ON THE INTEGRAL OF HARDY’S FUNCTION

ALEKSANDAR IVIĆ

To the memory of A.A. Lavrik (1964-2003)

Abstract. If \( Z(t) = \chi^{-1/2}(\frac{1}{2} + it)\zeta(\frac{1}{2} + it) \) denotes Hardy's function, where \( \zeta(s) = \chi(s)\zeta(1 - s) \), then it is proved that

\[
\int_0^T Z(t) \, dt = O_\varepsilon(T^{1/4 + \varepsilon}).
\]

Let as usual \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \) \((\sigma > 1)\) denote the Riemann zeta-function, where \( s = \sigma + it \) is a complex variable. The aim of this note is to provide a bound for the integral of Hardy’s function

\[
(1) \quad Z(t) = \chi^{-1/2}(\frac{1}{2} + it)\zeta(\frac{1}{2} + it), \quad \chi(s) = 2^s\pi^{s-1}\sin(\frac{1}{2}\pi s)\Gamma(1 - s),
\]

so that the functional equation for \( \zeta(s) \) has the form \( \zeta(s) = \chi(s)\zeta(1 - s) \). Since \( \chi(s)\chi(1 - s) = 1 \), it follows that \( |Z(t)| = |\zeta(\frac{1}{2} + it)| \), and that \( Z(t) \) is a real-valued function of \( t \). The function \( Z(t) \) plays an important role in the theory of the distribution of zeros of \( \zeta(s) \) on the “critical line” \( \Re s = \frac{1}{2} \) (see e.g., [1]–[3] and [5]–[6]).

The result on the integral of \( Z(t) \) is contained in the following

THEOREM. We have

\[
(2) \quad \int_0^T Z(t) \, dt = O_\varepsilon(T^{\frac{1}{4} + \varepsilon}).
\]

Proof. Here and later \( \varepsilon \) will denote arbitrarily small, positive constants, not necessarily the same ones at each occurrence. To prove (2) we shall make use of

1991 Mathematics Subject Classification. 11 M 06.
Key words and phrases. The Riemann zeta-function, Hardy’s function, saddle point.

Typeset by \textsc{AMS-\TeX}
the approximate functional equation

\begin{equation}
Z^k(t) = 2 \sum_{n \leq 2\tau} \rho \left( \frac{n}{\tau} \right) d_k(n)n^{-1/2} \cos \left( t \log \frac{\tau}{n} - \frac{k}{2} t - \frac{\pi k}{8} \right) + O(t^{\frac{k}{2} - 1} \log^{k-1} t),
\end{equation}

which is valid for any fixed integer \( k \geq 1 \) and \( t \geq 2 \). In (3) we have set for brevity \( \tau = \left( \frac{t}{2\pi} \right)^{k/2} \),

and further notation is as follows. The function \( d_k(n) \) represents the number of ways \( n \) may be represented as the product of \( k \) factors (\( d_1(n) \equiv 1, d_2(n) \equiv d(n) \)), the number of divisors of \( n \), while \( \rho(x) \) is a non-negative, smooth function supported in \( [0, 2] \), such that \( \rho(x) = 1 \) for \( 0 \leq x \leq 1/b \) for a fixed constant \( b > 1 \), and \( \rho(x) + \rho(1/x) = 1 \) for all \( x \). The author proved [4, Theorem 4.2] the approximate functional equation for \( \zeta^k(s) \), which gives (3) with \( x = y = \tau \), on using (1) and the asymptotic formula

\[
\chi(s) = \left( \frac{2\pi}{t} \right)^{\sigma+i\tau-1/2} e^{i(t+\pi/4)} \cdot \left( 1 + O \left( \frac{1}{t} \right) \right) \quad (0 \leq \sigma \leq 1, \ t \geq t_0 > 0).
\]

Taking \( k = 1 \) in (3) it follows that

\begin{equation}
\int_T^{2T} Z(t) \, dt = 2 \int_T^{2T} \sum_{n \leq 2\tau} \rho \left( \frac{n}{\tau} \right) n^{-1/2} \Re \left\{ e^{iF(t)} \right\} \, dt + O(T^{1/4}),
\end{equation}

where

\begin{equation}
\tau = \sqrt{\frac{t}{2\pi}}, \quad F(t) = t \log \frac{\tau}{n} - \frac{t}{2} - \frac{\pi}{8}.
\end{equation}

The reason that (3) was used is that the standard approximate functional equation for \( \zeta(s) \) (this is the Riemann-Siegel formula, see e.g., [1, Chapter 4]) has the error term \( O(t^{-1/4}) \), which is not sufficiently good to produce the bound in (2). For this reason we resorted to (3), which is a smoothed variant of the approximate functional equation with a sharp error term.

In view of (4), to prove (2) it clearly suffices to prove that

\begin{equation}
I(T) := \int_T^{2T} \sum_{n \leq 2\tau} \rho \left( \frac{n}{\tau} \right) n^{-1/2} \Re \left\{ e^{iF(t)} \right\} \, dt \ll \varepsilon T^{\frac{1}{2} + \varepsilon}.
\end{equation}
We have, in view of (5),

\[ I(T) = \sum_{n \leq 2\sqrt{T/\pi}} n^{-1/2} \Re \left\{ \int_{T_1}^{2T} \rho \left( \frac{n}{T} \right) e^{iF(t)} \, dt \right\} \]

(7)

\[ = \sum_1(T) + \sum_2(T) + \sum_3(T) + \sum_4(T) + \sum_5(T), \]

say, where

\[ T_1 = \max \left( T, 2\pi \left( \frac{n}{2} \right)^2 \right), \]

and the ranges of summation in \( \sum_j(T) \) \((j = 1, \ldots, 5)\) are respectively as follows:

- \( n \leq \sqrt{T/(2\pi)} - T^\varepsilon, \sqrt{T/(2\pi)} + T^\varepsilon < n \leq \sqrt{T/(2\pi)} + T^\varepsilon \)
- \( \sqrt{T/\pi} - T^\varepsilon < n \leq \sqrt{T/\pi} + T^\varepsilon \)
- \( \sqrt{T/\pi} - T^\varepsilon < n \leq \sqrt{T/\pi} + T^\varepsilon \)
- \( \sqrt{T/\pi} - T^\varepsilon < n \leq 2\sqrt{T/\pi} \).

We have

\[ F'(t) = \log \frac{\sqrt{t/(2\pi)}}{n}, \quad F''(t) = \frac{1}{2t}. \]

This means that, in \( \sum_1(T) \), we have

\[ F'(t) \geq \log \left( \frac{\sqrt{T/(2\pi)}}{n} \right), \]

hence \((m = \left\lfloor \sqrt{T/(2\pi)} \right\rfloor - n)\) by the first derivative test (see e.g., [1, Lemma 2.1])

\[ \sum_1(T) \ll T^{1/4} + \sum_{\frac{1}{2}\sqrt{T/(2\pi)} < n \leq \sqrt{T/(2\pi)} - T^\varepsilon} \frac{1}{\sqrt{n} \log \left( \left\lfloor \sqrt{T/(2\pi)} \right\rfloor / n \right)} \]

\[ \ll T^{1/4} + T^{-1/4} \sum_{\frac{1}{2}\sqrt{T/(2\pi)} < n \leq \sqrt{T/(2\pi)} - T^\varepsilon} \frac{n}{\left\lfloor \sqrt{T/(2\pi)} \right\rfloor - n} \]

\[ \ll T^{1/4} + T^{1/4} \sum_{m \leq \frac{1}{2}\sqrt{T/(2\pi)}} \frac{1}{m} \]

\[ \ll T^{1/4} \log T. \]

An analogous bound holds also for \( \sum_5(T) \).

To evaluate the sum \( \sum_3(T) \) in (7), which contains (for every \( n \) in the range of summation) a saddle point \( c \), namely the root of \( F'(c) = 0 \), so that \( c = c_n = 2\pi n^2 \),
one may use general results in the literature which for this purpose (see [1], [4] and [5]). A convenient one is [5, Lemma III.2], which says that

\begin{equation}
\int_a^b \varphi(x) \exp(2\pi i f(x)) \, dx = \frac{\varphi(c)}{\sqrt{f''(c)}} e^{2\pi i f(c)/4} + O(HAU^{-1})
+ O(H \min(|f'(a)|^{-1}, \sqrt{A})) + O(H \min(|f'(b)|^{-1}, \sqrt{A}),
\end{equation}

if \( f'(c) = 0, a \leq c \leq b \), and the following conditions hold: \( f(x) \in C^4[a, b], \varphi(x) \in C^2[a, b], f''(x) > 0 \) in \([a, b], f''(x) \asymp A^{-1}, f^{(3)}(x) \ll A^{-1}U^{-1}, f^{(4)}(x) \ll A^{-1}U^{-2}, \varphi^{(r)}(x) \ll HU^{-r} (r = 0, 1, 2) \) in \([a, b], 0 < H, A < U, 0 < b - a \leq U \).

We shall apply (8) with \( \varphi(t) = \rho(n/\tau), f(t) = (2\pi)^{-1} T/\tau, a = T_1, b = 2T, H = 1, U = T, A = T \). With \( c = c_n = 2\pi n^2 \) we have that \( c \in [a, b] \) for our range of \( n \), and furthermore \( \varphi(c_n) = \rho(1) = 1 \). Therefore the contribution of the first term on the right-hand side of (8) will be

\begin{equation}
\sum_{\sqrt{T/(2\pi)+T^\varepsilon < n < \sqrt{T/\pi-T^\varepsilon}}} n^{-1/2} \varphi(c_n)(f''(c_n))^{-1/2} \Re \{ \exp(2\pi i f(c_n) + \frac{1}{4} \pi i) \}
= \sqrt{8\pi} \sum_{\sqrt{T/(2\pi)+T^\varepsilon < n < \sqrt{T/\pi-T^\varepsilon}}} n^{-1/2} \Re \{ \frac{1}{8} \pi i - \pi in^2 \}
= \sqrt{8\pi} \cos \left( \frac{\pi}{8} \right) \sum_{\sqrt{T/(2\pi)+T^\varepsilon < n < \sqrt{T/\pi-T^\varepsilon}}} (-1)^n n^{1/2} = O(T^{1/4}),
\end{equation}

since the last sum is, in absolute value,

\begin{equation}
\leq \left| \sum_{\ell \ll K} (\sqrt{K + 2\ell} - \sqrt{K + 2\ell - 1}) \right| \ll \sum_{\ell \ll K} 1/\sqrt{K} \ll \sqrt{K} \quad (K \asymp \sqrt{T}).
\end{equation}

In the \( \sum_3(T) \) we have \( \sqrt{T/(2\pi)+T^\varepsilon < n < \sqrt{T/\pi-T^\varepsilon} \), hence similarly to the estimation of \( \sum_1(T) \), the total contribution of the error terms in (8) will be \( \ll \varepsilon T^{1/4+\varepsilon} \).

Finally, by using the second derivative test ([1, Lemma 2.2]), it follows that

\begin{equation}
\sum_2(T) + \sum_4(T) \ll \varepsilon T^{5/4} T^{-1/4} T^{1/2} = T^{1/2+\varepsilon}.
\end{equation}

Therefore, except for the bound in (10), we get the upper bound \( O_\varepsilon(T^{1/4+\varepsilon}) \) for our integral \( I(T) \) (see (6)). The reason for the range of summation over \( n \) in \( \sum_3(T) \) was the structure of the error terms in (8), namely if \( a \) or \( b \) is too near a
saddle point, then $\sqrt{A}$ is to be taken, which in our case is too large to produce (2).

To get around this obstacle, we shall employ the saddle point method directly, taking advantage of the particular structure of the exponential integrals in question, coupled with the summation over $n$ in (7). The main terms will be, of course, the same ones as those which appeared in (8), and the essential fact is the presence of $(-1)^n$ in the summation over $n$, which accounts for massive cancellation and leads to (2).

Henceforth we suppose that $n$ lies in the range covered by $\sum_j(T)$ ($j = 2, 3, 4$) in (7), namely

$$\sqrt{\frac{T}{2\pi}} - T^\varepsilon \leq n \leq \sqrt{\frac{T}{\pi}} + T^\varepsilon.$$  

For such $n$ let

$$J(T, n) = [2\pi n^2 - T^\varepsilon, 2\pi n^2 + T^\varepsilon], \quad K(T, n) = [T_1, 2T] \setminus J(T, n).$$

In dealing with

$$\int_{K(T, n)} \rho \left( \frac{n}{T} \right) e^{iF(t)} \, dt$$

we apply the first derivative test as before, obtaining after summation over $n$ a contribution which is $\ll T^{1/4+\varepsilon}$. In case $J(T, n)$ does not entirely lie in $[T, 2T]$, obvious modifications in the argument are to be made. To evaluate

$$\int_{J(T, n)} \rho \left( \frac{n}{T} \right) e^{iF(t)} \, dt,$$

we develop first $\rho(\frac{n}{T})$ by Taylor’s formula at the point $2\pi n^2$. Since each derivative of

$$\rho \left( \frac{n}{T} \right) = \rho \left( \frac{n}{\sqrt{T/2\pi}} \right),$$

as a function of $t$, decreases by a factor of $T$, and the measure of $J(T, n)$ is $\ll T^\varepsilon$, we first take so many terms in Taylor’s formula so that the contribution of the error term is negligible, namely $\ll T^{1/4+\varepsilon}$. The remaining integrals will be all of the same type, with the same exponential factor, and the largest one will be the first one, namely the one with ($c_n = 2\pi n^2$)

$$\rho \left( \frac{n}{\sqrt{2\pi}} \right) = \rho(1) = \frac{1}{2}.$$
since \( \rho(x) + \rho(1/x) = 1 \). Then we write, by Cauchy’s theorem,

\[
(13) \quad \int_{J(T,n)} e^{iF(z)} \, dz = \int_{L_1} e^{iF(z)} \, dz + \int_{L_2} e^{iF(z)} \, dz + \int_{L_3} e^{iF(z)} \, dz,
\]
say, where \( L_1 \) is the segment \( c_n - T^\varepsilon + ve^{-\frac{1}{4}\pi i}, 0 \leq v \leq \frac{1}{\sqrt{2}} T^\varepsilon \), \( L_2 \) is the segment \( c_n + ve^{\frac{1}{4}\pi i}, |v| \leq \frac{1}{\sqrt{2}} T^\varepsilon \), and \( L_3 \) is the segment \( c_n + T^\varepsilon - ve^{-\frac{1}{4}\pi i}, 0 \leq v \leq \frac{1}{\sqrt{2}} T^\varepsilon \). On \( L_2 \) we have

\[
(14) \quad iF(z) = iF(c_n) + i\frac{v^2}{2!} e^{\frac{3}{4}\pi i} F''(c_n) + i\frac{v^3}{3!} e^{\frac{3}{4}\pi i} F'''(c_n) + i\frac{v^4}{4!} F^{(4)}(c_n) + \cdots.
\]

Note that

\[
v^k F^{(k)}(c_n) \ll_{k,\varepsilon} T^{k\varepsilon} T^{1-k} = T^{1-k+k\varepsilon} \quad (k = 2, 3, \ldots).
\]

Hence if we choose \( K = K(\varepsilon) \) sufficiently large, then the terms of the series in (14) for \( k > K \), on using \( \exp z = 1 + O(|z|) \) for \( |z| \leq 1 \), will make a negligible contribution. Then we have

\[
\exp(iF(z)) = \exp(iF(c_n)) \exp\left(-\frac{1}{2} v^2 F''(c_n)\right) \exp\left(\sum_{k=3}^{K} d_k v^k F^{(k)}(c_n)\right)
\]

with \( d_k = \exp((k+2)\pi i)/k! \). The last exponential factor is expanded by Taylor’s series, and again the terms of the series (with \( v^k \)) for \( k > K \) will make a negligible contribution. In the remaining terms we restore integration over \( v \) to the whole real line, making a very small error. Then we use the classical integral (see e.g., the Appendix of [1])

\[
(15) \quad \int_{-\infty}^{\infty} \exp(Ax - Bx^2) \, dx = \sqrt{\pi B} \exp\left(\frac{A^2}{4B}\right) \quad (\Re B > 0).
\]

By differentiating (15) as a function of \( A \) we may explicitly evaluate integrals of the type

\[
\int_{-\infty}^{\infty} x^{2k} \exp(-Bx^2) \, dx \quad (\Re B > 0, \ k = 0, 1, 2, \ldots).
\]

It transpires that the largest contribution (\( = \sqrt{\pi} \)) will come from the integral with \( k = 0 \), which will coincide with the contribution of the main term in (8).
It remains to deal with the integrals over $L_1$ and $L_3$ in (13), which are estimated analogously, so only the former is considered. On $L_1$ we have

\[ iF(c_n - T^{1-\varepsilon} + ve^{-i\pi/4}) \]
\[ = i\left\{ F(c_n - T^{1-\varepsilon}) + F'(c_n - T^{1-\varepsilon})ve^{-i\pi/4} \right. \]
\[ + F''(c_n - T^{1-\varepsilon})\frac{v^2}{2!}e^{-2i\pi/4} + F'''(c_n - T^{1-\varepsilon})\frac{v^3}{3!}e^{-3i\pi/4} + \ldots \} \]
\[ = F_1(v; n, T) + iF_2(v; n, T), \]
say, with $F_1, F_2$ real. Then

\[ \frac{\partial F_1(v; n, T)}{\partial v} + i\frac{\partial F_2(v; n, T)}{\partial v} \]
\[ = \frac{1+i}{\sqrt{2}} F'(c_n - T^{1-\varepsilon}) + F''(c_n - T^{1-\varepsilon})v \]
\[ + \frac{1-i}{\sqrt{2}} F'''(c_n - T^{1-\varepsilon})\frac{v^2}{2!} + \ldots. \]

Therefore we find that

\[ \frac{\partial F_2(v; n, T)}{\partial v} \gg_{\varepsilon} T^{-\varepsilon}, \]

hence by the first derivative test the total contribution of the integral over $L_1$ is seen to be $\ll T^{1/4+\varepsilon}$. This finishes the proof of (2). However, the true order of the integral of $Z(t)$ remains elusive. In particular, it would be of interest to find an omega result for this quantity. Is it true that perhaps

\[ (16) \int_0^T Z(t) \, dt = \Omega(T^{1/4}) \quad (= \Omega_{\pm}(T^{1/4}))? \]

If yes, then the result of the Theorem would be (up to the factor "$\varepsilon$") best possible. The reason that (16) seems plausible is that $T^{1/4}$ is the order of the terms coming from the saddle points (see (8)), and in the evaluation of exponential integrals one usually expects the saddle points to produce the largest contribution.

**Note.** The revisions to the published version, made in July 2009, involve the corrections of some misprints. In the meantime M.A. Korolev, "On the primitive of the Hardy function $Z(t)$", Dokl. Math. 75, No. 2, 295-298 (2007); translation from Dokl. Akad. Nauk, Ross. Akad. Nauk 413, No. 5, 599–602 (2007), proved (16). Another proof is to be found in a forthcoming work of M. Jutila.
References

[1] A. Ivić, The Riemann zeta-function, John Wiley & Sons, New York, 1985.
[2] A. Ivić, On a problem connected with zeros of \( \zeta(s) \) on the critical line, Monatshefte Math. 104 (1987), 17-27.
[3] A. Ivić and M. Jutila, Gaps between consecutive zeros of the Riemann zeta-function, Monatshefte Math. 105 (1988), 59-73.
[4] A. Ivić, The mean values of the Riemann zeta-function, Tata Institute of Fundamental Research, Lecture Notes 82, Bombay 1991 (distr. Springer Verlag, Berlin etc.).
[5] A.A. Karatsuba and S.M. Voronin, The Riemann zeta-function, Walter de Gruyter, Berlin etc., 1992.
[6] A.A. Lavrik, Uniform approximations and zeros in short intervals of the derivatives of the Hardy function, Soviet Math. Dokl. 40 (1990), 20-22.

Katedra Matematike RGF-a Universiteta u Beogradu, Dušina 7, 11000 Beograd, Serbia (Yugoslavia)
E-mail address: aivic@matf.bg.ac.rs, ivic@rgf.bg.ac.rs