Beyond Submodular Maximization

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Abstract

While there are well-developed tools for maximizing a submodular function \( f(S) \) subject to a matroid constraint \( S \in \mathcal{M} \), there is much less work on the corresponding supermodular maximization problems. We develop new techniques for attacking these problems inspired by the continuous greedy method applied to the multi-linear extension of a submodular function. We first adapt the continuous greedy algorithm to work for general twice-continuously differentiable functions. Reminiscent of how the Lipschitz constant governs the convergence rate in convex optimization, the performance of the adapted algorithm depends on a new smoothness parameter. If \( F : [0, 1]^n \to \mathbb{R}_{\geq 0} \) is one-sided \( \sigma \)-smooth, then it yields an approximation factor depending only on \( \sigma \). We apply the new algorithm to a broad class of quadratic supermodular functions arising in diversity maximization. The case \( \sigma = 2 \) captures metric diversity maximization and general \( \sigma \) includes the densest subgraph problem. We also develop new methods for rounding quadratics over a matroid polytope. These are based on extensions to swap rounding and approximate integer decomposition. Together with the adapted continuous greedy this leads to a \( O(\sigma^{3/2}) \)-approximation. This is the best asymptotic approximation known for this class of diversity maximization and we give some evidence for why we believe it may be tight.

We then consider general (non-quadratic) functions. We give a broad parameterized family of monotone functions which include submodular functions and the just-discussed supermodular family of discrete quadratics. The new family is defined by restricting the one-sided smoothness condition to the boolean hypercube; such set functions are called \( \gamma \)-meta-submodular. We develop local search algorithms with approximation factors that depend only on \( \gamma \). We show that the \( \gamma \)-meta-submodular families include well-known function classes including meta-submodular functions (\( \gamma = 0 \)), proportionally submodular (\( \gamma = 1 \)), and diversity functions based on negative-type distances or Jensen-Shannon divergence (both \( \gamma = 2 \)) and (semi-)metric diversity functions.

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1 Introduction

In the past decade, the catalogue of algorithms available to combinatorial optimizers has been substantially extended to new settings which allow submodular objective functions. For instance, while classical work [42, 43, 25] already established a \( \frac{1}{2} \)-approximation for maximizing a non-negative monotone submodular function subject to a matroid constraint, it was not until recently when the work from [49, 12] achieved a tight \((1 - \frac{1}{e})\)-approximation for this problem. The latter required the development of new continuous optimization machinery for the associated multi-linear relaxation. These developments in submodular maximization were occurring at the same time that researchers found a wealth of new applications for these models [33, 39, 10, 36, 30, 40, 47, 41, 44, 18].

The related supermodular maximization models (submodular minimization) also offer an abundance of applications, but they appeared to be highly intractable even under simple cardinality constraints [48]. One exception came from a specific model for diversity maximization. Given a set function \( f(S) \) which measures the ‘diversity’ amongst elements of a set \( S \), a problem of broad interest is to find a set \( S \) of maximum diversity subject to a prescribed bound on its cardinality \(|S| \leq k\), or more generally, subject to a matroid \( M \) constraint:

\[
(\text{DivMax}) \quad \max \{f(S) : S \in M\}.
\]

One class of diversity functions that has wide applications in machine learning are the so-called remote-clique functions [11, 51, 26]. These are based on having a dis-similarity measure \( d(u,v) \) between each pair of objects \( u,v \) in the ground set. The corresponding max-sum problem is then to maximize \( f(S) := \sum_{u,v \in S} A(u,v) \) [37, 15]. If \( A(u,v) \geq 0 \), then one easily checks that \( f \) is supermodular. We sometimes abuse nomenclature and conflate \( A \) with its associated diversity function \( f \). These functions are essentially a special case of what we term discrete quadratic functions. Namely, a function which is the restriction of a quadratic \( x^T A x + b^T x \) to the boolean hypercube.

Throughout we assume that \( b \geq 0 \) (all our functions are non-negative) and the associated matrix is symmetric, non-negative and has 0 diagonal (so the quadratic is multi-linear).

Even for the subclass of discrete quadratic diversity functions, the problem DivMax is ostensibly intractable in the sense that it includes the densest subgraph problem [9]. However, for metric diversity functions (remote-clique function when \( A \) forms a metric), there is a 2-approximation subject to a cardinality constraint [45, 27]. Moreover, this has been generalized to the case of matroid constraint [11, 9]. Borodin et al. [7, 8] introduced the class of proportionally submodular (monotone) functions which include these metric diversity functions as well as monotone submodular functions. They give a 10.22-approximation for maximizing these functions subject to a matroid constraint.

The weaker notion of \( \sigma \)-semi-metric (that is, satisfying a \( \sigma \)-approximate triangle inequality for \( \sigma \geq 1 \)) is considered in [50]. They provide a \( 2\sigma \)-approximation under a cardinality constraint and a \( 2\sigma^2 \)-approximation under a matroid constraint.

The preceding results motivate the key impetus for our work, namely, to explain and explore the reasons for the fortunate cases when supermodular maximization is actually tractable. We argue that a one-sided smoothness parameter governs the degree to which we can approximate these problems. Two driving questions become:

\((\text{Div}^+)\) Find a parameterized family of supermodular functions which contains metric, and more generally \( \sigma \)-semi-metric, diversity functions and remains tractable in terms of \( \sigma \). A second motivating question is \((\text{SUB}+\text{DIV})\) to find a parameterized tractable family of monotone set functions which includes all monotone submodular functions and the aforementioned diversity functions.
2 Our Results

2.1 Nonlinear Maximization, One-Sided Smoothness and Matroid Rounding

In 1978 Fisher et al. \[42, 43, 25\] gave a 1/2-approximation for \(\max \{ f(S) : S \in \mathcal{M} \} \) where \(\mathcal{M}\) is a matroid and \(f\) is non-negative monotone submodular. In the special case of uniform matroids, \(\mathcal{M} = \{ S : |S| \leq k \}\), they gave a, provably tight, \((1 - 1/e)\)-approximation. Whether this ratio could be achieved for general matroids remained open for 35 years. Partly motivated by interest in the submodular welfare problem, Calinescu, Chekuri, Pál and Vondrak [49, 12] gave such a \((1 - 1/e)\)-approximation algorithm. This was based on a new (non-convex) relaxation followed by an elegant application of lossless pipage rounding of the fractional solution to a vertex of the matroid polytope. We examine both phases of their framework for clues to the question (\(\text{DIV}^+\)) on supermodular maximization.

At the heart of their approach is the problem of maximizing the multi-linear extension of a submodular set function over a downwards-closed polytope. Submodularity in this context ensures some nice properties for the multi-linear extension. For instance, concavity along a direction \(d \geq 0\) is used to bound a Taylor series expansion in the continuous greedy analysis [49]. Since non-submodular multi-linear extensions will not have this concavity property, we propose a "smoothness" condition which guarantees an alternative bound based on Taylor series. A continuously twice differentiable function \(F : [0, 1]^n \to \mathbb{R}\) is called one-sided \(\sigma\)-smooth at \(x \neq 0\) if for any \(u \in [0, 1]^n\)

\[
u^T \nabla^2 F(x) u \leq \sigma \cdot \frac{|u||u|}{||x||} u^T \nabla F(x).
\]

We call such a function \(F\) one-sided \(\sigma\)-smooth if it is \(\sigma\)-smooth at any non-zero point of its domain. As we see, approximation algorithms exist for maximizing these nonlinear functions due to a bound on their second derivatives in terms of their gradient. This is the essential ingredient in several of the main results - see Lemma [3].

We give an adaptation of continuous greedy which yields approximation factors that are upper-bounded by a function of the smoothness parameter \(\sigma\). These results are used in a 2-phase (relax and round) algorithm for maximizing a discrete quadratic function. Interestingly, however, one-sided smoothness also plays a role in the analysis of a local search algorithm discussed in the next section.

**Theorem 1** (Maximizing a One-Sided Smooth Function over Downwards-Closed Polytope). Let \(F : [0, 1]^n \to \mathbb{R}_+\) be a monotone one-sided \(\sigma\)-smooth function, and \(P \subseteq [0, 1]^n\) be a polytime separable, downwards-closed polytope. If we run the jump-start continuous greedy process (Algorithm [7]) with \(c = 1/2\), then \(x(1) \in P\) and \(F(x(1)) \geq \left[ 1 - \exp\left(\frac{-0.5}{3^r}\right) \right] \cdot \text{OPT} \geq \frac{0.5}{3^r+0.5} \cdot \text{OPT}\) where \(\text{OPT} = \max\{F(x) : x \in P\}\).

In the above result, the one-sided smoothness parameter \(\sigma\) governs the performance ratio of continuous greedy. This is somewhat reminiscent of how convergence rates in convex minimization can be tied to Lipschitz constants. As with Lipschitz conditions, we can improve our performance ratios by requiring smoothness on higher derivatives. This is encapsulated in the following, which shows that if the partials \(\nabla_i F\) are one-sided 0-smooth, then the approximation ratio improves to linear.

**Theorem 2.** Let \(F : [0, 1]^n \to \mathbb{R}_+\) be a monotone one-sided \(\sigma\)-smooth function with non-positive third order partial derivatives. Let \(c \in (0, 1)\) and \(P\) be a polytime separable, downwards-closed, polytope. If we run the jump-start continuous greedy process (Algorithm [7]) with \(c = 1/2\), then \(x(1) \in P\) and \(F(x(1)) \geq \left[ 1 - \exp\left(-\frac{1}{2\sigma+2}\right) \right] \cdot \text{OPT} \geq \frac{1}{2\sigma+3} \cdot \text{OPT}\), where \(\text{OPT} := \max\{F(x) : x \in P\}\).
By standard techniques (see [19,12]) one may discretize the continuous greedy process to obtain a finite algorithm, which deviates from the above guarantees by a $o(1)$ additive error.

We now return to the discrete setting and question (div$^+$). We focus on the tractability of the following class of supermodular functions: $f(S) = \sum_{\{u,v\} \subseteq S} A(u,v) + \sum_{v \in S} b(v)$, where $A,b \geq 0$ and $A$ is a symmetric 0-diagonal matrix.

This class is of interest for a variety of reasons. First, it is a natural family since these are just restrictions to the hypercube of quadratic forms $\frac{1}{2} x^T A x + b^T x$. This family also coincides with the class of second-order-modular functions introduced in [38] (see Lemma 5 in Appendix C.1). Second, in the special case when $b = 0$ and $A(u,v)$ forms a metric, this class corresponds to metric diversity functions and, as pointed out, the maximization problem over a matroid constraint has a 2-approximation [9,1]. Third, discrete quadratics have interesting behaviour with respect to their one-sided smoothness. The previous mentioned metric diversity functions have one-sided smoothness $\sigma = 2$. If $A$ is a negative type distance [4], then the corresponding problems have been shown to admit a PTAS [13,14]. Another well-known distance measure is the Jensen-Shannon divergence used to measure dis-similarity of two probability distributions. Both JS and negative-type distances have associated smoothness parameter $\sigma = 4$ (Propositions 5,6 in Appendix C).

For general $\sigma \geq 0$, let $O_{2\sigma}$ denote the family of discrete quadratic functions which are one-sided $\sigma$-smooth. One may show (Proposition 7 in Appendix C.1) that $O_{2\sigma}$ includes functions which are determined by a matrix $A$ which is a $\sigma$-semi-metric. That is, $A(u,v)$’s satisfy a $\sigma$-approximate triangle inequality - in this case, we refer to the associated discrete quadratic as a $\sigma$-semi-metric function. Generalizing the metric case, these semi-metric diversity problems admit a $2\sigma^2$-approximation for matroid constraint (and a $2\sigma$-approximation subject to a cardinality constraint) [50]. The next result, that relies on the hardness of the planted clique problem [3], shows that the approximation guarantee necessarily degrades as the smoothness parameter grows - see Appendix C.2 for the proof.

**Theorem 3.** Assuming the Planted Clique Conjecture: (1) for any constant $\sigma > 1$, it is hard to approximate the maximum of a $\sigma$-semi-metric function subject to a cardinality constraint within a factor of $2\sigma - \epsilon$ for any $\epsilon > 0$, and (2) for a super-constant $\sigma$, there is no constant factor (polynomial) approximation algorithm for maximizing a $\sigma$-semi-metric function subject to a cardinality constraint.

On the positive side, we show that by modifying the framework of Vondrak et al. we give an $O(\sigma^{3/2})$-approximation for the problem DivMax. This improves a known $O(\sigma^2)$-approximation [50] and we believe the exponent of $\frac{3}{2}$ may be tight (we discuss the reasons later).

**Theorem 4.** There is an $O(\sigma^{3/2})$-approximation algorithm for maximizing $f \in O_\sigma$ over a matroid.

This result is proved as follows. First, if $f \in O_\sigma$, then one may show that its multi-linear relaxation $F$ satisfies $\nabla^2 F = A$, and thus $\nabla^3 F = 0$. Hence Theorem 2 gives an $O(\sigma)$-approximation for continuous greedy applied to the multi-linear relaxation. We then show that for any matroid polytope $P_M$ and fractional $x^* \in P_M$, we may round to an integral vector in $P_M$ with a loss of at most $O(\sqrt{\sigma})$. The two phases together give the desired $O(\sigma^{3/2})$ bound for general matroids and, as discussed later, a tight $O(\sigma)$ bound for uniform matroids.

A key obvious ingredient is to bound the rounding phase. This is non-standard since we are dealing with quadratic objectives. This is achieved by two different types of rounding. We obtain a $O(\frac{1}{\sqrt{\sigma}})$ bound by a technique inspired by approximate integer decomposition methods (here $r$ denotes the rank of the matroid, and $c$ is the size of its smallest circuit$^4$). A second rounding

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1A symmetric, 0-diagonal matrix $A$ represents a negative-type distance if $x^T A x \leq 0$ for all $x$ such that $1^T x = 0$. These include $\ell_1, \ell_2$ and Jaccard distances.

2A circuit in a matroid is any minimal dependent set.
Theorem 5 (Quadratic Integrality Gap over Matroid). Let $f \in \mathcal{O}_\sigma$ be a set function and $F$ its multi-linear extension. Let $\mathcal{M}$ be a matroid of rank $r$, minimum circuit size $c$, and matroid polytope $P_\mathcal{M}$. Then there is a polytime algorithm which given $x^* \in P_\mathcal{M}$ produces an integral vector $1_I \in P_\mathcal{M}$ such that $F(x^*) \leq O(\min\{\frac{r}{\sqrt{c}}, 1 + \frac{c}{r}\})f(I) \leq O(\sqrt{\sigma})f(I)$.

The $O(\sqrt{\sigma})$ bound is pessimistic for some matroids. For instance, in uniform matroids (i.e., for cardinality constraint $|S| \leq k$) we have $r = k$ and $c = k + 1$. If $k \geq 2$, then the first rounding bound is $O(1)$. Hence the algorithm gives a tight $O(\sigma)$-approximation in this case. These observations, and the planted clique hardness result (Theorem 3) show that we cannot expect a $o(\sigma)$-approximation algorithm even for continuous greedy applied to the multi-linear extension of $f \in \mathcal{O}_\sigma$.

These integrality gap bounds are also tight in the sense that we give almost-matching lower bound examples - see Proposition 1. Now consider any algorithm which maximizes the multi-linear extension of $f \in \mathcal{O}_\sigma$ in a first phase, and then rounds the solution to an integral point. We have seen that the first phase should not asymptotically beat a $O(\sigma)$ factor and in the worst case, the (quadratic) integrality gap may be as bad as $\Omega(\sqrt{\sigma})$. Therefore, for this class of algorithms one may naively expect a best case approximation factor $O(\sigma^{3/2})$.

2.2 A Common Generalization of Submodular and Metric Diversity Functions

In this section we no longer restrict attention to discrete quadratic functions, and study more general monotone set functions. To motivate our approach we consider the definition of one-sided $\sigma$-smoothness restricted to only integral points of a function $F$ instead of its whole domain. Namely, for any non-empty $S \subseteq [n]$: $u^T \nabla^2 F(1_S)u \leq \sigma \cdot \frac{|u|}{|S|} u^T \nabla F(1_S)$. If we also limit our attention to directions $u = e_i + e_j$, the inequality becomes

$$\nabla^2 F(1_S) \leq \sigma \cdot \left( \frac{\nabla_i F(1_S) + \nabla_j F(1_S)}{|S|} \right). \tag{1}$$

Now suppose that $F$ is the multi-linear extension of a set function $f: 2^{[n]} \to \mathbb{R}_{\geq 0}$, and so $F(1_S) = f(S)$. One may show [49] that $\nabla_i F(1_S) = f(S + i) - f(S - i)$ and $\nabla^2 F(1_S) = f(S + i + j) - f(S + i - j) - f(S - i + j) + f(S - i - j)$, i.e., $\nabla_i F(1_S) = \nabla_j F(1_S)$ and $\nabla^2 F(1_S)$. To abbreviate notation we write $B_i(S) = \nabla_i F(1_S)$ and $A_{ij}(S) = \frac{B_i(S) + B_j(S)}{|S|}$ and so (1) becomes:

$$A_{ij}(S) \leq \sigma \cdot \frac{B_i(S) + B_j(S)}{|S|}. \tag{2}$$

We now call a set function $f$ $\sigma$-meta-submodular if it satisfies this inequality for any $S \neq \emptyset$. One may view this as the discrete analogue of bounding the second-order term of a Taylor series by the corresponding first-order term. We primarily focus on monotone functions and so we denote by $\mathcal{G}_\sigma$ the family of non-negative, monotone set functions which are $\sigma$-meta-submodular. Note that since the $B_i$’s are non-negative, we then have that $\mathcal{G}_\sigma \subseteq \mathcal{G}_{\sigma'}$ if $\sigma < \sigma'$.

We first discuss the structure around the meta-submodular family (see Fig. 1). Most importantly with respect to $(\text{SUB} + \text{DIV})$ is that $\mathcal{G}_\sigma$ includes all monotone submodular functions and $\sigma$-semi-metric diversity functions. More precisely, the 0-meta-submodular functions coincide with the class of meta-submodular functions defined by Kleinberg et al. [35] which properly includes all submodular functions - see Proposition 2 and 4 in Appendix [35]. A second property is that every proportional submodular function (cf. Borodin et al. [8]) is 1-meta-submodular (see Proposition 3 in Appendix [35]).

Given the performance guarantees of continuous greedy for smooth functions, it is natural to study the smoothness of multi-linear extensions from the meta-submodular families. First, one
can show that if the multi-linear extension of a set function is one-sided $\sigma$-smooth, then the set function itself is $\sigma$-meta-submodular (Proposition 8 in Appendix D). The converse is not necessarily true however: the multi-linear extension of a $\sigma$-meta-submodular function is not always one-sided $\sigma$-smooth. Hence, we prefer to use a different parameter $\gamma$ when referring to meta-submodularity. In other words we speak of $\gamma$-meta-submodular set functions and write $G_\gamma$. One may think of $\gamma$ as a discrete smoothness parameter. The following result shows that a set function’s multi-linear extension is one-sided smooth whenever a stronger probabilistic version of (2) is satisfied (see Appendix D for proof details). We call this the expectation inequality (3), where $R \sim x$ denotes a random set that contains element $i$ independently with probability $x_i$.

**Lemma 1 (Expectation Inequality).** Let $f$ be a non-negative, monotone set function and $F$ be its multi-linear extension. Let $x \in [0,1]^n$ and $\sigma \geq 0$. If for any $i,j \in [n]$ we have the following:

$$E_{R \sim x}(|R|) \cdot E_{R \sim x}[A_{ij}(R)] \leq \sigma \cdot (E_{R \sim x}[B_i(R)] + E_{R \sim x}[B_j(R)])$$

then $F$ is one-sided $2\sigma$-smooth at $x$.

We have proved that this inequality holds (modulo a constant factor) in the supermodular case, i.e. for the intersection of supermodular functions and $\gamma$-meta-submodular functions (see Lemma 7 in Appendix D). This yields the following.

**Theorem 6.** Let $f$ be a supermodular function in $G_\gamma$ and $F$ be its multi-linear extension. Then $F$ is one-sided $(\max\{6\gamma, 4\gamma + 2\})$-smooth.

We conjecture that this also holds for any $\gamma$-meta-submodular function with $\gamma > 0$.

**Conjecture 1.** Let $f \in G_\gamma$ and $F$ be its multi-linear extension where $\gamma > 0$. Then $F$ is one-sided $O(\gamma)$-smooth.

While we do not have the continuous greedy available to us for the general family $G_\gamma$, ironically one may use a weakened smoothness property to analyze a local search algorithm for the discrete problem $\max f(S) : S \in M$. The weakened property asks for $f$ to be one-sided smooth on a subdomain which dominates some integral point $1_S$.

**Theorem 7.** Let $f \in G_\gamma$ and $F$ be its multi-linear extension. Let $\alpha \geq 1$ and $S \subseteq [n]$ be non-empty. Then $F$ is one-sided $2\alpha\gamma$-smooth on $\{x | x \geq 1_S, ||x||_1 \leq \alpha|S|\}$.

This sub-domain smoothness property is used in a technical analysis to obtain the following approximation factor depending only on $\gamma$. This result provides a very general answer to question (SUB+DIV), and for low values of $\gamma$ we obtain a new tractable parameterized class of functions.

**Theorem 8.** Let $f \in G_\gamma$. Then there is an $O(\gamma^2 2^\gamma)$-approximation via local search for maximizing $f$ subject to a matroid constraint.
As with the continuous setting (Theorem 2), one can improve the performance ratios by requiring additional (discrete smoothness) conditions on higher order (first derivative) terms. As we have seen the discrete analog of $\nabla_i F$ is the marginal gain set function $B_i(S)$. The following result shows that if these set functions are submodular, then the exponential factor from Theorem 8 improves to a quadratic factor. We remark that submodularity of the $B_i$’s is just the notion of second-order-submodularity introduced in [38], and is also equivalent to the non-positivity of the third-order partial derivatives of the multi-linear extension.

**Theorem 9.** Let $f \in \mathcal{G}_\gamma$ such that $f$ is also second-order-submodular (that is, $f$’s marginal gains are submodular). Let $M = ([n], \mathcal{I})$ be a matroid of rank $r$. Then the modified local search algorithm (Algorithm 2) gives an $O(\gamma + \gamma^2/r)$-approximation for maximzing $f$ subject to $M$.

In order to achieve a sub-quadratic approximation matching Theorem 4 we also require the function to be supermodular. Moreover, the local search algorithm must be significantly adapted and find a maximum matching in the last step - see Algorithm 2. The full proof is included in Appendix F.4.

**Theorem 10.** If $f \in \mathcal{G}_\gamma$ is also second-order-submodular and supermodular, then Algorithm 2 gives an $O(\gamma^{3/2})$-approximation.

Let $\mathcal{S}_\gamma$ denote the class of functions $f \in \mathcal{G}_\gamma$ which are also supermodular and 2nd-order-submodular. Note that $\mathcal{S}_\gamma$ properly contains the family $\mathcal{O}_\gamma$ of discrete quadratic functions which are one-sided $\gamma$-smooth. By Theorem 10 there is an $O(\gamma^{3/2})$-approximation factor for functions in $\mathcal{S}_\gamma$, and hence this class provides our most general answer to question (DIV+).

### 2.3 Related Work

Other extensions of submodular functions with respect to some sliding parameter $\gamma$ (measuring how close a set function is to being submodular) have been considered in the literature. These include the class of $\gamma$-weakly submodular functions, introduced in [17] and further studied in [19, 24, 29, 16, 6]. The class of set functions with supermodular degree $d$ (an integer between 0 and $n - 1$ such that $d = 0$ if and only if $f$ is submodular), introduced in [21] and further considered in [22, 23]. The class of $\epsilon$-approximate submodular functions studied in [28]. The hierarchy over monotone set functions introduced in [20], where levels of the hierarchy correspond to the degree of complementarity in a given function. They refer to this class as MPH (Maximum over Positive Hypergraphs), and MPH-k denotes the $k$-th level in the hierarchy where $1 \leq k \leq n$. The highest level MPH-n of the hierarchy captures all monotone functions, while the lowest level MPH-1 captures the class of XOS functions (which include submodular).

We remark that our class of $\gamma$-meta-submodular functions differs from all the above extensions, since, for instance, none of them captures the class of metric diversity functions (in the sense of giving a good, say $O(1)$, approximation) while ours does. Moreover, a discussion about the one-sided smoothness and Lipschitz smoothness is provided in Appendix A.1.

Other adaptations of the original continuous greedy algorithm from [49] have been used in different submodular maximization settings, including non-monotone [24] and distributed [1] maximization.

### 2.4 Background, Notation, and Preliminary Results

We use $\{e_1, \ldots, e_n\}$ to denote the standard basis of $\mathbb{R}^n$ and $[n] := \{1, \ldots, n\}$ to refer to the ground set of a set function. If $R \subseteq [n]$ and $x = (x_1, \ldots, x_n) \in [0,1]^n$, $p_x(R)$ denotes the probability of picking set $R$ with respect to vector $x$. In other words, $p_x(R) = \prod_{v \in R} x_v \prod_{v \in [n] \setminus R} (1 - x_v)$. 


The multi-linear extension of a set function \( f : 2^{[n]} \rightarrow \mathbb{R} \) is \( F : [0,1]^n \rightarrow \mathbb{R} \), where
\[
F(x) = \sum_{R \subseteq [n]} f(R)p_x(R) = \mathbb{E}_{R \sim x}[f(R)].
\]

For a set \( R \subseteq [n] \), we denote by \( \mathbf{1}_R \) its characteristic vector. Given a vector \( x \) we denote its support by \( \text{supp}(x) \), i.e., the set of non-zero coordinates of \( x \). The following lemma describes the connection between the terms \( A_{ij} \) and \( B_i \) (see Appendix A for proof details).

**Lemma 2** (Discrete integral). Let \( f : 2^{[n]} \rightarrow \mathbb{R} \), \( i \in [n] \), and \( R = \{v_1, \ldots, v_r\} \subseteq [n] \). Moreover, let \( R_m = \{v_1, \ldots, v_m\} \) for \( 1 \leq m \leq r \) and \( R_0 = \emptyset \). Then \( B_i(R) = f(\{i\}) + \sum_{j=1}^{r} A_{iv_j}(R_{j-1}) \).

For vector \( x \in \mathbb{R}^n \) and \( i \in [n] \), we use the (somewhat unfortunate) notation \( x - i \in \mathbb{R}^{n-1} \) to denote the vector produced by eliminating the \( i \)-th coordinate of \( x \).

### 3 A Key Property of One-Sided Smoothness

The following result describes a property of one-sided smoothness that plays a key role in the analysis of both our continuous and discrete (local search) algorithms.

**Lemma 3.** Let \( x \in [0,1]^n \setminus \{\vec{0}\} \), \( u \in [0,1]^n \) and \( \epsilon > 0 \) such that \( x + \epsilon u \in [0,1]^n \). Let \( F : [0,1]^n \rightarrow \mathbb{R} \) be a non-negative, monotone function which is one-sided \( \sigma \)-smooth on \( \{y | x + \epsilon u \geq y \geq x\} \). Then
\[
\|x + \epsilon u\|_1 \leq \left( \frac{\|x + \epsilon u\|_1}{\|x\|_1} \right)^\sigma (u^T \nabla F(x)).
\]

**Proof.** Let \( g(t) := u^T \nabla F(x + tu) \). By the Chain Rule we have \( g'(t) = u^T \nabla^2 F(x + tu)u \).

By one-sided \( \sigma \)-smoothness on \( \{y | x + \epsilon u \geq y \geq x\} \), for any \( 0 \leq t \leq \epsilon \),
\[
g'(t) = u^T \nabla^2 F(x + tu)u \leq \sigma \frac{\|u\|_1}{\|x + tu\|_1} u^T \nabla F(x + tu) = \sigma \frac{\|u\|_1}{\|x + tu\|_1} g(t) \leq \sigma \frac{\|u\|_1}{\|x + tu\|_1} (g(t) + c),
\]
for any \( c > 0 \). Therefore, using that \( g(t) + c > 0 \) for all \( t \) (since \( g(t) \geq 0 \)), we have
\[
\frac{g'(t)}{g(t) + c} \leq \sigma \frac{\|u\|_1}{\|x + tu\|_1}. \tag{4}
\]

We integrate both sides of (4) with respect to \( t \). On the left hand side we get
\[
\int_0^\epsilon \frac{g'(t)}{g(t) + c} dt = \ln(g(t) + c) \bigg|_{0}^{\epsilon} = \ln \left( \frac{g(\epsilon) + c}{g(0) + c} \right),
\]
and on the right hand side we get
\[
\sigma \int_0^\epsilon \frac{\|u\|_1}{\|x + tu\|_1} dt = \sigma \ln(\|x + tu\|_1) \bigg|_{0}^{\epsilon} = \sigma \ln \left( \frac{\|x + \epsilon u\|_1}{\|x\|_1} \right),
\]
where we use that \( \|u\|_1 = \sum_i u_i = \frac{d}{dt} \sum_i (x_i + tu_i) = \frac{d}{dt} \|x + tu\|_1 \).

Therefore \( \ln \left( \frac{g(\epsilon) + c}{g(0) + c} \right) \leq \sigma \ln \left( \frac{\|x + \epsilon u\|_1}{\|x\|_1} \right) \), and hence \( g(\epsilon) + c \leq \left( \frac{\|x + \epsilon u\|_1}{\|x\|_1} \right)^\sigma (g(0) + c) \). Since this holds for any \( c > 0 \) taking the limit yields the desired result. \( \square \)
Algorithm 1: Jump-Start Continuous Greedy

1. **Input:** A monotone one-sided $\sigma$-smooth function $F : [0, 1]^n \rightarrow \mathbb{R}_{\geq 0}$, a polytime separable downward-closed polytope $P$, and $c \in (0, 1)$
2. $v^* \leftarrow \text{arg max}_{x \in P} ||x||_1$
3. $x(0) \leftarrow cv^*$
4. $v_{max}(x) \leftarrow \text{arg max}_{x \in P}\{v^T\nabla F(x)\}$
5. for $t \in [0, 1]$ do
   6. Solve $x'(t) = (1 - c)v_{max}(x(t))$ with boundary condition $x(0) = cv^*$
7. **return** $x(1)$

4 Continuous Greedy and One-Sided $\sigma$-Smoothness

In this section, we provide an adaptation of the **continuous greedy algorithm**, originally introduced in [49]. Algorithm 1 is for maximizing a monotone one-sided $\sigma$-smooth function over a polytime separable downward-close polytope. Unlike the classical continuous greedy, our algorithm starts from a non-zero point, which allows us to take advantage of Lemma 3. Because of this, we call our algorithm **jump-start continuous greedy**.

**Theorem 1.** Let $F : [0, 1]^n \rightarrow \mathbb{R}_{\geq 0}$ be a monotone one-sided $\sigma$-smooth function. Let $c \in (0, 1)$ and $P$ be a polytime separable, downward-closed, polytope. If we run the jump-start continuous greedy process (Algorithm 1) then $x(1) \in P$ and $F(x(1)) \geq [1 - \exp(-(1-c)(\frac{c}{c+1})^\sigma)] \cdot OPT$ where $OPT := \max\{F(x) : x \in P\}$.

The proof details are provided in Appendix E.1. Here we discuss the main idea of the proof. That is to show that moving in the $v_{max}$ direction guarantees a fractional progress proportional to $(\frac{c}{c+1})^\sigma$ towards $OPT$. Let $x^* \in P$ be such that $F(x^*) = OPT$. Also, let $x \in \{x(t) : 0 \leq t \leq 1\}$ and $u = (x^* - x) \lor 0$, i.e., $x^* \lor x = x + u$ (where $\lor$ denotes the component-wise maximum operation). We have by Taylor’s Theorem that for some $\epsilon \in (0, 1)$:

$$OPT \leq F(x^* \lor x) = F(x) + u^T\nabla F(x + \epsilon u) \leq F(x) + \left(\frac{||x + \epsilon u||_1}{||x||_1}\right)^\sigma u^T\nabla F(x),$$

where the last inequality follows from Lemma 3. By the choice of $x(0)$ we have that $||x(0)||_1 \geq c||w||_1$ for any $w \in P$, and then since $u \in P$ and $x(t)$ is non-decreasing in each component (because $v_{max}$ is always non-negative) we also have

$$\frac{||x + \epsilon u||_1}{||x||_1} \leq \frac{||x + u||_1}{||x||_1} = 1 + \frac{||u||_1}{||x||_1} \leq 1 + \frac{||u||_1}{||x(0)||_1} \leq 1 + \frac{1}{c} = \frac{c + 1}{c}.$$

By the choice of $v_{max}$ and above inequalities it follows that for any $x \in \{x(t) : 0 \leq t \leq 1\}$,

$$v_{max}(x) \cdot \nabla F(x) \geq u^T\nabla F(x) \geq \left(\frac{1}{||x + \epsilon u||_1}\right)^\sigma (OPT - F(x)) \geq \left(\frac{c}{c+1}\right)^\sigma (OPT - F(x)).$$

In Proposition 12 in Appendix E we provide an explicit expression for the best value of $c$ (in terms of $\sigma$) for Algorithm 1 when we are dealing with one-sided $\sigma$-smooth functions.

As discussed in Section 2.1, Theorem 2 if $F$ also satisfies the higher order smoothness condition of having non-positive third order partial derivatives, then the approximation factor of Algorithm 1 improves to $O(\sigma)$ (proof details are provided in Appendix E). Finally, we remark that by standard techniques (see [49][12]) one may discretize the continuous greedy process to obtain a finite algorithm, which deviates from the above guarantees by a $o(1)$ additive error.
5 Integality Gaps of Quadratic Objectives over Matroids

Let $M = ([n], I)$ be a matroid and $P_M$ be its polytope. In this section we consider the integrality gap for the quadratic program: $\max F(x) : x \in P_M$. Here $F$ is a non-negative, quadratic multi-linear function $F(x) = \frac{1}{2} x^T Ax + b^T x$ such that $A, b \geq 0$ and $A$ is a symmetric, zero diagonal matrix.

Gaps for such quadratic programmes may be unbounded even for graphic matroids if we allow parallel edges. Fortunately these large gaps transpire due to a simple reason, namely that the matroids have very small circuits. This is encapsulated in the following integrality gap upper bound.

**Theorem 11.** Let $F$ be a non-negative, quadratic multi-linear polynomial and $M$ be a matroid with rank $r$ and minimum circuit size $c \geq 3$. If $x^* \in P_M$, then there is an independent set $I$ of $M$ such that $(3 + \frac{2r}{c-2})F(1_I) \geq F(x^*)$.

We actually prove the following decomposition result. For $x^* \in P_M$, we define the coverage of a pair $u, v$ to be the quantity $x^*(u)x^*(v)$. Let $\text{Cov} \in \mathbb{R}^{|I|}$ be the vector with entries $\text{Cov}(u, v) = x^*(u)x^*(v)$. As $F$ is quadratic it is linear in these coverage values and the vector $x^*$: $F(x^*) = \sum_{u \neq v} \frac{1}{2} (\text{Cov}(u, v) + \sum_{v \neq v} b(v)x^*(v))$. For a set $X$ we say its coverage set is $\text{cov}(X) = \{u, v : u, v \in X, u \neq v\}$. A quadratic coverage of $x^*$ is a collection $C = \{1_{I_i}, \mu_i\}$ of weighted independent sets with properties (1) for each $u \neq v, \sum_{i: u \in I_i} \mu_i \geq \text{Cov}(u, v)$, and (2) for each $v, \sum_{i: v \in I_i} \mu_i \geq x^*(v)$. Recall that $A, b \geq 0$. It follows that $\sum \mu_i F(1_{I_i}) \geq F(x^*)$ and hence if the size $\sum \mu_i \leq K$, then some $I_i$ satisfies $F(1_{I_i}) \geq \frac{F(x^*)}{K}$. This bound depends on the fact that entries of $A$ are non-negative. By condition (1) of quadratic coverages, we have $\sum \mu_i 1_{\text{cov}(I_i)} \geq \text{Cov}$ and by condition (2), $\sum \mu_i 1_{I_i} \geq x^*$. Therefore, for such a collection we have $\sum \mu_i F(1_{I_i}) \geq F(x^*)$. This reasoning shows that to deduce Theorem 11 it suffices to find a quadratic coverage with $\sum \mu_i \leq (3 + \frac{2r}{c-2})$. We show how to do this in Appendix G.2 Theorem 17.

Our other rounding approach (Algorithm 7 in Appendix G.3) is inspired by swap rounding, and shows an integrality gap of at most $O(1 + \frac{1}{\varepsilon})$. It starts from a convex combination $x = \sum_{k=1}^n \lambda_k I_k$ of bases and at each step it merges two of them. Given two bases $I_k$ and $I_m$, the algorithm picks elements $i \in I_k \setminus I_m$ and $j \in I_m \setminus I_k$ and depending on the change of value in the objective, it either replaces $i$ by $j$ in $I_k$, or $j$ by $i$ in $I_m$. This is repeated until $I_k$ and $I_m$ merge into one basis. The set of pairs used to produce the merged basis form a matching $M = \{(i_1, j_1), \ldots, (i_t, j_t)\}$. We show (Lemma 14) that this process reduces the objective by at most $\lambda_m \wedge \sum_{i=1}^t A(i, j_i)$. Let $B$ be the final basis obtained during the merging process. We show that its overall loss is at most half the weight of the maximum of the matchings encountered. We then return to the bases $I_k, I_m$ corresponding to the maximum matching and do a merge on them to produce basis $B'$. The output of the algorithm is the better of $B, B'$ - see Theorem 18 in Appendix G.3.

We also have an almost matching lower bound to the integrality gaps in Theorem 5. See Appendix G.1 for proof details.

**Proposition 1.** Let $k, t \in \mathbb{N}$ with $1 \leq t \leq k$. There exists a $\sigma$-semi-metric with multi-linear extension $F$, and a matroid $M = ([2k], I)$ with rank $r = k + t - 1$ and minimum circuit size $c = 2t$, where the integrality gap of $F(x)$ over the matroid polytope $P_M$ is $\Omega(\min\{\frac{r}{c-2}, \frac{2}{\sigma}\})$.

6 Local Search

The main algorithmic result for general monotone $\gamma$-meta-submodular functions is as follows.
Theorem 8. Let \( f \in \mathcal{G}_\gamma \) and \( \mathcal{M} = ([n], \mathcal{I}) \) be a matroid of rank \( r \). Let \( A \in \mathcal{I} \) be an optimum set, i.e., \( A \in \arg \max_{R \in \mathcal{I}} f(R) \), and \( S \in \mathcal{I} \) be an \((1 + \frac{\epsilon}{n})\)-approximate local optima, i.e., for any \( i \) and \( j \) such that \( S - i + j \in \mathcal{I} \), \((1 + \frac{\epsilon}{n})f(S) \geq f(S - i + j)\), where \( \epsilon > 0 \) is a constant. Then if \( \gamma = O(r) \), \( f(A) \leq O(\gamma 2^{\gamma}) f(S) \) and if \( \gamma = \omega(r) \), \( f(A) \leq O(\gamma^2 2^{\gamma}) f(S) \).

This result does not need the last step of Algorithm 2 where we find a maximum matching. The analysis relies on two technical lemmas (Lemma 8, 9 in Appendix F) that use subdomain one-sided smoothness to bound the second term of the Taylor series in the right hand side of the following expression - see Appendix F for proof details.

\[
F(1_A) \leq F(1_A \lor 1_S) = F(1_S) + 1_{A \setminus S}^T \nabla F(1_S + \epsilon' 1_{A \setminus S}).
\]

We discuss the runtime of Algorithm 2 in Appendix F.2

Algorithm 2: Local search under matroid constraint

1. **Input:** A set function \( f \), a matroid \( \mathcal{M} = ([n], \mathcal{I}) \) with circuits of minimum cardinality \( c \), and \( \epsilon > 0 \).
2. \( S_0 \leftarrow \arg \max_{(v, v') \in \mathcal{I}} f((v, v')) \)
3. \( S \leftarrow \) a base of \( \mathcal{M} \) that contains \( S_0 \)
4. **while** \( S \) is not an approximate local optima **do**
5. \( \text{Find } i \in S \) and \( j \in [n] \setminus S \) such that \( S - i + j \in \mathcal{I} \) and \( f(S - i + j) \geq (1 + \frac{\epsilon}{n})f(S) \)
6. \( S \leftarrow S - i + j \)
7. **Create a complete weighted bipartite graph** \( G \) with node sets \( S \) and \([n] \setminus S\), and edge weights \( w(i, j) := A_{ij}(S) \) for each \( i \in S \) and \( j \notin S \). Find a maximum weighted matching \( M \) in \( G \) of (edge) cardinality \( \frac{\gamma}{2} \), and let \( S' \) denote the node set of \( M \).
8. **Return** \( \arg \max \{ f(S), f(S') \} \)

As discussed in Theorems 9 and 10 on Section 2.2 one can get improved approximation factors by requiring additional conditions on the marginal gains of the set function \( f \). More precisely, if \( f \in \mathcal{G}_\gamma \) has marginal gain terms \( B_i(S) = f(S + i) - f(S - i) \) which are submodular, then Algorithm 2 gives an improved \( O(\gamma + \frac{\gamma^2}{r})\)-approximation. If we go one step further and also require \( f \) to be supermodular, then the approximation factor becomes \( O(\min\{ \gamma + \frac{\gamma^2}{r}, \frac{\gamma \epsilon}{r} \}) \leq O(\gamma^{3/2}) \).

The analysis goes as follows. By Taylor’s theorem and non-positivity of third-order partial derivatives, we have

\[
F(1_A) \leq F(1_A \lor 1_S) = F(1_S) + 1_{A \setminus S}^T \nabla F(1_S) + \frac{1}{2} 1_{A \setminus S}^T \nabla^2 F(1_S) 1_{A \setminus S} \leq F(1_S) + (1 + \gamma \frac{|A \setminus S|}{|S|}) 1_{A \setminus S}^T \nabla F(1_S),
\]

where the last inequality follows from the definition of smoothness and Theorem 7 about subdomain smoothness (for \( \alpha = 1 \)). Writing it with other notation, we have

\[
f(A) \leq f(S) + (1 + \gamma \frac{|A \setminus S|}{|S|}) \sum_{i \in A \setminus S} B_i(S) \leq f(S) + (1 + \gamma) \sum_{i \in A \setminus S} B_i(S).
\]

Let \( g : A \setminus S \rightarrow S \setminus A \) be a bijective mapping where \( S - g(i) + i \in \mathcal{M} \) for any \( i \). Then by the above inequality, Lemma 2 (discrete integral) and approximate local optimality of \( S \), we have

\[
f(A) \leq f(S) + (1 + \gamma) \sum_{i \in A \setminus S} B_i(S) = f(S) + (1 + \gamma) \sum_{i \in A \setminus S} (B_i(S - g(i)) + A_{ig(i)}(S - g(i)))
\]

\[
\leq f(S) + (1 + \gamma) \sum_{i \in A \setminus S} (B_{g(i)}(S - g(i)) + \frac{\epsilon}{r^2} f(S) + A_{ig(i)}(S - g(i)))
\]

\[
\leq O(\gamma) f(S) + (1 + \gamma) \sum_{i \in A \setminus S} A_{ig(i)}(S - g(i)),
\]
where the last inequality follows from a technical lemma (Lemma 12) in Appendix F.4. Now, we can bound $\sum_{i \in A \setminus S} A_{ig(i)}(S - g(i))$ in two different ways: First, by using the definition of meta-submodularity, we obtain an $O(\gamma + \frac{\gamma^2}{r})$-approximation. Second, by using the maximum weighted matching (explained in Algorithm 2) with node sets $S'$ and returning the better of $S$ and $S'$, we obtain an $O(\frac{\gamma r}{c-1})$-approximation. We include a further discussion and proof details in Appendix F.4.

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A Appendix: Preliminaries

The following result describes the connection between the terms $A_{ij}$ and $B_i$. One can see it as a discrete integral formula.

**Lemma 2.** Let $f : 2^{[n]} \rightarrow \mathbb{R}$, $i \in [n]$, and $R = \{v_1, \ldots, v_r\} \subseteq [n]$. Moreover, let $R_m = \{v_1, \ldots, v_m\}$ for $1 \leq m \leq r$ and $R_0 = \emptyset$. Then

$$B_i(R) = f(\{i\}) + \sum_{j=1}^{r} A_{iv_j}(R_{j-1}).$$
Proof. First, we consider the case where $i \notin R$. Then $B_i(R) = f(R + i) - f(R)$ and the right hand side is equal to

$$
\begin{align*}
&f(R_{r-1} + i + v_r) - f(R_{r-1} - i + v_r) - f(R_{r-1} + i - v_r) + f(R_{r-1} - i - v_r) \\
&+ f(R_{r-2} + i + v_{r-1}) - f(R_{r-2} - i + v_{r-1}) - f(R_{r-2} + i - v_{r-1}) + f(R_{r-2} - i - v_{r-1}) \\
&+ \cdots \\
&+ f(R_1 + i + v_2) - f(R_1 - i + v_2) - f(R_1 + i - v_2) + f(R_1 - i - v_2) \\
&+ f(R_0 + i + v_1) - f(R_0 - i + v_1) - f(R_0 + i - v_1) + f(R_0 - i - v_1) \\
&+ f(\{i\}) \\
&= f(R + i) - f(R) - f(R_{r-1} + i) + f(R_{r-1}) \\
&+ f(R_{r-1} + i) - f(R_{r-1}) - f(R_{r-2} + i) + f(R_{r-2}) \\
&+ \cdots \\
&+ f(R_2 + i) - f(R_2) - f(R_1 + i) + f(R_1) \\
&+ f(R_1 + i) - f(R_1) - f(R_0 + i) + f(R_0) \\
&+ f(\{i\}) \\
&= f(R + i) - f(R)
\end{align*}
$$

The last equality holds because the third and the fourth elements of each line cancel out the first and the second element of the next line (except for the last two lines), respectively. For the last two lines, note that $f(R_0) = f(\emptyset) = 0$ and $f(R_0 + i) = f(\{i\})$.

Now, we consider the case that $i \in R$. Let $i = v_j$. Then $B_i(R) = f(R) - f(R - i)$ and the right
hand side is equal to

\[
\begin{align*}
& f(R_{r-1} + i + v_r) - f(R_{r-1} - i + v_r) - f(R_{r-1} + i - v_r) + f(R_{r-1} - i - v_r) \\
& + f(R_{r-2} + i + v_r) - f(R_{r-2} - i + v_r) - f(R_{r-2} + i - v_r) + f(R_{r-2} - i - v_r) \\
& + \cdots \\
& + f(R_j + i + v_{j+1}) - f(R_j - i + v_{j+1}) - f(R_j + i - v_{j+1}) + f(R_j - i - v_{j+1}) \\
& + f(R_{j-1} + i + v_j) - f(R_{j-1} - i + v_j) - f(R_{j-1} + i - v_j) + f(R_{j-1} - i - v_j) \\
& + f(R_{j-2} + i + v_{j-1}) - f(R_{j-2} - i + v_{j-1}) - f(R_{j-2} + i - v_{j-1}) + f(R_{j-2} - i - v_{j-1}) \\
& + \cdots \\
& + f(R_1 + i + v_2) - f(R_1 - i + v_2) - f(R_1 + i - v_2) + f(R_1 - i - v_2) \\
& + f(R_0 + i + v_1) - f(R_0 - i + v_1) - f(R_0 + i - v_1) + f(R_0 - i - v_1) \\
& + f(\{i\}) \\
& = f(R) - f(R - i) - f(R_{r-1} - i) \\
& + f(R_{r-1} - i) - f(R_{r-2} - i) + f(R_{r-2} - i) \\
& + \cdots \\
& + f(R_{j+1} - i - f(R_j) + f(R_j) \\
& + f(R_j) - f(R_{j-1}) + f(R_{j-1}) \\
& + f(R_{j-1}) - f(R_{j-2} + i) + f(R_{j-2}) \\
& + \cdots \\
& + f(R_2 + i - f(R_2) - f(R_1 + i) + f(R_1) \\
& + f(R_1 + i) - f(R_0 + i) + f(R_0) \\
& + f(\{i\}) \\
& = f(R) - f(R - i).
\end{align*}
\]

Like before the last equality holds because the last two terms of each line cancels out the first two terms of the next line except for the last two lines, the first \( f(R_j) \) line and the \( f(R_{j+1}) \) line. The terms of the first \( f(R_j) \) line cancel each other out, while the last two terms of the \( f(R_{j+1}) \) line cancel the first two terms of the second \( f(R_j) \) line.

The following result connects the first order difference \( (B_i) \) and the second order difference \( (A_{ij}) \) to the first and the second order partial derivatives of the multi-linear extension of a set function.

**Lemma 4** \(^{[19]}\). Let \( f \) be a set function and \( F \) its multi-linear function. Then for any \( x = (x_1, \ldots, x_n) \in [0, 1]^n \) and \( i, j \in [n] \),

\[
\nabla_i F(x) = \mathbb{E}_{R \sim x}[B_i(R)] = \sum_{R \subseteq [n]} B_i(R)p_x(R) = \sum_{R \subseteq [n]-i} [f(R + i) - f(R)] \prod_{v \in R} x_v \prod_{v \in [n] \setminus (R+i)} (1 - x_v),
\]

and,

\[
\nabla_{ij}^2 F(x) = \mathbb{E}_{R \sim x}[A_{ij}(R)] = \sum_{R \subseteq [n]} A_{ij}(R)p_x(R)
\]

\[
= \sum_{R \subseteq [n]-i-j} [f(R + i + j) - f(R + i) - f(R + j) + f(R)] \prod_{v \in R} x_v \prod_{v \in [n] \setminus (R+i+j)} (1 - x_v).
\]
Proof. First of all, note that if $i \notin R$ then $B_i(R + i) = B_i(R)$. Now, we write the multi-linear function

$$F(x) = \sum_{R \subseteq [n]} f(R) \prod_{v \in R} x_v \prod_{v \in [n] \setminus R} (1 - x_v)$$

$$= \sum_{R \subseteq [n] - i} (f(R + i)x_i + f(R)(1 - x_i)) \prod_{v \in R} x_v \prod_{v \in [n] \setminus (R + i)} (1 - x_v).$$

Therefore

$$\nabla_i F(x) = \sum_{R \subseteq [n] - i} (f(R + i) - f(R)) \prod_{v \in R} x_v \prod_{v \in [n] \setminus (R + i)} (1 - x_v)$$

$$= x_i \sum_{R \subseteq [n] - i} (f(R + i) - f(R)) \prod_{v \in R} x_v \prod_{v \in [n] \setminus (R + i)} (1 - x_v)$$

$$+ (1 - x_i) \sum_{R \subseteq [n] - i} (f(R + i) - f(R)) \prod_{v \in R} x_v \prod_{v \in [n] \setminus (R + i)} (1 - x_v)$$

$$= \sum_{R \subseteq [n] - i} (f(R + i) - f(R)) \prod_{v \in R} x_v \prod_{v \in [n] \setminus (R + i)} (1 - x_v)$$

$$+ \sum_{R \subseteq [n] - i} (f(R + i) - f(R)) \prod_{v \in R} x_v \prod_{v \in [n] \setminus R} (1 - x_v)$$

$$= \sum_{R \subseteq [n] - i} B_i(R + i)p_x(R + i) + \sum_{R \subseteq [n] - i} B_i(R)p_x(R)$$

$$= \sum_{R \subseteq [n]} B_i(R)p_x(R).$$

Now, to prove the other part of the lemma, we write the multi-linear function again.

$$F(x) = \sum_{R \subseteq [n]} f(R) \prod_{v \in R} x_v \prod_{v \in [n] \setminus R} (1 - x_v)$$

$$= x_ix_j \sum_{R \subseteq [n] - i - j} f(R + i + j) \prod_{v \in R} x_v \prod_{v \in [n] \setminus (R + i + j)} (1 - x_v)$$

$$+ x_i(1 - x_j) \sum_{R \subseteq [n] - i - j} f(R + i) \prod_{v \in R} x_v \prod_{v \in [n] \setminus (R + i + j)} (1 - x_v)$$

$$+ (1 - x_i)x_j \sum_{R \subseteq [n] - i - j} f(R + j) \prod_{v \in R} x_v \prod_{v \in [n] \setminus (R + i + j)} (1 - x_v)$$

$$+ (1 - x_i)(1 - x_j) \sum_{R \subseteq [n] - i - j} f(R) \prod_{v \in R} x_v \prod_{v \in [n] \setminus (R + i + j)} (1 - x_v).$$

Therefore, by using the fact that $x_ix_j + (1 - x_i)x_j + x_i(1 - x_j) + (1 - x_i)(1 - x_j) = 1$, and $A_{ij}(R + i + j) = A_{ij}(R + i) = A_{ij}(R + j) = A_{ij}(R) = f(R + i + j) - f(R + i) - f(R + j) + f(R)$
for $R \subseteq [n] - i - j$, we have

$$
\nabla^2_F(x) = \sum_{R \subseteq [n] - i - j} (f(R + i + j) - f(R + i) - f(R + j) + f(R)) \prod_{v \in R} x_v \prod_{v \in [n] \setminus (R + i + j)} (1 - x_v)
$$

$$
= x_i x_j \sum_{R \subseteq [n] - i - j} A_{ij}(R + i + j) \prod_{v \in R} x_v \prod_{v \in [n] \setminus (R + i + j)} (1 - x_v)
$$

$$
+ (1 - x_i)x_j \sum_{R \subseteq [n] - i - j} A_{ij}(R + j) \prod_{v \in R} x_v \prod_{v \in [n] \setminus (R + i + j)} (1 - x_v)
$$

$$
+ x_i (1 - x_j) \sum_{R \subseteq [n] - i - j} A_{ij}(R + i) \prod_{v \in R} x_v \prod_{v \in [n] \setminus (R + i + j)} (1 - x_v)
$$

$$
+ (1 - x_i)(1 - x_j) \sum_{R \subseteq [n] - i - j} A_{ij}(R) \prod_{v \in R} x_v \prod_{v \in [n] \setminus (R + i + j)} (1 - x_v)
$$

$$
= \sum_{R \subseteq [n] - i - j} A_{ij}(R + i + j) \prod_{v \in R} x_v \prod_{v \in [n] \setminus R} (1 - x_v)
$$

$$
+ \sum_{R \subseteq [n] - i - j} A_{ij}(R + j) \prod_{v \in R + j} x_v \prod_{v \in [n] \setminus (R + i)} (1 - x_v)
$$

$$
+ \sum_{R \subseteq [n] - i - j} A_{ij}(R + i) \prod_{v \in R + i} x_v \prod_{v \in V \setminus (R + i)} (1 - x_v)
$$

$$
+ \sum_{R \subseteq [n] - i - j} A_{ij}(R) \prod_{v \in R} x_v \prod_{v \in [n] \setminus R} (1 - x_v)
$$

$$
= \sum_{R \subseteq [n] - i - j} A_{ij}(R + i + j)p_x(R + i + j)
$$

$$
+ \sum_{R \subseteq [n] - i - j} A_{ij}(R + j)p_x(R + j)
$$

$$
+ \sum_{R \subseteq [n] - i - j} A_{ij}(R + i)p_x(R + i)
$$

$$
+ \sum_{R \subseteq [n] - i - j} A_{ij}(R)p_x(R)
$$

$$
= \sum_{R \subseteq [n]} A_{ij}(R)p_x(R).
$$

\( \square \)

### A.1 One-Sided Smoothness versus Lipschitz Smoothness

Lipschitz smoothness is an important, widely-used property in convex optimization and machine learning. One-sided $\sigma$-smoothness is different from Lipschitz smoothness (and other smoothness notions based on Holder’s or uniform continuity) and we believe it may also have applications to these areas.

A differentiable function is \textit{Lipschitz smooth} if its gradient is Lipschitz continuous. In other words, $f$ is Lipschitz smooth if there exists $L \geq 0$ such that for any $x$ and $y$, $\|\nabla f(x) - \nabla f(y)\|_2 \leq L\|x - y\|_2$ or equivalently for twice differentiable functions, $u^T \nabla^2 f(x)u \leq L\|u\|_2^2$. We then call $f$ \textit{L-Lipschitz smooth}. One could define the one-sided version of this smoothness if the above inequality holds for any $x \leq y$ (second definition/inequality holds for any $u \geq 0$). With this definition, it is easy
to see that submodular functions are one-sided 0-Lipschitz smooth. On the other hand one-sided \( \sigma \)-smoothness is not equivalent to one-sided \( L \)-Lipschitz smoothness. To see an important difference, consider \( g = cf \) function where \( c \) is a constant and \( f \) is one-sided smooth. We have \( \nabla g(x) = c \nabla f(x) \). Thus if \( f \) is one-sided \( L \)-Lipschitz smooth we may only assert that \( g \) is one-sided \( cL \)-Lipschitz smooth.

In particular, Lipschitz smoothness is not closed under multiplication. On the other hand, the one-sided \( \sigma \)-smooth functions form a cone. Intuitively, the reason is that in \( \sigma \)-smooth functions, the ratio of the gradients is bounded (as shown in Lemma 3) unlike Lipschitz smoothness where the difference of the gradients is bounded.

B Appendix: Meta-Submodular Family

In this section, we discuss the meta-submodularity parameter of the class of meta-submodular functions (defined by Kleinberg et al. [35]) and the class of proportionally submodular functions (defined by Borodin et al. [8]).

Proposition 2. \( f \) is 0-meta-submodular if and only if it is meta-submodular (by Kleinberg et al. definition [35]).

Proof. Kleinberg et al. [35] show that a set function \( f \) is meta-submodular if and only if

\[
f(S + i) - f(S) \geq f(T + i) - f(T), \quad \forall \emptyset \neq S \subseteq T, \forall i \notin T.
\]

The above is clearly equivalent to

\[
f(S + i) - f(S) \geq f(S + j + i) - f(S + j), \quad \forall S \neq \emptyset, \forall i \neq j \notin S.
\] (5)

Then

\[f \text{ is 0-meta submodular}\]
\[\iff A_{ij}(S) \leq 0, \quad \forall S \neq \emptyset, \forall i, j \in V\]
\[\iff f(S + i + j) - f(S + i) - f(S + j) + f(S) \leq 0, \quad \forall S \neq \emptyset, \forall i, j \in V\]
\[\iff f(S + i) - f(S) \geq f(S + j + i) - f(S + j), \quad \forall S \neq \emptyset, \forall i, j \in V\]
\[\iff f(S + i) - f(S) \geq f(S + j + i) - f(S + j), \quad \forall S \neq \emptyset, \forall i \neq j \notin S\]
\[\iff (5) \text{ holds}.
\]

Proposition 3. Any monotone proportionally submodular function is 1-meta-submodular.

Proof. The proof is by case analysis.

- If \( i, j \notin R \) then using weak submodularity property we have

\[
(|R| + 2)f(R) + (|R|)f(R + i + j) \leq (|R| + 1)f(R + i) + (|R| + 1)f(R + j),
\]

which means

\[
|R| \cdot (f(R) + f(R + i + j) - f(R + i) - f(R + j)) \leq f(R + i) + f(R + j) - 2f(R).
\]
Hence
\[
f(R + i + j) - f(R + i - j) - f(R + j - i) + f(R - i - j)
= f(R + i + j) - f(R + i) - f(R + j) + f(R)
\leq \frac{f(R + i) - f(R) + f(R + j) - f(R)}{|R|}
= \frac{f(R + i) - f(R - i) + f(R + j) - f(R - j)}{|R|}.
\]

- If \(i, j \in R\) then using weak submodularity property we have
  \[
  (|R| - 2)f(R) + (|R|)f(R - i - j) \leq (|R| - 1)f(R - i) + (|R| - 1)f(R - j),
  \]
  which means
  \[
  |R| \cdot (f(R) + f(R - i - j) - f(R - i) - f(R - j)) \leq 2f(R) - f(R - i) - f(R - j).
  \]
  Hence
  \[
  f(R + i + j) - f(R + i - j) - f(R + j - i) + f(R - i - j)
  = f(R) - f(R - j) - f(R - i) + f(R - i - j)
  \leq \frac{f(R) - f(R - i) + f(R) - f(R - j)}{|R|}
  = \frac{f(R + i) - f(R - i) + f(R + j) - f(R - j)}{|R|}.
  \]

- If \(i \in R\) and \(j \notin R\) then using weak submodularity property we have
  \[
  (|R| - 1)f(R + j) + (|R| + 1)f(R - i) \leq (|R|)f(R) + (|R|)f(R + j - i),
  \]
  which means
  \[
  |R| \cdot (f(R + j) + f(R - i) - f(R) - f(R + j - i)) \leq f(R + j) - f(R - i)
  = f(R + j) - f(R - j) + f(R + i) - f(R - i),
  \]
  where the equality is correct because \(f(R) = f(R - j) = f(R + i)\). Hence
  \[
  f(R + i + j) - f(R + i - j) - f(R + j - i) + f(R - i - j)
  = f(R + j) - f(R) - f(R + j - i) + f(R - i)
  \leq \frac{f(R + j) - f(R - i)}{|R|}
  = \frac{f(R + i) - f(R - i) + f(R + j) - f(R - j)}{|R|}.
  \]

**Proposition 4.** Any second-order-modular function with a \(\sigma\)-semi-metric distance function (\(\sigma \geq 1\)) and a non-negative modular function is a \(\sigma\)-meta submodular function.
Proof. Let $f(R) = \sum_{q \in R} g(q) + \sum_{\{q, q'\} \subseteq R} d(q, q')$ be a second-order modular function (by Lemma 5, it has this form). The proof is by case analysis.

- If $i, j \notin R$, we have

$$|R|A_{ij}(R) = |R|(f(R + i + j) - f(R + i - j) - f(R - i + j) + f(R - i - j))$$

$$= |R|(\sum_{q \in R} g(q) + \sum_{\{q, q'\} \subseteq R+i} d(q, q') - \sum_{q \in R} g(q) - \sum_{\{q, q'\} \subseteq R} d(q, q'))$$

$$- \sum_{q \in R+ i} g(q) - \sum_{\{q, q'\} \subseteq R+i} d(q, q') + \sum_{q \in R} g(q) + \sum_{\{q, q'\} \subseteq R} d(q, q'))$$

$$= |R||d(i, j).$$

We also have

$$\sigma(B_i(R) + B_j(R)) = \sigma(f(R + i) - f(R - i) + f(R + j) - f(R - j))$$

$$= \sigma(\sum_{q \in R+i} g(q) + \sum_{\{q, q'\} \subseteq R+i} d(q, q') - \sum_{q \in R} g(q) - \sum_{\{q, q'\} \subseteq R} d(q, q'))$$

$$+ \sum_{q \in R+j} g(q) + \sum_{\{q, q'\} \subseteq R+j} d(q, q') - \sum_{q \in R} g(q) - \sum_{\{q, q'\} \subseteq R} d(q, q'))$$

$$= \sigma g(i) + \sigma g(j) + \sigma \sum_{q \in R} d(i, q) + \sigma \sum_{q \in R} d(j, q).$$

Therefore $|R|A_{ij}(R) \leq \sigma(B_i(R) + B_j(R))$ because $g$ is non-negative and $d$ is non-negative $\sigma$-semi-metric.

- If $i, j \in R$, we have

$$|R|A_{ij}(R) = |R|(f(R + i + j) - f(R + i - j) - f(R - i + j) + f(R - i - j))$$

$$= |R|(\sum_{q \in R} g(q) + \sum_{\{q, q'\} \subseteq R} d(q, q') - \sum_{q \in R-j} g(q) - \sum_{\{q, q'\} \subseteq R-j} d(q, q'))$$

$$- \sum_{q \in R-i} g(q) - \sum_{\{q, q'\} \subseteq R-i} d(q, q') + \sum_{q \in R-j} g(q) + \sum_{\{q, q'\} \subseteq R-j} d(q, q'))$$

$$= |R||d(i, j).$$

We also have

$$\sigma(B_i(R) + B_j(R)) = \sigma(f(R + i) - f(R - i) + f(R + j) - f(R - j))$$

$$= \sigma(\sum_{q \in R} g(q) + \sum_{\{q, q'\} \subseteq R} d(q, q') - \sum_{q \in R-i} g(q) - \sum_{\{q, q'\} \subseteq R-i} d(q, q'))$$

$$+ \sum_{q \in R} g(q) + \sum_{\{q, q'\} \subseteq R} d(q, q') - \sum_{q \in R-j} g(q) - \sum_{\{q, q'\} \subseteq R-j} d(q, q'))$$

$$= \sigma g(i) + \sigma g(j) + 2\sigma d(i, j) + \sigma \sum_{q \in R-i-j} d(i, q) + \sigma \sum_{q \in R-i-j} d(j, q).$$

Therefore $|R|A_{ij}(R) \leq \sigma(B_i(R) + B_j(R))$ because $g$ is non-negative, $d$ is non-negative $\sigma$-semi-metric, and $\sigma \geq 1.$
• If $i \in R$ and $j \notin R$, we have

\[
|R|A_{ij}(R) = |R|((f(R + i) + f(R + i + j)) - f(R + i - j) - f(R - i + j) + f(R - i - j))
\]

\[
= |R|\left( \sum_{q \in R + j} g(q) + \sum_{\{q, q'\} \subseteq R + j} d(q, q') - \sum_{q \in R} g(q) - \sum_{\{q, q'\} \subseteq R} d(q, q') - \sum_{q \in R - i + j} g(q) - \sum_{\{q, q'\} \subseteq R - i + j} d(q, q') + \sum_{q \in R - i} g(q) + \sum_{\{q, q'\} \subseteq R - i} d(q, q') \right)
\]

\[
= |R|d(i, j).
\]

We also have

\[
\sigma(B_i(R) + B_j(R)) = \sigma(f(R + i) - f(R - i) + f(R + j) - f(R - i))
\]

\[
= \sigma\left( \sum_{q \in R} g(q) + \sum_{\{q, q'\} \subseteq R} d(q, q') - \sum_{q \in R - i} g(q) - \sum_{\{q, q'\} \subseteq R - i} d(q, q') + \sum_{q \in R + j} g(q) + \sum_{\{q, q'\} \subseteq R + j} d(q, q') - \sum_{q \in R} g(q) - \sum_{\{q, q'\} \subseteq R} d(q, q') \right)
\]

\[
= \sigma g(i) + \sigma g(j) + \sigma d(i, j) + \sigma \sum_{q \in R - i} d(i, q) + \sigma \sum_{q \in R - i} d(j, q).
\]

Therefore $|R|A_{ij}(R) \leq \sigma(B_i(R) + B_j(R))$ because $g$ is non-negative, $d$ is non-negative $\sigma$-semi-metric, and $\sigma \geq 1$.

\[\square\]

### C Appendix: Semi-Metric Diversity

In this section, we establish the smoothness parameter associated with several of the discrete quadratic functions discussed. In other words, we bound the approximate triangle inequality for their associated distance functions.

**Definition 1.** Let $d : [n] \times [n] \rightarrow \mathbb{R}_{\geq 0}$ be a distance function with the corresponding distance matrix $D \in \mathbb{R}_{\geq 0}^{n \times n}$ where $D_{a,b} = d(a,b)$. We say $d$ is a negative-type distance if for any $x \in \mathbb{R}^n$ with $\|x\|_1 = 0$ we have $x^T Dx \leq 0$.

**Proposition 5.** Any negative-type distance $d : [n] \times [n] \rightarrow \mathbb{R}_{\geq 0}$ is 2-semi-metric.

**Proof.** Let $x = 0.5e_a + 0.5e_b - e_c$. We know

\[
x^T Dx = 0.5d(a, b) - d(a, c) - d(b, c) \leq 0.
\]

Therefore $d(a, b) \leq 2d(a, c) + 2d(b, c)$ and $d$ is 2-semi metric. \[\square\]

Jensen-Shannon Divergence is a function which measures dis-similarity between probability distributions. It is well-known that if $d$ is a JS measure, then $\sqrt{d}$ is a metric. Hence JS distances form a 2-semi-metric by the following result.

**Proposition 6.** Let $d : [n] \times [n] \rightarrow \mathbb{R}_{\geq 0}$ be a distance function such that $\sqrt{d(\cdot, \cdot)}$ is a metric. Then $d(\cdot, \cdot)$ is a 2-semi-metric.

23
We also know that

\[ \sqrt{d(i,j)} \leq \sqrt{d(i,k)} + \sqrt{d(j,k)}. \]

Therefore,

\[ d(i,j) \leq d(i,k) + d(j,k) + 2\sqrt{d(i,k)d(j,k)}. \]

We also know that

\[ d(i,k) + d(j,k) - 2\sqrt{d(i,k)d(j,k)} = (\sqrt{d(i,k)} - \sqrt{d(j,k)})^2 \geq 0. \]

Hence,

\[ d(i,j) \leq 2(d(i,k) + d(j,k)). \]

\[ \square \]

C.1 Second-Order-Modular Functions

In this section, we describe the structure of second-order-modular functions (defined by Korula et al. [38]). We also discuss the smoothness parameter of quadratic functions defined on a \(\sigma\)-semi-metric distance. Moreover, we discuss the meta-submodularity parameter of the second-order-modular functions defined on a \(\sigma\)-semi-metric distance.

Definition 2 (38). A set functions \( f : 2^{[n]} \to \mathbb{R} \) is called second-order modular if \( B_i(S \cup R) - B_i(S) = B_i(T \cup R) - B_i(T) \) for any \( S \subseteq T, R \subseteq [n] \setminus T, \) and \( i \in [n] \setminus (T \cup R) \).

The following lemma characterize the structure of second-order modular functions.

Lemma 5. \( f \) is a second-order modular function if and only if there exist symmetric \( d : [n] \times [n] \to \mathbb{R} \), and \( g : [n] \to \mathbb{R} \) such that

\[ f(R) = \sum_{\{i,j\} \subseteq R} d(i,j) + \sum_{i \in R} g(i). \]

If \( f \) is also supermodular (submodular), then \( d \) is non-negative (non-positive).

Proof. Sufficiency is easy since

\[
B_i(S \cup R) - B_i(S) = (g(i) + \sum_{m \in S \cup R} d(m,i)) - (g(i) + \sum_{m \in S} d(m,i)) = \sum_{m \in R} d(m,i) \\
= (g(i) + \sum_{m \in T \cup R} d(m,i)) - (g(i) + \sum_{m \in T} d(m,i)) \\
= B_i(T \cup R) - B_i(T).
\]

To prove necessity, we first show that if \( i,j \in [n] \) and \( S \subseteq [n] - i - j \) then, by second-order modularity

\[ B_j(S + i) - B_j(S) = B_j([n] - j) - B_j([n] - i - j), \]

because \( S \subseteq [n] - i \) \( ([n] - i \) plays the role of \( T \) in the definition of second-order modular). Now, let

\[ d(i,j) = B_j([n] - j) - B_j([n] - i - j) \]

and \( g(i) = B_i(\emptyset) \). Note that \( d \) is symmetric because

\[
d(i,j) = B_j([n] - j) - B_j([n] - i - j) = (f([n]) - f([n] - j)) - (f([n] - i) - f([n] - i - j)) \\
= (f([n]) - f([n] - i)) - (f([n] - j) - f([n] - i - j)) = B_i([n] - i) - B_i([n] - i - j) = d(j,i).
\]

24
For any \( m \), let \( R_m = \{v_1, \ldots, v_m\} \), and set \( R_0 = \emptyset \). Consider a set \( R = \{v_1, \ldots, v_r\} \). Then we have

\[
f(R) = \sum_{m=0}^{r-1} (f(R_m + v_{m+1}) - f(R_m)) = \sum_{m=0}^{r-1} B_{v_{m+1}}(R_m)
\]

\[
= \sum_{m=0}^{r-1} \left( \sum_{t=1}^{m} (B_{v_{m+1}}(R_t) - B_{v_{m+1}}(R_{t-1})) + B_{v_{m+1}}(R_0) \right) \quad \text{telescoping sum}
\]

\[
= \sum_{m=0}^{r-1} \left( \sum_{t=1}^{m} (B_{v_{m+1}}([n] - v_{m+1}) - B_{v_{m+1}}([n] - v_t - v_{m+1})) + B_{v_{m+1}}(R_0) \right)
\]

\[
= \sum_{m=0}^{r-1} \sum_{t=1}^{m} d(v_t, v_{m+1}) + \sum_{m=0}^{r-1} g(v_{m+1}).
\]

If \( f \) is supermodular, \( i, j \in [n] \), and \( R \subseteq [n] - i - j \), we have

\[
f(R + i + j) - f(R + i) \geq f(R + j) - f(R).
\]

Therefore,

\[
g(j) + \sum_{v \in R+i} d(v, j) \geq g(j) + \sum_{v \in R} d(v, j),
\]

which means \( d(i, j) \geq 0 \). Similarly, if \( f \) is submodular, \( d \) is non-positive. \( \square \)

**Proposition 7.** Let \( A \in \mathbb{R}^{n \times n} \) be a symmetric, 0-diagonal matrix. Let \( b \in \mathbb{R}^n \) and \( b \geq 0 \). Then \( F(x) = \frac{1}{2}x^TAx + b^T x \) is one-sided \( 2\sigma \)-smooth if \( A \) is \( \sigma \)-semi-metric.

**Proof.** Note that \( \nabla^2 F(x) = A \) and \( \nabla F(x) = Ax + b \). Therefore,

\[
\sigma(\nabla_i F(x) + \nabla_j F(x)) \geq \sigma(\sum_{k=1}^{n} A(i, k)x_k + \sum_{k=1}^{n} A(j, k)x_k) = \sum_{k=1}^{n} \sigma(A(i, k) + A(j, k))x_k
\]

\[
\geq \sum_{k=1}^{n} A(i, j)x_k = ||x||_1 A(i, j) = ||x||_1 \nabla^2_{ij} F(x),
\]

where the first inequality follows from \( b \geq 0 \) and the last inequality holds because \( A \) is \( \sigma \)-semi-metric. Now by Lemma 11, we conclude that \( F \) is one-sided \( 2\sigma \)-smooth. \( \square \)

### C.2 Hardness of Approximation for \( \sigma \)-semi-metrics

In this section, we provide a hardness result for approximate maximization of remote-clique functions defined on a semi-metric distance.

**Theorem 3.** Assuming the Planted Clique Conjecture: (1) for any constant \( \sigma \geq 1 \), it is hard to approximate the maximum of a \( \sigma \)-semi-metric function subject to a cardinality constraint within a factor of \( 2\sigma - \epsilon \) for any \( \epsilon > 0 \) and (2) for a super-constant \( \sigma \), there is no constant factor (polytime) approximation algorithm for maximizing a \( \sigma \)-semi-metric function subject to a cardinality constraint.

**Proof.** Planted Clique problem asks for an algorithm to distinguish between the following graphs with probability of at least \( 3/4 \): 1) A graph drawn from \( G(n, 1/2) \), 2) A graph drawn from \( G(n, 1/2) \) and then a clique of size \( n^{1/2 - \delta} \) is planted in it (\( \delta > 0 \)) \[32\]. The planted clique conjecture states
that there is no polynomial time algorithm to do this task \cite{3, 31}. It has been shown that assuming the planted clique conjecture, it is hard to approximate the maximum of a metric diversity function within a factor better than 2 \cite{5, 9}.

Given a graph $G$, in the densest $k$-subgraph problem we need to find an induced subgraph of size $k$ with the maximum number of edges. Let $R$ be a subset of vertices of $G$ and $E(R)$ be the number of edges in the induced subgraph of $R$. The density of $R$ is defined as $\rho(R) = E(R)/\binom{|R|}{2}$. Alon et al. \cite{3} showed that if there is no polynomial time algorithm for the planted clique problem for a planted clique of size $n^{1/3}$, then there is no polynomial time algorithm for distinguishing between a graph $G_1$ of size $n$ that contains a clique of size $n^{1/3}$, and a graph $G_2$ of the same size in which the density of every subset of vertices of size $n^{1/3}$ is at most $\delta > 0$.

We can reduce the densest $k$-subgraph problem to $\sigma$-semi-metric function maximization in the following way. Consider an instance of densest $k$-subgraph ($k = n^{1/3}$) on graph $G$ with vertex set $[n]$. Create the distance function $d : [n] \times [n] \to \mathbb{R}$. If there is an edge between $i, j \in [n]$ in $G$, set $d(i, j) = 2\sigma$, otherwise set $d(i, j) = 1$. It is easy to see that this distance function is $\sigma$-semi-metric. Let $f(R) = \sum_{(i, j) \subseteq R} d(i, j)$. If $|R| = k$, we have

$$f(R) = 2\sigma E(R) + \binom{k}{2} - E(R).$$

We know $\binom{k}{2} \geq E(R)$. Therefore

$$2\sigma E(R) \leq f(R) \leq 2\sigma E(R) + \binom{k}{2},$$

and dividing both sides by $2\sigma \binom{k}{2}$ we get

$$\rho(R) \leq \frac{f(R)}{2\sigma \binom{k}{2}} \leq \rho(R) + \frac{1}{2\sigma}. \quad (6)$$

It is easy to see that

$$\arg \max_{R \subseteq [n]} \rho(R) = \arg \max_{|R| = k} f(R).$$

Now, assume that for some fixed constant $c \geq 1$ there is a $c$-factor approximate algorithm for finding the maximum of $\sigma$-semi-metric function ($\sigma$ is super-constant) and its output on $G$ is $S$. Also, let

$$\text{OPT} \in \arg \max_{R \subseteq [n]} \rho(R).$$

We have

$$\rho(\text{OPT}) \leq \frac{f(\text{OPT})}{2\sigma \binom{k}{2}} \leq \frac{cf(S)}{2\sigma \binom{k}{2}} \leq c\rho(S) + \frac{c}{2\sigma}.$$  

Since $\sigma \in \omega(1)$, for some $n$ large enough we have that $\frac{c}{2\sigma} \leq \frac{1}{2}$. Hence $\rho(\text{OPT}) \leq c\rho(S) + \frac{1}{2}$. Set $\delta = \frac{1}{4c}$ and note that $\delta > 0$ is a constant. If $G$ is a graph in which the density of every subset of vertices of size $k$ is at most $\delta$ then clearly $\rho(S) \leq \delta$. If $G$ is a graph that contains a clique of size $k$ then $1 = \rho(\text{OPT}) \leq c\rho(S) + \frac{1}{2}$, which means $\rho(S) \geq \frac{1}{2c} = 2\delta$. This means that our $c$-factor approximate algorithm can distinguish between these two graphs which is in contrast with the planted clique conjecture and Alon et al. result.
For the first part, given any constant \( \sigma \), assume there is a \((2\sigma - \epsilon)\)-factor approximate algorithm for some \( \epsilon > 0 \) for finding the maximum of \( \sigma \)-semi-metric function. Denote its output on \( G \) by \( S \), and let \( \text{OPT} \) be defined as above. We then have

\[
\rho(\text{OPT}) \leq \frac{f(\text{OPT})}{2\sigma(k^2)} \leq \frac{(2\sigma - \epsilon)f(S)}{2\sigma(k^2)} \leq (2\sigma - \epsilon)\rho(S) + \frac{2\sigma - \epsilon}{2\sigma}.
\]

Set \( \delta = \left(\frac{1}{2\sigma - \epsilon} - \frac{1}{2\sigma}\right)/2 = \frac{\epsilon}{4\sigma(2\sigma - \epsilon)} \), and note that \( \delta > 0 \) is a constant. If \( G \) is a graph in which the density of every subset of vertices of size \( k \) is at most \( \delta \) then clearly \( \rho(S) \leq \delta \). If \( G \) is a graph that contains a clique of size \( k \) then \( 1 = \rho(\text{OPT}) \leq (2\sigma - \epsilon)\rho(S) + \frac{2\sigma - \epsilon}{2\sigma} \) which means \( \rho(S) \geq \frac{1}{2\sigma - \epsilon} - \frac{1}{2\sigma} = 2\delta \). This means that our \((2\sigma - \epsilon)\)-factor approximate algorithm can distinguish between these two graphs which is in contrast with the planted clique conjecture and Alon et al. result.

\[\square\]

D Appendix: One-Sided Smoothness

In this section, we discuss the connection between meta-submodularity of a function and the smoothness of its multi-linear extension. We show that if a probabilistic version of (2) holds at a point \( x \), then the multi-linear extension of the functions is smooth at \( x \). We also show that the smoothness of the multi-linear extension results in the meta-submodularity of the corresponding set function.

Lemma \( \text{I} \). Let \( f \) be a non-negative, monotone set function and \( F \) be its multi-linear function. Let \( x \in [0,1]^n \) and \( \gamma \geq 0 \). If for any \( i, j \in [n] \) we have

\[
\mathbb{E}_{R \sim x}[|R|] \cdot \mathbb{E}_{R \sim x}[A_{ij}(R)] \leq \gamma \cdot (\mathbb{E}_{R \sim x}[B_i(R)] + \mathbb{E}_{R \sim x}[B_j(R)]),
\]

or equivalently (by Lemma \( \text{4} \),

\[
||x||_1 \nabla^2_{ij} F(x) \leq \gamma (\nabla_i F(x) + \nabla_j F(x)),
\]

then \( F \) is one-sided \( 2\gamma \)-smooth at \( x \).

Proof. We have

\[
u^T \nabla^2 F(x)u = \sum_{i=1}^{n} \sum_{j=1}^{n} u_i u_j \nabla^2_{ij} F(x) \leq \frac{\gamma}{||x||_1} \sum_{i=1}^{n} \sum_{j=1}^{n} u_i u_j (\nabla_i F(x) + \nabla_j F(x))
\]

\[
= \frac{\gamma}{||x||_1} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} u_i u_j \nabla_i F(x) + \sum_{i=1}^{n} \sum_{j=1}^{n} u_i u_j \nabla_j F(x) \right)
\]

\[
= \frac{\gamma}{||x||_1} \left( \sum_{i=1}^{n} u_i \nabla_i F(x) (\sum_{j=1}^{n} u_j) + \sum_{i=1}^{n} u_i (\sum_{j=1}^{n} u_j \nabla_j F(x)) \right)
\]

\[
= \frac{\gamma}{||x||_1} (||u||_1 \sum_{i=1}^{n} u_i \nabla_i F(x) + ||u||_1 \sum_{j=1}^{n} u_j \nabla_j F(x))
\]

\[
= 2\gamma \left( \frac{||u||_1}{||x||_1} \right) (u^T \nabla F(x)).
\]

\[\square\]
Proposition 8. Let $f$ be a set function and $F$ be its multi-linear extension. If $F$ is one-sided $\gamma$-smooth, then $f$ is $\gamma$-meta-submodular.

Proof. Let non-empty $R \subseteq [n]$ and $i, j \in [n]$. Consider the inequality of one-sided $\gamma$-smoothness for $u = \mathbb{1}_{\{i,j\}}$ and $x = \mathbb{1}_R$:

\[
2u_iu_j\nabla^2 F_{ij}(x) \leq \gamma \frac{u_i + u_j}{|x|_1} (u_i \nabla_i F(x) + u_j \nabla_j F(x))
\]

Since $u_i = u_j = 1$, $|x|_1 = |R|$, $\nabla^2 F_{ij}(x) = A_{ij}(R)$, and $\nabla_i F(x) + \nabla_j F(x) = B_i(S) + B_j(S)$ we obtain the $\gamma$-meta-submodular inequality. \hfill \Box

D.1 Smoothness of Supermodular $\gamma$-Meta-Submodular Functions

In this section, we show that the multi-linear extension of a supermodular $\gamma$-meta-submodular functions is one-sided $O(\gamma)$-smooth.

Lemma 6. Let $f : 2^{[n]} \to \mathbb{R}_+$ be a non-negative, monotone, supermodular, $\gamma$-meta-submodular set function. Let $x \in [0,1]^n \setminus \{\emptyset\}$ and $R \subseteq [n]$ such that $1 \leq |R| < |x|_1$. Then for all $i, j \in [n]$ we have

\[
(|x|_1 - |R|)A_{ij}(R)p_x(R) \leq 2\gamma \sum_{e \in [n] \setminus R} \frac{(B_i(R+e) + B_j(R+e))}{|R| + 1} p_x(R + e).
\]

Also, for the empty set,

\[
(|x|_1)A_{ij}(\emptyset)p_x(\emptyset) \leq 2(\gamma + 1) \sum_{e \in [n]} (B_i(\{e\}) + B_j(\{e\})) p_x(\{e\}).
\]

Proof. Let $|R| = r$. Note that $r < n$ because $|R| = r < |x|_1$. Also, note that if $x_e = 1$ for some $e \in [n] \setminus R$ then $p_x(R) = 0$, which means that the left hand side is zero. In that case, the inequality holds because $f$ is monotone and the right hand side is non-negative. Hence, we assume that $x_e < 1$ for all $e \in [n] \setminus R$. We know that

\[
\sum_{e \in [n]} x_e = |x|_1.
\]

Therefore, because each $x_e \leq 1$,

\[
\sum_{e \in [n] \setminus R} x_e = |x|_1 - \sum_{e \in R} x_e \geq |x|_1 - \sum_{e \in R} 1 = |x|_1 - |R|.
\]

Hence, since $0 < 1 - x_e \leq 1$ for all $e \in [n] \setminus R$, we get

\[
(|x|_1 - |R|)A_{ij}(R)p_x(R) \leq \sum_{e \in [n] \setminus R} x_e A_{ij}(R)p_x(R) \leq \sum_{e \in [n] \setminus R} \frac{x_e}{1 - x_e} A_{ij}(R)p_x(R) = \sum_{e \in [n] \setminus R} A_{ij}(R)p_x(R + e).
\]

Moreover, $2|R| \geq |R| + 1$ because $|R| \geq 1$, and we have

\[
\sum_{e \in [n] \setminus R} A_{ij}(R)p_x(R + e) \leq 2 \sum_{e \in [n] \setminus R} \frac{|R| A_{ij}(R)}{|R| + 1} p_x(R + e).
\]
Using the $\gamma$-meta-submodularity and supermodularity we have
\[
2 \sum_{e \in [n] \setminus R} \frac{|R| A_{ij}(R)}{|R| + 1} p_x(R + e) \leq 2\gamma \sum_{e \in [n] \setminus R} \frac{B_i(R) + B_j(R)}{|R| + 1} p_x(R + e)
\[
\leq 2\gamma \sum_{e \in [n] \setminus R} \frac{B_i(R + e) + B_j(R + e)}{|R| + 1} p_x(R + e)
\]
Combining all of these inequalities yields the first part of the lemma. For the second part of the lemma, we consider the set \{i, j, e\}. By Lemma 2 and the $\gamma$-meta-submodularity, we have
\[
f(\{i, j, e\}) = B_i(\{j, e\}) + B_j(\{e\}) + f(\{e\})
\[
= A_{ij}(\{\emptyset\}) + B_i(\{\emptyset\}) + B_j(\{\emptyset\}) + f(\{\emptyset\})
\]
\[
\leq (\gamma + 1)(B_i(\{\emptyset\}) + B_j(\{\emptyset\})) + f(\{\emptyset\}).
\]
Also, by Lemma 2 we have
\[
f(\{i, j, e\}) = B_i(\{j, e\}) + B_j(\{e\}) + f(\{e\})
\[
= A_{ie}(\{j\}) + A_{ij}(\emptyset) + f(\{i\}) + B_j(\{e\}) + f(\{e\}).
\]
Therefore
\[
A_{ie}(\{j\}) + A_{ij}(\emptyset) + f(\{i\}) + B_j(\{e\}) + f(\{e\}) \leq (\gamma + 1)(B_i(\{\emptyset\}) + B_j(\{\emptyset\})) + f(\{\emptyset\}).
\]
Hence, because $f$ is non-negative, monotone and supermodular, it follows that
\[
A_{ij}(\emptyset) \leq A_{ie}(\{j\}) + A_{ij}(\emptyset) + f(\{i\}) + B_j(\{e\}) \leq (\gamma + 1)(B_i(\{\emptyset\}) + B_j(\{\emptyset\})). \tag{7}
\]
Moreover, because $f$ is non-negative and monotone, we have
\[
A_{ij}(\emptyset) = f(\{i, j\}) - f(\{i\}) - f(\{j\}) + f(\emptyset) = B_j(\{i\}) - f(\{j\})
\]
\[
\leq B_j(\{i\}) + B_i(\{i\}) \leq (\gamma + 1)(B_i(\{\emptyset\}) + B_j(\{\emptyset\})),
\]
and
\[
A_{ij}(\emptyset) = f(\{i, j\}) - f(\{i\}) - f(\{j\}) + f(\emptyset) = B_i(\{j\}) - f(\{i\})
\]
\[
\leq B_i(\{j\}) + B_j(\{j\}) \leq (\gamma + 1)(B_i(\{\emptyset\}) + B_j(\{\emptyset\})).
\]
If $x_e = 1$ for an $e \in [n]$ then $p_x(\emptyset) = 0$ and the inequality holds because the left hand side is zero and the right hand side is non-negative (since $f$ is monotone). Therefore, we assume that $x_e < 1$ for all $e \in [n]$. Combining the above inequalities, we have
\[
(||x||_1) A_{ij}(\emptyset)p_x(\emptyset) = \sum_{e \in [n]} x_e A_{ij}(\emptyset)p_x(\emptyset)
\]
\[
\leq \sum_{e \in [n]} \frac{x_e}{1 - x_e} A_{ij}(\emptyset)p_x(\emptyset)
\]
\[
= \sum_{e \in [n]} A_{ij}(\emptyset)p_x(\{e\})
\]
\[
\leq (\gamma + 1) \sum_{e \in [n]} (B_i(\{e\}) + B_j(\{e\}))p_x(\{e\}),
\]
where the last inequality follows from (7). This completes the proof.
\[ \square \]
Lemma 7. Let $f$ be a non-negative, monotone, supermodular, $\gamma$-meta-submodular set function and $F$ be its multi-linear function. Then for any $x \in [0, 1]^n \setminus \{0\}$ and $i, j \in [n]$,

$$||x||_1 \nabla^2_{ij} F(x) \leq (\max\{3\gamma, 2\gamma + 1\})(\nabla_i F(x) + \nabla_j F(x)).$$

Proof. By using Lemma 4 for all the sets of size less than $||x||_1$, we can write

$$\sum_{R \subseteq [n]} (||x||_1) A_{ij}(\emptyset) p_x(\emptyset) + \sum_{1 \leq |R| < ||x||_1} (||x||_1 - |R|) A_{ij}(R) p_x(R)$$

$$\leq (\gamma + 1) \sum_{e \in [n]} (B_i(\{e\}) + B_j(\{e\})) p_x(\{e\}) + 2\gamma \sum_{2 \leq |R| < ||x||_1} \frac{(B_i(R + e) + B_j(R + e))}{|R| + 1} p_x(R + e)$$

$$= (\gamma + 1) \sum_{e \in [n]} (B_i(\{e\}) + B_j(\{e\})) p_x(\{e\}) + 2\gamma \sum_{R \subseteq [n]} (B_i(R) + B_j(R)) p_x(R)$$

$$\leq \max\{\gamma + 1, 2\gamma\} \sum_{R \subseteq [n]} (B_i(R) + B_j(R)) p_x(R) = \max\{\gamma + 1, 2\gamma\} (\nabla_i F(x) + \nabla_j F(x)), \quad (8)$$

where the equality follows from a simple counting argument, and in the last inequality we used the monotonicity of $f$ (i.e., the $B_i$’s are non-negative).

By $\gamma$-meta-submodularity, we also have that

$$\sum_{R \subseteq [n]} |R| A_{ij}(R) p_x(R) + \sum_{R \subseteq [n]} (||x||_1) A_{ij}(R) p_x(R)$$

$$\leq \sum_{|R| \geq 1} |R| A_{ij}(R) p_x(R) \leq \sum_{|R| \geq 1} \gamma(B_i(R) + B_j(R)) p_x(R)$$

$$\leq \sum_{R \subseteq [n]} \gamma(B_i(R) + B_j(R)) p_x(R) = \gamma(\nabla_i F(x) + \nabla_j F(x)). \quad (9)$$

By adding (8) and (9), we conclude that

$$\sum_{R \subseteq [n]} A_{ij}(R) p_x(R) = ||x||_1 \nabla^2_{ij} F(x) \leq \max\{2\gamma + 1, 3\gamma\} (\nabla_i F(x) + \nabla_j F(x)).$$

D.2 Smoothness of Submodular and 0-Meta-Submodular Functions

In this section, we provide results about the smoothness of the multi-linear extension of submodular functions and also sub-domain smoothness of the multi-linear extension of 0-meta-submodular functions.

Proposition 9. Let $f : 2^{[n]} \rightarrow \mathbb{R}$ and $F$ be its multi-linear extension. Then $f$ is submodular if and only if $F$ is one-sided 0-smooth.

Proof. A set function $f$ is submodular if and only if $A_{ij}(S) \leq 0$ for all $S \subseteq [n]$ and $i, j \in [n]$.

Let $f$ be submodular. Then $\nabla^2_{ij} F(x) = E_{R \sim x}[A_{ij}(R)] \leq 0$, for any $x \in [0, 1]^n$. It follows that $u^T \nabla^2 F(x) u \leq 0$ for any $u \in [0, 1]^n$, and thus $F$ is one-sided 0-smooth.
For the opposite direction, let $F$ be one-sided 0-smooth and let $u = e_i + e_j$. Then $u^T \nabla^2 F(x) u = 2 \nabla^2 F(x) u \leq 0$ for all $x \neq 0$. Moreover, by continuity of $\nabla^2 F(x)$, the inequality also holds at $x = 0$. We then have that $A_{ij}(S) = \nabla^2 F(1_S) \leq 0$ for all $S \subseteq [n]$, and thus $f$ is submodular. \qed

**Proposition 10.** Let $f$ be a non-negative, monotone, 0-meta-submodular function and $F$ be its multi-linear extension. Then for any $v \in [n]$, $F$ is one-sided 0-smooth on $\{x \in [0,1]^n : x \geq 1_{\{v\}}\}$.

**Proof.** By 0-meta-submodularity, for any set $R$, we have $|R| A_{ij}(R) \leq 0$. This means that for any non-empty $R$, $A_{ij}(R) \leq 0$. Since $x_v = 1$, the probability of picking a set that does not include $v$ is zero. Therefore, we have

$$\nabla^2 F(x) = \sum_{R \subseteq [n]} A_{ij}(R)p_x(R) = \sum_{R \subseteq [n] - v} A_{ij}(R + v)p_x(R + v) \leq 0.$$

Hence for $u \in [0,1]^n$,

$$u^T \nabla^2 F(x) u = 2 \sum_{\{i,j\} \subseteq [n]} u_i u_j \nabla^2 F(x) \leq 0.$$

\qed

### D.3 Sub-domain Smoothness of Meta-Submodular Functions and General Monotone Functions

In this section, we discuss the sub-domain smoothness of the multi-linear extension of general $\gamma$-meta-submodular functions and monotone set functions.

**Theorem 7**. Let $f$ be a non-negative, monotone, $\gamma$-meta-submodular set function and $F$ be its multi-linear extension. Let $c \geq 1$ and $S \subseteq [n]$ be non-empty. Then $F$ is one-sided $2c\gamma$-smooth on $\{x \mid x \geq 1_S, \|x\|_1 \leq c|S|\}$.

**Proof.** Let $y \in \{x \mid x \geq 1_S, \|x\|_1 \leq c|S|\}$. First, we show that

$$||y||_1 \nabla^2 F(y) \leq \gamma c(\nabla_i F(y) + \nabla_j F(y)).$$

We know $\nabla^2 F(y) = \sum_{R \subseteq [n]} A_{ij}(R)p_y(R)$. Since $y \geq 1_S$, $p_y(R) = 0$ for any $R$ that is not a superset of $S$. Therefore, $\nabla^2 F(y) = \sum_{R \subseteq [n] \setminus S} A_{ij}(S \cup R)p_y(S \cup R)$. We have

$$||y||_1 \nabla^2 F(y) = ||y||_1 \sum_{R \subseteq [n] \setminus S} A_{ij}(S \cup R)p_y(S \cup R) \leq c|S| \sum_{R \subseteq [n] \setminus S} A_{ij}(S \cup R)p_y(S \cup R)$$

$$\leq \sum_{R \subseteq [n] \setminus S} \frac{\gamma c|S|}{|S \cup R|} (B_1(S \cup R) + B_j(S \cup R))p_y(S \cup R)$$

$$\leq \sum_{R \subseteq [n] \setminus S} \gamma c(B_1(S \cup R) + B_j(S \cup R))p_y(S \cup R)$$

$$\leq \gamma c(\nabla_i F(y) + \nabla_j F(y)).$$

Now, by Lemma 1, we conclude that $F$ is one-sided $(2c\gamma)$-smooth at $y$. \qed

**Proposition 11.** Let $f : 2^{[n]} \to \mathbb{R}$ be a non-negative, monotone function and $F$ be its multi-linear extension. Let $x \in [0,1]^n$ such that $x_v > 0$ for each $v \in [n]$. Then there is a $\sigma \geq 0$, such that $F$ is one-sided $\sigma$-smooth at $x$. Moreover, let $z \in [0,1]^n$ whose smallest component value is $z_{\text{min}} > 0$. Then $F$ is $\frac{n}{z_{\text{min}}}$-smooth on $\{x : 1 \geq x \geq z\}$.
Proof. Let $i, j \in [n]$. By Lemma 4 we have
\[
\nabla^2_{ij} F(x) = \sum_{R \subseteq [n]} A_{ij}(R)p_x(R) = \sum_{R \subseteq [n]} (B_i(R + j) - B_i(R - j))p_x(R)
\]
\[
= \sum_{R \subseteq [n]} B_i(R + j)p_x(R) - \sum_{R \subseteq [n]} B_i(R - j)p_x(R).
\]
We first show that there is $\gamma_{ij} > 0$ such that
\[
||x||_1 \nabla^2_{ij} F(x) \leq \gamma_{ij} (\nabla_i F(x) + \nabla_j F(x)).
\] (10)
Since $f$ is monotone, the right hand side is non-negative. Hence, if $\nabla^2_{ij} F(x)$ is non-positive, the inequality holds for any $\gamma_{ij} > 0$. Therefore, we assume that $\nabla^2_{ij} F(x)$ is positive which implies that $\sum_{R \subseteq [n]} B_i(R + j)p_x(R) > 0$ by monotonicity. Hence
\[
0 < \nabla^2_{ij} F(x) \leq \sum_{R \subseteq [n]} B_i(R + j)p_x(R) = \sum_{R \subseteq [n] - j} B_i(R + j)p_x(R) + \sum_{R \subseteq [n] - j} B_i((R + j) + p_x(R + j))
\]
\[
= \sum_{R \subseteq [n] - j} B_i(R + j)(p_x(R) + p_x(R + j)) = \sum_{R \subseteq [n] - j} B_i(R + j)(\frac{1}{x_j} p_x(R + j) + p_x(R + j))
\]
\[
= \sum_{R \subseteq [n] - j} B_i(R + j)\frac{1}{x_j} p_x(R + j) = \frac{1}{x_j} \sum_{R \subseteq [n] - j} B_i(R + j)p_x(R + j)
\]
\[
\leq \frac{1}{x_j} (\sum_{R \subseteq [n] - j} B_i(R)p_x(R) + \sum_{R \subseteq [n] - j} B_i(R + j)p_x(R + j)) = \frac{1}{x_j} (\sum_{R \subseteq [n]} B_i(R)p_x(R) = \frac{1}{x_j} \nabla_i F(x).
\]
Hence, we conclude that $\nabla_i F(x) \geq \nabla^2_{ij} F(x)$ and so if $\nabla^2_{ij} F(x)$ is positive, then $\nabla_i F(x) + \nabla_j F(x)$ is also positive. Now, set $\gamma_{ij} = 0$ if $\nabla^2_{ij} F(x)$ is non-positive and otherwise we set
\[
\gamma_{ij} = \frac{||x||_1 \nabla^2_{ij} F(x)}{\nabla_i F(x) + \nabla_j F(x)} \leq \frac{||x||_1 \nabla^2_{ij} F(x)}{(x_i + x_j) \nabla^2_{ij} F(x)} = \frac{||x||_1}{x_i + x_j}.
\] (11)
Let $\gamma = 2 \max\{i, j\} \gamma_{ij}$. Then for $u \in [0, 1]^n$, we have by (11)
\[
u^T \nabla^2 F(x)u = \sum_{i=1}^n \sum_{j=1}^n u_iu_j \nabla_{ij} F(x) \leq \frac{1}{||x||_1} \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} u_iu_j (\nabla_i F(x) + \nabla_j F(x))
\]
\[
\leq \frac{\gamma}{2} \frac{1}{||x||_1} \sum_{i=1}^n \sum_{j=1}^n u_iu_j (\nabla_i F(x) + \nabla_j F(x))
\]
\[
= \frac{\gamma}{2} \frac{1}{||x||_1} (\sum_{i=1}^n \sum_{j=1}^n u_iu_j \nabla_i F(x) + \sum_{i=1}^n \sum_{j=1}^n u_iu_j \nabla_j F(x))
\]
\[
= \frac{\gamma}{2} \frac{1}{||x||_1} (\sum_{i=1}^n \sum_{j=1}^n u_i \nabla_i F(x) (\sum_{j=1}^n u_j) + \sum_{i=1}^n \sum_{j=1}^n u_i (\sum_{j=1}^n u_j \nabla_j F(x)))
\]
\[
= \frac{\gamma}{2} \frac{1}{||x||_1} (||u||_1 \sum_{i=1}^n u_i \nabla_i F(x) + ||u||_1 \sum_{j=1}^n u_j \nabla_j F(x))
\]
\[
= \gamma \left( \frac{||u||_1}{||x||_1} \right) (u^T \nabla F(x)).
\]
Now for the second part of the proof we must choose a $\gamma$ that works for all $x \geq z$ and each $i, j$. By (11) it is sufficient to choose $\gamma = \max_{i, j} \{ ||x||_1 : x \in [0, 1]^n, x \geq z \} \leq \frac{1}{z_{\min}}$. \qed
E Appendix: Jump-Start Continuous Greedy

In this section, we provide the omitted results and proofs about the jump-start continuous greedy algorithm.

E.1 Jump-Start Continuous Greedy for One-Sided $\sigma$-smooth functions

In this section, we provide the complete proof of the approximation bound of the jump-start continuous greedy algorithm for one-sided $\sigma$-smooth functions. We also discuss the optimum $c$ value for the algorithm when it runs on a one-sided $\sigma$-smooth function.

**Theorem**. Let $F : [0,1]^n \to \mathbb{R}_{\geq 0}$ be a monotone one-sided $\sigma$-smooth function. Let $c \in (0,1)$ and $P$ be a polytime separable, downward-closed, polytope. If we run the jump-start continuous greedy process (Algorithm 1) then $x(1) \in P$ and $F(x(1)) \geq [1 - \exp(-(1-c)(c+1))]:OPT$ where $OPT := \max\{F(x) : x \in P\}$.

**Proof.** For each $t \in [0,1]$ we have

$$x(t) = x(0) + (1-c) \int_0^t \nu_{\max}(x(\tau)) \, d\tau = cv^* + (1-c) \int_0^t \nu_{\max}(x(\tau)) \, d\tau. \quad (12)$$

Since $P$ is convex and $v^* \in P$, we have that $x(t) \in P$ as long as $y(t) := \int_0^t \nu_{\max}(x(\tau)) \, d\tau \in P$. Given that each $\nu_{\max}(x(\tau)) \in P$ and also $0 \in P$, it follows that $y(t)$ is a convex combination of points in $P$, and hence belongs to $P$.

Let $x^* \in P$ be such that $F(x^*) = OPT$. Also let $x \in \{x(t) : 0 \leq t \leq 1\}$ and $u = (x^* - x) \vee 0$, i.e., $x^* \vee x = x + u$. We have by Taylor’s Theorem that for some $\epsilon \in (0,1)$:

$$F(x^* \vee x) = F(x) + u^T \nabla F(x + \epsilon u) \leq F(x) + \left(\frac{\|x + \epsilon u\|_1}{\|x\|_1}\right)^\sigma u^T \nabla F(x) \leq F(x) + \left(\frac{\|x + u\|_1}{\|x\|_1}\right)^\sigma u^T \nabla F(x)$$

where the first inequality follows from Lemma 3. Hence

$$u^T \nabla F(x) \geq \frac{1}{\left(\frac{\|x + u\|_1}{\|x\|_1}\right)^\sigma}(F(x \vee x^*) - F(x)) \geq \frac{1}{\left(\frac{\|x + u\|_1}{\|x\|_1}\right)^\sigma}(OPT - F(x)), \quad (13)$$

where the last inequality follows from monotonicity since then $F(x \vee x^*) \geq F(x^*) = OPT$. We also have that

$$\nu_{\max}(x) \cdot \nabla F(x) \geq x^* \cdot \nabla F(x) \geq u \cdot \nabla F(x),$$

where the first inequality follows by definition of $\nu_{\max}$ and the fact that $x^* \in P$, and the second inequality from the fact that $x^* \geq u$ and $\nabla F \geq 0$. Combining this with (13) yields:

$$\nu_{\max}(x) \cdot \nabla F(x) \geq \frac{1}{\left(\frac{\|x + u\|_1}{\|x\|_1}\right)^\sigma}(OPT - F(x)). \quad (14)$$

By the choice of $x(0)$ we have that $\|x(0)\|_1 \geq c\|w\|_1$ for any $w \in P$. Since $u \in P$ and $x(t)$ is non-decreasing in each component (because $\nu_{\max}$ is always non-negative), we thus have

$$\frac{\|x + u\|_1}{\|x\|_1} \leq 1 + \frac{\|u\|_1}{\|x\|_1} \leq 1 + \frac{\|u\|_1}{\|x(0)\|_1} \leq 1 + \frac{1}{c} = \frac{c+1}{c}.$$
Hence we deduce that
\[
\frac{1}{\left(\frac{|x + y|}{|x|}\right)^\sigma} \geq \left(\frac{c}{c + 1}\right)^\sigma
\]  
(15)

for any \(x \in \{x(t) : 0 \leq t \leq 1\}\). Let us define \(\rho\) to be the righthand side quantity above. Intuitively, (14) indicates that the direction \(v_{max}\) makes at least a \(\rho\) “fractional progress” towards OPT.

Moreover, we can use the Chain Rule to get
\[
\frac{d}{dt}F(x(t)) = \nabla F(x(t)) \cdot x'(t) = \nabla F(x(t)) \cdot (1 - c)v_{max}(x(t)) \geq \rho(1 - c)[OPT - F(x(t))],
\]  
(16)

where the last inequality follows from (14) and (15).

We solve the above differential inequality by multiplying by \(e^{\rho(1-c)t}\).

\[
\frac{d}{dt}[e^{\rho(1-c)t} \cdot F(x(t))] = \rho(1 - c)e^{\rho(1-c)t} \cdot F(x(t)) + e^{\rho(1-c)t} \frac{d}{dt}F(x(t)) \\
\geq \rho(1 - c)e^{\rho(1-c)t} \cdot F(x(t)) + \rho \cdot e^{\rho(1-c)t}(1 - c)[OPT - F(x(t))] \\
= \rho(1 - c)e^{\rho(1-c)t} \cdot OPT.
\]

where the inequality follows from Equation (16).

Integrating the LHS and RHS of the above equation between 0 and \(t\) we get
\[
e^{\rho(1-c)t} \cdot F(x(t)) - e^{0} \cdot F(x(0)) \geq \rho(1 - c)OPT \int_{0}^{t} e^{\rho(1-c)t} \cdot d\tau \\
= \rho(1 - c)OPT \cdot \left[e^{\rho(1-c)t} - \frac{1}{\rho(1-c)}\right] = OPT \cdot [e^{\rho(1-c)t} - 1].
\]

Hence
\[
F(x(t)) \geq [1 - \frac{1}{e^{\rho(1-c)t}OPT}]OPT + \frac{F(x(0))}{e^{\rho(1-c)t}} \geq [1 - \frac{1}{e^{\rho(1-c)t}OPT}]OPT,
\]

where the last inequality follows from the fact that \(F\) is non-negative. Substituting \(t = 1\) and \(\rho = (\frac{c}{c+1})^\sigma\) gives the desired result. \(\square\)

**Proposition 12.** For any \(\sigma > 0\) the best approximation guarantee in Theorem 2 is attained at
\[
c = \frac{\sqrt{\sigma^2 + 6\sigma + 1} - (\sigma + 1)}{2}.
\]

**Proof.** We need to find the maximizer of \(g(c) = (1 - c)(\frac{c}{c+1})^\sigma\) where \(c \in [0,1]\). Hence, we solve \(g'(c) = 0\).

\[
g'(c) = \frac{\sigma c^{\sigma - 1}(c + 1)^\sigma - (\sigma + 1)c^\sigma(c+1) - (\sigma + 1)c^{\sigma - 1}c^\sigma + (c+1)^{\sigma - 1}c^{\sigma + 1}}{(c+1)^{2\sigma}} = 0
\]
\[
\Rightarrow \sigma c^{\sigma - 1}(c + 1)^{\sigma - 1} - \sigma c^\sigma(c+1)^{\sigma - 1} = c^\sigma(c+1)^\sigma
\]
\[
\Rightarrow \sigma c^{\sigma - 1}(c + 1)^{\sigma - 1}(1 - c) = c^\sigma(c+1)^\sigma
\]
\[
\Rightarrow \sigma(1 - c) = c(c+1) \Rightarrow c^2 + (1 + \sigma)c - \sigma = 0 \Rightarrow c = \frac{-(\sigma + 1) \pm \sqrt{\sigma^2 + 6\sigma + 1}}{2}
\]

The only solution in \((0,1)\) is \(\frac{-(\sigma + 1) + \sqrt{\sigma^2 + 6\sigma + 1}}{2}\) and this yields the proposition. \(\square\)
E.2 Jump-Start Continuous Greedy for second-order smooth functions

The following result improves the approximation factor of jump-start continuous greedy algorithm for smooth functions that also satisfy higher order smoothness conditions.

**Theorem 2** Let \( F : [0,1]^n \to \mathbb{R}_{\geq 0} \) be a monotone one-sided \( \sigma \)-smooth function with non-positive third order partial derivatives. Let \( c \in (0,1) \) and \( P \) be a polytime separable, downward-closed, polytope. If we run the jump-start continuous greedy process (Algorithm 1) then \( x(1) \in P \) and 
\[
F(x(1)) \geq [1 - \exp \left( -\frac{2c(1-c)}{2\sigma} \right)] \cdot \text{OPT}
\]
where \( \text{OPT} := \max \{ F(x) : x \in P \} \). In particular, taking \( c = 1/2 \) we get 
\[
F(x(1)) \geq [1 - \exp \left( -\frac{1}{2\sigma+3} \right)] \cdot \text{OPT}
\]
and so \( F(x(1)) \geq \frac{1}{2\sigma+3} \cdot \text{OPT} \) (since \( e^x \geq x+1 \) for \( x \leq 1 \)).

**Proof.** For each \( t \in [0,1] \) we have 
\[
x(t) = x(0) + (1-c) \int_0^t v_{\max}(x(\tau)) \, d\tau = c v^* + (1-c) \int_0^t v_{\max}(x(\tau)) \, d\tau.
\]
Since \( P \) is convex and \( v^* \in P \), we have that \( x(t) \in P \) as long as \( y(t) := \int_0^t v_{\max}(x(\tau)) \, d\tau \in P \). Given that each \( v_{\max}(x(\tau)) \in P \) and also \( 0 \in P \), it follows that \( y(t) \) is a convex combination of points in \( P \), and hence belongs to \( P \).

Let \( x^* \in P \) be such that \( F(x^*) = \text{OPT} \). Also let \( x \in \{ x(t) : 0 \leq t \leq 1 \} \) and \( u = (x^* - x) \vee 0 \), i.e., \( x^* \wedge x = x + u \). By Taylor’s Theorem and non-positivity of the third order derivatives of \( F \) we have 
\[
F(x^* \wedge x) \leq F(x) + u^T \nabla F(x) + \frac{1}{2} u^T \nabla^2 F(x) u \leq F(x) + \left( 1 + \frac{\sigma \|u\|}{2 \|x\|} \right) u^T \nabla F(x) \leq F(x) + \left( 1 + \frac{\sigma}{2c} \right) u^T \nabla F(x),
\]
where the second inequality follows from smoothness, and the third from the fact that \( \|x(t)\| \geq \|x(0)\| = c\|v^*\| \geq c\|u\| \). Thus
\[
u_{\max}(x) \cdot \nabla F(x) \geq \left( \frac{2c}{2c + \sigma} \right) \left( F(x^* \wedge x) - F(x) \right) \geq \left( \frac{2c}{2c + \sigma} \right) \left( \text{OPT} - F(x) \right),
\]
where the last inequality follows from monotonicity. We also have that
\[
v_{\max}(x) \cdot \nabla F(x) \geq x^* \cdot \nabla F(x) \geq u \cdot \nabla F(x),
\]
where the first inequality follows by definition of \( v_{\max} \) and the fact that \( x^* \in P \), and the second inequality from the fact that \( x^* \geq u \) and \( \nabla F \geq 0 \). Combining this with (13) yields:
\[
v_{\max}(x) \cdot \nabla F(x) \geq \left( \frac{2c}{2c + \sigma} \right) \left( \text{OPT} - F(x) \right),
\]
for any \( x \in \{ x(t) : 0 \leq t \leq 1 \} \). Let us denote \( \rho = 2c/(2c + \sigma) \). We can use the Chain Rule to get
\[
\frac{d}{dt} F(x(t)) = \nabla F(x(t)) \cdot x'(t) = \nabla F(x(t)) \cdot (1-c)v_{\max}(x(t)) \geq \rho(1-c) \left[ \text{OPT} - F(x(t)) \right],
\]
where the last inequality follows from (19).

We solve the above differential inequality by multiplying by \( e^{(1-c)t} \).
\[
\frac{d}{dt} \left[ e^{(1-c)t} \cdot F(x(t)) \right] = (1-c)e^{(1-c)t} \cdot F(x(t)) + e^{(1-c)t} \cdot \frac{d}{dt} F(x(t)) \\
\geq (1-c)e^{(1-c)t} \cdot F(x(t)) + \rho \cdot e^{(1-c)t} (1-c) \left[ \text{OPT} - F(x(t)) \right] \\
= (1-c)e^{(1-c)t} \cdot \text{OPT}.
\]
Linear extension. Let \( M \) be an independent set. Then, the continuous greedy process described in Algorithm 3 outputs a vector \( x \) satisfying

\[
\begin{align*}
F(x(t)) &\geq \rho(1-c)OPT \int_0^t e^{\rho(1-c)\tau} d\tau \\
&= \rho(1-c)OPT \cdot \left[ \frac{e^{\rho(1-c)t}}{\rho(1-c)} - \frac{1}{\rho(1-c)} \right] = OPT \cdot \left[ e^{\rho(1-c)t} - 1 \right].
\end{align*}
\]

Hence

\[
F(x(t)) \geq [1 - \frac{1}{e^{\rho(1-c)t}}]OPT + \frac{F(x(0))}{e^{\rho(1-c)t}} \geq [1 - \frac{1}{e^{\rho(1-c)t}}]OPT,
\]

where the last inequality follows from the fact that \( F \) is non-negative. Substituting \( t = 1 \) and \( \rho = 2c/(2c + \sigma) \) gives the desired result. \( \square \)

### E.3 Continuous Greedy and Pipage Rounding for 0-meta-submodular functions

In this section, we provide an adaptation of the continuous greedy algorithm for maximizing a 0-meta-submodular function over a polytime separable downward closed polytope. We also show that the pipage rounding algorithm can be used to bound the solution of the continuous greedy over a matroid polytope.

**Theorem 12.** There is a randomized \( (1 - \frac{1}{e} - o(1)) \)-approximation for maximizing a non-negative, monotone, 0-meta-submodular function subject to a matroid constraint.

Given a matroid \( M = ([n], \mathcal{I}) \), and an independent set \( R \subseteq \mathcal{I} \), we denote by \( M_R = ([n] - R, \mathcal{I}_R) \) the contraction of \( M \) by \( R \). That is, \( I \in \mathcal{I}_R \) if and only if \( R \cup I \in \mathcal{I} \). We denote by \( P_R \subseteq \{0,1\}^{[n]-R} \) its associated matroid polytope. We also define an extended version of \( P_R \), as \( \tilde{P}_R = \{x \in [0,1]^n : x|_R = 0, x|_{[n]-R} \in P_R \} \), where \( x|_R \in [0,1]^R \) denotes the restriction of \( x \) to the components in \( R \). That is, \( \tilde{P}_R \) is obtained by extending the contracted polytope \( P_R \) to the original space \([0,1]^n\), and setting all components \( x_i = 0 \) for \( i \in R \).

**Theorem 13.** Let \( f \) be a non-negative monotone 0-meta submodular function and \( F \) be its multilinear extension. Let \( M = ([n], \mathcal{I}) \) be a matroid, \( P(M) \) its corresponding polytope, and \( R \subseteq \mathcal{I} \) an independent set. Then, the continuous greedy process described in Algorithm 3 outputs a vector \( x \in P(M) \) satisfying \( x \geq 1_R \) and

\[
F(x) \geq [1 - e^{-1}] \cdot OPT_R
\]

where \( OPT_R := \max\{F(x) : x \geq 1_R\} \).

**Proof.** For each \( t \in [0,1] \) we have

\[
x(t) = x(0) + \int_0^t v_{\max}(x(\tau)) d\tau = 1_R + \int_0^t v_{\max}(x(\tau)) d\tau.
\]

Note that \( x \in \tilde{P}_R \) if and only if \( x \) is a convex combination \( x = \sum_{i=1}^m \lambda_i \mathbb{1}_{S_i} \) of some independent sets \( S_i \in \mathcal{I}_R \) (i.e. \( R \cup S_i \in \mathcal{I} \)). Thus, \( 1_R + x = 1_R + \sum_{i=1}^m \lambda_i \mathbb{1}_{S_i} = \sum_{i=1}^m \lambda_i [1_R + \mathbb{1}_{S_i}] \in P(M) \) since \( 1_R + \mathbb{1}_{S_i} \in P(M) \) for each \( i \in [m] \). Given that each \( v_{\max}(x(\tau)) \in \tilde{P}_R \) for each \( \tau \), it follows that \( \int_0^1 v_{\max}(x(\tau)) d\tau \in \tilde{P}_R \) and therefore \( x(t) \in P(M) \). Moreover, it is clear that \( x(t) \geq 1_R \).
Let \( U := \{ y + \mathbb{1}_R : y \in \tilde{P}_R \} \), or equivalently, \( U = \{ x \in P(\mathcal{M}) : x|_R = \mathbb{1}_R \} \). Let \( x, x^* \in U \) be such that \( F(x^*) = OPT_R \) and \( u = (x^* - x) \not\in 0 \), i.e., \( x^* \not\lor x \). By Theorem \( \square \) we know that \( F \) is one-sided \( 0 \)-smooth at \( U \). Hence, we have by Taylor’s Theorem that for some \( \epsilon \in (0, 1) \):

\[
F(x^* \lor x) = F(x) + u^T \nabla F(x + \epsilon u) \leq F(x) + \left( \frac{||x + \epsilon u||_1}{||x||_1} \right)^0 u^T \nabla F(x) = F(x) + u^T \nabla F(x)
\]

where the inequality follows from Lemma \( \square \). Hence

\[
u^T \nabla F(x) \geq F(x \lor x^*) - F(x) \geq OPT_R - F(x).
\]

We also have that

\[
\nu_{\text{max}}(x) \cdot \nabla F(x) \geq (x^* - \mathbb{1}_R) \cdot \nabla F(x) \geq u \cdot \nabla F(x),
\]

where the first inequality follows by definition of \( \nu_{\text{max}} \) and the fact that \( x^* - \mathbb{1}_R \in \tilde{P}_R \), and the second inequality from the fact that \( x^* - \mathbb{1}_R \geq u \) and \( \nabla \geq 0 \). Combining this with (22) yields:

\[
\nu_{\text{max}}(x) \cdot \nabla F(x) \geq OPT_R - F(x).
\]

We can now use the Chain Rule to get

\[
\frac{d}{dt} F(x(t)) = \nabla F(x(t)) \cdot x'(t) = \nabla F(x(t)) \cdot \nu_{\text{max}}(x(t)) \geq OPT_R - F(x(t)),
\]

where the last inequality follows from Equation (23).

We solve the above differential inequality by multiplying by \( e^t \).

\[
\frac{d}{dt}[e^t \cdot F(x(t))] = e^t \cdot F(x(t)) + e^t \cdot \frac{d}{dt} F(x(t)) \geq e^t \cdot F(x(t)) + e^t [OPT_R - F(x(t))] = e^t \cdot OPT_R.
\]

where the inequality follows from Equation (24).

Integrating the LHS and RHS of the above equation between 0 and \( t \) we get

\[
e^t \cdot F(x(t)) - e^0 \cdot F(x(0)) \geq OPT_R \int_0^t e^\tau d\tau = OPT_R \cdot [e^t - 1].
\]

Hence

\[
F(x(t)) \geq [1 - \frac{1}{e^t}]OPT_R + \frac{F(x(0))}{e^t} \geq [1 - \frac{1}{e^t}]OPT_R,
\]

where the last inequality follows from the fact that \( F \) is nonnegative. Taking \( t = 1 \) we get

\[
F(x(1)) \geq [1 - \frac{1}{e}]OPT_R.
\]

\( \square \)

Algorithm 3: Jump-Start Continuous Greedy for Contracted Matroids

1. **Input:** A monotone set function \( f \), its multi-linear extension \( F \), a matroid \( \mathcal{M} \), an independent set \( R \), and its extended contracted polytope \( \tilde{P}_R \)
2. \( x(0) \leftarrow \mathbb{1}_R \)
3. \( \nu_{\text{max}}(x) \leftarrow \arg \max_{u \in \tilde{P}_R} \{ u^T \nabla F(x) \} \)
4. for \( t \in [0, 1] \) do
5. \( \quad \) Solve \( x'(t) = \nu_{\text{max}}(x(t)) \) with boundary condition \( x(0) = \mathbb{1}_R \)
6. return \( x(1) \)

This now leads to the following result.

37
Then $\bar{B}$ multilinear extension. Let $M$ linear extension. Let $x^i$ denote the output of Algorithm 3 run with $R = \{i\}$, and let $\bar{x} = \arg\max_{i \in [n]} F(x^i)$. Then $\bar{x} \in P(M)$ and

$$F(\bar{x}) \geq [1 - e^{-1}] \cdot \max\{f(S) : S \in \mathcal{I}\}.$$

**Proof.** Let $O = \arg\max_{S \in \mathcal{I}} f(S)$ and $i \in O$. Then $1_O \geq 1_{\{i\}}$, and hence

$$F(\bar{x}) \geq F(x^i) \geq (1 - \frac{1}{e}) \cdot \max\{F(x) : x \geq 1_{\{i\}}\} \geq (1 - \frac{1}{e})F(1_O) = (1 - \frac{1}{e})f(O).$$

where the second inequality follows from Theorem 13. \hfill \Box

Hence, we can find a $(1 - 1/e)$-approximate fractional solution by running the continuous greedy process $n$ times. By standard techniques (see [49, 12]), one may discretize the continuous greedy process to obtain a finite algorithm achieving a $(1 - 1/e - o(1))$-approximation. In fact, it may be the case that a more careful analysis provides a clean $(1 - 1/e)$-approximation.

We now discuss a randomized technique that allows to round efficiently in the matroid polytope. This rounding technique was initially introduced by Ageev and Sviridenko [2], and later adapted for matroid polytopes by Calinescu et al. [11]. This rounding procedure is known as randomized pipage rounding and we describe it in Algorithm 4 (also note that it uses Algorithm 3 as a subroutine).

**Algorithm 4: Refinement Subroutine**

| **Input:** A vector $x \in [0, 1]^n$ and two components $i, j \in \{1, 2, \ldots, n\} |
|-------|
| 1. Let $S = \{S \subseteq V : i \in S, j \notin S\} |
| 2. Compute $S^* = \arg\min_{S \in S}[r(S) - x(S)]$ and let $\xi^* = r(S^*) - x(S^*) |
| 3. if $x_j < \xi^*$ then |
| 4. $x_i \leftarrow x_i + x_j$, $x_j \leftarrow 0$, $S' \leftarrow \{j\}$ |
| 5. else |
| 6. $x_i \leftarrow x_i + \xi^*$, $x_j \leftarrow x_j - \xi^*$, $S' \leftarrow S^*$ |
| 7. Output $(x, S') |

By monotonicity we may assume that the output $x^*$ of the continuous greedy algorithm (described in Section 4) is without loss of generality in the base polytope. We then have the following.

**Theorem 14.** Let $f : 2^{[n]} \to \mathbb{R}_{\geq 0}$ be a 0-meta-submodular set function and $F : [0, 1]^n \to \mathbb{R}_{\geq 0}$ its multilinear extension. Let $\mathcal{M}$ be a matroid and $x^* \in B(\mathcal{M})$ be the output of Corollary 1 over $\mathcal{M}$. Then Algorithm 4 outputs in polynomial time a random base $B$ of $\mathcal{M}$ such that $\mathbb{E}[1_B] = x^*$ and $\mathbb{E}[f(B)] \geq F(x^*)$.

**Proof.** It is well known [11] that the randomized pipage rounding algorithm finishes in polynomial time. We next argue that there is no loss (on expectation) in the objective value during the rounding. Let $x^*$ be the output of Corollary 1. Hence $x^*_i = 1$ for some $i^* \in [n]$, and by Proposition 10 it follows that $F$ is 0-smooth over the region $R := \{x \in [0, 1]^n : x_{i^*} = 1\}$, that is, $\nabla^2_x F(x) \leq 0$ for all $x \in R$.

Given any $x \in R$ and $i^* \neq i, j \in [n]$, let $\phi_x(t) := F(x + t(1_{\{i\}} - 1_{\{j\}}))$. Then $\phi''_x(t) = -2\nabla^2_x F(x + t(1_{\{i\}} - 1_{\{j\}})) \geq 0$, since $x + t(1_{\{i\}} - 1_{\{j\}}) \in R$. Hence $\phi_x$ is convex.

Let $x$ be the current point during the rounding procedure, and $i, j$ be the current changing coordinates. The next point is then given by $x' = x + t(1_{\{i\}} - 1_{\{j\}})$, where $t$ is a random variable such that $\mathbb{E}[t] = 0$. Then conditioning on the current point $x$ and changing coordinates $i, j$, by
Algorithm 5: Pipage Rounding

Input: A vector $x \in [0,1]^n$ and a matroid polytope $P(M)$

1. while $x$ not integral do
   2. $S \leftarrow V$
   3. while $S$ has fractional variables do
      4. Choose $i, j \in S$ fractional
      5. $(x^+, S^+) \leftarrow$ Refinement Subroutine $(x, i, j)$
      6. $(x^-, S^-) \leftarrow$ Refinement Subroutine $(x, j, i)$
      7. if $x = x^+ = x^-$ then
         8. $S \leftarrow S \cap S^+$
      else
         9. $p \leftarrow \frac{||x^+ - x||}{||x^+ - x^-||}$
         With probability $p$
         10. $x \leftarrow x^-$, $S \leftarrow S \cap S^-$
         Otherwise
         11. $x \leftarrow x^+$, $S \leftarrow S \cap S^+$
   12. Output $x$

Jensen’s inequality we get $E[F(x'|x, i, j)] = E[\phi_x(t)] \geq \phi_x(0) = F(x)$. Since this is true for any choice of $i, j$ that could be modified at that step, the result follows.

Note that Corollary 11 and Theorem 14 now prove Theorem 12.

E.4 Jump-Start Continuous Greedy for General Monotone Functions

In this section, we provide an adaptation of the jump-start continuous greedy algorithm that can be used for maximizing the multi-linear extension of a general monotone set function (Algorithm 6). This relies on the sub-domain smoothness result provided in Proposition 29 (Appendix D).

Theorem 15. Let $f : 2^{[n]} \rightarrow \mathbb{R}$ be a non-negative, monotone set function and $F$ be its multi-linear extension. Let $c \in (0, 1)$ and $P$ be a polytime separable, downward-closed, convex polytope such that $1_{\{i\}} \in P$ for any $i \in [n]$. Let $\sigma$ be the one-sided smoothness parameter on $\{y|y \geq c(\|v^*\|_1 + \frac{1}{n}) + \frac{\|v^*\|_1}{\|v^*\|_1 + v^*})\}$ where, $v^* = \arg \max_{x \in P} \|x\|_1$. Then Algorithm 6 outputs $x(1) \in P$ such that

$$F(x(1)) \geq [1 - \exp \left(-1 - c\left(\frac{c}{c+2}\right)\sigma\right)] \cdot OPT$$

where $OPT := \max\{F(x) : x \in P\}$.

Proof. We know that $1_{\{i\}} \in P$ for any $i \in [n]$ and so a convex combination of these is also in the polytope which means $\frac{1_{\{i\}}}{n} \in P$. Hence, since $v^* \in P$ and $P$ is convex,

$$(\frac{1}{\|v^*\|_1 + 1} + \frac{1_{[n]}}{n} + \frac{\|v^*\|_1}{\|v^*\|_1 + 1} v^*) \in P.$$}

For each $t \in [0, 1]$ we have

$$x(t) = x(0) + (1 - c) \int_0^t v_{max}(x(\tau)) d\tau \quad (25)$$
Since $P$ is convex and \( \left( \frac{1}{||v^{\ast}||} + \frac{2}{n} \right) + \frac{||v^{\ast}||}{||v^{\ast}||+1}v^{\ast} \in P \), we have that $x(t) \in P$ as long as $y(t) := \int_{0}^{t} v_{\max}(x(\tau)) \, d\tau \in P$. Given that each $v_{\max}(x(\tau)) \in P$ and also $\bar{0} \in P$, it follows that $y(t)$ is a convex combination of points in $P$, and hence belongs to $P$.

Let $x^{\ast} \in P$ be such that $F(x^{\ast}) = OPT$. Let $y \geq x(0)$ and $u = (x^{\ast} - y) \lor 0$, i.e., $x^{\ast} \lor y = y + u$. Note that all the coordinate of $x(0)$ are non-zero. We have by Taylor’s Theorem that for some $\epsilon \in (0, 1)$:

\[
F(x^{\ast} \lor y) = F(y) + u^{\top} \nabla F(y + \epsilon u) \leq F(y) + \left( \frac{||y + \epsilon u||}{||y||} \right)^{\sigma} u^{\top} \nabla F(y) \leq F(y) + \left( \frac{||y + u||}{||y||} \right)^{\sigma} u^{\top} \nabla F(y)
\]

where the first inequality follows from Proposition 11 and Lemma 3. Hence

\[
u_{\max}(y) \cdot \nabla F(y) \geq \left( \frac{||y + u||}{||y||} \right)^{\sigma}(OPT - F(y)),
\]

where the last inequality follows from monotonicity since then $F(y \lor x^{\ast}) \geq F(x^{\ast}) = OPT$.

The definition of $v_{\max}$ implies that $v_{\max}(y) \cdot \nabla F(y) \geq x^{\ast} \cdot \nabla F(y)$. Since $f$ is monotonic, $\nabla F \geq 0$. Hence since $u = (x^{\ast} - y) \lor 0 \leq x^{\ast}$, we also have $x^{\ast} \cdot \nabla F(y) \geq u \cdot \nabla F(y)$. Combining these with (26) yields:

\[
u_{\max}(y) \cdot \nabla F(y) \geq \left( \frac{||y + u||}{||y||} \right)^{\sigma}(OPT - F(y)).
\]

By the choice of $x(0)$ we have that for any $u \in P$,

\[
||x(0)|| = ||x(c) \left( \frac{1}{||v^{\ast}|| + 1} + \frac{||v^{\ast}||}{||v^{\ast}|| + 1}v^{\ast} \right)|| = \frac{1}{2} \frac{||v^{\ast}|| + 1}{||v^{\ast}|| + 1} + \frac{2}{||v^{\ast}|| + 1} = \frac{c}{2}||v^{\ast}|| + \frac{c}{2}||w||
\]

Since $u \in P$ and $x(t)$ is non-decreasing in each component (because $v_{\max}$ is always non-negative), we thus have

\[
\frac{||x(t) + u||}{||x(t)||} \leq 1 + \frac{||u||}{||x(t)||} \leq 1 + \frac{||u||}{||x(0)||} \leq 1 + \frac{2}{c} = \frac{c + 2}{c}.
\]

Hence we deduce that

\[
\left( \frac{||x(t) + u||}{||x(t)||} \right)^{\sigma} \geq \left( \frac{c}{c + 2} \right)^{\sigma}
\]

for all $x(t)$. Let us define $\rho$ to be the righthand side quantity above. Intuitively, (27) indicates that the direction $v_{\max}$ makes at least a $\rho$ "fractional progress" towards OPT.

Moreover, we can use the Chain Rule to get

\[
\frac{d}{dt} F(x(t)) = \nabla F(x(t)) \cdot x'(t) = \nabla F(x(t)) \cdot (1 - c)v_{\max}(x(t)) \geq \rho(1 - c)[OPT - F(x(t))],
\]

where the last inequality follows from Equation (27).

We solve the above differential inequality by multiplying by $e^{\rho(1 - c)t}$.

\[
\frac{d}{dt}[e^{\rho(1 - c)t} \cdot F(x(t))] = \rho(1 - c)e^{\rho(1 - c)t} \cdot F(x(t)) + e^{\rho(1 - c)t} \cdot \frac{d}{dt} F(x(t))
\]

\[
\geq \rho(1 - c)e^{\rho(1 - c)t} \cdot F(x(t)) + \rho \cdot e^{\rho(1 - c)t}(1 - c)[OPT - F(x(t))]
\]

\[
= \rho(1 - c)e^{\rho(1 - c)t} \cdot OPT.
\]
where the inequality follows from Equation (28).

Integrating the LHS and RHS of the above equation between 0 and t we get

\[ e^{\rho(1-c)t} \cdot F(x(t)) - e^{0} \cdot F(x(0)) \geq \rho(1-c)OPT \int_{0}^{t} e^{\rho(1-c)t} \cdot \frac{1}{\rho(1-c)} \cdot d\tau \]

\[ = \rho(1-c)OPT \cdot \left[ e^{\rho(1-c)t} - \frac{1}{\rho(1-c)} \right] = OPT \cdot [e^{\rho(1-c)t} - 1]. \]

Hence

\[ F(x(t)) \geq [1 - \frac{1}{e^{\rho(1-c)t}}]OPT + \frac{F(x(0))}{e^{\rho(1-c)t}} \geq [1 - \frac{1}{e^{\rho(1-c)t}}]OPT, \]

where the last inequality follows from the fact that \( F \) is nonnegative. Taking \( t = 1 \) we get

\[ F(x(1)) \geq [1 - \frac{1}{e^{\rho(1-c)}}]OPT. \]

Substituting \( \rho = (\frac{c}{e+c})^\sigma \) gives the desired result. \( \square \)

**Algorithm 6: Jump-Start Continuous Greedy for Monotone Functions**

1. **Input:** A monotone set function \( f \), its multi-linear extension \( F \), a polytime separable, downward-closed polytope \( P \subseteq [0, 1]^n \) and \( c \in (0, 1) \).
2. \( v^* \leftarrow \arg \max_{x \in P} ||x||_1 \)
3. \( x(0) \leftarrow c(\frac{1}{||v^*||_1+1}\frac{1}{n} + \frac{||v^*||_1}{||v^*||_1+1}v^*) \)
4. \( v_{\text{max}}(x) \leftarrow \arg \max_{x \in P} \{v^T \nabla F(x)\} \)
5. for \( t \in [0, 1] \) do
6. \quad Solve \( x'(t) = (1-c)v_{\text{max}}(x(t)) \) with boundary condition \( x(0) = c(\frac{1}{||v^*||_1+1}\frac{1}{n} + \frac{||v^*||_1}{||v^*||_1+1}v^*) \)
7. return \( x(1) \);

**F Appendix: Local Search**

In this section, we provide two key lemmas for bounding the Taylor series expansion for \( \gamma \)-meta-submodular functions. We later use these results to analyze the local search algorithm.

**Lemma 8.** Let \( f \) be a non-negative, monotone, \( \gamma \)-meta submodular function and \( F \) be its multi-linear extension. Let \( R \subseteq [n] \) such that \(|R| \geq 2\). Then

\[ 1_R^T \nabla F(1_R) = \sum_{i \in R} B_i(R-i) \leq 2(\frac{|R|}{2} + \frac{|R|}{2})^2 + 2|\gamma| + 2)f(R) \leq (9|\gamma| + 2)f(R) \]

**Proof.** Partition \( R \) into two sets of size \( \lfloor \frac{|R|}{2} \rfloor \) and of size \( \lceil \frac{|R|}{2} \rceil \) like \( S \) and \( T \). Using Theorem 7 we know that \( F \) is one-sided \((2(\lfloor \frac{|R|}{2} \rfloor + \lceil \frac{|R|}{2} \rceil) + 1)\gamma\)-smooth on \( \{y|1_S \leq y \leq 1_R\} \) and it is one-sided \((2(\lfloor \frac{|R|}{2} \rfloor + \lceil \frac{|R|}{2} \rceil + 1)\gamma\)-smooth on \( \{y|1_S \leq y \leq 1_R\} \). Let \( c = 2(\lfloor \frac{|R|}{2} \rfloor + \lceil \frac{|R|}{2} \rceil + 1)\gamma \). We show that

\[ \sum_{i \in T} B_i(R-i) \leq cf(R). \]
Let \( h(t) = F(1_S + t1_T) \) and \( g(t) = 1_T^T \nabla F(1_S + t1_T) \) where \( 0 \leq t \leq 1 \). Note that \( g(t) = h'(t) \) and \( 1_T^T \nabla^2 F(1_S + t1_T)1_T = g'(t) \). Since \( F \) is one-sided \( c \)-smooth at any given point \( 1_S \leq y \leq 1_R \), we have
\[
g'(t) = 1_T^T \nabla^2 F(1_S + t1_T)1_T \leq c\left( ||1_T||_1 \right)(1_T^T \nabla F(1_S + t1_T)) \leq \frac{1}{t}g(t).
\]
Therefore, \( tg'(t) \leq cg(t) \). Integrating both sides, we get
\[
\int_0^1 tg'(t)dt \leq \int_0^1 cg(t)dt.
\]
Applying the integration by parts formula to the left hand side, we get
\[
\int_0^1 tg(t)\left|^{1} - \int_0^1 g(t)dt \right. \leq c \int_0^1 g(t)dt.
\]
It follows that
\[
1 \cdot g(1) - 0 \cdot g(0) = 1_T^T \nabla F(1_S + 1_T) = 1_T^T \nabla F(1_R) = \sum_{i \in T} B_i(R - i) \leq (c + 1) \int_0^1 g(t)dt.
\]
By using \( g(t) = h'(t) \) we have
\[
\sum_{i \in T} B_i(R - i) \leq (c + 1) \int_0^1 h'(t)dt = (c + 1)(h(1) - h(0)) = (c + 1)(F(1_S + 1_T) - F(1_S))
\]
\[
\leq (c + 1)F(1_R) = (c + 1)f(R).
\]
This means that
\[
\sum_{i \in T} B_i(R - i) \leq 2(\frac{|R|}{2} + \frac{|R|}{2} + 1) \gamma + 1)f(R).
\]
With the same argument we can conclude that
\[
\sum_{i \in S} B_i(R - i) \leq 2(\frac{|R|}{2} + \frac{|R|}{2} + 1) \gamma + 1)f(R),
\]
and combining these inequalities yields the lemma. \( \square \)

**Lemma 9.** Let \( f \) be a non-negative, monotone, \( \gamma \)-meta-submodular function, \( F \) be its multi-linear function, \( R \subseteq [n] \), and \( x \in [0,1]^n \) such that \( ||x||_1 \leq |R| \). Let \( u = 1_R \vee x - 1_R \). Then for \( 0 \leq \epsilon \leq 1 \)
\[
u^T \nabla F(1_R + \epsilon u) \leq 2^{4\gamma}u^T \nabla F(1_R)
\]

**Proof.** By Theorem 2 we know that \( F \) is one-sided \( 4\gamma \)-smooth on \( A = \{y | y \geq 1_R, ||y||_1 \leq 2|R| \} \). Therefore \( F \) is one-sided \( 4\gamma \)-smooth on \( B = \{y | 1_R + \epsilon u \geq y \geq 1_R \} \) because \( B \subseteq A \). Therefore, the desired result yields by Lemma 3. \( \square \)
F.1 Local Search for $\gamma$-meta-submodular functions

**Theorem 8.** Let $f \in \mathcal{G}_{\gamma}$ and $\mathcal{M} = ([n], \mathcal{I})$ be a matroid of rank $r$. Let $A \in \mathcal{I}$ be an optimum set, i.e., $A \in \arg\max_{R \in \mathcal{I}} f(R)$, and $S \in \mathcal{I}$ be an $(1 + \frac{\gamma}{n})$-approximate local optima, i.e., for any $i$ and $j$ such that $S - i + j \in \mathcal{I}$, $(1 + \frac{\gamma}{n^2})f(S) \geq f(S - i + j)$, where $\epsilon > 0$ is a constant. Then if $\gamma = O(r)$, $f(A) \leq O(\gamma 2^{4\gamma})f(S)$ and if $\gamma = \omega(r)$, $f(A) \leq O(\gamma^{2\gamma})f(S)$.

Proof. Since $f$ is monotone, we assume that $|S| = |A| = r$. Given the exchangeability property of matroids, there is a bijective mapping \( \{46\} \) \( g : S \setminus A \rightarrow A \setminus S \) such that $S - i + g(i) \in \mathcal{I}$ where $i \in S \setminus A$. Since $S$ is a $(1 + \frac{\gamma}{n})$-approximate local optima, for all $i \in S \setminus A$ we have $(1 + \frac{\gamma}{n})f(S) \geq f(S - i + g(i))$. That is, $\frac{\gamma}{n} f(S) + B_i(S - i) \geq B_{g(i)}(S - i)$. Using this we get

$$B_{g(i)}(S) = B_{g(i)}(S - i) + A_{g(i)}(S - i) \leq B_{g(i)}(S - i) + \gamma \frac{B_{g(i)}(S - i) + B_i(S - i)}{r - 1}$$

$$\leq \frac{2\gamma + r - 1}{r - 1} B_i(S - i) + \gamma \frac{B_{g(i)}(S - i) + B_i(S - i)}{r - 1}$$

where the equality follows from Lemma 2 and the first inequality from $\gamma$-meta-submodularity. Therefore,

$$\sum_{i \in S \setminus A} B_{g(i)}(S) \leq \frac{2\gamma + r - 1}{r - 1} \sum_{i \in S \setminus A} B_i(S - i) + o(1)f(S).$$

Now, by Taylor’s Theorem, Lemma 3 and the above inequality, we have

$$f(S \cup A) = F(\mathbb{1}_S \cup \mathbb{1}_A) = F(\mathbb{1}_S + \mathbb{1}_A \setminus S) = F(\mathbb{1}_S) + \mathbb{1}_{A \setminus S}^T \nabla F(\mathbb{1}_S + \epsilon \mathbb{1}_{A \setminus S})$$

$$\leq F(\mathbb{1}_S) + 2^{4\gamma} \mathbb{1}_{A \setminus S}^T \nabla F(\mathbb{1}_S) = F(\mathbb{1}_S) + 2^{4\gamma} \sum_{i \in S \setminus A} B_{g(i)}(S)$$

$$\leq (1 + 2^{4\gamma} \cdot o(1))f(S) + \frac{2\gamma + r - 1}{r - 1} 2^{4\gamma} \sum_{i \in S \setminus A} B_i(S - i)$$

Therefore, using the monotonicity of $f$ and Lemma 3 we get

$$f(A) \leq f(S \cup A) \leq (1 + 2^{4\gamma} \cdot o(1))f(S) + \frac{2\gamma + r - 1}{r - 1} 2^{4\gamma} (9\gamma + 2) f(S)$$

$$= \left[ \frac{2\gamma + r - 1}{r - 1} 2^{4\gamma} (9\gamma + 2) + 1 + 2^{4\gamma} \cdot o(1) \right] f(S).$$

\[ \square \]

F.2 Runtime of the Local Search Algorithm

In this section, we analyze the runtime of the local search algorithm that finds an approximate local optima.

**Lemma 10.** Let $f$ be a non-negative, monotone, $\gamma$-meta submodular function and $\mathcal{M} = ([n], \mathcal{I})$ be a matroid of rank $r$. Let $A \in \mathcal{I}$ be an optimum set, i.e.,

$$A \in \arg\max_{R \in \mathcal{I}} f(R),$$

and

$$S_0 \in \arg\max_{\{v, v'\} \in \mathcal{I}} f(\{v, v'\}).$$

Then $f(A) \leq O(r(\gamma + 1)^{r-2})f(S_0)$. 

43
Proof. Let \( A = \{a_1, \ldots, a_r\} \) and \( A_i = \{a_1, \ldots, a_i\} \) for \( 1 \leq i \leq r \). By definition of \( S_0 \) we know that \( f(A_2) \leq f(S_0) \). Now by induction we show that for any \( 2 \leq i < j \leq n \), \( B_{a_j}(A_i) \leq O((\gamma + 1)^{i-1})f(S_0) \). The base case is \( i = 2 \). By definition of \( f(S_0) \), monotonicity and meta submodularity of \( f \), we have

\[
B_{a_j}(A_2) = B_{a_j}(A_1) + A_{a_2a_j}(A_1) \leq B_{a_j}(A_1) + \gamma(B_{a_j}(A_1) + B_{a_2}(A_1)) \leq (2\gamma + 1)f(S_0) \\
\leq O(\gamma + 1)f(S_0).
\]

Now assume that for \( k < j \leq n \), we have \( B_{a_j}(A_k) \leq O(\gamma^{k-1})f(S_0) \). We want to show that for \( k + 1 < j \leq n \), we have \( B_{a_j}(A_{k+1}) \leq O(\gamma^k)f(S_0) \).

\[
B_{a_j}(A_{k+1}) = B_{a_j}(A_k) + A_{a_{k+1}a_j}(A_k) \leq B_{a_j}(A_k) + \frac{\gamma}{k}(B_{a_{k+1}}(A_k) + B_{a_j}(A_k)) \\
\leq (1 + \frac{2\gamma}{k})O((\gamma + 1)^{k-1})f(S_0) \leq O((\gamma + 1)^k)f(S_0).
\]

We know that

\[
f(A) = f(A_2) + \sum_{i=3}^{r} B_{a_i}(A_{i-1}) \leq f(S_0) + \sum_{i=3}^{r} O((\gamma + 1)^{i-2})f(S_0) \leq O(r(\gamma + 1)^{r-2})f(S_0)
\]

Proposition 13. Local search algorithm (Algorithm 2) runs in \( O(n^4(\log(r) + r \log(\gamma + 1)/\epsilon)) \) time on a \( \gamma \)-meta submodular functions and a matroid of rank \( r \).

Proof. Cost of finding \( S_0 \) is \( O(n^2) \). Also, each iteration of the while loop costs \( O(n^2) \). Let \( S_k \) be the solution after \( k \) iterations and \( A \) be an optimum solution. By Lemma 10 we know

\[
f(S_k) \leq (1 + \frac{\epsilon}{n^2})^k f(S_0) \leq f(A) \leq O(r(\gamma + 1)^{r-2})f(S_0).
\]

Taking the logarithm, we have

\[
k \ln(1 + \frac{\epsilon}{n^2}) \leq O(\ln(r) + (r - 2) \ln(\gamma + 1)).
\]

Noting that \( \frac{\epsilon - 1}{x} \leq \ln x \) for any \( x > 0 \), we have

\[
k(\frac{\epsilon}{n^2})/(\frac{n^2 + \epsilon}{n^2}) \leq O(\ln(r) + (r - 2) \ln(\gamma + 1)).
\]

This yields the result.

F.3 Local Search for Set Functions with a Smooth Multi-Linear Extension

In this section, we first provide a key lemma for bounding the Taylor series expansion of smooth multi-linear extension. Then we show that the local search algorithm finds a solution which is within \( O(\sigma^2 2\sigma) \)-approximation of the optimal solution of the matroid polytope. One can also view this result as an integrality gap result for the matroid polytope.

Lemma 11. Let \( F : [0,1]^n \) be a one-sided \( \sigma \)-smooth function where \( F(\mathbf{0}) \geq 0 \). Then \( x^T \nabla F(x) \leq (\sigma + 1)F(x) \) and \( x^T \nabla^2 F(x)x \leq \sigma(\sigma + 1)F(x) \).
Proof. Given \( x \in [0,1]^n \), let \( h_x(t) = F(tx) \) and \( g_x(t) = x^T \nabla F(tx) \) where \( t \in \mathbb{R} \). Note that \( g_x(t) = h'_x(t) \) and \( x^T \nabla^2 F(tx)x = g'_x(t) \). Since \( F \) is one-sided \( \sigma \)-smooth, for \( 0 \leq t \leq 1 \) we have
\[
g'_x(t) = x^T \nabla^2 F(tx)x \leq \sigma \left( \frac{||x||_1}{||x||_1} \right)(x^T \nabla F(tx)) = \sigma \frac{1}{t} g_x(t).
\]
Therefore,
\[
tg'_x(t) \leq \sigma g_x(t),
\]
and integrating both sides, we get
\[
\int_0^1 tg'_x(t)dt \leq \int_0^1 \sigma g_x(t)dt.
\]
Applying the integration by parts formula to the left hand side, we get
\[
tg_x(t) \bigg|_0^1 - \int_0^1 g_x(t)dt \leq \sigma \int_0^1 g_x(t)dt.
\]
It follows that
\[
1 \cdot g_x(1) - 0 \cdot g_x(0) = x^T \nabla F(x) \leq (\sigma + 1) \int_0^1 g_x(t)dt.
\]
By using \( g_x(t) = h'_x(t) \) we have
\[
x^T \nabla F(x) \leq (\sigma + 1) \int_0^1 h'_x(t)dt = (\sigma + 1)(h_x(1) - h_x(0)) = (\sigma + 1)(F(x) - F(0)) = (\sigma + 1)F(x).
\]
By one-sided \( \sigma \)-smoothness we have
\[
x^T \nabla^2 F(x)x \leq \sigma x^T \nabla F(x).
\]
Hence,
\[
x^T \nabla^2 F(x)x \leq \sigma (\sigma + 1)F(x).
\]

\textbf{Theorem 16.} Let \( f \) be a non-negative, monotone set function such that its multi-linear extension \( F \) is one-sided \( \sigma \)-smooth, for some non-negative integer \( \sigma \). Let \( M = ([n], \mathcal{I}) \) be a matroid of rank \( r \) and \( P \) be its associated polytope. Let \( x \in P \) be such that \( ||x||_1 = c \) where \( c \in \{1, \ldots, r\} \). Let \( S \in \mathcal{I} \) of size \( c \) be an approximate local optima such that \( S \subseteq \text{supp}(x) \), i.e., for any \( a \in S \) and \( b \in \text{supp}(x) \setminus S \) such that \( S - a + b \in \mathcal{I} \),
\[
(1 + \frac{\epsilon}{n^2}) f(S) \geq f(S - a + b),
\]
where \( \epsilon > 0 \). Then if \( \sigma = O(c) \), \( F(x) \leq O(\sigma^2) f(S) \) and if \( \sigma = \omega(c) \), \( F(x) \leq O(\sigma^2 \omega) f(S) \).

Proof. Let \( u = (1_S \lor x) - 1_S \), i.e., \( 1_S \lor x = 1_S + u \). It follows that \( ||u||_1 \leq ||x||_1 = c \). By Taylor’s Theorem and Lemma 3 we have that for some \( \epsilon \in (0, 1) \)
\[
F(1_S \lor x) = F(1_S + u) = F(1_S) + u^T \nabla F(1_S + \epsilon u) \leq F(1_S) + u^T \nabla F(1_S) \left( \frac{||1_S + \epsilon u||_1}{||1_S||_1} \right)^\sigma.
\]
Using that \( |S| = c \), \( \epsilon \in (0, 1) \), and \( ||u||_1 \leq c \), we get
\[
F(x) \leq F(1_S \lor x) \leq F(1_S) + u^T \nabla F(1_S) \left( \frac{2c}{c} \right)^\sigma \leq f(S) + 2^\sigma u^T \nabla F(1_S).
\]
\( \Box \)
Let $e \in supp(u)$. Because of the exchange property, there is an $a \in S$ such that $S - a + e \in \mathcal{I}$. Because of the selection of $S$, we know that $(1 + \frac{1}{n^2})f(S) \geq f(S - a + e)$. Hence $\frac{1}{n^2}f(S) + B_a(S - a) \geq B_e(S - a)$. Therefore, we have

$$\nabla_e F(1_S) = B_e(S) = B_e(S - a) + A_{ae}(S - a) \leq B_e(S - a) + \sigma \frac{B_e(S - a) + B_a(S - a)}{c - 1} \leq \frac{c - 1 + 2\sigma}{c - 1} B_a(S - a) + \frac{(c - 1 + \sigma)\epsilon}{(c - 1)n^2} f(S)$$

Let $S = \{a_1, \ldots, a_c\}$ such that $B_{a_1}(S - a_1) \geq \cdots \geq B_{a_c}(S - a_c)$. Bounding $B_e(S)$ with $B_{a_i}(S - a_i)$ where $i$ is large is better. Let $R_i = \{e_1^i, \ldots, e_{k_i}^i\}$ be the set of elements in $supp(u)$ that are exchangeable with $a_i$ but are not exchangeable with any of $a_{i+1}, \ldots, a_c$. It is obvious that $R_i$’s partition $supp(u)$. Let $t_i = \sum_{e \in R_i} u_e$. By contradiction, we show that if $i \leq c - 1$ then $\sum_{j=1}^{i} t_j \leq i$. We know that for $R \subseteq [n]$ and $y \in P$ we have $\sum_{e \in R} y_e \leq r_M(R)$ where $r_M$ is the rank function of the matroid. If $\sum_{j=1}^{i} t_j > i$ then $r_M(\bigcup_{j=1}^{i} R_j) > i$. This means that there is $R \subseteq \bigcup_{j=1}^{i} R_j$ such that $|R| \geq i + 1$ and $R \in \mathcal{I}$. Now because of the exchange properties of matroids, we can add elements of $S$ to $R$ until they are the same size. Call this new set $R'$. Let $T_S = S \setminus R'$ and $T_R = R' \setminus S$. $|T_S| = |T_R| = i + 1$. Therefore, there is a perfect matching of exchangeability between $T_R$ and $T_S$. This contradicts our assumption because elements in $\bigcup_{j=1}^{i} R_j$ are only exchangeable with $a_1, \ldots, a_i$. Now, we have

$$u^T \nabla F(1_S) = \sum_{e \in supp(u)} u_e \nabla_e F(1_S) \leq \sum_{j=1}^{c} \sum_{e \in R_j} u_e \left( \frac{c - 1 + 2\sigma}{c - 1} B_{a_j}(S - a_j) + \frac{(c - 1 + \sigma)\epsilon}{(c - 1)n^2} f(S) \right)$$

$$= \sum_{j=1}^{c} t_j \left( \frac{c - 1 + 2\sigma}{c - 1} B_{a_j}(S - a_j) + \frac{(c - 1 + \sigma)\epsilon}{(c - 1)n^2} f(S) \right)$$

$$= \frac{c - 1 + 2\sigma}{c - 1} \left( \sum_{j=1}^{c} t_j B_{a_j}(S - a_j) \right) + \frac{c(c - 1 + \sigma)\epsilon}{(c - 1)n^2} f(S). \quad (30)$$

By Lemma [11] we know that

$$\sum_{j=1}^{c} B_{a_j}(S - a_j) = 1^T_S \nabla F(1_S) \leq (\sigma + 1)f(1_S).$$

We also know that $B_{a_1}(S - a_1) \geq \cdots \geq B_{a_c}(S - a_c)$, $\sum_{j=1}^{i} t_j = \|u\|_1 \leq c$, and $\sum_{j=1}^{i} t_j \leq i$ for $i = 1, \ldots, c - 1$. Now, we show that

$$\sum_{j=1}^{c} t_j B_{a_j}(S - a_j) \leq (\sigma + 1)f(S).$$

We try to find the maximizer of the above. Fix the value of $B_{a_j}(S - a_j)$’s. For any $j < k$, if we increase the value of $t_j$ by $\epsilon$ and decrease the value of $t_k$ by $\epsilon$, the value of the summation will increase. This means that the maximum happens when $t_1, \ldots, t_{\|u\|_1}$ are equal to one and $t_{\|u\|_1}$ is equal to $\|u\|_1 - \|u\|_1$. Therefore,

$$\sum_{j=1}^{c} t_j B_{a_j}(S - a_j) \leq \sum_{j=1}^{c} B_{a_j}(S - a_j) \leq (\sigma + 1)f(S).$$
Therefore, by (30), we have

\[ u^T \nabla F(1_S) \leq \frac{c - 1 + 2\sigma}{c - 1} (\sigma + 1)f(S) + \frac{c(c - 1 + \sigma}\epsilon}{(c - 1)n^2} f(S). \]

Hence, if \( \sigma = O(c) \) then \( u^T \nabla F(1_S) \leq O(\sigma)f(S) \) and if \( \sigma = \omega(c) \) then \( u^T \nabla F(1_S) \leq O(\sigma^2)f(S). \)

Combining this with (29) yields the result.

\[ \square \]

### F.4 Local Search for Second-Order-Submodular \( \gamma \)-Meta-Submodular Functions

In this section, we first provide a key lemma for bounding the Taylor series expansion of multi-linear extension of second-order-submodular functions. Then using this, we show that the modified local search algorithm (Algorithm 2) can be used to find an \( O \) \( \gamma \)-meta-submodular second-order-submodular subject to a matroid constraint.

**Lemma 12.** Let \( f : 2^n \rightarrow \mathbb{R} \) be a non-negative, second-order-submodular set function and \( F \) be its multi-linear extension. Then for any \( R \subseteq [n] \), \( \sum_{u \subseteq R} B_u(R) \leq 2f(R) \). If \( f \) is also monotone then \( x \in [0, 1]^n \), \( x^T \nabla^2 F(x)x \leq 2F(x) \).

**Proof.** For the first part, WLOG let \( R = [k] \) (we can always relabel the elements so that this is true) and \( R_i = [i] \). By Lemma 2, we have

\[ \sum_{i \in R} B_i(R) = \sum_{i=1}^{k} (f([i]) + \sum_{j=1}^{k} A_{ij}(R_{j-1})). \]

Since \( B_i(R_i) = B_i(R_{i-1}) \), and \( f(R_0) = f(\emptyset) = 0 \) we have

\[ 2f(R) = 2 \sum_{i=1}^{k} B_i(R_i) = 2 \sum_{i=1}^{k} (f([i]) + \sum_{j=1}^{i} A_{ij}(R_{j-1})). \]

Moreover, note that

\[ \sum_{i=1}^{k} \sum_{j=1}^{k} A_{ij}(R_{j-1}) \leq 2 \sum_{i=1}^{k} \sum_{j=1}^{i} A_{ij}(R_{j-1}) \]

since

\[ \sum_{i=1}^{k} \sum_{j=i+1}^{k} A_{ij}(R_{j-1}) = \sum_{j=1}^{k-1} \sum_{i=1}^{j+1} A_{ij}(R_{j-1}) = \sum_{j=1}^{k-1} \sum_{i=1}^{j} A_{ji}(R_{i-1}) \leq \sum_{j=1}^{k} \sum_{i=1}^{j} A_{ji}(R_{i-1}) = \sum_{i=1}^{k} A_{ii}(R_{i-1}) \]

\[ \sum_{i=1}^{k} \sum_{j=1}^{i} A_{ij}(R_{j-1}), \]

where the second equality follows from the fact that \( A_{ij}(S) = A_{ji}(S) \) for all \( i, j \in [n] \) and \( S \subseteq [n] \), and the third equality from the fact that \( A_{ii}(S) = 0 \) for all \( i \in [n] \) and \( S \subseteq [n] \). The inequality follows since \( R_{j-1} \supseteq R_{i-1} \) and \( f \) is second-order-submodular.

By non-negativity we also have that \( 2f([i]) \geq f([i]) \). This yields the first part of the lemma.

We now discuss the second part. By the Taylor’s Theorem, non-negativity, monotonicity and second-order-submodularity, we have

\[ F(x) = F(0) + x^T \nabla F(0) + \frac{1}{2} x^T \nabla^2 F(\epsilon x)x \geq \frac{1}{2} x^T \nabla^2 F(\epsilon x)x \geq \frac{1}{2} x^T \nabla^2 F(x)x. \]

\[ \square \]
Theorem 10. Let \( f \) be a \( \gamma \)-meta-submodular function which is also second order submodular (that is, \( f \)'s marginal gains are submodular). Let \( \mathcal{M} = ([n], \mathcal{I}) \) be a matroid of rank \( r \) and minimum circuit size \( c \). Let \( A \in \mathcal{I} \) be an optimum set, i.e., \( A \in \arg \max_{R \in \mathcal{I}} f(R) \), and \( S \in \mathcal{I} \) be an \( \left( 1 + \frac{\epsilon}{n^2} \right) \)-approximate local optima, i.e., for any \( i \) and \( j \) such that \( S - i + j \in \mathcal{I} \), \( (1 + \frac{\epsilon}{n^2})f(S) \geq f(S - i + j) \), where \( \epsilon > 0 \) is a constant. Then \( f(A) \leq O(\gamma + \frac{\epsilon^2}{r})f(S) \). So Algorithm 2 gives an \( O(\gamma + \frac{\epsilon^2}{r}) \)-approximation. If \( f \) is also supermodular then Algorithm 2 gives an \( O(\min\{\gamma + \frac{\epsilon^2}{r}, \frac{\epsilon}{n^2} \}) \) \( \leq O(\gamma^{3/2}) \)-approximation.

Proof. Since \( f \) is monotone, we assume that \(|S| = |A| = r\). Given the exchangeability property of matroids, there is a bijective mapping \( (45) \) \( g : S \setminus A \to A \setminus S \) such that \( S - i + g(i) \in \mathcal{I} \) where \( i \in S \setminus A \). Since \( S \) is a \( \left( 1 + \frac{\epsilon}{n^2} \right) \)-approximate local optima, for all \( i \in S \setminus A \) we have \( \left( 1 + \frac{\epsilon}{n^2} \right)f(S) \geq f(S - i + g(i)) \). That is,

\[
\frac{\epsilon}{n^2}f(S) + B_i(S - i) \geq B_{g(i)}(S - i).
\]

Using this we get

\[
B_{g(i)}(S) = B_{g(i)}(S - i) + A_{g(i)}(S - i) \leq B_{g(i)}(S - i) + \gamma\left(\frac{B_{g(i)}(S - i) + B_i(S - i)}{r - 1}\right)
\]

\[
\leq \frac{2\gamma + r - 1}{r - 1}B_i(S - i) + \frac{\epsilon(\gamma + r - 1)}{(r - 1)n^2}f(S) = \left(\frac{2\gamma}{r - 1} + 1\right)B_i(S) + \frac{\epsilon(\gamma + r - 1)}{(r - 1)n^2}f(S),
\]

where the first equality follows from Lemma [2] the first inequality from \( \gamma \)-meta-submodularity, and the last equality from \( B_i(S) = B_i(S - i) \) for all \( i \in [n] \) and \( S \subseteq [n] \). Thus,

\[
\sum_{i \in S \setminus A} B_{g(i)}(S) \leq \left(\frac{2\gamma}{r - 1} + 1\right)\sum_{i \in S \setminus A} B_i(S) + |S \setminus A| \cdot \frac{\epsilon(\gamma + r - 1)}{(r - 1)n^2}f(S)
\]

\[
\leq \left(\frac{2\gamma}{r - 1} + 1\right)\sum_{i \in S} B_i(S) + \frac{\epsilon(\gamma + r - 1)}{(r - 1)n}f(S)
\]

\[
\leq \left(\frac{4\gamma}{r - 1} + 2 + o(1)\right) \cdot f(S).
\]

where the second inequality follows from monotonicity (i.e. \( B_i(S) \geq 0 \)), and the last one follows from Lemma [12].

Now, by Taylor’s Theorem and the submodularity of the marginal gains of \( f \) (i.e. the non-positivity of the third order marginal gains), \( \gamma \)-meta submodularity, and the above inequality, we have

\[
f(A) \leq f(S \cup A) = F(1_S \vee 1_A) = F(1_S + 1_{A \setminus S}) \leq F(1_S) + 1^T_{A \setminus S} \nabla F(1_S) + \frac{1}{2} 1^T_{A \setminus S} \nabla^2 F(1_S) 1_{A \setminus S}
\]

\[
\leq F(1_S) + \left(1 + \frac{|A \setminus S|}{|S|}\right)1^T_{A \setminus S} \nabla F(1_S) \leq F(1_S) + (1 + \gamma)1^T_{A \setminus S} \nabla F(1_S)
\]

\[
= F(1_S) + (1 + \gamma)\sum_{i \in S \setminus A} B_{g(i)}(S) \leq \left(\frac{4\gamma^2}{r - 1} + \gamma\left(\frac{4}{r - 1} + 2 + o(1)\right) + 3 + o(1)\right)f(S)
\]

\[
= O\left(\frac{\gamma^2}{r} + \gamma\right)f(S).
\]
Now, we assume that $f$ is also supermodular. Let $S \cap S' = \{a_1, \ldots, a_p\}$ and $S' \setminus S = \{b_1, \ldots, b_p\}$ where $\{a_i, b_i\}$’s are the edges of the matching. Also, let $T_i = \{a_1, \ldots, a_i\}$ and $R_i = \{b_1, \ldots, b_i\}$. Then since $M$ is a maximum weighted matching, we have

\[
\sum_{i \in S \setminus A} A_{ig(i)}(S) \leq \frac{2 \cdot |S \setminus A|}{c-1} \sum_{i=1}^{p} A_{a_ib_i}(S) \leq \frac{2r}{c-1} \sum_{i=1}^{p} A_{a_ib_i}(S). \tag{32}
\]

We also have that

\[
f(S') = \sum_{i=1}^{p} \left( f(T_i \cup R_i) - f(T_{i-1} \cup R_{i-1}) \right) = \sum_{i=1}^{p} \left( B_{a_i}(T_{i-1} \cup R_{i-1}) + B_{b_i}(T_{i-1} \cup R_{i-1} + a_i) \right) = \sum_{i=1}^{p} \left( B_{a_i}(T_{i-1} \cup R_{i-1}) + B_{b_i}(T_{i-1} + a_i) \right).
\]

where the third equality follows from Lemma 2, the first inequality from monotonicity and supermodularity (i.e. all the $B_i$ and $A_{ij}$ terms are non-negative), and the last inequality from second-order-submodularity and the fact that $T_i \subseteq S$ for any $i = 1, \ldots, p$.

Hence, by combining (32) and (33), we get

\[
\sum_{i \in S \setminus A} A_{ig(i)}(S-i) = \sum_{i \in S \setminus A} A_{ig(i)}(S) \leq \frac{2r}{c-1} \sum_{i=1}^{p} A_{a_ib_i}(S) \leq \frac{2r}{c-1} f(S'). \tag{34}
\]

Using Taylor’s Theorem

\[
f(A) \leq f(S \cup A) = F(1_S \lor 1_A) = F(1_S + 1_A \setminus S) \leq F(1_S) + 1_A \setminus S \nabla F(1_S) + \frac{1}{2} 1_A \setminus S \nabla^2 F(1_S) 1_A \setminus S \leq F(1_S) + \left( 1 + \frac{|A \setminus S|}{|S|} \right) \frac{1}{2} T_{A \setminus S} \nabla F(1_S) \leq F(1_S) + (1 + \gamma) \frac{1}{2} T_{A \setminus S} \nabla F(1_S)
\]

\[
= f(S) + (1 + \gamma) \sum_{i \in S \setminus A} B_{g(i)}(S) = f(S) + (1 + \gamma) \sum_{i \in S \setminus A} B_{g(i)}(S-i) + \sum_{i \in S \setminus A} A_{ig(i)}(S-i)
\]

\[
\leq f(S) + (1 + \gamma) \left( \frac{r \gamma}{n^2} f(S) + \sum_{i \in S \setminus A} B_i(S-i) + \frac{2r}{c-1} f(S') \right) \leq O\left( \frac{\gamma r}{c-1} \right) \max\{f(S), f(S')\},
\]

where the second inequality follows from second-order-submodularity (i.e. the non-positivity of the third order derivatives), the third inequality from $\gamma$-meta submodularity, the fifth inequality from (31) and (34), and the second to last inequality from Lemma 12.

We then have that if $r \leq \sqrt{\gamma}$ then $\gamma r \leq \gamma^{3/2}$, and if $r \geq \sqrt{\gamma}$ then $\frac{\gamma^2}{r} + \gamma \leq \gamma^{3/2}$. Therefore, $f(A) \leq O(\gamma^{3/2}) \max\{f(S), f(S')\}$.

\[\square\]
Appendix: Integrality Gaps and Rounding Algorithms

In this section, we provide the omitted results and proofs about the integrality gap and different rounding techniques.

G.1 Integrality Gap Lower Bound

In this section, we describe an example that shows the integrality gap of a quadratic function with a $\sigma$-semi-metric distance over a matroid polytope is $\Omega(\min\{\frac{r}{c-2}, \frac{\sigma}{r}\})$ in the worst case, where $r$ is the rank of the matroid and $c$ is the size of the smallest circuit.

**Proposition** Let $k, t \in \mathbb{N}$ with $1 \leq t \leq k$. There exists a $\sigma$-semi-metric with multilinear extension $F$, and a matroid $\mathcal{M} = ([2k], \mathcal{I})$ with rank $r = k + t - 1$ and minimum circuit size $c = 2t$, where the integrality gap of $F(x)$ over the matroid polytope $P_{\mathcal{M}}$ is $\Omega(\min\{\frac{r}{c-2}, \frac{\sigma}{r}\})$.

**Proof.** Let $S_i = \{2i - 1, 2i\}$ for $1 \leq i \leq k$, and $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$. We define a matroid $\mathcal{M} = ([2k], \mathcal{I})$ in terms of its circuits as follows. A set $C$ is a circuit of $\mathcal{M}$ if and only if $C$ is the union of any $t$ sets $S_i$. It is then clear that the minimum size $c$ of a circuit is $2t$, and the rank $r$ of the matroid is $k + t - 1$. For example, $\mathcal{M}$ could be the graphic matroid corresponding to the graph in Figure[2]. Circuits here correspond to cycles of size 4, and the dashed lines show the non-zero coefficients of $F$.

Let $F(x) = \sum_{(u,v) \in \mathcal{S}} x_u x_v + \sum_{(u,v) \notin \mathcal{S}} \frac{1}{\sigma} x_u x_v$. It is straightforward to see that $F$ is the multilinear extension of a $\sigma$-semi-metric induced by a complete graph which has weight 1 on edges from $\mathcal{S}$ and weight $1/\sigma$ otherwise.

By definition of $\mathcal{M}$ and $F$, it is clear that any integral solution $x_I \in P_{\mathcal{M}}$ maximizing $F$ will pick $t - 1$ pairs from $\mathcal{S}$ and then singletons from other pairs. Therefore

$$F(x_I) := \max_{x \in P_{\mathcal{M}} \cap \{0, 1\}^{2k}} F(x) = (t-1) + \frac{1}{\sigma} \left( \binom{r}{2} - (t-1) \right) = (1 - \frac{1}{\sigma}) (t-1) + \frac{1}{\sigma} \binom{r}{2} = \frac{(\sigma - 1)(c-2) + r(r-1)}{2\sigma}.$$  

On the other hand, $x_0 = \frac{k+t-1}{2k} 1_{[2k]} \in P_{\mathcal{M}}$ and

$$F(x_0) = k \left( \frac{k+t-1}{2k} \right)^2 + \left( \frac{2k}{k} \right) - k \frac{1}{\sigma} \left( \frac{k+t-1}{2k} \right)^2 = k \left( \frac{k+t-1}{2k} \right)^2 \left( 1 + \frac{2(k-1)}{\sigma} \right).$$  

Using that $r = k + t - 1$ and $k = \frac{r}{2} + 1$ we have

$$k \left( \frac{k+t-1}{2k} \right)^2 = \frac{r^2}{4(r - \frac{t}{2} + 1)} = \frac{r^2}{2(2r - c + 2)} \geq \frac{r}{4},$$  

where the last inequality follows since $c \geq 2$. Hence, $F(x_0) \geq \frac{r}{4}(1 + \frac{2(k-1)}{\sigma})$. It follows that the integrality gap is at least

$$\frac{F(x_0)}{F(x_I)} \geq \frac{1}{2} \cdot \frac{\sigma r + 2r(k-1)}{(\sigma - 1)(c-2) + r(r-1)} \geq \frac{1}{2} \cdot \frac{\sigma r}{\sigma(c-2) + r^2} \geq \frac{1}{4} \cdot \min\{\frac{r}{c-2}, \frac{\sigma}{r}\}.$$  

\[\square\]
G.2 Quadratic Coverage Rounding

In this section, we provide the details about the quadratic coverage rounding.

We actually prove the following decomposition result. For \( x^* \in P_M \), we define the coverage of a pair \( u, v \) to be the quantity \( x^*(u)x^*(v) \). Let \( \text{Cov} \in \mathbb{R}^2 \) be the vector with entries \( \text{Cov}(u, v) = x^*(u)x^*(v) \). If \( F \) is quadratic it is linear in these coverage values and the vector \( x^* : F(x^*) = \sum_{u \neq v} (\frac{A(u,v)}{2}) \text{Cov}(u,v) + \sum_v b(v)x^*(v) \). For a set \( X \) we say its coverage set is \( \text{cov}(X) = \{ (u,v) : u,v \in X, u \neq v \} \). A quadratic coverage of \( x^* \) is a collection \( \mathcal{C} = \{ 1_{I_i}, \mu_i \} \) of weighted independent sets with properties (1) for each \( u \neq v, \sum_{i : (u,v) \in \text{cov}(I_i)} \mu_i \geq \text{Cov}(u,v) \), and (2) for each \( v, \sum_{i : I_i \ni v} \mu_i \geq x^*(v) \). We actually prove the following decomposition result. For \( x^* \) we define \( \mu_i \) to be the quantity \( \sum_{i : (u,v) \in \text{cov}(I_i)} \mu_i \). As \( F \) is quadratic it is linear in these coverage values and the vector \( x^* \).

**Theorem 17.** Let \( F(x) = \frac{1}{2}x^T Ax + b^T x \) be a non-negative, quadratic multi-linear polynomial and \( M \) be a matroid with rank \( r = r([n]) \) and minimum circuit size \( c \geq 3 \). If \( x^* \in P_M \), then it has a quadratic coverage of size at most \( 3 + \frac{2}{c-2} \).

**Proof.** We start with an arbitrary representation of \( x^* \) as a convex combination of independent sets: \( \sum_i \lambda_i 1_{B_i} \).

First note that \( \text{Cov}(u,v) = (\sum_{B_i \ni u} \lambda_i)(\sum_{B_j \ni v} \lambda_j) = \sum_{(i,j) : B_i \ni u, B_j \ni v} \lambda_i \lambda_j \). Hence an ordered pair \( (B_i, B_j) \) contributes \( \lambda_i \lambda_j \) to \( \text{Cov}(u,v) \) if \( u \in B_i, v \in B_j \). This implies that if \( B_i = B_j \), then this contributes exactly \( \lambda_i^2 \) for every \( u, v \in B_i \). If \( B_i \neq B_j \), then the unordered pair \( \{ B_i, B_j \} \) contributes to coverages as follows. It contributes \( 2 \lambda_i \lambda_j \) for each \( u, v \in B_i \cap B_j \) and \( \lambda_i \lambda_j \) for each \( uv \in \delta(B_i - B_j, B_j - B_i, B_i \cap B_j) \). Here for disjoint node sets \( X_1, X_2, \ldots, X_p \) we define \( \delta(X_1, X_2, \ldots, X_p) \) to be the set of edges which have endpoints in distinct sets from the \( X_i \)'s. Hence we can express the coverage vector \( \text{Cov} \) for \( x^* \) in \( \mathbb{R}^2 \) as:

\[
\sum_i \lambda_i^2 \cdot 1_{\text{cov}(B_i)} + \sum_{i < j} \lambda_i \lambda_j \cdot (2 \cdot 1_{\text{cov}(B_i \cap B_j)} + 1_{\delta(B_i - B_j, B_j - B_i, B_i \cap B_j)}).
\] (35)

We now define a quadratic coverage, that is, a weighted collection of independent sets satisfying conditions (1) and (2). In particular, for each \( i \leq j \) we define a family of independent sets \( \mathcal{T}_{i,j} \) which will take care of all coverages associated with terms \( \lambda_i \lambda_j \) in (35). In the case where \( i = j \), this is easy. We just include the set \( B_i \) with weight \( \mu_i = \lambda_i^2 \). Now consider the case where \( i < j \) which is trickier. For each set \( I \) in this family, we always associate the weight \( \mu_I = \lambda_i \lambda_j \) and so this amounts to finding a family which satisfies

\[
\sum_{I \in \mathcal{T}_{i,j}} 1_{\text{cov}(I)} \geq 2 \cdot 1_{\text{cov}(B_i \cap B_j)} + 1_{\delta(B_i - B_j, B_j - B_i, B_i \cap B_j)}.
\] (36)
We return to this construction later but we note that condition (2) will follow easily as long as we guarantee that for each \(v, i\) and \(j \neq i\), if \(B_i \supseteq v\), then the family \(T^{i,j}\) includes at least one set \(I\) which contains \(v\). Since we have \(\mu_I = \lambda_i \lambda_j\) for any such \(I\), we derive the desired inequality (2): \[\sum_{I \ni v} \mu_I \geq \sum_{B_i \ni v} \sum_{j} \lambda_i \lambda_j = \sum_{B_i \ni v} \lambda_i = x^*(v).\]

If we can achieve this construction so that \(|T^{i,j}| \leq K\) for each \(i, j\), then we have a quadratic coverage whose size is \(\sum_i \mu_i + \sum_{i<j} \sum_{I \ni T^{i,j}} \mu_i = \sum_i \lambda_i^2 + \sum_{i<j} \lambda_i \lambda_j |T^{i,j}| \leq \sum_i \lambda_i^2 + \sum_{i<j} \lambda_i \lambda_j K \leq 1 + K/2\). The last inequality follows since the \(\lambda_i\) are a convex combination.

We now define \(T^{i,j}\) for a fixed pair \(i, j\) and show how to find the desired independent sets \(T^{i,j} = \{I_k^{i,j} : k = 1, 2, \ldots, K\}\), where \(K\) is defined later. First, if \(|B_i \cap B_j| \geq 1\), then we include the sets \(B_i, B_j\). This takes care of the double-coverage of pairs in \(B_i \cap B_j\) as well as any pairs \(u, v\) with \(u \in B_i \cap B_j\) and \(v \in B_i \Delta B_j\). Let \(S_{ij} = B_i \setminus B_j\) and \(S_{ji} = B_j \setminus B_i\). Note that the excess coverage from these sets \(B_i, B_j\) is to contribute an extra \(\lambda_i \lambda_j\) to each pair in \(cov(S_{ij}) \cup cov(S_{ji})\). It now remains to cover the edges in \(\delta(S_{ij}, S_{ji})\).

Let \(t = \lfloor (c-1)/2 \rfloor\) and \(m = |B_i \cap B_j| \geq 0\). Decompose \(B_j \setminus B_i\) into \(t = \lfloor (r-m)/t \rfloor\) disjoint independent sets by ripping out sets of size \(t\) greedily, possibly the last being smaller than \(t\). Call these \(C_1, C_2, \ldots, C_t\). For each \(k \leq t\), we extend \(C_k\) to an independent set \(R_k^{i,j}\) in \(B_i \Delta B_j\) only adding elements from \(B_i \setminus B_j\). Hence this set will have used all elements of \(B_i\) except a subset, call it \(Z_k\), of size at most \(t\). Let \(C_k^{i,j} = Z_k \cup C_k\) and note that \(|C_k^{i,j}| \leq 2t \leq c-1\) and hence it is also independent.

We now examine the pairs covered by \(C_k^{i,j}, R_k^{i,j}\). Let \(u \in C_k, v \in B_i \setminus B_j\), then either \(u, v\) is covered by \(R_k^{i,j}\), or \(v \in Z_k\) in which case it is covered by \(C_k^{i,j}\).

Finally, we count the number of sets for a given family. There are two cases depending on whether \(B_i \cap B_j = \emptyset\) or not. If the intersection is empty, then we just build \(2\lfloor t^2 \rfloor\). Since \(t \geq \frac{c-2}{2}\), this is at most \(2 \cdot (1 + \frac{2r}{c-2})\). In the other case we have \(m \geq 1\), and we add the sets \(B_i, B_j\) up front and then we add \(2\lfloor \frac{t-m}{t} \rfloor\) more sets. Hence the overall number of sets in this case is at most \(2 + 2 \cdot \left(\frac{2r}{c-2} - \frac{2}{c-2} + 1\right)\).

It follows that \(K \leq 2 \cdot \left(2 + \frac{2r}{c-2}\right)\), and thus we have a quadratic coverage of size at most \(1 + \frac{K}{2} \leq 3 + \frac{2r}{c-2}\), as we wanted to show.

\[\square\]

### G.3 Swap Rounding

In this section, we analyze a modified version of the swap rounding algorithm (Algorithm 7) and we show that it finds an integral solution which is an \(O\left(1 + \frac{2}{\ell}\right)\)-approximation of the initial fractional solution.

First we define the following notation. \(d(S) = \sum_{(i,j) \subseteq S} d(i,j)\) and \(d(S, S') = \sum_{i \in S} \sum_{j \in S'} d(i,j)\). \(g(S) = \sum_{i \in S} g(i)\). The following result provides a decomposition of the multi-linear extension of a quadratic function based on the convex decomposition of a point to the bases of the matroid.

**Lemma 13.** Let \(f(S) = \sum_{i \in S} g(i) + \sum_{(i,j) \subseteq S} d(i, j)\) where \(g : [n] \to \mathbb{R}_{\geq 0}\) and \(d : [n] \times [n] \to \mathbb{R}_{\geq 0}\) with \(d(i, i) = 0\) for all \(i \in [n]\). Let \(b \in \mathbb{R}^n\) be a vector such that \(b_i = g(i)\) and \(A \in \mathbb{R}^{n \times n}\) be a matrix such that \(A_{ij} = d(i, j)\). Then the multi-linear extension of \(f\) is \(F(x) = \frac{1}{2}x^T Ax + x^T b\). Moreover, if \(x = \sum_{k=1}^{p} \lambda_k I_k\) for some scalars \(\lambda_k\)'s and subsets \(I_k \subseteq [n]\), then

\[F(x) = \sum_{k=1}^{p} \lambda_k g(I_k) + \sum_{k=1}^{p} \lambda_k^2 d(I_k) + \sum_{k=1}^{p-1} \sum_{\ell=k+1}^{p} \lambda_k \lambda_{\ell} d(I_k, I_\ell).\]  \hfill (37)
Proof. For the first part of the lemma note that

\[ F(x) = \sum_{S \subseteq [n]} f(S) \prod_{k \in S} x_k \prod_{k \in [n] \setminus S} (1 - x_k) = \sum_{S \subseteq [n]} (g(S) + d(S)) \prod_{k \in S} x_k \prod_{k \in [n] \setminus S} (1 - x_k) \]

\[ = \sum_{S \subseteq [n]} \left( \sum_{i \in S} g(i) \right) \prod_{k \in S} x_k \prod_{k \in [n] \setminus S} (1 - x_k) + \sum_{S \subseteq [n]} \left( \sum_{(i,j) \in S} d(i,j) \right) \prod_{k \in S} x_k \prod_{k \in [n] \setminus S} (1 - x_k) \]

\[ = \sum_{i \in [n]} g(i) \sum_{S \subseteq [n] \setminus i} \left( \prod_{k \in S} x_k \prod_{k \in [n] \setminus S} (1 - x_k) \right) + \sum_{S \subseteq [n] \setminus i} d(i,j) \sum_{(i,j) \in S} \left( \prod_{k \in S} x_k \prod_{k \in [n] \setminus S} (1 - x_k) \right) \]

\[ = \sum_{i \in [n]} g(i) x_i \sum_{S \subseteq [n] \setminus i} \left( \prod_{k \in S} x_k \prod_{k \in [n] \setminus S} (1 - x_k) \right) + \sum_{S \subseteq [n] \setminus i} d(i,j) x_i x_j \sum_{(i,j) \in S} \left( \prod_{k \in S} x_k \prod_{k \in [n] \setminus S} (1 - x_k) \right) \]

\[ = \sum_{i \in [n]} g(i) x_i + \sum_{(i,j) \subseteq [n]} d(i,j) x_i x_j = x^T b + \frac{1}{2} x^T Ax. \]

To see the second part, observe that

\[ b^T x = b^T \left( \sum_k \lambda_k 1_{I_k} \right) = \sum_k \lambda_k (b^T 1_{I_k}) = \sum_k \lambda_k g(I_k), \]

and

\[ x^T Ax = \left( \sum_{k=1}^p \lambda_k 1_{I_k} \right) A \left( \sum_{\ell=1}^p \lambda_\ell 1_{I_\ell} \right) = \sum_{k, \ell=1}^p \lambda_k \lambda_\ell 1_{I_k} 1_{I_\ell} = \sum_{k, \ell=1}^p \lambda_k \lambda_\ell d(I_k, I_\ell) \]

\[ = \sum_{k=1}^p \lambda_k^2 d(I_k, I_k) + 2 \sum_{k < \ell} \lambda_k \lambda_\ell d(I_k, I_\ell) = 2 \sum_{k=1}^p \lambda_k^2 d(I_k) + 2 \sum_{k=1}^{p-1} \sum_{\ell=k+1}^p \lambda_k \lambda_\ell d(I_k, I_\ell). \]

Lemma 14. Let \( M = ([n], \mathcal{I}) \) be a matroid and \( P \) be its corresponding base polytope. Let \( F(z) = \frac{1}{2} z^T Az + z^T b \) where \( A, b \geq 0 \) and \( A \) is a symmetric matrix such that its diagonal is zero. Let \( f(S) = F(1_S) \) for any \( S \subseteq [n] \). Let \( x = \sum_{i=1}^p \lambda_i 1_{I_i} \in P \) where \( I_i \)’s are bases of the matroid, \( \sum_{i=1}^p \lambda_i = 1 \), and \( \lambda_i \geq 0 \), for \( i = 1, \ldots, p \). Let \((I^1, M)\) be the output of \textsc{MergeBases} (defined in Algorithm 7) on \((I_1, \ldots, I_p)\) and \((\lambda_1, \ldots, \lambda_p)\). Let \( y = (\lambda_1 + \lambda_2) 1_{I_1} + \sum_{i=3}^p \lambda_i 1_{I_i} \). Then \( F(x) \leq F(y) + \lambda_1 \lambda_2 \sum_{(i,j) \in M} d(i,j) \).

Proof. Let \( I^0_1 = I_1 \) and \( I^0_2 = I_2 \) (the original inputs of the function). Let \( I^m_1 \) and \( I^m_2 \) be the resulting \( I_1 \) and \( I_2 \) after the \( m \)-th iteration of the while loop. Let \( x_m = \lambda_1 1_{I^m_1} + \lambda_2 1_{I^m_2} + \sum_{k=3}^p \lambda_k 1_{I_k} \). Let \( i_m, j_m \) be the elements we pick at the \( m \)-th iteration of the loop. We show that \( F(x_m) \leq F(x_{m-1}) + \lambda_1 \lambda_2 d(i_m, j_m) \) and this yields the desired result using a simple recursion argument. Without loss of generality, we assume

\[ g(i_m) + \lambda_1 d(i_m, I^m_1 - i_m) + \lambda_2 d(i_m, I^m_2 - j_m) + \sum_{k=3}^p \lambda_k d(i_m, I_k) \]

\[ \geq g(j_m) + \lambda_1 d(j_m, I^{m-1}_1 - i_m) + \lambda_2 d(j_m, I^{m-1}_2 - j_m) + \sum_{k=3}^p \lambda_k d(j_m, I_k) \quad (38) \]
We have
\[
F(x_{m-1}) = \lambda_1 g(I_1^{m-1}) + \lambda_2 g(I_2^{m-1}) + \sum_{k=3}^{p} \lambda_k g(I_k) + \lambda_1^2 d(I_1^{m-1}) + \lambda_2^2 d(I_2^{m-1}) + \sum_{k=3}^{p} \lambda_k^2 d(I_k)
\]
\[+ \lambda_1 \lambda_2 d(I_1^{m-1}, I_2^{m-1}) + \lambda_1 \sum_{k=3}^{p} \lambda_k d(I_1^{m-1}, I_k) + \lambda_2 \sum_{k=3}^{p} \lambda_k d(I_2^{m-1}, I_k) + \sum_{k=3}^{p} \sum_{k'=k+1}^{p} \lambda_k \lambda_{k'} d(I_k, I_{k'})
\]
\[= \lambda_1 g(I_1^{m-1}) + \lambda_2 g(I_2^{m-1} - j_m) + \sum_{k=3}^{p} \lambda_k g(I_k) + \lambda_1^2 d(I_1^{m-1}) + \lambda_2^2 d(I_2^{m-1} - j_m) + \sum_{k=3}^{p} \lambda_k^2 d(I_k)
\]
\[+ \lambda_1 \lambda_2 d(I_1^{m-1}, I_2^{m-1} - j_m) + \lambda_1 \sum_{k=3}^{p} \lambda_k d(I_1^{m-1}, I_k) + \lambda_2 \sum_{k=3}^{p} \lambda_k d(I_2^{m-1} - j_m, I_k)
\]
\[+ \sum_{k=3}^{p} \sum_{k'=k+1}^{p} \lambda_k \lambda_{k'} d(I_k, I_{k'}) + \lambda_2 g(j_m) + \lambda_2^2 d(j_m, I_2^{m-1} - j_m) + \lambda_1 \lambda_2 d(j_m, I_1^{m-1} - i_m)
\]
\[+ \lambda_2 \sum_{k=3}^{p} \lambda_k d(j_m, I_k) + \lambda_1 \lambda_2 d(i_m, j_m)
\]
\[\leq \lambda_1 g(I_1^{m-1}) + \lambda_2 g(I_2^{m-1} - j_m) + \sum_{k=3}^{p} \lambda_k g(I_k) + \lambda_1^2 d(I_1^{m-1}) + \lambda_2^2 d(I_2^{m-1} - j_m) + \sum_{k=3}^{p} \lambda_k^2 d(I_k)
\]
\[+ \lambda_1 \lambda_2 d(I_1^{m-1}, I_2^{m-1} - j_m) + \lambda_1 \sum_{k=3}^{p} \lambda_k d(I_1^{m-1}, I_k) + \lambda_2 \sum_{k=3}^{p} \lambda_k d(I_2^{m-1} - j_m, I_k)
\]
\[+ \sum_{k=3}^{p} \sum_{k'=k+1}^{p} \lambda_k \lambda_{k'} d(I_k, I_{k'}) + \lambda_2 g(i_m) + \lambda_2^2 d(i_m, I_2^{m-1} - j_m) + \lambda_1 \lambda_2 d(i_m, I_1^{m-1} - i_m)
\]
\[+ \lambda_2 \sum_{k=3}^{p} \lambda_k d(i_m, I_k) + \lambda_1 \lambda_2 d(i_m, j_m)
\]
\[= \lambda_1 g(I_1^m) + \lambda_2 g(I_2^m) + \sum_{k=3}^{p} \lambda_k g(I_k) + \lambda_1^2 d(I_1^m) + \lambda_2^2 d(I_2^m) + \sum_{k=3}^{p} \lambda_k^2 d(I_k)
\]
\[+ \lambda_1 \lambda_2 d(I_1^m, I_2^m) + \lambda_1 \sum_{k=3}^{p} \lambda_k d(I_1^m, I_k) + \lambda_2 \sum_{k=3}^{p} \lambda_k d(I_2^m, I_k)
\]
\[+ \sum_{k=3}^{p} \sum_{k'=k+1}^{p} \lambda_k \lambda_{k'} d(I_k, I_{k'}) + \lambda_1 \lambda_2 d(i_m, j_m) = F(x^m) + \lambda_1 \lambda_2 d(i_m, j_m).
\]

The inequality holds because of (33), and the first and the last equalities follow from Lemma 14. The second to the last equality uses that $I_1^m = I_1^{m-1}$ and $I_2^m = I_2^{m-1} - j_m + i_m$. □

**Theorem 18.** Let $\mathcal{M}([n], \mathcal{I})$ be a matroid of rank $r$ and $\mathcal{P}$ be its corresponding base polytope. Let $F(z) = \frac{1}{z} z^T A z + z^T b$ where $A, b \succeq 0$ and $A$ is a symmetric matrix with zero diagonal that satisfies the $\sigma$-semi-metric inequality, i.e., $A_{ij} \leq \sigma (A_{ik} + A_{jk})$. Let $f(S) = F(1_S)$ for any $S \subseteq [n]$. Let $x \in \mathcal{P}$ and $S$ be the output of the modified swap rounding (Algorithm 7) on $x$. Then $F(x) \leq O(1 + \frac{\sigma}{r}) f(S)$.

**Proof.** Let $x = \sum_{i=1}^{p} \lambda_i 1_{I_i} \in \mathcal{P}$ where $I_i$’s are bases of the matroid, $\sum_{i=1}^{p} \lambda_i = 1$, and $\lambda_i \geq 0$, for $i = 1, \ldots, p$. Let $S$ be the output of the swap rounding (Algorithm 7) if it starts from $(I_1, \ldots, I_p)$...
and \((\lambda_1, \ldots, \lambda_p)\). Let \(x_k\) denote the vector corresponding to \(I_k = (I_k', I_{k+1}, \ldots, I_p)\) and \(\lambda_k = (\lambda_{k'}, \lambda_{k+1}, \ldots, \lambda_p)\), i.e. \(x_k = \lambda_k' \mathbb{1}_{I_k'} + \sum_{i=k+1}^{p} \lambda_i \mathbb{1}_{I_i}\). By Lemma 14 for \(k = 1, \ldots, n - 1\), we have

\[
F(x_k) \leq F(x_{k+1}) + \lambda_k' \lambda_{k+1} \sum_{(i,j) \in M_k} d(i, j) \leq F(x_{k+1}) + \lambda_k' \sum_{(i,j) \in M_k} d(i, j),
\]

where \(t = \arg \max_{k=1,\ldots,p-1} \{ \sum_{(i,j) \in M_k} d(i, j) \}\). Therefore

\[
F(x_1) \leq F(x_p) + \left( \sum_{k=1}^{p-1} \lambda_k' \lambda_{k+1} \right) \sum_{(i,j) \in M_1} d(i, j) = F(x_p) + \left( \sum_{k=1}^{p-1} \sum_{m=1}^{k} \lambda_m \lambda_{k+1} \right) \sum_{(i,j) \in M_1} d(i, j),
\]

where the last inequality holds since \(2 \sum_{k=1}^{p-1} \sum_{m=1}^{k} \lambda_m \lambda_{k+1} \leq (\sum_{k=1}^{p} \lambda_k)^2 = 1\). Now, we bound the term \(\sum_{(i,j) \in M_1} d(i, j)\). By definition of \(M_1\), note that \(M_1 \subseteq I'_p \times I_{t+1}\). Using this and Lemma 13 it follows that

\[
\sum_{(i,j) \in M_1} d(i, j) \leq d(I'_1, I_{t+1}) \leq 4 \cdot F\left( \frac{1}{2} \mathbb{1}_{I'_p} + \frac{1}{2} \mathbb{1}_{I_{t+1}} \right).
\]

By Lemma 14 and the \(\sigma\)-semi-metric assumption, we also know that

\[
F\left( \frac{1}{2} \mathbb{1}_{I'_p} + \frac{1}{2} \mathbb{1}_{I_{t+1}} \right) \leq F(\mathbb{1}_{\sigma}) + \frac{1}{4} \sum_{(i,j) \in M^*} d(i, j) \leq F(\mathbb{1}_{\sigma}) + \frac{1}{4} \sum_{(i,j) \in M^*} \frac{\sigma}{r-1} (d(i, I'_1 - i) + d(j, I'_1 - i)).
\]

Note that none of the edges of \(M^*\) is present in the right hand side summation. Therefore

\[
\sum_{(i,j) \in M^*} (d(i, I'_1 - i) + d(j, I'_1 - i)) \leq d(I'_1) + d(I'_1, I_{t+1}) - \sum_{(i,j) \in M^*} d(i, j)
\]

\[
\leq 4 \cdot F\left( \frac{1}{2} \mathbb{1}_{I'_p} + \frac{1}{2} \mathbb{1}_{I_{t+1}} \right) - \sum_{(i,j) \in M^*} d(i, j) \leq 4F(\mathbb{1}_{\sigma}) = 4f(I^*).
\]

where the second inequality follows from Lemma 13 and the last inequality holds because of Lemma 14. Combining (40), (41), and (42), we get

\[
\sum_{(i,j) \in M_t} d(i, j) \leq (4 + \frac{4\sigma}{r-1}) f(I^*).
\]

Hence, by (39) and (43), we have

\[
F(x_1) \leq f(I'_p) + \left( 2 + \frac{2\sigma}{r-1} \right) f(I^*),
\]

and this yields the result. \(\square\)
Algorithm 7: Swap rounding for monotone second-order-modular functions under matroid constraints

1. Input: A matroid \( \mathcal{M} = ([n], \mathcal{I}) \), its base polytope \( P \), and a fractional solution \( x \in P \). A set function \( f(S) = \sum_{i \in S} g(i) + \sum_{(i,j) \in S} d(i,j) \).
2. Find \( \lambda_1 = (\lambda_1, 2, \ldots, \lambda_p) \) and \( I_1 = (I_1, I_2, \ldots, I_p) \) such that \( x = \sum_{i=1}^p \lambda_i I_i \), \( \lambda_i \geq 0 \) (for any \( i \)), \( \sum_{i=1}^p \lambda_i = 1 \), and \( I_i \)'s are bases of the matroid.
3. \( I'_1 \leftarrow I_1 \);
4. \( \lambda'_1 \leftarrow \lambda_1 \);
5. for \( k = 1, \ldots, p-1 \) do
6. \( (I'_{k+1}, M_k) \leftarrow \text{MergeBases}(I_k, \lambda_k) \);
7. \( \lambda'_{k+1} \leftarrow \lambda'_k + \lambda_{k+1} \);
8. \( I_{k+1} \leftarrow (I'_{k+1}, I_{k+2}, \ldots, I_p) \);
9. \( \lambda_{k+1} \leftarrow (\lambda'_{k+1}, \lambda_{k+2}, \ldots, \lambda_p) \);
10. \( t \leftarrow \arg \max_{k=1, \ldots, p-1} \{ \sum_{(i,j) \in M_k} d(i,j) \} \); 
11. \( (I^*, M^*) \leftarrow \text{MergeBases}((I_1, I_2, \ldots, I_p)) \);
12. return \( \arg \max \{ f(I^*), f(I'_p) \} \);
13. Function \( \text{MergeBases}(I = (I_1, I_2, \ldots, I_m), \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)) \):
14. \( M \leftarrow \emptyset \);
15. while \( I_1 \neq I_2 \) do
16. \( M \leftarrow M \cup \{(i,j)\} \); 
17. if \( g(i) + \lambda_1 d(i, I_1 - i) + \lambda_2 d(i, I_2 - j) + \sum_{k=3}^m \lambda_k d(i, I_k) \geq g(j) + \lambda_1 d(j, I_1 - i) + \lambda_2 d(j, I_2 - j) + \sum_{k=3}^m \lambda_k d(j, I_k) \) then
18. \( I_2 \leftarrow I_2 - j + i \); 
19. \( I_1 \leftarrow I_1 - i + j \); 
20. else return \( (I_1, M) \); 
21. End Function

G.4 Pipage Rounding

In section G.1 we provide super-constant lower bounds for rounding discrete quadratics over matroids. In this section we show that for uniform matroids, there is a constant-factor rounding algorithm even for the much more general class of second-order-submodular functions. We analyze the pipage rounding algorithm (Algorithm 8) for this purpose.

Recall that given a vector \( x \in [0,1]^n \) and \( i \in [n] \), we denote by \( x - i \) the vector resulting from setting the \( i \)th coordinate of \( x \) to zero. That is, \( (x - i)_j = x_j \) for all \( j \neq i \) and \( (x - i)_i = 0 \).

Lemma 15. Let \( f \) be a set function and \( F \) be its multi-linear extension. Let \( x \in [0,1]^n \) and \( i \neq j \in [n] \) such that \( \nabla_i F(x - i - j) \geq \nabla_j F(x - i - j) \). Consider the vector \( y = x + \epsilon (e_i - e_j) \), where \( e_i \) denotes the characteristic vector of \( i \in [n] \), and \( \epsilon = \min \{ x_j, 1 - x_i \} \). That is,

\[
y_k = \begin{cases} 
  x_i + \epsilon = \min \{ 1, x_i + x_j \}, & k = i \\
  x_j - \epsilon = \max \{ 0, x_i + x_j - 1 \}, & k = j \\
  x_k, & o.w.
\end{cases}
\]

Then \( F(y) + \max \{ 0, x_i x_j \nabla_{ij}^2 F(x) \} \geq F(x) \).
Proof. For any \( z \in [0, 1]^n \), we have

\[
F(z) = \sum_{R \subseteq [n]} f(R) \prod_{v \in R} z_v \prod_{v \notin R} (1 - z_v)
\]

\[
= z_i z_j \sum_{R \subseteq [n] - i - j} f(R + i + j) \prod_{v \in R} z_v \prod_{v \notin R + i + j} (1 - z_v)
+ z_i (1 - z_j) \sum_{R \subseteq [n] - i - j} f(R + i) \prod_{v \in R} z_v \prod_{v \notin R + i + j} (1 - z_v)
+ (1 - z_i) z_j \sum_{R \subseteq [n] - i - j} f(R + j) \prod_{v \in R} z_v \prod_{v \notin R + i + j} (1 - z_v)
+ (1 - z_i)(1 - z_j) \sum_{R \subseteq [n] - i - j} f(R) \prod_{v \in R} z_v \prod_{v \notin R + i + j} (1 - z_v)
\]

\[
= z_i z_j \sum_{R \subseteq [n] - i - j} \left( f(R + i + j) - f(R + i) - f(R + j) + f(R) \right) \prod_{v \in R} z_v \prod_{v \notin R + i + j} (1 - z_v)
+ z_i \sum_{R \subseteq [n] - i - j} \left( f(R + i) - f(R) \right) \prod_{v \in R} z_v \prod_{v \notin R + i + j} (1 - z_v)
+ z_j \sum_{R \subseteq [n] - i - j} \left( f(R + j) - f(R) \right) \prod_{v \in R} z_v \prod_{v \notin R + i + j} (1 - z_v)
+ \sum_{R \subseteq [n] - i - j} f(R) \prod_{v \in R} z_v \prod_{v \notin R + i + j} (1 - z_v)
\]

\[
= z_i z_j \nabla_{ij}^2 F(z - i - j) + z_i \nabla_i F(z - i - j) + z_j \nabla_j F(z - i - j) + F(z - i - j).
\]

Note that \( x - i - j = y - i - j \). Also, by definition of \( \epsilon \) we have \( \epsilon \geq x_j - x_i \), and hence

\[
y_i y_j = (x_i + \epsilon)(x_j - \epsilon) = x_i x_j + \epsilon(x_j - x_i - \epsilon) \leq x_i x_j.
\]

It follows that

\[
F(x) = x_i x_j \nabla_{ij}^2 F(x - i - j) + x_i \nabla_i F(x - i - j) + x_j \nabla_j F(x - i - j) + F(x - i - j)
\]

\[
= x_i x_j \nabla_{ij}^2 F(y - i - j) + x_i \nabla_i F(y - i - j) + x_j \nabla_j F(y - i - j) + F(y - i - j)
\]

\[
\leq x_i x_j \nabla_{ij}^2 F(y - i - j) + y_i \nabla_i F(y - i - j) + y_j \nabla_j F(y - i - j) + F(y - i - j)
\]

\[
= (x_i x_j - y_i y_j) \nabla_{ij}^2 F(y - i - j) + y_i y_j \nabla_{ij}^2 F(y - i - j) + y_i \nabla_i F(y - i - j)
+ y_j \nabla_j F(y - i - j) + F(y - i - j)
\]

\[
= (x_i x_j - y_i y_j) \nabla_{ij}^2 F(x - i - j) + F(y)
\]

\[
\leq (x_i x_j - y_i y_j) \max\{0, \nabla_{ij}^2 F(x - i - j)\} + F(y)
\]

\[
\leq x_i x_j \max\{0, \nabla_{ij}^2 F(x - i - j)\} + F(y)
\]

\[
= \max\{0, x_i x_j \nabla_{ij}^2 F(x)\} + F(y),
\]

where the first inequality follows from the assumption \( \nabla_i F(x - i - j) \geq \nabla_j F(x - i - j) \), and the last equality follows from \( \nabla_{ij}^2 F(x - i - j) = \nabla_{ij}^2 F(x) \) (see Lemma [1]). \( \qed \)

**Theorem 19.** Let \( f \) be a non-negative, monotone, second-order-submodular function and \( F \) be its multi-linear extension. Let \( x \in [0, 1]^n \) such that \( \|x\|_1 = k \). Then Algorithm 3 finds \( S \subseteq [n] \) such that \( |S| = k \) and \( 6f(S) \geq F(x) \).
Therefore, using this and (44), we have

\[
\begin{align*}
F(z) & \leq F(z') + \max\{0, z_i z_j \nabla_{ij}^2 F(z)\}. 
\end{align*}
\]  

(44)

Proof. Let \(z \in [0,1]^n\) and \(z^F\) be its fractional part (coordinates). Also let \(z'\) be \(z\) after one step of pipage rounding algorithm (Algorithm 19). By Lemma 15 we have

\[
F(z) \leq F(z') + \max\{0, z_i z_j \nabla_{ij}^2 F(z)\}. 
\]

By second-order-submodularity, Lemma 12 and monotonicity, we have

\[
\frac{1}{2} (z^F)^T \nabla^2 F(z)(z^F) \leq \frac{1}{2} (z^F)^T \nabla^2 F(z^F)(z^F) \leq F(z^F) \leq F(z). 
\]

Therefore,

\[
z_i z_j \nabla_{ij}^2 F(z) = \min_{\{q,q'\} \subseteq \text{supp}(z^F)} z_q z_{q'} \nabla_{qq'} F(z) \leq \frac{1}{\left(\frac{\left|\text{supp}(z^F)\right|}{2}\right)} F(z).
\]

Hence, by non-negativity of \(f\), we have

\[
\max\{0, z_i z_j \nabla_{ij}^2 F(z)\} \leq \frac{1}{\left(\frac{\left|\text{supp}(z^F)\right|}{2}\right)} F(z) 
\]

Using this and (44), we have

\[
\left(\frac{\left|\text{supp}(z^F)\right|}{2}\right) - \frac{1}{\left(\frac{\left|\text{supp}(z^F)\right|}{2}\right)} F(z) \leq F(z') 
\]

(45)
Let $x^1$ be the initial vector in Algorithm [19] and $x^{i+1}$ be the vector after $i$'th iteration of the loop. Also, let $n_i = |\text{supp}(x^i)|$. If the loop iterates $t$ times, we have $n \geq n_1 > n_2 > \ldots > n_t \geq 3$ because in each iteration, the number of integral coordinate increases by at least 1, and $\|x^f\|_1 > 2$ (the loop’s condition). By [15], for $i = 1, \ldots, t$, we have $F(x^{i+1}) \geq \frac{n_i^2 - n_i - 2}{n_i(n_i - 1)} F(x^i)$. Let $x^{t+2}$ be the final vector in the algorithm (it is integral). We show that $F(x^{t+2}) \geq \frac{1}{2} F(x^{t+1})$. Let $x^F$ be the fractional part of the $x^{i+1}$, $x^I$ be its integral part, $S = \text{supp}(x^I)$, and

$$\{i, j\} = \arg \max_{\{q, q'\} \subseteq \text{supp}(x^I)} (B_q(S) + B_{q'}(S) + A_{qq'}(S)).$$

Note that $\|x^F\|_1 = 2$ because the norm of the fractional part decreases by at most 1 at any iteration and also it is always an integer. Therefore, because of the selection of $i, j$, we have

$$\begin{align*}
& (\sum_{\{q, q'\} \subseteq \text{supp}(x^F)} x_q x_{q'})(B_i(S) + B_j(S) + A_{ij}(S)) \\
& \geq \sum_{\{q, q'\} \subseteq \text{supp}(x^F)} x_q x_{q'}(B_q(S) + B_{q'}(S) + A_{qq'}(S)) \\
& = \sum_{\{q, q'\} \subseteq \text{supp}(x^F)} x_q x_{q'} B_q(S) + \sum_{\{q, q'\} \subseteq \text{supp}(x^F)} x_q x_{q'} B_{q'}(S) + \sum_{\{q, q'\} \subseteq \text{supp}(x^F)} x_q x_{q'} A_{qq'}(S) \\
& = \sum_{q \in \text{supp}(x^F)} \sum_{q' \in \text{supp}(x^F), q \neq q'} x_q (2 - x_q) B_q(S) + \sum_{\{q, q'\} \subseteq \text{supp}(x^F)} x_q x_{q'} A_{qq'}(S) \\
& \geq \sum_{q \in \text{supp}(x^F)} x_q B_q(S) + \sum_{\{q, q'\} \subseteq \text{supp}(x^F)} x_q x_{q'} A_{qq'}(S) \\
& = (x^F)^T \nabla F(x^I) + \frac{1}{2} (x^F)^T \nabla F(x^I)(x^F)
\end{align*}$$

The second inequality holds because $\|x^I\|_1 = \sum_{q \in \text{supp}(x^F)} x_q = 2$ and $x_q$ is fractional, i.e., $x_q < 1$. By the Lagrange multipliers’ method and the fact that $\sum_{q \in \text{supp}(x^F)} x_q = 2$, we can conclude that

$$\sum_{\{q, q'\} \subseteq \text{supp}(x^F)} x_q x_{q'} \leq 2,$$

and the equality happens when all $x_q = 2/(|\text{supp}(x^F)|)$. Using non-negativity and monotonicity of $f$, the Taylor’s theorem, the above inequalities, and Lemma [2] we have

$$\begin{align*}
F(x^{t+1}) &= F(x^I) + (x^F)^T \nabla F(x^I) + \frac{1}{2} (x^F)^T \nabla F(x^I)(x^F) \\
& \leq 2F(x^I) + (\sum_{\{q, q'\} \subseteq \text{supp}(x^F)} x_q x_{q'})(B_i(S) + B_j(S) + A_{ij}(S)) \\
& = (2 - \sum_{\{q, q'\} \subseteq \text{supp}(x^F)} x_q x_{q'})(F(x^I) + \sum_{\{q, q'\} \subseteq \text{supp}(x^F)} x_q x_{q'})(F(x^I) + B_i(S) + B_j(S) + A_{ij}(S)) \\
& = (2 - \sum_{\{q, q'\} \subseteq \text{supp}(x^F)} x_q x_{q'})(F(x^I) + \sum_{\{q, q'\} \subseteq \text{supp}(x^F)} x_q x_{q'})(F(x^I) + \mathbf{1}_{(i, j)}) \\
& \leq 2F(x^I + \mathbf{1}_{(i, j)}) = 2F(x^{t+2})
\end{align*}$$

59
By the above inequalities, we have

\[
F(x^{t+2}) \geq \left( \prod_{i=1}^{t} \frac{(n) - 1}{(n)} \right) \frac{1}{2} F(x^1) \geq \left( \prod_{i=3}^{n} \frac{(i) - 1}{(i)} \right) \frac{1}{2} F(x^1) \geq \left( \prod_{i=3}^{n} \frac{i^2 - i - 2}{i(i - 1)} \right) \frac{1}{2} F(x^1)
\]

\[
= \left( \prod_{i=3}^{n} \frac{(i + 1)(i - 2)}{i(i - 1)} \right) \frac{1}{2} F(x^1) = \frac{n + 1}{3(n - 1)} \left( \prod_{i=3}^{n-1} \frac{(i - 1)(i + 1)}{(i + 1)(i - 1)} \right) \frac{1}{2} F(x^1)
\]

\[
= \frac{n + 1}{3(n - 1)} \frac{1}{2} F(x^1) \geq \frac{1}{6} F(x^1).
\]