Wheeler-De Witt equation for brane gravity

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Abstract

We consider the gravity in the system consisting of the Bogomol’nyi-Prasad-Sommerfield (BPS) D3-brane embedded in the flat background geometry, produced by the solutions of the supergravity. The effective action for this system is represented by the sum of the Hilbert-Einstein and DBI actions. We derive the Wheeler-De Witt equation for this system and obtain analytical solutions in some special cases. We also calculate tunneling probability from Planckian size of D3-brane to the classical regime.

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I. INTRODUCTION

The discovery of D branes in the string theory have opened the way to the realization of the universe as a braneworld embedded in an ambient space. In the type IIA, IIB and I strings there are many D branes sustaining gauge and matter fields, at least one of them should include the Standard Model of particle physics. This D-brane will be correspond to the our universe. All the matter and gauge fields are due to the open string sector. The strings from this sector have the end points constrained to move on the brane. The graviton and the dilaton are contained in the closed string sector and probe the extra dimensions. In the regions where the back-reaction of D-brane and stringy geometry can be ignored, the effective action for the worldvolume in the flat background is given by the DBI action coupled to the worldvolume gravity. Such models were used in cosmology in order to explain inflation from the string theory [1].

The DBI action describes the effects of virtual open strings at the tree level; it includes the effects of the background geometry and field strengths. But the nonlinear form of the DBI action is inconvenient for quantization. However, introducing an intrinsic worldvolume metric one can write down the equivalent action [2]. From the other side one can express the DBI action as sum of constraints [3-5], where Hamiltonian for the D-brane is the constrain. In the first case, where DBI action is represented by the intrinsic metric, one can consider this metric as a source of gravitation and the induced metric on the worldvolume as a field coupled to the gravity. This case will be considered. The physical sense of our study is that in the classical background of 10 dimensional spacetime we consider 4 dimensional quantum subsystem. In the low-energetic approximation the classical backgrounds are given by the solutions of the supergravity. One of the best known and well investigated are $AdS_k \times X_{10-k}$ backgrounds ($AdS/CFT$ correspondence). The other are backgrounds with warped metrics maintaining the four-dimensional Poincare symmetry. The most interesting backgrounds are responsible for inflation and leading to the de Sitter vacuum in four dimensions [6].

In this paper we extend the applicability of the models with the DBI action to the quantum regime. In this regime we have to take into account quantum effects as well as for the gravity and for the other fields from the string sector (or low-energetic approximation). These effects can be described by the Wheeler-De Witt equation. Motivated by the above remarks we consider the Friedmann-Robertson-Walker (FRW) model coupled to the DBI action. In section 2 we derive a classical action for the considered system. This action is the base for section 3 where we obtain the Wheeler-De Witt equation and give solutions of this equation in particular cases. Section 4 is devoted to conclusions.

II. DBI ACTION AND GRAVITY

The well-known form of DBI action for a D3-brane is (modulo WZ terms):

$$S_3 = -T_3 \int d^4x e^{-\phi} \sqrt{-\det(\gamma_{\alpha\beta} + 2\pi \alpha' F_{\alpha\beta} + B_{\alpha\beta})},$$

(2.1)

where $\gamma_{\alpha\beta}$ and $B_{\alpha\beta}$ are pull-backs of a background metric $g_{\alpha\beta}$ and NS field $B_{\alpha\beta}$ by an embedding field $X$, respectively. A strength $F_{\alpha\beta}$ of $U(1)$ gauge field is the intrinsic field on the worldvolume of D3-brane. As it is shown in [2] the above action can be expressed in the equivalent forms by an intrinsic auxiliary worldvolume tensor field $h_{\alpha\beta}$. We consider the simplest case when the fields $F_{\alpha\beta}$ and $B_{\alpha\beta}$ on the worldvolume vanish. Then the action (2.1) becomes the 4-dimensional Nabu-Goto, which takes the form:

$$S'_3 = -\frac{\Lambda T_3}{2} \int d^4x e^{-\phi} \sqrt{-\det(h_{\alpha\beta})} \left[ h^{\alpha\beta} \gamma_{\alpha\beta} - 2\Lambda \right],$$

(2.2)

where $h_{\alpha\beta}$ is the intrinsic auxiliary metric on the worldvolume and $\Lambda$ is a constant. We promote the metric $h_{\alpha\beta}$ to the dynamic degree of freedoms. Thus the metric $h_{\alpha\beta}$ is a field which is responsible for gravitation on the worldvolume and the metric $\gamma$ on the worldvolume is considered as a field induced by the background. Hence the system consists of gravitation on the worldvolume coupled to the fields originated from the background. Problems of the backreaction in this consideration are neglected. So the action for this system is obtained from
(2.2) by adding the Einstein-Hilbert term:

\[ S = \frac{m_p^2}{2} \int d^4x \sqrt{- \det (h_{\alpha\beta})} R(h) - \frac{\Lambda T_3}{2} \int d^4x e^{-\phi} \sqrt{- \det (h_{\alpha\beta})} \left[ h^{\alpha\beta} \gamma_{\alpha\beta} - 2\Lambda \right], \]

(2.3)

where \( m_p^2 = (8\pi G)^{-1} \). Next we apply the ADM construction for this system. In this construction the world-volume \( M \) is the product: \( \mathbf{R}^1 \times \Sigma_3 \) where \( \Sigma_3 \) is the 3-dimensional space-like slice of \( M \). Then the metric \( h_{\alpha\beta} \) is determined by a shift vector \( N^m \) and a lapse function \( N \) as follows:

\[ h_{00} = -N^2 + \overline{h}_{mn}N^mN^n, \quad h_{0m} = \overline{h}_{mn}N^n, \quad h_{mn} = \overline{h}_{mn}, \]

where \( \overline{h}_{mn} \) is the intrinsic metric on \( \Sigma_3 \). The matrix \( (h^{\alpha\beta}) \) has the entries:

\[ h^{00} = -1/N^2, \quad h^{0m} = N^m/N^2, \quad h^{mn} = \overline{h}^{mn} - N^mN^n/N^2. \]

Using the above relations we get:

\[ h^{\alpha\beta} \gamma_{\alpha\beta} = - \frac{1}{N^2} (\gamma_{00} + N^mN^n\gamma_{mn} - 2N^m\gamma_{0m} + \overline{h}^{mn}\gamma_{mn}). \]

The action (2.3) in the comoving coordinates \( (N^m = 0) \)

\[ h_{mn} = \overline{h}_{mn} \]

and for the FRW metric:

\[ ds^2 = -N^2 dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + R^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] \]

results in:

\[ S = \frac{m_p^2}{2} \int_{\Sigma_3} \mu \int_{\mathbf{R}^1} d\alpha^3 N \left[ \frac{k}{a^2} - \left( \frac{\dot{a}}{Na} \right)^2 \right] - \frac{\Lambda T_3}{2} \int_{\mathbf{R}^1} dt \int_{\Sigma_3} d\mu a^3 e^{-\phi} N \left[ -\frac{\gamma_{00}}{N^2} + \frac{1}{a^2} h_{mn} \gamma_{mn} - 2\Lambda \right], \]

(2.4)

where

\[ d\mu = \frac{6R^2 \sin \theta}{\sqrt{1 - kr^2}} \sqrt{r} d\phi d\theta \]

and

\[ h_{mn} \gamma_{mn} = \left( 1 - kR^2 \right) \gamma_{RR} + \frac{1}{R^2} \gamma_{\theta \theta} + \frac{1}{R^2 \sin^2 \theta} \gamma_{\phi \phi}. \]

For our purpose we assume that \( \Sigma_3 \) has \( O(4) \) symmetry and the metric \( h_{mn} \) is the metric on \( S^3 \) so \( k = +1 \). Hence the action reads:

\[ S = 6\pi^2 m_p^2 \int_{\mathbf{R}^1} d\alpha^3 N \left[ \frac{1}{a^2} - \left( \frac{\dot{a}}{Na} \right)^2 \right] + 6\pi^2 \Lambda T_3 \int_{\mathbf{R}^1} d\alpha^3 e^{-\phi} N \left[ \frac{\gamma_{00}}{N^2} + 2\Lambda \right] - \frac{\Lambda T_3}{2} \int_{\mathbf{R}^1} d\alpha \int_{S^3} d\mu e^{-\phi} NT \gamma_{00} + \Lambda_{T_3} \gamma_0 H_{(p-5)/4} + L, \]

(2.5)

where

\[ \gamma_{00} = -H_{1/2} \left( -1 + X^t X_t \right) + H_{1/2}^2 X^t X_t, \]

\[ \gamma_{mn} = H_{p-1/2} \delta_{mn}, \]

\[ \gamma_{RR} = H_{1/2} \gamma_{\theta \theta} + \frac{1}{R^2} \gamma_{\phi \phi} \]

The function \( N(t) \) can be arbitrarily chosen by a redefinition of time thus we will use the gauge \( N = 1 \).

This system we put in the flat background produced by \( N \) coincident BPS \( Dp \)-branes. The metric \( g_{MN} \), the dilaton \( \phi \) and RR-field \( C \) for this background have the form [7]:

\[ ds_{10}^2 = g_{MN} dx^M dx^N = H_{p-1/2} \eta_{\mu \nu} dx^\mu dx^\nu + H_{p-1/2}^2 dx_I dx^I, \]

(2.7)

\[ (\mu, \nu = 0, 1, ..., p \text{ and } I = p + 1, ..., 9), \]

\[ e^{2\phi} = H_{(3-p)/2}, \]

(2.8)

\[ C = (H_p^{-1} - 1) dx^0 \wedge ... \wedge dx^p, \]

(2.9)

where \( H_p \) is the harmonic function of the transverse coordinates \( (X_I) \) to the world-volume:

\[ H_p = 1 + \frac{N_{ds}}{r^2 - p^2}, \quad \left( r = (X_I X^I)^{1/2} \right) \]

(2.10)

and \( (\eta_{\mu \nu}) = diag(-1, +1, ..., +1) \).

Here we consider backgrounds produced by \( Dp \)-branes with \( p > 3 \). We also use gauge freedom choice for an embedding field \( X \) in the form (so called static gauge):

\[ X = (t, x^1, ..., x^3, X^4(t), ..., X^9(t)). \]

(2.11)

In this gauge the part of action related to RR field \( C \) vanishes. The induced metric in this embedding is:

\[ \gamma_{00} = H_{p-1/2} \left( -1 + X^t X_t \right) + H_{p-1/2}^2 X^t X_t, \]

\[ \gamma_{mn} = H_{p-1/2} \delta_{mn}, \]

(2.12)

where \( i = 4, ..., p \). The integral on \( S^3 \) in the last term of (2.6) gives:

\[ \int_{S^3} d\mu e^{-\phi} N \gamma_0 Tr(\gamma) = 12\pi^2 \left( 1 + 2 + \frac{1}{2} \ln 2 \right) H_{p-5/4}. \]

Thus the considered action (2.6) takes the form:

\[ S = 6\pi^2 \int_{\mathbf{R}^1} dt \left[ m_p a^3 \left( \frac{1}{a^2} - \left( \frac{\dot{a}}{a} \right)^2 \right) \right] - \frac{\Lambda T_3}{2} \gamma_0 H_{(p-5)/4} + L, \]

(2.13)

where \( L \) is given by the formula:

\[ L = \Lambda T_3 a^3 \left[ -H_{p-5/4} \left( 1 - X^t X_t - H_p X^t X_t \right) + 2\Lambda \right]. \]
and $\gamma = 1 + 2 + \frac{1}{2} \ln 2 = 3.3466$. In the directions spanned by $X_i$ and $X_I$ we introduce spherical coordinates:

$$X_i = \varphi f_i, \quad X_I = r h_I,$$

where $f_i f^i = h_I h^I = 1$. In the generic case the considered system (the probed D3-brane) has a non-trivial angular momentum in the transverse directions ($X_i$) and non-trivial angular momentum in the directions transverse ($X_I$) to D3-brane but longitudinal to the background branes. For the simplicity, we consider the non-rotating case: $\dot{f}_i = \dot{h}_I = 0$. Thus, the eq. (2.13) takes the following form:

$$L = \Lambda T_3 a^3 \left[ -H_p^{(p-5)/4} \left( 1 - \dot{\varphi}^2 - \dot{H}_p \right)^2 + 2 \Lambda \right].$$

Then the three dynamic fields ($a, \varphi, r$) span minisuper-space, and the action is:

$$S = 6\pi^2 \int_{\mathbb{R}^1} dt \left[ -m_p^2 a^2 + \Lambda T_3 a^3 H_p^{(p-5)/4} \left( \dot{\varphi}^2 + \dot{H}_p \right)^2 - \tilde{U}(a, r) \right], \tag{2.14}$$

where a potential $\tilde{U}$ is given by:

$$\tilde{U}(a, r) = a^3 \Lambda T_3 \left( H_p^{(p-5)/4} - 2 \Lambda \right) - a \left( m_p^2 + \Lambda T_3 \gamma H_p^{(p-5)/4} \right).$$

The metric $\tilde{G}_{\Sigma \Phi}$ on the minisuperspace is taken from (2.14):

$$\left( \tilde{G}_{\Sigma \Phi} \right) = \begin{pmatrix} -m_p^2 a & 0 & 0 \\ 0 & \Lambda T_3 a^3 H_p^{(p-5)/4} & 0 \\ 0 & 0 & \Lambda T_3 a^3 H_p^{(p-1)/4} \end{pmatrix}.$$ 

(2.16)

The equation of motion for $\varphi$ gives the following relation:

$$\frac{d}{dt} \left[ a^3 H_p^{(p-5)/4} \varphi \right] = 0.$$ 

In this way one obtains:

$$\dot{\varphi} = Ja^{-3} H_p^{(p-5)/4},$$

where $J$ is a constant. Thus eq. (2.14) is:

$$S = 6\pi^2 \int_{\mathbb{R}^1} dt \left[ -m_p^2 a^2 + \Lambda T_3 a^3 H_p^{(p-1)/4} \varphi^2 - U(a, r) \right], \tag{2.17}$$

and a potential $U$ is given by:

$$U(a, r) = a^3 \Lambda T_3 \left( H_p^{(p-5)/4} - 2 \Lambda \right) - a \left( m_p^2 + \Lambda T_3 \gamma H_p^{(p-5)/4} \right) + J^2 a^{-3} H_p^{(5-p)/4}. \tag{2.18}$$

Now, the mini-superspace is reduced and spanned by $a$ and $r$. The metric $G_{\Sigma \Phi}$ on this space is:

$$\left( G_{\Sigma \Phi} \right) = \begin{pmatrix} -m_p^2 a & \Lambda T_3 a^3 H_p^{(p-1)/4} \\ \Lambda T_3 a^3 H_p^{(p-1)/4} & \end{pmatrix}. \tag{2.19}$$

In the case when $r$ is fixed, the equations for $a$ are obtained from the Hamiltonian constraint and from the equation of motion for (2.17). These equations are given by:

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{\Lambda T_3}{m_p^2} \left( H_p^{(p-5)/4} - 2 \Lambda \right) - \frac{1}{a^2} \left( 1 + \frac{\Lambda T_3 \gamma}{m_p^2} H_p^{(p-5)/4} \right) + \frac{1}{a^6} J^2 H_p^{(5-p)/4}, \tag{2.20}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6m_p^2} \left( \Lambda T_3 \left( 2 \Lambda - H_p^{(p-5)/4} \right) + \frac{2}{a^6} J^2 H_p^{(5-p)/4} \right). \tag{2.21}$$

If one compares these equations to the standard Friedmann equations for a perfect fluid with an energy density $\rho$, a pressure $p$ and a curvature parameter $k$:

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{3m_p^2} \rho - \frac{k}{a^2}, \tag{2.22}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6m_p^2} \left( \rho + 3p \right), \tag{2.23}$$

then one can notice that the energy density $\rho$ is given by:

$$\rho = 3\Lambda T_3 \left( H_p^{(p-5)/4} - 2 \Lambda \right) + \frac{3}{a^6} J^2 H_p^{(5-p)/4}, \tag{2.24}$$

the pressure $p$ is expressed as follows:

$$p = -3\Lambda T_3 \left( H_p^{(p-5)/4} - 2 \Lambda \right) + \frac{3}{a^6} J^2 H_p^{(5-p)/4} \tag{2.25}$$

and the curvature parameter $k$ is given by:

$$k = 1 + \frac{\Lambda T_3 \gamma}{m_p^2} H_p^{(p-5)/4}. \tag{2.26}$$

The first term on r.h.s in (2.24) can be interpreted as a cosmological constant $\Lambda$:

$$\lambda = 3\Lambda T_3 \left( H_p^{(p-5)/4} - 2 \Lambda \right). \tag{2.27}$$

The state equation for this perfect fluid has the form:

$$w = \frac{p}{\rho} = -\left( 1 - \frac{2J^2 H_p^{(5-p)/4}}{\Lambda T_3 \left( H_p^{(p-5)/4} - 2 \Lambda \right) a^6 + J^2 H_p^{(5-p)/4}} \right). \tag{2.28}$$

Hence for $a \to \infty$ one obtains the following state equation:

$$w = \frac{p}{\rho} = -1. \tag{2.29}$$

Thus, for big $a$ the worldvolume of D3-brane will be dominated by the perfect fluid with the negative pressure and
the observer fixed to this worldvolume will see an accelerated expansion.

Let us notice that for \( p = 5 \) the action (2.17) is:

\[
S = 6\pi^2 \int_{\mathbb{R}^1} dt \left[ -m^2_p a^2 - \Lambda T_3 a^3 H_5 r^2 + a \left( m^2_p + \Lambda T_3 \sigma \right) - J^2 a^{-3} - a^3 \Lambda T_3 \left( 1 - 2\Lambda \right) \right],
\]

(2.30)

where

\[
H_5 = 1 + \frac{N g_s}{r^2}.
\]

If one replaces the field \( r \) by the following field \( \varphi \):

\[
\varphi = \sqrt{N g_s} + r^2 - N g_s \ln \left( \frac{N g_s + \sqrt{N g_s + r^2}}{r} \right),
\]

(2.31)

then the action takes the form:

\[
S = 6\pi^2 \int_{\mathbb{R}^1} dt \left[ -m^2_p a^2 - \Lambda T_3 a^3 \varphi^2 + a \left( m^2_p + \Lambda T_3 \gamma \right) - J^2 a^{-3} - a^3 \Lambda T_3 \left( 1 - 2\Lambda \right) \right].
\]

(2.32)

Thus for the background produced by 5-branes the action for 3-brane is reduced to the free scalar field \( \varphi \) coupled to the scale factor \( a \) with the potential \( v = -a \left( m^2_p + \Lambda T_3 \gamma \right) + J^2 a^{-3} + a^3 \Lambda T_3 \left( 1 - 2\Lambda \right) \). For \( J = 0 \) the action (2.32) is the action for the homogeneous and isotropic metric \( g \) coupled to the free scalar field with the cosmological constant \( \Lambda T_3 \left( 1 - 2\Lambda \right) \). One can see that the cosmological constant \( \Lambda \) is the function of intrinsic parameters describing 3-brane, namely the strength \( T_3 \) and the constant \( \Lambda \). Thus by a suitable choice of \( \Lambda \) one can obtain the real value (observed) of the cosmological constant. For \( p \neq 5 \) the cosmological constant is given by (2.27).

### III. THE WHEELER-DE WITT EQUATION

In the mini-superspace spanned by the fields \( Q = (Q_1, ..., Q_N) \) the general form of the Wheeler-De Witt (WD) equation has the form:

\[
-\frac{h^2}{2} \left[ \frac{\partial^2}{\partial a^2} - \frac{\gamma}{a} \frac{\partial}{\partial a} + \frac{\delta}{a^2} \right] \Psi + \frac{\nu (p-1)}{4H_p} \left( \frac{p + 3H_p^r}{H_p} - H_p^{''r} \right) \Psi + \frac{2m_p^2}{h^2} U_{eff} (a, r) \Psi = 0,
\]

(3.2)

where \( \gamma, \delta, \mu, \nu \) represent the factor ordering ambiguity and prime denotes derivative with respect to \( r \). Thus the WD equation is:

\[
\frac{d^2}{da^2} - \frac{\gamma}{a} \frac{d}{da} - \frac{\eta}{a^2} - \frac{2m_p^2}{h^2} a^2 - \frac{2m_p^2}{3h^2} a^4 \right] \Psi = 0,
\]

(3.3)

At the moment we assume that \( r \) is fixed and is considered as a parameter. Thus the potential \( U_{eff} \) and the wave function \( \Psi \) have fixed values for \( r \) so the eq. (3.2) results in:

\[
\frac{d^2}{da^2} - \frac{\gamma}{a} \frac{d}{da} - \frac{\eta}{a^2} - \frac{2m_p^2}{h^2} a^2 - \frac{2m_p^2}{3h^2} a^4 \right] \Psi = 0.
\]

(3.3)

where \( k, \lambda \) are given by (2.26-2.27) and \( \eta = -C - \delta + \nu D \) with \( C \) and \( D \) given by:

\[
C = \frac{2m_p^2}{h^2} J^2 H_p^{(5-p)/4}.
\]
The last coefficient $D$ has the form:

$$D (r) = - \frac{m_p^2 (p - 1)}{4 \Delta T_3 H_p^{(p+7)/4}} (p + 3) H_p' - 4 H_p H_p'' .$$


Then for the fixed value of $r_0$ the equation (3.7) becomes the Bessel equation:

$$\frac{d^2 F_0}{dz^2} + \frac{1}{z} \frac{d F_0}{dz} + \left( 1 - \frac{n^2}{z^2} \right) F_0 = 0. \quad (3.10)$$

The solution of (3.10) is given by:

$$F_0 (z) = \tilde{E} J_n (z) + \tilde{F} Y_n (z),$$

where $J_n$ and $Y_n$ are the Bessel functions of the first kind of the order $n$. Hence the wave function $\Psi$ obtained from (3.10) has the form:

$$\Psi (a; r_0) = a^{(1+\gamma)/2} \left[ E I_n \left( \frac{a^2}{l_{Pl}^2} \sqrt{\hbar^2 k^2 / 2} \right) + F K_n \left( \frac{a^2}{l_{Pl}^2} \sqrt{\hbar^2 k^2 / 2} \right) \right],$$

where $E$ and $F$ are constants, $I_n (z)$ and $K_n (z)$ are the modified Bessel functions of the first kind and the second kind. We introduced the Planck length $l_{Pl} = \hbar / m_P$. This wave function must satisfy given boundary conditions. It has to be regular everywhere, so using expansion of $I_n$ and $K_n$ near zero we obtain:

$$1 + \gamma \geq \sqrt{4 \eta + (\gamma + 1)^2}.$$  

It means that $4 \eta \leq 0$, so in explicit form we get:

$$- \frac{2m_P^2}{\hbar^2 H_p^{(p-5)/4} (r_0)} J^2 - \delta + \nu D (r_0) \leq 0, \quad (3.13)$$

where $D (r_0)$ is given by the equation (3.4). From this relation one can, in principle, determine the allowed positions of D3-brane in the background for which the wave function of the D3-brane is given by (3.12). For $z \to i \infty$ the asymptotics for $I_n$ and $K_n$ are following:

$$I_n (z) \sim \frac{1}{\sqrt{2 \pi z}} \exp (z),$$

$$K_n (z) \sim \sqrt{\frac{\pi}{2z}} \exp (-z).$$

Thus the wave function (3.12) for big $a$ reads:

$$\Psi (a; r_0) \sim \frac{l_{Pl}^2}{2} \left[ a \Gamma \left( \gamma + 1 \right) \Gamma \left( \gamma - 1 \right) \exp \left( \frac{a^2}{l_{Pl}^2} \sqrt{\hbar^2 k^2 / 2} \right) + \exp \left( -\frac{a^2}{l_{Pl}^2} \sqrt{\hbar^2 k^2 / 2} \right) \right].$$

The effects of the non-trivial D-brane background are included in $k$ and in the order $n$ of the Bessel functions. The regularity of the wave function near zero puts the constraint (3.13) on the positions of the D3-brane in the background. The predictions following from this wave function are well-known and broadly discussed with respect to the decay of a false vacuum [9].
The second exact solution of (3.6) is obtained for $s = (\gamma - 1)/4$ and $n^2 = 0$. In this case (3.6) takes the form:

$$
\frac{d^2 F}{dz^2} + (1 + mz) F = 0.
$$

(3.15)

It is then transformed to the Airy equation:

$$
\frac{d^2 \Phi}{dt^2} + \frac{1}{m^2} t \Phi = 0,
$$

(3.16)

where $t = 1 + mz$ and $F(z) = \Phi(1 + mz)$. The solutions $\Phi$ of the Airy equation depend on the sign of $m^2$. For $m^2 > 0$ and for $m^2 < 0$ the solutions are given by:

$$
\Phi_+ (t) = \frac{3}{\sqrt[4]{t}} \left[ A' J_{-1/3} \left( \frac{2}{3|m|} t^{3/2} \right) + B' J_{1/3} \left( \frac{2}{3|m|} t^{3/2} \right) \right],
$$

(3.17)

$$
\Phi_- (t) = \frac{3}{\sqrt[4]{t}} \left[ A I_{-1/3} \left( \frac{2}{3|m|} t^{3/2} \right) + B I_{1/3} \left( \frac{2}{3|m|} t^{3/2} \right) \right],
$$

(3.18)

respectively. Since $m^2 = - \left( \frac{l_{Pl}^2}{m^3} \right)^2 < 0$ the solution will be given by the second relation. Finally the wave function $\Psi$ has the form:

$$
\Psi (a; r_0) = e^{i ms} \frac{1}{\sqrt{l_{Pl}^2}} \left( \frac{h^2 k}{2} \right)^{r_0} a^{2s} \sqrt{1 - \frac{\lambda^4}{3h^2 k l_{Pl}^2}} \times
$$

$$
\times \left[ A I_{-1/3} \left( \frac{4}{l_{Pl}^2 \lambda} \left( \frac{h^2 k}{2} \right)^{3/2} \left( 1 - \frac{\lambda l_{Pl}^2 a^2}{3h^2 k l_{Pl}^2} \right)^{3/2} \right) + B I_{1/3} \left( \frac{4}{l_{Pl}^2 \lambda} \left( \frac{h^2 k}{2} \right)^{3/2} \left( 1 - \frac{\lambda l_{Pl}^2 a^2}{3h^2 k l_{Pl}^2} \right)^{3/2} \right) \right],
$$

(3.19)

and $s = (\gamma - 1)/4$. This wave function is valid only for $n^2 = 0$:

$$
\eta + (\gamma - 1) (\gamma + 3)/4 = 0.
$$

(3.20)

The above condition fixed the position $r_0$ of D3-brane in the background because $\eta$ is given by $H_p$ (see eqs. (3.4) and (3.5)). The wave function (3.19) has the oscillating character for $3h^2 k / (\lambda l_{Pl}^2) < a^2$ and for all $a \geq 0$ is regular. The (3.19) has the form of the Hartle-Hawking wave function for empty universe with the cosmological constant [11]. It is as should be since the condition (3.20) implies that the worldvolume is empty with the cosmological constant $\lambda$.

Thus we have obtained two particular solutions of the eq. (3.6) which correspond to the fixed positions of the D3-brane in the background. The first solution (3.12) is obtained for vanishing of the cosmological constant $\lambda$ on the worldvolume which is filled by the mirage fluid with the energy density $3J^2 H^{(5-p)/4}/a^6$. This solution is valid only in the region where the condition (3.13) holds. The second solution (3.19) is obtained for the position of D3-brane fixed by (3.20) and corresponds to the empty worldvolume with the cosmological constant $\lambda$.

Next we consider the tunneling probability in the potential (3.2a) for a fixed position $r_0$ of D3-brane with $m, n \neq 0$. In order to do this we choose $s = (\gamma - 1)/4$ in (3.6). Thus we get the one dimensional equation:

$$
\frac{d^2 F}{dz^2} + \left( 1 - \frac{n^2}{z^2} + mz \right) F = 0,
$$

(3.21)

where:

$$
n^2 = \eta/4 + (\gamma - 1) (\gamma + 3)/16.
$$

(3.22)

This equation can be considered as a Schrödinger equation in a potential $V(z)$ with a zero eigenvalue:

$$
\frac{d^2 F}{dz^2} - V(z) F = 0,
$$

(3.23)

where $V(z)$ is given by:

$$
V(z) = -1 + \frac{n^2}{z^2} - mz.
$$

(3.24)

For the real variable $y = - \frac{n^2 l_{Pl}^2}{a^2} \sqrt{h^2 k}/2 < 0$ so that $z = iy$ the potential $V$ takes the form

$$
V(y) = -1 - \frac{n^2}{y^2} + x n^2 \left( \frac{2}{h^2 k} \right)^{3/2} y,
$$

(3.25)

where $x = l_{Pl}^2 \lambda / (6n^2)$ and $l_{Pl}$ is the Planck length. This potential is sketched below:

The variable $y$ has physical meaning only for $y < 0$. The potential $V$ has a maximum for

$$
y_{\text{max}} = - \left( \frac{2}{x} \right)^{1/3} \left( \frac{h^2 k}{2} \right)^{1/2}
$$

with the value:

$$
V_{\text{max}} = -1 - 3n^2 \left( \frac{x}{2} \right)^{2/3} \frac{2}{h^2 k}.
$$

(3.26)
Thus, an integral the Planck length from the region where value of the scale factor takes the form:

\[ \int_0^{y_2} \sqrt{-E + V(y)} \, dy, \]

where the integration limits \( y_3, y_2 \) are given by the solution of the equation:

\[ E - V(y) = 0. \]

The roots \( y_3, y_2 \) are negative and ordered as follows: \( y_3 < y_2 < 0 \). Hence for \( E < V_{\text{max}} \leq 0 \) we obtain that:

\[ E - V(y) < 0. \]

Thus, an integral \( I = \int_{y_3}^{y_2} \sqrt{-E + V(y)} \, dy \) is real and has the form:

\[ I = n \sqrt{x} \left( \frac{2}{\hbar^2 k} \right)^{3/4} \int_{y_3}^{y_2} \frac{dy}{y} \sqrt{f(y)}, \]

a function \( f \) is given by a cubic polynomial:

\[ f(y) = y^3 + \sigma y^2 - w, \]

where \( \sigma = -w(E + 1)/n^2 \) and \( w = (\hbar^2 k/2)^{3/2}/x \). Now we have to determine the integration limits \( y_3, y_2 \) which depend on \( E \). We make the reasonable assumption (in the quasi-classical approximation) that the minimal value of the scale factor \( a \) on the worldvolume is given by the Planck length \( l_P = \hbar/m_P \). This minimal value corresponds to \( y_P = -\hbar \sqrt{k}/2 \). At this point the potential \( V \) has the value:

\[ V(y_P) = -1 - 2n^2 1 + x \frac{1}{\hbar^2 k}, \]

and \( V_{\text{max}} > V(y_P) \). The coefficient \( \sigma \) for \( E = V(y_P) \) takes the form:

\[ \sigma = \frac{\hbar^2 k}{2} (1 + \frac{1}{x}). \]

We evaluate \( \Gamma \) near the energy \( E = V(y_P) \). Thus:

\[ \sqrt{-E + V(y)} = \sqrt{-V(y_P) + V(y)} = n \sqrt{x} \left( \frac{2}{\hbar^2 k} \right)^{3/4} \frac{1}{y} \sqrt{f(y)}. \]

It means that the cubic polynomial \( f(y) \) has a root for \( y = y_P < 0 \) and the bottom integration limit is \( y_2 = y_P \). The other roots are given by:

\[ y_1 = \frac{1}{2x} \sqrt{\frac{\hbar^2 k}{2}} \left(-1 + \sqrt{1 + 4x} \right), \]

\[ y_3 = -\frac{1}{2x} \sqrt{\frac{\hbar^2 k}{2}} \left(1 + \sqrt{1 + 4x} \right) \]

and they are ordered in the following way \( y_3 < y_P < 0 \) and \( y_1 > 0 \). The upper integration limit is \( y_3 \) and the polynomial \( f = (y - y_3)(y - y_P)(y - y_1) \) is positive for \( y \in (y_3, y_P) \). The barrier width \( b \) which is equal to \( y_P - y_3 \):

\[ b = \sqrt{\frac{\hbar^2 k}{2}} \left( -1 + \frac{1 + \sqrt{1 + 4x} }{2x} \right) \]

is greater then zero for \( x \in (0, 2) \) and vanishes for \( x = 2 \).

The integral (3.28) is the elliptic integral and is expressed as follows:

\[ I = n \sqrt{x} \left( \frac{2}{\hbar^2 k} \right)^{3/4} \left( \frac{\sigma}{3} I_1 - w I_{-1} \right), \]

where:

\[ I_s = \int_{y_P}^{y_3} \frac{y^s}{\sqrt{f(y)}} dy \]

and \( s = -1, 1 \). Making the standard substitution \( y = y_3 + (y_P - y_3) \sin^2 \phi \) in the integrals \( I_s \) we obtain:

\[ I_1 = \frac{2 y_1}{\sqrt{y_1 - y_3}} K(k) + 2 \sqrt{y_1 - y_3} E(k), \]

\[ I_{-1} = -\frac{2}{y_3 \sqrt{y_1 - y_3}} \Pi(c, k), \]

where \( K(k), E(k) \) and \( \Pi(c, k) \) are the first kind, the second kind and the third kind complete elliptic integrals, respectively (see Appendix). The modulus \( c \) and \( \kappa \) are given by:

\[ \kappa = \frac{y_P - y_3}{y_1 - y_3} = \frac{1}{2} \left( 1 + \frac{1}{\sqrt{1 + 4x}} \right), \]

\[ c = \frac{y_P - y_3}{y_3} = \frac{1}{2} \left( 3 - \sqrt{1 + 4x} \right) > 0. \]

They have the range: \( 0 < \kappa < 1 \) and \( 0 < c < 1 \) for \( x \in (0, 2) \). Thus the integral \( I \) takes the form:

\[ I = n \sqrt{x} \left( \frac{2}{\hbar^2 k} \right)^{3/4} \left( -\frac{2\sigma y_1}{3 \sqrt{y_1}} K(k) + \frac{2\sigma \sqrt{y_1}}{3} E(k) + \frac{2w}{y_3 \sqrt{y_1 - y_3}} \Pi(c, k) \right). \]
where $y_{31} = y_1 - y_3 > 0$ and $y_{32} = y_2 - y_3 > 0$. Collecting the above formulas we get the function $I$ in the variables $x$, and $n$:

$$I(x, n) = \frac{n}{6x(1 + 4x)^{1/4}} \left[ 4\sqrt{1 + 4x} E(\kappa) + (1 - \sqrt{1 + 4x}) K(\kappa) \right].$$

(3.43)

This function tends to $\infty$ if $x \to 0$ and is decreasing function of $x$ with the minimum for $x = 2$

$$I(2, n) = \frac{\pi n}{4\sqrt{2}}.$$  

(3.44)

For the case $x = 2$ the barrier width $b$ vanishes and $n^2 = l_p^2 \lambda/12$. The maximum of the tunneling probability $\Gamma$ is given by:

$$\Gamma_{\text{max}} = \exp(-\frac{2}{\hbar} I(2, n)) = \exp\left(-\frac{\pi n}{2\sqrt{2n}}\right).$$ 

(3.45)

Our result agrees with [12] in the sense that maximum of the probability is achieved in the point where the barrier potential vanishes. Moreover $n$ is the function of $r$ and parameters related to the ordering (see eqs. (3.22), (3.4) and (3.5)). Thus $\Gamma_{\text{max}}$ takes the maximum for $n = 0$ with the cosmological constant $\lambda = 0$ on the world-volume. This case corresponds to the position $r_{\text{max}}$ for which:

$$n(r_{\text{max}}) = 0.$$

Thus $r_{\text{max}}$ gives the boundary value of the condition (3.13). In this position the D3-brane has the wave function given by (3.12).

IV. CONCLUSIONS

We have considered the gravity on the worldvolume of D3-brane embedded in the flat background produced by $N$ $p$-branes. This system is described by the sum of the Hilbert-Einstein and DBI actions. Although the DBI action is non-linear we have used equivalent form for it with an auxiliary metric. The obtained wave functions depend on the ordering of the conjugated variables. In general, the wave function depends on $a$ and $r$ but we can not find it in the explicit form. Even for fixed $r$ we get an equation which has not explicit solutions and is similar to the equation considered in the Stark effect. The perturbation method used in the Stark effect is inappropriate because we do not know which parameter is small in eq. (3.8). We made the assumption that the minimal value of the scale factor $a$ of the worldvolume is given by the Planck length and for this value we evaluated tunneling probability in the potential $V$. This probability is finite and depends on $\lambda$ and $r$. The maximum of the probability is for $x = 2$ and the cosmological constant is given by $\lambda = 12n^2l_p^{-4}$.

The considered model has the common features with a hydrogen atom in an external electric field. In our model the external field is represented by the background produced by D$p$-branes and the atom is represented by D3-brane. The tunneling from Planckian size of D3-brane to classical regime corresponds to the ionization of the hydrogen atom.

The above model is very simple and not quite realistic for several reasons. The first one is that we used FRW ansatz for the metric. Validity of this metric was extended to the pre-inflation epoch, when quantum gravity has dominating effects. In this regime the universe was not necessary isotropic and homogenous. The better approximation for the metric is IX Bianchi type. This case were considered in [13] for the empty universe with a cosmological constant. The second reason is that the stringy effects enter by the DBI action. This action is valid only on the tree level of the string theory and does not take into account backreactions. The other reasons are related to the special backgrounds which are classical solutions of the supergravity equations. The more general approach is if one consider the Wheeler-De Witt equation for the low-energetic approximation of string theory by supergravity and next compactificate this equation to 4-dimensions taking into account stringy corrections.

In the classical regime compactification to 4-dimensions for the Bianchi type-I cosmology in the presence the gravity, the dilaton, and the antisymmetric tensor field of the second rank, coupling to the gauge field strength living on the D3-brane was considered in [14]. It would be interesting to consider the quantum regime of this system, described by the Wheeler-DeWitt equation, and compare the results with the results of the model considered in this paper. We shall investigate this problem in the future.
V. APPENDIX

The complete elliptic integrals first kind $K$, second kind $E$ and third kind $\Pi$ are defined as follows [15]:

$$K(\kappa) = \int_{0}^{\pi/2} \frac{d\phi}{\sqrt{1 - \kappa \sin^2 \phi}},$$

$$E(\kappa) = \int_{0}^{\pi/2} \sqrt{1 - \kappa \sin^2 \phi} d\phi,$$

$$\Pi(c, \kappa) = \int_{0}^{\pi/2} \frac{d\phi}{(1 - c \sin^2 \phi) \sqrt{1 - \kappa \sin^2 \phi}}.$$

For $c = \kappa$ the elliptic integral $\Pi$ is expressed by $E$:

$$\Pi(\kappa, \kappa) = \frac{E(\kappa)}{1 - \kappa}.$$

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