Multi-channel phase-equivalent transformation and supersymmetry

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Phase-equivalent transformation of local interaction is generalized to the multi-channel case. Generally, the transformation does not change the number of the bound states in the system and their energies. However, with a special choice of the parameters, the transformation removes one of the bound states and is equivalent to the multi-channel supersymmetry transformation recently suggested in [20]. Using the transformation, it is also possible to add a bound state to the discrete spectrum of the system at a given energy $E < 0$ if the angular momentum at least in one of the coupled channels $l \geq 2$.

I. INTRODUCTION

Nucleon-nucleon, nucleon-cluster and cluster-cluster potentials are an input for various microscopic calculations of nuclear structure and reactions. Unfortunately, the exact form of the potentials describing these interactions is unknown. It is conventionally supposed that the interactions are local, that is, of course, an approximation only. However, the available scattering data and bound states properties can be fitted with approximately the same accuracy by different local potentials. For example, there is a lot of so-called realistic $NN$ potentials on the market describing $NN$ scattering and deuteron properties with high accuracy. More, a description of phenomenological data can be achieved with the potentials very different in structure. In particular, meson-exchange $NN$ potentials like the Nijmegen one [1], are known to have a short-range repulsive core in a triplet $s$ wave. The same high-quality description of the nucleon-nucleon data is provided by latest versions of Moscow potential [2,3] that does not have a repulsive core but instead is deeply-attractive in the triplet $s$ wave at short distances and supports an additional forbidden state. The possibility of alternative description of various cluster-cluster and nucleon-cluster interactions by means of repulsive-core and deeply-attractive potentials with forbidden states, is also well-known (see, e.g., the discussion in [2] and references therein).

Principally it is possible to distinguish experimentally between alternative potentials studying their off-shell properties in interaction with an additional particle. The simplest probe is the photon, and as it was shown in [4–6], the proton-proton bremsstrahlung reaction $pp \to pp\gamma$ at the energy range of 350–400 MeV can be used to discriminate between various nucleon-nucleon potentials. However the $pp \to pp\gamma$ reaction has not been examined experimentally in this energy range.

Another possibility is to study properties of three and four body systems bound by two-body potentials of interest. From this point of view, it looks like that we do not have at present satisfactory nucleon-nucleon, cluster-nucleon and cluster-cluster potentials. It is well-known that none of the realistic $NN$ potentials provides proper binding of tritium or $^3$He. There are successful attempts in generating phenomenological three-nucleon interactions tuned to fit the properties of light nuclei [7] (see also [8] and references therein). However, as it was shown in a detailed study of Picklesimer et al [4], the effect of three-nucleon forces consistent with realistic two-body ones on the binding energy of the triton is canceled by effects of the virtual excitation of $\Delta$ isobars, etc. Hence the trinucleon cannot be satisfactorily described using known realistic two-body potentials supplemented by three-body potentials consistent with them. All calculations within three-body cluster models also fail to
reproduce the correct binding energy of three-cluster nuclear systems with known local cluster-cluster and cluster-nucleon potentials fitted to the corresponding scattering data.

To design a potential consistent with two-body phenomenological data and providing the correct binding of few-body systems, it seems promising to make use of phase-equivalent transformations depending on a continuous parameter(s). Some attempts in this direction have been performed using non-local phase-equivalent transformations. The results of these attempts are encouraging: in Ref. [10] an oversimplified $NN$ potential providing a satisfactory description of $s$ wave $NN$ scattering data was fitted to reproduce exactly the triton binding energy, while in Ref. [11] realistic $n-\alpha$ potentials were tuned to reproduce various $^6\text{He}$ properties including the binding energy within the $\alpha+n+n$ cluster model. The interactions suggested in Ref. [10,11] are non-local. Various applications (see, e.g., [12,13]) of local phase-equivalent transformations to few-body problems were restricted to the supersymmetry transformation [14–16] that removes one of the bound states in a two body system. The supersymmetry transformation does not contain parameters and cannot be used for fine tuning of the interaction of interest.

A local phase-equivalent transformation which preserves the number of the bound states and depends on a continuous parameter, exists and is well-known in the inverse scattering theory [17]. Recently the effect of this transformation on the properties of three and four nucleon systems was studied in Ref. [18] using as an example a semi-realistic Malfliet-Tjon $NN$ interaction [19]. It was shown in Ref. [18] that a slight phase-equivalent modification of $NN$ interaction is enough to reproduce the trinucleon binding energy and to improve simultaneously the description of four-nucleon binding. However the local transformation was developed for a single-channel case only and cannot be applied without some approximations to realistic $NN$ interactions that mix triplet $s$ and $d$ partial waves. Another drawback of the transformation is that it involves a bound state wave function and thus cannot be used to modify $nn$ and $pp$ interactions and the $np$ interaction in all ‘non-deuteron’ partial waves.

Recently Sparenberg and Baye [20] suggested a multi-channel supersymmetry transformation. We use some ideas of Ref. [20] to derive in what follows a multi-channel phase-equivalent transformation which depends on continuous parameters. The transformation can be treated as a generalization both of the single-channel phase-equivalent transformation [17] and of the multi-channel supersymmetry transformation of Ref. [20]. Generally, the transformation does not change the number of the bound states in the system and their energies. However, with a special choice of the parameters, the transformation removes one of the bound states and becomes equivalent to the multi-channel supersymmetry transformation suggested in [20]. If the angular momenta in all coupled channels are less than 2, a parameter-dependent family of local interactions phase-equivalent to the given initial one can be constructed by means of the transformation even in the case when the system does not have a bound state. If the angular momentum at least in one of the coupled channels $l \geq 2$, the transformation can be used to add a bound state to the discrete spectrum of the system at a given energy $E < 0$. Having a bound state, one can construct a family of phase-equivalent potentials and afterwards remove the bound state by the supersymmetry version of the transformation. Thus, the suggested transformation can be used in a multi-channel case to produce phase-equivalent interactions without any restriction on the structure of the discrete spectrum of the system. In particular, the transformation can be applied to the realistic $NN$ interaction in all partial waves.

II. GENERAL FORM OF LOCAL MULTI-CHANNEL PHASE-EQUIVALENT TRANSFORMATION

Multi-channel scattering and bound states we describe by Schrödinger equation
\[
\sum_j (H_{ij} - E \delta_{ij}) \varphi_j(E, r) = 0, \tag{1}
\]
where indexes \(i\) and \(j\) label channels, \(E\) is the energy, the Hamiltonian
\[
H_{ij} = \frac{\hbar^2}{2m} \left[ -\frac{d^2}{dr^2} + \frac{l_i(l_i + 1)}{r^2} \delta_{ij} + V_{ij}(r) \right], \tag{2}
\]
\(m\) is the reduced mass, and \(l_i\) stands for the angular momentum in the channel \(i\). We suppose that the potential \(V_{ij}(r)\) (i) is Hermitian and (ii) at large distances it tends asymptotically to a diagonal constant matrix,
\[
V_{ij}(r) \xrightarrow{r \to \infty} \epsilon_i \delta_{ij}, \tag{3}
\]
where \(\epsilon_i\) is a threshold energy in the channel \(i\). We suppose that \(\epsilon_1 = 0\) and \(\epsilon_i \geq \epsilon_j\) if \(i > j\).

Boundary conditions for the wave functions are
\[
\varphi_i(E, 0) = 0, \tag{4}
\]
\[
\varphi_i(E, \infty) < \infty. \tag{5}
\]
Except for the discussion in section III C, we suppose that there is at least one bound state in the system at the energy \(E_0\). The corresponding wave function, \(\varphi_i(E_0, r)\), is supposed to be normalized,
\[
\sum_i \int_0^\infty \varphi_i^*(E_0, s) \varphi_i(E_0, s) \, ds = 1, \tag{6}
\]
where \(^*\) denotes the complex conjugation. Of course, \(\varphi_i(E_0, r)\) fits more severe boundary condition at \(r \to \infty\) than \(\varphi_i(E, \infty) < \infty\):
\[
\varphi_i(E_0, \infty) = 0. \tag{7}
\]
We define the transformed potential \(\tilde{V}_{ij}(r)\) as
\[
\tilde{V}_{ij}(r) = V_{ij}(r) + v_{ij}(r), \tag{8}
\]
where
\[
v_{ij}(r) = -2C \frac{\hbar^2}{2m} \frac{d}{dr} \frac{\varphi_i(E_0, r) \varphi_j^*(E_0, r)}{A + C \sum_k \int_a^r |\varphi_k(E_0, s)|^2 \, ds}, \tag{9}
\]
and \(A, C\) and \(a\) are arbitrary real parameters.

The main result of this paper can be formulated as the following statement.

• The wave function
\[
\tilde{\varphi}_i(E, r) = \varphi_i(E, r) - C \varphi_i(E_0, r) \frac{\sum_k \int_a^r \varphi_k^*(E_0, s) \varphi_k(E, s) \, ds}{A + C \sum_k \int_a^r |\varphi_k(E_0, s)|^2 \, ds} \tag{10}
\]
fits inhomogeneous multi-channel Schrödinger equation
\[
\sum_j \left( \tilde{H}_{ij} - E \delta_{ij} \right) \tilde{\varphi}_j(E, r) = C \frac{\hbar^2}{2m} \frac{\varphi_i(E_0, r)}{A + C \sum_k \int_a \varphi_k(E_0, s)^2 \, ds} \mathcal{W}(E_0, E; a),
\]

(11)

where the Hamiltonian

\[
\tilde{H}_{ij} = \delta_{ij} \frac{\hbar^2}{2m} \left[ -\frac{d^2}{dr^2} + \frac{l_i(l_i + 1)}{r^2} \right] + \tilde{V}_{ij}(r)
\]

(12)

and the quasi-Wronskian

\[
\mathcal{W}(E_0, E; a) \equiv \sum_k \left[ \varphi_k^*(E_0, a) \varphi_k'(E, a) - \varphi_k^*(E_0, a) \varphi_k(E, a) \right].
\]

(13)

We use prime to denote derivatives: \( f' \equiv \frac{d}{dr} f \).

To prove the statement, one can verify Eq. (11) by the direct calculation of \( \sum_j \left( \tilde{H}_{ij} - E \delta_{ij} \right) \tilde{\varphi}_j(E, r) \) using the definitions (8)–(10), (12) and (13) and other formulas given above as well as the fact that the interaction \( V_{ij}(r) \) is Hermitian, \( V_{ij}^*(r) = V_{ji}(r) \). The calculation is lengthy but straightforward.

It is clear from (10) and (7) that the suggested transformation is phase-equivalent at any energy \( E > 0 \); all the bound states supported by the initial potential \( V_{ij} \) are preserved by the transformation since the wave functions \( \tilde{\varphi}_i(E_b, r) \) for the corresponding energies \( E_b < 0 \) (including \( E_0 \)) fit both boundary conditions (4) and (7). However, the denominator in the last term in (10) should be non-zero at any distance \( r \), and therefore one should be accurate in assigning values to arbitrary parameters \( A, C \) and \( a \). This requirement can be easily satisfied in a wide and continuous range of parameter values.

III. PARTICULAR CASES OF THE PHASE-EQUIVALENT TRANSFORMATION

A. Homogeneous Schrödinger equation

Of course, we are mostly interested in phase-equivalent transformations that result in homogeneous Schrödinger equation

\[
\sum_j \left( \tilde{H}_{ij} - E \delta_{ij} \right) \tilde{\varphi}_j(E, r) = 0
\]

(14)

instead of the inhomogeneous Schrödinger equation (11). To derive the transformation leading to Eq. (14), we can fix the parameters \( A, C, \) and \( a \) in such a way that the r.h.s. of Eq. (11) will take zero value. The choice \( C = 0 \) brings us to the equivalent (contrary to phase-equivalent) transformation that is of no interest. Thus we should search for the parameters that fit the equation

\[
\mathcal{W}(E_0, E; a) = 0.
\]

(15)

Two obvious solutions of Eq. (15) are \( a = 0 \) and \( a = \infty \). Various other solutions of Eq. (13) can be found for particular potentials \( V_{ij}(r) \). However, the non-zero finite solutions \( a \) of Eq. (13) are energy-dependent. With the solutions \( a(E) \) of Eq. (13), we can obtain energy-dependent potentials \( \tilde{V}_{ij}(E; r) \) phase-equivalent to the initial energy-independent potential \( V_{ij}(r) \). It may be interesting for some
applications, but we shall not discuss the energy-dependent transformation and shall concentrate our attention on the solutions \( a = 0 \) and \( a = \infty \).

The case \( a = 0 \) presents a generalization of the single-channel phase-equivalent transformation of Ref. [17]. For the bound state at the energy \( E_0 \), the wave function obtained by means of the transformation is of the form:

\[
\tilde{\varphi}_i(E_0, r) = \frac{A \varphi_i(E_0, r)}{A + C \sum_j \int_0^r |\varphi_j(E_0, s)|^2 \, ds}.
\] (16)

The wave function (16) is not normalized. The normalization constant can be easily calculated. The normalized bound state wave function is

\[
\sqrt{\frac{A + C}{A}} \tilde{\varphi}_i(E_0, r) = \frac{\sqrt{A(A + C)} \, \varphi_i(E_0, r)}{A + C \sum_j \int_0^r |\varphi_j(E_0, s)|^2 \, ds}.
\] (17)

It is interesting that the components of the bound state wave function in all channels are modified by the transformation synchronically: all the components \( \varphi_i(E_0, r) \) are multiplied by the same multiplier \( \sqrt{A(A + C)} \left( A + C \sum_j \int_0^r |\varphi_j(E_0, s)|^2 \, ds \right)^{-1} \). Nevertheless the relative weight of the components \( \varphi_i(E_0, r) \) in the norm of the total multi-channel wave function can be changed by the transformation.

Now let us discuss the case \( a = \infty \). The transformed wave function in this case is of the form:

\[
\tilde{\varphi}_i(E, r) = \varphi_i(E, r) - \frac{C \varphi_i(E_0, r) \sum_j \int_0^r \varphi_j^*(E_0, s) \varphi_j(E, s) \, ds}{A + C \sum_j \int_\infty^r |\varphi_j(E_0, s)|^2 \, ds}.
\] (18)

If \( E \neq E_0 \), the functions \( \varphi_i(E, r) \) and \( \varphi_i(E_0, r) \) are orthogonal:

\[
\int_0^\infty \varphi_i^*(E_0, s) \varphi_i(E, s) \, ds = 0.
\] (19)

With the help of (19) and (18), we can rewrite (18) as

\[
\tilde{\varphi}_i(E, r) = \varphi_i(E, r) - \frac{C \varphi_i(E_0, r) \sum_j \int_0^r \varphi_j^*(E_0, s) \varphi_j(E, s) \, ds}{A - C + C \sum_j \int_0^r |\varphi_j(E_0, s)|^2 \, ds}.
\] (20)

It is seen from (20) that the case \( a = \infty \) is identical (up to the redefinition of the parameter \( A \to A + C \)) to the case \( a = 0 \) if \( E \neq E_0 \). It is clear, however, that after the redefinition of the parameter \( A \to A + C \), the potential \( v_{ij}(r) \) obtained with \( a = \infty \) becomes identical to the potential \( v_{ij}(r) \) corresponding to the case \( a = 0 \). Hence the the case \( a = \infty \) appears to be identical to the case \( a = 0 \) at any energy \( E \) including \( E = E_0 \). To demonstrate this explicitly, let us examine the wave function \( \tilde{\varphi}_i(E_0, r) \) in the case \( a = \infty \). Substituting \( E \) by \( E_0 \) in (18) we obtain:
\[ \tilde{\varphi}_i(E_0, r) = \frac{A \varphi_i(E_0, r)}{A + C \sum_j^\infty \int^r_0 |\varphi_j(E_0, s)|^2 \, ds} \]  

\[ = \frac{A \varphi_i(E_0, r)}{A - C + C \sum_j^\infty \int^0_0 |\varphi_j(E_0, s)|^2 \, ds}. \]  

Replacing \( A \) by \( A + C \) and normalizing the wave function (22), we obtain the expression (17).

**B. Supersymmetry**

Let us discuss a particular choice of parameters: \( C = 1, a = \infty, \) and \( A = 1. \) The wave function in this case is

\[ \tilde{\varphi}_i(E, r) = \varphi_i(E, r) + \frac{\varphi_i(E_0, r) \sum_j^\infty \int^r_0 |\varphi_j(E_0, s)|^2 \, ds}{\sum_j^r \int_0 |\varphi_j(E_0, s)|^2 \, ds} \]  

\[ = \varphi_i(E, r) - \frac{\varphi_i(E_0, r) \sum_j^r \int_0 |\varphi_j(E_0, s)|^2 \, ds}{\sum_j^r \int_0 |\varphi_j(E_0, s)|^2 \, ds}. \]  

Equation (23) can be used at any energy \( E \) while Eq. (24) is applicable only if \( E \neq E_0. \) In the case \( E = E_0, \) the wave function can be rewritten in a simpler form as

\[ \tilde{\varphi}_i(E_0, r) = \frac{\varphi_i(E_0, r)}{\sum_j^r \int_0 |\varphi_j(E_0, s)|^2 \, ds}. \]  

Equation (23) is just the Eq. (4) of Ref. [20]. In Ref. [20], Sparenberg and Baye suggested a multi-channel supersymmetry transformation. Thus equations (24)–(25) describe the multi-channel supersymmetry transformation, or, in other words, the multi-channel supersymmetry transformation is a particular case of the phase-equivalent multi-channel transformation discussed in this paper that corresponds to the particular choice of the parameters. Let us discuss how it works.

It is clear from Eq. (25) that \( |\tilde{\varphi}_i(E_0, r)| \to \infty \) as \( r \to 0. \) Hence at the energy \( E_0, \) the wave function \( \tilde{\varphi}_i(E_0, r) \) does not match the required boundary condition (4) at \( r = 0. \) At the same time, \( \tilde{\varphi}_i(E_0, r) \) fits the boundary condition (7) at \( r = \infty. \) Therefore it is impossible to construct another solution of the Schrödinger equation (14) consistent with both boundary conditions at the energy \( E = E_0. \) As a result the phase-equivalent transformation removes the bound state at \( E = E_0. \) At the same time, it is clear from (24) that for all energies \( E \neq E_0, \) zero in the denominator arising in the limit \( r \to 0 \) is canceled by the zero in the numerator and the wave function (24) matches the boundary conditions at the origin and at the infinity both at once. So, the transformation in this case removes the bound state at \( E = E_0 \) but not any of the other bound states, while the \( S \)-matrix at any energy \( E > 0 \) is unchanged.
Of course, the supersymmetry transformation can be also formulated in the case \( a = 0 \). It is interesting that the bound state in this case is removed by a different mechanism. Suppose \( A = 0 \) and \( C \) is arbitrary. The wave function at any energy \( E \) in this case may be written as

\[
\hat{\varphi}_i(E, r) = \varphi_i(E, r) - \frac{\varphi_i(E_0, r) \sum_j^r \int_0^r \varphi_j^*(E_0, s) \varphi_j(E, s) \, ds}{\sum_j^r \int_0^r |\varphi_j(E_0, s)|^2 \, ds}.
\]

It is seen that at \( E = E_0 \) the wave function \( \hat{\varphi}_i(E_0, r) \equiv 0 \).

We used the boundary condition (4) to construct the supersymmetry transformation: the bound state is removed because for some particular parameter values the wave function \( \hat{\varphi}_i(E_0, r) \) diverges at the origin and appears to be inconsistent with (4). One can suppose that it is also possible to use the \( a \) transformation. It is not so. Let us discuss the case of \( r \to \infty \) from (18), \( \tilde{\varphi} \) unchanged the potential can be found in Ref. [20].

An example of an application of the multi-channel supersymmetry transformation to the Moscow \( NN \) potential can be found in Ref. [20].

C. Inverse supersymmetry

A transformation that adds a bound state to the discrete spectrum of the system and leaves unchanged the \( S \)-matrix and the energies of all bound states supported by the initial Hamiltonian, we shall refer to as inverse supersymmetry transformation.

Let us suppose that there is no bound state at the energy \( E_0 < 0 \). By \( \varphi_i(E_0, r) \) we now denote the wave function at energy \( E_0 \) that matches the boundary condition (4) at infinity but diverges at the origin as \( r^{-l_i} \) (see, e.g. [4, 21]) where \( l_i \) is the angular momentum in the channel \( i \).

Having \( \varphi_i(E_0, r) \), we can use our transformation to obtain the homogeneous Schrödinger equation (17) in the case \( a = \infty \). The transformed wave function \( \tilde{\varphi}_i(E, r) \) is given by (18). It is seen from (18) that \( \tilde{\varphi}_i(E, r) \) does not diverge in the origin and matches the boundary conditions both at the origin and at infinity, at any energy \( E \neq E_0 \). For \( E = E_0 \), the transformed wave function \( \tilde{\varphi}_i(E_0, r) \) is given by (21). It is clear that \( \tilde{\varphi}_i(E_0, r) \) at the origin is proportional to \( r^{2L-l_i-1} \) where \( L = \max\{l_i\} \). Hence, \( \tilde{\varphi}_i(E_0, r) \) matches the boundary condition (4) if \( L \geq 2 \) and is not consistent with (4) if \( L \leq 1 \). Therefore our transformation with \( \varphi_i(E_0, r) \) irregular at the origin, is the inverse supersymmetry transformation in the case \( L \geq 2 \). In the case \( L \leq 1 \) the transformation appears to be a phase-equivalent transformation that does not make use of the bound state and can be applied to the system that does not support a bound state. If the transformation is applied to the free Hamiltonian with \( \bar{V}_{ij}(r) \equiv 0 \) in the \( s \) or \( p \) partial wave, it produces a non-zero ‘transparent’ potential \( \tilde{V}_{ij}(r) \) that provides phase shift \( \delta = 0 \) at any energy \( E \). The multi-channel version of the

\[ ^1 \]The \( r^{-l} \) divergence of the wave functions at the origin is derived in Ref. [2] for the single-channel case only. However the derivation of the \( r^{-l} \) rule of Ref. [21] can be easily generalized to the multi-channel case, at least for the potentials that do not diverge in the origin.
transformation couples $s$ and $p$ partial waves to produce a two-channel ‘transparent’ interaction that provides the $S$-matrix of the form $S_{ij} = \delta_{ij}$.

It is interesting that the inverse supersymmetry transformation is not unique: we have three parameters $E_0$, $A$ and $C$ that provide a family of inverse supersymmetry partner potentials. Contrary to it, the supersymmetry transformation is unique; however, it can be used in combination with the phase-equivalent transformation to construct a family of potentials phase-equivalent to the initial one but not supporting one of the bound state.

The multi-channel inverse supersymmetry transformation is discussed in more detail in a very recent paper of Leeb et al [22]. One can find in this paper examples of the applications of the transformation to realistic $NN$ potentials. This transformation is discussed in Ref. [23], too; in particular the authors of Ref. [23] also conclude that it is possible to create a new bound state by means of the phase-equivalent transformation only in the case $L \geq 2$.

IV. CONCLUSIONS

We derived a multi-channel phase-equivalent transformation that can be used without restrictions on the structure of the discrete spectrum of the system in various scattering problems like $NN$ scattering, nucleon-cluster or cluster-cluster scattering. The multi-channel supersymmetry and inverse supersymmetry transformations appear to be particular cases of the suggested general phase-equivalent transformation corresponding to particular choices of the parameter values. The inverse supersymmetry transformation is possible if only the orbital angular momentum $l_i \geq 2$ at least in one of the coupled channels. It is interesting to note that from the point of view of the $NN$ system, this means that a deep attractive $NN$ potential supporting an additional forbidden state like Moscow $NN$ potential, can be constructed by the inverse supersymmetry transformation of the realistic meson-exchange potential with repulsive core only due to the $d$ wave admixture in the deuteron wave function.

With the help of the suggested transformation, one can construct a family of phase-equivalent potentials depending on continuous parameters. Such families may be very useful for fine tuning of the interaction aimed to fit not only two-body observables but also three- and few-body ones. If the system has at least one bound state, the phase-equivalent potential family is constructed using directly formulas (8) and (9). One can construct phase-equivalent single- or multi-channel potential families also in the case when there are no bound states in the system: if all channel orbital angular momenta $l_i \leq 1$, one can apply directly the transformation with the irregular function $\varphi_i(E_0, r)$; if at least one of the channel orbital angular momenta $l_i \geq 2$, one can produce a bound state using inverse supersymmetry at the first stage and remove the bound state at the last stage with the help of supersymmetry version of the transformation. So, one can, for example, construct a family of phase-equivalent potentials for any combination of coupled partial waves in the $NN$ system.

We hope that the suggested transformation will be useful in various few-body applications.

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