CELLS AND REPRESENTATIONS OF RIGHT-ANGLED COXETER GROUPS

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Abstract. We study Kazhdan–Lusztig cells and the corresponding representations of right-angled Coxeter groups and Hecke algebras associated to them. In case of the infinite groups generated by reflections in the hyperbolic plane about the sides of right-angled polygons we obtain an explicit description of the left and two-sided cells. In particular, we prove that there are infinitely many left cells but they all form only three two-sided cells.

1. Introduction

A Coxeter group is said to be right-angled if for any two distinct simple reflections their product has order 2 or \(\infty\). Due to their special properties and rich structure, right-angled Coxeter groups often arise in different geometric and algebraic problems. In a sense, the most interesting right-angled Coxeter groups are those which can be presented as groups generated by reflections in hyperbolic spaces. Examples of geometric applications of such groups can be found in [D], [MV], [BGI]. These groups also occur as Weyl groups of certain Kac–Moody Lie algebras [KP]. We are interested in the representations of right-angled Coxeter groups \(W\) and the corresponding Hecke algebras \(\mathcal{H}\).

An important difference between hyperbolic reflection groups and affine or finite Coxeter groups is that the former do not usually have a local system of generators in the sense of [OV]. In particular, the groups \(P_n\) (see 3.1) that are generated by reflections in the hyperbolic plane about the sides of right-angled \(n\)-gons can be called anti-local since only adjacent generators in the canonical system of simple reflections commute. This means that we can hardly hope to construct the representations of these groups inductively using the approach suggested in [OV]. At the same time, we can still use the global methods of [KL] and try to describe the Kazhdan–Lusztig cells in our groups. It appears that certain symmetries of the initial groups are reflected in the structure of the partitions into cells making the cells trackable. Our main results concern the groups \(P_n\) but the methods can be applied to the other right-angled Coxeter groups as well.

We show that the partition of the group \(P_n\) \((n \geq 5)\) into left cells consists of infinitely many elements and give an explicit description of the cells. At the
same time, all the left cells in our groups form precisely three two-sided cells (one of which is, of course, the trivial cell). It was first shown in [B] that the number of one-sided cells of a hyperbolic Coxeter group can be infinite, but there the author used an implicit argument and obtained only a conjectural structure of the corresponding cell-partitions (see also [C] for discussion). The fact that infinitely many left cells can still fall into finitely many two-sided equivalence classes and hence give rise to only finitely many $\mathcal{H}$-bimodules seems to have been previously unnoticed.

The explicit description of the cells makes it possible to consider corresponding representations of the Coxeter groups and their Hecke algebras. Here we only start the related analysis leaving more detailed considerations for the future. We discuss the representations using $W$-graphs which were also introduced in [KL].

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2. Preliminaries

We recall some well-known facts about Coxeter groups and Hecke algebras associated with them. The basic reference is the fundamental paper [KL]. All the material cited here can be also found in the book [H].

2.1. Let $W$ be a Coxeter group and let $S$ be the corresponding set of simple reflections. With some ambiguity of language we shall also call by Coxeter group the Coxeter system $(W, S)$. The Hecke algebra $\mathcal{H}$ over the ring $\mathcal{A} = \mathbb{Z}[q^{1/2}, q^{-1/2}]$ of Laurent polynomials in $q^{1/2}$ is defined as follows. As an $\mathcal{A}$-module, $\mathcal{H}$ is free with basis $T_w$ ($w \in W$), the multiplication is defined by

\[
T_w T_{w'} = T_{w w'}, \text{ if } l(w w') = l(w) + l(w'), \\]

\[
T_s^2 = q + (q - 1)T, \text{ if } s \in S,
\]

where $l(w)$ is the length of $w$ in $(W, S)$.

It will be also convenient to define

\[
\overline{T}_w = q^{-l(w)/2}T_w.
\]

2.2. Let $a \rightarrow \overline{a}$ be the involution of the ring $\mathcal{A}$ defined by $q^{1/2} = q^{-1/2}$. This extends to an involution $h \rightarrow \overline{h}$ of the ring $\mathcal{H}$ given by

\[
\sum a_w T_w = \sum \overline{a_w} T_w^{-1}.
\]

Let $\preceq$ be the Bruhat order on $W$ (see [H], Ch. 5.9). Denote, as usual, $q_w = q^{l(w)}$, $\epsilon_w = (-1)^{l(w)}$ for all $w \in W$. In [KL] it was shown that for any $w \in W$ there...
exists a unique element element $C_w \in \mathcal{H}$ such that

\[
\overline{C}_w = C_w, \\
C_w = \epsilon_w q_w^{1/2} \sum_{y \leq w} \epsilon_y q_y^{-1} P_{y, w} T_y,
\]

where $P_{y, w} \in \mathcal{A}$ is a polynomial in $q$ of degree $\leq \frac{1}{2}(l(w) - l(y) - 1)$ for $y < w$, and $P_{w, w} = 1$.

Elements $C_w$ form a basis (called $C$-basis) of $\mathcal{H}$ as an $\mathcal{A}$-module. This basis and the polynomials $P_{y, w}$ (called Kazhdan–Lusztig polynomials) turn out to be of fundamental interest in the representation theory of Coxeter groups and Hecke algebras.

2.3. Given $y, w \in W$, we write $y \prec w$ if $y < w$ and $P_{y, w}$ is a polynomial in $q$ of degree exactly $\frac{1}{2}(l(w) - l(y) - 1)$ (which is, of course, possible only if $l(w) - l(y)$ is odd). In this case the coefficient of the highest power of $q$ in $P_{y, w}$ is denoted by $\mu(y, w)$. If $w < y$ we set $\mu(w, y) = \mu(y, w)$, otherwise (if neither $y < w$ nor $w < y$) let $\mu(y, w) = \mu(w, y) = 0$. We write $y - w$ if $\mu(x, y) \neq 0$.

For any $w \in W$ define subsets of $S$:

\[\mathcal{L}(w) = \{s \in S \mid sw < w\}, \quad \mathcal{R}(w) = \{s \in S \mid ws < w\}.\]

Now define $y \leq_L w$ to mean that there is a chain $y = y_0, y_1, \ldots, y_n = w$ such that $y_i - y_{i+1}$ and $\mathcal{L}(y_i) \not\subseteq \mathcal{L}(y_{i+1})$ for $0 \leq i < n$. Similarly, say that $y \leq_R w$ if $y^{-1} \leq_L w^{-1}$. Finally, define $y \leq_{LR} w$ to mean that there exists a chain $y = y_0, y_1, \ldots, y_n = w$ such that for each $i < n$ either $y_i \leq_L y_{i+1}$ or $y_i \leq_R y_{i+1}$.

Let $\sim_L, \sim_R, \sim_{LR}$ be the equivalence relations associated to the preorders $\leq_L, \leq_R, \leq_{LR}$, respectively. The corresponding equivalence classes are called left, right and two-sided cells of $W$.

2.4. We shall often make use of the following properties of the defined relations.

**Lemma.** Let $x, y \in W$ and $x < y$.

(i) If there exists $s \in S$ such that $x < sx$, $sy < y$, then $x < y$ if and only if $y = sx$.

(ii) If there exists $s \in S$ such that $x < xs$, $ys < y$, then $x < y$ if and only if $y = xs$.

Moreover, in each of the cases $\mu(x, y) = 1$.

(This statement can be found in the proof of Theorem 1.3 in [KL]; it follows from the formula for the action of the elements $T_s$ on the basis $\{C_w \mid w \in W\}$ of $\mathcal{H}$ and the definition of $C_w$.)

**Corollary.**

(i) If $x \leq_L y$, then $\mathcal{R}(x) \supset \mathcal{R}(y)$. Hence, $x \sim_L y$ implies $\mathcal{R}(x) = \mathcal{R}(y)$.

(ii) If $x \leq_R y$, then $\mathcal{L}(x) \supset \mathcal{L}(y)$. Hence, $x \sim_R y$ implies $\mathcal{L}(x) = \mathcal{L}(y)$.

(To prove the corollary it is enough to consider the case $x - y$ with $\mathcal{L}(x) \not\subseteq \mathcal{L}(y)$, details can be found in [KL].)
2.5. The main purpose of the partition of a Coxeter group $W$ into cells is that it gives rise to the representations of group $W$ and its Hecke algebra $\mathcal{H}$. It is convenient to describe these representations using $W$-graphs.

Let $X$ be a set. Consider an oriented graph $\Gamma$ whose set of vertices is $X$; for each $x \in X$ there is assigned a subset $I_x$ of $S$, and if $I_x \not\subset I_y$, then there is an edge $(x, y) \in X \times X$ labeled by an integer $\mu(x, y)$. Graph $\Gamma$ is called a $W$-graph if the map $s \rightarrow \tau_s$, such that
\[
\tau_s x = \begin{cases}
  -x & \text{if } x \in X, s \in I_x, \\
  qx + q^{1/2} \sum_{y \in X, s \in I_y} \mu(x, y) y & \text{if } x \in X, s \not\in I_x,
\end{cases}
\]
defines a representation of $\mathcal{H}$ on the free $A$-module $A(X)$.

In [KL] it was shown that $X = W$ with $I_x = L(x)$ and $\mu(x, y)$ defined by the polynomial $P_{x,y}$ as in 2.3 gives a $W$-graph. Moreover, its full subgraphs corresponding to the left cells with the same sets $I_x$ and the same function $\mu$ are $W$-graphs themselves. Finally, when $W$ is a symmetric group $S_n$ it was proved that all the irreducible representations of $\mathcal{H}$ are defined by the $W$-graphs associated to the left cells of $W$.

3. Right-angled Coxeter groups

3.1. Recall that Coxeter group $(W, S)$ is called a right-angled Coxeter group if for any $s \neq t$ in $S$ the product $st$ has order 2 or $\infty$. The Coxeter graph of $W$ has only edges labeled via $\infty$ (see examples in Figure 1).

The simplest example of a right-angled Coxeter group is the infinite dihedral group $D_\infty = < s_1, s_2 | s_1^2 = s_2^2 = 1 >$. We are mainly interested in right-angled Coxeter groups generated by reflections in the hyperbolic $n$-space. The important example for us are the groups $P_n$ having representations
\[
P_n = < s_1, s_2, \ldots, s_n | s_i^2 = 1, (s_j s_{j+1})^2 = 1, (s_n s_1)^2 = 1 >
\]
where $i = 1, \ldots, n; j = 1, \ldots n - 1$. If $n \geq 5$ the group $P_n$ can be presented as a group of isometries of the hyperbolic plane generated by reflections about the sides of a right-angled hyperbolic $n$-gon (for more information about this and other geometric facts mentioned here we refer the reader to [VS]).

3.2. If a Coxeter group $(W, S)$ is presented by isometries of the hyperbolic space, then it gives rise to a tessellation of the space by the fundamental chambers of the group. The dual graph $G$ of this tessellation endowed with the standard graph metric reflects the structure of the initial group: fixing a vertex $e \in G$ which will correspond to the identity element of $W$ and labelling all the edges with the corresponding simple reflections from $S$, we can associate to an element $w \in W$ represented by a word on $S$ a geodesic path starting from $e$ in $G$. Two paths define the same element if and only if they have the same ends. Reduced expressions in $W$ correspond to the shortest geodesics in $G$. The graph $G$ is called a Cayley graph.
of \((W, S)\); this graph can be defined for an arbitrary group with a fixed system of generators.

3.3. Let \((W, S)\) be an arbitrary Coxeter group. We call by *lines* elements \(w \in W\) that have unique reduced expressions. By a subword of a word \(s_1s_2\ldots s_n, s_i \in S\) we mean any expression of the form \(s_is_{i+1}\ldots s_j, 1 \leq i \leq j \leq n\). The subword \(u\) of word \(w_0uw_1\) is called a *segment* of the word and of the corresponding element \(w \in W\) if \(u\) represents a maximal line such that any reduced expression of \(w\) has the form \(w_0'uw_1'\) with \(w_0 = w_0', w_1 = w_1'\) as the elements of the group \(W\) (here "maximal" means that \(u\) is not contained in any other subword with the same properties).

Lines are exactly the unique shortest geodesics of a Cayley graph that start from \(e\). The same way, segments of \(w \in W\) correspond to the (maximal) geodesics of a Cayley graph that are contained in any shortest geodesic corresponding to \(w\). The only segment of a line is the line itself.

Let us give a characterization of the lines and segments in a right-angled Coxeter group.

**Proposition.** Let \((W, S)\) be a right-angled Coxeter group. Then

(i) a word \(s_1\ldots s_k\) represents a line in \(W\) if and only if for any \(i = 1, \ldots, k-1\) the product \(s_is_{i+1}\) has order \(\infty\);
(ii) a subword \( u \) of \( w_0 \) is a segment if and only if \( u \) gives a minimal line such that \( \#R(w_0) \neq 1 \) and \( \#L(w_1) \neq 1 \) (here “minimal” means that \( u \) does not contain any proper or empty subword with the same property).

The proof easily follows from the definitions.

The proposition implies that any element \( w \) in a right-angled Coxeter group can be written as \( w = u_0s_1u_1s_2u_2\ldots s_nu_n \) where all the subwords \( u_i \) are either trivial or segments in \( w \) and \( \#R(w_0) \neq 1 \) and \( \#L(w_1) \neq 1 \) (here “minimal” means that \( u \) does not contain any proper or empty subword with the same property).

4. Distinguished Involutions

We recall some definitions and results from \([L2], [L3]\).

As in Section 2 we fix a Coxeter group \( (W, S) \) and denote by \( H \) the corresponding Hecke algebra over \( A = \mathbb{Z}[q^{1/2}, q^{-1/2}] \). For any \( x, y, z \in W \) define elements \( f_{x,y,z}, g_{x,y,z}, h_{x,y,z} \) in \( A \) so that

\[
\tilde{T}_x \tilde{T}_y = \sum_z f_{x,y,z} \tilde{T}_z, \\
\tilde{T}_x C_y = \sum_z g_{x,y,z} C_z, \\
C_x C_y = \sum_z h_{x,y,z} C_z.
\]

4.1. Let \( A^+ = \mathbb{Z}[q^{1/2}] \). To go further we need the following assumptions about the Coxeter group:

- \((W, S)\) is crystallographic, which means that for any \( s \neq t \) in \( S \) the product \( st \) has order 2, 3, 4, 6 or \( \infty \);
- \((W, S)\) is bounded, which means that there exists an integer \( N \geq 0 \) such that \( q^{N/2} f_{x,y,z} \in A^+ \) for all \( x, y, z \in W \), or equivalently, \( q^{N/2} h_{x,y,z} \in A^+ \) for all \( x, y, z \in W \).

4.2. Right-angled Coxeter groups are crystallographic. We prove that they are bounded:

**Lemma.** If in right-angled Coxeter group \( (W, S) \) \( ts_1 \ldots s_n = \hat{t}s_1 \ldots \hat{s}_j \ldots s_n \) (\( t, s_i \in S \), \( i = 1, \ldots, n \)), then \( t = s_j \) and \( t \) commutes with \( s_1, \ldots, s_{j-1} \).

**Proof.** By \([T]\) there is a sequence of elementary M-operations taking word \( ts_1 \ldots s_n \) to \( \hat{t}s_1 \ldots \hat{s}_j \ldots s_n \) (\( t, s_i \in S \)). In case of right-angled groups the operations are

(i) \( ss \rightarrow 1 \);
(ii) \( st \rightarrow ts \).
(s, t ∈ S). Both operations preserve parity of the number of a simple reflection occurrences in the word, which follows t = s_j. We shall prove that t commutes with s_1, . . . , s_{j−1} by induction on j. If j − 1 = 1, then we have ts_1t = s_1; ts_1 = s_1t. Now, if ts_1 . . . s_n is equivalent to ts_1 . . . s_{j−1}t s_{j+1} . . . s_n and j > 2, then there is an M-operation which decreases the number of simple reflections between two t’s. It can be only an operation of type (ii), so t commutes with some s_i with i ∈ {1, . . . , j − 1}. It follows by the induction hypothesis that t also commutes with the remaining s_i.

**Theorem.** Let (W, S) be a right-angled Coxeter group. Then it is bounded with N = max l(w_0 S), where S’ runs through all subsets of S such that corresponding subgroup (W S’ S’) is finite and w_0 S’ denotes the longest element of W S’.

**Proof.** Take arbitrary x, y ∈ W and let s_k . . . s_1, t_1 . . . t_n be corresponding reduced expressions (s_i, t_j ∈ S, i = 1, . . . , k, j = 1, . . . , n). We have

\[ T_x T_y = T_{s_k} . . . T_{s_1} T_y. \]

If l(s_1y) = 1 + l(y), then \( \tilde{T}_{s_1} \tilde{T}_y = \tilde{T}_{s_1y} \); otherwise, \( l(s_1y) = l(y) − 1 \) and \( \tilde{T}_{s_1} \tilde{T}_y = (q^{1/2} − q^{-1/2}) \tilde{T}_y + \tilde{T}_{s_1y}. \)

Proceeding inductively we obtain (similarly to [L2]):

\[ \tilde{T}_x \tilde{T}_y = \sum_I (q^{1/2} − q^{-1/2})^{p_I} \tilde{T}_I \]

where I ranges over all subsets i_1 < · · · < i_p of \{1, . . . , k\} such that

\[ s_{i_1} . . . \tilde{s}_{i_{l−1}} . . . \tilde{s}_{i_1} s_1 y < \tilde{s}_{i_1} . . . \tilde{s}_{i_{l−1}} . . . \tilde{s}_{i_1} s_1 y \]

for \( l = 1, . . . , p_I \), and \( \tilde{T}_I = s_{l_k} . . . \tilde{s}_{i_p} . . . \tilde{s}_{i_1} s_1 y, p = p_I = \#I \).

From the lemma it follows that for any such I there exists a reduced expression of x in which \( i_j = j \) and \( s_{i_j} ∈ L(y) \) for \( j = 1, . . . , l \). Denoting by \( Γ_y \) the subgroup of W generated by simple reflections from L(y), we have \( s_{i_p} . . . s_{i_1} \) is a reduced expression of an element from \( Γ_y \). To complete the proof it remains to show that for any \( y ∈ W \) subgroup \( Γ_y ≤ W \) is finite. Really, let \( s ≠ t ∈ L(y) \). This means that y has a reduced expression \( su_2 . . . u_n (s, u_j ∈ S) \) and \( tsu_2 . . . u_n = tlsru_2 . . . u_k u_n, \) so again by the lemma \( st = ts \), which follows \( Γ_y ∼ Z_2^p (p = \#L(y)) \) is a finite group.

4.3. Let \( a(z) \) be the smallest integer such that \( q^{a(z)/2} h_{x,y,z} ∈ A^+ \) for any \( x, y ∈ W \). It follows that \( 0 ≤ a(z) ≤ N \) for any \( z ∈ W \). Here are some important properties of the function \( a \) obtained in [L2, L3].

**Properties of \( a(z) \):**

(i) \( a(w) = a(w^{-1}) \) for all \( w ∈ W \);

(ii) \( a(w) = 0 \) if and only if \( w = e \);

(iii) the function \( a \) is constant on the two-sided cells of \( W \);

(iv) \( a(w) ≤ l(w) \) for all \( w ∈ W \).
4.4. Define a subset $D \subset W$ as follows:
$$D = \{ z \in W \mid a(z) = l(z) - 2\delta(z) \},$$
where $\delta(z)$ is the degree of $P_{e,z}$ as a polynomial in $q$.

It can be shown that $d^2 = e$ for any $d \in D$. The elements of $D$ are called

distinguished involutions of $W$. The following theorem was proved in [L3]:

**Theorem.** Any left cell contains a unique $d \in D$.

We shall use this powerful result to distinguish left cells of the groups $P_n$ in
the next section.

5. **Cells**

We mainly consider left cells and corresponding left $H$-modules. The results
concerning right cells are entirely similar. Two-sided cells and $H$-bimodules are
obtained via combination of the left and right-sided ones.

Let us first suppose that $(W, S)$ is an arbitrary Coxeter group. We shall
always use $s, t$ (possibly with subscribed indexes which will be not connected with
the initial ordering of generators in the case of the group $P_n$) to denote the elements
from $S$.

5.1. Let $w_1, w_2 \in W$. We say that $w_1$ and $w_2$ belong to the same left precell and
write $w_1 \sim_{ip} w_2$ if there exists $w_L \neq 1$ such that $w_1, w_2$ have reduced expressions

$$w'_1 w_L, w'_2 w_L,$$

respectively, and

(a) if $l(w_L) > 1$, then $w'_1, w'_2$ are either trivial or segments in $w_1, w_2$, resp.;

(b) if $l(w_L) = 1$, then $w_1$ and $w_2$ are lines (with $R(w_1) = R(w_2) = w_L$).

Let us also suppose $1 \sim_{ip} 1$.

**Proposition.** Relation $\sim_{ip}$ has the following properties:

(i) it is an equivalence relation on $W$;

(ii) each left precell $\Gamma$ contains a unique shortest element $w_L = w_L(\Gamma)$ such
that $\Gamma = \{ w \mid w = w' w_L \}$ where $w'$ is either trivial, or a segment in $w$, or
$w' w_L$ is a line (as in the definition of $\sim_{ip}$).

**Proof.** To prove (i) we need only to check the transitivity of $\sim_{ip}$. Suppose

$$w_1 \sim_{ip} w_2, \ w_2 \sim_{ip} w_3.$$

Then we have

$$w_1 = w'_1 w_L 1, \ w_2 = w'_2 w_L 1 = w''_2 w_L 2, \ w_3 = w'_3 w_L 2,$$

with $w'_1, w'_2, w''_2, w'_3$ as in the definition of $\sim_{ip}$. We want to show $w_1 \sim_L w_3$ and
we can suppose that $w_1, w_2, w_3$ are all unequal since otherwise it is trivial. We
now need to do some routine case by case considerations:

1) $l(w_L 1) = 1$. Then $w_2$ is a line, so $w''_2$ cannot be its proper segment; consequently
we have only two possibilities:

a) $l(w_L 2) = 1; \ w_L 2 = w_L 1; \ w_3 \sim_L w_1.$
b) $w_2^3 = 1$; $w_3 = w_4 w_{2L_2} = w_4 w_2$, $w_3^3 \neq 1$ is a segment in $w_3$ but this is impossible by the definition of segment since $w_2$ is a line.

2) The case $l(w_{L,2}) = 1$ is entirely similar.

3) $l(w_{L,1}) > 1$, $l(w_{L,2}) > 1$. There are again two possibilities:

a) $w_2^3 = 1$; $w_1 = w_4 w_{L,1} = w_4 w_2 = w_4 w_{2L,2}$, $w_1 \neq 1$ is a segment in $w_1$. We have either $w_2^3 = 1$, which follows $w_3 \sim_L w_1$, or $w_2^3$ is a segment in $w_2$, which leads to a contradiction with the definition of segment.

b) $w_2^3$ is a segment in $w_2$ then either $w_2^3 = w_2^3$ and $w_3 \sim_L w_1$ or $w_2^3 = 1$: $w_3 = w_4 w_{L,2} = w_4 w_2 = w_4 w_{2L,1}$, which is impossible by the definition of segment.

So in all the possible cases we obtain $w_1 \sim_L w_3$.

To prove (ii) we can take for $w_L$ (any) shortest element of the equivalence class $\Gamma$. Then any $w$ in $\Gamma$ will have the required form by the definition and uniqueness of $w_L$ follows. It is also possible to deduce the existence and uniqueness of $w_L$ from the proof of (i).

The language of precells seems to be very appropriate for the description of the cells of right-angled Coxeter groups. We shall consider in detail the case $W = P_n$. Similar methods can be applied to the other right-angled Coxeter groups as well, we are going to study these cases elsewhere.

5.2. Lemma. A left cell in $W$ is a union of the left precells which are $\sim_L$-equivalent to each other.

Proof. By Lemma 2.4 if $w_1 \sim_{L_p} w_2$, then $w_1 \sim_L w_2$. The corresponding chains $x_0 - x_1 - \cdots - x_k$, joining $w_1$ ($w_2$) with $w_L$ and having property $L(x_i) \cap L(x_{i+1}) = \emptyset$ for any $i$ (which is actually stronger than it is required for $\sim_L$), are obtained from the lines $w_1^i (w_2^i)$ defined in Proposition 5.1 by the rule $x_0 = w_1^i w_L = t_1 \cdots t_k w_L$, $x_i = t_{i+1} \cdots t_k w_L$. Since both $\sim_L$ and $\sim_{L_p}$ are equivalence relations the remaining part of the statement follows easily. ■

The non-unit left cells of the group $P_n$ ($n \geq 5$) are:

(i) $n$ cells corresponding to the 1-dimensional left precells of $P_n$ defined by the generators of $P_n$.
(ii) infinitely many cells which are equivalence classes of the left precells with
the canonical representatives $\Gamma(w_L)$, such that $w_L = t_1 t_2 w'_L$ with $t_1 t_2 = t_2 t_1$ ($t_1, t_2 \in S$) and $w'_L$ is a segment in $w_L$.

Proof. 1) We first show that each left precell of $P_n$ belongs to an at least one left
cell defined in the statement. If $\dim(\Gamma) = 1$, then $\ell(w_L(\Gamma)) = 1$ (as it easily follows
from the definitions of the precell and $w_L$), so $\Gamma$ is one of the 1-dimensional precells
from (i). By the definition of the group $P_n$ the dimensions of its left precells are
not greater than 2 (this also follows from the representation of $P_n$ as a group of
isometries of the hyperbolic plane); thus it remains to consider a left precell $\Gamma$ with
$\dim(\Gamma) = 2$.

Let $w_L = w_L(\Gamma)$, we have $w_L = t_1, t_1, t_2, \ldots, t_2, u_2, \ldots, t_k, u_k$
where for any admissible $i, j$ the subwords $u_i$ are either trivial or segments in $w_L$,
and arbitrary subword $x$: $t_i, t_j, t_{i, j + 1} = t_{i, j + 1}, t_{i, j}$.

Define two elementary moves between reduced words:

A: $t_1 t_2 t_3 x \rightarrow t_2 t_3 x$ for $t_1 t_2 = t_2 t_3$ and arbitrary word $x$;
B: $s_1 s_2 u t_1 t_2 x \rightarrow t_1 t_2 x$ for $t_1 t_2 = t_2 t_1$, $s_1 s_2 = s_2 s_1$, $u$ is a segment or $u = 1$
and arbitrary $x$.

By applying this elementary moves to $w_L$ one can obtain the word $t_k, n_k - 1, t_k, n_k u_k$
($u_k$ is a segment) which defines a canonical precell in (ii). We shall show that the
moves produce $\sim_L$-equivalent words.

The equivalence $w = t_1 t_2 t_3 x \sim_L w_0 = t_2 t_3 x$ is easy: we have $w = t_1 w_0$,
$L(w_0) = \{t_1, t_2\}$, $L(w) = \{t_2, t_3\}$ and $t_3 \neq t_1$ (because all the expressions are
reduced), so $w \triangleright w_0 = L(w) \triangleright L(w)$ and $L(w) \not\subset L(w_0)$.

To prove that move B is a left equivalence we shall use a supplementary construction. Having
$w = s_1 s_2 u t_1 t_2 x$, $w_0 = t_1 t_2 x$, define $w^* = t_1 u^{-1} s_1 s_2 u t_1 t_2 x$.
Note that $u$ is a segment in $w$ implies $u^{-1}$ and $u$ are segments in $w^*$. We are going
to show the following relations:

$$w_0 \leq_L w^* \sim_L w^*_n \sim_L \ldots \sim_L w^*_1 \sim_L w$$

where $w_{i+1}$ is obtained from $w_i$ by adding at the left the next letter of $w^*$ and
$w^*_j$ is obtained from $w^*_j$ by deleting a letter at the left.

The difficult part is to prove $w_0 \leq_L w^*$ since all the other chains are just of
the form $x - y$ with $|l(x) - l(y)| = 1$ and so satisfy the definitions (one can also
note that $w^* \sim_L w$).

Let $u = u_1 \ldots u_k$ with $u_i \in S$ is a (the) reduced expression of $u$. It is enough
to show that the coefficient $\mu(\nu, w^* \nu)$ of the $q^{|l(w^*) - l(w)| - 1}/2 = q^{k+1}$ in $P_{w_0, w^*}$ is
not 0. Let us make use of the following formula for the polynomials $P_{y, w}$ obtained
in the proof of existence of the $C_w$-basis in [KL]:

$$P_{y,w} = q^{1-c}P_{sy,v} + q^vP_{y,v} - \sum_{y \leq z < w, s < z} \mu(z,v)q_z^{-1/2}q_w^{1/2}q^{1/2}P_{y,z} \quad (y \leq w),$$

where $w = sv$ with $l(w) = 1 + l(v)$, $c = 1$ if $sy < y$, $c = 0$ if $sy > y$ and $P_{x,v} = 0$ unless $x \leq v$.

We have

$$P_{w_0,w^*} = \sum_{t_1t_2x = t_1u^{-1}s_1s_2ut_1t_2x} = P_{t_2x,v} + qP_{t_1t_2x,v} - \Sigma_0$$

with $v = u^{-1}s_1s_2ut_1t_2x$ and $\Sigma_0 = \sum \mu(z,v)q_z^{-1/2}q_v^{1/2}q^{1/2}P_{y,z}$ where the summation is over $z$ such that $t_1t_2x \leq z < v$, $t_1z < z$.

$\mu(t_2x,v) = 0$ because $L(v) = u_k \notin L(t_2x)$ since $u$ is a segment in $w$, and so by Lemma 2.3 $P_{t_2x,v}$ has the maximal possible degree ($= k + 1$) if and only if $u^{-1}s_1s_2ut_1t_2x = u_k t_2x$, which is impossible. Consider the second term:

$$qP_{t_1t_2x,v} = q^2P_{u_k t_1t_2x,u_k-1...u_1s_1s_2u_1...u_k t_1t_2x} + qP_1 - q\Sigma_1,$$

defining for $i = 1, \ldots, k$:

$$P_i = P_{y_i,v_i},$$

$$\Sigma_i = \sum_{y_i \leq z < v_i, u_{k-i+1}z < z} \mu(z,v_i)q_z^{-1/2}q_{v_i}^{1/2}q^{1/2}P_{y_i,z},$$

$$y_i = u_{k-i+2}...u_k t_1t_2x,$$

$$v_i = u_{k-i}...u_1 s_1 s_2 u_1...u_k t_1t_2x$$

(with the conventions that $u_{k-i+2}...u_k = 1$ for $i = 1$ and $u_{k-i}...u_1 = 1$ for $i = k$).

The remarkable point is that the coefficients of $q^{k-i+2}$ in $qP_i$ and $\Sigma_{i-1}$ for $i = 1, \ldots, k$ are equal! Really, consider the sum $\Sigma_i$. We have $L(v_i) = u_{k-i} \notin L(z)$ (since $u_{k-i+1} \notin L(z)$ and it does not commute with $u_{k-i}$) so by Lemma 2.3 $z < v_i$ implies $v_i = u_{k-i}z$, $\mu(z,v_i) = 1$, $z = u_{k-i-1}...u_1 s_1 s_2 u_1...u_k t_1t_2x = v_{i+1}$ and this is possible only if

$$u_{k-i-1} = u_{k-i+1}. \quad (\ast)$$

In this case we have $P_{y_i,z} = P_{y_i,v_{i+1}} = P_{y_{i+1},v_{i+1}}$ (the last equality is the consequence of $L(v_i) = u_{k-i+2} \neq u_{k-i+1} = L(v_{i+1})$) and so $\Sigma_i = qP_{i+1}$. It remains to check that if $\mu(y_{i+1},v_{i+1}) \neq 0$, then the equality $(\ast)$ holds, which can be easily done by supposing on the contrary that $u_{k-i-1} \neq u_{k-i+1}$ and applying Lemma 2.3. The case of $\Sigma_0$ should be considered separately, of course, but appears to be very similar.

So the leading terms of $qP_i$ and $\Sigma_{i-1}$ repetitively cancel each other and we finally obtain that the coefficient of $q^{k+1}$ in $P_{w_0,w^*}$ is equal to the leading coefficient of $q^{k+1}P_{u_1 y_k,v_k}$, which is equal to 1 because $u_1 y_k < v_k$ and $l(v_k) - l(u_1 y_k) = 2$.

This proves $w_0 < w^*$ with $\mu(w_0, w^*) = 1$. 
2) It remains to show that the cells defined in the statement do not intersect. We shall use the distinguished involutions.

For the non-unit elements \( z \in P_n \) we have \( a(z) \in \{1, 2\} \). If \( z \) is in a cell of type (i), then there exists \( s \sim_L z \) and by the Properties 4.3 \( a(z) = a(s) = 1 \). Now let \( z \) is in a cell of type (ii). Using Moves A, B from the first part of this proof and their right-side analogs we see that \( z \) belongs to the same two-sided cell as \( st \) \( (s, t \in S, (st)^2 = 1) \). It is easy to show that \( a(st) = 2 \) (take \( x = y = st \) in the definition of the function \( a \)), so again by 4.3 \( a(z) = 2 \).

It immediately follows from the definitions that \( s_i \in D, i = 1, \ldots, n \), so we have the distinguished involutions for each of the cells of type (i). Now consider a cell of type (ii) with the representative \( \Gamma(w_L) \) as in the statement of the theorem. An element \( z = w'_L t_1 t_2 w'_L \in \Gamma(w_L) \) is an involution, we shall see that \( z \in D \).

We see that the argument which was used to show that \( w_0 \leq w \) in part (1) of the proof works without any changes in this case either and gives \( \deg(P_{c,z}) = k \).

So we have

\[
P_{c,z} = P_{c,u_1 \ldots u_k} = P_{u_k, u_1 \ldots u_k} = P_{u_k, u_1 \ldots u_k} t_1 t_2 u_1 \ldots u_k.
\]

and \( z \in D \) by the definition.

It remains to apply Theorem 4.4 to distinguish all the left cells. \( \square \)

5.4. It was pointed out to me by V. Ostrik that the cells of type (i) were previously considered in [L1]. There the cells and corresponding representations were constructed for an arbitrary Coxeter group and then thoroughly studied in the finite and affine cases.

Left cells of the group \( P_n \) can be visualized on the corresponding tessellation of the hyperbolic plane. Figure 2 presents the cells for the group \( P_5 \): the pentagon in the center is the unit cell, five shaded regions represent the cells of type (i) all giving a one two-sided cell (see Corollary 5.7), each white region represents a cell of type (ii) and altogether they form the third two-sided cell.

Below we give several corollaries from Theorem 5.3 and its proof.

5.5. Corollary. The distinguished involutions of the group \( P_n \) are

\[
D = \{1\} \cup \{s_1, \ldots, s_n\} \cup \{ustu^{-1} | (st)^2 = 1, u \text{ is a segment in } ustu^{-1}\}.
\]

The distinguished involutions are related with algebra \( J \) defined in [L3] which may be regarded as an asymptotic version of Hecke algebra \( H \). Using the methods from [L3] this corollary can be applied to retrieve a partial structure information about algebra \( J \) of the group \( P_n \).
Figure 2. Cells of the group $P_5$.\textsuperscript{1}

5.6. Corollary. $W$-graphs associated to the left cells of type (i) are infinite rooted trees (binary trees for the group $P_5$, see Figure 3), while the $W$-graphs associated to the type (ii) cells admit infinitely many different cycles.

One can see that all the $W$-graphs corresponding to the cells of type (i) of the group $P_n$ define equivalent representations of $H$, with the equivalences induced by the cyclic permutations of the simple reflections $s_i \in S$. We suppose that the representations corresponding to the cells of type (ii) are also all equivalent, but this does not readily follow from the above arguments.

5.7. Corollary. The partition of the group $P_n$ ($n \geq 5$) onto two-sided cells consists of 3 elements:
- the unit cell corresponding to the trivial representation of $H$;
- the union of the left cells of type (i) (a 1-dimensional cell), the corresponding $W$-graph is an infinite tree;

\textsuperscript{1} This picture uses W. Casselman’s PostScript library for hyperbolic geometry.
Figure 3. Example of a $W$-graph corresponding to a left cell of type (i) of the group $P_5$ (all $\mu(x,y) = 1$, the vertices are represented by circles with the corresponding subsets of $S$ inside).

- the union of the left cells of type (ii) (a 2-dimensional cell), the corresponding $W$-graph admits infinitely many different cycles.

The remarkable point about this corollary is that to establish it we actually need only the Moves A, B from the proof with their right-side analogs, and so we do not use part (2) of the argument which relies on certain very strong results about distinguished involutions.

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