The solar cycle was discovered by Schwabe in 1843 from observations of the sunspot number as a function of time. Schwabe determined a period for the solar cycle of approximately 10 yr and also described an irregular behavior, with fluctuations in the period and in the extent and intensity of the maxima. The sunspot number may vary as much as 100% from cycle to cycle, while the period can display fluctuations within the range of 2 yr. Many theoretical models connecting the solar magnetic fields with the activity cycle have been developed that adequately model the regularities observed in the solar cycle, such as the polar reversals and butterfly diagrams. These models involve many different mechanisms such as current generated by turbulent motions (Krause & Rädler 1980), Babcock-Leighton pictures (Leighton 1969; Durney 1995; Dikpati & Charbonneau 1999, and references therein), or buoyant instabilities of magnetic flux tubes within the base of the convective region (Ferriz-Mas, Schmitt, & Schüssler 1994; Caligari, Moreno-Insertis, & Schüssler 1995). However, the irregularities observed in the solar cycle still await satisfactory explanations.

Sunspots correspond to large concentrations of magnetic field on the solar surface. The generation of this field is explained by the dynamo effect, which is produced by the coupling of magnetic field with the subphotospheric velocity field. A dynamo model for the solar cycle should at least be able to explain the sunspot number mean period, the irregularities observed in the period and in the intensity of the maxima, the migration of the region of magnetic activity toward the equator as the cycle progresses, and the observed asymmetries in sunspot numbers between the northern and southern hemispheres.

One of the most remarkable features of the solar cycle is the long periods of strongly reduced activity in the solar cycle. The most famous of these is the Maunder minimum during the 17th century. Studies of $^{10}$Be records and other cosmogenic radioisotopes suggest that these minima have occurred with a certain regularity in the past (Wagner et al. 2001). Within this context, it must be pointed out that these periods of reduced activity do not need to correspond to epochs where the magnetic cycle shuts off completely (Beer, Tobias, & Weiss 1998).

To explain the nature of the irregularities displayed by the solar cycle, two different mechanisms have been invoked in the literature: chaos or stochasticity. In the chaotic scenario, the influence of the Lorentz force is considered in a dynamical fashion. Although there are theoretical models described by partial differential equations (Jennings & Weiss 1991), the complexity of the problem caused the effects of nonlinearities to be studied more extensively in truncated models (Zeldovich & Ruzmaikin 1990; Knobloch & Landsberg 1996; Knobloch, Tobias, & Weiss 1998, and references therein). This variety of models is due to the fact that formal truncations are only possible in special cases. More specifically, the mathematical algorithmic reduction to normal forms ensures the uniqueness of these projections only close to linear singularities (Holmes, Lumley, & Berkooz 1996; Guckenheimer & Holmes 1997). Therefore, the robustness of these truncations are expected to be limited to a restricted region in parameter space. Nonetheless, spatial truncations can still be used with the aim of generating a simple model to study the dynamics of the time series. We believe that comparisons between these theoretical models and observations can in fact provide a useful insight to explain the solar cycle variability.

However, the time series obtained are rarely compared with the observed sunspot time series. Also, it has been shown that the introduction of degrees of freedom in such truncated systems usually destroys the properties of the chaotic attractor (Weiss, Cattaneo, & Jones 1984; Cattaneo, Hughes, & Kim 1996; see, however, Tobias 1997). Moreover, there are no firm indicators of the existence of chaos in the sunspot number time series (Carbonell, Oliver, & Ballester 1994). Recent studies have shown that the
standard algorithms used to search for signatures of a low-dimensional chaotic attractor in time series (such as Lyapunov exponent estimators, correlation dimensions, and increase of a prediction error with a prediction horizon) can lead to spurious convergence when applied to limited time series (see Theiler 1986 and Dämming & Mitschke 1993 for a review). Also, it was shown that the increase in the prediction error with an increase in the prediction horizon can in fact be observed in systems with a deterministic skeleton and stochastic components (Ruelle 1990; Eckmann & Ruelle 1992).

On the other hand, it has been pointed out that the spatial and temporal complexity of the convective cells on the Sun can naturally give rise to a stochastic process underlying the dynamo equations. In turbulent dynamo theory, field fluctuations play a crucial role, since it is an essentially statistical theory (Choudhuri 1992; Hoyng 1993; Ossendrijver & Hoyng 1996, and references therein). In other pictures, such as Babcock-Leighton models, where the poloidal field is regenerated by the cumulative effects of a finite number of discrete flux emergence events, the idea carries over in a direct way (Charbonneau & Dikpati 2000). Within this framework, the spatial and temporal irregularities observed in the time series can be described as the end result of the stochastic process underlying the magnetic eruption to generate the spots.

Current nonlinear models do not account for this, and they suggest that the aperiodic behavior is caused by deterministic chaos. On the other hand, recent works provide evidence for a nonlinear oscillator underlying the dynamics of the sunspot cycle (Paluš & Novotná 1999). While stochastic models account for fluctuations in the mean fields, the corresponding equations are usually linear in the field components, and nonlinear effects are neglected (Hoyng, Schmitt, & Teuben 1994). In most cases, the equations are severely truncated, preserving just harmonic traveling waves or the first eigenmodes of the linearized equations (Hoyng 1993; Ossendrijver, Hoyng, & Schmitt 1996, and references therein). In cases where nonlinearities were considered, either the equations are integrated in a plane geometry (Choudhuri 1992; Ossendrijver & Hoyng 1996) or severe truncations are applied, which only preserve the temporal dynamics in a set of ordinary differential equations (Mini, Gómez, & Mindlin 2000, 2001). Also, important aspects of the problem such as the role of meridional circulation are usually not considered (see, however, Charbonneau & Dikpati 2000 for a study of stochastic fluctuations in a two-dimensional Babcock-Leighton model with meridional flow).

Given the strength of the magnetic field at the base of the convection zone, the importance of nonlinear effects in the dynamics of the solar dynamo cannot be neglected. Also, as mentioned above, the role of stochasticity in the solar case is nonnegligible. Many observed irregularities in the solar cycle, such as north-south asymmetries, have been interpreted by different authors in terms of either chaotic behavior (Tobias, Weiss, & Kirk 1995; Beer et al. 1998) or stochastic forcing (Ossendrijver, Hoyng, & Schmitt 1996). Some phenomena can be addressed successfully considering stochasticity, such as the well-known phase locking of the cycle (Charbonneau & Dikpati 2000). On the other hand, there are phenomena whose more natural explanation seems to be nonlinear dynamics, such as the well-known Maunder minimum, since it turns out to be rather difficult to produce periods of long-lasting reduced activity in stochastic models. However, see Moss et al. (1992) for a nonlinear dynamo driven by noise with long-term periodic behavior resembling Maunder minima occurrence. Therefore, it seems apparent that these phenomena can be more properly addressed by an adequate superposition of dynamical effects and stochastic forcing.

In this paper, we present a simple one-dimensional dynamo model in spherical coordinates, where nonlinear effects are responsible for the saturation of the magnetic field and the nonlinear mode coupling, while irregularities are excited using random forcing in the source term. The stochastic forcing is intended to provide a better treatment of the averaging process over the convective and turbulent motions. The intrinsically noisy behavior of these motions is then incorporated into the model, and its role in the irregularities of the solar cycle can therefore be analyzed. In this work, we focus on the study of the role of irregularities observed in the butterfly diagrams for the solar cycle. We find that the interaction of the deterministic dynamo equations with this stochastic forcing can provide a satisfactory explanation for some of the irregular features observed for the Sun, such as relaxation oscillations in the sunspot time series, variability in the amplitude and period, north-south asymmetries in the butterfly diagrams, and the existence of reverse polarity regions close to the equator. We believe that this treatment of the solar-cycle irregularities could in principle be extended to study long-time trends as well, since our model includes both stochastic and dynamical effects.

2. DYNAMO EQUATIONS

The complete mathematical description of the solar dynamo consists of the induction equation for the evolution of the magnetic field \( B \) coupled to the Navier-Stokes equation for the velocity field \( U \). The induction equation is

\[
\frac{\partial B}{\partial t} = \nabla \times (U \times B) + \eta \nabla^2 B ,
\]

where \( \eta \) is the magnetic diffusivity. In spherical coordinates, we write down the magnetic field in terms of its toroidal and poloidal components, i.e., \( B = B_\phi \phi + B_\theta r \), where \( B_\phi = \nabla \times (A_\phi, \phi) \). All quantities are assumed to be averages over the longitudinal angle \( \phi \), which leads to an axisymmetric set of dynamical equations. We will work under the kinematic approximation (i.e., we assume the velocity field as given), with the flow field given by \( U = u_r(r, \theta) \hat{r} + u_\theta(r, \theta) \hat{\theta} + r \sin \omega(r, \theta) \phi \hat{\phi} \). Therefore, we can write down the spherical components of the induction equation (1) as

\[
\frac{\partial B_\phi}{\partial t} = -\frac{1}{r} \left\{ -\frac{\partial}{\partial r} \left( r u_r B_\phi \right) + \frac{\partial}{\partial \theta} \left( \omega r \frac{\partial}{\partial \theta} (\sin \theta A_\phi) \right) \right\} + \frac{\omega^2}{r^2 \sin \theta} B_\phi ,
\]

\[
\frac{\partial A_\phi}{\partial t} = -\frac{1}{r} \left[ u_r \frac{\partial}{\partial r} (r A_\phi) + \frac{u_\theta}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\phi) \right] + \alpha B_\phi + \frac{\omega^2}{r^2 \sin \theta} A_\phi ,
\]

where \( \omega \) describes the differential rotation of the Sun.

A new term controlled by the coupling constant \( \alpha \) has been added to the induction equation to model the process
responsible for the conversion of toroidal into poloidal magnetic field. Note that although we use the notation $\alpha$ for the source term, we are not considering any particular model to explain its origin, such as the mean field theory or the Babcock-Leighton picture. The effect of stochastic fluctuations in the source term can be adapted to any of these scenarios. In mean field theory, the physical origin of the fluctuations is related to the turbulent motions in the convection zone, which consists of the rising and sinking of fluid elements. Within these convective cells, the velocity field is coherent. Since the number of cells is finite, spatially averaged quantities retain a fluctuating component. On Babcock-Leighton models, where the poloidal field is regenerated by the cumulative effects of a finite number of discrete events of flux emergence, the idea of stochasticity arises quite naturally (Leighton 1969; Charbonneau & Dikpati 2000).

The rotational shear observed in the Sun is small at the base of the tachocline, while in the upper part it displays a profile that remains approximately independent of the radius across the convective region. At the surface, the differential rotation profile is given by observations (see Fig. 1):

$$\omega(\theta) = a + b \cos^2(\theta) + c \cos^4(\theta) ,$$

where $a = 2.913 \times 10^{-6} \text{ rad s}^{-1}$, $b = -0.405 \times 10^{-6} \text{ rad s}^{-1}$, and $c = -0.422 \times 10^{-6} \text{ rad s}^{-1}$ (Beck 2000). Although a poleward meridional flow was observed at the solar surface, the internal return flow is essentially unknown. For simplicity, the meridional velocity field is assumed to be incompressible although in a stratified medium, and the mass flux is given by the following stream function (see Fig. 2):

$$\psi(\theta, \phi) = -\psi_0 \sin \phi \cos^q(\theta) \cos^q(\theta, \phi) [r^q(r - r_{\min})(r_{\max} - r)] ,$$

with $p = 2.5$, $q = 1$ to fit the observed meridional velocity field at the surface (Cameron & Hopkins 1998). We adopt a density profile in the convection zone given by $\rho(r) = \rho_0 r^{-n}$ and assume $n = 0.5$, $r_{\min} = 0.6 R_\odot$, and $r_{\max} = R_\odot$, where $R_\odot$ is the solar radius (Charbonneau & Dikpati 2000). Therefore, the meridional velocity components are given by $u = \rho^{-1} \nabla \times (\psi \hat{\phi})$.

3. THE CODE

Considering observational evidence that suggests that dynamo action takes place at a thin layer at the bottom of the convective region, we reduce the dynamo equations (2) to one-dimensional equations in latitude. Helioseismic measurements suggest that the meridional velocity flow varies in the radial direction much more smoothly than the magnetic field and differential rotation profiles. Therefore, we extract the meridional flow components out of the radial derivatives and approximate them by their radial mean values. The mean radial velocity is positive at the equator, changes sign at mid latitudes, and is negative at the poles. This is an even function of the latitudinal angle. The mean latitudinal velocity field is smaller than the mean radial velocity and is zero at the equator. In both hemispheres, the direction of this flow points toward the poles: that is to say, it is an odd function of latitude. Since the net mass flow for this stationary motion is zero, the velocity of the (relatively lighter) fluid closer to the surface is larger than the (denser) returning flow at the base of the convection zone. Also, the volume occupied by the poleward flow is larger because of the stratification. These two effects give rise to an average meridional flow pointing toward the poles. Also, the radial shear of the differential rotation is known to be maximum in the thin dynamo layer at the bottom of the convective region. The azimuthal shear is almost zero at the base of the tachocline, while in the upper part it displays a profile that remains approximately independent of the radius. To obtain a one-dimensional model, the remaining radial derivatives in the fields and in the differential rotation are replaced by $1/L_0$, where $L_0$ is the typical depth of the dynamo layer located at the bottom of the convection zone.

To write down the equations in dimensionless units, we define dimensionless variables with the aid of a typical time $T_0$ (approximately 1 yr), and a typical longitude $L_0 \ll 1 R_\odot$. The magnetic field is expressed in velocity units. Equation (2) becomes

$$\frac{\partial B_\phi}{\partial t} = - \left( u_r B_\phi + \epsilon \frac{\partial u_\phi}{\partial \theta} \right) B_\phi - \epsilon u_\theta \frac{\partial B_\phi}{\partial \theta} + \left( \omega \cos \theta - \sin \theta \frac{\partial \omega}{\partial \theta} \right) A_p + \omega \sin \theta \frac{\partial A_p}{\partial \theta}$$

$$+ \frac{1}{R_M} \left( - \frac{\epsilon^2}{\sin^2 \theta} + \epsilon^2 \cot \theta \frac{\partial \omega}{\partial \theta} + \epsilon^2 \frac{\partial^2}{\partial \theta^2} - 1 \right) B_\phi ,$$

$$\frac{\partial A_p}{\partial t} = - \left( u_r + \epsilon \cot \theta u_\theta \right) A_p - \epsilon u_\theta \frac{\partial A_p}{\partial \theta} + \alpha B_\phi$$

$$+ \frac{1}{R_M} \left( - \frac{\epsilon^2}{\sin^2 \theta} + \epsilon^2 \cot \theta \frac{\partial \omega}{\partial \theta} + \epsilon^2 \frac{\partial^2}{\partial \theta^2} - 1 \right) A_p ,$$

(5)
where $R_M = \frac{L^2_0}{(T_0 \eta)}$, $\epsilon = \frac{L_0}{R_0} \approx 0.1$, $u_0$ and $u_o$ are the radially averaged values of the meridional flow, and $\omega$ is the differential rotation. Note that the radial shear of the differential rotation is being considered, although in a rather simplified fashion.

If the magnetic energy becomes comparable to the kinetic energy, dynamic effects cannot be neglected. It is standard practice to introduce nonlinear terms in the induction equation to model the saturation of dynamo action due to the suppression of turbulence by the magnetic field. In the present paper, we introduce the back-reaction of the magnetic field on the $\alpha$-effect as

$$\alpha \rightarrow \alpha(B) = \frac{\alpha}{1 + \tau B^2} \sin(\theta) \cos(\theta),$$

where the latitude profile was set to fit estimates of kinetic helicity from solar observations (Kuzanyan, Bao, & Zhang 2000), and where $\tau$ is a free parameter of the model related to the saturation value of the magnetic field.

The equations were numerically integrated using a finite differences scheme with 500 points in latitude and a predictor-corrector method to evolve in time, with a time step of $T_0/500$. Since it is known that one-dimensional models display a bifurcation diagram that is different from those in higher dimensional models (Jennings et al. 1990; Roald & Thomas 1997), we did not make attempts at determining accurately the critical dynamo number or exploring the solutions for different values of the parameters. However, since we are interested in solar-like solutions, we used differential rotation and meridional flow profiles to represent as closely as possible their solar counterparts.

In Figures 3, 4, and 5, we show the result of integrating the equations using a maximum meridional velocity of $15$ m s$^{-1}$, a value close to the observed meridional velocity at the surface (see Cameron & Hopkins 1998 for a review), $\eta = 5.4 \times 10^{11}$ cm$^2$ s$^{-1}$, $\alpha = 5.7$ cm s$^{-1}$, and $\tau = 1.9 \times 10^{-7}$ s$^2$ cm$^{-2}$. Choudhuri (1992) estimates the $\alpha$ coefficient to lie between 3 and 30 cm s$^{-1}$, while Dikpati & Charbonneau (1999) estimate $\eta \approx 3 \times 10^{11}$ cm$^2$ s$^{-1}$. The value of $\tau$ corresponds to a saturation value for the magnetic field of about $10^4$ G, considering a mean density of $1.41$ g cm$^{-3}$. The solution displays a period of approximately 12 yr.
A magnetic cycle with a period close to 22 yr can be observed, with an inversion in the poloidal field every 11 yr, and 3 yr later than the moment of maximum poloidal field strength, which is in good agreement with observations (Harvey 1992). Also, we observe the migration of magnetic activity toward the equator as the solar cycle progresses. Both examples are reasonably solar-like, although the toroidal field peaks at a very high latitude in comparison to the solar butterflies.

The dynamo settles to a periodic mode with a given amplitude, since an increase in the amplitude of the magnetic field reduces $\alpha$ by the $\alpha$-quenching mechanism. In these equations, nonlinearities are therefore responsible for locking the system into a stable mode. In the next section, we show that the system displays relaxation oscillations that are in good agreement with the oscillations observed for the sunspot number (Mininni, Gómez, & Mindlin 2000).

The meridional flow speed has a profound influence in determining the time period of the cycle, while the amplitude of $\alpha$ does not. This is in good agreement with results arising from higher dimensional codes (Dikpati & Charbonneau 1999). A difficulty arising in our simple one-dimensional model is the strong magnetic field at high latitudes. Of the features observed for the Sun, which are not properly described by our simple one-dimensional model, one is that the region of strong magnetic activity is limited to latitudes smaller than $30^\circ$.

The simplicity and stability of the code allow us to make long runs and have reliable statistics to consider the effect of stochasticity and at the same time consider a reasonable geometry and realistic profiles for the velocity fields.

4. THE PHASE SPACE

To reconstruct the phase space of the observed data, we perform an embedding of the daily sunspot number time series. Since the physical quantity of interest in solar cycle studies is the magnetic field, we must obtain a field amplitude time series from the observational data.

Assuming that the sunspot number is approximately proportional to the square of the toroidal magnetic field strength (Leighton 1969; Stix 1976; Tobias et al. 1995; Schreiber 1999), we derive a time series for the spatial averaged magnetic intensity by following the following steps: (1) take the square root of the sunspot number series and (2) change sign at each minimum (following Bracewell 1953). Since the daily sunspot number time series has a considerable level of high-frequency fluctuations, we smooth out the series using a frequency filter, where all frequencies with amplitudes smaller than 2% of the fundamental mode are eliminated (see Mininni et al. 2000 for further details).

We reconstructed the phase space $B$ versus $dB/dt$ for the obtained time series by using a differential embedding (see Mindlin, Merener, & Boyd 1998; Gilmore 1998 for a review), where the temporal derivatives were calculated using centered finite-difference formulas. Figure 9 shows the phase space using a time step of 200 days in the finite-difference scheme.

Note the rectangular shape of the cycle, together with the slow dynamics in its lateral sides and the much faster evolution in its horizontal sides. This rectangular shape and the difference in the speed of the flow in different regions of the trajectory is associated with the well-known fast increase and slow decrease observed in each cycle of the sunspot number time series.

For a proper comparison of this trajectory in phase space and our simulations, we applied the same method to an average in latitude of the toroidal magnetic intensity. The phase space trajectory derived from the simulation is overlaid to the sunspot trajectory in Figure 9. The dynamical system has an attracting solution that correctly fits the shape of the average limit cycle and shares some of the features found in the time series data: for instance, (1) the difference of timescales in different portions of the trajectory and (2) the rectangular shape of the cycle. As a matter of fact, the difference in timescales and the rectangular shape is increased as the diffusivity of the model is reduced.

It was shown by Mininni et al. (2001) that a highly truncated model of the dynamo equations leads to a relaxation oscillator dynamical system, which adequately simulates the properties of the phase space shown in Figure 9. It is reassuring that this comparatively more complete theoretical model (which nonetheless is still a truncation of the full set of equations) also shares these properties.
The observed time series also has an irregular behavior, as shown in Figure 9. Considering the properties of the observed average limit cycle and the correct fit of the synthesized data, these irregularities can be understood as the end result of a stochastic process associated to the spatial and temporal complexity of the turbulent motions in the dynamo region.

5. Irregularities of the Solar Cycle

Although the idea of describing the solar cycle through a chaotic attractor of reduced dimensionality seems appealing, its very existence is still being debated. Searches for signatures of a low-dimensional chaotic attractor in the sunspot time series are not conclusive because of insufficient number of data (Carbonell, Oliver, & Ballester 1994).

As an alternative approach to explain the irregularities observed in the solar cycle, the scenario of a stochastically driven solar dynamo has been studied in a number of recent papers (Choudhuri 1992; Moss et al. 1992; Hoyng 1993; Ossendrijver & Hoyng 1996, and references therein). Stochastically forced models are conceptually more modest. They accept that part of the source of complexity lies beyond our modeling capabilities. The stochastic forcing can be regarded as the end result of the interaction of the deterministic model with modes in the microscale that are not being resolved.

In mean field theory, the ergodic assumptions used traditionally in the averaging procedure automatically yield the presence of stochastic coefficients in the dynamo equations. The ensemble average is replaced by a spatial average: strictly speaking, an average over a finite number of discrete convective cells. In each convective cell, physical parameters are approximately coherent. As a result of the limited statistics, fluctuations of quantities around their average values are nonnegligible (Choudhuri 1992; Hoyng 1993). Moreover, according to recent observations (Kuzanyan et al. 2000) and mean field estimates (Choudhuri 1992; Ossendrijver et al. 1996), the level of fluctuations is larger than their average values, as shown below. In other pictures, such as Babcock-Leighton models, this scenario carries on directly (Charbonneau & Dikpati 2000). In the particular case of the solar dynamo, the rising and sinking fluid motions in the convection zone play a crucial role in driving the system through the so-called $\alpha$-effect. The spatial and temporal complexity of these convective cells can be described naturally by adding a stochastic part to the velocity field, which in turn will drive stochastic magnetic field components and the $\alpha$-coefficient itself.

Recent papers suggest that the dynamics of the sunspot number time series can be explained by a relaxation oscillator driven by noise (Paluš & Novotná 1999; Mininni et al. 2000). In fact, the features observed in our reconstructed phase space suggest that nonlinearities in the dynamo equations are responsible for the relaxation of the system to a stable mode, while the irregularities can be explained considering a stochastic forcing term in the dynamo parameters that models the interaction with unresolved modes.

Although all of the parameters may fluctuate with a stochastic component, $\alpha$ controls the amplitude of the nonlinear term and dominates the dynamics of the system (Choudhuri 1992). Also, $\alpha$ may take opposite sign values in different cells, while, e.g., $\eta$ is a positive quantity and its fluctuations can be expected to be lower. Also, in higher dimensional models, it has been observed that fluctuations in $\alpha$ yield the correct correlation between the amplitude and the phase of the sunspot time series, while fluctuations in the poloidal field yield a correlation of opposite sign (Charbonneau & Dikpati 2000).

Within the framework of our model, the conversion of toroidal field into poloidal field is modeled through the nonlinear term proportional to $\alpha$ (see eq. [2]). We introduce a separation of $\alpha$ into a mean value plus a stochastic component:

$$\alpha = \bar{\alpha} + \alpha_s,$$  \hspace{1cm} (7)

where $\alpha_s$ is assumed to be a uniformly distributed stochastic process with dispersion equal to unity. Therefore, the dimensionless parameter $r$ is the rms value of the stochastic part of $\alpha$. Note that this stochastic forcing is also quenched by the nonlinear effect, which in this respect is different from the one used by Moss et al. (1992).

A spatial correlation of $2 \times 10^5$ km was assumed; $\alpha_s$ is spatially constant in each cell, and the value of $\alpha_s$ was changed randomly once every correlation time of 30 days. These values correspond to typical correlation numbers of giant cells (Choudhuri 1992). As an example, in Figure 10 we show the result of integrating the stochastic dynamo equations, with $r = 11.25$ m s$^{-1}$.

The imposed random fluctuations have a strength that is much larger than the $\alpha_0$ term ($r/\alpha_0 \approx 2.5$). Note that this allows for large variations of the source term. For instance, in a given cell, there is a high probability of having the sign of $\alpha$ to be opposite of the mean value. This is in good agreement with observational (Kuzanyan et al. 2000) and theoretical estimates (Choudhuri 1992; Ossendrijver et al. 1996, who estimate $\delta \alpha/\alpha_0 \approx 3$). As found by Moss et al. (1992) for $\alpha^2 \omega$ and $\alpha^2$ dynamos, even for quite strong perturba-
the solutions do not leave the neighborhood of the underlying attractor as long as the correlation time remains much shorter than the cycle period. When perturbed, the stable solution (see the synthesized time series $B^2$ in Fig. 9) remains near the attracting limit cycle. The addition of noise then kicks the system out of the attractor, making the solution irregular, resembling the one reconstructed for the sunspot number time series (see the observed time series $R$ in Fig. 9).

Note that the presence of a stochastic component in $\alpha$ leads to temporal and spatial variability in the butterfly diagrams. We find that the precise nature of the noise (its time and spatial correlation) is qualitatively inconsequential as far as the correlation time is much lower than the mean period of the cycle. This robustness is in good agreement with previous simulations in $\alpha^2 \omega$ and $\alpha^2$ dynamos by Moss et al. (1992).

Although the resulting diagrams resemble the ones obtained by Ossendrijver et al. (1996), note that the present model considers spherical geometry and nonlinear quenching, including meridional flow and the strong radial shear in the differential rotation suggested by helioseismic measurements. Asymmetries between north and south hemispheres can be observed, together with fluctuations of the dividing line between the two polarities around the equator, giving rise to reverse-polarity regions in the solar butterfly diagram. Also, some other isolated regions of reverse-polarity activity can be identified during the cycle. These kinds of solutions are different from the ones obtained by Charbonneau & Dikpati (2000), since they impose the parity of the solutions with respect to the equator.

To compare the irregularities generated by the dynamo equations driven by noise with the irregularities observed in the sunspot number time series, we study the statistical spatial and temporal behavior of the magnetic field.

In order to make a comparison between the irregularities observed in the time evolution of the sunspot data with our model, some assumptions must be made to relate the synthesized magnetic field with the sunspot number time series. To compare the variabilities of the sunspot number peak values and their cycle periods, we consider the sunspot number to be proportional to the square of the toroidal magnetic field (Leighton 1969; Stix 1976; Tobias et al. 1995). Also, time-series analysis suggest that sunspot numbers are in first approximation proportional to the square of the magnetic field strength (see Kugiumzis 1999; Schreiber 1999). Taking the square of the spatially averaged toroidal field, normalizing by its maximum, and multiplying it by the mean maximum value of the observed time series, we obtained a synthesized sunspot number time series. The synthesized series displays fast increase and slow decrease in the sunspot number, with a mean period of 10.6 yr, which agrees fairly well with the average times observed in the sunspot number time series.

Table 1 shows the mean values and deviations for the observational data and for the dynamical system driven by noise. Note that the dispersions of all of the parameters listed in Table 1 have been adjusted with only one parameter ($r$).

6. ASYMMETRIES BETWEEN HEMISPHERES

North-south asymmetry of solar activity has been studied by various authors (Carbonell, Oliver, & Ballester 1993; Oliver & Ballester 1994, 1996; Watari 1996, and references therein), and its existence and characteristics are reasonably well established. This asymmetry is measured by the asymmetry coefficient (Carbonell et al. 1993)

$$AS = \frac{N - S}{N + S},$$

where $N$ is the number of sunspots in the north hemisphere and $S$ is the number of sunspots in the south hemisphere.

| Source                  | Period (yr) | Deviation (yr) | Maximum Value (Number of Sunspots) | Deviation (Number of Sunspots) |
|-------------------------|-------------|----------------|------------------------------------|--------------------------------|
| Dynamo equations........ | 10.6        | 1.1            | 120                                | 36                             |
| Solar cycle............... | 10.7        | 0.8            | 113                                | 40                             |

Note.—Mean values and deviations for the observational data (daily sunspot number) and for the dynamical system driven by noise.
where $N$ and $S$ stand for north and south sunspot areas, respectively.

North-south asymmetries have been interpreted by different authors in terms of either chaotic behavior (Tobias et al. 1995; Beer et al. 1998) or stochastic forcing (Ossendrijver & Hoyng 1996). In our model, both dynamical and stochastic effects are present.

Carbonell et al. (1993) and Watari (1996) conclude from observations that the asymmetry is the result of stochasticity rather than chaoticity. It is also known that the asymmetry coefficient often peaks near minima of activity (Ossendrijver et al. 1996). Oliver & Ballester (1994, 1996) made a detailed analysis of the asymmetry time series and found statistical evidence of a long-term trend: a deterministic cycle with a period of 12.1 yr and a remaining component given by a time-correlated random process.

Note that the stochastic forcing term in the source term of the dynamo equations naturally excites asymmetries between the north and south hemispheres, qualitatively resembling the features observed in the Sun. To compare the observed asymmetries with our model, we compute the asymmetry coefficient using the synthesized toroidal magnetic field, as previously done in §4 and 5.

Figure 11 shows a fraction of the synthesized $AS$ time series for a time span of 300 yr. The minima of activity in the corresponding butterfly diagram are indicated by vertical bars. Note that the asymmetry between hemispheres often peaks near minima of activity. Figure 12 shows the power spectrum obtained for a time span of 400 yr. A slow trend can be identified, and a cycle close to 12 yr is revealed clearly in the spectrum. The high-frequency component of the time series is essentially featureless.

This results are in good agreement with observations and can be explained as follows. The stochastic nature of the asymmetry is a direct result of the stochastic forcing term in the dynamic equations, and the correlation of the noise is a consequence of the space and time integration. Within this context, it seems apparent that the asymmetry emerges more noticeably when the contribution of the deterministic part of the solution becomes very small.

On the other hand, the appearance of deterministic frequencies in the spectrum needs a detailed discussion. The intensity of the initial condition in the numerical integration is symmetric around the equator. However, the introduction of noise turns on overtones much weaker than the fundamental solution but with mixed parity. The nonlinear quenching term in the dynamo equations naturally couples these high-frequency mixed-parity modes with the fundamental mode, thus generating a deterministic cycle driven by the interaction of modes in the deterministic part of the dynamics. The end result of this interaction process is an asymmetric time series with deterministic components and a time- and space-correlated noisy background.

In other words, we can understand the dynamics of the asymmetries in our model as the combination of stochastic effects, which are responsible for the excitation of overtones, and the deterministic coupling and evolution of the mixed-parity excited modes. It is worth noticing that the role of stochasticity is not merely to render the solutions more noisy but more importantly to enhance nonlinear couplings between modes.

7. DISCUSSION

We present a simple one-dimensional $\alpha \omega$-dynamo code to study the role of source-term fluctuations in nonlinear kinematic dynamo models. We assume solar-like differential rotation in latitude and replace the radial derivative of the differential rotation by $1/L_0$, representing the strong shear at a thin shell at the base of the convective region. Given the strength of the magnetic field at the base of the convection zone, the importance of nonlinear effects in the dynamics of the solar dynamo cannot be neglected. Therefore, we introduce nonlinearities in the form of $\alpha$-quenching and consider the effect of meridional flow, as suggested by observations. The velocity profiles and parameter values were chosen to fit as closely as possible the observations of the Sun.

Since it is known that one-dimensional models display a bifurcation diagram that is different from those in higher dimensional models (Jennings et al. 1990; Roald & Thomas 1997), we did not make attempts at exploring the solutions for different values of the parameters. However, as the system includes nonlinearities, we expect dynamical effects to be present in our model.
Numerical butterfly diagrams display a solar-like structure with a full cycle period of approximately 22 yr and a maximum toroidal field of $10^4$–$10^5$ G. Inversion of the poloidal field occurs every 11 yr and approximately 3 yr after the time of maximum poloidal field strength, as observed for the solar cycle. Note, however, that unlike solar butterfly diagrams, in our simulations the strong toroidal-field region is not restricted to a branch below 30°. This limitation seems to be common to many other dynamo models (Charbonneau & Dikpati 2000). We also note that the meridional flow speed plays a crucial role in the determination of the time period of the cycle, while the amplitude of $\alpha$ does not. This is in good agreement with results arising from higher dimensional codes (Dikpati & Charbonneau 1999). Note that although the butterfly diagrams derived from our model resemble the ones obtained by Ossendrijver et al. (1996), the present model considers nonlinear effects and more realistic profiles.

We find that all solutions settle to a periodic mode with a given amplitude. Therefore, nonlinear effects in the absence of stochastic forcing are responsible for the saturation of the magnetic field. We show that the system displays relaxation oscillations that are in good agreement with the oscillations observed in the sunspot number. In phase space, this periodic mode can be identified with an attracting limit cycle. The observed time series seems to perform excursions in the neighborhood of this stable attractor as if it were perturbed by noise.

To explain the irregularities observed in the solar cycle, we introduce a random forcing term into the coefficient $\alpha$. This random forcing term is expected to model the spatial and temporal complexity of the convection zone and can be regarded as the end result of the interaction of the deterministic modes with microscopic modes that are not being resolved. This stochastic forcing is also quenched by nonlinear effects, which in this respect is different from other models. We find that the precise time and spatial correlation of the noise is irrelevant, at least for correlation times much smaller than the mean period of the cycle. As found by Moss et al. (1992) for $\alpha^2\omega$ and $\alpha^2$ dynamos, even for quite strong perturbations, the solutions do not leave the neighborhood of the underlying attractor.

In addition to a random component in the magnetic field, these fluctuations induce phenomena such as amplitude and cycle period variability, fluctuations in the dividing line between polarities around the equator, emergence of regions of reverse polarity, and north-south asymmetries.

We compare these irregularities with the irregularities observed in the sunspot number time series. For the time variabilities, we found a good agreement between the statistical properties of the synthesized sunspot number and the observational time series.

The stochastic forcing naturally introduces asymmetries between the northern and southern hemispheres, qualitatively resembling the features observed in the Sun. The synthesized asymmetry coefficient often peaks near activity minimum. A slow trend can be identified in the signal, and a cycle close to 12 yr was found in its spectrum. These results are in good agreement with observations and can be explained as the interaction of the deterministic part of the solution with overtones excited by noise in the dynamical equations. Therefore, the asymmetries in our model arise as the combination of stochastic effects (which are responsible for the excitation of the overtones) and deterministic coupling and evolution of mixed-parity excited modes. This coupling is not observed in the behavior of linear dynamo models excited by noise. Also, isolated regions of reversed polarity can be identified during the cycle. These kinds of solutions are different from the more complex model of Charbonneau & Dikpati (2000), since they solve the equations in a single solar quadrant and impose the parity of the solutions with respect to the equator.

We believe that simple dynamo models such as the one presented here and its comparison with observations might give us new insight into the ongoing physical processes that in turn can be used for the development of more realistic theoretical models. Within this framework, the inclusion of dynamic effects and the robustness of the attractor in the presence of noise can play a crucial role (Moss et al. 1992) to generate Maunder-like minima. We believe that this treatment of the solar-cycle irregularities could in principle be extended to study these long-time features, since our model includes both stochastic and dynamical effects. In any case, more research is needed to understand the physical reasons that lie behind these periods of reduced dynamo activity.

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