Multidimensional Homogeneous Cosmological Models in Wesson Theory of Gravitation

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Abstract

Higher dimensional solutions are obtained for a homogeneous, spatially isotropic cosmological model in Wesson theory of gravitation. Some cosmological parameter are also calculated for this model.

1 Introduction

Recently Wesson [1, 2] proposed a five dimensional theory of gravity where the rest masses varying with time. In this new theory, the space-time is no longer described by a four dimensional manifold but by a five dimensional space-time-mass (STM) Riemannian manifold where the mass plays the role of the fifth coordinate given by $x^4 = \frac{Gm}{c^2} (m = \text{mass})$. The Einstein theory is recovered when the velocity $\frac{G}{c^2} \frac{dm}{dt} = 0$, in other world, the mass is constant. In this regards the four dimensional theory of Einstein can be thought as embedded in a five dimensional STM theory.

This paper concerned with some possible exact solutions for a homogeneous, spatially isotropic n-dimensional ($n > 5$) cosmological model in a matter free space. Although some exact vacuum solutions for five dimensional cosmological model have been worked out by Chi [3] and Chaterjee [4]. We have taken the matter density to be zero because otherwise the higher dimensional field equations with both time and mass as a variable are too complicated to yield explicit solutions. Moreover for the particular case this empty space models provide instructive and transparent examples of various geometric possibilities. The paper ends with some comments and conclusions.

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This type of problem has been discussed in STM theory of gravity by Chaterjee [4] for five dimensional case. However in our solutions, it is pointed out that at this stage, the implications of higher dimension solutions and interpretation are preliminary in nature because of some conceptual problems associated with the new theory and because of the absence of similar higher dimensional solutions in literature so far.

1.1 Field Equations

The line element for a n-dimensional homogeneous and spatially isotropic cosmological model is taken as

\[ ds^2 = e^\nu dt^2 - e^\omega \sum_{i=1}^{(n-2)} dx_i^2 + e^\mu dm^2 \]  \hspace{1cm} (1)

where \( \mu, \omega \) and \( \nu \) are the functions of time and mass. Here the coordinate \( x^0 = t, x^1, x^2, \ldots, (n-2) \) (space coordinate) and \( x^{(n-1)} = m \). For simplicity we have set the magnitudes of both \( c \) and \( G \) to unity. By applying this metric to the Einstein field equation \( G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R = 0 \) with the assumption \( e^\nu = 1 \) we get

\[ G_{00} = -(n-2)(n-3) \frac{\dot{\omega}^2}{8} - (n-2) (\dot{\omega} \dot{\mu} - (n-2) e^{-\mu} \left( \frac{\ddot{\omega}}{2} - \frac{\dot{\omega} \dot{\mu}}{4} + (n-1) \frac{\ddot{\omega}^2}{8} \right) = 0 \]  \hspace{1cm} (2)

\[ G_{11} = e^\omega \left( \frac{\ddot{\mu}}{2} + \frac{\dot{\mu}^2}{4} \right) + (n-3)e^\omega \left( \frac{\ddot{\omega}}{2} + \frac{\dot{\omega} \dot{\mu}}{4} + (n-2) \frac{\ddot{\omega}^2}{8} \right) \]

\[ + (n-3)e^\omega \left( \frac{\ddot{\omega}}{2} - \frac{\dot{\omega} \dot{\mu}}{4} + (n-2) \frac{\ddot{\omega}^2}{8} \right) = 0 \]  \hspace{1cm} (3)

\[ G_{11} = G_{22} = G_{33} = \cdots = G_{(n-2)(n-2)} \]

\[ G_{0(n-1)} = (n-2) \left( \frac{\ddot{\omega}}{2} + \frac{\dot{\omega} \dot{\mu}}{4} - \frac{\ddot{\mu}}{4} \right) = 0 \]  \hspace{1cm} (4)

\[ G_{(n-1)(n-1)} = -(n-2)(n-3) \frac{\ddot{\omega}^2}{8} - (n-2) e^\mu \left( \frac{\ddot{\omega}}{2} + (n-1) \frac{\ddot{\omega}^2}{8} \right) = 0 \]  \hspace{1cm} (5)

where a dot and star denote, respectively partial derivative with respect to time and mass.

1.2 Solutions

By solving equation (4) we get

\[ e^\mu = \alpha_1(m) \ddot{\omega}^2 e^\omega \]  \hspace{1cm} (6)
where $\alpha_1(m)$ is an arbitrary function of mass. Using equation (6) in (5), we get
\[
\dot{\omega}^2 \left( \frac{(n-3)}{8} + \alpha_1(m)e^{\omega}(\frac{\ddot{\omega}}{2} + (n-1)\frac{\dot{\omega}^2}{8}) \right) = 0
\] (7)
since $\dot{\omega}^2$ is not equal to zero in higher dimensional metric it follows that
\[
\ddot{\omega} + \frac{(n-1)}{4} \dot{\omega}^2 = -\frac{(n-3)}{4\alpha_1(m)}e^{-\omega}
\] (8)
which yield the first integral
\[
\dot{\omega}^2 = -\frac{1}{\alpha_1} e^{-\omega} + C_1(m)e^{-(\frac{n-1}{2})\omega}
\] (9)
Equation (9) can also be written as
\[
\dot{X}^2 = -\frac{1}{4\alpha_1} + \frac{C_1}{4}X^{-2k+2}
\] (10)
where $X = e^{\frac{\omega}{2}}$ and $k = \frac{(n-1)}{2}$
\[
\text{or} \int \frac{dX}{\sqrt{C_1\alpha_1X^{2(1-k)} - 1}} = \int \frac{dt}{2\sqrt{\alpha_1}}
\] (11)
so that
\[
\frac{X\sqrt{1 - \alpha_1} C_1X^{3-n} F_1[\frac{1}{3-n}, \frac{1}{2}, 1 + \frac{1}{3-n}, \alpha_1 C_1X^{3-n}]}{\sqrt{C_1X^{3-n} - 1}} = \frac{1}{2\sqrt{\alpha_1}}t + C_2
\] (12)
where $F_1$ is a hypergeometric function and $C_2$ be the arbitrary function of mass. For $C_1 = 0$ in equation (9), we get
\[
e^{\omega} = \frac{-t^2 + 2bt - b^2}{4\alpha_1}
\]
\[
\text{or} \ e^{\omega} = -\alpha t^2 + \beta t + \gamma
\] (13)
where $\alpha = \frac{1}{4\alpha_1}$, $\beta = \frac{1}{2\alpha_1}$, $\gamma = -\frac{b^2}{4\alpha_1}$ are all functions of mass $m$ only.
Equation (13) is similar to the equation obtained by Chaterjee [4] for five dimensional case.

We can set $\mu = 0$ in the line element (1) instead of $\nu = 0$, and the resulting solution is
\[
\frac{X\sqrt{1 - ABX^{3-n} F_2[\frac{1}{3-n}, \frac{1}{2}, 1 + \frac{1}{3-n}, ABX^{3-n}]} F_2[\frac{1}{3-n}, \frac{1}{2}, 1 + \frac{1}{3-n}, ABX^{3-n}]}{\sqrt{ABX^{3-n} - 1}} = \frac{1}{2\sqrt{A}}m + C
\] (14)
where $F_2$ is hypergeometric function and $A$, $B$ and $C$ are all functions of time $t$ only.
1.3 Conclusion

In this paper we have considered the n-dimensional spatially homogeneous and isotropic cosmological model in STM theory of gravity. The main defect of the present model appears to be the arbitrary nature of mass functions. However, as the role of mass in the STM theory is not yet fully understood, neither in its geometrical concepts nor in its physical realm. We think that this new exact higher dimensional solutions together with cosmological considerations should bring some additional information, and as such, they need to be further investigated. It is our hope that the higher dimensional solutions presented here can be used as the starting point to investigate the behaviour of the rest of the particles in more realistic universe models.

References

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