Is the Bianchi identity always hyperbolic?

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Abstract

We consider $n + 1$ dimensional smooth Riemannian and Lorentzian spaces satisfying Einstein’s equations. The base manifold is assumed to be smoothly foliated by a one-parameter family of hypersurfaces. In both cases—likewise it is usually done in the Lorentzian case—Einstein’s equations may be split into ‘Hamiltonian’ and ‘momentum’ constraints and a ‘reduced’ set of field equations. It is shown that regardless of whether the primary space is Riemannian or Lorentzian, whenever the foliating hypersurfaces are Riemannian the ‘Hamiltonian’ and ‘momentum’ type expressions are subject to a subsidiary first order symmetric hyperbolic system. Since this subsidiary system is linear and homogeneous in the ‘Hamiltonian’ and ‘momentum’ type expressions, the hyperbolicity of the system implies that in both cases the solutions to the ‘reduced’ set of field equations are also solutions to the full set of equations provided that the constraints hold on one of the hypersurfaces foliating the base manifold.

Keywords: evolution, splitting, hyperbolic

1. Introduction

Consider a pair $(M, g_{ab})$, where $M$ is an $(n + 1)$-dimensional $(n \geq 2)$ smooth, paracompact, connected, orientable manifold endowed with a smooth metric $g_{ab}$ with a signature which is either Euclidean or Lorentzian.

Throughout this paper the geometry will be the focus of our main concern. In restricting the geometry we shall assume that Einstein’s equations

$$G_{ab} - 2\mathcal{R}_{ab} = 0, \quad (1.1)$$

1 All of our other conventions will be as in [7].
hold, where, for simplicity, the source term $\mathcal{G}_{ab}$ is assumed to have vanishing divergence. Note that whenever we have matter fields satisfying their field equations with energy-momentum tensor $T_{ab}$ and with cosmological constant $\Lambda$, the source term

$$\mathcal{G}_{ab} = 8\pi T_{ab} - \Lambda g_{ab}$$

(1.2)
suits the above requirements.

Concerning the topology of $M$ we shall assume that the manifold $M$ is foliated by a one-parameter family of hypersurfaces, i.e., $M \simeq \mathbb{R} \times \Sigma$, for some codimension one-manifold $\Sigma$. In other words, $M$ then possesses the structure of a trivial principal fiber bundle with structure group $\mathbb{R}$.

Note that this assumption is known to hold [3] for globally hyperbolic spacetimes, but we would like to emphasize that as the signature of the metric may not be Lorentzian or even if it was, in deriving our key results, we need not assume global hyperbolicity of the pertinent spacetime. Our assumptions on the topology of $M$ are known to be equivalent to the existence of a smooth function $\sigma : M \to \mathbb{R}$ with non-vanishing gradient $\nabla_\sigma \sigma$ such that the $\sigma = \text{const}$ level surfaces $\Sigma = \{ \sigma \} \times \Sigma$ comprise the one-parameter foliation of $M$.

Having the above generic setup it is natural to perform a $1 + n$ decomposition. In doing so, first a conventional $1 + n$ splitting of (1.1) will be done by generalizing conventional arguments (see, e.g., section 2.4 of [2]). This $1 + n$ splitting can be performed on equal footing in both the Lorentzian and Riemannian cases yielding ‘Hamiltonian’ and ‘momentum’ type expressions, along with a reduced set of equations referred to as an ‘evolutionary system’. By using the evolutionary system, a subsidiary system for the constraint expressions is derived. A remarkable and unexpected property of this subsidiary system is that regardless of whether the metric of the imbedding manifold is of Lorentzian or Euclidean signature—whenever the metric on the $\sigma = \text{const}$ level surfaces $\Sigma = \{ \sigma \} \times \Sigma$ comprises a first order symmetric hyperbolic system that is linear and homogeneous in the constraint expressions. This then guarantees that the constraint expressions vanish identically throughout domains where solutions to the evolutionary system exist, provided that they vanish on one of the level surfaces. These results are presented in section 2. Some useful relations are given in section 3 and in the appendix.

Having had a $1 + n$ type decomposition performed, it is natural to ask whether the analogous type of simplifications of the reduced equations in a succeeding $1 + (n - 1)$ splitting could also exist. The answer to this question requires—besides some obvious additional restrictions on the topology of the base manifold—the identification of those conditions that guarantee that the covariant divergence of the new source term $\mathcal{G}_{ab}$ vanishes. The corresponding analysis is carried out in section 4. The main conclusion here is that even though a formal $1 + (n - 1)$ splitting could be performed, in general, there is no room to acquire additional new simplifications. What can be done is nothing more than a redistribution of the simplifications associated with the primary splitting of the original field equations.

The paper is closed in section 5 by remarks on some of the implications of the derived new results.

2. The $1 + n$ decomposition

This section shows that a reduced set of the equations can be deduced from (1.1) such that, regardless of whether the metric of the imbedding manifold is of Lorentzian or Euclidean
signature, the solutions to this reduced system are also solutions to the full set (1.1) provided that the ‘constraints’ hold on one of the \( \sigma = \text{const} \) level surfaces.

We proceed by separating the ‘evolution’ and ‘constraint’ equations by adopting the strategy of the conventional 1 + 3 decomposition applied in spacetimes with the Lorentzian metric (see, e.g., [2]). In doing so, denote by \( n^a \) the ‘unit norm’ vector field that is normal to the \( \sigma = \text{const} \) level surfaces. To allow the simultaneous investigation of both spaces with either Euclidean or Lorentzian signature and timelike or spacelike level surfaces, the sign of the norm of \( n^a \) will not be fixed, i.e., it will be assumed that

\[
n^a n_a = \epsilon, \tag{2.1}
\]

where \( \epsilon \) takes the value \(-1\) or \(+1\).

The induced metric \( h_{ab} \) and the pertinent projection operator \( h^{ab} \) on the level surfaces of \( \sigma: M \to \mathbb{R} \) are then given as

\[
h_{ab} = g_{ab} - \epsilon n_a n_b, \quad \text{and} \quad h^{ab} = g^{ab} - \epsilon n^a n_b, \tag{2.2}
\]

respectively.

Denote by \( E_{ab} \) the left-hand-side of (1.1), i.e.,

\[
E_{ab} = G_{ab} - \partial_{ab}, \tag{2.3}
\]

and define the ‘Hamiltonian’ \( E^{(\text{H})} \) and ‘momentum’ \( E^{(\text{M})} \) expressions as

\[
E^{(\text{H})} = n^a n^b E_{ab}, \quad \text{and} \quad E^{(\text{M})} = n^a h^{ab} E_{bf}, \tag{2.4}
\]

respectively. Then, we have

\[
E_{ab} = h^a h^b E_{ab} + \epsilon \left[ n_a E^{(\text{M})}_b + n_b E^{(\text{M})}_a \right] + n_a n_b E^{(\text{n})}. \tag{2.5}
\]

Choose now as our ‘evolutionary’ system the combination

\[
E^{(\text{evol})}_{ab} = h^a h^b E_{ab} - \kappa h_{ab} E^{(\text{n})} = 0, \tag{2.6}
\]

where \( \kappa \) is some constant. Then, by combining (2.5) and (2.6), we get

\[
E_{ab} = \epsilon \left[ n_a E^{(\text{M})}_b + n_b E^{(\text{M})}_a \right] + \left[ (1 - \epsilon \kappa) n_a n_b + \kappa g_{ab} \right] E^{(\text{n})}. \tag{2.7}
\]

Taking now the \( V^a \) divergence of (2.7) and using our assumption concerning the vanishing of the covariant divergence \( V^a \partial_{ab} \), along with the twice contracted Bianchi identity we get

\[
e \left( V^a n_a \right) E^{(\text{M})}_b + \epsilon \left( n^a V^a E^{(\text{M})}_b \right) + e \left( E^{(\text{M})}_a V^a n_b \right) + \epsilon n_b \left( V^a E^{(\text{M})}_a \right) \]

\[
(1 - \epsilon \kappa) \left\{ (V^a n_a) n_b + (n^a V^a n_b) \right\} E^{(\text{n})} + n_b \left( n^a V^a E^{(\text{n})} \right) + \kappa \left( V_j E^{(\text{n})} \right) = 0. \tag{2.8}
\]

The ‘parallel’ and ‘orthogonal’ parts of (2.8) read then as

\[
n^a V_a E^{(\text{n})} + \epsilon h^a h^b E^{(\text{M})}_{bf} = (1 - \epsilon) \left( n^a V_a n^b \right) E^{(\text{M})}_b - \epsilon (1 - \epsilon \kappa) (V_a n^a) E^{(\text{n})}, \tag{2.9}
\]

\[= (1 - \epsilon) \left( n^a V_a n^b \right) E^{(\text{M})}_b - \epsilon (1 - \epsilon \kappa) (V_a n^a) E^{(\text{n})}, \tag{2.9}
\]
\[ h^{\alpha} n^\nu \nabla_\nu E_{\alpha} = -h^{\alpha} E_{\alpha} (\nabla^\nu n_\nu) - E_\nu (\nabla^\nu n_\nu) h^{\alpha} - \epsilon (1 - \kappa) h^{\alpha} (n^\nu \nabla_\nu n_\nu) E^{(\alpha)}, \tag{2.10} \]

where the relations \( \epsilon^2 = 1 \) and
\[ \nabla^\alpha E_{\alpha} = D^\alpha E_{\alpha} - \epsilon (n^\alpha \nabla_\alpha n^\beta) E^{(\beta)} \tag{2.11} \]

have been used, and \( D \) denotes the unique torsion free covariant derivative operator associated with \( h_{\mu \nu} \).

Although \((M, g_{ab})\) may not have anything to do with time evolution we shall refer to a vector field \( \sigma^a \) on \( M \) as an ‘evolution vector field’ if the relation \( \sigma^a \nabla_a \sigma = 0 \) holds. Notice that this condition guarantees that \( \sigma^a \) neither vanishes nor becomes tangent to the \( \sigma = \text{const} \) level surfaces. The unit normal \( n^\alpha \) to these level surfaces may always be decomposed as
\[ n^\alpha = \frac{1}{N} \left( \partial \alpha \right)^{\alpha} - N^\alpha, \tag{2.12} \]

where \( N \) and \( N^\alpha \) denote the ‘laps’ and ‘shift’ of the ‘evolution’ vector field \( \sigma^a = (\partial)^{\alpha} \sigma \) defined as
\[ N = \epsilon (\sigma^a n_a) \quad \text{and} \quad N^\alpha = h^\alpha, \tag{2.13} \]

respectively. Taking these relations into account, equations (2.9) and (2.10)—when writing them out explicitly in some local coordinates \((\sigma, x^1, \ldots, x^n)\) adopted to the vector field \( \sigma^a \) and the foliation \([\Sigma\sigma]\)—can be seen to take the form
\[ \begin{aligned}
\left( \begin{array}{cc}
\frac{1}{N} & 0 \\
0 & h^\alpha
\end{array} \right) \partial_\alpha + \left( \begin{array}{cc}
\frac{1}{N} N^\alpha & \epsilon h^\alpha \\
\epsilon \kappa h^\alpha & -\frac{1}{N} N^\alpha h^\alpha
\end{array} \right) \partial_\beta \begin{pmatrix} E^{(\alpha)} \\ E^{(\beta)} \end{pmatrix} = \begin{pmatrix} \mathcal{E} \\ \mathcal{E}' \end{pmatrix},
\end{aligned} \tag{2.14} \]

where, in virtue of (2.9) and (2.10), \( \mathcal{E} \) and \( \mathcal{E}' \) are linear and homogeneous expressions of \( E^{(\alpha)} \) and \( E^{(\alpha)} \). It follows immediately that the coefficient matrices of the partial derivatives are symmetric if \( \kappa = 1 \) and, in addition, the coefficient of \( \partial_\alpha \) is also positive definite provided that the induced metric \( h^{\alpha \beta} \) is positive definite.

Hereafter we shall assume that \( \kappa = 1 \) and that \( h^{\alpha \beta} \) is positive definite. The latter occurs if the \( \sigma \) level surfaces are spacelike allowing the signature of metric \( g_{ab} \) to be either Lorentzian, with \( \epsilon = -1 \), or Euclidean, with \( \epsilon = +1 \), respectively. In these cases (2.14) comprises a first order symmetric hyperbolic linear and homogeneous system
\[ \mathcal{A} \partial_\alpha v + \mathcal{B} v = 0 \tag{2.15} \]

for the vector valued variable \( v = \left( E^{(\alpha)}, E^{(\alpha)} \right)^T \). As these type of equations are guaranteed to have an identically vanishing solution for vanishing initial data, the ‘Hamiltonian’ and

\[ \begin{aligned}
\text{Remark}:\quad \text{The spatial indices of the pull backs of geometrical objects to the } \sigma = \text{const} \text{ slices yielded in the applied } 1 + n \text{ decomposition will be indicated by lowercase Latin indices from the second half and they will be assumed to take the values } 1, \ldots, n.
\end{aligned} \]
‘momentum’ expressions will be guaranteed to vanish throughout the domain of existence of solutions to the evolutionary system (2.6), with $\kappa = 1$, provided they vanish on one of the slides of the foliation $\{ \Sigma \}$.

By combining the above observations we have the following:

**Theorem 2.1.** Let $\left( M, g_{ab} \right)$ as described in section 1 such that the metric induced on the $\sigma = \text{const}$ level surfaces is Riemannian. Then, regardless of whether $g_{ab}$ is of Lorentzian or Euclidean signature, any solution to $E^{(\text{evol})}_{ab} = 0$, with $\kappa = 1$, is also a solution to the full set (1.1) provided that $E^{(\text{c})}_{a}$ and $E^{(\text{m})}_{a}$ vanish on one of the level surfaces.

It is a remarkable property of (2.14) that $\epsilon$ and $\kappa$ do not show up in the coefficient of $\partial_{\sigma}$, and once $\kappa = 1$ is chosen all the coefficients $B_{\sigma}$ in (2.15) are guaranteed to be symmetric regardless of the value of $\epsilon$.

3. The explicit forms

In exploring some of the consequences of theorem 2.1 we shall need the explicit forms of the constraint expressions and the evolutionary system. In spelling them out we shall refer to the extrinsic curvature $K_{ab}$, which is defined as

$$K_{ab} = h_{a}^{\gamma} \nabla_{\gamma} h_{b} = \frac{1}{2} \mathcal{L}_{n} h_{ab},$$

where $\mathcal{L}_{n}$ stands for the Lie derivative with respect to $n^{\sigma}$.

The ‘Gauss’ and ‘Codazzi’ relations take the form

$$h^{c}, h^{d} h^{b} h^{a} R_{c\delta} = (\text{c}) R_{abc}^{d} - \epsilon \left( K_{ab} K_{\delta}^{d} - K_{a}^{d} K_{\delta}^{b} \right),$$

$$h^{c}, n^{d} h^{b} h^{a} R_{c\delta}^{d} n \mathcal{L}_{n} h_{ab} = D_{b} K_{\delta}^{d} - D_{\delta} K_{b}^{d},$$

where $^{(\text{c})} R_{ab}$ stands for the $n$-dimensional Riemann tensor associated with $h_{ab}$.

The various projections of the full Ricci tensor—which can be derived either by contractions of the above two relations or that of the third non-trivial projection of the full Riemann tensor, $n^{b} h^{a} h^{c} n^{d} h^{e} R_{\sigma h}^{d}$—read as

$$h^{c}, n^{d} R_{c\delta} = (\text{c}) R_{cd} + \epsilon \left\{ -\mathcal{L}_{n} K_{ab} - K_{cd} K_{\delta}^{e} + 2 K_{cd} K_{\delta}^{e} - \frac{\epsilon}{N} D_{b} D_{\delta} N \right\}$$

$$h^{c}, n^{d} R_{\sigma c} = D_{b} K_{\delta}^{e} - D_{\delta} K_{b}^{e},$$

$$n^{d} R_{c\delta} = -\left\{ \mathcal{L}_{n} (K^{e})_{\sigma} + K_{e} K^{d} + \frac{\epsilon}{N} (D^{e} D_{\sigma} N) \right\},$$

where $^{(\text{c})} R_{ab}$ stands for the Ricci tensor associated with $h_{ab}$.

Taking all the above relations into account we have

$$E^{(\text{c})} = n^{d} n^{e} E_{c} = \frac{1}{2} \left\{ -\epsilon (\text{c}) R + (K^{e})_{\sigma}^{2} - K_{e} K^{d} - 2 \epsilon \right\},$$

(3.7)
\[
E^{(ab)}_b = h' h \nabla \hE_b = D_j K^j - D_j K_j^j - \epsilon \, p_a,
\]

\[
E^{(r)}_{ab} = \left( R_{ab} + \epsilon \left\{ -\mathcal{L}_a K_{ab} - (K^j) K_{ab} + 2 K_{ab} K^j - \frac{\epsilon}{N} D_j D_j N \right\} \right) - \left[ \mathcal{G}_{ab} - \epsilon \, h_{ab} \right] - \frac{1}{2} h_{ab} \left\{ \frac{2 \epsilon}{N} (1 - \epsilon) R - 2 \epsilon \mathcal{L}_a (K^j) + (1 - \epsilon) (K^j)^2 \right\} - (1 + \epsilon) K_{j} K^j - \frac{2}{N} D' D_j N \right],
\]

(3.9)

where \( \epsilon = n' n' \, \mathcal{G}_{ab} \), \( p_a = \epsilon \, h' n' \, \mathcal{G}_{ab} \) and \( \mathcal{G}_{ab} = h' j^j \, \mathcal{G}_{ab} \).

For certain cases (in particular, whenever \( \epsilon = -1 \)) it is rewarding to do some algebra by which it can be verified that

\[
E^{(r)}_{ab} = \frac{1}{n - 1} h_{ab} \left( E^{(r)}_b - h^j \right) = \tilde{E}^{(r)}_{ab},
\]

(3.10)

where

\[
\tilde{E}^{(r)}_{ab} = h' h \left[ R_{ab} - \left( \mathcal{G}_{ab} - \frac{1}{n - 1} g_{ab} \left[ \mathcal{G}_{ab} \mathcal{G}^{(r)} \right] \right) \right] + \frac{1 + \epsilon}{n - 1} h_{ab} E^{(a)}.
\]

(3.11)

In virtue of the above relations we have.

**Lemma 3.1.** The evolutionary system (2.6) holds if and only if either

(i) the right-hand side of (3.9), or

(ii) that of (3.11) vanishes.

The right-hand side of (3.11) can also be written as

\[
\tilde{E}^{(r)}_{ab} = \left( R_{ab} + \epsilon \left\{ -\mathcal{L}_a K_{ab} - (K^j) K_{ab} + 2 K_{ab} K^j - \frac{\epsilon}{N} D_j D_j N \right\} \right) - \left[ \mathcal{G}_{ab} - \frac{1}{n - 1} h_{ab} \left[ \mathcal{G}_{ab} \mathcal{G}^{(r)} \right] \right] + \frac{1 + \epsilon}{2(n - 1)} h_{ab} \left\{ -\epsilon R + (K^j)^2 - K_j K^j - 2 \epsilon \right\}.
\]

(3.12)

Note that by making use of the contractions \( \epsilon \), \( p_a \) and \( \mathcal{G}_{ab} \), our source term \( \mathcal{G}_{ab} \) can be decomposed as

\[
\mathcal{G}_{ab} = n_a n_b + n_a p_b + n_b p_a + \mathcal{G}_{ab},
\]

(3.13)

while its divergence \( V^a \mathcal{G}_{ab} \) takes the form (see also (A.8))

\[
V^a \mathcal{G}_{ab} = \epsilon (K^j) n_b + (K^j) p_b + n_b K^j - n_b \left( D^j p_j \right) + D \mathcal{G}_{ab} - \epsilon n_b \left( \mathcal{G}_{ab} \mathcal{G}^{(r)} \right)
+ n_b \epsilon + n_b \mathcal{L}_a e + \mathcal{L}_a p_b - n_b K^j - 2 \epsilon (n' p_b) n_a - \epsilon (n' \mathcal{G}_{ab}),
\]

(3.14)
where

\[
\dot{n}_a := n^c \nabla_c n_a = - \epsilon D_a \ln N.
\]  

(3.15)

Taking then the ‘parallel’ and ‘orthogonal’ parts of (3.14),

\[
\nabla \dot{\varphi}_{ab} = 0
\]  

(3.16)

we get (see also (A.9) and (A.10))

\[
\mathcal{L}_\epsilon p_a + D^a \bar{\epsilon}_{ab} = 0,
\]

(3.17)

\[
\mathcal{L}_\epsilon p_a + D^a \bar{\bar{\epsilon}}_{ab} = 0.
\]

(3.18)

Notice that in deriving (3.17) and (3.18) only the vanishing of the divergence \(\nabla \dot{\varphi}_{ab}\) has been used\(^3\). We may replace \(\varphi_{ab}\), for instance, by \(E_{ab}\). Accordingly, a simultaneous replacement of \(\bar{\epsilon}, \bar{\bar{\epsilon}}\) by \(E_{ab}\) and \(E^{(E\text{YOLG})}_{ab} + \kappa \, h_{ab} \, E^{(\sigma)}\), respectively, yields a system of equations which can be seen to be equivalent to (2.9) and (2.10) whenever \(E^{(E\text{YOLG})}_{ab} = 0\). Note also that if the term \(E^{(E\text{YOLG})}_{ab}\) is kept in these latter equations they can be used to justify the following statement which is complementary to that of theorem 2.1.

**Lemma 3.2.** If the constraint expressions \(E^{(\sigma)}_{ab}\) and \(E^{(M)}_{ab}\) vanish on all the \(\sigma = \text{const level surfaces then the relations}\)

\[
K^{ab} E^{(E\text{YOLG})}_{ab} = 0,
\]

(3.19)

\[
D^a E^{(E\text{YOLG})}_{ab} - \epsilon \, n^a E^{(E\text{YOLG})}_{ab} = 0,
\]

(3.20)

hold for the evolutionary expression \(E^{(E\text{YOLG})}_{ab}\).

### 4. Double decompositions

Once a \(1 + n\) splitting has been done, one may be interested in performing a succeeding \(1 + (n - 1)\) decomposition provided that the \(\sigma = \text{const level surfaces are guaranteed to be foliated by a one-parameter family of (} n - 1 \text{-dimensional hypersurfaces in } \Sigma\). Note, however, that before automatically adopting theorem 2.1 and the equations listed in the previous section, the validity of all the assumptions made in deriving them have to be inspected. The key requirement to be checked is the vanishing of the covariant divergence of \(\varphi_{ab}\). Therefore, once a \(1 + n\) decomposition had been done, before performing the succeeding \(1 + (n - 1)\) splitting, we need to check whether the new source term, \(\varphi_{ab}^{(n)}\), in

\[
\left( R_{ab} - \frac{1}{2} h_{ab} \right) \nabla_a \nabla_b - \varphi_{ab}^{(n)} = 0,
\]

(4.1)

\(^3\) Relations analogous to (3.17) and (3.18) were derived first by York in the context of the energy-momentum tensor \(T_{ab}\) in [8] (see also [4]).
does really have vanishing $D^a \left[ \sigma^{(a)}_{ab} \right]$ divergence. In doing so notice first that
\[ h^a h^f_b \left[ R_{ef} - \frac{1}{2} g_{ef} R \right] = h^a h^f_b R_{ef} - \frac{1}{2} h_{ab} R \] (4.2)
and—by substituting (1.1) to the left-hand side, whereas (3.4) and (A.1) to the right-hand side—
the source term can be seen to read as
\[ \sigma^{(a)}_{ab} = \Theta_{ab} - e \left\{ -L_a K_{ab} - (K^e_e) K_{ab} + 2 K_{ae} K^e_b - \frac{e}{N} D^a D_b N \right\} \]
\[ + h_{ab} \left[ L_a (K^e_e) + \frac{1}{2} (K^e_e)^2 + \frac{1}{2} K_{ef} K^{ef} + \frac{e}{N} D^a D_b N \right] \] (4.3)

Notice that all the tensor fields involved in (4.3) are apparently fields defined on the $\Sigma$ hypersurfaces, thereby to proceed it suffices to ensure the existence of a foliation of $\Sigma$ by a one-parameter family of homologous codimension-two surfaces.

Taking then the $D^a$-divergence of this relation and by commuting Lie and the covariant, as well as covariant derivatives, by a tedious but straightforward calculation, it can be verified that
\[ D^a \left[ \sigma^{(a)}_{ab} \right] = L_a p_a + D^a \Theta_{ab} + e L_a E^{(M)}_b + e (K^e_e) \left[ E^{(M)}_b + e p_b \right] \]
\[ + h_{ab} \left[ L_a (K^e_e) + K_{ef} K^{ef} + \frac{e}{N} D^a D_b N \right] \] (4.4)

By inspecting (3.4) and (3.6), and the coefficients of $n^a$ and $n_b$ in (4.4) it can be recognized that they are equal to $-e h^a h^f_b R_{ef}$ and $n^a n^f_b R_{ef}$, respectively. Taking then into account (1.1), along with $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$, we get
\[ h^a h^f_b R_{ef} = \Theta_{ab} - \frac{1}{n-1} h_{ab} \left[ \Theta_{ef} h^f + e \epsilon \right] \] (4.5)
\[ n^a n^f_b R_{ef} = e - \frac{e}{n-1} \left[ \Theta_{ef} h^f + e \epsilon \right]. \] (4.6)

These relations, along with (4.4), imply that
\[ D^a \left[ \sigma^{(a)}_{ab} \right] = 0 \] (4.7)
is equivalent to
\[ L_a p_a + D^a \Theta_{ab} + \left\{ -e \Theta_{ab} n^e + (K^e_e) p_b + e n_b \right\} + e \left[ L_a E^{(M)}_b + (K^e_e) E^{(M)}_b \right] = 0. \] (4.8)

In virtue of (3.18) and (4.8) the integrability condition (4.7) is guaranteed to hold whenever $h^f_b \nabla^a \sigma_{af} = 0$ and $E^{(M)}_b = 0$ on each of the $\sigma = const$ level surfaces.

In summarizing the above observations we have the following

**Proposition 4.1.** The integrability condition (4.7) holds on $\Sigma$ if $h^f_b \nabla^a \sigma_{af}$, the momentum constraint expression $E^{(M)}_b$ and its Lie derivative $L_a E^{(M)}_b$ vanish there.
In interpreting this result recall first that—by our assumptions concerning the source term for (1.1)—the projection \( h' \cdot V'' \partial_{\sigma} \) vanishes throughout \( \Sigma_{\sigma} \). In addition, in virtue of theorem 2.1 the Lie derivative of both the Hamiltonian and momentum constraint expressions vanish throughout \( \Sigma_{\sigma} \) if they themselves vanish on \( \Sigma_{\sigma} \) and the evolutionary system holds. Thus, as far as we prefer to solve first both the Hamiltonian and momentum constraints only on \( \Sigma_{\sigma} \), we have to solve the reduced evolutionary system in \( M \). In this case proposition 4.1 has no use as it can guarantee the integrability condition for the reduced system after the solution has been found.

Note, however, that proposition 4.1 allows a redistribution of the simplifications guaranteed by theorem 2.1. Namely, if we solve the momentum constraint on the entire base manifold in virtue of theorem 2.1 and proposition 4.1, besides solving the Hamiltonian constraint on \( \Sigma_{\sigma} \) and instead of solving the full reduced system on \( M \), it suffices to solve the second level of the Hamiltonian and momentum constraints on a codimension-two surface in \( M \), whereas the corresponding new reduced evolutionary system (formally only) on \( \Sigma_{\sigma} \). By repeating this type of formal splitting and always solving the yielded new momentum constraints, the entire process can be applied inductively provided that the product structure of the manifold allows it to be done. Applying this process, e.g., to the conventional Cauchy problem in the Lorentzian case, one may get on a suitable intermediate level mixed elliptic-hyperbolic systems from Einstein’s equations, as is done for a specific gauge choice in [1].

5. Final remarks

In answering the question raised in the title, the main results of this paper make it clear that some of the basic techniques developed for \( 1 + n \) splitting of Lorentzian spacetimes do also apply to spaces with a Riemannian metric. The most remarkable aspect here is that regardless of whether the metric is of Euclidean or Lorentzian signature, the subsidiary equations—these can be derived for the ‘Hamiltonian’ and ‘momentum’ type expressions, by making use of the Bianchi identity—are hyperbolic. This guarantees that in both cases the solutions to the ‘reduced’ set of equations are also solutions to the full set of Einstein’s equations (1.1) provided that the constraints hold on one of the hypersurfaces foliating the base manifold. Having been the first \( 1 + n \) type decomposition performed, it is important to know if there may be room for further simplifications in a succeeding \( 1 + ( n - 1 ) \) type decomposition. According to our findings there is no way to acquire new simplifications in a secondary splitting. It is remarkable that the new results apply regardless of whether the primary space is Riemannian or Lorentzian.

Having our results, it would be useful to know whether they can be applied in solving some specific problems. To indicate that even in one of the simplest possible setups some non-trivial implications may follow, let us consider the following example. Start with a four-dimensional Riemannian space foliated by a two-parameter family of homologous two-surfaces. Then—by making use of a suitable gauge fixing—the constraint equations can be seen to comprise a coupled parabolic–hyperbolic system, whereas the ‘evolutionary system’ is comprised by coupled elliptic equations. The system corresponding to the constraint equations is under-determined and it has to be solved on one of the \( \Sigma_{\sigma} \) hypersurfaces as a boundary-initial value problem with initial data specified on one of the codimension-two

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4 This could be done at most \( n \)-times, which is the number of equations involved in the original momentum constraint.
surfaces foliating $\Sigma$. The other elliptic system corresponding to the ‘evolutionary’ one in the present case has to be solved on the entire base manifold $M$ with boundary value yielded by the aforementioned parabolic–hyperbolic boundary-initial value problem on $\Sigma$ (see e.g., [5]).

The results covered by this paper also have applications in the conventional Cauchy problem and in the initial boundary value problem. In [5] it is demonstrated that the dynamics of four-dimensional spacetimes foliated by a two-parameter family of homologous two-surfaces can be interpreted as two-surface based ’geometrodynamics’, whereas in [6]—by making use of proposition 4.1, along with the fact that the results covered by sections 3 and 4 did not require any restriction on the signature of the metric induced on the $\Sigma$ hypersurfaces —some of the unsettled issues such as the geometric uniqueness in the metric based formulation of the initial boundary value problem will be addressed.

It is worth emphasizing that concerning the metric, only (1.1) had been used. This, besides the Riemannian or Lorentzian spaces satisfying Einstein’s equations, allows many other theories as well. In particular, our assumptions are satisfied by the ‘conformally equivalent representation’ of higher-curvature theories possessing a gravitational Lagrangian that is a polynomial of the Ricci scalar. Note also that the inclusion of metrics with Euclidean signature may significantly increase the variety of theories to be covered, although no attempt has been made here to explore these aspects.

Let us finally mention that irrespective of the simpleness of the observations made here, they may have interesting applications elsewhere. It would be useful to know whether they could be used in string and brane theories, and also in various other alternative higher dimensional Riemannian and Lorentzian metric theories of gravity.

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Appendix A

This section will provide some useful relations. These had been applied in deriving several relations in section 3, and their adopted form will also be applied in our upcoming papers [5, 6]. As our generic results are applicable in arbitrary dimension and to spaces with metrics of Euclidean or Lorentzian signature, we believe that these relations will find several applications.

It has been used implicitly in deriving (3.6) and plays some role elsewhere so it is useful to give the generic relation of the scalar curvatures which reads as

$$R = R + \epsilon \left\{ -2 L_{\alpha} (K^\alpha) - (K^\alpha)^2 - K_{\alpha}K^\alpha - \frac{2\epsilon}{N} D^\alpha D^\beta N \right\}. \quad (A.1)$$

Consider now a co-vector field $L_\alpha$ on $M$ foliated by the $\sigma = const$ hypersurfaces. Then $L_\alpha$ can be decomposed in terms of $n^\alpha$ and fields living on the $\sigma = const$ level surfaces as

Note also that the $n$-dimensional source term in (4.3) cannot be directly connected to matter fields.
\[ L_\nu = \delta^\alpha_\nu L_\alpha = (\delta^\alpha_\nu + \epsilon n^\nu n_\alpha) L_\alpha = \lambda n_\alpha + \mathbf{L}_\alpha, \]  
(A.2)

where

\[ \lambda = \epsilon n^\nu L_\nu \quad \text{and} \quad \mathbf{L}_\alpha = \delta^\alpha_\nu L_\nu. \]  
(A.3)

Making use of this decomposition the covariant derivative \( \nabla_\alpha L_\nu \) and the divergence \( \nabla^\nu L_\nu \) can be decomposed as

\[ \nabla_\alpha L_\nu = \left[ D\lambda + \epsilon n_\alpha \mathcal{L}_\alpha \lambda \right] n_\nu + \lambda \left( K_{\nu\alpha} + \epsilon n_\alpha n_\nu \right) + D\mathbf{L}_\nu - n_\nu n_\alpha \left( n^\nu L_\nu \right) + \epsilon \left\{ n_\nu \mathcal{L}_\alpha \mathbf{L}_\nu - n_\nu \mathbf{L}_\alpha K^{\nu\alpha} - n_\alpha \mathbf{L}_\alpha K^{\nu\lambda} \right\}. \]  
(A.4)

\[ \nabla^\nu L_\nu = (\delta^{\nu\nu} + \epsilon n^\nu n^\nu) \nabla_\nu L_\nu = \mathcal{L}_\lambda + \lambda \left( K^{\nu\nu} \right) + D\mathbf{L}_\nu - \epsilon \left( n^\nu L_\nu \right). \]  
(A.5)

Consider now a symmetric tensor \( P_{ab} \) defined on \( M \). Note first that \( P_{ab} \) can be decomposed in terms of \( n^\nu \) and fields living on the \( \sigma = \text{const} \) level surfaces as

\[ P_{ab} = \pi n_a n_b + \left[ n_a \mathbf{p}_b + n_b \mathbf{p}_a + \mathbf{P}_{ab} \right], \]  
(A.6)

where \( \pi = n^\nu n^\nu P_{ab} \), \( \mathbf{p}_a = \epsilon h^\nu n^\nu P_{ab} \), and \( \mathbf{P}_{ab} = \epsilon h^\nu \lambda P_{ab} \).

Then, the covariant derivative \( \nabla_\nu P_{ab} \) can be decomposed as

\[ \nabla_\nu P_{ab} = \pi \left[ \left( K_{ab} + \epsilon n_\alpha n_\beta \right) n_a + \left( K_{ab} + \epsilon n_\alpha n_\beta \right) n_b \right] + \left[ D\pi + \epsilon n_\nu \mathcal{L}_\nu \pi \right] n_a n_b + \left[ n_a D\mathbf{p}_b + n_b D\mathbf{p}_a + \left( K_{ab} + \epsilon n_\gamma n_\nu \right) \mathbf{p}_b - \left( K_{ab} + \epsilon n_\gamma n_\nu \right) \mathbf{p}_a \right] \]  
(A.7)

while the contraction \( \nabla^\nu P_{ab} \) reads as

\[ \nabla^\nu P_{ab} = \pi \left( K^{\nu\nu} \right) n_a + \left( K^{\nu\nu} \right) \mathbf{p}_b + n_b \left( D^\nu \mathbf{p}_a \right) + D^\nu \mathbf{p}_{ab} - \epsilon n_\nu \left( \mathbf{p}_a K^{\nu\gamma} \right) + n_\nu \pi + n_\nu \mathcal{L}_\nu \pi + \mathcal{L}_\nu \mathbf{p}_b - 2 \epsilon \left( n^\nu \mathbf{p}_a \right) n_b - \epsilon \left( n^\nu \mathbf{P}_{ab} \right). \]  
(A.8)

The parallel and orthogonal parts of (A.8) simplify as

\[ (\nabla^\nu P_{ab}) h^\nu_\alpha = \left( K^{\nu\nu} \right) \mathbf{p}_a + D^\nu \mathbf{p}_a + n_\nu \pi + \mathcal{L}_\nu \mathbf{p}_b - \epsilon n^\nu \mathbf{P}_{ab}, \]  
(A.9)

\[ (\nabla^\nu P_{ab}) n^\nu = \epsilon \left[ \pi \left( K^{\nu\nu} \right) + D^\nu \mathbf{p}_a - \mathbf{P}_a K^{\nu\gamma} + \mathcal{L}_\nu \pi - 2 \epsilon \mathbf{p}_a \right]. \]  
(A.10)

It also follows from (A.7) that

\[ \nabla_\nu P_\nu^{\gamma} = \epsilon \left[ D_\nu \pi + n_\nu \mathcal{L}_\nu \pi \right] + D_\nu \left( \mathbf{P}_\nu^{\gamma} \right) + \epsilon n_\nu \mathcal{L}_\nu \left( \mathbf{P}_\nu^{\gamma} \right), \]  
(A.11)

with parallel and orthogonal parts

\[ (\nabla_\nu P_\nu^{\gamma}) h_\nu^{\gamma} = \epsilon D_\nu \pi + D_\nu \left( \mathbf{P}_\nu^{\gamma} \right) \]  
(A.12)

\[ (\nabla_\nu P_\nu^{\gamma}) n^{\gamma} = \epsilon \mathcal{L}_\nu \pi + D_\nu \left( \mathbf{P}_\nu^{\gamma} \right). \]  
(A.13)
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