ON THE SPACE OF ORIENTED AFFINE LINES IN $\mathbb{R}^3$

BRENDAN GUILFOYLE AND WILHELM KLINGENBERG

Abstract. We introduce a local coordinate description for the correspondence between the space of oriented affine lines in Euclidean $\mathbb{R}^3$ and the tangent bundle to the 2-sphere. These can be utilised to give canonical coordinates on surfaces in $\mathbb{R}^3$, as we illustrate with a number of explicit examples.

The correspondence between oriented affine lines in $\mathbb{R}^3$ and the tangent bundle to the 2-sphere has a long history and has been used in various contexts. In particular, it has been used in the construction of minimal surfaces [2], solutions to the wave equation [3] and the monopole equation [1].

The Euclidean group of rotations and translations acts upon the space of oriented lines $\mathcal{L}$ and in this paper we freeze out this group action by introducing a particular set of coordinates on $\mathcal{L}$. Our aim is to provide a local coordinate representation for the correspondence, thereby making it accessible to further applications.

One application is the construction of canonical coordinates on surfaces $S$ in $\mathbb{R}^3$ which come from the description of the normal lines of $S$ as local sections of the tangent bundle of the 2-sphere. We illustrate this explicitly by considering the ellipsoid and the symmetric torus.

**Definition 1.** Let $\mathcal{L}$ be the set of oriented (affine) lines in Euclidean $\mathbb{R}^3$.

**Definition 2.** Let $\Phi : TS^2 \to \mathcal{L}$ be the map that identifies $\mathcal{L}$ with the tangent bundle to the unit 2-sphere in Euclidean $\mathbb{R}^3$, by parallel translation. This bijection gives $\mathcal{L}$ the structure of a differentiable 4-manifold.

Let $(\xi, \eta)$ be holomorphic coordinates on $TS^2$, where $\xi$ is obtained by stereographic projection from the south pole onto the plane through the equator, and we identify $(\xi, \eta)$ with the vector

$$\eta \frac{\partial}{\partial \xi} + \bar{\eta} \frac{\partial}{\partial \bar{\xi}} \in T_{\xi}S^2.$$  

**Theorem 1.** The map $\Phi$ takes $(\xi, \eta) \in TS^2$ to the oriented line given by

$$z = \frac{2(\eta - \bar{\eta} \xi^2) + 2\xi(1 + \bar{\xi})r}{(1 + \bar{\xi})^2}$$  

$$t = \frac{-2(\eta \bar{\xi} + \bar{\eta} \xi) + (1 - \xi^2 \bar{\xi}^2)r}{(1 + \bar{\xi})^2},$$  

*Date: December 7, 2002.*

1991 *Mathematics Subject Classification.* Primary: 51N20; Secondary: 53A55.

*Key words and phrases.* twistor, holomorphic coordinates.

The first author was supported by the Isabel Holgate Fellowship from Grey College, Durham and the Royal Irish Academy Travel Grant Scheme.
where \( z = x^1 + ix^2, \ t = x^3, \ (x^1, x^2, x^3) \) are Euclidean coordinates on \( \mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R} \) and \( r \) is an affine parameter along the line such that \( r = 0 \) is the point on the line that lies closest to the origin.

**Proof.** Stereographic projection from the south pole gives a map from \( \mathbb{C} \) to \( \mathbb{R}^3 \) by

\[
\begin{align*}
z &= \frac{2\xi}{1 + \xi \bar{\xi}} , \\
t &= \frac{1 - \xi \bar{\xi}}{1 + \xi \bar{\xi}} .
\end{align*}
\]  

(0.3)

The derivative of this map gives

\[
\frac{\partial}{\partial \xi} = \frac{2(1 + \xi \bar{\xi})}{(1 + \xi \bar{\xi})^2} \frac{\partial}{\partial z} - \frac{2\xi}{(1 + \xi \bar{\xi})^2} \frac{\partial}{\partial \bar{\xi}} - \frac{2\bar{\xi}}{(1 + \xi \bar{\xi})^2} \frac{\partial}{\partial t} ,
\]

and similarly for its conjugate.

Thus

\[
\eta \frac{\partial}{\partial \xi} + \bar{\eta} \frac{\partial}{\partial \bar{\xi}} = \frac{2(\eta - \bar{\eta} \xi^2)}{(1 + \xi \bar{\xi})^2} \frac{\partial}{\partial z} + \frac{2(\bar{\eta} - \eta \bar{\xi}^2)}{(1 + \xi \bar{\xi})^2} \frac{\partial}{\partial \bar{\xi}} - \frac{2(\eta \bar{\xi} + \bar{\eta} \xi)}{(1 + \xi \bar{\xi})^2} \frac{\partial}{\partial t} .
\]  

(0.4)

Consider the line in \( \mathbb{R}^3 \) given by equations (0.1) and (0.2). The direction of this line is

\[
\frac{2\xi}{1 + \xi \bar{\xi}} \frac{\partial}{\partial z} + \frac{2\bar{\xi}}{1 + \xi \bar{\xi}} \frac{\partial}{\partial \bar{\xi}} + \frac{1 - \xi \bar{\xi}}{1 + \xi \bar{\xi}} \frac{\partial}{\partial t} .
\]

When this unit vector is translated to the origin, it ends at the point \( \xi \in S^2 \) (cf. equation (0.3))

The fixed vector determining the line is seen to be (0.4), and, using the fact that the Euclidean inner product of the basis vectors is

\[
\left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{\xi}} \right) = \frac{1}{2} , \quad \left( \frac{\partial}{\partial \bar{\xi}}, \frac{\partial}{\partial \bar{\xi}} \right) = 1,
\]

\[
\left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{\xi}} \right) = \left( \frac{\partial}{\partial \bar{\xi}}, \frac{\partial}{\partial \bar{\xi}} \right) = \left( \frac{\partial}{\partial \bar{\xi}}, \frac{\partial}{\partial \bar{\xi}} \right) = \left( \frac{\partial}{\partial \bar{\xi}}, \frac{\partial}{\partial \bar{\xi}} \right) = 0 ,
\]

we compute that the line is orthogonal to the fixed vector given by (0.4). Thus \( r \) is an affine parameter along the line such that \( r = 0 \) is the point on the line that lies closest to the origin, and the proof is completed.

Consider the map \( \Psi : \mathcal{L} \times \mathbb{R} \rightarrow \mathbb{R}^3 \) which takes a line and a number \( r \) to a point on the line which is a parameter distance \( r \) from the point on the line closest to the origin.

**Proposition 1.** \( \Psi^{-1} \) takes a point \((z, t) \in \mathbb{R}^3\) to a sphere in \( \mathcal{L} \times \mathbb{R} \), the oriented lines containing the point:

\[
\begin{align*}
\eta &= \frac{1}{2}(z - 2t \xi - \bar{\xi}^2) , \\
r &= \frac{\bar{\xi}z + \xi \bar{\xi} + (1 - \xi \bar{\xi})t}{1 + \xi \bar{\xi}} .
\end{align*}
\]

Proof: This is comes from solving equations (0.1) and (0.2) for \( \eta \) and \( r \).

Alternatively, the second equation can be proved by finding the point \( p \) on the line with direction \( \xi \) through \((z, t) \in \mathbb{R}^3\) which minimises the distance to the origin. Then

\[
r^2 = |(z, t)|^2 - |p|^2 ,
\]

which gives the above expression for \( r \).
By throwing away the $r$ information, the above formula gives the holomorphic sphere of lines through a given point $(z,t) \in \mathbb{R}^3$, as described in [1]. These are a 3-parameter family of global sections of $T S^2$ and the associated line congruence in $\mathbb{R}^3$ is normal to round spheres about the given point.

More generally any oriented surface $S$ in $\mathbb{R}^3$ gives rise to a surface $\Sigma \subset \mathcal{L}$ through it’s normal line congruence. Such a $\Sigma$ will, in general, not be holomorphic, nor be given by global sections of the bundle. However, locally, a surface can often be given by local non-holomorphic sections and the following examples illustrate this for two well-known surfaces.

The examples can be verified by substitution in equations (0.1) and (0.2) and then checking that the resulting surface, parameterised by its normal direction coordinate $\xi$, is indeed the one claimed.

**Example 1.** The triaxial ellipsoid with semi-axes $a_1$, $a_2$ and $a_3$ can be covered by coordinates $\xi$ via

$$
\eta = \frac{a_1(\xi + \bar{\xi})(1 - \xi^2) + a_2(\xi - \bar{\xi})(1 + \xi^2) - 2a_3(1 - \xi \bar{\xi})}{2 \sqrt{a_1(\xi + \bar{\xi})^2 - a_2(\xi - \bar{\xi})^2 + a_3(1 - \xi \bar{\xi})^2}}
$$

$$
r = \sqrt{a_1 \left( \frac{\xi + \bar{\xi}}{1 + \xi \bar{\xi}} \right)^2 - a_2 \left( \frac{\xi - \bar{\xi}}{1 + \xi \bar{\xi}} \right)^2 + a_3 \left( \frac{1 - \xi \bar{\xi}}{1 + \xi \bar{\xi}} \right)^2}.
$$

These coordinates extend to $\xi \to \infty$ and so this is an example of a global non-holomorphic section of $\pi : T S^2 \to S^2$.

**Example 2.** The rotationally symmetric torus of radii $a$ and $b$ is given by

$$
\eta = \pm a \sqrt{\frac{\xi}{\xi(1 - \xi \bar{\xi})}}
$$

$$
r = b \pm 2a \sqrt{\frac{\xi}{1 + \xi \bar{\xi}}},
$$

This describes the torus as a double cover of the 2-sphere, branched at the north and south poles.

**References**

[1] N.J. Hitchin, *Monopoles and geodesics*, Comm. Math. Phys. 83 (1982), no. 4, 579-602.

[2] K. Weierstrass, *Untersuchungen über die Flächen, deren mittlere Krümmung überall gleich Null ist*, Monatsber. Akad. Wiss. Berlin (1866), 612-625.

[3] E. T. Whittaker, *On the partial differential equations of mathematical physics*, Math. Ann. 57 (1903), 333-355.

Brendan Guilfoyle, Department of Mathematics and Computing, Institute of Technology, Tralee, Clash, Tralee, Co. Kerry, Ireland.

E-mail address: brendan.guilfoyle@ittralee.ie

Wilhelm Klingenberg, Department of Mathematical Sciences, University of Durham, Durham DH1 3LE, United Kingdom.

E-mail address: wilhelm.klingenberg@durham.ac.uk