Entanglement dynamics of a bipartite system in squeezed vacuum reservoirs

Smail Bougouffa\(^1\) and Awatif Hindi\(^2\)

\(^1\)Department of Physics, Faculty of Science, Taibah University, PO Box 30002, Madinah, Saudi Arabia
\(^2\)Physics Department, College of Science, PO Box 22452, King Saud University, Riyadh 11495, Saudi Arabia

E-mail: sbougouffa@taibahu.edu.sa and sbougouffa@hotmail.com

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Abstract
Entanglement plays a crucial role in quantum information protocols; thus the dynamical behavior of entangled states is of great importance. In this paper, we suggest a useful scheme that permits a direct measure of entanglement in a two-qubit cavity system. It is realized through cavity-QED technology utilizing atoms as flying qubits. To quantify entanglement we use the concurrence. We derive the conditions that ensure that the state remains entangled in spite of the interaction with the reservoir. The phenomenon of entanglement sudden death in a bipartite system subjected to a squeezed vacuum reservoir is examined. We show that the sudden death time of the entangled states depends on the initial preparation of the entangled state and the parameters of the squeezed vacuum reservoir.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction
In contrast to classical theories, quantum mechanics ensures the existence of nonlocal correlations between two systems spatially separated and without any direct interactions, which Schrödinger named entanglement \([1]\). Entanglement is a key feature of various quantum information processes such as quantum teleportation \([2]\), quantum dense coding \([3]\), quantum cryptography \([4]\) and quantum computing \([5]\). Due to the crucial role of entanglement in quantum information processes, the study of entanglement has attracted much interest in recent years. With numerous studies on entanglement, the main question that may be posed is how to find out whether a quantum state is entangled or not. For a pure bipartite state, the Schmidt decomposition \([6]\) can be used to decide whether the state is entangled and the degree of entanglement can be quantified by the partial von Neumann entropy \([7]\). Hence, in principle, the problem of entanglement for pure states of a bipartite system has been completely solved. On the other hand, quantum systems predictably undergo decoherence processes and quantum systems are mostly in mixed states. For the density matrix of a quantum system consisting of two subsystems, some criteria on entanglement have been established \([8–17]\). Moreover, the generation of entangled states has been investigated in various systems for atoms or ions, photons and quadrature-phase amplitudes of the electromagnetic field \([18–36]\). It is clear that experimental and theoretical studies of bipartite systems have made great progress in recent years.

Real quantum systems are necessarily subjected to their environments, and these reciprocal interactions often result in the dissipative evolution of quantum coherence and loss of useful entanglement. Decoherence may be investigated in both local and global dynamics, which may lead to the eventual deterioration of entanglement \([37]\). Eberly and Yu \([38]\) have investigated the time evolution of entanglement of a bipartite qubit system undergoing various modes of decoherence. In particular, they found that, even when there is no interaction, there are certain states whose entanglement decays exponentially with time, while for other closely related states, the entanglement vanishes abruptly in a finite time, which depends upon the initial preparation of the qubits, a phenomenon termed entanglement sudden death (ESD) \([38]\) and was recently observed in two sophistically designed experiments with photonic qubits \([39]\) and atomic band \([40]\). Furthermore, it has also been observed in cavity QED and trapped ion systems \([41]\). On the other hand, the ESD phenomenon
has motivated many theoretical investigations into other bipartite systems involving pairs of atomic, photonic and spin qubits [42–45], multipartite systems [46, 47] and spin chains [48–50]. In addition, ESD has also been investigated for different environments [37, 38, 52, 53, 55].

Although, numerous investigations of ESD in a variety of systems have been performed so far; the question of ESD in interacting qubits remains open [54]. On the other hand, from the quantum technological point of view, states that show exponential decay of entanglement, and therefore maintain some trace of this considerable correlation for an infinite time, are of importance. Although the vanishingly small entanglement present in the exponential tail will be of limited practical importance, it is of interest to identify exactly in what situations ESD will occur [55, 56].

Over the last few years, numerous methods with which the entanglement of quantum systems can be detected and described have been suggested. Possibly the most prominent example is the concurrence to illustrate the degree of entanglement for any two-qubits, concurrence [2] offers a convenient measure of the entanglement of formation. This has provided a very useful tool for the measurement of experimental quantum states and is commonly used today in evaluating the abilities of emerging quantum information technologies.

The purpose of this paper is to propose an efficient scheme for quantum teleportation to generate entangled number states of a bipartite system under the influence of a squeezed vacuum reservoir. Thus we investigate the time evolution of these entangled states. We examine the problem of ESD for this proposed scheme for different initial entangled states and the parameters of the squeezed vacuum reservoir.

2. The bipartite model system

Recently, Zubairy et al [58] suggested a new scheme during their examination of the quantum disentanglement eraser. In this simple scheme, the concurrence can be directly measured from the visibility of an explicit class of entangled states. We propose here the same scheme but with some adjustment. A two-level atom with the upper level $|e\rangle$ and the lower level $|g\rangle$ passes consecutively through cavity A, a squeezed vacuum reservoir and a cavity B as shown in figure 1. The incident atom is initially prepared in the excited state $|e\rangle$ and the decay of the radiation field inside a cavity may be described by a model in which the mode of the field of interest is coupled to a whole set of reservoir modes. We assume that initially the two cavities are in vacuum state $|0\rangle$ and the atom always leaves the setup in the ground state $|g\rangle$.

In the interaction picture and the rotating-wave approximation, the Hamiltonian is simply

$$H(t) = \hbar \sum_{j=A,B} \sum_{k} \left[ g_{k}^{(j)} b_{k}^{†} a_{j} e^{-i(\nu_{k} t)} + g_{k}^{(j)} a_{j}^{†} b_{k} e^{i(\nu_{k} t)} \right],$$

(1)

where $a_{j}$ ($j = A, B$) and $a_{j}^{†}$ are the destruction and creation operators of the mode of the electromagnetic field of frequency $\nu$ and $b_{k}^{†}$ and $b_{k}$ are the modes of cavity $j$ of frequency $\nu_{k}$ that damp the field and $g_{k}^{(j)}$ is the coupling constant of the interaction between the electromagnetic field and the cavity.

3. Entanglement dynamics in squeezed reservoirs

Here we are concerned with the case in which cavity fields are exposed to broadband squeezed vacuum reservoirs. From the general analysis of system–reservoir interactions, when the modes $b_{k}^{†}$ are initially in a squeezed vacuum, with the Hamiltonian (1), and the squeezing bandwidths of the squeezed reservoirs are much larger than the atomic linewidths, we can directly obtain the master equation for the reduced density matrix for the field in the cavities as [59]

$$\rho(t) = \sum_{j=A,B} \left[ -\frac{\kappa^{(j)}}{2} (N_{j} + 1)(a_{j}^{†} a_{j}) \rho(t) - 2a_{j}^{†} \rho(t) a_{j}^{†} + \rho(t) a_{j}^{†} a_{j} \right]$$

$$- 2^{A} a_{j} a_{j}^{†} \rho(t) a_{j}^{†} + \rho(t) a_{j}^{†} a_{j}$$

$$- \frac{\kappa^{(j)}}{2} N_{j} (a_{j} a_{j}^{†} \rho(t) - 2 a_{j} a_{j}^{†} \rho(t) a_{j} + \rho(t) a_{j} a_{j}^{†}) + \frac{\kappa^{(j)}}{2} M_{j} (a_{j} a_{j}^{†} \rho(t) - 2 a_{j} \rho(t) a_{j} + \rho(t) a_{j} a_{j}^{†})$$

$$+ \frac{\kappa^{(j)}}{2} M_{j}^{∗} (a_{j} a_{j}^{†} \rho(t) - 2 a_{j}^{†} \rho(t) a_{j}^{†} + \rho(t) a_{j}^{†} a_{j}) \right].$$

(2)

where $\kappa^{(j)}$ ($j = A, B$) is the decay rate in the cavity, $N_{j} = \sinh^{2}(r_{j})$ and $M_{j} = \cosh(r_{j}) \sinh(r_{j}) \exp(-i\theta_{j})$, with $r_{j}$ being the squeeze parameter and $\theta_{j}$ being the reference phase for the squeezed fields that surround cavities A and B. If $N_{j} = M_{j} = 0$, the remaining terms are due to vacuum fluctuations.

To investigate the effect of interaction in the bipartite on decoherence, we have to investigate the dynamics of bipartite entanglement. Furthermore, the concept of concurrence was initiated by the original work of Hill and Wootters [57], where the closed expression for the entanglement of formation of a system of two qubits was derived. They established that the entanglement of formation is a convex monotonically increasing function of the concurrence. Here we use concurrence to illustrate the degree of entanglement for any
bipartite system. This measure satisfies the necessary and sufficient condition for being a good measure of entanglement for a $2 \times 2$ system. The concurrence varies from $C = 0$ for a separable state to $C = 1$ for a maximally entangled state. An explicit expression for concurrence can be written as

$$C(t) = \max(0, \sqrt{\lambda_1 - \sqrt{\lambda_2 - \sqrt{\lambda_3 - \sqrt{\lambda_4}}}}),$$

(3)

where $\lambda$’s are eigenvalues of the non-Hermitian matrix $\rho(t)\tilde{\rho}(t)$ arranged in decreasing order of magnitude. The matrix $\rho(t)$ is the density matrix for the bipartite and the matrix $\tilde{\rho}(t)$ is given by

$$\tilde{\rho}(t) = (\sigma^A_\gamma \otimes \sigma^B_\gamma) \rho^*(t)(\sigma^A_\gamma \otimes \sigma^B_\gamma),$$

(4)

where $\rho(t)^*$ is the complex conjugation of $\rho(t)$, and $\sigma_\gamma$ is the Pauli matrix given in quantum mechanics. In the general case, we consider the field states in the Fock basis in two identical high-$Q$ cavities A and B that represent a bipartite system surrounding the entangled field as

$$|\Psi\rangle_{AB}(0) = \alpha_1|0_A0_B⟩ + \alpha_2|0_A1_B⟩ + \alpha_3|1_A0_B⟩ + \alpha_4|1_A1_B⟩,$$

(5)

where $\alpha_i$ ($i = 1, 2, 3, 4$) are the probability amplitudes with $\sum_{i=1}^{4}|\alpha_i|^2 = 1$. We use the basis ($|1⟩ = |0_A0_B⟩$, $|2⟩ = |0_A1_B⟩$, $|3⟩ = |1_A0_B⟩$, $|4⟩ = |1_A1_B⟩$) to define the density matrix of the two-qubit system. The equations of motion in terms of density matrix elements can be obtained using master equation (2).

4. Results and conclusion

Here we will consider some interesting initial entangled states for the bipartite which can be prepared, and which have potential applications in quantum information processing tasks [56]. We will begin with an examination of the EPR states, which are perceptions in quantum information science, a vital part of quantum teleportation, and characterize the simplest possible examples of entanglement.

(i) Assume that the initially entangled state of the field in two cavities is in a NOON state given by

$$|\Psi\rangle_{AB}(0) = \alpha|0_A1_B⟩ + \sqrt{1 - \alpha^2}|1_A0_B⟩.$$  

(6)

This kind of state can be generated as we have mentioned in [56] and has potential applications in Heisenberg-limited metrology and quantum lithography [60]. The solutions of the master equation for this initial NOON state case are presented in the appendix.

(ii) Consider now the initially entangled bipartite to be in another EPR state given by

$$|\Psi\rangle_{AB}(0) = \alpha|0_A0_B⟩ + \sqrt{1 - \alpha^2}|1_A1_B⟩.$$  

(7)

This kind of state can be prepared as we have mentioned in [56]. States like these have been realized in experiments with trapped ions [61].

The solution of equation (2) depends on the initial state of the two-bipartite system. We can show that, for these two classes of initial states that were considered above, the solution of equation (2) has a matrix shape in the representation spanned by the two-bipartite states

$$\rho(t) = \begin{pmatrix} \rho_{11}(t) & 0 & 0 & \rho_{14}(t) \\ 0 & \rho_{22}(t) & \rho_{23}(t) & 0 \\ 0 & \rho_{32}(t) & \rho_{33}(t) & 0 \\ \rho_{41}(t) & 0 & 0 & \rho_{44}(t) \end{pmatrix}.$$  

(8)

With this form of the density matrix, we can show that the concurrence can be expressed as

$$C(t) = \max(0, \tilde{C}_1(t), \tilde{C}_2(t)).$$  

(9)

where

$$\tilde{C}_1(t) = 2[\sqrt{\rho_{23}(t)\rho_{32}} - \sqrt{\rho_{11}(t)\rho_{44}(t)}],$$  

(10)

$$\tilde{C}_2(t) = 2[\sqrt{\rho_{14}(t)\rho_{41}} - \sqrt{\rho_{22}(t)\rho_{33}(t)}].$$  

(11)

Using this formalism we can investigate the dynamics of entanglement for the two initial states that we considered above. However, in the case of squeezed reservoirs, we find that ESD always happens for two initial entangled states with $0 < \alpha < 1$. This is shown clearly by the numerical results plotted in figure 2. In figure 3, the time
evolution of concurrence is plotted for different values of the degree of squeezing and $\alpha = \frac{1}{2}$. (a) for the first initial NOON state and (b) for the second EPR initial state.

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Appendix. Solutions of equations of motion of the density matrix elements for squeezed vacuum reservoirs

The equation of motion of density matrix elements for the state (equation (6)) can be obtained and, for the sake of simplicity, we assume that the cavities are identical $k^{(A)} = k^{(B)} = k$. $N_A = N_B = N$ and $M_A = M_B = M$. On solving these equations of motion, we obtain the time evolution of the density matrix elements

$$
\rho_{11}(t) = \frac{a + 3}{8b^2} \left[ 1 + a + 2 \sinh(bk\alpha) - \frac{1 + a}{b} \cosh(bk\alpha) \right] e^{-\alpha kt},
$$

$$
\rho_{22}(t) = \frac{1}{16} \left[ (16\alpha^2 + 4b^2 - 4) + \left( 1 - \frac{1}{b^2} \right) \cosh(bk\alpha) \right] e^{-\alpha kt},
$$

$$
\rho_{33}(t) = \frac{1}{16} \left[ (-16\alpha^2 + 4b^2 + 12) + \left( 1 - \frac{1}{b^2} \right) \cosh(bk\alpha) \right] e^{-\alpha kt},
$$

$$
\rho_{44}(t) = \frac{1}{8b^2} \left[ a^2 - 1 + \frac{1}{4}(a - 1)(2b \sinh(bk\alpha)) \right] e^{-\alpha kt},
$$

$$
\rho_{14}(t) = -\frac{M}{|M|} \alpha \sqrt{1 - a^2} \sinh(|M|\alpha) e^{-\alpha kt},
$$

$$
\rho_{32}(t) = \alpha \sqrt{1 - a^2} \cosh(|M|\alpha) e^{-\alpha kt}
$$

and $\rho_{21}(t) = \rho_{14}(t) = 0$, $\rho_{31}(t) = \rho_{15}(t) = 0$, $\rho_{32}(t) = \rho_{53}(t)$, $\rho_{41}(t) = \rho_{14}(t)$, $\rho_{42}(t) = \rho_{15}(t) = 0$, $\rho_{31}(t) = \rho_{52}(t) = 0$, where $a = 4N + 1$ and $b^2 = 8N^2 + 8N + 1$.

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