STRUCTURE OF WORDS WITH SHORT 2-LENGTH IN A FREE PRODUCT OF GROUPS.

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Abstract. Howie and Duncan observed that a word in a free product with length at least two and which is not a proper power can be decomposed as a product of two cyclic subwords each of which is uniquely positioned. Using this property, they proved various important results about one-relator product of groups. In this paper, we show that similar results hold in a more general setting where we allow elements of order two.

1. INTRODUCTION

Let \( R \) be a cyclically reduced word which is not a proper power and has length at least two in the free group \( F = F(X) \). In \([5]\), Weinbaum showed that some cyclic conjugate of \( R \) has a decomposition of the form \( UV \), where \( U \) and \( V \) are non-empty cyclic subwords of \( R \), each of which is uniquely positioned in \( R \) i.e. occurs exactly once as a cyclic subword of \( R \). Weinbaum also conjectured that \( U \) and \( V \) can be chosen so that neither is a cyclic subword of \( R^{-1} \). A stronger version of his conjecture was proved by Duncan and Howie \([7]\). In this paper, a cyclic subword is uniquely positioned if it is non-empty, occurs exactly once as a subword of \( R \) and does not occur as a subword of \( R^{-1} \).

From now on \( R \) is a word in the free product of groups \( G_1 \) and \( G_2 \), which is not a proper power and has length at least two. Before we can continue, we need to define the notion of \( n \)-length of a word. We do this in the special case when \( n = 2 \) and the word is \( R \), but of course the definition can be generalised for any \( n > 1 \) and any word in a group.

For each element \( a \) of order 2 involved in \( R \), let \( D(a) \) denote its number of occurrence in \( R \). In other words suppose \( R \) has free product length of \( 2k \) for some \( k > 0 \). Then, \( R \) has an expression of the form

\[
R = \prod_{i=1}^{k} a_ib_i,
\]

with \( a_i \in G_1 \) and \( b_i \in G_2 \). If \( a^2 = 1 \), then we define \( D(a) \) to be the cardinality of the set \( \{i \in \{1,2,\ldots,k\} \mid a_i = a\} \). Denote by \( S_R \) the symmetrized closure of \( R \) in \( G_1 \ast G_2 \) i.e. the smallest subset of \( G_1 \ast G_2 \) containing \( R \) which is closed under cyclic permutations and inversion. Since \( D(a) \) is unchanged by replacing \( R \) with any other element in \( S_R \), we make the following definition.

**Definition 1.1.** The 2-length of \( S_R \), denoted by \( D_2(S_R) \), is the maximum \( D(a) \), such that \( a \) is a letter of order 2 involved in \( R \).

In this paper, we will be mostly concerned with the element \( R' \) in \( S_R \) of the form

\[
R' = \prod_{i=1}^{D_2(S_R)} aM_i,
\]

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We refer to Theorem 1.3. Let this paper.

As mentioned in the abstract, the authors of [7] observed that in the case when $D_2(S_R) = 0$, the word $R$ can be decomposed as a product of two uniquely positioned subwords. Using that they showed that every minimal picture over a one-relator product with relator $R^2$ satisfies $C(6)$, from which important results about the group were proved. In this paper we work in a more general setting where $D_2(S_R) \leq 2$. It is no longer always possible that $R$ has a decomposition into uniquely positioned subwords. However, we can show that $R$ has a certain structure which allows us to obtain similar results. This idea is captured in the following theorem.

**Theorem 1.2.** Let $R$ be a word in a free product of length at least 2 and which is not a proper power. Suppose also that $D_2(S_R) \leq 2$. Then either a cyclic conjugate of $R$ has a decomposition of the form $UV$ such that $U$ and $V$ are uniquely positioned or one of the following holds:

(a) $D_2(S_R) = 1$ and $R$ has a cyclic conjugate of the form $aXbX^{-1}$ or $aM$, where $a, b$ are letters of order 2 and $M$ does not involve any letter of order 2.

(b) $D_2(S_R) = 2$ and $R$ has a cyclic conjugate of the form $aXbX^{-1}$ where $a$ is a letter of order 2.

Note that in Theorem 1.2, the requirement that $D_2(S_R) \leq 2$ is optimal in the sense that there is no hope to obtain such result when $D_2(S_R) > 2$. To see why this is true, consider the word $S = \prod_{i=1}^{n} ab_1$, with $a \in G_1$ and $b_1 \in G_2$. Suppose that $b_i \neq b_j$ for $i \neq j$ and $a^2 = b_i^2 = 1$ for $i = 1, 2, \ldots, n$. It is easy to verify that $D_2(S_R) = n$ and Theorem 1.2 fails for $n > 2$. In other words, $S$ does not have a decomposition into two uniquely positioned subwords, nor does it have a decomposition of the form $xXyX^{-1}$ such that $x^2 = 1$.

Further analysis of the structure of $R$ leads us to the following theorem which is our main result in this paper.

**Theorem 1.3.** Let $R$ satisfy the conditions of Theorem 1.2. Then either any minimal picture over $G$ satisfies $C(6)$ or $R$ has the form (up to cyclic conjugacy) $aXbX^{-1}$ with $a^2 = 1 \neq b^2$.

The rest of the paper is arranged as follows. We begin in Section 2 by providing some literature on related results. We also recall only the basic ideas about pictures. In Section 3 we prove a number of Lemmas about word combinatorics and pictures. In particular we deduce Theorem 1.2. Furthermore, these Lemmas are applied in Section 4 to prove the main result and deduce a number of applications.

## 2. PRELIMINARIES

Let $G_1$ and $G_2$ be nontrivial groups and $w \in G_1 * G_2$ a cyclically reduced word. Let $G$ be the quotient of the free product $G_1 * G_2$ by the normal closure of $w$, denoted $N(w)$. Then $G$ is called a one-relator product and denoted by

$$G = (G_1 * G_2)/N(w).$$

We refer to $G_1, G_2$ as the factors of $G$, and $w$ as the relator. For us, $w = R^m$ such that $R$ is a cyclically reduced word which is not a proper power and $m \geq 3$. If $m \geq 4$, a number of results were proved in [2, 3, 4], about $G$. These results were also proved in [7] when $m = 3$ but not without the extra condition that $R$ involves no letter of order 2. We also mention that the case when $m = 2$ is largely open. For partial results in this case see [9, 10, 11]. The aim of this paper is to extend the result in [7] by allowing letters of order 2 in $R$. Also we require results about pictures over $G$, in particular the fact that $R^m$ satisfies the small cancellation condition $C(2m)$ when $R$ has a certain form. Pictures can be seen as duals of van Kampen diagrams and have been widely used to prove results about one-relator

with $D(a) = D_2(S_R)$ and $M_i \in G_1 * G_2$. It follows that each $M_i$ has odd length (as a reduced but not cyclically reduced word in the free product) and does not contain any letter equal to $a$. When we use the notation “=” for words, it will mean identical equality. We will use $\ell()$ to mean then length of a reduced free product word which is not necessarily cyclically reduced.
groups and one-relator products. Below, we recall only basic concepts on pictures over a one-relator product as given in [10]. For more details, the reader can see [2, 3, 4, 7, 11].

2.1. PICTURES. A picture $\Gamma$ over $G$ on an oriented surface $\Sigma$ is made up of the following data:

- a finite collection of pairwise disjoint closed discs in the interior of $\Sigma$ called vertices;
- a finite collection of disjoint closed arcs called edges, each of which is either: a simple closed arc in the interior of $\Sigma$ meeting no vertex of $\Gamma$, a simple arc joining two vertices (possibly same one) on $\Gamma$, a simple arc joining a vertex to the boundary $\partial \Sigma$ of $\Gamma$, a simple arc joining $\partial \Sigma$ to $\partial \Sigma$;
- a collection of labels (i.e words in $G_1 \cup G_2$), one for each corner of each region (i.e connected component of the complement in $\Sigma$ of the union of vertices and arcs of $\Gamma$) at a vertex and one along each component of the intersection of the region with $\partial \Sigma$. For each vertex, the label around it spells out the word $R_1^{\pm m}$ (up to cyclic permutation) in the clockwise order as a cyclically reduced word in $G_1 * G_2$. We call a vertex positive or negative depending on whether the label around it is $R_1^m$ or $R_1^{-m}$ respectively.

For us $\Sigma$ will either be the 2-sphere $S^2$ or 2-disc $D^2$. A picture on $\Sigma$ is called spherical if either $\Sigma = S^2$ or $\Sigma = D^2$ but with no arcs connected to $\partial D^2$. If $\Gamma$ is not spherical, $\partial D^2$ is one of the boundary components of a non-simply-connected region (provided, of course, that $\Gamma$ contains at least one vertex or arc), which is called the exterior. All other regions are called interior.

We shall be interested mainly in connected pictures. A picture is connected if the union of its vertices and arcs is connected. In particular, no arc of a connected picture is a closed arc or joins two points of $\partial \Sigma$, unless the picture consists only of that arc. In a connected picture, all interior regions $\Delta$ of $\Gamma$ are simply-connected, i.e topological discs. Just as in the case of vertices, the label around each region — read anticlockwise — gives a word which in a connected picture is required to be trivial in $G_1$ or $G_2$. Hence it makes sense to talk of $G_1$-regions or $G_2$-regions. Each arc is required to separate a $G_1$-region from a $G_2$-region. This is compatible with the alignment of regions around a vertex, where the labels spell a cyclically reduced word, so must come alternately from $G_1$ and $G_2$.

A vertex is called exterior if it is possible to join it to the exterior region by some arc without intersecting any arc of $\Gamma$, and interior otherwise. For simplicity we will indeed assume from this point that our $\Sigma$ is either $S^2$ or $D^2$. It follows that reading the label round any interior region spells a word which is trivial in $G_1$ or $G_2$. The boundary label of $\Gamma$ on $D^2$ is a word obtained by reading the labels on $\partial D^2$ in an anticlockwise direction. This word (which we may be assumed to cyclically reduced in $G_1 * G_2$) represents an identity element in $G$. In the case where $\Gamma$ is spherical, the boundary label is an element in $G_1$ or $G_2$ determined by other labels in the exterior region.

Two distinct vertices of a picture are said to cancel along an arc $e$ if they are joined by $e$ and if their labels, read from the endpoints of $e$, are mutually inverse words in $G_1 * G_2$. Such vertices can be removed from a picture via a sequence of bridge moves (see Figure 1 and [7] for more details), followed by deletion of a dipole without changing the boundary label. A dipole is a connected spherical sub-picture that contains precisely two vertices, does not meet $\partial \Sigma$, and such that none of its interior regions contain other components of $\Gamma$. This gives an alternative picture with the same boundary label and two fewer vertices.

We say that a picture $\Gamma$ is reduced if it cannot be altered by bridge moves to a picture with a pair of cancelling vertices. A picture $\Gamma$ on $D^2$ is minimal if it is non-empty and has the minimum number of vertices amongst all pictures over $G$ with the same boundary label as $\Gamma$. Clearly minimal pictures are reduced. Any cyclically reduced word in $G_1 * G_2$ representing the identity element of $G$ occurs as the boundary label of some reduced picture on $D^2$. 
Definition 2.1. Let $\Gamma$ be a picture over $G$. Two arcs of $\Gamma$ are said to be parallel if they are the only two arcs in the boundary of some simply-connected region $\Delta$ of $\Gamma$.

We will also use the term parallel to denote the equivalence relation generated by this relation, and refer to any of the corresponding equivalence classes as a class of $\omega$ parallel arcs or $\omega$-zone. Given a $\omega$-zone with $\omega > 1$ joining vertices $u$ and $v$ of $\Gamma$, consider the $\omega - 1$ two-sided regions separating these arcs. Each such region has a corner label $x_u$ at $u$ and a corner label $x_v$ at $v$, and the picture axioms imply that $x_u x_v = 1$ in $G_1$ or $G_2$. The $\omega - 1$ corner labels at $v$ spell a cyclic subword $s$ of length $\omega - 1$ of the label of $v$. Similarly the corner labels at $u$ spell out a cyclic subword $t$ of length $\omega - 1$ of the label of $u$. Moreover, $s = t^{-1}$. If we assume that $\Gamma$ is reduced, then $u$ and $v$ do not cancel. In the spirit of small-cancellation-theory, we refer to $t$ and $s$ as pieces.

As in graphs, the degree of a vertex in $\Gamma$ is the number of zones incident on it. For a region, the degree is the number corners it has. We say that a vertex $v$ of $\Gamma$ satisfies the (local) $C(m)$ condition if it is joined to at least $m$ zones. We say that $\Gamma$ satisfies $C(m)$ if every interior vertex satisfies $C(m)$.

3. TECHNICAL RESULTS

In this section we give a number of results on the structure of $R$ when $D_2(S_R) \leq 2$, from which Theorem 1.2 follows. It is assumed throughout that no element of $S_R$ has the form $UV$, where $U$ and $V$ are both uniquely positioned. In particular if $D(a) \geq 2$, there exists at most one $i \in \{1, 2, \ldots, D(a)\}$ such that $M_i$ uniquely positioned in the decomposition $R = \prod_{i=1}^{D(a)} aM_i$. We begin with the proof of part (a) of Theorem 1.2.

Lemma 3.1. If $D_2(S_R) = 1$, then $R$ has a cyclic conjugate of the form $aM$ or $aXbX^{-1}$, where $a, b$ are letters of order 2 and $M$ does not involve any letter of order 2.

Proof. Since $D_2(S_R) = 1$, we can assume without loss of generality that $R = aM$, where $M$ is a word in $G_1 \ast G_2$ which does not involve $a$. We now proceed to show that either $M$ does not involve any letter of order 2 or $M$ can be decomposed in the form $XbX^{-1}$, where $b \in G_1 \cup G_2$ is a letter of order 2 and $X$ is a (possibly empty) word in $G_1 \ast G_2$.

Suppose by contradiction that $M$ has a decomposition of the form $XbY$ with $b^2 = 1$ and $X \neq Y^{-1}$. Note that we can assume without loss of generality that $0 < \ell(X) < \ell(Y)$. Clearly, if $\ell(X) = \ell(Y) > 0$, then both $aX$ and $bY$ are uniquely positioned which is a contradiction. There is nothing to prove if $\ell(X) = \ell(Y) = 0$. Also if $\ell(X) = 0 \neq \ell(Y)$, we get a contradiction since $ab$ and $Y$ will be uniquely positioned. Hence the inequality $0 < \ell(X) < \ell(Y)$ holds.

Suppose that $X^2 = 1$ and $Y^2 = 1$ holds simultaneously. Then by setting $X = X_1 p X_1^{-1}$ and $Y = Y_1^{-1} q Y_1$, where $X_1, Y_1 \in G_1 \ast G_2$ and $p, q$ are distinct letters of order 2 in $G_1 \cup G_2$, we can replace $R$ with

$$R' = pX' q Y',$$
where $X' = (Y_1 b X_1)^{-1}$ and $Y' = Y_1 a X_1$. Since $a \neq b$, we have that $X' \neq Y'^{-1}$. Given that $\ell(X') = \ell(Y')$, we easily conclude that $p X'$ and $q Y'$ are uniquely positioned. This is a contradiction.

Suppose that $X^2 = 1 \neq Y^2$. By the assumption that $D_2(S_R) = 1$, we know that $X$ can not be equal to a segment of $Y$. Hence $aX$ and $bY$ are both uniquely positioned. This is a contradiction. Similarly, suppose that $X^2 \neq 1 = Y^2$. Since $\ell(X) < \ell(Y)$ and $D_2(S_R) = 1$, we have that both $bY$ and $Ya$ are uniquely positioned. Hence, neither $aX$ nor $Xb$ is uniquely positioned. This means that $X^{-1}$ is identically equal to an initial and a terminal segment of $Y$. Therefore, $X^2 = 1$. This is a contradiction.

Finally if $X^2 \neq 1 \neq Y^2$, then $aXb$ and $Y$ are both uniquely positioned. This contradiction completes the proof. \hfill \Box

**Lemma 3.2.** Suppose that $D_2(S_R) = 2$. Then $R$ has a cyclic conjugate of the form $aXbX^{-1}$ where $a$ is a letter of order 2. 

**Proof.** Since $D_2(S_R) = 2$, we can assume without loss of generality that

$$R = a M_1 a M_2,$$

where $M_1, M_2 \in G_1 \ast G_2$, and neither involves the letter $a$. By assumption $M_1$ and $M_2$ can not be uniquely positioned simultaneously. If $M_1^2 = 1$ and $M_2^2 = 1$ hold simultaneously, then by replacing $R$ with a cyclic conjugate, it can be shown that $R$ has the desired form. Without loss of generality, we can assume that $1 \leq \ell(M_1) \leq \ell(M_2)$.

Suppose that $\ell(M_1) = \ell(M_2)$. We can not have $M_1 = M_2$ since $R$ is not a proper power. Also if $M_1 = M_2^{-1}$, then there is nothing to prove. So we assume without loss of generality that $M_1^2 = 1$ and $M_2$ is uniquely positioned. If $\ell(M_1) = 1$, then there is nothing to prove since $M_1$ has order 2 and so $R$ has the desired form. Hence we assume that $\ell(M_1) = \ell(M_2) \geq 3$. Let $M_1 = XpX^{-1}$ and $M_2 = YqZ$, with $p, q \in G_1 \cup G_2$, $p^2 = 1$, $\ell(Y) = \ell(Z)$ and $Y \neq Z^{-1}$ (as otherwise there is nothing to prove). Then

$$R = a X p X^{-1} a Y q Z.$$

Set

$$U = a Y q, \quad U' = q Z a, \quad V = Z a X p X^{-1}, \quad V' = X p X^{-1} a Y.$$

Clearly, $V^2 \neq 1 \neq V'^2$ since $D(a) = 2$. Also since $Y \neq Z^{-1}$, it follows that $V$ and $V'$ are simultaneously uniquely positioned. Hence neither $U$ nor $U'$ is uniquely positioned. It is easy to see that this means that $U^2 = 1$ or $U'^2 = 1$ or $U' = U \pm 1$. However, any such occurrence will imply that $a = q$ or $Y = Z^{-1}$. This is a contradiction.

Now suppose that $\ell(M_i) \neq \ell(M_j)$, where $i, j \in \{1, 2\}$ with $i \neq j$. Note that it is not possible to have $M_i^2 \neq 1$ and $M_j^2 \neq 1$ holding simultaneously since that will imply that $a M_i a$ and $M_j$ are both uniquely positioned, assuming $\ell(M_i) < \ell(M_j)$. Suppose that $M_i^2 = 1$. Let $M_i = X p X^{-1}$ and $M_j = Y q Z$, with $p, q \in G_1 \cup G_2$, $p^2 = 1$, $\ell(Y) = \ell(Z)$ and $Y \neq Z^{-1}$. We claim that exactly one of $a Y$ or $Z a$ is uniquely positioned. This is because if both are uniquely positioned, then there is nothing to prove. Also if neither is uniquely positioned, then $Y = Z^{-1}$. In both cases we get a contradiction.

By symmetry we assume that $a Y$ is uniquely positioned, and hence $q Z a M_i$ is not. This leads to a contradiction when $\ell(Y) \geq \ell(M_i)$ since that will mean $Y = Z^{-1}$. Suppose then that $\ell(Y) < \ell(M_i)$. This implies that $M_i$ is an initial or terminal segment of $M_j$. Hence, we have that $M_j = M_j W$ or $M_j = W M_i$ for some $W \in G_1 \ast G_2$, depending on whether $M_i$ is an initial or terminal segment of $M_j$. Note that $\ell(W) = 2n$ for some integer $n > 0$. Now we replace $R$ by

$$R' = p M_p N,$$
where $M = X^{-1}aX$ and $N = X^{-1}WaX$ or $N = X^{-1}aWX$. We consider first the case when $N = X^{-1}WaX$. In this case, the initial segment $X^{-1}W$ of $N$ has length $\ell(X^{-1}W) \geq \ell(X) + 2$. Since $D_2(S_R) = 2$, $X^{-1}W$ neither involves $a$ nor $p$. It follows that $aXpXaX^{-1}p$ is uniquely positioned. Hence, $X^{-1}W$ is not uniquely positioned. The length condition on $X^{-1}W$ implies that $(X^{-1}W)^2 = 1$. Again since $D_2(S_R) = 2$, $X$ does not involve a letter of order 2. So $W = SzS^{-1}X$, for some (possibly empty) word $S$ and some letter $x$ of order 2. Hence

$$R' = pX^{-1}aXpX^{-1}SzS^{-1}XaX.$$ 

Consider the cyclic subwords $W_1 = S^{-1}XaXpX^{-1}aX$ and $W_2 = pX^{-1}Sz$. Clearly, $W_1^2 \neq 1$ as otherwise $S$ is empty and more importantly $X^2 = 1$, which is a contradiction. Also, $W_2^2 \neq 1$ since $p \neq x$. In fact, it is easy to see that both $W_1$ and $W_2$ are uniquely positioned. This is a contradiction. Similar argument works when $N = X^{-1}aWX$ by replacing $W_1$ and $W_2$ with their inverses. This completes the proof. 

By combining Lemmas 3.1 and 3.2 we obtain Theorem 1.2.

The remaining results in this section are consequences of results about a picture $\Gamma$ over $G$. First, we give a necessary and sufficient condition under which the word $R$ has a decomposition into a pair of uniquely positioned subwords when $D_2(S_R) = 1$.

**Lemma 3.3.** Let $r$ be a cyclically reduced word which is not a proper power in the free product $G_1 * G_2$ such that $D_2(S_r) = 1$. Then, $r$ has a decomposition into two uniquely positioned subwords if and only if $\ell(r) > 2$ and there exists $r' \in S_r$ such that $r' = aXxYyX^{-1}$ with $X, Y, x, y, a \in G_1 * G_2$, $\ell(Y) \geq 1$, $\ell(x) = \ell(y) = \ell(a) = 1$, $x \neq y^{-1}$ and $a^2 = 1$.

**Proof.** Suppose that $r$ has a decomposition into two uniquely positioned subwords $U$ and $V$. Since $D(S_r) = 1$, we have that $\ell(r) > 2$. Without loss of generality, it follows that a cyclic conjugate of $r$ has the form

$$r' = aU_2VU_1,$$

where $U = U_1aU_2$ and $a^2 = 1$. Hence $U_2VU_1 = XYX^{-1}$ for some words $X, Y \in G_1 * G_2$, where $X$ is possibly empty. Since $U$ and $V$ are uniquely positioned in $r$, we conclude that $\ell(Y) \geq 3$ and the first and last letters of $Y$ are not inverses. The result follows.

For the other direction, suppose $r' = aXxYyX^{-1}$ with $X, Y, x, y, a \in G_1 * G_2$, $\ell(x) = \ell(y) = \ell(a) = 1$, $x \neq y^{-1}$ and $a^2 = 1$. Then $aXx$ is clearly uniquely positioned in $r$ since $x \neq y^{-1}$. For the same reason, we deduce from part (a) of Theorem 1.2 that $XxYyX^{-1}$ has no element of order 2. In particular, this means that $YyX^{-1}$ and its inverse do not intersect (in an initial or terminal segment). We claim that this means that $YyX^{-1}$ is also uniquely positioned. We prove this by contradiction by assuming that $YyX^{-1}$ is not uniquely positioned and showing that $XxYyX^{-1}$ contains an element of order 2.

Let $XxYyX^{-1} = x_1x_2 \cdots x_n$, with $X = x_1x_2 \cdots x_p$. Suppose that $YyX^{-1}$ is not uniquely positioned. Then, $(YyX^{-1})^{\pm 1}$ is identically equal to some segment of $XxYyX^{-1}$. This segment must intersect $YyX^{-1}$. By above discussion, we have that $YyX^{-1}$ is identically equal to the segment

$$x_kx_{k+1} \cdots x_{\ell(YyX^{-1})-1},$$

with $k \leq p$. Hence, we have that the terminal segment of $XxYyX^{-1}$ of length $n + 1 - k$ has period $\lambda = p + 2 - k$. Consider the initial segment of this periodic segment given by

$$W_k = x_kx_{k+1} \cdots x_{n+1-k-(p+2)}.$$ 

In particular $W_k$ is of length $n - (p + 1)$. Note that $X^{-1} = x_p^{-1}x_{p-1}^{-1} \cdots x_1^{-1} = x_{n+1-p}x_{n+2-p} \cdots x_n$. If $x_i = x_i^{-1}$ for some $k \leq i \leq p$, then we are done. Suppose not. If $x_p$ (alternatively $x_k$) is identified
with \( x_i^{-1} \) for some \( k \leq i \leq p \), then \( x_{j+i}^{-1} = x_{j+i}^{-1} \) (alternatively \( x_{j+i} = x_{j+i}^{-1} \)). This is a contradiction. Otherwise, both \( x_k \) and \( x_p \) are identified with \( x_i^{-1} \) and \( x_i^{-1} \) respectively, where \( 1 \leq j \leq i < k-1 \) (since we are in a free product). In fact, \( j = i + k - p < 2k - 1 - p \). Choose \( j \) such that under this periodicity, \( x_j^{-1} \) is the letter that provides the first identification with \( x_p \). We claim that \( j + \lambda \) lies between \( k \) and \( p \). To verify this claim, it is enough to show that \( p \geq j + \lambda \). We have that \( j + \lambda < 2k - 1 - p + \lambda = k + 1 \). Therefore, \( j + \lambda \leq k \leq p \). Hence \( x_p = x_{j+\lambda}^{-1} \) and \( j + \lambda \leq p \). By the choice of \( j \), we must have that \( k \leq j + \lambda \leq p \). This is a contradiction. Hence \( Y Y X^{-1} \) is uniquely positioned. This completes the proof. 

Lemma 3.4. Let \( \Gamma \) be a reduced picture over \( G \) on \( D^2 \) where \( R = aXbX^{-1} \) and \( a^2 = b^2 = 1 \). Then either \( \Gamma \) is empty or \( \Gamma \) satisfies \( C(6) \).

Proof. Suppose \( \Gamma \) a non-empty picture over \( G \) which is reduced. Suppose also that \( \Gamma \) contains some interior vertex \( v \) of degree less than 6. Then \( v \) is connected to another vertex \( u \) by a zone containing \( a \) or \( b \). Using this zone, we can do bridge-moves so that \( u \) and \( v \) form a dipole. This contradicts the assumption that \( \Gamma \) was reduced. 

We obtain from Lemmas 3.1, 3.3 and 3.4 the following corollary.

Corollary 3.5. Let \( D_2(S_R) \leq 2 \) and \( R \neq aXbX^{-1} \) with \( a^2 = 1 \neq b^2 \). Then any non-empty reduced picture on \( D^2 \) over \( G \) satisfies \( C(6) \).

Proof. By Lemma 3.3, either \( R \) has a decomposition \( UV \) with \( U, V \) uniquely positioned in \( R \) or \( R \) has the form (up to cyclic conjugation) \( aXbX^{-1} \) with \( a^2 = b^2 = 1 \). For the first case, the proof is exactly as it is in [7] Lemma 3.1. For the latter case, the result follows from Lemmas 3.4. 

4. APPLICATIONS

In this section we deduce a number of applications of Theorem 1.3. But first, we recall the setting.

Let \( G_1 \) and \( G_2 \) be non-trivial groups and \( R \in G_1 * G_2 \) which is not a proper power and has length at least 2. We also require that no letter of order 2 involved in \( R \) appears more than twice i.e. \( D_2(S_R) \leq 2 \). For a natural number \( m \geq 3 \), \( G \) is the quotient of \( G_1 * G_2 \) by the normal closure of \( R^m \).

Proof of Theorem 1.3. This follows from Part (b) or Theorem 1.2 and Corollary 3.5.

When \( R \) has a conjugate of the form \( aXbX^{-1} \) and \( a^2 = 1 \neq b^2 \), we will call \( R \) exceptional. As mentioned earlier, there are results in the literature on the two classes and we list them without proof.

We begin the non-exceptional case. For this case the proofs can be found in [7].

Theorem 4.1. Suppose that \( G \) is as above and \( R \) is not exceptional. Then the following hold.

(a) Freiheitssatz. The natural homomorphisms \( G_1 \rightarrow G \) and \( G_2 \rightarrow G \) are injective.

(b) Weinbaum’s Theorem. No non-empty proper subword of \( R^m \) represents the identity element of \( G \).

(c) Word problem. If \( G_1 \) and \( G_2 \) are given by a recursive presentation with soluble word problem, then so is \( G \). Moreover, the generalized word problem for \( G_1 \) and \( G_2 \) in \( G \) is soluble with respect to these presentations.

(d) The Identity Theorem. \( N(R^m)/[N(R^m), N(R^m)] = ZG/(1 - R)ZG \) as a (right) \( ZG \)-module.
Corollary 4.2. There are natural isomorphisms for all $q > 3$:

$$H^q(G; -) \longrightarrow H^q(G_1; -) \times H^q(G_2; -) \times H^q(Z_m; -),$$

$$H_q(G; -) \leftarrow H_q(G_1; -) \oplus H^q(G_2; -) \oplus H^q(Z_m; -);$$

a natural epimorphism

$$H^2(G; -) \longrightarrow H^2(G_1; -) \times H^2(G_2; -) \times H^2(Z_m; -),$$

and a natural monomorphism

$$H^2(G; -) \leftarrow H^2(G_1; -) \oplus H^2(G_2; -) \oplus H^2(Z_m; -).$$

These are defined on the category of $\mathbb{Z}_G$-modules, $\mathbb{Z}_m$ is the cyclic subgroup of order $m$ generated by $R$, and all these maps are induced by restriction on each factor.

Next we consider the exceptional case. In this case, $G$ is called a one-relator product induced from the generalized triangle group $H$, described as follows. Let $A := \langle a \rangle$ and $X^{-1}BX := \langle b \rangle$ be the cyclic subgroups of $G_1$ or $G_2$ generated by $a$ and $b$ respectively. Then $H := (A \ast B)/N(R^m)$. Note that $G$ can be realized as a push-out of groups as shown in Figure 2.

![Figure 2. Push-out diagram.](image)

This pushout representation of $G$ is referred to as a generalized triangle group description of $G$, and we require it to be maximal in the sense [10]. Another technical requirement is that $(a, b)$ be admissible: whenever both $a$ and $b$ belong to same factor, say $G_1$, then either the subgroup of $G_1$ generated by $\{a, b\}$ is cyclic or $\langle a \rangle \cap \langle b \rangle = 1$. It is very easy to verify that these conditions are satisfied in our setting. Hence the results in [8] hold.

Theorem 4.3. Suppose that $G$ is as above and $R$ is exceptional. Then the following hold.

(a) Freiheitssatz. The natural homomorphisms $G_1 \to G$ and $G_2 \to G$ are injective.

(b) Weinbaum’s Theorem. No non-empty proper subword of $R^m$ represents the identity element of $G$.

(c) Membership problem. Assume that the membership problems for $\langle a \rangle$ and $\langle b \rangle$ in $G_1 \ast G_2$ are solvable. Then the word problem for $G$ is also solvable.

(d) Mayer-Vietoris. The pushout of groups in Figure 2 is geometrically Mayer-Vietoris in the sense of [8]. In particular it gives rise to Mayer-Vietoris sequences

$$\cdots \to H_{k+1}(G, M) \to H_k(A \ast B, M) \to$$

$$H_k(G_1 \ast G_2, M) \oplus H_k(H, M) \to H_k(G, M) \to \cdots$$

and

$$\cdots \to H^k(G, M) \to H^k(G_1 \ast G_2, M) \oplus H^k(H, M)$$

$$\to H^k(A \ast B, M) \to H^{k+1}(G, M) \to \cdots$$

for any $\mathbb{Z}_G$-module $M$. 
STRUCTURE OF WORDS WITH SHORT 2-LENGTH IN A FREE PRODUCT OF GROUPS.

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