Invariantly Admissible Policy Iteration for a Class of Nonlinear Optimal Control Problems

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Received: August, 2013

Abstract In this paper, we propose a generalized successive approximation method (SAM), called invariantly admissible policy iteration (PI), for finding the solution to a class of input-affine nonlinear optimal control problems by iterations. Unlike the existing SAM, the proposed method updates the domain of the next control and value function for admissibility (and invariance). In the existing SAM, the admissibility of the generated controls are guaranteed under the two implicit assumptions regarding Lyapunov’s theorem and invariance, both of which are presented and discussed in this paper and are generally not true. On the contrary, the proposed invariantly admissible PI guarantees the admissibility in a more refined manner, without such assumptions. The admissibility and invariance of the updated region, with respect to the corresponding controls, are mathematically prove under the specific invariant admissible update rule. We also provide monotonic decreasing and uniform convergence properties of the sequence of value functions under certain conditions. Finally, numerical simulations are presented to illustrate the proposed PI method and its effectiveness.

Keywords policy iteration · successive approximation · admissible control · nonlinear optimal control · nonlinear systems

AMS classifications: 93D30, 49L20, 49J15, 34K35, 93C10.

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1 Introduction

In nonlinear optimal control problems, it is well-known that the optimal solution is directly related to the solution of the underlying Hamilton-Jacobi-Bellman (HJB) equation [11,12,13]. However, solving the HJB equation has been a formidable task until recently; hence, many of the numerical algorithms have been proposed for efficiently calculating the solution to the HJB equation [14,15,16,17,18,19].

Among such numerical algorithms, the successive approximation method (SAM) given in [14,15,16,17] has provided one basic idea of recursively solving the HJB equation. The algorithm starts with an initial admissible control; during the recursions of the method, the agent finds the value function associated with the current control (policy evaluation), and then the control is updated using this associated value function (policy improvement). A class of algorithms using this idea is called policy iteration (PI), and many researchers have studied this idea in various ways and proposed their own algorithms from the perspectives of optimal control, adaptive (neuro-) dynamic programming, and reinforcement learning [14,15,16,17,18,19,20].

The PI method focused on in this paper is the SAM given by Beard, Saridis, and Wen [14,15,16], which can be considered the infinite-horizon special case of the SAM given by Leake and Liu [13], and becomes Newton method [22] in the case of linear quadratic regulation (LQR). Note that many of the PI methods were also developed within the same optimal control framework as the SAM [14,15,16], and ideally all of them generate the same sequences of value functions and controls [11,12,15,18,20]. In other words, those PI algorithms can be considered the equivalents, and hence can be indirectly studied by analyzing the SAM of Beard et al. [14,15,16] as a representative.

The admissibility of the controls generated by the SAM [14,15,16] is the motivation of this paper. Here, the admissibility of a control roughly implies that the control asymptotically stabilizes the system, and guarantees the finite value function on the domain of interest. In Theorem 5.3.1 in [14], it was stated that the controls generated by the SAM [14,15,16] are all admissible on the domain, and the sequence of the associated value functions is monotonically decreasing and converges to the optimal one, implying the improvement of the control up to the optimal one. The proof was conducted based on Lemma 5.2.4 in [14], which states the admissibility of the updated control and the pointwisely monotonic decreasing property of the associated value functions. However, the related Lyapunov’s theorem (Theorem 3.13 in [14]) used in its proof for the infinite-horizon case implicitly assumed that the domain of the Lyapunov function is a subset of the stabilizing region, and that the state trajectory generated by the nonlinear dynamics remains in that Lyapunov domain, so its existence is guaranteed for all future time. The problem here is that both implicit assumptions on the Lyapunov domain are not true in general, as discussed in this paper (see also Chapters 4.1, 4.2, and 8.2 in [23], and Theorem 3.3 in [23]). To the best authors’ knowledge, this problem does not happen only in the case of LQR since the stabilizing region becomes the entire \( \mathbb{R}^n \)-space and the state trajectory always exists for all time.

To solve the aforementioned admissibility problem related to the nonlinear SAM [14,15,16], this paper proposes a generalized SAM called invariantly admissible PI, which has an additional process to properly update the next admissible invariant re-
region after each policy improvement step. For this, we refine and generalize the notion of an admissible control (or policy) given in [7, 8, 9, 12]. Then, an invariantly admissible control is precisely defined with detailed discussions on its necessity, the relevant Lyapunov’s theorem, and the value functions for the underlying optimal control problem. From the discussions, a specific update rule for the invariantly admissible region in the proposed PI is presented. Without the aforementioned two implicit assumptions related to the Lyapunov’s theorem (Theorem 3.13 in [7]), it is proven in this paper that the next region generated by the update rule is invariant and admissible for the current and next controls, and the sequence of corresponding value functions is monotonically decreasing. The conditions for convergence to the optimal solution is also provided with detailed discussions. Finally, numerical simulations are presented to illustrate the proposed PI method and its effectiveness.

2 Notations and Mathematical Terminology

\( \mathbb{R}_+ \) denotes the set of all nonnegative real numbers, i.e., \( \mathbb{R}_+ = [0, \infty] \); the set of all \( n \times 1 \) real vectors and \( n \times m \) real matrices are denoted by \( \mathbb{R}^n \) and \( \mathbb{R}^{n \times m} \), respectively; \((\cdot)^T\) is the matrix transpose; \( \| \cdot \| \) denotes a norm on a vector space \( \mathbb{R}^n \). Throughout the paper, \( \Omega \) (resp. \( \Omega \)) denotes a subset of (resp. an invariant subset of) the given domain \( D \subseteq \mathbb{R}^n \) of the nonlinear dynamics. Here, the over-bar in \( \Omega \) means that it could be a compact set for some nice properties. The boundary of a subset \( \Omega \) is denoted by \( \partial \Omega \).

Note that all the mathematical notations including those given below will be clear and be precisely defined in this paper.

\( A(\Omega) \): the set of all controls that are admissible on a subset \( \Omega \);
\( A^I(\Omega) \): the set of all invariantly admissible controls on a subset \( \Omega \);
\( C^0(\Omega) \): the set of all continuous functions on a domain \( \Omega \);
\( C^1(\Omega) \): the set of all continuously differentiable functions on a domain \( \Omega \);
\( B_0(r) \): the closed ball in \( \mathbb{R}^n \) with radius \( r \). That is, \( B_0(r) := \{ x \in \mathbb{R}^n : \| x \| \leq r \} \);
\( \bar{R}_A(\mu) \): the region of attraction of the closed-loop system \( \dot{x} = f + g\mu \) in \( D \);

\( V \): a Lyapunov function for an asymptotically stable closed-loop system.
\( V^\mu \): a value function for an (invariantly) admissible control \( \mu \);
\( \nabla V^\mu \): the gradient column vector of a value function \( V^\mu \);
\( V^* \): the optimal value function;
\( \mu^* \): a control \( u = \mu(x) \) for the nonlinear system.
\( \mu \): the optimal control;

\( \Omega : \) the compact subset of the domain \( \Omega \) defined with \( \Omega : \Omega \to \mathbb{R}_+ \) by (5);
\( \bar{\Omega}^\mu : \) the compact subset of the domain \( \Omega \) defined with \( \bar{\Omega}^\mu : \Omega \to \mathbb{R}_+ \) by (7);
\( \Omega^* : \) the domain of \( V^* \) on which \( V^* \) is \( C^1 \) and satisfies the HJB equation.

The notations related to the invariantly admissible PI are summarized as follows.

\( \mu_i \): the updated control at \( i \)-th iteration;
\( V^\mu_i \): the value function for the control \( \mu_i \) obtained at \( i \)-th iteration;
\( \hat{V} \): the limit function to which \( \{ V^\mu_i \} \) converges;
\( \bar{\Omega}_{\mu_i}^\mu : \) a compact set defined as \( \bar{\Omega}_{\mu_i}^\mu := \{ x \in \mathbb{R}^n : V^\mu_i(x) \leq c_i \} \);
\( \Omega_i \): the updated region at \( i \)-th iteration such that \( \Omega_i \subseteq \Omega_{i-1} \);
\( \Omega \): the limit set of \( \Omega_i \) defined as \( \Omega = \bigcap_{i=0}^{\infty} \Omega_i \);
\( C_1^A(\Omega) \): the set of all continuously differentiable value functions \( V^\mu \) for \( \mu \in A(\Omega) \);

**Terminology.** All the subsets in \( \mathbb{R}^n \) (or in \( D \)) presented in this paper are assumed to contain a neighborhood of the origin, and without loss of generality, have no isolated region or point from the origin. Using the above notations, a positive definite (resp. negative definite) function is precisely defined as

**Definition 2.1** A function \( V : \Psi \to \mathbb{R}_+ \), where the domain \( \Psi \) is a subset of \( \mathbb{R}^p \) for some \( p \in \{1, 2, \cdots\} \) containing a neighborhood of the origin, is said to be positive definite (resp. negative definite) on \( \Psi \) if and only if it is continuous on \( \Psi \), \( V(0) = 0 \), and \( V(x) > 0 \) (resp. \( V(x) < 0 \)) for all \( x \in \Psi \setminus \{0\} \).

### 3 Preliminaries: Invariant Admissibility/Nonlinear Optimal Control Problems

In this paper, we consider the infinite-horizon nonlinear optimal control problem (1)–(2) for the following continuous-time nonlinear system for time \( t \in \mathbb{R}_+ \)

\[
\dot{x}(t) = f(x(t)) + g(x(t))u(x(t)), \quad x(0) = x_0 \in D \subseteq \mathbb{R}^n,
\]

where

\( x : \mathbb{R}_+ \to \mathbb{R}^n \): the system state for time \( t \in \mathbb{R}_+ \);
\( u : \mathbb{R}^n \to \mathbb{R}^m \): the control input function governed by a given control (or policy) \( \mu(x) \) (see Definition 3.1);
\( D \subseteq \mathbb{R}^n \): the domain of \( f \) and \( g \) containing a neighborhood of the origin;
\( f : D \to \mathbb{R}^n \): a given locally Lipschitz continuous nonlinear function that satisfies \( f(0) = 0 \);
\( g : D \to \mathbb{R}^n \times m \): a given locally Lipshitz continuous nonlinear function,

and the performance measure

\[
J(x_0, u(\cdot)) = \int_0^\infty r(\phi(\tau; x_0, u), u(\phi(\tau; x_0, u))) \, d\tau,
\]

where

- \( \phi(\tau; x_0, u) \): the state trajectory \( x(\tau) \) at time \( \tau \in \mathbb{R}_+ \) generated by (1) with the initial condition \( x_0 \in D \) and a given control \( u = \mu(x) \);
- \( r : D \times \mathbb{R}^m \to \mathbb{R}_+ \);
  the given positive definite cost function on \( D \times \mathbb{R}^m \) defined as
  \[
  r(x, u) := Q(x) + u^T R u
  \]
  for a positive definite function \( Q : D \to \mathbb{R}_+ \) on \( D \) and a positive definite matrix \( R \in \mathbb{R}^{m \times m} \).

Here, the notion of a control \( u = \mu(x) \) for the system (1) is precisely defined as follows.
Definition 3.1 A function $\mu : \Omega \rightarrow \mathbb{R}^m$ on a subset $\Omega \subseteq \mathcal{D}$ is said to be a control (on $\Omega$) if and only if $\mu$ is continuous (on $\Omega$) and satisfies $\mu(0) = 0$.

Note that the nonlinear dynamics [1], which has the origin ‘0’ as an equilibrium, can be regarded as the general description of the systems such as feedback linearizable systems [23], strict feedback systems [23], bilinear systems [7], and many practical nonlinear systems [24], all of which can be stabilized by a continuous feedback control $u = \mu(x)$ for the equilibrium ‘0’. For the existence of the solution $\phi(t; x_0, \mu)$ of the nonlinear dynamics [1] $\forall t \geq 0$, we assume that

Assumption 3.1 For any given control $u = \mu(x)$, $f(x) + g(x)\mu(x)$ is locally Lipschitz continuous on the domain $\mathcal{D}$.

If the control $\mu(x)$ is continuously differentiable on $\mathcal{D}$, i.e., $\mu \in C^1(\mathcal{D})$, then it can be easily shown that $f(x) + g(x)\mu(x)$ is locally Lipschitz continuous on $\mathcal{D}$, so Assumption 3.1 holds. In this paper, Assumption 3.1 suffices for the analysis, and we do not assume such a strict differentiability assumption on $\mu(x)$. Next, we precisely define a feasible trajectory and a stabilizing control on a given subset $\Omega$ of $\mathcal{D}$.

Definition 3.2 (Feasible trajectory.) For a given control $u = \mu(x)$, the state trajectory $\phi(t; x_0, \mu)$ is said to be feasible on a subset $\Omega \subseteq \mathcal{D}$ if and only if

$$x_0 \in \Omega \text{ implies } \phi(t; x_0, \mu) \in \mathcal{D} \text{ for all } t \geq 0.$$  \hspace{1cm} (3)

Definition 3.3 (Stabilizing control.) A control $u = \mu(x)$ is said to asymptotically stabilize the system $(f, g)$ on $\Omega \subseteq \mathcal{D}$ (or stabilizing on $\Omega$) if and only if

1. $\phi(t; x_0, \mu)$ exists for all initial condition $x_0 \in \Omega$ and all $t \geq 0$;
2. the equilibrium ‘0’ of the resulting closed-loop system $\dot{x} = f + g\mu$ is stable;
3. for all $x_0 \in \Omega$, $\lim_{t \to \infty} \phi(t; x_0, \mu) = 0$.

For a given stabilizing control $u = \mu(x)$, the region of attraction of the closed-loop system $\dot{x} = f(x) + g(x)\mu(x)$ is defined as

$$R_A(\mu) := \{ x_0 \in \mathcal{D} : \phi(t; x_0, \mu) \to 0 \text{ as } t \to \infty \};$$

Similarly, we define the value function $V^\mu(x_0)$ for $x_0 \in \mathcal{D}$, if it exists, as

$$V^\mu(x_0) := J(x_0, u(\cdot))|_{u=\mu(x)}.$$  

Since $Q(0) = 0$, $\mu(0) = 0$, and $\phi(t; x_0, \mu)|_{x_0=0} = 0$ for all $t \geq 0$, we have $V^\mu(0) = 0$. So, by the positive definiteness of $\tau(x, u)$ on $\mathcal{D} \times \mathbb{R}^m$, $V^\mu$ is always positive definite on its domain. Using Definitions 3.1-3.3, the notion of an admissible control (or policy) given by Beard et al. [8] for the existence of $V^\mu$ can be re-defined in a refined, generalized manner as follows.

Definition 3.4 (Admissible control.) A control (or policy) $u = \mu(x)$ is admissible on a subset $\Omega \subseteq \mathcal{D}$, denoted by $\mu \in \mathcal{A}(\Omega)$, if and only if

1. $u = \mu(x)$ asymptotically stabilizes the system $(f, g)$ on $\Omega$;
2. $\phi(t; x_0, \mu)$ is feasible on $\Omega$;
3. $V^\mu(x_0) < \infty$, $\forall x_0 \in \Omega$.

For the nonlinear dynamics (1), we assume the existence of an admissible control.

**Assumption 3.2** There exist a control $u = \mu(x)$ and a subset $\Omega \subseteq D$ for the nonlinear system (1) such that $\mu \in A(\Omega)$.

Note that $\mu \in A(\Omega)$ implies that $\mu$ is stabilizing on $\Omega$, and thereby, $\Omega \subseteq R_A(\mu)$. Compared with [8], the concept of admissibility in Definition 3.4 is refined and slightly generalized. First, it is defined on a subset $\Omega$ of $D$, so contains the previous definition as a special case 

$\Omega = D$ [8]; second, we assume that $\phi(t; x_0, \mu)$ $(t \geq 0)$ is feasible on $\Omega$, so $\phi(t; x_0, \mu)$ remains in the domain $D$ for all $t \geq 0$ and all $x_0 \in \Omega$. This condition is guaranteed if $D$ contains $R_A(\mu)$ or its invariant subset containing $\Omega$. However, such a domain $D$ is hard to determine (or even impossible) unless $D = R^n$ since both $R_A(\mu)$ and its invariant subset depend on the control $\mu$, and hence so does the determination of $D$. Therefore, instead of imposing such an unrealistic assumption on $D$, we introduce the concept of invariant admissibility as follows:

**Definition 3.5 (Invariantly admissible control.)** A control $u = \mu(x)$ is invariantly admissible on a subset $\bar{\Omega} \subseteq D$ containing a neighborhood of the origin, denoted by $\mu \in A_I(\bar{\Omega})$, if and only if

1. $\mu \in A(\bar{\Omega})$;
2. $\bar{\Omega}$ is invariant under the control $\mu$, i.e.,

$$\text{if } x_0 \in \bar{\Omega}, \text{ then } \phi(t; x_0, \mu) \in \bar{\Omega} \text{ for all } t \geq 0.$$  (4)

**Proposition 3.1** $\mu \in A_I(\bar{\Omega})$ implies $\mu \in A(\bar{\Omega})$.

Note that the invariance condition (4) in Definition 3.5 replaces the feasibility condition (3) in Definition 3.2 By Theorem 3.3 in [23] and Assumption 3.1, the invariance (4) also guarantees the existence of the unique solution $\phi(t; x_0, \mu)$ for all $x_0 \in \Omega$ and all $t \geq 0$ if $\Omega$ is compact. Related to these observations and invariant admissibility, we look inside a variant of the (local) Lyapunov's theorem for asymptotic stability (Theorem 4.1 in [23]) on a compact set $\bar{\Omega}_c$, defined as

$$\bar{\Omega}_c = \{x \in R^n : V(x) \leq c\},$$  (5)

where $V : \Omega \to R_+$ is a Lyapunov function for an asymptotically stable closed-loop system $\dot{x} = f(x) + g(x)\mu(x)$ on a domain $\Omega \subseteq D$, and $c$ is a constant determined in such a way that $\bar{\Omega}_c$ is contained by $\Omega$, i.e., $\bar{\Omega}_c \subseteq \Omega$.

**Theorem 3.1** For a subset $\Omega \subseteq D$, if there exists a function $V : \Omega \to R_+$ such that $V$ is positive definite on $\Omega$, $V \in C^1(\Omega)$, and $\dot{V} = \langle \partial V/\partial x \rangle^T(f + g\mu)$ is negative definite on $\Omega$, then

1. $u = \mu(x)$ asymptotically stabilizes the system $(f, g)$ on $\bar{\Omega}_c$;
2. $x_0 \in \bar{\Omega}_c$ implies $\phi(t; x_0, \mu) \in \bar{\Omega}_c \forall t \geq 0$. 

Proof See Theorem 4.1 in [23] and its proof. □

Theorem 3.1 provides an asymptotically stable invariant region $\bar{\Omega}_c$, which is a compact invariant subset of $R_A(\mu)$. On this invariant region $\bar{\Omega}_c$, existence and feasibility of the unique solution $\phi(t; x_0, \mu)$ are guaranteed. Therefore, if $V^\mu(x) < \infty$ holds for all $x \in \bar{\Omega}_c$, then the conditions in Theorem 3.1 imply $\mu \in A_\mathbb{Z}(\bar{\Omega}_c)$.

Remark 3.1 $\Omega \setminus \bar{\Omega}_c$ may not be a stabilizing region since $\Omega \setminus \bar{\Omega}_c \subseteq R_A(\mu)$ is not guaranteed. So, $\phi(t; x_0, \mu)$ for some $x_0 \in \Omega \setminus \bar{\Omega}_c$ may leave the domain $\mathcal{D}$ and even may diverge to $\infty$. In this situation, $\dot{V}(x) < 0$ and even the existence of $\phi(t; x_0, \mu)$ ($t \geq 0$) are not guaranteed (see Section 8.2 in [23] for more discussions). □

If $V^\mu \in C^1(\Omega)$, then it satisfies the Lyapunov equation for the system (1):

$$\nabla^T V^\mu(x) \cdot (f(x) + g(x)\mu(x)) = -r(x, \mu(x)), \quad \forall x \in \Omega,$$

(6)

which is the infinitesimal version of (2) and implies $\dot{V}^\mu(x) = -r(x, \mu) < 0$ along the trajectory $\phi(t; x_0, \mu)$. In this case, since $V^\mu$ is positive definite on its domain, (6) guarantees that $V^\mu$ is a Lyapunov function for the closed-loop system $\dot{x} = f + g\mu$ satisfying the conditions in Theorem 3.1. This provides the following converse lemma of Proposition 3.1 on a compact subset $\bar{\Omega}_c^\mu$ of $\bar{\Omega}_c$ defined similarly to $\bar{\Omega}_c$ by

$$\bar{\Omega}_c^\mu := \{x \in \mathbb{R}^n : V^\mu(x) \leq c\},$$   
(7)

where $c > 0$ is chosen such that $\bar{\Omega}_c^\mu$ is contained by the domain $\bar{\Omega} \subseteq \mathcal{D}$ of $V^\mu$, i.e., $\bar{\Omega}_c^\mu \subseteq \bar{\Omega} \subseteq \mathcal{D}$.

Lemma 3.1 If $\mu \in A(\Omega)$ and $V^\mu \in C^1(\Omega)$, then $\mu \in A_\mathbb{Z}(\bar{\Omega}_c^\mu)$.

Proof Apply Theorem 3.1 with the Lyapunov function $V = V^\mu$ satisfying (6) $\forall x \in \Omega$. □

Remark 3.2 The existence of the unique continuously differentiable $V^\mu$ on a subset of $R_A(\mu)$ is guaranteed under certain conditions, for example, if:

1. $Q(x)$ has second partial derivatives that are continuous, and all the real parts of the eigenvalues of $\nabla (f + g\mu)|_{x=0}$ are negative [24] (see also Theorem 3 in [25]);
2. the functions $\mu(x)$ and $Q(x)$ are continuously differentiable, and $\mu$ is admissible on the domain (Lemma 3.1.6 in [7]);
3. the functions $f(x)$, $g(x)$, and $Q(x)$ are all smooth on the domain, and all the real parts of the eigenvalues of $\nabla f(x)|_{x=0}$ are negative [15][21]. □

The next lemma is a refined, generalized version of Lemma 3.1.9 in [7], and states that the admissibility is preserved in a feasible stabilizing region.

Lemma 3.2 Assume $\mu \in A(\Omega)$ for a subset $\Omega \subseteq \mathcal{D}$ containing a neighborhood of the origin. Let $\bar{T} \subseteq \mathcal{D}$ be a feasible subset of $R_A(\mu)$. Then, $\mu \in A(\bar{T})$. 
Proof Let $N_0$ be the neighborhood of the origin contained by $\Omega$. Then, $\mu \in A(\Omega)$ implies $\mu \in A(N_0)$, so $V^\mu(x) < \infty$ for all $x \in N_0$. Since $T \subseteq D$ is a feasible subset of $R_A(\mu)$, we have “$x_0 \in T$ implies $\phi(t; x_0, \mu) \in D$ for all $t \geq 0$” and “$\lim_{t \to \infty} \phi(t; x_0, \mu) = 0$ for all $x_0 \in T$”. Here, the latter implies that there is a time $T > 0$ such that “$\phi(T; x_0, \mu) \in N_0$”. Therefore, we have $\mu \in A(T)$ since from (2) and the definition of $V^\mu$,

$$V^\mu(x_0) = \int_0^T r(\phi(\tau; x_0, \mu), \mu(\phi(\tau; x_0, \mu))) \, d\tau + \int_T^\infty r(\phi(\tau; x_0, \mu), \mu(\phi(\tau; x_0, \mu))) \, d\tau$$

$$< V^\mu(\phi(T; x_0, \mu)) < \infty \text{ holds for all } x_0 \in T. \quad \Box$$

Define the Hamiltonian $H : R^n \times R^m \times R^n \to R$ for the nonlinear optimal control problem (1)–(2) as

$$H(x, u, p) := r(x, u) + p^T (f(x) + g(x)u).$$

Then, the Lyapunov equation (6) can be represented as $H(x, \mu, \nabla V^\mu) = 0$, and minimizing $H(x, \mu, \nabla V^\mu)$ among all admissible controls $\mu$ yields the optimal control $u = \mu^*(x)$ below:

$$\mu^*(x) = -\frac{1}{2} R^{-1} g^T(x) \nabla V^*(x), \quad (8)$$

where $V^*$ is the optimal value function defined as $V^* := V^{\mu^*}$. Furthermore, substituting (8) into (6) and rearranging the equation yields the well-known HJB equation:

$$0 = Q(x) + \nabla V^* T(x)f(x) - \frac{1}{4} \nabla V^* T(x)g(x)R^{-1} g^T(x) \nabla V^*(x), \quad (9)$$

For the optimal solution $V^*$, we assume throughout the paper that

**Assumption 3.3** $V^*$ is positive definite and $C^1$ on a subset $\Omega^* \subseteq D$, and is the unique solution of the HJB equation (9) on $\Omega^*$.

### 4 Policy Iteration with Admissible Region Update: Invariantly Admissible PI

In this section, we focus on and discuss the invariant admissibility of the SAM [7], [8], [9]. Then, the advanced algorithm, called invariantly admissible PI in this paper, is proposed which determines not only the value function and the next control but its invariant admissible region at each iteration.

In Lemma 5.2.4 in [7], it was stated that the controls $\mu_i$’s generated by the SAM [7], [8], [9] with an initial admissible control $\mu_0 \in A(\Omega)$ are all admissible on $\Omega$. In the proof of the lemma, however,

(1) Lyapunov’s theorem (Theorem 3.13 in [7]) was applied under the implicit assumption that the domain $\Omega \subseteq D$ of the Lyapunov function $V^{\mu_i} : \Omega \to R_+$ for the $i$-th admissible control $\mu_i$ is a subset of $R_A(\mu_i)$;
Step 1: Initialization
Set $i = 0$ and $\varepsilon > 0$ be a small constant. Let $\mu_0$ be a given control that is admissible on a compact subset $\Omega_0 \subseteq D$, i.e., $\mu_0 \in \mathcal{A}(\Omega_0)$.

Step 2: Policy Evaluation
Find the value function $V^{\mu_i} \in C^1(\Omega_i)$ for $\mu_i \in \mathcal{A}(\Omega_i)$ such that
\begin{equation}
(\nabla V^{\mu_i}(x))^T (f(x) + g(x)\mu_i(x)) = -r(x, \mu_i(x)), \forall x \in \Omega_i. \tag{10}
\end{equation}

Step 3: Policy Improvement
Find the next control $\mu_{i+1} \in C^0(\Omega_i)$ such that for all $x \in \Omega_i$,
\begin{equation}
\mu_{i+1}(x) = -\frac{1}{2} R^{-1} g^T(x) \nabla V^{\mu_i}(x). \tag{11}
\end{equation}

Step 4: Invariant Admissible Region Update
Find the next compact region $\Omega_{i+1} \subseteq D$ such that $\mu_i \in \mathcal{A}_2(\Omega_{i+1})$ and $\mu_{i+1} \in \mathcal{A}_2(\Omega_{i+1})$.

Step 5: Convergence
Is $\sup_{x \in \Omega_{i+1}} \|\mu_{i+1} - \mu_i\| < \varepsilon$?

Fig. 1 The proposed invariantly admissible PI algorithm

the assumption is not true in general as mentioned in Remark 3.1. Moreover,

(2) the domain $D$, which was equal to $\Omega$ in [7,8,9], was arbitrarily given, not as an invariant estimate of $R_A(\mu_i)$, so that the trajectory $\phi(t; x_0, \mu_i)$ starting in $D$ may escape the domain $D$ and may not be feasible.

These two problems can be solved at the same time if the domain $D$ is given as an invariant estimate of the regions of attraction $R_A(\mu_i)$ for all the closed-loop systems
\[ \dot{x} = f + g \mu_i. \] That is, \( \forall i \in \mathbb{Z}_+, \mathcal{D} \subseteq R_A(\mu_i) \) and
\[ x_0 \in \mathcal{D} \text{ implies } \phi(t; x_0, \mu_i) \in \mathcal{D}, \quad \forall t \geq 0. \]

To determine such an invariant attraction domain \( \mathcal{D} \), however, the knowledge about all the updated controls \( \mu_i \) (\( i = 0, 1, 2, \cdots \)) in PI has to be given \textit{a priori}, which is impossible but \( i = 0 \) before the algorithm runs.

Instead of this unrealistic approach, this paper solves the addressed problems by using another technique, which is used in the proposed PI method and determines, for a given domain \( \mathcal{D} \) and at each \( i \)-th iteration, the next region \( \Omega_{i+1} \) such that both the current control \( \mu_i \) and the next control \( \mu_{i+1} \) are invariantly admissible on \( \Omega_{i+1} \), i.e., \( \mu_i, \mu_{i+1} \in A_\mathcal{D}(\Omega_{i+1}) \). Fig. 1 describes the whole process of the proposed PI algorithm, where the next invariant admissible domain \( \Omega_{i+1} \) is determined in the process of “invariant admissible region update” that is newly introduced for the safe learning of both the optimal solution \((V^*, \mu^*)\) and the corresponding invariant admissible region. Policy evaluation and improvement are the same as those in the SAM except that they are performed in the domain \( \Omega_i \), instead of in the whole domain \( \mathcal{D} \).

### 4.1 Invariant Admissibility and Monotonic Decreasing Properties

Related to policy evaluation and policy improvement, the following theorem states the invariant admissibility of the controls and the monotonic decreasing property of the sequence of associated value functions on the compact subset \( \Omega_{i+1} \subseteq \mathcal{D} \) defined with a positive constant \( c_i > 0 \) as
\[ \hat{\Omega}_{i+1}^{\mu_i} = \{ x \in \mathbb{R}^n : V^{\mu_i}(x) \leq c_i \}. \]

**Theorem 4.1** Assume \( \mu_i \in A(\Omega_i) \), and let \( \mathcal{T}_i \subseteq \mathcal{D} \) be any feasible subset of \( R_A(\mu_i) \). Then, \( \mu_i \in A(\mathcal{T}_i) \). Moreover, if \( V^{\mu_i} \) is continuously differentiable on \( \mathcal{T}_i \), \( \mu_{i+1} \) satisfies (11) for all \( x \in \mathcal{T}_i \), and \( c_i \) is chosen such that \( \hat{\Omega}_{i+1}^{\mu_i} \subseteq \mathcal{T}_i \), then
1. \( \mu_{i+1} \) is a control on \( \hat{\Omega}_{i+1}^{\mu_i} \);
2. \( \mu_i, \mu_{i+1} \in A_\mathcal{D}(\hat{\Omega}_{i+1}^{\mu_i}) \);
3. for all \( x \in \hat{\Omega}_{i+1}^{\mu_i} \), the next value function \( V^{\mu_{i+1}} \) satisfies
\[ 0 < V^{\mu_{i+1}}(x) \leq V^{\mu_i}(x) < \infty. \]  

**Proof** First, \( \mu_i \in A(\mathcal{T}_i) \) is easily proven by applying Lemma 3.2 with \( \Omega = \mathcal{T}_i \) and \( \mathcal{T} = \mathcal{T}_i \). Here, \( \mu_i \in A(\mathcal{T}_i) \) implies that \( V^{\mu_i}(x) \) is finite \( \forall x \in \mathcal{T}_i \). Next, assume \( V^{\mu_i} \) is continuously differentiable on \( \mathcal{T}_i \), and \( \hat{\Omega}_{i+1}^{\mu_i} \subseteq \mathcal{T}_i \). Then, we have \( \mu_i \in A_\mathcal{D}(\hat{\Omega}_{i+1}^{\mu_i}) \) by Lemma 3.1 with \( \Omega = \mathcal{T}_i \) and \( \hat{\Omega}_{i+1}^{\mu_i} = \hat{\Omega}_{i+1}^{\mu_i} \). For the remaining of the proof, assume further that \( \mu_{i+1} \) satisfies (11) for all \( x \in \mathcal{T}_i \).

We now show that \( \mu_{i+1} \) is a control on \( \hat{\Omega}_{i+1}^{\mu_i} \). Since \( V^{\mu_i} \) is \( C^1 \) and positive definite on the domain \( \mathcal{T}_i \), containing a neighborhood of the origin, \( 0 \in \mathbb{R}^n \) is the global minimum where \( \nabla V^{\mu_i}(0) = 0 \). Also note that \( g(x) \) and \( \nabla V^{\mu_i}(x) \) are continuous on the compact subset \( \hat{\Omega}_{i+1}^{\mu_i} \subseteq \mathcal{T}_i \subseteq \mathcal{D} \), \( g \in C^0(\mathcal{D}) \), and \( V^{\mu_i} \in C^1(\mathcal{T}_i) \). So, we have \( \mu_{i+1}(0) = 0 \) from (11) and \( \nabla V^{\mu_i}(0) = 0 \). From (11) and the continuity.
of $\nabla V^\mu_i(x)$ and $g(x)$ on the compact subset $\bar{\Omega}_{\mu_i}^i$, it can be also shown that $\mu_{i+1}$ is continuous on $\bar{\Omega}_{\mu_i}^i$. Therefore, $\mu_{i+1}$ is a control on $\bar{\Omega}_{\mu_i}^i$.

For the proof of $\mu_{i+1} \in A_d(\bar{\Omega}_{\mu_i}^i)$ and \[12\), consider $V^{\mu_i}$ as a Lyapunov function candidate for the system $\dot{x} = f(x) + g(x)\mu_{i+1}(x)$. Differentiating $V^{\mu_i}(x)$ with respect to the system $\dot{x} = f + g\mu_{i+1}$, we have

$$
\dot{V}^{\mu_i}(x) = \nabla^T V^{\mu_i}(x) \cdot (f(x) + g(x)\mu_{i+1}(x))
$$

$$
= -Q(x) - \mu_i^T R \mu_i - 2\mu_{i+1}^T R(\mu_{i+1} - \mu_i),
$$

(13)

where (10) and (11) are substituted in the second equality. Applying Young’s inequality $2x^T Ry \leq x^T Rx + y^T Ry$ for $x, y \in \mathbb{R}^m$ to \[13\), we obtain

$$
\dot{V}^{\mu_i}(x) \leq -r(x, \mu_{i+1}) < 0, \forall x \in \mathcal{T}_i.
$$

(14)

Therefore, by Theorem 3.1 with $\Omega = \mathcal{T}_i$ and $\bar{\Omega}_c = \bar{\Omega}_{\mu_i}^i \subseteq \mathcal{T}_i$, $\mu_{i+1}$ asymptotically stabilizes the system $(f, g)$ on $\bar{\Omega}_{\mu_i}^i$, and $\bar{\Omega}_{\mu_i}^i$ is invariant under $\mu_{i+1}$, i.e.,

$$
\text{if } x_0 \in \bar{\Omega}_{\mu_i}^i \text{ then } \phi(t; x_0, \mu_{i+1}) \in \bar{\Omega}_{\mu_i}^i \text{ for all } t \geq 0.
$$

(15)

Here, since we assume $\bar{\Omega}_{\mu_i}^i \subseteq \mathcal{T}_i$ and $V^{\mu_i} \in C^1(\mathcal{T}_i)$, the invariance \[15\) on $\bar{\Omega}_{\mu_i}^i$ implies that $V^{\mu_i}(\phi(t; x_0, \mu_{i+1}))$ and $\dot{V}^{\mu_i}(\phi(t; x_0, \mu_{i+1}))$ are finite for all $x_0 \in \bar{\Omega}_{\mu_i}^i$ and all $t \geq 0$. So, one can integrate \[14\) from $t = 0$ to $\infty$ to obtain

$$
0 < V^{\mu_{i+1}}(x_0) = \int_0^\infty r(t; \phi(t; x_0, \mu_{i+1}))(\phi(t; x_0, \mu_{i+1})) \, dt
$$

$$
\leq \int_0^\infty \dot{V}^{\mu_i}(\phi(t; x_0, \mu_{i+1})) \, dt = V^{\mu_i}(x_0) < \infty,
$$

where we have used in the second equality $\lim_{t \to \infty} V^{\mu_i}(\phi(t; x_0, \mu_{i+1})) = 0$, which holds $\forall x_0 \in \bar{\Omega}_{\mu_i}^i$ since $V^{\mu_i}(0) = 0$ and $\lim_{t \to \infty} \phi(t; x_0, \mu_{i+1}) = \phi(t; x_0, \mu_{i+1}) = 0$ for $x_0 \in \bar{\Omega}_{\mu_i}^i$ by asymptotic stability. Therefore, $V^{\mu_{i+1}}$ satisfies $0 < V^{\mu_{i+1}}(x_0) \leq V^{\mu_i}(x_0) < \infty$ for $\forall x_0 \in \bar{\Omega}_{\mu_i}^i$. This implies $\mu_{i+1} \in A_d(\bar{\Omega}_{\mu_i}^i)$, and we have $A_d(\bar{\Omega}_{\mu_i}^i)$ by \[15\). □

If a feasible subset $\mathcal{T}_i$ of $R_A(\mu_i)$ is given \textit{a priori}, it can be used to determine the invariantly admissible region $\bar{\Omega}_{\mu_i}^i \subseteq \mathcal{T}_i$. Moreover, if the domain $\mathcal{D}$ is extended to satisfy $R_A(\mu) \subseteq \mathcal{D}$ at $i$-th iteration, then $\mathcal{T}_i$ can be given as the largest attractive set $\mathcal{T}_i = R_A(\mu_i)$ for the control $\mu_i$. In this case, $\mathcal{T}_i$ is also a feasible subset of $\mathcal{D}$ since $R_A(\mu_i)$ is itself an invariant set \[23\] and contained by $\mathcal{D}$. However, calculating the region of attraction $R_A(\mu_i)$ at each $i$-th iteration or its feasible subset is not a trivial task and needs high computational burden. To avoid such difficulties, the admissible set $\bar{\Omega}_c \subseteq \mathcal{D}$ given \textit{a priori} can be used as a feasible subset $\mathcal{T}_i$ of $R_A(\mu_i)$, i.e., $\mathcal{T}_i = \bar{\Omega}_c$. The following corollary shows that under the assumption

**Assumption 4.1** For each $\mu_i \in A(\Omega_i)$, $V^{\mu_i}$ is continuously differentiable on $\Omega_i$;

**Corollary 4.1** Assume $\mu_i \in A(\Omega_i)$ and $c_i$ is chosen such that $\bar{\Omega}_{\mu_i}^i \subseteq \Omega_i$. Then,

1. $\mu_{i+1}$ is a control on $\bar{\Omega}_{\mu_i}^i$.
2. $\mu_i, \mu_{i+1} \in \mathcal{A}(\Omega_{ci}^i)$;
3. $V_{\mu_{i+1}}$ satisfies $0 < V_{\mu_{i+1}}(x) \leq V_{\mu_i}(x) < \infty$ for all $x \in \Omega_{ci}^i$.

**Proof** Apply Theorem 4.1 with $\bar{Y}_i = \Omega_i$ under Assumption 4.1.

Furthermore, the next theorem states that the (invariant) admissibility and the value function decreasing property are preserved under $\Omega_{i+1}$ determined by $\Omega_{i+1} = \bar{\Omega}_{ci}^i$ and $c_i > 0$ chosen such that $\bar{\Omega}_{ci}^i \subseteq \Omega_i$ is guaranteed.

**Theorem 4.2** Assume the initial control $\mu_0$ is admissible on $\Omega_0 \subseteq \mathcal{D}$. If the controls $\{\mu_i\}$ and the value functions $\{V_{\mu_i} \in C^1\}$ are generated by the proposed PI (Fig. 7) with $(\Omega_{i+1}, c_i)$ determined at each $i$-th step such that $\Omega_{i+1} = \Omega_{ci}^i$ and $\bar{\Omega}_{ci}^i \subseteq \Omega_i$, then for all $i \in \{0, 1, 2, \cdots\}$,

1. $\mu_{i+1}$ is a control on $\Omega_{i+1}$
2. $\mu_i$ and $\mu_{i+1}$ are invariantly admissible on $\Omega_{i+1}$
3. for all $x \in \Omega_{i+1}$,

$$0 < V_{\mu_{i+1}}(x) \leq V_{\mu_i}(x) \leq \cdots \leq V_{\mu_0}(x) \leq V_{\mu_0}(x).$$

**Proof** Since we assume $\mu_0 \in \mathcal{A}(\Omega_0)$ and $\Omega_1 = \bar{\Omega}_{c_0}^0 \subseteq \Omega_0$, Corollary 4.1 implies

1. $\mu_1$ is a control on $\Omega_1$;
2. $\mu_0, \mu_1 \in \mathcal{A}(\bar{\Omega}_1)$;
3. $V_{\mu_1}$ satisfies $0 < V_{\mu_1}(x) \leq V_{\mu_0}(x) < \infty$ for all $x \in \Omega_1$.

Then, we have again $\mu_1 \in \mathcal{A}(\Omega_1)$ by Proposition 3.1. Repeating this process $i$-times, we can prove the first and second parts; this process also proves that for any $j \in \{0, 1, 2, \cdots, i-1, i\}$,

$$0 < V_{\mu_{i+1}}(x) \leq V_{\mu_i}(x)$$

is satisfied for all $x \in \Omega_{j+1}$. Finally, since $\Omega_{i+1}$ satisfies

$$\Omega_{i+1} \subseteq \cdots \subseteq \Omega_{j+1} \subseteq \Omega_j \subseteq \cdots \subseteq \Omega_i \subseteq \Omega_0,$$

(17) also holds for all $x \in \Omega_{i+1}$ (i.e. $\Omega_{i+1} \subseteq \Omega_{j+1}$) and all $j \in \{0, 1, 2, \cdots, i\}$, which completes the proof of the third statement. $\square$

4.2 Convergence Analysis

Now, we analyze the convergence properties of the proposed PI method (Fig. [1]) under the assumptions that

**Assumption 4.2** The initial admissible region $\Omega_0$ satisfies $\Omega_0 \subseteq \Omega^*$, and $(\Omega_{i+1}, c_i)$ is determined for all $i \in \{0, 1, 2, \cdots\}$ such that $\Omega_{i+1} = \bar{\Omega}_{ci}^i$ and $\bar{\Omega}_{ci}^i \subseteq \Omega_i$. That is,

$$\Omega_{i+1} \subseteq \Omega_i \subseteq \cdots \subseteq \Omega_0 \subseteq \Omega^* \subseteq \mathcal{D}, \ \forall i \in \{0, 1, 2, \cdots\}.$$
Theorem 4.3 Consider the controls \( \{ \mu_i \} \) and the value functions \( \{ V^{\mu_i} \in C^1 \} \) generated by the invariantly admissible PI under an admissible initial control \( \mu_0 \). Assumptions \[ 3, 7, 8 \] and Assumptions \[ 4, 1, 2, 3 \]. If for some \( k \in \{ 0, 1, 2, \ldots \} \), \( V^{\mu_k}(x) = V^{\mu_{k+1}}(x) \) holds for all \( x \in \Omega_{k+1} \), then \( \mu_k = \mu_{k+1} = \mu^* \) and \( V^{\mu_k} = V^{\mu_{k+1}} = V^* \) on \( \Omega_{k+1} \).

Proof Substituting \( V^{\mu_k} = V^{\mu_{k+1}} \) and (11) for \( i = k \) into (10) for \( i = k + 1 \), we have

\[
0 = Q(x) + (\nabla V^{\mu_{k+1}}(x))^T f(x) - \frac{1}{4} (\nabla V^{\mu_{k+1}}(x))^T g(x) R^{-1} g^T(x) \nabla V^{\mu_{k+1}}(x),
\]

that holds for all \( x \in \Omega_{k+1} \) by Assumptions 4.1 and 4.2. Note that (18) is the HJB equation (9). Since \( \Omega_{k+1} \subseteq \Omega^* \) by Assumption 4.2 and \( V^* \) is the unique solution of the HJB equation (9) over \( \Omega^* \) by Assumption 3.3, we have \( V^{\mu_k} = V^{\mu_{k+1}} = V^* \) on \( \Omega_{k+1} \). Moreover, (8) and (11) with \( i = k \) and \( \mu_{k+1} = \mu_k \) proves \( \mu_k = \mu_{k+1} = \mu^* \) on \( \Omega_{k+1} \).

Theorem 4.3 states that if the process of invariantly admissible PI is terminated by convergence in a finite number of steps, then the solution is guaranteed to be optimal.

To investigate the general convergence conditions in case of that the process does not end in a finite number of steps, define the limit set \( \hat{\Omega} \) as \( \hat{\Omega} := \bigcap_{i=0}^{\infty} \Omega_i \). Then, from (16) and \( \Omega_{i+1} = \Omega_{i+1}^\ast \), one can see that, for the condition

\[
\hat{\Omega} \subseteq \cdots \subseteq \Omega_{i+1} \subseteq \Omega_i \subseteq \cdots \subseteq \Omega_1 \subseteq \Omega_0
\]

in Assumption 4.2 \( \{ c_i > 0 \} \) should be monotonically decreasing. In this case, since \( \{ c_i \} \) is bounded by zero, it converges with this decreasing condition to a limit point \( \hat{c} := \lim_{i \to \infty} c_i \) in a decreasing order

\[
0 < c_{i+1} \leq c_i \leq \cdots \leq c_1 \leq c_0.
\]

Also note that the limit set \( \hat{\Omega} \) is compact since arbitrary intersection of the closed and bounded sets \( \Omega_i \) is also closed and bounded.

Assumption 4.3 \( \hat{\Omega} \) contains a neighborhood of the origin.

Lemma 4.1 Under Assumptions \[ 3, 7, 8 \] Assumptions \[ 4, 1, 2, 3 \] and \( \mu_0 \in A(\Omega_0) \), there is a function \( \hat{V} : \Omega \to R_+ \) such that \( \{ V^{\mu_i} \in C^1 \} \) generated by the invariantly admissible PI pointwisely converges to \( \hat{V} \) on the limit set \( \hat{\Omega} \) as \( i \to \infty \), in a decreasing order

\[
0 \leq \hat{V}(x) \leq V^{\mu_{i+1}}(x) \leq V^{\mu_i}(x).
\]

Proof By (16) in Theorem 4.2 and \( \hat{\Omega} \subseteq \Omega_{i+1} \), we have \( 0 < V^{\mu_{i+1}}(x) \leq V^{\mu_i}(x) \) for all \( x \in \hat{\Omega} \) and all \( i \in \{ 0, 1, 2, \ldots \} \). Therefore, for any fixed \( x \in \hat{\Omega} \), \( \{ V^{\mu_i}(x) \} \) is decreasing and bounded by zero, implying the existence of \( \hat{V} \) to which \( \{ V^{\mu_i}(x) \} \) pointwisely converges in the decreasing order (21), which completes the proof.
The next theorem states the conditions for the uniform monotonic convergence of \( \{V^{\mu_i}\} \) and \( \mu_i \) to the optimal solution \( V^* \) and \( \mu^* \) on the limit set \( \hat{\Omega} \), respectively. For the discussion, we denote by \( C^1_A(\hat{\Omega}) \) the set of all continuously differentiable value functions for the controls that are admissible on the compact limit set \( \hat{\Omega} \). That is,

\[
C^1_A(\hat{\Omega}) := \{ V^{\mu} \in C^1(\Omega) : \mu \in A(\hat{\Omega}) \text{ and } \phi(t; x_0, \mu) \in \Omega \ \forall t \geq 0 \}.
\]

Note that \( V^{\mu_i} \in C^1(\Omega_i) \) generated by the proposed algorithm belongs to \( C^1_A(\hat{\Omega}) \) for any \( i \in \{0, 1, 2, \ldots \} \). This is because \( \hat{\Omega} \subseteq \Omega_i \) (see [19] or Assumption 4.2) and \( \mu_i \in A(\Omega_i) \) implies \( \mu_i \in A(\hat{\Omega}) \), so \( V^{\mu_i} \) is a value function for \( \mu_i \in A(\hat{\Omega}) \subseteq A(\hat{\Omega_i}) \).

Next, we define PI operator \( \mathcal{T} : C^1_A(\hat{\Omega}) \to C^1_A(\hat{\Omega}) \) as a composite mapping \( \mathcal{T} = \mathcal{T}^E \circ \mathcal{T}^I \), where \( \mathcal{T}^E \) and \( \mathcal{T}^I \) are policy evaluation and improvement operators defined as follows:

1. \( \mathcal{T}^E : A(\hat{\Omega}) \to C^1_A(\hat{\Omega}) \) is a map from an admissible control \( \mu \in A(\hat{\Omega}) \) to the corresponding value function \( V^{\mu} \in C^1_A(\hat{\Omega}) \) satisfying (6). That is,

\[
V^{\mu} = \mathcal{T}^E(\mu).
\]

2. \( \mathcal{T}^I : C^1_A(\hat{\Omega}) \to A(\hat{\Omega}) \) is a map from a value function \( V^{\mu} \in C^1_A(\hat{\Omega}) \) to the admissible control \( \mu^+ \in A(\hat{\Omega}) \) satisfying \( \mu^+ = -\frac{1}{2} R^{-1} g^T \nabla V^{\mu} \). That is,

\[
\mu^+ = \mathcal{T}^I(V^{\mu}).
\]

So, for a given value function \( V^{\mu} \in C^1_A(\hat{\Omega}) \) satisfying \( (\nabla V^{\mu})^T (f + g\mu) = 0 \), \( \mathcal{T} \) yields the value function \( V^{\mu^+} \in C^1_A(\hat{\Omega}) \) satisfying \( (\nabla V^{\mu^+})^T (f + g\mu^+) = 0 \) for the improved admissible control \( \mu^+ = -\frac{1}{2} R^{-1} g^T \nabla V^{\mu} \). In a compact form,

\[
V^{\mu^+} = \mathcal{T}(V^{\mu}).
\]

Note that \( \mathcal{T} \) represents one cycle of policy evaluation and improvement; the value functions \( V^{\mu_i} \) and \( V^{\mu_i+1} \) generated by the PI method satisfy \( V^{\mu_i+1} = \mathcal{T}(V^{\mu_i}) \). Also note that if \( V^{\mu_k} = V^{\mu_{k+1}} \), then \( \mathcal{T} \) satisfies \( \mathcal{T}(V^{\mu_k}) = V^{\mu_k} \). This implies that the fixed point of the operator \( \mathcal{T} \) corresponds to the optimal value function \( V^* \) since \( V^{\mu_k} = V^{\mu_{k+1}} \) implies \( V^{\mu_k} = V^{\mu_{k+1}} = V^* \) by Theorem 4.3.

**Theorem 4.4** The sequence of value functions \( \{V^{\mu_i}\} \) generated by the invariently admissible PI under \( \mu_0 \in A(\Omega_0) \), Assumptions 4.2, 4.3, and Assumptions 4.7, 4.8 uniformly converges to the optimal solution \( V^* \) on \( \Omega \) in a decreasing order

\[
0 < V^*(x) \leq \cdots \leq V^{\mu_{i+1}}(x) \leq V^{\mu_i}(x) \leq \cdots \leq V^{\mu_1}(x) \leq V^{\mu_0}(x)
\]  

if \( \mathcal{T} \) is continuous, and the limit function \( \hat{V} \) in (21) belongs to \( C^1_A(\hat{\Omega}) \).
Remark 4.1 In the convergence analysis [7] of the SAM given in [7,8,9], it was implicitly assumed that \( \hat{V} \) is continuously differentiable and that \( \hat{V} \) satisfies \( H(x, \hat{\mu}, \nabla \hat{V}) = 0 \) where \( \hat{\mu} \) is given by \( \hat{\mu} = -R^{-1} g^T \nabla \hat{V} \) (see Theorem 5.3.1 in [7] and its proof). The same assumptions exist in the convergence proofs of the variants [4,16,21]. However, the convergence of \( C^1 \) functions \( V^{\mu} \rightarrow \hat{V} \) proven in Lemma 4.1 and [4,7,16,21] does not imply the convergence \( \nabla V^{\mu} \rightarrow \nabla \hat{V} \) in general, and cannot guarantee even the differentiability of the limit function \( \hat{V} \) (see [26]). In Theorem 4.4 of this paper, we have exactly and rigorously stated the conditions for \( \hat{V} = V^* \) (and uniform convergence \( V^{\mu} \rightarrow V^* \)) regarding the proposed invariantly admissible PI; similar conditions were given only in [13] for the usual SAM without admissible region update.

4.3 Determination of \( c_i > 0 \) of \( \hat{\Omega}_i^{\mu} (= \Omega_{i+1}) \)

Under Assumption 4.2, the region \( \Omega_i \) becomes more conservative as the learning continues since \( \Omega_{i+1} = \hat{\Omega}_{i+1}^{\mu} \subseteq \Omega_i \). That is, as can be seen from Fig. 2, \( \hat{\Omega}_{i+1}^{\mu} \) (the blue region) is necessarily smaller than or equal to \( \hat{\Omega}_{i+1}^{\mu} \) (the gray region) to be contained by the set \( \hat{\Omega}_{i+1}^{\mu} \) (the green region) for invariant admissibility. Here, the set \( \hat{\Omega}_{i+1}^{\mu} \) (the gray region) is obviously larger than or equal to \( \hat{\Omega}_{i+1}^{\mu} \) (the green region) for the same \( c_i^{\mu} \), by the monotonic decreasing property (16).

We now propose an invariant admissible region update rule to alleviate the conservativeness of the next region \( \hat{\Omega}_{i+1} \). Under Assumption 4.2, the proposed update rule determines at each \( i \)-th iteration the largest region \( \hat{\Omega}_{i+1}^{\mu} \) (the red region) contained by \( \Omega_i \) (the green region for \( i \geq 1 \)). The update starts with an initial admissible region given by \( \Omega_0 = \hat{B}_0(r) \), where \( \hat{B}_0(r) \) is a closed-ball at the origin with \( r \) determined to satisfy \( \mu_0 \in A(\hat{B}_0(r)) \). Then, at each \( i \)-th iteration, the update rule determines the radius \( c_i^* \) of the next region \( \hat{\Omega}_{i+1} \) by

\[
c_i^* = \min \{ V^{\mu}(x) : x \in \partial \Omega_i \},
\]

where \( \partial \Omega_i \) is the boundary of \( \Omega_i \). With this \( c_i^* \), the next region is updated by \( \hat{\Omega}_{i+1} = \hat{\Omega}_{i+1}^{\mu} \) (the red region in Fig. 3).

The maximum region \( \hat{\Omega}_i^{\mu} \) (the red region) and the normal region \( \hat{\Omega}_i^{\mu} \) (the blue region) are shown in Fig. 2. Compared to the normal one (the blue region), \( \hat{\Omega}_i^{\mu} \) has the maximum radius \( c_i^* \) on the constrained set \( \Omega_i \) (the green region \( \hat{\Omega}_i^{\mu} \) in Fig. 2 for \( i \geq 1 \)) while satisfying \( \hat{\Omega}_i^{\mu} \subseteq \hat{\Omega}_i^{\mu} \) without violating \( \hat{\Omega}_{i+1} \subseteq \Omega_i \) in Assumption 4.2 as illustrated in Fig. 2.
Remark 4.2 Even if the region $\Omega_{i+1}$ is updated by $\Omega_{i+1} = \bar{\Omega}_{\mu_i}^{c_i}$ with (23), it may become very small or narrow in some cases by (19) as $i$ increases. In this case, $\Omega_{i+1}$ can be enlarged at some $i$-th update step by calculating a larger feasible subset $T_i$ of $R_A(\mu_i)$ such that $\Omega_i \subseteq T_i \subseteq D$ and then determining $\Omega_{i+1}$ by $\Omega_{i+1} = \bar{\Omega}_{\alpha_i}^{c_i}$ with $\alpha_i^*$ chosen by
\[
\alpha_i^* = \min \{ V^{\mu_i}(x) : x \in \partial T_i \}. \tag{24}
\]
In this case, $\mu_{i+1} \in A_T(\Omega_{i+1})$ is guaranteed by Theorem 4.1 and by $\Omega_i \subseteq T_i$, we have $c_i^* \leq \alpha_i^*$. Therefore, the next domain $\Omega_{i+1}$ updated by (24) is essentially larger than that updated by (23), resulting in the larger final domain $\hat{\Omega}$ at last.

Remark 4.3 As mentioned in Section 1, there are many PI algorithms \cite{11,12,15,18,20} for the optimal control problem (1)–(2), ideally generating the same control and value function sequences ($\{\mu_i\}_{i=0}^{\infty}$ and $\{V^{\mu_i}\}_{i=0}^{\infty}$) as the SAM \cite{7,8,9}. So, the proposed invariantly admissible PI method can be easily extended to those equivalent PI and reinforcement learning algorithms to improve the closed-loop stability, by incorporating the (invariant) admissible region update step and the update rules (23) and (24) into those algorithms.

5 Numerical Simulations

To illustrate the proposed PI method and its effectiveness, we performed the numerical simulations for the following nonlinear system:
\[
\begin{align*}
\dot{x}_1 &= -x_1 + x_2 \\
\dot{x}_2 &= -(x_1 + x_2)/2 + x_2(\sin^2 x_1)/2 + (\sin x_1)u
\end{align*}
\tag{25}
\]
Fig. 3 Simulation results for nonzero case $w_{1,0} = -1$, $w_{2,0} = 3$, and $w_{3,0} = 1.5$: (a) evolution of $\Omega_i$ (brown: $\Omega_1$, red: $\Omega_2$, blue: $\Omega_3$–$\Omega_6$), (b) evolution of the weights $w_j (j = 1, 2, 3)$.

and the performance measure 3 with $r(x, u) = x_1^2 + x_2^2 + u^2$. This optimal control problem was also shown in [18] to simulate their nonlinear integral PI method which, in ideal cases, generates the same sequences of policies and value functions to the SAM in [7,8,9]. Using the converse HJB approach [27], the optimal solution $(V^*, \mu^*)$ is given by

$$
\mu^*(x) = -x_2 \sin x_1 \quad \text{and} \quad V^*(x) = \frac{1}{2} x_1^2 + x_2^2.
$$

As in [13], the value function $V^*(x)$ is parameterized as $V^*(x) = w_1 x_1^2 + w_2 x_1 x_2 + w_3 x_2^2$, where $w_j (j = 1, 2, 3)$ are the weights to be determined in policy evaluation of the proposed PI method at every iteration.
In the simulations, the sample points $x$ used in the $i$-th policy evaluation step are collected only in the $i$-th admissible set $\Omega_i$ of the state space, with the same sampling interval $\Delta x_1 = \Delta x_2 = 10^{-2}$. Here, the initial admissible region is given by $\bar{\Omega}_0 = B_0(1)$ as in [18], where $B_0(1) = \{ x \in \mathbb{R}^2 : |x_1| \leq 1, |x_2| \leq 1 \}$. Then, the invariant admissible region update agent in the simulations finds, at each $i$-th step, the optimal radius $c^*_i$ satisfying (23) to determine the next largest invariant admissible region $\Omega_{i+1} = \bar{\Omega}_{\mu_i} \subseteq \Omega_i$ on the constraint set $\Omega_i$.

Fig. 4 illustrates the simulation result for nonzero initial weights $w_{1,0} = -1$, $w_{2,0} = 3$, and $w_{3,0} = 1.5$. As can be seen from Fig. 4(b), the weights $w_j$ ($j = 1, 2, 3$) converge to the optimal values as expected. In this case, the initial weights deviated
far from the optimal ones, and the rates of change of the weights are highest between $i = 0$ and 1. From Figs. 3(a) and (b), one can see that these initial characteristics cause the rapid changes of the principal axes of the ellipsoidal curve $V_{\mu_i}^*(x) = c_i^*$, making the next region $\Omega_2$ rather conservative. On the contrary, as shown in Fig. 3(b), the invariant admissible region $\Omega_i$ becomes stationary and converges as the weights $w_j$’s converge to the optimal ones. Here, the region $\Omega_i$ can be enlarged by providing the larger initial admissible domain $\Omega_0$, or using the method (24) in Remark 4.2 with a feasible larger subset $\mathcal{Y}_i$, or making the initial weights close to the optimal ones.

The simulation results for zero initial weights $w_{1,0} = w_{2,0} = w_{3,0} = 0$ are given in Fig. 4. Compared to the previous nonzero case, the initial weights were set close to the optimal ones, and the deviations of the weights are relatively small (Fig. 4(b)). These aspects result in the limit set $\hat{\Omega}$ shown in Fig. 4(a), being larger and less conservative than the limit set $\Omega$ in Fig. 3(a). As can be seen from Fig. 4(a), there is no significant change in the principal axes, making $\hat{\Omega}$ approximately equal to $\Omega_1$. While the existing PI generates $\mu_i$ over the whole domain $D$, which is time consuming and does not guarantee the admissibility on $D$, the proposed PI generates $\mu_i$ and the region $\Omega_i$, on which $\mu_i$ is invariantly admissible. As discussed earlier, the existing PI does not guarantee the admissibility on the whole domain $D$ unless the aforementioned strict assumptions regarding feasibility and stability are imposed.

6 Conclusions

This paper precisely defined an invariantly admissible control, the refined notion of an admissible control in terms of feasibility, closed-loop Lyapunov stability, and invariance. Then, as a generalization of the existing SAM [7,8,9], the invariantly admissible PI method was proposed that has the general update rule of the next region for invariant admissibility. The update rule for the next compact region based on the current value function was also proposed, and under this update rule, we mathematically showed the invariant admissibility of the generated controls and regions $(\mu_i, \Omega_i)$; the monotonic decreasing property and uniform convergence of the sequence of corresponding value functions were also presented under certain conditions. Unlike the existing SAM [7,8,9], the proposed PI method and the update rule did not implicitly assume the feasibility and the closed-loop stability on the Lyapunov domain while the algorithm runs. Finally, numerical simulations were provided to illustrate the proposed PI method and its effectiveness.

Acknowledgements This work has been supported by National Research Foundation of Korea Grant funded by the Korean Government (No. NRF-2013R1A1A2012851).

References

1. Bellman, R.: Dynamic programming. Princeton University Press, Princeton, NJ (1957)
2. Kirk, D.E.: Optimal control theory: an introduction. Dover Publications (2004)
3. Lewis, F.L., Syrmos, V.L.: Optimal control. John Wiley (1995)
4. Abu-Khalaf, M., Lewis, F.L.: Nearly optimal control laws for nonlinear systems with saturating actuators using a neural network HJB approach. Automatica 41(5), 779–791 (2005)
5. Adhyaru, D.M., Kar, I.N., Gopal, M.: Bounded robust control of nonlinear systems using neural network-based HJB solution. Neural Comput. Appl. 20(1), 91–103 (2011)
6. Alwardi, H., Wang, S., Jennings, L.S.: An adaptive domain decomposition method for the Hamilton-Jacobi-Bellman equation. J. Glob. Optim. 56(4) 1351–1373 (2012)
7. Beard, R.W.: Improving the closed-loop performance of nonlinear systems. Ph.D. thesis, Rensselaer Polytechnic Institute (1995)
8. Beard, R.W., Saridis, G.N., Wen, J.T.: Galerkin approximations of the generalized Hamilton-Jacobi-Bellman equation. Automatica 33(12), 2159–2177 (1997)
9. Beard, R.W., Saridis, G.N., Wen, J.T.: Approximate solutions to the time-invariant Hamilton-Jacobi-Bellman equation. J. Optim. Theory Appl. 96(3), 589–626 (1998)
10. Cheng, T., Lewis, F.L., Abu-Khalaf, M.: Fixed-final-time-constrained optimal control of nonlinear systems using neural network HJB approach. IEEE Trans. Neural. Netw. 18(6), 1725–1737 (2007)
11. Lee, J.Y., Park, J.B., Choi, Y.H.: Integral reinforcement learning with explorations for continuous-time nonlinear systems. In: Proc. Int. Jr. Conf. Neural Netw. (IJCNN), pp. 1042–1047 (2012)
12. Lewis, F.L., Vrabie, D.: Reinforcement learning and adaptive dynamic programming for feedback control. IEEE Trans. Circuits Syst. Mag. 9(3), 32–50 (2009)
13. Leake, R.J., Liu, R.W.: Construction of suboptimal control sequences. SIAM J. Control 5(1), 54–63 (1967)
14. Munos, R., Baird, L.C., Moore, A.W.: Gradient descent approaches to neural-net-based solutions of the Hamilton-Jacobi-Bellman equation. In: Proc. Int. Jr. Conf. Neural Netw. (IJCNN), vol. 3, pp. 2152–2157 (1999)
15. Murray, J.J., Cox, C.J., Lendaris, G.G., Saeks, R.: Adaptive dynamic programming. IEEE Trans. Syst. Man. Cybern. C Appl. Rev. 32(2), 140–153 (2002)
16. Saridis, G.N., Lee, C.S.G.: An approximation theory of optimal control for trainable manipulators. IEEE Trans. Syst. Man Cybern. 9(3), 152–159 (1979)
17. Spencer Jr., B.F., Timlin, T.L., Sam, M.K., Dyke, S.J.: Series solution of a class of nonlinear optimal regulators. J. Optim. Theory Appl. 91(2), 321–345 (1996)
18. Vrabie, D., Lewis, F.L.: Neural network approach to continuous-time direct adaptive optimal control for partially unknown nonlinear systems. Neural Netw. 22(3), 237–246 (2009)
19. Wang, S., Jennings, L.S., Teo, K.L.: Numerical solution of Hamilton-Jacobi-Bellman equations by an upwind finite volume method. J. Glob. Optim. 27(2-3), 177–192 (2003)
20. J. Y. Lee, J. B. Park, and Y. H. Choi. Integral $Q$-learning and explorized policy iteration for adaptive optimal control of continuous-time linear systems. Automatica, 48(11), 2850–2859 (2012)
21. Murray, J.J., Cox, C.J., Saeks, R.E.: The adaptive dynamic programming theorem. In: Stability and Control of Dynamical Systems with Applications, pp. 397-394, Springer (2003)
22. Kleinman, D.: On an iterative technique for Riccati equation computations. IEEE Trans. Automat. Contr. 13(1), 114–115 (1968)
23. Khalil, H.K.: Nonlinear systems. Prentice Hall (2002)
24. Knobloch, H., Kappel, F.: Gewöhnliche Differentialgleichungen. BG Teubner (1974)
25. Kaslik, E., Balint, A.M., Balint, S.: Methods for determination and approximation of the domain of attraction. Nonlinear Anal. Theory Methods Appl. 60(4), 703–717 (2005)
26. Marsden, J.E.: Elementary classical analysis. Macmillan (1993)
27. Nevišić, V., Primbs, J.A.: Constrained nonlinear optimal control: a converse HJB approach. California Institute of Technology Technical Report No. CTI-CDS 96-021 (1996)
