Oscillation inequalities on real and ergodic $H^1$ spaces

Sakin Demir
Agri Ibrahim Cecen University
Faculty of Education
Department of Basic Education
04100 Ağrı, Turkey
E-mail: sakin.demir@gmail.com

May 27, 2022

Abstract

Let $(x_n)$ be a sequence and $\rho \geq 1$. For a fixed sequences $n_1 < n_2 < n_3 < \ldots$, and $M$ define the oscillation operator

$$O_{\rho}(x_n) = \left( \sum_{k=1}^{\infty} \sup_{m \in M} \sup_{n_k \leq m \leq n_{k+1}} |x_m - x_{n_k}|^\rho \right)^{1/\rho}.$$

Let $(X, \mathcal{B}, \mu, \tau)$ be a dynamical system with $(X, \mathcal{B}, \mu)$ a probability space and $\tau$ a measurable, invertible, measure preserving point transformation from $X$ to itself.

Suppose that the sequences $(n_k)$ and $M$ are lacunary. Then we prove the following results for $\rho \geq 2$:

2020 Mathematics Subject Classification: Primary 42B20, 28D05; Secondary 42B30.

Key words and phrases: Oscillation Operator, Hardy Space, $H^1$ Space, Ergodic Hardy Space, Ergodic $H^1$ Space, Ergodic Average.
(i) Define $\phi_n(x) = \frac{1}{n} \chi_{[0,n]}(x)$ on $\mathbb{R}$. Then there exists a constant $C > 0$ such that

$$\|O_\rho(\phi_n * f)\|_{L^1(\mathbb{R})} \leq C \|f\|_{H^1(\mathbb{R})}$$

for all $f \in H^1(\mathbb{R})$.

(ii) Let

$$A_n f(x) = \frac{1}{n} \sum_{k=1}^{n} f(\tau^k x)$$

be the usual ergodic averages in ergodic theory. Then

$$\|O_\rho(A_n f)\|_{L^1(X)} \leq C \|f\|_{H^1(X)}$$

for all $f \in H^1(X)$.

(iii) If $[f(x) \log(x)]^+$ is integrable, then $O_\rho(A_n f)$ is integrable.

1 Preliminaries

Let $(X, \mathcal{B}, \mu)$ a totally $\sigma$-finite measure space and $\tau : X \to X$ be an ergodic measure preserving transformation. The function

$$f^*(x) = \sup_n \frac{1}{n} \sum_{i=0}^{n-1} |f(\tau^i x)|$$

is known as ergodic maximal function analogue to the Hardy-Littlewood maximal function $Mf$ on the real line $\mathbb{R}$ given by

$$Mf(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f(t)| \, dt,$$

where $I$ denotes an arbitrary interval in $\mathbb{R}$.

Let now $f$ be an integrable function and define

$$f^+(x) = \sup_n \frac{1}{n} \sum_{i=0}^{n-1} |f(\tau^i x) - T_n f(x)|$$

where

$$T_n f(x) = \frac{1}{n} \sum_{i=0}^{n-1} |f(\tau^i x)|.$$
Now recall that the space $H^1$ on the real line $\mathbb{R}$ can be characterized by

$$H^1(\mathbb{R}) = \left\{ f \in L^1(\mathbb{R}) : \tilde{H} f \in L^1(\mathbb{R}) \right\}$$

with the norm

$$\|f\|_{H^1} \sim \|f\|_1 + \|\tilde{H} f\|_1.$$

where $\tilde{H} f$ is the Hilbert transform on $\mathbb{R}$ defined by

$$\tilde{H} f(x) = \lim_{\epsilon \to 0^+} \int_{\epsilon}^{\infty} \frac{f(t + t) - f(x - t)}{t} dt.$$

Similar to the characterization by the maximal function we also define ergodic $H^1$ space by

$$H^1(X) = \left\{ f \in L^1(X) : H f \in L^1(X) \right\}$$

with the norm

$$\|f\|_{H^1} \sim \|f\|_1 + \|H f\|_1,$$

where $H f$ is the ergodic Hilbert transform defined by

$$H f(x) = \sum_{k=1}^{\infty} \frac{f(\tau^k x) - f(\tau^{-k} x)}{k}.$$

Similar to the classical case we can also identify the dual of ergodic $H^1$ spaces as ergodic bounded mean oscillation EBMO defined by the space of functions $f$ for which $f^\#$ is bounded with EBMO norm given by

$$\|f\|_{\text{EBMO}} = \|f^\#\|_\infty.$$

Let now $B$ be a Banach space and $p < \infty$, and let $f$ be a $B$-valued (strongly) measurable function defined on $\mathbb{R}$. Then the Bochner-Lebesgue space $L^p_B = L^p_B(\mathbb{R})$ is defined as

$$L^p_B = \left\{ f : \|f\|_{L^p_B} < \infty \right\}$$

where

$$\|f\|_{L^p_B} = \left( \int_\mathbb{R} \|f(x)\|_B^p \, dx \right)^{1/p}.$$

When $B$ is the scalar field, we simply write $L^p$ and $\| \cdot \|_p$. We also define the space $WL^p_B = \text{weak} - L^p_B$ as the space of all $B$-valued functions $f$ such that

$$\|f\|_{WL^p_B} = \sup_{\lambda > 0} \lambda \left( \int_\mathbb{R} \{ x \in \mathbb{R} : \|f(x)\|_B > \lambda \} \right)^{1/p} < \infty.$$
When we replace Lebesgue measure by \( w(x)dx \) for some positive weight \( w \) in \( \mathbb{R} \) we denote the corresponding spaces by \( L^p_B(w) \) and \( WL^p_B(w) \). When \( p = \infty \), we write
\[
L^\infty(B) = \{ f : \| f \|_{L^\infty(B)} < \infty \},
\]
where
\[
\| f \|_{L^\infty(B)} = \text{ess sup} \| f \|_B
\]
and the space of all compactly supported members of \( L^\infty(B) \) will be denoted by \( L^\infty_c(B) \).

For a locally integrable \( B \)-valued function \( f \), we define the maximal functions
\[
M_r f(x) = \sup_{x \in I} \left( \frac{1}{|I|} \int_I \| f(y) \|_B^r \, dy \right)^{1/r}, \quad 1 \leq r \leq \infty,
\]
and
\[
f^\sharp(x) = \sup_{x \in I} \frac{1}{|I|} \int_I \| f(y) - f_I \|_B \, dy,
\]
where \( I \) denotes an arbitrary interval in \( \mathbb{R} \) and
\[
f_I = \frac{1}{|I|} \int_I f(t) \, dt
\]
which is an element of \( B \).

Note that \( f^\sharp \) is the sharp maximal function in the classical case when \( B = \mathbb{R} \) and \( \| \cdot \|_B = | \cdot | \), \( M_1 f \) is the Hardy-Littlewood maximal function and \( M_\infty f \) is the constant function. Similar to the classical case we define the \( B \)-valued BMO space as
\[
\text{BMO}(B) = \{ f \in L^1_{\text{loc},B} : \| f \|_{\text{BMO}(B)} = \| f^\sharp \|_{L^\infty(B)} < \infty \}.
\]

Given a \( B \)-valued function \( f \), we obtain a non-negative function \( \| f \|_B \) defined by
\[
\| f \|_B(x) = \| f(x) \|_B,
\]
and it is important to point out that
\[
\| (\| f \|_B) \|_{\text{BMO}} \leq 2 \| f \|_{\text{BMO}(B)}.
\]

As usual a \( B \)-atom is a function \( a \in L^\infty(B) \) supported in an interval \( I \) and such that
\[
\| a(x) \|_B \leq \frac{1}{|I|}, \quad \int_I a(x) \, dx = 0
\]
and the space $H_B^1(\mathbb{R})$ such that

$$f(x) = \sum_j \lambda_j a_j(x); \quad (\lambda_j) \in l^1,$$

where $a_j$ are $B$-atoms with

$$\|f\|_{H_B^1} = \inf \sum_j |\lambda_j|.$$

Similar to the classical case given $B \in UMD$ we also have

$$H_B^1(\mathbb{R}) = \{f \in L_B^1(\mathbb{R}) : \tilde{H} f \in L_B^1(\mathbb{R})\}$$

and

$$\|f\|_{H_B^1} \sim \|f\|_{L_B^1} + \|\tilde{H} f\|_{L_B^1}.$$ 

**Definition 1.** A sequence $(n_k)$ of integers is called lacunary if there is a constant $\alpha > 1$ such that

$$\frac{n_{k+1}}{n_k} \geq \alpha$$

for all $k = 1, 2, 3, \ldots$.

**Definition 2.** Let $(x_n)$ be a sequence and $\rho \geq 1$. For a fixed sequences $n_1 < n_2 < n_3 < \ldots$, and $M$ define the oscillation operators

$$O_\rho(x_n) = \left( \sum_{k=1}^{\infty} \sup_{\substack{n_k \leq m < n_{k+1} \\text{m} \in M}} |x_m - x_{n_k}|^\rho \right)^{1/\rho}$$

and

$$O_\rho'(x_n) = \left( \sum_{k=1}^{\infty} \sup_{\substack{n_k \leq n \leq m < n_{k+1} \\text{m, n} \in M}} |x_m - x_n|^\rho \right)^{1/\rho}.$$ 

**Remark 1.** It is obvious that

$$O_2(x_n) \leq O_2'(x_n) \leq 2O_2(x_n).$$
Let \((X, \mathcal{B}, \mu, \tau)\) be a dynamical system with \((X, \mathcal{B}, \mu)\) a probability space and \(\tau\) a measurable, invertible, measure preserving point transformation from \(X\) to itself. Let \(\rho \geq 2\) and \(f \in L^1(X)\). Recall the usual ergodic averages

\[ A_n f(x) = \frac{1}{n} \sum_{k=1}^{n} f(\tau^k x). \]

The we can define the oscillation operator

\[ \mathcal{O}_\rho(A_n f)(x) = \left( \sum_{k=1}^{\infty} \sup_{m \in M, n_k \leq m < n_{k+1}} |A_m f(x) - A_{n_k} f(x)|^\rho \right)^{1/\rho}. \]

**Remark 2.** Let \(\rho \geq 1\). For a fixed sequences \(n_1 < n_2 < n_3 \ldots\), and \(M\), we can construct a Banach space \(B\) as an \(\ell^\rho\) sum of finite-dimensional \(\ell^\infty\) spaces with the following norm:

\[ \|b\|_B = \left( \sum_{k=1}^{\infty} \sup_{m \in M, n_k \leq m < n_{k+1}} |b_m - b_{n_k}|^\rho \right)^{1/\rho}. \]

Let us now define the kernel operator \(K : \mathbb{R} \to B\) as

\[ K(x) = \left( \left( \frac{1}{m} \chi_{[0,m]}(x) - \frac{1}{n_k} \chi_{[0,n_k]}(x) : n_k \leq m < n_{k+1}, m \in M \right) : k \geq 1 \right). \]

Let \(\phi_n(x) = \frac{1}{n} \chi_{[0,n]}(x)\) and define

\[ \mathcal{O}_\rho(\phi_n \ast f)(x) = \left( \sum_{k=1}^{\infty} \sup_{m \in M, n_k \leq m < n_{k+1}} |\phi_m \ast f(x) - \phi_{n_k} \ast f(x)|^\rho \right)^{1/\rho}. \]

It is clear that

\[ \mathcal{O}_\rho(\phi_n \ast f)(x) = \|K \ast f(x)\|_B, \]

where

\[ K \ast f(x) = \int K(x - y) \cdot f(y) \, dy. \]
2 The Results

Our first result is the following theorem which plays a key role in proving inequalities for the oscillation operators $O_\rho(\phi_n \ast f)$ and $O_\rho(A_nf)$:

**Theorem 1.** Let $(n_k)$ and $M$ be lacunary sequences, then there exists a constant $C > 0$ such that

$$\int_{|x| > 4|y|} \|K(x - y) - K(x)\|_B \, dx \leq C$$

where $C$ does not depend on $y \in \mathbb{R}$, i.e., $K$ satisfies the Hörmander condition.

**Proof.** For an $n \in \mathbb{N}$ let

$$\phi_n(x) = \frac{1}{n} \chi_{[0,n]}(x)$$

and

$$\Phi_{(m,k)}(x) = \phi_m(x) - \phi_{n_k}(x).$$

Then we have

$$\|K(x - y) - K(x)\|_B = \left( \sum_{k=1}^{\infty} \sup_{m \in M, n_k \leq m < n_{k+1}} \left| \Phi_{(m,k)}(x - y) - \Phi_{(m,k)}(x) \right|^\rho \right)^{1/\rho}.$$

Note that maximum is taken over all $m \in M$ between the related intervals as described in the definition of $O_\rho(\phi_n \ast f)$.

Let us first consider the case $x > 4y$, $y > 0$. We have

$$\Phi_{(m,k)}(x - y) - \Phi_{(m,k)}(x) = \phi_m(x - y) - \phi_{n_k}(x - y) - (\phi_m(x) - \phi_{n_k}(x))$$

$$= \phi_m(x - y) - \phi_m(x) - (\phi_{n_k}(x - y) - \phi_{n_k}(x)).$$

Since $x > 4y$, $y > 0$ we get

$$\phi_m(x - y) - \phi_m(x) = \frac{1}{m} \chi_{[0,m]}(x - y) - \frac{1}{m} \chi_{[0,m]}(x)$$

$$= \frac{1}{m} \chi_{[y,m+y]}(x) - \frac{1}{m} \chi_{[0,m]}(x)$$

$$= \frac{1}{m} \chi_{[4y,m+y]}(x) - \frac{1}{m} \chi_{[4y,m]}(x)$$

$$= \frac{1}{m} \chi_{[m,m+y]}(x).$$
Similarly, we have
\[
\phi_{n_k}(x - y) - \phi_{n_k}(x) = \frac{1}{n_k} \chi_{[n_k, n_k + y]}(x).
\]

We have
\[
\int_{x > 4y} \| K(x - y) - K(x) \|_B \, dx =
\]
\[
= \int_{x > 4y} \left( \sum_{k=1}^{\infty} \sup_{n_k \leq m < n_{k+1} \in M} \frac{1}{m} \chi_{[m, m+y]}(x) - \frac{1}{n_k} \chi_{[n_k, n_k + y]}(x) \right)^\rho \, dx
\]
\[
\leq \int_{x > 4y} \left( \sum_{k=1}^{\infty} \sup_{n_k \leq m < n_{k+1} \in M} \frac{1}{m} \chi_{[m, m+y]}(x) \right)^\rho \, dx +
\]
\[
+ \int_{x > 4y} \left( \sum_{k=1}^{\infty} \sup_{n_k \leq m < n_{k+1} \in M} \frac{1}{n_k} \chi_{[n_k, n_k + y]}(x) \right)^\rho \, dx
\]
\[
\leq \int_{x > 4y} \sum_{y \leq m} \sup_{n_k \leq m < n_{k+1} \in M} \frac{1}{m} \chi_{[m, m+y]}(x) \, dx +
\]
\[
+ \int_{x > 4y} \sum_{y \leq n_k} \sup_{n_k \leq m < n_{k+1} \in M} \frac{1}{n_k} \chi_{[n_k, n_k + y]}(x) \, dx
\]
\[
\leq y \sum_{y \leq m} \frac{1}{m} + y \sum_{y \leq n_k} \frac{1}{n_k}.
\]

On the other hand, we know that \((n_k)\) is a lacunary sequence, there is a constant \(\beta\) such that
\[
\frac{n_{k+1}}{n_k} \geq \beta > 1.
\]

Therefore, there is a constant \(C(\beta)\) such that
\[
\sum_{y \leq n_k} \frac{1}{n_k} \leq C(\beta) \frac{1}{y}.
\]
Similarly, since \( M \) is a lacunary sequence there is a constant \( \alpha > 1 \) and a constant \( C(\alpha) \) such that
\[
\sum_{y \leq n_k} \frac{1}{m} \leq \frac{C(\alpha)}{y}.
\]
As a conclusion we see that
\[
\int_{x>4|y|} \| K(x-y) - K(x) \|_B \, dx \leq C(\alpha) + C(\beta) = C(\alpha, \beta)
\]
and this shows that the kernel operator \( K \) satisfies the Hörmander condition.

Let us now consider the case \( y \leq 0 \) and \( x > 4|y| \),
\[
\phi_m(x-y) - \phi_m(x) = \frac{1}{m} \chi_{[y,y+m]}(x) - \frac{1}{m} \chi_{[0,m]}(x)
\]
we then have
\[
\phi_m(x-y) - \phi_m(x) = \frac{1}{m} \chi_{[m+y,m]}(x)
\]
and similarly, we have
\[
\phi_{n_k}(x-y) - \phi_{n_k}(x) = \frac{1}{n_k} \chi_{[n_k+y,n_k]}(x)
\]
for \( x > 4|y| \).

Again we see as in the previous case that
\[
\int_{x>4|y|} \| K(x-y) - K(x) \|_B \, dx \leq C(\alpha) + C(\beta) = C(\alpha, \beta)
\]
and this shows that \( K \) satisfies the the Hörmander condition in this case as well.

Suppose now that \( x < 0 \) and \( y > 0 \). Since \( |x| > 4|y| \), we see that \( x - y < 0 \) thus in this case
\[
\phi_m(x-y) - \phi_m(x) = \frac{1}{m} \chi_{[0,m]}(x-y) - \frac{1}{m} \chi_{[0,m]}(x) = 0
\]
and
\[
\phi_{n_k}(x-y) - \phi_{n_k}(x) = \frac{1}{n_k} \chi_{[0,n_k]}(x-y) - \frac{1}{n_k} \chi_{[0,n_k]}(x) = 0.
\]
Thus for any \( y \) we have
\[
\int_{|x|>4|y|} \| K(x-y) - K(x) \|_B \, dx = 0.
\]
We finally need to consider the case \( x < 0 \) and \( y < 0 \). In this case we have

\[
\phi_m(x - y) - \phi_m(x) = \frac{1}{m} \chi_{[y,y+m]}(x)
\]

and

\[
\phi_{n_k}(x - y) - \phi_{n_k}(x) = \frac{1}{n_k} \chi_{[y,y+n_k]}(x).
\]

Since we also have \(|x| > 4|y|\), we see that

\[
\phi_m(x - y) - \phi_m(x) = \frac{1}{m} \chi_{[y,y+m]}(x) = 0
\]

and

\[
\phi_{n_k}(x - y) - \phi_{n_k}(x) = \frac{1}{n_k} \chi_{[y,y+n_k]}(x) = 0.
\]

Thus for any \( y \) we have

\[
\int_{|x| > 4|y|} \|K(x - y) - K(x)\|_B \, dx = 0.
\]

Lemma 1. Suppose that \((n_k)\) is a lacunary sequence. Then there exists a constant \( C > 0 \) such that

\[
\|O_2(\phi_n \ast f)\|_{L^2(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}
\]

for all \( f \in L^2(\mathbb{R}) \).

Proof. When \((n_k)\) is lacunary it is proven in R. L. Jones et al [5]) (see Corollary 2.11) that there exists a constant \( C > 0 \) such that

\[
\|O_2'(A_n f)\|_{L^2(X)} \leq C\|f\|_{L^2(X)}
\]

for all \( f \in L^2(X) \). Remark 1 clearly implies that

\[
\|O_2(A_n f)\|_{L^2(X)} \leq \|O_2'(A_n f)\|_{L^2(X)}.
\]

Thus we get

\[
\|O_2(A_n f)\|_{L^2(X)} \leq C\|f\|_{L^2(X)}
\]

\( f \in L^2(X) \). The proof of our lemma follows by an application of Calderón transfer principle.
Theorem 2. There exists a constant $C > 0$ such

$$
\|O_\rho(\phi \ast f)\|_{L^2(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}
$$

for all $f \in L^2(\mathbb{R})$.

Proof. Let $\rho \geq 2$. Then we obtain

$$
\|O_\rho(\phi \ast f)\|_{L^2(\mathbb{R})}^2 = \int \left( \sum_{k=1}^{\infty} \sup_{m \in M} \frac{1}{n_k \leq m < n_{k+1}} |\phi_n \ast f(x) - \phi_{n_k} \ast f(x)|^\rho \right)^{2/\rho} \, dx
$$

$$
\leq \int \sum_{k=1}^{\infty} |\phi_n \ast f(x) - \phi_{n_k} \ast f(x)|^2 \, dx
$$

$$
= \int \left[ \left( \sum_{k=1}^{\infty} \sup_{m \in M} |\phi_n \ast f(x) - \phi_{n_k} \ast f(x)|^2 \right)^{1/2} \right]^2 \, dx
$$

$$
= \|O_2(\phi \ast f)\|_{L^2(\mathbb{R})}^2
$$

$$
\leq C\|f\|_{L^2(\mathbb{R})}^2
$$

for some constant $C > 0$ by Lemma 1.

We can now state and prove our next result:

Theorem 3. There exits a constant $C > 0$ such that

$$
\|O_\rho(\phi_n \ast f)\|_{L^1(\mathbb{R})} \leq C\|f\|_{H^1(\mathbb{R})}
$$

for all $f \in H^1(\mathbb{R})$.

Proof. It suffices to show that

$$
\|O_\rho(\phi_n \ast a)\|_{L^1(\mathbb{R})} \leq C
$$

for any atom $a$ with constant $C$ independent of the choice of $a$. We first consider a 1-atom centered at 0 with support support$(a) \subset I_R$, where $I_R$
denotes an interval centered at 0 with side length $2R$. Since

$$
\|a\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |a(x)|^2 \, dx
$$

$$
= \int_{I_R} |a(x)|^2 \, dx
$$

$$
\leq \frac{1}{|I_R|^2} \int_I dx
$$

$$
= \frac{1}{|I_R|}.
$$

we see that

$$
\|a\|_{L^2(\mathbb{R})} \leq \frac{1}{|I_R|^{1/2}}.
$$

Because of the definition of an atom we also have

$$
\int_{I_R} a(x) \, dx = 0.
$$

Hence

$$
\int_{|x|\geq 4R} |O_\rho(\phi_n * a)(x)| \, dx = \int_{|x|\geq 4R} \|K * a(x)\|_B \, dx
$$

$$
= \int_{|x|\geq 4R} \left| \int_{I_R} \|K(x-y) - K(x)\|_B \cdot a(y) \, dy \right| \, dx
$$

$$
\leq \int_{I_R} |a(y)| \, dy \int_{|x|\geq 4|y|} \|K(x-y) - K(x)\|_B \, dx
$$

$$
\leq C(\alpha, \beta) \int_{I_R} |a(y)| \, dy
$$

$$
\leq C(\alpha, \beta).
$$

On the other hand by Hölder’s inequality

$$
\int_{|x|<4R} |O_\rho(\phi_n * a)(x)| \, dx = \int_{|x|<4R} \|K * a(x)\|_B \, dx
$$

$$
\leq \left( \int \|K * a(x)\|_B^2 \right)^{1/2} 2|I_R|^{1/2}
$$

$$
\leq C_1 \|a\|_{L^2(\mathbb{R})}^2 |I_R|^{1/2}
$$

$$
\leq C.
$$
We obtain
\[ \| O_\rho (\phi_n \ast a) \|_{L^1(\mathbb{R})} \leq C \]
in both inequalities for any atom \( a \) centered at the origin.
Let now \( b \) be an atom centered at \( c \in \mathbb{R} \). Then \( a(x) = b(x - c) \) is an atom centered at 0. Moreover, we have
\[ \| O_\rho (\phi_n \ast a) \|_{L^2(\mathbb{R})} \leq C_1 \| a \|_{L^2(\mathbb{R})} \]
and we obtain as before
\[ \| O_\rho (\phi_n \ast a) \|_{L^1(\mathbb{R})} \leq C. \]
Hence we have
\[
\| O_\rho (\phi_n \ast b) \|_{L^1(\mathbb{R})} = \| O_\rho (\phi_n \ast a) \|_{L^1(\mathbb{R})} \\
\leq C.
\]

When we consider the maximal function characterization of the space \( H^1(\mathbb{R}) \), it is clear that \( \| f \|_{H^1(\mathbb{R})} \) is the \( L^1 \) norm of an operator of convolution type. That’s because we can apply the transfer principle of S. Demir [3] to Theorem 3 to find the following result with the same constant as in Theorem 3:

**Theorem 4.** There exits a constant \( C > 0 \) such that
\[
\| O_\rho (A_n f) \|_{L^1(X)} \leq C \| f \|_{H^1(X)}
\]
for all \( f \in H^1(X) \).

Note that in Theorem 4 \( H^1(X) \) denotes the ergodic Hardy space. We refer those readers who are not so familiar with these spaces to R. Caballero and A. de la Torre [1].

Recall the following well known theorem of D. Ornstein [6] which will be used in the proof of our last result:

**Lemma 2.** The ergodic maximal function
\[
f^*(x) = \sup_n \frac{1}{n} \sum_{i=0}^{n-1} | f(\tau^i x) |
\]
is integrable if and only if \( [f(x) \log(x)]^+ \) is integrable, where \( g^+ \) denotes the positive part of \( g \).
Our last result gives a condition for integrability of the oscillation operator $O_\rho(A_n f)$.

**Theorem 5.** If $[f(x) \log(x)]^+$ is integrable, then $O_\rho(A_n f)$ is integrable for $\rho \geq 2$.

**Proof.** It is proven in S. Demir [4] that 

$$\|f\|_{H^1(X)} \leq \|f^*\|_{L^1(X)}.$$ 

Now suppose that $[f(x) \log(x)]^+$ is integrable, then Lemma 2 implies the finiteness of $\|f^*\|_{L^1(X)}$, and thus the finiteness of $\|f\|_{H^1(X)}$. This means that $f \in H^1(X)$, and by Theorem 4 there exists a constant $C > 0$ such that

$$\|O_\rho(A_n f)\|_{L^1(X)} \leq C \|f\|_{H^1(X)}, \quad \rho \geq 2$$

thus we have 

$$\|O_\rho(A_n f)\|_{L^1(X)} < \infty$$

and this completes our proof. □

**References**

[1] R. Caballero and A. de la Torre, *An atomic theory of ergodic $H^p$ spaces*, Studia Math. 82 (1985) 39-69.

[2] A. P. Calderón, *Ergodic theory and translation-invariant operators*, Proc. Nat. Acad. Sci. USA. 59 (1968) 349-353.

[3] S. Demir, *A generalization of Calderón transfer principle*, Journal of Computer & Mathematical Sciences 9(5) (2018) 325-329.

[4] S. Demir, *$H^p$ spaces and inequalities in ergodic theory*, Ph.D Thesis, University of Illinois at Urbana-Champaign, USA, May 1999.

[5] R. L. Jones, R. Kaufman, J. M. Rosenblatt and Máté Wierdl, *Oscillation in ergodic theory*, Ergodic Th. & Dynam. Sys 18 (1998) 889-935.

[6] D. Ornstein, *A remark on Birkhoff ergodic theorem*, Illinois J. Math. 15 (1971) 77-79.