Oscillation analysis for vector hyperbolic equations based on the effect of impulse and delay

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Abstract. The oscillations of a class of impulsive vector hyperbolic partial differential equations with delays are studied. Our approach is to reduce the multi-dimensional oscillation problems to the nonexistence of positive solutions of one-dimensional impulsive delay differential inequalities by applying inner product reducing dimension method and differential inequality technique. Some sufficient conditions for H-oscillation of all solutions of the equations under Dirichlet boundary value condition are established, where H denotes a unit vector.

1. Introduction
The concept of $H$-oscillation is a new trenchancy tool in studying vector differential equations. In 1970, Domslak [1] introduced first the concept of $H$-oscillation in studying vector ordinary differential equations, where $H$ is a unit vector. For an excellent exposition of this concept and its applications, see [2]. Recently, Minchev and Yoshida [3], Li, Han and Meng [4] investigated the $H$-oscillation for vector parabolic partial differential equations with functional arguments and vector hyperbolic partial differential equations with deviating arguments, respectively. Li and Han [5], Luo and Yu [6] and Luo, Yang and Zeng [7] investigated the $H$-oscillation for impulsive vector delay parabolic partial differential equations and impulsive vector neutral parabolic partial differential equations. However, to the authors’ knowledge, there are few papers which concern the $H$-oscillation of impulsive vector hyperbolic partial differential equations. Our purpose of this paper is to study the $H$-oscillation of solutions of a class of impulsive vector delay hyperbolic partial differential equations, some interesting results are gained here. It should be noted that we could not find a work on $H$-oscillation of this kind of problem.

2. Formulation of the problem
In this paper, we study the $H$-oscillation of solutions of the following vector hyperbolic partial differential equations with effect of impulse and delay
\[
\frac{\partial}{\partial t} \left( b(t) \frac{\partial U(t,x)}{\partial t} \right) = a_0(t) \Delta U(t,x) + a(t) \Delta U(t-\tau,x) - p(t,x)U(t-\sigma,x),
\]
\[
(t,x) \in R_+ \times \Omega \equiv G, t \neq t_k,
\]
\[
U(t_k^+, x) - U(t_k^-, x) = b_k U(t_k^-), \quad k = 1,2,\Lambda,
\]
\[
U(t_k^+, x) - U(t_k^-, x) = c_k U_j(t_k^-), \quad k = 1,2,\Lambda,
\]
where \( U(t,x) \in C^2([t_0,\infty) \times \Omega; R^m) \) is a vector function, \( \Omega \) is a bounded domain in \( R^n \) with a piecewise smooth boundary \( \partial \Omega \), \( \Delta \) is the \( n \)-dimensional Laplacian in \( R^n \). \( R_+ = [0,\infty) \).

Consider the following Dirichlet boundary condition:
\[
U(t,x) = 0, \quad (t,x) \in R_+ \times \partial \Omega, \quad t \neq t_k,
\]
where 0 is the zero vector in \( R^m \).

Throughout this paper, we suppose that the following conditions hold:

\( (H_1) \) \( 0 < t_1 < t_2 < \Lambda < t_k < \Lambda \) and \( \lim_{k \rightarrow \infty} t_k = \infty \); \( \tau, \sigma \) are positive constant and \( b_k, c_k > -1, k = 1,2,\Lambda \);

\( (H_2) \) \( a_0(t), a(t) \in PC(R_+, R_+), p(t,x) \in PC(R_+ \times \Omega, R_+), P(t) = \min_{x \in \Omega} \{ p(t,x) \} \), where \( PC \) denotes the class of functions which are piecewise continuous in \( t \) with discontinuities of the first kind only at \( t = t_k, k = 1,2,\Lambda \) and left continuous at \( t = t_k, k = 1,2,\Lambda \);

\( (H_3) \) \( b(t) \in C(R_+, (0, \infty)) \).

**Definition 2.1** A vector function \( U(t,x) \in C^2([t_0,\infty) \times \Omega; R^m) \) is called a solution of the problem (1), (2), if the following conditions are satisfied:

(i) \( U(t,x) \) is a differentiable function with respect to \( t \), \( t \neq t_k, k = 1,2,\Lambda \);

(ii) \( U(t,x) \) is a piecewise continuous function with points of discontinuity of the first kind at \( t = t_k, k = 1,2,\Lambda \), and at the moments of impulse the following relations are satisfied:

\[
U(t_k^+, x) = U(t_k^-, x), \quad U_j(t_k^+, x) = U_j(t_k^-, x), \quad k = 1,2,\Lambda.
\]

(iii) \( U(t,x) \) is a twice differentiable function with respect to \( x \);

(iv) \( U(t,x) \) satisfies (1) in the domain \( G \) and the boundary condition (2) on \( R_+ \times \partial \Omega \).

**Definition 2.2** Let \( H \) be a fixed unit vector in \( R^m \). A solution \( U(x,t) \) of the problem (1), (2) is said to be \( H \)-oscillatory in the domain \( G \) if the inner product \( \langle U(x,t), H \rangle \) has a zero in \( \Omega \times (T, \infty) \) for any \( T > 0 \).

3. Reduction to scalar impulsive functional differential inequalities

In this section, we reduce the multi-dimensional oscillation problems for (1) to one-dimensional oscillation problems for scalar impulsive functional differential inequalities.

The following notation will be used:
\[
U_H(t,x) = \langle U(t,x), H \rangle,
\]
where \( \langle U,V \rangle \) denotes the inner product of \( U,V \in R^m \).

**Theorem 3.1** Let \( H \) be a fixed unit vector in \( R^m \) and \( U(t,x) \) be a solution of (1) and \( U_H(t,x) \) be eventually positive, then \( U_H(t,x) \) satisfies the scalar impulsive delay hyperbolic partial differential inequality...
The inner product of (1) and \(x\), from the impulsive conditions of (1), we have \(k\).

If the scalar impulsive functional differential \(\tau\) of the problem (1), \((-\).

If the scalar impulsive delay hyperbolic partial differential inequality (3) has no eventually positive solutions satisfying the boundary condition (2), then every differential inequality (3) has no eventually positive solutions satisfying the boundary condition (2').

Proof. Let \(U(t, x)\) be a solution of (1) and \(U_H(t, x)\) be eventually positive.

Case 1: \(t \neq t_k, k = 1,2,\Lambda\). The inner product of (1) and \(H\) yields the following:

\[
\frac{\partial}{\partial t}(b(t) \frac{\partial U_H}{\partial t}(t, x)) = a_0(t) \Delta U_H(t) + a(t) \Delta U_H(t - \tau) - P(t)U_H(t, x),
\]

From (H2), we have

\[
P(t, x)U_H(t, x) \geq P(t)U_H(t, x).
\]

It follows from (4) and (5) that

\[
\frac{\partial}{\partial t}(b(t) \frac{\partial U_H}{\partial t}(t, x)) \leq a_0(t) \Delta U_H(t) + a(t) \Delta U_H(t - \tau) - P(t)U_H(t, x).
\]

Case 2: \(t = t_k, k = 1,2,\Lambda\). From the impulsive conditions of (1), we have

\[
U_H(t_k^+, x) - U_H(t_k^-, x) = b_k U_H(t_k, x),
\]

\[
\frac{\partial U_H}{\partial t}(t_k^+, x) - \frac{\partial U_H}{\partial t}(t_k^-, x) = c_k \frac{\partial U_H}{\partial t}(t_k, x).
\]

Therefore, from (6)-(8), we immediately obtain (3), that is, \(U_H(t, x)\) satisfies (3). This completes the proof.

Associated with the boundary condition (2), we consider the following scalar boundary condition:

\[
U_H(t, x) = 0, \quad (t, x) \in R_+ \times \partial \Omega, \quad t \neq t_k.
\]

From Theorem 3.1, we can easily obtain the following result.

Theorem 3.2 Let \(H\) be a fixed unit vector in \(R^m\). If the scalar impulsive delay hyperbolic partial differential inequality (3) has no eventually positive solutions satisfying the boundary condition (2'), then every solution \(U(x, t)\) of the problem (1), (2) is \(H\)-oscillatory in \(G\).

The following fact shall be used later in the proof of Theorem 3.3. Consider the Dirichlet eigenvalue problem

\[
\begin{cases}
\Delta \omega(x) + \lambda \omega(x) = 0, & x \in \Omega, \\
\omega(x) = 0, & x \in \partial \Omega,
\end{cases}
\]

where \(\lambda\) is a constant. It is well known [8] that the first eigenvalue \(\lambda_0\) is positive and the corresponding eigenfunction \(\phi(x)\) may be chosen so that \(\phi(x) > 0\) in \(\Omega\).

We use the following notation:

\[
W(t) = \int_{\Omega} U_H(t, x)\phi(x)dx.
\]

Theorem 3.3 Let \(H\) be a fixed unit vector in \(R^m\). If the scalar impulsive functional differential inequality
\[
\begin{aligned}
(b(t)W'(t))' + \lambda_0 a_0(t)W(t) + \lambda_0 a(t)W(t - \tau) + P(t)W(t - \sigma) &\leq 0, \\
W(t_k^+) - W(t_k^-) &= b_k W(t_k) , \\
W'(t_k^+) - W'(t_k^-) &= c_k W'(t_k) ,
\end{aligned}
\tag{9}
\]

has no eventually positive solutions, then every solution \(U(x,t)\) of the problem (1), (2) is \(H\) oscillatory in \(G\).

**Proof.** By Theorem 3.2, under the assumptions of Theorem 3.3, we prove this fact that the problem (3), (2') has no eventually positive solutions. Suppose that there exists an eventually positive solution \(U_H(t,x)\) of (3), (2'), \((t,x) \in [T, \infty) \times \Omega, T \geq 0\).

Case 1: \(t \geq T, t \neq t_k, k = 1,2,\Lambda\). Multiplying both sides of the inequality (3) by \(\phi(x)\) and integrating with respect to \(x\) over the domain \(\Omega\), we have

\[
\frac{d}{dt} \left( b(t)W'(t) \right) + \lambda_0 a_0(t)W(t) + \lambda_0 a(t)W(t - \tau) + P(t)W(t - \sigma) \leq 0,
\]

\[
\int_{\Omega} \phi(x) \Delta U_H(t,x)dx = \int_{\Omega} \phi(x) \frac{\partial U_H(t,x)}{\partial N} - U_H(t,x) \frac{\partial \phi(x)}{\partial N} ds + \int_{\Omega} U_H(t,x) \Delta \phi(x)dx
\]

\[
= -\lambda_0 \int_{\Omega} U_H(t,x) \phi(x)dx ,
\]

\[
\text{where } N \text{ is the unit exterior normal vector to } \partial \Omega , \text{ } ds \text{ is the surface element on } \partial \Omega .
\]

Thus, combining (10)-(12), we have

\[
(b(t)W'(t))' + \lambda_0 a_0(t)W(t) + \lambda_0 a(t)W(t - \tau) + P(t)W(t - \sigma) \leq 0 ,
\]

\[
\int_{\Omega} U_H(t_k^+,x) \phi(x)dx - \int_{\Omega} U_H(t_k^-,x) \phi(x)dx = b_k \int_{\Omega} U_H(t_k,x) \phi(x)dx ,
\]

\[
\int_{\Omega} \frac{\partial U_H(t_k^+,x)}{\partial t} \phi(x)dx - \int_{\Omega} \frac{\partial U_H(t_k^-,x)}{\partial t} \phi(x)dx = c_k \int_{\Omega} \frac{\partial U_H(t_k,x)}{\partial t} \phi(x)dx .
\]

This implies

\[
W(t_k^+) - W(t_k^-) = b_k W(t_k) ,
\]

\[
W'(t_k^+) - W'(t_k^-) = c_k W'(t_k) .
\]

Therefore, from (13)-(15), we obtain that \(W(t) > 0\) is an eventually positive solution of the inequality (9). This is a contradiction. This completes the proof.

4. Sufficient conditions for \(H\)-oscillation

In this section, we shall establish some \(H\)-oscillation criteria for the impulsive vector delay hyperbolic partial differential equations (1) satisfying the boundary condition (2). The following lemmas will be needed.

**Lemma 4.1** Suppose that \(y(t) \in C^2([t_0, \infty), R)\) and \(y(t) > 0, y'(t) > 0, y''(t) < 0, t \geq t_0,\) then for every \(\theta \in (0,1),\) there exists \(t_1 \geq t_0,\) such that \(y(t) \geq \theta ty'(t)\) for \(t \geq t_1.\)
Lemma 4.2 ([10, Theorem 2]) Suppose that \( a(t), b(t) \in (R_+, R) \) are locally integrable functions and \( b(t) \geq 0; 0 < t_1 < t_2 < \Lambda < t_k < \Lambda \) and \( \lim_{k \to \infty} t_k = \infty \), \( y(t_k) = y(t_k^-) \); \( b_k > -1, k = 1,2, \Lambda \); \( \tau \) is a positive constant. If
\[
\lim_{t \to \infty}\int_{t-\tau}^{t} \frac{\theta(s) \exp(\int_{s-\sigma}^{t} \frac{a(r) \theta(r)}{b(r)} dr)}{b(s-\sigma)} \left(1 + c_k\right)^{-1} ds > \frac{1}{e},
\]
then the impulsive delay differential inequality
\[
\begin{align*}
\left\{ \begin{array}{l}
y'(t) + a(t)y(t) + b(t)y(t-\tau) \leq 0, \quad t \geq 0, t \neq t_k, k = 1,2,\Lambda \\
y(t_k^+) - y(t_k^-) = b_k y(t_k), \quad k = 1,2,\Lambda
\end{array} \right.
\]
has no eventually positive solution.

Now we are ready to prove the following results.

Theorem 4.1 Let \( H \) be a fixed unit vector in \( R^m \). If
\[
\int_{t_0}^{\infty} \frac{1}{b(t)} dt = \infty, t_0 > 0 \text{ and } b'(t) \geq 0, \quad t \in R_+ \quad (16)
\]
and
\[
\lim_{t \to \infty}\int_{t-\sigma}^{t} \frac{P(s) \theta(s-\sigma)}{b(s-\sigma)} \exp(\int_{s-\sigma}^{t} \frac{\lambda_0 a_0(r) \theta(r)}{b(r)} dr) \left(1 + c_k\right)^{-1} ds > \frac{1}{e} \quad (17)
\]
for every \( \theta \in (0,1) \), then every solution \( U(t,x) \) of the problem (1), (2) is \( H \)-oscillatory in \( G \).

Proof. Suppose that the problem (1), (2) has a \( H \)-nonoscillatory solution \( U(t,x) \), then \( U_H(t,x) \) is a nonoscillatory solution of (3), (2'). Without loss of generality, we may suppose that \( U_H(t,x) \) is eventually positive, \((t,x) \in [T,\infty) \times \Omega, T \geq 0 \) (if it is eventually negative, we can let \( V_H(t,x) = -U_H(t,x) \), the proof is similar).

Case 1: \( t \geq T, t \neq t_k, k = 1,2,\Lambda \). Similar to the proof of Theorem 3.3, we can get the inequality (13). Thus, from (13), we have
\[
(b(t)W'(t))' + \lambda_0 a_0(t) W(t) + P(t) W(t-\sigma) \leq 0. \quad (18)
\]
From (18), it is easy to know that \( (b(t)W'(t))' < 0, t \geq T \). We can claim that \( W'(t) > 0, t \geq T \). In fact, if this is not true, there exists \( T_1 \geq T \), such that \( W'(T_1) \leq 0 \). Furthermore, \( b(T_1)W'(T_1) < 0 \), \( t \geq T_1 \). This implies that
\[
W'(t) \leq \frac{1}{b(t)} b(T_1)W'(T_1) \text{ for } t \geq T_1,
\]
and
\[
W(t) \leq W(T_1) + b(T_1)W'(T_1) \int_{T_1}^{t} \frac{1}{b(s)} ds.
\]
Therefore, \( \lim_{t \to \infty} W(t) = -\infty \), which contradicts the fact that \( W(t) > 0 \).

Because \( W(t) > 0 \), \( W'(t) > 0 \), and \( W'(t) < -\frac{b'(t)}{b(t)} W'(t) < 0 \), form Lemma 4.1, there exists \( T_1 \geq T \), such that \( W(t) \geq \theta t W'(t) \), \( W(t-\sigma) \geq \theta(t-\sigma) W'(t-\sigma), t \geq T_1 \). Let \( Z(t) = b(t)W'(t) \), from (18), we have
\[
Z'(t) + \frac{\lambda_0 a_0(t) \theta t}{b(t)} Z(t) + \frac{P(t) \theta(t-\sigma)}{b(t-\sigma)} Z(t-\sigma) \leq 0, \quad t \geq T_1. \quad (19)
\]
Case 2: \( t \geq T , t = t_k , k = 1,2, \Lambda \). Similar to the proof of Theorem 3.3, we can get the equality (15). Thus, from (15), we have

\[
Z(t_k) - Z(t_{k+1}) = c_k Z(t_k).
\]

Hence we obtain that \( Z(t) > 0 \) is an eventually positive solution of the differential inequality (19), (20). But according to the condition (17) and Lemma 4.2, the differential inequality (19),(20) has no eventually positive solutions. This is a contradiction. This completes the proof.

From the differential inequality (13), we have

\[
(b(t)W'(t) + \lambda_0 a_0(t)W(t) + \lambda(t)W(t - \tau) \leq 0 , \ t \geq T , t \neq t_k.
\]

Similar to the proof of Theorem 4.1, we have the following result. It should be noted that the criterion in this theorem only depend on the diffusion coefficient \( a(t) \).

**Theorem 4.2** Let \( H \) be a fixed unit vector in \( R^m \). If (16) holds and

\[
\liminf_{t \to \infty} \int_{t-\tau}^{t} \frac{\lambda_0 a(s) \theta(s - \tau)}{b(s - \tau)} \exp\left(\int_{s-\tau}^{s} \frac{\lambda_0 a_0(r) \theta(r)}{b(r)} - dr\right) \prod_{s - \tau < \xi < s} (1 + c_k)^{-1} ds > \frac{1}{e} ,
\]

for every \( \theta \in (0,1) \), then every solution \( U(t, x) \) of the problem (1), (2) is \( H \)-oscillatory in \( G \).

5. Some remarks

**Remark 5.1** Let the conditions in Lemma 4.2 be replaced by the following conditions (see [10, Theorem 3]):

\[
\limsup_{t \to \infty} \int_{t-\tau}^{t} b(s) \exp\left(\int_{s-\tau}^{s} a(r)dr\right) \prod_{s - \tau < \xi < s} (1 + b_k)^{-1} ds > 1.
\]

Then we can still obtain the new results concerning the oscillation of solutions of the problem (1), (2), which are parallel to Theorem 4.1 and Theorem 4.2 of this paper.

**Remark 5.2** Using our ideas in this paper, we can consider other boundary conditions. For example, consider the following Robin boundary condition

\[
\frac{\partial U(t, x)}{\partial \mathbf{N}} + \beta(x) U(t, x) = 0 , \ (t, x) \in R_+ \times \partial \Omega , \ t \neq t_k ,
\]

where \( 0 \) is the zero unit vector in \( R^m \), \( \beta(x) \in C(\partial \Omega, (0, \infty)) \). It is not difficult to obtain some \( H \)-oscillation criteria of the problem (1), (21). Due to limited space, their statements are omitted here.

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