THE ENERGY AND SPECTRUM OF NON-COMMUTING GRAPHS

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Abstract. Let $G$ be a non-abelian group and $Z(G)$ be the center of $G$. The non-commuting graph $\Gamma(G)$ of $G$ is a graph with vertex set $G - Z(G)$ in which two vertices $x$ and $y$ are joined if and only if $xy \neq yx$. In this paper we calculate the energy, Laplacian energy and spectrum of non-commuting graph of dihedral group $D_{2n}$. Also we will obtain the energy of non-commuting graph of $D_{2n} \times D_{2n}$ and $G \times H$, where $G$ is a non-abelian finite group and $H$ is an abelian finite group.

Key words: Non-commuting graph, Energy of a graph, Spectrum.

2010 Mathematics Subject Classification: 20D99, 05C50.

1. Introduction and preliminaries

Let $G$ be a non-abelian finite group and $Z(G)$ be its center. The non-commuting graph $\Gamma(G)$ of $G$ is a graph whose vertex set is $G - Z(G)$ and two vertices $x$ and $y$ are joined if and only if $xy \neq yx$. Note that if $G$ is an abelian, then $\Gamma(G)$ is the null graph. The non-commuting graph $\Gamma(G)$ was first considered by Paul Erdős, when he posed the following problem in 1975 [7]: Let $G$ be a group whose non-commuting graph has no infinite complete subgraph. Is it true that there is a finite bound on the cardinalities of complete subgraphs of $\Gamma(G)$? B. H. Neumann answered positively Erdős’s question in [7]. The adjacency matrix of graph $\Gamma$ is the $(0,1)$ matrix $A$ indexed by the vertex set $V(\Gamma)$ of $\Gamma$, where $A_{xy} = 1$ when there is an edge from $x$ to $y$ in $\Gamma$ and $A_{xy} = 0$ otherwise. The characteristic polynomial of $A$, denoted by $P_A(x)$, is the polynomial defined by $P_A(x) = det(xI - A)$ where $I$ denotes the identity matrix. The spectrum of a finite graph $\Gamma$ is by definition the spectrum of the adjacency matrix $A$, that is, its set...
of eigenvalues together with their multiplicities. Assume that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_t$ are $t$ distinct eigenvalues of $\Gamma$ with the corresponding multiplicities $k_1, k_2, \ldots, k_t$. We denote by

$$spec(\Gamma) = \begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_t \\ k_1 & k_2 & \ldots & k_t \end{pmatrix}. $$

Let $\Gamma$ be an undirected graph without loops. The Laplacian matrix of $\Gamma$ is the matrix $L$ indexed by the vertex set of $\Gamma$, with zero row sums. If $D$ is the diagonal matrix, indexed by the vertex set of $\Gamma$ such that $D_{xx}$ is the degree of $x$ then $L = D - A$. The energy of a graph $\Gamma$, denoted by $E(\Gamma)$, is defined as

$$E(\Gamma) = \sum_{i=1}^{n} |\lambda_i|,$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are the eigenvalues of the adjacency matrix of $\Gamma$. This concept was introduced by Gutman and is intensively studied in chemistry, since it can be used to approximate the total π-electron energy of a molecule (see, e.g., [3, 4]).

Let $\Gamma$ be a graph with $n$ vertices and $m$ edges. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the Laplacian eigenvalues of $\Gamma$. The Laplacian energy of a graph $\Gamma$, is defined as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|.$$

**Theorem 1.1.** ([2, 6]). Let $K_{n_1, n_2, \ldots, n_p}$ denote the complete $p$-partite graph, $p \geq 1$, $n = n_1 + n_2 + \ldots + n_p$ and $n_1 \geq n_2 \geq \ldots \geq n_p > 0$. Then

$$P(K_{n_1, n_2, \ldots, n_p}, \lambda) = \lambda^{(n-p)} \left( 1 - \sum_{i=1}^{p} \frac{n_i}{\lambda + n_i} \right) \prod_{j=1}^{p} (\lambda + n_j).$$

The spectrum of $K_{n_1, n_2, \ldots, n_p}$ consist of the spectral radius $\lambda_1$ determined from the equation $\sum_{i=1}^{p} \frac{n_i}{\lambda + n_i} = 1$, eigenvalue 0 with multiplicity $n - p$ and $p - 1$ eigenvalues situated in the intervals $[-n_p, -n_{p-1}], \ldots, [-n_2, -n_1]$.

**Lemma 1.2.** ([5]) If $\lambda_1$ is the spectral radius of the complete multipartite graph $K_{n_1, n_2, \ldots, n_p}$, then
Lemma 1.3. ([5]). (1) If \( p = 2 \), then
\[
P(K_{n_1, n_2}, \lambda) = \lambda^{n-2}(\lambda^2 - n_1n_2),
\]
\[
\lambda_1(K_{n_1, n_2}) = \sqrt{n_1n_2}, \quad E(K_{n_1, n_2}) = 2\sqrt{n_1n_2}.
\]
(2) If \( p = 3 \), then
\[
P(K_{n_1, n_2, n_3}, \lambda) = \lambda^{(n-3)}(\lambda^3 - (n_1n_2 + n_2n_3 + n_3n_1)\lambda - 2n_1n_2n_3).
\]

2. The energy and Laplacian energy of non-commuting graphs of some special groups

In this section, we calculate the energy and Laplacian energy of non-commuting graph of dihedral group \( D_{2n} \).

Theorem 2.1.

(1) If \( n \) is even and \( n > 4 \), then
\[
spec(\Gamma_{D_{2n}}) = \begin{pmatrix}
-2 & 0 & \frac{(n-2)-\sqrt{5n^2-12n+4}}{2} & \frac{(n-2)+\sqrt{5n^2-12n+4}}{2} \\
\frac{n}{2} - 1 & \frac{3n}{2} - 3 & 1 & 1
\end{pmatrix}.
\]

(2) If \( n \) is odd, then
\[
spec(\Gamma_{D_{2n}}) = \begin{pmatrix}
-1 & 0 & \frac{(n-1)-\sqrt{5n^2-6n+1}}{2} & \frac{(n-1)+\sqrt{5n^2-6n+1}}{2} \\
(n-1) & (n-2) & 1 & 1
\end{pmatrix}.
\]

(3) If \( n = 4 \), then \( spec(\Gamma_{D_{2n}}) = \begin{pmatrix}
-2 & 0 & 4 \\
2 & 3 & 1
\end{pmatrix}.
\]

Proof. The adjacency matrix of \( \Gamma_{D_{2n}} \) is equal to
\[
A_{ij}(\Gamma_{D_{2n}}) = \begin{cases}
0 & 1 \leq i, j \leq n - 2 \\
0 & i = k + t, j = k + s, t, s = 0 \text{ or } 1 \\
1 & \text{and } k = n - 1, n + 1, \ldots, 2n - 3 \\
\end{cases}
o.w
when \( n \) is even, and
\[
A_{ij}(\Gamma_{D_{2n}}) = \begin{cases} 
0 & 1 \leq i, j \leq n - 1 \\
0 & i = j = n, n + 1, \ldots, 2n - 1 \\
1 & o.w
\end{cases}
\]
when \( n \) is odd. By direct calculations
\[
P_{\Gamma_{D_{2n}}}(x) = \begin{cases} 
(-x)^{\frac{n}{2}+3}(-x-2)^{\frac{n}{2}-1}(x^2-x(n-2)-n(n-2)) & \text{if } n \text{ is even} \\
(-x)^{n-2}(-x-1)^n-1(x^2-x(n-1)-n(n-1)) & \text{if } n \text{ is odd}
\end{cases}
\]
This completes the proof.

\[\blacksquare\]

**Corollary 2.2.**

\[
E(\Gamma_{D_{2n}}) = \begin{cases} 
(n-2) + \sqrt{5n^2 - 12n + 4} & \text{if } n \text{ is even} \\
(n-1) + \sqrt{5n^2 - 6n + 1} & \text{if } n \text{ is odd}
\end{cases}
\]

In Table 1, the energies of some non-commuting graphs of dihedral groups is given.

| groups | characteristic polynomial | eigenvalues | energy |
|--------|---------------------------|-------------|--------|
| \( D_6 \) | \( P_A(x) = (-x)(-x-1)^2(x^2-2x-6) \) | \((0)_1, (-1)_2, (1 + \sqrt{7}), (1 - \sqrt{7})\) | \(2 + 2\sqrt{7}\) |
| \( D_8 \) | \( P_A(x) = (-x)^3(-x+4)(-x-2)^2 \) | \((0)_3, (4)_1, (-2)_2\) | \(8\) |
| \( D_{10} \) | \( P_A(x) = (-x)^3(-x-1)^3(x^2-4x-20) \) | \((0)_3, (-1)_4, (2 - 2\sqrt{6}), (2 + 2\sqrt{6})\) | \(4 + 4\sqrt{6}\) |
| \( D_{12} \) | \( P_A(x) = (-x)^6(-x-2)^2(x^2-4x-24) \) | \((0)_6, (-2)_2, (2 + 2\sqrt{7}), (2 - 2\sqrt{7})\) | \(4 + 4\sqrt{7}\) |
| \( D_{16} \) | \( P_A(x) = (-x)^9(-x-2)^3(x^2-6x-48) \) | \((0)_9, (-2)_3, (3 + \sqrt{57}), (3 - \sqrt{57})\) | \(6 + 2\sqrt{57}\) |

**Table 1.** The characteristic polynomial, eigenvalues and energy of some graphs.

**Theorem 2.3.** Let \( G = GL(2, q) \), where \( q = p^n > 2 \) \((p \text{ is prime}). \) Then
\[
P_{\Gamma_G}(x) = x^{(n-1)}(x^3 + (-q^4 + q^3 + 4q^2 - 6q + 2)x^2 \\
+ (-2q^6 + 6q^5 - q^4 - 13q^3 + 15q^2 - 5q)x - (q - 1)^4q^2(q - 2)(q + 1)] \\
(x + (q - 1))^2(x + q(q - 1))^{\frac{x-2-x^2}{2}}(x + (q - 1)(q - 2))^{\frac{x^2-x^2}{2}}
\]
where \( t = q^2 + q + 1. \)
Proof. By [1] we have

\[ G - Z(G) = \bigcup_{g \in G} ((PZ(G)) D - Z(G)) \cup \bigcup_{g \in G} I Z(G) \cup \bigcup_{g \in G} D - Z(G), \]

such that \( D \) is the subgroup of all diagonal matrices in \( G \), \( I \) is a cyclic subgroup of \( G \) of order \( q^2 - 1 \) and \( P \) is the sylow \( p \)-subgroup of \( G \) containing all matrices as \( \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \). Moreover \(|D| = (q - 1)^2 \) and \(|PZ(G)| = q(q - 1)\).

Also the number of conjugates of \( D, I \) and \( PZ(G) \) is equal to \( \frac{q(q + 1)}{2}, \frac{q(q - 1)}{2} \) and \( q + 1 \), respectively. Since \( C_G(d) = D \) for any non-central element \( d \) of \( D, C_G(I) = I \) and \( PZ(G) = C_G(x) \) for any non-trivial element of \( P \), then the non-commuting graph of group \( G \) is a complete \( t \)-partite graph where

\[ t = \frac{q(q + 1)}{2} + \frac{q(q - 1)}{2} + q + 1 = q^2 + q + 1. \]

By Theorem 1.1, \( P_{\Gamma_G}(x) = x^{(n - t)} \left[ x^3 + (-q^4 + q^3 + 4q^2 - 6q + 2)x^2 \\
+ (-2q^6 + 6q^5 - q^4 - 13q^3 + 15q^2 - 5q)x - (q - 1)^4 q^2(q - 2)(q + 1) \right]
\]

\[ (x + (q - 1)^2)^q(x + q(q - 1))^{\frac{q^2 - q - 2}{2}}(x + (q - 1)(q - 2))^\frac{q^2 + q - 2}{2}. \]

\[ \square \]

Corollary 2.4.

\[ E(\Gamma_{GL(2,q)}) = |\gamma_1| + |\gamma_2| + |\gamma_3| + q(q - 1)^2 + \frac{q^2 - q - 2}{2}q(q - 1) + \frac{q^2 + q - 2}{2}(q - 1)(q - 2), \]

where \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) are roots of \( f(x) = x^3 + (-q^4 + q^3 + 4q^2 - 6q + 2)x^2 + (-2q^6 + 6q^5 - q^4 - 13q^3 + 15q^2 - 5q)x - (q - 1)^4 q^2(q - 2)(q + 1). \)

Proof. By Theorem 2.3, we have

\[ P_{\Gamma_G}(x) = x^{(n - t)} \left[ x^3 + (-q^4 + q^3 + 4q^2 - 6q + 2)x^2 \\
+ (-2q^6 + 6q^5 - q^4 - 13q^3 + 15q^2 - 5q)x - (q - 1)^4 q^2(q - 2)(q + 1) \right]
\]

\[ (x + (q - 1)^2)^q(x + q(q - 1))^{\frac{q^2 - q - 2}{2}}(x + (q - 1)(q - 2))^\frac{q^2 + q - 2}{2}. \]
Let \( f(x) = x^3 + (-q^4 + q^3 + 4q^2 - 6q + 2)x^2 + (-2q^6 + 6q^5 - q^4 - 13q^3 + 15q^2 - 5q)x - (q - 1)^4 q^2(q - 2)(q + 1) \), b = \((-q^4 + q^3 + 4q^2 - 6q + 2)\), c = \((-2q^6 + 6q^5 - q^4 - 13q^3 + 15q^2 - 5q)\) and \(d = -(q - 1)^4 q^2(q - 2)(q + 1)\).

Then we have \( f(x) = x^3 + bx^2 + cx + d \). It is convenient to make the translation \( x = y - \frac{b}{3} \), converting \( f(x) \) into \( g(y) = f(y - \frac{b}{3}) = y^3 + \alpha y + \beta \), where \( \alpha = \frac{b^2 - 2b}{3} + c \) and \( \beta = \frac{b^3 + 3\beta^2 - 9\beta c}{27} + d \). We have
\[
\Delta = \left(\frac{\alpha}{3}\right)^3 + \left(\frac{\beta}{2}\right)^2.
\]
Since \( \Delta < 0 \), \( g(y) \) has three real roots. Now let \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) be roots of \( g(y) \). Then
\[
E(\Gamma_{GL(2,q)}) = \sum_{i=1}^{n} |\lambda_i|
= |\gamma_1| + |\gamma_2| + |\gamma_3| + q \left| - (q - 1)^2 \right| + \frac{q^2 - q - 2}{2} |q(q - 1)|
+ \frac{q^2 + q - 2}{2} |q - 2| (q - 2)|
= |\gamma_1| + |\gamma_2| + |\gamma_3| + q(q - 1)^2 + \frac{q^2 - q - 2}{2} q(q - 1)
+ \frac{q^2 + q - 2}{2} (q - 1)(q - 2).
\]

\( \square \)

\textbf{Theorem 2.5.}

(1) If \( n \) is even and \( n > 4 \), then
\[
\text{spec}(L(\Gamma_{D_{2n}})) = \begin{pmatrix}
2n - 2 & 2n - 4 & n & 0 \\
\frac{n}{2} & \frac{n}{2} & n - 3 & 1
\end{pmatrix}.
\]

(2) If \( n \) is odd, then
\[
\text{spec}(L(\Gamma_{D_{2n}})) = \begin{pmatrix}
0 & n & 2n - 1 \\
1 & n - 2 & n
\end{pmatrix}.
\]

(3) If \( n = 4 \), then \( \text{spec}(L(\Gamma_{D_{2n}})) = \begin{pmatrix}
0 & 4 & 6 \\
1 & 3 & 2
\end{pmatrix} \).
Proof. By considering $L(\Gamma_{D_{2n}})$ and direct calculations, we have

$$P_{L(\Gamma_{D_{2n}})}(x) = \begin{cases} 
  x(x-n)^{(n-3)}(x-(2n-4))^{(2)}(x-(2n-2))^{(2)} & \text{if } n \text{ is even} \\
  (-x)(n-x)^{n-2}((2n-1)-x)^n & \text{if } n \text{ is odd}. 
\end{cases}$$

This completes the proof. \qed

Corollary 2.6.

$$LE(\Gamma_{D_{2n}}) = \begin{cases} 
  \frac{2n(n^2-4n+6)}{2n-2} & \text{if } n \text{ is even} \\
  3n(n-1) & \text{if } n \text{ is odd}. 
\end{cases}$$

In Table 2, the Laplacian characteristic polynomial and eigenvalues of non-commuting graphs of some dihedral groups is given.

| Groups | Characteristic Polynomial | Eigenvalues | Energy |
|--------|---------------------------|-------------|--------|
| $D_8$  | $P_A(x) = x(x-4)^3(x-6)^2$ | $(0)_1, (4)_3, (6)_2$ | 8 |
| $D_{10}$ | $P_A(x) = (-x)(x-5)^3(9-x)^5$ | $(0)_1, (5)_3, (9)_5$ | 60 |
| $D_{12}$ | $P_A(x) = x(x-6)^3(x-8)^3(x-10)_3$ | $(0)_1, (6)_3, (8)_3, (10)_3$ | $\frac{108}{5}$ |
| $D_{14}$ | $P_A(x) = (-x)(x-7)^5(13-x)^7$ | $(0)_1, (7)_5, (13)_7$ | 126 |
| $D_{16}$ | $P_A(x) = x(x-8)^5(x-12)^4(x-14)^4$ | $(0)_1, (8)_5, (12)_4, (14)_4$ | $\frac{304}{7}$ |

Table 2. The characteristic polynomial, eigenvalues and Laplacian energy of some graphs.

### 3. The energy of non-commuting graph of direct product of groups

In this section, we calculate the energy of non-commuting graph of $D_{2n} \times D_{2n}$ and $G \times H$, which $G$ is a non-abelian finite group and $H$ is an abelian finite group.

**Theorem 3.1.** Let $G$ be a non-abelian finite group and $H$ be an abelian group of order $n$. Then $E(\Gamma_{G \times H}) = nE(\Gamma_G)$.  

Proof. Let \( h_1, h_2, \ldots, h_n \) be elements of \( H \). Suppose \( A(G) \) and \( A(H) \) be adjacency matrices of non-commuting graph of groups \( G \) and \( H \), respectively. Then the adjacency matrix of \( \Gamma_{G \times H} \) as the following form

\[
A(\Gamma_{G \times H}) = \begin{bmatrix}
A(G) & A(G) & \ldots & A(G) \\
A(G) & A(G) & \ldots & A(G) \\
\vdots & \vdots & \ddots & \vdots \\
A(G) & A(G) & \ldots & A(G)
\end{bmatrix}.
\]

We have

\[
det(A(\Gamma_{G \times H}) - Ix) = \begin{vmatrix}
A(G) -Ix & A(G) & \ldots & A(G) \\
A(G) & A(G) -Ix & \ldots & A(G) \\
\vdots & \vdots & \ddots & \vdots \\
A(G) & A(G) & \ldots & A(G) -Ix
\end{vmatrix}.
\]

But

\[
det(A(\Gamma_{G \times H}) - Ix) = |(A(G) -Ix) + (n-1)A(G)| |(A(G) -Ix) - A(G)|^{n-1} \\
= |nA(G) -Ix| |-Ix|^{n-1}.
\]

Thus

\[
P_{A(\Gamma_{G \times H})}(x) = n |-Ix|^{n-1} P_{A(G)}(\frac{x}{n}).
\]

If \( \lambda_1, \lambda_2, \ldots, \lambda_t \) be eigenvalues of \( A(G) \), then

\[
x_1 = n\lambda_1, \ x_2 = n\lambda_2, \ldots, \ x_t = n\lambda_t,
\]

are eigenvalues of \( A(\Gamma_{G \times H}) \). Therefore

\[
E(\Gamma_{G \times H}) = \sum_{i=1}^{t} |\lambda_i| \\
= |n\lambda_1| + |n\lambda_2| + \ldots + |n\lambda_t| \\
= n(|\lambda_1| + \ldots + |\lambda_t|) \\
= nE(G).
\]

\(\square\)
Lemma 3.2. Let $G = D_8$. Then

$$P_{\Gamma_{G \times G}}(x) = (-x)^{45}(-x + 8)(-x - 4)^4(x^2 + 8x - 32)^4(x^2 - 40x - 128)$$

and

$$E(\Gamma_{G \times G}) = \sum_{i=1}^{60} |\lambda_i| = 8(3 + \sqrt{33} + 4\sqrt{3}).$$

Proof. For group $G$, we consider the adjacency matrix $A^0(G)$ similar to $A(G)$ by adding vertices $Z(G)$ to $A(G)$. Adjacency matrices of $A(G)$ and $A^0(G)$ differ only in some zero rows and zero columns. We have

$$A^0(\Gamma_{G \times G}) = \begin{bmatrix}
A^0(G) & A^0(G) & A^0(G) & A^0(G) & A^0(G) & A^0(G) & A^0(G) & A^0(G) & A^0(G)
A^0(G) & A^0(G) & A^0(G) & A^0(G) & A^0(G) & A^0(G) & A^0(G) & A^0(G) & A^0(G)
A^0(G) & A^0(G) & A^0(G) & A^0(G) & J & J & J & J & J
A^0(G) & A^0(G) & A^0(G) & A^0(G) & J & J & J & J & J
A^0(G) & A^0(G) & A^0(G) & A^0(G) & J & J & J & J & J
A^0(G) & A^0(G) & A^0(G) & A^0(G) & J & J & J & J & J
A^0(G) & A^0(G) & A^0(G) & A^0(G) & J & J & J & J & J
A^0(G) & A^0(G) & A^0(G) & A^0(G) & J & J & J & J & J
A^0(G) & A^0(G) & A^0(G) & A^0(G) & J & J & J & J & J
\end{bmatrix}.$$ 

By direct calculations

$$\det(A^0(\Gamma_{G \times G}) - Ix) = |-Ix|^4 \left| 2A(G)^0 - Ix - 2J \right|^2 \begin{vmatrix} 2A(G)^0 - Ix & 6A(G)^0 \\ 2A(G)^0 & 2A(G)^0 - Ix + 4J \end{vmatrix}.$$ 

Thus

$$\left| 2A(G)^0 - Ix - 2J \right|^2 =$$

$$\begin{vmatrix}
-x - 2 & -2 & 0 & 0 & 0 & 0 & -2 & -2 \\
-2 & -x - 2 & 0 & 0 & 0 & 0 & -2 & -2 \\
0 & 0 & -x - 2 & -2 & 0 & 0 & -2 & -2 \\
0 & 0 & -2 & -x - 2 & 0 & 0 & -2 & -2 \\
0 & 0 & 0 & -x - 2 & -2 & -2 & -2 & -2 \\
0 & 0 & 0 & 0 & -2 & -x - 2 & -2 & -2 \\
-2 & -2 & -2 & -2 & -2 & -2 & -x - 2 & -2 \\
-2 & -2 & -2 & -2 & -2 & -2 & -x - 2 & -2 \\
\end{vmatrix}^2 = (-x)^8(-x - 4)^4(x^2 + 8x - 32)^2.$$ 

Now assume that
\[
\begin{vmatrix}
2A(G)^0 - Ix & 6A(G)^0 \\
2A(G)^0 & 2A(G)^0 - Ix + 4J
\end{vmatrix} = M.
\]

By calculating
\[
detM = (-x)^9(x^2 + 8x - 32)^2(x^2 - 40x - 128)(-x + 8).
\]

According to the description of the beginning of the proof, we conclude that
\[
P_{\Gamma \times G}(x) = \frac{(-x)^{49}(-x + 8)(-x - 4)^4(x^2 + 8x - 32)^4(x^2 - 40x - 128)}{(-x)^4}
\]
\[
= (-x)^{45}(-x + 8)(-x - 4)^4(x^2 + 8x - 32)^4(x^2 - 40x - 128)
\]
and
\[
E(\Gamma \times G) = \sum_{i=1}^{60} |\lambda_i| = 8\left(3 + \sqrt{33} + 4\sqrt{3}\right).
\]

\[\square\]

**Theorem 3.3.** Let \(G = D_{2n}\). If \(n\) is even and \(n > 4\), then
\[
P_{\Gamma \times G}(x) = \frac{(-x)^{\frac{n^2}{4} - 2n - 3}(-x - 4)^{\frac{n^2}{4} - n + 1}}{(-x)^4}\left(-x^3 - (2n + 4)x^2 + 16n(n - 2)\right)^{n-2}f(x),
\]
where \(f(x)\) is a polynomial of degree 8. Also the spectrum of \(\Gamma \times G\) is equal to
\[
\text{spec}(\Gamma \times G) = 
\left(\begin{array}{cccccccccc}
\frac{n^2}{4} - n + 1 & 0 \\
\frac{15n^2}{4} - 2n - 7 & n - 2 & n - 2 & n - 2 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right),
\]
where \(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8\) and \(\alpha_1, \alpha_2, \alpha_3\) are roots of \(f(x)\) and \((-x^3 - (2n + 4)x^2 + 16n(n - 2))\), respectively.
Proof. By the proof of Lemma 3.2, we have

\[ A^0_{ij}(\Gamma_{G \times G}) = \begin{cases} 
A^0(G) & 1 \leq i \leq 2n, 1 \leq j \leq 2 \\
A^0(G) & 1 \leq i \leq 2, 1 \leq j \leq 2n \\
A^0(G) & 3 \leq i, j \leq n \\
A^0(G) & i = k + t, j = k + s; t, s = 0 \text{ or } 1 \\
& \quad \text{and } k = n + 1, \ldots, 2n - 1 \\
J & \text{o.w}
\end{cases} \]

By direct calculations

\[ P_{\Gamma_{G \times G}}(x) = \frac{|-Ix|^\frac{4n}{2} - 2}{(-x)^3} |2A^0(G) -Ix - 2J|^{\frac{n}{2} - 1} \]

\[ \begin{vmatrix} 
2A^0(G) -Ix & (n-2)A^0(G) & nA^0(G) \\
2A^0(G) & (n-2)A^0(G) -Ix & nJ \\
2A^0(G) & (n-2)J & 2A^0(G) -Ix + (n-2)J
\end{vmatrix}. \]

Thus

\[ |2A^0(G) -Ix - 2J|^{\frac{n}{2} - 1} = \left( (-x)^{\frac{4n}{2} - 2}(-x - 4)^{\frac{n}{2} - 1} \right)^{\frac{2}{n} - 1} \]

\[ \left( -x^3 - (2n+4)x^2 + 16n(n-2) \right)^{\frac{2}{n} - 1}. \]

Now assume that

\[ \begin{vmatrix} 
2A^0(G) -Ix & (n-2)A^0(G) & nA^0(G) \\
2A^0(G) & (n-2)A^0(G) -Ix & nJ \\
2A^0(G) & (n-2)J & 2A^0(G) -Ix + (n-2)J
\end{vmatrix} = M. \]

By direct calculations, we have

\[ \text{det}(M) = (-x)^{\frac{4n}{2} - 5} \]
where $a = (n - 2)^2$, $b = 2(n - 2)$ and $c = n(n - 2)$. Therefore

$$det(M) = \frac{1}{2} (-x)^{\frac{n}{2}} (-2x)^{2n+2}$$

Let

$$A = \begin{pmatrix}
-x & 4 & \ldots & 4 \\
b & -x & \ldots & 4 \\
\vdots & \ddots & \ddots & \vdots \\
b & 4 & \ldots & -x \\
-x & b & \ldots & b \\
\vdots & \vdots & \ddots & \vdots \\
-x & b & \ldots & b \\
\end{pmatrix},$$

$$B^T = \begin{pmatrix}
h & h & \ldots & h & h + 4c \\
h & h & \ldots & h & 6c \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
h & h & \ldots & h & 6c \\
\end{pmatrix},$$

$$C = \begin{pmatrix}
\frac{c}{2} & n \ldots n & (c+2a)x+cn & 4n & (3n-4)x+6c \\
\frac{c}{2} & n \ldots n & 2nx & 4c & (3n-4)x+6c \\
\end{pmatrix},$$

and

$$d = -x + \frac{bx + na + nc + 4c}{x}.$$
where $B^T$ is transpose of $B$. It follows that
\[
det(M) = \frac{1}{2}(-x)^{\frac{2n}{3} - 5}(-2x)^{\frac{2}{3} + 2}[(d - 1)\det(A) + \det(A - BC)]
\]
\[
= (-x)^{\frac{2n}{3} - 5}\left(-x^3 - (2n + 4)x^2 + 16n(n - 2)\right)^{\frac{5}{3} - 1} f(x),
\]
where $f(x)$ is a polynomial of degree 8. This implies that
\[
\begin{aligned}
P_{\Gamma_{G \times G}}(x) &= \frac{(-x)^{\frac{4n^2}{3} - 2n - 3}}{(-x)^{n+1}}\left(-x^3 - (2n + 4)x^2 + 16n(n - 2)\right)^{n-2} f(x).
\end{aligned}
\]
Now let $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7$ and $\gamma_8$ be roots of $f(x)$ in $P_{\Gamma_{G \times G}}(x)$. Suppose $\alpha_1$, $\alpha_2$ and $\alpha_3$ are roots of
\(( - x^3 - (2n + 4)x^2 + 16n(n - 2) )
\). Then
\[
spec(\Gamma_{G \times G}) = \begin{pmatrix} 
\frac{n^2}{2} - n + 1 & 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 \\
\end{pmatrix}.
\]
\[
\square
\]
**Corollary 3.4.**
\[
\mathcal{E}(\Gamma_{D_{2n} \times D_{2n}}) = (n^2 - 4n + 4) + (n - 2)\left(|\alpha_1| + |\alpha_2| + |\alpha_3|\right)
\]
\[
+ |\gamma_1| + |\gamma_2| + |\gamma_3| + |\gamma_4| + |\gamma_5| + |\gamma_6| + |\gamma_7| + |\gamma_8|.
\]

**Example 3.5.**
\[
P_{\Gamma_{D_{12} \times D_{12}}}(x) = (-x)^{384}(x^3 + 16x^2 - 384)^4(x + 4)^4(x + 24)(x^3 - 384x + 2304)
\]
\[
(x^4 - 104x^3 - 1152x^2 + 5376x + 55296)
\]
and
\[
\mathcal{E}(\Gamma_{D_{12} \times D_{12}}) = 40 + 4(|\alpha_1| + |\alpha_2| + |\alpha_3|) + (|\beta_1| + |\beta_2| + |\beta_3|)
\]
\[
+ (|\gamma_1| + |\gamma_2| + |\gamma_3| + |\gamma_4|).
\]
where $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ are roots of
\((x^3 + 16x^2 - 384)\) and \((x^3 - 384x + 2304)\), respectively. Also $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are roots of \((x^4 - 104x^3 - 1152x^2 + 5376x + 55296)\).
Theorem 3.6. Let $G$ be a finite non-abelian group, where $|Z(G)| = t$. Then

$$P_{\Gamma_{G \times S_3}}(x) = \frac{|-Ix|}{(-x)^t} \left| A^0(G) - Ix - J \right|^2$$

$$\begin{vmatrix}
A^0(G) - Ix & 2A^0(G) & 3A^0(G) \\
A^0(G) & 2A^0(G) - Ix & 3J \\
A^0(G) & 2J & A^0(G) - Ix + 2J
\end{vmatrix}.$$ 

Proof. By the proof of Lemma 3.2, the adjacency matrix of $\Gamma_{G \times S_3}$ as following

$$A^0_{ij}(\Gamma_{G \times S_3}) = \begin{cases}
A^0(G) & i = 1, 1 \leq j \leq 6 \\
A^0(G) & 1 \leq i \leq 6, j = 1 \\
A^0(G) & 2 \leq i, j \leq 3 \\
A^0(G) & i = j = 4, 5, 6 \\
J & \text{otherwise}
\end{cases}$$

Similar to the proof of Lemma 3.2, we have

$$P_{\Gamma_{G \times S_3}}(x) = \frac{|-Ix|}{(-x)^t} \left| A^0(G) - Ix - J \right|^2$$

$$\begin{vmatrix}
A^0(G) - Ix & 2A^0(G) & 3A^0(G) \\
A^0(G) & 2A^0(G) - Ix & 3J \\
A^0(G) & 2J & A^0(G) - Ix + 2J
\end{vmatrix}.$$ 

\[\square\]

Theorem 3.7. Let $G$ be a finite non-abelian group, where $|Z(G)| = t$. If $n$ is even, then

$$P_{\Gamma_{G \times D_{2n}}}(x) = \frac{|-Ix|}{(-x)^{\frac{n}{2}}t} \left| 2A^0(G) - Ix - 2J \right|^{\frac{n}{2}}$$

$$\begin{vmatrix}
2A^0(G) - Ix & (n - 2)A^0(G) & nA^0(G) \\
2A^0(G) & (n - 2)A^0(G) - Ix & nJ \\
2A^0(G) & (n - 2)J & 2A^0(G) - Ix + (n - 2)J
\end{vmatrix}.$$
Proof. By the proof of Lemma 3.2, we have

\[ A^0_{ij}(\Gamma \times D_{2n}) = \begin{cases} 
A^0(G) 
1 \leq i \leq 2n, 1 \leq j \leq 2 \\
A^0(G) 
1 \leq i \leq 2, 1 \leq j \leq 2n \\
A^0(G) 
3 \leq i, j \leq n \\
A^0(G) 
i = k + t, j = k + s; t, s = 0 \text{ or } 1 \\
\text{and } k = n + 1, \ldots, 2n - 1 \\
J \text{ o.w}
\end{cases} \]

By direct calculations

\[ P_{\Gamma \times D_{2n}}(x) = \frac{|-Ix|^{2n-2}}{(-x)^{2t}} \left| 2A^0(G) - Ix - 2J \right|^{n-1} \]

\[ \left| \begin{array}{ccc}
2A^0(G) - Ix & (n-2)A^0(G) & nA^0(G) \\
2A^0(G) & (n-2)A^0(G) - Ix & nJ \\
2A^0(G) & (n-2)J & 2A^0(G) - Ix + (n-2)J \\
\end{array} \right| . \]

□

Theorem 3.8. Let \( G \) be a finite non-abelian group, where \( |Z(G)| = t \). Then

\[ P_{\Gamma \times D_n}(x) = \frac{|-Ix|^4}{(-x)^{2t}} \left| 2A^0(G) - Ix - 2J \right|^2 \]

\[ \left| \begin{array}{ccc}
2A^0(G) - Ix & 6A^0(G) & 6A^0(G) \\
2A^0(G) & 2A^0(G) - Ix + 4J & 4A^0(G) - Ix - 4J \\
\end{array} \right| . \]

Proof. It is similar to the proof of Lemma 3.2. □

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