Self-Synchronization Theory of Circular Symmetrical Four Motors

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Abstract. The synchronization characteristics of the circularly symmetrical four-vibrators vibration system are studied. Firstly, the dynamic model of this type of vibration system is established, and the Lagrange equation is used to derive the differential equation of motion. Secondly, the steady-state solution of the vibration system is obtained by averaging the phase of the eccentric rotor of the four exciters in the vibration system, and Hamilton is used. The frame principle deduces the synchronization conditions and stability conditions for the four vibration exciters of this type of vibration system to achieve synchronous motion. Finally, the correctness of the theoretical analysis is verified by numerical simulation. The relevant research results can provide a theoretical basis for the design of the circularly symmetrical four-vibrators vibration machine.

1. Introduction
Synchronization is a common phenomenon in nature. Since the former Soviet scientist blekhma [1-2] put forward the self synchronization theory of vibration system in the 1960s, many domestic scholars have discussed and studied the self synchronization theory from different angles. In the 1970s, Wen [3-6] introduced the small parameter average perturbation method to obtain the conditions for system stability and synchronization. With the increase of relevant research, the research methods become more and more rigorous, and the research on the synchronization phenomenon of vibration system is gradually developing in the direction of multi machine, multi mass and mass spatial arrangement: Zhang [7-10] designed a plane single mass synchronous vibration system driven by three machines; Fang [11] studied the vibration synchronization mechanism of the space three machine vibration system and analyzed the dynamic characteristics of the vibration system. Huang [12] established the dynamic model of the vibration system excited by four exciters and proposed an accurate adjacent cross coupling strategy control method. Du [13] studied the vibration system of double eccentric rotor with nonlinear coupling and rotating in the same direction and its synchronization.

In this paper, a new type of four vibrator is designed, which can effectively enhance the amplitude and exciting force, and is suitable for more working conditions. The vibration system with circumferential symmetry design can also make the vibration more stable. This paper provides a theoretical basis for the research of vibration synchronization of four exciters, and has certain reference value and guiding significance for the development and analysis of similar systems.
2. System dynamics model

The circular symmetrical four vibrator vibration system is mainly composed of rigid body, vibrator and support spring. The built dynamic model is shown in Figure 1. The excitation motors 1, 2 and 3 are arranged symmetrically in the circumference, the excitation motor 4 is in the center of the circumference, and the support springs are installed on the fixed frame symmetrically. Under normal working conditions, the excitation motor 1 and excitation motors 2, 3 and 4 rotate in reverse at the same speed; The excitation motors 2, 3 and 4 rotate in the same direction at the same speed. In order to simplify the analysis, the excitation motor in Figure 1 is represented by an eccentric rotor. The mass of the four eccentric rotors is \( m_i \), \( i = 1, 2, 3, 4 \). The rotation center is \( O_i \), coincides with the barycenter \( O \), the radius of gyration are \( r_i \), the phases are \( \varphi_i \), the distances from the rotation center of eccentric rotor to the barycenter of the machine body are \( l_i \). The angles between the connection line from the rotation center of the three exciters to the barycenter of the vibration system and the x-axis are \( \beta_i \). The vibration system has three DOF, namely the horizontal direction \( x \) movement, the vertical direction \( y \) movement, and the movement \( \psi \) parallel to the \( Oxy \) plane.

![Figure 1. The dynamic model of the vibration system of the circularly symmetric four-vibrator](image)

The kinetic energy of the system:

\[
T = \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 \right) + \frac{1}{2} J m \dot{\varphi}^2 + \frac{1}{2} \sum_{i=1}^{4} J_i \dot{\varphi}_i^2 + \frac{1}{2} \sum_{i=1}^{4} m_i \left( \dot{x}_i^2 + \dot{y}_i^2 \right)
\]  

(1)

where: \( m \) — body mass; \( J_m \) — moment of inertia; \( J_i \) — moment of inertia of the eccentric block of the exciter around the respective center of rotation; \( x_i, y_i \) — the horizontal and vertical coordinates of the eccentric block of the exciter in the \( Oxy \) coordinate system.

The potential energy of the system:

\[
V = \frac{1}{4} k_x \left[ (x + \psi L_x)^2 + (x - \psi L_x)^2 \right] + \frac{1}{4} k_y \left[ (y + \psi L_y)^2 + (y - \psi L_y)^2 \right]
\]

(3)

where: \( k_x, k_y \) — spring stiffness. \( L_x, L_y \) — the distance between the center of mass of the device and the spring connection point in the \( x \) and \( y \) directions.

System energy dissipation function:

\[
D = \frac{1}{4} f_x \left[ (\dot{x} + \omega L_x)^2 + (\dot{x} - \omega L_x)^2 \right] + \frac{1}{4} f_y \left[ (\dot{y} + \omega L_y)^2 + (\dot{y} - \omega L_y)^2 \right] + \frac{1}{2} \sum_{i=1}^{4} f_i (\dot{\varphi}_i + \psi)
\]

(4)

where: \( f_x, f_y \) — spring damping coefficient; \( f_i \) — damping of shaft of excitation motor \( i \).

For the system, the Lagrange equation:

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i}
\]
Generalized force of the system:

\[
Q_i = \left\{ Q_{i,1}, Q_{i,2}, Q_{i,3}, Q_{i,4}, Q_{i,5}, Q_{i,6} \right\}^T
\]

where: \( Q_i = Q_j = Q_k = Q_l = 0 \), \( Q_m = T_e \), \( T_n \) —— electromagnetic torque of exciting motor.

Substituting the system kinetic energy, potential energy, and energy dissipation function expressions into Lagrange equations, the differential equations of motion in the x,y, \( \psi \) directions of the system and the rotation equations of the four eccentric rotors can be obtained:

\[
\begin{align*}
M\ddot{x} + f_\psi \dot{x} + k_x x &= m_r \left( \dot{\phi}_i \cos \phi_i + \dot{\phi}_i \sin \phi_i \right) - \sum_{i=1}^{d} m_r \left( \dot{\phi}_i^2 \cos \phi_i + \dot{\phi}_i \sin \phi_i \right) \\
M\ddot{y} + f_\psi \dot{y} + k_y y &= m_r \left( \dot{\phi}_i \sin \phi_i - \dot{\phi}_i \cos \phi_i \right) + \sum_{i=1}^{d} m_r \left( \dot{\phi}_i^2 \sin \phi_i - \phi_i \cos \phi_i \right) \\
J\ddot{\psi} + f_\psi \dot{\psi} + k_\psi \psi &= f_1 \left( \dot{\phi}_1 - \dot{\psi} \right) + l m_r \left[ \dot{\phi}_1^2 \sin (\phi_1 - \beta_1 - \psi) - \dot{\phi}_1 \cos (\phi_1 - \beta_1 - \psi) \right] \\
&+ \sum_{i=2}^{d} m_r \left[ \dot{\phi}_i^2 \sin (\phi_i - \beta_i + \psi) - \phi_i \cos (\phi_i - \beta_i + \psi) \right] - f_1 \left( \dot{\phi}_1 + \dot{\psi} \right) \\
\left( j_0 + m_r^2 \right) \dot{\phi}_i + f_1 \left( \dot{\phi}_i - \dot{\psi} \right) &= T_m - T_0 - m_r \left[ \dot{y} \cos \phi_1 - \dot{x} \sin \phi_1 - l \dot{\psi} \cos (\phi_1 - \beta_1 + \psi) \right] - l \dot{\psi}^2 \sin (\phi_1 - \beta_1 - \psi) (i = 1) \\
\left( j_0 + m_r^2 \right) \dot{\phi}_i + f_1 \left( \dot{\phi}_i + \dot{\psi} \right) &= T_m - T_0 - m_r \left[ \dot{y} \cos \phi_i - \dot{x} \sin \phi_i - l \dot{\psi} \cos (\phi_i - \beta_i - \psi) \right] + l \dot{\psi}^2 \sin (\phi_i - \beta_i + \psi) (i = 2, 3, 4)
\end{align*}
\]

where: \( M = m + \sum_{i=1}^{d} m_i \) —— the total mass of the body and the excitation motor; \( f_\psi = \frac{1}{2} \left( f_i l_i^2 + f_i l_i^2 \right) \) —— body rotation damping coefficient; \( J = J_m + \sum_{i=1}^{d} \left( \dot{x}_i^2 + \dot{y}_i^2 \right) \) —— the total moment of inertia of the body and the excitation motor; \( k_\psi = \frac{1}{2} \left( k_i l_i^2 + k_i l_i^2 \right) \) —— spring rotation stiffness.

3. System stability and synchronization analysis

3.1 Steady state response of vibration system

Suppose the phase of the eccentric rotor 4 in the circumferential center of the system is \( \phi \), and the phase difference of the eccentric rotor 1, 2, 3 and the eccentric rotor 4 are: \( 2\alpha_1 \), \( 2\alpha_2 \), \( 2\alpha_3 \). Suppose the excitation motor 4 is a standard vibration exciter, and its angular velocity is a constant constant. Within one rotation period of the eccentric rotor 4, we can get:

\[
\omega = \frac{1}{T_L} \int_{t}^{t+T_L} \dot{\phi}(t)dt = C
\]

The phase differences of eccentric rotors 1, 2, 3 and 4 are:

\[
\begin{align*}
\phi_1 - \phi &= 2\alpha_1 \\
\phi_2 - \phi &= 2\alpha_2 \\
\phi_3 - \phi &= 2\alpha_3
\end{align*}
\]

Substituting the above equation into equation 7 and ignoring the second-order derivative term in the differential equation of motion, the steady-state response solution of the differential equation of motion of the system is:
3.2 Synchronization conditions of the vibration system

In the vibration system, the Lagrange function is: 

\[ L = T - V \]

When the eccentric rotor rotates one circle in time \( t \), its Hamilton action can be expressed as:

\[ I = \int_0^T L \, dt = \frac{\pi}{2} \sum_{i=1}^{3} \left( C_{x,m^i,r^i}\sin\alpha_i + C_{y,m^i,r^i}\cos\alpha_i \right) + C_{y}\tan\Psi \]  \hspace{1cm} (11)

where,

\[ E = \sum_{i=0}^{4} \left( C_{x,m^i}\sin\phi_i + C_{y,m^i}\cos\phi_i \right) + C_{y}\tan\Psi \]

\[ C_x = \frac{\cos^2\alpha}{m_x}, \quad C_y = \frac{\cos^2\alpha}{m_y}, \quad C_{y} = \frac{\cos^2\alpha_y}{m_y} \]

According to the Hamilton principle, the integral sum of the variation of the Hamilton action of the system and the virtual work of the non-conservative force acting on the system in a period is zero, that

\[ \delta I + \int_{t=0}^{2\pi} \sum_{i=1}^{3} (F_i \delta q_i) dt = 0 \]  \hspace{1cm} (12)

\( F_i \) —— non-generalized force of the system, \( q_i \) —— generalized coordinates. Substitute \( I \) into the above formula

\[ \sin\alpha_i = D_i \quad (i = 1, 2, 3) \]  \hspace{1cm} (13)

where, \( D_i = C_{y}\tan\Psi \cdot \left( C_{x,m^i,r^i}\sin\alpha_i + C_{y,m^i,r^i}\cos\alpha_i \right) + (T_m - T_{m^i}) \)

because of \( |\sin\alpha_i| \leq 1 \), so \( D_i \leq 1 \). It is the condition for this type of vibration system to achieve self-synchronization.

3.3 Synchronous stability conditions of the vibration system

When the Hamilton effect \( I \) to \( \alpha_i \) has a second derivative greater than zero, which is \( \frac{d^2I}{d\alpha^2} > 0 \), corresponding to the stable operation of each motor. Find the second-order partial derivatives of \( \alpha_i \) separately for \( I \)

\[ \cos(2\alpha_i)(C_x - C_{x^i})m_{r^i} > 0 \quad (i = 1, 2, 3) \]  \hspace{1cm} (14)

If \( C_x - C_{x^i} > 0 \), then the synchronization stability condition of the vibration system is that the phase difference between the motor 1, 2, 3 and the standard motor is up to: \( \alpha_i \in \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \); if \( C_x - C_{x^i} < 0 \), then the synchronization stability condition of the vibration system is that the phase difference reaches:
\[ \alpha_i \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) ; \text{ if } C_i > C \Rightarrow = 0, \text{ then the vibration system cannot achieve synchronous and stable operation at this time.} \]

4. System Synchronous Numerical Simulation

In order to verify the rationality of the synchronous design of the system, Simulink is used to build a simulation model of the motor-vibration system, the Runge-Kutta algorithm and variable step length control strategy are used, and the system parameters in Table 1 are substituted for simulation analysis. When the phase difference between motors 1, 2, 3 and standard motor 4 becomes stable, the vibration system enters a synchronized state.

| M/kg (m₁~m₄)/kg | J/(kg m²) (J₁~J₄)/(kg m²) | (r₁~r₄)/m | (l₁~l₄)/m | l₄/m |
|------------------|---------------------------|-------------|-------------|-------|
| 148              | 3.5                       | 17          | 0.01        | 0.05  | 0.5   | 0       |

As shown in Figure 2(a), the angular velocity of the standard exciter is \( \omega_4 = 157 \text{ rad/s} \). In stable operation, the angular velocities of the four vibrator rotors are synchronized. And the four vibration exciters all change with approximately sine motion at 156.7~157.05 rad/s. Figure 2(b) is the time-varying curve of the phase difference between the center exciter and the outer exciters. It can be seen that in the process of reaching the synchronization state, the phase difference of the four exciters gradually stabilizes. Figure 3(c) to (e) shows the response of the vibration system in the three directions of \( x, y, \psi \), in the starting phase, the three-direction response changes irregularly; after entering the synchronization state, the three-direction response changes sinusoidally. Figure 2(f) shows that the trajectory of the system's center of mass is an ellipse. From the above results, it can be seen that the system has reached the synchronization state, and the synchronization design of this type of vibration system has certain rationality.

5. Conclusion

In this paper, a new type of four-vibration exciter was designed, and the distribution adopted a circularly symmetrical design of the vibration system. The Lagrange equation was introduced when solving the motion differential equation of this four-vibration system; the stability of the vibration system was obtained by derivation and calculation. State solution, and use the Hamiltonian principle to obtain the
variables that need to be controlled for the synchronization and stable operation of the four exciters of the system. The electromechanical coupling simulation model was built by Simulink to verify the rationality of the system design.

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