Trapped surfaces, energy conditions, and horizon avoidance in spherically-symmetric collapse

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We consider spherically-symmetric black holes in semiclassical gravity. For a collapsing radiating spherically symmetric thin shell we derive a sufficient condition on the exterior geometry that ensures that a black hole is not formed. This is also a sufficient condition for an infalling test particle to avoid the apparent horizon of an existing black hole and approach it only within a certain minimal distance. Taking the presence of a trapped region and its outer boundary — the apparent horizon — as the defining feature of black holes, we explore the consequences of their finite time of formation according to a distant observer. Assuming regularity of the apparent horizon we obtain the limiting form of the metric in its vicinity. This metric satisfies the sufficient condition for the horizon avoidance in the scenario we consider. Hence indeed a test particle cannot fall into a black hole and collapse of a thin evaporating shell with the exterior geometry modelled by this metric cannot lead to its formation.

I. INTRODUCTION

Event horizon — the null surface that bounds the spacetime region from which signals cannot escape — is the defining feature of black holes in general relativity [1–3]. This classical concept plays an important role in their quantum behaviour [4–6]. Emission of the Hawking radiation completes a thermodynamic picture of black holes, and its most straightforward derivation relies on the existence of an horizon [4]. This radiation is also one of the ingredients of the black hole information loss paradox [7], perhaps the longest-running controversy in theoretical physics [5–7].

Event horizons are global teleological entities that are in principle unobservable [8, 9]. Theoretical, numerical and observational studies therefore focus on other characteristic features of black holes [1,10]. A local expression of the idea of absence of communications with the outside world is provided by the notion of a trapped region. It is a domain where both outgoing and ingoing future-directed null geodesics emanating from a spacelike two-dimensional surface with spherical topology have negative expansion [1,3,17,18]. The apparent horizon is the outer boundary of the trapped region [1,10]. According to classical general relativity the apparent horizon is located inside the event horizons if the matter satisfies energy conditions [1,12].

Quantum states can violate energy conditions [12]. Black hole evaporation proceeds precisely because \( T_{\mu\nu} = \langle T_{\mu\nu} \rangle \) violates the null energy condition (NEC): there is a null vector \( k^\mu \) such that \( T_{\mu\nu} k^\mu k^\nu < 0 \). In this case the apparent horizon is outside the event horizon. In fact, the very existence of the latter is uncertain [13,14]. While existence of spacetime singularities is no longer prescribed, their appearance without the horizon cover (“naked”) is not excluded either. This situation motivated introduction of many models of the ultra-compact objects [9].

According to a distant observer formation of a classical black holes takes an infinite amount of time \( t \), even if effective blackening out happens very fast. Similarly, the plunge of a test particle into an existing black holes takes infinite amount of time \( t \), but finite proper time of a comoving observer. On the other hand, if quantum effects responsible for finite-time black hole evaporation allow for the formation of an apparent horizon, it happens in finite \( t \). The question is then if it is possible to fall into such a black hole.

Working in the framework of semiclassical gravity in a spherically symmetric case [16], we first consider the simplest model of black hole formation — a collapsing thin shell, discussing horizon formation in presence of evaporation. In this framework we use the classical concepts, and the standard curvature terms in the Einstein equations are equated to the expectation value of the energy-momentum tensor. We use \((-+++)\) signature of the metric and set \( c = \hbar = G = k_B = 1 \).

II. SPHERICAL SYMMETRY. SUFFICIENT CONDITION FOR HORIZON AVOIDANCE

A general spherically symmetric metric in Schwarzschild coordinates is given by

\[
ds^2 = -e^{2h(t,r)} f(t,r) dt^2 + f(t,r)^{-1} dr^2 + r^2 d\Omega^2
\]

(1)

The function \( f(t,r) = 1 - C(t,r)/r \) is coordinate-independent, where the function \( C(t,r) \) is the Misner-Sharp mass [3,15]. In an asymptotically flat spacetime \( t \) is the physical time of a distant observer.

Trapped regions exist only if the equation \( f(t,r) = 0 \) has a root [11]. This root (or, if there are several, the largest one) is the Schwarzschild radius \( r_g(t) \). Apparent horizons are in general observer-dependent entities. However they are unambiguously defined in the spherically symmetric case for all spherical-symmetry preserving foliations [15]. In this case the
apparent horizon is located at $r_g$. In the Schwarzschild spacetime $C(t, r) = 2M$ and $h = 0$, hence $r_g = 2M$.

In thin shell collapse models [2] the geometry inside the shell is given by the flat Minkowski metric. The matter content of the shell is given by the surface energy-momentum tensor. The trajectory of a massive shell is parameterized by its proper time $\tau$ and expresses as $(T(\tau), R(\tau))$ in the exterior Schwarzschild coordinates. Initially the shell is located outside its gravitational radius, $R(0) > r_g$. Its dynamics is obtained by using the so-called junction conditions [2, 18].

The first junction condition is the statement that the induced metric $h_{ab}$ on the shell $\Sigma$ is the same on both sides $\Sigma^\pm$, $d\Sigma = h_{ab}dy^a dy^b = -d\tau^2 + R^2 d\Omega^2$. Since for massive particles the four-velocity $u^\mu$ satisfies $u_\mu u^\mu = -1$, the shell’s trajectory obeys

$$\dot{T} = \frac{\sqrt{F + R^2}}{e^H F},$$

where $\dot{A} = dA/d\tau$, $H = h(T, R)$, $F = f(T, R)$. This condition is used to identify the radial coordinate of the shell in interior and exterior coordinates, $R_c \equiv R$.

Discontinuity of the extrinsic curvature $K_{ab}$ is described by the second junction condition [2, 18] that relates it to the surface energy-momentum tensor. Given the exterior metric the junction conditions result in the equations of motion for the shell. For a classical collapse in vacuum the exterior geometry is given by the Schwarzschild metric, and the resulting equation for $\dot{R}$ is simple enough to have an analytic solution $T(R)$, leading to the finite proper time $T(r_g)$ and infinite time $\dot{T}(r_g)$.

This equation of motion is modified for a general exterior metric and its solution has some remarkable features [13, 20]. Here we focus on the possibility of crossing the Schwarzschild sphere of an evaporating black hole ($r_g(t) < 0$) in finite proper time. For a finite evaporation time $t_e$ the finite proper crossing time is equivalent to having a finite time $t_e$ of a distant observer. By monitoring the gap between the shell and the Schwarzschild radius [16, 21],

$$X(\tau) := R(\tau) - r_g(T(\tau)),$$

we discover the sufficient condition for a thin shell to never cross its Schwarzschild radius. The same condition applies to the study of an infalling test particle into an existing black hole. The analysis is straightforwardly generalized to null shells and test particles.

The rate of approach to the Schwarzschild radius behaves as

$$\dot{X} = \dot{R} - r_g(T)\dot{T},$$

Close to the Schwarzschild radius we have $\dot{T} \approx -Re^{-H}/F$, and hence

$$\dot{X} \approx \dot{R}(1 - |r_g|e^{-H}/F).$$

If for a fixed $t$ the function $\exp(h)f$ goes to zero as $x := r - r_g \to 0$, then there is a stopping scale $\epsilon_*(r)$. If the shell comes to the Schwarzschild radius closer than $\epsilon_*$ the gap has to increase, $X > 0$, evidently indicating in this case that the shell never collapses to a black hole. It is so if, e.g., $h(t, r) \leq 0$. In particular, this is the case when the exterior geometry is given by the outgoing Vaidya metric. Then $\epsilon_* = 2C|dC/dU|$, where $U(\tau)$ is the retarded null coordinate of the shell [16, 21].

If we apply this analysis to a test particle plunging into an existing evaporating black hole with decreasing $r_g$, the conclusion remains the same. However, it is not a priori clear that in a general evaporating case this criterion is satisfied (and, moreover, $\epsilon^2 f \approx g(x, r_g)$ where $g(x, r_g) \to 0$ for small $x$).

### III. Metric Outside an Apparent Horizon

Using only one additional assumption it is possible to obtain the explicit form of the metric near $r_g$. In fact this metric satisfies the sufficient condition for the horizon avoidance. We consider an evaporating black hole that is formed at some distant observer’s finite time, i.e. its apparent horizon radius $r_h(t)$ is a decreasing function of time. In addition we assume that the horizon is regular (the standard “no drama at the horizon” postulate [6], where the established regularity of the classical results is assumed to hold in the quantum-dominated regime). The regularity is expressed by finite values of the curvature scalars that can be directly expressed in terms of the energy-momentum tensor [17], $T := T^\mu_\nu$ and $\Sigma := T^\mu_\nu T_{\mu\nu}$.

The existence of an apparent horizon and regularity assumptions strongly constrain the energy-momentum tensor, and consistency with the known results on the background of an eternal black hole [4, 22] specify its limiting form uniquely. The leading terms in the $(tr)$ block of the energy-momentum tensor turns out to be the functions of $\alpha^2 := r_g^2 r_e$, and the functions $C(t)$ and $h(t)$ take the following form [17],

$$C = r_g(t) - \alpha(t)\sqrt{x} + \frac{1}{3}x \ldots$$

and

$$h = -\frac{1}{2} \frac{\ln x}{r_g} + \ln \left(\frac{3 \alpha + 2\sqrt{x}}{r_g\sqrt{r_g}} + \frac{1}{1 + 2\sqrt{x/3} - \ln 2} \ldots \right),$$

where $x = r - r_g$. The constant in the function $h$ is set (using the freedom in re-defining the time variable) in such a way that $h \approx 0$ for a macroscopic BH when $x \to r_e$, i.e. far relative to the scale of quantum effects $\alpha^2$. The metric takes a particularly simple form in ingoing Vaidya coordinates [17].

The energy-momentum tensor that corresponds to this metric violates the null energy condition in the vicinity of the apparent horizon [17]. The comoving density and pressure at the apparent horizon are negative,

$$\rho = p = -\frac{r_g^2}{16\pi r^4 r_e^2},$$

where $r$ is the radial coordinate of the comoving observer [23]. We focus on the question of horizon avoidance. Return first to a collapsing thin shell problem where the exterior metric is
given now by Eq. (1) with the metric functions that are given above. Expanding Eqs. (5) and (7) in the range \( r_g \gg X \gg \alpha^2 \) the rate of approach becomes

\[
\dot{X} = \dot{R} - r'_g(T)\dot{T} \approx \dot{R} \left( 1 - \frac{3r_g| r'_g |}{2X} \right). \tag{9}
\]

Hence the approach to the apparent horizon stops at \( X = \epsilon_\ast := 3r_g| r'_g | / 2 \). For a macroscopic black hole \( 1 \gg \epsilon_\ast \gg \alpha^2 \), indicating consistency of the estimate. However, we also note that quantum energy inequalities may indicate that the required extent of the violation of the null energy condition is impossible for macroscopic black holes [17].

For a macroscopic black hole, i.e. a quasi-static exterior geometry, it makes sense to estimate the finite physical distance,

\[
L_\ast = \int_0^{\epsilon_\ast} f(t, r_g + x)^{-1} dx. \tag{10}
\]

For a finite evaporation rate \( r'_g \) the integral converges (for any upper bound, even if it is independent of \( r'_g \)). Assuming the steady-state evaporation rate \( r'_g = -\kappa / r^2_g \), where in the Planck units \( \kappa \ll 1, r_g \gg 1 \), we find that

\[
L_\ast \approx \frac{3}{2} r_g \left( -\log | r'_g | + \log(4/3) \right) + \ldots = \frac{3}{2} r_g \log \frac{r^2_g}{\kappa}, \tag{11}
\]

where the dots stands for the terms of the order of \( \sqrt{| r'_g |} \) and higher. Hence we see that the shell cannot cross its Schwarzschild radius but stays outside at some macroscopic distance from it.

The impossibility of a thin shell collapse to produce a trapped region in finite time \( t \), when the exterior geometry is modelled by the near-horizon metric of an existing black hole, does not exclude such an outcome in a more realistic setting. However, if a black hole forms in finite \( t \) a test particle cannot cross the apparent horizon. Hence the paradoxical situation that we have dismissed in Ref. [16], namely that the collapsing matter finds itself behind the horizon, while the subsequent infalling matter cannot cross it, actually is realized if a black hole is formed.

### IV. DISCUSSION

Thin shell models play an important role in many areas of gravitational physics. However, by collapse outcome is very sensitive to the assumptions about exterior geometry cases [19, 24], making imperative studies of more realistic models. Indeed, it is still unclear if a trapped region can form in finite time in more general models [23]. For most practical purposes it is immaterial, as even if it does not happen, only the null rays that are nearly radial within the small (and decreasing with mass) angle can leave the region bounded by \( r \approx C \) for a macroscopic distribution of mass.

Nevertheless, conceptual questions, such as entropy of the ultra-compact objects and the very formulation of the information loss paradox depend on its existence. Many important open questions for black hole formation (even in spherical symmetry) remain, and they will be addressed in the future work.

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