ON THE STRUCTURE OF $\infty$-HARMONIC MAPS

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Abstract. Let $H \in C^2(\mathbb{R}^{N \times n})$, $H \geq 0$. The PDE system

\[ A_\infty u := (H_P \otimes H_P + H[H_P]_\perp^\perp H_{PP}) \circ (Du) : D^2 u = 0 \]

arises as the “Euler-Lagrange PDE” of vectorial variational problems for the functional $E_\infty(u, \Omega) = \|H(Du)\|_{L^\infty(\Omega)}$, defined on maps $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$. The system (1) first appeared in the author’s recent work [K3]. The scalar case though has a long history initiated by Aronsson in [A1]. Herein we study the solutions of (1) with emphasis on the case of $n = 2 \leq N$ with $H$ the Euclidean norm on $\mathbb{R}^{N \times n}$, which we call the “$\infty$-Laplacian”. By establishing a rigidity theorem for rank-one maps of independent interest, we analyse a phenomenon of separation of the solutions to phases with qualitatively different behaviour. As a corollary, we extend to $N \geq 2$ the Aronsson-Evans-Yu theorem regarding non-existence of zeros of $|Du|$ and prove a Maximum Principle. We further characterise all $H$ for which (1) is elliptic and also study the initial value problem for the ODE system arising for $n = 1$ but with $H(\cdot, u, u')$ depending on all the arguments.

1. Introduction

Let $H \in C^2(\mathbb{R}^{N \times n})$ be a nonnegative function which we call Hamiltonian. In this paper we study the classical solutions $u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N$ of the PDE system

\[ A_\infty u := (H_P \otimes H_P + H[H_P]_\perp^\perp H_{PP}) \circ (Du) : D^2 u = 0. \]

Here $[H_P(P)]^\perp$ denotes the orthogonal projection on the nullspace of $H_P(P)^\top : \mathbb{R}^N \rightarrow \mathbb{R}^n$ and $H_P$ is the derivative matrix (for details see the Preliminaries 1.1). The system (1.1) arises as a sort of Euler-Lagrange PDE of vectorial variational problems in $L^\infty$ for the functional

\[ E_\infty(u, \Omega) := \|H(Du)\|_{L^\infty(\Omega)}. \]

Calculus of Variations in $L^\infty$ is very important for applications, since minimisation of the maximum value leads to more realistic models when compared to the more classical case of integral functionals in which case we minimise the average. (1.1) is a quasilinear 2nd order system in non-divergence form which was first formally derived by the author in the recent work [K3] as the limit of Euler-Lagrange equations of the functionals $\int_\Omega (H(Du))^p$ as $p \rightarrow \infty$. Herein particular emphasis will be given on the 2D case for $n = 2 \leq N$ with $H(P) = \frac{1}{2} |P|^2$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^{N \times n}$. In this case (1.1) simplifies to

\[ \Delta_\infty u := (Du \otimes Du + |Du|^2[Du]_\perp^\perp \otimes I) : D^2 u = 0. \]

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We call (1.3) the \( \infty \)-Laplacian and its solutions \( \infty \)-Harmonic maps. The name stems from its derivation which we now recall. After expansion and normalisation of the \( p \)-Laplace system \( \Delta_p u = \text{Div}(|Du|^{p-2}Du) = 0 \), we have

\[
Du \otimes Du : D^2u + \frac{|Du|^2}{p-2} \Delta u = 0.
\]

Let \([Du] \) and \([Du] \perp \) denote the orthogonal projections on the range of \( Du \) and the nullspace of \( Du \), respectively. Since \([Du] \perp + [Du] \perp = I\), by expanding \( \Delta u \) with respect to these projections, we get

\[
Du \otimes Du : D^2u + \frac{|Du|^2}{p-2} [Du] \perp \Delta u = 0.
\]

By orthogonality, right and left hand side of (1.5) are normal to each other. Hence, they both vanish and (1.5) actually decouples to 2 systems. By renormalising the right hand side of (1.5) and rearranging, we get

\[
Du \otimes Du : D^2u + |Du|^2 [Du] \perp \Delta u = -\frac{|Du|^2}{p-2} [Du] \top \Delta u.
\]

As \( p \to \infty \), (1.6) formally leads to (1.3). In the case of (1.3) the projection \([Du] \perp\) coincides with the projection on the geometric normal space of the image of the solution. When \( n = 1 \), the system simplifies to

\[
\Delta_\infty u = (u' \otimes u') u'' + |u'|^2 \left( I - \frac{u' \otimes u'}{|u'|^2} \right) u'' = |u'|^2 u''.
\]

In particular, it follows that \( \infty \)-Harmonic curves are affine and no interesting phenomena arise.

When \( N = 1 \), the normal coefficient \( |Du|^2 [Du] \perp \) vanishes identically and the same holds when \( u \) is submersion. The single \( \infty \)-Laplacian PDE \( D_i u D_j u D_{ij}^2 u = 0 \) and the related scalar \( L^\infty \)-variational problems have a long history. \( \Delta_\infty \) was first derived and studied by Aronsson in the ’60s in \([A3, A4]\) and has been extensively studied ever since (see for example Crandall \([C]\), Barron-Evans-Jensen \([BEJ]\) and references therein). A major difficulty in its study is its degeneracy and the emergence of singular solutions (see e.g. \([A6, A7, K1]\)). In the last 25 years the single PDE has been studied in the context of Viscosity Solutions.

A further difficulty of the vectorial case which is not present in the scalar case is that (1.1) has discontinuous coefficients even when the operator \( A_\infty \) is applied to \( C^\infty \) maps which are solutions. As an example consider

\[
u(x,y) := e^{ix} - e^{iy}, \quad u : \mathbb{R}^2 \to \mathbb{R}^2.
\]

In \([K3]\) we showed that (1.8) is a smooth solution of the \( \infty \)-Laplacian near the origin. However, the coefficient \( |Du|^2 [Du] \perp \) of (1.3) is discontinuous. The problem is that the projection \([Du] \perp\) “jumps” when the dimension of the image changes. Indeed, for (1.8) we have \( \text{rk}(Du) = 2 \) off the diagonal \( \{x = y\} \), while \( \text{rk}(Du) = 1 \) otherwise. Hence, the domain of (1.8) splits to 3 components, the “2D phase \( \Omega_2 \)”, whereon \( u \) is essentially 2D, the “interface \( S \)” where the coefficients of \( \Delta_\infty \) become discontinuous and the “1D phase \( \Omega_1 \)”, whereon \( u \) is essentially 1D (and in this case is empty). Much more intricate examples of smooth 2D \( \infty \)-Harmonic maps whose interfaces have triple junctions and corners are constructed in \([K6]\). For any
$K \in C^1(\mathbb{R})$ with $\|K\|_{L^{\infty}(\mathbb{R})} < \frac{\pi}{2}$, the formula

\begin{equation}
    u(x, y) := \int_y^x e^{iK(t)} dt
\end{equation}

defines a smooth $\infty$-Harmonic map whose phases are as shown in Figures 1(a), 1(b) below, when $K$ qualitatively behaves as shown in the Figures 2(a), 2(b) respectively.

Moreover, on $\Omega_1$ (1.9) is given by a scalar $\infty$-Harmonic function times a constant vector, and on $\Omega_2$ it is a solution of the vectorial Eikonal equation.

One of the principal results of this paper is that this phase separation is a general phenomenon for smooth 2D $\infty$-Harmonic maps. On each phase the dimension of the tangent space is constant and these phases are separated by interfaces whereon $|Du|^2$ becomes discontinuous. More precisely, in Section 3 we prove the next

**Theorem 1.1** (Structure of 2D $\infty$-Harmonic maps). Let $u : \Omega \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^N$ be an $\infty$-Harmonic map in $C^2(\Omega)^N$, that is a solution to (1.3). Let also $N \geq 2$. Then, there exists disjoint open sets $\Omega_1$, $\Omega_2 \subseteq \Omega$ and a closed nowhere dense set $S$ such that $\Omega = \Omega_1 \cup S \cup \Omega_2$ and:

(i) On $\Omega_2$ we have $rk(Du) = 2$ and the map $u : \Omega_2 \rightarrow \mathbb{R}^N$ is an immersion and solution of the vectorial Eikonal equation:

\begin{equation}
    |Du|^2 = c^2 > 0.
\end{equation}

The constant $c$ may vary on different connected components of $\Omega_2$.

(ii) On $\Omega_1$ we have $rk(Du) = 1$ and the map $u : \Omega_1 \rightarrow \mathbb{R}^N$ is given by an essentially scalar $\infty$-Harmonic function $f : \Omega_1 \rightarrow \mathbb{R}$:

\begin{equation}
    u = a + \xi f , \quad \Delta_\infty f = 0, \quad a \in \mathbb{R}^N, \quad \xi \in \mathbb{S}^{N-1}.
\end{equation}

The vectors $a, \xi$ may vary on different connected components of $\Omega_1$.

(iii) On $S$, $|Du|^2$ is constant and also $rk(Du) = 1$. Moreover if $S = \partial\Omega_1 \cap \partial\Omega_2$ (that is if both the 1D and 2D phases coexist) then $u : S \rightarrow \mathbb{R}^N$ is given by an essentially scalar solution of the Eikonal equation:

\begin{equation}
    u = a + \xi f , \quad |Df|^2 = c^2 > 0, \quad a \in \mathbb{R}^N, \quad \xi \in \mathbb{S}^{N-1}.
\end{equation}

We note that this phase separation is a genuinely vectorial phenomenon, which does not arise when the rank is one. By employing Aronsson’s result on the non-existence of zeros for the gradient of scalar $\infty$-Harmonic functions contained in [A4], we deduce the following consequence of Theorem 1.1:
Corollary 1.2 (∞-Harmonic maps have positive rank). Let \( u : \Omega \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^N \) be an \( \infty \)-Harmonic map in \( C^2(\Omega)^N \). Then, either \( |Du| > 0 \) on \( \Omega \) or \( |Du| \equiv 0 \) on \( \Omega \). Hence, non-constant \( \infty \)-Harmonic maps have positive rank.

Corollary 1.2 is an extension to the vector case of the aforementioned theorem of Aronsson, which has been subsequently improved by Evans [E] and Yu [Y]. Hence, \( \infty \)-Harmonic maps have positive rank but generally non-constant rank. As a corollary, in Section 3 we also establish a vectorial version of the Maximum Principle known as the Convex Hull Property, valid for \( n = N = 2 \):

Corollary 1.3 (Convex Hull Property). Suppose that \( u : \Omega \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is an \( \infty \)-Harmonic map. Then, for all \( \Omega' \Subset \Omega \), the image \( u(\Omega') \) is contained in the closed convex hull of the boundary values:

\[
\{ u(\Omega') \} \subseteq \overline{\text{co}(u(\partial\Omega'))}.
\]

Since a convex set coincides with the intersection of half-spaces containing it, (1.13) is just an elegant formulation of the Maximum Principle for all 1D projections of \( u \). It is well known in the context of Minimal Surfaces (see e.g. [CM], [O]) and more generally in Calculus of Variations (see [K2] and references therein). A topological consequence of Corollary 1.3 is

Corollary 1.4 (Absence of interfaces). Suppose that \( u : \Omega \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is an \( \infty \)-Harmonic map. Then:

(i) If \( \Omega_2 \Subset \Omega \), then \( \Omega_2 = \emptyset \) and \( \mathcal{S} = \emptyset \). Hence, either the set whereon \( u \) is a local diffeomorphism has a common boundary portion with \( \Omega \) or it is empty and \( u \) is everywhere essentially scalar without any interface \( \mathcal{S} \).

(ii) If \( \Omega \Subset \mathbb{R}^2 \) and \( u \) is essentially scalar near \( \partial\Omega \), then there is no interface \( \mathcal{S} \) inside \( \Omega \) and \( u \) is essentially scalar throughout \( \Omega \).

The main analytical machinery required for the proof of Theorem 1.1 is developed in Section 2 and is a rigidity result for maps with 1D range of independent interest. To begin with, consider a map \( u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N \) given as composition of a scalar function \( f \in C^2(\Omega)^N \) with a unit speed curve \( \nu : \mathbb{R} \rightarrow \mathbb{R}^N \), that is \( u = \nu \circ f \). Then, we have \( Du = (\dot{\nu} \circ f) \otimes Df \) and hence \( u \) is a Rank-One map, that is \( \text{rk}(Du) \leq 1 \) on \( \Omega \).

Interestingly, the class of Rank-One maps is rigid since a certain converse is true as well: all maps which satisfy \( \text{rk}(Du) \leq 1 \) arise as compositions of unit speed curves with scalar functions. More precisely,

Theorem 1.5 (Rigidity of Rank-One maps). Suppose \( \Omega \subseteq \mathbb{R}^n \) is open and contractible and \( u : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N \) is in \( C^2(\Omega)^N \). Then, the following are equivalent:

(i) \( u \) is a Rank-One map, that is \( \text{rk}(Du) \leq 1 \) on \( \Omega \) or equivalently there exist maps \( \xi : \Omega \rightarrow \mathbb{R}^N \) and \( w : \Omega \rightarrow \mathbb{R}^n \) with \( w \in C^1(\Omega)^n \) and \( \xi \in C^1(\Omega \setminus \{w = 0\})^N \) such that \( Du = \xi \otimes w \).
(ii) There exists $f \in C^2(\Omega)$, a partition $\{B_i\}_{i \in \mathbb{N}}$ of $\Omega$ to Borel sets where each $B_i$ equals a connected open set with a boundary portion and Lipschitz curves $\{\nu^i\}_{i \in \mathbb{N}} \subseteq W^{1,\infty}_l(\mathbb{R})^N$ such that on each $B_i$ $u$ equals the composition of $\nu^i$ with $f$:

\begin{equation}
    u = \nu^i \circ f, \quad \text{on } B_i \subseteq \Omega.
\end{equation}

Moreover, $|\nu^i| = 1$ on $f(B_i)$, $\nu^i \equiv 0$ on $\mathbb{R} \setminus f(B_i)$ and there exists $\tilde{\nu}^i$ on $f(B_i)$, interpreted as 1-sided on $\partial f(B_i)$, if any. Also,

\begin{equation}
    Du = (\nu^i \circ f) \otimes Df, \quad \text{on } B_i \subseteq \Omega,
\end{equation}

and the image $u(\Omega)$ is an $1$-rectifiable subset of $\mathbb{R}^N$:

\begin{equation}
    u(\Omega) = \bigcup_{i=1}^\infty \nu^i(f(B_i)) \subseteq \mathbb{R}^N.
\end{equation}

Theorem 1.5 is optimal. Without extra assumptions, there may not exist any $f$ globally defined on $\Omega$ and $u(\Omega)$ may bifurcate without being given by a single-valued curve $\nu$ for which $u = \nu \circ f$ (Corollary 2.1, Example 2.2). Theorem 1.5 has been motivated by the rigidity results of Rindler in [R1, R2]. Actually, we extend a part of his result from constant rank-one tensors $\xi \otimes w$ to variable rank-one $\xi(x) \otimes w(x)$ tensor fields. When compared to the rigidity results known in the literature (see e.g. Kirchheim [Ki]), it is somewhat surprising in that most rigidity phenomena appear for rank greater than 2. The idea is as follows: if $Du = \xi \otimes w$, then since $\text{Curl}(Du) \equiv 0$, we invoke Poincaré’s lemma to write $w = Df$ for a scalar $f$ and we also show that $\text{rk}(D\xi) \leq 1$. Then, we employ geodesic flows, Riemannian exponential maps and a curvilinear extension of “De Giorgi-type” arguments to show that $\xi$ and $f$ locally have the same level sets and hence $\xi = \tilde{\nu} \circ f$.

It seems that the natural setting for Theorem 1.5 is that of Lipschitz maps. Indeed, we provide such an extension in Theorem 2.3. Yet, this does not follow by a direct approximation argument and substantial complications arise. The problem is that the Rank-One property is not invariant under mollification: the mollification may “fatten” and its Hausdorff dimension may increase (Remark 2.4). We remedy this problem by imposing an extra approximation assumption.

In Section 4 we focus on the general system (1.1). We motivate our results by observing that (1.3) is quasilinear and degenerate elliptic, that is, for

\begin{equation}
    A_{\alpha i \beta j}(P) := P_{\alpha i}P_{\beta j} + |P|^2 |P|_{\alpha \beta \delta \gamma} \delta_{ij}
\end{equation}

we can rewrite the $\infty$-Laplacian (1.3) as $A(Du)_{\alpha i \beta j}D^2_{ij}u_{\beta} = 0$ and $A$ satisfies the symmetry condition and the Legendre-Hadamard condition:

\begin{equation}
    A_{\alpha i \beta j} = A_{\beta j \alpha i},
\end{equation}

\begin{equation}
    A_{\alpha i \beta j} \eta_\alpha a_i \eta_\beta a_j \geq 0, \quad \eta \in \mathbb{R}^N, a \in \mathbb{R}^n.
\end{equation}

However, the general system (1.1) is not degenerate elliptic since $[H_P]^\perp$ and $H_{PP}$ are symmetric but if $N \geq 2$ their product may not commute, not even when $H$ is strictly convex on $\mathbb{R}^{N \times n}$. For $N = 1$, though, Aronsson’s equation $H_P H_{PP} D^2_{ij}u = 0$ is trivially degenerate elliptic. In Theorem 4.1 we characterise the Hamiltonians which lead to elliptic systems as the “geometric” ones which depends on $Du$ via the Riemannian metric $Du^T Du$ on $u(\Omega) \subseteq \mathbb{R}^N$, that is when $H(P) = h(\frac{1}{2} P^T P)$. In the case of $\Delta_\infty$, we have $h(p) = \text{tr}(p)$. In dimensions $n \leq 3$, this is a complete equivalence. However, if $n \geq 4$ complicated structures in the higher order tensors $H_{P_{pq}P}$ appear and a necessary extra assumption is required for the full equivalence.
Without it, \( H \) can be written in this form up to an \( O(|P|^4) \) correction. In the case \( n = 1 \), we deduce that \( H \) is radially symmetric. This is very restrictive, but should be compared with the rigidity of Lipschitz extensions for maps in Kirszbraun’s theorem (see e.g. Federer [F], p. 201), in contrast to the flexibility of scalar Lipschitz extensions.

In this paper we also tackle two more independent topics related to the study of solutions to our system (1.1). In Section 5 we focus on the 1D case for \( n = 1 \leq N \) and we study the ODE system arising from Hamiltonian \( H \in C^2(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N) \) depending on all arguments \( H = H(x, u(x), u'(x)) \). The 1D case of vectorial Calculus of Variations in \( L^\infty \) provides an important model for Data Assimilation [K9]. We first formally derive the system in the limit as \( p \to \infty \) of the Euler-Lagrange equations of the respective \( L^p \)-functional (equation (5.8)). By imposing the condition of radial dependence in \( u' \), we obtain the degenerate elliptic version of the system:

\[
A_{\infty} u = |u'|^2 \left( h_p u'' - R_{u'} h_\eta \right) + h_x u' = 0.
\]

Here \( h \equiv h(\cdot, u, \frac{1}{2}|u'|^2) \), the arguments of \( h \) are \((x, \eta, p)\) and \( R_{u'} \) is the reflection operator with respect to the normal hyperplane \([u']^\perp\).

We note that although \( R_{u'} \) is discontinuous at critical points, in this case the coefficients of (1.20) are continuous. In Theorem 5.2 we study existence, uniqueness and \( W^{2,\infty}_{\text{loc}}(\mathbb{R})^N \) regularity of solutions to the initial value problem for (1.20).

Finally, motivated by Aronsson’s paper [A6], in Section 6 we analyse the class of solutions to (1.3) of the radial form \( u = \rho^k f(\kappa \theta) \) for \( k > 0 \) and \( f \) a curve in \( \mathbb{R}^N \). Interestingly, in Proposition 6.1 we prove that such solutions are very rigid since their image is contained in either an affine line or an affine plane.

We conclude this long introduction with some related results known in the literature. In [K4] we identified the variational principle characterising \( \infty \)-Harmonic maps for the model functional \( E_\infty(u, \Omega) = \| Du \|_{L^\infty(\Omega)} \). Surprisingly, the apt notion is not the obvious extension of Aronsson’s notion in higher dimensions but instead “a rank-one absolute minimal coupled by \( \infty \)-minimal area”. For details see [K4]. In [K5] we extended the results of [K3], [K4] to the subelliptic setting. In [K7], among other things, we proved that the Dirichlet problem for the \( \infty \)-Laplacian

\[
\begin{aligned}
\Delta_{\infty} u &= 0, & \text{in } \mathbb{B}^*, \\
u(x) &= x, & \text{on } \partial \mathbb{B}^*,
\end{aligned}
\]

surprisingly, has infinitely many smooth solutions \( u : \mathbb{B}^* \subseteq \mathbb{R}^n \to \mathbb{R}^n \) on the punctured unit ball \( \mathbb{B}^* = \{ x : 0 < |x| < 1 \} \), for all \( n \geq 2 \). The crucial observation is that smooth solutions the differential inclusion

\[
Du(\Omega) \subseteq \mathcal{K}, \quad \mathcal{K} := \left\{ P \in \mathbb{R}^{n \times n} : |P| = 1, \det(P) > 0 \right\}
\]
are \( \infty \)-Harmonic. In words, smooth solutions of the vectorial Eikonal equation which are local diffeomorphisms solve the \( \infty \)-Laplacian. For details see [K7]. It is worth mentioning that the maximum principle we establish herein is not a comparison principle and does not imply uniqueness. Ou, Troutman and Wilhelm in [OTW] and Wang and Ou in [WO] studied the “tangential part” of (1.3). Sheffield and Smart in [SS] used the nonsmooth operator norm on \( \mathbb{R}^{N \times n} \) as their Hamiltonian and derived a very singular variant of (1.3) which governs the so-called “tight maps”, that is vectorial optimal Lipschitz extensions. Our theorem 1.1 relates to an analogous phase separation of tight maps observed in [SS]. Capogna and Raich in [CR] used the dilation \( K(P) = \frac{|P|^n}{\det(P)} \) as Hamiltonian on \( \mathbb{R}^{n \times n} \) and developed an \( L^\infty \) variational approach to optimise Quasiconformal maps. They derived and studied a special important case of (1.1). Their results have been advanced by the author in [K8]. In the light of our general Theorem 4.1, it is not a coincidence that all Hamiltonians known in the literature depend on the gradient via the Riemannian metric \( Du^1Du \).

1.1. Preliminaries. Throughout this paper we reserve \( n, N \in \mathbb{N} \) for the dimensions of Euclidean spaces and \( S^{N-1} \) denotes the unit sphere of \( \mathbb{R}^N \). Greek indices \( \alpha, \beta, \gamma, ... \) run from 1 to \( N \) and Latin \( i, j, k, ... \) form 1 to \( n \). The summation convention will always be employed in repeated indices in a product. Vectors are always viewed as columns. Hence, for \( a, b \in \mathbb{R}^n \), \( a^\top b \) is their inner product and \( ab^\top \) equals \( a \otimes b \). If \( u = u_\alpha e_\alpha : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^N \) is a map, the gradient matrix \( Du \) is viewed as \( D_\alpha u e_\alpha \otimes e_i : \Omega \to \mathbb{R}^{N \times n} = \mathbb{R}^N \otimes \mathbb{R}^n \) and the Hessian tensor \( D^2u \) as \( D^2_{ij} u e_\alpha \otimes e_i \otimes e_j : \Omega \to \mathbb{R}^{N \times n} = \mathbb{R}^N \otimes \mathbb{S}(\mathbb{R}^n) \). If \( V \) is a vector space, then \( \mathbb{S}(V) \) denotes the symmetric linear operators \( T : V \to V \) for which \( T = T^\top \) and \( \mathbb{S}(V) \) is a map \( \mathbb{R}^{N \times n} \to \mathbb{R}^{N \times n} \). We will say that a \( q \)-th order tensor \( C \in \otimes(q) (\mathbb{R}^{N \times n}) \) is fully symmetric in all its arguments when

\[
C_{...\alpha i...\beta j...} = C_{...\alpha j...\beta i...} = C_{...\beta j...\alpha i...}
\]

We also introduce the following contract operation for tensors which extends the inner product \( P : Q = \text{tr}(P^\top Q) = P_{\alpha i} Q_{\alpha i} \) of \( \mathbb{R}^{N \times n} \). For, if \( C \in \otimes(q) (\mathbb{R}^{N \times n}) \) and \( A \in \otimes(p) (\mathbb{R}^{N \times n}) \) with \( p \leq q \), we define \( C : A \in \otimes(q-p)(\mathbb{R}^{N \times n}) \) by

\[
(C : A)_{\alpha q i_1...\alpha p+1 i_{p+1}} := C_{\alpha q i_1...\alpha i_1} A_{\alpha p i_p...\alpha i_1}.
\]

Let now \( P : \mathbb{R}^n \to \mathbb{R}^N \) be linear map. Upon identifying linear subspaces with orthogonal projections on them, we have the split \( \mathbb{R}^N = [P]^\top \oplus [P]^\perp \) where \( [P]^\top \) and \( [P]^\perp \) denote range of \( P \) and nullspace of \( P^\top \) respectively. Hence, if \( \xi \in S^{N-1} \), then \( [\xi]^\perp \) or simply \( \xi^\perp \) is (the projection on) the normal hyperplane \( I - \xi \otimes \xi \). Consequently, the \( \infty \)-Laplacian (1.3) in index form reads

\[
D_\alpha u \partial_{\alpha} D_j u_i D^2_{ij} u_\beta + |Du|^2 [Du]_{\alpha \beta}^\perp D^2_{ij} u_\beta = 0
\]
and the system (1.1) becomes
\[(1.25) \quad \left( H_{\alpha\beta} H_{\beta\gamma} + H f_{\alpha\gamma}\right)(Du)D_{ij}^2 u_{\beta} = 0.\]

For convenience we use a different scaling in (1.24) and (1.25) and we multiply the normal term of (1.24) by a factor 2 which is plausible since (1.25) consists of two systems normal to each other. Finally, \(H^k\) denotes the \(k\)-dimensional Hausdorff measure and for measure theoretic notions we use herein we refer to Simon [5].

2. Rigidity of Rank-One maps.

2.1. The case of smooth Rank-One maps. In this subsection we establish our Geometric Analysis rigidity result in the case of \(C^2\) maps.

Proof of Theorem 1.5. The implication (ii) \(\Rightarrow\) (i) is trivial and the whole proof is devoted to establish the reverse implication (i) \(\Rightarrow\) (ii). For, suppose there exist \(\xi : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^N\) and \(w : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n\) such that \(Du = \xi \otimes w\). By replacing \(\xi\) by \(\xi/|\xi|\) on \{|\xi| > 0\} and \(w\) by \(|\xi| w\) on \{|\xi| > 0\}, we may pass all the zeros of \(Du\) to \(w\) and assume that \(|\xi| \equiv 1\) on
\[(2.1) \quad \Omega_0 := \{|Du| > 0\} = \{|w| > 0\}.\]

By differentiating \(D_k u_{\alpha} = \xi_{\alpha} w_k\), we have
\[(2.2) \quad D_{ij}^2 u_{\alpha} = (D_j \xi_{\alpha}) w_i + \xi_{\alpha} (D_j w_i),\]
\[(2.3) \quad D_{ij}^2 u_{\alpha} = (D_i \xi_{\alpha}) w_j + \xi_{\alpha} (D_i w_j).\]

Since \(u \in C^2(\Omega)^N\), the curl of \(Du\) vanishes and we have
\[(2.4) \quad D_{ij}^2 u_{\alpha} = D_{ji}^2 u_{\alpha}.\]

Hence, by (2.2), (2.3), (2.4),
\[(2.5) \quad (D_j \xi) w_i - (D_i \xi) w_j = \xi(D_j w_i - D_i w_j).\]

Since \(|\xi|^2 = 1\) on \(\Omega_0\), we have \(D_k \xi \perp \xi = 0\) thereon. Hence, the two sides of (2.5) are normal to each other. By applying the projections \(\xi \otimes \xi\) and \([\xi]^\perp = I - \xi \otimes \xi\), (2.5) decouples on \(\Omega_0\) to
\[(2.6) \quad \text{Curl}(w)_{ij} = D_j w_i - D_i w_j \equiv 0,\]
\[(2.7) \quad (D_j \xi) w_i - (D_i \xi) w_j \equiv 0.\]

By (2.6), the curl of \(w : \Omega_0 \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n\) vanishes and by (2.1) \(w \equiv 0\) on \(\Omega \setminus \Omega_0\). Hence, since \(\Omega\) is contractible, by Poincaré’s Lemma \(w\) can be represented by the gradient of a scalar function \(f \in C^2(\Omega)\) : \(w = Df\). By (2.7), for all \(i, j \in \{1, ..., n\}\) for which \(\{w_i \neq 0\} \cap \{w_j \neq 0\} \neq \emptyset\), we have
\[(2.8) \quad \frac{D_j \xi_{\alpha}}{w_j} = \frac{D_i \xi_{\alpha}}{w_i}.\]

By (2.8), the quotient \(D_k \xi_{\alpha}/w_k\) is independent of \(k\). Hence, we may define
\[(2.9) \quad \eta := \frac{D_k \xi}{w_k} : \{w_k \neq 0\} \subseteq \Omega_0 \rightarrow \mathbb{R}^N.\]

By (2.8), \(\eta\) is well defined on all of \(\Omega_0\) since \(\bigcup_k \{w_k \neq 0\}\) is an open cover of \(\Omega_0 = \{|w| > 0\}\) and on the overlaps the different expressions coincide. By (2.9), we
ON THE STRUCTURE OF ∞-HARMONIC MAPS

have \( D_k \xi_\alpha = \eta_\alpha w_k \) on \( \{ w_k \neq 0 \} \). Actually, this extends to the whole of \( \Omega_0 \) since by (2.7) we get \( D_k \xi = 0 \) whenever \( w_k = 0 \). Thus,

\[
(2.10) \quad D\xi = \eta \otimes Df, \quad \text{on } \Omega_0,
\]

and also \( \eta \) is normal to \( \xi \), since \( \eta^\top \xi = \frac{1}{w_k} D_k (\frac{1}{2} |\xi|^2) = 0 \), on \( \{ w_k \neq 0 \} \). We now employ (2.10) to show that in a certain local sense \( \xi \) and \( f \) have the same level sets.

Fix \( \alpha \in \{ 1, \ldots, N \} \) and set

\[
(2.11) \quad A := \Omega_0 \cap \{|\eta_\alpha| > 0\},
\]

\[
(2.12) \quad g := \xi_\alpha, \quad \lambda := \eta_\alpha.
\]

We then obtain

\[
(2.13) \quad Dg = \lambda Df, \quad \text{on } A,
\]

while \( |Dg| > 0 \) and \( |\lambda| > 0 \) on \( A \). (2.13) says that the level hypersurfaces \( \{ f = f(x) \} \) and \( \{ g = g(x) \} \) passing through \( x \) have for all \( x \in A \) the same tangent spaces:

\[
(2.14) \quad [Dg]^\perp = [Df]^\perp = I - \frac{Df}{|Df|} \otimes \frac{Df}{|Df|}.
\]

Consider the level hypersurfaces of \( f, g \) as Riemannian submanifolds of \( A \) with the induced metrics from \( \mathbb{R}^n \). Since covariant derivatives coincide with tangential projections of derivatives in \( \mathbb{R}^n \), the geodesic equations for \( \chi, \psi \) with initial conditions \( \chi(0) = \psi(0) = x \in A \) and \( \dot{\chi}(0) = \dot{\psi}(0) = e \in [Df(x)]^\perp = [Dg(x)]^\perp \) are

\[
(2.15) \quad \begin{cases} [Df(\chi(t))]^\perp \ddot{\chi}(t) = 0, & t > 0, \\ \chi(0) = x, \quad \dot{\chi}(0) = e, \end{cases}
\]

\[
(2.16) \quad \begin{cases} [Dg(\psi(t))]^\perp \ddot{\psi}(t) = 0, & t > 0, \\ \psi(0) = x, \quad \dot{\psi}(0) = e. \end{cases}
\]

Since \( [Dg]^\perp = [Df]^\perp \), \( \chi \) and \( \psi \) satisfy the same ODEs with the same initial conditions. Hence, by uniqueness, \( \chi \equiv \psi \). Consequently, the exponential maps \( \exp^f_x \) and \( \exp^g_x \) of \( \{ f = f(x) \} \) and \( \{ g = g(x) \} \) coincide and hence \( (\exp^g_x)^{-1} \circ \exp^f_x \) equals the identity their common geodesically convex neighbourhood centered at \( x \). Hence, the level hypersurfaces of \( f, g \) within \( A \) coincide, but perhaps they are at different heights. Cover \( A \) by countably many balls whose radii are small enough to guarantee that the intersections of the level sets of \( f, g \) with each ball are connected.

Using this cover, we decompose \( A \) to a partition of connected Borel sets by writting \( A = \bigcup_i A_i \), where each \( A_i \) equals an open subset of the ball of the cover with possibly some boundary portion. Then, for each \( t \in \mathbb{R} \) and each \( i \in \mathbb{N} \) there is a
unique $\rho^i(t) \in \mathbb{R}$ such that \{f = t\} equals \{g = \rho^i(t)\} locally within $A_i$. Hence, there exists a unique bijection $\rho^i : f(A_i) \subseteq \mathbb{R} \rightarrow g(A_i) \subseteq \mathbb{R}$ such that
\[ (2.17) \quad \{g = \rho^i(t)\} = \{f = t\} = \{\rho^i \circ f = \rho^i(t)\}, \]
within $A_i \subseteq \Omega_0$. Equivalently,
\[ (2.18) \quad g = \rho^i \circ f, \quad \text{on } A_i, \quad i \in \mathbb{N}. \]

We extend $\rho^i$ from $f(A_i)$ to $\mathbb{R}$ by zero.

On $\Omega_0 \setminus A = \Omega_0 \setminus \bigcup_{i}^\infty A_i$, we have $Dg \equiv 0$. Hence, there exists a constant function $\rho^0 : f(\Omega_0 \setminus A) \subseteq \mathbb{R} \rightarrow \mathbb{R}$ such that
\[ (2.19) \quad g = \rho^0 \circ f, \quad \text{on } \Omega_0 \setminus \bigcup_{i}^\infty A_i. \]

We extend $\rho^0$ by zero on $\mathbb{R}$ as well. By recalling (2.11) and (2.12), we have shown that for any $\xi_\alpha$, $1 \leq \alpha \leq N$, there exists a partition of $\Omega_0$ to disjoint connected Borel sets $A^\alpha_i$ where each $A^\alpha_i$ equals an open set with possibly some boundary portion and also their complement $A^\alpha_0 := \Omega_0 \setminus \bigcup_{i}^\infty A^\alpha_i$. There also exist functions $\rho^\alpha_i : \mathbb{R} \rightarrow \mathbb{R}$ such that
\[ (2.20) \quad \xi_\alpha = \rho^\alpha_i \circ f, \quad \text{on } A^\alpha_i, \quad i = 0, 1, 2, \ldots. \]

Hence, by recalling that $|\xi| \equiv 1$ on $\Omega_0$, there exists a partition of $\Omega_0$ to connected Borel sets $\{B_i\}_{i \in \mathbb{N}}$ which are intersections of the $A_i$'s and respective bounded curves $\mu^i : \mathbb{R} \rightarrow \{0\} \cup S^{N-1} \subseteq \mathbb{R}^N$ which satisfy
\[ (2.21) \quad |\mu^i| \equiv 1 \text{ on } f(B_i), \quad \mu^i \equiv 0 \text{ on } \mathbb{R} \setminus f(B_i), \]
and are such that
\[ (2.22) \quad \xi = \mu^i \circ f, \quad \text{on } B_i, \]
for all $i \in \mathbb{N}$. We set
\[ (2.23) \quad \nu^i(t) := \int_0^t \mu^i(s) \, ds, \quad i \in \mathbb{N}. \]

Then, by (2.21) we have that $\nu^i \in W_{loc}^{1, \infty}(\mathbb{R})^N$, while $|\dot{\nu}^i| \equiv 1$ on the interval $f(B_i)$ and also $\dot{\nu}^i \equiv 0$ on $\mathbb{R} \setminus f(B_i)$. By (2.22) we have
\[ (2.24) \quad \xi = \nu^i \circ f, \quad \text{on } B_i. \]

Hence, (2.24) implies
\[ (2.25) \quad D\nu = \xi \otimes w = (\nu^i \circ f) \otimes Df = D(\nu^i \circ f), \]
on $B_i$. Thus, $u = \nu^i \circ f$ on each $B_i \subseteq \Omega_0$, up to an additive constant. By taking difference quotients in (2.24), comparing with (2.10) and passing to limits, we obtain
\[ (2.26) \quad D\xi = (\nu^i \circ f) \otimes Df, \]
and hence $\tilde{v}_i \circ f = D_k \xi_\alpha D_k f$, on $B_i$. Thus, $\tilde{v}_i$ exists on $f(B_i) \subseteq \mathbb{R}$ and is interpreted as 1-sided at the endpoints of this interval in case it is not open. Since $D u = 0$ and $D f = 0$ on $\partial(\Omega_0) \cap \Omega$, we can extend the partition $\cup_{i \in \mathbb{N}} B_i$ of $\Omega_0$ to $\overline{\Omega_0} \cap \Omega$ and further extend the families $\{B_i\}_{i \in \mathbb{N}}$ and $\{\nu^i\}_{i \in \mathbb{N}}$ by attaching the limit values and setting

\begin{align}
B_0 &:= \Omega \setminus \overline{\Omega_0}, \\
\nu^0 &:= u|_{\Omega \setminus \overline{\Omega_0}} = \text{const}.
\end{align}

Hence, since $u = \nu^i \circ f$ on each $B_i$ of the partition $\cup_{i \in \mathbb{N}} B_i = \Omega$, we conclude that $u$ is 1-rectifiable and the image $u(\Omega)$ equals a union of images of Lipschitz curves:

\begin{equation}
u^i(f(B_i)).\end{equation}

The theorem follows. \hfill \Box

As we have already mentioned in the Introduction, an extra assumptions is required in order to deduce that a rank-one map $u$ has the form $u = \nu \circ f$ for a unique single-valued unit speed curve $\nu$. This assumption guarantees “low complexity” for the direction field $\xi$.

Corollary 2.1 (Strong Rigidity of Rank-One maps). Suppose $\Omega \subseteq \mathbb{R}^n$ is open and contractible and $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^N$ is in $C^2(\Omega)$. Consider the following statements:

(i) $u$ is a strictly Rank-One map, that is $\text{rk}(D u) = 1$ on $\Omega$ or equivalently there exist $C^1$ maps $\xi : \Omega \to \mathbb{R}^N \setminus \{0\}$ and $w : \Omega \to \mathbb{R}^n \setminus \{0\}$ such that $D u = \xi \otimes w$. Moreover, the following condition holds

\begin{equation}
E := \Omega \cap \left(\bigcup_{\alpha=1}^N \partial \{|D \xi_\alpha| > 0\}\right) = \emptyset.
\end{equation}

(ii) $u$ equals the composition of a single curve $\nu \in W^{1,\infty}_{\text{loc}}(\mathbb{R})^N$ with a scalar function $f \in C^2(\Omega)$, without critical points that is $u = \nu \circ f$ with $|\dot{\nu}| \equiv 1$ on $f(\Omega)$, $\dot{\nu} \equiv 0$ on $\mathbb{R} \setminus f(\Omega)$. Moreover, $D u = (\dot{\nu} \circ f) \otimes D f$ on $\Omega$ and $u(\Omega)$ is 1-rectifiable, equal to $\nu(f(\Omega))$.

Then, (i) implies (ii) and also (ii) implies that $u$ is a strictly rank-one map, that is assertion (i) without \eqref{2.30}.

Proof of Corollary 2.1. In the setting of the proof of Theorem 1.1, if in addition the set $E$ given by \eqref{2.30} is empty and moreover $\text{rk}(D u) > 0$ on $\Omega$, then for all $\alpha \in \{1,\ldots,N\}$, either $D \xi_\alpha$ does not vanish anywhere inside $\Omega_0 = \Omega$ or it is identically constant. In both cases, the previous set $A$ is connected and coincides with $\Omega$. Hence, the curve $\nu$ constructed is unique and consequently $u = \nu \circ f$ with $|\dot{\nu}| \equiv 1$ on $f(\Omega)$ and $\dot{\nu} \equiv 0$ on $\mathbb{R} \setminus f(\Omega)$. The reverse implication is obvious. \hfill \Box

Example 2.2. The additional assumption \eqref{2.30} of Corollary 2.1 is necessary in order to obtain $u = \nu \circ f$. It reduces the complexity of $\xi$ and leads to the avoidance
of bifurcations in the curve \( \nu \). For, let \( u : \mathbb{R}^2 \to \mathbb{R}^2 \) be given by

\[
 u(x) := \begin{cases} 
 (f^4(x), f(x))^\top, & \text{on } \{ f > 0 \} \cap \{ x_1 > 0 \}, \\
 (-f^4(x), f(x))^\top, & \text{on } \{ f > 0 \} \cap \{ x_1 < 0 \}, \\
 (0, f(x))^\top, & \text{on } \{ f \leq 0 \},
\end{cases}
\]

where

\[
 f(x) := 1 - |x - e_1|^2 |x + e_1|^2.
\]

Then, \( u \) can not be written as \( u = \nu \circ f \) for a single-valued curve \( \nu \) since the unique \( \nu \) bifurcates and has two branches: \( \nu^\pm(t) = (\pm t^4 \chi_{(0,\infty)}(t), t)^\top \).

Figure 7.

2.2. Extension to Lipschitz Rank-One maps. In this subsection we extend Theorem 1.5 to the Lipschitz setting. As we have already explained, this does not follow by a standard mollification argument and an additional approximation property is required, which we introduce as assumption.

**Theorem 2.3** (Rigidity of Lipschitz Rank-One maps). Suppose \( \Omega \subseteq \mathbb{R}^n \) is open, bounded and contractible and \( u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^N \) is in \( W^{1,\infty}(\Omega)^N \).

We moreover assume that there exists a family \( \{ V^\varepsilon \}_{\varepsilon > 0} \) of rank-one smooth tensor fields in \( C^\infty(\Omega)^{Nn} \) where each \( V^\varepsilon \) is curl-free (that is \( \text{rk}(V^\varepsilon) \leq 1 \) and also \( D_j V^\varepsilon_{\alpha i} - D_i V^\varepsilon_{\alpha j} = 0 \)) such that

\[
 V^\varepsilon \rightharpoonup Du \text{ in } L^{\infty}(\Omega)^{Nn} \text{ and } V^\varepsilon \to Du \text{ a.e. on } \Omega, \text{ as } \varepsilon \to 0.
\]

Then, the following are equivalent:

(i) \( u \) is a Rank-One map, that is \( \text{rk}(Du) \leq 1 \) a.e. on \( \Omega \) or equivalently there exist \( L^\infty \) vector fields \( \xi : \Omega \to \mathbb{R}^N \) and \( w : \Omega \to \mathbb{R}^n \) such that \( Du = \xi \otimes w \) a.e. on \( \Omega \).

(ii) There exists \( f \in W^{1,\infty}(\Omega) \), a partition \( \{ B_i \}_{i \in \mathbb{N}} \) of \( \Omega \) to measurable sets which covers it a.e., that is \( |\Omega \setminus (\cup_i B_i) = 0 \) and Lipschitz curves \( \{ \nu^i \}_{i \in \mathbb{N}} \subseteq W^{1,\infty}_{\text{loc}}(\mathbb{R})^N \) such that on each \( B_i \) \( u \) equals the composition of \( \nu^i \) with \( f \):

\[
 u = \nu^i \circ f, \quad \text{on } B_i \subseteq \Omega.
\]

Moreover, \( \| \dot{\nu}^i \|_{L^\infty(\mathbb{R})} \leq 1 \) and \( \dot{\nu}^i = 0 \) a.e. on \( \mathbb{R} \setminus f(B_i) \). Also,

\[
 Du = (\dot{\nu}^i \circ f) \otimes Df, \quad \text{a.e. on } B_i \subseteq \Omega,
\]

and the image \( u(\Omega) \) is an 1-rectifiable subset of \( \mathbb{R}^n \):

\[
 \mathcal{H}^1 \left( u(\Omega) \setminus \bigcup_{i=1}^{\infty} \nu^i(\Omega(B_i)) \right) = 0.
\]
Remark 2.4. The extra approximation assumption (2.31) of Theorem 2.3 requires that $Du$ is in the intersection of the weak* and the pointwise closures in $L^\infty(\Omega)^{Nn}$ of the cone which consists of smooth rank-one curl-free tensor fields. Such an assumption is superfluous if either $\xi$ or $w$ is identically constant, since mollification of $Du = \xi \otimes w$ produces the desired approximations $V^\varepsilon$.

Generally, however, all standard mollification methods average at each point contributions from nearby points. As a result, if such a “partial affinity” of $u$ fails to hold and both $\xi$ and $w$ vary, the range $u(\Omega)$ may “fatten” and the mollification of $u$ may not be rank-one any more. Unfortunately, we have not been able neither to verify the necessity of the assumption nor to construct a proper mollification scheme allowing to drop it. Notwithstanding, this $W^{1,\infty}$-extension is not required for the phase separation theorem of the $\infty$-Laplacian.

Proof of Theorem 2.3. Is suffices to demonstrate the implication (i) $\Rightarrow$ (ii). Suppose $Du = \xi \otimes w$ a.e. on $\Omega$. By a rescaling of the form $Du = \left(\frac{1}{|\Omega|}\right) \otimes \left(|\xi| w\right)$ on $\{|\xi| > 0\}$, we may assume that $\xi : \Omega_0 \to S^{N-1}$, where $\Omega_0 := \{|Du| > 0\} \subseteq \Omega$ and also that $\xi = 0$ a.e. on $\Omega \setminus \Omega_0$. By assumption, we have $\text{rk}(V^\varepsilon) \leq 1$ and hence there exist $\xi^\varepsilon : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ and $w^\varepsilon : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ such that $V^\varepsilon = \xi^\varepsilon \otimes w^\varepsilon$. By an appropriate rescaling inside the products $\left(\frac{1}{|\xi^\varepsilon|} \otimes \left(|\xi^\varepsilon| w^\varepsilon\right)\right)$ on $\{|\xi^\varepsilon| > 0\}$, we may assume that $\xi^\varepsilon : \Omega_\varepsilon \to S^{N-1}$ where $\Omega_\varepsilon := \{|V^\varepsilon| > 0\} \subseteq \Omega$ and also that $\xi^\varepsilon \equiv 0$ on $\Omega \setminus \Omega_\varepsilon$.

We now claim that $\xi^\varepsilon \to \xi$ and also that $w^\varepsilon \to w$ as $\varepsilon \to 0$, both weakly* in $L^\infty(\Omega)$ and also a.e. on $\Omega$; indeed, there exists $\eta$ such that $\xi^\varepsilon \rightharpoonup \eta$ and hence by the $L^1(\Omega)^{Nn}$ strong convergence of $\xi^\varepsilon \otimes w^\varepsilon$ which follows by the Dominated Convergence theorem, we have

$$w^\varepsilon = (\xi^\varepsilon)^\top (\xi^\varepsilon \otimes w^\varepsilon) \rightharpoonup \eta^\top (\xi \otimes w) = (\eta^\top \xi) w,$$

as $\varepsilon \to 0$. Thus, by uniqueness of limits of $\xi^\varepsilon \otimes w^\varepsilon$ we have $((\eta \otimes \eta) \xi) \otimes w = \xi \otimes w$ a.e. on $\Omega$ and hence $\xi = \eta$. Since $\Omega$ is contractible, by Poincaré’s lemma, for any $\varepsilon > 0$ there exists a smooth map $\nu^\varepsilon : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ such that $V^\varepsilon$ can be represented as the gradient of $\nu^\varepsilon$: $Du^\varepsilon = \xi^\varepsilon \otimes w^\varepsilon$. Moreover, each $w^\varepsilon$ is a smooth rank-one map: by Theorem 1.5, there exist scalar functions $f^e \in C^\infty(\Omega)$, partitions of $\Omega$ to Borel sets $\{B^e_i\}_{i \in \mathbb{N}}$ with $\Omega = \bigcup^\infty_1 B^e_i$, families of Lipschitz curves $\{\nu^e_i\}_{i \in \mathbb{N}} \subseteq W^{1,\infty}(\mathbb{R})^N$ with $\|\nu^e_i\|_{L^\infty(\mathbb{R})} \leq 1$ and $\nu^e_i \equiv 0$ on $\mathbb{R} \setminus f^e(B^e_i)$ such that $w^\varepsilon = \nu^e \circ f^e$ on each $B^e_i \subseteq \Omega$, while the images $\nu^e(\Omega)$ are 1-rectifiable, equal to $\bigcup^\infty_1 \nu^e(B^e_i)$.

We will now show that appropriate normalised shifts of the maps $u^\varepsilon$ approximate $u$. Fix a point $\overline{x} \in \Omega$ and set $d := \text{diam}(\Omega)$. Since $Du^\varepsilon \rightharpoonup Du$ in $L^\infty(\Omega)^{Nn}$ as $\varepsilon \to 0$, for all $x, y \in \Omega$ and $\varepsilon > 0$ small we have

$$|u^\varepsilon(x) - u^\varepsilon(y)| \leq (\|Du\|_{L^\infty(\Omega)} + 1)|x - y|.$$

We further normalise $u^\varepsilon$ by considering appropriate shifts, denoted again by $u^\varepsilon$, such that $u^\varepsilon(\overline{x}) = u(\overline{x})$. By (2.36), we have

$$\|u^\varepsilon\|_{L^\infty(\Omega)} \leq d(\|Du\|_{L^\infty(\Omega)} + 1) + |u(\overline{x})|.$$

Hence, there exists $\nu$ such that $u^\varepsilon \rightharpoonup \nu$ in $W^{1,\infty}(\Omega)^N$ as $\varepsilon \to 0$. We will now show that $u \equiv \nu$. Since $Du^\varepsilon \rightharpoonup Du$ a.e. on $\Omega$, for $H^\sigma_{\nu}$-a.e. direction $\nu \in S^{n-1}$, we have that $Du^\varepsilon \to Du$ $H^\sigma$-a.e. on the set $(\overline{x} + \text{span}[\nu]) \cap \Omega = : I$. We fix such an $\nu$. By Egoroff’s theorem, for any $\sigma \in (0, 1)$, there is an $H^1$-measurable set $E_{\sigma} \subseteq I$
with $\mathcal{H}^1(E_\sigma) \leq \sigma$ such that $Du^\varepsilon \rightarrow Du$ uniformly on $I \setminus E_\sigma$ as $\varepsilon \to 0$. Since $u^\varepsilon(\bar{x}) = u(\bar{x})$, by the 1-dimensional Poincaré inequality, for $\varepsilon > 0$ small we have

$$\int_0^d |u^\varepsilon(\bar{x} + te) - u(\bar{x} + te)| \, dt \leq d \int_0^d |Du^\varepsilon(\bar{x} + te)e - Du(\bar{x} + te)e| \, dt \leq d^2 \sup_{\bar{x} \in E_\varepsilon} |Du^\varepsilon - Du| + d(2\|Du\|_{L^\infty(\Omega)} + 1)\mathcal{H}^1(E_\sigma).$$

(2.38)

Since $u^\varepsilon \rightharpoonup v$ in $C^0(\overline{\Omega})^N$ and $Du^\varepsilon \rightharpoonup Du$ in $C^0(I \setminus E_\sigma)^{Nn}$ as $\varepsilon \to 0$, by passing to the limit in (2.38) we obtain

$$\int_0^d |v(\bar{x} + te) - u(\bar{x} + te)| \, dt \leq d(2\|Du\|_{L^\infty(\Omega)} + 1)\sigma.$$  

(2.39)

By letting $\sigma \to 0$, by (2.39) we get $u \equiv v$ on $I \subseteq \Omega$. Since this holds for $\mathcal{H}^{n-1}$-a.e. direction $e \in S^{n-1}$, we get $u \equiv v$ on $\Omega$. Hence, $u^\varepsilon \rightharpoonup u$ in $W^{1,\infty}(\Omega)^N$ as $\varepsilon \to 0$. Since $Df^\varepsilon \rightharpoonup w$ in $L^\infty(\Omega)^n$, for $\varepsilon > 0$ small we have

$$|f^\varepsilon(x) - f^\varepsilon(y)| \leq (\|w\|_{L^\infty(\Omega)} + 1)|x - y|.$$  

(2.40)

We further normalise the family $f^\varepsilon$ by considering appropriate shifts denoted again by $f^\varepsilon$ such that $f^\varepsilon(\bar{x}) = f(\bar{x})$. By replacing also each $\nu^{\varepsilon,i}$ with the translate $\nu^{\varepsilon,i}(\cdot - (f(\bar{x}) - f^\varepsilon(\bar{x})))$, we do not affect the previous normalisation $u^\varepsilon(\bar{x}) = u(\bar{x})$. Consequently, (2.40) implies

$$\|f^\varepsilon\|_{L^\infty(\Omega)} \leq d(\|w\|_{L^\infty(\Omega)} + 1) + |f(\bar{x})|.$$  

(2.41)

As a result, there exists an $f$ such that $f^\varepsilon \rightharpoonup f$ in $W^{1,\infty}(\Omega)$ as $\varepsilon \to 0$.

Since $\nu^{\varepsilon,i} \circ f^\varepsilon = \xi^\varepsilon$ on $B_1^\varepsilon$ and $\nu^{\varepsilon,i} \circ f^\varepsilon = 0$ on $\Omega \setminus B_1^\varepsilon$, for $\varepsilon, \delta > 0$ small we have

$$\|B_1^\varepsilon \Delta B_1^\delta\| = \int_\Omega |\chi_{B_1^\varepsilon} - \chi_{B_1^\delta}| = \int_\Omega |\nu^{\varepsilon,i} \circ f^\varepsilon| - |\nu^{\varepsilon,i} \circ f^\delta| \leq \int_\Omega |\xi^\varepsilon - \xi^\delta|.$$  

(2.42)

Since $\xi^\varepsilon \rightarrow \xi$ in $L^1(\Omega)^N$, for each $i \in \mathbb{N}$ the family $\{B_1^\varepsilon\}_{\varepsilon > 0}$ is Cauchy in measure and hence has a measurable limit $B_i \subseteq \Omega$. Since for all $\varepsilon > 0$ we have $\Omega = \bigcup_{i=1}^\infty B_i^\varepsilon$ and $B_i^\varepsilon \cap B_j^\varepsilon = \emptyset$ for $i \neq j$, the limit family forms a cover of $\Omega$ except perhaps for a nullset: $|\Omega \setminus (\bigcup_{i=1}^\infty B_i)| = 0$. We recall that we have $u^\varepsilon = \nu^{\varepsilon,i} \circ f^\varepsilon$ on $B_1^\varepsilon$ and also $\|\nu^{\varepsilon,i}\|_{L^\infty(\mathbb{R})} \leq 1$ and $\nu^{\varepsilon,i} \equiv 0$ on $\mathbb{R} \setminus f(\Omega)$. 

Figure 8.
Hence, if \( \bar{\pi} \in B^*_i \), for any \( t \in \mathbb{R} \) we have
\[
|\nu^i(t)| \leq \|\dot{\nu}^i\|_{L^\infty(\mathbb{R})}|t - f^i(\overline{\pi})| + |\nu^i(f^i(\overline{\pi}))|
\]
(2.43)
\[= |t - f(\overline{\pi})| + |u(\overline{\pi})|.\]

If \( \bar{\pi} \notin B^*_i \), then \( f(\bar{\pi}) \) is in the complement of the interval \( f^i(B^*_i) \) and since \( |\nu^i| \) is constant on \( \mathbb{R} \setminus f^i(B^*_i) \), for any \( t \in \mathbb{R} \) we have
\[
|\nu^i(t)| \leq \|\dot{\nu}^i\|_{L^\infty(\mathbb{R})}|t - f(\overline{\pi})| + |\nu^i(f(\overline{\pi}))|
\]
(2.44)
\[\leq |t - f(\overline{\pi})| + \max \|\dot{\nu}^i(f^i(B^*_i))\| \leq |t - f(\overline{\pi})| + \|u^i\|_{L^\infty(\Omega)}.
\]

As a result, since the family \( u^i \) is uniformly bounded on \( \Omega \), for each \( i \in \mathbb{N} \) the family \( \{\nu^i\}_{\varepsilon > 0} \subseteq W^{1,\infty}_{\text{loc}}(\mathbb{R}) \) has a weak* limit \( \nu^i \in W^{1,\infty}_{\text{loc}}(\mathbb{R}) \) which satisfies \( \|\dot{\nu}^i\|_{L^\infty(\mathbb{R})} \leq 1 \). By passing to the limit as \( \varepsilon \to 0 \) we get \( u = \nu^i \circ f \) on \( B_i \subseteq \Omega \) and \( \nu^i = 0 \) on \( \mathbb{R} \setminus f(B_i) \). Finally, the image \( u(\Omega) \) is 1-rectifiable in \( \mathbb{R}^N \) and up to an \( \mathcal{H}^1 \)-nullset of \( \mathbb{R}^N \), we have \( u(\Omega) = \bigcup_i \nu^i(f(B_i)) \). The theorem follows. \( \square \)

3. The structure of 2-dimensional \( \infty \)-Harmonic maps.

In this section we use the Rigidity Theorem 1.5 proved in Section 2 to analyse the phase separation of classical solutions to (1.3) when \( n = 2 \) and \( N \geq 2 \).

**Proof of Theorem 1.1.** We begin by setting
\[
\Omega_1 := \text{int}\{\text{rk}(Du) \leq 1\},
\]
(3.1)
\[
\Omega_2 := \{\text{rk}(Du) = 2\},
\]
(3.2)
and let also \( S := \Omega \setminus (\Omega_1 \cup \Omega_2) \). Our PDE system (1.3) decouples to
\[
Du \left( \frac{1}{2} |Du|^2 \right) = 0,
\]
(3.3)
\[
|Du|^2 D^\perp u = 0.
\]
(3.4)
On \( \Omega_2 \), we have \( \text{rk}(Du) = 2 \) and hence \( u\big|_{\Omega_2} : \Omega_2 \to \mathbb{R}^N \) is an immersion. Thus, \( Du(x) \) possesses a left inverse \((Du(x))^{-1}\) for all \( x \in \Omega_2 \). Hence, (3.3) implies
\[
(Du)^{-1} Du \left( \frac{1}{2} |Du|^2 \right) = 0
\]
(3.5)
and hence \( D\left( \frac{1}{2} |Du|^2 \right) = 0 \) on \( \Omega_2 \), or equivalently
\[
|Du|^2 = \text{const.,}
\]
(3.6)
on each connected component of \( \Omega_2 \). Moreover, (3.6) holds on \( S \) as well, the common boundary of \( \Omega_2 \) and \( \Omega_1 \).

![Figure 9](image-url)
On the other hand, on $\Omega_1$ we have $\text{rk}(Du) \leq 1$. Hence, there exist vector fields $\xi : \Omega_1 \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^N$ and $w : \Omega_1 \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^n$ such that $Du = \xi \otimes w$. Suppose first that $\Omega_1$ is contractible. Then, by the Rigidity Theorem 1.5, there exists a function $f \in C^2(\Omega_1)$, a partition of $\Omega$ into $\{B_i\}_{i \in \mathbb{N}}$ and Lipschitz curves $\{\nu^i\}_{i \in \mathbb{N}} \subseteq W^{1,\infty}(\mathbb{R})^N$ with $|\nu^i| \equiv 1$ on $f(B_i)$, $|\nu^i| \equiv 0$ on $\mathbb{R} \setminus f(B_i)$ twice differentiable on $f(B_i)$, such that $u = \nu^i \circ f$ on each $B_i$ and hence $Du = (\nu^i \circ f) \otimes Df$ on $B_i$. By (3.3), we obtain

\[(\nu^i \circ f) \otimes Df \otimes (\nu^i \circ f) \otimes Df) = 0,\]

on $B_i \subseteq \Omega_1$. Since $|\nu^i| \equiv 1$ on $f(B_i)$, we have that $\tilde{\nu}^i$ is normal to $\nu^i$ and hence

\[(\nu^i \circ f) \otimes Df \otimes (\nu^i \circ f) \otimes Df) = 0,\]

on $B_i \subseteq \Omega_1$. Hence, by using again that $|\nu^i|^2 \equiv 1$ on $f(B_i)$ we get

\[(Df \otimes Df : D^2f)(\nu^i \circ f) = 0,\]

on $B_i \subseteq \Omega_1$. Thus, $\Delta_\infty f = 0$ on $B_i$. By (3.4) and again since $|\nu^i|^2 \equiv 1$ on $f(B_i)$, we have $|Du|^+ = |\nu^i \circ f|^+$ and hence

\[|Df|^2 |\nu^i \circ f|^+ \text{Div}(\nu^i \circ f) \otimes Df) = 0,\]

on $B_i \subseteq \Omega_1$. Hence,

\[|Df|^2 |\nu^i \circ f|^+ \left((\nu^i \circ f) \cdot Df|^2 + (\nu^i \circ f) \Delta f\right) = 0,\]

on $B_i$, which by using once again $|\nu^i|^2 \equiv 1$ gives

\[|Df|^4 (\nu^i \circ f) = 0,\]

on $B_i$. Since $\Delta_\infty f = 0$ on $B_i$, and $\Omega_1 = \bigcup_i B_i$, $f$ is $\infty$-Harmonic on $\Omega_1$. Thus, by Aronsson’s theorem in [A4], either $|Df| > 0$ or $|Df| \equiv 0$ on $\Omega_1$.

If the first alternative holds, then by (3.12) we have $\tilde{\nu}^i \equiv 0$ on $f(B_i)$ for all $i$ and hence $\nu^i$ is affine on $f(B_i)$, that is $\nu^i(t) = t\xi^i + a^i$ for some $|\xi^i| = 1, a^i \in \mathbb{R}^N$. Thus, since $u = \nu^i \circ f$ and $u \in C^2(\Omega_1)^N$, all $\xi^i$ and all $a^i$ coincide and consequently $u = \xi^i + a$, $\xi \in S^{N-1}$, with $a \in \mathbb{R}^N$ and $f \in C^2(\Omega_1)$.

If the second alternative holds, then $f$ is constant on $\Omega_1$ and hence by the representation $u = \nu^i \circ f$, $u$ is piecewise constant on each $B_i$. Since $u \in C^2(\Omega_1)^N$ and $\Omega_1 = \bigcup_i B_i$, necessarily $u$ is constant on $\Omega_1$. But then $|Du|_{\Omega_1} = |Df|_{\Omega_2} = 0$ and necessarily $\Omega_2 = \emptyset$. Hence, $|Du| \equiv 0$ on $\Omega_1$, that is $u$ is affine on each of the connected components of $\Omega_1$.

If $\Omega_1$ is not contractible, cover it with balls $\{B_m\}_{m \in \mathbb{N}}$ and apply the previous argument. Hence, on each $B_m$, we have $u = \xi^m f^m + a^m$, $\xi^m \in S^{N-1}, a^m \in \mathbb{R}^N$, and $f^m \in C^2(B_m)$ with $\Delta_\infty f^m = 0$ on $B_m$ and hence either $|Df^m| > 0$ or $|Df^m| \equiv 0$. Since $u \in C^2(\Omega_1)^N$, on the overlaps of the balls the different expressions of $u$ must coincide and hence we obtain $u = \xi f + a$ for $\xi \in S^{N-1}, a \in \mathbb{R}^N$ and $f \in C^2(\Omega_1)$ where $\xi$ and $a$ may vary on different connected components of $\Omega_1$. The theorem follows. \[\square\]

Theorem 1.1 implies a vectorial version of the Maximum Principle when $n = N = 2$, which we now prove.
Proof of Corollary 1.3. We begin by observing that (1.13) is an elegant restate-
ment of the Maximum Principle for all projections $\eta^\top u$ of $u$, that is, when for all
$\Omega' \subset \Omega$ and all directions $\eta \in S^{N-1}$ we have
\begin{equation}
\sup_{\Omega'} \eta^\top u \leq \max_{\partial \Omega'} \eta^\top u.
\end{equation}
Indeed, (3.13) says that $u(\Omega')$ is contained in the intersection of all halfspaces
containing $u(\partial \Omega')$. To see (3.13), fix $\Omega'$ and $\eta \in S^{N-1}$ and let $\Omega_1, \Omega_2, S$
respectively be the constant rank domains and the interface of $u$, as in Theorem 1.1. Suppose
that $u = \xi f + a$ on $\Omega_1 \cup S$, where $\xi \in S^{N-1}$, $a \in \mathbb{R}^N$ and $f \in C^2(\Omega_1 \cup S)$. Then,
\begin{equation}
|D(\eta^\top u)| = |\eta^\top Du|_\Omega_2 + |\eta^\top Du|_S \cup \Omega_1 = |\eta^\top Du|_\Omega_2 + |\eta^\top \xi||Df|_S \cup \Omega_1.
\end{equation}
If $|Df| \equiv 0$ on $\Omega_1$, then $\Omega_2 = \emptyset$ and $u$ is affine. Hence, (3.13) follows. Suppose now
$|Df| > 0$ on $\Omega_1$. Since $u|_{\Omega_2}$ is a local diffeomorphism, we have $|\eta^\top Du| > 0$ for all
$\eta \in S^{N-1}$.

Consequently, for all $\eta \in S^{N-1} \setminus [\xi]^\perp$, in view of (3.14) we have $|D(\eta^\top u)| > 0$
on $\Omega$. Hence, $\eta^\top u$ has no interior critical points inside $\Omega$ and consequently we have
\begin{equation}
\max_{\Omega'} \eta^\top u = \max_{\partial \Omega'} \eta^\top u,
\end{equation}
for all directions $\eta \not\perp \xi$. By letting $\text{dist}(\eta, [\xi]^\perp) \to 0$, (3.15) implies (1.13). □

4. Characterisation of the class of elliptic PDE systems.

In this section we focus on the general Aronsson system (1.1). As already explained in
the introduction, when $N \geq 2$ the normal coefficient $H[H_P]^\perp H_{PP}$ is not
symmetric and as a result the system generally is not degenerate elliptic, not
even for strictly convex Hamiltonians. In Theorem 4.1 below we establish that all
"geometric" Hamiltonians which depend on $Du$ via the induced Riemannian metric
$Du^\top Du$ lead to elliptic systems. Moreover, in low dimensions $n \leq 3$ the converse
is true as well for (normalised) analytic Hamiltonians with fully symmetric Hes-
sian tensor. When $n \geq 4$, there appear complicated structures in the minors of
forth and higher order derivatives and an additional assumption is required. The
constructive method of proof reveals that it is necessary. The main idea in the
reverse direction is to impose the commutativity relation $[H_P]^\perp H_{PP} = H_{PP}[H_P]^\perp$
and use power-series expansions of $H$ and induction, by a term-after-term blow-up
argument along inverse images under $H_P$ of rank-one directions.

Theorem 4.1 (Classification of Hamiltonians leading to elliptic systems (1.1)).
Suppose that $H \in C^2(\mathbb{R}^{N \times n})$ is a non-negative Hamiltonian with $n \geq 1$, $N \geq 2$.
Suppose also that $[H_P(P)]^\perp = [P]^\perp$ on $\mathbb{R}^{N \times n}$. Consider the following statements:
(i) There exists \( h \in C^2(S(\mathbb{R}^n)^+) \), with symmetric gradient \( h_p \), such that

\[
H(p) = h\left(\frac{1}{2} P^T P\right).
\]

(ii) The system

\[
A_{\infty} u := \left( H_P \otimes H_P + H [H_P]_+ H_{PP} \right) (Du) : D^2 u = 0
\]

is degenerate elliptic, that is, the tensor map

\[
A_{\alpha\beta\gamma}(p) := H_{P_{\alpha\beta}}(p) H_{P_{\gamma}}(p) + H(p)[H_P(p)]_+ H_{P_{\alpha\eta}} H_{P_{\beta\eta}}(p)
\]

satisfies the Legendre-Hadamard condition and the symmetry condition

\[
A(p) : (\eta \otimes w) \otimes (\eta \otimes w) \geq 0,
\]

\[
A(p) : (Q \otimes R - R \otimes Q) = 0,
\]

for all \( \eta \in \mathbb{R}^N \), \( w \in \mathbb{R}^n \) and \( P, Q, R \in \mathbb{R}^{N \times n} \).

Then, (i) implies (ii). If moreover \( H \) is analytic at 0 and satisfies

\[
\{ H = 0 \} = \{ H_P = 0 \} = 0, \quad H_{PP}(0) > 0 \& H_{PP} : (v \otimes w - w \otimes v) = 0,
\]

for \( v, w \in \mathbb{R}^n \), \( P \in \mathbb{R}^{N \times n} \), then, (ii) implies (i) when either

a) \( n \leq 3 \),

or

b) \( n \geq 4 \) and the \( q \)-th order derivative tensor \( H_{P_{\ldots}P}(0) \in \otimes^{(q)}(\mathbb{R}^n \otimes \mathbb{R}^n) \) is contained in the linear subspace \( \mathcal{L}^q \) which consists of fully symmetric tensors \( T \) for which the only non-trivial components are of the form \( T_{\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_k} \), where \( \alpha_m \in \{1, \ldots, N\} \), \( i, j, k \in \{1, \ldots, n\} \).

If \( n \geq 4 \) but \( \mathcal{H}_{\ldots}P(0) \notin \mathcal{L}^q \), then \( H \) has the form (4.1) up to a fourth order correction: \( H(P) = h\left(\frac{1}{2} P^T P\right) + O(|P|^4) \). If \( N \leq n \), it does not follow that \( h_p > 0 \).

In the case that \( H(P) \) equals \( h\left(\frac{1}{2} P^T P\right) \), the elliptic system takes the form

\[
A_{\infty} u = \left( Du h_p \otimes Du h_p + h[Du]_+ \otimes h_p \right) : D^2 u = 0
\]

with \( h = h\left(\frac{1}{2} Du^T Du\right) \).

The extra assumption \( \mathcal{H}_{\ldots}P(0) \in \mathcal{L}^q \) is necessary only in higher dimensions \( n \geq 4 \). It requires that \( \mathcal{H}_{\ldots}P(0) \) vanishes when more than 3 of its Latin indices are different to each other. The linear space \( \mathcal{L}^q \) can be described as

\[
\mathcal{L}^q := \left\{ T \in \otimes^{(q)}(\mathbb{R}^n \otimes \mathbb{R}^n) \mid T = T_{\alpha_1 \ldots \alpha_q i_1 i_2 \ldots i_q} \otimes e_{\alpha_1 i_1} \otimes \ldots \otimes e_{\alpha_q i_q} : \right. \]

\[
T_{\ldots \alpha_1 \ldots \beta_i \ldots} = T_{\ldots \beta_1 \ldots \alpha_2 \ldots} = T_{\ldots \beta_2 \ldots \alpha_3 \ldots} \ldots \]

\[
\left. \{i_1, \ldots, i_q\} \neq \{i, j, k, \ldots, k\} \implies T = 0 \right\}.
\]

If \( H_{\ldots}P(0) \notin \mathcal{L}^q \), then Hamiltonians with a little more complicated fourth and higher order derivatives also give rise to elliptic systems.

**Proof of Theorem 4.1.** We first prove the implication (i) \( \Rightarrow \) (ii). For, assume that the Hamiltonian \( H \) has the form (4.1). We begin by observing that the symmetry assumption \( h_{\alpha\beta} = h_{\beta\alpha} \) implies that second derivatives of \( h \) are fully symmetric in all indices: obviously since \( h \) is in \( C^2(S(\mathbb{R}^n)) \) we have \( h_{\alpha\beta\gamma} = h_{\beta\alpha\gamma} = h_{\beta\gamma\alpha} \) and also

\[
h_{\alpha\beta\gamma} = (h_{\alpha\beta})_{\gamma} = (h_{\beta\gamma})_{\alpha} = (h_{\gamma\alpha})_{\beta}.
\]
We set analyticity of $H_p(4.18)$ 

$$H_p(4.21)$$

and

$$H_p(4.17)$$

Also, by assumption $[H_P(P)]^\perp = [P]^\perp$. By (4.10) and (4.11), we have

$$H_P \otimes H_P + H[H_P]^\perp H_P P(4.12) = Ph_p \otimes Ph_p + h[P]^\perp (I \otimes h_p + Ph_p P^\top)$$

where $h = h(1/2 P^T P)$. Hence, in view of (4.3), equation (4.7) follows. Also, since $h \geq 0$ and $P$, $[P]^\perp$ are positive symmetric, conditions (4.4) and (4.5) follow as well:

$$A(P) : (\eta \otimes w) \otimes (\eta \otimes w) = P_{\alpha k} h_{\alpha} w_{\gamma} P_{\beta l} h_{\beta j} \eta_{\gamma} w_{\beta j} + h[P]^\perp \eta_{\alpha \beta} \eta_{\gamma} \eta_{\delta} \eta_{\beta j} \eta_{\delta i}$$

$$\geq 0,$$

$$A(P) : (Q \otimes R - R \otimes Q) = \pm P_{\alpha k} Q_{\alpha l} P_{\beta m} h_{\beta p} R_{\beta j} \pm h[P]^\perp Q_{\alpha l} h_{\beta p} R_{\beta j}$$

$$\geq 0,$$

for all $\eta, w \in \mathbb{R}^n$, $P, Q, R \in \mathbb{R}^{N \times n}$. Hence, (ii) follows.

Now we assume (ii) and prove the reverse implication. For, suppose $H$ is analytic at 0 and suppose that (4.4) - (4.8) hold. By (4.5), we have

$$H_P \otimes H_P + H[H_P]^\perp H_P P(4.15) = 0,$$

for all $P, Q, R \in \mathbb{R}^{N \times n}$. By symmetry of $H_P \otimes H_P$ and since by (4.6) we have $H > 0$ and $H_P \neq 0$ on $(\mathbb{R}^{N \times n}) \setminus \{0\}$, (4.15) gives

$$[H_P]^\perp H_P P(4.16) = 0.$$

By the identity $[H_P]^\perp = I - [H_P]^\top$ and since $I, H_P P$ and $[H_P]^\perp$ are symmetric, for $Q = e_\alpha \otimes e_i$ and $R = e_\beta \otimes e_j$, (4.16) gives the commutativity relation

$$[H_P]^\perp \alpha \gamma H_P, P_{\beta j} = H_P, P_{\beta j} [H_P]^\perp \alpha \gamma$$

on $(\mathbb{R}^{N \times n}) \setminus \{0\}$, that is

$$[H_P]^\perp H_P P = H_P P [H_P]^\perp.$$

We set $A_{\alpha \beta j} := H_{P_{\alpha i} P_{\beta j}}(0)$. By assumption (4.6), we have $A > 0$ in $S(\mathbb{R}^{N \times n})$. By analyticity of $H$ and since $H(0) = 0$ and $H_P(0) = 0$, we have

$$H(P) = \frac{1}{2} A : P \otimes P + O(|P|^3),$$

$$H_P(P) = A : P + O(|P|^2),$$

$$H_P P(4.19) = A + O(|P|).$$
as $|P| \to 0$. Since $A = H_{PP}(0) > 0$ and $H_{P}(0) = 0$, the map $H_{P}$ is a diffeomorphism between open neighbourhoods of zero in $\mathbb{R}^{N \times n}$. Hence, there is an $r > 0$ such that

$$H_{P} : \mathbb{B}_{r}^{N} := \{Q \in \mathbb{R}^{N \times n} : |Q| < r\} \to H_{P}(\mathbb{B}_{r}^{N}) \subseteq \mathbb{R}^{N \times n}$$

is a diffeomorphism. Hence, there is a $\rho > 0$ such that for $0 < t < \rho$, $\xi \in S^{n-1}$ and $\omega \in S^{n-1}$, there exists a unique $P(t) \in \mathbb{B}_{r}^{N}$ such that

$$t\xi \otimes \omega = H_{P}(P(t)).$$

Moreover, $|P(t)| \to 0$ as $t \to 0$. The path $P(\cdot)$ is the inverse image through $H_{P}$ of the rank-one line spanned by $\xi \otimes \omega$. By (4.23), we have

$$[H_{P}(P(t))]^{\top} = [t\xi \otimes \omega]^{\top} = \xi \otimes \xi.$$  

By evaluating (4.18) at $P(t)$ and using (4.24) and (4.21), we obtain

$$(\xi \otimes \xi)(A + o(1)) = (A + o(1))(\xi \otimes \xi),$$

at $t \to 0$. In the limit we get $(\xi \otimes \xi)A = A(\xi \otimes \xi)$, that is

$$(\xi_{\alpha}\xi_{\gamma}A_{\alpha\gamma}) = A_{\alpha\gamma}(\xi_{\alpha}\xi_{\gamma}),$$

for all $i, j \in \{1, \ldots, n\}$, $\alpha, \beta \in \{1, \ldots, N\}$. By the symmetry condition in assumption (4.6), for all $i, j$ fixed the matrix $A_{\alpha\gamma}$ commutes with all $1$-dimensional projections of $\mathbb{R}^{N}$. Hence, it is simultaneously diagonalisable with them and as such a multiple of the identity. Thus, there is a symmetric matrix $A_{\alpha\gamma}$ such that

$$A_{\alpha\gamma} = \hat{A}_{\alpha\gamma}\delta_{\alpha\gamma}.$$  

Consequently, $A : P \otimes P = \hat{A} : P^{\top}P$. We now set $B_{\alpha\gamma} := H_{P\alpha\gamma}(P_{\alpha\gamma}(0))$. Then, by (4.27), equations (4.20) and (4.21) become

$$H_{P}(P) = PA + O(|P|^{2}),$$

$$H_{PP}(P) = I \otimes \hat{A} + \frac{1}{2} B : P + O(|P|^{2}),$$

and hence by (4.28) and (4.23) we get

$$t\xi \otimes \omega = P(t)\hat{A} + O(|P(t)|^{2}).$$

Since $A > 0$ in $S(\mathbb{R}^{N \times n})$, we have $\hat{A} > 0$ in $S(\mathbb{R}^{n})$ as well. Thus, for $0 < t < \rho$, we have

$$\frac{P(t)}{|P(t)|} + O(|P(t)|) = \frac{t}{|P(t)|}\xi \otimes ((\hat{A}^{-1})^{\top}w).$$

As $t \to 0$, we have $|P(t)| \to 0$ and by compactness along an infinitesimal sequence $t_{m} \to 0$ there exists a $\bar{P}$ with $|\bar{P}| = 1$ such that $P(t_{m})/|P(t_{m})| \to \bar{P}$. By passing to the limit in (4.31) as $m \to \infty$ along $\{t_{m}\}$, we obtain that the limit of $t_{m}/|P(t_{m})|$ exists and

$$\lim_{m \to \infty} \frac{P_{\alpha\gamma}(t_{m})}{|P_{\alpha\gamma}(t_{m})|} = \bar{P} = \xi \otimes \left(\lim_{m \to \infty} \frac{t_{m}}{|P(t_{m})|}\right)^{\top}w.$$  

Since $\hat{A}^{-1} > 0$ and $|P| = 1$, for any $v \in S^{n-1}$, there is a $\omega \in S^{n-1}$ such that (4.32) becomes

$$\lim_{m \to \infty} \frac{P_{\alpha\gamma}(t_{m})}{|P(t_{m})|} = \bar{P} = \xi \otimes v.$$
By (4.18), (4.24), (4.28) (4.29), we have

\[
\xi \otimes \xi \left( I \otimes A + \frac{1}{2} B : P(t) + O(|P(t)|^2) \right)
\]

(4.34)

By cancelling the commutative term \( \xi \otimes \xi (I \otimes A) \), (4.34) gives

\[
\xi \otimes \xi \left( B : \frac{P(t)}{|P(t)|} + O(|P(t)|) \right) = \left( B : \frac{P(t)}{|P(t)|} + O(|P(t)|) \right) \xi \otimes \xi.
\]

(4.35)

By passing to the limit in (4.35) along \( t_m \to 0 \), in view of (4.33) we obtain

\[
\xi \otimes \xi (B : \xi \otimes v) = (B : \xi \otimes v) \xi \otimes \xi,
\]

for all \( \xi \in S^{n-1} \), \( v \in S^{n-1} \). Hence, (4.36) for \( v = e_k \) says

\[
\xi_\alpha (B_{\beta \lambda \mu \kappa} \xi_\mu \xi_\lambda) = (B_{\alpha \lambda \mu \kappa} \xi_\mu \xi_\lambda) \xi_\beta.
\]

(4.37)

By (4.37), \( B : \xi \otimes \xi \) is proportional to \( \xi \); hence, there is a map \( \hat{B} : \mathbb{R}^N \to \otimes^3 \mathbb{R}^n \) such that \( B : \xi \otimes \xi = \hat{B}(\xi) \otimes \xi \), or

\[
B_{\alpha \lambda \mu \kappa} \xi_\mu \xi_\lambda = \hat{B}_{\iota j k}(\xi) \xi_\iota.
\]

(4.38)

By assumption (4.6) and induction, all second and higher order derivatives are fully symmetric in all their indices. Hence, we may fix \( i, j, k \in \{1, \ldots, n \} \) and suppress the dependence in them to obtain \( B_{\alpha \kappa} \xi_\kappa \xi_\lambda = \hat{B}(\xi) \xi_\alpha \) with \( \hat{B} \in C^\infty (\mathbb{R}^N \setminus \{0\}) \).

The idea now is to differentiate in order to cancel both \( \xi \)'s contracted with \( B \) and then contract again with a vector which annihilates \( \xi \) from the right hand side. For, by differentiating we get

\[
D_\beta \hat{B}(\xi) \xi_\alpha = -\hat{B}(\xi) \delta_\alpha_\beta + 2B_{\alpha \beta \gamma} \xi_\gamma.
\]

(4.39)

By (4.39), we obtain that \( D \hat{B}(\xi) \otimes \xi \) is symmetric. Hence, we get that \( D \hat{B}(\xi) \otimes \xi = \xi \otimes D \hat{B}(\xi) \) and hence there exists \( \bar{B} \in C^\infty (\mathbb{R}^N \setminus \{0\}) \), such that \( D \bar{B}(\xi) = \hat{B}(\xi) \xi \).

Thus, (4.39) gives

\[
\bar{B}(\xi) \xi \otimes \xi + \hat{B}(\xi) I = 2B : \xi.
\]

(4.40)

By differentiating the expression \( D \hat{B}(\xi) = \bar{B}(\xi) \xi \), we get

\[
D \bar{B}(\xi) \otimes \xi = D^2 \bar{B}(\xi) - \bar{B}(\xi) I.
\]

(4.41)

By (4.41), we obtain that \( D \bar{B}(\xi) \otimes \xi \) is symmetric too. Hence, there exists \( \bar{B} \in C^\infty (\mathbb{R}^N \setminus \{0\}) \), such that \( D \bar{B}(\xi) = \bar{B}(\xi) \xi \) and hence (4.41) gives

\[
D^2 \bar{B}(\xi) = \bar{B}(\xi) \xi \otimes \xi + B(\xi) I.
\]

(4.42)

By differentiating (4.39) again and inserting (4.42) we get

\[
2B_{\alpha \beta \gamma} = D^2_{\beta \gamma} \bar{B}(\xi) \xi_\alpha + D_\beta \bar{B}(\xi) \delta_\alpha_\gamma + D_\gamma \bar{B}(\xi) \delta_\alpha_\beta \\
= \bar{B}(\xi) \xi_\alpha \xi_\beta \xi_\gamma + \bar{B}(\xi) \left( \xi_\alpha \delta_\beta_\gamma + \xi_\beta \delta_\alpha_\gamma + \xi_\gamma \delta_\alpha_\beta \right),
\]

for all \( \xi \in \mathbb{R}^N \setminus \{0\} \). Since \( N \geq 2 \), for each \( \eta \in \mathbb{R}^N \) we can choose a nonzero \( \xi \) normal to \( \eta \). Hence, by triple contraction in (4.43) we obtain

\[
B : \eta \otimes \eta \otimes \eta = \frac{1}{2} \left[ \bar{B}(\xi)(\xi^T \eta)^2 + \bar{B}(\xi)|\eta|^2 \right] (\xi^T \eta) = 0.
\]

(4.44)
Hence, by full symmetry in all indices we obtain \( H_{P_{\alpha}P_{\beta}P_{\gamma}P_{\delta}}(0) = B_{\alpha\beta\gamma\delta} = 0 \) and consequently third order derivatives vanish. We now set
\[
(4.45) \quad C_{\alpha\beta\gamma\delta\kappa\lambda} := H_{P_{\alpha}P_{\beta}P_{\gamma}P_{\delta}}(0)
\]
and then for \( 0 < t < \rho \), (4.29) and (4.35) become
\[
(4.46) \quad H_{PP}(P(t)) = I \otimes \hat{A} + \frac{1}{3!} C : P(t) \otimes P(t) + O(|P(t)|^3),
\]
\[
(4.47) \quad \xi \otimes \xi \left( C : \frac{P(t)}{|P(t)|} \otimes \frac{P(t)}{|P(t)|} + O(|P(t)|) \right) = \left( C : \frac{P(t)}{|P(t)|} \otimes \frac{P(t)}{|P(t)|} + O(|P(t)|) \right) \xi \otimes \xi.
\]
By setting \( t = t_m \) and letting \( m \to \infty \), in view of (4.33), we get
\[
(4.48) \quad \xi \otimes \xi \left[ C : (\xi \otimes v) \otimes (\xi \otimes v) \right] = \left[ C : (\xi \otimes v) \otimes (\xi \otimes v) \right] \xi \otimes \xi,
\]
for all \( \xi \in \mathbb{R}^N, v \in \mathbb{R}^n \). Hence, for \( v = e_k \),
\[
(4.49) \quad \xi_{\alpha} \left[ C_{\beta\gamma} \xi_{\beta} \xi_{\gamma} \right] = \left[ C_{\alpha\beta\gamma} \xi_{\beta} \xi_{\gamma} \right] \xi_{\alpha}.
\]
By (4.49), there exists a map \( \tilde{C} : \mathbb{R}^N \setminus \{0\} \to \otimes^4 \mathbb{R}^n \) with \( \tilde{C}_{ijk} \in C^\infty(\mathbb{R}^N \setminus \{0\}) \) such that
\[
(4.50) \quad C_{\alpha\beta\gamma\delta\kappa\lambda} \xi_{\alpha} \xi_{\beta} \xi_{\gamma} \xi_{\delta} = \tilde{C}_{ijk}(\xi) \xi_{\alpha}.
\]
By fixing again the indices \( i, j, k \), dropping them and arguing exactly as we did before for \( B_{\alpha\beta\gamma} \), there exist \( \tilde{C}, \tilde{C} \in C^\infty(\mathbb{R}^N \setminus \{0\}) \) such that
\[
(4.51) \quad 3! C_{\alpha\beta\gamma\delta} \xi_{\delta} = \tilde{C}(\xi) \xi_{\alpha} \xi_{\beta} \xi_{\gamma} + \tilde{C}(\xi) \left( \xi_{\alpha} \xi_{\beta} \xi_{\delta} + \xi_{\alpha} \xi_{\gamma} \xi_{\delta} + \xi_{\beta} \xi_{\gamma} \xi_{\delta} \right).
\]
By differentiating (4.51), we get
\[
3! C_{\alpha\beta\gamma\delta} - \tilde{C}(\xi) \left( \delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\gamma\beta} \delta_{\alpha\delta} + \delta_{\delta\gamma} \delta_{\alpha\beta} \right)
\]
\[
= \tilde{C}(\xi) \left( \xi_{\alpha} \xi_{\beta} \xi_{\gamma} \xi_{\delta} + \xi_{\alpha} \xi_{\gamma} \xi_{\delta} \xi_{\beta} + \xi_{\beta} \xi_{\gamma} \xi_{\delta} \xi_{\alpha} \right) + D_{\delta} \tilde{C}(\xi) \left( \xi_{\alpha} \xi_{\beta} \xi_{\gamma} \xi_{\delta} + \xi_{\alpha} \xi_{\gamma} \xi_{\delta} \xi_{\beta} + \xi_{\beta} \xi_{\gamma} \xi_{\delta} \xi_{\alpha} \right).
\]
Fix \( \eta \in \mathbb{R}^N \). Since \( N \geq 2 \), there exists \( \xi \perp \eta, \xi \neq 0 \). Then, (4.52) gives
\[
\left[ C_{\alpha\beta\gamma\delta} - \frac{\tilde{C}(\xi)}{3!} \left( \delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\gamma\beta} \delta_{\alpha\delta} + \delta_{\delta\gamma} \delta_{\alpha\beta} \right) \right] \eta_{\alpha} \eta_{\beta} \eta_{\gamma} \eta_{\delta} = O(|\eta|^3)
\]
\[
(4.53) \quad = 0.
\]
By (4.53), the function \( \tilde{C} \) is constant and moreover for all \( i, j, k \),
\[
(4.54) \quad C_{\alpha\beta\gamma\delta\kappa\lambda} = \frac{\tilde{C}_{ijk}}{3!} \left( \delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\gamma\beta} \delta_{\alpha\delta} + \delta_{\delta\gamma} \delta_{\alpha\beta} \right).
\]
If either \( n \leq 3 \) or \( n \geq 4 \) but \( H_{PPPP}(0) \in \mathcal{L}^4 \), where \( \mathcal{L}^4 \) is given by (4.8), then in view of (4.45), the tensor \( C_{\alpha\beta\gamma\delta\kappa\lambda} \) has no more than 3 different indices \( i, j, k, l \) for which it is non-zero. Hence, by full symmetry in all indices, (4.54) completely determines \( H_{PPPP}(0) \) and gives
\[
(4.55) \quad H_{PPPP}(0) \otimes^4 P = \frac{1}{2} \tilde{C}_{ijk} P_{\alpha i} P_{\beta j} P_{\delta k} P_{\gamma l} = \frac{\tilde{C}}{2} : (P^T P) \otimes (P^T P).
\]
Now we iterate the above arguments. The analog of (4.48) after blowing up along $t_m$ for $q$-th order derivatives is

\[(4.56) \quad \xi \otimes \xi [H_{P\ldots P}(0) : \otimes^{(q-2)}(\xi \otimes v)] = \left[H_{P\ldots P}(0) : \otimes^{(q-2)}(\xi \otimes v)\right] \xi \otimes \xi,\]

for all $\xi \in \mathbb{R}^N$, $v \in \mathbb{R}^n$. When $H_{P\ldots P}(0) \in \mathcal{L}^q$, the only components of the tensor $H_{P_{\alpha_1\ldots \alpha_q}}(0)$ which may not vanish are of the form

\[(4.57) \quad H_{P_{\alpha_1\ldots \alpha_k\ldots \alpha_q}}(0),\]

where $i,j,k \in \{1,\ldots,n\}$ and $\alpha_1,\ldots,\alpha_q \in \{1,\ldots,N\}$. Hence, (4.56), completely determines $H_{P\ldots P}(0)$. By induction, all odd order derivatives of $H$ vanish and all even order derivatives depend on $P$ via $P^\top P$: we have

\[(4.58) \quad H_{P\ldots P}(0) : \otimes^{(q)}P = \begin{cases} C_q : \otimes^{(q/2)}P^\top P, & q \in 2\mathbb{N}, \\
0, & q \in 2\mathbb{N} + 1, \end{cases}\]

for certain tensors $C_q \in \otimes^{(q/2)}\mathbb{R}^n$. Hence, by defining $h : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ by

\[(4.59) \quad h(p) := \sum_{m=1}^{\infty} 2^m C_{2m} : \otimes^{(m)}p,\]

we obtain

\[(4.60) \quad H(P) = h\left(\frac{1}{2}P^\top P\right).\]

Hence, $h \geq 0$ with $h \in C^\infty(\mathbb{S}(\mathbb{R}^n)^+)$ and also $h_p = h_{P_{\alpha_1\ldots \alpha_q}}$. Moreover, by assumption and (4.60) we have $[P]^\perp = [H_p(P)]^\perp = [Ph_p(\frac{1}{2}P^\top P)]^\perp$.

If finally $H_{P\ldots P}(0) \notin \mathcal{L}^q$, then $H$ has the form (4.60) up to a correction of order $O(|P|^4)$. This follows by decomposing each $H_{P\ldots P}(0)$ to the sum of a term in $\mathcal{L}^q$ and a term in the orthogonal complement of $\mathcal{L}^q$. The $O(|P|^4)$ function arises from the series consisting of the forth and higher order parts of $H_{P\ldots P}(0) : \otimes^{(q)}P$ in the orthogonal complements. The theorem follows. \[\square\]

5. The 1-dimensional case of ODE system with dependence on all arguments.

5.1. Formal derivation of the general ODE System. Let $H$ be a non-negative Hamiltonian in $C^2(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N)$, where $N \geq 2$ and we denote the arguments of $H$ by $H(x, \eta, P)$. Consider the integral functional

\[(5.1) \quad E_m(u, I) := \int_I \left[H(x, u(x), u'(x))\right]^m dx,\]

where $m \geq 2$ and $u : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^N$. The Euler-Lagrange equation of functional (5.1) is the ODE system

\[(5.2) \quad \left(H^{m-1}(\cdot, u, u')H_P(\cdot, u, u')\right)' = H^{m-1}(\cdot, u, u')H_\eta(\cdot, u, u')\]

which after expansion and normalisation gives

\[(5.3) \quad (H(\cdot, u, u'))'H_P(\cdot, u, u') + \frac{H(\cdot, u, u')}{m-1}\left((H_P(\cdot, u, u'))' - H_\eta(\cdot, u, u')\right) = 0,\]
on \( I \subseteq \mathbb{R} \). We define the following projections of \( \mathbb{R}^N \):

\[
[H(x, \eta, P)]^\top := \text{sgn}(H_P(x, \eta, P)) \otimes \text{sgn}(H_P(x, \eta, P)), \tag{5.4}
\]

\[
[H(x, \eta, P)\perp := I - [H(x, \eta, P)]^\top. \tag{5.5}
\]

Then, by employing (5.4) and (5.5) to expand the term in bracket of (5.3), we obtain

\[
(H(\cdot, u, u'))^\top H_P(\cdot, u, u') + \frac{H(\cdot, u, u')}{m - 1} [H_P(\cdot, u, u')]^\top \left((H_P(\cdot, u, u'))^\top - H_\eta(\cdot, u, u')\right)
\]

\[
= - \frac{H(\cdot, u, u')}{m - 1} [H_P(\cdot, u, u')]^{\perp} \left((H_P(\cdot, u, u'))^\top - H_\eta(\cdot, u, u')\right). \tag{5.6}
\]

By perpendicularity of the orthogonal projections (5.4) and (5.5), the left and right hand sides of (5.6) are normal to each other. Hence, they both vanish. By renormalising the right hand side and rearranging, we get

\[
(H(\cdot, u, u'))^\top H_P(\cdot, u, u') + H(\cdot, u, u') [H_P(\cdot, u, u')]^{\perp} \left((H_P(\cdot, u, u'))^\top - H_\eta(\cdot, u, u')\right)
\]

\[
= - \frac{H(\cdot, u, u')}{m - 1} [H_P(\cdot, u, u')]^\top \left((H_P(\cdot, u, u'))^\top - H_\eta(\cdot, u, u')\right). \tag{5.7}
\]

As \( m \to \infty \), we obtain the complete system of fundamental ODEs for a general Hamiltonian with dependence on all the arguments

\[
(H(\cdot, u, u'))^\top H_P(\cdot, u, u') + H(\cdot, u, u').
\]

\[
\cdot [H_P(\cdot, u, u')]^{\perp} \left((H_P(\cdot, u, u'))^\top - H_\eta(\cdot, u, u')\right) = 0, \tag{5.8}
\]

whose solutions are curves \( u : I \subseteq \mathbb{R} \to \mathbb{R}^N \).

### 5.2. Degenerate elliptic ODE systems.

We begin by observing that the Ellipticity Classification Theorem 4.1 readily extends to the case of \( H(x, \eta, P) \) with dependence on all arguments; the form (4.1) of the Hamiltonian modifies to

\[
H(x, \eta, P) = h \left(x, \eta, \frac{1}{2} P^\top P\right), \tag{5.9}
\]

and the PDE systems (4.2) and (4.7) modify by the appearance of first and lower order terms. In the case of ODEs where \( n = 1 \), the “geometric” Hamiltonians of the form (5.9) become the radially symmetric ones:

\[
H(x, \eta, P) = h \left(x, \eta, \frac{1}{2} |P|^2\right), \tag{5.10}
\]

where \( h \in C^2(\mathbb{R} \times \mathbb{R}^N \times [0, \infty)) \) and the degenerate elliptic ODE system takes a particularly important and simple form. We note that when we have lower order terms, the Hamiltonian

\[
H(x, \eta, P) = h \left(x, \eta, \frac{1}{2} |P - V(x, \eta)|^2\right)
\]

also leads to degenerate elliptic system, and this is important elsewhere [K9]. However, for simplicity herein we choose \( V \equiv 0 \). In the case of \( \Delta_\infty \), we have \( h(x, \eta, p) = p \). We now derive the ODEs in the elliptic case.
Suppose $h \in C^2(\mathbb{R} \times \mathbb{R}^N \times [0, \infty))$ with arguments denoted by $h(x, \eta, p)$ and define $H \in C^2(\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N)$ by means of (5.10). We henceforth assume
\begin{equation}
\{ h_p(x, \eta, \cdot) = 0 \} \subseteq \{ 0 \} = \{ h(x, \eta, \cdot) = 0 \},
\end{equation}
for all $(x, \eta) \in \mathbb{R}^{1+N}$. Assumption (5.11) is natural and will make the normal coefficient $H[H_p]^\perp$ of (5.8) continuous. By using (5.10) and supressing arguments, we calculate derivatives:
\begin{align}
H_P &= h_P P, & H_{PP} &= h_{PP} P \otimes P + h_P I, & H_\eta &= h_\eta, \\
H_{\eta P} &= P \otimes h_{P\eta}, & H_{Px} &= h_{Px} P, & H_x &= h_x.
\end{align}

By expanding derivatives in (5.8) and using (5.10), (5.12) and (5.13), we get
\begin{equation}
(h_p)^2(u' \otimes u')u'' + h_p(u' \otimes h_\eta)u' + h_x h_p u'
\end{equation}
\begin{equation}
+ h[p u']^\perp(h_{pp}(u' \otimes u')u'' + (u' \otimes h_{p\eta}) u' \\
+ h_{px} u' + h_p u'' - h_\eta) = 0,
\end{equation}
where $h = h(., u, \frac{1}{2}|u'|^2)$. By assumption, (5.11), we have \{ $h_p u' = 0 \} = \{ u' = 0 \} = \{ h = 0 \}$. Hence, we obtain that $[h_p u']^\perp = [u']^\perp$. On $\{ u' \neq 0 \}$, we multiply the normal term of (5.14) by $\frac{|u'|^2 h_x}{h}$ to obtain
\begin{equation}
(h_p)^2(u' \otimes u')u'' + h_p \left( (u' \otimes u')h_\eta + h_x u' \right)
\end{equation}
\begin{equation}
+ |u'|^2 h_p [u']^\perp \left( h_p u'' - h_\eta \right) = 0.
\end{equation}
Hence, by using the identity $|u'|^2 I = u' \otimes u' + |u'|^2 [u']^\perp$, (5.15) gives
\begin{equation}
(h_p)^2 |u'|^2 u'' - h_p \left( |u'|^2 \left( I - \frac{u'}{|u'|} \otimes \frac{u'}{|u'|} \right) h_\eta - h_x u' \right) = 0.
\end{equation}

By introducing the reflection operator $R_\xi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ with respect to the hyperplane $[\xi]^\perp$, $\xi \in \mathbb{R}^N \setminus \{ 0 \}$, given by
\begin{equation}
R_\xi := I - 2 \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|},
\end{equation}
the ODE system (5.16) becomes
\begin{equation}
A_\infty u := |u'|^2 \left( h_p u'' - R_u h_\eta \right) + h_x u' = 0,
\end{equation}
where $h = h(., u, \frac{1}{2}|u'|^2)$. In view of (5.11), the systems (5.18) and (5.8) are equivalent on $\{ u' = 0 \}$ as well. The system (5.18) comprises the degenerate elliptic ODE system.

**Remark 5.1.** We observe that in the special case where $h = h(\frac{1}{2}|u'|^2)$ and $h_\eta \equiv 0$, $h_x \equiv 0$, solutions of (5.18) trivialize to affine and actually (5.18) is equivalent to $A_\infty$. In the special case where $h = h(., \frac{1}{2}|u'|^2)$ and $h_\eta \equiv 0$, solutions of (5.18) become essentially scalar with affine rank-one range, that is $u(\mathbb{R})$ is contained in an affine line of $\mathbb{R}^N$ since $u''$ becomes proportional to $u'$ and (5.18) becomes essentially scalar. Consequently, (5.18) is most interesting when $h(x, u(x), \frac{1}{2}|u'(x)|^2)$ depends on $u(x)$ and hence $h_\eta \neq 0$. In this case the reflection operator $R_{u'}$ with respect to the normal hyperplane $[u']^\perp$ is discontinuous on $\{ u' = 0 \}$ at critical points of $u$,
but the product $|u'|^2 R_w$ is continuous. However, in all cases the system is always degenerate.

5.3. The initial value problem for the elliptic ODE systems. In this subsection we solve the initial value problem for ODE system (5.18) and consider some regularity questions.

**Theorem 5.2** (The initial value problem for the ODE system). Suppose that $h \in C^2(\mathbb{R} \times \mathbb{R}^N \times [0, \infty))$ satisfies $h, h_p \geq 0$ and also (5.11) and consider the following problem

$$
(5.19) \quad \begin{cases}
A_{\infty} u = |u'|^2 \left(h_p u'' - R_w h_\eta\right) + h_x u' = 0, \\
u(x_0) = u_0, \quad u'(x_0) = v_0, \quad x_0 \in \mathbb{R}.
\end{cases}
$$

Then:

(i) For any non-critical initial conditions $(u_0, v_0) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$, there exists a unique maximal smooth solution $u : (x_0 - r, x_0 + r) \to \mathbb{R}^N$ for some $r > 0$ which solves (5.19) and satisfies $|u'| > 0$.

(ii) For any critical initial condition $(u_0, 0) \in \mathbb{R}^N \times \{0\}$, there exists at least one solution to (5.19), one of them being the constant one $u \equiv u_0$.

(iii) If

$$
(5.20) \quad h_\eta(x, \eta, 0) \neq 0 \quad \text{and} \quad h_x(x, \eta, p) = O(p) \quad \text{as} \quad p \to 0,
$$

then bounded maximal solutions of (5.19) starting (in positive time) from non-critical data, either are defined on $[x_0, \infty)$ being smooth and satisfying $|u'| > 0$, or they reach a critical point $u' = 0$ and form a discontinuity in $u''$ in finite time.

(iv) If

$$
(5.21) \quad c \leq h_p \leq \frac{1}{c} \quad \text{for} \quad c > 0, \quad \text{and} \quad h_x(x, \eta, p) = O(p) \quad \text{as} \quad p \to 0,
$$

then bounded maximal solutions of (5.19) either are globally smooth or can be extended past singularities as $W^{2, \infty}_{loc}(\mathbb{R})^N$ strong solutions which satisfy (5.18) everywhere and are eventually constant.

The interpretation of $W^{2, \infty}_{loc}(\mathbb{R})^N$ solutions to (5.19) as strong everywhere solutions is the same as in Aronsson [A1, A2, A5]: at critical points of $u$ whereon $u''$ may not exist but is essentially bounded in a neighbourhood of $\{u' = 0\}$, the coefficient $|u'|^2$ vanishes.

**Example 5.3.** The solution of problem (5.19) is generally non-unique for critical initial conditions. Choose $h(x, \eta, p) := \frac{1}{2} |\eta|^2 + p$. Then, (5.18) takes the form

$$
(5.22) \quad |u'|^2 \left(u'' - R_w u\right) = 0
$$

and the Hamiltonian is $H(u, u') = \frac{1}{2}(|u|^2 + |u'|^2)$. In view of example 3 in Aronsson’s paper [A1], for essentially scalar solutions $u = \xi v$ where $\xi \in S^{N-1}$ and $v : \mathbb{R} \to \mathbb{R}$, (5.22) takes the form $|v'|^2 (v'' + v) \xi = 0$. Hence, for initial conditions $u(- \frac{\pi}{2}) = -e_1$, $u'( - \frac{\pi}{2}) = 0$ (5.23) admits the solutions $u_1(x) = e_1 \sin x$ and $u_2(x) = -e_1$. 
The non-uniqueness for critical data owes to that (5.18) is an 1-dimensional degenerate elliptic system and the initial value problem is not well-posed for it.

**Proof of Theorem 5.2.** All assertions follow directly by considering the following dynamical formulation of the ODE (5.18). For we write the N-dimensional second order degenerate implicit system (5.18) as a 2N-dimensional first order explicit system for a vector field defined off an N-dimensional “slice” of $\mathbb{R}^{2N}$. For $U=(u,v)^T \in \mathbb{R}^{2N}$, we set

$$U(x) := (u(x), u'(x))^T, \quad U : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^{2N}, \quad (5.23)$$

$$F(x,U) := \begin{bmatrix} v \\ \frac{1}{h_p(x,u,\frac{1}{2}|v|^2)} \left( R_{\rho} h_\eta(x,u,\frac{1}{2}|v|^2) - \frac{h_x(x,u,\frac{1}{2}|v|^2)}{|v|^2} v \right) \end{bmatrix}, \quad (5.24)$$

where

$$F : \mathbb{R} \times \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}) \rightarrow \mathbb{R}^{2N}. \quad (5.25)$$

Then, in view of (5.23) and (5.24), ODE system (5.18) can be written as

$$U(x) = F(x,U(x)), \quad U : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^{2N}. \quad (5.26)$$

We now merely observe that the equation

$$u'' = \frac{1}{h_p(\cdot,u,\frac{1}{2}|u'|^2)} \left( R_{\rho} h_\eta(\cdot,u,\frac{1}{2}|u'|^2) - \frac{h_x(\cdot,u,\frac{1}{2}|u'|^2)}{|u'|^2} u' \right) \quad (5.27)$$

which follows by (5.26), implies that under assumption (5.20) the first term in the bracket becomes discontinuous at critical points of $u$, while the second one vanishes. Solutions extend past critical points where $u''$ “jumps” by constant solutions. □

6. **Rigidity of radial 2-dimensional solutions.**

In this section we study a class of special solutions of the $\infty$-Laplacian, that of smooth $\infty$-Harmonic maps $u : \mathbb{R}^2 \rightarrow \mathbb{R}^N$, $N \geq 2$ of the form $u = \rho^k f(k\theta)$ in polar coordinates $(\rho, \theta)$. Here $k > 0$ is a parameter and $f : \mathbb{R} \rightarrow \mathbb{R}^N$ is a curve in $\mathbb{R}^N$. It follows that such solutions are very rigid, because if $k \neq 1$ they are essentially scalar and if $k = 1$ they always have affine image. The result here is

**Proposition 6.1 (Rigidity of radial 2D $\infty$-Harmonic maps).** Let $u : \mathbb{R}^2 \rightarrow \mathbb{R}^N$ be an $\infty$-Harmonic map of the form $u = \rho^k f(k\theta)$ in polar coordinates $(\rho, \theta) \in \mathbb{R}^2$, $k > 0$, $f \in C^\infty(\mathbb{R})^N$, $N \geq 2$. Then, $f$ solves the ODE systems

$$f' \otimes f'' + f + \frac{k-1}{k} (|f'|^2 + |f|^2) f = 0, \quad (6.1)$$

$$[(f', f)]_f f'' = 0. \quad (6.2)$$

Moreover:

(i) If $k \neq 1$, then all solutions have constant rank one, the image $u(\mathbb{R}^2)$ is contained into a line passing through the origin and $f$ can be represented as $f(\theta) = \xi g(\theta)$ for some $\xi \in S^{N-1}$ and $g \in C^\infty(\mathbb{R})$.

(ii) If $k = 1$, then all solutions have rank at most two and the image $u(\mathbb{R}^2)$ is contained into a 2-plane of $\mathbb{R}^N$ passing through the origin. On this plane $f$ can be represented as

$$f(\theta) = c \cos B(\theta) \left( \cos A(\theta), \sin A(\theta) \right)^\top, \quad (6.3)$$
where \( c \in \mathbb{R} \) and \( A, B \in C^\infty(\mathbb{R}) \) satisfy \(|B'|^2 + |A'|^2 \cot^2 B = 1 \) and \( 0 < B \leq \frac{\pi}{2} \).

**Proof of Proposition 6.1.** The derivation of the “tangential part” (6.1) of \( \Delta_\infty \) is entirely analogous to Aronsson’s derivation of its scalar counterpart in the paper [A6], p. 138. Hence, it suffices to outline the derivation of the “normal part” (6.2). Since for all \( \alpha \in \{1, ..., N\} \) we have \( u_\alpha = \rho^k f_\alpha(k\theta) \), we obtain

\[
\begin{bmatrix}
D_x u_\alpha \\
D_y u_\alpha
\end{bmatrix} = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
D_x u_\alpha \\
\frac{1}{\rho} D_y u_\alpha
\end{bmatrix}
\]

(6.4)

Hence, since \( O(\theta) = O(\theta)^{-1} \) we have

\[
N(Du^\top) = \{ \eta \in \mathbb{R}^N : \eta^\top (f, f') O(\theta)^\top = 0 \} = N((f, f')^\top).
\]

and consequently \([Du]^\perp = [(f, f')^\perp]^\perp\). Moreover,

\[
[Du]^\perp \Delta u = [(f, f')^\perp] \left( \frac{1}{\rho} D_x u + D_x^2 u + \frac{1}{\rho^2} D^2_{yy} u \right)
\]

(6.7)

By Corollary 1.2, we may require \(|Du| > 0\) and hence (6.2) follows by (3.4) and (6.7). Now, for (i) we have that if \( k \neq 1 \) then on \( \{|f| > 0\}\) (6.1) gives

\[
- \frac{k(f'' + f)^\top f'}{(k-1)(|f'|^2 + |f|^2)} f' = f.
\]

Consequently, \( f' \) is everywhere proportional to \( f \) and as a result \( f(\mathbb{R}) \) is contained into an 1-dimensional subspace of \( \mathbb{R}^N \).

For (ii), we have that if \( k = 1 \) then (6.2) implies \( f'' = \lambda f + \mu f' \) for some \( \lambda, \mu \in C^\infty(\mathbb{R}) \). Hence, \( f(\mathbb{R}) \) is contained into a 2-dimensional subspace of \( \mathbb{R}^N \), (6.1) gives the extra condition that \( f''(f' + f) = 0 \) which implies \(|f'|^2 + |f|^2 = c^2 \) for some \( c \in \mathbb{R} \). Hence, if \( c \neq 0 \) then \( \frac{1}{c}(|f'|, |f|)^\top \) is on the unit circle and as such \( |f| = c \cos B \) and \( |f'| = c \sin B \), for some \( B \) valued in \([0, \frac{\pi}{2}]\). Hence, \( f = c \cos B(\cos A, \sin A)^\top \) for some \( A \). The differential relation \(|B'|^2 + |A'|^2 \cot^2 B = 1 \) follows easily. \( \square \)

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