Integrable deformations of a polygon

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Abstract

The new integrable mapping with a simple geometric interpretation is presented. This mapping arise from the nonlinear superposition principle for the B"acklund transformations of some vector evolution equation.

1 Recuttings of the polygon

Let us consider a set of points $v_j \in \mathbb{R}^m$, $j \in \mathbb{Z}_N$, $N \geq 3$ and define the transformation $R_k$ as reflection of the point $v_k$ about the normal hyperplane passing through the middle of the segment $v_{k-1}v_{k+1}$ (all other points of the set remain unmovable). Intuitively, one cuts off the triangle $v_{k-1}v_kv_{k+1}$ from the polygon $v_1...v_N$ (that is closed broken line in $\mathbb{R}^m$), reverses it and glues it back in the same twodimensional plane. The formula which defines this transformation is

$$R_k : \tilde{v}_k = v_k + \frac{|v_{k+1} - v_k|^2 - |v_k - v_{k-1}|^2}{|v_{k+1} - v_{k-1}|^2}(v_{k+1} - v_{k-1}), \quad \tilde{v}_j = v_j, \ j \neq k. \quad (1)$$
Here $|a|^2 = \langle a, a \rangle$ denotes the square of the vector $a$ length, $\langle \cdot, \cdot \rangle$ is standard scalar product. If accidently $v_{k+1} = v_{k-1}$ then $R_k$ is defined as identity transformation. Thus some $N$-valued mapping $R : (\mathbb{R}^m)^N \to (\mathbb{R}^m)^N$ is defined and the problem is to investigate its iterations. Transformations $R_1, \ldots, R_N$ generate some group $G$ which acts on $(\mathbb{R}^m)^N$ and other setting of the problem is to investigate the dynamics of the vertices $v_j$ under this action.

This elementary geometric model possesses a number of remarkable properties and demonstrates very regular and beautiful behavior. In fact, the mapping $R$ turns out to be an example of integrable mappings, the general theory of which is actively developed now. Integrability means existence of the large enough set of invariants which make possible to apply some discrete version of the Liouville theorem (see e.g. [1], [2]). The next section provides a tool for generating of such a set, however the full analysis of its completeness is very complicated and I do not perform it. Instead of it I present some indirect evidences of integrability such as existence of continuous symmetries of the mapping and polynomial growth of the images number under its iterations.

The case $m = 2$ corresponding to the planar polygons was considered in [3], where connection with the Bäcklund transformation for the KdV equation was established. The approach of the present paper is more straightforward and is suitable for the general case.

In conclusion of this introductory section I mention the most obvious features of the model.

1. It is clear that transformation (1) permutes the quantities $\beta_k$ and $\beta_{k-1}$, where $\beta_j = |v_{j+1} - v_j|^2$. Therefore the polynomial

$$\prod_{1 \leq j \leq N} \left( \lambda + \beta_j \right)$$

is preserved under action of the group $G$ and provides $N$ invariants. Further we shall see that there exist other invariants in addition to these.

2. The fixed points of the mapping $R$ are obviously all equilateral polygons.

3. If initial values of all vertices $v_j$ lie on $m'$-dimensional hyperplane or sphere in $\mathbb{R}^m$ then dynamics will be restricted on this submanifolds. In particular we can take without loss
of generality that $m < N$.

4. Other example of invariant manifold come out if $N$ is even and odd vertices lie on one and even vertices on the other of two concentric spheres or parallel hyperplanes.

5. One-dimensional case (all vertices lie on a line or circle) is quite trivial and is reduced to the permutations of the polygons sides combined with shift or rotation.

2 Zero curvature representation

As in the continuous case the existence of some matrix representation like L-A pair is very important for studying of mappings. There exist several modifications of this notion for the discrete case, see e.g. [1], [3], [2]. We shall use the discrete version of the zero curvature representation presented in [3], [5]. Let $v_j$ be column vectors and

$$ W = \begin{pmatrix} -|v_1|^2 & 2v_1^t & 2 \\ -(\lambda + \beta)v_1 - |v_1|^2v & 2vv_1^t + (\lambda + \beta)I_m & 2v \\ \frac{1}{2}(\lambda + \beta)(\lambda + |v_1|^2 + |v|^2) + \frac{1}{2}|v_1|^2|v|^2 & -(\lambda + \beta)v^t - |v|^2v_1^t & -|v|^2 \end{pmatrix}. \quad (3) $$

where $\beta = |v_1 - v|^2$ and $I_m$ is $m \times m$ identity matrix. Let $W_j$ be the matrix $W$ in which $v_1$ and $v$ are substituted by $v_{j+1}$ and $v_j$ correspondingly. The following statement is proved directly.

**Proposition 1.** The transformation $R_k$ is equivalent to the following relations:

$$ R_k : \tilde{W}_k \tilde{W}_{k-1} = W_k W_{k-1}, \quad \tilde{W}_j = W_j, \quad j \neq k, k - 1. \quad (4) $$

This representation implies some important consequences.

**Corollary 1.** The characteristic polynomial

$$ d(\mu, \lambda) = \mu^{m+2} + d_{m+1}(\lambda)\mu^{m+1} + \ldots + d_0(\lambda) = \det(\mu I_m + \tilde{W}_1) \quad (5) $$

where $\tilde{W}_1 = W_N \ldots W_2 W_1$ is invariant under action of the transformations $R_j$. 

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Proof. Obviously the transformations $R_N, \ldots, R_2$ do not change $\tilde{W}_1$. The transformation $R_1$ acts as conjugation:

$$\tilde{W}_N \ldots \tilde{W}_2 \tilde{W}_1 = (\tilde{W}_N W_N^{-1}) \tilde{W}_1 (\tilde{W}_N W_N^{-1})^{-1}$$

and therefore does not change characteristic polynomial as well.

It is easy to check that $\det W = - (\lambda + \beta)^{n+2}$ so that invariant (2) is equivalent to $d_0$. The structure of the other terms in (3) is not clear to me now. Apparently this invariants do not exhaust all invariants of the mapping. It seems that $d(\mu, \lambda)$ is polynomial on the values $|v_i - v_j|^2$ and therefore it is invariant under shift $v \to v + a$. On the other hand, the mapping $R$ admits invariants which are not invariant under the shift and therefore are functionally independent with $d(\mu, \lambda)$. An example of such invariant is vector

$$J = \sum_{i=1}^{N} |v_j|^2 (v_{j+1} - v_{j-1}).$$

(6)

Applying the shift we obtain from it the new invariant

$$S = \sum_{i=1}^{N} \langle a, v_j \rangle (v_{j+1} - v_{j-1}),$$

(7)

where $a$ is arbitrary constant vector in $\mathbb{R}^m$. Obviously $S$ is orthogonal to $a$, so both vectors $J$ and $S$ provide $2m - 1$ scalar invariants. The invariants $S$ corresponding to the different vectors $a$ are equivalent under orthogonal group $SO(m)$.

Corollary 2. Transformations (1) satisfy identities

$$R_j^2 = (R_j R_{j+1})^3 = (R_j R_j)^2 = 1, \; i \neq j \pm 1.$$  

(8)

Proof. We have to prove only that $(R_j R_{j+1})^3 = 1$ since other identities are rather obvious. Denote $Q = (R_j R_{j+1})^3$ and $Q(W_i) = \tilde{W}_i$. Consider the product $P = W_{j+1} W_j W_{j-1}$. Accordingly to the proposition 1, transformation $Q$ does not change it, that is $\tilde{P} = P$. Moreover, it is clear that $Q$ acts identically on the determinants of the matrices $W_i$. Assume $\lambda = \beta_{j+1}$ then the first factors in both products $P$ and $\tilde{P}$ become degenerate. The image of the matrix $W_{j+1}$ is spanned over $m + 2$-vector $(2, 2v_{j+1}^t, -|v_{j+1}|^2)^t$ and since the matrix $W_j W_{j-1}$ is
nondegenerate, it coincides with the image of $P$. Analogously, the images of $\hat{W}_{j+1}$ and $\hat{P} = P$ coincide and therefore $\hat{v}_{j+1} = v_{j+1}$. The cancellation of the common factor $W_{j+1}$ implies equality $\hat{W}_{j} \hat{W}_{j-1} = W_{j} W_{j-1}$ and repetition of the argument proves the statement. 

This means that the group $\mathcal{R}$ is isomorphic to the affine Weyl group $\hat{A}_{N-1}$, or in other words, transformations $R_j$ define the nonlinear representation of the $\hat{A}_{N-1}$.

**Corollary 3.** If one performs recuttings $R_j$ of the polygon excepting one of them, say $R_1$, then all images of the points $v_j$ form finite set.

**Proof.** The subgroup generated by $R_j$, $j \neq 1$ is isomorphic to the symmetric group $S_N$ and therefore is finite. 

For the general $N$-valued mapping the number of initial data images under iterations grows exponentially. Accordingly to [1] one of the integrable mappings characteristic features is the polynomial growth of the images number. Using the identities (8) one can prove that this criterion is satisfied in our case. In fact, the transformations

$$T_j = (R_j N_1 \ldots R_j R_j)^{N(N-1)}, j = 1, \ldots, N - 1$$

generate the commutative subgroup with finite index in $G$ and therefore the number of images after $k$ iterations grows as $Ck^{N-1}$.

### 3 Continuous motions of the polygon

Some continuous symmetries of the presented mapping are quite obvious from its geometric description. Really, any shift or rotation of $\mathbb{R}^m$ commutes with transformations $R_j$ as well as reflections and homotheties. We have already used this fact for generating invariant (7) from the invariant (6). Analogy between the transformations (1) and examples considered in [3], [5] suggests that there exist some hidden symmetries as well. Now we shall demonstrate that it is really so. Let us consider the lattice

$$|v_{j+1} - v_j|^2(v_{j+1} + v_j)_x = 2(v_{j,x}, v_{j+1} - v_j)(v_{j+1} - v_j).$$  \hspace{1cm} (9)
Scalar product of (9) with \( v_{j+1} - v_j \) yields

\[
\langle v_{j+1,x} - v_{j,x}, v_{j+1} - v_j \rangle = 0,
\]  
(10)

that is the values \( \beta_j \) are the first integrals of the lattice. Then product with \( v_{j+1,x} - v_{j,x} \) yields

\[
|v_{j+1,x}|^2 = |v_{j,x}|^2,
\]

that is the velocities of all the points are of the same absolute value. We assume the normalization

\[
|v_{j,x}|^2 = 1, \quad j = 1, \ldots, N
\]  
(11)

without loss of generality since the lattice (9) is underdetermined and is invariant under change of independent variable \( \tilde{x} = \phi(x) \) with arbitrary \( \phi \).

The lattice (9) admits the zero curvature representation

\[
W_{j,x} = U_{j+1}W_j - W_jU_j
\]  
(12)

where \( W_j \) is given by (3) and the matrix \( U_j \) is defined by formula

\[
U = \frac{1}{\lambda} \begin{pmatrix}
-2(v, v_x) & 2v_x^t & 0 \\
|v|^2v_x - 2(v, v_x)v & 2vv_x^t - 2v_xv^t & -2v_x \\
0 & 2(v, v_x)v^t - |v|^2v_x^t & 2(v, v_x)
\end{pmatrix}
\]  
(13)

where \( v \) is substituted by \( v_j \).

Corollary 4. The characteristic polynomial (5) is the first integral of the lattice (9).

Proof. This is obvious from the Lax equation \( \hat{W}_{1,x} = [U_1, \hat{W}_1] \) which follows from (12).

For the examples considered in [5] the consistency of the transformations (1) and lattice (12) is proved by arguments which are based on special structure of the matrices \( W \) and \( U \) and fail in our case (the obstacle is the presence of derivatives \( v_x \) in \( U \)). Nevertheless, the following proposition can be proved by direct, although tedious computations.

Proposition 2. Transformations \( R_j \) act on the system (9), (11).
The representation \((12)\) suggests that the lattice \((9)\) can be regarded as sequence of the Bäcklund transformation (BT) for some partial differential equation with zero curvature representation

\[
U_t = V_x - [V, U]. \tag{14}
\]

The search of the matrix \(V\) brings us to

\[
V = \left( \frac{3}{2} \langle v_{xx}, v_{xx} \rangle - \frac{4}{\lambda} \right) U + \frac{2}{\lambda} V_0
\]

where

\[
V_0 = \begin{pmatrix}
-\langle v, v_{xxx} \rangle & v_{xxx}^t & 0 \\
u & v_{xxx}^t - v_{xxx} v_x^t + 2v_{xx} v_x^t - 2v_x v_{xx}^t & -v_{xxx} \\
0 & -u^t & \langle v, v_{xxx} \rangle
\end{pmatrix}
\]

and \(u = \frac{1}{2} |v|^2 v_{xxx} - \langle v, v_{xxx} \rangle v + 2 \langle v, v_{xx} \rangle v_x - 2 \langle v, v_x \rangle v_{xx} + 2v_x\).

The corresponding equation \((14)\) is equivalent to

\[
v_t = v_{xxx} + \frac{3}{2} \langle v_{xx}, v_{xx} \rangle v_x, \quad \langle v_x, v_x \rangle = 1. \tag{15}
\]

The following statement can be proved directly.

**Proposition 3.** The flows defined by the lattice \((9),(11)\) and equation \((13)\) commute. Transformations \((1)\) are consistent with the last equation.

One can easily check that the formula \((1)\) can be rewritten as

\[
v_{k+1} = v_{k-1} + \frac{|\hat{v}_k - v_{k-1}|^2 - |v_{k-1} - v_k|^2}{|\hat{v}_k - v_k|^2} (\hat{v}_k - v_k)
\]

(this is obvious also from geometric interpretation). In this form it can be considered as non-linear superposition formula which allows one to construct the solution \(v_{k+1}\) of the equation \((13)\) connected with the solution \(v_{k-1}\) by double BT by use of two solutions \(v_k\) and \(\hat{v}_k\) which are results of the single BT.

The equation \((13)\) appeared in [9] for the first time. In the case \(m = 1\) the lattice \((9),(11)\) and equation \((13)\) are trivial as well as the mapping \(R\). In the case \(m = 2\) denote \(v = (p, q)^t, \)

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then the equation (15) is reduced to the scalar equation

\[ p_t = p_{xxx} - \frac{3}{2} \frac{p_x p_{xx}^2}{p_x^2 - 1} \]

which is well known to be integrable.

4 Concluding remarks

Some problems concerning the presented models were disregarded. First of all, their Hamiltonian structures should be investigated. Possibly some analogies with the case of the dressing chain \[6\] will be useful here.

Other problem is to study higher symmetries of the equation (15). I hope that recursion operator can be found and the whole hierarchy constructed.

Recently a number of integrable equations was shown to describe motion of the non-stretching curves, see e.g. \[6\] and \[7\] where the condition that curve lies on \(m\)-dimensional sphere naturally arises. Possibly the presented model and its continuous symmetries can be considered as quantization of such motion.

Of course we can assume that number of points is infinite so that \(j \in \mathbb{Z}\). This corresponds to recuttings of nonclosed broken line. I do not know if there exists other boundary condition besides the periodic closure which leads to integrability of the mapping \(R\) and the lattice (9). Possibly some modification of the approach from \[6\], \[8\] can bring to the Painlevé type equations and their Bäcklund transformations.

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