Estimating the parameters of wind turbulence from spectra of radial velocity measured by a pulsed Doppler lidar

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Abstract. The paper proposes the method for estimating the turbulence energy dissipation rate and the variance of the vertical component of the wind velocity vector from the spectral density of radial velocity measured by a pulsed coherent Doppler lidar. The method takes into account the averaging of the radial velocity over the probing volume. It is shown that neglecting the spatial averaging leads to the understating the dissipation rate.

1. Introduction

Turbulence in the stably stratified atmosphere is a subject of study for many decades, but still remains poorly understood despite numerous publications. At the stable stratification in the atmospheric boundary layer (ABL), low level jet streams (LJS) and internal gravity waves (IGW) are formed, and turbulence is characterized by intermittence and not always obeys the Kolmogorov-Obukhov law.

The use of pulsed coherent Doppler lidars (PCDL) allows obtaining the information about wind, turbulence, and IGW in the whole ABL. Different PCDL measurement strategies (geometries) and methods for estimation of wind and turbulence parameters from raw lidar data have been developed [1-7]. In particular, the method of azimuthal structure function (ASF) for estimating the turbulence energy dissipation rate $\varepsilon$ and the variance $\sigma_\nu^2 = \langle V_v^2 \rangle - \langle V_v \rangle^2$ of radial velocity $V_v$ from PCDL data obtained with conical scanning by a lidar probing beam around the vertical axis was proposed [6]. However, during IGW this method appears to provide highly overstated values of $\varepsilon$ and $\sigma_\nu^2$. The modified ASF method was proposed in [8]. For the first time, it allowed obtaining unbiased estimates of the dissipation rate $\varepsilon$ at heights of jet streams during IGW propagation. However, the following experiments have shown that this modified ASF method not always gives correct results.

Other scenarios of lidar measurements are also used to study wind turbulence. For example, in [4] it was proposed to determine the turbulence energy dissipation rate from temporal spectra of the vertical component of wind velocity vector measured by PCDL at the probing vertically upward. This method was applied to examine variations of the dissipation rate in the atmospheric boundary layer [7]. However, when obtaining estimates of $\varepsilon$, the authors of [4,7] ignored the spatial averaging of the radial velocity along the probing beam axis, which can lead to significant underestimation of the dissipation rate.

This paper proposes a strategy of lidar measurements, which allows us to determine both the wind velocity vector and the temporal and spatial (along the mean wind direction) spectra of turbulent fluctuations of the vertical component of wind velocity vector from data of a single PCDL. The theoretical relations presented allow the averaging over the probing volume to be taken into account when estimating the spectra of fluctuations of the vertical velocity measured by the lidar. These
relations form the basis for the new method for estimation of the turbulence energy dissipation rate \( \varepsilon \) and the variance of vertical component of wind velocity \( \sigma_w^2 = \sigma_r^2 \) from the spectra of the vertical velocity measured by the lidar.

2. Measurement strategy and methodology of estimating the wind velocity vector and spectra of vertical velocity

For the data of a single lidar could be used for retrieving the wind velocity vector and calculating the temporal and spatial spectra of turbulence fluctuations of the vertical velocity, the following measurement strategy is applied. First, one conical scan by the probing beam at an elevation angle \( \varphi = 60^\circ \) for \( T_{\text{scan}} = 1 \) min is performed. Then the probing beam is directed vertically upward (elevation angle \( \varphi = 90^\circ \)) and the vertical probing is conducted for \( T_{\text{vert}} \). Then the elevation angle is changed from \( 90^\circ \) to \( 60^\circ \) and the procedure is repeated. The duration of one measurement cycle is \( T_c = T_{\text{scan}} + \delta t + T_{\text{vert}} + \delta t \), where \( \delta t \sim 10 \) s is the time needed to change the angle \( \varphi \). In the experiment, the time of vertical probing was taken \( T_{\text{vert}} = 500 \) s. The time for measurement of the radial velocity \( \Delta t \) is determined by the number of accumulated echo signals \( N_s \), which is set equal to 7500. At the pulse repetition frequency of StreamLine PCDL (Halo Photonics, Brockamin, Worcester, United Kingdom) \( f_p = 15 \) kHz, this corresponds to \( \Delta t = N_s / f_p = 0.5 \) s. Correspondingly, the number of measurements of the radial velocity was \( M_1 = T_{\text{vert}} / \Delta t = 1000 \) for the time \( T_{\text{vert}} = 500 \) s, and \( M_s = T_{\text{scan}} / \Delta t = 120 \) per one scan.

As a result, we obtained two arrays of data for estimating the signal-to-noise ratio \( \text{SNR}(m'\Delta t; R_k, \varphi_{\text{scan}}, n) \) and the radial velocity \( V_x(m'\Delta t; R_k, \varphi_{\text{scan}}, n) \) from measurements at the conical scanning and estimates of \( \text{SNR}(m\Delta t; R_k, \varphi_{\text{vert}}, n) \) and \( V_x(m\Delta t; R_k, \varphi_{\text{vert}}, n) \) from measurements in the vertical direction, where \( m' = 0,1,2,\ldots,M_s -1 \); \( m = 0,1,2,\ldots,M_1 -1 \) \( R_k = R_0 + k\Delta R \) is the distance from the lidar; \( k = 0,1,2,\ldots,K -1 ; \Delta R \) is the range step; \( \varphi_{\text{scan}} = 60^\circ \) and \( \varphi_{\text{vert}} = 90^\circ \) are the elevation angles of the conical and vertical scanning, respectively, and \( n = 1,2,3,\ldots \) is the number of cycle with duration \( T_c \).

The sine-wave fitting method [9] was applied to find the horizontal wind velocity \( U_i(h_k,n) \), wind direction angle \( \theta_i(h_k,n) \), and the velocity of vertical wind component \( V_z(h_k,n) \) averaged over the circle of the scanning cone at the height \( h_k = R_k \sin \varphi_{\text{scan}} \) for the time \( T_{\text{scan}} \) from the \( V_x(m'\Delta t; R_k, \varphi_{\text{scan}}, n) \) arrays obtained at the conical scanning. The \( V_x(m\Delta t; R_k, \varphi_{\text{vert}}, n) \) arrays obtained at the vertical scanning were used to determine the vertical component of the wind velocity vector \( w \) and to estimate the temporal \( \hat{S}_x(f_i) \) and spatial \( \hat{S}_x(\kappa_i) \) spectra of turbulent fluctuations of the vertical velocity, where \( f_i \) and \( \kappa_i = f_i / U \) are the temporal and spatial frequencies, respectively, and \( U \) is the average horizontal velocity at the height of spectral measurements.

To obtain estimates of wind turbulence parameters, data measured for half an hour are usually used. At \( T_{\text{vert}} = 500 \) s, the cycle duration is \( T_c \approx 10 \) min. Therefore, to estimate the spectral density of the vertical velocity measured by the lidar \( \hat{S}_x(f_i) \), the data of three consecutive measurement cycles were used. Correspondingly, the average velocity was estimated from four values of \( U_i \) as \( U(h_k,n) = (1/4) \sum_{i=0}^{3} U_i(h_k,n-1+i) \). The spectra \( \hat{S}_x(f_i) \) are estimated from the arrays of the radial
velocity $V_L(m\Delta t; R_k, \phi_{vert}, n)$ at heights $h_k = R_k \sin \phi_{vert} = R_k$, which differ from the heights of measurement of the average velocity $h_k = R_k \sin \phi_{scan}$. Therefore, to find the average velocity at the height of spectral measurements, the interpolation between the measured values of $U$ was applied. Experimental temporal spectra of the radial (vertical) velocity $\hat{S}_L(f_i)$ were calculated as

$$\hat{S}_L(f_i) = \frac{1}{3} \sum_{l=0}^{M} \frac{\Delta t}{M} \sum_{m=0}^{M-1} V_L(m\Delta t; R_k, \phi_{vert}, n) \exp \left( -2\pi j \frac{m}{M} f_i \right).$$

(1)

In Eqs. (1), $f_i = l\Delta f$; $l = 1, 2, 3, ..., M$ is the number of the spectral channel; $M = 1000$; $\Delta f = (M\Delta f)^{-1} = 0.002$ Hz is the width of the spectral channel; $0.002$ Hz $\leq f_i \leq 1$ Hz, and $j = \sqrt{-1}$ is the imaginary unit. The summation of spectral components over all frequencies gives the estimate of the variance of lidar estimate of the radial velocity

$$\sigma^2_L = 2\Delta f \sum_{l=1}^{M/2} \hat{S}_L(f_i).$$

(2)

The Taylor hypothesis of frozen turbulence allows us to pass from the temporal $\hat{S}_L(f_i)$ to spatial $\hat{S}_L(\kappa_i) = U\hat{S}_L(U\kappa_i)$ spectra, where $\kappa_i = l\Delta \kappa$, $\Delta \kappa = \Delta f / U$. Whence we can find

$$\hat{S}_L(f_i) = U^{-1}\hat{S}_L(f_i / U)$$

and

$$\sigma^2_L = 2\Delta \kappa \sum_{l=1}^{M/2} \hat{S}_L(\kappa_i).$$

3. Consideration of spatial averaging of the radial velocity

To take into account the averaging of radial velocity over the probing volume when estimating the dissipation rate $\varepsilon$ from the spectrum $\hat{S}_L(f_i)$ (or $\hat{S}_L(\kappa_i)$), it is necessary to know the low pass filter transfer function $H(\kappa_i)$ [5]. The equation for $H(\kappa_i)$ was derived in [5] for the case of estimating the radial velocity from the position of centroid of the Doppler spectrum. In the StreamLine PCDL with the low pulse energy, the signal-to-noise ratio SNR in the 50-MHz frequency bandwidth of the receiver usually does not exceed -10 dB. That is why the radial velocity is estimated from the position of maximum of the Doppler spectrum.

To make sure that the known equation for the transfer function $H(\kappa_i)$ [5] can be used to take into account the spatial averaging when estimating the radial velocity from the maximum of the Doppler spectrum, we have used numerical simulation to compare the transverse fluctuation spectra of the radial velocity estimated from Doppler spectra of the StreamLine lidar and the transverse spectra of the radial velocity averaged over the probing volume with the same averaging function as in [5]. The simulation was performed in the following way.

Assume that the longitudinal dimension of the probing volume $\Delta Z$ exceeds its transverse dimension $\Delta Y$ ($\Delta Y = U \Delta t$ at the vertical probing). Then, according to [5], the equation for the normalized correlation function of the useful component of the complex lidar signal $C_\delta(l'T_s, z, y)$ can be written as

$$C_\delta(l'T_s, z, y) = \int_{-\infty}^{\infty} dz' A(l', z')(-1)^{l'} \exp[2\pi j l' B^{-1} V_s(z + z', y)],$$

(3)

where $l' = 0, 1, 2, ..., 6$; $T_s = B^{-1}$; $B = 50$ MHz is the receiver frequency bandwidth; $z$ is the distance from the lidar to the center of the probing volume; $y = Ut$ is the transverse coordinate; $t$ is
time: \( A(l', z') = \frac{1}{7-l'} \sum_{m=0}^{6-l'} F(z'-(m'-3)\delta R) F(z'-(m'+l'-3)\delta R) \); \( \delta R = c T_f / 2 = 3 \text{ m} \); \( c \) is the speed of light; \( F(z') = (\sqrt{\pi} \Delta p)^{-1/2} \exp[-(z' / \Delta p)^2 / 2] \); \( \Delta p = c \sigma_p / 2 = 15.3 \text{ m} \); \( B_y = \lambda / (2 T_f) = 37.5 \text{ m/s} \); \( \lambda = 1.5 \mu \text{m} \) is probing beam wavelength, and \( V_r(z, y) \) is the 2D random distribution of the radial velocity over the plane \( \{ z, y \} \). Random realizations of \( V_r(z, y) \) at the nodes of the 1024×1024 computational grid with 3-m cells were simulated with the use of the von Karman model for the two-dimensional spectrum of wind velocity \( S_r(\kappa_r, \kappa_y) \) based on the algorithm described in [9].

The Doppler spectrum is calculated from \( C_r(l'T_f; z, y) \) with the Fast Fourier Transform (FFT) as

\[
S_D(f_k; z, y) = \text{Re} \left\{ \sum_{l=0}^{6} (2-\delta_r)(1-l'/7)C_r(l'T_f; z, y) \exp \left[ -2\pi f \frac{l'k'}{N} \right] \right\},
\]

where \( f_k = k'\delta f \) is the frequency; \( k' = 0, 1, 2, ..., N \) is the spectral channel number; \( \delta f = B / N \) is the spectral channel width, \( N = 1024 \) is the number of spectral channels, and \( \delta_r \) is the Kronecker delta (\( \delta_r = 1 \) and \( \delta_r = 0 \)). In Eq. (4), we can pass from the frequency distribution to the velocity distribution: \( V_k' = k'\delta V - B_y / 2 ; \delta V = B_y / N = 0.03662 \text{ m/s} \). The position of maximum in the Doppler spectrum is used to estimate the radial velocity \( \max(z, y) \) and to calculate the transverse spectrum of fluctuations of the radial velocity as

\[
S_{\max}(\kappa) = \int_{-\infty}^{+\infty} dr B_{\max} (r) \exp(-2\pi j \kappa r) ,
\]

where \( B_{\max}(r) = <\max'_r(z, y + r)\max'_r(z, y)> \) is the transverse correlation function of velocity fluctuations \( \max'_r = \max - <\max> \), and angular brackets denote the averaging over an ensemble of realizations.

The radial velocity \( \max(z, y) \) averaged over the lidar probing volume is given by the integral [5]

\[
\max(z, y) = \int_{-\infty}^{+\infty} dz'Q_r(z')\max'(z+z', y),
\]

where the filtering function (or averaging function) is described as

\[
Q_r(z') = \frac{1}{2\Delta R} \left[ \text{erf} \left( \frac{z'+\Delta R/2}{\Delta p} \right) - \text{erf} \left( \frac{z'-\Delta R/2}{\Delta p} \right) \right].
\]

In Eq. (7), \( \text{erf}(x) = (2 / \sqrt{\pi}) \int_{0}^{x} d\xi \exp(-\xi^2) \) is the probability integral. In our case, the range step is \( \Delta R = 6\delta R = 18 \text{ m} \), and the longitudinal dimension of the probing volume [5] is \( \Delta z = Q_{\perp}(0) = \Delta R / \text{erf} \left( \Delta R / (2\Delta p) \right) \approx 30 \text{ m} \). The transverse spectra \( S_r(\kappa) \) are calculated from the transverse correlation functions \( B_{\perp}(r) = <\max'_r(z, y + r)\max'_r(z, y)> \), where \( V'_r = \max - <\max> \), by the equation similar to Eq. (5).
Equations (3), (4), (6), and (7) are applied to calculate $V_{\text{max}}(z, y)$ and $V_{d}(z, y)$ as functions of the transverse coordinate $y$ at fixed $z$ from the simulated distributions $V'(z, y)$ and then to determine the transverse spectra $S_{\text{max}}(\kappa)$ and $S_{d}(\kappa)$. The averaging over an ensemble was performed with the use of no less than 1000 independent random realizations of $V'(z, y)$. For comparison, we also calculated the transverse spectra of the radial velocity without spatial and temporal averaging $S_{r}(\kappa) = \int dr B_{r}(r) \exp(-2\pi jkr)$, where $B_{r}(r) = \langle V''(z, y+r)V''(z, y) \rangle$, $V'' = V' - \langle V' \rangle$. The simulation of $V''$ was carried out with the use of the von Karman model (see equation (17) in [9]).

\[ S_{r}(\kappa) = \frac{4}{3} 0.07333 C_{k} \epsilon^{2/3} \kappa^{-5/3}, \]

where $C_{k} = 2$ is the Kolmogorov constant, is shown by the dashed curve. Whence for the von Karman model, we have [9]:

\[ L_{V} = 0.6973 \sigma_{r}^{3} / \epsilon, \]

where $\sigma_{r}^{2}$ is the variance of radial velocity. It can be seen from Fig. 1 that the spectra $S_{\text{max}}(\kappa)$ and $S_{d}(\kappa)$ completely coincide. This means that Eqs. (6) and (7) can be used to take into account the averaging of the radial velocity over the probing volume when estimating wind turbulence parameters from measurements by the StreamLine lidar. With the use of Eq. (7), the spatial low pass filter transfer function $H_{\parallel}(\kappa_{z})$ in the longitudinal direction has the form [5]

\[ H_{\parallel}(\kappa_{z}) = \left| \int_{-\infty}^{\infty} dz' Q_{z}(z') \exp(-2\pi j\kappa_{z} z') \right|^{2} = \left[ \exp\{-2\pi \Delta \rho \kappa_{z}^{2}\} \right]^{2}, \]

\[ \kappa_{z} = \frac{\pi}{\Delta R}, \]

\[
\text{Figure 1.} \text{ Transverse spectra of the radial velocity } S_{r}(\kappa) \text{ (black curve), } S_{\text{max}}(\kappa) \text{ (green curve), and } S_{d}(\kappa) \text{ (red curve). The blue dashed curve is for the Kolmogorov-Obukhov } \cdot 5/3 \text{ spectrum.}
\]

Figure 1 shows an example of the spectra $S_{r}(\kappa)$, $S_{\text{max}}(\kappa)$, and $S_{d}(\kappa)$ obtained in the numerical experiment with the following wind turbulence parameters given: variance of the radial velocity $\sigma_{r}^{2} = 1$ (m/s)$^{2}$ and the integral scale of turbulence $L_{V} = 100$ m. In addition, the transverse Kolmogorov-Obukhov $\cdot 5/3$ spectrum [10]

\[ S_{r}(\kappa) = \frac{4}{3} 0.07333 C_{k} \epsilon^{2/3} \kappa^{-5/3}, \]

where $S_{d}(\kappa)$ completely coincide. This means that Eqs. (6) and (7) can be used to take into account the averaging of the radial velocity over the probing volume when estimating wind turbulence parameters from measurements by the StreamLine lidar. With the use of Eq. (7), the spatial low pass filter transfer function $H_{\parallel}(\kappa_{z})$ in the longitudinal direction has the form [5]
where \( \text{sinc}(x) = \frac{\sin(x)}{x} \).

One can also see from Fig. 1 that at the frequencies \( \kappa \geq 1/(2L_v) = 0.005 \text{ m}^{-1} \) corresponding to the inertial range of turbulence, where the blue dashed curve and the black solid curve coincide, the averaging of the radial velocity within the probing volume along the axis \( z \) significantly overstates the spectral density. Thus, at the frequency \( \kappa = 0.1 \text{ m}^{-1} \) the amplitude of the spectrum \( S_a(\kappa) \) is sevenfold smaller than the amplitude of \( S_v(\kappa) \). This should necessarily be taken into account when estimating the dissipation rate \( \varepsilon \) from turbulent fluctuations of the velocity measured by the lidar.

We have conducted similar numerical experiments at different values of \( \sigma^2 \) (from 0.01 to 4 m\(^2\)/s\(^2\)) and \( L_v \) (from 30 to 300 m). In all the cases, very close results were obtained for \( S_{\text{max}}(\kappa) \) and \( S_a(\kappa) \), which confirms that Eqs. (6), (7), and (10) can be used for a good reason to take into account the averaging of the radial velocity measured by the StreamLine lidar over the probing volume.

4. Estimation of wind turbulence parameters

To estimate parameters of wind turbulence from the spectrum of radial velocity measured by the lidar (1), it is necessary to have the theoretically calculated estimate of the spectrum averaged over the ensemble of realizations, \( S_L(f_i) = \langle \hat{S}_L(f_i) \rangle \), with allowance made for the averaging over the probing volume and the instrumental error of measurement of the radial velocity. According to [3, 5], the unbiased lidar estimate of the radial velocity \( V_L(m\Delta t) \) can be represented as

\[
V_L(m\Delta t) = V_a(m\Delta t) + V_e(m\Delta t),
\]

where

\[
V_a(m\Delta t) = \frac{1}{\Delta t} \int_{-\Delta t/2}^{+\Delta t/2} dt' \int dz' Q_v(z') V_v(R_u + z', (m\Delta t + t')U)
\]

is the radial velocity averaged over the probing volume with the longitudinal dimension \( \Delta z \) and the transverse dimension \( \Delta y = U/\Delta t \); \( V_v(z,y) \) is the radial velocity at the point with coordinates \( \{z, y\} \); \( V_e(m\Delta t) \) is the random instrumental error (caused mostly by the noise component of the Doppler spectrum), which is characterized by the following statistical properties: \( <V_e> = <V_aV_e> = 0 \) and \( <V_e(m\Delta t)V_e(m'\Delta t)> = \sigma_e^2 \delta_{\Delta t=0} \); \( \sigma_e^2 <V_e^2> \).

Upon the substitution of Eq. (11) into Eq. (1) and averaging over the ensemble of realizations, with allowance made for the random instrumental error we obtain the equation

\[
S_L(f_i) = S_a(f_i) + S_N,
\]

where

\[
S_a(f_i) = \frac{\Delta t}{M_1} \sum_{m=0}^{M-1} V_a(m\Delta t) \exp \left( -2\pi f_j \frac{\ln M_1}{M_1} \right)^2
\]

is the spectral component carrying the information about wind turbulence, and

\[
S_N = \Delta t \sigma_e^2
\]

is the noise component (white noise) caused by the instrumental error of estimation of the radial velocity (\( \sigma_e \neq 0 \)).
On the assumption of isotropic turbulence for \( S_a(f_i) \) at frequencies \( f_i \geq U / (2L_v) \) corresponding to the inertial range, we can obtain from Eqs. (12) and (14) that

\[
S_a(f_i) = \int_{-\infty}^{+\infty} d\kappa_x d\kappa_y S_i(\kappa_x, \kappa_y) H_{||}(\kappa_y)[\text{sinc}(\pi U \Delta \kappa_y)]^2 P(f_i - U \kappa_y),
\]

where

\[
S_i(\kappa_x, \kappa_y) = 0.0163 \varepsilon^{2/3}(\kappa_x^2 + \kappa_y^2)^{-4/3} \left[ 1 + \frac{8 \kappa_y^2}{3 \kappa_x^2 + \kappa_y^2} \right]
\]

is the two-dimensional spatial spectrum of the radial velocity in the inertial range of turbulence (Kolmogorov-Obukhov spectrum) [9] and

\[
P(f_i - U \kappa_y) = \frac{\Delta t}{M_1} \frac{1 - \cos[2\pi \Delta t (f_i - U \kappa_y) M_1]}{1 - \cos[2\pi \Delta t (f_i - U \kappa_y)]}.
\]

In addition to low-pass filtering, Eq. (16) takes into account the phenomenon of frequency substitution, which is observed in experiments when the spectrum is nonzero at frequencies higher than the Nyquist frequency \( f_N = (2\Delta t)^{-1} \) [11]. If the frequency substitution is neglected, then the function \( P(f_i - U \kappa_y) \) can be represented as

\[
P(f_i - U \kappa_y) = U^{-1} \delta(\kappa_y - f_i / U), \tag{19}
\]

where \( \delta(x) \) is the delta function. Upon the substitution of Eq. (19) into Eq. (16), we obtain the approximate equation for the spectrum with the neglected frequency substitution

\[
S_a^{(1)}(f_i) = 0.0326(\varepsilon U)^{2/3} f_i^{-5/3} \int_0^\infty d\xi (1 + \xi^2)^{-4/3} [1 + (8/3) / (1 + \xi^2)] H_{||}(f_i \xi / U)[\text{sinc}(\pi \Delta f_i)]^2. \tag{20}
\]

The frequency substitution can be taken into account, if we use the equation [11]

\[
S_a(f_i) = S_a^{(1)}(f_i) + \sum_{n=1}^{N'} \left[ S_a^{(1)}(2nf_N - f_i) + S_a^{(1)}(2nf_N + f_i) \right]. \tag{21}
\]

The calculations of \( S_a(f_i) \) by Eqs. (16)-(18) and by Eqs. (20)-(21) demonstrate that already at \( N' = 1 \) the results coincide almost completely.

The spectrum \( S_a(f_i) \) is proportional to \( \varepsilon^{2/3} \), where \( \varepsilon \) is the sought parameter. Therefore, it is convenient to introduce the fitting function of the frequency of the function \( f_i \) and the average wind velocity \( U \) as

\[
G(f_i) = S_a(f_i) / \varepsilon^{2/3}. \tag{22}
\]

Equations (16) - (22) are valid for frequencies \( f_i \geq U / (2L_v) \) (inertial range of turbulence). Let \( U = 10 \) m/s and \( L_v = 100 \) m. Then \( f_i \geq 0.05 \) Hz. Consider the functions \( G(f_i) \), \( G'(f_i) = S_a^{(1)}(f_i) / \varepsilon^{2/3} \) (fitting function neglecting the frequency substitution), and

\[
G_k(f_i) = 0.0974U^{2/3} f_i^{-5/3}. \tag{23}
\]
(fitting function neglecting both the averaging of the radial velocity over the probing volume and the frequency substitution) in the frequency range $0.05 \leq f_i \leq 1$ Hz.

Figure 2 exemplifies the fitting functions $G_k(f_i)$, $G_l(f_i)$, and $G(f_i)$ calculated by Eqs. (20)-(23) at different values of the average wind velocity $U$. One can see that the effect of frequency substitution manifests itself mostly in the range 0.8 - 1 Hz (compare the red and green curves), where the decisive contribution to the spectrum $S_L(f_i)$ measured by the lidar is often due to the noise component of the spectrum $S_N$. Nevertheless, at the strong wind (and turbulence) and low noise ($S_N < S_n(f_i)$) the consideration of the frequency substitution can be important in the range from 0.8 to 1 Hz for correct determination of the instrumental error of estimation of the radial velocity $\sigma_e$. It also follows from Figure 2 that if the dissipation rate $\varepsilon$ is determined from the spectrum $S_L(f_i)$ measured by the lidar within the frequency range $0.05 \leq f_i \leq 0.2$ Hz at wind velocities $U$ higher than 10 m/s, then the use of $G_k(f_i)$, as in [4], can provide quite an acceptable result, since in this frequency range the difference between $G_k(f_i)$ and $G(f_i)$ is small (see Fig. 2(c,d)).

Using equation
$$\eta = \left( \sum_i G_k(f_i) / \sum_i G(f_i) \right)^{3/2},$$
where the summation is performed over frequencies from 0.05 to 0.2 Hz, we have calculated how many times the dissipation rate is underestimated when the effect of averaging the radial velocity over the probing volume is neglected. The results calculated are given in Table 1.

| $U$ [m/s] | 1  | 5  | 10 | 20 |
|-----------|----|----|----|----|
| $\eta$    | 14.1 | 2.2 | 1.45 | 1.16 |
As follows from Table 1, the estimates of the dissipation rate with the neglected averaging of the radial velocity over the probing volume at weak wind \((U = 1 \text{ m/s})\) turn out to be underestimated 14 times. With an increase of the average wind velocity, the bias of the estimate of dissipation rate decreases. However, even at \(U = 10 \text{ m/s}\), it remains significant.

According to [12], the dissipation rate \(\varepsilon\) can be found from the lidar estimate of the spectrum of radial velocity \(\hat{S}_L(f_i)\) by the maximal likelihood method from the maximum of the function

\[
\Phi(\varepsilon, S_N) = - \sum_{i=1}^{M/2} \left[ \ln \left( \varepsilon^{2/3} G(f_i) + S_N \right) + \frac{\hat{S}_L(f_i)}{\varepsilon^{2/3} G(f_i) + S_N} \right],
\]

where \(\Delta f l_3 = 0.05 \text{ Hz}\) and \(l_3 = 25\). To find the dissipation rate \(\varepsilon\) and the instrumental error \(\sigma_\varepsilon = \sqrt{S_N/\Delta t}\) based on Eq. (24), one can use the following iterative procedure. Assuming that in the frequency range from \(\Delta f l_1 = 0.8 \text{ Hz} (l_1 = 400)\) to \(\Delta f M_1/2 = 1 \text{ Hz} (M_1/2 = 500)\) the noise component is a major contributor to the spectrum \(\hat{S}_L(f_i)\) measured by the lidar, we can find the estimate \(S_N\) as

\[
\hat{S}_N^{(1)} = \frac{1}{M_1/2 - l_1 + 1} \sum_{i=l_1}^{M/2} \hat{S}_L(f_i).\]

Then we obtain the following estimate of the dissipation rate from the difference \(\hat{S}_L(f_i) - \hat{S}_N^{(1)}\) in the frequency range from \(\Delta f l_2 = 0.05 \text{ Hz}\) to \(\Delta f l_2 = 0.2 \text{ Hz} (l_2 = 100)\):

\[
\varepsilon^{(1)} = \left[ \frac{1}{l_2 - l_3 + 1} \sum_{i=l_3}^{l_2} \frac{\hat{S}_L(f_i) - \hat{S}_N^{(1)}}{G(f_i)} \right]^{3/2}.
\]

At the second step of the iterative procedure, \(S_N\) is estimated by the equation

\[
\hat{S}_N^{(2)} = \frac{1}{M_1/2 - l_1 + 1} \sum_{i=l_1}^{M/2} [\hat{S}_L(f_i) - \mu G(f_i)],
\]

where \(\mu = (\varepsilon^{(1)})^{2/3}\). Having substituted \(\hat{S}_N^{(2)}\) into Eq. (26) in place of \(\hat{S}_N^{(1)}\), we find the estimate of the dissipation rate at the second step of the iterative process as

\[
\varepsilon^{(2)} = \left[ \frac{1}{l_2 - l_3 + 1} \sum_{i=l_3}^{l_2} \frac{\hat{S}_L(f_i) - \hat{S}_N^{(2)}}{G(f_i)} \right]^{3/2}.
\]

The experiments have shown that one to two iterations are quite sufficient.

With allowance for Eqs. (11) - (23), the obtained estimates of the variance \(\hat{\sigma}_L^2\) given by Eq. (2), dissipation rate \(\hat{\varepsilon}\), and the noise component of the spectrum \(\hat{S}_N\) allow us to estimate the variance of radial velocity \(\hat{\sigma}_r^2\) as

\[
\hat{\sigma}_r^2 = \hat{\sigma}_L^2 - \hat{\sigma}_N/\Delta t + \hat{\varepsilon}^{2/3} 2\Delta f \sum_{i=1}^{M/2} [G_k(f_i) - G_l(f_i)].
\]
At vertical probing, the estimate of variance of radial velocity is the estimate of variance of the vertical component of the wind velocity vector $\sigma_w^2 = \langle w^2 \rangle - \langle w \rangle^2$. In contrast to the results obtained by other authors, Eq. (29) takes into account not only the instrumental error (second term from the right), but also the effect of averaging of the radial velocity over the probing volume on the estimate $\sigma_w^2$ (third term). With the obtained estimates $\hat{\epsilon}$ and $\hat{\sigma}_r^2$, it is possible to calculate the integral scale of turbulence $L_v$ ($L_v$ at vertical probing) by Eq. (9).

5. Conclusions

The paper proposes the strategy of measurements by PCDL, which allows us to obtain spatial (along the mean wind direction) spectra of turbulent fluctuations of the vertical component of the wind velocity vector. The proposed measurement strategy forms the basis of the method for estimating the turbulence energy dissipation rate and the variance of the vertical component of the wind velocity vector from the spectral density of radial velocity measured by PCDL. For the first time, the method takes into account the averaging of the radial velocity over the probing volume. It is shown that if the spatial averaging of the radial velocity is neglected, the lidar estimate of the dissipation rate is understated.

6. References

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