Composition of Two Potentials

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Abstract
Given two potentials \( V_0 \) and \( V_1 \) together with a certain nodeless solution \( \varphi_0 \) of \( V_0 \), we form a composition of these two potentials. If \( V_1 \) is exactly solvable, the composition is exactly solvable, too. By combining various solvable potentials in one-dimensional quantum mechanics, a huge variety of solvable compositions can be made.

1 Introduction
Some years ago, two of the present authors developed a method of composing two potentials \( V_0 \) and \( V_1 \) in a special setting [1]. The potentials were spherically symmetric and of short range and were explicitly solvable at zero energy. The starting potential \( V_0 \) should sustain no bound states. Then the composed potential was explicitly solvable at zero energy, too. The main motivation was applications to quantum mechanical scattering problems [2] and the radial Schrödinger equations.

In the present paper, we show that the composition of two potentials \( V_0 \) and \( V_1 \) in generic one dimensional quantum mechanics can be defined in a much more general setting. The only requirement is that \( V_0 \) should have a nodeless solution satisfying certain boundary conditions. If \( V_1 \) is exactly solvable, so is the composed potential \( V_C \). If \( V_1 \) has also a nodeless solution satisfying boundary conditions, a further composition with another potential \( V_2 \) can be made, as emphasised in the earlier papers [1]. We provide several examples of \( V_0 \),

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all exactly solvable, together with an infinite number of nodeless solutions satisfying the boundary conditions. These nodeless solutions are called virtual state wavefunctions [3, 4], which have played an important role in the construction of various rational deformations of exactly solvable potentials and the multi-indexed orthogonal polynomials [3]–[12] as the main part of the eigenfunctions of the deformed solvable systems.

This paper is organised as follows. In section 2, we formulate the composition of two potentials in the most general setting. Derivation is quite elementary. Section 3 provides explicit examples of nodeless solutions (virtual state wavefunctions) belonging to the well-known solvable potentials [13, 14]; the radial oscillator potential in §3.1, Pöschl-Teller potential in §3.2, hyperbolic Pöschl-Teller potential in §3.3, Rosen-Morse potential in §3.4 and Eckart potential in §3.5. The final section is for a short summary and comments.

2 General Composition

The main ingredients of the present theory are two quantum mechanical systems in one dimension with smooth potentials $V_0$ and $V_1$:

\[
\begin{align*}
\text{System 0 :} & \quad \mathcal{H}_0 = -\frac{d^2}{dx^2} + V_0(x), \quad a < x < b, \\
& \mathcal{H}_0\varphi_0(x) = \tilde{E}_0\varphi_0(x), \quad (2.2) \\
\text{System 1 :} & \quad \mathcal{H}_1 = -\frac{d^2}{dx^2} + V_1(x), \quad c < x < d, \\
& \mathcal{H}_1\phi_m(x) = E_m\phi_m(x), \quad m = 1, \ldots, \quad (2.4)
\end{align*}
\]

Since they are both quantum mechanical systems, a finite boundary can only be realised by a strong repulsive singularity. Otherwise, the wave functions would leak beyond that boundary and the quantum system is ill-defined. The System 0 needs not be solvable. The only requirement is that it has a nodeless real solution $\varphi_0(x)$ with proper boundary conditions (A) and (B), to be specified later. The System 0 may or may not have some discrete eigenstates, either finitely many or infinite in number. The System 1 is a general quantum system. Nothing special is required of $V_1(x)$ for the construction of the composition of the two potentials $V_0$ and $V_1$.

It is elementary to show that the following function

\[
\chi_0(x) \overset{\text{def}}{=} \varphi_0(x) \left( \alpha + \int_x^b \frac{dt}{\varphi_0^2(t)} \right), \quad \alpha \overset{\text{def}}{=} \frac{1}{d - c}, \quad (2.5)
\]
is another solution of $\mathcal{H}_0$ with the same energy $\tilde{\mathcal{E}}_0$:

$$\mathcal{H}_0 \chi_0(x) = \tilde{\mathcal{E}}_0 \chi_0(x).$$

(2.6)

Let us introduce a mapping function $\psi_0(x), a < x < b$,

$$\psi_0(x) \overset{\text{def}}{=} c + \frac{\varphi_0(x)}{\chi_0(x)} = c + \frac{1}{\alpha + \int_x^b \frac{dt}{\varphi_0'(t)}},$$

(2.7)

which is monotonously increasing

$$\frac{d\psi_0(x)}{dx} = \frac{1}{\chi_0'(x)} > 0.$$  

We assume the following boundary conditions on $\varphi_0(x)$:

$$x \to a \quad \int_x^b \frac{dt}{\varphi_0'(t)} \to \infty \quad \text{(A)},$$

$$x \to b \quad \int_x^b \frac{dt}{\varphi_0'(t)} \to 0 \quad \text{(B)},$$

which mean

$$\psi_0(a) = c, \quad \psi_0(b) = c + \frac{1}{\alpha} = d.$$  

(2.8)

Thus $\psi_0(x)$ is a 1:1 onto mapping from $(a, b)$ to $(c, d)$.

The composition of two potentials, $V_0$ and $V_1$ is achieved by $\chi_0(x)$ and $\psi_0(x)$:

$$\mathcal{H}_C = -\frac{d^2}{dx^2} + V_C(x), \quad V_C(x) \overset{\text{def}}{=} V_0(x) - \tilde{\mathcal{E}}_0 + \frac{1}{\chi_0'(x)} V_1(\psi_0(x)), \quad a < x < b.$$  

(2.9)

With a solution $\phi_m(x)$ of $\mathcal{H}_1$, let us define

$$\phi_m^C(x) \overset{\text{def}}{=} \chi_0(x) \phi_m(\psi_0(x)), \quad m = 1, \ldots.$$  

(2.10)

It is elementary to show that $\phi_m^C(x)$ satisfies

$$\mathcal{H}_C \phi_m^C(x) = \frac{\mathcal{E}_m}{\chi_0'(x)} \phi_m^C(x), \quad m = 1, \ldots.$$  

(2.11)

If $\{\phi_m(x)\}$ are eigenfunctions of $\mathcal{H}_1$:

$$\int_a^b \phi_m(x) \phi_n(x) dx = h_m \delta_{mn}, \quad h_m > 0, \quad m, n = 1, \ldots,$$  

(2.12)

then $\{\phi_m^C(x)\}$ are orthogonal with each other with respect to the weight function $\frac{1}{\chi_0(x)}$:

$$\int_a^b \phi_m^C(x) \phi_n^C(x) \frac{dx}{\chi_0'(x)} = \int_a^b \phi_m(\psi_0(x)) \phi_n(\psi_0(x)) \frac{dx}{\chi_0'(x)}.$$  

(2.13)
\[
\int_c^d \phi_m(\psi_0)\phi_n(\psi_0)d\psi_0 = h_m \delta_{mn}, \quad m, n = 1, \ldots . \tag{2.14}
\]

If \( \mathcal{H}_1 \) is exactly solvable, so is the composed Hamiltonian \( \mathcal{H}_C \).

As stressed in \cite{1}, further compositions are straightforward, so long as the proper nodeless solutions of can be found. Suppose we have another Hamiltonian system \( \mathcal{H}_2 \) with a potential \( V_2 \)

\[
\text{System 2:} \quad \mathcal{H}_2 = -\frac{d^2}{dx^2} + V_2(x), \quad e < x < f,
\]

\[
\mathcal{H}_2 \Phi_m(x) = E_m \Phi_m(x), \quad m = 1, \ldots ,
\]

and a nodeless solution \( \varphi_1 \) of \( \mathcal{H}_1 \) with proper boundary conditions. The next step is essentially the same as before:

\[
\mathcal{H}_1 \varphi_1(x) = \tilde{\mathcal{E}}_1 \varphi_1(x), \quad \chi_1(x) \overset{\text{def}}{=} \varphi_1(x) \left( \alpha_1 + \int_x^d \frac{dt}{\varphi_1^2(t)} \right), \quad \alpha_1 = (f - e)^{-1},
\]

\[
\mathcal{H}_1 \chi_1(x) = \tilde{\mathcal{E}}_1 \chi_1(x), \quad \psi_1(x) \overset{\text{def}}{=} e + \frac{\varphi_1(x)}{\chi_1(x)} = e + \frac{1}{\alpha_1 + \int_x^d \frac{dt}{\varphi_1^2(t)}}, \quad \frac{d\psi_1(x)}{dx} = \frac{1}{\chi_1^2(x)} > 0.
\]

The second composed Hamiltonian is

\[
\mathcal{H}_C^2 = -\frac{d^2}{dx^2} + V_C^2(x), \quad a < x < b,
\]

\[
V_C^2(x) \overset{\text{def}}{=} V_0(x) - \tilde{\mathcal{E}}_0 + \frac{1}{\chi_0^4(x)} \left( V_1(\psi_0(x)) - \tilde{\mathcal{E}}_1 \right) + \frac{1}{\chi_0^4(x)\chi_1^4(\psi_0(x))} V_2(\psi_1(\psi_0(x)));
\]

with its solutions

\[
\Phi_m^{C2}(x) \overset{\text{def}}{=} \chi_0(x)\chi_1(\psi_0(x))\Phi_m(\psi_1(\psi_0(x))), \quad m = 1, \ldots ,
\]

\[
\mathcal{H}_C^2 \Phi_m^{C2}(x) = \frac{E_m}{\chi_0^4(x)\chi_1^4(\psi_0(x))} \Phi_m^{C2}(x).
\]

The composition processes can go on indefinitely.

## 3 Explicit Examples

Several explicit examples of the compositions were demonstrated in \cite{1} in connection with certain scattering problems \cite{2}. For example, given two spherically symmetric and short range potentials for which the radial Schrödinger equations can be solved at zero energy, the composition can also be solved at zero energy. Here we give several explicit examples of the System 0, together with the proper nodeless solutions \( \varphi_0(x) \). Those examples presented in
will not be re-listed here. Although System 0 needs not be solvable, most examples listed below are, in fact, exactly solvable. The reason is quite trivial. The proper nodeless solutions can be most easily found when the system is solvable. The nodeless solutions listed below are called virtual state wavefunctions \[3, 4\]. Their energies are below the ground state energy and they together with their inverses are square non-integrable. Thus the conditions (A) and (B) are satisfied. In most cases, they are obtained from the eigenfunctions by discrete symmetry transformations. They have played an essential role in the rational deformations of solvable potentials, creating the multi-indexed and exceptional orthogonal polynomials \[3, 5–12\], which are new species of orthogonal polynomials satisfying second order differential equations but not the three term recurrence relations \[15\].

See for example \[13, 14\] for a general review of exactly solvable potentials. We follow \[13\] for the naming of solvable potentials. Detailed information of these systems, the symmetry, eigenfunctions, various virtual state wave functions and their generalisation can be found in \[3, 4\]. The first two examples have infinitely many discrete eigenstates.

### 3.1 Radial oscillator

The radial oscillator potential is

\[ V_0(x) \overset{\text{def}}{=} x^2 + \frac{g(g-1)}{x^2}, \quad 0 < x < \infty, \quad g > 1/2. \]  

The lower boundary \( x = 0 \) is a regular singularity with the characteristic exponents \( g \) and \( 1-g \). In terms of a discrete symmetry transformation \( x \rightarrow ix \), the eigenfunction with degree \( v \) is mapped to a virtual state wavefunction

\[ \varphi_{0,v}(x) \overset{\text{def}}{=} e^{x^2/2}x^gL_v^{(g-1/2)}(-x^2), \quad \tilde{E}_{0,v} = -4v - 2g + 3, \quad v = 0, 1, \ldots, \]  

in which \( L_n^{(a)}(x) \) is the Laguerre polynomial of degree \( n \) in \( x \). The nodelessness of \( \varphi_{0,v}(x) \) is obvious, since all the zeros of the Laguerre polynomial \( L_n^{(a)}(x) \) are positive. The boundary conditions (A) and (B) are satisfied. The Wronskian of these virtual state wave functions

\[ W[\varphi_{0,v_1}, \ldots, \varphi_{0,v_M}](x) \]  

can also be used for compositions, as demonstrated for the multi-indexed orthogonal polynomials \[3\]. The situation is the same for all the other examples below. The radial oscillator itself provides an infinite number of different compositions.
3.2 Pöschl-Teller

The Pöschl-Teller potential is

\[ V_0(x) = \frac{g(g-1)}{\sin^2 x} + \frac{h(h-1)}{\cos^2 x}, \quad 0 < x < \frac{\pi}{2}, \quad g, h > 1/2. \] (3.4)

The lower boundary \( x = 0 \) is a regular singularity with the characteristic exponents \( g \) and \( 1 - g \). The upper boundary \( x = \frac{\pi}{2} \) is also a regular singularity with the characteristic exponents \( h \) and \( 1 - h \). In terms of a discrete symmetry transformation \( h \rightarrow 1 - h \), the eigenfunction with lower degree \( v \) is mapped to a virtual state wave function

\[ \varphi_{0,v}(x) \equiv (\sin x)^g (\cos x)^{1-h} P_{\nu}^{(g-1/2,1/2-h)}(\cos 2x), \] (3.5)

\[ \tilde{E}_{0,v} \equiv (h - h + 1 + 2v)^2, \quad v = 0, 1, \ldots, [(h - 1/2)']. \] (3.6)

Here \( P_n^{(\alpha,\beta)}(x) \) is the Jacobi polynomial of degree \( n \) in \( x \) and \( [a]' \) denotes the greatest integer less than \( a \). It is easy to see that the boundary conditions (A) and (B) are satisfied. The expansion of the Jacobi polynomial can be used to demonstrate the nodelessness. (See (3.2) of [16].)

3.3 Hyperbolic Pöschl-Teller

The hyperbolic Pöschl-Teller potential is

\[ V_0(x) = \frac{g(g-1)}{\sinh^2 x} - \frac{h(h+1)}{\cosh^2 x}, \quad 0 < x < \infty, \quad h > g > 1/2, \] (3.7)

has \( [(h - g)/2]' + 1 \) discrete eigenstates. The lower boundary \( x = 0 \) is a regular singularity with the characteristic exponents \( g \) and \( 1 - g \). By the discrete symmetry transformation \( h \rightarrow -(h + 1) \), the eigenfunction with degree \( v \) is mapped to a virtual state wave function

\[ \varphi_{0,v}(x) \equiv (\sinh x)^g (\cosh x)^{h+1} P_{\nu}^{(g-1/2,h+1/2)}(\cosh 2x), \] (3.8)

\[ \tilde{E}_{0,v} \equiv -(h - g)^2 - (2v + 2g + 1)(2v + 2h + 1), \quad v = 0, 1, \ldots, [(h - g)/2]'. \] (3.9)

It is easy to see that the boundary conditions (A) and (B) are satisfied. The expansion of the Jacobi polynomial can be used to demonstrate the nodelessness.

The overshoot eigenfunctions [4] also provides the proper nodeless solutions. They have exactly the same form as the eigenfunctions but their degrees are much higher than that of the highest eigenstate so that their energies are lower than the ground state energy:

\[ \varphi_{0,v}(x) \equiv (\sinh x)^g (\cosh x)^{-h} P_{\nu}^{(g-1/2,-h-1/2)}(\cosh 2x), \] (3.10)
\[ \tilde{E}_{0,v} \overset{\text{def}}{=} -(h - g - 2v)^2, \quad v > h - g. \]  

(3.11)

It is easy to see that the boundary conditions (A) and (B) are satisfied. The expansion of
the Jacobi polynomial can be used to demonstrate the nodelessness.

3.4 Rosen-Morse

This potential is
\[ V_0(x) \overset{\text{def}}{=} -\frac{h(h + 1)}{\cosh^2 x} + 2\mu \tanh x, \quad -\infty < x < \infty, \quad h > \sqrt{\mu} > 0. \]

(3.12)

The system has finitely many discrete eigenstates \([h - \sqrt{\mu}]' + 1\). The overshoot eigenfunctions
\( h < v < h + \frac{\mu}{h} \) provides proper nodeless solutions:
\[ \varphi_{0,v}(x) \overset{\text{def}}{=} e^{-\frac{\mu}{h-v} x} \cosh x \cdot \frac{h-v}{h-v} P_v^{(\alpha_v,\beta_v)}(\tanh x), \]
\[ \alpha_v \overset{\text{def}}{=} h - v + \frac{\mu}{h-v}, \quad \beta_v \overset{\text{def}}{=} h - v - \frac{\mu}{h-v}, \]  

(3.13)

(3.14)

\[ \tilde{E}_{0,v} \overset{\text{def}}{=} -(h - v)^2 - \frac{\mu^2}{(h-v)^2}, \quad h < v < h + \frac{\mu}{h}. \]  

(3.15)

It is easy to see that the boundary conditions (A) and (B) are satisfied. The expansion of
the Jacobi polynomial can be used to demonstrate the nodelessness.

3.5 Eckart

This potential problem is also called Kepler problem in hyperbolic space:
\[ V_0(x) \overset{\text{def}}{=} -\frac{g(g - 1)}{\sinh^2 x} - 2\mu \coth x, \quad 0 < x < \infty, \quad \sqrt{\mu} > g > \frac{1}{2}. \]

(3.16)

It has a finite number of discrete eigenstates, \([\sqrt{\mu} - g]' + 1\). The lower boundary \( x = 0 \) is a regular singularity with the characteristic exponents \( g \) and \( 1 - g \). The overshoot eigenfunctions \( v > \frac{\mu}{g} - g \) provide proper nodeless solutions:
\[ \varphi_{0,v}(x) \overset{\text{def}}{=} e^{-\frac{\mu}{g-v} x} \sinh x \cdot \frac{g-v}{g-v} P_v^{(\alpha_v,\beta_v)}(\coth x), \]
\[ \alpha_v \overset{\text{def}}{=} -g - v + \frac{\mu}{g+v}, \quad \beta_v \overset{\text{def}}{=} -g - v - \frac{\mu}{g+v}, \]
\[ \tilde{E}_{0,v} \overset{\text{def}}{=} -(g + v)^2 - \frac{\mu^2}{(g+v)^2}, \quad v > \frac{\mu}{g} - g. \]  

(3.17)

(3.18)

(3.19)

It is easy to see that the boundary conditions (A) and (B) are satisfied. The expansion of
the Jacobi polynomial can be used to demonstrate the nodelessness.
General prescriptions for composing two potentials $V_0$ and $V_1$ in one dimensional quantum mechanics are presented by extending the original work in [1]. The virtual state wavefunctions [3] play an important role, as they have played in the rational deformations of solvable potentials [3]–[12]. The form of the mapping function $\psi_0(x)$ (2.7) is reminiscent of the Abraham-Moses transformations [17], [18], [19], or the so-called binary Darboux transformations [20]. The virtual state wavefunctions also play important roles in Abraham-Moses transformations [19].

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Appendix

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