New arithmetical proof of the reciprocity law for Dedekind sums

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Abstract. In this paper, for coprime numbers \( p \) and \( q \) we consider the Dedekind sums

\[
S(p, q) = \sum_{r=1}^{q-1} \left\{ \frac{r}{q} \right\} \left\{ \frac{rp}{q} \right\}.
\]

(0.1)

First, we give an improvement of the proof given by H. Rademacher and A. Whiteman [2], and we construct a new arithmetical proof for the reciprocity law

\[
S(p, q) + S(q, p) = \frac{p^2 + q^2 + 1}{12pq} + \frac{p + q}{4} - \frac{3}{4}.
\]

(0.2)

different of all the arithmetical proofs given until now.

Second, we found explicit formula of \( S(p, q) \) for

\[ q \equiv 1, b(p), 2, p - 2, 3, p - 3, p - 4 \text{ and } 4[p]. \]

1. Introduction and statement of main results

1.1. Introduction. In the literature there are several different proofs of the reciprocity law for Dedekind-Rademacher sums, H. Rademacher and E. Grosswald (in [3]) have constructed four proofs.

In this work, we are interested by the arithmetical ones. First we give an improvement of the proof of H. Rademacher and A. Whiteman [2, §3]. In the second time, using Euclidean division, we give a new arithmetical proof of such reciprocity law. Taking \( q \equiv b(p) \), the idea of the proof consists to write

\[
S(p, q) + S(q, p) = P(b),
\]

where \( P \) is a polynomial of degree 3. After we establish that \( P \) is a constant polynomial, and

\[
P(b) = \frac{p^2 + q^2 + 1}{12pq} + \frac{p + q}{4} - \frac{3}{4}
\]

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Finally from the reciprocity law and the expression of $S(p, q)$ we found explicit formula of $S(p, q)$ for $q \equiv 1, p - 1, 2, p - 2, 3, p - 3, p - 4$ and $4|p$.

In this work, we need the following two well-known results for finite sums

\[(1.1) \quad \sum_{r=1}^{q-1} r = \frac{q(q-1)}{2}, \]

\[(1.2) \quad \sum_{r=1}^{q-1} r^2 = \frac{q(q-1)(2q-1)}{6} \]

which can be proven by recursion.

1.2. Statement of main results. Let the first normalized Bernoulli function

\[(1.3) \quad B_1(t) := \left\{ \begin{array}{ll} \{t\} - \frac{1}{2}, & \text{if } t \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{array} \right. \]

Where $\{t\} = t - \lfloor t \rfloor$, and $\lfloor t \rfloor$ is the greater integer less than $t$.

For $p, q$ two coprime numbers, where $q$ is any integer, and $p$, is of course a positive integer consider the Dedekind sums

\[(1.4) \quad s(p, q) = \sum_{r=1}^{q-1} B_1 \left( \frac{r}{q} \right) B_1 \left( \frac{rp}{q} \right) \]

and

\[(1.5) \quad S(p, q) = \sum_{r=1}^{q-1} \left\{ \frac{r}{q} \right\} \left\{ \frac{rp}{q} \right\} \]

If the class modulo $p$ of $q$ is $b (1 \leq b \leq p - 1)$, then

$s(q, p) = s(b, q)$

and

$S(q, p) = S(b, p)$.

Without losing generality only we consider in this work $p < q$ and $p, q$ coprime. In this case

\[\sum_{r=1}^{q-1} B_1 \left( \frac{rp}{q} \right) = 0\]
then $s(p,q)$ can be transformed to

$$s(p,q) = \sum_{r=1}^{q-1} \left( \left\lfloor \frac{r}{q} \right\rfloor - \frac{1}{2} \right) B_1 \left( \frac{rp}{q} \right)$$

$$= \sum_{r=1}^{q-1} \left\lfloor \frac{r}{q} \right\rfloor B_1 \left( \frac{rp}{q} \right)$$

**Theorem 1.1.** For $p, q$ positive coprime numbers, we have

$$(1.6) \quad S(p, q) + S(q, p) = \frac{p^2 + q^2 + 1}{12pq} + \frac{p + q}{4} - \frac{3}{4}$$

The reciprocity law in (1.6) involves the reciprocity law of $s(p,q)$:

$$(1.7) \quad s(p, q) + s(q, p) = \frac{p^2 + q^2 - 3pq + 1}{12pq}$$

Specifically in the case $q \equiv 1, 2, 3$ and $4 \mod p$, we obtain

**Theorem 1.2.** For $p < q$ positive coprime numbers, we have

$$(1.8) \quad q \equiv 1 \mod p, \quad S(p, q) = \frac{p^2 + q^2 + 1}{12pq} + \frac{q}{4} - \frac{p}{12} - \frac{1}{6p} - \frac{1}{4}$$

$$(1.9) \quad q \equiv 2 \mod p, \quad S(p, q) = \frac{p^2 + q^2 + 1}{12pq} + \frac{q}{4} - \frac{p}{24} - \frac{5}{24p} - \frac{1}{4}$$

$$(1.10) \quad q \equiv 3 \mod p, \quad S(p, q) = \frac{p^2 + q^2 + 1}{12pq} + \frac{q}{4} - \frac{p}{36} - \frac{5}{18p} - \frac{1}{3} \left\lfloor \frac{p}{3} \right\rfloor - \frac{1}{12}$$

$$(1.11) \quad q \equiv 4 \mod p, \quad S(p, q) = \frac{p^2 + q^2 + 1}{12pq} + \frac{q}{4} - \frac{p}{48} - \frac{17}{48p} - \frac{1}{2} \left\lfloor \frac{p}{4} \right\rfloor$$

As a consequence we deduce for $q \equiv p - 1, p - 2, p - 3$ and $p - 4$ modulo $p$ that

**Corollary 1.1.** For $p < q$ positive coprime numbers, we have

$$(1.12) \quad q \equiv (p - 1) \mod p, \quad S(q - p, q) = \frac{q}{4} - \frac{p^2 + q^2 + 1}{12pq} + \frac{p}{12} + \frac{1}{6p} - \frac{1}{4}$$

$$(1.13) \quad q \equiv (p - 2) \mod p, \quad S(q - p, q) = \frac{q}{4} - \frac{p^2 + q^2 + 1}{12pq} + \frac{p}{24} + \frac{5}{24p} - \frac{1}{4}$$

$$(1.14) \quad q \equiv (p - 3) \mod p, \quad S(q - p, q) = \frac{q}{4} - \frac{p^2 + q^2 + 1}{12pq} + \frac{p}{36} + \frac{5}{18p} + \frac{1}{3} \left\lfloor \frac{p}{3} \right\rfloor - \frac{5}{12}$$

$$(1.15) \quad q \equiv (p - 4) \mod p, \quad S(q - p, q) = \frac{q}{4} - \frac{p^2 + q^2 + 1}{12pq} + \frac{p}{48} + \frac{17}{48p} + \frac{1}{2} \left\lfloor \frac{p}{4} \right\rfloor - \frac{1}{2}$$
2. Improvement of the short proof of Rademacher and Whiteman

Here we give an improvement of the short proof of Rademacher and Whiteman [2] different from the proof given by L. J. Mordell [1]. To do this we need the following lemma.

**Lemma 2.1.** For \( p, q \) two coprime numbers, we have

\[
q - 1 \sum_{r=1}^{p} \sum_{t < \frac{tp}{q}} t = \frac{(p - 1)(2pq - 3p + 2q)}{12} + ps(q, p)
\]

**Proof.** Since \((p, q) = 1\), \( \frac{tp}{q} \) is not an integer for \( 1 \leq t \leq p - 1 \). Then \( r < \frac{tp}{q} \) means that \( r \leq \left\lfloor \frac{tp}{q} \right\rfloor \) and then

\[
\sum_{r=1}^{q-1} \sum_{t < \frac{tp}{q}} t = \sum_{t=1}^{p} \sum_{r > \frac{tp}{q}} t
\]

\[
= \sum_{t=1}^{p} \left( \sum_{r=1}^{q-1} t - \sum_{r > \frac{tp}{q}} t \right)
\]

\[
= \sum_{t=1}^{p} \sum_{r=1}^{q-1} t - \sum_{t=1}^{p} \sum_{r > \frac{tp}{q}} t
\]

\[
= \frac{(q - 1)(p - 1)p}{2} - \sum_{t=1}^{p} t \left\lfloor \frac{tp}{q} \right\rfloor
\]

\[
= \frac{(q - 1)(p - 1)p}{2} - \sum_{t=1}^{p} \left( \frac{tp}{q} \right) \left\lfloor \frac{tp}{q} \right\rfloor
\]

\[
= \frac{(q - 1)(p - 1)p}{2} - q \sum_{t=1}^{p} t^2 + p \sum_{t=1}^{p} \left\lfloor \frac{tp}{q} \right\rfloor \left\lfloor \frac{tp}{q} \right\rfloor
\]

\[
= \frac{(q - 1)(p - 1)p}{2} - q(p - 1)(2p - 1) + \frac{p(p - 1)}{4} + ps(q, p)
\]

\[
= \frac{(2q - 1)(p - 1)p}{2} - q(p - 1)(2p - 1) + \frac{p(p - 1)}{4} + ps(q, p)
\]

\[
= \frac{(p - 1)(2pq - 3p + 2q)}{12} + ps(q, p)
\]

We compute the value of the sum \( \sum_{r=1}^{q-1} B_1 \left( \frac{rp}{q} \right)^2 \) with two different methods, and the comparison of the results gives the proof of the reciprocity law.
In one hand we have
\[
q^{-1} \sum_{r=1}^{q-1} B_1 \left( \frac{rp}{q} \right)^2 = \sum_{r=1}^{q-1} B_1 \left( \frac{r}{q} \right)^2 = \sum_{r=1}^{q-1} \left( \frac{r - \lfloor \frac{r}{q} \rfloor}{q} \right)^2 = \frac{1}{q^2} \sum_{r=1}^{q-1} r^2 - \frac{1}{4} \sum_{r=1}^{q-1} r + \frac{1}{4} \sum_{r=1}^{q-1} 1
\]
Then
\[
q^{-1} \sum_{r=1}^{q-1} B_1 \left( \frac{rp}{q} \right)^2 = \frac{(q-1)(2q-1)}{6q} - \frac{q-1}{2} + \frac{q-1}{4}
\]
Furthermore
\[
q^{-1} \sum_{r=1}^{q-1} B_1 \left( \frac{rp}{q} \right)^2 = \frac{(q-1)(2q-1)}{6q} - \frac{q-1}{4}
\]
In other hand
\[
q^{-1} \sum_{r=1}^{q-1} B_1 \left( \frac{rp}{q} \right)^2 = \left( \frac{q-p}{q} \right) - \frac{2q-2}{q} \left( \left\lfloor \frac{rp}{q} \right\rfloor + \frac{1}{2} \right) + \left( \left\lfloor \frac{rp}{q} \right\rfloor + \frac{1}{2} \right)^2
\]
Thus
\[
q^{-1} \sum_{r=1}^{q-1} B_1 \left( \frac{rp}{q} \right)^2 = \sum_{r=1}^{q-1} \left( \frac{rp}{q} \right)^2 - \frac{2rp}{q} \left( \left\lfloor \frac{rp}{q} \right\rfloor + \frac{1}{2} \right) + \left( \left\lfloor \frac{rp}{q} \right\rfloor + \frac{1}{2} \right)^2
\]
Then
\[
q^{-1} \sum_{r=1}^{q-1} B_1 \left( \frac{rp}{q} \right)^2 = -\frac{p^2}{q^2} \sum_{r=1}^{q-1} r^2 + 2p \sum_{r=1}^{q-1} \frac{r}{q} B_1 \left( \frac{rp}{q} \right) + \sum_{r=1}^{q-1} \left( \left\lfloor \frac{rp}{q} \right\rfloor + \frac{1}{2} \right)^2
\]
and
\[
q^{-1} \sum_{r=1}^{q-1} B_1 \left( \frac{rp}{q} \right)^2 = -\frac{p^2}{q^2} \sum_{r=1}^{q-1} r^2 + 2p \sum_{t=1}^{q-1} \frac{r}{q} B_1 \left( \frac{rp}{q} \right) + \sum_{r=1}^{q-1} \left( \left\lfloor \frac{rp}{q} \right\rfloor + \frac{1}{2} \right)^2
\]
But
\[
\left\lfloor \frac{rp}{q} \right\rfloor \left( \left\lfloor \frac{rp}{q} \right\rfloor + 1 \right) = 2 \sum_{t=1}^{q-1} t
\]
then
\[
q^{-1} \sum_{r=1}^{q-1} B_1 \left( \frac{rp}{q} \right)^2 = \frac{q-1}{4} - \frac{p^2(q-1)(2q-1)}{6q} + 2p \sum_{t=1}^{q-1} \frac{r}{q} \left( \left\lfloor \frac{rp}{q} \right\rfloor + 1 \right) + \frac{q-1}{4}
\]
and from the value of \( \sum_{r=1}^{q-1} \sum_{t=1}^{q/p} t \) in Lemma (2.1) we deduce that
\[
q^{-1} \sum_{r=1}^{q-1} B_1 \left( \frac{rp}{q} \right)^2 = \frac{q-1}{4} - \frac{p^2(q-1)(2q-1)}{6q} + 2p \sum_{t=1}^{q-1} \frac{r}{q} \left( \left\lfloor \frac{rp}{q} \right\rfloor + 1 \right) + \frac{q-1}{4}
\]
and from the value of \( \sum_{r=1}^{q-1} \sum_{t=1}^{q/p} t \) in Lemma (2.1) we deduce that
\[
q^{-1} \sum_{r=1}^{q-1} B_1 \left( \frac{rp}{q} \right)^2 = \frac{q-1}{4} - \frac{p^2(q-1)(2q-1)}{6q} + 2p \sum_{t=1}^{q-1} \frac{r}{q} \left( \left\lfloor \frac{rp}{q} \right\rfloor + 1 \right) + \frac{q-1}{4}
\]
and from the value of \( \sum_{r=1}^{q-1} \sum_{t=1}^{q/p} t \) in Lemma (2.1) we deduce that
\[
q^{-1} \sum_{r=1}^{q-1} B_1 \left( \frac{rp}{q} \right)^2 = \frac{q-1}{4} - \frac{p^2(q-1)(2q-1)}{6q} + 2p \sum_{t=1}^{q-1} \frac{r}{q} \left( \left\lfloor \frac{rp}{q} \right\rfloor + 1 \right) + \frac{q-1}{4}
\]
and then
\[ 2p (s(p, q) + s(q, p)) = -\frac{q - 1}{4} + \frac{p^2 (q - 1) (2q - 1)}{6q} - \frac{(p - 1)(2pq - 3p + 2q)}{6} + \frac{(q - 1)(2q - 1)}{6q} - \frac{q - 1}{4} \]

Finally
\[ 2p (s(p, q) + s(q, p)) = 1 - \frac{q - 1}{2} \]

Thus
\[ s(p, q) + s(q, p) = \frac{p^2 + q^2 - 3pq + 1}{12pq} \]

3. Proof of Theorem 1.1 and Theorem 1.2

We start this section with some useful preliminaries results.

3.1. finite sums involving fractional part function.

**Lemma 3.1.**

(3.1) \[ \sum_{r=1}^{q-1} \left\{ \frac{rp}{q} \right\} = \frac{q - 1}{2} \]

(3.2) \[ \sum_{r=1}^{q-1} \left\{ \frac{rp}{q} \right\}^2 = \frac{(q - 1)(2q - 1)}{6q} \]

(3.3) \[ \sum_{r=1}^{q-1} \left\lfloor \frac{rp}{q} \right\rfloor = \frac{(p - 1)(q - 1)}{2} \]

(3.4) \[ \sum_{r=1}^{q-1} \left\lfloor \frac{rp}{q} \right\rfloor^2 = \frac{(p^2 + 1)(q - 1)(2q - 1)}{6q} - 2pS(p, q) \]

**Proof.** For the first relation (3.1), let \( \bar{p} \) the inverse modulo \( q \) of \( p \) then

\[ \sum_{r=1}^{q-1} \left\{ \frac{r}{q} \right\} = \sum_{r=1}^{q-1} \left\{ \frac{r\bar{p}}{q} \right\} = \sum_{i=1}^{q-1} \left\{ \frac{ip}{q} \right\} \]

and

\[ \sum_{r=1}^{q-1} \left\lfloor \frac{r}{q} \right\rfloor = \frac{1}{q} \sum_{r=1}^{q-1} r = \frac{q - 1}{2} \]

For the second relation (3.2), we have

\[ \sum_{r=1}^{q-1} \left\lfloor \frac{rp}{q} \right\rfloor^2 = \sum_{r=1}^{q-1} \left\{ \frac{r}{q} \right\}^2 = \frac{1}{q^2} \sum_{r=1}^{q-1} r^2 = \frac{(q - 1)(2q - 1)}{6q} \].
The proof of the relation (3.3) is

\[
\sum_{r=1}^{q-1} \left\lfloor \frac{rp}{q} \right\rfloor = \sum_{r=1}^{q-1} \left( \frac{rp}{q} - \left\{ \frac{rp}{q} \right\} \right)
\]

\[
= \frac{p}{q} \sum_{r=1}^{q-1} r - \sum_{r=1}^{q-1} \left\{ \frac{rp}{q} \right\}
\]

\[
= \frac{p(q-1)}{2} - \frac{q-1}{2}
\]

Finally for the relation (3.4) we have

\[
\sum_{r=1}^{q-1} \left( \frac{rp}{q} \right)^2 = \sum_{r=1}^{q-1} \left( \frac{rp}{q} - \left\{ \frac{rp}{q} \right\} \right)^2
\]

\[
= \sum_{r=1}^{q-1} \left( \frac{r^2p^2}{q^2} - 2\frac{rp}{q} \left\{ \frac{rp}{q} \right\} + \left\{ \frac{rp}{q} \right\}^2 \right)
\]

\[
= \frac{p^2}{q^2} \sum_{r=1}^{q-1} r^2 + \sum_{r=1}^{q-1} \left\{ \frac{rp}{q} \right\}^2 - 2pS(p,q)
\]

\[
= \frac{p^2(q-1)(2q-1)}{6q} + \frac{(q-1)(2q-1)}{6q} - 2pS(p,q)
\]

\[
= \frac{(p^2+1)(q-1)(2q-1)}{6q} - 2pS(p,q)
\]

\[
\square
\]

**Corollary 3.1.**

(3.5) \( S(1,q) = \frac{(q-1)(2q-1)}{6q} \).

(3.6) \( s(p,q) = S(p,q) - \frac{q-1}{4} \).

**Proof.** Since

\[
S(1,q) = \sum_{r=1}^{q-1} \left\{ \frac{r}{q} \right\}^2
\]

then

\[
S(1,q) = \frac{1}{q^2} \sum_{r=1}^{q-1} r^2.
\]
From the relation (1.2) we deduce the result (3.5).

\[ s(p, q) = \sum_{r=1}^{q-1} B_1 \left( \frac{r}{q} \right) B_1 \left( \frac{rp}{q} \right) \]

\[ = \sum_{r=1}^{q-1} \left\{ \frac{r}{q} \right\} \left( \frac{rp}{q} - \frac{1}{2} \right) \]

\[ = S(p, q) - \frac{1}{2} \sum_{r=1}^{q-1} \left\{ \frac{rp}{q} \right\} \]

From the relation (3.1) Lemma 3.1 we deduce that

\[ s(p, q) = S(p, q) - \frac{q-1}{4}. \]

\[ \square \]

3.2. Some properties of the Dedekind sums \( S(p, q) \).

**Lemma 3.2.** For \( p, q \) coprime such that \( p < q \), we have

\[ S(q - p, q) = \frac{q-1}{2} - S(p, q) \]

**Proof.** Using the well known property of the fractional part function for any real \( x \):

\[ \{x\} + \{-x\} = 1 \]

we deduce that

\[ S(q - p, p) = \sum_{r=1}^{q-1} \left\{ \frac{r}{q} \right\} \left\{ \frac{r(q - p)}{q} \right\} \]

\[ = \sum_{r=1}^{q-1} \left\{ \frac{r}{q} \right\} \left\{ -\frac{rp}{q} \right\} \]

\[ = \sum_{r=1}^{q-1} \left\{ \frac{r}{q} \right\} \left( 1 - \left\{ \frac{rp}{q} \right\} \right) \]

\[ = \sum_{r=1}^{q-1} \left\{ \frac{r}{q} \right\} - S(p, q) \]

From the relation (3.1) lemma 3.1 we deduce that

\[ S(q - p, q) = \frac{q-1}{2} - S(p, q) \]

\[ \square \]

The following proposition gives a new expression of \( S(p, q) \) as a sum of three quantities.
Proposition 3.1. For $q > p$ positive coprime numbers, The Euclidean division of $q$ over $p$ gives $q = ap + b$ with $1 \leq b \leq p - 1$. Then we have

\[
S(p, q) = \frac{1}{q^2} \sum_{n=0}^{p-1} \sum_{t=1}^{a} (an + t)(pt - n); \quad \text{if } b = 1
\]

and

\[
S(p, q) = \frac{1}{q^2} \sum_{n=0}^{p-1} \sum_{t=1}^{a} (an + t)(tp - nb) + \frac{1}{q} \sum_{n=0}^{p-1} \sum_{t<\frac{b}{p}} (an + t) + \frac{1}{q^2} \sum_{t=1}^{b-1} (ap + t)(q + pt - pb); \quad \text{if } b \geq 2
\]

Proof.

Since $r$ lies to the set \{1, 2, 3, ..., $q - 1$\} and $q = ap + b$ we can write $r = an + t$ with $0 \leq n \leq p - 1$ and $1 \leq t \leq a$ for $r$ between 1 and $q - b$. And $r = ap + t$ for $1 \leq t \leq b - 1$ when $r$ lies to \{ap + 1, ..., ap + b\}. Then we distingue two cases

b=1: in this case $r$ lies to the set \{1, 2, ..., ap\} and then

\[
S(p, q) = \sum_{r=1}^{q-1} \left\{ \frac{r}{q} \right\} \left\{ \frac{rp}{q} \right\} = \sum_{n=0}^{p-1} \sum_{t=1}^{a} \left\{ \frac{an + t}{q} \right\} \left\{ \frac{p(an + t)}{q} \right\}
\]

One remarks that

\[
p(an + t) = (ap + 1)n + pt - n = qn + pt - n,
\]

\[
1 < pt - n \leq q - 1,
\]

and

\[
1 < an + t \leq q - 1.
\]

Then

\[
\left\{ \frac{p(an + t)}{q} \right\} = \left\{ \frac{pt - n}{q} \right\} = \frac{pt - n}{q}
\]

and we obtain

\[
S(p, q) = \frac{1}{q^2} \sum_{n=0}^{p-1} \sum_{t=1}^{a} (an + t)(pt - n)
\]
In this case we have
$$S(p, q) = \sum_{r=1}^{q-1} \left\{ \frac{r}{q} \right\} \left\{ \frac{rp}{q} \right\}$$
$$= \sum_{r=1}^{p-1} \left\{ \frac{r}{q} \right\} \left\{ \frac{rp}{q} \right\} + \sum_{r=ap+1}^{ap+b-1} \left\{ \frac{r}{q} \right\} \left\{ \frac{rp}{q} \right\}$$
$$= \sum_{n=0}^{p-1} \sum_{t=1}^{a} \left\{ \frac{an+t}{q} \right\} \left\{ \frac{p(an+t)}{q} \right\} + \sum_{r=1}^{b-1} \left\{ \frac{ap+t}{q} \right\} \left\{ \frac{(ap+t)p}{q} \right\}$$

Remark that
$$p(an+t) = (ap+b)n + pt - nb = qn + pt - nb,$$
$$p(ap+t) = qp + pt - pb,$$
$$pt - pb < 0, \ (1 < q + pt - pb < q - 1)$$

and
$$|pt - nb| < q.$$

Furthermore
$$\left\{ \frac{p(an+t)}{q} \right\} = \left\{ \frac{pt - nb}{q} \right\}$$

and
$$\left\{ \frac{p(ap+t)}{q} \right\} = \left\{ \frac{pt - pb}{q} \right\} = \left\{ \frac{q + pt - pb}{q} \right\} = \frac{q + pt - pb}{q}.$$

We deduce that
$$S(p, q) = \sum_{n=0}^{p-1} \sum_{t=1}^{a} \left\{ \frac{an+t}{q} \right\} \left\{ \frac{pt - nb}{q} \right\} + \frac{1}{q^2} \sum_{t=1}^{b-1} (ap+t) (q + pt - pb),$$

but
$$\sum_{n=0}^{p-1} \sum_{t=1}^{a} \left\{ \frac{an+t}{q} \right\} \left\{ \frac{pt - nb}{q} \right\} = \frac{1}{q^2} \sum_{n=0}^{p-1} \left( \sum_{t< \frac{nq}{p}} (an+t)(q+pt-np) + \sum_{t> \frac{nq}{p}} (an+t)(pt-nb) \right)$$
$$= \frac{1}{q^2} \sum_{n=0}^{p-1} \left( \sum_{t< \frac{nq}{p}} (an+t)(q+pt-np) + \sum_{t> \frac{nq}{p}} (an+t)(pt-nb) - \sum_{t< \frac{nq}{p}} (an+t)(pt-nb) \right)$$
$$+ \frac{1}{q^2} \sum_{n=0}^{p-1} \sum_{t=1}^{a} (an+t)(pt - nb)$$
$$= \frac{1}{q^2} \sum_{n=0}^{p-1} \sum_{t=1}^{a} (an+t)(pt - nb) + \frac{1}{q} \sum_{t=1}^{p-1} \sum_{n=0}^{\frac{tq}{p}} (an+t).$$

Then the result follows. □
The following lemma computes the three sums in the relation (3.9) of the Proposition 3.1.

**Lemma 3.3.**

\[
\begin{align*}
\sum_{n=0}^{p-1} \sum_{t=1}^{a} (an + t) (tp - nb) &= -\frac{1}{6p} (q - b)^2 b (p - 1) (2p - 1) \\
&\quad + \frac{1}{4p} (q - 2b) (q - b) (p - 1) (p + q - b) \\
&\quad + \frac{1}{6p} (q - b) (p + q - b) (2q + p - 2b)
\end{align*}
\]

\[
\sum_{n=0}^{p-1} \sum_{t=1}^{nb} (an + t) = \frac{1}{12p} (p - 1) (2p - 1) (2qb - b^2 + 1) + \frac{(p - 1) (b - 1)}{4} - qS (q, p)
\]

\[
\sum_{t=1}^{b-1} (ap + t) (q + pt - pb) = (q - b) (q - pb) (b - 1) + \frac{b (ap + q - 2pb) (b - 1)}{2} + \frac{pb (b - 1) (2b - 1)}{6}
\]

**Proof.** For the relation (3.10) we have

\[
\begin{align*}
(an + t) (tp - nb) &= -ab n^2 + (pa - b) nt = pt^2 = -ab n^2 + (q - 2b) nt + pt^2
\end{align*}
\]

and

\[
\begin{align*}
\sum_{t=1}^{a} t &= \frac{a (a + 1)}{2} \\
\sum_{t=1}^{a} t^2 &= \frac{a (a + 1) (2a + 1)}{6}
\end{align*}
\]

Thus

\[
\sum_{t=1}^{a} (an + t) (tp - nb) = -ba^2 n^2 + \frac{a (a + 1) (q - 2b) n}{2} + \frac{ap (a + 1) (2a + 1)}{6}
\]

and

\[
\begin{align*}
\sum_{n=0}^{p-1} \sum_{t=1}^{a} (an + t) (tp - nb) &= -ba^2 p (p - 1) (2p - 1) \\
&\quad + \frac{ap (a + 1) (q - 2b) (p - 1)}{4} \\
&\quad + \frac{ap^2 (a + 1) (2a + 1)}{6}
\end{align*}
\]

then
\[
\sum_{n=0}^{p-1} \sum_{t=1}^{a} (an + t) (tp - nb) = -\frac{b (ap)^2 (p - 1) (2p - 1)}{6p} + \frac{ap(ap + p) (q - 2b) (p - 1)}{4p} + \frac{ap(ap + p) (2ap + p)}{6p}
\]

Since \(ap = q - b, ap + p = q + p - b\) and \(2ap + p = 2q + p - 2b\), we obtain
\[
\sum_{n=0}^{p-1} \sum_{t=1}^{a} (an + t) (tp - nb) = -\frac{b (q - b)^2 (p - 1) (2p - 1)}{6p} + \frac{(q - b)(q + p - b) (q - 2b) (p - 1)}{4p} + \frac{(q - b)(q + p - b) (2q + p - 2b)}{6p}
\]

For the second relation (3.11) we have
\[
\sum_{n=0}^{p-1} \sum_{t<\frac{nb}{p}} (an + t) = a \sum_{n=0}^{p-1} n \left\lfloor \frac{nb}{p} \right\rfloor + \sum_{n=0}^{p-1} \sum_{t<\frac{nb}{p}} t
\]
\[
= a \sum_{n=0}^{p-1} n \left\lfloor \frac{nb}{p} \right\rfloor + \frac{1}{2} \sum_{n=0}^{p-1} \frac{nb}{p} \left( \left\lfloor \frac{nb}{p} \right\rfloor + 1 \right)
\]

But
\[
\sum_{n=0}^{p-1} \frac{nb}{p} \left( \left\lfloor \frac{nb}{p} \right\rfloor + 1 \right) = \sum_{n=0}^{p-1} \frac{nb}{p} + \sum_{n=0}^{p-1} \frac{nb}{p}^2
\]
\[
= \frac{(p - 1)(b - 1)}{2} + \frac{(b^2 + 1)(p - 1)(2p - 1)}{6p} - 2bS(b, p)
\]
and
\[
\sum_{n=0}^{p-1} n \left\lfloor \frac{nb}{p} \right\rfloor = \frac{b (p - 1) (2p - 1)}{6} - pS(b, p)
\]

Since
\[
\sum_{n=0}^{p-1} \sum_{t<\frac{nb}{p}} t = \frac{1}{2} \sum_{n=0}^{p-1} \frac{nb}{p} \left( \left\lfloor \frac{nb}{p} \right\rfloor + 1 \right)
\]
and
\[
S(b, p) = S(q, p)
\]
then

$$\sum_{n=0}^{p-1} \sum_{t<\frac{b}{p}} (an + t) = \frac{1}{12p} (p-1)(2p-1)(2qb-b^2+1) + \frac{(p-1)(b-1)}{4} - qS(q,p)$$

For the last one (3.12), we have

$$\sum_{t=1}^{b-1} (ap + t)(q + pt - pb) = \sum_{t=1}^{b-1} (ap(q - pb) + (ap^2 + q - pb)t + pt^2)$$
$$= \sum_{t=1}^{b-1} ((q - b)(q - pb) + (qp + q - 2pb)t + pt^2)$$
$$= (q - b)(q - pb)(b - 1) + \frac{b(qp + q - 2pb)(b - 1)}{2} + \frac{pb(b - 1)(2b - 1)}{6} \Box$$

The substitution of the relations (3.10), (3.11) and (3.12) of Lemma 3.3 in the relation (3.9) of the proposition 3.1 conduct to the following result.

**Corollary 3.2.**

(3.13)

$$S(p, q) + S(q, p) = -\frac{b(q - b)^2(p - 1)(2p - 1)}{6pq^2}$$
$$+ \frac{(q - b)(q + p - b)(q - 2b)(p - 1)}{4pq^2}$$
$$+ \frac{(q - b)(q + p - b)(2q + p - 2b)}{6pq^2}$$
$$+ \frac{(q - b)(q - pb)(b - 1)}{q^2} + \frac{b(qp + q - 2pb)(b - 1)}{2q^2} + \frac{pb(b - 1)(2b - 1)}{6q^2} + \frac{1}{12pq} (p - 1)(2p - 1)(2qb - b^2 + 1) + \frac{(p - 1)(b - 1)}{4q}$$

**3.3. Proof of Theorem 1.1.** In the case b=1. Taking b = 1 in the relation (3.10) Lemma 3.3 we obtain

$$\sum_{n=0}^{p-1} \sum_{t=1}^{a} (an + t)(tp - n) = -\frac{1}{6p} (q - 1)^2(p - 1)(2p - 1)$$
$$+ \frac{1}{4p} (q - 2)(q - 1)(p + q - 1)$$
$$+ \frac{1}{6p} (q - 1)(p + q - 1)(2q + p - 2),$$
and then

\[
S(p, q) = -\frac{1}{6pq^2} (q - 1)^2 (p - 1) (2p - 1) + \frac{1}{4pq^2} (q - 2) (q - 1) (p - 1) (p + q - 1) + \frac{1}{6pq^2} (q - 1) (p + q - 1) (2q + p - 2).
\]

Since

\[S(q, p) = S(1, p)\]

and from the relation (3.5) Corollary 3.1, we deduce that

\[S(q, p) = \frac{(p - 1) (2p - 1)}{6p}.\]

Then

(3.14)

\[
S(p, q) + S(q, p) = \frac{1}{6pq^2} (p - 1) (2p - 1) (2q - 1) + \frac{1}{4pq^2} (q - 2) (q - 1) (p - 1) (p + q - 1) + \frac{1}{6pq^2} (q - 1) (p + q - 1) (2q + p - 2).
\]

But

(3.15)

\[(2q - 1) (p - 1) (2p - 1) = 4p^2 - 2p^2 - 6pq + 3p + 2q - 1\]

(3.16)

\[(q - 2) (q - 1) (p - 1) (p + q - 1) = p^2q^2 + pq^3 - 3p^2q - 5pq^2 - q^3 + 4q^2 + 2p^2 + 8pq - 4p - 5q + 2\]

(3.17)

\[(q - 1) (p + q - 1) (2q + p - 2) = 2q^3 + p^2q + 3pq^2 - 6q^2 - p^2 - 6pq + 3p + 6q - 2\]

Substitute the relations (3.15), (3.16) and (3.17) in the relation (3.14) we get the result.

Case \(b \geq 2\)

The decomposition of the elements of the expression (3.13) of \(S(p, q) + S(q, p)\) in Corollary 3.2, conduct to

\[b(q - b)^2 = b^3 - 2qb^2 + q^2b\]

\[(q - b)(p + q - b)(q - 2b) = -2b^3 + (2p + 5q)b^2 - (3pq + 4q^2)b + pq^2 + q^3\]

\[(q - b)(p + q - b)(2q + p - 2b) = -2b^3 + (3p + 6q)b^2 - (p^2 + 6pq + 6q^2)b + p^2q + 3pq^2 + 2q^3\]

\[(q - b)(q - pb)(b - 1) = pb^3 - (pq + q + p)b^2 + (q^2 + pq + q)b - q^2\]
\[ b(b - 1)(pq + q - 2pb) = -2pb^3 + (pq + 2p + q)b^2 - (q + pq)b \]
\[ b(b - 1)(2b - 1) = 2b^3 - 3b^2 + b. \]

Then \( S(p, q) + S(q, p) \) is the polynomial
\[ P(b) = a_0b^3 + a_1b^2 + a_2b + a_3 \]
of degree 3; and its coefficients are

\[ a_0 = -\frac{(p - 1)(2p - 1)}{6pq^2} - \frac{p - 1}{2pq} - \frac{1}{3pq^2} + \frac{p}{q^2} - \frac{p}{3q^2} \]

\[ a_1 = \frac{q(p - 1)(2p - 1)}{3pq} + \frac{(p - 1)(2p + 5q)}{4pq^2} + \frac{3p + 6q}{2pq^2} + \frac{pq + p + q}{q^2} + \frac{pq + 2p + q}{2q^2} + \frac{p}{2q^2} \frac{(p - 1)(2p - 1)}{12pq} \]

\[ a_2 = -\frac{(p - 1)(2p - 1)}{6pq} \frac{q}{4pq^2} + \frac{(p - 1)(3pq + 4q^2)}{6pq^2} + \frac{pq + pq}{q^2} - \frac{pq}{2q^2} + \frac{p}{6q^2} + \frac{(p - 1)(2p - 1)}{6pq} \]

\[ a_3 = \frac{(p - 1)(pq^2 + q^3)}{4pq^2} + \frac{p^2q + 3pq^2 + 2q^3}{6pq^2} - 1 + \frac{(p - 1)(2p - 1)}{12pq} - \frac{p - 1}{4q} \]

\[ = \frac{1}{12pq} \left[ 3(p - 1)(pq + q^2) + 2(p^2 + 3pq + 2q^3) + (p - 1)(2p - 1) - 3p(p - 1) \right] - 1. \]

Then we obtain

\[ a_0 = a_1 = a_2 = 0 \]

and

\[ a_3 = \frac{p^2 + q^2 + 1}{12pq} + \frac{p + q}{4} - \frac{3}{4} \]

Furthermore the result follows.

### 3.4. Proof of Theorem 1.2 and Corollary 1.1.

To prove the Theorem 1.2 we need the following lemma

**Lemma 3.4.**

\[ S(p, 2) = \frac{1}{4} \]  
\[ S(p, 3) = \frac{1}{3} \left( 2 - \left\{ \frac{p}{3} \right\} \right) \]  
\[ S(p, 4) = 1 - \frac{1}{2} \left\{ \frac{p}{4} \right\} \]
Proof. The first relation (3.18) is trivial. For the others, first we remark that if \( q \equiv b \mod{p} \) then \( q = ap + b \) and \( p = a + \frac{b}{p} \), since \( 0 \leq b \leq p - 1 \) we deduce that \( \left\lfloor \frac{q}{p} \right\rfloor = a \) and then \( b = q - p \left\lfloor \frac{q}{p} \right\rfloor \) thus
\[
 b = p \left\{ \frac{q}{p} \right\}
\]

For the second relation (3.19), since \( p \equiv 1 \) or \( 2[3] \), we obtain
\[
 S(1, 3) = \frac{5}{9} \quad \text{and} \quad S(2, 3) = \frac{4}{9}
\]
We remark for \( b \in \{1, 2\} \) that
\[
 S(p, 3) = S(b, 3) = \frac{6 - b}{9}
\]
, since
\[
 b = 3 \left\{ \frac{p}{3} \right\}
\]
then
\[
 S(p, 3) = 2 \frac{1}{3} \left\{ \frac{p}{3} \right\}
\]
For the third relation (3.20), since \( p \equiv 1 \) or \( 3[4] \), we obtain
\[
 S(1, 4) = \frac{7}{8} \quad \text{and} \quad S(3, 4) = \frac{5}{8}
\]
We remark for \( b \in \{1, 3\} \) that
\[
 S(p, 4) = S(b, 4) = \frac{8 - b}{8}
\]
and then
\[
 S(p, 4) = 1 - \frac{1}{2} \left\{ \frac{p}{4} \right\}
\]
\[\square\]

Corollary 3.3. Let \( p \) any integer, then we have
if \( (p, 2) = 1 \) then
\[
 S(2, p) = \frac{7p}{24} + \frac{5}{24p} - \frac{1}{2}
\]
(3.21)
if \( (p, 3) = 1 \) then
\[
 S(3, p) = \frac{5p}{18} + \frac{5}{18p} + \frac{1}{3} \left\{ \frac{p}{3} \right\} - \frac{2}{3}
\]
(3.22)
and if \( (p, 4) = 1 \):
\[
 S(4, p) = \frac{13p}{48} + \frac{17}{48p} + \frac{1}{2} \left\{ \frac{p}{4} \right\} - \frac{3}{4}
\]
(3.23)
Proof. For the first relation (3.21), applying the reciprocity law (1.6) for \( p \) and 2 we get

\[
S(p, 2) + S(2, p) = \frac{p^2 + 5}{24p} + \frac{p + 2}{4} - \frac{3}{4}
\]

From the relation (3.18) Lemma 3.4 we deduce that

\[
S(2, p) = \frac{p^2 + 5}{24p} + \frac{p + 2}{4} - \frac{3}{4} = \frac{7p}{24} + \frac{5}{24} - \frac{1}{2}
\]

We do the same thing for the second and the third relation. \( \square \)

3.4.1. Proof of Theorem 1.2. The relation (1.8) is the consequence of the reciprocity law (1.6) and the expression (3.5) of \( S(1, q) \) in Corollary 3.1. To obtain the other relations we must combine the reciprocity law (1.6) and the respective results in Lemma 3.4.

3.4.2. Proof of the Corollary 1.1. The corollary 1.1 is the consequence of the Theorem 1.2 and the Corollary 3.3.

References

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