Non-Douglas-Kazakov phase transition of two-dimensional generalized Yang-Mills theories

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Abstract
In two-dimensional Yang-Mills and generalized Yang-Mills theories for large gauge groups, there is a dominant representation determining the thermodynamic limit of the system. This representation is characterized by a density the value of which should everywhere be between zero and one. This density itself is determined through a saddle-point analysis. For some values of the parameter space, this density exceeds one in some places. So one should modify it to obtain an acceptable density. This leads to the well-known Douglas-Kazakov phase transition. In generalized Yang-Mills theories, there are also regions in the parameter space where somewhere this density becomes negative. Here too, one should modify the density so that it remains nonnegative. This leads to another phase transition, different from the Douglas-Kazakov one. Here the general structure of this phase transition is studied, and it is shown that the order of this transition is typically three. Using carefully-chosen parameters, however, it is possible to construct models with phase-transition orders not equal to three. A class of these non-typical models are also studied.

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1 Introduction

The two-dimensional Yang-Mills theory ($\text{YM}_2$) is a laboratory for testing ideas and concepts about the Yang-Mills model in the real four dimensional world. The string picture of $\text{YM}_2$ is also interesting on its own, as an example for the nonperturbative analysis of a quantum field theory [1–3].

The starting point to make a correspondence between $\text{YM}_2$ and string theory, is to study the large–$N$ limit of $\text{YM}_2$. For example, as it is shown in [2], a gauge theory based on $\text{SU}(N)$ is split at large $N$ into two copies of a chiral theory, which encapsulate the geometry of the string maps. The chiral theory associated with the Yang–Mills theory on a two–manifold $M$ is a summation over maps from a two–dimensional world sheet (of arbitrary genus) to the manifold $M$. This leads to a $1/N$ expansion for the partition function and observables that is convergent for all of the values of area×coupling constant on the target space $M$, if the genus is one or greater.

The large-$N$ limit of the $\text{U}(N)$ $\text{YM}_2$ on a sphere has been studied in [4]. There the dominant (or classical) representation has been found and it has been shown that the free energy of the $\text{U}(N)$ $\text{YM}_2$ on a sphere with the surface area $A < A_c = \pi^2$ has a logarithmic behavior. In [5], the free energy has been calculated for areas $A > \pi^2$, from which it has been shown that the $\text{YM}_2$ on a sphere has a third-order phase transition at the critical area $A_c = \pi^2$. This is the well-known Douglas-Kazakov phase transition. It resembles the Gross-Witten-Wadia phase transition for the lattice two dimensional multicolour gauge theory [6].

The main characteristics of $\text{YM}_2$ are its invariance under area-preserving diffeomorphisms and the fact that there are no propagating degrees of freedom. These properties are not unique to $\text{YM}_2$, but rather are shared by a wide class of theories, called the generalized Yang-Mills theories ($\text{gYM}_2$’s) [7]. Various properties of $\text{gYM}_2$’s have been studied, including their phase structure [8], the large-$N$ behavior of the Wilson loops [9], and their double-scaling limit properties [10].

The phase structure of $\text{gYM}_2$’s is an interesting issue from various points of view. The reason the Douglas-Kazakov (DK) phase transition occurs in $\text{YM}_2$ and $\text{gYM}_2$’s, is that in $A < A_c$ areas (the so called the weak regime) the dominant density function determined through a saddle-point analysis ($\rho_w$) is everywhere less than one. But for $A > A_c$ areas (the strong regime) $\rho_w$ becomes greater than one in some points, which is not acceptable and so $\rho_w$ must be replaced by a new density ($\rho_s$). The transition from $\rho_w$ to $\rho_s$ in $\text{YM}_2$ induces a phase transition of order three. The DK phase transition has a richer structure in $\text{gYM}_2$’s and the order of which may be other than three, in the so called nontypical theories [8].

Another feature of the phase structure of $\text{gYM}_2$’s, which does not exist in $\text{YM}_2$, is the possibility that $\rho_w$ becomes negative somewhere. A negative density is not acceptable either, so it must be replaced by another density, which we denote it by $\rho_g$ ("g" for the “gaped phase”), in which the density function is zero in some region. In [11], a special $\text{gYM}_2$ has been studied, in which the action
is a combination of quartic and quadratic Casimirs, and a third order phase transition from the weak regime to the gaped regime has been observed. This observation is based on a classification done by Jurekiewicz and Zalewski [12].

In the present work, the structure of the transition between the weak regime and the gaped regime for arbitrary gYM\(_2\)'s is studied. The technique used is similar to that used in [8] and [9] to study the DK phase structure of gYM\(_2\)'s. In this method, only the behavior of \(\rho_w\) near the critical point is needed. This is important, since the functional form of \(\rho_w\) is explicitly known for all gYM\(_2\)'s, while that’s not the case for \(\rho_s\) and \(\rho_g\). It is shown that the non-DK phase transitions between the weak and gaped regimes are in almost all gYM\(_2\)'s of order three. There are, however, some special cases with fine-tuned couplings where one can have the phase transition of nontypical orders.

The scheme of the paper is the following. In section 2, gYM\(_2\)'s are briefly reviewed and the method of calculating the order of non-DK phase transitions is discussed. In section 3, this method is applied to the typical model discussed in [11] and it is proved that the order of this non-Dk phase transition is three. Finally, in section 4 a nontypical gYM\(_2\) is introduced the order of its weak-gaped transition is 5/2 and an outline is given on how to construct a general nontypical model with the non-DK phase transition of order \(2 + (1/k)\), where \(k\) is an integer greater than one.

## 2 The general method

The partition function of the generalized U(\(N\)) Yang-Mills theory on a sphere of area \(A\) is [7,13]

\[
Z = \sum_r d_r^2 e^{-A \Lambda(r)},
\]

(1)

where \(r\)'s label the irreducible representations of the group U(\(N\)), \(d_r\) is the dimension of the representation \(r\), and

\[
\Lambda(r) = \sum_{k=1}^{p} \frac{a_k}{N^{k-1}} C_k(r).
\]

(2)

\(C_k\) is the \(k\)'th Casimir of the gauge group, and \(a_k\)'s are arbitrary constants. For the partition function (1) to be convergent, it is necessary that \(p\) in (2) be even and \(a_p\) be positive. The representations of U(\(N\)) are parameterized by integers \(n_i\) such that \(n_1 \geq n_2 \geq \cdots \geq n_N\).

In the large-\(N\) limit, it is convenient to introduce the continuous variable

\[
\phi(x) := -n(x) - 1 + x,
\]

(3)

where

\[
0 \leq x := \frac{i}{N} \leq 1,
\]

\[
n(x) := \frac{n_i}{N}.
\]

(4)
The partition function (1) then becomes

$$Z = \int \prod_{0 \leq x \leq 1} d\phi(x) \ e^{S(\phi)},$$  \hspace{1cm} (5)$$

where

$$S(\phi) := N^2 \left\{ -A \int_0^1 dx \ G[\phi(x)] + \int_0^1 dx \int_0^1 dy \ \log |\phi(x) - \phi(y)| \right\},$$  \hspace{1cm} (6)$$

(apart from an unimportant constant), and

$$G(\phi) := \sum_{k=1}^P (-1)^k a_k \phi^k.$$  \hspace{1cm} (7)$$

Introducing the density function

$$\rho[\phi(x)] := \frac{dx}{d\phi(x)},$$  \hspace{1cm} (8)$$

it is seen that it satisfies

$$\int_{-a}^a dz \ \rho(z) = 1,$$  \hspace{1cm} (9)$$

where \([-a, a]\) is the interval corresponding to the values of \(\phi\). Here it is assumed that \(G(\phi)\) is even, and therefore \(\rho(z)\) is even as well. The condition \(n_1 \geq n_2 \geq \cdots \geq n_N\) demands

$$0 \leq \rho(z) \leq 1.$$  \hspace{1cm} (10)$$

As \(N\) tends to infinity, the only representation that contributes to the partition function \(5\) is the so called dominant (or classic) representation \([14]\), satisfying

$$g(z) = P \int_{-a}^a dz' \ \frac{\rho(z')}{z - z'},$$  \hspace{1cm} (11)$$

where \(P\) indicates the principal value of the integral, and

$$g(z) := \frac{A}{2} G'(z).$$  \hspace{1cm} (12)$$

The free energy of the theory is defined through

$$F := -\frac{1}{N^2} \ln Z.$$  \hspace{1cm} (13)$$

Using the standard method of solving the integral equation \(11\), the density function \(\rho\) is obtained, following \([14]\), as

$$\rho(z) = \frac{\sqrt{a^2 - z^2}}{\pi} \sum_{n,q=0}^{\infty} \frac{(2n-1)!!}{2^n n! (2n + 2q + 1)!!} a^{2n} z^{2q} g^{(2n+2q+1)}(0),$$  \hspace{1cm} (14)$$
where the parameter \( a \) satisfies
\[
\sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^n n! (2n-1)!} a^{2n} g^{(2n-1)}(0) = 1.
\] (15)

Here \( g^{(n)} \) is the \( n \)-th derivative of \( g \).

The density function in (14), which we call \( \rho_w \), should satisfy the conditions (10). It depends obviously on the area \( A \) and the parameters \( a_k \), and there could be regions in the parameter space (of \( A \) and \( a_k \)'s) where these conditions are violated. In such regions, the density corresponding to the dominant representation is not \( \rho_w \). The case where \( \rho_w(z) \) exceeds one for some \( z \) results in the well-known DK phase transition. Here we are interested in the case where \( \rho_w(z) \) becomes negative for some \( z \). In this case \( \rho_g \), the density corresponding to the dominant representation, is zero in some interval \( L_g \), the length of which is defined as \( 2b \). For the same parameters, \( \rho_w \) is negative in a region \( L_g' \), the length of which is of the order \( 2b \). Defining
\[
\alpha := |\min(\rho_w)|,
\] (16)
one can use exactly the same arguments used in [9] for the DK phase transition to show that
\[
H_g(z) - H_w(z) \sim b^{2m+2} \sim \alpha^{1+(1/m)}, \quad \text{for large } z.
\] (17)

Here
\[
H_{g,w}(z) := \int dy \frac{\rho_{g,w}(y)}{z-y},
\] (18)

\( 2m \) is the order of the first nonvanishing derivative of \( \rho_w \) at the point \( \rho_w \) attains its minimum, and by large \( z \) it is meant \( |z| >> a \).

Assume that the phase transition from \( \rho_w \) to \( \rho_g \) occurs at some critical value \( A = A_c \). If the order of the first nonvanishing derivative of \( \alpha \) with respect to \( A \) at the point \( A_c \) is \( l \), then (17) results in
\[
H_g - H_w (A - A_c)^{l[1+(1/m)]}.
\] (19)

Using (13), and the fact that the dominant representation maximizes \( S \), it is seen that
\[
F'(A) = \int_0^1 dx \ G[\phi(x)],
\]
\[
= \int dy \ \rho(y) G(y),
\]
\[
= \frac{1}{2 \pi i} \oint dz \ H(z) G(z),
\] (20)

where the last integration is over a large contour and in the last step (18) has been used. So one arrives at
\[
F_g'(A) - F_w'(A) \sim (A - A_c)^{l[1+(1/m)]},
\] (21)
and from that
\[ F_w(A) - F_w(A) \sim (A - A_c)^{1 + \frac{1}{l + \frac{1}{m}}}, \quad (22) \]
This is our desired relation. Eq. (22) shows that for typical theories where \( l = m = 1 \), the system exhibits a third order phase transition at \( A = A_c \), but for special theories (with fine-tuned coupling constants) this order can in principle be different from three. The situation is completely analogous to that of the DK phase transition.

One can extend this argument to the case where several phase transitions occur, either of them could correspond to a density exceeding one or becoming negative. Suppose that the density is already so that there are regions where it is identically one or zero, and varying the parameters creates a new region where the density either exceeds one or becomes negative. Let us call this density \( \rho_1 \), and the boundary which is violated (zero or one) \( B \). This density should be corrected so that the corrected density \( \rho_2 \) does not cross the boundary \( B \). For the density \( \rho_2 \), there is a new interval where \( \rho_2 \) is equal to \( B \). Defining \( \alpha \) as the maximum of \( \rho_1(z) - B \), and \( 2b \) as the width of the region where the value of \( \rho_1 \) is not in \([0, 1]\), it is seen that exactly similar arguments lead to something like (22), where the left-hand side is \( F_2(A) - F_1(A) \). So, even if there are several transitions of either kind (the density crossing zero or one), the order of each transition is similar to the case of DK phase transition. Specially, any transition which is typical \((l = m = 1)\) is a third order transition.

### 3 A typical model

Consider the following typical model
\[ G(z) = c_2 z^2 + c_4 z^4, \quad (23) \]
which has been investigated in [8, 11], where
\[ c_4 > 0. \quad (24) \]
Using (14) and (15), one finds
\[ \rho_w(z) = \frac{\mu}{\pi} \sqrt{a^2 - z^2} \left( z^2 + \frac{a^2}{2} + \beta \right), \quad (25) \]
\[ a^2 = -\frac{2 \beta}{3} + \frac{2}{3} \sqrt{\frac{6}{\mu} + \beta^2}, \quad (26) \]
where
\[ \beta := \frac{c_2}{2 c_4}, \quad (27) \]
\[ \mu := 2 A c_4. \]

There arise three distinct regions in the \((\beta - \mu)\) plane:
i $\beta > \sqrt{2/\mu}$.
Here $\rho_w(z)$ has a maximum at $z = 0$.

ii $-\sqrt{2/\mu} < \beta < \sqrt{2/\mu}$.
Here $\rho_w(z)$ has a positive minimum at $z = 0$ and two maxima at $\pm z_0 := \pm \sqrt{(3a^2 - 2\beta)/6}$.

iii $\beta < -\sqrt{2/\mu}$.
Here $\rho_w(z)$ has a negative minimum at $z = 0$ and two maxima at $\pm z_0 := \pm \sqrt{(3a^2 - 2\beta)/6}$.

Clearly in the region iii, one must replace $\rho_w$ with $\rho_g$, which is equal to zero in an even interval around $z = 0$, and the system undergoes a non-DK phase transition from the weak regime to the gaped regime on the critical curve

$$\beta = -\sqrt{2/\mu}.$$  \hspace{1cm} (28)

At this transition curve, the maximum value of $\rho_w(z)$ is

$$\rho_w(\pm z_0) = \left(\frac{213\mu}{3^3\pi^4}\right)^{1/4}.$$  \hspace{1cm} (29)

One should also make sure that at this point $\rho_w$ is still never greater than one, which sets a condition for $\beta$:

$$\beta \leq \beta_0 := -\frac{128}{27\pi^2}.$$  \hspace{1cm} (30)

So as $\mu$ increases, for $\beta < \beta_0$ the system goes from the weak regime to the gaped regime, while for $\beta > \beta_0$ the system goes from the weak regime to the strong regime, i.e. undergoes the ordinary DK phase transition.

To obtain the order of the above non-DK phase transition, one must determine the parameters $l$ and $m$ in (22). Using (25), it is seen that

$$\rho''(0)\big|_c = \left(\frac{128\mu^3}{\pi^4}\right)^{1/4},$$  \hspace{1cm} (31)

which is positive and shows that $m = 1$. To obtain $l$, one should investigate the behavior of $\alpha$ for fixed $\beta$ with respect to $\mu$. (Note that $\mu$ differs from $A$ by just a multiplicative positive constant, $2c_4$.) One has

$$\alpha = -\frac{\mu a}{\pi} \left(\frac{a^2}{2} + \beta\right).$$  \hspace{1cm} (32)

From (26), it is seen that the derivative of $a$ with respect to $\mu$ at fixed $\beta$ is a finite negative number. Noting that $(a^2 + 2\beta)$ vanishes at the transition, one obtains

$$\frac{\partial a}{\partial \mu} = -\frac{\mu a^2}{\pi} \frac{\partial a}{\partial \mu},$$  \hspace{1cm} (33)

which is positive. So $l = 1$ as well, and the order of the transition is 3. This has been pointed out in [11], based on different arguments.
4 Nontypical models

Consider the following potential.

\[ G_2(z) = \sum_{n=1}^{k+1} c_{2n} z^{2n}, \]  

(34)

where \( c_{2k+2} \) is positive. Defining

\[ \mu := (k + 1) A c_{2k+2}, \]

\[ \beta_n := \frac{k + 1 - n}{k + 1} \frac{c_{2k+2-2n}}{c_{2k+2}}, \]  

(35)

and using (14) and (15), one finds

\[ \rho_w(z) = \frac{\mu}{\pi} \sqrt{a^2 - z^2} \sum_{q=0}^{k} \sum_{n=0}^{k-q} \gamma_n \beta_{k-n-q} z^{2q}, \]  

(36)

\[ = \frac{\mu}{\pi} \sqrt{a^2 - z^2} \sum_{q=0}^{k} \delta_q z^{2q}, \]

and

\[ \mu \sum_{n=1}^{k+1} \gamma_n \beta_{k+1-n} = 1, \]  

(37)

respectively, where

\[ \gamma_n := \frac{(2n - 1)!!}{2^n n!} a^{2n}. \]  

(38)

The aim is to tune the coupling constants so that

\[ \sum_{q=0}^{k} \sum_{n=0}^{k-q} \gamma_n \beta_{k-n-q} z^{2q} \bigg|_{a=a_0} = z^k. \]  

(39)

This leads to

\[ \sum_{n=0}^{p} \gamma_n \beta_{p-n} \bigg|_{a=a_0} = \delta_{p,0}, \quad p \leq k. \]  

(40)

To solve this, one notices that

\[ \sum_{n=0}^{\infty} \gamma_n s^n = (1 - a^2 s)^{-1/2}. \]  

(41)

It is seen that if one defines \( \tilde{\beta}_n \) such that

\[ \sum_{n=0}^{\infty} \tilde{\beta}_n s^n = (1 - a_0^2 s)^{1/2}, \]  

(42)
then
\[ \sum_{n=0}^{p} \gamma_n \beta_{p-n} = \delta_{p0}. \]  \hspace{1cm} (43)

So \( \beta_n \) is the same as \( \tilde{\beta}_n \), for \( n \leq k \):
\[ \beta_n = -\frac{(2n-3)!!}{2^n n!} a_0^{2n}, \quad n \leq k. \]  \hspace{1cm} (44)

It is seen that \( \beta_0 \) is positive (in fact one), while other \( \beta_n \)'s are negative. Now consider the coefficient of \( z^2q \) in the summation (36). One has
\[ \delta_q = \gamma_{k-q} \left( 1 + \sum_{n=0}^{k-q-1} \frac{\gamma_n}{\gamma_{k-q}} \beta_{k-n-q} \right). \]  \hspace{1cm} (45)

It is seen that for \( q < k \), the derivative of the parenthesis with respect to \( a \) is positive. (Each term in the summation is a negative constant times a negative power of \( a \)). Knowing that it vanishes at \( a = a_0 \), one deduces that \( \delta_q \) is positive for \( a > a_0 \), negative for \( a < a_0 \), and there is a value \( a_q \) less than \( a_0 \), so that the derivative of \( \delta_q \) is positive for \( a > a_q \). One then concludes that \( \rho_w \) is nonnegative for \( a > a_0 \), and \( \rho_w(0) \) is negative for \( a < a_0 \). Also, the derivative of \( \rho_w(0) \) with respect to \( a \) is positive for \( a > a_0 \). One can summarize these like
\[ \rho_w(z) > 0, \quad a > a_0, \]  \hspace{1cm} (46)
\[ \rho_w(0) < 0, \quad a < a_0, \]  \hspace{1cm} (47)
\[ \frac{\partial \rho_w(0)}{\partial a} \bigg|_{a=a_0} > 0. \]  \hspace{1cm} (48)

Next consider (37). The aim is to prove that with the choice (44), to every \( a \) larger than \( a_0 \) there corresponds a positive \( \mu \), and for \( a \geq a_0 \) the derivative of \( a \) with respect to \( \mu \) is negative. One can rewrite (37) as
\[ \frac{1}{\mu} = -\tilde{\beta}_{k+1} + \sum_{n=0}^{k+1} \gamma_n \tilde{\beta}_{k+1-n}, \]
\[ = -\tilde{\beta}_{k+1} + \delta_{-1}. \]  \hspace{1cm} (49)

So, repeating exactly the same arguments used to arrive at (46) to (48), one concludes for that \( a \geq a_0 \) the right-hand side of (49) is positive and the derivative of the right-hand side of (49) with respect to \( a \) is positive:
\[ \mu(a) > 0, \quad a \geq a_0, \]  \hspace{1cm} (50)
\[ \frac{\partial \mu}{\partial a} \bigg|_{a=a_0} < 0, \quad a \geq a_0. \]  \hspace{1cm} (51)

The final picture is the following. Increasing \( \mu \) from zero, \( a \) decreases from infinity, so that at \( \mu = \mu_0 \) one arrives at \( a = a_0 \):
\[ \mu_0 := \frac{2^{k+1} (k + 1)!!}{(2k-1)!! a_0^{2k+2}}. \]  \hspace{1cm} (52)
For $\mu < \mu_0$, the density is nonnegative, while as $\mu$ exceeds $\mu_0$ the density becomes negative at $z = 0$. (48) and (51) show that the derivative of $\alpha$ with respect to $\mu$ is nonvanishing at $\mu = \mu_0$. Hence,

$$l = 1.$$  

(53)

To find the value of $m$, one notes that at the transition point the first nonvanishing derivative of $\rho_w$ at the origin is the $(2k)$'th derivative. So,

$$m = k.$$  

(54)

From these, one finds that the order of the transition is $[2 + (1/k)]$.

There remains one point to take into account, which is to make sure that for $\mu \leq \mu_0$ the density does not exceed one. To address this, one notices that the transformation

\[
\begin{align*}
a &\rightarrow \sigma a, \\
a_0 &\rightarrow \sigma a_0, \\
\mu &\rightarrow \sigma^{-2k-2} \mu, \\
z &\rightarrow \sigma z, \\
\rho &\rightarrow \sigma^{-1} \rho,
\end{align*}
\]

(55)

where $\sigma$ is an arbitrary positive constant, leaves all of the relations intact. So, using a sufficiently large $\sigma$ ensures that the DK transition does not occur before the transition from the weak regime to the gaped regime.

**Acknowledgement:** M.A. would like to thank the research council of the University of Tehran for partial financial support.
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