Non-existence of faithful isometric action of compact quantum groups on compact, connected Riemannian manifolds
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Abstract

Suppose that a compact quantum group $Q$ acts faithfully on a smooth, compact, connected manifold $M$, i.e. has a $C^*$ (co)-action $\alpha$ on $C(M)$, such that the action $\alpha$ is isometric in the sense of [10] for some Riemannian structure on $M$. We prove that $Q$ must be commutative as a $C^*$ algebra i.e. $Q \cong C(G)$ for some compact group $G$ acting smoothly on $M$. In particular, the quantum isometry group of $M$ (in the sense of [10]) coincides with $C(\text{ISO}(M))$.

Subject classification : 81R50, 81R60, 20G42, 58B34.
Keywords: Compact quantum group, quantum isometry group, Riemannian manifold, smooth action.

1 Introduction

It is a very important and interesting problem in the theory of quantum groups and noncommutative geometry to study ‘quantum symmetries’ of various classical and quantum structures. Indeed, symmetries of physical systems (classical or quantum) were conventionally modeled by group actions, and after the advent of quantum groups, group symmetries were naturally generalized to symmetries given by quantum group action. In this context, it is natural to think of quantum automorphism or the full quantum symmetry groups of various mathematical and physical structures. The underlying basic principle of defining a quantum automorphism group of a given mathematical structure consists of two steps: first, to identify (if possible) the group of automorphisms of the structure as a universal object in a suitable category, and then, try to look for the universal object in a similar but bigger category by replacing groups by quantum groups of appropriate type.

The formulation and study of such quantum symmetries in terms of universal Hopf algebras date back to Manin [21]. In the analytic set-up of compact quantum groups, it was considered by S. Wang who defined and studied quantum permutation groups of finite sets and quantum automorphism groups of finite dimensional algebras. Subsequently, such questions were taken up by a number of mathematicians including Banica, Bichon, Collins and others (see, e.g. [1], [5], [29]), and more recently in the framework of Connes’ noncommutative geometry ([7]) by Goswami, Bhowmick, Skalski, Banica, Soltan, De-Commer, Thibault and many others who have extensively studied the quantum group of isometries (or quantum isometry group) defined in [10] (see also [4], [28] etc.).

In this context, it is important to compute quantum symmetries of classical spaces. One may hope that there are many more quantum symmetries of a given classical space than classical group symmetries which will help one understand the space better. Indeed, it has been a remarkable discovery of S. Wang that for $n \geq 4$, a finite set of cardinality $n$ has an infinite dimensional compact quantum group $S^+_{n}$ (‘quantum permutation group’) of symmetries. For the relevance of quantum group symmetries in a wider and more geometric context, we refer

\footnote{1Partially supported by Swarnajayanti Fellowship and J C Bose Fellowship (2016 onwards) from D.S.T. (Govt. of India) and also acknowledges the Fields Institute, Toronto for providing hospitality for a brief stay when a small part of this work was done.}

\footnote{2Acknowledges support from CSIR and NBHM.}
the reader to the discussion on ‘hidden symmetry in algebraic geometry’ in Chapter 13 of [21]
where Manin made a remark about possible genuine Hopf algebra symmetries of classical smooth
algebraic varieties.

From a topological and geometric point of view, it is certainly more interesting to look for
examples of faithful continuous action by genuine (i.e. not of the form $C(G)$ for a compact group
$G$) compact quantum groups on $C(X)$ for a connected compact space $X$. Several examples of
such actions on have been constructed by H. Huang in [15]. One can also adapt an example
of [9] to get a faithful action of the group $C^*$ algebra $C^*(S_3)$ of the group of permutations
of 3 objects (which is a finite dimensional compact quantum group) on $C(X)$ where $X$ is the
wedge product of two copies of $[-1,1]$ identifying the point 0. However, none of the above
discuss are smooth manifolds. Motivated by the fact that a topological action of compact
group on a smooth manifold is smooth if and only if it is isometric w.r.t. some Riemannian
structure on the manifold, the first author of this paper and some of his collaborators and
students tried to compute quantum isometry groups for several classical (compact) Riemannian
manifolds including the spheres and the tori. Quite remarkably, in each of these cases, the
quantum isometry group turned out to be the same as $C(G)$ where $G$ is the corresponding
isometry group. On the other hand, Banica et al. ([2]) ruled out the possibility of (faithful)
isometric actions of a large class of compact quantum groups including $S^+_n$ on a connected
compact Riemannian manifold. All these led the first author of the present paper to make the
following conjecture in [11], where he also gave some supporting evidence to this conjecture
considering certain class of homogeneous spaces.

**Conjecture I:** It is not possible to have smooth faithful action of a genuine compact quantum
group on $C(M)$ when $M$ is a compact connected smooth manifold.

The aim of this article is to prove a very important case of this conjecture, namely, under the
condition that the action is isometric in the sense of [10] for some Riemannian metric on the
manifold.

More precisely, we have:

**Theorem I**

Suppose that a CQG $Q$ acts faithfully on a smooth, compact connected manifold $M$ such that
the action is isometric with respect to some Riemannian structure on $M$. Then $Q$ must be
commutative as a $C^*$ algebra i.e. $Q \cong C(G)$ for some compact group $G$ acting smoothly on $M$.

**Remark 1.1** Smoothness of $M$ and compactness of the quantum group $Q$ are quite crucial for
the above conjecture. We already mentioned counter-examples in case the space is non-smooth
and let us refer the reader to [20](Example 2.20) as well as [22] for faithful algebraic (co)-
action of Hopf-algebras corresponding to genuine non-compact quantum groups on commutative
domains associated with affine varieties. However, we do not yet have any example of genuine
CQG action on non-compact, smooth, connected manifold and believe that it may be possible to
extend our no-go result to this case as well.

**Remark 1.2** In some sense, our results indicate that one cannot possibly have a genuine ‘hid-

den quantum symmetry’ in the sense of Manin (Chapter 13 of [21]) for smooth connected vari-
eties coming from CQG Hopf algebras; i.e. one must look for such quantum symmetries given
by Hopf algebras of non-compact type only. From a physical point of view, it follows that for
a classical mechanical system with phase-space modeled by a compact connected manifold, the
generalized notion of symmetries in terms of (compact) quantum groups coincides with the con-
ventional notion, i.e. symmetries coming from group actions.

**Remark 1.3** In [9], Etingof and Walton obtained a somewhat similar result in the purely alge-

braic set up of finite dimensional Hopf algebra actions on commutative domains. However, their
proof does not seem to extend to the infinite dimension as it crucially depends on semisimplicity
and finite dimensionality of the Hopf algebra.
Remark 1.4 We should also mention here the attempts by several authors to formulate a notion of quantum isometry group in the purely metric space set-up (see [22], [1], [12] etc.) and the proof by A. L. Chirvasitu ([1]) of non-existence of genuine quantum isometry in the metric space set-up for geodesic metric of negatively curved, compact connected Riemannian manifold.

The following fact, which was observed in earlier works like [11], plays a crucial role in the proof of Theorem I. 

Fact: There does not exist any faithful action by a genuine CQG on a subset with nonempty interior of some Euclidean space $\mathbb{R}^n$ which is affine in the sense that the action leaves the linear span of the coordinate functions and the constant function 1 invariant.

We prove Theorem I by deducing first that the given isometric CQG action can be lifted to a faithful affine action on the closure of a suitable bounded Euclidean domain (open connected set with smooth boundary). In the classical case, i.e. a compact group action, one may take bases in sufficiently many spectral subspaces of $C^\infty(M)$ and use functions in these bases to embed $M$ into $\mathbb{R}^N$ for some $N$, so that the action becomes affine (equivalently linear, after a suitable shift of the origin) w.r.t. the coordinates of $\mathbb{R}^N$. This can be done for CQG actions as well. More precisely, the action (say $\alpha$) will satisfy

$$\alpha(X_i) = \sum_{j=1}^N X_j \otimes q_{ij} + c_i 1$$

for some generating set of self-adjoint elements $q_{ij}$ of the CQG, where $c_i \in \mathbb{R}$, $X_i$ is the restriction of the $i$-th coordinate function of $\mathbb{R}^N$ to the subset $M \subset \mathbb{R}^N$. However, the main difference between the classical and quantum situation is that an affine representation of a group $G$ on $\mathbb{R}^N$ can be lifted to an action on the algebra of continuous or smooth functions on $\mathbb{R}^N$ by ‘dualizing’ the point-wise action. Using compactness of the group, we can actually get action on some suitable Euclidean closed ball $B$ of large enough radius $r > 0$ (say) around $(c_1, \ldots, c_N)$ containing $M$ in the interior. This is not the case for a general CQG action.

An affine representation on $\mathbb{R}^N$ like the one obtained by restricting $\alpha$ to $\text{Sp}\{1, X_1, \ldots, X_N\}$ need not induce any action on $C(B)$. Indeed, existence of such an action would imply in particular that the algebra generated by $\{\alpha(\tilde{X}_i)(p), p \in B, \ i = 1, \ldots, N\}$ is commutative, where $\tilde{X}_i$ is the restriction of the $i$-th coordinate function to $B$. This is equivalent to the commutativity of the algebra generated by $\{\sum_j p_j q_{ij} : p \equiv (p_1, \ldots, p_N) \in B, i = 1, \ldots, N\}$. However, the fact that $\alpha$ is a $*-$homomorphism on $C^\infty(M)$ gives us only the commutativity of $\{\sum_j p_j q_{ij} : (p_1, \ldots, p_N) \in M, i = 1, \ldots, N\}$. Moreover, the coordinate functions restricted to $M$ need not be ‘quadratically independent’ in the sense of [11], so that we cannot deduce the commutativity of $q_{ij}$’s.

For this reason, we need to take a more elaborate route for lifting the given CQG action to an affine action on a bounded domain.

(a) We begin by lifting the isometric action to the total space $O(M)$ of the orthonormal frame bundle.

(b) As $O(M)$ is parallelizable, one can choose an isometric embedding of it in some Euclidean space where the corresponding normal bundle is trivial. Thus, the isometric action on $O(M)$ can be further lifted to an isometric action on some suitable tubular neighbourhood, say $N$, of $O(M)$.

(c) Finally, as $N$ is locally isometric to the flat Euclidean space, any isometric action on it must be affine in the corresponding coordinates. The connectedness of $M$ is used only in this step.

In some sense, the liftings in the above steps are achieved by adapting the classical line of arguments to the CQG set-up. However, this adaptation is rather non-trivial due to noncommutativity at every stage. Indeed, to start with, we only have the commutativity of $\alpha(f)(m)$ with $\alpha(g)(m)$ for different $f, g \in C^\infty(M)$, where $m \in M$ is fixed. Using the isometry condition, we first deduce that the first order partial derivatives of $\alpha(f)$ at $m$ (with respect to any
local coordinates) will commute among themselves as well as with \( \alpha(g)(m) \) for \( f, g \in C^\infty(M) \). This allows us to prove that the natural analogue of the differential of the action \( \alpha \), which is a representation of the CQG on the module of one and higher forms, is well-defined. This is necessary to lift the action to \( O(M) \). Then we prove that in a suitable sense the representation on the module of one-forms ‘commutes’ with the Levi-civita connection for the Riemannian metric. This in turn helps us to deduce ‘higher order commutativity’, namely the commutativity of all order partial derivatives of \( \alpha(f)(m) \) \( f \in C^\infty(M) \) at a given point \( m \) of the manifold. This fact has been crucially used in the proof of the step (c).

We begin by collecting some basic definitions, notations and standard facts in Section 2. In Section 3 we introduce smooth and inner product preserving action and prove that such an action can be lifted to the orthonormal frame bundle. Then we discuss the implications of isometric actions (i.e. actions commuting with the Hodge Laplacian) in Section 4. Finally, in Section 5, we state and prove the main result using embedding of \( O(M) \) in some \( \mathbb{R}^N \) with trivial normal bundle and the results of previous sections.

2 Preliminaries

2.1 Notational convention

In this paper all the Hilbert spaces are over \( \mathbb{C} \) unless mentioned otherwise. If \( V \) is a vector space over real numbers we denote its complexification by \( V_{\mathbb{C}} \). For a vector space \( V, V' \) stands for its algebraic dual. We denote the domain of a linear (possibly unbounded) map \( L \) by \( \text{Dom}(L) \). The algebraic tensor product of modules over an algebra \( \mathcal{C} \) is denoted by \( \otimes_{\mathcal{C}} \). In \( \mathcal{C} = \mathbb{C} \), i.e. the modules are vector spaces, we simply use \( \otimes \). For a complex \(*\)-algebra \( \mathcal{C} \), let \( \mathcal{C}_{s.a.} = \{ c \in \mathcal{C} : c^* = c \} \). We shall denote the \( \mathcal{C}^* \) algebra of bounded operators on a Hilbert space \( \mathcal{H} \) by \( B(\mathcal{H}) \) and the \( \mathcal{C}^* \) algebra of compact operators on \( \mathcal{H} \) by \( B_0(\mathcal{H}) \). \( Sp, \overline{Sp} \) stand for the linear span and closed linear span of elements of a vector space respectively, whereas \( \text{Im}(A) \) denotes the image of a linear map. \( \delta_{ij} \) is the Kronecker delta function. For an algebra \( \mathcal{C} \), let \( \sigma_{ij} : \mathcal{C} \otimes \mathcal{C} \otimes \cdots \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{C} \otimes \cdots \otimes \mathcal{C} \) (with \( \sigma_{12} \equiv 1 \) for \( n = 2 \)) denote the map which flips the \( i \) and \( j \)-th tensor copies.

We’ll use the notation \( \otimes \) for the minimal (spatial) tensor product between \( \mathcal{C}^* \) algebras or tensor product between \(*\)-homomorphisms (more generally, completely positive maps) between \( \mathcal{C}^* \) algebras. We will usually denote scalar valued inner product of Hilbert spaces by \( \langle \cdot, \cdot \rangle \) and some \(*\)-algebra valued inner product of Hilbert modules by \( \langle \langle \cdot, \cdot \rangle \rangle \). For two Hilbert \( \mathcal{A} \mathcal{B} \) and \( \mathcal{B} \mathcal{C} \) bimodules \( \mathcal{E} \) and \( \mathcal{E'} \), (hence in particular for Hilbert spaces) we denote by \( \overline{\mathcal{E} \otimes \mathcal{E'}} \) the Hilbert \( \mathcal{A} \mathcal{C} \) bimodule obtained by suitably quotienting and completing the algebraic tensor product \( \mathcal{E} \otimes_B \mathcal{E'} \) w.r.t. the \( \mathcal{C} \)-valued inner product defined by \( \langle \langle \xi \otimes_B \xi', \eta \otimes_B \eta' \rangle \rangle = \langle \langle \xi', \xi \rangle \rangle \eta' \rangle \) on the simple tensor elements and then extended naturally to their linear span. We’ll have occasions to use Hilbert bimodules over certain Fréchet \(*\)-algebras too.

2.2 Compact quantum groups, their representations and actions

We very briefly outline the notion of compact quantum groups (CQG) and their representations. The reader is referred to [20], [31] and the references therein for details. A compact quantum group (CQG for short) is a unital \( \mathcal{C}^* \) algebra \( \mathcal{Q} \) with a coassociative coproduct (see [20]) \( \Delta \) from \( \mathcal{Q} \) to \( \mathcal{Q} \otimes \mathcal{Q} \) such that each of the linear spans of \( \Delta(\mathcal{Q})(\mathcal{Q} \otimes 1) \) and that of \( \Delta(\mathcal{Q})(1 \otimes \mathcal{Q}) \) is norm-dense in \( \mathcal{Q} \otimes \mathcal{Q} \). From this condition, one can obtain a canonical dense unital \(*\)-subalgebra \( \mathcal{Q}_0 \) of \( \mathcal{Q} \) on which linear maps \( \kappa \) and \( \epsilon \) (called the antipode and the counit respectively) are defined making the above subalgebra a Hopf \(*\)-algebra.

It is known that there is a unique state \( \hat{h} \) on a CQG \( \mathcal{Q} \) (called the Haar state) which is bi invariant in the sense that \( (\text{id} \otimes \hat{h}) \circ \Delta(a) = (\hat{h} \otimes \text{id}) \circ \Delta(a) = h(a)1 \) for all \( a \in \mathcal{Q} \). The Haar
state need not be faithful in general, though it is always faithful on $Q_0$ at least. The image of $Q$ in the GNS representation of $h$, say $Q_\tau$, is called the reduced CQG corresponding to $Q$.

A unitary representation of a CQG $(Q, \Delta)$ on a Hilbert space $H$ is a $C^*$-linear map $U$ from $H$ to the Hilbert module $H \otimes Q$ such that

\begin{enumerate}
\item $\langle (U(\xi), U(\eta)) \rangle = \langle \xi, \eta \rangle_1$, where $\xi, \eta \in H$.
\item $(U \otimes \text{id}) U = (\text{id} \otimes \Delta) U$.
\item The right $Q$-linear span of $\text{Im}(U)$ is dense in $H \otimes Q$.
\end{enumerate}

In 2. above, the map $(U \otimes \text{id})$ denotes the extension of $U \otimes \text{id}$ to the completed tensor product $H \otimes Q$, which exists by the isometry condition 1. A closed subspace $H_1$ of $H$ is said to be invariant if $U(H_1) \subset H_1 \otimes Q$.

**Definition 2.1** An algebraic (co)-representation (or co-module) for a Hopf algebra $(H, \Delta, \epsilon, \kappa)$ on a vector space $V$ is a $C^*$-linear map $\alpha : V \to V \otimes H$ such that $(\alpha \otimes \text{id}) \circ \alpha = (\text{id} \otimes \Delta) \circ \alpha$ and $(\text{id} \otimes \epsilon) \circ \alpha = \text{id}$. In case $H$ is a Hopf $*$-algebra, and $V = A$ is a unital $*$-algebra and the (co)-representation $\alpha$ is also a unital $*$-algebra homomorphism, we say that it is a Hopf $*$-algebraic (co)-action of $H$ on $A$.

For an algebraic (co)-representation as above, one can prove that $\text{Sp}(\alpha(V)H) = V \otimes H$. For a Hopf algebra $H$ with the coproduct $\Delta$, we write $\Delta(q) = q(1) \otimes q(2)$ suppressing the summation notation (Sweedler’s notation). For an algebra (other than $H$ itself) or module $A$ and a $C^*$-linear map $\Gamma : A \to A \otimes H$ (typically a comodule map or a coaction) we shall also use an analogue of Sweedler’s notation and write $\Gamma(a) = a(0) \otimes a(1)$.

**Definition 2.2** A unital $*$-homomorphism $\alpha : C \to C \otimes Q$, where $C$ is a unital $C^*$-algebra and $Q$ is a CQG, is said to be an action of $Q$ on $C$ if

\begin{enumerate}
\item $(\alpha \otimes \text{id}) \alpha = (\text{id} \otimes \Delta) \alpha$ (co-associativity).
\item $\text{Sp} \alpha(C)(1 \otimes Q)$ is norm-dense in $C \otimes Q$.
\end{enumerate}

We say that the action is algebraic over a $*$-subalgebra $C_0 \subset C$ if $\alpha|_{C_0} : C_0 \to C_0 \otimes Q_0$ is a Hopf $*$-algebraic (co)-action.

The following result is proved by Podles ([22]).

**Proposition 2.3** Given an action $\alpha$ of a CQG $Q$ on $C$, there exists a norm-dense unital $*$-subalgebra $C_0$ of $C$ over which $\alpha$ is algebraic.

The action $\alpha$ is said to be faithful if the $*$-subalgebra of $Q$ generated by the elements of the form $(\omega \otimes \text{id})(\alpha(a))$, $a \in C_0$, $\omega \in C_Q$ is norm-dense in $Q$. When $C = C(M)$, faithfulness is equivalent to requiring the density of the subalgebra generated by elements of the form $\alpha(f)(m)$, $f \in C(M)$, $m \in M$.

### 2.3 $C^\infty(M)$ and bimodule of forms

Let $M$ be a smooth, $n$-dimensional compact manifold possibly with boundary. We denote the algebra of real (complex respectively) valued smooth functions on $M$ by $C^\infty(M)_\mathbb{R}$ ($C^\infty(M)_\mathbb{C}$ respectively). The natural Fréchet topology on $C^\infty(M)$ is given by the seminorms of the form $p^{U,K,\alpha}$,

\[ p^{U,K,\alpha}(f) = \sup_{x \in K} |\partial^\alpha f(x)|, \]

where $K$ is a compact subset contained in the domain of some coordinate chart $(U, (x_1, \ldots, x_n))$, $\alpha = (i_1, \ldots, i_k)$ a multi index and $\partial^\alpha = \partial^{i_1} \cdots \partial^{i_k}$, $i_j \in \{1, \ldots, n\}$. We can similarly define a Fréchet topology on $C^\infty(M, E)$, the space of smooth $E$-valued functions on $M$ for any Fréchet space $E$. A word on our notational convention: we denote by $T^n_m(M)$ the complexified cotangent space at $m$, whereas $(T^n_m(M))_\mathbb{R}$ will denote the corresponding real vector space.
space. Let \( \Omega^1(C^\infty(M)) \equiv \Lambda^1(C^\infty(M)) \) be the space of smooth complex valued 1 forms on the manifold \( M \), with the natural locally convex topology induced by the topology of \( C^\infty(M) \) given by a family of seminorms \( q^U,K,\alpha(\omega) = \sup_{x \in K, 1 \leq j \leq n} |\partial^\alpha f_j(x)| \), where \( K \subset U \), \( \alpha \) are as before, and \( \omega_U = \sum_{j=1}^n f_j dx_j \). It is clear from the definition that the differential map \( d : C^\infty(M) \to \Omega^1(C^\infty(M)) \) is Fréchet continuous. As \( M \) is compact, there is a Riemannian structure. Using the Riemannian structure on \( M \) we can equip \( \Omega^1(C^\infty(M)) \) with a \( C^\infty(M) \) valued inner product given by \( \langle (\omega,\eta) (m) = \langle \omega(m),\eta(m) \rangle_m \), where \((\cdot)_m\) denotes the complex-valued inner product on complexified cotangent space coming from the Riemannian structure. This makes \( \Omega^1(C^\infty(M)) \) a Fréchet-Hilbert module over \( C^\infty(M) \).

It can be shown that \( \Omega^1(C^\infty(M)) = \text{Sp}\{fdg : f,g \in C^\infty(M)\} \).

Given \( \omega \in \Omega^1(M) \), let us denote by \( X_\omega \) the (unique) smooth vector field satisfying

\[
X_\omega(f) = \langle \omega, df \rangle
\]

for any smooth function \( f \). Moreover, one has \( \langle X_\omega, X_\eta \rangle = \langle \eta, \omega \rangle \) from the relation between the Riemannian inner product on the tangent and cotangent spaces.

Similarly, we construct the Fréchet-Hilbert bimodules \( \Omega^k(C^\infty(M)) = \Omega^1(C^\infty(M)) \otimes_{C^\infty(M)} \cdots \otimes_{C^\infty(M)} \Omega^1(C^\infty(M)) \) by taking \( k \)-fold tensor product over \( C^\infty(M) \) and the inner product (denoted by \( \langle \langle \cdot, \cdot \rangle \rangle \)) coming from the Riemannian structure. In fact, we have \( \langle \langle \omega \otimes_{C^\infty(M)} \cdots \otimes_{C^\infty(M)} \omega_k, \eta_1 \otimes_{C^\infty(M)} \cdots \otimes_{C^\infty(M)} \eta_k \rangle \rangle = \langle \omega, \eta_1 \rangle \cdots \langle \omega_k, \eta_k \rangle \), using the commutativity of \( C^\infty(M) \) and the fact that \( \omega f = f \omega \) for all \( \omega \in \Omega^1(C^\infty(M)) \) and \( f \in C^\infty(M) \). Clearly, the group \( S_k \) of permutations of \( k \) objects acts on \( \Omega^k(C^\infty(M)) \) in an obvious way by permuting the tensor copies. For a character \( \chi \) of \( S_k \), let \( P_\chi \) be the corresponding spectral projection given by:

\[
P_\chi(f \omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \chi(\sigma) f \omega_{\sigma(1)} \otimes \omega_{\sigma(2)} \otimes \cdots \otimes \omega_{\sigma(k)},
\]

where \( f \in C^\infty(M), \omega_i \in \Omega^1(C^\infty(M)) \) \( \forall i \). The bimodule of smooth \( k \) forms \( \Lambda^k(C^\infty(M)) = \text{Sp} \{fdf_1 \wedge \cdots \wedge df_k : f, f_i \in C^\infty(M)\} \) is a complemented submodule \( P_{\text{sgn}}(\Omega^k(C^\infty(M))) \) of \( \Omega^k(C^\infty(M)) \), comprising the antisymmetric part, where \( \text{sgn}(\sigma) \) denotes the sign of the permutation \( \sigma \). It is easy to see that the \( S_k \)-action preserves the inner product on the product Hilbert space. Clearly, the above arguments go through if we replace \( C^\infty(M) \) by any unital *-subalgebra \( A \). In fact, the \( S_k \)-action comes from the fibre-wise permutation on the bundle \( T^*M \otimes T^*M \otimes \cdots \otimes T^*M \) (\( k \) copies). It also preserves the inner product of the product Hilbert space structure coming from the inner product on \( T^*_nM \) given by the Riemannian metric, hence becomes a unitary representation on each fibre. From this, it follows easily that the idempotent \( P_\chi \)'s are orthogonal projections satisfying \( \sum_\chi P_\chi = \text{Id} \), \( P_\chi P_{\chi'} = 0 \) if \( \chi \neq \chi' \). Therefore, we have a decomposition of \( \Omega^k(A) \) into mutually orthogonal (w.r.t. \( \langle \langle \cdot, \cdot \rangle \rangle \)) subspaces \( P_\chi(\Omega^k(A)) \).

In the special case \( k = 2 \), we write \( F_s(A) = (I - P_{\text{sgn}})(\Omega^2(A)) \) (‘symmetric submodule’) and \( \Lambda^2(A) = P_{\text{sgn}}(\Omega^2(A)) \) (‘antisymmetric submodule’). We have \( \Omega^2(A) = F_s(A) \oplus \Lambda^2(A) \), so that \( \Lambda^2(A) = F_s(A) \perp := \{\omega \in \Omega^2(A) : \langle \langle \omega, \eta \rangle \rangle = 0 \forall \eta \in F_s(A)\} \). We also note the following equivalent description of \( F_s(A) \) (see, e.g. [19], page 101):

**Proposition 2.4** The submodule \( F_s \) is spanned (as a right \( A \)-module) by \( \sum_i (df_i \otimes_A dg_i) \), with \( f_i, g_i \) such that \( \sum_i f_i g_i = 0 \) (finite sum).

We remark that if \( A \) is a Fréchet dense subalgebra of \( C^\infty(M) \), then \( \Lambda^k(A) := \text{Sp} \{fdf_1 \wedge \cdots \wedge df_k : f, f_i \in A\} \) is dense in the Fréchet Hilbert module \( \Lambda^k(C^\infty(M)) \) for all \( k = 1, \ldots, n \). Let \( E \) be a smooth, hermitian vector bundle over \( M \) and \( E \) be the \( C^\infty(M) \)-module of smooth sections of \( E \). For a \( C^* \) algebra \( Q \), there is a natural \( C^\infty(M,Q) \)-bimodule structure as well as \( C^\infty(M,Q) \)-valued inner product \( \langle \langle \cdot, \cdot \rangle \rangle \) on \( E \otimes Q \), satisfying the following:

\[
(f \otimes q_1)(s \otimes q') (g \otimes q_2) = f s g \otimes q_1 q_2 q', \quad \langle \langle s \otimes q, s' \otimes q' \rangle \rangle = \langle \langle s, s' \rangle \rangle c \otimes q' q',
\]
Thus defined by the condition $\Lambda$ of for every bounded linear functional $\omega$ and $\Omega = \sum U$ we define $X$ valued sections, i.e. $\Omega : M \to \bigcup \theta E_m \otimes Q$, for some $\Omega \in C^\infty(M, Q)$. We topologize $E \otimes Q$ with the weakest locally convex topology making the map $\Psi \mapsto \langle \langle \Psi, \Psi \rangle \rangle \in C^\infty(M, Q)$ continuous with respect to the Fréchet topology. The corresponding completion of $E \otimes Q$ will be denoted by $E \overline{\otimes} Q$ and it also inherits by continuity the $C^\infty(M, Q)$ bimodule structure as well as the $C^\infty(M, Q)$-valued inner product.

An element $X \in E \otimes Q$ can be thought of as a smooth, $Q$-valued section. To see this, fix $m \in M$ and choose smooth sections $s_1, \ldots, s_k$ (where $k$ is the rank of $E$) such that $\{s_1(m), \ldots, s_k(m)\}$ is an orthonormal basis of the fibre $E_m$ of $E$ at $m$. We define $X(m) := \sum s_i(m) \otimes X_i(m) \in E_m \otimes Q$, where $X_i(m) = \langle \langle s_i \otimes 1, X \rangle \rangle(m)$. It is easily seen that the definition does not depend on the choice of $s_1, \ldots, s_k$, except their values at $m$. One can check this in case $X$ is chosen from the algebraic tensor product $E \otimes Q$, and by density, it follows for a general $X$. In fact, we have the following:

**Lemma 2.5** There is a one-to-one correspondence between elements of $E \otimes Q$ and the smooth $Q$-valued sections, i.e. maps $X : M \to \bigcup \theta E_m \otimes Q$, such that $X(m) \in E_m \otimes Q$ for all $m$, and for every $\xi \in E$, $m \mapsto \langle \langle \xi(m) \otimes 1, X(m) \rangle \rangle \in Q$ is smooth.

**Proof:**
We have already seen one direction of this statement. To see the reverse direction, let $X$ be a smooth $Q$-valued section of the bundle $E$. Choose a finite open cover $U_\alpha, \alpha = 1, \ldots, l$ (say) such that $E|_{U_\alpha}$ is trivial. Let $f_\alpha, \alpha = 1, \ldots, l$ be the corresponding smooth partition of unity and for each $\alpha$, choose smooth sections $\{s_\alpha^i, i = 1, \ldots, k\}$ such that $\{s_\alpha^i(m), i = 1, \ldots, k\}$ is a basis of $E_m \forall m \in U_\alpha$. We can write $X(m)$ in terms of this basis, say, $X(m) = \sum_i s_\alpha^i(m) \otimes q^i(m)$, where $q^i(m) \in Q$. It is easily seen that $m \mapsto q^i(m)$ defines an element of $C^\infty(M, Q)$. Hence, $X = \sum \alpha_i f_\alpha(s_\alpha^i \otimes 1) q^i$, which is an element of $E \otimes Q$. □

In particular, we will identify elements $\Omega \in \Omega^1(C^\infty(M)) \otimes Q$ with $Q$-valued smooth one forms, i.e. $\Omega : M \to \bigcup_{m \in M} (T^*_m M \otimes Q)$, such that $\Omega(m) \in T^*_m M \otimes Q$ for all $m \in M$ and for any local coordinate chart $(U, (x_1, \ldots, x_n))$ around $m \in M$, we have $\Omega(x) = \sum_{i=1}^n dx_i(x) \otimes \Omega_i(x) \forall x \in U$, for some $\Omega_i \in C^\infty(M, Q), i = 1, \ldots, n$. We will usually write $dx_i(x) \otimes \Omega_i(x)$ as $dx_i(x) \Omega_i(x)$ and $\Omega = \sum_{i=1}^n dx_i \Omega_i$ on $U$.

For a smooth vector field $X$ defined on some open subset $\text{Dom}(X)$ of $M$ and $F \in C^\infty(M, Q)$, we define $X(F)(m) := \frac{d}{dt}|_{t=0} F(\gamma(t))$, where $m \in \text{Dom}(X)$ and $\gamma$ is the integral curve for $M$ passing through $m$. We also define $dF \equiv (d \otimes \text{id})(F) \in \Omega^1(C^\infty(M)) \otimes Q$ for $F \in C^\infty(M, Q)$ by the following:

$$(dF)(m) := \sum_{i=1}^n dx_i(m) \frac{\partial F}{\partial x_i}(m),$$

for $m \in M$ and for any local coordinate chart $(U, (x_1, \ldots, x_n))$ around $m$. Clearly this is uniquely defined by the condition

$$\omega(dF(m)) = d(F_\omega)(m)$$

for every bounded linear functional $\omega$ on $Q$, where $F_\omega \in C^\infty(M)$ is given by $F_\omega(m) := \omega(F(m))$. Thus $dF$ does not depend on the choice of the local coordinates.

Similarly, we consider $Q$-valued $k$-forms $\Lambda^k(M, Q) \equiv \Lambda^k(C^\infty(M)) \otimes Q$. We also define the differential and the wedge product of $Q$-valued forms. For $\Omega \in \Lambda^k(M, Q)$, $\Theta \in \Lambda^l(M, Q)$, we have $\Omega \wedge \Theta \in \Lambda^{k+l}(M, Q)$ given by $(\Omega \wedge \Theta)(m) := \Omega(m) \wedge \Theta(m)$. Where $\wedge : (C^\infty(Q) \times (C^\infty(Q) \to (C^k \wedge C^l) \otimes Q$ is defined by $(v \otimes q) \wedge (v' \otimes q') := v \wedge v' \otimes qq'$. For a $k$-form $\omega$, we denote by $\overline{\omega}$ the form obtained by fibre-wise complex conjugation. Let $\Lambda^k(M)_{\mathbb{R}}$ denote the module of forms $\omega$ such that $\overline{\omega} = \omega$. This is a module over $C^\infty(M)_{\mathbb{R}}$. We also have an analogue of complex conjugation on $\Lambda^k(M, Q)$ coming from the natural conjugation on $T^*_m(M) \otimes Q$, namely $(v \otimes q) = \overline{\sigma} \otimes q^*$. 7
2.4 Basics of the normal bundle

Let \( M \subseteq \mathbb{R}^N \) be a smooth embedded submanifold of \( \mathbb{R}^N \) without boundary. For each point \( x \in M \) define the space of normals to \( M \) at \( x \) to be

\[ N_x(M) = \{ v \in \mathbb{R}^N : v \perp T_x(M) \} \]

The total space \( N(M) \) of the normal bundle is defined to be

\[ N(M) = \{ (x, v) \in M \times \mathbb{R}^N ; v \perp T_x(M) \} \]

with the projection \( \pi \) on the first coordinate. Define \( N_\epsilon(M) = \{ (x, v) \in N(M) : ||v|| \leq \epsilon \} \) (see page no. 153 of [26]).

**Lemma 2.6**

(i) If a compact \( n \)-manifold \( M \) without boundary embedded \( \mathbb{R}^N \) has trivial normal bundle, then there exist an \( \epsilon > 0 \) and a global diffeomorphism \( F : M \times B^{N-n}_\epsilon(0) \to N_\epsilon(M) \subseteq \mathbb{R}^N \) given by

\[ F(x, u_1, u_2, ..., u_{N-n}) = x + \sum_{i=1}^{N-n} \xi_i(x) u_i \]

where \( B^{N-n}_\epsilon(0) \) is the closed ball in \( \mathbb{R}^{N-n} \) of radius \( \epsilon \) centered at 0, \( (\xi_1(x), ..., \xi_{N-n}(x)) \) is an orthonormal basis of \( N_x(M) \) for all \( x \), and \( x \mapsto \xi_i(x) \) is smooth \( \forall i = 1, ..., (N-n) \).

(ii) The map \( \pi_F : C^\infty(N_\epsilon(M)) \to C^\infty(M \times B^{N-n}_\epsilon(0)) \) given by \( \pi_F(f)(x, u_1, u_2, ..., u_{N-n}) = f(x + \sum_{i=1}^{N-n} \xi_i(x) u_i) \) is an algebra isomorphism.

**Proof:**

(i) is a consequence of the tubular neighbourhood lemma. For the proof see [26]. Proof of (ii) is straightforward. \( \square \).

We now introduce the notion of stably parallelizable manifolds.

**Definition 2.7**

A manifold \( M \) is said to be stably parallelizable if its tangent bundle is stably trivial.

From the discussion following Theorem 7.2 of Chapter 9 in [18] (see also [27]), we get the following:

**Proposition 2.8**

A manifold \( M \) is stably parallelizable if and only if there exists an embedding of \( M \) into some euclidean space with trivial normal bundle. Moreover, given a Riemannian structure on a stably parallelizable manifold \( M \), we can choose the embedding to be isometric.

We note that parallelizable manifolds (i.e. which have trivial tangent bundles) are in particular stably parallelizable. Moreover, given any compact Riemannian manifold \( M \), its orthonormal frame bundle \( O_M \) is parallelizable.

3 Smooth and inner product preserving actions of a CQG on a manifold

Throughout this section, let \( M \) be a compact, smooth manifold possibly with boundary, unless otherwise mentioned.
3.1 Smooth and inner product preserving actions

Definition 3.1 A C*-action $\alpha$ of a CQG $Q$ on $C(M)$ is called smooth if
(i) $\alpha(C(M)) \subseteq C(M, Q)$,
(ii) the $\mathbb{C}$-linear span of $\alpha(C(M))(1 \otimes Q)$ is Fréchet-dense in $C(M, Q)$.

In case $M$ has a smooth boundary $\partial M$, we also require that $\alpha$ maps the $C^*$ ideal $I := \{f \in C(M) : f|\partial M = 0\}$ into $I \otimes Q$. We’ll often say that $\alpha$ is a smooth action of $Q$ on $M$.

Remark 3.2 It follows from the Closed Graph Theorem that any smooth action is automatically continuous with respect to the Fréchet topologies on $C(M)$ and $C(M, Q)$.

An almost verbatim adaptation of arguments in [22] gives us the following analogue of Proposition [23]:

Proposition 3.3 A $C^*$ action $\alpha$ on $C(M)$ is smooth iff $\alpha(C(M)) \subset C(M, Q)$ and there is a Fréchet dense subalgebra $A$ of $C(M)$ over which $\alpha$ is algebraic.

Given a smooth action $\alpha$ let us introduce the following
Notation: For $x \in M$ let us denote by $Q_x$ the unital $\ast$-subalgebra of $Q$ generated by
$$\{\alpha(f)(x), ((\phi \otimes \text{id})\alpha(g))(x), f, g \in C(M), \phi \in \chi(M)\}, \tag{3}$$
where $\chi(M)$ is the set of smooth vector fields on $M$. Given a local coordinate $(x_1, \ldots, x_n)$ around a point $m$, by choosing a smooth vector field which agrees with $\frac{\partial}{\partial x_i}$ in a neighbourhood of $m$, we see that $\frac{\partial}{\partial x_i}|_m \alpha(f)$ belongs to $Q_m$.

Definition 3.4 Suppose that $M$ has a Riemannian structure with the corresponding $C(M)$ valued inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on $\Omega^1(C(M))$. A smooth action $\alpha$ on $M$ is said to preserve the inner product if
$$\langle\langle d\alpha(f), d\alpha(g) \rangle\rangle = \alpha(\langle\langle df, dg \rangle\rangle) \tag{4}$$
for all $f, g \in C(M)$.

It is easy to see, by the Fréchet continuity of the maps involved that it is enough to have (4) for $f, g$ varying in some Fréchet dense unital $\ast$-subalgebra of $C(M)$.

Theorem 3.5 Let $Q$ be a reduced CQG, i.e. the Haar state is faithful on $Q$. Given a smooth action $\alpha$ of a CQG $Q$ on $M$ the following are equivalent:
(i) For every $x \in M$, we have
$$\alpha(f)(x)(d \otimes \text{id})(\alpha(g))(x) = (d \otimes \text{id})(\alpha(g))(x)\alpha(f)(x), \tag{5}$$
for all $f, g \in C(M)$ and $x \in M$.
(ii) The manifold $M$ has a Riemannian structure such that $\alpha$ is inner product preserving.

Proof:
Fix a Fréchet-dense unital $\ast$-subalgebra $A$ over which $\alpha$ is algebraic. For proving the implication (ii) $\rightarrow$ (i), we consider $F = \alpha(f)\alpha(g) - \alpha(g)\alpha(f) \in \Omega^1(C(M)) \otimes Q$ and using (4), verify that $\langle\langle F, F \rangle\rangle = \alpha(\langle\langle df - dgf, df - dgf \rangle\rangle) = 0$. This proves $F = 0$, hence (i).

The proof of the converse is basically an adaptation of arguments in [14]. We indicate only the steps where a modification of the arguments of [14] is necessary. We note from [16] that the antipode $\kappa$ (say) is defined and norm bounded on the whole of $Q$. We first observe that the map $\Psi$ introduced in Lemma 3.2 of [14] can be actually defined on a larger set, namely,
for \( F \in C^\infty(M) \otimes Q_0 \). Indeed, as the multiplication map \( m \) is defined on \( Q \otimes Q_0 \), we set \( \Psi : C^\infty(M) \otimes Q_0 \to C^\infty(M) \) by

\[
\Psi(F)(x) = h \circ m \circ (\kappa \otimes \text{id}) ((\alpha \otimes \text{id})(F)(x)),
\]

where \( h \) denotes the Haar state. Then the proof of the complete positivity of \( \Psi \) as in Lemma 3.2 of [14] goes through verbatim. Next, choose any Riemannian metric on \( M \) and denote the corresponding \( C^\infty(M) \)-valued inner product by \( \langle \cdot, \cdot \rangle \). Define

\[
k(f, g) := \Psi((df_{(0)}, dg_{(0)}) \otimes f_{(1)}^* g_{(1)}),
\]

for \( f, g \in \mathcal{A} \). Arguing along the lines of the proof of Lemma 3.3 of [14] we can prove \( \Psi(F\alpha(f)) = \Psi(F)f \) for \( F \in C^\infty(M) \otimes Q_0, f \in \mathcal{A} \) from this, it follows that \( k(f, gh) = k(f, g)h + gk(f, h) \), i.e. for any fixed \( f \), the map \( g \mapsto k(f, g) \) is a derivation of \( \mathcal{A} \). As \( \mathcal{A} \) is Fréchet dense in \( C^\infty(M) \), for any \( x \in M \) we can find \( f_1, \ldots, f_n \) from \( \mathcal{A} \) such that \((f_1, \ldots, f_n)\) is a set of local coordinates in a neighbourhood \( U \) (say) of \( x \). The derivation \( k_f(\cdot) \) of the algebra generated by the coordinates \( f_1, \ldots, f_n \) must be of the form \( k(f, \cdot) = Y^{U,f_1,\ldots,f_n}_{(\cdot)}(\cdot) \), where \( Y^{U,f_1,\ldots,f_n}_{(\cdot)} \) a vector field defined and smooth on \( U \). The local vector fields \( Y^{U,f_1,\ldots,f_n}_f \) do patch up consistently to give a smooth globally defined vector field \( Y_f \) on \( M \). Indeed, given two such local coordinates \((U, (f_1, \ldots, f_n))\) and \((V, (g_1, \ldots, g_n))\) with \( U \cap V \) nonempty and \( f_i, g_j \in \mathcal{A} \) \( \forall i, j \), we have

\[
Y^{U,f_1,\ldots,f_n}_{(\cdot)}(\phi)(x) = Y^{V,g_1,\ldots,g_n}_{(\cdot)}(\phi)(x) = k(f, \phi)(x)
\]

for all \( \phi \in \mathcal{A}, x \in U \cap V \). By the Fréchet density of \( \mathcal{A} \) and the obvious Fréchet continuity of the locally smooth vector fields, they agree for all \( \phi \in C^\infty(M) \), i.e. \( Y^{U,f_1,\ldots,f_n}_{(\cdot)} = Y^{V,g_1,\ldots,g_n}_{(\cdot)} \) on \( U \cap V \).

Clearly, for any \( \eta \in \Omega^1(\mathcal{A}) \), the map \( \mathcal{A} \ni f \mapsto \eta(Y_f) \) is a derivation, and arguing as before, we get a globally defined smooth vector field \( Z_\eta \) (say) such that \( \eta(Y_f) = d\eta(Z_\eta) = df(Z_\eta) \). Define a sesquilinear form on \( \Omega^1(M) \) by \( \langle \langle \omega, \eta \rangle \rangle := \overline{\eta}(Z_\eta) \). Clearly,

\[
\langle \langle df, dg \rangle \rangle = d\overline{\eta}(Z_{dg}) = dg(Y_f) = Y_f(g) = k(f, g)
\]

for \( f, g \in \mathcal{A} \). The rest of the arguments for proving that \( \{\langle \langle \cdot, \cdot \rangle \rangle\}, x \in M \) indeed gives the required invariant Riemannian metric, are very similar to those in [14]. In particular, the proof of Lemma 3.4 of [14] goes through verbatim and in the proof of Theorem 3.1 of [14], it is easy to observe that the assumption \( \langle \langle df_{(0)}, dg_{(0)} \rangle \rangle \otimes f_{(1)}^* g_{(1)} \in \mathcal{A} \otimes Q_0 \) is not really necessary, as the arguments go through even if \( \langle \langle df_{(0)}, dg_{(0)} \rangle \rangle \otimes f_{(1)}^* g_{(1)} \in C^\infty(M) \otimes Q_0 \).

### 3.2 Lift of inner product preserving actions to bundles of one and higher forms

From now on throughout Sections 3 and 4, we fix a smooth, faithful action \( \alpha \) on \( M \) and a Fréchet-dense unital \(*\)-subalgebra \( \mathcal{A} \) on which \( \alpha \) is algebraic. In the present subsection we also assume that \( \alpha \) preserves some Riemannian inner product.

Using the condition (14), we easily see that, whenever \( \omega = \sum_{i=1}^l f_i dg_i = 0 \), where \( f_i, g_i \in \mathcal{A} \), we have \( \sum_{i} \alpha(f_i)(d \otimes \text{id})(\alpha(g_i)) = 0 \). Thus, we get a well-defined map \( \omega \mapsto d\alpha_{(1)}(\omega) := \sum_{i} \alpha(f_i)(d \otimes \text{id})(\alpha(g_i)) \) from \( \Omega^1(\mathcal{A}) \) to \( \Omega^1(\mathcal{A}) \otimes Q_0 \). Moreover, this has the properties that

\[
d\alpha_{(1)}(\omega f) = d\alpha_{(1)}(\omega)\alpha(f) = \alpha(f)d\alpha_{(1)}(\omega), \quad \langle \langle d\alpha_{(1)}(\omega), d\alpha_{(1)}(\eta) \rangle \rangle = \alpha(\langle \langle \omega, \eta \rangle \rangle)
\]

for all \( \omega, \eta \in \Omega^1(\mathcal{A}), f \in \mathcal{A} \). By Fréchet continuity of \( \alpha \), we can extend \( d\alpha_{(1)} \) to a continuous \( \mathbb{C} \)-linear map from \( \Omega^1(C^\infty(M)) \) to \( \Omega^1(C^\infty(M)) \otimes Q \) satisfying similar properties. This motivates the following definition.
Definition 3.6 Let $E$ be the module of smooth sections of a smooth hermitian bundle $E$ over $M$. A $\mathbb{C}$-linear map $\Gamma : E \to \mathcal{E} \otimes Q$ is said to be an $\alpha$-equivariant unitary representation of $Q$ on $\mathcal{E}$ if for all $\xi, \xi \in E$, $f \in C^\infty(M)$,
1. $\Gamma(f \xi) = \Gamma(\xi) \alpha(f) = \alpha(f) \Gamma(\xi)$,
2. $\langle \Gamma(\xi), \Gamma(\xi) \rangle = \alpha(\langle \xi, \xi \rangle)$,
3. $(\Gamma(\text{id} \otimes Q) \Gamma)(\xi)(m) = \Delta(\xi)(m)$ for all $\xi \in E, m \in M$ (co-associativity),
4. The right $Q$-linear span of $\Gamma(\xi)$ is Fréchet-dense in $\mathcal{E} \otimes Q$ (non degeneracy).

Note that the condition 2. in the definition above allows one to define $(\Gamma(\text{id} \otimes Q) \Gamma)$ as the completion of $\Gamma(\text{id})$ to the completion of $\mathcal{E} \otimes Q$. We have also used the identification of $\Gamma(\xi)$ as a $Q$-valued smooth section. Moreover, 3. can be interpreted (formally) as $(\Gamma(\text{id} \otimes Q) \Gamma) = (\text{id} \otimes \Delta)$. We also simply say ‘equivariant unitary representation’ if the action $\alpha$ is understood from the context.

In particular, when $E$ is the trivial $C^\infty(M)$-bimodule of rank $N$, we have the following:

Lemma 3.7 Given an $\alpha$ equivariant unitary representation $\Gamma$ of $Q$ on $\mathbb{C}^N \otimes C^\infty(M)$ such that $\Gamma(e_i \otimes 1) = \sum_N e_j \otimes b_{ij}, b_{ij} \in C^\infty(M, Q)$ for all $i, j = 1, ..., N$, where $\{e_i; i = 1, ..., N\}$ is an orthonormal basis of $\mathbb{C}^N$, then $B = ((b_{ij}))_{i,j=1,\ldots,N}$ is a unitary element of $M_N(C^\infty(M, Q))$.

Proof:
Clearly, $B$ can also be thought of as an element of the $C^*$ algebra $M_N(\mathcal{C})$, where $\mathcal{C} := C(M) \otimes Q$, which is isomorphic with the set of right $C$-linear, adjointable maps on the Hilbert module $\mathcal{F} := \mathbb{C}^N \otimes \mathcal{C}$. Viewing $B$ as a (right $C$-linear) map on $\mathcal{F}$, we first claim that $B^*B = I_\mathcal{F}$. Indeed, $\langle (\Gamma(e_i \otimes 1), \Gamma(e_j \otimes 1)) \rangle = \alpha(\langle e_i \otimes 1, e_j \otimes 1 \rangle)$, which implies
$$\sum_{k=1}^N b_{ki}^* b_{kj} = \alpha(\delta_{ij} 1) = \delta_{ij} 1_M,$$
proving the claim. It now suffices to prove that the range of $B$ is dense in $\mathcal{F}$. To this end, observe that
$$\Gamma(e_i \otimes f)q = \sum_j e_j \otimes b_{ji} \alpha(f)(1 \otimes q) = B(e_i \otimes 1_M) \alpha(f)(1 \otimes q) \subseteq \text{Im}(B),$$
for $f \in C^\infty(M, q \in \mathbb{Q})$. Thus, $\text{Im}(B)$ contains the right $Q$-linear span of $\Gamma(\mathbb{C}^N \otimes C^\infty(M))$, which is Fréchet-dense in $\mathbb{C}^N \otimes C^\infty(M, Q)$, hence norm-dense in $\mathcal{F}$ as well. $\square$

It is clear that $d \alpha(1)$ is an equivariant unitary representation on $\Omega^1(C^\infty(M))$. We now go further to construct similar representation on higher forms.

Lemma 3.8 For each $k \geq 1$, there is an equivariant unitary representation $U^{(k)}$ of $Q$ on each $\Omega^k(C^\infty(M))$, given on the dense subspace $\Omega^k(\mathcal{A})$ by
$$U^{(k)}(\omega_1 \otimes A \omega_2 \otimes A \cdots \otimes A \omega_k) = \omega_1(0) \otimes_A \cdots \otimes_A \omega_k(0) \otimes A \omega_1(1) \omega_2(1) \cdots \omega_k(1),$$
where $\omega_i \in \Omega^1(\mathcal{A}) \forall i$ and the Sweedler type notation for comodule maps has been used.

Proof:
For $k = 1$, $U^{(1)} = d \alpha(1)$ and there is nothing to prove. Let us prove the result for $k = 2$ only, as the proof for $k \geq 3$ is very similar. Using the condition 2. of Definition 3.6 for $d \alpha(1)$, we verify that $U^{(2)}$ is well defined. Indeed, for $\omega, \eta \in \Omega^1(\mathcal{A}), f \in \mathcal{A}$, we have
$$\langle \omega f \rangle(0) \otimes_A \eta(0) \otimes \langle \omega f \rangle(1) \eta(1) = \omega(0) f(0) \otimes_A \eta(0) \otimes \omega(1) f(1) \eta(1) = \omega(0) \otimes_A (f \eta)(0) \otimes \omega(1) (f \eta)(1).$$
Next, $U^{(2)}$ on $\Omega^2(A)$ is clearly an algebraic (co)-representation of $Q_0$, so it satisfies the co-associativity condition 3. and also $(id \otimes \epsilon) \circ U^{(2)} = id$. Moreover, using conditions 1. and 2. of Definition 3.6 for $U$, we have for $x \in M$,

$$(\langle U^{(2)}(x) \otimes A \eta, U^{(2)}(x' \otimes A \eta') \rangle)(x) = \langle \langle \omega(0), \omega'(0) \rangle(x), \langle \eta(0), \eta'(0) \rangle(x) \rangle_1 \omega(1) \omega'(1) \eta(1) \eta'(1) = \langle \langle \omega(0), \eta(0) \rangle(x) \eta(1) \omega(1) \eta'(1) \rangle(x) = \langle \langle \eta(0), \eta(0) \rangle(x) \eta'(1) \alpha(\langle \langle \omega, \omega' \rangle(x) \rangle_1 \eta(1) \rangle(x).$$

which, by (3), becomes

$$\alpha(\langle \langle \omega, \omega' \rangle(x) \rangle_1 \eta(1) \rangle(x).$$

This proves condition 2. for $U^{(2)}$ on $\Omega^2(A)$, hence $U^{(2)}$ extends to $\Omega^2(C^\infty(M))$ by continuity, and the extension is easily seen to satisfy the conditions 1, 2 and 3. Finally, as $U^{(2)}(\Omega^2(A) \otimes Q_0$ is total in $\Omega^2(A) \otimes Q_0$, by continuity we get the density condition of 4. This completes the proof. \hfill \Box

Recall the subalgebra $Q_x$ defined by (3) in Subsection 3.1.

**Lemma 3.9** For $x \in M$, $Q_x$ is commutative.

**Proof:**

Recall the decomposition of $\Omega^2(A) = F_s(A) \oplus \Lambda^2(A)$ and the description of $F_s(A)$ given by Proposition 2.4. Let $f_i, g_i \in A$ be such that $\sum_i f_idg_i = 0$. By applying $d\alpha(1)$ we get $\sum_i f_i(0)dg_i(0) \otimes f_i(1)g_i(1) = 0$, hence $\sum f_i(0)dg_i(0)\phi(f_i(1)g_i(1)) = 0$ for every linear functional $\phi$ on $Q_0$. So we have

$$(id \otimes \phi)(U^{(2)}(\sum_i df_i \otimes A dg_i)) = \sum_i df_i(0) \otimes A dg_i(0)\phi(f_i(1)g_i(1)) = 0.$$  

Thus, $U^{(2)}(F_s(A)) \subseteq F_s(A) \otimes Q_0$. Moreover, if $\omega \in \Lambda^2(A) = F_s(A)^\perp$, we have

$$\langle \langle U(\omega), U(\eta)q \rangle = \langle \langle \omega, \eta \rangle \rangle q = 0$$

for any $\eta \in F_s(A), q \in Q_0$. But $U^{(2)}|_{F_s(A)}$ is an algebraic (co)-representation which implies that $U^{(2)}(F_s(A))Q_0 = F_s(A) \otimes Q_0$. In particular, $\eta \otimes 1 \in U^{(2)}(F_s(A))Q_0$, which implies

$$\langle \langle \eta, (id \otimes \phi)(U^{(2)}(\omega)) \rangle = (id \otimes \phi)(\langle \langle \eta \otimes 1, U^{(2)}(\omega) \rangle) = 0$$

for $\omega \in \Lambda^2(A), \eta \in F_s(A), \phi \in Q_0$. Hence $U^{(2)}(\Lambda^2(A)) \subseteq \Lambda^2(A) \otimes Q_0$ and the restriction of $U^{(2)}$ to $\Lambda^2(A)$ gives a unitary equivariant representation of $Q_0$ on $\Lambda^2(A)$.

Now, choose smooth one-forms $\{\omega_1, \ldots, \omega_n\}$ such that they form a basis of $T^*M$ at $x$. Recall the notation $X_\omega$ from Subsection 2.3. Clearly, it is enough to prove that $F_l(x) := X_{\omega(1)}(\alpha(f))(x)$ and $G_j(x) := X_{\omega(j)}(\alpha(g))(x)$ commute for $f, g \in C^\infty(M)$ and $\forall i, j = 1, \ldots, n$. Indeed, $\alpha(f))(x) = \sum_i \omega_i(x)F_i(x), \alpha(g))(x) = \sum_i \omega_i(x)G_i(x)$. Now $U^{(2)}$ leaves invariant the submodules of symmetric and antisymmetric tensor product of $\Lambda^1(A)$, in particular, $C^{\alpha}_{ij} = C^{\alpha}_{ji}$, $C^{\alpha}_{ij} = -C^{\alpha}_{ij}$ for all $i, j$, where $C^{\alpha}_{ij}$ and $C^{\alpha}_{ij}$ denote the $Q$-valued coefficient of $w_i(x) \otimes w_j(x)$ in the expression of $U^{(2)}(df \otimes dg + dg \otimes df)|_x$ and $U^{(2)}(df \otimes dg - dg \otimes df)|_x$ respectively. By a simple calculation using these relations, we get the commutativity of $F_l(x), G_j(x)$ for all $i, j$. \hfill \Box

We denote the following $*$-subalgebra of $C^\infty(M, Q)$ by $C$, which is commutative by Lemma 3.9.

$$C := \{ F : F(x) \in Q_x \ \forall x \}.$$  

Viewing $d\alpha(1)(\omega)$ as a $Q$-valued smooth section, let us write $d\omega(0)(m) \otimes \omega(1) \in T^*_mM \otimes Q_0$ for $\omega \in \Omega^1(A), m \in M$. Then it follows from the above lemma that $\forall \omega, \eta \in \Omega^1(A),$  

$$\omega(0)(m) \otimes \eta(0)(m) \otimes \omega(1)\eta(1) = \omega(0)(m) \otimes \eta(0)(m) \otimes \eta(1)\omega(1).$$  

(6)
Proposition 3.10 If $\alpha$ is a Riemannian inner product preserving action, then for every $k \geq 0$, there exist an $\alpha$-equivariant representation $d\alpha(\cdot)$ of $\mathcal{Q}$ on $\Lambda^k(C^\infty(M))$ given by

$$d\alpha(\cdot)(f_0df_1 \wedge \ldots \wedge df_k) = \alpha(f_0)d(\alpha(f_1)) \wedge \ldots \wedge d(\alpha(f_k)).$$

Moreover, we have

$$d\alpha(k+i)(\omega \wedge \eta) = d\alpha(k)(\omega) \wedge d\alpha(i)(\eta)$$

for $\omega \in \Lambda^k(M, \mathcal{Q})$, $\eta \in \Lambda^i(M, \mathcal{Q})$. Here we have used the wedge product between $\mathcal{Q}$-valued forms introduced earlier.

Proof:

It follows from (6) that $U^{(k)}$ commutes with the action of $S_k$ on $\Omega^k(C^\infty(M))$, hence leaves each of the spectral subspaces for the $S_k$-action invariant, in particular the range of $P_{\text{sgn}}$, i.e. $\Lambda^k(C^\infty(M))$. We take the restriction of $U^{(k)}$ on $\Lambda^k(C^\infty(M))$ as the definition of $d\alpha(\cdot)$.

It is clear from the definition that $d\alpha(\cdot)(f\omega) = \alpha(f)d\alpha(\omega)$. We can now prove (7) by considering $\omega = df_1 \wedge \ldots \wedge df_k$, $\eta = gdg_1 \wedge \ldots \wedge dg_l$ and using the commutativity of $\mathcal{Q}_m$ $\forall m$. $\square$

3.3 Lift to the orthonormal frame bundle

Let $O(M)$ be the bundle of orthonormal frames of $M$. We can identify this as a subbundle of the direct sum of $n$ copies of cotangent bundle, say $E = T^*_m(M) \oplus \ldots \oplus T^*_m(M)$. Let $\xi$ denote the complex conjugate of a complexified co-tangent vector $\xi \in T^*_m(M)$, and for a (complexified) one-form $\omega$, define $\overline{\omega}(m) = \omega(m)$ for all $m \in M$.

Indeed,

$$O(M) = \{ (m, \omega) : m \in M, \omega = (\omega_1, \ldots, \omega_n), \omega_i \in T^*_m(M), \overline{\omega_i} = \omega_i \forall i < \omega_i, \omega_j > = \delta_{ij} \}.$$
Define $T^{U ω}_{ij} \equiv T^{U}_{ij} \in C^∞(O(M), Q)$ by:

$$T^{U}_{ij}(m, ω) := \langle (ω_i ⊗ 1_Q, dα(1)(ω_j))⟩^ε ⊕ Q(m),$$

where $⟨⟨, ⟩⟩^ε ⊕ Q$ denotes the $C^∞(M, Q)$ valued inner product as before and $ω = (ω_1, ..., ω_n)$. Clearly, $T^{U}_{ij}(e) ∈ Q(π(e)) \forall e ∈ O(M), \forall i, j = 1, ..., n$, hence $T^{U}_{ij}$’s commute among themselves.

Let us fix a coordinate neighbourhood $U$, $U$-orthonormal one forms $ω_j$’s and consider the corresponding $t^{U}_{ij}$, $T^{U}_{ij}$’s.

**Lemma 3.11** For any smooth real-valued function $χ$ supported in $U$, we have

$$(α(χ) ∘ π) \sum_j T^{U}_{ij}T^{U}_{ij} = (α(χ) ∘ π)δ_{il},$$

for all $i, l = 1, ..., n$.

**Proof:**

Fix $e = (m, (ω_1, ..., ω_n)) ∈ O(M)$, $m = π(e)$. Let $γ$ be a character (multiplicative linear functional) on the commutative $C^*$ algebra $Q_m$ and $u_j := (id ⊗ γ)(dα(1)(ω_j)(m)) ∈ T^*_m M$. By a simple calculation using the facts that $dα(1)$ is inner-product preserving and $ω_j$’s form an orthonormal basis of $T^* M$ at every point in the support of $χ$, we obtain

$$γ(α(χ)(m))^2(u_i, u_l) = γ\left(⟨⟨dα(1)(χω_i), dα(1)(χω_j)⟩⟩^ε ⊕ Q(m)\right) = γ(α(⟨⟨χω_i, ω_j⟩⟩^ε)(m)) = γ(α(χ^2)(m))δ_{il},$$

where $⟨⟨, ⟩⟩$ denotes the inner product of $T^*_m M$. Thus, in case $γ(α(χ)(m)))$ is nonzero, $u_1, ..., u_n$ is an orthonormal basis of $T^*_m M$ and by multiplicativity of $γ$ and self-adjointness of $T^{U}_{ij}$’s it is easy to see that $γ(T^{U}_{ij}(e)) = ⟨ω_i(π(e)), u_j⟩ = (u_j, ω_i(π(e)))$. We have

$$γ\left(α(χ^2)(π(e)) \sum_j T^{U}_{ij}(e)T^{U}_{ij}(e)\right) = γ(α(χ^2)(π(e))) \sum_j ⟨ω_i(π(e)), u_j⟩⟨u_j, ω_i(π(e))⟩ = γ(α(χ^2)(π(e)))(ω_i(π(e)), ω_i(π(e))) = δ_{il} γ(α(χ^2)(π(e))).$$

This equality is also true trivially when $γ(α(χ)(π(e))) = 0$. Hence $(α(χ)(π(e)))^2X(e) = δ_{il}(α(χ)(π(e)))^2$, where $X = \sum_j T^{U}_{ij}T^{U}_{ij}$. As $α(χ)(π(e))$ is self adjoint, the equality follows. $\square$

**Lemma 3.12** Let $U, V$ be two coordinate neighbourhoods, $U \cap V \neq ∅$. Also let $\{ω'_1, ..., ω'_n\}$ and $\{ω''_1, ..., ω''_n\}$ be $U$-orthonormal and $V$-orthonormal one-forms respectively. Then there are smooth functions $f_{jk}$ such that for every $f ∈ C^∞_p(U \cap V)$, we have

$$f(π)T^{V ω''}_{ij} = \sum_k (f(π)T^{V ω''}_{ik})T^{V ω''}_{kj}, \quad (9)$$

$$(α(f) ∘ π)T^{V ω''}_{ij} = \sum_k (α(fT^{V ω''}_{ik} ∘ π)T^{V ω''}_{ik}, \quad (10)$$

14
Proof:
As both \( \omega'_i \)'s and \( \omega''_i \)'s are bases of \( T^*_mM \) at each \( m \in U \), there are smooth functions \( f_{jk} \) such that
\[
f_{ij} = \sum_k f_{jk} \omega'_k
\]
for any \( f \in C^\infty_c(U) \). This implies (9). We obtain (10) by applying \( da(1) \) on both sides of (11). \( \square \)

Lemma 3.13 There is a well-defined, norm-contractive, *-homomorphism \( \eta_U : C^\infty_c(U) \to C^\infty(O(M), Q) \) satisfying
\[
\eta_U((f \circ \pi) I^U_{ij}) = (\alpha(f) \circ \pi) T^U_{ij}.
\]
for all \( f \in C^\infty_c(U) \). Moreover,
(i) \( \eta_U \) is continuous w.r.t. the Fréchet topology.
(ii) \( \eta_U \) does not depend on the choice of the sections \( \omega'_i \)'s.
(iii) For two coordinate neighbourhoods \( U \) and \( V \), with \( U \cap V \) nonempty and for \( f \in C^\infty_c(O(U)) \), \( G \in C^\infty_c(O(V)) \), we have \( \eta_U(FG) = \eta_U(F) \eta_U(G) \).

Proof:
The map \( \phi : O(U) \to U \times K \cong U \times O_n(\mathbb{R}) \) given by \( \phi(e) = (\pi(e), ((t^U_{ij}))_{i,j=1,\ldots,n}) \) is a diffeomorphism. For \( F \in C^\infty_c(O(U)) \), and a given \( \nu = (v_1, \ldots, v_n) \in K \), let \( F_\nu \in C^\infty_c(U) \subset C^\infty(M) \) given by \( F_\nu(m) = (F \circ \phi^{-1})(m, \nu) \), \( m \in U \) and \( 0 \) for \( m \notin U \). Let \( H_m(F) \in C^\infty(K, Q_m) \subset C^\infty(K, \mathbb{Q}) \) be defined by \( H_m(F)(\underline{v}) = \alpha(F_\nu)(m), m \in M \). Clearly, as the maps \( \nu \to F_\nu \) and \( \alpha \) are Fréchet continuous, \( \nu \to H_m(F_\nu) \) is Fréchet continuous too for any fixed \( m \). We want to define an element \( \eta_U(F)(e) \in \mathcal{E}_c(e) \) by specifying \( \eta_U(F)(e)(\gamma) \equiv \gamma(\eta_U(F)(e)) \) for \( \gamma \in \mathcal{Q}_c(e) \), where \( \mathcal{Q}_c(e) \) denotes the set of characters the commutative \( C^* \) algebra \( \mathcal{Q}_c(e) \cong C(\mathcal{Q}_c(e)) \), with the weak \* topology. Let \( \chi \in C^\infty_c(U) \) be such that \( 0 \leq \chi \leq 1 \) and \( \chi \equiv 1 \) on \( \pi(Supp(F)) \), so that \( (\chi \circ \pi)F = F \). Define \( \gamma(\eta_U(F)(e)) = 0 \) if \( \gamma(\alpha(\chi)(\pi(e))) = 0 \). Otherwise, by Lemma 3.13, \( \gamma(e)(\gamma(\eta_U(F)(e))(\gamma(\alpha(\chi)(\pi(e)))) \equiv \gamma(H_m(F)(\tau(e)))). \)

It follows from the continuity of \( H_m(F) \) that \( \gamma \to \gamma(\eta_U(F)(e)) \) is continuous and as \( \alpha \) and \( \gamma \) are *-homomorphisms, hence norm-contractive with
\[
\|\gamma(\eta_U(F)(e))\| \leq \|\chi\| \sup_{m \in M, \nu \in K} |F_\nu(m)|_{\infty} \leq \|F\|_{\infty}.
\]

This allows us to extend \( \eta_U \) as a norm-contractive map from \( C_c(E_U) \) to \( C(E, Q) \). It is also easy to see that \( F \to \gamma(\eta_U(F)(e)) \) is *-homomorphic for every \( \gamma \), which implies \( C_c(E_U) \ni F \to \eta_U(F) \subset C^\infty_c(E) \) is indeed *-homomorphic. To verify (12), it is enough to observe that, by definition, \( H_m((f \circ \pi) I^U_{ij})(\underline{v}) = \alpha(f)(m) v_{ij} \).

At this point, we claim that the definition of \( \eta_U \) does not really depend on the choice of \( \chi \). Indeed, if \( \chi' \) is another smooth function supported in \( U \), with \( \chi' \circ \pi)F = F \), denoting by \( \eta'_U \) the analogue of \( \eta_U \) using \( \chi' \) instead of \( \chi \), we'll have \( F_\nu' = \chi F_\nu = \chi' F_\nu' \) hence
\[
H_m(F) = \alpha(\chi)(m) H_m(F) = \alpha(\chi')(m) H_m(F).
\]
(13)

For a given \( e \) and \( \gamma \), if \( \gamma(\alpha(\chi')(\pi(e))) = 0 \) but \( \gamma(\alpha(\chi)(\pi(e))) \) is nonzero, we have from (13) that
\[
\gamma(H_{\pi(e)}(F)(\cdot)) = 0 = \gamma(\alpha(\chi')(\pi(e))) \gamma(H_{\pi(e)}(F)(\cdot)),
\]
so that \( \gamma(\eta_U(F)(e)) = \gamma(\eta'_U(F)(e)) = 0 \). Otherwise also, the equality of \( \gamma(\eta_U(e)) \) and \( \gamma(\eta'_U(e)) \) follows from (13). This proves our claim.

Next we verify that the definition is also independent of the choice of the one-forms \( \omega'_j, j = 1, \ldots, n \). Let \( \omega''_j, j = 1, \ldots, n \) be another such choice and let \( \gamma''_{ij} = (\theta''_{ij})^U, \Gamma''_{ij} = T''_{ij}^U \). Denote by
There exists a faithful smooth action \( \eta_U \) defined using \( \omega_j \)'s. Clearly, \( \zeta_U \) is uniquely determined by \( \zeta_U((f \circ \pi)\gamma_U^{ij}) \) for all \( f \in C^\infty_c(U) \), hence it suffices to prove \( \eta_U((f \circ \pi)\gamma_U^{ij}) = (\alpha(f) \circ \pi)\Gamma_U^{ij} \). However, this follows from Lemma 3.12 taking \( U = V \), first by applying \( \eta_U \) on the expression of \( \gamma_U^{ij} \) obtained from \([9]\) and then using \([10]\).

We can prove (iii) also by applying Lemma 3.12. In the notation of Lemma 3.12 it is enough to show

\[
\eta_U((f \circ \pi)(g \circ \pi)t_{ij}^{U\omega'_i}V_{kl}^{\omega'_l}) = \eta_U((f \circ \pi)t_{ij}^{U\omega'_i})\eta_U((g \circ \pi)t_{kl}^{U\omega'_l})
\]

\( \forall f \in C^\infty_c(U), \ g \in C^\infty_c(V) \). Substituting \( t_{ij}^{U\omega'_i} \) by \( t_{ij}^{V\omega'_i} \)'s using \([9]\) of Lemma 3.12 and the definition of \( \eta_U \), the left hand side is seen to be equal to \( (\alpha(f) \circ \pi)T_{ij}^U \sum_p(\alpha(g_{f_ip}) \circ \pi)T_{kp}^U \), which coincides with the right hand side by \([10]\) of Lemma 3.12 replacing \((ij)\) by \((kl)\).

To prove the Fréchet continuity of the map at a point \( e_0 \in U \), choose open set \( U_0 \) such that \( e_0 \in U_0 \subset \overline{U_0} \subset U \). Consider the embedding \( \psi \) (say) \( e \mapsto (x_1, \ldots, x_n, t_{ij}^U, i, j = 1, \ldots, n) \) of \( U \times K \subset \mathbb{R}^{n+n^2} \). As the map \( (m, \nu) \mapsto H(m, \nu) := H_m(F)(\nu) \) is smooth on the compact set \( U_0 \times K \) we can choose an open neighbourhood \( W \) of \( \psi(U_0 \times K) \subset \mathbb{R}^{n+n+2} \) such that \( H \circ \psi^{-1} \) extends as a smooth \( \mathcal{Q} \)-valued function \( H_1 \) (say) on \( W \). Choose \( \lambda \in C_c^\infty(\mathbb{R}^{n+n^2}) \) satisfying \( \lambda(z) = 1 \) for all \( z \in \psi(U_0 \times K) \), hence \( \lambda H_1 = H \circ \psi^{-1} \) on \( \psi(U_0 \times K) \). For \( e \in O(U) \), we claim the following:

\[
\eta(F)(e) := \alpha(\chi)(\pi(e)) \int \hat{\lambda H_1}(\mu, \xi)\exp\left(-2\pi i \sum_k |\mu_k x_k(\pi(e))| - 2\pi i \sum_{jl} \xi_{jl} T_{jl}^U(e)\right) d\mu d\xi,
\]

where the integration is over \( \mu, \xi \in \mathbb{R}^n \times \mathbb{R}^n \) and \( \hat{\lambda H_1} \) denotes the Fourier transform of the smooth, compactly supported function \( \lambda H_1 \). It is clear from the above expression and the smoothness of \( T_{ij}^U \) that \( \eta(F) \) is smooth. By adapting arguments of standard Fourier theory for Banach space valued, smooth, compactly supported functions, we have the following estimate by adapting part (d) of Theorem 7.4 of \([24]\):

\[
||\lambda H_1(\mu, \xi)|| \leq C_k(1 + \sum_r |\mu_r|^2 + \sum_{jl} |\xi_{jl}|^2)^{-k},
\]

for every \( k = 0, 1, 2, \ldots \), with some constants \( C_k \). This shows that the right hand side of \([14]\) converges absolutely. Let \( Z \) denote the right hand side of Equation \([14]\). For any given \( \gamma \in Q_\pi(e) \), applying the Fourier inversion formula to \( h := \lambda(\gamma \circ H_1) \in C_c(\mathbb{R}^{n+n^2}) \):

\[
h(x, \nu) = \int \hat{h}(\mu, \xi)\exp\left(-2\pi i \left(\sum_k |\mu_k x_k + \sum_{jl} \xi_{jl} T_{jl}^U(\pi(e))\right)\right) d\mu d\xi.
\]

If \( \gamma(\alpha(\chi)(\pi(e))) \) is zero then \( \gamma(Z) = 0 \) and by definition of \( \eta(F) \), \( \gamma(\eta(F)(e))(e) = 0 \) too. On the other hand, if \( \gamma(\alpha(\chi)(\pi(e))) \) is nonzero, putting \( \nu = \tau(e) = (\gamma(T_{ij}^U(e))) \), we have,

\[
\gamma(\eta(F)(e)) = \gamma(\alpha(\chi)(\pi(e)))h(x(e), \tau(e)) = \gamma(\alpha(\chi)(\pi(e))) \int \hat{h}(\mu, \xi)\exp\left(-2\pi i \left(\sum_k |\mu_k x_k(\pi(e)) + \sum_{jl} \xi_{jl} \gamma(T_{jl}^U(\pi(e)))\right)\right) d\mu d\xi = \gamma(Z),
\]

Note that we have used the estimate \([15]\) to justify interchanging \( \gamma \) with the integral sign, which proves \([14]\). As \( T_{ij}^U(e) \) and \( x_k(\pi(e)) \) are smooth functions of \( e \), we can easily see the Fréchet continuity of \( e \mapsto \eta(F)(e) \), using once again the bound \([15]\). \( \square \)

We continue to denote by \( \eta_U \) the unique \( * \)-homomorphic extension of \( \eta_U \) on \( C_c(E_U) \).

Theorem 3.14 There exists a faithful smooth action \( \eta \) of \( Q \) on \( O(M) \) satisfying \( \eta(F)(e) \in Q_\pi(e) \) for all \( F \in C^\infty(O(M)) \) and \( e \in O(M) \).
Proof:
Choose and fix a finite cover $U_1, \ldots, U_r$ of $M$ by coordinate neighbourhoods, a $C^\infty$ partition of unity $\chi_i, i = 1, \ldots, r$ subordinate to this cover and define

$$\eta(F) = \sum_{i=1}^{r} \eta_{U_i}((\chi_i \circ \pi)F),$$

for $F \in C(O(M))$. As each of the maps $\eta_{U_i}$ is Fréchet continuous on $C_c^\infty(O(U_i))$, $\eta$ too is Fréchet continuous.

For $F, G \in C(O(M))$, writing $\hat{\chi}_i = \chi_i \circ \pi$, $\eta_i = \eta_{U_i}$, we have $\eta(FG) = \sum_i \eta_i(\hat{\chi}_i FG) = \sum_{ij} \eta_i(\hat{\chi}_i \hat{\chi}_j FG) = \sum_{ij} \eta_i(\hat{\chi}_i F)\eta_j(\hat{\chi}_j G)$ by (iii) of Lemma 3.13. But this is nothing but

$$\left(\sum_i \eta_i(\hat{\chi}_i F)\right) \left(\sum_j \eta_j(\hat{\chi}_j G)\right) = \eta(F)\eta(G).$$

Similarly we can show $\eta(F^*) = \eta(F)^*$. Thus, $\eta$ is a $*$-homomorphism. Being Fréchet continuous, it maps $C^\infty(O(M))$ to $C^\infty(O(M), \mathbb{Q})$.

To complete the proof, consider the Fréchet dense $*$-subalgebra $A$ mentioned before, on which $\alpha$ is algebraic. Then the algebra $B$ generated by $\theta_{\omega}^i, 1 \leq i \leq n, \omega \in \Omega^1(A)$, is Fréchet dense in $C^\infty(O(M))$. We claim that for $\omega = f\omega_j', \omega \in C^\infty(U)$ for some coordinate neighbourhood $U$ and $(\omega_1', \ldots, \omega_n')$ are $U$-orthonormal one-forms, $\eta(\theta_{\omega}^i)(\epsilon) = (\langle (\beta_1 \otimes 1_\mathbb{Q}), d\alpha_{ij}(\omega) \rangle)(\pi(\epsilon)), e = (\pi(\epsilon), \beta_1, \ldots, \beta_n)$. This can be verified from the definition. However, such one-forms comprise a Fréchet dense subspace, hence we get the above for all $\omega$. From this, it is clear that

$$\eta(\theta_{\omega}^i) = \theta_{\omega(0)}^i \otimes \omega(1)$$

for all $\omega \in \Omega^1(A)$, using the Sweedler-type notation. As $\alpha$ is algebraic on $A$, we have $\eta(B) \subseteq B \otimes \mathbb{Q}_0$. It the co-associativity of $\eta$ on $B$ (hence also on $C(O(M))$ by density) follows from the co-associativity of $\alpha$ and moreover, we have $(\text{id} \otimes \epsilon) \circ \alpha = \text{id}$ on $A$ which implies $(\text{id} \otimes \epsilon) \circ \eta = \text{id}$ on $B$. That is, $\eta$ on $B$ is a Hopf-algebraic co-action, so in particular we have $\text{Sp}(\eta(B))(1 \otimes \mathbb{Q}_0) = B \otimes \mathbb{Q}_0$, which is Fréchet dense in $C^\infty(O(M), \mathbb{Q})$. Hence $\eta$ is a smooth action in our sense. To show faithfulness of $\eta$, consider $1 \leq p \leq r, f \in C^\infty(M)$ and $\chi \in C^\infty(U_p)$ such that $0 \leq \chi \leq 1$, $\chi \chi_p f = \chi_p f$. From Lemma 3.11 we get

$$\alpha(\chi_p f)(m) = \sum_{j=1}^{r} \eta((\chi_p \circ \pi)\nu_{1j}^p(e))\eta((\chi_f \circ \pi)\nu_{1j}^p(e))$$

for all $e \in O(M)$ with $\pi(e) = m$. Thus for all $1 \leq p \leq r$ and $f \in C^\infty(M)$, $\alpha(\chi_p f)(m)$ is contained in the $C^*$ algebra generated by $\{\eta(F)(e) | F \in C^\infty(O(M)), e \in O(M)\}$. Hence the $C^*$-algebra also contains the $C^*$ closure of $\{\alpha(f)(m) | f \in C^\infty(M), m \in M\}$ which is $\mathcal{Q}$ by faithfulness of $\alpha$. Finally, $\eta(F)(e) \in Q_{\pi(e)}$ because $T_{ij}^U(e) \in Q_{\pi(e)}$ by construction. \qed

4 Isometric actions, i.e. actions commuting with the Laplacian

4.1 Definition and smoothness of isometric action

We now discuss the apparently stronger condition of isometry, as in [10]. Throughout this section, let $M$ be a compact Riemannian manifold without boundary, with the Riemannian volume form $d\text{vol}$, $\mathcal{H} := L^2(M, d\text{vol})$ and let $\tau$ denote the functional on $C(M)$ given by $\tau(f) = \int_M f \text{vol}$. We denote by $\mathcal{L}$ the Hodge Laplacian $-d^*d$ to 0-forms, which is a self-adjoint operator on $\mathcal{H}$. It is known (see e.g. [8] and the references therein) that $\mathcal{L}$ has discrete spectrum given
by eigenvalues, say \( \{ \lambda_i, i \geq 1 \} \) having finite multiplicities and the eigenvectors are in fact smooth functions. Let \( \{ e_{ij} : j = 1, \ldots, d_i \} \) be the orthonormal eigenvectors of \( \mathcal{L} \) forming a basis for the eigenspace corresponding to the eigenvector \( \lambda_i \). We denote the linear span of \( \{ e_{ij} : 1 \leq j \leq d_i, i \geq 1 \} \) by \( \mathcal{A}_0^\infty \), which is a subspace of \( C^\infty(M) \). Clearly, \( \mathcal{L} \) maps \( C^\infty(M) \) to itself and we denote the restriction on \( \mathcal{L} \) to \( C^\infty(M) \) (which is a Fréchet continuous operator) again by the same symbol.

**Definition 4.1** An action \( \alpha \) of a CQG \( \mathcal{Q} \) on \( C(M) \) where \( M \) is a compact manifold \( M \) without boundary, is said to be isometric if \( \alpha(C^\infty(M)) \subseteq C^\infty(M, \mathcal{Q}) \) and for every state \( \psi \) on \( \mathcal{Q} \), the map \( (\text{id} \otimes \psi)\alpha \) commutes with \( \mathcal{L} = -d^*d \) on \( C^\infty(M) \).

We have the following:

**Theorem 4.2** Any isometric action \( \alpha \) is smooth, i.e. the linear span of \( \alpha(C^\infty(M))(1 \otimes \mathcal{Q}) \) is Fréchet dense in \( C^\infty(M, \mathcal{Q}) \).

**Proof:**
Clearly, \( \alpha \) is algebraic over \( \mathcal{A}_0^\infty \), hence \( \text{Sp} \alpha(\mathcal{A}_0^\infty)(1 \otimes \mathcal{Q}_0) = \mathcal{A}_0^\infty \otimes \mathcal{Q}_0 \). It suffices to show that \( \mathcal{A}_0^\infty \) is Fréchet dense in \( C^\infty(M) \). By Theorem 1.2 of [8] there are constants \( C \) and \( C' \) such that \( \|e_{ij}\|_\infty \leq C|\lambda_i|^{\frac{n}{2}} \) and \( d_i \leq C'|\lambda_i|^{\frac{n}{2}} \), where \( n \) is the dimension of the manifold. For \( f \in C^\infty(M) \) there are complex numbers \( f_{ij} \) such that \( \sum_{ij} f_{ij} e_{ij} \) converges to \( f \) in \( L^2 \) norm.

Now, \( \mathcal{L}(\sum_{i \leq N, j \leq d_i} f_{ij} e_{ij}) = \sum_{i \leq N, j \leq d_i} \lambda_i f_{ij} e_{ij} \) converges to \( \mathcal{L}(f) \) in the \( L^2 \) norm as \( N \to \infty \). By the Cauchy-Schwartz inequality,

\[
\left| \sum_{ij} \lambda_i |f_{ij}| ||e_{ij}||_\infty \right| \leq C(C')^{\frac{1}{2}} \left( \sum_{ij} |f_{ij}|^2 |\lambda_i|^{2k} \right)^{\frac{1}{2}} \left( \sum_i |\lambda_i|^{n-2k} \right) < \infty.
\]

Hence \( \lim_{N \to \infty} \| \mathcal{L}(\sum_{i \leq N, j \leq d_i} f_{ij} e_{ij}) - \mathcal{L}(f) \|_\infty = 0 \). Similarly we can show that

\[
\lim_{N \to \infty} \mathcal{L}^k(\sum_{i \leq N, j \leq d_i} f_{ij} e_{ij}) = \mathcal{L}^k(f)
\]

in the sup norm of \( C(M) \) for any \( k \geq 1 \). Hence \( \mathcal{A}_0^\infty \) is Fréchet dense in \( C^\infty(M) \). \( \square \)

We also have the following:

**Lemma 4.3** Any isometric action preserves the corresponding Riemannian inner product as well as the Riemannian volume measure.

**Proof:**
It is enough to prove the following:

\[
\langle \langle da(0), db(0) \rangle \rangle \otimes a_{(1)}^* b_{(1)} = \alpha(\langle \langle da, db \rangle \rangle),
\]

(16)

where \( \langle \langle df, dg \rangle \rangle = \mathcal{L}(\overline{f})g - \mathcal{L}(\overline{g})f - \overline{g} \mathcal{L}(f), \forall f, g \) belonging to the subalgebra \( \mathcal{A} \) as in Proposition 83. However, it is straightforward to verify (16) using \( \mathcal{L}(f(0)) \otimes f_{(1)} = \alpha(\mathcal{L}(f)) \) for \( f \in \mathcal{A} \) as well as the fact that \( \alpha \) is a \( * \)-homomorphism.

The proof of the statement about Riemannian volume measure preservation can be found in Lemma 2.5 of [10]. \( \square \)
4.2 Commutativity of higher order partial derivatives

Let $\nabla$ be the Levi-Civita connection viewed as a map from $\Omega^1(\mathcal{C}^\infty(M))$ to $\Omega^1(\mathcal{C}^\infty(M)) \otimes C^\infty(M)$ $\Omega^1(\mathcal{C}^\infty(M))$. We have the following, using the observations that $\omega(X_{dh}) = \langle \omega, dh \rangle$, $df(Z) = \langle \nabla X_{dh}, df \rangle$ and $\langle \nabla(df), dg \otimes dh \rangle = \langle \nabla X_{dh}(df), dg \rangle$:

$$\langle \nabla(df), dg \otimes C^\infty(M) dh \rangle = \frac{1}{2} \langle \langle df, d((dg, dh)) \rangle \rangle + \langle \langle dh, d((df, dg)) \rangle \rangle + \langle \langle dg, d((df, dh)) \rangle \rangle,$$

where $f, g, h \in \mathcal{C}^\infty(M)_\mathbb{R}$. This can be derived by a slightly long but straightforward calculation using the standard formula for the Levi-Civita connection on vector fields (see, for example, page 69 of [17]), as well as the formulae $\nabla_X(\omega)(Y) := X\omega(Y) - \omega(\nabla_X(Y))$ and $d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$ for all $X, Y \in \chi(M)$ (see page 54 of [17]).

**Lemma 4.4** Let $f, g, h \in \mathcal{A}_{s.a.}$. Then

$$\langle \langle dg(0), d((df(0), dh(0))) \rangle \rangle \otimes g(1)f(1)h(1) = \langle \langle dg(0), d((df(0), dh(0))) \rangle \rangle \otimes f(1)g(1)h(1) \tag{18}$$

$$\langle \langle dg(0), d((df(0), dh(0))) \rangle \rangle \otimes h(1)g(1)f(1) \tag{19}$$

**Proof:**
We have

$$\langle \langle dg(0), d((df(0), dh(0))) \rangle \rangle \otimes g(1)f(1)h(1)$$

$$= (\mathcal{L}(g(0))(\langle \langle df(0), dh(0) \rangle \rangle) - \mathcal{L}(g(0))\langle \langle df(0), dh(0) \rangle \rangle - g(0)\mathcal{L}(\langle \langle df(0), dh(0) \rangle \rangle)) \otimes g(1)f(1)h(1)$$

We compute the terms individually, using $\mathcal{L}(\phi(0)) \otimes \phi(1) = \alpha(\mathcal{L}(\phi))$.

$$\mathcal{L}(g(0)(\langle \langle df(0), dh(0) \rangle \rangle) \otimes g(1)f(1)h(1)$$

$$= (\mathcal{L} \otimes \text{id})\alpha(\langle \langle df, dh \rangle \rangle)$$

$$= (\mathcal{L} \otimes \text{id})\alpha(\langle \langle df, gh \rangle \rangle)$$

$$= (\mathcal{L} \otimes \text{id})\alpha(\langle \langle df \rangle \rangle)$$

Recall that $\tau(f) = \int f \text{dvol}$. As $\mathcal{L}$ is a self-adjoint operator on $\mathcal{H}$, for $\phi \in \mathcal{C}^\infty(M)$ we get

$$\langle \langle \mathcal{L}(\langle \langle df(0), dh(0) \rangle \rangle) g(0) \otimes g(1), f(1)h(1), \alpha(\phi) \rangle \rangle$$

$$= \tau(\mathcal{L}(\langle \langle df(0), dh(0) \rangle \rangle) g(0) \phi(0)) h(1)f(1)g(1)\phi(1)$$

$$= \tau(\langle \langle df(0), dh(0) \rangle \rangle \mathcal{L}(g(0)\phi(0)) h(1)f(1)g(1)\phi(1)) \text{ (by self-adjointness of } \mathcal{L})$$

$$= \tau(\langle \langle df(0), dh(0) \rangle \rangle \mathcal{L}(g(0)\phi(0)) + g(0)\mathcal{L}(\phi(0)) + \langle \langle df(0), d\phi(0) \rangle \rangle) h(1)f(1)g(1)\phi(1).$$

Using $\alpha(\langle \langle df, dh \rangle \rangle) = \langle \langle df(0), dh(0) \rangle \rangle \otimes f(1)h(1) = \langle \langle df(0), dh(0) \rangle \rangle \otimes h(1)f(1)$ as observed earlier, we get:

$$\tau(\langle \langle df(0), dh(0) \rangle \rangle \mathcal{L}(g(0)\phi(0)) h(1)f(1)g(1)\phi(1)$$

$$= (\tau \otimes \text{id})\alpha(\langle \langle df, dh \rangle \rangle \mathcal{L}(g)\phi(0))$$

$$= (\tau \otimes \text{id})\alpha(\langle \langle dh, df \rangle \rangle \mathcal{L}(g)\phi(0))$$

$$= (\tau \otimes \text{id})\alpha(\langle \langle dh, \mathcal{L}(g)df \rangle \rangle \phi(0))$$

$$= \tau(\langle \langle dh(0), df(0) \rangle \rangle \mathcal{L}(g(0)\phi(0)) h(1)f(1)g(1)\phi(1).$$
Similarly,
\[
\tau \left( \langle \langle df(0), dh(0) \rangle \rangle g(0) \mathcal{L}(\phi(0)) h(1) f(1) g(1) \phi(1) \right) = \tau \left( \langle \langle df(0), dh(0) \rangle \rangle g(0) \mathcal{L}(\phi(0)) \otimes h(1) g(1) f(1) \phi(1) \right).
\]

Also,
\[
\tau \left( \langle \langle df(0), dh(0) \rangle \rangle \langle \langle dg(0), d\phi(0) \rangle \rangle h(1) f(1) g(1) \phi(1) \right)
= (\tau \otimes \text{id}) \left( \langle \langle d \otimes \text{id} \rangle \alpha(h), (d \otimes \text{id}) \alpha(f) \right) \langle \langle d \otimes \text{id} \rangle \alpha(g), (d \otimes \text{id}) \alpha(\phi) \rangle \rangle.
\]

By (10),
\[
\tau \left( \langle \langle df(0), dh(0) \rangle \rangle \langle \langle dg(0), d\phi(0) \rangle \rangle h(1) f(1) g(1) \phi(1) \right)
= \tau \left( \langle \langle df(0), dh(0) \rangle \rangle \langle \langle dg(0), d\phi(0) \rangle \rangle h(1) g(1) f(1) \phi(1) \right),
\]

hence
\[
\langle \langle g(0) \mathcal{L}(\langle \langle df(0), dh(0) \rangle \rangle) \otimes g(1) f(1) h(1), \alpha(\phi) \rangle \rangle = \langle \langle g(0) \mathcal{L}(\langle \langle df(0), dh(0) \rangle \rangle) \otimes f(1) g(1) h(1), \alpha(\phi) \rangle \rangle
\]
for all \( \phi \in A \). As \( \text{Sp} \{ \alpha(\phi) q : \phi \in A, q \in \mathcal{Q} \} \) is dense in the Hilbert module \( \mathcal{H} \otimes \mathcal{Q} \), we get
\[
g(0) \mathcal{L}(\langle \langle df(0), dh(0) \rangle \rangle) \otimes g(1) f(1) h(1) = g(0) \mathcal{L}(\langle \langle df(0), dh(0) \rangle \rangle) \otimes f(1) g(1) h(1).
\]

Combining all these we get (15).

To prove the other equality, note that
\[
\langle \langle df(0), dh(0) \rangle \rangle \otimes f(1) h(1) = \langle \langle df(0), dh(0) \rangle \rangle \otimes h(1) f(1) = \langle \langle dh(0), df(0) \rangle \rangle \otimes h(1) f(1).
\]

Hence
\[
\langle \langle dg(0), df(0) \rangle \rangle \otimes g(1) f(1) h(1)
= \langle \langle dg(0), df(0) \rangle \rangle \otimes g(1) f(1) h(1)
= \langle \langle dg(0), df(0) \rangle \rangle \otimes h(1) g(1) f(1),
\]

where in the last step we have used (15) interchanging \( f \) and \( h \). \( \square \)

We consider \( (\nabla \otimes \text{id}_\mathcal{Q}) : \Omega^1(C^\infty(M)) \otimes \mathcal{Q} \to \Omega^2(C^\infty(M)) \otimes \mathcal{Q} \) as follows. Let \( m \in M, (U, x_1, \ldots, x_n) \) a local chart around \( m \) and \( \Omega \in \Omega^1(C^\infty(M)) \otimes \mathcal{Q} \) such that \( \Omega(x) = \sum_i dx_i(x) \Omega_i(x) \)
\( \forall x \in U \). Then we define
\[
(\nabla \otimes \text{id}_\mathcal{Q})(\Omega)(m) = \sum_i (\langle \nabla(dx_i)(m) \otimes 1 \rangle \Omega_i(m) + dx_i(m) \otimes (d\Omega_i)(m)).
\]

By standard arguments one can see that the above does not depend on the choice of local coordinates and \( (\nabla \otimes \text{id}_\mathcal{Q}) \) is Fréchet continuous.

**Corollary 4.5** For \( \omega \in \Omega^1(C^\infty(M)) \), we have
\[
(\nabla \otimes \text{id})(d\omega(1)) = U^{(2)}(\nabla(\omega)),
\]

where \( U^{(2)} \) is the equivariant unitary representation on \( \Omega^2(C^\infty(M)) \) constructed in Lemma 3.8.

**Proof:**

It suffices to prove
\[
(\nabla \otimes \text{id})(d\omega(1)(dg)) = U^{(2)}(\nabla(g dg))
\]
for \( g \in \mathcal{A}_{s.a.} \). Indeed, we can prove the identity \( (20) \) from \( (21) \) in two steps. First, we prove it for \( \omega = (dg)f \), where \( f, g \in \mathcal{A}_{s.a.} \), using the Leibniz rule for connection and the fact \( U^{(2)}(\Theta f) = \)...
$U^2(\Theta)\alpha(f) \forall \Theta \in \Omega^2(C^\infty(M))$. Then we use the Fréchet-continuity of $\nabla Id$ and Fréchet-density of the complex linear span of one-forms of the form $(dg)^{f}$, $f,g \in A_{sa}$ in $\Omega^1(C^\infty(M))$.

As the complex linear span of elements of the form $(df \otimes A dh)\phi$, where $f,h,\phi \in A_{sa}$, is dense in $\Omega^2(C^\infty(M))$, the complex linear span of $U^2((df \otimes A dh)\phi)q = df^{(0)} \otimes A dh^{(0)}\phi^{(0)} \otimes f^{(1)}h^{(1)}\phi^{(1)}q$, $q \in Q$, is dense in $\Omega^2(C^\infty(M))\otimes Q$, i.e. elements of the form $U^2(df \otimes A dh)$ constitute a right $C^\infty(M,Q)$-total subset. To prove (21) it is enough to show that $\langle\langle (\nabla(df^{(0)}),\Omega) = \langle\langle (U^2(\nabla(df)),\Omega) \rangle\rangle$ for all $\Omega$ in some subset whose right $C^\infty(M,Q)$-linear span is dense in $\Omega^2(C^\infty(M))\otimes Q$, in particular,

$$\Omega = U^2(df \otimes A dh) = df^{(0)} \otimes A dh^{(0)} \otimes f^{(1)}h^{(1)}.$$ 

But we have the following by combining the formula (17), Lemma 4.2 and the equivariance of $U$:

$$\langle\langle (\nabla(df^{(0)}),df^{(0)} \otimes A dh^{(0)})) \otimes g^{(1)}f^{(1)}h^{(1)} = \alpha(\langle\langle (\nabla(df),df \otimes A dh)\rangle\rangle). \tag{22}$$

Indeed, as $U^2$ is equivariant, the right hand side of the above is equal to

$$\langle\langle (U^2(\nabla(df))),U^2(df \otimes A dh)\rangle\rangle,$$

which completes the proof. $\square$

This leads to the following:

**Theorem 4.6** For any $m \in M$, local coordinates $(W,(x_1,\ldots,x_n))$ around $m$, a positive integer $k$ and $f \in C^\infty(M)$, we have

$$\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}} \alpha(f)(m) \in Q_m$$

$\forall i_l \in \{1,\ldots,n\}$.

**Proof:**

Let $A \subset C^\infty(M)$ be as in Proposition 8 and $\omega_1,\ldots,\omega_n$ be $W$-orthonormal smooth one forms where $W$ is some coordinate neighbourhood around $m$. Write $X_i = X_{\omega_i}$, using the notation introduced in Subsection 2.3. The result is clearly equivalent to the following: $(X_{i_1} \cdots X_{i_k}(\alpha(f)))(m) \in Q_m$ for $k \geq 1$, $1 \leq i_j \leq n$.

Let us define the following maps $\nabla^k$ from $\Omega^1(C^\infty(M))$ to $\Omega^{k+1}(C^\infty(M))$ for $k = 1,2,\ldots$.

Let $\sigma_{ij}$ denote the map which flips the $i$-th and the $j$-th copies of $\Omega^1(C^\infty(M))$. Define $\nabla^1 = \nabla$. Then, consider the $\mathbb{C}$-linear map $T : \Omega^1(C^\infty(M)) \otimes_{\mathbb{C}} \Omega^1(C^\infty(M)) \to \Omega^3(C^\infty(M))$ defined by

$$T(\omega \otimes C \eta) = \sigma_{23}(\nabla(\omega) \otimes C_{\infty}(M) \eta) + \omega \otimes C_{\infty}(M) \nabla(\eta).$$

Verify using the Leibniz rule for connections that $T(\omega f \otimes C \eta) = T(\omega \otimes C f \eta)$, hence it descends to a map say $\overline{T}$ on $\Omega^2(C^\infty(M))$. We define $\overline{\nabla}^2 = \overline{T} \circ \nabla$. In a similar way, for $k \geq 2$, define $\overline{\nabla}^k := (\sum_{l=1}^{k-1} \sigma_{l+k+1} \circ \nabla_l + \overline{\nabla}^k) \circ \overline{\nabla}^{k-1}$, where $\nabla_l$ is the map which acts on the $l$-th copy of $\Omega^1(C^\infty(M))$ by $\nabla$ leaving the other copies unaffected.

It follows by repeated application of (20) and the commutativity of $Q_m$ that

$$\overline{\nabla}^k(df^{(0)})(m) \otimes f^{(1)} = U^{(k+1)}(\overline{\nabla}^k(df))(m) \in (T_m^* M)^{\otimes k+1} \otimes Q_m.$$ 

On $W$, we can write $df^{(0)} \otimes f^{(1)} = \sum_{i=1}^{n} \omega_i X_i(f^{(0)}) \otimes f^{(1)}$, hence

$$\nabla(df^{(0)}) \otimes f^{(1)} = \sum_i \nabla(\omega_i) X_i(f^{(0)}) \otimes f^{(1)} + \sum_{i,j=1}^{n} \omega_i \otimes A \omega_j X_j X_i(f^{(0)}) \otimes f^{(1)}.$$

21
Evaluating at $m$ and noting that the first term belongs to $T^*_mM \otimes T^*_mM \otimes Q_m$ by definition of $Q_m$, we conclude that the second term must belong to this space too. Thus, taking inner product with $\omega_i(m) \otimes \omega_j(m)$ for any fixed $i, j$, we get $X_i X_j (\alpha(f))(m) \in Q_m$. Expanding $\nabla^2$ in a similar way and using $X_i \alpha(f)(x), X_i X_j \alpha(f)(x)$ are in $Q_x$ for all $x \in W$, we show $X_i X_j X_l \alpha(f)(m) \in Q_m$ for all $i, j, r$. Proceeding inductively, using the expansion of $\nabla^k (df(0)) \otimes f(1)$, the statement of the theorem follows for any $k \geq 1$. □

**Corollary 4.7** For any $\omega, \eta \in \Omega^1(M)$, we have $(\frac{\partial}{\partial x_i}(\langle \eta \otimes 1, d\alpha(\omega) \rangle))(m) \in Q_m$.

**Proof**: It is enough to prove it for $\omega = f dg, f, g \in A$. Indeed

\[ (\frac{\partial}{\partial x_i}(\langle \eta \otimes 1, d\alpha(f dg) \rangle))(m) = \frac{\partial}{\partial x_i}(\alpha(f)X_n(\alpha(g)))(m) \in Q_m, \]

as $\frac{\partial}{\partial x_i}(X \alpha(g))(m) \in Q_m$ for every smooth vector field $X$ by Theorem [[4.6](#)] □

## 5 Main result

**Lemma 5.1** Let $\Phi$ be a smooth action of a CQG $Q$ on a compact connected subset $W$ which is the closure of a bounded smooth domain, i.e. bounded, open connected subset of $R^N$ with smooth boundary. Suppose furthermore that the action preserves the usual (Euclidean) Riemannian inner product and for any $y \in W$, the algebra generated by $\{\Phi(g)(y), (\frac{\partial}{\partial y_{i_1}} \ldots \frac{\partial}{\partial y_{i_k}} \Phi(f))(y) : k \geq 1, 1 \leq i_j \leq N, f, g \in C^\infty(W)\}$ is commutative, where $y_1, \ldots, y_N$ denote the standard coordinates of $R^N$. Then $\Phi$ is affine i.e.

\[ \Phi(y_i) = 1 \otimes q_{i} + \sum_{j=1}^{N} y_j \otimes q_{ij}, \text{ for some } q_{ij}, q_i \in Q, \quad (23) \]

for all $i = 1, \ldots, N$.

**Proof**: Let $W = V$, where $V$ is a bounded, open connected set with smooth boundary. Note that $(dy_1, \ldots, dy_N)$ is an orthonormal basis of $T^*_mW$ at every point $m$. Let $D^k_{ij}(m) = \frac{\partial}{\partial y_i(y_k)}|_m \Phi(y_k), D^k_{ij}(m) = \frac{\partial^2}{\partial y_i \partial y_j}|_m \Phi(y_k)$. As $V$ is open and connected, it suffices to prove $D^k_{ij}(m) = 0$ for all $m \in V$. Using the arguments and discussion in the beginning of Subsection 3.2 (page 10), we get a $\Phi$-equivariant unitary representation $\Gamma := df(1)$ as in the Definition 3.6, satisfying $\Gamma(df) = df(1)$ for every $f \in C^\infty(W)$, in particular, $\Gamma(dy_i) = \sum_{j=1}^{N} dy_j D^1_{ij}.$ As the Hilbert $C^\infty(W)$-module of one-forms of $W$ is free of rank $N$ with the (orthonormal w.r.t. the Euclidean Riemannian structure) basis $dy_1, \ldots, dy_N$, we can adapt the arguments of Lemma 3.7 to conclude that $((D^1_{ij}))_{i,j=1}^{N}$ is a unitary element of $M_N(C^\infty(W))$. From the unitarity as well as self-adjointness of $D^1_{ij}$’s (which follows because $y_i$’s are self-adjoint), we get the following two equations:

\[ \sum_{l=1}^{N} D^1_{il} D^1_{lj} = \delta_{ij} 1_Q, \quad (24) \]

\[ \sum_{l=1}^{N} D^1_{il} D^1_{lj} = \delta_{ij} 1_Q. \quad (25) \]

Applying $\frac{\partial}{\partial y_{i}}$ to equation (24), and using the commutativity of $D^1_{ij}$ and $D^1_{ij}$ for all choices of the upper and lower indices, we get

\[ \sum_{l=1}^{N} (D^1_{ik} D^1_{lj} + D^1_{lj} D^1_{ik}) = 0. \quad (26) \]
Now we denote the $N^2 \times N$ matrix whose $(ij)k$-th entry $A_{(ij),k}$ is $D^k_{ij}(y)$ by $A$ and $N \times N$ matrix whose $ij$-th entry $B_{ij}$ is given by $D^j_i(y)$ by $B$. Then the $(ij)k$-th entry of the matrix $C = AB$ denoted by $C_{(ij)k}$ is given by $\sum_{l=1}^N A_{(ij),l} D^l_{k}$. From the equation (26), we get for all $i, j, k$,
$$
C_{(ik)j} + C_{(kj)i} = 0. \tag{27}
$$
Now we observe that $C_{(ij)k} = C_{(ji)k}$ for all $i, j, k$. Hence by repeated application of equation (27),
$$
C_{(ik)j} = C_{(ki)j} = -C_{(ij)k} = -C_{(ji)k} = C_{(jk)i}.
$$
Again by equation (27), we get $C_{(ik)j} = 0$ for all $i, j, k$ i.e. $C = 0$. As $B$ is unitary, we conclude that $A = 0$, i.e. $D^k_{ij}$ is zero for all $i, j, k$ completing the proof of the Lemma. \hfill \square

**Corollary 5.2** In the set up of Lemma 5.1, the $C^*$ algebra generated by $\{q_i, q_{kl} : i, k, l = 1, ..., n\}$ is commutative. In particular, if $\Phi$ is faithful, $Q$ must be commutative.

**Proof:**
Observe that $(\frac{\partial}{\partial y_j} \Phi(y_i))(x)$ and $\Phi(y_k)(x)$ must commute for $i, j, k = 1, ..., n$ and for all $x \in W$ as $\Phi$ is isometric. But we have $(\frac{\partial}{\partial y_j} \Phi(y_i)) = 1 \otimes q_{ij}$. Thus it follows from the expression of $\Phi(y_k)$ given by (24),
$$
1 \otimes q_{kq} + \sum_{i=1}^N y_i \otimes q_{ki} q_{ij} = 1 \otimes q_{ij} q_k + \sum_{i=1}^N y_i \otimes q_{ij} q_{kl}. \tag{28}
$$
Clearly the set $\{1, y_1, ..., y_N\}$ is a linearly independent set as $W$ has a non-empty interior, hence we get the commutativity of $q_k, q_{ij}$ and $q_{kl}, q_{ij}$ by comparing the coefficients of $\{1, y_1, ..., y_N\}$ on both sides of (28). \hfill \square

Now, we are in a position to prove the main theorem of the paper.

**Theorem 5.3** Let $\alpha$ be a faithful isometric action of a CQG $Q$ on a compact, connected, Riemannian $n$-manifold $M$. Then $Q$ is commutative.

**Proof:**
We break the proof into several steps. As the CQG $Q$ is commutative if and only if the corresponding reduced CQG $Q_r$ is so, we may assume without loss of generality that $Q = Q_r$ is reduced CQG so that Theorem 3.5 applies.

**Step 1**
Let $P = O(M)$ and let $\eta$ be the lift of the action on $P$ obtained by Theorem 3.14, which satisfies $\eta(\phi)(e) \in Q_{\pi(e)}$ for all $\phi \in C^\infty(P)$ and $e \in P$. We claim that $X(\eta(\phi))(e) \in Q_{\pi(e)}$ too for all $X \in \chi(P)$. Once this is proved, we can apply Theorem 3.5 (as $Q_{\pi(e)}$ is commutative) to get some Riemannian structure on $P$ for which $\eta$ is inner product preserving. To prove the claim, it is enough to consider $\phi$ of the form $(f \circ \pi)t_{ij}^{(U, \omega^U)}$ in the notation of Subsection 3.3. Moreover, as in the proof of Lemma 3.13, let us choose some local coordinate $(V, (x_1, ..., x_n))$ for $M$ around $\pi(e)$, $V$-orthonormal one-forms $\omega'_1, ..., \omega'_n$ and the corresponding embedding of $\pi^{-1}(V)$ into $\mathbb{R}^{n+n^2}$ using $x_r, t_{pq}^{(U, \omega^U)}$, $r, p, q = 1, ..., n$. Denoting the canonical coordinate functions of $\mathbb{R}^{n^2}$ by $y_{pq}, p, q = 1, ..., n$, it suffices to verify the claim for $X$ of the form $\partial_r \equiv \frac{\partial}{\partial x_r}$ and $\partial_{pq} \equiv \frac{\partial}{\partial y_{pq}}$. But
$$
\eta(\phi)(e) = \alpha(f)(\pi(e))T_{ij}^{(U, \omega^U)}(e) = \sum_{k=1}^n \alpha(f)(\pi(e))t_{ik}^{(V, \omega^V)}(e)\langle \omega'_k \otimes 1, d\alpha(1)(\omega'_j) \rangle(\pi(e)).
$$
From this expression, it is clear that $\partial_{pq} \eta(\phi)(e)$ is a complex linear combination of
$$\{\alpha(f)(\pi(e)), \langle \omega'_k \otimes 1, d\alpha(1)(\omega'_j) \rangle(\pi(e)), k, j = 1, ..., n\} \subseteq Q_{\pi(e)}.$$

23
On the other hand, $\partial_r \eta(\phi)(e)$ is a complex linear combination of $\partial_r ((\omega_k^r \otimes 1, a\alpha_{(1)}(\omega_j^r)))(\pi(e))$ as well as $\partial_r ((\omega_k^r \otimes 1, a\alpha_{(1)}(\omega_j^r)))(\pi(e))$ ($k, j = 1, \ldots, n$) which also belong to $Q_{\pi(e)}$ by Corollary 4.7.

Step 2:
We now fix a Riemannian structure on $P$ obtained by Step 1 and want to lift $\eta$ further to a tubular normal neighbourhood of $P$. Consider a Fréchet dense subalgebra $D$ of $C^\infty(P)$ over which $\eta$ is algebraic and $Sp(\eta(D)(1 \otimes Q_0)) = D \otimes Q_0$.

As $P$ is parallelizable, there is a Riemannian embedding (for the Riemannian structure of $P$ discussed in Step 1) into some $\mathbb{R}^N$, so that the corresponding normal bundle is trivial. Recall from Lemma 2.6 the global diffeomorphism $F$ and the corresponding isomorphism $\pi_F : C^\infty(\mathcal{N}_e P) \to C^\infty(P \times B^{N-r}_\varepsilon(0))$, where $r$ denotes the dimension of $P$. Define $\tilde{\eta} : C^\infty(P \times B^{N-r}_\varepsilon(0)) \to C^\infty(P \times B^{N-r}_\varepsilon(0), Q)$ by

$$\tilde{\eta}(G)(e, b) = \eta(G)(e),$$

where $G_b : P \to \mathbb{C}$ is given by $G_b(e) = G(e, b)$. It is clearly a $C^*$ action and also satisfies $\tilde{\eta} = \sigma_{23} \circ (\eta \otimes \text{id})$ on $D \otimes C^\infty(B^{N-r}_\varepsilon(0))$, from which it is clear that $\tilde{\eta}(D \otimes C^\infty(B^{N-r}_\varepsilon(0)))(1 \otimes Q_0) = D \otimes C^\infty(B^{N-r}_\varepsilon(0)) \otimes Q_0$. Thus, $\tilde{\eta}$ is indeed a smooth action. Now we have $\pi_{F-1} : C^\infty(M \times B^{N-r}_\varepsilon(0)) \to C^\infty(\mathcal{N}_e P)$. Define

$$\Phi := (\pi_{F-1} \otimes \text{id}) \circ \tilde{\eta} \circ \pi_F : C^\infty(\mathcal{N}_e P) \to C^\infty(\mathcal{N}_e P, Q).$$

We claim that $\Phi$ is a smooth action of $Q$ on $\mathcal{N}_e P$.

To see this, let $\tilde{D} := \pi_{F-1}(D \otimes C^\infty(B^{N-r}_\varepsilon(0)))$, which is a Fréchet dense subalgebra of $C^\infty(\mathcal{N}_e P)$. By construction, $\Phi$ is algebraic over $\tilde{D}$ and moreover, $Sp(\Phi(\tilde{D})(1 \otimes Q_0)) = \tilde{D} \otimes Q_0$. Also $\Phi$ is Fréchet continuous, hence smooth.

Step 3:
We claim that $\Phi$ preserves the Riemannian inner product of $\mathcal{N}_e P$. Denote the projection from $\mathcal{N}_e P$ to $P$ by $\rho$. As the normal bundle is trivial, choose a smoothly varying basis for normal space at each point of $P$. Let $y \in \mathcal{N}_e P$ and $\{e_i(y) : i = 1, \ldots, (N - r)\}$ be an orthonormal basis for the normal space to the manifold at the point $\rho(y)$ and let $u_1, u_2, \ldots, u_{N-r}$ be the components of $U(y) := (y - \rho(y))$ with respect to the basis $\{e_i(y) : i = 1, \ldots, (N - r)\}$. We introduce a local coordinate system for the manifold $\mathcal{N}_e P$ as follows:

$$G : \mathcal{N}_e P \xrightarrow{\rho^{-1}} P \times B^{N-r}_{\varepsilon}(0) \xrightarrow{\xi \times \text{id}} \mathbb{R}^N \quad (\xi \text{ is a coordinate map for } P):$$

$$y \rightarrow (\rho(y), U(y)) \rightarrow (y_1, \ldots, y_r, u_1, \ldots, u_{N-r}).$$

Clearly,

$$\left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right\rangle = 0$$

for all $i, j$, $i = 1, \ldots, N - r$, and $j = 1, \ldots, r$.

Consider $\phi, \psi \in \tilde{D}$ of the form $\phi = \xi \circ \rho$ and $\psi = \beta \circ U$, where $\xi, \beta$ are smooth functions. Then $\Phi(\phi)(y) = \eta(\xi)(\rho(y))$, $\Phi(\psi) = \psi \otimes 1$, which gives by (29)

$$\left\langle d\phi, d\psi \right\rangle = 0, \quad \left\langle d\Phi(1)(d\phi), d\Phi(1)(d\psi) \right\rangle = \Phi(\left\langle d\phi, d\psi \right\rangle) = 0.$$

A general element of $\tilde{D}$ is a $\mathbb{C}$-linear combination of the product functions of the form $(\xi \circ \rho)(\beta \circ U)$, $\xi \in C^\infty(P)$, $\beta \in C^\infty(B^{N-r}_{\varepsilon}(0))$. Hence it is enough to verify the following:

$$\left\langle d\Phi(1)(df_1), d\Phi(1)(df_2) \right\rangle = \Phi(\left\langle df_1, df_2 \right\rangle) \text{ for } f_i = (\xi_i \circ \rho)(\beta_i \circ U).$$
But we have $\Phi(f_i(y) = \eta(\xi_i(\rho(y))\beta_i(U(y))$ and $\langle d(\xi_i \circ \rho), d(\beta_j \circ U) \rangle = 0$ for $i, j = 1, 2$. Using this observation, Leibniz rule and the fact that $\eta$ is inner product preserving, we complete the proof of Step 3.

**Step 4:**
Finally, it is easy to observe from the construction of $\Phi$ that it satisfies the condition regarding commutativity of the partial derivatives as in the hypothesis of Corollary 5.2. Hence we conclude from that corollary that $Q$ must be commutative. □

Combining the above theorem with the techniques developed by Bhowmick and Goswami in [3] and by Joardar-Goswami in [13], one can prove the following:

**Corollary 5.4** The quantum isometry group of a noncommutative manifold obtained by cocycle twisting of a classical, connected, compact Riemannian manifold $M$ in the sense of [23] is a similar cocycle twisted version of $C(\text{ISO}(M))$, where $\text{ISO}(M)$ denotes the isometry group of $M$.

**Acknowledgement:** An initial version of this paper had Biswarup Das as a co-author. Later on the paper had undergone several rounds of revisions and corrections, in which Biswarup was not involved. In his opinion, his contribution to the present (final) draft does not remain significant enough to be a co-author and we have accepted his request to drop his name from the list of authors. However, we would like to thank him for useful discussion in the initial stage of this work, particularly concerning Step 3 of the proof of Theorem 5.3.

We also thank professors Pavel Etingof, Chelsea Walton and Marc Rieffel and Jyotishman Bhowmick and Subhrajit Bhattacharjee for encouragement, comments and discussion.

**References**

[1] Banica, T.: Quantum automorphism groups of small metric spaces, Pacific J. Math. 219 (2005), no. 1, 27-51.

[2] Banica, T., Bhowmick, J. and De Commer, K.: Quantum isometries and group dual subgroups, Ann. Math. Blaise Pascal 19 (2012), no.1, 1-27.

[3] Bhowmick, J., Goswami, D.: Quantum isometry groups: examples and computations, Comm. Math. Phys. 285 (2009), 421-444.

[4] Bhowmick, J., Skalski, A.: Quantum isometry groups of noncommutative manifolds associated with group $C^*$ algebras, Journal of Geometry and Physics, Volume 60, issue 10, Oct 2010, 1474-1489.

[5] Bichon, J.: Quantum automorphism groups of finite graphs, Proc. Amer. Math. Soc. 131 (2003), no. 3, 665-673.

[6] Chirvasitu, A. L.: Quantum rigidity of negatively curved manifolds, Comm. Math. Phys., 344 (2016), no. 1, 193-221.

[7] Connes, A.: “Noncommutative Geometry”, Academic Press, London-New York (1994).

[8] Donnelly, H.: Eigenfunctions of Laplacians on Compact Riemannian Manifolds, Asian J. Math. 10 (2006), no. 1, 115-126.

[9] Etingof, P., Walton, C.: Semisimple Hopf actions on commutative domains, Advances in Mathematics, Volume 251, 30 January 2014, Pages 47-61.

[10] Goswami, D.: Quantum Group of Isometries in Classical and Non Commutative Geometry, Comm. Math. Phys. 285 (2009), no. 1, 141-160.
[11] Goswami, D: Quadratic independence of coordinate functions of certain homogeneous spaces and action of compact quantum groups, Proc. Indian Acad. Sci. Math. Sci. 125 (2015), no. 1, 127-138.

[12] Goswami, D.: Existence and examples of quantum isometry groups for a class of compact metric spaces. Adv. Math. 280 (2015), 340-359.

[13] Goswami, D., Joardar S.: Quantum isometry groups of non commutative manifolds obtained by deformation using unitary 2-cocycle, SIGMA 10 (2014), 076, 18 pages.

[14] Goswami, D. and Joardar, S.: An averaging trick for smooth actions of compact quantum groups on manifolds, Indian J. Pure Appl. Math. 46(2015), no. 4, 477-488.

[15] Huang, H.: Faithful compact quantum group actions on connected compact metrizable spaces, Journal of Geometry and Physics, Volume 70, August 2013, 232-236.

[16] Huang, H.: Invariant subsets under compact quantum group actions, J. Noncommut. Geom. 10 (2016), no 2, 447-469.

[17] Lee, John M: “Riemannian manifolds. An introduction to curvature.”, Graduate Texts in Mathematics 176, Springer-Verlag, New York, 1997.

[18] Kosinski, Antoni A: “Differential manifolds”, Pure and Applied Mathematics 138, Academic Press, Inc., Boston, MA, 1993.

[19] Landi, G: “An Introduction to Non Commutative Spaces and their Geometry", Springer-Verlag, Berlin, Heidelberg, 1997.

[20] Maes, A. and Van Daele, A.: Notes on compact quantum groups, Nieuw Arch. Wisk. (4) 16 (1998), no. 1-2, 73-112.

[21] Manin, Y.: Quantum groups and noncommutative geometry, Universit de Montreal, Centre de Recherches Mathematiques, Montreal, QC, 1988.

[22] Podles, P.: Symmetries of Quantum Spaces, subgroups and quotient spaces of $SU(2)$ and $SO(3)$ groups, Comm. Math. Phys., 70(1):1995, 1-20.

[23] Rieffel, Mark A.: Deformation Quantization for actions of $R^d$, Memoirs of the American Mathematical Society, November 1993, Volume 106, Number 506.

[24] Rudin, W.: “Functional Analysis”, Second edition, International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991.

[25] Sabbe, M. and Quaegebeur, J.: Isometric coactions of compact quantum groups on compact quantum metric spaces, Proc. Indian Acad. Sci. Math. Sci. 122 (2012), no. 3, 351-373.

[26] Shastri, A.R.: “Elements of Differential Topology”, CRC Press (2011).

[27] Singhof, W. and Wemmer, D: Parallelizability of Homogeneous Spaces II, Math. Ann. 274, 157-176 (1986).

[28] Soltan, P., Dalecki, L., Jan: Quantum isometry groups of symmetric groups, Internat. J. Math. 23(2012), no 7, 1250074, 25 pages.

[29] Wang, S.: Quantum symmetry groups of finite spaces, Comm. Math. Phys., 195(1998), 195-211.
[30] Walton, C. and Wang X. : On quantum groups associated to non-Noetherian regular algebras of dimension 2, Math. Z. 284 (2016), no. 1-2, 543-574.

[31] Woronowicz, S.L.: Compact Matrix Pseudogroups, Comm. Math. Phys., 111(1987), 613-665.

[32] Woronowicz, S.L., Zakrzewski : Quantum ‘ax+b’ group, Reviews in Math. Phys., 14, Nos 7 & 8 (2002), 797-828.