On Coefficients Problems for Typically Real Functions Related to Gegenbauer Polynomials

Paweł Zaprawa, Magdalena Figiel and Anna Futa

Abstract. We solve problems concerning the coefficients of functions in the class $T(\lambda)$ of typically real functions associated with Gegenbauer polynomials. The main aim is to determine the estimates of two expressions: $|a_4 - 2a_2a_3|$ and $|a_{2a_4} - a_3^2|$. The second one is known as the second Hankel determinant. In order to obtain these bounds, we consider the regions of variability of selected pairs of coefficients for functions in $T(\lambda)$. Furthermore, we find the upper and the lower bounds of functionals of Fekete–Szegö type. Finally, we present some conclusions for the classes $T$ and $T(1/2)$.

Mathematics Subject Classification. 30C50.

Keywords. Gegenbauer polynomials, typically real functions, coefficient problems, Hankel determinants.

1. Introduction

Let $\Delta$ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $A$ be the class of all functions $f$ analytic in $\Delta$, normalized by the condition $f(0) = f'(0) - 1 = 0$. This means that $f \in A$ has the expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$  

In 1994 Szynal [15] introduced the class $T(\lambda)$, $\lambda \geq 0$ as the subclass of $A$ consisting of functions of the form

$$f(z) = \int_{-1}^{1} k(z, t)d\mu(t),$$  

where

$$k(z, t) = \frac{z}{(1-2tz+z^2)^{\lambda}}, \quad z \in \Delta, \quad t \in [-1, 1]$$  

(3)
and \( \mu \) is a probability measure on the interval \([-1,1]\). The collection of such measures on \([a,b]\) is denoted by \( P_{[a,b]} \).

The function \( k(z,t) \) has the Taylor series expansion

\[
k(z,t) = z + C^{(\lambda)}_1(t)z^2 + C^{(\lambda)}_2(t)z^3 + \cdots ,
\]

where \( C^{(\lambda)}_n(t) \) denotes the Gegenbauer polynomial of degree \( n \) (for details about the Gegenbauer polynomials, see [1,14]).

First polynomials of this type are following:

\[
\begin{align*}
C^{(\lambda)}_0(t) &= 1, \\
C^{(\lambda)}_1(t) &= 2\lambda t, \\
C^{(\lambda)}_2(t) &= 2\lambda(\lambda+1)t^2 - \lambda, \\
C^{(\lambda)}_3(t) &= \frac{4}{3}\lambda(\lambda+1)(\lambda+2)t^3 - 2\lambda(\lambda+1)t.
\end{align*}
\]

If \( f \in T(\lambda) \) is given by (2), then the coefficients of this function can be written as follows:

\[
a_n = \int_{-1}^{1} C^{(\lambda)}_{n-1}(t)d\mu(t).
\]

Note that \( T(1) = T \) is the well-known class of typically real functions (for details, see e.g., [2,8,9]). For \( \lambda = 1/2 \) we obtain the class of typically real functions related to Legendre polynomials \( P_n(t) = C^{(1/2)}_n(t) \). As it was shown in [15] (see also, [12]), this class is a proper superclass for \( S^*_R(1/2) \), the class of starlike functions of order 1/2 and for \( K_R(i) \), the class of convex functions in the direction of the imaginary axis; both of them consist of functions with real coefficients. More precisely, we have two chains of inclusions:

\[
K_R \subset K_R(i) \subset T(1/2)
\]

and

\[
K_R \subset S^*_R(1/2) \subset T(1/2).
\]

This observation is significant because it happens that the results in \( T(1/2) \) can be transferred to its subclasses (see [12,16]).

In this paper we consider various problems concerning the coefficients of functions in \( T(\lambda) \). We discuss the Fekete–Szegö functional \( a_3 - \mu a_2^2 \) and its modification, i.e., \( a_5 - \mu a_3^2 \). Moreover, we find the sharp bounds of two expressions: \(|a_4 - a_2 a_3|\) and \(|a_2 a_4 - a_3^2|\). The second is called the second Hankel determinant. It was Pommerenke the first who studied this determinant in the geometric theory of analytic functions ([6,7]).

Recently, the Hankel determinant has been studied by many mathematicians. They discussed the second Hankel determinants for various classes of univalent functions. Some results in this direction can be found in [3–5,10,11].
2. Regions of Variability of a Pair of Coefficients for $T(\lambda)$

Let $A_{n,m}$ denote the set of variability of the point $(a_n, a_m)$, where $a_n$ and $a_m$ are the coefficients of a given function $f \in T(\lambda)$ with the series expansion (1). The set $A_{n,m}$ coincides with the closed convex hull of the curve

$$
\gamma_{n,m} : [-1, 1] \ni t \mapsto \left( C_{n-1}^{(\lambda)}(t), C_{m-1}^{(\lambda)}(t) \right).
$$

By the Carathéodory theorem we conclude that it is sufficient to discuss only the functions

$$
f(z) = \alpha k(z, t_1) + (1 - \alpha) k(z, t_2),
$$

where $-1 \leq t_1 \leq t_2 \leq 1$, $\alpha \in [0, 1]$.

Now, we shall establish a few simple lemmas concerning the sets $A_{n,m}$ for initial integers $n$ and $m$.

**Lemma 2.1.** For the class $T(\lambda)$, $\lambda > 0$, we have

$$
A_{2,3} = \left\{ (x, y) \in \mathbb{R}^2 : -2\lambda \leq x \leq 2\lambda, \frac{\lambda + 1}{2\lambda} x^2 - \lambda \leq y \leq \lambda(2\lambda + 1) \right\}.
$$

**Proof.** Since $A_{2,3} = \text{conv}\{\gamma_{2,3}(t) : t \in [-1, 1]\}$, taking into account (5) we conclude that $\gamma_{2,3}([-1, 1])$ is a part of parabola $y = \frac{\lambda + 1}{2\lambda} x^2 - \lambda$ for $x \in [-2\lambda, 2\lambda]$. Hence, the boundary of the convex hull of $\gamma_{2,3}([-1, 1])$ consists of this curve and the line segment that connects two endpoints of this arc. □

The similar result can be obtained for $A_{3,5}$. It is enough to determine the convex hull of the arc of the parabola $\gamma_{3,5}([-1, 1])$. In this way we get the following lemma.

**Lemma 2.2.** For the class $T(\lambda)$, $\lambda > 0$, we have

$$
A_{3,5} = \left\{ (x, y) \in \mathbb{R}^2 : -\lambda \leq x \leq \lambda(2\lambda + 1), \frac{(\lambda + 2)(\lambda + 3)}{6\lambda(\lambda + 1)} x^2 - \frac{2\lambda(\lambda + 2)}{3(\lambda + 1)} x - \frac{\lambda(\lambda + 3)(2\lambda + 1)}{6(\lambda + 1)} \leq y \leq \frac{1}{3} \lambda(\lambda + 2)x + \frac{1}{6} \lambda(\lambda + 3)(2\lambda + 1) \right\}.
$$

**Remark 1.** In both lemmas the boundaries of the discussed sets consist of an arc of a parabola and a line segment. The points lying on the parabolas correspond to the coefficients of $k(z, t)$, $t \in [-1, 1]$. On the other hand, the points of the line segments are the coefficients of a function $\alpha k(z, -1) + (1 - \alpha) k(z, 1)$ with suitably taken $\alpha \in [0, 1]$.

**Lemma 2.3.** For the class $T(\lambda)$, $\lambda > 0$, we have

$$
A_{2,4} = \left\{ (x, y) \in \mathbb{R}^2 : -2\lambda \leq x \leq 2\lambda, -g(-x) \leq y \leq g(x) \right\},
$$

where

$$
g(x) = \begin{cases} 
\frac{(\lambda + 1)(\lambda + 2)}{6\lambda^2} x^2 - (\lambda + 1)x, & x \in [-2\lambda, -\lambda] \\
\frac{1}{2} \lambda(\lambda + 1)x + \frac{1}{3} \lambda(\lambda + 1)(\lambda + 2), & x \in [-\lambda, 2\lambda].
\end{cases}
$$
Lemma 2.3. It is easy to check that only one critical point of \( f \in T(\lambda) \) is in \([-\lambda, -\lambda]\), but only if \( \lambda \leq 1 \).

\[ a_2a_4 \leq \frac{4}{3} \lambda^2 (\lambda + 1) (2\lambda + 1) \]

and

\[ a_2a_4 \geq \begin{cases} 
-\frac{3\lambda^2(\lambda+1)}{2(\lambda+2)}, & \lambda \in (0,1] \\
-\frac{1}{18} \lambda (\lambda + 1)(\lambda + 2)^2, & \lambda \geq 1 
\end{cases} \]

Proof. The upper bound is obvious. In order to calculate the lower bound of \( a_2a_4 \), consider only these points \( (a_2, a_4) \) which belong to the boundary of \( A_{2,4} \). Since \( A_{2,4} \) is symmetric with respect to the origin, it is enough to discuss points lying on the curve \( y = g(x) \), where \( g \) is the function described in Lemma 2.3.

Let us denote by \( h_1(x) \) and \( h_2(x) \) two functions corresponding to the values of \( a_2a_4 \) on each part of the curve \( y = g(x) \). Namely,

\[ h_1(x) \equiv \frac{(\lambda + 1)(\lambda + 2)}{6\lambda^2} x^4 - (\lambda + 1) x^2, \quad x \in [-2\lambda, -\lambda] \]

and

\[ h_2(x) \equiv \frac{1}{2} \lambda (\lambda + 1) x^2 + \frac{1}{3} \lambda (\lambda + 1)(\lambda + 2) x \quad x \in [-\lambda, 2\lambda]. \]

Only one critical point \( x_1 = -\lambda \sqrt{\frac{3}{\lambda+2}} \) of \( h_1 \) is in \([-2\lambda, -\lambda] \), but only if \( \lambda \leq 1 \). Hence,

\[ \min \{ h_1(x) : x \in [-2\lambda, -\lambda] \} = \begin{cases} 
h_1(x_1) = -\frac{3\lambda^2(\lambda+1)}{2(\lambda+2)}, & 0 \leq \lambda \leq 1 \\
h_1(-\lambda) = \frac{1}{6} \lambda^2 (\lambda + 1)(\lambda - 4), & \lambda \geq 1. \end{cases} \]

(9)

It is easy to check that

\[ \min \{ h_2(x) : x \in [-\lambda, 2\lambda] \} = \begin{cases} 
h_2(-\lambda) = \frac{1}{6} \lambda^2 (\lambda + 1)(\lambda - 4), & 0 \leq \lambda \leq 1 \\
h_2(x_2) = -\frac{1}{18} \lambda (\lambda + 1)(\lambda + 2)^2, & \lambda \geq 1, \end{cases} \]

where \( x_2 = -(\lambda + 2)/3 \).

Combining (9) and (10) we obtain the desired result. \( \square \)

For \( \lambda = 1 \) this result reduces to the one proved in [17] in Theorem 13.
3. Functionals of the Fekete–Szegö Type

Now, we are ready to find the sharp bounds of two functionals defined for \( f \in \mathcal{T}(\lambda) \) of the form (1): \( a_3 - \mu a_2^2 \) and \( a_5 - \mu a_3^2 \). The first one is very well known and often discussed the so-called Fekete–Szegö functional.

**Theorem 3.1.** If \( f \in \mathcal{T}(\lambda), \lambda > 0 \), then the following sharp bounds hold:

\[
a_3 - \mu a_2^2 \leq \begin{cases} 
\lambda(2\lambda + 1 - 4\lambda \mu), & \mu \leq 0 \\
\lambda(2\lambda + 1), & \mu \geq 0
\end{cases} \tag{11}
\]

and

\[
a_3 - \mu a_2^2 \geq \begin{cases} 
-\lambda, & \mu \leq \frac{\lambda+1}{2\lambda} \\
\lambda(2\lambda + 1 - 4\lambda \mu), & \mu \geq \frac{\lambda+1}{2\lambda}
\end{cases} \tag{12}
\]

**Proof.** Let \( f \in \mathcal{T}(\lambda) \). From Lemma 2.1,

\[
a_3 - \mu a_2^2 \leq a_3 \leq \lambda(2\lambda + 1) \quad \text{for} \quad \mu \geq 0
\]

and

\[
a_3 - \mu a_2^2 \leq \lambda(2\lambda + 1) - \mu(2\lambda)^2 \quad \text{for} \quad \mu \leq 0.
\]

The equality holds if \( f(z) = \frac{1}{2} [k(z, -1) + k(z, 1)] = z + C_2^{(\lambda)}(t)z^3 + \cdots \) for \( \mu \geq 0 \) and \( k(z, \pm 1) \) for \( \mu \leq 0 \).

On the other hand, by Lemma 2.1,

\[
a_3 - \mu a_2^2 \geq \left( \frac{\lambda+1}{2\lambda} - \mu \right) a_2^2 - \lambda.
\]

Hence, for \( h(x) = (\frac{\lambda+1}{2\lambda} - \mu) x^2 - \lambda \), we get

\[
\min\{h(x) : x \in [-2\lambda, 2\lambda]\} = \begin{cases} 
h(0), & \mu \leq \frac{\lambda+1}{2\lambda} \\
h(\pm2\lambda), & \mu \geq \frac{\lambda+1}{2\lambda}
\end{cases}
\]

with equality for \( k(z, 0) = z - \lambda z^3 + \cdots \) in first case and for \( k(z, \pm 1) \) in the second one. \( \square \)

**Theorem 3.2.** If \( f \in \mathcal{T}(\lambda), \lambda > 0 \), then the following sharp bounds hold:

\[
a_5 - \mu a_3^2 \leq \begin{cases} 
\frac{1}{6} \lambda(2\lambda + 1) \left[ (\lambda + 1)(2\lambda + 3) - 6\lambda(2\lambda + 1)\mu \right], & \mu \leq \frac{\lambda+2}{6(2\lambda+1)} \\
\frac{1}{56} \lambda \left[ \frac{\lambda(\lambda+2)^2}{\mu} + 6(\lambda + 3)(2\lambda + 1) \right], & \mu \geq \frac{\lambda+2}{6(2\lambda+1)}
\end{cases} \tag{13}
\]

and

\[
a_5 - \mu a_3^2 \geq \begin{cases} 
-\frac{\lambda}{6(\lambda+1)} \left[ \frac{4\lambda^2(\lambda+2)^2}{(\lambda+2)(\lambda^2+3)-6\lambda(\lambda+1)\mu} + (\lambda + 3)(2\lambda + 1) \right], & \mu \leq \frac{\lambda+2}{6(2\lambda+1)} \\
\frac{1}{6} \lambda(2\lambda + 1) \left[ (\lambda + 1)(2\lambda + 3) - 6\lambda(2\lambda + 1)\mu \right], & \mu \geq \frac{\lambda+2}{6(2\lambda+1)}
\end{cases} \tag{14}
\]
Proof. Let \( f \in T(\lambda) \). Taking into account the bounds of \( a_5 \) proved in Lemma 2.2, we have \( a_5 - \mu a_3^2 \leq h_1(a_3) \), where

\[
h_1(x) = -\mu x^2 + \frac{1}{3} \lambda(\lambda + 2)x + \frac{1}{6} \lambda(\lambda + 3)(2\lambda + 1), \quad x \in [-\lambda, \lambda(2\lambda + 1)].
\]

If \( \mu \leq 0 \), then \( h_1(x) \) is \( \max \{h_1(-\lambda), h_1(\lambda(2\lambda + 1))\} \). But for \( \mu \leq 0 \) and \( \lambda > 0 \) we know that \( -\lambda^2 \mu \leq -\lambda^2(2\lambda + 1)^2 \mu \) and \( -\frac{1}{3} \lambda^2(2\lambda + 2) \leq \frac{1}{3} \lambda^2(\lambda + 2)(2\lambda + 1) \), so

\[
h_1(-\lambda) = -\lambda^2 \mu - \frac{1}{3} \lambda^2(\lambda + 2) + \frac{1}{6} \lambda(\lambda + 3)(2\lambda + 1)
\]

\[
< -\lambda^2(2\lambda + 1)^2 \mu + \frac{1}{3} \lambda^2(\lambda + 2)(2\lambda + 1) + \frac{1}{6} \lambda(\lambda + 3)(2\lambda + 1)
\]

\[
= h_1(\lambda(2\lambda + 1)).
\]

For \( \mu > 0 \) we obtain

\[
h_1(x) \leq \begin{cases} 
  h_1(x_1), & x_1 < \lambda(2\lambda + 1) \\
  h_1(\lambda(2\lambda + 1)), & x_1 \geq \lambda(2\lambda + 1),
\end{cases}
\]

where \( x_1 = \frac{\lambda(\lambda + 2)}{6\mu}, \ x_1 > 0 \). Combining these two cases we obtain (13).

By Lemma 2.2, \( a_5 - \mu a_3^2 \geq h_2(a_3) \), where

\[
h_2(x) = qx^2 - \frac{2\lambda(\lambda + 2)}{3(\lambda + 1)} x - \frac{\lambda(\lambda + 3)(2\lambda + 1)}{6(\lambda + 1)}, \quad x \in [-\lambda, \lambda(2\lambda + 1)]
\]

and \( q = \frac{(\lambda + 2)(\lambda + 3)}{6\lambda(\lambda + 1)} - \mu \).

If \( q \leq 0 \), then \( q\lambda^2 \geq q\lambda^2(2\lambda + 1)^2 \) and \( \frac{2\lambda^2(\lambda + 2)}{3(\lambda + 1)} > \frac{-2\lambda^2(\lambda + 2)(2\lambda + 1)}{3(\lambda + 1)} \). Hence,

\[
h_2(-\lambda) = q\lambda^2 + \frac{2\lambda^2(\lambda + 2)}{3(\lambda + 1)} - \frac{\lambda(\lambda + 3)(2\lambda + 1)}{6(\lambda + 1)}
\]

\[
> q\lambda^2(2\lambda + 1)^2 - \frac{2\lambda^2(\lambda + 2)(2\lambda + 1)}{3(\lambda + 1)} - \frac{\lambda(\lambda + 3)(2\lambda + 1)}{6(\lambda + 1)}
\]

\[
= h_2(\lambda(2\lambda + 1)).
\]

For \( q > 0 \),

\[
h_2(x) \geq \begin{cases} 
  h_2(x_2), & x_2 < \lambda(2\lambda + 1) \\
  h_2(\lambda(2\lambda + 1)), & x_2 \geq \lambda(2\lambda + 1),
\end{cases}
\]

where \( x_2 = \frac{\lambda(\lambda + 2)}{3(\lambda + 1)q}, \ x_2 > 0 \).

Once again, combining both cases for \( q \), the inequality (14) follows.

By Remark 1, the extremal functions in (13) are: \( k(z, 1) \) and a function \( \alpha_0 k(z, -1) + (1 - \alpha_0)k(z, 1) \) with suitably taken \( \alpha_0 \in [0, 1] \). The equality in (14) holds for \( k(z, t_0) \) for some \( t_0 \in [-1, 1] \) if \( \mu < \frac{(\lambda + 2)(2\lambda + 3)}{6\lambda(2\lambda + 1)} \) and for \( k(z, 1) \) if \( \mu \geq \frac{(\lambda + 2)(2\lambda + 3)}{6\lambda(2\lambda + 1)} \).

From Theorems 3.1 and 3.2, taking \( \lambda = 1 \) and \( \lambda = 1/2 \), we conclude the results for the class \( T \) and for \( T(1/2) \).
Corollary 1. For any $f \in T$ we have
\[
-1, \quad \mu \leq 1 \rightleftharpoons a_3 - \mu a_2^2 \leq \begin{cases} 
3 - 4\mu, & \mu \leq 0 \\
3, & \mu \geq 0
\end{cases}
\]
and
\[
-\frac{1}{4(1-\mu)} - 1, \quad \mu \leq 5/6 \rightleftharpoons a_5 - \mu a_3^2 \leq \begin{cases} 
5 - 9\mu, & \mu \leq 1/6 \\
2 + \frac{5}{4\mu}, & \mu \geq 1/6
\end{cases}
\]

Corollary 2. For any $f \in T(1/2)$ we have
\[
-1/2, \quad \mu \leq 3/2 \rightleftharpoons a_3 - \mu a_2^2 \leq \begin{cases} 
1 - \mu, & \mu \leq 0 \\
1, & \mu \geq 0
\end{cases}
\]
and
\[
-\frac{1}{18} \left( \frac{25}{35-18\mu} + 7 \right), \quad \mu \leq 5/3 \rightleftharpoons a_5 - \mu a_3^2 \leq \begin{cases} 
\frac{1}{7} + \frac{25}{576\mu}, & \mu \geq 5/24
\end{cases}
\]

4. Bounds of $|a_4 - a_2a_3|$

Let us denote by $\Omega_{n}(T(\lambda))$, $n \geq 1$ the region of variability of three succeeding coefficients of functions in $T(\lambda)$, i.e., the set $\{(a_n(f), a_{n+1}(f), a_{n+2}(f)) : f \in T(\lambda)\}$. Therefore, $\Omega_{n}(T(\lambda))$ is the closed convex hull of the curve
\[
\gamma_n : [-1, 1] \ni t \mapsto \left( C_{n-1}^{(\lambda)}(t), C_n^{(\lambda)}(t), C_{n+1}^{(\lambda)}(t) \right).
\]

Let $X$ be a compact Hausdorff space and $J_{\mu} = \int_X J(t)d\mu(t)$. Szapiel in [13] (Thm.1.40) proved the following theorem.

Theorem 4.1 ([13]). Let $J : [\alpha, \beta] \to \mathbb{R}^n$ be continuous. Suppose that there exists a positive integer $k$, such that for each non-zero $p$ in $\mathbb{R}^n$ the number of solutions of any equation $\langle J(t), \vec{p} \rangle = \text{const}$, $\alpha \leq t \leq \beta$ is not greater than $k$. Then, for every $\mu \in P_{[\alpha, \beta]}$ such that $J_{\mu}$ belongs to the boundary of the convex hull of $J([\alpha, \beta])$ the following statements are true:

1. if $k = 2m$, then
   \[(a) \quad |\text{supp}(\mu)| \leq m \text{ or } (b) \quad |\text{supp}(\mu)| = m + 1 \text{ and } \{\alpha, \beta\} \subset \text{supp}(\mu),\]

2. if $k = 2m + 1$, then
   \[(a) \quad |\text{supp}(\mu)| \leq m \text{ or } (b) \quad |\text{supp}(\mu)| = m + 1 \text{ and one of the points } \alpha, \beta \text{ belongs to } \text{supp}(\mu).\]

In the above, the symbol $\langle \vec{u}, \vec{v} \rangle$ means the scalar product of vectors $\vec{u}$ and $\vec{v}$, whereas the symbols $P_X$ and $|\text{supp}(\mu)|$ describe the set of probability measures on $X$ and the cardinality of the support of $\mu$, respectively.

According to Theorem 4.1, the boundary of the convex hull of $\gamma_2([-1, 1])$ is determined by atomic measures $\mu$ for which support consists of at most 2 points, where one of them is $-1$ or $1$. In this way we have the following lemma.
Lemma 4.1. The boundary of $\Omega_2(T(\lambda))$ consists of points $(a_2, a_3, a_4)$ that correspond to the following functions:

$$f(z) = \alpha \frac{z}{(1 - 2tz + z^2)^\lambda} + (1 - \alpha) \frac{z}{(1 - z)^{2\lambda}}, \quad \alpha \in [0, 1], \quad t \in [-1, 1]$$ \hspace{1cm} (15)

or

$$f(z) = \alpha \frac{z}{(1 - 2tz + z^2)^\lambda} + (1 - \alpha) \frac{z}{(1 + z)^{2\lambda}}, \quad \alpha \in [0, 1], \quad t \in [-1, 1].$$ \hspace{1cm} (16)

Now, we can establish the main theorem of this section.

Theorem 4.2. If $f \in T(\lambda)$ and $\lambda \geq 1$, then $|a_4 - a_2a_3| \leq \frac{2}{3} \lambda (4\lambda^2 - 1)$. The result is sharp.

Proof. The function (15) has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} \left[ \alpha C_{k-1}^{(\lambda)}(t) + (1 - \alpha)C_{k-1}^{(\lambda)}(1) \right] z^k.$$

Hence, $a_4 - a_2a_3 = h_1(\alpha, t)$, where

$$h_1(\alpha, t) = [\alpha C_3(t) + (1 - \alpha)C_3(1)] - [\alpha C_1(t) + (1 - \alpha)C_1(1)] \cdot [\alpha C_2(t) + (1 - \alpha)C_2(1)].$$ \hspace{1cm} (18)

From (5) we obtain

$$h_1(\alpha, t) = \frac{2}{3} \lambda (\lambda + 1)[\alpha(2\lambda(t^3 - 1) + (t - 1)(2t + 1)^2) + 2\lambda + 1] - 2\lambda^2(\alpha(t - 1) + 1)(2\alpha(\lambda + 1)(t^2 - 1) + 2\lambda + 1).$$ \hspace{1cm} (19)

Similarly, for the function given by (16), $a_4 - a_2a_3 = h_2(\alpha, t)$. Moreover, it is easy to conclude from (18) that $h_2(\alpha, t) = -h_1(\alpha, -t)$. Taking into account the symmetry of the range of variability of $t$, we obtain the same estimates of $|a_4 - a_2a_3|$ for the functions given by (15) and (16).

For the function (15), we can write

$$h_1(\alpha, t) = 4\lambda^2(\lambda + 1)\alpha(1 - \alpha)(1 - t)(1 - t^2) + \frac{8}{3} \lambda(\lambda^2 - 1)(1 - t^3)\alpha$$

$$+ 2\lambda(1 - t)\alpha + \frac{2}{3} \lambda(1 - 4\lambda^2).$$

First three expressions are positive, so $h_1(\alpha, t) \geq \frac{2}{3} \lambda(1 - 4\lambda^2)$. In this case the equality holds for $t = 1$ or $\alpha = 0$; consequently, the extremal function is $f(z) = \frac{z}{(1 - z)^{2\lambda}}$. Therefore, the estimate is sharp.

Now, we will prove that

$$h_1(\alpha, t) \leq \frac{2}{3} \lambda(4\lambda^2 - 1).$$ \hspace{1cm} (20)

Inequality (20) is equivalent to $w(\lambda) \leq 0$, where

$$w(\lambda) = B_3 \lambda^3 + B_2 \lambda^2 + B_1 \lambda.$$
and
\[
B_1 = -2\alpha t(1 - t^2) + \frac{2}{3}[2 - \alpha(1 - t^3)] ,
\]
\[
B_2 = 4\alpha(1 - \alpha)(1 - t)(1 - t^2) ,
\]
\[
B_3 = -4\alpha(1 - \alpha)(1 + t)t + \frac{4}{3}\alpha(1 - 3\alpha)t^3 - 4(1 - \alpha)^2 - \frac{4}{3}(1 + \alpha) .
\]

Obviously, \(B_2\) is non-negative. Since \(B_1\) can be rewritten as \(B_1 = \frac{4}{3} - \frac{2}{3}\alpha(1 - t)(1 + 2t)^2\), the minimum of \(B_1\) is attained for \(\alpha = 1\), so \(B_1 \geq \frac{2}{3}(1 + t)(1 - 2t)^2 \geq 0\).

We shall prove that \(B_3 \leq 0\). Indeed,
\[
B_3 = -\frac{4}{3}\alpha(1 - \alpha)(1 + t)^3 + \frac{8}{3}b(t) ,
\]
where
\[
b(t) = \alpha(1 - 2\alpha)t^3 - (2 - 3\alpha + 2\alpha^2) .
\]

But, if \(\alpha \in \left[0, \frac{1}{2}\right]\), then \(b(t) \leq b(1) = -2 + 4\alpha(1 - \alpha) \leq -1\) and if \(b \in \left[\frac{1}{2}, 1\right]\), then \(b(t) \leq b(-1) = -2(1 - \alpha) \leq 0\). Moreover, \(-\frac{4}{3}\alpha(1 - \alpha)(1 + t)^3 \leq 0\) for all \(t \in [-1, 1]\), \(\alpha \in [0, 1]\). It means that \(B_3 \leq 0\). The inequality \(B_3 \leq 0\) implies that in order to prove that \(w(\lambda) \leq 0\), it is enough to show \(w(1) \leq 0\).

Let us denote by \(P(\alpha, t)\), where \(\alpha \in [0, 1]\), \(t \in [-1, 1]\), the value \(w(1) = -8(1 - t)(1 - t^2)\alpha^2 + 2(1 - t)(5 - 4t^2)\alpha - 4\). We shall derive the greatest value of \(P(\alpha, t)\) for \(\alpha \in [0, 1]\), \(t \in [-1, 1]\).

Since \(\frac{\partial P}{\partial \alpha} = -16(1 - t)(1 - t^2)\alpha + 2(1 - t)(5 - 4t^2)\), as a function of a variable \(\alpha\), vanishes only for \(\alpha_0 = \frac{5 - 4t^2}{8(1 + t^2)}\), we conclude that \(P(\alpha, t) \leq \frac{(5 - 4t^2)^2}{8(1 + t^2)} - 4\), but only if \(\alpha_0 \in [0, 1]\). It occurs for \(t^2 \leq \frac{3}{4}\).

If we consider the function \(h(t) = \frac{(5 - 4t^2)^2}{8(1 + t^2)}\), we get max \(\{h(t) : t^2 \leq \frac{3}{4}\} = h(-\frac{1}{2}) = 4\). Hence, we obtain \(P(\alpha, t) \leq 0\) for \(t^2 \leq \frac{3}{4}\), \(\alpha \in [0, 1]\).

Now, we need to verify this inequality on the boundary of the set \([0, 1] \times [-1, 1]\). We have the following:
\[P(0, t) = -4,\]
\[P(1, t) = -2 - 2t \in [-4, 0],\]
\[P(\alpha, -1) = 4(\alpha - 1) \in [-4, 0],\]
\[P(\alpha, 1) = -4.\]
Considering both parts of this discussion we have \(P(\alpha, t) \leq 0\) in the whole set \([0, 1] \times [-1, 1]\), so \(w(1) \leq 0\) as we required. Therefore, the inequality (20) is also satisfied. Finally, we obtain \(|a_4 - a_2a_3| \leq \frac{2}{3}(4\lambda^2 - 1)\). □

From Theorem 4.2, taking \(\lambda = 1\), we conclude the result for the class \(T\).

**Corollary 3.** For any \(f \in T\), we have
\[|a_4 - a_2a_3| \leq 2 .\]
5. Bounds of \(|a_2a_4 - a_3^2|\)

**Theorem 5.1.** If \(f \in \mathcal{T}(\lambda)\) and \(\lambda \geq 1\), then \(|a_2a_4 - a_3^2| \leq \lambda^2(1 + 2\lambda)^2\). The result is sharp.

*Proof.* At the beginning, observe that for \((z, t)\) given by (3) we have

\[
a_2a_4 - a_3^2 = C_3^{(\lambda)}(t)(C_3^{(\lambda)}(t) - C_2^{(\lambda)}(t))^2 = -\frac{4}{3}\lambda^2(\lambda^2 - 1)t^4 - \lambda^2.
\]

(21)

The same argument as in the proof of Theorem 4.2 yields that only the functions given by (15) and (16) should be discussed in order to obtain the desired estimates.

Denote by \(g_1(\alpha, t)\) and \(g_2(\alpha, t)\) the second Hankel determinant for the functions (15) and (16), respectively. It is easy to see that \(g_1(\alpha, t) = g_2(\alpha, -t)\). For this reason we discuss only the first function.

From (15) it follows that

\[
g_1(\alpha, t) = \alpha^2[C_1(t)C_3(t) - [C_2(t)]^2] + (1 - \alpha)^2[C_1(1)C_3(1) - [C_2(1)]^2] + \alpha(1 - \alpha)[C_1(t)C_3(1) + C_1(1)C_3(t) - 2C_2(t)C_2(1)].
\]

Hence, (5) and (21) result in

\[
g_1(\alpha, t) = -\alpha^2\lambda^2\left[\frac{4}{3}(\lambda^2 - 1)t^4 + 1\right] - (1 - \alpha)^2\lambda^2\left[\frac{4}{3}(\lambda^2 - 1) + 1\right] + \alpha(1 - \alpha)w(t),
\]

(22)

where

\[
w(t) = \frac{8}{3}\lambda^2(\lambda + 1)(\lambda + 2)t^3 - 4\lambda^2(\lambda + 1)(2\lambda + 1)t^2 + \frac{8}{3}\lambda^2(\lambda^2 - 1)t + 2\lambda^2(2\lambda + 1).
\]

If \(\lambda \geq 1\), then \(w(t) \geq w(-1)\) and

\[
-\left[\frac{4}{3}(\lambda^2 - 1)t^4 + 1\right] \geq -\left[\frac{4}{3}(\lambda^2 - 1) + 1\right]
\]

with equality for \(t = -1\) or \(t = 1\). Thus, for \(\lambda \geq 1\), we have

\[
g_1(\alpha, t) \geq -\left[\frac{4}{3}(\lambda^2 - 1) + 1\right]\lambda^2(\alpha^2 + (1 - \alpha)^2) + \alpha(1 - \alpha)w(-1)
\]

\[
= \frac{1}{3}\lambda^2(2\lambda + 1)[1 - 2\lambda - 16\alpha(1 - \alpha)(1 + \lambda)]
\]

\[
\geq \frac{1}{3}\lambda^2(2\lambda + 1)[1 - 2\lambda - 4(1 + \lambda)] = -\lambda^2(2\lambda + 1)^2.
\]

Now, we shall show that \(g_1(\alpha, t) < \lambda^2(1 + 2\lambda)^2\). From (22) we obtain the rough inequality

\[
g_1(\alpha, t) < \alpha(1 - \alpha)\left[\frac{8}{3}\lambda^2(\lambda + 1)(\lambda + 2) + \frac{8}{3}\lambda^2(\lambda^2 - 1) + 2\lambda^2(2\lambda + 1)\right].
\]

Hence

\[
g_1(\alpha, t) < \frac{2}{3}\lambda^2(\lambda + 1)(\lambda + 2) + \frac{2}{3}\lambda^2(\lambda^2 - 1) + \frac{1}{2}\lambda^2(2\lambda + 1)
\]

\[
= \lambda^2(2\lambda + 1)\left(\frac{2}{3}\lambda + \frac{7}{6}\right),
\]

which is clearly less than \(\lambda^2(2\lambda + 1)^2\) for \(\lambda \geq 1\).
Therefore, for $\lambda \geq 1$, we have $|a_2a_4 - a_3^2| \leq \lambda^2(1 + 2\lambda)^2$. The equality holds if $t = 1$ and $\alpha = \frac{1}{2}$, so we obtain the extremal function $f(z) = \frac{1}{2} \left[ \frac{2}{(1+z)^{2\alpha}} + \frac{2}{(1-z)^{2\alpha}} \right]$. Thus, the estimate is sharp. □

From Theorem 5.1, taking $\lambda = 1$, we obtain the result for the class $T$ presented in [17].

**Corollary 4.** For any $f \in T$ we have

$$|a_2a_4 - a_3^2| \leq 9.$$ 

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Pawel Zaprawa
Department of Mathematics
Faculty of Mechanical Engineering
Lublin University of Technology
Nadbystrzycka 38D
20-618 Lublin
Poland
e-mail: p.zaprawa@pollub.pl

Magdalena Figiel and Anna Futa
Institute of Mathematics
Maria Curie-Skodowska University
Pl. Marii Curie-Skodowskiej 5
20-031 Lublin
Poland
e-mail: magdalena_figiel@wp.pl

Anna Futa
e-mail: anna-futa@wp.pl

Received: July 14, 2016.
Revised: January 16, 2017.
Accepted: January 25, 2017.