GEOMETRICAL LATTICE MODELS FOR $N = 2$ SUPERSYMMETRIC THEORIES IN TWO DIMENSIONS

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Abstract

We introduce in this paper two dimensional lattice models whose continuum limit belongs to the $N = 2$ series. The first kind of model is integrable and obtained through a geometrical reformulation, generalizing results known in the $k = 1$ case, of the $\Gamma_k$ vertex models (based on the quantum algebra $U_q sl(2)$ and representation of spin $j = k/2$). We demonstrate in particular that at the $N = 2$ point, the free energy of the $\Gamma_k$ vertex model can be obtained exactly by counting arguments, without any Bethe ansatz computation, and we exhibit lattice operators that reproduce the chiral ring. The second class of models is more adequately described in the language of twisted $N = 2$ supersymmetry, and consists of an infinite series of multicritical polymer points, which should lead to experimental realizations. The presence of $N = 2$ in that case is traced back to the Parisi Sourlas supersymmetry of the lagrangians usually used to replace $n \to 0$ limits. Boundary conditions as well as fermionic and bosonic variables are geometrically interpreted. Moreover it turns out that the exponents $\nu = (k + 2)/2(k + 1)$ for these multicritical polymer points coincide with the old phenomenological formulas of Flory. We therefore confirm that these formulas are exact in two dimensions, and suggest that their unexpected validity is due to non renormalization theorems for the $N = 2$ underlying theories. We also discuss the status of the much discussed theta point for polymers in the light of $N = 2$ renormalization group flows.

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1 Introduction

$N = 2$ theories are one of the most interesting example of conformal theories in two dimensions. A large subset of them possess in particular a very useful Landau Ginzburg description [1, 2, 3], whose efficiency is ensured by non renormalization theorems [4]. The Landau Ginzburg description makes also transparent the connection with singularity theory [2, 3], and explains nicely the origin of $ADE$ type classification.

Unfortunately these beautiful structures have not so far been much observed. It is known in principle how some special points of the $\Gamma_k$ vertex models [5], the XY (Gaussian) model or the Ashkin Teller [6, 7, 8] model possess $N = 2$ supersymmetry. However it has not been clear so far what is special about these points, and therefore how to tune the parameters to observe them. Also one would like the Landau Ginzburg description, and the chiral ring, to play some more illuminating role in these identifications, and make sense physically, as is partly the case for $N = 0$ theories [9].

In a preceding paper [10] we addressed these questions in the simplest case $k = 1$. We showed that the best point of view was to consider twisted $N = 2$ [11, 12], and that realizations of such theories were nicely provided by the geometrical problems of polymers and percolation. $N = 2$ supersymmetry was manifest in the structure of the correlators and the geometrical operators algebra. The presence of supersymmetry was traced back to the Parisi Sourlas [13] supersymmetry of the lagrangians (involving bosons and fermions) that are used in the description of geometrical models ($n \to 0$ limits). The knowledge of supersymmetry allowed us also to predict some new exponents, like the fractal dimension of the backbone of percolation in two dimensions.

The purpose of the present paper is to extend these observations to the case of arbitrary $k$. We find in particular lattice analogs of the chiral ring in integrable geometrical models, and families of multicritical polymer points connected by the $N = 2$ flow.

The organization is as follows. In the second section we work out the reexpression of $N = 2$ partition functions in terms of generalized Coulomb gas that was started in [5]. In section three we study the $\Gamma_k$ vertex models using a new geometrical formulation that generalizes the $k = 1$ case. This formulation involves bound states of strands carried by the edges of the square lattice. It exhibits very clearly the symmetry properties of the $q = \exp(i\pi/k + 2)$ points, at which the continuum limit is the $A_{k+1}$ modular invariant for the $N = 2$ series. For instance the free energy can be obtained in a simple way at that point, without using any Bethe ansatz computation. The continuum limit is indeed derived. We also identify a set of lattice operators that reproduce the chiral ring in an appropriate continuum limit. In the fourth section, we consider the duals (in the sense of Coulomb gas in two dimensions) of these models, and argue that they correspond to multicritical polymers. Generalizing the analysis of [10], the various sectors and the role of fermions and bosons, are geometrically interpreted. We discuss the coincidence of the exponents formulas with the so called Flory formulas [14].

\footnote{The $N = 0$ and $N = 1$ minimal series also possess in principle Landau Ginzburg descriptions, however these cannot really be used to obtain precise information on the system, like critical exponents.}
2 Generalized Coulomb Gas Expressions for the $N=2$ Partition Functions

The generalized lattice Coulomb gas was introduced in [5] to describe the continuum limit of SU(2) minimal coset models and of $\Gamma_k$ vertex models based on spin $j = k/2$ representations of $U_q\mathfrak{sl}(2)$. It involves as expected [15] a free bosonic field and $Z_k$ parafermions, which are coupled through boundary conditions. Introduce the partition function of a free bosonic field with coupling $g$ and windings $2m\pi, 2m'\pi$ along the generators $\omega_1, \omega_2$ of the torus, with $\tau = \omega_2/\omega_1$

$$Z_{mm'}(g) = \sqrt{\frac{g}{Im\tau}} \frac{1}{\eta} \exp \left( -\frac{\pi g}{Im\tau} |m - m'|^2 \right)$$

(1)

where $\eta$ is Dedekind’s function. Introduce also for a simply laced Lie algebra $G$ with Coxeter number $k+1$ the partition functions $Z_G^G(r,s)$ of the $Z_k$ model of [14, 17] with boundary conditions twisted by $e^{2i\pi r/k}, e^{2i\pi s/k}$ (we will not need their exact expression in the following). The generalized Coulomb partition function was then defined in [5]

$$Z_{gc}^G(g) = \sum_{r,s=0,...,k-1} Z_k^G(r,s) \sum_{m = r \mod k, m' = s \mod k} Z_{mm'}(g)$$

(2)

with central charge

$$c = \frac{3k}{k + 2}$$

(3)

In the following we shall concentrate on Lie algebras of $A$ type, and henceforth suppress the superscript $G$. The coulomb partition functions satisfy the duality relation

$$Z_{gc}(g) = Z_{gc} \left( \frac{1}{k^2 g} \right)$$

(4)

It was then shown in [5] that the $N=2$ partition functions for the minimal $A_{k+1}$ series with central charge (3) obtained in [18] could be rewritten as follows

$$Z_{N=2,k} = \frac{1}{2} (Z_{AP} + Z_{PA} + Z_{AA})$$

$$= \sum_{r,s=0,...,k-1} Z_k(r,s) \sum_{m = r \mod k, m' = s \mod k} \left( \frac{1}{2} + \delta_{m,m'} \mod 2 \right) Z_{mm'}(\frac{k+2}{2k})$$

(5)

The doubly periodic contribution is also known [11] to be

$$Z_{PP} = Tr(-)^F = k + 1$$

(6)

4The labels A (antiperiodic) and P (periodic) refer to the fermion boundary conditions on the torus
Notice that this index does not vanish.

We can now simplify the expression (5) using the following arguments. It was also shown in [5] how the generalized Coulomb gas can be used to reexpress the partition functions of minimal models

$$SU(2)_k \otimes SU(2)_{l-2}$$

with central charges

$$c = \frac{3k}{k+2} - \frac{6k}{l(l+k)}$$

This reexpression involves the generalized Coulomb gas with coupling

$$g = \frac{1}{k} - \frac{1}{l+k}$$

plus some interactions between the winding numbers. Consider now the particular case $l = 2$ that is associated with lattice models of type $A_{k+1}$. For such models $c = 0$ and one expects the partition function to be a constant, equal to $k + 1$ (the number of possible heights per lattice site). On the other hand, following [5] one gets

$$Z_{A_{k+1}} = \sum_{r,s=0,\ldots,k-1} Z_k(r,s) \sum_{m = r \mod k, m' = s \mod k} Z_{mm'} \left( \frac{2}{k(k+2)} \right)^{k+1} \sum_{n=1}^{\infty} \cos \left( \frac{2\pi n}{k+2} m \wedge m' \right)$$

The sum over exponents produces a $(k+2)\delta_{m \wedge m' \mod k+2 - 1}$. Analysis of each of the two terms and use of the duality symmetry gives therefore the formula

$$Z_{A_{k+1}} = k + 1 = \sum_{r,s=0,\ldots,k-1} Z_k(r,s) \sum_{m = r \mod k, m' = s \mod k} (2\delta_{m \wedge m' \mod 2 - 1}) Z_{mm'} \left( \frac{k+2}{2k} \right)$$

Notice that this can be reformulated as

$$k + 1 = \sum_{r,s=0,\ldots,k-1} Z_k(r,s) \sum_{m = r \mod k, m' = s \mod k} (-)^{m \wedge m'} Z_{mm'} \left( \frac{k+2}{2k} \right)$$

This identity generalizes the Euler’s identities. One deduces from (5) and (11)

$$Z'_{N=2,k} = Z_{gc} \left( \frac{k+2}{2k} \right) + \frac{k+1}{2} = Z_{gc} \left( \frac{2}{k(k+2)} \right) + \frac{k+1}{2}$$

The full $N=2$ partition functions with projection on odd fermion number writes therefore

$$Z_{N=2,k} = Z_{gc} \left( \frac{k+2}{2k} \right) = Z_{gc} \left( \frac{2}{k(k+2)} \right)$$
This identity was originally derived in [19] and independently in [20]. It provides a first clue concerning the lattice models whose continuum limit is \( N = 2 \) supersymmetric. Indeed the generalized Coulomb partition functions are the continuum partition functions for the \( \Gamma_k \) vertex models based on the quantum algebra \( U_q sl(2) \) and spin \( j = k/2 \). However this identification leaves totally obscure the origin of the \( N = 2 \) supersymmetry. At first sight, nothing distinguishes the \( \Gamma_k \) vertex model at the value \( q = \exp(i\pi/k + 2) \) from any other point on its critical line [3]. The next section is devoted to a geometrical formulation that will clarify this issue.

3 Geometrical Reformulation of the \( \Gamma_k \) Vertex Models

3.1 Generalities

The weights of the \( \Gamma_k \) vertex model are conveniently encoded in the \( \hat{R} \) matrix. This operator acts in a space \( (C^{k+1})^\otimes 2 \), and its elements are the Boltzmann weights of the corresponding configurations (figure 1). An important technical point is that several choices of weights are possible that are obtained by gauge transformations. Such transformations keep the Yang Baxter equation satisfied. Moreover they do not affect the torus partition function of the system. There is a special transformation that ensures commutation of the \( \hat{R} \) matrix with the generators of \( U_q sl(2) \), and in the following we always consider this \( \hat{R} \) matrix as giving the weights of the model. When the \( \hat{R} \) matrix is in the commutant of \( U_q sl(2) \), it satisfies some nice algebraic properties that define for instance in the \( j = 1/2 \) case the Temperley Lieb algebra. An advantage of focusing on these algebraic properties is that other representations can then be used, which make the physics of the problem more transparent.

Consider for instance the \( j = 1/2 \) case. The corresponding 6 vertex model has a \( \hat{R} \) matrix (called \( r \) in the following) that writes

\[
    r = I + f(u)e
\]

where

\[
    q = \exp(i\gamma), \gamma \in [0, \pi]
\]

\( u \) is the spectral parameter, and

\[
    f(u) = \frac{\sin \gamma}{\sin \gamma - u}
\]

The operator \( e \) writes in the usual spin \( j = 1/2 \) basis

\[
    e = \begin{pmatrix}
        0 & 0 & 0 & 0 \\
        0 & q^{-1} & -1 & 0 \\
        0 & -1 & q & 0 \\
        0 & 0 & 0 & 0
    \end{pmatrix}
\]

and satisfies the equations

\[
    e_i^2 = (q + q^{-1}) e_i \\
    e_i e_{i\pm 1} e_i = e_i \\
    [e_i, e_j] = 0, |i - j| > 1
\]
that define the Temperley Lieb algebra. In these equations, \(e_i\) is meant for the operator that acts as \(I\) in the \(i\)th and \(i+1\)th copies of \(C^2\), and identity otherwise.

It is however convenient to think of another representation provided independently by knot theory ideas [21], and by polymer and percolation problems. We notice that the Temperley Lieb algebra can be represented graphically as in figure 2. The operators act now in a space of lines, the identity just propagates the lines straight, while insertion of the Temperley Lieb matrix \(e\) attaches two neighbouring incoming and outgoing lines. Every time a closed loop is formed it is assigned the value \([2]_q = q + q^{-1}\). We can therefore think of the 6 vertex model as a loop model. There are now only two local configurations, represented in figure 3, which we call (this notation will be explained later), \(I, x\), with respective weights 1, \(f(u)\). A very important remark is that on a torus, the partition function of the loop model equals that of the original 6 vertex model if and only if the non contractible loops get a weight 2 instead of \([2]_q\).

The price to pay for having now only two local vertices is that the computation of the weight of a configuration requires in general a non local information, namely, when do pieces of lines attach together to form a loop, that should then be assigned a weight \([2]_q\). We recall that the loop evaluation used here is the geometrical equivalent of the Markov trace [22]. There is however a magic point where this non local information is not needed, it is obviously when \([2]_q = 1\). This corresponds to \(q = \exp(i\pi/3)\), \(g = 1 - 1/3 = 2/3\), ie it is the appropriate value for the \(N = 2\) point on the 6 vertex (Gaussian) line. Notice that at that point, the free energy can be evaluated in a trivial way: every vertex has two possible states, so in the large volume limit where we can neglect the special weight of non contractible loops we have

\[
f = \log 2
\]

This point was actually studied in [10] from the point of view of twisted \(N = 2\), and connected with the percolation problem. However, it does not seem that one can consider the other values of \(k\) in the series as representing some multicritical percolation points. The situation will be different for polymers, see below.

### 3.2 Symmetrizers and Fusion

The \(\Gamma_k\) vertex model can be obtained from the 6 vertex model by projecting \(k\) spins \(j = 1/2\) onto a spin \(k\) representation of \(U_q sl(2)\) [23–25]. The necessary object for doing that is the \(q\) symmetrizer, which has been widely studied [23–26]. It can be expressed using Temperley Lieb operators as follows. First recall that

\[
P_0 = \frac{1}{q + q^{-1}} e
\]

\[
P_1 = S_2 = 1 - P_0
\]

and introduce

\[
s = q^{-1} P_1 - q P_0
\]

In the following, \([x]_q\) denotes as usual the q-analog, equal to \((q^x - q^{-x})/(q - q^{-1})\)
Then one has

\[ S_k = \frac{q^{k(k-1)/2}}{|k|q} \sum_{\sigma} q^{-|I_\sigma|} \prod_{i \in I_\sigma} s(i) \]  

(23)

where the sum runs over permutations of \( k \) objects. Decomposing \( \sigma \) as a product of transpositions \( \tau_{i,i+1} \) gives the set of indices \( I_\sigma \), of minimal total number \( |I_\sigma| \). The symmetrizers satisfy some simple induction relations that make their determination easy. Now notice that \( S_2 \) can be represented graphically [28] as in figure 4. The property \( P_0P_1 = 0 \) translates into the fact that strands always have to pass through \( S_2 \), and cannot make a U turn. This property generalizes to any \( k \).

Now the fusion procedure for getting the Boltzmann weights of say the \( \Gamma_2 = 44 \) vertex model can be implemented in a variety of ways. We follow here the analysis of [26]. One finds in these references the proof that

\[ \hat{R}_{j=1} = S_2S_2r_2(u-\gamma)r_3(u)r_1(u)r_2(u+\gamma)S_2S_2 \]  

(24)

This a natural generalization of the well known formula used in conformal field theories or knot theory, that holds in the \( u \to i\infty \) limit. It corresponds to the picture of figure 5. Formula [26] generalizes easily to give the \( \hat{R} \) matrix of higher \( k \) vertex models as well. One finds

\[ \hat{R}_{j=k/2} = S_kS_k \{ r_k[u-(k-1)\gamma] \ldots r_{2k-1}[u] \} \times \{ r_{k-1}[u-(k-2)\gamma] \ldots r_{2k-2}[u+\gamma] \} \times \ldots \times \{ r_1[u] \ldots r_k[u+(k-1)\gamma] \} S_kS_k \]  

(25)

We remind the reader that \( \hat{R} \) has also an expression in terms of the quantum projectors [29]

\[ \hat{R}(u) = P_{j=k} + \frac{y^2 - q^{2k}}{1 - y^2q^{2k}} P_{j=k-1} + \ldots + \frac{y^2 - q^{2k}}{1 - y^2q^{2k}} \frac{y^2 - q^{2(k-1)}}{1 - y^2q^{2(k-1)}} \ldots \frac{y^2 - q^2}{1 - y^2q^2} P_{j=0} \]  

(26)

where \( y = \exp(-iu) \).

### 3.3 Geometrical Representation of the \( \Gamma_k \) Vertex Models

Let us start by discussing again the \( j = 1 \) \( \Gamma_2 = 44 \) vertex model. We consider the strand representation of the Temperley Lieb algebra introduced above, and follow it through fusion. First it is clear that the \( \hat{R} \) matrix acts now on \textbf{bound states} ie on states that have two lines incoming each leg of the four vertex. The symmetrizer acts on these lines, which is indicated by a transverse bar. Now we can compute the product (24) by picking one of the two terms in each of the \( r \) factors, and drawing accordingly either lines going straight or making some U turn. There are many allowed such configurations, that are represented in figure 6[4]. However after acting with the symmetrizers, only 3 configurations remain, which we call \( 1, x, x^2 \), see figure 7. Their respective weights are \( W_1, W_x, W_{x^2} \), given by

\[
\begin{align*}
W_1 & = 1 \\
W_2 & = f(u-\gamma)f(u+\gamma)f^2(u) \\
W_3 & = f(u-\gamma) + f(u+\gamma) + 2f(u-\gamma)f(u+\gamma)[f(u) + \cos\gamma]
\end{align*}
\]  

(27)

\[ ^{4}\text{This computation appears originally in some unpublished notes of Lou Kauffman devoted to "Multi strands calculus" and cabling in knot theory. The present formulation with insertion of symmetrizers leads to graphical rules for the SU(2) generalizations of the Jones polynomial (L.Kauffman, H.Saleur, work in progress) }\]
The isotropic point is, as for any $k$, obtained for $u = \gamma/2$, and one has then

\[
\begin{align*}
W_1 &= 1 \\
W_2 &= \frac{[2]_q}{1 + [2]_q} \\
W_3 &= 1
\end{align*}
\]  

(28)

so that the $W$ are invariant by a $90^\circ$ rotation. The three allowed configurations can be written in the Temperley Lieb algebra as

\[
\begin{align*}
1 &\leftrightarrow S_2 S_2 \\
x &\leftrightarrow S_2 S_2 e_2 S_2 S_2 \\
x^2 &\leftrightarrow S_2 S_2 e_2 e_1 e_3 e_2 S_2 S_2
\end{align*}
\]  

(29)

As for the 6 vertex model, although the interactions are much simpler in the geometric picture (44 vertices replaced by 3), the computation of the total Boltzmann weight of a configuration involves complicated evaluation of loops weights. In our case, a single contractible loop has weight $[2]_q$, while a symmetrized contractible double loop (see figure 8) has weight $[3]_q$. A single non contractible loop still has weight 2. The value of $q$ at which one can forget the loop weights, and the geometrical model becomes local is given by

\[ [3]_q = 1, \quad q = \exp(i\pi/4) \]  

(30)

There are various ways of proving this assertion. One is to argue that since the $q$ dimension of the spin one representation, which is the fundamental representation to consider in this case, is equal to one, the full Hilbert space onto which the transfer matrix is acting has also a $q$ dimension one, irrespective of the number of spin one representations that are considered. Therefore all quantum symmetric observables act trivially onto this space, and there cannot be any non local contribution. This can be made more precise, but is not very geometrical. Another proof is elementary and uses the very geometrical definition of the model. Consider a typical "fat" graph appearing in the computation of the partition function of a large system (see figure 9). Isolate in this graph an external "ear", and expand the symmetrizers that sit on the two branches. One gets the pictorial equation represented in figure 10. The first term has prefactor

\[ [2]_q - \frac{2}{[2]_q} = \frac{[3]_q - 1}{[2]_q} \]

that vanishes precisely at the point $q = \exp(i\pi/4) = \exp(i\pi/(k+2))$. At this point we can therefore simply cut each ear to evaluate the value of the graph, counting a factor $1/[2]_q^2$ for each suppressed ear. By repeating inductively this procedure we end with a graph that has no ear and therefore looks as the one of figure 11. We can decimate the loops using the pictorial equation of figure 12.

\[ \text{The model is truly local for an infinite system only. In a finite geometry, care has to be taken with non contractible loops. They however, for the present regime, are negligible for the evaluation of thermodynamic quantities.} \]
to get a factor \([2]_q - 1/[2]_q\) per loop, which equals \(1/[2]_q^2\) in that present case. The last loop gets a factor \([3]_q\). Therefore the total weight of the graph turns out to be

\[
w_{\text{graph}} = \left( \frac{1}{[2]_q} \right)^{\text{area}-1}
\]

Now notice that

\[
\text{area} = \text{number of vertices of type } x + 1
\]

Therefore we can evaluate the partition function by forgetting about the weights of contractible loops and multiplying the weights of each vertex by a factor

\[
\lambda_1 = 1, \lambda_2 = 1/[2]_q, \lambda_3 = 1
\]

It is also interesting to consider the correspondence between the geometrical vertices and the expansion of the \(\hat{R}\) matrix in terms of projectors. One finds

\[
\begin{align*}
1 & \leftrightarrow P_{j=0} + P_{j=1} + P_{j=2} \\
x & \leftrightarrow \left( [2]_q - \frac{2}{[2]_q} \right) P_{j=1} + \left( [2]_q - \frac{1}{[2]_q} \right) P_{j=0} \\
x^2 & \leftrightarrow \left( [2]_q^2 - 1 \right) P_{j=0}
\end{align*}
\]

The extension of these results to higher values of \(k\) is easy. In general, the legs of the vertices carry bound states made of \(k\) symmetrized lines. There remains only \(k + 1\) vertices, that can be labelled \(1, x, \ldots, x^k\) see figure 13. These vertices correspond to interactions that are once again more conveniently written graphically. We draw as in figure 14 a square with each face carrying an \(e\) matrix with the appropriate label, and multiply the \(e's\) from top to bottom, and inside a given line, from left to right. The precise form of the weights is not needed here (it involves use of quantum \(6j\) symbols\cite{30}). They have the property of being symmetric under 90° rotation for \(u = \gamma/2\). At this point one has therefore the symmetry

\[
W_{x^i} = W_{x^{k-i}}
\]

The symmetrized contractible \(k\) loops acquire a weight \([k+1]_q\) that becomes equal to one for

\[
q = \exp(i\pi/k + 2)
\]

At this point one can show that the geometrical model becomes local, with weights renormalized by the factors\cite{30}

\[
\lambda_{x^i} = \frac{1}{[i+1]_q}
\]

Notice that at the special point \(\lambda_{x^{k-i}}\) one has

\[
\lambda_{x^i} = \lambda_{x^{k-i}}
\]

Notice that, as in the \(k = 1\) case, the free energy can be simply evaluated at the \(N = 2\) point of the \(\Gamma_k\) vertex model. Since each vertex can be in one of the \(k + 1\) allowed states independently
of its neighbours, one gets in the large volume limit where the effect of non contractible loops is negligible

\[ f = \log \left( \sum_{i=1}^{k+1} \lambda_i W_{x^i} \right) \]  \hfill (39)

3.4 Torus partition functions

The loop model introduced above being equivalent to the \( \Gamma_k \) vertex model without charges at infinity, it is easy to derive its continuum limit using the generalized Coulomb g as mapping. Following the results of [5] one has

\[ g = \frac{1}{k} - \gamma = \frac{2}{k(k+2)} \]  \hfill (40)

The coupling between the parafermions and free bosonic field quantum numbers is built in in the model. The only remaining point concerns the allowed range of \( m, m' \). Because we want the \( x^i \) to be true operators acting in a transfer matrix formalism, we need the square lattice at hand to be described diagonally as in figure 15. There is therefore an even number of edges crossed when describing one of the generators \( \omega_1, \omega_2 \), so that following [5], \( m, m' \) are integers. Formula (14) therefore follows.

3.5 Lattice Candidates For Chiral Primaries

We now present some arguments that the \( x^i \) operators, that become local operators at the magic point \( q = \exp(i\pi/k + 2) \) for the \( j = k/2, k \) strands case, are lattice candidates for the chiral primaries [31]. Of course as lattice operators, the \( x^i \) have a left and a right moving part, that behave identically. It is more proper to identify their lattice algebra with the direct product of left and right chiral rings of the continuum limit. For simplicity we think of the left moving sector only in the following.

It is straightforward to perform the multiplication \( x^i . x^j \). For this one simply connects all the outgoing lines of a \( x^j \) vertex to the incoming lines of an \( x^i \) one, and expands the intermediate symmetrizers (figure 16). The result takes the form

\[ x^i . x^j = \sum_{l=0}^{\min(i+j,k)} c_l x^l \]  \hfill (41)

The precise form of the \( c_l \) coefficients is not needed in the following. The important point is that the right member truncates and does not contain terms with exponents greater than \( i + j \). The reason for this is not totally obvious. To form a term \( x^l \) requires the introduction of \( l \) matrices \( e_k \) (figure 17). On the other hand, \( x^i \) and \( x^j \) contain respectively \( i \) and \( j \) such terms, while the symmetrizers do not contain any. Therefore there are not enough \( e_k \) terms to form an \( x^l \) when \( l > i + j \) in the product. The fact that \( i + j \) cannot exceed \( k \) is simply because we start with \( k \) strands on each leg, so we cannot connect more than \( 2k \) of them.

Now we can try to transform such a product into a meaningful expression in the continuum limit. For convenience we suppose that we work on a cylinder with some strands states propagating
in the time direction (figure 18). After the conformal mapping from the plane to the torus, fields become operators acting on the strands. Due to the conformal factors, these operators have matrix elements that scale like \( L^{-2h} \), where \( L \) is the radius of the cylinder. On the other hand, the discrete operators \( x^i \) introduced above have matrix elements that scale like 1. Therefore we define rescaled operators

\[
X^i_L = L^{-2h} x^i
\]

so the product (41) reads now

\[
X^i_L . X^j_L = \sum_{l=0}^{i+j} c_l L^{2i-h_i-h_j} X^l_L
\]

This equation has a (non trivial) limit when \( L \) becomes large if and only if

\[
h_{i+j} = h_i + h_j
\]

In that case (42) becomes in the large \( L \) limit

\[
X^i . X^j = X^{i+j} \text{ for } i + j \leq k, \ 0 \text{ otherwise}
\]

This algebra is precisely the chiral ring [2, 3], while the condition (44) occurs as well in the study of continuum \( N = 2 \) theories when one requires the short distance expansion to reproduce the chiral ring. One finds then

\[
h_{x^i} = i/2(k + 2)
\]

Another point concerns the \( U(1) \) charge of the \( x^i \) operators. Following [10] let us define it as proportional to the maximum number of non contractible loops (in the time direction) the insertion of \( x^i \) can create. One finds

\[
Q_{x^i} \propto i
\]

again a result known to hold for \( N = 2 \) theories where

\[
Q_{x^i} = i/(k + 2)
\]

It is also possible to consider the \( x^i \) as antichiral primaries. They satisfy the same algebra, but we define their charge as proportional to minus the maximum number of non contractible loops (in the time direction) their insertion can make contractible (figure 19). Notice also that the 90° rotation formally correspond to the so called Poincare duality [2, 3]. Finally it is clear that \( I \) behaves indeed exactly as the identity in the transfer matrix formalism, and therefore has vanishing conformal weight. We discuss the conformal weights of the other \( x^i \) fields in the next section.

3.6 Discussion

First we discuss the case \( j = 1/2 \). We therefore have contractible contours with weight \([2] q = 1 \) for \( q = \exp i \pi/3 \), non contractible contours with weight two, and two vertices of interaction, \( I, x \).
with the same weight one. We consider the model on a cylinder. It is easy to isolate the doubly periodic sector by considering for a while configurations where the non contractible loops also have a weight one. In that case the partition function is obviously equal to a constant, in agreement with $Z_{PP} = Z_{\tilde{R}} = 2$. This last value is merely an overall normalization problem when taking the continuum limit in the Coulomb gas mapping. It can be found by noticing that the loop configurations so isolated just compute the partition function of the $A_2$ model, which corresponds in turn to taking yet another representation of the Temperley Lieb algebra of solid on solid type, with elements for the face configuration of figure 20

$$e_{l_1,l_2,l_3} = \delta_{l_1,l_3} \sqrt{\frac{[l_2]_q [l'_2]_q}{[l_1]_q}}$$  \hspace{1cm} (49)$$

We can therefore picture the Ramond ground state as a frustration line running across the cylinder in time direction, such that every loop crossing it gets a weight 1 instead of 2. In the Coulomb gas mapping, this corresponds to having a pair of electric charges at the extremities of the cylinder, with value $e_0 = \pm 1/3$. This has dimension $h = e_0^2/4g = 1/24$ as expected. Let us now consider the full theory, with non contractible loops having weight 8. Because insertion of $I$ acts identically on the propagating strands, its dimension is certainly

$$h_I = 0$$  \hspace{1cm} (50)$$

Consider now the operator $x$. It is reasonable to assume that the connected correlation function with its antichiral conjugate is obtained by creating a pair of non contractible loops where $x$ is inserted, and destroying it when the conjugate is inserted: in other words by selecting the configurations where a loop connects the two insertions as in figure 21. It is well known how to evaluate this dimension in the Coulomb gas mapping \[32\]. The corresponding operator is magnetic with charge $m = 1$, and therefore

$$h_x = g(m = 1)^2 \frac{1}{4} = \frac{1}{6}$$  \hspace{1cm} (51)$$

as expected. This dimension is also what would be observed if we were to compute the correlation function in the antiperiodic (NS) sector.

This picture essentially generalizes to the higher $k$ case. The doubly periodic sector is reproduced by using the restricted solid on solid model representation of the Temperley Lieb algebra for $A_{k+1}$. The Witten index is $k + 1$ corresponding to the number of nodes of the diagram. The Ramond ground state is obtained by giving to the non contractible loops across the cylinder a weight $[2]_q$. Finally if we admit the likely hypothesis that connected correlation functions of $x^i$ operators and their conjugates are described by $i$ loops joining the two insertions we find, using the generalized Coulomb gas mapping, the weights $h_{x^i} = i/2(k + 2)$.

### 4 D type invariants

It is well known how the $D_{2+k/2}$ theories can be obtained from the $A_{k+1}$ theories by a $Z_2$ orbifold. In our case it is easy to form symmetric combinations of the diagrams $x^i$ and $x^{k-i}$. However we

Unfortunately we do not know what is the microscopic definition of all the sectors in that case. In that respect, the polymer situation described in the next section is more favorable
do not know what the equivalent of the $y$ field should be. Hints in the identification of $D$ lattice models can be obtained from consideration of the partition functions. Using once again the results of one finds first that the expression (14) generalizes to $D$ or $E$ invariants, but with the appropriate parafermions partition functions. Then since $D$ parafermions are themselves orbifold of the $A$ ones one has

$$Z_{k}^{D_{2+k/2}}(r, s) = \frac{1}{2} [Z_{k}(r, s) + (-)^{r} Z_{k}(r, s + k/2) + (-)^{s} Z_{k}(r + k/2, s) + (-)^{r+s} Z_{k}(r + k/2, s + k/2)]$$

The peculiar combination of sectors of the $\Gamma_{k}$ vertex model reproducing the $D_{2+k/2}$ invariant can obviously be read from the above equation. But it still lacks a natural interpretation.

5 $K^{th}$ Critical Polymer Models and Twisted $N = 2$ with Level $k$

There is another, and physically more interesting, way of interpreting the $N = 2$ partition functions. Recall that the $\Gamma_{k}$ vertex models are expected to have in fact two phases. In the first one, the model is integrable with Boltzmann weights given by the usual formulas, and coupling constant as above

$$g = \frac{1}{k} - \gamma, \quad q = e^{i\gamma}, \quad \gamma \in [0, 1/k]$$

The second phase is not very well known except for $k = 1$. It is supposed to be obtained by adding some vacancies. There are however some strong arguments that the corresponding renormalized coupling constant is

$$g = \frac{1}{k} + \gamma$$

In the case $k = 1$ the first regime corresponds to the critical Potts model with $Q = (q + q^{-1})^2$, the second regime to the critical $O(n)$ model with $n = q + q^{-1}$. For higher $k$ it is likely that the second regime describes a sort of multicritical $O(n)$ model with in particular the formation of bound states, ie up to $k$ lines colliding on a same edge, allowed. Now choose $\gamma = 1/2$. This value corresponds to $n = 0$ ie to $k^{th}$ critical polymers. The associated coupling constant is

$$g = \frac{k + 2}{2k}$$

dual to the coupling constant for the vertex model that we discussed in the preceding section. In the case of polymers, we can however generalize the discussion of for the dense and critical case, and provide a natural geometrical interpretation of the various quantities so far encountered. The natural point of view here is to consider twisted $N = 2$ theories. Recall that twisting the theory is obtained by adding a term $\frac{1}{2} \partial J$ to the stress energy tensor. The new central charge vanishes. The partition function of the twisted model is however the same as the one of the untwisted model, so twisting merely amounts to a change of point of view. The former ground state of the Ramond sector is now considered as the true $SL_2$ invariant ground state, while the
former true ground state is now considered as a state with negative dimension. Notice also that, because of twisting, the fermions acquire integer dimensions and therefore have identical boundary conditions for the cylinder and the plane.

5.1 Polymers Partition Functions

First we notice that for polymers also the bulk free energy is easy to evaluate. Polymers occupying a vanishing fraction of the available space one has $f = 0$, as is usual for $n \to 0$ limit models.

As in [10] we think of the $n \to 0$ limit as obtained by associating to each polymer loop a bosonic or a fermionic variable. In this way all contractible loops on a torus disappear, while non contractible loops can get a weight zero or two depending on their winding and the fermions boundary conditions (figure 22). One has, if the labels A or P refer to these discrete fermions boundary conditions

**Definition**

$Z_{AP} (\text{resp. } P_A, \text{resp. } AA) = \text{Sum over configurations with an even number of non contractible } k^{th} \text{ critical polymers of total length } \mathcal{L}, \text{ that cross } \omega_2 \text{ (resp. } \omega_3, \text{ resp. } \omega_1 + \omega_2) \text{ an odd number of times, with weight } 2^{\text{number of polymers}} \mu^{-\mathcal{L}} \times \text{ the Boltzmann weights induced by the } k^{th} \text{ critical interactions}$

where $\mu$ is the appropriate inverse connectivity constant. One can also define the manifestly modular invariant quantity

$$Z^e = \left\{ \text{Sum over configurations with an even number of non contractible } k^{th} \text{ critical \}

\text{ polymers of total length } \mathcal{L} \text{ with weight } 2^{\text{number of polymers}} \mu^{-\mathcal{L}} \times \text{ the Boltzmann weights induced by the } k^{th} \text{ critical interactions} \right\} \quad (56)$$

One checks easily from these definitions that

$$Z^e = \frac{1}{2} (Z_{AP} + Z_{PA} + Z_{AA} - Z_{PP}) \quad (57)$$

Notice that projection on odd fermion number occurs naturally on geometrical grounds.

On the other hand the analysis with the generalized lattice Coulomb gas is such that topological properties (magnetic defects, electric charges, winding numbers...) do not depend on the peculiar $k^{th}$ critical regime. We can therefore use the same expressions as the ones in [10] to express in the continuum limit

$$Z_{PP} = \sum_{r,s=0,...,k-1} Z_k(r,s) \sum_{m = r \mod k} (-)^m Z_{mm'} \quad Z_{AP} = \sum_{r,s=0,...,k-1} Z_k(r,s) \sum_{m = r \mod k} (-)^m (-)^{m \wedge m'} Z_{mm'}$$

\footnote{Recall that the two winding numbers of a non contractible loop are coprimes}
\[ Z_{PA} = \sum_{r,s=0,\ldots,k-1} Z_k(r,s) \sum_{m = r \mod k} (-m')^m \left( -m \wedge m' \right) Z_{mm'} \]

\[ Z_{AA} = \sum_{r,s=0,\ldots,k-1} Z_k(r,s) \sum_{m = r \mod k} \sum_{m' = s \mod k} (-m + m')^m \left( -m \wedge m' \right) Z_{mm'} \]  \hspace{1cm} (58)

One checks that each of these expressions coincides respectively with the known partition functions of the \( N = 2 \) theories. In that case we therefore have found a geometrical interpretation for all the sectors of the theory. Let us emphasize again that due to twisting, the fermions acquire integer dimensions and therefore have the same boundary conditions in the plane and on the torus. Summing these four contributions one gets

\[ Z^e = Z_{gc} \left( \frac{k+2}{2k} \right), \text{ for } k^{th} \text{ critical polymers} \]  \hspace{1cm} (59)

As in the \( k = 1 \) case, this partition function is not consistent physically because it does not contain the identity (for the twisted theory), as expected since we projected on odd fermion number. The physical polymer partition function is rather obtained by projection on even fermion number

\[ Z^{phy} = Z^e + k + 1 \to Z^e + k + 1 \]  \hspace{1cm} (60)

The ground states in the twisted theory describe simply configurations with no polymers. That they appear with some multiplicity occurs probably because of the additional degrees of freedom introduced to make the polymers multicritical (vacancies and their generalizations \[10\]). As in the \( k = 1 \) case we notice that R and NS correctly describe observables with an \textbf{even} number of polymers only. The full polymer theory would need in addition the consideration of a sector with \( Z_4 \) twists \[10\], which is easily studied via the spectral flow. Also we recall that the BRS cohomology used to obtain topological theories out of twisted \( N = 2 \) \[12\] is not the right procedure to extract physical states from the polymer point of view. On the contrary, \( Q_{BRS} \) turns out to be the operator that creates polymers out of the vacuum.

### 5.2 Some Polymer Exponents and the Flory Formula

The most important quantity is the exponent \( \nu \) that controls the mean size of polymers. Considering a polymer loop of \( \mathcal{L} \) monomers one has asymptotically

\[ \langle R_i^2 \rangle \propto \mathcal{L}^{2\nu} \]  \hspace{1cm} (61)

where \( R_i \) is the radius of gyration. If we introduce the 2 legs polymer operator, of conformal weights \( h = \overline{h} \), whose correlations are defined by summing over configurations of a polymer loop attached at two different points, one has, by standard scaling arguments

\[ \frac{1}{\nu} = 2 - 2h \]  \hspace{1cm} (62)

\[ ^{10} \text{In the preceding paper \[10\] we referred to R and NS as the boundary conditions in the plane} \]
This operator was identified in [10] to be the first non trivial operator in the periodic sector (X in the Landau Ginzburg picture), with weight after twisting given by

\[ h = \frac{1}{k + 2} \]  

(63)

Using formula (62) one finds

\[ \nu = \frac{k + 2}{2(k + 1)} \]  

(64)

These exponents have already appeared in the literature and they are known as the Flory exponents [14] for \( k \)th multicritical polymers (in our conventions, ordinary polymers are \( k = 1 \)). The Flory formula is obtained by making crude, and notably wrong [34], assumptions for the free energy of a polymer. One usually writes \( F \) as the sum of an elastic and energetic contributions, and minimizes with respect to say \( \mathcal{L} \) at \( R_G \) fixed to obtain \( \nu \). The elastic contribution is argued to be \( F_{el} \propto R_G^2 / \mathcal{L} \), and the energy contribution for a \( k \) multicritical point \( F_{en} \propto R_G^d (\mathcal{L}/R_G)^{k+1} \) where \( d \) is the space dimension [35]. The argument for this last expression mimics field theory where for ordinary polymers, the \( \phi^4 \) term dominates, corresponding to two pieces of polymers that come in contact (figure 23), for tricritical polymers the \( \phi^6 \) term dominates corresponding to three pieces of polymers coming in contact [36]...

Writing therefore

\[ F \propto R_G^2 / \mathcal{L} + R_G^d (\mathcal{L}/R_G)^{k+1} \]

(65)

one finds after minimization

\[ \nu = \frac{k + 2}{2 + dk} \]  

(66)

One could not really take this formula seriously based only on the Flory "derivation". However in the case \( k = 1 \) it is in fact exact for \( d = 1, 2, 4 \) and very close to the numerical estimates for \( d = 3 \). \( \nu = 1/2 \) for \( d = 4 \) which one can therefore correctly identify as the upper critical dimension. Formula (66) should not be applied above the critical dimension, where the exponent \( \nu \) is expected to remain equal to 1/2. Some explanations of approximate validity of Flory’s formula have been proposed [37]. What is fascinating however is that it can be exact in most cases. So far higher \( k \)’s had not been really considered. We notice however that if we put \( d = 2 \) in (66) we recover the exponents (64). This shows they are also exact for the peculiar kind of multicritical polymers at hand, and strongly suggests that the Flory formula can be exact because of non renormalization theorems due to the hidden \( N = 2 \) supersymmetry in polymers. As a final argument in that direction, notice that \( \nu \) in (66) becomes equal to 1/2 for

\[ d = 2 + \frac{2}{k} \]  

(67)

which is precisely the upper critical dimension for \( k \)th critical \( N = 2 \) Landau Ginzburg theories.
5.3 Renormalization Group Flow

It is worth considering the problem from the point of view of renormalization group flow [9, 1]. We remark that because twisting is only a change of point of view on the system, the flow results established for the untwisted case should hold also for polymers. Indeed perturbing for instance by $X$ the $k$ theory means shifting the geometrical weight of the polymer to $(\mu + \delta \mu)^{-L}$, independently of whether we consider the ground state as the state without polymers, or the state with large non contractible loops that have weight two.

For the $N = 2$ series, the "minimal" supersymmetry preserving flow is a flow from the theory with $k$ to the next theory with $k - 2$ ($X^{k-1}$ is redundant). It is generated by the superpartner of the top component of the chiral primary field $X^k$, with expression in the generalized Coulomb gas, where $\psi_1$ is the parafermionic field of lowest dimension,

$$
\Phi_{\text{pert}} = \psi_1 \bar{\psi}_1 exp \left( \frac{2i}{k} \phi(z, \bar{z}) \right)
$$

and dimension $h = \frac{k}{k+2}$. After twisting, since $G^+$ acquires weight one, this field has the same dimension as $X^k$ in the twisted theory, which can also be computed by adding the proper charge at infinity for the field $\phi$ in the generalized Coulomb gas. One finds

$$
h = \frac{k}{k+2}
$$

This perturbation is still integrable in the twisted theory as discussed in [12]. The above dimension provides the value of the exponent $\nu_u$ for the multicritical polymers at hand [43]

$$
\nu_u = \frac{k+2}{4}
$$

and therefore the crossover exponent

$$
\varphi = \frac{\nu}{\nu_u} = \frac{2}{k+1}
$$

We recall that the renormalization group flow in the $N = 2$ series is more subtle than in the $N = 0$ or $N = 1$ series. The analysis of the superpotential $X^{k+2} + \lambda X^k$ can easily be done, at least naively, since there is only wave function renormalization [11]. However two points with different values of $k$ are infinitely far away in the sense of [12], so the flow cannot be studied in the conformal perturbation framework [11]. This is because the index takes value $k+1$ for the $k^{th}$ critical polymers, and therefore has to jump by two units between the $k$ and the $k-2$ theory. As long as $0 < k < \infty$ one does not expect spontaneous breaking of supersymmetry.

The perturbation with the operator $X$ is also integrable [44], and physically interesting. Recall that $X$ was identified with the two polymer legs operator, and therefore adding an $X$ perturbation is like changing the geometrical weight of polymers. Two behaviours can occur depending on the

11 From the polymer point of view, and owing to experimental and numerical knowledge of multicritical polymer systems, there is little doubt that such flow occurs.
sign of the coupling. For the sign corresponding to $\delta \mu > 0$, the theory flows to a trivial fixed point. This describes so small polymers that they disappear at large scale. For the sign corresponding to $\delta \mu < 0$, we expect physically the polymers to become so large that they occupy a finite fraction of the available volume, with fractal dimension exactly equal to two, that is an exponent $\nu = 1/2$. This means the system should flow to the dense polymers phase, that is described by an $\eta, \xi$ system $[10]$, and that supersymmetry is spontaneously broken. The decoupling of the ground state has a very transparent physical interpretation. Indeed for the state without polymers $Z \propto 1$, while as soon as one dense polymer is allowed $Z \propto e^{\text{free energy}}$ for $f$ some non vanishing free energy. Notice that in the dense phase, the free energy is now a non trivial number, which is not known exactly except in special cases (hamiltonians walks on the Manhattan lattice for instance).

For $k = 1$ (ordinary self avoiding polymers), the perturbations $X^k$ and $X$ coincide. The "bottom" of the series of multicritical points is therefore dense polymers, where the supersymmetry is spontaneously broken.

5.4 $k \to \infty$ limit

Another insight can be obtained by considering the $k \to \infty$ limit. In that limit the exponent $\nu$ tends to $1/2$, the exponent of dense polymers $[10]$. This corresponds to polymers that are collapsed onto themselves, due to strong attractive interactions, or compression from the exterior. We therefore find again dense polymers, which are indeed expected to be the end point of the multicritical polymers series when described in the direct renormalization or $O(n), n \to 0$ multicritical field theory. We can here make the identification of the $k \to \infty$ limit of our theories with dense polymers very precise. Indeed it was shown in $[10]$ that dense polymers are conveniently described by an $\eta, \xi$ system with central charge $c = -2$. This corresponds to breaking the symmetry between bosons and fermions in say the free field representation of $N = 2$ twisted theories. By summing over various sectors, it was shown that the entire partition function of dense polymers with an even number of non contractible loops on the torus is a gaussian partition function

$$Z_{\text{dense}} = Z_c[1/2]$$

where by $Z_c$ we mean an expression similar to $Z_{gc}$ but with the partition functions of the $Z_1$ parafermions taken to be equal to one. On the other hand it is possible to work out the $k \to \infty$ limit of the partition functions $[14]$. For this purpose notice first that the coupling constant goes to a finite value $g = 1/2$ in that limit. For $k$ large enough, the $m, m'$ excitations are rapidly damped out in the second summation in the expression $[2]$ of the generalized Coulomb partition function, so we can truncate this summation to $m = r, m' = s$. Moreover as $k$ becomes large, the frustrations of the $Z_k$ model become negligible for the values of $r, s$ such that $Z_{rs}$ is not itself too small. One can therefore write

$$Z_{gc} \left( \frac{k + 2}{2k} \right) \propto Z_k(0,0)Z_c[1/2], \text{ as } k \to \infty$$

Now such a result can also be established for the level $k$ Wess Zumino partition functions which can be rewritten as a generalized Coulomb partition function for a coupling $g = 1/k$. On the other $^{12}\nu = 1/2$ is of course also the exponent of brownian walks, but this is not apparently the same problem, although some other features are common with the dense case.
hand, using reexpression of these same WZW partition functions in terms of 3 bosonic partition functions as worked out in [38] leads to

\[ Z_{WZW} \propto \left( \frac{1}{\sqrt{\text{Im}\tau\eta}} \right)^3 \]  

(74)

Combining these results we get finally

\[ Z_{gc} \left( \frac{k + 2}{2k} \right) \propto \left( \frac{1}{\sqrt{\text{Im}\tau\eta}} \right)^2 Z_c[1/2] \text{ as } k \to \infty \]  

(75)

Therefore in the large \( k \) limit we find indeed the dense polymer partition function, up to a free bosonic field factor that helps maintaining central charge at a value zero. As in the preceding study of the \( X \) perturbation, one expects in the \( k \to \infty \) that spontaneous braking of supersymmetry will occur, leaving only the \( \eta, \xi \) system with partition function \( Z_c[1/2] \). Notice that the untwisted model corresponding to the \( k \to \infty \) limit is the \( c = 3 N = 2 \) theory, which is well known to have vanishing index. Therefore dense polymers occur both at the "top" and the "bottom" of the multicritical series. That dense polymers appear both in the \( k \to \infty \) and in the low temperature phase is due to the existence of two independent mechanisms for obtaining them. When \( k \to \infty \), more and more attractive interactions are added, so the polymer is forced to collapse onto itself for internal reasons. On the other hand, the breaking of symmetry of the \( O(n) \), \( n \to 0 \) model actually reduces the available volume, so the polymer collapses for topological reasons.

5.5 Ordinary theta point

Can we identify one of the multicritical points discussed above with the usual theta or theta' [35, 40] point? We recall that the theta point, which was solved in [10] involves a set of attractions between nearest and second nearest neighbours on the honeycomb lattice, with a partition function, for an even number of non contractible polymer chains, that was supposed so far to equal \( Z_c[2/3] \). The point \( k = 6 \) in our series has also \( g = 2/3 \), but a rather different operator content due to the \( Z_6 \) parafermions and the boundary conditions couplings. The exponent \( \nu = 4/7 \) is the same. For the theta point it is expected, and rather well established numerically, that the cross over exponent is \( \varphi = 3/7 \). This means that the perturbation physically identified as changing the attraction between monomers has dimension

\[ h = 1 - \frac{\varphi}{2\nu} = \frac{5}{8} \]  

(76)

This is the dimension of the field \( X^5 \) after twisting, while for \( k = 6 \) the theory has an \( X^8 \) potential. Under this perturbation the theory flows to the \( k = 3 \) theory however, not the \( k = 1 \) theory! This can be interpreted in two ways. We can of course believe that the present set of multicritical points has nothing in common with the theta point, and that the possible concordance of some exponents is an accident. We can also try to minimize the available set of polymer systems by assuming that \( k = 6 \) is indeed the theta point, and that something was not understood in [10]. This would

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13 The distinction of theta and theta' points is a technical matter that we cannot discuss here. See [40] and subsequent comments and replies in Phys. Rev. Lett.
explain why the numerical analysis of the problem is so difficult. Indeed in the present scheme, we can go from $k = 6$ to $k = 1$ with a crossover exponent $\varphi = 3/7$ at the $k = 6$ point only by passing through the $k = 3$ point, with $\nu = 5/8$. The mixture of these behaviours for the lattice polymers may indeed lead to very confusing measurements. We summarize some of the properties of the polymer multicritical series in figure 24.

6 Conclusions

In conclusion the two kinds of models we have introduced illustrate various aspects of $N = 2$ theories. The geometrical reformulation of the $\Gamma_k$ vertex model leads to lattice analogs of the chiral ring. The multicritical polymers are not precisely defined yet \footnote{We do not know their exact Boltzmann weights, as is usually the case for multicritical systems}. But they provide a nice interpretation of the role of bosonic and fermionic degrees of freedom, a geometrical understanding of the boundary conditions, and hopefully physical realizations of $N = 2$ flow and spontaneous supersymmetry breaking. For both models, the bulk free energy can be readily obtained, without Bethe ansatz type computations.

This work opens many physical questions. Among them, one would like to find an explicit $N = 2$ formulation of the Flory approximations, which maybe could be generalized to higher dimensions. Also, the series of multicritical polymer points seems a good place to apply the ideas of \footnote{We do not know their exact Boltzmann weights, as is usually the case for multicritical systems}. Notice that polymers can give rise to experiments in two dimensions \footnote{We do not know their exact Boltzmann weights, as is usually the case for multicritical systems}, and therefore it should be possible to observe in such systems consequences of $N = 2$ supersymmetry.

On the more mathematical side, the study of interrelations between the singularity structure and the geometrical lattice models should be fruitful.

Finally, it seems from our study that the most natural description of lattice models with $N = 2$ continuum limit is geometrical. This is maybe not totally unexpected. For instance that the free energy can be obtained without computation, which is the kind of nice property one expects for a $N = 2$ model, seems to imply a geometrical setting where the non trivial observables are defined by topological properties, for instance the connectivity in polymers \footnote{We do not know their exact Boltzmann weights, as is usually the case for multicritical systems}. Maybe by pursuing this route one could indeed give lattice interpretations of the magic numbers that have been computed for topological theories \footnote{We do not know their exact Boltzmann weights, as is usually the case for multicritical systems}.

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Figure Captions

Figure 1: The weights of vertices are encoded in the $\hat{R}$ matrix.
Figure 2: Graphical representation of the Temperley Lieb algebra.
Figure 3: The two vertices in the geometrical reformulation of the 6 vertex model.
Figure 4: Geometrical representation of the symmetrizer on two strands.
Figure 5: Geometrical representation of the fusion formula for $\hat{R}$ matrices.
Figure 6: The various configurations that enter in the decomposition of the $j = 1$ fusion before applying the symmetrizers.
Figure 7: The three remaining configurations after application of the symmetrizers.
Figure 8: The value of the double contractible loop is $|3\rangle_q$.
Figure 9: A typical fat graph entering the computation of the partition function.
Figure 10: Factorization rule for one ”ear” of the diagram.
Figure 11: The same diagram after factorizing out the ears.
Figure 12: Factorization of "handles".
Figure 13: The four vertices of interaction for $k = 3$.
Figure 14: Squares with $e$ matrices on each face encode conveniently the algebraic form of the $x^i$ interaction terms. This figure corresponds to the interaction of figure 17.
Figure 15: Diagonal propagation for the square lattice.
Figure 16: Graphical computation of $x^2$ for $k = 2$.
Figure 17: Formulation of $x^4$ in the Temperley Lieb algebra.
Figure 18: Strands propagating on a cylinder.
Figure 19: Insertion of $x$ reduces the number of non contractible loops by two.
Figure 20: $e$ matrix in the solid on solid basis.
Figure 21: Configurations that contribute to the connected correlation function for the $X$ operator.
Figure 22: If the wiggly line represents anti periodic boundary conditions for fermions, a non contractible loop that crosses it once gets a factor $1 + 1 = 2$, while a non contractible loop that crosses it twice gets $1 - 1 = 0$.
Figure 23: Multicritical polymers are obtained by adding multipieces interactions.
Figure 24: Some features of the renormalization group flow in the multicritical polymers series.
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