COMPLEX SYMMETRIC COMPOSITION OPERATORS

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Abstract. Let \( \varphi \) be a linear fractional self-map of the open unit disk \( \mathbb{D} \) and \( H^2 \) the Hardy space of analytic functions on \( \mathbb{D} \). The goal of this article is to characterize the linear fractional composition operators \( C_\varphi f = f \circ \varphi \) on \( H^2 \) that are complex symmetric.

1. Introduction

We say that a bounded linear operator \( T \) on a separable Hilbert space \( \mathcal{H} \) is complex symmetric if \( T \) has a self-transpose matrix representation with respect to some orthonormal basis of \( \mathcal{H} \). This is equivalent to the existence of a conjugation (i.e., a conjugate-linear, isometric involution) \( C \) on \( \mathcal{H} \) such that \( T = CT^*C \). In this case \( T \) is called \( C \)-symmetric. The general study of complex symmetric operators on Hilbert spaces was initiated by Garcia, Putinar and Wogen ([12],[13],[15],[16]).

Let \( \varphi \) be a holomorphic self-map of the open unit disk \( \mathbb{D} \). The composition operator induced by \( \varphi \) is defined on the Hardy space \( H^2 \) by \( C_\varphi f = f \circ \varphi \). It is a non-trivial fact that composition operators are always bounded on \( H^2 \). The text [9] contains an encyclopedic treatment of these operators. The study of complex symmetry of composition operators was recently initiated by Garcia and Hammond [11]. They posed the problem of characterizing all complex symmetric composition operators on the Hardy space \( H^2 \). The main goal of this article is to characterize the linear fractional composition operators that are complex symmetric.

Main Result. Let \( \varphi \) be a non-constant linear fractional self-map of the disk \( \mathbb{D} \). If \( C_\varphi \) is complex symmetric on \( H^2 \), then either \( C_\varphi \) is normal or \( \varphi \) is an elliptic automorphism of finite order.

The plan of the paper is the following. In Section 2 we collect all the necessary preliminaries for our work. In Section 3, we use the hypercyclicity of composition operators induced by hyperbolic maps and parabolic automorphisms to show that these composition operators are not complex symmetric. In Section 4, we prove that composition operators induced by parabolic non-automorphisms are also not complex symmetric. Finally in Sections 5 and 6, we study the complex symmetry of composition operators induced by non-automorphisms with an interior fixed point and elliptic automorphisms respectively. In particular, in Section 6 we show that the converse of the main result does not hold by proving that most finite order elliptic automorphisms induce composition operators that are not complex symmetric. Combining all these sections will establish the main result stated above.

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2. Preliminaries

When studying $C$-symmetric operators, the bilinear form $[f,g] = \langle f,Cg \rangle$ for $f,g \in \mathcal{H}$ is as important as the standard sesquilinear form on $\mathcal{H}$. We say that $(u_n)_{n \in \mathbb{N}}$ is a complete $C$-orthogonal system in $\mathcal{H}$ if the linear span of vectors $u_n$ is dense in $\mathcal{H}$ and if $[u_i,u_j] = 0$ for all $i \neq j$. The completeness of $(u_n)_{n \in \mathbb{N}}$ implies that $[u_n, u_m] \neq 0$ for all $n \in \mathbb{N}$. We say that a vector $f$ is isotropic if $[f,f] = 0$. Hence a complete $C$-orthogonal system consists of non-isotropic vectors. Garcia and Putinar [14] made a systematic study of $C$-orthogonal systems and showed that they appear naturally as eigenfunctions of certain classes of complex symmetric operators. In particular, we shall need the following result from [10]:

**Lemma 2.1.** The eigenvectors of a $C$-symmetric operator $T$ corresponding to distinct eigenvalues are $C$-orthogonal.

Recall that a holomorphic function $f$ on $\mathbb{D}$ belongs to the Hardy space $H^p$ for some $0 < p < \infty$ if

$$
||f||_p = \sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty.
$$

If $1 \leq p < \infty$ then $H^p$ is a Banach space with norm $|| \cdot ||_p$, while if $0 < p < 1$ then $H^p$ is a $p$-Banach space [2]. Our work will be concerned with the Hardy space $H^2$.

For each $\omega \in \mathbb{D}$ and $n \in \mathbb{N}$, we let $K^{(n)}(\omega)$ denote the unique function in $H^2$ which satisfies $\langle f, K^{(n)}(\omega) \rangle = f^{(n)}(\omega)$ where $f^{(n)}$ denotes the $n$-th derivative for any $f \in H^2$. Note that for each $\omega \in \mathbb{D}$, the span of $(K^{(n)}(\omega))_{n \in \mathbb{N}}$ is dense in $H^2$; because if for some $f \in H^2$ we have $f \perp K^{(n)}(\omega)$ for all $n \in \mathbb{N}$, then $f \equiv 0$ in some neighborhood of $\omega$.

A non-constant map $\varphi : \mathbb{D} \to \mathbb{D}$ is called a linear fractional map if it can be written as

$$
\varphi(z) = \frac{az + b}{cz + d}
$$

where $a,b,c,d \in \mathbb{C}$. Linear fractional maps are automorphisms of the Riemann sphere $\hat{\mathbb{C}}$ and they have at least one and at most two fixed points in $\hat{\mathbb{C}}$. We note that the automorphisms of $\mathbb{D}$ are all linear fractional maps. Let $\varphi^{[n]}$ denote the $n$-th composite of $\varphi$. If $\alpha \in \overline{\mathbb{D}}$ is a fixed point for $\varphi$ such that the sequence $\varphi^{[n]}(z)$ converges uniformly on compact subsets of $\mathbb{D}$ to $\alpha$, then $\alpha$ is said to be an attractive fixed point for $\varphi$. Thus, depending on the position of these fixed points, linear fractional maps fall into one of the following three classes:

- **Maps with an interior fixed point.** By the Schwarz Lemma the interior fixed point $\alpha \in \mathbb{D}$ is either attractive with $\varphi'(\alpha) \in \mathbb{D}$, or the map is an elliptic automorphism with $\varphi'(\alpha) \in \mathbb{T}$.
- **Parabolic maps.** These maps $\varphi$ have a unique attractive fixed point $\alpha \in \mathbb{T}$. Furthermore, $\varphi$ is a parabolic automorphism if and only if for each $z \in \mathbb{D}$ the orbit $(\varphi^{[n]}(z))_{n \in \mathbb{N}}$ is separated in the hyperbolic metric.
- **Hyperbolic maps.** These maps have an attractive fixed point $\alpha \in \mathbb{T}$ and a second fixed point $\beta \in \mathbb{C} \setminus \mathbb{D}$. Furthermore, $\varphi$ is a hyperbolic automorphism if and only if $\beta \in \mathbb{T}$.

Normal operators are examples of complex symmetric operators (see [12]), and normal composition operators $C_\varphi$ are precisely those with $\varphi(z) = \beta z$ for $|\beta| \leq 1$ (see [9]).
The following result (Proposition 2.5, [11]) gives a necessary condition for the complex symmetry of composition operators:

**Proposition 2.2.** If $\varphi$ is a non-constant self-map of $\mathbb{D}$ and $C_\varphi$ is complex symmetric, then $\varphi$ is univalent.

We state a result from [7] that will allow us to choose standard forms for the composition operators under study.

**Lemma 2.3.** Let $\varphi$ be a holomorphic self-map of $\mathbb{D}$. For $\theta$ a real number, let $U_\theta$ be the unitary composition operator $(U_\theta f)(z) = f(e^{i\theta}z)$ for $f \in H^2$. Then

$$U_\theta^* C_\varphi U_\theta = C_{\tilde{\varphi}}$$

where $\tilde{\varphi}(z) = e^{i\theta}\varphi(e^{-i\theta}z)$.

### 3. Hypercyclic Composition Operators

A bounded operator $T$ on a separable Hilbert space $\mathcal{H}$ is said to be hypercyclic if there is an $f \in \mathcal{H}$ such that the orbit $(T^n f)_{n \in \mathbb{N}}$ is dense in $\mathcal{H}$. The reason for our interest in hypercyclic operators is the next result.

**Proposition 3.1.** If $T$ is a complex symmetric operator on $\mathcal{H}$ with non-empty point spectrum, then it is not hypercyclic.

*Proof.* Let $T$ be $C$-symmetric and $\lambda$ an eigenvalue for $T$. If $h \in \ker(T - \lambda I)$ then the identity $C(T - \lambda I) = (T^*- \bar{\lambda} I)C$ implies that $g = Ch \in \ker(T^* - \bar{\lambda} I)$. Hence if $f \in \mathcal{H}$ is an arbitrary element, then

$$\langle g, T^n f \rangle = \langle T^* g, f \rangle = \bar{\lambda}^n \langle g, f \rangle$$

for all $n \in \mathbb{N}$. The sequence $(\bar{\lambda}^n \langle g, f \rangle)_{n \in \mathbb{N}}$ is not dense in $\mathbb{C}$ for any $f \in \mathcal{H}$. It follows that $(T^n f)_{n \in \mathbb{N}}$ is not dense in $\mathcal{H}$ for any $f \in \mathcal{H}$. Hence $T$ is not hypercyclic. $\square$

Bourdon and Shapiro [4] gave the following characterization of hypercyclicity for composition operators induced by linear fractional maps: Let $\varphi$ be a linear fractional self-map of $\mathbb{D}$. Then $C_\varphi$ is hypercyclic on $H^2$ if and only if $\varphi$ is either a hyperbolic map or a parabolic automorphism. Hence by Proposition 3.1 and the fact that 1 is an eigenvalue for any composition operator, we get the main result of this section:

**Theorem 3.2.** Let $\varphi$ be a linear fractional self-map of $\mathbb{D}$. If $\varphi$ is a hyperbolic map or a parabolic automorphism, then $C_\varphi$ is not complex symmetric.

### 4. Parabolic Non-Automorphisms

Let $\varphi$ be a parabolic non-automorphism. Assume that $\varphi(e^{i\theta_0}) = e^{i\theta_0}$ for some real number $\theta_0$. Using Lemma 2.3 we see that $C_\varphi$ is unitarily equivalent to $C_{\tilde{\varphi}}$ with $\tilde{\varphi}(z) = e^{-i\theta_0}\varphi(e^{i\theta_0}z)$. Then $\tilde{\varphi}$ is also a parabolic non-automorphism and $\tilde{\varphi}(1) = 1$. Since complex symmetry is preserved by unitary equivalence, we assume from now on that $\varphi(1) = 1$. Let $\tau$ denote the Cayley map

$$\tau(z) = \frac{1 + z}{1 - z}$$

which maps $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{H}^+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$. The map $\Phi := \tau \circ \varphi \circ \tau^{-1}$ is then a linear fractional map that takes $\mathbb{H}^+$ into itself and fixes $\infty$. Hence $\Phi(z) = z + a$ for some $a \in \mathbb{C}$ with $\text{Im } a \geq 0$, and for all $z \in \mathbb{H}^+$. 


Note that $\phi$ is an automorphism of $D$ if and only if $\Phi$ is an automorphism of $H^+$; and this happens precisely when $a$ is real. Hence we must take $\text{Im } a > 0$. Denote by $\psi_t$ the singular inner function on $D$ defined by

$$\psi_t(z) = \exp[it\tau(z)] = \exp \left( \frac{t}{z-1} \right)$$

for each $t \geq 0$ and it follows that

$$C_\phi \psi_t = \exp[it(\phi \circ \varphi)] = \exp[it(\Phi \circ \tau)] = \exp[it(\tau + a)] = e^{iat} \psi_t.$$

Hence each $\psi_t$ is an eigenvector corresponding to eigenvalue $e^{iat}$ for all $t \geq 0$. In fact, Cowen [8] showed that $(e^{iat})_{t \geq 0} \cup \{0\}$ is the spectrum of $C_\phi$. We note that since $\text{Im } a > 0$, each eigenvalue $e^{iat}$ is distinct. We will need the following result

$$\text{span}(\psi_t)_{t \geq 0} = H^2$$

that is contained in Ahern and Clark’s work [1]. We are ready to prove the main result of this section.

**Theorem 4.1.** If $\phi$ is a linear fractional parabolic non-automorphism, then $C_\phi$ is not complex symmetric.

**Proof.** Since $H^2$ is a separable Hilbert space and $\text{span}(\psi_t)_{t \geq 0} = H^2$, it follows that there must exist a sequence $(t_n)_{n \in \mathbb{N}}$ of distinct non-negative real numbers such that $(\psi_{t_n})_{n \in \mathbb{N}}$ is complete in $H^2$. Now choose $t \geq 0$ so that $t \notin (t_n)_{n \in \mathbb{N}}$ and denote by $\Gamma$ the sequence $\{\psi_t\} \cup (\psi_{t_n})_{n \in \mathbb{N}}$ in $H^2$. Now suppose that $C_\phi$ is $J$-symmetric for some conjugation $J$ on $H^2$. Since $\Gamma$ is a sequence of eigenvectors of $C_\phi$ corresponding to distinct eigenvalues, it follows that $\Gamma$ is a $J$-orthogonal system by Lemma 2.1. Hence $JT$ is a biorthogonal sequence to $\Gamma$, and the existence of a biorthogonal sequence is equivalent to the minimality of $\Gamma$ (Lemma 3.3.1, [5]). In particular this implies that

$$\psi_t \notin \text{span}(\psi_{t_n})_{n \in \mathbb{N}}.$$

This contradicts the completeness of $(\psi_{t_n})_{n \in \mathbb{N}}$ in $H^2$. \hfill $\square$

5. NON-AUTOMORPHISMS WITH AN INTERIOR FIXED POINT

Let $\varphi$ be a non-constant holomorphic self-map of $D$ such that $\varphi(\alpha) = \alpha$ for some $\alpha \in D$ and $\varphi'(\alpha)$ satisfies $0 < |\varphi'(\alpha)| < 1$. In 1884, Koenigs [18] showed that if $f : D \to \mathbb{C}$ is a non-constant holomorphic solution to Schroeder’s functional equation

$$f \circ \varphi = \lambda f$$

then there exists a holomorphic map $\kappa$ on $D$, called the Koenigs eigenfunction of $\varphi$, such that $f$ is a constant multiple of $\kappa^n$ and $\lambda = \varphi'(\alpha)^n$ for some $n \in \mathbb{N}$.

The Koenigs eigenfunction $\kappa$ for $\varphi$ does not in general belong to any $H^p$ space. Results relating the operator-theoretic properties of composition operators with the $H^p$ membership of $\kappa$ have been obtained by Bourdon and Shapiro [2,3] and Poggi-Corradini [20]. In particular, they showed that $\kappa \in \bigcap_{p<\infty} H^p$ if and only if $C_\varphi : H^2 \to H^2$ satisfies

$$\lim_{n \to \infty} ||C_\varphi^n||_{p \to p}/n = 0$$

where $|| \cdot ||_p$ is the essential norm. Such operators are called Riesz operators. One should note that $\kappa \in \bigcap_{p<\infty} H^p$ is equivalent to $(\kappa^n)_{n \in \mathbb{N}} \subset H^2$; which easily follows from the fact that $H^q \subset H^p$ for all $0 < p < q \leq \infty$. 
One of the main results in [11] was to show that complex symmetric composition operators with symbols that have an attractive fixed point in \( \mathbb{D} \) are Riesz operators:

**Proposition 5.1.** Let \( \varphi \) be a non-constant holomorphic self-map of \( \mathbb{D} \) such that \( \varphi(\alpha) = \alpha \) for some \( \alpha \in \mathbb{D} \) with \( 0 < |\varphi'(\alpha)| < 1 \). If \( C_\varphi \) is a complex symmetric operator on \( H^2 \), then \( (\kappa^n)_{n \in \mathbb{N}} \subset H^2 \).

Let \( \varphi \) be a holomorphic self-map of \( \mathbb{D} \) with \( \varphi(\alpha) = \alpha \) for some \( \alpha \in \mathbb{D} \) and \( \lambda = \varphi'(\alpha) \). The point spectrum of \( C^*_\varphi \) contains \( (\lambda^n)_{n \in \mathbb{N}} \) [9]. Denote by \( v_n \) the eigenvector of \( C^*_\varphi \) corresponding to \( \lambda^n \) for \( n \in \mathbb{N} \). In the proof of Proposition 2.6 [11], Garcia and Hammond established the following fact which we need: If \( \varphi(\alpha) = \alpha \) for some \( \alpha \in \mathbb{D} \), then for all \( n \in \mathbb{N} \)

\[
K^{(n)}_\alpha \in \text{span}\{v_0, v_1, \ldots, v_n\}.
\]

Since \( \varphi \) must be univalent if \( C_\varphi \) is complex symmetric, the requirement \( \varphi'(\alpha) \neq 0 \) is unnecessary in what follows. The main result in this section shows that normal composition operators are the only composition operators induced by symbols with an attractive fixed point in \( \mathbb{D} \), that are complex symmetric.

**Theorem 5.2.** Let \( \varphi \) be a non-constant holomorphic self-map of \( \mathbb{D} \) such that \( \varphi(\alpha) = \alpha \) for some \( \alpha \in \mathbb{D} \) with \( |\varphi'(\alpha)| < 1 \). If \( C_\varphi \) is complex symmetric, then it must be normal. In particular, if \( \alpha \neq 0 \) then \( C_\varphi \) is not complex symmetric.

**Proof.** Our first goal is to prove that the complex symmetry of \( C_\varphi \) implies that \( (\kappa^n)_{n \in \mathbb{N}} \) is a complete \( J \)-orthogonal system in \( H^2 \). Suppose that \( C_\varphi \) is complex symmetric. Hence the Koenigs eigenfunctions \((\kappa^n)_{n \in \mathbb{N}} \) belong to \( H^2 \) by Proposition 5.1 and there exists a conjugation \( J \) such that \( C_\varphi \) is \( J \)-symmetric.

By the \( J \)-symmetry of \( C_\varphi \), the identity \( J(C^*_\varphi - \lambda^n I) = (C_\varphi - \lambda^n I)J \) implies that \( J \) is an isometric conjugate-linear isomorphism between \( \ker(C^*_\varphi - \lambda^n I) \) and \( \ker(C_\varphi - \lambda^n I) \) for all \( n \in \mathbb{N} \). Therefore \( Jv_n \) belongs to \( \ker(C_\varphi - \lambda^n I) \) and hence is a constant multiple of \( \kappa^n \) by Koenigs’s theorem. Therefore, applying \( J \) to the identity

\[
K^{(n)}_\alpha \in \text{span}\{v_0, v_1, \ldots, v_n\}
\]

mentioned earlier gives us \( JK^{(n)}_\alpha \in \text{span}\{1, \kappa, \ldots, \kappa^n\} \) for all \( n \in \mathbb{N} \). Since \( \text{span}(K^{(n)}_\alpha)_{n \in \mathbb{N}} \) is dense in \( H^2 \) and \( J \) is a surjective isometry, it follows that the span of \( (\kappa^n)_{n \in \mathbb{N}} \) is dense in \( H^2 \). Hence \( (\kappa^n)_{n \in \mathbb{N}} \) is complete. The \( J \)-orthogonality of \( (\kappa^n)_{n \in \mathbb{N}} \) follows by Lemma 2.1, since the eigenvalues \((\lambda^n)_{n \in \mathbb{N}} \) are distinct.

We now show that the complete \( J \)-orthogonality of \((\kappa^n)_{n \in \mathbb{N}} \) implies that \( C_\varphi \) is normal. Consider the orthogonal decomposition

\[
[\ker(C_\varphi - \lambda^n I) \cap \text{clos}(\text{ran}(C_\varphi - \lambda^n I))] \oplus [\ker(C_\varphi - \lambda^n I) \cap \ker(C^*_\varphi - \bar{\lambda}^n I)]
\]

of \( \ker(C_\varphi - \lambda^n I) \). The eigenspace \( \ker(C_\varphi - \lambda^n I) \) is 1-dimensional and is generated by \( \kappa^n \). The subspace \( \ker(C_\varphi - \lambda^n I) \cap \text{clos}(\text{ran}(C_\varphi - \lambda^n I)) \) consists entirely of isotropic vectors (Theorem 6, [11]). The complete \( J \)-orthogonality of \((\kappa^n)_{n \in \mathbb{N}} \) implies that they are all non-isotropic vectors. Hence it follows that

\[
\kappa^n \in \ker(C_\varphi - \lambda^n I) \cap \ker(C^*_\varphi - \bar{\lambda}^n I)
\]

for each \( n \in \mathbb{N} \). Therefore \( C^*_\varphi C_\varphi \kappa^n = |\lambda|^{2n} \kappa^n = C_\varphi C^*_\varphi \kappa^n \) for all \( n \in \mathbb{N} \) and the completeness of \((\kappa^n)_{n \in \mathbb{N}} \) implies that \( C_\varphi \) is normal. \( \square \)
Let \((e_n)_{n \in \mathbb{N}}\) be an orthonormal basis in a separable Hilbert space \(\mathcal{H}\) and \(C\) the conjugation that fixes \((e_n)_{n \in \mathbb{N}}\). Then \((e_n)_{n \in \mathbb{N}}\) is clearly \(C\)-orthogonal. This is in general not true for Riesz bases. For any \(\alpha \in \mathbb{D}\), denote by \(\varphi_\alpha\) the involutive disk automorphism

\[
\varphi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}
\]

for \(z \in \mathbb{D}\). The sequence \((\varphi_\alpha^n)_{n \in \mathbb{N}} \subset H^2\) is a Riesz basis since it is the image of the orthonormal basis \((z^n)_{n \in \mathbb{N}}\) under the invertible composition operator \(C_{\varphi_\alpha}\).

**Corollary 5.3.** For \(\alpha \in \mathbb{D} \setminus \{0\}\), the sequence \((\varphi_\alpha^n)_{n \in \mathbb{N}}\) is not \(J\)-orthogonal for any conjugation \(J\).

**Proof.** Define the map \(\psi = \varphi_\alpha \circ \omega \circ \varphi_\alpha\) for \(0 < |\omega| < 1\). Then \(\psi(\alpha) = \alpha\) with \(\psi'(\alpha) = \omega\) and \(C_{\psi} \varphi_\alpha^n = \omega^n \varphi_\alpha^n\). Hence \((\varphi_\alpha^n)_{n \in \mathbb{N}} \subset H^2\) are the Koenigs eigenfunctions for \(\psi\). It follows from the second part of the proof of Theorem 5.2 that the complete \(J\)-orthogonality of \((\varphi_\alpha^n)_{n \in \mathbb{N}}\) will imply that \(C_{\psi}\) is normal. Hence \((\varphi_\alpha^n)_{n \in \mathbb{N}}\) is not \(J\)-orthogonal for any conjugation \(J\). \(\square\)

6. **ELLiptic Automorphisms**

If \(\varphi\) is an elliptic automorphism and there exists an \(N \in \mathbb{N}\) such that \(\varphi^{[N]}\) is the identity, then \(\varphi\) is said to have finite order. Otherwise \(\varphi\) is said to have infinite order.

**Theorem 6.1.** If \(\varphi\) is an infinite order elliptic automorphism and is not a rotation, then \(C_{\varphi}\) is not complex symmetric.

**Proof.** Let \(\alpha \in \mathbb{D} \setminus \{0\}\) be the fixed point of \(\varphi\). Since \(\varphi\) is of infinite order, we must have \(\varphi = \varphi_\alpha \circ \lambda z \circ \varphi_\alpha\) where \(\lambda = e^{2\pi i \theta}\) with \(\theta\) irrational. It follows that \((\lambda^n)_{n \in \mathbb{N}}\) is a sequence of distinct complex numbers. We note that \(C_\varphi \varphi_\alpha^n = \lambda^n \varphi^n_\alpha\) implies that \((\varphi^n_\alpha)_{n \in \mathbb{N}}\) is a system of eigenvectors for \(C_\varphi\) corresponding to distinct eigenvalues. Suppose \(C_{\varphi}\) is \(J\)-symmetric. It then follows that \((\varphi^n_\alpha)_{n \in \mathbb{N}}\) is a \(J\)-orthogonal system by Lemma 2.1. This contradicts Corollary 5.3. \(\square\)

The elliptic automorphisms of order 2 are the involutive disk automorphisms \(\varphi_\alpha\) for \(\alpha \in \mathbb{D}\). The unique fixed point of \(\varphi_\alpha\) that lies in \(\mathbb{D}\) is

\[
\beta = \frac{1 - \sqrt{1 - |\alpha|^2}}{\alpha}
\]

It follows that \(C_{\varphi_\alpha}\) is complex symmetric, since \(C_{\varphi_\alpha}^2 = C_{\varphi_\alpha} \circ \varphi_\alpha = I\) and operators that are algebraic of degree 2 are complex symmetric (Theorem 2, [16]). Garcia and Hammond [11] posed the problem of finding an explicit conjugation \(J\) that would symmetrize \(C_{\varphi_\alpha}\). In [19], the author found such a conjugation. Denote by \(J\) the conjugation defined on \(H^2\) by \((J f)(z) = f(\bar{z})\) for \(z \in \mathbb{D}\). Let \(W_\alpha\) be the unitary part in the polar decomposition of \(C_{\varphi_\alpha}\); and \(U_\theta = C_{e^{i\theta}}\) the unitary composition operator with \(\theta \in \mathbb{R}\) chosen so that \(\bar{\alpha} = e^{i\theta} \alpha \in \mathbb{R}\). Then

**Theorem 6.2.** \(C_{\varphi_\alpha}\) is \(J_\alpha\)-symmetric where \(J_\alpha = U_\theta J W_\alpha U_\alpha^*\).

We note that if a bounded operator \(T\) is \(C\)-symmetric for some conjugation \(C\), then \(T\) is \(\lambda C\)-symmetric for all \(\lambda \in \mathbb{T}\). Our goal is to show that the conjugation \(J_\alpha\) is unique up to scalar multiples. This is in fact true in greater generality:
Theorem 6.3. Suppose $\varphi$ is an inner function that is not a rotation and $C_{\varphi}$ is $J$-symmetric. Then $J$ is unique up to scalar multiples.

Proof. We first note that if $\alpha := \varphi(0) = 0$, then the complex symmetry of $C_{\varphi}$ implies that $C_{\varphi}$ is normal by Theorem 5.2 hence $\varphi(z) = \beta z$ for some $|\beta| \leq 1$. Since $\varphi$ is inner, it must then be a rotation. Therefore $\alpha \neq 0$. Let us assume that the sequence $(\alpha^n)_{n \in \mathbb{N}}$ does not lie on a diameter of $D$. It then follows by a result of Jury (Theorem 2.6, [17]) that the unilateral shift operator $T_z : H^2 \to H^2$ belongs to $C^*(C_{\varphi})$; the $C^*$-algebra generated by $C_{\varphi}$. A well-known theorem of Coburn [6] shows that the $C^*$-algebra generated by $T_z$ contains the $C^*$-algebra of compact operators $\mathcal{K}$ and there is an exact sequence of $C^*$-algebras

$$0 \to \mathcal{K} \to C^*(T_z) \to \mathcal{C}(\mathbb{T}) \to 0$$

where $\mathcal{C}(\mathbb{T})$ is the algebra of continuous functions on the unit circle. Therefore $\mathcal{K}$ is contained in $C^*(C_{\varphi})$. Now suppose that $J_1$ and $J_2$ are two conjugations that both symmetrize $C_{\varphi}$. Then the unitary operator $U = J_1 J_2$ satisfies $UC_{\varphi} = C_{\varphi} U$ and $UC_{\varphi}^* = C_{\varphi} U$. Hence $U$ commutes with each element of $C^*(C_{\varphi})$, and in particular with every compact operator on $H^2$. Our goal is to show that the only unitary operators that commute with all compact operators are $U = \lambda I$ for $\lambda \in \mathbb{T}$. Denote by $P_1$ the orthogonal projection of $H^2$ onto the 1-dimensional subspace generated by $f \in H^2$. Since $U$ commutes with $P_{\varphi^n}$ for all $n \in \mathbb{N}$, we get $Uz^n = \lambda_n z^n$ for $\lambda_n \in \mathbb{T}$ and $n \in \mathbb{N}$. Suppose $\lambda_i \neq \lambda_j$ for some $i \neq j$. Then $UP_{\varphi_{i} + \varphi_{j}} = P_{\varphi_{i} + \varphi_{j}} U$ implies that $U(z^i + z^j) = \lambda_i z^i + \lambda_j z^j$ is a multiple of $z^i + z^j$. A contradiction. Therefore $U = \lambda I$ and hence $J_1 = \lambda J_2$ for some $\lambda \in \mathbb{T}$.

For the general case if $(\alpha^n)_{n \in \mathbb{N}}$ lies on a diameter of $D$, we can rotate via the unitary operator $U_{\theta} = C_{e^{i\theta}}$ for $\theta \in \mathbb{R}$ to get $U_{\theta}^* C_{\varphi} U_{\theta} = C_{\tilde{\varphi}}$ where $\tilde{\varphi} = e^{i\theta} \varphi(e^{-i\theta} z)$ by Lemma 2.3. So $\theta$ can be chosen so that $\tilde{\alpha} = \tilde{\varphi}(0) = e^{i\theta} \alpha$ is such that $(\tilde{\alpha}^n)_{n \in \mathbb{N}}$ does not lie on a diameter of $D$. Hence for $T \in C^*(C_{\tilde{\varphi}})$, the map $T \mapsto U_{\theta}^* T U_{\theta}$ is a $C^*$-algebra isomorphism from $C^*(C_{\tilde{\varphi}})$ onto $C^*(C_{\varphi})$ that leaves $\mathcal{K}$ invariant. The result now follows from the previous case since if $J_1$ and $J_2$ both symmetrize $C_{\tilde{\varphi}}$, then $U_{\theta}^* J_1 U_{\theta}$ and $U_{\theta}^* J_2 U_{\theta}$ both symmetrize $C_{\tilde{\varphi}}$. So $U_{\theta}^* J_1 U_{\theta} = \lambda U_{\theta}^* J_2 U_{\theta}$ and hence $J_1 = \lambda J_2$.

We now consider other elliptic automorphisms of finite order. We say that an elliptic automorphism $\varphi$ is of order $N$ and denote $\text{ord}(\varphi) = N$, if $N$ is the smallest integer such that $\varphi^N$ is the identity. Our next result shows that certain sums of elliptic composition operators of finite order are always complex symmetric:

Proposition 6.4. Let $\varphi$ be an elliptic automorphism of order $N \geq 2$. Then the operator $S_{\varphi} = C_{\varphi} + C_{\varphi}[z] + \ldots + C_{\varphi}[N-1]$ is complex symmetric on $H^2$.

Proof. Let $T = I + S_{\varphi} = I + C_{\varphi} + C_{\varphi}[z] + \ldots + C_{\varphi}[N-1]$. Then it is easy to compute that $T^2 = NT$. So $T$ is algebraic of order 2 and hence is complex symmetric (Theorem 2, [16]). Therefore $S_{\varphi} = T - I$ is also complex symmetric.

We note that Proposition 6.4 generalizes the result discussed earlier that $C_{\varphi_{\alpha}}$ is complex symmetric for $\alpha \in \mathbb{D}$. Our final result shows that even among the elliptic automorphisms of finite order, very few induce composition operators that are complex symmetric:
Theorem 6.5. Let $p$ be a prime integer and $\beta \in \mathbb{D} \setminus \{0\}$. Denote by $E_{\beta,p}$ the collection of all elliptic automorphisms $\varphi$ that fix $\beta$ with $\text{ord}(\varphi)$ a multiple of $p$ and $C_\varphi$ complex symmetric. Then $E_{\beta,p}$ has at most finitely many elements.

Proof. Assume on the contrary that some $E_{\beta,p}$ has infinitely many elements. Our first goal is to show that there is a conjugation $J$ such that $C_\varphi$ is $J$-symmetric for all $\varphi \in E_{\beta,p}$. Choose an arbitrary $\varphi \in E_{\beta,p}$. Then $\text{ord}(\varphi) = pN$ for some $N \in \mathbb{N}$ and there exists a conjugation $J$ such that $C_\varphi$ is $J$-symmetric. It follows that $\psi = \varphi^{[N]}$ is an elliptic automorphism of order $p$ and $C_\psi$ is $J$-symmetric. We note that the elliptic automorphisms of order $p$ are precisely $\psi, \psi^{[2]}, \ldots, \psi^{[p-1]}$, each corresponding to a $p$th root of unity, and hence $C_\psi$ is $J$-symmetric for each $k = 1, \ldots, p - 1$. Since $\varphi$ was arbitrary in $E_{\beta,p}$, it follows by Theorem 6.3 and the remark after Theorem 6.2 that we can choose a conjugation $J$ such that $C_\varphi$ is $J$-symmetric for all $\varphi \in E_{\beta,p}$. Now since $E_{\beta,p}$ is infinite, there is a sequence $(\psi_k)_{k \in \mathbb{N}}$ in $E_{\beta,p}$ and an increasing sequence of integers $(N_k)_{k \in \mathbb{N}}$ such that $\psi_k$ is of order $N_k$. Hence we must have $\psi_k = \varphi^\beta \circ \lambda_k \circ \varphi^\beta$ for $\lambda_k \in \mathbb{T}$ such that $N_k$ is the smallest integer such that $\lambda_k^{N_k} = 1$. Now for any pair $i, j \in \mathbb{N}$ such that $i \neq j$, choose $k$ so that $N_k > \max\{i, j\}$. Hence it follows that $\lambda_k^i \neq \lambda_k^j$ and

$$
\lambda_k^i (\varphi_\beta^i, J \varphi_\beta^i) = \langle C_{\psi_k} \varphi_\beta^i, J \varphi_\beta^i \rangle = \langle \varphi_\beta^i, C_{\psi_k}^* J \varphi_\beta^i \rangle = \lambda_k^i (\varphi_\beta^i, J \varphi_\beta^i).
$$

Therefore $(\varphi_\beta^i, J \varphi_\beta^i) = 0$ for all $i \neq j$. So $(\varphi_\beta^i)_{i \in \mathbb{N}}$ is a $J$-orthogonal system. But this again contradicts Corollary 5.3.

\[ \square \]

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