An analysis on the convergence of equal-time commutators and the closure of the BRST algebra in Yang-Mills theories

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Abstract

In renormalizable theories, we define equal-time commutators (ETC’S) in terms of the equal-time limit and investigate its convergence in perturbation theory. We find that the equal-time limit vanishes for amplitudes with the effective dimension $d_{\text{eff}} \leq -2$ and is finite for those with $d_{\text{eff}} = -1$ but without nontrivial discontinuity. Otherwise we expect divergent equal-time limits. We also find that, if the ETC’s involved in verifying an Jacobi identity exist, the identity is satisfied. Under these circumstances, we show in the Yang-Mills theory that the ETC of the 0 component of the BRST current with each other vanishes to all orders in perturbation theory if the theory is free from the chiral anomaly, from which we conclude that $[Q, Q] = 0$, where $Q$ is the BRST charge. For the case that the chiral anomaly is not canceled, we use various broken Ward identities to show that $[Q, Q]$ is finite and $[Q, [Q, Q]]$ vanishes at the one-loop level and that they start to diverge at the two-loop level unless there is some unexpected cancellation mechanism that improves the degree of convergence.

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1 Introduction

In the first attempts to develop quantum field theory, the equal-time canonical commutation relations played an important role in the canonical quantization procedure. It, however, turned out that equal-time commutators (ETC’s) are singular in general and that it is possible to formulate relativistic quantum field theory without assuming the existence of the equal-time canonical commutation relations among interacting quantum fields [1, 2]. Besides, ETC’s have lost more and more their meanings for phenomenological applications in renormalizable quantum field theories.

Nevertheless, it is certainly wrong to underestimate the phenomenological as well as theoretical usefulness of ETC’s. They have been a fundamental concept in current algebra, and have taken undoubtedly a special place in investigating and understanding of anomalies [3].

During the mathematization of anomalies, especially soon after the works of Stora [4], Zumino [5] and Baulieu [6] in connection to the chiral anomalies, Faddeev [7] succeeded to reveal the relation between the gauge group cohomology and the anomalous Schwinger term [8] in the commutator algebra of the Gauß law operators in chiral Yang-Mills theories. Based on the cohomological technique, he predicted an explicit form of the anomalous Schwinger term, and later many concrete calculations [9] verified his mathematical result [7]. In the course of these explicit calculations, one re-discovered [10, 11] the old result [12, 13, 14] that various ETC’s may violate the Jacobi-identity, which rose the question of whether there exist three-cocycle anomalies, i.e., Jacobi-identity violating, anomalous equal-time commutation relations among the generators of a symmetry transformation. Although some applications of three-cocycles were found in quantum mechanics [15], that question in field theory has remained unanswered [3].

An anomaly has to satisfy a certain algebraic property, the Wess-Zumino consistency conditions [16]. It were the Wess-Zumino consistency conditions on the chiral anomalies, because of which the so-called descent equations [4, 5, 6], discovered as one of consequences from the mathematization of the chiral anomalies mentioned above, have an application in

\[^1\text{See ref. [3].}\]
physics. The Wess-Zumino consistency conditions (which can be rigorously established in renormalization theory \[17\] by applying the action principle \[18\] and the normal product algorithm \[19, 20\]) have been a basic tool in investigating the algebraic properties of different types of anomalies.\(^3\)

The Jacobi identity of ETC’s can also be used to derive the Wess-Zumino consistency conditions on anomalous Schwinger terms, and many applications of this idea have been reported in recent years \[25\]–\[28\]. In particular, it has been shown \[26, 27\] that the Jacobi identity for the BRST algebra in a BRST quantized theory leads to the Hamiltonian descent equations. It seems that the algebraic approach to anomalies based on the Jacobi identity of ETC’s is an alternative way to investigate the algebraic structure of anomalies. Unfortunately, this approach suffers from the essential disadvantage, because ETC’s do not always exist, as we emphasized at the beginning. But this general remark does not necessarily prevent us from dealing with ETC’s. Before applying this idea, we have to check whether the ETC’s involved in verifying an Jacobi identity do really exist or not, and if not, at which order in perturbation theory they start to be ill-defined. Once their existence is established, even in lower orders in perturbation theory, the Jacobi identity of ETC’s can play a powerful tool to study the algebraic nature of anomalies, as experienced in many examples \[25\]–\[28\].

Our concern in this paper is as follows:

(i) How to compute perturbatively ETC’s in a renormalizable field theory,

(ii) under what conditions they can be given a well-defined meaning, and

(iii) whether the Jacobi identity of ETC’s is satisfied.

Obviously, these three points are closely related, and there are partially answers to them. In particular, the Bjorken-Johnson-Low (BJL) method \[29, 13\] to compute ETC’s has been used over more than two decades. In many applications of the BJL method, one encountered “divergent” ETC’s \[13\]. The appearance of those divergent ETC’s certainly reflects the above mentioned fact that ETC’s are singular in general. However, it has not been carefully investigated whether there is one-to-one correspondence between the

\(^2\)See also refs. \[21\], and references therein. For renormaliation of descent equations, see ref. \[22\].

\(^3\)See, for instance, ref. \[23, 24\]
appearance of a divergent ETC in the BJL method and the non-existence of the ETC.

In section 2, we will begin by formally discussing the singular nature of the equal-time limit, and then we will investigate its convergence in momentum space by considering the BJL method carefully and introducing the notion of the effective dimension $d_{\text{eff}}$. It will turn out that the equal-time limit vanishes for amplitudes with $d_{\text{eff}} \leq -2$ and is finite for tree-like amplitudes (those without nontrivial discontinuity) with $d_{\text{eff}} = -1$. Amplitudes with $d_{\text{eff}} \geq 1$ yield divergent ETC’s unless there is some unexpected cancellation mechanism. We will then consider double ETC’s and the Jacobi identity of ETC’s in section 3, and will find that the double ETC’s have to satisfy for their existence similar power counting rules as the single ETC’s. It will be shown that, if the convergence condition is satisfied for the ETC’s involved in verifying an Jacobi identity, the Jacobi identity is automatically satisfied. It should be emphasized that the ECT’s defined in terms of the equal-time limit is generally deformed in that they do not always satisfy the product rule.

Under those circumstances, it is still possible to use ETC’s to analyze quantum symmetries and their anomalies, especially BRST symmetries and anomalies. This is because “0” of an ETC can be shown to all orders (we have to show that $d_{\text{eff}} \leq -2$) and the the diagrams that are influenced by an anomaly and yield a nontrivial Schwinger term to “0”, can behave like tree diagrams in the lowest order. In section 4, we will apply the power counting rules in the Yang-Mills theory to investigate the existence of the ETC’s of the BRST current $J_{\mu}$. We will find that $[J_{\mu}(x), J_{\nu}(0)]_{\text{ETC}}$ only for $\mu = \nu = 0$ vanishes to all orders in perturbation theory if the theory is free from the chiral anomaly so that we conclude $[Q, Q] = 0$, where $Q$ is the BRST charge. (For other components, they are power-counting divergent.) We will also find that in anomalous Yang-Mills theories the $O(\hbar^2)$ term of $[Q, Q]$ is finite and the $O(\hbar^4)$ term of $[Q, [Q, Q]]$ vanishes, but they are ill-defined in higher orders. This justifies the assumption of ref. [27] from which a set of various consistency conditions on the anomalous Schwinger terms in chiral Yang-Mills theories has been derived. We understand this way why the explicit computations of the algebra of the Gauß law operators in the one-loop order [4] have yielded finite results.

Finally, we would like to emphasize that there are no fundamental principles that justify renormalization or regularization of ETC’s. A theory with ill-defined ETC’s is not
necessarily sick. This is so for the ultraviolet as well as infrared singularities. Our power counting rules concern the ultraviolet behaviors only, and so ETC’s which satisfy our convergence condition might still suffer from infrared singularities. The problem of these infrared singularities is beyond the scope of the present paper, and should be carefully investigated elsewhere.

2 Convergence criterion for the equal-time limit

2.1 Singularities in the equal-time limit

We consider a renormalizable theory, and assume that the theory is renormalized in some renormalization scheme, e.g., the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) scheme [30], the minimal subtraction (MS) scheme [31, 32, 33], etc. For definiteness, we work in $D = 4$ dimensions. $\phi_A(x)$ stands for a local, renormalized, elementary Heisenberg field, and $A$ includes Lorentz and also those corresponding to internal degrees of freedom.

The time-ordered product, T-product, of $\phi$’s

$$T \phi_{A_1}(x_1)\phi_{A_2}(x_2)\cdots\phi_{A_N}(x_N)$$

$$\equiv \phi_{A_{j_1}}(x_{j_1})\phi_{A_{j_2}}(x_{j_2})\cdots\phi_{A_{j_N}}(x_{j_N}) \text{ for } x^0_{j_1} > x^0_{j_2} > \cdots > x^0_{j_N}$$

(1)

is defined in terms of the Green’s function $\langle \alpha | T \phi_{A_1}(x_1)\phi_{A_2}(x_2)\cdots\phi_{A_N}(x_N) | \beta \rangle$, which is given a well-defined meaning as a temperate distribution by assumption. Similarly, composite operators are well-defined in perturbation theory: In the BPHZ renormalization scheme, they are defined in terms of Zimmermann’s normal products [19, 20], which can be appropriately extended for MS scheme [34]. As in the case of T-products, their precise meaning in perturbation theory is given by Green’s functions containing those composite operators.

4$x^\mu$ ($\mu = 0, 1, 2, 3$) are coordinates, and we employ the metric convention $(+, -, -, -)$.

5Its Grassman parity is assumed to be even for simplicity.

6See also references cited in ref. [35].
Let denote $A(x)$ and $B(y)$ renormalized, local operators, elementary or composite. We assume that the theory is invariant under Lorentz transformations as well as translations and rotations, and recall the definition of the T-product of $A(x)$ and $B(y)$ to consider the equal-time limit,

$$\lim_{x^0 \to 0^+} \{ T A(\vec{x}, x^0) B(0) - T A(\vec{x}, -x^0) B(0) \}. \quad (2)$$

If the limit has a well-defined meaning, we may define an ETC of $A(x)$ and $B(y)$. Note that because of local commutativity,

$$[A(x), B(y)] = 0 \quad \text{for} \quad (x-y)^2 < 0, \quad (3)$$

the expressions (2) are different from zero only in the region $\vec{x} \simeq \vec{0}$. Therefore, it may be convenient to apply the operator product expansion method \cite{36, 37} to consider the equal-time limits.

As known, the Wilson coefficients appearing in the operator expansion method are temperate distributions in general and so the equal-limits do not always have a sensible meaning. Therefore, in order for the ETC to exist, we require that the smeared expression,

$$\hat{G}(x^0) = \begin{cases} G(x^0) \text{ for } x^0 > 0 \\ -G(-x^0) \text{ for } x^0 < 0 \end{cases}, \quad (4)$$

$$G(x^0) = \int d^3 \vec{x} \chi(\vec{x}) < \alpha | \{ T A(\vec{x}, x^0) B(0) - T A(\vec{x}, -x^0) B(0) \} | \beta >,$$

is a function of $x^0$ and has an well-defined equal-time limit, where $< \alpha |$ and $| \beta >$ denote arbitrary states \footnote{This is the most conservative standpoint on the equal-time limits. See ref. \cite{14} for a more relaxed treatment of the equal-time limits.}

There are two types of singularities in renormalizable field theories that prevent $\lim_{x^0 \to 0} \hat{G}(x^0)$ to be well-defined \footnote{We do not consider infrared singularities throughout this paper as already announced.}. The one is harmless, and is related to the ambiguity of Green’s functions at the same points. Because of this arbitrariness, it is always possible to add to two-point Green’ functions a quasi-local distribution of the form \cite{38},

$$P(\partial/\partial x) \delta^4(x-y),$$

where $P(\partial/\partial x)$ is some polynomial in $\partial/\partial x$ with constant coefficients.
Therefore, \( G(x^0) \) may involve the singularities like \( \delta(x^0 = 0) \). The arbitrariness mentioned here corresponds to different choices of adding local counterterms. Therefore, the \( \delta \)-singularities can be canceled by local counterterms and are harmless.

The real singular nature in the \( x^0 \to 0 \) limit originates from the fact that the quantum fields smeared in the spatial coordinates only (as we have done above) may still suffer from some singularities \([1, 2]\). These singularity has a non-local nature like

\[
\lim_{x^0 \to 0} \tilde{G}(x^0) \sim \frac{\ln^p x^0}{(x^0)^r}.
\]

in the equal-time limit.

The absence of such singularities can not be ensured in general. But there are many examples in lower orders in perturbation theory that suggest the existence of various equal-time commutators in lower orders in perturbation theory. Those computations have been performed in momentum space by using the BJL method \([10]\). It is therefore appropriate to carefully investigate the relation between the divergences that appear in the Bjorken limit and the singularities above.

### 2.2 Power counting rules

We use momentum space representation to consider the equal-time limit (2) \([11]\):

\[
\tilde{G}(\vec{p}, x^0) = (+) \int_{-\infty}^{+\infty} \frac{dp^0}{(2\pi)^3} \left( e^{-ip^0 x_0} - e^{+ip^0 x_0} \right) T(\vec{p}, p^0) \quad \text{for} \quad x^0 > (\text{<}) 0,
\]

\[
T(\vec{p}, p^0) = \int d^4x e^{ip\cdot x} <\alpha| T A(x) B(0) |\beta> ,
\]

where the Fourier transform of the equal-time limit, i.e.,

\[
\int \frac{d^3\vec{p}}{(2\pi)^3} (\exp i\vec{p} \cdot \vec{x}) \lim_{x^0 \to +0} \tilde{G}(\vec{p}, x^0) ,
\]

\(\text{In addition to this problem, one could encounter another unwelcome situation that there exist some inequivalent representations of the canonical commutation relations, which may be related to Haag’s theorem }[2]\)

\(\text{It is certainly possible to compute the ETC’s directly in coordinate space. The computations might become even more economic if one could apply the technique of the asymptotic expansions (see, for instance, ref. }[35]\).

\(\text{We suppress the dependence of the external states in the amplitude.}\)
is exactly equal to (2). One can convince oneself from (6) that only the odd part of amplitude,

\[ T^{\text{odd}}(\vec{p}, p^0) \equiv \frac{1}{2} \{ T(\vec{p}, p^0) - T(\vec{p}, -p^0) \}, \]  

(8)

contributes to (6). Our task here is to find a sufficient condition on the amplitude \( T(p) \) for \( \tilde{G}(\vec{p}, x^0) \) to be well-defined in the \( x^0 \to 0 \) limit.

Feynman amplitude may be analytically continued into the complex plane. So we assume that \( T^{\text{odd}}(\vec{p}, z) \) (which is supposed to be real in some interval on the real axis) is an analytic function of \( z \) with possible singularities no worse than some discontinuities on the real axis and poles. We may also assume that all the external state momenta as well as the spatial components \( p_i \) of \( \tilde{G}(\vec{p}, x^0) \) are kept finite so that all the poles in \( z \) are located in a finite domain on the complex plane, and that they do not lie on the branch cuts. Accordingly, we extend the \( z \) integral to the lower half plane for the first term in the parenthesis and to the upper half plane for the second term because \( x^0 > 0 \). We then write the complex integrals, respectively, as a sum of two integrals:

\[
\int_{-\infty}^{+\infty} \frac{dz^0}{(2\pi)} e^{-i z x_0} T^{\text{odd}}(\vec{p}, z) \\
= \int_{C_R^+(-)} dz^0 (2\pi) e^{-i z x_0} T^{\text{odd}}(\vec{p}, z) + \int_{L_R^+(-)} dz^0 (2\pi) e^{-i z x_0} T^{\text{odd}}(\vec{p}, z),
\]

where the contours \( C_R^+ \), \( L_R^+ \) are shown in fig. 1(a). (Recall that the contours along the real axis must be carefully chosen so that it corresponds to the \( \epsilon \) prescription of the Feynman propagators.)

We first consider the second integral in the \( R \to \infty \) limit, and require that

\[
\lim_{R \to \infty} \int_{L_R^+(-)} \frac{dz^0}{(2\pi)} e^{-i z x_0} T^{\text{odd}}(\vec{p}, z) = 0. \]

(9)

This limit exists if the amplitude \( T^{\text{odd}}(\vec{p}, z) \) vanishes as \( |z| \) approaches infinity and \( |\partial T^{\text{odd}}(\vec{p}, p_0) / \partial p_0| \) decreases at most like \( |p_0|^{-\delta + 1} \) with \( \delta \geq 0 \) as \( |p_0| \to \infty \). For non-oscillating amplitudes (which is the case in general), it means

\[
\lim_{|p_0| \to \infty} T^{\text{odd}}(\vec{p}, p_0) \leq K |p_0|^{-\delta}, \]

(10)

where \( K \) is some positive (real) number.
To investigate the integrals on $C^+_R$ and $C^-_R$, we further divide the integration contours; closed contours which do not contain the branch cuts but the poles and those along the branch cuts. For the first integrals (i.e., those on the closed contours), we are allowed to interchange the equal-time limit and the $z$ integration. This is because the $z$ integrals produce only a sum of functions assuming the form like

$$(x^0)^m \exp i( f(\vec{p})x^0) , \ m = 0, 1, \cdots.$$ 

We therefore set $x^0$ equal to zero and add the contributions from the two contour integrations to obtain

$$- \oint_{C_R} \frac{dz}{(2\pi)} \ T^{\text{odd}}(\vec{p}, z) , \tag{11}$$

where the contour $C_R$ is shown in fig. 1(b), where the fade lines on the real axis indicate the branch cuts. As for the integrals along the branch cuts, we use the formula

$$\lim_{\epsilon \to +0} \left\{ T^{\text{odd}}(\vec{p}, p_0 + i\epsilon) - T^{\text{odd}}(\vec{p}, p_0 - i\epsilon) \right\} = 2i \Im \lim_{\epsilon \to +0} T^{\text{odd}}(\vec{p}, p_0 + i\epsilon) , \tag{12}$$

and arrive at the final expression:

F.T. \[ \lim_{x^0 \to \pm 0} <\alpha|\{ T \mathcal{A}(\vec{x}, x^0) \mathcal{B}(0) - T \mathcal{A}(\vec{x}, -x^0) \mathcal{B}(0) \} |\beta > \] \[ = \lim_{x^0 \to +0} \lim_{R \to \infty} \{ \int_{I^+_R} \frac{dp_0}{2\pi} e^{-ip_0x^0} + \int_{I^-_R} \frac{dp_0}{2\pi} e^{+ip_0x^0} \} \Im \lim_{\epsilon \to +0} T^{\text{odd}}(\vec{p}, p_0 + i\epsilon) \tag{12} \]

where $I^+_R$ denote the line intervals along the cuts, and we have assumed (9). If the last term vanishes either in the $R \to \infty$ or $x^0 \to +0$ limit, we are left with the first integral, which is the formula derived by Johnson and Low \[ \text{[13]} \] in 1966.

Since the first contour integral of (12) is independent of $R$ if $R$ is sufficiently large so that all the poles are encircled, it is sufficient for the equal-time limit of left-hand side of (12) to be defined that the last term has an well-defined meaning. However, we observe that the integrals along the branch cuts are infinite in general, even if the Bjorken limit condition,

$$\lim_{p_0 \to \infty} \left[ p_0 \Im T^{\text{odd}}(p) \right] = \text{finite} , \tag{13}$$
is satisfied, because

$$\lim_{x^0 \to 0} \int_{-\infty}^{\infty} dp_0 f(p_0) \cos p_0 x^0 = \infty,$$

if \( f(p_0) \) behaves like \( p_0^{-1} \) as \( p_0 \to \infty \). If the Bjorken limit (13) vanishes like \( p_0^{-\delta} \) (\( \delta > 0 \)), we may change the \( x^0 \to +0 \) and \( R \to \infty \) limits to obtain

$$+ 2i \lim_{x^0 \to +0} \lim_{R \to \infty} \left\{ \int_{I^+} \frac{dp_0}{2\pi} + \int_{I^-} \frac{dp_0}{2\pi} \right\} \text{Im} \lim_{\epsilon \to +0} T^{\text{odd}}(\vec{p}, p_0 + i\epsilon),$$

which cancels the integral (11) along the branch cuts on \( C_R \). The integral on the arc of \( C_R \) vanishes in the \( R \to \infty \) limit so that the equal-time limit in question vanishes.

Therefore, the convergence condition of the equal-time limit of the integrals along the branch cuts is a very strong condition on the Feynman amplitude, and may be satisfied only for tree-like amplitudes, i.e., those without nontrivial discontinuity, in accord with the general statement [1, 2] that ETC’s do not always exist. As for tree-like diagrams, the convergence condition can be directly translated to a power counting rules for Feynman diagrams because the large \( p^0 \) behavior of an amplitude is basically the same as its large \( p^\mu \) behavior. Therefore, the equal-time limit (12) exists if the (tree-like) amplitude \( T^{\text{odd}} \) has a negative canonical dimension, for instance. Clearly, not every amplitude with negative dimension contributes, and the “lower limit” has already been found above. So we conclude that the amplitudes having

\[
\begin{align*}
\text{(i)} & \quad d_{\text{eff}}[T(p)] = -1 \\
\text{(ii)} & \quad d_{\text{eff}}[T(p)] \leq -2,
\end{align*}
\]

can give well-defined ETC’s if there is no nontrivial discontinuity, and those with

\[
\begin{align*}
\text{(i)} & \quad d_{\text{eff}}[T(p)] = -1 \\
\text{(ii)} & \quad d_{\text{eff}}[T(p)] \leq -2 ,
\end{align*}
\]

do not contribute to the equal-time limit (12) to all orders in perturbation theory, where the effective dimension \( d_{\text{eff}} \) is the dimension which one obtains form the canonical dimension if we do not count the powers of the spatial components \( p_i \) as well as the external state momenta that are multiplied with the amplitudes. If \( d_{\text{eff}} = 0 \) for an amplitude, we assign \( d_{\text{eff}} = -1 \) to the amplitude because only the odd part \( T^{\text{odd}} \) contributes to (12). For amplitudes with \( d_{\text{eff}}[T(p)] \geq 1 \), we can not expect an well-defined equal-time limit in general.
Note that “0” on the left-hand side of (12) can be proven in perturbation theory and has an well-defined meaning. This is why BRST symmetries can be investigated in terms of ETC’s because the BRST algebra is abelian, as we will demonstrate in section .

3 The violation of the Jacobi identity

There are many examples of Jacobi-identity violating ETC’s [10]–[14]. Its origin is of course the singular nature of the equal-time limit. Here we would like to investigate this problem in coordinate space as well as momentum space in some detail.

To discuss the Jacobi identity, we have to consider double equal-time commutators which always involves two independent equal-time limits. Because of the singular nature of the equal-time limit, the order of these two limits can not be changed in general.

3.1 An well-known example revised

In perturbation theory, we are mostly dealing with linear operators, which associate by definition and so have to satisfy the Jacobi identity

\[ \left[ A(x), \left[ B(y), C(z) \right]_{ETC} \right] + \text{cyclic permutations} = 0. \] (16)

In fact, Green’s functions are computed in perturbation theory, regardless of the order of multiplication of linear operators, whether they are elementary or composite. So why there are Jacobi-identity violating ETC’s? This is the question we will address below.

We consider a double ETC, \([A(x), [B(y), C(z)]_{ETC}]_{ETC}\), along with its cyclic permutations. If

\[
J_{ETC}(A(x), B(y), C(z)) \equiv \left[ A(x), [B(y), C(z)]_{ETC} \right]_{ETC} + \left[ C(z), [A(x), B(y)]_{ETC} \right]_{ETC} + \left[ B(y), [C(z), A(x)]_{ETC} \right]_{ETC}
\] (17)

vanishes, the Jacobi identity is satisfied. We have written all the three terms to emphasize that the orders of the equal-time limits for three double ETC’s are different. This is why (16) does not automatically imply the Jacobi identity of ETC’s, on the one hand, and on the other hand, it suggests that the Jacobi identity of ETC’s is satisfied if we can change
the order of the different equal-time limits. This is exactly the origin of the violation of Jacobi identity, as we will see this more explicitly in a simple example below.

We consider an well-known Jacobi-identity violating ETC \[10, 11\], the equal-time commutators among the vector and axial vector currents, \(V^\mu(x)\) and \(A^\mu(x)\) \((\mu = 0, \cdots, 3)\), in the theory of a free massless fermion field in \(D = 4\) dimensions, to illustrate the observation above.

We denote the spinor field by \(\psi\), and follow the Bjorken-Drell notation for the gamma matrices \(\gamma^\mu\) and \(\gamma^5\) and also the singular functions. The currents

\[
V^\mu(x) \equiv : \bar{\psi}(x)\gamma^\mu\psi(x) : , \quad A^\mu(x) \equiv : \bar{\psi}(x)\gamma_5\gamma^\mu\psi(x) : .
\]

are normal ordered as indicated by \(:\) , and we do not need any other specification of the regularization to compute commutators because the products of the singular functions we will encounter are well-defined distributions \[12\]. We then consider

\[
J_{ETC}( A^0(x), V^i(y), V^j(z)) , \quad i, j = 1, 2, 3 ,
\]

where \(J_{ETC}\) is defined in (17) (zero of which means the Jacobi identity).

To calculate ETC’s for the present case, we use the Wick theorem to derive

\[
\begin{align*}
[.: \bar{\psi}(x)\Gamma^\alpha\psi(y) : , .: \bar{\psi}(z)\Gamma^\beta\psi(w) : ]
&= -:.\bar{\psi}(z)\Gamma^\beta(-i)S(w - x)\Gamma^\alpha\psi(y) : + .: \bar{\psi}(x)\Gamma^\alpha(-i)S(y - z)\Gamma^\beta\psi(w) : \\
&= -Tr\Gamma^\alpha S^+(y - z)\Gamma^\beta S^-(w - x) + Tr\Gamma^\alpha S^-(y - z)\Gamma^\beta S^+(w - x) ,
\end{align*}
\]

where

\[
S^{\pm}(x - y) = \pm[\psi^{(\pm)}(x), \bar{\psi}^{(\mp)}(y)] , \quad S(x - y) = iS^+(x - y) - iS^-(x - y) ,
\]

\[
S^{\pm}(x) = i\gamma^\mu\partial_\mu \Delta^{\pm}(x) ,
\]

\[
\Delta^{\pm}(x) = -(\frac{1}{4\pi^2})\frac{1}{x^2 \mp i0x^0} = \frac{1}{(2\pi)^3} \int d^4p \theta(\pm) \delta(p^2) e^{-ip \cdot x} .
\]

\[12\] The calculation based on the BJL method requires a regularization because there are superficially divergent diagrams.

\[13\] We use the same symbol for the commutators and anticommutators.
In the limit, \( x = y \to z = w \), we encounter the products of the singular functions of the type \( S^-(x) \Gamma S^+(x) \) which, in contrast to the product of two propagators, are well-defined distributions [38] as announced. For instance,

\[
\Delta^-(x) \Delta^+(x) = \frac{-i}{8 \pi} \int \frac{d^4p}{(2\pi)^4} \theta(p^0) \theta(p^2) \exp(-i|\vec{p}|x^0 + i\vec{p} \cdot \vec{x}) ,
\]

from which we obtain its equal-time limit

\[
\lim_{x^0 \to 0} \Delta^-(x) \Delta^+(x) = \frac{i}{16 \pi^2 x^0} \delta^3(\vec{x}) \cdots .
\]

From similar calculations we find:

\[
\lim_{x^0 \to 0} S^-(x) \Gamma S^+(x) = \frac{-i}{96 \pi^2} \left[ \gamma^0 \Gamma \gamma^0 \left( \frac{6}{x^0_0} - \frac{1}{x^0} \nabla^2 + \cdots \right) + \left\{ \gamma^0 \Gamma \gamma^j + \gamma^j \Gamma \gamma^0 \right\} \left( -\frac{2}{x^2_0} + \nabla^2 \right) \partial_j + \cdots \right] + \gamma^j \Gamma \gamma^k \left( -\frac{2}{x^2_0} \delta_{jk} + \frac{1}{x^0} (\nabla^2 \delta_{jk} + 2 \partial_j \partial_k) + \cdots \right) \delta^3(\vec{x}) .
\]

Using this formula, we find that, in accord with the known results [10], the equal-time commutators between \( A^0(x) \) and \( V^i(y) \), and between \( V^i(y) \) and \( V^j(z) \) are canonical, i.e., identical to \( i\times \) the Poisson brackets:

\[
[A^0(x), V^i(y)]_{ETC} = 0 ,
\]

\[
[V^i(x), V^j(y)]_{ETC} = 2i \epsilon_{ijk} A^k \delta^3(\vec{x} - \vec{y}) ,
\]

and that the ETC of \( A^0(x) \) and \( A^i(y) \) is divergent:

\[
\lim_{x^0 \to y^0, y^0 \to z^0} [A^0(x), A^k(y)] = \frac{-i}{3 \pi^2} \left[ \frac{1}{(x_0 - y_0)^2} \partial^2_k - \frac{1}{2} \vec{\nabla}^2 \partial^x_k + \cdots \right] \delta^3(\vec{x} - \vec{y}) .
\]

Therefore, the Jacobi identity \( J_{ETC}(A^0(x), V^i(y), V^j(z)) = 0 \) is violated.

This is a typical example in which one sees that the order of the equal-time limits cannot be freely changed. One namely finds that

\[
\lim_{x^0 \to z^0} \lim_{y^0 \to z^0} [V^i(x), [V^j(y), A^0(z)]] = 0 ,
\]
whereas

\[
\lim_{\rho \to 0} \lim_{\rho_0 \to 0} \left\{ [V^i(x), [V^j(y), A^0(z)] ] + [V^j(y), [A^0(z), V^i(x)] ] \right\} \\
= \lim_{\rho \to 0} \lim_{\rho_0 \to 0} \left\{ \text{Tr} [\gamma^i S^+(x - y) \gamma^j S^-(y - z) \gamma^0 \gamma_5 S^-(z - x) \\
+ \gamma^0 \gamma_5 S^-(z - y) \gamma^j S^+(y - x) \gamma^i S^-(x - z)] - \text{Tr}[+ \leftrightarrow -] \\
- \gamma^j S^+(y - x) \gamma^i S^-(x - z) \gamma^0 \gamma_5 S^-(z - y) \\
- \gamma^0 \gamma_5 S^-(z - x) \gamma^j S^+(x - y) \gamma^i S^-(y - z)] - \text{Tr}[+ \leftrightarrow -] \right\} \\
= \lim_{\rho \to 0} \lim_{\rho_0 \to 0} \left\{ \text{Tr} [\gamma^j (-i) S(y - x) \gamma^i S^+(x - z) \gamma^0 \gamma_5 S^-(z - y) \\
+ \gamma^0 \gamma_5 S^+(z - x) \gamma^i (-i) S(x - y) \gamma^j S^-(y - z)] - \text{Tr}[+ \leftrightarrow -] \right\} \\
= -\frac{2}{3\pi^2} \epsilon_{ijk} \left[ \frac{1}{(y_0 - z_0)^2} \partial_k (y - z) - \frac{1}{2} \nabla^2 (y - z) \partial_k (y - z) + \frac{x_0 - y_0}{y_0 - z_0} \nabla^2 (y - z) \partial_k (x - y) \\
+ \cdots \right] \delta^3 (\vec{y} - \vec{z}) \delta^3 (\vec{x} - \vec{y}). \quad (24)
\]

To derive (24), we have applied (19) twice and (20). By using (21),(22) and (23), one can easily confirm that the result above is consistent with the Jacobi identity of the commutator (16)\(^{14}\), i.e.,

\[
\lim_{\rho_0 \to 0} \lim_{\rho \to 0} J( A^0(x), V^i(y), V^j(z) ) = 0, \quad i, j = 1, 2, 3.
\]

The example treated here is rather simple because we can use the Wick theorem (19) and only such products of singular functions appear that are well-defined distributions. In more realistic cases where renormalizable interactions are present, we have to consider \(T\)-products which are generally more singular than commutators. In the next section, we will consider the violation of the Jacobi identity of ETC’s in momentum space.

### 3.2 Momentum space consideration

Double equal-time commutators have of course an integral representation similar to (12):

\[
< \alpha | [A(x), [B(y), C(0)]_{ETC}]_{ETC} | \beta > \\
\equiv \lim_{x^0 \to 0} \lim_{y^0 \to 0} < \alpha | \{ T A(\vec{x}, x^0) B(\vec{y}, y^0) C(0) - T A(\vec{y}, -y^0) B(\vec{x}, x^0) C(0) \\
- T A(\vec{x}, x^0) B(\vec{x}, x^0) C(0) + T A(\vec{y}, y^0) B(\vec{y}, y^0) C(0) \} | \beta >
\]

\(^{14}\)This is the origin of the observation of ref. [1] that the Jacobi identity can be recovered by appropriately changing the order of Bjorken limits.
\[-T \mathcal{A}(\vec{x}, -x^0) \mathcal{B}(\vec{y}, y^0) \mathcal{C}(0) + T \mathcal{A}(\vec{x}, -x^0) \mathcal{B}(\vec{y}, -y^0) \mathcal{C}(0) \}\vert \beta >
\]
\[
= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} e^{+ip \cdot \vec{x} + iq \cdot \vec{y}} \oint_{C^w} \frac{dz}{2\pi} \oint_{C^w} \frac{dw}{2\pi} T^{\text{odd}}(\vec{p}, z, \vec{q}, w) ,
\]
(25)

where
\[
< \alpha | T \mathcal{A}(x) \mathcal{B}(y) \mathcal{C}(0) | \beta > = \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} e^{-ip \cdot x - iq \cdot y} T(p, q) ,
\]
\[
T^{\text{odd}}(\vec{p}, p^0, \vec{q}, q^0) = \frac{1}{4} \{ T(\vec{p}, p^0, \vec{q}, q^0) - T(\vec{p}, p^0, \vec{q}, -q^0)
\]
\[
- T(\vec{p}, -p^0, \vec{q}, q^0) + T(\vec{p}, -p^0, \vec{q}, -q^0) \} .
\]
(26)

Since the equal-time limit of the integrals along the branch cuts are either 0 or \(\infty\), we have assumed that the amplitude has no nontrivial discontinuity in the \(z\) and \(w\) planes. Note that the \(w\) integration should be first performed in such a way that \(C^w\) encloses all the poles of the amplitude. If the order is reversed we will obtain
\[
< \alpha | [\mathcal{B}(y), [\mathcal{A}(x), \mathcal{C}(0)]_{\text{ETC}}]_{\text{ETC}} | \beta > .
\]
(27)

The formula (25) may be easily guessed from (12), but our concern is the question of when the last equation of (25) really exhibits the corresponding double ETC.

First of all, the inner ETC, \([\mathcal{B}(y), \mathcal{C}(0)]_{\text{ETC}}\), which basically corresponds to the \(w\) integral, has to exist. Applying the convergence condition (14) for the inner ETC, it means that \(d_{\text{eff}}\) of \(T^{\text{odd}}(\vec{p}, p^0, \vec{q}, q^0)\) with respect to \(q^0\) must be equal to \(-1\). The second limit, \(x^0 \to +0\), exists if the \(w\) integration produces a function of \(z\) with the effective dimension \(\leq -1\) with respect to \(z\). From these observations, we conclude that the amplitudes having

\[
(iii) \quad d_{\text{eff}}[T(p, q)] = -2
\]
(28)

can give well-defined double ETC’s if the amplitudes have no nontrivial discontinuity, and those with

\[
(iv) \quad d_{\text{eff}}[T(p, q)] \leq -3
\]
(29)
do not contribute to the double equal-time limits (25), where \(d_{\text{eff}}\) can be obtained from the canonical dimension if we do not count to the dimension the powers of the spatial components \(p_i, q_i\) as well as the external-state momenta that are multiplied with the amplitudes. Because of the odd nature of the amplitudes (26), we may assign the amplitude

15
with \( d_{\text{eff}} = -1 \) to \( d_{\text{eff}} = -2 \). For amplitudes with \( d_{\text{eff}} [T(p, q)] \geq 0 \), we can not expect an well-defined double equal-time limit in general.

We now would like to come to investigate whether the Jacobi identity is satisfied. All the ETC’s involved in verifying the Jacobi identity should be assumed t o exist (otherwise, we can not give a sensible meaning to the violation of the Jacobi identity). One can compute those ETC’s from the same matrix element by changing the equal-time limits, as we have mentioned in concluding (27). Another limit we can obtain is

\[
(t^0 \to +0) (s^0 \to +0) , \quad \text{with } s^0 \equiv x^0 - y^0 , \ t^0 \equiv - y^0 ,
\]

which corresponds to the double ETC

\[
< \alpha | [ [ \mathcal{A}(x) , \mathcal{B}(y) ]_{\text{ETC}} , \mathcal{C}(0) ]_{\text{ETC}} | \beta > . \tag{30}
\]

Rewriting the exponent \( \exp \left[ -ix^0 z - iy^0 w \right] \) as \( \exp \left[ -is^0 z - it^0 v \right] \) with \( v \equiv - z - w \), one can easily find that

\[
(30) = - \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} e^{-i\vec{k} \cdot \vec{y} + i\vec{p} \cdot (\vec{x} - \vec{y})}
\times \oint_{C^{z}} \frac{dv}{2\pi} \oint_{C^{z}} \frac{dz}{2\pi} T_{\text{odd}} (\vec{p}, z, -\vec{k} - \vec{p}, -v - z) , \vec{k} \equiv - \vec{p} - \vec{q} .
\]

Therefore, the Jacobi identity corresponds to

\[
0 = \oint_{C^{z}} \frac{dz}{2\pi} \oint_{C^{z}} \frac{dw}{2\pi} T_{\text{odd}} (\vec{p}, z, \vec{q}, w) + \oint_{C^{z}} \frac{dv}{2\pi} \oint_{C^{z}} \frac{dz}{2\pi} T_{\text{odd}} (\vec{p}, z, -\vec{k} - \vec{p}, -v - z)
\]

\[
- \oint_{C^{z}} \frac{dw}{2\pi} \oint_{C^{z}} \frac{dz}{2\pi} T_{\text{odd}} (\vec{p}, z, \vec{q}, w) . \tag{31}
\]

To see that (31) is indeed satisfied, we express the amplitude in the Low representation.

We first consider the case of the single ETC (6) and write the amplitude \( T(p) \) in the Low representation:

\[
T(\vec{p}, p_0) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dp'_0 \left[ \frac{\rho_{AB}(\vec{p}, p'_0)}{p_0 - p'_0 + i\epsilon} - \frac{\rho_{BA}(\vec{p}, p'_0)}{p_0 - p'_0 - i\epsilon} \right] ,
\]

where

\[
\rho_{AB}(p) = \int d^4x e^{ip \cdot x} < \alpha | \mathcal{A}(x) \mathcal{B}(0) | \beta > ,
\]

\[
\rho_{BA}(p) = \int d^4x e^{ip \cdot x} < \alpha | \mathcal{B}(0) \mathcal{A}(x) | \beta > . \tag{32}
\]
Since we assume that the equal-time limit is finite, the amplitude must be tree-like. So the spectral functions \( \rho \)'s are basically \( \delta \)-function distributions. We may further assume that the singularities are located in a finite domain of \( p'_0 \) because the spatial components \( p_i \) and the external-state momenta are kept finite. Applying the Johnson-Low formula (12), one finds that

\[
\lim_{x^0 \to 0} \tilde{G}(\vec{p}, x^0) = -i \int_{CR} \frac{dz}{2\pi^2} \int_{-\infty}^{\infty} \frac{dp'_0}{2\pi} \left[ \rho_{AB}(\vec{p}, p'_0) - \rho_{BA}(\vec{p}, -p'_0) \right] \left( z - p'_0 + i\epsilon \right) \left( z + p'_0 - i\epsilon \right),
\]

where \( \tilde{G}(\vec{p}, x^0) \) in this case is given in (6) and \( CR \) is a circle with radius \( R \). Since the whole result is supposed to be independent of (sufficiently large) \( R \), \( R \) can be so chosen that all the singularities of \( \rho \)'s are located in the interval \((-R', R')\) with \( R' < R \). It is then obvious the \( z \) and \( p'_0 \) integrations in (33) may be changed to obtain

\[
\lim_{x^0 \to 0} \tilde{G}(\vec{p}, x^0) = \int_{-\infty}^{\infty} \frac{dp'_0}{2\pi} \left[ \rho_{AB}(\vec{p}, p'_0) - \rho_{BA}(\vec{p}, p'_0) \right].
\]

(34)

Remembering the definition of \( \rho \)'s (32), we see that the right-hand side of (34) is exactly the Fourier transform of

\[
<\alpha | A(\vec{x}, 0) B(0) - B(0) A(\vec{x}, 0) | \beta > = <\alpha | [A(x), B(0)]_{ETC} | \beta >.
\]

(35)

This means that \( \tilde{G}(\vec{p}, x^0) \) may be regarded as a continuous function in \( x^0 \) with the value (35) at \( x^0 = 0 \).

So we write the amplitude \( T(p, q) \) (26) in the Low representation as we did above:

\[
T(p, q) = -\int_{-\infty}^{\infty} \frac{dp'_0}{2\pi} \int_{-\infty}^{\infty} \frac{dq'_0}{2\pi} \left[ \rho_{ABC}(\vec{p}, p'_0, \vec{q}, q'_0) \cdot \right.
\]

\[
\rho_{ACB}(\vec{p}, p'_0, \vec{q}, q'_0)
\]

\[
- \rho_{BCA}(\vec{p}, p'_0, \vec{q}, q'_0)
\]

\[
- \rho_{CBA}(\vec{p}, p'_0, \vec{q}, q'_0)
\]

\[
\left. \left\{ \frac{(p_0 - p'_0 - i\epsilon)(q_0 - q'_0 - i\epsilon)}{(p_0 - p'_0 + i\epsilon)(q_0 - q'_0 + i\epsilon)} \right\} + \frac{(p_0 + q_0 - p'_0 - q'_0 + i\epsilon)(q_0 - q'_0 + i\epsilon)}{(p_0 + q_0 - p'_0 - q'_0 - i\epsilon)(q_0 - q'_0 - i\epsilon)} \right].
\]

(36)

where

\[
\rho_{ABC}(p, q) \equiv \int d^4x \, d^4y e^{ip \cdot x + iq \cdot y} <\alpha | A(x) B(y) C(0) | \beta >,
\]

and similarly for other \( \rho \)'s. Inserting (the odd part of) this expression into (31) and recalling the result that the order of the contour and the \( p'_0 \) and \( q'_0 \) integrations may be
changed (in the absence of nontrivial discontinuities), one easily finds that the right-hand side of (31) indeed vanishes \[15\]

Therefore, we conclude that the Jacobi identity is satisfied for tree-like amplitudes. Such amplitudes can of course appear in higher orders in perturbation theory. This is why one can use ETC’s to analyze quantum anomalies in terms of ETC’s, as we will see in the next section.

4 An application: The closure of the BRST algebra in Yang-Mills theories

4.1 Gauge-fixed theory

Before we go to the quantized theory of a Yang-Mills theory, we stay for a while in the classical approximation, and discuss some Poisson bracket structures in the theory. We begin by writing down the Lagrangian in the Landau gauge:

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + i \bar{\psi}_L \gamma^\mu D_\mu \psi_L + B^a \partial_\mu A^{a\mu} - \partial^\mu \overline{c}^a D_\mu c^b ,
\]

\[
D_\mu = \partial_\mu - ig A_\mu^a T^a , D^{ab}_\mu = \partial_\mu \delta^{ab} + g f^{abc} A_\mu^c , \tag{37}
\]

where \(\psi_L\) is a left-handed Weyl field in some representation of the gauge group \(G\) and is minimally coupled to the gauge fields \(A^{a\mu}\), and \(c^a (\overline{c}^a)\) are the Faddeev-Popov (anti-) ghost fields \[16\].

The Lagrangian is invariant under the BRST transformation \[17\]

\[
\delta A_\mu^a = D^{ab}_\mu c^b , \delta c^a = -\frac{g}{2} f^{abc} c^b c^c ,
\]

\[
\delta \overline{c}^a = -B^a , \delta B^a = 0 . \tag{38}
\]

They can be generated, at least at the level of the Poisson brackets, by the BRST charge:

\[
\delta \cdot = -\{ Q , \cdot \}_{PB} ,
\]

\[15\]One finds that the convergence condition (28) is not satisfied for the case of section 3.1. The discussion above is similar to that of ref. \[11\], but emphasize the importance of the absence of nontrivial discontinuities in the amplitudes.

\[16\]The generators \(T^a\) are assumed to be hermitian and satisfy \([T^a , T^b] = i f^{abc} T^c\).
where
\[ Q = \int d^3 \vec{x} \{ c^a \varphi^a - \frac{g}{2} f^{abc} \varphi^a c^b c^c + B^a D_0^{ab} c^b \}, \tag{39} \]
and \( \varphi^a \) are the Gauß law constraints
\[ \varphi^a = -D_i^{ab} E^{bi} - g \bar{\psi}_L \gamma^0 T^a \psi_L, \quad E^{ai} = -F^{abi}. \]
They satisfy the Poisson bracket algebra
\[ \{ \varphi^a(\vec{x}, x^0), \varphi^b(\vec{y}, x^0) \}_{PB} = g f^{abc} \varphi^c(\vec{x}, x^0) \delta(\vec{x} - \vec{y}), \]
which ensures the “nilpotencey” of \( Q \) at the classical level \[ 17 \]:
\[ \{ Q, Q \}_{PB} = 0. \tag{40} \]

For our purpose, it is more convenient, by means of the equation of motion, to rewrite the BRST charge (39) as \[ 40 \]
\[ Q = \int d^3 \vec{x} J_0(x), \]
\[ J_\mu = -c^a \partial_\mu B^a + \frac{g}{2} f^{abc} \partial_\mu \varphi^a c^b c^c + B^a D_\mu^{ab} c^b, \tag{41} \]
so that \( Q \) is expressed as an integral of the \( \mu = 0 \) component of a conserved current, the BRST current \( J_\mu \).

### 4.2 The nilpotency of \( Q \)

We will show that the quantum generalization of the Poisson bracket algebra (40), \([ Q, Q ] = 0\), is satisfied to all orders in perturbation theory if the theory is free from the chiral anomaly \[ 18 \].

Feynman rules are conventional \[ 19 \], but we remind ourselves that the \( A - B \) propagator, which is expressed by the line of fig. 2, is given by
\[ < A_\mu^a(x) B^b(y) > = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} \frac{-p_\mu \delta^{ab}}{p^2 + i\epsilon}, \]
\[ 17 \]We use the same symbol for the symmetric Poisson brackets as for the antisymmetric ones. We will do so for the commutators and anticommutators too.
\[ 18 \]Remember that the nilpotency of \( Q \) plays an important role to ensure unitarity in the operator formalism \[ 18 \]. Here we give a perturbative proof for the existence of a nilpotent BRST charge.
\[ 19 \]See, for instance, ref. \[ 14 \].
in coordinate space and also that

\[
< B^a(x) B^b(y) > = 0 .
\]

We employ the dimensional regularization of ref. [31, 33], and the composite operators contained in the BRST charge and current are defined as normal products in the MS scheme. We however would like to emphasize that the discussion below is regularization independent in character because we are basically applying the power counting rules and Ward identities of renormalized amplitudes only.

Obviously, it is sufficient to consider ETC’s among the BRST currents \(J_\mu\), and it is also more convenient to do so because we can rely on the Lorentz covariance. To begin with, we assume that the theory is free from the chiral anomaly so that all the Ward identities are satisfied and also the BRST current \(J_\mu\) is conserved. Later we will take into account the presence of the anomaly. So we consider

\[
[J_\mu(x), J_\nu(0)]_{ETC} .
\] (42)

All the diagrams which may contribute to the ETC (42) are shown in fig. 3. We will show that for \(\mu = \nu = 0\) the effective dimension of all the diagrams are equal to or less than two, which means, according to the convergence condition of (45), that the ETC (42) identically vanishes for \(\mu = \nu = 0\).

**Diagram (a)**

The diagram (a) has the canonical dimension of two so that it would yield a divergent ETC if there would be no restriction on the amplitude. The amplitude is the Fourier transform of

\[
< 0| T J_\mu(x) J_\nu(0) |c(k_a), c(k_b) > ,
\]

which we denote by

\[
\delta^{ab} T^{\mu\nu}(p, k_a, k_b) ,
\] (43)

where we have factorized the group index structure. To investigate its large momentum behavior (which we need for the convergence criterion), we expand the amplitude in
small external-state momenta, \( k_a \) and \( k_b \), and require that at each order in the expansion \( \partial_\mu J^\mu = 0 \) be satisfied:

\[
p_\mu T^{\mu\nu}(p, k_a, k_b) = q_\nu T^{\mu\nu}(p, k_a, k_b) = 0 ,
\]

\[
q = -(p + k_+) , \quad k_\pm = k_a \pm k_b .
\]

The amplitude should also respect antisymmetry

\[
T^{\mu\nu}(p, k_a, k_b) = -T^{\mu\nu}(p, k_b, k_a) ,
\]

which follows from the Fermi statistics of the ghost fields.

After some algebraic calculations, one finds that up to and including \( O(k^3) \) there are eleven independent terms, \( T^{\mu\nu}_{(aN)} \) \((N = 1, \cdots, 11)\), which are explicitly written in appendix. All the terms are consistent with (44) and (45) up to and including that order in \( k \). There are remarkable cancellations among the terms for \( \mu = \nu = 0 \); for instance,

\[
T^{00}_{(a1)} = \{ (k_+ k_-) \{ -(p_0 p_0) (\vec{p}^2 + \vec{p} \cdot \vec{k}_+) + p_0 k_{+0} \vec{p}^2 + \vec{p}^2 \vec{k}_+^2 \} T_{(a1)}(p^2) ,
\]

where the scalar amplitude \( T_{(a1)}(p^2) \) has dimension of \(-4\) so that it behaves like \( (p_0)^{-4} \) as \( p_0 \to \infty \). Therefore,

\[
T^{00}_{(a1)} \sim (p_0)^{-2} \text{ as } p_0 \to \infty ,
\]

so that \( d_{\text{eff}} \leq -2 \) and so according to the convergence condition (14) it does not contributes to the ETC (42) for \( \mu = \nu = 0 \). For the other amplitudes, i.e., \( T^{00}_{aN} \) \((N = 1, \cdots, 11)\), one observes the similar cancellations which ensures that the diagram (a) of fig. 3 can not contributes to \([J_0(x), J_0(0)]_{\text{ETC}}\).

**Diagram (b)**

This diagram corresponds to the Green function

\[
< 0 \mid T J_\mu(x) J_\nu(0) \mid c(k_a), c(k_b), A_\alpha^c(k) > ,
\]

and its Fourier transform

\[
f^{abc} T^{\mu\nu\alpha}(p, k_a, k_b, k) \]

(46)
and has the canonical dimension of one so that it is potentially dangerous. The amplitude
(44) has to satisfy the identities like (46) and the symmetry

\[ T^{\mu\nu}(p, k_a, k_b, k) = T^{\mu\nu}(p, k_b, k_a, k) , \]  

and also the Ward identity which follows from

\[ 0 = \delta < 0 | T J_\mu(x) J_\nu(0) \bar{\phi}^a(u) \bar{\phi}^b(v) \bar{\phi}^c(w)|0 > \\
- < 0 | T J_\mu(x) J_\nu(0) B^a(u) \bar{\phi}^b(v) \bar{\phi}^c(w)|0 > \\
+ < 0 | T J_\mu(x) J_\nu(0) B^b(v) \bar{\phi}^a(u) \bar{\phi}^c(w)|0 > \\
- < 0 | T J_\mu(x) J_\nu(0) \bar{\phi}^a(u) \bar{\phi}^b(v) B^c(w)|0 > , \]

where \( \delta \) is the BRST variation (32). Expressed in terms of \( T^{\mu\nu\alpha}(p, k_a, k_b, k) \), it means that

\[ 0 = k_\alpha T^{\mu\nu\alpha}(p, k_a, k_b, k) + k_\alpha T^{\mu\nu\alpha}(p, k, k_a) + k_\beta T^{\mu\nu\alpha}(p, k_a, k, k_b) . \]  

Up to and including \( O(k^2) \), there are exactly eight independent terms, \( T^{\mu\nu\alpha}(bN) \) \( (N = 1, \cdots, 8) \), that satisfy all the requirements above, and they are explicitly given in appendix. As in the previous case, one finds that for \( \mu = \nu = 0 \) there are cancellations that reduce \( d_{\text{eff}} \) of the diagram (b) at least down to \(-2\) so that it does not contribute to the ETC for \( \mu = \nu = 0 \).

**Diagram (c)**

As previously, we impose the antisymmetry of the ghost lines and the conservation of the BRST current to restrict the form of the amplitude in the \( p \to \infty \) limit. One finds:

\[ T^{\mu\nu\alpha \beta}(p, k_a, k_b, k_c) \rightarrow [\delta^{ab} \delta^{cd}(k_a - k_b)p + \delta^{ac} \delta^{bd}(k_c - k_a)p \\
\delta^{bc} \delta^{ad}(k_b - k_c)p] (p_\mu p_\nu - p^2 g_{\mu\nu}) T(c)(p^2) , \]

which is the lowest order expression in the large \( p \) expansion. Again one sees the cancellation for \( \mu = \nu = 0 \).

**Diagram (d)**

There are three independent terms as far as the group index structure is concerned:

\[ \delta^{ab} \delta^{cd}(k_a - k_b) , (\delta^{ac} \delta^{db} - \delta^{ad} \delta^{bc})(k_c - k_d) , (\delta^{ac} \delta^{db} + \delta^{ad} \delta^{bc})(k_a - k_b) , \]
where we have suppressed the Lorentz index. If one further impose the conservation of the BRST current, one finds that in the large \( p \) limit the amplitude must be proportional to \( p_\mu p_\nu - p^2 g_{\mu\nu} \). For \( \mu = \nu = 0 \), it means the reduction of \( d_{\text{eff}} \) by two. The diagrams (e) and (g) must also contain the same factor to satisfy the conservation of the BRST current. So the diagrams (d), (e) and (g) do not contribute to \([J_0(x), J_0(0)]_{\text{ETC}}\).

**Diagram (f)**

At \( O(k^0) \) there are seven terms that are consistent with \( \partial_\mu J^\mu = 0 \):

\[
\{ (-p_\mu p_\nu + p^2 g_{\mu\nu}) p_\gamma g_{\alpha\beta} \}, \quad (49)
\]

and (49) with \( \{ \gamma \to \alpha, \alpha \to \beta, \beta \to \gamma, \gamma \to \alpha \} \),

\[
\{ (2p_\alpha p_\mu p_\nu - p^2 p_\nu g_{\alpha\mu} - p^2 p_\mu g_{\alpha\nu}) p_\beta p_\gamma \}
\]

\[
-(p_\alpha p_\nu - p^2 g_{\alpha\nu}) p^2 p_\gamma, g_{\beta\mu} - (p_\alpha p_\mu - p^2 g_{\alpha\mu}) p^2 p_\gamma g_{\beta\nu} \}, \quad (50)
\]

and (50) with \( \{ \gamma \to \alpha, \alpha \to \beta, \beta \to \gamma, \beta \to \alpha \} \),

\[
p_\alpha p_\beta p_\gamma (p_\mu p_\nu - p^2 g_{\mu\nu}) .
\]

Since the amplitude has the canonical dimension of \(-1\), we multiply the terms above with \( p_0^{-6} \) to consider the \( p_0 \to \infty \) limit. One easily finds that, for instance, the term (50) for \( \mu = \nu = 0 \) becomes

\[
2 \vec{p}^2 \vec{p}^2 \vec{p}_\gamma \quad \text{for} \quad \alpha = \beta = 0 , \quad 2 p_0 p_j \vec{p}^2 \vec{p}_\gamma \quad \text{for} \quad \alpha = 0, \beta = j ,
\]

\[
2 p_0^2 p_i p_j \vec{p}_\gamma \quad \text{for} \quad \alpha = i, \beta = j ,
\]

and that \( d_{\text{eff}} \) is reduced at least by 2 because we do not count the spatial components \( p_i \) to the dimension.

We may summarize our findings by concluding that the nilpotency condition of the BRST charge

\[
Q^2 \equiv \frac{1}{2} [Q, Q] = \frac{1}{2} \int d^3 \vec{x} d^3 \vec{y} [J_0(x), J_0(y)]_{\text{ETC}} = 0 \quad (51)
\]

is satisfied if the BRST symmetry is intact. The result is of course in accord with expectation. But we would like to emphasize that the ETC’s defined in terms of the equal-time limit are not canonical and there is no compelling reason for that to be true in perturbation theory in general: ETC’s are generally ill-defined and there is no guaranty for the
generators of a symmetry transformation to form a closed algebra under the equal-time commutators.

4.3 Schwinger term for \([Q, Q]\)

Until now we have assumed that the BRST current \((41)\) is conserved and the various Ward identities are satisfied. In the presence of the chiral anomaly, this is no longer the case. To investigate the effects of the anomaly, we recall that according to the action principle [18] the violations of conservation laws and Ward identities manifest themselves in certain insertions in Green’s functions. For the diagram (a) of fig. 3, for example, we have to consider

\[
< 0 | \Delta_\mu(x) J_\nu(0) | c(k_a), c(k_b) > ,
\]

where \(\Delta_\mu\) is a local insertion and given by [17] (in a geometric notation)

\[
\Delta_\mu dx^\mu = \frac{i}{24\pi^2} \text{Tr} C d[A dA + \frac{1}{2} A^3] ,
\]

where \(d\) is the exterior derivative in four-dimensions. One easily observes that there is no tree diagram for \((52)\). This means that for the diagram (a) the violation of the BRST symmetry effectively appears at earliest at \(O(\bar{\hbar}^2)\) because \(\Delta_\mu\) is of \(O(\hbar)\).

From similar considerations one finds that only the diagrams (c) and (d) suffer from the anomaly at \(O(\hbar)\) as shown in fig. 4(a) and (b), that is, the effect of the anomaly appears at the one-loop level for those diagrams. Recalling that their dimensions are \(-1\) and \(-2\), respectively, and that they happen to be tree-like diagrams, we conclude that the \(O(\hbar^2)\) term of \([Q, Q]\) has to be finite. That is, \(\Omega\) defined by

\[
[Q, Q] = i\hbar^2 \Omega + O(\hbar^3) ,
\]

has an well-defined meaning while the higher order terms are presumably ill-defined.

The violation of the Jacobi identity may be studied as follows. Since \([J_0(x), J_0(0)]_{ETC}\) vanishes, the effective dimension of the amplitude for

\[
< 0 | T J_0(x) J_0(y) J_0(0) | c(k_a), c(k_b), c(k_c) >
\]

24
has to be equal to or less that $-3$ if there is no anomaly. This is because for each equal-time limit the inner ETC is the ETC of two $J_0$'s which vanishes identically if the BRST symmetry is intact. In the presence of the chiral anomaly, the effective dimension will be altered, and we have to consider the large momentum behavior of the Fourier transform of Green’s functions of the form

$$< 0| T J_0(x) J_0(y) J_0(0) | c(k_a), c(k_b), c(k_c), \chi > ,$$

where $\chi$ denotes additional external lines to (55). But we know from the previous investigation that only those with two and three external gauge boson lines can have contributions at the lowest non-trivial order in $\hbar$. The corresponding amplitudes thus have dimensions of $0$ and $-1$, respectively, and moreover they are tree-like diagrams (the corresponding amplitudes do not have nontrivial discontinuities). Therefore, according to the conclusion of section 3.2, the Jacobi identity has to be satisfied in the lowest order. That is,

$$[Q, [Q, Q]] = O(\hbar^4) ,$$

(57)

where the $O(\hbar^4)$ term above is probably divergent.

This justifies the assumption of ref. [27] (at least to the lowest non-trivial order in $\hbar$) in which a set of various consistency conditions on the anomalous Schwinger terms for the BRST algebra has been derived and exhaustively solved. The basic idea there was that, starting from (54) and assuming (57), one derives the consistency condition on $\Omega$,

$$\{Q, \Omega\}_{PB} = 0 .$$

(58)

This defines a classical cohomology problem, and the solution, unique up to cohomologically trivial terms, is given by [27]

$$\Omega = \frac{i}{24 \pi^2} \int \text{Tr} \left\{ C^2 (\hat{A} \hat{d} \hat{A} + \hat{d} \hat{A} \hat{A} + \hat{A}^3 + C \hat{A} C \hat{d} \hat{A}) \right\} ,$$

$$\hat{A} = -ig T^a A^a_k \hat{d} x^k \ (k = 1, 2, 3) ,$$

where $\hat{d}$ is the exterior derivative in three-dimensions, and $\hat{A}$ is a three-dimensional one-form.
5 Summary

The closure of the algebra of first-class constraints under the Poisson brackets is the expression of the presence of local symmetries in the classical Hamiltonian formalism \cite{42}. Since the Poisson brackets are replaced by the equal-time commutators (ETC’s) in the canonical quantization procedure, one might expect that the corresponding constraint operators form the same algebra under the equal-time commutators. However, we have seen that one cannot obtain a sensible, finite equal-time limit if the amplitude has nontrivial discontinuities. Since higher order amplitudes have discontinuities in general, it is unlikely possible to exhibit the Poisson bracket algebra in terms of ETC’s in renormalizable field theories. However, any constraints algebra, closed or open, can be expressed as a BRST algebra under the Poisson brackets \cite{43}. Since the algebra is abelian and “0” of an ETC can be proven in perturbation theory, it is possible to study quantum BRST symmetries in terms of certain equal-time commutators. We have in fact shown that the nilpotency condition on the BRST charge in Yang-Mills theories is satisfied if the chiral anomaly is canceled.

Even if ETC’s exist, they are often “deformed” in the sense that the product rule is violated so that they differ from the canonical results. So another important question is whether these ETC’s have the derivative property, i.e., they are Jacobi-identity satisfying, which is needed to form an associative algebra. We have found that, if the the convergence condition for double ETC’s are satisfied, the ETC’s have the derivative property. This observation can be applied to investigate the algebraic structure of anomalous Schwinger terms, as we have done in anomalous Yang-Mills theories.

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Appendix

The independent terms in the small external-state momenta, that are consistent with various Ward identities, are listed. The calculation has been performed by using the Mathematica package "Tracer" [44].

Diagram of fig. 3 (a): \( T_{aN}^{\mu\nu} \)

\[
T_{a1}^{\mu\nu} = [-p_\mu p_\nu + (p k_+) p_\mu p_\nu - p_\mu^2 k_{\mu+} + (p^2) g_{\mu\nu}] p^2 (k_+ k_-) T_{a1}(p^2),
\]

\[
T_{a2}^{\mu\nu} = [p_\mu p_\nu - p_\nu k_{\mu+} + (p^2) g_{\mu\nu} + (p k_+) g_{\mu\nu}] (p k_-) T_{a2}(p^2),
\]

\[
T_{a3}^{\mu\nu} = [-p_\mu p_\nu + (p k_+) p_\mu p_\nu - p_\mu^2 k_{\mu+} - (p k_+) p_\mu k_{\nu+} + p^2 k_{\mu+} k_{\nu+} + (p^2) g_{\mu\nu}] (p k_-) T_{a3}(p^2),
\]

\[
T_{a4}^{\mu\nu} = \{ -p_\mu p_\nu + p^2 g_{\mu\nu} \} \{ (k_+^2 (p k_-), (p k_+) (k_+ k_-), (p k_-) k_+^2 \} T_{a5,6,7}(p^2),
\]

\[
T_{a8}^{\mu\nu} = \{ -(p k_+) p_\nu k_{\mu+} - (p k_+) p_\mu k_{\nu+} + p^2 k_{\mu+} k_{\nu+} + (p k_+^2 g_{\mu\nu}) (p k_-) T_{a8}(p^2),
\]

\[
T_{a9}^{\mu\nu} = \{ (p k_-) p_\nu k_{\mu-} - (p k_+) p_\mu k_{\nu-} + p^2 k_{\mu-} k_{\nu-} + (p k_-^2 g_{\mu\nu}) (p k_-) T_{a9}(p^2),
\]

\[
T_{a10}^{\mu\nu} = \{ (p k_+)^2 p_\mu p_\nu - p^2 (p k_+) p_\nu k_{\mu+} - p^2 (p k_+) p_\mu k_{\nu+} + (p^2) k_{\mu+} k_{\nu+} \} (p k_-) T_{a10}(p^2),
\]

\[
T_{a11}^{\mu\nu} = \{ (p k_-)^2 p_\mu p_\nu - p^2 (p k_-) p_\nu k_{\mu-} - p^2 (p k_-) p_\mu k_{\nu-} + (p^2) k_{\mu-} k_{\nu-} \} (p k_-) T_{a11}(p^2).
\]

Diagram fig. 3(b): \( T_{bN}^{\mu\nu\alpha} \)

\[
T_{b1}^{\mu\nu\alpha} = \{ k_\nu p_\mu p_\alpha + [ p_\alpha k_{\nu}, - p_{\nu} k_{\alpha}, + p_\alpha k_{\nu}, + p_{\nu} k_{\alpha}, + k_\nu p_\alpha - k_\alpha p_\nu - p_\alpha p_\nu \} k_\mu p^2
\]

\[
+[(k p) k_\alpha - k_\nu p_\alpha - (k k_\alpha) p_\alpha - (k k_\alpha) p_\nu - (k p) k_{\alpha}, + (k p) k_{\alpha}, \} p_\mu p_\nu
\]

\[
+[-(k p) k_\nu - k_\nu p_\nu + k_\nu^2 + (k p) p_\nu + (k k_\alpha) p_\nu + (k k_\alpha) p_\nu
\]

\[
-(k p) k_{\alpha}, - (k p) k_{\alpha}, \} p^2 g_{\alpha\mu} + [ k_\mu p^2 - (k p) p_\mu \} p^2 g_{\alpha\nu} \} T_{b1}(p^2),
\]

\[
T_{b2}^{\mu\nu\alpha} = \{ (p k_{b}) k_{\alpha} p_\mu + (k k_{b}) k_{\alpha} p_\mu + (k k_{b}) k_{\alpha} p_\mu + (k k_{b}) k_{\alpha} p_\mu - (k k_{b}) p_\alpha
\]

\[
-p_\alpha k_{\alpha} k_{\alpha} + k_\mu p_\mu + k_\mu p_\mu + k_\mu k_{\alpha} k_{\alpha} + k_\mu p_\mu k_{\alpha} - p_\alpha k_\mu k_{\alpha} - p_\alpha k_\mu k_{\alpha}
\]

\[
-k_\alpha p_\mu k_\mu + p_\alpha p_\mu k_{\alpha} + p_\alpha k_{\alpha} k_{\alpha} - p_\nu k_\nu k_{\alpha} - k_\alpha p_\mu k_{\alpha} - p_\alpha p_\mu k_{\alpha}
\]

\[
27
\]
\[-p_\nu k_{\alpha\alpha} k_{b\mu} + p_\alpha k_{\alpha\alpha} k_{b\mu} - p_\nu k_{\beta\alpha} k_{b\mu} + k_\mu p_\alpha k_{b\nu} + p_\alpha p_\mu k_{b\nu} + p_\alpha k_{\alpha\mu} k_{b\nu} + p_\alpha k_{b\mu} k_{b\nu} \] \[= (k k_\alpha) p_\nu + (k k_\beta) p_\nu + (p k_\alpha) p_\nu + (p k_\beta) p_\nu + p_\nu k_\alpha^2 + 2 p_\nu(k_\alpha k_b) - (k p) k_{a\nu} \]
\[-p_\nu^2 k_{a\nu} - (p k_\alpha) k_{a\nu} - (p k_\beta) k_{a\nu} + p_\nu k_\beta^2 - (k p) k_{b\nu} - p_\nu^2 k_{b\nu} \]
\[-(p k_\alpha) k_{b\nu} - (p k_\beta) k_{b\nu} \] \[= (k p) k_{a\mu} + p_\nu^2 k_{a\mu} + (k p) k_{b\mu} + p_\nu^2 k_{b\mu} \] \[= T_{b3}(p^2) , \]
\[T_{b4}^{\mu\nu} = \{ [(k p) k_\alpha - k^2 p_\alpha] p_\mu p_\nu - [(k p) k_\alpha + k^2 p_\alpha] p^2 g_{\mu\nu} \} T_{b3}(p^2) , \]
\[T_{b5(6)}^{\mu\nu} = \{ [k_\alpha (p k_\alpha) + k_\alpha (p k_\beta) - p_\alpha k_\alpha^2 + 2 p_\alpha k_\alpha k_b + (k p) k_{b\alpha} + (k p) k_{b\beta}] p_\mu p_\nu \]
\[= (k k_\alpha) p_\nu + (k k_\beta) p_\nu + (p k_\alpha) p_\nu + (p k_\beta) p_\nu + p_\nu k_\alpha^2 + 2 p_\nu(k_\alpha k_b) - (k p) k_{a\nu} \]
\[-p_\nu^2 k_{a\nu} - (p k_\alpha) k_{a\nu} - (p k_\beta) k_{a\nu} + p_\nu k_\beta^2 - (k p) k_{b\nu} - p_\nu^2 k_{b\nu} \]
\[-(p k_\alpha) k_{b\nu} - (p k_\beta) k_{b\nu} \] \[= (k p) k_{a\mu} + p_\nu^2 k_{a\mu} + (k p) k_{b\mu} + p_\nu^2 k_{b\mu} \] \[= T_{b3}(p^2) , \]
\[T_{b7}^{\mu\nu} = \{ [-k_\nu k_{a\mu} + k_\mu k_{a\nu} - k_\nu k_{b\mu} + k_\mu k_{b\nu}] p_\alpha + [k_\nu (p k_\alpha) + k_\nu (p k_\beta) - (k p) k_{a\nu} \]
\[-(k p) k_{b\nu}] g_{\alpha\nu} + [-k_\mu (p k_\alpha) - k_\mu (p k_\beta) + (k p) k_{a\mu} + (k p) k_{b\mu}] g_{\alpha\nu} \} T_{b7}(p^2) , \]
\[T_{b8}^{\mu\nu} = \{ [-k_\nu (p k_\alpha) k_{a\alpha} - (k p) k_{b\alpha} + (p k_\beta) k_{b\alpha}] p_\mu p_\nu + [(k p) k_{a\alpha} \]
\[-(p k_\alpha) k_{a\alpha} + (k p)^2 k_{b\alpha} g_{\mu\nu} - (p k_b) k_{b\alpha}] p^2 g_{\mu\nu} \} T_{b8}(p^2) , \]

where \( T_{a,bN}(p^2) \) are scalar functions of \( p^2 \).
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Figure Captions

Fig. 1 Integration contours.

Fig. 2 $A - B$ propagator.

Fig. 3 Diagrams that contribute to $[Q,Q]$.

Fig. 4 Tree level contributions of the insertion $\Delta_\mu$, which is indicated by $\bullet$. 
This figure "fig1-1.png" is available in "png" format from:

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