Lepton Flavor Mixing and CP Violation in the Minimal Type-(I+II) Seesaw Model with a Modular $A_4$ Symmetry

Xin Wang $^{a, b}$

$^a$Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China
$^b$School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

Abstract

In this paper, we study the implications of the modular $A_4$ flavor symmetry in constructing a supersymmetric minimal type-(I+II) seesaw model, in which only one right-handed neutrino and two Higgs triplets are introduced to account for the tiny neutrino masses, flavor mixing and CP violation. The right-handed neutrino as well as the Higgs triplets in this model are assigned into the trivial one-dimensional irreducible representation of the modular group $A_4$. We show that the individual contributions to the neutrino masses from the right-handed neutrino and the Higgs triplet are comparable. We also find that the neutrino mass matrix can possess an approximate $\mu - \tau$ reflection symmetry for some specific values of free model parameters. Moreover, our model predicts relatively large masses of three light neutrinos, thus can be easily tested in future neutrino experiments.

*E-mail: wangx@ihep.ac.cn
1 Introduction

Neutrino oscillation experiments in the past two decades have provided us with the very solid evidence that neutrinos are massive and lepton flavor mixing indeed exists [1,2]. In order to generate tiny neutrino masses, one can extend the standard model (SM) by adding a few new particles and allowing for the lepton number violation, and then the tiny masses of light neutrinos can be attributed to the introduced heavy degrees of freedom. This is the so-called seesaw mechanism. For example, in the typical type-I seesaw mechanism [3–6], three right-handed neutrinos, which are singlets under the $SU(2)_L \times U(1)_Y$ gauge symmetry of the SM, are introduced and the smallness of light neutrino masses can thus be explained by the heavy mass scale of the right-handed neutrinos.

Another interesting realization of the seesaw mechanism is the type-II seesaw mechanism [7–12], in which an additional Higgs triplet under $SU(2)_L$ is added into the SM. Therefore, the gauge-invariant Lagrangian relevant for lepton masses and flavor mixing can be written as

$$-L_{\text{lepton}} = \overline{\ell}_L Y_l H E_R + \frac{1}{2} \overline{\ell}_L Y_\Delta \Delta i \sigma_2 \ell_L^C + \text{h.c.},$$

where $\ell_L$ and $E_R$ denote the left-handed lepton doublet and the right-handed charged-lepton singlet, $H$ and $\Delta$ are the Higgs doublet and triplet, respectively. Note that in Eq. (1.1), $\ell_L^C \equiv C \ell_L^T$ with $C = i \gamma^2 \gamma^0$ being the charge-conjugation matrix has been defined. After the spontaneous symmetry breaking, we can obtain the charged-lepton and neutrino mass matrices as $M_l = Y_l v / \sqrt{2}$ and $M_\nu = Y_\Delta v_\Delta$ respectively, where $v = \sqrt{2} \langle H^0 \rangle \approx 246$ GeV with $\langle H^0 \rangle$ being the vacuum expectation value (vev) of the neutral component of $H$ and $v_\Delta$ is the vev of the neutral component of $\Delta$. The explicit form of $v_\Delta$ can be determined from the following potential which involves both the Higgs doublet and triplet

$$V(H, \Delta) = -\mu^2 H^\dagger H + \lambda (H^\dagger H)^2 + \frac{1}{2} M_\Delta^2 \text{Tr}(\Delta^\dagger \Delta) - (\lambda_\Delta M_\Delta H^T i \sigma_2 \Delta H + \text{h.c.}),$$

where $\mu$, $\lambda$ and $\lambda_\Delta$ are the coupling coefficients and $M_\Delta$ denotes the mass of the Higgs triplet. Then the vev’s $v = \sqrt{\mu^2 / (\lambda - 2 \lambda_\Delta^2)}$ and $v_\Delta = \lambda_\Delta v^2 / M_\Delta$ can be derived from Eq. (1.2). The small value of $v_\Delta$, which is suppressed by the large mass scale of $M_\Delta$, can also explain the observed tiny neutrino masses. In addition, there is another possibility that the light neutrino masses are not originated from one single seesaw mechanism. For instance, one can consider a combination of the type-I and type-II seesaw mechanism, i.e., the type-(I+II) seesaw mechanism, in which both right-handed neutrinos and the Higgs triplet are introduced. In this case, the Lagrangian in Eq. (1.1) becomes

$$-L_{\text{lepton}} = \overline{\ell}_L Y_l H E_R + \overline{\ell}_L Y_\nu \tilde{H} N_R + \frac{1}{2} \overline{N}_R^C M_R N_R + \frac{1}{2} \overline{\ell}_L Y_\Delta \Delta i \sigma_2 \ell_L^C + \text{h.c.},$$

where $N_R$ and $M_R$ denote the right-handed neutrinos and their Majorana mass matrix respectively. Note that in Eq. (1.3), $\tilde{H} \equiv i \sigma_2 H^*$ and $N_R^C \equiv C N_R^T$ have been defined.

Although the seesaw model provides us with an elegant way to explain the tiny neutrino masses, it can not account for the flavor structures existing in the lepton mass matrices. As a consequence, the model is in general lacking of predictive power for lepton mass spectra, flavor
mixing pattern and CP violation \cite{13}. On this account, non-Abelian discrete flavor symmetries have been implemented in the seesaw model to explain the flavor mixing in recent literature, e.g., Refs. \cite{14,19}. To be specific, one can first assume that the Lagrangian maintains an overall discrete flavor symmetry at some high-energy scale. Next a few of gauge-singlet scalar fields which are called flavons are introduced to break down the whole symmetry into distinct residual symmetries in the charged-lepton and neutrino sectors \cite{20,25}. Then the flavor structures will be determined by the vev’s of these flavons. However, the introduction of flavons will inevitably bring a large number of free parameters into the model and how to experimentally prove the existence of the flavons is also a tough problem.

Recently, a new and attractive approach to solve the flavor mixing problem, which is based on the modular invariance, has been proposed in Ref. \cite{26}. Within the framework of modular symmetries, the Yukawa couplings are regarded as the modular forms with even weights, which transform as the multiplets under some finite modular symmetry groups $\Gamma_N$. For a given value of $N$, $\Gamma_N$ is isomorphic to the well-known non-Abelian discrete symmetry group, e.g., $\Gamma_2 \simeq S_3$ \cite{27,30}, $\Gamma_3 \simeq A_4$ \cite{31,40}, $\Gamma_4 \simeq S_4$ \cite{41,44} and $\Gamma_5 \simeq A_5$ \cite{45,47}. Modular forms are the functions of the modulus $\tau$, and the modular symmetry is broken and the flavor mixing pattern is generated right after the value of $\tau$ is fixed. Therefore the flavon field is not necessary in the framework of modular symmetries. Except the references we have mentioned above, there are also plenty of works related to other interesting aspects of modular symmetries, such as the combination of modular symmetries and the CP symmetry \cite{48,49}, multiple modular symmetries \cite{50,51}, the double covering of modular groups \cite{52}, the $A_4$ symmetry from the modular $S_4$ symmetry \cite{53,54}, the modular residual symmetry \cite{55,56}, the unification of quark and lepton flavors with modular invariance \cite{57}, the realization of texture zeros via the modular symmetry \cite{58,59} and the applications of modular symmetries on other types of seesaw models \cite{60,61}.

In this paper, we investigate the minimal supersymmetric type-(I+II) seesaw model, where only one right-handed neutrino and two Higgs triplets are introduced, with the modular $A_4$ symmetry and explore its implications for lepton mass spectra, flavor mixing pattern and CP violation. The pure type-II seesaw model with two additional Higgs triplets under the modular $A_4$ symmetry has already been investigated in Ref. \cite{60}. In such a framework, usually one has to require a large number of free model parameters and higher weights of the modular forms in order to find the suitable parameter space consistent with current experimental data. However, in our minimal type-(I+II) seesaw model, only a few of free parameters are introduced and most of the modular forms involved in are with the lowest non-trivial weights. We show that our model predicts relatively large masses of three light neutrinos, and the individual contributions to the neutrino masses from the right-handed neutrino and the Higgs triplet turn out to be comparable. In addition, we also find that the neutrino mass matrix can possess an approximate $\mu-\tau$ reflection symmetry \cite{62} for some specific values of free model parameters.

The remaining part of this paper is organized as follows. In Sec. 2 a brief summary of the modular $A_4$ symmetry is given. The concrete type-(I+II) seesaw model with the modular $A_4$ symmetry is then proposed in Sec. 3. The low-energy phenomenology of lepton mass spectra, flavor mixing pattern and CP violation in our model are discussed in Sec. 4. Finally, we summarize our main conclusions in Sec. 5. Some properties of the modular $A_4$ symmetry group are presented...
in Appendix A.

2 Modular $A_4$ Symmetry

The basics of modular symmetries have been expounded in previous works (See, e.g., Ref. [26]). In this section, we shall only give a brief review on the modular symmetry.

In a supersymmetric theory, the action $S$ keeps invariant under the modular transformation

$$\gamma : \tau \rightarrow \frac{a\tau + b}{c\tau + d},$$

(2.1)

where $\gamma$ is the element of the modular group $\Gamma$ with $a$, $b$, $c$ and $d$ being integers satisfying $ad - bc = 1$ and $\tau$ is an arbitrary complex number in the upper complex plane. As a consequence, the Kähler potential $K(\tau, \bar{\tau}, \chi, \bar{\chi})$ with $\chi$ being the supermultiplet is invariant up to the Kähler transformation $K(\tau, \bar{\tau}, \chi, \bar{\chi}) \rightarrow K(\tau, \bar{\tau}, \chi, \bar{\chi}) + f(\tau, \chi) + f(\bar{\tau}, \bar{\chi})$ where $f(\tau, \chi)$ itself is invariant under the modular transformation. Meanwhile, the superpotential $W(\tau, \chi)$ is invariant as well and can be expanded in terms of the supermultiplets as follows

$$W(\tau, \chi) = \sum_n \sum_{\{I_1, \ldots, I_n\}} Y_{I_1 \ldots I_n}(\tau) \chi^{(I_1)} \cdots \chi^{(I_n)},$$

(2.2)

where the coefficients $Y_{I_1 \ldots I_n}(\tau)$ take the modular forms, transforming under the finite modular group $\Gamma_N \equiv \Gamma/\Gamma(N)$ (with $\Gamma(N)$ being the principal congruence subgroup of $\Gamma$) as

$$Y_{I_1 \ldots I_n}(\tau) \rightarrow (c\tau + d)^{k_Y} \rho_Y(\gamma) Y_{I_1 \ldots I_n}(\tau),$$

(2.3)

where the even integer $k_Y$ is the weight of $Y_{I_1 \ldots I_n}(\tau)$ and $\rho_Y$ is the representation matrix of $\Gamma_N$. In addition, $k_Y$ and $\rho_Y$ must satisfy $k_Y = k_{I_1} + \cdots + k_{I_n}$ and $\rho_Y \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_n} \ni 1$, respectively.

For the symmetry group $\Gamma_3 \simeq A_4$ of our interest, there are three linearly independent modular forms of the lowest non-trivial weight $k_Y = 2$, denoted as $Y_i(\tau)$ for $i = 1, 2, 3$, which form a triplet $3$ under the modular $A_4$ symmetry transformations [26], namely,

$$Y_3(\tau) \equiv \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}.$$  

(2.4)

The exact expressions of $Y_i(\tau)$ (for $i = 1, 2, 3$) are presented in Appendix A. Based on the modular forms $Y_i(\tau)$ of weight $k_Y = 2$, one can construct the modular forms of higher weights, such as $k_Y = 4$ and $k_Y = 6$. For $k_Y = 4$, there are totally five independent modular forms, which transform as $1$, $1'$ and $3$ under the $A_4$ symmetry [26,37,58], namely,

$$Y_1^{(4)} = Y_1^2 + 2Y_2Y_3, \quad Y_1'^{(4)} = Y_3^2 + 2Y_1Y_2, \quad Y_3^{(4)} = \begin{pmatrix} Y_1^2 - Y_2Y_3 \\ Y_3^2 - Y_1Y_2 \\ Y_2^2 - Y_1Y_3 \end{pmatrix},$$

(2.5)

The most general Kähler potential consistent with the modular symmetry may contain additional terms, as recently pointed out in Ref. [63]. However, for a phenomenological purpose, we consider only the simplest form of the Kähler potential.
where the argument $\tau$ of all the modular forms is suppressed. For $k_\tau = 6$, we have seven independent modular forms, whose assignments under the $A_4$ symmetry are as follows \cite{26,37,58}

$$Y_1^{(6)} = Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3,$$

$$Y_3^{(6)} = (Y_1^2 + 2Y_2Y_3) \left( \begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \end{array} \right), \quad Y_3^{(6)} = (Y_3^2 + 2Y_1Y_2) \left( \begin{array}{c} Y_3 \\ Y_1 \\ Y_2 \end{array} \right). \tag{2.6}$$

## 3 The Minimal Type-(I+II) Seesaw Model

In this section, we are going to construct a minimal type-(I+II) seesaw model with the modular $A_4$ symmetry. To begin with, let us first make some general remarks on the model building.

- A criterion for the model building is that our model should be economical enough, which means that the number of free model parameters should be as small as possible. To be specific, we have eight low-energy observables, including three charged-lepton masses \{\(m_e, m_\mu, m_\tau\)}, two independent neutrino mass-squared differences \{\(\Delta m_{21}^2, \Delta m_{31}^2\)} in the normal mass ordering (NO) case where \(m_1 < m_2 < m_3\) or \{\(\Delta m_{21}^2, \Delta m_{32}^2\)} in the inverted mass ordering (IO) case where \(m_3 < m_1 < m_2\) and three mixing angles \{\(\theta_{12}, \theta_{13}, \theta_{23}\)}.

Therefore the number of free model parameters should be no more than eight in order to have predictive power for the other parameters, such as the CP-violating phases.

- As the modular symmetry is intrinsically working in the supersymmetric framework, we should introduce one chiral superfield \(\hat{N}_C\)\footnote{In this paper, we use the “hat” symbol to denote the chiral superfield.} which contains the right-handed neutrino singlet and a pair of SU(2)_L triplet Higgs superfields \{\(\hat{\Delta}_1, \hat{\Delta}_2\)} with the hypercharges \{+1, -1\} defined as

\[
\hat{\Delta}_1 \equiv \sqrt{2} \left( \frac{\hat{\Delta}_1^+}{\sqrt{2}}, \frac{\hat{\Delta}_1^{++}}{\sqrt{2}}, -\hat{\Delta}_1^+/\sqrt{2} \right), \quad \hat{\Delta}_2 \equiv \sqrt{2} \left( \frac{\hat{\Delta}_2^-}{\sqrt{2}}, \frac{\hat{\Delta}_2^0}{\sqrt{2}}, -\hat{\Delta}_2^-/\sqrt{2} \right), \tag{3.1}
\]

where \(\hat{\Delta}_1^{++} (\hat{\Delta}_2^-), \hat{\Delta}_1^+ (\hat{\Delta}_2^-)\) and \(\hat{\Delta}_1^0 (\hat{\Delta}_2^0)\) denote the doubly-charged, singly-charged and neutral components of \(\hat{\Delta}_1\) (\(\hat{\Delta}_2\)), respectively. \(\hat{N}_C, \hat{\Delta}_1\) and \(\hat{\Delta}_2\) are all arranged to be the trivial singlet under the modular $A_4$ symmetry in our model for simplicity. Furthermore, the superfields for Higgs doublets \{\(\hat{H}_u, \hat{H}_d\)} with the hypercharges \{+1/2, -1/2\} are also assigned into \(\mathbf{1}\) under the modular $A_4$ symmetry. As a consequence, we do not need to change the remaining part of the MSSM irrelevant for leptonic flavor mixing.

- The superfields for three lepton doublets \{\(\hat{L}_1, \hat{L}_2, \hat{L}_3\)} are arranged as a triplet \(\mathbf{3}\) under the $A_4$ symmetry, while the superfields for three charged-lepton singlets \{\(\hat{E}_1^C, \hat{E}_2^C, \hat{E}_3^C\)} should be assigned into three different singlets of $A_4$ (e.g., \(\hat{E}_1^C \sim \mathbf{1}, \hat{E}_2^C \sim \mathbf{1}'\) and \(\hat{E}_3^C \sim \mathbf{1}\)). Otherwise, it will be difficult to explain the strong mass hierarchy of three charged leptons, namely, \(m_e \ll m_\mu \ll m_\tau\).
and type-II seesaw mechanisms can be read from the neutrino sector. In Eq. (3.2), the individual contributions to neutrino masses from the type-I where $g$ parameter $W$ weights listed in the last row.

Table 1: The charge assignment of the chiral superfields and the relevant couplings under the $SU(2)_L$ symmetry and the modular $A_4$ symmetry in our model, with the corresponding modular weights listed in the last row.

|        | $\hat{L}$ | $\hat{E}^C_1$, $\hat{E}^C_2$, $\hat{E}^C_3$ | $\hat{N}^C$ | $\hat{H}_u$, $\hat{H}_d$ | $\hat{A}_1$, $\hat{A}_2$ | $f_\tau(\tau)$, $f_\mu(\tau)$, $f_\tau(\tau)$, $f_\Delta(\tau)$ | $f_D(\tau)$ | $f_R(\tau)$ |
|--------|-----------|--------------------------------------------|-------------|-----------------|----------------------------|-------------------------------------------------|------------|------------|
| $SU(2)$ | 2         | 1                                         | 1           | 1               | 3                         | 1                                                | 1          | 1          |
| $A_4$  | 3         | 1, 1', 1''                               | 1           | 1               | 1                         | 3                                                | 3          | 1          |
| $-k_f$ | -1        | -1                                        | -3          | 0               | 0                         | $k_{\epsilon,\mu,\tau,\Delta} = 2$              | $k_D = 4$  | $k_R = 6$  |

- The modular forms relevant for lepton masses and flavor mixing can be exactly determined from the two identities $k_Y = k_{I_1} + \cdots + k_{I_N}$ and $\rho_r \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_N} \supseteq 1$ after the weights and representations of the superfields are fixed. Note that since both the right-handed neutrino and Higgs triplets are introduced into our model, more terms will appear in the whole superpotential. Consequently, there remains less freedom for us to adjust the weights and representations under the modular $A_4$ symmetry of all the superfields as well as the modular forms. In Table 1 we show the charge assignments of the chiral superfields and the couplings under the $SU(2)_L$ gauge symmetry and the modular $A_4$ symmetry for our model, and the corresponding modular weights are listed in the last row. Note that $k_D = 4$ and $k_R = 6$ are the lowest weights which the modular forms $f_D$ and $f_R$ can take respectively under the premise that $k_Y = k_{I_1} + \cdots + k_{I_N}$ should be satisfied in each superpotential.

Keeping these assignments above in mind, now it is straightforward for us to write down the modular $A_4$ invariant superpotential $\mathcal{W}$, which can be decomposed into three parts $\mathcal{W} = \mathcal{W}_I + \mathcal{W}_I + \mathcal{W}_{II}$ with

$$\mathcal{W}_I = \alpha_1 \left[ \left( \hat{L} \hat{E}^C_1 \right)_3 (f_e(\tau))_3 \right]_1 \hat{H}_d + \alpha_2 \left[ \left( \hat{L} \hat{E}^C_2 \right)_3 (f_\mu(\tau))_3 \right]_1 \hat{H}_d + \alpha_3 \left[ \left( \hat{L} \hat{E}^C_3 \right)_3 (f_\tau(\tau))_3 \right]_1 \hat{H}_d ,$$

$$\mathcal{W}_I = g_1 \left[ \left( \hat{L} \hat{N}^C \right)_3 (f_D(\tau))_3 \right]_1 \hat{H}_u + \frac{1}{2} \Lambda \left[ \left( \hat{N}^C \hat{N}^C \right)_1 (f_R(\tau))_1 \right]_1 ,$$

$$\mathcal{W}_{II} = \frac{1}{2} g_2 \left[ \left( \hat{L} \hat{L} \right)_3 (f_\Delta)_3 \right]_1 \left( i\sigma_2 \hat{A}_1 \right) ,$$

(3.2)

where $\alpha_1$, $\alpha_2$ and $\alpha_3$ are three coupling coefficients in the charged-lepton sector which we can set to be real and positive without loss of generality while $g_1$, $g_2$ and $\Lambda$ are the coupling coefficients in the neutrino sector. In Eq. (3.2), the individual contributions to neutrino masses from the type-I and type-II seesaw mechanisms can be read from $\mathcal{W}_I$ and $\mathcal{W}_{II}$, respectively. When the modulus parameter $\tau$ is fixed, the modular symmetry is broken down and the superpotential reads

$$\mathcal{W} = \lambda_1 \hat{L} \hat{H}_d \hat{E}^C + \lambda_2 \hat{H}_u \hat{N}^C + \frac{1}{2} \lambda_\Sigma \hat{N}^C \hat{N}^C + \frac{1}{2} \lambda_{\Sigma} \hat{L} \left( i\sigma_2 \hat{A}_1 \right) \hat{L} ,$$

(3.3)

where $\lambda_1$ and $\lambda_2$ turn out to be the charged-lepton and Dirac neutrino Yukawa coupling matrices, respectively, $\lambda_\Sigma$ becomes the right-handed neutrino mass matrix and $\lambda_{\Sigma}$ is the neutrino mass matrix induced by the type-II seesaw.

On the other hand, the superpotential relevant for the couplings between the Higgs doublets and triplets, which is just the supersymmetric version of Eq. (1.2), can be written as

$$\mathcal{W}_{II-\Delta} = \mu \hat{H}_u \hat{H}_d + \lambda_1 \hat{H}_d \left( i\sigma_2 \hat{A}_1 \right) \hat{H}_d + \lambda_2 \hat{H}_u \left( i\sigma_2 \hat{A}_2 \right) \hat{H}_u + \frac{1}{2} M_\Delta \text{Tr} \left( \hat{A}_1 \hat{A}_2 \right) ,$$

(3.4)
where $\mu$, $\lambda_1$ and $\lambda_2$ are the coupling coefficients. After the supersymmetry breaking and the SU(2)$_L$ × U(1)$_Y$ gauge symmetry breakdown, all the Higgs fields get their own vev’s and one can then obtain the lepton mass terms from Eq. (3.2). It has been indicated in Ref. [44] that there exists the following correspondence between the lepton mass matrices and the Yukawa coupling matrices in the MSSM framework under the left-right convention for the fermion mass terms

\[ M_l = v_d \lambda_1^*/\sqrt{2}, \quad M_D = v_u \lambda_D^*/\sqrt{2}, \quad M_R = \lambda_R^*, \quad M_{\Pi} = v_1 \lambda_{\Pi}^*. \]  

(3.5)

where $v_d = v \cos \beta$ and $v_u = v \sin \beta$ are respectively the vev of the neutral scalar component field of $\tilde{H}_d$ to that of $\tilde{H}_u$, with $\tan \beta \equiv v_u/v_d$ being their ratio, and $v_1 = \lambda_2 v_0^2/M_\Delta$ is the vev of the neutral scalar component field of $\Delta_1$ and can be derived from Eq. (3.4). Note that here we use "*" to denote the complex conjugation. Therefore, by using the product rules of the $A_4$ symmetry group collected in Appendix A, we can obtain the charged-lepton mass matrix

\[ M_l = \frac{v_d}{\sqrt{2}} \begin{pmatrix} Y_1 & Y_2 & Y_3 \\ Y_3 & Y_1 & Y_2 \\ Y_2 & Y_3 & Y_1 \end{pmatrix}^* \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \]  

(3.6)

and the Dirac neutrino mass matrix

\[ M_D = \frac{v_u g^*_1}{\sqrt{2}} \begin{pmatrix} Y_1^2 - Y_2 Y_3 \\ Y_2^2 - Y_1 Y_3 \\ Y_3^2 - Y_1 Y_2 \end{pmatrix}^*. \]  

(3.7)

Since only one right-handed neutrino is introduced, the Majorana mass matrix will degenerate to a complex number

\[ M_R = \Lambda(Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1 Y_2 Y_3)^*. \]  

(3.8)

After applying the type-I seesaw mechanism $M_I \approx -M_D M_R^{-1} M_D^T$, we arrive at the neutrino mass matrix from the type-I seesaw

\[ M_I = -\frac{v_d^2 (g^*_1)^2}{2\Lambda(Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1 Y_2 Y_3)^*} \times \begin{pmatrix} (Y_1^2 - Y_2 Y_3)^2 & (Y_1^2 - Y_2 Y_3)(Y_2^2 - Y_1 Y_3) & (Y_1^2 - Y_2 Y_3)(Y_3^2 - Y_1 Y_2) \\ (Y_2^2 - Y_1 Y_3)^2 & (Y_2^2 - Y_1 Y_3)(Y_3^2 - Y_1 Y_2) & (Y_2^2 - Y_1 Y_3)(Y_3^2 - Y_1 Y_2) \\ (Y_3^2 - Y_1 Y_2)^2 & (Y_3^2 - Y_1 Y_2)(Y_2^2 - Y_1 Y_3) & (Y_3^2 - Y_1 Y_2)(Y_3^2 - Y_1 Y_2) \end{pmatrix}^*. \]  

(3.9)

Meanwhile, the neutrino mass matrix induced by the type-II seesaw can be expressed as

\[ M_{\Pi} = \frac{\lambda_2 v_0^2 g^*_2}{3M_\Delta} \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix}^*. \]  

(3.10)

Then the whole effective neutrino mass matrix should be a combination of $M_I$ and $M_{\Pi}$. Since the overall phase of any lepton mass matrix is irrelevant for lepton masses and flavor mixing, one
can take $g_1$ in Eq. (3.7) to be real and it is convenient to define a new complex parameter as $2\lambda_2 g_2 \Lambda / (3g_1^2 M_\Lambda) \equiv \tilde{g} = g e^{i\phi_g}$ with $g = |\tilde{g}|$ and $\phi_g \equiv \arg(\tilde{g})$. Therefore $M_\nu$ can be written as

$$M_\nu = M_I + M_{II}$$

$$-\frac{v_d^2 g_1^2}{2\Lambda} \left[ \frac{1}{Y_1^2 + Y_2^2 + Y_3^2 - 3Y_1 Y_2 Y_3} \right.$$  

$$\times \begin{pmatrix}
(Y_1^2 - Y_2 Y_3)^2 & (Y_1^2 - Y_2 Y_3)(Y_2^2 - Y_1 Y_3) & (Y_1^2 - Y_2 Y_3)(Y_3^2 - Y_1 Y_2) \\
(Y_1^2 - Y_2 Y_3)(Y_2^2 - Y_1 Y_3) & (Y_2^2 - Y_1 Y_3)^2 & (Y_2^2 - Y_1 Y_3)(Y_3^2 - Y_1 Y_2) \\
(Y_1^2 - Y_2 Y_3)(Y_3^2 - Y_1 Y_2) & (Y_2^2 - Y_1 Y_3)(Y_3^2 - Y_1 Y_2) & (Y_3^2 - Y_1 Y_2)^2 
\end{pmatrix}$$

$$\left. \quad -\tilde{g} \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix} \right] ^* .$$

From Eq. (3.11) we can find that each element in the mass matrix $M_{II}$ contains only one single modular form with a weight of 2, whereas every element in the matrix generated by $M_D M_D^T$ is a multiplication of two modular forms of weight 4, thus having a total modular weight of 8. However, $M_R^{-1}$ contributes another weight of $-6$, therefore the overall weight of $M_I$ is also 2, same as the weight of $M_{II}$.

4 Low-energy Phenomenology

Next we discuss the low-energy phenomenology of our model. As can be seen from the previous section, there are totally eight free model parameters, which are a complex modulus $\tau$ (or equivalently two real parameters $\text{Re} \tau$ and $\text{Im} \tau$), three real parameters $v_d \alpha_3 / \sqrt{2}$, $\alpha_1 / \alpha_3$ and $\alpha_2 / \alpha_3$ in the charged-lepton sector and two complex parameters $g$ and $\phi_g$ as well as an overall factor $v_d^2 g_1^2 / (2\Lambda)$ in the neutrino sector. The number of free parameters is the same as that of the low-energy observables. As a result, our model should be predictive. Then we proceed to explore the phenomenological implications for lepton mass spectra, flavor mixing and CP violation. We carry out a numerical analysis of our model and demonstrate that the predictions are consistent with the experimental data only in the NO case at the 1\sigma level. The main strategy for numerical analysis is analogous to what we have done in Ref. [44]. Here we list it as follows.

- First of all, the modulus parameter $\tau$ is randomly generated in the right-hand part of the fundamental domain $\mathcal{G}$, which is defined as

$$\mathcal{G} = \{ \tau \in \mathbb{C} : \text{Im} \tau > 0, |\text{Re} \tau| \leq 0.5, |\tau| \geq 1 \} .$$

This domain can be identified by using the basic properties of the modular forms as clearly explained in Ref. [42]. One can also notice that if the replacement $\tau \rightarrow -\tau^*$ is made in Eq. (A.8), the modular forms $Y_i(\tau)$ will change to their complex-conjugate counterparts, i.e., $Y_i(-\tau^*) = Y_i^*(\tau)$. If we further replace $\tilde{g}$ with $\tilde{g}^*$ in the neutrino sector, all the lepton mass matrices will become their complex-conjugate counterparts. Under such a transformation,
Table 2: The best-fit values, the 1σ and 3σ intervals, together with the values of σ_i being the symmetrized 1σ uncertainties, for three neutrino mixing angles \{\theta_{12}, \theta_{13}, \theta_{23}\}, two mass-squared differences \{\Delta m^2_{21}, \Delta m^2_{31} or \Delta m^2_{32}\} and the Dirac CP-violating phase δ from a global-fit analysis of current experimental data \cite{65}.

| Parameter | Best fit | 1σ range | 3σ range | σ_i |
|-----------|----------|-----------|-----------|-----|
| \sin^2 \theta_{12} | 0.310 | 0.298 — 0.323 | 0.275 — 0.350 | 0.0125 |
| \sin^2 \theta_{13} | 0.02241 | 0.02176 — 0.02307 | 0.02046 — 0.02440 | 0.000655 |
| \sin^2 \theta_{23} | 0.558 | 0.525 — 0.578 | 0.427 — 0.609 | 0.0265 |
| δ [°] | 222 | 194 — 260 | 141 — 370 | 33 |
| \Delta m^2_{21} [10^{-5} \text{ eV}^2] | 7.39 | 7.19 — 7.60 | 6.79 — 8.01 | 0.205 |
| \Delta m^2_{31} [10^{-3} \text{ eV}^2] | +2.523 | +2.493 — +2.555 | +2.432 — +2.618 | 0.031 |

| Parameter | Best fit | 1σ range | 3σ range | σ_i |
|-----------|----------|-----------|-----------|-----|
| \sin^2 \theta_{12} | 0.310 | 0.298 — 0.323 | 0.275 — 0.350 | 0.0125 |
| \sin^2 \theta_{13} | 0.02261 | 0.02197 — 0.02328 | 0.02066 — 0.02461 | 0.000655 |
| \sin^2 \theta_{23} | 0.563 | 0.537 — 0.582 | 0.430 — 0.612 | 0.0225 |
| δ [°] | 285 | 259 — 309 | 205 — 354 | 25 |
| \Delta m^2_{21} [10^{-5} \text{ eV}^2] | 7.39 | 7.19 — 7.60 | 6.79 — 8.01 | 0.205 |
| \Delta m^2_{32} [10^{-3} \text{ eV}^2] | -2.509 | -2.539 — -2.477 | -2.603 — -2.416 | 0.031 |

In the charged-lepton sector, once we randomly choose the values of \{\text{Re} \tau, \text{Im} \tau\}, the parameters \nu_i \alpha_i/\sqrt{2}, \alpha_1/\alpha_3 and \alpha_2/\alpha_3 can be calculated from the following identities

\[
\text{Tr} \left( M_l M_l^\dagger \right) = m_e^2 + m_\mu^2 + m_\tau^2 , \quad (4.2)
\]
\[
\text{Det} \left( M_l M_l^\dagger \right) = m_e^2 m_\mu^2 m_\tau^2 , \quad (4.3)
\]
\[
\frac{1}{2} \left[ \text{Tr} \left( M_l M_l^\dagger \right) \right]^2 - \frac{1}{2} \text{Tr} \left[ (M_l M_l^\dagger)^2 \right] = m_e^2 m_\mu^2 + m_\mu^2 m_\tau^2 + m_\tau^2 m_e^2 , \quad (4.4)
\]

where we take \(m_e = 0.511 \text{ MeV}, m_\mu = 105.7 \text{ MeV}\) and \(m_\tau = 1776.86 \text{ MeV}\) for the observed charged-lepton masses \cite{2}. Notice that Eqs. (4.2)-(4.4) have multiple solutions, corresponding to the different hierarchies of \(\alpha_1, \alpha_2\) and \(\alpha_3\). Later we will see there exist two kinds of hierarchies \((\alpha_1 \ll \alpha_3 \ll \alpha_2 \ll \alpha_2 \ll \alpha_3)\) which can lead to the realistic mixing pattern, with different predictions to the value of \(\theta_{23}\). So far all the parameters in \(M_l\) have been determined. It is then easy to diagonalize the charged-lepton mass matrix via \(U_l^\dagger M_l M_l^\dagger U_l = \text{Diag} \left\{ m_e^2, m_\mu^2, m_\tau^2 \right\} \), from which the unitary matrix \(U_l\) can be obtained.
Figure 1: Allowed ranges of the model parameters \(\{\Re \tau, \Im \tau\}\) and \(\{g, \phi_g\}\) in the NO case, where the 1\(\sigma\) (yellow dots) and 3\(\sigma\) (red dots) ranges of neutrino mixing parameters and mass-squared differences from the global-fit analysis of neutrino oscillation data have been input \cite{65}. The best-fit values are indicated by the black stars. In the left panel, the horizontal dashed line separates the parameter space of \(\{\Re \tau, \Im \tau\}\) into two regions: **Region A** and **Region B**.

- Next the values of the other two parameters \(g \in (0, 10]\) and \(\phi_g \in [0^\circ, 360^\circ]\) are randomly generated. Therefore, the effective neutrino mass matrix \(M_\nu\) is determined up to the overall scale parameter \(v_u^2g_1^2/(2\Lambda)\). We introduce a ratio \(r\) defined as \(r \equiv \Delta m^2_{21}/\Delta m^2_{31}\) in the NO case or \(r \equiv \Delta m^2_{21}/|\Delta m^2_{32}|\) in the IO case which is irrelevant to this overall scale parameter, and this ratio can help us restrict the values of \(\Re \tau, \Im \tau, g\) and \(\phi_g\). The overall parameter \(v_u^2g_1^2/(2\Lambda)\) can be determined right after we fix the value of the lightest neutrino mass. After diagonalizing \(M_\nu\) via \(U_1^\dagger M_\nu U_1^\dagger = \text{Diag}\{m_1, m_2, m_3\}\), we get the unitary matrix \(U_\nu\). Finally, the lepton flavor mixing matrix \(U = U_1^\dagger U_\nu\) can be calculated by using \(U_1\) and \(U_\nu\). In the standard parametrization \cite{2}, we have

\[
U = \begin{pmatrix} 
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
    -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
    s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix} \begin{pmatrix} e^{i\rho} & 1 \\
    e^{i\sigma} & 1
\end{pmatrix}, \quad (4.5)
\]

where \(c_{ij} \equiv \cos \theta_{ij}\) and \(s_{ij} \equiv \sin \theta_{ij}\) (for \(ij = 12, 13, 23\)) have been defined and \(\delta, \rho\) and \(\sigma\) are the Dirac and two Majorana CP-violating phases, respectively.

To find out the allowed parameter space of \(\{\Re \tau, \Im \tau\}\) and \(\{g, \phi_g\}\), we implement the global-fit results from NuFIT 4.1 \cite{65} without including the atmospheric neutrino data from Super-Kamiokande. The best-fit values of three neutrino mixing angles \(\{\theta_{12}, \theta_{13}, \theta_{23}\}\), two neutrino mass-squared differences \(\{\Delta m^2_{21}, \Delta m^2_{31}\}\) (or \(\{\Delta m^2_{21}, \Delta m^2_{32}\}\)), the Dirac CP-violating phase \(\delta\), together with their 1\(\sigma\) and 3\(\sigma\) ranges in the NO (or IO) case are summarized in Table 2.

As we have mentioned before, the predictions of our model are consistent with the experimental data only in the NO case at the 1\(\sigma\) level. The allowed parameter space of \(\{\Re \tau, \Im \tau\}\), \(\{g, \phi_g\}\) and \(\{\alpha_1/\alpha_3, \alpha_2/\alpha_3\}\) has been shown in Figs. 1\cite{1} where the 1\(\sigma\) (3\(\sigma\)) range is denoted by the yellow (red) dots. As one can see from the left panel of Fig. 1\cite{1} almost all the range \([0, 0.5]\) of \(\Re \tau\) is allowed at the 3\(\sigma\) level while the value of \(\Im \tau\) is restricted to be larger than 1.75. Note that here we artificially cut off the parameter space of \(\Im \tau\) at \(\Im \tau = 4\), since in the range where
Figure 2: Allowed ranges of two ratios \(\{\alpha_1/\alpha_3, \alpha_2/\alpha_3\}\) in the charged-lepton sector for the NO case, where the 1\(\sigma\) (yellow dots) and 3\(\sigma\) (red dots) ranges of neutrino mixing parameters and mass-squared differences from the global-fit analysis of neutrino oscillation data have been input\cite{gepar}. The best-fit value is indicated by the black star. The left panel corresponds to the hierarchy \(\alpha_1 \ll \alpha_3 \ll \alpha_2\) where only the 3\(\sigma\) range is allowed and the right panel is related to \(\alpha_1 \ll \alpha_2 \ll \alpha_3\).

Im \(\tau > 4\), we find that the predicted values of mixing angles and CP-violating phases tend to be stable and the sum of three light neutrino masses \(\sum m_\nu = m_1 + m_2 + m_3 > 2\, \text{eV}\), which has already been far away from the favored region of the latest Planck observations\cite{planck}, thus being out of our interest. Actually we can separate the parameter space of \(\{\text{Re}\,\tau, \text{Im}\,\tau\}\) into two regions depending on the values of \(\text{Im}\,\tau\), to be specific, Region A with \(1.75 < \text{Im}\,\tau < 2.07\) and Region B with \(2.07 < \text{Im}\,\tau < 4\). An important feature to distinguish these two regions is that only the hierarchy \(\alpha_1 \ll \alpha_3 \ll \alpha_2\) is permitted in Region A while both the hierarchies \(\alpha_1 \ll \alpha_3 \ll \alpha_2\) and \(\alpha_1 \ll \alpha_2 \ll \alpha_3\) are allowed in Region B. The reason for this fact will be discussed in detail later. On the other hand, in Region A the value of \(\text{Im}\,\tau\) can only change in a narrow region. However \(\text{Re}\,\tau\) can vary in a wide range, from 0.04 to 0.5. On the contrary, in Region B the value of \(\text{Re}\,\tau\) is about 0.03 while \(\text{Im}\,\tau\) can reach very large values. The constraints on \(g\) and \(\phi_g\) within the 3\(\sigma\) level are \(0.82 < g < 0.92\) and \(1.92^\circ < \phi_g < 21.8^\circ\) respectively, as can be seen from the right panel of Fig. 1. The value of \(\tilde{g}\) measures the individual contributions to the neutrino masses from the type-I and type-II seesaw mechanisms, and \(g \sim 1\) means that their individual contributions are comparable to each other.

To determine the model parameters from neutrino oscillation data and describe how well the model is consistent with observations, we construct the \(\chi^2\)-function by regarding the best-fit values \(q_j^{bf}\) of the oscillation parameters \(q_j \in \{\sin^2 \theta_{12}, \sin^2 \theta_{13}, \sin^2 \theta_{23}, \Delta m_{21}^2, \Delta m_{31}^2\}\) from the global analysis in Ref.\cite{gepar} as experimental measurements, namely,

\[
\chi^2(p_i) = \sum_j \left( \frac{q_j(p_i) - q_j^{bf}}{\sigma_j} \right)^2,
\]

where \(p_i \in \{\text{Re}\,\tau, \text{Im}\,\tau, g, \phi_g\}\) stand for the free model parameters and \(q_j(p_i)\) denote the model predictions for observables with \(\sigma_j\) being the symmetrized 1\(\sigma\) uncertainties from the global-fit analysis, which has already been given in Table 2. Since the current constraint on \(\delta\) from the global-fit results is rather weak, we will not include the information of \(\delta\) in the \(\chi^2\)-function. The
allowed ranges of model parameters can be obtained by following the standard $\chi^2$-fit approach. Based on the $\chi^2$-fit analysis, we find that the minimum $\chi^2_{\text{min}} = 0.232$ is obtained in the NO case with the following best-fit values of the model parameters

$$\text{Re}\, \tau = 0.0365 , \quad \text{Im}\, \tau = 2.33 , \quad g = 0.834 , \quad \phi_g = 19.2^\circ ,$$

(4.7)

which together with the charged-lepton masses $m_\alpha$ (for $\alpha = e, \mu, \tau$) lead to $v_4\alpha_3/\sqrt{2} = 1.77$ GeV, $\alpha_1/\alpha_3 = 2.88 \times 10^{-3}$ and $\alpha_2/\alpha_3 = 5.96 \times 10^{-2}$. Once the model parameters are known, we can get the constraints on the observables $q_i$ and the predictions for the CP-violating phases $\delta, \rho$ and $\sigma$, as well as the effective neutrino mass $m_\beta$ for beta decays and $m_{\beta\beta}$ for neutrinoless double-beta decays, which will be presented in Table 3.

The relation between $\sum m_\nu$ and $\text{Im}\, \tau$ is presented in Fig. 3 from which we can find that the value of $\sum m_\nu$ is tightly related to $\text{Im}\, \tau$. To be specific, as the value of $\text{Im}\, \tau$ increases, $\sum m_\nu$ will also increase. This can be understood in the following way. Since $\text{Im}\, \tau$ in our model is larger than 1.75, $|q| = e^{-2\pi\text{Im}\, \tau} < 4 \times 10^{-3}$ in the Fourier expansions of modular forms $Y_i(\tau)$ (for $i = 1, 2, 3$) in Eq. (A.8) is a small parameter. Therefore we can retain only the leading order terms in the expansions of $Y_i(\tau)$ and Eq. (A.8) will change to

$$Y_1 \approx 1 , \quad Y_2 \approx -6q^{1/3} \equiv t , \quad Y_3 \approx -18q^{2/3} \equiv -\frac{1}{2} t^2 ,$$

(4.8)

where $t \equiv -6q^{1/3} = -6e^{-2\pi\text{Im}\, \tau/3}$ has been defined. Then the neutrino mass matrix $M_\nu$ can be expressed in an approximate form up to $O(t^3)$

$$M_\nu \approx \frac{-v_2g_2}{2 \lambda} \begin{pmatrix} 1 - 2\bar{g} - \frac{3}{2} t^3 & \frac{1}{2} (3 - \bar{g}) t^2 & -(1 - \bar{g}) t \\ \frac{1}{2} (3 - \bar{g}) t^2 & -2\bar{g} t & \bar{g} - \frac{3}{2} t^3 \\ -(1 - \bar{g}) t & \bar{g} - \frac{3}{2} t^3 & (1 + \bar{g}) t^2 \end{pmatrix}^* .$$

(4.9)
has already exceeded the upper bound on the sum of neutrino masses \( \alpha \) the Planck observations \cite{66}. However this upper bound is cosmological model dependent and the NO case while the correlation between \( \sum m_\nu > 0.6 \) eV which is excluded by the latest Planck observations \cite{66} while the blue shaded region denotes the range of \( \theta_{23} \) within the 1\( \sigma \) level. The horizontal (vertical) dashed line in the left (right) panel refers to \( \theta_{23} = 45^\circ \).

All the elements except \( (M_\nu)_{11}, (M_\nu)_{23} \) and \( (M_\nu)_{32} \) in the right-hand side of Eq. \cite{49} are suppressed by the higher order terms of \( t \). As the value of \( \text{Im} \tau \) becomes larger, \( (M_\nu)_{11}, (M_\nu)_{23} \) and \( (M_\nu)_{32} \) will dominant the eigenvalues of \( M_\nu \). Given the parameter space of \( \tilde{g} \), we can find that the modulus of \( (M_\nu)_{11} \) is close to that of \( (M_\nu)_{23} \) (Note that \( (M_\nu)_{23} = (M_\nu)_{32} \) exactly holds due to the nature of the Majorana mass matrix) especially when \( \text{Im} \tau \) is large enough, implying a quasi-degeneracy among \( m_1, m_2 \) and \( m_3 \). The high degeneracy of three neutrino masses requires a large \( \sum m_\nu \), which is why the value of \( \sum m_\nu \) increases with the rise of \( \text{Im} \tau \).

Fig. \ref{fig:3} also indicates that the minimal value of \( \sum m_\nu \) predicted in our model is 0.16 eV, which has already exceeded the upper bound on the sum of neutrino masses \( \sum m_\nu < 0.12 \) eV from the Planck observations \cite{66}. However this upper bound is cosmological model dependent and obtained by combining other experimental data such as the baryon acoustic oscillation (BAO), the gravitational lensing of galaxies and the high multipole TT, TE and EE polarization spectra. If only the BAO data and the cosmic microwave background (CMB) lensing reconstruction power spectrum are taken into consideration, the restriction to \( \sum m_\nu \) can be liberalized to \( \sum m_\nu < 0.6 \) eV \cite{66}. Therefore, our model can still be compatible with the Planck observations in the disfavored region where \( 0.12 \) eV \( < \sum m_\nu < 0.6 \) eV. In addition, since our model predicts relatively large values of three neutrino masses, it can be easily tested in future neutrino experiments.

A salient feature of our model is that the predicted value of \( \theta_{23} \) shows a strong dependence on the free model parameters especially \( \text{Im} \tau \), as can be seen from the left panel of Fig. \ref{fig:4} where the red and blue curves correspond to the hierarchy \( \alpha_1 \ll \alpha_3 \ll \alpha_2 \) and \( \alpha_1 \ll \alpha_2 \ll \alpha_3 \), respectively. Some useful remarks are as follows.

- There are two branches in the allowed range of \( \{ \text{Im} \tau, \theta_{23} \} \), depending on which hierarchy of \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) is taken into consideration. In the hierarchy \( \alpha_1 \ll \alpha_3 \ll \alpha_2 \), \( \theta_{23} \) is located in the first octant where \( \theta_{23} < 45^\circ \) while in the hierarchy \( \alpha_1 \ll \alpha_2 \ll \alpha_3 \), \( \theta_{23} \) is in the second octant, which is preferred by the latest global-fit analysis within 1\( \sigma \) level. If further long
baseline experiments such as DUNE \cite{67} and Hyper-Kamiokande \cite{68} can give more precise measurements on the octant of $\theta_{23}$, it will be promising to determine the hierarchy of $\alpha_1$, $\alpha_2$, and $\alpha_3$ in our model unambiguously.

- In order to illustrate how the value of $\theta_{23}$ is connected with the hierarchies of $\alpha_1$, $\alpha_2$ and $\alpha_3$, we express the mass matrix $M_l$ in an approximate form by substituting Eq. (4.8) into Eq. (3.6)

$$M_l \approx \frac{v_1}{\sqrt{2}} \begin{pmatrix} 1 & t & -\frac{1}{2}t^2 \\ -\frac{1}{2}t^2 & 1 & t \\ t & -\frac{1}{2}t^2 & 1 \end{pmatrix}^* \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}. \quad (4.10)$$

Since $t$ can be regarded as a small parameter and there is a strong hierarchy of $\alpha_1$, $\alpha_2$ and $\alpha_3$ (namely $\alpha_1 \ll \alpha_3 \ll \alpha_2$ or $\alpha_1 \ll \alpha_2 \ll \alpha_3$), it is possible to obtain the approximate analytical form of the unitary matrix $U_l$. To be specific, in the hierarchy $\alpha_1 \ll \alpha_3 \ll \alpha_2$, the unitary matrix $U_l^{(1)}$ can be written as

$$U_l^{(1)} \approx \begin{pmatrix} -1 + \frac{1}{2}|t|^2 & -\frac{3}{2}(t^*)^2 & t^* \\ t & \frac{1}{2}t^* & 1 - \frac{1}{2}|t|^2 \\ -\frac{3}{2}t^* & \frac{1}{2}t^* & 1 - \frac{1}{2}(t^*)^2 \end{pmatrix}, \quad (4.11)$$

where we retain only the terms up to $O(t^2)$. Then the lepton mixing matrix $U^{(1)}$ turns out to be $U^{(1)} = (U_l^{(1)})^\dagger U_\nu$, and $\sin^2 \theta_{23}^{(1)}$ can be obtained from $\sin^2 \theta_{23}^{(1)} = |U_{\mu_3}^{(1)}|^2/(1 - |U_{e_3}^{(1)}|^2)$. While in the hierarchy $\alpha_1 \ll \alpha_2 \ll \alpha_3$, the unitary matrix $U_l^{(2)}$ is

$$U_l^{(2)} \approx \begin{pmatrix} -1 + \frac{1}{2}|t|^2 & t^* & -\frac{1}{2}(t^*)^2 \\ t & 1 - |t|^2 & t^* \\ -\frac{3}{2}t^* & -t & 1 - \frac{1}{2}|t|^2 \end{pmatrix}. \quad (4.12)$$

The lepton mixing matrix $U^{(2)}$ and $\sin^2 \theta_{23}^{(2)}$ then can be expressed as $U^{(2)} = (U_l^{(2)})^\dagger U_\nu$ and $\sin^2 \theta_{23}^{(2)} = |U_{\mu_3}^{(2)}|^2/(1 - |U_{e_3}^{(2)}|^2)$, respectively. We can use another unitary matrix $P$ to connect $U_l^{(1)}$ and $U_l^{(2)}$ via $U_l^{(2)} = U_l^{(1)} P$. Now that we have already obtained the approximate expressions of $U_l^{(1)}$ and $U_l^{(2)}$, it is easy to write down the explicit form of $P$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -t + \frac{1}{2}(t^*)^2 & 1 - \frac{1}{2}|t|^2 \\ 0 & 1 - \frac{1}{2}|t|^2 & t^* \end{pmatrix}. \quad (4.13)$$

With the expression of $P$ in Eq. (4.13), we can express $U_{e_2}^{(2)}$, $U_{e_3}^{(2)}$ and $U_{\mu_3}^{(2)}$ by using the elements of $U^{(1)}$ as

$$U_{e_2}^{(2)} = U_{e_2}^{(1)}, \quad U_{e_3}^{(2)} = U_{e_3}^{(1)}, \quad U_{\mu_3}^{(2)} = \left(-t^* + \frac{1}{2}t^2\right) U_{\mu_3}^{(1)} + \left(1 - \frac{1}{2}|t|^2\right) U_{\tau_3}^{(1)}. \quad (4.14)$$
So finally we arrive at the expression of \( \sin^2 \theta_{23}^{(2)} \)

\[
\sin^2 \theta_{23}^{(2)} = |t|^2 \sin^2 \theta_{23}^{(1)} + \left( 1 - \frac{1}{2} |t|^2 \right) \cos^2 \theta_{23}^{(1)} - \frac{2 \Re \left[ (t^* - t^2/2) U_{\mu 3}^{(1)} (U_{\tau 3}^{(1)})^* \right]}{1 - |U_{e 3}^{(1)}|^2}.
\]  \( (4.15) \)

If \( \Im \tau \) is sufficiently large, all the higher order terms of \( t \) can be neglected, and we will arrive at \( \sin^2 \theta_{23}^{(1)} \approx \cos^2 \theta_{23}^{(2)} \), i.e., \( \theta_{23}^{(1)} + \theta_{23}^{(2)} \approx 90^\circ \). However when \( \Im \tau \) is not large enough, we should take the modification from higher order terms of \( t \) into consideration, especially the term of \( \mathcal{O}(t) \). The numerical calculation shows that \( \Re \left[ t^* U_{\mu 3}^{(1)} (U_{\tau 3}^{(1)})^* \right] < 0 \), indicating \( \sin^2 \theta_{23}^{(2)} \approx \cos^2 \theta_{23}^{(1)} \), as can be seen from the left panel of Fig. 4. Let us consider a critical case where \( \Re \tau = 0.0398 \), \( \Im \tau = 2.07 \), \( g = 0.825 \) and \( \phi_g = 17.5^\circ \). In the hierarchy \( \alpha_1 \ll \alpha_3 \ll \alpha_2 \) we have

\[
\theta_{23}^{(1)} \simeq 43.3^\circ, \quad \frac{2 \Re \left[ t^* U_{\mu 3}^{(1)} (U_{\tau 3}^{(1)})^* \right]}{1 - |U_{e 3}^{(1)}|^2} \simeq -0.0775.
\]  \( (4.16) \)

Then from Eq. \( (4.15) \) we can obtain the value of \( \theta_{23}^{(2)} \approx 51.3^\circ \), which is exactly the upper bound of the 3\( \sigma \) range from the global-fit results of \( \theta_{23} \). Therefore, if \( \Im \tau < 2.07 \), we could not find the proper value of \( \theta_{23}^{(2)} \) located in the 3\( \sigma \) range any more. That is why only the hierarchy \( \alpha_1 \ll \alpha_3 \ll \alpha_2 \) is allowed in Region A.

- The asymptotic behavior of \( \theta_{23} \) and \( \delta \) when \( \Im \tau \) is extremely large deserves some more discussion. As can be seen from Fig. 4, \( \theta_{23} \approx 45^\circ \) and \( \delta \approx 90^\circ \) or 270\(^\circ \) hold excellently when \( \Im \tau \sim 4 \). The distinctive values of \( \theta_{23} \) and \( \delta \) imply that the neutrino mass matrix \( M_\nu \) might possess an approximate \( \mu - \tau \) reflection symmetry \[62\], which means

\[
X_{\mu \tau}^T M_\nu X_{\mu \tau} = M_\nu^*,
\]  \( (4.17) \)

where

\[
X_{\mu \tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\]  \( (4.18) \)

or equivalently \( |U_{\mu i}| = |U_{\tau i}| \) (for \( i = 1, 2, 3 \)). Actually as we have mentioned before, except the \( (M_\nu)_{11}, (M_\nu)_{23} \) and \( (M_\nu)_{32} \), all the other elements in Eq. \( (4.9) \) are vanishing in the limit where \( \Im \tau \) tends to infinity. Under this limit it is easy to identify Eq. \( (4.17) \) is satisfied and \( M_\nu \) can be regarded as having a trivial \( \mu - \tau \) reflection symmetry. However, the non-zero value of \( t \) is required to slightly break down this symmetry and generate non-trivial values of the mass-squared differences as well as other mixing angles to fit the experiment data. In order to illustrate this approximate \( \mu - \tau \) reflection symmetry indeed exists, we give a specific example of the numerical result for \( |U_\nu| \) as

\[
|U_\nu| = \begin{pmatrix} 0.816 & 0.599 & 0.146 \\ 0.408 & 0.586 & 0.700 \\ 0.409 & 0.587 & 0.699 \end{pmatrix},
\]  \( (4.19) \)
Figure 5: In the left panel, the correlation between $m_\beta$ and $m_1$ is shown for the NO case where the $1\sigma$ (yellow dots) and $3\sigma$ (red dots) ranges of neutrino mixing parameters and mass-squared differences from the global-fit analysis of neutrino oscillation data have been input \cite{65}. While the right panel is for the correlation between $m_{\beta\beta}$ and $m_1$. The gray shaded regions represent the range $\sum m_\nu > 0.6$ eV which is excluded by the latest Planck observations \cite{66} while the light gray shaded region in the right panel denotes the upper bound on $m_{\beta\beta}$ from the KamLAND-Zen experiment \cite{71}. The pink (purple) boundary in the right panel is obtained by using the $1\sigma$ ($3\sigma$) ranges of $\{\theta_{12}, \theta_{13}\}$ and $\{\Delta m^2_{21}, \Delta m^2_{31}\}$ from the global-fit analysis.

where the free model parameters are set to be

$$\text{Re} \, \tau = 0.0275, \quad \text{Im} \, \tau = 3.95, \quad g = 0.856, \quad \phi_g = 20.8^\circ. \quad \text{(4.20)}$$

Hence we can find $|U_{\mu\mu}| \approx |U_{\tau\tau}|$, which also indicates that there is an approximate $\mu - \tau$ reflection symmetry in $M_L$.

- As can be seen in Eq. (4.14), the elements $U_{e2}$ and $U_{e3}$ in the lepton mixing matrix $U$ keep invariant under the conversion from one hierarchy to another. Therefore different from $\theta_{23}$, the values of $\theta_{12}$ and $\theta_{13}$ do not depend on the hierarchies of $\alpha_1, \alpha_2$ and $\alpha_3$.

On the other hand, once the neutrino mass spectrum and the mixing parameters are known, we can predict the effective neutrino mass for beta decays,

$$m_\beta \equiv \sqrt{m_1^2 |U_{e1}|^2 + m_2^2 |U_{e2}|^2 + m_3^2 |U_{e3}|^2} . \quad \text{(4.21)}$$

In the case where three neutrino masses are quasi-degenerate, $m_\beta$ is approximately proportional to $m_1$, as can be seen from the left panel of Fig. 5 The latest result from the KATRIN experiment, where the electron energy spectrum from tritium beta decays is precisely measured, indicates $m_\beta < 1.1$ eV at the 90\% confidence level \cite{69,70}. With more data accumulated in KATRIN, the upper bound will be improved to $m_\beta < 0.2$ eV. Then it will for sure provide us with some clues to test whether the corresponding parameter space in our model is still consistent with the experiment data or not.

Furthermore, three light neutrinos are Majorana particles in the seesaw model, indicating that the neutrinoless double-beta decays of some even-even heavy nuclei could take place. The effective
Table 3: The best-fit values with the minimum $\chi^2_{\text{min}} = 0.232$, together with the $1\sigma$ and $3\sigma$ ranges of all the free model parameters and observables for the NO case in our model.

| Free model parameters | Best fit | $1\sigma$ range | $3\sigma$ range |
|-----------------------|----------|-----------------|-----------------|
| $\text{Re } \tau$    | 0.0365   | 0.0303 – 0.0394 | 0.0260 – 0.5    |
| $\text{Im } \tau$    | 2.33     | 2.24 – 2.80     | 1.75 – 4        |
| $g$                   | 0.834    | 0.829 – 0.847   | 0.815 – 0.924   |
| $\phi_g \, [^\circ]$ | 19.2     | 18.6 – 20.7     | 1.92 – 21.8     |
| $v_d \alpha_3/\sqrt{2} \, [\text{GeV}]$ | 1.775 | 1.774 – 1.777 | 0.1056 – 0.1058 $\oplus$ 1.771 – 1.777 |
| $\alpha_1/\alpha_3$  | 0.00288  | 0.00288 – 0.00289 | 0.00483 – 0.00492 $\oplus$ 0.00288 – 0.00290 |
| $\alpha_2/\alpha_3$  | 0.0596   | 0.0595 – 0.0596 | 16.6 – 16.8 $\oplus$ 0.0595 – 0.0597 |
| $v_u^2 g_1^2/(2\Lambda) \, [\text{eV}]$ | 0.134 | 0.122 – 0.221 | 0.0687 – 0.778 |

| Observables | | | |
|-------------|-------------|-------------|-------------|
| $m_1 \, [\text{eV}]$ | 0.106 | 0.096 – 0.183 | 0.0478 – 0.659 |
| $m_2 \, [\text{eV}]$ | 0.107 | 0.096 – 0.183 | 0.0486 – 0.659 |
| $m_3 \, [\text{eV}]$ | 0.118 | 0.108 – 0.190 | 0.0687 – 0.661 |
| $\theta_{12} \, [^\circ]$ | 33.8 | 33.1 – 34.6 | 31.6 – 36.3 |
| $\theta_{13} \, [^\circ]$ | 8.60 | 8.48 – 8.74 | 8.22 – 8.99 |
| $\theta_{23} \, [^\circ]$ | 48.8 | 46.4 – 49.4 | 41.7 – 51.3 |
| $\delta \, [^\circ]$ | 255 | 251 – 260 | 90 – 270 |
| $\rho \, [^\circ]$ | 89.5 | 89.3 – 89.8 | 0 – 180 |
| $\sigma \, [^\circ]$ | 90.6 | 90.2 – 90.7 | 0 – 180 |
| $m_\beta \, [\text{eV}]$ | 0.107 | 0.0956 – 0.183 | 0.0486 – 0.659 |
| $m_{\beta\beta} \, [\text{eV}]$ | 0.107 | 0.0960 – 0.185 | 0.0453 – 0.659 |

The right panel of Fig. 5 shows that the predicted values of $m_{\beta\beta}$ in our model have already reached the upper bound from the KamLAND-Zen experiment $[71]$, $m^{\text{upper}}_{\beta\beta} = 0.061 – 0.165 \, \text{eV}$, which is currently the best experimental constraint on $m_{\beta\beta}$. Hence our model is quite testable and can be easily ruled out in the next-generation neutrinoless double-beta decay experiments $[72]$.

As a summary of this section, we list the best-fit values, together with the $1\sigma$ and $3\sigma$ ranges of all the free model parameters and observables in our model in Table 3.

5 Summary

The modular symmetry is a very attractive and interesting way to understand lepton flavor mixing. In this paper, we consider the application of the modular $A_4$ symmetry to the supersymmetric minimal type-(I+II) seesaw model, where only one right-handed neutrino and two Higgs triplets are introduced beyond the particle content of the SM. We successfully construct a model to account
for lepton mass spectra and the flavor mixing, which is consistent with current neutrino oscillation data in the NO case.

In order to keep our model simple and economical enough, we assign the right-handed neutrino, two Higgs doublets and two Higgs triplets to be the trivial $A_4$ singlet $1$, and implement the minimal set for the weights of modular forms $(k_e, k_\mu, k_\tau, k_\Delta, k_D, k_R) = (2, 2, 2, 2, 4, 6)$ under the premise that the sum of weights in each superpotential should be vanishing. We construct the mass matrices in both the charged-lepton and neutrino sectors under such a setup of weights. After performing the numerical analysis, we find out that our model is consistent with the latest global-fit results of neutrino oscillation data at the 1$\sigma$ level only in the NO case and the individual contributions to the neutrino masses from the right-handed neutrino and the Higgs triplet are comparable. The allowed parameter space of the model parameters, namely, the modulus parameter $\tau = \text{Re}\tau+i\text{Im}\tau$, three real parameters $v_d\alpha_3/\sqrt{2}$, $\alpha_1/\alpha_3$ and $\alpha_2/\alpha_3$ in the charged-lepton sector, together with the coupling coefficient $\tilde{g} = ge^{i\phi}$ in the neutrino sector has been obtained. Moreover, we also give the constrained regions of three light neutrino masses $\{m_1, m_2, m_3\}$, three neutrino mixing angles $\{\theta_{12}, \theta_{13}, \theta_{23}\}$ and three CP-violating phases $\{\delta, \rho, \sigma\}$, as well as the predictions for the effective neutrino masses $m_\beta$ in beta decays and $m_{\beta\beta}$ in neutrinoless double-beta decays.

An interesting feature of our model is that the octant of $\theta_{23}$ strongly depends on which hierarchy of $\alpha_1$, $\alpha_2$ and $\alpha_3$ is taken into consideration. To be specific, the hierarchy $\alpha_1 \ll \alpha_3 \ll \alpha_2$ corresponds to the first octant of $\theta_{23}$ while the hierarchy $\alpha_1 \ll \alpha_2 \ll \alpha_3$ corresponds to the second octant of $\theta_{23}$. Furthermore, when the value of $\text{Im}\tau$ is sufficiently large, the neutrino mass matrix $M_\nu$ will possess an approximate $\mu - \tau$ reflection symmetry, indicating $\theta_{23} \approx 45^0$ and $\delta \approx 90^0$ or $270^0$. While the small parameter $t \equiv -6q^{1/3} = -6e^{2\pi i r/3}$ slightly breaks down this symmetry and generate realistic values for the mass-squared differences and other mixing angles. Besides, since our model predicts relatively large values of $\sum m_\nu = m_1 + m_2 + m_3$, $m_\beta$ and $m_{\beta\beta}$, it is very likely to be tested in the further neutrino experiments.

We stress that the hybrid seesaw models where not only one kind of seesaw mechanism is involved may lead to some new scenarios about flavor mixing and CP violation, and it deserves more attention to discuss the applications of the modular symmetry in such kinds of models. It is also interesting to study the type-(I+II) seesaw model with other kinds of finite modular symmetries. We hope to come back to these issues in the future works.

**Acknowledgements**

I am greatly indebted to Prof. Shun Zhou for suggesting this work and carefully reading this manuscript. I would also like to thank Guo-yuan Huang, Dr. Biswajit Karmakar and Di Zhang for helpful discussions. This work was supported in part by the National Natural Science Foundation of China under grant No. 11775232 and No. 11835013.
A The $\Gamma_3 \simeq A_4$ Symmetry Group

The permutation symmetry group $A_4$ has twelve elements and four irreducible representations, which are denoted as $1$, $1'$, $1''$ and $3$. In the present work, we choose the complex basis which is used in Ref. [26] for the representation matrices of two generators $S$ and $T$, namely,

\[
\begin{align*}
1 & : \quad \rho(S) = 1, \quad \rho(T) = 1, \\
1' & : \quad \rho(S) = 1, \quad \rho(T) = \omega, \\
1'' & : \quad \rho(S) = 1, \quad \rho(T) = \omega^2, \\
3 & : \quad \rho(S) = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}.
\end{align*}
\]

(A.1)

In this basis, we can explicitly write down the decomposition rules of the Kronecker products of any two $A_4$ multiplets.

- For the Kronecker products of two $A_4$ singlets:

\[
1 \otimes 1 = 1, \quad 1' \otimes 1' = 1'', \quad 1'' \otimes 1'' = 1', \quad 1' \otimes 1'' = 1;
\]

(A.2)

- For the Kronecker products of two $A_4$ triplets:

\[
\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix}_3 \otimes \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}_3 = (\zeta_1 \xi_1 + \zeta_2 \xi_3 + \zeta_3 \xi_2)_1 \oplus (\zeta_3 \xi_3 + \zeta_1 \xi_2 + \zeta_2 \xi_1)_1' \oplus (\zeta_2 \xi_2 + \zeta_1 \xi_3 + \zeta_3 \xi_1)_1'' \\
\oplus \frac{1}{3} \begin{pmatrix} 2\zeta_1 \xi_1 - \zeta_2 \xi_3 - \zeta_3 \xi_2 \\ 2\zeta_3 \xi_3 - \zeta_1 \xi_2 - \zeta_2 \xi_1 \\ 2\zeta_2 \xi_2 - \zeta_1 \xi_3 - \zeta_3 \xi_1 \end{pmatrix}_3 \oplus \frac{1}{2} \begin{pmatrix} \zeta_2 \xi_3 - \zeta_3 \xi_2 \\ \zeta_3 \xi_1 - \zeta_1 \xi_3 \\ \zeta_1 \xi_2 - \zeta_2 \xi_1 \end{pmatrix}_3.
\]

(A.3)

With the above decomposition rules and the assignments of relevant fields and modular forms, one can immediately find out the Lagrangian invariant under the modular $A_4$ symmetry group.

As has been mentioned in Sec. 2, there exist three linearly independent modular forms of the lowest non-trivial weight $k_Y = 2$, denoted as $Y_i(\tau)$ for $i = 1, 2, 3$. They transform as a triplet $3$ under the $A_4$ symmetry, namely,

\[
Y_3(\tau) \equiv \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}.
\]

(A.4)

In fact, the exact expressions of the modular forms can be derived with the help of the Dedekind $\eta$ function

\[
\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),
\]

(A.5)
with \( q \equiv e^{2\pi i \tau} \), and its derivative

\[
Y(a_1, \ldots, a_4|\tau) \equiv \frac{d}{d\tau} \left[ a_1 \log \eta \left( \frac{\tau}{3} \right) + a_2 \log \left( \frac{\tau + 1}{3} \right) + a_3 \log \left( \frac{\tau + 2}{3} \right) + a_4 \log \eta(3\tau) \right],
\]

(A.6)

with the coefficients \( a_i \) (for \( i = 1, 2, \ldots, 4 \)) fulfilling \( a_1 + \cdots + a_4 = 0 \). More explicitly, we have

\[
Y_1(\tau) \equiv \frac{i}{2\pi} \left[ \frac{\eta' \left( \frac{\tau}{3} \right)}{\eta \left( \frac{\tau}{3} \right)} + \frac{\eta' \left( \frac{\tau+1}{3} \right)}{\eta \left( \frac{\tau+1}{3} \right)} + \frac{\eta' \left( \frac{\tau+2}{3} \right)}{\eta \left( \frac{\tau+2}{3} \right)} - \frac{27\eta' \left( 3\tau \right)}{\eta(3\tau)} \right],
\]

\[
Y_2(\tau) \equiv \frac{-i}{\pi} \left[ \frac{\eta' \left( \frac{\tau}{3} \right)}{\eta \left( \frac{\tau}{3} \right)} + \omega \frac{\eta' \left( \frac{\tau+1}{3} \right)}{\eta \left( \frac{\tau+1}{3} \right)} + \omega^2 \frac{\eta' \left( \frac{\tau+2}{3} \right)}{\eta \left( \frac{\tau+2}{3} \right)} \right],
\]

\[
Y_3(\tau) \equiv \frac{-i}{\pi} \left[ \frac{\eta' \left( \frac{\tau}{3} \right)}{\eta \left( \frac{\tau}{3} \right)} + \omega \frac{\eta' \left( \frac{\tau+1}{3} \right)}{\eta \left( \frac{\tau+1}{3} \right)} + \omega^2 \frac{\eta' \left( \frac{\tau+2}{3} \right)}{\eta \left( \frac{\tau+2}{3} \right)} \right],
\]

(A.7)

which can be expanded as the Fourier series, i.e.,

\[
Y_1(\tau) = 1 + 12q + 36q^2 + 12q^3 + \cdots,
\]

\[
Y_2(\tau) = -6q^{1/3}(1 + 7q + 8q^2 + \cdots),
\]

\[
Y_3(\tau) = -18q^{2/3}(1 + 2q + 5q^2 + \cdots).
\]

(A.8)
References

[1] Z. z. Xing and S. Zhou, “Neutrinos in particle physics, astronomy and cosmology,” Springer-Verlag, Berlin Heidelberg (2011).

[2] M. Tanabashi et al. [Particle Data Group], “Review of Particle Physics,” Phys. Rev. D 98 (2018) no.3, 030001.

[3] P. Minkowski, “μ → eγ at a Rate of One Out of 10^9 Muon Decays?,” Phys. Lett. 67B (1977) 421.

[4] T. Yanagida, in Proc. Workshop on the Baryon Number of the Universe and Unified Theories, edited by O. Sawada and A. Sugamoto (1979), p. 95.

[5] M. Gell-Mann, P. Ramond, and R. Slansky, in Supergravity , edited by P. van Nieuwenhuizen and D. Freedman (1979), p. 315.

[6] R. N. Mohapatra and G. Senjanovic, “Neutrino Mass and Spontaneous Parity Nonconservation,” Phys. Rev. Lett. 44 (1980) 912.

[7] W. Konetschny and W. Kummer, “Nonconservation of Total Lepton Number with Scalar Bosons,” Phys. Lett. 70B (1977) 433.

[8] M. Magg and C. Wetterich, “Neutrino Mass Problem and Gauge Hierarchy,” Phys. Lett. 94B (1980) 61.

[9] J. Schechter and J. W. F. Valle, “Neutrino Masses in SU(2) x U(1) Theories,” Phys. Rev. D 22 (1980) 2227.

[10] T. P. Cheng and L. F. Li, “Neutrino Masses, Mixings and Oscillations in SU(2) x U(1) Models of Electroweak Interactions,” Phys. Rev. D 22 (1980) 2860.

[11] R. N. Mohapatra and G. Senjanovic, “Neutrino Masses and Mixings in Gauge Models with Spontaneous Parity Violation,” Phys. Rev. D 23 (1981) 165.

[12] G. Lazarides, Q. Shafi and C. Wetterich, “Proton Lifetime and Fermion Masses in an SO(10) Model,” Nucl. Phys. B 181 (1981) 287.

[13] Z. z. Xing, “Flavor structures of charged fermions and massive neutrinos,” arXiv:1909.09610.

[14] G. Altarelli and F. Feruglio, “Discrete Flavor Symmetries and Models of Neutrino Mixing,” Rev. Mod. Phys. 82 (2010) 2701 arXiv:1002.0211.

[15] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada and M. Tanimoto, “Non-Abelian Discrete Symmetries in Particle Physics,” Prog. Theor. Phys. Suppl. 183 (2010) 1 arXiv:1003.3552.

[16] S. F. King and C. Luhn, “Neutrino Mass and Mixing with Discrete Symmetry,” Rept. Prog. Phys. 76 (2013) 056201 arXiv:1301.1340.
[17] S. F. King, A. Merle, S. Morisi, Y. Shimizu and M. Tanimoto, “Neutrino Mass and Mixing: from Theory to Experiment,” New J. Phys. 16 (2014) 045018 [arXiv:1402.4271].

[18] S. F. King, “Unified Models of Neutrinos, Flavour and CP Violation,” Prog. Part. Nucl. Phys. 94 (2017) 217 [arXiv:1701.04413].

[19] F. Feruglio and A. Romanino, “Neutrino Flavour Symmetries,” [arXiv:1912.06028] [hep-ph].

[20] C. S. Lam, “Determining Horizontal Symmetry from Neutrino Mixing,” Phys. Rev. Lett. 101 (2008) 121602 [arXiv:0804.2622].

[21] C. S. Lam, “The Unique Horizontal Symmetry of Leptons,” Phys. Rev. D 78 (2008) 073015 [arXiv:0809.1185].

[22] S. F. Ge, D. A. Dicus and W. W. Repko, “$Z_2$ Symmetry Prediction for the Leptonic Dirac CP Phase,” Phys. Lett. B 702 (2011) 220 [arXiv:1104.0602].

[23] S. F. Ge, D. A. Dicus and W. W. Repko, “Residual Symmetries for Neutrino Mixing with a Large $\theta_{13}$ and Nearly Maximal $\delta_D$,” Phys. Rev. Lett. 108 (2012) 041801 [arXiv:1108.0964].

[24] D. Hernandez and A. Y. Smirnov, “Lepton mixing and discrete symmetries,” Phys. Rev. D 86 (2012) 053014 [arXiv:1204.0445].

[25] F. Feruglio, C. Hagedorn and R. Ziegler, “Lepton Mixing Parameters from Discrete and CP Symmetries,” JHEP 1307 (2013) 027 [arXiv:1211.5560].

[26] F. Feruglio, “Are neutrino masses modular forms?,” [arXiv:1706.08749]

[27] T. Kobayashi, K. Tanaka and T. H. Tatsuishi, “Neutrino mixing from finite modular groups,” Phys. Rev. D 98 (2018) no.1, 016004 [arXiv:1803.10391].

[28] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, T. H. Tatsuishi and H. Uchida, “Finite modular subgroups for fermion mass matrices and baryon/lepton number violation,” Phys. Lett. B 794 (2019) 114 [arXiv:1812.11072].

[29] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto and T. H. Tatsuishi, “Modular $S_3$ invariant flavor model in SU(5) GUT,” [arXiv:1906.10341]

[30] H. Okada and Y. Orikasa, “Modular $S_3$ symmetric radiative seesaw model,” Phys. Rev. D 100 (2019) no.11, 115037 [arXiv:1907.04716].

[31] T. Kobayashi, N. Omoto, Y. Shimizu, K. Takagi, M. Tanimoto and T. H. Tatsuishi, JHEP 1811 (2018) 196 [arXiv:1808.03012].

[32] J. C. Criado and F. Feruglio, “Modular Invariance Faces Precision Neutrino Data,” SciPost Phys. 5 (2018) no.5, 042 [arXiv:1807.01125].

[33] F. J. de Anda, S. F. King and E. Perdomo, “$SU(5)$ Grand Unified Theory with $A_4$ Modular Symmetry,” [arXiv:1812.05620]
[34] H. Okada and M. Tanimoto, “CP violation of quarks in $A_4$ modular invariance,” Phys. Lett. B 791 (2019) 54 [arXiv:1812.09677].

[35] T. Nomura and H. Okada, “A modular $A_4$ symmetric model of dark matter and neutrino,” Phys. Lett. B 797 (2019) 134799 [arXiv:1904.03937].

[36] T. Nomura and H. Okada, “A two loop induced neutrino mass model with modular $A_4$ symmetry,” arXiv:1906.03927.

[37] G. J. Ding, S. F. King and X. G. Liu, “Modular $A_4$ symmetry models of neutrinos and charged leptons,” JHEP 1909 (2019) 074 [arXiv:1907.11714].

[38] T. Nomura, H. Okada and O. Popov, “A modular $A_4$ symmetric scotogenic model,” arXiv:1908.07457.

[39] H. Okada and Y. Orikasa, “A radiative seesaw model in modular $A_4$ symmetry,” arXiv:1907.13520.

[40] T. Asaka, Y. Heo, T. H. Tatsuishi and T. Yoshida, “Modular $A_4$ invariance and leptogenesis,” arXiv:1909.06520.

[41] J. T. Penedo and S. T. Petcov, “Lepton Masses and Mixing from Modular $S_4$ Symmetry,” Nucl. Phys. B 939 (2019) 292 [arXiv:1806.11040].

[42] P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov, “Modular $S_4$ models of lepton masses and mixing,” JHEP 1904 (2019) 005 [arXiv:1811.04933].

[43] H. Okada and Y. Orikasa, “Neutrino mass model with a modular $S_4$ symmetry,” arXiv:1908.08409.

[44] X. Wang and S. Zhou, arXiv:1910.09473

[45] P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov, “Modular $A_5$ symmetry for flavour model building,” JHEP 1904 (2019) 174 [arXiv:1812.02158].

[46] G. J. Ding, S. F. King and X. G. Liu, “Neutrino mass and mixing with $A_5$ modular symmetry,” Phys. Rev. D 100 (2019) no.11, 115005 [arXiv:1903.12588].

[47] J. C. Criado, F. Feruglio, F. Feruglio and S. J. D. King, “Modular Invariant Models of Lepton Masses at Levels 4 and 5,” arXiv:1908.11867.

[48] P. P. Novichkov, J. T. Penedo, S. T. Petcov and A. V. Titov, “Generalised CP Symmetry in Modular-Invariant Models of Flavour,” JHEP 1907 (2019) 165 [arXiv:1905.11970].

[49] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, T. H. Tatsuishi and H. Uchida, arXiv:1910.11553.

[50] I. De Medeiros Varzielas, S. F. King and Y. L. Zhou, “Multiple modular symmetries as the origin of flavour,” arXiv:1906.02208.
[51] S. F. King and Y. L. Zhou, “Trimaximal TM1 mixing with two modular S4 groups,” arXiv:1908.02770.

[52] X. G. Liu and G. J. Ding, “Neutrino Masses and Mixing from Double Covering of Finite Modular Groups,” JHEP 1908 (2019) 134 [arXiv:1907.01488].

[53] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto and T. H. Tatsuishi, “New A4 lepton flavor model from S4 modular symmetry,” arXiv:1907.09141.

[54] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto and T. H. Tatsuishi, “A4 lepton flavor model and modulus stabilization from S4 modular symmetry,” Phys. Rev. D 100 (2019) 115045 [arXiv:1909.05139].

[55] P. P. Novichkov, S. T. Petcov and M. Tanimoto, “Trimaximal Neutrino Mixing from Modular A4 Invariance with Residual Symmetries,” Phys. Lett. B 793 (2019) 247 [arXiv:1812.11289].

[56] G. J. Ding, S. F. King, X. G. Liu and J. N. Lu, “Modular S4 and A4 symmetries and their fixed points: new predictive examples of lepton mixing,” JHEP 1912 (2019) 030 [arXiv:1910.03460].

[57] H. Okada and M. Tanimoto, “Towards unification of quark and lepton flavors in A4 modular invariance,” arXiv:1905.13421.

[58] D. Zhang, “A modular A4 symmetry realization of two-zero textures of the Majorana neutrino mass matrix,” arXiv:1910.07869.

[59] J. N. Lu, X. G. Liu and G. J. Ding, “Modular symmetry origin of texture zeros and quark lepton unification,” arXiv:1912.07573.

[60] T. Kobayashi, T. Nomura and T. Shimomura, arXiv:1912.00637

[61] T. Nomura, H. Okada and S. Patra, arXiv:1912.00379

[62] P. F. Harrison and W. G. Scott, “mu - tau reflection symmetry in lepton mixing and neutrino oscillations,” Phys. Lett. B 547 (2002) 219 [hep-ph/0210197].

[63] M. C. Chen, S. Ramos-Sanchez and M. Ratz, “A note on the predictions of models with modular flavor symmetries,” arXiv:1909.06910.

[64] A. Rossi, “Supersymmetric seesaw without singlet neutrinos: Neutrino masses and lepton flavor violation,” Phys. Rev. D 66 (2002) 075003 [hep-ph/0207006].

[65] I. Esteban, M. C. Gonzalez-Garcia, A. Hernandez-Cabezudo, M. Maltoni and T. Schwetz, “Global analysis of three-flavour neutrino oscillations: synergies and tensions in the determination of $\theta_{23}, \delta_{CP}$, and the mass ordering,” JHEP 1901 (2019) 106 [arXiv:1811.05487].

[66] N. Aghanim et al. [Planck Collaboration], “Planck 2018 results. VI. Cosmological parameters,” arXiv:1807.06209
[67] R. Acciarri et al. [DUNE Collaboration], “Long-Baseline Neutrino Facility (LBNF) and Deep Underground Neutrino Experiment (DUNE) : Conceptual Design Report, Volume 2: The Physics Program for DUNE at LBNF,” arXiv:1512.06148.

[68] K. Abe et al. [Hyper-Kamiokande Collaboration], “Hyper-Kamiokande Design Report,” arXiv:1805.04163.

[69] M. Aker et al. [KATRIN Collaboration], “An improved upper limit on the neutrino mass from a direct kinematic method by KATRIN,” arXiv:1909.06048.

[70] M. Aker et al. [KATRIN Collaboration], “First operation of the KATRIN experiment with tritium,” arXiv:1909.06069.

[71] A. Gando et al. [KamLAND-Zen Collaboration], “Limit on Neutrinoless $\beta\beta$ Decay of $^{136}$Xe from the First Phase of KamLAND-Zen and Comparison with the Positive Claim in $^{76}$Ge,” Phys. Rev. Lett. 110 (2013) no.6, 062502 [arXiv:1211.3863].

[72] M. J. Dolinski, A. W. P. Poon and W. Rodejohann, “Neutrinoless Double-Beta Decay: Status and Prospects,” arXiv:1902.04097.