PERTURBATIONS OF BASIC DIRAC OPERATORS ON RIEMANNIAN FOLIATIONS

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Abstract. Using the method of Witten deformation, we express the basic index of a transversal Dirac operator over a Riemannian foliation as the sum of integers associated to the critical leaf closures of a given foliated bundle map.

1. Introduction

It is well-known that the index of the de Rham operator
\[ D = d + d^* : \Omega^{\text{even}}(M) \to \Omega^{\text{odd}}(M) \]
defined by
\[ \text{ind}(D) = \dim \ker(D|_{\Omega^{\text{even}}}) - \dim \ker(D|_{\Omega^{\text{odd}}}) \]
is the Euler characteristic of the Riemannian manifold \( M \) (compact, no boundary). In [25], Witten replaced \( d \) with the deformed differential \( d_s = e^{-sf}d e^{sf} \), where \( s > 0 \) and \( f \) is a smooth real-valued function. This leads to a one-parameter family of deformed Dirac operators
\[ D_s = d_s + d_s^* = D + sZ, \]
where \( Z = df \wedge + (df \wedge)^* \). This family of Fredholm operators has the same index for all \( s \), since the index is invariant under homotopy. Witten’s idea was that each eigenvalue of \( D_s^2 \) has an asymptotic expansion as \( s \to \infty \) with the leading term computable from the local data at the critical set of \( f \) (where \( df = 0 \)). In particular, if \( f \) is a Morse function, one can show that
\[ \text{ind}(D) = \sum (-1)^p m_p, \]
where \( m_p \) is the number of critical points of \( f \) of Morse index \( p \).

In [13], the authors expanded the method of Witten deformation and provided the formula for the index in all cases when \( D \) is a Dirac-type operator and \( Z \) is an admissible bundle map satisfying certain nondegeneracy conditions (similar to the Morse conditions).

The purpose of this paper is to find an expression for the basic index of a transversal Dirac operator over a Riemannian foliation in terms of local quantities associated to the singular set of a foliated bundle map satisfying admissibility and nondegeneracy conditions.

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We use the method of Witten deformation to achieve localization (see Corollary 4.6). Our main result (Theorem 4.12) is the formula

\[ \text{ind}_b(D_b) = \sum \dim \left( \bigcap_j \left( \bigoplus_{\lambda < 0} E_\lambda \left( L_j^+ (\vec{x}) \right) \right)^{H_{\vec{x}}} \right) - \dim \left( \bigcap_j \left( \bigoplus_{\lambda < 0} E_\lambda \left( L_j^- (\vec{x}) \right) \right)^{H_{\vec{x}}} \right). \]

Here, \( D_b \) is a basic Dirac operator, \( \ell \) is a critical leaf closure for a basic bundle map \( Z \), and \( \vec{x} \) is an arbitrary point of \( \ell \). The linear maps \( L_j^\pm (\vec{x}) \) are obtained from the linearization of the Clifford form of \( Z \) at \( \vec{x} \) (see Section 4.2), the space \( E_\lambda \) is the eigenspace associated to the eigenvalue \( \lambda \), and \( H_{\vec{x}} \) is the infinitesimal holonomy group associated to \( \vec{x} \). The Hopf index theorem for Riemannian foliations proved in [4] can be easily derived from this formula.

We now briefly explain the setup for the formula; precise definitions are in Section 2.1. The reader may consult the introduction in [11] and Section 3 of [19] for more complete expositions concerning basic Dirac operators, and more information on Riemannian foliations is contained in [24] and [16]. A Dirac operator has the form \( D = \sum c(e_j) \nabla^E e_j \), where \( \nabla^E \) is a Clifford connection on a Hermitian vector bundle \( E \) over \( M \), \( \{ e_j \} \) is a local orthonormal frame of \( TM \), and \( c \) denotes Clifford multiplication. Suppose now that \( M \) has the additional structure of a Riemannian foliation \( F \), i.e. a layering of \( M \) by immersed submanifolds (leaves), and a transverse Riemannian metric that is invariant along the leaves. A simple example of this structure is that of the orbits of a compact Lie group action, where all the orbits have the same dimension, and where the invariant transverse metric can be obtained as the average of an arbitrary transverse metric along the orbits (leaves).

On a Riemannian foliation, there exist natural operators called transversal Dirac operators. Consider a Hermitian vector bundle \( E \) over \( M \) that is a module over the complexified Clifford algebra of the normal bundle \( NF \) to the foliation. The formula for the transversal Dirac operator is the same as that of the ordinary Dirac operator, but the sum is merely over a local orthonormal frame of the normal bundle \( NF \subset TM \). This operator restricts to an operator on \( \Gamma_b(M,E,F) \), the space of basic sections of \( E \). A section \( s \) of \( E \) is called basic if it is invariant under parallel translation along the leaves, i.e. if \( \nabla^F_X s = 0 \) for all \( X \in \Gamma(TM) \). The leaf space \( M/F \) may be quite singular; these basic sections provide a type of desingularization of the leaf space. On a Riemannian foliation, a transverse Dirac operator maps the space \( \Gamma_b(M,E,F) \) to itself, but it is not necessarily symmetric with respect to the \( L^2 \) inner product. A basic Dirac operator \( D_b \) is the symmetric operator defined as

\[ D_b = \sum c(e_j) \nabla^E_{e_j} - \frac{1}{2} c(\kappa_b), \]

where the sum is over an orthonormal frame of \( NF \) and \( \kappa_b \) is the basic component of the mean curvature one-form.

The basic index of a basic Dirac operator \( D_b \) is

\[ \text{ind}_b(D_b) = \dim \ker D_b^+ - \dim \ker(D_b^-), \]

where one restricts to the subspace of basic sections of the graded Hermitian bundle \( E = E^+ \oplus E^- \). It turns out that this basic index is a Fredholm index, implying that the dimensions are finite and that the index is stable under perturbations. Using Molino theory (see Section 3), one may show that this index is equivalent to the equivariant index of a certain Dirac operator on an \( O(n) \)-manifold.
It is an important problem to express the basic index in geometric terms. We refer to [9, Problem 2.8.9], where this problem was first stated explicitly. One example of an index formula is the Gauss-Bonnett theorem for Riemannian foliations, proved in [6]. Also, in [6], J. Brüning, F. W. Kamber, and Richardson use the equivariant index theorem of [5] to give a geometric formula for the basic index of a basic Dirac operator in terms of Atiyah-Singer type integrands and eta invariants. In [10], A. Gorokhovsky and J. Lott proved a different formula for the basic index of a basic Dirac operator, in the case where all the infinitesimal holonomy groups of the foliation are connected tori and when Molino’s commuting sheaf is abelian and has trivial holonomy. In contrast, in our formula the index is a sum of integers, each of which is computed at a single leaf closure.

A related natural question is whether the basic index of a transversally elliptic operator on a Riemannian foliation has similar properties to those of an elliptic operator. Until now, it was not known whether this index is always zero for transversally elliptic differential operators on a Riemannian foliation of odd codimension. Previously, this seemingly elementary fact had only been proved for the basic Euler characteristic (in [12]), and the general case is now proved in Corollary 4.8.

In Section 2.1, we review properties of foliated vector bundles and Dirac operators over Riemannian foliations. In Sections 2.2 and 2.3, we establish conditions on $Z$ so that the index computation localizes to small neighborhoods of critical leaf closures. The necessary condition for localization is proved in Theorem 2.5. We show in Section 3 how to reduce basic differential operator calculations on a tubular neighborhood of a leaf closure to equivariant differential operator calculations in Euclidean space. In Section 4.1, we define the model operator and prove the localization theorem. We show in Section 4.2 that $Z$ can be deformed to a special form called Clifford form, which is used in our formula for $\text{ind}_b(D_b)$. We prove the main result (Theorem 4.12) in Section 4.3. Finally, in Section 5, we apply Theorem 4.12 to compute the basic Euler characteristic and basic signature of some foliations. In particular, Example 5.3 shows that localization is possible even when $D_bZ + ZD_b$ is a first order differential operator.

2. Perturbing basic Dirac operators

2.1. Preliminaries and Notational Conventions. Let $(M, \mathcal{F})$ be a closed, connected, smooth manifold $M$ equipped with a smooth foliation $\mathcal{F}$ of codimension $q$. We assume that $(M, \mathcal{F})$ has additional geometric structure. That is, let $Q = TM/T\mathcal{F} \to M$ be the normal bundle of $\mathcal{F}$, and let $g_Q$ be a metric on $Q$. We assume that $g_Q$ is holonomy-invariant, meaning that its Lie derivative in directions tangent to $\mathcal{F}$ is zero. A foliation along with such a $g_Q$ is called a Riemannian foliation. When $(M, \mathcal{F}, g_Q)$ is a Riemannian foliation, one may always choose a metric $g$ on $M$ such that the restriction of $g$ to $(T\mathcal{F})^\perp$ agrees with $g_Q$; such metrics are called bundle-like metrics. Given a bundle-like metric, the leaves of $\mathcal{F}$ are locally equidistant. Throughout this paper, we assume that we have chosen a bundle-like metric $g$ on $M$ compatible with $g_Q$. See [20], [16], and [24] for standard facts about Riemannian foliations.

We now recall the definitions (see [13] and [16]) of foliated bundle and basic connection. Let $G$ be a compact Lie group. We say that a principal $G$–bundle $P \to (M, \mathcal{F})$ is a foliated principal bundle if it is equipped with a foliation $\mathcal{F}_P$ (the lifted foliation) such that the distribution $T\mathcal{F}_P$ is invariant under the right action of $G$, is transversal to the tangent space to the fiber, and projects onto $T\mathcal{F}$. A connection $\omega$ on $P$ is called adapted to $\mathcal{F}_P$ if the
associated horizontal distribution contains $T\mathcal{F}_P$. An adapted connection $\omega$ is called a **basic connection** if it is basic as a $g$-valued form on $(P, \mathcal{F}_P)$; that is, $i_X \omega = 0$ and $i_X d\omega = 0$ for every $X \in T\mathcal{F}$, where $i_X$ denotes the interior product with $X$. Note that in [13] the authors showed that basic connections always exist on a foliated principal bundle over a Riemannian foliation.

Similarly, a complex vector bundle $E \to (M, \mathcal{F})$ of rank $k$ is **foliated** if $E$ is associated to a foliated principal bundle $P \to (M, \mathcal{F})$ via a representation $\rho$ from $G$ to $U(k)$. Let $\Omega(M, E)$ denote the space of forms on $M$ with coefficients in $E$. If a connection form $\omega$ on $P$ is adapted, then we say that an associated covariant derivative operator $\nabla^E$ on $\Omega(M, E)$ is **adapted** to the foliated bundle. We say that $\nabla^E$ is a **basic connection** on $E$ if in addition the associated curvature operator $(\nabla^E)^2$ satisfies $i_X (\nabla^E)^2 = 0$ for every $X \in T\mathcal{F}$. Note that $\nabla^E$ is basic if $\omega$ is basic.

For $m \in M$, let $\mathbb{C}l(Q_m)$ be the complex Clifford algebra associated to the vector space $Q$ and quadratic form $g_Q$. We say that $(M, \mathcal{F}, g_Q)$ is **transversally spin** if $Q$ is spin$^c$. That is, there exists a complex spinor $\mathbb{C}l(Q)$-bundle $\mathcal{S}$ over $M$ such that, for all $m \in M$, the action of $\mathbb{C}l(Q_m)$ on $\mathcal{S}_m$ is an irreducible representation of $\mathbb{C}l(Q_m)$. A simple example shows that this condition is necessary even for spin$^c$ manifolds; if $X$ is any manifold that is not spin$^c$, then the product foliation $X \times X$ would not be transversally spin$^c$ even though $X \times X$ always admits a complex structure and thus is spin$^c$.

Let in addition $E \to M$ be a Hermitian $\mathbb{C}l(Q)$ Clifford bundle. We assume that the basic connection $\nabla^E$ is compatible with these additional structures.

Let $\Gamma(M, E)$ denote the space of smooth sections of $E$ over $M$. Let $\Gamma_b(M, E, \mathcal{F})$ denote the space of **basic sections** of $E$, meaning that

$$\Gamma_b(M, E, \mathcal{F}) = \{ u \in \Gamma(M, E) : \nabla^E_V u = 0 \text{ for all } V \in \Gamma(T\mathcal{F}) \}.$$ 

Similarly, a bundle endomorphism $A$ is basic if $\nabla^A_{\text{End}(E)}A = 0$ for all $X \in \Gamma(T\mathcal{F})$.

Now we define the transversal Dirac operator, which is similar to the standard Dirac operator but is elliptic only when restricted to directions orthogonal to the tangent bundle $T\mathcal{F}$ of the foliation. Formally, the **transversal Dirac operator** $D_{tr}$ is the composition of the maps

$$\Gamma(E) \xrightarrow{\nabla^E} \Gamma(Q^* \otimes E) \xrightarrow{\cong} \Gamma(Q \otimes E) \xrightarrow{c} \Gamma(E),$$

where the last map denotes Clifford multiplication, and the operator $\nabla^E$ is the projection of $\nabla^E$ to $\Gamma(Q^* \otimes E)$. Clifford multiplication by an element $v \in Q_x$ on the fiber $E_x$ is denoted by $c(v)$. Clifford multiplication by cotangent vectors in $Q^*$ will use the same notation: $c(\alpha) := c(\alpha^\#)$, where $Q^* \xrightarrow{\#} Q$ is the metric isomorphism. The transversal Dirac operator fixes the basic sections but is not symmetric on this subspace. By modifying $D_{tr}$ by a bundle map, we obtain a symmetric and essentially self-adjoint operator $D_b$ on $\Gamma_b(E)$. To define $D_b$, first let $H = \sum_{i=1}^p \pi(\nabla^{TM}_{f_i} f_i)$ be the mean curvature of the foliation, where $\pi : TM \to Q$ denotes the projection and $\{ f_i \}_{i=1, \ldots, p}$ is a local orthonormal frame of $T\mathcal{F}$. Let $\kappa = H^p$ be the corresponding 1-form, so that $H = \kappa^2$. Let $\kappa_b := P_b \kappa$ be the $L^2$-orthogonal projection of $\kappa$ onto the space of basic forms (see [1], [17]). We now define the **basic Dirac operator** by

$$D_b u := \frac{1}{2}(D_{tr} + D_{tr}^*) u = \sum_{i=1}^q c(e_i) \nabla_{e_i}^E u - \frac{1}{2}c(\kappa_b) u,$$
where \( \{ e_i \}_{i=1}^{q} \) is a local orthonormal frame of \( Q \). A direct computation shows that \( D_b \) preserves the basic sections. The operator \( D_b : \Gamma_b(E) \to \Gamma_b(E) \) is transversely elliptic, has

discrete spectrum, and is Fredholm (\cite{19}). The spectrum of \( D_b \) was shown to depend only on \( g_Q \) and not on \( g \) in \cite{11}, in spite of the fact that the mean curvature and \( L^2 \) inner product certainly depend on the choice of \( g \).

A grading on \( E \) is induced by the action of the chirality operator \( \gamma \) (in the transverse direction). Recall that if \( e_1, ..., e_q \) is an oriented orthonormal basis of \( Q_\times \), then the chirality operator is multiplication by

\[
\gamma = i^k c(e_1) ... c(e_q) \in \text{End} (E_x),
\]

where \( k = q/2 \) if \( q \) is even and \( k = (q + 1)/2 \) if \( q \) is odd. The \( \pm 1 \) eigenspaces of \( \gamma \) determine a grading of \( E = E^+ \oplus E^- \). This grading is called the natural grading. The other possible gradings are classified in \cite{18}, pp. 319ff. Observe that the chirality operator is a basic bundle map, because for \( X \in \Gamma(T\mathcal{F}), u \in \Gamma(E) \), and a local framing \( (e_1, ..., e_q) \) of \( Q \),

\[
\nabla^\text{End(E)}_{X} \gamma = i^k \sum_{j=1}^{q} c(e_1) ... c(\nabla^E_{X} e_j) ... c(e_q) = 0,
\]

because \((M, \mathcal{F}, g_Q)\) is Riemannian and we may choose the local framing so that each \( e_j \) is a local basic section. Let \( D_b^\pm : \Gamma_b(M, E^\pm, \mathcal{F}) \to \Gamma_b(M, E^\mp, \mathcal{F}) \) denote the restrictions of \( D_b \) to smooth even and odd sections. The operator \( D_b^- \) is the adjoint of \( D_b^+ \) with respect to the \( L^2 \)-metric on the space of basic sections \( \Gamma_b(M, E, \mathcal{F}) \) defined by \( g \) and the Hermitian metric on \( E \).

### 2.2. Admissible basic perturbations

We wish to perturb the basic Dirac operator by a basic bundle map. Let \( Z^+ : \Gamma(M, E^+) \to \Gamma(M, E^-) \) be a smooth basic bundle map, and we let \( Z^- \) denote the adjoint of \( Z^+ \). The operator \( Z \) on \( \Gamma(M, E) \), defined by \( Z(v^+ + v^-) = Z^-v^- + Z^+v^+ \) for any \( v^+ \in E^+_x \) and \( v^- \in E^-_x \), is self-adjoint. Let \( D_s \) denote the perturbed basic Dirac operator

\[
D_s = (D_b + sZ) : \Gamma_b(M, E, \mathcal{F}) \to \Gamma_b(M, E, \mathcal{F}), \tag{2.1}
\]

and define the operators \( D_s^\pm \) by restricting in the obvious ways.

The basic index \( \text{ind}_b(D_b) \) depends only on the homotopy type of the principal transverse symbol of \( D_b \) and satisfies

\[
\text{ind}_b(D_b) = \dim \ker ( (D_s)^2 |_{\Gamma_b(M, E^+, \mathcal{F})} ) - \dim \ker ( (D_s)^2 |_{\Gamma_b(M, E^-, \mathcal{F})} ).
\]

Thus, we need to study the operator

\[
(D_s)^2 = D_b^2 + s( ZD_b + D_bZ ) + s^2Z^2.
\]

As will be shown later, the leading order behavior of the eigenvalues of this operator as \( s \to \infty \) is determined by combinatorial data at the singular set of the operator \( Z \). This “localization” allows one to compute \( \text{ind}_b(D_b) \) in terms of that data. A sufficient condition for localization techniques to work is the requirement that the operator \( ZD_b + D_bZ \) restricts to a bounded operator on the space of smooth basic sections \( \Gamma_b(M, E, \mathcal{F}) \). We need the following lemmas.

A foliation \( \mathcal{F} \) of codimension \( q \) is called transversally parallelizable if there exists a global basis of \( Q \) consisting of basic vector fields. For any Riemannian foliation \( \mathcal{F} \), the
lifted foliation $\hat{F}$ on the orthonormal transverse frame bundle $\hat{M}$ is always transversally parallelizable (see [16]).

**Lemma 2.1.** If $F$ is transversally parallelizable, then the space $\Gamma_b(M, E, F)$ is a finitely-generated module over the space $C^\infty(M, F)$ of basic functions.

**Proof.** If $F$ is transversally parallelizable and $(M, F)$ is Riemannian, then the leaf closures are the fibers of a Riemannian submersion for any choice of bundle-like metric. There is a set $\{s_1, ..., s_k\}$ of basic sections of $E$ such that for every $x \in M$, $\{s_1(x), ..., s_k(x)\}$ is a basis of $E_x$. Then every basic section can be written as $\sum f_j(x) s_j(x)$ for some functions $f_j(x)$. For any $X$ tangent to $F$, $\nabla^E_F (\sum f_j(x) s_j(x)) = 0$ implies $\sum (X f_j)(x) s_j(x) = 0$, so that $X f_j$ for every $j$ and every $X \in TF$. Thus, the functions $f_j$ must be basic. □

**Lemma 2.2.** If $F$ is transversally parallelizable and if $V$ is a basic vector field that is tangent to every leaf closure of $(M, F)$, then $\nabla^E_V : \Gamma_b(M, E, F) \rightarrow \Gamma_b(M, E, F)$ is a bounded $C^\infty(M, F)$-linear operator.

**Proof.** As in the last proof, any section can be written as $\sum f_j(x) s_j(x)$, and for $V$ basic and tangent to the leaf closures, we have $V f_j = 0$, and $\nabla^E_F (\sum f_j(x) s_j(x)) = \sum f_j(x) \nabla^E_V s_j(x)$. Since $\nabla^E_V s_j$ must be basic, it is a linear combination $(\nabla^E_V s_j)(x) = \sum a_{jk}(x) s_k(x)$. Thus, $\nabla^E_V$ is bounded, since $a_{jk}$ is bounded on the compact $M$. □

**Lemma 2.3.** If $(M, F)$ is any Riemannian foliation and if $V$ is a basic vector field that is tangent to every leaf closure of $(M, F)$, then $\nabla^E_V : \Gamma_b(M, E, F) \rightarrow \Gamma_b(M, E, F)$ is a bounded $C^\infty(M, F)$-linear operator.

**Proof.** Let $V$ be a basic vector field that is tangent to every leaf closure of $F$, and let $\hat{V}$ be its horizontal lift in the orthonormal transverse frame bundle $\hat{M}$. Observe that horizontal lifts of basic vector fields are basic for the lifted foliation, so that $\hat{V}$ is a basic vector field. Since the leaf closures of $\hat{F}$ are principle bundles over the leaf closures of $F$, $\hat{V}$ must be tangent to the leaf closures of $\hat{F}$. Since $\left(\hat{M}, \hat{F}\right)$ is transversally parallelizable, $\nabla^\hat{F} V$ is bounded as a linear operator on the space of all basic sections $\Gamma_b(\hat{M}, \pi^* E, \hat{F})$. Since for every section $s \in \Gamma_b(M, E, F)$, $\pi^* s \in \Gamma_b(\hat{M}, \pi^* E, \hat{F})$, $\nabla^\hat{F} V$ is bounded as an operator on $\pi^* \Gamma_b(M, E, F)$. Finally, $\nabla^E_V s = \nabla^\hat{F} V \pi^* s$ for all $s \in \Gamma_b(M, E, F)$, and thus $\nabla^E_V : \Gamma_b(M, E, F) \rightarrow \Gamma_b(M, E, F)$ is a bounded operator. Further, $\nabla^E_V$ is $C^\infty(M, F)$-linear for the same reasons as in the last lemma. □

**Lemma 2.4.** If $(M, F)$ is any Riemannian foliation and if $x \in M$, then there exists a local orthonormal frame $\{e_1, ..., e_{\overline{q}}, e_{\overline{q}+1}, ..., e_q\}$ of $NF$ near $x$ consisting of basic vector fields such that $e_1, ..., e_{\overline{q}}$ is a local transverse orthonormal frame for the leaf closure containing $x$, and in a neighborhood of $x$ the fields $e_1, ..., e_{\overline{q}}$ remain tangent to the leaf closures of $F$.

**Proof.** From [16], there exists a local orthonormal frame of $NF$ near $x$ consisting of basic vector fields. It remains to show that the frame may be chosen to be adapted to the leaf closures near $x$. Let $B$ be a transversal submanifold to the leaf through $x$. By [16] Ch. 1, 5, App. D], the orbits of the closure of the holonomy pseudogroup acting on $B$ are exactly the intersections of the leaf closures with $B$, and there exist a smooth family of local isometries that generate the leaf closures near $x$. This family is at least $\overline{q}$-dimensional, where $\overline{q}$ is the
Suppose Theorem 2.5. \( \xi \in \text{bounded on } \Gamma \) covector \( \xi \)

Definition 2.9. nondegeneracy conditions on the perturbation. Note that the perturbations of the basic de

Remark 2.6. \( ZD_b + D_b Z \) can be a first order operator and still be bounded on \( \Gamma_b(M, E, F) \).

Remark 2.7. The condition on \( Z \) is weaker than the standard nonfoliated case (\cite{18} Section

Remark 2.8. Such a bundle endomorphism \( Z \) satisfying this condition does not always exist. For example, if \( M \) is an even dimensional spin manifold, \( D \) is a spin Dirac operator, and \( M \) is foliated by points, then no such \( Z \) exists. See \cite{18} Proposition 2.4].

2.3. Proper perturbations of basic Dirac Operators. In this section, we state the nondegeneracy conditions on the perturbation. Note that the perturbations of the basic de Rham operators in \cite{2} and \cite{14} are special cases.

Definition 2.9. Let \( Z : E \to E \) be a smooth basic bundle map on \( E \to (M, F) \).
Remark 2.13. The first condition is satisfied if
\[ x \] if and only if every point of \( \ell \).

Suppose \( c > 0 \). Compactness of \( Z \) implies the inequality holds in tubular neighborhoods of \( \sigma \).

\[ \sum \sigma_j Z_j \alpha \]

where \( \parallel \cdot \parallel_y \) is the pointwise norm on \( E_y \) and where \( d(y, \ell) \) is the distance from \( y \) to the leaf closure \( \ell \).

Definition 2.10. Let \( \ell \) be a leaf closure of \((M, \mathcal{F})\). Suppose that a neighborhood \( U \) of a point \( p \in \ell \) is diffeomorphic to an open set of the form \( U_1 \times U_2 \), where \( U_1 = \ell \cap U \) and \( U_2 \) is a open ball in \( \mathbb{R}^7 \) centered at the origin. The corresponding coordinates \((x, y)\) are called the \textit{adapted coordinates} if each coordinate \( y_j \) is a locally defined basic function for \( \mathcal{F} \), and \( \{(x, y) : y = 0\} = \ell \cap U \). In this case, we say that \( U \) is \textit{adapted to} \( \ell \).

Since \( \ell \) is an embedded submanifold of \( M \), there exists an adapted neighborhood of every point in \( \ell \).

Lemma 2.11. A critical leaf closure \( \ell \) of \((M, \mathcal{F})\) is nondegenerate for a basic bundle map \( Z \) if and only if every adapted neighborhood \( U \) of \( \ell \) has an adapted neighborhood \( U_1 \times U_2 \) such that \( \sigma_j Z_j \) for \( 1 \leq j \leq q \) over \( U \) such that \( Z = \sum_j y_j Z_j \) on \( U \), and \( Z \) is invertible over \( U \setminus \ell \).

Proof. Suppose \( U \) is an adapted neighborhood of a point of \( \ell \), a critical leaf closure for \( Z \). Since \( Z \) is smooth and vanishes at each \((x, 0)\), \( Z = \sum_j y_j Z_j \) for some bundle maps \( Z_j \). Let \( U \) be the closure of \( U \); we assume that we have chosen \( U \) to be small enough so that \( U \) is diffeomorphic to a product of compact sets in \( \ell \) and \( \mathbb{R}^7 \). The inequality in the definition of nondegenerate above is equivalent to

\[ \left\| \sum_j \sigma_j Z_j \alpha \right\|_{(x,y)}^2 \geq c^2 \]

for every \( \sigma \in S^{q-1} \), \( \alpha \in \Gamma(U, E) \), \((x, y) \in \overline{U} \) such that \( \|\alpha\|_{(x,y)} = 1 \). Since the left hand side of the inequality is a continuous function of \( \sigma \) and \( \alpha \) over the compact set \( S^{q-1} \times \Gamma \), its infimum is attained. It follows that on each such neighborhood, \( Z \) is invertible away from \( \ell \) if and only if the inequality holds with \( c > 0 \). Compactness of \( \ell \) implies the inequality holds in tubular neighborhoods of \( \ell \).

Definition 2.12. Let \( D^\pm_b : \Gamma(M, E^\pm) \to \Gamma(M, E^\mp) \) be the basic Dirac operator associated to a bundle of graded Clifford modules. Let \( D_s = D_b + sZ \) for \( s \in \mathbb{R} \), where \( Z = (Z^+, (Z^+)^*) \in \Gamma_b(M, \text{End}(E^+, E^-)) \). We say that \( Z \) is a \textit{proper perturbation} of \( D_b \) if

1. \( (D_s)^2 - D_b^2 \) is an operator that is bounded on \( \Gamma_b(M, E, \mathcal{F}) \).
2. All critical leaf closures for \( Z \) are nondegenerate.

Remark 2.13. The first condition is satisfied if \( Z \circ \sigma_{D_b} (x, \xi) + \sigma_{D_b} (x, \xi) \circ Z = 0 \) on \( E_x \) for every \( x \in M \), and every covector \( \xi \in (N\mathcal{F})^* \); see Theorem 2.3.

The following lemma shows that only certain ranks of vector bundles may admit proper perturbations.
Lemma 2.14. (in [18]) For any positive integer \( k \), there exists a linear map \( L : \mathbb{R}^k \to M, (\mathbb{C}) \) that satisfies \( L(x) \in \text{Gl}(r, \mathbb{C}) \) for \( x \neq 0 \) if and only if \( r = m2^{-1} \) for some positive integer \( m \).

The following result shows that nonsingular proper perturbations always exist on Clifford modules over a Riemannian foliation of odd codimension.

Proposition 2.15. Suppose that the codimension \( q \) of \((M,F)\) is odd. Let \( E \) be a bundle of basic graded Clifford modules over \( M \), and let \( D_b \) be the corresponding basic Dirac operator. Then there always exists a proper basic perturbation \( Z \) of \( D_b \); in particular the perturbation may be chosen to be invertible.

Proof. Near a point of \( M \), let \((e_1, \ldots, e_q)\) be a local basic orthonormal frame of the normal bundle. Let \( Z = i^{q(q+1)/2} c(e_1) c(e_2) \cdots c(e_q) \), which is well-defined independent of the choice of orthonormal frame, and it is a basic bundle map from \( E^\pm \) to \( E^\mp \). It satisfies the conditions of Theorem 2.14. Since the adjoint of \( Z \) is

\[
Z^* = i^{q(q+1)/2} (-1)^{q(q+1)/2} (-1)^q c(e_q) c(e_{q-1}) \cdots c(e_1) = i^{q(q+1)/2} c(e_1) c(e_2) \cdots c(e_q) = Z,
\]

we have a proper basic perturbation of \( D_b \) that is globally invertible. \( \square \)

3. Vector bundles over foliations and compact group actions

In this section, we restrict to a subset of foliated vector bundles over \((M,F)\). We will discuss \( \mathcal{F} \)-equivariant vector bundles; most of the interesting cases of geometrically constructed foliated bundles are also \( \mathcal{F} \)-equivariant.

Let \((M,F,g_Q)\) be a Riemannian foliation with bundle-like metric \( g \), let \( NF = (TF)^\perp \cong Q \). Let \( M/\mathcal{F} \) denote the space of leaf closures.

Definition 3.1. The bundle \( E \to M \) is called an \( \mathcal{F} \)-vector bundle if the following holonomy lifting property is satisfied. For any holonomy diffeomorphism \( h : U \to V \) associated to a path \( \gamma \) in a leaf from \( x \in D \) to \( y \in D \), where \( D \) is a submanifold at \( x \) transverse to \( \mathcal{F} \) and \( U \) and \( V \) are open subsets of \( D \) such that \( x, y \in U \cap V \), \( h \) lifts to a bundle isomorphism \( \tilde{h} : E|_U \to E|_V \), and all such maps satisfy

1. \( \tilde{h}_1 \tilde{h}_2 = \tilde{h}_1 \tilde{h}_2 \) (when both sides are defined)
2. \( \text{id}_D = \text{id}_E \)
3. \( \tilde{h}|_W = \tilde{h}|_W \) for any proper open subset \( W \) of the domain of \( h \).

Such a vector bundle is often called equivariant with respect to the foliation groupoid, and these vector bundles are examples of foliated vector bundles (see [13]), as described in Section 3.1. They come equipped with a partial connection \( \nabla^E \), which may be completed to a basic connection (with the same name). When \( E \) is an \( \mathcal{F} \)-vector bundle, the basic sections are those sections that satisfy \( \tilde{h}(s_x) = s_y \) for every holonomy diffeomorphism \( h \) as above with \( h(x) = y \), i.e. the holonomy-invariant sections. Let \( \text{Diff}_b(M,E,F) \) denote the space of differential operators that preserve the basic sections, and let \( \text{Diff}_b^*(M,E,F) \) denote the space of their restrictions to \( \Gamma_b(M,E,F) \). In the above, we call an \( \mathcal{F} \)-vector bundle an Hermitian \( \mathcal{F} \)-vector bundle if it is equipped with a holonomy-invariant Hermitian inner product.

Let \( \pi : \tilde{M} \to M \) denote the bundle of orthonormal transverse frames over \((M,F,g_Q)\). Let \( \pi^* \mathcal{F} \) be the pullback foliation, which has dimension \( \dim(F) + \dim(O(q)) \). The connection
on $\hat{M}$ is flat along the leaves, since $(M, F, g_Q)$ is Riemannian. The lifted foliation $\hat{F}$ is defined on $\hat{M}$ by horizontally lifting the leaves of $F$ locally. The tangent space of $\hat{F}$ is the intersection of the tangent space to the pullback foliation and the horizontal space coming from the adapted connection. The lifted foliation $\hat{F}$ has the same dimension as $F$ and is transversally parallelizable, and this implies that the leaf closure space $W = \hat{M}/\hat{F}$ is a manifold, called the basic manifold (see [16]). The quotient map $p : \hat{M} \rightarrow W$ is a Riemannian submersion for a natural choice of metric on $\hat{M}$. For any $O(q)$-equivariant vector bundle $F \rightarrow W$, let $\text{Diff}_{O(q)}(W, F)$ denote the space of $O(q)$-equivariant differential operators, and let $\text{Diff}_{O(q)}^*(W, F)$ denote the space of their restrictions to $O(q)$-invariant sections of $F$.

For each Hermitian $F$-vector bundle $\pi : E \rightarrow M$, we construct a canonically associated Hermitian $O(q)$-vector bundle $\pi' : E' \rightarrow W$ that has similar geometric and analytic properties, as follows. Specifically, the bundle $E'$ over $W$ is defined by

$$E'_w = \Gamma_b \left( p^{-1}(w), \pi^*E|_{p^{-1}(w)} , \hat{F}|_{p^{-1}(w)} \right),$$

and it is a $O(q)$-equivariant Hermitian vector bundle of finite rank (see [9] section 2.7). It is also true that $p^*E' = \pi^*E$, and we have the algebra isomorphisms

$$\Gamma_b (M, E, F) \rightarrow \Gamma_b \left( \hat{M}, \pi^*E, \pi^*F \right) \rightarrow \Gamma (W, E')^{O(q)}.$$

The isomorphisms are determined by the correspondences

$$s \in \Gamma_b (M, E, F) \rightarrow \text{section } \tilde{s} = \pi^*s \in \Gamma_b \left( \hat{M}, \pi^*E, \pi^*F \right)$$

$$\leftarrow \tilde{s}|_{p^{-1}(w)} \in \Gamma_b \left( p^{-1}(w), \pi^*E|_{p^{-1}(w)}, \hat{F}|_{p^{-1}(w)} \right) \text{ for each } w \in W$$

$$\leftarrow \tilde{s} \in \Gamma (W, E')^{O(q)}, \tilde{s}(w) = \tilde{s}|_{p^{-1}(w)}.$$

Given a differential operator $P \in \text{Diff}_b^*(M, E, F)$, it induces an operator $\tilde{P}$ on $\Gamma (W, E')^{O(q)}$ by

$$\tilde{P}(\tilde{s})(w) = \pi^* (Ps)|_{p^{-1}(w)},$$

where $s$ is the unique section in $\Gamma_b (M, E, F)$ such that $\tilde{s}(w) = \pi^*s|_{p^{-1}(w)}$. The following proposition from [2] gives the important properties of this construction.

**Proposition 3.2.** ([2] variant of Theorem 2.1) For each Hermitian $F$-vector bundle $\pi : E \rightarrow M$, there is a canonically associated Hermitian $O(q)$-vector bundle $\pi' : E' \rightarrow W$ and a canonical isomorphism of algebras

$$\text{Diff}_b^*(M, E, F) \cong \text{Diff}_{O(q)}^*(W, E').$$

The isomorphism preserves transverse ellipticity, and there is a metric on $W$ so that the isomorphism preserves formal adjoints with respect to the $L^2$ inner products.

**Corollary 3.3.** The previous proposition is also valid locally. Namely, if $M$ is replaced with an $\varepsilon$-tubular neighborhood $T_\varepsilon$ of a leaf closure $\ell \subset M$ and $W$ is replaced by a tubular neighborhood $T'_\varepsilon$ of the $O(q)$-orbit $p(\pi^{-1}(\ell))$, then $\text{Diff}_b^*(T_\varepsilon, E|_{T_\varepsilon}, F) \cong \text{Diff}_{O(q)}^*(T'_\varepsilon, E'|_{T'_\varepsilon}).$
Let $C_b^\infty (M, \mathcal{F})$ denote the space of smooth basic functions, and let $C^\infty (W)^O(q)$ denote the space of smooth $O(q)$-invariant functions. The following proposition is similar to \cite[Propositions 3.3 and 3.4]{elliptic}, which considers the special case of differential forms. As above, let $\text{Diff}_G \left( T_{\varepsilon}^\prime, E'|_{T_x^G} \right)$ be the set of $G$-equivariant differential operators on $\Gamma \left( T_{\varepsilon}^\prime, E'|_{T_x^G} \right)$. Two such operators are called equivalent if their restrictions to $G$-invariant sections are identical. Let $\text{Diff}_G^* \left( T_{\varepsilon}^\prime, E'|_{T_x^G} \right)$ be the set of restrictions of elements of $\text{Diff}_G \left( T_{\varepsilon}^\prime, E'|_{T_x^G} \right)$ to $G$-invariant sections; note that equivalent $G$-equivariant operators yield a single element of $\text{Diff}_G^* \left( T_{\varepsilon}^\prime, E'|_{T_x^G} \right)$.

**Proposition 3.4.** Let $T_{\varepsilon}^\prime$ be a tubular neighborhood of an orbit $O_x$ of a point $x$ in a Riemannian $G$-manifold $W$. Let $B_\varepsilon$ be the image of $\varepsilon$-ball in the normal space $(T_x O_x)^\perp$ under the exponential map at $x$. Then there is a canonical isomorphism of algebras

$$\text{Diff}_G^* \left( T_{\varepsilon}^\prime, E'|_{T_x^G} \right) \cong \text{Diff}_H^* \left( B_\varepsilon, E'|_{B_\varepsilon} \right),$$

where $H$ is the isotropy subgroup of $G$ at $x$. Further, the isomorphism puts transversally elliptic operators in $\text{Diff}_G^* \left( T_{\varepsilon}^\prime, E'|_{T_x^G} \right)$ in one-to-one correspondence with elliptic operators in $\text{Diff}_H^* \left( B_\varepsilon, E'|_{B_\varepsilon} \right)$. In addition, there is a Riemannian metric on $B_\varepsilon$ such that isomorphism preserves formal adjoints with respect to $L^2$ inner products. Finally, if $P^* \in \text{Diff}_G^* \left( T_{\varepsilon}^\prime, E'|_{T_x^G} \right)$ and $P'$ is the corresponding element of $\text{Diff}_H^* \left( B_\varepsilon, E'|_{B_\varepsilon} \right)$, and if $s \in \Gamma \left( T_{\varepsilon}^\prime, E'|_{T_x^G} \right)$, then

$$P(\Sigma)|_{B_\varepsilon} = P'(\Sigma)|_{B_\varepsilon}.$$

**Proof.** First, note that sections in $\Gamma \left( T_{\varepsilon}^\prime, E'|_{T_x^G} \right)^G$ are in one-to-one correspondence with sections of $\Gamma \left( B_\varepsilon, E'|_{B_\varepsilon} \right)^H$. To see this, given a section $s \in \Gamma \left( T_{\varepsilon}^\prime, E'|_{T_x^G} \right)^G$, the restriction $\Phi(s) = s|_{B_\varepsilon}$ is an element of $\Gamma \left( B_\varepsilon, E'|_{B_\varepsilon} \right)^H$. Note that $\Phi$ is a linear map that is one-to-one, because each section of $\Gamma \left( T_{\varepsilon}^\prime, E'|_{T_x^G} \right)^G$ is determined by its restriction to $B_\varepsilon$, because $B_\varepsilon$ intersects every orbit in $T_{\varepsilon}^\prime$. Next, if $u \in \Gamma \left( B_\varepsilon, E'|_{B_\varepsilon} \right)^H$, $u$ extends to $T_{\varepsilon}^\prime$ by $u'(gx) := g u(x)$ for $g \in G$ and $x \in B_\varepsilon$. Note that if $g_1 x = g_2 x$, then $g_2^{-1} g_1 x = x$, so that $g_2^{-1} g_1$ is an element of the isotropy subgroup at $x$. Thus $u'(g_2 x) = g_2 u(x) = g_2 u \left( g_2^{-1} g_1 x \right) = g_1 u(x) = u'(g_1 x)$, so that the extension is well-defined. Thus, $\Phi : \Gamma \left( T_{\varepsilon}^\prime, E'|_{T_x^G} \right)^G \to \Gamma \left( B_\varepsilon, E'|_{B_\varepsilon} \right)^H$ is an isomorphism.

Let $S$ be a small open ball complementary and transverse to the $H$-orbits through the identity in $G$. Let $\ell_x = \{ s x : s \in S \}$ for $x \in B_\varepsilon$. Note that $\ell_x \cap \ell_y = \emptyset$ if $x \neq y$ and $x, y \in B_\varepsilon$; if $s_1 x = s_2 y$ with $s_1, s_2 \in S$ implies $s_2^{-1} s_1 x = y$, so that $s_2^{-1} s_1 \in H$, $s_1 \in s_2 H$, so that $s_1 = s_2$ since $S$ is transverse to the $H$-orbits in $G$. The leaves $\{ \ell_x : x \in B_\varepsilon \}$ form a trivial Riemannian foliation $\mathcal{F}$ of a neighborhood $N$ of $B_\varepsilon$ in $T_{\varepsilon}^\prime$. Note that the restriction of $\Phi \left( T_{\varepsilon}^\prime, E'|_{T_x^G} \right)^G$ to $N$ is a subspace of the space of basic sections of $E'|_{N}$, which is a foliated vector bundle over $(N, \mathcal{F})$. Note that $\Sigma$ is a basic section if and only if $\nabla^E_X u = 0$ for all leafwise vector fields $X$. 


Given an operator \( P^* \in \text{Diff}_G^* \left( T^\epsilon, E'|T^\epsilon \right) \), it is represented by a \( G \)-equivariant differential operator \( P \) in \( \text{Diff}_G \left( T^\epsilon, E'|T^\epsilon \right) \). Then \( P|_{\Gamma(N,E')} \) is a differential operator that maps \( \Gamma_b(N,E') \), the space of basic sections \( E' \) over \( N \), to itself, because for \( g \in S \), and \( u \in \Gamma \left( T^\epsilon, E'|T^\epsilon \right) \), \( P \left( g \cdot u \right) = g \cdot P(u) \). We choose a framing of \( TN \) adapted to \( \mathcal{L} \): \( (e_1, \ldots, e_p, e_{p+1}, \ldots, e_n) \), where \( (e_1, \ldots, e_p) \) is tangent to \( \mathcal{L} \) and \( (e_{p+1}, \ldots, e_n) \) is perpendicular to \( \mathcal{L} \). Then we may write \( P|_{\Gamma_b(N,E')} \) as a polynomial over \( C^\infty(N) \) in the covariant derivatives \( \nabla_{e_j}^{E'} \), such that

\[
P = P_1 + \sum_{j=1}^{p} P_{2j} \nabla_{e_j}^{E'},
\]

where \( P_1 \) is a differential operator whose only derivatives are of the form \( \nabla_{e_j}^{E'} \) with \( j > p \).

Here we have used the fact that operators of the form \( \nabla_a^{E'} \nabla_b^{E'} - \nabla_b^{E'} \nabla_a^{E'} \) are first-order differential operators for \( a \) and \( b \) coordinate vector fields. Thus the restriction satisfies

\[
P|_{\Gamma_b(N,E')} = P_1|_{\Gamma_b(N,E')}.
\]

Thus, \( P \) restricted to basic sections may be expressed as a differential operator on the local quotient \( N/\mathcal{L} \), which is diffeomorphic to \( B_\epsilon \). Since \( P \) is \( H \)-equivariant, the corresponding operator on \( B_\epsilon \) must be \( H \)-equivariant. If we let \( P' \) be that operator, the proof is complete.

\[\square\]

**Remark 3.5.** If \( E \) comes equipped with a grading, a similar grading is induced on \( E' \), and the constructions and isomorphisms above preserve those gradings.

4. Localization and Consequences

4.1. Localization to the model operator. In this section \( G = O(q) \). We start with an operator of the form \( D_b + sZ \in \text{Diff}_b^* (M, E, \mathcal{F}) \), where \( D_b \) is a basic Dirac operator and \( Z \) is an \( \mathcal{F} \)-basic bundle map. Let \( D' + sZ' \in \text{Diff}_{O(q)}^* (W, E') \) be the \( O(q) \)-equivariant operator on \( W \) corresponding to \( D_b + sZ \) via the isomorphism in Proposition 3.2. The critical leaf closures of \( Z \) on \( M \) correspond exactly to critical orbits of \( Z' \) on \( W \).

Let

\[
H_s = \frac{1}{s} \left( D' + sZ' \right)^2.
\]

We wish to study the asymptotics of the \( O(q) \)-invariant part of the spectrum of \( H_s \) in the semiclassical limit as \( s \to \infty \). The goal of this section is to construct the model operator, whose spectrum approximates the spectrum of \( H_s \), under the following conditions:

1. \( (D')^2 \) is a second order, transversally elliptic differential operator with positive definite principal transverse symbol.
2. \( (D'Z' + Z'D')|_{\Gamma(W,E')} \) is a bounded operator.
3. The bundle map \( (Z')^2 \) satisfies \( (Z'(x))^2 \geq 0 \) for all \( x \in M \), and at each point \( \overline{x} \) where \( Z'(\overline{x}) \) is singular, we have \( Z'(\overline{x}) = 0 \), and there exists \( c > 0 \) such that

\[
(Z'(x))^2 \geq c \cdot d(O_x, \overline{x})^2 \mathbf{1}
\]

in a neighborhood of \( \overline{x} \), where \( d(O_x, \overline{x}) \) is the distance between the orbit \( O_x \) of \( x \) to \( \overline{x} \).
Remark 4.1. These conditions are satisfied for \( D' \) and \( Z' \) if and only if \( Z \) is a proper perturbation of \( D_b \), where in particular \( Z \circ \sigma_D(x, \xi) + \sigma_D(x, \xi) \circ Z = 0 \) on \( E_x \) for every \( x \in M \), and every covector \( \xi \in (N\mathcal{F})^* \). See Theorem 2.5.

The singular set of \( Z' \) is a union of isolated orbits; each such critical orbit corresponds to a critical leaf closure of \( Z \). We will choose an arbitrary point \( \overline{\sigma} \) on each critical orbit, labeled \( \mathcal{O}_{\overline{\sigma}} \). Then the model operator is a direct sum \( K^G = \bigoplus_{\text{critical}} c_{\overline{\sigma}} K_{\overline{\sigma}}^{H_{\overline{\sigma}}} \), where the operators \( K_{\overline{\sigma}}^{H_{\overline{\sigma}}} \) are constructed below. Because all operators are \( O(q) \)-equivariant, the spectrum of \( K_{\overline{\sigma}}^{H_{\overline{\sigma}}} \) is independent of the choice of \( \overline{\sigma} \) on a fixed orbit.

At each \( \overline{\sigma} \) where \( Z'(\overline{\sigma}) \) is singular, the corresponding operators \( (H_s)' \) from Proposition 3.4 are \( G_{\overline{\sigma}} \)-equivariant on the transverse balls \( B_{\overline{\sigma}, \overline{\sigma}} \), where \( G_{\overline{\sigma}} \) is the isotropy subgroup at \( \overline{\sigma} \). Note that the quadratic condition (3) on \( Z' \) is equivalent to the condition of nondegeneracy for a leaf closure \( \ell \) that is critical for \( Z \) in Definition 2.9.

Let \( H_s^G \) denote the restriction of \( H_s \) to \( \Gamma(W, E')^G \). It is known (see [3, p. 12-13]) that the operator \( H_s^G \) has discrete spectrum.

Near each critical orbit \( \mathcal{O}_{\overline{\sigma}} \) of \( (Z')^2 \), we choose coordinates \( x = (x_1, \ldots, x_m) \) for the transverse ball \( B_{\overline{\sigma}, \overline{\sigma}} \) such that \( \overline{\sigma} \) corresponds to the origin, \( N_{\overline{\sigma}} \mathcal{O}_{\overline{\sigma}} = \mathbb{R}^m \), and the volume form at the origin is \( dx_1 \wedge \ldots \wedge dx_m \). Let \( H_{\overline{\sigma}} \) be the isotropy subgroup at \( \overline{\sigma} \). We choose a trivialization of \( E' \) near \( \overline{\sigma} \). Then \( A = (D')^2 \), \( B = D'Z' + Z'D' \), and \( C = (Z')^2 \) become differential operators with matrix coefficients. We define the model operator \( K_{\overline{\sigma}}^{H_{\overline{\sigma}}} : \Gamma(\mathbb{R}^m, E_{\overline{\sigma}})^{H_{\overline{\sigma}}} \to \Gamma(\mathbb{R}^m, E_{\overline{\sigma}})^{H_{\overline{\sigma}}} \) by

\[
K_{\overline{\sigma}}^{H_{\overline{\sigma}}} = \tilde{A} + \tilde{B} + \tilde{C},
\]

where

\[
\tilde{A} = \text{the principal part of } A \text{ at } \overline{\sigma},
\]

\[
\tilde{B} = B|_{\Gamma(W, E)^G}(\overline{\sigma}),
\]

\[
\tilde{C} = \sum x_i x_j (\nabla_i \nabla_j C)|_{\overline{\sigma}} = \text{the quadratic part of } C \text{ at } \overline{\sigma},
\]

where \( \nabla \) is the induced connection on \( E' \otimes E'^* \). It is easy to check that \( \tilde{C} \) is independent of the coordinates and connection chosen. Let \( dg \) denote the differential of the action of \( g \in H_{\overline{\sigma}} \) at \( \overline{\sigma} \), so we write \( dg : \mathbb{R}^m \to \mathbb{R}^m \). Let the action of \( g \) on \( \mathbb{R}^m \times E_{\overline{\sigma}} \) be defined as

\[
(x, v_{\overline{\sigma}}) \mapsto g = (dg(x), g \cdot v_{\overline{\sigma}}).
\]

Lemma 4.2. The operator \( K_{\overline{\sigma}}^{H_{\overline{\sigma}}} \) is equivariant with respect to this \( H_{\overline{\sigma}} \)-action.

Proof. Since \( H_s \) is equivariant with respect to \( G \) for each \( s > 0 \), it is easy to show that each of the operators \( A, B, \) and \( C \) is \( G \)-equivariant. Then the principal symbol of \( A \) is \( G \)-equivariant, and in particular the principal symbol of \( A \) at \( \overline{\sigma} \) is \( H_{\overline{\sigma}} \)-invariant. Thus, \( A \) is \( H_{\overline{\sigma}} \)-invariant. Next, since \( B|_{\Gamma(W, E)^G} \) is equivariant, its restriction \( \tilde{B} \) to \( \overline{\sigma} \) is also. Finally, since \( C \) is \( G \)-equivariant and the connection is \( G \)-equivariant, it follows that \( \tilde{C} \) is \( H_{\overline{\sigma}} \)-equivariant. \( \square \)

Lemma 4.3. The operator \( K_{\overline{\sigma}}^{H_{\overline{\sigma}}} : \Gamma(\mathbb{R}^m, E_{\overline{\sigma}})^{H_{\overline{\sigma}}} \to \Gamma(\mathbb{R}^m, E_{\overline{\sigma}})^{H_{\overline{\sigma}}} \) has discrete spectrum.

Proof. Consider the extended operator \( K_{\overline{\sigma}}^{H_{\overline{\sigma}}} : \Gamma(\mathbb{R}^m, E_{\overline{\sigma}}) \to \Gamma(\mathbb{R}^m, E_{\overline{\sigma}}) \). This operator is elliptic and essentially self-adjoint, and the operator is bounded below by \( (C_1 + C_2 \cdot |x|^2) \mathbf{1} \), where \( C_1 \in \mathbb{R} \) and \( C_2 > 0 \). Since this bound goes to infinity as \( x \to \pm \infty \), the operator \( K_{\overline{\sigma}}^{H_{\overline{\sigma}}} - (C_1 - 1) \mathbf{1} \) has a compact resolvent. Thus, the restriction of \( K_{\overline{\sigma}}^{H_{\overline{\sigma}}} \) to \( \Gamma(\mathbb{R}^m, E_{\overline{\sigma}})^{H_{\overline{\sigma}}} \) also has a compact resolvent. \( \square \)
Remark 4.4. The dimension \( m \) above may depend on \( \varpi \), even though this is not obvious from the notation.

We define the model operator \( K^G \) by

\[
K^G = \bigoplus_{\text{critical } \varpi} K^{H^\varpi}_{\varpi}.
\]

Clearly, this operator has discrete spectrum.

Let \( \lambda_1^G(s) \leq \lambda_2^G(s) \leq \cdots \) be the eigenvalues of \( H^G_s \), repeated according to multiplicity, correspond to the orthonormal basis of eigensections \( \omega_1^G(s), \omega_2^G(s), \ldots \). Let \( \mu_1^G \leq \mu_2^G \leq \cdots \) be the eigenvalues of the model operator \( K^G \), repeated according to multiplicity, correspond to the \( L^2 \)-orthonormal basis of eigensections \( \phi_1^G, \phi_2^G, \ldots \). Then we have the following result.

**Theorem 4.5. (Equivariant Localization Theorem)** Assume that the singular set of \( Z \) is not empty. Then, for each fixed \( N > 0 \), there exists \( C > 0 \) and \( s_0 > 0 \) such that for any \( s > s_0 \) and any \( j \leq N \), \( |\lambda_j^G(s) - \mu_j^G| \leq C_s^{-1/5} \). If the singular set of \( Z \) is empty, then there is a \( c > 0 \) such that for \( s \) sufficiently large, \( \lambda_j^G(s) \geq cs \).

**Proof.** This proof is a generalization of Theorem 1.1 in [23] to the equivariant setting. We identify the parameter \( s \) in our theorem with \( \frac{1}{\varpi} \) in [23].

To obtain an upper bound for the eigenvalues of \( H^G_s \) (or a lower bound on the spectral counting function of \( H^G_s \)), we use eigensections of the model operator \( K^G \) to produce test sections for \( H^G_s \) in the Rayleigh quotient. Suppose that \( \psi \) is an eigensection of \( K^{H^\varpi}_\varpi : \Gamma (\mathbb{R}^m, E^\varpi_\varpi) \to \Gamma (\mathbb{R}^m, E^\varpi_\varpi) \) corresponding to the eigenvalue \( \lambda \). Let \( J \in C_0^\infty (\mathbb{R}^m) \) be a radial function defined such that \( 0 \leq J \leq 1 \), \( J(x) = 1 \) if \( |x| \leq 1 \), \( J(x) = 0 \) if \( |x| \geq 2 \). For any \( s > 0 \), let \( J^{(s)}(x) = J(s^{2/\varpi} x) \). Then the section

\[
\phi(x) = J^{(s)}(x) s^{n/2} \psi(s^{1/2} x)
\]

is in \( \Gamma (\mathbb{R}^m, E^\varpi_\varpi) \) as well, because \( J^{(s)} \) is \( G \)-invariant. We produce a corresponding element \( \tilde{\phi} \in \Gamma (B_{\varepsilon, \varpi}, E')^H_{\varpi} \) that has support in a small neighborhood \( B_{\varepsilon, \varpi} \) of \( \varpi \), as follows. Let \( \gamma \) be the unit speed geodesic from \( \varpi \) to \( p \in B_{\varepsilon, \varpi} \), let \( x_p \) be the geodesic normal coordinates of \( p \), and let \( P_\gamma : E^\varpi_\varpi \to E^\varpi_p \) denote parallel translation along \( \gamma \). We define

\[
\tilde{\phi}(p) = P_\gamma \phi(x_p).
\]

Clearly, \( \tilde{\phi} \in \Gamma (B_{\varepsilon, \varpi}, E') \). Because the connection on \( E' \) is \( G \)-equivariant, parallel translation commutes with the action of \( H_{\varpi} \), and \( \tilde{\phi} \in \Gamma (B_{\varepsilon, \varpi}, E')^H_{\varpi} \). Abusing notation, we also denote by \( \tilde{\phi} \) the corresponding \( G \)-invariant section of \( T^G \) (and thus of \( W \)) using the isomorphism in Proposition 3.4. This specific trivialization of \( E' \) produces test sections that can be used as in [23] to obtain the upper bounds for the eigenvalues of \( H^G_s \). We denote \( \Phi : \Gamma (\mathbb{R}^m, E^\varpi_\varpi) \to \Gamma (W, E')^G \) to be the trivialization \( \phi \to \tilde{\phi} \). We may extend \( \Phi \) to act on the direct sum of the spaces \( \Gamma (\mathbb{R}^m, E^\varpi_\varpi) \) (with fixed \( \varpi \) in each critical orbit \( O_{\varpi} \)).

To obtain a lower bound on the eigenvalues of \( H^G_s \) (or an upper bound on the spectral counting function of \( H^G_s \)), we proceed exactly as in [23]. The functions in the partition of unity are chosen so that those corresponding to neighborhoods of critical points are radial; then the partition of unity will consist of invariant functions. Next, the IMS localization formula allows us to localize to these small neighborhoods, comparing the operators \( \Phi^{-1} H^G \Phi \) and \( K^G \).

\( \Box \)
Corollary 4.6. If $D_b \in \text{Diff}^*_b (M, E, F)$ is a basic Dirac operator, and let $K^G$ be the model operator constructed above, when the singular set of $Z$ is nonempty. Then

$$\text{ind}_b (D_b) = \dim \ker \left( (K^G)^+ \right) - \dim \ker \left( (K^G)^- \right).$$

When the singular set of $Z$ is empty, $\text{ind}_b (D_b) = 0$.

Proof. For each $s > 0$, operators $H^+_s = s^{-1} (D'_s)^- (D'_s)^+$ and $H^-_s = s^{-1} (D'_s)^+ (D'_s)^-$ are positive transversally elliptic self-adjoint operators acting on sections of vector bundles over the compact smooth manifold $W$. Therefore the operators $H^+_s$ and $H^-_s$ have discrete spectra $\sigma (H^+_s) \subset [0, +\infty)$ with finite multiplicities. By Lemma 4.3 and Theorem 4.5, the spectra of $(K^G)^+$ and $(K^G)^-$ are also discrete and nonnegative.

Choose any real number $r > 0$, so that $r$ is strictly less than the least positive number in the union of the spectra of $(K^G)^+$ and $(K^G)^-$. Then for any $s > 0$ we have

$$\text{ind}_b (D_b) = \dim \ker \left( s^{-1} (D_s)^2 \right)_{\Gamma_b(M,E^+,F)} - \dim \ker \left( s^{-1} (D_s)^2 \right)_{\Gamma_b(M,E^-,F)} + \dim \ker H^+_s - \dim \ker H^-_s = \# \{ \sigma (H^+_s) \cap [0, r) \} - \# \{ \sigma (H^-_s) \cap [0, r) \} + \dim \ker H^+_s - \dim \ker H^-_s = \dim \ker \left( (K^G)^+ \right) - \dim \ker \left( (K^G)^- \right).$$

because $D^+_s$ is an isomorphism between the eigenspaces of $H^+_s$ and of $H^-_s$ corresponding to nonzero eigenvalues. By choosing $s$ sufficiently large in the formula above and applying Theorem 4.5, we obtain

$$\text{ind} (D_b) = \dim \ker \left( (K^G)^+ \right) - \dim \ker \left( (K^G)^- \right).$$

□

Remark 4.7. With the notation of Section 2.1, if $Z := (Z^+, (Z^+)*) \in \Gamma (M, \text{End} (E^+ \oplus E^-))$ is a smooth basic bundle map that has no critical leaf closures and anticommutes with Clifford multiplication by vectors orthogonal to leaf closures, then the corollary implies that the index of the basic Dirac operator $D_b$ must be zero. This is clear for several reasons, for instance

$$\ker \left( (D_s)^2 \right)_{\Gamma_b(M,E^{\pm},F)} = \ker \left( (D_b^2 + s (ZD_b + D_bZ) + s^2 Z^2) \right)_{\Gamma_b(M,E^\pm\pm,F)}.$$

There exists $c > 0$ such that for sufficiently large $s$, $s^2 Z^2 + s (ZD_b + D_bZ) > cs^2 \mathbf{1}$, and the kernel is empty.

Corollary 4.8. The index of a basic Dirac operator corresponding to a basic Clifford bundle over a Riemannian foliation of odd codimension is zero.

Proof. Proposition 2.15 implies that there exists a basic perturbation $Z$ that is everywhere invertible. □

4.2. Clifford form of proper perturbations. In this section, we show that in a neighborhood of a critical leaf closure, the operator $Z$ may be continuously deformed so that it has a special form, called Clifford form, near this leaf closure.

Definition 4.9. Suppose that $Z$ is a proper perturbation of $D_b$. Then $Z$ is said to be of Clifford form if near every critical leaf closure $\ell$, it has the form $\gamma \otimes Z^\dagger$ (or in odd
codimension $1 \otimes \begin{pmatrix} 0 & Z^\dagger \\ -Z^\dagger & 0 \end{pmatrix}$, and in coordinates $x_k$ in a disk normal to $\ell$, 

$$Z^\dagger = \sum x_k Z_k^\dagger,$$

where

$$Z_j^\dagger Z_k^\dagger + Z_k^\dagger Z_j^\dagger = -2 \langle \partial_j, \partial_k \rangle$$

at the origin $\pi \in \ell$.

**Remark 4.10.** If $Z$ is of Clifford form, then near $\ell$ each $Z_j$ anticommutes with each $c(\partial_k)$, and the operators $L_j = c(\partial_j) Z_j$ commute with each other at the origin.

In the following, we use the terminology **stable homotopy** of endomorphisms and zeroth order operators in $K$-theory sense.

**Proposition 4.11.** Let $Z$ be a proper perturbation of $D_b$. Then there exists stable homotopy $\Phi_t$ of proper perturbations of $D_b$ such that

1. $\Phi_0 = Z$,
2. $\Phi_t = Z$ outside a small neighborhood of the critical leaf closure,
3. $\Phi_1$ has the same singular set as $Z$, and
4. $\Phi_1$ is in Clifford form.

**Proof.** We show that this homotopy can simultaneously be performed in the neighborhood of each critical leaf closure $\ell$ of $Z$. Given a small tubular neighborhood of $\ell$, let $U$ be the open neighborhood of the origin in $\mathbb{R}^m$ identified with a ball orthogonal to $\ell$ at a point. We are given an isometric action of the compact Lie group $H$ on $U$ and on $F = E|_U$, corresponding to the holonomy of the given leaf. We have $F \cong S \otimes W$, where the Clifford action of $\Cl(\mathbb{R}^m)$ on the bundle is of the form $c \otimes 1$, and where $S$ is the irreducible spinor space for $\Cl(\mathbb{R}^m)$. Since $W = \Hom_{\Cl(\mathbb{R}^m)}(S, F)$, the actions of $H$ on $S$ and on $F$ induce the action on $W$. The group $H$ commutes with the action of the chirality operator $\gamma \otimes 1$ (for $\Cl(\mathbb{R}^m)$) and with the equivariant bundle map $Z$. As in [18] Propositions 2.7 and 2.10, $Z$ has the form $\gamma \otimes Z^\dagger$ or $1 \otimes \begin{pmatrix} 0 & Z^\dagger \\ -Z^\dagger & 0 \end{pmatrix}$. Then $Z^\dagger$ is equivariant with respect to the action on $W$ (or $W^\dagger$ in the odd case with $W = W^\dagger \oplus W^\dagger$). Since $H$ acts on $U$ by isometries, then if $U$ is even-dimensional, the equivariant $K$-theory satisfies

$$K_H(pt) \cong K_H(U) \cong R(H),$$

the representation ring of $H$, with generators given by $[S^+ \otimes V_\rho, S^- \otimes V_\rho, c \otimes \rho] \in K_H(\mathbb{C}^{n/2})$, where $c(x)$ is Clifford multiplication by $x \in \mathbb{C}^{n/2}$ and $\rho : H \rightarrow U(V_\rho)$ is an irreducible unitary representation. In more generality (even in odd dimensions), $K_H(\mathbb{R}^{n})$ is generated by such triples; see [8] for the specific details. Therefore, we may stably homotope $Z^\dagger$ in a neighborhood of the origin to a Clifford multiplication-type operator, so that $Z^\dagger$ satisfies the conditions in the definition above.

The homotopy is performed as follows for the even-dimensional case, and the odd-dimensional case is similar. We consider on a single even-dimensional disk normal $U$ to the critical leaf closure. Restrict $Z^\dagger$ to a sphere of radius $r$ in the normal disk, where $Z^\dagger$ is nonsingular. It defines an element of $K_H(\mathbb{C}^{n/2})$, and thus is in the same class as $[S^+ \otimes V_\rho, S^- \otimes V_\rho, c \otimes \rho]$ as above. After stabilizing, there exists an equivariant homotopy between the two endomorphisms. Let $\tilde{Z}^\dagger(\tau)$ satisfy $\tilde{Z}^\dagger(r) = Z^\dagger$, $\tilde{Z}^\dagger(\tau) = c \otimes \rho$ for $\tau \leq \frac{1}{2r}$, and
otherwise \( \widetilde{Z}^\dagger (\tau) \) is the \( H \)-equivariant homotopy between the two operators for \( r \geq \tau \geq \frac{1}{2} r \).

Then the homotopy between \( Z \) and \( \gamma \otimes \widetilde{Z}^\dagger \) is \( \Phi_t = (1 - t) Z + t (\gamma \otimes \widetilde{Z}^\dagger) \) near the critical leaf closure and is constant outside the tubular neighborhood of radius \( r \). This homotopy introduces no additional critical leaf closures by construction. \( \square \)

4.3. Local calculations and the main theorem. In this section we use Corollary 4.6 to prove the main theorem, Theorem 4.12. Given any proper perturbation \( Z \) of \( D_b \), we stably deform it to a proper perturbation in Clifford form using Proposition 4.11. We consider \( H_s = \frac{1}{s} (D_b^2 + sZ')^2 \in \text{Diff}_{\text{O}(q)} (W, E') \), with the local form of the model operator at \( \tau = 0 \) given by

\[
K_\tau = \left( \sum_j c (\partial_j) \partial_j + \sum_k x_k Z_k \right)^2 : \Gamma (\mathbb{R}^m, E'_\tau) \rightarrow \Gamma (\mathbb{R}^m, E'_\tau),
\]

with each \( Z_k \) a constant endomorphism that anticommutes with \( c (\partial_j) \) for each \( j \), and such that \( Z_j Z_k + Z_k Z_j = -2x_{jk} \), as described in the previous section.

As a consequence, the Hermitian operators \( L_j = c (\partial_j) Z_j \) commute with each other. Let \( K_{\tau}^{H_\tau} \) be the restriction of \( K_\tau \) to \( \Gamma (\mathbb{R}^m, E'_\tau)^{H_\tau} \). Then

\[
K_\tau = - \sum_{j=1}^m \partial_j^2 + \sum_{j=1}^m c (\partial_j) Z_j + \sum_{j=1}^m x_j^2 Z_j^2
= \sum_{j=1}^m (- \partial_j^2 + L_j + x_j^2 L_j^2).
\]

The operators \( L_j \) can be diagonalized simultaneously. Let \( v \) be a common eigenvector; let \( \lambda_j \) be the eigenvalue of the operator \( L_j \) corresponding to \( v \). Letting \( f \) be a scalar function of \( x \), we have

\[
K_\tau (fv) = \left( \sum_{j=1}^m \left( - \partial_j^2 + \lambda_j + \lambda_j^2 x_j^2 \right) f \right) v.
\]

The section \( fv \) is in the kernel of \( K_\tau \) if and only if each \( \lambda_j \) is negative, and, up to a constant, \( fv = \exp \left( \frac{1}{2} \sum_j \lambda_j x_j^2 \right) v \). The kernel \( K_{\tau}^{H_\tau} \) is the \( H_\tau \)-invariant subspace of \( \ker K_\tau \). The kernel of \( K_\tau \) is the intersection of the direct sum of eigenspaces \( E_{\lambda} (L_j) \) of \( L_j \) corresponding to negative eigenvalues. Note that \( L_j \) maps \( E^+ \) to itself (call the restriction \( L_j^+ \)), so that the dimension of \( \ker K_{\tau}^{H_\tau} \) is simply the dimension of \( \bigcap_j \left( \bigoplus_{\lambda < 0} E_{\lambda} (L_j^+) \right)^{H_\tau} \).

The calculation above and Corollary 4.6 imply the following theorem.

**Theorem 4.12.** Let \( D_b \in \text{Diff}_K^+ (M, E, F) \) be a basic Dirac operator. Suppose that there exists a proper perturbation (Definition 2.12) \( Z \) of \( D_b \). Then

\[
\text{ind}_b (D_b) = \sum_{\ell} \text{ind} (Z, \ell),
\]

where for each critical leaf closure \( \ell \),

\[
\text{ind} (Z, \ell) = \left( \dim \left( \bigcap_j \left( \bigoplus_{\lambda < 0} E_{\lambda} (L_j^+ (\tau)) \right)^{H_\tau} \right) \right) - \dim \left( \bigcap_j \left( \bigoplus_{\lambda < 0} E_{\lambda} (L_j^- (\tau)) \right)^{H_\tau} \right),
\]
where each critical leaf closures \( \ell \) of \( Z \) corresponds to a fixed point \( \overline{x} \) on the basic manifold \( W \).

5. Examples

The perturbations \( Z \) in this section are already in Clifford form at the critical leaf closures.

Example 5.1. Consider the one dimensional foliation obtained by suspending an irrational rotation on the standard unit sphere \( S^2 \). On \( S^2 \) we use the cylindrical coordinates \((\theta, z)\), related to the standard rectangular coordinates by \( x' = \sqrt{(1 - z^2)} \cos \theta, \ y' = \sqrt{(1 - z^2)} \sin \theta, z' = z, \ \theta \in \mathbb{R} \mod 2\pi, \ z \in [-1, 1] \). Let \( \alpha \) be a fixed irrational multiple of \( 2\pi \), and let the three–manifold \( M_\alpha = S^2 \times [0, 1] / \sim \), where \((\theta, z, 0) \sim (\theta + \alpha, z, 1) \). Endow \( M_\alpha \) with the product metric on \( T(\theta,z,t)M_\alpha \cong T(\theta,z)S^2 \times T_1\mathbb{R} \). Let the foliation \( F_\alpha \) be defined by the immersed submanifolds \( \{(\theta', z, \tau): \theta' = \theta + n\alpha, \ n \in \mathbb{Z}, \ \tau \in \mathbb{R} \mod 1\} \) through points \((z, \theta, t)\).

The leaf closures for \( |z| < 1 \) are two dimensional, and the closures corresponding to the poles \((z = \pm 1)\) are one dimensional. In the natural metric, the foliation is Riemannian. We wish to calculate the basic Euler characteristic of this foliation by using our theorem.

Let \( d_b : \Omega^* (M_\alpha, F_\alpha) \rightarrow \Omega^* (M_\alpha, F_\alpha) \) be the exterior derivative restricted to basic forms, and let \( \delta_b \) be the \( L^2 \)-adjoint of \( d_b \). The operator \( D_b = d_b + \delta_b \) acting on even degree basic forms is called the basic de Rham operator. The basic Laplacian is \( \Delta_b = (d_b + \delta_b)^2 \), and its kernel consists of basic harmonic forms. Since the Hodge theorem is valid for Riemannian foliations, the standard argument shows that the index of \( D_b \) is the basic Euler characteristic \( \chi (M_\alpha, F_\alpha) = \sum_{j \geq 0} (-1)^j \dim (H^j_b (M_\alpha, F_\alpha)) \). Here, \( H^j_b (M_\alpha, F_\alpha) = \ker d_b|_{\Omega^j_b} / \operatorname{Im} d_b|_{\Omega^{j-1}_b} \) is the basis cohomology group (see [10, 20]).

For any bundle-like metric, the basic de Rham operator is

\[
D_b = (d_b + \delta_b) = d_b + \delta_T + \kappa_b \wedge ,
\]

where \( \kappa_b \) is the mean curvature one-form and \( \alpha \wedge = (\alpha \wedge)^* \) for one-forms \( \alpha \). Let the perturbation be

\[
Z = dz \wedge + dz_\wedge .
\]

The reader may verify that \( Z \) anticommutes with the principal symbol of \( D_b \); as explained in Section 2.2, this implies that \( D_b Z + Z D_b \) is zeroth order. The critical leaf closures correspond to the poles \( z = \pm 1 \). In the coordinates \( x, y \) near the poles, \( Z = xZ_1 + yZ_2 \), where

\[
Z_1 (\pm 1) = \mp (dx \wedge + dx_\wedge), \ Z_2 (\pm 1) = \mp (dy \wedge + dy_\wedge).
\]

Then

\[
L_j = c (\partial_j) Z_j (\pm 1),
\]

\[
L_1 = \mp (dx \wedge - dx_\wedge) (dx \wedge + dx_\wedge) = \mp (dx \wedge dx_\wedge - dx_\wedge dx \wedge)
\]

\[
L_2 = \mp (dy \wedge - dy_\wedge) (dy \wedge + dy_\wedge) = \mp (dy \wedge dy_\wedge - dy_\wedge dy \wedge)
\]

At the north pole \( z = +1 \),

\[
E_{-1} (L_1) = \text{span} \left\{ dx, dx \wedge dy \right\}, \ E_{-1} (L_2) = \text{span} \left\{ dy, dx \wedge dy \right\}
\]

\[
E_{-1} (L_1) \cap E_{-1} (L_2) = \text{span} \left\{ dx \wedge dy \right\}.
\]

Similarly, at the south pole \((z = -1)\),

\[
E_{-1} (L_1) \cap E_{-1} (L_2) = \text{span} \left\{ 1 \right\}
\]
Here \( E^+ = \Lambda^\text{even} T^*_0 \mathbb{R}^2 \), \( E^- = \Lambda^\text{odd} T^*_0 \mathbb{R}^2 \), and the vector subspaces found are \( O(2) \)-invariant, so that

\[
\text{ind} (Z, z = \pm 1) = \left( \dim \left[ \bigcap_j \left( \bigoplus_{\lambda < 0} E_\lambda \left( L^+_j (\mathcal{X}) \right) \right)^{H_\pi} \right] \right) - \dim \left[ \bigcap_j \left( \bigoplus_{\lambda < 0} E_\lambda \left( L^-_j (\mathcal{X}) \right) \right)^{H_\pi} \right]
= 1 - 0 = 1.
\]

\( \chi (M_\alpha, \mathcal{F}_\alpha) = \text{ind} (D_b) = 1 + 1 = 2. \)

We now directly calculate the Euler characteristic of this foliation. Since the foliation is taut, the standard Poincare duality works \([14] [15]\), and \( H^1_b (M) \cong H^2 (M) \cong \mathbb{R} \). It suffices to check the dimension \( h^1 \) of the cohomology group \( H^1_b (M) \). Then the basic Euler characteristic is \( \chi (M_\alpha, \mathcal{F}_\alpha) = 1 - h^1 + 1 = 2 - h^1 \). It was shown in \([7]\) Example 10.4] that \( h^1 = 0 \), so indeed \( \chi (M_\alpha, \mathcal{F}_\alpha) = 2. \)

**Example 5.2.** We will compute the basic Euler characteristic of the Carrière example from \([7]\) in the 3-dimensional case. Let \( A \) be a matrix in \( \text{SL}_2 (\mathbb{Z}) \) with \( \text{tr} (A) > 2 \). We denote respectively by \( V_1 \) and \( V_2 \) the eigenvectors associated with the eigenvalues \( \lambda \) and \( \frac{1}{\lambda} \) of \( A \) with \( \lambda > 1 \) irrational. Let the hyperbolic torus \( \mathbb{T}^3_A \) be the quotient of \( \mathbb{T}^2 \times \mathbb{R} \) by the equivalence relation which identifies \( (m, t) \) to \( (A(m), t + 1) \). The flow generated by the vector field \( V_1 \) can be made into a Riemannian foliation with a bundle-like metric \( g \), as follows. Let \( (x, y, t) \) denote the local coordinates in \( V_1 \), \( V_2 \), and \( \mathbb{R} \) directions, respectively, and let

\[ g = \lambda^2 dx^2 + \lambda^{-2} dy^2 + dt^2. \]

Next, let

\[ Z = \cos (2\pi t) dt \land + \cos (2\pi t) dt. \land. \]

The operator \( Z \) is a proper perturbation, as in the previous example. The critical leaf closures are those points of \( \mathbb{T}^3_A \) corresponding to \( t = \frac{1}{3}, \frac{2}{3} \). The bundle map has the local form \( Z (t) = -2\pi \left( t - \frac{1}{3} \right) (dt \land + dt.\land) \) near \( t = \frac{1}{3} \) and \( Z (t) = 2\pi \left( t - \frac{2}{3} \right) (dt \land + dt.\land) \) near \( t = \frac{2}{3} \) on the one-dimensional orthogonal disk, and so that

\[
Z_1 = \begin{cases} 
-2\pi (dt \land + dt.\land) & \text{if } t = \frac{1}{3} \\
2\pi (dt \land + dt.\land) & \text{if } t = \frac{2}{3}
\end{cases}
\]

and thus

\[
L_1 = c (\partial_t) Z_1 \]
\[
= \begin{cases} 
-2\pi (dt \land - dt.\land) (dt \land + dt.\land) & \text{if } t = \frac{1}{3} \\
2\pi (dt \land - dt.\land) (dt \land + dt.\land) & \text{if } t = \frac{3}{4}
\end{cases}
\]
\[
= \begin{cases} 
-2\pi (dt \land dt.\land - dt.\land dt \land) & \text{if } t = \frac{1}{3} \\
2\pi (dt \land dt.\land - dt.\land dt \land) & \text{if } t = \frac{3}{4}
\end{cases}
\]

There is no holonomy, so in both cases \( H_\pi \) is trivial. We compute the eigenvalues of \( L_1 \) on \( E^+ = \text{span} \{ 1, dy , dt \} \) and on \( E^- = \text{span} \{ dy, dt \} \). (\( dt \land dt.\land - dt.\land dt \land \) has eigenvalues \( -1, 1 \) on \( E^+ \), \( -1, 1 \) on \( E^- \). Thus, in the index formula,

\[
\text{ind} \left( Z, t = \frac{1}{4} \text{ or } \frac{3}{4} \right) = \left( \dim [E_{-1} (L^+_1)] - \dim [E_{-1} (L^-_1)] \right)
= 1 - 1 = 0,
\]
An easy computation shows that invariant self-dual forms
so that
\[ \chi(T^3_A, \mathcal{F}) = \text{ind}(D_b) = 0 + 0 = 0. \]

It was shown in [6 Example 10.8]) that
\[ H^0_b(T^3_A, \mathcal{F}) \cong H^1_b(T^3_A, \mathcal{F}) \cong \mathbb{R}, H^2_b(T^3_A, \mathcal{F}) \cong 0, \]
which verifies our computation.

Example 5.3. (Localization when \(D_bZ + ZD_b\) is a first order operator.) We will examine a transverse signature operator for a one-dimensional foliation. The manifold is the suspension of a \(\mathbb{Z}\)-action on \(\mathbb{C}P^2\). Specifically, let \((\theta_0, \theta_1, \theta_2) \in T^3 = S^1 \times S^1 \times S^1\) be a generator of a discrete dense subgroup of \(T^3\), and let \(\phi([z_0, z_1, z_2]) = (\theta_0, \theta_1, \theta_2) \cdot [z_0, z_1, z_2] = [\exp (i\theta_0) z_0, \exp (i\theta_1) z_1, \exp (i\theta_2) z_2] \) for all \([z_0, z_1, z_2] \in \mathbb{C}P^2\). The suspension is \(M = (\mathbb{C}P^2 \times \mathbb{R}) / \sim\), where \((z, t) \sim (\phi(z), t + 1)\) for all \(z \in \mathbb{C}P^2\), \(t \in \mathbb{R}\). The t-parameter curves form a Riemannian foliation of \(M\) with a natural bundle-like metric. Next, let \(D_b\) be the transverse signature operator, the pullback of the signature operator on \(\mathbb{C}P^2\) via the local projections \(\pi_1: U \to \mathbb{C}P^2\) for \(U\) open in \(M\). We may identify the bundle \(Q\) with \(TCP^2\).

Then \(c : Q^* \to \text{End}(\Lambda^* Q^*)\) is the standard Clifford action by cotangent vectors, so that the basic Dirac operator on \(M\) is the pullback the operator \(D_b = d + d^*\) on \(T^3\)-invariant forms on \(\mathbb{C}P^2\). The grading on the bundle \(E = \Lambda^* Q^*\) is given by the standard signature involution operator on \(\mathbb{C}P^2\). Next, let \(Z = ic(V^*)\), where \(V\) is the infinitesimal generator of the \(S^1\) action \(t \mapsto [\exp (i\alpha_0) z_0, \exp (i\alpha_1) z_1, \exp (i\alpha_2) z_2] \) on \(\mathbb{C}P^2\); we assume that \(\alpha_0, \alpha_1, \alpha_2\) are linearly independent over \(\mathbb{Q}\). The reader may check that \(D_bZ + ZD_b\) is a first order differential operator but is bounded on the basic forms.

The critical leaf closures are the circles corresponding to the points \([1, 0, 0], [0, 1, 0], [0, 0, 1]\) of \(\mathbb{C}P^2\). In the coordinates \([1, z_1, z_2]\) near \(\overline{x} = [1, 0, 0]\), \(V = Z = ic(V^*) = x_1Z_1 + y_1Z_2 + x_2Z_3 + y_2Z_4\), where
\[
\begin{align*}
Z_1(\overline{x}) &= i (\alpha_1 - \alpha_0) (dy_1 - dy_1) = i (\alpha_1 - \alpha_0) c(dy_1) \\
Z_2(\overline{x}) &= -i (\alpha_1 - \alpha_0) c(dx_1) \\
Z_3(\overline{x}) &= i (\alpha_2 - \alpha_0) c(dy_2) \\
Z_4(\overline{x}) &= -i (\alpha_2 - \alpha_0) c(dx_2)
\end{align*}
\]

Then
\[
\begin{align*}
L_1 &= c(dx_1) Z_1(\overline{x}) \\
&= i (\alpha_1 - \alpha_0) c(dx_1) c(dy_1) = L_2 \\
L_3 &= L_4 = i (\alpha_2 - \alpha_0) c(dx_2) c(dy_2).
\end{align*}
\]

An easy computation shows that invariant self-dual forms \(E^+\) and invariant anti-self-dual forms \(E^-\) at \(\overline{x}\) are
\[
E^\pm = \text{span}\left\{ dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \mp 1, dx_1 \wedge dy_1 \pm dx_2 \wedge dy_2 \right\}
\]

We see that
\[
\begin{align*}
ic(dx_q) c(dy_q) (1 \pm idx_q \wedge dy_q) &= \pm (1 \pm idx_q \wedge dy_q) \\
L_1 (1 \pm idx_1 \wedge dy_1) &= L_2 (1 \pm idx_1 \wedge dy_1) = \pm (\alpha_1 - \alpha_0) (1 \pm idx_1 \wedge dy_1) \\
L_3 (1 \pm idx_2 \wedge dy_2) &= L_4 (1 \pm idx_2 \wedge dy_2) = \pm (\alpha_2 - \alpha_0) (1 \pm idx_2 \wedge dy_2)
\end{align*}
\]
Thus,
\[
\bigcap_j \left( \bigoplus_{\lambda < 0} E_\lambda (L_j (\mathfrak{T})) \right) = \text{span} \left\{ (1 - \text{sgn} (\alpha_1 - \alpha_0) i dx_1 \wedge dy_1) \wedge (1 - \text{sgn} (\alpha_2 - \alpha_0) i dx_2 \wedge dy_2) \right\}.
\]

If \(\text{sgn} (\alpha_1 - \alpha_0) = -\text{sgn} (\alpha_2 - \alpha_0)\), then
\[
\bigcap_j \left( \bigoplus_{\lambda < 0} E_\lambda^+ (L_j (\mathfrak{T})) \right) = \{0\},
\]
\[
\bigcap_j \left( \bigoplus_{\lambda < 0} E_\lambda^- (L_j (\mathfrak{T})) \right) = \text{span} \left\{ (1 - \text{sgn} (\alpha_1 - \alpha_0) i dx_1 \wedge dy_1) \wedge (1 - \text{sgn} (\alpha_2 - \alpha_0) i dx_2 \wedge dy_2) \right\}.
\]

If \(\text{sgn} (\alpha_1 - \alpha_0) = \text{sgn} (\alpha_2 - \alpha_0)\), then
\[
\bigcap_j \left( \bigoplus_{\lambda < 0} E_\lambda^+ (L_j (\mathfrak{T})) \right) = \text{span} \left\{ (1 - \text{sgn} (\alpha_1 - \alpha_0) i dx_1 \wedge dy_1) \wedge (1 - \text{sgn} (\alpha_2 - \alpha_0) i dx_2 \wedge dy_2) \right\},
\]
\[
\bigcap_j \left( \bigoplus_{\lambda < 0} E_\lambda^- (L_j (\mathfrak{T})) \right) = \{0\}.
\]

Therefore, by Theorem 4.12,
\[
\text{ind} (Z, [1, 0, 0]) = \begin{cases} 
1 & \text{if } \text{sgn} (\alpha_1 - \alpha_0) = \text{sgn} (\alpha_2 - \alpha_0) \\
-1 & \text{if } \text{sgn} (\alpha_1 - \alpha_0) = -\text{sgn} (\alpha_2 - \alpha_0)
\end{cases}
\]
\[
\text{ind} (Z, [0, 1, 0]) = \begin{cases} 
1 & \text{if } \text{sgn} (\alpha_0 - \alpha_1) = \text{sgn} (\alpha_2 - \alpha_1) \\
-1 & \text{if } \text{sgn} (\alpha_0 - \alpha_1) = -\text{sgn} (\alpha_2 - \alpha_1)
\end{cases}
\]
\[
\text{ind} (Z, [0, 0, 1]) = \begin{cases} 
1 & \text{if } \text{sgn} (\alpha_0 - \alpha_2) = \text{sgn} (\alpha_1 - \alpha_2) \\
-1 & \text{if } \text{sgn} (\alpha_0 - \alpha_2) = -\text{sgn} (\alpha_1 - \alpha_2)
\end{cases}
\]

Thus,
\[
\text{ind}_b (D_b) = \sum_{\ell} \text{ind} (Z, \ell) = 1 + 1 - 1 = 1
\]
in all cases. This calculation agrees with the fact that the signature of \(\mathbb{C}P^2\) is 1, and the space of degree-two harmonic forms has dimension one and is also invariant under the isometry group of \(\mathbb{C}P^2\).
References

[1] J. A. Álvarez-López, The basic component of the mean curvature of Riemannian foliations, Ann. Global Anal. Geom. 10 (1992), 179–194.
[2] J. A. Álvarez-López, Morse inequalities for pseudogroups of local isometries, J. Differential Geom. 37 (1993), no. 3, 603–638.
[3] M. F. Atiyah, Elliptic operators and compact groups, Lecture Notes in Math. 401, Springer-Verlag, Berlin, 1974.
[4] V. Belfi, E. Park, and K. Richardson, A Hopf index theorem for foliations, Differential Geom. Appl. 18 (2003), 319-341.
[5] J. Brüning, F. W. Kamber, and K. Richardson, The eta invariant and equivariant index of transversally elliptic operators, preprint arXiv:1005.3845v1 [math.DG].
[6] J. Brüning, F. W. Kamber, and K. Richardson, Index theory for basic Dirac operators on Riemannian foliations, Contemporary Mathematics 546 (2011), 39–81.
[7] Y. Carrière, Flots riemanniens, Astérisque 116 (1984), 31–52.
[8] S. Echterhoff and O. Pfante, Equivariant K-theory of finite dimensional real vector spaces, Münster J. Math. 2 (2009), 65–94.
[9] A. El Kacimi-Alaoui, Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications, Compositio Math. 73 (1990), 57–106.
[10] A. Gorokhovsky and J. Lott, The index of a transverse Dirac-type operator: the case of abelian Molino sheaf, preprint arXiv:1005.0161 [math.DG].
[11] G. Habib and K. Richardson, A brief note on the spectrum of the basic Dirac operator, Bull. Lond. Math. Soc. 41 (2009), no. 4, 683–690.
[12] G. Habib and K. Richardson, Modified differentials and basic cohomology for Riemannian foliations, J. Geom. Anal., published online, reference is DOI 10.1007/s12220-011-9289-6.
[13] F. W. Kamber and Ph. Tondeur, Foliated bundles and characteristic classes, Lecture Notes in Math. 493, Springer-Verlag, Berlin–New York 1975.
[14] F. W. Kamber and Ph. Tondeur, Foliations and metrics, in: Proc. of a Year in Differential Geometry, University of Maryland, Birkhäuser Progr. Math. 32 (1983), 103–152.
[15] F. W. Kamber and Ph. Tondeur, Duality theorems for foliations, Transversal structure of foliations (Toulouse, 1982), Astérisque 116 (1984), 108–116.
[16] P. Molino, Riemannian foliations, Progress in Mathematics 73, Birkhäuser, Boston 1988.
[17] E. Park and K. Richardson, The basic Laplacian of a Riemannian foliation, Amer. J. Math. 118 (1996), 1249–1275.
[18] I. Prokhorenkov and K. Richardson, Perturbations of Dirac operators, J. Geom. Phys. 57 (2006), no. 1, 297–321.
[19] I. Prokhorenkov and K. Richardson, Natural Equivariant Dirac Operators, Geom. Dedicata 151(2011), 411–429.
[20] B. Reinhart, Differential Geometry of Foliations, Ergebnisse der Mathematik und ihrer Grenzgebiete 99, Springer-Verlag, Berlin, 1983.
[21] K. Richardson, The transverse geometry of $G$-manifolds and Riemannian foliations, Ill. J. Math. 45 (2001), no. 2, 517–535.
[22] K. Richardson, Traces of heat operators on Riemannian foliations, Trans. Amer. Math. Soc. 362 (2010), no. 5, 2301-2337.
[23] M. A. Shubin, Semiclassical asymptotics on covering manifolds and Morse inequalities, Geom. Funct. Anal. 6 (1996), no. 2, 370–409.
[24] Ph. Tondeur, Geometry of foliations, Monographs in Mathematics 90, Birkhäuser Verlag, Basel 1997.
[25] E. Witten, Supersymmetry and Morse Theory, J. Differ. Geometry, 17, 661-692 (1982).

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