Bayes Pre-Test Shrinkage Estimators of Scale Parameter for Maxwell Distribution under Squared Loss Functions

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Abstract The Maxwell distribution is a lifetime model, and it is used in many applications of physics and chemistry. In this paper, we suggest and study the pre-test Shrinkage Bayesian estimators of scale parameter for Maxwell distribution. The equation of risk function and relative risk with respect classical estimator for the proposed estimators under squared error loss function (SELF). Using simulations, these estimators are compared in terms of the relative risk, which is accounted using the programming language R and the numerical results that showed performance of our estimators are compared with Bayesian classical estimator.

1. Introduction
The Maxwell distribution is a continuous probability distribution, that having a scale parameter (θ) always positive, which defines the probability density function (PDF) of the distribution. The Maxwell distribution is used in many applications in physics and chemistry and forms the basis for the theory of gas movement that represents the properties of gases, including gas pressure, diffusion and speed of particles as well as thermodynamic equilibrium. The Maxwell distribution was first described by Scottish physicist (James Clark Maxwell in 1860), and then described by (Boltzmann) with a few assumptions.

In our study, we will use different estimating methods. Firstly, maximum likelihood Estimating method (MLE) which assumes that x is a variable random sample of size (n) from the failure function distribution \( f(x, \theta) \). The \( \theta \) parameter is unknown constant. Secondly, the Bayesian method which assumes that the parameters are random variables and not constant quantities. Thirdly, the shrinkage estimator method which is used when there is availability of the previous information about the parameter which is considered as initial values correlated by \( \theta_0 \). It is called a “shrinkage estimation”. The research paper aims to estimate the scaling parameter of the Maxwell distribution using the Bayesian shrinkage estimator and to study its properties within SELF.
2. Model Description Maxwell Distribution

The PDF for the Maxwell distribution of the scaling parameter is the following:

\[ f(x, \theta) = \frac{4}{\sqrt{\pi} \theta^3} x^2 e^{-\frac{x^2}{\theta^2}} \quad x > 0, \theta > 0 \]  

(1)

While the cumulative distribution function (C.D.F) has the following form:

\[ F(X, \theta) = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_{0}^{x} e^{-u} u^{\frac{1}{2}} du \quad x > 0 \]  

(2)

Where: \( \Gamma(x, \alpha) = \int_{0}^{\infty} e^{-u} u^{\alpha-1} du \quad (x, \alpha > 0) \) is incomplete gamma function.

We can summarize the most important properties by the following table: (Wolfram Language, 2008)

**TABLE 1**, The most important characteristics of the Maxwell distribution

| Properties      | Formula |
|-----------------|---------|
| Mean            | \( 2 \sqrt{\frac{\theta}{\pi}} \) |
| Mode            | \( \sqrt{\theta} \) |
| Median          | \( \sqrt{\theta/\pi} ((\sqrt{\pi} + 4)/3) \) |
| Variance        | \( \theta^2 \frac{(3\pi - 8)}{\pi} \) |
| Skewness        | 2 \sqrt{2(5\pi - 16)} \frac{(3\pi - 8)^{\frac{3}{2}}}{(3\mu - 8)^2} |
| Kurtosis        | \( \frac{4(96 - 40\pi + 3\pi^2)}{2^{4+n} \theta^n \Gamma\left(\frac{1}{2}(3 + n)\right)} \) |
| \((n)^{th}\) moment | \( \frac{\sqrt{\pi}}{2^{4+n} \theta^n \Gamma\left(\frac{1}{2}(3 + n)\right)} > -3 \) |

3. Parameters Estimating Methods

To find the joint probability function distribution of the random variable, we use estimation parameters which is considered one of the most important branches of statistical inference because it gives a clear definition of “distributions with unknown parameters” (Mood, 1974).

3.1 Classical Methods

When information about parameters is not available, we use classical methods such as, the ordinary least squares method, moments method, Maximum likelihood estimation (MLE), etc.
3.1.1 Maximum Likelihood Estimation.

Let \( x = (x_1, x_2, \ldots, x_n) \) is a variable of a random sample of size \( n \) from the distribution of the failure function \( f(x, \theta) \) and that the parameter \( \theta \) is unknown. \( \theta \) is the estimated MLE if it is more than the likelihood function. We obtain MLE by solving the equation (Wackerly, 2014). We obtain MLE by solving the equation:

\[
L(x|\theta) = \prod_{i=1}^{n} f(x_i, \theta) = \ln L(x, \theta) = \frac{\ln L(x, \theta)}{\theta} = 0 \tag{3}
\]

Where \( L(x|\theta) \) is the probability function.

3.2 Bayesian Methods

Assuming that the parameters are random variables and are not fixed quantities that have an initial probability distribution, and do not depend on the data of the current sample, represented by the initial probability density function whose determination is important in clarifying the initial information available to the researcher about the parameters through available previous data.

3.3 Shrinkage Methods

Assuming that the previous information is initial values correlated by \( \theta^o \), when previous information is available about the parameter it is called a shrinkage estimator, and it was determined by (James R. Thompson, 1968) based on a linear combination between the classical estimate and its estimated value \( \theta^o \).

The shrinkage coefficient is defined as: a linear combination of the pre-value \( \theta^o \) and the estimated value \( \theta \) based on the shrinkage weight function which is denoted \( (k) \) (Handa, 1990). The shrinkage coefficient is found by the following equation:

\[
\theta = k\hat{\theta} + (1 - k)\theta^o \tag{4}
\]

Where \( K \) is a constant, and its value is limited to \( 0 < k < 1 \) which is the shrinkage coefficient.

When there no guarantee that the true value \( \theta \) is close to the pre-value \( \theta^o \), the researchers propose to conduct a pre-test depending on the test hypothesis of the null hypothesis \( H_o: \theta = \theta^o \) versus the alternative hypothesis \( H_1: \theta \neq \theta^o \). If the null hypothesis is accepted, our estimator becomes \( k\hat{\theta} + (1 - k)\theta^o \), otherwise the classical estimator is our estimator.

The pre-test assessor is defined as follows:

\[
\hat{\theta} = \begin{cases} 
  k\hat{\theta} + (1 - k)\theta^o, & \text{if } H_o \text{ is excepted} \\
  \hat{\theta}, & \text{otherwise}
\end{cases} \tag{5}
\]

Assuming that \( \hat{\theta} \) is an unknown estimator of \( \theta \), it can be said that \( \hat{\theta} \) is a loss function to \( \theta \) when the function has a non-negative real value and the following conditions are met. [Islam, (2011)]

\[
L(\hat{\theta}, \theta) > 0 , \forall (\hat{\theta} \neq \theta)
\]

\[
L(\hat{\theta}, \theta) = 0 , \forall (\hat{\theta} = \theta)
\]
Squared Error Loss Function.

It is one of the most important symmetric loss functions and is used by researchers for easy calculation of the BES estimator under this function, and it is determined through the following formula [Islam, (2011)]:

\[ L(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2 \] (6)

4. Literature Review

The estimation of the parameters using the shrinkage method was intensively studied by many researchers:

In (1987), Park presented a study of some of the estimators of the Bayesian shrink of the reliability function in the left truncated exponential distribution assuming the MVUE and Bayes estimators of the reliability function with the previous unknown distributions using the “Bayesian estimator” instead of the estimated value using the Monte Carlo, MVUE and MSE methods to compare the relative efficiencies of Bayesian shrinkage estimators.

Dhawiyah et al. (2008) presented a proposal to use the shrinkage estimator to estimate the shape parameter of the amputated failure samples from the two-parameters Whipple distribution in case of availability the prior estimation of scale includes the shape parameter. The shape parameter has been estimated by linear estimation and shrinkage estimator which are compared by the least MSE and the estimator relative efficiency. The study concluded that the use of shrinkage estimator in order to estimate the shape parameter is biased in comparison of other estimators and has the lowest MSE.

In (2015), researcher Sanku derived the estimator of the Bayesian shrinkage estimator of Rayleigh's coefficient and the function of associated risk based on a previous coupling under the general entropy loss function assumption for the gradual data of the second type subject to control. The risk function was derived and compared to estimate MSE, Bayes, and Bayesian Shrink.

Al Kutubi suggested in (2017), a new shrinkage determinant using the Whipple distribution by Bayes and estimating the MSE using a linear combination between the two methods, and used a simulation study to compare the two estimators, to find the best through the lowest MSE with the different sample sizes. The study conducted that the new estimators is the best for having the lowest average MSE as well as the percentage of error decreases with increasing sample size.

Hassoun and Hussein (2020) suggested using the shrink method to estimate the scale parameter of weighted Rayleigh distribution by using the classical and Bayesian estimation methods.

Al-Qatif (2020) proposed to study the Bayesian properties of a pre-test of the shrinkage estimator of the scale coefficient of the Rayleigh distribution. The equation for the relative risk and risk function was derived of the classical estimator of the proposed estimators under the SELF and the Linex loss function. The numerical results showed that the shrinkage estimator performed more than the traditional Bayesian estimator.

5. The Bayesian Estimator of the Maxwell Distribution

Let \( x = (x_1, x_2, \ldots, x_n) \) be a random sample has size \( n \) of Maxwell and it has the previous PDF for generalized uniform distribution for the following parameter: (Nadia, 2012)
\[ g(\theta) = \frac{(c-1)(\alpha \beta)^{a-1}}{\beta^{c-1} \Gamma(c-1)} \frac{1}{\theta^c}, \quad 0 < \theta \leq \beta < \infty, \alpha > 0 \quad (7) \]

Then the posterior distribution of \( \theta \) given the data \((x_1, x_2, \ldots, x_n)\) is follows

\[ \pi(\theta | x_1, x_2, \ldots, x_n) = \frac{g(\theta) \prod_{i=1}^{n} f(x_i, x_2, \ldots, x_n | \theta)}{\int_{0}^{\beta} g(\theta) \prod_{i=1}^{n} f(x_i, x_2, \ldots, x_n | \theta) d\theta} \quad (8) \]

After calculating the integrals and simplifications we get the posterior density function of \( \theta \) as the following:

\[ \pi(\theta | x_1, x_2, \ldots, x_n) = \frac{\left( \frac{3n}{2c} \right) x^{c-1} e^{-\frac{\nu x^2}{\theta}}}{\theta^{\frac{3n}{2}} r^2 \Gamma \left( \frac{3n}{2} + c - 1 \right)} \quad (9) \]

We calculate the Bayes we use SELF so the Bayesian given estimator is:

\[ \hat{\theta}_B = E(\theta) = \int_{0}^{\beta} \theta \pi(\theta | x) \, d\theta \]

\[ \hat{\theta}_B = \int_{0}^{\beta} x^{c-1} e^{-\frac{\nu x^2}{\theta}} \, d\theta \]

Using the transformations method, we assume that: \( \theta = \frac{\nu x^2}{y}, y = \frac{\nu x^2}{\theta}, \frac{d\theta}{dy} = -\frac{\nu x^2}{y^2} \, dy = |J| \)

After calculating the integrals and simplifications we get:

\[ \hat{\theta}_B = \frac{\nu x^2}{2c-2} \quad (10) \]

supposing that \( z = \frac{3n}{2} + c - 2 \)

\[ \theta = \frac{\nu x^2}{z} \quad (11) \]

If \( c = 2 \) then we obtain the Maximum likelihood estimation of \( \theta \):

\[ \theta = \frac{2 \nu x^2}{3n} \quad (12) \]

To find the distribution of the PDF for the estimator \( \hat{\theta}_B \)

The researcher uses the method of transfers, supposes that

\[ \frac{dx}{dt} = \frac{1}{2\sqrt{t}} \, dt, \quad t > 0 \Rightarrow x = \sqrt{t} \]

\[ h(t, \theta) = \frac{4 t}{\sqrt{n} \theta^2} \frac{\nu x^2}{\theta} \frac{1}{2\sqrt{t}} \]

\[ h(t, \theta) = \left( \frac{2 \sqrt{t} \theta^{\frac{3n}{2}} \nu x^2}{\theta^2} \right) \sim Gamma \left( \frac{3n}{2}, \frac{1}{\theta} \right) \quad (13) \]
Therefore, the probability function

\[ f(T, \theta) = \frac{T^{3n-1} e^{-\frac{T}{\theta}}}{\Gamma \left( \frac{3n}{2} \right) \theta^{3n}} \quad t > 0, \theta > 0 \]  

(14)

The researcher uses the transformations method to find the parameters of the distribution, supposes that

\[ K = \frac{2T}{\theta} \Rightarrow T = \frac{K0}{2} \cdot \theta \]

\[ f(k, \theta) = \frac{\left(\frac{k0}{2}\right)^{2n-1} e^{-\frac{k0}{2\theta}}}{\Gamma \left( \frac{3n}{2} \right) \theta^{3n}} \]

\[ f \left( k = \frac{2T}{\theta} \right) \sim \chi^2 \frac{2n-1}{2} - \chi^2 \frac{3n}{2} \]  

(15)

To find the PDF \( f(\theta_{SB}, \theta) \), the transformations method from equations (14) and (11) is used. We get the following results

\[ \theta_B = \frac{T}{z} \]  

(16)

Using the method of transformations from equation (16), we get \( T = \theta_B \cdot z \), \( \frac{\partial T}{\partial \theta_B} = z = \| \) So the density function is as follows

\[ f(\theta_{BS}, \theta) = \begin{cases} \frac{z(\theta_B)^{2n-1} e^{-\frac{\theta_B}{\theta}}}{\Gamma \left( \frac{3n}{2} \right) \theta^{3n}}, & \theta_B > 0 \\ 0, & \text{otherwise} \end{cases} \]  

(17)

6. Proposed Pre-Test Shrinkage Estimators

Assuming that the previous information about an unknown parameter \( \theta \) as an estimate value of \( \theta_o \), it is important to use it in our estimator and depending on the PDF \( \theta_{BS} \) and \( T \) above, we suggest the following pre-test estimators. The first suggested estimator \( \theta_{BS} \):

\[ \theta_{BS} = \left\{ \begin{array}{ll} k_1 \theta_B + (1 - k_1) \theta_o & \text{if } H_0 \text{-excepted} \\
\theta_B & \text{otherwise} \end{array} \right. \]  

(18)

Where \( k_1 \) is the shrink factor and its value is \( 0 < k_1 < 1 \).

And R is the pre-test area for testing the null hypothesis when \( H_0: \theta = \theta_o \), against the alternative hypothesis \( H_1: \theta \neq \theta_o \) at the level of \( \alpha \) significance.

The proposed second estimator was identified for(\( \theta_{B1} \)) as the following:

\[ \theta_{B1} = \left\{ \begin{array}{ll} k_2 \theta_B + (1 - k_2) \theta_o & \text{if } H_0 \text{-excepted} \\
\theta_B & \text{otherwise} \end{array} \right. \]  

(19)
Where $k_1$ is the shrinkage coefficient depending on the statistical test of the null hypothesis when $H_0: \theta = \theta_0$, against the alternative hypothesis $H_1: \theta \neq \theta_0$ at the level of $\alpha$ significance.

When $\theta = \theta_0$, then the acceptance region of $H_0$ is the following:

$$\frac{\theta_0}{2} \leq \chi^2 \leq \frac{2 \theta_0}{ x_{3n - a}^2}$$

They are the limits of $\chi^2$ at a degree of freedom $3n$, so:

$$r_1 \leq T \leq r_2$$

$$r_1 = \frac{\theta_0}{2} \chi^2 \leq r_2 = \frac{\theta_0}{2} \chi^2$$

Then

$$r_1 \leq T \leq r_2$$

$$k_2 = \frac{T - r_1}{r_2 - r_1}$$

We can find the terms defined by the region $R$ as following:

$$C = \left[ \frac{\theta_0}{2}, \frac{\theta_0}{2} \chi^2 \right]$$

7. Risk Function of Shrinking Estimators $\tilde{\theta}_{B1}$ Proposed For Pretesting of Self

$$R(\tilde{\theta}_{B1}, \text{SELF}) = E(\tilde{\theta}_{B1} - \theta)^2$$

(20)

$$E(\tilde{\theta}_{B1}) = \int z(k_1 \tilde{\theta}_{B1} + (1 - k_1)\theta_0)f(\tilde{\theta}_{B1})d\tilde{\theta}_{B1} + \int \tilde{\theta}_{B1} f(\tilde{\theta}_{B1})d\tilde{\theta}_{B1}$$

$$E(\tilde{\theta}_{B1}) = \int z(k_1 \tilde{\theta}_{B1} + (1 - k_1)\theta_0) \frac{(\theta_{B2})^{2n-1} exp \frac{\theta_{B2}}{r(\frac{3n}{2})\theta^{2n}}}{\tilde{\theta}_{B1}}d\tilde{\theta}_{B1} + \int \tilde{\theta}_{B1} \frac{(\theta_{B2})^{2n-1} exp \frac{\theta_{B2}}{r(\frac{3n}{2})\theta^{2n}}}{\tilde{\theta}_{B1}}d\tilde{\theta}_{B1}$$

$$E(\tilde{\theta}_{B1}) = \int z(k_1 \tilde{\theta}_{B1} + (1 - k_1)\theta_0) \frac{(\theta_{B2})^{2n-1} exp \frac{\theta_{B2}}{r(\frac{3n}{2})\theta^{2n}}}{\tilde{\theta}_{B1}}d\tilde{\theta}_{B1} + \int \tilde{\theta}_{B1} \frac{(\theta_{B2})^{2n-1} exp \frac{\theta_{B2}}{r(\frac{3n}{2})\theta^{2n}}}{\tilde{\theta}_{B1}}d\tilde{\theta}_{B1}$$

$$f(\tilde{\theta}_{B1}) = \int \frac{\theta_{B2}}{\theta_0} \frac{(\theta_{B2})^{2n-1} exp \frac{\theta_{B2}}{r(\frac{3n}{2})\theta^{2n}}}{\tilde{\theta}_{B1}}d\tilde{\theta}_{B1} + \int \tilde{\theta}_{B1} \frac{(\theta_{B2})^{2n-1} exp \frac{\theta_{B2}}{r(\frac{3n}{2})\theta^{2n}}}{\tilde{\theta}_{B1}}d\tilde{\theta}_{B1}$$
Using the transformations method, suppose that 

\[
E(\tilde{\theta}_B) = \frac{z}{I\left(\frac{3n}{2}\right)\sigma^{2n}} \int_{a}^{b} (k_1 + (1 - k_1)\theta_o)\int_{y} \frac{2^{n-1}}{\sigma} \exp\left(-\frac{y\theta}{\sigma}\right) dy + \frac{z}{I\left(\frac{3n}{2}\right)\sigma^{2n}} \int_{\theta}^\infty \frac{y\theta}{\sigma} \int_{y} \frac{2^{n-1}}{\sigma} \exp\left(-\frac{y\theta}{\sigma}\right) dy - \int_{a}^{b} \frac{z}{I\left(\frac{3n}{2}\right)\sigma^{2n}} \int_{y} \frac{2^{n-1}}{\sigma} \exp\left(-\frac{y\theta}{\sigma}\right) dy
\]

After calculating the integrals and simplifications, we get

\[
E(\tilde{\theta}_B) = k_1 J_1(a^*, b^*) + \theta_o J_0(a^*, b^*) - k_1 J_1(a^*, b^*) + \frac{\theta}{\sigma I\left(\frac{3n}{2}\right)} \Gamma\left(\frac{3}{2}n + 1\right) - \frac{\theta}{\sigma I\left(\frac{3n}{2}\right)} J_1(a^*, b^*)
\]

(22)

\[
J_m(a^*, b^*) = \frac{1}{I\left(\frac{3n}{2}\right)} \int_{a}^{b} (j) \tau^{n-1} \exp\left(-\frac{\lambda}{\tau}\right) \tau d\tau
\]

, \hspace{1em} m = 0, 1 (23)

\[
a^* = \frac{1}{2} \chi^2 3n, b^* = \frac{1}{2} \chi^2 3n + \frac{a}{z}
\]

\[
\lambda = \frac{\theta}{\sigma} \chi^2 3n - \frac{1}{2} \chi^2 3n + \frac{a}{z}
\]

Are respectively lower and higher \(\frac{\theta}{\sigma}\) the percentage value of the chi-square distribution with 3n degrees of freedom.

When accepting the null hypothesis \(H_0: \theta = \theta_o\) against \(H_1: \theta \neq \theta_o\), we also calculate

\[
E(\tilde{\theta}_B) = \int_{k} z(k_1 \theta_B + (1 - k_1)\theta_o)^n \frac{\theta}{\sigma I\left(\frac{3n}{2}\right)} \frac{\partial^2 \theta}{\partial \theta^2} \exp\left(-\frac{\lambda}{\tau}\right) \tau d\tau + \int_{0}^{\infty} \frac{z(\theta_B)^n \exp\left(-\frac{\lambda}{\tau}\right) \tau d\tau}{I\left(\frac{3n}{2}\right)}
\]

(24)

\[
E(\tilde{\theta}_B) = \int_{a}^{b} \frac{z}{I\left(\frac{3n}{2}\right)} \frac{\partial^2 \theta}{\partial \theta^2} \exp\left(-\frac{\lambda}{\tau}\right) \tau d\tau + \int_{0}^{\infty} \frac{z(\theta_B)^n \exp\left(-\frac{\lambda}{\tau}\right) \tau d\tau}{I\left(\frac{3n}{2}\right)}
\]

Using the same transformation and previous assumptions for \(E(\tilde{\theta}_B)\)

\[
= \frac{1}{I\left(\frac{3n}{2}\right)} \int_{a}^{b} \frac{k_1^2 y z^{2n+1}}{z} + \frac{2k_1 \theta_o y z^{2n+1}}{z} + \theta_o y z^{2n+1} - 2k_1 \theta_o y z^{2n+1} + k_1^2 \theta o \left(y z^{2n+1}\right) \exp\left(-\frac{\lambda}{\tau}\right) \tau d\tau + \frac{1}{I\left(\frac{3n}{2}\right)} \int_{a}^{b} \frac{k_1^2 \theta o \left(y z^{2n+1}\right) \exp\left(-\frac{\lambda}{\tau}\right) \tau d\tau}{z^2 I\left(\frac{3n}{2}\right)}
\]

And using a simplification, we get

\[
E(\tilde{\theta}_B) = k_1 \left[\frac{2}{z} J_2(a^*, b^*) + \theta_o J_0(a^*, b^*) - \frac{2\theta o J_1(a^*, b^*)}{z}\right] + 2k_1 \left[\frac{\theta o J_1(a^*, b^*)}{z}\right] - \theta_o J_0(a^*, b^*) - \frac{\theta o J_1(a^*, b^*)}{z^2} J_2(a^*, b^*) + \frac{\theta o}{z^2} J_2(2n+1) + \theta o J_0(a^*, b^*) - \frac{\theta o}{z^2} J_2(a^*, b^*) + \frac{\theta o}{z^2} J_2(2n+1)
\]

(24)

We substitute equations (24) and (22) with equation (21)
8. Risk Function of the Shrinkage Estimators $\hat{\theta}_{B2}$ Proposed for Pre-testing of the Self

$$R(\hat{\theta}_{B1}\mid\text{SELF}) = \theta^2\left\{k_1^2\left[\frac{1}{z^2} f_2(a^*, b^*) + \lambda^2 f_6(a^*, b^*) - \frac{2z}{z^2} f_1(a^*, b^*)\right] + 2k_1\left[\frac{1}{\lambda^2}(\lambda - 1)f_1(a^*, b^*) - \lambda(\lambda - 1)\right]\right\}$$

$$R(\hat{\theta}_{B2}\mid\text{SELF}) = \theta(\hat{\theta}_{B2} - \theta)^2$$

After calculating the integrals and simplifications, we get

$$\int (\hat{\theta}_{B2} - \theta)^2 f(\hat{\theta}_{B2})d\hat{\theta}_{B2} = \frac{1}{(r_2-r_1)^2}\int_c (\hat{\theta}_{B2} - \theta)^2 f(\hat{\theta}_{B2})d\hat{\theta}_{B2} - 2r_1\int_c (\hat{\theta}_{B2} - \theta)^2 f(\hat{\theta}_{B2})d\hat{\theta}_{B2} + r_1^2\int_c (\hat{\theta}_{B2} - \theta)^2 f(\hat{\theta}_{B2})d\hat{\theta}_{B2} + \int_c (\hat{\theta}_{B2} - \theta)^2 f(\hat{\theta}_{B2})d\hat{\theta}_{B2}$$

Using the same previous transformation and assumptions, we get

$$\int (\hat{\theta}_{B2} - \theta)^2 f(\hat{\theta}_{B2})d\hat{\theta}_{B2} = \frac{1}{(r_2-r_1)^2}\int_c (\hat{\theta}_{B2} - \theta)^2 f(\hat{\theta}_{B2})d\hat{\theta}_{B2} - 2r_1\int_c (\hat{\theta}_{B2} - \theta)^2 f(\hat{\theta}_{B2})d\hat{\theta}_{B2} + r_1^2\int_c (\hat{\theta}_{B2} - \theta)^2 f(\hat{\theta}_{B2})d\hat{\theta}_{B2} + \int_c (\hat{\theta}_{B2} - \theta)^2 f(\hat{\theta}_{B2})d\hat{\theta}_{B2}$$

After calculating the integrals and simplifications, we get

$$R(\hat{\theta}_{B2}\mid\text{SELF}) = \theta^2\left[\frac{1}{(r_2-r_1)^2} z^2 f_2(a^*, b^*) - \frac{2}{(r_2-r_1)^2} z^2 f_6(a^*, b^*) - \frac{2}{(r_2-r_1)^2} z^2 f_1(a^*, b^*)\right] + 2r_1\int_c (\hat{\theta}_{B2} - \theta)^2 f(\hat{\theta}_{B2})d\hat{\theta}_{B2} + r_1^2\int_c (\hat{\theta}_{B2} - \theta)^2 f(\hat{\theta}_{B2})d\hat{\theta}_{B2} + \int_c (\hat{\theta}_{B2} - \theta)^2 f(\hat{\theta}_{B2})d\hat{\theta}_{B2}$$

9. Relative Risk

To compare the proposed estimates with respect to the Bayes $\hat{\theta}$ estimator, we use the relative risk. Therefore, we must find the risk function of $\hat{\theta}_{B}$ in terms of SELF as following:

$$E(\hat{\theta}_{B}) = \int_0^\infty \hat{\theta} f(\hat{\theta}_{B})d\hat{\theta}_{B} = \int_0^\infty \theta(\hat{\theta}_{B})\frac{2^{\frac{1}{2}} e^{-\frac{1}{2}\theta_{B}\gamma}}{\Gamma\left(\frac{3n}{2}\right)} d\theta_{B}$$

After using the transformation $\gamma = \frac{\hat{\theta}_{B}}{\theta}$ we get

$$E(\hat{\theta}_{B}) = \frac{3n\theta}{2z}$$

$$E(\hat{\theta}_{B}) = \frac{3n\theta}{2z}$$

(27)
Using the same previous transformation and assumptions for $E(\theta)$, we get the following:

$$E(\theta)^2 = \int_0^\infty (\theta)^2 f(\theta) d\theta = \int_0^\infty (\theta)^2 \frac{e^{-\theta}}{\Gamma(n) \theta^n} d\theta$$

Using the same previous transformation and assumptions for $E(\hat{\theta})$, we get the following:

$$E(\hat{\theta})^2 = \frac{z}{\alpha^2 \Gamma\left(\frac{3n}{\alpha}\right)} \int_0^\infty \frac{(\gamma \theta)^{2n-2} e{-\gamma \theta}}{z^2} dy$$

$$E(\hat{\theta})^2 = \frac{\theta^2 (3n/\alpha + 1)}{z^2}$$

By using the equation (27) and (28), we get the following:

$$R(\hat{\theta}_B \backslash SELF) = \theta^2 \left(\frac{3n/\alpha + 1}{z^2} - \frac{3n}{z} + 1\right)$$

Therefore, the relative risk can be evaluated with respect to the Bayesian estimator $\hat{\theta}_B$ of the proposed pre-test reducer estimator with respect to the Bayesian estimator $\hat{\theta}_B$, denoted by $R(\cdot)$ of $\hat{\theta}_{BS1}$; $\hat{\theta}_{BS2}$, for SELF as the following:

$$R(\hat{\theta}_{BS1} \backslash SELF) = \frac{R(\hat{\theta}_B \backslash SELF)}{R(\hat{\theta}_{BS1} \backslash SELF)}$$

$$R(\hat{\theta}_{BS2} \backslash SELF) = \frac{R(\hat{\theta}_B \backslash SELF)}{R(\hat{\theta}_{BS2} \backslash SELF)}$$

10. Conclusion

According to what was previously studied, the researcher made the simulation, assuming the following values of the equations

- $n=5, 15, 25, 35, 45$
- $\lambda = 0.3, 0.6, 0.9, 1, 1.2, 1.6, 1.9$
- $\alpha = 0.01, 0.05$
- $\kappa = 0.01, 0.5, 0.9$
- $c = 1, 2, 3$
- $\theta_o = 1$

Using the R program, we get the following results:

i. The proposed estimates $\hat{\theta}_{BS1}$ and $\hat{\theta}_{BS2}$ give a high relative risk under the squared error loss function with respect to the $\hat{\theta}_B$ estimator when is equal to one, that is, the true value is equal to the pre-value.

ii. The relative risk of the first estimator $\hat{\theta}_{BS1}$ with respect to the $\hat{\theta}_B$ estimator is considered better than the second estimator $\hat{\theta}_{BS2}$ under the squared error loss function, as shown in Fig. (3), (4).

iii. The relative risk increases with $c$ for the estimator $\hat{\theta}_B$ under the quadratic error loss function, as shown in Fig. (5), with respect to the estimator of $\hat{\theta}_B$, except for the estimator $\hat{\theta}_{BS2}$ under SELF with respect to the estimator $\hat{\theta}_B$ has high relative risks at $c = 1$ and relative risks low at $c = 2$ as shown in Figure (1).

iv. The relative risk of estimator $\hat{\theta}_B$ under SELF with respect to the estimator of Bayes $\hat{\theta}_B$, increases with $n$, as shown in Figure (5), except for the relative risk of estimator $\hat{\theta}_{BS1}$ under SELF with respect to Bayes estimator $\hat{\theta}_B$ decreases with $n$ as in Figure (2).
FIGURE 1, $(\theta_{BS1,\text{SELF}})_{k=0.1, \alpha=0.01, n=5}$

FIGURE 2, $R(\theta_{BS1}\text{SELF})_{k=0.1, \alpha=0.01, c=1}$

FIGURE 4, $R(\theta_{BS1,2}\text{SELF})_{k=0.1, \alpha=0.01, n=5}$

FIGURE 3, $R(\theta_{BS1,2}\text{SELF})_{k=0.1, \alpha=0.01, n=5}$

FIGURE 6, $R(\theta_{BS2}\text{SELF})_{k=0.1, \alpha=0.01, c=1}$

FIGURE 5, $R(\theta_{BS2}\text{SELF})_{k=0.1, \alpha=0.01, n=5}$
### Table No. (2) The Relative Risk of Function $SELF\theta_{BS_1}$

| $n=5,k=0.1$ | $n=5,k=0.5$ | $n=5,k=0.9$ | $n=15,k=0.1$ | $n=15,k=0.5$ | $n=15,k=0.9$ |
|-------------|-------------|-------------|---------------|---------------|---------------|
| $c$ | $\alpha=0.05$ | $\alpha=0.01$ | $\alpha=0.05$ | $\alpha=0.01$ | $\alpha=0.05$ | $\alpha=0.01$ | $\alpha=0.05$ | $\alpha=0.01$ | $\alpha=0.05$ | $\alpha=0.01$ |
| 0.3 | 0.912935 | 0.8072707 | 0.954093 | 0.897361 | 0.991424 | 0.981138 | 0.999009 | 0.9944922 | 0.9994723 | 0.9971159 | 0.9998991 | 0.999459 |
| 0.6 | 0.796698 | 0.8239003 | 0.953533 | 1.082206 | 1.00894 | 1.057232 | 0.650302 | 0.4912436 | 0.8023291 | 0.7048851 | 0.9626502 | 0.950213 |
| 0.9 | 2.247069 | 4.903494 | 1.836396 | 2.818926 | 1.137033 | 1.204312 | 1.69592 | 2.851625 | 1.622465 | 2.409439 | 1.120909 | 1.193089 |
| 1 | 3.434529 | 10.08351 | 2.159825 | 3.149067 | 1.157461 | 1.209024 | 3.470348 | 10.44959 | 2.170497 | 3.175386 | 1.158234 | 1.209999 |
| 1.2 | 2.890297 | 4.186279 | 1.998545 | 2.437204 | 1.14572 | 1.17828 | 1.167888 | 1.305654 | 1.329951 | 1.587327 | 1.090827 | 1.138943 |
| 1.6 | 0.684138 | 0.6428494 | 0.944333 | 0.997789 | 1.02851 | 1.064786 | 0.31834 | 0.2139693 | 0.5220954 | 0.4244767 | 0.8863508 | 0.881708 |
| 1 | 1.9 | 0.383912 | 0.313216 | 0.622455 | 0.592507 | 0.939576 | 0.9694934 | 0.351161 | 0.1640667 | 0.5421601 | 0.3229108 | 0.8830105 | 0.785785 |
| 0.3 | 0.904959 | 0.776038 | 0.948221 | 0.873593 | 0.989929 | 0.9748367 | 0.999021 | 0.9943583 | 0.9994754 | 0.9970207 | 0.9998989 | 0.999436 |
| 0.6 | 0.763946 | 0.7166432 | 0.907775 | 0.943745 | 0.993734 | 1.020501 | 0.651075 | 0.4763653 | 0.7968008 | 0.6780498 | 0.9592888 | 0.936513 |
| 0.9 | 2.222619 | 4.346027 | 1.760656 | 2.539545 | 1.12536 | 1.186164 | 1.717873 | 2.681659 | 1.578073 | 2.197103 | 1.111228 | 1.174074 |
| 1 | 3.309373 | 9.68002 | 2.1216 | 3.118315 | 1.154637 | 1.207865 | 3.429092 | 10.32053 | 2.158173 | 3.16627 | 1.157341 | 1.209663 |
| 1.2 | 2.149058 | 3.197055 | 1.96964 | 2.657909 | 1.159431 | 1.209408 | 1.012525 | 1.172811 | 1.313193 | 1.658294 | 1.103783 | 1.171345 |
| 1.6 | 0.459636 | 0.4449956 | 0.816571 | 0.937353 | 1.042506 | 1.125931 | 0.274405 | 0.186997 | 0.4827445 | 0.4032268 | 0.8812199 | 0.901777 |
| 2 | 1.9 | 0.255752 | 0.213613 | 0.504814 | 0.507692 | 0.935424 | 0.102276 | 0.307546 | 0.142953 | 0.501852 | 0.293841 | 0.8719336 | 0.784556 |
| 0.3 | 0.92553 | 0.7988337 | 0.959138 | 0.885075 | 0.991964 | 0.9766018 | 0.999131 | 0.9947608 | 0.9995316 | 0.9972136 | 0.9999092 | 0.999468 |
| 0.6 | 0.854828 | 0.7503089 | 0.949893 | 0.923084 | 0.997775 | 1.003659 | 0.680615 | 0.4990295 | 0.813322 | 0.6804308 | 0.9618436 | 0.932274 |
| 0.9 | 2.681396 | 4.665101 | 1.851359 | 2.415253 | 1.129011 | 1.17246 | 1.867746 | 2.713944 | 1.592065 | 2.067342 | 1.107582 | 1.157869 |
| 1 | 3.904896 | 11.38764 | 2.291332 | 3.236766 | 1.16655 | 1.212218 | 3.613795 | 10.92438 | 2.217198 | 3.207478 | 1.161541 | 1.211169 |
| 1.2 | 2.035044 | 3.145924 | 2.243775 | 3.4011 | 1.197916 | 1.260397 | 0.95651 | 1.149232 | 1.373004 | 1.914975 | 1.128312 | 1.214612 |
| 1.6 | 0.406889 | 0.405558 | 0.867603 | 1.108827 | 1.112253 | 1.253319 | 0.257012 | 0.1785113 | 0.4763756 | 0.4134294 | 0.8940815 | 0.946794 |
| 3 | 1.9 | 0.224696 | 0.1916679 | 0.50904 | 0.551133 | 0.998058 | 1.166613 | 0.289305 | 0.1357323 | 0.4897649 | 0.2966591 | 0.8743038 | 0.885896 |
| c  | λ   | α=0.05 | α=0.01 | α=0.05 | α=0.01 | α=0.05 | α=0.01 | α=0.05 | α=0.01 |
|----|-----|---------|---------|---------|---------|---------|---------|---------|---------|
| 1  | 0.3 | 0.9997  | 1       | 0.9998  | 1       | 1       | 1       | 1       | 1       |
|    | 0.6 | 0.74618 | 0.52862 | 0.85746 | 0.71124 | 0.9721  | 0.94045 | 0.958448 | 0.985664 |
|    | 0.9 | 1.37957 | 2.00807 | 1.45675 | 2.08013 | 1.10478 | 1.17914 | 1.16499  | 1.089658 |
|    |    | 1.27303 | 3.18146 | 1.15842 | 1.21022 | 1.06706 | 1.10535 | 2.17416  | 3.18415  |
| 1  | 0.958247 | 0.93101 | 0.89831 | 0.81849 | 0.958854 | 0.993972 | 0.96613  | 0.7663  | 0.99379  | 0.95389 |
| 0.3 | 0.9996 | 0.52422 | 0.85771 | 0.70201 | 0.97152 | 0.9355 | 1.47329 | 1.081709 | 1.30499  | 1.71005  | 1.08171  | 1.14373 |
|    | 1.0529 | 1.3165 | 1.5789 | 1.21003 | 1.21018 | 1.21018 | 1.21018 | 1.21018  | 1.21018  |
|    | 1.2 | 0.69725 | 0.72835 | 1.01235 | 1.2288 | 1.05624 | 1.13112 | 0.8278491 | 0.927175 | 0.84764  | 0.97195  | 0.10762  | 0.91975 |
|    | 1.6 | 0.3557 | 0.17338 | 0.53233 | 0.34729 | 0.88373 | 0.8183 | 0.9519111 | 0.9935  | 0.96301  | 0.7539  | 0.99351  | 0.95191 |
| 2  | 1.9 | 0.65305 | 0.26867 | 0.80062 | 0.45967 | 0.9610 | 0.85636 | 1       | 1       | 1       | 1       | 1       | 1       |
|    | 0.3 | 0.9997 | 0.99998 | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
|    | 0.6 | 0.76644 | 0.53579 | 0.86657 | 0.70698 | 0.97302 | 0.935  | 1.131932 | 1.077072 | 1.30628  | 1.6302  | 1.077072 | 1.13193 |
|    | 0.9 | 1.48598 | 1.94231 | 1.42928 | 1.82082 | 1.09147 | 1.14471 | 1.210836 | 1.159953 | 1.21935  | 1.31829 | 1.159953 | 1.21884 |
|    | 1.2 | 0.6652 | 0.71507 | 0.10371 | 1.33195 | 1.07408 | 1.16817 | 0.833721 | 0.926662 | 0.66219  | 0.39366 | 0.92666  | 0.83372 |
|    | 1.6 | 0.32059 | 0.16723 | 0.52302 | 0.34733 | 0.8855 | 0.83448 | 0.9519253 | 0.993327 | 0.96134  | 0.74917 | 0.99333  | 0.95193 |
| 3  | 1.9 | 0.61375 | 0.25946 | 0.79274 | 0.45415 | 0.96029 | 0.85937 | 1       | 1       | 1       | 1       | 1       | 1       |

| c  | λ   | α=0.05 | α=0.01 | α=0.05 | α=0.01 | α=0.05 | α=0.01 |
|----|-----|---------|---------|---------|---------|---------|---------|
| 1  | 0.3 | 0.937038 | 0.7803087 | 0.9669081 | 0.8791796 | 0.9938146 | 0.9768611 |
|    | 0.6 | 1.030248 | 1.275596 | 1.225255 | 1.633903 | 1.075587 | 1.150988 |
|    | 0.9 | 3.484893 | 10.59814 | 2.174798 | 3.185665 | 1.158544 | 1.210376 |
|    | 1.2 | 0.508274 | 0.4535833 | 0.7603482 | 0.790264 | 0.9817578 | 1.033715 |
|    | 1.6 | 0.9911799 | 0.8864533 | 0.9958663 | 0.9430018 | 0.9992679 | 0.9901529 |
| 1  | 0.3 | 0.937038 | 0.7803087 | 0.9669081 | 0.8791796 | 0.9938146 | 0.9768611 |
|    | 0.6 | 1.030248 | 1.275596 | 1.225255 | 1.633903 | 1.075587 | 1.150988 |
|    | 0.9 | 3.484893 | 10.59814 | 2.174798 | 3.185665 | 1.158544 | 1.210376 |
|    | 1.2 | 0.508274 | 0.4535833 | 0.7603482 | 0.790264 | 0.9817578 | 1.033715 |
|    | 1.6 | 0.9911799 | 0.8864533 | 0.9958663 | 0.9430018 | 0.9992679 | 0.9901529 |
### Table 3, the Relative risk of a function \( \hat{\theta}_{B52} \):

| c | n=5  | n=15 | n=25 | n=35 | n=45 |
|---|------|------|------|------|------|
|   | \( \alpha=0.05 \) | \( \alpha=0.01 \) | \( \alpha=0.05 \) | \( \alpha=0.01 \) | \( \alpha=0.05 \) | \( \alpha=0.01 \) | \( \alpha=0.05 \) | \( \alpha=0.01 \) |
| 1 | 0.9848098 | 0.962829 | 0.9990007 | 0.999494 | 0.999997 | 0.999997 | 1 | 1 | 1 |
| 0.3 | 0.9810875 | 1.015483 | 0.9322427 | 0.87769 | 0.9612831 | 0.9078806 | 0.983293 | 0.949295 | 0.993711 | 0.976516 |
| 0.6 | 1.42227 | 1.866099 | 1.370886 | 1.68069 | 1.304396 | 1.535792 | 1.243906 | 1.422427 | 1.192 | 1.332996 |
| 0.9 | 1.590444 | 2.186297 | 1.655564 | 2.24552 | 1.687706 | 2.286132 | 1.707211 | 2.312284 | 1.72607 | 2.330835 |
| 1 | 1.479873 | 1.903887 | 0.9557091 | 1.10555 | 0.7135737 | 0.7626251 | 0.583202 | 0.580103 | 0.50579 | 0.47078 |
| 1.2 | 0.621648 | 0.602083 | 0.3135849 | 0.21197 | 0.3582548 | 0.183153 | 0.499542 | 0.221258 | 0.68594 | 0.310244 |
| 1.6 | 0.3609296 | 0.303552 | 0.3397093 | 0.15629 | 0.6679662 | 0.2744078 | 0.928345 | 0.580143 | 0.99103 | 0.876723 |
| 1.9 | 0.9851461 | 0.956099 | 0.9990004 | 0.999484 | 0.999997 | 0.999997 | 1 | 1 | 1 | 1 |
| 0.3 | 1.007108 | 0.997112 | 0.937643 | 0.87179 | 0.9631629 | 0.9062334 | 0.98396 | 0.9498069 | 0.99394 | 0.976311 |
| 0.6 | 1.625637 | 2.112408 | 1.490189 | 1.79311 | 1.384099 | 1.597871 | 1.302339 | 1.460782 | 1.23667 | 1.35795 |
| 0.9 | 1.840688 | 2.62706 | 1.85122 | 2.59211 | 1.851536 | 2.577306 | 1.851156 | 2.568551 | 1.85071 | 2.562565 |
| 1 | 1.466882 | 1.983138 | 0.916777 | 1.10609 | 0.6812914 | 0.7508724 | 0.557467 | 0.567814 | 0.48466 | 0.45984 |
| 1.2 | 0.4544051 | 0.449 | 0.272544 | 0.18666 | 0.329395 | 0.168293 | 0.473621 | 0.470575 | 0.66863 | 0.297598 |
| 1.6 | 0.2533188 | 0.21187 | 0.2972959 | 0.13604 | 0.6378259 | 0.2533423 | 0.921043 | 0.559357 | 0.99021 | 0.86883 |
| 1.9 | 0.9915632 | 0.965559 | 0.9991797 | 0.99953 | 0.9999998 | 0.999997 | 1 | 1 | 1 | 1 |
| 0.3 | 1.097464 | 1.05946 | 0.9534842 | 0.88323 | 0.9679227 | 0.9190296 | 0.985448 | 0.95101 | 0.99459 | 0.77145 |
| 0.6 | 2.14717 | 2.750272 | 1.68886 | 1.98486 | 1.503046 | 1.69674 | 1.383701 | 1.521593 | 1.29708 | 1.398903 |
| 0.9 | 2.568555 | 3.89144 | 2.199637 | 3.20283 | 2.105457 | 3.024023 | 2.058782 | 2.934663 | 2.02992 | 2.879121 |
| 1 | 1.78309 | 2.626201 | 0.9454639 | 1.19932 | 0.6804884 | 0.7762343 | 0.550636 | 0.575688 | 0.47648 | 0.461641 |
| 1.2 | 0.4266314 | 0.432701 | 0.257045 | 0.17937 | 0.3141341 | 0.1623984 | 0.459342 | 0.201389 | 0.65523 | 0.293042 |
| 1.6 | 0.2272161 | 0.194022 | 0.2795726 | 0.129099 | 0.6216752 | 0.2442079 | 0.916573 | 0.548824 | 0.98986 | 0.864382 |
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