Energy Momentum Tensor in Conformal Field Theories
Near a Boundary

D.M. McAvity and H. Osborn

Department of Applied Mathematics and Theoretical Physics,
Silver Street, Cambridge, UK

The requirements of conformal invariance for the two point function of the energy momentum tensor in the neighbourhood of a plane boundary are investigated, restricting the conformal group to those transformations leaving the boundary invariant. It is shown that the general solution may contain an arbitrary function of a single conformally invariant variable $v$, except in dimension 2. The functional dependence on $v$ is determined for free scalar and fermion fields in arbitrary dimension $d$ and also to leading order in the $\varepsilon$ expansion about $d = 4$ for the non Gaussian fixed point in $\phi^4$ theory. The two point correlation function of the energy momentum tensor and a scalar field is also shown to have a unique expression in terms of $v$ and the overall coefficient is determined by the operator product expansion. The energy momentum tensor on a general curved manifold is further discussed by considering variations of the metric. In the presence of a boundary this procedure naturally defines extra boundary operators. By considering diffeomorphisms these are related to components of the energy momentum tensor on the boundary. The implications of Weyl invariance in this framework are also derived.
1 Introduction

Besides the usual bulk critical phenomena in a statistical mechanical system there are theoretically interesting and experimentally measurable effects associated with the presence of a surface, or the consequences of finite size, which are inevitably present in any realisable physical system. There are possible phase transitions corresponding to surface ordering such that, at or close to a critical point where the correlation length is large and microscopic details are unimportant, there are additional critical exponents describing the behaviour of correlation functions of operators on or in the neighbourhood of the boundary surface. These may also be discussed within the framework of continuum scale invariant quantum field theories just as for conventional considerations of bulk critical phenomena [1]. Furthermore, as Cardy has shown [2], it is similarly natural to impose also the consequences of conformal invariance which is present at the critical point assuming the trace of the energy momentum tensor vanishes. This restricts the functional form of correlation functions in the neighbourhood of the boundary although the constraints are less than in the bulk since, assuming a plane boundary for $d$-dimensional Euclidean space with $d > 2$, the symmetry group which leaves the boundary invariant is reduced from the full conformal group $O(d + 1, 1)$ to $O(d, 1)$. Hence in general even two point functions of primary operators are not determined uniquely up to a constant factor but may contain an arbitrary function.

Nevertheless Cardy also subsequently obtained significant relations [3] between critical exponents and the coefficients of the universal terms in the Casimir energy for simple geometries. In part his arguments depended on the assumption that the functional dependence of the two point function for the energy momentum tensor was essentially unique for conformal theories in the neighbourhood of a boundary for general $d$. This derivation has recently been criticised [4] since the required assumption was incompatible with the results of explicit calculations to $O(\varepsilon)$ at the non Gaussian fixed point present in scalar $\phi^4$ theory when $d = 4 - \varepsilon$.

In this paper we re-analyse the implications of conformal invariance for two point functions involving the energy momentum tensor in the presence of a plane boundary. Although the conditions we derive are necessary rather than sufficient they allow for an arbitrary function to be present in the two point function, except when $d = 2$, and this freedom appears to be required to accommodate the results of calculations in specific conformal field theories for $d > 2$ (even for free scalar fields the two point function depends on the particular boundary conditions, compatible with conformal invariance, obeyed by the scalar field).

In the next section we therefore derive the consequences of conformal invariance for the two point function of the energy momentum tensor near a boundary by obtaining differential equations for the dependence on a variable $v$, formed from the spatial arguments $x, x'$ and which is an invariant under conformal transformations leaving the boundary invariant. For either $x$ or $x'$ on the boundary $v = 1$ whereas for both $x, x'$ at large distances from the boundary compared with their separation $v \to 0$. In this limit the two point function may be related to that for the bulk critical system neglecting any boundary
effects. For \( d = 2 \) the equations have a unique functional solution but not if \( d > 2 \). We also consider the two point function of the energy momentum tensor and a scalar field of arbitrary dimension which may be non zero in the presence of a boundary and for which there is a unique functional solution for the dependence on \( v \) for any \( d \). In section 3 we determine the functional dependence on \( v \) for free scalar fields, with either Dirichlet or Neumann boundary conditions which maintain conformal invariance, corresponding to the simple Gaussian fixed point in scalar field theories for any \( d \). We also calculate the corresponding expressions to \( O(\varepsilon) \) at the non trivial fixed point in the \( O(\varepsilon) \) theory which is found in the \( \varepsilon = 4 - d \) expansion around \( d = 4 \). Restricting to the two point function of \( T_{nn} \), where \( n \) denotes the normal component on the boundary, we are able to recover the results of Eisenriegler et al. [4].

In section 4 we further consider general results for the energy momentum tensor \( T_{\mu\nu} \) as defined by variations with respect to the metric \( g_{\mu\nu} \) on a curved manifold with a smooth boundary when variations of the induced metric, and also related geometric quantities on the boundary, are taken into account. The implications of diffeomorphism invariance, which now go beyond the usual conservation equation \( \nabla^\mu T_{\mu\nu} = 0 \), and Weyl invariance under local rescalings of the metric, which extend the traceless condition \( g^{\mu\nu}T_{\mu\nu} = 0 \), are derived. These conditions relate components of the energy momentum tensor on the boundary to new local boundary operators present in any field theory defined for arbitrary smooth boundaries. The results are verified for a general scalar field theory treated classically or to zeroth order in the loop expansion. For completeness a brief summary of the essential results used here arising from a covariant geometrical treatment for smoothly curved boundary surfaces is given in appendix A. Appendix B contains a discussion of the extension of the treatment in section 4 to fermion fields while appendix C contains some of the salient details of the \( O(\varepsilon) \) calculations necessary to obtain the results in section 3.

2 Conformal Invariance with a Plane Boundary

For a flat \( d \)-dimensional space with coordinates \( x_\mu = (x_1, x) \) we assume a plane boundary at \( x_1 = 0 \). It is then necessary to restrict the conformal group to the subgroup leaving \( x_1 = 0 \) invariant [2]. Besides \( d - 1 \) dimensional translations, \( O(d - 1) \) rotations and scale transformations

\[
    x_i \rightarrow x_i + a_i, \quad x_i \rightarrow R_{ij}x_j, \quad x_\mu \rightarrow \lambda x_\mu,
\]

where \( i = 2, 3, \ldots \), this restriction also allows special conformal transformations

\[
    x_\mu \rightarrow \frac{x_\mu + b_\mu x^2}{\Omega(x)}, \quad \Omega(x) = 1 + 2b \cdot x + b^2 x^2,
\]

so long as \( b_1 = 0 \). In this case it is easy to see that for two points \( x_\mu, x_\mu' \)

\[
    (x - x')^2 \rightarrow \frac{(x - x')^2}{\Omega(x)\Omega(x')}, \quad x_1 \rightarrow \frac{x_1}{\Omega(x)}, \quad x_1' \rightarrow \frac{x_1'}{\Omega(x')},
\]

so that it is straightforward to form an invariant under the restricted conformal group which we take as

\[
    \nu^2 = \frac{(x - x')^2}{(x - x')^2 + (x_1 + x_1')^2}.
\]
In consequence for a scalar operator $O(x)$ of dimension $\eta$ the connected two point function has the general form, as required by conformal invariance near a boundary,

$$\langle O(x)O(x') \rangle = \frac{1}{(x-x')^{2\eta}} F(v) ,$$

(2.5)
so that for $x_1 = 0$ or $x'_1 = 0$ the magnitude is given by $F(1)$ (if $O(x)$ obeys Dirichlet boundary conditions $F(1) = 0$) whereas the scale of the bulk amplitude neglecting surface effects is given by $F(0)$.

For the energy momentum tensor $T_{\mu\nu}$ under a general conformal transformation $\tilde{x} \rightarrow x$

$$T_{\mu\nu}(\tilde{x}) \rightarrow \Omega(\tilde{x})^d R_{\mu\alpha}(\tilde{x})R_{\nu\beta}(\tilde{x})T_{\alpha\beta}(\tilde{x}) \quad \text{and} \quad R_{\mu\alpha}(\tilde{x}) = \Omega(\tilde{x}) \frac{\partial x_{\mu}}{\partial \tilde{x}_{\alpha}} , \quad R_{\mu\alpha}R_{\nu\alpha} = \delta_{\mu\nu} .$$

(2.6)

It is convenient to write the two point function for the energy momentum tensor in the form (in this semi-infinite geometry $\langle T_{\mu\nu} \rangle = 0$)

$$\langle T_{\mu\nu}(x)T_{\sigma\rho}(x') \rangle = \frac{1}{(x-x')^{2d}} A_{\mu\nu\sigma\rho}(s, x_1, x'_1) , \quad s = x - x' .$$

(2.7)

**figure 1**

By a translation and an appropriate special conformal transformation of the form (2.2), with $b_1 = 0$, we may choose $x = x' = 0$ so that $x, x'$ lie on a perpendicular to the boundary as in fig. 1. In this case, by invariance under rotations and scale transformations as in (2.1) which preserve this perpendicular geometry, we may write

$$A_{1111} = \alpha , \quad A_{ij11} = \beta \delta_{ij} , \quad A_{11k1} = \beta' \delta_{k1} , \quad A_{i1k1} = \gamma \delta_{ik} , \quad A_{ijk\ell} = \delta \delta_{ij} \delta_{k\ell} + \epsilon(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) ,$$

(2.8)
with other components given by symmetry \((A_{\mu\nu\sigma\rho} = A_{\nu\mu\sigma\rho} = A_{\mu\nu\rho\sigma})\). The coefficients \(\alpha, \beta, \beta', \gamma, \delta, \epsilon\) are functions of

\[
v = \frac{y - y'}{y + y'}, \quad x_1 = y > x'_1 = y'.
\]

Assuming the form (2.8) it is trivial to obtain the conditions for tracelessness

\[
\alpha + (d - 1)\beta = 0, \quad \beta = \beta', \quad (2.10a)
\]

\[
\beta + (d - 1)\delta + 2\epsilon = 0.
\]

(2.10b)

In addition it is necessary to impose the conservation equation \(\partial_\mu \langle T_{\mu\nu} T_{\sigma\rho} \rangle = 0\). To derive these extra conditions* it is necessary to extend the form (2.8), valid for \(s = 0\), to \(s \neq 0\). For an infinitesimal transformation leaving \(x'_\mu = (y', 0)\) invariant

\[
\delta x_i = b_i x^2 - 2x_i b \cdot x - b_i y'^2, \quad \delta x_1 = -2x_1 b \cdot x,
\]

\[
\delta (y', 0) = 0, \quad \delta (y, 0) = (0, \xi), \quad \xi = b (y^2 - y'^2),
\]

then the \(O(d)\) transformation matrices as defined by (2.6) at \(x\) and \(x'\) respectively are

\[
R_{\mu\alpha} = \begin{pmatrix}
1 & -2b_j y \\
2b_i y & \delta_{ij}
\end{pmatrix}, \quad R'_{\mu\alpha} = \begin{pmatrix}
1 & -2b_j y' \\
2b_i y' & \delta_{ij}
\end{pmatrix}.
\]

(2.12)

Hence

\[
A_{i111}(\xi, y, y') = \frac{2}{y^2 - y'^2} \xi_i (y\alpha - y\beta - 2y'\gamma) + O(\xi^2)
\]

(2.13)

and the condition \(\partial_i \langle T_{i1} T_{11} \rangle + \partial_1 \langle T_{11} T_{11} \rangle = 0\) gives, using (2.10a),

\[
(y^2 - y'^2)\partial_y \beta = 2y'(d\beta - \gamma).
\]

(2.14)

Similarly

\[
A_{i1k\ell}(\xi, y, y') = \frac{2}{y^2 - y'^2} (\delta_{k\ell} \xi_i (y\beta - y\delta) - (\delta_{ik} \xi_\ell + \delta_{i\ell} \xi_k) (y\epsilon - y'\gamma)) + O(\xi^2),
\]

\[
A_{ijk1}(\xi, y, y') = \frac{2}{y^2 - y'^2} (\delta_{ij} \xi_k (y'\beta - y'\delta) + (\delta_{jk} \xi_i + \delta_{ik} \xi_j) (y\gamma - y'\epsilon)) + O(\xi^2),
\]

(2.15)

and the associated conservation equations, using (2.10a,b), lead to (2.14) again and also

\[
(y^2 - y'^2)\partial_y \gamma = 2y'(d\gamma - \beta + \delta + d\epsilon).
\]

(2.16)

* The following arguments are an adaptation of a similar discussion of conformal invariance requirements on three point functions for general \(d\) [5].
(2.14) and (2.16) exhaust the restrictions following from restricted conformal invariance and the conservation equation for \( T_{\mu\nu} \) in the perpendicular geometry assumed for (2.8). In terms of the variable \( v \) in (2.9) (2.14) and (2.16) simplify to

\[
v \frac{d}{dv} \beta = d \beta - 2 \gamma , \quad v \frac{d}{dv} \gamma = d \gamma - \beta + \delta + d \epsilon . \tag{2.17}
\]

Even after using (2.10a,b) to eliminate \( \alpha \) and, for instance, \( \epsilon \) there remain two coupled linear differential equations amongst three unknowns so in general an arbitrary function of \( v \) appears in the solution. This is exemplified by particular cases subsequently. At large distances from the boundary the solutions should tend smoothly to

\[
\alpha(0) = (d - 1)C , \quad \beta(0) = \delta(0) = -C , \quad -\gamma(0) = \epsilon(0) = \frac{1}{2} dC . \tag{2.18}
\]

For \( d = 2 \) \( C \) is proportional to the Virasoro central charge while for \( d = 4 \) \( C \) is related to the coefficient of the Weyl tensor squared in the trace of the energy momentum tensor on a curved space background [6].

However for \( d = 2 \) the index \( i \) is restricted to just \( i = 2 \) and in (2.8) \( A_{\mu\sigma\rho} \) depends only on the combination \( \delta + 2 \epsilon \). This is reflected in eqs. (2.10b) and (2.17) so that they can be solved to give

\[
\alpha = -\beta = \delta + 2 \epsilon = C(1 + v^4) , \quad \gamma = -C(1 - v^4) . \tag{2.19}
\]

The form of \( \gamma \) corresponds to \( T_{12} = 0 \) on the boundary \( x_1 = 0 \).

As a further application we may consider the correlation function of the energy momentum tensor with a scalar field \( \mathcal{O} \) of dimension \( \eta \). In this case as well as (2.6) for \( \tilde{x} \to x \)

\[
\mathcal{O}(\tilde{x}) \to \Omega(\tilde{x})^\eta \mathcal{O}(\tilde{x})
\]

and instead of (2.7)

\[
\langle T_{\mu\nu}(x)\mathcal{O}(x') \rangle = \frac{1}{(x-x')^{d+\eta}} S_{\mu\nu}(s, x_1, x'_1) . \tag{2.20}
\]

In the perpendicular geometry \( x_\mu = (y, 0) \), \( x'_\mu = (y', 0) \), \( y > y' \), then imposing tracelessness gives

\[
S_{11} = w(v) , \quad S_{ij} = -\frac{1}{d-1} w(v) \delta_{ij} . \tag{2.21}
\]

Following a similar discussion to before

\[
S_{i1}(\xi, y, y') = 2\xi_1 \frac{y}{y^2 - y'^2} \frac{d}{d-1} w(v) + O(\xi^2)
\]

and the conservation equation \( \partial_i T_{i1} + \partial_1 T_{11} = 0 \) leads to

\[
\frac{d}{dv} w(v) + \frac{d-\eta}{1-v} w - \frac{\eta}{v} w = 0 . \tag{2.22}
\]

\[
5
\]
This has the solution
\[ w(v) = Sv^n(1 - v)^{d-\eta}, \quad y > y'. \tag{2.23} \]
Conversely if \( y' > y \) then, with now \( v = (y' - y)/(y + y') \), the solution becomes
\[ w(v) = Sv^n(1 + v)^{d-\eta}. \tag{2.24} \]

The coefficient \( S \) appearing in (2.23,24) may be related to \( \langle O(x) \rangle \) by using the operator product expansion of \( T_{\mu\nu}(x) \) and \( O(x') \) for \( x \to x' \) which with our normalisations has the form
\[
T_{\mu\nu}(y,0)O(y',0) \sim \frac{1}{|y-y'|^d} A_{\mu\nu} O(y',0),
\]
\[
A_{11} = -A_{ii}, \quad A_{ij} = -\frac{\eta}{S_d} \delta_{ij}, \quad S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{1}{2}d)}.
\]  

Hence from (2.20,21) and (2.23) or (2.24) with \( v \to 0 \)
\[ \langle O(y,0) \rangle = -\frac{S_d}{\eta} \frac{S}{(2y)^\eta}. \tag{2.26} \]

Cardy [3] derived further relations by considering analogous short distance expansions in the neighbourhood of the boundary where \( O(x) \) is expanded in terms of low dimension boundary operators. Supposing that the expansion is restricted to the unit operator and the energy momentum operator \( T_{\mu\nu} \) itself on the boundary (where \( T_{i1} = 0 \)) he assumed
\[ O(y,0) \sim \langle O(y,0) \rangle (1 + b y^d T_{11}(0,0) + \ldots), \tag{2.27} \]
then from the solution (2.25) for \( y' \to 0 \) and also (2.7,8)
\[ \langle T_{11}(y,0)T_{11}(0,0) \rangle = \alpha(1) \frac{1}{y^{2d}}, \quad \alpha(1) = -2^d \eta \frac{1}{S_d} b. \tag{2.28} \]

This shows that the coefficient \( b \) does not depend on the particular operator \( O \). In two dimensions from the solutions (2.19) \( \alpha(1) = 2C \) and this leads to so called hyperscaling relations [3] for \( b \) which is then independent of particular boundary conditions and which appears in universal terms in the expression for the Casimir energy for parallel plate geometries. For \( d > 2 \) there appears to be no general relation between \( \alpha(0) \) and \( \alpha(1) \).

Alternatively if in (2.20) with the solution (2.24) we take \( y \to 0 \) then in this case the limit is non singular and from (2.26) we obtain
\[ \langle T_{11}(0,0)O(y',0) \rangle = -\frac{\eta}{S_d} \frac{2^d}{y^d} \langle O(y',0) \rangle. \tag{2.29} \]

More general configurations than the perpendicular geometry discussed above may be obtained by conformal transformation. If \( x'_\mu = (y',0) \) is fixed then we may take, generalising the infinitesimal transformation (2.11),
\[
(y,0) \to (x_1, x), \quad x_1 = \frac{1 + b^2 y^2}{1 + b^2 y^2} y, \quad x = \frac{y^2 - y'^2}{1 + b^2 y^2} b \tag{2.30}.
\]
\begin{equation}
\mathcal{R}_{\mu\alpha} = \left( \frac{1+b^2y^2}{1+b^2y^2} \right) - \frac{2by}{1+b^2y^2} \delta_{ij} - \frac{2b_1y}{1+b^2y^2} \delta_{ij},
\end{equation}

and correspondingly for \( \mathcal{R}'_{\mu\alpha} \) with \( y \to y' \). As the magnitude of \( b \) varies \( x_\mu \) moves on a semi-circle from \((y,0)\) to \((y'/y,0)\) for \( b^2 \to \infty \), as shown in fig. 1, and \( v \to -v \). It is easy to verify from (2.30) that \( (y-y')^2 \to (x-x')^2 \) and \( v^2 = (y-y')^2/(y+y')^2 \to v^2 \) as given by (2.4). Hence \( \langle T_{\mu\nu}(x)T_{\sigma\rho}(x') \rangle \) may be found for arbitrary \( x, x' \) in terms of \( \alpha, \beta, \gamma, \delta, \epsilon \). Assuming (2.6) and (2.7,8) with (2.30,31), taking \( x_1 = y' \), gives

\begin{equation}
\langle T_{\mu\nu}(x)T_{\sigma\rho}(x') \rangle = \frac{1}{s^{2d}} \left\{ \delta(v) \delta_{\mu\nu} \delta_{\sigma\rho} + \epsilon(v) (I_{\mu\sigma}(s)I_{\nu\rho}(s) + I_{\mu\rho}(s)I_{\nu\sigma}(s))
\right.
\end{equation}

\begin{equation}
\left. + (\beta(v) - \delta(v))(X_\mu X_\nu \delta_{\sigma\rho} + X'_\sigma X'_\rho \delta_{\mu\nu})
\right.
\end{equation}

\begin{equation}
\left. - (\gamma(v) + \epsilon(v))(X_\mu X'_\sigma I_{\nu\rho}(s) + X_\nu X'_\rho I_{\mu\sigma}(s))
\right.
\end{equation}

\begin{equation}
\left. + X_\mu X'_\rho I_{\nu\sigma}(s) + X_\nu X'_\sigma I_{\mu\rho}(s) \right\},
\end{equation}

where \( v \) is given by (2.4), \( s = x - x' \) and

\begin{equation}
I_{\mu\sigma}(s) = \delta_{\mu\sigma} - 2 \frac{s_\mu s_\sigma}{s^2},
\end{equation}

\begin{equation}
X_\mu = \mathcal{R}_{\mu\alpha} n_\alpha = N(x_1^2 - x_1'^2 - s^2, 2x_1 s),
\end{equation}

\begin{equation}
X'_\sigma = - \mathcal{R}'_{\sigma\alpha} n_\alpha = I_{\sigma\mu}(s)X_\mu = N(x_1'^2 - x_1^2 - s^2, -2x_1 s),
\end{equation}

\begin{equation}
N^{-2} = s^2(s^2 + 4x_1 x'_1) = (s^2)^2/v^2,
\end{equation}

for \( n_\alpha = (1,0) \) defining the normal to the boundary. It is easy to check that (2.32) is consistent with \( T_{\mu\nu} = 0 \), using (2.10a,b), since \( X, X' \) are unit vectors. At large distances from the boundary, when \( v \to 0 \), all terms containing \( X \) or \( X' \) vanish by virtue of (2.18) and the usual result [6] for no boundary is obtained.

As a special case we may obtain

\begin{equation}
\langle T_{11}(0,x)T_{11}(0,x') \rangle = \alpha(1) \frac{1}{(x-x')^{2d}},
\end{equation}

which may also be found more directly from (2.28). In a similar fashion from (2.29)

\begin{equation}
\langle T_{11}(0,0)\mathcal{O}(y,x) \rangle = - \frac{n}{S_d} \left( \frac{2y}{x^2 + y^2} \right)^d \langle \mathcal{O}(y,x) \rangle.
\end{equation}

3 Calculations in Specific Models

The simplest conformal field theory for arbitrary \( d \) is perhaps that corresponding to a free scalar field \( \phi \) for which the energy momentum tensor is

\begin{equation}
T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} \frac{1}{d-1} ((d-2)\partial_\mu \partial_\nu + \delta_{\mu\nu} \partial^2) \phi^2.
\end{equation}
This is conserved and traceless on the equations of motion \( \partial^2 \phi = 0 \). In order to maintain conformal invariance with a plane boundary at \( x_1 = 0 \) it is sufficient to impose Neumann \( \partial_1 \phi(0, 0) = 0 \) or Dirichlet \( \phi(0, 0) = 0 \) boundary conditions. In the perpendicular geometry described in the previous section the basic two point function is

\[
\langle \phi(y, 0) \phi(y', 0) \rangle = \frac{1}{S_d(d-2)} \left( \frac{1}{|y-y'|^{d-2} \pm \frac{1}{(y+y')^{d-2}}} \right) \tag{3.2}
\]

and we may easily find

\[
\langle \partial_t \phi(y, 0) \partial_j \phi(y', 0) \rangle = -\langle \partial_i \partial_j \phi(y, 0) \phi(y', 0) \rangle = \frac{1}{S_d} \delta_{ij} \left( \frac{1}{|y-y'|^d \pm \frac{1}{(y+y')^d}} \right),
\]

\[
\langle \partial_i \partial_j \phi(y, 0) \partial_k \partial_\ell \phi(y', 0) \rangle = \frac{d}{S_d d!} \left( \delta_{ij} \delta_{k\ell} + \delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk} \right) \left( \frac{1}{|y-y'|^{d+2} \pm \frac{1}{(y+y')^{d+2}}} \right). \tag{3.3}
\]

With these results it is a matter of straightforward calculation to find

\[
S_d^2 \alpha(v) = 1 + v^{2d} \pm \frac{1}{2} (d-2) d \frac{d+1}{d-1} \quad v^{d-2} (1-v^2)^2,
\]

\[
S_d^2 \gamma(v) = -\frac{d}{2(d-1)} \left( 1 - v^{2d} \pm \frac{1}{2} (d-2) d \frac{d+1}{d-1} \quad v^{d-2} (1-v^4) \right), \tag{3.4}
\]

\[
S_d^2 \epsilon(v) = \frac{d}{2(d-1)} (1 + v^{2d}) \pm \frac{1}{4} \frac{d}{(d-1)^2} \left( (d-2) (v^{d-2} + v^{d+2}) + 2d v^d \right),
\]

with \( \beta, \delta \) determined from (2.10a,b). It is easy to verify that these results satisfy the differential equations (2.17). For \( v \to 0 \) (3.4) gives \( C = (d-1)^{-1} S_d^{-2} \) in (2.18).

For the scalar operator \( \phi^2 \), of dimension \( d-2 \) in this free theory, then from (3.2) after subtraction of the usual short distance divergence

\[
\langle \phi^2(y, 0) \rangle = \mp \frac{1}{(d-2) S_d} \frac{1}{(2y)^{d-2}} \tag{3.5}
\]

and it is easy to verify that this is compatible with \( \langle T_{\mu\nu}(y, 0) \phi^2(y', 0) \rangle \) as determined from (2.20,21) and (2.23) or (2.24).

An equally simple conformal field theory is that provided by free fermion fields for which the energy momentum tensor is

\[
T_{\mu\nu} = \frac{1}{2} \bar{\psi} (\gamma_\mu \partial_\nu + \gamma_\nu \partial_\mu ) \psi.
\tag{3.6}
\]

The basic two point function is taken as

\[
\langle \psi(x) \bar{\psi}(x') \rangle = \frac{1}{S_d} \left( \gamma \cdot (x-x') \right) \frac{1}{(x-x')^d} + U \frac{\gamma \cdot (\bar{x} - x')}{(\bar{x} - x')^d}, \quad \bar{x}_\mu = (-x_1, \mathbf{x}) \tag{3.7}
\]

where \( U \) is a matrix satisfying

\[
U \gamma_1 = -\gamma_1 \bar{U}, \quad U \gamma_i = \gamma_i \bar{U}, \quad U^2 = \bar{U}^2 = 1. \tag{3.8}
\]
This corresponds to boundary conditions

\[(1 - U)\psi|_{\partial M} = 0, \quad \bar{\psi}(1 - U)|_{\partial M} = 0.\]  

\[(3.9)\]

It is straightforward to show that in this case

\[\epsilon(v) = -\frac{1}{2}d\delta(v), \quad \beta(v) = \delta(v) = -\frac{1}{2}\frac{2 + d}{S_d^2}(1 + v^2), \quad \gamma(v) = -\frac{1}{4}\frac{2 + d}{S_d^2}(1 - v^2),\]  

\[(3.10)\]

which is similar to the scalar field case after averaging over Neumann and Dirichlet boundary conditions.

For a less trivial calculation of the universal functions defining the two point function of the energy momentum tensor we consider the well known critical point in scalar field theories with a $\phi^4$ interaction where $g_* = O(\epsilon)$. At this non Gaussian fixed point the theory is conformally invariant and moreover critical exponents and other universal quantities may be calculated as an expansion in $\epsilon = 4 - d$ using standard perturbative loop expansions. Thus for $\phi$ an $n$-component field with an $O(n)$ invariant interaction

\[V(\phi) = \frac{1}{24g} (\phi^2)^2\]  

\[(3.11)\]

we consider the $O(g)$ contributions to $\epsilon$ and $\gamma$. These involve only off-diagonal components of $T_{\mu\nu}$ so that the expression (3.1) is still sufficient. The relevant Feynman diagrams are calculated in appendix C for either Neumann or Dirichlet boundary conditions. The leading terms in the $\epsilon$ expansion are given by (3.4) while the order $O(\epsilon)$ corrections from (C.7a,b) and (C.9a,b) are

\[\epsilon^{(1)}(v) = K\epsilon \left\{ \mp v^2(1 + 4v^2 + v^4)\ln \frac{(1 - v^2)^2}{v^2} \mp \frac{9}{2}v^2(1 + v^4) \mp 2v^4ight.\]

\[+ (3(1 - v^8) + 8v^2(1 - v^4))\ln \frac{(1 - v^2)^2}{v^2}
- 3(1 - v^8)\ln v^2 + \frac{4}{3}v^2(1 - v^4)\left(1 - \frac{2v^2}{(1 - v^2)^2} + \frac{6v^4}{(1 - v^2)^4}\right)\ln v^2
- 5v^2(1 + v^4) - \frac{8}{5}v^4 + \frac{12}{5}\frac{v^6}{(1 - v^2)^2}\right\},\]

\[\gamma^{(1)}(v) = K\epsilon \left\{ \pm 5v^2(1 - v^4)\left(\ln \frac{(1 - v^2)^2}{v^2} - \frac{1}{2}\right)
+ (3(1 - v^8) - v^2(1 - v^4))\ln \frac{(1 - v^2)^2}{v^2}
+ (3(1 + v^8) - v^2(1 + v^4))\ln v^2
+ \frac{2v^6}{(1 - v^2)^2}\ln v^2 + \frac{v^2(1 + v^6)}{1 - v^2}\right\},\]

\[(3.12a, b)\]
where

\[ K = \frac{1}{S_4^2} \frac{n}{9} \frac{n + 2}{n + 8} \cdot \quad (3.13) \]

It is straightforward to verify that these results satisfy (2.17), together with (2.10a,b), which in this context become

\[ (v \frac{d}{dv} - 4) \gamma^{(1)} = -\frac{4}{3} \beta^{(1)} + \frac{10}{3} \epsilon^{(1)} \], \quad \left( v \frac{d}{dv} - 4 \right) \beta^{(1)} = -2 \gamma^{(1)}. \quad (3.14) \]

From these equations

\[ \beta^{(1)}(v) = K \epsilon \left\{ \pm 5 v^2 (1 - v^2)^2 \ln \frac{(1 - v^2)^2}{v^2} \mp \frac{15}{2} v^2 (1 + v^4) \pm 10 v^4 \right. \]

\[ + \frac{1}{2} (1 - v^2)^2 (3 + 4 v^2 + 3 v^4) \ln \frac{(1 - v^2)^2}{v^2} \]

\[ + \frac{3}{2} (1 - v^8) \ln v^2 - \frac{v^2 (1 + v^6)}{1 - v^2} \ln v^2 \]

\[ - 2 v^2 (1 + v^4) + 2 v^4 \left\} \right. \cdot \quad (3.15) \]

All quantities \( \epsilon^{(1)}, \gamma^{(1)}, \beta^{(1)} \) vanish as \( v \to 0 \) since there is no change in \( C \), given by (2.18), at this order. As required by the boundary condition \( T_{11} = 0 \) for \( x_1 = 0 \) \( \gamma^{(1)}(1) = 0 \). However \( \epsilon^{(1)}(v) \) is singular as \( v \to 1 \) although \( \beta^{(1)}(v) \) is well behaved in this limit. In the next section it is shown that \( T_{11} = -T_{ii} \) is a well defined operator on the boundary \( x_1 = 0 \) but this does not restrict the traceless part of \( T_{ij} \) to be finite in general on the boundary. Of course for \( d = 2 \) all components of \( T_{\mu \nu} \) are non singular near the boundary. From the above expressions for \( \beta \) for \( v = 1 \) it is easy to see that the leading terms in the \( \epsilon \) expansion give

\[ \alpha(1) = \frac{n}{S_4^d} \left( \frac{2}{3} \frac{n + 2}{n + 8} \epsilon \right), \quad (3.16) \]

which coincides exactly with the results of Eisenriegler et al. [4] who calculated the two point function of \( T_{11} \) at \( x_1 = 0 \) as in (2.34).

**4 Energy Momentum Tensor on Curved Manifolds with Boundary**

In a quantum field theory defined on a manifold \( \mathcal{M} \) with an arbitrary smooth metric \( g_{\mu \nu} \) the energy momentum tensor may be defined as a finite local composite operator by considering variations with respect to \( g_{\mu \nu} \). For \( x^\mu \in \mathcal{M} \) then if \( \mathcal{M} \), of dimension \( d \), has a boundary \( \partial \mathcal{M} \), of dimension \( d - 1 \), this may be determined by \( x^b_\mu(\hat{x}) \) for \( \hat{x} \) arbitrary coordinates on \( \partial \mathcal{M} \). On \( \partial \mathcal{M} \) the induced metric is defined by

\[ \hat{g}_{ij}(\hat{x}) = g_{\mu \nu}(x_b) e^\mu_i(\hat{x}) e^\nu_j(\hat{x}), \quad e^\mu_i(\hat{x}) = \frac{\partial x^\mu_b}{\partial x^i} \cdot \quad (4.1) \]
and we may also define related geometric quantities such as the unit inward normal \( n^\mu(\hat{x}) \), satisfying \( n_\mu e^\mu_1 = 0 \), \( n_\mu n^\mu = 1 \), and the extrinsic curvature \( K_{ij}(\hat{x}) \) given by (A.1) which has dimension 1.

For any quantum field theory the vacuum energy functional \( W(g, x_b) \), which here depends on the metric \( g_{\mu\nu} \) and the embedding of \( \partial M \) in \( M \) given by \( x_b \), may be defined by a functional integral of the generic form

\[
e^W = \int d[\phi] e^{-S_0(\phi)}
\]

for fields \( \phi \) on \( M \) satisfying suitable boundary conditions on \( \partial M \) and \( S_0 \) the bare action including all necessary counterterms in an appropriate regularisation scheme so that \( W \) is a finite functional for arbitrary smooth metrics \( g_{\mu\nu} \) and boundaries \( x^\mu_b \). Recently [7] we have shown how this may be achieved to two loops using dimensional regularisation in four dimensions for renormalisable scalar field theories with quantum fields obeying either Dirichlet or generalised Neumann boundary conditions.

In such a framework we may write under variations in the metric

\[
-\delta_g W = \int_M dv \frac{1}{2} \delta g^{\mu\nu} \langle T_{\mu\nu} \rangle + \int_{\partial M} dS \left( \frac{1}{2} \delta \hat{\gamma}_{ij} \langle B_{ij} \rangle + \delta n^\mu \langle \lambda_\mu \rangle + \frac{1}{2} \delta K_{ij} \langle C^{ij} \rangle + \ldots \right),
\]

where \( dv = d^d x \sqrt{g} \), \( dS = d^{d-1} \hat{x} \sqrt{\gamma} \). This relation defines insertions of local operators \( B_{ij}(\hat{x}) \), \( \lambda_n(\hat{x}) \), \( \lambda_i(\hat{x}) \) and \( C^{ij}(\hat{x}) \) on \( \partial M \). The neglected surface terms in (4.3) involve variations of higher dimension geometric tensors on \( \partial M \), such as components of the curvature tensor like \( R_{nijn} = n^\mu e^i_\mu n^\sigma e^j_\sigma R_{\mu\nu\sigma\rho} \), but these are absent in simple theories.

To derive conservation equations we assume invariance under diffeomorphisms, or reparameterisations of the coordinates, as given by \( \delta_v x^\mu = -v^\mu(x) \). On the boundary \( \partial M \) we also require \( \delta_v \hat{x}^i = -v^i(\hat{x}) \) where \( v^\mu(x_b) = e^\mu_1(\hat{x}) v^1(\hat{x}) + n^\mu(\hat{x}) v_n(\hat{x}) \). On \( M \) the external metric is required to transform as

\[
\delta_v g_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu.
\]

The corresponding variations on \( \partial M \) in \( \hat{\gamma}_{ij} \), \( n^\mu \) and \( K_{ij} \) induced by \( \delta_v g_{\mu\nu} \) in (4.4) are obtained in the appendix to be

\[
\delta_v \hat{\gamma}_{ij} = \hat{\nabla}_i v_j + \hat{\nabla}_j v_i - 2K_{ij} v_n,
\]
\[
\delta_v n^\mu = - \nabla_n v^\mu - e^{\mu i}(\delta_i v_n + K_{ij} v^j),
\]
\[
\delta_v K_{ij} = L_v K_{ij} + (R_{nijn} - K_{ik} K^k_j) v_n + \hat{\nabla}_i \partial_j v_n,
\]
\[
L_v K_{ij} = v^k \hat{\nabla}_k K_{ij} + \hat{\nabla}_i v^k K_{kj} + \hat{\nabla}_j v^k K_{ik},
\]

for \( \hat{\nabla}_i \) the covariant derivative acting on tensor fields over \( \partial M \) with the Christoffel connection prescribed by the metric \( \hat{\gamma}_{ij} \). For a general diffeomorphism it is necessary to assume also a shift in the boundary surface represented by \( \delta_v x^\mu_b(\hat{x}) = -n^\mu(\hat{x}) v_n(\hat{x}) \).
By integration by parts invariance under diffeomorphisms on $\mathcal{M}$ gives the usual conservation equation

$$\nabla^{\mu} \langle T_{\mu\nu} \rangle = 0 \, ,$$

(4.6)

(for simplicity we neglect contributions from any sources for other local operators on which $W$ may also depend). On the boundary invariance gives from the results (4.5)

$$\langle \lambda_{\mu} \rangle = 0 \, ,$$

$$\langle T_{ni} \rangle = - \hat{\nabla}^{j} \langle B_{ij} \rangle - \frac{1}{2} \hat{\nabla}_{i} K_{jk} \langle C^{jk} \rangle + \hat{\nabla}_{j} \langle K_{ik} \langle C^{jk} \rangle \rangle \, .$$

(4.7)

In addition, after setting $\langle \lambda_{\mu} \rangle = 0$, we may also obtain

$$- n^{\mu} \frac{\delta}{\delta x^{\mu}} W = \langle T_{nn} \rangle + K^{ij} \langle B_{ij} \rangle + \frac{1}{2} \left( R_{ninj} - K_{ik} K_{j}^{k} \right) \langle C^{ij} \rangle + \frac{1}{2} \hat{\nabla}_{i} \hat{\nabla}_{j} \langle C^{ij} \rangle \, .$$

(4.8)

(4.7) and (4.8) ensure that $T_{nn} = n^{\mu} n^{\nu} T_{\mu\nu}$ and $T_{ni} = n^{\mu} e^{\nu}_{i} T_{\mu\nu}$ are well defined local operators on $\partial \mathcal{M}$.

If we also assume Weyl invariance under local rescalings of the metric

$$\delta_{\sigma} g^{\mu\nu} = 2 \sigma g^{\mu\nu}$$

(4.9)

then on $\mathcal{M}$ this requires as usual

$$g^{\mu\nu} \langle T_{\mu\nu} \rangle = 0 \, .$$

(4.10)

From (4.9) the induced variations on the boundary are

$$\delta_{\sigma} \hat{\gamma}^{ij} = 2 \sigma \hat{\gamma}^{ij} \, , \quad \delta_{\sigma} n^{\mu} = \sigma n^{\mu} \, , \quad \delta_{\sigma} K_{ij} = \sigma K_{ij} + \hat{\gamma}_{ij} \partial_{n} \sigma \, .$$

(4.11)

Weyl invariance then also requires

$$\hat{\gamma}^{ij} \langle B_{ij} \rangle + \frac{1}{2} K_{ij} \langle C^{ij} \rangle + \langle \lambda_{n} \rangle = 0 \, ,$$

$$\hat{\gamma}_{ij} \langle C^{ij} \rangle = 0 \, .$$

(4.12)

For $d = 2$ and a plane boundary $x_{1} = 0$ (4.7) is equivalent to the operator relation on the boundary $T_{1x} = - \partial_{x} B$, for $x_{2} = x$, while from (4.12) for conformal invariance $B = 0$.

As an illustration of these results we a consider a simple scalar field theory specified by bulk and surface contributions to the action of the form

$$S(\phi) = \int_{\mathcal{M}} dv \mathcal{L} \, , \quad \mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{2} R \phi^{2} + V(\phi) \, ,$$

$$\hat{S}(\phi) = \int_{\partial \mathcal{M}} dS \left( Q(\phi) + \frac{1}{2} \rho K \phi^{2} \right) \, .$$

(4.13)

* Similar equations have been obtained by Cardy [8].
where $R$ is the scalar curvature on $\mathcal{M}$ and $K = \hat{\gamma}^{ij} K_{ij}$. In the corresponding quantum field theory to lowest order in the loop expansion we may write
\begin{equation}
W(g, x_b)^{(0)} = -S(\phi) - \hat{S}(\phi) , 
\end{equation}
where $\phi$ is a solution of the classical field equations and boundary conditions such that $S + \hat{S}$ is stationary,
\begin{equation}
\nabla^2 \phi - \tau R \phi - V'(\phi) = 0 , \quad (\partial_n \phi - \rho K \phi - Q'(\phi))\big|_{\partial \mathcal{M}} = 0 .
\end{equation}

Under a variation in the metric, since $\delta_g R = \delta g^{\mu \nu} R_{\mu \nu} - (\nabla_{\mu} \nabla_{\nu} - g_{\mu \nu} \nabla^2) \delta g^{\mu \nu}$,
\begin{align}
\delta_g S(\phi) &= \int_\mathcal{M} dv \frac{1}{2} \delta g^{\mu \nu} T_{\mu \nu} \\
&\quad + \int_{\partial \mathcal{M}} dS \frac{1}{2} \tau \left( (-h^{\mu \nu} \nabla_n \delta g^{\mu \nu} + n_{\mu} e_{\nu} \nabla_i \delta g^{\mu \nu}) \phi^2 + h_{\mu \nu} \delta g^{\mu \nu} \partial_n \phi^2 - n_{\mu} e_{\nu} \delta g^{\mu \nu} \partial_i \phi^2 \right) ,
\end{align}
for $h_{\mu \nu} = g_{\mu \nu} - n_{\mu} n_{\nu}$ and
\begin{equation}
T_{\mu \nu} = \partial_{\mu} \phi \partial_{\nu} \phi + \tau R_{\mu \nu} \phi^2 - \tau (\nabla_{\mu} \nabla_{\nu} - g_{\mu \nu} \nabla^2) \phi^2 - g_{\mu \nu} \mathcal{L} .
\end{equation}

It is easy to check that $\nabla^\mu T_{\mu \nu} = 0$ using the equation of motion (4.15). For $V = 0$ and $\tau = 1/(d - 2)/(d - 1)$ it is also straightforward to verify that $g^{\mu \nu} T_{\mu \nu} = 0$ on flat space and that $T_{\mu \nu}$ is then equivalent to that assumed in (3.1) for the free scalar conformal field theory. Using (4.16) and the results in the appendix it is possible to write
\begin{equation}
\delta_g (S + \hat{S}) = \int_\mathcal{M} dv \frac{1}{2} \delta g^{\mu \nu} T_{\mu \nu} + \int_{\partial \mathcal{M}} dS \left( \frac{1}{2} \delta \hat{\gamma}^{ij} B_{ij} + \delta n^\mu \lambda_\mu + \frac{1}{2} \delta K_{ij} C^{ij} \right) ,
\end{equation}
where, using the boundary conditions on $\phi$ in (4.15),
\begin{align}
B_{ij} &= \hat{\gamma}^{ij} (-Q(\phi) + 2 \tau Q'(\phi) + 2 \tau \rho K \phi^2 - \frac{1}{2} \rho K \phi^2) + (\rho - \tau) K_{ij} \phi^2 , \\
C^{ij} &= (\rho - 2 \tau) \hat{\gamma}^{ij} \phi^2 , \quad \lambda_\mu = 0 .
\end{align}

The general results derived above may now be verified directly. On the boundary from (4.17)
\begin{equation}
T_{ni} = \partial_n \phi \partial_i \phi + \tau R_{ni} \phi^2 - \tau \partial_i \partial_n \phi^2 - \tau K_{ij} \partial_j \phi^2 ,
\end{equation}
where $\partial_n \phi$ may be eliminated from (4.15). Using the Gauss-Codazzi equation $R_{ni} = \partial_i K - \nabla_j K_{ij}$ it is then easy to show that
\begin{equation}
T_{ni} = -\hat{\nabla}_j B_{ij} - \frac{1}{2} \hat{\nabla}_i K_{jk} C^{jk} + \hat{\nabla}_j (K_{ik} C^{jk}) ,
\end{equation}
which is in accord with (4.7) as expected. On the boundary also
\begin{equation}
T_{nn} = \partial_n \phi \partial_n \phi - \frac{1}{2} \hat{\nabla}_i \phi \partial_i \phi - \frac{1}{2} \rho R \phi^2 - V(\phi) + \tau R_{nn} \phi^2 + \tau \hat{\nabla}^2 \phi^2 - \tau K \partial_n \phi^2 ,
\end{equation}
and using (A.7,8,9) we may verify (4.8) in this case as well

\[ n^\mu \frac{\delta}{\delta x^b} (S + \hat{S}) = T_{nn} + K^{ij} B_{ij} + \frac{1}{2} (R_{ninj} - K_{ik} K^k_j) C^{ij} + \frac{1}{2} \hat{\nabla}_i \hat{\nabla}_j C^{ij}. \]  

(4.23)

For Weyl invariance from the second of the conditions in (4.12) \(\hat{\gamma}_{ij} C^{ij} = 0\) we obtain \(\rho = 2\tau\) and then

\[ \hat{\gamma}^{ij} B_{ij} = (d - 1) (-Q(\phi) + 2\tau Q'(\phi)\phi) + \tau (4(d - 1)\tau - (d - 2)) K\phi^2. \]  

(4.24)

Hence \(\hat{\gamma}^{ij} B_{ij} = 0\) gives the same result for \(\tau\) as \(g^{\mu\nu} T_{\mu\nu} = 0\). If \(Q(\phi) = \frac{1}{2} c\phi^2\) the remaining condition is satisfied if either \(c = 0\), and \(\partial_n \phi|_{\partial\mathcal{M}} = 0\), or if \(c \to \infty\), \(c\phi = O(1)\), so that \(\phi|_{\partial\mathcal{M}} = 0\), showing how both Neumann and Dirichlet boundary conditions on scalar fields are separately compatible with conformal invariance.

5 Conclusion

It is clear from the results of this paper that two-point correlation functions of the energy momentum tensor in conformal field theories with appropriate boundary conditions preserving conformal invariance may have a dependence on the conformal invariant variable \(v\), defined here by (2.4), which depends on the particular conformally invariant theory and its boundary conditions whenever \(d > 2\). An interesting question, beyond the scope of our considerations here, is whether such functions are measurable in realistic statistical physics models at a critical point when we may take \(d = 3\). For models in the same universality class as the Ising model then our \(\varepsilon\) expansion results would perhaps be relevant setting \(\varepsilon = 1\). For such applications it would be desirable to find functions which agreed with our results to \(O(\varepsilon)\) but remained solutions of the equations (2.10a,b) and (2.17) for general \(d\) since they would then extrapolate to the unique functional form for \(d = 2\) given by (2.19).

We are grateful to John Cardy for stimulating conversations and sending us a copy of ref. 4.
Appendix A

For application in section 4 we here summarise the essential results in a geometrical treatment of a boundary $\partial M$, parameterised by coordinates $\tilde{x}^i$, where $\partial M$ is specified in terms of the coordinates for $M$ by $x^i_b(\tilde{x})$. A natural tangent frame basis on $\partial M$ is given by $e^\mu_i(\tilde{x})$, $n^\mu(\tilde{x})$, with $n^\mu$ the unit inward normal, and the extrinsic curvature on $\partial M$ is defined as in (4.1). The symmetric tensor $K_{ij}$ forming the extrinsic curvature and a connection $\hat{\Gamma}^k_{ij}$ are defined on $\partial M$ by

$$\nabla_i e^\mu_j = \partial_i e^\mu_j + \Gamma^\mu_{\sigma\rho} e^\sigma_i e^\rho_j - e^\mu_k \hat{\Gamma}^k_{ij} = n^\mu K_{ij},$$
$$\nabla_i n^\mu = \partial_i n^\mu + \Gamma^\mu_{\sigma\rho} e^\sigma_i n^\rho = -K_{ij} e^\mu_j, \quad \partial_i = e^\mu_i \partial_\mu,$$  \hfill (A.1)

where $\Gamma^\mu_{\sigma\rho}$ the usual Christoffel connection formed from $g_{\mu\nu}$. Acting on tensors on $\partial M$ a covariant derivative $\hat{\nabla}_i$ may be defined by the connection $\hat{\Gamma}^k_{ij} = \hat{\Gamma}^k_{ji}$ and from (A.1) and (4.1) $\hat{\nabla}_i \hat{\gamma}^k_{jk} = 0$ so that $\hat{\Gamma}^k_{ij}$ is just the Christoffel connection formed from $\hat{\gamma}^k_{ij}$. From (A.1) it is straightforward to derive the Gauss-Codazzi equations relating the Riemann curvature tensor $R_{\mu\nu\sigma\rho}$ for $x \in \partial M$, with zero and one component along the normal $n^\mu$, to the intrinsic Riemann curvature $\hat{R}_{ijk\ell}$ of $\partial M$ associated with the covariant derivative $\hat{\nabla}_i$ and also the extrinsic curvature $K_{ij}$. To derive the implications for $\hat{\gamma}^k_{ij}$, $n^\mu$ and $K_{ij}$ of the variation of the basic metric (4.5) induced by a diffeomorphism we first note that from (A.1) $e^\mu_i e^\nu_j \nabla_\mu v_\nu = \hat{\nabla}_i v_j - K_{ij} v_n$ which leads directly from the definition (4.1) to $\delta_v \hat{\gamma}^k_{ij}$ in (4.6). The variation of the normal vector $n^\mu$ in (4.6) may be determined from the equations $\delta_v n^\mu e^\mu_i = 0$ and $\delta_v n^\mu n^\mu = -\frac{1}{2} \delta_v g_{\mu\nu} n^\mu n^\nu = n^\mu n^\nu \nabla_\mu v_\nu$. For the variation in the extrinsic curvature $K_{ij} = n_\mu \nabla_i e^\mu_j$ from (A.1) in order to obtain the result in (4.6) we use

$$\delta_v K_{ij} = n^\mu n^\nu \nabla_\mu v_\nu K_{ij} + n_\mu \delta_v \Gamma^\mu_{\sigma\rho} e^\sigma_i e^\rho_j ,$$
$$\delta_v \Gamma^\mu_{\sigma\rho} = g^{\mu\nu} \nabla_{(\rho} v_{\sigma)} - v^\lambda R_{\lambda(\rho|\sigma)}^{\mu} , \quad R_{k(ij)} = \hat{\nabla}_k K_{ij} = \hat{\nabla}_i \hat{\nabla}_j v_n - K_{ik} K_{jk} v_n + \hat{\nabla}_i v^k K_{jk} + \hat{\nabla}_j v^k K_{ik} + \hat{\nabla}_i (K_{jk} v^k) .$$

To express the variations in the metric and its derivatives on $\partial M$ in terms of variations in $\hat{\gamma}^k_{ij}$, $n^\mu$ and $K_{ij}$, as required for the simplification of (4.16), we use

$$\delta g^{\mu\nu} = \delta n^\mu n^\nu + n^\mu \delta n^\nu + e^\mu_i e^\nu_j \delta \hat{\gamma}^{ij} .$$  \hfill (A.2)

It is easy to see that, for $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$,

$$h_{\mu\nu} \delta g^{\mu\nu} = \hat{\gamma}^{ij} \delta \hat{\gamma}^{ij} , \quad n_\mu e^\nu_i \delta g^{\mu\nu} = e^\nu_i \delta n^\nu ,$$  \hfill (A.3)

and also, using (A.1),

$$n_\mu e^\nu_i \nabla_i \delta g^{\mu\nu} = \hat{\nabla}_i (e^\nu_i \delta n^\nu) - 2 K n_\mu \delta n^\mu + K_{ij} \delta \hat{\gamma}^{ij} .$$  \hfill (A.4)
To determine a corresponding expression for \( h_{\mu\nu} \nabla_n \delta g^{\mu\nu} \) we first use \( K_{ij} = -e^\mu_i e^\mu_j \nabla_\mu \delta g^{\mu\nu} \) to find

\[
\delta K_{ij} = -e^\mu_i e^\mu_j \nabla_\mu \delta n^\mu - e^\mu_i (\hat{\nabla}_j \delta n^\mu + K_{ij} n^\mu \delta n^\mu - \hat{n}_k (i K_j) \delta \hat{\gamma}^{jk}) + \frac{1}{2} e^\mu_i e^\nu_j \nabla_n \delta g^{\mu\nu} .
\]

From this it is straightforward to obtain

\[
h_{\mu\nu} \nabla_n \delta g^{\mu\nu} = 2\delta K - 2K n^\mu \delta n^\mu + 2 \hat{\nabla}_i (e^\mu_i \delta n^\mu) .
\] (A.5)

Finally we consider the variations induced by a shift in the boundary surface along a normal as given by \( \delta t x^\mu_b = -n^\mu \delta t \). We let

\[
\delta_t e^\mu_i = \partial_t \delta_t x^\mu_b = -n^\mu \delta t \Gamma^\mu_{\sigma\nu} e^\nu_i = \delta t K_{ij} e^\mu_j - n^\mu \partial_t \delta t ,
\]

\[
\delta_t n^\mu = \partial_t n^\mu - n^\nu \Gamma^\nu_{\sigma\mu} n^\sigma \delta t = e^\mu_i \partial_i \delta t .
\]

(A.6)

With the induced metric given by (4.1) we have

\[
\delta_t \hat{\gamma}_{ij} = 2K_{ij} \delta t ,
\]

(A.7)

and using \( K_{ij} = n^\mu \nabla_\mu e^\nu_j \) from (A.1)

\[
\delta_t K_{ij} = e^\nu_k \partial_t \delta t \nabla_i e^\nu_j - n^\mu [\nabla_n, \nabla_i] e^\mu_j + n^\mu \nabla_i \delta_t e^\mu_j
\]

\[
= - \hat{\nabla}_i \partial_t \delta t - (R_{n^j n^i} - K_{jk} K^k_i) \delta t .
\]

(A.8)

For integrals over a local scalar function \( f \) on \( \mathcal{M} \) or, restricted to the boundary, on \( \partial \mathcal{M} \)

\[
\delta_t \int_{\mathcal{M}} dv f = \int_{\partial \mathcal{M}} dS \delta t f|_{\partial \mathcal{M}} ,
\]

\[
\delta_t \int_{\partial \mathcal{M}} dS f|_{\partial \mathcal{M}} = \int_{\partial \mathcal{M}} dS (\delta t (-\partial_n f + K f) + \delta_t f)|_{\partial \mathcal{M}} .
\] (A.9)

**Appendix B**

The discussion in section 4 of the energy momentum tensor and related boundary operators, as defined through variations of the metric for a curved space background, seemingly requires modification if fermion fields are present. As is well known [9] it is then necessary for a consistent treatment to introduce a tangent frame basis \( V^\mu_a \) such that \( g^{\mu\nu} = V^\mu_a V^\nu_a \) and a corresponding connection \( \omega_{\mu ab} = -\omega_{\mu ba} \) defined by \( \nabla_\mu V^\nu_a = \partial_\mu V^\nu_a + \Gamma^\nu_{\mu\sigma} V^\sigma_a + \omega_{\mu ab} V^\nu_b = 0 \). The conventional Dirac action is

\[
S_D = \int_{\mathcal{M}} dv \bar{\psi} (\gamma^\mu \hat{\nabla}_\mu + M) \psi , \quad \gamma^\mu = V^\mu_a \gamma_a , \quad \nabla_\mu = \partial_\mu + \frac{1}{2} \omega_{\mu ab} \sigma_{ab} ,
\]

(B.1)
for $\gamma_a$ the usual Dirac matrices $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$, $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$. On the classical field equations, $(\gamma^\mu \nabla_\mu + M)\psi = 0$, $\tilde{\psi}(\gamma^\mu \tilde{\nabla}_\mu - M) = 0$, then $S^D = 0$.

When considering variations in $V^\mu_a$, which gives a variation in the metric $g^{\mu\nu}$, it is also necessary to allow for contributions proportional to

$$\ell^{\mu\nu} = \frac{1}{2}(\delta V^\mu_a V^\nu_a - \delta V^\nu_a V^\mu_a) .$$

By straightforward calculation, using $\delta \omega_{ab} = V_\nu[a \nabla_\mu \delta V^\nu_b] - V^\sigma[a V^\rho_b] \nabla_\sigma \delta g_{\sigma\mu}$, assuming $\psi, \tilde{\psi}$ satisfy the equations of motion but without any specific boundary conditions then

$$\delta V S^D = \int_M dv \left( \frac{1}{2} \delta g^{\mu\nu} T^{D\mu\nu}_\rho + \frac{1}{2} \ell^{\mu\nu} \tilde{\psi}[\sigma_{\mu\nu}, M]\psi \right) - \int_{\partial M} dS \frac{1}{2} \ell^{ij} \tilde{\psi} \gamma_n \sigma_{ij} \psi ,$$

$$T^{D\mu\nu}_\rho = \tilde{\psi} \gamma(\mu \tilde{\nabla}_\nu) \psi .$$

As usual [9], assuming invariance under local rotations which in this context requires $[\sigma_{ab}, M] = 0$, terms proportional to $\ell^{\mu\nu}$ in $\delta V S^D$ on $M$ are absent. Clearly from (B.3), for the simple Dirac action $S^D$, there are no contributions under a variation of the metric proportional to $\delta \gamma^{ij}, \ldots$ as in (4.3) or (4.18).

Under variations in $\psi, \tilde{\psi}$, subject to the field equations on $M$,

$$\delta \psi S^D = \int_{\partial M} dS \left( \delta \tilde{\psi} \gamma_n \psi - \tilde{\psi} \gamma_n \delta \psi \right) .$$

Conventionally [10] linear boundary conditions on $\tilde{\psi}', \psi$ are chosen, as in (3.9), so that

$$\tilde{\psi}' \gamma_n \psi |_{\partial M} = 0 ,$$

and hence $i \gamma^\mu \nabla_\mu$ is a symmetric operator and in (B.4) we also have $\delta \psi S^D = 0$.

The basic equations derived in section 4 from invariance under diffeomorphisms giving expressions for $T_{ni}, T_{nn}$ on $\partial M$ in terms of boundary operators now require extension to take account of terms proportional to $\ell^{\mu\nu}$ as well as terms involving $\delta \gamma^{ij}, \ldots$ from variations of the metric. Corresponding to a diffeomorphism, leading to (4.4), we require

$$\delta_v V^\mu_a = v^\sigma \partial_\sigma V^\mu_a - \partial_\sigma v^\mu_a V^\sigma_a = -\nabla_\sigma v^\mu_a V^\sigma_a - v^\sigma \omega_{\sigma ab} V^\mu_b ,$$

$$\delta_v \psi = \tilde{\psi}' \gamma_n \psi = v^\sigma \nabla_\sigma \psi - \frac{1}{2} v^\sigma \omega_{\sigma ab} \sigma_{ab} \psi ,$$

and hence from the definition (B.2)

$$\ell^{\mu\nu} = \frac{1}{2} (\nabla^\mu v^\nu - \nabla^\nu v^\mu) + v^\sigma \omega_{\sigma ab} V^\mu_a V^\nu_b .$$

The essential identity of section 4 has the form in this case

$$\delta_v S^D = - \int_{\partial M} dS \left( v^n T^{Dn}_{nn} + v^i T^{Di}_{ni} \right)$$

$$+ \frac{1}{2} \int_{\partial M} dS \left( v^n \omega_{nab} \tilde{\psi} \frac{1}{2} \{ \gamma_n, \sigma_{ab} \} \psi + \tilde{\psi} v^i \tilde{\psi} \gamma_n \sigma_{ij} \psi \right)$$

$$+ \frac{1}{2} \int_{\partial M} dS \left( \delta_v \tilde{\psi} \gamma_n \psi - \tilde{\psi} \gamma_n \delta_v \psi \right) .$$
This may be verified using \( \delta_v \bar{\psi}, \delta_v \psi \) from (B.6) and from (B.3)

\[
T_{nn}^D = \frac{1}{2} \bar{\psi} \gamma_n \nabla_n \psi - \frac{1}{2} \bar{\psi} \gamma_n \nabla_n \psi , \\
T_{ni}^D = \frac{1}{2} \bar{\psi} \gamma_n \nabla_i \psi - \frac{1}{2} \bar{\psi} \nabla_i \gamma_n \psi + \frac{1}{2} \nabla^j (\bar{\psi} \gamma_n \sigma_{ij} \psi) .
\]  

(B.9)

Since \( S^D = 0 \) there are no terms involving \( n^\mu \delta S^D / \delta x_\mu \) as in (4.23). To derive (B.9) it is useful to note that we may write the tangential components of the spinor covariant derivatives on \( \partial M \) in the form

\[
\nabla_i = \tilde{\nabla}_i + \frac{1}{2} K_{ij} \gamma^j , \quad \tilde{\nabla}_i = \partial_i + \tilde{\omega}_i , \\
\tilde{\nabla}_i \gamma_n = \partial_i \gamma_n + [\tilde{\omega}_i, \gamma_n] = 0 , \quad \tilde{\nabla}_i \gamma_j = 0 .
\]

(B.10)

If the boundary conditions on \( \bar{\psi}, \psi \) imply also \( \bar{\psi} \gamma_n \sigma_{ij} \psi = 0 \), besides (B.5), then the additional surface terms in (B.3), (B.8) are zero and \( T_{nn}^D = T_{ni}^D = 0 \) on \( \partial M \) and the discussion in section 4 does not require any modification.

**Appendix C**

Here we outline the essential steps in the calculation of the two point function of the energy momentum tensor at the critical point to \( O(\varepsilon) \). We calculate only those contributions to \( \epsilon \) and \( \gamma \), as defined by (2.8), which involve off diagonal elements of \( T_{\mu\nu} \) since these are not affected by interaction terms in this theory. By virtue of (2.10a,b) and (2.17) knowledge of \( \epsilon \) and \( \gamma \) is clearly sufficient to determine all other pieces of the the two point function for \( T_{\mu\nu} \) as well as providing a convenient consistency check.

The two basic graphs whose contributions need to be calculated to \( O(g) \) are shown in fig. 2a,b. Corresponding to fig. 2a it is sufficient to find the one loop corrections to
\( \langle \phi(x)\phi(x') \rangle \). It is easy to see that

\[
\langle \phi(y, 0)\phi(y', 0) \rangle^{(1)} = -\frac{1}{6} (n + 2) g \frac{1}{S^2_0(d - 2)^3} \left( G(y, y') + G(y, -y') \right),
\]

\[
G(y, y') = \int_{-\infty}^{\infty} \int d^{d-1} x \frac{1}{X^{d-2}X'^{d-2}} \left( \frac{1}{4\pi^2} \right)^{\frac{1}{2}(d-2)},
\]

\[X^2 = x^2 + (y - z)^2, \quad X'^2 = x^2 + (y' - z)^2.\]  

Since the critical coupling \( g_s = O(\varepsilon) \) it is sufficient to evaluate \( G(y, y') \) for \( d = 4 \). The integral in (C.1) then has a linear divergence which may be regularised by imposing a cut-off \( |z| > \varepsilon \) and we find

\[
G(y, y') = \pi^2 \left\{ \frac{1}{(y + y')^2} \ln \frac{4|y||y'|}{(y - y')^2} + \frac{1}{\varepsilon y} \ln \frac{1}{|y| + |y'|} \right\}. \quad (C.2)
\]

For the Dirichlet case the divergence as \( \varepsilon \to 0 \) cancels while in the Neumann case it may be removed by a surface counterterm in the action \( \propto \phi^2 \). At the critical point to this order \( g_s/16\pi^2 = 3\varepsilon/(n + 8) \) and the result for \( \langle \phi(y, 0)\phi(y', 0) \rangle \) is compatible with (2.5), since \( \eta = \frac{1}{2}(d - 2) + O(\varepsilon^2) \), where the function of the conformal invariant \( v \) is given by

\[
S_d(d - 2)F(v)^{(1)} = -\frac{1}{2} \frac{n + 2}{n + 8} \varepsilon \left( v^2 \ln \frac{1 - v^2}{v^2} \pm \ln(1 - v^2) \right). \quad (C.3)
\]

This result is equivalent to that obtained by Gompper and Wagner [11]. For use in calculating the contributions corresponding to fig. 2a to \( \langle T_{ij}(y, 0)T_{k\ell}(y', 0) \rangle \) we also require, as in (3.3) for free fields,

\[
\langle \partial_i\phi(y, 0)\partial_j\phi(y', 0) \rangle = -\langle \partial_i\partial_j\phi(y, 0)\phi(y', 0) \rangle = \delta_{ij} \frac{1}{|y - y'|^{2n+2}} F_1(v), \quad (C.4)
\]

\[
\langle \partial_i\partial_j\phi(y, 0)\partial_k\partial_l\phi(y', 0) \rangle = (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \frac{1}{|y - y'|^{2n+4}} F_2(v),
\]

where \( F_1, F_2 \) are determined in terms of \( F \) by

\[
F_1(v) = 2\eta F(v) - v(1 - v^2)F'(v), \quad F_2(v) = 2\eta(2\eta + 2)F(v) - v(1 - v^2)(4\eta + 1 + 3v^2)F'(v) + v^2(1 - v^2)F''(v). \quad (C.5)
\]

Corresponding to (C.3) we have, taking \( \eta \to 1, \)

\[
S_d(d - 2)F_1(v)^{(1)} = -\frac{n + 2}{n + 8} \varepsilon \left( v^4 \ln \frac{1 - v^2}{v^2} \pm \ln(1 - v^2) + v^2 \pm v^2 \right),
\]

\[
S_d(d - 2)F_2(v)^{(1)} = -4 \frac{n + 2}{n + 8} \varepsilon \left( v^6 \ln \frac{1 - v^2}{v^2} \pm \ln(1 - v^2) + v^4 + \frac{1}{2}v^2 \pm v^2 \pm \frac{1}{2}v^4 \right). \quad (C.6)
\]
With these results we may find the contributions arising from fig. 2a to \( \epsilon \) and \( \gamma \) just as for free fields in section 3. In the case of \( \gamma \) the required correlation functions involving \( \partial_1 \phi \) may be found from \( \langle \phi(y,0)\phi(y',0) \rangle \) and \( \langle \partial_i \phi(y,0)\partial_j \phi(y',0) \rangle \) by differentiation with respect to \( y, y' \). Hence we obtain

\[
S_{4\epsilon a}^{(1)} = \frac{n}{9} \frac{n+2}{n+8} \varepsilon \left\{ \pm v^2 (1 + 4v^2 + v^4) \ln \left( \frac{1 - v^2}{v^2} \right) - \frac{9}{2} v^2 (1 + v^4) + 2v^4 

- 6v^8 \ln \frac{1 - v^2}{v^2} - 6 \ln (1 - v^2) - 5v^2 (1 + v^4) - v^4 \right\}, \quad (C.7a)
\]

\[
S_{4\gamma a}^{(1)} = \frac{n}{9} \frac{n+2}{n+8} \varepsilon \left\{ \pm 5v^2 (1 - v^4) \left( \ln \left( \frac{1 - v^2}{v^2} \right) - \frac{1}{2} \right) 

- 6v^8 \ln \frac{1 - v^2}{v^2} + 6 \ln (1 - v^2) + v^2 (1 - v^4) \right\}. \quad (C.7b)
\]

For the graph in fig. 2b the corresponding contributions relevant for determining \( \epsilon \) and \( \gamma \) are

\[
\langle T_{ij}(y,0)T_{kl}(y',0) \rangle_b^{(1)} = - (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl}) \frac{1}{3} n(n+2) g \left( \frac{d}{d-1} \right)^2 \frac{1}{d^2 - 1} \frac{32}{S_d^4} \]

\[
\times y^2 y'^2 \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dx \int_{x^2 + x'^2}^{\infty} \frac{1}{X^{d+2} X'^{d+2}} \frac{1}{\bar{X}^{d+2} \bar{X}'^{d+2}} 

+ \text{terms proportional to } \delta_{ij} \delta_{kl}, \quad (C.8a)
\]

\[
\langle T_{i1}(y,0)T_{k1}(y',0) \rangle_b^{(1)} = - \delta_{ik} \frac{1}{3} n(n+2) g \left( \frac{d}{d-1} \right)^2 \frac{1}{d-1} \frac{8}{S_d^4} \]

\[
\times y y' \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dx \int_{x^2 + x'^2}^{\infty} \frac{1}{X^{d+2} X'^{d+2}} \frac{1}{\bar{X}^{d+2} \bar{X}'^{d+2}} 

\times (x^2 + z^2 - y^2)(x^2 + z^2 - y'^2), \quad (C.8b)
\]

with \( X, X' \) as given in (C.1) and \( \bar{X}^2 = x^2 + (y + z)^2, \bar{X}'^2 = x^2 + (y' + z)^2 \). It is clear from (C.8) that in the absence of a boundary the one loop graphs for \( \langle T_{\mu\nu}(x)T_{\sigma\rho}(x') \rangle \) are zero [12] so there is no change in the bulk coefficient \( C \) in (2.18) to this order and hence the results of all calculations for \( \epsilon, \gamma \) at this order should vanish as \( v \to 0 \). The integrals are more tedious to evaluate in this case but there are no singularities requiring regularisation even when \( d = 4 \). In this case we obtain

\[
S_{4\epsilon b}^{(1)} = \frac{n}{9} \frac{n+2}{n+8} \varepsilon \left\{ v^2 (1 - v^2)^2 \ln \left( \frac{1 - v^2}{v^2} \right) - 3v^4 + \frac{12v^6}{(1 - v^2)^2} 

+ v^2 (1 - v^4) \left( \frac{2v^2}{(1 - v^2)^2} + \frac{6v^4}{(1 - v^2)^4} \right) \ln v^2 \right\}, \quad (C.9a)
\]

20
\[ S_4^{(1)} = \frac{n}{9} \frac{n + 2}{n + 8} \varepsilon \left\{ -v^2 (1 - v^4) \ln \frac{(1 - v^2)^2}{v^2} + \frac{v^4 (1 + v^2)}{(1 - v^2)} \right. \\
- v^2 (1 + v^4) \left(1 - \frac{v^2}{(1 - v^2)^2}\right) \ln v^2 - v^4 \ln v^2 \left\}. \quad (C.9b) \]

We should note that (C.7) and (C.9) are in accord with the symmetry \( \epsilon(v) = v^{2d} \epsilon(v^{-1}) \), \( \gamma(v) = -v^{2d} \gamma(v^{-1}) \) with \( d = 4 \). It is easy to see that the apparent singularities as \( v \to 1 \) cancel and in fact \( \gamma_b^{(1)} = 0 \) for \( v = 1 \).

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