DEHN TWISTS ON KAUFFMAN BRACKET SKEIN ALGEBRAS

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Abstract. We give an explicit formula for the action of the Dehn twist along a simple closed curve in a compact connected oriented surface on the completion of the filtered skein modules. To do this, we introduce filtrations of the Kauffman bracket skein algebra and the Kauffman bracket skein modules on the surface.

1. Introduction

Recently it has come to light that the Goldman Lie algebra of a surface plays an important role in the study of the mapping class group of the surface. See [3], [4] and [8] for details. Before that, Turaev [12] drew an analogy between the Goldman Lie algebra and some skein algebra. Hence it is important to establish some explicit connection between the Kauffman bracket skein algebra and the mapping class group. This new connection motivates much of the interest in the mapping class group of a surface and the link theory. In fact, skein algebras give us a new way of studying the mapping class group. Furthermore, we expect that this connection will bring us some information about 3-manifolds including the Casson invariant.

The aim of this paper is to explain a new relationship between the Kauffman bracket skein algebra and the mapping class group. Let \( \Sigma \) be a compact connected oriented surface, \( I \) the closed interval \([0,1]\) and \( \mathbb{Q}[A,A^{-1}] \) the ring of Laurent polynomials over \( \mathbb{Q} \) in an indeterminate \( A \). The Kauffman bracket skein algebra \( S(\Sigma) \) is defined to be the quotient of the free \( \mathbb{Q}[A,A^{-1}] \)-module with basis the set of unoriented framed links in \( \Sigma \times I \) by the skein relation which defines the Kauffman bracket. Let \( J \) be a finite subset of \( \partial \Sigma \). The Kauffman bracket skein module \( S(\Sigma,J) \) is defined to be the quotient of the free \( \mathbb{Q}[A,A^{-1}] \)-module with basis \( \mathcal{T}(\Sigma,J) \), where we denote by \( \mathcal{T}(\Sigma,J) \) the set of unoriented framed tangles with the base point set \( J \times \{ \frac{1}{2} \} \). For details, see the subsection [3.1]. The Kauffman skein algebra \( S(\Sigma) \) has the structure of an associative algebra and a Lie algebra over \( \mathbb{Q}[A,A^{-1}] \). The Kauffman bracket skein module \( S(\Sigma,J) \) has the structure of a \( S(\Sigma) \)-bimodule. Furthermore, we define the action \( \sigma \) of \( S(\Sigma) \) on \( S(\Sigma,J) \) such that \( S(\Sigma,J) \) is \( S(\Sigma) \)-module under the action

\[
t_c = \exp(\sigma(\frac{1}{2}(|\log(c)|)^2)) : \hat{\mathbb{Q}}\pi_1(\Sigma) \to \hat{\mathbb{Q}}\pi_1(\Sigma).
\]

Our goal in this paper is to establish a skein algebra version of this formula.

Let \( \Sigma \) be a compact connected oriented surface, \( I \) the closed interval \([0,1]\) and \( \mathbb{Q}[A,A^{-1}] \) the ring of Laurent polynomials over \( \mathbb{Q} \) in an indeterminate \( A \). The Kauffman bracket skein algebra \( S(\Sigma) \) is defined to be the quotient of the free \( \mathbb{Q}[A,A^{-1}] \)-module with basis the set of unoriented framed links in \( \Sigma \times I \) by the skein relation which defines the Kauffman bracket. Let \( J \) be a finite subset of \( \partial \Sigma \). The Kauffman bracket skein module \( S(\Sigma,J) \) is defined to be the quotient of the free \( \mathbb{Q}[A,A^{-1}] \)-module with basis \( \mathcal{T}(\Sigma,J) \), where we denote by \( \mathcal{T}(\Sigma,J) \) the set of unoriented framed tangles with the base point set \( J \times \{ \frac{1}{2} \} \). For details, see the subsection [3.1]. The Kauffman skein algebra \( S(\Sigma) \) has the structure of an associative algebra and a Lie algebra over \( \mathbb{Q}[A,A^{-1}] \). The Kauffman bracket skein module \( S(\Sigma,J) \) has the structure of a \( S(\Sigma) \)-bimodule. Furthermore, we define the action \( \sigma \) of \( S(\Sigma) \) on \( S(\Sigma,J) \) such that \( S(\Sigma,J) \) is \( S(\Sigma) \)-module under the action

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σ when we regard $\mathcal{S}(\Sigma)$ as a Lie algebra. For details, see the subsection 3.2. In this paper, we introduce the filtration \{$F^n\mathcal{S}(\Sigma)$\}_{n\geq 0} of $\mathcal{S}(\Sigma)$ and the filtration \{$F^n\mathcal{S}(\Sigma, J)$\}_{n\geq 0} of $\mathcal{S}(\Sigma, J)$ defined by an augmentation ideal. These operations are continuous in the topologies of $\mathcal{S}(\Sigma)$ and $\mathcal{S}(\Sigma, J)$ induced by these filtrations. We remark that there is some relationship between the completion of the group ring of the fundamental group of $\Sigma$ and these filtrations of $\mathcal{S}(\Sigma)$ and $\mathcal{S}(\Sigma, J)$ which will appear in \[1\]. We denote the completions of $\mathcal{S}(\Sigma)$ and $\mathcal{S}(\Sigma, J)$ in these topologies by $\mathcal{S}(\Sigma)$ and $\mathcal{S}(\Sigma, J)$ respectively. For details, see the subsection 3.3. The main result of the paper is the formula for the action of the Dehn twist along a simple closed curve $c$

$$t_c = \exp\left(\frac{-A + A^{-1}}{4\log(-A)} \left(\text{arccosh}\left(-\frac{c}{2}\right)^2\right)\right) : \mathcal{S}(\Sigma, J) \to \mathcal{S}(\Sigma, J).$$

which is a skein version of the formula \[4\]. Here $\log(-A) = \sum_{i=1}^{\infty} \frac{-1}{i}(A + 1)^i \in \mathbb{Q}[[A + 1]]$ and $(\text{arccosh}(-\frac{c}{2}))^2 = \sum_{i=0}^{\infty} \frac{1}{2^i(2i + 1)(2i + 3)}(1 - \frac{c^2}{4})^i \in \mathbb{Q}[[c + 2]]$. This skein version does not follow from the original one \[3\] \[4\] \[8\].

In section 5, we prove the following properties.

1. Let $\Sigma$ be a compact connected oriented surface with non-empty boundary. The topology on $\mathcal{S}(\Sigma, J)$ introduced by the filtration is Hausdorff, in other words, we have $\cap_{n=0}^{\infty} F^n\mathcal{S}(\Sigma, J) = 0$.

2. Let $\Sigma$ and $\Sigma'$ be two oriented compact connected surfaces satisfying $\pi_1(\Sigma) \simeq \pi_1(\Sigma')$, $J$ and $J'$ finite subsets of $\partial \Sigma$ and $\partial \Sigma'$, respectively, satisfying $\emptyset J = \emptyset J'$. There exists a diffeomorphism $\xi : (\Sigma \times I, J \times I) \to (\Sigma' \times I, J' \times I)$. Then we have $\xi(F^n\mathcal{S}(\Sigma, J)) = F^n\mathcal{S}(\Sigma', J')$. But the induced map $\xi : \mathcal{S}(\Sigma) \to \mathcal{S}(\Sigma')$ does not seem to be an algebra homomorphism.

3. We have

$$\sum_{L' \subseteq L} (-1)^{|L'|} (-2)^{-|L'|} [L'] \in (\ker \epsilon)^n$$

for a link $L$ in $\Sigma \times I$ having components more than $n$, where the sum is over all sublinks $\emptyset \subseteq L' \subseteq L$ and $|L|$ the number of components of $L$. In other words, for a link $L$ in $\Sigma \times I$, $(-2)^{-|L'|}[L'] \mod (\ker \epsilon)^n$ is a finite type invariant of order $n$ in the sense of Le \[7\] (3.2).

We need the first property to prove that the action of the mapping class group of a compact connected oriented surface with non-empty boundary on the completed skein algebra of the surface is faithful. The second and third properties follows from Lemma 5.3. Using the second property and Lickorish’s theorem \[6\] (Theorem 5.6), we prove the first property. In subsequent papers, we need all the above properties.

In particular, we need the third property to prove that the action of the mapping class group of a compact connected oriented surface with non-empty boundary on the completed skein algebra of the surface.

In subsequent papers, using this formula of Dehn twists, we obtain an embedding of the Torelli group into the completed skein algebra defined in this paper. This embedding gives a construction of the first Johnson homomorphism and a new filtration consisting of normal subgroups in the mapping class group. Furthermore, it gives an invariant for integral homology 3-spheres inducing a finite type invariants of any order $n$. The details will appear elsewhere \[11\].
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2. Definition of tangles in $\Sigma \times I$

In this section, let $\Sigma$ be a compact connected oriented surface.

We define the set of tangles in $\Sigma \times I$.

Definition 2.1. Let $J$ be a finite subset of $\partial \Sigma$. We define $\mathcal{E}(\Sigma, J)$ to be the set consisting of all the injective map $E = \bigcup_i \tau_i \cup \bigcup_j v_j$ from a domain $D$ consisting of a finite collection of strips $\bigcup_i I \times (-\epsilon, \epsilon)$ and annuli $\bigcup_j S^1 \times (-\epsilon, \epsilon)$ into $\Sigma \times (0, 1)$ satisfying the following four conditions.

1. Each $v_j$ is an embedding into $\Sigma \times (0, 1)$.
2. The restriction of each $\tau_i$ to $(0, 1) \times (-\epsilon, \epsilon)$ is an embedding into $\Sigma \times (0, 1)$.
3. The restriction of each $\tau_i$ to $\{0, 1\} \times (-\epsilon, \epsilon)$ is an orientation preserving embedding into $J \times I$.
4. For $j \in J$, $\tau(D) \cap (j \times I)$ is not empty and is connected.

Two elements $E_0$ and $E_1$ of $\mathcal{E}(\Sigma, J)$ which have same domain $D$ are unoriented-isotopic if there exists a continuous map $H : D \times I \to \Sigma \times I$ such that $H(D \times \{0\}) = E_0(D)$, $H(D \times \{1\}) = E_1(D)$ and $H(\cdot, t) \in \mathcal{E}(\Sigma, J)$ for $t \in I$. We denote by $T(\Sigma, J)$ the set of unoriented-isotopy classes of elements of $\mathcal{E}(\Sigma, J)$. We denote by $\langle \cdot \rangle$ the quotient map $\mathcal{E}(\Sigma, J) \to T(\Sigma, J)$. If $J = \emptyset$, we simply denote $T(\Sigma, J)$ and $\mathcal{E}(\Sigma, J)$ by $T(\Sigma)$ and $\mathcal{E}(\Sigma)$.

The definition of ‘tangles’ is similar to the definition of ‘link’ of marked surfaces in [9]. But, a tangle in this definition has one arc on each point of $J$.

Definition 2.2. Let $J$ be a finite subset of $\partial \Sigma$. An element $E$ of $\mathcal{E}(\Sigma, J)$ is generic if $E : (\bigcup_i I \cup \bigcup_j S^1) \times (-\epsilon, \epsilon) \to \Sigma \times I$ satisfies the following two conditions.
For $x \in \bigsqcup_i I \sqcup \bigsqcup_j S^1$, $(-\epsilon, \epsilon) \to I, t \mapsto p_2 \circ E(x, t)$ is an orientation-preserving embedding map, where we denote by $p_1$ the projection $\Sigma \times I \to I$.

(2) $\bigsqcup_i I \sqcup \bigsqcup_j S^1 \to \Sigma, x \mapsto p_1 \circ E(x, 0)$ is an immersion such that the intersections of the image consist of transverse double points, where we denote by $p_2$ the projection $\Sigma \times I \to \Sigma$.

It is convenient to present tangles in $\Sigma \times I$ by tangle diagrams on $\Sigma$ in the same fashion in which links in $\mathbb{R}^3$ may be presented by planar link diagrams.

**Definition 2.3.** Let $J$ be a finite subset of $\partial \Sigma$, $T$ an element of $\mathcal{T}(\Sigma, J)$ and $E : (\bigsqcup_i I \sqcup \bigsqcup_j S^1) \times (-\epsilon, \epsilon) \to \Sigma \times I$ an element of $\mathcal{E}(\Sigma, J)$ such that $E$ is generic and that $\langle E \rangle = T$. The tangle diagram of $T$ is $p_1 \circ E((\bigsqcup_i I \sqcup \bigsqcup_j S^1) \times \{0\})$ together with height-information, i.e., the choice of the upper branch of the curve at each crossing. The chosen branch is called an over crossing; the other branch is called an under crossing.

**Proposition 2.4** (see, for example, [1]). Let $J$ be a finite subset of $\partial \Sigma$. Let $T$ and $T'$ be two elements of $\mathcal{T}(\Sigma, J)$ and $d$ and $d'$ tangle diagrams of them respectively. Then, $T$ equals $T'$ if and only if $d$ can be transformed into $d'$ by a sequence of isotopies of $\Sigma$ and the RI, RII, RIII moves shown in Figure 1, 2, and 3.

Let $J$ and $J'$ be two finite subsets of $\partial \Sigma$ with $J \cap J' = \emptyset$. Here $e_1$ and $e_2$ denote the embedding maps from $\Sigma \times I$ to $\Sigma \times I$ defined by $e_1(x, t) = (x, \frac{t+1}{2})$ and $e_2(x, t) = (x, \frac{t}{2})$. We define $\boxtimes : \mathcal{T}(\Sigma, J) \times \mathcal{T}(\Sigma, J') \to \mathcal{T}(\Sigma, J \cup J')$ by

$$\langle E \rangle \boxtimes \langle E' \rangle \overset{\text{def.}}{=} \langle e_1 \circ E \sqcup e_2 \circ E' \rangle$$

for $E \in \mathcal{E}(\Sigma, J)$ and $E' \in \mathcal{E}(\Sigma, J')$.

Let $J$ be a finite subset of $\partial \Sigma$, $T$ an element of $\mathcal{T}(\Sigma, J)$ represented by $E \in \mathcal{E}(\Sigma, J)$ and $\xi$ an element of $\mathcal{M}(\Sigma)$ represented by a diffeomorphism $X_\xi$, where we denote by $\mathcal{M}(\Sigma)$ the mapping class group of $\Sigma$ preserving the boundary. We denote by $\xi T$ an element of $\mathcal{T}(\Sigma, J)$ represented by $(X_\xi \times \text{id}_I) \circ E \in \mathcal{E}(\Sigma, J)$. 

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**Fig 1. RI: Reidemester move I**

**Fig 2. RII: Reidemeister move II**

**Fig 3. RIII: Reidemeister move III**
3. Kauffman bracket skein modules

Throughout this section, let $\Sigma$ be a compact connected oriented surface.

3.1. Definition of Kauffman bracket skein modules. In this subsection, we define Kauffman bracket skein modules.

First of all, we define a Kauffman triple.

Definition 3.1. Let $J$ be a finite subset of $\partial \Sigma$. A triple of three tangles $T_1$, $T_\infty$ and $T_0 \in \mathcal{T}(\Sigma,J)$ is a Kauffman triple if there exist $E_1$, $E_\infty$ and $E_0 \in \mathcal{E}(\Sigma,J)$ whose domains are $D_1$, $D_\infty$ and $D_0$ satisfying the following two conditions.

- We have $\langle E_1 \rangle = T_1$, $\langle E_\infty \rangle = T_\infty$ and $\langle E_0 \rangle = T_0$.
- The three images $E_1(D_1)$, $E_\infty(D_\infty)$ and $E_0(D_0)$ are identical except for some neighborhood of a point, where they differ as shown in the figure.

\[
\begin{align*}
T_1 &: E_1(D_1) \quad T_\infty &: E_\infty(D_\infty) \quad T_0 &: E_0(D_0)
\end{align*}
\]

In other words, there exist three tangle diagrams $d_1$, $d_\infty$ and $d_0$ which present $T_1$, $T_\infty$ and $T_0$ respectively such that $d_1$, $d_\infty$ and $d_0$ are identical except for some neighborhood of a point, where they differ as shown in Figure 4, Figure 6 and Figure 7 respectively.

**Fig 4.**  \hspace{1cm} **Fig 5.**  \hspace{1cm} **Fig 6.**  \hspace{1cm} **Fig 7.**

We define Kauffman bracket skein modules.

Definition 3.2 (Kauffman bracket skein module). Let $J$ be a finite subset of $\partial \Sigma$. We define $S(\Sigma,J)$ to be the quotient of the free module $\mathbb{Q}[A,A^{-1}]\mathcal{T}(\Sigma,J)$ by the skein relation, i.e., by the submodule of $\mathbb{Q}[A,A^{-1}]\mathcal{T}(\Sigma,J)$ generated by

\[
\{ -T_1 + A T_\infty + A^{-1} T_0 | (T_1, T_\infty, T_0) \text{ is a Kauffman triple} \} \cup \{ T \boxtimes O + (A^2 + A^{-2}) T | T \in \mathcal{T}(\Sigma,J) \}
\]

where $O \in \mathcal{T}(\Sigma)$ is the trivial knot. Following [12], the element of $S(\Sigma,J)$ represented by $T \in \mathcal{T}(\Sigma,J)$ is denoted by $[T]$. We simply denote $S(\Sigma,\emptyset)$ by $S(\Sigma)$.

In [9], Muller also defined the skein modules for a surface with boundary. We, however, do not need ‘the boundary skein relation’ and ‘the value of a contractible arc’.

Let $J$ and $J'$ be two finite subset of $\partial \Sigma$ satisfying $J \cap J' = \emptyset$. The $\mathbb{Q}[A,A^{-1}]$ bilinear homomorphism $\boxtimes : S(\Sigma,J) \times S(\Sigma,J') \to S(\Sigma,J \cup J')$ is defined by $[T] \boxtimes
Let $J$ be a finite subset of $\partial \Sigma$. We denote by $\mathcal{E}_0(\Sigma, J)$ the set of 1-dimensional submanifolds of $\Sigma$ with boundary $J$ and no inessential components. Here a connected 1-dimensional submanifold of $\Sigma$ is inessential if it is a boundary of a disk in $\Sigma$. We denote the set of isotopy classes in $\mathcal{E}_0(\Sigma, J)$ by $\mathcal{T}_0(\Sigma, J)$.

**Theorem 3.3.** Let $J$ be a finite subset of $\partial \Sigma$. The skein module $S(\Sigma, J)$ is the $\mathbb{Q}[A, A^{-1}]$-free module with basis $\mathcal{T}_0(\Sigma, J)$.

In the case when $J = \emptyset$, this is proved by Przytycky [10]. For the general case, it is proved in a similar way to [10].

**Corollary 3.4.** We have $S(S^1 \times I) = \mathbb{Q}[A, A^{-1}][l]$ where $l$ is the element represented by the link whose diagram is $S^1 \times \{\frac{1}{2}\}$.

**Corollary 3.5.** Let $J$ be a finite subset of $\partial \Sigma$. The $\mathbb{Q}[A, A^{-1}]$-module homomorphism $-A + A^{-1} : S(\Sigma, J) \to S(\Sigma, J), x \mapsto (-A + A^{-1})x$ is an injective map.

**Lemma 3.6.** Let $J$ be a finite subset of $\partial \Sigma$. Let $T_1, T_2, T_3$ and $T_4$ be four elements of $T(\Sigma, J)$ presented by four diagrams which are identical except for some neighborhood of a point, where they differ as shown in Figure 4, Figure 5, Figure 6 and Figure 7 respectively. Then we have $[T_1] - [T_2] = (A - A^{-1})([T_3] - [T_4])$.

**Proof.** We have

$[T_1] - [T_2] = (A[T_3] + A^{-1}[T_4]) - (A^{-1}[T_3] + A[T_4]) = (A - A^{-1})([T_3] - [T_4])$.

In Definition 3.8, we introduce a Lie bracket in $S(\Sigma)$ by using the following proposition.

**Proposition 3.7.** Let $J$ and $J'$ be two finite subsets of $\partial \Sigma$ satisfying $J \cap J' = \emptyset$. We have $v \boxtimes v' - v' \boxtimes v \in (A - A^{-1})S(\Sigma, J \cup J')$ for $v \in S(\Sigma, J)$ and $v' \in S(\Sigma, J')$.

**Proof.** Let $T$ be an element of $T(\Sigma, J)$ and $T'$ an element of $T(\Sigma, J')$. Choose tangle diagrams $d$ and $d'$ presenting $T$ and $T'$, respectively, such that the intersections of $d$ and $d'$ consist of transverse double points $P_1, P_2, \cdots, P_m$. For $i = 1, 2, \cdots, m$, let $d(1, i)$ and $d(-1, i)$ be two tangle diagrams satisfying the following conditions.

- The two tangle diagrams $d(1, i)$ and $d(-1, i)$ equal $d \cup d'$ with the same height-information as $d$ and $d'$ except for the neighborhoods of the intersections of $d$ and $d'$.
- The branches of $d(1, i)$ and $d(-1, i)$ in the neighborhood of $P_j$ belonging to $d'$ are overcrossings for $j = 1, \cdots, i - 1$.
- The branches of $d(1, i)$ and $d(-1, i)$ in the neighborhood of $P_j$ belonging to $d$ are overcrossings for $j = i + 1, \cdots, m$.
- The two tangle diagrams $d(1, i)$ and $d(-1, i)$ are as shown in Figure 8 and Figure 9 respectively, in the neighborhood of $P_i$. 

For $i = 1, 2, \cdots, m$, we have

$\mathcal{E}_0(\Sigma, J \cup J') \ni (d(1, i), d(-1, i)) \mapsto v = (\boxtimes d(1, i)) - (\boxtimes d(-1, i)) \in (A - A^{-1})S(\Sigma, J \cup J')$.
We denote by $T(1,i)$ a tangle presented by $d(1,i)$ and by $T(-1,i)$ a tangle presented by $d(-1,i)$. Using lemma 3.6, we have

$$[T] \boxtimes [T'] - [T'] \boxtimes [T] = (A - A^{-1}) \sum_{i=1}^{m} ([T(1,i)] - [T(-1,i)]).$$

This proves the proposition.

Definition 3.8. Let $J$ be a finite subset of $\partial \Sigma$. We define a bracket $[,]$ of $S(\Sigma)$ by

$$[x,y] \overset{\text{def}}{=} \frac{1}{-A + A^{-1}} (xy - yx)$$

for $x$ and $y \in S(\Sigma)$. We define an action $\sigma$ of $S(\Sigma)$ on $S(\Sigma,J)$ by

$$\sigma(x)(v) \overset{\text{def}}{=} \frac{1}{-A + A^{-1}} (xv - vx)$$

for $x \in S(\Sigma)$ and $v \in S(\Sigma,J)$.

It is easy to prove the following proposition.

Proposition 3.9. Let $J$ be a finite subset of $\partial \Sigma$. The bracket $[,] : S(\Sigma) \times S(\Sigma) \to S(\Sigma)$ makes $S(\Sigma)$ a Lie algebra. The action $\sigma : S(\Sigma) \times S(\Sigma,J) \to S(\Sigma,J)$ makes $S(\Sigma,J)$ a $S(\Sigma)$-module when we regard $S(\Sigma)$ as a Lie algebra. Furthermore, for $x, y$ and $z \in S(\Sigma)$ and $v \in S(\Sigma,J)$, we have the Leibniz rules:

$$[xy, z] = x[y, z] + [x, z]y,$$

$$\sigma(xy)(v) = x\sigma(y)(v) + \sigma(x)(vy),$$

$$\sigma(x)(yv) = [x, y]v + y\sigma(x)(v),$$

$$\sigma(x)(vy) = \sigma(x)(vy) + v[x, y].$$

Let $J$ and $J'$ be two finite subsets of $\partial \Sigma$. We have

$$\sigma(x)(v \boxtimes v') = \sigma(x)(v) \boxtimes v' + v \boxtimes \sigma(x)(v')$$

for $x \in S(\Sigma), v \in S(\Sigma,J)$ and $v' \in S(\Sigma,J')$. 
3.3. Filtrations and completions. We introduce filtrations of Kauffman bracket skein modules and define the completed Kauffman bracket skein modules.

We define an augmentation map \( \epsilon : S(\Sigma) \to \mathbb{Q} \) by \( A \mapsto -1 \) and \( |L| \mapsto (-2)^{|L|} \) for \( L \in T(\Sigma) \) where \(|L|\) is the number of components of \( L \).

**Proposition 3.10.** The augmentation map \( \epsilon \) is well-defined.

**Proof.** Let \( T_1, T_\infty \) and \( T_0 \) be three elements of \( T(\Sigma) \) such that \((T_1, T_\infty, T_0)\) is a Kauffman triple. There are three cases,

\[
|T_1| - 1 = |T_\infty| = |T_0|, \\
|T_1| = |T_\infty| - 1 = |T_0|, \\
|T_1| = |T_\infty| = |T_0| - 1.
\]

In each case, we have \( \epsilon([T_1] - A[T_\infty] - A^{-1}[T_0]) = 0 \). For \( T \in T(\Sigma) \), we have \( \epsilon([T \boxtimes \mathcal{O}] + (A^2 + A^{-2})T)) = 0 \). This proves the lemma.

**□**

**Lemma 3.11.** We have \( [S(\Sigma), S(\Sigma)] \subset \ker \epsilon \).

**Proof.** Since \( S(\Sigma) \) is generated by the sets of elements represented by knots, it suffices to show that \([T, T'] \in \ker \epsilon \) for any two elements \( T \) and \( T' \) of \( T(\Sigma) \) satisfying \(|T| = 1 \) and \(|T'| = 1 \). Choose tangle diagrams \( d \) and \( d' \) presenting \( T \) and \( T' \), respectively, such that the intersections of \( d \) and \( d' \) consist of transverse double points \( P_1, P_2, \ldots, P_m \). For \( i = 1, 2, \ldots, m \), let \( d(1, i) \) and \( d(-1, i) \) be two tangle diagrams satisfying the following conditions.

- The two tangle diagrams \( d(1, i) \) and \( d(-1, i) \) equal \( d \cup d' \) with the same height-information as \( d \) and \( d' \) except for the neighborhoods of the intersections of \( d \) and \( d' \).
- The branches of \( d(1, i) \) and \( d(-1, i) \) in the neighborhood of \( P_j \) belonging to \( d' \) are over crossings for \( j = 1, \ldots, i - 1 \).
- The branches of \( d(1, i) \) and \( d(-1, i) \) in the neighborhood of \( P_j \) belonging to \( d \) are over crossings for \( j = i + 1, \ldots, m \).
- The two tangle diagrams \( d \) and \( d' \) are as shown in Figure 8 and Figure 9 respectively, in the neighborhood of \( P_1 \).

We denote by \( T(1, i) \) a tangle presented by \( d(1, i) \) and by \( T(-1, i) \) a tangle presented by \( d(-1, i) \). Using lemma 3.6, we have

\[
[[T], [T']] = - \sum_{i=1}^{m} ([T(1, i)] - [T(-1, i)]).
\]

We remark that \( T(1, i) \) and \( T(-1, i) \) are knots for \( i \in \{1, \ldots, m\} \). We have \( \epsilon(- \sum_{i=1}^{m} ([T(1, i)] - [T(-1, i)])) = 0 \). This proves the proposition.

**□**

Let \( J \) be a finite subset of \( \partial \Sigma \). We define a filtration of \( S(\Sigma) \) by \( F^n S(\Sigma) = (\ker \epsilon)^n \) and a filtration of \( S(\Sigma, J) \) by \( F^n S(\Sigma, J) = (F^n S(\Sigma))S(\Sigma, J) \).

**Theorem 3.12.** (1) Let \( J \) be a finite subset of \( \partial \Sigma \). We have

\[
F^n S(\Sigma) F^m S(\Sigma) \subset F^{n+ m} S(\Sigma), \\
F^n S(\Sigma) F^m S(\Sigma, J) \subset F^{n+ m} S(\Sigma, J), \\
F^n S(\Sigma, J) F^m S(\Sigma) \subset F^{n+ m} S(\Sigma, J),
\]
for $n$ and $m \in \mathbb{Z}_{\geq 0}$.

(2) we have $[F^iS(\Sigma), F^jS(\Sigma)] \subset F^{\text{max}(i+j-1,i,j)}S(\Sigma)$ and $\sigma(F^iS(\Sigma))(F^jS(\Sigma, J)) \subset F^{\text{max}(i+j-1,i-1,j)}S(\Sigma, J)$ for $i$ and $j \in \mathbb{Z}_{\geq 0}$.

Proof. In order to show, for $i$ and $j \in \mathbb{Z}_{\geq 0}$,

\[ F^iS(\Sigma)F^jS(\Sigma, J) \subset F^{i+j}S(\Sigma, J), \]
\[ F^jS(\Sigma, J)F^iS(\Sigma) \subset F^{i+j}S(\Sigma, J), \]

it suffices to prove

\[(\ker \epsilon)S(\Sigma, J) = S(\Sigma, J)(\ker \epsilon),\]

which is obvious by Proposition 3.7. This proves (1).

Using the Leibniz rule, Lemma 3.11 show, for $i$ and $j \in \mathbb{Z}_{\geq 0}$,

\[ [F^iS(\Sigma), F^jS(\Sigma)] \subset F^{\text{max}(i+j-1,i,j)}S(\Sigma), \]
\[ \sigma(F^iS(\Sigma))(F^jS(\Sigma, J)) \subset F^{\text{max}(i+j-1,i-1,j)}S(\Sigma, J). \]

This proves (2).

Let $J$ be a finite subset of $\partial \Sigma$. We define an action of $M(\Sigma)$ on $S(\Sigma, J)$ by \(\xi[T] = [\xi T]\) for $\xi \in M(\Sigma)$ and $T \in T(\Sigma, J)$. We have

\[ \xi(F^nS(\Sigma)) = F^nS(\Sigma), \]
\[ \xi(F^nS(\Sigma, J)) = F^nS(\Sigma, J) \]

for $\xi \in M(\Sigma)$ and $n \in \mathbb{Z}_{\geq 0}$.

**Remark 3.13.** We have $\dim \mathbb{Q}(F^nS(\Sigma, J)/F^{n+1}S(\Sigma, J)) < \infty$. The proof will appear in [11].

Let $J$ be a finite subset of $\partial \Sigma$. We consider the topology on $S(\Sigma)$ induced by the filtration $\{F^nS(\Sigma)\}_{n \geq 0}$, and denote its completion by $\widehat{S(\Sigma)} \overset{\text{def}}{=} \lim_{n \to \infty}S(\Sigma)/F^nS(\Sigma)$.

We call $\widehat{S(\Sigma)}$ the completed skein algebra. We also consider the topology on $S(\Sigma, J)$ induced by the filtration $\{F^nS(\Sigma, J)\}_{n \geq 0}$, and denote its completion by $\widehat{S(\Sigma, J)} \overset{\text{def}}{=} \lim_{n \to \infty}S(\Sigma, J)/F^nS(\Sigma, J)$. We call $\widehat{S(\Sigma, J)}$ the completed skein module. The completed skein algebra $\widehat{S(\Sigma)}$ has a filtration $\widehat{S(\Sigma)} = F^0\widehat{S(\Sigma)} \supset F^1\widehat{S(\Sigma)} \supset F^2\widehat{S(\Sigma)} \supset \cdots$ such that $\widehat{S(\Sigma)}/F^n\widehat{S(\Sigma)} \simeq S(\Sigma)/F^nS(\Sigma)$ for $n \in \mathbb{Z}_{\geq 0}$. The completed skein module $\widehat{S(\Sigma, J)}$ also has a filtration $\widehat{S(\Sigma, J)} = F^0\widehat{S(\Sigma, J)} \supset F^1\widehat{S(\Sigma, J)} \supset F^2\widehat{S(\Sigma, J)} \supset \cdots$ such that $\widehat{S(\Sigma, J)}/F^n\widehat{S(\Sigma, J)} \simeq S(\Sigma, J)/F^nS(\Sigma, J)$ for $n \in \mathbb{Z}_{\geq 0}$. We remark that the completed skein algebra $\widehat{S(\Sigma)}$ is an associative $\mathbb{Q}[[A + 1]]$-algebra and that the completed skein module $\widehat{S(\Sigma, J)}$ is a $\mathbb{Q}[[A + 1]]$-module. The set $\{(S(\Sigma, J), \{F^nS(\Sigma, J)\}_{n \geq 0})|J \subset \partial \Sigma, |J| < \infty\}$ is denoted by $\Theta(\Sigma)$.

We denote by $\mathcal{M}(\Sigma) \subset M(\Sigma)$ the subset consisting of elements $\xi$ satisfying the condition that, for any finite subset $J$ of $\partial \Sigma$, any non-negative integer $m$ and any element $v \in F^mS(\Sigma, J)$, there exists a non-negative integer $N$ such that $j \geq N \Rightarrow (1 - \xi)^j(v) \in F^{m+1}S(\Sigma, J)$.
For $\xi \in \hat{\mathcal{M}}(\Sigma)$ and a finite subset $J$ of $\partial \Sigma$, a $\mathbb{Q}[A + 1]$-module homomorphism $\log(\xi) : \mathcal{S}(\Sigma, J) \to \mathcal{S}(\Sigma, J)$ is defined by $\log(\xi)(v) = \sum_{i=1}^{\infty} -1 \frac{1}{i^2}(\text{id} - \xi)^i(v)$. For $\xi \in \hat{\mathcal{M}}(\Sigma)$, $x \in \hat{\mathcal{S}}(\Sigma)$ and $z \in \hat{\mathcal{S}}(\Sigma, J)$, since $\xi(xz) = \xi(x)\xi(z)$ and $\xi(zx) = \xi(z)\xi(x)$, $\log(\xi)$ satisfies the Leibniz rule

$$\log(\xi)(xz) = \log(\xi)(x)z + x\log(\xi)(z),$$

$$\log(\xi)(zx) = \log(\xi)(z)x + z\log(\xi)(x).$$

**Definition 3.14.** For $\xi \in \hat{\mathcal{M}}(\Sigma, \partial \Sigma)$, an element $x_\xi \in \hat{\mathcal{S}}(\Sigma)$ has a skein representative of $\xi$ by $((\mathcal{S}(\Sigma), \{\mathcal{S}(\Sigma)\}_{n \geq 0}), \Theta(\Sigma))$ if we have

$$\log(\xi) = \sigma(x_\xi) : \mathcal{S}(\Sigma, J) \to \mathcal{S}(\Sigma, J),$$

in other words

$$\xi(\cdot) = \exp(\sigma(x_\xi)) : \mathcal{S}(\Sigma, J) \to \mathcal{S}(\Sigma, J),$$

for a finite subset $J$ of $\partial \Sigma$.

4. Dehn twists

In this section we show the following theorem.

**Theorem 4.1.** Let $\Sigma$ be a compact connected oriented surface and $c$ a simple closed curve. We also denote by $c$ an element of $\mathcal{S}(\Sigma)$ represented by a knot presented by the simple closed curve $c$. Then we have $t_c \in \hat{\mathcal{M}}(\Sigma)$, and $\frac{A + A^{-1}}{4\log(-A)}(\text{arccosh}(-\frac{c}{A}))^2 \in \hat{\mathcal{S}}(\Sigma)$ has a skein representative of $t_c \in \hat{\mathcal{M}}(\Sigma)$ by $((\mathcal{S}(\Sigma), \{F^n\mathcal{S}(\Sigma)\}_{n \geq 0}), \Theta(\Sigma))$ in the sense of Definition 3.14. Here $\frac{A + A^{-1}}{4\log(-A)}$ is an element of $\mathbb{Q}[A + 1]$ and $(\text{arccosh}(-\frac{c}{A}))^2$ is $\sum_{i=0}^{\infty} \frac{\text{id}}{(i+1)(2i+1)}(1 - \frac{c^2}{4})^{i+1} \in \mathbb{Q}[c + 2]$. We denote by $S^1 \overset{\text{def}}{=} \mathbb{R}/\mathbb{Z}$, by $c_t$ a simple closed curve $S^1 \times \{\frac{t}{2}\}$ in $S^1 \times I$, by $t$ the Dehn twist along $c_t$ and by $l$ an element of $\mathcal{S}(S^1 \times I)$ represented by the knot presented by $c_t$. We fix an embedding $\iota : S^1 \times I \to \Sigma$ such that $\iota(c_t) = c$.

We assume that $\iota(S^1 \times I)$ separate $\Sigma$ into two surfaces $\Sigma^1$ and $\Sigma^2$. For a finite set $J' \subset S^1$, we consider the trilinear map

$$\varpi_{J'} : \mathcal{S}(\Sigma^1, (J \cap \partial \Sigma^1) \cup \iota(J' \times \{1\})) \times \mathcal{S}(S^1 \times I, J' \times \{0, 1\}) \times \mathcal{S}(\Sigma^2, (J \cap \partial \Sigma^1) \cup \iota(J' \times \{0\})) \to \mathcal{S}(\Sigma, J)$$

defined by $\varpi_{J'}([T_1], [T_2], [T_3]) = [T_1T_2T_3]$ for $T_1 \in \mathcal{T}(\Sigma, (J \cap \partial \Sigma^1) \cup \iota(J' \times \{1\}))$, $T_2 \in \mathcal{T}(S^1 \times I, J' \times \{0, 1\})$ and $T_3 \in \mathcal{T}(\Sigma^2, (J \cap \partial \Sigma^1) \cup \iota(J' \times \{0\}))$. Here we denote by $T_1T_2T_3$ the tangle presented by $d_1 \cup \iota(d_2) \cup d_3$, respectively, where $d_1$, $d_2$ and $d_3$ present $T_1$, $T_2$ and $T_3$, respectively. We remark that $d_1 \cup \iota(d_2) \cup d_3$ must be smoothed out in the neighborhood of $\iota(S^1 \times \{0, 1\})$. Then we have the followings.

- The set

$$\bigcup_{J'} \varpi_{J'}(\mathcal{S}(\Sigma^1, (J \cap \partial \Sigma^1) \cup \iota(J' \times \{1\})) \times \mathcal{S}(S^1 \times I, J' \times \{0, 1\}) \times \mathcal{S}(\Sigma^2, (J \cap \partial \Sigma^1) \cup \iota(J' \times \{0\})))$$

generates $\mathcal{S}(\Sigma, J)$ as $\mathbb{Q}[A, A^{-1}]$-module.
The map \( \varpi_{J'} \) preserves the filtrations, in other words,
\[
\varpi_{J'}(S(\Sigma^1, (J \cap \partial \Sigma^1) \cup \iota(J' \times \{1\})) \times F^n S(S^1 \times I, J' \times \{0, 1\}))
\times S(\Sigma^2, (J \cap \partial \Sigma^1) \cup \iota(J' \times \{0\}))) \subset F^n S(\Sigma, J).
\]

We have \( t_c \circ \varpi_{J'} = \varpi_{J'} \circ (\text{id}, t_c, \text{id}) \) and \( \sigma(\iota(x)) \circ \varpi_{J'} = \varpi_{J'} \circ (\text{id}, \sigma(x), \text{id}) \)
for \( x \in S(S^1 \times I) \).

We assume that \( \Sigma \setminus \iota(S^1 \times \{0, 1\}) \) is a connected surface \( \Sigma^1 \). For a finite set \( J' \subset S^1 \), we consider the bilinear map
\[
\varpi_{J'} : S(\Sigma^1, J \cup \iota(J' \times \{0, 1\})) \times S(S^1 \times I, J' \times \{0, 1\}) \rightarrow S(\Sigma, J)
\]
defined by \( \varpi_{J'}([T_1], [T_2]) = [T_1 T_2] \) for \( T_1 \in T(J \cup \iota(J' \times \{0, 1\})) \) and \( T_2 \in T(S^1 \times I, J' \times \{0, 1\}) \).

Here we denote by \( T_1 T_2 \) the tangle presented by \( d_1 \cup \iota(d_2) \), respectively, where \( d_1 \) and \( d_2 \) present \( T_1 \) and \( T_2 \), respectively. We remark that \( d_1 \cup \iota(d_2) \) must be smoothed out in the neighborhood of \( \iota(S^1 \times \{0, 1\}) \). Then we have the followings.

- The set
  \[
  \bigcup_{J'} \varpi_{J'}(S(\Sigma^1, J \cup \iota(J' \times \{0, 1\})) \times S(S^1 \times I, J' \times \{0, 1\}))
  \]
  generates \( S(\Sigma, J) \) as \( \mathbb{Q}[A, A^{-1}] \)-module.

- The map \( \varpi_{J'} \) preserves the filtrations, in other words,
  \[
  \varpi_{J'}(S(\Sigma^1, J \cup \iota(J' \times \{0, 1\})) \times F^n S(S^1 \times I, J' \times \{0, 1\})) \subset F^n S(\Sigma, J).
  \]

- We have \( t_c \circ \varpi_{J'} = \varpi_{J'} \circ (\text{id}, t_c) \) and \( \sigma(\iota(x)) \circ \varpi_{J'} = \varpi_{J'} \circ (\text{id}, \sigma(x)) \)
  for \( x \in S(S^1 \times I) \).

Hence, it suffices to show the following lemma.

**Lemma 4.2.** Fix a positive integer \( m \). Choose points \( p_1 = \frac{1}{2m}, \ldots, p_i = \frac{i}{2m}, \ldots, p_m = \frac{m}{2m} \) in \( S^1 \). We denote by \( r^0_i \) an element of \( S(S^1 \times I, \{(p_i, 0), (p_i, 1)\}) \) represented by the tangle presented by \( \{p_i\} \times I \). Then we have the following.

- (1) We have \((t-1)^{2n+m}(r^0_1 \boxtimes r^0_2 \boxtimes \cdots \boxtimes r^0_m) \) in \( F^n S(S^1 \times I, \{p_1, \ldots, p_m\} \times \{0, 1\}) \).

- (2) We have
  \[
  \log(t)(r^0_1 \boxtimes r^0_2 \boxtimes \cdots \boxtimes r^0_m) = \sigma \left( \frac{-A + A^{-1}}{4\log(-A)} \right) (\text{arccosh}(-\frac{l}{2})) (r^0_1 \boxtimes r^0_2 \boxtimes \cdots \boxtimes r^0_m).
  \]

**Proof of Lemma 4.2 (1).** We need a \( \mathbb{Q}[A, A^{-1}] \)-bilinear map \( (\cdot)(\cdot) : S(S^1 \times I, J \times \{0, 1\}) \times S(S^1 \times I, J \times \{0, 1\}) \rightarrow S(S^1 \times I, J \times \{0, 1\}) \) defined by \([T_1][T_2] = [T_1 T_2] \) for any finite subset \( J \subset S^1 \). Here we denote by \( T_1 T_2 \) the tangle presented by \( \mu_1(D_1) \cup \mu_2(D_2) \) where we choose tangle diagrams \( D_1 \) and \( D_2 \) presenting \( T_1 \) and \( T_2 \), respectively, and embedding maps \( \mu_1 \) and \( \mu_2 : S^1 \rightarrow S^1 \) defined by \( \mu_1(\theta, t) = (\theta, \frac{t+1}{2}) \) and \( \mu_2(\theta, t) = (\theta, \frac{t}{2}) \). We remark that \( \mu_1(D_1) \cup \mu_2(D_2) \) must be smoothed out in the neighborhood of \( \iota_1 \). By definition we have \( (F^k S(S^1 \times I, J \times \{0, 1\}) \times F^l S(S^1 \times I, J \times \{0, 1\} \subset F^{k+l} S(S^1 \times I, J \times \{0, 1\}) \) for \( k, l \in \mathbb{Z}_{\geq 0} \) and \( xy = yx \) for \( x, y \in S(S^1 \times I, \{p\} \times \{0, 1\}) \) for \( p \in S^1 \). For \( i = 1, \ldots, m \), we denote by \( x_i \) the tangle \( r^0_1 \boxtimes \cdots \boxtimes r^0_{i-1} \boxtimes t(r^0_{i+1}) \boxtimes r^0_{i+2} \boxtimes \cdots \boxtimes r^0_m \) and by \( x_i^{-1} \) the tangle \( t^{-1}(r^0_i) \boxtimes r^0_{i+1} \boxtimes \cdots \boxtimes r^0_m \). We simply denote \( \text{id} = r^0_1 \boxtimes r^0_2 \boxtimes \cdots \boxtimes r^0_m \). We remark that \( x_i x_i^{-1} = \text{id} \).
Since \((t(r_i^n) - r_i^0)^2 = -(l + 2)t(r_i^0) + (A + 1)t^2(r_i^0) + (A^{-1} + 1)r_i^0 \in F^1 S(S^1 \times I, p_i \times \{0, 1\})\), we have \((t - 1)^{2n+m}(id) \in F^r S(S^1 \times I, \{P_1, \ldots, P_m\} \times \{0, 1\})\). This proves the part (1) of the lemma.

To prove Lemma 4.2 (2), we need the following lemma.

**Lemma 4.3.** We have \(\sigma \frac{A + A^{-1}}{4 \log(-A)} (\arccosh(-\frac{1}{2})) (r_i^0)^2 = \log(t(r_i^0)) \) for \(i = 1, \ldots, m\).

For \(n = 0, 1, \ldots\), we define the Chebyshev polynomial \(T_n(X) \in \mathbb{Z}[X]\) by setting \(T_0(X) = 2\) and \(T_{n+1}(X) = XT_n(X) - T_{n-1}(X)\). We denote by \((T + 1)_n(x) \overset{\text{def}}{=} \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} T_i(X)\). It is obvious that \((T + 1)_n(q + q^{-1}) = (q + q^{-1})^n + (q^{-1} + 1)^n\). Since

\[
(T + 1)_n(x) = (\sqrt{x} + 2)^n (\frac{\sqrt{x} + 2 - \sqrt{x} - 2}{2})^n + (\frac{\sqrt{x} + 2 + \sqrt{x} - 2}{2})^n,
\]

we have the following proposition.

**Proposition 4.4.** We have

\[
(T + 1)_{2n}(x) \in (x + 2)^n \mathbb{Z}[x],
\]

\[
(T + 1)_{2n+1}(x) \in (x + 2)^n \mathbb{Z}[x].
\]

We define a sequence \(\{a_n\}_{n \geq 2}\) by \((\log(-x))^2 = \sum_{n=2}^{\infty} a_n(x + 1)^n \in \mathbb{Q}[[x + 1]]\).

We remark that \(2(\arccosh(-\frac{X}{2})) = (\log(-T))^2(X) \overset{\text{def}}{=} \sum_{n=2}^{\infty} a_n(T + 1)_n(X) \in \mathbb{Q}[[X + 2]]\).

**Proof of Lemma 4.3.** We have

\[
\sigma \frac{A + A^{-1}}{4 \log(-A)} (\arccosh(-\frac{1}{2})) (r_i^0)^2
\]

\[
= \sigma \frac{A + A^{-1}}{8 \log(-A)} (\log(-T))^2(l)(r_i^0)^2
\]

\[
= \frac{1}{8 \log(-A)} ((\log(-T))^2(l)(r_i^0)^2 - (r_i^0)(\log(-T))^2(l))
\]

\[
= \frac{1}{8 \log(-A)} \sum_{k=1}^{\infty} (a_k(T + 1)_k(l) r_i^0 - a_k r_i^0(T + 1)_k(l))
\]

\[
= \frac{1}{8 \log(-A)} \sum_{k=1}^{\infty} (a_k(-Ar_i^1 + r_i^0)k + a_k(-A^{-1}r_i^{-1} - r_i^0)k - a_k(-A^{-1}r_i^{-1} - r_i^0)k - a_k(-Ar_i^{-1} - r_i^0)k)
\]

\[
= \frac{1}{8 \log(-A)} (\log^2(Ar_i^1) + \log^2(-A^{-1}r_i^{-1}) - \log^2(-A^{-1}r_i^1) - \log^2(-Ar_i^{-1}))
\]

\[
= \frac{1}{8 \log(-A)} (((2(\log(-A))^2 r_i^0 + 2 \log(-A) \log r_i^1 + (\log r_i^1)^2))
\]

\[- 2((\log(-A))^2 r_i^0 - 2 \log(-A) \log r_i^1 + (\log r_i^1)^2))
\]

\[
= \log r_i^1 \overset{\text{def}}{=} \log(t(r_i^0)).
\]
Here we denote by $r_i \overset{\text{def}}{=} t(r_i^0)$, by $r_i^{-1} \overset{\text{def}}{=} t^{-1}(r_i^0)$, $\log r_i \overset{\text{def}}{=} \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} (r_i^1 - r_i^0)$ and by $\log^2 ((-A)^{r_i^2} r_i^{r_i^2}) \overset{\text{def}}{=} \sum_{n=2}^{\infty} a_n ((-A)^{r_i^2} r_i^{r_i^2})^n$ for $\epsilon_1 \in \{-1,0,1\}$ and $\epsilon_2 \in \{-1,1\}$. This proves the lemma.

Proof of Lemma 4.2(2). We have

\begin{align*}
\sigma & (\frac{-A + A^{-1}}{4 \log(-A)} (\arccosh(\frac{l}{2}))^2)(r_1^0 \boxtimes r_2^0 \boxtimes \cdots \boxtimes r_m^0) \\
& = \sum_{i=1}^{m} r_1^0 \boxtimes \cdots \boxtimes r_{i-1}^0 \boxtimes \sigma(\frac{-A + A^{-1}}{4 \log(-A)} (\arccosh(\frac{l}{2}))^2(r_i^0) \boxtimes r_{i+1}^0 \boxtimes \cdots \boxtimes r_m^0) \\
& = \sum_{i=1}^{m} \log(t(r_i^0)) \boxtimes r_{i+1}^0 \boxtimes \cdots \boxtimes r_m^0 \\
& = \sum_{i=1}^{m} \log(x_i) \\
& = \log(x_1 x_2 \cdots x_m) \\
& = \log(t(\id)).
\end{align*}

This proves the lemma. \qed

Remark 4.5. Let $\Sigma$ be a compact connected oriented surface with non-empty connected boundary and let $\mathcal{I}(\Sigma) \subset \mathcal{M}(\Sigma)$ be the Torelli group of $\Sigma$. Then we have

1. $\mathcal{I}(\Sigma) \subset \hat{\mathcal{M}}(\Sigma)$.

2. For any $\xi \in \mathcal{I}(\Sigma)$, there exists $x_\xi \in \mathcal{S}(\Sigma)$ satisfying that $x_\xi$ is a skein representative of $\xi \in \mathcal{I}(\Sigma) \subset \hat{\mathcal{M}}(\Sigma)$ by $((\mathcal{S}(\Sigma), \{F^n \mathcal{S}(\Sigma)\}_{n \geq 0}), \Theta(\Sigma))$ in the sense of Definition 3.4.

The proof will appear in \cite{??}.

5. Filtrations

5.1. The filtrations depend only on the underlying 3-manifold. In this subsection, we prove the following theorem. The proof of the theorem is analogous to that of \cite{??} Proposition 6.10.

Theorem 5.1. Let $\Sigma$ and $\Sigma'$ be two oriented compact connected surfaces, $J$ a finite subset of $\partial \Sigma$ and $J'$ a finite subset of $\partial \Sigma'$ such that there exists a diffeomorphism $\xi : (\Sigma \times I, J \times I) \to (\Sigma' \times I, J' \times I)$. Then we have $\xi(F^n \mathcal{S}(\Sigma, J)) = F^n \mathcal{S}(\Sigma', J')$ for $n \geq 0$.

To prove it, we need new filtrations of the Kauffman skein modules.

Let $\mathbb{Q}[A, A^{-1}]\mathcal{T}(\Sigma, J)$ be the free module with basis $\mathcal{T}(\Sigma, J)$ over $\mathbb{Q}[A, A^{-1}]$ and $\langle \cdot \rangle$ the natural surjection $\mathbb{Q}[A, A^{-1}]\mathcal{T}(\Sigma, J) \to \mathcal{S}(\Sigma, J)$. For a tangle $T \in \mathcal{T}(\Sigma, J)$ and closed components $L_1, L_2, \ldots, L_m$ of $T$, we denote by

\begin{align*}
(T, \bigcup_{i=1}^{m} L_i) & \overset{\text{def}}{=} \sum_{j=0}^{m} \sum_{\{i_1, i_2, \ldots, i_j\} \subset \{1, 2, \ldots, m\}} 2^{m-j} \mathcal{T}^{j} \cup \bigcup_{h=1}^{j} L_{i_h} \in \mathbb{Q}[A, A^{-1}]\mathcal{T}(\Sigma, J)
\end{align*}
where \( T' \cup \bigcup_{i=1}^{n} L_i = T \). We denote by \( F^*S(\Sigma, J) \) \( \overset{\text{def}}{=} S(\Sigma, J) \) and by \( F^{*n}S(\Sigma, J) \) the \( \mathbb{Q}[A, A^{-1}] \) submodule generated by \((A+1)F^{*n-1}\) and the set of \( S(\Sigma, J) \) consisting of the elements \( \langle\langle T, \bigcup_{i=1}^{n} K_i \rangle\rangle \in S(\Sigma, J) \) for \( T \in \mathcal{T}(\Sigma, J) \) and closed components \( K_1, K_2, \ldots, K_n \) of \( T \) for \( n \geq 1 \). Similarly, the filtration \( \{F^{*n}S(\Sigma', J')\}_{n \geq 0} \) is defined as \( \{F^{*n}S(\Sigma, J)\}_{n \geq 0} \).

**Lemma 5.2.** Let \( E : \mathcal{D} \sqcup \prod_{i=1}^{n} (S^1)_i \to \Sigma \) be an immersion whose intersections consist of transverse double points, and suppose that \( E(\partial D) = J \). We denote by \( D = \prod_{i=1}^{n} I_i \sqcup \prod_{i=1}^{M} (S^1)_{k+i''} \). Fix an intersection \( P \) of \( E(D \sqcup \prod_{i=1}^{n} (S^1)_i) \). Let \( d(1) \) and \( d(2) \) be two tangle diagrams that equal \( E(D \sqcup \prod_{i=1}^{n} (S^1)_i) \) with identical height-information except the neighborhood of \( P \), where they looks as in Figure 4 and Figure 5, respectively. We denote by \( T(1) \) and \( T(2) \) the two tangles presented by \( d(1) \) and \( d(2) \), respectively. For \( i = 1, \ldots, m \), let \( K(i)_i \) be the component of \( T(1) \) presented by \( E((S^1)_i) \) and \( K(i)_i \) the component of \( T(2) \) presented by \( E((S^1)_i) \). Then we have \( \langle\langle T(1), \bigcup_{i=1}^{n} K(1)_i \rangle\rangle = \langle\langle T(2), \bigcup_{i=1}^{n} K(2)_i \rangle\rangle \in (A+1)F^{*n-1}S(\Sigma, J) \).

**Proof.** We denote by

\[
T'(1) \overset{\text{def}}{=} T(1) \setminus \bigcup_{i \in \{1, \ldots, n\}} K(1)_i,
\]

\[
T'(2) \overset{\text{def}}{=} T(2) \setminus \bigcup_{i \in \{1, \ldots, n\}} K(2)_i.
\]

Let \( T(0) \) and \( T(\infty) \) be two elements of \( \mathcal{T}(\Sigma, J) \) presented by tangle diagrams \( d(0) \) and \( d(\infty) \) which equal \( d(1) \) except \( D \), where they are shown in Figure 4 and Figure 5, respectively.

There are three cases.

1. \( P \) is a crossing of \( K(1)_k \) and \( K(1)_l \) for some \( 1 \leq k < l \leq m \).
2. \( P \) is one of the crossings of \( K(1)_k \) and the crossings of \( K(1)_k \) and \( T'(1) \) for some \( 1 \leq k \leq n \).
3. \( P \) is a crossing of \( T'(1) \)

(1) Assume \( P \) is a crossing of \( K(1)_k \) and \( K(1)_l \) for some \( 1 \leq k < l \leq m \). Let \( K(0)_kl \) be the component of \( T(0) \) satisfying \( T(0) \setminus K(0)_kl = T(1) \setminus (K(1)_k \cup K(1)_l) \) and \( K(\infty)_kl \) be the component of \( T(\infty) \) satisfying \( T(\infty) \setminus K(\infty)_kl = T(1) \setminus (K(1)_k \cup K(1)_l) \). For \( i \neq k, l \), let \( K(i)_i \) be the component of \( T(0) \) presented by \( E((S^1)_i) \) and \( K(i)_i \) the component of \( T(\infty) \) presented by \( E((S^1)_i) \). We denote by

\[
T'(0) \overset{\text{def}}{=} T(0) \setminus \bigcup_{i \in \{1, \ldots, n\} \setminus \{k,l\}} K(0)_i \cup K(0)_kl,
\]

\[
T'(\infty) \overset{\text{def}}{=} T(\infty) \setminus \bigcup_{i \in \{1, \ldots, n\} \setminus \{k,l\}} K(\infty)_i \cup K(\infty)_kl.
\]

For a subset \( \{i_1, \ldots, i_j\} \subset \{1, \ldots, n\} \) not including \( k \) or \( l \), we have

\[
\langle T'(1) \cup \bigcup_{h \in \{i_1, \ldots, i_j\}} K(1)_h \rangle - \langle T'(2) \cup \bigcup_{h \in \{i_1, \ldots, i_j\}} K(2)_h \rangle = 0.
\]
Using Lemma 3.6 for a subset \( \{i_1, \cdots, i_j\} \subset \{1, \cdots, n\} \) including \( k \) and \( l \), we have
\[
\langle T'(1) \cup \bigcup_{h \in \{i_1, \cdots, i_j\}} K(1)_{ih} \rangle - \langle T'(2) \cup \bigcup_{h \in \{i_1, \cdots, i_j\}} K(2)_{ih} \rangle
\]
\[
= (A - A^{-1})((T'(0) \cup \bigcup_{h \in \{i_1, \cdots, i_j\} \setminus \{k, l\}} K(0)_{ih} \cup K(0)_{kl} )
- (T'(0) \cup \bigcup_{h \in \{i_1, \cdots, i_j\} \setminus \{k, l\}} K(0)_{ih} \cup K(0)_{kl} )).
\]
Hence we have
\[
\langle (T(1), \bigcup_{i \in \{1, \cdots, n\}} K(1)_{i}) \rangle - \langle (T(2), \bigcup_{i \in \{1, \cdots, n\}} K(2)_{i}) \rangle
\]
\[
= (A - A^{-1})((\langle T(0), (\bigcup_{i \in \{1, \cdots, n\} \setminus \{k, l\}} K(0)_{i} \cup K(0)_{kl} ) \rangle
- (\langle T(\infty), (\bigcup_{i \in \{1, \cdots, n\} \setminus \{k, l\}} K(\infty)_{i} \cup K(\infty)_{kl} ) \rangle).
\]

(2) We assume \( P \) is one of the crossings of \( K(1)_{k} \) and the crossings of \( K(1)_{k} \) and \( T'(1) \) for some \( 1 \leq k \leq n \). For \( i \neq k \), let \( K(0)_{i} \) be the component of \( T(0) \) presented by \( E((S_{i}^{1})) \) and \( K(\infty)_{i} \) the component of \( T(\infty) \) presented by \( E((S_{i}^{1})) \). Using Lemma 3.6 we have
\[
\langle (T(1), \bigcup_{i \in \{1, \cdots, n\}} K(1)_{i}) \rangle - \langle (T(2), \bigcup_{i \in \{1, \cdots, n\}} K(2)_{i}) \rangle
\]
\[
= (A - A^{-1})((\langle T(0), (\bigcup_{i \in \{1, \cdots, n\} \setminus \{k\}} K(0)_{i} \rangle
- \langle T(\infty), (\bigcup_{i \in \{1, \cdots, n\} \setminus \{k\}} K(\infty)_{i} \rangle).
\]

(3) We assume that \( P \) is a crossing of \( T'(1) \). For \( i = 1, \cdots, n \), let \( K(0)_{i} \) be the component of \( T(0) \) presented by \( E((S_{i}^{1})) \) and \( K(\infty)_{i} \) the component of \( T(\infty) \) presented by \( E((S_{i}^{1})) \). Using Lemma 3.6 we have
\[
\langle (T(1), \bigcup_{i \in \{1, \cdots, n\}} K(1)_{i}) \rangle - \langle (T(2), \bigcup_{i \in \{1, \cdots, n\}} K(2)_{i}) \rangle
\]
\[
= (A - A^{-1})((\langle T(0), (\bigcup_{i \in \{1, \cdots, n\}} K(0)_{i} \rangle
- \langle T(\infty), (\bigcup_{i \in \{1, \cdots, n\}} K(\infty)_{i} \rangle).
\]

We conclude that \( \langle (T(1), \bigcup_{i=1}^{n} K(1)_{i}) \rangle - \langle (T(2), \bigcup_{i=1}^{n} K(2)_{i}) \rangle \in (A+1)F^{*}(n-1)S(\Sigma, J) \). This proves the lemma.

\[\square\]

Lemma 5.3. Let \( \Sigma \) be a compact connected oriented surface, and \( J \) a finite subset of \( \partial \Sigma \). We have \( F^{*}S(\Sigma, J) = F^{*}S(\Sigma, J) \) for any non-negative integer \( n \). Furthermore, We have
\[
\sum_{L' \subset L} (-1)^{|L'|}(-2)^{-|L'|}|L'| \in (\ker e)^n
\]
for a link \( L \) in \( \Sigma \times I \) having components more than \( n \), where the sum is over all sublinks \( L' \subset L \) including the empty link and we denote by \(|L|\) the number
Proof. We prove the lemma by induction on \( \text{mod} (\ker \text{of components of } L) \). In other words, for a link \( L \in \Sigma \times I \), \( (-1)^{|L'|(-2)^{|L'|}|L'|} \) \( \text{mod} (\ker \text{of components of } L) \) is a finite type invariant of order \( n \) in the sense of Le \cite{Le} (3.2).

For any tangle \( T \in T(\Sigma, J) \) and knots \( K_1, K_2, \ldots, K_n \in T(\Sigma) \), we have
\[
\langle (K_1+2)(K_2+2)\cdots(K_n+2)\rangle_T = \langle (K_1\otimes K_2\otimes\cdots\otimes K_n\otimes T, K_1\otimes K_2\otimes\cdots\otimes K_n) \rangle
\]
Hence we have \( F^*S(\Sigma, J) \supset F^nS(\Sigma, J) \). Using Lemma \ref{lem:|L'|}, for any tangle \( T \) and closed components \( K_1, K_2, \ldots, K_n \) of \( T \), we have
\[
\langle (K_1\otimes K_2\otimes\cdots\otimes K_n\otimes T', K_1\otimes K_2\otimes\cdots\otimes K_n) \rangle = \langle (T, \bigcup_{i=1}^n K_i) \rangle
\]
\( \in (A - A^{-1})F^{*(n-1)}S(\Sigma, J) = (A - A^{-1})F^{n-1}S(\Sigma, J) \subset F^nS(\Sigma, J) \).

Here we denote by \( T' \overset{\text{def}}{=} T(\bigcup_{i=1}^n K_i) \). Since \( \langle (K_1\otimes K_2\otimes\cdots\otimes K_n\otimes T', K_1\otimes K_2\otimes\cdots\otimes K_n) \rangle = \langle (K_1+2)(K_2+2)\cdots(K_n+2)\rangle_T \in F^nS(\Sigma, J) \), we have \( \langle (T, \bigcup_{i=1}^n K_i) \rangle \in F^nS(\Sigma, J) \). This proves the lemma.

Proof of Theorem \ref{thm:K}. By the definition, we have \( \xi(F^*S(\Sigma, J)) = F^*S(\Sigma', J') \). Using Lemma \ref{lem:|L'|} we have \( \xi(F^nS(\Sigma, J)) = \xi(F^nS(\Sigma, J')) = F^nS(\Sigma', J') = F^nS(\Sigma', J') \). This proves the theorem.

In this paper, we define the Kauffman bracket \( \mathcal{K} : \{ \text{unoriented framed links } S^3 \} \rightarrow \mathbb{Q}[A, A^{-1}] \) by \( \langle L \rangle = \mathcal{K}(L)(\emptyset) \in S(I \times I) \) for \( L \in \{ \text{links in } S^3 \} = T(I \times I) \). For an unoriented framed link \( L \) in \( S^3 \) and components \( K_1, K_2, \ldots, K_m \), we define \( \mathcal{K}(L, \bigcup_{i=1}^m K_i) \overset{\text{def}}{=} \sum_{j=0}^m \sum_{i_1, i_2, \ldots, i_j} \in \{ 1, 2, \ldots, m \} 2^{m-j} \mathcal{K}(L' \cup \bigcup_{i=1}^j K_{i_k}) \). Here we denote by \( L' \overset{\text{def}}{=} L \setminus \bigcup_{i=1}^m K_i \).

Using Lemma \ref{lem:|L'|}, we have the following corollary

**Corollary 5.4.** For an unoriented framed link \( L \) in \( S^3 \) and some components \( K_1, K_2, \ldots, K_m \), we have \( \mathcal{K}(L, \bigcup_{i=1}^m K_i) \in (A + 1)^m \mathbb{Q}[A, A^{-1}] \).

**Proof.** Since \( \ker \epsilon = (A+1)^m \mathbb{Q}[A, A^{-1}]\emptyset \), we have \( F^mS(I \times I) = (A+1)^m \mathbb{Q}[A, A^{-1}]\emptyset \). It is clear that \( (A+1)^m \mathbb{Q}[A, A^{-1}]\emptyset = F^mS(I \times I) = F^mS(I \times I) \). We have \( \langle (L, \bigcup_{i=1}^m K_i) \rangle = \mathcal{K}(L, \bigcup_{i=1}^m K_i) \emptyset \). This proves the corollary.

5.2. **Filtrations are Hausdorff.** In this subsection, we prove the following theorem.

**Theorem 5.5.** Let \( \Sigma \) be a compact connected oriented surface with non-empty boundary and \( J \) a finite subset of \( \partial \Sigma \). We have \( \bigcap_{\Sigma} F^nS(\Sigma, J) = 0 \), in other words, the natural homomorphism \( S(\Sigma, J) \rightarrow S(\Sigma, J) \) is injective.

We denote by \( V \) be the subset of \( \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \) consisting of the triples \((a, b, c) \in \mathbb{Z}_{\geq 0}^3\) which satisfies \( a + b + c \in 2\mathbb{Z}_{\geq 0} \) and \( |b - c| \leq a \leq b + c \). For \((a, b, c) \in V\), we denote the following right figure by the following left figure.
Let $\Sigma_{0,g+1}$ be the surface $D^2 \setminus \bigcup_{i=1}^g d_i$ where we denote by $D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and by $d_i$ the open disk $\{(x,y) \in \mathbb{R}^2 \mid x^2 + (y+1 - \frac{i}{g+1})^2 < \left(\frac{1}{4g+1}\right)^2\}$ for $1 \leq i \leq g$. We denote by $V(g)$ the set of consisting of all $(i_1, i_2, \cdots, i_{3g-3})$ which satifying

$$(i_{3j-3}, i_{3j-2}, i_{3j-1}), (i_{3j-1}, i_{3j}, i_{3j+1}) \in V$$

for $j = 1, \cdots, g - 1$. Here we define $i_0 \overset{\text{def}}{=} i_1$ and $i_{3g-2} \overset{\text{def}}{=} i_{3g-3}$. We denote by $\lambda(g,0)(i_1, i_2, \cdots, i_{3g-3})$ the element of $T(\Sigma_{0,g+1})$ presented by Figure 10 for $(i_1, i_2, \cdots, i_{3g-3}) \in V(g)$.

Fix an orientation preserving embedding $e_3 : D^2 \times I \to S^3$ and a diffeomorphism $e_4 : \Sigma_{0,g+1} \times I \to \overline{S^3 \setminus e_3(\Sigma_{0,g+1} \times I)}$ where we denote the closure of $\overline{S^3 \setminus e_3(\Sigma_{0,g+1} \times I)}$ by $\overline{S^3 \setminus e_3(\Sigma_{0,g+1} \times I)}$. Then we define a bilinear map $(\cdot, \cdot) : S(\Sigma_{0,g+1}) \times S(\Sigma_{0,g+1}) \to \mathbb{Q}[A, A^{-1}]$ by $(\langle L_1 \rangle, \langle L_2 \rangle) = K(e_3(L_1) \cup e_4(L_2))$ for $L_1$ and $L_2 \in T(\Sigma_{0,g+1})$. The bilinear map induces $(\cdot, \cdot) : C \otimes S(\Sigma_{0,g+1}) \times C \otimes S(\Sigma_{0,g+1}) \to C[A, A^{-1}]$. Here we denote by $C[A, A^{-1}]$ the ring of Laurent polynomials over $C$. For a primitive $2r$-th root $\gamma$, the bilinear map induces $(\cdot, \cdot) : S^\gamma(\Sigma_{0,g+1}) \times S^\gamma(\Sigma_{0,g+1}) \to C$ where we denote by $S^\gamma(\Sigma_{0,g+1}) \overset{\text{def}}{=} C \otimes S(\Sigma_{0,g+1})/(A - \gamma)C \otimes S(\Sigma_{0,g+1})$. The bilinear map induces the linear map $\psi : S(\Sigma_{0,g+1}) \to \text{Hom}_\mathbb{Q}[A, A^{-1}](S(\Sigma_{0,g+1}), \mathbb{Q}[A, A^{-1}])$ by $v \mapsto (u \mapsto (v, u))$. It induces the linear maps

$$\psi : C \otimes S(\Sigma_{0,g+1}) \to \text{Hom}_\mathbb{C}[A, A^{-1}](C \otimes S(\Sigma_{0,g+1}), \mathbb{C}[A, A^{-1}]),$$

$$\psi^\gamma : S^\gamma(\Sigma_{0,g+1}) \to \text{Hom}_\mathbb{C}(S^\gamma(\Sigma_{0,g+1})).$$

We denote by $|A| = 0$ the quotient map $C \otimes S(\Sigma_{0,g+1}) \to S^\gamma(\Sigma_{0,g+1})$.

For a surface $\Sigma$ and any finite subset $J \subset \partial \Sigma$, we recall that we denote by $T_0(\Sigma, J)$ the set of isotopy classes of 1-dimensional submanifolds of $\Sigma$ whose boundary is $J$ and that $S(\Sigma, J)$ is freely generated by $T_0(\Sigma, J)$ as a $\mathbb{Q}[A, A^{-1}]$-module.

**Theorem 5.6** (Lickorish [6], P.347, Theorem). (1) The map $\lambda(g,0) : V(g) \to T_0(\Sigma_{0,g+1})$ is bijective.

(2) For a primitive $4r$-th root $\gamma$, $S^\gamma(\Sigma_{0,g+1})/\ker \psi$ is a free $C$-module with basis

$$\{\lambda(g,0)(i_1, \cdots, i_{3g-3})(i_{3j-3}, i_{3j-2}, i_{3j-1}), (i_{3j-1}, i_{3j}, i_{3j+1}) \in V, 2r - 4 \geq i_{3j-3} + i_{3j-2} + i_{3j-1}, 2r - 4 \geq i_{3j-1} + i_{3j} + i_{3j+1}\}.$$
We remark that Lickorish gave another basis in [6]. Using Theorem in [6], we have \( \lambda(g,0) \) is injective. It is proved in a similar way to the proof of Lemma 5.10 in this paper that \( \lambda(g,0) \) is surjective.

**Lemma 5.7.** (1) The \( \mathbb{C}[A,A^{-1}] \) module homomorphism

\[
\psi : \mathbb{C} \otimes S(\Sigma_{0,g+1}) \to \text{Hom}_{\mathbb{C}[A,A^{-1}]}(\mathbb{C} \otimes S(\Sigma_{0,g+1}), \mathbb{C}[A,A^{-1}])
\]

is injective.

(2) The \( \mathbb{Q}[A,A^{-1}] \) module homomorphism

\[
\psi : S(\Sigma_{0,g+1}) \to \text{Hom}_{\mathbb{Q}[A,A^{-1}]}(S(\Sigma_{0,g+1}), \mathbb{Q}[A,A^{-1}])
\]

is injective.

**Proof.** Let \( x \) be an element of \( \mathbb{C} \otimes S(\Sigma_{0,g+1}) \setminus \{0\} \). Using Theorem 5.6, we have \( \psi(x|_{A=\gamma}) \neq 0 \) for some primitive 4r-th root \( \gamma \). In other words, we have \( (x|_{A=\gamma}, y) \neq 0 \) for some \( y \in S^\gamma(\Sigma_{0,g+1}) \). We regard \( y \) as an element of \( \mathbb{C} \otimes S(\Sigma_{0,g+1}) \) by \( S^\gamma(\Sigma_{0,g+1}) = \mathbb{C} T_0(\Sigma_{0,g+1}) \hookrightarrow \mathbb{C}[A,A^{-1}]T_0(\Sigma_{0,g+1}) = \mathbb{C} \otimes S(\Sigma_{0,g+1}) \). Since \( (x,y)|_{A=\gamma} = (x|_{A=\gamma}, y) \neq 0 \), we have \( (x,y) \neq 0 \). This proves (1).

Let \( x \) be an element of \( S(\Sigma_{0,g+1}) \setminus \{0\} \). We regard \( x \) as an element of \( \mathbb{C} \otimes S(\Sigma_{0,g+1}) \). By (1), we have \( x \sum_{j=1}^{m}(a_j + b_j \sqrt{-1})c_j \neq 0 \) for some \( a_j \) and \( b_j \in \mathbb{R} \) and \( c_j \in T_0(\Sigma_{0,g+1}) \). Let \( k \) be an integer satisfying the coefficient of \( A^k \) in

**Fig 10.** \( \lambda(g,0)(i_1, i_2, \cdots, i_{3g-3}) \)
(x, \sum_{j=1}^m (a_j + b_j \sqrt{-1})c_j) \text{ is not 0. We denote by } \omega(u_1, \ldots, u_m) \text{ the coefficient of } A^k \text{ in } (x, \sum_{j=1}^m u_j c_j) \text{ for } u_j \in \mathbb{R}. \text{ Then, } \omega : \mathbb{R}^m \to \mathbb{R}, (u_1, \ldots, u_m) \mapsto \omega(u_1, \ldots, u_m) \text{ is a linear map. Since } \omega \text{ is linear, } \omega \text{ is continuous. By the definition, we have } \\
\omega(a_1, \ldots, a_m) \neq 0 \text{ or } \omega(b_1, \ldots, b_m) \neq 0. \text{ By the density of } Q \text{ in } \mathbb{R}, \text{ we have } \omega(q_1, \ldots, q_m) \neq 0 \text{ for some } q_1, \ldots, q_m \in Q. \text{ Hence we obtain } (x, \sum_{j=1}^m q_j c_j) \neq 0. \text{ This proves (2).} \]

\[ \square \]

To prove Theorem 5.5 in the case \( J = \emptyset \), we need the following lemma.

**Lemma 5.8.** Let \( \Sigma \) be a compact connected oriented surface with non-empty boundary. We have
\[ \psi(F^k(S(\Sigma))) = \psi(F^k(S(\Sigma))) \subset (A + 1)^k \text{Hom}_{\mathbb{Q}[A, A^{-1}]}(S(\Sigma), \mathbb{Q}[A, A^{-1}]) \]
for \( k \in \mathbb{Z}_{\geq 0} \).

**Proof.** By Theorem 5.1, it is sufficient to prove the lemma in the case \( \Sigma = \Sigma_0,g+1 \). Let \( L \) and \( L' \) be links in \( \Sigma_0,g+1 \times I \) and \( K_1, \ldots, K_k \) components of \( L \). By Corollary 5.4, we have \( K(e_3(L) \cup e_4(L'), \bigcup_{i=1}^ke_3(K_i)) \in (A + 1)^k \mathbb{Q}[A, A^{-1}] \). This proves the lemma.

\[ \square \]

**Lemma 5.9** (A special case of Theorem 5.5). Let \( \Sigma \) be a compact connected oriented surface with non-empty boundary. We have \( \cap_{i=1}^\infty F^n(S(\Sigma)) = 0 \).

**Proof.** By Theorem 5.1, it is sufficient to prove the lemma in the case \( \Sigma = \Sigma_0,g+1 \). By Lemma 5.8, we have
\[ \psi(\cap_{i=1}^\infty F^n(S(\Sigma_0,g+1))) \subset \cap_{i=1}^\infty (A + 1)^i \text{Hom}_{\mathbb{Q}[A, A^{-1}]}(S(\Sigma), \mathbb{Q}[A, A^{-1}]) = 0. \]
Since \( \psi \) is injective, we have \( \cap_{i=1}^\infty F^n(S(\Sigma_0,g+1)) = 0 \). This proves the lemma.

\[ \square \]

For \( g \geq 1 \) and \( m \geq 1 \), we denote by \( \mathcal{V}(g,m) \) the set of consisting of all \( (i_1, i_2, \ldots, i_{2m-1}, j_1, j_2, \ldots, j_{3g-2}) \) which satisfy \((i_{k-1}, i_k, 1) \in \mathcal{V} \) for \( k = 1, \ldots, 2m-1 \), \((j_{3k-2}, j_{3k-1}, j_{3k}) \), \((j_{3k+1}, j_{3k+2}) \) \( \in \mathcal{V} \) for \( k = 1, \ldots, g-1 \) and \( (i_{2n-1}, j_1, j_2) \) where we define by \( i_0 \overset{\text{def}}{=} 1 \) and \( i_{3g-1} \overset{\text{def}}{=} i_{3g-2} \). Let \( J \) be a finite subset of \( \partial D^2 \subset \partial \Sigma_0,g+1 \) satisfying \( \sharp J = 2m \). We define a map \( \lambda(g,m) : \mathcal{V}(g,m) \to \mathcal{T}(\Sigma_0,g+1, J) \) by \( \lambda(g,m)(i_1, i_2, \ldots, i_{2m-1}, j_1, j_2, \ldots, j_{3g-2}) \) looks as in Figure 5.2 for any \( (i_1, i_2, \ldots, i_{2m-1}, j_1, j_2, \ldots, j_{3g-2}) \in \mathcal{V}(g,m) \).

**Lemma 5.10.** For \( g \geq 1 \) and \( m \geq 1 \), \( \lambda(g,m) : \mathcal{V}(g,m) \to \mathcal{T}(\Sigma_0,g+1, J) \) is surjective.

**Proof.** We use the following proposition. For any \( L \in \mathcal{T}(\Sigma_0,g+1, J) \), there exists \( \hat{L} \) representing \( L \) and satisfying the above conditions in Proposition 5.11. This proves the lemma.

\[ \square \]

Let \( I_1, \ldots, I_{2m-1}, J_1, \ldots, J_{3g-2} \) be one-dimensional submanifolds of \( \Sigma_0,g+1 \) as in Figure 12. We denote by \( \hat{L} \overset{\text{def}}{=} (\bigcup_{q=1}^{2m-1} I_q) \cup (\bigcup_{r=1}^{3g-2} J_r) \).

We prove the following proposition by induction on \( n \).

**Proposition 5.11 (n).** Let \( \hat{L} \) be a one-dimensional submanifold of \( \Sigma_0,g+1 \) satisfying the following conditions.
Fig 11. $\lambda(g, m)(i_1, i_2, \ldots, i_{2m-1}, j_1, j_2, \ldots, j_{3g-2})$

Fig 12. $L$
There is no closed disk \( d \) in \( \Sigma_{0,g+1} \) such that \( \partial d \subset \hat{L} \).

We have \( \partial \hat{L} = J \).

The intersections \( \hat{L} \cap \bar{L} \) consist transverse double points.

We denote by \( \delta \) the set consisting of all \( P \in \hat{L} \cap \bar{L} \) satisfying the following condition

\[
e((x, y) \in \partial D^2 | x \geq 0) \subset \hat{L}, e((x, y) \in \partial D^2 | x \leq 0) \subset \bar{L}, e(1, 0) = P .
\]

for some \( e : D^2 \to \Sigma_{0,g+1} \). Then we have \( \sharp \delta(\alpha(\hat{L})) \leq n \Rightarrow L \in \lambda(g, m)(\mathcal{V}(g, m)) \)

where we denote by \( L \) the isotopy class of \( \hat{L} \).

Proof. By definition, we have Proposition \( \text{5.11} (0) \). We assume \( e \in \partial \Sigma_{0,g+1} \) satisfying the above conditions and \( \sharp \delta(\alpha(\hat{L})) = n \). Since \( \sharp \delta(\alpha(\hat{L})) > 0 \), there exists an embedding \( e : D^2 \to \Sigma_{0,g+1} \) such that \( e((x, y) \in \partial D^2 | x \geq 0) \subset \hat{L}, e((x, y) \in \partial D^2 | x \leq 0) \subset \bar{L} \). Choose \( \hat{L}' \) a one-dimensional submanifold of \( \Sigma_{0,g+1} \) which is \( \hat{L} \) except for the neighborhood of \( e(D) \), where it looks as shown in the figure.

Since \( \hat{L} \simeq \hat{L}' \) and \( \sharp \delta(\alpha(\hat{L}')) < n \), we have \( L \in \lambda(g, m)(\mathcal{V}(g, m)) \) where we denote by \( L \) the isotopy class of \( \hat{L} \). This proves Proposition \( \text{5.11}(n) \) for any \( n \geq 0 \).

We define an injective map

\[
i(g, m) : \mathcal{V}(g, m) \to \mathcal{V}(g + m)
\]

\[
(i_1, i_2, \cdots, i_{2m-1}, j_1, j_2, \cdots, j_{3g-2}) \mapsto
\]

\[
(1, i_1, 1, i_2, 3, 1, \cdots, i_{2m-3}, 1, i_{2m-2}, i_{2m-1}, j_1, j_2, \cdots, j_{3g-2}.
\]

for \( m \geq 1 \) and \( g \geq 1 \). Let \( J \) be a finite subset of \( \partial D^2 \subset \partial \Sigma_{0,g+1} \) satisfying \( \sharp J = 2m \). We define the \( \mathbb{Q}[\mathcal{A}, \mathcal{A}^{-1}] \)-module homomorphism \( i(g, m) : S(\Sigma_{g+1}, J) \to S(\Sigma_{0,g+1} + 1) \) by \( (\lambda(g, m)(v)) \to (\lambda(g + m, 0)(i(g, m)(v))) \). Using Theorem \( \text{5.6}(1) \), we have the following proposition.

**Proposition 5.12.** The \( \mathbb{Q}[\mathcal{A}, \mathcal{A}^{-1}] \)-module homomorphism \( i(g, m) : S(\Sigma_{0,g+1}, J) \to S(\Sigma_{0,g+1} + 1) \) is well-defined and injective.

**Corollary 5.13.** The map \( \lambda(g, m) : \mathcal{V}(g, m) \to \mathcal{T}_0(\Sigma_{0,g+1}, J) \) is bijective.

Let \( \Sigma \) be a compact connected oriented surface with non-empty boundary, \( J \) a finite subset of \( \partial \Sigma \) and \( P_1 \) and \( P_2 \) two points of \( J \). We choose two orientation preserving embeddings \( \delta_1, \delta_2 : I \to \partial \Sigma \) such that \( \delta_1(I) \cap J = \delta_1(\frac{1}{2}) = P_1 \) and that \( \delta_2(I) \cap J = \delta_2(\frac{1}{2}) = P_2 \). We define a surface \( \Sigma(P_1, P_2) \) by gluing \( \Sigma \) and \( I \times \Sigma \) by \( (0, 1-t) = \delta_1(t) \) and \( (1, t) = \delta_2(t) \). We introduce \( \iota(P_1, P_2) : \mathcal{T}_0(\Sigma, J) \to \mathcal{T}_0(\Sigma(P_1, P_2), J \setminus \{P_1, P_2\}) \) such that \( \iota(P_1, P_2)(L) \) is the isotopy class of \( \hat{L} \cup \{\frac{1}{2} \} \subset I \times I(J \in I) \) where \( \hat{L} \) represents \( L \). The map \( \iota(P_1, P_2) \) induces a \( \mathbb{Q}[\mathcal{A}, \mathcal{A}^{-1}] \)-module homomorphism \( \iota(P_1, P_2) : S(\Sigma, J) \to S(\Sigma(P_1, P_2), J \setminus \{P_1, P_2\}) \).
Lemma 5.14. Let \( \Sigma \) be a compact connected oriented surface with non-empty boundary, \( J \) a finite subset of \( \partial \Sigma \) and \( P_1 \) and \( P_2 \) two points of \( J \). Then \( \iota(P_1, P_2) : \mathcal{T}_0(\Sigma, J) \to \mathcal{T}_0(\Sigma(P_1, P_2), J\{P_1, P_2\}) \) is injective. In other words, \( \iota(P_1, P_2) : \mathcal{S}(\Sigma, J) \to \mathcal{S}(\Sigma(P_1, P_2), J\{P_1, P_2\}) \) is injective.

Proof. Let \( J = \{P_1, P_2, \ldots, P_{2m-1}, P_{2m}\} \). We denote by \( \eta = \iota(P_{2m-1}, P_{2m}) \circ \cdots \circ \iota(P_3, P_4) \circ \iota(P_1, P_2) \) and by \( \Sigma = \Sigma(P_1, P_2)(P_3, P_4) \cdots (P_{2m-1}, P_{2m}) \). For some integer \( g \) and some finite subset \( J' \subset \partial \Sigma^2 \subset \partial \Sigma_{0, g+1} \), we choose a diffeomorphism \( \chi : (\Sigma \times I, J \times I) \to (\Sigma_{0, g+1} \times I, J' \times I) \) and \( \chi' : \Sigma \times I \to \Sigma_{0, g+m+1} \times I \) satisfying \( \chi_* \circ \iota(g, m) \circ (\chi'_*)^{-1} = \eta \). Here we denote by \( \chi_* : \mathcal{S}(\Sigma, J) \to \mathcal{S}(\Sigma_{0, g+1}, J') \) and \( \chi'_* : \mathcal{S}(\Sigma) \to \mathcal{S}(\Sigma_{0, g+m+1}) \) the \( \mathbb{Q}[A, A^{-1}] \)-module automorphisms induced by \( \chi \) and \( \chi' \), respectively. Since \( \iota(g, m) \) is injective, \( \eta \) is also injective. Hence \( \iota(P_1, P_2) \) is injective. This proves the lemma.

\( \square \)

Proof of Theorem 5.9 in general cases. We suppose \( J \neq \emptyset \). Let \( J = \{P_1, \ldots, P_{2m}\} \). We denote by \( \eta = \iota(P_{2m-1}, P_{2m}) \circ \cdots \circ \iota(P_1, P_2) \) and by \( \Sigma = \Sigma(P_1, P_2)(P_3, P_4) \cdots (P_{2m-1}, P_{2m}) \). By definition, we have \( \eta(F^n \mathcal{S}(\Sigma, J)) \subset F^n \mathcal{S}(\Sigma) \) for any \( n \in \mathbb{Z} \geq 0 \). Using Lemma 5.9, we have \( \bigcap_{n=0}^{\infty} F^n \mathcal{S}(\Sigma, J) \subset \bigcap_{n=0}^{\infty} F^n \mathcal{S}(\Sigma) = 0 \). Since \( \eta \) is injective, we have \( \bigcap_{n=0}^{\infty} F^n \mathcal{S}(\Sigma, J) = 0 \). This proves the theorem.

\( \square \)

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