Wyner-Ziv Coding Based on Multidimensional Nested Lattices

Cong Ling, Su Gao, and Jean-Claude Belfiore

Abstract

Distributed source coding (DSC) addresses the compression of correlated sources without communication links among them. This paper is concerned with the Wyner-Ziv problem: coding of an information source with side information available only at the decoder in the form of a noisy version of the source. Both the theoretical analysis and code design are addressed in the framework of multi-dimensional nested lattice coding (NLC). For theoretical analysis, accurate computation of the rate-distortion function is given under the high-resolution assumption, and a new upper bound using the derivative of the theta series is derived. For practical code design, several techniques with low complexity are proposed. Compared to the existing Slepian-Wolf coded nested quantization (SWC-NQ) for Wyner-Ziv coding based on one or two-dimensional lattices, our proposed multi-dimensional NLC can offer better performance at arguably lower complexity, since it does not require the second stage of Slepian-Wolf coding.

Index Terms

Distributed source coding, multi-dimensional lattices, nested lattices, rate-distortion function, Wyner-Ziv coding.

I. INTRODUCTION

In wireless sensor networks, energy is a major constraint to individual node performance. Compressing data prior to transmission and minimizing inter-node communications can effec-

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tively reduce energy consumption. The problem is well-known as multi-terminal source coding or distributed source coding (DSC) [1], where one or more of the sensor nodes (or terminals, information sources) compress data separately (i.e. without communication with each other) before transmission. The joint decoder at the central node recovers the transmitted data either losslessly or up to a prescribed distortion. For the lossless case, Slepian and Wolf [2] proposed the theoretical framework. A common scenario for the lossy case is the Wyner-Ziv problem, which considers encoding a source where the decoder has access to some side information [3].

Code designs for Wyner-Ziv coding have emerged in recent years. A structured binning scheme using (random) nested lattice codes (NLC) was introduced [4], [5], which can asymptotically achieve the Wyner-Ziv limit [3] as the dimension of the lattice approaches infinity. The first practical code design based on this idea was done in [6], where preliminary analysis for particular nested lattices was presented. A general distortion analysis for \( n \)-dimensional lattices was given by Liu et al. [7]. In particular, a lower bound on the distortion was derived, which, for given \( n \), exhibits an increasing gap to the Wyner-Ziv limit as the rate grows. For this reason, [7] focused on 1- and 2-dimensional lattices and used a second stage of coding, namely, Slepian-Wolf coding, to further exploit the correlation between quantized data. This technique was termed as “Slepian-Wolf coded nested quantization” (SWC-NQ) in [7]. The Slepian-Wolf coding in [7] relied on channel capacity-achieving codes such as low-density parity check (LDPC) codes, resulting in considerable implementation complexity.

In this paper, we take a new approach by directly using multi-dimensional NLC, which is conceptually simpler. It was known that the distortion performance improves with the lattice dimension [5], [7]. Our work is inspired by this result, and we aim to develop less complicated practical codes for the Wyner-Ziv problem. The contributions of this paper are two-fold. On one hand, we complement the theoretical analysis of the rate-distortion function in [7]. Specifically, we develop a technique to compute the distortion, which is accurate under the high-resolution assumption, and a new upper bound based on the derivative of the theta series. It is worth mentioning that, in practice, an upper bound is a safer guideline of system design than a lower bound. On the other hand, we implement NLC based on a variety of techniques to obtain sublattices, and show that our implementation gains over the conventional one and two-dimensional schemes. These techniques include clean similar sublattices of dimensions greater than two [8], the random ensemble of nested lattices in [9], and scaling and rotation of a lattice
to obtain its sublattices. Recently, a similar idea using the scaling-rotating technique to obtain nested lattices was proposed independently in [10] for the synchronous multiple-access channel.

The obtained distortion performance is close to the Wyner-Ziv limit, especially at rates < 10 bits/sample. These simulation results are consistent with theoretical analysis which predicts improved performance as dimension increases, although the law of diminishing returns will apply in high dimensions. Meanwhile, the complexity of our proposed scheme is arguably lower than that of SWC-NQ, as we do not use turbo or LDPC codes [11] [12].

II. PRELIMINARIES

Consider Wyner-Ziv coding in the two-source case. Let \( \{(X_i, Y_i)\}_{i=1}^{\infty} \) be a sequence of independent and identically distributed (i.i.d.) drawings of a pair of correlated random variables \( X \) and \( Y \), and let \( d(X_i, \tilde{X}_i) \) denote a single-letter distortion measure between the source \( X_i \) and its reconstructed version \( \tilde{X}_i \) at the decoder. Wyner-Ziv coding [3] asks the question of how many bits are needed to encode source \( X \) under the constraint that the average distortion \( E[d(X_i, \tilde{X}_i)] \) is not greater than a given target distortion \( D \), assuming the side information \( Y \) is available at the decoder but not at the encoder.

We specifically consider the quadratic Gaussian case of the correlation model \( X = Y + Z \), where \( Z \) is the Gaussian noise with distribution \( Z \sim N(0, \sigma_Z^2) \), and \( Y \) is independent of \( Z \). Mean-square error (MSE) is used for distortion measure. The quadratic Gaussian case is special since it has no rate loss, namely, the rate-distortion function is given by [7]

\[
D_{WZ}(R) = \sigma_Z^2 2^{-2R}
\]  

which will be referred to as the ‘Wyner-Ziv limit’.

A. Lattices, nested lattices and similar sublattices

For a set of \( n \) linearly independent basis vectors \( \{m_1, m_2, \ldots, m_n\} \), an \( n \)-dimensional lattice \( \Lambda \) is composed of all integral combinations of the basis vectors:

\[
\Lambda = \{ l = M \cdot i : i \in \mathbb{Z}^n \} \tag{2}
\]

where \( M = [m_1, m_2, \ldots, m_n] \) is the generator matrix and \( \mathbb{Z} = \{0, \pm1, \pm2, \ldots\} \) is the set of integers. For a vector \( x = [x_1, x_2, \ldots, x_n]^T \), the nearest-neighbor quantizer associated with \( \Lambda \)
is $Q_\Lambda(x) = \arg \min_{l \in \Lambda} ||x - l||$. The basic Voronoi cell of $\Lambda$, defined by $\mathcal{V} = \{x : Q_\Lambda(x) = 0\}$, specifies the nearest-neighbor decoding region. Important quantities for $\mathcal{V}$ include the cell volume $V = \int_{\mathcal{V}} dx$, the second moment $\sigma^2 = \frac{1}{nV} \int_{\mathcal{V}} ||x||^2 dx$ and the normalized second moment $G(\Lambda) = \frac{\sigma^2}{V^2}$. The minimum of $G(\Lambda)$ of all the $n$-dimensional lattices is denoted as $G_n$.

From [13], $G_n \geq 1/2\pi e$, $\forall n$ and $\lim_{n \to \infty} G_n = 1/2\pi e$.

Let $\Lambda_F$ be a fine lattice with a generator matrix $M_F$. Similarly, let $\Lambda_C$ be a coarse lattice with a generator matrix $M_C$. A pair of $n$-dimensional lattices $(\Lambda_F, \Lambda_C)$ is nested in the sense of $\Lambda_C \subset \Lambda_F$, if $M_C = M_F \cdot P$, where $P$ is an $n \times n$ integer matrix with determinant greater than one (in absolute value). We define the nesting ratio $N = V_C/V_F$, where $V_F$ and $V_C$ are the cell volumes of the fine and coarse lattice, respectively.

There is one special case where $\Lambda_C$ is geometrically similar to $\Lambda_F$, which means that $\Lambda_C$ can be obtained from $\Lambda_F$ by applying a similarity transform [8] including a rotation, change of scale and possibly a reflection. $\Lambda_C$ is strictly similar to $\Lambda_F$ when reflection is not used. We also refer to $\Lambda_C$ as a similar sublattice to $\Lambda_F$.

### B. Encoding and Decoding Scheme

Throughout the paper, we follow the high-resolution assumption in [7], which means $V_F$ is sufficiently small such that the probability density function (pdf) of $Z$ is approximately constant over each Voronoi cell of the fine lattice $\Lambda_F$.

The encoding and decoding scheme is the same as in [7], which is simplified from [5] under the high-resolution assumption. The scheme is described as follows:

- The encoder quantizes $x$ to $x_{Q\Lambda_F} = Q_{\Lambda_F}(x)$, compute the coset leader $s = x_{Q\Lambda_F} - Q_{\Lambda_C}(x_{Q\Lambda_F})$, and transmits the index corresponding to $s$.
- The decoder receives $s$ and reconstructs $x$ as $\hat{x} = s + Q_{\Lambda_C}(y - s)$.

Under this formulation, the rate per dimension is given by $R = \frac{1}{n} \log_2 \left( \frac{V_C}{V_F} \right)$. In our practical design, we also use minimum MSE estimation as described in [7].
C. Theta series and even unimodular lattices

The development of this paper is heavily based on the theta series. Given a lattice \( \Lambda \), the theta series \([13]\) is defined as

\[
\Theta_\Lambda(q) = \sum_{x \in \Lambda} q^{\|x\|^2},
\]

where \( q = e^{j\pi\tau} (\Im(\tau) > 0) \). Using the change of variable \( j\tau \to \tau \) (\( \tau > 0 \) real), it can alternatively be expressed as

\[
\Theta_\Lambda(\tau) = \sum_{x \in \Lambda} e^{-\pi\tau\|x\|^2}.
\]

Theta series of some standard lattices are given in \([13]\). In general, theta series are not easy to evaluate, and their derivatives are even more complex to calculate. Fortunately, there is a class of lattices for which the computation of such functions is tractable, the even unimodular lattices. A detailed description of such lattices can be found in \([14]\).

A lattice \( \Lambda \) is unimodular \([13]\) if \( \Lambda \) is integral, i.e., \( M^T M \) is an integer matrix, and \( | \det \Lambda | = 1 \) or equivalently \( \Lambda \) is equal to its dual \( \Lambda^* \), i.e., \( \Lambda = \Lambda^* \). If \( \Lambda \) is integral, then \( \|l\|^2 \) is necessarily an integer for all \( l \in \Lambda \). Further, if \( \|l\|^2 \) is an even integer for all \( l \in \Lambda \), then \( \Lambda \) is called an even unimodular lattice. Many exceptional lattices such as the Gosset lattice \( E_8 \) or the Leech lattice \( \Lambda_{24} \) are even unimodular.

We introduce now the three Jacobi theta functions \([13]\),

\[
\begin{align*}
\vartheta_2(q) &= \sum_{n=-\infty}^{+\infty} q^{(n+\frac{1}{2})^2} \\
\vartheta_3(q) &= \sum_{n=-\infty}^{+\infty} q^{n^2} \\
\vartheta_4(q) &= \sum_{n=-\infty}^{+\infty} (-1)^n q^{n^2}
\end{align*}
\]

Now, consider the Eisenstein series,

\[
E_{2k}(q) = 1 + \frac{2}{\zeta(1-2k)} \sum_{m=1}^{+\infty} \frac{m^{2k-1} q^m}{1 - q^m}
\]

where \( \zeta(s) \) is the Riemann zeta function,

\[
\zeta(s) = \sum_{k=1}^{+\infty} \frac{1}{k^s}.
\]

We can relate \( E_4(q) \) and the Jacobi theta functions through the relation,

\[
E_4(q^2) = \frac{1}{2} \left( \vartheta_2^8(q) + \vartheta_3^8(q) + \vartheta_4^8(q) \right).
\]
Another fundamental series is the so-called modular discriminant defined as,

\[ \Delta(q) = \frac{1}{12} \left( E_4^3(q) - E_6^2(q) \right) \tag{6} \]

which is also related to the three Jacobi theta functions through the identity

\[ \Delta(q^2) = \frac{1}{256} \vartheta_2^8(q) \vartheta_3^8(q) \vartheta_4^8(q) \cdot \]

Remarkably, theta series of all even unimodular lattices can be expressed as polynomials in the two variables \( E_4 \) and \( \Delta \):

**Proposition 1** ([13]). *If \( \Lambda \) is an even unimodular lattice of dimension \( n \) then,*

1) \( n \) *is a multiple of 8, \( n = 24m + 8k \) with \( k = 0, 1, 2 \) and \( m \) being any non-negative integer;*

2) *its theta series is given by

\[ \Theta_\Lambda(\tau) = E_4^{3m+k} \left( e^{-2\pi \tau} \right) + \sum_{j=1}^{m} b_j E_4^{3(m-j)+k} \left( e^{-2\pi \tau} \right) \Delta^j \left( e^{-2\pi \tau} \right), \quad b_j \in \mathbb{Q}. \tag{7} \]

**III. New Analysis of Rate-Distortion Function**

Knowing the suitability of lattices to this problem through theoretical analysis is essential for the code design. Our starting point is the rate-distortion function of nested-lattice Wyner-Ziv coding derived in [7]. It was shown there that the distortion \( D \) is comprised of a ‘source-coding component’ \( D_S \) and a ‘channel-coding component’ \( D_C \). More precisely, the distortion per dimension for the Wyner-Ziv coding at high resolution is given by [7]

\[ D_n = \min_{V_C > 0} \left\{ D_S + D_C \right\} \]

\[ = \min_{V_C > 0} \left\{ G(\Lambda_F) V_F^2 + \frac{1}{n} E_Z \| Q_{\Lambda_C}(Z) \|^2 \right\} \]

\[ = \min_{V_C > 0} \left\{ G(\Lambda_F) V_C^2 2^{-2R} + \frac{1}{n} \sum_{l_c \in \Lambda_C} \| l_c \|^2 P(z \in V_C(l_c)) \right\} \tag{8} \]

where \( D_S \) and \( D_C \) are implicitly defined, and \( V_C(l_C) \) is the Voronoi cell associated with the lattice point \( l_C \in \Lambda_C \).

There is a standard way to handle \( D_S \): choosing a fine lattice that is good for quantization, i.e., a lattice with \( G(\Lambda_F) \) as small as possible. The second component \( D_C \) is not so easy to handle, though. This is the main problem we want to solve in this paper.
A lower bound on $D_n$ was given in [7], where the Voronoi region $V_C(l_C)$ is replaced with the packing sphere $S$ [13]. It was pointed out in [7] that this lower bound is asymptotically tight since the shape of $V_C(l_C)$ will approach a sphere as $n$ increases for lattices with the best rate-distortion performance. However, for finite $n$, the Voronoi regions may not be close to a sphere. We observe that for these lattices, the lower bound results in a non-negligible gap.

A. Accurate Calculation Under High-Resolution Assumption

We present an accurate calculation of the rate-distortion, based on the high-resolution assumption.

**Theorem 1** (Accurate Calculation Under High-Resolution Assumption). Let $l_F \in \Lambda_F$ be a fine lattice point and let

$$f_Z(z) = \frac{1}{(\sqrt{2\pi}\sigma_Z)^n} \exp\left(-\frac{\|z\|^2}{2\sigma_Z^2}\right)$$

be the pdf of a vector $z$ whose elements $z_i$’s are i.i.d. Gaussian with variance $\sigma_Z$. Then, under the high-resolution assumption, the channel coding component $D_C$ of the distortion function for nested-lattice Wyner-Ziv coding is given by

$$D_C = \frac{V_F}{n} \sum_{l_C \in \Lambda_C} \|l_C\|^2 \sum_{l_F \in V_C} f_Z(\|l_F + l_C\|) .$$

**Proof:** Based on the high-resolution assumption [7], the pdf $f_Z(z)$ is uniform over each Voronoi cell of the fine lattice. Thus $P(z \in V_C(l_C))$ in (8) can be approximated by

$$P (z \in V_C(l_C)) = \sum_{l_F \in V_C} f_Z(\|l_F + l_C\|) V_F .$$

Then the theorem follows.

To compute (10), we need to enumeration all lattice points $l_C \in \Lambda_C$, and those points $l_F \in V_C$. The latter may be done by checking all points $l_F$ within the covering radius of $\Lambda_C$: $l_F \in V_C$ if and only if $l_F$ is decoded to 0 in $\Lambda_C$.

In Fig. 1 we give the rate-distortion performance of fine lattice $Z_8$ and $E_8$. The sublattice is generated from the expand-rotate method described in the next Section. We also give the corresponding lower bound in [7] for comparison. From Fig. 1 we can see in dimension eight, the lower bound in [7] is tight for the best quantization lattice $E_8$ in this dimension, but has a gap to our theoretical calculation for the $Z_8$ lattice. We find this is also true for other dimensions.
B. An Upper Bound

Albeit accurate under the high-resolution assumption, the method in the preceding subsection unfortunately does not lend much insight into the selection criterion of a good coarse lattice. Let us reexamine the $D_C$ component in (8)

$$D_C = \frac{1}{n} \sum_{l_C \in \Lambda_C} \|l_C\|^2 \int_{V_C} \frac{1}{(\sqrt{2\pi}\sigma_Z)^n} \exp \left( -\frac{\|l_C + z\|^2}{2\sigma_Z^2} \right) dz,$$

which should be minimized. Of course, this problem itself is not tractable due to the integration over the basic Voronoi region.

To deal with this problem, the common approach is to choose a coarse lattice to maximize $P(z \in V_C)$, which is reduced to the standard approach to channel coding. However, this standard approach does not quite capture the essence of Wyner-Ziv coding, since the metric in (12) is clearly different. Alternatively, one could derive either the upper or lower bound by using the covering or packing radius, the latter of which has already been done by Liu et al. [7]. Again, in addition to the computational complexity, it is not insightful.

Here, we propose a new upper bound using the the derivative of the theta series.

**Theorem 2 (Upper Bound on Distortion).** The component $D_C$ is bounded by

$$D_C \leq -\frac{1}{2\pi n} \Theta'_{\Lambda_C} \left( \frac{1}{8\pi \sigma_Z^2} \right),$$

where $\Theta'_{\Lambda}$ denotes the derivative of the theta series (with respect to $\tau$):

$$\Theta'_{\Lambda}(\tau) = -\sum_{l \in \Lambda} \pi \|l\|^2 e^{\tau \|l\|^2}.$$

**Proof:** The idea is to upper-bound the integral over $V_C$ by that over the half plane whose boundary is equally far from lattice points 0 and $l_C$, similar to the technique widely used in the union bound on the decoding error probability. That is, $P(z \in V_C(l_C)) \leq P(\|z\| > \|l_C\|/2) = Q\left( \frac{\|l_C\|}{2\sigma_Z} \right)$, where $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{t^2}{2} \right) dt$ for $x > 0$ is the Gauss Q-function. Then, we have

$$D_C \leq \frac{1}{n} \sum_{l_C \in \Lambda_C} \|l_C\|^2 Q\left( \frac{\|l_C\|}{2\sigma_Z} \right) \leq \frac{1}{2n} \sum_{l_C \in \Lambda_C} \|l_C\|^2 \exp \left( -\frac{\|l_C\|^2}{8\sigma_Z^2} \right),$$

where $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{t^2}{2} \right) dt$ for $x > 0$ is the Gauss Q-function.
TABLE I
THETA SERIES AND DERIVATIVES OF WELL-KNOWN LATTICES.

| Lattice | Theta Series | Derivative |
|---------|-------------|------------|
| \( Z^n \) | \( \vartheta^n \) | \( n\vartheta^n \theta' \) |
| \( D_n \) | \( \frac{1}{2}(\vartheta^n + \vartheta^{n+2}) \) | \( \frac{1}{2}(n\vartheta^n \vartheta' + n\vartheta^{n+2} \vartheta') \) |
| \( D_n^* \) | \( \vartheta^n + \vartheta^{n+2} \) | \( n\vartheta^n \vartheta' + n\vartheta^{n+2} \vartheta' \) |
| \( E_8 \) | \( \frac{1}{4}(\vartheta^n + \vartheta^{n+2} + \vartheta^{n+4}) \) | \( \frac{1}{4}(8\vartheta^n \vartheta' + 8\vartheta^{n+2} \vartheta' + 8\vartheta^{n+4} \vartheta') \) |

where the second inequality follows from the fact \( Q(x) \leq \frac{1}{2} \exp(-x^2/2) \) for \( x > 0 \).

Now we may recognize the right-hand side of (15) as the derivative of the theta series. Accordingly, the distortion can be expressed in the desired form.

**Remark 1.** The bound may be slightly improved by applying the alternative expression of the \( Q \)-function \( Q(x) = \frac{1}{\pi} \int_0^{\pi/2} \exp \left( -\frac{x^2}{2\sin^2 \theta} \right) d\theta \) for \( x > 0 \).

Now the problem reduces to that of finding a coarse lattice which maximizes the derivative of the theta series (note that its derivative is negative). We formalize this new criterion as the following Proposition:

**Proposition 2** (Design Criterion of the Coarse Lattice). A coarse lattice is good for Wyner-Ziv coding if the derivative of its theta series is large.

**Example 1.** Table I shows the theta series and associated derivatives of some well-known lattices.

C. Even Unimodular Lattices

As the theta series of even unimodular lattices are polynomial in \( E_4 \) and \( \Delta \), it is enough to find the derivatives of these two series. The Ramanujan system [16] gives the answer. It expresses the derivatives of \( E_2 \), \( E_4 \) and \( E_6 \) as functions of themselves:

\[
\begin{align*}
q \frac{dE_2(q)}{dq} &= \frac{1}{12} (E_2^2(q) - E_4(q)) \\
q \frac{dE_4(q)}{dq} &= \frac{1}{3} (E_2(q)E_4(q) - E_6(q)) \\
q \frac{dE_6(q)}{dq} &= \frac{1}{2} (E_2(q)E_6(q) - E_3^2(q))
\end{align*}
\]
As we are interested in the functions $E_k(e^{-2\pi \tau})$ for $k = 4, 6$, we get
\begin{align*}
\frac{dE_4(e^{-2\pi \tau})}{d\tau} &= -\frac{2\pi}{3} \left( E_2(e^{-2\pi \tau}) E_4(e^{-2\pi \tau}) - E_6(e^{-2\pi \tau}) \right) \\
\frac{dE_6(e^{-2\pi \tau})}{d\tau} &= -\pi \left( E_2(e^{-2\pi \tau}) E_6(e^{-2\pi \tau}) - E_4^2(e^{-2\pi \tau}) \right)
\end{align*}
and so, combining (6), (16) and (17), we obtain the derivative of the modular discriminant with respect to $\tau$:
\begin{equation}
\frac{d}{d\tau} \Delta(e^{-2\pi \tau}) = -2\pi E_2(e^{-2\pi \tau}) \Delta(e^{-2\pi \tau}) = -\frac{\pi}{6 \times 12^2} E_2(e^{-2\pi \tau}) \left( E_4^3(e^{-2\pi \tau}) - E_6^2(e^{-2\pi \tau}) \right).
\end{equation}
This means that, using (7), (16) and (18), we are able to get the derivative of the theta series of an even unimodular lattice as a function of $E_2$, $E_4$ and $E_6$.

**Example 2.** The theta series of the Gosset lattice $E_8$ is
\[ \Theta_{E_8}(\tau) = E_4(e^{-2\pi \tau}). \]
Using (16), we get
\[ \Theta'_{E_8}(y) = -\frac{2\pi}{3} \left( E_2(e^{-2\pi \tau}) E_4(e^{-2\pi \tau}) - E_6(e^{-2\pi \tau}) \right). \]

**Example 3.** The theta series of the Leech lattice $\Lambda_{24}$ is
\[ \Theta_{\Lambda_{24}}(y) = E_4^3(e^{-2\pi \tau}) - 720 \Delta(e^{-2\pi \tau}) = \frac{1}{12} \left( 7E_4^3(e^{-2\pi \tau}) + 5E_6^2(e^{-2\pi \tau}) \right). \]
From this expression, and using (16) and (17), we deduce the derivative with respect to $y$:
\[ \Theta'_{\Lambda_{24}}(y) = -\frac{\pi}{12} \left( \frac{2}{7} E_2(e^{-2\pi \tau}) E_4^2(e^{-2\pi \tau}) + 5E_2(e^{-2\pi \tau}) E_6^2(e^{-2\pi \tau}) - \frac{37}{7} E_4^2(e^{-2\pi \tau}) E_6(e^{-2\pi \tau}) \right). \]

1) The average behavior: By the Siegel-Weyl formula, the average theta series of an even unimodular lattice of dimension $4k$ is $E_{2k}(e^{-2\pi \tau})$ [14]. This series is a polynomial in $E_4$ and $E_6$, and so, its derivative is a polynomial in $E_2$, $E_4$ and $E_6$. We give, here, a method of calculating this polynomial.

Set
\[ G_{2k} = 2\zeta(2k) E_{2k} \]
and
\[ d_k = (2k + 3) k! G_{2k+4}. \]

Then, \( d_k \) satisfies the recurrence relation,
\[ \sum_{k=0}^{n} \binom{n}{k} d_k d_{n-k} = \frac{2n + 9}{3n + 6} d_{n+2} \]  \hspace{1cm} (19)

with \( d_0 = 3G_4 = 6\zeta(4)E_4 \) and \( d_1 = 5G_6 = 10\zeta(6)E_6 \). From (19), we can compute all Eisenstein series \( E_{2k} \) as functions of \( E_4 \) and \( E_6 \). Then, use (16) and (17) to calculate \( dE_{2k}(e^{-2\pi \tau})/d\tau \).

Suppose \( \sigma_Z^2 = 0.01 \). Figure 2 shows the upper bound on the overall distortion \( D_n \) as a function of \( R \) for some lattices \( (n = 8, 24, 80) \), and the average over all even unimodular lattices of dimension \( n = 2400 \) computed by using the derivative with respect to \( \tau \) of the Eisenstein series \( E_{1200}(e^{-2\pi \tau}) \). For the ‘source coding’ component \( D_S \), we use Zador’s upper bound on \( G_n \) [13, p. 58]. It can be seen that the upper bound on \( D_n \) approaches the Wyner-Ziv limit as \( n \) increases, meaning the bound is asymptotically tight.

IV. CODE DESIGN

In this Section, we present a variety of methods to obtain nested lattices and the associated simulation results. In our simulation, both \( X \) and \( Y \) are Gaussian. Also, \( X = Y + Z \) with \( Y \sim N(0, 1) \) and \( Z \sim N(0, 0.01) \). The nested lattices in our design may or may not be similar.

A. Clean Similar Sublattices

Recall that a clean similar sublattice means the boundary of its Voronoi cell does not touch any fine lattice points [8]. It is known that clean similar sublattices perform better than non-clean ones [7]. In this subsection we aim to find clean similar sublattices. A similar technique was used in solving multiple-description problem [17]. In a multiple-description framework proposed in [18], a number of clean similar sublattice constructions were given by extending the result in [8]. We will form our own constructions to be suitably used in nested lattices based on their constructions.

In the two-dimensional case, we introduce a complex-valued multiplying factor \( \xi = a + bi \), \( a, b \in \mathbb{Z} \), which is multiplied to the fine lattice points to obtain the coarse lattice points [18]. For various \( \xi \), this calculation should generate both clean and non-clean similar sublattices. The
clean similar sublattices were addressed in [18], where the authors proved the sublattice $\xi \mathbb{Z}^2$ is clean if and only if the nesting ratio $a^2 + b^2$ is odd, and the sublattice $\xi A_2$ is clean if and only if $a$ and $b$ are relatively prime.

Next, we give the nested lattice construction based on $\mathbb{Z}^4$. We use the ring of Lipschitz integer quaternions $\mathbb{H}_0 = \{a + bi + cj + dk; a, b, c, d \in \mathbb{Z}\}$, where $i, j, k$ are unit quaternions. Let an arbitrary point in a four-dimensional fine lattice $\Lambda_F$ be denoted as $l_F = x + yi + zj + wk$. Then set the multiplying factor $\xi$ as $\xi = a + bi + cj + dk$. Multiplying $\xi$ with the fine lattice points $l_F$ gives the coarse lattice points $l_C$. The calculated $l_C$ is equivalent to the product shown below:

\[
\begin{pmatrix}
 a & -b & -c & -d \\
 b & a & -d & c \\
 c & d & a & -b \\
 d & -c & b & a \\
\end{pmatrix} \cdot \begin{pmatrix}
 x \\
 y \\
 z \\
 w \\
\end{pmatrix} = (20)
\]

So if the generator matrix of the fine lattice is the identity matrix, the first matrix above can be seen as the generator matrix of the coarse lattice.

More generally, it was shown in [18] that for lattices $\Lambda = \mathbb{Z}^d$, $d = 1, 2, 3 \ldots$, if there exists a geometrically similar sublattice of nesting ratio $N$, then $N$ should be of the form $m^{n/2}$ for some integer $m$, where $n = 4l$ is the dimension. From [18], the nesting ratio $N$ must be odd to make sure the similar sublattices are clean. From our simulation, the nesting ratios which are odd perfect squares indeed give better performance than others. The simulation results are shown in Fig. 3.

B. Obtaining Similar Sublattices by Scaling and Rotation

In this subsection, we present a more general method to obtain sublattices by scaling and rotation of the fine lattice. The obtained sublattices are not necessarily clean. We set a parameter, $\beta$, as the expanding factor. The way to generate the similar coarse lattice is as follows:

- First expand the fine lattice by multiplying the expanding factor $\beta$, such that the lattice points with the smallest norm greater than zero in the fine lattice become the outermost points in the basic Voronoi cell of the coarse lattice. When $\beta$ is an integer, there is always a less (or equal) number of coarse lattice points on each shell (formed by points of equal norm) of the coarse lattice than in the fine lattice. This requirement can be fulfilled by using integer expanding factors.
- For non-integer expanding factors, some expanded lattice points may not be a subset of the fine lattice. This case does not only happen to only one point, but also to a group of points with the same norm. The angle between these points and their fine lattice counterparts can be detected and calculated. Due to symmetry, these angles for points with the same norm are the same. So we can rotate the expanded lattice points by this angle, to match the positions of those fine lattice points. Also, the rotation for all the norms should be the same since the expanded lattice is similar to the fine one; so the rotated coarse lattice remains the similar shape as the original fine lattice.

The values of $\beta$ giving rise to sublattices are those norms of the fine lattices. So $\beta$ is not necessarily an integer. By exploiting tables of theta series in [13], we can find a sublattice.

Example 4. We give an example in the two-dimensional case for both integer and non-integer expanding factors. Let the fine lattice be a hexagonal lattice generated by the following generator matrix:

$$M_F = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Then the fine lattice is expanded (and rotated when necessary) to form the coarse lattice $M_C = \beta M_F$. For fine and coarse lattices with integer expanding factors (e.g. $\beta = 3$ and $\beta = 4$), rotation is not needed. When $\beta = \sqrt{7}$, rotation is needed to make the coarse lattice points form a subset of the fine lattice, as shown in Fig. 4.

Using this method, we now give the simulation results of the rate-distortion performance for nested lattice Wyner-Ziv coding for $D_n$, $E_8$ and $\Lambda_{24}$.

In the three-dimensional case, we implement the scheme using both $D_3$ and $D_3^*$. The rate-distortion performance is shown in Fig. 5. Compared to the one- and two-dimensional case, the three-dimensional scheme gives less distortion. Also notice NLC using $D_3^*$ is better than the one using $D_3$.

The simulation results using $D_4$ and $E_8$ are similar, and are both better than the three-dimensional case. At rate 1.5 bits per sample, their distortion gaps to the Wyner-Ziv limit are 2.52 dB and 1.07 dB, respectively.

Moreover, for $\Lambda_{24}$, our scheme gives distortion performance closer to the Wyner-Ziv limit than the SWC-NQ scheme proposed in [7], especially at low rates. Results in [7] show a gap
of distortion 1.53 dB from the Wyner-Ziv limit while our gap is less than 1.28 dB at rate less than three bits per sample for $\Lambda_{24}$. See Fig. 6.

One can also see that the “increasing gap” between the rate-distortion curve of NLC and the Wyner-Ziv limit [7] indeed exists in concrete implementation. Nonetheless, from Fig. 6 this widening gap as rate increases can be handled by increasing the dimension. In the meantime, the rate can not be too high in sensor network applications. So the gap is acceptable with 8- or 24-dimensional lattices. By increasing the dimension of NLC as the rate increases, a constant gap from the Wyner-Ziv limit can be maintained.

C. Ensemble of Random Nested Lattices

Code designs above are restricted to those proposed in [13]. Although they have acceptable performance and low complexity, it is better to have a scheme existing in any dimensions. Hence, we use the ensemble of good nested lattice codes proposed in [9] based on the concept of random lattices. The random lattice ensemble in [9] can be generated as follows.

1. Take $p$ to be prime.
2. Define a $k \times n$ generator matrix $M$, where $M_{i,j}$ is uniformly distributed on $(0, \ldots, p-1)$, $i = 1, \ldots, k$; $j = 1, \ldots, n$.
3. Apply Construction A in [9] to obtain the lattice $\Lambda'_F$.

The $n$-dimensional cubic lattice $\mathbb{Z}^n$ can be viewed as a sublattice of the random lattice $\Lambda'_F$ [9]. Hence, for any dimension, various nested lattices can be obtained by simply applying different linear transformations $G$ to both $\mathbb{Z}^n$ and $\Lambda'_F$. Obviously, the resultant sublattice is not necessarily similar. The nest ratio $N = p^{n-k}$.

To make the nested ensemble good for the Wyner-Ziv problem, the fine lattice should be good for source coding and the coarse lattice good for channel coding. This is proved in [19] by extending the results in [9]. Also, both the coarse lattice $G\mathbb{Z}^n$ and the fine lattice $G\Lambda'_F$ should be good for quantization. To make $G\mathbb{Z}^n$ good for quantization, $G$ should be the generator matrix of a good quantizing lattice [13], e.g., the hexagonal lattice in dimension two and $E_8$ in dimension eight.

However, such lattices are not easy to decode. We use the sphere decoding algorithm described in [20] for the quantization to the random lattices, whose speed is tolerable if $n \leq 24$. 
Examples for dimensions 2, 4, 8 and 24 are given in Fig. 7. The rate-distortion performance approaches the Wyner-Ziv limit as dimension increases and the gap is only 1.87 dB within the limit for dimension twenty-four. The performance is very close to explicit lattices like the Leech lattice.

Since the elements of $\mathbf{M}$ are uniformly distributed over $\mathbb{Z}_p$ for large $p$ and since the decoding is not easy, this ensemble is of theoretic value only.

V. Conclusions

In this paper, we have investigated the rate-distortion function and practical code design of Wyner-Ziv coding based on multi-dimensional nested lattices. Under the high-resolution assumption, an accurate calculation was developed, and an upper bound expressed in terms of the derivative of the theta series was derived. These results can be used to judge the performance and serve as a practical guide for choosing good lattices for Wyner-Ziv coding. Several practical code designs were presented using multi-dimensional NLC. High-dimensional schemes gave performance close to the Wyner-Ziv limit. The performance is even better than the SWC-NQ scheme [7]. Compared to [7], our scheme does not require extra Slepian-Wolf coding based on powerful error correction codes, thereby enjoying lower complexity. Hence, the proposed scheme may be attractive for applications in sensor networks where simple coding schemes are needed.

This work leaves some open problems. Firstly, there may be room to improve the upper bound. Secondly, the derivative of the theta series arising from the upper bound, which is to be maximized, is a new problem for lattice researchers. Last but not the least, a more systematic approach to low-complexity code design is to be pursued.

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Fig. 1. Comparison of proposed accurate calculation and the lower bound in [7] for fine lattices $Z^8$ and $E^8$. $Y \sim N(0, 1)$ and $Z \sim N(0, 0.01)$. 
Fig. 2. Upper bound on distortion $D_n$ versus rate for some lattices ($n = 8, 24, 80$), and an average lattice ($n = 2400$).

Fig. 3. Distortion versus rate for clean similar sublattices of fine lattices $Z^4$ and $Z^8$. 
Fig. 4. Example of two-dimensional coarse lattice: $\beta = \sqrt{7}$. □ coarse lattice points, • fine lattice points.

Fig. 5. Distortion versus rate for fine lattices $D_3$ and $D_3^*$. 

Distortion (dB)

Rate (bits/sample)
Fig. 6. Distortion versus rate for fine lattice $\Lambda_{24}$.

Fig. 7. Distortion versus rate for the ensemble of random nested lattices.