An optimal quantum algorithm to approximate the mean and its application for approximating the median of a set of points over an arbitrary distance

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Abstract

We describe two quantum algorithms to approximate the mean value of a black-box function. The first algorithm is novel and asymptotically optimal while the second is a variation on an earlier algorithm due to Aharonov. Both algorithms have their own strengths and caveats and may be relevant in different contexts. We then propose a new algorithm for approximating the median of a set of points over an arbitrary distance function.

Keywords: Quantum computing, Mean, Median, Amplitude estimation.

1 Introduction

Let $F : \{0, \ldots, N - 1\} \rightarrow [0, 1]$ be a function and $m = \frac{1}{N} \sum_{i=1}^{N} F(i)$ be its mean. When $F$ is given as a black box (i.e. an oracle), the complexity of computing the mean can be measured by counting the number of queries made to this black box. The first quantum algorithm to approximate the mean was given by Grover, whose output of the estimate $\tilde{m}$ was such that $|m - \tilde{m}| \leq \epsilon$ after $O(\frac{1}{\epsilon} \log \log(\frac{1}{\epsilon}))$ queries to the black box [7]. Later, Nayak and Wu [10] have proven that to get such a precision, $\Omega(1/\epsilon)$ calls to $F$ are necessary, which still left a gap between the lower and upper bounds for
this problem. In this paper, we close this gap by presenting an asymptotically optimal algorithm to approximate the mean. We also describe a second algorithm that is a variation of Aharonov’s algorithm [1], which may be more suitable than the first one in some contexts.

Afterwards, these two algorithms to approximate the mean are used in combination with the quantum algorithm for finding the minimum of Dürr and Høyer [6] to obtain a quantum algorithm for approximating the median among a set of points with arbitrary black-box distance function between these points. The median, which is defined as the point with minimum average (or total) distance to the other points, can be thought of as the point that is the most representative of all the other points. Note that this is very different from the simpler problem of finding the median of a set of values, which has already been solved by Nayak and Wu [10]. Our median-finding algorithm combines the amplitude estimation technique of Brassard, Høyer, Mosca and Tapp [3] with the minimum-finding algorithm of Dürr and Høyer [6].

The outline of the paper is as follows. In Section 2, we present all the tools that we need, including Grover’s algorithm, the quantum algorithm for computing the minimum of a function and the amplitude estimation technique. In Section 3, we describe two efficient algorithms to approximate the mean value of a function, which we use in Section 4 to develop our novel quantum algorithm for approximating the median of an ensemble of points for which distances between them are given by a black box. Finally, we conclude in Section 5 with open questions for future work.

2 Preliminaries

In this section, we briefly review the quantum information processing notions that are relevant for understanding our algorithms. A detailed account of the field can be found in the book of Nielsen and Chuang [11].

As is often the case in the analysis of quantum algorithms, we shall assume that the input to the algorithms is given in the form of a black box (or “oracle”) that can be accessed in quantum superposition. In practice, the quantum black box will be implemented as a quantum circuit that can have classical inputs and outputs. We shall count as our main resource the number of calls (also called “evaluations”) that are required to that black box.
**Theorem 2.1** (Search [8, 2]). There exists a quantum algorithm that takes an arbitrary function \( F : \{0, \ldots, N - 1\} \rightarrow \{0,1\} \) as input and finds some \( x \) such that \( F(x) = 1 \) if one exists or outputs “void” otherwise. Any such \( x \) is called a “solution”. The algorithm requires \( O(\sqrt{N}) \) evaluations of \( F \) if there are no solutions. If there are \( s > 0 \) solutions, the algorithm finds one with probability at least \( 2/3 \) after \( O(\sqrt{N/s}) \) expected evaluations of \( F \). This is true even if the value of \( s \) is not known ahead of time.

Following Grover’s seminal work, Dür and Høyer [6] have proposed a quantum algorithm that can find the minimum of a function with a quadratic speed-up compared to the best possible classical algorithm.

**Theorem 2.2** (Minimum Finding [6, 5]). There exists a quantum algorithm \( \text{minimum} \) that takes an arbitrary function \( F : \{0, \ldots, N - 1\} \rightarrow Y \) as input (for an arbitrary totally ordered range \( Y \)) and returns a pair \((i, F(i))\) such that \( F(i) \) is the minimum value taken by \( F \). The algorithm finds a correct answer with probability at least \( 3/4 \) after \( O(\sqrt{N}) \) evaluations of \( F \).

Another extension of Grover’s algorithm makes it possible to approximately count the number of solutions to a search problem [4]. It was subsequently formulated as follows.

**Theorem 2.3** (Counting [3]). There exists a quantum algorithm \( \text{count} \) that takes an arbitrary function \( F : \{0, \ldots, N - 1\} \rightarrow \{0,1\} \) as input as well as some positive integer \( t \). If there are \( s \) values of \( x \) such that \( F(x) = 1 \), algorithm \( \text{count}(F, t) \) outputs an integer estimate \( \tilde{s} \) for \( s \) such that

\[
|s - \tilde{s}| < 2\pi \frac{\sqrt{s(N - s)}}{t} + \frac{\pi^2 N}{t^2}
\]

with probability at least \( 8/\pi^2 \) after exactly \( t \) evaluations of \( F \). In special case \( s = 0 \), \( \text{count}(F, t) \) always outputs perfect estimate \( \tilde{s} = 0 \).

The following theorem on amplitude amplification is also adapted from [3]. Its statement is rather more technical than that of the previous theorems.

**Theorem 2.4** (Amplitude estimation [3]). There exists a quantum algorithm \( \text{amplitude estimation} \) that takes as inputs two unitary transformations \( A \) and \( B \) as well as some positive integer \( t \). If

\[
A |0\rangle = \alpha |\psi_0\rangle + \beta |\psi_1\rangle
\]

(where \( |\psi_0\rangle \) and \( |\psi_1\rangle \) are orthogonal states and \( |0\rangle \) is of arbitrary dimension) and

\[
B |\psi_0\rangle |0\rangle = |\psi_0\rangle |0\rangle \quad \text{and} \quad B |\psi_1\rangle |0\rangle = |\psi_1\rangle |1\rangle ,
\]

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then \textbf{amplitude estimation}(A, B, t) outputs $\tilde{a}$, an estimate of $a = \|\beta\|^2$, such that

$$|\tilde{a} - a| \leq 2\pi \frac{\sqrt{a(1-a)}}{t} + \frac{\pi^2}{t^2}$$

with probability at least $8/\pi^2$ at a cost of doing $t$ evaluations each of $A$, $A^{-1}$ and $B$.

We shall also need the following technical result, which we derive using standard Chernoff bound arguments.

\textbf{Theorem 2.5} (Majority). Let $B$ be a quantum black box that approximates some function

$$F : \{0, \ldots, N - 1\} \to \{0, \ldots, M - 1\}$$

such that its output is within $\Delta$ of the true value with probability at least $2/3$, i.e.

$$B |i\rangle |0\rangle = \sum_j \alpha_{ij} |i\rangle |x_{ij}\rangle \text{ and } \sum_{\{j:|x_{ij} - F(i)| \leq \Delta\}} |\alpha_{ij}|^2 \geq 2/3$$

for all $i$. Then, for all $n$ there exists a quantum black box $B_n$ that computes $F$ with its output within $2\Delta$ of the true value with probability at least $1 - 1/n$, i.e.

$$B_n |i\rangle |0\rangle = \sum_j \beta_{ij} |i\rangle |y_{ij}\rangle \text{ and } \sum_{\{j:|y_{ij} - F(i)| \leq 2\Delta\}} |\beta_{ij}|^2 \geq 1 - 1/n$$

for all $i$. Algorithm $B_n$ requires $O(\log n)$ calls to $B$.

\textbf{Proof}. Given an input index $i$, $B_n$ calls $k$ times black box $B$ with input $i$, where $k = \lceil(\log n)/D(\frac{2}{5}\|\frac{2}{3}\|)\rceil$ and $D(\cdot\|\cdot)$ denotes the standard Kullback-Leibler divergence [9] (sometimes called the relative entropy). If there exists an interval of size $2\Delta$ that contains at least $3/5$ of the outputs, then $B_n$ outputs the midpoint of that interval. If there is no such interval (a very unlikely occurrence), then $B_n$ outputs 0. If at least $3/5$ of the outputs are within $\Delta$ of $F(i)$, then the output of $B_n$ cannot be further than $2\Delta$ from $F(i)$ since the interval selected by $B_n$ must contain at least one of those points. By the Chernoff bound, this happens with probability at least $1 - 2^{kD(\frac{2}{5}\|\frac{2}{3}\|)} \geq 1 - 1/n$. □

Hereinafter, we shall denote by \textbf{majority}(B, n) the black box $B_n$ that results from using this algorithm on black box $B$ with parameter $n$. Note that $D(\frac{2}{5}\|\frac{2}{3}\|) > 1/100$, hence \textbf{majority}(B, n) requires less than 100 $\log n$ calls to $B$. Note also that the number of calls to $B$ does not depend on $\Delta$.  

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3 Two Efficient Algorithms to Approximate the Mean

We present two different algorithms to compute the mean value of a function. In both algorithms, let $F : \{0, \ldots, N-1\} \rightarrow [0, 1]$ be a black-box function and let $m = \frac{1}{N} \sum_x F(x)$ be the mean value of $F$, which we seek to approximate. Without loss of generality, we assume throughout that $N$ is a power of 2. The first algorithm assumes that $F(x)$ can be obtained with arbitrary precision at unit cost while the second algorithm considers that the output of function $F$ is given with $\ell$ bits of precision.

**Algorithm 1 mean1($F, N, t$)**

Let

$$A'|x\rangle |0\rangle = |x\rangle (\sqrt{1 - F(x)} |0\rangle + \sqrt{F(x)} |1\rangle)$$

and

$$A = A' (H^{\otimes \lg N} \otimes \text{Id}),$$

where $H$ is the Walsh–Hadamard transform and $\text{Id}$ denotes the identity transformation on one qubit.

Let

$$B |x\rangle |0\rangle |0\rangle = |x\rangle |0\rangle |0\rangle$$

and

$$B |x\rangle |1\rangle |0\rangle = |x\rangle |1\rangle |1\rangle$$

return amplitude_estimation($A, B, t$)

Note that in Algorithm 1 it is easy to implement $A'$ (and therefore $A$ as well as $A^{-1}$) with only two evaluations of $F$. First, $F$ is computed in an ancillary register initialized to $|0\rangle$, then the appropriate controlled rotations are performed, and finally $F$ is computed again to reset the ancillary register back to $|0\rangle$. (In practice, this transformation will be approximated to a prescribed precision.) The following theorem formalizes the result obtained by this algorithm.

**Theorem 3.1.** Given a black-box function $F : \{0, \ldots, N-1\} \rightarrow [0, 1]$ and its mean value $m = \frac{1}{N} \sum_x F(x)$, algorithm mean1 outputs $\tilde{m}$ such that $|\tilde{m} - m| \in O(1/t)$ with probability at least $8/\pi^2$. The algorithm requires $4t$ evaluations of $F$.

**Proof.** Using the same definition as in Theorem 2.4, we have that

$$|\psi_1\rangle = \sum_x \sqrt{\frac{F(x)}{\sum_y F(y)}} |x\rangle |1\rangle$$

and

$$\beta = \sqrt{\frac{\sum_x F(x)}{N}}.$$

The algorithm amplitude_estimation($A, B, t$) returns an estimate $\tilde{m} = \tilde{a}$ of $a = \|\beta\|^2 = \frac{1}{N} \sum_x F(x) = m$ and thus $\tilde{m}$ is directly an estimate of $m$. The error
$|\tilde{m} - m|$ is at most

$$2\pi \sqrt{m(1-m)} t + \frac{\pi^2}{t^2} \in O\left(\frac{1}{t}\right)$$

(1)

with probability at least $8/\pi^2$. This requires $4t$ evaluations of $F$ because each of the $t$ calls on $A$ and on $A^{-1}$ requires 2 evaluations of $F$.

This theorem states that the error goes down asymptotically linearly with the number of evaluation of $F$. This is optimal according to Nayak and Wu [10], who have proven that in the general case in which we have no a priori knowledge of the possible distribution of outputs, an additive error of $\varepsilon$ requires an amount of work proportional to $1/\varepsilon$ in the worst case when the function is given as a black box. Note that for $t < (1 + \sqrt{2})\pi/\sqrt{m}$, when our bound on the error exceeds the targeted mean (which is rather bad), the error goes down quadratically (which is good).

We now present a variation on an algorithm of Aharonov [1] and analyse its characteristics. This algorithm is also based on amplitude estimation, but it relies on the fact that points in real interval $[0,1]$ can be represented in binary as $\ell$-bit strings, where $\ell$ is the precision with which we wish to consider the output of black-box function $F$. The algorithm estimates the number of 1s in each binary position. The difference between our algorithm (mean2) and Aharonov’s original algorithm is that we make sure that the estimates of the counts in every bit position are all simultaneously within the desired error bound. For each $i$ between 1 and $\ell$, let $F_i(x)$ represent the $i^{th}$ bit of the binary expansion of $F(x)$, so that $F(x) = \sum_i F_i(x)2^{-i}$, with the obvious case $F_i(x) = 1$ for all $i$ when $F(x) = 1$.

**Algorithm 2 mean2($F$,$N$)**

```plaintext
for $i = 1$ to $\ell$
    $\tilde{m}_i = \text{majority}(\text{count}(F_i(x), 5\pi\sqrt{N}), n = \lceil \frac{3}{2}\ell \rceil)$
end for
return $\tilde{m} = \frac{1}{N} \sum_{i=1}^{\ell} \tilde{m}_i 2^{-i}$
```

**Theorem 3.2.** Given a black-box function $F : \{0, \ldots, N-1\} \rightarrow [0,1]$ where the output of $F$ has $\ell$ bits of precision, algorithm mean2 outputs an estimate $\tilde{m}$ such that $|\tilde{m} - m| \leq \frac{1}{N} \sum_i \sqrt{m_i} 2^{-i}$, where $m_i = \sum_x F_i(x)$, with probability at least $2/3$. The algorithm requires $O(\sqrt{N} \ell \log \ell)$ evaluations of $F$.

**Proof.** The proof is a straightforward corollary of Theorems 2.3 and 2.5. Using count on each column with $s = 5\pi\sqrt{N}$ yields an error of

$$|m_i - \tilde{m}_i| \leq \frac{2}{5} \sqrt{m_i} + \frac{1}{25}$$
with probability at least \(8/\pi^2\), and hence with probability at least \(2/3\), where \(\hat{m}_i\) denotes \(\text{count}(F_i(x), 5\pi\sqrt{N})\). Using \text{majority} with \(n = \lceil 3/2 \ell \rceil\) on this, we obtain

\[
|m_i - \hat{m}_i| \leq \frac{4}{5} \sqrt{m_i} + \frac{2}{25}
\]

with probability at least \(1 - 2/3\ell\). When \(m_i \geq 1\), this is bounded by \(\sqrt{m_i}\). Furthermore, \text{count} makes no error when \(m_i = 0\). Hence, the error in each column is bounded by \(\sqrt{m_i}\) with probability at least \(1/2\). The union bound, all of our estimates for the columns are simultaneously within the above error bounds with probability at least \(2/3\), and the error bound on our final estimate is \(|\tilde{m} - m| \leq \frac{1}{N} \sum \sqrt{m_i} 2^{-i}\). It is straightforward to count the number of evaluations of \(F\) from Theorems 2.3 and 2.5.

The choice of which among algorithms \text{mean1} or \text{mean2} is more appropriate depends on the particular characteristics of the input function. For example, one can consider the situation in which \(F(x) = 2^{-N}\) for all \(x\), hence the mean \(m = 2^{-N}\) as well. In this case, if we choose \(t = cN^{3/2} \lg N\) in \text{mean1} and \(\ell = N\) in \text{mean2}, where constant \(c\) is chosen so that both algorithms call function \(F\) the same number of times, the first algorithm is in the regime \(t \ll (1 + \sqrt{2})\pi/\sqrt{m}\) where it performs badly because the error on the estimated mean is expected to be much larger than the mean itself. On the other hand, the expected error produced by the second algorithm is bounded by \(m/\sqrt{N}\), which is much smaller than the targeted mean. At the other end of the spectrum, if \(F(x) = 1/2\) for all \(x\), hence the mean \(m = 1/2\) as well, and if \(t \gg 2\pi\sqrt{N}\), then the error produced by \text{mean1} is much smaller than \(m/\sqrt{N}\) according to Equation (1). With the same parameters, the error produced by \text{mean2}, which is again bounded by \(m/\sqrt{N}\), is strictly unaffected by the choice of \(\ell\), so that the second algorithm can work arbitrarily harder than the first, yet produce a less precise estimate of the mean.

4 Approximate Median Algorithm

Let \(\text{dist}: \{0, \ldots, N-1\} \times \{0, \ldots, N-1\} \rightarrow [0,1]\) be an arbitrary black-box distance function.

**Definition 4.1** (Median). The median is the point within an ensemble of points whose average distance to the other points is minimum.

Formally, the median of a set of points \(Q = \{0, \ldots, N-1\}\) is

\[
\text{median}(Q) = \arg \min_{z \in Q} \sum_{j=0}^{N-1} \text{dist}(z, j).
\]
The median can be found classically by going through each point \( z \in Q \), computing the average distance from \( z \) to all the other points in \( Q \), and then taking the minimum (ties are broken arbitrarily). This process requires a time of \( O(N^2) \). In the general case, in which there are no restrictions on the distance function used and no structure among the ensemble of points that can be exploited, no technique can be more efficient than this naïve algorithm. Indeed, consider the case in which all the points are at the same distance from each other, except for two points that are closer than the rest of the points. These two points are the medians of this ensemble. In this case, classically we would need to query the oracle for the distances between each and every pair of points before we can identify one of the two medians. (We expect to discover this special pair after querying about half the pairs on the average but we cannot know that there isn’t some other even closer pair until all the pairs have been queried.) This results in a lower bound of \( \Omega(N^2) \) calls to the oracle.

In Algorithm 3, \textit{mean} stands for either one of the two algorithms given in the previous section (in case \textit{mean1} is used, parameter \( t \) must be added) but it is repeated \( O(\log N) \) times in order to get all the means within the desired error bound with a constant probability via our \textit{majority} algorithm (Theorem 2.5). Here, \( d_i = \frac{1}{N} \sum_j \text{dist}(i, j) \) and \( d_{\min} = d_k \) for any \( k \) such that \( d_k \leq d_i \) for all \( i \).

**Algorithm 3** median(dist)

\begin{itemize}
  \item For each \( i \), define function \( F_i(x) = \text{dist}(i, x) \)
  \item For each \( i \), define \( \tilde{d}_i = \text{majority}(\text{mean}(F_i, N), n = N^2) \)
  \item return minimum(\( \tilde{d}_i \))
\end{itemize}

**Theorem 4.2.** For any black-box distance function

\[
\text{dist} : \{0, \ldots, N - 1\} \times \{0, \ldots, N - 1\} \to [0, 1],
\]

when \textit{mean1} is used with parameter \( t \), algorithm \textit{median} outputs an index \( j \) such that \( |d_j - d_{\min}| \in O(1/t) \) with probability at least \( \frac{2}{3} \). The algorithm requires \( O(t \sqrt{N} \log N) \) evaluations of \textit{dist}.

**Proof.** This result is obtained by a straightforward combination of Theorems 2.2, 2.5 and 3.1. The procedure \textit{majority} is used with parameter \( n = N^2 \) to ensure that all the \( d_i \)'s computed by the algorithm (in superposition) are simultaneously within the bound given by Theorem 3.1 except with probability \( o(1) \). Note that with parameter \( n = N^2 \), the number of repetitions is still in \( O(\log N) \). The success probability of the algorithm follows from the fact that \( \frac{2}{3}(1 - o(1)) > \frac{2}{3} \). In this case, the error is in \( O(1/t) \) and the number of evaluations of \textit{dist} is in \( O(t \sqrt{N} \log N) \). \qed
By replacing \texttt{mean1} by \texttt{mean2} in the \texttt{median} algorithm we obtain the following theorem.

**Theorem 4.3.** For any black-box distance function

\[ \text{dist} : \{0, \ldots, N - 1\} \times \{0, \ldots, N - 1\} \rightarrow [0, 1], \]

when \texttt{mean2} is used, algorithm \texttt{median} outputs an index \( j \) such that

\[ |d_j - d_{\text{min}}| \leq \frac{1}{N} \sum_{i=1}^{\ell} \sqrt{m_i} 2^{-i} \]

with probability at least \( \frac{2}{3} \). (See algorithm \texttt{mean2} for a definition of \( m_i \) and \( \ell \).) The algorithm requires \( O(N \log N) \) evaluations of \texttt{dist}.

## 5 Conclusion

We have described two quantum algorithms to approximate the mean and their applications to approximate the median of a set of points over an arbitrary distance function given by a black box. We leave open for future work an in-depth study on how the different behaviour of the two algorithms impact the quality of the median they return. For instance, we know that the behaviour of both algorithms for the mean depends on the distribution of data points and the distances between points, but we still have to investigate more precisely the exact context where it matters. Of course, understanding the behaviour of the algorithms in different contexts is important, but a more interesting question is to tailor the algorithm to obtain better results on different data distributions of interest.

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