Asymptotic behaviour and the Nahm transform of doubly periodic instantons with square integrable curvature

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Abstract

We study the asymptotic behaviour of doubly periodic instantons with square-integrable curvature. Then, we establish the equivalence given by the Nahm transform between the doubly periodic instantons with square integrable curvature and the wild harmonic bundles on the dual torus.

1 Introduction

Let $T := \mathbb{C}/L$, where $L$ is a lattice of $\mathbb{C}$. The product $T \times \mathbb{C}$ is equipped with the standard metric $dz d\overline{z} + dw d\overline{w}$, where $(z, w)$ is the standard local coordinate of $T \times \mathbb{C}$. In this paper, we shall study any $L^2$-instanton $(E, \nabla, h)$ on $T \times \mathbb{C}$, i.e., the curvature $F(\nabla)$ satisfies the equation $\Lambda F(h) = 0$, and it is $L^2$.

There is another natural decay condition around $\infty$. That is the quadratic curvature decay, i.e., $|F(\nabla)| = O(|w|^{-2})$ with respect to $h$ and the Euclidean metric $dz d\overline{z} + dw d\overline{w}$. M. Jardim [27] studied the Nahm transform of some kind of harmonic bundles with tame singularity on the dual torus $T^\vee$ to produce instantons on $T \times \mathbb{C}$ satisfying the quadratic curvature decay. O. Biquard and Jardim [8] studied the asymptotic behaviour of such instantons with rank 2. Based on the results, the inverse transform was constructed in [28], i.e., the Nahm transform of such instantons on $T \times \mathbb{C}$ to produce some type of harmonic bundles with tame singularity on $T^\vee$.

It is our purpose in this paper to generalize their results. Namely, we will study the asymptotic behaviour of $L^2$-instantons, and establish the equivalence between the $L^2$-instantons on $T \times \mathbb{C}$ and harmonic bundles with wild singularity on $T^\vee$. We shall also introduce algebraic counterparts of the transforms. They are useful to describe the induced transformations of the singular data, for example.

1.1 Asymptotic behaviour of $L^2$-instanton

1.1.1 The dimensional reduction due to Hitchin

Briefly speaking, it is our goal in the study of the asymptotic behaviour of $L^2$-instantons, to show that they behave like wild harmonic bundles around $\infty$. For the explanation, let us recall the dimensional reduction due to N. Hitchin. Let $U$ be any open subset of $\mathbb{C}$. Let $(V, \bar{\nabla}_V)$ be a holomorphic vector bundle on $U$ with a Higgs field $\theta$. Let $h$ be a hermitian metric of $V$. We have the Chern connection $\nabla_{V,h} = \bar{\nabla}_V + \partial_{V,h}$. We have the adjoint $\theta^\dagger$ of $\theta$ with respect to $h$. The tuple $(V, \bar{\nabla}_V, h, \theta)$ is called a harmonic bundle, if the Hitchin equation $F(\nabla_{V,h}) + [\theta, \theta^\dagger] = 0$ is satisfied.

Let $p : T \times U \to U$ be the projection. We have the expression $\theta = f dw$ and $\theta^\dagger = f^\dagger d\overline{w}$, where $f$ is a holomorphic endomorphism of $V$, and $f^\dagger$ is the adjoint of $f$. We set $(E, h_E) := p^*(V, h)$. Let $\nabla_E$ be the unitary connection given by $\nabla_E = p^*(\nabla_{V,h}) + f d\overline{z} - f^\dagger dz$. Then, $(E, h_E, \nabla_E)$ is an instanton, if and only if $(V, \bar{\nabla}_V, h, \theta)$ is a harmonic bundle. Indeed, we have the equivalence between harmonic bundles on $U$, and $T$-equivariant instantons on $T \times U$, which is due to Hitchin.

1.1.2 Examples and remarks

We set $U := \{ w \in \mathbb{C} \mid |w| > R \}$. We shall make $R$ larger without mention. Let $a$ be any holomorphic function on $U$. We have the harmonic bundle $\mathcal{L}(a)$ obtained as the tuple of the trivial line bundle $\mathcal{O}_U e$, the trivial metric
Let \( (E, \nabla, h) \) be an \( L^2 \)-instanton on \( T \times U \). Let \( (E, \overline{\nabla}_E) \) denote the underlying holomorphic vector bundle on \( T \times U \). By using a theorem of Uhlenbeck, we obtain \( F(\nabla) = o(1) \). It implies that the restrictions \( (E, \overline{\nabla}_E)\) to \( T \times \{w\} \) are semistable of degree 0 if \( |w| \) is sufficiently large. Hence, the relative Fourier-Mukai transform of \( (E, \overline{\nabla}_E) \) gives an \( \mathcal{O}_{T \times U} \)-module whose support \( Sp(E) \) is relatively 0-dimensional over \( U \). The first important issue in the study is the following.

**Theorem 1.1 (Theorem 5.10)** \( Sp(E) \) is extended to a complex analytic subvariety \( \tilde{Sp}(E) \) in \( T \times U \).

We use an effective control of the spectrum of semistable bundles of degree 0, in terms of the eigenvalues of the monodromy transformations of unitary connections with the small curvature (Corollary 4.10). If we fix an embedding \( \text{Sym}^{\text{rank} E}(T^\vee) \subset \mathbb{P}^N \), the spectrum induces a holomorphic map from \( U \) to \( \mathbb{P}^N \), which we regard as a harmonic map. We will observe that the energy of the harmonic map is dominated by the curvature. Then, we obtain the desired extendability of the spectral curve from the regularity theorem of J. Sacks and K. Uhlenbeck [10] for harmonic maps with finite energy.

Let \( \pi : T \times U \to U \) denote the projection. We fix a lift of \( \tilde{Sp}(E) \) to \( \tilde{Sp}(E) \subset \mathbb{C} \times U \). Then, we obtain a holomorphic vector bundle \( V \) on \( U \) with an endomorphism \( g \), with a \( C^\infty \)-isomorphism \( \pi^* V \cong E \) such that (i) \( \pi^* \overline{\nabla}_E + g \overline{\nabla}_E = \overline{\nabla}_E \), (ii) \( Sp(g) = \tilde{Sp}(E) \). By the identification \( E = \pi^* V \), we obtain a \( T \)-action on \( E \).

We set \( \tilde{Sp}_{\infty}(E) := (\mathbb{C} \times \{\infty\}) \cap \tilde{Sp}(E) \). We have the decomposition \( (V, g) = \bigoplus_{\alpha \in \tilde{Sp}_{\infty}(E)} (V_{\alpha}, g_{\alpha}) \) such that the eigenvalues of \( g_{\alpha}(w) \) go to \( \alpha \) when \( w \to \infty \). We have the corresponding decomposition \( E = \bigoplus_{\alpha \in \tilde{Sp}_{\infty}(E)} E_{\alpha} \). This completes the proof.
The hermitian metric $h$ of $E$ is decomposed into the sum $h = \sum h_{\alpha,\beta}$, where $h_{\alpha,\beta}$ are the sesqui-linear pairings of $E_{\alpha}$ and $E_{\beta}$. By using the Fourier expansion, we decompose $h_{\alpha,\beta}$ into the $T$-invariant part and the complement. Let $h^*$ denote the $T$-invariant part of $\sum h_{\alpha,\alpha}$. We shall prove that the complement $h^\perp := h - h^*$ and its derivatives have exponential decay.

**Theorem 1.2 (Theorem 5.11)** For any polynomial $P(t_1, t_2, t_3, t_4)$ of non-commutative variables, there exists $C > 0$ such that

$$P(\nabla_z, \nabla_{\bar{z}}, \nabla_w, \nabla_{\bar{w}}) h^\perp = O(\exp(-C|w|)).$$

We have a hermitian metric $h_V$ of $V$ induced by $h^*$. As a result, $(\mathcal{O}_T, h_V, g dw)$ satisfies the Hitchin equation up to an exponentially small term (Proposition 5.13). Such a tuple $(\mathcal{O}_T, h_V, g dw)$ can be studied as in the case of wild harmonic bundles [40] with minor modifications. (See to obvious ambiguity.

**Theorem 1.3 (Theorem 5.14)** There exists $\rho > 0$ such that the following holds:

$$F(h) = O\left(\frac{dz \, d\bar{z}}{|w|^2 (-\log |w|)^2}\right) + O\left(\frac{dw \, d\bar{w}}{|w|^4 (1+\rho)}\right) + O\left(\frac{dz \, d\bar{w}}{|w|^4 (1+\rho)}\right)$$

In particular, $F(\nabla) = O(|w|^{-1-\rho})$ for some $\rho > 0$ with respect to $h$ and the Euclidean metric $dw \, d\bar{w} + dz \, d\bar{z}$.

The estimate [3] implies that $(E, \nabla_E, h)$ is acceptable, i.e., $F(\nabla)$ is bounded with respect to $h$ and the Poincaré like metric $|w|^{-2} (\log |w|)^{-2} dw \, d\bar{w} + dz \, d\bar{z}$ on $T \times U$. By applying a general result in [40], we obtain the following prolongation result.

**Corollary 1.4 (Corollary 5.16)** The holomorphic bundle $(E, \nabla_E)$ is naturally extended to a filtered bundle $\mathcal{P}_* E$ on $(T \times U, T \times \{\infty\})$.

Here, the filtered bundle $\mathcal{P}_* E$ on $(T \times U, T \times \{\infty\})$ is an increasing sequence $(\mathcal{P}_a E | a \in \mathbb{R})$ of locally free $\mathcal{O}_{T \times U}$-modules such that (i) $\mathcal{P}_a(E)|_{T \times U} = E$, (ii) $\mathcal{P}_a(E)/\mathcal{P}_{a+1}(E)$ are locally free $\mathcal{O}_{T \times \{\infty\}}$-modules, where $\mathcal{P}_a E = \sum_{b \leq a} \mathcal{P}_b E$, (iii) $\mathcal{P}_a(E) = \mathcal{P}_{a+\epsilon}(E)$ for some $\epsilon > 0$, (iv) $\mathcal{P}_{a+1}(E) = \mathcal{P}_a(E) \otimes \mathcal{O}_{T \times \{\infty\}}(T \times \{\infty\})$. The sheaves $\mathcal{P}_a E$ are obtained as the space of the holomorphic sections of $E$ whose norms with respect to $h$ have growth order $O(|w|^{a+\epsilon})$ for any $\epsilon > 0$.

The filtered bundle is useful in the study of the instanton. For example, it turns out that $\frac{1}{2\pi^2} \int_{T \times C} \text{Tr}(F(h)^2)$ is equal to $\int_{T \times C} c_2(\mathcal{P}_a E)$ for any $a \in \mathbb{R}$, where $c_2$ denotes the second Chern class. (Proposition 6.6) In particular, the number $\frac{1}{2\pi^2} \int_{T \times C} \text{Tr}(F(h)^2)$ is an integer. (See [50] for this kind of integrality in a more general situation, which was informed by the referee.)

We can also use this filtered bundle to characterize the metric, i.e., a uniqueness part of the so-called Kobayashi-Hitchin correspondence. The stability condition for this type of filtered bundles is defined, as in [8]. (See [2.4.4] Note that it is not a standard (slope-)stability condition for filtered bundles.)

**Proposition 1.5 (Proposition 6.4, Proposition 6.5)** The associated filtered bundle $\mathcal{P}_* E$ is poly-stable of degree 0. The metric $h$ is uniquely determined as a Hermitian-Einstein metric of $(E, \nabla_E)$ adapted to $\mathcal{P}_* E$, up to obvious ambiguity.

We observe that, we need only a weaker assumption on the curvature decay, if we assume the prolongation of the spectral curve.

**Theorem 1.6 (Theorem 5.17)** Suppose that $F(\nabla) \rightarrow 0$ when $|w| \rightarrow \infty$, and that the spectral curve $\mathcal{S}p(E)$ is extended to a complex subvariety of $T \times U$. Then, $(E, \nabla_h)$ is an $L^2$-instanton. In particular, we may apply the results on the asymptotic behaviour of $L^2$-instantons.

More precisely, we directly prove the claims of Theorem 1.2 and Theorem 1.3 under the assumption, without considering $L^2$-condition.
1.1.4 Some remarks

In [8], Jardim and Biquard showed that an instanton of rank 2 with quadratic decay is an exponentially small perturbation of a tuple \((V,\bar{\nabla}_V, gdw, h_V)\) which satisfies the Hitchin equation up to an exponentially small term. Our result could be regarded as a generalization of theirs. But, the methods are rather different. To obtain a decomposition into the \(T\)-invariant part and the complement, they started with the construction of a global frame satisfying some nice property, which is an analogue of the Coulomb gauge of Uhlenbeck. Their method seems to require a stronger decay condition than \(L^2\), for example the quadratic decay condition as they imposed. We use a more natural decomposition induced by a standard method of the Fourier-Mukai transform in complex geometry, which allows us to consider \(L^2\)-instantons, once we deal with the issue of the prolongation of the spectral curve. (See also [12] for the \(L^2\)-property and the quadratic decay property of doubly periodic instantons.)

As mentioned above, we shall establish that an \(L^2\)-instanton is an exponentially small perturbation of \((V,\bar{\nabla}_V, h_V, \theta_V)\) which satisfies the Hitchin equation up to exponentially small term. Interestingly to the author, we can obtain a more refined result. Namely, we can naturally construct a harmonic metric \(h'_V\) of \((V,\bar{\nabla}_V, g dw)\) defined on a neighbourhood of \(\infty\), from the \(L^2\)-instanton. It is an analogue of the reductions from wild harmonic bundles to tame harmonic bundles studied in [10]. We consider a kind of meromorphic prolongation of the holomorphic vector bundle on the twistor space associated to \(T \times \mathbb{C}\), then we encounter a kind of infinite dimensional Stokes phenomena. By taking the graduation with respect to the Stokes structure, we obtain a wild harmonic bundle. Relatedly, in this paper, we consider only the product holomorphic structure of \(T \times \mathbb{C}\). From the viewpoint of twistor theory, the holomorphic vector bundle with respect to the other holomorphic structures should also be studied. The prolongation of the twistor family of the holomorphic structure is related with the issue in the previous paragraph. The author hopes to return to this deeper aspect of the study elsewhere.

Although we do not use it explicitly, we prefer to regard an instanton on \(T \times U\) as an infinite dimensional harmonic bundle on \(U\), which is suggested by the reduction of Hitchin. This heuristic is useful in our study of the asymptotic behaviour of \(L^2\)-instantons. From the viewpoint, Theorem 1.2 and Theorem 1.6 can be naturally regarded as a variant of Simpson’s main estimate [49]. (See also [38] and [40].)

1.2 Nahm transform for wild harmonic bundles and \(L^2\)-instantons

1.2.1 Nahm transforms and algebraic Nahm transforms

As an application of the study of the asymptotic behaviour, we shall establish the equivalence between \(L^2\)-instantons on \(T \times \mathbb{C}\) and wild harmonic bundles on \(T \times \mathbb{C}\) given by the Nahm transforms, which is a differential geometric variant of the Fourier-Mukai transform. (See [4] and [30] for the long history of a various versions of the Nahm transforms.)

Once we understand the asymptotic behaviour of an \(L^2\)-instanton \((E, \nabla, h)\), we can prove the desired property of the associated cohomology groups and harmonic sections. Hence, the standard \(L^2\)-method allows us to construct the Nahm transform \(\text{Nahm}(E, \nabla, h)\) which is a wild harmonic bundle on \((T', S_{p, \infty}(E)))\). (See [40] and [41].) Conversely, we may construct the Nahm transform of any wild harmonic bundle \((\mathcal{E}, \bar{\nabla}_E, \theta, h_E)\) on \((T', D)\) to \(L^2\)-instantons \(\text{Nahm}(\mathcal{E}, \bar{\nabla}_E, \theta, h_E)\) on \(T \times \mathbb{C}\), by using the result on wild harmonic bundles on curves ([40], [44] and [57]), although we need some estimates to establish the \(L^2\)-property (see [47]).

To study their more detailed properties, we introduce the algebraic Nahm transforms for filtered Higgs bundles on \((T', D)\) and filtered bundles on \((T \times \mathbb{C}, T \times \{\infty\})\), which do not necessarily come from wild harmonic bundles or \(L^2\)-instantons. The constructions are based on the Higgs interpretation of the Nahm transforms. It could be regarded as a filtered version of the Fourier transform for Higgs bundles studied in [9], although we restrict ourselfs to the case that the base space is an elliptic curve.

As mentioned in [13,14] we obtain the filtered bundle \(\mathcal{P}_* E\) on \((T \times \mathbb{P}^1, T \times \{\infty\})\) associated to any \(L^2\)-instanton \((E, \nabla, h)\), and the metric \(h\) is determined by \(\mathcal{P}_* E\) essentially uniquely. We have the good filtered Higgs bundle \((\mathcal{P}_* \mathcal{E}, \theta)\) on \((T', D)\) associated to any wild harmonic bundle \((\mathcal{E}, \bar{\nabla}_E, \theta, h_E)\), and the metric \(h_E\) is determined by \((\mathcal{P}_* \mathcal{E}, \theta)\) essentially uniquely [7], [40]. So, it is significant to describe the induced transformation
between the underlying filtered bundles on $T \times \mathbb{P}^1$ and the underlying good filtered Higgs bundles on $(T^\vee, D)$, that is given by the algebraic Nahm transform. In particular, it is useful to understand how the singular data are transformed. It is also useful to prove that the Nahm transforms are mutually inverse.

### 1.2.2 Algebraic Nahm transform for filtered Higgs bundles

Let us briefly explain how the algebraic Nahm transform is constructed for filtered Higgs bundles $(\mathcal{P}, \mathcal{E}, \theta)$ on $(T^\vee, D)$. (The details will be given in §3 after the preliminary in §2.) We should impose several conditions to the filtered Higgs bundles.

**Goodness and admissibility** One of the conditions is the compatibility of the filtered bundle $\mathcal{P}, \mathcal{E}$ and the Higgs field $\theta$ at each $P \in D$. Suppose that the filtered Higgs bundle $(\mathcal{P}, \mathcal{E}, \theta)$ comes from a good wild harmonic bundle. Let $U_P$ be a small neighbourhood of $P$ with a coordinate $\zeta_P$ with $\zeta_P(P) = 0$. If we take a ramified covering $\varphi : U_P \to U_P$ given by $\varphi(u) = u^p$, for an appropriate $p$, then we have the following decomposition:

$$\varphi^*_{\mathcal{P}}(\mathcal{P}, \theta) = \bigoplus_{a \in u^{-1}[a^{-1}]} (\mathcal{P}, \mathcal{E}, \theta_{a})$$

Here, $\theta_{a} - da$ are logarithmic in the sense that $(\theta_{a} - a)\mathcal{P}, \mathcal{E} \subset \mathcal{P}, \mathcal{E} du/u$. Such a filtered Higgs bundle is called good. This kind of filtered Higgs bundles are closely related with wild harmonic bundles and $L^2$-instantons.

But, it seems natural to consider an algebraic Nahm transform for a wider class of filtered Higgs bundles. For $(p, m) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ with $\text{g.c.d.}(p, m) = 1$, we say that a filtered Higgs bundle has type $(p, m)$ at $P$, if we have that $u^m - \varphi^*_P \theta$ gives a morphism of filtered bundles $\varphi^*_P \mathcal{P}, \mathcal{E} \to \varphi^*_P \mathcal{P}, \mathcal{E} du/u$ on $U_P$, and that it is an isomorphism in the case $(p, m) \neq (1, 0)$. We say that $(\mathcal{P}, \mathcal{E}, \theta)$ is admissible at $P$, if it is a direct sum $\bigoplus (\mathcal{P}, \mathcal{E}^{(p, m)}_P, \theta^{(p, m)}_P)$ of the filtered bundles of type $(p, m)$, after $U_P$ is shrank appropriately. We say that its slope is smaller (resp. strictly smaller) than $\alpha$, if $\mathcal{E}^{(p, m)}_P = 0$ for $m/p > \alpha$ (resp. $m/p \geq \alpha$).

Each $(\mathcal{P}, \mathcal{E}^{(p, m)}_P, \theta^{(p, m)}_P)$ has a refined decomposition, as explained in §3.1. In particular, $(\mathcal{P}, \mathcal{E}^{(1, 0)}_P, \theta^{(1, 0)}_P)$ has a decomposition

$$(\mathcal{P}, \mathcal{E}^{(1, 0)}_P, \theta^{(1, 0)}_P) = \bigoplus_{\alpha \in \mathbb{C}} (\mathcal{P}, \mathcal{E}^{(1, 0)}_{P, \alpha}, \theta^{(1, 0)}_{P, \alpha}).$$

Here, for the expression $\theta^{(1, 0)}_{P, \alpha} = f^{(1, 0)}_P d\zeta_P/\zeta_P$, the eigenvalues of $f^{(1, 0)}_P$ goes to $\alpha$ when $\zeta_P \to 0$. On $U_P$, we set

$$C^0(\mathcal{P}, \mathcal{E}, \theta)_P = \bigoplus_{(p, m) \neq (1, 0)} \mathcal{P}_{-1/2-m/p} \mathcal{E}^{(p, m)}_P \bigoplus \mathcal{P}_{0} \mathcal{E}^{(1, 0)}_P \bigoplus \mathcal{P}_{0} \mathcal{E}^{(1, 0)}_{P, 0}$$

and

$$C^1(\mathcal{P}, \mathcal{E}, \theta)_P = \bigoplus_{(p, m) \neq (1, 0)} \mathcal{P}_{1/2-m/p} \mathcal{E}^{(p, m)}_P \bigoplus \mathcal{P}_{0} \mathcal{E}^{(1, 0)}_P \bigoplus \Omega_{T^\vee} \bigoplus \mathcal{P}_{1} \mathcal{E}^{(1, 0)}_P \bigoplus \mathcal{P}_{0} \mathcal{E}^{(1, 0)}_{P, 0}$$

The Higgs field $\theta$ gives a morphism $C^0(\mathcal{P}, \mathcal{E}, \theta)_P \to C^1(\mathcal{P}, \mathcal{E}, \theta)_P$. Thus, we obtain a complex $\mathcal{C}^\bullet(\mathcal{P}, \mathcal{E}, \theta)$ on $U_P$, which is an extension of $\mathcal{E} \to \mathcal{E} \otimes \Omega^1$ on $U_P \setminus P$.

We say that $(\mathcal{P}, \mathcal{E}, \theta)$ on $(T^\vee, D)$ is admissible, if its restriction to a neighbourhood of each $P \in D$ is admissible. By considering the extension at each $P \in D$, we obtain a complex $\mathcal{C}^\bullet(\mathcal{P}, \mathcal{E}, \theta)$ on $T^\vee$, as an extension of $\mathcal{E} \to \mathcal{E} \otimes \Omega^1$ on $T^\vee \setminus D$.

### Vanishing of some cohomology groups

For each $w \in \mathbb{C}$ and a holomorphic line bundle $L$ of degree 0 on $T^\vee$, we obtain a complex $\mathcal{C}^\bullet_{w, L}(\mathcal{P}, \mathcal{E}, \theta) := \mathcal{C}^\bullet(\mathcal{P}, \mathcal{E} \otimes L, \theta + w d\zeta)$. To consider algebraic Nahm transform $(\mathcal{P}, \mathcal{E}, \theta)$, it is natural to impose the following vanishing:

$(A0)$ $H^i(T^\vee, \mathcal{C}^\bullet_{w, L}) = 0$ unless $i = 1$, for any $w \in \mathbb{C}$ and a holomorphic line bundle $L$ of degree 0 on $T^\vee$.

For $I \subset \{1, 2, 3\}$, let $p_I$ denote the projection of $T^\vee \times T \times \mathbb{P}^1$ onto the product of the $i$-th components $(i \in I)$. Let $\text{Poin}$ denote the Poincaré bundle on $T^\vee \times T$. We consider the following complex on $T^\vee \times T \times \mathbb{P}^1$:

$$\mathcal{C} := p^*_1 \mathcal{C}^0 \otimes p^*_2 \text{Poin} \otimes p^*_3 O_{\mathbb{P}^1}(-1) \xrightarrow{\theta \cdot w \cdot d\zeta} \mathcal{C}^1 := p^*_1 \mathcal{C}^1 \otimes p^*_2 \text{Poin}$$

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Then, it turns out that $N(\mathcal{P}, \mathcal{E}, \theta) := R^1p_{23*}\hat{\mathcal{O}}$ is a locally free $\mathcal{O}_{T \times \mathbb{P}^1}$-module on $T \times \mathbb{P}^1$. In particular, we obtain a locally free $\mathcal{O}_{T \times \mathbb{P}^1}(\ast(T \times \{\infty\}))$-module

$$\text{Nahm}(\mathcal{P}, \mathcal{E}, \theta) := N(\mathcal{P}, \mathcal{E}, \theta) \otimes \mathcal{O}_{T \times \mathbb{P}^1}(\ast(T \times \{\infty\})).$$

**Filtered bundles on** $(T \times U, T \times \{\infty\})$ The algebraic Nahm transform of $(\mathcal{P}, \mathcal{E}, \theta)$ is defined to be a filtered bundle over the meromorphic bundle $\text{Nahm}(\mathcal{P}, \mathcal{E}, \theta)$, i.e., an increasing sequence of $\mathcal{O}_{T \times \mathbb{P}^1}$-submodules of $\text{Nahm}(\mathcal{P}, \mathcal{E}, \theta)$. For the construction of such a filtration, it would be convenient to have a description of filtered bundles $\mathcal{P}, \mathcal{E}$ on $(T \times U, T \times \{\infty\})$ satisfying the following condition, where $U$ denotes a neighbourhood of $\infty$:

(A1) $\text{Gr}_P^0(\mathcal{E})$ are semistable bundles of degree 0 on $T$ for any $c \in \mathbb{R}$.

If $U$ is shrank, $\mathcal{P}_c(\mathcal{E})|_{T \times \{w\}}$ are semistable of degree 0 for any $w \in U$ and for any $c \in \mathbb{R}$. We may assume it from the beginning. We set $\mathcal{S}_p(E) := \mathcal{S}(\mathcal{P}, \mathcal{E})|_{T \times \{\infty\}} \subset T^\vee$, which is independent of $c \in \mathbb{R}$. We fix a lift $\tilde{\mathcal{S}}_{p\infty}(E) \subset \mathbb{C}$ of $\mathcal{S}_p(E)$, i.e., $\tilde{\mathcal{S}}_{p\infty}(E)$ is mapped bijectively to $\mathcal{S}_p(E)$ by the projection $\mathbb{C} \rightarrow T^\vee$. Then, we have a filtered bundle $\mathcal{P}, \mathcal{V}$ on $(U, \infty)$ with an endomorphism $g$ such that $\mathcal{S}(g|_\infty) = \tilde{\mathcal{S}}_{p\infty}(E)$ corresponding to $\mathcal{P}, \mathcal{E}$. Namely, we have a $C^\infty$-isomorphism $\mathcal{P}, \mathcal{E} \simeq \pi^*\mathcal{P}, \mathcal{V}$, under which $\tilde{\mathcal{S}}_{\mathcal{P}, \mathcal{E}} = \pi^*(\tilde{\mathcal{S}}_{\mathcal{P}, \mathcal{V}}) + g \, \bar{\zeta}$, where $\pi : T \times U \rightarrow U$ denotes the projection. The filtered bundle with an endomorphism $(\mathcal{P}, \mathcal{V}, g)$, or equivalently, the filtered Higgs bundle $(\mathcal{P}, \mathcal{V}, g \, dw)$ completely determines $\mathcal{P}, \mathcal{E}$. We have the decomposition $(\mathcal{P}, \mathcal{V}, g) = \bigoplus_{a \in \tilde{\mathcal{S}}_{p\infty}(E)} (\mathcal{P}, \mathcal{V}, g_a)$ with $\mathcal{S}(g_a|_\infty) = \{a\}$. The filtered bundle $\mathcal{P}, \mathcal{E}$ satisfying (A1) is called admissible, if the following holds:

(A2) The filtered Higgs bundles $(\mathcal{P}, \mathcal{V}, (g_a - \alpha \, dw))$ are admissible for any $\alpha \in \tilde{\mathcal{S}}_{p\infty}(E)$.

The slope of $(\mathcal{P}, \mathcal{V}, (g_a - \alpha \, dw))$ is strictly smaller than 1 by construction.

**Local algebraic Nahm transform and algebraic Nahm transform** The local algebraic Nahm transform $\mathcal{N}^{0, \infty}$ is a transform from the germs of admissible filtered Higgs bundles to the germs of admissible filtered Higgs bundles whose slopes are strictly smaller than 1. It is an analogue of the local Fourier transform $\mathcal{F}^{0, \infty}$ for meromorphic flat bundles on $\mathbb{P}^1$ in [13] and [22]. (More precisely, it is an analogue of local Fourier transform of minimal extension of meromorphic flat bundles.) It gives a procedure to produce an admissible filtered bundle $\mathcal{P}, \mathcal{E}_P$ on $(T \times U, T \times \{\infty\})$ such that $\mathcal{S}(\mathcal{P}, \mathcal{E}_P) = \{P\}$, from an admissible filtered Higgs bundle $(\mathcal{P}, \mathcal{E}, \theta)|_{U_P}$ on $(U_P, P)$. From the local Nahm transform $\bigoplus_{P \in D} \mathcal{P}, \mathcal{E}_P$ and the meromorphic bundle $\text{Nahm}(\mathcal{P}, \mathcal{E}, \theta)$, we obtain a filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$, denoted by $\text{Nahm}(\mathcal{P}, \mathcal{E}, \theta)$, that is the algebraic Nahm transform for admissible filtered Higgs bundles.

1.2.3 **Algebraic Nahm transform for admissible filtered bundles**

Let $\mathcal{P}, \mathcal{E}$ be an admissible filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$. To define the algebraic Nahm transform of $\mathcal{P}, \mathcal{E}$, we impose the following vanishing condition:

(A3) $H^0(T \times \mathbb{P}^1, \mathcal{P}, \mathcal{E} \otimes L) = 0$ and $H^2(T \times \mathbb{P}^1, \mathcal{P}, \mathcal{E} \otimes L) = 0$ for any holomorphic line bundle $L$ of degree 0 on $T$.

We set $D := \tilde{\mathcal{S}}_{p\infty}(E)$. It is easy to observe that the condition (A3) implies that $H^i(T \times \mathbb{P}^1, \mathcal{P}, \mathcal{E} \otimes L^\vee) = 0$ ($i \neq 1$) for any $c \in \mathbb{R}$, unless $L \in D$. For any $I \subset \{1, 2, 3\}$, let $p_I$ denote the projection of $T^\vee \times T \times \mathbb{P}^1$ onto the product of the $i$-th components ($i \in I$). We define

$$\text{Nahm}(\mathcal{P}, \mathcal{E}) := R^1p_{1*}(p_{12}^*\text{Poin}^\vee \otimes p_{23}^*\mathcal{P}, \mathcal{E}) \ast D \simeq R^1p_{1*}(p_{12}^*\text{Poin}^\vee \otimes p_{23}^*\mathcal{P}, \mathcal{E}) \ast D.$$
Let \((P_\ast V_P, g_P - \tilde{P}) = \bigoplus (P_\ast V_P^{(p,m)}, g_P^{(p,m)})\), where \((P_\ast V_P^{(p,m)}, g_P^{(p,m)} \, dw)\) has slope \((p, m)\) with \(m/p < 1\), and \(\tilde{P} \in \tilde{D}\) is a lift of \(P\). Moreover, we have the decomposition

\[
(P_\ast V_P^{(1,0)}, g_P^{(1,0)}) = \bigoplus_{\alpha \in \mathbb{C}} (P_\ast V_P^{(1,0)}|_{P,\alpha}, g_P^{(1,0)}|_{P,\alpha})
\]

It turns out that we have a decomposition of the meromorphic bundle

\[
\text{Nahm}(P_\ast E)|_{U_P} = \text{Nahm}(P_\ast E)^{(1,0)}_{P,0} \oplus \bigoplus_{\alpha \neq 0} \text{Nahm}(P_\ast E)^{(1,0)}_{P,\alpha} \oplus \bigoplus_{(p, m) \neq (1, 0)} \text{Nahm}(P_\ast E)^{(m,p)}_{P},
\]

and \(\text{Nahm}(P_\ast E)^{(1,0)}_{P,\alpha} (\alpha \neq 0)\) and \(\text{Nahm}(P_\ast E)^{(p,m)}_{P} ((p, m) \neq (1, 0))\) are determined by \((P_\ast V_P^{(1,0)}, g_P^{(1,0)})\) and \((P_\ast V_P^{(p,m)}, g_P^{(p,m)})\).

We have the local algebraic Nahm transform \(N^{\infty,0}\), which is a transform of admissible Higgs bundles \((P, V, \theta)\) such that (i) the slopes are strictly smaller than 0, (ii) \(P_\ast V_0^{(1,0)} = 0\). It is an analogue of the local Fourier transform \(\mathfrak{F}^{\infty,0}\) in [15] and [22]. It is an inverse of \(N^{0,\infty}\) except for the part \((p, m) = (1, 0)\) and \(\alpha = 0\). We may introduce filtrations of \(\text{Nahm}(P_\ast E)^{(1,0)}_{P,\alpha} (\alpha \neq 0)\) and \(\text{Nahm}(P_\ast E)^{(p,m)}_{P} ((p, m) \neq (1, 0))\) by using the local algebraic Nahm transform \(N^{\infty,0}\). As for the part \((p, m) = (1, 0)\) and \(\alpha = 0\), we have an injection \(\mathcal{P}_0 V_0^{(1,0)} \subset \mathcal{P}_{1}\mathcal{P}_{0} \mathcal{P}^{\mathbb{C}} \mathcal{P}_{23}^{1} \mathcal{P}_{1} E^{(1,0)}_{P,0,P}\), by which we can introduce a filtration on \(\text{Nahm}(P_\ast E)^{(1,0)}_{P}\). Thus, we obtain a filtered Higgs bundle denoted by \(\text{Nahm}_{\ast}(P_\ast E)\). We obtain the following correspondence.

**Theorem 1.7 (Proposition 3.13, Proposition 3.22, Proposition 3.25)**
The Nahm transforms \(\text{Nahm}_{\ast}\) give an equivalence of the following objects, and they are mutually inverse:

- Admissible filtered Higgs bundles on \((T^\vee, D)\) satisfying the condition \((A0)\).
- Admissible filtered bundles \(P_\ast E\) on \((T \times \mathbb{P}^1, T \times \{\infty\})\) with \(S_{p,\infty}(E) = D\) satisfying the condition \((A3)\).

We can also that the Nahm transforms preserve the parabolic degrees (Proposition 3.17).

As already mentioned, the filtered Higgs bundles associated to wild harmonic bundles satisfy a stronger condition called goodness. Similarly, it turns out that the filtered bundles associated to \(L^2\)-instantons are also good, in the sense that the corresponding filtered Higgs bundles are good. We can observe that the algebraic Nahm transforms preserve the goodness conditions.

**Theorem 1.8 (Theorem 3.27)**
The Nahm transforms \(\text{Nahm}_{\ast}\) gives an equivalence of the following objects:

- Good filtered Higgs bundles on \((T^\vee, D)\) satisfying the condition \((A0)\).
- Good filtered bundles \(P_\ast E\) on \((T \times \mathbb{P}^1, T \times \{\infty\})\) with \(S_{p,\infty}(E) = D\) satisfying the condition \((A3)\).

### 1.2.4 Application of the algebraic Nahm transform

We have the following compatibility of the Nahm transform and the algebraic Nahm transform.

**Theorem 1.9 (Theorem 7.12, Theorem 7.13)**

- Let \((E, \nabla, h)\) be an \(L^2\)-instanton on \(T \times \mathbb{C}\). Let \(P_\ast E\) be the associated filtered bundle on \((T \times \mathbb{P}^1, T \times \{\infty\})\). Then, the underlying filtered Higgs bundle of the wild harmonic bundle \(\text{Nahm}(E, \nabla, h)\) on \((T^\vee, S_{p,\infty}(E))\) is naturally isomorphic to the algebraic Nahm transform \(\text{Nahm}_{\ast}(P_\ast E)\).
- Let \((\mathcal{E}, \mathcal{D}_E, \theta, h_E)\) be a wild harmonic bundle on \((T^\vee, D)\). Let \((P_\ast \mathcal{E}, \theta)\) be the associated good filtered Higgs bundle on \((T^\vee, D)\). Then, the underlying filtered bundle of the \(L^2\)-instanton \(\text{Nahm}(\mathcal{E}, \mathcal{D}_E, h_E, \theta)\) is naturally isomorphic to the algebraic Nahm transform \(\text{Nahm}_{\ast}(P_\ast \mathcal{E}, \theta)\).

As an application, we obtain the inversion property of the Nahm transforms.
Corollary 1.10 (Corollary [7,14]) For an $L^2$-instanton $(E, \nabla, h)$ on $T \times \mathbb{C}$, we have an isomorphism
\[
\text{Nahm}(\text{Nahm}(E, \nabla, h)) \simeq (E, \nabla, h).
\]

For a wild harmonic bundle $(\mathcal{E}, \overline{\partial}_{\mathcal{E}}, \theta, h_{\mathcal{E}})$ on $(T^\vee, D)$, we have an isomorphism
\[
\text{Nahm}(\text{Nahm}(\mathcal{E}, \overline{\partial}_{\mathcal{E}}, \theta, h_{\mathcal{E}})) \simeq (\mathcal{E}, \overline{\partial}_{\mathcal{E}}, \theta, h_{\mathcal{E}})
\]
Indeed, it follows from Theorem 1.9 and the uniqueness of the Hermitian-Einstein metric (resp. the harmonic metric) adapted to the filtered bundle (resp. filtered Higgs bundle).

As another application of the compatibility, we can easily compute the characteristic classes of the bundles obtained as the algebraic Nahm transform, which allows us to obtain the rank and the second Chern class of the bundle obtained as the Nahm transform. The local algebraic Nahm transform also leads us a rather complete understanding of the transformation of singularity data by the Nahm transform.

1.2.5 Some remarks
Recall that the hyperkahler manifold $T \times \mathbb{C}$ has a twistor deformation. Namely, for any complex number $\lambda$, we have the moduli space $\mathcal{M}_\lambda$ of the line bundles of degree 0 with flat $\lambda$-connection on $T^\vee$. We have $\mathcal{M}^0 = T \times \mathbb{C}$. The spaces $\mathcal{M}_\lambda$ can also be regarded as the deformation associated to the hyperkahler structure of $T \times \mathbb{C}$. An instanton on $T \times \mathbb{C}$ naturally induces a holomorphic vector bundle. If the instanton is $L^2$, the holomorphic bundle with the metric induces a filtered bundle on $(\mathcal{M}^\lambda, T^\lambda_\infty)$, where $\mathcal{M}^\lambda$ is a natural compactification of $\mathcal{M}_\lambda$, and $T^\lambda_\infty \simeq T$ is the infinity. A wild harmonic bundle has the underlying good filtered $\lambda$-flat bundle for each complex number $\lambda$. It is also natural to study the transformation of the underlying filtered bundles on $(\mathcal{M}^\lambda, T^\lambda_\infty)$ and the underlying filtered $\lambda$-flat bundles. It should be a filtered enhancement of the generalized Fourier-Mukai transform for elliptic curves due to G. Laumon and M. Rothstein. We would like to study this interesting aspect elsewhere.

If we consider a counterpart of the algebraic Nahm transform for the other non-product holomorphic structure of $T \times \mathbb{C}$ underlying the hyperKahler structure, it is essentially a filtered version of the generalized Fourier-Mukai transform in [35] and [43]. Interestingly to the author, we have an analogue of the stationary phase formula even in this case. The details will be given elsewhere.

In this paper, we consider transforms between filtered bundles on $T \times \mathbb{P}^1$ and filtered Higgs bundles on $T^\vee$. We may introduce similar transforms for filtered Higgs bundles on $\mathbb{P}^1$ with an additional work on the local Nahm transform $\mathcal{N}^{\infty, \infty}$, which is an analogue of the local Fourier transform $\mathcal{F}^{\infty, \infty}$. It should be the Higgs counterpart of the Nahm transforms between wild harmonic bundles on $\mathbb{P}^1$, which is given by the procedure for wild pure twistor $D$-modules established in [40].

Relatedly, Sz. Szabó [53] studied the Nahm transform for some interesting type of harmonic bundles on $\mathbb{P}^1$. He also studied the transform of the underlying parabolic Higgs bundles, which looks closely related with the regular version of ours in [32], K. Aker and Szabó [1] introduced a transformation of more general parabolic Higgs bundles on $\mathbb{P}^1$, which they call the algebraic Nahm transform. Their method to define the transform is different from ours, and the precise relation is not clear at this moment.

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2 Preliminaries on filtered objects

2.1 Semistable bundles of degree 0 on elliptic curves

2.1.1 Elliptic curve and Fourier-Mukai transform

For a variable $z$, let $\mathbb{C}_z$ denote a complex line with the standard coordinate $z$. For a $\mathbb{C}$-vector space $V$ and a $C^\infty$-manifold $X$, let $\mathcal{L}_X$ denote the product bundle $V \times X$ over $X$. If $X$ is a complex manifold, the natural holomorphic structure of $\mathcal{L}_X$ is denoted just by $\mathcal{D}$.

We have a real bilinear map $\mathbb{C}_z \times \mathbb{C}_\zeta \rightarrow \mathbb{R}$ given by $(z, \zeta) \mapsto \text{Im}(z\zeta)$. Let $\tau = \tau_1 + \sqrt{-1}\tau_2$ ($\tau_1 \in \mathbb{R}$) be a complex number such that $\tau_2 \neq 0$. Let $L := \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}_z$. In this paper, the dual lattice $L^\vee$ means the following:

$$L^\vee := \{ \zeta \in \mathbb{C}_\zeta \mid \text{Im}(\chi\zeta) \in \pi\mathbb{Z} \quad (\forall \chi \in L) \} = \left\{ \frac{\pi}{\tau_2}(n + m\tau) \mid n, m \in \mathbb{Z} \right\}$$

We obtain the elliptic curves $T := \mathbb{C}_z/L$ and $T^\vee := \mathbb{C}_\zeta/L^\vee$.

For any $\nu \in L^\vee$, we have $\rho_\nu \in C^\infty(T)$ given by $\rho_\nu(z) := \exp(2\sqrt{-1}\text{Im}(\nu z)) = \exp(\nu z - \overline{\nu}z)$. We have $\overline{\partial}z\rho_\nu = \rho_\nu \nu d\overline{\zeta}$ and $\partial z\rho_\nu = -\rho_\nu \overline{\nu} dz$.

We can naturally regard $T^\vee$ as the moduli space $\text{Pic}_0(T)$ of holomorphic line bundles of degree 0 on $T$. Indeed, $\zeta$ gives a holomorphic bundle $\mathcal{L}_\zeta = (\mathcal{O}_T, \overline{\partial} + \zeta d\overline{\zeta})$. It induces an isomorphism $T^\vee \simeq \text{Pic}_0(T)$. We have the isomorphism $\Phi : \mathcal{L}_\zeta \simeq \mathcal{L}_{\zeta + \nu}$ given by $\Phi(f) = \rho_{\nu-z}.f$.

We have the unitary flat connection associated to $\mathcal{L}_\zeta$ with the trivial metric, which is $d - \zeta dz + \zeta d\overline{\zeta}$. The monodromy along the segment from 0 to $\chi \in L$ is $\exp\left(\int_0^1 (-\overline{\zeta} \chi + \chi \overline{\zeta}) dt \right) = \exp(2\sqrt{-1}\text{Im}(\zeta \chi))$.

We recall a differential geometric construction of the Poincaré bundle on $T \times T^\vee$, following [23]. On $T \times \mathbb{C}_\zeta$, we have the holomorphic line bundle $\mathcal{P}\text{oin} = (\mathcal{O}_T \otimes \mathcal{O}, \overline{\partial} + \zeta d\overline{\zeta})$. The $L^\vee$-action on $T \times \mathbb{C}_\zeta$ is naturally lifted to the action on $\mathcal{P}\text{oin}$ given by $\nu(z, \zeta, v) = (z, \zeta + \nu, \rho_{\nu-z}(z)v)$. Thus, a holomorphic line bundle is induced on $T \times T^\vee$, which is the Poincaré bundle denoted by $\mathcal{P}\text{oin}$. The dual bundle $\mathcal{P}\text{oin}^\vee$ is induced by $\mathcal{P}\text{oin}^\vee = (\mathcal{O}_T \otimes \mathcal{O}, \overline{\partial} - \zeta d\overline{\zeta})$ with the action $\nu(z, \zeta, v) = (z, \zeta + \nu, \rho_{\nu-z}(z)v)$.

Let $S$ be any complex analytic space. For $I \subset \{1, 2, 3\}$, let $p_I$ denote the projection of $T \times T^\vee \times S$ onto the product of the $i$-th components ($i \in I$). For an object $\mathcal{M} \in D^b(\mathcal{O}_{T \times S})$, we set

$$\mathcal{R}\mathcal{F}M_\pm(\mathcal{M}) := Rp_{23*}\left(p_{13*}(\mathcal{M}) \otimes p_{12*}\mathcal{P}\text{oin}^{\pm 1}\right)[1] \in D^b(\mathcal{O}_{T^\vee \times S}).$$

For an object $\mathcal{N} \in D^b(\mathcal{O}_{T^\vee \times S})$, we set

$$\mathcal{R}\mathcal{F}M_\pm(\mathcal{N}) := Rp_{13*}\left(p_{23*}(\mathcal{N}) \otimes p_{12*}\mathcal{P}\text{oin}^{\pm 1}\right) \in D^b(\mathcal{O}_{T \times S}).$$

Recall that we have a natural isomorphism $\mathcal{R}\mathcal{F}M_+ \circ \mathcal{R}\mathcal{F}M_- \simeq \mathcal{M}$.

2.1.2 Semistable bundle of degree 0

For a holomorphic vector bundle $(E, \mathcal{D}E)$ on $T$, we have the degree $\deg(E) := \int_T c_1(E)$ and the slope $\mu(E) := \deg(E)/\text{rank}(E)$. A holomorphic vector bundle $E$ on $T$ is called semistable, if $\mu(F) \leq \mu(E)$ holds for any non-trivial holomorphic subbundle $F \subset E$. In the following, we shall not distinguish a holomorphic vector bundle and the associated sheaf of holomorphic sections.

Let $E$ be a semistable bundle of degree 0 on $T$. It is well known that the support $\mathcal{S}p(E)$ of $\mathcal{R}\mathcal{F}M_- (E)$ are finite points. Indeed, $E$ is obtained as an extension of the line bundles $\mathcal{L}_\zeta (\zeta \in \mathcal{S}p(E))$. We call $\mathcal{S}p(E)$ the spectrum of $E$. We have the decomposition $E = \bigoplus_{\alpha \in \mathcal{S}p(E)} E_\alpha$, where the support of $\mathcal{R}\mathcal{F}M_- (E_\alpha)$ is $\{\alpha\}$. It is called the spectral decomposition of $E$. We call a subset $\mathcal{S}p(E) \subset \mathbb{C}$ is a lift of $\mathcal{S}p(E)$, if the projection $\Phi : \mathbb{C} \rightarrow T^\vee$ induces the bijection $\mathcal{S}p(E) \simeq \mathcal{S}p(E)$. If we fix a lift, an $\mathcal{O}_\mathbb{C}$-module $\mathcal{M}(E)$ is determined (up to canonical isomorphisms) by the conditions (i) the support of $\mathcal{M}(E)$ is $\mathcal{S}p(E)$, (ii) $\Phi_*\mathcal{M}(E) \simeq \mathcal{R}\mathcal{F}M_- (E)$. Such $\mathcal{M}(E)$ is called a lift of $\mathcal{R}\mathcal{F}M_- (E)$. The multiplication of $\zeta$ on $\mathcal{M}(E)$ induces an endomorphisms of $\mathcal{R}\mathcal{F}M_- (E)$ and $E$. The endomorphism of $E$ is denoted by $f_\zeta$. 

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Let $S$ be any complex analytic space. Let $E$ be a holomorphic vector bundle on $T \times S$. It is called semistable of degree 0 relatively to $S$, if $E|_{T \times \{s\}}$ are semistable of degree 0 for any $s \in S$. The support of $\text{RFM}_- (E)$ is relatively 0-dimensional over $S$. It is denoted by $\mathcal{S}p (E)$, and called the spectrum of $E$. If we have a hypersurface $\tilde{S}p (E) \subseteq \mathbb{C} \times S$ such that the projection $\Phi : \mathbb{C} \times S \to T' \times S$ induces $\tilde{S}p (E) \simeq \mathcal{S}p (E)$, we call $\tilde{S}p (E)$ a lift of $\mathcal{S}p (E)$. If we have a lift of $\mathcal{S}p (E)$, we obtain a lift $\mathcal{M} (E)$ of $\text{RFM}_- (E)$ as in the case that $S$ is a point. We also obtain an endomorphism $f_\zeta$ of $E$ induced by the multiplication of $\zeta$ on $\mathcal{M} (E)$.

### 2.1.3 Equivalence of categories

For a vector space $V$, let $\mathcal{V}$ denote the product bundle $T \times V$ over $T$, and let $\mathcal{V}_0$ denote the natural holomorphic structure of $\mathcal{V}$. For any $f \in \text{End} (V)$, we have the associated holomorphic vector bundle $\mathcal{G} (V, f) := (\mathcal{V}, \overline{\partial}_0 + if \partial)$. We have a natural isomorphism $\mathcal{G} (V, f) \simeq \mathcal{G} (V, f + \nu \text{id}_V)$ for each $\nu \in \mathcal{L}$, induced by the multiplication of $\nu$. Let $\mathcal{S}p (f)$ denote the set of the eigenvalues of $f$.

**Lemma 2.1** $\mathcal{G} (V, f)$ is semistable of degree 0, and we have $\mathcal{S}p (\mathcal{G} (V, f)) = \phi (\mathcal{S}p (f))$ in $T'$, where $\phi : \mathbb{C} \to T'$ denotes the projection.

**Proof** We have only to consider the case $f$ has a unique eigenvalue $\alpha$. In that case, $\mathcal{G} (V, f)$ is an extension of the line bundle $L_\alpha$. Then, the claim is clear.

Let $VS'$ denote the category of finite dimensional $\mathbb{C}$-vector spaces with an endomorphism, i.e., an object in $VS'$ is a finite dimensional vector space $V$ with an endomorphism $f$, and a morphism $(V, f) \to (W, g)$ in $VS'$ is a linear map $\varphi : V \to W$ such that $g \circ \varphi = \varphi \circ f$. For a given subset $\mathfrak{s} \subseteq \mathbb{C}$, let $VS' (\mathfrak{s}) \subseteq VS'$ denote the full subcategory of $VS'$.

Let $VB_0^0 (T)$ denote the category of semistable bundles of degree 0 on $T$, i.e., an object in $VB_0^0 (T)$ is a semistable vector bundle of degree 0 on $T$, and a morphism $V_1 \to V_2$ in $VB_0^0 (T)$ is a morphism of coherent sheaves. For a given subset $\mathfrak{s} \subseteq T'$, let $VB_0^0 (T, \mathfrak{s}) \subseteq VB_0^0 (T)$ denote the full subcategory of semistable bundles of degree 0 whose spectrum are contained in $\mathfrak{s}$.

We have the functor $\mathcal{G} : VS' \to VB_0^0 (T)$ given by the above construction. If $\mathfrak{s}$ is mapped to $\mathfrak{s}$ by the projection $\phi : \mathbb{C} \to T'$, it induces a functor $\mathcal{G} : VS' (\mathfrak{s}) \to VB_0^0 (T, \mathfrak{s})$.

**Proposition 2.2** If $\phi : \mathbb{C} \to T'$ induces a bijection $\mathfrak{s} \simeq \mathfrak{s}$, then $\mathcal{G}$ gives an equivalence of the categories $VS' (\mathfrak{s}) \simeq VB_0^0 (T, \mathfrak{s})$.

**Proof** Let us show that it is fully faithful. We set $E_f := \mathcal{G} (V, f)$. We will not distinguish $E_f$ and the associated sheaf of holomorphic sections. Suppose that $f$ has a unique eigenvalue $\alpha$ such that $\alpha \neq 0$ modulo $L'$. Because $E_f$ is obtained as an extension of the holomorphic line bundle $L_\alpha$, we have $H^0 (T, E_f) = H^1 (T, E_f) = 0$. In particular, we obtain the following.

**Lemma 2.3** Assume that $f_i \in \text{End} (V)$ has a unique eigenvalue $\alpha_i$ for $i = 1, 2$. If $\alpha_1 \neq \alpha_2$ modulo $L'$, any morphism $E_{f_1} \to E_{f_2}$ is 0.

Suppose that $f$ is nilpotent. We have the natural inclusion $V \to C^\infty (T, E_f)$ as constant functions. We have a linear map $V \to C^\infty (T, E_f \otimes \Omega_{T}^{0,1})$ given by $s \mapsto s df$. They induce a chain map $\iota$ from $C_1 = (V \to V)$ to the Dolbeault complex $C^\infty (T, E_f \otimes \Omega^{0,*}_T)$ of $E_f$.

**Lemma 2.4** $\iota$ is a quasi-isomorphism.

**Proof** Let $W$ be the monodromy weight filtration of $f$. It induces filtrations of $C_1$ and $C^\infty (T, E_f \otimes \Omega_{T}^{0,*})$, and $\iota$ gives a morphism of filtered chain complex. It induces a quasi-isomorphism of the associated graded complexes. Hence, $\iota$ is a quasi-isomorphism.

We obtain the following lemma as an immediate consequence.

**Lemma 2.5** Assume that $f_i \in \text{End} (V)$ are nilpotent ($i = 1, 2$). Then, holomorphic morphisms $E_{f_1} \to E_{f_2}$ naturally correspond to holomorphic morphisms $\phi : E_0 \to E_0$ such that $f_2 \circ \phi - \phi \circ f_1 = 0$.

In particular, if $f$ is nilpotent, holomorphic sections of $\text{End} (E_f)$ bijectively correspond to holomorphic sections $g$ of $\text{End} (E_0)$ such that $[f, g] = 0$. 

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The fully faithfulness of the functor $\mathfrak{G}$ follows from Lemma 2.3 and Lemma 2.5. Let us show the essential surjectivity of $\mathfrak{G}$. Let $E \in VB^s(T, s)$. We have the $O_{C_\kappa}$-module $M(E)$ and the endomorphism $f_\zeta$ of $E$ as in [2,1,2]. We have a natural isomorphism $RF_{M_+} \circ RF_{M_-}(E) \simeq E$. The functor $RF_{M_+}$ is induced by the holomorphic line bundle on $T \times T^\vee$, obtained as the descent of $P\text{oin} = (C_\kappa, \partial_0 + \zeta d\bar{\zeta})$. Let $p$ and $q$ denote the projections $T \times C_\kappa \to T$ and $T \times C_\kappa \to C_\kappa$. We have $E \simeq p_* (q^*(M) \otimes P\text{oin})$, and the latter is naturally isomorphic to $(H^0((C_\kappa, M(E)), \partial_0 + f_\zeta d\bar{\zeta}))$. Thus, we obtain the essential surjectivity of $\mathfrak{G}$. The proof of Proposition 2.2 is finished.

As appeared in the proof of Proposition 2.2, we have another equivalent construction of $\mathfrak{G}$. Let $N^0(V, f)$ denote the cokernel of the endomorphism $\zeta \text{id} - f$ on $V \otimes O_{C_\kappa}$. It naturally induces an $O_{T^\vee}$-module $N(V, f)$. We obtain $RF^+_{M_+} (N(V, f))$, which is naturally isomorphic to $\mathfrak{G}(V, f)$. We obtain a quasi-inverse of $\mathfrak{G}$ as follows. Let $E$ be a semistable bundle of degree 0 on $T$. We obtain a vector space $H^0(T^\vee, RF_{M_-}(E))$. If we fix a lift of $SP(E)$ to $\tilde{S}P(E) \subset C$, then the multiplication of $\zeta$ induces an endomorphism $g_\zeta$ of $H^0(T^\vee, RF_{M_-}(E))$. The construction of $(H^0(T^\vee, RF_{M_-}(E)), g_\zeta)$ from $E$ gives a quasi-inverse of $\mathfrak{G}$.

Let $(E, \tilde{\zeta})$ be a semistable bundle of degree 0 on $T$. Let $\tilde{s}$ be a subset of $C$ which is injectively mapped to $T^\vee = C/L^\vee$.

**Corollary 2.6** We have a unique decomposition $\tilde{\zeta}_E = \tilde{\zeta}_{E,0} + f d\bar{\zeta}$ with the following property:

- $(E, \tilde{\zeta}_{E,0})$ is holomorphically trivial, i.e., isomorphic to a direct sum of $O_T$.

- $f$ is a holomorphic endomorphism of $(E, \tilde{\zeta}_{E,0})$. We impose the condition that $SP(H^0(f)) \subset \tilde{s}$, where $H^0(f)$ is the induced endomorphism of the space of the global sections of $(E, \tilde{\zeta}_{E,0})$.

**Proof** The existence of such a decomposition follows from the essential surjectivity of $\mathfrak{G}$. Let us show the uniqueness. By considering the spectral decomposition, we have only to consider the case $\tilde{s} = \{0\}$. Suppose that $\tilde{\zeta}_E = \tilde{\zeta}_{E,0} + g d\bar{\zeta}$ is another decomposition with the desired property. Because $f$ is holomorphic with respect to $\tilde{\zeta}_E$, we have $\tilde{\zeta}_{E,0} f = 0$ and $[f, g] = 0$ by Lemma 2.5. We put $h = f - g$, which is also nilpotent. The identity induces an isomorphism $(E, \tilde{\zeta}_{E,0} + h) \simeq (E, \tilde{\zeta}_{E,0})$. Because $\mathfrak{G}$ is fully faithful, we obtain $h = 0$.

**The family version** We have a family version of the equivalence. Let $S$ be any complex manifold. Let $\pi_S : T \times S \to S$ denote the projection. Let $VB^s(S)$ denote the category of pairs $(V, f)$ of coherent locally free $O_S$-module $V$ and its endomorphism $f$. A morphism $(V, f) \to (V', f')$ in $VB^s(S)$ is a morphism of $O_S$-modules $g : V \to V'$ such that $f' \circ g = g \circ f$. Such $(V, f)$ naturally induces an $O_{C_\kappa \times S}$-module $M(V, f)$. The support is denoted by $SP(f)$. When we are given a divisor $\tilde{s} \subset C_\kappa \times S$ which is finite and relatively 0-dimensional over $S$, then $VB^s(S, \tilde{s})$ denote the full subcategory of $(V, f) \in VB^s(S)$ such that $SP(f) \subset \tilde{s}$.

Let $VB^s_0(T \times S/S)$ denote the full subcategory of $O_{T^\vee \times S}$-modules, whose objects are semistable of degree 0 relative to $S$. When we are given a divisor $s \subset T^\vee \times S$ which is relatively 0-dimensional over $S$, then let $VB^s_0(T \times S/S, s)$ denote the full subcategory of $E \in VB^s_0(T \times S/S)$ such that $SP(E) \subset s$.

Let $V$ be a holomorphic vector bundle on $S$ with a holomorphic endomorphism $f$. The $C^\infty$-vector bundle $\pi_S^{-1}V$ is equipped with a naturally induced holomorphic structure obtained as the pull back, denoted by $\tilde{\partial}_0$. We obtain a holomorphic vector bundle $\mathfrak{G}(V, f) := (\pi_S^{-1}V, \tilde{\partial}_0 + f d\bar{\zeta})$. By Lemma 2.1, $\mathfrak{G}$ gives a functor $VB^s(S) \to VB^s_0(T \times S/S)$. If we are given $s \subset T^\vee \times S$ and its lift $\tilde{s} \subset C_\kappa \times S$, it gives an equivalence of the categories $VB^s(S, s) \to VB^s_0(T \times S/S, \tilde{s})$.

We have another equivalent description of $\mathfrak{G}$. Let $(V, f) \in VB^s(S)$. We have the naturally induced $O_{C_\kappa \times S}$-module $M(V, f)$, which induces an $O_{T^\vee \times S}$-module $N(V, f)$. We have a natural isomorphism $\mathfrak{G}(V, f) \simeq RF_{M_-}(N(V, f))$.

If we are given $s \subset T^\vee \times S$ with a lift $\tilde{s} \subset C_\kappa \times S$, for an object $E \in VB^s_0(T \times S/S, s)$, we obtain an $O_{C_\kappa \times S}$-module $M(E)$ such that (i) the support of $M(E)$ is contained in $\tilde{s}$, (ii) $\Phi_* M(E) \simeq RF_{M_-}(E)$. The multiplication of $\zeta$ induces an endomorphism of $RF_{M_-}(E)$, and hence an endomorphism of $\pi_S^{-1}(RF_{M_-}(E))$, denoted by $g_\zeta$, where $\pi_S : T^\vee \times S \to S$. The construction of $(\pi_S^{-1}(RF_{M_-}(E), g_\zeta)$ from $E$ gives a quasi-inverse of $\mathfrak{G}$.
2.1.4 Differential geometric criterion

We recall a differential geometric criterion in terms of the curvature for a metrized holomorphic vector bundle to be semistable of degree 0. Let $(E, \overline{\nabla}_E)$ be a holomorphic vector bundle on $T$ with a hermitian metric $h$. Let $F(h)$ denote the curvature. We use the standard metric $dz \overline{dz}$ of $T$.

Lemma 2.7 There exists a constant $\epsilon > 0$, depending only on $T$ and rank $E$, with the following property:

- If $|F(h)|_h \leq \epsilon$, then $(E, \overline{\nabla}_E, h)$ is semistable of degree 0.

Proof The number $\text{deg}(E) = \int \text{Tr} F(h)$ is an integer. We have $\int |\text{Tr} F(h)| \leq |T| \text{rank} E \epsilon$, where $|T|$ is the volume of $T$. Hence, we have $\int \text{Tr} F(E_\epsilon) = 0$, if $\epsilon$ is sufficiently small. For any subbundle $E' \subset E$, by using the decreasing property of the curvature of subbundles, we also obtain $\text{deg}(E') < 1$ and hence $\text{deg}(E') \leq 0$.  

2.2 Filtered bundles

2.2.1 Filtered sheaves

Let us recall the notion of filtered sheaves and filtered bundles. Let $X$ be a complex manifold with a smooth hypersurface $D$. (We restrict ourselves to the case that $D$ is smooth, because we are interested in only the case in this paper.) Let $E$ be a coherent $\mathcal{O}_X(*)D$-module. Let $D = \bigsqcup_{i \in \Lambda} D_i$ be the decomposition into the connected components. A filtered sheaf $\mathcal{P}_aE$ over $E$ is a sequence of coherent $\mathcal{O}_X$-submodules $\mathcal{P}_aE \subset E$ indexed by $\mathbb{R}^\Lambda$ satisfying the following.

- $\mathcal{P}_aE|_{X \setminus D} = E|_{X \setminus D}$. We have $\mathcal{P}_aE \subset \mathcal{P}_bE$, if $a_i \leq b_i$ ($i \in \Lambda$), where $a = (a_i \mid i \in \Lambda)$ and $b = (b_i \mid i \in \Lambda)$.

- On a small neighbourhood $U$ of $D_i$ ($i \in \Lambda$), $\mathcal{P}_aE|_U$ depends only on $a_i$, which we denote by $\mathcal{P}_{a_i}(E|_U)$, or $\mathcal{P}_{a_i}(E)$, when we emphasize $i$.

- For each $i$ and $c \in \mathbb{R}$, there exists $\epsilon > 0$ such that $\mathcal{P}_c(E|_U) = \mathcal{P}_{c+\epsilon}(E|_U)$.

- We have $\mathcal{P}_{a+n}E = \mathcal{P}_aE(\sum n_i D_i)$, where $n = (n_i) \in \mathbb{Z}^\Lambda$.

The tuple $(E, \{\mathcal{P}_aE \mid a \in \mathbb{R}^\Lambda\})$ is denoted by $\mathcal{P}_aE$. The filtration $\{\mathcal{P}_aE \mid a \in \mathbb{R}^\Lambda\}$ is also denoted by $\mathcal{P}_aE$. We say that $E$ is the $\mathcal{O}_X(*)D$-module underlying $\mathcal{P}_aE$.

For a small neighbourhood $U$ of $D_i$, we set $\mathcal{P}_{<a}(E|_U) := \sum_{b<a} \mathcal{P}_b(E|_U)$. We also put $\mathcal{P}_a(E|_{D_i}) := \mathcal{P}_a(E|_U)|_{D_i}$, and $\mathcal{P}_{<a}(E) := \mathcal{P}_a(E|_U)/\mathcal{P}_{<a}(E|_U)$, which are coherent $\mathcal{O}_{D_i}$-modules. We set

$$\mathcal{P}ar(\mathcal{P}_aE, i) := \{ b \in [a_i - 1, a_i] \mid \mathcal{P}_{Gr}^a(E) \neq 0 \}, \quad \mathcal{P}ar(\mathcal{P}_aE, i) := \bigcup_{a \in \mathbb{R}^\Lambda} \mathcal{P}ar(\mathcal{P}_aE, i).$$

A morphism of filtered sheaves $\mathcal{P}_aE_1 \rightarrow \mathcal{P}_aE_2$ is a morphism of $\mathcal{O}_X$-modules which is compatible with the filtrations. A subobject $\mathcal{P}_aE_1 \subset \mathcal{P}_aE$ is a subsheaf $E_1 \subset E$ satisfying $\mathcal{P}_a(E_1) \subset \mathcal{P}_a(E)$ for any $a \in \mathbb{R}^\Lambda$. It is called strict, if $\mathcal{P}_a(E_1) = E_1 \cap \mathcal{P}_a(E)$ for any $a \in \mathbb{R}^\Lambda$.

2.2.2 Filtered bundles and basic operations

A filtered sheaf $\mathcal{P}_aE$ is called a filtered bundle, if (i) $\mathcal{P}_aE$ are locally free $\mathcal{O}_X$-modules, (ii) $\mathcal{P}_{<a}(E)$ are locally free $\mathcal{O}_{D_i}$-modules for any $i \in \Lambda$ and $a \in \mathbb{R}$. In that case, for any $b \in [a_i - 1, a_i]$, we set

$$F_b(\mathcal{P}_a(E)|_{D_i}) := \text{Im}\left(\mathcal{P}_b(E)|_{D_i} \rightarrow \mathcal{P}_a(E)|_{D_i}\right).$$

It is called the parabolic filtration.

The direct sum of filtered bundles $\mathcal{P}_aE_i$ ($i = 1, 2$) is defined as the locally free $\mathcal{O}_X(*)D$-module $E_1 \oplus E_2$ with the $\mathcal{O}_X$-submodules $\mathcal{P}_a(E_1 \oplus E_2) = \mathcal{P}_aE_1 \oplus \mathcal{P}_aE_2$ ($a \in \mathbb{R}^\Lambda$). The tensor product of filtered bundles $\mathcal{P}_aE_i$ ($i = 1, 2$) is defined as the $\mathcal{O}_X(*)D$-module $E_1 \otimes E_2$ with the $\mathcal{O}_X$-submodules $\mathcal{P}_a(E_1 \otimes E_2) = \sum_{b+c=a} \mathcal{P}_b(E_1) \otimes \mathcal{P}_c(E_2)$. The inner homomorphism is defined as the $\mathcal{O}_X(*)D$-module $\text{Hom}_{\mathcal{O}_X(*)D}(E_1, E_2)$ with the $\mathcal{O}_X$-submodules $\mathcal{P}_a\text{Hom}(E_1, E_2) = \{ f \in \text{Hom}(E_1, E_2) \mid f(\mathcal{P}_bE_1) \subset \mathcal{P}_{b+a}E_2 \}$.  

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The most typical example is the $\mathcal{O}_X(\ast D)$-module $\mathcal{O}_X(\ast D)$ with the $\mathcal{O}_X$-submodules $\mathcal{P}_a(\mathcal{O}_X(\ast D)) := \mathcal{O}(\sum [a_i] D_i)$, where $\{a_i\} := \max n \in \mathbb{Z} \mid |a| \leq a$. The filtered bundle is denoted just by $\mathcal{O}_X(\ast D)$. For any filtered bundle $\mathcal{P}_a \mathcal{E}$, the dual $\mathcal{P}_a \mathcal{E}^\vee$ is defined as $\mathcal{H}om(\mathcal{P}_a \mathcal{E}, \mathcal{O}_X(\ast D))$. We have a natural isomorphism $\mathcal{P}_a(\mathcal{E}^\vee) \simeq \mathcal{P}_{\leq -a + \delta}(\mathcal{E}^\vee)$, where $\delta = (1, \ldots, 1)$.

Let $\varphi : (X', D') \rightarrow (X, D)$ be a ramified covering with $D' = \bigsqcup_{i \in \Lambda} D'_i$ and $D = \bigsqcup_{i \in \Lambda} D_i$. Let $e_i$ be the degree of the ramification. Let $\mathcal{P}_a \mathcal{E}$ be a filtered bundle over $\mathcal{E}$. The pull back of a filtered bundle is defined as the $\mathcal{O}_X(\ast D')$-module $\varphi^* \mathcal{E}$ with the $\mathcal{O}_X$-submodules $\mathcal{P}_a \varphi^* \mathcal{E} = \sum_{e_i + n \leq a} \varphi^*(\mathcal{P}_b \mathcal{E}) \otimes \mathcal{O}_X(\sum n_i D'_i)$, where $e_b = (e_i \mid i \in \Lambda)$. The filtered bundle is denoted by $\varphi^*(\mathcal{P}_a \mathcal{E})$.

Let $\mathcal{P}_a \mathcal{E}'$ be a filtered bundle on $(X', D')$. Then, we obtain a locally free $\mathcal{O}_X(\ast D')$-module $\varphi_{*} \mathcal{E}'$ with the $\mathcal{O}_X$-submodules $\mathcal{P}_a(\varphi_{*} \mathcal{E}')$ such that $\mathcal{P}_a(\varphi_{*} \mathcal{E}')[U] = \varphi_*(\mathcal{P}_{a, \psi^{-1}(U, j)})$. The filtered bundle is denoted by $\varphi_*(\mathcal{P}_a \mathcal{E})$. Suppose that $\varphi : (X', D') \rightarrow (X, D)$ is a Galois covering with the Galois group $\text{Gal}(\varphi)$, and that $\mathcal{P}_a \mathcal{E}'$ be a Gal($\varphi$)-equivariant filtered bundle, then $\varphi_*(\mathcal{P}_a \mathcal{E})$ is equipped with an induced Gal($\varphi$)-action. The Gal($\varphi$)-invariant part is called the descent of $\mathcal{P}_a \mathcal{E}'$ with respect to $\varphi$.

### 2.2.3 The parabolic first Chern class

Let $\mathcal{P}_a \mathcal{E}$ be a filtered sheaf on $(X, D)$. Suppose that $\mathcal{E}$ is torsion-free. The parabolic first Chern class of $\mathcal{P}_a \mathcal{E}$ is defined as

$$\text{par-}c_1(\mathcal{P}_a \mathcal{E}) = c_1(\mathcal{P}_a \mathcal{E}) - \sum_{i \in \Lambda} \sum_{b \in \text{Par}(\mathcal{P}_a \mathcal{E}, i)} b \dim \text{Gr}^b_1(\mathcal{E}) [D_i].$$

Here, $[D_i]$ is the cohomology class of $D_i$. It is independent of the choice of $\mathcal{a}$.

Let $U_i$ be a small neighbourhood of $D_i$. Suppose that we are given a decomposition $\mathcal{P}_a \mathcal{E}_{|U_i} = \bigsqcup_{k \in \Lambda(i)} \mathcal{P}_{a(i,k)} \mathcal{E}_{i,k}$ for each $i \in \Lambda$. Let $U$ be a locally free $\mathcal{O}_X$-submodule of $\mathcal{E}$ such that $U_{|U_i} = \bigsqcup_{k \in \Lambda(i)} \mathcal{P}_{a(i,k)} \mathcal{E}_{i,k}$, where $a(i, k) \in \mathbb{R}$. It is easy to check the following equality:

$$\text{par-}c_1(\mathcal{P}_a \mathcal{E}) = c_1(U) - \sum_{i \in \Lambda} \sum_{k \in \Lambda(i)} \delta(\mathcal{P}_a \mathcal{E}_{i,k}, a(i, k)),
$$

$$\delta(\mathcal{P}_a \mathcal{E}_{i,k}, a(i, k)) := \sum_{b \in \text{Par}(\mathcal{P}_{a(i,k)} \mathcal{E}_{i,k})} b \text{ rank } \text{Gr}^b_1(\mathcal{E}_{i,k}) [D_i].$$

### 2.2.4 Compatible frames

For simplicity, we consider the case that $X$ is a neighbourhood of $0$ in $\mathbb{C}$, and $D = \{0\}$. Let $\mathcal{P}_a \mathcal{E}$ be a filtered bundle on $(X, D)$. For any section $f$ of $\mathcal{E}$, we set $\deg^P(f) := \min \{a \in \mathbb{R} \mid f \in \mathcal{P}_a \mathcal{E}\}$. Let $v = (v_1, \ldots, v_r)$ be a frame of $\mathcal{P}_a \mathcal{E}$. We say that it is compatible with the parabolic structure, if for any $b \in \text{Par}(\mathcal{P}_a \mathcal{E})$, the set $\{v_i \mid \deg(v_i) = b\}$ induces a base of $\text{Gr}^b_1(\mathcal{E})$.

Let $\varphi : (X', D') \rightarrow (X, D)$ be a ramified covering given by $\varphi(u) = u^p$. Let $\mathcal{P}_a \mathcal{E}$ be a filtered bundle on $(X, D)$. Let $v$ be a compatible frame of $\mathcal{P}_a \mathcal{E}$. Let $c_i := \deg^P(v_i)$. We set $n_i := \max \{n \in \mathbb{Z} \mid n + pc_i \leq a\}$, and $w_i := u^{-n_i} \varphi^* v_i$. Then, $w = (w_1, \ldots, w_r)$ is a compatible frame of $\varphi^*(\mathcal{P}_a \mathcal{E})$, such that $\deg^P(w_i) = n_i + pc_i$.

Let $\mathcal{P}_a \mathcal{E}'$ be a filtered bundle on $(X', D')$. Let $v'$ be a compatible frame of $\mathcal{P}_a \mathcal{E}'$. Let $c_j := \deg^P(v'_j)$. For $0 \leq j < p$, we set $w'_{ij} := u^i v'_j$. They naturally induce sections of $\mathcal{P}_{a/p}(\varphi_{*} \mathcal{E}')$, denoted by $\tilde{w}'_{ij}$. Then, $\tilde{w}' := (\tilde{w}'_{ij} \mid 1 \leq i \leq \text{rank } \mathcal{E}', 0 \leq j < p)$ gives a compatible frame of $\mathcal{P}_{a/p}(\varphi_{*} \mathcal{E})$ such that $\deg^P(\tilde{w}'_{ij}) = (c_i - j)/p$.

### 2.2.5 Adapted metric

Let us return to the setting in 2.2.4. Let $V$ be a holomorphic vector bundle on $X \setminus D$ with a hermitian metric $h$. Recall that, for any $a \in \mathbb{R}^{\Lambda}$, we obtain a natural $\mathcal{O}_X$-module $\mathcal{P}_a(\mathcal{h} V)$ on $X$ as follows. Let $U$ be any open subset of $X$. For any $P \in U$, we take a holomorphic coordinate neighbourhood $(X_P, z_1, \ldots, z_n)$ around $P$ such that (i) $X_P$ is relatively compact in $U$, (ii) $X_P \cap D = X_P \cap D_i$ for some $i \in \Lambda$, (ii) $X_P \cap D = \{z_1 = 0\}$. Then, let $\mathcal{P}_a(\mathcal{h} U)$ denote the space of holomorphic sections $f$ of $V_{|U \setminus D}$ such that $|f|_{X_P} = O(|z_1|^{-n-\epsilon})$ for any $\epsilon > 0$ and for any $P \in U$. In general, $\mathcal{P}_a V$ are not $\mathcal{O}_X$-coherent.
Suppose that we are given a filtered bundle \( \mathcal{P}_s V \) on \((X,D)\), and that \( V := \mathcal{P}_s V|_{X\setminus D} \) is equipped with a hermitian metric \( h \) such that \( \mathcal{P}_s V = \mathcal{P}_s V \). In that case, \( h \) is called adapted to \( \mathcal{P}_s V \).

### 2.3 Filtered Higgs bundles

Let us recall the notion of filtered Higgs bundles on curves. Let \( X \) be a complex curve with a discrete subset \( D \). Let \( \mathcal{P}_s \mathcal{E} \) be a filtered sheaf on \((X,D)\). Let \( \theta \) be a Higgs field of \( \mathcal{E} \), i.e., \( \theta \) is an \( \mathcal{O}_X \)-homomorphism \( \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X \).

Then, \((\mathcal{P}_s \mathcal{E}, \theta)\) is called a filtered Higgs bundle.

We shall consider two conditions on the compatibility of \( \theta \) and the filtration \( \mathcal{P}_s \mathcal{E} \). One is the admissibility, and the other is goodness. The latter is what we are really interested in, because it is closely related with wild harmonic bundles and \( L^2 \)-instantons. The former is easier to handle, and more natural when we consider algebraic Nahm transforms. We shall explain the easier one first.

The conditions are given locally around each point of \( D \). So, we shall explain them in the case \( X := \{ z \in \mathbb{C} \mid |z| < \rho_0 \} \) and \( D := \{ 0 \} \).

#### 2.3.1 Admissible filtered Higgs bundles

For each positive integer \( p \), let \( \varphi_p : X^{(p)} = \{ |z| < \rho_0^{1/p} \} \to X \) be given by \( \varphi_p(z_p) = z_p^p \). Let \( \mathcal{P}_s V \) be a filtered bundle on \((X,D)\) with a Higgs field \( \theta \). Let \( m \in \mathbb{Z}_{\geq 0} \) and \( p \in \mathbb{Z}_{>0} \) such that \( \gcd(p,m) = 1 \). We say that \((\mathcal{P}_s V, \theta)\) has slope \((p,m)\), if

\[
\langle \mathrm{Res}(z^{m}_p \theta^{(p)}), \mathcal{P}_s V^{(p)} \rangle \cong \bigoplus_{\alpha \in \mathcal{O}_{(p,m)}} (\mathcal{P}_s V^{(p,m)}, \theta^{(p,m)}_\alpha)
\]

is good, if there exists a ramified covering \( \varphi : (X^{(p)}, D^{(p)}) \to (X,D) \) given by \( \varphi_p(z_p) = z_p^p \) with a

**Lemma 2.8** Let \((\mathcal{P}_s V, \theta)\) be an admissible filtered Higgs bundle on \((X,D)\). Let \( \mathcal{P}_s V' \) be a strict filtered Higgs subbundle, i.e., it is a strict filtered subbundle such that \( \theta(V') \subset V' \otimes \Omega^1_X \). The restriction of \( \theta \) to \( V' \) is denoted by \( \theta' \). Then, \((\mathcal{P}_s V', \theta')\) is admissible.

#### 2.3.2 Good filtered Higgs bundles

We have a stronger condition. Let \( X \) and \( D \) be as in \([2.3.1]\). We say that a filtered Higgs bundle \((\mathcal{P}_s V, \theta)\) on \((X,D)\) is good, if there exists a ramified covering \( \varphi_p : (X^{(p)}, D^{(p)}) \to (X,D) \) given by \( \varphi_p(z_p) = z_p^p \) with a
such that $\theta_a^{(p)} - da id_{V_a^{(p)}}$ is logarithmic in the sense that it gives a morphism $\mathcal{P}_a V_a^{(p)} \to \mathcal{P}_a V_a^{(p)} dz_p/z_p$. Let $\text{Irr}(\varphi^* \theta)$ denote the set of $a$ such that $V_a^{(p)} \neq 0$. The Galois group $\text{Gal}(\varphi_p)$ naturally acts on $\varphi^*_p (\mathcal{P}, V, \theta)$ and $\text{Irr}(\varphi^*_p \theta)$. The quotient set $\text{Irr}(\varphi^*_p \theta)/\text{Gal}(\varphi_p)$ is denoted by $\text{Irr}(\varphi^*_p \theta)$. We have the orbit decomposition $\text{Irr}(\varphi^*_p \theta) = \bigsqcup_{o \in \text{Irr}(\varphi^*_p \theta)} o$. We set $(\mathcal{P}_o V_o^{(p)}, \theta_o^{(p)}) := \bigoplus_{a \in o} (\mathcal{P}_a V_a^{(p)}, \theta_a^{(p)})$. We obtain a Gal$(\varphi_p)$-equivariant decomposition $\varphi^*_p (\mathcal{P}, V, \theta) = \bigoplus_{o \in \text{Irr}(\varphi^*_p \theta)} (\mathcal{P}_o V_o^{(p)}, \theta_o^{(p)})$. By the descent, we obtain a decomposition
\[
(\mathcal{P}, V, \theta) = \bigoplus_{o \in \text{Irr}(\varphi^*_p \theta)} (\mathcal{P}_o V_o, \theta_o).
\] (5)

If we have a factorization $\varphi_p = \varphi_{p_1} \circ \varphi_{p_2}$ such that $\varphi_{p_2}^* (\mathcal{P}, V, \theta)$ has a decomposition as above, $\varphi_{p_1}$ gives a bijection $\text{Irr}(\varphi^*_p \theta) \simeq \text{Irr}(\varphi^*_{p_1} \theta)$. It induces a bijection of the quotient sets by the Galois groups. By the identification, we denote them by $\text{Irr}(\theta)$ and $\text{Irr}(\theta)$. The decomposition (5) is independent of the choice of $\varphi_p$.

For each $o \in \text{Irr}(\theta)$, there exists a minimum $p_o$ among the numbers $p$ such that $\varphi^*_p (\mathcal{P}_o V_o, \theta_o)$ has a decomposition such as (4). In this case, we have $|p_o| = p_o$. We set $X^o := X^{(p_o)}, \varphi_o := \varphi_{p_o}$ and $z_o := z_{p_o}$. We have the decomposition on $X^o$:
\[
\varphi^*_o (\mathcal{P}_o V_o, \theta_o) = \bigoplus_{a \in o} (\mathcal{P}_a V_a^{(p)}, \theta_a^{(p)}).
\] (6)

For any $a \in o$, we have a natural isomorphism $(\mathcal{P}_a V_a, \theta_a) \simeq \varphi_{a*} (\mathcal{P}_o V_o^{(p)}, \theta_o^{(p)})$. We set $m_o := (\text{ord}_{z_{p_o}^{-1}} a)$ which is independent of $a \in o$. In this paper, if $(\mathcal{P}, V, \theta) = (\mathcal{P}_o V_o, \theta_o)$, we say that $(\mathcal{P}, V, \theta)$ has pure irregularity $o$.

If $X$ is shrank appropriately, we have the following decomposition, which is a refinement of (4):
\[
\varphi^*_p (\mathcal{P}, V, \theta) = \bigoplus_{a \in p^{-1}C[z_p^{-1}]} \bigoplus_{a \in C} (\mathcal{P}_a V_a^{(p)}, \theta_a^{(p)})
\]
such that the eigenvalues of the residues $\text{Res}(\theta_a^{(p)} - (da + podz_p/z_p) id_{V_a^{(p)}})$ are 0. Let $(\mathcal{P}_a V_{o,a}, \theta_{a,o})$ be the descent of $\bigoplus_{a \in o} (\mathcal{P}_a V_a^{(p)}, \theta_a^{(p)})$ to $X$. We obtain a decomposition:
\[
(\mathcal{P}, V, \theta) = \bigoplus_{o \in \text{Irr}(\theta)} \bigoplus_{a \in C} (\mathcal{P}_a V_{o,a}, \theta_{a,o})
\]

On $X^o$, we have a decomposition:
\[
\varphi^*_o (\mathcal{P}_o V_{o,a}, \theta_{a,o}) = \bigoplus_{a \in o} (\mathcal{P}_a V_{o,a}^{(p)}, \theta_{a,o}^{(p)})
\]

**Lemma 2.9** Let $(\mathcal{P}, V, \theta)$ be a good filtered Higgs bundle. Let $\mathcal{P}_* V'$ be a strict Higgs subbundle. The restriction of $\theta$ to $V'$ is denoted by $\theta'$. Then, $(\mathcal{P}_* V', \theta')$ is also good.

**Proof** Suppose $(\mathcal{P}, V, \theta)$ is unramified with the decomposition $(\mathcal{P}, V, \theta) = \bigoplus (\mathcal{P}_a V_a, \theta_a)$. Because $\theta(V') \subset V' \otimes \Omega^1_X$, we have $V' = \bigoplus (V' \cap V_a)$. By the strictness, we obtain $\mathcal{P}_* V' = \bigoplus (V' \cap V_a)$. Hence, $(\mathcal{P}_* V', \theta')$ is also good. The ramified case can be reduced to the unramified case by the descent. □

Take $p \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}_{\geq 0}$ with g.c.d.$(p, m) = 1$. Let $\text{Irr}(\theta, p, m) := \{ o \in \text{Irr}(\theta) | p_o/m_o = p/m \}$. We have
\[
\mathcal{P}_a V^{(p,m)} = \bigoplus_{o \in \text{Irr}(\theta, p, m)} \mathcal{P}_a V_o.
\]
For any $o \in \mathcal{O}(p, m)$, we have $\text{Irr}(\theta, p, m, o) \subset \text{Irr}(\theta, p, m)$ such that
\[
\mathcal{P}_{\alpha}V_{\alpha}^{(p, m)} = \bigoplus_{\alpha \in \text{Irr}(\theta, p, m, o)} \mathcal{P}_{\alpha}V_{\alpha}.
\]
Take any $\alpha \in o$. For each $o \in \text{Irr}(\theta, p, m, o)$, we have $a \in o$ such that
\[
\mathcal{P}_{\alpha}V_{\alpha}^{(p)} = \bigoplus_{o \in \text{Irr}(\theta, p, m, o)} (\varphi(p_{o}/p) \ast (\mathcal{P}_{\alpha}V_{\alpha}^{o})).
\]
Here, $\varphi_{p_{o}/p}$ is the ramified covering $X^{o} \to X^{(p)}$ given by $\varphi_{p_{o}/p}(z_{o}) = z_{o}^{p_{o}/p}$. Let $c \in \mathbb{R}$. We take a frame $v_{o} = (v_{o,i})$ of $\mathcal{P}_{p_{o}}V_{o}^{*}$ compatible with the parabolic structure. Then, the tuple of the sections
\[
\left\{z_{o}^{i}v_{o,i} \mid o \in \text{Irr}(\theta, p, m, o), \ 1 \leq i \leq \text{rank}V_{o}^{*}, \ 0 \leq j < p_{o}/p\right\}
\]
gives a frame of $\mathcal{P}_{p_{o}}V_{o}^{(p)}$.

2.3.3 Filtered bundles with an endomorphism

Let $U_{\tau}$ be a small neighbourhood of 0 in $\mathbb{C}_{\tau}$. Let $\mathcal{P}_{\tau}V$ be a filtered bundle on $(U_{\tau}, 0)$ with an endomorphism $g$. We say that $(\mathcal{P}_{\tau}V, g)$ has type $(p, m, o)$ (slope $(p, m)$), if $(\mathcal{P}_{\tau}V, -\tau^{-2}g\tau)$ has type $(p, m, o)$ (resp. slope $(p, m)$). The condition implies $p \geq m$. We say that $(\mathcal{P}_{\tau}V, g)$ is admissible, if $(\mathcal{P}_{\tau}V, -\tau^{-2}g\tau)$ is admissible. If $(\mathcal{P}_{\tau}V, g)$ is admissible, we have the type decomposition $(\mathcal{P}_{\tau}V, g) = \bigoplus(\mathcal{P}_{\tau}V_{\alpha}^{(p, m)}, g_{\alpha}^{(p, m)})$ and the slope decomposition $(\mathcal{P}_{\tau}V_{\alpha}^{(p, m)}, g_{\alpha}^{(p, m)})$)

Similarly, $(\mathcal{P}_{\tau}V, g)$ is called good, if $(\mathcal{P}_{\tau}V, -\tau^{-2}g\tau)$ is a good filtered Higgs bundle.

Remark 2.10 We regard $U_{\tau}$ as a neighbourhood of $\infty$ in $\mathbb{P}^{1}$. Of course, $-\tau^{-2}d\tau = dw$ for $w = \tau^{-1}$.

Remark 2.11 We shall be interested in the case that $(\mathcal{P}_{\tau}V, g)$ is decomposed into $\bigoplus_{\alpha \in \mathbb{C}}(\mathcal{P}_{\tau}V_{\alpha}, g_{\alpha})$ such that (i) $\mathcal{S}(g_{\alpha}|_{0}) = \{\alpha\}$, (ii) $(\mathcal{P}_{\tau}V_{\alpha}, g_{\alpha} - \alpha)$ is admissible. In that case, in the slope decomposition $(\mathcal{P}_{\tau}V_{\alpha}, g_{\alpha} - \alpha) = \bigoplus(\mathcal{P}_{\tau}V_{\alpha}^{(p, m)}, g_{\alpha}^{(p, m)})$, we have $m/p < 1$ for $V_{\alpha}^{(p, m)} \neq 0$.

2.4 Filtered bundles on $(T \times \mathbb{P}^{1}, T \times \{\infty\})$

2.4.1 Local conditions

Let $U \subset \mathbb{P}^{1}$ be a small neighbourhood of $\infty$. We introduce some conditions on filtered bundles $\mathcal{P}_{\tau}E$ on $(T \times U, T \times \{\infty\})$.

(A1) $\mathcal{P}_{c}E|_{T \times \infty}$ are semistable of degree 0 for any $c \in \mathbb{R}$.

The condition is equivalent to that $\text{Gr}^{	ext{P}}_{c}(E)$ are semistable of degree 0 for any $c \in \mathbb{R}$. Let $\mathcal{S}\mathcal{P}_{\infty}(E) \subset T' \subset \mathbb{C}$. Let $\mathcal{S}\mathcal{P}_{\infty}(E) \subset T' \subset \mathbb{C}$. Let $\mathcal{S}\mathcal{P}_{\infty}(E) \subset T' \subset \mathbb{C}$. Then, as observed in [2.1] for a small neighbourhood $U'$ of $\infty \in \mathbb{P}^{1}$, we obtain the corresponding filtered bundle $\mathcal{P}_{\tau}V$ with an endomorphism $g$ on $(U', \infty)$ such that $\mathcal{S}\mathcal{P}(g|_{\infty}) = \mathcal{S}\mathcal{P}_{\infty}(E)$. We have the decomposition
\[
(\mathcal{P}_{\tau}V, g) = \bigoplus_{P \in \mathcal{S}\mathcal{P}_{\infty}(E)} (\mathcal{P}_{\tau}V_{P}, g_{P})
\]
such that $\mathcal{S}\mathcal{P}(g_{P}) \cap (\mathbb{C} \times \{\infty\}) = \{P\}$ is the lift of $P$. A filtered bundle satisfying (A1) is called admissible, if it satisfies the following condition, which is independent of the choice of $\mathcal{S}\mathcal{P}_{\infty}(E)$.

(A2) $(\mathcal{P}_{\tau}V_{P}, g_{P} - \tilde{P} \text{id})$ is admissible in the sense of [2.3, 3] for any $P \in \mathcal{S}\mathcal{P}_{\infty}(E)$. 

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We have the type decomposition \( (\mathcal{P}, V_{\mathcal{P}}, g_{\mathcal{P}} - \tilde{P}_{\text{id}}) = \bigoplus_{p,m,o} (\mathcal{P}, V_{\mathcal{P},o}^{(p,m)}, g_{\mathcal{P},o}^{(p,m)}) \). We have the corresponding decomposition \( \mathcal{P}E = \bigoplus_{\mathcal{P}} \bigoplus_{p,m,o} \mathcal{P}E_{\mathcal{P},o}^{(p,m)} \), which is called the type decomposition of \( \mathcal{P}E \). The following lemma is clear.

**Lemma 2.12** If \( \mathcal{P}E \) satisfies the condition \( (A1) \) (resp. the admissibility), then the dual \( \mathcal{P}E^\vee \) also satisfies the condition \( (A1) \) (resp. the admissibility).

We also have the following condition.

**Good** Let \( \mathcal{P}E \) be a filtered bundle on \( (T \times \mathbb{P}, T \times \{ \infty \}) \) satisfying \( (A1) \). Take any lift \( \tilde{\mathcal{S}}_{\mathcal{P}E}(E) \subset \mathbb{C} \) of \( \mathcal{S}_{\mathcal{P}E}(E) \). Then, the filtered bundle is called good, if the corresponding filtered bundle \( \mathcal{P}V \) with an endomorphism \( g \) is good in the sense of [2.5.3].

### 2.4.2 Some remarks on the cohomology

Let \( \mathcal{P}E \) be a filtered bundle on \( (T \times \mathbb{P}, T \times \{ \infty \}) \) satisfying \( (A1) \). Let \( U \) be any \( O_{T \times \mathbb{P}} \)-submodule of \( \mathcal{P}E \) for some \( c \in \mathbb{R} \), such that (i) \( U_{T \times \{ c \}} = \mathcal{P}E_{T \times \{ c \}} \), (ii) \( U_{T \times \{ \infty \}} \) is semistable of degree 0. We give some remarks on the cohomology groups of \( U \).

**Lemma 2.13** Suppose \( 0 \not\in \mathcal{S}_{\mathcal{P}E}(\mathcal{P}E) \). Then, we have \( H^j(T \times \mathbb{P}, U) = H^j(T \times \mathbb{P}, \mathcal{P}E) \).

**Proof** Let \( \pi : T \times \mathbb{P} \to \mathbb{P} \) be the projection. By the assumption, we obtain \( R\pi_* (U \otimes L) \simeq R\pi_* (\mathcal{P}E \otimes L) \), because both of them vanish around \( \infty \). Then, the claim of the lemma follows.

Suppose that \( \mathcal{P}E \) is admissible, i.e., it satisfies \( (A1,2) \), and we give some refinement. We have the decomposition \( U = \bigoplus_{\mathcal{P}} \bigoplus_{p,m,o} U_{\mathcal{P},o}^{(p,m)} \) around \( T \times \{ \infty \} \), where \( U_{\mathcal{P},o}^{(p,m)} := U \cap \mathcal{P}E_{\mathcal{P},o}^{(p,m)} \). Let \( U' \subset \mathcal{P}E \) be a subsheaf satisfying the above conditions (i,ii). If \( 0 \not\in \mathcal{S}_{\mathcal{P}E}(\mathcal{P}E) \), we have \( H^i(T \times \mathbb{P}, U') \simeq H^i(T \times \mathbb{P}, \mathcal{P}E) \) by Lemma 2.13.

**Lemma 2.14** Suppose \( 0 \in \mathcal{S}_{\mathcal{P}E}(\mathcal{P}E) \), and that \( U_{i,0}^{(1,0)} = U_{i,0}^{(1,0)} \). Then, we have natural isomorphisms \( H^i(T \times \mathbb{P}, U) \simeq H^i(T \times \mathbb{P}, U') \) for \( i = 0,2 \).

**Proof** We have only to consider the case that \( U \subset U' \), and we shall prove that the natural morphisms \( H^i(T \times \mathbb{P}, U) \to H^i(T \times \mathbb{P}, U') \) are isomorphisms. Let \( \varphi \in H^0(T \times \mathbb{P}, U') \). Around \( T \times \{ \infty \} \), we have the decomposition \( \varphi = \sum_{p,m,o} \varphi_{\mathcal{P},o}^{(p,m)} \). It is easy to observe that \( \varphi_{\mathcal{P},o}^{(p,m)} \) is zero unless \( (p,m,o) = (0,1,0) \). Hence, we obtain that \( H^0(T \times \mathbb{P}, U) \to H^0(T \times \mathbb{P}, U') \) is an isomorphism. The duals \( U \) and \( (U')^\vee \) are subsheaves of \( \mathcal{P}^c(E^\vee) \) for some \( c' \), and satisfy the conditions (i,ii). Hence, by using the Serre duality, we obtain that \( H^2(T \times \mathbb{P}, U) \to H^2(T \times \mathbb{P}, U') \) is an isomorphism.

### 2.4.3 Vanishing condition

Let \( (\mathcal{P}, E, \theta) \) be a filtered bundle on \( (T \times \mathbb{P}, T \times \{ \infty \}) \). We will be concerned with the following condition on vanishing of the cohomology:

(A3) \( H^0(T \times \mathbb{P}, \mathcal{P}E \otimes p^* L) = 0 \) and \( H^2(T \times \mathbb{P}, \mathcal{P}E \otimes p^* L) = 0 \) for any line bundle \( L \) of degree 0 on \( T \), where \( p \) denotes the projection \( T \times \mathbb{P} \to T \).

We shall often omit to denote \( p^* \), if there is no risk of confusion.

**Lemma 2.15** If \( \mathcal{P}E \) satisfies the condition (A3), the dual \( \mathcal{P}E^\vee \) also satisfies the condition (A3).

**Proof** Note \( \mathcal{P}^c(E^\vee)^\vee \otimes \Omega_{\mathbb{P}} \simeq \mathcal{P}^c_{\leq -1}(E) \). Hence, by the Serre duality, \( H^0(T \times \mathbb{P}, \mathcal{P}^c(E^\vee) \otimes L^\vee) \) is the dual space of \( H^2(T \times \mathbb{P}, \mathcal{P}^c_{\leq -1}(E) \otimes L) \), and \( H^2(T \times \mathbb{P}, \mathcal{P}^c_{\leq -1}(E^\vee) \otimes L^\vee) \) is the dual space of \( H^0(T \times \mathbb{P}, \mathcal{P}^c_{\leq -1}(E^\vee) \otimes L) \). The claim of the lemma follows.

**Lemma 2.16** Let \( L \) be any holomorphic line bundle on \( T \) of degree 0. If \( \mathcal{P}E \) satisfies (A3), we have \( H^0(T \times \mathbb{P}, \mathcal{P}E \otimes L) = 0 \) for any \( c \leq 0 \) and \( H^2(T \times \mathbb{P}, \mathcal{P}E \otimes L) = 0 \) for any \( c \geq -1 \).

**Proof** We have only to consider the case \( L = \mathcal{O}_T \). For \( c \leq 0 \), we have \( H^0(T \times \mathbb{P}, \mathcal{P}E) \subset H^0(T \times \mathbb{P}, \mathcal{P}E) = 0 \). For \( c \geq -1 \), the support of the quotient \( \mathcal{P}^c(E)/\mathcal{P}^c_{\leq -1}(E) \) is one dimensional. Hence, the morphism \( 0 = H^2(T \times \mathbb{P}, \mathcal{P}^c_{\leq -1}(E) \to H^2(T \times \mathbb{P}, \mathcal{P}^c_{\leq -1}(E)) \to H^2(T \times \mathbb{P}, \mathcal{P}^c_{\leq -1}(E)) \) is surjective.
2.4.4 Stability condition

We introduce a stability condition for filtered bundles satisfying (A1) on \((T \times \mathbb{P}^1, T \times \{\infty\})\), by following [3]. Note that this is not the same as the standard slope stability condition for filtered bundles on projective varieties \([37]\).

Let \(\omega_T \in H^2(T \times \mathbb{P}^1, \mathbb{Z})\) denote the pull back of the fundamental class of \(T\) by the projection \(T \times \mathbb{P}^1 \to T\). For any filtered torsion-free sheaf \(\mathcal{P}_* \mathcal{E}\) on \((T \times \mathbb{P}^1, T \times \{\infty\})\), we define the degree of \(\mathcal{P}_* \mathcal{E}\) by

\[
\deg(\mathcal{P}_* \mathcal{E}) := \int_{T \times \mathbb{P}^1} \text{par-c}_1(\mathcal{P}_* \mathcal{E}) \omega_T = \int_{\{z\} \times \mathbb{P}^1} \text{par-c}_1(\mathcal{P}_* \mathcal{E}).
\]

We set \(\mu(\mathcal{P}_* \mathcal{E}) := \deg(\mathcal{P}_* \mathcal{E})/\text{rank} \mathcal{E}\). We say that a filtered bundle \(\mathcal{P}_* \mathcal{E}\) is stable (semistable), if \(\mu(\mathcal{P}_* \mathcal{E}) < \mu(\mathcal{P}_* \mathcal{E})\) (resp. \(\mu(\mathcal{P}_* \mathcal{E}) \leq \mu(\mathcal{P}_* \mathcal{E})\)) for any \(\mathcal{P}_* \mathcal{E} \subset \mathcal{P}_* \mathcal{E}\) such that (i) \(0 < \text{rank} \mathcal{E} < \text{rank} \mathcal{E}\), (ii) \(\mathcal{P}_* \mathcal{E}\) also satisfies (A1) around \(T \times \{\infty\}\). We say that a semistable filtered bundle \(\mathcal{P}_* \mathcal{E}\) is polystable, if it has a decomposition \(\mathcal{P}_* \mathcal{E} = \bigoplus \mathcal{P}_* \mathcal{E}_i\) such that each \(\mathcal{P}_* \mathcal{E}_i\) is stable. The following lemma is clear and standard.

**Lemma 2.17** Let \(\mathcal{P}_* \mathcal{E}\) be a filtered bundle satisfying (A1) on \((T \times \mathbb{P}^1, T \times \{\infty\})\). If \(\mathcal{P}_* \mathcal{E}\) is stable, then \(\mathcal{P}_* \mathcal{E}^{\vee}\) is also stable.

It is standard to obtain the vanishing of some cohomology groups under the assumption of the stability and the degree 0.

**Lemma 2.18** Let \(\mathcal{P}_* \mathcal{E}\) be a filtered bundle satisfying (A1) on \((T \times \mathbb{P}^1, T \times \{\infty\})\). If \(\mathcal{P}_* \mathcal{E}\) is stable with \(\deg(\mathcal{P}_* \mathcal{E}) = 0\) and \(\text{rank} \mathcal{P}_* \mathcal{E} > 1\), it satisfies the condition (A3).

**Proof** Because \(\mathcal{P}_* \mathcal{E}\) is stable of degree 0, we obtain that \(H^0(T \times \mathbb{P}^1, \mathcal{P}_* \mathcal{E}) = 0\) for any \(c \leq 0\). Indeed, a non-zero section induces a filtered strict subsheaf \(\mathcal{P}_* \mathcal{O} \subset \mathcal{P}_* \mathcal{E}\) with \(\deg(\mathcal{P}_* \mathcal{O}) \geq 0\) and \(0 < \text{rank} \mathcal{O} < \text{rank} \mathcal{E}\). Because \((\mathcal{P}_* \mathcal{E})^{\vee}\) is also stable of degree 0, we obtain the vanishing \(H^2(T \times \mathbb{P}^1, \mathcal{P}_* \mathcal{E})\) by using the Serre duality.

**Remark 2.19** Filtered bundles \(\mathcal{P}_* \mathcal{E}\) satisfying (A1) with \(\deg(\mathcal{P}_* \mathcal{E}) = 0\) and \(\text{rank} \mathcal{P}_* \mathcal{E} = 1\) naturally correspond to line bundles of degree 0 on \(T\). Indeed, there exists a line bundle of degree 0 on \(L\) on \(T\) such that \(\mathcal{P}_* \mathcal{E} \simeq p^*L \otimes \mathcal{O}([a](T \times \{\infty\}))\) for each \(a \in \mathbb{R}\), where \([a] := \max\{n \in \mathbb{Z} \mid n \leq a\}\). In this case, (A3) is not satisfied for \(L^{-1}\).

3 Algebraic Nahm transforms

3.1 Local algebraic Nahm transforms

3.1.1 Complex

Let \(X := \{z \in \mathbb{C} \mid |z| < \rho_0\}\) and \(D := \{0\}\). In the following of this subsection, we shall shrink \(X\) without mentioning it. We shall use the notation in [2,3.1]. We define a complex of sheaves associated to an admissible filtered Higgs bundle \((\mathcal{P}_* \mathcal{V}, \theta)\) on \((X, D)\). First, let us consider the case that \((\mathcal{P}_* \mathcal{V}, \theta)\) has type \((p, m, o)\). Suppose \((p, m, o) \neq (1, 0, 0)\). For each \(c \in \mathbb{R}\), let \(\mathcal{P}_c(V \otimes \Omega^\bullet_X, \theta)\) denote the complex

\[
\mathcal{P}_{c-m/p}V \longrightarrow \mathcal{P}_{c+1}V dz
\]

where the first term sits in the degree 0. Take any \(a \in o\). For each \(c \in \mathbb{R}\), let \(\mathcal{P}_c(V^{(p)}_a \otimes \Omega^\bullet_{X(p)}, \theta^{(p)}_a)\) denote the following complex on \(X^{(p)}\):

\[
\mathcal{P}_{c-m}V^{(p)}_a \longrightarrow \mathcal{P}_cV^{(p)}_a \otimes \frac{dz_a}{z_a}.
\]

We have a natural isomorphism \(\mathcal{P}_c(V \otimes \Omega^\bullet_X, \theta) \simeq \varphi_{c,p}\mathcal{P}_c(V^{(p)}_a \otimes \Omega^1_{X(p)}, \theta^{(p)}_a)\). It is also isomorphic to the descent of \(\bigoplus_{\alpha \in o} \mathcal{P}_{c,p}(V^{(p)}_a \otimes \Omega^1_{X(p)}, \theta^{(p)}_a)\). For \(c \leq c'\), the natural inclusion \(\mathcal{P}_{c}(V \otimes \Omega^\bullet_X, \theta) \to \mathcal{P}_{c'}(V \otimes \Omega^\bullet_X, \theta)\) is a quasi-isomorphism. We set \(\mathcal{C}^i(\mathcal{P}_* \mathcal{V}, \theta) := \mathcal{P}_{c+1/2}(V \otimes \Omega^i_X, \theta).\) In the case \((p, m, o) = (1, 0, 0)\), we set \(\mathcal{C}^0(\mathcal{P}_* \mathcal{V}, \theta) := \mathcal{P}_0V\) and

\[
\mathcal{C}^1(\mathcal{P}_* \mathcal{V}, \theta) := \mathcal{P}_{-1/2}V \otimes \Omega^1_X + \theta(\mathcal{P}_0V) \subset \mathcal{P}_1V \otimes \Omega^1_X.
\]
Thus, we obtain the complex $\mathcal{C}^\bullet(P, V, \theta)$, when $(P, V, \theta)$ has type $(p, m, o)$.

For a general admissible filtered Higgs bundle $(P, V, \theta)$, the complex $\mathcal{C}^\bullet(P, V, \theta)$ is defined as the extension of the complex $(V \to V \otimes \Omega^1_X)$ on $X \setminus D$ to a complex on $X$, such that it is $\bigoplus_{(m, p, o)} \mathcal{C}^\bullet(P, V_{o\to(p, m)}, \theta_{o\to(p, m)})$ around $D$, according to the type decomposition.

**Lemma 3.1** If $(P, V, \theta)$ comes from a wild harmonic bundle $(E, \overline{\mathcal{D}}_E, \tau, h)$ on $(X, D)$, then $\mathcal{C}^\bullet(P, V, \theta)$ is naturally quasi-isomorphic to the complex of square-integrable sections of the Higgs complex $E \otimes \Omega^\bullet$.

**Proof** By the descent, we have only to consider the unramified case. We omit to denote $p$. We have the naturally defined map $\pi_c: P_c V \to \text{Gr}_c^p(V)$. Let $W$ be the weight filtration of the nilpotent part of the endomorphism $\text{Res}(\theta)$ on $\text{Gr}_c^p(V)$. We set $W_k P_c V := \pi_c^{-1}(W_k \text{Gr}_c^p(V))$.

Suppose that $(P, V, \theta)$ has type $(m, o)$. If $(m, o) \neq (0, 0)$, let $C^\bullet_{L_2}(P, V, \theta)$ be the following complex:

$$W_{-2} P_{-m} V \to W_{-2} P_0 V \otimes \Omega_X^1(\log D)$$

It is easy to check that the following natural morphisms are quasi-isomorphisms:

$$C^\bullet_{L_2}(P, V, \theta) \to P_0 (V \otimes \Omega^\bullet, \theta) \leftarrow C^\bullet(P, V, \theta)$$

If $(m, o) = (0, 0)$, let $C^\bullet_{L_3}(P, V, \theta)$ be the following complex:

$$W_0 P_0 V \to W_{-2} P_0 V \otimes \Omega_X^1(\log D)$$

It is easy to check that the natural inclusion $C^\bullet_{L_2}(P, V, \theta) \to C^\bullet(P, V, \theta)$ is a quasi-isomorphism.

In general, we define $C^\bullet_{L_2}(P, V, \theta) = \bigoplus C^\bullet_{L_3}(P, V_{o\to(m)}, \theta_{o\to(m)})$ by using the type decomposition. According to the result in §5.1 of [40], $C^\bullet_{L_2}(P, V, \theta)$ is naturally quasi-isomorphic to the complex of square-integrable sections of the Higgs complex $E \otimes \Omega^\bullet$. Thus, we obtain the claim of the lemma.

3.1.2 Transform

We shall construct some transformations for filtered Higgs bundles, which are analogue to the local Fourier transform in [15] and [22].

In the following, for a variable $x$, let $U_x$ denote a small neighbourhood of 0 in $\mathbb{C}_x$. For two variables $x$ and $y$, let $U_{x,y} := U_x \times U_y$, and let $\pi_1: U_{x,y} \to U_x$ and $\pi_2: U_{x,y} \to U_y$ denote the projections.

Let $(P, V, \theta)$ be an admissible filtered Higgs bundle on $(U_\zeta, 0)$. Let us define a filtered bundle $\mathcal{N}^0_{\zeta, \tau}(P, V, \theta)$ with an endomorphism $g$ on $U_\tau$. We consider the following complex on $U_{\zeta, \tau}$:

$$\pi_1^* C^0(P, V, \theta) \xrightarrow{\tau \theta + d\zeta} \pi_1^* C^1(P, V, \theta)$$

Let $Q$ be the quotient. We define

$$\mathcal{N}^0_{\zeta, \tau}(P, V, \theta) := \pi_2_* Q, \quad \mathcal{N}^0_{\zeta, \tau}(P, V, \theta) := \mathcal{N}^0_{\zeta, \tau}(P, V, \theta)(\ast \tau).$$

Here, $(\ast \tau)$ means the localization with respect to $\tau$. If $U_\tau$ is sufficiently small, then the support of $Q$ is proper and relatively 0-dimensional over $U_\tau$. Indeed, $Q \cap \{(0, 0) \times \zeta\} = \{(0, 0)\}$. Hence, $\mathcal{N}^0_{\zeta, \tau}(P, V, \theta)$ is coherent. Let us check that $\mathcal{N}^0_{\zeta, \tau}(P, V, \theta)$ is torsion-free. Let $v$ be a section of $\pi_1^* C^1(P, V, \theta)$, such that there exists a section $u$ of $\pi_1^* C^0(P, V, \theta)$ satisfying $\tau v = (\tau \theta + d \zeta) u$. We obtain that $d \zeta \cdot u$ is contained in $\tau \pi_1^* C^1(P, V, \theta)$.

Then, we obtain that $u = \tau u'$ for some section $u'$ of $\pi_1^* C^0(P, V, \theta)$, and we have $v = (\tau \theta + d \zeta) u'$. It implies that $\mathcal{N}^0_{\zeta, \tau}(P, V, \theta)$ is torsion-free. Hence, $\mathcal{N}^0_{\zeta, \tau}(P, V, \theta)$ is a locally free $O_{U_{\zeta, \tau}}$-module. In particular, $\mathcal{N}^0_{\zeta, \tau}(P, V, \theta)$ is a locally free $O_{U_{\zeta, \tau}}(\ast \tau)$-module. The multiplication of $\zeta$ induces the endomorphism $g$. By setting $\psi := -g \tau^{-2} d\tau$, we obtain a Higgs field of $\mathcal{N}^0_{\zeta, \tau}(P, V, \theta)$. We shall introduce a filtered bundle $\mathcal{N}^0_{\zeta, \tau}(P, V, \theta) = (\mathcal{N}^0_{\zeta, \tau}(P, V, \theta) | a \in \mathbb{R})$ over $\mathcal{N}^0_{\zeta, \tau}(P, V, \theta)$.

If $(P, V, \theta)$ has type $(p, m, o) \neq (1, 0, 0)$, we consider the following complexes on $U_{\zeta, \tau}$ for any $c \in \mathbb{R}$:

$$\pi_1^* P_{c-m/p}(V) \xrightarrow{\tau \theta + d\zeta} \pi_1^* P_c(V)(d \zeta / \zeta)$$

(7)
Let $Q_c$ denote the quotient. We define

$$\mathcal{N}^0_{\kappa(p, m, c)}(P, \theta) := \pi_2^* Q_c, \quad \kappa_1(p, m, c) := \frac{2pc - m}{2(p + m)}$$

By construction, we have $\mathcal{N}^0_{\kappa_1/2}(P, \theta) = \mathcal{N}(P, \theta)$ in this case. It is easy to check that $\mathcal{N}^0\mathcal{N}(P, \theta)$ are locally free $O_U$-module of finite rank. We have a naturally induced map $\mathcal{N}^0\mathcal{N}(P, \theta) \to \mathcal{N}^0\mathcal{N}(P, \theta)$. For $a' \leq a$. Its restriction to $\{ \tau \neq 0 \}$ is an isomorphism, and hence is injective. We also obtain $\mathcal{N}^0\mathcal{N}(P, \theta)(\ast \tau) = \mathcal{N}^0\mathcal{N}(P, \theta)$. For $c' := c - (1 + m/p)$, the images of $\tau \cdot \pi_1^* P_c V(\rho / \zeta)$ and $\pi_1^* P_c V(\rho / \zeta)$ are the same in the quotient of $Q_c$. It implies $\tau \mathcal{N}^0\mathcal{N}(P, \theta) = \mathcal{N}^0_{\kappa_1}(P, \theta)$ for any $a \in \mathbb{R}$. Hence, $\mathcal{N}^0\mathcal{N}(P, \theta)$ give a filtered bundle over $\mathcal{N}^0\mathcal{N}(P, \theta)$.

If $(P, \theta)$ has type $(p, m, o) = (1, 0, 0)$, we define $\mathcal{N}^0_{o, \infty}(P, \theta) := \mathcal{N}^0\mathcal{N}(P, \theta)$. We have the following natural morphisms:

$$\mathcal{N}^0_{o, \infty}(P, \theta)|_{o} \simeq \mathcal{C}^1(P, \theta)/\mathcal{C}^0(P, \theta) d\zeta \to (P_0 V)|_o$$

Here, the subscript "o" means the fiber of the vector bundle over 0, and the latter map is given by the residue, which is injective. Hence, the parabolic filtration of the right hand side induces a parabolic filtration of $\mathcal{N}^0_{o, \infty}(P, \theta)|_{o}$ indexed by $[-1, 0]$. It induces a filtered bundle $\mathcal{N}^0_{o, \infty}(P, \theta)$ over $\mathcal{N}^0_{o, \infty}(P, \theta)$.

If $(P, \theta)$ is admissible, we replace $U_\zeta$ with smaller neighbourhoods so that it has the type decomposition, and we define

$$\mathcal{N}^0_{o, \infty}(P, \theta) := \bigoplus_{p, m, o} \mathcal{N}^0_{o, \infty}(P, V^{(p, m, o)}).$$

The construction $\mathcal{N}^0_{o, \infty}$ gives a functor from the category of the germs of admissible filtered Higgs bundles to the category of the germs of filtered Higgs bundles. We set $\mathcal{N}^0_{o, \infty}(P, \theta) := \sum_{b < c} \mathcal{N}^0_{o, \infty}(P, \theta)$.

**Lemma 3.2** Suppose $(P, \theta)$ has type $(p, m, o)$. The rank of $\mathcal{N}^0_{o, \infty}(P, \theta)$ is $(p + m)$ rank $V/p$ in the case $(p, m, o) \neq (1, 0, 0)$, or rank $V_\theta$ minus Ker $G_{P_\theta}^0(\theta)$ in the case $(p, m, o) = (1, 0, 0)$.

**Proof** The rank is equal to the dimension of $\mathcal{C}^1(P, \theta)/\mathcal{C}^0(P, \theta) d\zeta$ as a $C$-vector space. Then, the claim can be checked by a direct computation. (See also the proof of Proposition 3.3 below for the case $(p, m, o) \neq (1, 0, 0)$.)

**Proposition 3.3** $(\mathcal{N}^0_{o, \infty}(P, \theta), \psi)$ is admissible. If $(P, \theta)$ has type $(p, m, o)$, then $(\mathcal{N}^0_{o, \infty}(P, \theta), \psi)$ has type $(p + m, m', o')$ for some $o'$.

**Proof** We have only to consider the case that $(P, \theta)$ has type $(p, m, o)$. Let us consider the case $(p, m, o) = (1, 0, 0)$. For the expression $\theta = f d\zeta / \zeta$, $f$ gives an endomorphism of $P_c V$ for any $c$, and $f_0$ is nilpotent. We have $\psi = -\tau^{-1} g(d\zeta / \zeta)$, and $-\tau^{-1} g$ is induced by $f$ by construction. Hence, it preserves $\mathcal{N}^0_{o, \infty}(P, \theta)$. If we regard $\mathcal{N}^0_{o, \infty}(P, \theta)|_{o}$ as a subspace of $\mathcal{P}_0 V|_{o}$ above, $-\tau^{-1} g|_{o}$ is the restriction of $f_0|_{o}$. Hence, it is nilpotent, and it preserves the parabolic filtration, i.e., $(\mathcal{N}^0_{o, \infty}(P, \theta), \psi)$ is admissible of type $(1, 0, 0)$.

Let us consider the case $(p, m, o) \neq (1, 0, 0)$. Fix $\alpha \in o$. We consider the following on $U_{\zeta_{p, \tau}}$:

$$\pi_1^* P_{V^{(p)}} \frac{\tau\theta_{\beta} + d\zeta}{d\zeta} \pi_1^* P_{V^{(p)}} (d\zeta / \zeta)$$

The quotient is denoted by $Q'_{\zeta}$. We have a natural isomorphism $\pi_2^* Q'_{\zeta} \simeq \mathcal{N}^0_{\kappa_1}(P, \theta)$. The natural map $Q'_{\zeta} \to Q'_{\zeta}$ for $c' \leq c$ is injective. We set $Q'_{\zeta} := \bigcup_{b < c} Q'_{\zeta}$. We have the following exact sequence:

$$0 \to \pi_1^* \mathcal{G}_{V^{(p)}}^{c} \frac{\theta_{\beta} + d\zeta}{d\zeta} \to \pi_1^* \mathcal{G}_{V^{(p)}}^{c} (d\zeta / \zeta) \to Q'_{\zeta} / Q'_{\zeta} \to 0$$

It induces the following isomorphism of $C$-vector space for any $c \in \mathbb{R}$:

$$\mathcal{G}_{V^{(p)}}^{c} \simeq \frac{\mathcal{N}^0_{\kappa_1}(P, \theta)}{\mathcal{N}^0_{\kappa_1}(P, \theta)}$$

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Let $v = (v_i)$ be a frame of $\mathcal{P}_{pc}V^{(p)}_\alpha$. We set $c_i := \min\{ a \in \mathbb{R} | v_i \in \mathcal{P}_a V^{(p)} \}$. We assume that $v$ is compatible with the parabolic structure in the sense that the induced tuple $\{ [v_i] | c_i = d \}$ of elements in $Gr_d^P(V^{(p)}_\alpha)$ is a base for any $d \in [pc - 1, pc]$. We set $\nu_i := \zeta_p^j v_j (d_{sp}/\zeta_p)$ for $(0 \leq i \leq p + m - 1, 1 \leq j \leq \text{rank } V^{(p)}_\alpha)$. The induced sections of $\mathcal{N}_{k_1(p,m,c)}(\mathcal{P}_V, \theta)$ are also denoted by the same symbols. Because they induce a base of $\mathcal{N}_{k_1(p,m,c)}^0(\mathcal{P}_V, \theta)/\mathcal{T} \mathcal{N}_{k_1(p,m,c)}^0(\mathcal{P}_V, \theta)$, they give a frame of $\mathcal{N}_{k_1(p,m,c)}^0(\mathcal{P}_V, \theta)$ on a neighbourhood of $0$. (In particular, the rank of $\mathcal{N}_{k_1(p,m,c)}^0(\mathcal{P}_V, \theta)$ is $(p + m) \text{rank } V^{(p)}_\alpha = p^{-1}(p + m) \text{rank } V$.) Moreover, by the isomorphism $\mathbb{G}$, the frame is compatible with the parabolic structure of $\mathcal{N}_{k_1(p,m,c)}^0(\mathcal{P}_V, \theta)$.

We take a ramified covering $\varphi : U_{\eta} \rightarrow U_{\tau}$ by $\varphi(\eta) = \eta^{p + m}$. Let $\mathcal{P}_\tau V$ be the filtered bundle on $(U_{\tau}, 0)$ obtained as the pull back of $\mathcal{N}_{k_1(p,m,c)}^0(\mathcal{P}_V, \theta)$ by $\varphi$. The tuple of the sections $\tilde{\nu}_j := \eta^{-i} \varphi^* \nu_j$ gives a frame $\tilde{v}$ of $\mathcal{P}_{pc - m/2} V$ which is compatible with the parabolic structure. By the frames $v$ and $\tilde{v}$, we obtain an isomorphism of $\mathcal{P}_{pc - m/2} V^{(p)} |_0$ to $\mathcal{P}_{pc} V^{(p)}(0) \otimes \mathbb{C}^{p + m}$.

Let us show that $\psi = -\tau^{-2} g d\tau$ has type $(p + m, m')$ for some $m'$. Note that $g$ is induced by the multiplication of $\zeta = \zeta_p^p$. Let $g_1$ be the endomorphism of $\mathcal{N}_{k_1(p,m,c)}^0(\mathcal{P}_V, \theta)$ induced by the multiplication of $\zeta$. We have $g_1(\mathcal{N}_{k_1(p,m,c)}^0(\mathcal{P}_V, \theta)) \subset \mathcal{N}_{k_1(p,m,c)}^0(\mathcal{P}_V, \theta)$. Hence, we obtain $\eta^{-1} g_1$ gives an endomorphism of $\mathcal{P}_{pc - m/2} V$. In particular, $\eta^{-p} g$ gives an endomorphism of $\mathcal{P}_{pc - m/2} V$. Let us show that the restriction $(\eta^{-p} g) |_0$ has a unique non-zero eigenvalue modulo the action of $\text{Gal}(\varphi)$.

We have the parabolic filtration $F$ of $\mathcal{P}_{pc} V^{(p)} |_0$, indexed by $|pc - 1, pc|$. Let $W$ denote the monodromy weight filtration of the nilpotent part of $\text{Res}(\zeta_p^p \theta_a^{(p)})$ on $Gr^F(\mathcal{P}_{pc} V^{(p)} |_0)$. Let $\pi_a : F_a(\mathcal{P}_{pc} V^{(p)} |_0) \rightarrow Gr^F_a(\mathcal{P}_{pc} V^{(p)} |_0)$ denote the projection. Let $M := \min\{ |a - b| | a, b \in \text{Par}(\mathcal{P}_{pc} V^{(p)}), a \neq b \}$. We take a small positive number $\delta$ such that $\delta \text{rank } \mathcal{P}_{pc} V^{(p)} < M/100$. We set $\bar{F}_{a + \delta} := \pi_a^{-1}(W_k)$. Then, we obtain a filtration $\bar{F}$ of $\mathcal{P}_{pc} V^{(p)} |_0$ indexed by $|pc - 1 + \epsilon, pc + \delta|$ for some small $\epsilon > 0$. Then, $\bar{F}$ is preserved by $\text{Res}(\zeta_p^p \theta_a^{(p)})$, and the induced endomorphism on the associated graded space $Gr_{\bar{F}}$ is semisimple. We may assume that the frame $v$ is compatible with $\bar{F}$.

Let $\bar{F}'$ be a filtration of $\mathcal{P}_{pc - m/2} V^{(p)} |_0 \simeq \mathcal{P}_{pc} V^{(p)} |_0 \otimes \mathbb{C}^{p + m}$ indexed by $|pc - 1 + \delta, pc + \delta|$, determined by the condition $\text{deg}_{\bar{F}'} (\tilde{v}_j) = \text{deg}_{\bar{F}} (v_j)$. The multiplication of $\eta^{-1} \zeta_p$ induces an endomorphism of $\mathcal{P}_{pc - m/2} V$. We have $(\eta^{-1} \zeta_p) \tilde{\nu}_j = \tilde{\nu}_{i + j}$ for $i < p + m - 1$, and $(\eta^{-1} \zeta_p) \tilde{\nu}_{p + m - 1, j}$ is equal to the section $s$ induced by

$$p^{-1} \theta_a^{(p)} (\zeta_p^{m} v_j) = \left( -\frac{\alpha}{p} v_j + \sum_{\text{deg}^F (v_j) < \text{deg}^F (v_j)} \gamma_k \cdot v_k + \zeta_p u \right) (d_{sp}/\zeta_p).$$

Here, $\gamma_k$ are complex numbers, and $u$ is a section of $\mathcal{P}_{pc} V^{(p)}$. If $\text{deg}_{\bar{F}} (v_j |_0) = a$, then $s_{j0} + (\alpha/p) \tilde{\nu}_{0, j} |_0 \in \bar{F}_{\zeta_p^a}$. The endomorphism $\eta^{-1} g$ of $\mathcal{P}_{pc - m/2} V$ is induced by the multiplication of the $p$-th power of $\eta^{-1} \zeta_p$. Hence, $(\eta^{-p} g) |_0$ is compatible with $\bar{F}'$, and the induced endomorphism on $Gr_{\bar{F}'}$ is represented by the following matrix:

$$\sum_{i=1}^{m} I \otimes E_{p+i,i} + \sum_{i=1}^{p} -\frac{\alpha}{p} I \otimes E_{i,m+i}$$

Here, $I$ is the identity matrix and $E_{i,j}$ denote the $(p + m)$-square matrix whose $(k, \ell)$-entry is $1$ if $(k, \ell) = (i, j)$, and $0$ otherwise. Then, the set of the eigenvalues are $\epsilon^{2\pi \sqrt{-1}/(p+m)} \alpha^p$ $(j = 0, \ldots, p + m - 1)$. Thus, we are done.

**Corollary 3.4** The construction $\mathcal{N}_{k_1(p,m,c)}^0$ gives a functor from the category of the germs of admissible filtered Higgs bundles to the category of the germs of admissible filtered Higgs bundles whose slopes are strictly less than 1.

**3.1.3 Inverse transform**

Let $\mathcal{P}_\tau V$ be a filtered bundle on $(U_{\tau}, 0)$ with an endomorphism $g$, which is admissible in the sense of $\overset{2.5.3}{\text{ }}$. In this subsection, we impose the following vanishing:
(C0) \( V^{(1,0)}_0 = 0 \) and \( V^{(p,m)} = 0 \) unless \( p > m \).

Note that the eigenvalues of \( g(\tau) \) goes to 0 when \( \tau \to 0 \) under the assumption (C0).

If \( (P, V, g) \) has slope \( (p, m) \), we consider the following complex on \( U_{c, \zeta} \):

\[
\pi_1^* P_c V \xrightarrow{g - \zeta} \pi_1^* P_c V
\]

The quotient is denoted by \( \mathcal{M}_c \). If \( U_{c, \zeta} \) is sufficiently small, the support of \( \mathcal{M}_c \) is proper over \( U_{c, \zeta} \). We define

\[
N_{a, p, m, c} := \pi_2^* \mathcal{M}_c, \quad \kappa_2(p, m, c) := \frac{2pc + m}{2(p - m)}.
\]

They are locally free \( \mathcal{O}_{U_{c, \zeta}} \)-modules. For \( a \leq a' \), we have a naturally defined map \( N_{a, p, m, c} \to N_{a', p, m, c} \) which induces \( N_{a, p, m, c} \to N_{a', p, m, c} \) via \( \zeta \). We have \( N_{a, p, m, c} = \zeta N_{a', p, m, c} \) for any \( a \in \mathbb{R} \).

Thus, we obtain a filtered bundle \( N_{a, p, m, c} \) on \( (U_{c, \zeta}, 0) \). In the general case, we define \( N_{a, p, m, c} := \bigoplus N_{a, p, m, c} \) by using the slope decomposition of \( (P, V, g) \). The multiplication of \( -\tau^2 \) for some \( \tau \), the rank is \( (p - m) \) rank \( V/p \).

**Proposition 3.5** \( (N_{a, p, m, c}, \theta) \) is admissible. If \( (P, V, g) \) has type \( (p, m, a) \), then \( N_{a, p, m, c} \) has type \( (p - m, m, a') \) for some \( a' \) and the rank is \( (p - m) \) rank \( V/p \).

**Proof** We have only to consider the case that \( (P, V, g) \) has type \( (p, m, a) \). Let \( \varphi_p : U_{\eta} \to U_{\zeta} \) be given by \( \varphi_p = u \) for some \( u \). We use the decomposition \( \varphi_p = g \) if \( \varphi_p \) is admissible. If \( \varphi_p \) is admissible, then \( \varphi_p \) is a \( \mathbb{C} \)-vector bundle.

We use the decomposition \( \varphi_p = g \). We consider the following complex on \( U_{\eta, \zeta} \):

\[
\pi_1^* P_{pc} V^{(p)} = \bigoplus_{a \in \mathbb{R}} (P_{\alpha} V^{(p)}) \xrightarrow{g^{(p)} - \zeta} \pi_1^* P_{pc} V^{(p)}
\]

The quotient is denoted by \( \mathcal{M}'_c \). We have \( \pi_2^* \mathcal{M}'_c \simeq N_{a, p, m, c} \) for any \( a \in \mathbb{R} \). We have the following exact sequence, as in the case of \( N_0, -\infty \) (see the proof of Proposition 3.3):

\[
0 \longrightarrow \pi_1^* \operatorname{Gr} P_{pc} V^{(p)} \longrightarrow \varphi_p \alpha^* \operatorname{Gr} P_{pc} V^{(p)} \longrightarrow \mathcal{M}'_c \longrightarrow 0
\]

It induces the following isomorphism of \( \mathbb{C} \)-vector spaces:

\[
G_{I_p} (V^{(p)}) \simeq \frac{N_{a, p, m, c} \oplus (P, V, g)}{N_{a, p, m, c} \oplus (P, V, g)}
\]

We take a frame \( \mathbf{v} \) on \( P_{pc} V^{(p)} \) compatible with the parabolic structure. We set \( \mu_{ij} := \eta^i \nu_{ij} \). By the isomorphism (10), we induce a frame of \( N_{a, p, m, c} \) on \( P_{pc} V^{(p)} \) compatible with the parabolic structure. We set \( \tilde{\nu}_{ij} := u^{-1} \eta^i \nu_{ij} \). The tuple \( \tilde{\nu} \) induces a frame of \( P_{pc} V^{(p)} \) on \( (U_{\eta, \zeta}, 0) \) compatible with the parabolic structure.

We consider the endomorphism \( h := \eta^{-p} u \) on \( P_{pc} V^{(p)} \), which is invertible. We have \( \eta^{-p+m} u^{-m} = h \) on \( V \). Let \( k \) be the integer determined by the condition \( 0 \leq -p + k(p - m) < p - m \). We set \( a := -p + k(p - m) \). We have \( \eta^{-p+m} u^{-m} = h^{-k-1} \). We have

\[
u^{a}_{ij} := \left\{ \begin{array}{ll}
\eta^{a+i} u^{-(a+i)} h^{k} (\nu_{ij}) & (a + i < p - m) \\
\eta^{a+i-(p-m)} u^{-(a+i)+p-m} h^{k-1} (\nu_{ij}) & (a + i \geq p - m)
\end{array} \right.
\]

Hence, \( \nu^{-p} \eta^{-p} \) preserves \( P_{pc} V^{(p)} \).

By the frames \( \mathbf{v} \) and \( \tilde{\nu} \), we have an isomorphism \( P_{pc} V^{(p)} \otimes \mathbb{C}^{p-m} \). We take a refinement \( \tilde{F} \) of the parabolic filtration of \( P_{pc} V^{(p)} \) such that (i) \( \tilde{F} \) is preserved by \( h \), (ii) the induced endomorphism on
Gr$^\beta$ is semisimple with a unique eigenvalue $\beta$. It induces a filtration $\tilde{F}'$ of $P_{p(c+1)-m/2}V_0$. (See the proof of Proposition 3.3 for a concrete construction.) On Gr$^\tilde{F}'$, $w^{p-n}p$ is expressed by the matrix

$$\sum E_{n+i} \otimes \beta^k I + \sum E_{i+p-m-n} \otimes \beta^{k-1} I$$

with respect to an appropriate base. Then, we obtain that $(P_*N^{\infty,0}(P_*V, g), \theta)$ has type $(p-m, m', \alpha')$ for some $\alpha'$.

**Corollary 3.6** The construction $N^{\infty,0}_*$ gives a functor from the category of the germs of admissible filtered Higgs bundles satisfying (C0) to the category of the germs of admissible filtered Higgs bundles.

We denote $(N^{0,\infty}_*(P_*V, \theta), g)$ in [3.1.2] by $N^{0,\infty}_*(P_*V, \theta)$ for simplicity. We also denote $(N^{\infty,0}_*(P_*V, g), \theta)$ by $N^{\infty,0}_*(P_*V, g)$.

**Proposition 3.7**

- Suppose that $(P_*V, \theta)$ is admissible such that $V_0^{(1,0)} = 0$ in the type decomposition. Then, we have a natural isomorphism of germs of filtered Higgs bundles $N^{\infty,0}_*N^{0,\infty}_*(P_*V, \theta) \simeq (P_*V, \theta)$.

- Suppose that $(P_*V, g)$ is admissible and satisfies the condition (C0). Then, we have a natural isomorphism of germs of filtered bundles with endomorphisms $N^{0,\infty}_*N^{\infty,0}_*(P_*V, g) \simeq (P_*V, g)$.

**Proof** Suppose that $(P_*V, \theta)$ has type $(p, m)$. Note that, if we set $d := \kappa_1(p, m, c)$, then we have $\kappa_2(p + m, m, d) = c$. Let $p_1$ be the projection of $U_\zeta \times U_\tau \times U_{\zeta'}$ onto the $i$-th component. We have the following diagram on $U_\zeta \times U_\tau \times U_{\zeta'}$:

$$\begin{array}{c}
p_1^*P_{c-m/p}(V) \quad \tau^\theta + d\zeta \quad p_1^*P_{c}(V)d\zeta/\zeta \\
\downarrow \zeta - c' \downarrow \zeta - c' \\
p_1^*P_{c-m/p}(V) \quad \tau^\theta + d\zeta \\
\downarrow p_1^*P_{c-m/p}(V) \quad \tau^\theta + d\zeta \\
\end{array}$$

We regard it as a double complex, where the left upper $p_1^*P_{c-m/p}(V)$ sits in the degree $(0, 0)$. Let $C^*$ denote the associated total complex. By the construction, $N^{c+1}_*N^{c}_*(P_*V, \theta)$ is obtained as $p_3H^2(C^*)$. We can observe that it is isomorphic to the push-forward of $Q_c$ in [3.1.2] by the projection $U_\zeta \longrightarrow U_\zeta$, which is naturally isomorphic to $P_cV d\zeta/\zeta \simeq P_{c+1}V$. The action of $-\tau^{-1}$ is equal to $f$ for the expression $\theta = f d\zeta$. Hence, we obtain the desired isomorphism $N^{\infty}_*N^{\infty}_*(P_*V, \theta) \simeq (P_*V, \theta)$.

Suppose that $(P_*V, g)$ has type $(p, m)$ with $p > m$. Let $p_1$ denote the projection of $U_\tau \times U_\zeta \times U_{\zeta'}$ onto the $i$-th component. We have the following commutative diagram of the sheaves on $U_\tau \times U_\zeta \times U_{\zeta'}$:

$$\begin{array}{c}
p_1^*P_{c-1}V \quad g-d\zeta \quad p_1^*P_{c-1}V \\
\downarrow (-\tau^\tau(d+1)) \downarrow (-\tau^\tau(d+1))d\zeta \\
p_1^*P_{c}V d\zeta \quad g-d\zeta \\
\downarrow p_1^*P_{c}V d\zeta \\
\end{array}$$

We regard it as the double complex, where the left upper $p_1^*P_{c-1}V$ sits in the degree $(0, 0)$. Let $C^*$ denote the associated total complex. By the construction, $N^{0,\infty}_*N^{\infty,0}_*(P_*V, g)$ is naturally isomorphic to $p_3H^2(C^*)$. We can observe that it is naturally isomorphic to the push-forward of $M_c$ in [3.1.3] by the projection $U_\tau \times U_\zeta \longrightarrow U_\tau$, which is naturally isomorphic to $P_cV$. The action of $\zeta$ is given by $g$. Hence, we obtain the desired isomorphism $N^{0,\infty}_*N^{\infty,0}_*(P_*V, g) \simeq (P_*V, g)$.

**3.1.4 Description of the functors**

Let $(P_*V, \theta)$ be a filtered Higgs bundle with slope $(p, m) \neq (1, 0)$ on $U_\zeta$. Suppose that there exist a ramified covering $\varphi_q : U_{\zeta_q} \longrightarrow U_\zeta$ and a filtered Higgs bundle $(P_*V', \theta')$ on $U_{\zeta_q}$ with an isomorphism $\varphi_{q*}(P_*V', \theta') \simeq (P_*V, \theta)$. For $c \in \mathbb{R}$, we consider the following morphism on $U_{\zeta_q, \tau'}$:

$$\begin{array}{c}
P_{q(c-m/p)}V' \quad \tau^\theta + d\zeta_q \quad P_{q}V' d\zeta_q/\zeta_q \\
\end{array}$$
The quotient is denoted by $Q'_c$. The following lemma is clear by construction.

**Lemma 3.8** $\pi_2\mathcal{O}'_c$ is naturally isomorphic to $\mathcal{N}^{0,\infty}_{k_1(p,m,c)}(\mathcal{P}_eV,\theta)$.

Let $(\mathcal{P}_eV,\psi)$ be a filtered Higgs bundle with slope $(p,m)$ on $U_\tau$, such that $(p,m) \neq (1,0)$ and $p > m$. Suppose that there exist a ramified covering $\varphi_q : U_{\tau_q} \to U_\tau$ and a filtered Higgs bundle $(\mathcal{P}_eV',\psi')$ on $U_{\tau_q}$ with an isomorphism $\varphi_{q*}(\mathcal{P}_eV',\psi') \simeq (\mathcal{P}_eV,\psi)$. Let $\psi' = g' \varphi^*(-\tau^{-2} d\tau)$. For $c \in \mathbb{R}$, we consider the following morphism on $U_{\tau_q,\xi}$:

$$\mathcal{P}_{q*}V' \xrightarrow{g'-\xi} \mathcal{P}_{q*}V'$$

Let $M'_c$ denote the quotient. The following is clear by construction.

**Lemma 3.9** $\pi_2\mathcal{M}'_c$ is naturally isomorphic to $\mathcal{N}^{\infty,0}_{k_2(p,m,c)+1}(\mathcal{P}_eV,\theta)$.

### 3.2 Algebraic Nahm transform for admissible filtered Higgs bundle

#### 3.2.1 Construction of the transform

Let $T' := \mathbb{C}/L'$. Let $D \subset T'$ be an effective reduced divisor. Let $(\mathcal{P}_eE,\theta)$ be a filtered Higgs bundle on $(T',D)$. Suppose that $E$ is admissible around each point of $D$ in the sense of §23.1. We shall construct a filtered bundle Nahm $(\mathcal{P}_eE,\theta)$ on $(T \times \mathbb{P}^1, T \times \{\infty\})$. We begin with a construction of an object $N(\mathcal{P}_eE,\theta)$ in $D^b(\mathcal{O}_T^{\mathbb{P}^1})$.

For $I \subset \{1,2,3\}$, let $p_I$ be the projections of $T' \times T \times \mathbb{P}^1$ onto the product of the $i$-th components ($i \in I$). Let $\mathcal{P}oin$ be the Poincaré bundle on $T' \times T$. Applying the construction in §3.1.1 around each point of $D$, we extend $E$ and $E \otimes \Omega^1_X$ on $X \setminus D$ to $\mathcal{O}_T^{\mathbb{P}^1}(\mathcal{P}_eE,\theta)$ and $\mathcal{O}_T^{\mathbb{P}^1}(\mathcal{P}_eE,\theta)$, respectively. We set

$$\tilde{C}^0(\mathcal{P}_eE,\theta) := p_1^*C^0(\mathcal{P}_eE,\theta) \otimes p_{12}^*\mathcal{P}oin \otimes p_3^*\mathcal{O}_{\mathbb{P}^1}(-1), \quad \tilde{C}^1(\mathcal{P}_eE,\theta) := p_1^*C^1(\mathcal{P}_eE,\theta) \otimes p_{12}^*\mathcal{P}oin.$$

Let $\zeta$ be the standard coordinate of $\mathbb{C}$, which induces local coordinates of $T'$. We have the holomorphic 1-form $d\zeta$ on $T'$. Let $w$ be the standard coordinate of $\mathbb{C} \subset \mathbb{P}^1$, which we can naturally regard as a section of $\mathcal{O}_{\mathbb{P}^1}(1)$. Then, we have the following morphism:

$$\theta + w d\zeta : \tilde{C}^0(\mathcal{P}_eE,\theta) \longrightarrow \tilde{C}^1(\mathcal{P}_eE,\theta). \quad (11)$$

Thus, we obtain a complex $\tilde{C}^\bullet(\mathcal{P}_eE,\theta)$ on $T' \times T \times \mathbb{P}^1$. We define

$$N(\mathcal{P}_eE,\theta) := Rp_{23*}(\tilde{C}^\bullet(\mathcal{P}_eE,\theta))[1].$$

**Lemma 3.10** There exists a neighbourhood $U$ of $\infty$ in $\mathbb{P}^1$, such that $\mathcal{H}^i(N(\mathcal{P}_eE,\theta))_{|T \times U} = 0$ unless $i \neq 0$. Moreover, $\mathcal{H}^0(N(\mathcal{P}_eE,\theta))_{|T \times \{P\}}$ are semistable bundles of degree 0 for any $P \in U$.

**Proof** Let $\pi_i$ denote the projection of $T' \times \mathbb{P}^1$ onto the $i$-th component. We have the following complex $\tilde{C}^\bullet(\mathcal{P}_eE,\theta)$ on $T' \times \mathbb{P}^1$:

$$\pi_1^*\tilde{C}^0(\mathcal{P}_eE,\theta) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\theta + w d\zeta} \pi_1^*\tilde{C}^1(\mathcal{P}_eE,\theta)$$

By the construction, $N(\mathcal{P}_eE,\theta)$ is isomorphic to $\widehat{RFM}_+(\tilde{C}^\bullet(\mathcal{P}_eE,\theta))[1]$. If $U$ is sufficiently small, $\theta + w d\zeta$ is injective on $T' \times U$, and the support of the cokernel is relatively 0-dimensional over $U$. Then, the claim of the lemma follows.

We consider the following vanishing condition.

**A0** $H^i(T', \tilde{C}^\bullet(\mathcal{P}_eE \otimes L, \theta + wd\zeta)) = 0$ unless $i = 1$, for any $w \in \mathbb{C}$ and any holomorphic line bundle $L$ of degree 0 on $T'$. 

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Under the assumption (A0), we naturally identify $N(\mathcal{P}_*\mathcal{E}, \theta)$ with the 0-th cohomology sheaf $H^0(N(\mathcal{P}_*\mathcal{E}, \theta))$, which is a locally free sheaf on $T \times \mathbb{P}^1$. Indeed, $\mathcal{C}^*(\mathcal{P}_*\mathcal{E} \otimes L, \theta + w d\zeta)$ is naturally identified with the specialization of $\mathcal{C}^*(\mathcal{P}_*\mathcal{E}, \theta)$ to $T^\vee \times \{ (L, w) \}$. Note that we always have $\mathbb{H}^i(T^\vee, \mathcal{C}^*(\mathcal{P}_*\mathcal{E} \otimes L, d\zeta)) = 0$ unless $i = 1$, for any $L$, which corresponds to the specialization at $w = \infty$. We define

$$\text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta) := N(\mathcal{P}_*\mathcal{E}, \theta) \otimes O_{T \times \mathbb{P}^1}((T \times \{ \infty \})).$$

We shall define a filtered bundle $\text{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta)$ over $\text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta)$.

By Lemma 3.10 there exists a neighbourhood $U$ of $\infty \in \mathbb{P}^1$ such that $N(\mathcal{P}_*\mathcal{E}, \theta)|_{T \times \{ \tau_1 \}}$ are semistable of degree 0 for any $\tau_1 \in U$. Let $s \subset T^\vee \times U$ denote the spectrum. We have $s \cap (T^\vee \times \{ \infty \}) \subset D$. We fix a lift of $D$ to $D \subset \mathbb{C}$. Then, if $U$ is sufficiently small, we may have a lift of $s$ to $\tilde{s} \subset \mathbb{C} \times U$. We obtain the corresponding holomorphic vector bundle $V$ with an endomorphism $g$ such that $Sp(g) \subset \tilde{s}$. (See (2.6.3)) We have the decomposition

$$(V, g) = \bigoplus_{P \in D} (V_P, g_P),$$

where $Sp(g_P) \cap (\mathbb{C} \times \{ \infty \})$ is the lift of $P$. We have the induced decomposition on $T \times U$:

$$\text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta) = \bigoplus_{P \in D} \text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta)_P.$$

Let $U_P \subset T^\vee$ be a small neighbourhood of $P \in D$. We use the coordinate $\zeta_P := \zeta - \tilde{P}$. By the construction, we have a natural isomorphism $V_P \cong N^{0,\infty}(\mathcal{P}_*(\mathcal{E}, \theta)|_{U_P})$. We have $g_P = g'_P + \tilde{P}$, where $g'_P$ is the endomorphism induced by $\zeta_P$. Thus, we obtain a filtered bundle $\text{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta)_P$ over $\text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta)_P$, by transferring $N^{0,\infty}(\mathcal{P}_*(\mathcal{E}, \theta)|_{U_P})$. By taking the direct sum, we obtain a filtered bundle $\text{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta)$ over $\text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta)$.

**Remark 3.11** We obtain a different transformation by replacing $\text{Poin}$ and $wd\zeta$ with $\text{Poin}^\vee$ and $-wd\zeta$, respectively, for which we can argue in a similar way.

**Remark 3.12** In [3], the Fourier transform for Higgs bundles on smooth projective curves are studied. The algebraic Nahm transform in this paper may be regarded as a filtered variant, although we consider only the case where the base space is an elliptic curve. We also remark that this construction is an analogue of the Fourier transform of the minimal extension of algebraic meromorphic flat bundles on affine lines.

### 3.2.2 Some property

Let $(\mathcal{P}_*\mathcal{E}, \theta)$ be a filtered Higgs bundle on $(T^\vee, D)$ satisfying (A0).

**Proposition 3.13** The filtered bundle $\text{Nahm}_*(\mathcal{P}_*\mathcal{E}, \theta)$ is admissible and satisfies the condition (A3).

**Proof** Let $(\mathcal{P}_*\mathcal{E}, \theta)$ be an admissible Higgs bundle on $(T^\vee, D)$. Clearly, $\text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta)$ satisfies (A1). It satisfies (A2) by Proposition 3.3.

Let $L \in T^\vee$. We set $N_L := N(\mathcal{P}_*\mathcal{E}, \theta) \otimes L^\vee$. We have the type decomposition $N_L = \bigoplus_P \bigoplus_{P,m,o} (N_L)^{(p,m)}_{P,o}$. By the construction, we have

$$(N_L)^{(p,m)}_{P,o} = \begin{cases} \mathcal{P}_{-1/2} \text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta)_{P,o} \otimes L^\vee & \text{if } (p, m, o) \neq (1, 0, 0) \\ \mathcal{P}_0 \text{Nahm}(\mathcal{P}_*\mathcal{E}, \theta)_{P,o} \otimes L^\vee & \text{if } (p, m, o) = (1, 0, 0) \end{cases}$$

Here, $P \otimes L \in T^\vee$ denotes the multiplication of $P, L \in T^\vee$ in the group $T^\vee$. We shall study the cohomology of $N_L$ and its variant. Let us consider the following complex on $T^\vee \times T \times \mathbb{P}^1$:

$$\mathcal{C}^0 := \mathcal{C}^0 \oplus p_2^*L^\vee \xrightarrow{\theta + w d\zeta} \mathcal{C}^1 := \mathcal{C}^0 \oplus p_2^*L^\vee$$

By the construction, we have $N_L \cong R^1p_{23!}\mathcal{C}^0$. We have $R^0p_{12*}\mathcal{C}_L^0 \cong R^1p_{12*}\mathcal{C}_L^0[-1] \cong \mathcal{C}(\mathcal{P}_*\mathcal{E}, \theta \otimes \text{Poin} \otimes L^\vee)[-1]$ on $T^\vee \times T$. For the projection $\pi : T^\vee \times T \longrightarrow T^\vee$, we have $R^0\pi_*\mathcal{C}(\mathcal{P}_*\mathcal{E}, \theta \otimes \text{Poin} \otimes L^\vee) = 1^1(\mathcal{P}_*\mathcal{E}, \theta \otimes \text{Poin} \otimes L^\vee)$, which is a skyscraper sheaf $\mathcal{C}(\mathcal{P}_*\mathcal{E}, \theta)_{L}$ at $L$. Hence, we have

$$\mathbb{H}^i(T^\vee \times T \times \mathbb{P}^1, \mathcal{C}^0) \cong \begin{cases} 0 & \text{if } i \neq 2 \\ \mathcal{C}(\mathcal{P}_*\mathcal{E}, \theta)_{L} & \text{if } i = 2 \end{cases}$$

(12)
We obtain $H^i(T \times \mathbb{P}^1, N_L) = 0$ unless $i = 1$, and $H^1(T \times \mathbb{P}^1, N_L) \simeq C^1(\mathcal{P}, \mathcal{E}, \theta)|_L$. We have
\[
p_{12*}((C^*_L \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq R^1p_{12*}((C^*_L \otimes \mathcal{O}_{\mathbb{P}^1}(-1))[1] \simeq C^0(\mathcal{P}, \mathcal{E}, \theta) \otimes \text{Poin} \otimes L^\vee[-1]
\]
on $T \times T^\vee$. Hence, we have
\[
H^i(T^\vee \times T \times \mathbb{P}^1, (C^*_L \otimes \mathcal{O}_{\mathbb{P}^1}(-1))) \simeq \begin{cases} 0 & (i \neq 2) \\ C^0(\mathcal{P}, \mathcal{E}, \theta)|_L & (i = 2) \end{cases}
\]
We obtain $H^i(T \times \mathbb{P}^1, N_L \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$ unless $i = 1$, and $H^1(T \times \mathbb{P}^1, N_L \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq C^0(\mathcal{P}, \mathcal{E}, \theta)|_L$.

**Lemma 3.14** The map $H^1(T \times \mathbb{P}^1, N_L \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$ injects into $H^1(T \times \mathbb{P}^1, N_L)$ induced by the multiplication of $w$ is equal to the map $C^0(\mathcal{P}, \mathcal{E}, \theta)|_L \rightarrow C^1(\mathcal{P}, \mathcal{E}, \theta)|_L$ induced by $\theta$, up to signatures.

**Proof** Let $V_i (i = 0, 1)$ be vector spaces with morphisms $f_0, f_\infty \in \text{Hom}(V_0, V_i)$. Let $\alpha_\infty : \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^1}$ be morphisms induced as $\mathcal{O}_{\mathbb{P}^1}(-1) \simeq \mathcal{O}_{\mathbb{P}^1}(-\{ \kappa \}) \rightarrow \mathcal{O}_{\mathbb{P}^1}$. The induced morphisms $\mathcal{O}_{\mathbb{P}^1}(-m - 1) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-m)$ are also denoted by $\alpha_\infty$.

We consider a complex $C^*$ on $\mathbb{P}^1$ given as $C^0 = V_0 \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ and $C^1 = V_1 \otimes \mathcal{O}_{\mathbb{P}^1}$ with $f_0\alpha_0 - f_\infty \alpha_\infty$. The morphisms $\alpha_0$ induce $C^* \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow C^*$. We have $H^1(\mathbb{P}^1, C^*) \simeq V_1$ and $H^1(\mathbb{P}^1, (C^* \otimes \mathcal{O}(-1))) \simeq V_0$. Let us prove that the induced map $\alpha : H^1(\mathbb{P}^1, C^*) \rightarrow H^1(\mathbb{P}^1, (C^* \otimes \mathcal{O}(-1)))$ is equal to $f_\infty$ up to signatures under the identifications, which implies the claim of the lemma.

Although we can check it by a direct computation, we may also use the following argument. We consider a double complex given as follows: We set $C^{00} = V_0 \otimes \mathcal{O}(-2)$, $C^{01} = V_1 \otimes \mathcal{O}(-1)$, $C^{10} = V_0 \otimes \mathcal{O}(-1)$ and $C^{11} = V_1 \otimes \mathcal{O}$. The morphisms $C^{0i} \rightarrow C^{1i}$ are given by $\alpha_0$, and the morphisms $C^{0i} \rightarrow C^{1i}$ are given by $f_0\alpha_0 - f_\infty \alpha_\infty$.

For $i = 0, 1$, we set $D^{ij}_i = C^{ij}$ and $D^{kj}_i = 0$ for $k \neq i$. Then, we have an exact sequence of the double complexes $0 \rightarrow D^{ij}_1 \rightarrow C^{ij} \rightarrow D^{ij}_0 \rightarrow 0$. Similarly, we set $E^{ij}_i = C^{ij}$ and $E^{kj}_i = 0$ for $k \neq i$. Then, we have an exact sequence $0 \rightarrow E^{ij}_1 \rightarrow C^{ij} \rightarrow E^{ij}_0 \rightarrow 0$. We set $F^{ij}_0 = C^{00}$ and $F^{ij}_0 = 0$ for $(i, j) \neq (0, 0)$. We set $F^{ij}_1 = C^{ij}$ for $(i, j) \neq (0, 0)$ and $F^{ij}_1 = 0$. Then, we have an exact sequence $0 \rightarrow F^{ij}_1 \rightarrow C^{ij} \rightarrow F^{ij}_0 \rightarrow 0$.

We have the following commutative diagrams:
\[
\begin{array}{cccc}
D^{ij}_1 & \rightarrow & C^{ij} & \rightarrow & D^{ij}_0 \\
\downarrow & & \downarrow & & \downarrow \\
F^{ij}_1 & \rightarrow & C^{ij} & \rightarrow & F^{ij}_0 \\
\end{array}
\]
\[
\begin{array}{cccc}
E^{ij}_1 & \rightarrow & C^{ij} & \rightarrow & E^{ij}_0 \\
\downarrow & & \downarrow & & \downarrow \\
F^{ij}_1 & \rightarrow & C^{ij} & \rightarrow & F^{ij}_0 \\
\end{array}
\]

The natural morphisms $H^*(\mathbb{P}^1, \text{Tot } D^{ij}_1) \rightarrow H^*(\mathbb{P}^1, \text{Tot } E^{ij}_1)$ are isomorphisms. The map $\alpha$ is regarded as the connecting homomorphism of the long exact sequence associated to $0 \rightarrow \text{Tot } D^{ij}_1 \rightarrow \text{Tot } C^{ij} \rightarrow \text{Tot } D^{ij}_0 \rightarrow 0$. The cokernel of $C^{0i} \rightarrow C^{1i}$ are the skyscraper sheaf at $\infty$, whose fibers are $V_i$.

Hence, the connecting homomorphism for $0 \rightarrow \text{Tot } E^{ij}_1 \rightarrow \text{Tot } C^{ij} \rightarrow \text{Tot } E^{ij}_0 \rightarrow 0$ is $f_\infty$ up to signature. Thus, we are done.

For any $Y$, let $\iota_\infty : Y \times \{ \infty \} \rightarrow Y \times \mathbb{P}^1$. The morphism $N_L \rightarrow \iota_\infty^*N_L|_{T \times \{ \infty \}}$ is obtained as the push-forward of $C_L^1 \rightarrow \iota_\infty^*(C^1/C^0 \otimes \text{Poin} \otimes L^\vee)$. Hence, $H^1(T \times \mathbb{P}^1, N_L) \rightarrow H^1(T, N_L|_{T \times \{ \infty \}})$ is identified with $C^1_L \rightarrow (C^1/C^0)|_L$. By the construction, the parabolic filtration of $((N_L)^{1,0}_{L,0})_{T \times \{ \infty \}}$ is induced by the isomorphism $((N_L)^{1,0}_{L,0})_{T \times \{ \infty \}} \simeq (C^1/C^0)|_L \otimes \mathcal{O}_T$.

We have the following commutative diagrams:
\[
\begin{array}{ccc}
H^1(T \times \mathbb{P}^1, N_L \otimes \mathcal{O}(-1)) & \xrightarrow{b_1} & H^1(T \times \mathbb{P}^1, \text{Gr}^{-1}_T((N_L)^{1,0}_{L,0})) \\
\downarrow b_2 & & \downarrow b_3 \\
H^1(T \times \mathbb{P}^1, N_L) & \xrightarrow{b_1} & H^1(T \times \mathbb{P}^1, \text{Gr}^0_T((N_L)^{1,0}_{L,0}))
\end{array}
\]
Here, $b_2$ and $b_3$ are induced by the multiplication of $w$. By the previous consideration, the composite $b_4 \circ b_2$ is identified with $\mathcal{C}_L^1 \rightarrow \text{Gr}^P((\mathcal{C}^1/\mathcal{C}^0)_L)$, which is surjective. Hence, $b_1$ is surjective. Let $\mathcal{N}_L$ denote the kernel of $\mathcal{N}_L \otimes \mathcal{O}(1) \rightarrow \text{Gr}^P(\mathcal{N}_L|_{0,0})$. We obtain $H^2(T \times \mathbb{P}^1, \mathcal{N}_L) = 0$ from the surjectivity of $b_1$ and $H^2(T \times \mathbb{P}^1, \mathcal{N}_L \otimes \mathcal{O}(1)) = 0$. By the construction, $\mathcal{N}_L \subset P_0 \text{Nahm}(\mathcal{P}_*, \theta) \otimes L'$ satisfies the conditions (i,ii) in [2.4.2] and we have

$$(\mathcal{N}_L|_{0,0}) = P_{<1}(\text{Nahm}(\mathcal{P}_*, \theta)) \otimes L'|_{0,0}^{(1,0)}$$

Hence, by using Lemma [2.14] we obtain that $\text{Nahm}(\mathcal{P}_*, \theta)$ satisfies the condition (A3).

3.2.3 Characteristic number

For compact complex manifolds $Z_i$ ($i = 1, 2$), let $\omega_{Z_i} \in H^*(Z_1 \times Z_2)$ denote the pull back of the fundamental class of $Z_i$ by the projection.

Let $(\mathcal{P}_*, \theta)$ be a filtered Higgs bundle on $(T^\vee, D)$ satisfying the condition (A0). We shall study the characteristic numbers of $\text{Nahm}(\mathcal{P}_*, \theta)$.

Lemma 3.15 We have $\int_{T \times \mathbb{P}^1} c_1(\text{Nahm}_a(\mathcal{P}_*, \theta))|_{\mathbb{P}^1} = 0$ for any $a \in \mathbb{R}$.

Proof It follows from that $\text{Nahm}_a(\mathcal{P}_*, \theta)/\text{Nahm}_{<a}(\mathcal{P}_*, \theta)$ is of degree 0 for any $a \in \mathbb{R}$.

The following lemma can be checked easily.

Lemma 3.16 $c_2(\text{Nahm}_a(\mathcal{P}_*, \theta))$ is independent of $a \in \mathbb{R}$. We denote it by $c_2(\text{Nahm}_a(\mathcal{P}_*, \theta))$.

We have the type decomposition $(\mathcal{P}_*, \theta)|_{U_P} = \bigoplus_{(p,m,o)} (\mathcal{P}_i(\mathcal{E}^{(p,m)}_{p,o}, \theta_{p,o}))$ on a small neighbourhood $U_P$ of each $P \in D$. We set

$$\ell_P := \dim \text{Cok} \left( \text{Res}(\theta) : \text{Gr}^P_0(\mathcal{E}^{(1,0)}_{p,0}) \rightarrow \text{Gr}^P_0(\mathcal{E}^{(1,0)}_{p,0}) \right).$$

We put $r^{(p,m)}_{p,o} = \text{rank}(\mathcal{E}^{(p,m)}_{p,o})/p$ and $r^{(p,m)}_{p,o} := \sum_o r^{(p,m)}_{p,o}$. We have $\sum_{p,m} r^{(p,m)}_{p,o} p = \text{rank} \mathcal{E}$.

Proposition 3.17 We have the following equalities:

$$\text{rank} \text{Nahm}(\mathcal{P}_*, \theta) = \sum_P \sum_{p,m} r^{(p,m)}_{p,o} (p + m) - \sum_P \ell_P$$

(13)

$$\int_{T \times \mathbb{P}^1} c_1(\text{Nahm}_a(\mathcal{P}_*, \theta))|_{\mathbb{P}^1} = \deg(\mathcal{P}_*, \theta)$$

(14)

$$\int_{T \times \mathbb{P}^1} c_2(\text{Nahm}_a(\mathcal{P}_*, \theta)) = \text{rank} \mathcal{E}$$

(15)

Proof Let us prove (13) and (14). We have only to consider the rank and the degree of $\text{Nahm}_a(\mathcal{P}_*, \theta)|_{(0) \times \mathbb{P}^1}$. Let $\mathcal{V} \subset \mathcal{P}_1 \mathcal{E}$ be the subsheaf determined by the following conditions:

- $\mathcal{V} = \mathcal{P}_1 \mathcal{E}$ on the complement of $D$.

- It has a decomposition $\mathcal{V} = \bigoplus_P \mathcal{V}^{(p,m)}_{P,o}$ around each $P \in D$.

- We have $\mathcal{V}^{(p,m)}_{P,o} = \mathcal{P}_{1/2} \mathcal{E}^{(p,m)}_{P,o}$ for $(p, m, o) \neq (1, 0, 0)$, and $\mathcal{V}^{(1,0)}_{P,0} = \mathcal{P}_{1} \mathcal{E}^{(1,0)}_{P,0}$.

Let $\pi_i$ denote the projection of $T^\vee \times \mathbb{P}^1$ onto the $i$-th component. We have the following $K$-theoretic description:

$$\left( \mathcal{C}(\mathcal{P}_*, \theta) - \mathcal{C}^0(\mathcal{P}_*, \theta) \right)|_{T^\vee \times \{0\} \times \mathbb{P}^1} = \pi_1^* \left( \mathcal{V} - \sum_{P \in D} \mathcal{O}^{p+2}_P \right) - \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-1) \cdot \pi_1^* \left( \mathcal{V} - \sum_{P \in D} \sum_{p,m,o} \mathcal{O}^{r^{(p,m)}_{P,o}}_P (p + m) \right)$$

(16)
The Chern character of \((16)\) is equal to the following:

\[
\pi_1^* \text{ch}(\mathcal{V}) - \sum_{p \in D} \ell_p \omega_{\mathcal{V}^p} - (1 - \omega_{\mathcal{V}^1}) \left( \pi_1^* \text{ch}(\mathcal{V}) - \sum_{p, m} r_p^{(p, m)}(p + m) \omega_{\mathcal{V}^p} \right) \\
= \left( \sum_{p, m} r_p^{(p, m)}(p + m) - \sum_{p} \ell_p \omega_{\mathcal{V}^p} + \omega_{\mathcal{V}^1} \pi_1^* \text{ch}(\mathcal{V}) - \omega_{\mathcal{V}^1} \sum_{p, m} r_p^{(p, m)}(p + m) \omega_{\mathcal{V}^p} \right) \\
(17)
\]

Hence, the Chern character of \(N(\mathcal{P}, \mathcal{E}, \theta)_{[0]} \times \mathbb{P}^1\) is

\[
\sum_{p, m} r_p^{(p, m)}(p + m) - \sum_{p} \ell_p + \omega_{\mathcal{V}^1} \left( \text{deg}(\mathcal{V}) - \sum_{p, m} r_p^{(p, m)}(p + m) \right) \\
= \sum_{p, m} r_p^{(p, m)}(p + m) - \sum_{p} \ell_p + \omega_{\mathcal{V}^1} \left( \text{deg}(\mathcal{V}(-D)) - \sum_{p, m} r_p^{(p, m)}(p + m) \right) \\
(18)
\]

In particular, we obtain \((13)\). We also obtain the following equality:

\[
\text{deg}(\mathcal{N}(\mathcal{P}, \mathcal{E}, \theta)_{[0]} \times \mathbb{P}^1) = \text{deg}(\mathcal{V}(-D)) - \sum_{p, m} r_p^{(p, m)}(p + m)
\]

We set \((a(p, m, o) = -1/2)\) if \((a, m, o) \neq (1, 0, 0)\), and \(a(1, 0, 0) := 0\). For the parabolic characteristic numbers, we have the following expressions:

\[
\text{deg}(\mathcal{P}, \mathcal{E}) = \text{deg}(\mathcal{V}(-D)) - \sum_{p, m} \delta(\mathcal{P}, \mathcal{E}_{\alpha}(p, m), a(p, m, o))
\]

\[
\text{deg}(\text{Nahm}(\mathcal{P}, \mathcal{E}, \theta)_{[0]} \times \mathbb{P}^1) = \text{deg}(\mathcal{N}(\mathcal{P}, \mathcal{E}, \theta)_{[0]} \times \mathbb{P}^1) - \sum_{p, m} \delta(\text{Nahm}(\mathcal{P}, \mathcal{E}, \theta)_{[0]} \times \mathbb{P}^1, a(p, m, o))
\]

Here, \(\delta(B, a(p, m, o))\) denote the contributions of the locally given filtered bundles \(B\) to the parabolic degree. (See \((2.2.3)\)) In the following, we omit \(a(p, m, o)\). In the case \((p, m, o) = (1, 0, 0)\), we have

\[
\delta(\text{Nahm}(\mathcal{P}, \mathcal{E}, \theta)_{[0]}^{(1, 0)}) = \sum_{c \geq 0} \dim \text{Gr}_c \text{Nahm}(\mathcal{P}, \mathcal{E}, \theta)_{[0]}^{(1, 0)} = \sum_{c \geq 0} \dim \text{Gr}_c \mathcal{E}_{[0]}^{(1, 0)} = \delta(\mathcal{P}, \mathcal{E}_{[0]}^{(1, 0)}).
\]

Let us consider the case \((p, m, o) \neq (1, 0, 0)\). Let \(\varphi_p : U_u \rightarrow U_p\) be given by \(\varphi_p(u) = u^p\). We have the decomposition \(\varphi_p^{(p, m)}(\mathcal{P}, \mathcal{E}_{\alpha}(p, m)) = \bigoplus_{\alpha \in o}(\mathcal{P}, \mathcal{E}_{\alpha}(p, m))\). For any \(c \in \mathbb{R}\), we put

\[
r_{\alpha}(p, m) := \dim \text{Gr}_c V_{\alpha}.
\]

It is independent of the choice of \(\alpha \in o\). We have the following equality:

\[
\delta(\mathcal{P}, \mathcal{E}_{\alpha}(p, m)) = \sum_{-p/2 \leq c \leq -p/2} \sum_{0 \leq j \leq p-1} r_{\alpha}(p, m) \frac{c-j}{p} = \sum_{-p/2 \leq c \leq -p/2} \sum_{0 \leq j \leq p-1} r_{\alpha}(p, m) \left( c - \frac{1}{2}(p - 1) \right)
\]

We also have the following equality from the expression of the parabolic structure of \(N_{[0]}^{[0]}(\mathcal{P}, \mathcal{E}, \theta)\) in the proof of Proposition \(3.3\)

\[
\delta(\text{Nahm}(\mathcal{P}, \mathcal{E}, \theta)_{[0]}^{(p+m, m)}) = \sum_{c \geq 0} \sum_{0 \leq j \leq p/2} \frac{2c - 2j - m}{2(p + m)} = \sum_{c \geq 0} \sum_{0 \leq j \leq p/2} \frac{c - m - \frac{1}{2}(p - 1)}{2(p + m)}
\]

Then, the equality \((13)\) follows from \(\sum_{c, \alpha} r_{\alpha}(p, m) = r_p^{(p, m)}\).
Let us prove (15). We have $\int_{T \times \mathbb{P}^1} c_2(\text{Nahm}_*(\mathcal{P}, \theta)) = \int_{T \times \mathbb{P}^1} c_2(\mathcal{N}(\mathcal{P}, \theta))$. We also have the following:

$$
\int_{T \times \mathbb{P}^1} c_2(\mathcal{N}(\mathcal{P}, \theta)) = -\int_{T \times \mathbb{P}^1} \mathrm{ch}_2(\mathcal{N}(\mathcal{P}, \theta)) = -\int_{T^\vee \times T \times \mathbb{P}^1} \mathrm{ch}_3(\mathcal{C}^1 - \mathcal{C}^0)
$$

We have $\mathrm{ch}_3(\mathcal{C}^1) = 0$. We have $c_1(\text{Poin})^2 = -2\omega_T \omega_{T^\vee}$. We also have

$$
\int_{T^\vee \times T \times \mathbb{P}^1} \mathrm{ch}_3(\mathcal{C}^0) = \int_{T^\vee \times T \times \mathbb{P}^1} \text{rank}(\mathcal{V})\omega_T \omega_{T^\vee} \omega_{\mathbb{P}^1} = \text{rank}(\mathcal{V}).
$$

Hence, we obtain (15).

### 3.2.4 Stable filtered Higgs bundles of degree 0

We consider the standard stability condition for filtered Higgs bundles on $(T^\vee, D)$. For any filtered bundle $(\mathcal{P}, \mathcal{E}, \theta)$ on a projective curve $(X, D)$, we define the slope $\mu(\mathcal{P}, \mathcal{E}) := \int_X \text{par-c}_1(\mathcal{P}, \mathcal{E})/\text{rank} \mathcal{E}$. It is called stable (resp. semistable) if $\mu(\mathcal{P}, \mathcal{E}) < \mu(\mathcal{P}, \mathcal{E})$ (resp. $\mu(\mathcal{P}, \mathcal{E}) \leq \mu(\mathcal{P}, \mathcal{E})$) for any non-trivial filtered subbundle $\mathcal{P}, \mathcal{E}' \subset \mathcal{P}, \mathcal{E}$ such that $\theta(\mathcal{E}') \subset \mathcal{E}' \otimes \Omega^1$. A semistable filtered Higgs bundle is called poly-stable, if it is a direct sum of stable ones. The following lemma is easy to see.

**Lemma 3.18** If $(\mathcal{P}, \mathcal{E}, \theta)$ be a stable Higgs bundle on $(T^\vee, D)$, then its dual is also stable.

The following proposition is standard.

**Proposition 3.19** Let $(\mathcal{P}, \mathcal{E}, \theta)$ be a stable admissible filtered bundle on $(T^\vee, D)$ with $\text{deg}(\mathcal{P}, \mathcal{E}) = 0$. Then, if $\text{rank} \mathcal{E} > 1$, it satisfies the condition (A0).

**Proof** Indeed, an element of $H^0(T^\vee, \mathcal{C}^\bullet(\mathcal{P}, \mathcal{E} \otimes L, \theta + wd\xi))$ corresponds to a morphism $(\mathcal{O}_{T^\vee}(*D), 0) \rightarrow (\mathcal{P} \otimes L, \theta)$. By the stability with $\text{deg}(\mathcal{P}, \mathcal{E}) = 0$ and $\text{rank} \mathcal{E} > 1$, we obtain that such a morphism has to be 0. We obtain the vanishing of $H^2$ from the following lemma.

**Lemma 3.20** $H^i(T^\vee, \mathcal{C}(\mathcal{P}, \mathcal{E}^\vee, -\theta^\vee))$ is naturally isomorphic to the dual space of $H^{2-i}(T^\vee, \mathcal{C}(\mathcal{P}, \mathcal{E}, \theta))$.

**Proof** We use the natural identification $\Omega^1_{T^\vee} \simeq \mathcal{O}_{T^\vee}$. Let $P \in D$. We have $\left(\mathcal{P}_0 {\mathcal{E}}^{(1,0)}_P\right)^* \simeq \mathcal{P}_c(\mathcal{E}^{(1,0)}_P, -\theta^{(1,0)}_P) := C^1(\mathcal{P}, \mathcal{E}^\vee, -\theta^\vee)_{P,0}^{(1,0)}$. Let $\pi$ denote the projection $\mathcal{P}_0(\mathcal{E}^\vee)_{P,0} \rightarrow \text{Gr}^P_0((\mathcal{E}^\vee)_{P,0}^{(1,0)})$. We have a subspace

$$
\text{Ker}(\text{Gr}^P_0(\mathcal{E}^{(1,0)}_P)^*) \subset \text{Gr}^P_0((\mathcal{E}^\vee)_{P,0}^{(1,0)}).
$$

We have a natural isomorphism

$$
C^0(\mathcal{P}, \mathcal{E}^\vee, -\theta^\vee)_{P,0}^{(1,0)} := \pi^{-1} \left(\text{Ker}(\text{Gr}^P_0(\mathcal{E}^{(1,0)}_P))^*\right) \simeq (C^1(\mathcal{P}, \mathcal{E}, \theta)_{P,0}^{(1,0)})^*.
$$

The Higgs field $-\theta^\vee$ induces $C^0(\mathcal{P}, \mathcal{E}^\vee, -\theta^\vee)_{P,0}^{(1,0)} \rightarrow C^1(\mathcal{P}, \mathcal{E}^\vee, -\theta^\vee)_{P,0}^{(1,0)}$. The complex $C^\bullet(\mathcal{P}, \mathcal{E}^\vee, -\theta^\vee)_{P,0}^{(1,0)}[1]$ is the dual of $C^\bullet(\mathcal{P}, \mathcal{E}, -\theta^\vee)_{P,0}^{(1,0)}$. The natural inclusions induce a quasi isomorphism $C^\bullet(\mathcal{P}, \mathcal{E}^\vee, -\theta^\vee)_{P,0}^{(1,0)} \rightarrow C^\bullet(\mathcal{P}, \mathcal{E}^\vee, -\theta^\vee)_{P,0}^{(1,0)}$. For $(p, m, o) \neq (1, 0, 0)$, the dual of the complex $C^\bullet(\mathcal{P}, \mathcal{E}, \theta)_{P, o}^{(p,m)}$ is

$$
\mathcal{P}_{<1/2}(\mathcal{E}^\vee)_{P, -o}^{(p,m)} \rightarrow \mathcal{P}_{<3/2 + m/p}(\mathcal{E}^\vee)_{P, -o}^{(p,m)}
$$

where the first term sits in the degree $-1$. It is naturally quasi-isomorphic to $C^\bullet(\mathcal{P}, \mathcal{E}^\vee, -\theta^\vee)_{P, -o}^{(p,m)}[1]$. Then, the claim of the lemma follows from Serre duality. Thus, we complete the proof of Lemma 3.20 and Proposition 3.19.
3.2.5 Filtered Higgs bundles of rank 1 on \(T^\vee, D\)

Filtered Higgs bundles of rank 1 are always admissible and stable. Let \((\mathcal{P}, \mathcal{E}, \theta)\) be a filtered Higgs field of rank 1 on \((T^\vee, D)\). For each \(P \in D\), we have the complex number \(\text{Res}_P(\theta)\). We also have \(a(P) \in \mathbb{R}\) such that \(\text{Par}(\mathcal{P}, \mathcal{E}, P) = \{a(P) + n | n \in \mathbb{Z}\}\). Such \(a(P)\) is uniquely determined in \(\mathbb{R}/\mathbb{Z}\). We say that \(P\) is a non-trivial singularity of \((\mathcal{P}, \mathcal{E}, \theta)\), if \((\text{Res}_P \theta, a(P)) \neq (0, 0)\) in \(\mathbb{C} \times (\mathbb{R}/\mathbb{Z})\). If \(P\) is a trivial singularity, i.e., \((\text{Res}_P \theta, a(P)) = (0, 0)\), we obtain a filtered Higgs bundle on \((T^\vee, D \setminus \{P\})\) by considering the lattice \(\mathcal{P}_0(\mathcal{E})\) around \(P\). The following proposition is clear.

**Proposition 3.21** Let \((\mathcal{P}, \mathcal{E}, \theta)\) be a filtered Higgs bundle of rank 1 on \((T^\vee, D)\).

- If each \(P \in D\) is a trivial singularity of \((\mathcal{P}, \mathcal{E}, \theta)\), then \((\mathcal{P}, \mathcal{E}, \theta) \simeq \langle L(*D), \alpha \, d\zeta \rangle\) for some \(\alpha \in \mathbb{C}\) and some line bundle \(L\) of degree 0. Here the parabolic structure of \(L(*D)\) is given in a typical way as in \(2.7.2\).

- If one of \(P \in D\) is a non-trivial singularity of \((\mathcal{P}, \mathcal{E}, \theta)\), then \((\mathcal{P}, \mathcal{E}, \theta)\) satisfies \((A0)\).

3.3 Algebraic Nahm transform for admissible filtered bundles

3.3.1 Construction of the transform

For \(I \subset \{1, 2, 3\}\), let \(p_I\) be the projection of \(T^\vee \times T \times \mathbb{P}^1\) onto the product of the \(i\)-th components \((i \in I)\). Let \(\mathcal{P}_{\text{Poin}}\) denote the Poincaré bundle on \(T^\vee \times T\).

Let \(\mathcal{P}, \mathcal{E}\) be an admissible filtered bundle on \((T \times \mathbb{P}^1, T \times \{\infty\})\) satisfying the conditions \((A3)\). We put \(D := S_{p,\infty}(\mathcal{P}, \mathcal{E})\). We define

\[
\text{Nahm}(\mathcal{P}, \mathcal{E}) := R^1p_{1*}(p_{12}^*\mathcal{P}_{\text{Poin}}^\vee \otimes p_{23}^*\mathcal{P}_{-1} \mathcal{E}) \otimes \mathcal{O}_{T^\vee}(*D).
\]

By \((A3)\), \(\text{Nahm}(\mathcal{P}, \mathcal{E})\) is a locally free \(\mathcal{O}_{T^\vee}(\mathcal{P}, \mathcal{E})\)-module. By Lemma 2.13, we have a natural isomorphism

\[
\text{Nahm}(\mathcal{P}, \mathcal{E}) \simeq R^1p_{1*}(p_{12}^*\mathcal{P}_{\text{Poin}}^\vee \otimes p_{23}^*\mathcal{P}_0 \mathcal{E}) \otimes \mathcal{O}_{T^\vee}(\mathcal{P}, \mathcal{E})\).
\]

Let \(w\) be the standard coordinate of \(\mathbb{C} \subset \mathbb{P}^1\). It naturally gives a section of \(\mathcal{O}_{\mathbb{P}^1}(1)\). The multiplication of \(-w\) induces an endomorphism \(f\) of \(\text{Nahm}(\mathcal{P}, \mathcal{E})\). We obtain a Higgs field \(\theta := f \, d\zeta\) of \(\text{Nahm}(\mathcal{P}, \mathcal{E})\). We shall define a filtered bundle \(\text{Nahm}_*(\mathcal{P}, \mathcal{E}) = (\text{Nahm}_*(\mathcal{P}, \mathcal{E}) \mid a \in \mathbb{R})\) over \(\text{Nahm}(\mathcal{P}, \mathcal{E})\).

We have the type decomposition \(\mathcal{P}, \mathcal{E} = \bigoplus_{P \in D} \bigoplus_{p,m,a} \mathcal{P}_{a}^{(p,m)}\) on a neighbourhood of \(T \times \{\infty\}\). Let \(\mathcal{U} \subset \mathcal{P}, \mathcal{E}\) be an \(\mathcal{O}_{T \times \mathbb{P}^1}\)-submodule for some large \(c \in \mathbb{R}\), satisfying the conditions (i,ii) in \(2.4.2\). We suppose \(\mathcal{P}, \mathcal{E} = \mathcal{P}_{a}^{(p,m)}\) for any \(P \in D\). We define \(N(\mathcal{U}) := R^1p_{1*}(p_{12}^*\mathcal{P}_{\text{Poin}}^\vee \otimes p_{23}^*\mathcal{P}_0 \mathcal{E})\). By Lemma 2.13 and Lemma 2.14 we have \(R^1p_{1*}(p_{12}^*\mathcal{P}_{\text{Poin}}^\vee \otimes p_{23}^*\mathcal{P}_0 \mathcal{E}) = 0\), and \(N(\mathcal{U})\) is a locally free sheaf on \(T^\vee\).

We have the following object in \(D^b(\mathcal{O}_{T^\vee \times \mathbb{P}^1})\):

\[
\text{RFM}_-(\mathcal{U}) := R^{p_{13}}(p_{12}^*\mathcal{P}_{\text{Poin}}^\vee \otimes p_{23}^*\mathcal{P}_0 \mathcal{E})[1]
\]

We can express \(\text{RFM}_-(\mathcal{U})\) as a two term complex of locally free \(\mathcal{O}_{T^\vee \times \mathbb{P}^1}\) modules \(\mathcal{N}_{-1} \to \mathcal{N}_0\). Because \(a\) is generically isomorphism, it is injective. Hence, we have \(\text{RFM}_-(\mathcal{U}) \simeq \mathcal{H}^0(\text{RFM}_-(\mathcal{U}))\). We will not distinguish them.

Suppose \(0 \in D\). Let \(U_0 \subset T^\vee\) denote a small neighbourhood of 0. Let \(W_\infty \subset \mathbb{P}^1\) be a small neighbourhood of \(\infty\). We have the following decomposition:

\[
\text{RFM}_-(\mathcal{U})|_{U_0 \times W_\infty} = \bigoplus_{p,m,a} \text{RFM}_-(\mathcal{U}|_{U_0 \times W_\infty}^{(p,m)}).
\]

If \((p, m, a) \neq (1, 0, 0)\), the support of \(\text{RFM}_-(\mathcal{U}|_{U_0 \times W_\infty}^{(p,m)})\) is proper over \(U_0\). Hence, we have the following decomposition:

\[
\text{RFM}_-(\mathcal{U})|_{U_0 \times \mathbb{P}^1} = \bigoplus_{(p,m,a) \neq (1,0,0)} \text{RFM}_-(\mathcal{U}|_{U_0 \times \mathbb{P}^1}^{(p,m)}) \oplus \mathcal{M}(\mathcal{U}).
\]
Here, \( \mathcal{M}(\mathcal{U})|_{U_0 \times W_\infty} = \text{RFM}_-(\mathcal{U}^{(1,0)}_{0,0}) \). We have similar decompositions for any \( P \in D \).

We have \( N(\mathcal{U})(+D) = \text{Nahm}(\mathcal{P}, E) \). We obtain the following decomposition around any \( P \in D \) induced by the decomposition \([19]\) considered for \( P \):

\[
N(\mathcal{U}) = \bigoplus_{p,m,\alpha} N(\mathcal{U})^{(p,m)}_{p,\alpha}
\]

In particular, we have the following decomposition around any \( P \in D \):

\[
\text{Nahm}(\mathcal{P}, E) = \bigoplus_{p,m,\alpha} \text{Nahm}(\mathcal{P}, E)^{(p,m)}_{P,\alpha}
\]

(20)

We fix a lift \( \tilde{P} \in \mathbb{C} \) of any \( P \in D \), and we use a local coordinate \( \zeta_P := \zeta - \tilde{P} \) around \( P \). Let \( W_\infty \) be a small neighbourhood of \( \infty \). We have the filtered bundles with an endomorphism \( (\mathcal{P}, V^0_{\mathcal{P},\alpha} \oplus g^0_{\mathcal{P},\alpha}) \) on \( (W_\infty, \infty) \), as in \([24,1]\). If \( (p, m, \alpha) \neq (1, 0, 0) \), we have a natural isomorphism \( \text{Nahm}(\mathcal{P}, E)^{(p,m)}_{P,\alpha} \simeq N^{(\infty,0)}(\mathcal{P}, V^0_{\mathcal{P},\alpha} \oplus g^0_{\mathcal{P},\alpha}) \). Under the isomorphism, we define

\[
\text{Nahm}_a(\mathcal{P}, E)^{(p,m)}_{P,\alpha} := N^{\infty,0}(\mathcal{P}, V^0_{\mathcal{P},\alpha} \oplus g^0_{\mathcal{P},\alpha}).
\]

Let us consider the case \( (p, m, \alpha) = (1,0,0) \). First, we define

\[
\text{Nahm}_0(\mathcal{P}, E)^{(1,0)}_{P,0} := N(\mathcal{P}^{-1} E)^{(1,0)}_{P,0}.
\]

We set \( \mathcal{E}_P := \mathcal{P}_0 E^{(1,0)}_{P,0} / \mathcal{P}^{-1} E^{(1,0)}_{P,0} \). We have the following exact sequence around \( P \):

\[
0 \to N(\mathcal{P}^{-1} E)^{(1,0)}_{P,0} \to N(\mathcal{P}_0 E)^{(1,0)}_{P,0} \to R^1 p_{12}^* \text{Poin} \otimes p_{23}^* \mathcal{E}_P \to 0
\]

We may regard \( \mathcal{E}_P \) as a locally free sheaf on \( T \), and then it is isomorphic to a direct sum of some copies of the line bundle corresponding to \( P \). Hence, the multiplication of \( \zeta_P \) on \( R^1 p_{12}^* \text{Poin} \otimes p_{23}^* \mathcal{E}_P \) is 0. We obtain the induced surjection:

\[
N(\mathcal{P}_0 E)^{(1,0)}_{P,0} := N(\mathcal{P}_0 E)^{(1,0)}_{P,0} \otimes \mathcal{O}_P \to R^1 p_{12}^* \text{Poin} \otimes p_{23}^* \mathcal{E}_P
\]

Let \( K \) denote the kernel. We have the following morphisms:

\[
R^1 p_{12}^* \text{Poin} \otimes p_{23}^* \mathcal{E}_P \simeq N(\mathcal{P}_0 E)^{(1,0)}_{P,0} / K \to N(\mathcal{P}^{-1} E)^{(1,0)}_{P,0}
\]

Here, \( h \) is the injection induced by the multiplication of \( \zeta_P \). We have a natural isomorphism of \( \mathbb{C} \)-vector spaces:

\[
R^1 p_{12}^* \text{Poin} \otimes p_{23}^* \mathcal{E}_P \simeq \mathcal{P}_0 V^{(1,0)}_{P,0|\infty}
\]

Hence, for any \(-1 < c < 0\), we define

\[
F_c(\text{Nahm}_0(\mathcal{P}, E)^{(1,0)}_{P,0} \otimes \mathcal{O}_P) := F_c(\mathcal{P}_0 V^{(1,0)}_{P,0|\infty}).
\]

We also set \( F_0(\text{Nahm}_0(\mathcal{P}, E)^{(1,0)}_{P,0} \otimes \mathcal{O}_P) = \text{Nahm}_0(\mathcal{P}, E)^{(1,0)}_{P,0} \otimes \mathcal{O}_P \). The filtration of \( \text{Nahm}_0(\mathcal{P}, E)^{(1,0)}_{P,0} \otimes \mathcal{O}_P \) indexed by \([1, 0]\) induces a filtered bundle \( \text{Nahm}_a(\mathcal{P}, E)^{(1,0)}_{P,0} \) over \( \text{Nahm}(\mathcal{P}, E)^{(1,0)}_{P,0} \). In all, we obtain a filtered bundle \( \text{Nahm}_a(\mathcal{P}, E) \) over \( \text{Nahm}(\mathcal{P}, E) \).

**Proposition 3.22** \( \text{Nahm}_a(\mathcal{P}, E) \) with \( \theta \) is admissible, and satisfies the condition \((A0)\). Moreover, the complex \( N(\mathcal{P}^{-1} E) \to N(\mathcal{P}_0 E) \) is naturally identified with \( \mathbb{C}^* \text{Nahm}_a(\mathcal{P}, E) \).

**Proof** Suppose \((p, m, \alpha) \neq (1, 0, 0)\). By Proposition \([8,3]\), \( \text{Nahm}_a(\mathcal{P}, E)^{(p,m)}_{P,\alpha} \) with \( \theta \) is admissible. Moreover, \( N(\mathcal{P}^{-1} E)^{(p,m)}_{P,\alpha} \to N(\mathcal{P}_0 E)^{(p,m)}_{P,\alpha} \) is naturally identified with \( \mathbb{C}^* \text{Nahm}_a(\mathcal{P}, E)^{(p,m)}_{P,\alpha} \) by the construction.
Lemma 3.23 \( \text{Nahm}_s(\mathcal{P}_*, E)_{P}^{(1,0)} \) with \( \theta \) is admissible of type \((1,0,0), \) and \( N(\mathcal{P}_{-1} E)_{P,0}^{(1,0)} \xrightarrow{w} N(\mathcal{P}_0 E)_{P,0}^{(1,0)} \) is naturally identified with \( C^*\left(\text{Nahm}(\mathcal{P}_*, E)_{P,0}^{(1,0)}\right) \).

Proof The morphism \( f \) induces \( f_0 : N(\mathcal{P}_{-1} E)_{P,0}^{(1,0)} \to N(\mathcal{P}_0 E)_{P,0}^{(1,0)} \). The endomorphism \( f_0 \circ f \) on \( N(\mathcal{P}_0 E)_{P,0}^{(1,0)}/K \) is identified with \(-w g_{P,0}^{(1,0)} \) on \( P_0 V_{P,0}^{(1,0)} \). It is nilpotent, and it preserves the parabolic filtration on \( P_0 V_{P,0}^{(1,0)} \). By the construction of the parabolic filtration, \( h \circ f \) preserves the parabolic filtration on \( N(\mathcal{P}_{-1} E)_{P,0}^{(1,0)} \). Thus, we obtain that \( \text{Nahm}_s(\mathcal{P}_*, E)_{P,0}^{(1,0)} \) is admissible of type \((1,0,0)\).

Clearly, we have a natural isomorphism \( N(\mathcal{P}_{-1} E)_{P,0}^{(1,0)} \cong C^0(\text{Nahm}(\mathcal{P}_*, E)_{P,0}^{(1,0)}) \) by the construction. We have the natural morphism \( A : N(\mathcal{P}_0 E)_{P,0}^{(1,0)} \to N(\mathcal{P}_{-1} E)_{P,0}^{(1,0)} \otimes \Omega^1(\mathcal{P}) \). Let \( \rho \) denote the natural map \( N(\mathcal{P}_0 E)_{P,0}^{(1,0)} \to N(\mathcal{P}_0 E)_{P,0}^{(1,0)}/K \cong P_0 V_{P,0}^{(1,0)} \). By the construction of \( \text{Nahm}_s(\mathcal{P}_*, E)_{P,0}^{(1,0)} \), the image of \( \rho^{-1}(\mathcal{P}_{<0}) \) by \( A \) is equal to \( \mathcal{P}_{<0} \text{Nahm}_s(\mathcal{P}_*, E)_{P,0}^{(1,0)} \otimes \Omega^1(\mathcal{P}) \). By the construction of \( \theta \), we obtain that \( \text{Im}(A) \) also contains \( \theta(N(\mathcal{P}_{-1} E)_{P,0}^{(1,0)}) \). Hence, \( C^1(\text{Nahm}_s(\mathcal{P}_*, E)_{P,0}^{(1,0)}) \) is contained in \( \text{Im}(A) \). We remark that the following morphism is surjective, because \( H^2(T \times \mathbb{P}^1, \mathcal{P}_{<1} E \otimes L) = 0 \) for any holomorphic line bundle \( L \) of degree 0 on \( T \):

\[
N(\mathcal{P}_{-1} E)_{P,0}^{(1,0)} \longrightarrow R^1 p_{1*} \left( p_{12}^* \text{Poin} \otimes p_{23}^* \text{Gr}_{-1}^P(E_{P,0}^{(1,0)}) \right) \xrightarrow{w} R^1 p_{1*} \left( p_{12}^* \text{Poin} \otimes p_{23}^* \text{Gr}_{1}^P(E_{P,0}^{(1,0)}) \right)
\]

(21)

It implies that the morphism \( N(\mathcal{P}_{-1} E)_{P,0}^{(1,0)} \to P_0 V_{P,0}^{(1,0)}/\mathcal{P}_{<0} \) induced by \( \theta \) is surjective. Then, we obtain that \( \mathcal{I} = C^1(\text{Nahm}_s(\mathcal{P}_*, E)_{P,0}^{(1,0)}) \). The proof of Lemma 3.23 is finished.

Let us prove that \( \text{Nahm}_s(\mathcal{P}_*, E) \) with \( \theta \) satisfies the condition (A0). For \( I \subset \{1,2,3\} \), let \( p_I \) denote the projection of \( T^I \times T \times \mathbb{P}^1 \) onto the product of the \( i \)-th components \((i \in I)\). For any \( a \in \mathbb{C} \) and a line bundle \( L \) of degree 0 on \( T^I \), we consider the complex

\[
\tilde{C} := \left( p_{23}^* \mathcal{P}_* E \otimes p_{12}^* \text{Poin} \otimes p_{1}^* L \xrightarrow{w+a} p_{23}^* \mathcal{P}_0 E \otimes p_{12}^* \text{Poin} \otimes p_{1}^* L \right)
\]

where the first term sits in the degree \(-1\). Because \( R p_{1*} \tilde{C} \) is the complex \( N(\mathcal{P}_{-1}(E)) \otimes L \xrightarrow{-w+a} N(\mathcal{P}_0 E) \otimes L \) on \( T^I \), which is identified with \( C^*\left(\text{Nahm}_s(E) \otimes L, \theta + ad\zeta\right) \), we have

\[
\mathbb{H}^1(T^I \times T \times \mathbb{P}^1, \tilde{C}) \cong \mathbb{H}^1(T^I, C^*(\text{Nahm}_s(E) \otimes L, \theta + ad\zeta))
\]

Because \( R p_{23*} \tilde{C} \) is quasi-isomorphic to \( \mathcal{P}_{-1} E_{(I)} \times \mathbb{P}^1 \to \mathcal{P}_0 E_{(I)} \times \mathbb{P}^1 \), where the first term sits in the degree 0, we have \( \mathbb{H}^1(T^I \times T \times \mathbb{P}^1, \tilde{C}) = 0 \) unless \( i = 1 \). Thus, we obtain that \( \text{Nahm}_s(\mathcal{P}_*, E) \) with \( \theta \) satisfies (A0), and the proof of Proposition 3.22 is finished.

We denote the filtered Higgs bundle \( \left(\text{Nahm}_s(\mathcal{P}_*, E), \theta\right) \) just by \( \text{Nahm}_s(\mathcal{P}_*, E) \).

Remark 3.24 We obtain a slightly different transformation by replacing \( \text{Poin} \) with \( \text{Poin}^\vee \), for which we can argue in a similar way.

3.3.2 Inversion

Proposition 3.25

- Let \( (\mathcal{P}_*, \mathcal{E}, \theta) \) be an admissible filtered Higgs bundle on \((T^\vee, D)\) satisfying (A0). Then, we have a natural isomorphism \( \text{Nahm}_s(\text{Nahm}_s(\mathcal{P}_*, \mathcal{E}, \theta)) \cong (\mathcal{P}_*, \mathcal{E}, \theta) \).
- Let \( \mathcal{P}_* \mathcal{E} \) be an admissible filtered bundle on \((T \times \mathbb{P}^1, T \times \{\infty\})\) satisfying the conditions (A3). Then, we have a natural isomorphism \( \text{Nahm}_s(\text{Nahm}_s(\mathcal{P}_*, \mathcal{E})) \cong \mathcal{P}_* \mathcal{E} \).
Proof For any $I \subset \{1, 2, 3, 4\}$, let $p_I$ denote the projection of $T^\vee \times T \times \mathbb{P}^1 \times T^\vee$ onto the product of the $i$-th components ($i \in I$). We set $\mathcal{C}^i := C^i(\mathcal{P}_\mathcal{E}, \theta)$. We consider the following complex on $T^\vee \times T \times \mathbb{P}^1$:

$$p_1^* \mathcal{C}^0 \otimes p_1^* \mathcal{P} \text{oin} \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_2^* \mathcal{P} \text{oin}^\vee \xrightarrow{\theta + \text{wd} \zeta} p_1^* \mathcal{C}^1 \otimes p_1^* \mathcal{P} \text{oin} \otimes p_2^* \mathcal{P} \text{oin}^\vee$$

The complex is denoted by $\mathcal{C}^\vee$. We can observe that $R\mathcal{p}_{144} \mathcal{C}^\vee$ is quasi-isomorphic to $p_1^* \mathcal{C}^1 \otimes \mathcal{O}_\Delta[-2]$, where $p_1 : T^\vee \times T^\vee \to T^\vee$ denotes the projection onto the first component, and $\mathcal{O}_\Delta$ denote the structure sheaf of the diagonal. Hence, $R\mathcal{p}_{44} \mathcal{C}^\vee$ is naturally isomorphic to $\mathcal{C}^1[-2]$. We can also observe that $R\mathcal{p}_{2344} \mathcal{C}^\vee$ is quasi isomorphic to $q_{12}^* N(\mathcal{P}_\mathcal{E}, \theta) \otimes q_{13}^* \mathcal{P} \text{oin}^\vee[-1]$, where $q_I$ denotes the projection of $T \times \mathbb{P}^1 \times T^\vee$ onto the product of the $i$-th components ($i \in I$). Hence, we have $R\mathcal{p}_{44} \mathcal{C}^\vee (*) \to \mathcal{N}$ is quasi isomorphic to $\mathcal{N}(\mathcal{N}(\mathcal{P}_\mathcal{E}, \theta))$. We obtain

$$\mathcal{N}(\mathcal{N}(\mathcal{P}_\mathcal{E}, \theta)) \otimes \mathcal{O}(* \to \mathcal{N}) \simeq (\mathcal{E}, \theta).$$

If $(p, m, \alpha) \neq (1, 0, 0)$, we obtain the comparison of the filtered bundles over $\mathcal{E}^{(p, m)}_{\alpha}$ from Proposition 3.7. We obtain the comparison of the filtered bundles over $\mathcal{E}^{(1, 0)}_{\alpha}$ directly from the construction. Thus, we obtain the first claim.

Let $\mathcal{P}_E$ be an admissible filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$ satisfying (A3). Let $\mathcal{V} \subset \mathcal{P}_E$ be an $\mathcal{O}_{T \times \mathbb{P}^1}$-submodule satisfying the conditions (i, ii) in (2.4.2) and the following:

$$\mathcal{V}^{(p, m)}_{\alpha} = \begin{cases} \mathcal{P}_0 E^{(1, 0)}_{\alpha}, & ((p, m, \alpha) = (1, 0, 0)), \\ \mathcal{P}_{-1/2} E^{(p, m)}_{\alpha}, & \text{(otherwise)}. \end{cases}$$

By Proposition 3.22, we have $\mathcal{C}^0(\mathcal{N}(\mathcal{P}_E)) = N(\mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$ and $\mathcal{C}^1(\mathcal{N}(\mathcal{P}_E)) = N(\mathcal{V}) d\xi$. The differential $\mathcal{C}^0 \to \mathcal{C}^1$ is induced by the multiplication of $-w$. We shall rewrite the complex

$$\mathcal{C}^0(\mathcal{N}(\mathcal{P}_E)) \xrightarrow{\theta + \text{wd} \zeta} \mathcal{C}^1(\mathcal{N}(\mathcal{P}_E)). \quad (22)$$

For $I \subset \{1, 2, 3, 4, 5\}$, let $p_I$ denote the projection of $T \times \mathbb{P}^1 \times T^\vee \times T \times \mathbb{P}^1$ onto the product of the $i$-th components ($i \in I$). We set

$$C_0 := p_{12}^*(\mathcal{V} \otimes \mathcal{P}_{\mathbb{P}^1}(-1)) \otimes p_{13}^* \mathcal{P} \text{oin}^\vee \otimes p_{34}^* \mathcal{P} \text{oin} \otimes p_5^* \mathcal{O}_{\mathbb{P}^1}(-1)$$

$$C_1 := p_{12}^* \mathcal{V} \otimes p_{13}^* \mathcal{P} \text{oin}^\vee \otimes p_{34}^* \mathcal{P} \text{oin}$$

We regard $\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_{\mathbb{P}^1}(\{\infty\})$, and let $i : \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(\{\infty\})$ be the natural inclusion. Let $G : C_0 \to C_1$ be induced by $-p_2^* w \otimes p_5^* t + p_2^* t \otimes p_5^* w$. Then, (22) is naturally isomorphic to $R^3 p_{3454*} \left( C_0 \xrightarrow{G} C_1 \right)$.

For $I \subset \{1, 2, 3, 4\}$ let $q_I$ denote the projection of $T \times T \times \mathbb{P}^1$ onto the product of the $i$-th components ($i \in I$). The complex $p_{1345*} \left( C_0 \xrightarrow{G} C_1 \right)$ is quasi isomorphic to

$$q_{14}^* \mathcal{V} \otimes q_{12}^* \mathcal{P} \text{oin}^\vee \otimes q_{23}^* \mathcal{P} \text{oin}[-1].$$

For $I \subset \{1, 2, 3\}$, let $s_I$ denote the projection of $T \times T \times \mathbb{P}^1$ onto the product of the $i$-th components ($i \in I$). We have the following natural isomorphism

$$q_{1345} \left( q_{14}^* \mathcal{V} \otimes q_{12}^* \mathcal{P} \text{oin}^\vee \otimes q_{23}^* \mathcal{P} \text{oin}[-1] \right) \simeq s_{12}^* \mathcal{V} \otimes s_{12}^* \mathcal{O}_\Delta[-2]$$

Here, $\mathcal{O}_\Delta$ denote the structure sheaf of the diagonal in $T \times T$. Then, we obtain a natural isomorphism

$$\mathcal{V} \simeq N(\mathcal{N}(\mathcal{P}_E, \theta))$$

as $\mathcal{O}_{T \times \mathbb{P}^1}$-modules. If $(p, m, \alpha) \neq (1, 0, 0)$, we obtain the comparison of the filtered bundles over $\mathcal{V}^{(*) (T \times \{\infty\})}_{(p, m)}$ from Proposition 3.7. The comparison in the case $(p, m, \alpha) = (1, 0, 0)$ follows directly from the construction.

**Corollary 3.26** Let $\mathcal{P}_E$ be an admissible filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$ satisfying the conditions (A3). We have

$$\deg(\mathcal{P}_E) = \deg(\mathcal{N}(\mathcal{N}(\mathcal{P}_E)))$$

**Proof** It follows from Proposition 3.17 and Proposition 3.26.
3.4 Refinement for good filtered Higgs bundles

3.4.1 A stationary phase formula

We have the following type of stationary phase formula for the local Nahm transform, which is analogue of the stationary phase formula for the local Fourier transforms. (See [17], [20], [23], [34], [36], and [45].) We shall prove it in 3.4.4 after preliminaries in 3.4.2–3.4.3.

Theorem 3.27 Let \( U_\zeta \) be a small neighborhood of 0 in \( \mathbb{C}_\zeta \). Let \((\mathcal{P}, V, \theta)\) be an admissible filtered Higgs bundle on \( U_\zeta \).

- \((\mathcal{P}, V, \theta)\) is good, if and only if \( \mathcal{N}_e^{0,\infty}(\mathcal{P}, V, \theta) \) is good.
- Suppose \((\mathcal{P}, V, \theta) \simeq \varphi_{p*}(\mathcal{P}, V', \theta')\), where \( \theta' - da \id \) is logarithmic for some \( a \in \zeta^{-1}_p \mathbb{C}[\zeta^{-1}_p] \) with deg\( \zeta^{-1}_p \) \( a = m > 0 \). Then, there exists \((\mathcal{P}, W', \psi')\) on \( U_{\tau_p+m} \) such that (i) \( \psi' - db \) is logarithmic for some \( b \in \tau^{-1}_{p+m} \mathbb{C}[\tau^{-1}_{p+m}] \) with deg\( \tau^{-1}_{p+m} \) \( b = m \), (ii) we have an isomorphism \( \varphi_{p+m*}(\mathcal{P}, W', \psi') \simeq \mathcal{N}_e^{0,\infty}(\mathcal{P}, V, \theta) \).

Moreover, we have an isomorphism \( \text{Gr}_c^P(V') \simeq \text{Gr}_{c-m/2}^P(W') \) under which \( \text{Res}(\varphi^*_p \theta) = \text{Res}(\varphi^*_{p+m} \psi') \).

(The choice of \( b \) will be explained in the proof.)

- If \((\mathcal{P}, V, \theta)\) is logarithmic, \((\mathcal{P}, W, \psi) := \mathcal{N}_e^{0,\infty}(\mathcal{P}, V, \theta)\) is also logarithmic. Moreover, we have an isomorphism

\[
\text{Gr}_c^P(W) \simeq \begin{cases} 
\text{Gr}_c^P(V) & (1 < c < 0) \\
\text{Im}(\text{Res}(\theta) : \text{Gr}_0^P(V) \to \text{Gr}_0^P(V)) & (c = 0)
\end{cases}
\]

Under the isomorphism, we have \( \text{Res}(\psi) = \text{Res}(\theta) \).

We obtain the following corollary from Theorem 3.27. (Recall the notion of good filtered bundle in 2.4.1.)

Corollary 3.28

- Let \((\mathcal{P}, \mathcal{E}, \theta)\) be a good filtered Higgs bundle on \((T^\vee, D)\) satisfying (A0). Then, \( \text{Nahm}_e(\mathcal{P}, \mathcal{E}, \theta) \) is a good filtered bundle on \((T \times \mathbb{P}^1, T \times \{\infty}\))

- Let \( \mathcal{P}, \mathcal{E} \) be a good filtered bundle on \((T \times \mathbb{P}^1, T \times \{\infty}\)) satisfying (A3) with \( S_{p,\infty}(E) = D \). Then, \( \text{Nahm}_e(\mathcal{P}, \mathcal{E}) \) is a good filtered Higgs bundle on \((T^\vee, D)\).

3.4.2 Description of the parabolic structure of \( \mathcal{N}_e^{0,\infty}(\mathcal{P}, V, \theta) \)

Let \((\mathcal{P}, V, \theta)\) be a good filtered Higgs bundle on \((U_\zeta, 0)\). For simplicity, we assume that \((\mathcal{P}, V, \theta)\) has slope \((p, m)\). We take \( a \in \mathfrak{a} \) for each \( \mathfrak{a} \in \text{Irr}(\theta) \). Let \( c \in \mathbb{R} \). We take a frame \( v_{\mathfrak{a}} = (v_{\mathfrak{a},1}, \ldots, v_{\mathfrak{a}, \rho_{\mathfrak{a}}} \mathbb{P}^1) \) of \( \mathcal{P} \mathbb{P}^1 \mathfrak{a} \) compatible with the parabolic structure. Each \( \zeta_{\mathfrak{a}} v_{\mathfrak{a},i} dz / z \) induces a section of \( \mathcal{N}_e^{0,\infty}(\mathcal{P}, V, \theta) \), denoted by \( [\zeta_{\mathfrak{a}} v_{\mathfrak{a},i} dz / z] \). The following lemma is clear by the construction of the filtered bundle \( \mathcal{N}_e^{0,\infty}(\mathcal{P}, V, \theta) \). (See the proof of Proposition 3.3)

Lemma 3.29 The tuple

\[
\left\{ [\zeta_{\mathfrak{a}} v_{\mathfrak{a},i} dz / z] : \mathfrak{a} \in \text{Irr}(\theta), \ 0 \leq j < p_{\mathfrak{a}} + m_{\mathfrak{a}}, \ 1 \leq i \leq \text{rank} V_{\mathfrak{a}} \right\}
\]

is a frame of \( \mathcal{N}_e^{0,\infty}(\mathcal{P}, V, \theta) \), compatible with the parabolic structure. If the parabolic degree of \( v_{\mathfrak{a},i} \) is \( b \), the parabolic degree of \( [\zeta_{\mathfrak{a}} v_{\mathfrak{a},i} dz / z] \) is \( (b - j - m_{\mathfrak{a}}/2)(p_{\mathfrak{a}} + m_{\mathfrak{a}})^{-1} \).
3.4.4 Proof of Theorem 3.27

Let us return to the situation in 3.3.1. Let us begin with the third claim. We obtain the isomorphism of the associated graded vector spaces by the construction of \( P \). We have the expression \( f = d\zeta/\zeta \), where \( f \) is an endomorphism of \( P \). It naturally induces an endomorphism \( f' \) of \( P \). We have \( f = f' d\zeta/\zeta \) by the construction. Thus, we obtain the third claim.

Let us consider the second claim. Our argument is close to that in [17]. To simplify the notation, we set \( \eta := \tau_{p+m} \) and \( u := \zeta_p \). We set \( G(u) := u\partial_u(u) = \sum_{j=1}^m \alpha_j u^{-j} \). Let \( \omega := e^{2\pi \sqrt{-1} (p+m)} \). We have holomorphic functions \( u^{(i)}(\eta) \) \((i = 0, \ldots , p + m - 1)\) on \( U_{\eta} \), satisfying \( \partial_u u^{(i)}(0) = \partial_u u^{(0)}(0) \omega^i \) and

\[
G(u^{(i)}(\eta)) + pu^{(i)}(\eta)\rho/p + p+m = 0.
\]

For any \( c \in \mathbb{R} \), we consider \( P_{c-m/2} := P_{c-m/2} \varphi_{p+m}^{0,\infty}(P,\theta) \). We take a frame \( v \) of \( P' \) compatible with the parabolic structure. We put \( \tilde{v}_{ij} := (\eta^{-1}u)^{ij} \) \((0 \leq i \leq p + m - 1, 1 \leq j \leq \text{rank } V') \). They induce a frame of \( P_{c-m/2} \), which is compatible with the parabolic structure. By the frames, for \( c - 1 < \delta \leq c \), we obtain an isomorphism

\[
\text{Gr}_{d-m/2}^P(P,\varphi_{p+m}) \cong \text{Gr}_{d'}^P(V') \otimes \mathbb{C}^{p+m}.
\]

The following lemma can be checked by a direct computation.

**Lemma 3.31** \( \eta^{-1}u \) gives an endomorphism \( F \) of \( P \). On \( \text{Gr}_{d-m/2}^P(V) \), we have

\[
F(\tilde{v}_{i,j}) = \begin{cases} 
\tilde{v}_{i+1,j} & (i < p + m - 1) \\
-\rho \tilde{u}_{i,j} & (i = p + m - 1)
\end{cases}
\]

The eigenvalues of \( F \) on \( \text{Gr}^P \) are \( \partial_u u^{(i)}(0) \) \((i = 0, \ldots , p + m - 1)\). By the lemma, we obtain the decomposition \( (P, V, F) = \bigoplus_{j=0}^{p+m-1} (P, V^{(j)}, F^{(j)}) \) such that \( F^{(j)} \) has a unique eigenvalue \( \partial_u u^{(j)}(0) \). Note that \( \mathcal{N}^{0,\infty}(P, V, \theta) \cong \varphi_{p+m}^{0}(P, V^{0}, \zeta(-\tau^{-2}d\tau)) \). We also have an isomorphism \( \text{Gr}_{c}^P(V') \cong \text{Gr}_{c-m/2}^P(V^{0}) \).

We have the expression \( \theta_u = (G(u) + f) du/u \), where \( f \) is an endomorphism of \( P \). On \( P_{c-m/2} \), we have \( \eta^m(G(u) + pu^p/\eta^{p+m}) = -\eta^m f \). We have the following decomposition:

\[
\eta^m(G(u) + pu^p/\eta^{p+m}) = \left( \eta^{-1}u - \eta^{-1}u^{(0)}(\eta) \right) \times \prod_{i=1}^{p+m-1} \left( \eta^{-1}u - \eta^{-1}u^{(i)}(\eta) \right) \eta^{-1}u^{-m}
\]
Because \(\eta^{-1}u - \eta^{-1}u^{(j)}(\eta)\) \((1 \leq j < p - m)\) are invertible on \(P_{c-m/2}V^{(0)}\), we obtain the following on \(P_{c-m/2}V^{(0)}\):

\[
\eta^{-1}u - \eta^{-1}u^{(0)}(\eta) = -p^{-1}\eta^m \cdot f \cdot \prod_{j=1}^{p+m-1} (\eta^{-1}u - \eta^{-1}u^{(j)}(\eta))^{-1} (\eta^{-1}u)^m
\]

Let \(Q_k(x, y) = \sum_{i+j=k} x^i y^j\). We have

\[
\zeta/\tau - \frac{u^{(0)}(\eta)^p}{\eta^{p+m}} = \eta^{-m} (\eta^{-1}u - \eta^{-1}u^{(0)}(\eta)) \cdot Q_{p-1}(\eta^{-1}u, \eta^{-1}u^{(0)}(\eta))
\]

\[
= -f \prod_{j=1}^{p+m-1} (\eta^{-1}u - \eta^{-1}u^{(j)}(\eta))^{-1} (\eta^{-1}u)^m p^{-1}Q_{p-1}(\eta^{-1}u, \eta^{-1}u^{(0)}(\eta))
\]

(24)

Hence, we obtain that \((\zeta/\tau - u^{(0)}(\eta)^p \eta^{-p-m})P_\star V^{(0)} \subset P_\star V^{(0)}\). On \(Gr^P_\star(V^{(0)})\), the endomorphisms \(u/\eta \) and \(u^{(0)}(\eta)/\eta\) are the multiplication of \(\partial_u u^{(0)}(0)\). Hence, \((\zeta/\tau - u^{(0)}(\eta)^p \eta^{-p-m})\) acts as \(-(p+m)^{-1}f\) on \(Gr^P(V)\). We set \(P_\star W' := P_\star V^{(0)}\) and \(\psi' := -\zeta \tau^{-2}d\tau = -(\zeta/\tau)(p+m)d\eta/\eta\). We have \(b \in \eta^{-1}\mathbb{C}\eta^{-1}\) uniquely determined by the condition that \(\eta \partial_u b = \) is equal to the polar part of \(-(p+m)u^{(0)}(\eta)^p \eta^{-p-m}\). Then, \(\psi' - db\) is logarithmic. The residue acts as \(f\). Hence, the second claim of Theorem 3.34 follows. It also implies the “only if” part in the first claim.

Let us prove the “if” part of the first claim. We use the inverse transform. Let \((P_\star W, \psi)\) be a good filtered Higgs bundle on \((U_\tau, 0)\) which is isomorphic to \(\varphi_{p*}(P_\star W', \psi')\), where \(\psi' - db\) id is logarithmic for some \(b \in \tau_{-1}^{-1}\mathbb{C}[\tau_{-1}^{-1}]\) with deg_{-1} \(b = m < p\). If \(p = 1\), we assume that any eigenvalue of \(\text{Res}(\psi')\) is not 0. The claim of Theorem 3.34 follows from the next proposition.

**Proposition 3.32** There exists \((P_\star V', \theta')\) on \(U_{\zeta_{-m}}\) such that \((i)\) \(\theta' - da\) is logarithmic for some \(a \in \zeta_{-m}^{-1}\mathbb{C}[\zeta_{-m}^{-1}]\), \((ii)\) we have an isomorphism \(\varphi_{p-m*}(P_\star V', \theta') \simeq N^{\infty, 0}_<(P_\star W, \psi)\).

**Proof** To simplify the notation, we set \(\eta := \tau_p\) and \(u := \zeta_{-m}\). We have the expression

\[
\psi' = (G(\eta) \text{id} + \eta^p f') \varphi_p^*(-\tau^{-2}d\tau),
\]

such that \((i)\) \(G(\eta) = \sum_{j=1}^m \beta_j \eta^{p-j}\) with \(\beta_m \neq 0\), \((ii)\) \(f\) is an endomorphism of \(P_\star W'\). We fix a holomorphic function \(\eta^{(0)}(u)\) such that \(G(\eta^{(0)}(u)) = u^{p-m} = 0\) such that \(0 < C_1 \leq |\eta^{(0)}(u)| \leq C_2\) for some constants \(C_i\).

We set \(P_{c+p-m/2}V := \cap_{P_{c+p-m/2}}^P N^{\infty, 0}(P_\star W, \psi)\). Let \(\nu\) be a frame of \(P_\star W'\) compatible with the parabolic structure. We set \(\nu_{ij} = u^{-1} \varphi_{ij}\) \((0 \leq i \leq p - m - 1, 1 \leq j \leq \text{rank}(W')\). They induce a frame of \(P_{c+p-m/2}V\) compatible with the parabolic structure. By using the frame, for any \(-1 < \delta \leq c\), we obtain an isomorphism \(Gr^P_{c+p-m/2}(V) \simeq Gr^P_{d+c}(W') \otimes \mathbb{C}^{p-m}\). The following lemma can be checked directly.

**Lemma 3.33** \(u^{-1} \eta\) gives an endomorphism \(F\) of \(P_\star V,\) preserving the parabolic structure, and the induced endomorphism on \(Gr^P(V)\) is given by \(F(\nu_{ij}) = \nu_{i+1,j}\) \((i = 0, \ldots, p - m - 2)\) and \(F(\nu_{p-m-1,j}) = -\beta_{m-1} \nu_{0,j}\). The eigenvalues are \(\omega_i^\partial_u \eta^{(0)}(0)\) \((i = 0, \ldots, p - m - 1)\), where \(\omega = e^{2\pi \sqrt{-1} \tau/(p-m)}\).

We obtain the decomposition \((P_\star V, F) = \bigoplus_{i=0}^{p-m-1} (P_\star V^{(i)}, F^{(i)})\) such that \(F^{(i)}_{0}\) has a unique eigenvalue \(\omega_i^\partial_u \eta^{(0)}(0)\). We have an isomorphism \(\varphi_{p-m*}(P_\star V^{(0)}, -\tau^{-1}d\zeta) \simeq N^{\infty, 0}_<(P_\star W, \psi)\). We also have an isomorphism \(Gr^P_{c+p-m/2}(V^{(0)}) \simeq Gr^P_{c}(W')\).

We have \(G(\eta) - u^{p-m} = -\eta^p f\) on \(V\). Note that \(u^{-(p-m-1)} \sum_{j=1}^m \beta_j Q_{p-j-1}(\eta^{(0)}(u), \eta)\) is invertible on \(P_{c+p-m/2}V^{(0)}\). Hence, we obtain the following on \(P_{c+p-m/2}V^{(0)}\):

\[
u^{p-m-1}(\eta^{(0)}(u) - \eta) = \eta^p f \cdot \prod_{j=1}^m \beta_j Q_{p-j-1}(\eta^{(0)}(u), \eta)^{-1} u^{p-m-1}
\]

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We have the following:

\[ u^{p-m}(\eta^p - \eta^{(0)}(u)^{-p}) = f \eta^p Q_{p-1}(\eta^{(0)}(u)^{-1}, \eta^{-1}) \eta^{(0)}(u)^{-1} \left( \sum_{j=1}^{m} \beta_j Q_{p-j-1}(\eta^{(0)}(u), \eta) \right) \]

Hence, we obtain that \( u^{p-m}(\eta^p - \eta^{(0)}(u)^{-p}) \) is an endomorphism of \( \mathcal{P}_*(\mathcal{V}^{(0)}) \). We set \( \mathcal{P}_*V := \mathcal{P}_*(\mathcal{V}^{(0)}) \) and \( \theta' := -\tau^{-1} \phi^* \partial_m d \kappa = -\eta^p (p-m) u^{p-m}(du/\eta) \). We have \( a \in u^{-1}C[u^{-1}] \) uniquely determined by the condition \( u \theta a = -\eta^{(0)}(u)^{-p}(p-m) u^{p-m} \). Then, \( \theta' - da \) is logarithmic. Thus, the proof of Proposition 3.32 and Theorem 3.27 are finished.

4 Family of vector bundles on torus with small curvature

4.1 Small perturbation

We use the notation in \( \mathcal{G}_k \). We use the metric \( dz \overline{d\sigma} \) of \( T \). For any finite dimensional vector space \( V \), let \( L_k^p(V) \) be the space of \( V \)-valued \( L_k^p \)-functions on \( T \), and let \( L_k^p(V \otimes \Omega^j) \) be the space of \( V \)-valued \( L_k^p \)-differential \((i,j)\)-forms. We have the linear map \( \int f : L_k^p(V) \rightarrow V \) given by \( \int f := |T|^{-1} \int_T f \overline{dz} \overline{d\sigma} \), where \( |T| \) denotes the volume of \( T \). The kernel is denoted by \( L_k^p(V)_0 \). We have a natural inclusion \( V \rightarrow L_k^p(V) \) as constant functions. We have the decomposition \( L_k^p(V) = L_k^p(V)_0 \oplus V \) as topological vector spaces.

Suppose that \( V \) is \( r \)-dimensional and equipped with a hermitian metric \( h_V \). Let \( p \geq 2 \). Let \( \mathcal{G}_k(V) \) be the space of \( L_{k+2}^p \)-maps from \( T \) to \( GL(V) \). We set \( \mathcal{G}_k^p(V) := \{ \partial_0 + A \mid A \in L_{k+1}^p(End(V) \otimes \Omega^{0,1}) \} \), i.e., the space of \((0,1)\)-type differential operators of the endomorphism \( V \) of the form \( \partial_0 + A \) (\( A \in L_{k+1}^p(End(V) \otimes \Omega^{0,1}) \)). We have the natural right \( \mathcal{G}_k^p \)-action on \( \mathcal{G}_k^p(V) \) given by \( g \circ \mathcal{G}_k^p := g^{-1} \circ \mathcal{G}_k^p \circ g = \mathcal{G}_k^p + g^{-1} \mathcal{G}_k^p \).

Let \( \Gamma \) be an endomorphism of \( V \). Let \( U_1 \subset L_{k+2}^p(End(V))_0 \) be a sufficiently small neighbourhood of 0 such that \( 1 + U_1 \subset \mathcal{G}_k^p \). Let \( U_2 \) be a neighbourhood of 0 in \( End(V) \). We consider the map \( \Psi : U_1 \times U_2 \rightarrow \mathcal{G}_k^p(V) \) given by

\[ \Psi(a, b) := (1+a) \cdot \left( \partial_0 + (\Gamma + b) d\sigma \right). \]

We use the norm on \( L_{k+2}^p(End(V)) \) such that \( L_{k+2}^p(End(V)) \simeq L_{k+2}^p(End(V))_0 \oplus End(V) \) is an isometry, and the norm on \( L_{k+1}^p(End(V)) \) such that \( L_{k+1}^p(End(V)) \rightarrow L_{k+1}^p(End(V))_0 \oplus End(V) \) is an isometry.

**Proposition 4.1** Fix \( \delta > 0 \). Suppose that \( \Gamma \) is decomposed as \( \Gamma = \Gamma_0 + \Gamma_1 \) satisfying the following conditions:

- \( \Gamma_0 \) is commutative with its adjoint \( \Gamma_0^\dagger \), i.e., it is diagonalizable, and the eigen spaces are orthogonal with respect to \( h_V \). Moreover, there exists \( \zeta_0 \in \mathbb{C} \) such that \( Sp(\Gamma) \) is contained in

\[ K_1(L, \zeta_0) := \{ \zeta \in \mathbb{C} \mid 0 \leq \Im(\zeta - \zeta_0) \leq (1-\delta)\pi, \ 0 \leq \Im((\zeta - \zeta_0)^{-1}) \leq (1-\delta)\pi \}. \]

- \( |\Gamma_1|_{h_V} \leq \delta/100 \).

Then, there exist positive constants \( C_i \) (\( i = 1, 2 \)), independently from \( \Gamma \) and \( \zeta_0 \), such that the following holds:

- For \( B \in L_{k+1}^p(End(V) \otimes \Omega^{0,1}) \) with \( |B| \leq C_1 \), there exists a unique \((a, b) \in U_1 \times U_2 \) with \( |a| + |b| \leq C_2 |B| \) satisfying \( \partial_0 + \Gamma d\sigma + B d\sigma = \Psi(a, b) \).

**Proof** We set \( K(L) := \{ \zeta \in \mathbb{C} \mid |\Im(\zeta)| \leq (1-\delta)\pi, \ |\Im(\zeta^{-1})| \leq (1-\delta)\pi \}. \) We have \( Sp(ad(\Gamma_0)) \subset K(L) \). In the following, \( C_i \) will be positive constants which are independent from \( \Gamma \) and \( \zeta_0 \).

We have a morphism \( \Phi_T : L_{k+2}^p(End(V)) = L_{k+2}^p(End(V))_0 \oplus End(V) \rightarrow L_{k+1}^p(End(V) \otimes \Omega^{0,1}) \) given by

\[ \Phi_T(A, B) = \partial_0 + [\Gamma, A] d\sigma + B d\sigma, \]

where \( A \in L_{k+2}^p(End(V))_0 \) and \( B \in End(V) \). We have \( \Phi_0(A, B) = \partial_0 + B d\sigma \), which is an isometry by our choice of the norms.

**Lemma 4.2** \( \Phi_T \) is a homeomorphism.
We have \( \Phi_0 \circ \Phi = C_3 \) and \( |\Phi_0^{-1} \circ \Phi_0| \leq C_3 \), independently from \( \Gamma \), where \(| \cdot |\) denotes the operator norm.

Proof Let \( S \) be the set of \( \Gamma \) satisfying the conditions of the proposition. It is compact. For any fixed \((A, B) \in L^p_k(\text{End}(V)) \otimes \text{End}(V)\), the map \( \Gamma \mapsto \Phi_1^{-1} \circ \Phi_0(A, B) \) gives a continuous map from \( S \) to \( L^p_{k+2}(\text{End}(V)) \otimes \text{End}(V) \), and hence bounded. Then, we obtain the claim for \( \Phi_1^{-1} \circ \Phi_0 \) by the uniform boundedness principle. We obtain the claim for \( \Phi_0^{-1} \circ \Phi_1 \) similarly.

We set \( A(a, b) := \Psi(a) - \Psi(0, 0) \in L^p_{k+1}(\text{End}(V) \otimes \Omega^{k, 1}) \), i.e.,

\[
A(a, b) = (1 + a)^{-1}(\bar{\partial}_a + [\Gamma, a]) + \text{Ad}(1 + a) b \bar{\partial}.
\]

We have \(|A(a, b)| = O(|a| + |b|)\), independently from \( \Gamma \). The derivative \( T_{(a, b)} \Psi \) of \( \Psi \) at any \((a, b) \in U_1 \times U_2\) is given by

\[
T_{(a, b)} \Psi(X, Y) = \Phi(X, Y) + \left[ A(a, b), (1 + a)^{-1}X \right] - \left[ \Psi(0, 0), (1 + a)^{-1}a X \right] + \left( \text{Ad}(1 + a) - 1 \right) Y.
\]

Hence, we obtain an estimate \( |\Phi^{-1}_1 \circ T_{(a, b)} \Psi - \text{id}| \leq C_4(|a| + |b|)\), which is independent from \( \Gamma \). Then, the claim of Proposition 4.3 follows from the classical inverse function theorem ([31], for example).

Corollary 4.4 \( \Psi \) gives a diffeomorphism of a neighbourhood of \((0, 0)\) in \( U_1 \times U_2 \) and a neighbourhood of \( \partial_0 + \Gamma \bar{\partial} \) in \( \mathfrak{X}_p(V) \).

4.2 Frames

4.2.1 Preliminary

We set \( U_1 := \{(x_1, x_2) \mid 0 \leq x_i \leq 1\} \) and \( U_2 := \{(\xi_1, \ldots, \xi_{n-2}) \mid |\xi_i| \leq 1\} \). Let \( T_0 = \mathbb{R}^2 / \mathbb{Z}^2 \). Let \( U_1 \times U_2 \to T_0 \times U_2 \) denote the natural projection. We also use the variables \( t_i = x_i \) (\( i = 1, 2 \)) and \( t_i = \xi_{i-2} \) (\( i = 3, \ldots, n \)). We also use \( x = x_1, y = x_2 \).

For any positive integer \( k \), we set \( S_1(k) := \{(m_1, m_2) \mid m_1 + m_2 = k, m_i \geq 0\} \). We also set \( S_2(k) := \{(m_1, \ldots, m_{n-2}) \mid \sum m_i = k, m_i \geq 0\} \). We set \( S(k_1, k_2) := S_1(k_1) \times S_2(k_2) \). We put \( \partial_{x_i}^m := \prod \partial_{x_i}^{m_i} \) and \( \partial_{x_i}^m := \prod \partial_{\xi_i}^{m_i} \). We put \( N_i(k) := |S_i(k)| \) and \( N(k_1, k_2) := N_1(k_1) \times N_2(k_2) \). Let \( V \) be a vector space. For \( f \in C^\infty(U_1 \times U_2, V) \), we set

\[
D_{x_i}^m D_{x_j}^n(f) := \left( \partial_{x_i}^m \partial_{x_j}^n f \right) \left( \mathbf{m}_1, \mathbf{m}_2 \right) \in C^\infty(S(k_1, k_2)).
\]

Formally, we set \( D_0^m f := f \in C^\infty(U_1 \times U_2, V) \). We use similar notations for the functions on \( T_0 \times U_2 \) and \([0, 1] \times U_2 \).

4.2.2 Orthonormal frame

Let \( E \) be a topologically trivial \( C^\infty \)-vector bundle on \( T_0 \times U_2 \) with a hermitian metric \( h \) and a unitary connection \( \nabla \). We set \( r := \text{rank } E \). Let \( F \) denote the curvature of \( \nabla \). For any frame \( v \) of \( E \), let \( A^v = \sum_{i=1}^n A^v_i dt_i \) denote the connection form of \( \nabla \) with respect to \( v \). We put \( 1A^v := A^v_1 dt_1 + A^v_2 dt_2 \) and \( 2A^v := \sum_{i=3}^n A^v_i dt_i \). Similarly \( F^v = \sum F^v_j dt_j \) denote the curvature form with respect to \( v \).

Fix a positive number \( M \). Let \( \epsilon \) be a small positive number. Assume that \( |D_{x_i}^m D_{x_j}^n F|_{\epsilon} \leq \epsilon \) for any \( k_1, k_2 \leq M \).

Lemma 4.5 If \( \epsilon \) is sufficiently small, there exist an orthonormal frame \( v \) of \((E, h)\) on \( T_0 \times U_2 \) and anti-hermitian matrices \( A^{(x)}, A^{(v)} \) such that the following holds:
(A1) For $\kappa = x, y$, there exist $0 \leq \theta_\kappa < 2\pi$ such that any eigenvalue $\sqrt{-1}\alpha$ of $\Lambda^{(\kappa)}$ satisfies $|\alpha - \theta_\kappa| \leq \pi(2r - 1)/2r$. They satisfy $|[\Lambda^{(x)}, \Lambda^{(y)}]| \leq C\varepsilon$.

(A2) $|A \nu - \Lambda| \leq C\varepsilon$, and $|D^x_k D^y_k (A \nu)| \leq C\varepsilon$ for any $0 \leq k_1 \leq M$ and $0 \leq k_2 \leq M$ with $(k_1, k_2) \neq (0, 0)$, where $\Lambda = \Lambda^{(x)} dx + \Lambda^{(y)} dy$.

(A3) $|D^x_k D^y_k (A \nu)| \leq C\varepsilon$ for any $0 \leq k_1, k_2 \leq M$.

Here, the constant $C$ may depend only on $r$ and $M$.

**Proof** We shall indicate an outline of the construction, although it is elementary. We say that a quantity $P$ is $O(\varepsilon)$, if $P \leq C\varepsilon$ for some constant $C$ which may depend only on $r$ and $M$. Let $[a, b]_\mathbb{Z}$ denote the set of integers $k$ such that $a \leq k \leq b$. For $j \geq 1$, let $H_j$ be the subset of $U_1 \times U_2$ determined by the condition $t_i = 0$ ($i \in [1, j]_\mathbb{Z}$). We set $H_0 := U_1 \times U_2$.

Let $u$ be an orthonormal frame of $\pi^*(E, h)$ on $U_1 \times U_2$ satisfying $\nabla_i u = 0$ on $H_{i-1}$ for any $i$. We have $A_0 u = 0$ on $H_{i-1}$ by the construction. For $j < p$, we have $\partial_{t_j} A_0 u = \partial_{t_j} F_0 u$ on $H_{j-1}$. For a monomial $P$ of $\partial_{t_{j+1}} \ldots \partial_{t_{k}},$ we have $\partial_{t_j}^{a+1} P A_0 u = \partial_{t_j} P F_0 u$ on $H_{j-1}$. Hence, for $j \leq p$ and for a monomial $\tilde{P} = \prod_{i=1}^m \partial_{t_i}^{m_i}$ satisfying $m_1 + m_2 \leq M$ and $\sum_{i=2}^m m_i \leq M$, we obtain $\tilde{P} A_0 u = O(\varepsilon)$ on $H_{j-1}$ by a descending induction. In particular, we obtain $D_0^x D_0^y A_0 u = O(\varepsilon)$ for any $(k_1, k_2) \in [0, M^2_\mathbb{Z})$.

Let $G^{(x)} : H_1 \rightarrow U(r)$ be determined by $u|_{(1, y, \xi)} = u|_{(0, y, \xi)} G^{(x)}(y, \xi)$, where $U(r)$ denotes the $r$-th unitary group. By the equation

$$\partial_{t_j} G^{(x)}(t_2, \ldots, t_n) - G^{(x)}(t_2, \ldots, t_n) A_0 u_{t_1(1,t_2,\ldots,t_n)} + A_0 u_{t_1(0,t_2,\ldots,t_n)} G^{(x)}(t_2, \ldots, t_n) = 0,$$

we obtain $|D^x_k G^{(x)}| + |D^y_k G^{(x)}| = O(\varepsilon)$. By an easy induction, we obtain $|D^x_k D^y_k G^{(x)}| = O(\varepsilon)$ for any $(k_1, k_2) \in [0, M^2_\mathbb{Z}) \setminus \{(0, 0)\}$. We also have $|G^{(x)}(y, \xi) - G^{(x)}(y', \xi')| = O(\varepsilon)$.

Let $G^{(y)}(x, \xi)$ be determined by $u|_{(x, 0, \xi)} = u|_{(x, 0, \xi)} G^{(y)}(x, \xi) \xi$. Similarly, we have $|D^x_k D^y_k G^{(y)}| = O(\varepsilon)$ for any $(k_1, k_2) \in [0, M^2_\mathbb{Z}) \setminus \{(0, 0)\}$, and $|G^{(y)}(x, \xi) - G^{(y)}(x', \xi')| = O(\varepsilon)$. Because $G^{(y)}(0, \xi) G^{(x)}(1, \xi) = G^{(x)}(0, \xi) G^{(y)}(0, \xi) G^{(x)}(1, \xi) = G^{(y)}(0, \xi) G^{(y)}(0, \xi) G^{(y)}(0, \xi) = O(\varepsilon)$. We set $G^{(y)} := G^{(y)}(0, \xi)$ and $G^{(x)} := G^{(x)}(0, \xi)$.

Let $I^\kappa$ denote the set of the eigenvalues of $\tilde{G}^{(\kappa)}$ for $\kappa = x, y$. Let $dS_1$ denote the standard distance on $S^1 = \{ e^{\sqrt{-1} \theta} | \theta \in \mathbb{R} \}$ induced by the metric $d\theta d\bar{\theta}$. There exist $\gamma_\kappa \in S^1$ such that $dS_1 (\gamma_\kappa, \gamma) \geq \pi/(2r)$ for any $\gamma \in I^\kappa$. Let $\theta_\kappa$ be determined by $e^{\sqrt{-1} \theta_\kappa} = e^{\sqrt{-1} \gamma_\kappa}$ and $0 \leq \theta_\kappa < 2\pi$. For any $\gamma \in I^\kappa$, we can take $\alpha_i$ satisfying (i) $e^{\sqrt{-1} \alpha_i} = \gamma_i$, (ii) $|\theta_\kappa - \alpha_i| \leq \pi(2r - 1)/2r$. We remark that, for any $\gamma_i, \gamma_j \in I^\kappa$, we have

$$|\alpha_i - \alpha_j| = O\left(\left|\gamma_i - \gamma_j\right|\right)$$

We have the eigen decompositions $C^r = \bigoplus_{\gamma_i} V^{(x)}_{\gamma_i}$ for $\tilde{G}^{(\kappa)}$. We set $\Lambda^{(\kappa)} = \bigoplus_{\gamma_i} \sqrt{-1} \alpha_i id_{V^{(x)}_{\gamma_i}}$. By the construction, we have $\exp(\Lambda^{(\kappa)}) = G^{(\kappa)}$.

**Lemma 4.6 We have $[\Lambda^{(x)}, \Lambda^{(y)}] = O(\varepsilon)$.**

**Proof** According to the decomposition $C^r = \bigoplus_{\gamma_i} V^{(x)}_{\gamma_i}$, we have the decomposition $\tilde{G}^{(y)} = \sum_{\gamma_i, \gamma_j} \tilde{G}^{(y)}_{\gamma_i, \gamma_j}$, where $\tilde{G}^{(y)}_{\gamma_i, \gamma_j} \in \text{Hom}(V^{(x)}_{\gamma_i}, V^{(x)}_{\gamma_j})$. We have $\{ (\gamma_i - \gamma_j) \} \tilde{G}^{(y)}_{\gamma_i, \gamma_j} = O(\varepsilon)$ by using (25), we obtain $[\tilde{G}^{(y)}, \Lambda^{(x)}] = O(\varepsilon)$. By using a similar consideration, we obtain $[\Lambda^{(y)}, \Lambda^{(x)}] = O(\varepsilon)$.

Let us return to the proof of Lemma 4.5. We put $g^{(x)}(x) := \exp(-x \Lambda^{(x)})$, $g^{(y)}(y) := \exp(-y \Lambda^{(y)})$, and $g(x, y) := g^{(x)}(x) g^{(y)}(y)$. We obtain an orthonormal frame $u' := u g(x, y)$ of $\pi^*(E, h)$. Let $A' := A u'$. We have $|A' - \Lambda| = O(\varepsilon)$ and $|D_0^x D_0^y A'| = O(\varepsilon)$ for any $(k_1, k_2) \in [0, M^2_\mathbb{Z}) \setminus \{(0, 0)\}$.

Let $G^{(x)}(y, \xi)$ and $G^{(y)}(x, \xi)$ be determined by

$$u'|_{(1, y, \xi)} = u'|_{(0, y, \xi)} G^{(x)}(y, \xi), \quad u'|_{(x, 1, \xi)} = u'|_{(x, 0, \xi)} G^{(y)}(x, \xi),$$

where $u'|_{(1, y, \xi)}$ and $u'|_{(x, 1, \xi)}$ are determined by $u|_{(1, y, \xi)} = u|_{(x, 1, \xi)} G^{(x)}(y, \xi)$ and $u|_{(1, y, \xi)} = u|_{(x, 1, \xi)} G^{(y)}(x, \xi)$. For a monomial $P$ of $\partial_{t_{j+1}} \ldots \partial_{t_{k}},$ we have $\partial_{t_j}^{a+1} P u|_{(x, 1, \xi)} = \partial_{t_j} P u|_{(x, 1, \xi)} G^{(x)}(y, \xi)$ on $H_{j-1}$. Hence, for $j \leq p$ and for a monomial $\tilde{P} = \prod_{i=1}^m \partial_{t_i}^{m_i}$ satisfying $m_1 + m_2 \leq M$ and $\sum_{i=2}^m m_i \leq M$, we obtain $\tilde{P} u|_{(x, 1, \xi)} = O(\varepsilon)$ on $H_{j-1}$ by a descending induction. In particular, we obtain $D_0^x D_0^y u|_{(x, 1, \xi)} = O(\varepsilon)$ for any $(k_1, k_2) \in [0, M^2_\mathbb{Z})$. We have $|D_0^x D_0^y u|_{(x, 1, \xi)}| \leq C\varepsilon$ for any $(k_1, k_2) \in [0, M^2_\mathbb{Z})$.
We have \( G'(x)(y, \xi) = g^{(y)}(y)^{-1} G'(x)(y, \xi) (\overline{G'(x)})^{-1} g^{(y)}(y) \) and hence \( |G'(x) - 1| = O(\epsilon) \). We have

\[
dG'(x) = g^{(y)}(y)^{-1} dG'(x)(y, \xi) (\overline{G'(x)})^{-1} g^{(y)}(y) - [g^{(y)}(y)^{-1} dG'(y) (y, \xi), (G'(x) - 1)].
\]

Hence, we have \( |D_y G'(x)| = O(\epsilon) \) and \( |D_{\xi} G'(x)| = O(\epsilon) \). By an easy induction, we obtain \( |D_y D_{\xi} G'(x)| = O(\epsilon) \) for \((k_1, k_2) \in [0, M]_2 \setminus \{(0, 0)\}\). We have \( G'(y) = \overline{g^{(y)}}(x)^{-1} G'(y)(x, \xi) g^{(y)}(x)(\overline{G'(y)})^{-1} \). We obtain

\[
G'(y) - 1 = g^{(x)}(x)^{-1} \left( G'(y)(x, \xi) \overline{G'(y)}(y) - 1 \right) - g^{(x)}(x)^{-1} g^{(x)}(y) \left[ \overline{G'(y)}(y)^{-1}, g^{(x)}(x) \right] = O(\epsilon).
\]

As in the case \( \kappa = x \), we also obtain \( |D_y D_{\xi} G'(x)| = O(\epsilon) \) for \((k_1, k_2) \in [0, M]_2 \setminus \{(0, 0)\}\). Hence, the proof of Lemma 4.5 is finished.

Let \( \chi(x) \) be a non-negative valued \( C^\infty \)-function on \([0, 1]\) such that \( \chi(x) = 0 \) \((x \leq 1/3)\) and \( \chi(x) = 1 \) \((x \geq 2/3)\). We put \( h_2(x, y, \xi) := \chi(x) \exp^{-1}(G'(x)(y, \xi)) \). By construction, we have \( |D_x h_2| \leq O(\epsilon) \) for \((k_1, k_2) \in [0, M]_2 \setminus \{(0, 0)\}\).

We give only an indication. With an appropriate change of orthonormal base, we may assume that \( \Gamma \) is upper triangular. By the base, we identify matrices and endomorphisms. Let \( \Gamma \) be a decomposition \( \Gamma = \Gamma_0 + \Gamma_1 \) such that \( (i) \) \( |\Gamma_0, \Gamma_1| = 0 \) and \( \mathcal{S} \gamma_0(\Gamma_0) = \mathcal{S} \gamma(\Gamma) \) \( \mathcal{S} \gamma_1 = O(\epsilon^{1/2}) \). Moreover, if \( \delta > 0 \) is sufficiently smaller than \( 1/\epsilon \), but independently from \( \epsilon \), \( \mathcal{S} \gamma_1 = O(\epsilon^{1/2}) \). Let \( \chi \) denote the \( (i, j) \)-entry of \( \Gamma \). Then, the \((k, k)\)-entries of \( [\Gamma, \Gamma'] \) is \( \sum_{i > k} |\gamma_{k, i}|^2 - \sum_{i < k} |\gamma_{k, i}|^2 \). Then, we obtain the desired estimate for \( \Gamma_1 \).
from $[\Gamma, \Gamma'] = O(\epsilon)$, which follows from the condition $(A1)$ in Lemma 4.5. Thus, we obtain the first condition. We also obtain the second condition from $(A1)$ in Lemma 4.5.

By Proposition 4.1, if $\epsilon$ is sufficiently small, there exist functions $a : U_2 \to L^p_{M+1}(M_r(\mathbb{C}))$ and $b : U_2 \to M_r(\mathbb{C})$ satisfying the following:

1. $\|D_\xi \Phi\|_{L^p_{M+1}} = O(\epsilon)$ for $k \in [0, M]_\mathbb{Z}$, and $\|D_\xi \gamma\| = O(\epsilon)$ for $k \in [0, M]_\mathbb{Z}$.
2. $(1 + a) \cdot (\nabla_{\tau, 0} + (\Gamma + b) d\tau) = \nabla_\tau$, where $\nabla_{\tau, 0}$ is given by $\nabla_{\tau, 0}v = 0$.

Let $u := v(1 + a)$. By construction, we have $\nabla_{\tau} u = u(\Gamma + b)$. The other estimates for $A_\Phi^2$ and $2A_\Phi^2$ are also satisfied. Because $H(h, u) = (1 + a)(1 + a)$, we obtain the estimate for $H(h, u)$.

**Remark 4.9** If $A_\Phi^2$ is constant along the $T$-direction, such a frame $w$ is called a partially almost holomorphic frame, in this paper.

### 4.2.4 Spectra

Let $E_\xi$ denote the holomorphic bundle on $T$ given by $E_{T \times \xi}$ with $\nabla_{T \times \xi}$. According to Lemma 4.7, if $\epsilon$ is sufficiently small, $E_\xi$ are semistable of degree 0 for any $\xi \in U_2$. We have the spectrum $\text{Sp}(E_\xi) \subset T^\vee$. We regard it as a point in $\text{Sym}^r T^\vee$. The point is denoted by $[\text{Sp}(E_\xi)]$. Let $\Gamma$ be as in 4.2.3. The eigenvalues of $\Gamma$ give a point in $\text{Sym}^r \mathbb{C}$, denoted by $[\text{Sp}(\Gamma)]$. The quotient map $\Phi : \mathbb{C} \to T^\vee$ induces $\text{Sym}^r \mathbb{C} \to \text{Sym}^r T^\vee$, denoted by $\Phi$. Recall that $\text{Sym}^r T^\vee$ is naturally a smooth complex manifold. Let $d_{\text{Sym}^r T^\vee}$ be a distance induced by a $C^\infty$-Riemannian metric.

**Corollary 4.10** There exist $\epsilon_0 > 0$ and $C > 0$, depending only on $r$, such that the following holds if $\epsilon \leq \epsilon_0$:

$$d_{\text{Sym}^r T^\vee}([\text{Sp}(E_\xi)], [\Phi[\text{Sp}(\Gamma)]] \leq C\epsilon$$

In particular, for $\xi, \xi' \in U_2$, we have $d_{\text{Sym}^r T^\vee}([\text{Sp}(E_\xi)], [\text{Sp}(E_{\xi'})]) \leq 2C\epsilon$.

**Proof** Let $u$ be a frame as in Proposition 4.1. Recall that $\text{Sym}^r \mathbb{C}$ is naturally a complex manifold. We take a distance $d_{\text{Sym}^r \mathbb{C}}$ induced by a $C^\infty$-Riemannian metric. We have $d_{\text{Sym}^r \mathbb{C}}([\text{Sp}(\Gamma)], [\text{Sp}(A_\Phi^2)]) \leq C_1\epsilon$. There exists $\zeta_0 \in \mathbb{C}$ such that $\text{Sp}(\Gamma)$ and $\text{Sp}(A_\Phi^2)$ are contained in $K_1(L, \zeta_0)$. Note that the restriction of $\Phi$ to $\text{Sym}^r K_1(L, \zeta_0)$ is Lipschitz continuous, and the Lipschitz constant is uniform for $\zeta_0$. Then, the claim of the corollary follows.

### 4.3 Estimates

#### 4.3.1 Preliminary

We continue to use the setting in 4.2. We impose additional assumptions.

**Assumption 4.11**

- We are given a finite subset $Z \subset \mathbb{C}$ and a positive number $\rho > 0$ with the following property:
  - $Z$ is contained in $K_1(L, \zeta_0)$, where $K_1(L, \zeta_0)$ is as in 4.1, for some appropriate $\delta > 0$.
  - For any distinct points $\nu_1, \nu_2 \in Z$, $d_C(\nu_1, \nu_2) > 100\rho^2$.
  - For any $\kappa \in \text{Sp}(E_\xi)$, there exists $\nu \in Z$ such that $d_{T^\vee}(\Phi(\nu), \kappa) < \rho$, where $\Phi : \mathbb{C} \to T^\vee$ denotes the projection.
  - $\epsilon$ is sufficiently small so that $E_\xi$ is semistable of degree 0 for any $\xi \in U_2$. We also assume that $\epsilon$ is sufficiently smaller than $\rho^2$.
We have the spectral decomposition $E_{ξ} = \bigoplus_{ν′ ∈ T^*} E_{ξ,ν′}$. Let $E_{ν,ξ}$ be the direct sum of $E_{ξ,ν′}$, where $ν′$ is contained in a $ρ$-ball of $Φ(ν)$. We obtain a decomposition $E_{ξ} = \bigoplus_{ν∈ Z} E_{ν,ξ}$. It induces a $C^∞$-decomposition $E = \bigoplus_{ν∈ Z} E_{ν}$, which is compatible with $∇_{ξ}$. We may assume that the partially almost holomorphic frame $u$ in Proposition 4.7 is compatible with the decomposition.

We have the decomposition $∇_{ξ} = ∇_{ξ,0} + f$ such that (i) $(E,∇_{ξ,0}|T^*ξ)$ are holomorphically trivial for any $ξ ∈ U_2$, (ii) $∇_{ξ}(f) = 0$, (iii) $Sp(f)$ is contained in the union of the $ρ$-balls around $ν ∈ Z$. For each $ξ ∈ U_2$, we obtain the vector space $V_ξ$ of the holomorphic global sections of $(E,∇_{ξ,0}|T^*ξ)$. It is easy to see that $V_ξ$ ($ξ ∈ U_2$) naturally gives a $C^∞$-vector bundle $V$ on $U_2$, and that we have a natural isomorphism $p^*V ≃ E$ as $C^∞$-bundles. We identify them by the isomorphism. A $C^∞$-section $s$ of $p^*V$ is constant along the $T$-direction, if and only if $∇_{ξ,0}s = 0$ under the identification. It can be regarded as a section of $V$. We have the decomposition $V = \bigoplus_{ν∈ Z} V_ν$, corresponding to $E = \bigoplus_{ν∈ Z} E_ν$.

### 4.3.2 Spaces of functions

Let $C^MξL^p_{M,ξ}$ denote the space of $C^M$-functions $U_2 → L^p_M(T)$. Let $C^MξL^p_{M,ξ}(E)$ denote the sections $f = \sum f_i u_i$ of $E$ such that $f_i ∈ C^MξL^p_{M,ξ}$, where $u_i = (u_i)$ is a frame as in Proposition 4.7. It is independent of the choice of $u$. We have the naturally defined integration $∫_T : C^MξL^p_{M,ξ}(E) → C^M(U_2,V)$. The kernel is denoted by $C^MξL^p_{M,ξ}(E)_0$. Similar spaces are defined for $End(E)$ and $Hom(E_ν, E_μ)$. We set

$$C^MξL^p_{M,ξ}(End(E))^\circ := \bigoplus_{ν} C^M(U_2, End(V_ν))$$

$$C^MξL^p_{M,ξ}(End(E))^\perp := \bigoplus_{ν} C^M(U_2, End(V_ν))_0 \oplus \bigoplus_{ν\neq μ} C^MξL^p_{M,ξ}(Hom(E_ν, E_μ)).$$

We have a decomposition $C^MξL^p_{M,ξ}(End(E)) = C^MξL^p_{M,ξ}(End(E))^\circ \oplus C^MξL^p_{M,ξ}(End(E))^\perp$. For any $s ∈ C^MξL^p_{M,ξ}(End(E))$, the corresponding decomposition is denoted by $s = s^\circ + s^\perp$. We use similar notations for sections of $End(E) ⊗ Ω^j_T$.

### 4.3.3 Some estimates

Let $u$ be a frame as in Proposition 4.7. We set $H(h, u)_{i,j} := h(u_i, u_j)$, and we obtain a function $H(h, u)$ from $T × U_2$ to the space $H$ of positive definite hermitian $r$-th matrices. Each entry is $C^MξL^p_{M,ξ}$-class. Let $H_1$ be a function of $U_2$ to $H$ determined by $(H_1)^2 = ∫_T H(h, u)$. Then, we have $|H_1 - I| = O(ε)$ and $|D^pξH_1| = O(ε)$ for $k ∈ [1,M]Z$. Note that $u^* := u H_1$ also has the property in Proposition 4.7. So, we may assume that $∫_T H(h, u) = 1$ from the beginning.

We set $g := H(h, u)$. We have $∥g - I∥_{L^p_{M+1}} = O(ε)$, $∥D^pξg∥_{L^p_{M+1}} = O(ε)$ ($k ∈ [1,M]Z$), and $∫_T g = I$.

**Lemma 4.12** There exist $C > 0$ and $ε_0 > 0$, such that $∥g - I∥_{L^p_{M+2}} \leq C ∥F^⊥_{ξ}|ξ|L^2$ holds if $ε < ε_0$. In particular, $sup_{T × ξ}|g - I| ≤ O(C^r∥F^⊥_{ξ}|ξ|L^2$ for some $C^r > 0$.

**Proof** We put $B := A^u_ξ$. Let $Γ_2$ be the diagonal matrix whose $(i,i)$-entry $ν_i$ is determined by $u_i ∈ E_ν$. Let $Γ$ be as in 4.2.3, which is decomposed $Γ = Γ_0 + Γ_1$ as in Lemma 4.8. We have $|Γ_0 - Γ_1| ≤ r ρ$ and $|Γ_1| = O(ε^{1/2})$. We have $|A^u_ξ - Γ| = O(ε)$. Hence, if $ε$ is sufficiently small, we may have $|A^u_ξ - Γ_2| ≤ 2 r ρ$.

We have $A^u_ξ = -g^{-1}(Tg) + g^{-1}∂_z g$. Let $B_{ξz}$ be the matrix-valued function determined by $F_{ξz}u = u B_{ξz}$. We have $B_{ξz} = (∂_z A^u_ξ - ∂_z A^u_ξ + [A^u_ξ, A^u_ξ]$. Hence, we have the following equation:

$$B_{ξz} = [g^{-1}(Tg) + g^{-1}∂_z g] - g^{-1}∂_z g [g^{-1}(Tg) + g^{-1}∂_z g] - [g^{-1}(Tg), B] - B, g^{-1}∂_z g$$

(26)

Let $b := g - I$. We have a polynomial $Q(t_1, t_2, t_3, t_4, t_5, t_6, t_7) = \sum Q_{j_1,...,j_m} t_{j_1} t_{j_2} \cdots t_{j_m}$ in non-commutative variables $t_i$ such that (i) if $Q_{j_1,...,j_m} ≠ 0$ then $m_1 + m_2 + m_3 ≥ 2$, where $m_i = \{ k | j_k = i \}$, (ii) the following equality holds:

$$(∂_+ - ad(B)) (∂_z - ad(T))]b = -gB_{ξz} - [T, B] + Q(b, ∂_z b, ∂_z b, (1 + b)^{-1}, B, T).$$

(27)
By taking the $\perp$-part, we obtain the following
\[
(\partial_\tau + \text{ad}(B)) \circ (\partial_\tau - \text{ad}(\overline{B})) b = -\langle bE,\overline{\tau} \rangle + Q(b, \partial_\tau b, \partial_{\overline{\tau}}b, (1 + b)^{-1}, B, \overline{B})^\perp. \tag{28}
\]
We obtain
\[
\|b\|_{L^p_{m+2}} \leq C_2 \|F^\perp_\tau\|_{L^p_m} + C_2 \|b\|_{L^p_{m+2}}.
\]
Hence, we obtain $\|b\|_{L^p_{m+2}} \leq C_3 \|F^\perp_\tau\|_{L^p_m}$. 

**Lemma 4.13** Let $a_1$ and $a_2$ be sections of $\text{End}(E)_{|T \times \{\xi\}}$. Assume that $a_1 = a_1^\perp$ and $a_2 = a_2^\perp$. Then, we have
\[
\left| \int_T h(a_1, a_2) \right| \leq \|a_1\|_{L^2} \cdot \|a_2\|_{L^2} \cdot \|\langle F^\perp_\tau \rangle_{|T \times \{\xi\}}\|_{L^2}.
\]

**Proof** It follows from Lemma 4.12 and $H(h, u) = \tilde{g}$. 

**Lemma 4.14** Let $P$ be an endomorphism of $E$, and let $P^\dagger$ denote the adjoint with respect to $h$. Let $R$ (resp. $R^\dagger$) be the matrix representing $P$ (resp. $P^\dagger$) with respect to $u$. Then, we have
\[
(R^\dagger)^0 = (R^0)^0 + O(|R^\dagger| \cdot \|F^\perp_\tau\|_{L^2}) + O(\|F^\perp_\tau\|_{L^2}^2 \cdot |R^0|)
\]
\[
(R^\dagger)^\perp = (R^\perp)^\perp + O(|R^\dagger| \cdot \|F^\perp_\tau\|_{L^2}) + O(\|F^\perp_\tau\|_{L^2} \cdot |R^\perp|).
\]
In particular, we have $|(R^\dagger)^\perp| = |R^\perp| + O(|R| \cdot \|F^\perp_\tau\|_{L^2})$.

**Proof** Let $H = H(h, u)$. We have $R^\dagger = \overline{H^{-1}(R)} H$. Then, the claim follows from the estimate for $H$.

**Lemma 4.15** For $k \in [1, M]_{\mathbb{Z}}$, we have $\|D_x^k g\|_{L^p_{m+2}} = O\left(\sum_{j=0}^k \|D_x^j F^\perp_\tau\|_{L^p_m}\right)$.

**Proof** We obtain the estimate from (28) by a standard inductive argument.

## 5 Estimates for $L^2$-Instantons

### 5.1 Preliminary

Let $\tau$ be a complex number such that $\text{Im} \tau > 0$. Let $T$ be a complex torus obtained as the quotient of $\mathbb{C}$ by a lattice $\mathbb{Z} + Z \tau$. Let $z$ be the standard coordinate of $\mathbb{C}$. It also gives a local coordinate of a small open subset in $T$, once we fix a lift of the open subset in $\mathbb{C}$. We shall use the metric $dz \overline{dz}$ for $\mathbb{C}$ and $T$ unless otherwise specified.

For any open subset $W \subset T_w$, we use the metric $dw \overline{dw}$ on $W$, and the metric $dz \overline{dz} + dw \overline{dw}$ on $T \times W$ unless otherwise specified. Let $\omega$ denote the associated Kähler form. For $w \in W$, we put $T_w := T \times \{w\} \subset T \times W$.

Let $E$ be a complex $C^\infty$-vector bundle on $T \times W$ with a hermitian metric $h$ and a unitary connection $\nabla$. Let $F(\nabla)$ denote the curvature of $\nabla$. We shall often denote it simply by $F$. The $(0,1)$-part and the $(1,0)$-part of $\nabla$ are denoted by $\partial_E$ and $\partial_{\overline{E}}$ respectively. The restrictions of $(E, h)$ to $T_w$ are denoted by $(E_w, h_w)$.

Recall that $(E, \nabla, h)$ is called an instanton, if $\Lambda_w F(\nabla) = 0$. For the expression $F(\nabla) = F_w dz \overline{dz} + F_{w\overline{w}} dz \overline{dw} + F_{w\overline{w}}^w dz \overline{dz} + F_{w\overline{w}}^w dz \overline{dw} + F_w^w dz \overline{dw}$, the equation is $F_w^w = F_{w\overline{w}} = 0$. We have the following equalities:
\[
(\nabla_z \overline{\nabla}_\tau + \nabla_w \nabla_{\overline{w}}) F_{w\overline{w}} = - (\nabla_z \overline{\nabla}_\tau + \nabla_w \nabla_{\overline{w}}) F_z^\tau = [F_w^\tau, F_{w\overline{w}}] \tag{29}
\]
\[
(\nabla_z \overline{\nabla}_\tau + \nabla_w \nabla_{\overline{w}}) F_{wz} = [F_{w\overline{w}}, F_{wz}], \quad (\nabla_z \overline{\nabla}_\tau + \nabla_w \nabla_{\overline{w}}) F_{w\overline{w}} = [F_{w\overline{w}}, F_{w\overline{w}}] \tag{30}
\]
5.1.1 Hitchin’s equivalence

Let us recall the relation between harmonic bundles on an open subset $W \subset \mathbb{C}_w$ and instantons on $T \times W$ due to Hitchin. Let $(E, \nabla, h, \theta)$ be a harmonic bundle on $W$. Let $\nabla^0 := \overline{\nabla}_E + \partial_E$ be the Chern connection. Let $\theta^\dagger$ be the adjoint of $\theta$. Let $p : T \times W \to W$ be the projection. The pull back $p^* (E, \nabla^0, h)$ is denoted by $(E_1, \nabla^1, h_1)$. We set $\nabla := \nabla^1 + f d\bar{z} - fdz$. Then, $(E_1, \nabla, h_1)$ is an instanton on $T \times W$.

Conversely, let $(E_2, \nabla^2, h_2)$ be a $T$-equivariant instanton on $T \times W$. By considering $T$-equivariant sections, we obtain a vector bundle $E$ on $W$ such that $p^* E \simeq E_2$. It is naturally equipped with a connection $\nabla^0$ such that $p^* \nabla^0 = \nabla^2$, where $v$ denotes the natural horizontal lift of vector fields on $W$. By using the $T$-equivariance of $\nabla^2$, we have the expression $\nabla^2 - p^* \nabla^0 = p^* f d\bar{z} - p^* f dz$, where $f$ is a section of $\text{End}(E)$. Then, $(E, \overline{\nabla}_E, h, f dz)$ is a harmonic bundle. In summary, we have the following.

**Proposition 5.1 (Hitchin)** Harmonic bundles on $W$ naturally correspond to $T$-equivariant instantons on $T \times W$.

5.2 Local estimate

Let $U$ be a closed disc $\{ |w - w_0| \leq 1 \}$ of $\mathbb{C}$. Let $(E, \nabla, h)$ be an instanton on $T \times U$.

**Assumption 5.2** We assume that $|F(\nabla)| \leq \epsilon$ for a given positive small number $\epsilon$. We also impose Assumption 4.11.

We use the notation in 4.12. Note that we have $|D^k_w D^k_{\bar{w}} F| \leq C_{k_1, k_2} \epsilon$, where $C_{k_1, k_2}$ are constants depending only on $(k_1, k_2)$.

5.2.1 Estimates of the $\perp$-part of the connection form

Let $u$ be a partially almost holomorphic frame as in Proposition 4.7. We also assume that $f, H(h, u) = 1$, as in 4.3.3. Let $A$ be the connection form of $\nabla$ with respect to $u$. We use $B_{\sigma \tau}$ and $B_{u \sigma \tau}$ in similar meanings.

We prepare a notation in a general situation. Let $V$ be any vector bundle with a hermitian metric $h_V$ on $U$. Let $\pi : T \times U \to U$ be the projection. Let $p \geq 2$. For any section $\tilde{f}$ of $\pi^* V$ on $T \times U$, let $|f|_p$ denote the function on $U$ given by $|f|_p(w) = \left( \int_{T \times \{w\}} |f|^p_{h_V} \right)^{1/p}$.

**Lemma 5.3** We have $\| A_w \|_p = O(\| F_{\bar{w}}^{-1} \|_p)$ and $\| \partial_{\bar{w}} A_w \|_p = O(\| \nabla_{\bar{w}} F_{\bar{w}} \|_p) + O(\epsilon \| F_{\bar{w}} \|_p)$.

**Proof** Because $\partial_{\bar{w}} A_w - \partial_{\bar{w}} A_w + [A_w, A_w] = B_{w \sigma \tau}$, we have the following equalities:

$$\partial_{\bar{w}} A_w + [A_w, A_w] = -B_{w \sigma \tau} \quad (31)$$

Then, we obtain the first estimate. We also obtain the following equation:

$$\partial_{\bar{w}} \partial_{\bar{w}} A_w + [A_w, A_w] = -\partial_{\bar{w}} B_{w \sigma \tau} - [\partial_{\bar{w}} A_w, A_w]$$

Because $\partial_{\bar{w}} A_w = O(\epsilon)$, we obtain the second estimate.

**Lemma 5.4** We have $\| A_w \|_p = O(\| F_{\bar{w}} \|_p + \| F_{\bar{w}} \|_p + \| \nabla_{\bar{w}} F_{\bar{w}} \|_p)$. We also have

$$\| \partial_{\bar{w}} A_w \|_p = O(\| \nabla_{\bar{w}} F_{\bar{w}} \|_p + \| F_{\bar{w}} \|_p + \| F_{\bar{w}} \|_p + \| \nabla_{\bar{w}} F_{\bar{w}} \|_p)$$

**Proof** We set $\tilde{g} := H(h, u)$. We have $A_{\bar{w}} = -\tilde{g}^{-1} (A_w) \tilde{g} + \tilde{g}^{-1} \partial_{\bar{w}} \tilde{g}$. Hence, the first claim follows from Lemma 5.3 and Lemma 4.12. We have $\partial_{\bar{w}} A_{\bar{w}} - \partial_{\bar{w}} A_w + [A_w, A_w] = B_{w \sigma \tau}$. Hence, we have

$$\| \partial_{\bar{w}} A_w \|_p = O(\| \partial_{\bar{w}} A_w \|_p) + O(\| A_w \|_p + \| A_w \|_p + \| F_{\bar{w}} \|_p)$$

Then, the second claim follows.
5.2.2 Estimate of the $\perp$-part of the curvature

We prepare a notation in a general situation. Let $V$ be any vector bundle with a hermitian metric $h_V$ on $T \times U$. Let $\pi : T \times U \to U$ be the projection. For any section $f$ of $V$ on $T \times U$, let $\|f\|$ denote the function on $U$ given by $\left(\int_T |f|^2 h_V\right)^{1/2}$. For any sections $f$ and $g$ of $V$, let $\langle f, g \rangle$ denote the function on $U$ given by $\int_T h_V(f, g)$.

**Proposition 5.5** We have the following:

$$\Delta \|F_{\perp}\|^2 \leq -\|\nabla_{\perp}F_{\perp}\|^2 - \|\nabla_{\perp}F_{\perp}\|^2 - \|\nabla_{\perp}F_{\perp}\|^2 - \|\nabla_{\perp}F_{\perp}\|^2$$

$$+ O\left(\epsilon \|F_{\perp}\|^2 + \epsilon \|F_{\perp}\| |F_{\perp}\| + \epsilon \|\nabla_{\perp}F_{\perp}\| |F_{\perp}\| + \epsilon \|\nabla_{\perp}F_{\perp}\| |F_{\perp}\|\right)$$

$$+ O\left(\epsilon \|\nabla_{\perp}F_{\perp}\| |F_{\perp}\| + \epsilon \|F_{\perp}\|^2 + \epsilon \|F_{\perp}\| |\nabla_{\perp}F_{\perp}\|\right)$$

(32)

**Proof** We have the following equation:

$$\Delta \|F_{\perp}\|^2 = -(\nabla_{\perp} \nabla_{\perp} F_{\perp} + (F_{\perp}^\perp \nabla_{\perp} F_{\perp}) - (\nabla_{\perp} F_{\perp}^\perp \nabla_{\perp} F_{\perp}) - (\nabla_{\perp} \nabla_{\perp} F_{\perp}^\perp) + (\nabla_{\perp} \nabla_{\perp} F_{\perp}^\perp)$$

Let us consider the estimate of $(\nabla_{\perp} \nabla_{\perp} F_{\perp}^\perp)$. The endomorphism $\nabla_{\perp} \nabla_{\perp} F_{\perp}^\perp$ is represented by the following with respect to $u$:

$$\partial_u \partial_u B_{\perp}^2 + [A_w, \partial_u B_{\perp}^2] + \partial_w [A_w, B_{\perp}^2] + [A_w, [A_w, B_{\perp}^2]]$$

Recall Lemma 4.13. We have the following estimates:

$$\left(\partial_u \partial_u B_{\perp}^2, B_{\perp}^2\right) = O\left(\|\partial_u \partial_u B_{\perp}^2\| \|B_{\perp}^2\| \|F_{\perp}^\perp\|\right)$$

(33)

$$\left([A_w, \partial_u B_{\perp}], B_{\perp}^2\right) = O\left(\|A_w \| \|\partial_u B_{\perp}^2\| \|B_{\perp}^2\| \|F_{\perp}^\perp\|\right)$$

(34)

We obtain the following estimate for $(\nabla_{\perp} \nabla_{\perp} F_{\perp}^\perp)$ from (33)–(37) with Lemma 5.3

$$\langle \nabla_{\perp} \nabla_{\perp} F_{\perp}^\perp, F_{\perp}^\perp\rangle = O\left(\epsilon \|F_{\perp}^\perp\|^2 + \epsilon \|F_{\perp}^\perp\| |F_{\perp}^\perp\| + \epsilon \|\nabla_{\perp} F_{\perp}^\perp\| |F_{\perp}^\perp\| + \epsilon \|\nabla_{\perp} F_{\perp}^\perp\| |F_{\perp}^\perp\|\right)$$

(38)

We have

$$- \langle \nabla_{\perp} \nabla_{\perp} F_{\perp}^\perp, F_{\perp}^\perp\rangle = \langle \nabla_{\perp} \nabla_{\perp} F_{\perp}^\perp, F_{\perp}^\perp\rangle + \langle [F_{\perp}^\perp, F_{\perp}^\perp], F_{\perp}^\perp\rangle = -\langle \nabla_{\perp} F_{\perp}^\perp, \nabla_{\perp} F_{\perp}^\perp\rangle + \langle [F_{\perp}^\perp, F_{\perp}^\perp], F_{\perp}^\perp\rangle$$

(39)

We have

$$- \langle \nabla_{\perp} F_{\perp}^\perp, \nabla_{\perp} F_{\perp}^\perp\rangle = -\langle \nabla_{\perp} F_{\perp}^\perp, \nabla_{\perp} F_{\perp}^\perp\rangle - \langle \nabla_{\perp} F_{\perp}^\perp, \nabla_{\perp} F_{\perp}^\perp\rangle$$

$$= -\langle \nabla_{\perp} F_{\perp}^\perp, \nabla_{\perp} F_{\perp}^\perp\rangle + O\left(\|\nabla_{\perp} F_{\perp}^\perp\| |\nabla_{\perp} F_{\perp}^\perp\| \|F_{\perp}^\perp\|\right)$$

(40)
We also have the following:

\[
\left(\left[[F_{z,w}, F_{z,w}]^\perp, F_{z,w}^\perp\right]\right) = O\left(||[F_{z,w}, F_{w,z}]^\perp||F_{z,w}^\perp||F_{w,z}^\perp|| + O\left(||F_{z,w}^\perp|| ||F_{w,z}^\perp|| ||F_{z,w}^\perp||\right) + O\left(||F_{w,z}^\perp|| ||F_{z,w}^\perp|| ||F_{w,z}^\perp||\right) + O\left(||F_{w,z}^\perp|| ||F_{z,w}^\perp|| ||F_{w,z}^\perp||\right)
\]

(41)

We have a similar estimate for the contribution of \(-\left(F_{z,w}^\perp, \nabla F_{z,w}^\perp\right)\). In all, we obtain the claim of Proposition 5.5

**Proposition 5.6** We have the following inequality:

\[
\Delta_w ||F_{z,w}^\perp||^2 \leq -||\nabla F_{z,w}^\perp||^2 - ||\nabla w F_{z,w}^\perp||^2 - ||\nabla F_{w,z}^\perp||^2 - ||\nabla F_{w,z}^\perp||^2
\]

\[
+ O\left(\epsilon ||F_{z,w}^\perp|| ||F_{w,z}^\perp|| + \epsilon||\nabla F_{w,z}^\perp|| ||F_{z,w}^\perp|| + ||\nabla F_{w,z}^\perp|| + \epsilon||\nabla F_{z,w}^\perp|| ||F_{w,z}^\perp||\right)
\]

\[
+ O\left(\epsilon||\nabla F_{z,w}^\perp|| ||F_{z,w}^\perp|| + \epsilon||F_{w,z}^\perp|| ||F_{w,z}^\perp||\right)
\]

(42)

**Proof** We have the following:

\[
- \partial_w \partial \nabla F_{z,w}^\perp = -||\nabla F_{z,w}^\perp||^2 - ||\nabla w F_{z,w}^\perp||^2 - (\nabla w \nabla F_{z,w}^\perp, F_{z,w}^\perp) - (F_{z,w}^\perp, \nabla w \nabla F_{z,w}^\perp)
\]

(43)

We have

\[
- (\nabla w \nabla F_{z,w}^\perp, F_{z,w}^\perp) = - (\nabla w \nabla F_{z,w}^\perp, F_{z,w}^\perp) + (\nabla w \nabla F_{z,w}^\perp, F_{z,w}^\perp)
\]

(44)

Let us look at the contribution of \((\nabla w \nabla F_{z,w}^\perp, F_{z,w}^\perp)\). Let \(B_{z,w}\) express \(F_{z,w}\) with respect to \(u\) as in the proof of Proposition 5.5 Then, \(\nabla w \nabla F_{z,w}^\perp\) is represented by the following:

\[
\partial_w \partial \nabla B_{z,w}^\perp + [\partial_w A_{z,w}, B_{z,w}^\perp] + [A_{z,w}, \partial_w B_{z,w}^\perp] + [A_w, \partial w B_{z,w}^\perp] + [A_w, [A_{z,w}, B_{z,w}^\perp]]
\]

We have the following estimates:

\[
- (\partial_w \partial \nabla B_{z,w}^\perp, B_{z,w}^\perp)_h = O\left(||\partial_w \partial \nabla B_{z,w}^\perp||_h ||B_{z,w}^\perp||_h ||F_{z,w}^\perp||_h\right)
\]

(45)

\[
\left(\left[B_{z,w}^\perp, B_{z,w}^\perp\right]\right)_h = O\left(||[B_{z,w}^\perp, B_{z,w}^\perp]||_h ||B_{z,w}^\perp||_h ||F_{z,w}^\perp||_h\right) + O\left(||[B_{z,w}^\perp, B_{z,w}^\perp, B_{z,w}^\perp]||_h \right)
\]

(46)

\[
\left([A_{z,w}, B_{z,w}^\perp], B_{z,w}^\perp\right)_h = O\left(||[A_{z,w}, B_{z,w}^\perp, B_{z,w}^\perp, B_{z,w}^\perp]|_{||B_{z,w}^\perp||_h} ||F_{z,w}^\perp||_h\right) + O\left(||[A_{z,w}, B_{z,w}^\perp]|_{||B_{z,w}^\perp||_h} ||B_{z,w}^\perp||_h\right)
\]

(47)

\[
\left([A_{z,w}, B_{z,w}^\perp, B_{z,w}^\perp]|_{||B_{z,w}^\perp||_h} \right)_h = O\left(||[A_{z,w, B_{z,w}^\perp, B_{z,w}^\perp]}|_{||B_{z,w}^\perp||_h} ||B_{z,w}^\perp||_h\right)
\]

(48)

\[
\left([A_{z,w}, [A_{z,w}, B_{z,w}^\perp, B_{z,w}^\perp]|_{||B_{z,w}^\perp||_h} \right)_h = O\left(||[A_{z,w, [A_{z,w}, B_{z,w}^\perp, B_{z,w}^\perp]}]|_{||B_{z,w}^\perp||_h} ||B_{z,w}^\perp||_h\right) + O\left(||[A_{z,w, B_{z,w}^\perp, B_{z,w}^\perp}]|_{||B_{z,w}^\perp||_h} ||B_{z,w}^\perp||_h\right) + O\left(||[A_{z,w, B_{z,w}^\perp, B_{z,w}^\perp}]|_{||B_{z,w}^\perp||_h} ||B_{z,w}^\perp||_h\right)
\]

(49)

Hence, we obtain the following:

\[
\left(\nabla w \nabla F_{z,w}^\perp, F_{z,w}^\perp\right) = O\left(\epsilon ||F_{z,w}^\perp|| ||F_{w,z}^\perp||_h + \epsilon||\nabla F_{w,z}^\perp|| ||F_{z,w}^\perp||_h + \epsilon||F_{w,z}^\perp|| ||F_{z,w}^\perp||_h + \epsilon||F_{z,w}^\perp|| ||F_{w,z}^\perp||_h\right)
\]

(50)

We have

\[
\left(\left(\nabla w \nabla F_{z,w}^\perp, F_{z,w}^\perp\right)\right) = \left(\nabla w \nabla F_{z,w}^\perp, F_{z,w}^\perp\right) - 2\left(\left(F_{w,z}^\perp, F_{z,w}^\perp\right)\right)
\]

\[
= -\left(\nabla F_{z,w}^\perp, \nabla F_{z,w}^\perp\right) - 2\left(\left(F_{w,z}^\perp, F_{z,w}^\perp\right)\right)
\]

(51)
We have
\[-(\nabla w F_{2w}, \nabla F_{2w}^\perp) = -(\nabla w F_{2w}^\perp, \nabla F_{2w}^\perp) - (\nabla w F_{2w}^\perp, \nabla F_{2w}^\perp)\]
\[= - (\nabla w F_{2w}^\perp, \nabla F_{2w}^\perp) + O \left( \|w\| \|\nabla F_{2w}^\perp\| \right) \]  
(52)

We also have
\[\left( \|w\|, F_{2w}^\perp \right) = O \left( \|w\| \|F_{2w}^\perp\| \right) + O \left( \|w\| \|F_{2w}^\perp\| \right) + O \left( \|w\| \|F_{2w}^\perp\| \right) \]
\[+ O \left( \|w\| \|F_{2w}^\perp\| \right) + O \left( \|w\| \|F_{2w}^\perp\| \right) \]
(53)

We have a similar estimate for the contribution of \(-(F_{2w}^\perp, \nabla w F_{2w}^\perp)\). In all, we obtain the desired estimate \((\ref{eq:52})\). \hfill \blacksquare

**Proposition 5.7** There exist \(C > 0\) and \(\epsilon_0 > 0\) such that the following inequality holds if \(\epsilon < \epsilon_0\):

\[\Delta_w (\|F_{2w}^\perp\|^2 + \|F_{2w}^\perp\|^2) \leq - C \left( \|F_{2w}^\perp\|^2 + \|F_{2w}^\perp\|^2 \right) - C \left( \|F_{2w}^\perp\|^2 + \|\nabla F_{2w}^\perp\|^2 + \|w\| \|F_{2w}^\perp\|^2 \right) \]
\[- C \left( \|F_{2w}^\perp\|^2 + \|\nabla F_{2w}^\perp\|^2 + \|w\| \|F_{2w}^\perp\|^2 + \|\nabla F_{2w}^\perp\|^2 \right) \]
(54)

**Proof** There exist \(C_1 > 0\) such that \(\|\nabla s \| \geq C_1 \|s\|\) and \(\|\nabla s \| \geq C_1 \|s\|\) for any section of \(\text{End}(E)\) such that \(s = s^\perp\). Then, the claim follows from Proposition 5.5 and Proposition 5.6 \hfill \blacksquare

5.2.3 Higher derivative

Assume that \(\|F_{2w}^\perp\|^2 + \|F_{2w}^\perp\|^2 \leq \delta^2\) for some \(\delta << \epsilon\). For \(\rho < 1\), we set \(U(\rho) = \{w \mid \|w - w_0\| \leq \rho\} \subset U\).

**Proposition 5.8** For any \(k, \rho\), there exists \(C > 0\) such that

\[\|F_{2w}^\perp\|_{L^\infty(T \times U(\rho))} \leq C \delta, \quad \|F_{2w}^\perp\|_{L^\infty(T \times U(\rho))} \leq C \delta.\]

**Proof** It can be shown by a standard bootstrapping argument. We give only an indication. We take \(\rho < \rho' < 1\). In the following, we shall replace \(\rho'\) with smaller one. Let \(\kappa\) denote \(z, \mathfrak{z}, w\) and \(\mathfrak{w}\). By Proposition 5.7, we obtain \(\|\nabla \kappa F_{2w}^\perp\|_{L^2(T \times U(\rho'))} = O(\delta)\) and \(\|\nabla \kappa F_{2w}^\perp\|_{L^2(T \times U(\rho'))} = O(\delta)\).

With respect to the frame \(u\), the endomorphism \(-\nabla \mathfrak{w} \nabla w F_{2w}^\perp\) is represented by

\[- \partial_w \partial_{\mathfrak{w}} B_{2\mathfrak{w}} - \partial_w A_{\mathfrak{w}}, B_{2\mathfrak{w}} - [A_{\mathfrak{w}}, \partial_{\mathfrak{w}} B_{2\mathfrak{w}}] + [A_{\mathfrak{w}}, \partial_{\mathfrak{w}} B_{2\mathfrak{w}}] + [A_{\mathfrak{w}}, [A_{\mathfrak{w}}, B_{2\mathfrak{w}}]],\]
(55)

and the endomorphism \(-\nabla w \nabla F_{2w}^\perp\) is represented by

\[- \partial_z \partial_{\mathfrak{z}} B_{2\mathfrak{z}} - [A_{\mathfrak{z}}, \partial_{\mathfrak{z}} B_{2\mathfrak{z}}] + [A_{\mathfrak{z}}, \partial_{\mathfrak{z}} B_{2\mathfrak{z}}] + [A_{\mathfrak{z}}, [A_{\mathfrak{z}}, B_{2\mathfrak{z}}]].\]
(56)

The sum of (55) and (56) is equal to \([B_{\mathfrak{w} \mathfrak{w}}, B_{\mathfrak{w} \mathfrak{w}}]\). By looking at the \(\perp\)-part of the equation, we obtain the following equation:

\[\text{The } \perp\text{-part of (55) + The } \perp\text{-part of (56) = } [B_{\mathfrak{w} \mathfrak{w}}, B_{\mathfrak{w} \mathfrak{w}}]\perp\]
(57)

By using Lemma 5.3 and Lemma 5.4, we obtain \(\|\partial_{\mathfrak{w}} \partial_{\mathfrak{w}} + \partial_{\mathfrak{z}} \partial_{\mathfrak{z}} B_{2\mathfrak{w}}\|_{L^2(T \times U(\rho'))} = O(\delta)\). Similarly, we obtain \(\|\partial_{\mathfrak{w}} \partial_{\mathfrak{w}} + \partial_{\mathfrak{z}} \partial_{\mathfrak{z}} B_{2\mathfrak{w}}\|_{L^2(T \times U(\rho'))} = O(\delta)\). It follows that

\[\|F_{2w}^\perp\|_{L^4(T \times U(\rho'))} + \|F_{2w}^\perp\|_{L^4(T \times U(\rho'))} = O(\delta)\]
\[\|\nabla \kappa F_{2w}^\perp\|_{L^2(T \times U(\rho'))} + \|\nabla \kappa F_{2w}^\perp\|_{L^2(T \times U(\rho'))} = O(\delta)\]

By using Lemma 5.3, Lemma 5.4 and (57), we obtain \(\|\partial_{\mathfrak{w}} \partial_{\mathfrak{w}} + \partial_{\mathfrak{z}} \partial_{\mathfrak{z}} B_{2\mathfrak{w}}\|_{L^4(T \times U(\rho'))} = O(\delta)\). Similarly, we obtain \(\|\partial_{\mathfrak{w}} \partial_{\mathfrak{w}} + \partial_{\mathfrak{z}} \partial_{\mathfrak{z}} B_{2\mathfrak{w}}\|_{L^4(T \times U(\rho'))} = O(\delta)\). By the same argument, we obtain the following for any \(p:\)

\[\|F_{2w}^\perp\|_{L^p(T \times U(\rho'))} + \|F_{2w}^\perp\|_{L^p(T \times U(\rho'))} + \|\nabla \kappa F_{2w}^\perp\|_{L^p(T \times U(\rho'))} + \|\nabla \kappa F_{2w}^\perp\|_{L^p(T \times U(\rho'))} = O(\delta)\]
Then, \( \phi \) is holomorphic. We can also regard it as a harmonic map between Kähler manifolds. Let \( \phi \) be the derivative of \( \phi \), be the closure of \( \mathcal{L} \). By Corollary 4.10, for any \( (x, w) \) of Uhlenbeck [54], for any \( (x, w) \), we have \( |F(x, w)| = O(\|F|_{T \times \mathcal{B}_w(2)}\|_{L^2}) = O(\epsilon) \). In particular, we may assume that \( E_w \) are semistable if \( w \in \mathcal{Y}_{2R_1} \). Because we are interested in the behaviour around infinity, we may assume that \( (E_w, \mathcal{D}_{E_w}) \) are semistable of degree 0 for any \( w \in \mathcal{Y}_{R_1} \), from the beginning.

5.3.2 Prolongation of the spectral curve

We consider the relative Fourier-Mukai transform \( RFM_\phi(E, \mathcal{D}_E) \), which is a coherent sheaf on \( T^\vee \times \mathcal{Y}_{R_1} \). The support is relatively 0-dimensional over \( \mathcal{Y}_{R_1} \), denoted by \( \mathcal{S}_p(E) \). It is called the spectral curve of \( (E, \mathcal{D}_E) \). Let \( \mathcal{Y}_{R_1} \) be the closure of \( \mathcal{Y}_{R_1} \) in \( \mathbb{P}^1 \), i.e., \( \mathcal{Y}_{R_1} = \mathcal{Y}_{R_1} \cup \{ \infty \} \).

Theorem 5.10 \( \mathcal{S}_p(E) \) is extended to a closed subvariety \( \mathcal{S}_p(E) \) in \( T^\vee \times \mathcal{Y}_{R_1} \).

Proof Let \( \rho \) denote the rank of \( E \). We have the holomorphic map \( \phi : \mathcal{Y}_{R_1} \rightarrow \text{Sym}^\rho T^\vee \) induced by \( \mathcal{S}_p(E) \). We have only to show that it is an extension of a holomorphic map \( \mathcal{Y}_{R_1} \rightarrow \text{Sym}^\rho T^\vee \). We fix a closed immersion \( \text{Sym}^\rho T^\vee \subset \mathbb{P}^N \) for a sufficiently large \( N \), and we regard \( \phi \) as a holomorphic map \( \mathcal{Y}_{R_1} \rightarrow \mathbb{P}^N \). Let \( d_p \) denote the distance of \( \mathbb{P}^N \), induced by the Fubini-Study metric.

Take any \( w_0 \in \mathcal{Y}_{2R_1} \). By Corollary 1.11 for any \( w_1, w_2 \in \mathcal{B}_{w_0}(1/2) \), we have

\[
\|d_{p,N}(\phi(w_1), \phi(w_2))\| = O(\|F|_{T \times \mathcal{B}_w(2)}\|_{L^2}).
\]

(58)

Note that \( \phi \) is holomorphic. We can also regard it as a harmonic map between Kähler manifolds. Let \( T_w \phi \) be the derivative of \( \phi \), and \( |T_w \phi| \) denote the norm of \( T_w \phi \) with respect to the Euclidean metric \( dw \, d\overline{w} \) and the Fubini-Study metric of \( \mathbb{P}^N \). For any \( w \in \mathcal{B}_{w_0}(1/4) \), we obtain the following estimate from (58) by using Cauchy’s formula for differentiation in complex analysis:

\[
\|T_w \phi\| = O(\|F|_{T \times \mathcal{B}_w(2)}\|_{L^2})
\]

(59)

Hence, we obtain the following finiteness of the energy of the harmonic map \( \phi \):

\[
\int_{\mathcal{Y}_{2R_1}} |T_w \phi|^2 \, |dw \, d\overline{w}| < C \|F|_{Y_{R_1}}\|_{L^2}^2 < \infty
\]

Then, \( \phi \) is extended on \( \mathcal{Y}_{R_1} \), according to Theorem 3.6 in [10].

The intersection \( \mathcal{S}_p(E) \cap (T^\vee \times \{\infty\}) \) is denoted by \( \mathcal{S}_p_{\infty}(E) \).
5.3.3 Asymptotic decay

We fix a lift of $\overline{\partial p}(E)$ to a closed subvariety $\overline{\partial p}(E)_1 \subset Y_R \times \mathbb{C}_\zeta$, which induces an action of $\zeta$ on $\text{RFM}_-(E, \overline{\partial E})$. (See [2.1]) Let $f_{\zeta}$ be the corresponding holomorphic endomorphism of $E$. We set $\overline{\partial}_0 := \overline{\partial E} - f_{\zeta}d\overline{\zeta}$, which gives a holomorphic structure of $E$. For each $w$, the restriction of $E' := (E, \overline{\partial}_0)$ to $T_z \times \{w\}$ is holomorphically trivial. It is naturally isomorphic to $p^*P(E')$, where $p : X_R \to Y_R$ denotes the natural projection. We obtain the decomposition $h = h^0 + h^\perp$ as in 4.4.3.3

**Theorem 5.11** For any polynomial $P(t_1, t_2, t_3, t_4)$ of non-commutative variables, there exists $C > 0$ such that

$$P(\nabla_z, \nabla_{\zeta}, \nabla_w, \nabla_{\overline{\zeta}})h^\perp = O(\exp(-C|w|)).$$

**Proof** Let $\epsilon > 0$ be any sufficiently small number. We may assume that $\|F_{|X_R}| < \epsilon$ for some $R_1 > 0$. By Theorem 5.10 we may assume that Assumption 5.2 is satisfied for the restriction of $(E, \nabla, h)$ to any disc contained in $X_{R_1}$. In particular, we can apply Proposition 5.7 to $(E, \nabla, h)|_{X_{R_1}}$. We obtain the following inequality for some $C_1 > 0$:

$$\Delta_w(\|F_{\zeta}^w\|^2 + \|F_{\zeta}^{\overline{\zeta}}\|^2) \leq -C_1(\|F_{\zeta}\|^2 + \|F_{\zeta}^{\overline{\zeta}}\|^2)$$

We obtain the following lemma by a standard argument.

**Lemma 5.12** We have $\|F_{\zeta}^w\|^2 + \|F_{\zeta}^{\overline{\zeta}}\|^2 = O\left(\exp(-C_2|w|)\right)$ for some $C_2 > 0$.

**Proof** This is a variant of a lemma of Ahlfors ([2], [29]). We give only an indication. We put $G := \|F_{\zeta}^w\|^2 + \|F_{\zeta}^{\overline{\zeta}}\|^2$. We put $f_{\zeta} := C_1\exp(-2C_1^2/|w|) + \epsilon$, where $\epsilon > 0$ and $C_3 > 0$. We have the inequality $\Delta_w f_{\zeta} \geq -C_1 f_{\zeta}$. If $C_3$ is sufficiently large, we have $f_{\zeta} > G$ on $|w| = R_1$. For each $\epsilon > 0$, we have $f_{\zeta} > G$ outside a compact subset. We put $U := \{w \mid f_{\zeta}(w) < G(w)\}$. Then, $U$ is relatively compact, and we have $f_{\zeta} = G$ on the boundary of $U$. On $U$, we have $\Delta_w (G - f_{\zeta}) \leq -C(G - f_{\zeta}) \leq 0$. By the maximum principle, we have $\sup_U (G - f_{\zeta}) \leq \max_{\partial U} (G - f_{\zeta}) = 0$. Hence, we obtain that $U$ is empty. It means $G \leq f_{\zeta}$ on $Y_R$ for any $\epsilon$. We obtain the desired inequality by taking the limit $\epsilon \to 0$.

Then, the claim of Theorem 5.11 follows from Corollary 5.9.

5.3.4 Reduction to asymptotic harmonic bundles

Let $p : X_R \to Y_R$ denote the projection. By using the push-forward of $\mathcal{O}$-modules, we obtain a holomorphic vector bundle $V := p_*E'$ on $Y_R$. It is equipped with a Higgs field $\theta_V := f_{\zeta}dw$. For any $s_i \in V_w$ ($i = 1, 2$), the corresponding holomorphic section of $E'_{|s_w}$ is denoted by $\tilde{s}_i$. We set $h_V(s_1, s_2) := \int_{X_R} h(\tilde{s}_1, \tilde{s}_2)$. We have the Chern connection $\overline{\partial}V + \theta_V$ with respect to $h_V$. Let $\theta_V^+$ denote the adjoint of $\theta_V$.

**Proposition 5.13** There exists $C > 0$ such that the following holds:

$$F(h_V) + [\theta_V, \theta_V^+] = O(\exp(-C|w|)). \quad (60)$$

**Proof** We identify $p^*V = E'$. According to Theorem 5.11 the difference $h - p^*h_V$ and its derivatives are $O(\exp(-C_1|w|))$. (The constant $C_1$ may depend on the order of derivatives.) We also have $\overline{\partial}_E = p^*\overline{\partial}V + f_{\zeta}d\overline{\zeta}$. Hence, $(p^*V, p^*\overline{\partial}_V + f_{\zeta}d\overline{\zeta}, p^*h_V)$ satisfies $\Delta_w F(p^*h_V) = O(\exp(-C_2|w|))$, which is equivalent to (60).

5.3.5 Estimate of the curvature

**Theorem 5.14** There exist $\rho > 0$ such that the following holds:

$$F(h) = O\left(\frac{dz d\overline{\zeta}}{|w|^2(-\log|w|)^2}\right) + O\left(\frac{dw d\overline{w}}{|w|^2(-\log|w|)^2}\right) + O\left(\frac{dw d\overline{w}}{|w|^{1+\rho}}\right) + O\left(\frac{dz d\overline{w}}{|w|^{1+\rho}}\right)$$

49
Corollary 5.15 Let \( \varphi : \Delta_u = \{ |u| < R^{-1/\epsilon} \} \rightarrow \overline{\mathbb{Y}}_R \) be given by \( \varphi(u) = u^\epsilon \). For the expression \( \theta = f_z dw = f_z(\varepsilon u^{-\epsilon-1} du) \), according to Theorem 5.10 the spectral curve \( \mathcal{S}p(f_z) \subset \mathbb{C} \times Y_R \) is contained in \( \{|z| \leq R'\} \times Y_R \) for some \( R' \), and the closure in \( \mathbb{C} \times \overline{\mathbb{Y}}_R \) is a complex variety. Hence, we may assume that \( \varphi^* (E, \overline{\mathcal{D}}_E, \theta) \) has the following decomposition as in (61):

\[
\varphi^* E = \bigoplus_{a \in \text{Irr}(\varphi^* \mathcal{D}_V)} E_a
\]

Moreover, we have \( \deg_{u-1} a \leq e \) for any \( a \in \text{Irr}(\varphi^* \mathcal{D}_V) \).

We set \( (V', \overline{\mathcal{D}}_V, \theta_V, h') := \varphi^{-1}(V, \overline{\mathcal{D}}_V, \theta_V, h_V) \). According to Proposition 5.13 it satisfies (62). By Corollary 5.19 we have

\[
|F(h_V)|_h = O(|u|^{-2}(\log |u|)^{-2})
\]

Hence, we have \( |F(h_{\mathcal{U}, h'})|_h = |F(h_{\mathcal{U}, h'})|_h = O(|w|^{-2}(\log |w|)^{-2}) \).

We take a frame \( v \) of \( \mathcal{P}_w V' \) as in (62.5.2). Let \( \Theta \) be determined by \( \varphi^* f_z v = v \Theta \). Let \( C_w \) be determined by \( \varphi^*(\partial_w) v = v C_w \). We have \( \varphi^*(\partial_w) v = v \Theta \) (6.2.1). We have the expression

\[
\Theta = \bigoplus (\varphi^*(\partial_u a - e^{-1} \alpha u^\epsilon) I_{a, \alpha} - e^{-1} u^\epsilon \Theta_{a, \alpha}),
\]

where the entries of \( \Theta_{a, \alpha} \) are holomorphic at \( u = 0 \). The norm of the endomorphism determined by \( v \) and \( \Theta_{a, \alpha} \) is \( O((\log |w|)^{-1}) \) by Proposition 5.15. Note that \( \varphi^*(\partial_u) = -e^{-1} u^{\epsilon+1} \partial_u \) and \( \varphi^*(\partial^2_u) a = O(|\varphi^*(a)|^{-1 - \rho}) \) for some \( \rho > 0 \). Hence, the contribution of \( \varphi^*(\partial_u) \Theta \) to \( \varphi^*(\partial_u) \) is dominated as \( O(|\varphi^*(a)|^{-1 - \rho}) \) for some \( \rho > 0 \). Let \( G_w \) be the endomorphism determined by \( v \) and \( C_w \). By using Lemma 5.24, we obtain \( |G_w, \varphi^* f_z| = O(|\varphi^*(a)|^{-2}) \). Hence, we obtain \( |\partial_w G_w f|_h = O(|w|^{-1 - \rho}) \) for some \( \rho > 0 \). Then, we obtain \( |F_{\mathcal{U}, \mathcal{D}_V, h'}|_{h'} = |F_{\mathcal{U}, \mathcal{D}_V, h'}|_{h'} = O(|w|^{-1 - \rho}) \) for some \( \rho > 0 \).

Corollary 5.15 \( (E, \overline{\mathcal{D}}_E, h) \) is acceptable, i.e., the curvature \( F(h) \) is bounded with respect to \( h \) and the Poincaré metric \( |w|^{-2}(\log |w|)^{-2} dw d\overline{w} + dz d\overline{z} \) on \( X_R \) around \( T \times \{ \infty \} \).

5.3.6 Prolongation to a filtered bundle

We set \( \overline{\mathbb{X}}_R := T \times \overline{\mathbb{Y}}_R \).

Corollary 5.16 The holomorphic vector bundle \( (E, \overline{\mathcal{D}}_E) \) is naturally extended to a filtered bundle \( \mathcal{P}_a E \) on \( (\mathbb{X}_R, T \times \{ \infty \}) \). (See 2.4.1 for filtered bundles.) Moreover, the filtered bundle is good in the sense of 2.4.1.

Proof Because \( (E, \overline{\mathcal{D}}_E, h) \) is acceptable, we obtain the first claim from Theorem 21.31 of [40]. As explained in 5.5.2 we obtain a filtered bundle \( \mathcal{P}_a V \) on \( (\overline{\mathbb{Y}}_R, \infty) \) from the Higgs bundle with hermitian metric \( (V, \overline{\mathcal{D}}_V, h_V, \theta_V) \). By Proposition 5.15 we obtain that the filtered Higgs bundle \( \mathcal{P}_a V, \theta_V \) is good. It implies the claim of the corollary.

We obtain the spectral curve \( \mathcal{S}p(\mathcal{P}_a E) \subset T^\vee \times \overline{\mathbb{Y}}_R \) of \( \mathcal{P}_a E \). It is equal to \( \overline{\mathcal{S}p}(E) \) in Theorem 5.10 and independent of the choice of \( a \in \mathbb{R} \).

5.4 An estimate in a variant case

We continue to use the notation in 5.3. Let \( (E, \nabla, h) \) be an instanton on \( X_R \). Let \( F = F(\nabla) \) be its curvature. We suppose the following:

- \( |F(z, w)| \rightarrow 0 \) when \( |w| \rightarrow \infty \), i.e., for any \( \delta > 0 \), there exists \( R_\delta > 0 \) such that \( |F(z, w)|_h \leq \delta \) for any \( |w| \geq R_\delta \). In particular, we obtain \( \mathcal{S}p(E, \overline{\mathcal{D}}_E) \subset T^\vee \times Y_{R_\delta} \), if \( \delta \) is sufficiently small.

- The closure of \( \mathcal{S}p(E) \) in \( T^\vee \times \overline{\mathbb{Y}}_{R_\delta} \) is a complex subvariety.

We denote the closure by \( \overline{\mathcal{S}p}(E) \), and we set \( \mathcal{S}p_{\infty}(E) := \overline{\mathcal{S}p}(E) \cap (T^\vee \times \{ \infty \}) \). We obtain the following theorem.

Theorem 5.17 Under the assumption, \( (E, \nabla, h) \) is an \( L^2 \)-instanton.
We put $\text{Irr}(\theta, \ell)$ for the expression

$$
\text{Irr}(\theta, \ell) = \sum_{\alpha \in \mathbb{C}} (E_{a, \alpha}, \theta_{a, \alpha})
$$

such that, for any $\alpha \in \mathbb{C}$, $E_{a, \alpha}$ is the induced map. For any $a(z) = \sum_{j \geq -N} a_j z^j$ with $a_{-N} \neq 0$, we set $\text{ord}(a) := -N$. We also set $\text{ord}(0) := 0$. We take a negative number $p$ satisfying $p < \min\{\text{ord}(a - b) | a, b \in \text{Irr}(\theta), a \neq b\}$.

Let $h$ be a hermitian metric of $E$. Let $\theta^\dagger$ denote the adjoint of $\theta$ with respect to $h$. Let $F(h)$ denote the curvature of $(E, \overline{\nabla}_E, \theta)$. We impose the following condition for some $C_0 > 0$ and $\epsilon_0 > 0$:

$$
|F(h) + [\theta, \theta^\dagger]|_{h, \theta^\dagger} \leq C_0 \exp(-\epsilon_0 |z|^p)
$$

### 5.5.1 Asymptotic orthogonality and acceptability

We have the following version of Simpson’s main estimate.

**Proposition 5.18** Suppose that $(E, \overline{\nabla}_E, \theta, h)$ satisfies (62).

- If $a \neq b$, there exists $\epsilon > 0$ such that $E_{a, \alpha}$ and $E_{b, \beta}$ are $O(\exp(-\epsilon |z|^{|\text{ord}(a-b)|}))$-asymptotically orthogonal, i.e., there exists $C > 0$ such that, for any $u, v \in E_0$, we have $|h(u, v)| \leq C \exp(-\epsilon |z|^{|Q|^{\text{ord}(a-b)}})$.

- If $\alpha \neq \beta$, there exists $\epsilon > 0$ such that $E_{a, \alpha}$ and $E_{a, \beta}$ are $O(|z|^\epsilon)$-asymptotically orthogonal.

- $\theta_{a, \alpha} - (da + \alpha dz/z) \text{id}_{E_{a, \alpha}}$ is bounded with respect to $h$ and the Poincaré metric $g_p$.

**Proof** By considering the tensor product with a harmonic bundle of a rank one, we may assume $p < \min\{\text{ord}(a) | a \in \text{Irr}(\theta)\}$. We have a map $\eta_{\ell} : z^{-1}\mathbb{C}[z^{-1}] \rightarrow \mathcal{I}_\ell := z^{-\ell}\mathbb{C}[z^{-1}]$ by forgetting the terms $\sum_{j \geq -\ell + 1} a_j z^j$. For each $b \in \mathcal{I}_\ell$, we set $E_b^{(\ell)} := \bigoplus_{a \in \mathbb{C}} E_{a, \alpha}$. Let $\pi_a^{(\ell)}$ denote the projection of $E$ onto $E_b^{(\ell)}$ with respect to the decomposition $E = \bigoplus E_b^{(\ell)}$. In the case $\ell = 1$, we omit the superscript (1).

Let $\text{Irr}(\theta, \ell)$ be the image of $\text{Irr}(\theta)$ by $\eta_{\ell}$. We take a total order $\leq'$ on $\text{Irr}(\theta, \ell)$ for each $\ell$ such that the induced map $\text{Irr}(\theta, 1) \rightarrow \text{Irr}(\theta, \ell)$ is order-preserving. Let $E_b^{(\ell)}$ be the orthogonal complement of $\bigoplus_{c \leq b} E_c$ in $\bigoplus_{c \leq b} E_c$. Let $\pi_b^{(\ell)}$ be the orthogonal projection onto $E_b^{(\ell)}$. In the case $\ell = 1$, we omit the superscript (1). We have $\pi_b^{(1)} = \sum_{a \leq b} \eta_{\ell}^{-1}(a)' \eta_{\ell}^{-1}(a) \pi_a^{(1)}$.

We put $\zeta := \eta_{\ell} - \eta_{\ell+1}$. We have the expression $\theta = f dz$. We put $f^{(\ell)} := f - \sum_{a} \partial_z \eta_{\ell+1}(a) \pi_a$, $\mu^{(\ell)} := f^{(\ell)} - \sum_{a} \partial_z \zeta(a) \pi_a^{(1)}$ and $R_b^{(\ell)} := \pi_b^{(\ell)} - R_b^{(\ell)}$. We consider the following claims.

$(P_2)$ $|f^{(\ell)}|_h = O(|z|^{-\ell - 1})$ for $\ell' \geq \ell$. 

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Corollary 5.19 \((Q_e)\) \(|\mu^{(\ell')}|_h = O(|z|^{-\ell'})\) for \(\ell' \geq \ell\).

\((R_e)\) \(|\mathcal{R}^{(\ell')}|_h = O(\exp(-C|z|^{-\ell'}))\) for \(\ell' \geq \ell\) and for \(b \in \text{Irr}(\theta, \ell')\).

The asymptotic orthogonality of \(E_{a,\alpha}\) and \(E_{a,\beta}\) (\(\alpha \neq \beta\)) follows from \((R_1)\).

In the proof of Theorem 7.2.1 of [40], we proved the claims for a wild harmonic bundle by using a descending induction on \(\ell\). The essentially same argument can work. We give an indication for a modification in this situation.

We have the expression \(\theta^t = f^t d\zeta\). Let \(\Delta := -\partial_z \partial_{\bar{z}}\). If a holomorphic section \(s\) of \(\text{End}(E)\) satisfies \([s, f] = 0\), we obtain the following inequality from (62):

\[
\Delta \log |s|^2 \leq -\frac{|[f^t, s]|_h^2}{|s|_h^2} + C_0 \exp(-\varepsilon_0 |z|^p)
\]

(63)

Let \(f^{(t)}\) denote the adjoint of \(f^{(t)}\) with respect to \(h\). Suppose \(P_{\ell+1}, Q_{\ell+1}\) and \(R_{\ell+1}\). By applying (63) to \(f^{(t)}\), we obtain the following, as in (99) of [40]:

\[
\Delta \log |f^{(t)}|_h^2 \leq -\frac{|[f^{(t)}, f^{(t)}]|_h^2}{|f^{(t)}|_h^2} + C_1
\]

Then, by the same argument as that in \(\S 7.3.2-\S 7.3.3\) of [40], we obtain \(P_{\ell}\) and \(Q_{\ell}\). We put

\[
k^{(t)}_b := \log \left(\frac{|\pi^{(t)}_b|^2}{|\pi^{(t)}_b|^2}\right) = \log \left(1 + \frac{\mathcal{R}^{(t)}|_{\partial_h}}{|\pi^{(t)}_b|^2}\right).
\]

By applying (63) to \(\pi^{(t)}_b\), we obtain

\[
\Delta \log k^{(t)}_b \leq -\frac{|[f^{(t)}, \pi^{(t)}_b]|_h^2}{|\pi^{(t)}_b|^2} + C_0 \exp(-\varepsilon_0 |z|^p).
\]

There exists \(C_1 > 0\) and \(R_1 > 0\) such that the following holds for any \(|z| < R_1\):

\[
\Delta \exp(-A|z|^{-\ell}) \geq -\exp(-A|z|^{-\ell}) \left(\frac{\ell^2}{4} A^2|z|^{-2(\ell+1)}\right)
\]

\[
\geq -\exp(-A|z|^{-\ell}) \frac{\ell^2}{4} A^2 C_1 |z|^{-2(\ell+1)} + C_0 \exp(-\varepsilon_0 |z|^p)
\]

(64)

Hence, we obtain \(R_{\ell}\) by using the argument in \(\S 7.3.4\) of [40]. Similarly, we obtain the asymptotic orthogonality of \(E_{a,\alpha}\) and \(E_{a,\beta}\) (\(\alpha \neq \beta\)), and the boundedness of \(\theta_{a,\alpha} = -(\bar{\alpha}a + a\bar{\alpha}z) \text{id}_{E_{a,\alpha}}\) by using the argument in \(\S 7.3.5-\S 7.3.7\) of [40] with (63).

We obtain the following corollary. (See \(\S 7.2.5\) of [40] for the argument.)

**Corollary 5.19** \((E, \overline{\partial}_E, h)\) is acceptable, i.e., the curvature \(F(h)\) is bounded with respect to \(h\) and \(g_p\).

### 5.5.2 Prolongation and the norm estimate

For any \(U \subset X\) and for any \(a \in \mathbb{R}\), let \(\mathcal{P}_a E(U)\) denote the space of holomorphic sections \(s\) of \(E_{|U\setminus D}\) such that \(|s|_h = O(|z|^{-\alpha})\) \((\forall \varepsilon)\) locally around any point of \(U\). (See \(\S 2.2.3\).) According to a general theory of acceptable bundles, we obtain a locally free \(O_X\)-module \(\mathcal{P}_a E\), and a filtered bundle \(\mathcal{P}_a E = (\mathcal{P}_a E | a \in \mathbb{R})\). (See \(\S 2.2\) for a review of filtered bundles.) The decomposition (61) is extended to a decomposition of \(\mathcal{P}_a E:\)

\[
\mathcal{P}_a E = \bigoplus \mathcal{P}_a E_{a,\alpha}
\]

We set \(\mathcal{P} E := \bigcup_{a \in \mathbb{R}} \mathcal{P}_a E\) and \(\mathcal{P} E_{a,\alpha} := \bigcup_{a \in \mathbb{R}} \mathcal{P}_a E_{a,\alpha}\). We set \(\text{Gr}^P_a(E) := \mathcal{P}_a E / \mathcal{P}_{<a} E\), which we naturally regard as \(\mathbb{C}\)-vector spaces.
By Proposition 5.18, θ gives a section of \( \text{End}(PE) \otimes \Omega_1^k \), which preserves the decomposition \( PE = \bigoplus PE_{a,\alpha} \). By the estimate in Proposition 5.18, \( \theta_{a,\alpha} - (\text{ad} + \text{ad} z/\zeta) \text{id}_{E_{a,\alpha}} \) is logarithmic with respect to the lattice \( PE_{a,\alpha} \). Hence, we have the induced endomorphism \( \text{Res}(\theta_{a,\alpha}) \) of \( Gr^PE_{a,\alpha} \), which has a unique eigenvalue \( \alpha \). We set \( \text{Res}(\theta) = \bigoplus \text{Res}(\theta_{a,\alpha}) \). Let \( W Gr^P(E) \) be the monodromy weight filtration of the nilpotent part of \( \text{Res}(\theta) \).

For each section \( s \) of \( PE \), let \( \deg^P(s) := \min \{ a \mid s \in PE_a \} \). For any \( g \in Gr^P(E) \), let \( \deg^W(g) := \min \{ m \mid g \in W_m \} \). Let \( v = (v_i) \) be a frame of \( PE \) which is compatible with the decomposition \( PE = \bigoplus PE_{a,\alpha} \), the parabolic filtration and the weight filtration, i.e., each \( v_i \) is a section of a direct summand \( E_{a,\alpha} \), the tuple \( v^{(b)} := (v_i \mid \deg^P v_i = b) \) induces a base \([v^{(b)}] \) of \( Gr^P(E) \) for any \( a - 1 < b \leq a \), and the tuple \([v^{(b)},m] := (v_i^{(b)}) \ deg^W v_i^{(b)} = m \) induces a base of \( Gr^W Gr^P(E) \). We set \( a_i := \deg^P(v_i) \) and \( k_i := \deg^W(v_i) \). Let \( h_0 \) be the metric of \( E \) determined by \( h_0(v_i, v_i) = |z|^{-2\min(-\log |z|)^{k_i}} \) and \( h_0(v_i, v_j) = 0 \) \( (i \neq j) \). The following proposition can be shown by the argument in §8.1.2 of [10].

**Proposition 5.20** \( h \) and \( h_0 \) are mutually bounded.

### 5.5.3 Connection form

Let \( v \) be a frame of \( PE \), which is compatible with the decomposition \( PE = \bigoplus PE_{a,\alpha} \), the parabolic filtration and the weight filtration. Let \( G \) be the endomorphism of \( E \) determined by \( G(v_i) dz = \partial v_i \) for \( i = 1, \ldots, \text{rank } E \). We can show the following by the arguments of Lemma 7.5.5, Lemma 10.1.3 and Proposition 10.3.3 of [10].

**Lemma 5.21** We have \( |G|_h = O(|z|^{-1}) \). For the decomposition \( G = \sum G_{a,\alpha}^{(b,\beta)} \) according to \( E = \bigoplus E_{a,\alpha} \), we have the following estimate for some \( \epsilon > 0 \):

\[
|G_{a,\alpha}^{(b,\beta)}|_h = \begin{cases} 
O(\exp(-\epsilon |z|^{\text{ord}(a-b)})) & (a \neq b) \\
O(|z|^{-1+\epsilon}) & (a = b, \alpha \neq \beta)
\end{cases}
\]

We have the expression \( \theta = f dz \). Let us consider \( \partial_h f \). Let \( \Theta \) be determined by \( f v = v \Theta \). Let \( C \) be determined by \( \partial_h f v = v C \). We have \( \{\partial_h f\} v = v (\partial_h \Theta dz + [C, \Theta]) \) and \( [G, f v = v C, \Theta] \). We have the decompositions \( \partial_h f = \sum (\partial_h f)^{a,\alpha}_{(b,\beta)} \) and \( \overline{f}^t = \sum (\overline{f}^t)^{a,\alpha}_{(b,\beta)} \) according to \( E = \bigoplus E_{a,\alpha} \).

**Corollary 5.22** Let \( m := \min \{ \text{ord}(a) \mid a \in \text{Irr}(\theta) \} \). If \( m < 0 \), we have \( \partial_h f = O(|z|^{-2+m} dz) \) with respect to \( h \) and \( dz d\bar{z} \). We have

\[
|\langle \partial_h f \rangle_{(a,\alpha)}^{(b,\beta)}|_h = \begin{cases} 
O(\exp(-\epsilon |z|^{\text{ord}(a-b)})) & (a \neq b) \\
O(|z|^{-2}) & (a = b, \alpha \neq \beta)
\end{cases}
\]

We also have the following:

\[
|\langle \overline{f}^t \rangle_{(a,\alpha)}^{(b,\beta)}|_h = \begin{cases} 
O(\exp(-\epsilon |z|^{\text{ord}(a-b)})) & (a \neq b) \\
O(|z|^{-2}) & (a = b, \alpha \neq \beta)
\end{cases}
\]

**Proof** It follows from Lemma 5.21

### 5.5.4 Some estimate

Let \( t \) be a \( C^\infty \)-endomorphism of \( E \). According to the decomposition \( E = \bigoplus E_{a,\alpha} \), we have the decomposition \( t = \sum t^{a,\alpha}_{(b,\beta)} \), where \( t^{a,\alpha}_{(b,\beta)} \in \text{Hom}(E_{b,\beta}, E_{a,\alpha}) \). Let \( \mathcal{C} \) be the set of \( C^\infty \)-endomorphisms \( t \) such that the following holds for some \( \epsilon > 0 \) which may depend on \( t \):

\[
|t^{a,\alpha}_{(b,\beta)}|_h = \begin{cases} 
O(|z|^{\epsilon \text{exp}(-\epsilon |z|^{\text{ord}(a-b)})}) & ((a, \alpha) \neq (b, \beta)) \\
O(1) & \text{(otherwise)}
\end{cases}
\]

Note that \( \mathcal{C} \) is closed under the addition and the composition.
Proposition 5.23 Suppose $t$ and $|z|^2\partial_z\partial t$ are contained in $C$. Then, $z\partial_z t$ and $\overline{\partial}_t t$ are also contained in $C$.

Proof Let $\Psi : \mathbb{H} := \{ u \in \mathbb{C} \mid \text{Im} u > 0 \} \longrightarrow \{ z \in \mathbb{C} \mid 0 < |z| < 1 \}$ be given by $\Psi(u) = \exp(\sqrt{-1}u)$. Because $\Psi^t$ and $\partial_t \partial_t \Psi(t)$ are bounded, we obtain that $\partial_z \Psi^t$ and $\overline{\partial}_t \Psi^t$ are also bounded.

In the following argument, positive constants $\epsilon$ can change. We use the notation in the proof of Proposition 5.18. We clearly have $\partial_z \pi^{(t)}_b = 0$. We have $\partial_z \pi^{(t)}_b = O(\exp(-|\epsilon|z^{-1}))$ by Lemma 5.21. We also have $\partial_z \partial_z \pi^{(t)}_b = [F(h), \pi^{(t)}_b] = O(\exp(-|\epsilon|z^{-1}))$.

We have the decomposition $t = \sum t^{(t)}_{a,b}$ according to the decomposition $E = \bigoplus E^{(t)}_a$. We have $t^{(t)}_{a,b} = O(\exp(-|\epsilon|z^{-1}))$ if $a \neq b$. Hence, we have

$$[t, \pi^{(t)}_b] = \sum_{a \neq b} t^{(t)}_{a,b} - \sum_{a \neq b} t^{(t)}_{b,a} = O(\exp(-|\epsilon|z^{-1}))$$

We also have $|z|^2\partial_z \partial_z [t, \pi^{(t)}_b] = [z|^2\partial_z \partial_z t, \pi^{(t)}_b] + [z|^2\partial_z \partial_z \pi^{(t)}_b] + [t, |z|^2\partial_z \partial_z \pi^{(t)}_b] = O(\exp(-|\epsilon|z^{-1}))$. Hence, we obtain $z\partial_z [t, \pi^{(t)}_b] = O(\exp(-|\epsilon|z^{-1}))$ and $\overline{\partial}_t [t, \pi^{(t)}_b] = O(\exp(-|\epsilon|z^{-1}))$. Therefore, we obtain $z\partial_z \pi^{(t)}_b = O(\exp(-|\epsilon|z^{-1}))$ and $\overline{\partial}_t \pi^{(t)}_b = O(\exp(-|\epsilon|z^{-1}))$ for $a \neq b$.

We have $z\partial_z \pi_{a,a} = O(|\epsilon|)$ and $|z|^2\partial_z \partial_z \pi_{a,a} = O(|\epsilon|)$ by Lemma 5.21. Then, we obtain $z\partial_z t_{(a,a), (b, \beta)} = |z|^\epsilon$ and $\overline{\partial}_t t_{(a,a), (b, \beta)} = |z|^\epsilon$ for $\alpha \neq \beta$. If $a \neq b$ with $\ell = \text{ord}(a - b)$, we obtain the desired estimate by using $t_{(a,a), (b, \beta)} = \pi_{a,a} \circ t_{\eta(a), \eta(b) \circ \pi_{b,\beta}}$.

5.5.5 Refined asymptotic orthogonality

We obtain an asymptotic orthogonality of the derivative by assuming the following with respect to $h$ and $dz dt$ in addition to (62):

$$\partial_z \partial_z (F(h) + [\theta, \theta^t]) = O(\exp(-|\epsilon_0|z^{\mu})).$$

Let $\nu$ be a holomorphic frame of $P_0E$, compatible with the decomposition $P_0E = \bigoplus P_0E_{a, \alpha}$, the parabolic filtration and the weight filtration. Let $(a_i, \alpha_i)$ be determined by $v_i \in P_0E_{a_i, \alpha_i}$. We say that a matrix valued function $B = (B_{ij})$ satisfies the condition $C_1$, if the following holds for some $\epsilon > 0$ which may depend on $B$:

$$B_{ij} = \begin{cases} O(|v_{ij}|^{\epsilon \text{ord}(a_{ij})}) & \text{if } (a_i, \alpha_i) \neq (a_j, \alpha_j) \\ O(|v_{ij}|^{\epsilon \text{ord}(a_{ij})}) & \text{otherwise} \end{cases}$$

Let $H$ be the matrix valued function determined by $H_{ij} = h(v_{ij})$. Lemma 5.21 implies that $z\partial_z H$ and $\overline{\partial}_t H$ satisfy the condition $C_1$.

Proposition 5.24 $(|z|^2\partial_z \partial_z t)^2 H$ satisfies the condition $C_1$.

Proof Let $G(A)$ denote the endomorphism determined by $\nu$ and a matrix-valued function $A$. By Lemma 5.21, we have

$$G(H^{-1}z\partial_z H), G(H^{-1}\overline{\partial}_t H), G(H^{-1}z\partial_z H), G(H^{-1}\overline{\partial}_t H) \in C.$$

Because $G(\overline{\partial}_t H)$ is $|z|^2 F(h) \in C$, we have $G(\overline{\partial}_t H) \in C$. We have the expression $\theta = f dz$. We have $\partial_z \partial_z [f, f^t] = [[F(h)_{\overline{\partial}_t}, f], f^t] + [\partial_z f, \overline{\partial}_t f^t]$. It gives an estimate for $\partial_z \partial_z [f, f^t]$ by Corollary 5.22 from which we can deduce that $|z|^2\partial_z \partial_z [f, f^t] \in C$. By Proposition 5.22, we obtain $z\partial_z [f, f^t] \in C$ and $\overline{\partial}_t [f, f^t] \in C$. We obtain $G(\overline{\partial}_t H), G(\overline{\partial}_t H) \in C$. Then, we obtain $G(H^{-1}z\partial_z H), G(H^{-1}z\partial_z H) \in C$ from $|z|^2\partial_z \partial_z [f, f^t] \in C$. It implies the claim of the lemma.

Corollary 5.25 $(z\partial_z H)^2 H$ satisfies the condition $C_1$.

Remark 5.26 The estimate as in Corollary 5.25 will be used in the study for the extension of the associated twistor family, which will be discussed elsewhere.
6 \(L^2\)-instantons on \(T \times \mathbb{C}\)

6.1 Some standard property

6.1.1 Instantons of rank one

Let \((E, \nabla, h)\) be an \(L^2\)-instanton on \(T \times \mathbb{C}\) with rank \(E = 1\).

**Lemma 6.1** \((E, \nabla, h)\) is a unitary flat bundle.

**Proof** Because rank \(E = 1\), we have \((\nabla_z \nabla_\tau + \nabla_w \nabla_\tau)F_\tau = 0\) and \((\nabla_z \nabla_\tau + \nabla_w \nabla_\tau)F_\omega = 0\). We obtain the following inequalities:

\[-(\partial_{\tau} \partial_\tau + \partial_w \partial_w) |F_\tau|^2 \leq 0, \quad -(\partial_{\omega} \partial_\omega + \partial_z \partial_z) |F_\omega|^2 \leq 0.\]

We use the notation in \([5, 2, 2]\). By applying the fiber integral for \(T \times \mathbb{C} \rightarrow \mathbb{C}\), we obtain \(-\partial_w \partial_\omega \|F_\tau\|^2 \leq 0\) and \(-\partial_w \partial_\omega \|F_\omega\|^2 \leq 0\). Because the functions \(\|F_\tau\|^2\) and \(\|F_\omega\|^2\) are \(L^1\) on \(\mathbb{C}_w\), they are 0.

**Corollary 6.2** Let \((E, \nabla, h)\) be an \(L^2\)-instanton on \(T \times \mathbb{C}\) of an arbitrary rank. Then, \(\det (E, \nabla, h)\) is a flat unitary bundle, i.e., we have \(\text{Tr} F(\nabla) = 0\).

If we do not impose the \(L^2\)-property, there exist much more instantons of rank one on \(T \times \mathbb{C}\).

(i) Let \(a\) be any holomorphic function on \(\mathbb{C}\). Then, the trivial holomorphic line bundle \(O_\mathbb{C}\) with the trivial metric and the Higgs field \(da\) gives a harmonic bundle \(L(a)\) on \(\mathbb{C}\). By the equivalence of Hitchin, we have the associated instanton on \(T \times \mathbb{C}\).

(ii) Let \(\rho\) be an \(\mathbb{R}\)-valued harmonic function on \(T \times \mathbb{C}\). Then, the trivial holomorphic line bundle \(O_{\mathbb{T} \times \mathbb{C}} e\) with the metric \(h_\rho\) given by \(\log h_\rho(e, e) = \rho\) gives an instanton \(L(\rho)\) on \(T \times \mathbb{C}\). Note that there exist many harmonic functions which is not the real part of a holomorphic function on \(T \times \mathbb{C}\). We can construct such a function by using a Bessel function \(I_0(r) = \int_1^\infty \cosh(rt)(t^2 - 1)^{-1/2} dt\) which satisfies \(I_0'' + r^{-1}I_0' - I_0 = 0\). It is a \(C^\infty\)-function on \(\mathbb{R}\), satisfying \(I_0(r) = I_0(-r)\). In particular, \(\kappa(w) := I_0(|w|)\) gives a \(C^\infty\)-function on \(\mathbb{C}\) satisfying \((-\partial_w \partial_\omega + 4)\kappa = 0\). By using the Fourier expansion on \(T \times \mathbb{C}\) in a standard way, we can construct a harmonic function \(\rho\) on \(T \times \mathbb{C}\) from \(\kappa\) such that \(\rho\) is not constant along \(T\). (See [28].) It is not the real part of any holomorphic function.

In general, any instanton of rank one \((E, \overline{\nabla}_E, h)\) can be expressed as the tensor product of instantons of types (i) and (ii). Indeed, by considering the support \(RFM_-(E, \overline{\nabla}_E)\), we obtain a holomorphic function \(C \rightarrow T'\). Because \(\mathbb{C}\) is simply connected, it is lifted to a holomorphic function \(b : \mathbb{C} \rightarrow \mathbb{C}\). We have a holomorphic function \(a\) such that \(\partial_w a = b\). Then, we can observe that \((E, \overline{\nabla}_E, h)\) is isomorphic to \(L(a) \otimes L(\rho)\) for a harmonic function \(\rho\) on \(T \times \mathbb{C}\).

6.1.2 Polystability of the associated filtered bundle

Let \((E, \nabla, h)\) be an \(L^2\)-instanton on \(T \times \mathbb{C}\). Let \((E, \overline{\nabla}_E)\) be the underlying holomorphic vector bundle on \(T \times \mathbb{C}\).

For a saturated \(O_{\mathbb{T} \times \mathbb{C}}\)-subsheaf \(F \subset E\), let \(h_F\) denote the induced hermitian metric of the smooth part of \(\mathcal{F}\).

Let \(F(h_F)\) denote the curvature. As in [8] and [48], we set

\[\deg(\mathcal{F}, h) := \sqrt{-1} \int_{T \times \mathbb{C}} \text{Tr} (\Lambda F(h_F)) \ d\text{vol}_{T \times \mathbb{C}}.\]

Let \(\pi_F\) denote the orthogonal projection of \(E\) to \(\mathcal{F}\), where it is considered only on the smooth part of \(\mathcal{F}\). By the Chern-Weil formula [48], we have

\[\deg(\mathcal{F}, h) = -\int_{T \times \mathbb{C}} |\bar{\nabla}_h|^2 \ d\text{vol}_{T \times \mathbb{C}}.\]

**Lemma 6.3** \(\deg(\mathcal{F}, h)\) is finite, if and only if (i) the degree of \(\mathcal{F}|_{T \times \{w\}}\) are 0 for any \(w \in \mathbb{C}\), (ii) \(\mathcal{F}\) is extended to a subsheaf \(P_0 \mathcal{F}\) of \(P_0 E\). In that case, we have \(\deg(\mathcal{F}, h) = \int_{z \times \mathbb{P}^1} \text{par-c}_1(P_0 \mathcal{F})\), where \(P_0 \mathcal{F}\) denotes \(P_0 \mathcal{F}\) with the induced parabolic structure.
Proposition 6.4 $P,E$ is polystable. We have $\deg(P,E) = 0$. (See [2.4.1] for the stability condition in this case.)

Proof The second claim directly follows from Lemma 6.3 and Corollary 6.2. Let $\mathcal{P},\mathcal{F}$ be a filtered subsheaf $\mathcal{P},\mathcal{E}$ satisfying (A1,2) in [2.3.1]. Let $\mathcal{F}$ be its restriction to $X \times \mathbb{C}$. By Lemma 6.3, we have $\mu(\mathcal{P},\mathcal{F}) = \mu(\mathcal{F},h) \leq 0$. Moreover, if it is 0, the orthogonal projection onto $\mathcal{F}$ is holomorphic. Hence, the orthogonal decomposition $E = \mathcal{F} \oplus \mathcal{F}^\perp$ is holomorphic. It is extended to a decomposition $\mathcal{P},\mathcal{E} = \mathcal{P},\mathcal{F} \oplus \mathcal{P},\mathcal{F}^\perp$. Both $\mathcal{F}$ and $\mathcal{F}^\perp$ with the induced metrics are $L^2$-instantons. Hence, we obtain the first claim of the corollary by an easy induction on the rank.

6.1.3 Uniqueness of the $L^2$-instanton adapted to a filtered bundle

Let $(E,\nabla,h)$ be an $L^2$-instanton on $T \times \mathbb{C}$. We have the associated filtered bundle $\mathcal{P},\mathcal{E}$ on $(T \times \mathbb{P}^1,T \times \{\infty\})$. Let $h'$ be a hermitian metric of $E$, and let $\nabla_{h'}$ be a unitary connection of $(E,h')$ such that (i) $(E,\nabla_{h'},h')$ is an $L^2$-instanton, (ii) the $(0,1)$-parts of $\nabla_{h'}$ and $\nabla_h$ are equal, (iii) $h'$ is adapted to $\mathcal{P},\mathcal{E}$. (See [2.2.5] for adaptedness.)

Proposition 6.5 We have a holomorphic decomposition $(E,\overline{\partial}_E) = \bigoplus_i (E_i,\overline{\partial}_{E_i})$ such that (i) it is orthogonal with respect to both $h$ and $h'$, (ii) for each $i$, there exists $\alpha_i > 0$ such that $h_{|E_i} = \alpha_i h'_{|E_i}$. In particular, we have $\overline{\partial}_h = \overline{\partial}_{h'}$.

Proof Let $s$ be the self-adjoint endomorphism determined by $h' = h s$. According to [48], we have the following inequality (see p.876 of [48]):

$$-\partial_E z + \partial_E w \partial_{\overline{w}} \operatorname{Tr}(s) + |\mathcal{O}(s)s^{-1/2}|^2_h \leq 0$$

By taking the fiber integral for $T \times \mathbb{C} \rightarrow \mathbb{C}$, we obtain

$$-\partial_E \partial_{\overline{w}} \int_T \operatorname{Tr}(s) + \int_T |\mathcal{O}(s)s^{-1/2}|^2_h \leq 0$$

It implies that $\int_T \operatorname{Tr}(s)$ is a subharmonic function on $\mathbb{C}_w$. By using the norm estimate for asymptotically harmonic bundle (Proposition 5.20), we obtain that $h$ and $h'$ are mutually bounded, i.e., $s$ and $s^{-1}$ are bounded with respect to both $h$ and $h'$. Hence, we obtain that $\int_T \operatorname{Tr}(s)$ is constant. We obtain $\int_T |\mathcal{O}(s)s^{-1/2}|^2_h = 0$, which implies $\mathcal{O}(s) = 0$. Then, the claim of the proposition follows.

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6.1.4 Instanton number

Let \((E, \nabla, h)\) be an \(L^2\)-instanton on \(T \times \mathbb{C}\). We have the associated filtered bundle \(\mathcal{P}_aE\) on \((T \times \mathbb{P}^1, T \times \{0\})\). Note that the second Chern class of \(\mathcal{P}_aE\) is independent of \(a \in \mathbb{R}\).

**Proposition 6.6** We have the following equality for any \(a \in \mathbb{R}\).

\[
\frac{1}{8\pi^2} \int_{T \times \mathbb{C}} \text{Tr}(F(h)^2) = \int_{T \times \mathbb{P}^1} c_2(\mathcal{P}_aE)
\]

**Proof** Let \(U \subset \mathbb{P}^1\) be a small neighbourhood of \(\infty\) such that \(\mathcal{P}_aE|_{T \times w}\) are semistable of degree 0 for any \(w \in U\). In the following argument, we will shrink \(U\). We have the filtered Higgs bundle \((\mathcal{P}_aV, gdw)\) on \((U, \infty)\) corresponding to \(\mathcal{P}_aE\).

Let \(p : T \times U \rightarrow U\) be the projection. We have a natural \(C^\infty\)-isomorphism \(\mathcal{P}_aE \simeq p^* (\mathcal{P}_aV)\), and the holomorphic structure of \(\mathcal{P}_aE\) is described as \(p^* (\mathcal{T}_{\mathcal{P}_aV}) + g d\sigma\).

We take a holomorphic structure \(\mathcal{T}_{\mathcal{P}_aE}\) of \(\mathcal{P}_aE\) such that its restriction to \(T \times (\mathbb{P}^1 \setminus U)\) is \(\mathcal{T}_E\), and its restriction to \(T \times U\) is \(p^* (\mathcal{T}_{\mathcal{P}_aV})\).

We take a holomorphic frame \(v\) of \(\mathcal{P}_aV\) which is compatible with the parabolic structure. It induces a \(C^\infty\)-frame \(u\) of \(\mathcal{P}_aE\) on \(T \times U\). We take a \(C^\infty\)-metric \(h_0\) of \(\mathcal{P}_aE\) such that \(u\) is orthonormal with respect to \(h_0|_{T \times U}\). Let \(\nabla_0\) be the Chern connection determined by \(h_0\) and \(\partial\mathcal{T}_{\mathcal{P}_aE}\). We have \(\nabla_0 u = 0\) on \(T \times U\). The curvature \(F(\nabla_0)\) vanishes on \(T \times U\). We set \(A := \nabla - \nabla_0\).

Let \(J\) be the endomorphism of \(E|_{T \times \{\infty\}}\) determined by \(\nabla_w u_i = J(u_i)\) \((i = 1, \ldots, \text{rank } E)\). According to Lemma [5.21] and Theorem [5.11] we have \(J = O(|w|^{-1})\) with respect to \(h\). On \(T \times U\), we have

\[
A = J dw + g d\sigma - g_h^t dz.
\]

Here, \(g_h^t\) denotes the adjoint of \(g\) with respect to \(h\). We have \(|g|_h = |g^t|_h = O(1)\). According to Theorem [5.11] and Proposition [5.18] we have \([g, g_h^t] = O(|w|^{-2} (\log |w|)^{-2})\) with respect to \(h\). According to Lemma [5.21] and Theorem [5.11] we have \([g, J] = O(|w|^{-2})\) and \([g^t, J] = O(|w|^{-2})\) with respect to \(h\). Hence, we have

\[
A^2 = O(|w|^{-2}) dw d\sigma + O(|w|^{-2}) dw dz + O(|w|^{-2}) dz d\sigma.
\]

We set \(\nabla_t := t \nabla + (1 - t) \nabla_0\) for \(0 \leq t \leq 1\). On \(T \times (U \setminus \{\infty\})\), we have the following estimate for some \(\rho > 0\), which is uniform for \(t\):

\[
F(\nabla_t) = t F(\nabla) + (t^2 - t) A^2 = O(|w|^{-2}) dw d\sigma + O(|w|^{-2}) dw dz + O(|w|^{-2}) dz d\sigma + O(|w|^{-2}) dz d\sigma
\]

We obtain the following estimate, which is uniform for \(t\):

\[
\text{Tr}(F(\nabla_t) A) = O(|w|^{-2}) dw d\sigma dz + O(|w|^{-2}) dw d\sigma dz + O(|w|^{-2}) dw dz d\sigma + O(|w|^{-2}) dw dz d\sigma
\]

We obtain the following:

\[
-\frac{1}{8\pi^2} \int_{T \times \mathbb{C}} \text{Tr}(F(h)^2) = -\frac{1}{8\pi^2} \int_{T \times \mathbb{P}^1} \text{Tr}(F(\nabla_0)^2) = \int_{X \times \mathbb{P}^1} \text{ch}_2(\mathcal{P}_aE) = -\int_{X \times \mathbb{P}^1} c_2(\mathcal{P}_aE)
\]

Thus, we are done.

### 6.2 Cohomology

Let \((E, \nabla, h)\) be an \(L^2\)-instanton on \(X := T \times \mathbb{C}\). The \((0, 1)\)-part of \(\nabla\) is denoted by \(\overline{\nabla}_E\). Let \(X := T \times \mathbb{P}^1\). We put \(D := T \times \{\infty\}\). Let \(\mathcal{A}^{0,1}(E)\) denote the space of \(C^\infty\)-sections of \(E \otimes \Omega^{0,1}\) on \(X\) with compact supports. Its cohomology group is denoted by \(H^0_c(X, E)\). Let \(\mathcal{A}^{0,1}(\mathcal{P}_aE)\) denote the space of \(C^\infty\)-sections of \(\mathcal{P}_aE \otimes \Omega^{0,1}\) on \(X\). Its cohomology group is \(H^1(X, \mathcal{P}_aE)\). In this subsection, we suppose that

\[0 \notin \mathcal{S}_{p_{\infty}}(E)\]
The natural map $H^0_\omega(X, E) \rightarrow H^0_\omega(\overline{X}, \mathcal{P}_a E)$ is an isomorphism for any $a \in \mathbb{R}$.

**Proof** There exists $R > 0$ such that, if $|w| > R$, $E|_{T \times \{w\}}$ is semistable of degree 0, and $0 \notin \mathcal{S}(E|_{T \times \{w\}})$. We have two consequences for a $C^\infty$-section $s$ of $\mathcal{P}_a E$ on $X_R$.

- There exists a $C^\infty$-section $t$ of $\mathcal{P}_a E$ on $X_R$ such that $\nabla_{\mathcal{P}} s = s$.
- If $\nabla_{\mathcal{P}} s = 0$, then $s = 0$.

Then, the claim can be shown easily.

Let $A^0_{L^2}(E)$ be the space of $L^2$-sections $s$ of $E \otimes \Omega^0_\omega$ on $X$ such that $\mathcal{F}_E s$ is also $L^2$. The cohomology group of the complex $(A^0_{L^2}(E), \mathcal{F}_E)$ is denoted by $H^0_{L^2}(X, E)$.

**Proposition 6.8** The natural map $H^0_\omega(X, E) \rightarrow H^0_{L^2}(X, E)$ is an isomorphism.

**Proof** Let $A^0_{L^2,c}(E) \subset A^0_{L^2}(E)$ be the subspace of the sections with compact supports. It gives a subcomplex, and its cohomology is denoted by $H^0_{L^2,c}(X, E)$.

**Lemma 6.9** The natural map $H^0_{L^2,c}(X, E) \rightarrow H^0_{L^2}(X, E)$ is an isomorphism.

**Proof** For any $L^2$-section $s$ of $E$ on $X_R$, there exists an $L^2$-section $t$ of $E$ on $X_{R'}$ ($R' > R$) such that $\nabla_{\mathcal{P}} s = s$ on $X_R$. If an $L^2$-section $s$ of $E$ on $X_R$ satisfies $\nabla_{\mathcal{P}} s = 0$, then we have $s = 0$. Then, the claim of the lemma can be shown.

We take a smooth Kähler metric of $\overline{X}$. Let $B^0_{L^2}(\mathcal{P}_a E)$ be the space of $L^2$-sections $\omega$ of $\mathcal{P}_a E$ on $\overline{X}$ such that $\overline{\partial} \omega$ is $L^2$. Let $B^0_{L^2,c}(\mathcal{P}_a E) \subset B^0_{L^2}(\mathcal{P}_a E)$ denote the subspace of the sections whose support is contained in $X$. By the same argument, the natural map $B^0_{L^2,c}(\mathcal{P}_a E) \rightarrow B^0_{L^2,c}(\mathcal{P}_a E)$ is a quasi isomorphism. We have a natural identification $B^0_{L^2,c}(\mathcal{P}_a E) = A^0_{L^2,c}(E)$ as $\mathbb{C}$-linear spaces. By the $L^2$-Dolbeault theorem, the cohomology group of $B^0_{L^2}(\mathcal{P}_a E)$ is naturally isomorphic to $H^i(\overline{X}, \mathcal{P}_a E)$. Then, the claim of Proposition 6.8 follows.

**Corollary 6.10** $H^0_{L^2}(\overline{X}, E)$ is finite dimensional.

**Proposition 6.11** We have $H^0(\overline{X}, \mathcal{P}_a E) = H^2(\overline{X}, \mathcal{P}_a E) = 0$.

**Proof** Clearly $H^0(\overline{X}, \mathcal{P}_a E)) = 0$. Let $p : \overline{X} \rightarrow \mathbb{P}^1$ be the projection. We have $p_* E = 0$, and the support of $R^1 p_* E$ is 0-dimensional. Then, we obtain $H^2(\overline{X}, \mathcal{P}_a E) = 0$.

### 6.3 Exponential decay of harmonic sections

#### 6.3.1 Statement

Let $(E, \nabla, h)$ be an $L^2$-instanton on $T \times \mathbb{C}$. Let $\overline{\partial}_E$ denote the $(0, 1)$-part of $\nabla$, and let $\overline{\partial}^c_E$ denote the formal adjoint with respect to $h$ and $d z \overline{a}^c + dw a^c$. We set $\Delta_E := \overline{\partial}_E \overline{\partial}_E + \overline{\partial}_E \overline{\partial}_E$.

**Proposition 6.12** Assume that $0 \notin \mathcal{S}_{P_\infty}(E)$. Let $\omega$ be an $L^2$-section of $E \otimes \Omega^0_\omega$ on $T \times \mathbb{C}$ such that $\Delta_E \omega = 0$. Then, we have $|\omega| = O(\exp(-C|w|))$ for some $C > 0$.

#### 6.3.2 An estimate

Take $R > 0$, and put $Y_R := \{ |w| \geq R \}$ and $X_R := T \times Y_R$. Let $(E, \nabla, h)$ be an $L^2$-instanton on $X_R$.

**Lemma 6.13** Assume that $0 \notin \mathcal{S}_{P_\infty}(E)$. Suppose that $\omega$ is an $L^2$-section of $E \otimes \Omega^0_\omega$ on $Y_R$ such that $\overline{\partial}_E \omega = \overline{\partial}_E \omega = 0$. Then, there exists $C > 0$ such that $|\omega| = O(\exp(-C|w|))$. 

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The following equalities:

\[
\nabla_w \nabla \varphi f = \nabla_w (\nabla \varphi g) = \nabla \varphi \nabla w g + F_w \varphi g = -\nabla \varphi \nabla z f + F_w \varphi g = -\nabla \varphi \nabla z f + F_{\varphi} f + F_w \varphi g
\]

\[
\nabla_w \nabla \varphi g = F_w \varphi g + \nabla \varphi \nabla w g = F_w \varphi g + \nabla \varphi (-\nabla z f) = F_w \varphi g - \nabla \varphi \nabla z f + F_{\varphi} f = F_w \varphi g - \nabla \varphi \nabla z g + F_{\varphi} f
\]

We obtain the following

\[-(\partial_w \partial \varphi + \partial \varphi) \varphi (f, f) \leq - (\nabla \varphi \nabla \varphi f) - 2 \text{Re} \left( (\nabla_w \nabla \varphi + \nabla \varphi \nabla w) f, f \right) = - (\nabla \varphi \nabla \varphi f) - 2 \text{Re} \left( F_{\varphi} f + F_w \varphi g, f \right) \quad (66)\]

Using the notation in \ref{5.3.2} we obtain

\[-\partial_w \partial \varphi \| f \|^2 \leq - \| \nabla \varphi f \|^2 + O \left( \| F \| (\| f \|^2 + \| g \|^2) \right)\]

Similarly, we obtain

\[-\partial_w \partial \varphi \| g \|^2 \leq - \| \nabla \varphi g \|^2 + O \left( \| F \| (\| f \|^2 + \| g \|^2) \right)\]

By the assumption \( 0 \notin S_{p,w}(E) \), there exist \( R_1 > R \) and \( C_1 > 0 \) such that, if \( |w| \geq R_1 \), we have \( \| \partial \varphi g \| \geq C_1 \| g \| \) and \( \| \partial \varphi f \| \geq C_1 \| f \| \). Hence, there exist \( \epsilon > 0 \) and \( R_2 > R \) such that the following holds if \( |w| > R_2 \):

\[-\partial_w \partial \varphi (\| f \|^2 + \| g \|^2) \leq - \epsilon (\| f \|^2 + \| g \|^2) \quad (67)\]

In general, if \( \varphi \) is a positive \( L^1 \)-subharmonic function on \( Y_{R_2} \), Then \( \varphi(w) = O(|w|^{-2}) \). Indeed, by the mean value property, we have

\[\varphi(w) \leq \frac{4}{\pi(|w| - R_2)^2} \int_{|w-w'| \leq |w| - R_2} \varphi(w') \leq \frac{C_2}{(|w| - R_2)^2}\]

Hence, we have \( \| f \|^2 + \| g \|^2 = O(|w|^{-2}) \). Then, by a standard argument with \ref{5.3.2}, we obtain \( \| f \|^2 + \| g \|^2 = O(\exp(-C_4 |w|)) \). (See the proof of Lemma \ref{5.12}) By a bootstrapping argument, we obtain \( |f(z,w)| = O(\exp(-C_4 |w|)) \) and \( |g(z,w)| = O(\exp(-C_4 |w|)) \).

6.3.3 Finiteness

We continue to use the notation in \ref{5.3.2}. Let \( \omega \) be a \( C^\infty \)-section of \( E \otimes \Omega^{0,1} \) on \( X_R \). Suppose that the support of \( \omega \) is contained in \( T \times \{ |w| \geq R + 1 \} \). We set \( D := \overline{E} + \overline{\partial} \). Let \( d\text{vol} \) denote the volume form induced by the Euclidean metric.

Lemma 6.14 Assume that \( \omega \) and \( \Delta E \omega \) are \( L^2 \). Then, \( \overline{\partial} \omega \) and \( \overline{\partial} E \omega \) are \( L^2 \), and we have

\[\int h(\omega, \Delta E \omega) \text{dvol} = \int |D\omega|^2_h \text{dvol} \]

Proof Let \( g := dz \, d\bar{z} + dw \, d\bar{w} \). Let \( | \cdot |_{h,g} \) denote the norm of sections of \( E \otimes \Omega \) induced by \( h \) and \( g \). Let \( \chi(t) \) be a non-negative valued \( C^\infty \)-function such that \( \chi(t) = 1 \) \( (t \leq 0) \) and \( \chi(t) = 0 \) \( (t \geq 1) \), and that \( \partial_t \chi / \chi^{1/2} \) is also \( C^\infty \). For a large \( N \), we put \( \chi_N(w) := \chi(\log |w| - N) \). There exists \( C_1 > 0 \) such that \( |\partial_w \chi_N| \leq C_1 |w|^{-1} \), \( |\partial \chi_N| \leq C_1 |w|^{-1} \), and \( |\partial_{w} \partial \chi_N| \leq C_1 |w|^{-2} \). We have

\[\left| \int \chi_N h(\omega, \Delta E \omega) \text{dvol} - \int \chi_N |D\omega|^2_h \text{dvol} \right| \leq \left( \int |\nabla \chi_N|^2 \chi_N^{-1} |\omega|^2_{h,g} \text{dvol} \right)^{1/2} \left( \int \chi_N |D\omega|^2_{h,g} \text{dvol} \right)^{1/2} \]

\[+ \left( \int |\nabla \chi_N|^2 \chi_N^{-1} |\omega|^2_{h,g} \text{dvol} \right)^{1/2} \left( \int \chi_N |D\omega|^2_{h,g} \text{dvol} \right)^{1/2} \quad (68)\]

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There exist $C_i > 0$ ($i = 2, 3$) such that the following holds:

$$\int \chi_N |\mathcal{D} \omega|^2_{h,g} \, d\text{vol} \leq C_2 \left( \int \chi_N |\mathcal{D} \omega|^2 \, d\text{vol} \right)^{1/2} + C_3$$

Then, the first claim of Lemma 6.14 follows. We have

$$\left| \int h(\chi_N \omega, \Delta E \omega) \, d\text{vol} - \int \chi_N |\mathcal{D} \omega|^2_{h,g} \, d\text{vol} \right| \leq C_4 \int |\mathcal{D} \chi_N|_{g} |\mathcal{D} \omega|_{h,g} \, d\text{vol} + C_4 \int |\nabla \chi_N|_{g} |\mathcal{D} \omega|_{h,g} \, d\text{vol}$$

for some $C_4 > 0$. By the first claim, the integrands of the right hand side are dominated by some integrable functions, independently from $N$. By taking the limit, we obtain the second claim.

63.4 Proof of Proposition 6.12

Let us return to the setting in 6.3.4 According to Lemma 6.13, we have only to show the following lemma to establish Proposition 6.12.

Lemma 6.15 $\overline{\nabla} \mathcal{D} \omega = \overline{\nabla} E \omega = 0$.

Proof By the first claim of Lemma 6.14 we obtain that $\mathcal{D} \omega$ is $L^2$. By the argument in the proof of the second claim of the same lemma, we obtain $\int |\mathcal{D} \omega|^2_{h,g} \, d\text{vol} = 0$, i.e., $\mathcal{D} \omega = 0$.

6.4 Nahm transform for $L^2$-instantons

Let $(E, \nabla, h)$ be an $L^2$-instanton on $T \times \mathbb{C}$ with rank $R > 1$. Let $D := \mathcal{D}_{\infty}(E)$. We shall construct a harmonic bundle on $T^\vee \setminus D$ with the method in [14] and [28]. For any $\zeta \in T^\vee \setminus D$, let $L^{-\zeta} = (\mathcal{C}, \overline{\partial} T - \zeta d\overline{\partial})$ denote the corresponding line bundle on $T$ with the natural hermitian metric. Let $\text{Nahm}(E, \nabla)_{\zeta}$ denote the space of $L^2$-harmonic sections of $E \otimes L^{-\zeta} \otimes \Omega^{0,1}$. It is finite dimensional, and naturally isomorphic to $H^1(T \times \mathbb{P}^1, \mathcal{P}_{-1} E \otimes L^{-\zeta}) \cong H^1(T \times \mathbb{P}^1, \mathcal{P}_{0} E \otimes L^{-\zeta})$. (See [6.2]) The Euclidean metric $dz \overline{\sigma} + dw \overline{\sigma} \overline{\partial}$ of $T \times \mathbb{C}$ and the hermitian metric $h$ induce a metric $h_1(\zeta)$ of $\text{Nahm}(E, \nabla)_{\zeta}$. The multiplication of $-w \in \mathcal{O}(1)$ induces an endomorphism $F_w(\zeta)$ of $\text{Nahm}(E, \nabla)_{\zeta}$. It is also described as $-P_1 \circ w$, where $P_1$ denotes the orthogonal projection of the space of $L^2$-sections of $E \otimes L^{-\zeta} \otimes \Omega^{0,1}_{\nabla}$ onto $\text{Nahm}(E, \nabla)_{\zeta}$. (Note Proposition 6.12)

Let $A^{\mathcal{P},\mathcal{Q}}(E \otimes L^{-\zeta})$ denote the space of $L^2$-sections of $E \otimes L^{-\zeta} \otimes \Omega^{\mathcal{P},\mathcal{Q}}_{\nabla}$-bundle. Let $\overline{\mathcal{D}}_{E,\zeta}$ denote the $\overline{\partial}$-operator of $E \otimes L^{-\zeta}$. Let $\overline{\mathcal{D}}_{E,\zeta}$ denote its adjoint. Let $\mathcal{D}_{\zeta} := \overline{\mathcal{D}}_{E,\zeta} + \overline{\mathcal{D}}_{E,\zeta}$ be the closed operator $A^{0,0}(E \otimes L^{-\zeta} \otimes \Omega^{0,1}_{\nabla}) \rightarrow A^{0,1}(E \otimes L^{-\zeta})$. Let $\mathcal{D}_{\mathcal{P}} := \mathcal{D}_{E,\zeta} + \overline{\mathcal{D}}_{E,\zeta}$ denote the adjoint $A^{0,1}(E \otimes L^{-\zeta}) \rightarrow A^{0,0}(E \otimes L^{-\zeta} \otimes \Omega^{0,1}_{\nabla}) = A^{0,1}(E \otimes L^{-\zeta})$. By the results in [6.2] we obtain that $\mathcal{D}_{\mathcal{P}}$ is surjective. We have $\text{Ker}(\mathcal{D}_{\zeta}) = \text{Ker}(\mathcal{D}) = \text{Ker}(\mathcal{D})$. It induces a metric $h_1(\zeta)$ of $\text{Nahm}(E, \nabla)_{\zeta}$ on $T^\vee \setminus D$. It is equipped with a $C^\infty$-bundle $\text{Nahm}(E, \nabla)$ on $T^\vee \setminus D$. $F_w$ is holomorphic. We set $\theta_1 := F_w \overline{\partial}$. The $(0,1)$-part of $\overline{\partial}$ is equal to the $\overline{\partial}$-operator of $\text{Nahm}(E, \nabla)$.

Proposition 6.16 $(E_1, \overline{\mathcal{D}}_{E_1}, \theta_1, h_1)$ is a wild harmonic bundle.

Proof Because the argument is rather standard, we give only an indication for the convenience of the readers. For $I \subset \{1, 2, 3\}$, let $p_I$ denote the projection of $T^\vee \times T \times \mathbb{P}^1$ onto the product of the $i$-th components. By the construction, we have a natural isomorphism $(E_1, \overline{\mathcal{D}}_{E_1}) \simeq Rp_1((p_{23}^* P_{-1} E \otimes p_{12}^* \text{Poin}^{-1})|_{T^\vee \setminus D})$. The endomorphism $F_w$ is equal the multiplication of $-w : Rp_1((p_{23}^* P_{-1} E \otimes p_{12}^* \text{Poin}^{-1})|_{T^\vee \setminus D}) \rightarrow Rp_2((p_{23}^* P_{0} E \otimes p_{12}^* \text{Poin}^{-1})|_{T^\vee \setminus D})$. Hence, we obtain that $\theta$ is a wild Higgs field in the sense that, for the local expression $\theta = f \overline{d} \zeta$ around $P_1 \in D$, the coefficients of the characteristic polynomial $\det(id - f)$ are meromorphic at $P_1$.

Let us prove that $(E_1, \overline{\mathcal{D}}_{E_1}, \theta_1, h_1)$ is a harmonic bundle. Although we follow a standard argument, we give rather details for the convenience of readers. Let $\Delta_E$ denote the Laplacian on $A^{0,0}(E)$, i.e., $\Delta_E = \overline{\mathcal{D}}_{E,\zeta} \overline{\mathcal{D}}_{E,\zeta} = -\sqrt{-1} \Delta \overline{\partial} E \overline{\partial} E$. We have

$$\Delta_E \psi = -2(\nabla_z \nabla_w + \nabla_w \nabla_z) \psi.$$
On $A^{0,2}(E)$, the Laplacian is given by $\overline{\partial}_E \overline{\partial}_E = (-\sqrt{-1}) \overline{\partial}_E \Lambda \partial_E$. We have

$$\overline{\partial}_E \overline{\partial}_E (\psi d\zeta d\overline{\omega}) = -2(\nabla_\zeta \nabla_\omega + \nabla_\omega \nabla_\zeta) \psi d\zeta d\overline{\omega}.$$ 

Because $F_{\zeta} + F_{\omega} = 0$, it is equal to $\Delta_E (\psi) d\zeta d\overline{\omega}$. Hence, under the natural identification $A^{0,0}(E) \oplus A^{0,2}(E) \cong A^{0,0}(E) \oslash \langle 1, d\zeta d\overline{\omega} \rangle$, the Laplacian $\Delta^* \Delta$ acts as $\Delta_E \otimes \id$, where $\langle a, b \rangle$ denotes the 2-dimensional vector space generated by $a, b$. The Green operator of $\Delta^* \Delta$ acts as $G_E \otimes \id$, where $G_E$ denotes the Green operator for $\Delta_E$ on $A^{0,0}(E)$.

We naturally identify $A^{p,q}(E \otimes \mathcal{L}_- \langle \zeta \rangle)$ with $A^{p,q}(E)$. For a differential form $\tau$, let $\mu(\tau)$ be an endomorphism of $\bigoplus A^{p,q}(E)$ given by $\mu(\tau)(\varphi) = \tau \wedge \varphi$. We have $\overline{\partial}_E \zeta = \overline{\partial}_E - \zeta \mu(\zeta)$ and $\overline{\partial}_E \zeta = \overline{\partial}_E + (\sqrt{-1}) \zeta \Lambda \circ \mu(d\zeta)$. Let $d_{T^\vee}$ denote the trivial connection of the product vector bundle $A^{0,1}(E) \times (T^\vee \setminus D)$ over $T^\vee \setminus D$. We have the following relation for the operators on the space of the sections $T^\vee \setminus D \rightarrow A^{0,1}(E) \times (T^\vee \setminus D)$:

$$[d_{T^\vee}, \zeta + d\zeta] = d\zeta \mu(d\zeta), \quad [d_{T^\vee}, (\zeta + d\zeta)^*] = \sqrt{-1} d\zeta \Lambda \circ \mu(d\zeta).$$

We set $\Omega := d\zeta \mu(d\zeta) + d\zeta \sqrt{-1} \Lambda \circ \mu(d\zeta)$. Let $P_\zeta$ denote the orthogonal projection of $A^{0,1}(E)$ onto the kernel of $\Delta_\zeta$. Let $\Delta_\zeta = \overline{\partial}_E \zeta \overline{\partial}_E \zeta$ denote the Laplacian on $A^{0,0}(E)$ for $E \otimes \mathcal{L}_- \langle \zeta \rangle$. Let $G_\zeta$ denote the Laplacian for $\Delta_\zeta$ on $A^{0,0}(E)$, i.e., $G_\zeta = \id_{A^{0,0}(E)}$. The Green operator $G_\zeta$ for $\Delta_\zeta$ on $A^{0,0} \oplus A^{0,2}$ is given by $G_\zeta \otimes \id$. We have $P_\zeta = 1 - D_\zeta \circ G_\zeta \circ D_\zeta$. Let $G_\zeta \otimes \id$ also denote the naturally induced operator on $A^{0,1} \cong A^{0,0} \otimes \langle d\zeta, d\overline{\omega} \rangle$.

Let $\langle \cdot, \cdot \rangle$ denote the inner product of $A^{0,1}(E)$ induced by $h$ and $d\zeta \circ d\overline{\omega}$. By a standard computation, the curvature $F$ of the connection $\nabla_1$ is described as follows, for any sections $\psi_i (i = 1, 2)$ of Nahm $(E, \nabla)$:

$$\langle \psi_1, F \psi_2 \rangle = \langle \psi_1, d_{T^\vee} \circ P_\zeta (d_{T^\vee} \psi_2) \rangle = -\langle \psi_1, d_{T^\vee} \circ D_\zeta \circ G_\zeta \circ D_\zeta^* (d_{T^\vee} \psi_2) \rangle$$

$$= \langle d_{T^\vee} \psi_1, D_\zeta \circ G_\zeta \circ D_\zeta^* (d_{T^\vee} \psi_2) \rangle = \langle D_\zeta (d_{T^\vee} \psi_1), G_\zeta \circ D_\zeta^* (d_{T^\vee} \psi_2) \rangle = \langle \Omega \psi_1, G_\zeta \Omega \psi_2 \rangle$$

$$= d\zeta \langle [\overline{\partial}_E \psi_1, d\zeta (G_\zeta \circ \id) \psi_2] - \langle \Lambda (d\zeta \psi_1), \Lambda (d\zeta (G_\zeta \circ \id) \psi_2) \rangle \rangle \quad (69)$$

We have $\theta(\psi) = P_\zeta (w \psi) d\zeta$ and $\theta^i(\psi) = P_\zeta (\overline{\partial}_E \psi) d\zeta$. We have

$$\langle \psi_1, (P_\zeta w \circ P_\zeta \overline{\omega} - P_\zeta \overline{\omega} \circ P_\zeta w) \psi_2 \rangle d\zeta d\overline{\zeta} = -\langle \psi_1, (w(P_\zeta - 1) \overline{\omega} - \overline{\omega} (P_\zeta - 1) w) \psi_2 \rangle d\zeta d\overline{\zeta}$$

$$= \langle \{ \overline{\partial}_E \psi_1, D_\zeta [G_\zeta \circ D_\zeta^* (w \psi_2)] - \langle w \psi_1, D_\zeta [G_\zeta \circ D_\zeta^* (w \psi_2)] \rangle \rangle d\zeta d\overline{\zeta}$$

$$= \langle [D_\zeta, \overline{\partial}_E \psi_1, G_\zeta [D_\zeta, \overline{\partial}_E \psi_2] - \langle [D_\zeta, \psi_1, G_\zeta [D_\zeta, \psi_2] \rangle \rangle d\zeta d\overline{\zeta} \quad (70)$$

We have $[D_\zeta, \overline{\partial}_E] = \mu(d\overline{\omega})$ and $[D_\zeta, w] = -\sqrt{-1} \Lambda \circ \mu(dw)$. Hence, we obtain the following expression:

$$\langle \psi_1, (P_\zeta w \circ P_\zeta \overline{\omega} - P_\zeta \overline{\omega} \circ P_\zeta w) \psi_2 d\zeta d\overline{\zeta} \rangle = \langle d\zeta \psi_1, d\overline{\zeta} (G_\zeta \circ \id) \psi_2 \rangle - \langle \Lambda (d\zeta \psi_1), \Lambda (d\zeta (G_\zeta \circ \id) \psi_2) \rangle \rangle d\zeta d\overline{\zeta} \quad (71)$$

By using $\langle d\zeta d\omega, d\zeta d\overline{\omega} \rangle = \langle \Lambda dw \overline{\omega}, \Lambda dw \overline{\omega} \rangle = \langle \Lambda dz \overline{\omega}, \Lambda dz \overline{\omega} \rangle$ for the metric on $\Omega^*_{T \times \mathbb{C}}$, we obtain the following:

$$\langle \psi_1, \big( F + (P \circ w d\zeta) \circ (P \circ \overline{\partial}_E \zeta) \big) \psi_2 \rangle = 0$$

Namely, the Hitchin equation is satisfied. Thus, the proof of Proposition 6.10 is finished.

**Remark 6.17** We obtain a different transformation by replacing $\mathcal{L}_- \langle \zeta \rangle$ with $\mathcal{L}_\zeta$, for which we do not need any essential change.

**Remark 6.18** We use the operators, which looks natural in the complex geometry, instead of the Dirac operator itself.
7 \textit{L}^2\text{-instantons and wild harmonic bundles}

7.1 Nahm transform for wild harmonic bundles on \(T^\vee\)

7.1.1 Construction

Let \(D\) be a non-empty finite subset of \(T^\vee\). We fix a Kähler metric \(g_{T^\vee \setminus D}\) of \(T^\vee \setminus D\), which is Poincaré like around \(D\). Let \((E, \overline{\nabla}_E, \theta, h)\) be a wild harmonic bundle on \((T^\vee, D)\). We assume that \((E, \overline{\nabla}_E, \theta, h)\) has a singularity at each point of \(D\), i.e., \(\theta\) has a pole, or the parabolic structure is non-trivial. We shall construct an \(L^2\)-instanton from \((E, \overline{\nabla}_E, \theta, h)\) with the method in [13] and [27]. Let \(H^1_{L^2}(E, \overline{\nabla}_E, \theta, h)\) denote the \(i\)-th \(L^2\)-cohomology group of \((E, \overline{\nabla}_E, \theta, h)\). As recalled in Lemma 3.1, they are isomorphic to the hypercohomology groups of the complex \(C^*(\mathcal{P}_E \otimes \Omega^\bullet, \theta)\), where \((\mathcal{P}_E, \theta, h)\) is the associated good filtered Higgs bundle. In particular, they are finite dimensional, and isomorphic to the space of \(L^2\)-harmonic \(i\)-forms of \((E, \overline{\nabla}_E, \theta, h)\). We have \(H^1_{L^2}(E, \overline{\nabla}_E, \theta, h) = 0\) unless \((E, \overline{\nabla}_E, \theta, h)\) is \(\mathcal{O}_{T^\vee}\) with the trivial metric and the trivial Higgs field.

Remark 7.1 If \(D\) is empty, \((E, \overline{\nabla}_E, \theta, h)\) is isomorphic to a direct sum \(\bigoplus (L_i, \overline{\nabla}_{L_i}, \theta_i, h_i)\) such that rank \(L_i = 1\). So we exclude the case \(D = \emptyset\).

For any \((z, w)\), let \(L_{z, w}\) denote the harmonic bundle of rank one given by \((z, w) + z d\zeta\) with trivial metric and the Higgs field \(w d\zeta\). Let \((E, \overline{\nabla}_E, \theta_w, h)\) denote \((E, \overline{\nabla}_E, \theta, h) \otimes L_{z, w}\). Let \(\text{Nahm}(E, \overline{\nabla}_E, \theta, h)_{(z, w)}\) be the space of \(L^2\)-harmonic \(1\)-forms of \((E, \overline{\nabla}_E, \theta_w, h)\). It is independent of the choice of the Poincaré like metric \(g_{T^\vee \setminus D}\). It is finite dimensional, and naturally isomorphic to \(\text{Nahm}(\mathcal{P}_E, \theta)_{(z, w)}\). It is naturally equipped with the metric \(h_i\) induced by \(h\).

Let \(A^{p,q}(E)\) denote the space of \(L^2\)-sections of \(E \otimes \Omega^p \wedge \Omega^q_{T^\vee \setminus D}\). Let \(\overline{\nabla}_{E,z} : A^{p,q} \to A^{p,q-1}\) denote the adjoint of the closed operator \(\overline{\nabla}_{E,z} : A^{p,q} \to A^{p,q+1}\). Let \(\theta_{w}^1 : A^{p,q} \to A^{p-1,q}\) denote the adjoint of \(\theta_{w} : A^{p,q} \to A^{p+1,q}\). We have \(\overline{\nabla}_{E,z} := -\sqrt{-1}[\Lambda, \overline{\nabla}_E - zd\zeta]\) and \(\theta_{w}^1 = -\sqrt{-1}[\Lambda, \theta_w^1] = -\sqrt{-1}[\Lambda, \theta^1 + w d\zeta]\).

We set \(S^+ := A^{0,0}(E) \oplus A^{1,1}(E)\) and \(S^- := A^1(E) = A^{0,1}(E) \oplus A^{1,0}(E)\). Let \(D_{z, w} := \overline{\nabla}_{E,z} + \theta_{w} + \overline{\nabla}_E + \theta_w^1\) be the closed operator \(S^+ \to S^-(z, w)\) and let \(D_{z, w}^* := \overline{\nabla}_{E,z} + \theta_{w} + \overline{\nabla}_E + \theta_w^1\) denote its adjoint \(S^- \to S^+(z, w)\). We have \(\text{Ker} D_{z, w} = \text{Nahm}(E, \overline{\nabla}_E, \theta_w, h)_{(z, w)}\). (See [10].) By the vanishing \(H^1_{L^2}(E, \overline{\nabla}_E, \theta_w, h, i = 0, 2)\), we obtain that \(D^*\) is surjective. Hence, the family \(\bigcup_{(z, w)} \text{Nahm}(E, \overline{\nabla}_E, \theta_w, h)_{(z, w)}\) gives a \(C^\infty\)-vector bundle on \(T \times \mathbb{C}\). (See [13].) It is equipped with an induced \(C^\infty\)-metric \(h_1\) and an induced unitary connection \(\nabla_1\). Because the \(C^\infty\)-bundle is also constructed as a family of the cohomology of the complexes \((A^*(E), \overline{\nabla}_E + \theta_w d\zeta)\), it is equipped with a naturally induced holomorphic structure, which is equal to the (0, 1)-part of \(\nabla_1\). By the construction, the holomorphic bundle is naturally isomorphic to \(\text{Nahm}(\mathcal{P}_E, \theta)_{|T \times \mathbb{C}}\). (See [6, 2] for more details on this isomorphism.) We shall give the proof of the following theorem in \(\text{[4,3]}\) after preliminaries.

Theorem 7.2 \((\text{Nahm}(E, \overline{\nabla}_E, \theta, h_1, \nabla_1)\) is an \(L^2\)-instanton.

We give a remark on the proof. It is rather easy and standard to prove that \((\text{Nahm}(E, \overline{\nabla}_E, \theta, h_1, \nabla_1)\) is an instanton by using the twistor property of instantons and harmonic bundles. But, we do not give such an argument in the following. Instead, we follow another standard argument to use a description of the curvature \(F(\nabla_1)\) in terms of the Green operator. Because we need an estimate for the decay of \(F(\nabla_1)\), we need the description, anyway.

7.1.2 Preliminary

Let \(X\) be a torus \(\mathbb{C}^\mathbb{C}/L\). Let \(D \subset X\) be a finite set. Let \(g_A = Ad\zeta d\zeta\) be a Kähler metric of \(X \setminus D\) for some positive valued function \(A\), which is Poincaré like around \(D\). Let \((E, \overline{\nabla}_E, \theta, h)\) be a wild harmonic bundle on \(X \setminus D\). We set \(D := \overline{\nabla}_E + \theta\). Let \(D^*_A\) (resp. \(D^*_1\)) denote the formal adjoint of \(D\) with respect to \(h\) and \(g_A\) (resp. \(d\zeta d\zeta\)). We set \(\Delta_A = D^*_A D\) and \(\Delta_1 = D^*_1 D\). We have \(\Delta_A = A^{-1} \Delta_1\).
Lemma 7.3 Let $\varphi$ be a section of $E$ on $X \setminus D$ such that

$$\int |\varphi|^2_h A d\zeta d\overline{\zeta} + \int |\Delta_1 \varphi|^2_h A d\zeta d\overline{\zeta} < \infty.$$ 

Then, we have the following finiteness:

$$\int |\varphi|^2_h |d\zeta d\overline{\zeta}| + \int |\Delta_A \varphi|^2_h A |d\zeta d\overline{\zeta}| < \infty$$  \hspace{1cm} (72)

$$\int h(\varphi, \Delta_1 \varphi) |d\zeta d\overline{\zeta}| = \int h(\varphi, \Delta_A \varphi) A |d\zeta d\overline{\zeta}| = \int |\mathcal{D} \varphi|^2_h < \infty$$  \hspace{1cm} (73)

Proof The finiteness (72) is clear. In (73), the first equality is trivial. The second equality and finiteness can be shown by an argument in the proof of Lemma 6.14.

We set $\mathcal{D}^\dagger := \partial_E + \vartheta^\dagger$. Let $(\mathcal{D}^\dagger)^*_A$ (resp. $(\mathcal{D}^\dagger)^*_1$) denote the formal adjoint of $\mathcal{D}$ with respect to $g_A$ (resp. $d\zeta d\overline{\zeta}$). We have $\Delta_1 = (\mathcal{D}^\dagger)_1^* \mathcal{D}^\dagger$ and $\Delta_A = (\mathcal{D}^\dagger)_A^* \mathcal{D}^\dagger$.

Lemma 7.4 Let $\varphi$ be as in Lemma 7.3. Then, we have

$$\int h(\varphi, \Delta_1 \varphi) |d\zeta d\overline{\zeta}| = \int h(\varphi, \Delta_A \varphi) A |d\zeta d\overline{\zeta}| = \int |\mathcal{D}^\dagger \varphi|^2_h < \infty.$$  \hspace{1cm} (74)

Proof The first equality is trivial. For the second, we have only to apply Lemma 7.3 to a harmonic bundle $(E, \partial_E, \theta^\dagger, h)$ on $X \setminus D$.

7.1.3 Estimate

Let $X$ be a torus $\mathbb{C}^\times / L$ with a non-empty finite subset $D$. We use the Euclidean metric $d\zeta d\overline{\zeta}$ of $X$. Let $d\text{vol}_X = |d\zeta d\overline{\zeta}|$ denote the associated volume form. Let $(E, \overline{\partial}_E, \theta, h)$ be a wild harmonic bundle on $(X, D)$. Assume that the harmonic bundle has a singularity at each point of $D$.

Let $\nabla_h = \overline{\partial}_E + \partial_E$ be the Chern connection. Let $\mathcal{H}_w$ be the space of the sections of $E$ on $X \setminus D$ such that

$$\int_X |\varphi|^2_h d\text{vol}_X + \int_X \left( |\nabla_h \varphi|^2_h + |(\theta + w d\zeta) \varphi|^2_h \right) < \infty.$$ 

Proposition 7.5 There exist positive constants $R > 0$, $C > 0$ and $\rho > 0$ such that, if $|w| > R$, the following holds for any $\varphi \in \mathcal{H}_w$:

$$\int_X \left( |\nabla_h \varphi|^2_h + |(\theta + w d\zeta) \varphi|^2_h \right) \geq C |w|^\rho \int_X |\varphi|^2_h d\text{vol}_X$$

(See also a refined estimate in Proposition 7.9 below.)

Proof We use an argument in §2.4 of [63] with an adjustment to our situation. We use the standard distance on $X$. We take small neighbourhoods $B_P$ of $P \in D$. There exists $R_1 > 0$ and $C_1 > 0$ such that, if $|w| \geq R_1$, then we have $||\theta + w d\zeta| \varphi|^2_h \geq C_1 |w|^2 |\varphi|^2_h d\text{vol}_X$ on $X \setminus \bigcup_{P \in D} B_P$. We have only to show the estimate on each $B_P$. We may assume $P = 0$, and $B_P$ is an $\epsilon$-ball $B_\epsilon = \{ |\zeta| \leq \epsilon \}$.

We have a ramified covering $\psi : (B_\epsilon, 0) \longrightarrow (B, 0)$ given by $\psi(u) = u^p$ such that $\psi^*(E, \overline{\partial}_E, \theta, h)$ is unramified, i.e., we have the decomposition

$$\psi^*(E, \overline{\partial}_E, \theta) = \bigoplus_{a \in u^{-1} \mathbb{C}[u^{-1}]} (E_a, \overline{\partial}_{E_a}, \theta_a),$$

where the Higgs field $\theta_a - da \text{id}_{E_a}$ are tame. Let $\ell := \max\{ \deg_{a^{-1}} a \mid E_a \neq 0 \}$.

Lemma 7.6 There exists $R' > 0$, $C'_i > 0$ ($i = 1, 2$) such that $|\theta \varphi|_h \geq C'_1 |w| |d\zeta| |\varphi|_h$ on $B_\epsilon \setminus \{ |\zeta| < C'_2 |w|^{-p/(\ell + p)} \}$, if $|w| \geq R'$.  

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Proof. We have only to estimate each $\theta_a$ on $B'_\epsilon$. Let us consider the case $a \neq 0$. We set $n := \deg_{a^{-1}}a$. For each $w$, we have the solutions $b_i(w)$ ($i = 0, \ldots, n + p - 1)$ of the following equation:

$$\partial_a a(u) + p u w^{p-1} = 0$$

We have the equality $u^{p+1} \partial_a a(u) + p w = \alpha \prod_{i=0}^{n+p-1} (u - b_i(w))^{-1}$ for some $\alpha \in \mathbb{C} \setminus \{0\}$. We have

$$\theta_a = \partial_a a \, \text{id}_{E_a} \, du + g_a \, du,$$

where $|g_a| \leq C_1 |u|^{-1}$. We have $R_2 > 0$ and $C_2 > 0$ such that the following holds if $|w| > R_2$:

$$C_2^{-1} \leq |b_i(w)| |w|^{1/(n+p)} \leq C_2$$

We take $C_3 >> C_2$. We set $W_1 := \{|w| \leq C_3 |w|^{-1/(n+p)}\}$.

On $B'_\epsilon \setminus W_1$, we have $|g_a| \leq (C_1/C_3) |w|^{1/(n+p)}$. We also have

$$|u^{p+1} \partial_a a(w) - b_i(w)^{-1}| \geq |b_i(w)^{-1}| - |u| \geq (C_2^{-1} - C_3^{-1}) |w|^{1/(n+p)}$$

for any $i$, and hence $|u^{p+1} \partial_a a + p w| \geq |\alpha| (C_2^{-1} - C_3^{-1}) |w|$.

Hence, if $C_3$ is sufficiently larger than $C_2$, there exist $R_4 > 0$ and $C_4 > 0$ such that the following holds if $|w| > R_4$:

$$|\partial_a a + p u w^{p-1}| \text{id}_{E_a} + g_a | \geq C_4 |w|^{p-1}$$

Hence, we obtain the desired inequality for the integral over $B'_\epsilon \setminus W_1$ in the case $a \neq 0$.

Let us consider the case $a = 0$. We have the expression $\theta_0 = g_0 \, du$, and $|g_0| \leq C_10 |u|^{-1}$ for some $C_10 > 0$. We take $C_11 > C_10$, and we consider $W := \{|w| \leq C_11 |w|^{-1/(n+p)}\}$. On $B'_\epsilon \setminus W$, we have $|w u^{p-1}| \geq C_11^{-1} |w|^{1/p}$. We also have $|g_0| \leq (C_10/C_11) |w|^{1/p}$. Hence, if $C_11$ is sufficiently larger than $C_10$, we have the following for some $C_12 > 0$:

$$|p u^{p-1} \text{id}_{E_0} + g_0| \geq C_12 |w u^{p-1}|$$

Hence, we obtain the desired inequality in the case $a = 0$. Thus, the proof of Lemma 7.6 is finished.

Let $\varphi$ be an $L^2$-section of $E$ on $B_\epsilon$ with respect to $\text{dvol}_X$, such that

$$\int_{B_\epsilon} (|\nabla_k \varphi|^2 + |(\theta + w d\zeta) \varphi|^2) \, d\text{vol}_X < \infty.$$

We set $W_1 := \{|\zeta| < 2C_2 |w|^{-p/(\ell + p)}\}$ and $W_2 := \{|\zeta| < C'_2 |w|^{-p/(\ell + p)}\}$. We have the following type of Poincaré inequality, i.e., there exist $C''' > 0$ and $R'' > 0$ such that the following holds if $|w| > R''$ (see [7], and (2.12) of [53]):

$$|w|^{2p/(\ell + p)} \int_{W_1} |\varphi|^2 |d\zeta d\overline{\varphi}| \leq C''' \left( \int_{W_1} |\varphi|^2 |d\zeta d\overline{\varphi}| + |w|^{2p/(n+p)} \int_{W_1 \setminus W_2} |\varphi|^2 |d\zeta d\overline{\varphi}| \right)$$

There exists $C'''$ such that the right hand side is dominated by

$$C''' \left( \int_{W_1} |\nabla_k \varphi|^2 + \int_{W_1 \setminus W_2} |(\theta + w d\zeta) \varphi|^2 \right)$$

Thus, the proof of Proposition 7.5 is finished.

Let $D := \overline{\partial}_E + \theta$. Let $D^*_1$ denote the adjoint with respect to the Euclidean metric $d\zeta d\overline{\varphi}$. Let $\Delta_1 := D^*_1 \circ D$. Let $g_X \setminus D$ be a Kähler metric of $X \setminus D$ which is Poincaré like around $D$. Let $\text{dvol}_X \setminus D$ be the volume form associated to $g_X \setminus D$.

Corollary 7.7. There exist $\rho > 0$ and $C > 0$ such that the following holds:
• Let $\varphi$ be a section of $E$ such that
\[ \int |\varphi|^2_d \text{vol}_{X \setminus D} + \int |\Delta_1 \varphi|^2_h \text{vol}_X < \infty. \] (75)

Then, we have the following inequality:
\[ C|w|^p \left( \int |\varphi|^2_d \text{vol}_X \right)^{1/2} \leq \left( \int |\Delta_1 \varphi|^2_h \text{vol}_X \right)^{1/2} \] (76)

(See Corollary 7.11 for a refinement.)

**Proof** Let $D^1 = \partial E + \theta^1$. From (75), Lemma 7.3 and Lemma 7.4, we obtain $\int |D\varphi|^2_0 < \infty$ and $\int |D^1\varphi|^2_h < \infty$. By using the same lemmas and Proposition 7.5, we obtain
\[ C|w|^p \int |\varphi|^2_d \text{vol}_X \leq \int |D\varphi|^2_0 + \int |D^1\varphi|^2_h = 2 \int h(\varphi, \Delta_1 \varphi) \text{vol}_X. \]

Then, the claim of the corollary follows.

7.1.4 Proof of Theorem 7.2

Let $\omega_{T^\vee \setminus D}$ be the Kähler form associated to the metric $g_{T^\vee \setminus D}$. The multiplication of $\omega_{T^\vee \setminus D}$ induces an isomorphism $A^{0,0}(E) \simeq A^{1,1}(E)$. It gives an identification $S^+ \simeq A^{0,0}(E) \otimes \mathbb{C}[1, \omega_{T^\vee \setminus D}]$, where $\mathbb{C}[1, \omega_{T^\vee \setminus D}]$ denotes the 2-dimensional vector space generated by 1 and $\omega_{T^\vee \setminus D}$. By a general theory of harmonic bundles, the Laplacian $D^*_zw Dzw$ on $S^+$ is identified with $\Delta_z \otimes \text{id}$ on $A^{0,0}(E) \otimes \mathbb{C}[1, \omega_{T^\vee \setminus D}]$, where $\Delta_z := (\overline{\partial}_{E,z} + \theta^1_w) \circ (\overline{\partial}_{E,z} + \theta^1_w)$ on $A^{0,0}(E)$. (See [50]. In this case, it can be easily checked directly.) The Green operator $Gzw$ for $D^*_zw Dzw$ is identified with $Gzw \otimes \text{id}$, where $Gzw$ is the Green operator of $\Delta_z$ on $A^{0,0}(E)$.

For a differential form $\tau$ on $T^\vee$, let $\mu(\tau)$ be an endomorphism of $\bigoplus A^{p,q}(E)$ given by $\mu(\tau)(\varphi) = \tau \wedge \varphi$. Let $d_{T \times \mathbb{C}}$ denote the trivial connection of the product vector bundle $S^+ \times (T \times \mathbb{C})$ over $T \times \mathbb{C}$. We have the following relation for the operators on the space of the sections $T \times \mathbb{C} \rightarrow S^+ \times (T \times \mathbb{C})$:
\[ [d_{T \times \mathbb{C}}, \overline{\partial} + z \, d\zeta] = dz \mu(d\zeta), \quad [d_{T \times \mathbb{C}}, \theta + w \, d\zeta] = dw \mu(d\zeta), \]
\[ [d_{T \times \mathbb{C}}, (\overline{\partial} + z \, d\zeta)^*] = d\overline{\tau} \left( \sqrt{-1} \Lambda \mu(d\zeta) \right), \quad [d_{T \times \mathbb{C}}, (\theta + w \, d\zeta)^*] = d\overline{\tau} \left( \sqrt{-1} \Lambda \mu(d\zeta) \right). \]

We set $\Omega := dz \mu(d\zeta) + dw \mu(d\zeta) + d\overline{\tau} \left( \sqrt{-1} \Lambda \mu(d\zeta) \right) + d\overline{\tau} \left( \sqrt{-1} \Lambda \mu(d\zeta) \right)$.

Let $F(\nabla_1)$ be the curvature of the transformed bundle $\text{Nahm}(E, \overline{\partial}_E, \theta, h)$ with the metric and the unitary connection. Let $Pzw$ denote the orthogonal projection of $S^+$ onto $\text{Nahm}(E, \overline{\partial}_E, \theta, h)(z,w)$. Let $\psi_i$ be sections of $\text{Nahm}(E, \overline{\partial}_E, \theta, h)$. Let $\langle \cdot, \cdot \rangle$ denote the hermitian pairing on $A^{p,q}(E)$ induced by $h$ and $\omega_{T^\vee \setminus D}$. We have the following standard computation:
\[ \langle \psi_1, F(\nabla_1) \psi_2 \rangle = \langle \psi_1, Pzw \circ (d \circ Pzw) \psi_2 \rangle = \langle \psi_1, d \circ (Pzw - 1) \circ d \psi_2 \rangle = -\langle d \psi_1, (Pzw - 1) \circ d \psi_2 \rangle = \langle d \psi_1, Dzw \circ Gzw \circ D^*_zw \psi_2 \rangle. \]
\[ = \langle d, D^*_zw \psi_1, Gzw[d, D^*_zw] \psi_2 \rangle = \langle \Omega \psi_1, Gzw \Omega \psi_2 \rangle \] (77)

We have the expression $\psi_1 = \psi_{11} d\zeta + \psi_{12} d\overline{\zeta}$. We have
\[ \Omega \psi_1 = dz \psi_{11} d\zeta d\zeta + dw \psi_{12} d\zeta d\overline{\zeta} - \sqrt{-1} d\overline{\tau} \psi_{12} \Lambda(d\zeta d\zeta) + \sqrt{-1} d\overline{\tau} \psi_{12} \Lambda(d\zeta d\zeta). \]

Let $A$ be determined by $g_{T^\vee \setminus D} = A d\zeta d\overline{\zeta}$. We have the following:
\[ Gzw \Omega \psi_2 = dz Gzw \left( A^{-1} \psi_{21} \right) A d\zeta d\zeta + dw Gzw \left( A^{-1} \psi_{22} \right) A d\overline{\zeta} d\overline{\zeta} - \sqrt{-1} d\overline{\tau} Gzw \left( \psi_{21} \Lambda(d\zeta d\zeta) \right) + \sqrt{-1} d\overline{\tau} Gzw \left( \psi_{22} \Lambda(d\zeta d\zeta) \right). \] (78)
We have the following:

$$\langle \psi_{11} d\zeta d\zeta, AGzw(A^{-1}\psi_{21}) d\zeta d\zeta \rangle = \langle \psi_{11} \Lambda(d\zeta d\zeta), Gzw(\psi_{21} \Lambda(d\zeta d\zeta)) \rangle = 4 \int |\psi_{11}, Gzw(A^{-1}\psi_{21})| d\text{vol}_{T^\vee}$$  \hspace{1cm} (79)

$$\langle \psi_{12} d\zeta d\zeta, AGzw(A^{-1}\psi_{22}) d\zeta d\zeta \rangle = \langle \psi_{12} \Lambda(d\zeta d\zeta), Gzw(\psi_{22} \Lambda(d\zeta d\zeta)) \rangle = 4 \int |\psi_{12}, Gzw(A^{-1}\psi_{22})| d\text{vol}_{T^\vee}$$  \hspace{1cm} (80)

From these equalities, we obtain $(dz d\tau + dw d\bar{w}) \wedge \psi_1, F(\nabla_1) = 0$, which means that Nahm $(E, \overline{\theta}_E, \theta, h)$ with the induced metric $h_1$ and connection $\nabla_1$ is an instanton.

Let us show that it is an $L^2$-instanton. Let $(\overline{\theta}_{E,z} + \theta_w)$ denote the formal adjoint of $\overline{\theta}_{E,z} + \theta_w$ with respect to $h$ and $d\zeta d\bar{\zeta}$. We set $A := (\overline{\theta}_{E,z} + \theta_w)(\overline{\theta}_{E,z} + \theta_w)$. Because $A_{zw,1} = A_{zw,1} (\theta_{zw}(A^{-1}\psi_{21})) = \psi_{21}$. We have

$$\int |Gzw(A^{-1}\psi_{21})|_h^2 d\text{vol}_{T^\vee} + \int |\psi_{21}|_h^2 d\text{vol}_{T^\vee} < \infty$$

By Corollary 7.7, we have the following for some $\rho > 0$ and $C > 0$:

$$C|w|^{2\rho} \int_{T^\vee} |Gzw(A^{-1}\psi_{21})|_h^2 d\text{vol}_{T^\vee} < \int_{T^\vee} |\psi_{21}|_h^2 d\text{vol}_{T^\vee}$$

Hence, we obtain

$$|\langle \psi_{11} d\zeta d\zeta, AGzw(A^{-1}\psi_{21}) d\zeta d\zeta \rangle| = |\langle \psi_{11} \Lambda(d\zeta d\zeta), Gzw(\psi_{21} \Lambda(d\zeta d\zeta)) \rangle| < C|w|^{-\rho} \left( \int |\psi_{11} d\zeta d\zeta|_h^2 \right)^{1/2} \left( \int |\psi_{21} d\zeta d\zeta|_h^2 \right)^{1/2}$$  \hspace{1cm} (81)

We have a similar estimate for $|\langle \psi_{12} d\zeta d\zeta, Gzw(\psi_{22}) d\zeta d\zeta \rangle| = |\langle \psi_{12} \Lambda(d\zeta d\zeta), Gzw(\psi_{22} \Lambda(d\zeta d\zeta)) \rangle|$. From these estimates, we obtain $|F(\nabla_1)| = O(|w|^{-\rho})$ for some $\rho > 0$. Because $\text{Nahm}(E, \overline{\theta}_E, \theta, h) \simeq \text{Nahm}(P, E, \theta)|_{T^\vee C}$, we can apply Theorem 7.11 and hence we obtain that $F(\nabla_1)$ is $L^2$. Thus, the proof of Theorem 7.2 is finished.

**Remark 7.8** We can directly prove that the curvature is $L^2$ by using Corollary 7.11 below.

### 7.1.5 Refined estimates (Appendix)

We refine the estimates in 7.1.3 i.e., we show that $\rho$ can be replaced with $1 + \rho$. Although we do not use it in this paper, this type of argument seems useful in the study of a different type of Nahm transform, and so we would like to keep it.

**Proposition 7.9** There exist positive constants $R > 0$, $C > 0$ and $\rho > 0$ such that, if $|w| > R$, the following holds for any $\varphi \in \mathcal{H}_w$:

$$\int_X \left( |\nabla_{\mathcal{H}} \varphi|^2_h + |(\theta + w d\zeta) \varphi|^2_h \right) \geq C |w|^{1+\rho} \int_X |\varphi|^2_h d\text{vol}_X$$

**Proof** We again use the argument in §2.4 of 6.3 with an adjustment to our situation. We use the standard distance on $X$. We take small neighbourhoods $B_P$ of $P \in D$. There exists $R_1 > 0$ and $C_1 > 0$ such that, if $|w| \geq R_1$, then we have $|(\theta + w d\zeta) \varphi|^2_h \geq C_1 |w|^2 |\varphi|^2_h d\text{vol}_X$ on $X \setminus \bigcup_{P \in D} B_P$. We have only to show the estimate on each $B_P$. We may assume $P = 0$, and $B_P$ is an $\epsilon$-ball $B_\epsilon = \{|\zeta| \leq \epsilon\}$.

We have a ramified covering $\psi : (B_\epsilon^0) \rightarrow (B_\epsilon^0)$ given by $\psi(u) = u^p$ such that $\psi^*(E, \overline{\theta}_E, \theta, h)$ is unramified, i.e., we have the decomposition

$$\psi^*(E, \overline{\theta}_E, \theta) = \bigoplus_{a \in u^{-1} \mathbb{C}[u^{-1}]} (E_a, \overline{\theta}_{E_a}, \theta_a)$$  \hspace{1cm} (82)
where the Higgs field $\theta_a - da \text{id}_{E_a}$ are tame. Let $h' = \bigoplus h|_{E_a}$, and let $\nabla_{h'}$ denote the unitary connection associated to $\psi^*(E, \overline{\nabla}_{E})$ with $h'$. By the asymptotic orthogonality of the decomposition (22) with respect to $h$ (see [40]), we have the following inequality:

$$
\int_{B'_i} \left( |\nabla_{h'} \varphi|^2_{h'} + |(\theta + w d\zeta) \varphi|^2_{h'} \right) \geq C_2 \int_{B'_i} \left( |\nabla_{h'} \varphi|^2_{h'} + |(\theta + w d\zeta) \varphi|^2_{h'} \right)
$$

$$
\int_{B'_i(p)} |\varphi|^2_{h'} \psi^* \text{dvol}_x \leq C_3 \int_{B'_i(p)} |\varphi|^2_{h'} \psi^* \text{dvol}_x
$$

Hence, we need only the estimate with respect to the metric $h'$.

Let us begin with the estimate for sections of $E_a$ with $a \neq 0$. We set $n := \deg_{E_a} a$.

**Lemma 7.10** There exist constants $R' > 0$ and $C' > 0$ such the following holds if $|w| \geq R'$:

- Let $\varphi$ be an $L^2$-section of $E_a$ on $B'_i$ with respect to $\psi^* \text{dvol}_x$, such that

$$
\int_{B'_i} \left( |\nabla_{h'} \varphi|^2_{h'} + |(\theta_a + w d\zeta) \varphi|^2_{h'} \right) \psi^* \text{dvol}_x < \infty.
$$

Then, we have

$$
|w|^\epsilon \int_{B'_i} |\varphi|^2_{h'} \psi^* \text{dvol}_x \leq C' \int_{B'_i} \left( |\nabla_{h'} \varphi|^2_{h'} + |(\theta_a + w d\zeta) \varphi|^2_{h'} \right) \psi^* \text{dvol}_x.
$$

Here, $\epsilon = 1 + p/(n + p) > 1$.

**Proof** For each $w$, we have the solutions $b_i(w)$ ($i = 0, \ldots, n + p - 1$) of the following equation:

$$
\partial_a a(u) + pwu^{p-1} = 0
$$

We have the equality $u^{-p+1} \partial_a a(u) + pw = \alpha \prod_{i=0}^{n+p-1} (u^{-1} - b_i(w)^{-1})$ for some $\alpha \in \mathbb{C} \setminus \{0\}$. We have

$$
\theta_a = \partial_a a \text{id}_{E_a} \text{d}u + g_a \text{d}u,
$$

where $|g_a|_{h'} \leq C_1 |u|^{-1}$. We have $R_2 > 0$ and $C_2 > 0$ such that the following holds if $|w| > R_2$:

$$
C_2^{-1} \leq |b_i(w)| \leq C_2
$$

We take $C_3 >> C_2$. We set $U_1 := \{|w| \leq C_3^{-1} |w|^{-1/(n+p)} \}$ and $U_2 := \{|w| \leq C_3 |w|^{-1/(n+p)} \}$.

Let us consider the estimate on $B'_i \setminus U_2$. We have $|g_a|_{h'} \leq (C_1/C_3) |w|^{1/(n+p)}$. We also have

$$
|u^{-1} - b_i(w)^{-1}| \geq |b_i(w)^{-1}| - |u^{-1}| \geq (C_2^{-1} - C_3^{-1}) |w|^{1/(n+p)}
$$

for any $i$, and hence $|u^{-p+1} \partial_a a + pw| \geq |\alpha|(C_2^{-1} - C_3^{-1})^{n+p} |w|$. Hence, if $C_3$ is sufficiently larger than $C_2$, we have the following for some $C_4 > 0$:

$$
\left| (\partial_a a + pwu^{p-1}) \varphi + g_a(\varphi) \right|_{h'} \geq C_4 |w| |\varphi|_{h'} |u|^{p-1}
$$

Hence, we obtain the inequality for the integral over $B'_i \setminus U_2$.

Let us consider the estimate on $U_1$. There exist $C_5 > 0$ and $R_5 > 0$ such that

$$
\left| (\partial_a a + pwu^{p-1}) \varphi \right|_{h'} \geq C_5 |w|^{-n-1} |\varphi|_{h'} |du|.
$$

We also have $|g_a(\varphi) du|_{h'} \leq C_1 |u|^{-1} |\varphi|_{h'} |du|$. Hence, there exists $C_6 > 0$ such that

$$
\left| (\theta_a + w du^{p-1}) \varphi \right|_{h'}^2 \geq C_6 \left| \varphi \right|_{h'}^2 |u|^{-2(n+p)} |u|^{2(p-1)} |du| |\text{d}u| \geq C_6 C_3 |\varphi|_{h'}^2 |u|^{2(p-1)} |du| |\text{d}u|.
$$
Therefore, we have the desired inequality for the integral over $\mathcal{U}_1$.

Let us consider the estimate on $\mathcal{U}_2 \setminus \mathcal{U}_1$. For each $i = 0, \ldots, n+p-1$, we set $\tilde{V}_i := \{|u - b_i(\omega)| \leq \epsilon_1|\omega|^{-1/(n+p)}\}$ for some $\epsilon_1 > 0$. Let $u \in \mathcal{U}_2 \setminus (\mathcal{U}_1 \cup \bigcup_i \tilde{V}_i)$. We have

$$|u|^{-p+1} \partial_w a + pw = |pw| |u|^{-p-n} \prod_{i=0}^{n+p-1} |u - b_i(\omega)| \geq pC_3^{-1}|\omega|^2 \prod_{i=0}^{n+p-1} |u - b_i(\omega)| \geq pC_3^{-1}\epsilon_1^{p+n} |\omega|$$

We also have the following:

$$|g_\alpha \varphi|_{h'} \leq C_1|u|^{p+1-\alpha}|\varphi|_{h'} \cdot |u|^{-p} \leq C_1|u|^{p+1-\alpha}|\varphi|_{h'} \cdot C_3|\omega|^{p/(n+p)} = C_1|u|^{p+1-\alpha}|\varphi|_{h'} |\omega| \cdot C_3|\omega|^{-n/(n+p)}$$

Hence, there exists $C_7 > 0$ and $R_7 > 0$ such that the following holds on $\mathcal{U}_2 \setminus (\mathcal{U}_1 \cup \bigcup_i \tilde{V}_i)$, if $|\omega| \geq R_7$:

$$|(\partial_w a + p e w^p)^{\alpha}) \varphi du + g_\alpha \varphi du|_{h'} \geq C_7|\omega| |\varphi|_{h'} |u|^{p+1-\alpha} |du|$$

We set $a := (n+2)/(n+p)$. We put $V_i := \{|u - b_i(\omega)| \leq \epsilon_1|\omega|^{-a}\}$ and $V'_i := \{|u - b_i(\omega)| \leq \epsilon_1|\omega|^{-a}/2\}$. On $\tilde{V}_i \setminus V_i$, we have

$$|u|^{-p+1} \partial_w a + pw = pC_3^{-1}|\omega|^2 \prod_{i=0}^{n+p-1} |u - b_i(\omega)| \geq pC_3^{-1}|\omega|^2(\epsilon_1|\omega|^{-1/(n+p)})^{n+p-1} \times (\epsilon_1|\omega|^{-a}/2) \geq pC_3^{-1}\epsilon_1^{p+n} |\omega|^{1+1/(n+p)-a} \quad (83)$$

We also have $|ga| \leq C_1C_3|\omega|^{1/(n+p)}$. Because $-(p-1)/(n+p) + 1 + 1/(n+p) - a > 1/(n+p)$, there exist $C_3 > 0$ and $R_3 > 0$ such that the following holds on $\tilde{V}_i \setminus V_i$, if $|\omega| \geq R_3$:

$$|\theta_a + p e u^{p-1}) \varphi|_{h'} \geq C_3|\omega|^{1/(n+p)-a}|u|^{p-1} |du| |\varphi|_{h'} = C_3|\omega|^{(n+2p)/(n+p)} |u|^{p-1} |du| |\varphi|_{h'}$$

We have the following type of Poincaré inequality, i.e., there exist $C_0 > 0$ and $R_0 > 0$ such that the following holds on $V_i$, if $|\omega| \geq R_0$ (see (7) and (2.12) of [53]):

$$|u|^{(n+2)/(n+p)} \int_{V_i} |\varphi|^{2}_{h'} |du| \leq C_0 \left( \int_{V_i} |d| \varphi|^{2}_{h'} |du| + |u|^{(n+2)/(n+p)} \int_{V_i \setminus V'_i} |\varphi|^{2}_{h'} |du| \right)$$

We also have the following inequalities:

$$|u|^{(n+2p)/(n+p)} \int_{V_i} |\varphi|^{2}_{h'} |u|^{2(p-1)} |du| \leq C_3^{2(p-1)}|u|^{(n+2p)/(n+p)} \int_{V_i} |\varphi|^{2}_{h'} |du| \quad (84)$$

$$|u|^{(n+2)/(n+p)} \int_{V_i \setminus V'_i} |\varphi|^{2}_{h'} |du| \leq C_3^{2(p-1)}|u|^{(n+2p)/(n+p)} \int_{V_i \setminus V'_i} |\varphi|^{2}_{h'} |u|^{2(p-1)} |du| \leq C_3 C_3^{2(p-1)} \int_{V_i \setminus V'_i} |(\theta_a + p e u^{p-1}) \varphi|^{2}_{h'} \quad (85)$$

Then, we obtain the desired inequality for the integral over $V_i$. Thus, the proof of Lemma 7.10 is finished.  

Let us consider the case $a = 0$. Because this part is essentially contained in [53], we give just an indication. We take a positive number $C_{10}$ which is sufficiently larger than $|\alpha|$ for any eigenvalues of the residue of $\theta_0$. We may assume $|g_0| \leq (C_{10}/10)|\omega|^{-1}$ on $B_\epsilon$. Take $R_{10} > 0$ sufficiently larger than $C_{10}$. For $|\omega| \geq R_{10}$, let $U := \{|\omega| \leq C_{10}|\omega|^{-1}\}$ and $U' := \{\omega|_{\omega} \leq C_{10}|\omega|^{-1}/2\}$. On $B_\epsilon \setminus U'$, we have

$$|(\theta_0 + w d \zeta) \varphi|_{h'} \geq |w| |\varphi|_{h'} |d\zeta| - |g_0| |\varphi|_{h'} |d\zeta| \geq \frac{4}{5} |w| |\varphi|_{h'} |d\zeta| \quad (86)$$
There exist $C_{11} > 0$ and $R_{11} > 0$ such that the following holds on $U$, if $|w| \geq R_{11}$:

$$|w|^2 \int_U |\varphi|^2_h |d\zeta d\overline{\zeta}| \leq C_{11} \int_U |d|\varphi|^2_h|^2 + |w|^2 \int_{U \cup U'} |\varphi|^2_h |d\zeta d\overline{\zeta}| \leq \int_U \left( C_{11} |\nabla_h \varphi|^2_h + 4 |\theta_0 + wd\zeta| \varphi|^2_h \right)$$  \hspace{1cm} (87)

We obtain the desired inequality for sections of $E_0$ from (86) and (87). Thus, the proof of Proposition 7.9 is finished.

The following is a refinement of Corollary 7.7.

**Corollary 7.11** There exist $\rho > 0$ and $C > 0$ such that the following holds:

- Let $\varphi$ be a section of $E$ such that
  $$\int |\varphi|^2_h \text{dvol}_X + \int |\Delta_1 \varphi|^2_h \text{dvol}_X < \infty.$$  \hspace{1cm} (88)

Then, we have the following inequality:

$$C|w|^{1+\rho} \left( \int |\varphi|^2_h \text{dvol}_X \right)^{1/2} \leq \left( \int |\Delta_1 \varphi|^2_h \text{dvol}_X \right)^{1/2}$$  \hspace{1cm} (89)

**Proof** It is shown by the argument for Corollary 7.7 by using Proposition 7.9 instead of Proposition 7.5.

### 7.2 Comparison with the algebraic Nahm transform

#### 7.2.1 Statements

Let $(E, \overline{\Theta}_E, \theta, h)$ be a wild harmonic bundle on $(T^\vee, D)$. Let $P_* E$ be the associated filtered bundle on $(T^\vee, D)$. Let $(E_1, h_1, \nabla_1)$ be the $L^2$-instanton on $T \times \mathbb{C}$ obtained as the Nahm transform of $(E, \overline{\Theta}_E, \theta, h)$ (see §7.1). Let $P_* E_1$ be the associated filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$.

**Theorem 7.12** There is a natural isomorphism of the filtered bundles $P_* E_1 \simeq \text{Nahm}_* (P_* E, \theta)$.

Conversely, let $(E_1, \nabla_1, h_1)$ be an $L^2$-instanton on $T \times \mathbb{C}$. Let $P_* E_1$ be the associated filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$. Let $(E, \overline{\Theta}_E, \theta, h)$ be the wild harmonic bundle on $(T^\vee, D)$ obtained as the Nahm transform of $(E_1, \nabla_1, h_1)$ (see §6.4). Let $(P_* E, \theta)$ be the associated filtered Higgs bundle.

**Theorem 7.13** There is a natural isomorphism of the filtered Higgs bundles $(P_* E, \theta) \simeq \text{Nahm}_* (P_* E_1)$.

We obtain the involutivity of the Nahm transform in the following sense.

**Corollary 7.14** For an $L^2$-instanton $(E_1, \nabla_1, h_1)$ on $T \times \mathbb{C}$, we have an isomorphism

$$\text{Nahm}(\text{Nahm}(E_1, \nabla_1, h_1)) \simeq (E_1, \nabla_1, h_1).$$

For a wild harmonic bundle $(E, \overline{\Theta}_E, \theta, h)$ on $T^\vee$, we have an isomorphism

$$\text{Nahm}(\text{Nahm}(E, \overline{\Theta}_E, \theta, h)) \simeq (E, \overline{\Theta}_E, \theta, h).$$

**Proof** It follows from Proposition 3.25 Theorem 7.12 Theorem 7.13 and the uniqueness of the harmonic metric or Hermitian-Einstein metric adapted to the filtered bundle. (See Proposition 6.10 for the uniqueness of Hermitian-Einstein metric. See §7 for the uniqueness of the harmonic metric. See also §11.) See also [22.2.3] for adaptedness of metrics and filtered bundles.)
7.2.2 Proof of Theorem 7.12

Let us construct an isomorphism \((E_1, \mathcal{J}_{E_1}) \simeq \text{Nahm}(\mathcal{P}_E, \theta)|_{T \times \mathbb{C}}\). We recall the monad construction of \(E_1 = \text{Nahm}(E, \mathcal{J}_E, \theta, h)\) [14]. We use the notation in [7.1.1]. Let \(g_{T \times D}\) be a \(\text{Poincaré} \) like \(\text{Kähler metric of } T^\vee \setminus D\). Let \(\mathcal{A}_i(E, \mathcal{J}_E, \theta, h)\) denote the space of sections \(\varphi\) of \(E \otimes \Omega^i\) on \(T^\vee \setminus D\) such that \(\varphi\) and \((\mathcal{J}_E + \theta)\varphi\) are \(L^2\) with respect to \(h\) and \(g_{T \times D}\). Note that the conditions also imply \((\mathcal{J}_E + \theta)\varphi = L^2\) for any \((z, w) \in T \times \mathbb{C}\). Let \(\mathcal{A}_i\) denote the sheaf of holomorphic sections of the product bundle \(\mathcal{A}_i(E, \mathcal{J}_E, \theta, h) \times T \times \mathbb{C}\) over \(T \times \mathbb{C}\). We have the morphisms \(\delta^i : \mathcal{A}_i \rightarrow \mathcal{A}_{i+1}\) induced by \(\mathcal{J}_{E, z} + \theta_w\), and the sheaf of holomorphic sections of \(E_1\) is isomorphic to \(\ker \delta^i / \text{Im} \delta^i\).

Applying the construction in the proof of Lemma 3.1 around each point of \(D\), we extend \(E\) and \(E \otimes \Omega^1\) to \(\mathcal{C}_1^0(\mathcal{P}_E, \theta)\) and \(\mathcal{C}_1^0(\mathcal{P}_E, \theta)\). Let \(\mathcal{C}_1^0(\mathcal{P}_E, \theta)\) denote the Dolbeault resolution of \(\mathcal{C}_1^0(\mathcal{P}_E, \theta)\).

For \(I \subset \{1, 2, 3\}\), let \(p_I\) denote the projection of \(T^\vee \times T \times \mathbb{C}\) onto the product of the \(i\)-th components \((i \in I)\). On \(T^\vee \times T \times \mathbb{C}\), we set \(\mathcal{C}_1^0 := \bigoplus_{k + \ell = i} p_i^{-1} \mathcal{C}_1^0(\mathcal{P}_E, \theta) \otimes p_i^{-1} \mathcal{O}_T \mathcal{P}\text{oin}\)

We have \(\delta^i : \mathcal{C}_1^0 \rightarrow \mathcal{C}_1^{i+1}\) induced by \(\mathcal{J}_{E, z} + \theta_w\). We have a natural inclusion \(\Phi : p_{23}\mathcal{C}_1^0(\mathcal{P}_E, \theta) \rightarrow \mathcal{A}_i\) of complexes on \(T \times \mathbb{C}\). According to the results in §5.1 of [40], \(\Phi\) is a quasi-isomorphism. We also have the following natural isomorphisms in \(D^b(\mathcal{O}_T)\):

\[
p_{23}\mathcal{C}_1^0 \simeq R_{p_{23}}\left(\mathcal{C}_1^0(\mathcal{P}_E, \theta) \otimes p_{12} \mathcal{P}\text{oin} \right) \simeq R_{p_{23}}\left(\mathcal{C}_1^0(\mathcal{P}_E, \theta) \otimes p_{12} \mathcal{P}\text{oin} \right) \simeq \text{Nahm}(\mathcal{P}_E, \theta)|_{T \times \mathbb{C}}
\]

(See Lemma 3.1 for \(\Psi\).) Thus, we obtain the desired isomorphism \(E_1 \simeq \text{Nahm}(\mathcal{P}_E, \theta)|_{T \times \mathbb{C}}\), by which we shall identify them.

Lemma 7.15 To prove Theorem 7.12 we have only to show \(\text{Nahm}_a(\mathcal{P}_E, \theta) \subset \mathcal{P}_a E_1\) for any \(a\).

Proof By Proposition 3.17 we have \(\text{deg}(\text{Nahm}_a(\mathcal{P}_E, \theta)) = \text{deg}(\mathcal{P}_a E_1) = 0\). By Proposition 6.4 we also have \(\text{deg}(\mathcal{P}_a E_1) = 0\). Hence, \(\text{Nahm}_a(\mathcal{P}_E, \theta) \subset \mathcal{P}_a E_1\) implies \(\text{Nahm}_a(\mathcal{P}_E, \theta) = \mathcal{P}_a E_1\).

To show \(\text{Nahm}_a(\mathcal{P}_E, \theta) \subset \mathcal{P}_a E_1\), we need an estimate of the upper bound of the norms of sections of \(\text{Nahm}_a(\mathcal{P}_E, \theta)\). We use an argument of scaling in [53]. Because we need only the upper bound, we will not consider more precise estimates for harmonic representatives or their approximation.

Let \(U_\tau \subset \mathbb{P}^1\) be a neighbourhood of \(\infty\) with the coordinate \(\tau = w^{-1}\). If \(U_\tau\) is sufficiently small, we have the decomposition \(\text{Nahm}_a(\mathcal{P}_E, \theta) = \bigoplus_{P \in D} \text{Nahm}_a(\mathcal{P}_E, \theta)_P\) by the spectrum on \(T \times U_\tau\). We have the refined decomposition

\[
\text{Nahm}_a(\mathcal{P}_E, \theta)_P = \bigoplus_{\sigma \in \mathcal{I}(\theta) \cap P} \text{Nahm}_a(\mathcal{P}_E, \theta)_{P, \sigma, \alpha} \subset \mathcal{C}_1^0,
\]

to the decomposition of the filtered Higgs bundle \((\mathcal{P}_E, \theta) = \bigoplus_{\sigma \in \mathcal{I}(\theta) \cap P} \mathcal{C}_1^0(\mathcal{P}_E, \theta)_{P, \sigma, \alpha}\) around each \(P \in D\). We have only to show that \(\text{Nahm}_a(\mathcal{P}_E, \theta)_{P, \sigma, \alpha} \subset \mathcal{P}_a E_1\). We shall argue the case \(P = \{0\}\) in the following. The other case can be established similarly. We omit the subscript \(P\). We take a small neighbourhood \(U_\zeta \subset T^\vee\) of \(\{0\}\).

Let us consider the case \((\sigma, \alpha) \neq (0, 0)\). Take \(a \in \mathfrak{a}\). For each \(c \in \mathbb{R}\), we have the frame of \(\text{Nahm}_c(\mathcal{P}_E, \theta)_{P, \sigma, \alpha}\) in Lemma 3.29. We have only to show

\[
\left|\zeta_\alpha \mathcal{P}\text{oin} \mathcal{O}_{a, \alpha} / \mathcal{O}_{a, \alpha}\right|_{h_1} = O(|w|^{(b-j-m_\alpha + 1/2)(p_\sigma + m_\sigma) + 1/2})
\]

(90)

Here, \(b\) is the parabolic degree of \(\mathfrak{a}_{a, \alpha}^\alpha\).

We give a preliminary. We have the expression \(\zeta_\alpha \partial_{\zeta_\alpha} a + p_\alpha = \sum_{j=0}^{m_\alpha} \alpha_j \zeta_\alpha^{-j} = : G(\zeta_\alpha)\). We fix a complex number \(\gamma\) such that \(\alpha_m + p_\alpha \gamma^{p_\sigma + m_\sigma} = 0\). Take a covering \(U_\eta \rightarrow U_\tau\) given by \(\eta \mapsto \eta^{p_\sigma + m_\sigma}\). If \(U_\tau\) is sufficiently small, we can take holomorphic functions \(u_{i, 0}^{(i)}(\eta)\) \((i = 1, \ldots, p_\sigma + m_\sigma)\) satisfying the following:

\[
G(u_{i, 0}^{(i)}(\eta)) + p_\sigma u_{i, 0}^{(i)}(\eta)\eta^{-p_\sigma - m_\sigma} = 0, \quad \lim_{\eta \rightarrow 0} u_{i, 0}^{(i)}(\eta) / \eta = \gamma \exp(2\pi \sqrt{-1} i / (m_\sigma + p_\sigma))
\]
There exist $C_1 > 0$ and $\epsilon_1 > 0$ such that
\[ |\eta^{-1} u_0^{(i)}(\eta) - \gamma \exp(2\pi \sqrt{-1}i/(m_\phi + p_\phi)) | \leq C_1 |\eta|^{\epsilon_1}. \]

**Lemma 7.16** Let $Z_\eta$ denote the support of $\text{Cok}(\nu^{p_\phi + m_\phi} \theta_\alpha^0 + p_\phi \xi^{p_\phi + m_\phi} d\zeta_o/\zeta_o)$ on $U_\zeta$. If $U_\zeta$ and $U_\tau$ are sufficiently small, there exists a decomposition $Z_\eta = \bigsqcup_{\eta \in Z_\eta^{(i)}}$ such that the following holds for any $u \in Z_\eta^{(i)}$:
\[ |u_\eta^{(i)}(\eta) - u| \leq C|\eta|^{1 + m_\phi + \epsilon}. \]

**Proof** Take $u_1 \in Z_\eta$. There exists a possibly multi-valued holomorphic 1-form $\nu(\zeta_o) d\zeta_o/\zeta_o$ obtained as the eigenvalue of $\theta_\alpha^0$, such that $\nu(u_1) + \nu^{-p_\phi - m_\phi} p_\phi u_1^{p_\phi} = 0$. Because $\nu(\zeta_o) - G(\zeta_o) = O(\zeta_o^\epsilon)$, there exist $C_2 > 0$ and $\epsilon_2 > 0$, independently from $\eta$, such that the following holds for some unique $i$:
\[ \left| \eta^{-1} u_1 - \gamma \exp(2\pi \sqrt{-1}i/(p_\phi + m_\phi)) \right| \leq C_2 |\eta|^{\epsilon_2}. \] (91)

We obtain a decomposition of $Z_\eta = \bigsqcup_{\eta \in Z_\eta^{(i)}}$ by the condition $[91]$. Let $u_1 \in Z_\eta^{(i)}$. We set $Q_\eta(x, y) := \sum_{i+j=q} x^i y^j$. We have
\[ \left( u_0^{(i)}(\eta)/\eta \right)^{-1} - (u_1/\eta)^{-1} = \sum_{j=1}^{m_\phi} \alpha_i \eta^{m_\phi-j} Q_{j-1} \left( u_0^{(i)}(\eta)/\eta, u_1/\eta \right) \]
\[ = O(\eta^{m_\phi+\epsilon}) \] (92)

We obtain $|\left( u_0^{(i)}(\eta)/\eta \right)^{-1} - (u_1/\eta)^{-1} | = O(|\eta|^{m_\phi+\epsilon})$. Then, we obtain the desired estimate.

Let $\rho$ be an $\mathbb{R}_{\geq 0}$-valued function on $\mathbb{C}_{\nu}$ such that $\rho(\eta) = 1$ for $|\eta| < 1/2$ and $\rho(\eta) = 0$ for $|\eta| > 1$. We set $u_0 := u_0^{(0)}$. We consider the following $C^\infty$-sections of $E_{a, \alpha}^0 \otimes \Omega_\mathcal{X}^1$:
\[ \mu_1(v_\alpha, i, \xi) := \rho \left( |\xi|^{1 + m_\phi/2} (\zeta_o - u_0(\xi)) \right) v_\alpha i d\zeta_o/\zeta_o \]
\[ \mu_2(v_\alpha, i, \xi) := (\theta_\alpha^0 + \xi^{p_\phi + m_\phi} d\zeta_o)\rho \mu_1(v_\alpha, i, \xi) \]

By Lemma[7.16] if $|\xi|$ is sufficiently large, $\rho \left( |\xi|^{1 + m_\phi/2} (\zeta_o - u_0(\xi)) \right)$ is constantly 1 around $\zeta$. Hence, the tuple $\mu(v_\alpha, i, \xi) = (\mu_1(v_\alpha, i, \xi), \mu_2(v_\alpha, i, \xi))$ gives a representative of $[v_\alpha i d\zeta_o/\zeta_o]$.

By an elementary change of variables, we obtain the following for any $\delta > 0$:
\[ \int |\mu_1(v_\alpha, i, \xi)|^2_h \leq \int \rho(|\xi|^{1 + m_\phi/2} (\zeta_o - u_0(\xi)))^2 |\zeta_o|^{-2(b+\delta)-2} |d\zeta_o| \leq C_{1\delta} |\xi|^{2(b+\delta) - m_\phi} \]
Note that we have $|\zeta_o - u_0(\xi)| \sim |\xi|^{1-m_\phi/2}$ for $\zeta_o$ such that $\overline{\partial}\rho \left( |\xi|^{1 + m_\phi/2} (\zeta_o - u_0(\xi)) \right) \neq 0$. Hence, we also have the following:
\[ \int |\mu_2(v_\alpha, i, \xi)|^2_h \leq C_{2\delta} \int |\overline{\partial}\rho \left( |\xi|^{1 + m_\phi/2} (\zeta_o - u_0(\xi)) \right)|^2 |\zeta_o|^{-2(b+\delta)} \frac{1}{|\zeta_o|^{2(m_\phi + 1 + 2(1 + m_\phi)/2)} = C_{3\delta} |\xi|^{2(b+\delta) - m_\phi} \]
By the construction of $h_\delta$, we have $[v_\alpha i d\zeta_o/\zeta_o]^2_h \leq \int \left( |\mu_1(v_\alpha, i, \xi)|^2_h + |\mu_2(v_\alpha, i, \xi)|^2_h \right)$. Hence, we obtain the desired estimate $[90]$ for $[v_\alpha i d\zeta_o/\zeta_o]$. We obtain the estimate for $[v_\alpha i \zeta_o d\zeta_o/\zeta_o]$ similarly.

Let us consider the case $(\alpha, \alpha) = (\{0\}, 0)$. The following lemma is easy to see.
Lemma 7.17 Let $Z_w$ denote the support of $\text{Cok}(\theta_{0,0} + wd\zeta)$. There exist $C > 0$ and $\epsilon > 0$ such that $|u| \leq C|w|^{-1-\epsilon}$ holds for any $u \in Z_w$.

For a holomorphic section $s$ of $C^1(P_*E_{P,\{0\}} \otimes \Omega^*,\theta)$ (see [3.1]), we consider the following $C^\infty$-sections of $E_{P,\{0\}} \otimes \Omega^1$:

$$\mu_1(s,w) := (\rho(\zeta) - \rho(w\zeta)) s d\zeta/\zeta, \quad \mu_2(s,w) := \left(\theta_{P,\{0\}} + w d\zeta\right)^{-1}(\overline{\partial}\mu_1(s))$$

$$\mu'_1(s,w) := \rho(\zeta) s d\zeta/\zeta, \quad \mu'_2(s,w) := \left(\theta_{P,\{0\}} + w d\zeta\right)^{-1}(\overline{\partial}\mu'_1(s))$$

By Lemma 7.17, $\mu_2$ and $\mu'_2$ are well defined. The tuples $\mu(s,w) = (\mu_1(s,w),\mu_2(s,w))$ and $\mu'(s,w) = (\mu'_1(s,w),\mu'_2(s,w))$ naturally induce the same holomorphic section of Nahm($P_*E$)$_P$. If $s$ is a section of $P_*E_{P,\{0\}}$, then it is elementary to show the following for any $\delta > 0$:

$$\int |\mu(s,w)|^2_{\theta} \leq C|w|^{2(c+\delta)}$$

We obtain $|\mu(s,w)|^2_{\theta} \leq C|w|^{c+\delta}$ for any $\delta > 0$. By the construction of Nahm($P_*E$)$_{P,\{0\}}$, we obtain Nahm($P_*E$)$_{P,\{0\}} \subset P_*E_1$. Thus, the proof of Theorem 7.12 is finished.

7.2.3 Proof of Theorem 7.13

Let us construct an isomorphism of the Higgs bundles $(E,\overline{\partial}E,\theta) \simeq \text{Nahm}(P_*E_1)|_{T^\vee \setminus D}$. Let $\mathcal{A}^0$ denote the space of sections $\varphi$ of $E_1$ on $T^\times \mathbb{C}$, such that $\varphi$ and $\overline{\partial}_{E_1}\varphi$ are $L^2$ with respect to $h_1$ and the Euclidean metric. Let $\mathcal{A}^{0,i}$ denote the sheaf of holomorphic sections of the product bundle $\mathcal{A}^{0,i} \times (T^\vee \setminus D)$ over $T^\vee \setminus D$. We have the morphism $\delta^i : \mathcal{A}^{0,i} \rightarrow \mathcal{A}^{0,i+1}$ induced by $\overline{\partial}_{E_1} - \zeta d\zeta$, and the sheaf of holomorphic sections of $(E,\overline{\partial}E)$ is isomorphic to $\text{Ker} \delta^i/\text{Im} \delta^0$.

For $I \subset \{1,2,3\}$, let $p_I$ denote the projection of $T^\vee \times T^\times \mathbb{C}$ onto the product of the $i$-th components. By the construction, we have a natural morphism $R_{P,*}(p_{P\{1\}P\{2\}P\{3\}}P_{\infty}E_1) \otimes p_{P\{1\}P\{2\}P\{3\}}P_{\infty}E_1) \rightarrow \mathcal{A}^0$. By the results in [3.2], it is a quasi-isomorphism. Hence, we obtain a holomorphic isomorphism $E \simeq \text{Nahm}(P_*E_1)|_{T^\vee \setminus D}$, by which we identify them. The Higgs fields are equal, because they are induced by the multiplication of $-w$.

We give a preliminary. Let $U \subset \mathbb{P}^1$ be a small neighbourhood of $\infty$. On $T \times U$, we have the following decomposition

$$P_*E_1 = \bigoplus_{P \in SP_{\infty}(E_1) \cup \{0\}} P_*E_1(P_{\varphi})_{P,\varphi} \cup \cdots \cup P_*E_1(P_{\varphi})_{P,\varphi}.$$ (94)

Fix a lift of $S_{\inf}(E_1) \subset T^\vee$ to $\tilde{S}_{\inf}(E_1) \subset \mathbb{C}$. We have the filtered bundles with an endomorphism $(P_*V,g)$ on $U$, corresponding to $P_*E_1$. It has a decomposition $(P_*V,g) = \bigoplus (P_*V_{P,\varphi})_{P,\varphi}$. Let $\mathcal{U} \subset P_{-1}(E_1)$ be the subsheaf such that $\mathcal{U}|_{T \times \mathbb{C}} = \bigoplus_{P \in \mathcal{U}} P_{-1}(E_1)_{P,\varphi}$ and

$$\mathcal{U} = \bigoplus_{P \in \mathcal{U}} P_{-1}(E_1)_{P,\varphi} \bigoplus_{\varphi \notin \{0\}, \varphi} P_{-1}(E_1)_{P,\varphi}.$$ (95)

around $T \times \{\infty\}$. We use the notation in [3.3.1]

Lemma 7.18 We have $N(\mathcal{U}) \subset P_{0}E$.

Proof We give an argument around $0 \in T^\vee$, by supposing $0 \in D$. The other case can be proved similarly. We may suppose the lift of $0 \in D$ is $0 \in \mathbb{C}$. Let $t$ be a holomorphic section of $N(\mathcal{U})$ around $0 \in T^\vee$.

We have to show $|t|_h = O(|\zeta|^{-\delta})$ for any $\delta > 0$. It is represented by a family of $C^\infty$-sections $\kappa(\zeta) = \kappa^1(\zeta)d\zeta + \kappa^2(\zeta)d\zeta$ of $P_{-1}E_1 \otimes \Omega^1_{T \times \mathbb{P}^1} \otimes L_{\zeta}^{-1}$. According to the decomposition (95), we have

$$\kappa^i(\zeta) = \sum_{P,\varphi} \kappa^i(\zeta)_{P,\varphi}.$$
If $P \neq 0$, we may assume $\kappa^i(\zeta)_{P,a} = 0$ on $U$. (See the proof of Proposition 6.7) Let $\text{dvol} := |dzd\tau d\sigma d\varpi|$. We take a $C^\infty$-metric $h_1$ of $U$. Note $h_1 = O(h_2|w|^{-2+4\delta})$ for any $\delta > 0$ on $P_{-1}(E_1)_{P,a}$, and $h_1 = O(h_2|w|^{-1-\epsilon})$ for some $\epsilon > 0$ on $P_{-1}(E_1)_{P,a}$ for $(a,a) \neq (\{0\},0)$. If $P = 0$ and $(a,a) \neq (\{0\},0)$, we have the following finiteness uniformly for $\zeta$:

$$
\int_{T \times U} |\kappa^i(\zeta)_{0,a} |^2_{h_1} \text{dvol} \leq C_0 \int_{T \times U} |\kappa^i(\zeta)_{0,a} |^2_{h_2} |w|^{-2-\epsilon} \text{dvol} < \infty
$$

We have $|q_{0,0,0,0}|_{h_1} \leq C_1|w|^{-1}$ for some $C_1$. We take a sufficiently small $C_2 > 0$, and we put $H_{\zeta} := \{w \mid |w|^{-1} < C_2|\zeta|\}$. We can find a unique family of $C^\infty$-sections $\mu(\zeta)$ of $P_{-1}E \otimes L_{\zeta}^{-1}$ on $H_{\zeta}$ such that

$$
(\partial_{E} + \zeta d\sigma)\mu(\zeta) = \left(\kappa^1(\zeta)_{0,0,0} d\tau + \kappa^2(\zeta)_{0,0,0} d\varpi\right)|_{H_{\zeta}}.
$$

There exists $C_3 > 0$ such that the following holds:

$$
\int_{T \times U} |\mu(\zeta)|^2_{h_2} |dzd\tau| \leq C_3|\zeta|^{-2} \int_{T \times U} |\kappa^1(\zeta)_{0,0,0} |^2_{h_2} |dzd\tau|
$$

Let $\chi(\varpi)$ be an $\mathbb{R}_{\geq 0}$-valued $C^\infty$-function such that $\chi(\varpi) = 1$ if $|\varpi|^{-1} \leq C_2/4$ and $\chi(\varpi) = 0$ if $|\varpi|^{-1} \geq C_2/2$. We set

$$
\mathcal{K}^1(\zeta) = \kappa^1(\zeta)_{0,0,0} - \partial_{E}(\chi(\varpi)\mu(\zeta)) = (1 - \chi(\varpi))\kappa^1(\zeta)_{0,0,0}
$$

$$
\mathcal{K}^2(\zeta) = \kappa^2(\zeta)_{0,0,0} - \partial_{E}(\chi(\varpi)\mu(\zeta)) = (1 - \chi(\varpi))\kappa^2(\zeta)_{0,0,0} - (\partial_{E}\chi)(\varpi) \cdot \mu(\zeta)
$$

For any $\delta > 0$, we have the following finiteness, which is uniform for $\zeta$:

$$
\int_{T \times U} (|\mathcal{K}^1(\zeta)|^2_{h_2} + |\mathcal{K}^2(\zeta)|^2_{h_2}) |dzd\tau| \frac{|dw d\varpi|}{|w|^{2+\delta}} \leq C_1,\delta
$$

Hence, we have the following for any $\delta > 0$:

$$
\int_{T \times U} (|\mathcal{K}^1(\zeta)|^2_{h_1} + |\mathcal{K}^2(\zeta)|^2_{h_2}) |dzd\tau dwd\varpi| \leq C_{2,\delta} \int_{T \times U} (|\mathcal{K}^1(\zeta)|^2_{h_2} + |\mathcal{K}^2(\zeta)|^2_{h_2}) |dzd\tau| \frac{|dw d\varpi|}{|w|^{2+\delta}} |\zeta|^{-2\delta} \leq C_{3,\delta}|\zeta|^{-2\delta}
$$

Hence, we obtain $t(\zeta)|_{h} \leq C_{4,\delta}|\zeta|^{-\delta}$ for any $\delta > 0$. Thus, the proof of Lemma 7.18 is finished.

Let us prove $\text{Nahm}_a(P_{-1}E) = P_{a}E$. We have the following, which is similar to Lemma 7.15

**Lemma 7.19** We have only to show $\text{Nahm}_a(P_{-1}E) \subset P_aE$ for any $a$. Around each $P \in D$, we have the decomposition

$$
\text{Nahm}_a(P_{-1}E) = \bigoplus_{a,a} \text{Nahm}_a(P_{-1}E)_{P,a,a},
$$

according to the decomposition [11]. We have only to prove $\text{Nahm}_a(P_{-1}E)_{P,a,a} \subset P_aE$. We shall argue the case $P = 0$ in the following. The other case can be proved similarly. We shall omit the subscript $P$. We take a small neighbourhood $U_{\zeta}$ of $P$.

Let us consider the case $(a,a) \neq (0,0)$. Let $U_\tau \subset \mathbb{P}^1$ be a small neighbourhood of $\infty$ with the coordinate $\tau = w^{-1}$. Take $a \in a$. For each $c \in \mathbb{R}$, we have the frame of Lemma 3.30. We have only to show

$$
| [r_{a,a}^{i}] |_{h_1} = O( |\zeta|^{-(b+j+p_a-m_a/2)(p_a-m_a)^{-1}} )
$$

Here, $b$ is the parabolic degree of $r_{a,a}$.

We give a preliminary. We take a ramified covering $U_a \longrightarrow U_\zeta$ given by $\zeta = u^{p_a-m_a}$. We put $G(\tau_0) := \partial_{a,a}(\tau_0) - \alpha_{a,a} \tau^{p_a}$, where $\alpha_{a,a} = \sum_{j=0}^{p_a} \beta_j \tau^{p_a-j}$. We take a complex number $\gamma$ such that $\beta_{p_a}\gamma^{p_a-m_a} = 1$. Let $U_{\zeta}$ be sufficiently small, we can take holomorphic functions $n_0^{(i)}(u)$ $(i = 1, \ldots, p_a - m_a)$ on $U_{\zeta}$ satisfying

$$
G(n_0^{(i)}(u)) = u^{p_a-m_a}, \quad \lim_{u \to 0} u^{-1} n_0^{(i)}(u) = \gamma \exp(2\pi \sqrt{-1}u/(p_a-m_a))
$$
There exist $C_1 > 0$ and $\epsilon_1 > 0$ such that \[ |u^{-1}\eta_0^{(i)}(u) - \gamma \exp(2\pi \sqrt{-1}/(p \cdot m))| \leq C_1|u|^{\epsilon_1}. \] The following lemma is similar to Lemma 7.10.

**Lemma 7.20** Let $Z_u$ denote the support of $\text{Cok}(g_{a,a} - u^{p_{s-m_s}})$ on $U_{\tau}$. If $U_{\tau}$ and $U_{\zeta}$ are sufficiently small, we have a decomposition $Z_u = \prod_{i=1}^{p_{s-m_s}} Z_u^{(i)}$ and positive constants $C$ and $\epsilon$ such that \[ |\eta_0^{(i)}(u) - \eta_i| \leq C|u|^{1 + m_s + \epsilon} \] for any $\eta_i \in Z_u^{(i)}$.

We set $d := 1 + m_s/2$. We consider the following sections of $E_a^a \otimes \Omega^{0,1}$:

\[ \mu_1(v, a, u) := \rho \left( |u|^d (\tau - \eta_0(u)) \right) v_a \, d\tau \]

\[ \mu_2(v, a, u) := (g_{a,a} - u^{p_{s-m_s}})^{-1} \left( \tau \rho \left( |u|^d (\tau - \eta_0(u)) \right) \right) v_a, \]

The tuple $(\mu(v, a, u))$ defines a section of $\text{Nahm}(P_{s}E_1)_{\rho, a}$. By Lemma 7.20, $\rho(|u|^d (\tau - \eta_0(u)))$ is constantly $1$ around $Z_u$. Hence, $(\mu(v, a, u))$ induces $[v_a]$.

By an elementary change of variables, we obtain the following for any $\delta > 0$:

\[ \int |\mu_1(v, a, u)|_h^2 \leq \int \rho(|u|^d (\tau - \eta_0(u)))^2 |\tau|^{-(2b+\delta)} |dz \, d\tau| \leq C_1 \delta |u|^{-2(b+\delta) - 2p_s - 2 + 2d} = C_1 \delta |u|^{-2(b+\delta + p_s - m_s/2)} \quad (98) \]

We also have the following:

\[ \int |\mu_2(v, a, u)|_h^2 \leq \int \rho(|u|^d (\tau - \eta_0(u)))^2 |\tau|^{-(2b+\delta)} \frac{1}{|\partial_{a}(\tau) - \alpha \zeta^{p_{s-m_s}}|^2} \leq C_2 \delta |u|^{-2(b+\delta) - 2p_s + 1 - 2d} = C_2 \delta |u|^{-2(b+\delta + p_s - m_s/2)} \quad (99) \]

Hence, we obtain the estimate $|v|_h$.

Let us consider the case $(\alpha, a) = (\{0\}, 0)$. Note that $N(P_{-1}E_1)_{\{0\}, 0} = N(\Omega)_{\{0\}, 0} \subset \text{Nahm}(P_{s}E_1)$. Let $\nu \in \text{Nahm}_{1+c}(P_{s}E_1)_{\{0\}, 0}/N(P_{-1}E_1)_{\{0\}, 0}$ for $-1 < c \leq 0$. We take $v \in P_{s}V_{\{0\}, 0}$ which represents $\nu$. We naturally regard $v$ as a $C^\infty$-section of $P_{s}E_1$. Fix a sufficiently small number $b > 0$, and let $\rho$ be a $\mathbb{R}_{>0}$-valued $C^\infty$-function on $C_T$ such that $\rho(\tau) = 1$ if $|\tau| \leq b/2$ and $\rho(\tau) = 0$ if $|\tau| \geq b$. We obtain a $C^\infty$-section $\overline{\rho}(\tau)v \, d\tau$ of $P_{-1}(E_1)_{\{0\}} \otimes \Omega^{0,2}$. By using $H^2(T \times \mathbb{P}^1, \mathcal{O}[L - \zeta]) = 0$ for any $\zeta$, we can take a holomorphic family of $C^\infty$-forms $\kappa(\zeta) = \kappa(\zeta)dz + \kappa(\zeta)d\tau$ of $\mathcal{O}[L - \zeta]$ such that $\partial_{\tau} \zeta \kappa(\zeta) = \overline{\rho}(\tau)v \, d\tau$. Then, $\overline{\rho}(\tau)v \, d\tau - \kappa(\zeta)$ induces a holomorphic section $\overline{\nu}$ of $\text{Nahm}_{1+c}(P_{s}E_1)$ around $P$ which induces $\nu$ in $\text{Nahm}_{1+c}(P_{s}E_1)/N(\Omega)$.

We consider the following sections:

\[ \mu_1(v, \zeta) := (\rho(\tau) - \rho(\zeta^{-1})\tau)v \, d\tau \]

\[ \mu_2(v, \zeta) := (\overline{\rho}(\rho(\tau) - \rho(\zeta^{-1})))(g_{0,0}\zeta^{-1})^{-1}(v) \]

Then, $\mu_1(v, \zeta) + \mu_2(v, \zeta) - \kappa(\zeta)$ induces the same section $\overline{\nu}$.

We have the following for any $\delta > 0$:

\[ \int |\mu_1|_{h_1}^2 |d\tau| \leq C_1 \int |\tau|^{-(2c+\delta) - 4} |d\tau| \leq C_1 |\zeta|^{-2(c+1+\delta)} \]

We also have the following:

\[ \int |\mu_2|_{h_1}^2 |d\tau| \leq C_1 \int |\overline{\rho}(\tau)\zeta^{-1}\tau|^{-2(c+\delta)} \leq C_1 |\zeta|^{-2(c+1+\delta)} \]

Because the support of $\overline{\rho}(\tau)v \, d\tau$ is compact, we obtain $\int |\sigma|_{h_1}^2 d\nu = O(|\zeta|^{-\delta})$ for any $\delta > 0$, by the argument in the proof of Lemma 7.18. We obtain $|\overline{\nu}|_h \leq C_1 |\zeta|^{-(c+1+\delta)}$ for any $\delta > 0$. Thus, we obtain $\text{Nahm}_{1+c}(P_{s}E_1)_{\{0\}, 0} \subset P_{1+c}E_1$, and the proof of Theorem 7.19 is finished.
7.3 Kobayashi-Hitchin correspondence for $L^2$-instantons

7.3.1 Statements

Let $\mathcal{P}_*E_1$ be a good filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$ of degree 0 satisfying the conditions (A3). (See §2.4.1 for good filtered bundles.)

**Proposition 7.21** $\mathcal{P}_*E_1$ is stable, if and only if $\text{Nahm}_*(\mathcal{P}_*E_1)$ is a stable filtered Higgs bundle. (See §2.4.1 for the stability condition of $\mathcal{P}_*E_1$.)

Before going to the proof, we give a consequence.

**Theorem 7.22** Let $\mathcal{P}_*E_2$ be a stable good filtered bundle on $(T \times \mathbb{P}^1, T \times \{\infty\})$ with $\text{deg}(\mathcal{P}_*E_2) = 0$. We set $E_2 := (\mathcal{P}_*E_2)_{|T \times C}$ which is independent of $a$. Then, there exists a Hermitian-Einstein metric $h$ of $E_2$ on $T \times C$ such that (i) its curvature is $L^2$ with respect to $h$ and the Euclidean metric, (ii) it is adapted to the filtered bundle $\mathcal{P}_*E_2$. (See §2.2.7 for adaptedness.) Such a metric is unique up to the multiplication of positive constants.

**Proof** If rank $E_2 = 1$, then $E_2$ is the pull back of a line bundle $L$ of degree 0 on $T$ by the projection $T \times C \to T$, and the parabolic structure is the natural one, as in Remark 2.10. The flat metric of $L$ induces the Hermitian-Einstein metric of $E_2$ adapted to $\mathcal{P}_*E_2$.

Suppose rank $E_2 > 1$. By Proposition 7.21 $\text{Nahm}(\mathcal{P}_*E_2)$ is stable. By Corollary 3.26 we have

$$\text{deg} \text{Nahm}(\mathcal{P}_*E_2) = \text{deg}(\mathcal{P}_*E_2) = 0.$$ 

By Corollary 3.28 $\text{Nahm}(\mathcal{P}_*E_1)$ is a good filtered Higgs bundle. Hence, by the Kobayashi-Hitchin correspondence for wild harmonic bundles on curves [7], we obtain an adapted harmonic metric for $\text{Nahm}(\mathcal{P}_*E_1)$. Its Nahm transform induces a Hermitian-Einstein metric of $E_1$ adapted to the filtered bundle $\mathcal{P}_*E_1$, by Theorem 7.12 and Proposition 3.25.

Note that the converse is given in Proposition 6.4.

**Remark 7.23** This proof of Theorem 7.22 is based on the idea mentioned in Remark 5.13 of [8].

7.3.2 Proof of Proposition 7.21

Let us prove the “if” part in Proposition 7.21. Suppose $\text{Nahm}(\mathcal{P}_*E_1)$ is stable. By the Kobayashi-Hitchin correspondence for wild harmonic bundles on curves [7] (see also [40] for the case of good filtered flat bundles), we have an adapted harmonic metric for $\text{Nahm}(\mathcal{P}_*E_1)$. By Theorem 7.12 its Nahm transform gives an adapted Hermitian-Einstein metric for $\mathcal{P}_*E_1$. By Proposition 6.3 $\mathcal{P}_*E_1$ is polystable. If it is not stable, the decomposition into the stable components induces a decomposition of $\text{Nahm}(\mathcal{P}_*E_1)$, which contradicts with the stability of $\text{Nahm}(\mathcal{P}_*E_1)$. Hence, $\mathcal{P}_*E_1$ is stable.

Let us show the “only if” part in Proposition 7.21. Let $(\mathcal{P}_*E, \theta) := \text{Nahm}(\mathcal{P}_*E_1)$. Let $(\mathcal{P}_*E', \theta') \subset (\mathcal{P}_*E, \theta)$ be a strict filtered Higgs subbundle with $0 < \text{rank} E' < \text{rank} E$. We obtain a subcomplex $\mathcal{C}^* (\mathcal{P}_*E', \theta') \subset \mathcal{C}^* (\mathcal{P}_*E, \theta)$ on $T^\vee \times T \times \mathbb{P}^1$. Let $\mathcal{Y}^* := (\mathcal{Y}^0 \to \mathcal{Y}^1)$ be the quotient complex.

**Lemma 7.24** The induced morphism $R^1 p_{23*} (\mathcal{C}^* (\mathcal{P}_*E', \theta')) \to R^1 p_{23*} (\mathcal{C}^* (\mathcal{P}_*E, \theta))$ is injective.

**Proof** By the construction, $\mathcal{Y}^0$ is locally free. Hence, we obtain that $R^0 p_{23*} \mathcal{Y}^0$ is torsion-free. Because

$$R^0 p_{23*} \mathcal{Y}^* \to R^0 p_{23*} \mathcal{Y}^0$$

is injective, we obtain that $R^0 p_{23*} \mathcal{Y}^*$ is torsion-free. On a small neighbourhood $U \subset \mathbb{P}^1$ of $\infty$, we have $R^i p_{23*} (\mathcal{C}^* (\mathcal{P}_*E', \theta')) = R^i p_{23*} (\mathcal{C}^* (\mathcal{P}_*E, \theta)) = 0$ unless $i = 1$. It is easy to check that

$$R^1 p_{23*} (\mathcal{C}^* (\mathcal{P}_*E', \theta'))_{|T \times \{\infty\}} \to R^1 p_{23*} (\mathcal{C}^* (\mathcal{P}_*E, \theta))_{|T \times \{\infty\}}$$

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is injective. Hence, \( R^1p_{23\ast}(\mathcal{C}^\ast(\mathcal{P}_E', \theta')) \big|_{T \times U} \rightarrow R^1p_{23\ast}(\mathcal{C}^\ast(\mathcal{P}_E, \theta)) \big|_{T \times U} \) is injective. Because

\[
0 \rightarrow R^0p_{23\ast}\mathcal{J}^\ast \rightarrow R^1p_{23\ast}(\mathcal{C}^\ast(\mathcal{P}_E', \theta')) \rightarrow R^1p_{23\ast}(\mathcal{C}^\ast(\mathcal{P}_E, \theta))
\]

is exact, we obtain \( R^0p_{23\ast}\mathcal{J}^\ast = 0 \), and \( R^1p_{23\ast}(\mathcal{C}^\ast(\mathcal{P}_E', \theta')) \rightarrow R^1p_{23\ast}(\mathcal{C}^\ast(\mathcal{P}_E, \theta)) \) is injective.

We define the parabolic structure of \( R^1p_{23\ast}(\mathcal{C}^\ast(\mathcal{P}_E', \theta')) \) as in \([3.2.1]\). The filtered sheaf is denoted by \( \mathcal{P}_\mathcal{V}_1 \). We have a naturally defined injective morphism \( \mathcal{P}_\mathcal{V}_1 \rightarrow \mathcal{P}_E \). Hence, we have \( \deg(\mathcal{P}_\mathcal{V}_1) \leq 0 \). By the argument in \([3.2.3]\) we obtain

\[
\int_{T \times \mathbb{P}^1} c_1(\mathcal{P}_\mathcal{V}_1)\omega_T - \int_{T \times \mathbb{P}^1} c_1(R^2p_{23\ast}\mathcal{C}^\ast(\mathcal{P}_E', \theta'))\omega_T = \deg(\mathcal{P}_E')
\]

Because \( R^2p_{23\ast}\mathcal{C}^\ast(\mathcal{P}_E', \theta') \) is a torsion sheaf, we obtain \( \int_{T \times \mathbb{P}^1} c_1(R^2p_{23\ast}\mathcal{C}^\ast(\mathcal{P}_E', \theta'))\omega_T \geq 0 \). Hence, we obtain \( \deg(\mathcal{P}_E') \leq 0 \), i.e., \( (\mathcal{P}_E, \theta) \) is semistable.

We have \( (\mathcal{P}_E', \theta') \subset (\mathcal{P}_E, \theta) \) such that \( (\mathcal{P}_E', \theta') \) is stable of degree 0. If \( (\mathcal{P}_E', \theta') \) has no singularity, it is isomorphic to \( O_T^r \) with a Higgs field \( \alpha d\zeta (\alpha \in \mathbb{C}) \), and hence \( R^1p_{23\ast}\mathcal{C}^\ast(\mathcal{P}_E', \theta') \) is a non-zero torsion subsheaf of \( E_1 \), which contradicts with Lemma \([7.24]\). Therefore, \( (\mathcal{P}_E', \theta') \) has a singularity, and \( \text{Nahm}(\mathcal{P}_E', \theta') \neq 0 \) is a good filtered subbundle of \( \mathcal{P}_E \). By the stability of \( \mathcal{P}_E \), we have \( \text{rank} \text{Nahm}(\mathcal{P}_E', \theta') = \text{rank} E_1 \). Because \( \deg \text{Nahm}(\mathcal{P}_E', \theta') = \deg(\mathcal{P}_E) \), we have \( \text{Nahm}(\mathcal{P}_E', \theta') = \mathcal{P}_E \) in codimension one. Because both of them are filtered bundles, we have \( \text{Nahm}(\mathcal{P}_E', \theta') = \mathcal{P}_E \) on \( T \times \mathbb{P}^1 \). Then, we obtain \( (\mathcal{P}_E', \theta') = (\mathcal{P}_E, \theta) \) by the involutivity of the algebraic Nahm transforms.

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