Gluing of completely positive maps

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Abstract. Gluings of completely positive maps (CPMs) are defined and investigated. As a brief description of this concept consider a pair of ‘evolution machines’, each with the ability to evolve the internal state of a ‘particle’ inserted into its input. Each of these machines is characterized by a channel describing the operation the internal state has experienced when the particle is returned at the output. Suppose a particle is put in a superposition between the input of the first and the second machine. Here it is shown that the total evolution caused by a pair of such devices is not uniquely determined by the channels of the two machines. Such ‘global’ channels describing the machine pair are examples of gluings of the two single machine channels. Under the limiting assumption that all involved Hilbert spaces are finite-dimensional, an expression which generates all subspace preserving gluings of a given pair of CPMs, is derived. The nature of the non-uniqueness of gluings and its relation to a proposed definition of subspace locality, is discussed.

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1. Introduction

Completely positive maps (CPMs) and trace preserving completely positive maps are useful tools to describe operations in quantum mechanics. This investigation appears in a family of papers [1], [2] devoted to the study of completely positive maps with respect to properties tied to orthogonal sum decomposition of the Hilbert spaces of quantum systems. In [2] the concept of subspace preserving CPMs is introduced. In [1] a definition of subspace locality is proposed, while here the concept of gluings of CPMs is introduced.

To give an intuitive picture of the concept of gluing, imagine an apparatus which evolves the state of quantum systems. Imagine a ‘particle’ of some kind and a machine constructed such that when the particle is inserted in the input of the machine, the internal state of the particle is evolved and the particle is returned at the output. The operation of the machine is modeled by a ‘channel’, i.e. a trace preserving CPM. Imagine now that we have two such machines, each characterized by a channel and suppose we have one single particle. The question is: suppose this single particle is put in superposition between the two inputs of the two evolution machines, what would be the output, when the two machines act on this superposition? At first glance it may seem as if the evolution of the superposition is trivially determined by the two channels characterizing the action of the two machines. This is however not the case,
as shall be demonstrated here. The two channels do not provide sufficient information to uniquely determine the total evolution.

To rephrase the problem in more mathematical terms, there are two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, the first representing the pure input states of machine 1, the second representing pure input states of machine 2. The total input state Hilbert space $\mathcal{H}$ can be described as the orthogonal sum of the two separate state spaces $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

For the quite general type of evolution we are considering here, allowing interaction with ancillary quantum systems, it is necessary to use density operators to describe states of quantum systems. A channel maps density operators to density operators. The channel $\Phi_1$ characterizing machine 1 maps density operators on $\mathcal{H}_1$ to density operators on $\mathcal{H}_1$. Likewise the channel $\Phi_2$ of machine 2 maps density operators on $\mathcal{H}_2$ to density operators on $\mathcal{H}_2$. The operation caused by the two machines acting in combination on one single input, can be described as a channel which maps density operators on $\mathcal{H}$ to density operators on $\mathcal{H}$. To rephrase the above question: does $\Phi_1$ and $\Phi_2$ determine $\Phi$ uniquely? The negative answer to this question implies that there exist several channels $\Phi$ which, in some sense, are ‘compatible’ with the two channels $\Phi_1$ and $\Phi_2$. We call such a channel $\Phi$ a gluing of $\Phi_1$ and $\Phi_2$. One of the purposes of this investigation is to find an explicit expression for describing all possible gluings of given channels $\Phi_1$ and $\Phi_2$. This is achieved by first proving that all trace preserving gluings of channels have to be subspace preserving \cite{2} and using tools developed in \cite{2}.

One further question is: what happens if one imposes the restriction that the two evolution machines should act independently of each other? Hence, there should be no interaction, communication, or sharing of resources like entangled or correlated quantum systems, which would enable a correlated action. This restriction is added by using a definition of subspace local channels suggested in \cite{1}. The set of gluings which satisfies this additional condition is deduced. One may perhaps imagine that the non-uniqueness of the gluings has its cause in the freedom of the machines to interact (or to share correlated systems), and that if all such ‘dependencies’ are cut away, the non-uniqueness of the gluings would disappear. When the gluings are restricted to be subspace local, the non-uniqueness get reduced, but some non-trivial non-uniqueness actually does remain. Hence, even if the machines are acting independently, the CPMs of the two machines are still not sufficient to determine the joint action of the two devices. This is resolved by noting that the channel $\Phi_1$ for device 1 (and $\Phi_2$ for device 2) is actually not a full description of the action of this machine, in this context. By providing the ‘missing parts’, the gluings are uniquely determined.

The derivations performed here are made under the limiting assumption that all Hilbert spaces appearing are finite-dimensional. The author believes that much of the material derived here do have generalizations, with suitable technical modifications, to separable Hilbert spaces. This question will, however, not be treated here.

The structure of this article is the following. In section 2 gluings of CPMs is introduced. Explicit expressions to generate all possible gluings of two given CPMs are deduced. In section 3 we turn to the special case of SP gluings which fulfills the additional condition of being subspace local. In section 4 the theory is illustrated with some simple examples of gluings. In section 5 we discuss some conceptual aspects of the non-uniqueness of gluings and its relation to subspace locality. A summary is presented in section 6.
2. Gluings

We begin by introducing some notation, terminology and conventions to be used throughout this article.

\( \mathcal{H} \) (with various subscripts) denotes a finite-dimensional complex Hilbert space. Completely positive maps (CPMs) take trace class operators of one Hilbert space to trace-class operators on another Hilbert space. On a finite-dimensional Hilbert space the set of trace class operators coincides with the set of linear operators. Since this study is restricted to finite-dimensional Hilbert spaces we let CPMs operate on the set of linear operators on the Hilbert space in question.

The set of linear operators on \( \mathcal{H} \) is denoted \( \mathcal{L}(\mathcal{H}) \). The set of linear operators from \( \mathcal{H}_S \) to \( \mathcal{H}_T \) is denoted \( \mathcal{L}(\mathcal{H}_S, \mathcal{H}_T) \).

If a CPM \( \phi \) maps elements in \( \mathcal{L}(\mathcal{H}_S) \) to elements in \( \mathcal{L}(\mathcal{H}_T) \), we say that \( \mathcal{H}_S \) is the source space (or just source) of \( \phi \), and that \( \mathcal{H}_T \) is the target space (or just target) of \( \phi \). When discussing a CPM the spaces \( \mathcal{H}_S \) and \( \mathcal{H}_T \) are always assumed to be the source and the target space of the CPM in question, unless otherwise stated.

CPMs can always be constructed via Kraus representations \([3]\). Given a CPM \( \phi \) there exists some set of operators \( \{V_k\}_k \subset \mathcal{L}(\mathcal{H}_S, \mathcal{H}_T) \) such that \( \phi(Q) = \sum_k V_k Q V_k^\dagger \) for all \( Q \in \mathcal{L}(\mathcal{H}_S) \). We refer to such a set \( \{V_k\}_k \) as a Kraus representation of \( \Phi \).

To every CPM there exists a linearly independent Kraus representation (proposition 6 in \([2]\)). The number of elements in a linearly independent Kraus representation of a CPM \( \phi \) is called the Kraus number (proposition 6 in \([2]\)) of the CPM and is denoted \( k(\phi) \).

\( \mathcal{H}_{s1} \) and \( \mathcal{H}_{s2} \) denote subspaces of \( \mathcal{H}_S \). They are assumed to be orthogonal complements of each other, such that \( \mathcal{H}_S = \mathcal{H}_{s1} \oplus \mathcal{H}_{s2} \) (\( \oplus \) denotes orthogonal sum). Analogously \( \mathcal{H}_T = \mathcal{H}_{t1} \oplus \mathcal{H}_{t2} \). Finally it is assumed that each of the subspaces \( \mathcal{H}_{s1}, \mathcal{H}_{s2}, \mathcal{H}_{t1}, \mathcal{H}_{t2} \) are at least one-dimensional. To all these subspaces are orthogonal projectors associated. To \( \mathcal{H}_{s1} \) belongs the projector \( P_{s1} \), to \( \mathcal{H}_{t1} \) belongs \( P_{t1} \), etc.

Given an operator \( V : \mathcal{H}_{s1} \to \mathcal{H}_{t1} \), we will in some expressions handle it as if it was an operator \( V' : \mathcal{H}_S \to \mathcal{H}_T \), where \( V' \) acts as \( V \) on \( \mathcal{H}_{s1} \) and as the zero operator on \( \mathcal{H}_{s2} \) and is extended linearly to whole \( \mathcal{H}_S \). For notational simplicity we do not differ between \( V \) and \( V' \). Another abuse of notation, in the same spirit as the previous one, concerns CPMs. Given a CPM \( \phi \) with source space \( \mathcal{H}_{s1} \) and target space \( \mathcal{H}_{t1} \), we will in some expressions handle it as if it had source space \( \mathcal{H}_S \) and target \( \mathcal{H}_T \). If \( \phi \) has a Kraus representation \( \{V_k\}_k \), then this ‘extended’ CPM can be constructed as \( \{V_k'\}_k \) with \( V_k' \) as described above. We will not make any difference between these CPMs. It is to be noted that if \( \Phi \) is a trace preserving CPM with source \( \mathcal{H}_{s1} \) and target \( \mathcal{H}_{t1} \), then it is not trace preserving if regarded as having source \( \mathcal{H}_S \) and target \( \mathcal{H}_T \).

Let \( \phi \) be a CPM with source space \( \mathcal{H}_S \) and target space \( \mathcal{H}_T \) and let \( \mathcal{H}_{s1} \) be a subspace of \( \mathcal{H}_S \). We let \( \phi_{s1} \) be defined as the restriction (in the ordinary sense) of \( \phi \) to the subset \( L(\mathcal{H}_{s1}) \). We say that \( \phi_{s1} \) is the restriction in source space to \( \mathcal{H}_{s1} \). Since the set of density operators on the subspace \( \mathcal{H}_{s1} \) is a subset of the set of density operators on \( \mathcal{H}_S \), follows that \( \phi_{s1} \) is a positive map. Let \( \mathcal{H}_n \) be n-dimensional and let \( I_n \) be the identity CPM with source and target \( \mathcal{H}_n \). That \( \phi_{s1} \) is completely positive follows since \( \phi_{s1} \otimes I_n \) is the restriction in source space of \( \phi \otimes I_n \) to \( \mathcal{H}_{s1} \otimes \mathcal{H}_n \). Above we concluded that the restriction in source space of a positive map is positive, hence \( \phi_{s1} \otimes I_n \) is positive for each \( n \). Hence, by definition \([3]\), \( \phi_{s1} \) is completely positive.

Next we define restriction in target space. Given a subspace \( \mathcal{H}_{t1} \) of the target space of a CPM \( \phi \), we define the mapping \( \phi'(Q) = P_{t1}(Q)P_{t1}, \forall Q \in \tau(\mathcal{H}_S) \). In this
Proposition 2 will be used to derive an expression for all the SP gluings of two CPMs with known \(\{\phi, P\}\). Moreover, this restriction is trace preserving. An analogous reasoning holds for \(H_Φ(\text{see proposition 2 in [2]})\) from the two last equations follows, by definition [2], that \(Φ\) is SP from \(H|\). Since the mapping \(η(Q) = P_{t1}QP_{t1}\) is written in the Kraus representation form (no matter the exact status of \(P_{t1}\)) it follows that \(η\) is a CPM. Hence, \(ϕ'\) is a composition of two CPMs and hence is a CPM. We can conclude:

**Lemma 1** • The restriction in source space of a CPM is a CPM. Moreover, the restriction in source space of a trace preserving CPM is trace preserving.

• The restriction in target space of a CPM is a CPM.

Note that in difference with restriction in source space, a restriction in target space of a trace preserving CPM is, in general, not trace preserving.

**Definition 1** Let \(φ_1\) be a CPM with source \(H_{s1}\) and target \(H_{t1}\) and let \(φ_2\) be a CPM with source \(H_{s2}\) and target \(H_{t2}\). A CPM \(ϕ\) with source \(H_S\) and target \(H_T\) is said to be a gluing of \(φ_1\) and \(φ_2\), if \(φ_1\) is the result of restriction in target to \(H_{t1}\) and in source to \(H_{s1}\) of \(ϕ\), and if \(φ_2\) is the result of restriction in target to \(H_{t2}\) and in source to \(H_{s2}\) of \(ϕ\). If moreover \(ϕ\) is subspace preserving (SP) from \(|H_{s1}, H_{s2}\rangle\) to \(|H_{t1}, H_{t2}\rangle\) then we say that \(ϕ\) is an SP gluing of \(φ_1\) and \(φ_2\).

For the trace preserving CPMs an especially simple relation holds for gluings and SP gluings.

**Proposition 1** Let \(Φ\) be a trace preserving CPM with source \(H_S\) and target \(H_T\). \(Φ\) is SP from \(|H_{s1}, H_{s2}\rangle\) to \(|H_{t1}, H_{t2}\rangle\) if and only if \(Φ\) is a trace preserving gluing of two trace preserving CPMs, \(Φ_1\) with source \(H_{s1}\) and target \(H_{t1}\), and \(Φ_2\) with source \(H_{s2}\) and target \(H_{t2}\).

**Proof.** We begin to prove the “if” part of the proposition. From \(Φ_1\) being trace preserving it follows that \(\text{Tr}(P_{t1}Φ(|ψ⟩⟨ψ|)) = \text{Tr}(Φ_1(|ψ⟩⟨ψ|)) = 1\), for all normalized \(|ψ⟩\in H_{s1}\). Since \(Φ\) is trace preserving, it follows that \(\text{Tr}(P_{t2}Φ(|ψ⟩⟨ψ|)) = 0\), for all \(|ψ⟩\in H_{s1}\). By an analogous argument follows \(\text{Tr}(P_{t1}Φ(|ψ⟩⟨ψ|)) = 0\), for all \(|ψ⟩\in H_{s2}\). From the two last equations follows, by definition [2], that \(Φ\) is SP from \(|H_{s1}, H_{s2}\rangle\) to \(|H_{t1}, H_{t2}\rangle\).

The “only if” part follows from the fact that any SP CPM can be decomposed as (see proposition 2 in [2])

\[Φ(Q) = P_{t1}Φ(P_{t1}QP_{t1})P_{t1} + P_{t1}Φ(P_{t2}QP_{t2})P_{t2} + P_{t2}Φ(P_{s2}QP_{s2})P_{t1} + P_{t2}Φ(P_{s2}QP_{s2})P_{t2}.\]

The CPM \(P_{t1}Φ(P_{s2}QP_{s2})P_{t1}\) is essentially (with a purely technical modification of the source and target spaces) the restriction in source to \(H_{s1}\) and in target to \(H_{t1}\). Moreover, this restriction is trace preserving. An analogous reasoning holds for \(P_{t2}Φ(P_{s2}QP_{s2})P_{t2}\). Hence, \(Φ\) is a trace preserving gluing of two channels. \(\square\)

In [2] an expression for the set of all SP CPMs has been derived. This expression will be used to derive an expression for all the SP gluings of two CPMs with known linearly independent Kraus representations. For the sake of convenience proposition 10 in [2] is restated here.

**Proposition 2** Let \(\{V_k\}_{k=1}^K\) be a basis of \(L(H_{s1}, H_{t1})\), \(K = \dim H_{s1} \dim H_{t1}\), and let \(\{W_l\}_{l=1}^L\) be a basis of \(L(H_{s2}, H_{t2})\), \(L = \dim H_{s2} \dim H_{t2}\). The mapping \(ϕ\), defined by

\[ϕ(Q) = \sum_{kk'} A_{k,k'} V_k Q V^\dagger_{k'} + \sum_{ll'} B_{l,l'} W_l Q W^\dagger_{l'} + \sum_{kl} C_{kl} V_k Q W^\dagger_{l} + \sum_{kl} C^*_{kl} W_l Q V^\dagger_{k}.\]
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for all \( Q \in \mathcal{L}(\mathcal{H}_S) \), is an SP CPM from \((\mathcal{H}_{s1}, \mathcal{H}_{s2})\) to \((\mathcal{H}_{t1}, \mathcal{H}_{t2})\) if and only if the matrices\( A = [A_{k,k'}]_{k,k'}^{K} = 1, \) \( B = [B_{l,l'}]_{l,l'}^{L} = 1, \) and \( C = [C_{k,l}]_{k,l=1}^{K,L} \) fulfill the relations
\[
A \geq 0, \quad B \geq 0, \quad P_{A,0} = 0, \quad C P_{B,0} = 0, \quad A \geq CB^\ominus C^\dagger, \tag{3}
\]
where \( B^\ominus \) denotes the Moore-Penrose pseudo inverse \([4],[5],[6]\) of \( B \). \( P_{A,0} \) denotes the orthogonal projector onto the zero eigenspace of \( A \) and analogously for \( P_{B,0} \).

Moreover, \([4]\) defines a bijection between the set of all CPMs which are SP from \((\mathcal{H}_{s1}, \mathcal{H}_{s2})\) to \((\mathcal{H}_{t1}, \mathcal{H}_{t2})\), and the set of all triples of matrices \( A,B,C \) fulfilling the conditions \([2]\) and \([3]\).

The proposition above gives an expression for all possible SP CPMs with respect to given decompositions of the source and target space. We are interested, not in all such CPMs, but only those SP CPMs which are gluings of two given CPMs. Given the knowledge of linearly independent Kraus representations of the CPMs \( \phi_1 \) and \( \phi_2 \), it is possible to construct a rather compact expression for the set of SP gluings.

**Proposition 3** Let \( \phi_1 \) be a CPM with source space \( \mathcal{H}_{s1} \) and target space \( \mathcal{H}_{t1} \). Let \( \phi_2 \) be a CPM with source space \( \mathcal{H}_{s2} \) and target space \( \mathcal{H}_{t2} \). Let \( \{V_n\}_n^{N} \) be a linearly independent Kraus representation \([3]\) of \( \phi_1 \) and let \( \{W_m\}_m^{M} \) be a linearly independent Kraus representation of \( \phi_2 \). Then \( \phi \) is an SP gluing of \( \phi_1 \) and \( \phi_2 \), if and only if \( \phi \) can be written
\[
\phi(Q) = \sum_{n=1}^{N} V_n Q V_n^\dagger + \sum_{m=1}^{M} W_m Q W_m^\dagger
+ \sum_{m=1,n=1}^{N,M} C_{m,n} V_m Q W_n^\dagger + \sum_{m=1,n=1}^{N,M} C_{n,m}^* W_m Q V_n^\dagger \quad \forall Q \in \mathcal{L}(\mathcal{H}_S), \tag{4}
\]
where the matrix \( C = [C_{n,m}]_{n=1,m=1}^{N,M} \) fulfills the condition
\[
I_N \geq C C^\dagger, \tag{5}
\]
where \( I_N \) denotes the \( N \times N \) identity matrix. Moreover, \([4]\) defines a bijection between the set of all SP gluings of \( \phi_1 \) and \( \phi_2 \), and the set of all matrices \( C \) fulfilling condition \([4]\).

The condition \( I_N \geq C C^\dagger \) is equivalent to \( I_M \geq C^\dagger C \), with \( I_M \) the \( M \times M \) identity matrix. These conditions in turn are equivalent to the condition that the largest singular value of \( C \) should be less than or equal to one. These comments can be derived by using singular value decomposition \([4]\) of \( C \). (Use \( C = U_1^\dagger C U_2 \) where these operators are defined as in the proof of proposition \([4]\)).

The matrix \( C \) in the above proposition we refer to as the gluing matrix. Note that the choices of linearly independent Kraus representations are arbitrary. The gluing matrix depends on this choice, but not the set of gluings.

If \( \phi_1 \) and \( \phi_2 \) are trace preserving, then proposition \([4]\) gives all the trace preserving gluings. This follows from proposition \([4]\).

**proof.** We begin to prove that any CPM on the form \([4]\) is an SP gluing of \( \phi_1 \) and \( \phi_2 \). One can check that any CPM which can be written on the form \([3]\) has \( \phi_1 \) as restriction in source to \( \mathcal{H}_{s1} \) and in target to \( \mathcal{H}_{t1} \), and has \( \phi_2 \) as restriction in source to \( \mathcal{H}_{s2} \) and in target to \( \mathcal{H}_{t2} \). Hence, \( \phi \) is a gluing of \( \phi_1 \) and \( \phi_2 \). It remains to show that every mapping \( \phi \) defined by \([4]\) is an SP CPM. (The main part is to prove that...
φ is a CPM.) Complete the set \( \{V_n\}_{n=1}^N \) into a basis \( \{V_k\}_{k=1}^K \) of \( \mathcal{L}(\mathcal{H}_{A_1}, \mathcal{H}_{A_1}) \), in such a way that the first \( N \) elements form the linearly independent Kraus representation \( \{V_n\}_{n=1}^N \). Define the matrix \( A = [A_{k,k'}]_{k,k'=1}^K \) by
\[
A_{k,k'} = \delta_{k,k'}, \quad \forall k, k' \leq N, \quad \text{and} \quad A_{k,k'} = 0 \text{ else.} \tag{6}
\]
Clearly \( \phi_1(Q) = \sum_{k,k'=1}^K A_{k,k'}V_kQV_{k}'\). Similarly, complete the set \( \{W_m\}_{m=1}^M \) into a basis of \( \mathcal{L}(\mathcal{H}_{A_2}, \mathcal{H}_{A_2}) \) and construct the matrix \( B \) similarly as \( A \) was constructed. Let the matrix \( \tilde{C} = [\tilde{c}_{k,l}]_{k=1,l=1}^L \) be defined as \( \tilde{c}_{k,l} = C_{k,l} \) for \( k \leq N \) and \( l \leq M \), and \( \tilde{c}_{k,l} = 0 \) else. One can verify that the triple \((A, B, \tilde{C})\) fulfills the conditions \( \ref{eq:2} \) and \( \ref{eq:3} \) of proposition \( \ref{prop:2} \) Moreover, \( \phi \) is the result when \((A, B, \tilde{C})\) are used in \( \ref{eq:11} \). Hence, according to proposition \( \ref{prop:2} \) \( \phi \) is an SP CPM.

It is to be shown that every SP gluing of \( \phi_1 \) and \( \phi_2 \) can be written on the form \( \ref{eq:4} \). Complete (like above) the linearly independent Kraus representations \( \{V_n\}_{n=1}^N \) and \( \{W_m\}_{m=1}^M \) into bases. Since we are searching for all SP gluings of \( \phi_1 \) and \( \phi_2 \), all of them can be written (since they are SP) on the form \( \ref{eq:11} \), with respect to the above chosen bases, where each SP CPM corresponds to a triple \((A, B, \tilde{C})\). For such an SP CPM to be a gluing of \( \phi_1 \) and \( \phi_2 \), one can see that a necessary condition is that \( \phi_1(Q) = \sum_{k,k'=1}^K A_{k,k'}V_kQV_{k}' \) and \( \phi_2(Q) = \sum_{i,l=1}^L B_{i,l}QW_iQW_i' \). Since these matrices are uniquely determined by the choice of bases (see proposition 5 in \( \ref{prop:2} \)) it follows that the matrix \( A = [A_{k,k'}]_{k,k'=1}^K \) is the one defined in \( \ref{eq:4} \). A similar reasoning holds for \( B \).

Using the conditions \( \ref{eq:9} \) it follows that only a sub-matrix of the matrix \( \tilde{C} \) is non-zero. This sub-matrix is the sub-matrix defined by \( C = [c_{n,m}]_{n=1,m=1}^{N,M} \). Moreover one can check that the conditions \( \ref{eq:3} \) on \((A, B, \tilde{C})\) imply that \( C \) fulfills \( \ref{eq:5} \).

That equation \( \ref{eq:4} \) defines a bijection between the set of all SP gluings of \( \phi_1 \) and \( \phi_2 \) and the set of all matrices \( C \) fulfilling condition \( \ref{eq:5} \), follows from the bijectivity stated in proposition \( \ref{prop:2} \).

One may wonder how the gluing matrix \( C \) of proposition \( \ref{prop:4} \) changes if one makes other choices of linearly independent Kraus representations of \( \phi_1 \) and \( \phi_2 \). From proposition \( \ref{prop:7} \) it is known that there is a bijective correspondence between the set of \( K(\phi) \times K(\phi) \) unitary matrices and the set of linearly independent Kraus representations of \( \phi \). From proposition \( \ref{prop:7} \) it follows that if the linearly independent Kraus representations are changed, then there exists a unitary \( K(\phi_1) \times K(\phi_1) \) matrix \( U_1 \), and a unitary \( K(\phi_2) \times K(\phi_2) \) matrix \( U_2 \), such that the new gluing matrix \( C' \) relates to the old as \( C' = U_1CU_1' \). One may note that this implies that the set of singular values of the gluing matrix is independent of the choice of linearly independent Kraus representations.

The following proposition shows that the Kraus number \( \ref{eq:2} \) of an SP gluing, is constrained by the Kraus numbers of the CPMs which are glued.

**Proposition 4** If the CPM \( \phi \) is an SP gluing of the CPMs \( \phi_1 \) and \( \phi_2 \), then
\[
\max(K(\phi_1), K(\phi_2)) \leq K(\phi) \leq K(\phi_1) + K(\phi_2). \tag{7}
\]
The left equality holds if and only if the gluing matrix \( C \) given by proposition \( \ref{prop:5} \) has \( \min(K(\phi_1), K(\phi_2)) \) singular values with value 1, counted with multiplicity. Moreover, \( K(\phi_1) + K(\phi_2) - K(\phi) \) is the number of singular values with value 1 of the matrix \( C \), counted with multiplicity.

**proof.** Let \( K_1 = K(\phi_1) \) and \( K_2 = K(\phi_2) \). By proposition 6 in \( \ref{prop:2} \) the Kraus number of a CPM \( \phi \) is equal to the number of non-zero eigenvalues of a representation matrix.
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Let \( F \), given by proposition 5 in [1], This is true regardless of the choice of basis of \( \mathcal{L}(H_T, H_S) \). Given linearly independent Kraus representations of \( \phi_1 \) and \( \phi_2 \), the same construction as in the proof of proposition 3 can be made. Hence, we complete the two sets of linearly independent Kraus representations to become bases. With respect to this choice of bases we find that the representation matrix only has a sub-matrix which is non-zero. This sub-matrix has the form

\[
F = \begin{bmatrix}
I_1 & C \\
C^\dagger & I_2
\end{bmatrix},
\]

where \( I_1 \) denotes the \( K_1 \times K_1 \) identity matrix, and \( I_2 \) the \( K_2 \times K_2 \) identity matrix. The number of non-zero eigenvalues of the total representation matrix is the number of non-zero eigenvalues of the sub-matrix \( F \). Without loss of generality we may assume \( K_1 \leq K_2 \). The matrix \( C \) can be transformed into an especially simple form by applying a singular value decomposition [1]. There exists a \( K_1 \times K_1 \) unitary matrix \( U_1 \) and a \( K_2 \times K_2 \) unitary matrix \( U_2 \) such that \( U_1 C U_2^\dagger = \tilde{C} \), where \( \tilde{C} \) is a \( K_1 \times K_2 \) matrix which is composed from a diagonal \( K_1 \times K_1 \) matrix, with the singular values as diagonal elements, and a \( K_1 \times (K_2 - K_1) \) zero matrix. (Singular values are always non-negative.)

\[
\tilde{C} = \begin{bmatrix}
r_1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
r_{K_1} & 0 & \ldots & 0
\end{bmatrix}.
\]

(8)

Let

\[
\tilde{F} = (U_1 \oplus U_2) F (U_1^\dagger \oplus U_2^\dagger) = \begin{bmatrix}
I_1 & \tilde{C} \\
\tilde{C}^\dagger & I_2
\end{bmatrix}.
\]

(9)

Since \( \tilde{F} \) is obtained from \( F \) by a unitary transformation, both have the same set of eigenvalues. One can check that the unitary matrix

\[
\tilde{U} = \begin{bmatrix}
\frac{1}{\sqrt{2}} I_{K_1} & \frac{1}{\sqrt{2}} I_{K_1} & 0 \\
\frac{1}{\sqrt{2}} I_{K_1} & -\frac{1}{\sqrt{2}} I_{K_1} & 0 \\
0 & 0 & I_{K_2 - K_1}
\end{bmatrix}
\]

(10)

diagonalizes \( \tilde{F} \), in such a way that the first \( K_1 \) eigenvalues are \( 1+r_k \) for \( k = 1, \ldots, K_1 \), the next \( K_1 \) eigenvalues are \( 1-r_k \) for \( k = 1, \ldots, K_1 \), and the remaining \( K_2 - K_1 \) eigenvalues are all 1. Since \( r_k \geq 0 \), the number of non-zero eigenvalues of \( \tilde{F} \) is \( K_1 + K_2 - N \), where \( N \) is the number of singular values with value 1. Since the number of non-zero singular values of \( C \) maximally can be \( \min(K_1, K_2) \), and since \( K_1 + K_2 - \min(K_1, K_2) = \max(K_1, K_2) \), the proposition follows.

The set of SP gluings of two given CPMs forms a convex set. For the rest of this section we find out the extreme points of this set. Let \( I(N, M) \) denote the set of all complex \( N \times M \) matrices \( C \) such that \( CC^\dagger \leq I_N \). For the sake of simplicity it is assumed that \( N \leq M \). Let \( EI(N, M) \) denote the set of complex \( N \times M \) matrices with \( CC^\dagger = I_N \).

**Lemma 2** \( I(N, M) \) is a convex set. \( EI(N, M) \) is the set of extreme points of \( I(N, M) \).

A complex \( N \times M \) matrix belongs to \( EI(N, M) \) if and only it has precisely \( N \) non-zero singular values, all of value 1 (if \( N \leq M \)).

**proof.** First note that the lemma is true if \( N = 1 \). Hence, we may in the following assume \( N \geq 2 \).
Gluing of completely positive maps

We have to prove that every element of \( I(N,M) \) can be formed as a convex combination of elements in \( EI(N,M) \). Let \( \{ \overline{t}_j \}_{j=1}^N \) be an orthonormal basis of \( \mathbb{C}^N \) and let \( \{ \overline{a}_k \}_{k=1}^N \) be an orthonormal set in \( \mathbb{C}^M \). (Regard \( t_j \) and \( a_k \) as column vectors.) Let \( D^{(0)} = \sum_{k=1}^N t_k a_k^\dagger \). For \( 1 \leq n \leq N-1 \), let \( D^{(n)} = - \sum_{k=n+1}^N a_k^\dagger t_k + \sum_{k=1}^{n-1} \overline{a}_k \overline{t}_k \). Then, \( D^{(N)} = -D^{(0)} \). All the matrices \( D^{(n)} \), \( n = 0, \ldots, N \) belong to \( EI(N,M) \). Define \( H^{(n)} = \frac{1}{n+1} D^{(0)} + \frac{n}{n+1} D^{(n)} \), for \( n = 0, \ldots, N \). By construction each \( H^{(n)} \) is in the convex hull of \( EI(N,M) \).

Let \( C \) be an arbitrary \( N \times M \) complex matrix. As such it can be decomposed using a singular value decomposition \( \overline{c} \). There exist non-negative real numbers \( r_n \), \( n = 1, \ldots, N \), an orthonormal basis \( \{ \overline{c}_k \}_{k=1}^N \) of \( \mathbb{C}^N \), and an orthonormal set \( \{ \overline{d}_k \}_{k=1}^N \) in \( \mathbb{C}^M \), such that \( C = \sum_{k=1}^N r_k \overline{c}_k \overline{d}_k^\dagger \). (Here the assumption \( N \leq M \) is used.) One can check that \( C \in EI(N,M) \) if and only if \( r_k \leq 1 \) for all \( k = 1, \ldots, N \). Assume \( C \in I(N,M) \). Without loss of generality we may assume that the orthonormal sets are ordered in such a way that \( r_1 \leq r_2 \leq \ldots \leq r_N \). Let \( \lambda_0 = r_1 \) and \( \lambda_n = r_{n+1} - r_n \) for \( n = 1, \ldots, N-1 \). Let \( \lambda_N = 1 - r_N \). By construction it follows that \( \lambda_n \geq 0 \). Moreover, \( \sum_{n=0}^N \lambda_n = 1 \). One can check that \( \sum_{n=0}^N \lambda_n H^{(n)} = C \). Hence, \( C \) is in the convex hull of \( EI(N,M) \).

Now it is to be proved that no element in \( EI(N,M) \) can be written as a non-trivial convex combination of two elements in \( I(N,M) \). Suppose \( C \in EI(N,M) \) is such that there exists \( C_1, C_2 \in I(N,M) \) and \( 0 < p < 1 \), such that \( C = pC_1 + (1-p)C_2 \). Then,

\[
I_N = CC^\dagger = (pC_1 + (1-p)C_2)(pC_1 + (1-p)C_2)^\dagger,
\]

since \( C \in EI(N,M) \). Moreover, since \( C_1, C_2 \in I(N,M) \) it follows that

\[
(pC_1 + (1-p)C_2)(pC_1 + (1-p)C_2)^\dagger \leq p^2 I_N + (1-p)^2 I_N + p(1-p)(C_2 C_1^\dagger + C_1 C_2^\dagger).
\]

By combining (11) and (12) and using \( 0 < p < 1 \) one finds that

\[
2I_N \leq C_2 C_1^\dagger + C_1 C_2^\dagger.
\]

If \( \overline{c} \) is an arbitrary normalized element of \( \mathbb{C}^N \), then it follows from (14) that

\[
1 \leq \Re(\overline{c} C_1 C_2^\dagger \overline{c}^\dagger).
\]

From \( C_1, C_2 \in I(N,M) \) and \( ||\overline{c}|| = 1 \) it follows that \( ||C_1 \overline{c}|| \leq 1 \) and \( ||C_2 \overline{c}|| \leq 1 \). The two last inequalities together with (15) can be true only if \( C_1 \overline{c} = C_2 \overline{c} \). Since \( \overline{c} \) is an arbitrary normalized vector it follows that \( C_1 = C_2 \). Hence, \( C = pC_1 + (1-p)C_2 = C_1 = C_2 \). This is a trivial convex combination. Hence, there is no element in \( EI(N,M) \) which is a non-trivial convex combination of elements in \( I(N,M) \). \( \square \)

**Proposition 5** The set of all SP gluings of two given CPMs \( \phi_1 \) and \( \phi_2 \) is convex. An SP gluing \( \phi \) in this set is an extreme point if and only if the matrix \( C \) of proposition B has precisely \( \min(K(\phi_1), K(\phi_2)) \) singular values with value 1, which occurs if and only if \( K(\phi) = \max(K(\phi_1), K(\phi_2)) \).

**proof.** Without loss of generality we may assume \( K(\phi_1) \leq K(\phi_2) \). The set of gluings matrices of proposition B with respect to some arbitrary choices of linearly independent Kraus representations of \( \phi_1 \) and \( \phi_2 \), is \( I(K(\phi_1), K(\phi_2)) \). According to lemma 2 this is a convex set. From lemma 2 we know that the set of extreme points is \( EI(K(\phi_1), K(\phi_2)) \). A matrix \( C \) is an element of \( EI(K(\phi_1), K(\phi_2)) \) if and only if \( C \)
has precisely $K(\phi_1)$ non-zero singular values, all with value 1. Since we have assumed $K(\phi_1) \leq K(\phi_2)$, the general condition is that there should be $\min(K(\phi_1), K(\phi_2))$ singular values, all with value 1. From proposition 6, we know that this occurs if and only if $K(\phi) = \max(K(\phi_1), K(\phi_2))$. □

3. Local subspace preserving gluings

In [1] the concepts of subspace local (SL) channels and local subspace preserving (LSP) channels, are introduced. Here these are used to construct gluings which are subspace local. In terms of the ‘evolution machines’ this corresponds to a combination of two machines which do not interact or share any correlated systems. This is an attempt to formalize the concept of a combination of two independently acting devices. LSP gluings are defined as follows:

**Definition 2** If $\Phi_1$ is a channel with source $\mathcal{H}_{s_1}$ and target $\mathcal{H}_{t_1}$, and if $\Phi_2$ is a channel with source $\mathcal{H}_{s_2}$ and target $\mathcal{H}_{t_2}$ then a trace preserving gluing of $\Phi_1$ and $\Phi_2$ is called a LSP gluing if $\Phi$ is SL from $(\mathcal{H}_{s_1}, \mathcal{H}_{s_2})$ to $(\mathcal{H}_{t_1}, \mathcal{H}_{t_2})$.

The reason why these gluings are called “LSP gluings” and not “SL gluings” is that (see proposition 5 in [1]).

The intersection between the set of SP channels and the set of subspace local channels, is the set of LSP channels (see proposition 5 in [1]).

By combining proposition 6, proposition 5 and proposition 2 in [1] the following proposition is obtained.

**Proposition 6** Let $\Phi_1$ be a trace preserving CPM with source space $\mathcal{H}_{s_1}$ and target space $\mathcal{H}_{t_1}$. Let $\Phi_2$ be trace preserving CPM with with source space $\mathcal{H}_{s_2}$ and target space $\mathcal{H}_{t_2}$. Let $\{V_n\}_{n=1}^N$ be a linearly independent Kraus representation of $\Phi_1$. Let $\{W_m\}_{m=1}^M$ be a linearly independent Kraus representation of $\Phi_2$. Then $\Phi$ is an LSP gluing of $\Phi_1$ and $\Phi_2$ if and only if $\Phi$ can be written

$$
\Phi(Q) = \sum_{n=1}^N V_n Q V_n^\dagger + \sum_{m=1}^M W_m Q W_m^\dagger + Q W Q^\dagger + W Q V^\dagger, \quad \forall Q \in \mathcal{L}(\mathcal{H}_S),
$$

with $V = \sum_{n=1}^N c_{1,n} V_n$, $W = \sum_{m=1}^M c_{2,m} W_m$, where the vectors $c_1 = [c_{1,n}]_{n=1}^N$ and $c_2 = [c_{2,m}]_{m=1}^M$ fulfill the conditions $||c_1||^2 = \sum_{n=1}^N |c_{1,n}|^2 \leq 1$ and $||c_2||^2 = \sum_{m=1}^M |c_{2,m}|^2 \leq 1$.

**Proposition 7** A trace preserving gluing of two channels is an LSP gluing if and only if the matrix $C$ given by proposition 6 has at most one non-zero singular value, counted with multiplicity.

One can note that the vectors $c_1$ and $c_2$ are not uniquely determined by the LSP gluing. However, the gluing matrix $C_{nm} = c_{1,n} c_{2,m}$ is. The same non-uniqueness affects the operators $V$ and $W$. However, the non-uniqueness is rather ‘mild’, since it is essentially limited to variations of a single complex number.

**Proof.** We begin with the “only if” part. By comparing proposition 6 and 5 one can see that if $c_1 = [c_{1,n}]_{n=1}^N$ and $c_2 = [c_{2,m}]_{m=1}^M$ fulfill the conditions of proposition 6 then the matrix $C$ of proposition 6 is $C = c_1 c_2^\dagger$. ($c_1$ and $c_2$ are regarded as column vectors.) Since a matrix on this form has at most one non-zero singular value counted with multiplicity, the “only if” part follows.
Gluing of completely positive maps

For the “if” part, let $C$ be the matrix resulting from proposition 8. If $C$ has at most one non-zero singular value (counted with multiplicity), then there exist vectors $\tilde{c}_1$ and $\tilde{c}_2$ such that $C$ can be written $C = \tilde{c}_1 \tilde{c}_2^\dagger$. In case $C = 0$ the condition for being an LSP gluing is clearly fulfilled. Hence, without loss of generality we may assume $C$ is not the zero matrix. Hence, $\tilde{c}_1$ and $\tilde{c}_2$ must be non-zero. The matrix $C$ fulfills $CC^\dagger \leq I_N$. This condition is, for the special form of $C$ considered here, translated into $||\tilde{c}_1||^2 ||\tilde{c}_2||^2 \leq 1$. Let $c_1 = \tilde{c}_1(||\tilde{c}_2||/||\tilde{c}_1||)^{1/2}$ and $c_2 = \tilde{c}_2( ||\tilde{c}_1||/||\tilde{c}_2||)^{1/2}$, then $c_1$ and $c_2$ fulfill the conditions of proposition 8.

**Proposition 8** Let $\Phi_1$ be a trace preserving CPM and let $\Phi_2$ be a trace preserving CPM such that $K(\Phi_2) = 1$. Every trace preserving gluing of $\Phi_1$ and $\Phi_2$ is an LSP gluing.

As a direct corollary of this proposition it follows that every trace preserving gluing of a trace preserving CPM and an identity CPM is an LSP gluing.

**proof.** The matrix $C$ of proposition 8 is a $K(\Phi_1) \times K(\Phi_2)$ matrix. Since $K(\Phi_2) = 1$ this matrix can have at most one non-zero singular value counted with multiplicity. Hence, by proposition 8 the the statement of the proposition follows.

Being an LSP gluing puts a very stringent condition on the relation between the Kraus numbers of the gluing and the glued channels, as the following proposition shows. This proposition follows directly by combining proposition 4 and proposition 8.

**Proposition 9** If a trace preserving CPM $\Phi$ is an LSP gluing of the trace preserving CPMs $\Phi_1$ and $\Phi_2$, then $K(\Phi_1) + K(\Phi_2) - 1 \leq K(\Phi) \leq K(\Phi_1) + K(\Phi_2)$.

4. Some illustrations

In this section the theory is illustrated with some simple examples. In addition to serving as illustration, some of these derivations also indicate possible directions for future studies. We attempt to compare the ability of LSP and SP gluings to preserve superposition, in some sense.

A quantum channel is called unitary on $H_{s1}$ if it can be written $\Phi_1(Q) = U_1QU_1^\dagger$, where $U_1$ is unitary operator on $H_{s1}$. Given a unitary channel $\Phi_1$ on $H_{s1}$ and a unitary channel $\Phi_2$ on $H_{s2}$, what is the set of trace preserving gluings of these channels? Both the channels have Kraus number 1. Hence, all gluings of them have to be LSP (proposition 8). By proposition 8 all those gluings can be written

$$\Phi(Q) = U_1QU_1^\dagger + U_2QU_2^\dagger + cU_1QU_1^\dagger + c^*U_2QU_1^\dagger,$$

where $|c| \leq 1$. Using a polar decomposition of $c$ into $c = re^{i\theta}$, (17) can be rewritten as

$$\Phi(Q) = (1-r)U_1QU_1^\dagger + (1-r)U_2QU_2^\dagger + r(U_1 + e^{-i\theta}U_2)Q(U_1 + e^{-i\theta}U_2)^\dagger.$$  (18)

It is straightforward to check that $U_1 + e^{-i\theta}U_2$ is a unitary operator. Let us focus on the two extreme cases $r = 0$ and $r = 1$ to see what happens with an initial superposition of states localized in the two subspaces. In case $r = 1$, $\Phi_{r=1}$ is unitary and hence maps pure states to pure states. In some sense these channels preserve the superposition between the two subspaces. To see this, let $|\psi\rangle \in H_S$ be normalized but else arbitrary. It can be written $|\psi\rangle = \alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle$, where $|\psi_1\rangle$ is some normalized state in $H_{s1}$ and $|\psi_2\rangle$ some normalized state in $H_{s2}$. Then $\Phi(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi'|$, where $|\psi'| = \alpha_1U_1|\psi_1\rangle + \alpha_2e^{-i\theta}U_2|\psi_2\rangle$. Hence, the weights
$|\alpha_1|^2$ and $|\alpha_2|^2$ of the two subspaces in the superposition, have not changed under the mapping. In some sense the ‘amount’ of superposition is preserved under these types of channels. In case $r = 0$, the channel $\Phi_{r=0}$ completely destroys the superposition, in the sense that any superposition between states localized in the two subspaces is turned into a mixture of states localized in the two subspaces. $\Phi_{r=0}(|\psi\rangle\langle\psi|) = |\alpha_1|^2U_1|\psi_1\rangle\langle\psi_1|U_1^\dagger + |\alpha_2|^2U_2|\psi_2\rangle\langle\psi_2|U_2^\dagger$. Hence, in one extreme $r = 1$ superposition is preserved, and in the other extreme $r = 0$ superposition is completely destroyed. From (18) follows that intermediate choices of $r$ give a partial destruction of superposition. It has to be emphasized that all these channels, no matter the choice of $r$, are possible to perform subspace locally. Hence, in this specific case, the question of whether the channel preserve superposition or not, is independent of whether the channel is possible to perform subspace locally or not. We will see however, that in the case of other more complicated gluings, LSP gluings seems to be less able to preserve superposition, compared to general SP gluings.

It is to be noted that a channel do not need to be implemented subspace locally just because it is subspace local. The subspace locality only states that it can be implemented subspace locally. An example is $\Phi_r$ with $0 < r < 1$. Since $\Phi_r = (1 - r)\Phi_{r=0} + r\Phi_{r=1}$, and since $r$ can be interpreted as a probability, follows that $\Phi_r$ can be implemented by using a random generator choosing the channel $\Phi_{r=0}$ with probability $r$ and channel $\Phi_{r=1}$ with probability $(1 - r)$. This is not a local implementation since the outcome of the random generator has to be distributed to both locations. However, this channel is subspace local, as have been demonstrated above.

For comparison we here consider a channel which is SP but not LSP. Let the unitary channels $\Phi_a$ and $\Phi_b$ be defined by

$$\Phi_a(Q) = U_aQU_a^\dagger, \quad \Phi_b(Q) = U_bQU_b^\dagger,$$

$$U_a = V_{s1} + P_{s1}, \quad U_b = V_{s2} + P_{s2}, \quad (19)$$

where $V_{s1}$ and $V_{s2}$ satisfy, $V_{s1}V_{s1}^\dagger = V_{s1}^\dagger V_{s1} = P_{s1}$ and $V_{s2}V_{s2}^\dagger = V_{s2}^\dagger V_{s2} = P_{s2}$. Moreover, we assume that $\{V_{s1}, P_{s1}\}$ is a linearly independent set, and that $\{V_{s2}, P_{s2}\}$ is a linearly independent set. (Both $\Phi_a$ and $\Phi_b$ are LSP gluings of the type considered in the previous example.) Consider the convex combination $\Phi = \frac{1}{2}\Phi_a + \frac{1}{2}\Phi_b$. The mapping $\Phi$ can be realized with a random generator determining which of the operations $\Phi_a$ or $\Phi_b$ is performed, each with probability one half. Hence, with probability one half a unitary channel is operating locally in subspace $H_{s1}$ and the identity CPM acts on $H_{s2}$. With probability one half the opposite happens. Because of the construction with the shared outcome of the random generator, this implementation is not subspace local. As the previous example has shown, the channel may still be subspace local. However, it is possible to prove that $\Phi$ is not an LSP channel. The channel $\Phi = \frac{1}{2}\Phi_a + \frac{1}{2}\Phi_b$ can be written

$$\Phi(Q) = \Phi_1(Q) + \Phi_2(Q) + \frac{1}{2}V_{s1}QP_{s2} + \frac{1}{2}P_{s1}QV_{s2}^\dagger + \frac{1}{2}V_{s2}QP_{s1} + \frac{1}{2}P_{s2}QV_{s1}^\dagger, \quad (20)$$

where $\Phi_1$ and $\Phi_2$ are given by $\Phi_1(Q) = \frac{1}{2}V_{s1}QV_{s1}^\dagger + \frac{1}{2}P_{s1}QV_{s1}^\dagger$ and $\Phi_2(Q) = \frac{1}{2}V_{s2}QV_{s2}^\dagger + \frac{1}{2}P_{s2}QP_{s2}$. $\Phi_1$ can be regarded as a trace preserving CPM with source and target $H_{s1}$ (but it is not trace preserving if regarded as a CPM with source and target $H_{s1}$). Likewise $\Phi_2$ is trace preserving with source and target $H_{s2}$. The set $\{\frac{1}{\sqrt{2}}V_{s1}, \frac{1}{\sqrt{2}}P_{s1}\}$ is a linearly independent Kraus representation of $\Phi_1$, and $\{\frac{1}{\sqrt{2}}V_{s2}, \frac{1}{\sqrt{2}}P_{s2}\}$ is a linearly independent Kraus representation of $\Phi_2$. Equation (20)
is on the form required by proposition \( \Psi \) with matrix \( C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). This matrix has two non-zero singular values (both are 1). By proposition \( \Phi \) it follows that \( \Phi \) cannot be an LSP gluing. We conclude that \( \Phi \) is an SP gluing, but not an LSP gluing, of the two channels \( \Phi_1 \) and \( \Phi_2 \). Hence, \( \Phi \) is not subspace local.

A very simple type of channel maps every state (pure or mixed) to a fixed pure state. Let \( |\psi_1\rangle \in \mathcal{H}_{s_1} \) be normalized. Define \( \Phi_1 \) to be the channel which maps any density operator on \( \mathcal{H}_{s_1} \) to the density operator \( |\psi_1\rangle\langle\psi_1| \). If \( \{ |s_{1k}\rangle \}_k \) is an arbitrary orthonormal basis of \( \mathcal{H}_{s_1} \), then a linearly independent Kraus representation of \( \Phi_1 \) is \( \{ |\psi_1\rangle\langle s_{1k}| \}_k \). Likewise a similar channel \( \Phi_2 \), with source and target \( \mathcal{H}_{s_2} \), which maps all states to a pure state \( |\psi_2\rangle \in \mathcal{H}_{s_2} \), has a linearly independent Kraus representation \( \{ |\psi_2\rangle\langle 2| \}_l \). All trace preserving gluings of \( \Phi_1 \) and \( \Phi_2 \) can be written
\[
\Phi(Q) = |\psi_1\rangle\langle\psi_1| \text{Tr}(P_{s_1}Q) + |\psi_2\rangle\langle\psi_2| \text{Tr}(P_{s_2}Q)
+ |\psi_1\rangle\langle\psi_2| \sum_{kl} C_{kl} \langle s_{1k}|Q|s_{2l}\rangle + |\psi_2\rangle\langle\psi_1| \sum_{kl} C_{kl}^* \langle s_{2l}|Q|s_{1k}\rangle, \tag{21}
\]
for all \( Q \in \mathcal{L}(\mathcal{H}_S) \). In the special case of LSP gluings, \( \Phi \) takes the form \( \Phi(Q) = |\psi_1\rangle\langle\psi_1| \text{Tr}(P_{s_1}Q) + |\psi_2\rangle\langle\psi_2| \text{Tr}(P_{s_2}Q) + |\psi_1\rangle\langle\psi_2| (\alpha|a\rangle\langle b| + |\beta\rangle\langle\beta|Q|a\rangle + |\beta\rangle\langle\beta|Q|a\rangle\langle b| + |\beta\rangle\langle\beta|Q|a\rangle) \), where \( |a\rangle \in \mathcal{H}_{s_1} \) and \( |b\rangle \in \mathcal{H}_{s_2} \) fulfill \( ||a|| \leq 1 \) and \( ||b|| \leq 1 \), but are otherwise arbitrary. If we choose \( |a\rangle = |\psi_1\rangle \) and \( |b\rangle = |\psi_2\rangle \), the result is a channel of the form
\[
\Phi(Q) = |\psi_1\rangle\langle\psi_1| \text{Tr}(P_{s_1}Q) + |\psi_2\rangle\langle\psi_2| \text{Tr}(P_{s_2}Q)
+ |\psi_1\rangle\langle\psi_2| |\psi_2|\langle\psi_1|Q|\psi_1| \langle\psi_1|Q|\psi_2| \psi_2 \rangle \tag{22}
\]
Any pure state on the form \( |\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle \) is left intact by channel \( \Phi \). Hence, in this case the superposition between the two subspaces is preserved. Let \( |\psi'\rangle = \alpha|\psi_1^+\rangle + \beta|\psi_2^+\rangle \), where \( |\psi_1^+\rangle \) is any state in \( \mathcal{H}_{s_1} \) which is orthogonal to \( |\psi_1\rangle \), and analogously for \( |\psi_2^+\rangle \). (Hence, \( \mathcal{H}_{s_1} \) and \( \mathcal{H}_{s_2} \) are at least two-dimensional.) It follows that \( \Phi(|\psi'\rangle\langle\psi'|) = |\alpha|^2 |\psi_1\rangle\langle\psi_1| + |\beta|^2 |\psi_2\rangle\langle\psi_2| \). In this case the superposition is destroyed and leaves a mixture of states localized in the two subspaces. Hence, this channel preserves the superposition only for a quite limited set of input states. Now we turn to the more general SP gluings. Assume \( \dim(\mathcal{H}_{s_1}) = \dim(\mathcal{H}_{s_2}) \). Then it is possible to choose \( C_{kl} = \delta_{kl} \). This gluing is such that any pure input state on the form \( \alpha|s_{1k}\rangle + \beta|s_{2k}\rangle \) is mapped to the pure state \( \alpha|\psi_1\rangle + \beta|\psi_2\rangle \). Hence, in general the states will not be preserved, but in some sense the ‘amount’ of superposition is preserved. In this case a number of families of input states are mapped in a way that ‘preserves’ the superposition, in contrast with the case of LSP gluings, where there is only one such family. This indicates that SP gluings have better abilities to preserve superposition than do LSP gluings.

Finally we consider two rather simple channels to see under what conditions these are SP or LSP. Let \( \mathcal{H_a} \) be some arbitrary finite-dimensional Hilbert space and let \( \sigma \) be some density operator on \( \mathcal{H_a} \). Let \( \rho \) denote density operators on a space \( \mathcal{H}_S \), and define the channel \( \Lambda(\rho) = \rho \otimes \sigma \). Hence, \( \Lambda \) has source space \( \mathcal{H}_S \) and target space \( \mathcal{H}_S \otimes \mathcal{H_a} \). Given an arbitrary decomposition \( \mathcal{H}_S = \mathcal{H}_{s_1} \otimes \mathcal{H}_{s_2} \), \( (\mathcal{H}_{s_1} \otimes \mathcal{H}_{s_2} \otimes \mathcal{H}_a) \) have at least one-dimensional), is \( \Lambda \) SP from \( (\mathcal{H}_{s_1} \otimes \mathcal{H}_{s_2} \otimes \mathcal{H}_a) \) to \( (\mathcal{H}_{s_1} \otimes \mathcal{H}_a, \mathcal{H}_{s_2} \otimes \mathcal{H}_a) \) Moreover, is it LSP? The channel \( \Lambda \) is SP since \( \text{Tr}(P_{s_1} \otimes I_a \Lambda(Q)) = \text{Tr}(P_{s_1}Q \otimes \sigma) = \text{Tr}(P_{s_1}Q) \), for any \( Q \in \mathcal{L}(\mathcal{H}_S) \). Hence, by proposition 4 in \( \Psi_1 \) is an SP channel. We wish to find out whether \( \Lambda \) is an LSP channel, or not. Since \( \Lambda \) is SP, it has to be a gluing of two trace preserving CPMs. One may verify that these two are \( \Psi_1(Q) = P_{s_1}Q P_{s_1} \otimes \sigma \) and \( \Psi_2(Q) = P_{s_2}Q P_{s_2} \otimes \sigma \). Let \( \{ \lambda_k \}_{k=1}^\infty \) denote the non-zero eigenvalues and
\( \{ |\lambda_k \rangle \}_{k=1}^K \) a corresponding orthonormal set of eigenvectors of \( \sigma \). In a slightly odd notation \( \{ \sqrt{\lambda_k} |\lambda_k \rangle P_{s1} \}_{k=1}^K \) is a linearly independent Kraus representation of \( \Phi_1 \), and \( \{ \sqrt{\lambda_k} |\lambda_k \rangle P_{s2} \}_{k=1}^K \) of \( \Phi_2 \). (A more strict notation would be \( \{ \sum \sqrt{\lambda_k} |s_k \rangle \langle s_k| \}_{k=1}^K \) for some orthonormal basis \( \langle s_k| \rangle \) of \( \mathcal{H}_a \) ) When applying proposition 5 to \( \Phi \), with these choices of linearly independent Kraus representations, it is found that \( \Phi = I_{K} \), where \( I_{K} \) denotes the \( K \times K \) identity matrix. Hence, by using proposition 7 it is found that \( \Phi \) can be LSP from \( (\mathcal{H}_s \otimes \mathcal{H}_a, \mathcal{H}_s \otimes \mathcal{H}_a) \) to \( (\mathcal{H}_s \otimes \mathcal{H}_a, \mathcal{H}_s \otimes \mathcal{H}_a) \), and if only if \( K = 1 \). Hence, \( \Phi \) is LSP if and only if the density operator \( \sigma \) represents a pure state.

The, in a sense, ‘opposite’ channel to \( \Lambda \) is the partial trace. \( \text{Tr}_a \) is SP from \( (\mathcal{H}_s \otimes \mathcal{H}_a, \mathcal{H}_s \otimes \mathcal{H}_a) \) to \( (\mathcal{H}_s, \mathcal{H}_s) \), since \( \text{Tr}(P_{s1} \text{Tr}_a(Q)) = \text{Tr}(P_{s1} \otimes I_1, Q) \) for all \( Q \in \mathcal{L}(\mathcal{H}_S \otimes \mathcal{H}_a) \). Again we refer to proposition 4 of 2 to conclude that partial trace is SP. As such it is a trace preserving gluing of two trace preserving CPMs, one with linearly independent Kraus representation \( \{ P_{s1}(a_k)| \}_{k=1}^K \) and one with linearly independent Kraus representation \( \{ P_{s2}(a_k)| \}_{k=1}^K \), where \( \{ |a_k\rangle \}_{k=1}^K \) is an arbitrary orthonormal basis of \( \mathcal{H}_a \). The matrix \( C \) given by proposition 8 with respect to the chosen linearly independent Kraus representations, is \( C = I_K \). Hence, the partial trace is LSP if and only if \( \mathcal{H}_a \) is one-dimensional. We can conclude the following.

**Proposition 10** Let \( \mathcal{H}_a \) be finite-dimensional and let \( \sigma \) be a density operator on \( \mathcal{H}_a \).

- \( \Lambda(Q) = Q \otimes \sigma \), \( \forall Q \in \mathcal{L}(\mathcal{H}_S) \) is an SP channel from \((\mathcal{H}_s, \mathcal{H}_a)\) to \((\mathcal{H}_s \otimes \mathcal{H}_a, \mathcal{H}_s \otimes \mathcal{H}_a)\). It is LSP from \((\mathcal{H}_s, \mathcal{H}_a)\) to \((\mathcal{H}_s \otimes \mathcal{H}_a, \mathcal{H}_s \otimes \mathcal{H}_a)\) if and only if \( \sigma \) is pure.

- \( \text{Tr}_a \) is an SP channel from \((\mathcal{H}_s \otimes \mathcal{H}_a, \mathcal{H}_s \otimes \mathcal{H}_a)\) to \((\mathcal{H}_s, \mathcal{H}_a)\). It is LSP from \((\mathcal{H}_s \otimes \mathcal{H}_a, \mathcal{H}_s \otimes \mathcal{H}_a)\) to \((\mathcal{H}_s, \mathcal{H}_a)\) if and only if \( \dim(\mathcal{H}_a) = 1 \).

Except for providing some examples of subspace non-local channels, the channels of proposition 11 can be used as building blocks to construct the set of SP channels from the set of LSP channels. Proposition 11 and 12 both show that every SP channel can be constructed from LSP channels and a final partial trace. The reason why both have been included is that while proposition 11 highlights the role of the partial trace, proposition 12 is perhaps more intuitively accessible, since it constructs the channel as a more clearcut action on a system-ancilla decomposition.

**Proposition 11** A channel \( \Phi \) is SP from \((\mathcal{H}_s, \mathcal{H}_a)\) to \((\mathcal{H}_s, \mathcal{H}_a)\) if and only if there exists a Hilbert space \( \mathcal{H}_a \) and a channel \( \Psi \), which is LSP from \((\mathcal{H}_s, \mathcal{H}_a)\) to \((\mathcal{H}_s \otimes \mathcal{H}_a, \mathcal{H}_s \otimes \mathcal{H}_a)\) and such that

\[
\Phi(Q) = \text{Tr}_a \Psi(Q), \quad \forall Q \in \mathcal{L}(\mathcal{H}_S).
\]

**Proof.** Since \( \Phi \) is a composition of two SP channels, the “if” part follows. To prove the “only if” part, we use that every Kraus representation of an SP CPM can be written \{ \( V_{1,k} + V_{2,k} \) \}_{k} \), where \( P_{11} V_{1,k} P_{s1} = V_{1,k} \) and \( P_{12} V_{2,k} P_{s2} = V_{2,k} \) (proposition 1 in 2). This Kraus representation can be chosen to be finite if the source and target space are finite-dimensional (see proposition 6 in 2), which they are by assumption. If the Kraus representation contains \( K \) elements, let \( \mathcal{H}_a \) be a \( K \)-dimensional Hilbert space and let \( \{ |a_k\rangle \}_{k=1}^K \) be an arbitrary orthonormal basis of \( \mathcal{H}_a \). Let the operators \( V_1 \) and \( V_2 \) be defined by \( V_1 = \sum_{k=1}^K |a_k\rangle V_{1,k} \) and \( V_2 = \sum_{k=1}^K |a_k\rangle V_{2,k} \). One can show that the CPM \( \Psi \) defined by the Kraus representation \{ \( V_1 + V_2 \) \} is trace preserving and fulfills equation 23. \( \Psi \) is a gluing of a trace preserving CPM with linearly independent Kraus representation \{ \( V_1 \) \} (with source \( \mathcal{H}_s \) and target \( \mathcal{H}_s \otimes \mathcal{H}_a \) ) and a trace preserving CPM with linearly independent Kraus representation \{ \( V_2 \) \}. Since \( \Psi \)
Hence, we may have single particle states, the vacuum state, as well as various linear combinations of these, as input states. The same construction is made for the input of the first device. The total Hilbert space is \( H_1 \otimes H_2 \). By construction of subspace gluings of two arbitrary channels, concentrate on gluings of a channel and an identity channel. The identity channel may correspond to a machine which does nothing with the particle. We moreover assume identical source and target spaces, and identical decompositions of these. Hence, we consider gluings of the channel \( \Phi_1 \) with source and target \( H_1 \), and the identity channel \( I_2 \) with source and target \( H_2 \). The relevant part of the occupation state space of the input of the first device is essentially \( H_1 = H_1 \otimes \text{Sp}\{\{0\}\} \). The state \( \{0\} \) is the 'vacuum state', with no particle present. Hence, we may have single particle states, the vacuum state, as well as various linear combinations of these, as input states. The same construction is made for the input of the other machine. The total Hilbert space is \( H_1 \otimes H_2 \).

**Proposition 12** A channel \( \Phi \) is SP from \( (H_{a1}, H_{a2}) \) to \( (H_{t1}, H_{t2}) \) if and only if there exists a Hilbert space \( H_a \), a normalized state \( \{a\} \in H_a \), and a channel \( \Psi' \), which is LSP from \( (H_{a1} \otimes H_a, H_{a2} \otimes H_a) \) to \( (H_{t1} \otimes H_a, H_{t2} \otimes H_a) \) and such that

\[
\Phi(Q) = \text{Tr}_a \Psi'(Q \otimes |a\rangle\langle a|), \quad \forall Q \in \mathcal{L}(H_S).
\]

**Proof.** The "if" part follows as in the proof of proposition 11. For the "only if" part, let \( H_a \), \( \{\{a\}\}_i \), \( V_1 \), and \( V_2 \) be as in the proof of proposition 11. The set \( \{V_i|a_i\rangle\}_i \) is a linearly independent Kraus representation of a trace preserving CPM with source \( H_{a1} \otimes H_a \) and target \( H_{t1} \otimes H_a \). Likewise \( \{V_2|a_i\rangle\}_i \) is a linearly independent Kraus representation of a trace preserving CPM with source \( H_{a2} \otimes H_a \) and target \( H_{t2} \otimes H_a \). We construct \( \Psi' \) as a trace preserving gluing of these two channels. Let \( |a\rangle \in H_a \) be an arbitrary normalized state. Define \( \Psi' \) by

\[
\Psi'(Q) = \sum_{i=1}^K V_i |a_i\rangle \langle a_i| V_1^\dagger + \sum_{i=1}^K V_2 |a_i\rangle \langle a_i| V_2^\dagger + V_1 |a\rangle \langle a| V_1^\dagger + V_2 |a\rangle \langle a| V_2^\dagger \quad \text{for all} \quad Q \in \mathcal{L}(H_S \otimes H_a).
\]

Since \( \{a\} = \sum_{k=1}^K c_k |a_k\rangle \) for some complex numbers \( \{c_k\}_k \) such that \( \sum_{k=1}^K |c_k|^2 = 1 \), it follows by proposition 6 that \( \Psi' \) is an LSP gluing from \( (H_{a1} \otimes H_a, H_{a2} \otimes H_a) \) to \( (H_{t1} \otimes H_a, H_{t2} \otimes H_a) \). One can check that \( \Psi' \) fulfills equation (24). \( \square \)

5. The non-uniqueness of gluings

The fact that gluings are not unique indicates that there is some aspect of the joint evolution which is not captured by the two channels alone. If we return to the picture of two evolution machines and a single particle, it means that although we know precisely how each of the machines alone handles a particle, that knowledge is not enough to deduce how the two machines act jointly on a superposition. In the case of SP gluings this is perhaps not very surprising, since the SP gluing allows the machines to interact with each other, or share some correlated resources like entangled pairs of particles. If we accept the definition of subspace locality put forward in \(1\), the two machines in an LSP gluing should truly be independent of each other. It may seem a reasonable guess that the LSP gluing should be uniquely determined by the channels \( \Phi_1 \) and \( \Phi_2 \). However, by comparison of propositions 9 and 10, one finds that the set of gluings is reduced when the assumption of subspace locality is added, but there is still non-trivial non-uniqueness. To understand the remaining non-uniqueness we change perspective on this problem.

The definition of subspace locality is based on a second quantization of the Hilbert space of the system, as described in \(1\). To simplify the discussion we, instead of LSP gluings of two arbitrary channels, concentrate on gluings of a channel and an identity channel. The identity channel may correspond to a machine which does nothing with the particle. We moreover assume identical source and target spaces, and identical decompositions of these. Hence, we consider gluings of the channel \( \Phi_1 \) with source and target \( H_1 \), and the identity channel \( I_2 \) with source and target \( H_2 \). The relevant part of the occupation state space of the input of the first device is essentially \( H_1 = H_1 \otimes \text{Sp}\{\{0\}\} \). The state \( \{0\} \) is the 'vacuum state', with no particle present. Hence, we may have single particle states, the vacuum state, as well as various linear combinations of these, as input states. The same construction is made for the input of the other machine. The total Hilbert space is \( H_1 \otimes H_2 \). By construction of subspace
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locality \[1\] there corresponds a channel \(\tilde{\Phi}_1 \otimes \tilde{I}_2\) to every gluing of the channels \(\Phi_1\) and \(I_2\). The channel \(\tilde{\Phi}_1\) has source and target \(\tilde{\mathcal{H}}_1\), and \(\tilde{I}_2\) is the identity channel with source and target \(\tilde{\mathcal{H}}_2\). The channel \(\tilde{\Phi}_1 \otimes \tilde{I}_2\) describes the action on the occupation state. If \(\Phi_1\) has linearly independent Kraus representation \(\{V_k\}_k\), it can be shown that \(\tilde{\Phi}_1\) has to be on the form

\[
\tilde{\Phi}_1(Q) = \sum_k V_k Q V_k^\dagger + |0\rangle\langle 0| Q^\dagger Q, \quad \forall Q \in \mathcal{L}(\mathcal{H}_s \oplus \text{Sp}\{|0\rangle\}),
\]

(25)

where the operator \(V\) is

\[
V = \sum_k c_k V_k,
\]

(26)

where the complex numbers \(c_k\) fulfill \(\sum_k |c_k|^2 \leq 1\). (Equation (26) is equation (6) in the proof of proposition 2 in \[1\].) In words, \(\tilde{\Phi}_1\) is a trace preserving gluing of \(\Phi_1\) and a CPM which maps the vacuum state to the vacuum state. Hence, the operator \(V\) results from the gluing of these two channels, which explains the form of \(V\).

In view of equation (26) one can see what is missing in the description of the device. The channel \(\tilde{\Phi}_1\) only describes what happens when there is a particle ‘fully present’ in the input of the non-trivial device. The missing part is provided by the operator \(V\), which describes what happens to linear combinations of single particle states and the vacuum state. Hence, a complete description of the machine in this context is the channel \(\tilde{\Phi}_1\). This channel can equivalently be described as the pair \((\Phi_1, V)\). In an LSP gluing (see proposition \[3\], two such pairs, one for each of the two machines, uniquely determine the gluing. This reflects the fact that when an evolution device acts on a superposition, there is, in some sense, an additional ‘degree of freedom’ involved, namely the presence non-presence of the particle. The action on this additional degree of freedom has to be specified. When this is done, the LSP gluing is uniquely determined. This restores the intuitive notion that the joint action of two independently acting devices should be possible to describe using the knowledge of the action of the two devices alone, and explains the non-uniqueness of LSP gluings. For more examples and discussions, in a more specific context, the reader is referred to \[7\], where these matters are discussed in terms of single-particle two-path interferometry.

6. Summary

The concept of gluing of completely positive maps is introduced. To give a brief description of this concept, consider a quantum system with corresponding Hilbert space \(\mathcal{H}\). This state space is decomposed in an orthogonal sum of two subspaces \(\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2\). Operations on the state of the quantum system is described by trace preserving CPMs (channels), which map density operators on \(\mathcal{H}\) to density operators on \(\mathcal{H}\). Suppose channel \(\Phi\) is such that when an input state is localized in \(\mathcal{H}_1\), the returned state is again localized in subspace \(\mathcal{H}_1\). By ‘localized’ is intended that \(P_1 \rho P_1 = \rho\) where \(\rho\) is the density operator and \(P_1\) is the projection operator onto \(\mathcal{H}_1\). Assume that this restricted mapping can be described by the channel \(\Phi_1\). Likewise if the initial state is localized in subspace \(\mathcal{H}_2\), the returned state is localized in \(\mathcal{H}_2\), and this mapping is described by the channel \(\Phi_2\). The ‘global’ channel \(\Phi\) is an example of a gluing of the two channels \(\Phi_1\) and \(\Phi_1\). The channel \(\Phi\) is not uniquely
determined by $\Phi_1$ and $\Phi_2$. As shown here, there exist several possible trace preserving gluings of $\Phi_1$ and $\Phi_2$.

An expression is derived (proposition 3), by which it is possible to generate all subspace preserving (SP) gluings of two given CPMs, with respect to arbitrary linearly independent Kraus representations of these. From this follows the construction of all trace preserving gluings of given pairs of channels.

Using a proposed definition of subspace locality of quantum operations [1], it is possible to express all trace preserving gluings of trace preserving CPMs which also fulfill the property of being subspace local (proposition 6). Intuitively an operation which is subspace local can be regarded as being caused by two independent evolution machines. It is intended that these two devices do not interact or share any correlated resources like entangled pairs of particles. The fact that gluings are not uniquely determined by the glued channels, is discussed. We focus on the difference between the general trace preserving gluings of channels and the subspace local gluings. It is shown that even if the gluing is subspace local, the gluings are still not uniquely determined. It is argued that this non-uniqueness is due to that the channels $\Phi_1$ and $\Phi_2$, given as characterizations of the two machines, are not full descriptions of these two devices, in this context. It is suggested that a more complete description of each of the two machines is given by a pair $(\Phi_1, V)$ where $V$ is a linear map from the source space of $\Phi_1$ to its target space. It is shown that a subspace local gluing is characterized by two such pairs, one for each device. Except for providing insight in the nature of gluings, this discussion also serves as a conceptual illustration of the results obtained in [1]. The developed theory is illustrated with some examples of gluings.

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