Twisted Classical Poincaré Algebras

Jerzy Lukierski$^{1,4)}$, Henri Ruegg$^4$ and Valerij N. Tolstoy$^{2,4)}$

Dept. de Physique Theorique, Université de Geneve, 24, qui Ernest-Ansermet, 1211 Geneve 4, Switzerland

Anatol Nowicki$^3$)

Physikalisches Inst., Universität Bonn, Nussallee 12, 53115 Bonn, Germany

Abstract

We consider the twisting of Hopf structure for classical enveloping algebra $U(\hat{g})$, where $\hat{g}$ is the inhomogenous rotations algebra, with explicit formulae given for $D = 4$ Poincaré algebra ($\hat{g} = \mathcal{P}_4$). The comultiplications of twisted $U^F(\mathcal{P}_4)$ are obtained by conjugating primitive classical coproducts by $F \in U(\hat{c}) \otimes U(\hat{c})$, where $\hat{c}$ denotes any Abelian subalgebra of $\mathcal{P}_4$, and the universal $R$-matrices for $U^F(\mathcal{P}_4)$ are triangular. As an example we show that the quantum deformation of Poincaré algebra recently proposed by Chaichian and Demiczev is a twisted classical Poincaré algebra. The interpretation of twisted Poincaré algebra as describing relativistic symmetries with clustered 2-particle states is proposed.

$^1)$ On leave of absence from the Institute for Theoretical Physics, University of Wroclaw, pl. Maxa Borna 9, 50-204 Wroclaw, Poland.

$^2)$ On leave of absence from Institute of Nuclear Physics, Moscow State University, 119899 Moscow, Russia.

$^3)$ On leave of absence from the Institute of Physics, Pedagogical University, Pl. Słowiński 6, 65-029 Zielona Góra, Poland.

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1. Introduction

Let us consider Poincaré algebra $\mathcal{P}_4$ with the generators $\hat{g} = (P_\mu, M_{\mu\nu})$ as a classical Hopf algebra. We supplement the well-known algebraic relations

\[
[M_{\mu\nu}, M_{\rho\tau}] = i(g_{\mu\tau}M_{\nu\rho} - g_{\nu\tau}M_{\mu\rho} + g_{\nu\rho}M_{\mu\tau} - g_{\mu\rho}M_{\nu\tau})
\]

\[
[M_{\mu\nu}, P_\rho] = i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu)
\]

\[
[P_\mu, P_\nu] = 0
\]

by the "primitive" coproduct relations

\[
\Delta_0(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}
\]

\[
\Delta_0(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu
\]

and the antipode $S_0(\hat{g}) = -\hat{g}$ ($\hat{g} \in \mathcal{P}_4$). The relations (1.1) lead to the well known Wigner theory of representations of Poincaré algebra [1,2] which are spanned by the Hilbert vectors $|m, s; p_\mu, S_3 \rangle$, where $m$ and $s$ describe respectively the eigenvalues of mass and relativistic spin (Pauli-Lubanski) Casimir, $p_\mu$ is the fourmomentum and $S_3$ ($-S \leq S_3 \leq S$) describe the spin projection values. The coproduct formula dictates how to calculate the action of the Poincaré generators on tensor product.

The quantum deformations of Poincaré algebra are described by the modifications of the relations (1.1-2) preserving the Hopf algebra structure (for general framework see e.g. [3,4]). In this paper we would like to consider the mildest quantum deformations of (1.1-2) obtained by the twisting procedure [5-9]. Following Drinfeld [5] two Hopf algebras $\mathcal{A} = (A, \Delta, S, \varepsilon)$ and $\mathcal{A}^F = (A, \Delta^F, S^F, \varepsilon)$ are related by twisting if there exists an invertible function $F = \sum_i f_i \otimes f^i \in \mathcal{A} \otimes \mathcal{A}^\otimes$ satisfying the "cocycle" condition [5,7,8]

\[
F_{23}(1 \otimes \Delta)F = F_{12}(\Delta \otimes 1)F
\]

1 Strictly speaking we consider below $F$ belonging to an extension of $\mathcal{A} \otimes \mathcal{A}$. 
and \((\varepsilon \otimes 1)F = (1 \otimes \varepsilon)F = 1\). In such a case \(\Delta^F\) and \(\Delta\) are related as follows
\[(a \otimes b \cdot c \otimes d = ac \otimes bd)\]
\[
\Delta^F(a) = F \cdot \Delta(a) \cdot F^{-1}
\]
Introducing \(U = \sum_i f_i \cdot S(f^i)\) one obtains also that
\[
S^F(a) = US(a)U^{-1}
\]
If \(\mathcal{A}\) is the quasitriangular Hopf algebra and the relations (1.3) are replaced by [5,6]
\[
(\Delta \otimes 1)F = F_{12}F_{23} \quad (1 \otimes \Delta)F = F_{13}F_{12}
\]
the universal \(R\)-matrices for \(\mathcal{A}\) and \(\mathcal{A}^F\) are related by the formulae (\(\bar{F} = \sigma \cdot F = \sum_i f^i \otimes f_i\))
\[
R^F = F^{-1} \cdot R \cdot \bar{F}
\]
For the complex simple Lie algebras \(\hat{g}\) there were considered twistings described by
\[
F = \exp f \quad f \in \hat{\mathfrak{c}} \otimes \hat{\mathfrak{c}}
\]
where \(\hat{\mathfrak{c}}\) is the commutative subalgebra of \(\hat{g}\) (Cartan subalgebra in [6], Borel subalgebra in [8]). Indeed it is easy to check that if \(f \in \hat{\mathfrak{c}} \otimes \hat{\mathfrak{c}}\), and \(\hat{\mathfrak{c}}\) is abelian, the conditions (1.5) are valid.

In this paper we shall consider the twisting of physically important case of inhomogeneous rotation algebras \(\hat{g} = O(D - k, k) \supset T_D\), in particular the D=4 Poincaré algebra \(\hat{g} = O(3, 1) \supset T_4\). In such nonsimple algebras one can select the commutative subalgebra \(C_m\) in several ways, e.g.

- a) Cartan subalgebra \((h_1, \ldots, h_n)\) \((n = \frac{D}{2}\) for D even, \(n = \frac{D-1}{2}\) for D odd\)
- b) Translation generators \((P_1 \ldots P_D)\)
- c) “Mixed” Cartan–translation algebra \(C_k\) \(k \leq \frac{N}{2}\)
\[
C_k = (h_1 \ldots h_k, P_{2k+1} \ldots P_D)
\]

The aim of this paper is

- a) to describe the twistings of \(U_q(\mathfrak{p}_4)\) depending on Cartan generators and
translation generators.
b) to provide an interesting example.

In Sect.2 we shall consider in explicite way the twisted D=4 Poincaré algebras $U^F(\mathcal{P}_d)$ with the choice of the algebra $\hat{g}$ (see (1.7)) described by the formula (1.8) with $k = 0, 1, 2$. Further generalization in the presence of central generators $Z_i$ ([$Z_i, \hat{g}$] = 0 for $\hat{g} \in \mathcal{P}_d$) is also given. In Sect.3 we shall discuss as an example of classical twisted Poincaré algebra the quantum Poincaré algebra considered recently by Chaichian and Demiczev [10]. In Sect.4 we shall discuss the elements of the representation theory of twisted Poincaré algebras, and present an outlook: some generalizations as well unsolved problems.

2. Twisting of the classical Poincaré algebra.

Let us denote the basis of the commutative algebra $\hat{c}$ ($F \in \hat{c} \otimes \hat{c}$) by $(c_1 \ldots c_n)$. We define

$$F = F_+ F_- \quad F_\pm = \exp f_\pm$$

(2.1)

where $F_\pm = \pm \sigma \cdot f_\pm$ ($\sigma$ is the exchange map: $\sigma(c_i \otimes c_j) = c_j \otimes c_i$), and

$$f^{(\pm)} = \frac{1}{2} \alpha^{(\pm)}_{ij}(c_i \otimes c_j \pm c_j \otimes c_i)$$

(2.2)

i.e. one can assume that $\alpha_{\pm ij} = \pm \alpha_{\pm ji}$.

If we twist the coproducts of classical Lie algebra we obtain from the commutativity of $\hat{c}$ that

$$U = \sum f_i \otimes S(f^i) = \exp(-\alpha_{ij} c_i c_j)$$

(2.3)

and after using (1.6) the $R$–matrix takes the particular form:

$$R = \exp(-2f_-) = (F_-)^{-2}$$

(2.4)

The formulae for the coproduct $\Delta^F$ depend on the particular choice of the algebra $\hat{c}$. We shall further specify our algebra for the case of classical Poincaré algebra (1.1), and we shall consider following three types of the
twist function:
a) \( \hat{c} = (M_3 = M_{12}, \ N_3 = M_{30}) \)
We postulate

\[
\begin{align*}
f_+ &= \alpha_+ M_3 \otimes M_3 + \beta_+ (M_3 \otimes N_3 + N_3 \otimes M_3) + \gamma_+ N_3 \otimes N_3 \\
f_- &= \beta_- (M_3 \otimes N_3 - N_3 \otimes M_3)
\end{align*}
\]

(2.5)

One gets \( (M_i \equiv \frac{1}{2} \epsilon_{ijk} M_{jk}; \ M_\pm \equiv M_1 \pm i M_2; \ N_i \equiv M_{i0}; \ P_\pm \equiv P_1 \pm i P_2) \)

\[
\Delta^F(M_\pm) = M_\pm \otimes e^{\pm A_1} \cos(B_1) + e^{\pm A_2} \cos(B_2) \otimes M_\pm
\]

\[
\pm N_\pm \otimes e^{\pm A_1} \sin(B_1) + e^{\pm A_2} \sin(B_2) \otimes N_\pm
\]

\[
\Delta^F(M_3) = M_3 \otimes 1 + 1 \otimes M_3
\]

(2.6)

\[
\Delta^F(N_\pm) = N_\pm \otimes e^{\pm A_1} \cos(B_1) + e^{\pm A_2} \cos(B_2) \otimes N_\pm
\]

\[
\mp M_\pm \otimes e^{\pm A_1} \sin(B_1) \mp e^{\pm A_2} \sin(B_2) \otimes M_\pm
\]

\[
\Delta^F(N_3) = N_3 \otimes 1 + 1 \otimes M_3
\]

\[
\Delta^F(P_\pm) = P_\pm \otimes e^{\pm A_1} + e^{\pm A_2} \otimes P_\pm
\]

\[
\Delta^F(P_3) = P_3 \otimes \cos(B_1) + \cos(B_2) \otimes P_3 + i P_0 \otimes \sin(B_1) + i \sin(B_2) \otimes P_0
\]

\[
\Delta^F(P_0) = P_0 \otimes \cos(B_1) + \cos(B_2) \otimes P_0 + i P_3 \otimes \sin(B_1) + i \sin(B_2) \otimes P_3
\]

where
\[ A_k = \alpha_+ M_3 + (\beta_+ - (-1)^k \beta_-) N_3 \]
\[ B_k = \gamma_+ N_3 + (\beta_+ + (-1)^k \beta_-) M_3 \]

b) \( \hat{c} = (M_3 = M_{12}, P_3, P_0) \)
We assume that \((r, s = 3, 0)\)

\[ f_+ = \alpha_+ M_3 \otimes M_3 + \delta^r_+ (M_3 \otimes P_r + P_r \otimes M_3) + \rho^r s P_r \otimes P_s \]
\[ f_- = \delta^r_- (M_3 \otimes P_r - P_r \otimes M_3) \]

One obtains
\[ \Delta^F(M_{\pm}) = M_{\pm} \otimes e^{\pm A_1} + e^{\pm A_2} \otimes M_{\pm} \pm P_{\pm} \otimes B_3 e^{\pm A_1} \pm e^{\pm A_2} C_3 \otimes P_{\pm} \]
\[ \Delta^F(M_3) = M_3 \otimes 1 + 1 \otimes M_3 \]
\[ \Delta^F(N_{\pm}) = N_{\pm} \otimes e^{\pm A_1} + e^{\pm A_2} \otimes N_{\pm} - i P_{\pm} \otimes B_0 e^{\pm A_1} + C_0 e^{\pm A_2} \otimes P_{\pm} \]
\[ \Delta^F(N_3) = N_3 \otimes 1 + 1 \otimes N_3 - i P_3 \otimes B_0 + C_0 \otimes P_3 + P_0 \otimes B_3 + C_3 \otimes P_0 \]
\[ \Delta^F(P_1) = P_1 \otimes \cosh(A_1) + \cosh(A_2) \otimes P_1 + i P_2 \otimes \sinh(A_1) + i \sinh(A_2) \otimes P_2 \]
\[ \Delta^F(P_2) = P_2 \otimes \cosh(A_1) + \cosh(A_2) \otimes P_2 - i P_1 \otimes \sinh(A_1) - i \sinh(A_2) \otimes P_1 \]
\[ \Delta^F(P_3) = P_3 \otimes 1 + 1 \otimes P_3 \]
\[ \Delta^F(P_0) = P_0 \otimes 1 + 1 \otimes P_0 \]

where:
\[ A_1 = \alpha_+ M_3 + (\delta^r_+ + \delta^-) P_r \quad B_r = (\delta^r_+ - \delta^-) M_3 + \rho^r s P_s \]
\[ A_2 = \alpha + M_3 + (\delta r - \delta r^*) P_r \quad C_r = \rho^* P_a + (\delta r + \delta r^*) M_3 \]

c) \( \hat{c} = (P_1, P_2, N_3 = M_{30}) \)

Putting \( a, b = 1, 2 \)

\[ f_+ = \rho_{\mp}^{ab} P_a \otimes P_a + \xi_+^a (N_3 \otimes P_a + P_a \otimes N_3) + \gamma_+ N_3 \otimes N_3 \]

\[ f_- = \xi_-^a (N_3 \otimes P_a - P_a \otimes N_3) \]

one gets

\[ \Delta^F(M_\pm) = M_\pm \otimes \cos(A_1) + \cos(A_2) \otimes M_\pm \pm \{N_\pm \otimes \sin(A_1) + \sin(A_2) \otimes N_\pm \} \]

\[ \mp \{P_{3} \otimes (B_1 \pm i B_2) \cos(A_1) + (C_1 \pm i C_2) \cos(A_2) \otimes P_3 \} \]

\[ \mp i \{P_0 \otimes (B_1 \pm i B_2) \sin(A_1) + (C_1 \pm i C_2) \sin(A_2) \otimes P_0 \} \]

\[ \Delta^F(M_3) = M_3 \otimes 1 + 1 \otimes M_3 \quad (2.10) \]

\[ -i \{P_2 \otimes B_1 + C_1 \otimes P_2 \} + i \{P_1 \otimes B_2 + C_2 \otimes P_1 \} \]

\[ \Delta^F(N_\pm) = N_\pm \otimes \cos(A_1) + \cos(A_2) \otimes N_\pm \mp \{M_\pm \otimes \sin(A_1) + \sin(A_2) \otimes M_\pm \} \]

\[ -i \{P_0 \otimes (B_1 \pm i B_2) \cos(A_1) + (C_1 \pm i C_2) \cos(A_2) \otimes P_0 \} \]

\[ + P_3 \otimes (B_1 \pm i B_2) \sin(A_1) + (C_1 \pm i C_2) \sin(A_2) \otimes P_3 \]

\[ \Delta^F(N_3) = N_3 \otimes 1 + 1 \otimes N_3 \]

\[ \Delta^F(P_\pm) = P_\pm \otimes 1 + 1 \otimes P_\pm \]
\[ \Delta^F(P_3) = P_3 \otimes \cos(A_1) + \cos(A_2) \otimes P_3 + iP_0 \otimes \sin(A_1) + i\sin(A_2) \otimes P_0 \]

\[ \Delta^F(P_0) = P_0 \otimes \cos(A_1) + \cos(A_2) \otimes P_0 + iP_3 \otimes \sin(A_1) + i\sin(A_2) \otimes P_3 \]

where

\[ B_1 \pm iB_2 = (\rho_1^b \pm i\rho_2^b)P_b + (\xi_1^1 \pm i\xi_2^1 - (\xi_1^1 \pm i\xi_2^1))N_3 \]

\[ C_1 \pm iC_2 = (\rho_1^b \pm i\rho_2^b)P_b + (\xi_1^1 \pm i\xi_2^1 + (\xi_1^1 \pm i\xi_2^1))N_3 \]

d) \( \hat{c} = (P_1, P_2, P_3, P_0) \)

one can write \( (\hat{\rho}_\pm^{\mu\nu} = \pm \rho_\pm^{\mu\nu}) \)

\[ f_+ = \hat{\rho}_+^{\mu\nu}(P_\mu \otimes P_\nu + P_\nu \otimes P_\mu) \]

\[ f_- = \hat{\rho}_-^{\mu\nu}(P_\mu \otimes P_\nu - P_\nu \otimes P_\mu) \]

(2.11)

Because the split Casimir

\[ C_2^{\text{split}} \equiv \Delta(P_\mu P^\mu) - P_\mu P^\mu \otimes 1 - 1 \otimes P_\mu P^\mu = 2P_\mu \otimes P^\mu \]

(2.12)

commutes with \( \Delta(\hat{a}) \) for any \( \hat{a} \in U(P_4) \), one can assume further that \( \rho_+^{\mu\nu} \eta_{\mu\nu} = \rho_-^{\mu\nu} = 0 \) \( (\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)) \).

The formulae for the coproduct take the form

\[ \Delta^F(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} + (\alpha_{+\mu} P_\nu - \alpha_{+\nu} P_\mu) \otimes P_\rho \]

\[ + P_\rho \otimes (\alpha_{+\mu} P_\nu - \alpha_{+\nu} P_\mu) + (\alpha_{-\mu} P_\nu - \alpha_{-\nu} P_\mu) \otimes P_\rho - \]

\[ - P_\rho \otimes (\alpha_{-\mu} P_\nu - \alpha_{-\nu} P_\mu) \]

(2.13)

\[ \Delta^F(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu \]
In general we assume that the Poincaré algebra is the complex one, and the twist function parameters are also complex. The reality condition imposed on the Poincaré generators imply the reality conditions for the coefficient in the formulae (2.5), (2.7), (2.9) and (2.11). For simplicity we shall consider the last example of the twist function, given by (2.11). It is known that if the real structure is an antihomomorphism in the algebra sector, still one can impose on the generators of twisted Poincaré algebra two types of reality conditions [10,17]:

a) Standard one, denoted in [17] by +. In the case of the formulae (2.13) one obtains

\[(\Delta(M_{\mu\nu}))^+ = \Delta(M_{\mu\nu}) \implies \alpha^{\rho\tau} \text{ real} \quad (2.14a)\]

b) nonstandard one, used e.g. in [18], and denoted in [10] by ⊕. In such a case

\[(\Delta(M_{\mu\nu}))^{\oplus} = \Delta(M_{\mu\nu}) \implies \alpha^{\rho\tau} = (\alpha^{\tau\rho})^* \quad (2.14b)\]

i.e. the matrix \(\alpha \equiv (\alpha^{\rho\tau})\) is Hermitean.

Finally we consider the extension of \(\hat{g}\) by an Abelian algebra \(\hat{z}\) (\(\hat{g} \rightarrow \hat{g} \oplus \hat{z}\)), with \(z_A \quad (A = 1, \ldots, m)\) describing the central charges. The formulae (2.2) determining twist function can be extended as follows:

\[f_\pm \rightarrow f_\pm^{(z)} = f_\pm + \frac{1}{2} \beta_{\pm A}(c_i \otimes Z_A \pm Z \otimes c_i) \quad (2.15)\]

The candidates for \(Z_A\) are the central charges as well as the Casimir operators. As an example we shall consider the case d) with one central charge \(Z\), i.e. we assume that the formulae (2.11) is extended as follows:

\[f_\pm \rightarrow f_\pm^{(z)} = f_\pm + \rho_\pm^\mu(P_\mu \otimes Z \pm Z \otimes P_\mu) \quad (2.16)\]

The formulae (2.13) for twisted coproduct is modified as follows:

\[\Delta^F(M_{\mu\nu}) \rightarrow \Delta^F(M_{\mu\nu}) + \rho_+^\mu(P_\mu \otimes Z + Z \otimes P_\mu) + \rho_-^\mu(P_\mu \otimes Z - Z \otimes P_\mu) \quad (2.17)\]
With the choice (2.16) the explicit formulae for the universal $R$–matrix is the following:

$$ R = \exp(-2f^{(z)}) = \exp(-2\rho_{\mu}^\nu (P_\mu \otimes P_\nu - P_\nu \otimes P_\mu)) = $$

$$ \exp(-2\rho_{\mu}^\nu (P_\mu \otimes Z - Z \otimes P_\mu)) \quad (2.18) $$

The invariant tensor (2.3) takes the form

$$ U = \exp(-2\alpha_+^{\mu\nu} P_\mu \cdot P_\nu - 2\rho_{\mu}^\nu P_\mu \cdot Z) \quad (2.19) $$

and using the formulae $S^F = US_0U^{-1}$ one gets

$$ S^{F(z)}(P_\mu) = S_0(P_\mu) = -P_\mu $$

$$ S^{F(z)}(M_{\mu\nu}) = -M_{\mu\nu} - 2(\alpha_+^{\mu} P_\rho - \alpha_+^{\rho} P_\mu P_\rho) \quad (2.20) $$

$$ -(\rho_+^{\mu} P_\nu - \rho_+^{\nu} P_\mu) \cdot Z $$

The reality conditions for the parameters $\rho_{\pm}^{\mu}$ take the form:

a) $+$ – involution: $\rho_+^{\mu}$ real

b) $\oplus$ – involution: $$(\rho_+^{\mu})^* = \rho_-^{\mu} \quad (2.21)$$

In this Section we considered classical twisted Poincaré algebras, parametrized by multiparameter twist functions. These Hopf algebras by duality relations determine multiparameter deformations of the functions of the Poincaré group. Using the duality relation between multiplication and comultiplication

$$ <a \cdot b, c> = <a \otimes b, \Delta(c)> \quad (2.22)$$

one sees easily that all the antisymmetric contributions to the twisted coproducts (see e.q.(2.13)) lead to noncommutativity of the generators of the corresponding dual quantum Poincaré group.

It is an interesting exercise to classify the quantum Poincaré groups dual to the classical twisted Poincaré algebras.
3. An example: Chaichian-Demiczev quantum Poincaré algebra

We shall show that the example of $q$–Poincaré algebra given in [10] is isomorphic as a Hopf algebra to twisted classical Poincaré algebra. We shall describe firstly the complexified classical Lorentz algebra $SO(4; C) = SO(3; C) \oplus SO(3; C)$ as follows:

$$[e_i, e_{-j}] = \delta_{ij} h_i$$

$$[h_i, h_j] = 0$$

$$[h_i, e_{\pm j}] = \pm 2 \delta_{ij} e_{\pm j}$$

(3.1)

where $(e_1, e_{-1}, h_1)$ and $(e_2, e_{-2}, h_2)$ describe two $O(3; C)$ sectors.

Introducing

$$L_1 = e_{-1} \quad L_2 = e_{-2} \quad L_5 = \frac{1}{2}(h_1 + h_2)$$

$$L_3 = e_{+2} \quad L_4 = e_{+1} \quad L_6 = \frac{1}{2}(h_2 - h_1)$$

(3.2)

one obtains the relations

$$[L_1, L_5] = L_1 \quad [L_2, L_5] = L_2$$

$$[L_1, L_6] = -L_1 \quad [L_2, L_6] = L_2$$
\[ [L_1, L_4] = L_6 - L_5 \quad [L_2, L_3] = L_6 + L_5 \]

\[ [L_3, L_5] = -L_3 \quad [L_4, L_5] = -L_4 \]

\[ [L_3, L_6] = -L_3 \quad [L_4, L_6] = L_4 \]

where of course (\( a = 1 \ldots 6 \))

\[ \Delta(L_a) = L_a \otimes I + I \otimes L_a \]  \( (3.4) \)

Let us perform the twist of this coproduct

\[ F = q^{h_2 \otimes h_1} = q^{(L_5 + L_6) \otimes (L_5 - L_6)} \]

One gets \( (\Delta^F(L_a) = F \cdot \Delta(L_a) \cdot F^{-1}) \)

\[ \Delta^F(L_1) = L_1 \otimes I + q^{-2(L_5 + L_6)} \otimes L_1 \]

\[ \Delta^F(L_2) = I \otimes L_2 + L_2 \otimes q^{-2(L_5 - L_6)} \]

\[ \Delta^F(L_3) = I \otimes L_3 + L_3 \otimes q^{2(L_5 - L_6)} \]  \( (3.5) \)

\[ \Delta^F(L_4) = L_4 \otimes I + q^{2(L_5 + L_6)} \otimes L_4 \]

\[ \Delta^F(L_5) = \Delta(L_5) \quad \Delta^F(L_6) = \Delta(L_6) \]
Introducing

\[ \tilde{L}_1 = L_1 \quad \tilde{L}_2 = q^{-2} L_2 q^{-2(L_5 - L_6)} \quad \tilde{L}_3 = q^{-2} L_3 q^{2(L_5 - L_6)} \]

\[ \tilde{L}_4 = L_4 \quad \tilde{L}_5 = L_5 \quad \tilde{L}_6 = L_6 \]  \hspace{1cm} (3.6)

one can identify the transformed classical Lorentz algebra (3.3) with the \( q \)-deformed Lorentz algebra proposed in [10], with the coproduct

\[ \Delta^F(\tilde{L}_1) = \tilde{L}_1 \otimes I + q^{-2(\tilde{L}_5 + L_6)} \otimes \tilde{L}_1 \]

\[ \Delta^F(\tilde{L}_2) = I \otimes \tilde{L}_2 + \tilde{L}_2 \otimes q^{-2(\tilde{L}_5 - L_6)} \]

\[ \Delta^F(\tilde{L}_3) = I \otimes \tilde{L}_3 + \tilde{L}_3 \otimes q^{2(\tilde{L}_5 - L_6)} \]

\[ \Delta^F(\tilde{L}_4) = \tilde{L}_4 \otimes I + q^{2(\tilde{L}_5 + L_6)} \otimes \tilde{L}_4 \]

\[ \Delta^F(\tilde{L}_5) = \tilde{L}_5 \otimes I + I \otimes \tilde{L}_5 \]

\[ \Delta^F(\tilde{L}_6) = \tilde{L}_6 \otimes I + I \otimes \tilde{L}_6 \] \hspace{1cm} (3.7)

Introducing fourmomentum operators, which in the basis (3.2) will satisfy the following covariance relations with \( L_5, L_6 \)

\[ [P_1, L_5] = P_1 \quad [P_2, L_5] = 0 \quad [P_3, L_5] = -P_3 \quad [P_4, L_5] = 0 \]

\[ [P_1, L_6] = 0 \quad [P_2, L_6] = P_2 \quad [P_3, L_6] = 0 \quad [P_4, L_6] = -P_4 \] \hspace{1cm} (3.8)

one obtains after the nonlinear transformation
\[ \tilde{P}_1 = q^{L_5-L_6} P_1 \quad \tilde{P}_2 = q^{L_5-L_6} P_2 \]
\[ \tilde{P}_3 = q^{L_6-L_5} P_3 \quad \tilde{P}_4 = q^{L_6-L_5} P_4 \]

the relations
\[ [\tilde{P}_1, \tilde{P}_2]_{q^2} = [\tilde{P}_4, \tilde{P}_1]_{q^2} = [\tilde{P}_2, \tilde{P}_3]_{q^2} = [\tilde{P}_3, \tilde{P}_4]_{q^2} = 0 \]

\[ [\tilde{P}_1, \tilde{P}_3] = [\tilde{P}_2, \tilde{P}_4] = 0 \]

and the coproducts
\[ \Delta^F(\tilde{P}_1) = \tilde{P}_1 \otimes 1 + q^{-2L_6} \otimes \tilde{P}_1 \]
\[ \Delta^F(\tilde{P}_2) = \tilde{P}_2 \otimes 1 + q^{2L_5} \otimes \tilde{P}_2 \]
\[ \Delta^F(\tilde{P}_3) = \tilde{P}_3 \otimes 1 + q^{2L_6} \otimes \tilde{P}_3 \]
\[ \Delta^F(\tilde{P}_4) = \tilde{P}_4 \otimes 1 + q^{-2L_5} \otimes \tilde{P}_4 \]

The relations (3.10-11) describe the translation sector of Chaichian–Demiczev quantum algebra.

Let us recall that recently the quantum Lorentz groups have been classified by Worononowicz and Zakrzewski [11], where besides the Drinfeld–Jimbo parameter \( q \) a new parameter \( t \) has been introduced. It can be shown that the quantum deformation, proposed by Chaichian and Demiczev corresponds to \( q = 1 \). This condition as the necessary requirement for the existence of nontrivial quantum deformation of Poincaré algebra, with the Lorentz part as the Hopf subalgebra, has been obtained in [12] (see also [13]).

It should be stressed that in [11] there were given also other examples of the quantum deformations of the Lorentz group, which satisfy the condition \( q = 1 \) and can be extended to the quantum deformations of the Poincaré
algebra without supplementing an eleventh dilatation generator. It would be interesting to prove the conjecture that all quantum deformations of Poincaré algebra which do have the deformed Lorentz algebra as its Hopf subalgebra are classical twisted Poincaré algebras.

We would like finally to mention that it is possible to obtain the Poincaré quantum group as well as Poincaré quantum algebra with Drinfeld–Jimbo deformation parameter $q \neq 1$ if we assume braided structure of the tensor products, i.e. we consider the deformations in the framework of braided quantum groups and algebras (see e.g. [14]). In such a case the parameter $q$ enters into the definition of braided tensor product of the Lorentz generators and the translation generators [12] (see also ref. [15,16]). In this paper we assume however the standard “bosonic” relations for the tensor categories.

4. Discussion

i) Representation theory of twisted Poincaré algebra.

The theory of irreducible representations of twisted Poincaré algebras is described by the conventional Wigner representations for the Poincaré algebra [1,2]. The twisting can be interpreted as the modification of the tensor products for relativistic free particle states, in particular the 2-particle sectors in a relativistic Fock space. The tensor product $|1 \otimes 2>$ of two free one-particle states (i=1,2)

$$|i> = |m^{(i)}, s^{(i)}; p^{(i)}, s_3^{(i)}>$$

one modifies as follows

$$|1 \otimes F|2>= F(c^{(1)}, c^{(2)})|1 \otimes 2>$$

(4.2a)

where ($\alpha = \alpha_+ + \alpha_-$)

$$F(c^{(1)}, c^{(2)}) = \exp \alpha_{ij} c^{(1)}_i c^{(2)}_j$$

(4.2b)

Let us denote by $\hat{\alpha}$ the algebra describing the levels of the representation space (for (4.1) $\hat{\alpha} = (P_\mu, S_3)$, where $S_\mu = 1/2\epsilon_{\mu
u\rho\tau} M^{\nu\rho} P^\tau$), and by
\( \hat{O} \) the Casimirs parametrizing by its eigenvalues the representations \((\hat{O} = (P_{\mu}P^\mu, S_{\mu}S^\mu) \text{ for } \mathcal{P}_4)\). One can distinguish the following two cases:

i1) \([c_i, \hat{\alpha}] = 0\).

This corresponds to our choice d) (see (2.11), (2.13)). In such a case the twisted tensor product of two representations (4.1) describe the fixed four-momenta components of the wave packet

\[ |1, 2 \rangle_F = \exp(\alpha^{\mu\nu}(p_1^{(1)}p_2^{(2)})|1 \rangle \otimes |2 \rangle \]  

(4.3)

For dimensional reasons one should put \(\alpha^{\mu\nu} = \frac{1}{\kappa^2} a^{\mu\nu} \) (\(\kappa\)-masslike parameter). If we assume that \(a^{\mu\nu}\) has negative eigenvalues, one obtains from (4.3) the Gauss-like 2-particle wave function.

ii) \([c_i, \hat{\alpha}] \neq 0\)

Such a case is described by the choices a), b), c) of the twist function as well as the example described in Sect.3. In such a case twisted two-particle states described by (4.2) are not eigenvalues of the “two-particle observable” \(\Delta^F(\hat{\alpha})\), because

\[ \Delta^F(\hat{\alpha}) = F \cdot \Delta(\hat{\alpha}) \cdot F^{-1} \neq \Delta(\hat{\alpha}) \]  

(4.4)

For the fourmomentum operators the additivity of the fourmomenta eigenvalues is modified by the formula

\[ \Delta^F(P_{\mu}) = F \cdot (P_{\mu} \otimes 1 + 1 \otimes P_{\mu}) \cdot F^{-1} \]  

(4.5)

In our example in Sect.3 the formulae (4.5) take the form (3.11). The physical interpretation of generalized wave packets (4.2a) with modified addition for the fourmomenta is not clear.

ii) Twisted Poincaré algebra from the contraction of \(U_q(O(4, 2))\).

In recent paper [17] two of the present authors proposed the contraction of \(U_q(O(4, 2))\) to quantum Poincaré algebra. It can be shown that the result of the contraction is a twisted Poincaré algebra with the twist function depending on the fourmomenta and one central charge \(Z\) (see (2.16)), obtained
from the contraction of the dilatation generator in the conformal algebra.

iii) Nonabelian choice of twist functions.

It is interesting to consider more general classes of twisting functions, with $F$ spanned by nonabelian sectors of the algebra. In particular such a twisting function is provided by the universal $R$–matrix, which interchanges two noncocomutative coproducts $\Delta$ and $\Delta' = \sigma \cdot \Delta$ of a quantum algebra. It is known that for Drinfeld–Jimbo deformations $U_q(\hat{g})$ of simple Lie algebras the universal $R$–matrix can be decomposed into the product \cite{18,19}

$$R = \prod_{\alpha \in \Delta^{(+)} } R_\alpha \cdot K$$

where

$$R_\alpha = \exp q_\alpha (a_\alpha(q) e_\alpha \otimes e_{-\alpha})$$

and $K$ depends only on the Cartan generators. It appears that any component (4.7) of the product (4.6) can be used as a twist function $F$ \cite{20}. Because $a_\alpha(q)$ is proportional to $q - q^{-1}$, the twisting with $F = R_\alpha$ can be introduced only for genuine quantum algebras ($q \neq 1$). It is interesting to find nontrivial twist functions for quantum $\kappa$–Poincaré algebra, proposed in \cite{21,22}. Because the universal $\hat{R}$–matrix for $\kappa$–Poincaré algebra is not known, the type of twisting proposed in \cite{20} can not be applied.
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References

1. E.P. Wigner, Ann. Math. 40, 149 (1939)
2. A.S. Wightman, 1960 Les Houches Summer School, ”Relations de dispersion et particules elementaries”, p.161
3. V.G. Drinfeld, Quantum groups, Proc.Intern.Congress of Mathematics, (Berkeley,USA, 1986) p.798
4. L. Faddeev, N. Reshetikhin and L. Takhtajan, Alg.Anal. 1, 178 (1989)
5. V.G. Drinfeld, Leningrad Math. Journ. 1, 1419 (1990)
6. N. Reshetkhin, Lett. Math. Phys. 20, 331 (1990)
7. G. Gurevich and S. Majid, “Branded Groups of Hopf Algebras obtained by Twistig”, Cambridge Univ. preprint DAMTP 91-49
8. B. Enriquez, Lett. Math. Phys. 25, 111 (1992)
9. A. Kempf, “Multiparameter R-matrices, Subquantum Groups and Generalized Twisting Method”, München Univ. prep. LMU-TPW 91-4
10. M. Chaichian and A.P. Demiczev, “Quantum Poincaré Group, Algebra and Quantum Geometry of Minkowski Space”, Helsinki Univ. preprint HU-TFT-93-24, March 1993 and Phys.Lett., B304, 220 (1993)
11. S.L. Woronowicz and S. Zakrzewski, “Quantum Deformations of Lorentz group: Hopf ∗-algebra level”, Warsaw Univ. preprint 1992, Compositio Mathematica, in press
12. S. Majid, Journ. Math. Phys. 34, 2045 (1993)

13. P. Podleś, and S.L. Woronowicz, private communication

14. S. Majid, J. Math. Phys. 34, 1176 (1993)

15. J. Rembieliński,”Quantum braided Poincaré group”, Lodz Univ. preprint KFT UL 7/93

16. J.A. de Azcaraga, P. Kulish and F. Rodenas,”Reflection equations and q-Minkowski space algebras”, hep-th 9309036

17. J. Lukierski, A. Nowicki, Phys. Lett. B279, 299 (1992)

18. S.M. Khoroshkin and V.N. Tolstoy, Comm. Math. Phys. 141, 559 (1991)

19. S. Levendorskii, Y.Soibelman, Comm.Math.Phys., 139, 141 (1991)

20. S.M. Khoroshkin and V.N. Tolstoy,”Twisting of quantum (super)algebras. Connection of Drinfeld’s and Cartan-Weyl realizations for quantum affine algebras”, preprint MPI, Bonn 1993

21. J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. B264, 331 (1991)

22. J. Lukierski, A. Nowicki and H. Ruegg, Phys.Lett. B293, 344 (1992)