Restricted convolution inequalities, multilinear operators and applications

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Abstract. For $1 \leq k < n$, we prove that for functions $F, G$ on $\mathbb{R}^n$, any $k$-dimensional affine subspace $H \subset \mathbb{R}^n$, and $p, q, r \geq 2$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, one has the estimate

$$\| (F \ast G) |H| \|_{L^r(H)} \leq \| F \|_{\Lambda_{2,p}^H(\mathbb{R}^n)} \cdot \| G \|_{\Lambda_{2,q}^H(\mathbb{R}^n)},$$

where the mixed norms on the right are defined by

$$\| F \|_{\Lambda_{2,p}^H(\mathbb{R}^n)} = \left( \int_{H^⊥} \left( \int_{H} |\hat{F}|^2 \, dH^⊥_\xi \right)^{\frac{p}{2}} \, d\xi \right)^{\frac{1}{p}},$$

with $dH^⊥_\xi$ the $(n-k)$-dimensional Lebesgue measure on the affine subspace $H^⊥ := \xi + H^⊥$.

Dually, one obtains restriction theorems for the Fourier transform for affine subspaces. Applied to $F(x^1, \ldots, x^m) = \prod_{j=1}^m f_j(x^j)$ on $\mathbb{R}^{md}$, the diagonal $H_0 = \{ (x, \ldots, x) : x \in \mathbb{R}^d \}$ and suitable kernels $G$, this implies new results for multilinear convolution operators, including $L^p$-improving bounds for measures, an $m$-linear variant of Stein’s spherical maximal theorem, estimates for $m$-linear oscillatory integral operators, certain Sobolev trace inequalities, and bilinear estimates for solutions to the wave equation.

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1. Introduction

Convolution inequalities play a central role in harmonic analysis and related areas. Perhaps the most classical of these is Young’s inequality: if $F, G \in \mathcal{S}(\mathbb{R}^n)$, then, for $p, q, r \geq 1$,

$$
\|F * G\|_{L^r(\mathbb{R}^n)} \leq \|F\|_{L^p(\mathbb{R}^n)} \|G\|_{L^q(\mathbb{R}^n)}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.
$$

(1.1)

Many of the applications of this inequality take the following form. (See, e.g., [24, 1] and the references there.) Let $F(x) = f(x)$, a function, and $g(x) = K(x)$, a suitable kernel or measure. Then (1.1) implies Lebesgue spaces mapping properties of the linear convolution operator,

$$
Tf(x) = \int f(x - y)K(y)dy.
$$

In this paper we prove an analogue of Young’s inequality, playing a similar role for multilinear convolution operators as Young’s inequality plays for linear ones. This will follow from a result of independent interest, concerning restrictions of convolutions to affine subspaces; to state that, we need to introduce some notation.

For $1 \leq k \leq n$, let $H \subset \mathbb{R}^n$ be a $k$-dimensional linear subspace, $i_H : H \hookrightarrow \mathbb{R}^n$ the inclusion map, and $i^*_H : (\mathbb{R}^n)^* \rightarrow H^*$ the dual restriction map. $(\mathbb{R}^n)^*$ foliates into a union of $(n-k)$-dimensional affine subspaces, parallel to $(i_H^*)^{-1}(0) = H^\perp$.

$$
(\mathbb{R}^n)^* = \bigcup_{\xi \in H^\perp} (i_H^*)^{-1}(\xi).
$$

(1.2)

Via the Euclidean metric, one identifies $(i_H^*)^{-1}(\xi)$ with $H^\perp_\xi := \xi + H^\perp$. The Lebesgue measure $d\zeta$ on $(\mathbb{R}^n)^*$ thus decomposes as $d\zeta = dH^\perp_\xi(\zeta) d\xi$, where $dH^\perp_\xi$ and $d\xi$, resp., are the Lebesgue measures of dimensions $n-k$ and $k$ on $H^\perp_\xi$ and $H^\perp$, resp. Relative to this decomposition of the Fourier variables, we define a scale of mixed-norm spaces:

**Definition 1.1.** For $1 \leq r, p \leq \infty$ and $F \in \mathcal{S}(\mathbb{R}^n)$, let

$$
\|F\|_{\Lambda^H_{r,p}(\mathbb{R}^n)} = \left( \left( \int_{H^*} \left( \int_{H^\perp_\xi} \left| \hat{F}(\zeta) \right|^r dH^\perp_\xi(\zeta) \right)^{\frac{p}{r}} d\xi \right)^{\frac{r}{p}} \right)^{\frac{1}{r}},
$$

(1.3)

and $\Lambda^H_{r,p} := \Lambda^H_{r,p}(\mathbb{R}^n)$ be the completion of $\mathcal{S}(\mathbb{R}^n)$ with respect to this norm.

**Remark 1.2.** If $k = n$ or $r = p$, $\|F\|_{\Lambda^H_{r,p}} = \|\hat{F}\|_{L^p}$, and if $r = p = 2$, this equals $\|F\|_{L^2}$. Furthermore, the norm is translation-invariant, so it is natural, if $\hat{H} \subset \mathbb{R}^n$ is an affine $k$-plane, to define $\|\cdot\|_{\Lambda^H_{r,p}}$ by using $H$ and $H^\perp_\xi$ on the right hand side of (1.3), where $H$ is the translate of $\hat{H}$ passing through 0.

We now state our first result, which concerns restrictions of convolutions to affine subspaces.

**Theorem 1.3.** Let $1 \leq k < n$, $H \subset \mathbb{R}^n$ a $k$-dimensional affine subspace, and $F, G \in \mathcal{S}(\mathbb{R}^n)$. Then, for $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $p, q, r \geq 2$,

$$
\|(F * G)|_H\|_{L^r(H)} \leq \|F\|_{\Lambda^H_{r,p}(\mathbb{R}^n)} \cdot \|G\|_{\Lambda^H_{q,r}(\mathbb{R}^n)}.
$$

(1.4)
Moreover, the best constant in (1.4), considering $F \rightarrow F \ast G|_H$ as a linear operator acting on $F$, is

$$\|F \rightarrow F \ast G\|_{L^2(\mathbb{R}^n)\rightarrow L^2(H)} = C := \|G\|_{A^H_{\infty}(\mathbb{R}^n)}.$$ 

One motivation for considering restricted convolution inequalities such as (1.4) is the study of multilinear convolution operators. For $d \geq 1$, $m \geq 2$, denote elements of $\mathbb{R}^{md}$ by $\vec{x} := (x^1, \ldots, x^m)$ and elements of $(\mathbb{R}^{md})^*$ by $\vec{\xi} := (\xi^1, \ldots, \xi^m)$. For a kernel $K(\vec{x}) \in D'(\mathbb{R}^{md})$, define an $m$-linear convolution operator $T : (D(\mathbb{R}^d))^m \rightarrow \mathcal{E}(\mathbb{R}^d)$,

$$T(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^{md}} f_1(x - u^1) \cdots f_m(x - u^m) K(\vec{u}) \, du^1 \cdots du^m, \quad x \in \mathbb{R}^d.$$ 

Then, since

$$T(f_1, \ldots, f_m)(x) = (f_1 \otimes \cdots \otimes f_m) * K(x, \ldots, x),$$

multilinear bounds for $T$ can be obtained from Thm. 1.3, with $H$ being the diagonal of $\mathbb{R}^{md}$, $H_0 := \{(x, \ldots, x) : x \in \mathbb{R}^d\}$. An immediate consequence is:

**Corollary 1.4.** Let $K \in \mathcal{E}'(\mathbb{R}^{md})$, $f_1, \ldots, f_m \in \mathcal{S}(\mathbb{R}^d)$, and $2 \leq r \leq \infty$. Then, the multilinear convolution operator defined by (1.6) satisfies

$$\|T(f_1, \ldots, f_m)\|_{L^r(\mathbb{R}^d)} \lesssim \|K\|_{A^{H_0}_{2,\frac{2}{m}}} \cdot \prod_{j=1}^m \|f_j\|_{L^2(\mathbb{R}^d)}.$$ 

**Remark 1.5.** In Sec. 2, we show how Cor. 1.4 and some additional machinery can be applied to obtain $L^p$-improving results for multilinear convolution with measures, analogous to those in the linear setting.

Note that $H^{-1}_0$ can be identified with $\{\left(\frac{\xi^1}{m}, \ldots, \frac{\xi^d}{m}\right) : \xi \in (\mathbb{R}^d)^*\}$, while on the anti-diagonal, $H_0^+ = \{\vec{\eta} := (\eta^1, \ldots, \eta^m) : \sum \eta^j = 0\}$, we can solve for $\eta^m$ in terms of $(\eta^1, \ldots, \eta^{m-1})$. This give rise to a making parametrization concrete the decomposition (1.2),

$$\rho_{H_0} : \mathbb{R}^d_\xi \times \mathbb{R}^{(m-1)d}_\eta \rightarrow (\mathbb{R}^{md})^*, \quad \rho_H(\xi, \vec{\eta}) = \left(\frac{\xi}{m} + \eta^1, \ldots, \frac{\xi}{m} + \eta^{m-1}, \frac{\xi}{m} - \sum_{j=1}^{m-1} \eta^j\right).$$

(In the bilinear case $(m = 2)$, it is convenient to use the more symmetric parametrization of $\rho_{H_0}(\xi, \eta) = (\frac{\xi - \eta}{2}, \frac{\xi + \eta}{2}).$) One can then express the norm of $K$ in (1.7) as

$$\|K\|_{A^{H_0}_{2,\frac{2}{m}}} = c \left[\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^{(m-1)d}} \left|\hat{K}(\rho_{H_0}(\xi, \eta))\right|^2 \, d\eta\right)^{\frac{m-1}{m}} \, d\xi\right]^{\frac{m}{m-1}}.$$ 

Under the Fourier transform, Theorem 1.3 becomes a bilinear Fourier restriction estimate. Interchanging the roles of $F, G$ and $\hat{F}, \hat{G}$, consider $H$ as a subspace of $(\mathbb{R}^n)^*$ and, as with the

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1The notation $\vec{x}, \vec{\xi},$ etc., will occasionally also be used for $l$-tuples of vectors in $\mathbb{R}^d$ for $l < m$. 

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diagonal $H_0 \subset \mathbb{R}^{nd}$ above, introduce linear coordinates $u \in \mathbb{R}^k$, $v \in \mathbb{R}^{n-k}$ on $H^*$, $H^\perp$. This gives rise to a linear isomorphism $\rho_H : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^n$. Thm. 1.3 then yields:

**Corollary 1.6.** Let $H \subset (\mathbb{R}^n)^*$ be a $k$-dimensional subspace, $0 \leq k \leq n$, and $F,G \in \mathcal{S}(\mathbb{R}^n)$. Then, for $p,q,r \geq 2$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, we have

$$||\hat{F}G||_{L^r(H)} \lesssim ||F \circ \rho_H||_{L^p_{\mathbb{R}^k}L^q_{\mathbb{R}^{n-k}}} \cdot ||G \circ \rho_H||_{L^s_{\mathbb{R}^k}L^t_{\mathbb{R}^{n-k}}}.$$  

where $L^p_{\mathbb{R}^k}L^q_{\mathbb{R}^{n-k}}$ denotes the mixed-norm space with norm $\left(\int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^{n-k}} |F(u,v)|^p dv\right)^{\frac{r}{p}} du\right)^{\frac{1}{r}}$.

**Remark 1.7.** Note that the extreme cases in Cor. 1.6 of $\text{dim}(H) = n$ or 0 correspond to standard facts: For $H = \mathbb{R}^n$, one has $H^\perp = \{0\}$ and $L^p_{\mathbb{R}^k}L^q_{\mathbb{R}^{n-k}} = L^p_{\mathbb{R}^n}$, so that $FG \in L^r$ by Hölder and then $\hat{F}G \in L^r$ by Hausdorff-Young, while for $H = \{0\}$, $H^\perp = \mathbb{R}^n$, $L^p_{\mathbb{R}^k}L^q_{\mathbb{R}^{n-k}} = L^p_\mathbb{R}$ and $FG \in L^1 \implies \hat{F}G(0)$ is well-defined.

Starting with a single function $F$, replacing $F$ and $G$ in Cor. 1.6 by $|F|^\frac{p}{2}$ and $|F|^\frac{p}{2} \cdot \frac{F}{|F|}$, resp., convolving each with an approximate identity to restore membership in $\mathcal{S}$, and passing to the limit, one obtains a linear restriction theorem for linear subspaces:

**Corollary 1.8.** If $H \subset (\mathbb{R}^n)^*$ is a $k$-dimensional subspace and $F \in \mathcal{S}(\mathbb{R}^n)$, then, for $p,q,r \geq 2$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$,

$$||\hat{F}||_{L^r(H)} \lesssim ||F \circ \rho_H||_{L^p_{\mathbb{R}^k}L^q_{\mathbb{R}^{n-k}}} \cdot ||G \circ \rho_H||_{L^s_{\mathbb{R}^k}L^t_{\mathbb{R}^{n-k}}},$$  

where $0 < \ell < \frac{n-k}{k}$

and

$$||\hat{F}||_{L^r(H)} \lesssim ||F \circ \rho_H||_{L^p_{\mathbb{R}^k}L^q_{\mathbb{R}^{n-k}}},$$  

$$r = \frac{p}{p-2}, \ p \geq 2.$$

Corollary 1.8 shows that the Fourier transform of a function can be restricted to a linear subspace, provided that the mixed norm is finite; this is in contrast with the well-known fact that the Fourier transforms of general functions in a standard $L^p$, $p > 1$, cannot be restricted to subspaces.

Thm. 1.3 also has a maximal operator variant:

**Theorem 1.9.** Let $F,G \in \mathcal{S}(\mathbb{R}^n)$ and, for $t > 0$, define $G_t(x) = t^{-n}G\left(\frac{x}{t}\right)$. Let $H \subset \mathbb{R}^n$ be a $k$-dimensional subspace. Suppose that, for some $\gamma > \frac{k}{2}$ and any $t > 0$,

$$||G_t||_{L^\infty_{\mathbb{R}^n}} := \sup_{\xi \in H^*} \left(\int_{H^*} |\hat{G}_j(t\xi)|^2 dH^*_\xi(\xi)\right)^{\frac{1}{2}} \lesssim 2^{-j\gamma},$$  

where, for $j \geq 0$, $G_j$ is the Littlewood-Paley component of $G$ at frequency scale $2^j$ relative to a dyadic decomposition of $(\mathbb{R}^n)^*$, and the same estimate holds for $\nabla G_j$. Then,

$$\left(\sup_{t > 0} ||F \ast G_t||_{L^2_{\mathbb{R}^n}}\right) \lesssim ||F||_{L^2(\mathbb{R}^{md})}.$$
Remark 1.10. Approximating $F$ by a continuous function plus a function with a small $L^2$ norm, a standard argument then shows that, for $F \in L^2(\mathbb{R}^n)$,

$$\lim_{t \to 0} \int_{\mathbb{R}^n} F(x-u)G_t(u) \, du = c_G \cdot F(x) \text{ a.e., with } c_G := \int_{\mathbb{R}^n} G(u) \, du.$$ 

Returning to the multilinear setting and taking $F(\vec{x}) = \prod_{j=1}^m f_j(x^j)$, $G(\vec{x}) = K(\vec{x})$, $H = H_0$, and $p = 2$ in Thm. 1.9, we obtain the following bound for maximal $m$-linear convolution operators.

**Corollary 1.11.** If $K$ is a finite Borel measure on $\mathbb{R}^{md}$, $f_j \in S(\mathbb{R}^d)$, $1 \leq j \leq m$, and $t > 0$, define

$$B_t(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^{md}} f_1(x-tu^1) \cdots f_m(x-tu^m) \, dK(\vec{u}),$$

and the associated maximal operator,

$$B(f_1, \ldots, f_m)(x) = \sup_{t > 0} |B_t(f_1, \ldots, f_m)(x)|.$$

Then

$$(1.14) \quad \|B(f_1, \ldots, f_m)\|_{L^2(\mathbb{R}^d)} \lesssim \prod_{j=1}^m \|f_j\|_{L^2(\mathbb{R}^d)},$$

provided that $K$ and $\nabla K$ satisfy the Littlewood-Paley decay condition (1.12) w.r.t. $\Lambda_{H_0}^{H_{2,\infty}}$.

In Sec. 2.3 below, we use Cor. 1.14 and some additional machinery to establish $m$-linear analogues of Stein’s spherical maximal theorem [23].

We point out that there have been other works on restrictions of convolutions (e.g., [3, 2]) and estimates for multilinear convolution operators [15, 25], but the results there seem to be of a different nature than those we obtain.

The paper is organized as follows. In Sec. 2 we use Thm. 1.3 to prove estimates for multilinear analogues of both Stein’s spherical maximal theorem and classical $L^p$-improving estimates for convolution. The proofs of Theorems 1.3 and 1.9 are then given in Sec. 3. We show in Sec. 4 how these results imply certain Sobolev trace inequalities and estimates for solutions of the heat and wave equations. In Sec. 5 we use our method to derive bounds for multilinear oscillatory integral operators, and Sec. 6 contains some further discussion and open problems.

### 2. $L^p$ improving and maximal estimates for multilinear operators:

Multilinear operators play a key role in many aspects of harmonic analysis and partial differential equations. See, for example, [24] and the references there for a comprehensive description and history of this subject. The key starting point for us is the work of Coifman and Meyer [9], who initiated the comprehensive study of Calderón-Zygmund theory in the multilinear setting. They showed that an $m$-linear multiplier operator

$$M : L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_m}(\mathbb{R}^d) \to L^r_{\text{loc}}(\mathbb{R}^d)$$
if
\[ \frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{r} \]
and the multiplier \( m(\xi) \) of \( M \) satisfies
\[ |D^\alpha m(\xi^1, \ldots, \xi^m)| \leq C_\alpha (1 + |\xi^1| + \cdots + |\xi^m|)^{-s-|\alpha|}, \forall \alpha \in \mathbb{Z}_{md}^+ \).

This estimate has many important and interesting applications. However, in many multilinear situations the symbolic decay conditions on the multiplier and its derivatives are violated. For example, if the multiplier is the Fourier transform of the Lebesgue measure on a smooth hypersurface, the derivatives satisfy the same decay estimates as the multiplier, and no better. Another example which is not covered by this result is the case of multilinear fractional integration, handled in \[19\] using different methods; see also \[13\].

A systematic study of multilinear analogues of linear generalized Radon transforms was begun in \[12\], focussing on applications to the study of finite point configurations problems in geometric measure theory. These problems have their roots in the work of Falconer on the distance problem in the continuous setting and of the Erdős school on finite point configurations in the discrete setting. See, e.g., \[6\] and the references there. The main bilinear estimate in \[12\] can be stated as follows.

**Theorem 2.1.** Let \( K \) be a non-negative integrable function on \( \mathbb{R}^{2d} \), and define
\[ B(f, g)(x) = \int f(x-u)g(x-v)K(u,v)du dv. \]
Then for \( 1 \leq p \leq 2 \),
\[ \|B(f, g)\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^2(\mathbb{R}^d)} \cdot \|g\|_{L^2(\mathbb{R}^d)} \cdot \left( \int |\hat{K}(\xi, -\xi)|^{p'} d\xi \right)^{1/p}. \]

While this result was useful in establishing geometric and combinatorial estimates, a weakness that limits wider applications, is the positivity assumption on the kernel. This prevents one from exploiting, e.g., Fourier decay information by using Littlewood-Paley type decompositions. We resolve this problem to a significant extent in Thm. 1.3 and Cor. 1.4, opening the door to a variety of applications illustrated in this section.

### 2.1. \( L^p \)-improving measures.

A finite measure \( \mu \) on \( \mathbb{R}^d \) is said to be \( L^p \)-improving if
\[ \|f \ast \mu\|_{L^q(\mathbb{R}^d)} \leq C_{p, q}\|f\|_{L^p(\mathbb{R}^d)} \]
for some \( q > p \).

There has been much interest in determining the pairs \((p, q)\) for which (2.1) holds. Two representative results in this area are due to Strichartz \[26\] and Littman \[18\]; see Christ \[8\] for more recent work. The result of Littman and Strichartz says that if \( \mu \) is the Lebesgue measure on the unit sphere in \( \mathbb{R}^d \), \( d \geq 2 \), then (2.1) holds if and only if \((1/p, 1/q)\) is contained in the closed triangle with endpoints \((0, 0), (1, 1)\) and \(\left(\frac{d}{d+1}, \frac{1}{d+1}\right)\).

In the multilinear setting, the notion of \( L^p \)-improving may be expressed in the following way, indicating an improvement over the Hölder inequality exponents governing pointwise multiplication.

**Definition 2.2.** Let \( B(f_1, \ldots, f_m) \) be an \( m \)-linear operator. We say that \( B \) is \( \text{Lebesgue space improving} \) if there exist \( p_1, \ldots, p_m, r > 0 \) with
\[
\frac{1}{p_1} + \cdots + \frac{1}{p_m} > \frac{1}{r}
\]
such that
\[
B : L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_m}(\mathbb{R}^d) \to L^r(\mathbb{R}^d).
\]

The following result is a simple adaptation of Cor. 1.4.

**Theorem 2.3.** Suppose that \( \nu(\vec{u}) \) is a smooth multiple of surface measure on a compact, codimension \( l \) submanifold of \( \mathbb{R}^{md} \). For \( j \geq 0 \), let \( \nu_j \) be the \( j \)th Littlewood-Paley component of \( \nu \), and suppose that, for some \( \gamma > 0 \),

\[
\|\nu_j\|_{A^2_{\infty, \infty}} \lesssim 2^{-j\gamma}.
\]

Define
\[
B\nu(f_1, \ldots, f_m)(x) := \int_{\mathbb{R}^{md}} f_1(x - u^1) \cdots f_m(x - u^m) \, d\nu(\vec{u}).
\]

Then
\[
\|B\nu(f_1, \ldots, f_m)\|_{L^2(\mathbb{R}^d)} \lesssim \prod_{j=1}^m \|f_j\|_{L^2(\mathbb{R}^d)}
\]
and
\[
\|B\nu(f_1, \ldots, f_m)\|_{L^{p'}(\mathbb{R}^d)} \lesssim \prod_{j=1}^m \|f_j\|_{L^p(\mathbb{R}^d)} \text{ for } p > \frac{2(l + \gamma)}{l + 2\gamma}.
\]

**Remark 2.4.** It is not difficult to check that if \( |\hat{\nu}(\vec{\xi})| \leq C(1 + |\vec{\xi}|)^{-\alpha} \), then (2.2) holds with the exponent \( \gamma = \alpha - \frac{(m-1)d}{2} \).

**Corollary 2.5.** Let \( \nu \) be surface measure on the sphere \( S^{md-1} = \{ \vec{u} \in \mathbb{R}^{md} : |u^1|^2 + \cdots + |u^m|^2 = 1 \} \). Then
\[
B\nu : L^p(\mathbb{R}^d) \times \cdots \times L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d) \text{ for } \frac{d+1}{d} < p \leq 2.
\]

Cor. 2.5 follows from Thm. 2.3, since the codimension \( l = 1 \) and the estimate (2.2) holds with \( \gamma = \frac{md-1}{2} - \frac{d}{2} \), using the Remark 2.4 and the well-known decay (see, e.g., [24]) that, if \( \sigma \) denotes surface measure on the sphere in \( \mathbb{R}^n \), \( n \geq 2 \), then

\[
|\hat{\sigma}(\vec{\xi})| \leq C(1 + |\vec{\xi}|)^{-\frac{d-1}{2}}.
\]

One can in fact improve Cor. 2.5 to obtain the endpoint \( p = \frac{d+1}{d} \) as follows. Define
\[
B_{\nu}^z(f_1, \ldots, f_m)(x) := \int_{\mathbb{R}^{md}} f_1(x - u^1) \cdots f_m(x - u^m) \, d\nu^z(\vec{u}),
\]
where

\[ \nu^z(\vec{u}) = \frac{1}{\Gamma(z)} (1 - |u|^2)^{z-1}. \]

When \( Re(z) = -\frac{d-1}{2} \), \( B_\nu : L^2(\mathbb{R}^d) \times \cdots \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) via a direct appeal to Cor. 1.4 and the observation above that we may take all the way up to \( \gamma = \frac{(m-1)d-1}{2} \) in (2.2). When \( Re(z) = 1 \), \( B_\nu : L^1(\mathbb{R}^d) \times \cdots \times L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d) \) trivially since the kernel becomes bounded. Stein’s analytic interpolation theorem then yields the conclusion of Cor. 2.5 without the loss of the endpoint.

This estimate is basically sharp, as can be seen by the following adaptation of the usual linear sharpness example which we illustrate in the bilinear setting. Let \( f \) be the characteristic function of the unit ball and let \( g \) be the characteristic function of the ball of radius \( \delta \). Then \( \|g\|_p \approx \delta^\frac{d}{2} \) and \( B_\nu(f, g) \approx \delta^{d-1} \) on the annulus of thickness \( \approx \delta \) and radius \( \approx 1 \). Sharpness of the estimate above, up to the endpoint, follows.

In the bilinear case, if \( \{(u^1, u^2) : |u^1|^2 + |u^2|^2 = 1\} \) is replaced by \( \{(u^1, u^2) : u^1 \cdot u^2 = t\} \) with \( t \neq 0 \) fixed, one obtains the same exponents as in Cor. 2.5. This surface arose in [12] in connection with the study of areas of triangles determined by fractal subsets of Euclidean space.

### 2.2. Proof of Theorem 2.3.

The estimate (2.3) follows from Thm. 1.3 and (2.2). In fact, since \( B_\nu(f_1, \ldots, f_m) = (f_1 \otimes \cdots \otimes f_m) \ast \nu_{H_0} \), we have \( (I - \Delta_x)^{\gamma/2} B_\nu(f_1, \ldots, f_m) = (f_1 \otimes \cdots \otimes f_m) \ast \hat{V}_{H_0} \), where \( \hat{V}(\xi_1, \ldots, \xi_m) = (1 + |\xi|^2)^{\gamma/2}|\hat{\nu}|. \) From (2.2), we see that \( \|V\|_{L^q_{\|u\|}} \lesssim 1 \), and hence applying Thm. 1.3 with \( r = p = 2 \), \( q = \infty \) yields (2.3).

To prove (2.4), define as above the \( j \)th Littlewood-Paley piece \( \nu_j \) of \( \nu \), \( 0 \leq j < \infty \), and set

\[ B_j(f_1, \ldots, f_m)(x) = \int \cdots \int f_1(x - u^1) \cdots f_m(x - u^m) \nu_j(\vec{u}) d\vec{u}. \]

It follows from (2.3) that

\[ B_j : L^2(\mathbb{R}^d) \times \cdots \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \]

with norm \( 2^{-jn} \).

On the other hand,

\[ \|B_j(f_1, \ldots, f_m)\|_{L^\infty(\mathbb{R}^d)} \leq \|f_1\|_{L^1(\mathbb{R}^d)} \cdots \|f_m\|_{L^1(\mathbb{R}^d)} \cdot \|\nu_j\|_{L^\infty(\mathbb{R}^m d)} \cdot \|\nu\|_{L^\infty(\mathbb{R}^m d)}, \]

By a direct calculation (see, e.g., [29]), using the codimension \( l \) of the support of \( \nu \),

\[ \|\nu_j\|_{L^\infty(\mathbb{R}^m d)} \leq C2^j. \]

The Riesz-Thorin interpolation theorem then yields (2.4).

**Corollary 2.6.** Let \( \nu \) be surface measure on \( S^{d-1} \times \cdots \times S^{d-1} \subset \mathbb{R}^{md} \). Then

\[ B_\nu : L^p(\mathbb{R}^d) \times \cdots \times L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \]

for \( \frac{(m^2 - 2m + 2)d + m^2}{(m^2 - 2m + 2)d} \leq p \leq 2 \).

To prove this, define \( \nu^z(\vec{u}) = \prod_{j=1}^m \frac{1}{\Gamma(z)} (1 - |u_j|^2)^{z-1} \) and proceed as in the proof of Cor. 2.5.
2.3. Bilinear analogues of Stein’s spherical maximal theorem. The classical (linear) spherical maximal operator is given by

$$\mathcal{M}f(x) = \sup_{t > 0} \left| \int_{S^{d-1}} f(x - ty) d\sigma(y) \right|,$$

where $\sigma$ is the Lebesgue measure on the unit sphere in $\mathbb{R}^d$, $d \geq 2$. It was established by Stein [23] in three dimensions and higher, and by Bourgain [4] in two dimensions that

$$\mathcal{M} : L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d) \text{ for } p > \frac{d}{d-1}.$$  

The result is best possible, as can be shown by taking

$$f(x) = |x|^{-d+1} \log \left( \frac{1}{|x|} \right) \chi_{B(0, \frac{1}{2})}(x) \text{ with } t = |x|.$$  

The cornerstone of the general theory of maximal averages in dimension $d \geq 3$ is the following result; see Bourgain [5], Carbery [7], Cowling-Mauceri [10] and Sogge-Stein [21].

**Theorem 2.7.** Let $\mu$ be a finite, compactly supported measure on $\mathbb{R}^d$, such that, for some $\epsilon > 0$,

$$|\hat{\mu}(\xi)| \leq C (1 + |\xi|)^{-\frac{d}{2} - \epsilon}. $$

Define

$$M_\mu f(x) = \sup_{t > 0} \left| \int f(x - ty) d\mu(y) \right|.$$  

Then, $M_\mu : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$.

The techniques introduced above yield multilinear analogues of these maximal theorems. For simplicity, we restrict ourselves to the bilinear case. Given a finite Borel measure $\nu$ on $\mathbb{R}^{2d}$, define the bilinear averaging operators,

$$B_t(f, g)(x) = \int \int f(x - tu)g(x - tv)d\nu(u, v), 0 < t < \infty,$$

and the associated maximal operator, $B_\nu(f, g)(x) = \sup_{t > 0} |B_t(f, g)(x)|$. The following bilinear analogue of Thm. 2.7 is an immediate restatement of Cor. 1.11.

**Theorem 2.8.** Suppose that, for all $t > 0$ and some $\epsilon > 0$,

$$\sup_{\xi \in \mathbb{R}^d} \left( \int_{2^j \leq |t\xi|^2 + |t\eta|^2 \leq 2^{j+1}} \left| \hat{\nu} \left( \frac{t\xi + t\eta}{2}, \frac{t\xi - t\eta}{2} \right) \right|^2 d\eta \right)^{\frac{1}{2}} \leq C 2^{-j(\frac{d}{2} + \epsilon)},$$

and the same estimate holds for $|\nabla \nu|$. Then $B_\nu : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$.

Using additional geometric information on the support of $\nu$, one can improve the exponents in Thm. 2.8 as follows.

**Corollary 2.9.** Suppose that $\nu$ is supported on a smooth compact submanifold of $\mathbb{R}^d \times \mathbb{R}^d$ of codimension $l$. Suppose that, for all $t > 0$ and some $\gamma > \frac{1}{2}$,
2.8 \[ \sup_{\xi \in \mathbb{R}^d} \left( \int_{2^j \leq |\xi|^2 + |\eta|^2 \leq 2^{j+1}} \left| \hat{\varphi}\left(\frac{t\xi + t\eta, t\xi - t\eta}{2}\right) \right|^2 d\eta \right)^{\frac{1}{2}} \leq C2^{-j\gamma}, \; j \geq 0 \]

and the same holds for the gradient. Then

2.9 \[ \mathcal{B}_\nu : L^p(\mathbb{R}^d) \times L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d) \text{ with } p > \frac{2l + 2\gamma - 1}{l + 2\gamma - 1}. \]

Remark 2.10. By a standard argument involving the approximation of an \( L^p \) function by a continuous function plus a function with a small \( L^p \) norm, Thm. 2.8 and Cor. 2.9 yield differentiation theorems. For example, Cor. 2.9 implies that if \( f, g \in L^p(\mathbb{R}^d) \) with \( p \) as in (2.9), then

\[ \lim_{t \to 0} \int \int f(x - tu)g(x - tv)dv(u,v) = f(x)g(x) \text{ a.e.} \]

Example 2.11. Let \( \nu \) be the Lebesgue measure on the sphere \( \{(u, v) \in \mathbb{R}^d \times \mathbb{R}^d : |u|^2 + |v|^2 = 1\} \), \( d \geq 2 \). Then

\[ \mathcal{B}_\nu : L^p(\mathbb{R}^d) \times L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d) \text{ for } p > \frac{d}{d - 1}. \]

To see that the exponent \( \frac{d}{d - 1} \) is optimal, take \( g \equiv 1 \). What we get is essentially the maximal averaging operator of \( f \) over the sphere in \( \mathbb{R}^d \) of radius \( 1 - |v|^2 \). The needed restriction follows from Stein’s sharpness example for the linear spherical maximal operator, namely, \( f(x) = |x|^{-d+1} \log \left( \frac{1}{|x|} \right) \chi_{B(0,1)}(x) \).

2.4. Proof of Corollary 2.9. By a direct calculation,

\[ |B_j^i(f, g)(x)| \leq \|f\|_1 \cdot \|g\|_1 \cdot \|K^j\|_\infty \leq C\|f\|_1 \cdot \|g\|_1 \cdot 2^j. \]

The proof of Thm. 2.8 above gives

\[ \mathcal{B}^j : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \text{ with norm } \approx 2^{-j(\gamma - \frac{1}{2})}, \]

and using Riesz-Thorin again yields Cor. 2.9.

3. Proofs of Theorems 1.3 and 1.9

3.1. Proof of Thm. 1.3. By the translation-invariance of the norms in (1.4), one may assume that \( H \subset \mathbb{R}^n \) is a \( k \)-dimensional linear subspace, which, by rotation covariance, can be put in the form \( H = \{(x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k} \mid x'' = 0\} \sim \mathbb{R}^k \), so that, for \( \xi' \in H^* \), \( H_{\xi}^* = \{\langle \xi', \xi'' \rangle \mid \xi'' \in \mathbb{R}^{n-k}\} \).

Fixing \( G \), define the restricted convolution operator \( TF = (F * G)|_H \), i.e., \( TF(x') = (F * G)(x', 0) \).

Then the formal adjoint is given by

\[ T^* h(u', u'') = \int_{\mathbb{R}^k} \overline{G(x' - u', -u'')} h(x') dx', \]
from which one sees that $TT^*$ is a translation-invariant operator on $\mathbb{R}^k$, given in Fourier multiplier form by

$$\widetilde{TT^*}h'(\xi) = \left( \int_{\mathbb{R}^{n-k}} |\hat{G}(\xi', \xi'')|^2 d\xi'' \right) \hat{h}(\xi').$$

By Parseval,

$$||TT^*||_{L^2(\mathcal{H}) \rightarrow L^2(\mathcal{H})} = \sup_{\xi \in \mathcal{H}^s} \left( \int_{H_{\xi'}^s} |\hat{G}(\xi')|^2 dH_{\xi'}^s(\xi') \right) = ||G||_{L^2(\mathcal{H})}^2,$$

and thus $||T||_{L^2(\mathcal{H}) \rightarrow L^2(\mathcal{R}^n)} = ||G||_{L^2(\mathcal{H})}$, yielding (1.5), and therefore (1.3) for $p = 2$, $q = \infty$, $r = 2$. On the other hand,

$$||F \ast G||_{L^2(\mathcal{H})} \leq ||F \ast G||_{C_0(\mathcal{R}^n)} \leq ||\hat{F} \hat{G}||_{L^2(\mathcal{R}^n)} \leq ||\hat{F}||_{L^2} \cdot ||\hat{G}||_{L^2} \leq ||F||_{L^2} \cdot ||G||_{L^2},$$

which is (1.3) for $p = q = 2$, $r = \infty$. Interpolation then gives (1.3) in general $p, q, r$.

### 3.2. Proof of Thm. 1.9.

It is sufficient to establish

$$|| \sup_{R \leq t \leq 2R} |F \ast G_t||_{L^2(\mathcal{H})}^2 \leq ||F||_2^2$$

where $R$ ranges over all dyadic numbers. For fixed $R$, make a Littlewood-Paley decomposition with respect to that scale. First define

$$\hat{F}_j(\xi) = \psi \left( 2^{-j} |\xi| \right) \hat{F}(\xi),$$

where $\psi$ is a smooth cut-off function supported in the annulus in $\mathbb{R}^{md}$ of inner radius $\frac{1}{2}$ and outer radius 4, identically equal to 1 in the annulus of inner radius 1 and outer radius 2, and satisfying $\sum_j \psi(2^{-j} \cdot) \equiv 1$. Letting $[]$ denote the greatest integer function, we now aim to estimate

$$\sup_{R \leq j \leq 2R} |F^{j+\log_2(R)} \ast G_t|.$$

We shall need the following elementary observation.

**Lemma 3.1.** Let $F$ be a differentiable function on $[R, 2R]$. Then

$$\sup_{t \in [R, 2R]} |F(t)|^2 \leq |F(R)|^2 + 2 \left( \int_R^{2R} |F(t)|^2 dt \right)^\frac{1}{2} \cdot \left( \int_R^{2R} (F'(t))^2 dt \right)^\frac{1}{2}.$$

To prove the lemma, just observe that by the fundamental theorem of calculus,

$$F(t)^2 = F(R)^2 + 2 \int_R^t 2F(s)F'(s) ds$$

and apply the Cauchy-Schwarz inequality. In order to use Lemma 3.1, observe that

$$\frac{d}{dt} (F \ast G_t)(x) = \int e^{2\pi i x \cdot (\xi_1 + \cdots + \xi_m)} \hat{F}(\xi) \frac{d}{dt} \hat{G}(t \xi) d\xi$$

$$= \int e^{2\pi i x \cdot (\xi_1 + \cdots + \xi_m)} \hat{F}(\xi) \langle \nabla \hat{G}(t \xi), \xi \rangle d\xi,$$
\[ = \int e^{2\pi ix \cdot (\xi^1 + \cdots + \xi^m)} \hat{F}(\xi) t^{-1} \hat{G}(t\xi) d\xi, \]

where
\[ |\hat{G}(\xi)| \lesssim |\xi| \cdot |\nabla \hat{G}(\xi)|. \]

Note that
\[ \left(F^j + \left[ \log_2 (R) \right]^* G_t \right)(x) = \int e^{2\pi ix \cdot (\xi^1 + \cdots + \xi^m)} \hat{F}(\xi) \psi \left(2^{-j} R|\xi| \right) \hat{G}(t\xi) d\xi \]
\[ = \int e^{2\pi ix \cdot (\xi^1 + \cdots + \xi^m)} \hat{F}(\xi) \psi \left(2^{-j} t|\xi| \right) \hat{G}(t\xi) d\xi \]
\[ = \left(F \ast G^j_t \right)(x) \]

when \( R \leq t \leq 2R \), where \( G^j_t \) is given by \( \hat{G}^j_t(\xi) = \psi \left(2^{-j} t|\xi| \right) \hat{G}(t\xi) \). Similar calculations hold for \( G^* \).

Applying Lemma 3.1, we see that
\[ \sup_{t \in [R, 2R]} |F^j + \left[ \log_2 (R) \right]^* G_t|_H^2 \leq ||F \ast G^j_R|_H||^2_{L^2(H)} \]
\[ + 2 \left( \frac{1}{R} \int_R^{2R} \int |(F \ast G^j_t)(x)|^2 dx dt \right)^{1/2} \]
\[ \times \left( R \int_R^{2R} t^{-2} \int |(F \ast (G^* \psi^j))(x)|^2 dx dt \right)^{1/2}. \]

Now appealing to Thm. 1.3 and using estimate (1.12), we see that this expression is
\[ \lesssim ||F||^2_2 \cdot 2^{-j(1+2\epsilon)} + ||F||^2_2 \cdot 2^{-j(\frac{3}{2}+\epsilon)} \cdot 2^{-j(\frac{3}{2}+\epsilon)2j}. \]

We could have made the argument above stronger by replacing the \( \psi \) factor with \( \hat{\psi} \left(2^{-j} R|\xi| \right) \hat{\psi} \left(2^{-j} R|\xi| \right) \), where \( \hat{\psi} \) is a slight widening of \( \psi \). The argument as above would then yield
\[ \sup_{t \in [R, 2R]} |F^j + \left[ \log_2 (R) \right]^* G_t|_H^2 \lesssim 2^{-j\epsilon} ||F^j + \left[ \log_2 (R) \right]|_2^2, \]

where the Littlewood-Paley decomposition on the right hand side is made with respect to the dilates of \( \hat{\psi} \). Summing the corresponding geometric series and bounding a supremum by a square function, we obtain equation (3.1) and thus the conclusion of Theorem 1.9.

### 4. Applications to PDE

We now show that the results established so far can be used to obtain a variety of estimates related to Sobolev spaces and partial differential equations.
4.1. Sobolev traces on subspaces. The classical Sobolev trace inequality from \( \mathbb{R}^n \) to \( \mathbb{R}^{n-1} \) (see, e.g., [27, Prop. 1.6]) says that if the restriction (or trace) map \( \tau \) is defined by \( \tau(u) = f \), where \( f(x') = u(0, x') \), \( x = (x_1, x') \), \( x' = (x_2, \ldots, x_n) \), then

\[
\|\tau(u)\|_{L^2_s(\mathbb{R}^{n-1})} \leq C_s\|u\|_{L^2_s(\mathbb{R}^n)}, \quad s > \frac{n}{2}.
\]

Iterating (4.1) \( n-k \) times yields, for any \( k \)-dimensional subspace \( H \subset \mathbb{R}^n \),

\[
\|\tau(u)\|_{L^2_s(\mathbb{R}^{n-k}(H))} \leq C'_s\|u\|_{L^2_s(\mathbb{R}^{n-k})}, \quad s > \frac{n-k}{2}.
\]

We shall use Thm. 1.3 to prove an endpoint version of (4.2) with the best constant, and then give some applications to partial differential equations.

**Theorem 4.1.** If \( H \subset \mathbb{R}^n \) is an affine subspace with \( \text{dim}(H) = k \), and \( s > \frac{n-k}{2} \), then

\[
\|u\|_{L^2_s(H)} \leq C_{s,H}\|u\|_{L^2_s(\mathbb{R}^n)}, \quad \forall u \in S(\mathbb{R}^n).
\]

Furthermore, the optimal constant is

\[
C_{s,n-k} = \left\| \left(1 + |\xi|^2\right)^{-\frac{s}{2}} \right\|_{L^\infty_s} = \sqrt{|S^{n-k-1}|} \int_0^\infty (1+r^2)^{-s/2} r^{n-k-1} dr.
\]

**Remark 4.2.** Thm. 4.1 is result is an endpoint version of (4.2) in the sense that we take \( L^2(H) \) on the left hand side instead of \( L^2_s(\mathbb{R}^{n-k}) \) for \( s > \frac{n-k}{2} \). One cannot pin the endpoint on both sides, as boundedness \( \tau : L^2_s(\mathbb{R}^{2d}) \rightarrow L^2(H) \) does not hold. Clearly, if \( C_{s,n-k} \) is as in (4.4) then

\[
\lim_{s \searrow \frac{n-k}{2}} C_{s,n-k} = \infty.
\]

**Proof of Theorem 4.1.** As in the proof of Thm. 1.3, by translation- and rotation-invariance of the function spaces involved, we can assume that \( H = \{(x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k} \mid x'' = 0\} \), so that \( H^* \simeq \mathbb{R}^k \) and \( H^*_\xi = \{(\xi', \xi'') \mid \xi'' \in \mathbb{R}^{n-k}\} \). One can rewrite (4.3) in the form

\[
\left\| \left( -\Delta \right)^{-\frac{s}{2}} u \right\|_{L^2_s(H)} \leq C_s\|u\|_{L^2_s(\mathbb{R}^{2d})}, \quad s > \frac{d}{2}.
\]

The left hand side can be rewritten as

\[
\|u * G_s(H)\|_{L^2_s(H)} \quad \text{with} \quad \tilde{G}_s(\xi') = (1 + |\xi'|^2 + |\xi''|^2)^{-\frac{s}{2}}.
\]

By Thm. 1.3 we conclude that (4.3) holds, with optimal constant

\[
C_{s,n-k} = \left\| \left(1 + |\xi|^2 + |\xi''|^2\right)^{-\frac{s}{2}} \right\|_{L^\infty_s}.
\]

The \( \sup_{C_s} \) in the norm on the right is attained at \( \xi' = 0 \), yielding

\[
C^2_{s,n-k} = |S^{n-k-1}| \int_0^\infty (1+r^2)^{-s} r^{n-k-1} dr < \infty, \quad s > \frac{n-k}{2},
\]

completing the proof of (4.3) with constant (4.4).

We shall also give a simple proof of the following product Sobolev estimate [28]:
Theorem 4.3. Let $u, v \in S(\mathbb{R}^d)$. Suppose that $\gamma = r + s - \frac{d}{2}$ and $r, s \geq 0$. Then

$$||(uv)||_{L^2(\mathbb{R}^d)} \leq C ||u||_{L^2(\mathbb{R}^d)} \cdot ||v||_{L^2(\mathbb{R}^d)}.$$  

Remark 4.4. Unfortunately, we are not able to obtain the best constant in (4.5), since we cannot be certain that the optimizing function for the inequality (1.5) is realizable as a product function.

Proof of Theorem 4.3. We make use of the bilinear convolution estimates from Cor. 1.4. Let $f = (-\triangle)^{\frac{r}{2}} u$ and $g = (-\triangle)^{\frac{s}{2}}$. We can rewrite (4.5) in the form

$$||(uv)||_{L^2(\mathbb{R}^d)} \leq C ||f||_{L^2(\mathbb{R}^d)} \cdot ||g||_{L^2(\mathbb{R}^d)}.$$  

Let $B(f, g)(x) = c_{r,s} \int \int f(x-u)g(x-v)|u|^{-d+r}|v|^{-d+s} dudv$ and observe that, with the appropriate choice of $c_{r,s}$, one has $B(f, g) = uv$. Applying Cor. 1.4, we see that

$$||(uv)||_{L^2(\mathbb{R}^d)} \leq ||f||_{L^2(\mathbb{R}^d)} \cdot ||g||_{L^2(\mathbb{R}^d)} \cdot \sup_{\xi, \eta \in \mathbb{R}^d} \left( \int \left| \hat{K} \left( \frac{\xi + \eta}{2}, \frac{\xi - \eta}{2} \right) \right|^2 d\eta \right)^{\frac{1}{2}},$$  

where $\hat{K}(\xi, \eta) = |\xi|^{-r}|\eta|^{-s}$. Since

$$\left( \int_{|\xi|+|\eta| \approx 2^j} \left| \hat{K} \left( \frac{\xi + \eta}{2}, \frac{\xi - \eta}{2} \right) \right|^2 d\eta \right)^{\frac{1}{2}} \approx 2^{-j(r+s-\frac{d}{2})},$$  

the result follows.

We now give some applications of Thms. 4.1 and 4.3 to a priori estimates in partial differential equations.

4.2. The heat equation: restriction to subspaces. Consider the heat equation on $\mathbb{R}^n \times \mathbb{R}_+$,

$$\frac{\partial u}{\partial t} = \triangle u, \ (x, t) \in \mathbb{R}^n \times \mathbb{R}_+, \ \ u(x, 0) = F(x) \in L^2(\mathbb{R}^n),$$  

with solution given by $u(x, t) = (F \ast \Phi_t)(x)$, where

$$\Phi_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}.$$  

We have the following result concerning the restriction of the heat semi-group to affine subspaces.

Theorem 4.5. Suppose that $u$ is the solution to (4.6). Then, for any $k$-dimensional affine subspace $H \subset \mathbb{R}^n$,

$$||u(\cdot, t)||_H^{L^2(H)} \leq (4\pi t)^{-\frac{n-k}{2}} \cdot ||F||_{L^2(\mathbb{R}^n)},$$  

and the constant $(4\pi t)^{-\frac{n-k}{2}}$ is optimal.
In other words, the initial data $F$ might blowup along $H$ for $t$ small, resulting in the solution $u$ being large along $H$ for small values of $t$, but after the fixed time $T = \frac{1}{4\pi}$, independent of $F$, the $L^2(H)$ norm of the solution drops below the $L^2(\mathbb{R}^n)$ norm of $F$.

**Proof of Theorem 4.5.** By invariance of (4.6) under the Euclidian motion group, as in the proof of Thm. 4.1 $H$ may be taken to be $H = \{(x',0) \in \mathbb{R}^n \mid x' \in \mathbb{R}^k\}$. Since $\hat{\Phi}_t(\xi) = e^{-4\pi^2 t|\xi|^2} = e^{-4\pi^2 t(|\xi'|^2 + |\xi''|^2)}$, one has

$$||\Phi_t||_{L^2_H} = \sup_{\xi \in \mathbb{R}^n} e^{-2\pi^2 t|\xi'|^2} \left(\int_{\mathbb{R}^{n-k}} e^{-4\pi^2 t|\xi''|^2} d\xi''\right)^{\frac{1}{2}}$$

$$= \left(\int_{\mathbb{R}^{n-k}} e^{-4\pi^2 t|\xi''|^2} d\xi''\right)^{\frac{1}{2}}$$

$$= \left(2\pi t\right)^{-\frac{n-k}{2}} \left(\int_{\mathbb{R}^{n-k}} e^{-|\xi''|^2} d\xi''\right)^{\frac{1}{2}} = (4\pi t)^{-\frac{n-k}{4}}.$$ 


4.3. **The wave equation: restriction to subspaces.** We now consider solutions to the Cauchy problem for the wave equation,

\begin{equation}
(4.7) \quad u_{tt} = \Delta u, \ (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \ u(x,0) = 0, \ u_t(x,0) = F(x,y) \in L^2(\mathbb{R}^n).
\end{equation}

As is well-known, the map $F(x) \rightarrow u(x,t)$ is bounded between a variety of function spaces; see, e.g., [22, 24] and the references there. We show here that for fixed $t \neq 0$ the solution $u(x,t)$ restricts in a natural way to any $k$-plane.

By the standard integral representation of the solution to (4.7),

$$u(x,t) = F \ast K_t(x) := c_n \left(\frac{1}{t} \frac{d}{dt}\right)^{\frac{n-3}{2}} \left\{t^{n-2} A_t F(x,y)\right\},$$

with, as before,

$$A_t F(x,y) = \int_{S^{n-1}} F(x - tu) \, d\sigma(u),$$

where $d\sigma$ is the surface measure on the unit sphere in $\mathbb{R}^n$ and $K_t$ is the (fixed time) fundamental solution. The following is an immediate consequence of Thm. 1.3.

**Theorem 4.6.** Let $H$ be a $k$-dimensional affine subspace of $\mathbb{R}^n$. If $s < 1 - \frac{n-k}{2}$, then, for all $t > 0$,

$$||| (I - \Delta_x)^{\frac{s}{2}} u(\cdot,t) |||_{L^2(H)} \leq C||| F |||_{L^2(\mathbb{R}^n)},$$

with the optimal constant given by

$$C_{s,n-k,t} = ||| (I - \Delta_x)^{\frac{s}{2}} K_t |||_{L^2(\mathbb{R}^n)},$$

$$\text{and}$$

$$(4\pi t)^{\frac{n-k}{4}}.$$
**Proof.** Since \( \left| \left( (I - \triangle_x)^s K_t \right)(\xi) \right| \lesssim (1 + |\xi|)^{s-1} \), after putting \( H \) into the same form \( \{(x',0) \in \mathbb{R}^n \mid x' \in \mathbb{R}^k \} \) as used above, we see that, for \( s < 1 \),

\[
\left\| (I - \triangle_x)^s K_t \right\|_{L^2_{\infty}} < \infty \quad \text{iff} \quad \int_{\mathbb{R}^{n-k}} (1 + |\xi''|^2)^{2s-2} d\xi'' < \infty,
\]

which holds off \( 2s - 2 < -(n - k) \), i.e., \( s < 1 - \frac{n-k}{2} \).

**4.4. The wave equation: product of solutions.** Consider solutions \( u \) and \( v \) on \( \mathbb{R}^{3+1} \) to

\[
u_{tt} = \triangle u, \quad u(x,0) = 0, \quad u_t(x,0) = f(x), \quad \text{and} \quad v_{tt} = \triangle v, \quad v(x,0) = 0, \quad v_t(x,0) = g(x).
\]

We have the following estimate for products of solutions to these; see [11] for similar results with an additional average in time.

**Theorem 4.7.** With the notation above, for any fixed \( t \),

\[
(u(\cdot,t)v(\cdot,t))_t \lesssim C\|u(\cdot,t)v(\cdot,t))\|_{L^2_{\infty}(\mathbb{R}^3)} \leq C\|f\|_{L^2(\mathbb{R}^3)} \cdot \|g\|_{L^2(\mathbb{R}^3)},
\]

\[
||u(\cdot,t)v(\cdot,t)||_{L^p(\mathbb{R}^3)} \leq C_{p,t}\|f\|_{L^p(\mathbb{R}^3)} \cdot \|g\|_{L^p(\mathbb{R}^3)} \text{ for } \frac{5}{3} < p \leq 2.
\]

**Remark 4.8.** One then obtains \( L^p \times L^p \rightarrow L^q \) estimates for \( (\frac{1}{p}, \frac{1}{q}) \) in a nonclosed triangle with vertices at \( (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{3}) \) and \( (\frac{2}{3}, \frac{2}{3}) \).

To prove Thm. 4.7, observe that

\[
u(x,t) = \frac{t}{4\pi} \int_{S^2} f(x-ty)d\sigma(y),
\]

so that estimate (4.9) follows immediately from Cor. 2.6. On the other hand, (4.8) follows from Thm. 4.3. In fact, since \( |\hat{\sigma}(t\xi)| \leq C(1 + t|\xi|)^{-1} \), it follows that

\[
||u(\cdot,t)||_{L^2_{\infty}(\mathbb{R}^3)} \leq C\|f\|_{L^2(\mathbb{R}^3)} \quad \text{and} \quad ||v(\cdot,t)||_{L^2_{\infty}(\mathbb{R}^3)} \leq C\|g\|_{L^2(\mathbb{R}^3)},
\]

with \( C \) independent of \( t \). Let \( r = s = 1 \), so that \( \gamma = 2 - \frac{4}{3} = \frac{2}{3} \). Thm. 4.3, followed by (4.10), implies

\[
||u(\cdot,t)v(\cdot,t)||_{L^2(\mathbb{R}^3)} \leq C\|u\|_{L^2(\mathbb{R}^3)}\|v\|_{L^2(\mathbb{R}^3)} \leq C\|f\|_{L^2(\mathbb{R}^3)} \cdot \|g\|_{L^2(\mathbb{R}^3)}.
\]

One then applies Sobolev embedding in \( \mathbb{R}^3 \) to obtain the first inequality in (4.8).

**5. Bilinear oscillatory integral operators**

Let \( \phi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \) be a smooth phase function and \( \psi \in C_0^\infty(\mathbb{R}^{2d}) \) a fixed amplitude. Define

\[
T_\phi^\lambda F(x,y) = \int \int F(x-u,y-v)e^{2\pi i \lambda \phi(u,v)} \psi(u,v)dudv.
\]

If \( \phi \) is nondegenerate, i.e., the determinant of the Hessian matrix of \( \phi \) does not vanish on the support of \( \psi \), then it is a result of Hörmander that

\[
T_\phi^\lambda : L^2(\mathbb{R}^{2d}) \rightarrow L^2(\mathbb{R}^{2d}) \text{ with norm } \leq C\lambda^{-d}.
\]
See [24] and the references there for this and related results. What we describe above works equally well in any $\mathbb{R}^n$, but this formulation leads naturally to the context of trace inequalities, which we now present.

**Definition 5.1.** The bilinear surface associated with $\phi$ is

$S_\phi = \{(u, v, z, \delta_z \phi(u, v)) : (u, v) \in U; z \in Z\} \subset \mathbb{R}^{3d+1},$

where $U := \text{supp}(\psi)$, $Z = \pi_1(U) - \pi_1(U)$, with $\pi_1(u, v) = u$, and

$$\delta_z \phi(u, v) = \phi(u, v) - \phi(u - z, v - z).$$

Let $\sigma_\phi$ denote the induced surface measure on $S_\phi$. We shall see below that Thm. 1.3 implies the following result.

**Theorem 5.2.** Let $H$ be a $d$-dimensional plane in $\mathbb{R}^{2d}$. Then

$$|| (T^\lambda_{\phi} F)(x) ||_{L^2(H)} \leq \sqrt{\sup_{\xi \in \mathbb{R}^d} |\hat{\sigma}_\phi(\xi, \lambda)|} \cdot ||F||_{L^2(\mathbb{R}^{2d})}.$$

We now apply our results to bilinear oscillatory integral operators, defined by

$$M^\lambda_{\phi}(f, g)(x) = \int \int f(x - u)g(x - v)e^{2\pi i \lambda \phi(u, v)}\psi(u, v)dudv.$$

We have the following bound on the $L^2$-norm of $M^\lambda_{\phi}$, which follows at once from Thm. 5.2.

**Corollary 5.3.** Let $M^\lambda_{\phi}$ be as above. Then

$$||M^\lambda_{\phi}(f, g)||_2 \leq \sqrt{\sup_{\xi \in \mathbb{R}^d} |\hat{\sigma}_\phi(\xi, \lambda)|} \cdot ||f||_2 ||g||_2.$$

**Corollary 5.4.** Suppose that the determinant of the Hessian matrix of $\phi$ does not vanish on the support of $\psi$. Then

$$||M^\lambda_{\phi}(f, g)||_2 \leq C\lambda^{-\frac{d}{2}} ||f||_2 ||g||_2.$$

**Remark 5.5.** The power of $\lambda$ in the estimate (5.3) cannot, in general, be improved. This can be checked by taking $\phi(u, v) = |u|^2 + |v|^2$ and running a simple scaling argument.

**Example 5.6.** Let $\phi(u, v) = u \cdot v$. Then the conditions of Cor. 5.4 are satisfied and the conclusion holds.
Proof of Theorem 5.2. By Thm. 1.3, we need to compute the square root of
\[
\sup_{\xi \in \mathbb{R}^d} \left| \int \int e^{-2\pi i \left( \frac{\xi}{\sqrt{2}} \cdot u + \frac{\lambda}{\sqrt{2}} \cdot v + \lambda \phi(u, v) \right)} \psi(u, v) \, du \, dv \right|^2 \, d\eta
\]
\[
= \sup_{\xi \in \mathbb{R}^d} \int \int \ldots \int e^{-2\pi i \left( \frac{\xi}{\sqrt{2}} \cdot (u-u') + \frac{\lambda}{\sqrt{2}} \cdot (v-v') + \lambda \phi(u, v) - \phi(u', v') \right)} \psi(u, v) \psi(u', v') \, du \, dv \, du' \, dv' \, d\eta
\]
\[
= \sup_{\xi \in \mathbb{R}^d} \int \int \ldots \int e^{-2\pi i \left( \frac{\xi}{\sqrt{2}} \cdot (u-u') + \frac{\lambda}{\sqrt{2}} \cdot (v-v') + \lambda \phi(u, v) - \phi(u', v') \right)} \psi(u, v) \psi(u', v') \delta_0((u-u') - (v-v')) \, du \, dv \, du' \, dv',
\]
since the integral with respect to \( \eta \) in the previous line yields \( \delta_0((u-u') - (v-v')) \), where \( \delta_0 \) denotes the \( \delta \)-distribution at the origin. Making the change of variables \( z = u - u', u = u, v = v \), we obtain
\[
\sup_{\xi \in \mathbb{R}^d} \int \int e^{-2\pi i (\xi \cdot \lambda \delta_z \phi(u, v))} \psi(u, v) \psi(u - z, v - z) \, du \, dv \, dz = \sup_{\xi \in \mathbb{R}^d} \tilde{\sigma}_\phi(\xi, \lambda),
\]
and the proof is complete.

Proof of Corollary 5.4. By Theorem 5.2, we must estimate
\[
\sup_{\xi \in \mathbb{R}^d} \int \int \int e^{-2\pi i (z \cdot \lambda \delta_z \phi(u, v))} \psi(u, v) \psi(u - z, v - z) \, du \, dv \, dz,
\]
and by the method of stationary phase (see, e.g., [24]) our claim would follow if we could show that the rank of the \( 3d \times 3d \) Hessian matrix of \( \Phi(u, v, z) = \delta_z \phi(u, v) \) has rank \( \geq 2d \). Since the argument is local, we can use the Morse lemma and establish the result with
\[
\phi(u, v) = u_1^2 \pm u_2^2 \pm \cdots \pm u_d^2 \pm v_1^2 \pm v_2^2 \pm \cdots \pm v_d^2,
\]
for which
\[
\Phi(u, v, z) = \sum_{i=1}^d \pm \left( (z_i - u_i)^2 - u_i^2 \right) + \sum_{j=1}^d \pm \left( (z_j - v_j)^2 - v_j^2 \right),
\]
and Cor. 5.4 follows.

6. Further thoughts and open problems

The key idea of this paper is that estimates for restrictions of convolutions (Theorem 1.3) provide a mechanism to obtain multilinear operator bounds, in the same way as the classical Young’s inequality yields estimates for linear operators. This approach does not rely on positivity, which was a drawback of the technique in [12]. Despite the applications we have presented in this paper, serious issues remain, especially regarding the bounds for bilinear operators arising in the geometric context. In [12], \( L^2 \times L^2 \to L^1 \) bounds were established for bilinear operators, with applications to Erdős/Falconer problems in geometry. These bounds relied heavily on the positivity of the kernel, which created significant obstacles, but the bounds were established in the arguably more natural range of exponents. While the problem of positivity was largely resolved in the current paper, the range of boundedness is centered around \( L^2 \times L^2 \to L^2 \). It would be extremely interesting...
to reconcile the methods in this paper and [12] and come up with a unified set of bounds that do not rely on positivity of the kernel. We now state some concrete open problems.

**Problem 6.1.** It is proved in [12] that if $B(f, g)$ is defined as above, then for $1 \leq r \leq 2$ and $K$ a non-negative finite measure,

$$
||B(f, g)||_{L^r(\mathbb{R}^d)} \leq ||f||_{L^2(\mathbb{R}^d)} \cdot ||g||_{L^2(\mathbb{R}^d)} \cdot \left( \int \left| \hat{K}(\xi, -\xi) \right|^{r'} \, d\eta \right)^{\frac{1}{r'}}.
$$

On the other hand, Cor. 1.7 above shows that, if $2 \leq r \leq \infty$,

$$
||B(f, g)||_{L^r(\mathbb{R}^d)} \leq ||f||_{L^2(\mathbb{R}^d)} \cdot ||g||_{L^2(\mathbb{R}^d)} \cdot \left[ \int \left( \int \left| \hat{K} \left( \frac{\xi - \eta}{2}, \frac{\xi + \eta}{2} \right) \right|^2 \, d\eta \right]^{\frac{1}{r'}} \, d\xi \right]^\frac{r'}{r}. \tag{6.2}
$$

The two estimates agree at $r = 2$ and the positivity of $K$ is clearly irrelevant there. The question we ask is whether it is possible to reconcile (6.1) and (6.2) without assuming that $K$ is positive.

Our second problem attempts to further address the issue raised in Cor. 1.8.

**Problem 6.2.** It follows from (1.10) that

$$
\left( \int_{\mathbb{R}^d} |\hat{F}(x, x)|^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} \left| F \left( \frac{x - y}{2}, \frac{x + y}{2} \right) \right|^2 \, dy \right] \, dx \right)^{\frac{1}{2}}. \tag{1.10}
$$

At least in two dimensions, it would be interesting to generalize this estimate to a universal $L^2$-restriction theorem where the left hand side is

$$
\left( \int |\hat{F}(x, \phi(x))|^2 \, dx \right)^{\frac{1}{2}},
$$

with $\phi$ a suitably regular function of $x$, and the right hand side is a mixed norm depending on $\phi$. In the special case when $\phi(x) = x^2$, say, the right hand side should be comparable to the $||F||_{L^6(\mathbb{R}^2)}$, consistent with the Stein-Tomas restriction theorem.

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