Reduction of Open String Amplitudes by Mostly BRST Exact Operators

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Abstract

We study two and three-point tree-level amplitudes of open strings. These amplitudes are reduced from higher-point correlation functions by using mostly BRST exact operators for gauge fixing. For simplicity, we focus on an open string tachyon. The two-point amplitude of open string tachyons is reduced from a three-point function of two tachyon vertex operators and one mostly BRST exact operator. Similarly the three-point amplitude is from a four-point function of three tachyon vertex operators and one mostly BRST exact operator. One can also obtain the two-point amplitude from the four-point function of two tachyon vertex operators and two mostly BRST exact operators. In these derivation from four-point functions, moduli integrals are significant. We discuss the overall signs of amplitudes which are indefinite in this formalism.

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1 Introduction

String scattering amplitudes is one of important origins of string theories. It has been studied in detail for a long time and even now provides unexpected and interesting subjects for us. In bosonic open string theory, the Veneziano amplitude given by the Euler beta function was discovered to explain stringy nature such as duality, and in a modern formulation, more generic string amplitudes are given as correlation functions of vertex operators in the Polyakov path integral, where the measure has to be divided by the volume of diffeomorphism and Weyl symmetry. For open string tree amplitudes, correlator has to be divided by the volume of $PSL(2, \mathbb{R})$ on an upper-half plane and so the one and two-point amplitudes were thought to be vanished due to the infinite volume of the residual symmetry. However, this view has been overturned by [1] for two-point amplitudes. They have shown that the infinite volume and the divergence of energy-conserving delta function $\delta(0)$ are canceled each other for the two-point amplitudes and then the standard free particle expression can be obtained.

Motivated by their work, the present authors introduced the operator which fixes a part of the $PSL(2, \mathbb{R})$ gauge symmetry in the BRST operator formalism. The operator is defined by

$$\mathcal{E}(z) \equiv \frac{1}{i} \int_{-\infty}^{\infty} dq \left( c \partial X^0 e^{-i q X^0} - i \alpha' q (\partial c) e^{-i q X^0} \right), \quad (1.1)$$

where $X^0(z, \bar{z})$ is the 0-th string coordinate and $c(z)$ is the ghost field, and $z$ is a point on the boundary.\(^1\) This is a BRST invariant operator and moreover it is written by using the BRST charge $Q_B$:

$$\mathcal{E}(z) = \frac{1}{i} \int_{-\infty}^{\infty} dq \frac{i}{q} \delta_B e^{-i q X^0}, \quad \delta_B e^{-i q X^0} = [Q_B, e^{-i q X^0}]. \quad (1.2)$$

\(^1\)In this paper, we multiply the operator $V_0$ in [2] by the factor $i \alpha'$ to define the mostly BRST operator $\mathcal{E}$. 
We call it a *mostly* BRST exact operator, since the integrand for $q \neq 0$ is indeed BRST exact, but it is not so at the singular point $q = 0$. In [2], the two-point amplitude is reproduced by calculating the three point function in which two arbitrary on-shell open string vertex operators and the operator (1.1) are inserted on an upper-half plane.

The concept of “mostly BRST exactness” is very important. The correlation function including the integrand of (1.1) should behave as a kind of distributions supported at $q = 0$, since it becomes zero for $q \neq 0$ due to BRST exactness. Then, performing the $q$-integration, we should have a non-zero result, which is applicable to two-point amplitudes. Mostly BRST exactness also assures that the operator satisfies Lorentz and conformal invariance, although depending only on the time-direction. Moreover, the concept can be widely applied in the BRST formalism. Indeed, a mostly BRST exact operator is constructed in the pure spinor formalism of superstring theory [3], and then two-point superstring amplitudes have been found to agree with the corresponding expression in the field theories.

An insertion of the mostly BRST exact operator fixes one degree of freedom of $PSL(2, \mathbb{R})$, since the operator is a BRST extension of the gauge fixing delta functional and the Fadeev-Popov determinant discussed in [1]. Then, it is natural to ask whether general $n$-point amplitudes can be obtained by fixing a part of $PSL(2, \mathbb{R})$ in terms of the mostly BRST exact operator, as mentioned in the last part of [2]. The main purpose of this paper is to report on the results of the $n$-point amplitudes by inserting this gauge fixing operator. Consequently, it is confirmed that the mostly BRST exact operator works well for gauge fixing of $PSL(2, \mathbb{R})$.

In the derivation of the two-point amplitudes in [2], the energy-conserving delta function becomes $\delta(q)$, if two states have the same mass and one of them is an incoming state and the other outgoing. After the $q$-integration, we can find the correct expression of two-point amplitudes not involving the delta function for energy. However, if we attempt to evaluate three-point amplitudes by an additional insertion of the gauge fixing operator, the positions of the two matter vertex operators is fixed, but the position of the remaining vertex has to be integrated. Compared with the conventional three-point amplitudes, this expression includes, as extra components, the moduli and $q$ integrations. In this paper, we indicate that the Feynman $i\varepsilon$ plays a key role in elimination of these integrals to derive the correct amplitudes.

In the Veneziano amplitude, there are poles for the Mandelstam variables and, precisely, the poles are defined by the $i\varepsilon$ prescription as well as Minkowski amplitudes [4]. This prescription is related to the fact that the string worldsheet hides Lorentzian nature although it is treated as Euclidean space generically [5]. Similarly, the moduli integral of our amplitudes generates poles which include the delta function of the energy zero mode $q$. Then, we will find that, after the $q$-integration, the conventional expression of three-point amplitudes is derived from the additional insertion of the gauge fixing vertex.

We should comment on the indefiniteness of the overall sign of the amplitude given by our gauge fixing operators. First, we remind that even for conventional amplitudes in the BRST
formalism, the overall sign is indefinite and it is fixed by referring to the Jacobian factor in the Fadeev-Popov procedure.\(^2\) This indefiniteness arises from removing the absolute value from the Fadeev-Popov determinant to introduce the ghost fields. The same extract of the absolute value causes uncertainty in the amplitudes with our gauge fixing operator. Then, we will see the signature factor depending on momenta of open string states even in the two-point amplitude.

More precisely, we will show that conventional open string amplitudes can be reproduced by using our gauge fixing operator for \(PSL(2, \mathbb{R})\), except the overall sign which can be positive, negative or zero. The sign factor is expressed by using momenta of external strings and we may attempt to interpret this expression in terms of a signed intersection number associated with the gauge fixing condition. This interpretation is essentially same as the discussion of a subtlety relating to a gauge choice in [6], which is inspired by the result of [1].

We begin in the section 2 by illustrating the derivation of the two and three-point amplitudes fixed a part of \(PSL(2, \mathbb{R})\) by the insertion of the mostly BRST exact operator. For simplicity, we choose open string tachyons as external states. We then verify in the subsection 2.3 the overall sign factor by interpreting it as a signed intersection number. In the section 3, we fixes two degrees of freedom in \(PSL(2, \mathbb{R})\) by the mostly BRST exact operators to derive two-point amplitudes for open string tachyons. This result justifies our gauge fixing procedure of \(PSL(2, \mathbb{R})\) and the interpretation of the overall sign in terms of the intersection. Finally, we provide concluding remarks in the section 4.

## 2 Reduction by one \(E\)

### 2.1 Two-point amplitude from three-point function

In [2], it is shown that the mostly BRST exact operator leads to correct two-point amplitudes for any open string vertex operators. Here, we illustrate how to derive a two-point amplitude in terms of open string tachyons in the 26 dimensional flat Minkowski spacetime.

We consider an upper-half plane as the world-sheet and the real axis is regarded as the open string boundary. If a part of \(PSL(2, \mathbb{R})\) is fixed by the insertion of \(E\), the amplitude for two tachyon vertex operators is given by

\[
A_2 = ig_6^2 C_{D_2} \langle 0 | E(y_0) eV_1(y_1) eV_2(y_2) | 0 \rangle.
\]  

(2.1)

where the tachyon vertex operator \(V_i\) is defined as

\[
V_i(y) \equiv e^{i p_i \cdot X}(y),
\]  

(2.2)

\(^2\)The sign can also be determined by deriving the amplitude from string field theories based on the BRST formalism.
and \( C_{D_2} \) is a normalization factor for disk amplitudes: \( C_{D_2} = 1/(\alpha' g_0^2) \) [4]. The positions \( y_i \) are along the real axis and we set \( y_0 < y_1 < y_2 \). The tachyon vertex operators include a factor of the open string coupling \( g_o \), but \( g_o \) is not assigned to the operator \( E \), because \( E \) does not add an extra string to the process. \( |0\rangle \) is the \( SL(2, \mathbb{R}) \) invariant vacuum normalized as \( \langle 0|0 \rangle = (2\pi)^{26} \delta^{26}(0) \).

Calculating the correlation function, we obtain

\[
A_2 = ig_o^3 C_{D_2} \frac{\alpha'(p_1^0 - p_2^0)}{i} (2\pi)^{25} \delta^{25}(p_1 + p_2) \int_{-\infty}^{\infty} dq \delta(q + p_1^0 + p_2^0)|y_{01}|^{-2\alpha' q^2}\delta^2(y_{12})|y_{21}|^{\alpha' q^2}.
\]

where \( y_{ij} \) is defined as \( y_{ij} \equiv y_i - y_j \). The integrand with the exponent including \( q \) implies that the operator in the \( q \)-integration of \( E \) apparently breaks conformal invariance for \( q \neq 0 \), however it will be found immediately that the resulting amplitude has the invariance and no dependence of the position of the vertex operators.

By the on-shell condition \( \alpha'(p_i)^2 = 1 \), we have \( p_1^0 = \pm \sqrt{(p_0)^2 + (-1/\alpha')} \) and so the amplitude vanishes if \( p_1^0 \) and \( p_2^0 \) have the same signature due to the factor \( p_1^0 - p_2^0 \). In general, this vanishing property is followed from the BRST invariance, since only the \( q = 0 \) part is not BRST exact in \( E \) and so energies for two tachyons should be conserved in the non-zero amplitude. Actually, if \( p_1^0 \) and \( p_2^0 \) have opposite sign, we find that the integral depends only on the contribution from \( q = 0 \), namely including \( \delta(q) \), and then the amplitude becomes non-zero. In this case, one tachyon corresponds to an incoming state and another is outgoing.

The resulting amplitude is given by

\[
A_2 = \frac{p_1^0}{|p_1^0|} \times 2|p_1^0|(2\pi)^{25} \delta^{25}(p_1 + p_2).
\]

Thus, we can obtain the two-point amplitude in the standard free particle expression, however it should be noted that the signature is not determined by this gauge fixing procedure, as mentioned in the introduction. This sign factor is depend on a momentum of external states.

### 2.2 Three-point amplitude from four-point function

Let us consider a three open tachyon amplitude by fixing the two vertex operators to positions \( y_1, y_2 \) on the real axis. To fix the residual gauge symmetry of \( PSL(2, \mathbb{R}) \), we insert the operator \( E \) to a position \( y_0 \).

\[
A_3 = ig_o^3 C_{D_2} \langle 0| E(y_0) cV_1(y_1) cV_2(y_2) \int_{-\infty}^{\infty} dy_3 V_3(y_3) |0 \rangle.
\]

Although it is a three point amplitude, the position \( y_3 \) of the third vertex operator is integrated as a result of the insertion of \( E \). Since \( y_3 \) varies from \(-\infty \) to \( \infty \), this expression includes two
ordering of the three tachyons and so it is not needed to add an extra contribution from the exchange, $p_1 \leftrightarrow p_2$, unlike a conventional case in [4].

Here, for convenience, introducing a covariant expression for the momentum $q$ in $E$ defined by (1.2): $p_0 = (q, 0, \ldots, 0)$, we rewrite $\delta_B e^{-iqX^0}$ as $\delta_B e^{ip_0 X}$. We fix the positions as $y_0 < y_1 < y_2$, since the ordering of $y_0$, $y_1$ and $y_2$ is related only to the overall sign. The correlation function in the amplitude (2.5) can be calculated as

$$\mathcal{F}_1(q) = \langle 0 | \left( \frac{i}{\pi q} \delta_B e^{ip_0 X} \right) cV_1(y_1) cV_2(y_2) \int_{-\infty}^{\infty} dy_3 V_3(y_3) | 0 \rangle$$

$$= \left| \frac{y_{01} y_{02}}{y_{12}} \right|^{-\alpha'(p_0)^2} (2\pi)^{26} \delta^{26}(\sum_j p_j) I ,$$

$$I = \frac{i}{\pi q} \int_{-\infty}^{\infty} dx \frac{1}{2} \left( \alpha'(p_0)^2 + 2\alpha' p_0 \cdot p_1 - 2\alpha' p_0 \cdot p_3 \frac{x}{1-x} \right) |x|^{2\alpha' p_2 p_3} |1 - x|^{2\alpha' p_0 p_3} , \quad (2.6)$$

where $x$ denotes a moduli parameter given by the cross ratio,

$$x = \frac{y_{01} y_{23}}{y_{02} y_{13}} . \quad (2.7)$$

The prefactor with the exponent $-\alpha'(p_0)^2 = \alpha' q^2$ implies that $E$ apparently breaks $PSL(2, \mathbb{R})$ invariance for $q \neq 0$, but it is not problematic since the final result should depend only on the $q = 0$ part.

The amplitude is written in terms of the Mandelstam variables;

$$s = -(p_0 + p_1)^2, \quad t = -(p_0 + p_2)^2, \quad u = -(p_0 + p_3)^2 . \quad (2.8)$$

However, unlike the Veneziano amplitude, these are independent since $p_0$ does not satisfy an on-shell condition, i.e., $s + t + u = q^2 - 3\alpha'$. The moduli integral (2.6) is rewritten as

$$I(q) = \frac{-i}{2\pi q} \int_{-\infty}^{\infty} dx \left( (\alpha's + 1) \frac{1}{1-x} + (\alpha't + 1) \frac{x}{1-x} \right) |x|^{-\alpha's-2} |1-x|^{\alpha'(s+t)+2} . \quad (2.9)$$

Noting that the moduli integral $I$ splits into three ranges, i.e., $I = I(i) + I(ii) + I(iii)$, for

$$\begin{align*}
(i) & \quad -\infty < y_3 < y_0, \ y_2 < y_3 < \infty \\
(ii) & \quad y_0 < y_3 < y_1 \\
(iii) & \quad y_1 < y_3 < y_2 ,
\end{align*} \quad (2.10)$$

or equivalently (i) $0 < x < 1$, (ii) $1 < x < \infty$, (iii) $-\infty < x < 0$, the amplitude becomes

$$\mathcal{A}_3 = ig_0^3 C_{D_2} \int_{-\infty}^{\infty} dq \mathcal{F}_1 = ig_0^3 C_{D_2} \int_{-\infty}^{\infty} dq \left| \frac{y_{01} y_{02}}{y_{12}} \right|^{-\alpha'(p_0)^2} (2\pi)^{26} \delta^{26}(\sum_j p_j) (I(i) + I(ii) + I(iii)) , \quad (2.11)$$

where

$$I(i) = \frac{-i}{2\pi q} \left\{(\alpha's + 1)B(-\alpha's - 1, \alpha'(s+t) + 2) + (\alpha't + 1)B(-\alpha's, \alpha'(s+t) + 2) \right\} , \quad (2.12)$$

$$I(ii) = \frac{-i}{2\pi q} \left\{-(\alpha's + 1)B(-\alpha't, \alpha'(s+t) + 2) - (\alpha't + 1)B(-\alpha't - 1, \alpha'(s+t) + 2) \right\} , \quad (2.13)$$

$$I(iii) = \frac{-i}{2\pi q} \left\{(\alpha's + 1)B(-\alpha's - 1, -\alpha't) - (\alpha't + 1)B(-\alpha's, -\alpha't - 1) \right\} . \quad (2.14)$$
The Euler beta function is defined by

\[ B(u, v) = \int_0^1 dx \, x^{u-1}(1-x)^{v-1}. \] (2.15)

If the momentum variables are in the convergence region of the integral \( B \), we find that \( I_{(i)}, I_{(ii)} \) and \( I_{(iii)} \) vanish. This reflects the mostly BRST exactness of \( \mathcal{L} \), however, it does not necessarily imply that the amplitude becomes trivially zero. To derive the physical result, we should study the \( q = 0 \) part of the amplitude, which is given as the divergence of the integral.

To study the divergence, we have to introduce a convergence factor which corresponds to the Feynman \( i\varepsilon \). For instance, one of the integrals in \( I_{(i)} \) has the convergence factor \( \exp(-i\varepsilon(\log x + \log(1-x))) \) [5] and it should be precisely defined as

\[ B(-\alpha'(s-1, \alpha'(s+t)+2) = \int_0^1 dx x^{-\alpha'-2-i\varepsilon}(1-x)^{\alpha'(s+t)+1-i\varepsilon}. \] (2.16)

To extract the singularity at \( q = 0 \), we have only to expand it in rational fractions and to use the formula,

\[ \frac{1}{x-i\varepsilon} = \text{P} \frac{1}{x} + \pi i \delta(x), \] (2.17)

where \( \text{P} \) denotes the principal value. In this example, the singularity can be evaluated as

\[ B(-\alpha'(s-1, \alpha'(s+t)+2) = \frac{1}{-\alpha'(s-1-i\varepsilon)} + \frac{1}{\alpha'(s+t)+2-i\varepsilon} + \cdots = \frac{\pi i}{2\alpha'} \left( \frac{1}{|p_1^0|} + \frac{1}{|p_2^0|} \right) \delta(q) + \cdots. \] (2.18)

These terms are generated from the singular configuration of the worldsheet with \( y_3 \) approaching to \( y_0 \) or \( y_2 \).

Similarly, by extracting \( \delta(q) \), the moduli integration gives the following results for each integration range:

\[ I_{(i)} \sim \frac{1}{2} \left( \frac{p_1^0}{|p_1^0|} - \frac{p_3^0}{|p_3^0|} \right) \delta(q), \quad I_{(ii)} \sim \frac{1}{2} \left( -\frac{p_2^0}{|p_2^0|} + \frac{p_3^0}{|p_3^0|} \right) \delta(q), \quad I_{(iii)} \sim \frac{1}{2} \left( \frac{p_1^0}{|p_1^0|} - \frac{p_2^0}{|p_2^0|} \right) \delta(q), \] (2.19)

where \( \sim \) stands for equal up to analytic terms. By substituting these results into (2.11) and performing the \( q \)-integration, we obtain the final expression of the amplitude:

\[ A_3 = \frac{1}{2} \left( \frac{p_1^0}{|p_1^0|} - \frac{p_2^0}{|p_2^0|} \right) \times \frac{2ig_0}{\alpha'} (2\pi)^{26} \delta^{26}(p_1 + p_2 + p_3). \] (2.20)

This result agrees with the correct three string amplitude except the prefactor taking the value \( \pm 1 \) or 0, which can not be determined in this gauge fixing as mentioned above.
2.3 Overall sign of amplitudes

The resulting three-point amplitude includes the sign factor

\[ \frac{1}{2} \left( \frac{p_0^1}{|p_1^0|} - \frac{p_0^2}{|p_2^0|} \right), \tag{2.21} \]

which is \( \pm 1 \) if \( p_0^1 \) and \( p_0^2 \) have opposite signs each other, and becomes zero if they are the same signs. Let us verify the validity of this factor.

The operator \( \mathcal{E} \) corresponds to the gauge fixing \( X^0(y_0) = 0 \), where the point \( y_0 \) is on the segment of the boundary of the disk. In the case \( y_0 < y_1 < y_2 \), the segment consists of the regions, \(-\infty < y < y_1 \) and \( y_2 < y < \infty \), noting that \( \pm \infty \) are identified. The tachyon vertex operators are inserted at the points \( y_1 \) and \( y_2 \).

From the path integral formulation, we integrate all over the function \( X^0(y) \) on the boundary to derive the amplitude. However, \( X^0(y) \) should approach \( +\infty \) at the inserted points of the vertex operators if the open string corresponds to outgoing states, and \( -\infty \) for incoming states, because \( X^0 \) is the time coordinate of the target space.

Suppose that the operator at \( y_2 \) is incoming and the one at \( y_1 \) is outgoing, which implies \( p_2^0 > 0 \) and \( p_1^0 < 0 \). In this case, \( X^0(y_2) = -\infty \) and \( X^0(y_1) = +\infty \). Since the equation \( X^0(y) = 0 \) has necessarily roots, it is possible to impose the gauge fixing condition \( X^0(y_0) = 0 \). In Fig. 1, we show the function \( X^0(y) \) satisfying the gauge choice \( X^0(y_0) = 0 \) for the case that the total number of roots is three.

![Figure 1](image-url)  

Figure 1: The function \( X^0(y) \) with \( y \) in the region, \(-\infty < y < y_1 \) or \( y_2 < y < \infty \), for the case that the total number of roots is three.

the total number of roots is three. For the three roots, there are three possibilities how to choose intersections at \( y_0 \), namely I, II and III in Fig. 1. As mentioned in the introduction, the operator \( \mathcal{E} \) originates in the Fadeev-Popov determinant for the gauge fixing, but we have to define \( \mathcal{E} \) by using the signed determinant. Therefore, for the three possibilities, \( \partial X^0(y_0) \) has different signatures: positive for I and III and negative for II. Then, the amplitude becomes positive for I and III and negative for II. Consequently, after summing over all \( X^0 \) in the case of
the three intersection, the amplitude is given as a positive result if I, II and III lead to the same
absolute value. So, we expect that the sign of the amplitude is given as the signed intersection
number of the graph $u = X^0(y)$ with $u = 0$ for any number of the roots.

The signed intersection number is $+1$ for $p_0^2 > 0$ and $p_1^0 < 0$, and it is $-1$ for $p_0^2 < 0$ and
$p_1^0 > 0$. If $p_0^2$ and $p_1^0$ have the same sign, both of the tachyon states are incoming or outgoing
and so the signed intersection number becomes zero. These results agree with the signature of
(2.21).

There is the case where the point $y_3$ enters the same segment of $y_0$, but the signed intersection
number is unchanged regardless of whether $X^0(y_3)$ is infinity or minus infinity.

The two point amplitude is given by (2.4), which is given by the gauge choice $X^0(y_0) = 0$
and the vertex insertions at $y_1$ and $y_2$. So, we apply the same discussion as the above to the
signature of (2.4). The only difference is that the sum of $p_0^1$ and $p_0^2$ is zero for the two point
amplitude, and so the signature (2.21) equals to $p_0^1/|p_0^1|$, which agrees with the sign of (2.4).

Thus, we can verify the validity of the signature of the amplitude from the point of view of
the signed intersection number.

3 Reduction by two $E$’s

3.1 Two-point amplitude from four-point function

Let us consider two open tachyon amplitudes by inserting the mostly BRST exact operators
at two points, $y_0$ and $y_1$. To fix $PSL(2, \mathbb{R})$ gauge symmetry, we fix the one tachyon vertex
operator at the position $y_2$ and so the position $y_3$ of another vertex operator is integrated. The
amplitude is given by

$$A_2 = \frac{1}{2} \times i g_0^2 C_{D_2} \langle 0| E(y_0)E(y_1)cV_2(y_2) \int_{-\infty}^{\infty} dy_3 V_3(y_3) |0\rangle. \quad (3.1)$$

Here, the two mostly BRST exact operators at $y_0$ and $y_1$ are indistinguishable each other and so
we should divide the amplitude by the statistical factor 2 from the perspective of the Feynman
diagram of four-point amplitudes. This factor can be also understood in world-sheet language.
The metric on the upper-half plane is given by

$$ds^2 = \frac{dzd\bar{z}}{|z - \bar{z}|^2}, \quad (3.2)$$

which is invariant under $PSL(2, \mathbb{R})$ transformation: $z \rightarrow (az + b)/(cz + d)$, $ad - bc = 1$. In
addition, the metric is invariant under the discrete transformation $z \rightarrow -\bar{z}$. These symmetries
generate the conformal Killing group (CKG). In the usual gauge fixing, the positions of three
different vertex operators are fixed and so any conformal Killing invariance is not remained.
However, if the two positions are fixed by the indistinguishable operators, a $Z_2$ invariance as a part of the CKG is unfixed. More explicitly, the $Z_2$ transformation is given by

$$z \to f(z) = \frac{- (y_0 y_1 - y_2^2) \bar{z} + y_2 (2y_0 y_1 - y_0 y_2 - y_1 y_2)}{-(y_0 + y_1 - 2y_2) \bar{z} + (y_0 y_1 - y_2^2)}. \quad (3.3)$$

This mapping function satisfies $f(y_0) = y_1$, $f(y_1) = y_0$, $f(y_2) = y_2$ and $f(f(z)) = z$. Therefore, the $Z_2$ symmetry should be fixed by restricting the integration range of $y_3$ to half of the boundary, or by dividing the factor 2 while the $Z_2$ symmetry remains unfixed. In the expression (3.1), we adopt the latter procedure to fix the CKG.

Now, let us evaluate the correlation function in (3.1) according to the discussion in the previous section. After tedious calculation, we obtain

$$\mathcal{F}_2(q, q') = \langle 0 | \left( \frac{i}{\pi q} \delta_B e^{ip_0 X} (y_0) \right) \left( \frac{i}{\pi q'} \delta_B e^{ip_1 X} (y_1) \right) c e^{ip_2 X} (y_2) \int_{-\infty}^{\infty} dy_3 e^{ip_3 X} (y_3) | 0 \rangle$$

$$= \frac{y_{12}}{y_{01} y_{02}} |^{\alpha'(p_0)^2} \left( \frac{y_{02}}{y_{01} y_{12}} \right)^{\alpha'(p_1)^2} (2\pi)^2 \delta^2(\sum_{j=0}^{3} p_j) \cdot J,$$

$$J = \frac{1}{\pi^2 q q'} \int_{-\infty}^{\infty} dx \frac{1}{2} \left( \alpha' p_0 \cdot p_1 - (\alpha')^2 (p_0 + p_1)^2 (p_0 \cdot p_1) \right.\right.$$

$$\left. - (\alpha')^2 ((p_0)^2 + 2p_0 \cdot p_1) (p_1 \cdot p_3) x + (\alpha')^2 ((p_1)^2 + 2p_0 \cdot p_1) (p_0 \cdot p_3) \frac{x}{1-x} \right.$$

$$\left. + 2(\alpha')^2 (p_0 \cdot p_3) (p_1 \cdot p_3) \frac{x^2}{1-x} \right) |x|^{2\alpha' p_2 \cdot p_3} |1-x|^{2\alpha' p_0 \cdot p_3},\quad (3.4)$$

where $p_0 = (q, 0, \ldots, 0)$, $p_1 = (q', 0, \ldots, 0)$. $q$ and $q'$ are integrated later in $\mathcal{E}(y_0)$ and $\mathcal{E}(y_1)$, respectively. Due to the BRST invariance, the $q \neq 0$ part does not contribute to the amplitude and so it can be easily seen that the $x$-integration of (3.4) vanishes in the convergence region. Then we have only to extract the singularity of $q = 0$ as the previous case.

We should comment on a different type of singularity proportional to $\delta((q + q')^2)$, which seems to arise from (3.4), since there are some terms in the moduli integration which have the pole at $2\alpha' p_2 \cdot p_3 + 2 = -\alpha'(q + q')^2 = 0$. This pole corresponds to a massless open string state as the intermediate one of the two tachyon collision. However, this pole should vanish since the amplitude has twist symmetry for two tachyons. Indeed, gathering together factors of this pole, using on-shell condition and momentum conservation, we find that they totally cancel each other and then the dangerous distribution does not appear in the amplitude.

Now, we show the final expression of the amplitude by extracting $\delta(q)$ and $\delta(q')$. Taking $y_0 < y_1 < y_2$, the integration (3.4) for three regions of (2.10) leads to $J = J_{(i)} + J_{(ii)} + J_{(iii)}$ with

$$J_{(i)} \sim \frac{i \alpha'}{2\pi} |p_0^0| \delta(q), \quad J_{(ii)} \sim \frac{i \alpha'}{2\pi} |p_0^0| \left( \delta(q) + \delta(q') \right), \quad J_{(iii)} \sim \frac{i \alpha'}{2\pi} |p_0^0| \delta(q'). \quad (3.5)$$

For the Veneziano amplitude, this $Z_2$ symmetry is remained if two momenta are identical, however the identical case has measure zero and so this factor is irrelevant to the amplitude at generic momenta.
Unless $p_2^0 + p_3^0 = 0$, the amplitude vanishes as well as the two-point amplitude reduced from the three-point function. Finally, in the case of $p_2^0 + p_3^0 = 0$, the resulting amplitude is given by

$$\mathcal{A}_2 = \frac{i g_5^2 C D_2}{2} \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dq' \mathcal{F}_2 = -1 \times 2 |p_3^0| (2\pi)^2 \delta^{25} (p_2 + p_3). \tag{3.6}$$

Similar to the previous cases, the signature of the amplitude can not be determined in this gauge fixing. However, contrastingly, the signature of (3.6) does not depend on momenta of the vertex operators. This can be also understood in terms of the signed intersection. For simplicity, suppose that $p_2^0 > 0$ and $p_3^0 < 0$, and so $X^0(y_2) = -\infty$ and $X^0(y_3) = +\infty$. We set $y_0 < y_1 < y_2$. If $y_1 < y_3$ or $y_3 < y_0$, the intersection number of $X^0$ on the segment in which $y_0$ and $y_1$ are located becomes an odd number. By assigning these intersections to $y_0$ and $y_1$ and adding the contributions of all combination, the corresponding amplitude is expected to be zero since $\sum \text{sgn}(\partial X^0(y_0))\text{sgn}(\partial X^0(y_1)) = 0$ in this case. If $y_0 < y_3 < y_1$, the intersection number on each segment for $y_0$ and $y_1$ is odd and moreover we find that $\sum \text{sgn}(\partial X^0(y_0)) = +1$ and $\sum \text{sgn}(\partial X^0(y_1)) = -1$. Then, the signature factor is $+1 \times (-1) = -1$ in this case. It can be easily seen that the same signature is obtained for $p_2^0 < 0$ and $p_3^0 > 0$, and therefore the signature is independent of momenta in the two-point amplitude reduced from the four-point function.

4 Concluding remarks

We have shown that the two and three-point tree-level amplitudes of open string tachyons are reduced from the three and four-point correlation functions by inserting the mostly BRST exact operators $\mathcal{E}$. Firstly we have obtained the two-point amplitude by calculating the three-point correlation function with one $\mathcal{E}$: $\langle \mathcal{E} c V_1 c V_2 \rangle$. This two-point amplitude has the same expression as the standard free particle, but its overall sign depends on the sign of the momentum of external state.

Similarly we have shown that the three-point amplitude is reduced from the four-point function of three tachyon vertex operators and one $\mathcal{E}$: $\langle \mathcal{E}(y_0) c V_1(y_1) c V_2(y_2) \int dy_3 V_3(y_3) \rangle$. In this computation, the moduli integration plays an important role, so that $\delta(q)$ appears from the singularities of the Euler beta functions. Combining it with the $q$-integration, the four-point function leads to the correct three-point amplitude up to the overall sign. The operator $\mathcal{E}$ implies the gauge fixing, $X^0(y_0) = 0$. $X^0(y)$ at $y_1$ and $y_2$, where tachyon vertex operators are inserted, are equal to $\pm \infty$, of which sign depends on the choice of incoming or outgoing for each external state. The configuration of $X^0(y)$ with $X^0(y_0) = 0$ and $X^0(y_{1,2}) = \pm \infty$ fixes the total signed intersection number of the graph $u = X^0(y)$ with $u = 0$. One can recognize that the overall sign of the two-point amplitude is given by this intersection number.

We have also studied the two insertions of $\mathcal{E}$. The four-point function $\langle \mathcal{E} \mathcal{E} c V_1 \int V_2 \rangle$ is reduced to the two-point amplitude. The two $\mathcal{E}$’s provide $q$ and $q'$-integrals. In the similar way
as the computation for the reduction from four-point to three, the moduli integration yields \( \delta(q) \) and \( \delta(q') \). Due to the \( q \) and \( q' \)-integration with these delta functions, we can obtain the correct two-point amplitude. There again appear a problem of the overall sign, but in this case the signature does not depend on the external momenta.

In this paper, by focusing only on open string tachyons, we have explicitly evaluated the correlation functions, and, as a result, we have found they are reduced to the lower-point amplitudes. In the case of two-point amplitudes of general vertex operators, Refs. [1, 2] have shown that the two-point amplitudes can be derived from the three-point function. However, for the case of higher-point amplitudes, we should understand more how the operator \( E \) acts in the higher-point functions of general vertex operators.

It is not straightforward to apply our formalism using the operator \( E \) for closed string amplitudes. For instance, if we extend \( E \) to \( E_{\text{closed}} \sim \int \frac{dq}{q} [Q^c_B, e^{-iqX^0(z,\bar{z})}] \), where \( Q^c_B \) is the BRST charge of closed string, it causes a problem of ghost number. Therefore, we need to find a new operator, which plays a same role as \( E \) in the closed string theory [7].

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