Abstract. We show Mckay correspondence of Betti numbers of Chen-Ruan cohomology for omnioriented quasitoric orbifolds. In previous articles with M. Poddar [8, 9], we proved the correspondence for four dimension and six dimensions. Here we deal with the general case.

1. Quasitoric orbifolds

In this section we review the combinatorial construction of quasitoric orbifolds. We also construct an explicit orbifold atlas for them and list a few important properties. The notations established here will be important for the rest of the article. This material has been taken from [9].

1.1. Construction. Fix a copy $N$ of $\mathbb{Z}^n$ and let $T_N := (N \otimes \mathbb{Z} \mathbb{R})/N \cong \mathbb{R}^n/N$ be the corresponding $n$-dimensional torus. A primitive vector in $N$, modulo sign, corresponds to a circle subgroup of $T_N$. More generally, suppose $M$ is a submodule of $N$ of rank $m$. Then

\begin{equation}
T_M := (M \otimes \mathbb{Z} \mathbb{R})/M
\end{equation}

is a torus of dimension $m$. Moreover there is a natural homomorphism of Lie groups $\xi_M : T_M \to T_N$ induced by the inclusion $M \hookrightarrow N$.

Definition 1.1. Define $T(M)$ to be the image of $T_M$ under $\xi_M$. If $M$ is generated by a vector $\lambda \in N$, denote $T_M$ and $T(M)$ by $T_\lambda$ and $T(\lambda)$ respectively.

Usually a polytope is defined to be the convex hull of a finite set of points in $\mathbb{R}^n$. To keep our notation manageable, we will take a more liberal interpretation of the term polytope.

Definition 1.2. A polytope $P$ will denote a subset of $\mathbb{R}^n$ which is diffeomorphic, as manifold with corners, to the convex hull $Q$ of a finite number of points in $\mathbb{R}^n$. Faces of $P$ are the images of the faces of $Q$ under the diffeomorphism.

Let $P$ be a simple polytope in $\mathbb{R}^n$, i.e. every vertex of $P$ is the intersection of exactly $n$ codimension one faces (facets). Consequently every $k$-dimensional face $F$ of $P$ is the intersection of a unique collection of $n-k$ facets. Let $\mathcal{F} := \{F_1, \ldots, F_m\}$ be the set of facets of $P$.

Definition 1.3. A function $\Lambda : \mathcal{F} \to N$ is called a characteristic function for $P$ if $\Lambda(F_{i_1}), \ldots, \Lambda(F_{i_k})$ are linearly independent whenever $F_{i_1}, \ldots, F_{i_k}$ intersect at a face in $P$. We write $\lambda_i$ for $\Lambda(F_i)$ and call it a characteristic vector.
Remark 1.1. In this article we assume that all characteristic vectors are primitive. Corresponding quasitoric orbifolds have been termed primitive quasitoric orbifold in [11]. They are characterized by the codimension of singular locus being greater than or equal to four.

Definition 1.4. For any face $F$ of $P$, let $\mathcal{I}(F) = \{i | F \subset F_i\}$. Let $\Lambda$ be a characteristic function for $P$. The set $\lambda_F := \{\lambda_i : i \in \mathcal{I}(F)\}$ is called the characteristic set of $F$. Let $N(F)$ be the submodule of $N$ generated by $\lambda_F$. Note that $\mathcal{I}(P)$ is empty and $N(P) = \{0\}$.

For any point $p \in P$, denote by $F(p)$ the face of $P$ whose relative interior contains $p$. Define an equivalence relation $\sim$ on the space $P \times T_N$ by

$$ (p, t) \sim (q, s) \text{ if and only if } p = q \text{ and } s^{-1}t \in T(N(F(p))) $$

Then the quotient space $X := P \times T_N / \sim$ can be given the structure of a $2n$-dimensional orbifold. We refer to the pair $(P, \Lambda)$ as a model for the quasitoric orbifold. The space $X$ inherits an action of $T_N$ with orbit space $P$ from the natural action on $P \times T_N$. Let $\pi : X \to P$ be the associated quotient map.

The space $X$ is a manifold if the characteristic vectors $\lambda_{i_1}, \ldots, \lambda_{i_k}$ generate a unimodular subspace of $N$ whenever the facets $F_{i_1}, \ldots, F_{i_k}$ intersect. The points $\pi^{-1}(v) \in X$, where $v$ is any vertex of $P$, are fixed by the action of $T_N$. For simplicity we will denote the point $\pi^{-1}(v)$ by $v$ when there is no confusion.

1.2. Orbifold charts. Consider open neighborhoods $U_v \subset P$ of the vertices $v$ such that $U_v$ is the complement in $P$ of all edges that do not contain $v$. Let

$$ X_v := \pi^{-1}(U_v) = U_v \times T_N / \sim $$

For a face $F$ of $P$ containing $v$ there is a natural inclusion of $N(F)$ in $N(v)$. It induces an injective homomorphism $T_{N(F)} \to T_{N(v)}$ since a basis of $N(F)$ extends to a basis of $N(v)$. We will regard $T_{N(F)}$ as a subgroup of $T_{N(v)}$ without confusion. Define an equivalence relation $\sim_v$ on $U_v \times T_{N(v)}$ by $(p, t) \sim_v (q, s)$ if $p = q$ and $s^{-1}t \in T_{N(F)}$ where $F$ is the face whose relative interior contains $p$. Then the space

$$ \tilde{X}_v := U_v \times T_{N(v)} / \sim_v $$

is $\theta$-equivariantly diffeomorphic to an open set in $\mathbb{C}^n$, where $\theta : T_{N(v)} \to U(1)^n$ is an isomorphism, see [4]. There exists a diffeomorphism $f : \tilde{X}_v \to B \subset \mathbb{C}^n$ such that $f(t \cdot x) = \theta(t) \cdot f(x)$ for all $x \in \tilde{X}_v$. This will be evident from the subsequent discussion.

The map $\xi_{N(v)} : T_{N(v)} \to T_N$ induces a map $\xi_v : \tilde{X}_v \to X_v$ defined by $\xi_v([(p, t)]^{\sim_v}) = [(p, \xi_{N(v)}(t))]^{\sim_v}$ on equivalence classes. The kernel of $\xi_{N(v)}$, $G_v = N/N(v)$, is a finite subgroup of $T_{N(v)}$ and therefore has a natural smooth, free action on $T_{N(v)}$ induced by the group operation. This induces smooth action of $G_v$ on $\tilde{X}_v$. This action is not free in general. Since $T_N \cong T_{N(v)}/G_v$, $X_v$ is homeomorphic to the quotient space $\tilde{X}_v/G_v$. An orbifold chart (or uniformizing system) on $X_v$ is given by $(\tilde{X}_v, G_v, \xi_v)$.

We define a homeomorphism $\phi(v) : \tilde{X}_v \to \mathbb{R}^{2n}$ as follows. Assume without loss of generality that $F_1, \ldots, F_n$ are the facets of $U_v$. Let the equation of $F_i$ be $p_i(v) = 0$. 

Assume that $p_i(v) > 0$ in the interior of $U_v$ for every $i$. Let $\Lambda(v)$ be the corresponding matrix of characteristic vectors

\[ \Lambda(v) = [\lambda_1 \ldots \lambda_n]. \] 

If $q(v) = (q_1(v), \ldots, q_n(v))^t$ are angular coordinates of an element of $T_N$ with respect to the basis $\{\lambda_1, \ldots, \lambda_n\}$ of $N \otimes \mathbb{R}$, then the standard coordinates $q = (q_1, \ldots, q_n)^t$ may be expressed as

\[ q = \Lambda(v)q(v). \]

Then define the homeomorphism $\phi(v) : \tilde{X}_v \to \mathbb{R}^{2n}$ by

\[ x_i = x_i(v) := \sqrt{p_i(v)} \cos(2\pi q_i(v)), \quad y_i = y_i(v) := \sqrt{p_i(v)} \sin(2\pi q_i(v)) \quad \text{for } i = 1, \ldots, n \]

**Remark 1.2.** The square root over $p_i$ is necessary to ensure that the orbit map is smooth.

We write

\[ z_i = x_i + \sqrt{-1}y_i, \quad \text{and} \quad z_i(v) = x_i(v) + \sqrt{-1}y_i(v) \]

Now consider the action of $G_v = N/N(v)$ on $\tilde{X}_v$. An element $g$ of $G_v$ is represented by a vector $\sum_{i=1}^n a_i \lambda_i$ in $N$ where each $a_i \in \mathbb{Q}$. The action of $g$ transforms the coordinates $q_i(v)$ to $q_i(v) + a_i$. Therefore

\[ g \cdot (z_1, \ldots, z_n) = (e^{2\pi \sqrt{-1}a_1}z_1, \ldots, e^{2\pi \sqrt{-1}a_n}z_n). \]

We define

\[ G_F := ((N(F) \otimes \mathbb{Z} \mathbb{Q}) \cap N)/N(F). \]

We denote the space $X$ with the above orbifold structure by $X$.

1.3. **Invariant suborbifolds.** The $T_N$-invariant subset $X(F) = \pi^{-1}(F)$, where $F$ is a face of $P$, has a natural structure of a quasitoric orbifold \[\square\]. This structure is obtained by taking $F$ as the polytope for $X(F)$ and projecting the characteristic vectors to $N/N^*(F)$ where $N^*(F) = (N(F) \otimes \mathbb{Z} \mathbb{Q}) \cap N$. With this structure $X(F)$ is a suborbifold of $X$. It is called a characteristic suborbifold if $F$ is a facet. Suppose $\lambda$ is the characteristic vector attached to the facet $F$. Then $\pi^{-1}(F)$ is fixed by the circle subgroup $T(\lambda)$ of $T_N$. We denote the relative interior of a face $F$ by $F^o$ and the corresponding invariant space $\pi^{-1}(F^o)$ by $X(F^o)$. Note that $v^o = v$ if $v$ is a vertex.

1.4. **Orientation.** Quasitoric orbifolds are oriented. For more detailed discussion see 2.8 \[\square\]. A choice of orientations for $P \subset \mathbb{R}^n$ and $T_N$ induces an orientation for $X$. 
1.5. **Omniorientation.** An omniorientation is a choice of orientation for the orbifold as well as an orientation for each characteristic suborbifold. At any vertex $v$, the $G_v$-representation $T_0\tilde{X}$ splits into the direct sum of $n G_v$-representations corresponding to the normal spaces of $z_i(v) = 0$. Thus we have a decomposition of the orbifold tangent space $T_vX$ as a direct sum of the normal spaces of the characteristic suborbifolds that meet at $v$. Given an omniorientation, we say that the sign of a vertex $v$ is positive if the orientations of $T_v(X)$ determined by the orientation of $X$ and orientations of characteristic suborbifolds coincide. Otherwise we say that sign of $v$ is negative. An omniorientation is then said to be positive if each vertex has positive sign.

It is easy to verify that reversing the sign of any number of characteristic vectors does not affect the topology or differentiable structure of the quasitoric orbifold. There is a circle action of $T_{\lambda_i}$ on the normal bundle of $X(F_i)$ producing a complex structure and orientation on it. This action and orientation varies with the sign of $\lambda_i$. Therefore, given an orientation on $X$, omniorientations correspond bijectively to choices of signs for the characteristic vectors. We will assume the standard orientations on $P$ and $T^n$ so that omniorientations will be solely determined by signs of characteristic vectors. Also under this choice the sign of $v$ equals the sign of $\det(\Lambda(v))$.

2. **Blowdowns**

This material has been taken from [9]. Topologically the blowup will correspond to replacing an invariant suborbifold by the projectivization of its normal bundle. Combinatorially we replace a face by a facet with a new characteristic vector. Suppose $F$ is a face of $P$. We choose a hyperplane $H = \{\hat{p}_0 = 0\}$ such that $p_0$ is negative on $F$ and $\hat{P} := \{\hat{p}_0 > 0\} \cap P$ is a simple polytope having one more facet than $P$. Suppose $F_1,\ldots,F_m$ are the facets of $P$. Denote the facets $F_i \cap \hat{P}$ by $F_i$ without confusion. Denote the extra facet $H \cap P$ by $F_0$.

Without loss of generality let $F = \bigcap_{j=1}^k F_j$. Suppose there exists a primitive vector $\lambda_0 \in \mathbb{N}$ such that

\[(2.1)\quad \lambda_0 = \sum_{j=1}^k b_j \lambda_j, \quad b_j > 0 \forall j.
\]

Then the assignment $F_0 \mapsto \lambda_0$ extends the characteristic function of $P$ to a characteristic function $\tilde{\Lambda}$ on $\hat{P}$. Denote the omnioriented quasitoric orbifold derived from the model $(\hat{P}, \tilde{\Lambda})$ by $Y$.

**Definition 2.1.** We define blowdown a map $\rho$ from $Y \mapsto X$ which is inverse of a blow-up. Such maps have been constructed in [9].

**Lemma 2.1.** (Lemma 4.2 [9]) If $X$ is positively omnioriented, then so is a blowup $Y$.

**Definition 2.2.** A blowdown is called crepant if $\sum b_j = 1$. 
3. Chen-Ruan Cohomology

This material has been taken from [3]. The Chen-Ruan cohomology group is built out of the ordinary cohomology of certain copies of singular strata of an orbifold called twisted sectors. The twisted sectors of orbifold toric varieties was computed in [10]. The determination of such sectors for quasitoric orbifolds is similar in essence. Another important feature of Chen-Ruan cohomology is the grading which is rational in general. In our case the grading will depend on the omniorientation.

Let $X$ be an omnioriented quasitoric orbifold. Consider any element $g$ of the group $G_F$ ([10]). Then $g$ may be represented by a vector $\sum_{j \in I(F)} a_j \lambda_j$. We may restrict $a_j$ to $[0, 1) \cap \mathbb{Q}$. Then the above representation is unique. Then define the degree shifting number or age of $g$ to be

$$\iota(g) = \sum a_j.$$  

For faces $F$ and $H$ of $P$ we write $F \leq H$ if $F$ is a sub-face of $H$, and $F < H$ if it is a proper sub-face. If $F \leq H$ we have a natural inclusion of $G_H$ into $G_F$ induced by the inclusion of $N(H)$ into $N(F)$. Therefore we may regard $G_H$ as a subgroup of $G_F$. Define the set

$$G^\circ_F = G_F - \bigcup_{F < H} G_H$$

Note that $G^\circ_F = \{\sum_{j \in I(F)} a_j \lambda_j | 0 < a_j < 1\} \cap N$, and $G^\circ_P = G_P = \{0\}$.

**Definition 3.1.** We define the Chen-Ruan orbifold cohomology of an omnioriented quasitoric orbifold $X$ to be

$$H^*_\text{CR}(X, \mathbb{R}) = \bigoplus_{F \leq P} \bigoplus_{g \in G^\circ_F} H^{*-2\iota(g)}(X(F), \mathbb{R}).$$

Here $H^*$ refers to singular cohomology or equivalently to de Rham cohomology of invariant forms when $X(F)$ is considered as the orbifold $X(F)$. The pairs $(X(F), g)$ where $F < P$ and $g \in G^\circ_F$ are called twisted sectors of $X$. The pair $(X(P), 1)$, i.e. the underlying space $X$, is called the untwisted sector.

First we introduce some notation. Consider a codimension $k$ face $F = F_1 \cap \ldots \cap F_k$ of $P$ where $k \geq 1$. Define a $k$-dimensional cone $C_F$ in $N \otimes \mathbb{R}$ as follows,

$$C_F = \{\sum_{j=1}^k a_j \lambda_j : a_j \geq 0\}$$

The group $G_F$ can be identified with the subset $Box_F$ of $C_F$, where

$$Box_F := \{\sum_{j=1}^k a_j \lambda_j : 0 \leq a_j < 1\} \cap N.$$
Consequently the set $G^\circ_F$ is identified with the subset

\[ Box^\circ_F := \{ \sum_{j=1}^{k} a_j \lambda_j : 0 < a_j < 1 \} \cap N \]

of the interior of $C_F$. We define $Box_F = Box^\circ_F = \{ 0 \}$.

Suppose $v = F_1 \cap \ldots \cap F_n$ is a vertex of $P$. Then $Box_v = \bigcup_{v \leq F} Box^\circ_F$. This implies

\[ G_v = \bigcup_{v \leq F} G^\circ_F \]

An almost complex orbifold is $SL$ if the linearization of each $g$ is in $SL(n, \mathbb{C})$. This is equivalent to $\iota(g)$ being integral for every twisted sector. Therefore, to suit our purposes, we make the following definition.

**Definition 3.2.** A quasitoric orbifold is said to be quasi-$SL$ if the age of every twisted sector is an integer.

**Lemma 3.1.** (Lemma 8.2 \[9\]) The crepant blowup of a quasi-$SL$ quasitoric orbifold is quasi-$SL$.

### 4. Correspondence of Betti numbers

#### 4.1. Singularity and lattice polyhedron

Following the discussions in sections \[3\], a singularity of a face $F$ is defined by a cone $C_F$ formed by positive linear combinations of vectors in its characteristic set $\lambda_1, \ldots, \lambda_d$ where $d$ is the codimension of the face in the polytope. The elements of the local group $G_F$ are of the form $g = \text{diag}(e^{2\pi \sqrt{-1} \alpha_1}, \ldots, e^{2\pi \sqrt{-1} \alpha_d})$, where $\sum_{i=1}^{d} \alpha_i \lambda_i \in N$, and $0 \leq \alpha_i < 1$. Recall that the age

\[ \iota(g) = \alpha_1 + \ldots + \alpha_d \]

is integral in quasi-$SL$ case by definition \[3.2\].

The singularity along the interior of $F$ is of the form $\mathbb{C}^d/G_F$. These singularities are same as Gorenstein toric quotient singularities in complex algebraic geometry. Now let $N_v$ be the lattice formed by $\{ \lambda_1, \ldots, \lambda_n \}$, the characteristic vectors at a vertex $v$ contained in the face $F$. Let $m_v$ be the element in the dual lattice of $N_v$ such that its evaluation on each $\lambda_i$ is one. Now from Lemma 9.2 of \[3\] we know that the cone $C_v$ contains an integral basis, say $e_1, \ldots, e_n$. Suppose $e_i = \sum a_{ij} \lambda_j$. By \[1.4\] $e_i$ corresponds to an element of $G_v$, and since the singularity is quasi-$SL$, $\sum a_{ij}$ is integral. Hence $m_v$ evaluated on each $e_j$ is integral. So $m_v$ an element of the dual of the integral lattice $N$.

Consider the $(n-1)$-dimensional lattice polyhedron $\Delta_v$ defined as $\{ x \in C_v \mid \langle x , m_v \rangle = 1 \}$. Note that $\Delta_v = \{ \sum_{i=1}^{n} a_i \lambda_i \mid a_i \geq 0, \sum a_i = 1 \}$. For any face $F$ containing $v$ we define $\Delta_F = \Delta_v \cap C_F$. If $\{ \lambda_1, \ldots, \lambda_d \}$ denote the characteristic set of $F$, then $\Delta_F = \{ \sum_{i=1}^{d} a_i \lambda_i \mid a_i \geq 0, \sum a_i = 1 \}$. Hence $\Delta_F$ is independent of the choice of $v$. 
Remark 4.1. An element $g \in G$ of an $SL$ orbifold singularity can be diagonalized to the form $g = \text{diag}(e^{2\pi\sqrt{-1}\alpha_1}, \ldots, e^{2\pi\sqrt{-1}\alpha_d})$, where $0 \leq \alpha_i < 1$ and $\iota(g) = \alpha_1 + \ldots + \alpha_d$ is integral.

We make some definitions following [3].

Definition 4.1. Let $G$ be a finite subgroup of $SL(d, \mathbb{C})$. Denote by $\psi_i(G)$ the number of the conjugacy classes of $G$ having $\iota(g) = i$. Define
\[
W(G; uv) = \psi_0(G) + \psi_1(G)uv + \ldots + \psi_{d-1}(G)(uv)^{d-1}
\]

Definition 4.2. We define height$(g) = \text{rank}(g-I)$

Definition 4.3. Let $G$ be a finite subgroup of $SL(d, \mathbb{C})$. Denote by $\tilde{\psi}_i(G)$ the number of the conjugacy classes of $G$ having the height $= d$ and $\iota(g) = i$. Define
\[
\tilde{W}(G; uv) = \tilde{\psi}_0(G) + \tilde{\psi}_1(G)uv + \ldots + \tilde{\psi}_{d-1}(G)(uv)^{d-1}
\]

Definition 4.4. For a lattice polyhedron $\Delta_F$ defining a $SL$ singularity $\mathbb{C}^d/G_F$, we define the following:
\[
W(\Delta_F; uv) = W(G_F; uv)
\]
\[
\psi_i(\Delta_F) = \psi_i(G_F)
\]
\[
\tilde{W}(\Delta_F; uv) = \tilde{W}(G_F; uv)
\]
\[
\tilde{\psi}_i(\Delta_F) = \tilde{\psi}_i(G_F)
\]

Definition 4.5. A finite collection $\tau = \{\theta\}$ of simplices with vertices in $\Delta_F \cap N$ is called a triangulation of $\Delta_F$ if the following properties are satisfied.

1. If $\theta'$ is a face of $\theta \in \tau$ then $\theta' \in \tau$
2. The intersection of any two simplices $\theta', \theta'' \in \tau$ is either empty, or a common face of both of them;
3. $\Delta_F = \bigcup_{\theta \in \tau} \theta$

4.2. Blowdown and triangulation of polyhedron. A crepant blowup gives rise to triangulation of the polyhedrons corresponding to some of the faces. Suppose we blow up about a face $F$. Then it is clear that new characteristic vector is an integral vector lying in the interior of the polyhedron $\Delta_F$. Note that $\Delta_F$ is a simplex. The crepant blow up induces a barycentric subdivision of $\Delta_F$ with the new characteristic vector as barycenter. We denote this triangulation of $\Delta_F$ by $\tau_F$. For the faces $F'$ contained in $F$, $\Delta_{F'}$ is triangulated as follows. Let $K_{F'} = \lambda_{F'} - \lambda_F$ be difference of two characteristic sets. The triangulation $\tau_{F'}$ consists of simplices with vertex set of the form $\theta \cup \beta$ where $\theta$ are the vertices of a simplex of $\tau_F$ and $\beta \subset K_{F'}$. To see that this process takes care of all the faces lost and created we make the following comments. First of all the faces lost are $F$ and its subfaces. This means there will be no simplex with vertex set having $\lambda_F$ as a subset. This is exactly what happens here. The new faces created are subfaces of the intersection of new facet (created by the blowup) with faces having as vertex one of the vertices of $F$. These faces intersected
F prior to the blow up in some $F'$ and so the new faces formed correspond to the simplices with vertex set that are subset of the union $\theta \cup \beta$ discussed above.

4.3. **E-polynomial.** The following has been taken from the paper of Batyrev and Dais [3]. Let $X$ be an algebraic variety over $\mathbb{C}$ which is not necessarily compact or smooth. Denote by $H^{p,q}(H^k_c(X))$ the dimension of the $(p, q)$ Hodge component of the $k$-th cohomology with compact supports. We define

$$e^{p,q}(X) = \Sigma_{k \geq 0} (-1)^k H^{p,q}(H^k_c(X)).$$

The polynomial

$$E(X; u, v) := \Sigma_{p,q} e^{p,q}(X) u^p v^q$$

is called $E$-polynomial of $X$.

**Remark 4.2.** If the Hodge structure is pure, for example in the case of smooth projective toric varieties, then the coefficients $e^{p,q}(X)$ of the $E$-polynomial of $X$ are related to the usual Hodge numbers by $e^{p,q}(X) = (-1)^{p+q} h^{p,q}(X)$.

We state the following theorem without proof.

**Theorem 4.3.** (Proposition 3.4 in [3]) Let $X$ be a disjoint union of locally closed subvarieties $X_j$, $j \in J$, where $J \subset \mathbb{N}$. Then

$$E(X; u, v) = \Sigma_{j \in J} E(X_j; u, v)$$

4.4. **Ehrhart power series.** Let $\Delta$ be a lattice polyhedron and $k\Delta := \{kx \mid x \in \Delta\}$. Let $l(k\Delta)$ be the number of lattice points of $k\Delta$. Then the Ehrhart power series

$$P_{\Delta}(t) = \Sigma_{k \geq 0} l(k\Delta) t^k$$

**Definition 4.6.** Let $\Delta_F$ be a $(d-1)$ dimensional lattice polyhedron defining a $d$-dimensional toric singularity. It is well-known (see, for instance, [3], Theorem 5.4) that $P_{\Delta_F}(t)$ can be written in the form,

$$P_{\Delta_F}(t) = \frac{\psi_0(\Delta_F) + \psi_1(\Delta_F) t + \ldots + \psi_{d-1}(\Delta_F) t^{d-1}}{(1 - t)^d}$$

where $\psi_0(\Delta_F), \ldots, \psi_{d-1}(\Delta_F)$ are non-negative integers defined in (1.5).

4.5. **More on quasi-SL orbifolds.** Let $X$ be a compact quasi-SL $2n$-dimensional quasitoric orbifold. Let $\text{Sing}(X)$ be the set of singular points of $X$. Consider the set $I = \{i \in \mathbb{N} \mid i \leq \text{number of faces in the polytope of } X\}$. We can index the set of faces by the set $I$. Call the inverse image of the interior of the face $F_i$ as $X_i$. It can be easily seen that this gives a stratification of the orbifold where each stratum $X_i$ is diffeomorphic to a complex torus.

It is easily seen that

$$W(\Delta_F, uv) = \Sigma_{X_j \supset X_i} \tilde{W}(\Delta_{F_j}, uv)$$

where $X_j \supset X_i$ if $X_j \supset X_i$. The above result is true because the coefficient of each term in the left hand side can be broken in to ones with different heights (see definition (1.2), equations (4.6), (4.3) and (4.7). The ones with height equal to the
codimension of \(X_i\) contribute to \(\tilde{W}(\Delta_{F_i}, uv))\). These come from \(G_{\circ F_i}\). Use the decomposition \(G_{F_i} = \bigsqcup_{F_j \supseteq F_i} G_{\circ F_j}\) to observe that terms with lesser heights correspond to higher \(X_j\).

4.6. Poincaré Polynomial. Recall that

\[
H^*_CR(X, \mathbb{R}) = \bigoplus_{F \leq P} \bigoplus_{g \in G_{\circ F}} H^{*-2\ell(g)}(X(F), \mathbb{R})
\]

where \(X(F)\) is the inverse image of the face \(F\).

**Definition 4.7.** The Poincaré polynomial of a cohomology of \(X\) is a polynomial \(P(X)(t)\) where the coefficient of \(t^d\) is the rank of the degree \(d\) cohomology group. We denote by \(PP(X)(v)\) the Poincaré polynomial of the ordinary singular cohomology and \(PP_{CR}(X)(v)\) as the Poincaré polynomial of the Chen-Ruan cohomology of \(X\).

Now if \(X\) is a projective toric orbifold, it has pure Hodge structure. Since the Zariski closure of the \(X_i\) are the suborbifolds corresponding to the faces, from (4.14), (4.6) and (4.3), we have

\[
PP_{CR}(X)(v) = \sum_{i \in I} PP(X_i)(v) \tilde{W}(\Delta_{F_i}, v^2).
\]

4.7. Correspondence in quasitoric orbifolds. Take a quasi-SL quasitoric orbifold \(X\). A slight perturbation makes the polytope \(P\) associated with the orbifold into a rational polytope (see section 5.1.3 in [4]), and with suitable dilations make it into an integral polytope \(P'\) which is combinatorially equivalent to \(P\). From the normal fan of \(P'\) we get a projective toric orbifold \(X'\) whose polytope is \(P'\). (The orbifold structure of \(X'\) is determined by its analytic structure and we may conveniently refrain from using bold-face notation.) Putting \(u = v\) in Theorem (4.3) we have

\[
E(X'; v, v) = \sum_{i \in I} E(X_i'; v, v)
\]

In the left hand side the coefficient of \(v^k\) is the sum of \(e^{p,q}(X')\) where \(p + q = k\). Since \(X'\) is Kahler the Hodge structure is pure and from remark (1.2) it follows \(e^{p,q}(X') = (-1)^{p+q} h^{p,q}(X')\). Since toric orbifolds (see section 4 of [11]) have zero odd cohomology only the coefficient of \(v^{2k}\) terms are nonzero. By Baily’s Hodge decomposition (see [1]), the Hodge numbers \(h^{p,q}\) for \(p + q = 2k\) add up to the \(2k\)-th Betti number of singular cohomology group. So the left hand side is the Poincaré polynomial of the ordinary cohomology, giving

\[
PP(X')(v) = \sum_{i \in I} E(X_i'; v, v).
\]

It is known from section 4 of [11] that the Betti numbers depend on the combinatorial equivalence class of the polytope \(P'\). As \(P'\) is combinatorially equivalent to \(P\), the left hand side equals the Poincaré polynomial of the quasitoric orbifold \(X\). The right hand side is a sum of \(E\)-polynomials of a number of tori. Since the number of tori of each dimension is the same by combinatorial equivalence of the polytopes, we have,

\[
PP(X)(v) = \sum_{i \in I} E(X_i, v, v)
\]
where $\bigsqcup X_i$ is the stratification by tori of the quasitoric orbifold $X$. Now from (4.18) we get,
\begin{equation}
PP_{CR}(X)(v) = \sum_{i \in I} PP(X_i)(v) \tilde{W}(\Delta_{F_i}, v^2)
\end{equation}
Using (4.18) we have
\begin{equation}
PP_{CR}(X)(v) = \sum_{i \in I} \sum_{X_j \leq X_i} E(X_j, v, v) \tilde{W}(\Delta_{F_i}, v^2)
\end{equation}
Interchanging the order of summation, and using (4.13) we have
\begin{equation}
PP_{CR}(X)(v) = \sum_{j \in I} E(X_j, v, v) W(\Delta_{F_j}, v^2)
\end{equation}

**Theorem 4.4.** Suppose $X$ is a quasi-SL quasitoric orbifold, and $\hat{X}$ a crepant blowup. Then
\begin{equation}
PP_{CR}(X)(v) = PP_{CR}(\hat{X})(v)
\end{equation}

**Proof.** Let $\rho : \hat{X} \to X$ be a crepant blowdown. We set $\hat{X}_i := \rho^{-1}(X_i)$. Then $\hat{X}_i$ has a natural stratification by products $X_i \times ((\mathbb{C}^*)^{\text{codim}(\theta)})$ induced by the triangulation,
\begin{equation}
\Delta_{F_i} = \bigcup_{\theta \in \tau_i} \theta
\end{equation}
where $\tau_i$ consists of all simplices which intersect the interior of $\Delta_{F_i}$, and $\text{codim}(\theta)$ denotes the codimension of $\theta$ in $\Delta_{F_i}$.

Note that the $E$-polynomial of a $k$-dimensional complex torus is $(v^2 - 1)^k$.

From (4.12) we have
\begin{equation}
W(\Delta_{F_i}; v^2) = P_{\Delta_{F_i}}(v^2)(1 - v^2)^d
\end{equation}
where $d$ is the dimension of the face $F_i$. Consider the triangulation (4.23) of $\Delta_{F_i}$. By counting lattice points using (4.12) and applying the inclusion exclusion principle we have
\begin{equation}
P_{\Delta_{F_i}}(v^2) = \sum_{\theta \in \tau_i} (-1)^{\text{codim}(\theta)} P_{\theta}(v^2) = \sum_{\theta \in \tau_i} (-1)^{\text{codim}(\theta)} W(\theta, v^2)(1 - v^2)^{-\text{dim}(\theta)}
\end{equation}

Multiplying both sides by $(1 - v^2)^d$, we obtain
\begin{equation}
W(\Delta_{F_i}; v^2) = \sum_{\theta \in \tau_i} (-1)^{\text{codim}(\theta)} W(\theta, v^2)
\end{equation}

Since we are dealing with simplices $\theta$ which intersect the interior of $\Delta_{F_i}$ each stratum of $\hat{X}$ is counted once. This is because each stratum corresponds to the interior of a face and for each face we have a simplex and it will lie in the interior of exactly one of the original (pre-triangulation) polyhedrons. Thus the equation (4.21) applied to $\hat{X}$ gives
\begin{equation}
PP_{CR}(\hat{X})(v) = \sum_{i \in I} E(X_i; v, v) \sum_{\theta \in \tau_i} (-1)^{\text{codim}(\theta)} W(\theta, v^2)
\end{equation}

Now using (4.20)
\begin{equation}
PP_{CR}(\hat{X})(v)) = \sum_{i \in I} E(X_i; v, v) W(\Delta_{F_i}; v^2) = PP_{CR}(X)(v)
\end{equation}

$\square$
REFERENCES

[1] Walter L. Baily, Jr The Decomposition Theorem for V-Manifolds American Journal of Mathematics, Vol. 78, No. 4 (1956), 862-888.
[2] V. V. Batyrev: Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs, J. Eur. Math. Soc. (JEMS) 1 (1999), no. 1, 5-33.
[3] V. V. Batyrev and D. I. Dais: Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry, Topology 35 (1996), no. 4, 901-929.
[4] V. M. Buchstaber and T. E. Panov : Torus actions and their applications in topology and combinatorics, University Lecture Series 24, American Mathematical Society, Providence, RI, 2002.
[5] W. Chen and Y. Ruan: A new cohomology theory of orbifold, Comm. Math. Phys. 248 (2004), no. 1, 1-31.
[6] C.-H. Cho and M. Poddar: Holomorphic orbidiscs and Lagrangian Floer cohomology of symplectic toric orbifolds, arXiv:1206.3994
[7] M. W. Davis and T. Januszkiewicz: Convex polytopes, Coxeter orbifolds and torus actions, Duke Math. J. 62 (1991), no.2, 417-451.
[8] S. Ganguli and M. Poddar: Blowdowns and McKay correspondence on four dimensional quasitoric orbifolds, to appear in Osaka J. Math., preprint arXiv:0911.0766v3
[9] S. Ganguli and M. Poddar: Almost complex structure, blowdowns and McKay correspondence in quasitoric orbifolds, to appear in Osaka J. Math., preprint arXiv:1202.5578
[10] M. Poddar: Orbifold cohomology group of toric varieties. Orbifolds in mathematics and physics (Madison, WI, 2001), 223-231, Contemp. Math., 310, Amer. Math. Soc., Providence, RI, 2002.
[11] M. Poddar and S. Sarkar: On quasitoric orbifolds, Osaka J. Math. 47 (2010) No. 4, 1055-1076.

DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE LOS ANDES, BOGOTA, COLOMBIA

E-mail address: saibalgan@gmail.com