Higher spin holography with Galilean symmetry in general dimensions

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Abstract

We construct Schrödinger-like solutions of the Vasiliev higher spin theory in $D > 3$ dimension. Symmetries of such solutions and the linearized equation of motion for the scalar on such backgrounds are analyzed. We further propose Galilean invariant bosonic and fermionic field theories that could be dual to the two parity invariant higher spin theories on the Schrödinger-like background respectively. The discussion is phrased mainly in $D = 4$ dimension, while similar constructions follow straightforwardly in higher dimensions.

Keywords: non-relativistic holography, Schrödinger, higher-spin

1. Introduction

Theories with higher spin symmetry have proved to be ideal playgrounds that help broaden and deepen our understandings of the holographic principle and String theory. The gauge/gravity duality with higher spin symmetry is a weak–weak duality; therefore, it is possible to compare results from perturbative computations on the both sides, which helps understand the duality better [1–23]. In addition, remarkable progress has been made to discover the connection between higher spin theory and String theory [24–29], which supports the general belief that the higher spin theory should be identified as a subsector of String theory in the tensionless limit [30–36].

Previous examples of higher spin (super)gravity in $D \geq 3$ dimension are mostly on the background of maximally symmetric spacetime with non-zero cosmological constant, especially the Anti de-Sitter spacetime [37–39]. In $D > 3$ dimension, there are other classical solutions [32, 40–44] of the Vasiliev higher spin theory, but the geometric meanings of most of these solutions are less clear. However, in $D = 3$ dimension, solutions with different

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geometries [45, 46] are found thanks to the Chern–Simons formulation of the higher spin theory [47–50]. In particular, such solutions include spacetime with the Galilean isometry [51–54], whose metrics read

$$ds^2 = -\frac{dr^2}{r^{2z}} + \frac{dr^2 + 2d\xi d\zeta + \sum_{i=1}^{D-3} dx_i^2}{r^2},$$  \hspace{1cm} (1.1)

where \( D = 3 \) and \( z > 1 \) is the dynamical scaling. Translation between the Chern–Simons formalism and the metric like formalism is discussed in [49, 55–59]. Further discussion of this Schrödinger geometry in holographic theories can be found in [60].

In general dimensions, the Schrödinger spacetime (1.1) with \( z = 2 \) has the Galilean group in \( D - 3 \) spatial dimensions as isometry group. For \( z = 2 \), the isometry group is enhanced to the Schrödinger group, which is the non-relativistic version of the conformal group [61–65]. This symmetry governs many non-relativistic systems, such as unitary fermions, via the gauge/gravity duality [51]. Properties of the Schrödinger geometries have been investigated in different contexts [60, 66, 67], and several ways of embedding the Schrödinger geometry into String theory have been considered [68–70].

At zero temperature, Schrödinger spacetime solutions in \( D = 3 \) can be supported by massive vector fields, massive graviton [60] and higher spin fields [59], while in \( D > 3 \), only massive vector fields are known to support Schrödinger spacetime. At finite temperature, it can also be supported by heavy objects [68–70]. In this paper, we construct Schrödinger-like solutions, whose metric has a form of Schrödinger geometry (1.1) with integer dynamical exponent \( z \), to the Vasiliev equations in general \( D > 3 \) dimension. This is an exact solution of Vasiliev equation. This provides another interesting example of the higher spin holography in addition to the relatively well understood higher spin systems on the AdS background. The idea of our construction is, in the language of general relativity, to turn on higher spin fields which back-react on the geometry and support the non-maximally symmetric Schrödinger geometry. In practice, this amounts to solving the Vasiliev system in the ground state with non-vanishing (finite) higher spin fields. This type of solution provides us a perfect example to examine properties of the original Vasiliev system. For example, it demonstrates what is the precise effect of the higher spin fields on the geometry. In addition, it could make possible to extract more information of higher spin interactions by isolating some spin-\( s \) fields in the Vasiliev equation. Moreover, since most of the interesting condensed matter systems with Schrödinger symmetry live in \( d_s \geq 1 \) spatial dimension, the corresponding bulk geometry should live in \( D = d_s + 3 \geq 4 \) dimension; this also motivates our construction.

Our paper is organized as follows. In section 2, we review the Schrödinger spacetime as a solution of \( D = 3 \) dimensional \( hs(\lambda) \) higher spin theory in the Chern–Simons formulation. In section 3, we discuss how to find \( D = 4 \) dimensional Schrödinger-like solution in the spinorial [37, 71] language. We show that the spacetime symmetry of the solution does not possess the whole Schrödinger symmetry but only a subgroup of it. Therefore, the terminology ‘Schrödinger solution’ simply refers to solutions whose corresponding metrics are of the form of (1.1). In addition, linearized scalar equation of motion is analysed. In section 4, we briefly discuss the Schrödinger geometry in general dimension in the vectorial language [38, 39]. Field theories dual to these Schrödinger solutions are proposed in section 5. We conclude our paper in section 6 and in particular we comment on one realization of \( D = 3 \) dimensional Lifshitz higher spin theory with \( z = 2 \) from dimensional reduction of Schrödinger spacetime with \( z = 0 \) in \( D = 4 \) dimension [72, 73].
2. Review of 3D Schrödinger solution

2.1. Chern–Simons formulation

We start with higher spin theory in the Chern–Simons formulation, which is defined as the difference of two Chern–Simons actions:

\[ S_{EH} = S_{CS}[A] - S_{CS}[\tilde{A}], \quad S_{CS} = \frac{k}{4\pi} \int_M \text{Tr}\left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \]

where \( A^a \) and \( \tilde{A}^a \) take values in some Lie algebra.

The equations of motion are the flatness conditions

\[ F = dA + A \wedge A = 0, \quad \tilde{F} = d\tilde{A} + \tilde{A} \wedge \tilde{A} = 0. \]

Metric-like fields in 3D (1.1) are obtained by

\[ g_{\mu\nu} = \frac{1}{2} \text{Tr}(E_{\mu}E_{\nu}), \quad \phi_{\mu\nu\rho} = \frac{1}{6} \text{Tr}(E_{\mu}E_{\nu}E_{\rho}), \] with \( E_{\mu} = \frac{1}{2} (A_{\mu} - \tilde{A}_{\mu}) \).

A gauge transformation takes the connection to the form

\[ A = e^{-i\rho a}(a(z) + \partial_z)e^{i\rho a}, \quad \tilde{A} = e^{i\rho\alpha}(\tilde{a}(\zeta) - \partial_\zeta)e^{-i\rho\alpha}. \]

Schrödinger solutions with integer dynamical exponent \( z \) [45, 59] can be constructed as

\[ a = (L_1 + \sigma W_\mu)dr, \quad \tilde{a} = \sigma W_\mu dr + 2L_{-1}d\xi, \]

where \( L_{0, \pm 1} \) are the modes of the Virasoro generator and the \( W_\mu \) satisfy

\[ [W_\pm, L_0] = \pm z W_\pm, \quad [W_\pm, L_{\pm 1}] = 0, \quad \text{tr}(W_\mu W_\nu) \neq 0. \]

We can simply take \( W_\mu \) to be the \( V_0^{(z+1)} \) modes in the higher spin algebra. In \( z = 2 \) case, the spacetime metric and the spin-3 fields are

\[ ds^2 = -\sigma^2 e^{i\rho} dr^2 + 2e^{i\rho} d\rho d\xi + d\rho^2, \]

\[ \phi_{\xi\xi} = \frac{\sigma}{3} e^{i\rho}, \quad \phi_{\mu\mu} = -\frac{\sigma}{4} e^{i\rho}. \]

These metric like fields solve the Einstein equation perturbatively [59].

2.2. Vasiliev formulation

Our normalization in this section is slightly different from [76] but is self-consistent. Let us introduce oscillators \( \hat{\xi}_\alpha \) (\( \alpha = 1, 2 \)) fulfilling

\[ [\hat{\xi}_\alpha, \hat{\xi}_\beta] = \frac{1}{2} \epsilon_{\alpha\beta}(1 + \nu(k), \quad k\hat{\xi}_\alpha = -\hat{\xi}_\alpha k, \quad k^2 = 1, \]

We only focus on integer \( z \) since it is recently argued [59] that Schrödinger solution with fractional dynamical exponent \( z \) in 3D higher spin theory may not have well-defined metric like description. The reason is that given the frame field \( E_\mu \) defined above, one cannot solve spin-connection \( \omega \) uniquely from torsion free equation [56]

\[ dr + \sigma \wedge \omega - \omega \wedge e = 0. \] This implies that there is some information in the metric like fields that cannot be retrieved from Chern–Simons gauge fields. In addition, we do not consider the Lifshitz spacetime [74]

\[ ds^2 = \frac{dr^2}{\sigma^2} + dr^2 + d\xi^2, \]

in higher spin context [75] with any \( z \) for the same reason.
where $\nu$ is a free parameter and $k$ is the Klein operator. Define bilinear oscillators $T_{\alpha\beta}$

$$T_{\alpha\beta} = \{\hat{\psi}_{\alpha}, \hat{\psi}_{\beta}\},$$

that generate a $s(2)$ algebra

$$[T_{\alpha\beta}, T_{\gamma\sigma}] = \epsilon_{\beta\gamma} T_{\alpha\sigma} + T_{\beta\sigma} \epsilon_{\alpha\gamma} + T_{\alpha\gamma} \epsilon_{\beta\sigma} + \epsilon_{\alpha\sigma} T_{\beta\gamma}.$$  \hfill (2.13)

Higher (symmetric) powers of these oscillators give the higher spin generators. The connection with the Chern–Simons formulation is explained in appendix A.

In the current case, the gravitational connection

$$W = \omega + \frac{1}{l} \psi e, \quad \psi^2 = 1, \quad [\psi, \hat{\psi}] = 0,$$

where $\psi$ is the central involutive element and $l$ is the AdS radius, satisfies the equation of motion [76]

$$dW + W \wedge W = 0.$$  \hfill (2.14)

The $z = 2$ Schrödinger gauge fields (2.7) translate to the oscillator form

$$e = l \left( \frac{1}{4} r T_{11} + \frac{\sigma}{8} r^2 T_{11} T_{22} - \frac{\sigma}{8} r^2 T_{22} T_{11} \right) dt + \frac{l}{2} r T_{22} d\xi + \frac{l}{2r} T_{12} dr,$$

$$\omega = l \left( \frac{1}{4} r T_{11} + \frac{\sigma}{8} r^2 T_{11} T_{22} + \frac{\sigma}{8} r^2 T_{22} T_{11} \right) dt + \frac{l}{2} r T_{22} d\xi.$$  \hfill (2.15)

via (A.6). It is then trivial to check that they solve the above equation of motion (setting $l = 1$), which in component form reads

Torsion free equations

$$\psi T_{\alpha\beta} : de^{\alpha\beta} + e^{\alpha\kappa} \wedge \omega^{\beta\gamma} e_{\kappa\gamma} + e^{\kappa\beta} \wedge \omega^{\gamma\alpha} e_{\kappa\gamma} = 0,$$  \hfill (2.17)

$$\psi \sigma T_{\alpha\beta} T_{\gamma\kappa} : de^{\alpha\beta}\gamma\kappa} + 2 e^{\alpha\kappa\gamma\kappa} \wedge \epsilon_{\alpha\beta} e_{\kappa\gamma} + 2 e^{\alpha\beta\kappa\kappa} \wedge \epsilon_{\alpha\beta} e_{\kappa\gamma} + 2 e^{\alpha\beta\kappa\kappa} \wedge \epsilon_{\alpha\beta} e_{\kappa\gamma} = 0,$$  \hfill (2.18)

$$\psi T_{\alpha\beta} T_{\gamma\kappa} T_{\mu\nu} : \omega^{\alpha\beta\gamma\kappa} \wedge \epsilon^{\mu\nu cd} = 0,$$  \hfill (2.19)

and Curvature equations

$$T_{\alpha\beta} : d\omega^{\alpha\beta} + \omega^{\alpha\kappa} \wedge \omega^{\beta\gamma} e_{\kappa\gamma} + \frac{1}{l^2} e^{\alpha\kappa} \wedge \omega^{\beta\gamma} e_{\kappa\gamma} = 0,$$  \hfill (2.20)

$$\sigma T_{\alpha\beta} T_{\gamma\kappa} : d\omega^{\alpha\beta\gamma\kappa} + 2 \omega^{\alpha\kappa\gamma\kappa} \wedge \epsilon_{\alpha\beta} e_{\kappa\gamma} + 2 \omega^{\alpha\beta\kappa\kappa} \wedge \epsilon_{\alpha\beta} e_{\kappa\gamma} + 2 \omega^{\alpha\beta\kappa\kappa} \wedge \epsilon_{\alpha\beta} e_{\kappa\gamma} = 0,$$  \hfill (2.21)

$$T_{\alpha\beta} T_{\gamma\kappa} T_{\mu\nu} : \omega^{(4)} \wedge \epsilon^{(4)} + \epsilon^{(4)} \wedge \epsilon^{(4)} = 0.$$  \hfill (2.22)

This solution has no non-trivial holonomy, so one can do a large gauge transformation to relate this solution to empty AdS [59].

2.3. Scalar equations

In this section, we consider the motion of a scalar in the above 3D Schrödinger background, characterized by
We briefly review the analysis of [77] in terms of the lone-star product in this subsection. The notation and its relation with the previously mentioned oscillator formalism is explained in appendix A.

All the fields take value in the higher spin algebra
\[
C = \sum_{s=0}^{\infty} \sum_{|m|<s} C_m^s V_m^s, \quad A = \sum_{s=2}^{\infty} \sum_{|m|<s} A_m^s V_m^s, \quad \tilde{A} = \sum_{s=2}^{\infty} \sum_{|m|<s} \tilde{A}_m^s V_m^s
\]
(2.24)
with \( C_0 \) being the physical scalar. We now extract the equation of motion of \( C_0 \).

If \( A \) and \( \tilde{A} \) span pure AdS3 gravity, equation (2.23) reduces to Klein–Gordon equation. Now consider \( z^i = \frac{1}{2} \) Schrödinger spacetime [45, 46]
\[
A = (\sigma e^{2\rho} V_2^0 + e^{\rho} V_2^1) \, dt + V_0^2 \, d\rho, \quad \tilde{A} = \sigma e^{2\rho} V_2^1 \, dt + 2 e^{\rho} V_2^2 \, d\xi - V_0^2 \, d\rho,
\]
(2.25)
where the constant source \( \sigma \) parametrizes the higher spin deformation. Plugging these expansions into the scalar equation (2.23) we get an infinite set of equations, one from each term proportional to \( \sigma \).

### 3. 4D solution with Schrödinger isometry

#### 3.1. Star product in 4D

Most of the notation in this section will follow [71, 78], where \( x^\mu (\mu = 0, 1, 2, 3) \) denote spacetime Poincaré coordinates with \( x_3 = r \). In this coordinate, the AdS spacetime metric is
\[
dx^2 = -\frac{dr^2 + dx^2 + dx^2 + dx^2}{r^2}.
\]
(3.1)

The internal twistor space is parametrized by spinors \((Y, Z) = (y^a, \tilde{y}^\alpha, z^\alpha, \tilde{z}^\alpha)\), \(a, \alpha = 1, 2\). Here \( z^\alpha, \tilde{z}^\alpha \) are auxiliary coordinates; physical fields are those with constraints \( \epsilon^\mu = \tilde{\epsilon}^\alpha = 0 \).

The star product of two spinor-valued functions can be defined as [71]
\[
f(Y, Z) * g(Y, Z) = f(Y, Z) \exp [\epsilon^{a\beta}(\tilde{\partial}_y + \tilde{\partial}_{\tilde{y}} - \partial_z - \partial_{\tilde{z}})] g(Y, Z)
+ \epsilon^{a\beta}(\tilde{\partial}_y + \tilde{\partial}_{\tilde{y}} - \partial_z - \partial_{\tilde{z}})] f(Y, Z).
\]
(3.2)

There are in addition Klein operators \( K(t) = e^{2\rho} \) and \( \tilde{K} (t) = e^{2\rho} \).

Vasiliev master fields include a gravitational connection \( W = W_0^i(x|y, \tilde{y}, z, \tilde{z}) \) and a scalar field

\[\begin{align*}
A^i &= e^{a\beta}A^a_j; \quad A_{ij} = A^a_e\epsilon_{a\beta}^i; \quad e^i_{12} = e^{12} = 1.
\end{align*}\]
The equations of motion that determine the dynamics of the system are

\[ \begin{align*}
  d_1 W + W * \wedge W &= 0, \\
  d_2 W + d_1 S + [W, S]_* &= 0, \\
  d_2 S + S * S &= B * K dz^2 + B * \tilde{K} dz^2, \\
  d_1 B + W * B - B * \pi(W) &= 0, \\
  d_2 B + S * B - B * \pi(S) &= 0,
\end{align*} \]

where \( \pi(H) \) flips the signs of unbarred spinors \((y, z, dz)\) in \(H\) while it preserves the signs of barred coordinates \((\bar{y}, \bar{z}, d\bar{z})\). These master fields also satisfy

\[ [R, W]_b = [R, S]_b = [R, B]_b = 0, \]

where \( R = K \tilde{K} \). This implies \( W, B \) are even functions of \((Y, Z)\) while \( S \) is an odd function of \((Y, Z)\).

In this section, we will discuss the vacuum solutions of master equation (3.3), i.e. \( \bar{B} = 0, S = dz^a \bar{z}^a + dz^\alpha \bar{z}_\alpha \) and \( W(Y, Z) = W(Y) \) from (3.3b).

### 3.2. AdS solution in lightcone coordinate

Vacuum AdS4 spacetime

\[ B = 0, \quad S = dz^a \bar{z}^a + dz^\alpha \bar{z}_\alpha, \quad W = e_{a\beta} y^a y^\beta + \omega_{a\beta\gamma} y^a y^\beta + \omega_{a\beta\gamma} y^a y^\beta, \]

is a solution to the Vasiliev equations (3.3), which reduces to the component form

\[ \begin{align*}
  y^a y^\alpha : d e_{a\alpha} + 4 e_{a\alpha} \wedge \omega_{b\beta} \epsilon^{\beta\gamma} - 4 e_{a\alpha} \wedge \omega_{b\gamma} \epsilon^{\gamma} &= 0, \\
  y^a y^\beta : d \omega_{a\beta} + e_{\gamma\delta} \wedge e_{a\beta} \epsilon^{\gamma\delta} + 4 \omega_{a\beta} \wedge \omega_{\gamma\delta} \epsilon^{\gamma\delta} &= 0, \\
  y^a y^\gamma : d \omega_{a\gamma} - e_{a\alpha} \wedge e_{a\gamma} \epsilon^{a\delta} + 4 \omega_{a\gamma} \wedge \omega_{a\delta} \epsilon^{a\delta} &= 0.
\end{align*} \]

Explicitly, we have

\[ e_{a\beta} = \frac{1}{4} e^a (\sigma_2)_{a\beta}, \quad \omega_{a\beta\gamma} = -\omega^{a}(\sigma_2)_{a\beta\gamma}, \quad \omega_{a\beta} = -\omega^{a}(\sigma_2)_{a\beta\gamma}, \]

where \( e^a = \frac{\partial}{\partial x^a} \) are the vielbein and the spin connection of AdS spacetime (3.1) in the lightcone Poincaré coordinate

\[ ds^2 = \frac{2d\xi^2 + dr^2 + dz^2}{r^2}, \quad \xi = x_1 - x_0 \sqrt{2}, \quad t = x_1 + x_0 \sqrt{2}, \quad x = x_3. \]

We have further employed Pauli matrices in the lightcone coordinate in (3.9)

\[ \begin{align*}
  \sigma_1 &= \frac{a_0 + a_1}{\sqrt{2}}, \quad \sigma_2 = -\frac{a_0 + a_1}{\sqrt{2}}, \quad \sigma_3 = \sigma_3, \quad \sigma_4 = \sigma_3, \\
  \sigma_{ij} &= \frac{a_0 + a_1}{\sqrt{2}}, \quad \sigma_{ij} = \frac{a_0 + a_1}{\sqrt{2}}, \quad \sigma_{ij} = \frac{a_0 + a_1}{\sqrt{2}}, \quad \sigma_{ij} = \frac{a_0 + a_1}{\sqrt{2}}.
\end{align*} \]

Further notice that we work in the Minkowski signature, so the Pauli matrices are the familiar ones that are hermitian. As a consequence, the parity action is our convention is then \( y_{\alpha} \leftrightarrow \bar{y}_\alpha, \quad z_\alpha \leftrightarrow \bar{z}_\alpha \), and further accompanied with hermitian conjugation of the coefficients of the oscillators.
3.3. Schrödinger solution with \( z = 2 \)

We are now ready to construct 4D Schrödinger geometry (1.1) in Vasiliev higher spin theory. The simplest non-trivial example is the \( z = 2 \) Schrödinger geometry which turns out to be supported by extra \( s = 3 \) higher spin fields. We consider a variant form of the Schrödinger metric

\[
ds^2 = -\frac{\sigma^2 dt^2}{r^2} + 2 dr d\xi + dr^2 + dx^2, \quad z = 2, \quad \sigma \in \mathbb{R}, \ \sigma \neq 0,
\]

which can be converted from (1.1) by field redefinition \( t \rightarrow \sigma t, \ \xi \rightarrow \xi \sigma^{-1} \).

3.3.1. General solution

We try to find a ground state solution to (3.3) of the form

\[
B = 0, \quad S = d\xi^\alpha \bar{z}_\alpha + d\bar{\xi}^\alpha z_\alpha, \quad W(Y, Z|x) = W(Y|x),
\]

with some spin-3 fields turned on in \( W \). We simply take \( W = W_2 + W_3 \), where \( W_2 \) is the spin-2 piece (3.5) and (3.9), and \( W_3 \) encodes spin-3 fields that are quartic in the \( y, \bar{y} \) oscillators

\[
W_3 = \omega_{(y)jk}\gamma^0 y^j y^k + \omega_{(y)jk}\gamma^0 y^j y^k + \omega_{(y)jk}\gamma^0 y^j y^k
\]

\[
+ \omega_{(y)jk}\gamma^0 y^j y^k + \omega_{(y)jk}\gamma^0 y^j y^k
\]

The only nontrivial equation (3.3a) decomposes schematically to

\[
y^2 : \quad d_4 W_2 + W_2 * \land W_2 = 0, \quad \text{(3.15a)}
\]

\[
y^4 : \quad d_4 W_3 + W_2 * \land W_3 + W_3 * \land W_2 = 0, \quad \text{(3.15b)}
\]

\[
y^6 : \quad W_3 * \land W_3 = 0. \quad \text{(3.15c)}
\]

The equation (3.15a) simply means we can take \( W_2 \) as the AdS connection (3.5) and (3.9). The equation (3.15c) is very restrictive and can only be solved due to the wedge product: we take \( W_3 \) to be proportional to \( dt \) in the light of our aimed solution (1.1). The only remaining equation to be solved, namely (3.15b), decomposes to

\[
y^4 : \quad d_4 \omega_{(y)jk} + 2 e_{jk} \land \omega_{(y)pb} e^{pb} + 16 \omega_{(y)k} \land \omega_{(y)pb} e^{pb} = 0,
\]

\[
y^6 : \quad d_4 \omega_{(y)jk} + 6 e_{jk} \land \omega_{(y)pb} e^{pb} + 6 e_{jk} \land \omega_{(y)pb} e^{pb}
\]

\[
+ 8 \omega_{(y)k} \land \omega_{(y)pb} e^{pb} + 8 \omega_{(y)k} \land \omega_{(y)pb} e^{pb} = 0,
\]

\[
y^3 : \quad d_4 \omega_{(y)jk} + 4 e_{jk} \land \omega_{(y)pb} e^{pb} + 8 e_{jk} \land \omega_{(y)pb} e^{pb}
\]

\[
+ 4 \omega_{(y)k} \land \omega_{(y)pb} e^{pb} + 12 \omega_{(y)k} \land \omega_{(y)pb} e^{pb} = 0,
\]

Considering only time independent, spherical symmetric solution, this set of equations is solved to get
\[
\begin{align*}
\omega^2_{2222} &= \frac{C_1}{r^7}, & \omega^2_{2222} &= \frac{-i C_1}{r^7}, & \omega^2_{2222} &= \frac{-3 C_1}{3 r^7}, & \omega^2_{2222} &= \frac{i C_1}{3 r^7}, & \omega^2_{2222} &= \frac{C_1}{3 r^7}, \\
\omega^2_{2221} &= \frac{-C_2}{6 r^7}, & \omega^2_{2221} &= \frac{2 C_2}{3 r^7}, & \omega^2_{2221} &= \frac{2 C_2}{3 r^7}, & \omega^2_{2221} &= \frac{C_1}{3 r^7}, & \omega^2_{2221} &= \frac{-C_1}{6 r^7}, \\
\omega^2_{2221} &= \frac{C_1}{r^7}, & \omega^2_{2221} &= \frac{-2 C_2}{3 r^7}, & \omega^2_{2221} &= \frac{-2 C_2}{3 r^7}, & \omega^2_{2221} &= \frac{-C_1}{6 r^7}, & \omega^2_{2221} &= \frac{C_1}{6 r^7}, \\
\omega^2_{1111} &= \frac{C_1}{r^7}, & \omega^2_{1111} &= \frac{-i C_1}{r^7}, & \omega^2_{1111} &= \frac{-3 C_1}{3 r^7}, & \omega^2_{1111} &= \frac{i C_1}{3 r^7}, & \omega^2_{1111} &= \frac{C_1}{3 r^7}, \\
\omega^2_{1122} &= \frac{-C_2}{6 r^7}, & \omega^2_{1122} &= \frac{2 C_2}{3 r^7}, & \omega^2_{1122} &= \frac{C_1}{3 r^7}, & \omega^2_{1122} &= \frac{-2 C_2}{3 r^7}, & \omega^2_{1122} &= \frac{-C_1}{6 r^7}, \\
\omega^2_{1122} &= \frac{C_1}{r^7}, & \omega^2_{1122} &= \frac{-2 C_2}{3 r^7}, & \omega^2_{1122} &= \frac{-2 C_2}{3 r^7}, & \omega^2_{1122} &= \frac{C_1}{6 r^7}, & \omega^2_{1122} &= \frac{-C_1}{6 r^7}, \\
\omega^2_{1112} &= \frac{-C_2}{6 r^7}, & \omega^2_{1112} &= \frac{2 C_2}{3 r^7}, & \omega^2_{1112} &= \frac{C_1}{3 r^7}, & \omega^2_{1112} &= \frac{-2 C_2}{3 r^7}, & \omega^2_{1112} &= \frac{C_1}{6 r^7}, \\
\omega^2_{1112} &= \frac{C_1}{r^7}, & \omega^2_{1112} &= \frac{-2 C_2}{3 r^7}, & \omega^2_{1112} &= \frac{-2 C_2}{3 r^7}, & \omega^2_{1112} &= \frac{-C_1}{6 r^7}, & \omega^2_{1112} &= \frac{C_1}{6 r^7},
\end{align*}
\]

(3.16)

where \(C_i\) \((i = 1, \ldots, 5)\) are arbitrary real constants. Furthermore, this solution is manifestly parity invariant.

We would like to remark that, in general, once spin-3 generators in \(D > 3\) dimensional higher spin theory are included, one is forced to include the infinite tower of higher spin fields to solve the equation. This problem is avoided in our construction since the spin-3 fields are only turned on in the \(t\) direction and \(dt \wedge dt = 0\). For this reason we are able to isolate a single spin-\(s\) field, which back-reacts and supports the \(z = s - 1\) Schrödinger spacetime. The spinorial index structure of \(\omega_{a1}\) fields implies that the above solution can be expanded in a basis consisting of tensors of two Pauli matrices. Making use of the identity [79]

\[
\sigma^\mu_{\alpha \gamma} \sigma^\nu_{\beta \delta} + \sigma^\nu_{\alpha \gamma} \sigma^\mu_{\beta \delta} = \eta^{\mu \nu} \sigma_{\alpha \beta} \sigma_{\gamma \delta} + 4 (\sigma^\mu \epsilon)_{\alpha \beta} (\sigma^\nu \epsilon)_{\gamma \delta},
\]

the \(W_3\) field can be recast into

\[
W_3 = (\epsilon^{ab} \sigma_a \sigma_b - H_{ew} \sigma_a (\sigma_2 \epsilon) + H_{ew} \sigma_a (\sigma_2 \epsilon) + H_{ew} (\sigma_2 \epsilon) (\sigma_2 \epsilon) + H_{ew} (\sigma_2 \epsilon) (\sigma_2 \epsilon)) dt
\]

We have checked that the \(\epsilon^{ab}, H_{ew}\) fields can be determined for the Schrödinger spacetime (3.16). However, the result is not much simpler than (3.16) and is not very illuminating so we do not show them explicitly.

Another comment is that given a generalized vielbein

\[
E = e_{\alpha \beta} y^\alpha y^\beta + \omega_{\gamma \beta \epsilon} y^\gamma y^\beta y^\epsilon,
\]

(3.19)

which means fixing the \(C_i, i = 1, \ldots, 5\) parameters, the \(W\) field is fully determined. This is equivalent to the statement that (generalized) spin-connection can be fully determined by the (generalized) vielbein from ‘torsion free’ equations. Therefore, our \(z = 2\) Schrödinger solution is free from degeneracy problem [59].

3.3.2. The metric. As we have briefly explained in the previous section, we do not treat the spin-3 fields as probe but take their backreaction on the geometry into account. We thus propose the following formula to compute the metric from the (generalized) vielbein

\[
g = \text{Tr}(E \ast E),
\]

(3.20)

where the trace is defined in (A.5). Notice that this definition reduces to the more familiar definition \(g = \text{Tr}(e \ast e)\) in general relativity when the higher spin fields are turned off.

This formula is determined by requiring the invariance of the metric under generalized local Lorentz transformations that rotate the local Lorentz indices and thus the local basis.
This idea was first proposed in three-dimensional [49] and we simply generalize it to higher dimension. To justify our proposal, we start with the general gauge transformation of any solution of the set of Vasiliev equations (3.3)

\[ \delta W = d\epsilon + [W, \epsilon]_h, \quad \delta B = B \ast \pi (\epsilon) - \epsilon \ast B, \quad \delta S = [S, \epsilon]_h. \] (3.21)

Since we have \( B = 0 \) and \( \epsilon = \epsilon (Y|x) \), we only consider the first transformation. From which we can read off the general transformation \( \delta E \) of our definition \( E \) (3.20). Then we want to decompose the gauge transformation as

\[ \epsilon = \xi + \Lambda + \Lambda_{\text{extra}}, \] (3.22)

where \( \xi \) parametrizes the generalized diffeomorphisms, \( \Lambda \) parametrizing the generalized local Lorentz transformations and \( \Lambda_{\text{extra}} \) parametrizes the extra gauge transformation associated to the extra auxiliary fields and other terms from higher spin generators\(^7\). The difference between the latter two is that the \( \Lambda \) only rotates the index in the first row in the two-row Young tableaux notation while \( \Lambda_{\text{extra}} \) rotates indices in both the two rows. We thus require the metric to be invariant under all transformations parametrized by\(^8\) \( \Lambda \). It can be explicitly checked that our proposal (3.20) fulfills this requirement: the extra variation of the vielbeins under the local higher spin transformation is cancelled by the variation of the generalized vielbeins \( \omega_{\alpha j;\beta k}. \) In fact, there is a much easier way to demonstrate this invariance. The variation takes a nice form \( \delta E = [E, \Lambda]_h \), then it is trivial to verify the invariance of the metric by cyclicity of the trace\(^9\)

\[ \delta_{\text{tr}} = \text{Tr}(E, \Lambda)_h \ast E + E \ast [E, \Lambda]_h = 0. \] (3.23)

With this definition, the solution we have found gives the following metric

\[ ds^2 = -(72C_4^2 - 64C_2C_5 + 144C_1C_3) \frac{dr^2}{r^4} + \frac{2d\xi d\zeta + dr^2 + dx^2}{r^2}. \] (3.24)

### 3.3.3. Higher spin fields

The spin-3 metric like field can be determined similarly

\[ \Phi = \text{Tr}(E \ast E \ast E), \] (3.25)

which is again invariant under the higher spin generalization of the local Lorentz transformation. Linearizing the above spin-3 field leads to traceless symmetric tensor

\[
\Phi_{\mu
\nu\rho} \sim \text{Tr} (e_{\alpha_1 \beta_1} y^{\alpha_1 \beta_1} \hat{y}^{\hat{\alpha_1} \hat{\beta_1}} \epsilon^{\gamma_1 \delta_1} \hat{y}^{\gamma_1 \delta_1} \hat{y}^{\gamma_1 \delta_1} \omega_{\alpha_2 \beta_2 \gamma_2 \delta_2} y^{\alpha_2 \beta_2} \hat{y}^{\gamma_2 \delta_2} \hat{y}^{\gamma_2 \delta_2} y_{\mu\nu\rho})
\sim \hat{\sigma}_{\mu_1 \gamma_1} \hat{\sigma}_{\nu_2 \delta_2} \omega_{\rho_1 \nu_2 \delta_2 \gamma_2 \delta_2},
\] (3.26)

which agrees with the expression given in [71] up to normalization. The authors are acknowledged there are some nontriviality with this definition. However, (3.25) is shown to be invariant under local Lorentz transformation. Considering it matches the known result at linearized level, the definition is a potential candidate for spin-3 field at least in this Schrödinger vacuum case.

---

\(^7\) Although the equation of motion is truncated by wedge product, the symmetry group is not truncated. Commutator between spin-3 generator in master field \( W \) and gauge transformation \( \Lambda \) can result in terms with spin \( s > 3 \).

\(^8\) The metric does transform under \( \Lambda_{\text{extra}} \), which is the higher dimensional analogue of phenomena discussed in, e.g. [80, 81].

\(^9\) We thank Stefen Theisen to point this out to us.
We can further evaluate the fully nonlinear spin-3 fields (3.25) explicitly

\[
\begin{align*}
\phi_{rr} &= \frac{3((3G + 8G_3 + 3C_3 - 12C_3 - 8C_2) r^2 + 512(4C_1^2 - 9C_1 C_2) r^2 + 8(2C_1 C_2 C_3 - 8C_1 C_3 - 6G C_2^2 - 6C_1^2 C_3))}{2r^2}, \\
\phi_{r\zeta} &= \frac{-4G_3 - 3(G + G_3)}{r^2}, \\
\phi_{\zeta\zeta} &= \frac{-3C_3 + 8G - G_3 + 12C_3 - 8C_2}{2r^4}, \\
\phi_{\zeta\xi} &= \frac{3C_3 - 4G_3 - 3C_2 + 4C_3}{2r^4},
\end{align*}
\]

(3.27)

with all other components vanish. Notice that in most of the terms the power at the boundary is exactly the dimension \(\Delta = 4\) of a conserved spin-3 currents in the dual field theory. The only exception is the \(r^{-6}\) term in \(\Phi_{tr}\) which has cubic coefficients \(C_i C_j C_k\); both its scaling behavior and its coefficient structure indicate the nonlinear nature of this term. We will discuss more about this \(r^{-6}\) power in section 5.

As we have shown explicitly, the metric and the spin-3 metric like fields can be uniquely determined. To determine metric like higher spin fields with \(s > 3\), more information is needed, which is similar to what happens in 3D [55], in addition to the requirement of local Lorentz invariance and the correct linearization limit. This is because there are more than one combinations of vielbeins satisfying the above constraints. For example, for \(s = 4\), \((tr(E \ast E))^2\) and \(tr(E \ast E \ast E \ast E)\) are both local Lorentz invariant. Only a linear combination of these two terms gives the right Fronsdal field

\[
\Phi^{(4)} = tr((E \ast E \ast E \ast E)_t) + c \, tr(E \ast E)tr(E \ast E),
\]

where \((a \ast b)_t = a \ast b + b \ast a\) is the totally symmetric star product. The coefficient \(c\) can be fixed by imposing the double-traceless condition or by imposing a Fefferman–Graham-like gauge condition \(\Phi_{rr} = 0\) [82]. Remarkably, the two conditions lead to the same value \(^{10}c = -\frac{1}{2}\). This result agrees with our expectation and also agrees with what happens in 3D.

We comment here that even though we only turn on spin-2 and spin-3 components of the frame like field \(W(3.14)\), there can be a nonzero spin-4 metric like field as constructed above. This property can only be seen at the fully nonlinear level; the linearized spin-4 field, defined similarly as (3.26), vanishes. Moreover, we believe the whole tower of the metric like fields of arbitrary spin are nonzero unless protected by some hidden symmetries.

3.3.4. Symmetries of the solution. Relation with the AdS spacetime. One immediate question is if the solution we have got is gauge equivalent to the AdS vacuum. This is a reasonable question since both of them are solutions of equation (3.3a). However, we can show that the two solutions are physically distinct.

• Indeed, the following transformation

\[
\delta W = d \epsilon + [W, \epsilon]_\ast,
\]

maps a solution \(W\) of (3.3a) into another solution \(W + \delta W\) of (3.3a). For the case we are interested in, the AdS solution can be mapped to our Schrödinger solution with the parameter \(\epsilon = \epsilon_{abcd}y^a y^b y^c y^d\).

\(^{10}\) The exact value of \(c\) depends on our definition of the trace, but the conclusion that the two conditions lead to the same value is independent of our definition of the trace; the latter can be checked explicitly.
\[ \varepsilon_{2222} = \frac{1}{4} \varepsilon_{2222} = \frac{1}{6} \varepsilon_{2222} = -\frac{1}{6} \varepsilon_{2222} = \varepsilon_{2222} = \frac{C_3 + d_1}{4r^2} \]
\[ \varepsilon_{1222} = \frac{1}{4} \varepsilon_{1222} = \frac{1}{6} \varepsilon_{1222} = -\frac{1}{6} \varepsilon_{2212} = -\frac{1}{6} \varepsilon_{1222} = \frac{C_3 + d_4}{4r^2} \]
\[ \varepsilon_{1111} = \frac{1}{4} \varepsilon_{1111} = \frac{1}{6} \varepsilon_{1111} = \frac{1}{6} \varepsilon_{1111} = C_3 + d_1 \]
\[ \varepsilon_{1112} = \frac{1}{4} \varepsilon_{1112} = \frac{1}{6} \varepsilon_{1112} = \frac{1}{6} \varepsilon_{1112} = C_3 + d_4 \]
\[ \varepsilon_{1112} = \frac{1}{4} \varepsilon_{1112} = \frac{1}{6} \varepsilon_{1112} = \frac{1}{6} \varepsilon_{1112} = \frac{C_3 + d_5}{6r^2} \]
\[ \varepsilon_{1112} = \frac{1}{4} \varepsilon_{1112} = \frac{1}{6} \varepsilon_{1112} = \frac{1}{6} \varepsilon_{1112} = \frac{C_3 + d_5}{6r^2}. \]

However, as discussed in [24, 71], any transformation relating two solutions with different boundary falloff behavior is not a true gauge transformation. The Schrödinger solution we found has \( r \) component being

\[ W = W_2 + W_3 \rightarrow \frac{1}{r^2} \sim W_3 \quad \text{as} \quad r \rightarrow 0 \]

which is different from AdS boundary condition. This fact can also be seen from the parameters characterizing the transformation (3.29); the parameters diverge at the boundary \( r = 0 \), which means they are non-trivial on the boundary. Such transformation relates two different physical solutions, which means our Schrödinger solution is not equivalent to the AdS solution.

- It is intuitive to have an interpretation of the fields in terms of Einstein classical gravity theory. It is confirmed [56, 59] by perturbation calculations that 3D Einstein equation can be solved by \( z = 2 \) Schrödinger metric and its spin-3 matter fields. In the current 4D example, we again expect the spin-3 fields to be responsible for supporting the non-AdS metric solutions

\[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} + \Lambda g_{\mu \nu} = T_{\mu \nu}. \]

The solution in Vasiliev frame equation is a strong evidence indicating that the higher spin fields give the correct stress-energy tensor \( T_{\mu \nu} \), although it is not simple to compute it explicitly due to the lack of an action. This nonvanishing \( T_{\mu \nu} \) tensor also indicates that this solution is physically different from AdS vacuum solution. We expect this solution can be a simple model to study the interaction between spin-2 metric and higher spin fields.\( ^{11} \)

**Space–time symmetry.** We can find the spacetime symmetry of the full solution by finding all the Killing vectors of both the metric and the higher spin metric like fields. By definition, the Lie derivative of the fields along the direction of any Killing vector \( \chi^\mu \) vanishes

\[ \mathcal{L}_\chi g_{\mu \nu} = 0, \quad \mathcal{L}_\chi \phi^{\mu \nu} = 0, \quad \mathcal{L}_\chi \phi^{\mu \nu \rho} = 0 \ldots \]

Solving the first equations, we find the follow Killing vectors generating the Schrödinger isometry of the spacetime in our \( z = 2 \) example

\( ^{11} \) In another known example, Schrödinger spacetime in \( D \geq 4 \) can be obtained by coupling a gauge field \( A_\mu \) to the Einstein gravity and then turning on finite background \( A_\mu \) field [51]. (Notice this gauge field also only has non-vanishing component in \( r \) direction.)
Applying the Lie derivatives associated with these vectors to the spin-3 fields, we find in general only $H$, $MP$ remain symmetry of the spin-3 fields. However, for special choice of the parameters $C_i$, the symmetry of the system could get enhanced. These extra enhanced symmetries can be summarized in table 1 where the coefficients take the following values in different cases:

\[
\begin{align*}
\chi_D &= 2t \partial_t + x \partial_x + r \partial_r, \\
\chi_C &= t^2 \partial_t - \frac{1}{2}(\alpha^2 + r^2) \partial_x + tx \partial_x + tr \partial_r.
\end{align*}
\]

(3.32)  
(3.33)

Applying the Lie derivatives associated with these vectors to the spin-3 fields, we find in general only $H$, $M$, $P$ remain symmetry of the spin-3 fields. However, for special choice of the parameters $C_i$, $i = 1, \ldots, 5$, the symmetry of the system could get enhanced. These extra enhanced symmetries can be summarized in table 1 where the coefficients take the following values in different cases:

\[
\begin{align*}
(a): & \quad C_2 \rightarrow -\frac{3}{2}C_3, \quad C_1 \rightarrow C_3, \quad C_4 \rightarrow -\frac{3}{2}C_5, \quad C_5 \rightarrow \frac{3}{2}C_3, \\
(b): & \quad C_2 \rightarrow 0, \quad C_1 \rightarrow C_3, \quad C_4 \rightarrow \frac{1}{2}C_3, \quad C_5 \rightarrow 0, \\
(c): & \quad C_1 \rightarrow 0, \quad C_2 \rightarrow C_3, \quad C_4 \rightarrow -\frac{1}{4}C_3, \quad C_5 \rightarrow \frac{3}{4}C_3, \\
(d): & \quad C_2 \rightarrow \frac{3}{4}(C_1 \mp \sqrt{C_1C_3}), \quad C_4 \rightarrow \frac{1}{4}(C_1 \mp 4\sqrt{C_1C_3} - C_3), \quad C_5 \rightarrow \frac{3}{4}(\pm \sqrt{C_1C_3} + C_3).
\end{align*}
\]

(3.34)

Thus we see that in case (a) the boost $K$ generator restores and the symmetry is enhanced to a Galilean group\(^ {12} \). For another choice of the parameters (b), the scaling symmetry is respected. Furthermore, it is possible for some other choices of parameters (c) and (d) that a linear combination of boost and scaling becomes a symmetry. But it is impossible that both of them become symmetry simultaneously; there are at most four generators in the symmetry of the solution.

The solutions (a)–(c) have different boundary behavior and hence are different physical solutions. While in case (d) the parameter $C_1$ is a gauge parameter that relates the solutions (d) to (c).

We then consider the symmetries of the spin-4 metric like fields. Astoundingly, the previously determined symmetries of the metric and spin-3 metric like fields are all symmetries of the spin-4 metric like field as well. This is very likely to be a consequence of the fact that in the frame like field $W$, only spin-3 components of the higher spin fields are

\(^ {12} \) In our convention, the Galilean group is generated by translations, rotations and boosts. One could also add in a dilatation generator, but the particle number will not be conserved under this scaling transformation for $z = 2$. Therefore in this paper we do not include this dilatation generator to be part of the Galilean group and consider it as part of the extension to the Schrödinger group at $z = 2$. 

---

**Table 1. Symmetry enhancement and metric like fields.**

| Killing vectors | $-g_{ij}r^4$ | spin-3 fields |
|-----------------|--------------|---------------|
| (a) $\chi_K$    | $162C_3^2$   | $\phi_{tt} = \frac{72C_3}{r^4}$ |
| (b) $\chi_D$    | $162C_3^2$   | $\phi_{tt} = \frac{60\sqrt{C_1C_3}}{r^4}, \phi_{tt} = \frac{3C_3}{r^4}, \phi_{tt} = \frac{3C_3}{r^4}$ |
| (c) $\chi_D - \sqrt{2}\chi_K$ | $\frac{3}{2}C_3^2$ | $\phi_{tt} = \frac{6C_3\sqrt{C_1C_3}}{r^4}, \phi_{tt} = \frac{6C_3}{r^4}$ |
| (d) $\chi_D + \frac{2\sqrt{C_1\sqrt{C_1\sqrt{C_1}}}C_3}{\sqrt{C_1} + C_3} + \frac{9}{2}(C_1^2 + 3C_3C_1 + C_3^2)$ | $\phi_{tt} = \frac{6C_3\sqrt{C_1C_3} - 2\sqrt{C_1C_3}}{r^4}, \phi_{tt} = \frac{6\sqrt{2C_1 - C_3}}{r^4}$ |

---

\[\chi_H = \partial_t, \quad \chi_M = \partial_x, \quad \chi_P = \partial_x, \quad \chi_K = x \partial_x - t \partial_t, \quad \chi_D = 2t \partial_t + x \partial_x + r \partial_r, \quad \chi_C = t^2 \partial_t - \frac{1}{2}(\alpha^2 + r^2) \partial_x + tx \partial_x + tr \partial_r.\]
turned on; even though the spin $s > 3$ metric like fields are non-vanishing, they do not carry new physical information. Therefore we believe the symmetries we have found previously are symmetries of the full solution that we have constructed.

Global internal symmetry. Global symmetry of a vacuum solution to the Vasiliev equation can be extracted from the equation

$$\text{d} \epsilon (y|x) + [W, \epsilon (y|x)]_b = 0,$$

which determines how does a given symmetry parameter $\epsilon_0(y)$ at any fixed spacetime point extend to a small neighborhood around this point. Since $W$ is a solution to the flatness equation, it is always possible to rewrite the vacuum solution in the form of a pure gauge in this neighborhood.

$$W = g^{-1}(y|x) * \text{d} \epsilon_0(y|x).$$

The solution to the equation (3.35) in this gauge can be trivially solved as

$$\epsilon (y|x) = g^{-1}(y|x) * \epsilon_0(y) * g(y|x),$$

where $\epsilon_0(y)$ does not depend on spacetime coordinates and fully determines the global (internal) symmetry. It is concluded in [59] that the symmetry of Schrödinger higher spin solution in 3D Chern–Simons theory is just $\text{SL}(N, R) \times \text{SL}(N, R)$ by applying the gauge function method above. In the current higher dimensional case, one could also conclude that $\epsilon_0(y)$ exhausts the whole Vasiliev higher spin symmetry group.

3.4. Solutions with other scaling factors

As we have mentioned in the introduction, $z = 2$ Schrödinger spacetime has a larger isometry group than Schrödinger spacetime with $z \neq 2$. To demonstrate that our construction is universal for all integer $z$ rather than merely a result of the larger symmetry group at $z = 2$, we have also constructed the $z = 3$ Schrödinger spacetime in a similar way. The $z = 3$ Schrödinger spacetime turns out to be supported by spin-4 fields in the $t$ direction. We spare the reader from the tedious expression since it is not particularly illuminating. From the construction, we find explicitly that the back-reaction of spin-$s$ fields `deforms' AdS$_4$ to Schrödinger spacetime in 4D with $z = s - 1$.

A general spin-$s$ field $W_{(2, -2)} = \{\omega_{\alpha_1 \ldots \alpha_2 z}, \ldots, \omega_{\alpha_1 \ldots \alpha_2 z}\}$ has $N_i = \frac{s(s^2 - 1)}{4}$ independent components, which is the same as the number of independent equations in (3.16). In other words, if one specifies a group of parameters as `boundary conditions' of the differential equations, all the components of master field $W$ can be uniquely determined. Furthermore, if this group of parameters can be fixed from a given set of generalized vielbein, as in our spin-3 example, there is no degeneracy problem. This property can only be checked case by case.

3.5. RG flows

In the previous sections, we have considered solution to the Vasiliev equation that corresponds to spacetime with Schrödinger isometry. These solutions are derived by turning on higher spin fields with one given spin. One immediate question is what if we turn on fields with different spins in a similar manner.

From the above construction, we notice that the higher spin fields only enter equation (3.15b) and hence fields with different spins are in general independent to each other.

We thank Wei Li for a discussion on similar situations in 3D.

We thank Matthias Gaberdiel for pointing this direction to us.
other. Therefore, the general solution with different higher spin fields turned on is simply a linear combination of the previous solutions where only one single higher spin field is turned on. Thus the general solution gives the following metric

\[
\text{ds}^2 = \left( \sum_{i=\min}^{\max} f_i \frac{1}{r^{2i-2}} \right) \text{d}r^2 + \frac{2\text{d}r \text{d}z + \text{d}z^2 + \text{d}x^2}{r^2},
\]

(3.38)

where the index \(i_{\min}\) and \(i_{\max}\) are the minimal and maximal spins we have turned on in the \(t\)-direction. The number of independent parameter \(f_i\) agrees with the number of different higher spin fields. Higher spin Fronsdal fields can be similarly determined.

Geometrically, these solutions interpolate between Schrödinger-like geometries with different dynamic exponents. This can be easily verified not only for the metric but also the higher spin Fronsdal fields. The existence of this RG type solution is due to the presence of higher spin fields, as well studied in the pure AdS case \([80, 85]\).

From the dual field theory point of view, these solutions correspond to RG flows between \(U(N)\) models with different deformations, which we discuss in detail in section 5, resembling the RG flows between different Landau-Ginzburg models or minimal models in 2D. In the cases where the solutions respect the scaling symmetry, the dual RG flow is also interesting since that relates theories where the time direction scales differently.

3.6. Linearized scalar equations

We have discussed new exact ground-state solutions to Vasiliev equation (3.3) in 4D. In this section, we consider the motion of the scalar fields. We find the linearized scalar equation of motion on this Schrödinger background to be ‘deformation’ of the free Klein–Gordon equation with extra radius-dependent source due to the spin-3 fields. Explicit calculations of correlation functions are left for future work \([86]\).

The ground state configuration is

\[
W_0 = W_2 + W_3, \quad B_0 = 0, \quad S_0 = \text{d}z^\alpha z_\alpha + \text{d}\bar{z}^\alpha \bar{z}_\alpha.
\]

(3.39)

Linearized perturbation around it means

\[
W = W_0(x|Y) + W_1(x|Y, Z); \quad S = S_0 + S_1(x|Y, Z); \quad B = B_1(x|Y, Z).
\]

(3.40)

The linearized Vasiliev equations are

\[
D_0 W_1 = 0, \quad \bar{D}_0 B_1 = 0, \quad \text{d}_x W_1 + D_0 S_1 = 0, \quad \text{d}_x S_1 = e^{i\theta_0} B_1 * K \text{d}x^2 + e^{-i\theta_0} B_1 * \bar{K} \text{d}x^2, \quad \text{d}_z B_1 = 0,
\]

(3.41)

where \(\theta_0\) is a parameter corresponding to the parity of the scalar. From the last equation of (3.41), \(B_1\) is independent of \(Z\). Then higher spin fields are

\[
C(x|Y) = B|_{Z=0} = B_1(x|Y).
\]

We consider the linearized equation of the scalar

\[
\bar{D}_0 C = \text{d}C + W_0 * C - C * \pi(W_0) = 0,
\]

(3.42)

where \(\pi(W_0(Y, \bar{y})) = W_0(-y, \bar{y})\), which flips the sign of the coefficients of any odd number of \(y\) oscillator. Therefore, it is useful to separate our background spin-3 field \(W_3\) into two pieces that has even or odd number of (un)barred oscillators, namely \(W_3 = W_3^e + W_3^o\),
respectively. In the direction other than $t$, the background gauge field is the same as that of the AdS spacetime, while in the $t$ direction, there are spin-3 fields turned on. Therefore we have
\[
\nabla_{\mu}^L C(x|Y) + 2(e_{\mu})_{\gamma}^{\alpha} \gamma^{\alpha} \bar{y}^\beta C(x|Y) + 2(e_{\mu})_{\gamma}^{\alpha} \partial_{\gamma} \partial_{\bar{y}^\beta} C(x|Y) = N_{\gamma} \delta_{\mu \nu},
\]
where
\[
N_{\gamma} = N_{\gamma}^3 + N_{\gamma}^3 = -([W_{\gamma}^3, C(x|Y)] + [W_{\gamma}^3, C(x|Y)]).
\]

Following the notation of [71], $C^{(n,m)}$ represents the terms in $C(x, Y)$ of degree $n$ in $y$ and degree $m$ in $\bar{y}$. Scalar fields are contained in $C^{(n,0)}$. The $C^{(0,0)}$ equation receives higher spin corrections
\[
\partial_{\mu} C(x|0,0) + 2(e_{\mu})_{\gamma}^{\alpha} \frac{\partial}{\partial y_{\alpha}} C^{(1,1)} = N_{\gamma}^3 \delta_{\mu \nu},
\]
where the commutator term in (3.44) vanishes and hence we only keep the odd piece $N_{\gamma}^3$ of $N_{\gamma}$. Similarly, for the $C^{(1,1)}$ equation, we have
\[
\nabla_{\mu}^L C^{(1,1)} + 2(e_{\mu})_{\gamma}^{\alpha} \gamma^{\alpha} \bar{y}^\beta C^{(0,0)} + 2(e_{\mu})_{\gamma}^{\alpha} \partial_{\gamma} \partial_{\bar{y}^\beta} C^{(2,2)} = N_{\gamma}^3 \delta_{\mu \nu}.
\]
Expanding the above equation in component form
\[
C^{(1,1)} = C^{(1,1)}_{\alpha \beta} y^{\alpha} \bar{y}^{\beta}, \quad C^{(2,2)} = C^{(2,2)}_{\alpha \beta} y^{\alpha} \bar{y}^{\beta} y^r,
\]
we get
\[
C^{(1,1)}_{\gamma \delta} = 4(e_{\mu})_{\gamma \delta} \partial_{\mu} C(x|0,0) - 4N_{\gamma}^3 (e^\gamma)_{\gamma},
\]
\[
(e^\mu)_{\gamma} \nabla_{\mu}^L C^{(1,1)} = \frac{1}{4} y^\gamma \bar{y}^\delta C^{(0,0)} - \frac{1}{2} (e^\mu)_{\gamma \delta} y^\gamma \bar{y}^\delta C^{(2,2)} = (N_{\gamma}^3)_{\gamma \delta} y^\gamma \bar{y}^\delta (e^\gamma)_{\gamma}.
\]
Eliminating $C^{(2,2)}$ term by acting the equation with $\partial_{\gamma} \partial_{\bar{y}^\delta}$, we get our final deformed Klein–Gordon equation
\[
-\frac{1}{2}(\nabla_{\mu} \partial^{\mu} + 2) C^{(0,0)} = (e^\mu)_{\gamma} (e^\gamma)_{\gamma} (4(\partial_{\mu} N_{\gamma}^3) + (N_{\gamma}^3)_{\gamma} (e^\gamma)_{\gamma}.
\]
This equation is simply the normal Klein–Gordon equation sourced by the known function $N_{\gamma}$. We can solve it as the motion of the scalar under a classical potential $N_{\gamma}$ due to the spin-3 fields. We will report the detailed analysis in the near future [86].

4. Schrödinger solution in general dimension

Our construction is in fact applicable in any dimension where the Vasiliev higher spin theory is defined. One interesting example is Schrödinger solutions in the 6D higher spin gravity since the isometry group of the metric in that case is $\text{Sch}(3)$, which is the same symmetry governing the 3D unitary fermion theory that can be used to describe the cold atom system [51, 52, 87, 88].

We use the vectorial formalism [38, 39] for the construction in general $d$-dimension. In particular, we have the generators of the AdS spacetime isometry
\[
T^{AB} = -T^{BA} = \epsilon^{i|k} Y_{i}^{A} Y_{i}^{B},
\]
where $i, j = 1, 2$ are the $sp(2)$ indices and $A, B = t, \xi, x_1, \ldots, x_{d-2}, \hat{r}$ are the (extended) spacetime index. The $\hat{d}$ is an auxiliary direction with negative signature. The master gauge
field $W$ encodes the vielbein and spin connections

$$W \supset W^{(2)} = W_{ab} T^{ab}, \quad e^a = W^{ad} , \quad w^{ab} = W_{ab}.$$  \hfill (4.2)

The AdS solution is simply

$$e_a = W_{ad} \frac{\delta_{am} dx^a}{r}, \quad w_{ab} = - w_{ba} \frac{\delta_{am} \delta_{bn} - \delta_{an} \delta_{bm}}{r}.$$  \hfill (4.3)

One special feature of the vectorial formalism is the requirement of factoring out the ideal that is generated by the $sp(2)$ generators [38, 39]. This process puts the set of equations on mass-shell. However, this process of factoring out the ideal can be fairly complicated in general dimension. We have constructed solutions in 4D and 6D vector like formalism by turning on spin-3 fields in the $t$-direction, but the solutions are still off mass-shell at the current stage. We will put complete analysis in future work [86].

As a further evidence, we can realize the Schrödinger symmetry generators in terms of the vectorial oscillators

$$M^i = i T^i, \quad P^i = i T^{*i}, \quad K^i = i T^{*i} - i T^i, \quad H = \frac{1}{\sqrt{2}} (T^{+} + T^{-}), \quad d = \frac{1}{\sqrt{2}} (T^{+} - T^{-}),$$

where we have defined a second pair of light-cone coordinates $\zeta = r \pm \tilde{r}$.

As in the four-dimensional case, we expect the symmetry of the full solution to be the Galilean group, with possible mixing with scaling transformation.

5. Dual field theory

There are two parity invariant Vasiliev theories on AdS$_4$ that are dual to bosonic [2] and fermionic [1, 3] versions of $O(N)$ or $U(N)$ vector models respectively (see e.g. [71] for a review). Since our background can be explicitly check to be parity invariant and we can put the same two types of Vasiliev theories on our background, we believe that there should also be a bosonic and a fermionic version of the holographic dual field theories. In this section, we discuss such bosonic and fermionic field theories, which are simply free $U(N)$ field theories with spin-3 operators in the action that respect the same symmetries as the bulk solutions. They are valid candidates as the dual theories since our solutions in the bulk are constructed in the same manner by turning on spin-3 fields on top of the AdS geometry. The bulk scalar can have different boundary behaviors; the corresponding dual theories can be derived from the theories we proposed below by adding double-trace operators and flow to the IR/UV fix points. This is in parallel to the AdS case so we will not go into the details.

5.1. Theories with Galilean symmetry

We have shown that a class of our solutions respects the Galilean symmetry. The corresponding field theories on the boundary can be obtained from a CFT by turning on extra spin-3 current operators

$$S = S_{CFT} + \int d\tau dx \tilde{\xi} \Phi^{\nu\rho} \mathcal{F}_{\mu\nu\rho}.$$  \hfill (5.1)

that break the relativistic conformal symmetry but preserve the same symmetry as in the bulk. In particular, the field theory should preserve the Galilean boost symmetry.
\[ \xi' = \xi + vx - \frac{1}{2}v^2t, \quad x' = x - vt, \quad \forall v. \]  

(5.2)

As it is discussed in section 3.3.4, only \( F \sim F \sim r^{-6} \sim r^{-4} \) component is present in the bulk. Thus we propose that the dual bosonic field theory with the Galilean symmetry has the action

\[ S_\theta = \int \mathrm{d}x^4 \mathrm{d}^3 \xi (\partial_\xi \tilde{\phi}^{\mu} \partial_\xi \phi^\mu + \partial_\xi \tilde{\phi}^{\alpha} \partial_\xi \phi^\alpha + \partial_\xi \tilde{\phi}^{\nu} \partial_\xi \phi^\nu - \Sigma \tilde{\phi}^{\mu} \partial_\xi^2 \phi^\mu), \]

(5.3)

where \( a \) is the \( U(N) \) index and \( \Sigma \) has mass dimension \(-1\). The equation of motion is

\[ H_B \phi^\alpha = (2\partial_x \xi + \partial_x^2 + \Sigma \partial_x^2) \phi^\alpha = 0. \]

(5.4)

It is easy to check that the equation is preserved by time translation \( H \), momentum \( P \), non-relativistic mass \( M \) and Galilean boost \( K \), i.e. the Galilean group.

Like the case of AdS holography, we expect another fermionic theory to be dual to the bulk higher spin gravity with a parity odd scalar. Following the same reasoning as the bosonic theory (5.3), it would be natural to propose a spin-3 ‘deformed’ fermionic free \( U(N) \) theory, which can be defined by the following action

\[ S_F = \int \mathrm{d}x^4 \mathrm{d}^3 \xi (i\tilde{\psi}^\mu \Gamma^\mu \psi^\mu - i\Sigma \tilde{\psi}^\alpha \Gamma^\alpha \partial_\xi^2 \psi^\alpha), \]

(5.5)

where \( \mu \) runs over all the spacetime indices and we have used the following definition of the \( D \)-dimensional gamma matrices in the lightcone coordinates

\[ \Gamma^\nu = \frac{1}{\sqrt{2}}(\Gamma^0 + \Gamma^1), \quad \Gamma^\xi = \frac{1}{\sqrt{2}}(-\Gamma^0 + \Gamma^1), \]

(5.6)

where \( \Gamma^\nu, \Gamma^\xi \) satisfy

\[ \Gamma^\nu \Gamma^\nu = \Gamma^\xi \Gamma^\xi = 0; \quad \{ \Gamma^\nu, \Gamma^\xi \} = 2I. \]

(5.7)

In 3D, a representation of these matrices can be chosen to be \( \Gamma^0 = \sigma_x, \Gamma^1 = \sigma_y, \Gamma^2 = \sigma_z \). The equation of motion is

\[ H_F \psi^\alpha = (\Gamma^\nu \partial_\mu - \Sigma \Gamma^\nu \partial_\xi^2) \psi^\alpha = 0. \]

(5.8)

The action can be explicitly shown to be invariant under the Galilean symmetry group, which is not hard to understand since \( H_B = H_B^2 \).

### 5.2. Theories with non-relativistic scaling symmetry

Another family of solutions with enhanced non-relativistic scaling symmetry has the following spin-3 fields

\[ \Phi_{\mu\nu} = -\frac{3072C^3}{r^6}, \quad \Phi_{\mu\xi} = -\frac{8C_1}{r^4}, \quad \Phi_{\xi\xi} = \frac{8C_1}{r^4}. \]

(5.9)

Note that \( \Phi_{\mu\nu} \) has distinct dimension from the other terms, which makes writing down an action for the dual field theory more challenging since we expect a spin-3 conserved current to have dimension \( \Delta = 4 \) in a 3D CFT. In the language of field theory, this difficulty is that the \( \tilde{\psi}^{\mu} \partial_\xi^2 \psi^\mu \) component in the higher spin currents does not respect the non-relativistic scaling. Therefore, there seems to be no straightforward way to embed non-relativistic scaling symmetry into higher spin symmetry in the current construction unless severe modification is made. This result is somewhat consistent with the result obtained from 3D Chern–Simons theory. However, the \( r^{-6} \) power hints on the possibility of terms that are not components of the higher spin currents, such as the multi-trace operator \( (\tilde{\psi}^{\mu} \partial_\xi \psi^\mu)^3 \), to appear in the action.
This term is possible in light of the nonlinear nature of bulk higher spin field $\phi_{\mu\nu}$. Notice that this multi-trace operators can be understood, at least in the large $N$ limit, to ‘run’ the theory to some UV conformal fixed point, as in the more familiar AdS case [2, 9, 12, 24, 89].

On the other hand, there are known examples where the non-relativistic scaling is incorporated into the symmetry of the theory; these are the well known theories with the Schrödinger symmetry, which is an extension of the Galilean symmetry by a non-relativistic dilatation and a special conformal transformations. One can construct such theories by ‘deforming’ a free CFT with spin-3 current operators

- **Bosonic**
  \[
  S_{\text{schr}}^B = \int dx^{D-1} (\partial_{\mu} \bar{\phi}^{\mu} \partial_{\nu} \phi^{\nu} + \partial_{\mu} \bar{\phi}^{\mu} \partial_{\nu} \phi^{\nu} + \partial_{\mu} \bar{\phi}^{\mu} \partial_{\nu} \phi^{\nu} - \Sigma \bar{\phi}^{\nu} (\partial_{\mu} \partial_{\nu} - 2 \partial_{\mu} \partial_{\nu}) \phi^{\nu}),
  \]
  (5.10)

- **Fermionic**
  \[
  S_{\text{schr}}^F = \int dx^{D-1} (i \bar{\psi}^{\mu} \gamma^{\nu} \partial_{\mu} \psi^{\nu} - i \Sigma \bar{\psi}^{\nu} \gamma^{\nu} (\partial_{\mu} \partial_{\nu} - 2 \partial_{\mu} \partial_{\nu}) \psi^{\nu}).
  \]
  (5.11)

It is easy to check directly that the above theories have the Schrödinger symmetry. There is a qualitative way to understand the presence of this symmetry. Taking the action (5.10) as example, the corresponding equation of motion reads

\[
(1 + \Sigma \partial_{\mu})(2 \partial_{\nu} \partial_{\mu} + \partial_{\nu} \partial_{\mu}) \phi^{\mu} = 0.
\]

(5.12)

From which we see explicitly that the symmetry of this equation of motion are the subset of the symmetries of the Klein–Gordon equation that further commute with the $1 + \Sigma \partial_{\mu}$ factor. This subset is the centralizer of $\partial_{\mu} + \frac{1}{\Sigma}$, which is nothing but the Schrödinger group by construction of the light-cone reduction [69, 87]. Moreover, since the $\partial_{\mu}$ plays the role of the non-relativistic mass generator $M$, the meaning of $\Sigma$ in the action (5.10) is then the (minus) inverse mass in the theory. This interpretation also holds in action with only Galilean symmetry, namely (5.3). A similar argument applies to the fermionic action. This action then suggests that $\phi_{\mu\nu}$ term vanishes in the bulk and boundary values of $\phi_{\mu\nu}$ and $\phi_{\mu\nu}$ differ by a factor of 2. We do not observe this in the Schrödinger solutions we have constructed. It would be interesting to construct a solution of the Vasiliev higher spin theory that is dual to the above Schrödinger invariant field theory. Furthermore, it is interesting to see if there are higher spin solutions with the Galilean conformal symmetry [54, 90].

### 6. Discussion

In this paper, we have constructed solutions of the Vasiliev higher spin theory with Galilean symmetry in general dimensions. We show that the spacetime symmetry group can be the Galilean group or a non-relativistic scaling symmetry group. We further conjecture a bosonic and a fermionic field theory that could be dual to the type-A and type-B Vasiliev theories living on the Schrödinger background that we have constructed. The difference of the two types is only visible when considering the motion of the scalar probes, whose linearized equation of motion is also derived. Therefore the immediate next step is to consider correlation functions of the bulk higher spin system on the Schrödinger background and in the dual field theories we have proposed. This would provide another piece of strong evidence of whether our proposal is sensible or not. This is currently under investigation [86].
One general property of the higher spin gravity is that some usual geometric quantities such as event horizon might not remain physical observable in higher spin theory \cite{80,91}. In fact, there are even different ways of identifying the gravity sector in a given higher spin system, which leads to interesting observations \cite{80,85}. One proposal due to these special properties is the resolution of black hole or cosmic singularities by performing higher spin gauge transformations \cite{81,92}. It is argued in \cite{59} that this method cannot be used to resolve IR tidal force singularity in 3D Lifshitz \cite{93,94} and $1 < z < 2$ Schrödinger spacetime \cite{65} because degeneracy problem makes the spacetime interpretation problematic. However it is possible to construct $z = 2$ 3D Lifshitz spacetime by dimensional reduction. One can show that if one adds a constant one-form $\eta = \eta_i \, dt$ to the AdS gravitational connection
\begin{equation}
 e = e_{(\alpha)} y^\alpha \bar{y}^\beta + \eta; \quad \omega = \omega_{(\alpha)} y^\alpha \bar{y}^\beta + \omega_{(\alpha)} y^\alpha \bar{y}^\beta,
\end{equation}
the master field $W$ still solves Vasiliev equation. It turns out that the corresponding metric represents the $z = 0$ Schrödinger spacetime
\begin{equation}
d s^2 = -\eta^2 \, dt^2 + \frac{2 \, d\xi \, d\tau + dr^2 + dx^2}{r^2}.\end{equation}
To proceed, we use the fact that $D - 1$ dimensional $z = 2$ Lifshitz spacetime emerges from $z = 0$ Schrödinger spacetime in $D \geq 4$ dimension by dimensional reduction in the $t$ direction \cite{72,73}. Those 3D Lifshitz spacetimes are solutions of Einstein equation with supporting matter fields and therefore safe from degeneracy problem in higher spin theory \cite{59}. One may be able to study how higher spin transformation operates on the Lifshitz geometry, and understand the physical meaning of IR singularity \cite{95}. The recent development of invariant functional \cite{96} for the Vasiliev theory could help make progress in this direction.

Last but not the least, it would be interesting to know whether Schrödinger black hole solution exists in 4D Vasiliev theory. The known higher spin solution in three-dimension \cite{91,97}, the charged black hole solution with asymptotic Schrödinger geometry \cite{68–70} together with the reformulation of AdS$_4$ Kerr black hole solution into the unfolding formalism \cite{41} hint on possibility of finding black hole solutions with asymptotic Schrödinger geometry in higher spin theory. We will leave this for future work.

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Appendix A. Higher spin algebra in $D = 3$

We will follow the notation in \cite{50,98}. The higher spin algebra $hs [\lambda]$ generator $V^\alpha_m$ are defined to be
\begin{equation}
 V^\alpha_m = (-1)^{s-1-m} \frac{(s+m-1)!}{(2s-2)!} \left[ V^2_{(2s)} \cdots [V^2_{(m)} [V^2_{(1)} (V^{(s-1)}_{(1)})]] \right],
\end{equation}
where
\[ V_1^2 = L_1, \quad V_0^2 = L_0, \quad V_{-1}^2 = L_{-1}. \]
If \( \lambda = N \), the algebra is truncated to \( sl(N) \) and all the \( s > N \) generators can be removed. The lone star product between generators has a closed form
\[ V_m^* \ast V_n^* = \frac{1}{2} \sum_{u=1}^{s+t-|s-t|-1} g_u^\ast (m, n; \lambda) V_{m+n}^{s+t-u}, \tag{A.2} \]
with
\[ g_u^\ast (m, n; \lambda) = \left( \frac{1}{4} \right)^{u-2} \frac{1}{2(u-1)!} \phi^\ast_u (\lambda) N_u^\ast (m, n), \]
where
\[ N_u^\ast (m, n) = \sum_{k=0}^{u-1} (-1)^k {u \choose k} \left[ s - 1 + m \right]_{n-1-k} \left[ s - 1 - m \right]_{n-1-k} \left[ t - 1 + n \right]_k \left[ t - 1 - n \right]_{n-1-k}, \]
\[ \phi^\ast_u (\lambda) = \binom{1/2 + \lambda k - \lambda, 1/2 - \lambda/2, 1/2 - \lambda, 1/2 + s + t - u}{3/2 - s, 3/2 - t, 1/2 + s + t - u}. \]
Here \( [a]_n = a(a-1)...(a-n+1) \) are the descending Pochhammer symbol. The commutator of two generators are defined as
\[ [X, Y] = X \ast Y - Y \ast X. \tag{A.3} \]
\( V_m^\ast \) transforms in the \((2s-1)\) dimensional representation of \( sl(2) \) Lie algebra
\[ [V_m^2, V_n^2] = (-n + m(s - 1))V_{m+n}^\ast, \tag{A.4} \]
which is also one of the useful formulas used in verifying Schrödinger solution. The trace of lone star product is defined to be
\[ \text{tr}(X \ast Y) = X \ast Y|_{V_0^2=0,t>0}. \tag{A.5} \]
The relation with the oscillator realization is via the identification
\[ V_1^2 = \frac{1}{2} T_{11}, \quad V_0^2 = \frac{1}{2} T_{12}, \quad V_{-1}^2 = \frac{1}{2} T_{22}. \tag{A.6} \]
Other higher spin generators \( V_m^\ast \) are related to \( T_{\alpha \beta} \) via equation (A.1).

**Appendix B. Prove local Lorentz invariance of metric-like fields in 4D**

We are going to show in the section that metric-like fields defined in section 3.1 are invariant under generalized local Lorentz transformation. Take the following ansatz:
\[ E = e_2 + e_3 = e_{\alpha \beta} y^{\alpha \beta} g^{\gamma \delta} + \omega_{\alpha \beta \gamma \delta} y^{\alpha \beta} g^{\gamma \delta} \]
We will take spin-2 metric-like fields \( g_{\alpha \beta} \) as example. Invariance of higher spin fields can be proved in similar way, but requires more texts to explain.

It is very straightforward to check \( g = \text{Tr}(e_2 \ast e_3) \) is invariant under local Lorentz transformation \( \Lambda_2 \) if only spin-2 fields are involved. In this case, we can confirm
\[ e_2 = e_{\alpha\beta} y^\alpha y^\beta; \quad \omega = \omega_{\alpha\beta} y^\alpha y^\beta + \omega_{\alpha\beta} \tilde{y}^\alpha \tilde{y}^\beta \] (B.1)
\[ \xi = \tilde{e}_{\alpha\beta} y^\alpha y^\beta; \quad \Lambda_2 = \tilde{e}_{\alpha\beta} y^\alpha y^\beta + \tilde{e}_{\alpha\beta} \tilde{y}^\alpha \tilde{y}^\beta. \] (B.2)

Then
\[ \delta e_2 = d\xi + [e_2, \Lambda_2] + [\omega, \xi], \]
\[ \delta g = \text{Tr}(e_2, \Lambda_2) + e_2 + e_2 + [e_2, \Lambda_2] = 0. \]

Spin-3 case is more complicated. For clarity, we will try to prove its invariance under the basis of oscillator \( y^\alpha \). Denote the generalized local Lorentz transformation as \( \Lambda = \Lambda_2 + \Lambda_3 \). \( \Lambda_3 \) are those terms whose commutator with \( E \) would vary it by \( dE \)
\[ \Lambda_3 \sim e_{\alpha\beta\gamma\delta} y^\alpha y^\beta y^\gamma y^\delta + \tilde{e}_{\alpha\beta\gamma\delta} y^\alpha y^\beta y^\gamma y^\delta. \]

We take the calculation of first term as example. By calculation,
\[ [e_2, \Lambda_3] + e_2 + [e_2, \Lambda_3] = e_2 + [e_2, \Lambda_3] + e_2, \]
\[ + [e_2, \Lambda_3] + e_2 + [e_2, \Lambda_3] + e_2 + [e_2, \Lambda_3] + e_2 + [e_2, \Lambda_3] + e_2. \]

Note finally, we need to prove \( \delta g = 0 \). The trace structure helps us simplify the calculation. Note all the 4 terms in \( \delta g \) would not have contribution to \( dE \). Take first term as an example. The commutator results in terms with odd numbers of \( y \) tensor, so \( e_{\alpha\beta\gamma\delta} y^\alpha y^\beta y^\gamma y^\delta \) only has terms with two \( y \). The trace contraction of \( y_2 \) and \( y_4 \) by star product is always zero.

We are interested in those spin-3 gauge transformation terms whose commutator with \( E \) change the value of \( E \). These terms are
\[ \Lambda_3 \sim e_{\alpha\beta\gamma\delta} y^\alpha y^\beta y^\gamma y^\delta + \tilde{e}_{\alpha\beta\gamma\delta} y^\alpha y^\beta y^\gamma y^\delta. \]

We take the calculation of first term as example. By calculation, \( [e_2, \Lambda_3] + e_2 + [e_2, \Lambda_3] \) has a term
\[ [e_2, \Lambda_3] + e_2 + [e_2, \Lambda_3] = e_2 + [e_2, \Lambda_3] + e_2 + [e_2, \Lambda_3] + e_2 + [e_2, \Lambda_3] + e_2. \]

The second term above is not important since it vanishes after taking trace. The first term results in
\[ \text{Tr}(6e_{\alpha\beta\gamma\delta} e_{\gamma\delta\epsilon\zeta} y^\alpha y^\beta y^\epsilon y^\zeta) = 96e_{\alpha\beta\gamma\delta} e_{\gamma\delta\epsilon\zeta} y^\alpha y^\beta y^\epsilon y^\zeta. \] (B.6)

This term is exactly cancelled by its counter partners in \( [e_2, \Lambda_3] + e_2 + [e_2, \Lambda_3] \). Since
\[ [e_2, \Lambda_3] + e_2 + [e_2, \Lambda_3] = \mathcal{O}(y_6) + 48e_{\alpha\beta\gamma\delta} e_{\gamma\delta\epsilon\zeta} y^\alpha y^\beta y^\epsilon y^\zeta. \]

The first term has no influence on the result. The second term contracts with \( e_2 \) and gives
\[ 96e_{\alpha\beta\gamma\delta} e_{\gamma\delta\epsilon\zeta} e_{\epsilon\zeta\delta\beta} e_{\delta\beta\alpha\gamma} y^\alpha y^\beta y^\epsilon y^\zeta, \]
which exactly cancels the term (B.6). The cancellation of other term related to \( \tilde{e}_{\alpha\beta\gamma\delta} \) can be shown in similar way. The other two terms in (B.4) can be trivially cancelled by counter partners in \( E = [E, \Lambda_3] \). Putting all these results together, we prove the metric defined by star-product trace has local Lorentz transformation invariance.

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