Hom-Lie Superalgebras and Hom-Lie admissible Superalgebras

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Abstract

The purpose of this paper is to study Hom-Lie superalgebras, that is a superspace with a bracket for which the superJacobi identity is twisted by a homomorphism. This class is a particular case of \( \Gamma \)-graded quasi-Lie algebras introduced by Larsson and Silvestrov. In this paper, we characterize Hom-Lie admissible superalgebras and provide a construction theorem from which we derive a one parameter family of Hom-Lie superalgebras deforming the orthosymplectic Lie superalgebra. Also, we prove a \( \mathbb{Z}_2 \)-graded version of a Hartwig-Larsson-Silvestrov Theorem which leads us to a construction of a \( q \)-deformed Witt superalgebra.

Key words: Hom-Lie superalgebra, Hom-Associative superalgebra, Hom-Lie admissible superalgebra, Lie admissible superalgebra, \( q \)-Witt superalgebra

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Introduction

The motivations to study Hom-Lie structures are related to physics and to deformations of Lie algebras, in particular Lie algebras of vector fields. The paradigmatic examples are \( q \)-deformations of Witt and Virasoro algebras constructed in pioneering works (see [4, 5, 6, 16, 11]).

Larsson and Silvestrov introduced, in [15], the class of \( \Gamma \)-graded quasi-Lie algebras which incorporates as special case \( \Gamma \)-graded hom-Lie algebras and \( \Gamma \)-graded quasi-hom-Lie algebras (qhl-algebras). As a particular case, it contains Lie superalgebras, Lie algebras as well as Hom-Lie superalgebras and Hom-Lie algebras. The main feature of quasi-Lie algebras, quasi-hom-Lie algebras and Hom-Lie algebras is that the skew-symmetry and the Jacobi identity are twisted by several deforming twisting maps and also in quasi-Lie and quasi-hom-Lie algebras the Jacobi identity in general contains 6 twisted triple bracket terms. On the other hand these algebras correspond to new single and multi-parameter families of algebras obtained using twisted derivations and constituting deformations and quasi-deformations of universal enveloping algebras of Lie and color Lie algebras and of algebras of vector-fields.

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The Hom-Lie algebras were discussed intensively in \([10, 12, 13, 14]\) while the graded case was mentioned in \([15]\). Hom-associative algebras were introduced in \([17]\), where it is shown that the commutator bracket of a Hom-associative algebra gives rise to a Hom-Lie algebra and where a classification of Hom-Lie admissible algebras is established. Given a Hom-Lie algebra, there is a universal enveloping Hom-associative algebra (see \([22]\)). Dualizing Hom-associative algebras, one can define Hom-coassociative coalgebras, Hom-bialgebras and Hom-Hopf algebras (see \([18, 20]\)). It is shown in \([24]\) that the universal enveloping Hom-associative algebra carries a structure of Hom-bialgebra. See also \([2, 3, 8, 19, 23, 25, 26]\) for other works on twisted algebraic structures.

This paper focuses on \(\mathbb{Z}_2\)-graded Hom-algebras. Mainly, we introduce and characterize Hom-Lie admissible superalgebras and prove a \(\mathbb{Z}_2\)-graded version of a Hartwig-Larsson-Silvestrov Theorem which leads us to construct a \(q\)-deformed Witt superalgebra. Also, we provide a construction theorem from which we derive a one parameter family of Hom-Lie superalgebras deforming the orthosymplectic Lie superalgebra \(osp(1,2)\).

In Section 1 of this paper, we summarize the main Hom-algebra structures and recall the framework of quasi-hom-Lie algebras. In Section 2, we introduce Hom-associative superalgebras and Hom-Lie superalgebras. We provide a different way for constructing Hom-Lie superalgebras by extending the fundamental construction of Lie superalgebras from associative superalgebras via supercommutator bracket. We show that the supercommutator bracket defined using the multiplication in a Hom-associative superalgebra leads naturally to a Hom-Lie superalgebra. We also extend a Yau’s theorem \([23]\) on Hom-Lie algebras to Hom-Lie superalgebras. We show that starting with an ordinary Lie superalgebra and a superalgebra endomorphism, we may construct a Hom-Lie superalgebra. Moreover, we construct a one parameter family of Hom-Lie superalgebras deforming the orthosymplectic superalgebra \(osp(1,2)\). In Section 3, we introduce Hom-Lie admissible superalgebras and more general \(G\)-Hom-associative superalgebras, where \(G\) is a subgroup of the permutation group \(S_3\). We show that Hom-Lie admissible superalgebras are \(G\)-Hom-associative superalgebras. As a corollary, we obtain a classification of Lie admissible superalgebras. These results generalize the classifications obtained in the ungraded case in \([17]\), and in classical Lie case in \([8]\). Section 4 is devoted to \(\mathbb{Z}_2\)-graded version of a Hartwig-Larsson-Silvestrov Theorem (see \([10]\), Theorem 5). A more general graded version of this theorem was mentioned in \([15]\). We aim to state it explicitly and prove it in the superalgebra case. This leads us, in the last Section, to construct \(q\)-deformed Witt superalgebras. We should point out that all these results could naturally be extended to color Lie algebras.

1. Hom-algebras and graded quasi-hom-Lie algebras

Throughout this paper \(\mathbb{K}\) denotes a field of characteristic 0. We summarize in the following the ungraded definitions of Hom-associative, Hom-Leibniz and Hom-Lie algebras (see \([17]\)). Also, we recall the definition of quasi-Lie algebras (see \([13]\)) which border Hom-Lie algebras, Hom-Lie superalgebras and color Lie algebras, as well as quasi-hom-Lie algebras appearing naturally in the study of \(\sigma\)-derivations.

**Definition 1.1.** A Hom-associative algebra is a triple \((V, \mu, \alpha)\) consisting of a linear space \(V\) over \(\mathbb{K}\), a bilinear map \(\mu : V \times V \to V\) and a homomorphism \(\alpha : V \to V\) satisfying

\[
\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z))
\]
A class of quasi-Leibniz algebras was introduced in [13] in connection to general quasi-Lie algebras following the standard Loday’s conventions for Leibniz algebras (i.e. right Loday algebras). In the context of the subclass of Hom-Lie algebras one gets a class of Hom-Leibniz algebras.

**Definition 1.2.** A Hom-Leibniz algebra is a triple \((V, [\cdot, \cdot], \alpha)\) consisting of a linear space \(V\), bilinear map \([\cdot, \cdot] : V \times V \to V\) and a homomorphism \(\alpha : V \to V\) satisfying

\[
[x, y], \alpha(z)] = [x, [z, \alpha(y)] + [\alpha(x), [y, z]]
\] (1.1)

In terms of the (right) adjoint homomorphisms \(Ad_Y : V \to V\) defined by \(Ad_Y(x) = [x, Y]\), the identity (1.1) can be written as

\[
Ad_{\alpha(z)}([x, y]) = [Ad_{\alpha(y)}(x), \alpha(y)] + [\alpha(x), Ad_{\alpha(z)}(y)]
\] (1.2)

or in pure operator form

\[
Ad_{\alpha(z)} \circ Ad_Y = Ad_{\alpha(y)} \circ Ad_z + Ad_{Ad_{\alpha(z)}(y)} \circ \alpha
\] (1.3)

The Hom-Lie algebras were initially introduced by Hartwig, Larson and Silvestrov in [10] motivated initially by examples of deformed Lie algebras coming from twisted discretizations of vector fields.

**Definition 1.3.** A Hom-Lie algebra is a triple \((V, [\cdot, \cdot], \alpha)\) consisting of a linear space \(V\), bilinear map \([\cdot, \cdot] : V \times V \to V\) and a linear space homomorphism \(\alpha : V \to V\) satisfying

\[
[x, y] = -[y, x] \quad \text{(skew-symmetry)}
\] (1.4)

\[
\bigcirc_{x,y,z} [\alpha(x), [y, z]] = 0 \quad \text{(Hom-Jacobi identity)}
\] (1.5)

for all \(x, y, z\) from \(V\), where \(\bigcirc_{x,y,z}\) denotes summation over the cyclic permutation on \(x, y, z\).

Using the skew-symmetry, one may write the Hom-Jacobi identity in the form (1.2). Thus, if a Hom-Leibniz algebra is skewsymmetric, then it is a Hom-Lie algebra.

We recall in the following the definitions of quasi-Lie and quasi-hom-Lie algebras.

**Definition 1.4 ([12]).** A quasi-Lie algebra is a tuple \((V, [\cdot, \cdot], \alpha, \beta, \omega, \theta)\) where

- \(V\) is a linear space over \(K\),
- \([\cdot, \cdot] : V \times V \to V\) is a bilinear map that is called a product or bracket in \(V\);
- \(\alpha, \beta : V \to V\) are linear maps,
- \(\omega : D_\omega \to L_K(V)\) and \(\theta : D_\theta \to L_K(V)\) are maps with domains of definition \(D_\omega, D_\theta \subseteq V \times V\), and where \(L_K(V)\) denotes the set of linear maps on \(V\) over \(K\),

such that the following conditions hold:

- (\(\omega\)-symmetry) The product satisfies a generalized skew-symmetry condition
  \[
  [x, y]_V = \omega(x, y)[y, x]_V, \quad \text{for all } (x, y) \in D_\omega;
  \]
Remark 2.3. Morphisms of Hom-associative superalgebras are defined similarly.

The class of Hom-Lie superalgebras is obtained by specifying superalgebras from associative superalgebras via supercommutator bracket.

The class of Quasi-Lie algebras incorporates as special cases Hom-Lie algebras and Hom-associative algebras by extending the fundamental construction of Lie superalgebras (see [10]) and related deformations of infinite-dimensional and finite-dimensional Lie algebras. To get the class of qhl-algebras one specifies \( \theta = \omega \) and restricts attention to maps \( \alpha \) and \( \beta \) satisfying the twisting condition \( [\alpha(x), \alpha(y)]_V = \beta \circ \alpha(x, y)_V \). Specifying this further by taking \( D_\omega = V \times V \), \( \beta = \text{id} \) and \( \omega = -\text{id} \), one gets the class of Hom-Lie algebras including Lie algebras when \( \alpha = \text{id} \).

The class of Hom-Lie superalgebras is obtained by specifying \( \beta = 0 \), \( \omega(x, y) = (-1)^{|x||y|} \) and \( \theta(x, y) = (-1)^{|x||y|} \), where \( x, y \) are elements of \( \mathbb{Z}_2 \)-graded space \( V \) and \( |x|, |y| \) their parities.

2. Hom-associative Superalgebras and Hom-Lie Superalgebras

In this Section, we focus on Hom-associative and Hom-Lie superalgebras and provide a different way for constructing Hom-Lie superalgebras by extending the fundamental construction of Lie superalgebras from associative superalgebras via supercommutator bracket.

Now, let \( V \) be a linear superspace over \( K \) that is a \( \mathbb{Z}_2 \)-graded linear space with a direct sum \( V = V_0 \oplus V_1 \). The element of \( V_j, j = \{0, 1\} \), are said to be homogenous and of parity \( j \). The parity of a homogeneous element \( x \) is denoted by \( |x| \).

Definition 2.1. A Hom-associative superalgebra is a triple \((V, \mu, \alpha)\) consisting of a superspace \( V \), an even bilinear map \( \mu : V \times V \to V \) and an even homomorphism \( \alpha : V \to V \) satisfying

\[
\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z))
\]

Definition 2.2. A Hom-Lie superalgebra is a triple \((V, [\cdot, \cdot], \alpha)\) consisting of a superspace \( V \), an even bilinear map \([\cdot, \cdot] : V \times V \to V \) and an even superspace homomorphism \( \alpha : V \to V \) satisfying

\[
[x, y] = -(−1)^{|x||y|} [y, x] \tag{2.1}
\]

\[
(−1)^{|x||z|} [\alpha(x), [y, z]] + (−1)^{|z||y|} [\alpha(z), [x, y]] + (−1)^{|y||x|} [\alpha(y), [z, x]] = 0 \tag{2.2}
\]

for all homogeneous element \( x, y, z \) in \( V \).

Let \((V, [\cdot, \cdot], \alpha)\) and \((V', [\cdot', \cdot'], \alpha')\) be two Hom-Lie superalgebras. An even homomorphism \( f : V \to V' \) is said to be a morphism of Hom-Lie superalgebras if

\[
[f(x), f(y)]' = f([x, y]) \quad \forall x, y \in V \tag{2.3}
\]

\[
f \circ \alpha = \alpha' \circ f \tag{2.4}
\]

Morphisms of Hom-associative superalgebras are defined similarly.

Remark 2.3. We recover the classical Lie superalgebra and associative superalgebra when \( \alpha = \text{id} \). The Hom-Lie algebra and Hom-associative algebras are obtained when the part of parity one is trivial.
Example 2.4 (2-dimensional abelian Hom-Lie superalgebra). Every bilinear map \( \mu \) on a 2-dimensional linear superspace \( V = V_0 \oplus V_1 \), where \( V_0 \) is generated by \( x \) and \( V_1 \) is generated by \( y \) and such that \( [x, y] = 0 \) defines a Hom-Lie superalgebra for any homomorphism \( \alpha \) of superalgebra. Indeed, the graded Hom-Jacobi identity is satisfied for any triple \( (x, y, z) \).

Example 2.5 (Affine Hom-Lie superalgebra). Let \( V = V_0 \oplus V_1 \) be a 3-dimensional superspace where \( V_0 \) is generated by \( e_1, e_2 \) and \( V_1 \) is generated by \( e_3 \). The triple \( (V, [,], \alpha) \) is a Hom-Lie superalgebra defined by \( [e_1, e_2] = e_1, [e_1, e_3] = e_2, e_3 = [e_3, e_3] = 0 \) and \( \alpha \) is any homomorphism.

In the following, we show that the supercommutator bracket defined using the multiplication in a Hom-associative algebra leads naturally to Hom-Lie superalgebra.

Proposition 2.6. Given any Hom-associative superalgebra \((V, \mu, \alpha)\) one can define the supercommutator on homogeneous elements by

\[
[x, y] = \mu(x, y) - (-1)^{|x||y|} \mu(y, x)
\]

and then extending by linearity to all elements. Then \((V, [,], \alpha)\) is a Hom-Lie superalgebra.

Proof. The bracket is obviously supersymmetric and with a direct computation we have

\[
(-1)^{|x||z|}[\alpha(x), [y, z]] + (-1)^{|z||y|}[\alpha(z), [x, y]] + (-1)^{|y||z|}[\alpha(y), [z, x]] = 0
\]

\[
(-1)^{|x||z|}\mu(\alpha(x), \mu(y, z)) - (-1)^{|x||z|+|y||z|}\mu(\alpha(x), \mu(z, y)) - (-1)^{|x||y|}\mu(\mu(y, z), \alpha(x)) + (-1)^{|x||y|+|y||z|}\mu(\mu(z, y), \alpha(x)) + (-1)^{|y||x|+|z||x|}\mu(\mu(z, y), \alpha(x)) = 0
\]

We extend in the following the Yau’s theorem (see [23]) into the graded case. The following theorem gives a way to construct Hom-Lie superalgebras, starting from a Lie superalgebra and an even superalgebra endomorphism.

Theorem 2.7. Let \((V, [,])\) be a Lie superalgebra and \( \alpha : V \to V \) be an even Lie superalgebra endomorphism. Then \((V, [,], \alpha)\), where \([x, y]_{\alpha} = \alpha([x, y])\), is a Hom-Lie superalgebra.

Moreover, suppose that \((V', [,])\) is another Lie superalgebra and \( \alpha' : V' \to V' \) is a Lie superalgebra endomorphism. If \( f : V \to V' \) is a Lie superalgebra morphism that satisfies \( f \circ \alpha = \alpha' \circ f \) then

\[
f : (V, [,], \alpha) \longrightarrow (V', [,], \alpha')
\]

is a morphism of Hom-Lie superalgebras.

Proof. We show that \((V, [,], \alpha)\) satisfies the Hom-super-Jacobi identity [22]. Indeed

\[
\circ_{x,y,z}(\alpha(x), [y,z]) = \circ_{x,y,z}(\alpha(x), [y,z]) = \circ_{x,y,z}(\alpha(x), [y,z]) = 0
\]
The second assertion follows from
\[f \circ [\cdot, \cdot]_\alpha = f \circ \alpha' \circ [\cdot, \cdot] = \alpha' \circ f \circ [\cdot, \cdot] = \alpha' \circ [\cdot, \cdot]' \circ f = [\cdot, \cdot]'_\alpha \circ f.\]

\(\square\)

**Example 2.8.** We construct an example of Hom-Lie superalgebra, which is not a Lie superalgebra starting from the orthosymplectic Lie superalgebra. We consider in the sequel the matrix realization of this Lie superalgebra.

Let \(osp(1, 2) = V_0 \oplus V_1\) be the Lie superalgebra where \(V_0\) is generated by:
\[
H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]

and \(V_1\) is generated by:
\[
F = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.
\]

The defining relations (we give only the ones with non zero values in the right hand side) are
\[
[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H,
\]
\[
[Y, G] = F, \quad [X, F] = G, \quad [H, F] = -F, \quad [H, G] = G,
\]
\[
[G, F] = H, \quad [G, G] = -2X, \quad [F, F] = 2Y.
\]

Let \(\lambda \in \mathbb{R}^*\), we consider the linear map \(\alpha_\lambda : osp(1, 2) \rightarrow osp(1, 2)\) defined by:
\[
\alpha_\lambda(X) = \lambda^2 X, \quad \alpha_\lambda(Y) = \frac{1}{\lambda^2} Y, \quad \alpha_\lambda(H) = H, \quad \alpha_\lambda(F) = \frac{1}{\lambda} F, \quad \alpha_\lambda(G) = \lambda G.
\]

We provide a family of Hom-Lie superalgebras \(osp(1, 2)_\lambda = (osp(1, 2), [\cdot, \cdot]_{\alpha_\lambda}, \alpha_\lambda)\) where the Hom-Lie superalgebra bracket \([\cdot, \cdot]_{\alpha_\lambda}\) on the basis elements is given, for \(\lambda \neq 0\), by:
\[
[H, X]_{\alpha_\lambda} = 2\lambda^2 X, \quad [H, Y]_{\alpha_\lambda} = -\frac{2}{\lambda^2} Y, \quad [X, Y]_{\alpha_\lambda} = H,
\]
\[
[Y, G]_{\alpha_\lambda} = \frac{1}{\lambda} F, \quad [X, F]_{\alpha_\lambda} = \lambda G, \quad [H, F]_{\alpha_\lambda} = -\frac{1}{\lambda} F, \quad [H, G]_{\alpha_\lambda} = \lambda G,
\]
\[
[G, F]_{\alpha_\lambda} = H, \quad [G, G]_{\alpha_\lambda} = -2\lambda^2 X, \quad [F, F]_{\alpha_\lambda} = \frac{2}{\lambda^2} Y.
\]

These Hom-Lie superalgebras are not Lie superalgebras for \(\lambda \neq 1\). Indeed, the left hand side of the superJacobi identity (2.2), for \(\alpha = id\), leads to
\[
[X, [Y, H] - [H, [X, Y]] + [Y, [H, X]] = \frac{2(1 - \lambda^4)}{\lambda^2} Y,
\]
and also
\[
[H, [F, F] - [F, [H, F]] + [F, [H, F]] = \frac{4(\lambda - 1)}{\lambda^4} Y.
\]

Then, they do not vanish for \(\lambda \neq 1\).
3. Hom-Lie-Admissible Superalgebras

We introduce and discuss in this Section the Hom-Lie admissible superalgebras. The Lie admissible algebras were introduced by A. A. Albert in 1948. We extend to graded algebras the concept of Hom-Lie-Admissible algebras studied in [17]. This study borders also an extension to graded case of the Lie-admissible algebras discussed in [9].

Let $A = (V, \mu, \alpha)$ be a Hom-superalgebra, that is a superspace $V$ with an even bilinear map $\mu$ and an even linear map $\alpha$ satisfying eventually identities. Let $[x, y] = \mu(x, y) - (-1)^{|x||y|}\mu(y, x)$, for all homogeneous element $x, y \in V$, be the associated supercommutator. We call $\alpha$-associator of a multiplication $\mu$ a trilinear map $\text{as}_\alpha$ on $V$ defined for $x_1, x_2, x_3 \in V$ by

$$\text{as}_\alpha(x_1, x_2, x_3) = \mu(\alpha(x_1), \mu(x_2, x_3)) - \mu(\mu(x_1, x_2), \alpha(x_3)).$$

**Definition 3.1.** Let $A = (V, \mu, \alpha)$ be a Hom-superalgebra on $V$ defined by an even multiplication $\mu$ and an even homomorphism $\alpha$. Then $A$ is said to be Hom-Lie admissible superalgebra over $V$ if the bracket defined for all homogeneous element $x, y \in V$ by

$$[x, y] = \mu(x, y) - (-1)^{|x||y|}\mu(y, x)$$

satisfies the Hom-superJacobi identity 223.

**Remark 3.2.** Since the supercommutator bracket (3.1) is always supersymmetric, this makes any Hom-Lie admissible superalgebra into a Hom-Lie superalgebra.

**Remark 3.3.** As mentioned in in the proposition 2.0, any associative superalgebra is a Hom-Lie admissible superalgebra.

**Lemma 3.4.** Let $A = (V, \mu, \alpha)$ be a Hom-superalgebra and $[-, -]$ be the associated supercommutator then

\[
\text{as}_\alpha(x, y, z) = \mu(\alpha(x), \text{as}_\alpha(y, z)) + (-1)^{|x||z|}\mu(y, \text{as}_\alpha(x, z)) + (-1)^{|z||x|}\mu(z, \text{as}_\alpha(x, y)) - (-1)^{|y||z|}\mu(y, \text{as}_\alpha(x, z)) - (-1)^{|x||y|}\mu(x, \text{as}_\alpha(y, z)) - (-1)^{|x||z|}\mu(x, \text{as}_\alpha(y, z)) - (-1)^{|y||z|}\mu(y, \text{as}_\alpha(x, z))
\]

**Proof.** By straightforward calculation, we have

\[
\text{as}_\alpha(x, y, z) = \mu(\alpha(x), \text{as}_\alpha(y, z)) + (-1)^{|x||z|}\mu(y, \text{as}_\alpha(x, z)) + (-1)^{|z||x|}\mu(z, \text{as}_\alpha(x, y)) - (-1)^{|y||z|}\mu(y, \text{as}_\alpha(x, z)) - (-1)^{|x||y|}\mu(x, \text{as}_\alpha(y, z)) - (-1)^{|x||z|}\mu(x, \text{as}_\alpha(y, z)) - (-1)^{|y||z|}\mu(y, \text{as}_\alpha(x, z))
\]
for any homogenous elements \( x, y, z \).

Proof. We have

\[
S(x, y, z) := (-1)^{|x||y|+|x||z|+|y||z|}a_\alpha(x, y, z) + (-1)^{|y||z|}a_\alpha(y, z, x) + (-1)^{|z||x|}a_\alpha(z, x, y).
\]

Then, we have the following properties.

**Proposition 3.6.** Let \( \mathcal{A} = (V, \mu, \alpha) \) be a Hom-superalgebra, then \( \mathcal{A} \) is a Hom-Lie admissible superalgebra if and only if it satisfies

\[
S(x, y, z) = (-1)^{|x||y|+|x||z|+|y||z|}S(x, z, y)
\]

for any homogenous elements \( x, y, z \in V \).

Proof. We have

\[
S(x, y, z) - (-1)^{|x||y|+|x||z|+|y||z|}S(x, z, y)
\]

\[
= (-1)^{|x||z|}a_\alpha(x, y, z) + (-1)^{|y||x|}a_\alpha(y, z, x) + (-1)^{|z||x|}a_\alpha(z, x, y)
\]

\[
- (-1)^{|x||y|+|x||z|+|y||z|}((-1)^{|x||y|}a_\alpha(x, z, y) + (-1)^{|z||x|}a_\alpha(z, y, x) + (-1)^{|y||z|}a_\alpha(y, x, z))
\]

\[
= (-1)^{|x||z|}a_\alpha(x, y, z) + (-1)^{|y||x|}a_\alpha(y, z, x) + (-1)^{|z||x|}a_\alpha(z, x, y)
\]

\[
- (-1)^{|x||y|+|x||z|+|y||z|}a_\alpha(x, z, y) - (-1)^{|x||y|+|x||z|+|y||z|}a_\alpha(z, y, x) - (-1)^{|x||y|+|x||z|+|y||z|}a_\alpha(y, x, z)
\]

\(
\Rightarrow \tag{3.3}
\)

Then the Hom-superJacobi identity (2.2) is satisfied if and only if the condition (3.3) holds. \( \square \)

In the following, we provide a classification of Hom-Lie admissible superalgebras using the symmetric group \( S_3 \). We extend to graded case the notion of \( G \)-Hom-associative algebras which was introduced in the classical ungraded Lie case in (3) and developed for the Hom-Lie case in (17).

Let \( S_3 \) be the permutation group generated by \( \sigma_1, \sigma_2 \). We extend a permutation \( \tau \in S_3 \) to a map \( \tau : V^3 \to V^3 \) defined for \( x_1, x_2, x_3 \in V \) by \( \tau(x_1, x_2, x_3) = (x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}) \). We keep for simplicity the same notation. In particular, \( \sigma_1(x_1, x_2, x_3) = (x_2, x_1, x_3) \) and \( \sigma_2(x_1, x_2, x_3) = (x_1, x_3, x_2) \).

We introduce a notion of a parity of transposition \( \sigma_i \) where \( i \in \{1, 2\} \), by setting

\[
|\sigma_i(x_1, x_2, x_3)| = |x_i||x_{i+1}|.
\]

We assume that the parity of the identity is 0 and for the composition \( \sigma_i\sigma_j \), it is defined by

\[
|\sigma_i\sigma_j(x_1, x_2, x_3)| = |\sigma_j(x_1, x_2, x_3)| + |\sigma_i(\sigma_j(x_1, x_2, x_3))|
\]

\[
= |\sigma_j(x_1, x_2, x_3)| + |\sigma_i(\sigma_j(1), \sigma_j(2), \sigma_j(3))|
\]

We define by induction the parity for any composition. For the elements id, \( \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_2\sigma_1\sigma_2 \)
of $S_3$, we obtain
\[
\begin{aligned}
| \text{id}(x_1, x_2, x_3) | &= 0, \\
| \sigma_1(x_1, x_2, x_3) | &= |x_1||x_2|, \\
| \sigma_2(x_1, x_2, x_3) | &= |x_2||x_3|, \\
| \sigma_1 \sigma_2(x_1, x_2, x_3) | &= |x_2||x_3| + |x_1||x_3|, \\
| \sigma_2 \sigma_1(x_1, x_2, x_3) | &= |x_1||x_2| + |x_1||x_3|, \\
| \sigma_2 \sigma_1 \sigma_2(x_1, x_2, x_3) | &= |x_2||x_3| + |x_1||x_3| + |x_1||x_2|.
\end{aligned}
\]

Now, we express the condition of Hom-Lie admissibility of a Hom-superalgebra using permutations.

**Lemma 3.7.** A Hom-superalgebra $A = (V, \mu, \alpha)$ is a Hom-Lie admissible superalgebra if the following condition holds
\[
\sum_{\tau \in S_3} (-1)^{\varepsilon(\tau)}(-1)^{|\tau(x_1, x_2, x_3)|} \alpha_{\tau} \circ \tau(x_1, x_2, x_3) = 0 \tag{3.4}
\]
where $x_i$ are in $V$, $(-1)^{\varepsilon(\tau)}$ is the signature of the permutation $\tau$ and $|\tau(x_1, x_2, x_3)|$ is the parity of $\tau$.

**Proof.** By straightforward calculation, the associated supercommutator bracket satisfies
\[
\circ_{x_1, x_2, x_3} (-1)^{|x_1||x_3|} [\alpha(x_1), [x_2, x_3]] = (-1)^{|x_1||x_3|} \sum_{\tau \in S_3} (-1)^{\varepsilon(\tau)}(-1)^{|\tau(x_1, x_2, x_3)|} \alpha_{\tau} \circ \tau(x_1, x_2, x_3).
\]

It turns out that, for the associated supercommutator of a Hom-superalgebra, the Hom-superJacobi identity \((2.2)\) is equivalent to
\[
\sum_{\tau \in S_3} (-1)^{\varepsilon(\tau)}(-1)^{|\tau(x_1, x_2, x_3)|} \alpha_{\tau} \circ \tau(x_1, x_2, x_3) = 0.
\]

We introduce the notion of $G$-Hom-associative superalgebra, where $G$ is a subgroup of the permutations group $S_3$.

**Definition 3.8.** Let $G$ be a subgroup of the permutations group $S_3$, a Hom-superalgebra on $V$ defined by the multiplication $\mu$ and a homomorphism $\alpha$ is said to be $G$-Hom-associative superalgebra if
\[
\sum_{\tau \in G} (-1)^{\varepsilon(\tau)}(-1)^{|\tau(x_1, x_2, x_3)|} \alpha_{\tau} \circ \tau(x_1, x_2, x_3) = 0 \tag{3.5}
\]
where $x_i$ are in $V$, $(-1)^{\varepsilon(\tau)}$ is the signature of the permutation and $|\tau(x_1, x_2, x_3)|$ is the parity of $\tau$ defined above.

The following result is a graded version of the results obtained in \((9, 17)\).

**Proposition 3.9.** Let $G$ be a subgroup of the permutations group $S_3$. Then any $G$-Hom-associative superalgebra is a Hom-Lie admissible superalgebra.
Proof. The supersymmetry follows straightway from the definition.

We have a subgroup $G$ in $S_3$. Take the set of conjugacy class $\{gG\}_{g \in I}$ where $I \subseteq G$, and for any $\tau_1, \tau_2 \in I, \sigma_1 \neq \tau_2 \Rightarrow \tau_1G \cap \tau_2G = \emptyset$. Then

$$\sum_{\tau \in S_3} (-1)^{\varepsilon(\tau)}(-1)^{|\tau(x_1,x_2,x_3)|}a_\alpha \circ \tau(x_1,x_2,x_3) =$$

$$\sum_{\tau_1 \in I} \sum_{\tau_2 \in \tau_1G} (-1)^{\varepsilon(\tau_2)}(-1)^{|\tau_2(x_1,x_2,x_3)|}a_\alpha \circ \tau_2(x_1,x_2,x_3) = 0$$

where $(x_1,x_2,x_3) \in V$, with $V$ the underlaying superspace of the $G$-Hom-associative superalgebra.

It follows that in particular, we have:

**Corollary 3.10.** Let $G$ be a subgroup of the permutations group $S_3$. Then any $G$-associative superalgebra is a Lie admissible superalgebra.

Now, we provide a classification of the Hom-Lie admissible superalgebras through $G$-Hom-associative superalgebras. The ungraded $G$-associative algebras in classical case was studied in [9], then extended to $G$-Hom-associative algebras in ([17]).

The subgroups of $S_3$ are

$$G_1 = \{Id\}, G_2 = \{Id, \sigma_1\}, G_3 = \{Id, \sigma_2\}, G_4 = \{Id, \sigma_2 \sigma_1\}, G_5 = A_3, G_6 = S_3,$$

where $A_3$ is the alternating group.

We obtain the following type of Hom-Lie admissible superalgebras.

- The $G_1$-Hom-associative superalgebras are the Hom-associative superalgebras defined above.
- The $G_2$-Hom-associative superalgebras satisfy the identity
  $$\mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) = (-1)^{|x||y|}(\mu(\alpha(y), \mu(x, z)) - \mu(y, x, \alpha(z)))$$
- The $G_3$-Hom-associative superalgebras satisfy the identity
  $$\mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) = (-1)^{|y||z|}(\mu(\alpha(x), \mu(z, y)) - \mu(x, z, \alpha(y)))$$
- The $G_4$-Hom-associative superalgebras satisfy the identity
  $$\mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) = (-1)^{|x||y|+|x||z|+|y||z|}(\mu(\alpha(z), \mu(y, x)) - \mu(z, y, \alpha(x)))$$
- The $G_5$-Hom-associative superalgebras satisfy the identity
  $$\mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) + (-1)^{|x||y|+|x||z|}(\mu(\alpha(y), \mu(z, x) + \mu(y, z), \alpha(x))) =$$
  $$-(-1)^{|x||z|+|y||z|}(\mu(\alpha(z), \mu(x, y)) + \mu(z, x, \alpha(y)))$$
- The $G_6$-Hom-associative superalgebras are the Hom-Lie admissible superalgebras.
Remark 3.11. Moreover, if in the previous identities we consider \( \alpha = \text{id} \), then we obtain a classification of Lie-admissible superalgebras.

Remark 3.12. One may call \( G_2 \)-Hom-associative (resp. \( G_2 \)-associative) superalgebras Hom-Vinberg superalgebras (resp. Vinberg superalgebras) and \( G_3 \)-Hom-associative (resp. \( G_3 \)-associative) superalgebras Hom-pre-Lie superalgebras (resp. pre-Lie superalgebras). Notice that a Hom-pre-Lie superalgebra is the opposite algebra of a Hom-Vinberg superalgebra. Therefore, they actually form a same class.

4. \( Z_2 \)-Graded Hartwig-Larsson-Silvestrov theorem

In this Section, we describe and prove the Hartwig-Larsson-Silvestrov theorem (see [10], theorem 5) in the case of \( Z_2 \)-graded algebra. A more general graded version of this theorem was mentioned in [12]. We aim to consider it deeply for superalgebras case.

Let \( \mathcal{A} = A_0 \oplus A_1 \) be an associative superalgebra. We assume that \( \mathcal{A} \) is supercommutative, that is for homogeneous elements \( a, b \), the identity \( ab = (-1)^{|a||b|}ba \) holds. For example, \( A_0 = \mathbb{C}[t, t^{-1}] \) and \( A_1 = \theta \cdot A_0 \) where \( \theta \) is the Grassman variable (\( \theta^2 = 0 \)). Let \( \sigma \) be an even superalgebra endomorphism of \( \mathcal{A} \). Then, \( \mathcal{A} \) is a bimodule over itself, the left (resp. right) action is defined by \( a \cdot b = \sigma(a)b \) (resp. \( b \cdot a = ba \)). For simplicity, we denote the module multiplication by a dot and the superalgebra multiplication by juxtaposition. In the sequel the elements of \( \mathcal{A} \) are supposed to be homogeneous.

Definition 4.1. Let \( i \in \{0, 1\} \). A \( \sigma \)-derivation \( D_i \) on \( \mathcal{A} \) is an endomorphism satisfying:

\[
D_i(ab) = D_i(a)b + (-1)^{|a|i}|\sigma(a)|D_i(b)
\]

where \( a, b \in \mathcal{A} \) are homogeneous element and \( |a| \) is the parity of \( a \).

A \( \sigma \)-derivation \( D_0 \) is called even \( \sigma \)-derivation and \( D_1 \) is called odd \( \sigma \)-derivation. The set of all \( \sigma \)-derivations is denoted by \( \text{Der}_\sigma(\mathcal{A}) \). Therefore, \( \text{Der}_\sigma(\mathcal{A}) = \text{Der}_\sigma(\mathcal{A})_0 \oplus \text{Der}_\sigma(\mathcal{A})_1 \), where \( \text{Der}_\sigma(\mathcal{A})_0 \) (resp. \( \text{Der}_\sigma(\mathcal{A})_1 \)) is the space of even (resp. odd) \( \sigma \)-derivations. The structure of \( \mathcal{A} \)-supermodule of \( \text{Der}_\sigma(\mathcal{A}) \) is as usual. Let \( D \in \text{Der}_\sigma(\mathcal{A}) \), the annihilator \( \text{Ann}(D) \) is the set of all \( a \in \mathcal{A} \) such that \( a \cdot D = 0 \). We set \( \mathcal{A} \cdot D = \{ a \cdot D : a \in \mathcal{A} \} \) be an \( \mathcal{A} \)-subsupermodule of \( \text{Der}_\sigma(\mathcal{A}) \).

Let \( \sigma : \mathcal{A} \to \mathcal{A} \) be a fixed even endomorphism, \( \Delta \) an even \( \sigma \)-derivation (\( \Delta \in \text{Der}_\sigma(\mathcal{A})_0 \)) and \( \delta \) be an element in \( \mathcal{A} \).

Theorem 4.2. If

\[
\sigma(\text{Ann}\Delta) \subset \text{Ann}\Delta
\]

holds then the map \([-,-]_\sigma : \mathcal{A}\Delta \times \mathcal{A}\Delta \to \mathcal{A}\Delta \) defined by setting

\[
[a \cdot \Delta, b \cdot \Delta]_\sigma = (\sigma(a) \cdot \Delta) \circ (b \cdot \Delta) - (-1)^{|a||b|}\sigma(b) \circ (a \cdot \Delta) \quad \text{for} \quad a, b \in \mathcal{A}
\]

where \( \circ \) denotes the composition of functions, is a well-defined superalgebra bracket on the superspace \( \mathcal{A} \cdot \Delta \) and satisfies the following identities for \( a, b, c \in \mathcal{A} \)

\[
[a \cdot \Delta, b \cdot \Delta]_\sigma = (\sigma(a)\Delta(b) - (-1)^{|a||b|}\sigma(b)\Delta(a)) \cdot \Delta
\]

\[
[a \cdot \Delta, b \cdot \Delta]_\sigma = -(-1)^{|a||b|}[b \cdot \Delta, a \cdot \Delta]_\sigma
\]

In addition, if

\[
\Delta(\sigma(a)) = \delta\sigma(\Delta(a)) \quad \text{for} \quad a \in \mathcal{A}
\]
holds, then
\[
\circ_{a,b,c} (-1)^{|a||c|} ([\sigma(a) \cdot \Delta, [b \cdot \Delta, c \cdot \Delta]_\sigma] \circ + \delta[a \cdot \Delta, [b \cdot \Delta, c \cdot \Delta]_\sigma]) = 0
\]
(4.6)

**Proof.** We show first that \([-,-]_\sigma\) is a well defined function. That is if \(a_1 \cdot \Delta = a_2 \cdot \Delta\) then \([a_1 \cdot \Delta, b \cdot \Delta]_\sigma = [a_2 \cdot \Delta, b \cdot \Delta]_\sigma\) and \([b \cdot \Delta, a_1 \cdot \Delta]_\sigma = [b \cdot \Delta, a_2 \cdot \Delta]_\sigma\) for \(a_1, a_2, b \in \mathcal{A}\)

We compute
\[
[a_1 \cdot \Delta, b \cdot \Delta]_\sigma = (\sigma(a_1) \cdot \Delta) \circ (b \cdot \Delta) - (-1)^{|a_1||b|}\sigma(b) \cdot \Delta) \circ (a_1 \cdot \Delta) = (\sigma(a_2) \cdot \Delta) \circ (b \cdot \Delta) + (-1)^{|a_2||b|}\sigma(b) \cdot \Delta) \circ (a_2 \cdot \Delta) = (\sigma(a_1) - a_2) \cdot \Delta) \circ (b \cdot \Delta) - (-1)^{|a_1||b|}\sigma(b) \cdot \Delta) \circ ((a_1 - (a_1)(a_2 || a_1) \cdot b \cdot \Delta) \circ (a_2 \cdot \Delta)
\]

Obviously, \(a_1 \cdot \Delta = a_2 \cdot \Delta\) is equivalent to \((a_1 - a_2) \in Ann(\Delta)\). Hence, using the assumption (4.1), we also have \(\sigma(a_1) - a_2 \in Ann(\Delta)\). Then, since \(|a_2| - |a_1| = 0\) and \(\sigma(a_1) - a_2 \in Ann(\Delta)\), we obtain
\[
[a_1 \cdot \Delta, b \cdot \Delta]_\sigma = [a_2 \cdot \Delta, b \cdot \Delta]_\sigma = 0.
\]

Similarly, we have \([b \cdot \Delta, a_1 \cdot \Delta]_\sigma = [b \cdot \Delta, a_2 \cdot \Delta]_\sigma\).

Next, we show that the superspace \(\mathcal{A} \Delta\) is closed under \([-,-]_\sigma\). Indeed,
\[
[a \cdot \Delta, b \cdot \Delta]_\sigma(c) = (\sigma(a) \cdot \Delta) \circ (b \cdot \Delta)(c) - (-1)^{|a||b|}\sigma(b) \cdot \Delta \circ (a \cdot \Delta)(c) = (\sigma(a) \cdot \Delta) \circ (b \Delta(c)) - (-1)^{|a||b|}\sigma(b) \cdot \Delta(a \Delta(c)) = \sigma(a)(\Delta(b) \Delta(c)) = (\sigma(a) \Delta(b)) \Delta(c) + (\sigma(a) \sigma(b)) \Delta(\sigma(a)) \Delta(c)
\]

Since \(\mathcal{A}\) is supercommutative the second term vanishes. Therefore, we obtain formula (4.3) which shows that the bracket is closed.

Now, we prove that formula (4.6) holds. First, we consider part given by the first term. part.
\[
\circ_{a,b,c} (-1)^{|a||c|} [\sigma(a) \cdot \Delta, [b \cdot \Delta, c \cdot \Delta]_\sigma]
\]

\[
= \circ_{a,b,c} (-1)^{|a||c|} [\sigma(a) \cdot \Delta, \sigma(b) \Delta(c) - (-1)^{|b||c|}\sigma(c) \Delta(b)] \cdot \Delta
\]

\[
= \circ_{a,b,c} (-1)^{|a||c|} (\sigma^2(c) \Delta(\sigma(b)) \Delta(c) + \sigma^2(b) \Delta^2(c) - (-1)^{|b||c|}\sigma^2(c) \sigma(\Delta(b)) \Delta(\sigma(a))) \cdot \Delta
\]

\[
= \circ_{a,b,c} (-1)^{|a||c|} (\sigma^2(b) \sigma(\Delta(b)) \Delta(\sigma(a))) \cdot \Delta
\]

where \(\sigma^2 = \sigma \circ \sigma\) and \(\Delta^2 = \Delta \circ \Delta\).

Applying cyclic summation to the second and fourth terms, they vanish. Using assumption and doing the same with the fifth and sixth in the previous identity we get also 0.
Finally, it remains
\[
\mathcal{O}_{a,b,c} (-1)^{[a][c]} [\sigma(a) \cdot \Delta, [b \cdot \Delta, c \cdot \Delta)]_\sigma = \langle 4.7 \rangle \\
\mathcal{O}_{a,b,c} (-1)^{[a][c]} (\sigma^2(a) \Delta(\sigma(b)) \Delta(c) - (-1)^{[b][c]} \sigma^2(a) \Delta(\sigma(c)) \Delta(b)) \cdot \Delta
\]

Now, we consider the part in \(\langle 4.7 \rangle\) given by the second term.
\[
\mathcal{O}_{a,b,c} (-1)^{[a][c]} \delta [a \cdot \Delta, [b \cdot \Delta, c \cdot \Delta)]_\sigma
\]
\[
= \mathcal{O}_{a,b,c} (-1)^{[a][c]} \delta [a \cdot \Delta, ((\sigma(b) \Delta(c) - (-1)^{[b][c]} \sigma(c) \Delta(b)) \cdot \Delta)]_\sigma
\]
\[
= \mathcal{O}_{a,b,c} (-1)^{[a][c]} \delta (\sigma(a) \Delta(\sigma(b)) \Delta(c) - (-1)^{[b][c]} \sigma(c) \Delta(b) \cdot \Delta
\]
\[
- (-1)^{[a][b][c]} (\sigma^2(b) \sigma(c) \Delta(c)) + (-1)^{[b][c]} \sigma^2(c) \sigma(\Delta(b)) \Delta(a)) \cdot \Delta
\]
\[
= \mathcal{O}_{a,b,c} (-1)^{[a][c]} \delta (\sigma(a) \Delta(\sigma(b)) \sigma(c) - \sigma(a) \Delta(\Delta(b)) \sigma(c))
\]
\[
- (-1)^{[a][b][c]} \sigma^2(b) \sigma(\Delta(c)) \sigma(\Delta(a)) \cdot \Delta
\]
\[
= \mathcal{O}_{a,b,c} (-1)^{[a][c]} \delta (-1)^{[b][c]} \sigma(a) \sigma(\Delta(b)) \sigma(\Delta(c))
\]
\[
- (-1)^{[a][b][c]} \sigma^2(b) \sigma(\Delta(c)) \Delta(a) + (-1)^{[b][c]} \sigma^2(c) \sigma(\Delta(b)) \Delta(a)) \cdot \Delta
\]

The four first terms vanish up to cyclic summation. It remains
\[
\mathcal{O}_{a,b,c} (-1)^{[a][c]} \delta [a \cdot \Delta, [b \cdot \Delta, c \cdot \Delta)]_\sigma
\]
\[
= \mathcal{O}_{a,b,c} (-1)^{[a][b]} \sigma(a) \Delta(\sigma(c)) \sigma(\Delta(a)) \cdot \Delta
\]
\[
= \mathcal{O}_{a,b,c} (-1)^{[a][c]} \delta (\sigma(a) \Delta(\sigma(b)) \sigma(\Delta(c))
\]
\[
\Delta
\]

Which is, up to permutation, the opposite of expression \(\langle 4.7 \rangle\). This ends the proof.

\[\square\]

5. A q-deformed Witt superalgebra

In this Section, we provide an example of infinite dimensional Hom-Lie superalgebra. We construct using the previous theorem a realization of the q-deformed Witt superalgebra, which carries a structure of Hom-Lie superalgebra for a suitable choice of the twist map.

Let \(\mathcal{A}\) be the complex superalgebra \(\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1\) where \(\mathcal{A}_0 = \mathbb{C}[t, t^{-1}]\) is the Laurent polynomials in one variable and \(\mathcal{A}_1 = \theta \mathbb{C}[t, t^{-1}],\) where \(\theta\) is the Grassman variable (\(\theta^2 = 0\)). We assume that \(t\) and \(\theta\) commute. The generators of \(\mathcal{A}\) are of the form \(t^n\) and \(\theta t^n\) for \(n \in \mathbb{Z}\).

Let \(q \in \mathbb{C}\setminus\{0, 1\}\) and \(n \in \mathbb{N}\), we set \(\{n\} = \frac{1 - q^n}{1 - q}\), a q-number. The q-numbers have the following properties \(\{n + 1\} = 1 + q\{n\}\) and \(\{n + m\} = \{n\} + q^n\{m\}\).

Let \(\sigma\) be the algebra endomorphism on \(\mathcal{A}\) defined by
\[
\sigma(t^n) = q^n t^n \quad \text{and} \quad \sigma(\theta) = q \theta.
\]

Let \(\partial_t\) and \(\partial_\theta\) be two linear maps on \(\mathcal{A}\) defined by
\[
\partial_t(t^n) = \{n\} t^n, \quad \partial_t(\theta t^n) = \{n\} \theta t^n, \\
\partial_\theta(t^n) = 0, \quad \partial_\theta(\theta t^n) = q^n t^n.
\]

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Lemma 5.1. The linear map $\Delta = \partial_t + \theta \partial_\theta$ on $A$ is an even $\sigma$-derivation.

Hence,

$$\Delta(t^n) = \{n\} t^n,$$
$$\Delta(\theta t^n) = \{n + 1\} \theta t^n.$$  

Proof. We show that

$$\Delta(t^n m) = \{n\} t^n m + \sigma(t^n) \Delta(t^m),$$  

(5.1)

$$\Delta(t^n \theta) = \{n\} t^n \theta + \sigma(t^n) \Delta(\theta),$$  

(5.2)

$$\Delta(\theta t^n) + \sigma(\theta) \Delta(t^n).$$  

(5.3)

Indeed, we have $\Delta(t^n m) = \{n + m\} t^n m$. On the other hand

$$\Delta(t^n m) + \sigma(t^n) \Delta(t^m) = \{n\} t^n m + q^n t^m \{m\} t^n m = \{n + m\} t^n m.$$  

Also, we have $\Delta(\theta t^n) = \{n + 1\} \theta t^n$. On the other hand

$$\Delta(\theta t^n) + \sigma(\theta) \Delta(t^n) = \{n\} \theta t^n + q^n \theta t^n = (\{n\} + q^n) \theta t^n = \{n + 1\} \theta t^n,$$

and

$$\Delta(\theta t^n) + \sigma(\theta) \Delta(t^n) = \theta t^n + q^n \theta \{n\} t^n = (1 + q^n \{n\}) \theta t^n = \{n + 1\} \theta t^n.$$

The conditions (4.1) and (4.5) of the theorem (4.2) are satisfied by the $\sigma$ derivation $\Delta$, with $\delta = 1$. Therefore we may construct a Hom-Lie superalgebra on the superspace $V = A \cdot \Delta$.

Let $V = A \cdot \Delta$, be a superspace generated by the elements $X_n = t^n \cdot \Delta$ of parity 0 and the elements $G_n = \theta t^n \cdot \Delta$ of parity 1.

Let $[-,-]$ be a bracket on the superspace $V$ defined by

$$[X_n, X_m] = \{m\} - \{n\} X_{n+m},$$

$$[X_n, G_m] = q^n \{m + 1\} - q^{n+1} \{n\} G_{n+m}.$$  

The others brackets are obtained by supersymmetry or are 0.

Let $\alpha$ be an even linear map on $V$ defined on the generators by

$$\alpha(X_n) = (1 + q^n) X_n,$$
$$\alpha(G_n) = (1 + q^{n+1}) G_n.$$  

Proposition 5.2. The triple $(V, [-,-], \alpha)$ is a Hom-Lie superalgebra.

The supersymmetry follow from the definition and the Hom-superJacobi identity follows from the theorem (4.2). One can also check directly that

$$\circ \ X_n, X_m, G_p \ [\alpha(X_n), [X_m, G_p]] = 0.$$  

The algebra constructed is a realization of the $q$-deformed Witt algebra.
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