phi_0^2)^{-1/2} \frac{da}{a} \) gives

\[
|\phi| W^2 \phi^{-\beta_p} = -4 \rho_{m0} \int_1^a x^{-4} (-2 \beta_p \ln(x) + \phi_0^2)^{-\beta_p} dx
\]  

(27)

Equation 27 can also be written as

\[
|\phi| W^2 = -4 \rho_{m0} \phi^{\beta_p} \int_1^a x^{-4} \phi^{-\beta_p} dx
\]  

(28)

Since \( \phi(a) = (-2 \beta_p \ln(a) + \phi_0^2)^{1/2} \) the super potential as a function of \( a \) is

\[
W(a) = \{-4 \rho_{m0} (-2 \beta_p \ln(a) + \phi_0^2)^{\beta_p} \int_1^a x^{-4} (-2 \beta_p \ln(x) + \phi_0^2)^{-\beta_p} dx + W_0^2 (\phi(a) \phi_0)^{1/2}
\]  

(29)

The integral in eqn. 20 is solved by two changes of variable. The first change is to let \( z = (-2 \beta_p \ln(a) + \phi_0^2) \) which yields

\[
-(1/2) \exp(-3 \phi_0^2 / 2 \beta_p) \int z^{-2} \exp(z) dz
\]  

(30)

The second change of variable is \( y = -3 \phi_0^2 / 2 \beta_p \) which produces the integral

\[
-1/3 \left(-\frac{2 \beta_p}{3}\right) \frac{\beta_p}{2} \exp(-3 \phi_0^2 / 2 \beta_p) \int y^{-2} \exp(-y) dy
\]  

(31)

The integral in \( y \) in eqn. 31 is the incomplete Gamma function \( \Gamma(1 - \beta_p, 3 \ln(a) - \phi_0^2 / 2 \beta_p) \). The formal solution for the super potential in terms of the scale factor is

\[
W_p(a) = \{-4 \rho_{m0} (-2 \beta_p)^{-\beta_p} \exp(-3 \phi_0^2 / 2 \beta_p) (\phi(a))^{\beta_p} \}
\]

\[
\{\Gamma(1 - \beta_p / 2, 3 \ln(a) - \phi_0^2 / 2 \beta_p) - \Gamma(1 - \beta_p / 2, 3 \phi_0^2 / 2 \beta_p)\}
\]

\[
+ W_0^2 (\phi(a) \phi_0)^{1/2}
\]  

(32)

The negative square root is chosen since \( W(a) \) is a negative quantity.

The solution for \( W(a) \) in the inverse power law case is very similar to the power law. The integrating factor is \( \phi^{\beta_i} \) rather than \( \phi^{-\beta} \). The equivalent to eqn. 28 is

\[
|\phi| W^2 = -4 \rho_{m0} \phi^{\beta_p} \int_1^a x^{-4} \phi^{\beta_p} dx
\]  

(33)

and the formal solution for \( W(a) \) for the inverse power law case is

\[
W_i(a) = \{-4 \rho_{m0} (2 \beta_i / 3)^{-\beta_i} \exp(3 \phi_0^2 / 2 \beta_i) (\phi_i(a))^{-\beta_i} \}
\]

\[
\{\Gamma(1 + \beta_i / 2, 3 \ln(a) + 3 \phi_0^2 / 2 \beta_i) - \Gamma(1 + \beta_i / 2, 3 \phi_0^2 / 2 \beta_i)\}
\]

\[
+ W_0^2 (\phi_i(a) \phi_0)^{1/2}
\]

(34)

7 THE EVOLUTION OF COSMOLOGICAL PARAMETERS

Establishing the analytic functions for the super potential \( W \) as a function of the scale factor \( a \) provides the means for calculating the evolution of cosmological parameters. It is obvious from its definition (eqns. 10) that super potential determines the Hubble parameter. Normally the discussion of the cosmological parameters would center on the Hubble parameter but the super potential is again used here to be consistent with existing literature on the beta function formalism.

In the following the parameters are presented as a function of \( \phi_p, \phi_i \) with eqn. 4 providing the proper equations for the scalar \( \phi \) as a function of the scale factor \( a \). This convention is adopted to preserve the dependence of the parameters on the scalar \( \phi \) while providing the means to calculate the parameters as a function of the scale factor \( a \). An exception to this convention is the matter density where eqn. 28 explicitly show that the density varies as \( a^{-3} \). Although it should be obvious from the context the scalar will be written as \( \phi_p(a) \) for the power law and \( \phi_i(a) \) for the inverse power law but the current value of \( \phi \) will still be written as \( \phi_0 \) since it is the same for both cases.

7.1 The Evolution of the Hubble Factor and the Onset of Acceleration

Two observable quantities are the evolution of the Hubble factor \( H(a) \) and the onset of the acceleration of the expansion of the universe. Since \( H(a) = -\frac{W(a)}{a} \) eqns. 29 and 31 specify the evolution of the Hubble factor for the power and inverse power law potentials. Figure 12 shows the evolution of \( H(a) \) for all of the cases considered in this study including \( \Lambda CDM \). All of the solutions plotted in fig. 12 conform to the limits on \( a \) in tables 1 and 2. Remarkably the solutions for both the power and inverse power law as well as \( \Lambda CDM \) all overlap each other at the resolution of fig. 12 making \( H(a) \) insensitive to either the power of the potential or the current value of the dark energy equation of state, including \( \omega_0 = -1 \), for the cases considered here.
7.1.1 The insensitivity of $H(a)$ to the Potential and $w_0$

At first glance the insensitivity of the Hubble value $H(a)$ to the power of the potential and the value of $w_0$ seems remarkable but further examination shows that it is due to a combination of factors. The first is that all solutions must have the same initial value $H_0$ which is set by observation, independent of $w_0$. A second factor is that at early times when the evolution is matter dominated the common $\rho_m = \rho_{m0}a^{-3}$ term makes the evolution the same for all cases. Thirdly the last term in both eqns. [29] and [31] is proportional to either $a^{3w}$ for the power law or $a^{-w}$ for the inverse power law. Examination of fig. 3 shows that the power and inverse power law scalars are decreasing and increasing respective with $a$ making both late time evolutions decreasing with increasing $a$. Finally examination of eqns. [28] and [33] reveal that the integrals are multiplied by opposing positive and negative powers of $\phi$ inside and outside of the integral. Since the change in $\phi$ is small the positive and negative powers of $\phi$ effectively cancel each other.

7.1.2 A simple common equation for $H(a)$

Equations. [28] and [33] suggest that the integral over $x$ with the $\phi$ term held constant may be an excellent approximation for describing $H(a)$. That approximation is given by

$$H(a) = -\frac{1}{2} \sqrt{\frac{4}{3} \rho_{m0}(a^{-3} - 1) + W_0^2 \left(\frac{\phi(a)}{\phi_0}\right)^{3p}}$$

(35)

for the power law case and

$$H(a) = \frac{1}{2} \sqrt{\frac{4}{3} \rho_{m0}(a^{-3} - 1) + W_0^2 \left(\frac{\phi_0}{\phi(a)}\right)^{3p}}$$

(36)

for the inverse power law case. Equation [7] provides the appropriate $\phi(a)$. Equations [35] and [36] give $H(a)$ solutions that are indistinguishable from the suite of solutions shown in figure [12] at the resolution of the plot. It is interesting to note that $\Lambda$CDM is the $\beta_{p,i} = 0$ case for either equation making $\Lambda$CDM a member of the family of solutions. This is an indication of the universality of the formalism.

7.1.3 The onset of acceleration

In a universe with mass the onset of the acceleration of the expansion is delayed until the matter density is low enough that dark energy begins to dominate. The onset of acceleration is marked by an increase in the expansion rate $\ddot{a} = aH(a)$. Figure [13] shows the track of $\ddot{a}$ versus $a$. The acceleration begins at a scale factor of $\approx 0.6$ (z $\approx 0.7$) which is consistent with observations eg. [Avssianishvili et al. 2011, 2013]. Given the insensitivity of $H(a)$ to $\beta_{p,i}$ and $w_0$ the only adjustable parameters are $H_0$ and $\rho_{m0}$ which are set by observation.

7.1.4 Comparison with Observations

We have shown that the Hubble factor $H(a)$ is remarkably insensitive to either the power of the potential, $\beta_{p,i}$ or $w_0$ and is identical to the $\Lambda$CDM $H(a)$ solution. This makes the Hubble factor a poor parameter for discriminating between static and dynamical dark energy. It, however, offers an excellent opportunity for determining $H_0$ for both cosmologies. The recently compiled $H(a)$ observations by [Jesus et al. 2017] provide an example of such a measurement. Using eqn. [35] as the model with $H_0$ as the only variable a chi squared analysis determined that the most likely value of $H_0$ for the example data set is $H_0 = 66.5$ (km/sec)/Mpc. Figure [14] shows the run of $\chi^2$ versus $H_0$. This is not a result, just an example for the particular data set.

Figure [15] shows the example $H_0 = 70$ and the best fit $H_0 = 66.5$ $H(a)$ evolution superimposed on the [Jesus et al. 2017] data set. The dashed curve for the $H_0 = 70$ case is just barely resolved above the solid line. The minimum chi square of about 5.6 is not a high quality measurement but is probably consistent with the scatter in the data set providing evidence that the beta function calculations have more than sufficient accuracy for comparison with observations.

7.2 The Dark Energy Equation of State

One of the most important observable cosmological parameters is the dark energy equation of state $w$. The static
ACDM cosmology predicts that \( w \) equals minus one for all time whereas dynamical cosmologies predict values deviant from minus one. It should be noted that \( w \) need not vary to produce dynamical cosmological parameters, it just needs to be different from minus one. Section 7.3 on fundamental constants is an example of such a case. From Cicciarella and Pieroni (2017) the dark energy equation of state is given by

\[
1 + w(\phi) = \frac{\beta^2}{3} \left( 1 - \frac{4\rho_{m0}}{3W^2} \right)^{-1} = \frac{\beta^2}{3} (1 - \Omega_m)^{-1} = \frac{\beta^2}{3} \Omega_\phi^{-1}
\]  

(37)

for a flat universe where the terms after the first equality are provided by the author for clarity. The second equality shows that \( 1 + w(\phi) \) is proportional to \((1 - \Omega_m)^{-1}\). In the matter dominated era \( \Omega_m \) approaches one making \((1 - \Omega_m)^{-1}\) very susceptible to small errors in \( \Omega_m \). For this reason the analytic solutions for \( (1 + w) \) employ eqns. 32 and 54 for \( W(a) \) rather than the approximations for \( W(a) \) and \( H(a) \) in eqns. 55 and 56. In terms of the scale factor \( a \) the dark energy equation of state \( w(a) \) is given by

\[
1 + w(a) = \frac{\beta_{p,i}^2}{3\phi_{p,i}(a)} \left( 1 - \frac{4\rho_{m0}}{3\alpha^2W_{p,i}(a)} \right)^{-1} \quad (38)
\]

In eqn. 58 \( \phi_{p,i}(a) \) is given by eqns. 4 and \( W_{p,i}(a) \) by eqns. 32 and 53.

Figure 15 shows the evolution of \( 1 + w(a) \) for \( \beta_{p,i} = 3.0 \) with the five values of \( w_0 \). As expected the evolution of \( (1 + w(a)) \) is slowly freezing toward \( w_0 \) for scale factors larger than 0.5 while there is significant evolution for scale factors smaller than 0.5. At increased deviations of \( w_0 \) from minus one the inverse power law cases increasingly deviates from the power law cases.

Figure 16 shows the evolution of \( 1 + w(a) \) with \( w_0 = -0.94 \) for the five different values of \( \beta_{p,i} \). For both the power law (solid) and the inverse power law (dashed) cases the degree of evolution decreases slightly with increasing \( \beta_{p,i} \). All of the inverse power law tracks lie above the power law tracks.

\[
\Omega = 0.7, \; H_0 = 70, \; w_0 = -0.94
\]

7.3 The Fundamental Constants

Beyond the cosmological parameters the fundamental constants provide important and generally under utilized information to discriminate between static and dynamic dark energy. Fundamental constants are pure numbers with no dimensions and therefore invariant to the system of units. The constants considered here are the proton to electron mass ratio, \( \mu \), and the fine structure constant, \( \alpha \). The standard model, with the cosmological constant as dark energy, predicts that the fundamental constants are temporally and spatially invariant. Quintessence and most other rolling scalar field cosmologies predict a temporal variation of the constants that is proportional to the deviation of \( w \) from minus one (Calabrese et al. 2011; Thompson 2012). This connection occurs because the scalar \( \phi \) that provides dark energy also interacts with other sectors beyond the gravitational sector.

In the absence of special and finely tuned symmetries it is very difficult to restrict a scalar field that interacts with gravity from interacting with the weak, electromagnetic and strong sectors as well eg. (Carroll 1998; Avelino et al. 2006). In this scenario the same field \( \phi \) that serves as dark energy also produces changes in the fundamental constants and particle physics parameters through interactions in sectors other than gravity. The values of the fundamental constants such as the proton to electron mass ratio \( \mu \) and the fine structure constant \( \alpha \) are set by the values of the particle physics parameters such as the Quantum Chromodynamic
The coupling to the constant \( c \) is \( \zeta_c \), which may be either positive or negative. The linear dependence of the variance of the constants on \( \phi \) can be thought of as the first term in a Taylor series expansion of a more complicated coupling. Since the limits on observed changes in the constants are on the order of \( 10^{-6} \) or less the linear dependence is a good approximation. Although \( \zeta_c \) is written as a single term it is actually a combination of the individual couplings to the QCD scale, the Higgs VEV and the Yukawa couplings as discussed above, in Thompson (2017), and at the end of Section 7.3.1. It is clear from eqn. (39) that once the beta function is defined and the boundary condition selected the evolution of the fundamental constants is completely defined. This is one of the significant advantages of the beta function formalism. Using the connection between \( w \) and \( \phi \) given in eqn. (39) the evolution of the fundamental constants can also be written as

\[
\frac{\Delta c}{c} = \zeta_c \int_1^a \sqrt{3 \Omega_\phi (w+1)} x^{-1} dx
\]  

(Calabrese et al. 2011; Thompson 2012) which shows that whenever \( w \) is different from minus one the fundamental constants are expected to vary making \( \mu \) and \( \alpha \) \( w \) in the universe and excellent discriminators between static and dynamic dark energy. The beta function, however, provides a much simpler method for predicting the evolution of the constants as a function of the cosmology and the dark energy potential.

Since the proton to electron mass ratio \( \mu \) has the most reliable and tightest restriction on its temporal variance it is used as the example in this discussion. The discussion for the fine structure constant \( \alpha \) is for the most part exactly the same for the substitution of \( \zeta_\alpha \) instead of \( \zeta_c \) in eqn. (39) or (40). In both cases the coupling \( \zeta_\mu, \alpha \) is considered a constant. The evolution of \( \mu \) as a function of the scale factor is simply

\[
\frac{\Delta \mu}{\mu} = \zeta_\mu \left( \sqrt{-2 \beta_p \ln(a) + \phi_0^2 - \phi_0} \right), \quad \zeta_\mu \left( \sqrt{2 \beta_0 \ln(a) + \phi_0^2 - \phi_0} \right)
\]

for the power law, \( \beta_p \) or the inverse power law, \( \beta_0 \), dark energy potentials.

As an example \( \zeta_\mu \) is set to \( 10^{-6} \) and \( \beta_p,i \) to 3.0. Figure 18 shows the evolution for both the power law (solid lines) and the inverse power law (dashed) line cases. Since the coupling constant can be either positive or negative the sign of \( \Delta \mu/\mu \) is not a discriminator between the power and inverse power law potentials unless the sign of the coupling is somehow determined. The sensitivity to \( w_0 \) is evident in the figure. Figure 19 shows the evolution of \( \Delta \mu/\mu \) with \( w_0 \) held constant at \(-0.94\) with the five different values of \( \beta_p,i \). As expected the evolution of \( \Delta \mu/\mu \) is largely insensitive to \( \beta_p,i \) since it is proportional to \( \Delta \phi \) which is also largely independent of the power of the power laws as shown in fig. 7.

7.3.1 Observational Constraints on \( \Delta \mu/\mu \)

Observations of molecular absorption lines from cold gas along the line of sight to distant quasars provide the constraints on \( \Delta \mu/\mu \). Changes in \( \mu \) alter the energy levels of molecules according to the quantum numbers of the upper and lower states of the transition (Thompson 1973) changing the wavelengths of the transitions in a manner that can not be mimicked by a redshift. The majority of constraints arise from the observation of molecular hydrogen absorption lines of the Lyman and Werner bands at redshifts greater than 2. More recently radio observations of methanol and ammonia absorption lines at redshifts less than one have provided more stringent constraints. The tightest constraints come from methanol lines in the spectrum of PKS1830-211 at a redshift of 0.88582 by Bagdonaite et al. (2013) and Kanekar et al. (2013) finding \( \frac{\Delta \mu}{\mu} = (-2.9 \pm 5.7) \times 10^{-6} \). Concerns about common lines of sight has raised the \( 1 \sigma \) error to \( \pm 10^{-7} \) which will be used here. Figure 18 shows all of the measurements to date. All of the measurements at redshifts greater than one are not optical observations of molecular hydrogen redshifted into the visible region. The radio constraints at redshifts less than one are not visible at the scale of this plot.

The PKS1830-211 constraint is shown in fig. 21 at expanded scale to make the constraint visible. For \( \beta_p,i = 3.0 \) and \( \zeta_c = 10^{-6} \) the constraint requires \( (w_0 + 1) \) to be 0.02 or less and is of course consistent with the \( \Lambda \)CDM value of zero. If the error bar was centered on zero then \( (w_0 + 1) \) would have to be less than 0.02.

Any constraint on \( \Delta \mu/\mu \) or \( \Delta \mu/\mu \) can be met by either adjusting a cosmological parameter \( (w_0 + 1) \) or a particle...
The allowed and forbidden regions in the \( \zeta_\mu \) vs \( (w_0+1) \) plane imposed by the limit on \( \frac{\Delta \mu}{\mu} \) shown in fig. 21. The combined limits on the fractional variation of \( \Lambda_{QCD} \), the Higgs Vacuum Expectation Value, \( \mu \), and \( \nu \) then place limits on \( \frac{\Delta \alpha}{\alpha} < 7.9 \times 10^{-5} \) and the sum \( (\frac{\Delta \mu}{\mu} + \frac{\Delta \alpha}{\alpha}) < 8.0 \times 10^{-5} \) that can not be duplicated by laboratory measurements (Thompson 2017).

8 Relevant but not directly observable parameters

There are several cosmological parameters that are relevant but not directly observable. Here three parameters, the time derivative of the scalar field, the dark energy density, and the dark energy pressure, are calculated as functions of the scale factor \( a \).

8.1 The Evolution of the Time Derivative of the Scalar

The time derivative of the scalar \( \phi \) is an important cosmological parameter that appears in both the dark energy pressure and density equations. Since the beta function is the derivative of the scalar with respect to the natural log of the scale factor the time derivative of the scalar is simply the Hubble parameter times the beta function.

\[
\dot{\phi} = a \frac{d\phi}{da} = \beta H = -\frac{1}{2} \beta W
\]

Figure 23 shows the evolution of \( \dot{\phi} \) with respect to the scale factor \( a \). Since \( H(a) \) is essentially invariant to either the power of the dark energy potential or the value of \( w_0 \) the dependence on \( w_0 \) is entirely due to the beta functions' dependence on \( \beta_{p,i} \) and \( w_0 \). Figures 4 and 5 show that the main dependence is on \( w_0 \) as opposed to \( \beta_{p,i} \).

An analytic expression for \( \phi \) is obtained by multiplying either eqn. 23 or 24 by \(-\frac{1}{2}\) with \( \beta(\phi) \) given by the appropriate functions in eqns. 6 and 7. An alternative is to use the functions in eqns. 45 and 46 for \( H(a) \) resulting in

\[
\phi_p(a) = \frac{\beta_p}{2p_v(a)} W_p(a)
\]
for the power law potential and

$$\dot{\phi}_i(a) = -\beta_{\phi,i} W_i(a)$$  \hspace{1cm} (44)

for the inverse power law. In eqns (43) and (44) the approximate forms of $H(a)$ can also be used. Using the Gamma function forms is slightly more accurate.

It is obvious from fig. 24 that although there is significant early time evolution of $\dot{\phi}$ the late time evolution is a slow approach to zero. This indicates that power and inverse power law quintessence predicts very small time variations of the fundamental constants at the present time. This is a general characteristic of most freezing cosmologies where $w$ is initially different from minus one and evolves toward minus one with time. The power law values of $\dot{\phi}$ are negative since the scalar is decreasing while the inverse power law values are positive since the scalar is increasing with time for this case.

8.2 The Evolution of the Dark Energy Density and Pressure

From eqn. (21) it is clear that

$$\rho_\phi = 3H^2 - \rho_m = 3H^2(a) - \frac{\rho_{m0}}{a^3}$$  \hspace{1cm} (45)

which is consistent with eqn. 3.8 from Cicciarella and Pieroni (2017) which gives the total potential with mass as

$$V = \rho_\phi = \frac{1}{2} \dot{\phi}^2 = 3H^2 - \rho_m = \frac{1}{2} \dot{\phi}^2$$  \hspace{1cm} (46)

Figure 22 shows the evolution of the densities using the Gamma function equations (52) and (53) to compute $H(a)$. The matter density, shown by the dash dot line, is also plotted to indicate the crossover from matter to dark energy dominated evolution. For values of $(w_0 + 1)$ close to zero the power and inverse power law plots nearly overlap but as $(w_0 + 1)$ diverges from zero the inverse power law cases have slightly higher densities at scale factors less than 0.5. All cases converge to the boundary condition on the density at a scale factor of one. Most of the evolution of $\rho_\phi$ occurs at scale factor less than 0.3, consistent with the previous plots of the evolution of $(w + 1)$ in fig. 16 and $\dot{\phi}^2$ in fig. 23.

The dark energy pressure is also given in eqns. (5)

$$p_\phi(a) = \dot{\phi}^2 - 3H(a) = \frac{\rho_{m0}}{a^3}$$  \hspace{1cm} (47)

Figure 25 shows the evolution of the dark energy pressure. As was the case with the dark energy density the inverse power law dark energy potential case has more evolution than the power law case, particularly for $w_0$ values further from minus one. Both the dark energy density and the dark energy potential are not significantly dependent on the power $\beta_{\phi,i}$ for a given value of $w_0$.

9 SUMMARY

The beta function formalism is demonstrated using the example of the quintessence cosmology with power and inverse power law dark energy potentials. Simple beta functions were found, $\beta(\phi) = \pm \frac{\beta_{\phi,i}}{\rho_m}$ where $\beta_{\phi,i}$ is a constant equal to the power. The minus sign applies to the power law, $p$, and the positive sign to the inverse power law, $i$. From the beta functions the scalar $\phi$, as a function of the scale factor $a$ is calculated with a boundary condition supplied by the current value of the dark energy equation of state $w$. This provides an easy transition from functions of the generally unobservable scalar $\phi$ to functions of the easily observable scale factor $a = \frac{1}{1+z}$. Beta potentials are produced that reproduce the model dark energy potentials to better than one percent. These potentials produce actions that accurately represent the actions with the model potentials. The extra beta function combined with the quintessence equations for the dark energy pressure and density plus the usual cosmological equations provide the means to calculate an analytic function for the super potential, $W = \frac{-1}{2}H$ where $H$ is the Hubble parameter.

The super potential automatically provides the Hubble parameter as a function of the scale factor. It is found that the Hubble parameter is essentially insensitive to the power of the potential or $w_0$ and includes the $\beta_{\phi,i} = 0$ case which corresponds to the $\Lambda$CDM cosmology. This demonstrates that the Hubble parameter is not a good indicator to discriminate between static and dynamical dark energy. It is confirmed that the transition from matter dominated to dark energy dominated epochs occurs at the proper time and that the evolution of the Hubble parameter matches a randomly selected current list of $H(z)$ measurements. The
measurements also provide a best fit value of $H_0$ for the selected data set.

Additional observable parameters, the dark energy equation of state, and the variation of the fundamental constants $\mu$ and $\alpha$ in a rolling scalar field are calculated. The limits on the variation of the constants imposes allowed and forbidden regions in the two dimensional $w + 1$, $\zeta_{\mu}$ plane in a balance between cosmological and elementary particle physics parameters. Analytic expressions for three not directly observable parameters, $\dot{\phi}$, $\rho_{\phi}$ and $p_{\phi}$ are also calculated. It is generally noted that the parameter evolution is more sensitive to the current value of the dark energy equation of state $w_0$ than the power of the potentials $\beta_{p,i}$.

This work demonstrates of the power of the beta function formalism to produce accurate predictions for comparison with observation. The formalism is expandable to other forms of the dark energy potential and other cosmologies which will be the subject of future work.

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