Self-similar motion of melting sphere

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Abstract. In the framework of the Stokes approximation the problem of the self-similar motion of a melting solid sphere in its own viscous incompressible melt at a given difference of temperature of the melt far away from the sphere and the melting temperature. The force acting on the sphere is calculated. The equation of the sphere motion until complete melting is obtained. It is shown that a reactive force significantly accelerates the sphere at sufficiently large temperature difference.

1. Introduction

It is known [1] that an accelerating force \( F = -\dot{m}v \) is acting on a point source of strength \( \dot{m}(t) < 0 \) in the flow of an incompressible ideal fluid with velocity \( v(x,t) \) and it is essentially related to the theory of a point of variable mass. In the work of L.I. Sedov [2] this effect was put in the basis of the theory of drag reduction of bodies moving in a fluid.

This formula was also used to construct the theory of gravitational interaction of bodies [3–5], both within the framework of the Newtonian mechanics and the theory of relativity, which made it possible to reveal two new effects: the recently discovered particles system acceleration at large times and the increase in angular momentum of a body moving along an orbit.

In [6] reactive force acting on spherical particle and viscous force were compared, and the superiority of the reactive force was observed at a sufficiently large relative difference between the temperature of the melt and the melting point of the sphere (for metals it is of the order of 0.1 within the Kelvin scale). The effect is intensified by the fact that the viscosity of the melts usually decreases noticeably (in several times [7]) at the indicated increase in temperature.

The random behavior of a Brownian melting particle was investigated in [8]. The solution of the stochastic differential equation of particle motion is given. The average square speed and the equation of motion is calculated, the Fokker-Planck equation is derived as well.

In this paper we consider more detailed solution of the self-similar problem (independent of the initial data) of the melting spherical particle motion and corresponding thermal problem. To construct a solution, the method of invariant-group analysis of differential equations is used, which was developed in the works of L.V. Ovsyannikov [9].

2. Self-similar motion of particle

Let consider the slow motion (corresponding Reynolds number is small) of a spherical melting particle of radius \( R(t) \) in he melt. The problem of the motion of variable radius sphere can be solved separately, in particular, as a self-similar one if there is only one dimensional constant,
the kinematic viscosity coefficient $\nu$. As it is known, the mechanical quantities can be neglected in the calculating of temperature distribution for small Prandtl number $\nu/k$ [10] and small Reynolds number $[11]$, where $k$ is the thermal diffusivity, which is typical for metals [7]. In this case, the dimension of temperature is separated from the dimension of the squared velocity and the thermal problem of melting becomes self-similar. Based on this fact, the equations of motion of a particle in an ideal incompressible fluid were derived in [5] under the assumption that the fluid is in quasi-equilibrium state. A more accurate approach with regard to radial motion is given in [6], but qualitatively leads to similar results.

2.1. Formulation of the problem
Let consider the melt as an incompressible viscous fluid. We associate the inertial frame of reference with the Cartesian coordinates $x^i$ with the resting melt at infinity, $x^i_0(t) = \delta^i_3 z_0(t)$ are coordinates of the particle center of mass moving along the axis $x^3$. Let introduce relative coordinates $y^i = x^i - x^i_0(t)$ and note that derivatives by the coordinates $y^i$ and $x^i$ are equivalent. In the case of self-similar motions $x^i/\sqrt{\nu|t|}$ and time $t < 0$ are invariant independent variables. The coordinate of center of mass is $z_0 = -z_1/\sqrt{\nu|t|}$, its velocity is $V_0(t) = (z_1/2)\sqrt{\nu/|t|}$, particle radius is $R = R_1/\sqrt{\nu|t|}$, where $R_1$ depends on the temperature difference and other thermal constants. We also introduce the relative radial coordinate $r = |y|$.

Stokes equations for the fluid velocity field $\mathbf{v}$ and pressure $p$ at constant melt density $\rho$ disregarded by quadratic terms have the form

$$\mathbf{v}_t + \frac{1}{\rho} \nabla p = \nu \Delta \mathbf{v}, \quad \nabla_i v^i = 0. \tag{1}$$

Consider the boundary conditions. The fluid is at rest at infinity ($v^i = 0$). The velocity on the surface of the particle $r = R(t)$ is made up of the translational motion $V_0 \delta^i_3$ and the velocity of the radial motion $Q(t)/(4\pi R^2)$, where $Q = Q_1 \nu^{1/2}|t|^{1/2}$ is volume flow, which is related to the radial velocity of the particle surface $\dot{R}$ by the law of mass flow conservation

$$\rho_0 \dot{R} = \rho \left( \dot{R} - \frac{Q}{4\pi R^2} \right), \tag{2}$$

where $\rho_0$ is the density of the particle, also constant.

We assume that the quantity $Q/(4\pi R^2)$ is of the order of small velocity $V_0$ and $V_0/\dot{R} \ll 1$. This leads to smallness of the relative differences of densities of solid metal and its melt, which is usually is actually executed [7]. Attempt to count $\dot{R}$ in the order of $Q/(4\pi R^2)$ would lead to the need to solve equations of motion in the Oseen approximation taking into account the convective members.

2.2. Construction of the solution
Using the method of constructing the solution of equations Stokes on the stationary motion of a sphere, described in [10], suitable and for the self-similar solution due to the absence of delay, we set

$$v^i = \delta^i_3 \Delta B - \nabla^i (\nabla_3 B + Q/(4\pi r)), \quad B = V_0(\alpha Rr + \beta R^3/r).$$

The boundary conditions on the sphere give $\alpha = 3/4$, $\beta = 1/4$. So

$$v^i = 3\delta^i_3 V_0 R/2r + \nabla^i \left( \frac{y_3 V_0 R}{4r} \left( \frac{R^2}{r^2} - 3 \right) - \frac{Q}{4\pi r} \right) =$$

$$= V_0 \delta^i_3 \left( \frac{3R}{4r} + \frac{R^3}{4r^3} \right) + 3y^i y_3 V_0 R/4r^3 \left( 1 - \frac{R^2}{r^2} \right) + y^i Q/4\pi r^3.$$
The vorticity tensor has the form
\[ \omega_{ij} = \frac{1}{2} \left( \nabla_i v_j - \nabla_j v_i \right) = \frac{-3V_0 R}{4r^3} (y_i \delta_j^3 - y_j \delta_i^3). \]

Obviously, in this approximation \( \omega_{ij,t} = 0, \Delta \omega_{ij} = 0 \), as in the stationary case, due to the constancy of the product \( V_0 R \). Note that \( y' \) included in expression of \( z_0(t) \) gives an error of order \( V_0^2 \).

Then we calculate the pressure gradient using equation (1). As a result, considering the pressure at infinity to be zero, we get
\[ \frac{p}{\rho} = \frac{y_3 V_0 R}{2r^3} \left( 3\nu - R \dot{R} \right) + \frac{Q}{4\pi r}. \]

Obviously, \( \Delta p = 0 \). When \( v^i_t \) are calculated members of the second order of smallness also are discarded.

2.3. Force acting on a particle

From the dimensional analysis it can be seen that the force acting on the particle is constant. Discarding terms of the second order of smallness, we calculate the surface force acting on the sphere from the fluid in projection on the \( x^3 \) axis.

\[ F = -\int_{S_R} \left( pn_3 - 2\mu e_{3i} n^i \right) d\sigma, \]

where in spherical coordinates \( d\sigma = R^2 \sin \theta d\theta d\varphi \), and the normal is
\[ n = y/R = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \]

and the coefficient \( \mu = \rho \nu \). Calculating the components of the strain-rate tensor gives
\[ e_{3i} n^i = (1/2) (\nabla_3 v_i + \nabla_i v_3) n^i = \frac{3V_0}{4R} \left( \frac{y_3^2}{R^2} - 1 \right) - \frac{Q y_3}{2\pi R^4}. \]

As a result, using (2) for \( Q \), we obtain
\[ F = \frac{2\pi}{3} \rho V_0 R^2 \dot{R} - 6\pi \mu V_0 R. \]

By virtue of the equality resulting from the problem self-similarity \( V_0 \dot{R} = -\dot{V}_0 R \) the first term gives the added mass of the sphere \( m/2 \), and the last member is a Stokes drag force. For the final determination of drag force, one have to solve the thermal problem for finding \( R \) at a given melting point \( T_0 \) and a temperature at infinity \( T_\infty \).

2.4. Thermal problem

From the law of conservation of energy flow at melting front, neglecting kinetic energy transfer and work done by internal surface forces, we obtain the heat flux which can be represented by Fourier’s law:
\[ q = \rho_0 \dot{R} (T_0(c - c_0) + |\lambda|) = -\rho c k T_r, \]

where \( c \) and \( c_0 \) are the specific heat capacities of the melt and solid bodies, respectively, \( |\lambda| \) is the value of latent heat of the melting. The difference in heat capacities is usually neglected.
After neglecting of small hydrodynamic terms of the order of $V_0$ the heat equation will have following form in case of spherical symmetry

$$T_t = k \left( T_{rr} + \frac{2}{r} T_r \right).$$

Suppose self-similar character of temperature distribution $T(\xi)$, where $\xi = r/\sqrt{\nu|t|}$. Due to the smallness of the Prandtl number $\nu/k$ simple quasistatic solution can be derived

$$T = T_\infty + \frac{R}{r}(T_0 - T_\infty),$$

which sufficient for further research. Then, condition (9) gives an equation for $R(t)$

$$R \dot{R} = -\frac{\nu R^2}{2} = -\frac{\rho c k(T_\infty - T_0)}{\rho_0((c - c_0)T_0 + |\lambda|)}.$$

(3)

2.5. Sphere motion

Finally, the equation of motion for the center of the melting sphere with mass $m_0 = (4\pi/3)\rho_0 R^3$ can be written:

$$(d/dt)(m_0 V_0) = F.$$

By opening the brackets and dividing by the total mass $m_0 + m/2$, we get sphere acceleration (at $\rho \approx \rho_0$)

$$\dot{V}_0 = -\frac{V_0}{R^2} \left( 2R \dot{R} + 3 \nu \right), \quad R^2 = R_1^2 \nu |t|.$$

Now, excluding $\dot{R}$ from the relation (3), we can estimate temperature difference at which the acceleration of $V_0$ is positive despite of the viscous drag force. So counting $V_0 > 0$, we get the inequality

$$\frac{2c(T_\infty - T_0)}{T_0(c - c_0) + |\lambda|} > \frac{3\nu}{k}.$$

Moreover, due to the self-similarity of the problem, the power dependence of non-zero velocity $V_0 \sim |t|^{-1/2}$ gives a certain value of $R_1^2 = 6$. Therefore, this solution is possible only with special melt temperature

$$T_\infty = T_0 + \frac{3\nu(T_0(c - c_0) + |\lambda|)}{ck}.$$

In this case, the dimensionless quantity $z_1$ remains arbitrary.

Other values of this temperature lead to non-self-similar problem statement and require a separate study. In the future, we will try to find a solution of the problem of the moving sphere with variable radius problem without self-similarity using Boussinesq’s approach.

Conclusion

In the framework of the Stokes approximation, the problem of the self-similar motion of a melting solid sphere in its own viscous incompressible melt is solved for a given melting point and melt temperature far from the sphere. The force acting on the sphere from the liquid side is calculated, the equation of the sphere motion is obtained up to moment of complete melting. It is shown that with a sufficiently large temperature difference a constant reactive force appears, significantly accelerating the body. The solution of the problem indicates possible explosive phenomena in mixtures of the melt and many melting particles.

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