On the stability of time-discrete dynamic multiple network poroelasticity systems arising from second-order implicit time-stepping schemes

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Abstract

The classical Biot’s theory provides the foundation of a fully dynamic poroelasticity model describing the propagation of elastic waves in fluid-saturated media. Multiple network poroelastic theory (MPET) takes into account that the elastic matrix (solid) can be permeated by one or several ($n \geq 1$) superimposed interacting single fluid networks of possibly different characteristics; hence the single network (classical Biot) model can be considered as a special case of the MPET model.

We analyze the stability properties of the time-discrete systems arising from second-order implicit time stepping schemes applied to the variational formulation of the MPET model and prove an inf-sup condition with a constant that is independent of all model parameters. Moreover, we show that the fully discrete models obtained for a family of strongly conservative space discretizations are also uniformly stable with respect to the spatial discretization parameter. The norms in which these results hold are the basis for parameter-robust preconditioners.

Keywords— Fully dynamic Biot model, multiple network poroelastic theory, MPET equations, second-order implicit time stepping scheme, inf-sup stability, parameter-robust preconditioners

1 Fully dynamic poroelasticity models: The continuous case

In this section we formulate the continuous dynamic models whose stable and mass-conservative discretization we will address in the further course of this work.

1.1 The dynamic Biot model

Let us start with the single network model, which we will also refer to as dynamic Biot problem. For an open domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, the unknown physical variables in the dynamic Biot problem we are going to consider are the displacement $u$ of the solid matrix occupying $\Omega$, the relative displacement $w := \varphi(v - u)$ of the fluid, denoting by $v$ the displacement of the fluid and by $\varphi \in (0, 1)$ the porosity of the solid$^1$, and the fluid pressure $p$, cf. [1].

In the regime of linear elasticity (assuming Hook’s law) we have the relations

$$
\begin{align*}
\sigma(u) &= 2\mu \epsilon(u) + \lambda \text{div}(u) I, \\
\epsilon(u) &= \frac{1}{2} \left( \nabla u + (\nabla u)^T \right)
\end{align*}
$$

(1a)

(1b)

between the total stress $\sigma = \sigma(u)$, the strain $\epsilon = \epsilon(u)$ and the displacement field $u$. Defining the total density $\rho$ of the fluid-saturated porous medium by

$$
\rho := \varphi \rho_f + (1 - \varphi) \rho_s
$$

(2)

$^1$According to [1] the porosity $\varphi$ is defined as $\varphi = \frac{V_p}{V_b}$ where $V_p$ is the volume of the pores contained in a sample of bulk volume $V_b$. 

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in terms of the fluid density \( \rho_f \) and the solid density \( \rho_s \), the first equation of motion reads

\[
- \text{div} \mathbf{\sigma} + \rho_f \ddot{\mathbf{u}} + \rho_f \mathbf{\dot{w}} + \alpha \nabla p = \mathbf{f}, \quad \text{in } \Omega \times (0, T)
\]  

(3)

where \( \alpha \in [\varphi, 1] \) denotes the Biot-Willis parameter\(^2\) and \( \mathbf{f} \) the body force density, cf. [2].

The second equation of motion, describing the momentum balance of the fluid component, is given by

\[
\rho_f \mathbf{\dot{u}} + \rho_m \mathbf{\dot{w}} + \mathbf{K}^{-1} \mathbf{w} + \nabla p = -\mathbf{\dot{f}}, \quad \text{in } \Omega \times (0, T)
\]  

(4)

where \( \rho_m := \rho_f / \varphi \) is the effective fluid density, \( \mathbf{K} := \kappa / \eta \) the hydraulic conductivity of the medium for a fluid with viscosity \( \eta \), and \( \kappa \) the permeability tensor, which for simplicity here will be assumed to be of the form \( \kappa = \kappa I \) for a scalar permeability coefficient \( \kappa \).\(^3\) If the only body forces are due to gravity then the total body force and fluid body force are given by \( \mathbf{f} = \rho \mathbf{b} \) and \( \mathbf{\dot{f}} = \rho_f \mathbf{b} \) with \( \mathbf{b} \) denoting the gravitational acceleration.

The system is closed by the mass conservation equation

\[
- \alpha \text{div} \mathbf{u} - \text{div} \mathbf{w} - c_p \dot{p} = 0, \quad \text{in } \Omega \times (0, T)
\]  

(5)

where \( c_p \) is the constrained specific storage coefficient. Note that if both the elastic solid and the fluid are incompressible one has \( c_p = 0 \), cf [4], a situation which is also covered by the analysis presented in section 4 which provides stability in this case as well.

For symmetry reasons, it is convenient to transform the equations (3)–(5) into an equivalent system which after variational formulation produces a saddle point problem in each step of an implicit time integration method. This can be achieved by first inserting the right hand side of the definition \( \mathbf{w} := \varphi (\mathbf{v} - \mathbf{u}) \) of \( \mathbf{w} \) in (3)–(5), then multiplying equation (4) with \( -\varphi \) and adding it to equation (3) to obtain a new equation replacing (3). Finally, substituting \( \varphi \mathbf{v} \) with \( \mathbf{v} \) and denoting \( 0 \leq \alpha := \tilde{\alpha} - \varphi \) one ends up with the new system

\[
- \text{div} \mathbf{\sigma} + ((1 - \varphi) \rho_s - \varphi \rho_f + \varphi^2 \rho_m) \mathbf{\dot{u}} + \varphi^2 \mathbf{K}^{-1} \mathbf{\dot{u}} + (\rho_f - \varphi \rho_m) \mathbf{\dot{w}} - \varphi \mathbf{K}^{-1} \mathbf{\dot{w}} + \alpha \nabla p = \mathbf{f},
\]

(6a)

\[
(\rho_f - \varphi \rho_m) \mathbf{\dot{u}} - \varphi \mathbf{K}^{-1} \mathbf{\dot{u}} + \rho_m \mathbf{\dot{w}} + \mathbf{K}^{-1} \mathbf{\dot{w}} + \nabla p = \mathbf{g},
\]

(6b)

\[
- \alpha \text{div} \mathbf{u} - \text{div} \mathbf{w} - c_p \dot{p} = 0.
\]

(6c)

where we have also used the notation \( \mathbf{f} := \mathbf{\dot{f}} + \varphi \mathbf{\dot{f}} \) and \( \mathbf{g} := -\mathbf{\dot{f}} \).

The dynamic Biot problem (6) has to be complemented by proper initial conditions at time \( t = t_0 \), e.g., prescribing \( \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^{(0)}(\mathbf{x}), \mathbf{v}(\mathbf{x}, 0) = \mathbf{v}^{(0)}(\mathbf{x}), p(\mathbf{x}, 0) = p^{(0)}(\mathbf{x}), \mathbf{\dot{u}}(\mathbf{x}, 0) = \mathbf{u}^{(1)}(\mathbf{x}), \mathbf{\dot{v}}(\mathbf{x}, 0) = \mathbf{v}^{(1)}(\mathbf{x}) \) at time \( t_0 = 0 \) as well as proper boundary conditions at any time \( t > t_0 \), e.g.,

\[
\begin{align*}
p(\mathbf{x}, t) &= p_D(\mathbf{x}, t) \quad &\text{for } &\mathbf{x} \in \Gamma_{p,D}, & t &> 0, \\
K \frac{\partial p(\mathbf{x}, t)}{\partial n} &= q_N(\mathbf{x}, t) \quad &\text{for } &\mathbf{x} \in \Gamma_{p,N}, & t &> 0, \\
\mathbf{u}(\mathbf{x}, t) &= \mathbf{u}_D(\mathbf{x}, t) \quad &\text{for } &\mathbf{x} \in \Gamma_{u,D}, & t &> 0, \\
(\mathbf{\sigma}(\mathbf{x}, t) - \alpha p I) \mathbf{n}(\mathbf{x}) &= \mathbf{g}_N(\mathbf{x}, t) \quad &\text{for } &\mathbf{x} \in \Gamma_{u,N}, & t &> 0,
\end{align*}
\]  

(7a–7d)

where \( \Gamma_{p,D} \cap \Gamma_{p,N} = \emptyset, \Gamma_{p,D} \cup \Gamma_{p,N} = \Gamma = \partial \Omega \) and \( \Gamma_{u,D} \cap \Gamma_{u,N} = \emptyset, \Gamma_{u,D} \cup \Gamma_{u,N} = \Gamma \). A more detailed derivation of the system (6) and some fundamental results regarding its well-posedness can be found in [5, 6, 7, 8].

In compact notation, the system (6a)–(6c) can be written in the form

\[
\mathcal{M} \ddot{\mathbf{y}} + \mathcal{D} \dot{\mathbf{y}} + \mathcal{L} \mathbf{y} = \mathbf{F}
\]  

(8)

\(^2\)For physical reasons it is natural to assume that \( \varphi \leq \tilde{\alpha} \leq 1 \), cf. [2].

\(^3\)The latter assumption is valid for isotropic porous media, cf. [3].
with operators $\mathcal{M}$, $\mathcal{D}$, $\mathcal{L}$, right hand side $\mathcal{F}$, and unknown vector $\mathbf{y}$ given by

$$
\mathcal{M} = \begin{bmatrix}
(1 - \varphi) \rho_s - \varphi \rho_f + \varphi^2 \rho_m & (\rho_f - \varphi \rho_m) I \\
(\rho_f - \varphi \rho_m) I & \rho_m I
\end{bmatrix},
$$

$$
\mathcal{D} = \begin{bmatrix}
\varphi^2 K^{-1} & -\varphi K^{-1} & 0 \\
-\varphi K^{-1} & K^{-1} & 0 \\
-\alpha \text{div} & -\text{div} & -c_p I
\end{bmatrix},
$$

$$
\mathcal{L} = \begin{bmatrix}
-2 \mu \text{div} - \lambda \nabla \text{div} & 0 & \alpha \nabla \\
0 & 0 & \nabla \\
0 & 0 & 0
\end{bmatrix},
$$

and

$$
\mathcal{F} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} u \\ v \\ p \end{bmatrix}.
$$

Many problems in structural dynamics can be represented in the abstract form (8). Note that in case of the dynamic Biot model $\mathcal{M} + \mathcal{L} + \mathcal{D}$ is a self-adjoint and invertible linear operator with an inverse $(\mathcal{M} + \mathcal{L} + \mathcal{D})^{-1}$ defined on the dual space $W^* := U^* \times U^* \times P^*$ of an appropriate product space $W := U \times U \times P$. However, the operators $\mathcal{M} : W \to W^*$ and $\mathcal{L} : W \to W^*$ are not invertible individually. This is the reason why many popular standard implicit time integration schemes, for instance the Crank Nicolson method, cf [9], which requires the invertibility of $\mathcal{M}$, can not be applied straightforwardly. However, further refined/combined methods have already been considered in the present context as early as in [10]. Before we will also address this issue we will generalize the system (6) in order to present the dynamic MPET model as subject of further discussions.

### 1.2 The dynamic MPET model

A basic assumption in the MPET model is that the elastic solid matrix is permeated by $n \geq 1$ fluid networks each of which being described by its individual fluid displacement $\mathbf{v}_i$, relative fluid displacement $\mathbf{w}_i = \varphi_i (\mathbf{v}_i - \mathbf{u})$ and fluid pressure $p_i$, where $\varphi_i$ denotes the porosity of the solid induced by the $i$-th network. For consistency reasons we may assume that

$$
\sum_{i=1}^{n} \varphi_i = \varphi \in (0, 1) \quad \text{and} \quad \varphi_i \in (0, 1) \quad \text{for all } i = 1, 2, \ldots, n.
$$

The system of two momentum and one mass balance equations for $n$ fluid networks reads

$$
-\text{div} \mathbf{\sigma} + \rho \ddot{\mathbf{u}} + \sum_{i=1}^{n} \rho_i \dot{\mathbf{w}}_i + \sum_{i=1}^{n} \alpha_i \nabla p_i = \mathbf{f}, \quad \text{in } \Omega \times (0,T),
$$

$$
\rho_i \ddot{\mathbf{u}} + \rho_m \dot{\mathbf{w}}_i + K_i^{-1} \dot{\mathbf{w}}_i + \nabla p_i = -\mathbf{f}_i, \quad \text{in } \Omega \times (0,T), \quad \text{for all } i = 1, \ldots, n,
$$

$$
-\alpha_i \text{div} \mathbf{u} - \text{div} \mathbf{w}_i - c_p \dot{\mathbf{p}}_i - \sum_{j=1}^{n} \beta_{ij} (p_i - p_j) = 0, \quad \text{in } \Omega \times (0,T), \quad \text{for all } i = 1, \ldots, n,
$$

herewith generalizing (3)–(5), where $\rho := \sum_{i=1}^{n} \varphi_i \rho_i + (1 - \varphi) \rho_s$ again denotes the total density and $\mathbf{f}$ the body force; The mass densities of the solid and the $i$-th fluid component are denoted by $\rho_s$ and $\rho_i$, respectively, the Biot-Willis parameter of the $i$-th network by $\alpha_i \in [\varphi_i, 1]$. Further, $\mathbf{f}_i$ is the body force associated with the $i$-th fluid compartment. Moreover, each fluid is characterized by its effective density $\rho_m \geq \frac{\alpha_i}{1 - \varphi}$, cf [4], and viscosity $\eta_i$ resulting in a hydraulic conductivity $K_i := \kappa_i/\eta_i$ of the $i$-th network, where $\kappa_i$ denotes its permeability.

Note that the additional term $\sum_{j \neq i} \beta_{ij} (p_i - p_j)$ in (11c) models mass exchange between the networks due to pressure differences, cf [11][12].
Applying a symmetrization procedure analogous to the one that has lead to (6) we obtain

\[-\text{div}\sigma + ((1 - \varphi)\rho_s - \sum_{i=1}^{n} \varphi_i \rho_i + \sum_{i=1}^{n} \varphi_i^2 \rho_{m_i}) \hat{u} + \sum_{i=1}^{n} \varphi_i^2 K_i^{-1} \hat{u} + \sum_{i=1}^{n} ((\rho_i - \varphi_i \rho_{m_i}) \hat{v}_i - \varphi_i K_i^{-1} \hat{v}_i) + \sum_{i=1}^{n} \alpha_i \nabla p_i = \mathbf{f},\]

\[(p_i - \varphi_i \rho_{m_i}) \hat{u} - \varphi_i K_i^{-1} \hat{u} + \rho_{m_i} \hat{v}_i + K_i^{-1} \hat{v}_i + \nabla p_i = \mathbf{g}_i, \quad i = 1, \ldots, n,\]

\[-\alpha_i \text{div}\hat{u} - \text{div}\hat{v}_i - c_{pi} \hat{p}_i - \sum_{j=1}^{n} \beta_{ij} (p_i - p_j) = 0, \quad i = 1, \ldots, n,\]

where \(0 \leq \alpha_i := \bar{\alpha}_i - \varphi_i\) and \(\mathbf{f} := \tilde{\mathbf{f}} + \sum_{i=1}^{n} \varphi_i \tilde{\mathbf{f}}_i\) and \(\mathbf{g}_i := -\tilde{\mathbf{f}}_i\).

Again, the system (12) can be represented in the form (8) but now with operators \(\mathcal{M}, \mathcal{D}, \mathcal{L}\), right hand side \(\mathcal{F}\), and unknown vector \(\mathbf{y}\) given by

\[\mathcal{M} = \begin{bmatrix}
((1 - \varphi)\rho_s - \sum_{i=1}^{n} \varphi_i (\rho_i - \varphi_i \rho_{m_i})) I & (p_1 - \varphi_1 \rho_{m_1}) I & \cdots & (p_n - \varphi_n \rho_{m_n}) I & 0 & \cdots & 0 \\
(p_1 - \varphi_1 \rho_{m_1}) I & \rho_{m_1} I & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(p_n - \varphi_n \rho_{m_n}) I & 0 & \cdots & \rho_{m_n} I & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{bmatrix},\]

\[\mathcal{D} = \begin{bmatrix}
\sum_{i=1}^{n} \varphi_i^2 K_i^{-1} & -\varphi_i K_i^{-1} & \cdots & -\varphi_n K_n^{-1} & 0 & \cdots & 0 \\
-\varphi_1 K_1^{-1} & K_1^{-1} & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-\varphi_n K_n^{-1} & 0 & \cdots & \rho_{m_n} I & 0 & \cdots & 0 \\
-\alpha_1 \text{div} & -\text{div} & \cdots & 0 & -c_{p1} I & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-\alpha_n \text{div} & 0 & \cdots & -\text{div} & 0 & \cdots & -c_{pn} I
\end{bmatrix},\]

\[\mathcal{L} = \begin{bmatrix}
-2\mu \text{div} - \lambda \nabla \text{div} & 0 & \cdots & 0 & \alpha_1 \nabla & \cdots & \alpha_n \nabla \\
0 & 0 & \cdots & 0 & \nabla & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \nabla & \cdots & \nabla \\
0 & 0 & \cdots & 0 & -\beta_{11} I & \cdots & \beta_{1n} I \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \beta_{n1} I & \cdots & -\beta_{nn} I
\end{bmatrix},\]
Then, starting from known initial values for \( u \) and \( y \), we can approximate the vector of unknowns \( y \) for the unknowns \( v \) and \( p \) by solving an operator equation of the form

\[
\mathcal{F} = \begin{bmatrix} f \\ g_1 \\ \vdots \\ g_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad y = \begin{bmatrix} u \\ v_1 \\ \vdots \\ v_n \\ p_1 \\ \vdots \\ p_n \end{bmatrix}.
\]  

(14)

Note that, as before, \( (\mathcal{M} + \mathcal{L} + \mathcal{D}) \) is self-adjoint. In the next section, we will use (block) operators composed of submatrices of \( \mathcal{M}, \mathcal{L} \) and \( \mathcal{D} \). For this reason we define

\[
\mathcal{M} := \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}, \quad \mathcal{D} := \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix}, \quad \mathcal{L} := \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}
\]

(15)

where the three-by-three partitioning of \( \mathcal{M}, \mathcal{L} \) and \( \mathcal{D} \) corresponds with the partitioning of the unknown vector \( y \) into \( u, v := (v_1^T, \ldots, v_n^T)^T \) and \( p = (p_1, \ldots, p_n)^T \). The definition of \( M_{i,j}, L_{i,j} \) and \( D_{i,j} \) for \( 1 \leq i, j \leq 3 \) then follows from equating corresponding blocks in (15) and (13a)–(13c).

2 Discretization

In this section we present first a second order time discretization method for the dynamic MPET problem and then recall a family of mixed finite element methods for space discretization that provide mass conservation in a strong, that is, pointwise, sense.

2.1 Time discretization

To start with, consider the equation (8) with the operators \( \mathcal{M}, \mathcal{D}, \mathcal{L} \), right hand side \( \mathcal{F} \), and unknown vector \( y \) defined by (13) and (14). A second-order accurate implicit time integration method that can be represented in the form (8) is the Crank Nicolson method method. We want to use it in the present context in which we have to resolve the issue that the operator \( \mathcal{M} \) is not invertible.

Let us consider a time interval \( [0, T] \), for simplicity partitioned into \( n \) equidistant subintervals of length \( \tau \), i.e., \( \tau = T/n \). Then, starting from known initial values for \( u, v := (v_1^T, \ldots, v_n^T)^T \) and \( p = (p_1, \ldots, p_n)^T \) at time \( t = 0 \), which we will denote by \( u^0, v^0 \) and \( p^0 \) and collect in a vector \( y^0 \) the time-stepping scheme we wish to construct should produce a time-discrete approximation of the vector of unknowns \( y \) at time \( t_{k+1} = t_k + \tau \) denoted by \( y^{k+1} \) from the time-discrete approximation \( y^k \) at time \( t_k \) by solving an operator equation of the form

\[
\mathcal{A}y^{k+1} = \mathcal{G}^{k+1}
\]

(16)

where the right hand side \( \mathcal{G}^{k+1} \) is defined in terms of computable quantities at time \( t_k \) and \( t_{k+1} \), including, for instance, approximations of the time derivatives \( \dot{u} \) and \( \dot{v} \) at time \( t_k \) if available.\(^4\)

Assuming for a moment that we know \( p \) we can consider a restricted dynamical problem

\[
\mathcal{M}\ddot{r} + \mathcal{D}\dot{r} + \mathcal{L}r = \mathcal{F}
\]

(17)

for the unknowns \( u \) and \( v \) which we collect in the vector \( r \), i.e., \( r = (u^T, v^T)^T \), where the operators \( \mathcal{M}, \mathcal{D}, \mathcal{L} \) and right hand side \( \mathcal{F} \) are defined by

\[
\mathcal{M} := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad \mathcal{D} := \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}, \quad \mathcal{L} := \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, \quad \mathcal{F} := \begin{bmatrix} f \\ g \\ \vdots \end{bmatrix} - \begin{bmatrix} L_{13} \\ L_{23} \end{bmatrix} p.
\]

(18)

\(^4\)Note that the exact values of \( u \) and \( v \) are given at time \( t_0 = 0 \).
Introducing the new variable \( s := r \), the second-order system (17) can be rewritten in form of the following equivalent first-order system:

\[
\begin{align*}
\dot{r} &= s, \quad (19a) \\
\dot{s} &= -\mathcal{M}^{-1}\mathcal{D}s - \mathcal{M}^{-1}\mathcal{L}r + \mathcal{M}^{-1}\mathcal{F} =: \mathcal{H}.
\end{align*}
\]

Applying Crank-Nicolson method to (19) one computes the approximation \( r^{k+1} \) at time \( t_{k+1} \) from the system

\[
\begin{align*}
r^{k+1} &= r^k + \tau s^k + \frac{\tau}{2} s^{k+1}, \quad (20a) \\
s^{k+1} &= s^k + \frac{\tau}{2} (\mathcal{H}(r^k, s^k, t_k) + \mathcal{H}(r^{k+1}, s^{k+1}, t_{k+1})) =: s^k + \frac{\tau}{2} (\mathcal{H}^k + \mathcal{H}^{k+1}).
\end{align*}
\]

Using the definition of \( \mathcal{H}^k \) and \( \mathcal{H}^{k+1} \) according to (19b) yields

\[
(\mathcal{M} + \frac{\tau}{2} \mathcal{D})s^{k+1} = (\mathcal{M} - \frac{\tau}{2} \mathcal{D})s^k - \frac{\tau}{2} (\mathcal{L}r^{k+1} + \mathcal{L}r^k) + \frac{\tau}{2} (\mathcal{F}^k + \mathcal{F}^{k+1})
\]

Next, inserting (21) in (20a). Collecting terms, the time-step equation for the restricted dynamical problem (17) is given by

\[
\begin{align*}
\mathcal{M} + \frac{\tau}{2} \mathcal{D} + \frac{\tau^2}{4} \mathcal{L} r^{k+1} &= \frac{\tau^2}{4} (\mathcal{F}^k + \mathcal{F}^{k+1}) + (\mathcal{M} + \frac{\tau}{2} \mathcal{D} - \frac{\tau^2}{4} \mathcal{L}) r^k + \tau \mathcal{M} s^k \\
\frac{\tau}{2} s^{k+1} - r^{k+1} &= -r^k - \frac{\tau^2}{4} s^k
\end{align*}
\]

For symmetry reasons, multiplying equation (22b) with \((-1)\) and adding it to equation (22a), then multiplying equation (22b) with \( \frac{\tau}{2} \), we obtain

\[
\begin{align*}
(\mathcal{M} + \mathcal{I} + \frac{\tau}{2} \mathcal{D} - \frac{\tau}{2} \mathcal{L}) r^{k+1} &= \frac{\tau}{2} (\mathcal{F}^k + \mathcal{F}^{k+1}) + (\mathcal{M} + \mathcal{I} + \frac{\tau}{2} \mathcal{D} - \frac{\tau^2}{4} \mathcal{L}) r^k + \tau (\mathcal{M} + \frac{1}{2} \mathcal{I}) s^k \\
\frac{\tau}{2} s^{k+1} - r^{k+1} &= -r^k - \frac{\tau^2}{4} s^k
\end{align*}
\]

As a consequence of the presence of \( p^k \) and \( p^{k+1} \) in (23a), i.e.,

\[
\mathcal{F}^k + \mathcal{F}^{k+1} = \begin{bmatrix} f^k \\ g^k \end{bmatrix} + \begin{bmatrix} f^{k+1} \\ g^{k+1} \end{bmatrix} - \begin{bmatrix} L_{13} \\ L_{23} \end{bmatrix} p^k + \begin{bmatrix} L_{13} \\ L_{23} \end{bmatrix} p^{k+1},
\]

it is not possible to apply (23) as a stand-alone scheme. This is why we will couple (23) to a second time-step equation obtained from the mass balance equation.

We use the operators defined in the previous section to rewrite (12c) in the form

\[
D_{31} u + D_{32} v + D_{33} p + L_{33} p = 0,
\]

or, equivalently,

\[
\dot{p} = -L_{33} p.
\]

where we have introduced the new variable \( \dot{p} := D_{31} u + D_{32} v + D_{33} p \). Application of the Crank-Nicolson scheme to (26) results in

\[
\begin{align*}
p^{k+1} &= p^k + \frac{\tau}{2} \left( p^k + p^{k+1} \right) = p^k - \frac{\tau}{2} (L_{33} p^k + L_{33} p^{k+1}),
\end{align*}
\]

which can also be expressed as

\[
D_{31} u^{k+1} + D_{32} v^{k+1} + (\frac{\tau}{2} L_{33} + D_{33}) p^{k+1} = D_{31} u^k + D_{32} v^k - (\frac{\tau}{2} L_{33} - D_{33}) p^k.
\]
To ensure Symmetry of all System, multiply the above equation with $\frac{T}{2}$ yields

$$
\frac{T}{4}D_{31}u^{k+1} + \frac{T}{4}D_{32}v^{k+1} + \frac{T}{4}(L_{33}^2 + D_{33})p^{k+1} = \frac{T}{4}D_{31}u^k + \frac{T}{4}D_{32}v^k - \frac{T}{4}(L_{33}^2 - D_{33})p^k.
$$

(28)

The combined scheme is now defined based on (22), which we slightly rearrange in the form

$$(\mathcal{M} + I + \frac{T}{2}D + \frac{T}{4}L)r^{k+1} - \frac{T}{2}I s^{k+1} + \frac{T}{4}\left[\mathcal{L}_{13}\right]p^{k+1} = \frac{T}{4}\left[f^{k}\right] + \frac{T}{4}\left[g^{k+1}\right] - \frac{T}{4}\left[\mathcal{L}_{13}\right]p^k + \left(\mathcal{M} + I + \frac{T}{2}D - \frac{T}{4}L\right)r^k + \tau(M + \frac{T}{2}I)s^k$$

(29a)

$$
-\frac{T}{4}Is^{k+1} = \frac{T}{2}Ir^k - \frac{T}{4}Is^k
$$

(29b)

in order to collect all unknown quantities referring to time $t_{k+1}$ on the left hand side and (28). It finally can be represented in the modified form (16) where the operator $A$ and right hand side vector $G^{k+1}$ are given by

$$
A := \begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} & -\frac{T}{4}I & 0 & \frac{T}{2}L_{13} \\
\hat{A}_{21} & \hat{A}_{22} & 0 & -\frac{T}{4}I & \frac{T}{2}L_{23} \\
-\frac{T}{4}I & 0 & \frac{T}{4}I & 0 & \frac{T}{2}L_{33} \end{bmatrix}, \quad y = \begin{bmatrix} u \\ v \\ \dot{u} \end{bmatrix}
$$

(30)

where

$$
\begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} \\
\hat{A}_{21} & \hat{A}_{22}
\end{bmatrix} := \begin{bmatrix}
\mathcal{M}_{11} + I + \frac{T}{2}D_{11} + \frac{T}{4}L_{11} & \mathcal{M}_{12} + \frac{T}{2}D_{12} + \frac{T}{4}L_{12} \\
\mathcal{M}_{21} + \frac{T}{2}D_{21} + \frac{T}{4}L_{21} & \mathcal{M}_{22} + I + \frac{T}{2}D_{22} + \frac{T}{4}L_{22}
\end{bmatrix}
$$

(31)

and

$$
\dot{v} := (v_1^T, \ldots, v_n^T)^T
$$

and

$$
G^{k+1} := \begin{bmatrix}
\frac{T}{2}f^k + \frac{T}{2}f^{k+1} + \hat{A}_{11}u^k + \hat{A}_{12}v^k - \frac{T}{2}(L_{11}u^k + L_{12}v^k) - \frac{T}{2}L_{13}p^k + \tau((\mathcal{M}_{11} + \frac{T}{2}I)u^k + \mathcal{M}_{12}v^k) \\
\frac{T}{2}g^k + \frac{T}{2}g^{k+1} + \hat{A}_{21}u^k + \hat{A}_{22}v^k - \frac{T}{2}(L_{21}u^k + L_{22}v^k) - \frac{T}{2}L_{23}p^k + \tau((\mathcal{M}_{21} + \frac{T}{2}I)u^k + \mathcal{M}_{22}v^k) \\
\frac{T}{2}\dot{u}^k + \frac{T}{2}\dot{v}^k - \frac{T}{2}u^k - \frac{T}{2}v^k \\
\frac{T}{2}D_{31}u^k + \frac{T}{2}D_{32}v^k - \frac{T}{2}(L_{33}^2 - D_{33})p^k
\end{bmatrix}
$$

(32)

We see that the right hand is defined in terms of the quantities $u^k, v^k, u^k, v^k$ and $p^k$ that are known from the previous time step, and additionally $f^k, g^k$ and $f^{k+1}, g^{k+1}$, which can be evaluated at any time moment due to the known right hand side of (12).

In summary, we have defined a time-stepping scheme that requires in each time step the solution of an equation of the form (16) with a self-adjoint operator $A$ defined in (30), and we introduce the new abbreviations

$$
\gamma_i := -\left((\rho_i - \varphi_i\rho_{m_i}) - \frac{T}{2}\varphi_iK_i^{-1}\right), \quad (33a)
$$

$$
\gamma_u := \left((1 - \varphi)\rho_u + 1 + \sum_{i=1}^n \varphi_i\gamma_i\right), \quad \gamma_{ui} := \frac{\rho_u + \gamma_u}{\varphi_i} + 1 \quad (33b)
$$

$$
\beta_{ij} := \frac{T}{8}\hat{\beta}_{ij}, \quad 1 \leq i, j \leq n, \ i \neq j, \quad \beta_{ii} := \sum_{j=1, i \neq j}^n \frac{T}{8}\hat{\beta}_{ij} + \frac{T^2}{4}c_{p_i}, \quad 1 \leq i \leq n. \quad (33c)
$$
and assumed that \( K = K I \). Then a self-adjoint operator \( A \) defined in (30), can be written as follows,

\[
A := \begin{pmatrix}
-\frac{\tau^2}{4} \text{div} \sigma + \gamma_u & -\gamma_1 & \cdots & -\gamma_n & -\frac{\tau}{2} & 0 & \cdots & 0 & \frac{\tau}{2} \alpha_1 \nabla & \cdots & \frac{\tau}{2} \alpha_n \nabla \\
-\gamma_1 & \gamma_v,1 & \cdots & 0 & 0 & -\frac{\tau}{2} & \cdots & 0 & \frac{\tau}{2} \nabla & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-\gamma_n & 0 & \cdots & \gamma_v,n & 0 & 0 & \cdots & -\frac{\tau}{2} & 0 & \cdots & -\frac{\tau}{2} \nabla \\
-\frac{\tau}{2} & 0 & \cdots & 0 & \frac{\tau}{4} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & -\frac{\tau}{2} & \cdots & 0 & 0 & \frac{\tau^2}{4} & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & -\frac{\tau}{2} & 0 & 0 & \cdots & \frac{\tau^2}{4} & 0 & \cdots & 0 \\
-\frac{\tau^2}{4} \alpha_1 \text{div} & -\frac{\tau^2}{4} \text{div} & \cdots & 0 & 0 & 0 & \cdots & 0 & -\beta_{11} & \cdots & \beta_{1n} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-\frac{\tau^2}{4} \alpha_n \text{div} & 0 & \cdots & -\frac{\tau^2}{4} \text{div} & 0 & 0 & \cdots & 0 & \beta_{n1} & \cdots & -\beta_{nn}
\end{pmatrix}
\]

3 Mass conserving space discretization

3.1 Space discretization of continuous Problem

The weak formulation of system (29) and (28):

Find \((u; v; \tilde{u}; \tilde{v}; p) \in \bar{U} \times U \times \bar{V} \times V \times P\), such that for any \((w; z; \tilde{w}; \tilde{z}; q) \in \bar{U} \times U \times \bar{V} \times V \times P\) then

\[
\begin{align*}
\frac{\mu \tau^2}{2} (\epsilon(u), \epsilon(w)) + \frac{\lambda \tau^2}{4} (\text{div} u, \text{div} w) + \gamma_u(u, w) - \frac{\tau}{2} (\tilde{u}, w) + (A_{12} v, w) - \frac{\tau^2}{4} (\alpha p, \text{Div} w) &= (G_1, w), \\
(A_{31} u, z) + (A_{22} v, z) - \frac{\tau}{2} (v, z) - \frac{\tau^2}{4} (p, \text{Div} z) &= (G_2, z), \\
-\frac{\tau}{2} (\tilde{u}, \tilde{w}) + \frac{\tau^2}{4} (\tilde{u}, \tilde{w}) &= (G_3, \tilde{w}), \\
-\frac{\tau}{2} (v, \tilde{z}) + \frac{\tau^2}{4} (\tilde{v}, \tilde{z}) &= (G_4, \tilde{z}), \\
-\frac{\tau^2}{4} (\alpha \text{Div} u, q) - \frac{\tau^2}{4} (\text{Div} u, q) + ((\frac{3}{8} L_{33} + \frac{\tau^2}{4} D_{33}) p, q) &= (G_5, q),
\end{align*}
\]

where

\[
\text{Div} = \begin{pmatrix}
\text{div} v_1 \\
\vdots \\
\text{div} v_n
\end{pmatrix}
\text{ for all } v \in U, \quad \text{Div} u = \begin{pmatrix}
\text{div} u \\
\vdots \\
\text{div} u
\end{pmatrix}
\text{ for all } u \in \bar{U}, \quad \alpha := \begin{pmatrix}
\alpha_1 & 0 & \cdots & 0 \\
0 & \alpha_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha_n
\end{pmatrix}
\]

8
Consider the Hilbert spaces $\tilde{U} = H_0^1(\Omega)^d$, $\tilde{V} = H_0(\text{div}, \Omega)$, $P = (L_2^2(\Omega))^n$ and $U, V = (H_0(\text{div}, \Omega))^n$ with parameter-dependent norms $\| \cdot \|_{\tilde{U} \times U \times \tilde{V} \times V}$, $\| \cdot \|_P$ induced by the inner product

$$
((u, v; \bar{u}, \bar{v}), (w, z, \bar{w}, \bar{z}))_{\tilde{U} \times U \times \tilde{V} \times V} = \frac{\mu^2}{2}(\epsilon(u), \epsilon(w)) + \frac{\lambda r^2}{4}(\text{div}u, \text{div}w) + (\Lambda_{uv} \begin{pmatrix} u_v \bar{u} \\ v \bar{v} \end{pmatrix}, \begin{pmatrix} w \bar{w} \\ z \bar{z} \end{pmatrix})
$$

$$
+ \frac{\tau^2}{4}(\Lambda^{-1}(\text{Div}u + \alpha\text{Div}w), \text{Div}z + \alpha\text{Div}w),
$$

(35a)

$$(p, q)_P = \frac{\tau^2}{4}(\Lambda p, q)$$

(35b)

where

$$\Lambda := \Lambda_1 + \Lambda_2 + \Lambda_3, \quad \Lambda_1 := \frac{\tau^2}{4}A_{22}, \quad \Lambda_2 := \frac{\tau^2}{4}A_{11}, \quad \Lambda_3 := \frac{\tau^2}{4}\alpha\Lambda_1\alpha, \quad \gamma := \max\left\{ \frac{\tau^2}{2}, \frac{\tau^2}{4}\lambda, \gamma_0 \right\}
$$

$$
\Lambda_4 := \begin{bmatrix}
1 & 1 & \cdots & \cdots & 1 \\
1 & 1 & \cdots & \cdots & 1 \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
1 & 1 & \cdots & \cdots & 1
\end{bmatrix}, \quad \Lambda_{uv} := \begin{bmatrix}
\gamma_0 & A_{12} & -\frac{\tau}{2}I & 0 \\
A_{21} & A_{22} & 0 & -\frac{\tau}{2}I \\
-\frac{\tau}{2}I & 0 & \frac{\tau^2}{4}I & 0 \\
0 & -\frac{\tau}{2}I & 0 & \frac{\tau^2}{4}I
\end{bmatrix}
$$

The matrices $\Lambda_1$ and $\Lambda_{uv}$ are positive semi-definit, because they are sum of positive matrices.

From System of equations (34) introduce the bilinear form

$$
\mathcal{A}((u; v; \bar{u}; \bar{v}; p), (w, z, \bar{w}, \bar{z}; q)) = \frac{\mu^2}{2}(\epsilon(u), \epsilon(w)) + \frac{\lambda r^2}{4}(\text{div}u, \text{div}w) + (\Lambda_{uv} \begin{pmatrix} u_v \bar{u} \\ v \bar{v} \end{pmatrix}, \begin{pmatrix} w \bar{w} \\ z \bar{z} \end{pmatrix})
$$

$$
- \frac{\tau^2}{4}(p, \alpha\text{Div}w + \text{Div}z) - \frac{\tau^2}{4}(\alpha\text{Div}u + \text{Div}v, q) - \frac{\tau^2}{4}(\Lambda_1 p, q)
$$

(36)

### 3.2 Preliminaries and notation

#### 3.2.1 Preliminaries and notation

Let $T_h$ be a shape-regular triangulation of mesh-size $h$ of the domain $\Omega$ into triangles $\{T\}$ and define the set of all interior edges (or faces) of $T_h$ by $E^I_h$ and the set of all boundary edges (or faces) by $E^B_h$. Let $E_h = E^I_h \cup E^B_h$.

For $s \geq 1$, we introduce the spaces

$$H^s(T_h) = \{ \phi \in L^2(\Omega), \text{ such that } \phi|_T \in H^s(T) \text{ for all } T \in T_h \}.
$$

We further define some trace operators. Denote by $e = \partial T_1 \cap \partial T_2$ the common boundary (interface) of two subdomains $T_1$ and $T_2$ in $T_h$, and by $n_1$ and $n_2$, the unit normal vectors to $e$ that point to the exterior of $T_1$ and $T_2$, correspondingly.

For any $e \in E^I_h$ and $q \in H^1(T_h), v \in H^1(T_h)^d$ and $\tau \in H^1(T_h)^{d \times d}$, the averages are defined as

$$
\{ v \} = \frac{1}{2}(v|_{\partial T_1 \cap e} \cdot n_1 - v|_{\partial T_2 \cap e} \cdot n_2), \quad \{ \tau \} = \frac{1}{2}(\tau|_{\partial T_1 \cap e} n_1 - \tau|_{\partial T_2 \cap e} n_2),
$$

and the jumps are given by

$$[q] = q|_{\partial T_1 \cap e} - q|_{\partial T_2 \cap e}, \quad [v] = v|_{\partial T_1 \cap e} - v|_{\partial T_2 \cap e}.
When \( e \in \mathcal{E}_h^B \), then the above quantities are defined as
\[
\{v\} = v|_e \cdot n, \quad \{\tau\} = \tau|_e \cdot n, \quad \{q\} = q|_e, \quad \{v\} = v|_e.
\]
If \( n_T \) is the outward unit normal to \( \partial T \), it is easy to show that, for \( \tau \in H^1(\Omega)^d \times d \) and for all \( \upsilon \in H^1(\mathcal{T}_h)^d \), we have
\[
\sum_{T \in \mathcal{T}_h} \int_{\partial T} (\tau n_T) \cdot \upsilon ds = \sum_{e \in \mathcal{E}_h} \int_e (\tau) \cdot [\upsilon] ds.
\]

### 3.2.2 DG discretization

In the present context, we define the local spaces \( \tilde{\mathcal{P}}_i \). Similar to the continuous problem, we denote
\[
\tilde{\mathcal{U}}_h = \{u \in H(\text{div}; \Omega) : u|_T \in \tilde{\mathcal{U}}(T), \; T \in \mathcal{T}_h; \; u \cdot n = 0 \text{ on } \partial \Omega\},
\]
\[
\tilde{\mathcal{U}}_i,h = \{u \in H(\text{div}; \Omega) : u|_T \in \tilde{\mathcal{U}}(T), \; T \in \mathcal{T}_h; \; u \cdot n = 0 \text{ on } \partial \Omega\}, \quad i = 1, \cdots, n.
\]
\[
\tilde{\mathcal{V}}_h = \{\tilde{v} \in H(\text{div}; \Omega) : \tilde{v}|_T \in \tilde{\mathcal{V}}(T), \; T \in \mathcal{T}_h; \; \tilde{v} \cdot n = 0 \text{ on } \partial \Omega\}, \quad i = 1, \cdots, n.
\]
\[
\tilde{\mathcal{P}}_i,h = \{p \in L^2(\Omega) : p|_T \in Q_i(T), \; T \in \mathcal{T}_h; \; \int_{\Omega} p dx = 0\}, \quad i = 1, \cdots, n.
\]

The discretization we analyze in the present context, define the local spaces \( \tilde{\mathcal{U}}(T)/\tilde{\mathcal{U}}(T)/\tilde{\mathcal{V}}(T)/\tilde{\mathcal{V}}(T)/\tilde{\mathcal{P}}_i(T) \) via \( BDM_i(T)/BDM_1(T)/RT_{i-1}(T)/RT_0(T)/P_{i-1}(T) \), or \( BDM_1(T)/BDM_1(T)/RT_{i-1}(T)/RT_0(T)/P_{i-1}(T) \) for \( i \geq 1 \). Note that, for each of these choices, the important condition \( \text{div}\tilde{\mathcal{U}}(T) = \text{div}\tilde{\mathcal{U}}(T) = \text{div}\tilde{\mathcal{V}}(T) = \text{div}\tilde{\mathcal{V}}(T) = Q_i(T) \) is satisfied, cf [13][14][15]

Note that the normal component of any \( u \in \tilde{\mathcal{U}}_h \) is continuous on the internal edges and vanishes on the boundary edges. Then, for all \( e \in \mathcal{E}_h \) and for all \( \tau \in H^1(T)^d \), \( u \in \tilde{\mathcal{U}}_h \) it holds
\[
\int_e [u_i] \cdot \tau ds = 0, \quad \text{implying that} \quad \int_e [u] \cdot \tau ds = \int_e [u_\tau] \cdot \tau ds,
\]
where \( u_\tau \) and \( u_i \) denote the normal and tangential component of \( u \), respectively.

Similar to the continuous problem, we denote
\[
\tilde{u}_h = (\tilde{u}_1,h, \cdots, \tilde{u}_n,h), \quad \tilde{v}_h = (\tilde{v}_1,h, \cdots, \tilde{v}_n,h), \quad p_h = (p_1,h, \cdots, p_n,h), \quad z_h = (z_1,h, \cdots, z_n,h),
\]
\[
\tilde{z}_h = (\tilde{z}_1,h, \cdots, \tilde{z}_n,h), \quad q_h = (q_1,h, \cdots, q_n,h), \quad \mathcal{V}_h = \mathcal{V}_1 \times \cdots \times \mathcal{V}_n,h, \quad \mathcal{P}_h = \mathcal{P}_1 \times \cdots \times \mathcal{P}_n,h.
\]

With this notation at hand, the discretization of the variational problem (34) is given as follows:

Find \( (\tilde{u}_h; \tilde{v}_h; \tilde{v}_i; p_h, q_h) \in \tilde{\mathcal{U}}_h \times \tilde{\mathcal{U}}_h \times \tilde{\mathcal{U}}_h \times \tilde{\mathcal{V}}_h \times \tilde{\mathcal{P}}_i \) such that, for any \( (u_h; v_h; \tilde{u}_h; \tilde{v}_h; z_h; q_h) \in \tilde{\mathcal{U}}_h \times \tilde{\mathcal{U}}_h \times \tilde{\mathcal{U}}_h \times \tilde{\mathcal{V}}_h \times \tilde{\mathcal{P}}_i \)

\[
\frac{\mu}{2} u_h(u_h, w_h) + \frac{\lambda}{4} \text{div}(u_h, \text{div}w_h) + \gamma(u_h, w_h) - \frac{\tau}{2} (\tilde{u}_h, \tilde{v}_h) + (\tilde{z}_h, z_h) = \frac{\tau}{4} \alpha p_h, \text{div}w_h, (G_1, w_h), \quad \text{(39a)}
\]

\[
(\bar{A}_{21} u_h, \bar{z}_h) + (\bar{A}_{22} v_h, z_h) - \frac{\tau}{2} (\tilde{v}_h, \tilde{z}_h) = \frac{\tau}{4} (p_h, \text{div}z_h) = (G_2, z_h), \quad \text{(39b)}
\]

\[
- \frac{\tau}{2} (u_h, \tilde{u}_h) + \frac{\tau}{4} (\tilde{u}_h, \tilde{u}_h) = (G_3, \tilde{u}_h), \quad \text{(39c)}
\]

\[
- \frac{\tau}{2} (v_h, \tilde{v}_h) + \frac{\tau}{4} (\tilde{v}_h, \tilde{v}_h) = (G_4, \tilde{v}_h), \quad \text{(39d)}
\]

\[
- \frac{\tau}{4} (\alpha \text{div}u_h, q_h) = \frac{\tau}{4} (\text{div}v_h, q_h) = \frac{\tau}{4} (\Lambda_1 p_h, q_h) = (G_5, q_h), \quad \text{(39e)}
\]

where
\[
a_h(u, w) = \sum_{T \in \mathcal{T}_h} \int_T \epsilon(u) : \epsilon(w) dx - \sum_{e \in \mathcal{E}_h} \int_e \{\epsilon(u)\} \cdot [w] ds - \sum_{e \in \mathcal{E}_h} \int_e \{\epsilon(w)\} \cdot [u] ds + \sum_{e \in \mathcal{E}_h} \int_e \eta_h^{-1}[u] \cdot [w] ds,
\]
we now summarize several results on well-posedness and approximation properties of the DG formulation:

\[
\|u\|_h^2 = \sum_{T \in \mathcal{T}_h} \|\varepsilon(u)\|_{0,T}^2 + \sum_{e \in E_h} h_e^{-1} \|[\varepsilon]_e\|_{0,e}^2, \quad \|u\|_{1,h}^2 = \sum_{T \in \mathcal{T}_h} \|\nabla u\|_{0,T}^2 + \sum_{e \in E_h} h_e^{-1} \|[\varepsilon]_e\|_{0,e}^2.
\]

the "DG"-norm

\[
\|u\|_{DG}^2 = \sum_{T \in \mathcal{T}_h} \|\nabla u\|_{0,T}^2 + \sum_{e \in E_h} h_e^{-1} \|[\varepsilon]_e\|_{0,e}^2 + \sum_{T \in \mathcal{T}_h} h_T^2 |u|_{2,T}^2,
\]

and, finally, the mesh-dependent norm \(\|(\cdot,\cdot,\cdot)\|_{\mathcal{U}_h \times \mathbb{U} \times \mathbb{V} \times \mathbb{V}}\) by

\[
\|(u;v;\tilde{u};\tilde{v})\|_{\mathcal{U}_h \times \mathbb{U} \times \mathbb{V} \times \mathbb{V}}^2 = \frac{\mu \tau^2}{2} \|u\|_{DG}^2 + \frac{\tau^2 \lambda}{4} \|\text{div}u\| + \|\Lambda_u\| \begin{pmatrix} u \\ v \\ \tilde{u} \\ \tilde{v} \end{pmatrix} \|^2 + \frac{\tau^2}{4} \|\text{Div} + \alpha \text{Div}u\|^2.
\]

We now summarize several results on well-posedness and approximation properties of the DG formulation:

- From the discrete version of Korn's inequality we have that the norms \(\|\cdot\|_{DG}, \|\cdot\|_h\) and \(\|\cdot\|_{1,h}\) are equivalent on \(\mathcal{U}_h\), namely,

\[
\|u\|_{DG} \approx \|u\|_h \approx \|u\|_{1,h}, \quad \text{for all} \quad u \in \mathcal{U}_h.
\]

- The bilinear form \(a_h(\cdot,\cdot)\), introduced in (40) is continuous and we have

\[
|a_h(u,w)| \leq c_2 \|u\|_{DG} \|w\|_{DG}, \quad \text{for all} \quad u, w \in H^2(\mathcal{T}_h)^d.
\]

- The discrete Poincare inequality, cf [16]

\[
\|u\| \leq c_3 \|u\|_{1,h}, \quad \text{for all} \quad u \in \mathcal{U}_h.
\]

- For our choice of the finite element spaces \(\mathcal{U}_h, \mathbb{U}_h\) and \(\mathbb{P}_h\) we have the following inf-sup conditions,

\[
\inf_{q_h \in P_h} \sup_{u_h \in \mathcal{U}_h} \frac{\text{div}u_h \cdot \sum_{i=1}^n q_i h_i}{\|u_h\|_{1,h} \sum_{i=1}^n q_i h_i} \geq \beta_s, h, \quad \inf_{q_h \in P_h} \sup_{v_h \in \mathcal{V}_h} \frac{\text{div}v_h \cdot \sum_{i=1}^n q_i h_i}{\|u_h\|_{1,h} \sum_{i=1}^n q_i h_i} \geq \beta_v, h,
\]

where \(\beta_s, h\) and \(\beta_v, h\) are positive constant independent of all parameters, the network scale \(n\) and the mesh size \(h\).

- The coercivity of \(a_h(\cdot,\cdot)\)

\[
a_h(u_h, w_h) \geq \alpha_a \|u_h\|_{1,h}^2, \quad \text{for all} \quad u_h \in \mathcal{U}_h.
\]

where \(\alpha_a\) is a positive constant independent of all parameters, the network scale \(n\) and the mesh size \(h\).

Related to the discrete problem (30) and from the definition of the matrix \(\Lambda_{uv}\), we define the bilinear form

\[
A_h((u_h; v_h; \tilde{u}_h; \tilde{v}_h; p_h), (w_h; z_h; \tilde{w}_h; \tilde{z}_h; q_h)) = \frac{\mu \tau^2}{2} a_h(u_h, w_h) + \frac{\tau^2 \lambda}{4} (\text{div}u_h, \text{div}w_h) + (\Lambda_{uv} \begin{pmatrix} u_h \\ v_h \\ \tilde{u}_h \\ \tilde{v}_h \end{pmatrix}, \begin{pmatrix} u_h \\ v_h \\ \tilde{u}_h \\ \tilde{v}_h \end{pmatrix}) - \frac{\tau^2}{4} (p_h, \alpha \text{Div}w_h + \text{Div}z_h) - \frac{\tau^2}{4} (\alpha \text{Div}u_h + \text{Div}v_h, q_h) - \frac{\tau^2}{4} (\lambda_1 p_h, q_h)
\]
4 Stability analysis

4.1 Stability of the time-discrete problem

The main result of this section is a proof of the uniform well-posedness, of problem (34) under the norms induced by (35). Before we study the full dynamic MPET equations, we recall the following well known results, cf [17][18], and two help Lemmas:

Lemma 4.1. There exists a constant $\beta_v > 0$ such that

$$\inf_{q \in P, v \in V} \sup_{\|v\| = 1} \left( \text{div} u, q \right) \geq \beta_v, \quad i = 1, \ldots, n. \quad (49)$$

Lemma 4.2. There exists a constant $\beta_s > 0$ such that

$$\inf_{q \in P} \sup_{u \in U} \left( \text{div} u, q \right) \geq \beta_s \quad (50)$$

Lemma 4.3. The determinant of the following matrix

$$A := \begin{bmatrix} -b_1 & -b_2 & \ldots & \ldots & -b_n \\ a & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & a & 0 \end{bmatrix} \quad \text{n x n}$$

is $\det(A) = (-1)^n \cdot a^{n-1} \cdot b_n$

Proof. By using induction method. For $n = 1$ we have $\det(A) = -b_1$. Assume the induction hypothesis is true for $(n-1)$ and we prove for $n$. By using the Laplace's formula for the last row it follow

$$\det(A) = (-1)^{n+1-n-1} \cdot a \cdot \begin{bmatrix} -b_1 & -b_2 & \ldots & \ldots & -b_n \\ a & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & a & 0 \end{bmatrix} = (-1)^n \cdot a^{n-1} \cdot b_n$$

Lemma 4.4. The determinant of the following matrix

$$B := \begin{bmatrix} c & -b_1 & -b_2 & \ldots & b_n \\ -b_1 & a & \ldots & 0 \\ -b_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ -b_n & 0 & \ldots & a \end{bmatrix} \quad \text{(n+1) x (n+1)}$$

is $\det(B) = a^{n-1} \left( a \cdot c - \sum_{i=1}^{n} b_i^2 \right)$
Proof. By using induction method. For \( n = 1 \) we have \( \det(A) = a \cdot c - b^2_1 \). Assume the induction hypothesis is true \( n \) and we proof for \((n + 1)\). By using the Laplace's formula for the last row it follow

\[
\begin{vmatrix}
-b_1 & -b_2 & \cdots & \cdots & -b_n \\
a & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & a & 0 \\
\end{vmatrix}
= (-1)^{n+1} \cdot (b_n) \\
\begin{vmatrix}
c & -b_1 & -b_2 & \cdots & b_{n-1} \\
-b_1 & a & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
-b_{n-1} & 0 & \cdots & a & 0 \\
\end{vmatrix}
\]

The following theorem shows the boundedness of \( \mathcal{A}((\cdot; \cdot; \cdot; \cdot), (\cdot; \cdot; \cdot; \cdot)) \) in the norm induced by (35):

**Theorem 4.5.** There exists a constant \( C_b \) independent of all parameters and the network scale \( n \), such that for any \((u; v; u; v; p) \in \tilde{U} \times U \times V \times V \times P, (w; z; \tilde{w}; z; q) \in \tilde{U} \times U \times V \times V \times P \)

\[
|\mathcal{A}((u; v; u; v; p), (w; z; \tilde{w}; z; q))| \leq C_b (\|\mathcal{A}(u, v, u, v)\|_{D \times U \times V \times V} + \|p\|_p) \cdot (\|\mathcal{A}(w, z, \tilde{w}, \tilde{z})\|_{D \times U \times V \times V} + \|q\|_p).
\]  

(51)

Proof. By applying Cauchy-Schwarz inequality on the bilinear form (36) we obtain

\[
\mathcal{A}((u; v; p), (w; z; q)) \leq \frac{\mu \tau^2}{2} \|\mathbf{e}(u)\| \cdot \|\mathbf{e}(w)\| + \frac{\tau^2 \lambda}{4} \|\text{div} u\| \cdot \|\text{div} w\| + \|\Lambda^\frac{1}{2}_v\| \cdot \|\Lambda^\frac{1}{2}_v\| \cdot \|\Lambda^\frac{1}{2}_v\| \cdot \|\Lambda^\frac{1}{2}_v\| \cdot \|\Lambda^\frac{1}{2}_v\| \cdot \|\Lambda^\frac{1}{2}_v\|
\]

\[
+ \frac{\tau^2}{4} \|\Lambda^\frac{1}{2}_v p\| \cdot \|\Lambda^\frac{1}{2}_v p\| \cdot \|\Lambda^\frac{1}{2}_v q\| + \|\Lambda^\frac{1}{2}_v (\alpha \text{div} u + \text{div} v)\| \cdot \|\Lambda^\frac{1}{2}_v q\| + \|\Lambda^\frac{1}{2}_v (\alpha \text{div} u + \text{div} v)\| \cdot \|\Lambda^\frac{1}{2}_v q\| + \tau^2 \|\Lambda^\frac{1}{2}_v p\| \cdot \|\Lambda^\frac{1}{2}_v p\|
\]

We obtain (51), by applying again Cauchy-Schwarz inequality.

The following theorem shows the inf-sup-condition (LBB) of \( \mathcal{A}((\cdot; \cdot; \cdot; \cdot), (\cdot; \cdot; \cdot; \cdot)) \) in the norm induced by (35):

**Theorem 4.6.** There exists a constant \( \omega > 0 \) independent of all parameters and the network scale \( n \), such that

\[
\inf_{(u; v; u; v; p) \in \tilde{U} \times U \times V \times V \times P} \sup_{(w; z; \tilde{w}; z; q) \in \tilde{U} \times U \times V \times V \times P} \frac{\mathcal{A}((u; v; u; v; p), (w; z; \tilde{w}; z; q))}{\|\mathcal{A}(u, v, u, v)\|_{D \times U \times V \times V} + \|p\|_p + \|\mathcal{A}(w, z, \tilde{w}, \tilde{z})\|_{D \times U \times V \times V} + \|q\|_p} \geq \omega.
\]

Proof. For any \((u; v; u; v; p) \in \tilde{U} \times U \times V \times V \times P \), by Lemma 4.1, there exist

\[v_0 \in U \text{ such that } \text{Div} v_0 = \frac{\tau}{2} \Lambda^\frac{1}{2}_v p \text{ and } \|v_0\|_{\text{div}} \leq \beta_v^{-1} \|\Lambda^\frac{1}{2}_v p\|, \]

(52)

and by Lemma 4.2, there exists

\[u_0 \in \tilde{U} \text{ such that } \text{Div} u_0 = \frac{\tau}{2} \Lambda^\frac{1}{2}_v \alpha \text{ and } \|u_0\|_1 \leq \frac{\tau \beta_v^{-1}}{2} \|\Lambda^\frac{1}{2}_v \alpha \| = \beta_v^{-1} \|\Lambda^\frac{1}{2}_v \alpha \|, \]

(53)

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Choose
\[ w = \delta u - \frac{\tau}{2\sqrt{\gamma}} u_0, \quad z = \delta v - \frac{\tau}{2} \bar{\lambda} \bar{\lambda}^{\frac{1}{2}} v_0, \quad \bar{w} = \delta \bar{u}, \quad \bar{z} = \delta \bar{v}, \quad q = -\delta p - \frac{\tau^2}{4} \Lambda^{-1}(\text{Div} v + \alpha \text{Div} u), \] (54)
where \( \delta \) is a positive constant to be determined later. Before we verify the boundedness of \((w; z; \bar{w}; \bar{z}; q)\) by \((u; v; \bar{u}; \bar{v}; p)\), we try to estimate \(\|\Lambda^\frac{1}{2} \left( \begin{array}{c} \frac{\tau}{2\sqrt{\gamma}} u_0 \\ \bar{\lambda} \bar{\lambda}^{\frac{1}{2}} v_0 \\ 0 \\ 0 \end{array} \right) \|_2^2\):
\[ \|\Lambda^\frac{1}{2} \left( \begin{array}{c} \frac{\tau}{2\sqrt{\gamma}} u_0 \\ \bar{\lambda} \bar{\lambda}^{\frac{1}{2}} v_0 \\ 0 \\ 0 \end{array} \right) \|_2^2 = \frac{\tau^2}{4} \left( \begin{array}{c} c \\ \bar{\lambda} \bar{\lambda}^{\frac{1}{2}} \bar{\lambda} \bar{\lambda}^{\frac{1}{2}} v_0 \\ 0 \\ 0 \end{array} \right) \left( \begin{array}{c} u_0 \\ v_0 \\ 0 \\ 0 \end{array} \right) \]
\[ \leq \frac{\tau^2}{4} \lambda_{\text{max}}(G)(\|u_0\|^2 + \|v_0\|^2) \leq \frac{\tau^2}{4} \lambda_{\text{max}}(G) \left( \beta_2^{-2} \|\Lambda^\frac{1}{2} p\|^2 + \beta_3^{-2} \|\Lambda^\frac{1}{2} p\|^2 \right) \] (55)
where \( c := \frac{\tau^2}{4} \leq 1 \), and let \(-b_i := \left( \frac{1}{\bar{\lambda}} \bar{\lambda} \bar{\lambda}^{\frac{1}{2}} \right) i = -\gamma_i \frac{1}{\gamma_i + \gamma_i}, i = 1, \ldots, n, \) then
\[ \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} \left( \gamma_i \frac{\varphi_i}{\rho_i + \varphi_i + \gamma_i} \right) \leq \sum_{i=1}^{n} \left( \frac{\gamma_i \varphi_i}{(1 - \varphi)(\rho_i + \varphi_i + (\sum_{i=1}^{n} \varphi_i + \gamma_i))} \right) \leq 1 \] (56)
To find the eigenvalues of the matrix \( G \), we using Lemma 4.4:
\[ \det(G - \lambda I) = \begin{vmatrix} c - \lambda & -b_1 & \cdots & -b_n \\ -b_1 & 1 - \lambda & 0 & \cdots & 0 \\ \vdots & 0 & 1 - \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -b_n & 0 & \cdots & \cdots & 1 - \lambda \end{vmatrix} = (1 - \lambda)^{n-1} \left( (1 - \lambda)(c - \lambda) - \sum_{i=1}^{n} b_i \right) = 0 \]
which implies
\[ \lambda_1 = 1, \quad \lambda_{2,3} = \frac{(1 + c) \pm \sqrt{(1 - c)^2 + 4 \sum_{i=1}^{n} b_i^2}}{2}, \]
\[ \lambda_{\text{max}}^2 = \frac{(1 + c) + \sqrt{(1 - c)^2 + 4 \sum_{i=1}^{n} b_i^2}}{4} \leq \frac{2(1 + c)^2 + 2(1 - c)^2 + 8 \sum_{i=1}^{n} b_i^2}{4} \leq \frac{4 + 4c^2 + 8}{4} \leq 4 \quad \Rightarrow \lambda_{\text{max}}(G) \leq 2 \] (57)
finally from (55) we obtain:
\[ \|\Lambda^\frac{1}{2} \left( \begin{array}{c} \frac{\tau}{2\sqrt{\gamma}} u_0 \\ \bar{\lambda} \bar{\lambda}^{\frac{1}{2}} v_0 \\ 0 \\ 0 \end{array} \right) \|_2^2 \leq \frac{\tau^2}{2} \left( \beta_2^{-2} \|\Lambda^\frac{1}{2} p\|^2 + \beta_3^{-2} \|\Lambda^\frac{1}{2} p\|^2 \right) \] (58)
Let now verify the boundedness of \((w; z; \bar{w}; \bar{z}; q)\) by \((u; v; \bar{u}; \bar{v}; p)\). Firstly for \((w; z; \bar{w}; \bar{z})\) we have,

\[
\| (w; z; \bar{w}; \bar{z}) \|_{U \times V}^2 = \| (\delta u - \frac{\tau}{2\sqrt{\gamma}} u_0; \delta v - \frac{\tau}{2\sqrt{\gamma}} \bar{v}_0; \delta \bar{u}; \delta \bar{v}) \|_{U \times V}^2 \\
= \frac{\tau^2 \mu}{2} \| e(\delta u - \frac{\tau}{2\sqrt{\gamma}} u_0) \|^2 + \frac{\tau^2 \lambda}{4} \| \text{div}(\delta u - \frac{\tau}{2\sqrt{\gamma}} u_0) \|^2 + \| \Lambda_\alpha \left( \begin{array}{c} u \\ u \\ \nu \\ \nu \\ \end{array} \right) \|_{V}^2 \\
+ \frac{\tau^2}{4} \| \Lambda^{-\frac{1}{2}} (\text{Div} \nu + \alpha \text{Div} u) \|_{V}^2 + \frac{\tau^2}{2} \| \Lambda^{-\frac{1}{2}} (\frac{\tau}{2\sqrt{\gamma}} \bar{\lambda}_2 \bar{v}_0 + \frac{\tau}{2\sqrt{\gamma}} \alpha \text{Div} u_0) \|_{V}^2,
\]

by applying triangle inequality it follows

\[
\leq \tau^2 \rho^2 \| e(u) \|^2 + \tau^2 \rho \| e(\frac{\tau}{2\sqrt{\gamma}} u_0) \|^2 + \frac{\tau^2 \lambda}{2} \| \text{div} u \|^2 + \frac{\tau^2 \lambda}{2} \| \text{div} \nu \|^2 + \frac{\tau^2}{2} \| \Lambda^{-\frac{1}{2}} (\frac{\tau}{2\sqrt{\gamma}} \bar{\lambda}_2 \bar{v}_0 + \frac{\tau}{2\sqrt{\gamma}} \alpha \text{Div} u_0) \|^2.
\]

by (52), (53), (58) and definition of \(\gamma\), we have

\[
\leq \tau^2 \rho^2 \| e(u) \|^2 + \frac{\tau^2 \beta_{s}^{-2}}{2} \| \Lambda^{-\frac{1}{2}} p \|^2 + \frac{\tau^2 \gamma_{s}^{-2}}{2} \| \Lambda^{-\frac{1}{2}} p \|^2 + \frac{\tau^2}{2} \| \Lambda^{-\frac{1}{2}} (\text{Div} \nu + \alpha \text{Div} u) \|^2 + \tau^2 \| \Lambda^{-\frac{1}{2}} (\Lambda_2 + \Lambda_3) p \|^2
\]

Secondly for \(q\) we have

\[
\| - \delta p - \Lambda^{-1} (\text{Div} + \alpha \text{Div} u) \|_{P}^2 = \frac{\tau^2}{4} (\Lambda ( - \delta p - \Lambda^{-1} (\text{Div} + \alpha \text{Div} u)), - \delta p - \Lambda^{-1} (\text{Div} + \alpha \text{Div} u))
\]

by applying triangle inequality it follows

\[
\leq \frac{\tau^2 \delta^2}{2} \| \Lambda^{-\frac{1}{2}} p \|^2 + \tau^2 \| \Lambda^{-\frac{1}{2}} (\text{Div} + \alpha \text{Div} u) \|^2
\]

Collecting the estimates (59) and (60), we obtain

\[
\| (w; z; \bar{w}; \bar{z}) \|^2_{U \times V} + \| q \|^2_P \leq (2\delta^2 + 2 + 8\beta_{s}^{-2} + 4\delta_{s}^{-2}) (\| (u; v; \bar{u}; \bar{v}) \|^2_{U \times V} + \| p \|^2_P)
\]

Stays to show the coercivity of \(A((u; v; \bar{u}; \bar{v}; p), (w; z; \bar{w}; \bar{z}; q))\). Using the definition of \(A((u; v; \bar{u}; \bar{v}; p), (w; z; \bar{w}; \bar{z}; q))\) and
(w; z; į; q) from (54), it follow
\[
\mathcal{A}((u; v; u; v; p), (w, z; į; q)) = \frac{\mu \tau^2}{2}(\epsilon(u), \epsilon(w)) + \frac{\lambda \tau^2}{4}(\text{div}u, \text{div}w) + (\Lambda_{uv} \left( \begin{array}{c} u \\ v \\ u \\ v \end{array} \right), \left( \begin{array}{c} w \\ z \\ į \end{array} \right))
\]
\[
- \frac{\tau^2}{4}(p, \alpha\text{div}w + \text{Div}z) - \frac{\tau^2}{4}(\alpha\text{div}u + \text{Div}q) - \frac{\tau^2}{4}(\Lambda_1 p, q)
\]
\[
= \frac{\mu \tau^2}{2}(\epsilon(u), \epsilon(\delta u - \frac{\tau}{2\sqrt{\gamma}}u_0)) + \frac{\lambda \tau^2}{4}(\text{div}u, \text{div}(\delta u - \frac{\tau}{2\sqrt{\gamma}}u_0)) + (\Lambda_{uv} \left( \begin{array}{c} u \\ v \\ u \\ v \end{array} \right), \left( \begin{array}{c} \delta u - \frac{\tau}{2\sqrt{\gamma}}u_0 \\ \delta v - \frac{\tau}{2\sqrt{\gamma}}v_0 \\ \delta u - \frac{\tau}{2\sqrt{\gamma}}u_0 \\ \delta v - \frac{\tau}{2\sqrt{\gamma}}v_0 \end{array} \right))
\]
\[
- \frac{\tau^2}{4}(p, \alpha\text{div}(\delta u - \frac{\tau}{2\sqrt{\gamma}}u_0)) + \text{Div}(\delta v - \frac{\tau}{2}\Lambda_2 \frac{\delta}{\delta v} v_0)) - \frac{\tau^2}{4}(\alpha\text{div}u + \text{Div}v, -\delta p - \Lambda^{-1}(\text{Div}v + \alpha\text{div}u))
\]
\[
- \frac{\tau^2}{4}(\Lambda_1 p, -\delta p - \Lambda^{-1}(\text{Div}v + \alpha\text{div}u))
\]
from (52) and (53), it follow,
\[
= \frac{\delta \mu \tau^2}{2}\|\epsilon(u)\|^2 - \frac{\mu \tau^2}{2}(\epsilon(u), \epsilon(\frac{\tau}{2\sqrt{\gamma}}u_0)) + \frac{\delta \lambda \tau^2}{4}\|\text{div}u\|^2 - \frac{\lambda \tau^2}{4}(\text{div}u, \text{div}(\frac{\tau}{2\sqrt{\gamma}}u_0))
\]
\[
+ \delta\|\Lambda_\frac{1}{2}\left( \begin{array}{c} u \\ v \\ u \\ v \end{array} \right)\|^2 - (\Lambda_{uv} \left( \begin{array}{c} u \\ v \\ u \\ v \end{array} \right), \Lambda_{uv} \left( \begin{array}{c} \frac{\tau}{2\sqrt{\gamma}}u_0 \\ \frac{\tau}{2\sqrt{\gamma}}v_0 \\ \frac{\tau}{2\sqrt{\gamma}}u_0 \\ \frac{\tau}{2\sqrt{\gamma}}v_0 \end{array} \right)) + \frac{\tau^2}{4}(p, (\Lambda_2 + \Lambda_3)p) + \frac{\tau^2}{4}\|\Lambda^{-\frac{1}{2}}(\text{Div}v + \alpha\text{div}u)\|^2 + \frac{\delta \tau^2}{4}\|\Lambda_\frac{1}{2} p\|^2
\]
\[
+ \frac{\tau^2}{4}(\Lambda_1 p, \Lambda^{-1}(\text{Div}v + \alpha\text{div}u))
\]
by using Young's inequality, we obtain,
\[
\geq \frac{\delta \mu \tau^2}{2}\|\epsilon(u)\|^2 - \frac{\mu \tau^2}{2}\|\epsilon(u)\|^2 - \frac{\mu \tau^2}{2}(\epsilon(u_0)) + \frac{\delta \lambda \tau^2}{4}\|\text{div}u\|^2 - \frac{\lambda \tau^2}{4}(\text{div}u, \text{div}(\epsilon(\frac{\tau}{2\sqrt{\gamma}}u_0)))
\]
\[
+ \delta\|\Lambda_\frac{1}{2}\left( \begin{array}{c} u \\ v \\ u \\ v \end{array} \right)\|^2 - \frac{\epsilon_3}{2}\|\Lambda_\frac{1}{2} \left( \begin{array}{c} u \\ v \\ u \\ v \end{array} \right)\|^2 - \frac{1}{\epsilon_3}\|\Lambda_\frac{1}{2} \left( \begin{array}{c} \frac{\tau}{2\sqrt{\gamma}}u_0 \\ \frac{\tau}{2\sqrt{\gamma}}v_0 \\ \frac{\tau}{2\sqrt{\gamma}}u_0 \\ \frac{\tau}{2\sqrt{\gamma}}v_0 \end{array} \right)\|^2 + \frac{\tau^2}{4}\|\Lambda_\frac{1}{2} p\|^2 + \frac{\tau^2}{4}\|\Lambda_\frac{1}{2} p\|^2
\]
\[
+ \frac{\tau^2}{4}\|\Lambda^{-\frac{1}{2}}(\text{Div}v + \alpha\text{div}u)\|^2 + \frac{\delta \tau^2}{4}\|\Lambda_\frac{1}{2} p\|^2 - \frac{\tau^2}{8}\|\Lambda^{-\frac{1}{2}}(\text{Div}v + \alpha\text{div}u)\|^2
\]
by using again (53) , (58) and the definition of γ and Λ, we obtain,
\[
\geq \frac{\delta \mu \tau^2}{2}\|\epsilon(u)\|^2 - \frac{\mu \tau^2}{4}\|\epsilon(u)\|^2 - \frac{\mu \tau^2}{8\epsilon_3}\|\Lambda_\frac{1}{2} p\|^2 + \frac{\delta \lambda \tau^2}{4}\|\text{div}u\|^2 - \frac{\lambda \tau^2}{8}\|\text{div}u\|^2 - \frac{\beta \tau^2}{8}\|\Lambda_\frac{1}{2} p\|^2
\]
\[
+ \delta\|\Lambda_\frac{1}{2}\left( \begin{array}{c} u \\ v \\ u \\ v \end{array} \right)\|^2 - \frac{\epsilon_3}{2}\|\Lambda_\frac{1}{2} \left( \begin{array}{c} u \\ v \\ u \\ v \end{array} \right)\|^2 - \frac{1}{\epsilon_3}\|\Lambda_\frac{1}{2} \left( \begin{array}{c} \frac{\tau}{2\sqrt{\gamma}}u_0 \\ \frac{\tau}{2\sqrt{\gamma}}v_0 \\ \frac{\tau}{2\sqrt{\gamma}}u_0 \\ \frac{\tau}{2\sqrt{\gamma}}v_0 \end{array} \right)\|^2 + \frac{\tau^2}{4}\|\Lambda_\frac{1}{2} p\|^2 + \frac{\tau^2}{4}\|\Lambda_\frac{1}{2} p\|^2
\]
\[
+ \frac{\tau^2}{4}\|\Lambda^{-\frac{1}{2}}(\text{Div}v + \alpha\text{div}u)\|^2 + \frac{\delta \tau^2}{4}\|\Lambda_\frac{1}{2} p\|^2 - \frac{\tau^2}{8}\|\Lambda^{-\frac{1}{2}}(\text{Div}v + \alpha\text{div}u)\|^2
\]
Let $\epsilon_1 = 2\beta_1^2$, $\epsilon_2 = 2\beta_2^2$, $\epsilon_3 = 4\max\{\beta_1^2, \beta_2^2\} := 4\beta_2^2$, we obtain
\[
\delta \geq \frac{\delta - \beta_2^2}{2} \|\hat{e}(u)\| - \frac{\tau^2}{16}\|\Lambda_2^{-1}p\|^2 + \frac{\delta - \beta_2^2}{4} \|\text{div}u\|^2 - \frac{\tau^2}{16}\|\Lambda_3^{-1}p\|^2 + \frac{3}{4}\|\text{div}(\lambda^2 u)^\alpha\|^2
\]
\[
- \frac{\tau^2}{16}\|\Lambda_2^{-1}p\|^2 + \frac{\tau^2}{4}\|\Lambda_2^{-1}p\|^2 + \frac{\tau^2}{4}\|\Lambda_1^{-1}p\|^2 + \frac{3}{4}\|\text{div}(\lambda^2 u)^\alpha\|^2 - \frac{\tau^2}{8}\|\Lambda_1^{-1}p\|^2
\]

Let $\delta := 2\beta^2 + \frac{1}{4}$ we obtain. Finally,
\[
A((u; v; \hat{u}; \hat{v}; p), (w; z; \hat{w}; \hat{z}; q)) \geq \frac{1}{4}\|((u; v; \hat{u}; \hat{v})\|_{\bar{U} \times U \times \bar{V} \times V} + \|p\|_{\bar{P}}^2)
\]

The above theorem implies the following stability estimate.

**Corollary 4.7.** Let $(u; v; \hat{u}; \hat{v}) \in \bar{U} \times U \times \bar{V} \times V \times P$ be the solution of (34). Then there holds the estimate
\[
\|((u; v; \hat{u}; \hat{v})\|_{\bar{U} \times U \times \bar{V} \times V} + \|p\|_{\bar{P}} \leq C_1\|((G_1; G_2; G_3; G_3))\|_{\bar{U} \times U \times \bar{V} \times V} + ||G_3||_{\bar{P}^*},
\]
where $C_1$ is a constant independent of all parameters and the network scale $n$ and
\[
\|((G_1; G_2; G_3; G_3))\|_{\bar{U} \times U \times \bar{V} \times V} = \sup_{(w; z; \hat{w}; \hat{z}) \in \bar{U} \times U \times \bar{V} \times V} \|((G_1; G_2; G_3; G_3))\|_{\bar{U} \times U \times \bar{V} \times V} + ||G_3||_{\bar{P}^*} = \sup_{q \in \bar{P}} \|q\|_{\bar{P}} = ||A^{-\frac{1}{2}}G_3||.
\]

**4.2 Stability of the fully discrete problem**

The main result of this section is a proof of the uniform well-posedness of problem (39) under the norms induced by (42) and (35).

**Theorem 4.8.** There exists a constant $C_4$ independent of all parameters, the network scale $n$ and the mesh size $h$, such that for any $(u_h; v_h; \hat{u}_h; \hat{v}_h; p_h), (w_h; z_h; \hat{w}_h; \hat{z}_h; q_h) \in \bar{U}_h \times U_h \times \bar{V}_h \times V_h \times P_h$
\[
|A_h((u_h; v_h; \hat{u}_h; \hat{v}_h; p_h), (w_h; z_h; \hat{w}_h; \hat{z}_h; q_h))| \leq C_4\|((u_h; v_h; \hat{u}_h; \hat{v}_h))\|_{\bar{U}_h \times U_h \times \bar{V}_h \times V_h} + ||p_h||_{\bar{P}}
\]
\[
\|((w_h; z_h; \hat{w}_h; \hat{z}_h))\|_{\bar{U}_h \times U_h \times \bar{V}_h \times V_h} + ||q_h||_{\bar{P}}.
\]

**Proof.** The proof of this theorem can be obtained by following the proof of Theorem 4.5 

The following theorem shows the inf-sup-condition (LBB) of $A_h((u; v; \hat{u}; \hat{v}); (w; z; \hat{w}; \hat{z}))$

**Theorem 4.9.** There exists a constant $\omega_h > 0$ independent of all parameters, the network scale $n$ and the mesh size $h$, such that
\[
\inf_{(u_h; v_h; \hat{u}_h; \hat{v}_h; p_h) \in \bar{U}_h \times U_h \times \bar{V}_h \times V_h \times P_h} \sup_{(w_h; z_h; \hat{w}_h; \hat{z}_h; q_h) \in \bar{U}_h \times U_h \times \bar{V}_h \times V_h \times P_h} A_h((u_h; v_h; \hat{u}_h; \hat{v}_h; p_h), (w_h; z_h; \hat{w}_h; \hat{z}_h; q_h))
\]
\[
\|((u; v; \hat{u}; \hat{v}; p_h))\|_{\bar{U}_h \times U_h \times \bar{V}_h \times V_h} + ||p_h||_{\bar{P}}(\|((w; z; \hat{w}; \hat{z}; q_h))\|_{\bar{U}_h \times U_h \times \bar{V}_h \times V_h} + ||q_h||_{\bar{P}}) \geq \omega_h.
\]
Proof. For any \((u_h; v_h; \tilde{u}_h; \tilde{v}_h; p_h) \in \tilde{U}_h \times U_h \times \tilde{V}_h \times V_h \times P_h\), from 46, there exist
\[
v_{0,h} \in U \text{ such that } \text{Div} v_{0,h} = \frac{\tau}{2} \tilde{A}_{22}^{\frac{1}{2}} p_h \text{ and } \|v_{0,h}\|_{\text{div}} \leq \beta_{v,h}^{-1} \|\Lambda_{2}^{\frac{1}{2}} p_h\|,
\]
also, there exists
\[
u_{0,h} \in \tilde{U}_h \text{ such that } \text{Div} \nu_{0,h} = \frac{\tau}{2} A_{22}^{\frac{1}{2}} p_h, \|\nu_{0,h}\|_1 \leq \frac{\tau \beta_{\nu,h}^{-1}}{2} \|\Lambda_{2}^{\frac{1}{2}} p_h\| = \beta_{\nu,h}^{-1} \|\Lambda_{2}^{\frac{1}{2}} p_h\|.
\]
Choose
\[
w_h = \delta u_h - \frac{\tau}{2} \sqrt{7} u_{0,h}, \quad z_h = \delta v_h - \frac{\tau}{2} \tilde{A}_{22}^{\frac{1}{2}} v_{0,h}, \quad \tilde{w}_h = \delta \tilde{u}_h, \quad \tilde{z}_h = \delta \tilde{v}_h, \quad q_h = -\delta p_h - \frac{\tau^2}{4} \Lambda^{-1} (\text{Div} \nu_h + \alpha \text{Div} u_h),
\]
where \(\delta\) is a positive constant to be determined later.

Following the proof of Theorem 4.6, we try to estimate \(\|\Lambda_{2}^{\frac{1}{2}} \left( \begin{array}{ccc} \frac{\tau}{2} A_{22}^{\frac{1}{2}} v_{0,h} \\ 0 \\ 0 \end{array} \right) \|^2\):
\[
\|\Lambda_{2}^{\frac{1}{2}} \left( \begin{array}{ccc} \frac{\tau}{2} A_{22}^{\frac{1}{2}} v_{0,h} \\ 0 \\ 0 \end{array} \right) \|^2 \leq \frac{\tau^2}{4} \lambda_{\text{max}}(G) (\|u_h\|_2^2 + \|v_{0,h}\|_2^2) \leq \frac{\tau^2}{4} \lambda_{\text{max}}(G) \left( c_2 \beta_{v,h}^{-2} \|\Lambda_{2}^{\frac{1}{2}} p_h\|^2 + \beta_{\nu,h}^{-2} \|\Lambda_{2}^{\frac{1}{2}} p_h\|^2 \right)
\]
\[
\leq \frac{\tau^2}{4} \left( c_3 \beta_{v,h}^{-2} \|\Lambda_{2}^{\frac{1}{2}} p_h\|^2 + \beta_{\nu,h}^{-2} \|\Lambda_{2}^{\frac{1}{2}} p_h\|^2 \right)
\]
Let now verify the boundedness of \((w_h; z_h; \tilde{w}_h; \tilde{z}_h; q_h)\) by \((u_h; v_h; \tilde{u}_h; \tilde{v}_h; p_h)\). Firstly for \((w_h; z_h; \tilde{w}_h; \tilde{z}_h; q_h)\) we have,
\[
\|(w_h; z_h; \tilde{w}_h; \tilde{z}_h; q_h)\|_{\tilde{U}_h \times U \times \tilde{V} \times V}^2 = \left\| \left( \begin{array}{cc} \delta u_h - \frac{\tau}{2} \sqrt{7} u_{0,h} & \delta v_h - \frac{\tau}{2} \tilde{A}_{22}^{\frac{1}{2}} v_{0,h} \\ \delta \tilde{u}_h & \delta \tilde{v}_h \end{array} \right) \right\|_{\tilde{U}_h \times U \times \tilde{V} \times V}^2
\]
\[
= \frac{\tau^2}{2} \|\delta u_h - \frac{\tau}{2} \sqrt{7} u_{0,h}\|_{\text{DG}}^2 + \frac{\tau^2}{4} \|\text{Div} \delta u_h - \frac{\tau}{2} \sqrt{7} u_{0,h}\|_2^2 + \|\Lambda_{2}^{\frac{1}{2}} \left( \begin{array}{ccc} \delta u_h - \frac{\tau}{2} \sqrt{7} u_{0,h} \\ \delta v_h - \frac{\tau}{2} \tilde{A}_{22}^{\frac{1}{2}} v_{0,h} \end{array} \right) \|^2
\]
\[
+ \frac{\tau^2}{4} \|\Lambda^{-\frac{1}{2}} \left( \begin{array}{c} \text{Div} \delta u_h - \frac{\tau}{2} \sqrt{7} u_{0,h} \end{array} \right) + \alpha \text{Div} \delta u_h - \frac{\tau}{2} \sqrt{7} u_{0,h} \|^2,
\]
by applying triangle inequality it follows
\[
\leq \frac{\tau^2}{2} \|\delta u_h\|_{\text{DG}}^2 + \frac{\tau^2}{2} \|\text{Div} \delta u_h\|_2^2 + \frac{\tau^2}{2} \|\delta \tilde{v}_h\|_2^2 + \frac{\tau^2}{2} \|\text{Div} \delta \tilde{v}_h\|_2^2 + 2 \delta^2 \|\Lambda_{2}^{\frac{1}{2}} \left( \begin{array}{ccc} u_h \\ v_h \\ \tilde{u}_h \\ \tilde{v}_h \end{array} \right) \|^2
\]
\[
+ 2 \|\Lambda_{2}^{\frac{1}{2}} \left( \begin{array}{ccc} \frac{\tau}{2} \sqrt{7} u_{0,h} \\ 0 \\ 0 \\ 0 \end{array} \right) \|^2 + \frac{\tau^2}{2} \|\Lambda^{-\frac{1}{2}} (\text{Div} \nu_h + \alpha \text{Div} u_h)\|^2 + \frac{\tau^2}{2} \|\Lambda^{-\frac{1}{2}} (\frac{\tau}{2} \tilde{A}_{22}^{\frac{1}{2}} \text{Div} v_{0,h} + \frac{\tau}{2} \sqrt{7} \alpha \text{Div} u_{0,h})\|^2.
\]
by (62), (63), (65), (43) and definition of $\gamma, \Lambda$, we have

$$
\leq \tau^2 \mu^2 \|u_h\|_{DG}^2 + \frac{c_0 \tau^2 \beta_{s+2}^2}{2} \|\Lambda_3^2 p_h\|^2 + \frac{\tau^2 \beta_{s+2}^2}{2} \|\text{div} u_h\|^2 + \frac{\tau^2 \beta_{s+2}^2}{2} \|\Lambda_3^2 p_h\|^2 + 2\delta^2 \|\Lambda_{s,v}^+\| \left( \begin{array}{c} u_h \\ v_h \\ u_h \\ v_h \end{array} \right) \|^2
$$

$$
+ \tau^2 \left( c_3 \beta_{s+2}^2 \|\Lambda_3^2 p_h\|^2 + \beta_{s+2}^2 \|\Lambda_3^2 p_h\|^2 \right) + \frac{\tau^2 \delta^2}{2} \|\Lambda^{-\frac{1}{2}} (\text{Div} v_h + \alpha \text{Div} u_h)\|^2 + \frac{\tau^2}{2} \|\Lambda^{-\frac{1}{2}} ((\Lambda_2 + \Lambda_3) p_h)\|^2
$$

(66)

Secondly for $q_h$ we have

$$
\| - \delta p_h - \Lambda^{-1} (\text{Div} v_h + \alpha \text{Div} u_h) \|^2 \leq \frac{\tau^2}{4} (\Lambda (-\delta p_h - \Lambda^{-1} (\text{Div} v_h + \alpha \text{Div} u_h)), -\delta p_h - \Lambda^{-1} (\text{Div} v_h + \alpha \text{Div} u_h))
$$

$$
\leq \frac{\tau^2}{4} \|\Lambda^\frac{1}{2} p_h\|^2 + \frac{\tau^2}{4} \|\Lambda^{-\frac{1}{2}} (\text{Div} v_h + \alpha \text{Div} u_h)\|^2
$$

(67)

Collecting the estimates (66) and (67), we obtain

$$
\| (w_h, \tilde{w}_h, \tilde{z}_h) \|_{L^2(W \times V \times V)}^2 + \|q_h\|^2 \leq \left( 2\delta^2 + (2 + 2c_0 + 4c_3) \beta_{s+2}^2 + 4\beta_{s+2}^2 \right) \| (u_h, v_h, w_h, \tilde{z}_h, q_h) \|_{L^2(W \times U \times V \times V)}^2 + \| p_h \|^2
$$

(68)

Stays to show the coercivity of $\mathcal{A}(u_h; v_h; \tilde{w}_h; \tilde{z}_h; p_h; (w_h, \tilde{w}_h, \tilde{z}_h, q_h))$. Using the definition of $\mathcal{A}(u_h; v_h; \tilde{w}_h; \tilde{z}_h; p_h; (w_h, \tilde{w}_h, \tilde{z}_h, q_h))$ and (64), it follow

$$
\mathcal{A}(u_h; v_h; \tilde{w}_h; \tilde{z}_h, p_h, (w_h, \tilde{w}_h, \tilde{z}_h, q_h)) = \frac{\mu^2 \tau^2}{2} a_h(u_h, w_h) + \frac{\lambda \tau^2}{4} (\text{div} u_h, \text{div} w_h) + (\Lambda_{uv} \left( \begin{array}{c} u_h \\ v_h \\ u_h \\ v_h \end{array} \right), \left( \begin{array}{c} w_h \\ z_h \\ \tilde{w}_h \\ \tilde{z}_h \end{array} \right))
$$

$$
- \frac{\tau^2}{4} (p_h, \alpha \text{Div} w_h + \text{Div} z_h) - \frac{\tau^2}{4} (\alpha \text{Div} u_h + \text{Div} q_h, p_h) - \frac{\tau^2}{4} (\Lambda_1 p_h, q_h)
$$

$$
= \frac{\mu^2 \tau^2}{2} a_h(u_h, \delta u_h) - \frac{\tau}{2\sqrt{\gamma}} (u_{0,h}, \delta u_h) + \frac{\lambda \tau^2}{4} (\text{div} u_h, \text{div} (\delta u_h - \frac{\tau}{2\sqrt{\gamma}} u_{0,h})) + (\Lambda_{uv} \left( \begin{array}{c} u_h \\ v_h \\ u_h \\ v_h \end{array} \right), \left( \begin{array}{c} \delta u_h - \frac{\tau}{2\sqrt{\gamma}} u_{0,h} \\ \delta v_h - \frac{\tau}{2\sqrt{\gamma}} v_{0,h} \end{array} \right))
$$

$$
- \frac{\tau^2}{4} (p_h, \alpha \text{Div} (\delta u_h - \frac{\tau}{2\sqrt{\gamma}} u_{0,h}) + \text{Div} (\delta v_h - \frac{\tau}{2\sqrt{\gamma}} v_{0,h})) - \frac{\tau^2}{4} (\alpha \text{Div} u_h + \text{Div} v_h, -\delta p_h - \Lambda^{-1} (\text{Div} v_h + \alpha \text{Div} u_h))
$$

$$
- \frac{\tau^2}{4} (\Lambda_1 p_h, -\delta p_h - \Lambda^{-1} (\text{Div} v_h + \alpha \text{Div} u_h))
$$

from (62) and (63), it follow,

$$
= \frac{\delta \mu^2 \tau^2}{2} a_h(u_h, u_h) - \frac{\mu^2 \tau^2}{2} a_h(u_h, \frac{\tau}{2\sqrt{\gamma}} u_{0,h}) + \frac{\delta \lambda \tau^2}{4} (\text{div} u_h, \text{div} (\frac{\tau}{2\sqrt{\gamma}} u_{0,h})) + (\Lambda_{uv} \left( \begin{array}{c} u_h \\ v_h \\ u_h \\ v_h \end{array} \right), \left( \begin{array}{c} u_h \\ v_h \\ u_h \\ v_h \end{array} \right))
$$

$$
- (\Lambda_{uv} \left( \begin{array}{c} u_h \\ v_h \\ u_h \\ v_h \end{array} \right), \left( \begin{array}{c} \frac{\tau}{2\sqrt{\gamma}} u_{0,h} \\ \frac{\tau}{2\sqrt{\gamma}} v_{0,h} \\ 0 \\ 0 \end{array} \right)) + \frac{\tau^2}{4} (p_h, (\Lambda_2 + \Lambda_3) p_h) + \frac{\tau^2}{4} \|\Lambda^{-\frac{1}{2}} ((\Lambda_2 + \Lambda_3) p_h)\|^2 + \frac{\tau^2}{4} \|\Lambda^{-\frac{1}{2}} (\text{Div} v_h + \alpha \text{Div} u_h)\|^2
$$

$$
+ \frac{\tau^2}{4} (\Lambda_1 p_h, -\Lambda^{-1} (\text{Div} v_h + \alpha \text{Div} u_h)))
$$

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by using Young’s inequality, (44) and (47) we obtain,
\[
\begin{align*}
&\geq \frac{\delta_\alpha \mu^2 \tau^2}{2} \| u_h \|^2 - \frac{\mu^2 \epsilon_1}{4} \| u_h \|^2_D - \frac{\mu^2 \epsilon_2}{4} \| u_h \|^2_{D^2} + \frac{\delta_\lambda \tau^2}{4} \| \delta \alpha \|_{\phi} \| u_h \|^2 - \frac{\lambda \tau^2}{8} \| \delta \alpha \|_{\phi} \| u_h \|^2 - \frac{\lambda \tau^2}{8} \| \delta \alpha \|_{\phi} \| u_h \|^2 \\
&+ \delta \| A_{\phi} \|_{\phi} \left( \begin{array}{c} u_h \\ v_h \\ u_h \\ v_h \\ v_h \end{array} \right) \|^2 \geq \frac{\delta_\alpha}{2} \| A_{\phi} \|_{\phi} \left( \begin{array}{c} u_h \\ v_h \\ u_h \\ v_h \\ v_h \end{array} \right) \|^2 - \frac{\lambda \tau^2}{8} \| \delta \alpha \|_{\phi} \| u_h \|^2 - \frac{\lambda \tau^2}{8} \| \delta \alpha \|_{\phi} \| u_h \|^2 \end{align*}
\]

by using again (63), (65), (43), and the definition of γ, we obtain,
\[
\begin{align*}
&\geq \frac{\alpha_1 \delta \mu^2 \tau^2}{2} \| u_h \|^2 - \frac{\mu^2 \epsilon_1 \epsilon_2}{4} \| u_h \|^2_{D^2} - \frac{\mu^2 \epsilon_2}{4} \| u_h \|^2_{D^2} + \frac{\delta_\lambda \tau^2}{8} \| \delta \alpha \|_{\phi} \| u_h \|^2 - \frac{\lambda \tau^2}{8} \| \delta \alpha \|_{\phi} \| u_h \|^2 - \frac{\lambda \tau^2}{8} \| \delta \alpha \|_{\phi} \| u_h \|^2 \\
&+ \delta \| A_{\phi} \|_{\phi} \left( \begin{array}{c} u_h \\ v_h \\ u_h \\ v_h \\ v_h \end{array} \right) \|^2 \geq \frac{\delta_\alpha}{2} \| A_{\phi} \|_{\phi} \left( \begin{array}{c} u_h \\ v_h \\ u_h \\ v_h \\ v_h \end{array} \right) \|^2 - \frac{\lambda \tau^2}{8} \| \delta \alpha \|_{\phi} \| u_h \|^2 - \frac{\lambda \tau^2}{8} \| \delta \alpha \|_{\phi} \| u_h \|^2 \end{align*}
\]

Let \( \gamma_1 = 2 \beta_{\phi, u}^2 c_2 \epsilon_2, \epsilon_2 = 2 \beta_{\phi, u}^2 \epsilon_3 = 4 \max \{ \beta_{\phi, u}^2, \beta_{\phi, u}^2 \} = 4 \beta_{\phi, u}^2, \) we obtain
\[
\begin{align*}
&\geq \frac{\alpha_1 \delta \mu^2 \tau^2}{2} \| u_h \|^2_{D^2} - \frac{\tau^2}{16} \| \Lambda_{\phi} \|^2 \| u_h \|^2 - \frac{\tau^2}{16} \| \Lambda_{\phi} \|^2 \| u_h \|^2 + \frac{\delta \lambda \tau^2}{4} \| \delta \alpha \|_{\phi} \| u_h \|^2 - \frac{\lambda \tau^2}{4} \| \delta \alpha \|_{\phi} \| u_h \|^2 - \frac{\lambda \tau^2}{4} \| \delta \alpha \|_{\phi} \| u_h \|^2 \\
&- \frac{\tau^2}{16} \| \Lambda_{\phi} \|^2 \| u_h \|^2 + \| \Lambda_{\phi} \|^2 \| u_h \|^2 \end{align*}
\]

Let \( \delta := \frac{\max \{ \beta_{\phi, u}^2 c_2 \epsilon_2, \beta_{\phi, u}^2 \} \tau^2}{\min \{ \gamma_1, \epsilon_3 \}} + \frac{\delta}{2} \) we obtain Finally,
\[
\mathcal{A}(u_h; v_h; \dot{u}_h; \dot{v}_h; p_h) \geq \frac{1}{4} \| \|u_h; v_h; \dot{u}_h; \dot{v}_h \|_{D^2 \times U \times \dot{U} \times \dot{V} \times \dot{V}} + \| p_h \|^2
\]

The following stability estimate is a consequence of the above theorem.

**Corollary 4.10.** Let \((u_h; v_h; \dot{u}_h; \dot{v}_h; p_h) \in \hat{U}_h \times \hat{V}_h \times \hat{V}_h \times \hat{P}_h\) be the solution of (39), then we have the estimate
\[
\|(u_h; v_h; \dot{u}_h; \dot{v}_h)\|_{D^2 \times U \times \dot{U} \times \dot{V} \times \dot{V}} + \| p_h \|_{P} \leq C_2 \|(G_1; G_2; G_3; G_4)\|_{D^2 \times U \times \dot{U} \times \dot{V} \times \dot{V}} + \| G_5 \|_{P^*},
\] (68)
holds where
\[
\|(G_1; G_2; G_3; G_4)\|_{D^2 \times U \times \dot{U} \times \dot{V} \times \dot{V}} = \sup \|G_1; G_2; G_3; G_4\|_{\hat{W}_h \times \hat{U}_h \times \hat{V}_h \times \hat{V}_h}, \quad \| G_5 \|_{P^*} = \sup \|G_5; q\|_{P} = \| \Lambda_{\phi} \|^2 \| G_5 \|_{P^*}
\]
and \(C_2\) is a constant independent of all parameters the network scale \(n\), and the mesh size \(h\).
4.3 Consequences

4.3.1 Norm-equivalent preconditioner

Remark 4.11. Let \( \Lambda = (\gamma_{ij})_{n \times n}, \Lambda^{-1} = (\tilde{\gamma}_{ij})_{n \times n} \). Define

\[
B := \begin{bmatrix} B_{w}^{-1} & 0 \\ B_{p}^{-1} & \Lambda \end{bmatrix}
\]

where

\[
B_{w} = \frac{\tau^2}{4} \begin{bmatrix} \bar{B}_{w} & 0 \\ 0 & 0 \end{bmatrix} + \Lambda_{w}, \quad B_{p} = \frac{\tau^2}{4} \begin{bmatrix} \tilde{\gamma}_{11} I & \tilde{\gamma}_{12} I & \ldots & \tilde{\gamma}_{1n} I \\ \tilde{\gamma}_{21} I & \tilde{\gamma}_{22} I & \ldots & \tilde{\gamma}_{2n} I \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\gamma}_{n1} I & \tilde{\gamma}_{n2} I & \ldots & \tilde{\gamma}_{nn} I \end{bmatrix}
\]

\[
\bar{B}_{w} := \begin{bmatrix} 2\mu \text{div} - \lambda \text{div} - \sum_{i,j=1}^{n} \alpha_{ij} \gamma_{ij} \text{div} & -\sum_{i=1}^{n} \alpha_{i} \gamma_{i1} \text{div} & \ldots & -\sum_{i=1}^{n} \alpha_{i} \gamma_{in} \text{div} \\ -\sum_{i=1}^{n} \alpha_{i} \gamma_{i1} \text{div} & -\gamma_{11} \text{div} & \ldots & -\gamma_{1n} \text{div} \\ \vdots & \vdots & \ddots & \vdots \\ -\sum_{i=1}^{n} \alpha_{i} \gamma_{in} \text{div} & -\gamma_{n1} \text{div} & \ldots & -\gamma_{nn} \text{div} \end{bmatrix}
\]

Emulating from the theory presented in [19], Theorems 4.5 and 4.6 imply that the operator \( B \) in (69) defines a norm-equivalent (canonical) block-diagonal preconditioner for the operator \( A \) which is robust in all model parameters.

Remark 4.12. Let \( W_h := \bar{U}_h \times U_h \times \bar{V}_h \times V_h \times P_h \) be equipped with the norm \( \| \cdot \|_{\bar{W}_h} := \| \cdot \|_{\bar{E}_h \times U_h \times \bar{V}_h \times V_h} + \| \cdot \|_{\bar{P}_h} \) and consider the operator

\[
A_h := \begin{bmatrix} -\text{div}_{h} \sigma_h + \gamma_u & -\gamma_1 & \ldots & -\gamma_n & -2\tau^{-1} & 0 & \ldots & 0 & \alpha_1 \nabla_h & \ldots & \alpha_n \nabla_h \\ -\gamma_1 & \gamma_{v,1} & \ldots & 0 & 0 & -2\tau^{-1} & \ldots & 0 & \nabla_h & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\gamma_n & 0 & \ldots & \gamma_{v,n} & 0 & 0 & \ldots & -2\tau^{-1} & 0 & \ldots & \nabla_h \\ -2\tau^{-1} & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & -2\tau^{-1} & \ldots & 0 & 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -2\tau^{-1} & 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \\ -\alpha_1 \text{div}_{h} & -\text{div}_{h} & \ldots & 0 & 0 & 0 & \ldots & 0 & -\beta_{11} & \ldots & \beta_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_n \text{div}_{h} & 0 & \ldots & -\text{div}_{h} & 0 & 0 & \ldots & 0 & \beta_{n1} & \ldots & -\beta_{nn} \end{bmatrix}
\]

induced by the bilinear form (48). Clearly, \( A_h \) is self-adjoint and indefinite on \( W_h \). Moreover, Theorems 4.8 and 4.9 imply that it is a uniform isomorphism in the sense of being bounded and having a bounded inverse with bounds independent of the mesh size, the network scale, and the model parameters. Following the framework in the study of Mardal et al. [19], we define the self-adjoint positive definite operator

\[
B_h := \begin{bmatrix} B_{h,w}^{-1} & 0 \\ 0 & B_{h,p}^{-1} \end{bmatrix}
\]
where

\[ B_{h,wv} = \frac{\tau^2}{4} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + A_{wv}, \quad B_{ph} = \frac{\tau^2}{4} \begin{bmatrix} \tilde{\gamma}_{11} I & \tilde{\gamma}_{12} I & \cdots & \tilde{\gamma}_{1n} I \\ \tilde{\gamma}_{21} I & \tilde{\gamma}_{22} I & \cdots & \tilde{\gamma}_{2n} I \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\gamma}_{n1} I & \tilde{\gamma}_{n2} I & \cdots & \tilde{\gamma}_{nn} I \end{bmatrix}, \]

\[ \tilde{B}_{h,wv} := \begin{bmatrix} -2\mu \text{div} \cdot \nabla_h \text{div} \cdot - \lambda \nabla_h \text{div} \cdot - \sum_{i,j=1}^{n} \alpha_i \tilde{\gamma}_{ij} \nabla_h \text{div} \cdot & - \sum_{i=1}^{n} \alpha_i \tilde{\gamma}_{i1} \nabla_h \text{div} \cdot & \cdots & - \sum_{i=1}^{n} \alpha_i \tilde{\gamma}_{in} \nabla_h \text{div} \cdot \\ - \sum_{i=1}^{n} \alpha_i \tilde{\gamma}_{1i} \nabla_h \text{div} \cdot & - \tilde{\gamma}_{11} \nabla_h \text{div} \cdot & \cdots & - \tilde{\gamma}_{1n} \nabla_h \text{div} \cdot \\ \vdots & \vdots & \ddots & \vdots \\ - \sum_{i=1}^{n} \alpha_i \tilde{\gamma}_{ni} \nabla_h \text{div} \cdot & - \tilde{\gamma}_{n1} \nabla_h \text{div} \cdot & \cdots & - \tilde{\gamma}_{nn} \nabla_h \text{div} \cdot \end{bmatrix}. \]

It is obvious that

\[ \langle B_{h}^{-1} x_h, x_h \rangle \approx \| x_h \|_{W_h}, \]

where \( x_h = (u_h, v_h, \bar{u}_h, \bar{v}_h, p_h) \in W_h \) "\( \sim \" stands for a norm equivalence, uniform with respect to model and discretization parameters; and \( \langle \cdot, \cdot \rangle \) expresses the duality pairing between \( W_h \) and \( W_h^* \), that is, \( B_{h}^{-1} \) is a uniform isomorphism. By using the properties of \( B_h \) and \( A_h \) when solving the generalized eigenvalue problem \( A_h x_h = \xi B_{h}^{-1} x_h \), the condition number \( \kappa(B_h A_h) \) is easily shown to be uniformly bounded with respect to the all parameter, the network scale \( n \), and the mesh size \( h \). Therefore, \( B_h \) defines a uniform preconditioner.

5 Conclusions

In this paper, we analyze the stability properties of the time-discrete systems arising from second-order implicit time stepping schemes applied to the variational formulation of the MPET model and prove an inf-sup condition with a constant that is independent of all model parameters. Moreover, we show that the fully discrete models obtained for a family of strongly conservative space discretizations are also uniformly stable with respect to the spatial discretization parameter. The norms in which these results hold are the basis for parameter-robust preconditioners The transfer of the canonical (norm-equivalent) operator preconditioners from the continuous and the discrete level lays the foundation for optimal and fully robust iterative solution methods.

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