Ricci solitons on Lorentzian manifolds with large isometry groups

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Abstract

We show that Lorentzian manifolds whose isometry group is of dimension at least \( \frac{1}{2} n(n - 1) + 1 \) admit different vector fields resulting in expanding, steady and shrinking Ricci solitons. Moreover, it is proved that those Ricci solitons are gradient (only) in the steady case. This provides examples of complete locally conformally flat and symmetric Lorentzian gradient Ricci solitons that are not rigid.

1. Introduction

A Ricci soliton is a pseudo-Riemannian manifold \((M, g)\) that admits a smooth vector field \(X\) on \(M\) such that

\[ \mathcal{L}_X g + \text{Ric} = \lambda g, \]

where \(\mathcal{L}_X\) denotes the Lie derivative in the direction of \(X\), \(\text{Ric}\) is the Ricci tensor and \(\lambda\) is a real number \((\lambda = (1/n)(2 \text{div} X + \text{Sc})\), where \(n = \text{dim} M\) and \(\text{Sc}\) denotes the scalar curvature of \((M, g)\). A Ricci soliton is said to be shrinking, steady or expanding, if \(\lambda > 0\), \(\lambda = 0\) or \(\lambda < 0\), respectively. Moreover, we say that a Ricci soliton \((M, g)\) is a gradient Ricci soliton if it admits a vector field \(X\) satisfying \(X = \text{grad} h\), for some potential function \(h\). In such a case equation (1) can be written in terms of \(h\) as

\[ 2 \text{Hess}_h + \text{Ric} = \lambda g, \]

where \(\text{Hess}_h\) denotes the Hessian of \(h\). A gradient Ricci soliton is rigid if it is isometric to a quotient of \(N \times \mathbb{R}^k\), where \(N\) is an Einstein manifold and the potential function is a generalization of the Gaussian soliton (that is, \(h = (\lambda/2)\|x\|^2\) on the Euclidean factor). Rigid solitons have been systematically studied \[15\] and further investigated in \[8, 10\].

Although Ricci solitons exist on many Lie groups and homogeneous spaces, all homogeneous gradient Ricci solitons are rigid \[16\] in the Riemannian setting. This is based on the existence of splitting results originated by Killing vector fields on gradient Ricci solitons. Indeed, Petersen and Wylie showed that, for any Killing vector field \(Z\) on a gradient Ricci soliton with potential function \(h\), one has that either \(Z(h) = 0\) or the metric splits off a Euclidean factor. This splitting is not guaranteed in the pseudo-Riemannian setting due to the fact that \(\text{grad} \nabla_Z h\) may be a null vector, in which case \((M, g)\) becomes a Walker manifold (see \[3\] for information on Walker geometry).

For any gradient Ricci soliton with potential function \(h\) the following equation is satisfied (see, for instance, \[9\]):

\[ \text{Sc} + 2\|\text{grad} h\|^2 - 2\lambda h = C, \]

where \(C\) is a constant. Hence, if the scalar curvature is constant, then after rescaling the potential function, one may assume \(\|\text{grad} h\|^2 = \lambda h\), which shows that \(\text{grad} h\) is a null vector.
field on \((M, g)\) if and only if the soliton is steady. Further, in this case \(\text{Ric}(\text{grad} \, h) = 0\) (since \(\text{Ric}(\text{grad} \, h, \cdot) = d\text{Sc}(\cdot)\) holds true for any gradient Ricci soliton) and then one has that 
\[
g(\nabla_{\text{grad} \, h} \text{grad} \, h, Y) = 0 \quad \text{for all vector fields } Y,
\]
thus showing that \(\text{grad} \, h\) is a geodesic vector field. On the other hand, note that the gradient of the potential function is a recurrent vector field (that is, the plane field \(\text{span}\{\text{grad} \, h\}\) is parallel) if and only if \(\nabla_Y \text{grad} \, h = \omega(Y) \text{grad} \, h\) for some 1-form \(\omega\). This means that the \((1, 1)\)-Hessian tensor \(h_{\omega}(Y) = \nabla_Y \text{grad} \, h = \omega(Y) \text{grad} \, h\) for all vector fields \(Y\) and thus the constancy of the scalar curvature gives \(\text{Ric}(\text{grad} \, h) = -h_{\omega}(\text{grad} \, h) = 0\), showing that if the gradient of the potential function is recurrent, then the Ricci operator and the Hessian tensor are 2-step nilpotent.

The situation described above occurs, for instance, when studying pseudo-Riemannian manifolds with large isometry groups. Riemannian manifolds admitting a group of isometries of dimension at least \(\frac{1}{2} n(n - 1) + 1\) are either of constant curvature or products of an \((n - 1)\)-dimensional space of constant curvature with a line or a circle. Lorentzian metrics allow other solutions that are related to the existence of null submanifolds on \(M\) and thus non-homogeneous examples exist.

The complete classification of Lorentzian manifolds with an isometry group of this size was given for any dimension \(n \geq 4\), \(n \neq 7\), by Patrangenaru \([14]\). Besides the spaces of constant curvature \(M^n(c)\) and manifolds reducible as products \(N^{n-1}(c) \times \mathbb{R}\), the following are the examples that they all are expanding, steady and shrinking Ricci solitons, but only steady Ricci solitons may be gradient ones. The \(\varepsilon\)-spaces are then obtained as the locally conformally flat Cahen–Wallach Lorentzian symmetric spaces.

(i) **Egorov spaces**: Lorentzian manifolds \((\mathbb{R}^{n+2}, g_f)\), where \(f\) is a positive function of a real variable

\[
g_f(u, v, x_1, \ldots, x_n) = du \, dv + f(u) \sum_{i=1}^{n} (dx_i)^2. \tag{3}
\]

(ii) **\(\varepsilon\)-spaces**: Lorentzian manifolds \((\mathbb{R}^{n+2}, g_\varepsilon)\), where

\[
g_\varepsilon(u, v, x_1, \ldots, x_n) = \varepsilon \sum_{i=1}^{n} x_i^2(du)^2 + du \, dv + \sum_{i=1}^{n} (dx_i)^2. \tag{4}
\]

The geometry of Egorov spaces and \(\varepsilon\)-spaces has been investigated in the literature (see, for example, \([1, 4, 6, 7, 13]\) and references therein), where it is shown that both are Walker manifolds. It is worth emphasizing here that although \(\varepsilon\)-spaces are locally symmetric, Egorov spaces are not homogeneous in general. However, both families are locally conformally flat and have 2-step nilpotent Ricci operator.

Recall that spaces of constant curvature are Einstein and thus trivial Ricci solitons and, moreover, products \(N^{n-1}(c) \times \mathbb{R}\) are rigid Ricci solitons. The first purpose of this note is to show that both Egorov spaces and \(\varepsilon\)-spaces are (expanding, steady and shrinking) Ricci solitons. Secondly, we show that both families are gradient Ricci solitons that are necessarily steady, and therefore non-rigid. This leads to new examples of Lorentzian locally conformally flat Ricci solitons without Riemannian analogue (see \([17]\) for the Riemannian case).

This work is structured as follows. Egorov spaces are studied in Section 2. In Section 3, we consider the Cahen–Wallach Lorentzian symmetric spaces that describe the indecomposable but not irreducible Lorentzian symmetric spaces \([5, 4]\), and that generalize \(\varepsilon\)-spaces. We show that they all are expanding, steady and shrinking Ricci solitons, but only steady Ricci solitons may be gradient ones. The \(\varepsilon\)-spaces are then obtained as the locally conformally flat Cahen–Wallach Lorentzian symmetric spaces.
2. Egorov spaces

Let \((\mathbb{R}^{n+2}, g_f)\), \(n \geq 1\) be an Egorov space. As proved in [1], with respect to the basis of coordinate vector fields \(\{ \partial_u = \partial/\partial u, \partial_v = \partial/\partial v, \partial_i = \partial/\partial x_i \}\), with \(i = 1, \ldots, n\), for which \(g_f\) adopts expression (3), the non-vanishing covariant derivatives of coordinate vector fields are given by

\[
\nabla_{\partial_i} \partial_i = -\frac{f'}{2} \partial_v, \quad \nabla_{\partial_i} \partial_u = \frac{f'}{2f} \partial_i, \quad i = 1, \ldots, n.
\]

Hence, observe that \(\partial_v\) is a parallel null vector field and thus that Egorov spaces are Walker metrics [6, 3]. By an explicit calculation on the geodesic equations, it has been shown in [1] that Egorov spaces are geodesically complete. The curvature tensor \(R\), which is given by

\[
R(X,Y) = \nabla_X \nabla_Y - \nabla_{\nabla_X Y} - [X,Y],
\]

is determined by

\[
R_{vui} = \frac{1}{4f} [(f')^2 - 2ff''], \quad R_{iuu} = -\frac{1}{4f^2} [(f')^2 - 2ff''], \quad i = 1, \ldots, n.
\]

The Ricci tensor \(\text{Ric}(X,Y) = \text{trace}\{Z \rightarrow R(X,Z)Y\}\) satisfies

\[
\text{Ric}_{uu} = \frac{n}{4f^2} [(f')^2 - 2ff''],
\]

being zero otherwise. This shows that the Ricci operator is 2-step nilpotent.

**Remark 1.** In contrast with \(\varepsilon\)-spaces, Egorov spaces are not homogeneous in general. However, the Ricci tensor is recurrent and so is the curvature tensor since they are locally conformally flat (see [1, 6]).

2.1. Soliton equations

Next we will show that all Egorov metrics are Ricci solitons. Let \(X = \sum_{i=u,v,1,\ldots,n} X_i \partial_i\) be an arbitrary vector field on \((\mathbb{R}^{n+2}, g_f)\). Then (1) becomes

\[
\begin{align*}
\partial_i X_j + \partial_j X_i &= 0, & 1 \leq i \neq j \leq n, \\
\partial_i X_u + f \partial_u X_i &= 0, & 1 \leq i \leq n, \\
\partial_i X_v + f \partial_v X_i &= 0, & 1 \leq i \leq n, \\
X_u f' + 2f \partial_i X_i &= \lambda f, & 1 \leq i \leq n, \\
\text{Ric}_{uu} + 2 \partial_u X_v &= 0, \\
\partial_u X_u + \partial_v X_v &= \lambda, \\
\partial_v X_u &= 0.
\end{align*}
\]

Now it follows by a straightforward calculation that the metric \(g_f\) is a Ricci soliton since the vector field

\[
X = \left( -\frac{1}{2} \int \text{Ric}_{uu} du + \lambda v \right) \partial_v + \sum_{i=1}^n \frac{\lambda}{2} x_i \partial_i
\]

satisfies (8). Note that \(\lambda\) is the constant of equation (1) and can be chosen with absolute freedom. Therefore, we obtain the following.

**Theorem 2.** All Egorov spaces \((\mathbb{R}^{n+2}, g_f)\) are expanding, steady and shrinking Ricci solitons.
Remark 3. By a standard process of integration, one finds that the general solution of (8) is

\[ X_u = a + bu, \]
\[ X_v = c_0 + (\lambda - b)v - \frac{1}{2} \int \text{Ric}_{u\nu} du + \sum k_i x_i + \left( \sum_i x_i^2 \right) K, \]
\[ X_i = c_i - \int \frac{k_i}{f} du + \left( \frac{\lambda}{2} - (a + bu) \frac{f'(u)}{2f(u)} \right) x_i + \sum_{j \neq i} A_{ij} x_j, \]

where \( a, b \) and \( K \) are constants that satisfy the equation

\[ bf'(u) + (a + bu) \left( f''(u) - \frac{f'(u)^2}{f(u)} \right) = 4K, \]

\((A_{ij})\) is an arbitrary skew-symmetric matrix; \( c_0, c_i \) and \( k_i \) are arbitrary constants for \( i = 1, \ldots, n \).

Remark 4. The general solution of (8) obtained in Remark 3 can be determined from the particular solution (9) using the following observation. Any two vector fields \( \partial X \) satisfy equations \( (8) \) satisfying (1) \( (L_{X_1} g + \text{Ric} = \lambda_1 g, i = 1, 2) \) differ in a conformal vector field with constant divergence (that is, a homothetic vector field) since

\[ L_{X_1 - X_2} g - (\lambda_1 - \lambda_2) g = L_{X_1} g - \lambda_1 g - L_{X_2} g + \lambda_2 g = 0. \]

Conversely, for any Ricci soliton \( X_1 \), adding a homothetic vector field gives another Ricci soliton. A special case of the above occurs if the manifold is compact, where an immediate application of the divergence theorem shows that any two Ricci solitons differ in a Killing vector field.

2.2. Gradient Ricci solitons

Now, let \( X = \text{grad} h \) be an arbitrary gradient vector field on \((\mathbb{R}^{n+2}, g_f)\) with potential function \( h \) (which is given by \( \text{grad} h = (\partial_u h, \partial_u h, (1/f)\partial_1 h, \ldots, (1/f)\partial_n h) \) in the coordinates (3)). By standard calculations we get from (8) that \((\mathbb{R}^{n+2}, g_f)\) is a gradient Ricci soliton if and only if the following holds:

\[
\begin{align*}
\left\{ 
\begin{array}{ll}
f' \partial_i h + 2\partial_{ii}^2 h = \lambda f, & 1 \leq i \leq n, \\
2\partial_{ii}^2 h - \frac{f'}{f} \partial_i h = 0, & 1 \leq i \leq n, \\
2\partial_{uu}^2 h + \text{Ric}_{uu} = 0, & \\
\partial_{uu}^2 h = \frac{\lambda}{2}, & \\
\partial_{ij}^2 h = \partial_{ii}^2 h = \partial_{jj}^2 h = 0, & 1 \leq i \neq j \leq n.
\end{array}
\right.
\tag{10}
\end{align*}
\]

Integrating the equations \( \partial_{ii}^2 h = \partial_{ii}^2 h = 0 \) in (10), we obtain that the potential function splits as \( h(u, v, x_1, \ldots, x_n) = h_0(u, x_1, \ldots, x_n) + h_1(u)v \) for some functions \( h_0, h_1 \). Moreover, equations \( \partial_{yy}^2 h = \lambda/2 \) and \( \partial_{ij}^2 h = 0 \) now show that \( h(u, v, x_1, \ldots, x_n) = h_0(u) + \sum_i h_i(u, x_i) + ((\lambda/2)u + \kappa)v \) for some constant \( \kappa \) and functions \( h_i, i = 1, \ldots, n \). Hence, (10) reduces to

\[
\begin{align*}
\left\{ 
\begin{array}{ll}
\left( \kappa + \frac{\lambda}{2} \right) f' + 2\partial_{ii}^2 h_i = \lambda f, & 1 \leq i \leq n, \\
2f \partial_{ii}^2 h_i = f' \partial_i h_i, & 1 \leq i \leq n, \\
\text{Ric}_{uu} + 2h_0'' + 2 \sum_i \partial_{ii}^2 h_i = 0.
\end{array}
\right.
\tag{11}
\end{align*}
\]
Integrating the first equations in (11), one gets
\[ h_i(u, x_i) = \frac{1}{2} x_i^2 (2\lambda f - (2\kappa + u\lambda)f') + x_i h_i(u) + k_i(u), \]
for some functions \( h_i(u) \) and \( k_i(u) \). Substituting the above into (11) and deriving the second equation \( 2f h''_i(u) = f' h'' h_i(u) \), one gets
\[ \left( \frac{\lambda}{2} u + \kappa \right) ((f')^2 - 2ff'') = 0, \]
which shows that \( \lambda = \kappa = 0 \) unless the manifold is flat. Hence, the potential function becomes \( h(u, v, x_1, \ldots, x_n) = h_0(u) + \sum x_i h_i(u) \). Now the second equation in (11) reduces to
\[ 2fh'_i = f' h_i, \]
and hence \( h_i(u) = c_i \sqrt{f(u)} \) for some constants \( c_i \). Now, deriving the last equation in (11) with respect to \( x_i \), one gets \( 0 = c_i((f')^2 - 2ff'') \) and thus all \( c_i \) must vanish. Therefore, the potential function reduces to a function of the single variable \( u \), \( h(u, v, x_1, \ldots, x_n) = h_0(u) \), and hence the only remaining constraint in (11) is \( R_{uu} + 2h_0'' = 0 \). Therefore, we have shown the following corollary.

**Corollary 5.** A non-flat Egorov space is a gradient steady Ricci soliton where the potential function \( h(u) \) is given by \( h'' = -\frac{1}{2} R_{uu} \).

**Remark 6.** Note that for any function \( h(u) \), \( \text{grad} h = h'(u) \partial_u \) is a null vector field and moreover \( \nabla_{\partial_u} \text{grad} h = h''(u) \partial_u \) (the other derivatives being zero). Therefore, \( \text{grad} h \) is a null geodesic vector field. Further observe that \( \text{grad} h \) is a recurrent vector field in the direction of the parallel null vector \( \partial_u \).

### 3. Cahen–Wallach symmetric spaces

Recall that the notion of irreducibility (the holonomy group does not stabilize any non-trivial subspace) is very strong in the pseudo-Riemannian setting. Indeed, irreducible Lorentzian symmetric spaces are necessarily of constant sectional curvature [4]. A pseudo-Riemannian manifold is said to be **indecomposable** if the holonomy group acting at each point \( p \in M \) stabilizes only non-trivial degenerate subspaces \( V \subset T_p M \) (that is, the restriction of the metric to \( V \times V \) is degenerate).

Clearly, any irreducible Lorentzian symmetric space is a trivial (Einstein) Ricci soliton. Our purpose in this section is to show that indecomposable (not irreducible) Lorentzian symmetric spaces are non-trivial Ricci solitons. Indecomposable Lorentzian symmetric spaces are either irreducible or the so-called Cahen–Wallach symmetric spaces which are given as follows [5, 4]. Take \( M = \mathbb{R}^{n+2} \) and define a metric tensor by
\[
g_{cu}(u, v, x_1, \ldots, x_n) = \left( \sum_{i=1}^{n} \kappa_i x_i^2 \right) (du)^2 + dv du + \sum_{i=1}^{n} (dx_i)^2, \tag{12} \]
where \( \kappa_i, i = 1, \ldots, n \), are non-zero real numbers. The Levi-Civita connection is determined by the non-zero Christoffel symbols:

\[
\nabla_{\partial_u} \partial_u = -\sum_i \kappa_i x_i \partial_1, \quad \nabla_{\partial_u} \partial_i = \nabla_{\partial_1} \partial_u = \kappa_i x_i \partial_v.
\]

Hence, the only non-zero components of the \((0, 4)\) curvature tensor are given by
\[
R(\partial_u, \partial_i, \partial_a, \partial_t) = -\kappa_i, \quad i = 1, \ldots, n.
\]
The Ricci tensor satisfies $\text{Ric}_{uu} = -\sum_{i=1}^{n} \kappa_i$, the other terms being zero. Hence, the Ricci operator is 2-step nilpotent (and thus the scalar curvature vanishes).

Cahen–Wallach symmetric spaces are locally conformally flat if and only if $\kappa_1 = \ldots = \kappa_n$, in which case the resulting manifold is an $\varepsilon$-space (we refer to [5, 4] for more information on the geometry of Cahen–Wallach spaces and to [1, 6] for $\varepsilon$-spaces).

3.1. Soliton equations

Let $X = X_u \partial_u + X_v \partial_v + \sum_i X_i \partial_i$ be an arbitrary vector field on $\mathbb{R}^{n+2}$. The Lie derivative of the metric $g$ is given by

$$\mathcal{L}_X g = \begin{pmatrix} 2 \left( \sum_i \kappa_i x_i \right) & \left( \sum_i \kappa_i x_i^2 \right) \\ \frac{\partial}{\partial u} X_u & \frac{\partial}{\partial v} X_u \\ \frac{\partial}{\partial u} X_v & \frac{\partial}{\partial v} X_v \\ \frac{\partial}{\partial u} X_j & \frac{\partial}{\partial v} X_j \\ \frac{\partial}{\partial u} X_k & \frac{\partial}{\partial v} X_k \end{pmatrix},$$

where

$$A_{uv} = \frac{\partial}{\partial u} X_u + \frac{\partial}{\partial v} X_v + \left( \sum_i \kappa_i x_i^2 \right) \frac{\partial}{\partial v} X_u,$$

$$A_{uj} = \left( \sum_i \kappa_i x_i^2 \right) \frac{\partial}{\partial u} X_u + \frac{\partial}{\partial v} X_v + \frac{\partial}{\partial u} X_j$$

and

$$A_{jk} = \frac{\partial}{\partial j} X_k + \frac{\partial}{\partial k} X_j.$$

Thus, the Ricci soliton equation (1) is equivalent to (note that $\lambda = (2/(n+2)) \text{div}(X)$ and $\text{div}(X) = \frac{\partial}{\partial u} X_u + \frac{\partial}{\partial v} X_v + \sum_i \partial_i X_i$)

$$\begin{cases} \sum_i \kappa_i - 2 \sum_i \kappa_i x_i - 2 \frac{\partial}{\partial u} X_v - \left( \sum_i \kappa_i x_i^2 \right) \left( 2 \frac{\partial}{\partial u} X_u - \lambda \right) = 0, \\ \frac{\partial}{\partial v} X_u = 0, \\ \left( \sum_i \kappa_i x_i^2 \right) \frac{\partial}{\partial u} X_u + \left( \frac{\partial}{\partial v} X_u + \frac{\partial}{\partial v} X_v \right) - \lambda = 0, \\ \left( \sum_i \kappa_i x_i^2 \right) \frac{\partial}{\partial u} X_v + \frac{\partial}{\partial v} X_j = 0, \\ \frac{\partial}{\partial j} X_u + \frac{\partial}{\partial v} X_j - 2 \frac{\partial}{\partial j} X_j - \lambda = 0, \\ \frac{\partial}{\partial j} X_k + \frac{\partial}{\partial k} X_j = 0, \end{cases}$$

(13)

where $j, k = 1, \ldots, n$ and $j \neq k$.

First of all, observe that a particular solution of the previous system is

$$X_{(u,v,x_1,\ldots,x_n)} = \left( 0, \frac{1}{2} \left( \sum_i \kappa_i \right) u + \lambda v, \frac{\lambda}{2} x_1, \ldots, \frac{\lambda}{2} x_n \right),$$

(14)

where $\lambda$ is the constant of equation (1) and can be chosen with absolute freedom. The choice of this particular family of spacelike vector fields shows that $(\mathbb{R}^{n+2}, g_{cw})$ is an expanding, steady or shrinking Ricci soliton, and therefore proves the following.

**Theorem 7.** Indecomposable Lorentzian symmetric spaces are expanding, steady and shrinking Ricci solitons.
Remark 8. Next we consider the general solution of (13) assuming that $\kappa_i \neq \kappa_j$ for some $i, j \in \{1, \ldots, n\}$ (the case $\kappa_1 = \ldots = \kappa_n$ will be considered in Section 3.2). After a standard integration process, one gets that all Ricci solitons are given by vector fields $X = (X_u, X_v, X_1, \ldots, X_n)$ of the form

$$X_u = a,$$
$$X_v = b + \lambda v + \frac{1}{2} \left( \sum_i \kappa_i \right) x_i^2 - \sum_i x_i h'_i(u),$$
$$X_j = \frac{\lambda}{2} x_j + h_j(u) + \sum_{i \neq j} c_{ij} x_i, \quad j = 1, \ldots, n,$$

where $a$ and $b$ are arbitrary constants, $c_{ij}$ is a skew-symmetric matrix of constants satisfying $c_{ij}(\kappa_i - \kappa_j) = 0$ and $h_i$ are functions that satisfy $\kappa_i h_i - h''_i = 0$ (hence note that if $\kappa_i > 0$, then $h_i(u) = d_i e^{-u\sqrt{\kappa_i}} + d_i' e^{u\sqrt{\kappa_i}}$ while if $\kappa_i < 0$, then $h_i(u) = d_i e^{u\sqrt{-\kappa_i}} + d_i' e^{-u\sqrt{-\kappa_i}}$ for $d_i, d_i' \in \mathbb{R}$).

A straightforward calculation shows that the causal character of $X$ depends on the point and the value of $a, b, c_{ij}, h_j, \kappa_i$ and $\lambda$ ($i, j = 1, \ldots, n$).

3.2. Gradient Ricci solitons

Let $h$ be a function on $\mathbb{R}^{n+2}$. Then the gradient with respect to the metric (12) is given by

$$\text{grad}(h) = \left( \partial_v h, -\left( \sum_i \kappa_i x_i^2 \right) \partial_u h + \partial_u h, \partial_1 h, \ldots, \partial_n h \right),$$

and thus (13) becomes (where $j, k = 1, \ldots, n$, $j \neq k$)

$$\begin{align*}
\sum_i \kappa_i &- 2 \sum_i \kappa_i x_i \partial_i h - 2 \partial_{uv}^2 h + \left( \sum_i \kappa_i x_i^2 \right) \lambda = 0, \\
\partial_{v}^2 h & = 0, \\
2 \partial_{uv}^2 h &- \lambda = 0, \\
\kappa_j x_j \partial_v h &- \partial_{uj}^2 h = 0, \\
\partial_{v}^2 h & = 0, \\
2 \partial_{jj}^2 h &- \lambda = 0, \\
\partial_{jj}^2 h & = 0.
\end{align*} \hspace{1cm} (15)$$

Using that $\partial_{v}^2 h = \partial_{uj}^2 h = \partial_{jk}^2 h = 0, (j \neq k)$, the function $h$ separates variables $v, x_1, \ldots, x_n$ and the previous system reduces to

$$\begin{align*}
\sum_i \kappa_i &- 2 \sum_i \kappa_i x_i \partial_i h - 2 \partial_{uv}^2 h + \left( \sum_i \kappa_i x_i^2 \right) \lambda = 0, \\
2 \partial_{uv}^2 h &- \lambda = 0, \\
\kappa_j x_j \partial_v h &- \partial_{uj}^2 h = 0, \\
2 \partial_{jj}^2 h &- \lambda = 0,
\end{align*} \hspace{1cm} (16)$$

where $j = 1, \ldots, n$.

Differentiate the third equation with respect to $u$ to obtain that $\kappa_j x_j \partial_{uv}^2 h - \partial_{uuj}^3 h = 0$ and, by the second equation, we have

$$2 \partial_{uuj}^3 h = \kappa_j x_j \lambda. \hspace{1cm} (17)$$
Now differentiate the first equation with respect to \( x_j \) to obtain
\[-2\kappa_j \partial_j h - 2\kappa_j x_j \partial^2_{jj} h + 2\kappa_j x_j \lambda - 2\partial^3_{uuu_j} h = 0.\]

Use equations (16) and (17) to simplify the previous identity and get that
\[\kappa_j \partial_j h = 0.\]

This shows that \( h \) does not depend on the variables \( x_j \) (\( j = 1, \ldots, n \)). Simplify now the third equation of (16) to conclude that \( \partial_v h = 0 \), so \( h \) is only a function of the variable \( u \), \( h = h(u) \), and (15) reduces to
\[\sum_i \kappa_i - 2h''(u) = 0.\]

We integrate this equation to obtain the following function as the general solution of the system of equations (15):
\[h(u) = \alpha + \beta u + \frac{1}{4} \sum_i \kappa_i u^2.\]

Note that \( \text{grad}(h) \) is a null vector field that satisfies equation (1) for \( \lambda = 0 \); indeed \( \lambda = 2 \text{div}(\text{grad}(h))/(n + 2) = \Delta h = 0 \). Therefore, we have proved the following theorem.

**Theorem 9.** Indecomposable Lorentzian symmetric spaces are steady gradient Ricci solitons.

**Remark 10.** Finally, observe that, proceeding as in Remark 6, \( X = \text{grad}(h) = (\beta + \frac{1}{2} u \sum_i \kappa_i) \partial_u \) is a null geodesic vector field that is recurrent. Hence, the line field \( \mathcal{L} = \text{span}\{\text{grad}(h)\} \) is a parallel degenerate one-dimensional plane field, which agrees with the Walker structure of Cahen–Wallach spaces.

**Remark 11.** Riemannian complete shrinking Ricci solitons with bounded curvature can be made gradient by adding a Killing vector field [11]. Theorems 7 and 9 (for a suitable choice of \( f \) which makes the curvature bounded) show that this result does not hold in the Lorentzian setting due to the existence of complete shrinking or expanding Ricci solitons which cannot be made gradient by adding any Killing vector field.

### 3.3. \( \varepsilon \)-spaces

Locally conformally flat Cahen–Wallach symmetric spaces are precisely the \( \varepsilon \)-spaces introduced in Section 1. In this case, the metric is given by (12) with \( \kappa_1 = \ldots = \kappa_n = \varepsilon \).

For \( X = (X_u, X_v, X_1, \ldots, X_n) \) we particularize the solutions in Remark 8 to obtain
\[X_u = a, \quad X_v = b + \lambda v + \frac{n\varepsilon}{2} u - \left( \sum_i x_i h'_i(u) \right), \quad X_j = \frac{\lambda}{2} x_j + h_j(u) + \sum_{i \neq j} c_{ij} x_i, \quad j = 1, \ldots, n,
\]

where \( a \) and \( b \) are arbitrary constants, \( c_{ij} \) is a skew-symmetric matrix of constants and \( h_i \) are functions satisfying \( \varepsilon h_i - h_i'' = 0 \). Therefore, \( \varepsilon \)-spaces are expanding, steady and shrinking Ricci solitons that are symmetric and locally conformally flat.
Moreover, Theorem 9 shows that \( \varepsilon \)-spaces are steady gradient Ricci solitons with potential function 
\[
h(u) = \alpha + \beta u + (n/4)\varepsilon u^2.
\]

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