Spatial patterns of random walkers under evolution of the attractiveness: persistent nodes, degree distribution, and spectral properties

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Abstract
In this paper we explore the features of a graph generated by random walkers with nodes that have evolutionary attractiveness and Boltzmann-like transition probabilities that depend both on the euclidean distance between the nodes and on the ratio $\beta$ of the attractiveness between them. We show that persistent nodes, i.e., nodes that never been reached by random walker in asymptotic times are possible in the stationary case differently from the case where the attractiveness is fixed and equal to one for all nodes ($\beta = 1$). Simultaneously, we also investigate the spectral properties and statistics related to the attractiveness and degree distribution of the evolutionary network. Finally, we study a crossover between persistent phase and no persistent phase and we also show the existence of a special type of transition probability that leads to a power law behaviour for the time evolution of the persistence.

Contents

1 Introduction 1
2 The model and methods 5
    2.1 The model 5
    2.2 Monte Carlo Simulations and some numerical analysis 6
    2.3 Mean-field regime 8
3 Simulation Results 9
    3.1 Persistent nodes: degree and attractiveness distributions 9
    3.2 Spectral analysis 14
    3.3 Crossover in $\delta$ 16
    3.4 Comparing persistence for different transition probabilities 16
4 Summaries and Conclusions 17

1. Introduction
The idea of preferential attachment of Barabasi and Albert [1, 2] has brought a revolution to the study of complex systems mainly by the simplicity of the idea and its wide applicability which goes from web,
epidemics, metabolic networks, scientific collaborations, human mobility and so on. Thus, a really wide scope of the extensions of this and many other ideas in network science have been developed.

An interesting point in the theory of evolutionary networks is related to how random walks with properties on the edges or on the nodes can perform this evolution starting from initial condition where only nodes exist and are distributed in a two-dimensional surface.

In this context certain known properties of random walks as first time passage \[3\] and persistence \[4, 5, 6\] should be closely linked with properties of the generated graph formed by the path (sequence of the visited edges and nodes) of the random walker taking into account for a dependence on the spatial distribution of the nodes and their peculiarities. Following this idea, the walker should be guided by the euclidean distance between the nodes. Moreover these nodes also should have a kind attractiveness characterized for example by the frequency which they are visited along the time evolution, remembering in some sense of preferential attachment.

In this context the internet is a good example. We can imagine that individuals have some distance in relation to some topics. For example, one has some preference list and navigates in the internet, reading news about science, maybe politics. Thus we consider that she(he) keeps a short distance in relation to these topics. On the other hand, she (he) keeps high distances in relation to topics such as soccer (not me), religion, social gossips, so that the access probability is initially small. However since some topics have been accessed, they call attention even of non-usual users increasing the access probabilities and contributing to the formation of the so-called “trending topics”.

Looking at the graph properties, we can highlight some basic properties as the degree distribution, distribution of accesses but also spectral properties as the density of eigenvalues of matrices related to adjacency matrix of the random walk generated graph. The idea of considering a random walk in a set of random points in a two-dimensional space brings a lot of interesting discussion in Physics and an important question refers to the problem of a particular node not being visited at large times, or translating to a spin system the question change to what the probability that a particular spin does not change its state until time \(t\).

Such concept was deeply studied by many authors, considering dynamics at temperature \(T = 0\), known as coarsening dynamics (see \[5\]). A simple dynamics in this context, for example considers that if a flip of a particular spin state interacting only with its first neighbours, decreases the energy system, its state must be changed otherwise it changes with probability \(1/2\). In this case the fraction of spins with unaltered state

\[\text{http://barabasi.com/networksciencebook/}\]
from the beginning to the time $t$ in one, two, or three-dimensional lattices decays as power law as function of time

$$\text{Pers}(t) \sim t^{-\theta}.$$  \hspace{1cm} (1)

Such behaviour is exponentially stretched for $T \neq 0$. This concept is known as local persistence. Similarly, this concept has a global version [4]. In this case, it can be shown that for systems at critical temperature, the probability of magnetization (essentially the sum of spin states divided by the system size) does not have changed its sign until time $t$ also decays as a power law given in Eq. [1]

So we are interested in persistent sites in the graph formed by the random walk, or more precisely if we have a power law decay for the persistence which means that some nodes are persistent at large times. Alternatively, if the decay is exponential, in the process we have a fixed fraction of persistent sites in the limit:

$$\lim_{t \to \infty} \text{Pers}(t) = p_\infty$$  \hspace{1cm} (2)

with $p_\infty > 0$. Starting from this idea, let us imagine that a navigation of a particular people in the web can be mimicked as a random walk with some idiosyncrasies. In this random walk, in general, we suppose that this person has some initial distances in relation to the topics, however after a visit to a particular topic, the attractiveness of this site, which was not even the most important, has changed given its visit, and the probability of visit to this topic now, not only depends on the distance but also depends of its attractiveness that initially was the same for all sites.

So if we consider that walk is governed initially by the distances of the walker in relation to nodes and after some time the incidence of the walker on the nodes make them more attractive which works as a mechanism to change the effects of the distance the transition probabilities among the different pairs of nodes, what is the properties of the graph formed during the evolution of this peculiar random walk? In a more abstract point of view, we can imagine a mathematical modelling where a walker paints an edge (if this does not exist) when transits between two nodes and the growing graph obtained from this process (nodes + painted edges) can be studied.

In the literature some authors have explored similar models such for example the step-by-step random walk network model described in [7], and preferential network random walk studied in [8]. All these models have a similar dynamics but different aims. Differently from our original idea to this current work, these models start from a fully connected network of a certain number of nodes $m$. In [7], a new node is created
which will be connected to the certain number of nodes $n \leq m$. These nodes which will linked to the new created node are determined by a step by step self-avoiding random walk defined on the current network. Actually this model is a simple mechanism for generating scale-free networks.

Alternatively, [8] considers multiple random walkers over a fully connected network. At time-step, each walker chooses one of its neighbours with a certain probability and transits to the chosen neighbour. When the walker passess by the link the transition probability is decreased (destructive approach) or enlarged (constructive approach).

In this work, our idea is quite different. We explored the properties of a random walk where the distance between nodes as well as their attractiveness are taken into account and its connexion with the spectral properties of the adjacency matrix of the generated network and other properties of this network.

More precisely, we consider $N$ points randomly distributed in a unity square. Starting from an initial point randomly chosen, a single random walk can transit to other randomly chosen node. But differently from [8] we consider that potential (or attractiveness) of the nodes (not links as in [8]) achieved by the random walker are increased by the incidence of the walker and the transition probability between two nodes follows a Boltzmann form which depends not only on the distance between these nodes but also on the ratio of the potentials of these two nodes.

In our analysis, we performed Monte Carlo (MC) simulations. First, we performed an analysis where the attractiveness of the nodes remain fixed and the same for all nodes. In this case only the distance is important and we compared the existence of persistent nodes with the situation where the attractiveness is incremented according to the incidence on the nodes.

Additionally, we study the spectral properties of a matrix that depends on the adjacency matrix of formed graph obtained by our walker with this special random walk. More precisely we obtained the density of eigenvalues of this matrix comparing with expected law (a variation of semi-circle law) valid for random graphs (Erdős–Rényi model) exactly for connexion probability $p = 1/2$. Finally we also study the effects on the existence of the persistent nodes, by considering some modifications of the transition probability with Boltzmann and non-Boltzmann functional dependence.

In the next section (Sec. 2) we present the model, the numerical methods and some details of the random matrices theory (RMT) to be employed in the analysis of the results of this work. Additionally, we describe a simple mean-field method to describe the persistence probability for a comparison with numerical results.

In section 3 we present our results obtained via numerical simulations. Finally in Sec. 4 we present our summaries and the main conclusions.
2. The model and methods

In the next subsection we present the model to be used to describe the spatial exploration with our random walk based on the distance and attractiveness ratio among the pairs of nodes. Following, we also explore some details of analysis and numerical methods to be used in the analysis of the problem and finally in the last subsection, a simple mean field result is obtained when the attractiveness is constant and the same for all nodes in order to compare with numerical results for the existence of the persistent nodes.

2.1. The model

Let us consider a graph generated by the time evolution of a random walk defined on the nodes \((x_i, y_i)\), with \(i = 1, 2, \ldots N\), randomly distributed in the two-dimensional unity square \([0, 1]^2\). There are no edges initially.

In this scenario, we add a special ingredient, where all nodes start with initial potential \(\phi_i = 1\), which means (by thinking in the context of social physics, or web applications and so on) its attractiveness. In each run we choose a node \(j = 1, \ldots, N\), where our walker starts. Another node \(k = 1, \ldots, N\) is randomly chosen and the walk jumps to the node \(k\) with probability which follows a Boltzmann weight:

\[
p_{jk} = p(j \rightarrow k) = e^{-N^\delta \beta_{jk} d_{jk}} \tag{3}
\]

where \(d_{jk} = d_{kj} = \sqrt{(x_j - x_k)^2 + (y_j - y_k)^2}\), with \(0 \leq \delta\). This indicates that probability depends on Euclidean distance between the nodes. A walker, for which the jump from one site to another is performed with probability proportional to \(\exp(-E(d)/T)\) where \(E(d)\) is an arbitrary cost that depends on the hop distance \(d\) was studied in \([9]\). These authors show to exist a glass transition for a critical value of \(T\). In other work \([10]\), considering deterministic random walks in one-dimensional environment (the walker goes to the nearest site) with a memory \(\mu\), the authors show a crossover between localized and extended regimes at the critical memory \(\mu_c = \log_2 N\).

Thus, we propose to consider a kind of “local network temperature”, considering the thermodynamic motivation of the problem:

\[
\beta_{jk} = \frac{\phi_j}{\phi_k} \tag{4}
\]

i.e., the higher is the attractiveness of the arrival node is in relation to starting node, the smaller is the \(\beta\), and therefore larger is the probability to jump between the nodes.

So by finally capturing the idea of preferential attachment, nodes reached by the walk become more probable to be reached, since every time that a node is crossed by random walk we increment the potential
in one unit: \( \phi_{\text{new}} = \phi_{\text{old}} + 1 \). This modelling should be extended for example in the modelling of knowledge acquisition dynamics by professionals in a work network, since in general, a scientist for example, has a set of topics which keeps a certain distances from them, as we evolve in the studied topic and the nodes are visited with some frequency, in general, users tend to visit the same topics that other people (or even them) have already been accessed.

Thus in this work, we propose such model which combines two important ideas: the affinity that people have with topics in a network, for example defined by the euclidean distances \( d(i, j) \) which is compensated by the individual popularity of node (its attractiveness) which is modelled by the number of accesses.

Finally, in Eq. 3, we consider a scaling factor \( N^\delta \) which controls the importance of distance and attractiveness effects in the argument of the Boltzmann weight. If for example \( \delta = 0 \), we have, initially since \( \phi_i(t = 0) = 1 \) for all sites, the inequality: \( e^{-\sqrt{2}} < p(j \rightarrow k) < 1 \). Since \( e^{-\sqrt{2}} \approx 0.24 \), we have large probabilities to jump which is very different from \( \delta = 1/2 \). In this case, for \( N = 100 \) sites, we have \( p_{\min} = e^{-10\sqrt{2}} \approx 7.2 \times 10^{-7} \).

2.2. Monte Carlo Simulations and some numerical analysis

In this paper, we perform numerical MC simulations to study the statistical fluctuations related to properties of our preferential random walk and consequently of generated graph. An initial set of points is random and uniformly sorted in the two-dimensional unit square. So we choose an initial random node, and the graph is built as the nodes (and edges) are travelled by the random walk.

Here is appropriated to consider simulation divided by turns, i.e., the system evolves and for each jump we paint the edge which governs our network evolution. After \( N \) jumps (or simply attempts) one turn is completed in a numerical simulation for example. So one time unit, \( t = 1, 2, 3, \ldots, t_{\text{max}} \) (\( t \) turns) supposes that \( \tau = Nt \) jumps were executed by the walker. For each turn you can imagine that one other walker starts a new walk considering that the previous walker has evolved the network built up to that moment according to your path.

In order to compute averages, we repeat the \( N_{\text{run}} \) different evolutions, and the averages of the different amounts were obtained over different sources of variations: different set of \( N \) points distributed in the unit square, different initial selected point, and different random numbers used during the evolution to perform the transitions among the nodes.

Our model, which adds two important concepts: spatial exploration and network formation by a random walk, suggests important questions about the effects of the attractiveness of nodes on the network properties generated and about statistics of accesses to these nodes. Among the very interesting statistics we can ask
about the probability a node has of having not been reached up to time \( t \), which is quantified by the previously described amount, known by statistical physicists as local persistence, which from this computational scenario is here calculated by

\[
p(t) = \frac{1}{N} \sum_{j=1}^{N_{\text{run}}} n_j(t)
\]

where \( n_j(t) \) is the number of unreachable nodes at time \( t \), at in the \( j \)-th run. Intuitively, obtaining such amount it must bring information about the evolution and the final stage of the emergent network.

Simultaneously, in this paper we also analyze the spectral properties of generated graph by random walker. So, for a better numerical results, we propose to consider the information from the more appropriate matrix that describes our graph, which is defined by:

\[
C = \frac{1}{N} A^T A
\]

were \( A \) is a simple mapping from adjacency matrix \( M \), where \( M_{ij} = 1 \) if node \( i \) is linked to node \( j \) and 0 otherwise, defined by \( A_{ij} = 2M_{ij} - 1 = \pm 1 \), and in this particular case \( C = \frac{1}{N} A^2 \). If \( A_{ij} \) is a random variable with \( A_{ij} = 0 \) and \( \langle A_{ij}^2 \rangle - \langle A_{ij} \rangle^2 = 1 \), we will enunciate which is a known extension of the famous semi-circle law. Considering the cumulative distribution of eigenvalues of \( C \),

\[
F_N(\lambda) = \frac{1}{N} \#\{\lambda_i, \lambda_i < \lambda\}
\]

where \( \#\{\lambda_i, \lambda_i < \lambda\} \) denotes the number of eigenvalues which are smaller than \( \lambda \), the density of eigenvalues defined by:

\[
\rho_C(\lambda) = \lim_{N \to \infty} \frac{dF_N(\lambda)}{d\lambda}
\]

is given by \([11]\)

\[
\rho_C(\lambda) = \begin{cases} 
\frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{2\pi \lambda} & \lambda_- \leq \lambda \leq \lambda_+ \\
0 & \lambda < \lambda_- \text{, } \lambda > \lambda_+
\end{cases}
\]

where \( \lambda_- = 0 \) and \( \lambda_+ = 4 \).

So by using the Jacobi’s method to diagonalize \( C^2 \) we compute the eigenvalues as the graph generated by the random walker evolves. In order to compute the numerical \( \rho_C(\lambda; t) \) we accumulate for each instant \( t \)

\[\text{Using } C \text{ and not } A \text{ directly is more interesting since according to method employed to determine the eigenvalues, } A \text{ seems to be more sensitive than } C \text{ and some numerical problems were detected in the first case.}\]
of MC simulation, $N_{\text{run}}$ sets of $N$ eigenvalues.

The idea is to observe the similarity and distortions between the real eigenvalues distribution and theoretical prediction. This last supposes that graph is a genuine Erdos-Renny graph \cite{12} with parameters $N$ and $p$, i.e., there is a edge between any pair of nodes of the network with probability $p$ and a total of $N$ nodes. We believe that our graph must pass by this situation (where the nodes have approximately the same coordination in average) for intermediate times of the evolution. But in general we must observe the eigenvalues that scape from the law \cite{6} but how the eigenvalues scape from this theoretical bulk, is deeply related to walker cover the nodes and compose the graph.

2.3. Mean-field regime

Let us to explore the process in a first special kind of spacial exploration of our walker where $\beta$ is constant. Particularly when the attractiveness is constant and equal for all nodes during the time evolution corresponds to $\beta = 1$. Particularly, we consider that in a special regime where we exchange $d_{ij}$ by $\langle d \rangle$. At time $t > 0$, the probability of any site to be reached by the walk, in a kind of “mean field regime” is given by

$$p = \frac{e^{-\beta \langle d \rangle}}{N}$$

since we randomly choose a site we jump to it with probability $e^{-\beta \langle d \rangle}$, where

$$\langle d \rangle = \lim_{N \to \infty} \frac{1}{N(N-1)/2} \sum_{i<j} d_{ij}$$

$$= \frac{1}{0} \frac{1}{0} \frac{1}{0} \int dx_i \int dx_j \int dy_i \int dy_j \left[ (x_i - x_j)^2 + (y_i - y_j)^2 \right]^{1/2}$$

$$\approx 0.519$$

If our time is computed in turns of $N$ jump trials, the probability of a site is not reached by the walk up to time $\tau = tN$, is

$$\text{Pr}(\tau = tN) = \hat{p} = (1 - p)^{tN}$$

and the probability that $N_{\text{pers}}$ sites are persistent at time is

$$\text{Pr}(N_{\text{pers}}) = \frac{N!}{N_{\text{pers}}!(N - N_{\text{pers}})!} \hat{p}^{N_{\text{pers}}} (1 - \hat{p})^{N - N_{\text{pers}}}$$

and the average number of persistence sites at time $\tau = tN$, is $N_{\text{pers}} = N \hat{p}$ and a good approximation to the
probability of a site is persistent at time \( \tau = tN \) is the fraction

\[
\text{Pers}(t) = \frac{N_{\text{pers}}}{N} = \hat{p}
\]

\[
\approx \left( 1 - e^{-0.519\beta} \right)^N t
\]

For fixed \( N \) we have an exponential decay for \( \text{Pers}(t) \) and the sites must be explored after some time and we do not expect persistent sites for \( t \to \infty \). For example, particularly for \( \delta = 0 \), we have

\[
\text{Pers}(t) \sim \exp(-e^{-0.519\beta} t).
\]

This simple mean field analysis suggests that no persistent sites are expected at large times. The important question if this mean field regime corresponds to similar behaviour to be found in the case where \( \beta \) evolves according to evolution of potentials. The answer is no! And our numerical simulations will corroborate such study by showing that we expect persistent sites when the nodes have different attractiveness which changes over time by the incidence.

We also complete this study in the next section looking the effects of \( \delta \) on the persistent sites as well as the evolution of formed graphs by the walker. Moreover we study the spectral properties of this graphs under light of random matrix theory and finally some results about other transition probabilities are presented.

3. Simulation Results

In this section we present our numerical results.

3.1. Persistent nodes: degree and attractiveness distributions

In this paper we performed MC simulations considering initial set of points (nodes) randomly distributed in the two-dimensional unit square. So we choose an initial random node, and the graph is built as the nodes (and edges) are travelled by the random walk and we perform \( t_{\text{max}} \) turns (each turn composed by \( N \) jumps or attempts). This process are repeated \( N_{\text{run}} \) times to average the quantities obtained in this paper.

As suggested by preliminary results obtained in mean-field regime, if we keep \( \beta \) constant (\( \phi_i \) constant and the same for all nodes) the random walker explores the space and no persistent site is expected at large times. Sure the exploration time (necessary time for all sites to be visited) depends on \( \delta \). By starting our simulations we elaborated a plot that computes the persistence considering \( N_{\text{run}} = 400 \) runs.
We compute the persistence $p(t_{\text{max}})$ as function of $\delta$, in order to qualitatively check the mean-field results. Here $t_{\text{max}}$ is the last turn of evolution.

In Fig. 1 we plot the value of persistence $p(t_{\text{max}})$, estimated as function of $\delta$ for different $t_{\text{max}}$. We can see that after a $\delta_c(t_{\text{max}})$, we have $p(t_{\text{max}}) > 0$. We observe that $p(t_{\text{max}})$ point is larger as $t_{\text{max}}$ increases. For example, in the worst case, $\delta = 1$, for $t_{\text{max}} = 1000$, $p(t_{\text{max}}) \approx 1$. But according to mean-field prescription, $p(t_{\text{max}})$ must decreases as function of $t_{\text{max}}$ as can be observed in the inset plot however the decay is very slow and we expected $p(t_{\text{max}}) = 0$ for $\tau \approx t N \approx 10^8$ steps of the walk since the slope is $O(10^{-6})$. For these simulations we used $\beta = 1$ and $N = 100$. For estimating $p(t_{\text{max}})$ we used $N_{\text{run}} = 400$ different runs for a good estimate.

So the question how the evolutionary attractiveness of the nodes affects the graph evolution. Now we consider $\beta$ given in Eq. 4. We start our analysis looking to the graph at large times ($t_{\text{max}} = 1000$ steps).
As can be observed in Fig. 2 we show the graphs obtained for a specific run and in all situations we have persistent sites even for $\delta = 0$. This suggests that persistence has no exponential decay to zero at large times and we must expect an asymptotic convergence to value $p_\infty > 0$.

Figure 2: Generated graphs for $t_{\text{max}} = 1000$ turns considering evolutionary $\beta$ for different values of $\delta$. Full circles correspond to persistent sites while the open circles correspond to sites which already were visited.

It is important to call the attention that for $\delta = 1/2$, the walker is confined in no more than 3 sites. So let us now concentrate our study in the worst case for the study of the persistence, $\delta = 0$. So we elaborate a plot of the time evolution of $p(t)$, for different size systems (number of nodes), for $\delta = 0$. Here $p(t)$ was estimated by using $N_{\text{run}} = 1000$ runs.

Time evolution of the persistence, for $\delta = 0$, for different size systems in log-log scale is shown in Fig. 3. We can observe for large times the persistence asymptotically goes to a constant $p_\infty > 0$. The upper inset plot shows the number of asymptotic persistent sites ($n_\infty = Np_\infty$) as function of $N$. We observe a linear dependence (linear fit: continuous curve in red) with a slope $\approx 0.04$. Just for a comparison, the
Figure 3: Time evolution of the persistence, for $\delta = 0$, for different size systems in log-log scale. We can observe that for large times the persistence asymptotically goes to a constant $p_\infty > 0$. The upper inset plot shows the number of asymptotic persistent sites ($n_\infty = Np_\infty$) as function of $N$. We note a linear dependence (linear fit corresponds to red line) with a slope $\approx 0.04$. Just for a comparison, the time evolution of persistence is also shown for some different fixed $\beta$ values (lower inset plot) which corroborates the exponential explorations of space, as predicted in mean-field regime.

The model is also suggesting an important point, the distance can delay the space exploration but cannot confine the random walk. It only happens if the attractiveness of the nodes evolves according to the dynamics here proposed. Thus, it is interesting to better investigate the statistics related to distribution of attractiveness and degree distributions of the nodes in the network. Let us start by the degree distribution. Defining $f(k)$ the frequency of nodes with degree equal to $k$, we look at distribution for different turns. For that we separate our simulations again in a) $\beta = 1$ and b) $\beta$ evolves according to increment of the
potentials, since we want to check the differences and similarities of $f(k)$ in both situations.

In Fig. 4 we show the degree distribution for three different instants: $t = 2$, 50, and 300 turns. In (a) we can observe (in mono-log scale) that for $t = 50$ we have a gaussian distribution of nodes, which suggests a formation of a random graph (Erdős–Rényi model) with $p = 1/2$, i.e., the probability that two randomly chosen nodes are connected is equal to 1/2.

![Figure 4: Degree distribution for different times. (a) $\beta = 1$ (attractiveness fixed and equal to all nodes) (b) $\beta$ evolves over time according to the attractiveness evolution.](image)

We can observe that this stage is not observed in the evolution of degree distribution related to evolution of attractiveness (plot (b) in Fig. 4). We used $N_{\text{run}} = 1000$ runs and $N = 100$, which means that we have 1000 sets of 100 values of degrees to elaborate these histograms. Actually, in the beginning of evolution, in both cases the degrees are similarly distributed with low values, since theoretically $f(k, t = 0) = \delta(k - 0)$. As the walker evolves, what occurs is different: In (a) we transit to a random graph with $p = 1/2$ up to converge (at large times) to a complete graph (all nodes are connected to all other nodes) since $f(k, t \to \infty) = \delta(k - N)$.
Differently in (b) the system does not transit to the random graph with \( p = \frac{1}{2} \) in intermediate times and does not converge to a complete graph, since a selection of nodes as the system evolves over time, generates an interesting shapes for distribution where nodes with high degree and low degree are highly frequent with interesting valley for the intermediate degrees of the nodes.

Figure 5: Attractiveness distribution for different times in log-log scale. The distribution is governed by a power law distribution but as time increases, the increment of attractiveness leads to a strong deviation of this power law since some of them become “trading topics” in the generated network.

The reason of this peculiar distribution found in (b) can be corroborated looking for the attractiveness distribution. The distribution is governed by a power law distribution as the system evolves, however the increment of attractiveness leads to a strong deviation from this power law for nodes with high attractiveness as can be observed in Fig. 5

3.2. Spectral analysis

We observed that when the attractiveness evolves over time we do not observe a transition from \( f(k, t = 0) = \delta(k - 0) \) to \( f(k, t \to \infty) = \delta(k - N) \) passing through an intermediate stage where the degrees of the
nodes follow a Gaussian distribution. We consider that a non-conventional but an interesting and alternative way to describe the features of a graph originated by the random walk is to analyze the spectral properties of a matrix $C$ built from adjacency matrix as discussed in section 2.2.

Fig. 6 (a) shows the time evolution of the density of eigenvalues of the matrix $C$ obtained of the graph generated by random walk with transitions where the attractiveness is fixed and equal to all nodes during the evolution ($\beta = 1$). We can observe that for $t \approx 70$ turns we have $\rho(\lambda)$ fitting the prescription of Eq. 6 (green continuous curve), which indicates that we have all entries of adjacency matrix following approximately a symmetric probability distribution, i.e., given two nodes there is an edge with probability 1/2. After that the eigenvalues escape from bulk and at large times all nodes are connected to all nodes (complete graph). In this situation we have

$$C_\infty = \frac{1}{N} \begin{pmatrix} -1 & 1 & \cdots & 1 \\ 1 & -1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & -1 \end{pmatrix}^2$$

that have only two distinct eigenvalues: $\lambda_{\text{min}} = \frac{4}{N}$ and $\lambda_{\text{max}} = \frac{(N-2)^2}{N}$ and for large times we have: $\rho_\infty(\lambda) = c_1 \delta(\lambda - \lambda_{\text{min}}) + c_2 \delta(\lambda - \lambda_{\text{max}})$. For $N = 100$, we have $\lambda_{\text{min}} = 0.04$ and $\lambda_{\text{max}} \approx 96$. We can observe two peaks exactly in these two values for $t = 400$ turns.

Differently when the attractiveness evolves over time, plot (b) in Fig. 6, $\rho(\lambda)$ does not follow the law described by Eq. 6 at any time since the evolution of attractiveness leads to a situation where a lot of eigenvalues escape the bulk. At large times we have not a complete graph since there are persistent sites and the final distribution is not composed by two pronounced peaks as expected in conditions of plot (a). More precisely we can understand that for $\beta$ fixed, the graph can be mapped by random graph from connexion probability $p = 0$ until $p = 1$ where different $p$ values corresponds to different times of the evolution. Differently when $\beta$ evolves over the time, this map is not obtained and a split of the main bulk of eigenvalues can be observed differently from the case which $\beta$ is fixed. The dynamics of eigenvalues scattering is also different and does not lead to two peaks at large times since the network does not converge to a complete graph. It is important to notice that the idea to characterize networks via spectral analysis is not novel (see for example [2]). The scape of eigenvalues of the bulk for example were analysed in
Transition probability & Formulae \\
\hline
I & \( p_{ij} = \exp(-0.519 \frac{\phi_i}{\phi_j}) \) \\
II & \( p_{ij} = \exp(-\frac{\phi_i}{\phi_j} d_{ij}) \) \\
III & \( p_{ij} = \min\{1, \frac{\phi_j}{\phi_i}\} \) \\
IV & \( p_{ij} = \frac{\phi_j}{\phi_i+\phi_j} \) \\
V & \( p_{ij} = \exp(-d_{ij}) \) \\
\hline

| Table 1: Transition probabilities |
|----------------------------------|

...the context of stock market analysis [13]. There, the authors use this deviates from the expected law to characterize genuine correlations in the stock market time series. Here we use to characterize the existence of persistent nodes and consequently the deviation from the symmetric \((p = 1/2)\) random graph behaviour.

3.3. Crossover in \( \delta \)

In previous sections of this paper, we fixed \( \delta = 0 \) and studied the properties of the graphs generated by the spatial diffusion with evolution of the attractiveness which leads to the existence persistent nodes. In order to better understand the effects of \( \delta \), we analyze two important amounts, \( p_\infty \) and the variance of attractiveness distribution of the nodes, \( \text{var}_\infty(\phi) = \langle \phi^2 \rangle - \langle \phi \rangle^2 \) in the steady state, for different systems size.

We can observe in Fig. 7 the effects of \( \delta \) on \( p_\infty \) and \( \text{var}(\phi) \). First it is interesting to observe in plot (a) that curves \( p_\infty \times \delta \) for different system sizes, have a crossover in \( \delta \approx 0.09 \). This crossover corresponds exactly to the maximum in plot (b) for \( \text{var}(\phi) \) versus \( \delta \).

3.4. Comparing persistence for different transition probabilities

An important question is to observe if the distance has some importance to the existence of persistence nodes at large times as well as the other transition probabilities. So we analyze different points:

1. Instead of \( p_{ij} = \exp(-\frac{\phi_i}{\phi_j} d_{ij}) \) we consider \( p_{ij} = \exp(-\frac{\phi_i}{\phi_j} \langle d \rangle) \approx \exp(-0.519 \frac{\phi_i}{\phi_j}) \), in order to observe if the effect of different distances has some importance on the persistence;

2. We alternatively test non-Boltzmannian transition probabilities: \( p_{ij} = \min\{1, \frac{\phi_j}{\phi_i}\} \) and \( p_{ij} = \frac{\phi_j}{\phi_i+\phi_j} \), which also follows the same idea: how bigger \( \phi_j \) in relation to \( \phi_i \), bigger is \( p_{ij} \). However this functional dependence has no dependence on this distance between nodes.
So we plot the time evolution of persistence considering different transition probabilities.

Fig. 8 shows \( \text{Pers}(t) \times t \) for different transition probabilities (defined in Table 1). We can observe for exponential transition probabilities with \( \beta_{ij} = \varphi_j / \varphi_i \) that \( \text{Pers}(t \to \infty) = p_{\infty} \) (for both cases: I and II). However we have \( p^{(I)}_{\infty} \), which corresponds to the situation where \( d_{ij} \) is changed by \( \langle d \rangle \) in the formula, is larger than \( p^{(II)}_{\infty} \), where the euclidean distances are indeed considered. This test suggests that distance has importance for example on the changing of the number of persistent sites in the steady state but not on the decay behaviour that seems to be originated from two combined aspects: the influence of attractiveness evolution in the transition probability in an exponential Boltzmann formulae.

At this point an important question should be related to the existence of power law decay as Eq. 1 for some \( p_{ij} \)? Thus, we consider a non-Boltzmann probability transition: \( p^{(III)}_{ij} = \min\{1, \varphi_j / \varphi_i\} \). In this case we can observe a power law decay for the persistence, which suggests that some nodes should take a very large time to be reached by the random walk according to this prescription.

Exponential decays are observed for \( p^{(IV)}_{ij} \) and \( p^{(V)}_{ij} \). It is important to see, that \( p^{(IV)}_{ij} \) also corresponds to the previously case where the attractiveness does not change (\( \beta \) fixed equal to 1). The case IV, corresponds to an alternative simple test to III of non-Boltzmann probability transition. In this case as well as the case III, only the attractiveness is important and the distances are not taken into account. It is interesting to observe that we also obtain an exponential decay for the persistence differently from III, but weaker than V where the attractiveness is not important and only the distance is considered in the Boltzmann rate.

4. Summaries and Conclusions

In this paper we explore the properties of an interesting preferential attachment random walk over the points randomly distributed in a two dimension unit square where transition probability depends on the distance between nodes and also on attractiveness of the nodes which increases according to the incidence of the random walks on these nodes.

We show that stationary persistent nodes (nodes that never were reached by the random walk) appear when we modulate the transition probability with a ratio \( \beta \), which is given by attractiveness of arrival node divided by the attractiveness of the departure node. When \( \beta \) is fixed the persistence exponentially decays to zero and no persistent node is observed. This reason is deeply related to the fact that when \( \beta \) is fixed the network generated evolves passing to have characteristics of a symmetric random graph up to reach a complete graph. Differently when \( \beta \) evolves considering the evolution of attractiveness of the nodes, the graph neither passes through a symmetric random graph during the evolution nor reach a complete
graph as $t \to \infty$. This is corroborated by distortions in the density of eigenvalues, degree distribution, and attractiveness distribution.

We also find a crossover for the fraction of persistent sites for $t \to \infty$ between $p_\infty = 0$ and $p_\infty = 1$, for the parameter $\delta$ that modulates the argument of the exponential that describes the probability transition between two nodes prescribed by Eq. [3]. Our results suggest $\delta_c \approx 0.08$ for this crossover which is corroborated with maximum of the variance of the attractiveness of the nodes.

Finally we show that persistence can follow a power law decay for a specific choice of non-Boltzmannian transition probability which does take into account distances between the nodes and only the attractiveness of these nodes. It is important to mention that another non-Boltzmann transition probability used in this paper leads to exponential persistence exactly as the Boltzmann one with $\beta$ fixed and taking into account the distances between the nodes.

**Acknowledgments** – This research work was in part supported financially by CNPq (National Council for Scientific and Technological Development). We would like to thank Prof. S. D. Prado (IF-UFRGS) for kindly reading this manuscript and pointing out some interesting observations.

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Figure 6: Time evolution of the density of eigenvalues of the matrix $C$, for $N = 100$, in two situations: a) No attractiveness evolution: $\beta = 1$  b) With $\beta$ evolving over time according to Eq. 4.
Figure 7: (a) $p_\infty$ versus $\delta$ for different systems size. (b) variance of attractiveness as function of $\delta$. Both quantities were obtained in the steady state.
Figure 8: Pers(t) × t for different transition probabilities (see Table 1). We can observe for exponential transition probabilities with $\beta_{ij} = \varphi_j / \varphi_i$ that Pers$(t \to \infty) = p_{\infty}$ (I and II). However we have that asymptotic value $p_{\infty}^{(I)}$, which corresponds to the situation where $d_{ij}$ is changed by $\langle d \rangle$ in the formula, is larger than $p_{\infty}^{(II)}$. Finally we find power law decay of the persistence for a peculiar non-Boltzmann transition probability (III). Exponential decays are observed for $p_{ij}^{(IV)}$ and $p_{ij}^{(V)}$. The choice $p_{ij}^{(V)}$ corresponds to the case where the attractiveness does not change ($\beta$ fixed equal to 1). The case IV, corresponds to other simple test of non-exponential probability transition. This case, which also depends only on attractiveness and not on the distances as the case III, leads to an exponential decay and not a power law, although weaker than case V.