A convection-diffusion problem with a small variable diffusion coefficient

Hans-G. Roos and Martin Schopf

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Abstract

Consider a singularly perturbed convection-diffusion problem with a small, variable diffusion. Based on certain a priori estimates for the solution we prove robustness of a finite element method on a Duran-Shishkin mesh.

Key words: singular perturbation, finite element method, layer-adapted mesh

MSC (2000) 65N30

1 Introduction

Consider the one dimensional boundary value problem

$$L_\varepsilon u := -(\varepsilon u')' - bu' + cu = f \quad \text{in } (0,1),$$
$$u(0) = 0,$$
$$u(1) = 0,$$

(1.1)

with smooth functions $\varepsilon, b, c, f : [0,1] \to \mathbb{R}$, satisfying

$$0 < \beta < b(x)$$
$$0 < \underline{\varepsilon} \leq \varepsilon(x) \leq \bar{\varepsilon} \ll 1 \quad \text{for } x \in [0,1].$$

(1.2)

Moreover we assume

$$c \geq 0, \quad c + b'/2 \geq \gamma > 0,$$

(1.3)

which can be ensured using the assumptions (1.2) and the transformation $u = \hat{u} e^{\delta x}$ with suitably chosen constant $\delta$, see, for instance, [6].

We do not know any results concerning robust numerical methods for such problems, the only exceptions are [2, 3], where $\varepsilon(x)$ has piecewise the special form $\varepsilon_i p_i(x)$ in $\Omega_i$ with different parameters $\varepsilon_i$.

Assuming additionally $\varepsilon' > -\beta$, we have an outflow boundary layer at $x = 0$. It is relatively technical to prove a priori estimates for derivatives of $u$ and to prove the existence of a solution decomposition into a smooth part and a layer part. But this can be done with well known techniques (Kellogg/Tsan; use of extended domains), see the Appendix.

Under additional assumptions ($\varepsilon'$ is nonnegative and bounded; moreover conditions on $\varepsilon''$, see Theorem 10 and Remark 4) we have: There exists a solution decomposition

$$u = S + E$$

with

$$|S^{(k)}(x)| \leq C \quad \text{for } k = 0, 1, 2,$$

(1.4a)
and
\[ |E^{(k)}(x)| \leq C \frac{1}{\epsilon(x)^k} e^{-\beta \epsilon(x)} \quad \text{for } k = 0, 1, 2, \]  
(1.4b)

here
\[ e(x) = \int_0^x \frac{1}{\epsilon(t)} dt. \]

Based on the solution decomposition we are going to analyze the finite element method on a special mesh. The weak formulation of the problem uses the bilinear form
\[ a(v, w) := (\epsilon v', w') - (bv', w) + (cv, w). \]  
(1.5)

Let \( V_h \in H^1_0(0, 1) \) be the space of linear finite elements. We look for \( u_h \in V_h \) such that
\[ a(u_h, v_h) = (f, v_h) \quad \text{for all } v_h \in V_h. \]  
(1.6)

Define an energy norm by
\[ \| v \|_2^2 := \| \epsilon^{1/2} v' \|_0^2 + \| v \|_0^2. \]

Then we ask: on which layer adapted mesh can we prove an (almost) robust error estimate for our finite element method in that energy norm?

## 2 The mesh and the interpolation error

Near the layer we use a fine graded mesh, otherwise an equidistant mesh with the step size \( h \).

First we introduce a point \( \tau^* \) satisfying
\[ e(\tau^*) = -\frac{2}{\beta} \ln h. \]  
(2.1)

Observe that as \( e(0) = 0 \) and \( e \) is strictly increasing (2.1) has a unique solution.

Since \( e \) is strictly increasing the choice (2.1) also implies
\[ e^{-\beta \epsilon(x)} \leq e^{-\beta \epsilon(\tau^*)} \leq h^2 \quad \text{for } x \geq \tau^*. \]  
(2.2)

Moreover, \( \tau^* \) satisfies
\[ -\frac{2}{\beta} \epsilon \ln h \leq \tau^* \leq -\frac{2}{\beta} \epsilon \ln h. \]  
(2.3)

Following [1], we introduce near \( x = 0 \) the graded mesh (D-L mesh)
\[
\begin{aligned}
& x_0 = 0, \\
& x_1 = h \delta, \\
& x_{i+1} = x_i + h x_i, \quad \text{for } 1 \leq i \leq N^*. 
\end{aligned}
\]  
(2.4)

We choose \( N^* \) in such a way that \( \tau = x_{N^*+1} \) is the first point with \( \tau \geq \tau^* \). Then, \( \tau \) has similar properties as \( \tau^* \). In the subinterval \([\tau, 1]\) we use an equidistant mesh with a mesh size of order \( O(h) \).

To simplify the notation, we introduce the symbol \( \preceq \) and note \( A \preceq B \), if there exists a constant \( C \) independent of \( \epsilon \), such that \( A \leq CB \).

Because the smooth part \( S \) satisfies \( \| S'' \| \leq C \), we have for the interpolation error of the piecewise linear interpolant
\[ \| S - S^I \|_0 \preceq h^2, \quad \| S - S^I \|_1 \preceq h. \]

On \([\tau, 1]\) we obtain for the layer component
\[ \| E - E^I \|_{0,[\tau,1]} \preceq \| E \|_{\infty,[\tau,1]} \preceq h^2. \]
Moreover, by an inverse inequality
\[
\|\varepsilon^{1/2}(E - E^I)\|_{0,[\tau,1]}^2 \leq \int_\tau^1 \frac{1}{\varepsilon(x)} e^{-2\beta \varepsilon(x)} + \frac{1}{h^2} \|\varepsilon^{1/2} E^I\|_{0,[\tau,1]}^2 \leq h^2. \tag{2.5}
\]

Next we study the interpolation error on the fine subinterval \([0,\tau]\), using the definition of the mesh, the estimate of \(E^I\) and \(x \leq \varepsilon(x)\): \[
\|\varepsilon^{-1/2}(E - E^I)\|_{0,[0,\tau]}^2 = \int_0^{\tau} \varepsilon^{-1}(E - E^I)^2 + \frac{1}{h^4} \sum_{i=0}^{N^*} \int_{x_i}^{x_{i+1}} \varepsilon^{-1}(E - E^I)^2 \leq h^4 + \frac{1}{h^4} \int_0^\tau \varepsilon^{-5} x^4 e^{-2\beta \varepsilon(x)} \leq h^4 \left(1 + \int_0^\tau \varepsilon^{-1}(x)\right) \tag{2.6}
\]
\[
\leq h^4 \left(1 + \int_0^\infty s^4 e^{-2\beta s}\right) \leq h^4.
\]
Thus we obtain \[
\|\varepsilon^{-1/2}(E - E^I)\|_{0,[0,\tau]} \leq h^2 \quad \text{and} \quad \|E - E^I\|_{0,[0,\tau]} \leq h^2 \|\varepsilon^{-1/2} E^I\|_{0,[0,\tau]} \tag{2.7}
\]
Similarly we get
\[
\|\varepsilon^{1/2}(E - E^I)'\|_{0,[0,\tau]}^2 = \int_0^{\tau} \varepsilon((E - E^I)')^2 + \frac{1}{h^4} \sum_{i=0}^{N^*} \int_{x_i}^{x_{i+1}} \varepsilon((E - E^I)')^2 \leq h^4 \left(1 + \int_0^\infty s^2 e^{-2\beta s}\right), \tag{2.8}
\]
resulting in
\[
\|\varepsilon^{1/2}(E - E^I)'\|_{0,[0,\tau]} \leq h. \tag{2.9}
\]

### 3 The discretization error

So far we proved \(\|u - u^I\|_\varepsilon \leq h\) and start now to estimate \(\|u_h - u^I\|_\varepsilon\). As usual, we have based on the coercivity of our bilinear form in the given norm
\[
\|u^I - u_h\|_h^2 = a(u^I - u_h, u^I - u_h) = a(u^I - u, u^I - u_h) = (\varepsilon(u^I - u), v_h) - \frac{1}{h} \|E - E^I\|_0 \|v_h\|_\varepsilon,
\]
with \(v_h = u^I - u_h\). The first and the third term can be easily estimated, only the convection term needs some care. We use integration by parts and on the finite part of the mesh
\[
|(E - E^I, (v_h)')| \leq \|\varepsilon^{-1/2}(E - E^I)\|_0 \|v_h\|_\varepsilon,
\]
while on the coarse part an inverse inequality yields
\[
|(E - E^I, (v_h)')| \leq \frac{1}{h} \|E - E^I\|_0 \|v_h\|_0 \leq \frac{1}{h} \|E - E^I\|_0 \|v_h\|_\varepsilon.
\]
Using (2.7), we get finally

**Theorem 1.** If there exists a solution decomposition with the properties (1.4), then the finite element approximation with linear elements on our DL-Shishkin mesh satisfies
\[
\|u_h - u\|_\varepsilon \leq h. \tag{3.2}
\]

Remark that our result is not fully robust: the number of mesh points used is of order \(O(\psi(\varepsilon, h)^{1/4})\), where \(\psi(\varepsilon, h)\) can be estimate by \(\ln((\varepsilon)/|\varepsilon|) + \ln((-\ln h)/h)\).
4 Appendix

Consider the one dimensional boundary value problem

\[ \mathcal{L}_\varepsilon u := -(\varepsilon u')' - bu' + cu = f \quad \text{in} \ (0, 1), \]
\[ u(0) = 0, \]
\[ u(1) = 0. \quad (4.1) \]

Assume (1.2) and (1.3).

The differential equation in (4.1) can be rewritten in the equivalent form

\[ -\varepsilon u'' - (b + \varepsilon')u' + cu = f. \quad (4.2) \]

Thus the first derivative of \( \varepsilon \) has a crucial influence on the behavior of the exact solution: If for instance \( \varepsilon' < -b \) then the outflow boundary will shift to the point \( x = 1 \) leading to the formation of an exponential boundary layer at that point. We shall consider the case \( \varepsilon' > -\beta \geq -b \) leaving the outflow boundary point at the origin of the unit interval.

**Lemma 2.** Let \( u \) be the solution of (4.1) and \( T \) be the coordinate transformation

\[ \xi = T(x) = \int_0^x \sqrt{\frac{\varepsilon}{\varepsilon(t)}} \, dt, \quad (4.3) \]

mapping the domain \((0, 1)\) to \((0, T(1))\). Then in the transformed variable \( \xi \) it holds

\[ |\hat{u}^{(k)}(\xi)| \leq C \left( 1 + \varepsilon^{-k} e^{-\frac{2\beta}{\sqrt{\varepsilon}} \xi} \right) \quad (4.4) \]

with \( \hat{u} := u \circ T^{-1} \) and \( \sigma := \min_{x \in [0, 1]} \varepsilon'(z) > -\beta \).

**Proof.** Let \( T \) be the coordinate transformation defined by (4.3). As strict monotone mapping \( T \) is injective and therefore \( T^{-1} : [0, T(1)] \to [0, 1], \xi \mapsto x \) exists. The chain rule yields for \( \hat{u}(T(x)) = u(x) \) and \( x \in (0, 1) \):

\[ u'(x) = \frac{d}{dx} \hat{u}(T(x)) = \hat{u}'(T(x)) T'(x), \]
\[ u''(x) = \hat{u}''(T(x)) (T'(x))^2 + \hat{u}'(T(x)) T''(x). \]

Thus the differential equation (4.2) is transformed into

\[ \varepsilon \hat{u}''(T(x)) - \left( (b(x) + \varepsilon'(x)) \sqrt{\frac{\varepsilon}{\varepsilon(x)}} - \frac{\varepsilon(x) \varepsilon'(x)}{2 \varepsilon(x)} \right) \hat{u}'(T(x)) + c(x) \hat{u}(T(x)) = f(x). \]

Note that the coefficient of the highest derivative of \( \hat{u} \) is the constant \( \varepsilon \) and that the functions \( \hat{c} := c \circ T^{-1} \) and \( \hat{f} := f \circ T^{-1} \) remain bounded. Therefore rewriting (4.2) in the new variable \( \xi \) yields:

\[ \varepsilon \hat{u}''(\xi) - \hat{b}(\xi) \hat{u}'(\xi) + \hat{c}(\xi) \hat{u}(\xi) = \hat{f}(\xi), \quad \xi \in [0, T(1)] \]

\( \hat{b} = \frac{2b \circ T^{-1} + \varepsilon' \circ T^{-1}}{2 \sqrt{\varepsilon \circ T^{-1}}} \sqrt{\frac{\varepsilon}{\varepsilon(x)}}. \)

In order to obtain bounds on the derivatives of \( \hat{u} \) we need an estimate \( \beta \leq \hat{b}(\xi) \) for \( \xi \in [0, T(1)] \) with a constant \( \beta > 0 \). Equivalently, we provide an estimate \( \beta \leq \hat{b}(T(x)) \) for \( x \in [0, 1] \):

\[ \hat{b}(T(x)) = \frac{2b(x) + \varepsilon'(x)}{2} \sqrt{\frac{\varepsilon}{\varepsilon(x)}} \geq \sqrt{\frac{\varepsilon}{\varepsilon(x)}} \frac{\sigma + 2\beta}{2} \]
\[ \geq \sqrt{\frac{\varepsilon}{\varepsilon(x)} \frac{\sigma + 2\beta}{2}} =: \beta > 0. \]
Remark that \( \varepsilon / \varepsilon(t) \leq 1 \) implies \( T(x) \leq x \) and hence \( T(1) \leq 1 \):

\[
T(x) = \int_0^x \frac{1}{\sqrt{\varepsilon(t)}} \, dt \leq \int_0^x \, dt = x.
\]

Thus, we can apply well known a-priory estimates for the case when \( \varepsilon \) is a constant to obtain

\[
|\tilde{u}^{(k)}(\xi)| \leq C \left( 1 + \varepsilon^{-k} e^{-\frac{\beta}{2\sqrt{\varepsilon}}} \right) \leq C \left( 1 + \varepsilon^{-k} e^{-\frac{\varepsilon+2\varepsilon}{2\sqrt{\varepsilon}}} \right).
\]

\( \square \)

**Lemma 3.** Set

\[
\tilde{c}(x) := \int_0^x \frac{1}{\sqrt{\varepsilon(t)}} \, dt
\]

**The solution** \( u \) **of Problem (4.1)** satisfies

\[
|u(x)| \leq C \left( 1 + e^{-\frac{\varepsilon+2\varepsilon}{2\sqrt{\varepsilon}}} \tilde{c}(x) \right),
\]

\( 4.6a \)

\[
|u'(x)| \leq C \sqrt{\frac{\varepsilon}{\varepsilon(x)}} \left( 1 + \varepsilon^{-1} e^{-\frac{\varepsilon+2\varepsilon}{2\sqrt{\varepsilon}}} \tilde{c}(x) \right),
\]

\( 4.6b \)

\[
|u''(x)| \leq C \left( \frac{\varepsilon}{\varepsilon(x)} \left( 1 + \varepsilon^{-2} e^{-\frac{\varepsilon+2\varepsilon}{2\sqrt{\varepsilon}}} \tilde{c}(x) \right) + \frac{\varepsilon'(x)}{2(\varepsilon(x))^{2}} \left( 1 + \varepsilon^{-1} e^{-\frac{\varepsilon+2\varepsilon}{2\sqrt{\varepsilon}}} \tilde{c}(x) \right) \right).
\]

\( 4.6c \)

with \( \sigma := \min_{x \in [0,1]} \tilde{c}'(x) > -\beta \).

**Proof.** Lemma 2 yields

\[
\left| (u \circ T^{-1})^{(k)}(\xi) \right| = |\tilde{u}^{(k)}(\xi)| \leq C \left( 1 + \varepsilon^{-k} e^{-\frac{\varepsilon+2\varepsilon}{2\sqrt{\varepsilon}}} \tilde{c}(x) \right).
\]

The transformation \( \xi = T(x) \) gives

\[
\left| (u \circ T^{-1})^{(k)}(T(x)) \right| \leq C \left( 1 + \varepsilon^{-k} e^{-\frac{\varepsilon+2\varepsilon}{2\sqrt{\varepsilon}}} T(x) \right) = C \left( 1 + \varepsilon^{-k} e^{-\frac{\varepsilon+2\varepsilon}{2\sqrt{\varepsilon}}} \tilde{c}(x) \right).
\]

\( 4.7 \)

We use (4.7) to deduce our proposition. First (4.6a) is an immediate consequence of (4.7) for \( k = 0 \). Next we want to verify (4.6b). A simple calculation yields

\[
(u \circ T^{-1})'(\xi) = u'(T^{-1}(\xi)) \left( T^{-1} \right)'(\xi) = u'(T^{-1}(\xi)) \frac{1}{T'(T^{-1}(\xi))}.
\]

With \( \xi = T(x) \) we conclude

\[
(u \circ T^{-1})'(T(x)) = u'(x) \frac{1}{T'(x)}.
\]

\( 4.8 \)

Collecting (4.7) with \( k = 1 \) and (4.8) the estimate (4.6b) follows. Same techniques yield

\[
\left| (u \circ T^{-1})''(T(x)) \right| = \left| u''(x) \frac{1}{(T'(x))^2} - u'(x) \frac{T''(x)}{(T'(x))^3} \right|
\]

\[
\geq \left| u''(x) \frac{1}{(T'(x))^2} - u'(x) \right| \frac{|T''(x)|}{(T'(x))^2}
\]

\( 4.9 \)

Combining (4.7) with \( k = 2 \) and (4.9) we obtain

\[
|u''(x)| \leq C \left( T'(x) \right)^2 \left( 1 + \varepsilon^{-2} e^{-\frac{\varepsilon+2\varepsilon}{2\sqrt{\varepsilon}}} \tilde{c}(x) \right) + |u'(x)| |T''(x)|
\]

\[
\leq C \frac{\varepsilon}{\varepsilon(x)} \left( 1 + \varepsilon^{-2} e^{-\frac{\varepsilon+2\varepsilon}{2\sqrt{\varepsilon}}} \right) + |u'(x)| \frac{\varepsilon'(x)}{2 \sqrt{\varepsilon(x)} (\varepsilon(x))^{2}}.
\]

Using (4.7) with \( k = 1 \) for the second term the proof is complete. \( \square \)
Remark 1. In the classical constant setting \( \varepsilon \equiv \bar{\varepsilon} = \bar{\tau} \) the formulas (4.6) reduce to the well-known form

\[
|u^{(k)}(x)| \leq C \left( 1 + \varepsilon^{-k} e^{-\frac{\bar{\varepsilon}}{2} x} \right), \quad k = 0, 1, 2. \tag{4.10}
\]

Unfortunately, all summands of the right hand side of the bounds (4.6b) and (4.6c) have a large multiplier if \( \varepsilon \) changes on a huge scale. Moreover the exponential decay in the estimates (4.6) appears to be suboptimal. In order to provide better bounds we will use the following Lemmas.

Lemma 4. The differential operator \( \mathcal{L}_\varepsilon \) obeys the following maximum principle: For any function \( v \in C^2(a, b) \cap C[a, b] \)

\[
\begin{cases}
\mathcal{L}_\varepsilon v \leq 0 & \text{in } (a, b), \\
v(a) \leq 0, \\
v(b) \leq 0
\end{cases} \quad \Rightarrow \quad v \leq 0 \quad \text{on } [a, b].
\]

Proof. A proof can be found e.g. in [5]. \( \square \)

The maximum principle applied to \( v_1 - v_2 \) also yields a comparison principle.

Lemma 5. Let \( a < x, \ell \in \mathbb{N}_0 \), suppose \( \varepsilon' \geq \sigma_0 \geq 0 \) on \( [a, x] \) and set \( e_a(t) := \int_a^t 1/\varepsilon(z)dz \). Then

\[
\int_a^x \varepsilon(t)\varepsilon_{\gamma a}(t)dt \leq \frac{1}{\gamma + (\ell + 1)\sigma_0} (\varepsilon(x)\varepsilon^{\ell+1}_{\gamma a}(x) - \varepsilon(a)\varepsilon^{\ell+1}(a)). \tag{4.11}
\]

for \( \gamma > -(\ell + 1)\sigma_0 \).

Proof. Since \( \sigma_0 \leq \varepsilon'(t) \) for \( t \in [a, x] \) multiplication with \( \varepsilon(t)\varepsilon^{\ell+1}_a(t) > 0 \) and integration yields

\[
\sigma_0 \int_a^x \varepsilon(t)\varepsilon^{\ell+1}_a(t)dt \leq \int_a^x \varepsilon^{\ell+1}_a(t)\varepsilon(t)\varepsilon'(t)dt.
\]

Integration by parts gives

\[
\sigma_0 \int_a^x \varepsilon(t)\varepsilon^{\ell+1}_a(t)dt \leq \frac{1}{\ell + 1} \left( \varepsilon(x)\varepsilon^{\ell+1}_{\gamma a}(x) - \varepsilon(a)\varepsilon^{\ell+1}(a) - \int_a^x \varepsilon(t)\varepsilon^{\ell+1}_{\gamma a}(t)e_a'(t)dt \right). \tag{4.12}
\]

Inserting

\[
- \int_a^x \varepsilon(t)\varepsilon^{\ell+1}_{\gamma a}(t)e_a'(t)dt = -\gamma \int_a^x \varepsilon(t)\varepsilon^{\ell+1}_{\gamma a}(t)dt
\]

into (4.12) we obtain

\[
\left( \frac{\gamma}{\ell + 1} + \sigma_0 \right) \int_a^x \varepsilon(t)\varepsilon^{\ell+1}_a(t)dt \leq \frac{1}{\ell + 1} \left( \varepsilon(x)\varepsilon^{\ell+1}_{\gamma a}(x) - \varepsilon(a)\varepsilon^{\ell+1}(a) \right)
\]

and (4.11) follows. \( \square \)

Next, we want to proof some pointwise bounds for the solution of the following problem in a possibly extended domain \((a, 1)\) with \( a \leq 0 \):

\[
-(\varepsilon^* w')' - b^* w' + c^* w = f^* \quad \text{in } (a, 1), \quad w(a) = 0, \quad w(1) = u_1, \tag{4.13}
\]

with smooth functions \( \varepsilon^*, b^*, c^* \) and \( f^* \) defined on \((a, 1)\) and satisfying

\[
0 \leq \varepsilon^*(x) \leq \bar{\varepsilon}, \quad \beta \leq b^*(x) \quad \text{for } x \in [a, 1], \quad 0 \leq c^*(x) \tag{4.14}
\]

Next, we want to proof some pointwise bounds for the solution of the following problem in a possibly extended domain \((a, 1)\) with \( a \leq 0 \):

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\]

with smooth functions \( \varepsilon^*, b^*, c^* \) and \( f^* \) defined on \((a, 1)\) and satisfying

\[
0 \leq \varepsilon^*(x) \leq \bar{\varepsilon}, \quad \beta \leq b^*(x) \quad \text{for } x \in [a, 1], \quad 0 \leq c^*(x) \tag{4.14}
\]
Lemma 6. Suppose $0 \leq (\varepsilon^*)'$ on $[a, 1]$. Then the solution $w$ of problem (4.13) satisfies
\begin{equation}
|w(x)| \leq C, \quad x \in [a, 1].
\end{equation}

Proof. Using the comparison principle induced by Lemma 4 with the barrier functions $\psi^\pm$ defined by
\begin{equation*}
\psi^\pm(x) := \pm \frac{1}{\beta} \|f^*\|_\infty (1 - x) \pm |u_1|
\end{equation*}
one obtains the result, because
\begin{align*}
(L_\varepsilon \psi^+) (x) &= \frac{b^*(x) + (\varepsilon^*)'(x)}{\beta_\varepsilon} \|f^*\|_\infty + c^*(x) \left(\frac{1}{\beta}(1 - x) + |u_1|\right) \geq \|f^*\|_\infty \geq (L_\varepsilon w) (x) \quad \text{in} \ (a, 1), \\
\psi^+(a) &= \frac{1}{\beta} \|f^*\|_\infty (1 - a) + |u_1| \geq 0 = w(a), \\
\psi^+(1) &= |u_1| \geq u_1 = w(1).
\end{align*}
Hence $w \leq \psi^+ \leq C$ on $[a, 1]$. The other bound follows similarly with $\psi^-$. \hfill \Box

The following argument is an extension of [4].

Lemma 7. Suppose $0 \leq (\varepsilon^*)'$ on $[a, 1]$ and set $e_\varepsilon(t) := \int_a^t 1/\varepsilon^*(z)dz$. Then the solution $w$ of problem (4.13) satisfies
\begin{equation}
|w'(x)| \leq C \left(1 + \frac{1}{\varepsilon^*(x)} e^{-\beta e_\varepsilon(x)}\right), \quad x \in [a, 1].
\end{equation}

Proof. For the sake of readability, we drop the star from the notation of the functions $\varepsilon^*$, $b^*$, $c^*$ and $f^*$ within this proof. Set $h := f - cw$. The problem
\begin{equation*}
w''(x) - \frac{b(x) + \varepsilon'(x)}{\varepsilon(x)} w'(x) = \frac{h(x)}{\varepsilon(x)}, \quad w(a) = 0, \ w(1) = u_1
\end{equation*}
is equivalent to problem (4.13). It’s solution $w$ admits the representation
\begin{equation*}
w(x) = w_p(x) + K_1 + K_2 \int_a^x e^{-\left(\frac{B(t) - B(a)}{e_\varepsilon(t)}\right)} dt,
\end{equation*}
where
\begin{align*}
w_p(x) &:= - \int_a^x z(t) dt, \quad z(x) := \int_a^x \frac{h(t)}{\varepsilon(t)} e^{-\left(\frac{B(x) - B(t)}{e_\varepsilon(t)}\right)} dt, \\
B(x) &:= \int_a^x \frac{b(t) + \varepsilon'(t)}{\varepsilon(t)} dt = \int_a^x \frac{b(t)}{\varepsilon(t)} dt + \ln \left(\varepsilon(x)\right) - \ln \left(\varepsilon(a)\right),
\end{align*}
i.e. $B$ is an indefinite integral of $(b + \varepsilon')/\varepsilon$. The constants $K_1$ and $K_2$ may depend on $\varepsilon$. The boundary condition $w(a) = 0$ yields $K_1 = 0$ whereas the other boundary condition $w(1) = u_1$ gives
\begin{equation}
\begin{aligned}
u_1 - w_p(1) &= K_2 \int_a^1 e^{-\left(\frac{B(t) - B(a)}{e_\varepsilon(t)}\right)} dt = K_2 \int_a^1 e^{-\int_a^t \frac{B'(z)}{e_\varepsilon(z)} dz + \ln \left(\frac{e_\varepsilon(t)}{e_\varepsilon(a)}\right)} dt \\
&= K_2 e^{(a)} \int_a^1 \frac{1}{\varepsilon(t)} e^{-\int_a^t \frac{B'(z)}{e_\varepsilon(z)} dz + \ln e^{a_\varepsilon(a)}} dt.
\end{aligned}
\end{equation}
Because of Lemma 6 we know \( \|w\|_\infty \leq C \). Thus

\[
|z(x)| \leq C \int_a^x \frac{1}{\varepsilon(t)} e^{-(B(x)-B(t))} dt. \tag{4.18}
\]

For \( t \leq x \) a simple calculation yields

\[
\frac{1}{\varepsilon(t)} e^{-(B(x)-B(t))} = e^{-B(x)+B(t)-\ln \left(\varepsilon(t)\right)} = e^{-\int_a^x \frac{b(z)}{\varepsilon(z)} dz + \int_a^t \frac{\varepsilon(z)}{\varepsilon(t)} dz - \ln \left(\varepsilon(t)\right)}
\]

\[
= \frac{1}{\varepsilon(x)} e^{-\int_a^t \frac{\varepsilon(z)}{\varepsilon(t)} dz} \leq \frac{1}{\varepsilon(x)} e^{-\int_a^t \frac{\varepsilon(z)}{\varepsilon(t)} dz} = \frac{1}{\varepsilon(x)} e^{-\varepsilon \left( \varepsilon(x) - e_a(t) \right)}. \nonumber
\]

Inserting this estimate into (4.18) and applying Lemma 5 with \( \ell = 0 \) we obtain

\[
|z(x)| \leq \frac{C}{\varepsilon(x)} \int_a^x e^{-\varepsilon \left( \varepsilon(x) - e_a(t) \right)} dt \leq \frac{C}{\varepsilon(x)} e^{-\varepsilon e_a(x)} \int_a^x e^{\beta e_a(t)} dt \leq \frac{C}{\varepsilon(x)} e^{-\varepsilon e_a(x)} \frac{1}{\beta + \sigma_0} e^{\beta e_a(x)} \leq C.
\]

Moreover \( |z(x)| \leq C \) for all \( x \in [0,1] \) implies \( |w_p(1)| \leq C \). We still need to estimate

\[
\int_a^1 \frac{1}{\varepsilon(t)} e^{-\int_a^t \frac{\varepsilon(z)}{\varepsilon(t)} dz} dt \geq \int_a^1 \frac{1}{\varepsilon(t)} e^{-\|b\|_\infty e_a(t)} dt = \int_0^{e_a(1)} e^{-\|b\|_\infty s} ds
\]

\[
= \frac{1}{\|b\|_\infty} \left( 1 - e^{-\|b\|_\infty e_a(1)} \right) \geq C.
\]

Here we used the substitution \( s = e_a(t) \) with \( ds/dt = e_a'(t) = 1/\varepsilon(t) \). Thus, with (4.17) we get

\[
|K_2| \leq \frac{1}{\varepsilon(a)}. \nonumber
\]

Combining this with

\[
w'(x) = -z(x) + K_2 e^{-\left( B(x)-B(a) \right)}, \tag{4.19}
\]

we obtain

\[
|w'(x)| \leq |z(x)| + C \frac{1}{\varepsilon(x)} e^{-\varepsilon e_a(x)} \leq C \left( 1 + \frac{1}{\varepsilon(x)} e^{-\varepsilon e_a(x)} \right)
\]

and (4.16) is verified. \( \square \)

**Remark 2.** Again in the classical case where \( \varepsilon^* \) is constant the formula (4.16) reduces to (4.10) which is known to be optimal.

**Remark 3.** An inspection of the proof of (4.16) shows that the assumption \( \varepsilon \ll 1 \) can be dropped provided \( w \) remains uniformly bounded and \( e_a(1) \) is sufficiently large — Remark that \( e_a \) is a strictly increasing function.

With (4.16) we readily obtain a pointwise estimate for \( w'' \).

**Lemma 8.** Let \( 0 \leq (\varepsilon^*)' \) on \( [a, 1] \) and set \( e_a(t) := \int_a^t 1/\varepsilon^*(z) dz \). Then the solution \( w \) of problem (4.13) satisfies

\[
|w''(x)| \leq C \frac{1 + (\varepsilon^*)'(x)}{\varepsilon^*(x)} \left( 1 + \frac{1}{\varepsilon^*(x)} e^{-\varepsilon e_a(x)} \right), \quad x \in [a, 1]. \tag{4.20}
\]

**Proof.** The result is an immediate consequence of (4.13), \( \|w\|_\infty \leq C \) and (4.16). \( \square \)
Lemma 9. Suppose $0 \leq (\varepsilon^*)'$ on $[a, 1]$. Then for the solution $w$ of Problem (4.13)

$$|w''(x)| \leq C \left(1 + K_4(x) + \frac{1}{\varepsilon^*} (1 + (\varepsilon^*)'(a) + \| (\varepsilon^*)'' \|_{L^1(a,x)}) e^{-\beta_a(x)} \right), \quad x \in [a, 1] \quad (4.21)$$

holds with $e_a(t) := \int_a^t 1/\varepsilon^*(z)dz$ and $K_4(x) \leq C \min \left\{ \| (\varepsilon^*)'' \|_{L^1(a,x)}, \frac{1}{\sqrt{\varepsilon^*(x)}}, \| (\varepsilon^*)'' \|_{L^1(a,x)} \right\}$.

Proof. In order to simplify the illustration we again drop the star from the notation of the functions $\varepsilon^*, b^*, c^*$ and $f^*$ within this proof. A differentiation of (4.13) yields

$$-w^{(3)} - \frac{b + 2\varepsilon'}{\varepsilon} w'' = \frac{f' + (b' + \varepsilon'' - c)w' - c'w}{\varepsilon} =: g.$$}

Thus, we obtain a differential equations for $\omega := w''$, indeed $-\omega' - (b + 2\varepsilon')/\varepsilon = g$. Setting

$$\hat{B}(x) := \int_a^x \frac{b(t) + 2\varepsilon'(t)}{\varepsilon(t)} dt = \int_a^x \frac{b(t)}{\varepsilon(t)} dt + 2 \ln \left(\varepsilon(x)\right) - 2 \ln \left(\varepsilon(a)\right)$$

(i.e. $\hat{B}$ is an indefinite integral of $(b + 2\varepsilon')/\varepsilon$) the function $\omega$ can be represented as

$$\omega(x) = K_3 e^{-\hat{B}(x) - \hat{B}(a)} - \int_a^x g(t)e^{-\hat{B}(x) + \hat{B}(t)} dt. \quad (4.22)$$

Here the constant $K_3$ may depend on $\varepsilon$. Because of the identity

$$e^{-\hat{B}(x) - \hat{B}(t)} = \left(\frac{\varepsilon(t)}{\varepsilon(x)}\right)^2 e^{-f' \frac{\varepsilon(t)}{\varepsilon(x)} dt}$$

and $K_3 = \omega(a) = w''(a)$ the representation (4.22) implies

$$|\omega(x)| \leq |w''(a)| \left(\frac{\varepsilon(a)}{\varepsilon(x)}\right)^2 e^{-\beta_a(x)} + \int_a^x |g(t)| \left(\frac{\varepsilon(t)}{\varepsilon(x)}\right)^2 e^{-\beta(\varepsilon_a(x) - \varepsilon_a(t))} dt. \quad (4.23)$$

Because $|g(t)| \leq |\frac{b'(t) + \varepsilon'(t) - c(t)}{\varepsilon(t)}| w'(t)| + |\frac{c'(t)}{\varepsilon(t)}| |w'(t)| + \frac{|f'(t)|}{\varepsilon(t)}$ the integral in (4.23) is dominated by the sum of the two integrals $I_0(x)$ and $I_1(x)$ with

$$I_0(x) := \int_a^x \left(\frac{f'(t)}{\varepsilon(t)} + \frac{\varepsilon'(t)}{\varepsilon(t)} |w(t)|\right) \left(\frac{\varepsilon(t)}{\varepsilon(x)}\right)^2 e^{-\beta(\varepsilon_a(x) - \varepsilon_a(t))} dt,$$

$$I_1(x) := \int_a^x \left|\frac{b'(t) + \varepsilon'(t) - c(t)}{\varepsilon(t)}\right| |w'(t)| \left(\frac{\varepsilon(t)}{\varepsilon(x)}\right)^2 e^{-\beta(\varepsilon_a(x) - \varepsilon_a(t))} dt.$$}

Using $\|w\|_{\infty} \leq C$ and applying Lemma 5 with $\ell = 1$ we see that

$$I_0(x) \leq C \frac{1}{\varepsilon(x)^2} e^{-\beta_a(x)} \int_a^x \varepsilon(t) e^{\beta_a(t)} dt \leq C \frac{1}{\varepsilon(x)^2} e^{-\beta_a(x) \varepsilon(x)} e^{\beta_a(x)} \varepsilon(x) \leq C. \quad (4.24)$$

For $I_1$ the bound (4.16) yields with Lemma 5 ($\ell = 1$)

$$I_1(x) \leq C \frac{1}{\varepsilon(x)^2} e^{-\beta_a(x)} \int_a^x \left(1 + |\varepsilon''(t)|\right) \left(1 + \frac{1}{\varepsilon(t)} e^{-\beta_a(t)}\right) \varepsilon(t) e^{\beta_a(t)} dt \leq C \frac{1}{\varepsilon(x)^2} e^{-\beta_a(x)} \int_a^x \varepsilon(t) e^{\beta_a(t)} + |\varepsilon''(t)| \varepsilon(t) e^{\beta_a(t)} + 1 + |\varepsilon''(t)| dt \leq C \frac{1}{\varepsilon(x)^2} e^{-\beta_a(x)} \left(\varepsilon(x)^2 e^{\beta_a(x)} + K_4(x) + (1 + \|\varepsilon''\|_{L^1(a,x)})\right) \leq C \left(1 + K_4(x) + \frac{1}{\varepsilon(x)^2} (1 + \|\varepsilon''\|_{L^1(a,x)}) e^{-\beta_a(x)}\right) \quad (4.25)$$

\[9\]
with \( K_4(x) := \int_a^x |\varepsilon''(t)|\varepsilon(t)e^{\beta \varepsilon_a(t)}dt \). Hölder’s inequality and the Cauchy-Schwarz inequality yield for \( K_4(x) \) with Lemma 5

\[
K_4(x) \leq \frac{C}{\varepsilon(x)^2} e^{-\beta \varepsilon_a(x)} \|\varepsilon''\|_{\infty, (a,x)} \int_a^x \varepsilon(t)e^{\beta \varepsilon_a(t)}dt \leq C\|\varepsilon''\|_{\infty, (a,x)}
\]

\[
K_4(x) \leq \frac{C}{\varepsilon(x)^2} e^{-\beta \varepsilon_a(x)} \|\varepsilon''\|_{0, (a,x)} \left( \int_a^x \varepsilon(t)^2 e^{2\beta \varepsilon_a(t)}dt \right)^{1/2} \leq C \frac{1}{\sqrt{\varepsilon(x)}} \|\varepsilon''\|_{0, (a,x)}
\]

From (4.20) we deduce the bound

\[
|\varepsilon''(a)| \leq C \frac{1 + \varepsilon'(a)}{\varepsilon(a)^2}
\]

We conclude our proposition by collecting (4.23), (4.24), (4.25), (4.26) and (4.27). \(\square\)

**Theorem 10** (Solution Decomposition). Suppose \( 0 \leq \varepsilon' \) on \([0,1]\) and define \( e(t) := \int_0^t 1/\varepsilon(z)dz \). Then there exists a constant \( S_0 \) with \( |S_0| \leq C \) such that the solution \( u \) of (4.1) can be decomposed into the sum of a smooth part \( S \) and an exponential boundary layer component \( E^{BL} \), i.e. \( u = S + E^{BL} \) such that \( S \) and \( E \) solve the boundary-value problems

\[
\mathcal{L}_\varepsilon S = f \quad \text{in} \quad (0,1), \quad S(0) = S_0, \quad S(1) = 0,
\]

\[
\mathcal{L}_\varepsilon E^{BL} = 0 \quad \text{in} \quad (0,1), \quad E^{BL}(0) = -S_0, \quad E^{BL}(1) = 0.
\]

Moreover there exists a constant \( C \) such that for \( x \in [0,1] \)

\[
|S^{(k)}(x)| \leq C \quad \text{for} \quad k = 0, 1,
\]

\[
|S''(x)| \leq C \frac{1 + \varepsilon'(x)}{\varepsilon(x)},
\]

\[
|\varepsilon(x)S''(x) + \varepsilon'(x)S'(x)| \leq C,
\]

and

\[
|(E^{BL})^{(k)}(x)| \leq C \frac{1}{\varepsilon(x)^k} e^{-\beta \varepsilon(x)} \quad \text{for} \quad k = 0, 1,
\]

\[
|(E^{BL})''(x)| \leq C \frac{1 + \varepsilon'(x)}{\varepsilon(x)^2} e^{-\beta \varepsilon(x)}.
\]

**Proof.** We start off with the regular solution component \( S \): Fix \( a < \frac{1}{\beta} \ln \frac{1}{\underline{\varepsilon}} < 0 \). On the the interval \((a,1)\) choose smooth extension \( \varepsilon^*, b^*, c^* \) and \( f^* \) of \( \varepsilon, b, c \) and \( f \) such in a way that \( \varepsilon^* \) is non-decreasing and the assumptions (4.14) are met. Thus we can apply Lemma 6 and Lemma 7 to the boundary value problem

\[-(\varepsilon^*(S^*)')' - b^*(S^*)' + c^*S^* = f^* \quad \text{in} \quad (a,1), \quad S^*(a) = 0, \quad S^*(1) = u_1\]

to obtain

\[
|S^*(x)| \leq C \quad \text{and} \quad |(S^*)'(x)| \leq C \left( 1 + \frac{1}{\varepsilon^*(x)} e^{-\beta \varepsilon_a(x)} \right) \quad \text{for} \quad x \in [a,1].
\]

Since \( \varepsilon \) is non-decreasing we can set \( \underline{\varepsilon} := \varepsilon(0) = \varepsilon^*(0) \). This implies for all \( x \in (a,0] \) that

\[ C_{\underline{\varepsilon}} \leq \varepsilon^*(x) \leq \underline{\varepsilon}. \]

Hence

\[
e^{-\beta \varepsilon_a(0)} = e^{-\beta} \int_0^0 \frac{1}{\varepsilon(z)}dz \leq e^{\frac{-\beta a}{2}} < \underline{\varepsilon}.
\]

The fact that \( e_a \) is a strictly increasing function implies \( e_a(x) > e_a(0) \). Thus we arrive at

\[
e^{-\beta \varepsilon_a(x)} \leq e^{-\beta \varepsilon_a(0)} < \underline{\varepsilon} \quad \text{for} \quad x \in [0,1).
\]
From (4.29) it now follows that
\[ |(S^*)^{(k)}(x)| \leq C \quad \text{for } k = 0, 1 \text{ and } x \in [0, 1] \]

since \( \varepsilon^*|_{(0,1)} = \varepsilon \geq \varepsilon_0 \). Setting \( S := S^*|_{(0,1)} \) it satisfies the boundary value problem (4.28a) because \( b^*|_{(0,1)} = b, \quad c^*|_{(0,1)} = c \) as well as \( f^*|_{(0,1)} = f \). The bound on \( S^* + (S^*)' \) yields (4.28c), in particular \( S(0) = S_0 := S^*(0) \) with \( |S_0| \leq C \). Since (4.28d) and (4.28e) are immediate consequences of (4.28a) and (4.28c) all propositions for the regular part \( S \) are verified.

To bound the layer component \( E^{BL} \) we use the barrier functions \( \phi^\pm \) defined by
\[ \phi^\pm(x) = \pm |E^{BL}(0)| e^{-b\varepsilon(x)} \]

A simple calculation yields
\[ (L_\varepsilon \phi^+)(x) = |E^{BL}(0)| \left( \beta b(x) + \varepsilon'(x) - \beta - \varepsilon'(x) \right) + c(x) e^{-\varepsilon(x)} \geq 0 = (L_\varepsilon E^{BL})(x), \]
\[ \phi^+(0) = |E^{BL}(0)| \geq E^{BL}(0), \]
\[ \phi^+(1) = |E^{BL}(0)| e^{-\beta E(1)} \geq 0 = E^{BL}(1). \]

Hence \( E^{BL}(x) \leq \phi^+(x) \leq C e^{-\varepsilon(x)} \). Because \( \phi^- = -\phi^+ \) the estimate (4.28f) for \( k = 0 \) follows.

For the first derivative of the boundary layer term \( E^{BL} \) we use the representation
\[ E^{BL}(x) = \int_x^1 z(s) ds + K \int_x^1 e^{-B(s)} ds \]
with
\[ z(x) = -\int_0^x \frac{b(s) + \varepsilon'(s)}{\varepsilon(s)} E^{BL}(s) e^{-B(s)} ds, \]
\[ B(x) = \int_0^x \frac{b(s) + \varepsilon'(s)}{\varepsilon(s)} ds = \int_0^x \frac{b(s)}{\varepsilon(s)} ds + \ln \left( \varepsilon(x) \right) - \ln \left( \varepsilon(0) \right). \]

The estimate (4.28f) with \( k = 0 \) yields for \( z \):
\[ |z(x)| \leq C \int_0^x \frac{b(s) + \varepsilon'(s)}{\varepsilon(s)} e^{-\beta E(s)} \frac{\varepsilon(s)}{\varepsilon(x)} e^{-\beta \left( \varepsilon(x) - \varepsilon(s) \right)} ds \]
\[ \leq C \frac{1}{\varepsilon(x)} e^{-\beta \varepsilon(x)} \int_0^x \left( 1 + \varepsilon'(s) \right) ds \leq C \frac{1}{\varepsilon(x)} e^{-\beta \varepsilon(x)}. \]

The constant \( K \) is governed by the boundary condition \( E^{BL}(0) = u_0 - S_0 \):
\[ K \int_0^1 e^{-B(s)} ds = u_0 - S_0 - \int_0^1 z(s) ds. \quad (4.30) \]

Using the substitution \( t = \varepsilon(s) \) with \( \frac{dt}{ds} = e'(s) = \frac{1}{\varepsilon(s)} \) we obtain
\[ \int_0^1 e^{-B(s)} ds = \int_0^1 \frac{\varepsilon(0)}{\varepsilon(s)} e^{-\beta \varepsilon(s)} \frac{b(\varepsilon(s))}{\varepsilon(s)} ds \geq \int_0^1 \frac{\varepsilon(0)}{\varepsilon(s)} e^{-\|b\|_{\infty} \varepsilon(s)} ds = \varepsilon(0) \int_0^{\varepsilon(1)} e^{-\|b\|_{\infty} t} dt \geq C \varepsilon(0) \]
and
\[ \int_0^1 z(s) ds \leq C \int_0^1 \frac{\varepsilon(0)}{\varepsilon(s)} e^{-\beta \varepsilon(s)} ds \leq C \int_0^{\varepsilon(1)} e^{-\beta t} dt \leq C \int_0^{\infty} e^{-\beta t} dt \leq C . \]

Hence (4.30) gives \( |K| \leq \frac{C}{\varepsilon(0)} \) and because
\[ (E^{BL})'(x) = -z(x) - Ke^{-B(x)} \]
we can estimate
\[
\left|(E^{BL})'(x)\right| \leq |z(x)| + |K|e^{-B(x)} \leq C \frac{1}{\varepsilon(x)} e^{-\beta \varepsilon(x)} + \frac{C}{\varepsilon(0)}\frac{\varepsilon(0)}{\varepsilon(x)} e^{-\beta \varepsilon(x)}
\]
which is (4.28f) for \( k = 1 \). For the remaining result (4.28g) we use the differential equation (4.28b) and the bounds (4.28f):
\[
\left|(E^{BL})''(x)\right| \leq \frac{|h(x)| + \varepsilon'(x)}{\varepsilon(x)} \left|E^{BL}'(x)\right| + \frac{|c(x)|}{\varepsilon(x)} \left|(E^{BL})(x)\right| \leq C \frac{1 + \varepsilon'(x)}{\varepsilon(x)^2} e^{-\beta \varepsilon(x)}.
\]

\[\square\]

Remark 4. It is possible to use Lemma 9 to prove the existence of a solution decomposition \( u = S + E \) with the better bounds (1.4a) and (1.4b) for the derivatives of \( S \) and \( E \). This requires the additional assumption that \( \varepsilon' \) is bounded, moreover additional assumptions concerning smooth extensions of \( \varepsilon'' \).

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