A note on inequalities for the masses of the lightest $\pi\pi$ resonances in large $N_c$ QCD

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Abstract

We derive and analyse inequalities relating masses of the lightest $\pi\pi$ resonances ($\rho$ and $\sigma$) to low energy couplings of the effective chiral Lagrangian in the limit of large number of colours.

1. The issue of the existence and interpretation of the light scalar resonance (we call it as $\sigma$ in what follows) is one of the most controversial questions in the meson spectroscopy (for a review of the scalar meson spectroscopy see the note on scalar mesons by S. Spanier and N. Tornqvist in Review of Particle Physics \(^{\dagger}\)). Far from complete list of experimental and theoretical papers devoted to the $\sigma$-meson \(^{3}\)-\(^{10}\) (see also \(^{2}\) and references therein) shows that this topic attracts considerable interest.

In these notes we analyse the sum rules relating low energy constants (LECs) of the effective chiral Lagrangian (EChL) to the resonance spectrum parameters in the limit of large number of colours \(^{11}\), \(^{12}\). \(^{\S}\) We shall show that from these sum rules one can derive a set of inequalities, e.g. such as:

$$M^2_\sigma(3L_2 + L_3) + M^2_\rho L_2 \leq \frac{F^2_\pi}{4},$$

where $L_i$ are the coupling constants of the fourth order EChL \(^{15}\), $M_\rho$ is the mass of the lightest isovector resonance ($\rho$ meson), $M_\sigma$ is the mass of the lightest isoscalar resonance.

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\(^{\dagger}\)Related problems are also discussed in the review article \(^{16}\).

\(^{\S}\)In \(^{13}\) it is shown that those sum rules can be derived from the general postulates of the effective theory without referring to large-$N_c$ limit; see also \(^{14}\).
(σ meson), and $F_\pi \approx 93$ MeV is the pion decay constant. This inequality, apart from applications for estimates of the σ-meson mass from above, demonstrates that properties of the resonance spectrum are in close relations with properties of chiral symmetry breaking. Below we give derivation of the inequality (1) as well as its enhancements.

2. In the ref. [41, 42] the following set of the large $N_c$ sum rules relating the constants of the effective chiral Lagrangian $L_i$ to the parameters of resonance spectrum has been derived:

\begin{align}
1 + O(m_\pi^4) &= \sum \frac{F_0^2 V_0}{[M_0^2 - 2m_\pi^2]^2} + \sum \frac{F_0^2 V_1}{[M_1^2 - 2m_\pi^2]^2}, \\
3L_2 + L_3 + am_\pi^2 + O(m_\pi^4) &= \frac{F_0^4}{4} \sum \frac{V_0}{[M_0^2 - 2m_\pi^2]^3}, \\
L_2 + \beta m_\pi^2 + O(m_\pi^4) &= \frac{F_0^4}{4} \sum \frac{V_1}{[M_1^2 - 2m_\pi^2]^3}. \tag{2}
\end{align}

Here $M_I$ are the masses of pion-pion resonances with isospin $I$, and $V_I$ — the corresponding residues. The latter are related to the $\pi\pi$ resonance width $\Gamma(R \to \pi\pi)$ via:

\begin{align}
V_0 &= \frac{2}{3} 16\pi (2J + 1) \frac{M_0^2}{\sqrt{M_0^2 - 4m_\pi^2}} \Gamma(R \to \pi\pi), \tag{3} \\
V_1 &= 16\pi (2J + 1) \frac{M_1^2}{\sqrt{M_1^2 - 4m_\pi^2}} \Gamma(R \to \pi\pi), \tag{4}
\end{align}

where $J$ is the resonance spin. The constant $F_0 \approx 88$ MeV is the pion decay constant in the chiral limit. The constants $\alpha, \beta$ are related to low energy coefficients (LECs) of the sixth order EChL. We use estimates for the LECs of the sixth order EChL obtained in ref. [53, 54] from the chiral expansion of the dual (string) models:

\begin{align}
\alpha m_\pi^2 &\approx 0.18 \cdot 10^{-3}, \quad \beta m_\pi^2 \approx 0.05 \cdot 10^{-3}. \tag{5}
\end{align}

From the sum rules eqs. (2) one can immediately obtain the following obvious inequalities:

\begin{align}
3L_3 + L_2 + am_\pi^2 > 0, \\
L_2 + \beta m_\pi^2 > \frac{V_\rho F_0^4}{4(M_\rho^2 - 2m_\pi^2)^3} \approx 1.66 \cdot 10^{-3}, \tag{6}
\end{align}

where $M_\rho$ and $V_\rho$ stand for the mass and residue of the lightest isovector $\pi\pi$ resonance ($\rho$-meson). Further noting that

\begin{align}
\sum_{j=0}^{k} \frac{V_0}{[M_0^2 - 2m_\pi^2]^{k+j}} > \left[ M_\sigma^2 - 2m_\pi^2 \right] \sum_{j=0}^{k} \frac{V_0}{[M_0^2 - 2m_\pi^2]^{k+j+1}}, \tag{7}
\end{align}

where $k \geq 2$ and $M_\sigma$ is a mass of the lightest isoscalar (scalar) resonance, we obtain the following inequality:

\begin{align}
M_\sigma^2 \left( 3L_3 + L_2 + am_\pi^2 \right) + M_\rho^2 \left( L_2 + \beta m_\pi^2 \right) < \frac{F_0^2}{4} + 2m_\pi^2 \left( 4L_2 + L_3 \right). \tag{8}
\end{align}

*Note that LECs $L_i$ are scale independent in the large $N_c$ limit.
This inequality provides us with a nice example of nontrivial relations between the parameters of resonance spectrum and low energy constants of EChL. The model independent large-$N_c$ inequality (8) can be used for the estimates of the $\sigma$--meson mass from above (see below), as well as for consistency checks of various models of low–energy QCD in the large-$N_c$ limit.

Parameters of the EChL in the large-$N_c$ limit have been calculated in various models of the low–energy QCD [46, 47, 48, 49, 50, 51]. We shall use parameters from the analysis of the EChL coupling constants in the large-$N_c$ limit done in [51] (the error bars take into account different values of the constants obtained in the fits performed in [51]):

$$L_2 = (1.6 \pm 0.1) \cdot 10^{-3},$$  
$$L_3 = -(4 \pm 1) \cdot 10^{-3},$$  

(9)

These values are close to those obtained from the phenomenological analysis [45, 52], what shows that the $1/N_c$ corrections to low energy coefficients $L_i$ are rather small.

Due to the inequality eq. (6) the value of $L_2$ can not be below $1.63 \cdot 10^{-3}$, therefore we shall use this minimal value of $L_2 = 1.63 \cdot 10^{-3}$ lying in the range given by the eq. [1].

The error of calculation of $L_3$ is bigger. Also, the errors of $L_2$ and $L_3$ are strongly correlated. In order to make an estimations of the $M_\sigma$ based on inequality eqs. (8) we use first the relation $2L_2 + L_3 = 0$ which follows from integration of the non–topological chiral anomaly [46, 47, 48] and from the low–energy limit of the dual–resonance (string) models [53]. Using the above values of $L_2$ and $L_3$ we obtain from eqs. (8):

$$M_\sigma < 770 \text{ MeV}, \quad \text{if} \quad 2L_2 + L_3 = 0.$$  

(10)

This is the upper bound for the lightest isoscalar resonance if one assumes the relation $2L_2 + L_3 = 0$. To consider the more general case, we derive the upper limit on $M_\sigma$ as a function of the parameter $\Delta$ defined as follows:

$$\Delta = - \frac{2L_2 + L_3}{L_2}. \quad \text{ (11)}$$

The value of this parameter is zero for EChL obtained by integration of non-topological chiral anomaly [46, 47, 48, 49] as well as for EChL obtained by chiral expansion of the dual–resonance (string) models [53]. In the large-$N_c$ based model of ref. [50] the value of $\Delta$ is fixed in terms of gluon condensate and constituent quark mass $m_Q \approx 0.35 \text{ GeV}$ as $\Delta = \frac{\pi^2 (\alpha_s G^2)}{8N_c m_Q^4} \approx 0.3$. The value of LECs obtained in ref. [55] corresponds to $\Delta = 5/8 = 0.625$. In any case the value of $\Delta$ can not exceed unity due to the inequality (6). Experimentally, the parameter $\Delta$ is constrained by the ratio of the D-wave pion scattering lengths:

$$\Delta = -3 \frac{a_0^2}{a_2^2} + O(m_\pi^2) \approx -0.2 \pm 0.6,$$  

(12)

where we took the experimental values of the D-wave scattering lengths ref. [56].

\[\text{Note that for larger values of } L_2 \text{ the bounds on } M_\sigma \text{ discussed below are stronger}\]
Now it is easy to derive from the inequality (8) the upper bound for the \( \sigma \) meson mass as a function of the parameter \( \Delta \). This function at small values of \( \Delta \) takes the form

\[
M_\sigma < 770 \left[ 1 + 0.42\Delta + 0.29\Delta^2 + O(\Delta^3) \right] \text{ MeV}.
\]  

(13)

We see that the upper bound for the \( \sigma \)-meson mass is sensitive to the sign of the parameter \( \Delta \) (see definition (11)). Therefore the values of LECs of the fourth order EChL can give us a valuable information about the lightest scalar meson in the spectrum of QCD.

3. In the case when one possesses an additional information (masses and widths of resonances) on the excited meson spectrum (mesons heavier than \( \sigma \) in the isoscalar channel and \( \rho \) in isovector one) the inequality (8) can be enhanced. Let us call the excited resonances for which we have additional information about their masses and widths as known. With this additional information the inequality (8) can be enhanced as follows:

\[
M_\sigma^2 \left( 3L_3 + L_2 + \alpha m^2_\pi - \sum_{\text{known}} \frac{F^4_0 V_0}{4 [M^2_0 - 2m^2_\pi]^3} \right) + M_\rho^2 \left( L_2 + \beta m^2_\pi - \sum_{\text{known}} \frac{F^4_0 V_1}{4 [M^2_1 - 2m^2_\pi]^3} \right)
\]

\[
< \frac{F^2_0}{4} - \sum_{\text{known}} \frac{F^4_0 V_0}{4 [M^2_0 - 2m^2_\pi]^3} - \sum_{\text{known}} \frac{F^4_0 V_1}{4 [M^2_1 - 2m^2_\pi]^3}
\]

\[
+ 2m^2_\pi \left( 4L_2 + L_3 - \sum_{\text{known}} \frac{F^4_0 V_0}{2 [M^2_0 - 2m^2_\pi]^3} - \sum_{\text{known}} \frac{F^4_0 V_1}{2 [M^2_1 - 2m^2_\pi]^3} \right). 
\]

(14)

For the numerical estimates we take as the known resonances \( f_2(1275) \) in the isoscalar channel and \( \rho_3(1690) \) in the isovector channel. We do not include other scalar and vector mesons as their nature is not well established and it is not clear whether their dynamics is “leading” in large \( N_c \). Taking the masses and \( \pi \pi \) widths of \( f_2(1275) \) and \( \rho_3(1690) \) from \[1\] we obtain the enhancement of the inequality (13)

\[
M_\sigma < 665 \left[ 1 + 0.44\Delta + 0.33\Delta^2 + O(\Delta^3) \right] \text{ MeV}.
\]

(15)

Obviously the inclusion of other resonances, e.g. \( f_0(980), f_0(1370), f_0(1500), \rho, f_4 \), etc. would lead to lower bound on the mass of \( \sigma \)-meson.

4. To summarize, we derive the inequalities for the masses of the lightest \( \pi \pi \) resonances in the limit of large number of colours (\( N_c \to \infty \)), see eqs. (8-13). These inequalities put an upper bound on the mass of \( \sigma \)-meson in terms of pion decay constant \( F_\pi \) and the low–energy constants of effective chiral Lagrangian \( L_2 \) and \( L_3 \). Analysis of these inequalities favours the presence of the light (mass < 750 MeV) scalar state in the meson spectrum of the multicolour QCD.

As a final remark we note that the sum rules (2) are derived in the limit of large number of colours, this implies that the exotic mesons (glueballs, four-quark states) do not contribute to the sum rules because their contributions are suppressed by powers of \( 1/N_c \). This observation shows that the sum rules (2) can be used for identification of the nature of various mesons, in particular the low–lying ones. For example, the sum rules in eq. (2) tell us that the leading large–\( N_c \) part of the width (read the width of the \( q\bar{q} \) and hybrid parts) and the mass of the \( \sigma \)-meson should satisfy the following constraint:
\[
\frac{32\pi F_0^2 M_\sigma^2 \Gamma (\sigma \rightarrow \pi \pi)}{3\sqrt{M_\sigma^2 - 4m_\pi^2 [M_\sigma^2 - 2m_\pi^2]^2}} \leq 1 - \frac{4 (M_\rho^2 - 2m_\pi^2)}{F_0^2} L_2 .
\]  
(16)

Obviously, other sum rules in eq. (3) and an additional information about resonance spectrum would provide more sophisticated constraints on the parameters of \( q\bar{q} \) component of the \( \sigma \)-meson. We shall analyse them elsewhere.

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