Variance of Entropy Decreases Under the Polar Transform

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Abstract

We consider the behavior of entropy of data elements as they are operated on by polar transforms. The data elements under consideration are pairs \((X, Y)\) where \(X\) is a binary random variable and \(Y\) is some side information about \(X\). The entropy random variable for such a data element is defined as \(h(X|Y) = -\log p_{X|Y}(X|Y)\). The variance of entropy (varentropy) is defined as \(\text{Var}(h(X|Y))\). A polar transform of order two is a mapping that takes two independent data elements and produces two new data elements (that are in general correlated). We show that the sum of the varentropies of the output data elements is less than or equal to the sum of the varentropies of the input data elements, with equality if and only if at least one of the input data elements has zero varentropy. This result is then extended to polar transforms of higher orders and an asymptotic analysis is given for an important special case in which the average output varentropy decreases monotonically to zero as the transform size increases.

I. INTRODUCTION

Polar coding is a method for constructing capacity-achieving channel and source codes [1], [2]. Polar codes are constructed by manipulating a number of independent copies of a given channel or source through certain transformations so as to synthesize new channels or sources that are more suitable for coding. This paper focuses on the behavior of the variance of entropy as the channels or sources undergo a polar transform. In this section, we will first define the problem more precisely. Then, we will state the basic result of the paper. This will be followed by some examples. The section will end with an outline of the rest of the paper.

A. Varentropy

Given any pair of discrete random variables \((X, Y)\), define the conditional entropy random variable as

\[
h(X|Y) \triangleq -\log p_{X|Y}(X|Y).
\]

The average conditional entropy is denoted as usual by

\[
H(X|Y) \triangleq \mathbb{E}h(X|Y)
\]

while the variance of entropy will be denoted as

\[
V(X|Y) \triangleq \text{Var}(h(X|Y)).
\]
All logarithms will be to the base 2.

Henceforth, we will use the term “varentropy” to refer to the variance of entropy. This term appears to have been coined only recently by Kontoyiannis and Verdú [3], although the concept has been in the literature for a long time. The recognition that varentropy is a key parameter in determining the fine asymptotic behavior of the probability of error in source and channel coding problems goes back to Strassen [4]. In more recent work, Polyanskiy, Poor and Verdú [5] gave a comprehensive treatment of the subject and elucidated the theoretical and practical significance of varentropy and related parameters for estimating the performance of codes at finite block lengths.

B. Polar transform

Polar transforms are in essence operations on data elements of the form \((X, Y)\). The data elements under consideration in this paper will be such that \(X\) takes values in \(\mathcal{X} = \{0, 1\}\) while \(Y\) will take values in a finite but otherwise arbitrary alphabet \(\mathcal{Y}\). In a channel coding context, \(X\) will represent the input to a binary-input channel and \(Y\) the channel output. In a source coding context, \(X\) will represent a Bernoulli random variable and \(Y\) some side information about \(X\). A data element \((X, Y)\) will be called extreme if \(H(X|Y)\) equals 0 or 1. We will write \((X; Y)\) to denote a data element when \(Y\) consists of a list of random variables. Throughout the paper we will develop alternative representations of a data element as the need arises.

The specific polar transform considered in this paper takes as input a pair of independent data elements \((X_1, Y_1), (X_2, Y_2)\) and produces as output another pair of data elements \((X_1 \oplus X_2, Y), (X_2; X_1 \oplus X_2, Y)\), where \(Y = (Y_1, Y_2)\) and \(\oplus\) denotes addition modulo 2. We will prefer to write the output data elements as \((U_1, Y)\) and \((U_2; U_1, Y)\) by defining

\[
U_1 = X_1 \oplus X_2 \quad \text{and} \quad U_2 = X_2.
\]  

Note that while the data elements at the input are independent by assumption, the data elements at the output are in general correlated. The creation of such correlations is in fact the underlying mechanism for creating the desired “polarization” effects as the results of this paper will show in a more quantitative manner.

A basic property of the polar transform is conservation of entropy:

\[
H(U_1|Y) + H(U_2|U_1, Y) = H(X_1|Y_1) + H(X_2|Y_2).
\]  

A second basic property is polarization in the sense that

\[
H(U_1|Y) \geq \max\{H(X_1|Y_1), H(X_2|Y_2)\}
\]  

\[
H(U_2|U_1, Y) \leq \min\{H(X_1|Y_1), H(X_2|Y_2)\}.
\]  

This means that the data elements at the output are “more extreme” than those at the input. The entropy conservation and polarization properties of polar transforms were exploited in [1], [2] to construct capacity-achieving channel and source codes by considering recursive extensions of the basic transform [1] that enhanced the basic polarization effect to the extent that almost all data elements became extreme, asymptotically as the transform size grew.
C. The basic result

**Theorem 1.** The varentropy decreases under the polar transform \( \mathbf{1} \) in the sense that

\[
V(U_1|Y) + V(U_2|U_1, Y) \leq V(X_1|Y_1) + V(X_2|Y_2),
\]

(5)

with equality iff either \((X_1, Y_1)\) or \((X_2, Y_2)\) is extreme.

This theorem is proved in Section IV. Note that the theorem is stated and will be proved with the restriction that the alphabets \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \) are finite. This finiteness condition has been imposed to keep the mathematical detail at a minimum. It is possible to generalize the results to more general alphabets, as will be discussed in Section IV-E.

Theorem \( \mathbf{1} \) can be interpreted either from a channel coding or a source coding viewpoint. For the former, we regard the data elements \((X_1, Y_1)\) and \((X_2, Y_2)\) as independent binary-input memoryless channels. Then, the polar transform \( \mathbf{1} \) may be seen as creating two synthetic channels, a first channel \((U_1, Y)\) with input \(U_1\) and output \(Y\), and a second channel \((U_2; U_1, Y)\) with input \(U_2\) and output \((U_1, Y)\). Theorem \( \mathbf{1} \) states that the sum of the varentropies for the synthetic channels is smaller than the sum of the varentropies of the original channels, except when at least one of the channels at the input side is extreme. If we take a source coding viewpoint, then we regard each data element \((X_i, Y_i)\) as comprising a Bernoulli source \(X_i\) with side information \(Y_i\). In this case, the polar transform creates two new sources with side information, again leading to a shrinkage in varentropy.

Although the motivation of studying varentropy in this paper is related to polar coding applications, we wish to explain the relevance of Theorem \( \mathbf{1} \) to the main body of literature on varentropy. As shown in [5], under optimal coding techniques at block length \( N \), the amount of back-off from channel capacity, to achieve a block error probability of \( \epsilon \), is given essentially by \( \sqrt{V/NQ^{-1}(\epsilon)} \) where \( V \) is channel varentropy (or “dispersion” as it is called in [5]) and \( Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \). Now consider two application scenarios. In the main scenario, we use a binary-input memoryless channel \( W = (X, Y) \) with varentropy \( V \) and apply optimal block coding methods at length \( N \). In the alternative scenario, we first apply polar transforms of order two to create from each pair of \( W \) a pair of new binary-input channels, \( W' = (X', Y') \) with varentropy \( V' \) and \( W'' = (X'', Y'') \) with varentropy \( V'' \). Thus, we create \( N/2 \) copies of \( W' \) and \( N/2 \) copies of \( W'' \) from \( N \) copies of \( W \). The \( W' \) channels are independent among themselves. The \( W'' \) channels are also independent among themselves. However, there is coupling between the two sets of channels since they are created in correlated pairs. Ignoring this correlation and the practical details emanating from its presence, consider applying optimal coding techniques at block length \( N/2 \) to \( W' \) and \( W'' \) separately. To achieve a given probability of error \( \epsilon \), the back-off in the main scenario is \( B = c\sqrt{V/N} \) with \( c = Q^{-1}(\epsilon) \). The back-off in the alternative scenario is \( B' = c\sqrt{2V'/N} \) for \( W' \) and \( B'' = c\sqrt{2V''/N} \) for \( W'' \). Theorem \( \mathbf{1} \) states that \( \sqrt{(B')^2 + (B'')^2} \leq B \) with equality iff \( W \) is an extreme channel. Thus, roughly speaking, the closer \( W \) is to an extreme channel, the smaller is the loss as measured by the back-off from channel capacity.

Before we end this section, we give an alternative formulation of the main result. Let us introduce the following shorthand notation for the entropy random variables at the input and output side of the polar transform:

\[
h_{\text{in}, 1} \triangleq h(X_1|Y_1), \quad h_{\text{in}, 2} \triangleq h(X_2|Y_2),
\]

(6)
\[ h_{\text{out},1} \triangleq h(U_1|Y), \quad h_{\text{out},2} \triangleq h(U_2|U_1,Y). \] (7)

This notation allows rewriting inequality (5) as

\[ \text{Var}(h_{\text{out},1}) + \text{Var}(h_{\text{out},2}) \leq \text{Var}(h_{\text{in},1}) + \text{Var}(h_{\text{in},2}). \] (8)

However, the real reason for introducing the new notation is to write down the identity

\[ h_{\text{out},1} + h_{\text{out},2} = h_{\text{in},1} + h_{\text{in},2}, \] (9)

which is the analog of (2) and expresses an entropy conservation on a per-sample basis. From (9), we have

\[ \text{Var}(h_{\text{out},1} + h_{\text{out},2}) = \text{Var}(h_{\text{in},1} + h_{\text{in},2}). \]

Since \( h_{\text{in},1} \) and \( h_{\text{in},2} \) are independent,

\[ \text{Var}(h_{\text{in},1} + h_{\text{in},2}) = \text{Var}(h_{\text{in},1}) + \text{Var}(h_{\text{in},2}); \]

while by a general identity \( \text{Var}(h_{\text{out},1} + h_{\text{out},2}) \) equals

\[ \text{Var}(h_{\text{out},1}) + \text{Var}(h_{\text{out},2}) + 2 \text{Cov}(h_{\text{out},1}, h_{\text{out},2}). \]

Thus, we obtain the following reformulation of Theorem 1.

**Theorem 1**. The entropy terms \( h_{\text{out},1} \) and \( h_{\text{out},2} \) at the output of the polar transform satisfy the covariance inequality

\[ \text{Cov}(h_{\text{out},1}, h_{\text{out},2}) \geq 0, \] (10)

with equality iff either \((X_1, Y_1)\) or \((X_2, Y_2)\) is extreme.

This form makes it clear that any reduction in varentropy can be attributed entirely to the creation of a positive correlation between the entropy random variables \( h_{\text{out},1} \) and \( h_{\text{out},2} \) at the output of polar transform. Our proof of Theorem 1 will be directed at showing that (10) is true.

**D. Examples**

In this section, we give two examples to illustrate Theorem 1. In both examples, \((X_1, Y_1)\) and \((X_2, Y_2)\) are independent copies of a given data element \((X, Y) \sim p(x,y)\). The terminology in both examples reflects a channel coding viewpoint, although each example has a dual source coding interpretation as well.

**Example 1** (Binary Symmetric Channel). Suppose that the data element \((X,Y)\) represents a BSC with crossover probability \(0 \leq \epsilon \leq \frac{1}{2}\). In other words, assume that \(X\) and \(Y\) take values in the set \(\{0,1\}\) and that

\[
p(x,y) = \begin{cases} \frac{1}{2}(1-\epsilon) & \text{if } x = y; \\ \frac{1}{2}\epsilon & \text{if } x \neq y. \end{cases}
\]

A straightforward calculation gives the varentropy of such a BSC as \(\epsilon(1-\epsilon)(\log \frac{1-\epsilon}{\epsilon})^2\). This gives the curve \(\text{Var}(h_{\text{in}})\) in Fig. 1. The figure also displays the output varentropy terms \(\text{Var}(h_{\text{out},1})\) and \(\text{Var}(h_{\text{out},2})\), and the
covariance term Cov(h_{out,1}, h_{out,2}). The non-negativity of the covariance is an indication that the varentropy is reduced on average by the polar transform.

Example 2 (Binary Erasure Channel). Suppose the data element (X, Y) represents a BEC with erasure probability $\epsilon$. In this case, X takes values in \{0, 1\}, Y takes values in \{0, 1, 2\}, and

$$p(x, y) = \begin{cases} \frac{1}{2}(1 - \epsilon) & \text{if } x = y; \\ \frac{1}{2} \epsilon & \text{if } y = 2. \end{cases}$$

Let $h_{in} \triangleq -\log p_X|Y(X|Y)$. For this channel, the following simple formulas are obtained for the varentropy terms.

$\text{Var}(h_{in,1}) = \text{Var}(h_{in,2}) = \text{Var}(h_{in}) = \epsilon(1 - \epsilon)$, $\text{Var}(h_{out,1}) = (2\epsilon - \epsilon^2)(1 - \epsilon)^2$, $\text{Var}(h_{out,2}) = \epsilon^2(1 - \epsilon^2)$. The covariance term is given by $\text{Cov}(h_{out,1}, h_{out,2}) = \epsilon^2(1 - \epsilon)^2$. The corresponding curves are plotted in Fig. 2.

Remark 1. The above examples show that there is no analog of the polarization relations (3) and (4) for varentropy. The only order relation exhibited by the variance terms in the above examples is that $\text{Var}(h_{in}) \geq \min\{\text{Var}(h_{out,1}), \text{Var}(h_{out,2})\}$, which is indeed a consequence of Theorem 7.

Remark 2. While the entropy function satisfies $H(X|Y) \leq H(X)$, there is no general ordering between the varentropy terms $V(X|Y)$ and $V(X) \triangleq \text{Var}(-\log p(X))$. For example, if $(X, Y)$ is a BSC with $\epsilon = \frac{1}{4}$, then $V(X) = 0$ while $V(X|Y) > 0$. On the other hand if $(X, Y)$ is such that $Y = X$, then $V(X|Y) = 0$ while $V(X)$ can be non-zero.

E. Organization

The rest of the paper is organized as follows. In Section II, we define two parameters $A$ and $B$ to represent a given data element $(X, Y)$. These parameters serve as a “sufficient statistic” for the types of problems considered.
Fig. 2. Varentropy and covariance for BEC under polar transform.

in this paper. In Section III we summarize some known inequalities about correlations of monotone functions. Section IV contains the proof of Theorem II. Section V considers the behavior of varentropy under higher order polar transforms. The paper concludes with Section VI where an asymptotic analysis and some final remarks are given.

II. PARAMETRIC REPRESENTATION OF DATA ELEMENTS

Until now, the problem formulation has been given in terms of data elements $(X,Y)$ where $X$ takes values in $\mathcal{X} = \{0, 1\}$ while $Y$ takes values in an arbitrary (finite) alphabet $\mathcal{Y}$. The arbitrary nature of $\mathcal{Y}$, which is assumed for the sake of generality, complicates the analysis unnecessarily. The information measures that we are concerned with are determined solely by the joint probability assignment on $(X,Y)$ and the specific details of $\mathcal{Y}$ play no role. Hence, it is possible and desirable to re-parametrize the problem so that $Y$ is replaced with an equivalent random variable that takes values over a canonical alphabet. Such canonical representations have been given for Binary Memoryless Symmetric (BMS) channels in [6]. Here, the class of data elements $(X,Y)$ is more general than BMS channels but similar ideas apply.

We associate to each $y \in \mathcal{Y}$ the parameter

$$\alpha(y) \triangleq p_{X|Y}(1|y)$$

and define a random variable $A \triangleq \alpha(Y)$. The alphabet of possible values of the $\alpha$-parameter is given by $\mathcal{A} \triangleq \{\alpha(y) : y \in \mathcal{Y}\} \subset [0, 1]$. The $\alpha$ parameter takes the original representation $(X,Y)$ to a canonical representation $(X,A)$. The new representation $(X,A)$ hides irrelevant details by using an alphabet $\mathcal{A}$ that is always a subset of $[0, 1]$ and merges any two symbols $y, y' \in \mathcal{Y}$ into a common symbol whenever $\alpha(y) = \alpha(y')$. 

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We give some examples to illustrate the canonical representations. For the BSC \((X, Y)\) in Example 1 we have \(\alpha(0) = \epsilon, \alpha(1) = 1 - \epsilon, A = \{\epsilon, 1 - \epsilon\}\) and
\[
p_{X,A}(x, a) = \begin{cases} 
\frac{1}{2}(1 - \epsilon) & \text{if } (x, a) \in \{(0, \epsilon), (1, 1 - \epsilon)\}; \\
\frac{1}{2}\epsilon & \text{if } (x, a) \in \{(1, \epsilon), (0, 1 - \epsilon)\}.
\end{cases}
\]
In the case of the BEC in Example 2 we have \(\alpha(0) = 0, \alpha(1) = 1, \alpha(2) = \frac{1}{2}, A = \{0, \frac{1}{2}, 1\}\) and
\[
p_{X,A}(x, a) = \begin{cases} 
\frac{1}{2}(1 - \epsilon) & \text{if } (x, a) \in \{(0, 0), (1, 1)\}; \\
\frac{1}{2}\epsilon & \text{if } (x, a) \in \{(1, \frac{1}{2}), (0, \frac{1}{2})\}.
\end{cases}
\]
As a third example, consider the Z-channel, characterized by the joint distribution
\[
p_{X,Y}(x, y) = \begin{cases} 
q(1 - \epsilon) & \text{if } (x, y) \in \{(1, 1)\}; \\
q\epsilon & \text{if } (x, y) \in \{(1, 0)\}; \\
(1 - q) & \text{if } (x, y) \in \{(0, 0)\}.
\end{cases}
\]
where \(\epsilon q = p_{X}(1)\) are parameters in the range \([0, 1]\). Now, we have \(\alpha(0) = \delta \overset{\Delta}{=} q\epsilon/(q(1 - q)), \alpha(1) = 1, A = \{\delta, 1\}, \) and
\[
p_{X,A}(x, a) = \begin{cases} 
q(1 - \epsilon) & \text{if } (x, a) \in \{(1, 1)\}; \\
q\epsilon & \text{if } (x, a) \in \{(1, \delta)\}; \\
(1 - q) & \text{if } (x, a) \in \{(0, \delta)\}.
\end{cases}
\]
Although the canonical form \((X, A)\) gets rid of inessential features of \((X, Y)\), there is need for an even more compact representation for the type of problems considered in the sequel. This more compact representation is obtained by associating to each \(y \in Y\) a second parameter
\[
\beta(y) \overset{\Delta}{=} \min\{\alpha(y), 1 - \alpha(y)\} = \min\{p_{X|Y}(0|y), p_{X|Y}(1|y)\},
\]
(12)
and defining an associated random variable \(B \overset{\Delta}{=} \beta(Y)\). The range of possible values of \(B\) is given by \(B = \{\beta(y) : y \in Y\} \subset [0, \frac{1}{2}]\). \(B\) may be thought of as the probability of MAP decision error in deciding \(X\) when \(Y\) is supplied.

We now show that the main information-theoretic measures of interest about a data element \((X, Y)\) can be expressed in terms of the parameters \(A\) and \(B\). First, the conditional entropy can be written as
\[
H(X|Y = y) = H(X|A = \alpha(y)) = H(\alpha(y)) = H(\beta(y))
\]
where \(H(p) \overset{\Delta}{=} -p \log(p) - (1 - p) \log(1 - p), p \in [0, 1],\) is the binary entropy function. Taking expectations, we obtain
\[
H(X|Y) = H(X|A) = H(X|B) = \mathbb{E} H(A) = \mathbb{E} H(B).
\]
The second moment of \( h(X|Y) \) can be expressed as
\[
\mathbb{E}[h(X|Y)^2] = \mathbb{E}[A \log^2 A + (1 - A) \log^2 (1 - A)]
= \mathbb{E}[B \log^2 B + (1 - B) \log^2 (1 - B)].
\]
Thus, the varentropy \( V(X|Y) \) is fully determined by \( A \) or \( B \). In fact, one can express the moment generating function of \( h(X|Y) \) as
\[
\mathbb{E} 2^{sh(X|Y)} = \mathbb{E}[A^{1-s} + (1 - A)^{1-s}]
= \mathbb{E}[B^{1-s} + (1 - B)^{1-s}].
\]
For \( a \in [0, 1] \), let \( g_a(s) \triangleq a^{1-s} + (1 - a)^{1-s} \). For each \( a \in [0, 1] \), \( g_a(s) \) is an analytical function of \( s \leq 1 \); hence, derivatives \( g_a^{(n)}(0) = d^n g_a(s)/ds^n|_{s=0} \) of all orders \( n \geq 1 \) exist. Furthermore, \( g_a^{(n)}(0) \) is a continuous function of \( a \in [0, 1] \) for any fixed \( n \); so, the maximum of \( g_a^{(n)}(0) \) over \( a \in [0, 1] \) exists and is finite. Hence, the entropy random variable \( h(X|Y) \) has finite moments of all orders \( n \geq 1 \). In particular, the second moment is bounded by
\[
E[h(X|Y)^2] \leq \max_{0 \leq x \leq 1} [x \log^2(x) + (1 - x) \log^2 (1 - x)]
\leq 2 \max_{0 \leq x \leq 1} [x \log^2(x)] = 8e^{-2}\log^2(e) \approx 2.2434.
\]
Since \( V(X|Y) \leq E[h(X|Y)^2] \), the varentropy is also bounded universally by \( 8e^{-2}\log^2(e) \). This is a loose bound but it will be sufficient for our purposes, in particular, at a certain point in Section \[\text{VI}\]

It may appear that \( B \) is superfluous since anything that can be expressed in terms of \( B \) can also be expressed in terms of \( A \). The real reason for introducing \( B \) will become clear later in the paper when certain correlation inequalities are considered. We will in fact prefer to use \( B \) rather than \( A \) since \( B \) is more compact and certain functions that are monotone in \( B \) need not be so in \( A \). For example, \( \mathcal{H}(B) \) is an increasing function over the range \( B \subset [0, \text{\frac{1}{2}}] \) of \( B \), but \( \mathcal{H}(A) \) need not be monotone over the range \( A \subset [0, 1] \) of \( A \).

As a convention, in the following analysis, we will retain the given data element \((X, Y)\) as the original representation of the problem, but carry out most calculations using the parameter \( B \). We will classify a data element \((X, Y)\) in terms of the characteristics of the range of \( B \) as follows. A data element \((X, Y)\) will be called pure if \(|B| = 1\), mixed otherwise. A pure \((X, Y)\) will be called extreme if \( B = \{0\} \) or \( B = \{\frac{1}{2}\} \). (This definition is consistent with the earlier definition of the term “extreme”.) An extreme \((X, Y)\) will be called purely deterministic if \( B = \{0\} \) and purely random if \( B = \{\frac{1}{2}\} \). In channel coding terms, a pure \((X, Y)\) is the equivalent of a BSC. An example of a mixed \((X, Y)\) is a BEC with an erasure probability \( \epsilon \in (0, 1) \). Such a BEC has \( B = \{0, \frac{1}{2}\} \) and is a mixture of \((1 - \epsilon)\)-part purely deterministic channel and \( \epsilon \)-part purely random channel. In general, a data element \((X, Y)\) with \( B = \{0, \frac{1}{2}\} \) will be called an erasure data element.

### III. Covariance decomposition and an inequality

In this part, we give a formula for covariance decomposition and a correlation inequality which will be useful for the proof of Theorem \[\text{I}\] in the next section. We will use the following notational conventions. Let \((S, T)\) be a
joint ensemble consisting of two random vectors $S$ and $T$. We will write $E_{S|T}$ and $\text{Cov}_{S|T}$ to denote expectation and covariance operators with respect to the conditional distribution of $S$ given $T$; $E_T$ and $\text{Cov}_T$ will denote expectation and covariance with respect to the marginal distribution of $T$; and, $E$ and $\text{Cov}$ will denote expectation and covariance with respect to the full ensemble.

The first result we wish to recall is the following formula for decomposing a covariance.

**Lemma 1.** Let $S$, $T$ be jointly distributed random vectors, with dimensions $m$ and $n$, respectively. Let $f, g : \mathbb{R}^{m+n} \to \mathbb{R}$ be functions such that $\text{Cov}[f(S, T), g(S, T)]$ exists, i.e., $E_f(S, T)g(S, T)$, $E_f(S, T)$, and $E_g(S, T)$ all exist. Then,

$$\text{Cov}[f(S, T), g(S, T)] = E_T \text{Cov}_{S|T}[f(S, T), g(S, T)] + \text{Cov}_T[E_{S|T}f(S, T), E_{S|T}g(S, T)].$$

Although this is an elementary result, we give a proof here mainly for illustrating the notation. Our proof follows [7].

**Proof:** We will omit the arguments of the functions for brevity.

$$\text{Cov}(f, g) = E_{S,T}fg - E_{S,T}f \cdot E_{S,T}g = E_T E_{S|T}fg - E_T[E_{S|T}f \cdot E_{S|T}g] + E_T[E_{S|T}f \cdot E_{S|T}g] - E_T E_{S|T}f \cdot E_T E_{S|T}g$$

$$= E_T \text{Cov}_{S|T}(f, g) + \text{Cov}_T[E_{S|T}f, E_{S|T}g].$$

We now give a correlation inequality. The subject of correlation inequalities is a rich one, with inequalities of various forms and varieties. Here, we select an inequality due to Esary, Proschan, and Walkup [7] which is directly applicable to the specific problems considered in this paper. The interested reader is referred to [8, Ch. 5] for a comprehensive survey of early results on the subject, and to [9, Ch. 6] for more recent results. Our presentation below follows [7].

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called increasing if, for all $x, y \in \mathbb{R}^n$, $f(x) \leq f(y)$ whenever $x_i \leq y_i$ for all $i = 1, \ldots, n$. A collection of random variables $T = (T_1, T_2, \ldots, T_n)$ is called associated if

$$\text{Cov}[f(T), g(T)] \geq 0$$

for all increasing functions $f, g : \mathbb{R}^n \to \mathbb{R}$ for which $E_f(T)$, $E_g(T)$, and $E_f(T)g(T)$ exist.

In [7] various sufficient conditions are given for a set of random variables to be associated. For our purposes the following characterization is all that is needed.

**Lemma 2** (Theorem 2.1 in [7]). Independent random variables are associated.

For the proof, we refer to [7]. A concise proof of Lemma 2 can also be found in [10, Sect. 9.7].

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In the univariate case ($T = T_1$), Lemma 2 is equivalent to the well-known Chebyshev correlation inequality [11 p. 43]. In that case, it is known that equality holds in (14) for a specific pair of increasing functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ iff either $f(T_1)$ or $g(T_1)$ is constant. (Here and elsewhere, when we say that a random variable is “constant”, we mean “constant almost surely”.)

Unfortunately, in the multivariate case, characterizing necessary and sufficient conditions for equality in (14) is not so simple. To see some of the difficulties, consider the bivariate case $T = (T_1, T_2)$ with independent $T_1$ and $T_2$. As a first example, let $f(T) = T_1$ and $g(T) = T_2$. Then, $\text{Cov}[f(T), g(T)] = 0$. Thus, the covariance may be zero although neither $f(T)$ nor $g(T)$ is constant. As a second example, take $f(T) = T_1$ and $g(T) = T_1 + T_2$. Then, $\text{Cov}[f(T), g(T)] = \text{Var}(T_1)$, which may be strictly positive. This second example shows that the covariance may be strictly positive even when one of the functions ($f$ in this case) is conditionally constant given one of the variables ($T_i$).

Instead of trying to identify the necessary and sufficient conditions for equality in (14) in general, we will take a pragmatic approach and consider only the particular case of interest in the rest of the paper. The following lemma addresses this case.

**Lemma 3.** Let $T = (T_1, T_2)$ be any pair of independent random variables. Let $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be any pair of increasing functions such that $E_f(T), E_g(T)$, and $f(T)g(T)$ exist. Then, $\text{Cov}[f(T), g(T)] = 0$ iff the following two conditions are simultaneously satisfied:

1) $f(T)$ is independent of $T_2$ or $g(T)$ is independent of $T_2$.

2) $E_{T_1} f(T)$ or $E_{T_1} g(T)$ is constant.

Alternatively, $\text{Cov}[f(T), g(T)] = 0$ iff the above conditions are true with $T_2$ in the role of $T_1$ and vice versa.

**Proof:** Use the conditional covariance formula (13) to write

$$\text{Cov}[f(T), g(T)] = E_{T_1} \text{Cov}_{T_2}[f(T), g(T)]$$

$$+ \text{Cov}_{T_1}[E_{T_2} f(T), E_{T_2} g(T)].$$

(15)

Note that here we used the independence of $T_1$ and $T_2$ to write $E_{T_2}$ and $\text{Cov}_{T_2}$ instead of $E_{T_2|T_1}$ and $\text{Cov}_{T_2|T_1}$, respectively. The covariances on the right side of (15) are of scalar type and Chebyshev’s correlation inequality applies to each, together with the necessary and sufficient conditions for the covariance to be zero. To be more specific, note that $f(t_1, t_2)$ and $g(t_1, t_2)$ are increasing in $t_2$ conditional on a fixed $t_1$; hence $\text{Cov}_{T_2}[f(t_1, T_2), g(t_1, T_2)]$ is non-negative for any $T_1 = t_1$. Note also that $E_{T_2} f(t_1, T_2)$ and $E_{T_2} g(t_1, T_2)$ are increasing in $t_1$; hence $\text{Cov}_{T_1}[E_{T_2} f(T), E_{T_2} g(T)]$ is also non-negative. The conditions stated in the lemma are simply Chebyshev’s necessary and sufficient conditions for each covariance term on the right side of (15) to be zero.

**IV. Proof of Theorem 1**

In this section we give a proof of Theorem 1. This will also be a proof of Theorem 1 since the two results are equivalent. We begin the section by recalling the set-up and notation. Then, we split the covariance $\text{Cov}(h_{\text{out},1}, h_{\text{out},2})$
into two parts using the covariance decomposition formula, which is followed by the proof that each term in the decomposition is non-negative. The section concludes with some complementary remarks.

A. Notation

Throughout this section, we use the problem formulation and notation of Section I-B. In particular, we will have a joint ensemble consisting of two independent data elements \((X_1, Y_1)\) and \((X_2, Y_2)\). We will denote the probability measure over this ensemble by \(P\) and expectations by \(E\). Partial and conditional expectations and covariances will be denoted as \(E_{X|Y}, \text{Cov}_{X|Y}, \text{Cov}_{X,Y}, \text{etc.}\), with \(X = (X_1, X_2)\) and \(Y = (Y_1, Y_2)\).

A number of other random variables will appear throughout the section; all such variables will be defined as functions of the primary variables \((X_1, Y_1)\) and \((X_2, Y_2)\). In particular, we will have \(U_1 = X_1 \oplus X_2, U_2 = X_2\), as defined in (1), and the canonical parameters \(A_i \triangleq \alpha(Y_i)\) and \(B_i \triangleq \beta(Y_i), i = 1, 2\). Due to the independence of \(Y_1\) and \(Y_2\), \(A_1\) and \(A_2\) will be independent; likewise, \(B_1\) and \(B_2\) will be independent. For shorthand, we will write \(U = (U_1, U_2), A = (A_1, A_2), B = (B_1, B_2)\). Note that, due to the 1-1 nature of the correspondence between \(U\) and \(X,\) expectation and covariance operators such as \(E_{U|Y}\) and \(\text{Cov}_{U|Y}\) will be equivalent to \(E_{X|Y}\) and \(\text{Cov}_{X|Y}\), respectively, and we will prefer the latter notation in which the primary random variables appear explicitly.

For \(0 \leq p \leq 1\), we will write \(\overline{p}\) to denote \(1 - p\). For \(0 \leq p, q \leq 1\), we define the convolution operation by

\[ p \ast q \triangleq pq + \overline{pq}. \]

B. Covariance decomposition step

As the first step of the proof of Theorem 1, we use the covariance decomposition formula (13) to write

\[
\text{Cov}(h_{\text{out},1}, h_{\text{out},2}) = E_Y \text{Cov}_{X|Y}(h_{\text{out},1}, h_{\text{out},2}) + \text{Cov}_Y(E_{X|Y}h_{\text{out},1}, E_{X|Y}h_{\text{out},2}).
\]

(16)

For brevity, we will use the notation

\[
\text{Cov}_1 \triangleq E_Y \text{Cov}(h_{\text{out},1}, h_{\text{out},2})
\]

\[
\text{Cov}_2 \triangleq \text{Cov}_Y(E_{X|Y}h_{\text{out},1}, E_{X|Y}h_{\text{out},2})
\]

to denote the two terms on the right hand side of (16). Our proof of Theorem 1 will consist of proving the following two statements.

**Proposition 1.** We have \(\text{Cov}_1 \geq 0\), with equality iff either \((X_1, Y_1)\) or \((X_2, Y_2)\) is an erasure data element.

**Proposition 2.** We have \(\text{Cov}_2 \geq 0\), with equality iff either (i) one of the data elements \((X_1, Y_1)\) or \((X_2, Y_2)\) is extreme, or (ii) both data elements are pure.

These propositions will be proved in the following two subsections. Here, we will momentarily assume that they are correct and complete the proof of Theorem 1.
Proof of Theorem 1: The covariance inequality (10) is an immediate consequence of (16) and Propositions 1 and 2. We have Cov(h_{out,1}, h_{out,2}) = 0 iff both Cov 1 and Cov 2 are zero. However, the necessary and sufficient conditions of Propositions 1 and 2 can be satisfied simultaneously iff one of the data elements (X_1, Y_1) and (X_2, Y_2) is an extreme data element. This completes the proof.

C. Proof of Proposition 7

For p, q ∈ [0, 1], define

\[ f(p, q) \triangleq (p * q)(p * \bar{q}) \log \left( \frac{p * q}{p * \bar{q}} \right) \times \left[ H \left( \frac{p \bar{q}}{p * \bar{q}} \right) - H \left( \frac{pq}{p * q} \right) \right]. \tag{17} \]

We will give soon a formula for Cov_1 in terms of this function. First, a number of properties of f(p, q) will be listed. The following symmetry properties are immediate:

\[ f(p, q) = f(\bar{p}, q) = f(p, \bar{q}) = f(\bar{p}, \bar{q}), \tag{18} \]
\[ f(p, q) = f(q, p). \tag{19} \]

Lemma 4. We have f(p, q) ≥ 0 for all p, q ∈ [0, 1] with equality iff p ∈ \{0, \frac{1}{2}, 1\} or q ∈ \{0, \frac{1}{2}, 1\}.

Proof: We use (18) to write

\[ f(p, q) = f(r, s) \tag{20} \]

where \( r \triangleq \min\{p, \bar{p}\} \) and \( s \triangleq \min\{q, \bar{q}\} \). Thus, instead of proving \( f(p, q) \geq 0 \), it suffices to prove \( f(r, s) \geq 0 \) for \( 0 \leq r, s \leq \frac{1}{2} \). In fact, using (19), it suffices to prove \( f(r, s) \geq 0 \) for \( 0 \leq r \leq s \leq \frac{1}{2} \). Assuming \( 0 \leq r \leq s \leq \frac{1}{2} \), it is straightforward to show that

\[ r * s \geq r * \bar{s} \quad \text{and} \quad \frac{rs}{r * s} \leq \frac{r \bar{s}}{r * \bar{s}} \leq \frac{1}{2}. \tag{21} \]

Thus, if we write out the expression for \( f(r, s) \), as in (17) with \( (r, s) \) in place of \( (p, q) \), we can see easily that each of the four factors on the right hand side of that expression are non-negative. More specifically, the logarithmic term is non-negative due to the first inequality in (21) and the bracketed term is non-negative due to the second inequality in (21). This completes the proof that \( f(p, q) \geq 0 \) for all \( p, q \in [0, 1] \).

Next, we identify the necessary and sufficient conditions for \( f(p, q) \) to be zero over \( 0 \leq p, q \leq 1 \). Clearly, \( f(p, q) = 0 \) iff one of the four factors on the right hand side of (17) equals zero. By straightforward algebra, one can verify the following statements. The first factor \( p * q \) equals zero iff \( (p, q) \in \{(0, 1), (1, 0)\} \). The second factor \( p * \bar{q} \) equals zero iff \( (p, q) \in \{(0, 0), (1, 1)\} \). The log term equals zero iff \( p = \frac{1}{2} \) or \( q = \frac{1}{2} \). Finally the difference of the entropy terms equals zero iff \( pq/p * q = \bar{q}/p * \bar{q} \) or \( pq/p * q = 1 - \bar{q}/p * \bar{q} \) which in turn is true iff \( p \in \{0, \frac{1}{2}, 1\} \) or \( q \in \{0, \frac{1}{2}, 1\} \). Taking the logical combination of these conditions we conclude that \( f(p, q) = 0 \) iff \( p \in \{0, \frac{1}{2}, 1\} \) or \( q \in \{0, \frac{1}{2}, 1\} \).
Lemma 5. We have

\[ \text{Cov}_1 = \mathbb{E} f(A) = \mathbb{E} f(B). \]  

**Proof:** Fix a sample \( y = (y_1, y_2) \). Note that

\[ \text{Cov}_{X|Y}(h_{\text{out},1}, h_{\text{out},2}) = \text{Cov}_{X|Y}(h(U_1|y), h(U_2|U_1, y)) \]

\[ = \mathbb{E}_{X|Y} \left[ h(U_1|y) - \mathbb{E}_{X|Y} h(U_1|y) \right] h(U_2|U_1, y) \]

\[ = \sum_{u_1, u_2} p(u_1, u_2|y) \left[ h(u_1|y) - \mathbb{E}_{X|Y} h(U_1|y) \right] h(u_2|u_1, y) \]

\[ = \sum_{u_1} p(u_1|y) \left[ h(u_1|y) - \mathbb{E}_{X|Y} h(U_1|y) \right] H(U_2|u_1, y), \]

where \( H(U_2|u_1, y) = \sum_{u_2} p(u_2|u_1, y) h(u_2|u_1, y) \). The term \( [h(u_1|y) - \mathbb{E} h(U_1|y)] \) simplifies to

\[ (1 - p(u_1|y)) \log \frac{1 - p(u_1|y)}{p(u_1|y)}. \]

Substituting this in the preceding equation and writing out the sum over \( U_1 \) explicitly, we obtain

\[ \text{Cov}_{X|Y}(h_{\text{out},1}, h_{\text{out},2}) = p_{U_1|Y}(0|y)p_{U_1|Y}(1|y) \log \frac{p_{U_1|Y}(0|y)}{p_{U_1|Y}(1|y)} \times \left[ H(U_2|U_1 = 1, y) - H(U_2|U_1 = 0, y) \right]. \]

Expressing each factor on the right side of the above equation in terms of \( a_i = \alpha(y_i), \ i = 1, 2 \), we obtain that it equals \( f(a_1, a_2) \). Taking expectations, we obtain \( \text{Cov}_1 = \mathbb{E} f(A) \). The alternative formula \( \text{Cov}_1 = \mathbb{E} f(B) \) follows from the fact that \( f(B) = f(A) \) due to the symmetries 18.

We now complete the proof of Proposition 1.

**Proof of Proposition 1** We have \( \text{Cov}_1 \geq 0 \) since \( f(a_1, a_2) \geq 0 \) for all \( a_1, a_2 \in [0, 1] \) by Lemma 4. By the same lemma, strict positivity, \( \mathbb{E} f(A) > 0 \), is possible iff the events \( A_1 \notin \{0, 1\} \) and \( A_2 \notin \{0, 1\} \) can occur simultaneously with non-zero probability; i.e.,

\[ P(A_1 \notin \{0, \frac{1}{2}, 1\}) P(A_2 \notin \{0, \frac{1}{2}, 1\}) > 0, \]  

(23)

since \( A_1 \) and \( A_2 \) are independent. Condition 23 is in turn equivalent to having

\[ P(B_1 \notin \{0, \frac{1}{2}\}) P(B_2 \notin \{0, \frac{1}{2}\}) > 0, \]  

(24)

which is to say that either \( (X_1, Y_1) \) or \( (X_2, Y_2) \) is an erasure data element.

**D. Proof of Proposition 2**

The main tool in this part will be Lemma 2. We define two functions that will play the role of \( f \) and \( g \) in that lemma. Let \( g_1(p, q) \overset{\Delta}{=} \mathcal{H}(p \ast q) \) and \( g_2(p, q) \overset{\Delta}{=} \mathcal{H}(p) + \mathcal{H}(q) - \mathcal{H}(p \ast q) \) for \( p, q \in [0, 1] \). Soon, we will express \( \text{Cov}_2 \) in terms of these functions.

The following symmetry properties are immediate (for \( i = 1, 2 \)):

\[ g_i(p, q) = g_i(\bar{p}, \bar{q}) = g_i(\bar{p}, q) = g_i(p, \bar{q}), \]

(25)
Lemma 6. We have, for \( i = 1, 2 \),
\[
\mathbb{E}_{X|Y} h_{\text{out},i} = g_i(A) = g_i(B).
\] (27)

**Proof:** Fix \( Y = y \) and let \( a_i = \alpha(y_i) \), \( i = 1, 2 \). By direct calculation, we obtain
\[
\mathbb{E}_{X|Y} h_{\text{out},1} = H(U_1|y) = \mathcal{H}(a_1 * a_2) = g_1(a_1, a_2),
\]
\[
\mathbb{E}_{X|Y} h_{\text{out},2} = H(U_2|U_1, y) = H(U_1, U_2|y) - H(U_1|y)
\]
\[
= H(X_1, X_2|y) - H(U_1|y)
\]
\[
= H(X_1|y_1) + H(X_2|y_2) - H(U_1|y)
\]
\[
= \mathcal{H}(a_1) + \mathcal{H}(a_2) - \mathcal{H}(a_1 * a_2) = g_2(a_1, a_2).
\]

We thus obtain the first formula in (27), which involves \( A \). The second formula in terms of \( B \) follows from the symmetry properties (25).

As a corollary to Lemma 6, we obtain
\[
\text{Cov}_2 = \text{Cov}[g_1(A), g_2(A)],
\] (28)
\[
\text{Cov}_2 = \text{Cov}[g_1(B), g_2(B)].
\] (29)

In order to prove that \( \text{Cov}_2 \geq 0 \), we will apply Lemma 2 to (29). First, we need to establish the following fact.

**Lemma 7.** \( g_1, g_2 : [0, \frac{1}{2}]^2 \rightarrow \mathbb{R}^+ \) are increasing in the sense defined in Section IV.

**Proof:** First consider \( g_1 \). We wish to prove that \( g_1(b_1, b_2) \) is increasing as a function of (i) \( b_1 \in [0, \frac{1}{2}] \) for fixed \( b_2 \in [0, \frac{1}{2}] \) and (ii) \( b_2 \in [0, \frac{1}{2}] \) for fixed \( b_1 \in [0, \frac{1}{2}] \). In fact since \( g_1(b_1, b_2) = g_1(b_2, b_1) \) it suffices to prove only one of these statements. Accordingly, fix \( b_2 \in [0, \frac{1}{2}] \) and consider \( g_1(b_1, b_2) \) as a function of \( b_1 \in [0, \frac{1}{2}] \). Recall that \( g_1(b_1, b_2) = \mathcal{H}(b_1 * b_2) \). Recall also the well-known facts that the function \( \mathcal{H}(p) \) over \( p \in [0, 1] \) is a strictly concave non-negative function, symmetric around \( p = \frac{1}{2} \), attaining its minimum value of 0 at \( p \in \{0, 1\} \), and its maximum value of 1 at \( p = \frac{1}{2} \). It is readily verified that, for any fixed \( b_2 \in [0, \frac{1}{2}] \), as \( b_1 \) ranges from 0 to \( \frac{1}{2} \), \( b_1 * b_2 \) decreases from \( b_2 \) to \( \frac{1}{2} \), hence \( \mathcal{H}(b_1 * b_2) \) increases from \( \mathcal{H}(b_2) \) to \( \mathcal{H}(\frac{1}{2}) = 1 \), with strict monotonicity if \( b_2 \neq \frac{1}{2} \).

It follows (by the symmetry mentioned above) that \( g_1 \) is increasing on \( [0, \frac{1}{2}]^2 \), as claimed.

Next, consider \( g_2 \). Due to the symmetry \( g_2(b_1, b_2) = g_2(b_2, b_1) \), we need only show that \( g_2(b_1, b_2) \) is increasing in \( b_1 \in [0, \frac{1}{2}] \) for fixed \( b_2 \in [0, \frac{1}{2}] \). Recall that \( g_2(b_1, b_2) = \mathcal{H}(b_1) + \mathcal{H}(b_2) - \mathcal{H}(b_1 * b_2) \). Exclude the constant term \( \mathcal{H}(b_2) \) and focus on the behavior of \( I(b_1) = \mathcal{H}(b_1) + \mathcal{H}(b_2) - \mathcal{H}(b_1 * b_2) \) over \( b_1 \in [0, \frac{1}{2}] \). Observe that \( I(b_1) \) is the mutual information between the input and output terminals of a BSC with crossover probability \( b_1 \) and a Bernoulli-\( b_2 \) input. The mutual information between the input and output of a discrete memoryless channel is a convex function of the set of channel transition probabilities for any fixed input probability assignment [12] p. 90. So, \( I(b_1) \) is convex in
$b_1 \in [0, \frac{1}{2}]$. Since $I(0) = \mathcal{H}(b_2)$ and $I(\frac{1}{2}) = 0$, it follows from the convexity property that $I(b_1)$ is decreasing in $b_1 \in [0, \frac{1}{2}]$, and strictly decreasing if $b_2 \neq 0$. Thus, $g_2(b_1, b_2)$ is increasing in $b_1 \in [0, \frac{1}{2}]$ for fixed $b_2 \in [0, \frac{1}{2}]$. It follows (by the symmetry property) that $g_2$ is increasing on $[0, \frac{1}{2}]^2$.

We now complete the proof of Proposition 2.

**Proof of Proposition 2.** The inequality $\text{Cov}_2 \geq 0$ follows as a corollary to Lemmas 2 and 7. The only remaining issue is to identify the conditions for $\text{Cov}_2$ to be zero.

It is easily seen that the conditions stated in Proposition 2 are sufficient for $\text{Cov}_2$ to be zero: (i) If both data elements are pure, then both $g_1$ or $g_2$ are constant, with $g_1(B) = \mathcal{H}(b_1 + b_2)$ and $g_2(B) = \mathcal{H}(b_1) + \mathcal{H}(b_2) - \mathcal{H}(b_1 + b_2)$ where $b_1$ and $b_2$ are the only possible values of $B_1$ and $B_2$, respectively. (ii) If one of the data elements is purely random, then $g_1(B) \equiv 1$. (iii) If one of the data elements is purely deterministic, then $g_2(B) \equiv 0$. In all three cases, $\text{Cov}(g_1, g_2) = 0$ since either $g_1$ or $g_2$ is constant.

To prove that the conditions of the proposition are also necessary for $\text{Cov}_2$ to be zero, we will use Lemma 3. Assume, by contraposition, that (i) at least one of the two data elements $(X_1, Y_1)$ and $(X_2, Y_2)$ is mixed, and (ii) neither data element is extreme. We may assume, without loss of generality, that $(X_1, Y_1)$ is mixed, since both $g_1$ and $g_2$ are invariant under exchange of $(B_1, B_2)$ with $(B_2, B_1)$. Let $b_1, b'_1 \in B_1$ be any two distinct elements; let $b_2 \in B_2$ be such that $b_2 \notin \{0, \frac{1}{2}\}$. Then, we have $g_1(b_1, b_2) \neq g_1(b'_1, b_2)$ and $g_2(b_1, b_2) \neq g_2(b'_1, b_2)$ since both $g_2(b_1, b_2)$ and $g_2(b'_1, b_2)$ are strictly increasing in $b_1$ for fixed $b_2 \in (0, \frac{1}{2})$, as proved in Lemma 7. Thus, by Lemma 3, $\text{Cov}_2$ is strictly greater than zero. This completes the proof that the stated conditions are necessary.

**E. Complementary remarks**

**Interpretation of $\text{Cov}_1$ and $\text{Cov}_2$:** What do the individual covariance terms $\text{Cov}_1$ and $\text{Cov}_2$ measure? To address this question, let us call each possible value of the $B$ parameter for a given data element $(X, Y)$ a “mode.” For example, a BSC has only one mode. A BEC has two modes: $B = 0$ and $B = \frac{1}{2}$, which are both extreme.

The formula $\text{Cov}_1 = \mathbb{E}_f(B_1, B_2)$ of Lemma 5 can be interpreted as saying that $\text{Cov}_1$ is a weighted measure of cross-coupling between individual pairs of modes, one mode from each data element. When one of the data elements is an erasure element, $\text{Cov}_1$ equals zero since an erasure data element has extreme modes only, and an extreme mode is incapable of cross-coupling with any mode of the other data element.

To give an interpretation to $\text{Cov}_2$, we expand $\text{Cov}_2 = \text{Cov}[g_1(B), g_2(B)]$ by the conditional covariance formula to write it as the sum of $\mathbb{E}_{B_1} \text{Cov}_{B_2|B_1}(g_1, g_2)$ and $\text{Cov}_{B_1}(\mathbb{E}_{B_2|B_1} g_1, \mathbb{E}_{B_2|B_1} g_2)$. The first term $\mathbb{E}_{B_1} \text{Cov}_{B_2|B_1}(g_1, g_2)$ can be interpreted as a measure of average coupling among modes $B_2$ of the second data element $(X_2, Y_2)$ when the mode $B_1$ of the first data element $(X_1, Y_1)$ is fixed. The second term $\text{Cov}_{B_1}(\mathbb{E}_{B_2|B_1} g_1, \mathbb{E}_{B_2|B_1} g_2)$ is a measure of coupling among modes $B_1$ of $(X_1, Y_1)$ when the mode $B_2$ of $(X_2, Y_2)$ is randomized. Thus, $\text{Cov}_2$ as a whole can be interpreted roughly as a measure of $\text{intra}$-coupling among modes of individual data elements entering the polar transform. If both data elements have a single mode (like a BSC), then $\text{Cov}_2$ equals zero since the there can be no intra-coupling. If one of the data elements is an extreme one, then $\text{Cov}_2$ is again zero since conditional on an extreme mode at one data element, the modes of the other data element do not couple with each other.
Table IV-E summarizes the extreme cases of Cov\(_1\) and Cov\(_2\), using a channel coding terminology. The first two columns “Ch. 1” and “Ch. 2” refer to the type of channel corresponding to the data elements \((X_1, Y_1)\) and \((X_2, Y_2)\), respectively. The first three rows of the table are obtained from Lemma 8 below; the fourth row follows from Lemma 5. The parameter \(\delta\) in the table is defined as \(\delta = \mathbb{E}(\mathcal{H}(B))\) where \(B\) is the parameter relating to the channel labeled as “any”.

**TABLE I**

| Ch. 1 | Ch. 2 | Cov\(_1\) | Cov\(_2\) |
|-------|-------|------------|------------|
| BEC(\(\epsilon_1\)) | BEC(\(\epsilon_2\)) | 0 | \(\epsilon_1\mathbb{I}\{\epsilon_2\leq\tfrac{1}{2}\}\) |
| BEC(\(\epsilon_1\)) | any | 0 | \(\epsilon_1\mathbb{I}\{\epsilon\leq\tfrac{1}{2}\}\) |
| any | BEC(\(\epsilon_2\)) | 0 | \(\delta\mathbb{I}\{\epsilon_1\geq\tfrac{1}{2}\}\) |
| BSC(\(\epsilon_1\)) | BSC(\(\epsilon_2\)) | \(f(\epsilon_1, \epsilon_2)\) | 0 |

**Lemma 8.** Let \((X_1, Y_1)\) be an erasure data element with probability of erasure \(0 \leq \epsilon \leq 1\). Let \((X_2, Y_2)\) be arbitrary. Then,

\[
\text{Cov}(g_1, g_2) = \epsilon(1-\epsilon)\delta(1-\delta)
\]

where \(\delta = \mathbb{E}(\mathcal{H}(B))\).

**Proof:** For \((X_1, Y_1)\) an erasure data element,
\[
g_1(B_1, B_2) = \begin{cases} 
\mathcal{H}(B_2), & B_1 = 0; \\
1, & B_1 = \tfrac{1}{2}; 
\end{cases}
\]
\[
g_2(B_1, B_2) = \begin{cases} 
0, & B_1 = 0; \\
\mathcal{H}(B_2), & B_1 = \tfrac{1}{2}. 
\end{cases}
\]

The claim is obtained by simply computing the covariance of these two random variables.

**Generalizations:** There are several directions in which one may try to generalize the preceding results. First, one may try to lift the restriction that the alphabets \(\mathcal{Y}_1\) and \(\mathcal{Y}_2\) are finite. Second, one may consider a non-binary alphabet for the variables \(X_1\) and \(X_2\). Third, one may consider the case where \((X_1, Y_1)\) and \((X_2, Y_2)\) are not independent. The first generalization can be accomplished without any essential difficulty since the \(\mathcal{Y}_1\) and \(\mathcal{Y}_2\) are replaced in effect by the alphabets \(\mathcal{B}_1\) and \(\mathcal{B}_2\), which in turn are subsets of a compact interval \([0, \tfrac{1}{2}]\). Density functions over \([0, \tfrac{1}{2}]\) can be approximated arbitrarily by discrete distributions for purposes of computing the covariances. The second and third questions require further research work.
V. Varentropy under Higher Order Transforms

For any \( n \geq 1 \), there is a polar transform of order \( N = 2^n \). A polar transform of order \( N \) is a mapping \( \psi_N \) that takes in \( N \) data elements and puts out \( N \) data elements. The input data elements will be assumed to be of the form \((X_i, Y_i)\) where \( X_i \) takes values in \( \mathcal{X} = \{0, 1\} \) and \( Y_i \) takes values in some finite alphabet \( \mathcal{Y}_i \), \( i = 1, \ldots , N \). We will assume that the input data elements are independent but not necessarily identically distributed. The output data elements will be of the form \((U_i; U_i^{-1}, Y)\), \( i = 1, \ldots , N \), where \( Y = (Y_1, \ldots , Y_N) \) and \( U_i^{-1} = (U_1, \ldots , U_i-1) \).

The binary data vector \( U = (U_1, \ldots , U_N) \) at the output is related to the binary data vector \( X = (X_1, \ldots , X_N) \) at the input by the linear transformation

\[
U = XG_N, \quad G_N \triangleq F \otimes n, \quad F \triangleq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},
\]

where the “\( \otimes n \)” in the exponent denotes the \( n \)th Kronecker power.

Consider the transform-domain entropy random variables

\[
h(U_i|U_i^{-1}, Y) = -\log p(U_i|U_i^{-1}, Y)
\]

and their averages

\[
H(U_i|U_i^{-1}, Y) = \mathbb{E} h(U_i|U_i^{-1}, Y).
\]

As in the case of \( N = 2 \) considered earlier, one has the conservation laws:

\[
\sum_{i=1}^{N} h(U_i|U_i^{-1}, Y) = \sum_{i=1}^{N} h(X_i|Y_i) \tag{30}
\]

and

\[
\sum_{i=1}^{N} H(U_i|U_i^{-1}, Y) = \sum_{i=1}^{N} H(X_i|Y_i) \tag{31}
\]

In [2], the conservation law (31) was used with i.i.d. data elements \((X_i, Y_i) \sim (X, Y)\) to set up a martingale and prove that the entropy terms \( H(U_i|U_i^{-1}, Y) \) polarize to 0 or 1. More precisely, it was shown that, for any \( \delta > 0 \), as \( N \to \infty \),

\[
\frac{1}{N} \left| \left\{ i : \delta < H(U_i|U_i^{-1}, Y) < 1 - \delta \right\} \right| \to 0. \tag{32}
\]

Here, we focus on the law (30) and generalize the varentropy results of the preceding sections. We will use the notation introduced in Section II and write \( V(U_i|U_i^{-1}, Y) \) to denote the varentropy of \( U_i \) given \((U_i^{-1}, Y)\).

**Theorem 2.** The varentropy decreases under the polar transform in the sense that

\[
\sum_{i=1}^{N} V(U_i|U_i^{-1}, Y) \leq \sum_{i=1}^{N} V(X_i|Y_i). \tag{33}
\]

Before proving this theorem, we will bring out the recursive nature of the polar transform by giving a more abstract formulation. To begin, let us recall that a polar transform of order two is essentially a mapping of the form \((B_{in,1}, B_{in,2}) \to (B_{out,1}, B_{out,2})\), where \( B_{in,1} \) and \( B_{in,2} \) are the \( \beta \)-parameters of the input data elements \((X_1, Y_1)\) and \((X_2, Y_2)\), and \( B_{out,1} \) and \( B_{out,2} \) are the \( \beta \)-parameters of the output data elements \((U_1, Y)\) and \((U_2; U_1, Y)\).
Alternatively, polar transform may be viewed as an operation in the space of cumulative distribution functions (CDFs) of \( \beta \)-parameters and represented in the form

\[
(F_{out,1}, F_{out,2}) = \psi_2(F_{in,1}, F_{in,2})
\]

where \( F_{in,i} \) and \( F_{out,i} \) are the CDFs of \( B_{in,i} \) and \( B_{out,i} \), respectively. Let \( \mathcal{F} \) denote the class of CDFs belonging to random variables that take values in the interval \([0, \frac{1}{2}]\); this is the class of all possible CDFs for \( \beta \)-parameters. Each data element may be thought of abstractly as a point in the space \( \mathcal{F} \) and the polar transform of order two can be regarded as an operator \( \psi_2 : \mathcal{F}^2 \to \mathcal{F}^2 \) mapping one pair of points from \( \mathcal{F} \) to another pair of points in \( \mathcal{F} \). We will define higher order polar transforms following this viewpoint.

For each \( i = 1, \ldots, N \), let \( B_{in,i} \) denote the \( \beta \)-parameter corresponding to the input data element \((X_i, Y_i)\) and \( F_{in,i} \) the CDF of \( B_{in,i} \). Likewise, for each \( i = 1, \ldots, N \), let \( B_{out,i} \) denote the \( \beta \)-parameter for the output data element \((U_i, U_i^{-1}, Y)\) and \( F_{out,i} \) the CDF of \( B_{out,i} \). Let \( F_{in} = (F_{in,1}, \ldots, F_{in,N}) \) and \( F_{out} = (F_{out,1}, \ldots, F_{out,N}) \). We will represent a polar transform of order \( N \) abstractly as

\[
F_{out} = \psi_N(F_{in}).
\]

The recursive formula defining a polar transform of order \( N \) in terms of a polar transforms of order \( N/2 \) is

\[
F_{out} = (F'_{out}, F''_{out}), \quad F'_{out} = \psi_{N/2}(F'_{in}), \quad F''_{out} = \psi_{N/2}(F''_{in})
\]

where \( F'_{in} = (F'_{in,1}, \ldots, F'_{in,N/2}) \), \( F''_{in} = (F''_{in,1}, \ldots, F''_{in,N/2}) \) are obtained from \( F_{in} \) through a series of size-2 transforms

\[
(F'_{in,i}, F''_{in,i}) = \psi_2(F_{in,i}, F_{in,i+N/2}), \quad i = 1, \ldots, N/2.
\]

The proof that the above recursive definition of a polar transform is equivalent to the algebraic definition given earlier is straightforward and will be omitted. We are now ready to prove the theorem.

**Proof:** The claim \([33]\) is true for \( N = 2 \) by Theorem \([1]\) We will use induction to prove the claim for \( N \geq 4 \). Assume the claim is true for transforms of order \( N/2 \) or smaller. We will show that the claim is true for order \( N \). We will write \( V(F) \) to denote the varentropy \( V(X|Y) \) of a data element \((X, Y)\) whose \( \beta \)-parameter has CDF \( F \). By the induction hypothesis, we have (using the notation introduced above)

\[
\sum_{i=1}^{N/2} V(F'_{out,i}) \leq \sum_{i=1}^{N/2} V(F'_{in,i})
\]

and

\[
\sum_{i=1}^{N/2} V(F''_{out,i}) \leq \sum_{i=1}^{N/2} V(F''_{in,i}).
\]

Using the induction hypothesis again,

\[
V(F'_{in,i}) + V(F''_{in,i}) \leq V(F_{in,i}) + V(F_{in,i+N/2})
\]

for all \( i = 1, \ldots, N/2 \). Thus,

\[
\sum_{i=1}^{N} V(F_{out,i}) \leq \sum_{i=1}^{N} V(F_{in,i}),
\]

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which is the desired inequality.

VI. ASYMPTOTIC BEHAVIOR

The next logical step in the development is to consider the asymptotic behavior of varentropy as the transform size grows. The following result addresses this question for the important special case where the data elements at the transform input are i.i.d.

**Theorem 3.** Let \((X_i, Y_i), i \geq 1,\) be a sequence of independent copies of a given data element \((X, Y).\) Then,

\[
\frac{1}{N} \sum_{i=1}^{N} V(U_i|U_i^{i-1}, Y) \to 0, \quad \text{as} \ N \to \infty.
\]

**(Proof):** Let \(S_n \triangleq \frac{1}{2^n} \sum_{i=1}^{2^n} V(U_i|U_i^{i-1}, Y)\) for \(n \geq 1,\) and \(S_0 \triangleq V(X|Y).\) The sequence \(\{S_n\}\) is non-negative, and monotone decreasing by Theorem 2. So, \(S_n\) converges to a limit \(c \geq 0.\) To prove that \(c = 0,\) we invoke the polarization result (32), which states that in the limit all channels at the transform output, with the possible exception of an asymptotically vanishing fraction, become extreme. Since varentropy is bounded by a constant from above, the asymptotically vanishing fraction of varentropy terms that do not converge to zero have no effect on average varentropy. Hence, \(c\) has to be zero.

It would be desirable to give a proof of Theorem 3 in a more self-contained manner, without appeal to previous polarization theorems. Such a proof could in fact be an alternate and valuable tool for proving polarization theorems. We propose this as a topic for future study.

One of the implications of the convergence of average varentropy to zero is that the entropy random variables “concentrate” around their means along almost all trajectories of the polar transform. This concentration phenomenon provides a theoretical basis for understanding why polar decoders that operate with quantized versions of the entropy random variables do not show a significant performance degradation.

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