Absence of stress energy tensor in CFT\textsubscript{2} models

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Abstract

We prove: the local quantum theories defined by conformally covariant derivatives of the $U(1)$-current algebra in $1+1$ dimensions do not contain a stress-energy tensor in the sense of the theorem of Lüscher and Mack.

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1 Introduction

Much of the present understanding of quantum field theories was achieved by methods related to internal and space-time symmetries. There are reasons to be interested in objects connected with symmetries which are of a local nature, and in this work we concern ourselves with densities generating specific space-time symmetries.

Within the classical framework the relation between local objects and continuous symmetries of a LAGRANGEan field theory is canonical by NOETHER’s theorem: to each such symmetry we have an explicitly known conserved current, whose integrals over space, the charges, generate the corresponding symmetry transformation. In quantum field theory the situation is less satisfactory. If one quantises a classical LAGRANGEan field theory, it may happen that some symmetries do not survive at all because of renormalisation effects. Moreover, there is no a priori knowledge of densities connected with continuous symmetries of a general quantum field theory, although it is possible to characterise such fields abstractly, of course. The nature of conserved currents connected to symmetries at the quantum level (and of their charges in particular) is hard to clarify in general.

These problems are more accessible for the global conformal space-time symmetry in $1+1$ dimensions. Here we have an abundance of models for which explicit constructions of a conserved \textsc{Wightman} quantum field are known, which serves
as a density for the conformal symmetry. When smeared out with suitable test functions, this field actually generates the conformal symmetry in the sense of integrable Lie algebra representations. Its interpretation as a stress-energy tensor is in direct analogy with the classical object.

Depending on weak assumptions LüSCHER AND MACK found that stress-energy tensors of conformally covariant quantum field theories in 1+1 dimensions always yield local formulation of the VIRASORO algebra \([\text{FST89}]\) (theorem 3.1).

We prove: No such stress-energy tensor exists in a class of completely well-behaved conformal theories in 1+1 dimensions, the conformally covariant derivatives of the \(U(1)\) current. These are constructed as fields on MINKOWSKI space and possess conformally covariant extensions on their own FOCK space, but they do not transform covariantly with respect to the transformations implementing global conformal symmetry of the \(U(1)\) current.

YNGVASON \([\text{Yng94}]\) studied the conformally covariant derivatives as part of a broader class of derivatives of the \(U(1)\) current and established, among other things, that they do not fulfill HAAG duality on MINKOWSKI space\(^1\). GUIDO, LONGO AND WIESBROCK \([\text{CLAW98}]\) studied locally normal representations of these models and found representations of the first derivative, which do not allow an implementation of global conformal symmetry. In a closing side remark they noted that this contradicts, by unpublished results of D’ANTONI AND FREDENHAGEN, presence of diffeomorphism symmetry in these models. A general version of these results is available in \([\text{Kös03a}]\), but at this point we are interested in a straightforward argument excluding presence of a stress-energy tensor for the derivative models.

As a first step we establish the conformally covariant derivatives of the \(U(1)\) current as conformally covariant local quantum theories, following a method introduced by BUCHHOLZ AND SCHULZ-MIRBACH \([\text{BSM90}]\).

## 2 The \(\Phi^{(n)}\)-models

The \(U(1)\) current in 1+1 dimensions decomposes into two independent chiral components, the chiral currents, and we shall discuss one of these only. The derivatives of the chiral current \(j\) are given as fields on the light-ray by \(\Phi^{(n)}(x) := \partial_x^n j(x)\), where we used \(\partial_x := d/dx\). These fields are covariant with scaling dimension \(n + 1\) when acted upon by the implementation of the stabiliser group of \(\infty\) for the \(U(1)\)-current theory. By looking at their WIGHTMAN functions one recognises that the \(\Phi^{(n)}\) possess conformally covariant extensions, if restricted to their own FOCK space; the corresponding unitary representation of \(\text{PSL}(2, \mathbb{R})\) implementing global conformal symmetry leaves invariant the vacuum

\(^1\)Essential duality, which is another name for HAAG duality on the conformal covering of MINKOWSKI space, is a consequence of conformal symmetry.
and fulfills the spectrum condition. Each of these extensions transforms covariantly with respect to a different representation of the global conformal group and lives on a different Fock space. From now on, we look at the fields as operators in their cyclic subrepresentation equipped with their own representation of the global conformal group and we use the symbol $\Phi^{(n)}$ in this sense.

By construction, the derivative fields $\Phi^{(n)}$ obey the following commutation relation as fields on the light-ray:

$$\left[\Phi^{(n)}(x), \Phi^{(n)}(y)\right] = \frac{i}{2\pi}(-)^n \delta^{(2n+1)}(x - y) \mathbb{1}.$$  

We want to calculate the corresponding commutation relations for the modes of the conformally extended fields on the compactified light-ray, $S^1$. These fields will be denoted $\tilde{\Phi}^{(n)}$. The test-functions of fields on $S^1$ and their images living on the light-ray are connected by a transformation $f \mapsto \hat{f}$ depending on the scaling dimension of the respective field; its definition is induced by $\tilde{\Phi}^{(n)}(f) \equiv \Phi^{(n)}(\hat{f})$.

The inverse transformation is denoted as $\hat{f} \mapsto \tilde{\hat{f}} = f$.

**Proposition 1** The modes $\Phi^{(n)}_m := \tilde{\Phi}^{(n)}([z^{n+m}])$ have the following commutation relations:

$$\left[\Phi^{(n)}_m, \Phi^{(n)}_{m'}\right] = \delta_{m,-m'} \Pi^{(n)}(m) ,$$  

if we set $\Pi^{(n)}(m) := \prod_{k=0}^{2n}(m - n + k)$.

Remark: These relations imply that the modes $\Phi^{(n)}_m$, $|m| \leq n$, are central, which in turn means that all $L_0$-eigenspaces for eigenvalues $1, \ldots, n$ are null.

**Proof:** We use the shorthand notations $\zeta := (1 + z)$, $d/dz = \partial_\zeta$ and arrive at:

$$\left[\tilde{\Phi}^{(n)}(f), \tilde{\Phi}^{(n)}(g)\right] \equiv \left[\Phi^{(n)}(\hat{f}), \Phi^{(n)}(\hat{g})\right] = (-)^{2n+1} \oint \frac{dz}{2\pi i} f(z) \zeta^{-2(n+1)} \left(\zeta^2 \frac{d}{d\zeta}\right)^{2n+1} \zeta^{-2n} g(z)$$

$$= \oint \frac{dz}{2\pi i} g(z) \left(\frac{d}{dz}\right)^{2n+1} f(z) .$$

The identity of the two integration kernels as distributions may be proved inductively. Applying the induction assumption we see that we have to prove: $\zeta^{-2n} \partial_\zeta \zeta^2 \partial_\zeta \zeta^{2n-1} \zeta^{-2} = \partial_\zeta^{2n+1}$. One may verify this identity for $n = 1$ explicitly. Then one proves by induction on $n$:

$$\zeta^{-2(n+1)} \partial_\zeta \zeta^2 \partial_\zeta \zeta^{2(n+1)} \partial_\zeta^{2n+1} \zeta^{-2}$$

$$= \zeta^{-2} \partial_\zeta^{2n+1} \left(\zeta^2 \partial_\zeta^2 - 2(2n + 1)\zeta^2 \partial_\zeta \zeta^{-1} + (2n + 1)(2n)\right) = \partial_\zeta^{2n+3} .$$

□
As we can see by looking at their canonical commutation relations, the derivative fields may be treated as local quantum theories of bounded operators in terms of Weyl operators and their relations (cf [GLW98]). We take another approach which was introduced by Buchholz and Schulz-Mirbach [BSM90] for the nets of the stress-energy tensor and the $U(1)$ current: By establishing linear energy bounds referring to the conformal Hamiltonian $L_0$ the Haag-Kastler axioms follow from Wightman’s set of axioms. In particular the fields are essentially self-adjoint on the Wightman domain, their bounded functions fulfill locality and the local algebras generate a dense subspace from the vacuum. The local algebras are generated by unitaries $W(f) := \exp(i\tilde{\Phi}^{(n)}(f)^-)\tilde{\Phi}^{(n)}(f)^-$ self-adjoint and $\text{supp}(f) \in S^1$. The $W(f)$ are concrete representations of the Weyl operators.

**Proposition 2** The following defines the local algebras of the chiral net generated by the $\Phi^{(n)}$ fields:

$$A_{\Phi^{(n)}}(I) := \left\{ \tilde{\Phi}^{(n)}(f)^-, \text{supp}(f) \subset I, f = \tilde{f} \right\}, I \in S^1.$$  \hspace{1cm} (3)

**Proof:** The proof follows the lines indicated in [BSM90]. If $\psi_N$ denotes an arbitrary eigenvector of $L_0$ with energy $N$ and norm 1, then $\Phi^{(n)}_m\psi_N, m > 0$, is a multiple of a unit vector of energy $N - m$, which we call $\psi_{N-m}$. Making use of a general estimate for positive, linear functionals $\eta$ [Buc90]: $|\eta(Q)|^2 \leq \eta(Q^*Q)$, we are led to the following bound:

$$\|\Phi^{(n)}_m\psi_N\|^4 \leq \|\Phi^{(n)}_m\psi_N\|^2\|\Phi^{(n)}_m\psi_{N-m}\|^2 + \Pi^{(n)}(m)\|\Phi^{(n)}_m\psi_N\|^2.$$  

For $m \neq 0$ we set $\Pi^{(n)}(m) := \frac{1}{m}\Pi^{(n)}(m)$ and we prove inductively using the spectrum condition: $\|\Phi^{(n)}_m\Psi_N\|^2 \leq N\Pi^{(n)}(m), m \geq 1$.

For the generating modes we have:

$$\|\Phi^{(n)}_m\Psi_N\|^2 = \|\Phi^{(n)}_m\Psi_N\|^2 + \Pi^{(n)}(m) \leq (N + m)\Pi^{(n)}(m).$$

The zeroth mode is central in the theory and is, therefore, a multiple $q$ of the identity. So we have: $\|\Phi^{(n)}_0\Psi_N\|^2 = q^2, \Re q$. For general $\Phi^{(n)}(f), f \in C^\infty(S^1)$, and a vector $\Psi$ from the Wightman domain we have the following estimate:

$$\|\Phi^{(n)}(f)\Psi\| \leq \|(L_0 + 1)\Psi\| \sum_{m \in \mathbb{Z}} |f_m|(|m| + \Pi^{(n)}(m) + |q| + 1).$$  \hspace{1cm} (4)

This is the linear energy bound from which the Haag-Kastler axioms follow as discussed in [BSM90].

Thus shortly on the nuclearity condition for the conformally covariant derivatives of the $U(1)$ current. Since null vectors reduce the multiplicity of $L_0$ eigenvalues, the trace of $e^{-\beta L_0}, \beta > 0$, of the vacuum representation of the derivative
models is dominated by the $L_0$ character for the $U(1)$ current, which is given by the combinatorial partition function $p(e^{-\beta})$. The following discussion applies for the same reason to all theories defined by a stress-energy tensor and to the $U(1)$-current algebra. $p(e^{-\beta})$ is directly connected to Dedekind’s $\eta$-function:

$$p(e^{-\beta})^{-1} = \prod_{m \geq 1}(1 - e^{-\beta m}) = e^{-\frac{\beta}{2\pi i} \eta(i\beta/(2\pi))}.$$  

For the nuclearity condition we have to check the asymptotic behaviour for $\beta \searrow 0$. This behaviour is determined by the transformation law of $\eta$ for $\tau \rightarrow 1/\tau$. It reads [Sch74] (III.§3): $\sqrt{\beta/2\pi} \eta(i\beta/(2\pi)) = \eta(i2\pi/\beta)$. We have with $\beta_0 > -1 + \pi^2/6$ and $n = 1$:

$$\lim_{\beta \searrow 0} p(e^{-\beta})e^{-(\frac{\beta_0}{2})} = 0 .$$ (5)  

This estimate is a special form of a nuclearity condition and ensures the split property for all models under consideration by arguments as given in [GE93] (Lemma 2.12.).

3 No stress-energy tensor in $\Phi^{(n)}$ models

We seek for a stress-energy tensor in the theories defined by conformally covariant derivatives of degree $n$ of the $U(1)$ current in 1+1 dimensions. We assume the stress-energy tensor to deserve its name and therefore it should be a local, covariant, conserved, symmetric, traceless quantum field $\Theta$ of scaling dimension 2, which is relatively local to the $\Phi^{(n)}$ under consideration and a density for its infinitesimal conformal transformations. Because all models involved factorise into chiral components, we shall discuss the situation on the compactified light-ray, i.e. the fields live on $S^1$.

According to the analysis of Lüscher and Mack the commutation relations of $\tilde{\Theta}$ have a very specific form [LM76, Mac88, FST89 theorem 3.1]. $\tilde{\Theta}$ is a Lie field with an extension proportional to $c$, the central charge of $\tilde{\Theta}$:

$$\frac{c}{12} \oint \frac{dz}{2\pi i} f'''(z)g(z) = \left[\tilde{\Theta}(f), \tilde{\Theta}(g)\right] - \tilde{\Theta}(f'g - fg').$$

$c/2$ is the normalisation constant of the two point function of $\tilde{\Theta}$, hence we have $c \in \mathbb{R}_+$, and, by the Reeh-Schlieder theorem, $\tilde{\Theta} = 0$ if and only if $c = 0$.

Proposition 3 The conformally covariant derivatives of the $U(1)$ current in 1+1 dimensions do not contain a stress-energy tensor.

Proof: Looking at the commutation relations of the modes of $\Phi^{(n)}$ (equation 1), we learn that the eigenspaces of the conformal Hamiltonian $L_0$ associated
with energy 1, . . . , n are all null. If n ≥ 2 this yields for \( L_{-2} = \tilde{\Theta}([z^{-1}]) \): \( c/2 = \| L_{-2}\Omega \|^2 = 0 \), and hence \( \tilde{\Theta} = 0 \).

In the case \( n = 1 \) all vectors of energy 2 are multiples of \( \Phi^{(1)}_{-2}\Omega \). If there is a stress-energy tensor \( \tilde{\Theta} \), we have: \( \gamma L_{-2}\Omega = \Phi^{(1)}_{-2}\Omega, \ c|\gamma|^2 = 12 \). Obviously, \( \tilde{\Phi}^{(1)} - \gamma \tilde{\Theta} \) is a quasi-primary field and its two-point function is determined by conformal covariance up to a constant, \( C \geq 0 \):

\[
\langle \Omega, (\tilde{\Phi}^{(1)}(z) - \gamma \tilde{\Theta}(z))(\tilde{\Phi}^{(1)}(w) - \gamma \tilde{\Theta}(w)) \Omega \rangle = C(z > w)^{-4}.
\]

In particular, we have: \( C = \| (\Phi^{(1)}_{-2} - \gamma L_{-2})\Omega \|^2 = 0 \). By the REEH-SCHLIEDER theorem, the field \( \tilde{\Phi}^{(1)} - \gamma \tilde{\Theta} \) is zero. Since \( \gamma^{-1} \tilde{\Phi}^{(1)} \) is not a stress-energy tensor, the claim holds for \( n = 1 \) as well.

\[\square\]

4 Discussion

We have shown that a quantum field theory of conformally covariant derivatives of the \( U(1) \) current in 1 + 1 dimensions does not contain a stress-energy tensor. This adds another detail to their character as archetypes of conformal theories in 1 + 1 dimensions: In spite of being simple and completely well behaved, they do not exhibit special properties of other comparatively simple models such as strong additivity or presence of a stress-energy tensor.

If there is a local density associated in some sense with the conformal symmetry of these models, it has to be of a different nature.

We mention just one reason why such densities are desirable. Looking at a chiral conformal theory \( \mathcal{B} \) in its vacuum representation and at a covariant subnet \( \mathcal{A} \subset \mathcal{B} \), the question arises, whether the local relative commutants \( \mathcal{A}(I)' \cap \mathcal{B}(I) \) define a subtheory as well. The problem is to show that the relative commutants increase with \( I \), i.e. to prove isotony for this set. If there is a sufficiently well behaved local density for the dilatations in the globally inner representation \( U^\mathcal{A} \), this can actually be confirmed. This program has been carried out in presence of stress-energy tensors [Kös03b], but it should be feasible in more general settings as well: There ought to be sufficiently many local observables to answer such questions.

A general quantum version of NOETHER’s theorem exists [BDL86] on grounds of the split property, which is established easily for the conformally covariant derivatives (see section 2). Here, symmetries are implemented on local algebras by operators which are localised in a somewhat enlarged region. These local implementers are densities of a different nature than eg stress-energy tensors, since they define a representation of the respective symmetry group with the same spectral properties as the original one, but it has been established that they provide approximations for the global implementation of the respective symmetry
It is not clear, however, whether these local implementers will prove sufficiently well behaved, if we want to apply them to the isotony problem.

Carpi [Car99] reconstructed the stress-energy tensor of some models by taking point-like limits of the local implementers applying methods of Jörß and Fredenhagen [FJ96] and hence gave an explicit account of the relation of local implementers to densities as needed in [Kös03b]. It appears important to study the local implementers as densities for the conformal transformations in more detail and starting with a look at the conformally covariant derivatives seems to be promising.

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