Hecke eigenvalues of Siegel modular forms of “different weights”

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Given two Siegel eigenforms of different weights, we determine explicit sets of Hecke eigenvalues for the two forms that must be distinct. In degree two, and under some additional conditions, we determine explicit sets of Fourier coefficients of the two forms that must be distinct.

Keywords: Siegel modular forms; Hecke operators; Fourier coefficients.

1 Introduction

Our story revolves around the

Motivating question: How similar can two Hecke eigenforms of different weights be?

The case of elliptic modular forms has been investigated in some depth, both in characteristic zero and in positive characteristic; see [17], [10], [12], [7]. Here the eigenforms are compared via the Fourier coefficients, or via the Hecke eigenvalues, which are equivalent (after normalization).

In this paper we treat the case of Siegel modular forms in characteristic zero. While this is motivated by the elliptic case, the richness of the Siegel theory makes for a significantly more diverse picture. On one hand, Fourier coefficients encode more information than the Hecke eigenvalues: while the Fourier expansion completely determines the form for a fixed weight and level (by a generalization of the $q$-expansion principle [10, Proposition V.1.8]), the set of Hecke eigenvalues does not determine the form—there exist distinct eigenforms of the same level and weight that have the same Hecke eigenvalues for $p$ not dividing the level (see the introduction of [20]). On the other hand, even when staying entirely on the local Hecke algebra side, the structure is delicate enough that there are several choices of Hecke operators to consider: the “standard” generators $T(p), T_1(p^2), \ldots, T_n(p^2)$ described by Andrianov [3]; the “averaged” generators $\tilde{T}(p), \tilde{T}_1(p^2), \ldots, \tilde{T}_n(p^2)$ defined and studied by Hafner and Walling [14]; or the operators $T(p^r)$ that appear more naturally in geometric situations such as the work of Bergström, Faber and van der Geer [6].

Our investigation touches upon most of these aspects, with varying degrees of success and generality. The most direct way of approaching our Motivating question is via the operator $T_n(p^2)$: this acts on a Siegel modular form of level coprime to $p$ and weight given by $(\lambda_1 \geq \ldots \geq \lambda_n)$ as multiplication by the scalar $p^{\sum \lambda_j-n(n+1)}$. It follows that two forms that have the same eigenvalue for $T_n(p^2)$ must have the same $\sum \lambda_j$. The contrapositive version of this statement is: if for two forms $F$ and $G$ of level coprime to $p$ the corresponding integers $\sum \lambda_j$ differ (this is the sense in which the “different weights” of the title should be understood), then their eigenvalues under the operator $T_n(p^2)$ must also differ. This basic
observation forms the basic leitmotif of the paper, and the variations thereupon may be described informally as follows:

• in Section 4, we show that the eigenvalues of $F$ and $G$ for at least one of the operators $T(p), T(p^2), T_1(p^2), \ldots , T_n(p^2)$ must be distinct;

• in Section 5, we consider the special case of degree 2 and show that the eigenvalues of $F$ and $G$ for at least one of the operators $T(p^r), r = 1, \ldots , 6,$ must be distinct;

• in Section 6, we show that, subject to a number of conditions, there exists a Fourier expansion index $S$ of explicitly-bounded determinant such that the coefficient of $F$ at $S$ is distinct from the coefficient of $G$ at $S$.

These results are preceded by a review of the basic theory of Hecke operators on Siegel modular forms in Section 2, and by the derivation of a formula for $T(p^2)$ (inspired by work of Hafner-Walling) in Section 3.

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2 Hecke operators on Siegel modular forms

We gather here some definitions and basic results on Siegel modular forms and their Hecke action. For more leisurely expositions of various parts of this material, the reader is invited to consult [23] or [11].

2.1 Siegel modular forms

Let $n \geq 1$ be an integer and let $R$ be a commutative ring. Consider the $R$-module $R^{2n}$ with generators $e_1, \ldots , e_n, f_1, \ldots , f_n$, equipped with a symplectic form $\langle \cdot , \cdot \rangle$ which is defined on the generators by the rules

$$\langle e_i, e_j \rangle = 0, \quad \langle f_i, f_j \rangle = 0 \quad \text{and} \quad \langle e_i, f_j \rangle = \delta_{ij} \quad \text{(Kronecker’s delta)}$$

and extended by $R$-bilinearity.

Definition 2.1. The symplectic group $Sp(2n, R)$ is the automorphism group of the pair $(R^{2n}, \langle \cdot , \cdot \rangle)$:

$$Sp(2n, R) = \{ \alpha \in \text{GL}(R^{2n}) \mid \langle \alpha(u), \alpha(v) \rangle = \langle u, v \rangle \text{ for all } u, v \in R^{2n} \} .$$

We also work with the group of symplectic similitudes

$$GSp(2n, R) = \{ \alpha \in \text{GL}(R^{2n}) \mid \text{there exists } \eta(\alpha) \in R^\times \text{ such that } \langle \alpha(u), \alpha(v) \rangle = \eta(\alpha) \langle u, v \rangle \text{ for all } u, v \in R^{2n} \} .$$

With respect to the basis comprising the $e_i$ and $f_i$, the elements of $Sp(2n, R)$ and $GSp(2n, R)$ are represented by matrices

$$Sp(2n, R) = \{ \gamma \in \text{GL}(2n, R) \mid \gamma J \gamma^t = J \}$$

$$GSp(2n, R) = \{ \gamma \in \text{GL}(2n, R) \mid \gamma J \gamma^t = \eta(\gamma) J \text{ for some } \eta(\gamma) \in R^\times \} ,$$

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where

\[ J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \]

The Siegel modular group of degree \( n \) is \( \Gamma = \text{Sp}(2n, \mathbb{Z}) \). We shall be principally concerned with a family of so-called congruence subgroups of \( \Gamma \).

**Definition 2.2.** Let \( N \geq 1 \) be an integer. The congruence subgroup \( \Gamma_0(N) \) of level \( N \) is

\[ \Gamma_0(N) = \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma \ \middle| \ C \equiv 0 \pmod{N} \right\}. \]

**Definition 2.3.** Let \( n \geq 1 \) be an integer. The Siegel upper half space \( \mathbb{H}^n \) is the set of symmetric \( n \times n \) complex matrices having positive-definite imaginary part:

\[ \mathbb{H}^n = \left\{ z \in \text{Mat}(n, \mathbb{C}) \ \middle| \ z^t = z \text{ and } \text{Im}(z) > 0 \right\} \]

The set \( \mathbb{H}^n \) is an open subset of the complex manifold \( \text{Mat}(n, \mathbb{C}) \cong \mathbb{C}^{n^2} \).

The group \( \Gamma \) acts on \( \mathbb{H}^n \) by generalized Möbius transformations:

\[ \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \cdot z := (Az + B)(Cz + D)^{-1}. \]

**Definition 2.4.** Let \( n \geq 2 \) and \( N \geq 1 \) be integers and let \( \rho: \text{GL}_n(\mathbb{C}) \to \text{GL}(V) \) be a polynomial representation. A Siegel modular form of degree \( n \), weight \( \rho \) and level \( N \) is a holomorphic function \( F: \mathbb{H}^n \to V \) such that

\[ F(\gamma \cdot z) = \rho(Cz + D)F(z) \]

for all \( \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(N) \) and all \( z \in \mathbb{H}^n \). The \( \mathbb{C} \)-vector space of all such forms is denoted \( M_{\rho}(\Gamma_0(N)) \).

**Remark 2.5.** The case \( n = 1 \) requires an extra condition, holomorphicity at the cusps, which is automatically satisfied for \( n \geq 2 \) (by the Koecher principle).

**Remark 2.6.** We may commit various abuses of notation regarding the weight \( \rho \) of a Siegel modular form:

- Finite-dimensional representations \( \rho: \text{GL}_n(\mathbb{C}) \to \text{GL}(V) \) are indexed by nonincreasing \( n \)-tuples of integers \( (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n) \), and we will refer to such a tuple as the weight.

- The special case \( \lambda_1 = \lambda_2 = \ldots = \lambda_n = k \) corresponds to scalar-valued Siegel modular forms; here we call \( k \) the weight.

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\(^1\)A finite-dimensional representation \( \rho: \text{GL}_n(\mathbb{C}) \to \text{GL}(V) \) is polynomial if it is given by polynomial functions in the entries of the matrices in \( \text{GL}_n(\mathbb{C}) \).
2.2 The Hecke algebra

We recall the construction of some interesting elements of the local Hecke algebra of $\text{Sp}(2n)$ at a prime $p$; for this, we combine the approach and notation from [13, Section 2] and [23, Section 16].

Let $G = \text{Sp}(2n, \mathbb{Q}_p)$ and $K = \text{Sp}(2n, \mathbb{Z}_p)$. The (local) Hecke algebra $\mathcal{H}$ is the Hecke algebra of the pair $(G, K)$, i.e.

$$\mathcal{H} = \{ f : G \to \mathbb{Z} \mid f \text{ locally constant, compactly supported, bi-$K$-invariant} \}.$$ 

The multiplication is given by convolution of such functions; for all $x \in G$, we set

$$(f \cdot g)(x) = \int_{\gamma \in G} f(\gamma)g(\gamma^{-1}x) \, d\gamma,$$

where $d\gamma$ is the unique Haar measure on $G$ which is normalized so that $K$ has volume 1.

The prototypical examples of elements of $\mathcal{H}$ are provided by the characteristic functions of double cosets $K\gamma K$ with $\gamma \in G$:

$$1_{K\gamma K}(x) = \begin{cases} 1 & \text{if } x \in K\gamma K \\ 0 & \text{otherwise.} \end{cases}$$

In fact, every element of $\mathcal{H}$ is a finite $\mathbb{Z}$-linear combination of characteristic functions $1_{K\gamma K}$.

For any $r \geq 1$, consider the set of matrices

$$O_n(p^r) = \{ \gamma \in \text{Mat}(2n, \mathbb{C}) \mid \gamma^t \eta = p^r \gamma \}.$$

**Definition 2.7.** Set

$$T(p) = 1_{K\gamma K} \quad \text{where } \gamma = \text{diag}(1, \ldots, 1, p, \ldots, p)$$

and, for $j = 0, \ldots, n$:

$$T_j(p^2) = 1_{K\gamma K} \quad \text{where } \gamma = \text{diag}(1, \ldots, 1, p, \ldots, p, p^2, \ldots, p^2, \ldots, p),$$

and finally, for $r \geq 1$:

$$T(p^r) = \sum_{\gamma \in O_n(p^r)} 1_{K\gamma K}$$

(Equation (2.3) is consistent with Equation (2.1) when $r = 1$.)

The algebra $\mathcal{H}$ can be written as $\mathcal{H} = \mathcal{H}^0 \left[1/T_n(p^2)\right]$, where $\mathcal{H}^0$ is the subalgebra generated by the characteristic functions of double cosets of matrices with entries in $\mathbb{Z}_p$. Moreover, a set of generators for the algebra $\mathcal{H}^0$ is given by the elements $T(p), T_1(p^2), T_2(p^2), \ldots, T_n(p^2)$.

2.3 Action of the Hecke algebra on Siegel modular forms

Let $F$ be a Siegel modular form of level $\Gamma_0(N)$ and weight $\rho$ given by $(\lambda_1 \geq \ldots \geq \lambda_n)$. Given $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GSp}(2n, \mathbb{Q})$ with positive determinant, we set

$$F|_{\gamma, \rho}(z) = \eta(\gamma) \sum \lambda_j - n(n+1)/2 \rho(Cz + D)^{-1} F(\gamma \cdot z).$$
Let $p$ be a prime not dividing $N$ and let $K = \text{Sp}(2n, \mathbb{Z}_p)$ as in Section 2.2. Given a double coset $K\gamma K$ and its decomposition into right cosets

$$K\gamma K = \prod_{i=1}^{h} K\gamma_i,$$

we set

$$\tag{2.5} (K\gamma K)(F)(z) = \sum_{i=1}^{h} F|_{\gamma_i}\rho(z).$$

The result is a Siegel modular form of the same weight and level as $F$, and independent of the choice of double coset representative $\gamma$ and right coset representatives $\gamma_i$.

Finally, we extend (2.5) by $\mathbb{Z}$-linearity to an action of the local Hecke algebra $H$ on $M_\rho(\Gamma_0(N))$. In particular, we can think of the elements $T(p), T_j(p^2), T(p')$ from Definition 2.7 as operators on the space $M_\rho(\Gamma_0(N))$.

In order to obtain explicit expressions for the action of these Hecke operators, we need explicit decompositions of the relevant double cosets into right cosets. Such decompositions for the generators $T(p), T_1(p^2), \ldots, T_n(p^2)$ are given in [3, Lemma III.3.32]. We will only need the decomposition of $T_n(p^2)$, which takes a particularly simple form and leads to the following description:

**Lemma 2.8.** Let $F \in M_\rho(\Gamma_0(N))$, with $\rho$ given by $(\lambda_1 \geq \ldots \geq \lambda_n)$. Let $p$ be a prime not dividing $N$. Then $T_n(p^2)F = p^{\sum \lambda_j - n(n+1)/2}F$.

**Proof.** We know that $T_n(p^2)$ is given by the double coset

$$KpI_{2n}K = KpI_{2n} = K\gamma \quad \text{with} \quad \gamma = \begin{pmatrix} pI_n & 0 \\ 0 & pI_n \end{pmatrix}.$$

The claim follows from Equation (2.5), since:

$$\eta(pI_{2n}) = p^2, \quad \rho(pI_n) = p^{\sum \lambda_j}, \quad \text{and} \quad (pI_n) \cdot z = z.$$

### 3 A formula for $T(p^2)$

In [14], Hafner and Walling introduced a new set of generators $T(p)$, $\tilde{T}_j(p^2)$ ($j = 1, \ldots, n$) for the local Hecke algebra at $p$. Their motivation was that these generators act on the Fourier expansions of Siegel modular forms in a much simpler way than the standard Hecke operators. We are interested in the $\tilde{T}_j(p^2)$ because they satisfy the simple relation (3.1) given below; we use this to deduce the formula (3.2) for $T(p^2)$, which will play a crucial role in Section 4.
**Definition 3.1.** For \( n \geq 1, j = 0,1,\ldots, n \) and \( k \geq n + 1 \), set
\[
T(p) = p^{n(k-n-1)/2} T(p)
\]
\[
T_j(p^2) = T_{n-j}(p^2)
\]
\[
\tilde{T}_j(p^2) = p^{j(k-n-1)} \sum_{t=0}^{j} \binom{n-t}{j-t} T_t(p^2)
\]
where
\[
\binom{m}{\ell}_p = \# \text{Gr}(\ell,m)(\mathbb{F}_p) = \prod_{i=1}^{\ell} \frac{p^{m-i+1} - 1}{p^i - 1}
\]
is the number of \( \ell \)-dimensional subspaces of \( \mathbb{F}_p^m \).

**Remark 3.2.** Hafner and Walling [14] write the operators \( T_j(p^2) \), etc., simply as \( T_j(p^2) \). We have preferred to differentiate them typographically from the standard generators. We have also chosen to replace the notation \( \beta_p(m, \ell) \) from [14] with the more established \( q \)-binomial coefficient notation \( \binom{m}{\ell}_p \), which has the additional advantage that it is rather suggestive of the properties of these numbers that we will exploit below.

**Remark 3.3.** Note that
\[
\tilde{T}_n(p^2) = p^{n(k-n-1)} \sum_{t=0}^{n-1} T_{n-t}(p^2) = p^{n(k-n-1)} T(p^2).
\]

The operators \( T(p) \) and \( \tilde{T}_j(p^2) \) satisfy the following relation:

**Theorem 3.4** (Hafner-Walling [14] Proposition 5.1).
\[
\tilde{T}_n(p^2) = T(p^2) - \sum_{j=0}^{n-1} p^{j(n-j)+j(j+1)/2-n(n+1)/2} \tilde{T}_j(p^2).
\]

We will make crucial use of this identity, which we first translate into a statement about the operators \( T(p) \), \( T(p^2) \), \( T_j(p^2) \):

**Theorem 3.5.** The following relation holds in the local Hecke algebra at \( p \):
\[
T(p^2) = T(p^2) - \sum_{s=1}^{n} c_s T_s(p^2),
\]
where the coefficients \( c_s \) are positive and given by
\[
c_s = \sum_{i=1}^{s} p^{i(i+1)/2} \binom{s}{i}_p = \prod_{i=1}^{s} (p^i + 1) - 1.
\]

**Proof.** Combine Equation (3.1) with the relations from Definition 3.1 and divide by the normalizing factor \( p^{n(k-n-1)} \) to get
\[
T(p^2) - T(p^2) = -\sum_{j=0}^{n-1} p^{j(n-j)(n-j+1)/2} \sum_{t=0}^{j} \binom{n-t}{j-t} T_{n-t}(p^2).
\]
We substitute \( i = n - j \) and \( s = n - t \):
\[
T(p^2) - T(p^2) = -\sum_{i=1}^{n} p^{i(i+1)/2} \sum_{s=i}^{n} \binom{s}{s-i} T_s(p^2),
\]

after which we interchange the two summations and use \((s)_p^{s-1} = \binom{s}{t}_p\) to get Equation 5.5 with coefficients

\[
c_s = \sum_{i=1}^{s} p^{(i+1)/2} \binom{s}{i}_p.
\]

It remains to show that these coefficients can be simplified to give the product on the right hand side of Equation 5.5. For this we use Gauss’s binomial formula (see, for instance, Equation 5.5): \((x+a)_q^n = \sum_{j=0}^{n} \binom{n}{j}_q a^j x^{n-j}\).

With our notation \((x=1, a=p, q=p, n=s)\), this gives precisely

\[
c_s + 1 = \sum_{i=0}^{s} p^{(i+1)/2} \binom{s}{i}_p = (1+p)_p^s = \prod_{j=1}^{s} (1+p^j),
\]

which concludes the proof.

\[
\square
\]

4 Distinguishing eigenforms via the operators \(T(p), T(p^2), T_j(p^2)\)

Given a Siegel eigenform \(F\) of degree \(n\), let \(a_F(T)\) denote its eigenvalue for \(T \in \mathcal{H}\), and set

\[
E_F(p) = (a_F(T(p)), a_F(T(p^2)), a_F(T_1(p^2)), a_F(T_2(p^2)), \ldots, a_F(T_{n-1}(p^2))).
\]

**Theorem 4.1.** Let \(F\) and \(G\) be Siegel eigenforms of degree \(n\) on \(\Gamma_0(N)\), of respective weights \((\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n)\) and \((\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n)\), satisfying

\[
\sum_{j=1}^{n} \lambda_j \neq \sum_{j=1}^{n} \mu_j.
\]

If \(p\) is a prime number not dividing \(N\), then

\[
E_F(p) \neq E_G(p).
\]

**Proof.** Assume, on the contrary, that \(E_F(p) = E_G(p)\). Theorem 3.5 gives

\[
c_n T_n(p^2) = T(p)^2 - T(p^3) - \sum_{s=1}^{n-1} c_s T_s(p^2),
\]

so for our two forms \(F\) and \(G\) we get

\[
c_n a_F(T_n(p^2)) = a_F(T(p))^2 - a_F(T(p^2)) - \sum_{s=1}^{n-1} c_s a_F(T_s(p^2))
\]

\[
c_n a_G(T_n(p^2)) = a_G(T(p))^2 - a_G(T(p^2)) - \sum_{s=1}^{n-1} c_s a_G(T_s(p^2)).
\]

Our hypothesis then tells us that the two right hand sides are equal, so \(c_n a_F(T_n(p^2)) = c_n a_G(T_n(p^2))\), and since \(c_n \neq 0\), we get \(a_F(T_n(p^2)) = a_G(T_n(p^2))\).

Finally, since \(p\) does not divide the level \(N\), Lemma 2.8 gives us

\[
a_F(T_n(p^2)) = p^{(\sum_{j=1}^{n-1} \lambda_j) - n(n+1)/2}
\]

\[
a_G(T_n(p^2)) = p^{(\sum_{j=1}^{n-1} \mu_j) - n(n+1)/2},
\]

from which we conclude that \(\sum \lambda_j = \sum \mu_j\), a contradiction. \(\square\)
A special case of interest is that of scalar-valued forms:

**Corollary 4.2.** Under the same notation as in Theorem 4.1, suppose that \( F \) and \( G \) are scalar-valued, of respective weights \( k_1 \neq k_2 \). Then \( \mathcal{E}_F(p) \neq \mathcal{E}_G(p) \).

## 5 Distinguishing degree 2 eigenforms via the operators \( T(p^r) \)

The structure of the spaces of scalar-valued Siegel modular forms of degree 2 and level 1 is well-known thanks to results of Igusa; these results, together with the interplay between Jacobi modular forms and Siegel modular forms, have allowed the explicit decomposition of these spaces into eigenspaces for the Hecke operators. This approach was introduced by Skoruppa \[22\] and exploited and refined by a number of authors, most recently Raum \[19\]. Bases for these spaces are computed as sets of explicit Fourier expansions, and the effect of the operators \( T(p) \), \( T_j(p^2) \) on these Fourier expansions can then be computed using the explicit formulas from \[3\] or \[14\]. The results of the last section fit naturally in this setting.

However, the Fourier-expansion based approach is not the only way of obtaining Hecke eigenforms. In fact, when working with vector-valued Siegel modular forms of weight \( 3 \) \( \text{Sym}^3 \otimes \det^k \), this approach only works for small values of \( j \). In a series of papers \([8\), \[9\], \[5\], \[6\] \]), Bergström, Faber and van der Geer have developed a completely different way of computing eigenforms, based on the cohomology of local systems on the moduli space of abelian varieties, and counting curves of certain type over finite fields. Their method naturally works with vector-valued Siegel modular forms, and has been implemented successfully to the study of forms of level 1 and degree 2 or 3, and to forms of level \( \Gamma_0(2) \) and degree 2. The eigenvalues that it produces are attached to the Hecke operators \( T(p^r) \), rather than the \( T_j(p^2) \) that we have been considering so far.

In this section, we exhibit a result (Theorem 5.2) of the same flavour as Theorem 4.1, but for the operators \( T(p^r) \). As we will see, the situation is more complicated here, so we prefer to restrict our treatment to forms of degree 2. The same method should apply to other small degrees, at the expense of increasingly tedious algebraic manipulations.

We start by gathering some useful relations between the eigenvalues \( a_F(p^r) \) of an eigenform \( F \).

**Lemma 5.1.** Let \( F \) be a Siegel eigenform of degree 2 on \( \Gamma_0(N) \), of weight \( (\lambda_1 \geq \lambda_2) \). Let \( p \) be a prime not dividing \( N \). Let \( a_F(p^r) \) be the Hecke eigenvalues of \( F \) under the operators \( T(p^r) \). Then the following identity holds:

\[
(5.1) \quad a_F(p^3) - 2a_F(p)a_F(p^2) + a_F(p^3) - p^{\lambda_1 + \lambda_2 - 4}(p + 1)a_F(p) = 0
\]

Further, if \( a_F(p) = 0 \) then

\[
(5.2) \quad a_F(p^{2i+1}) = 0 \quad \text{for all} \quad i = 0, 1, 2, \ldots,
\]

\[
(5.3) \quad a_F(p^4) - a_F(p^2)^2 - p^{\lambda_1 + \lambda_2 - 4}a_F(p^2) + p^{2\lambda_1 + 2\lambda_2 - 6} = 0,
\]

\[
(5.4) \quad a_F(p^6) - a_F(p^4)a_F(p^2) - p^{\lambda_1 + \lambda_2 - 4}a_F(p^2) + p^{2\lambda_1 + 2\lambda_2 - 6}a_F(p^2) = 0.
\]

**Proof.** By definition, \( T(p^r)F = a_F(p^r)F \) for all \( r \). Moreover, if \( p \nmid N \), then by Lemma 2.8 we have

\[
(5.5) \quad T_2(p^2)F = p^{\lambda_1 + \lambda_2 - 6}F.
\]

\[\text{In the context of vector-valued Siegel modular forms of degree 2, the weight is often expressed as the actual representation} \ \text{Sym}^3(C^2) \otimes \det^k, \ \text{rather than via the highest weight vector notation} (\lambda_1 \geq \lambda_2). \text{We prefer} \]

\[\text{to stick with the latter for the sake of consistency with the degree n situation treated in the previous} \]

\[\text{sections. The reader who prefers to think in terms of} \ \text{Sym}^j \otimes \det^k \text{can use the dictionary} \ j = \lambda_1 - \lambda_2, \ k = \lambda_2. \]
By [21, Theorem 2], the Hecke operators at $p$ satisfy the relations summarised by

\begin{equation}
(5.6) \quad z^4 f(1/z) \sum_{i=0}^{\infty} T(p^i)z^i = 1 - p^2 T_2(p^2)z^2
\end{equation}

where $f$ is the degree 4 polynomial (aka local Euler factor at $p$)

\[ f(X) = X^4 - T(p)X^3 + [T(p)^2 - T(p^2) - p^2 T_2(p^2)] X^2 - p^3 T_2(p^2)X + p^6 T_2(p^2)^2. \]

Upon expansion of (5.6) and inspection of the degree 3 terms, we obtain

\[ T(p^3) - 2T(p)(p^2) + T(p)^3 - p^2(p+1)T(p)T_2(p^2) = 0, \]

which gives rise to (5.1).

For any $i \geq 0$ we can equate the coefficient of $z^{2i+1}$ on either side of (5.6) to find that

\[ T(p^{2i+1}) - T(p)(p^2) + [T(p)^2 - T(p^2) - p^2 T_2(p^2)] T(p^{2(i-1)+1}) \]

\[ - p^3 T(p)T_2(p^2)T(p^{2i-2}) + p^6 T_2(p^2)T(p^{2(i-2)+1}) = 0 \]

(we harmlessly define $T(p^i)$ for $i < 0$). The claimed equality (5.2) now follows by induction on $i \geq 0$ under the additional hypothesis $a_F(p) = 0$.

The terms of degree 4 and 6 respectively in the identity (5.6) are

\[ T(p^4) - T(p)(p^2)T(p^3) + T(p^2)X^2 - T(p^2)^2 
= \]

\[ - p^2 T_2(p^2) - p^3 T_2(p^2)X + p^6 T_2(p^2)^2 = 0, \]

\[ T(p^6) - T(p) [T(p)^2 - T(p^2) + p^3 T_2(p^2)] 
- T(p)^2 T(p^4) - p^2 T_2(p^2)^2 T(p^4) + p^6 T_2(p^2)^2 T(p^2) = 0. \]

Equalities (5.3) and (5.4) follow from these expressions, together with the relation (5.6) and the assumption $a_F(p) = 0$.

\[ \square \]

**Theorem 5.2.** Let $F$ and $G$ be Siegel eigenforms of degree 2 on $\Gamma_0(N)$, of respective weights $(\lambda_1, \lambda_2)$ and $(\mu_1, \mu_2)$, with

\[ \lambda_1 + \lambda_2 \neq \mu_1 + \mu_2. \]

Let $p$ be a prime not dividing $N$. Let $a_F(p^r)$, $a_G(p^r)$ denote the Hecke eigenvalues of $F$, $G$ for the operators $T(p^r)$.

(i) If $a_F(p)$ and $a_G(p)$ are not both zero, then

\[ (a_F(p), a_F(p^2), a_F(p^3)) \neq (a_G(p), a_G(p^2), a_G(p^3)). \]

(ii) If $a_F(p) = a_G(p) = 0$, then

\[ (a_F(p^2), a_F(p^4), a_F(p^6)) \neq (a_G(p^2), a_G(p^4), a_G(p^6)). \]

**Proof.**

(i) Assume $a_F(p) \neq 0$. Suppose, on the contrary, that $a_F(p^r) = a_G(p^r)$ for each $r = 1, 2, 3$.

By (5.1) from Lemma 5.1

\[ a_F(p^3) - 2a_F(p)a_F(p^2) + a_F(p)^3 - p^{\lambda_1 + \lambda_2 - 4}(p+1)a_F(p) = 0, \]

\[ a_G(p^3) - 2a_G(p)a_G(p^2) + a_G(p)^3 - p^{\mu_1 + \mu_2 - 4}(p+1)a_G(p) = 0. \]
This implies that
\[(p + 1)a_F(p)(p^{\lambda_1 + \lambda_2 - 4} - p^{\mu_1 + \mu_2 - 4}) = 0\]
and under the assumption that \(a_F(p) \neq 0\) we conclude that \(\lambda_1 + \lambda_2 = \mu_1 + \mu_2\), a contradiction.

(ii) Assume that \(a_F(p_r) = a_G(p_r) =: a(p_r)\) for each \(r \in \{1, 2, 4, 6\}\) and that \(a(p) = 0\).
Equality \(5.3\) of Lemma \(5.1\) then implies that
\[a(p)^2(p^{\mu_1 + \mu_2} - p^{\lambda_1 + \lambda_2}) = \frac{1}{p^2}(p^{2\mu_1 + 2\mu_2} - p^{2\lambda_1 + 2\lambda_2}).\]

Recognising the difference of perfect squares in the right hand expression, we may (under our standing assumption that \(\lambda_1 + \lambda_2 \neq \mu_1 + \mu_2\)) conclude that
\[a(p) = \frac{p^{\lambda_1 + \lambda_2} + p^{\mu_1 + \mu_2}}{p^2}.\]

Under the same assumptions, an identical analysis beginning with equality \(5.4\) of Lemma \(5.1\) leads to the identity
\[a(p^4) = \frac{a(p^2)(p^{\lambda_1 + \lambda_2} + p^{\mu_1 + \mu_2})}{p^2} = a(p^2)^2.\]

Applying \(5.3\) to \(F\) and \(G\) respectively we find that
\[a(p^2) = a_F(p^2) = \frac{p^{2\lambda_1 + 2\lambda_2 - 6}}{p^{\lambda_1 + \lambda_2 - 4}} = p^{\lambda_1 + \lambda_2 - 2}\]
\[a(p^2) = a_G(p^2) = \frac{p^{2\mu_1 + 2\mu_2 - 6}}{p^{\mu_1 + \mu_2 - 4}} = p^{\mu_1 + \mu_2 - 2}.\]

Once again we have contradicted the assumption \(\lambda_1 + \lambda_2 \neq \mu_1 + \mu_2\).

\[\Box\]

**Corollary 5.3.** Let \(F\) and \(G\) be as in Theorem \(5.2\). There exists \(m \leq (2 \log(N) + 2)^6\), such that \(a_F(m) \neq a_G(m)\).

*Proof.* Use the fact that given \(N\), there exists a prime \(p \leq 2 \log(N) + 2\) that does not divide \(N\). (For a proof, see [12, Theorem 1].) \(\Box\)

### 6 Distinguishing degree 2 eigenforms via Fourier coefficients

Let \(\rho: \text{GL}_2 \rightarrow \text{GL}(V)\) be a polynomial representation. Any \(F \in M_\rho(\Gamma_0(N))\) has a multivariate Fourier expansion of the form
\[F(z) = \sum_{S \in F(2)} c_F(S)q^S \quad \text{with} \quad c_F(S) \in V,\]
where
- the variable \(z\) is in \(\mathbb{H}^2\), i.e. a symmetric \(2 \times 2\) complex matrix with positive-definite imaginary part;
• the index set $F(2)$ consists of all matrices $S \in \text{GL}(2, \mathbb{Q})$ that are symmetric, positive-semidefinite and \textit{half-integral}, that is, $S = (s_{ij})$ with $2s_{ij} \in \mathbb{Z}$ and $s_{ii} \in \mathbb{Z}$;

• we set
\[ q^S = e^{2\pi i \text{Tr}(S)}. \]

Arakawa [4] obtained results on Euler products for vector-valued Siegel modular forms of degree 2 (extending Andrianov’s investigation of the scalar-valued case in [1]). In particular, he gave simple explicit formulas for the Hecke action on \textit{certain} Fourier coefficients:

\textbf{Theorem 6.1} (Arakawa [4, Proposition 2.3]). Let $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in F(2)$ be:

• primitive, that is, $\gcd(a, b, c) = 1$;

• such that $d = b^2 - 4ac$ is the discriminant of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$;

• such that $K$ has class number 1.

Let $p$ be a rational prime that is inert in $K$, and let $m$ be a positive integer coprime to $p$. Then, for $F \in M_p(\Gamma_0(N))$ and any $r$, we have
\[ c_{T(p^r)F} (mS) = c_F(p^r mS). \]

\textbf{Corollary 6.2.} Let $p \equiv 3 \pmod{4}$, $F \in M_p(\Gamma_0(N))$ and $r \geq 1$, then
\[ c_{T(p^r)F} (I) = c_F(p^r I). \]

\textit{Proof.} The identity matrix $I$ corresponds to the quadratic form $x^2 + y^2$, which gives the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-1})$. \qed

\textbf{Theorem 6.3.} Let $F$ and $G$ be Siegel eigenforms of degree 2 on $\Gamma_0(N)$, of respective weights $(\lambda_1 \geq \lambda_2)$ and $(\mu_1 \geq \mu_2)$ satisfying
\[ \lambda_1 + \lambda_2 \neq \mu_1 + \mu_2. \]

Suppose that at least one of the Fourier coefficients $c_F(I)$ and $c_G(I)$ is nonzero. Let $p$ be a prime $\equiv 3 \pmod{4}$ not dividing $N$. Then there exists $r$ with $0 \leq r \leq 6$ such that
\[ c_F(p^r I) \neq c_G(p^r I). \]

\textit{Proof.} We proceed by contradiction: suppose
\[ c_F(p^r I) = c_G(p^r I) \quad \text{for } 0 \leq r \leq 6. \]

(In particular, $c_F(I) = c_G(I) \neq 0$.) By Corollary [6,2]
\[ a_F(p^r) = \frac{c_{T(p^r)F}(I)}{c_F(I)} = \frac{c_{T(p^r)G}(I)}{c_G(I)} = a_G(p^r) \quad \text{for } 0 \leq r \leq 6. \]

This contradicts Theorem [5,2]. \qed

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Remark 6.4. The assumption that at least one of \( c_F(I) \) and \( c_G(I) \) is nonzero is essential to the proof. It is likely that the \( I \)-th coefficient of any Siegel eigenform is nonzero, but there are no general results in this direction. It has been conjectured that the first Fourier-Jacobi coefficient of a Siegel eigenform \( F \) is nonzero, and it is conceivable that the \((n = 1, r = 0)\)-th coefficient of a Jacobi eigenform is also nonzero, which would imply our condition \( c_F(I) \neq 0 \).

A discussion of this issue features in Arakawa’s work on the \( L \)-functions attached to Siegel eigenforms, where he also gives a construction of some eigenforms \( F \) such that \( c_F(I) \neq 0 \), see [4, Section 4].

Corollary 6.5. Let \( F \) and \( G \) be as in Theorem 6.3. There exists a matrix \( S \in \mathcal{F}(2) \) such that
\[
\det(S) \leq (3\log(N) + 4)^{12} \quad \text{and} \quad c_F(S) \neq c_G(S).
\]

This follows from the following estimate:

Proposition 6.6. Let \( N \geq 1 \) be an integer. Let \( p \) be the smallest prime \( \equiv 3 \pmod{4} \), not dividing \( N \). Then
\[
p \leq 3\log(N) + 4.
\]

Proof. The cases \( 1 \leq N < 40 \) are settled by a quick case-by-case computation.

So we can assume \( N \geq 40 \). We proceed by contradiction: suppose \( N \) is divisible by all primes \( \equiv 3 \pmod{4} \) that are less than or equal to \( 3\log(N) + 4 \). Then
\[
N \geq \prod_{p \leq 3\log(N) + 4 \atop p \equiv 3 \pmod{4}} p,
\]
so that
\[
\log(N) \geq \sum_{p \leq 3\log(N) + 4 \atop p \equiv 3 \pmod{4}} \log(p) = \theta_3(3\log(N) + 4) = \theta_3(3\log(N)),
\]
where \( \theta_3 \) denotes the following modification of Chebyshev’s function:
\[
\theta_3(x) = \sum_{p \leq x \atop p \equiv 3 \pmod{4}} \log(p).
\]

If \( N \geq 40 \) then \( \log(N) \geq 11/3 \), so by Lemma 6.7 the right hand side of Equation (6.1) is \( > \log(N) \), which is a contradiction.

Lemma 6.7. The function \( \theta_3 \) satisfies
\[
\theta_3(3x) > x \quad \text{for all} \quad x \geq \frac{11}{3}.
\]

Proof. Ramaré and Rumely give the following explicit estimate for \( \theta_3 \) (see [18, Theorems 1 and 2]):
\[
|\theta_3(x) - x/2| \leq \begin{cases} 0.001119x & \text{for} \ x \geq 10^{10} \\ 1.780719\sqrt{x} & \text{for} \ x < 10^{10}. \end{cases}
\]

We can lower the bound \( 10^{10} \) at the expense of a weaker estimate:
\[
|\theta_3(x) - x/2| \leq \begin{cases} 0.16188x & \text{for} \ x \geq 11 \\ 1.780719\sqrt{x} & \text{for} \ x < 11. \end{cases}
\]

So
\[
\theta_3(3x) \geq \frac{3x}{2} - 0.16188 \cdot 3x \cong 1.014x > x \quad \text{for} \ 3x \geq 11.
\]
References

[1] A. N. Andrianov. Euler products that correspond to Siegel’s modular forms of genus 2. *Uspehi Mat. Nauk*, 29(3 (177)):43–110, 1974.

[2] A. N. Andrianov. *Quadratic forms and Hecke operators*, volume 286 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1987.

[3] A. N. Andrianov and V. G. Zhuravlev. *Modular forms and Hecke operators*, volume 145 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1995. Translated from the 1990 Russian original by Neal Koblitz.

[4] T. Arakawa. Vector-valued Siegel’s modular forms of degree two and the associated Andrianov $L$-functions. *Manuscripta Math.*, 44(1-3):155–185, 1983.

[5] J. Bergström, C. Faber, and G. van der Geer. Siegel modular forms of genus 2 and level 2: cohomological computations and conjectures. *Int. Math. Res. Not. IMRN*, 2008.

[6] J. Bergström, C. Faber, and G. van der Geer. Siegel modular forms of degree three and the cohomology of local systems. *Selecta Math.*, to appear.

[7] S. Chow and A. Ghitza. Distinguishing eigenforms modulo a prime ideal. *arXiv:1304.1832*, 2013.

[8] C. Faber and G. van der Geer. Sur la cohomologie des systèmes locaux sur les espaces de modules des courbes de genre 2 et des surfaces abéliennes. I. *C. R. Math. Acad. Sci. Paris*, 338(5):381–384, 2004.

[9] C. Faber and G. van der Geer. Sur la cohomologie des systèmes locaux sur les espaces de modules des courbes de genre 2 et des surfaces abéliennes. II. *C. R. Math. Acad. Sci. Paris*, 338(6):467–470, 2004.

[10] G. Faltings and C.-L. Chai. *Degeneration of abelian varieties*, volume 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1990. With an appendix by David Mumford.

[11] E. Freitag. *Siegelsche Modulfunktionen*, volume 254 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1983.

[12] A. Ghitza. Distinguishing Hecke eigenforms. *Int. J. Number Theory*, 7(5):1247–1253, 2011.

[13] B. H. Gross. On the Satake isomorphism. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 223–237. Cambridge Univ. Press, Cambridge, 1998.

[14] J. L. Hafner and L. H. Walling. Explicit action of Hecke operators on Siegel modular forms. *J. Number Theory*, 93(1):34–57, 2002.

[15] V. Kac and P. Cheung. *Quantum calculus*. Universitext. Springer-Verlag, New York, 2002.

[16] W. Kohnen. On Fourier coefficients of modular forms of different weights. *Acta Arith.*, 113(1):57–67, 2004.
[17] M. Ram Murty. Congruences between modular forms. In Analytic number theory (Kyoto, 1996), volume 247 of London Math. Soc. Lecture Note Ser., pages 309–320. Cambridge Univ. Press, Cambridge, 1997.

[18] O. Ramaré and R. Rumely. Primes in arithmetic progressions. Math. Comp., 65(213):397–425, 1996.

[19] M. Raum. Efficiently generated spaces of classical Siegel modular forms and the Böcherer conjecture. J. Aust. Math. Soc., 89(3):393–405, 2010.

[20] R. Schulze-Pillot. Siegel modular forms having the same $L$-functions. J. Math. Sci. Univ. Tokyo, 6(1):217–227, 1999.

[21] G. Shimura. On modular correspondences for $\text{Sp}(n, \mathbb{Z})$ and their congruence relations. Proc. Nat. Acad. Sci. U.S.A., 49:824–828, 1963.

[22] N.-P. Skoruppa. Computations of Siegel modular forms of genus two. Math. Comp., 58(197):381–398, 1992.

[23] G. van der Geer. Siegel modular forms and their applications. In The 1-2-3 of modular forms, Universitext, pages 181–245. Springer, Berlin, 2008.