Approximating open quantum system dynamics in a controlled and efficient way: A microscopic approach to decoherence

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We demonstrate that the dynamics of an open quantum system can be calculated efficiently and with predefined error, provided a basis exists in which the system-environment interactions are local and hence obey the Lieb-Robinson bound. We show that this assumption can generally be made. Defining a dynamical renormalization group transformation, we obtain an effective Hamiltonian for the full system plus environment that comprises only those environmental degrees of freedom that are within the effective light cone of the system. The reduced system dynamics can therefore be simulated with a computational effort that scales at most polynomially in the interaction time and the size of the effective light cone. Our results hold for generic environments consisting of either discrete or continuous degrees of freedom.

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Introduction

The interaction with its environment causes a quantum system to lose energy and phase – this is termed decoherence [1]. Since a quantum system can never completely be isolated from its environment, decoherence poses a severe challenge to the application of quantum technologies such as quantum information and communication [2]. The effect of the environment is not necessarily detrimental. For example, it can be used for robust implementation of quantum processes [3]. Whether one wants to fight or exploit the effect of the environment, a profound understanding of the system-environment interaction and the ensuing open quantum system dynamics is indispensable.

The challenge for a rigorous treatment of open quantum system dynamics is due to the system and environmental degrees of freedom (DOF) becoming entangled because of their interaction. This entanglement is neglected when invoking the Markov approximation [1]. Non-Markovian effects are abundant, in particular in the condensed phase. Examples of current interest include solid state devices that interact with defects modelled as two-level fluctuators [4, 5] or excitons of the light-harvesting complex immersed by proteins [6]. While under specific assumptions, theoretical methods to tackle non-Markovian environments can be derived [7-9], a numerically feasible and generally applicable method to study the dynamics and control of open quantum system is lacking.

A complete microscopic quantum description of both system and environment,

\[ \hat{H} = \hat{H}_S \otimes \mathbb{1}_B + \hat{H}_{SB} + \mathbb{1}_S \otimes \hat{H}_B, \]  

accounts for any type of system-environment interaction. Using Eq. [1] is, however, prohibitive due to the exponential scaling of the Hilbert space dimension with the number of DOF. Typically a complete description is also not needed. In particular, if one is interested in the reduced dynamics of the system,

\[ \frac{d\hat{\rho}_S(t)}{dt} = -i \text{tr}_B \left\{ [\hat{H}, \hat{\rho}(t)] \right\}, \]  

at finite times, for example in femtosecond photochemistry, the complete description can faithfully be replaced by a 'surrogate Hamiltonian' [10-12]. This is due to the fact that at finite times, the system can only resolve a finite part of its environment.

Here we combine the concepts of the 'surrogate Hamiltonian' and the Lieb-Robinson bound [13, 14] to devise a general approach to model decoherence starting from a microscopic description of the system interacting with its environment. Our existential proof yields an efficient algorithm to calculate the reduced dynamics. We use the Lieb-Robinson bound to translate the inherent locality of quantum dynamics into a quantitative estimate for the information propagation speed [15]. The notion of an effective light cone can thus be used to set up a dynamical renormalization procedure for the generator of the 'surrogate' evolution.

Graph representation of the full Hamiltonian

Assuming the interactions between system and environment to be bilinear and bounded and the environment to be comprised of discrete DOF, the Hamiltonian (1) can be defined on a generic lattice in arbitrary dimensions,

\[ \hat{H} = \hat{H}_S + \sum_{i=1}^{N_S} \sum_{j=1}^{N_B} \hat{\Phi}_{ij}^{S\rightarrow B} + \sum_{i=1}^{N_S} \sum_{j=1}^{N_B} \hat{\Phi}_{ij}^{B\rightarrow S}, \]  

with \( N_S \) system and \( N_B \) environmental (or 'bath') DOF (\( N_B \to \infty \)). We will later generalize our results to continuous environments, e.g., comprised of harmonic oscillators, and unbounded interactions. In Eq. (3), each degree of freedom is defined on a finite-dimensional Hilbert space, \( \mathcal{H}_i \), with \( N_S^{\text{int}} \left( N_B^{\text{int}} \right) \) denoting the number of those system (environment) DOF that interact with the environment (system). The interactions \( \hat{\Phi}_{ij} \) can be expressed.
in terms of operators $\hat{O}_i$ belonging to the algebra $B(H_i)$ of bounded linear operators on $H_i$,

$$\hat{\Phi}_{ij} = \sum_{\mu=0}^{\dim(B(H_i))-1} \sum_{\nu=0}^{\dim(B(H_j))-1} J_{ij}^{\mu\nu} \hat{O}_i^\mu \hat{O}_j^\nu,$$

where the diagonal terms represent on-site interactions and $|J_{ij}^{\mu\nu}| < \infty$. The Hamiltonian $H = \sum_{i,j} J_{ij}^{\mu\nu}$ defines a graph, i.e., an ordered pair $G = (\Gamma, E)$ of nodes and edges. The set of nodes $\Gamma = \{N_S + N_B\}$ consists of all the system plus environment DOF and hence can have an infinite number of elements. The set of edges $E = \{J_{ij} \neq 0\}$ represents all non-zero couplings $J_{ij} = \left[\sum_{\mu\nu} (J_{ij}^{\mu\nu})^2\right]^{1/2}$. The bare structure of the graph is encoded in the adjacency matrix $A = A(G)$ whose entries $A_{ij}$ ($A_{ij} = 0, 1$) are the number of edges connecting $i$ and $j$.

**Regularity of the graph** To apply a renormalization procedure to Hamiltonian (3), the associated dynamics has to be quasi-local. This corresponds to the existence of a non-increasing function $F$, $F(d(i, j)) : [0, \infty) \to [0, \infty)$, acting on a metric $d(i, j)$ defined on $G$, that is uniformly integrable, $\|F\| = \sup_{i \in \Gamma} \sum_{j \in \Gamma} F(d(i, j)) < \infty$, and obeys the convolution property, $\sup_{i, j \in \Gamma} \sum_{z \in \Gamma} F(d(i, z))F(d(z, j)) \leq CF(d(i, j))$ with $C<\infty$. The proof requires us to assume that the graph, although infinite, is locally finite. Defining the connectivity $c_i$ of the $i$th node as the number of edges emanating from it, local finiteness implies $c_i < \infty$ for each $i$. The connectivity encodes both the dimensionality and the range of interactions on the physical lattice which therefore have to be finite. Local finiteness of the graph is necessary and sufficient for the adjacency matrix $A$ to be bounded in the operator norm on $G$. $\|A\| \leq \sup_{i \in \Gamma} c_i = \sup_{i \in \Gamma} \sum_{j=1}^{\infty} A_{ij} = \hat{c} [13]$. The boundedness of $A$ ensures boundedness of the coupling matrix $J$. We can thus define the metric on $G$ as the shortest path connecting two nodes,

$$d(i, j) := \min \{n \in N_0 : [A^n]_{ij} \neq 0\},$$

which does not necessarily coincide with the distance between sites on the physical lattice. A walk of length $n$ from node $i$ to $j$ can then be defined as a sequence $\pi_n(i, j) = [i = i_0, i_1, \ldots, i_n = j]$ of $n$ adjacent nodes. Defining $J = \|J\|$, the weight of the walk $\pi_n(i, j)$ becomes $w(\pi_n(i, j)) = \prod_{k=1}^{n-1} J_{i_k, i_{k+1}}$, with the weight of the zero-length walk set equal to 1. The set $\Pi_n(i, j)$ of all paths of length $n$ between nodes $i$ and $j$ contains $|\Pi_n(i, j)| = [A^n]_{ij}$ elements and weights $w(\Pi_n(i, j)) = [J^n]_{ij}$. The weight of the shortest path(s) between nodes $i$ and $j$ is $[J^{d(i, j)}]_{ij}$ and can be used to define the desired non-increasing function $F$,

$$F_r(d(i, j)) = \frac{[J^{d(i, j)}]_{ij}}{r^{d(i, j)}},$$

with $r > 1$ a damping factor penalizing longer paths. It follows from the boundedness of $J$ that $F_r(d(i, j))$ satisfies the uniform integrability condition, $\|F_r\| \leq \sum_{d(i, j)} = 0 \left[\|J\|d(i, j) r^{-1}\right] \|d(i, j) \leq \frac{1}{r^{-1}}$. For a given damping $r$, $\|F_r\|$ quantifies the maximum influence of the graph on a single node and vice versa [20]. $F_r(d(i, j))$ clearly satisfies the convolution property with convolution constant $C$ equal to one.

**Lieb-Robinson bound** To assess the quasi-locality of the dynamics generated by Hamiltonian (3), consider a finite subset $X$ of $G$, surrounding the system nodes $S$, together with the truncation of $\hat{H}$ on this subset, $\hat{H}_X = \sum_{\mu, \nu} \sum_{i, j \in X} \hat{\Phi}_{ij}^{\mu\nu}$. Denoting the boundary of the region $X$ (containing the system nodes) by $\partial S$ and that of the region $X$ (containing the environment nodes closest to $S$) by $\partial X$, the graph distance between $\partial S$ and $\partial X$ defines the radius of $X$, $R^S_X = d(\partial S, \partial X)$. The boundary $\partial X$ consists of those elements of $X$ that interact with DOF outside of $X$. Approximating Eq. (3) by the truncated Hamiltonian $\hat{H}_X$ gives rise, in the Heisenberg picture, to an approximated time evolution, $\hat{O}^S_X(t)$, for any system operator $\hat{O}_S$. The error introduced by the approximation is bounded,

$$\left\|\hat{O}_S(t) - \hat{O}^S_X(t)\right\| \leq 2 \left\|\hat{O}_S\right\| \|\partial S\| e^{-2(R^S_X-v) \cdot t},$$

where $v_r = 2\|\hat{\Phi}\|_r \epsilon$ and $\|\hat{\Phi}\|_r = \sup_{i, j \in \Gamma} \|\hat{\Phi}_{ij}\| |F_r(d(i, j))|^{-1}$. The difference is taken in the operator norm and $\|\partial S\|$ denotes the number of elements of $\partial S$. Equation (5) states the Lieb-Robinson bound [13–15] for open quantum system dynamics (see the supplementary material for details), reflecting the approximate locality of the dynamics generated by $\hat{H}$. At any fixed time, the full generator can therefore be approximated by a truncated one. The approximation improves exponentially with increasing radius, $R^S_X$, of the effective support, i.e., the contribution of environmental DOF outside of the effective light cone of $S$ to the system dynamics is exponentially small. The bound (5) is fully general since it relies only on the cut-off imposed by the discrete graph structure [30]. For a prespecified error $\epsilon_1$, Eq. (5) defines a time window,

$$t \in \left[0, t_{R^S_X} \right] = \max \left\{0, \left(\ln(C_r) + R^S_X / v_r\right)\right\},$$

with $C_r = \epsilon_1 (2\|\hat{\Phi}\|_r \|\partial S\|)^{-1} \in (0, 1)$. The value of $r$ can be set to 2 by maximization of $t_{R^S_X}$ and the subscript $r = 2$ will be dropped below. Within the time window (6), the full generator, $\hat{H}$, can be replaced by a surrogate one, $\hat{H}_X$. The bound $t_{R^S_X}$ accounts for the worst-case relation between interaction time and radius of the effective support. Therefore, the full environment is needed at most in the limit of infinite interaction time.

**Truncation of the environment: dynamical renormalization group** The ability to truncate $\hat{H}$ for finite times provides the basis for a renormalization procedure to determine, given $t$ and $\epsilon_1$, the ‘surrogate Hamiltonian’ $\hat{H}_X$.  

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[13–15] Supplementary material: open quantum system dynamics
Reordering the environmental DOF according to their graph distance \( d \) from \( S \) in an onion-like fashion, the full Hamiltonian (5) is rewritten,

\[
\hat{H} = \sum_{d=0}^{\infty} (\hat{h}_d + \hat{h}_{d,d+1}) ,
\]

where \( \hat{h}_d \) accounts for the interactions between all DOF at a distance \( d \) from \( S \), i.e., within the layer \( \partial X_d = X_d \setminus X_{d-1} \), while interactions between two successive layers, \( \partial X_d \) and \( \partial X_{d+1} \) are grouped within \( \hat{h}_{d,d+1} \). The lowest order terms include the system DOF, \( \hat{h}_0 = \hat{H}_S \), \( h_{0,1} = \hat{H}_{SB} \). After the reordering, the DOF in \( X_1 \) are labelled by indices \( i = (1, \ldots, N_S^{\text{int}}) \), those in \( \partial X_1 \) by \( i = (N_S^{\text{int}}+1, \ldots, N_S^{\text{int}}+|\partial X_2|) \), and those in the \( d_{th} \) layer \( \partial X_d \) by \( i = (\sum_{j=1}^{d-1} |\partial X_j|+1, \ldots, \sum_{j=1}^{d} |\partial X_j|) \). Truncating the sum in Eq. (7) yields a set

\[
\hat{H}_{X_d} = \sum_{k=1}^{d} (\hat{h}_k + \hat{h}_{k-1,k}) ,
\]

which approaches \( \hat{H} \) in the limit \( d \to \infty \). Correspondingly, the bound (6) defines a sequence of increasing times \( \{t_d\}_{d=0}^{\infty} \) with \( t_d = \max \left\{ 0, (\ln(C) + d) / \epsilon \right\} \).

Then the following dynamical renormalization group transformation for the generator holds

\[
t = t_d \Rightarrow \hat{H}_{X_{d-1}} = \hat{H}_{X_d} + \hat{h}_d + \hat{h}_{d-1,d} \Rightarrow \| \hat{O}_S(t) - \hat{O}_S^d(t) \| \leq \epsilon_1 \forall d \in \mathbb{N}_0 , t \in [t_d, t_{d+1}) ,
\]

which ensures that the error in the simulation is kept fixed by retaining, within each time interval, the corresponding minimal set of environmental DOF, cf. Fig. 1. Clearly, the number of environmental DOF required at each step is model-dependent.

Efficient simulation For a given \( t \), the computational complexity of simulating the open quantum system dynamics can be further reduced by applying the Suzuki-Trotter expansion: For \( t_d \leq t < t_{d+1} \), the effective propagator defined by \( \hat{H}_{X_d} \) can be approximated as

\[
e^{-i\hat{H}_{X_d}t} \approx \left( \prod_{\{i,j\} \in X_d} e^{-i\phi_{ij}\Delta t} \right)^{m_d} ,
\]

where \( \Delta t = t/m_d \) and the product runs over all the interacting pairs \( \{i,j\} \) belonging to \( X_d \) taken in arbitrary order. The generator \( \hat{H}_{X_d} \) contains \( K_d \leq |X_d|^2 \) terms. The error introduced by approximating \( e^{-i\hat{H}_{X_d}\Delta t} \) within each \( \Delta t \) by a product of \( K_d \) terms is of the order \( \epsilon_2 \leq \frac{1}{2} c_2^2 K_d^2 (\Delta t)^2 \) with \( c_2^2 = \sup_{(ij) \in X_{d-1}} \| \Phi_{ij} \| \).

A prespecified error \( \epsilon_2/2 \) for the whole time \( t \) is achieved by taking \( m_d = c_2^2 t^2 K_d^2 / \epsilon_2 \) Trotter steps, i.e.,

\[
\Delta t = \epsilon_2/(t c_2^2 K_d^2) .
\]

The characteristics of the microscopic description (5), dimensionality and interaction range, determine only \( K_d \). An increase in either one of them therefore increases only the number of Trotter steps \( m_d \) but not the size of the operators. For any time \( t \) the full dynamics of system and environment, \( |\Psi_{X_d}(t)\rangle \), can thus be determined according to Eqs. (9) and (10). The state of the system, \( \hat{\rho}_S(t) \), is obtained as \( t \hat{B}_S [\hat{\Psi}_{X_d}(t)] [\hat{\Psi}_{X_d}(t)]^\dagger \) and differs from the true one evolved according to Eq. (2) by at most \( \epsilon_1 + \epsilon_2/2 \).

Extension to continuous environments The above renormalization scheme generalizes to both fully connected and unbounded interactions. Indeed two standard models of decoherence, the central spin model and the harmonic bath [1], fall into these categories. As a benchmark example, consider a system interacting with a continuous bosonic environment,

\[
\hat{H} = \hat{H}_S + \hat{O}_S \int_0^{\omega_c} h(\epsilon) (\hat{a}_\epsilon + \hat{a}_\epsilon^\dagger) d\epsilon + \int_0^{\omega_c} g(\epsilon) \hat{a}_\epsilon \hat{a}_\epsilon^\dagger d\epsilon ,
\]

where \( \hat{a}_\epsilon (\hat{a}_\epsilon^\dagger) \) denotes the annihilator (creator) for a bath mode of energy \( \epsilon \), and \( \omega_c \) is a cutoff frequency. The bath dispersion \( g(\epsilon) \) and the system-bath coupling \( h(\epsilon) \) are related to the spectral density \( J(\epsilon) = \pi d^{-1}(\epsilon) h^2(g^{-1}(\epsilon)) \).

By a proper choice of basis using orthogonal polynomials, every Hamiltonian of the form (11) is unitarily
equivalent to a semi-infinite chain \[20, 21\],
\[
\hat{H} = \hat{H}_S + J_0 \hat{O}_S (\hat{b}_0 + \hat{b}_0^\dagger) + \sum_{n=0}^{\infty} \omega_n \hat{b}_n^\dagger \hat{b}_n + t_n (\hat{b}_n^\dagger \hat{b}_{n+1} + h.c.).
\]
(12)

This establishes local finiteness for continuous environments and holds also for discrete Hamiltonians of the functional form \[11\]. Since \[11\] is quadratic in the bath operators, a Lieb-Robinson bound holds even for unbounded interactions \[22\]. In the supplementary material, we present a direct derivation of the bound. It is unbounded interactions \[22\]. In the supplementary material, we present a direct derivation of the bound. It is based on \(\hat{H}_B\) being quadratic such that the free evolution of the environmental DOF is mapped into that of wave functions \(f \in l^2(\Gamma)\) on the graph, \(b_n(t) = \sum_{k=0}^{\infty} f_{nk}(t) \hat{b}_k\), with \(f_{nk}(t) = [e^{-iJ_{B}t}]_{nk}\) and \(J_B\) the bath coupling matrix. The wave functions are bounded,
\[
|f_{nk}(t)| \leq F_r |(n-k)| e^{-(|n-k|-v_r)t},
\]
(13)

with \(v_r = er \|J_B\| / J_{\min,B}^{-1}, J_{\min,B} = \min(J_{B,ij}) / \|J_B\|\). In the interaction picture, \(\hat{O}^I(t) = e^{iH_B t} \hat{O} e^{-iH_B t}\), the generator becomes
\[
\hat{H}^I(t) = \hat{H}_S + H_{SB} = \hat{H}_S + J_0 \hat{O}_S \sum_{k=0}^{\infty} (f_{ok}(t) \hat{b}_k + f_{ok}^\dagger(t) \hat{b}_k^\dagger),
\]
(14)

and, using the Suzuki-Trotter expansion, the corresponding propagator is written as \(\hat{V}^I(t) = \lim_{m \to \infty} \prod_{j=0}^{m} V(t_j, t_j + \Delta t)\), with \(V(t_j, t_j + \Delta t) = e^{-iH_S \Delta t} e^{-iH_{SB}(j \Delta t)}\). As shown in the supplementary material, the bound \[13\] yields the following estimate for the accuracy of the effective propagator defined by \(H^I_d(t)\) accounting only for the first \(d\) bosons in the chain \[14\],
\[
\|V^I(t) - V^I_d(t)\| \leq \sum_{k,d} \int_0^t |f_{ok}(s)|^2 ds \leq \|F^2_r\| (2v_r)^{-1} e^{-2(d-v_r)t}.
\]
(15)

This is a Lieb-Robinson bound analogous to Eq. \[5\] with \(r = 2\) again and serves as the starting point to set up the renormalization transformation \[9\] for the harmonic oscillator bath. Since the interactions are nearest-neighbour in a one dimensional chain, no reordering of the environmental DOF is required prior to renormalization.

**Conclusions** We have shown that the reduced dynamics of a generic open quantum system can be determined efficiently and with arbitrary precision. The only assumption to be fulfilled is that each degree of freedom interacts with a finite number of other DOF. This holds for interactions that decay with distance or, in particular for continuous systems, if a finite energy cut-off exists. The computational effort scales at most polynomially in time and size of the effective support of the environment. The memory required to store the surrogate Hilbert space vector \(|\Psi_{X_d}\rangle\) scales exponentially with the number of DOF in the effective light cone. This scaling can be made polynomial by restriction to excitation subspaces \[11\]. Our result can be extended to account for arbitrary time-dependent bounded interactions applying available generalizations of the Suzuki-Trotter decomposition \[23\]. While generalizations of the Lieb-Robinson bound to dissipative dynamics exist \[24, 25\], they have never been applied to the full system-bath evolution. Our work thus provides a novel and very natural application of the bound.

The bipartition of system and environment intrinsic to our renormalization scheme makes it more powerful for modelling decoherence than t-DMRG \[26, 28\]. This is due to the existence of an effective light cone that can be fully exploited in terms of an optimized truncation of the environmental DOF if one only cares about the dynamics of a small subsystem. Our renormalization scheme allows to go beyond one-dimensional lattices with nearest-neighbor interactions while keeping the quality of the approximation throughout time. By resorting to hyper-graph theory, our proof is extended to arbitrary \(k\)-local interactions (as shown in the supplementary material). Our method thus opens the way to study a huge range of open quantum systems, including those in non-Markovian environments, and their control.

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[13] E. H. Lieb and D. W. Robinson, Commun. Math. Phys. 28, 251 (1972).
[14] B. Nachtergaele and R. Sims, in New Trends in Mathematical Physics. Selected contributions of the XVth International Congress on Mathematical Physics, edited by V. Sidoravicius (Springer, 2009), pp. 591–614.
[15] S. Bravyi, M. B. Hastings, and F. Verstraete, Phys. Rev. Lett. 97, 050401 (2006).
[16] B. Mohar and W. Woess, Bull. London Math. Soc. 21, 209 (1989).
[17] H. F. Trotter, Proc. Am. Math. Soc. 10, 545 (1959).
[18] M. Suzuki, Proc. Japan. Acad. 69, 161 (1993).
[19] R. Bulla, T. A. Costi, and T. Pruschke, Rev. Mod. Phys. 80, 395 (2008).
[20] J. Prior, A. W. Chin, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. 105, 050404 (2010).
[21] A. W. Chin, Ángel Rivas, S. F. Huelga, and M. B. Plenio, J. Math. Phys. 51, 092109 (2010).
[22] B. Nachtergaele, B. Schlein, R. Sims, S. Starr, and V. Zagrebnov, Rev. Math. Phys. 22, 207 (2010).
[23] D. Poulin, A. Qarry, R. Somma, and F. Verstraete, Phys. Rev. Lett. 106, 170501 (2011).
[24] D. Poulin, Phys. Rev. Lett. 104, 190401 (2010).
[25] M. B. Hastings, Phys. Rev. Lett. 93, 140402 (2004).
[26] G. Vidal, Phys. Rev. Lett. 93, 040502 (2004).
[27] S. R. White and A. E. Feiguin, Phys. Rev. Lett. 93, 076401 (2004).
[28] G. De Chiara, M. Rizzi, D. Rossini, and S. Montangero, J. Comp. Theor. Nanosci. 5, 1277 (2008).
[29] In graph theory, $\|F_r\|$ represents the maximum Katz centrality of the nodes [L. Katz, Psychometrika. 18, 39-43 (1953)].
[30] In some cases tighter model-dependent bounds can be derived [Burrell and Osborne, Phys. Rev. Lett. 99, 167201 (2007)]
QUASI-LOCALITY OF QUANTUM DYNAMICS

In this section, the proof of Eq. (5) in the main text is sketched for the sake of self-containedness, following Refs. [14] [15]. Two steps are required: (i) proof of the Lieb-Robinson bound in its original formulation as a bound on the commutator of two observables taken at different times [14], and (ii) demonstration that this bound implies that the time-evolution of any observable can be well approximated by an operator acting only on degrees of freedom within its light cone [15].

Lieb-Robinson bound

Consider a set $\Gamma$ of degrees of freedom, labeled by the index $i$, each corresponding to a finite Hilbert space $\mathcal{H}_i$. The full Hilbert space of the system is then $\mathcal{H}_\Gamma = \bigotimes_{i \in \Gamma} \mathcal{H}_i$. For each degree of freedom, the observables belong to the algebra $\mathcal{B}(\mathcal{H}_i)$ of bounded operators on $\mathcal{H}_i$. Given a finite subset $X \subset \Gamma$, the algebra of observables over $X$ is defined as $\mathcal{A}_X = \bigotimes_{i \in X} \mathcal{B}(\mathcal{H}_i)$. By identifying an operator $\hat{A} \in \mathcal{A}_X$ with $\hat{A} \otimes \mathbb{1}_{\Gamma \setminus X} \in \mathcal{A}_\Gamma$, it then follows that $\mathcal{A}_X \subset \mathcal{A}_\Gamma$. If the set $\Gamma$ is infinite, $\mathcal{A}_\Gamma$ is defined by the inductive limit $\mathcal{A}_\Gamma = \bigcup_{X \subset \Gamma} \mathcal{A}_X$. The support of an observable $\hat{A} \in \mathcal{A}_\Gamma$ is defined as the minimal set $\Gamma$ for which $\hat{A}$ can be written $\hat{A} = \hat{A}' \otimes \mathbb{1}_{\Gamma \setminus X}$ with $\hat{A}' \in \mathcal{A}_X$. A local observable is therefore an observable whose support is a finite subset of $\Gamma$.

The interactions among elements of $\Gamma$ can in general be defined as a map from the set of subsets of $\Gamma$, $\{X\}_{X \subset \Gamma}$, to the algebra $\mathcal{A}_\Gamma$, $\Phi : \{X\}_{X \subset \Gamma} \rightarrow \mathcal{A}_\Gamma$, which obeys the following properties

- $\hat{\Phi}(X) \in \mathcal{A}_X$;
- $\hat{\Phi}(X) = \hat{\Phi}(X)^*$.

A generic Hamiltonian on $\Gamma$ can then be written

$$\hat{H} = \sum_{X \in \Gamma} \hat{\Phi}(X)$$

(16)

and, for any observable $\hat{A} \in \mathcal{A}_\Gamma$, the dynamics generated by $\hat{H}$ is expressed as

$$\tau_t(\hat{A}) = e^{it\hat{H}} \hat{A} e^{-it\hat{H}}.$$  

(17)

Intuitively, the spread of the support of an observable $\hat{A}$ due to its dynamical evolution should depend on the surface area rather than the volume of its support. The surface of a set $X \subset \Gamma$ is the subset of $\Gamma$ having non-zero intersection both with $X$ and its complement $X^c$,

$$S(X) = \{ Z \subset \Gamma : Z \cap X \neq \emptyset \text{ and } Z \cap X^c \neq \emptyset \}.$$  

The boundary of a set $\partial X$ is then defined as the subset of $X$ whose elements interact with degrees of freedom outside of $X$, 

$$\partial X = \{ x \in X : \exists Z \in S(X) \text{ with } x \in Z \text{ and } \Phi(Z) \neq 0 \}.$$  

Finally, given a metric $d$ on $\Gamma$, the distance between two sets $X \subset \Gamma$ and $Y \subset \Gamma$ is defined by $d(X,Y) = \min_{x \in X, y \in Y} d(x,y)$. Equipped with these definitions the following holds [13]:

**Lieb-Robinson bound.** Given a set $\Gamma$ and a metric $d$ on it, if there exists a non-increasing function $F : [0, \infty) \rightarrow [0, \infty)$ with the following properties

1. **uniform integrability:**

   $$\|F\| = \sup_{i \in \Gamma} \sum_{j \in \Gamma} F(d(i,j)) < \infty,$$

2. **convolution:**

   $$\exists C \in \mathbb{R}^+ < \infty : \sup_{i,j \in \Gamma} \sum_{z \in \Gamma} \frac{F(d(i,z))F(d(z,j))}{F(d(i,j))} < C,$$

then for any $\mu \in \mathbb{R}^+ \cup \{0\}$ and any pair of local observables $\hat{A} \in \mathcal{A}_X$ and $\hat{B} \in \mathcal{A}_Y$ with $d(X,Y) > 0$,

$$\|[\tau_t(\hat{A}), \hat{B}]\| \leq \frac{2\|\hat{A}\|\|\hat{B}\|}{C_\mu} \|F\| \min\{|\partial X|, |\partial Y|\} e^{-\mu(d(X,Y)-2\|\Phi\|_\mu C_\mu |t|)}$$

(18)

with
1. \( \| \Phi \|_\mu = \sup_{i,j \in \Gamma} \sum_{\Gamma, j \in X} e^{\mu d(i,j)} \| \Phi(X) \| (F(d(i,j)))^{-1} \),
2. the convolution constant \( C_\mu \) corresponding to \( F_\mu(d(i,j)) = e^{-\mu d(i,j)} F(d(i,j)) \).

Equation (18) states that the support of a given observable spreads at a velocity \( v \) with
\[
v \leq \inf_{\mu \geq 0} \frac{2 \| \Phi \|_\mu C_\mu}{\mu}.
\]
To prove Eq. (18) the following quantity needs to be estimated
\[
C_B(X; t) = \sup_{A \in A_X} \left\| \tau_t(\hat{A}, \hat{B}) \right\| \left\| A \right\|
\]
assuming that \( \sum_{i \in \partial X} \sum_{j \in Y} F(d(i,j)) \leq \sum_{j \in \partial Y} \sum_{i \in X} F(d(i,j)) \) (if this assumption is not fulfilled the proof below needs to be carried out for \( \left\| \tau_{-t}(\hat{B}, \hat{A}) \right\| \)).

The dynamics generated by
\[
\hat{H} = \sum_{i \in \Gamma} \hat{h}_i + \sum_{Z \in X} \hat{\Phi}(X) = \hat{H}^{loc} + \hat{H}^\Phi.
\]
leaves the algebra \( A_X \) of observables on \( X \) invariant, i.e., the propagated operator defined for \( \hat{A} \in A_\Gamma \),
\[
\tau_t^{loc}(\hat{A}) = e^{i(\hat{H}^{loc} + \hat{H}^\Phi) t} \hat{A} e^{-i(\hat{H}^{loc} + \hat{H}^\Phi) t},
\]
will again be in \( A_X \), \( \tau_t^{loc}(\hat{A}) \in \hat{A} X \), if \( \hat{A} \in A_X \). Using the property of the commutator, one can then write
\[
f(t) = [\tau_t(\hat{A}), \hat{B}] = [\tau_t(\tau_t^{loc}(\hat{A})), \hat{B}].
\]
Differentiating \( f(t) \) with respect to time yields the following differential equation,
\[
\dot{f}(t) = i \sum_{Z \in S(X)} \left[ \tau_t(\hat{\Phi}(Z), f(t) \right] - i \sum_{Z \in S(X)} \left[ \tau_t(\tau_t^{loc}(\hat{A})), \left[ \tau_t(\hat{\Phi}(Z)), \hat{B} \right] \right].
\]

The first term on the right hand side of Eq. (19) is norm preserving. Using the properties of first order inhomogeneous differential equations, one obtains
\[
\| f(t) \| \leq \| [\hat{A}, \hat{B}] \| + 2 \| \hat{A} \| \sum_{Z \in S(X)} \int_0^t \| [\tau_s(\hat{\Phi}(Z)), \hat{B}] \| ds.
\]
Due to inequality (20) and the fact that \( \tau_t^{loc}(\cdot) \) is norm preserving,
\[
C_B(X, t) \leq C_B(X, 0) + 2 \sum_{Z \in S(X)} \| \hat{\Phi}(Z) \| \int_0^{|t|} C_B(Z, s) ds,
\]
where \( C_B(X, 0) = 0 \) since we assumed \( d(X, Y) > 0 \). Iterating the same argument used in (21) for the integrated function yields
\[
C_B(X, t) \leq 2 \| \hat{B} \| \sum_{n=0}^\infty \frac{(2 |t|)^n a_n}{n!},
\]
with coefficients \( a_n (n \geq 1) \)
\[
a_n = \sum_{Z_1 \in S(X)} \sum_{Z_2 \in S(Z_1)} \cdots \sum_{Z_n \in S(Z_{n-1})} \delta_Y(Z_n) \prod_{i=1}^n \| \hat{\Phi}(Z_i) \|
\]
where \( \delta_Y(\cdot) = 1 \) if \( Y \cap \cdot \neq 0 \) and zero otherwise. To put an upper bound on the coefficients \( a_n \) requires existence of the function \( F(d(i,j)) \). After some algebra and employing the convolution property of \( F \), the following estimate is obtained,
\[
a_n \leq \| \hat{\Phi} \| n C^{n-1} \sum_{i \in \partial_X} \sum_{j \in Y} F(d(i,j)),
\]
where \( \| \hat{\Phi} \| = \sup_{i,j \in \Gamma} \sum_{i,j \in X} \| \hat{\Phi}(X) \| (F(d(i,j)))^{-1} \). Inserting (23) into (21), one obtains
\[
\| [\tau_t(\hat{A}), \hat{B}] \| \leq \| [\hat{A}, \hat{B}] \| (e^{2C} |t|) - 1) \sum_{i,j \in \partial X} \sum_{j \in Y} F(d(i,j))
\]
which is the first form of the Lieb-Robinson bound. Note that if the supports of the observables \( \hat{A} \) and \( \hat{B} \) have a
non-empty intersection, i.e. \( d(X,Y) = 0 \), then one only needs to substitute \( e^{-2|i|} ||\hat{\Phi}||_{1}|t| \) by \( e^{-2|i|} ||\hat{\Phi}||_{1} \) in \( \text{(24)} \). To recast \( \text{(24)} \) in the form of \( \text{(15)} \), it suffices to note that the properties of \( F(d(i,j)) \) are obeyed also by any function of the form \( F_\mu(d(i,j)) = e^{-\mu d(i,j)} F(d(i,j)) \) with \( \mu > 0 \). Indeed, \( \text{(15)} \) is obtained by replacing \( F \) by \( F_\mu \) in Eq. \( \text{(22)} \) and, correspondingly, \( ||\hat{\Phi}||_\mu \) by \( ||\hat{\Phi}||_\mu \),

\[
||\hat{\Phi}||_\mu = \sup_{i,j \in \Gamma} \sum_{i,j \in X} ||\hat{\Phi}(X)||(F_\mu(d(i,j)))^{-1},
\]

and noting that

\[
\min \left \{ \sum_{i \in \partial_Y X} \sum_{j \in Y} F_\mu(d(i,j)) \sum_{j \in \partial_Y Y} F_\mu(d(i,j)) \right \}
\leq e^{-\mu d(X,Y)} ||F|| \min \left \{ \partial_Y X, \partial_Y Y \right \} .
\]

**Approximate locality of quantum dynamics**

The second step consists in showing that the bound \( \text{(18)} \) implies quasi-locality of the dynamics generated by \( H \) of Eq. \( \text{(16)} \). In other words, the dynamical evolution \( \tau_t(\hat{A}) \) of a local operator \( \hat{A} \) with support in \( X \subset \Gamma \) can be well approximated by an operator acting only on the degrees of freedom within the effective light cone of \( \hat{A} \). As shown in Ref. \[1\], this operator can then be identified with the time evolution \( \hat{A} \) generated by those terms in \( \text{(16)} \) that are supported on the effective light cone. In the following section, we will establish an analogous result for continuous systems.

Denoting by \( Y^c \subset \Gamma \) the set of degrees of freedom having at least distance \( d \) from \( X \), the (normalized) partial trace of \( \tau_t(\hat{A}) \) over \( Y^c \) is an operator acting on \( \mathcal{H}_Y \subset \mathcal{H}_X \) defined by

\[
\langle \tau_t(\hat{A}) \rangle_Y = \frac{\text{tr}_{Y^c}(\tau_t(\hat{A})) \otimes I_{Y^c}}{\text{tr}_{Y^c}(I_{Y^c})} \in \mathcal{A}_Y .
\]

(25)

It is convenient to represent the partial trace as an invariant integration over the group of unitary transformations \( U(Y^c) \) acting on the Hilbert space corresponding to \( Y^c \),

\[
\langle \tau_t(\hat{A}) \rangle_Y = \int_{U(Y^c)} (I_Y \otimes \hat{U}) \hat{A}(I_Y \otimes \hat{U}) d \mu(U) \quad (26)
\]

with normalized Haar measure \( d \mu(U) \),

\[
\int_{U(Y^c)} d \mu(U) = \text{dim}(\mathcal{H}_Y) .
\]

Equations \( \text{(25)} \) and \( \text{(26)} \) define indeed the same operator since both expressions, starting from \( \tau_t(\hat{A}) \), uniquely define an operator which commutes with all operators of the form \( \mathcal{I}_Y \otimes \hat{O} \). Using \( \text{(18)} \) and \( \text{(26)} \) one obtains

\[
\|\tau_t(\hat{A}) - \langle \tau_t(\hat{A}) \rangle_Y \|_2 
\leq \frac{1}{C_\mu} \int_{U(Y^c)} \|\tau_t(\hat{A}), U\| d \mu(U)
\leq \frac{2}{C_\mu} \|F\| \|\partial X\| e^{-\mu d(X,Y) - 2\|\Phi\|_{\nu} C_\mu |t|} ,
\]

which is precisely a Lieb-Robinson bound of the form of Eq. \( \text{(4)} \) of the main text.

**LIEB-ROBINSON BOUND FOR HARMONIC ENVIRONMENTS**

This section is devoted to the proof of Eq. \( \text{(13)} \) in the main text. In Ref. \[1\] a Lieb-Robinson bound for quadratic Hamiltonians with unbounded interactions has been obtained. Here we present an alternative simpler proof of the bound tailored to the kind of quadratic Hamiltonians that is used to model decoherence due to an environment consisting of harmonic oscillators.

The goal is to obtain an explicit estimate of the quasi-locality of the dynamics generated by the following Hamiltonian

\[
\hat{H}_S + \hat{H}_{SB} + \hat{H}_B = \hat{H}_S + J_t \hat{O}_S \left( \hat{b}_0 + \hat{b}_0^\dagger \right) + \sum_{n=0}^{\infty} \omega_n \hat{b}_n^\dagger \hat{b}_n + t_n \left( \hat{b}_n^\dagger \hat{b}_{n+1} + h.c. \right) .
\]

(27)

where \( \hat{H}_B \) can be written as \( \hat{H}_B = \hat{b} J_B \hat{b}^\dagger \) with \( \hat{b} = (\hat{b}_0, \cdots, \hat{b}_n) \) and \( J_B \) the coupling matrix of the environmental degrees of freedom with entries \( \omega_n \) and \( t_n \). Then the normal mode representation of \( \hat{H}_B \),

\[
\hat{H}_B = \sum_\lambda \lambda_I \hat{a}_I^\dagger \hat{a}_I,
\]

can simply be obtained by diagonalizing \( J_B \),

\[
\lambda_I = (U J_B U^\dagger)_I , \quad \hat{b}_k = \sum_\lambda U_{kI} \hat{a}_I , \quad \hat{a}_I = \sum_\lambda U_{I k}^\dagger \hat{b}_k .
\]

Since the normal mode free evolution is given by

\[
\tau_t(\hat{a}_I) = e^{-i \lambda_I t} \hat{a}_I ,
\]

the operators in the original basis \( \text{(27)} \) evolve according to

\[
\tau_t(\hat{b}_k) = \sum_n \left[ e^{-i J_B t} \right]_{k n} \hat{b}_n
\]

\[
= \sum_n \left[ U \text{diag}(e^{-i \lambda_I t} U) \right]_{k n} \hat{b}_n
\]

\[
= \sum_n \left( \psi_n^c(k) \right) \hat{b}_n + \psi_n^c(k) \hat{b}_n ,
\]

(28)
where \( \lambda_t \) denote the eigenvalues of \( J_B \). Hence, for any initial linear superposition of bosonic operators,
\[
b(f) = \sum_k (f_k b_k + f^*_k b^*_k),
\]
with \( f = (f_1, \cdots, f_\infty) \in l^2(\Gamma) \), one can write
\[
\tau_t(b(f)) = b(T_t f) = b \left( U \text{diag}(e^{-i \lambda_t t}) U^\dagger \right) f
\]
\[
= \sum_{k,n} \psi_{nk}(t) f_k b_n + \psi_{nk}^*(t) f_k^* b^*_n
\]
\[
= \sum_{k,n} f_{nk}(t) \tilde{b}_n + f_{nk}^*(t) \tilde{b}^*_n. \tag{29}
\]

It follows from Eq. \((29)\) that the existence of normal modes allows for mapping the dynamical evolution of operators on the Hilbert space \( \mathcal{H}_\Gamma \) into that of wave functions on \( \Gamma \).

The task is now to bound each \( |f_{nk}(t)| \). To do so we set
\[
F_r(d(i,j)) = \left| \frac{J_B^{d(i,j)}(j)}{J_B^{d(i,j)}} \right|, \tag{30}
\]
where \( J_B = J_B/\|J_B\|, r > 1 \)
\[
d(i,j) = \min\{n \in \mathbb{N}_0 : [A^n]_{ij} \neq 0 \}
\]
with \( A \) the adjacency matrix of \( J_B \) and \( \|J_B\| \leq \|A\| \leq 2 \) since \( A \) is tri-diagonal. Defining
\[
F_r(m) = \sup_{d(i,j) = m} F_r(d(i,j)), \tag{31}
\]
we can estimate
\[
(F_r(m))^{-1} \leq \left( \frac{r}{\tilde{J}_{\min}} \right)^m \phi^m \tag{32}
\]
with \( \tilde{J}_{\min} = \min_{d(i,j)} J_B, d(i,j) \neq 0 \). Using Eqs. \((30)\) and \((31)\), the following bound is obtained
\[
|f_{0k}(t)| = \left| e^{-i \theta_{0k} t} f_{nk}(0) \right|
\]
\[
\leq \left| \sum_{m=0}^{\infty} \frac{(-i t)^m}{m!} \left[ J_B^m \right]_{nk} f_k(0) \right|
\]
\[
\leq \sum_{m=0}^{\infty} \frac{t^m}{m!} \left| J_B^m \right| e^{-\mu m} F_r(m) e^{i m \left( F_r(m) \right)^{-1}}
\]
\[
\leq e^{-\mu|n-k|} F_r(|n-k|) \sum_{m=|n-k|}^{\infty} \frac{t^m}{m!} (\|J_B\| e^{\mu F_r})^m
\]
\[
\leq F_r(|n-k|) e^{-\mu(n-k)} e^{-i \mu t}. \tag{33}
\]

In the interaction picture defined by
\[
\hat{O}^I(t) = e^{i H_B t} \hat{O} e^{-i \hat{H}_B t}, \tag{34}
\]
the generator of the evolution of the states becomes
\[
\hat{H}^I(t) = \hat{H}_S + J_0 \hat{O}_S \sum_{k=0}^{\infty} (f_{0k}(t) \hat{b}_k + f_{0k}^*(t) \hat{b}_k^*)
\]
\[
= \hat{H}_S + \hat{H}_{SB}(t). \tag{35}
\]

Employing the Suzuki-Trotter expansion \cite{17,18}, the corresponding propagator can be written
\[
\hat{V}^I(t) = \lim_{m \to \infty} \prod_{j=0}^{m} e^{-i \hat{H}_S \Delta t} e^{-i \hat{H}_{SB}(j \Delta t)}
\]
\[
= \lim_{m \to \infty} \prod_{j=0}^{m} \hat{V}(t_j, t_j + \Delta t), \tag{36}
\]
where \( \Delta t = t/m \). The propagator associated to the truncated generator,
\[
\hat{H}_{SB}(t) = J_0 \hat{O}_S \sum_{k=0}^{d} (f_{0k}(t) \hat{b}_k + f_{0k}^*(t) \hat{b}_k^*)
\]
is expressed analogously. Setting
\[
\hat{H}^d_{SB}(t) = J_0 \hat{O}_S \sum_{k=d+1}^{\infty} \left( f_{0k}(t) \hat{b}_k + f_{0k}^*(t) \hat{b}_k^* \right),
\]
the error made by employing the truncated generator within an infinitesimal time interval can be estimated as
\[
\| \hat{V}(t_j, t_j + \Delta t) - \hat{V}_d(t_j, t_j + \Delta t) \|
\]
\[
\leq \| e^{-i \hat{H}_{SB}(j \Delta t)} - e^{-i \hat{H}^d_{SB}(j \Delta t)} \|
\]
\[
= \| e^{-i \hat{H}_{SB}(j \Delta t)} - e^{-i \hat{H}_{SB}(j \Delta t)} e^{-i \hat{H}^d_{SB}(j \Delta t)} \|
\]
\[
= \| e^{-i \hat{H}^d_{SB}(j \Delta t)} [e^{i \hat{H}^d_{SB}(j \Delta t)} e^{-i \hat{H}_{SB}(j \Delta t)}] \|
\]
\[
\leq \| e^{i \hat{H}^d_{SB}(j \Delta t)} e^{-i \hat{H}_{SB}(j \Delta t)} \|. \tag{37}
\]

Recognizing that \( e^{i \hat{H}^2_{SB}(j \Delta t)} \) and \( e^{-i \hat{H}_{SB}(j \Delta t)} \) are two Weyl operators since they are of the form
\[
W(f) = \exp \left[ ib(f) + b^*(f) \right]
\]
with \( f \in l^2(\Gamma) \) and hence obey the Weyl relation
\[
W(f) W(g) = W(f + g) e^{-i \text{Im} \langle f, g \rangle}
\]
with \( \langle \cdot, \cdot \rangle \) the inner product on \( l^2(\Gamma) \), one obtains the following estimate for Eq. \((37)\)
\[
\| e^{i \hat{H}^d_{SB}(j \Delta t)} e^{-i \hat{H}_{SB}(j \Delta t)} \| \leq | \langle f_0, g_0 \rangle |. \tag{38}
\]
with $f_0(\Delta t) = \{ f_{0k}(\Delta t) \}_{k=0}^{\infty}, g_0(\Delta t) = \{ f_{0k}(\Delta t) \}_{k=0}^{\infty}$. Together with Eq. (33), Eq. (38) yields the following estimate

$$
\| [e^{iH_S B(j \Delta t)}, e^{-iH_S B(j \Delta t)}] \| \leq \sum_{k \geq d} |f_{0k}(\Delta t)|^2 \\
\leq \| F^2_r \| e^{-2\mu(k-\nu) \Delta t} = \epsilon(\Delta t).
$$

Taking the limit $m \rightarrow \infty$, one obtains Eq. (15) of the main text,

$$
\| V(t) - V_d(t) \| \leq \int_0^t \epsilon(s) ds \leq \| F^2_r \| (2\nu_r)^{-1} e^{-2(d-\nu_r) t}.
$$

**GENERALIZATION TO $k$-LOCAL INTERACTIONS**

We now show how the renormalization procedure generalizes to Hamiltonians exhibiting arbitrary $k$-local interactions. Since the Lieb-Robinson bound is fully general, the only step required is to define an appropriate metric $d$ on the set of degrees of freedom $\Gamma$ and a non-increasing function $F(d(i,j))$ obeying both uniform integrability and the convolution property.

Consider a generic $k$-local Hamiltonian

$$
\hat{H} = \hat{H}_S + \sum_{i=1}^{N_1^{(k)}} \sum_{j_1, \ldots, j_{k-1}=1}^{N_1^{(k)}} \hat{\Phi}_{i,j_1, \ldots, j_{k-1}} + \sum_{j_1, \ldots, j_k=1}^{N_1} \hat{\Phi}_{j_1, \ldots, j_k},
$$

(39)

where, without loss of generality, we have assumed all $\hat{\Phi}$ to be $k$-local (instead of at most $k$-local),

$$
\hat{\Phi}_{i_1, \ldots, i_k} = \sum_{\{\mu_1\}, \ldots, \{\nu_k\}} J_{i_1, \ldots, i_k}^{\mu_1, \ldots, \mu_k} \prod_{j=1}^{k} \hat{O}_{i_j}. 
$$

(40)

In Eq. (40), $\{\mu_i\}$ denotes that each index $\mu_i$ varies between 0 and the dimension of the finite Hilbert space of the $(j_i)_{i}$th degree of freedom, $\dim(H_{j_i})$, and $\hat{O}_{i_j}$ stands for the tensor product of the operator $\hat{O}_{i_j} \in B(H_{j_i})$ and the identity over the other degrees of freedom. The Hamiltonian (39) defines a hypergraph, i.e., an ordered pair $\mathcal{G} = (\Gamma, E)$ where the set of nodes $\Gamma$ represents the degrees of freedom and $E$ is the set of non-empty subsets of $\Gamma$, called hyperedges or links, for which $J_{i_1, \ldots, i_k}^{\mu_1, \ldots, \mu_k} \neq 0$. Since we have assumed all the interactions to be $k$-local, all hyperedges have size $k$ and the hypergraph is $k$-uniform. A graph can therefore be regarded as a 2-uniform hypergraph. The adjacency matrix $A_{ij}$ of an hypergraph $\mathcal{G}$ is defined as the matrix whose entries $A_{ij}^k$ correspond to the number of hyperedges containing both degrees of freedom $i$ and $j$. The connectivity of a node $c_i$ is given by the number of hyperedges involving the node, $c_i = \sum_j A_{ij}$. If all nodes have the same connectivity $c$, then the hypergraph is $c$-regular, in analogy to the notion of homogenous graphs. Analogously to the adjacency matrix of a graph, $A^h$ is bounded if and only if the hypergraph is locally finite, i.e., $c_i < \infty$ for each $i \in \Gamma$, and we obtain $\| A^h \| \leq \bar{c} = \sup_{i \in \Gamma} \sum_{j=1}^{\infty} A_{ij}^h$. The metric on the hypergraph can then be defined in terms of the shortest path connecting two nodes,

$$
d(i,j) := \min\{n \in \mathbb{N}_0 : (A^h)^n_{ij} \neq 0\}.
$$

One choice for $F(d(i,j))$ is then

$$
F_r(d(i,j)) = \frac{[(A^h)^n]_{ij}}{r^{d(i,j)}}
$$

with $\bar{A}^h = A^h/\| A^h \|$. With these definitions we can proceed analogously to the case of bilinear interactions and define the dynamical renormalization group transformation (Eq. (8) in the main text). Once an effective propagator for $t_d < t < t_{d+1}$ has been obtained, it can be approximated using the Suzuki-Trotter decomposition [17][18].

$$
e^{-i\hat{H}_{X_d} t} \approx \left( \prod_{\{i_1, \ldots, i_k\} \in X_d} e^{-i\Phi_{i_1, \ldots, i_k} t} \right)^{m_d},
$$

where the product is over all time intervals $\Delta t = t/m_d$ and all $k$-tuples belonging to $X_d$. We find that for a prespecified error $\epsilon_2/2$ the effective propagator for time $t$ can be approximated as a product of $n_d = (\epsilon_2 t K_d)^3/\epsilon_2$ terms where $\epsilon_2 = \| \Phi_{i_1, \ldots, i_k} \|$ and $K_d \leq |X_d|^k$ the number of $k$-local unitaries. One can then use the Solovay-Kitaev algorithm [3][23] to further decompose each $k$-unitary into a product of one- and two-body unitaries chosen from a suitable set. To achieve an accuracy $\epsilon$ for each $k$-unitary transformation, $n_{SK} = a \log_2(\epsilon^{-1})$ operations are required with $a, b$ constants. If one chooses $\epsilon = \epsilon_2/(2n_d)$, the effective propagator is simulated with an accuracy $\epsilon_2$ employing $n_d = a n_d \log_2(n_d/\epsilon_2)$ one- and two-body unitaries, i.e., with a computational effort that scales polynomially in time and size $|X_d|$ of the effective light cone.

[1] B. Nachtergaele, in XVIIth International Congress on Mathematical Physics, edited by P. Exner (World Scientific, 2009), pp. 391–396.
[2] J. A. Rodriguez, Appl. Math. Lett. 22, 916 (2009).
[3] C. M. Dawson and M. A. Nielsen, Quant. Inf. Comput. 6 (2006).