The Structure of Magnetically Dominated Energy Extracting Black Hole Magnetospheres: Dependencies on Field Line Angular Velocity

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If the inner light surface of a force-free black hole magnetosphere lies within the ergoregion that magnetosphere can extract the black hole’s rotational energy and transmit it to distant observers. We take the distribution of magnetic field line angular velocity on the horizon to be a useful proxy for inner light surface location and study how different distributions affect the structure of energy-extracting magnetospheres. We find that distributions that most naturally correspond to the presence of nearby accreting matter structures also give rise to magnetosphere structures compatible with the presence of a jet, suggesting that in practical terms black hole energy extraction might often be coupled to jet launching. We also find over a factor of 100 in jet-like luminosity differences that are directly attributable to variations in magnetosphere structure and independent of black hole spin.

I. INTRODUCTION

Rotating black holes can carry enormous amounts of extractable energy. Christodoulou and Ruffini [1] calculated that on average 6% of an uncharged black hole’s mass could be extracted, peaking at 29% in the limit of extreme rotation. Blandford and Znajek [2] later demonstrated that electromagnetic braking through an appropriately configured magnetosphere can be a practical method of extracting that rotational energy and transmitting it to distant observers. Ever since energy-extracting black hole magnetospheres have been cited as potential drivers of some of the most energetic astrophysical objects, ranging from gamma-ray bursts to active galactic nuclei.

Within the context of stationary, axisymmetric, and ideal magnetohydrodynamics Takahashi et al. [3] showed that for a net energy outflow to occur along an isolated magnetic field line the Alfvén point of the plasma inflow along that magnetic field line must occur within the ergoregion. This dependency on some type of ergoregion behavior is a common feature of mechanisms that extract black hole rotational energy. The mechanical method proposed by Penrose [4] (reprinted as [5]), for example, relies upon the creation of particles with negative energies via collisions or other processes within the ergoregion.

While the importance of the Alfvén point of an isolated magnetic field line is known, the practical significance of the Alfvén surface of a collection of magnetic field lines forming a complete magnetosphere is far less certain. This is largely due to the analytic intractability of the equations governing plasma flows in curved spacetimes. In our previous work [6] we began to explore potential implications of Alfvén surface location for near horizon magnetospheres by considering force-free magnetospheres as the limit of magnetically dominated magnetospheres. In force-free magnetospheres the ingoing Alfvén surface coincides with the inner light surface, the location of which is determined by the magnetosphere’s rotational profile.

We found that relatively slowly rotating magnetospheres with inner light surfaces near the outer limits of the ergoregion resulted in the bending of magnetic field lines towards the azimuthal axis and the formation of jet-like structures. Relatively rapidly rotating magnetospheres with inner light surfaces near the horizon resulted in the bending of magnetic field lines towards the equatorial plane, compatible with a direct connection between the horizon and a disk or similar nearby accreting matter structure. Due to the complex nature of the equations involved those results were arrived at numerically, but such tendencies can also be seen in a more restricted form analytically [2].

In our previous numerical work we made multiple assumptions [6], one being a uniformly rotating magnetosphere. That assumption was convenient in that it allowed us to know a priori exactly where the inner light surface of a given magnetosphere would lie, and it allowed us to focus on zeroth-order effects of magnetosphere rotation. Despite those conveniences, however, uniformly rotating magnetospheres are likely to be fairly crude approximations of real black hole magnetospheres.

The goal of this work was to relax the assumption of uniform magnetosphere rotation and study rotational profiles that might more closely correspond to astrophysical black hole magnetospheres. We are primarily focused on near-horizon behaviors where the effects of a rotating spacetime are the strongest, so we took the event horizon as a natural place to specify the rotation of a magnetosphere. Specifically we chose to study distributions of field line angular velocity $\Omega_F$ on the horizon corresponding to the first two terms of a series expansion of an arbitrary distribution:

$$\Omega_F|_{\mathcal{H}} = (A + B \sin \theta) \omega_H.$$

Here $\omega_H$ is the angular velocity of the horizon (i.e. the
angular velocity of a zero angular momentum observer on the horizon), while $A$ and $B$ are unitless constants. Field line angular velocity $\Omega_F$ corresponds to a magnetosphere’s rotation in that it may be thought of as a measure of the rotational boost velocity to the plasma rest frame (an explicit definition of $\Omega_F$ may be found in Equation 4 of the next section). Our previous work studied uniformly rotating magnetospheres with $A = [0 \ldots 1]$ and $B = 0$ for a full range of black hole spins and corresponding horizon angular velocities $\omega_H$.

In this work we studied $A = [0 \ldots 1]$ and $B = (-1 \ldots 1)$, as those values most completely encompass arbitrarily rotating energy-extracting black hole magnetospheres. Values of $A$ less than 0 or greater than or equal to 1 were not considered as they would not generally correspond to energy-extracting magnetospheres. We applied the condition $0 \leq A + B < 1$ so that the magnetospheres would extract rotational energy along almost every magnetic field line (field lines along the azimuthal axis or corresponding to $A + B \sin \theta = 0$ being the few exceptions). We focused exclusively on a spacetime with black hole spin parameter $a = 0.8m$, selected as being large enough to be interesting without being overly extreme and potentially less widely representative.

From our previous work we expected (and found) that the most interesting $A$ and $B$ values would fall into a fairly narrow range corresponding to low field angular velocities near the azimuthal axis and high field line angular velocities near the equatorial plane, as those values would form magnetospheres with jet-like structures aligned with the azimuthal axis and structures reminiscent of horizon-disk connections at lower latitudes. Such magnetospheres (which we will label as “Jet-Disk” magnetospheres in Section IIIA) are of special interest because they combine nearby matter structure compatibility with the formation of collimated jet-like structures without directly relying on anything except near-horizon magnetosphere conditions. The horizon field line angular velocity distributions corresponding to those magnetospheres are also intrinsically compatible with conditions that might be imposed by the presence of nearby accreting matter structures (discussed in detail in Section IVC), further increasing the likelihood of their astrophysical interest.

While some of the $A$ and $B$ pairs we studied are of special interest, many are deliberately naive and likely not very relevant to plausible astrophysical black hole magnetospheres. Nonetheless we still felt that their calculation was important. Not only do they form a more complete set when viewing Equation 4 as a generic expansion of arbitrary magnetosphere rotation, they also place the more astrophysically relevant distributions of field line angular velocity on the horizon in a more complete context. So while many of the magnetospheres we calculate are likely to be mostly mathematical curiosities, their illumination of more interesting black hole magnetospheres still gives them value.

In Section III we report on our results for all $A$ and $B$ pairs, but with greater emphasis placed on the pairs that are more likely to correspond to astrophysical black hole magnetospheres. Before that in Section II we provide brief reviews of the assumptions we have made and the numerical techniques we have used. In Section IV we discuss some potential implications of our results before concluding.

## II. Assumptions and Numerical Techniques

The primary difference between this work and our previous work [6] is the relaxation of the condition of uniform field line angular velocity. Therefore we will primarily provide summaries of the assumptions and numerical techniques used and direct those interested in more detailed discussions to our previous work.

### A. Assumptions

#### 1. Core Assumptions

We assume a black hole whose surrounding spacetime is adequately described by the Kerr metric in Boyer-Lindquist coordinates, corresponding to the line element:

\[
\begin{align*}
\text{ds}^2 &= \left(1 - \frac{2mr}{\Sigma} \right) dt^2 + \frac{4mar\sin^2 \theta}{\Sigma} dtd\phi - \frac{\Sigma}{\Delta} dr^2, \\
- \Sigma d\theta^2 &- A \sin^2 \theta \frac{\Sigma}{\Delta} d\phi^2,
\end{align*}
\]

where

\[
\begin{align*}
\Sigma &= r^2 + a^2 \cos^2 \theta, \\
\Delta &= r^2 - 2mr + a^2, \\
A &= (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta.
\end{align*}
\]

We also assume that the black hole is surrounded by a perfectly conducting plasma that is both stationary and axisymmetric, with the axis of symmetry corresponding to that of the black hole. We additionally take the magnetically dominated force-free limit and assume that plasma inertial effects may be discarded, such that the magnetosphere can be completely described by three parameters: the toroidal vector potential $A_\phi$, the field line angular velocity $\Omega_F$, and the toroidal magnetic field $\sqrt{-g} F^{\theta r}$. The toroidal vector potential $A_\phi$ and the field line angular velocity $\Omega_F$ are related to the field strength tensor $F^{\alpha \beta}$ by:

\[
\begin{align*}
F_{\alpha \beta} &= A_{\beta, \alpha} - A_{\alpha, \beta}, \\
F_{r\theta} \Omega_F &= F_{1r}, \\
F_{\theta \phi} \Omega_F &= F_{\theta \phi}.
\end{align*}
\]

Here a comma denotes a partial derivative. The toroidal vector potential $A_\phi$ is conserved along magnetic field lines.
and as such is a useful flux function for the poloidal magnetic field. The field line angular velocity and toroidal magnetic field are also conserved along magnetic field lines, respectively statements that field lines rotate rigidly and that energy and angular momentum Poynting fluxes are conserved. The conservation of field line angular velocity is a consequence of stationarity, axisymmetry, and a perfectly conducting plasma (expressed as the vanishing contraction of the field strength tensor and its dual $F_{\alpha\beta}F_{\alpha\beta} = 0$) and as such is also a conserved quantity when plasma inertial effects are considered. The conservation of the toroidal field is the force-free limit of angular momentum flux conservation. Further discussion of force-free conserved quantities may be found in Blandford and Znajek [2]; further discussion of conserved quantities when plasma inertial effects are present may be found in Bekenstein and Oron [8], Camenzind [9], and Takahashi et al. [3].

2. Boundary Conditions

We assume reflection symmetry across the equator so that we need only solve for the structure of the magnetosphere in the upper half of the poloidal plane. We then apply boundary conditions along the azimuthal axis and equatorial plane that are compatible with a “monopolar” magnetic field, in the sense that we assume fixed magnetic field lines tracing the azimuthal axis and equatorial plane (mathematically $A_\phi(\theta = 0) = A_{\phi\text{max}}$ and $A_\phi(\theta = \pi/2) = A_{\phi\text{min}}$ where $A_{\phi\text{max}}$ and $A_{\phi\text{min}}$ are fixed constants). We use the term “monopolar” to describe those boundary conditions while noting that has the potential to be misleading. There will often be a substantial toroidal component of the magnetic field, resulting in a helical magnetic field that only resembles a monopole when projected onto the poloidal plane. Additionally, non-zero magnetic flux through a closed surface surrounding the black hole demands a reversal of the direction of the magnetic field below the equatorial plane, so “split-monopolar” would be a more appropriate term when the entire poloidal plane is considered. In short “monopolar” should only be taken to describe the boundary conditions on the upper half of the poloidal plane where we conduct our numerical calculations, not the magnetosphere as a whole.

A field line tracing the azimuthal axis is not a significant restriction as stationarity and axisymmetry already imply a single magnetic field line extending straight upward from the pole. However a single magnetic field line tracing the equatorial plane is a more severe restriction. Although it might be physically reasonable in regions close to the horizon, the further away one gets the less reasonable it is likely to become. This is because significant amounts of matter would generally be expected near the equatorial plane at least as close as the innermost stable circular orbit. That matter would likely anchor many different magnetic field lines and be better described by higher order multipoles in the equatorial plane.

Despite their potential deficiencies, monopolar boundary conditions are still very useful in explorations of basic interactions between the electromagnetic fields and the background spacetime. Other boundary conditions intrinsically assume a specific matter distribution outside the black hole and by extension a specific astrophysical context. Any such assumed context not only introduces its own assumptions but also has the potential to introduce arbitrarily large forcings on the magnetosphere that might obscure interactions between the electromagnetic fields and the background spacetime. In short the assumption of monopolar boundary conditions is a deliberate compromise; we are favoring a more fundamental exploration of black hole magnetospheres over direct applicability to any specific astrophysical context.

3. Limited Domain

We do not extend our magnetospheres past the outer light surface (pulsar light cylinder analog) or $r = 20m$, whichever occurs sooner. That choice is not made completely freely, as diffusive numerical techniques are generally incapable of finding magnetospheres that pass smoothly through both inner and outer light surfaces once a specific distribution of horizon field line angular velocity has been specified. This is because the character of the equations involved changes across a light surface; the prefactor on the second derivatives of the vector potential changes sign. This means that numerical schemes can fail to find many valid solutions, as for stability they must necessarily evolve the magnetosphere differently on either side of a light surface. Finding solutions numerically then generally reduces to “matching” minimum energy solutions across a light surface by adjusting field-aligned conserved quantities. Matching three regions across two light surfaces would require two different conserved quantities, but once a distribution of horizon field line angular velocity has been specified only the toroidal field remains as a free variable. This means that numerical techniques will generally be incapable of finding matched solutions across both inner and outer light surfaces once a distribution of horizon field line angular velocity has been specified, even if such solutions exist.

For example, within the monopolar boundary conditions we have set infinitely many solutions are known to exist that pass smoothly through both inner and outer light surfaces (e.g. Menon and Dermer [10]). However when numerical techniques are applied by matching across light surfaces only a single highly monopolar (minimum energy) solution is found (e.g. Contopoulos et al. [11]). As we are primarily interested in near-horizon behaviors in this work, we have chosen to limit ourselves to regions interior to the outer light surface.

Despite the fact that the outer light surface is a numerical limitation, our selection of a limited domain in-
terior to \( r = 20m \) or the outer light surface has physical motivations. For example, we have assumed that the field-aligned conserved quantities are rigidly conserved across the entire magnetosphere, from the horizon to the outer boundary. While the assumptions of stationarity, axisymmetry, and a force-free plasma that led to those conserved quantities might be approximately valid over any given region, as a field line grows in length small deviations from those assumptions can grow in significance. Our limited domain can be thought of as the assumption that field-aligned quantities are only approximately conserved, and that near horizon values might differ significantly from more distant values.

Additionally, when the force-free limit is viewed as the magnetically-dominated limit of an ideal plasma flow, then consideration must be made of the “separation surface” between the inner and outer light surfaces that separates a plasma inflow from a plasma outflow (by demarcating the change in dominance from inward gravitational forces to outward centripetal forces). Near the separation surface plasma effects can become significant and deviations from our assumptions can be expected, to some extent decoupling inner and outer magnetospheres near the separation surface. We note that the separation surface is a suggestion, however, and not a rule - problematic effects (mathematically sourced by a diminishing Alfvén Mach number and concurrent increase in plasma density) do not have to emerge there, but if they do not emerge at or interior to the separation surface they are guaranteed to emerge as the outer light surface is approached. As such we are not concerned with addressing the numerical difficulties that emerge at the outer light surface, as in the problem space of ideal plasma flows we necessarily demand physical changes to the problem (from a plasma inflow to a plasma outflow) before arriving there. Although in general we expect the separation surface to be a much more physically relevant indication of changing physics than the outer light surface, it is numerically trivial to extend the magnetosphere past the separation surface to the outer light surface, and we see no harm in doing so.

It is possible that the selection of the outer boundary might influence (or drive) the structure of the magnetosphere contained within it (e.g. spherical outer boundaries at \( r = 3m \), \( r = 4m \), and \( r = 5m \) might result in different solutions). We studied that behavior in initially developing the numerical techniques applied in our previous work \([6]\), and found that such differences could indeed emerge if the treatment of the outer boundary was poorly implemented. If done appropriately, however, the solutions obtained are identical and the outer boundary does not influence the solution. In this work we continued to verify the apparent invariance of the solutions with outer boundary, and (in addition to other tests) calculated every magnetosphere a minimum of two times: once with a spherical outer boundary (typically between \( r = 3m \) and \( r = 4m \)) completely interior to the outer light surface, and once with an outer boundary limited by the outer light surface (or \( r = 20m \)). The solutions obtained were always identical, regardless of outer boundary. For this reason we do not view the outer light surface as a “boundary condition”, but rather the point at which the rigid application of stationary and axisymmetric force-free magnetohydrodynamics (extended from the horizon) has definitively broken down. In a specific model that breakdown might be avoided or diminished by appealing to additional physics, which is a reason why Blandford and Znajek \([2]\) and others have discussed “spark gaps” and other plasma injection mechanisms. A more generic exploration of the solution space of ingoing magnetospheres has no such luxury, however, and is restricted to a domain interior to the outer light surface (e.g. MacDonald and Thorne \([12]\)).

### B. Numerical Techniques

The only significant change to our numerical techniques from our previous work \([6]\) is our method of kink reduction across the inner light surface, which we have improved to allow for significantly reduced error levels there. That error reduction does not modify our final results in any appreciable way but does allow for more efficient computation of magnetospheres.

#### 1. Magnetofrictional Method

To calculate magnetospheres we apply a relativistic extension of the magnetofrictional method developed by Yang et al. \([13]\), similar to the method used by Uzdensky \([14]\). This method takes an initial guess for the structure of a magnetosphere and then calculates the divergence of the stress energy tensor. If that divergence does not vanish the configuration is invalid (or at least inconsistent). To find a valid configuration the invalid excess momentum fluxes are converted to the velocity \( v \) of a fictitious plasma via empirically determined “friction”. The magnetic fields \( B \) are then modified by the relativistic analog of the ideal induction equation \( \partial_t B = \nabla \times (v \times B) \). The end result is that the vector potential \( A_\phi \) is evolved via a simple advection equation until a solution is found:

\[
A_{\phi,t} = -v^A A_{\phi,A}. 
\]  

Here the uppercase Latin indices denote poloidal directions (\( r \) and \( \theta \)) and the magnetofrictional velocity \( v^A \) includes empirically determined weighting factors for convenience and stability as outlined in our previous work \([6]\). A demonstration that application of the magnetofrictional method will always result in a valid force-free magnetosphere under fairly general assumptions may also be found there.

We have found that our numerical procedures always converge to the same solution regardless of the initial guess for the structure of the magnetosphere (initial \( A_\phi \),
potential, opposite directions on either side. To find a smooth solution, we adjust the toroidal field as a function of the vector potential, \( \sqrt{-g F^{\theta r}} \), until that kink disappears. To make modifications to that function, we first shoot across the inner light surface from the outside such that the near horizon region provides an inner boundary condition for the region outside the inner light surface. We then modify the toroidal field corresponding to the shot grid squares by measuring how close to being force-free those squares are. Specifically, we use the error level of the shot grid squares (calculation of that error is discussed in the next section) to gradually correct the functional form of the toroidal field:

\[
\sqrt{-g F^{\theta r}} \bigg|_{\text{New}} = \sqrt{-g F^{\theta r}} \bigg|_{\text{Old}} - \lambda \cdot \text{Error}. \quad (6)
\]

Here \( \lambda \) is an empirically determined constant; optimal values vary widely, depending primarily upon grid resolution and current error level.

The above method of kink reduction differs slightly from our previous work \[6\]. Previously we adjusted the toroidal field by evolving both sides of the inner light surface separately, then measured the magnitude of the difference in \( A_\phi \) across the light surface as an input in adjusting the toroidal field. While that method works reasonably well, as the kink in \( A_\phi \) becomes smaller it can become very difficult to accurately quantify and therefore reduce the error level of the final solution.

By directly using the error level to modify the toroidal field we are able to reduce the error along the inner light surface significantly from what was obtainable in our previous work. Error levels of at least 0.001% are now fairly easy to obtain along the entire extent of the inner light surface (computation time being the primary limiting factor) while our previous method would sometimes struggle to significantly exceed 0.1% as a worst case.

Despite the advantages in error reduction, the primary motivation for the change in kink reduction method was to enable more rapid convergence to a solution. Enhanced error reduction was merely a side effect that did not change our results in any appreciable fashion.

Both our previous and current methods for kink reduction are identical in basic principle to the method developed by Contopoulos \textit{et al.} \[15\] for pulsar magnetospheres and applied more recently by others such as Contopoulos \textit{et al.} \[11\], Nathanail and Contopoulos \[16\], and Pan \textit{et al.} \[17\] to black hole magnetospheres.

### 3. Error Determination

Our magnetospheres can contain both strongly monopolar regions as well as regions with very small current, so most commonly used measures of force-freeness that rely directly on some physical attribute of the fields (such as the ratio of Lorentz force and electric current, as in McKinney and Gammie \[18\] and others) can yield unreliable results due to one or more of the measured physical attributes becoming vanishingly small. We have therefore developed a more mathematical technique for measuring the error of our solutions. A valid and self-consistent solution will have a stress energy tensor with vanishing divergence, so we measure how close to zero the divergence of a magnetosphere’s stress energy tensor is in order to determine the magnetosphere’s error level. Specifically, we first separate the divergence into seven terms:

\[
T^{\alpha \beta} i_{\beta} \sim \sum_{i=1}^{7} D_i = \delta. \quad (7)
\]

The exact form of \( D_i \) we use is detailed in our previous work \[6\]: their sum is not completely equivalent to the divergence of the stress energy tensor because we apply overall weighting factors for convenience. When \( \delta \) is close enough to zero a solution has been found, close enough being determined by comparing \( \delta \) to the largest of the \( D_i \) terms:

\[
|\delta| < \epsilon \cdot \text{Max}(|D_i|). \quad (8)
\]

We have set \( \epsilon = 1\% \) over the entire domain as an adequate error level, but in practice most of the domain will be significantly less. Typically the largest \( r \) values are the last regions to achieve the 1\% level, at which time averages of 0.0001\% inside the ergosphere are common. Magnetospheres with an error level of 10\% are generally not substantively different from those at 1\%, which are in turn effectively indistinguishable from those at 0.1\% and below. We chose 1\% as an error level in order to remove as much numerical uncertainty as possible while avoiding excessive amounts of computation time.

### 4. Computational Specifics

The vector potential \( A_\phi \) is calculated over a rectangular \((r, \theta)\) grid with 200 evenly spaced grid squares in \( \theta \).
and on average around 1000 variably spaced grid squares in \( r \). The radial spacing varies from magnetosphere to magnetosphere; magnetospheres with inner light surfaces near the horizon have tighter spacing there in order to adequately resolve the inner light surface. The radial grid extends from just inside the horizon to \( r = 20m \) or the outer light surface, whichever is smaller, with radial spacing of around \( 0.1m \) near \( r = 20m \).

The toroidal field is implemented as a function of \( A_\phi \) with over 1000 points of varied spacing between \( A_{\phi_{\text{min}}} \) on the equatorial plane and \( A_{\phi_{\text{max}}} \) on the azimuthal axis. That spacing is determined by convenience, as convergence can be optimized by using very fine sampling in regions where the toroidal field as a function of \( A_\phi \) is steep.

To evolve the advection equation for \( A_\phi \) we use an upwind differencing algorithm similar to the one described in Hawley et al. \[19\]. One-sided finite difference approximations appropriate to that algorithm are made to evolve \( A_\phi \), but centered finite difference approximations appropriate to the local grid spacing are used to determine the magnetofrictional velocity \( v^A \).

The azimuthal axis and equatorial plane are taken as fixed boundaries. At all other boundaries (\( r_{\text{min}}, r_{\text{max}}, \) and/or along the outer light surface) we shoot outwards using a quadratic fit after every time step, an approach that is largely equivalent to using one-sided derivatives on those boundaries.

5. Performance

We computed all magnetospheres on a single desktop computer with a 6-core Intel Haswell CPU assisted by Nvidia Kepler GPUs, which can generally find a magnetosphere at the 1% error level within a few hours. Exact time to completion can vary widely, though, from well under an hour to days in extreme cases. Computational time is highly dependent upon how good the initial guess was, how tight the grid spacing is, and how optimal various empirical tunings (strength of friction, modification to the toroidal field, etc.) are. At a high level our algorithm is conceptually similar to finding the root of a computationally expensive function by crawling along that function, and as such can be susceptible to large inefficiencies similar to those found when over or under evaluating an expensive function and from taking steps that are either too big or too small. Significantly improving the speed of our code and its algorithmic inefficiencies should be possible, but we found the current performance level to be adequate for this work.

III. RESULTS

We divide our results into three sections. First we explore the general structure of the magnetospheres obtained as a function of field line angular velocity, measured by the \( A \) and \( B \) parameters in \( \Omega_F(r_H, \theta) = (A + B \sin \theta)\omega_H \). We then explore the rates of energy and angular momentum extraction from the black hole. Lastly we explore the behavior of magnetospheres containing both jet-like regions and structures resembling horizon-disk connections in more detail, as those are likely to be the magnetospheres of greatest astrophysical interest.

\[
\Omega_F = (0.20 + 0.65 \sin \theta)\omega_H
\]

FIG. 1. The structure of a magnetosphere with horizon field line angular velocity \( \Omega_F = (0.20 + 0.65 \sin \theta)\omega_H \). The black poloidal magnetic field lines are spaced evenly on the horizon. The green lines trace the inner and outer light surfaces, the red line traces the boundary of the ergosphere, and the cyan line traces the separation surface (the point at which gravitational and centripetal forces are balanced). The dotted magenta line traces the monopolar separatrix between field lines bending towards the axis and field lines bending towards the equatorial plane. This magnetosphere is classified as a Jet-Disk magnetosphere, denoted by the “J-D” text inside the horizon. The background shading denotes the magnitude of the conserved field-aligned Poynting flux; \( E = (1/4\pi)\sqrt{-g}F^{\mu \nu}\Omega_F \). A plot of this magnetosphere’s ergoregion is shown in Figure 2, and an additional 12 magnetospheres are displayed in similar fashion in Figure 3.

A. General Structure

We calculated 400 distinct magnetospheres with different horizon field line angular velocities \( \Omega_F = (A + B \sin \theta)\omega_H \) using a spacing of 0.05 in both \( A \) and \( B \) over the ranges \( A = [0 \ldots 1] \) and \( B = [-1 \ldots 1] \) under the condition that \( 0 \leq A + B < 1 \). It would be impractical to show all 400 in detail, so instead we classify magne-
tospheres based upon their general structure and show some representative types.

![Diagram of field lines and light surfaces](image)

**FIG. 2.** The structure of a magnetosphere with horizon field line angular velocity distribution \( \Omega_F = (0.20 + 0.65 \sin \theta) \omega_H \), the same as shown in Figure 1. The inner light surface is shown in green, the boundary of the ergosphere is marked in red, and the monopolar separatrix between Jet and Disk behaviors is shown as a dotted magenta line. The black magnetic field lines are spaced evenly on the horizon. The three blue magnetic field lines rotate with field line angular velocities \( \omega_H \), \( \Omega_F = 0.4 \omega_H \), and \( \Omega_F = 0.5 \omega_H \). The shading to the left of the horizon \((r_H = 1.6m)\) is a measure of the poloidal magnetic field strength on the horizon; \( |B_P^2| \sim \omega_H \csc \theta \). The shading outside the horizon is a measure of both the toroidal magnetic field and conserved angular momentum Poynting flux; \( L = \langle \omega_H \rangle \sqrt{\epsilon F^{\epsilon \epsilon}} \). The three dotted blue lines correspond to the inner light surfaces of uniformly rotating magnetospheres with \( \Omega_F = 0.5 \omega_H \) (closest to the horizon), \( \Omega_F = 0.5 \omega_H \), and \( \Omega_F = 0.4 \omega_H \) (furthest from the horizon). An additional 12 magnetospheres are displayed in similar fashion in Figure 3.

In the poloidal plane there are only three things that a magnetic field line can do: bend upwards toward the azimuthal axis, remain straight, or bend downwards towards the equatorial plane. We classify each of those three tendencies as being “jet-like”, “monopole-like”, or “disk-like”, respectively, and then classify magnetospheres by the typical behaviors of their field lines in high latitudes and in low latitudes. For example a purely monopolar magnetosphere is classified as “Monopole-Monopole” while a magnetosphere with field lines that bend upwards in high latitudes and downwards in low latitudes is classified as a “Jet-Disk” magnetosphere. The classification of a magnetosphere is accomplished by subjective inspection; as such the boundary between high and low latitudes and what is more “monopolar” than not varies from magnetosphere to magnetosphere.

In Figure 1 we plot a Jet-Disk magnetosphere with horizon field line angular velocity distribution \( \Omega_F = (0.20 + 0.65 \sin \theta) \omega_H \). The magnetosphere is limited to the region interior to the outer light surface, shown as a green line. The separation surface is shown as a cyan line and might be considered as a more realistic outer boundary, as it delineates the region where the forces on the plasma shift from being dominated by outward centripetal forces to being dominated by inward gravitational forces. As such a large accumulation of plasma might be expected near the separation surface, breaking the assumption of a force-free plasma and the rigid conservation of field-aligned quantities. In Figure 2 we plot the same magnetosphere using an \((r, \theta)\) grid focused on the ergoregion. The shading in Figure 2 corresponds to the outward momentum flux (or toroidal magnetic field); the strength of the poloidal field on the horizon is shown as a colorbar immediately inside the horizon, allowing for a comparison of the relative strengths of the two magnetic fields. The inner light surface is shown as a green line while the inner light surfaces of uniformly rotating magnetospheres with field line angular velocities of \( 0.4 \omega_H \), \( 0.5 \omega_H \), and \( 0.6 \omega_H \) are shown as dotted blue lines, allowing for a rough determination of correlation of inner light surface location with magnetosphere behavior. Such correlations and other effects are discussed in more detail below, and an additional twelve magnetospheres of various different types are plotted in Figure 3 using methods identical to Figures 1 and 2.

Monopole-Jet magnetospheres were the only type not found. The other eight types were present, although as shown in Figure 2 some types were much more common than others. By far the most common type was Jet-Jet, followed by Jet-Disk and Disk-Disk. The boundaries and transitions between different types of magnetospheres are slightly fuzzy due to the subjective nature of their classification, but within a given classification region the behaviors are robust.

As a general rule the structure of a magnetosphere is predictable by considering the average field line angular velocity of small collections of field lines. If the average is less than half of the horizon’s angular velocity, \( \langle \Omega_F \rangle < 0.5 \omega_H \), then the field lines will bend upwards toward the azimuthal axis. If the average is greater than half of the horizon’s angular velocity, \( \langle \Omega_F \rangle > 0.5 \omega_H \), then the field lines will bend towards the equatorial plane. The strength of either bending increases the further away from \( 0.5 \omega_H \) the average field line angular velocity becomes.

The exception to the above rule is when contradictory preferences are present, such as when high latitude groups want to bend downwards and low latitude groups want to bend upwards. In that case one group will generally dominate over the other and cause the entire magnetosphere to be either completely Jet-Jet or
FIG. 3. Twelve different magnetospheres displayed using the same conventions as in Figures 1 and 2. The main plot shows the structure of the magnetosphere on a Cartesian grid colored by outward energy flux per unit field line $E = (1/4\pi) \sqrt{-g} F^{\theta r} \Omega_F$. The inset plot shows the structure of the ergoregion on an $(r, \theta)$ grid colored by outward momentum flux per unit field line $L = (1/4\pi) \sqrt{-g} F^{\theta r}$; the colorbar to the left of the horizon denotes poloidal magnetic field strength on the horizon using $|B^p_H| \sim A_{\phi, \theta} \csc \theta$. The inner and outer light surfaces are shown in green, the boundary of the ergosphere is marked in red, and the separation surface is shown in cyan. The black field lines are spaced evenly on the horizon; the most monopolar field line, if relevant, is marked in dotted magenta. The magnetosphere classification type (Jet, Monopole, Disk) of high and low latitude regions is denoted by text inside the horizon. The inset plots show the locations of the inner light surfaces of magnetospheres with uniform field line angular velocities $0.6 \omega_H$, $0.5 \omega_H$, and $0.4 \omega_H$ (furthest from the horizon) as dotted blue lines. The inset plots on the middle two rows and middle of the bottom row (d, e, f, g, h, i, and k) mark the field lines rotating at $0.6 \omega_H$, $0.5 \omega_H$, and $0.4 \omega_H$ in blue. All 400 calculated magnetospheres are listed by type in Figure 4; any magnetosphere of the same type as one of the 12 above is qualitatively similar in structure. Monopole-Disk and Disk-Monopole magnetospheres are the only type not shown above as they may be easily imagined as a combination of (a) and (b) or (b) and (c).
FIG. 4. Classification of magnetospheres according to their high and low latitude behaviors as a function of field line angular velocity on the horizon. The boundaries between regions are moderately susceptible to subjective interpretation, but the classification within a region is robust. The cyan shading denotes magnetospheres with at least 90% of the maximum luminosity (cf. Figure 5); the magenta shading denotes magnetospheres with at least 90% of the maximum rate of angular momentum extraction (cf. Figure 6). The green numbers denote the Jet-Disk magnetosphere numbering scheme used in Section III C. The blue circles mark the magnetospheres shown in Figure 3 the red circle marks the magnetosphere shown in Figures 1 and 2.

There is a third potential indicator of magnetosphere structure in addition to average field line angular velocity and the location of the inner light surface. The colorbars to the left of the horizon in Figure 2 and the inset plots of Figure 3 are measures of poloidal magnetic field strength, while the shading outside the horizon is a measure of the toroidal magnetic field (as the conserved angular momentum Poynting flux). Comparison of the two indicates that large-scale field line bending could also be predicted via the relative strengths of poloidal and toroidal magnetic fields on the horizon. A strong toroidal field relative to the poloidal field generally causes bending towards the azimuthal axis, while a weak toroidal field and stronger poloidal field results in bending towards the equatorial plane.

In Jet-Disk and Disk-Jet magnetospheres a single monopolar field line can be defined as the separatrix between the two regions of opposite bending. We determined that separatrix by doing a Cartesian \((x, z)\) transformation from the \((r, \theta)\) computational grid near the outer boundary of the magnetosphere followed by finding the absolute minimum of the second derivative in \(x\) along different field lines. The resulting monopolar field line was then visually verified in comparison with the entire magnetosphere to ensure reasonableness. In general the field line angular velocity of that field line falls between 0.5\(\omega_H\) and 0.6\(\omega_H\), compatible with the notion that 0.5\(\omega_H\) field lines “want” to be straight. Not much more than compatibility should be concluded, however, as a careful inspection of the monopolar field lines drawn in magenta in Figure 5 makes it clear that the determi-
nation of monopolarity can be somewhat arbitrary and dependent upon the region of the magnetosphere chosen for analysis.

Jet-Disk magnetospheres are perhaps the most interesting in terms of astrophysical relevance and are generally predicted by two features. First, the field line angular velocity on the azimuthal axis must be less than or equal to 0.4ωH. Second, the field line angular velocity must increase from the azimuthal axis to the equatorial plane with an ultimate value greater than or equal to 0.6ωH. This is again compatible with the general rule of low/high ΩF jet/disk bending, and the more extreme the difference between azimuthal axis and equatorial plane field line angular velocities the more obvious the “jet-disk” behavior becomes. There is significant variation in the amount of energy and angular momentum flowing into either the “jet” or to the “disk”, as shown by the shading in the middle two rows of Figure 3; we explore that variation in more detail below in Section III.C.

B. Energy and Angular Momentum Extraction

We measure the net rate of black hole energy extraction via the dimensionless parameter χ, calculated as an integral over the horizon (cf. Lasota et al. [20]):

\[ \chi = \frac{1}{2} \frac{a^2}{(r_+^2 + a^2)} \int_0^\pi Q (1 - Q) \frac{A_{\phi,0}^2 \sin \theta}{r_+^2 + a^2 \cos^2 \theta} d\theta. \]  

(9)

Here Q is a unitless scaling of the field line angular velocity on the horizon; \( Q = A + B \sin \theta \). In terms of χ, the net luminosity is given by:

\[ P = \int_{r_+} T^\nu T^\rho \sqrt{-g} \delta \eta d\phi d\theta \]

\[ = 6.5 \times 10^{20} \cdot \chi \cdot r_+^4 \cdot B_x^2 \cdot m^2 \cdot \text{erg} / s. \]  

(10)

Here \( a_+ \) and \( r_+ \) are dimensionless measures of black hole spin and horizon radius; \( a = a_+ m \) and \( r_H = r_H m \). The quantity \( B_x \) corresponds to monopolar magnetic field strength at dimensionless radius \( r_+ \), in the sense that in the Newtonian limit of a monopole we would have a magnetic field that in spherical orthonormal coordinates is given by:

\[ B = \frac{B_x}{r^2} \hat{r}. \]  

(11)

We emphasize that the definitions of \( B_x \) and \( r_+ \) are made purely for convenient compatibility with our monopolar boundary conditions and should be taken to be nothing more than a rough average of magnetic field strength as our magnetospheres are neither Newtonian nor generally truly monopolar. The rate at which a given magnetosphere extracts energy in terms of χ is shown in Figure 5.

![Figure 5](https://via.placeholder.com/150)

**FIG. 5.** The net rate of black hole energy extraction for all magnetospheres in terms of the dimensionless parameter χ. The top panel plots lines of constant \( A \); the bottom panel plots χ as a function of average field line angular velocity on the horizon. The cyan shading denotes the region within 90% of the maximum luminosity. That region includes every type of observed magnetosphere, as shown by the compatible cyan shading in Figure 4.

We measure the net rate of angular momentum extraction via the dimensionless parameter \( \varphi \) in almost identical fashion to the measurement of χ:

\[ \varphi = \frac{1}{2} \frac{a_+}{r_+^2} \int_0^\pi (1 - Q) \frac{A_{\phi,0}^2 \sin \theta}{r_+^2 + a^2 \cos^2 \theta} d\theta. \]  

(12)

In terms of \( \varphi \), the net rate of black hole angular momentum extraction is given by:

\[ K = -\int_{r_+} T^\nu T^\phi \sqrt{-g} \delta \eta d\phi d\theta \]

\[ = 3.2 \times 10^{15} \cdot \varphi \cdot r_+^4 \cdot B^2 \cdot m^3 \cdot \text{erg}. \]  

(13)

The rate at which a given magnetosphere extracts momentum in terms of \( \varphi \) is shown in Figure 6.

Perhaps the most striking feature of the net rate of energy extraction in Figure 5 is the very broad peak. The maximum luminosity corresponds to a horizon field line angular velocity of \( \Omega_F = (0.5 + 0.05 \sin \theta) \omega_H \), but there are a large number of magnetospheres that have effectively equivalent luminosities within 90% of that maximum. Those magnetospheres encompass every single type of observed magnetosphere, as shown by the cyan shading in Figure 4. The bottom panel of Figure 5 indicates that average field line angular velocity is not very predictive of a maximally luminous magnetosphere; anything from \( \langle \Omega_F \rangle = 0.3 \omega_H \) to \( \langle \Omega_F \rangle = 0.8 \omega_H \) can yield a very close to maximum luminosity magnetosphere. Averages near 0.5ωH could be assumed to be relatively luminous and averages closer to the outer limits of energy
extracting magnetospheres ($\langle \Omega_F \rangle = 0$ or $\langle \Omega_F \rangle = \omega_H$) could be assumed to be relatively dim, but anything else would require closer analysis.

The rate of energy extraction calculated by Blandford and Znajek [2] for a monopolar geometry corresponds to a $\chi$ parameter of:

$$\chi_{BZ77} = \frac{1}{8} \frac{a_*^2}{r_{++}^2} + a_*^2 \int_0^\pi \sin^3 \theta \left( r_{++}^2 + a_*^2 \cos^2 \theta \right) d\theta.$$  

(14)

In this work we used a dimensionless black hole spin parameter $a_* = 0.8$ and corresponding horizon radius $r_{++} = 1.6$, yielding $\chi_{BZ77} = 0.0124$. Despite being well outside the "low spin" assumption used to derive that value, it is still only about 10% over our maximum value of $\chi$. This is primarily a result of our solutions concentrating more horizon poloidal magnetic flux near the azimuthal axis than a monopole would, as shown by the middle of the top row of Figure 3.

The closeness of such a crude approximation indicates that any estimate of the net luminosity of an energy-extracting black hole magnetosphere based in some way on the assumptions of a Blandford and Znajek monopole (such as made in McKinney and Gammie [18], Tchekhovskoy et al. [21], and Pan and Yu [22], among others) will in general be successful. There are a wide range of magnetospheres within 10% or so of the near energy maximum such an estimate would yield, including essentially every magnetosphere with an average horizon field line angular velocity $\langle \Omega_F \rangle = 0.5 \omega_H$, typically used in such estimates.

The net rate of angular momentum extraction is more discriminating in its behavior than the net rate of energy extraction, as shown in Figure 9. Very low field line angular velocity magnetospheres always extract more angular momentum than others. Those magnetospheres also always correspond to Jet-Jet magnetospheres, as shown by the magenta shading in Figure 4. The spread in the rate of angular momentum extraction for a given average horizon field line angular velocity is typically not very large, so it is also generally safe to assume that if average field line angular velocity is increased then the net rate of angular momentum extraction will be decreased.

So far we have only considered net rates of energy and angular momentum extraction. However the direction of those flows could in many instances be of far greater importance than the net values, as indicated by comparing the luminosities of Figure 5 with magnetosphere type in Figure 4 and magnetosphere structure in Figure 3. Magnetospheres near the luminosity maximum are those that are closest to exhibiting monopolar behaviors; magnetospheres that bend most tightly towards either the azimuthal axis or the equatorial plane are the dimmest. When compared with the rates of angular momentum extraction, we find that magnetospheres that would most rapidly spin down a black hole will be fairly underluminous in a global sense, but that most of that energy will be very tightly directed along the azimuthal axis. Magnetospheres that would take the longest amount of time to spin down a black hole will be similarly underluminous in a global sense, but most of that energy will be transmitted into a small nearby region in the equatorial plane. So while some magnetospheres might be dimmer overall than others, they might direct that energy in a far more efficient fashion to specific regions of interest while at the same time correlating with other behaviors due to their different rates of angular momentum extraction.

We examine the importance of the direction of energy and angular momentum flows in more depth in the next section within the context of Jet-Disk magnetospheres.

C. Structure of Jet-Disk Magnetospheres

Jet-Disk disk magnetospheres are probably the most interesting type of magnetosphere we found, as they most closely resemble what might often be expected in an astrophysical environment: open field lines aligned with the azimuthal axis and lower latitude structures compatible with a connection to nearby accreting matter. In light of that interest, in this section we divide the fluxes of energy and angular momentum in Jet-Disk magnetospheres into their "jet" and "disk" components and compare their magnitudes.

Of the 400 calculated magnetospheres, 69 are classified as Jet-Disk, with $A$ parameters in the distribution of horizon field line angular velocity $\Omega_F = (A + B \sin \theta) \omega_H$ ranging from 0.0 to 0.4. In order to discuss the general tendencies of Jet-Disk magnetospheres we first number each

\[
\chi_{BZ77} = \frac{1}{8} \frac{a_*^2}{r_{++}^2} + a_*^2 \int_0^\pi \sin^3 \theta \left( r_{++}^2 + a_*^2 \cos^2 \theta \right) d\theta.
\]
of the 69 magnetospheres by their different $A$ parameters as shown in Figure 4 magnetospheres with $A = 0.0$ and $B = [0.60..0.95]$ are numbered by increasing $B$ from 1 to 8, those with $A = 0.05$ and $B = [0.55..0.90]$ are numbered by increasing $B$ from 9 to 16, and so on.

We separate magnetospheres into jet and disk regions by using the most monopolar field line (i.e. the line with minimal bending in the poloidal plane) as the separatrix between the two regions. The determination of the most monopolar field line is slightly arbitrary in that it is dependent upon the parameters and regions used to measure bending, which in Section III A led to some variability in the field line angular velocity of the most monopolar field line. In this case that variability is not significant, as both the ratios of energy and angular momentum flow to jet and disk regions as well as the trends in their magnitudes from magnetosphere to magnetosphere do not significantly change over the spread of what could reasonably be called a valid separatrix between jet and disk regions.

The total rates of energy and angular momentum extraction don’t vary that much across all of the Jet-Disk magnetospheres, but as the average field line angular velocity goes up very little of either gets sent into the jet region.

![Figure 7](image7.png)

**FIG. 7.** The rates of energy and angular momentum flow into the jet and disk regions for all 69 Jet-Disk magnetospheres. The magnetosphere numbering scheme is shown in Figure 4; the vertical lines separate magnetospheres into $A$ parameter sections, while $B$ parameter increases across a section ($\Omega_p = (A + B \sin \theta) \omega_H$). The total rates of energy and angular momentum extracted are fairly constant but the ratio between jet and disk regions can be large. The maximum rate of energy flow into the jet is $\chi_{\text{Jet Max}} \approx 3 \times 10^{-3}$ while the minimum is $\chi_{\text{Jet Min}} \approx 2 \times 10^{-5}$, around 130 times smaller. The blue circles mark the magnetospheres shown in Figure 4; the red circle marks the magnetosphere shown in Figures 1 and 2.

Figure 5 shows the rates of energy and angular momentum flux into both jet and disk regions for all 69 Jet-Disk magnetospheres, grouped by the $A$ parameter of their horizon field line angular velocities. In general much more energy flows into the disk region than into the jet region. Similarly much more angular momentum flows into the disk region than into the jet region, although the relative difference is generally smaller. The amounts of energy and angular momentum flowing into the disk region per unit of energy flowing into the jet region in Jet-Disk magnetospheres. The magnetosphere numbering scheme is shown in Figure 4 (the same as used in Figure 3). For any assumed energy flow into the jet-like structure aligned with the azimuthal axis the concurrent amounts of energy and angular momentum flowing towards the equatorial plane can vary widely. The blue circles mark the magnetospheres shown in Figure 3; the red circle marks the magnetosphere shown in Figures 1 and 2.

![Figure 8](image8.png)

**FIG. 8.** The amount of energy and angular momentum flowing into the disk region per unit of energy flowing into the jet region in Jet-Disk magnetospheres. For an assumed jet energy up to 200 times more energy could be flowing into the disk region, with a median of around 10 times more energy. For that same assumed jet energy there is also a large range of possible momentum fluxes into the disk region; the maximum is almost 60 times larger than the minimum amount. Those ranges are coupled to the strength of the jet in terms of how sharply field lines bend towards the axis, as illustrated by the two middle rows of Figure 3. The tightest jet-like structures have the smallest ratios between jet energy and momentum and disk energy and momentum, while the loosest jet-like structures have the largest ratios between jet and disk energies and momenta.

Putting the ratio of jet energy and disk energy and momentum flows into more concrete terms, for every erg of energy flowing into the jet region there will be between 2 and 200 ergs of energy flowing into the disk region and between $1 \times 10^{-4}m/M_\odot$ and $5 \times 10^{-3} m/M_\odot$ erg-seconds of angular momentum flowing into the disk region, with...
median values given by:

\[ 1 \text{ Jet erg} \sim 10 \text{ Disk erg} \]
\[ \sim 3 \times 10^{-4} \frac{m}{M_{\odot}} \text{ Disk erg-s.} \quad (15) \]

Potential implications of those ranges are discussed below in Section IV C.

IV. DISCUSSION

In this section we discuss three topics in more depth. First we compare our results with our previous work in calculating uniform field line angular velocity magnetospheres. Second we discuss the original concept that led to this work, namely a desire to determine if the inner Alfvén surface of a magnetosphere might convey any additional useful information beyond the potential for an individual field line crossing it to extract a black hole’s rotational energy. Lastly we explore how reasonable the Jet-Disk magnetospheres we found might be, and what potential implications they might have for astrophysical objects.

A. Comparison with Uniform Field Line Angular Velocity

In our previous numerical work [6] we calculated the structure of energy extracting force-free black hole magnetospheres as a function of uniform field line angular velocity using the same monopolar boundary conditions along the azimuthal axis and equatorial plane that were used here. We found that rapidly rotating magnetospheres (referenced to the horizon’s angular velocity) with inner light surfaces near the horizon had poloidal magnetic field lines that bent towards the equatorial plane, while slowly rotating magnetospheres with inner light surfaces near the outer boundary of the ergoregion had poloidal magnetic field lines that bent upwards towards the azimuthal axis. Such behavior may also be seen analytically [7], though in a more limited form.

In this work we largely found the same behaviors. If a collection of field lines had a relatively small average field line angular velocity they bent upwards towards the azimuthal axis; if they had a relatively large average field line angular velocity they bent downwards towards the equatorial plane. The only exceptions to that behavior were in cases where adjacent groups had incompatible bending preferences; in those cases the group with most extreme field line angular velocity would generally “win” and bend the other group.

Our previous work speculated that consideration of the direction of energy and angular momentum flows might be critical to any consideration of black hole energy extraction as a plausible central engine driving astrophysical phenomena. In that speculation we were hampered by the crudity of our assumptions, perhaps most notably that of uniform field line angular velocity, but nonetheless were able to use the timescale of a transient object as an example of how the direction of energy and angular momentum flows could potentially modify observed behaviors.

Having now solved for more realistic distributions of field line angular velocity we are more strongly convinced that the direction of energy flow should be a primary consideration in determining the applicability of black hole energy extraction to any given astrophysical object. Within the context of Jet-Disk magnetospheres, for example, any assumed amount of jet energy would need to be coupled to a consideration of the effects of the concurrent flows of energy and angular momentum into nearby accreting matter, something we discuss in more detail below in Section IV C.

Lastly, our previous work speculated that changes in magnetosphere structure could be more significant in varying luminosity than changing either black hole spin or magnetic field strength. That is what we found here; within Jet-Disk magnetospheres we found that jet energies varied by a factor of 130 (Figure 7). As a general rule the rate of energy extraction varies with black hole spin as $a^2$ (e.g. Equation 14 or 9), although for very high spin better approximations can be made (e.g. Tchekhovskoy et al. [21]). If we select what might be a reasonable range of black hole spins for active galactic nuclei, 0.3m to 0.95m (e.g. Volonteri et al. [23]), then an $a^2$ estimate only yields a factor of 10 variation in luminosity due to changing black hole spin. Our previous numerical work [6] suggests that a factor of 30 variation might be a better estimate, but that’s still sub dominant to the factor of 130 variation within Jet-Disk magnetospheres found here. When both effects are combined, jet luminosity variations in excess of a factor of 1000 could be expected for the jets of Jet-Disk magnetospheres over a reasonable range of black hole spins, a variation that would be correlated with both the degree of jet collimation and the amount of energy and angular momentum concurrently flowing into the disk region.

B. Alfvén Surface Location

In the introduction and elsewhere we have made reference to the inner light (Alfvén) surface as being an important element of black hole energy extraction that might influence magnetosphere structure. In this section we discuss in what sense we make those statements and how robust they might be.

As noted in the introduction, under fairly general assumptions if a given magnetic field line of an ideal plasma flow has an ongoing Alfvén point within the ergoregion then it is possible for black hole rotational energy to be extracted along that field line [3]. Further, the exact position of the Alfvén point within the ergoregion restricts the allowed relationships between the conserved
plasma parameters associated with that field line (Takahashi [24]). Those restrictions necessarily couple Alfvén point location to both global magnetosphere structure and the rate of black hole energy extraction.

Quantifying any coupling between Alfvén point location and global magnetosphere structure could be a useful tool in exploring black hole magnetospheres. The equation describing force balance perpendicular to magnetic field lines, the general relativistic Grad-Shafranov (trans-field) equation (Nitta et al. [25]), is almost completely analytically intractable. As such analytic explorations of black hole magnetospheres almost universally assume some poloidal magnetic field line structure and/or focus on explorations of the interactions between plasma parameters along a single magnetic field line (e.g. Pu et al. [26], Globus and Levinson [27], Takahashi and Tomimatsu [28]). If a given Alfvén point location and its concurrent restrictions on plasma parameters were known to be most compatible with an assumed poloidal magnetic field line structure, that knowledge could provide a valuable constraint on analytic explorations of black hole magnetospheres. It is from this perspective that we make reference to Alfvén surface location being coupled to the structure of black hole magnetospheres.

However, it should be understood that when we make statements about Alfvén (or light) surface location coupling to magnetosphere structure that we are making those statements within a specific context. When considering a single magnetic field line, for example, there are infinitely many relationships between the plasma parameters passing through a given Alfvén point, and as such infinitely many magnetospheres that might be associated with that single magnetic field line. A collection of magnetic field lines is similar; there can be infinitely many magnetospheres associated with any given Alfvén surface. In other words Alfvén surface location cannot possibly encode complete information as to the structure of a magnetosphere - additional restrictions or information must be given for context. In this work that context should largely be taken to be force-free magnetospheres in a minimum energy state that are compatible with monopolar boundary conditions near the horizon.

The magnetospheres we calculated in this work did not exhibit any significant bending of field lines within the ergoregion, making it impossible to decouple inner light surface location from horizon field line angular velocity distribution. Additionally, large scale structure could just as easily be attributed to the ratio of poloidal to toroidal magnetic field strength, which is coupled to field line angular velocity on the horizon via the Znajek regularity condition (Znajek [29]). The 3-way equivalence between inner light surface location, horizon field line angular velocity, and horizon toroidal and poloidal magnetic fields allows for many interpretations as to what “causes” global magnetosphere behavior. Our view is that inner light surface location is potentially the most fundamentally interesting and that horizon field line angular velocity is the most useful descriptor, but we would in no way suggest that as being a universally correct or useful perspective.

It is possible that the addition of plasma inertia effects might help decouple Alfvén surface location from field line angular velocity or otherwise break the 3-way equivalence in interpretation of the source of global magnetosphere structure. However, the relationships between plasma parameters at the Alfvén point intimately involve field line angular velocity and as such it might ultimately be impossible to disentangle their respective effects within the context of reasonable magnetospheres. Our belief (or more accurately hope, biased by our perspective and background in examining 1D plasma flows along single magnetic field lines) is that some decoupling will occur and that Alfvén surface location will prove to be useful, but additional study is required to determine if that is the case.

C. Jet-Disk Magnetospheres

In this section we explore some potential implications of the Jet-Disk magnetospheres that were found. We first examine whether or not the distributions of horizon field line angular velocities leading to those magnetospheres are reasonable, then examine what restrictions Jet-Disk magnetospheres might place on black hole energy extraction in astrophysical contexts.

To determine how reasonable the distributions of Jet-Disk horizon field line angular velocities might be, we consider what might be expected of the black hole’s nearby environment. An isolated black hole cannot support a magnetic field, so the magnetic flux that we assume exists on the horizon must be maintained by nearby matter. A likely configuration of such matter compatible with our assumption of stationarity is matter rotating near the equatorial plane with an angular velocity distribution corresponding to centripetal forces roughly balancing gravitational forces. For convenience we will call that configuration of matter a “disk” (a choice made to aid discussion, not to imply preference for a thin/thick disk, torus, or other structure). The disk is likely to be highly conductive, meaning that the magnetic field will rotate with the disk and possess a field line angular velocity compatible with the disk’s angular velocity. This means that near the equatorial plane the angular velocity of magnetic field lines should be largest near the black hole and decrease as the distance from the black hole increases, formally vanishing infinitely far away.

The field lines on the horizon near the equatorial plane should connect to the disk in nearby regions, and by virtue of rotating with the disk should have $$\Omega_F \approx \omega_H$$. Proceeding up the horizon field lines will connect with the disk further and further away, resulting in a gradual diminishing of field line angular velocity towards the azimuthal axis. That is exactly the type of horizon field line angular velocity distribution that results in Jet-Disk magnetospheres, and has been used by Yuan et al. [30] to
calculate magnetospheres that directly connect the horizon to a nearby disk (the problem setup used there prohibits the emergence of open field lines connected to the horizon, however, making the emergence of a Jet-Disk magnetosphere impossible).

In other words the simplest of assumptions one could make about the black hole’s environment are both compatible with and intrinsically imply the presence of jet-like structures aligned with the azimuthal axis and disk-like connections near the equatorial plane. It indicates that at a basic level asking how a jet didn’t form in a given black hole magnetosphere could be a far more difficult question to answer than how it did in another.

![Diagram of Jet-Disk magnetospheres](image)

**FIG. 9.** Cartoon depiction of Jet-Disk magnetospheres. The field line angular velocities of magnetic field lines in the disk decrease from \( \Omega_F \approx 0.0\omega_H \) far from the black hole to \( \Omega_F \approx \omega_H \) at the inner edge of the disk. With rigid field line rotation the simplest assumption for horizon field line angular velocity distributions mirrors the disk distribution, starting with \( \Omega_F \approx 0.0\Omega_F \) on the pole and ending with \( \Omega_F \approx \omega_H \) near the equator. That assumption leads directly to Jet-Disk magnetospheres, with the separatrix between jet and disk connected field lines being a roughly monopolar field line with \( \Omega_F \approx 0.5\omega_H \). The large amounts of energy and angular momentum flowing outward along field lines between \( 0.5\omega_H \leq \Omega_F \leq \omega_H \) will be deposited into a relatively small disk region, creating a natural reservoir from which to launch an energetic jet along upward bending field lines.

The last feature implied by our assumptions is shown in cartoon form in Figure 9. In Section III C we noted that for a given amount of energy flowing upward into the jet region, a large amount of energy and angular momentum will also flow into the disk region. That energy will be absorbed by nearby disk anchoring those field lines, providing a natural energy reservoir to fuel a jet and help convert the “jet-like” structure into a bona fide jet. In simple terms the terminal Lorentz factor of such a jet would be given by (derived in Appendix B):

\[
u_t \text{ Final} = \nu_t \text{ Initial} + \kappa \frac{\Omega_F}{4\pi\mu |\eta|} \sqrt{-\frac{\eta F_{\theta r}}{\mu}}.
\tag{16}
\]

Here \( \nu_t \) is the temporal component of the plasma’s four velocity (\( \sim \) Lorentz factor), \( \kappa \) a measure of the efficiency of the conversion of electromagnetic Poynting flux to plasma energy, \( \mu \) is the relativistic enthalpy of the plasma (particle mass in the limit of a cold flow), \( \eta \) is the particle flux per unit flux tube, and we have taken absolute values for ease of interpretation. So long as the toroidal field at the base of the jet is large relative to plasma inertia (\( \sim \mu \eta \)) a highly relativistic jet should be expected. The physical acceleration mechanism is largely the same as the acceleration of a bead along a rotating wire, as used in Blandford and Payne [31] to describe disk-launched jets. This can be noted from the fact that conversion of electromagnetic Poynting flux into plasma energy flux simultaneously demands transference of angular momentum to the plasma (cf. Equation [31]). A more detailed analytic treatment of how magnetic energy can be transformed into plasma kinetic energy and launch a jet may be found in Tomimatsu and Takahashi [32], numerical simulations using similarly “monopolar” field lines as those used here including more discussion of acceleration mechanisms may be found in Komissarov et al. [33] and Tchekhovskoy et al. [34], and observational evidence for a highly magnetized jet base using the event horizon telescope may be found in Kino et al. [35].

For convenience we have used the word “disk” to describe the structure of nearby accreting matter, not intending to imply a demand for a geometrically thin disk close to the black hole. Nonetheless the assumption of a thin disk is useful to gauge how compatible a Jet-Disk horizon field line angular velocity distribution might be with a “disk” by converting the specific energies and angular momenta of a thin disk into a field line angular velocity distribution. The results for the same black hole spin parameter \( a = 0.8\mu \) we used to calculate our magnetospheres are shown in Figure 10.

The radius of the disk corresponding to \( \Omega_F = 0.95\omega_H \) (the maximum value used in calculating magnetospheres) corresponds to what might reasonably be called the inner edge of the disk. It is interior to the innermost stable circular orbit (ISCO), but a hard disk cutoff there is not necessarily to be expected. A more reasonable inner edge would be the point at which the specific angular momentum required to maintain an orbit begins to noticeably increase before diverging at the radius of a co-rotating photon orbit. Subjectively that point might be taken to be near the point where \( \Omega_F = 0.95\omega_H \).

In the absence of direct observational evidence of field line angular velocity in real Jet-Disk type magnetospheres we can also consider the results of numerical simulations. Within that context it can be difficult to quantify field line angular velocity (it is most relevant to stationary and axisymmetric plasmas) and as such is unfortunately often not reported in detail in the literature.
depth is beyond the scope of this work. In qualitative form, however, they can be summarized by the statement that the term “Blandford-Znajek mechanism” might often be more accurately described as the “Blandford-Znajek/Blandford-Payne mechanism” - the launching of a self-collimated jet could be intrinsically implied by some of the most basic assumptions of the structure of a realistic energy-extracting black hole magnetosphere. Additionally, if an extreme jet luminosity associated with black hole energy extraction is indicated then variability in that jet might be expected as a fundamental feature, and any observed absence might indicate that an exploration of how angular momentum is being absorbed by the accreting matter near the black hole could be interesting.

V. CONCLUSIONS

We calculated 400 energy-extracting black hole magnetospheres with varying horizon field line angular velocity distributions given by \( \Omega_F = (A + B \sin \theta) \omega_H \), corresponding to the first two terms of a series expansion of an arbitrary horizon field line angular velocity distribution.

We found that horizon field line angular velocity and the location of the inner light surface are equally predictive of large scale magnetosphere structure. Groups of field lines with field line angular velocity \( \Omega_F \leq 0.5 \omega_H \) (inner light surfaces closer to the outer limits of the ergoregion) tend to bend towards the azimuthal axis. Groups of field lines with field line angular velocity \( \Omega_F \geq 0.5 \omega_H \) (inner light surfaces closer to the horizon) tend to bend towards the equatorial plane. The strength of either bending increases if the further away from \( 0.5 \omega_H \) the field line angular velocity becomes (the closer the inner light surface gets to the horizon or outer limit of the ergoregion).

We also found that the horizon field line angular velocity distribution perhaps most compatible with conditions introduced by nearby accreting matter naturally correspond to magnetospheres that both connect the horizon to that matter and might easily launch a jet. This implies that near-horizon jet launching might be expected as a general feature of energy-extracting black hole magnetospheres. The structure of these magnetospheres further implies that temporal jet variability might be expected as a necessary and intrinsic feature of high luminosity jets. Finally, varying black hole spin from \( a = 0.3m \) to \( a = 0.95m \) coupled to variations in magnetosphere structure could easily lead to a factor of 1000 or more difference in the luminosity of black hole jets. Much of that variation would be due to changes in magnetosphere structure, both in terms of the degree of jet collimation and in terms of highly variable rates of energy and angular momentum deposition into small equatorial regions near the black hole’s innermost stable circular orbit.

However McKinney and Gammie [18] is an exception, finding a Jet-Disk type magnetosphere and reporting a horizon field line angular velocity distribution that increased from \( 0.4 \omega_H \) on the azimuthal axis to \( 0.8 \omega_H \) near the equatorial plane, compatible with our Jet-Disk distributions.

The radius of the disk corresponding to \( \Omega_F = 0.5 \omega_H \) is roughly only a single gravitational radii away from the radius corresponding to \( \Omega_F = 0.95 \omega_H \), meaning that only a very small portion of the disk would have to absorb the potentially enormous amounts of energy and angular momentum required for a given amount of energy directed into the jet. This could have at least two useful effects. First, it might assist in jet launching by heating the plasma near the base of the jet. Second, it might provide the increase in specific angular momentum required for disk material to continue orbiting in relatively stable fashion inside the ISCO. However, disk energy and angular momentum deposition could also be harmful to jet formation, in that the rate of angular momentum deposition could become too large for a stationary disk to absorb. In that case the disk could be blown away and the jet halted until accretion was able to resume, implying that the most luminous jets might necessarily be intrinsically variable.

Quantifying the behaviors discussed above in more
Appendix A: Disk Energy, Momentum, and Field Line Angular Velocity

In this appendix we provide a brief review of the specific energy, angular momentum, and field line angular velocity associated with a disk in the equatorial plane of a Kerr spacetime in Boyer-Lindquist coordinates. Using a (+, −, −, −) metric signature, the conserved specific energy $e$ and angular momentum $l$ of a particle with four velocity $u^\alpha$ are given by:

\begin{equation}
\begin{align*}
& e = k^\alpha u_\alpha, \\
& l = -l^\alpha u_\alpha.
\end{align*}
\end{equation}

Here $k^\alpha$ and $l^\alpha$ are the Killing vectors associated with the temporal and azimuthal symmetries of the spacetime. The normalization of the four velocity $u_\alpha u^\alpha = 1$ with $u^\prime = u^\prime = 0$ then yields:

\begin{equation}
-\frac{1}{\rho_\omega^2} (g_{\phi\phi}e^2 + 2g_{\phi\theta}el + g_{tt}l^2) = 1.
\end{equation}

Here $\rho_\omega$ is the cylindrical radius, defined as $\rho_\omega^2 \equiv g_{\phi\phi} - g_{tt}g_{\phi\phi} = \Delta \sin^2 \theta$. We then demand that the centripetal force on the particle balance the gravitational force in...
the radial direction to find:

\[ 0 = \Gamma_{\alpha\beta} u^\alpha u^\beta = -\frac{g^{rr}}{r^2} \left[ g_{t\phi,\phi} \left( g_{\phi\phi} c + g_{\phi\phi} \right) l \right]^2 - 2 g_{t\phi,\phi} \left( g_{\phi\phi} c + g_{\phi\phi} \right) \left( g_{t\phi} c + g_{t\phi} l \right) + g_{t\phi,\phi} \left( g_{\phi\phi} c + g_{\phi\phi} \right)^2 \]. \tag{A3}

Exploiting the definition of the cylindrical radius \( r_\rho \) this reduces to:

\[ 0 = -\frac{g^{rr}}{2r_\rho^2} \left[ g_{\phi\phi,\phi} e^2 + 2 g_{\phi\phi,\phi} l c + g_{\phi\phi,\phi} l^2 + \rho_{\phi,\phi} \right]. \tag{A4} \]

For the gravitational and centripetal forces to balance the quantity in the square brackets must vanish. Inserting the expressions for the metric components in the equatorial plane, the velocity normalization condition and force balance condition then yield two conditions that \( e \) and \( l \) must satisfy:

\[ 0 = 2r \left( 1 - e^2 \right) - 2m + \frac{2m \left( l - ace \right)^2}{r^2}, \]
\[ 0 = -2mr + \left( 1 - e^2 \right) \left( r^2 + a^2 \right) + l^2 - \frac{2m \left( l - ace \right)^2}{r}. \tag{A5} \]

These equations may be solved for \( e \) and \( l^2 \) to find:

\[ e = \pm \sqrt{1 + \frac{l^2 - 4mr}{3r^2 + a^2}}, \]
\[ l^2 = \frac{C_1 \pm C_2}{C_3}. \tag{A6} \]

The functions \( C_i \) are given by:

\[ C_1 = mr^6 - 3n^2r^5 + 2ma^2r^4 + 6m^2a^2r^3 + ma^2 \left( a^2 - 12m^2 \right) r^2 + 5m^2a^4r, \]
\[ C_2 = 2ma \left( a^2 + 3r^3 \right) \Delta \sqrt{mr}, \]
\[ C_3 = r^2 \left( 6m^2r + 9m^2r - 4a^2 \right). \tag{A7} \]

What signs are chosen in \( e \) and \( l \) depend on what conditions are desired:

\[ e > 0 \rightarrow l = -\sqrt{\ldots} \]
\[ e > 0 \rightarrow l = +\sqrt{\ldots} \]
\[ e < 0 \rightarrow l = +\sqrt{\ldots} \]
\[ e < 0 \rightarrow l = -\sqrt{\ldots} \tag{A8} \]

In order to determine the field line angular velocity associated with the disk we use the condition that the plasma is a perfect conductor, \( F_{\alpha\beta} u^\beta = 0 \), to find two expressions for \( u^i \) and \( u^\phi \) under the assumption that \( u^r = u^\phi = 0 \):

\[ F_{tr} u^t + F_{\phi r} u^\phi = 0, \]
\[ F_{\phi t} u^t + F_{\phi\phi} u^\phi = 0. \tag{A9} \]

When applying either definition of field line angular velocity, \( F_{tr} \equiv -F_{\phi r} \Omega_F \) or \( F_{\phi t} \equiv -F_{\phi\phi} \Omega_F \), both of the above expressions yield the same condition: \( u^\phi = \Omega_F u^t \). Replacing \( u^i \) and \( u^\phi \) with \( e \) and \( l \) from their definitions in terms of Killing vectors from Equation [A1] and solving for \( \Omega_F \) yields:

\[ \Omega_F = -\frac{g_{t\phi,\phi} c - g_{t\phi} l}{g_{\phi\phi,\phi} c + g_{\phi\phi} l}. \tag{A10} \]

Inserting the appropriate metric components in the equatorial plane, we finally conclude that:

\[ \Omega_F = \frac{2mrH}{a} \left( \frac{2ma + (r - 2m)}{\left(r^3 + a^2r + 2ma^2 \right)} e - 2ma \right) \Omega_H. \tag{A11} \]

We introduced a factor of \( r_H = m + \sqrt{m^2 - a^2} \) in order to scale by the angular velocity of the horizon \( \Omega_H = a/2mr_H \). The distributions of \( e \), \( l \) and \( \Omega_F \) for a co-rotating positive energy disk around a black hole with spin parameter \( a = 0.8m \) are shown in Figure [10].

**Appendix B: Terminal Jet Lorentz Factor**

In this appendix we provide a brief derivation of the terminal Lorentz factor of material along a magnetic field line. Robust definitions and explanations of the plasma parameters necessary for the following discussion would not be appropriate here, but may be found in Takahashi *et al.* [3]: we will simply state their meanings.

When plasma inertia effects are important, there are four important field-aligned conserved quantities: the energy flux \( E \), the angular momentum flux \( L \), the particle flux \( \eta \), and the field line angular velocity \( \Omega_F \). In terms of the relativistic enthalpy of the plasma \( \mu \), they are related by:

\[ E = \mu \eta u_t + \frac{1}{4\pi} \sqrt{-g} F^{\phi\phi} \Omega_F, \]
\[ L = -\mu \eta u_\phi + \frac{1}{4\pi} \sqrt{-g} F^{\phi r}. \tag{B1} \]

In other words the the flow’s total flux of energy or angular momentum is conserved, but the ratio of the energy or momentum carried by the plasma or transported as a Poynting flux can vary. If for simplicity we assume a cold flow, then the relativistic enthalpy \( \mu \) reduces to the mass of the plasma’s constituent particles and is conserved. Conservation of energy then yields a restriction between
the initial (I) state of the flow, presumed to be near the black hole, and the final (F) state, presumed to be far away:

\begin{equation}
\mu \eta u_t(I) + \frac{1}{4\pi} B^\phi(I) \Omega_F = \mu \eta u_t(F) + \frac{1}{4\pi} B^\phi(F) \Omega_F. \tag{B2}
\end{equation}

Here we have replaced \( \sqrt{-g} F^{\phi r} \) with \( B^\phi \) to highlight the importance of the toroidal magnetic field; the exact correspondence in flat space is \( \sqrt{-g} F^{\phi r} = r \sin \theta B^\phi \), where \( B^\phi \) corresponds to a standard orthonormal spherical coordinate system. If we assume that some percentage \( \kappa \) of the electromagnetic Poynting flux is converted into plasma energy, then we have \( B^\phi(F) = (1 - \kappa) B^\phi(I) \). Solving for \( u_t(F) \) we then find:

\begin{equation}
 u_t(F) = u_t(I) + \kappa \left| \frac{B^\phi(I)}{\mu \eta} \right| \frac{\Omega_F}{4\pi}. \tag{B3}
\end{equation}

We took an absolute value for simplicity of interpretation; both \( \eta \) and \( B^\phi \) are directed but in this instance combine to form a positive quantity. In flat space far from the black hole \( u_t \) corresponds to the Lorentz factor of the flow. In other words so long as the toroidal field is large enough and the plasma flow isn’t overly massive almost arbitrarily large Lorentz factors can be achieved. While remaining within the context of stationary and axisymmetric ideal plasma flows the primary limiting factor of such a “bead on a rotating wire” acceleration mechanism will be the geometry of the field lines.