Near-Linear Time Approximation Schemes for Clustering in Doubling Metrics

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Abstract
We consider the classic Facility Location, $k$-Median, and $k$-Means problems in metric spaces of doubling dimension $d$. We give nearly-linear time approximation schemes for each problem. The complexity of our algorithms is $2^{O((1/\varepsilon)\log^{d+2} n)} n \log^4 n + 2^{O(d)} n \log^3 n$, making a significant improvement over the state-of-the-art algorithm which runs in time $n^{(d/\varepsilon)^{O(d)}}$.

1 Introduction

The $k$-Median and $k$-Means problems are classic clustering problems that are highly popular for modeling the problem of computing a “good” partition of a set of points of a metric space into $k$ parts so that points that are “close” should be in the same part. Since a good clustering of a dataset allows to retrieve information from the underlying data, the $k$-Median and $k$-Means problems are the cornerstone of various approaches in data analysis and machine learning. The design of efficient algorithms for these clustering problems has thus become an important challenge.

The input for the problems is a set of points in a metric space and the objective is to identify a set of $k$ centers $C$ such that sum of the $p$th power of the distance from each point of the metric to its closest center in $C$ is minimized. In the $k$-Median problem, $p$ is set to 1 while in the $k$-Means problem, $p$ is set to 2. In general metric spaces both problems are known to be APX-hard, and this hardness even extends to Euclidean spaces of any dimension $d = \Omega(\log n)$ [3], both problems remain NP-hard for points in $\mathbb{R}^d$ (see Section 1.3 for more related work).

Thus to bypass these hardness results, researchers have considered low-dimensional inputs like Euclidean spaces of fixed dimension or more generally metrics of fixed doubling dimension. There has been a large body of work to design good tools for clustering in metrics of fixed doubling dimension, from the general result of Talwar [26] to very recent coreset constructions for clustering problems [19]. In their seminal work, Arora et al. [2] gave a polynomial time approximation scheme (PTAS) for $k$-Median in $\mathbb{R}^2$ which generalizes to a quasi-polynomial time approximation scheme (QPTAS) for inputs in $\mathbb{R}^d$. This result was improved in two ways. First by Talwar [26] who generalized the result to any metric space of fixed doubling dimension. Second by Kolliopoulos and Rao [21] who obtained an $f(\varepsilon, d) \cdot n \log^{d+6} n$ time algorithm for $k$-Median in $d$-dimensional Euclidean space. Unfortunately, Kolliopoulos and Rao’s algorithm relies on the Euclidean structure

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of the input and does not immediately generalize to low doubling metric. Thus, until recently the only result known for $k$-Median in metrics of fixed doubling dimension was a QPTAS. This was also the case for slightly simpler problems such as Uniform Facility Location. Moreover, as pointed out in [10], the classic approach of Arora et al. [2] cannot work for the $k$-Means problem. Thus no efficient algorithms were known for the $k$-Means problem, even in the plane.

Recently, Friggstad et al. [15] and Cohen-Addad et al. [12] showed that the classic local search algorithm for the problems gives a $(1 + \varepsilon)$-approximation in time $n^{1/\varepsilon O(d)}$ in Euclidean space and planar graphs [12] and in time $n^{(d/\varepsilon)O(d)}$ in metrics of fixed doubling dimension [15]. More recently Cohen-Addad [10] showed how to speed up the local search algorithm for Euclidean space to obtain a PTAS with running time $nk(\log n)^{(d/\varepsilon)O(d)}$.

Nonetheless, obtaining an efficient approximation scheme (namely an algorithm running in time $f(\varepsilon, d)\text{poly}(n)$) for $k$-Median and $k$-Means in metrics of fixed doubling dimension has remained a major challenge.

1.1 Our results

We obtain the first near-linear time algorithms for the Facility Location, $k$-median and $k$-means problems in metrics of fixed doubling dimension. More precisely, we show the following theorems.

Let $f(\varepsilon) = (1/\varepsilon)^{1/\varepsilon}$.

**Theorem 1.1.** For any $0 < \varepsilon < 1/3$, there exists a randomized $(1 + \varepsilon)$-approximation algorithm for Facility Location in metrics of doubling dimension $d$ with running time is $f(\varepsilon)^{O(d^2)} \cdot n + 2^{O(d)} n \log n$ and success probability at least $1 - \varepsilon$.

For the $k$-Median and $k$-Means problems we also obtain near-line time approximation schemes.

**Theorem 1.2.** For any $0 < \varepsilon < 1/3$, there exists a randomized $(1 + \varepsilon)$-approximation algorithm for $k$-Median in metrics of doubling dimension $d$ with running time is $f(\varepsilon)^{O(d^2)} n \log^4 n + 2^{O(d)} n \log^9 n$ and success probability at least $1 - \varepsilon$.

**Theorem 1.3.** For any $0 < \varepsilon < 1/3$, there exists a randomized $(1 + \varepsilon)$-approximation algorithm for $k$-Means in metrics of doubling dimension $d$ with running time is $f(\varepsilon)^{2O(d^2)} n \log^6 n + 2^{O(d)} n \log^9 n$ and success probability at least $1 - \varepsilon$.

Note that the double-exponential dependency in $d$ is unavoidable unless $P = \text{NP}$, since the problems are APX-Hard in Euclidean space of dimension $d = O(\log n)$. For Euclidean inputs, our algorithms for the $k$-means and $k$-median problems outperform the ones of Cohen-Addad [10], removing in particular the dependency in $k$, and the one of Kolliopoulos and Rao [21] when $d > 3$, by removing the dependency in $\log^{d+6} n$. Interestingly, for $k = \omega(\log^9 n)$ our algorithm for the $k$-Means problem is faster than popular heuristics like $k$-Means++ which runs in time $O(nk)$ in Euclidean space.

We note that the success probability can be boosted to $1 - \varepsilon^\delta$ by repeating the algorithm $\log \delta$ times and outputing the best solution encountered.

1.2 Techniques

Our starting point is the split-tree decomposition of Talwar [26] together with the notion of badly cut vertices of Cohen-Addad [9] for capacitated version of the problem. The reader who is not familiar with the split-tree decomposition of Talwar may refer to Section 2.2.1 for a quick introduction or directly to [26]. The goal is to define, as in [2, 21, 26], a dynamic program that works bottom-up in the decomposition. The dynamic program makes use of portal in each level of the decomposition: any points $u, v$ that are in different parts of the decomposition can connect through
portals, this incurs some detour. The objective is then to bound the number of portals used to have a small running time, and to bound the overall detour to have a good approximation ratio.

In the standard approach of [2], the number of portals required in order for the total detour to be \(\varepsilon \cdot \text{cost}(\text{OPT})\) is \(\Omega (\log^{d-1} n)\) which leads to the QPTAS result. This is a classic barrier that has been overcome for some problems such as the traveling salesperson problem [6], or for Steiner forest [8] but which has not been overcome for clustering.

We bypass this barrier as follows using the notion of badly cut vertices of [9]. We start with the following observation, consider a center \(f\) of the optimal solution and assume it is separated from one of its client \(c\) at a level of the decomposition that is at most \(\log \dist(c, f) + f(\varepsilon, d)\), or in other words when the diameter of the ball is at most \(f(\varepsilon, d) \cdot \dist(c, f)\). Then using a portal set of size \(f(\varepsilon, d)/\varepsilon\) incurs a maximum detour of \(\varepsilon \dist(c, f)\) which is fine. Moreover, this holds even when paying the squared distance and applies to the \(k\)-means problem. Now, how to handle pairs of clients and facilities that are separated at higher levels?

To do so, as in [9], we make use of an approximate solution \(L\) (here we can use an \(O(1)\)-approximation, whereas [9] has to start with an \(O(\log n)\)-approximation) and proceed as follows. Consider a client \(c\) and the center \(L(c)\) serving \(c\) in \(L\) (i.e., \(L(c)\) is closest to \(c\) among the centers in \(L\)). If the client \(c\) is separated from \(L(c)\) at a higher level than \(\log(\dist(L(c), \text{OPT}(L(c)))) + f(\varepsilon, d)\), we will simply relocate \(c\) to \(L(c)\). We proceed similarly for all clients and then work in the resulting instance. The intuition is that because this happens with a tiny probability any solution obtained in the new instance can be lifted back to the original instance with a tiny cost increase, namely \(\varepsilon \cdot \text{cost}(L) = O(\varepsilon \cdot \text{cost}(\text{OPT}))\).

Our next step is to show that if \(L(c)\) is separated from its closest center \(\text{OPT}(L(c))\) in \(\text{OPT}\) at a level higher than \(\log(\dist(L(c), \text{OPT}(L(c)))) + f(\varepsilon, d)\), then we can simply decide that \(L(c)\) is now open. This is a critical property and a major challenge since we need to argue that there exists a near-optimal solution using at most \(k\) centers that contains all the facilities of \(L(c)\) that are separated from \(\text{OPT}(L(c))\) at a too high level.

The outcome of all this is that one can show that

- any solution \(S\) has a cost in the modified instance that is within a \((1 + \varepsilon)\) factor of its cost in the original instance,
- there exists a solution \(S^*\) that contains all the centers of \(L\) that are separated from their closest center in \(\text{OPT}\) at a too high level, and
- in the modified instance, each client \(c\) is separated from the closest\(^1\) center of \(S^*\) at a level of at most \(\log(20(\dist(c, L(c)) + \dist(c, \text{OPT}(c)))) + f(\varepsilon, d)\). Thus, the detour to reach this center is at most \(\varepsilon(\dist(c, L(c)) + \dist(c, \text{OPT}(c)))\). Summing up over all clients, this is at most \(\varepsilon \cdot \text{cost}(\text{OPT})\).

Therefore, we can use a similar dynamic-programming procedure to identify the best solution in the modified instance that is forced to go through a portal and we are guaranteed that in the original instance this solution has cost at most \((1 + \varepsilon)\) times the cost of the optimal solution.

Our result on Facility Location in Section 4 gives a simple use of these ideas. Our main result on \(k\)-Median and \(k\)-Means is described in Section 6 and requires more elaborate ideas.

### 1.3 Further related work

**On clustering problems.** The three clustering problems considered in this paper are known to be NP-hard, even restricted to inputs lying in the Euclidean plane (see Mahajan et al. [23] or Dasgupta and Freund [14] for \(k\)-Means, Megiddo and Supowit [24] for the other two). The problems of Facility Location and \(k\)-Median have been studied since a long time in graphs, see e.g. [20]. The current best approximation ratio for metric Facility Location is 1.488, due to Li [22], whereas it is 2.67 for \(k\)-Medians by Byrka et al. [7].

\(^1\)To be precise a center that is at distance at most \((1 + \varepsilon)\) times its distance to the closest client.
The problem of $k$-Means in general graphs also received a lot of attention (see e.g., Kanungo et al. [20]) and the best approximation ratio is 6.357, by Ahmadian et al. [1].

**On doubling dimension.** Despite their hardness in general metrics, these problems admit a PTAS when the input is restricted to a low dimensional metric space: Friggstad et al [15] showed that local search gives a $(1 + \varepsilon)$-approximation. However, the running time of their algorithm is $n^{(d/\varepsilon)\Omega(d)}$ in metrics with doubling dimension $d$.

A long line of research tries to fill the gap between results for Euclidean spaces and metrics with bounded doubling dimension. This started with the work of Talwar [26], who gave QPTASs for a long list of problems. The complexity for some of these problems was improved later on: for the Traveling Salesperson problem, Gottlieb [16] gave a near-linear time approximation scheme, Chan et al. [8] gave a PTAS for Steiner Forest, and Gottlieb [16] described an efficient spanner construction.

## 2 Preliminaries

### 2.1 Definitions

Consider a metric space $(V, \text{dist})$. For a vertex $v \in V$ and an integer $r \geq 0$, we define the ball $\beta(v, r) = \{w \in V \mid \text{dist}(v, w) \leq r\}$ around $v$ with radius $r$. The doubling dimension of a metric is the smallest integer $d$ such that any ball of radius $2r$ can be covered by $2^d$ balls of radius $r$.

Given a set of points called *clients* and a set of points called *candidate centers* in a metric space, the goal of the $k$-Median problem is to output a set of $k$ centers chosen among the candidate centers that minimizes the sum of the distances from each client to its closest centers. More formally, an instance to the $k$-Median problem is a 4-tuple $(C, F, \text{dist}, k)$, where $(C \cup F, \text{dist})$ is a metric space and $k$ is a positive integer. The goal is to find a set $S \subseteq F$ such that $|S| \leq k$ and $\sum_{c \in C} \min_{f \in S} \{\text{dist}(c, f)\}$ is minimized. Let $n = |C \cup F|$. The $k$-Means problem is identical except from the objective function which is $\sum_{c \in C} \min_{f \in S} \{\text{dist}(c, f)\}^2$.

In the Facility Location problem, the number of centers in the solution is not limited but there is a cost for each candidate center $f$ and the goal is to find a solution $S$ minimizing $\sum_{c \in C} \min_{f \in S} \{\text{dist}(c, f)\} + \sum_{f \in S} c_f$.

A $\delta$-net of $V$ is a set of points $X \subseteq V$ such that for all $v \in V$ there is an $x \in X$ such that $\text{dist}(v, x) \leq \delta$, and for all $x, y \in X$ we have $\text{dist}(x, y) > \delta$. A net is therefore a set of points not too close to each other, such that every point of the metric is close to a net point. A useful lemma bounds the cardinality of a net in metrics of low doubling dimension:

**Lemma 2.1** (from Gupta et. al [17]). Let $(V, d)$ by a metric space with doubling dimension $d$ and diameter $\Delta$, and let $X$ be a $\delta$-net of $V$. Then $|X| \leq 2^{d \lfloor \log_2(\Delta/\delta) \rfloor}$.

We will also make use the following lemma.

**Lemma 2.2** ([11]). Let $p \geq 0$ and $1/2 > \varepsilon > 0$. For any $a, b, c \in A \cup F$, we have $\text{dist}(a, b)^p \leq (1 + \varepsilon)^p \text{dist}(a, c)^p + \text{dist}(c, b)^p (1 + 1/\varepsilon)^p$.

### 2.2 Decomposition of metric spaces

We define a hierarchical decomposition (sometimes simply a decomposition) of a metric with distance function $\text{dist}$ on a set $S$ of points as a collection $D = \{B_0, \ldots, B_{|D|}\}$ that satisfies the following:

- each $B_i$ is a partition of $S$,
- $B_i$ is a refinement of $B_{i+1}$, namely for each part $B \in B_i$ there exists a part $B' \in B_{i+1}$ that contains $B$,
\begin{itemize}
  \item $B_0$ contains a singleton set for each $v \in S$, while $B_{|D|}$ is a trivial partition that contains only one set, namely $S$.
  
  We define the \textit{ith level} of the decomposition to be the partition $B_i$, and call $B \in B_i$ a level-$i$ cluster. If $B' \in B_{i-1}$ is such that $B' \subset B$, we say that $B'$ is a subcluster of $B$.
  
  For a given decomposition $D = \{B_0, \ldots, B_{|D|}\}$, we say that a vertex $u$ is \textit{cut from} $v$ at level $j$ if $j$ is the maximum integer such that $v$ is in some $B \in B_j$ and $u$ is in some $B' \in B_j$ with $B \neq B'$. For a vertex $v \in F$ we define the ball $\beta_v = \beta(v, 2^i)$ and we say that the ball $\beta_v$ is \textit{cut} by $D$ at level $j$ if there is at least one vertex of the ball that is cut from $v$ at level $j$.
  
  In the following, we will aim at finding a decomposition $D$ of metric spaces together with sets of \textit{portals} with the following properties. Let $d$ be a parameter of the metric (say the doubling dimension), $\varepsilon$ be an error parameter, and $f_1, f_2$ any computable functions.
  
  1. \textbf{Scaling probability:} For any $v \in V$, radius $r$, and level $i$, we have
   \[ \Pr[D \text{ cuts } \beta(v, r) \text{ at a level } i] \leq f_1(d, \varepsilon) \cdot r/2^i. \]

  2. \textbf{Concise and precise portal set:} For any set $B \in B_i$ where $B_i \in D$, there is a set of portals $\mathcal{P}_B$ such that,
   \[ (a) \text{ concise: } |\mathcal{P}_B| \leq f_2(d, \varepsilon); \text{ and} \]
   \[ (b) \text{ precise: } \text{for any ball } \beta(v, r) \subseteq B \text{ cut by } C_T \text{ at level } i \text{ and pair of distinct sets } B_1, B_2 \in B_{i-1} \text{ on level } i - 1, \text{ we have for any } u \in B_1 \cap \beta(v, r), \text{ and } w \in B_2 \cap \beta(v, r), \]
   \[ \min_{p \in \mathcal{P}_B} \{\text{dist}(u, p) + \text{dist}(p, w)\} \leq \text{dist}(u, w) + O(\varepsilon 2^i). \]

\end{itemize}

\subsection{2.2.1 Structural results on metrics of bounded doubling dimension}

We make the common assumption to have access to the distances of the graph in constant time, as, e.g., in [13, 16, 18]. This assumption is discussed in Bartal et al. [5].

For low doubling metrics, Talwar [26] proved that it is possible to find such a hierarchical decomposition. This decomposition is the following. First, assume that the smallest distance in the metric is 1, and let $\Delta$ be the diameter of the metric. Then define a hierarchy $Y_0 := V, \ldots, Y_{\log(\Delta)}$ such that $Y_i$ is a $2^{-i/2}$-net of $Y_{i-1}$. Then pick a random order on the points $V$ and a random number $\tau \in [1/2, 1)$. Let $V$ be the single set at level $\log(\Delta)$. Then, to decompose a cluster $C$ at level $i$ into cluster at level $i - 1$, do the following: for each $y \in Y_{i-1}$ in the random order, define $C \cap B(y, \tau 2^i)$ to be a cluster at level $i - 1$ and remove $C \cap B(y, \tau 2^i)$ from $C$.

When we assume access to the distances through an oracle, it is possible to construct the hierarchy in time $2^{O(d)} n \log(\Delta)$ [13, 18]. Then, to divide a level $i$ cluster $C$ into subclusters, we consider the points in $Y_{i-1} \cap C$ in the random order (by Lemma 2.1 there are $2^{O(d)}$ of them) and for each of them we find in $O(|C|)$ time the points of $C$ at distance at most $\tau 2^i$ of it. Since the clusters form a partition of the input, the total running time of constructing a level is $2^{O(d)} n$. Since there are $\log(\Delta)$ levels, the total complexity of constructing a split-tree is $2^{O(d)} n \log(\Delta)$.

Since we are aiming at $(1 + \varepsilon)$-approximations, in all our applications we can perturb the input such that $\Delta = \text{poly}(n)$ (taking for instance an $(\varepsilon \Delta / n^2)$-net of the input), and therefore the complexity of computing the split-tree is considered to be $2^{O(d)} n \log n$.

\textbf{Lemma 2.3 ([4, 26])}. \textit{For any metric $S$ of doubling dimension $d$, there is a randomized hierarchical decomposition $D$ such that the diameter of a part $B \in B_i$ is at most $2^{i+1}$, $|D| \leq \lceil \log_2(\text{diam}(S)) \rceil$, and:}

1. \textbf{Scaling probability:} for any $v \in V$, radius $r$, and level $i$, we have
   \[ \Pr[D \text{ cuts } \beta(v, r) \text{ at a level } i] = 2^{O(d)} r/2^i. \]
2. **Concise and precise portal set:** For any level \(i\) there exists a set of portals \(\mathcal{P}_i\) such that 
\(\mathcal{P}_i \cap B\) respects the concise property with \(f(d, \varepsilon) = 1/\varepsilon^d\) and the precise property.

Moreover, this decomposition can be found in near-linear time.

**Proof.** We prove here that Talwar’s hierarchical decomposition has the required properties. The diameter of each part is bounded by \(2^{i+1}\) by construction; therefore to have property 2 it is enough to make \(\mathcal{P}_i\) an \((\varepsilon 2^{i+1})\)-net of \(V\). The property 2.1 ensures the conciseness, and the definition of a net ensures the preciseness. Proving the scaling property requires a bit more work.

Recall the construction of Talwar’s hierarchical decomposition. The level \(i\) clusters are formed by balls of radius \(\tau 2^i\) centered on points of a set \(Y_i\), with \(\tau\) chosen uniformly at random in \([1/2, 1]\). Thus the diameter of any cluster is at most \(2^{i+1}\). Also, the set \(Y_i\) is a \(2^{i-3}\)-packing, meaning that the minimum distance in \(Y_i\) is bigger than \(2^{i-3}\).

These two properties are enough in order to prove our lemma. Let \(i\) be a level such that \(2^i \leq r\): then \(r/2^i = \Omega(1)\) so there is nothing to prove. Otherwise, we proceed in two steps. First, let us count the number of level \(i\) clusters that could possibly cut a ball \(\beta(x, r)\). A level \(i\) cluster is included in a ball \(\beta(y, 2^i)\) for some \(y \in Y_i\); therefore if \(\text{dist}(x, y) > r + 2^i\) then \(y\)’s cluster cannot cut \(\beta(x, r)\). So it is required that \(\text{dist}(x, y) \leq r + 2^i \leq 2 \cdot 2^i\). But since \(Y_i\) is a \(2^{i-3}\)-packing and has doubling dimension \(d\), 
\[|Y_i \cap \beta(x, 2 \cdot 2^i)| = 2^d \log(2^{2^i}/2^{i-3}) = 2^O(d).\]

Thus there is only a bounded number of clusters to consider.

We prove for each of them that the probability that it cuts \(\beta(x, r)\) is \(O(r/2^i)\). A union-bound on all the possible clusters is then enough to conclude. Let therefore \(y \in Y_i \cap \beta(x, 2 \cdot 2^i)\), and \(x_m\) and \(x_M\) be the respective closest and farthest point of \(\beta(x, r)\) from \(y\). A necessary condition for \(y\)’s cluster to cut \(\beta(x, r)\) is that the diameter of the cluster is in the open interval \((d(y, x_m), d(y, x_M))\).

Since \(x_m, x_M \in \beta(x, r)\) this interval has size \(2r\), and the radius of the cluster is picked uniformly in \([2^i/2, 2^i]\). Therefore the probability that the radius of the cluster falls in \((d(y, x_m), d(y, x_M))\) is at most \(4r/2^i\). And finally, the probability that \(y\)’s cluster cuts \(\beta(x, r)\) is indeed \(O(r/2^i)\).

By a union-bound over all the clusters that could possibly cut \(\beta(x, r)\) we obtain the claimed probability 
\[\Pr[\mathcal{C} \text{ cuts } \beta(x, r) \text{ at a level } i] = 2^{O(d)} r/2^i.\]

As mentioned before, the complexity of constructing this decomposition is near-linear in doubling metrics. This concludes the lemma.

Another property from doubling metrics that will be useful for our purpose is the existence of low-stretch spanners with a linear number of edges. Their construction is evoked in Har-Peled and Mendel [18] and takes time \(2^{O(d)} n\). More precisely, they show that we can find a graph in the input metric such that distances in the graph approximates the original distances up to a constant factor. Since we make use of these edges only for computing constant-factor approximations of our problems, using a spanner is enough. Therefore we can assume that the number of edges is \(m = 2^{O(d)} n\).

3 **Introduction to the notion of badly cut vertices**

This section introduces the key notion of *badly cut vertices*, and show some key structural result for the differentes clustering problems tackled in this paper. Our notion is similar in spirit of the notion of badly cut for capacitated clustering problems of [9]. However, applying the notion of [9] directly only allows to bound the portal set by \(O(f(\varepsilon, d) \log^d n)\) which, for the design of a near-linear time algorithm is not sufficient. Here our goal is to make use of it in order to obtain a portal set of size \(O(f(\varepsilon, d))\). We thus refine this notion by also applying it to clients (not only to facilities as in [9]) and defining badly cut facilities of the approximate solution relatively to the closest facility in the optimal solution.
Preprocessing. In the following, we will work with the slightly more general version of the clustering problems where there is some demand on each vertex: there is a function $\chi : C \rightarrow \{1, \ldots, n\}$ and the goal is to minimize $\sum_{c \in C} \chi(c) \cdot \min_{f \in S} \text{dist}(c, f) + \sum_{f \in S} q f$ for the Facility Location problem, or $\sum_{c \in C} \chi(c) \cdot \min_{f \in S} \text{dist}(c, f)^p$ for the $k$-clustering problem, with $p = 1$ or 2.

To be able to quickly compute the split-tree decomposition, we will first ensure that the aspect-ratio is $n^{O(1)}$. Our algorithm preprocesses the input in the following way. It first computes an $O(1)$-approximation to the problem and uses it to establish a lower bound on the cost of the optimum solution $OPT$. We can use this bound on $OPT$ to replace distances $d$ by $\max(d, 2OPT)$. Distances longer than $2OPT$ will indeed never be used by any near-optimal solution, so that the diameter of the metric is at most $O(OPT)$.

Then, the algorithm defines an instance of the slightly more general version of the clustering problem by taking an $(\epsilon \cdot \text{cost}(OPT)/n)$-net of the input graph, and for $c$ a net point it sets $\chi(c)$ to be the number of clients such that $c$ is their closest net point. This transformation adds an error of $\epsilon \cdot \text{cost}(OPT)$ to the optimal solution, and ensures that the smallest distance in the graph is at least $\epsilon \cdot \text{cost}(OPT)/n$. The aspect-ratio is therefore $O(n^2/\epsilon)$. We call this modified instance $I$.

The algorithm then computes the split-tree decomposition $D$ described in Lemma 2.3, stopping the recursive calls on a given cluster if the diameter of the cluster is at most $\epsilon \cdot \text{cost}(OPT)/n$. Since the radii of the clusters are geometrically decreasing and the aspect-ratio of the graph is polynomial, the number of levels of $D$ is at most $O(\log(n/\epsilon))$.

Finally, each cluster $B$ of the decomposition is equipped with a set of portals, which is a $\rho \cdot \text{diam}(B)$-net of $B$, for some parameter $\rho$ determined later.

Key definitions. For a client $c$, an approximate solution $L$ and an optimum solution $OPT$, let $L(c) = \arg\min_{f \in L} \text{dist}(c, f)$, $OPT(c) = \arg\min_{f \in OPT} \text{dist}(c, f)$, where ties are broken arbitrarily. Moreover, let $OPT_c = \text{dist}(c, OPT(c))$, and $L_c = \text{dist}(c, L(c))$.

We say that a vertex $v$ is badly cut if there exists an integer $i$ such that $2^i \in [\epsilon L_c, L_c/\epsilon]$ and $\beta_v^i$ is cut at some level $j$ greater than $i + O(d) + 2 \log \left(\frac{(p+1)^p}{\epsilon} \right)$, where the $O(d)$ is the same as in Property 1. In the following, we denote $\kappa(\epsilon, p) = \frac{\epsilon^{p+2}}{(p+1)!}$ and $\tau(\epsilon, d) = O(d) + 2 \log \left(\frac{p+1}{\epsilon^{p+2}} \right)$.

We say that a center $f$ of $L$ is badly cut if there exists an integer $i$ such that $2^i \in [\epsilon OPT_f, OPT_f/\epsilon]$ and $\beta_f^i$ is cut at some level $j$ greater than $i + \tau(\epsilon, d)$, where $OPT_f$ is the distance from $f$ to the closest facility of $OPT$.

We provide some intuition about the notion of badly cut. Consider for example a badly cut facility $f \in L$. The notion of badly cut means that $\text{dist}(f, OPT(f))$ may not be well approximated by the portal set. Namely, if we force $f$ to make a detour to the closest portal at the level where $f$ and $OPT(f)$ are separated, the cost incurred by the detour would be higher than $\epsilon \cdot \text{dist}(f, OPT(f))$.

The following lemma is essential and bounds the probability of being badly cut.

Lemma 3.1. Fix a vertex $v \in C \cup F$. The probability that $v$ is badly cut is $O(\kappa(\epsilon, p))$

Proof. By Property 1, the probability that the ball $\beta(v, 2^i)$ is cut at level at least $j$ is at most $2^{O(d)2^j}/2^i$. Hence the probability that $\beta(v, 2^i)$ is cut at a level $j$ greater than $i + O(d) + 2 \log (1/\kappa(\epsilon, p))$ is at most $\kappa(\epsilon, p)^2$. Now, since the balls we are looking at have radius in $[\epsilon^2 L_v, L_v/\epsilon^2]$ if $v \in C$ or in $[\epsilon OPT_v, OPT_v/\epsilon]$ if $v \in F$, there are only $O(\log(1/\epsilon))$ of them. Therefore, taking a union bound over all such balls, we conclude that the probability that $v$ is badly cut is at most $O(\log(1/\epsilon) \cdot \kappa(\epsilon, p)^2) = O(\kappa(\epsilon, p))$. \hfill $\square$

3.1 Structure theorem

Starting from an instance $I$ with polynomial aspect-ratio, the first step of the algorithm is to compute an approximate solution $L$, together with a randomized hierarchical decomposition $D$, and
to transform the instance: every badly cut client \( c \) is moved to \( L(c) \), namely, there is no more client at \( c \) and we add an extra client at \( L(c) \). We let \( \mathcal{I}_\mathcal{D} \) denote the resulting instance and note that \( \mathcal{I}_\mathcal{D} \) is a random variable that depends on the randomness of \( \mathcal{D} \). Moreover, we let \( B_\mathcal{D} \) be the set of badly cut facilities of \( L \). We call \( \text{cost}_\mathcal{I} \) the cost of a solution in the original instance, and \( \text{cost}_{\mathcal{I}_\mathcal{D}} \) the one in \( \mathcal{I}_\mathcal{D} \).

We let \( \nu_{\mathcal{I}_\mathcal{D}} = \max_{\text{solution } S} (\text{cost}_\mathcal{I}(S) - (1 + 3\varepsilon)\text{cost}_{\mathcal{I}_\mathcal{D}}(S), (1 - 3\varepsilon)\text{cost}_{\mathcal{I}_\mathcal{D}}(S) - \text{cost}_\mathcal{I}(S)) \). We say that an instance \( \mathcal{I}_\mathcal{D} \) has small distortion if \( \sum_{f \in B_\mathcal{D}} c_f \leq \varepsilon \text{cost}(L) \) and \( \nu_{\mathcal{I}_\mathcal{D}} \leq \varepsilon \text{cost}(L) \).

**Lemma 3.2.** The probability that \( \mathcal{I}_\mathcal{D} \) has small distortion is at least \( 1 - \varepsilon \).

**Proof.** To show the lemma, we will show that \( \text{Pr}(\mathcal{I}_\mathcal{D}) \leq \varepsilon^2 \text{cost}(L)/2 \) and \( \text{Pr}(\nu_{\mathcal{I}_\mathcal{D}} \leq \varepsilon^2 \text{cost}(L)/2) \). Then, by Markov’s inequality and taking a union bound over the probabilities of failure yields the lemma. Note that \( \text{E} \left[ \sum_{f \in B_\mathcal{D}} c_f \right] = \sum_{f \in L} \text{Pr}[f \text{ badly cut}] \cdot c_f \leq \varepsilon^2 \text{cost}(L)/2 \) is immediate from Lemma 3.1.

We thus aim at showing that \( \text{E}[\nu_{\mathcal{I}_\mathcal{D}}] \leq \varepsilon^2 \text{cost}(L)/2 \). By definition, we have that for any solution \( S \),

\[
\text{cost}(S) - \text{cost}_{\mathcal{I}_\mathcal{D}}(S) \leq \sum_{\text{badly cut client } c} \text{dist}(c, S)^p - \text{dist}(S, L(c))^p
\]

\[
\leq \sum_{\text{badly cut client } c} (1 + 3\varepsilon)\text{dist}(S, L(c))^p + \frac{\text{dist}(c, L(c))^p}{(\varepsilon/(p + 1))^p} - \text{dist}(S, L(c))^p,
\]

using Lemma 2.2 with parameter \( \varepsilon/p \). This is at most

\[
\sum_{\text{badly cut client } c} 3\varepsilon \cdot \text{dist}(S, L(c))^p + \frac{\text{dist}(c, L(c))^p}{(\varepsilon/(p + 1))^p},
\]

and so, we have

\[
\text{cost}(S) - (1 + 3\varepsilon)\text{cost}_{\mathcal{I}_\mathcal{D}}(S) \leq \sum_{\text{badly cut client } c} \frac{\text{dist}(c, L(c))^p}{(\varepsilon/(p + 1))^p}
\]

Similarly, we have that

\[
\text{cost}_{\mathcal{I}_\mathcal{D}}(S) - \text{cost}(S) \leq \sum_{\text{badly cut client } c} \text{dist}(S, L(c))^p - \text{dist}(c, S)^p
\]

\[
\leq \sum_{\text{badly cut client } c} (1 + 3\varepsilon)\text{dist}(c, S)^p + \frac{\text{dist}(c, L(c))^p}{(\varepsilon/(p + 1))^p} - \text{dist}(c, S)^p
\]

\[
\leq \sum_{\text{badly cut client } c} 3\varepsilon \cdot \text{dist}(c, S)^p + \frac{\text{dist}(c, L(c))^p}{(\varepsilon/(p + 1))^p}
\]

and we conclude

\[
(1 - 3\varepsilon)\text{cost}_{\mathcal{I}_\mathcal{D}}(S) - \text{cost}(S) \leq \sum_{\text{badly cut client } c} \frac{\text{dist}(c, L(c))^p}{(\varepsilon/(p + 1))^p}
\]

Therefore, the expected value of \( \nu_{\mathcal{I}_\mathcal{D}} \) is

\[
\text{E}[\nu_{\mathcal{I}_\mathcal{D}}] \leq \sum_{\text{client } c} \text{Pr}[c \text{ badly cut}] \cdot \frac{\text{dist}(c, L(c))^p}{(\varepsilon/(p + 1))^p}.
\]

8
Applying Lemma 3.1 and using $\kappa(\varepsilon, p) = \frac{2p^k}{(p+1)^p}$, we conclude $E[\nu_{I_D}] = O(\varepsilon^2 \cdot \text{cost}(L))$, the lemma follows for a sufficiently small $\varepsilon$.

4 A near-linear time approximation scheme for non-uniform Facility Location

To illustrate the framework of the former section, we show that it leads to a near-linear time approximation scheme for Facility Location in metrics of bounded doubling dimension. In this context we refer to centers in the set $F$ of the input as facilities.

We first prove that for an instance $I_D$ that has small distortion, there exists a solution $OPT'$ with small cost such that each client is cut from its serving facility at a bounded level. Together with the structure theorem, this proves that there exists a portal-respecting solution which is near-optimal, with a good probability.

4.1 Portal respecting solution

Let $OPT' = OPT \cup B_D$. Recall that $L_c$ and $OPT_c$ are the distances from the original position of $c$ to $L$ and OPT, but $c$ may have been moved to $L(c)$.

**Lemma 4.1.** Condition on $I_D$ having small distortion. For any client $c$ in $I_D$, let $OPT'(c)$ be the closest facility to $c$ in $OPT'$. Then $c$ and $OPT'(c)$ are separated in $D$ at level at most $\log(5(L_c + OPT_c)) + \tau(\varepsilon, d)$.

**Proof.** Let $c$ be a client. To find the level at which $c$ and $OPT'(c)$ are separated, we distinguish between two cases: either $c$ is badly cut, or not.

If $c$ is badly cut, then it is now located at $L(c)$ in the instance $I_D$. In that case, either:

1. $L(c)$ is also badly cut, and therefore $L(c) \in OPT'$ and so $OPT'(c) = L(c)$. It follows that $c$ and $OPT'(c)$ are never separated.

2. $L(c)$ is not badly cut. Then $d(c, OPT'(c)) \leq OPT_{L(c)}$. We bound the level at which $c$ and $OPT'(c)$ are separated. Since $L(c)$ is not badly cut, the path between $L(c)$ and $OPT(L(c))$ is cut at a level at most $\log(OPT_{L(c)}) + \tau(\varepsilon, d)$ By triangle inequality, $OPT_{L(c)} = \text{dist}(L(c), OPT(L(c))) \leq L_c + OPT_c$, and thus $c$ and $OPT'(c)$ are also separated at level at most $\log (L_c + OPT_c) + \tau(\varepsilon, d)$.

In the other case where $c$ is not badly cut, all of the rings in $[\varepsilon L_c, L_c/\varepsilon]$ are not badly cut. Note that in that case $c$ is not moved to $L_c$. We make a case distinction according to $OPT_c$.

1. If $L_c \leq \varepsilon OPT_c$, then we have the following. If $L(c)$ is badly cut, $L(c)$ is open and therefore $OPT'(c) = L_c$. Moreover, since $c$ is not badly cut the ball $\beta(c, L_c)$ is cut at level at most $\log(L_c) + \tau(\varepsilon, d)$. Therefore $c$ and $OPT'(c)$ are separated at level at most $\log(L_c) + \tau(\varepsilon, d)$.

In the case where $L(c)$ is not badly cut, both $c$ and $OPT'(c)$ lie in the ring centered at $L(c)$ and of diameter $2OPT_{L(c)}$. Indeed,

\[
\text{dist}(c, L(c)) \leq \varepsilon \text{dist}(c, OPT(c)) \leq \varepsilon \text{dist}(c, OPT(L(c))) \\
\leq \varepsilon \text{dist}(c, L(c)) + \varepsilon \text{dist}(L(c), OPT(L(c)))
\]
And therefore $\text{dist}(c, L(c)) \leq \frac{\varepsilon}{4} \text{OPT}_{L(c)} \leq 2 \text{OPT}_{L(c)}$ for any $\varepsilon \leq 2/3$. On the other hand,

$$\text{dist}(\text{OPT}'(c), L(c)) \leq \text{dist}(\text{OPT}'(c), c) + \text{dist}(c, L(c)) \leq \text{dist}(c, \text{OPT}(L(c))) + \text{dist}(c, L(c)) \leq 2\text{dist}(c, L(c)) + \text{dist}(L(c), \text{OPT}(L(c))) \leq \left(1 + \frac{2\varepsilon}{1 - \varepsilon}\right) \text{OPT}_{L(c)},$$

which is smaller than $2 \text{OPT}_{L(c)}$ for any $\varepsilon \leq 1/3$. Hence we have $c, \text{OPT}'(c) \in \beta(L(c), 2 \text{OPT}_{L(c)})$. To apply the definition of badly cut, we need to consider rings with radius power of 2: let us therefore consider $i$ such that $2 \text{OPT}_{L(c)} \in (2^{i-1}, 2^i]$ (note that $2^i \leq 4 \text{OPT}_{L(c)}$). Since $L(c)$ is not badly cut, this ring is not badly cut either. Therefore $c$ and $\text{OPT}'(c)$ are separated at level at most $i + \tau(\varepsilon, d)$. Since $\text{dist}(L(c), \text{OPT}(L(c))) \leq \text{dist}(L(c), \text{OPT}(c)) \leq \text{dist}(L(c), c) + \text{dist}(c, \text{OPT}(c)) \leq (1 + \varepsilon)\text{OPT}_c$, we have that $i \leq \log(4 \text{OPT}_{L(c)}) \leq \log(5 \text{OPT}_c)$, which is smaller than what we want.

2. If $L_c \geq \text{OPT}_c/\varepsilon$ then, since $c$ is not badly cut, the ball centered at $c$ and of radius $\varepsilon L_c$ is not badly cut. Since we have $\text{dist}(c, \text{OPT}'(c)) \leq \text{OPT}_c \leq \varepsilon L_c$, $c$ and $\text{OPT}'(c)$ lie in the ball $\beta(c, \varepsilon L_c)$ and are thus cut at level at most $\log(\varepsilon L_c) + \tau(\varepsilon, d)$.

3. If $\varepsilon L_c \leq \text{OPT}_c \leq L_c/\varepsilon$, then since $c$ is not badly cut the ball $\beta(c, \text{OPT}(c))$ is cut at level at most $\log(2 \text{OPT}_c) + \tau(\varepsilon, d)$. Moreover, $\text{OPT}'(c)$ lies in this ball.

This concludes the proof. \qed

We aim at making every path of the solution portal-respecting, meaning that they go in and out of clusters only at portals. We define a portal-respecting solution to be a solutions such that each path from a client to its facility is portal-respecting. This portal respecting property helps the dynamic program to find a near-optimal solution.

Let $u$ and $v$ be two vertices separated at level $i$ by the decomposition $D$. We note a property of the decomposition that will simplify our calculations. For the path between $u$ and $v$ to be portal-respecting, there needs to be a detour at every level below $i$, with an error $\sum_{j \leq i} \varepsilon 2^j \leq \varepsilon 2^{i+2}$. In the remainder of the paper, we will thus bound the error incurred by the detour by $4\varepsilon \cdot 2^i$, where $i$ is the level at which $u$ and $v$ are separated.

Lemma 4.2. Condition on $I_D$ having small distortion. The best portal-respecting solution $S$ in $I_D$ is such that $\text{cost}_I(S) \leq (1 + O(\varepsilon))\text{cost}_I(\text{OPT}) + O(\varepsilon \text{cost}_I(L))$.

Proof. Consider solution $\text{OPT}'$. Since $I_D$ has small distortion, we have that the facility cost of $\text{OPT}'$ is at most the facility cost of $\text{OPT}$ plus $\varepsilon \text{cost}(L)$. Furthermore, again since $I_D$ has small distortion we have that $\text{cost}_{I_D}(\text{OPT}') \leq (1 + O(\varepsilon))\text{cost}_I(\text{OPT}) + O(\varepsilon \text{cost}_I(L))$.

We now bound the cost of making $\text{OPT}'$ portal respecting by applying Lemma 4.1. Since each client $c$ of $I_D$ is separated from $\text{OPT}'(c)$ at level at most $\log(5(L_c + \text{OPT}_c)) + \tau(\varepsilon, d)$, we have that the detour for making the assignment of $c$ to $\text{OPT}'(c)$ portal-respecting is at most $\rho 2^{\tau(\varepsilon, d)} 5(L_c + \text{OPT}_c)$. Choosing $\rho = 2^{-\tau(\varepsilon, d)}$ ensures that the detour is at most $20\varepsilon(L_c + \text{OPT}_c)$.

Therefore, the best portal-respecting solution $S$ in $I_D$ is such that

$$\text{cost}_{I_D}(S) \leq \text{cost}_{I_D}(\text{OPT}') + 20\varepsilon(\text{cost}_I(\text{OPT}) + \text{cost}_I(L)) \leq (1 + O(\varepsilon))\text{cost}_I(\text{OPT}) + O(\varepsilon \text{cost}_I(L))$$

Since $I_D$ has small distortion, we conclude that

$$\text{cost}_I(S) \leq (1 + O(\varepsilon))\text{cost}_{I_D}(S) \leq (1 + O(\varepsilon))\text{cost}_I(\text{OPT}) + O(\varepsilon \text{cost}_I(L))$$ \qed
4.2 The algorithm.

To compute a constant-factor approximation $L$, we use Meyerson’s algorithm [25]. The approximate value of $\text{OPT}$ can then be used to reduce the aspect-ratio of the input, as described earlier. It is then possible to compute a split-tree decomposition, as explained in the preliminaries, with parameter $\rho = \varepsilon 2^{-\gamma(\varepsilon,d)}$.

Given $L$ and the decomposition, it is then possible to find all the badly cut clients: for a client $c$, only $O(\log(1/\varepsilon))$ balls have to be considered, namely the balls centered at $c$ and with radius in $[\varepsilon L_c, L_c/\varepsilon]$. For each such ball $\beta$, one has to check whether the decomposition cuts $\beta$ at a level that is too high, making $c$ badly cut. This is possible by verifying whether $c$ is at distance smaller than $L_c/\varepsilon$ to such a cluster of too high level. Thus, the algorithm is able to find all the badly cut clients in near-linear time.

The next step of the algorithm is to compute instance $I_D$ by moving every badly cut client $c$ to its facility in $L$.

We can now turn on to the description of the dynamic program (DP) for obtaining the best portal-respecting solution of $I_D$.

For a given cluster of the decomposition, each element of the net (i.e.: portal) stores its approximate distance to the closest facility in $\text{OPT} - L$ that is inside the cluster and outside the cluster. Those distances are called the parameters, and an entry of the DP table is a cluster with a set of parameters. The value for this entry is the minimal cost for a solution with facilities respecting the constraint set on the portals.

To fill the table, we use a dynamic program following the lines of Arora et al. [2] or Kolliopoulos and Rao [21]. If a cluster has no descendant (meaning the cluster contains a single point), computing the solution given the parameters is straightforward: either a center is opened on this point or not, and it is easy to check the consistency with the parameters. At a higher level of the decomposition, a solution is simply obtained by going over all the sets of parameter values for all the children clusters. It is immediate to see whether sets of parameter values for the children can lead to a consistent solution. A direct induction yields the theorem.

This dynamic program would have a complexity polylogarithmic in $n$, since there would be $O(\log n)$ possible values for a rounded distance. However, using the notion of badly cut, one can shave off the logarithms from the complexity.

More precisely, suppose we are at a level where the diameter of the clusters is $\Delta$. Then we can afford a detour of $\varepsilon \Delta$: if two vertices are cut at this level, this amount is the detour incurred by the decomposition, and our analysis showed that the total detour is affordable. Hence we don’t have to consider distances smaller than $\varepsilon \Delta$.

Now, suppose that the closest facility outside the cluster is at distance greater than $\Delta/\varepsilon$, and that there is no facility inside the cluster. Then, since the diameter is $\Delta$, w.l.o.g., we have that all the points of the cluster are assigned to the same facility. So we don’t want to guess the precise distances and just have a special flag saying we are in this regime. We can then treat this whole cluster as a single client (weighted by the number of clients inside the cluster), and consider it at higher levels (say $\log(1/\varepsilon)$ levels higher).

On the other hand, if there is some facility inside the cluster and the closest facility outside the cluster is at distance at least $2\Delta$, then each client of the cluster should be served by a facility inside the cluster in any optimal assignment. Thus we don’t have to guess the distances outside the cluster, and we can just put a flag saying that the facilities outside are too far to be useful for this cluster.

Assuming that the closest facility is at distance less than $\Delta/\varepsilon$, we have that for any portal of the cluster the closest facility is at distance at most $\Delta/\varepsilon + \Delta$ (since $\Delta$ is the diameter of the cluster). Therefore the range of distances of interest is $[\varepsilon \Delta, \Delta/\varepsilon + \Delta]$. Moreover, it is possible to round every distances to the closest multiple of $\varepsilon \Delta$: this gives an additional error $\varepsilon \Delta$, which is again affordable.
There are \(1/\varepsilon^2 + 1/\varepsilon\) multiples of \(\varepsilon \Delta\) in the range \([\varepsilon \Delta, \Delta/\varepsilon + \Delta]\), Hence the number of possible rounded distances for a given element of the net is bounded by \(O(1/\varepsilon^2)\).

**Analysis.** The preprocessing step (computing \(L\), the split-tree decomposition \(D\), and the instance \(I_D\)) has a running time \(O(n \log n)\), as all the steps can be done with this complexity: a fast implementation of Meyerson’s algorithm \([25]\) tailored for graphs can compute \(L\) in time \(O(m \log n)\). Using it on the spanner computed with \([18]\) gives a \(O(1)\)-approximation in time \(O(n \log n)\). As explained earlier, the split-tree and the instance \(I_D\) can also be computed with this complexity. 

The DP has a linear-time complexity: in a cluster of diameter \(\Delta\), the portal set is a \(\varepsilon 2^{-\tau(\varepsilon,d)} \Delta\)-net, and hence has size \(2^{d \log(2\tau(\varepsilon,d)/\varepsilon)}\) by Lemma 2.1. Since \(\tau(\varepsilon,d) = O(d) + 2 \log \frac{(p+1)^p}{\varepsilon \cdot \varepsilon} \cdot \Delta\), this number can be simplified to \(2^{O(d^2)}/\varepsilon\). Since each portal stores a distance that can take only \(1/\varepsilon^2\) values, there are at most \(E = (1/\varepsilon^2) 2^{O(d^2)}/\varepsilon = (1/\varepsilon) 2^{O(d^2)/\varepsilon}\) possible table entries for a given cluster.

To fill the table, notice that a cluster has at most \(2^{O(d)}\) children, due to the properties of the split-tree. Going over all the sets of parameter values for all the children clusters therefore takes time \(E 2^{O(d)} = (1/\varepsilon) 2^{O(d^2)/\varepsilon}\). This dominates the complexity of the dynamic program, which is therefore \(n(1/\varepsilon) 2^{O(d^2)/\varepsilon}\).

The total complexity of the algorithm is thus

\[
\left(\frac{1}{\varepsilon}\right)^{2^{O(d^2)/\varepsilon}} \cdot n + 2^{O(d) n \log n}
\]

Moreover, the dynamic program computes the best portal-respecting solution. Following Lemma 4.2, the cost of this solution is at most \((1 + O(\varepsilon)) \text{cost}_D(\text{OPT}) + O(\varepsilon \text{cost}_D(L))\). Since \(L\) is a constant-factor approximation, this cost turns out to be \((1 + O(\varepsilon)) \text{cost}_D(\text{OPT})\). This concludes the proof of Theorem 1.1.

## 5 A structured near-optimal solution for \(k\)-Median and \(k\)-Means

We aim at using the same approach as for Facility Location. If we were to find a bicriteria approximation, our analysis immediately works: by opening \(ek\) additional centers of \(L\), we can define a solution \(\text{OPT}'\) such that in instance \(I_D\), the distance from each client \(c\) to a facility \(\text{OPT}'(c)\) is nearly optimal and uses a dynamic program to solve the instance.

Our main challenge is to avoid dealing with bicriteria solutions. We show that if a center of \(L\) is badly cut, then there exists some center of \(\text{OPT}\) that we can remove and such that the resulting solution is of cost at most \((1 + \varepsilon) \text{cost}(\text{OPT}) + \varepsilon \text{cost}(L)\).

We focus the presentation on \(k\)-Median, and only later show how to adapt everything to \(k\)-Means. The outcome of this section is used in the next section to prove Theorems 1.2 and 1.3, which state that these problems can be solved in near-linear time in doubling metrics. The algorithm is given in detail in the next section.

**Preliminaries and notations for \(k\)-Median.** We consider the \(k\)-Median problem: given a metric space \((V, \text{dist})\), an integer \(k\), a set of candidate centers \(F \subseteq V\) and a set of clients \(C \subseteq V\), the goal is to identify a set \(S \subseteq F\) of size at most \(k\) that minimizes \(\sum_{c \in C} \min_{f \in S} \text{dist}(c, f)\). Thus, a solution is defined by a subset of \(F\) of size at most \(k\). We refer to the elements of any solution as **centers** or **facilities**.

In the following, we will work with the slightly more general version of the \(k\)-Median problem where in addition there is a function \(\chi : C \rightarrow \{1, \ldots, n\}\) and the goal is to minimize \(\sum_{c \in C} \chi(c)\).
min_{f \in S} dist(c, f). As described earlier, we will work with a randomized decomposition $D$ as described in Lemma 2.3 for the case of doubling metrics.

Preprocessing. Our algorithm preprocesses the input graph as described in Section 3, to obtain a modified instance $I$ of the more general version of $k$-Median, such that the aspect-ratio of the graph is polynomial.

As in Section 3, the first step of the algorithm is to compute an approximate solution $L$. To be able to apply the framework from Section 3, we need to construct a near-optimal solution $S^*$ such that all badly-cut facilities of $L$ are also facilities of $S^*$. For this, and to respect the constraint on the number of centers, we need to make some room in the optimal solution by removing a few centers. We will first present how to achieve this goal, and use it to prove a structure theorem similar to Lemma 3.2.

Removing few centers. Let OPT be an optimal solution. We consider the mapping of the facilities of OPT to $L$ defined as follows. We extend the function $L$ to the facilities of OPT (it was only defined for clients yet): for any $f \in OPT$, let $L(f)$ denote the facility of $L$ that is the closest to $f$. For any facility $\ell$ of $L$, define $\psi(\ell)$ to be the set of facilities of OPT that are mapped to $\ell$, namely, $\psi(\ell) = \{ f \in OPT \mid L(f) = \ell \}$. Define $L^1$ to be the set of facilities $\ell$ of $L$ that are such that there exists a unique $f \in OPT$ such that $L(f) = \ell$, namely $L^1 = \{ \ell \mid |\psi(\ell)| = 1 \}$. Let $L^0 = \{ \ell \mid |\psi(\ell)| = 0 \}$, and $L^2 = L - L^1 - L^0$. Similarly, define $OPT^1 = \{ f \in OPT \mid L(f) \in L^1 \}$ and $OPT^2 = \{ f \in OPT \mid L(f) \in L^2 \}$. Note that $|OPT^2| = |L^0| + |L^2|$, since the function $L$ is a bijection from $OPT^1$ to $L^1$ and, w.l.o.g., $|OPT| = |L| = k$.

We define a new solution $S^*_D$ (which we also refer to as $S^*$) in 3 steps. The first one is described here, and the other two later on. Start with $S^* = OPT$.

- **Step 1.** Among the facilities of $OPT^2$ that are not the closest of their corresponding facility in $L^2$, remove from $S^*$ the subset $OPT^0$ of size $\lfloor \varepsilon \cdot |OPT^2|/2 \rfloor$ that yields the smallest cost increase.

This construction ensures that $S^*$ has a near-optimal cost.

Lemma 5.1. After step 1, $S^*$ has cost $(1 + O(\varepsilon))cost(OPT) + O(\varepsilon)cost(L)$

Proof. We show that the cost increase is at most $O(\varepsilon)(cost(OPT) + cost(L))$.

Let $H \subseteq OPT^2$ be the set of facility of $OPT^2$ that are not the closest to their corresponding facility in $L^2$, i.e., $f \in H$ if and only if $f \in \psi(\ell)$ for some $\ell \in L^2$ and $dist(f, \ell) > \min_{f' \in \psi(\ell)} dist(f', \ell)$ (breaking ties arbitrarily). First note that $|H| \geq |OPT^2|/2$: the only elements in $OPT^2 - H$ are the ones closest to their corresponding facilities. Hence for every facility of $L^2$ such that $|\psi(f)| \geq 2$ there is therefore exactly 1 facility in $OPT^2 - H$, and at least 2 in $OPT^2$; and if $|\psi(f)| = 0$ $f$ does not correspond to any facility at all in $OPT^2$. Therefore $|H| \geq |OPT^2|/2$.

We claim that for a client $c$ served by $f \in H$ in the optimum solution OPT, i.e., $f = OPT(c)$, the detour entailed by the deletion of $f$ is $O(OPT_c + L_c)$. Indeed, let $f'$ be the facility of OPT that is closest to $L(f)$, and recall that $L(c)$ is the facility that serves $c$ in the solution $L$. Since $f' \notin H$, the cost to serve $c$ after the removal of $f$ is at most $dist(c, f')$, which can be bounded by $dist(c, f') \leq dist(c, f) + dist(f, L(f)) + dist(L(f), f')$. But by definition of $f'$, $dist(f', L(f)) \leq dist(L(f), f)$, and by definition of the function $L$ we have $dist(L(f), f) \leq dist(L(c), f)$, so that $dist(c, f') \leq dist(c, f) + 2dist(f, L(c))$. Using the triangle inequality finally gives $dist(c, f') \leq 3dist(c, f) + 2dist(c, L(c))$ which is $O(OPT_c + L_c)$. For a facility $f$ of OPT, we denote $C(f)$ the set of clients served by $f$, i.e., $C(f) = \{ c \in C \mid OPT(c) = f \}$. The total cost incurred by the removal of $f$ is then $O(cost_{OPT}(C(f)) + cost_L(C(f)))$.

Recall that in Step 1 we remove the set $OPT^0$ of size $\lfloor \varepsilon |OPT^2| \rfloor$ from $H$, such that $OPT^0$ minimizes the cost increase. We use an averaging argument to bound the cost increase: the sum among all facilities $f \in H$ of the cost of removing the facility $f$ is less than $O(cost(OPT) + cost(L))$, and
\(|H| = O(1/\varepsilon) \cdot |\varepsilon|OPT^2|\). Therefore removing \(OPT^0\) increases the cost by at most \(O(\varepsilon)(\text{cost}(OPT) + \text{cost}(L))\), so that Step 2 is not too expensive.

We can therefore use this solution as an approximation of \(OPT\), and henceforth we will denote this solution by \(OPT\). In particular, the badly-cut facilities are defined from this solution and not from the original \(OPT\).

**Structure theorem.** As in Section 3, the algorithm computes a randomized hierarchical decomposition \(D\), and transforms the instance of the problem. Every badly cut client \(c\) is moved to \(L(c)\), namely, there is no more client at \(c\) and we add an extra client at \(L(c)\). Again, we let \(I_D\) denote the resulting instance and note that \(I_D\) is a random variable that depends on the randomness of \(D\).

Moreover, as for Facility Location, we let \(B_D\) be the set of badly cut centers of \(L\). We call \(\text{cost}_I\) the cost of a solution in the original instance, and \(\text{cost}_{I_D}\) the one in \(I_D\). We let \(\nu_{I_D} = \max_{\text{solution}} \{\text{cost}_I(S) - (1 + 3\varepsilon)\text{cost}_{I_D}(S), (1 - 3\varepsilon)\text{cost}_{I_D}(S) - \text{cost}_I(S)\}\). We say that an instance \(I_D\) has small distortion if \(\nu_{I_D} \leq \varepsilon\text{cost}(L)\) and there exists a solution \(S\) that contains \(B_D\) and such that \(\text{cost}_I(S) \leq (1 + \varepsilon)\text{cost}_I(OPT) + \varepsilon\text{cost}_I(L)\).

We go on with the construction of \(S^*\), after the Step 1.

- **Step 2.** For each badly-cut facility \(f \in L\) for which \(\psi(f) \neq \emptyset\), let \(f' \in \psi(f)\) be the closest to \(f\). Replace \(f'\) by \(f\) in \(S^*\).

- **Step 3.** Add all badly cut facility \(f'\) of \(L^0\) to \(S^*\)

We show in the next lemma that \(S^*\) satisfies the conditions for \(I_D\) to have small distortion with a good probability.

**Proposition 5.2.** The probability that \(I_D\) has small distortion is at least \(1 - \varepsilon\).

**Proof.** The proof that \(\nu_{I_D} \leq \varepsilon\text{cost}(L)\) with probability at least \(1 - \varepsilon/2\) is identical to the one in Lemma 3.2. We thus turn to bound the probability that the solution \(S^*\) satisfies the requirement.

Our goal is to show that this happens with probability is at least \(1 - \varepsilon/2\). Then, taking a union bound over the probabilities of failure yields the proposition.

By definition, we have that \(S^*\) contains \(B_D\). We prove in the following lemmas some properties on \(S^*\).

**Lemma 5.3.** With probability at least \(1 - \varepsilon/4\), the set \(S^*\) is an admissible solution, i.e., \(|S^*| \leq k\).

**Proof.** We let \(b\) be the number of facilities of \(L^0\) that are badly cut. By Lemma 3.1, we have that \(\mathbb{E}[b] \leq \varepsilon^2|L|/4\). By Markov’s inequality, the probability that \(b\) is such that \(b > \varepsilon|L^0|/2\) is at most \(\varepsilon/2\).

Now, condition on the event that \(b \leq \varepsilon|L^0|/2\). Since \(|L^0| + |L^2| = |\text{OPT}^2|\), we have that \(b \leq \varepsilon|\text{OPT}^2|/2\). Moreover, the 3 steps ensure that \(|S^*| \leq k + b - \varepsilon|\text{OPT}^2|/2|\).

Combining the two inequalities gives \(|S^*| \leq k\). \(\square\)

**Lemma 5.4.** With probability at least \(1 - \varepsilon/4\), \(\text{cost}(S^*) \leq (1 + O(\varepsilon))\text{cost}(\text{OPT}) + O(\varepsilon \cdot \text{cost}(L))\).

**Proof.** We showed in Lemma 5.1 that the cost increase due to Step 1 is at most \(O(\varepsilon)(\text{cost}(\text{OPT}) + \text{cost}(L))\).

We show now that Step 2 leads to a cost increase of \(O(\varepsilon \cdot (\text{cost}(\text{OPT}) + \text{cost}(L)))\) with good probability. For that, let \(\text{OPT}_{\text{close}} := \{f \in \text{OPT} : f\text{ is the closest facility to }L(f)\}\). We show that the cost of replacing all \(f \in \text{OPT}_{\text{close}}\) by \(L(f)\) in \(L\) is \(O(\text{cost}(\text{OPT}) + \text{cost}(L))\). In order to prove this, we call the *mixed solution* the solution with facilities where every facility of \(f \in \text{OPT}_{\text{close}}\) is replaced by \(L(f)\). Note that \(L(\text{OPT}_{\text{close}}) = L - L^0\).

For that, let \(c\) be a client that is served in \(OPT\) by a facility \(f\) of \(\text{OPT}_{\text{close}}\). If \(c\) is served in \(L\) by a facility of \(L - L^0\), then the facility appears in the mixed solution and the serving cost
of c is dist(c, L). On the other hand, if c is served by a facility f₀ of L⁰ in L, then it is possible to serve it by the L(f) that replaces f in the mixed solution. The serving cost is therefore dist(c, L(f)) ≤ dist(c, f) + dist(f, L(f)) ≤ dist(c, f) + dist(f, f₀), using the definition of L(f) for the last inequality. Using again the triangle inequality, this cost is at most 2dist(c, f) + dist(f, f₀). Moreover, any client served by a facility of OPT−OPT_close is served by its optimal facility in the mixed solution, with cost dist(c, OPT). Hence the cost of the mixed solution is at most 2cost(OPT) + cost(L).

Moreover, the probability of replacing f ∈ OPT_close by L(f) ∈ L − L⁰ in Step 2 is the probability that L(f) is badly cut, which is O(κ(ε, p)) by Lemma 3.1. Finally, with linearity of expectation, the expected cost to add the badly cut facilities f ∈ L − L⁰ instead of their closest facility of OPT in Step 2 is O(κ(ε, p)(cost(OPT) + cost(L))). Markov’s inequality thus implies that the cost of the first step is at most O(ε · (cost(OPT) + cost(L))) with probability 1 − \frac{O(κ(ε, p))}{ε} ≥ 1 − ε/4, since κ(ε, p) ≤ ε²/4.

Proposition 5.2 follows from taking a union bound over the probabilities of failure of Lemmas 5.3 and 5.4.

Condition now on I₆ having a small distortion, and let OPT' be the solution containing B₆ with cost (1 + ε)cost_I₆(OPT) + εcost_I₆(L). We have to prove the same structure theorem as for Facility Location, to say that there exists a portal-respecting solution with cost close to cost(OPT').

Recall that L_c and OPT_c are the distances from the original position of c to L and OPT; but c may have been moved to L(c). Recall that OPT is defined after removing some centers in Step 1.

**Lemma 5.5.** Condition on I₆ having small distortion. For any client c in I₆, let OPT'(c) be the closest facility to c in OPT'. Then c and OPT'(c) are separated in D at level at most log(7(L_c + OPT_c)) + τ(ε, d).

**Proof.** The proof of this lemma is very similar to the one of Lemma 4.1. However, since some facilities of OPT were removed in step 2, we need to adapt the proof carefully.

Let c be a client. If OPT(c) was removed in Step 2, it was replaced by a facility f such that dist(OPT(c), f) ≤ dist(OPT(c), L(c)) (because L(OPT(c)) = f means that f is the facility of L closest to OPT(c)). Therefore

\[\text{dist}(c, f) ≤ 2\text{dist}(c, \text{OPT}(c)) + \text{dist}(c, L(c)).\]  \hspace{1cm} (1)

To find the level at which c and OPT'(c) are separated, we distinguish between two cases: either c is badly cut, or not.

If c is badly cut, then it is now located at L(c) in the instance I₆. In that case, either:

1. L(c) is also badly cut, and therefore L(c) ∈ OPT' and so OPT'(c) = L(c). It follows that c and OPT'(c) are never separated.

2. L(c) is not badly cut. Then dist(c, OPT'(c)) = dist(L(c), OPT'(L(c)). OPT(L(c)) is not necessarily in OPT: in that case, it was replaced by a facility f that verifies dist(c, f) ≤ 2dist(c, OPT(c)) + dist(c, L(c)), by Property (1). Since dist(c, L(c)) = 0, we have (either if OPT(L(c)) ∈ OPT' or not) that dist(c, OPT'(c)) ≤ 2OPT_L(c).

Since L(c) is not badly cut, the ball β(L(c), 2OPT_L(c)) is cut at level at most \log(4OPT_L(c)) + τ(ε, d). By triangle inequality, OPT_L(c) = dist(L(c), OPT(L(c))) ≤ L_c + OPT_c, and thus c and OPT'(c) are also separated at level at most \log(4L_c + 4OPT_c) + τ(ε, d).

In the other case where c is not badly cut, all of the rings in [εL_c, L_c/ε] are not badly cut, and c is not moved to L_c. We make a case distinction according to OPT_c.
1. If $L_c \leq \varepsilon \text{OPT}_c$, then we have the following. If $L(c)$ is badly cut, $L(c)$ is open and therefore $\text{OPT}'(c) = L_c$. Moreover, since $c$ is not badly cut the ball $\beta(c, L_c)$ is cut at level at most $\log L_c + \tau(\varepsilon, d)$. Therefore $c$ and $\text{OPT}'(c)$ are separated at level at most $\log L_c + \tau(\varepsilon, d)$.

In the case where $L(c)$ is not badly cut, both $c$ and $\text{OPT}'(c)$ lie in the ring centered at $L(c)$ and of diameter $3\text{OPT}_{L(c)}$. Indeed,

$$\text{dist}(c, L(c)) \leq \varepsilon \text{dist}(c, \text{OPT}(c)) \leq \varepsilon \text{dist}(c, \text{OPT}(L(c)))$$

$$\leq \varepsilon \text{dist}(c, L(c)) + \varepsilon d(L(c), \text{OPT}(L(c)))$$

And therefore $\text{dist}(c, L(c)) \leq \frac{1}{1-\varepsilon} \text{OPT}_{L(c)} \leq 3\text{OPT}_{L(c)}$ for any $\varepsilon \leq 3/4$. On the other hand,

$$\text{dist}(\text{OPT}'(c), L(c)) \leq \text{dist}(\text{OPT}'(c), c) + \text{dist}(c, L(c))$$

$$\leq 2\text{dist}(c, \text{OPT}(c)) + 2\text{dist}(c, L(c)) \quad (\text{using Property (1)})$$

$$\leq 2\text{dist}(c, \text{OPT}(L(c))) + 2\text{dist}(c, L(c))$$

$$\leq 4\text{dist}(c, L(c)) + 2\text{dist}(c, \text{OPT}(L(c)))$$

$$\leq \left(2 + \frac{4\varepsilon}{1-\varepsilon}\right) \text{OPT}_{L(c)},$$

which is smaller than $3\text{OPT}_{L(c)}$ for any $\varepsilon \leq 1/2$. Hence we have $c, \text{OPT}'(c) \in \beta(L(c), 3\text{OPT}_{L(c)})$.

To apply the definition of badly cut, we need to consider rings with radius power of 2: let us therefore pick $i$ such that $3\text{OPT}_{L(c)} \in (2^{i-1}, 2^{i}]$ (note that $2^i \leq 6\text{OPT}_{L(c)}$). Since $L(c)$ is not badly cut, this ring is not badly cut either and thus $c$ and $\text{OPT}'(c)$ are separated at level at most $i + \tau(\varepsilon, d)$. Since $\text{dist}(L(c), \text{OPT}(L(c))) \leq \text{dist}(L(c), \text{OPT}(c)) \leq \text{dist}(L(c), c) + \text{dist}(c, \text{OPT}(c)) \leq (1 + \varepsilon)\text{OPT}_c$, we have that $i \leq \log(6\text{OPT}_{L(c)}) \leq \log(7\text{OPT}_c)$, which is smaller that what we want.

2. If $L_c \geq \text{OPT}_c/\varepsilon$ then, since $c$ is not badly cut, the ball centered at $c$ and of radius $\varepsilon L_c$ is not badly cut. Since we have $\text{dist}(c, \text{OPT}'(c)) \leq 2\text{OPT}_c + L_c \leq 2L_c$, $c$ and $\text{OPT}'(c)$ lie in the ball $\beta(c, 2L_c)$ and are thus cut at level at most $4L_c + \tau(\varepsilon, d)$.

3. If $\varepsilon L_c \leq \text{OPT}_c \leq L_c/\varepsilon$ then, since $c$ is not badly cut, the ball $\beta(c, 2\text{OPT}_c + L_c)$ is cut at level at most $\log(4\text{OPT}_c + 2L_c) + \tau(\varepsilon, d)$. Moreover, $\text{OPT}'(c)$ lies in this ball, which concludes the lemma.

Equipped with these two lemmas, we can prove the following lemma, which concludes the section:

**Lemma 5.6.** Condition on $\mathcal{I}_D$ having small distortion. The best portal-respecting solution $S$ in $\mathcal{I}_D$ is such that $\text{cost}_\mathcal{I}(S) \leq (1 + O(\varepsilon))\text{cost}_\mathcal{I}(\text{OPT}) + O(\varepsilon \text{cost}_\mathcal{I}(L))$.

**Proof.** The proof of this theorem follows exactly the one of Lemma 4.2, using the definition of small distortion, $\text{OPT}'$, and Lemma 5.5.

**Extension to $k$-Means** The adaptation to $k$-Means can be essentially captured by the following inequality: $(x + y)^2 \leq 2(x^2 + y^2)$. Indeed, taking the example of **Lemma 5.4**, the detour $\text{dist}(c, f') \leq 3\text{dist}(c, f) + 2\text{dist}(c, l)$ gives a cost $\text{dist}(c, f')^2 = O(\text{dist}(c, f)^2 + \text{dist}(c, l)^2 + \text{dist}(c, f) \cdot \text{dist}(c, l)) = O(\text{dist}(c, f)^2 + \text{dist}(c, l)^2)$. This follows through all the other lemmas, and therefore the structure theorem holds also for $k$-Means.
6 An algorithm to find a near-optimal portal-respecting solution for \(k\)-Median and \(k\)-Means

In this section, we detail how to obtain a near-optimal solution by combining the structural results obtained in Section 4 and Section 5.

This section is dedicated to the proof of the theorems on \(k\)-Median and \(k\)-Means, that we restate here for convenience.

**Theorem 1.2** For any \(0 < \varepsilon < 1/3\), there exists a randomized \((1 + \varepsilon)\)-approximation algorithm for \(k\)-Median in metrics of doubling dimension \(d\) with running time \(f(\varepsilon)2^{O(d^2)}n \log^4 n + 2^{O(d)}n \log^9 n\) and success probability at least \(1 - \varepsilon\).

**Theorem 1.3**

For any \(0 < \varepsilon < 1/3\), there exists a randomized \((1 + \varepsilon)\)-approximation algorithm for \(k\)-Means in metrics of doubling dimension \(d\) with running time \(f(\varepsilon)2^{O(d^2)}n \log^5 n + 2^{O(d)}n \log^9 n\) and success probability at least \(1 - \varepsilon\).

In the following, we consider a randomized hierarchical decomposition \(\mathcal{D}\) (with parameter \(\rho = \varepsilon 2^{-\tau(\varepsilon,d)}\) as in Facility Location) together with a constant factor approximation \(L\). We provide a dynamic program that outputs a solution \(S\) of cost at most \((1 + \varepsilon)\text{OPT}\), conditioned on the event that the instance \(I_\mathcal{D}\) has small distortion. By Proposition 5.2, this happens with probability at least \(1 - \varepsilon\). Henceforth, we condition on \(I_\mathcal{D}\) having small distortion.

### 6.1 The algorithm

The first steps of the algorithm are identical to those of the algorithm presented in Section 4. Namely, it computes an \(O(1)\)-approximation, and a decomposition \(\mathcal{D}\) together with the instance \(I_\mathcal{D}\). In the rest of this section, we design an algorithm that directly operates on instance \(I_\mathcal{D}\).

**Dynamic programming.** The algorithm proceeds bottom up, along the levels of the decomposition. We give an overview of the dynamic program which is a slightly refined algorithm compared to the one presented for Facility Location in Section 4. We make use of two additional ideas.

To avoid the dependency on \(k\) we proceed as follows. In the standard approach, a cell of the dynamic program is defined by a cluster of the decomposition \(\mathcal{D}\), the portal parameters (as defined in Section 4), and a value \(k_0 \in [k]\). The value of an entry in the table is then the cost of the best solution that uses \(k_0\) centers, given the portal parameters. We define a cell of the dynamic program by a cluster \(B\), the portal parameters and a value \(c_0\) in \([\text{OPT}/n; (1 + \varepsilon)\text{OPT}]\). The entry of the cell is equal to the minimum number of centers \(k_0\) that needs to be placed in cluster \(B\) in order to achieve a cost of \(c_0\) given the portal parameters. Moreover, we only consider values for \(c_0\) that are powers of \((1 + \varepsilon)/\log n\). It follows that the number of DP cells is bounded by the total range of the portal parameters times \(O(\log^2 n)\) times the number of clusters \(n\). It is easy to see that the table entry for the root of the decomposition that has value at most \(k\) and that has the smallest value \(c_0\), corresponds to a \((1 + \varepsilon)\)-approximation to the best portal respecting solution.

We now explain how to obtain the claimed running time. The table can be computed the following way. For the clusters that have no descendant, computing the best clustering given a set of parameters can be done easily: there is at most one client in the cluster, and verifying that the parameter values for the centers inside the cluster are consistent can be done easily. At a higher level of the decomposition, a solution is simply obtained by going over all the sets of parameter values for all the children clusters. Since there are at most \(2^{O(d)}\) of them, this gives a running time
of $q^{O(d)}$, where $q$ is the total number of values for the parameter values. It is immediate to see whether sets of parameter values for the children can lead to a consistent solution (similar to [2, 21]).

This strategy would lead to a running time of $f(\varepsilon, d)n \log^{O(d)} n$. We can however treat the children in order, instead of naively testing all parameter values for them. We use a classical transformation of the dynamic program, in which the first table is filled using an auxiliary dynamic program. A cell of this auxiliary DP is a cluster $C$, one of its children $C_i$, portal parameters for the portals of $C$ and all its children before $C_i$ and a value $c_0$ in $[\text{OPT}/n; (1 + \varepsilon) \text{OPT}]$. The entry of the cell is equal to the minimum number of centers $k_0$ that need to be placed in the children clusters after $C_i$ in order to achieve a cost of $c_0$ given the portal parameters. To fill this table, one just has to try all possible sets of parameters for the next children, see whether they can lead to a consistent solution, and compute the minimum value among them.

There are $2^{O(d)}$ possible children, so the number of portals in this table is $2^{O(d)} \cdot 2^{O(d^2)} \varepsilon = 2^{O(d^2)} \varepsilon$ (with the same computation as for Facility Location). The overall complexity is therefore $n \cdot 2^{O(d)} \cdot (1/\varepsilon) 2^{O(d^2)} \varepsilon \cdot \log^4 n$.

**Running time of the preprocessing steps.** We need to bound the running time of three steps: computing an approximation, computing the split-tree decomposition, and running the dynamic program.

For $k$-Median, a constant-factor approximation can be computed in time $O(m \log^9 n) = 2^{O(d)} n \log^9 n$ with Thorup’s algorithm [27]. The split-tree decomposition can be found in $2^{O(d)} n \log n$ time as explained in Section 2. Moreover, as explained in the former paragraph, the dynamic program runs in time $f(\varepsilon, d)n \log^4 n$, ending the proof of the Theorem 1.2.

Another step is required for $k$-Means. It is indeed not known how to find a constant-factor approximation in near-linear time. However, one can notice that a $c$-approximation for $k$-Median is an $nc$-approximation for $k$-Means, using the Cauchy-Schwarz inequality. Moreover, notice that starting from a solution $S$, our algorithm finds a solution with cost $(1 + O(\varepsilon)) \text{cost(OPT)} + O(\varepsilon) \text{cost(S)}$ in time $f(\varepsilon, d)n \log^4 n$, as for $k$-Median.

Repeating this algorithm $N$ times, and in step $i + 1$ using the solution given by step $i$, therefore gives a solution of cost $(1 + O(\varepsilon)) \text{cost(OPT)} + O(\varepsilon^N) \text{cost(S)}$. Starting with $\text{cost}(S) = O(n) \text{cost(OPT)}$ and taking $N = O(\log n)$ ensures to find a solution for $k$-Means with cost $(1 + O(\varepsilon)) \text{cost(OPT)}$.

The complexity for $k$-Means is therefore the same as for $k$-Median, with an additional $\log n$ factor. This concludes proof of Theorem 1.3.

**References**

[1] Sara Ahmadian, Ashkan Norouzi-Fard, Ola Svensson, and Justin Ward. Better guarantees for $k$-means and euclidean $k$-median by primal-dual algorithms. In Foundations of Computer Science (FOCS), 2017 IEEE 58th Annual Symposium on, pages 61–72. Ieee, 2017.

[2] Sanjeev Arora, Prabhakar Raghavan, and Satish Rao. Approximation schemes for euclidean $k$-medians and related problems. In Proceedings of the Thirtieth Annual ACM Symposium on Theory of Computing, STOC ’98, pages 106–113, New York, NY, USA, 1998. ACM. ISBN 0-89791-962-9. doi: 10.1145/276698.276718. URL http://doi.acm.org/10.1145/276698.276718.

[3] Pranjal Awasthi, Moses Charikar, Ravishankar Krishnaswamy, and Ali Kemal Sinop. The hardness of approximation of euclidean $k$-means. In 31st International Symposium on Computational Geometry, SoCG 2015, June 22-25, 2015, Eindhoven, The Netherlands, pages 754–767, 2015.
[4] Yair Bartal and Lee-Ad Gottlieb. A linear time approximation scheme for euclidean TSP. In 54th Annual IEEE Symposium on Foundations of Computer Science, FOCS, 2013.

[5] Yair Bartal, Lee-Ad Gottlieb, Tsvi Kopelowitz, Moshe Lewenstein, and Liam Roditty. Fast, precise and dynamic distance queries. In Proceedings of the twenty-second annual ACM-SIAM symposium on Discrete Algorithms, pages 840–853. Society for Industrial and Applied Mathematics, 2011.

[6] Yair Bartal, Lee-Ad Gottlieb, and Robert Krauthgamer. The traveling salesman problem: low-dimensionality implies a polynomial time approximation scheme. SIAM Journal on Computing, 45(4):1563–1581, 2016.

[7] Jaroslaw Byrka, Thomas Pensyl, Bartosz Rybicki, Aravind Srinivasan, and Khoa Trinh. An improved approximation for k-median, and positive correlation in budgeted optimization. In Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2015, San Diego, CA, USA, January 4-6, 2015, pages 737–756, 2015.

[8] TH Hubert Chan, Shuguang Hu, and Shaofeng H-C Jiang. A ptas for the steiner forest problem in doubling metrics. In Foundations of Computer Science (FOCS), 2016 IEEE 57th Annual Symposium on, pages 810–819. IEEE, 2016.

[9] Vincent Cohen-Addad. Approximation schemes for capacitated clustering in doubling metrics. CoRR, abs/1812.07721, 2018. URL http://arxiv.org/abs/1812.07721.

[10] Vincent Cohen-Addad. A fast approximation scheme for low-dimensional k-means. In Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’18, pages 430–440, Philadelphia, PA, USA, 2018. Society for Industrial and Applied Mathematics. ISBN 978-1-6119-7503-1. URL http://dl.acm.org/citation.cfm?id=3174304.3175298.

[11] Vincent Cohen-Addad and Chris Schwiegelshohn. On the local structure of stable clustering instances. In 58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17, 2017, pages 49–60, 2017. doi: 10.1109/FOCS.2017.14. URL https://doi.org/10.1109/FOCS.2017.14.

[12] Vincent Cohen-Addad, Philip N. Klein, and Claire Mathieu. Local search yields approximation schemes for k-means and k-median in euclidean and minor-free metrics. In IEEE 57th Annual Symposium on Foundations of Computer Science, FOCS 2016, 9-11 October 2016, Hyatt Regency, New Brunswick, New Jersey, USA, pages 353–364, 2016. doi: 10.1109/FOCS.2016.46. URL https://doi.org/10.1109/FOCS.2016.46.

[13] Richard Cole and Lee-Ad Gottlieb. Searching dynamic point sets in spaces with bounded doubling dimension. In Proceedings of the thirty-eighth annual ACM symposium on Theory of computing, pages 574–583. ACM, 2006.

[14] Sanjoy Dasgupta and Yoav Freund. Random projection trees for vector quantization. IEEE Transactions on Information Theory, 55(7):3229–3242, 2009.

[15] Zachary Friggstad, Mohsen Rezapour, and Mohammad R Salavatipour. Local search yields a ptas for k-means in doubling metrics. In Foundations of Computer Science (FOCS), 2016 IEEE 57th Annual Symposium on, pages 365–374. IEEE, 2016.

[16] Lee-Ad Gottlieb. A light metric spanner. In Symposium on Foundations of Computer Science, FOCS, 2015.
[17] Anupam Gupta, Robert Krauthgamer, and James R. Lee. Bounded geometries, fractals, and low-distortion embeddings. In Proceedings of the 44th Annual IEEE Symposium on Foundations of Computer Science, FOCS ’03, 2003.

[18] Sariel Har-Peled and Manor Mendel. Fast construction of nets in low-dimensional metrics and their applications. SIAM Journal on Computing, 35(5):1148–1184, 2006.

[19] Lingxiao Huang, Shaofeng H-C Jiang, Jian Li, and Xuan Wu. $\varepsilon$-coresets for clustering (with outliers) in doubling metrics. Proceedings of FOCS 2018.

[20] Tapas Kanungo, David M Mount, Nathan S Netanyahu, Christine D Piatko, Ruth Silverman, and Angela Y Wu. A local search approximation algorithm for k-means clustering. Computational Geometry, 28(2-3):89–112, 2004.

[21] Stavros G Kolliopoulos and Satish Rao. A nearly linear-time approximation scheme for the euclidean k-median problem. SIAM Journal on Computing, 37(3):757–782, 2007.

[22] Shi Li. A 1.488 approximation algorithm for the uncapacitated facility location problem. Inf. Comput., 222:45–58, 2013.

[23] Meena Mahajan, Prajakta Nimbhorkar, and Kasturi R. Varadarajan. The planar k-means problem is np-hard. Theor. Comput. Sci., 442:13–21, 2012.

[24] Nimrod Megiddo and Kenneth J Supowit. On the complexity of some common geometric location problems. SIAM journal on computing, 13(1):182–196, 1984.

[25] Adam Meyerson. Online facility location. In Foundations of Computer Science, 2001. Proceedings. 42nd IEEE Symposium on, pages 426–431. IEEE, 2001.

[26] K. Talwar. Bypassing the embedding: algorithms for low dimensional metrics. In Proceedings of the thirty-sixth annual ACM symposium on Theory of computing, pages 281–290. ACM, 2004. doi: 10.1145/1007352.1007399.

[27] Mikkel Thorup. Quick k-median, k-center, and facility location for sparse graphs. SIAM Journal on Computing, 34(2):405–432, 2005.