The Heavy Quark Free-Energy at $T \leq T_c$ in AdS/QCD

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Starting with the modified AdS/QCD metric developed in Ref. \[1\] we use the Nambu-Goto action to obtain the free energy of a quark-antiquark pair at $T < T_c$, for which we show that the effective string tension goes to zero at $T_c = 154$ MeV.

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I. INTRODUCTION

As shown in Ref. \[2\] the free energy of a static, infinitely massive quark-antiquark pair $F$ is given by

\[ e^{-\beta F} = \langle L(\vec{r}) L^\dagger(\vec{r}) \rangle , \]

where $L(\vec{r})$ is the Wilson-Line

\[ L(\vec{r}) = \frac{1}{N} \text{tr} \, T \exp \left[ i \int_0^\beta d\tau \hat{A}^0(\vec{r}, \tau) \right] , \]

$\beta = 1/T$ is the inverse temperature, $\hat{A}^0$ is the gluon field in the fundamental representation and $\tau$ denotes imaginary time. According to the holographic dictionary [3] the right hand side of equation (1.1) is equal to the string partition function on the AdS$_5$ space with the integration contours on the boundary of AdS$_5$. In saddle-point approximation:

\[ e^{-\beta F} = \langle L(\vec{r}) L^\dagger(\vec{r}) \rangle \approx e^{-S_{NG}} , \]

where $S_{NG}$ is the Nambu-Goto action

\[ S_{NG} = \frac{1}{2\pi l_s^2} \int d^2\xi \sqrt{\det h_{ab}} , \]

with the induced worldsheet metric

\[ h_{ab} = G_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} . \]

As described in reference [4], due to the symmetry of the problem, we may set up a cylindrical coordinate system in five-dimensional Euclidean space. Then the five coordinates are:

- $t$ - time
- $z$ - the bulk coordinate (extra 5th dimension)
- $x, r, \phi$ - three spatial coordinates

We parameterize an element of the surface connecting the two Wilson loops with $d\xi_1 = rd\phi$ and $d\xi_2 = dx$, set $t = 0$ and reinterpret $d\tau \equiv rd\phi$ as Euclidean time. We use the modified metric $G_{\mu\nu}$ developed in reference [1] effectively for $N_c = 3$ and $N_f = 4$. This metric has been further analysed as a possible solution of 5-d gravity in Ref. [5]

\[ ds_{Euc}^2 = \frac{h(z)L^2}{z^2} (r^2 d\phi^2 + dr^2 + dx^2 + dz^2) , \]

\[ h(z) = \frac{\log(\epsilon)}{\log((Lz)^2 + \epsilon)} , \]

\[ \Lambda = L^{-1} = 264\text{MeV} , \]

\[ \epsilon = \frac{l_s^2}{L^2} = 0.48 . \]

The Nambu-Goto action determines the string surface with $l_s$ as string length:
follows from Noether’s theorem that there exists a con-
explicitly, it is invariant under variations

dervative:

\[
\text{Eq. (3.1)} \]

The Euler-Lagrange equations corresponding to the
Lagrangian \( S_{\text{NG}} \) does not depend on \( x \) explicitly, it is invariant under variations \( x \to x + \delta x \). It follows from Noether’s theorem that there exists a con-
served quantity \( k \) given by

\[
k = \frac{h(z) \cdot r}{z^2} \frac{1}{1 + \left( z' \right)^2 + \left( r' \right)^2} . \quad (2.1)
\]

The Euler-Lagrange equations corresponding to the
Nambu-Goto action (1.10) can be simplified with
Eq. (2.1) (see Ref. [1]):

\[
r'' - \frac{h^2(z) \cdot r}{k^2 z^4} = 0 ,
\]

\[
z'' - \frac{h(z) \cdot r^2 \cdot (z \partial_z h(z) - 2h(z))}{k^2 z^5} = 0 . \quad (2.2)
\]

The boundary conditions are

\[
r(\pm d/2) = R = \frac{\beta}{2\pi} = \frac{1}{2\pi T} ,
z(\pm d/2) = 0 . \quad (2.3)
\]

But the conditions Eqs. (2.3) and (3.1) are not given at
the same point. It is more convenient to find conditions
for the functions and their derivatives at the same point.
Analysis of Eq. (2.2) shows that \( r' \) and \( z' \) must diverge
near the boundary \( x \to \pm \frac{d}{2} \). In order to obtain some
stable numerical solution, we have studied the behavior
of \( r(x) \) and \( z(x) \) near the boundary, cf. Ref. [3]. Numerically,
a small cutoff \( \nu \) is applied, then \( r(-d/2 + \nu) \), \( z(-d/2 + \nu) \), \( r'(-d/2 + \nu) \) and \( z'(-d/2 + \nu) \) can be calculated
asymptotically and used as initial conditions. We
do not prescibe the value of \( d \). For a fixed value of \( R \), we
give an arbitrary value to \( k > 0 \), then calculate \( r, z \) and
consequently the Nambu-Goto action \( S_{\text{NG}} \) for this value
of \( k \). By changing the value of \( k \) we obtain the Nambu-
Goto action as a function of \( k \), denoted by \( S_{\text{NG}}(k) \). We
can express \( d \) as a function of \( k \), and determine the
distance \( d \) associated with the Nambu-Goto action \( S_{\text{NG}}(k) \).
In principle, the constructed numerical solutions \( r'(x) \) and \( z'(x) \) can vanish at different points due to the small
difference between our asymptotic solution and the real
solution. We adjust the initial conditions at the point \( v \)
keeping the value of \( k \) fixed in such a way that \( r'(0) \) and
\( z'(0) \) vanish at the same point. The Nambu-Goto action is
given by

\[
S_{\text{NG}} = \frac{2}{\epsilon} \int_{-d/2}^{0} dz \frac{h(z) \cdot r}{z^2} \frac{1}{1 + \left( z' \right)^2 + \left( r' \right)^2} = \frac{2}{\epsilon} \int_{-d/2}^{0} dz \frac{h(z) \cdot r}{z^2} \frac{1}{1 + \left( z' \right)^2 + \left( r' \right)^2} + \frac{2}{\epsilon} \int_{-d/2}^{0} dz \frac{h(z) \cdot r}{z^2} \frac{1}{1 + \left( z' \right)^2 + \left( r' \right)^2} , \quad (3.2)
\]

where the subscript “a” denotes “asymptotic solution”
near \( x = -d/2 \), while “n” means “numerical solution”. 
In the last expression, the first integral is divergent at
\( x = -d/2 \). But, as we have the explicit form of the

\[
\begin{align*}
S_{\text{NG}} &= \frac{1}{2\pi l_s^2} \int_0^{2\pi} \int_{-d/2}^{d/2} dz \frac{L^2 h(z)}{z^2} r \sqrt{1 + \left( z' \right)^2 + \left( r' \right)^2} \\
&= \frac{1}{\epsilon} \int_{-d/2}^{d/2} dz \frac{h(z)}{z^2} r \sqrt{1 + \left( z' \right)^2 + \left( r' \right)^2} \\
&= \frac{1}{\epsilon} \int_{-d/2}^{d/2} dz \mathcal{L}[z(x), z'(x), r(x), r'(x)] , \quad (1.10)
\end{align*}
\]

The quark and antiquark are separated by a distance
\( d \) along the x-axis. The two Polyakov loops are approx-
imated by Wilson loops of radius \( R = \beta/2\pi \). \( \mathcal{L} \) is the
Lagrangian density and the prime (') denotes the deri-

cative with respect to \( x \). The configuration in shown in
figure [1]

\[
\text{II. EULER-LAGRANGE EQUATIONS}
\]

\[
\text{III. NUMERICAL SOLUTIONS}
\]

Analysis of symmetry (cf. Fig. [1]) gives for the first
derivatives:

\[
r'(0) = 0 , 
z'(0) = 0 . \quad (3.1)
\]

\[
\begin{align*}
S_{\text{NG}} &= \frac{1}{2\pi l_s^2} \int_0^{2\pi} \int_{-d/2}^{d/2} dz \frac{L^2 h(z)}{z^2} r \sqrt{1 + \left( z' \right)^2 + \left( r' \right)^2} \\
&= \frac{1}{\epsilon} \int_{-d/2}^{d/2} dz \frac{h(z)}{z^2} r \sqrt{1 + \left( z' \right)^2 + \left( r' \right)^2} \\
&= \frac{1}{\epsilon} \int_{-d/2}^{d/2} dz \mathcal{L}[z(x), z'(x), r(x), r'(x)] , \quad (1.11)
\end{align*}
\]

\[
\text{FIG. 2: The Euler-Lagrange equations (2.2) have only so-
}
\]

\[
\text{olutions for temperature } T \text{ and quark-antiquark separation } d
\text{ within the shaded area.}
\]
asymptotic solutions \( r_a(x) \) and \( z_a(x) \), we can expand the first integrand into power series near \( x = -d/2 \), and remove the divergent terms. To compensate this removal, we should add the antiderivative of the divergent terms at \( x = -d/2 + v \). This way, we obtain the regularized value of \( S_{\text{NG}} \).

Integrating the Euler-Lagrange equations \(^{22}\) for a wide range of initial values suggests that solutions exist only for a specific range of temperature \( T \) and quark–antiquark separation \( d \) (Fig. 2).

Figure 3 defines the regularized Nambu-Goto action \( S_{\text{NG,reg}} \) for several temperatures \( T \) as a function of quark–antiquark separation \( d \). A fit to the numerical calculations gives

\[
S_{\text{NG,reg}}^{\text{fit}} = \frac{-0.48}{Td} + d \left( \frac{-7.46}{\text{fm}} + \frac{5.84}{T \text{fm}^2} \right). \tag{3.3}
\]

**IV. THERMODYNAMIC QUANTITIES**

Using \( S_{\text{NG,reg}}^{\text{fit}} \) it is easy to calculate the free energy of the \( QQ \)-system as a function of the \( QQ \) separation:

\[
F = T \cdot S_{\text{NG,reg}}^{\text{fit}} = \frac{-0.48}{d} + d \left( \frac{-7.46}{\text{fm}} T + \frac{5.84}{\text{fm}^2} \right). \tag{4.1}
\]

Fig. 4 shows the free energy. One can recognize the flattening of \( F \) when increasing temperature \( T \) for large distances \( d \). The term linear in \( d \) yields the effective string tension:

\[
\sigma_{\text{effective}} = \frac{-7.46}{\text{fm}} T + \frac{5.84}{\text{fm}^2}. \tag{4.2}
\]

The entropy writes as

\[
S = \frac{\partial F}{\partial T} = \frac{7.46}{\text{fm}} d, \tag{4.3}
\]

and does not depend on temperature. Such an entropy is well known for strong coupling QCD on a 3-dimensional lattice, where \( S = \ell \cdot \log (2D - 1) \), with \( D = 3 \) and \( \ell \) is the length of the random path in lattice units connecting the quark and antiquark \(^{6}\). Inner energy and string tension also do not depend on temperature and are given by

\[
E = F + TS = \frac{-0.48}{d} + \frac{5.84}{\text{fm}^2} d, \tag{4.4}
\]

and

\[
\sigma = \frac{5.84}{\text{fm}^2} = \sigma_{\text{effective}}(T = 0), \tag{4.5}
\]

respectively.

**V. CONFINEMENT AND PHASE TRANSITION**

The free energy Eq. (4.1) contains a linear term and a Coulomb-like term, which is absent in strong coupling lattice QCD. The linear term provides confinement: when it becomes zero there will be no confinement. From Eq. (4.2) we can estimate the critical temperature for the confinement/deconfinement phase transition, which is roughly \( T_c = 154 \text{ MeV} \). This value is in agreement with lattice QCD results, i.e. \( T_c^{lattice}(N_c = 3, N_f = 3) = 155 \pm 10 \text{ MeV} \) \(^{8}\), but one must admit that we only solve a pure gluon theory without dynamical quarks, where, however, the input \( T = 0 \) potential has been fitted for \( N_f = 4 \).

**VI. CONCLUSION**

We have shown that the modified metric of AdS/QCD proposed in Ref. \(^{4}\) can also be applied to the heavy quark potential. Since we are not using the black hole metric this theory is restricted to the \( T < T_c \) regime. We
found that the modified AdS$_5$-metric produces confinement and the short distance Coulombic behavior in this region. In previous works on loop-loop correlators [8, 9], these two features had to be added by hand, whereas here they follow from one action. We also can determine $T_c$ by demanding that the effective string tension vanishes.

Because of the singularity in the metric at $z_{IR} = 1/\sqrt{T - \epsilon} \approx 0.54$ fm the Euler-Lagrange equations (2.2) have only solution for a very limited range of boundary condition. In particular for large $T$ they yield solutions only for very small $q\bar{q}$ separations making the fit (3.3) more hypothetical.

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