Quantum Time-Frequency Transforms

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Abstract

Time-frequency transforms represent a signal as a mixture of its time domain representation and its frequency domain representation. We present efficient algorithms for the quantum Zak transform and quantum Weyl-Heisenberg transform.

1 Introduction

The Fourier transform is an operator that expresses a time-dependent signal as a sum (or integral) of periodic signals. In other words the Fourier transform changes a function of time $s(t)$ into a function of frequency $S(\omega)$. If a signal is a function of time it said to be in the “time domain” and if it is a function of frequency it is said to be in the “frequency domain”. For signals whose spectrum is changing in time, i.e. nonstationary signals, sometimes the best description is a mixture of the time and frequency components. Signal representations which mix the time and frequency domains are called, naturally enough, “time-frequency representations” and are often used to describe time-varying signals for which the pure frequency or Fourier representation is inadequate [2],[3]. A familiar example of a time-frequency representation is a musical score, which describes when (time) certain notes (frequency) are to be played.

Formally speaking for our present purposes, a quantum signal is simply a quantum state $|\psi\rangle$ where the Hilbert space is the group algebra $\mathbb{C}[G]$ of a finite abelian group $G$. The Quantum Fourier Transform (QFT) is central to the important quantum algorithms for factoring and discrete logarithm. Mathematically speaking, the Quantum Fourier Transform is a linear operator on the Hilbert Space $\mathbb{C}[G]$ which is a change of basis from

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the basis of group elements \(\{g_1, \ldots, g_{|G|}\}\) to the basis of characters of \(G\), \(\{|\chi_1\rangle, |\chi_2\rangle, \ldots, |\chi_{|G|}\rangle\}\).

We present efficient algorithms for quantum versions of the Zak and Weyl-Heisenberg transforms. Both these time-frequency transforms can be seen as generalizations of Fourier transforms and the quantum algorithms make heavy use of the Quantum Fourier Transform. We follow the theory and notation of [4] and recommend this book as background to this material.

2 Zak Transforms

2.1 Background

Let \(A\) be a finite, abelian group, \(A^*\) the group of characters of \(A\) (note: in this paper \(*\) does not mean conjugation), \(B \leq A\) a subgroup of \(A\), \(B_s = \{a^* \in A^*: a^*(b) = 1, b \in B\}\) the dual to \(B\), and \(f \in C[A]\), the group algebra of \(A\). Define

\[
Z(B)f \in C[A \times A^*]
\]

by the formula

\[
Z(B)f(a, a^*) = \sum_{b \in B} f(a + b)\overline{a^*(b)}.
\]

\(F = Z(B)f\) is called the Zak transform of \(f\) over \(B\). A simple calculation shows that \(F(a + b, a^* + b_s) = a^*(b)F(a, a^*)\) where \(b \in B\) and \(b_s \in B_s\). Therefore \(F\) is determined by its values on a set of coset representatives of \(B \times B_s\) in \(A \times A^*\) and thus conceptually we may think of \(F\) as a function on \(T\) where \(T\) is a set of coset representatives. Since

\[
\frac{|A \times A^*|}{|B \times B_s|} = \frac{|A|^2}{|B||B_s|} = |A|
\]

we have the same number of degrees of freedom with which we started. Notice that if \(B\) contains only the identity, i.e. is the trivial subgroup, then \(Z(B)f(a, a^*) = f(a)\) and is basically the identity map. Also notice that if \(B = A\) then \(Z(A)f(0, a^*) = \langle a^*|f\rangle\) and therefore \(Z(A)f\) is basically the Fourier transform of \(f\). So the Zak transform mediates between the time domain and frequency domain depending on the subgroup \(B\).

Consider the function \(f(a_0) = \delta(x - a_0)\) which is 1 on \(a_0\) and 0 otherwise. Applying the above formula for the Zak transform yields \(F(a, a^*) = a^*(a - a_0)\) for \(a \in a_0 + B, a^* \in A^*\) and 0 otherwise. But since \(F\) is determined by its values on a set of coset representatives of \(B \times B_s\) in
Let us introduce such a set of representatives \( T = T_1 \times T_2 = \{(x_i, a^*_j)\} \) where \( T_1 = \{x_i\} \) is a set of coset representatives of \( B \) in \( A \) and \( T_2 = \{a^*_j\} \) is a set of coset representatives of \( B^* \) in \( A^* \). Bearing in mind the above transformation of a delta function, we now offer our definition of the Quantum Zak Transform (QZT) (with respect to \( T \)) by

\[
|a\rangle \mapsto \frac{1}{\sqrt{|B|}} \sum_{a^*_j \in T_2} a^*_j (x_a - a)|x_a\rangle|a^*_j\rangle.
\]

where \( x_a \in T_1 \) is the coset representative of \( a \). Now notice that \( x_a - a \in B \). Therefore \( a^*_j \) is restricted to \( B \) and therefore can be considered to be a character of \( B \), i.e. an element of \( B^* \), and this restriction is independent of the choice of coset representative, i.e. it is natural or canonical. Therefore an equivalent formulation of the QZT is given by

\[
|a\rangle \mapsto \frac{1}{\sqrt{|B|}} \sum_{b^* \in B^*} b^*(x_a - a)|x_a\rangle|b^*\rangle.
\]

The only difference in these two formulations is in the interpretation of the observed content of the second register.

2.2 The Quantum Algorithm

We now show that the QZT is efficiently implementable. Define \( P(B) \) to be the transform

\[
P(B)|a\rangle = |x_a\rangle|x_a - a\rangle
\]

which decomposes \( a \) into its coset representative and the corresponding element of \( B \). \( P \) is clearly unitary and efficiently implementable. After applying \( P(B) \) we apply the Quantum Fourier Transform (over the group \( B \), denoted \( F_B \)) to the second register. This results in the state

\[
\frac{1}{\sqrt{|B|}} \sum_{b^* \in B^*} b^*(x_a - a)|x_a\rangle|b^*\rangle.
\]

Therefore the QZT is simply \( Z(B) = (I \otimes F_B) \circ P(B) \).

3 Weyl-Heisenberg Transforms

3.1 Background

Define \( g_{(x,x^*)}(a) = g(a - x)x^*(a) \) to be the time-frequency translate of \( g \) by \((x,x^*)\) where \( g \in C[A] \). We will use time-frequency translates to form
orthonormal bases so we also require $|g| = 1$. Let $\Delta = B \times B_*$ and $(g, \Delta) = \{g(x,x^*) : (b,b_*) \in \Delta\}$. We call $(g, \Delta)$ a W-H system over $\Delta$ with window $g$. A basic result ([4], Theorem 12.1 corrected version) is that $(g, \Delta)$ is an orthonormal basis of $C[\Lambda]$ if and only if for all $(a,a^*) \in A \times A^*$ we have $|G(a,a^*)| = \sqrt{\frac{|B|}{|A|}}$ where the Zak tranform is taken over $B$. Because von Neumann measurements must be unitary we will restrict our attention to window functions $g$ which satisfy this constraint. Utilizing POVMs one could consider implementing nonorthonormal W-H systems but we will not address this in this note. This orthogonality constraint together with the earlier observation that $G$ is determined by its values on a set of coset representatives of $B \times B_*$ in $A \times A^*$ implies that orthonormal W-H systems are in bijective correspondence with the set of all $|A|$-tuples of complex numbers with modulus $\sqrt{\frac{|B|}{|A|}}$. In this note we will restrict the W-H systems under consideration by assuming that for each $(a,a^*) \in A \times A^*$ the phase of $G(a,a^*)$ is a rational fraction of $2\pi$ which we can compute in polynomial time. Whether or not this last assumption is excessively restrictive would depend on the intended application. Notice that if $g$ is the constant function $g = \frac{1}{\sqrt{|A|}}$ and $\Delta = \{0\} \times A^*$ then $(g, \Delta)$ is the (normalized) Fourier basis, $G(a,a^*) = \frac{1}{\sqrt{|A|}}$ and this restriction holds trivially.

We define the Quantum Weyl-Heisenberg Transform (QWHT) by

$$|\psi\rangle \mapsto \sum_{(b,b^*) \in \Delta} \langle \psi|g(b,b^*)\rangle |b,b^*\rangle.$$ 

In other words, the QWHT expresses $|\psi\rangle$ in the orthonormal basis of time-frequency translates of the window function.

### 3.2 The Quantum Algorithm

Let

$$f = \sum_{(b,b^*) \in \Delta} \alpha(b,b_*) g(b,b^*)$$

i.e. the $\alpha$’s are the coefficients of the WH-expansion of $f$. Define

$$P(a,a^*) = \sum_{(b,b^*) \in \Delta} \alpha(b,b_*) b_*(a) a^*(b).$$

Notice that $P$ is $\Delta$-periodic and that the $\alpha$’s are, by definition, the Fourier coefficients (over $A \times A^*$) of $P$. A fundamental result ([4], Theorem 7.5)
states that $F = GP$. This result suggests an algorithm for computing the WH-coefficients of $f$, namely compute the Fourier coefficients of $P = \frac{F}{G}$.

Define $\Phi(g)$ to be the unitary transformation which acts on the Hilbert space $C[T]$ (recall $T$ is the set of coset representatives of $B \times B^*$ in $A \times A^*$) by

$$ |x_i\rangle|a_j^*\rangle \mapsto \frac{1}{G(x_i, a_j^*)} |x_i\rangle|a_j^*\rangle. $$

Since the phase of $G(x_i, a_j^*)$ is, by assumption, a rational fraction of $2\pi$ computable in polynomial time we may efficiently implement $\Phi(g)$ by the phase kickback technique described in [1]. Finally in order to complete our description of the algorithm, we must assume that we are given an explicit isomorphism between $A$ and $A^*$. These groups are isomorphic, though not canonically so. Therefore in any computational situation we provide an explicit isomorphism by choosing an explicit computational representation of the groups $A$ and $A^*$. This isomorphism induces explicit isomorphisms between $B$ and $B^*$ and between the factor group $A/B$ and $B^*$. We will see shortly how we will employ these three interrelated isomorphisms. We will highlight this interrelation, and abuse notation, by using the symbol $\phi$ to refer to all three of these isomorphisms, allowing for context to make the usage clear. As in the case of the Zak transformation, these isomorphisms are simply reinterpretations of the contents of the registers.

Our QWHT is the sequence $F_{B \times B} \circ \Phi(g) \circ Z(B)$. Let us see how this unitary transformation acts on $|a\rangle$. We have

$$ Z(B)|a\rangle = \frac{1}{\sqrt{|B|}} \sum_{a_j^* \in T_2} a_j^*(x_a - a)|x_a\rangle|a_j^*\rangle $$

and then after applying $\Phi(g)$ we obtain:

$$ \frac{1}{\sqrt{|B|}} \sum_{a_j^* \in T_2} a_j^*(x_a - a) \frac{G(x_a, a_j^*)}{G(x_a, a_j^*)} |x_a\rangle|a_j^*\rangle $$

which by the fundamental result discussed above equals:

$$ \frac{1}{\sqrt{|B|}} \sum_{b^*} P(x_a, b^*)|x_a\rangle|b^*\rangle $$

where we are now considering the contents of the second register to be an element of $B^*$. We now utilize our explicit isomorphisms to reinterpret the
contents of the first register as an element of $B_*$ and the contents of the second register as an element of $B$:

$$\frac{1}{\sqrt{|B|}} \sum_b P(b, b) |b\rangle |b\rangle = \frac{1}{\sqrt{|B|}} \sum_{\phi(a)} P(\phi(x_a), \phi(b^*)) |\phi(x_a)\rangle |\phi(b^*)\rangle.$$ 

By applying the final transformation in the sequence $F_{B_*xB}$ we obtain our desired expansion:

$$\sum_{(b, b^*) \in \Delta} \langle a | g(b, b^*) |b\rangle |b\rangle.$$ 

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References

[1] Cleve, Richard, Artur Ekert, Chiara Macchiavello, and Michele Mosca, Quantum algorithms revisited, Proceedings of the Royal Society of London, Series A, Volume 454, Number 1969, pages 339-354.

[2] Cohen, Leon, Time-Frequency Analysis, Prentice-Hall, Upper Saddle River, NJ, 1995.

[3] Qian, Shie, Dapang Chen, Joint Time-Frequency Analysis—Methods and Applications, Prentice-Hall, Upper Saddle River, NJ, 1996.

[4] Tolimieri, Richard, Myoung An, Time-Frequency Representations. Birkhauser, Boston, 1998.