STATE AND CONTROL PATHS-DEPENDENT STOCHASTIC ZERO-SUM DIFFERENTIAL GAMES: DYNAMIC PROGRAMMING PRINCIPLE AND VISCOSITY SOLUTION OF PATHS-DEPENDENT HAMILTON-JACOBI-ISAACS EQUATION

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Abstract. In this paper, we consider state and control paths-dependent stochastic zero-sum differential games, where the dynamics and the running cost include both the state and control paths of the players. Using the notion of nonanticipative strategies, we define lower and upper value functionals, which are functions of the initial state and control paths of the players. We prove that the value functionals satisfy the dynamic programming principle. The associated lower and upper Hamilton-Jacobi-Isaacs (HJI) equations from the dynamic programming principle are state and control paths-dependent nonlinear second-order partial differential equations. We apply the functional Itô calculus to prove that the lower and upper value functionals are viscosity solutions of (lower and upper) state and control paths-dependent HJI equations, where the notion of viscosity solutions is defined on a compact subset of an $\kappa$-Hölder space introduced in [45, 52]. For the state path-dependent case, the uniqueness of viscosity solutions and the Isaacs condition imply the existence of the game value, and under additional assumptions we prove the uniqueness of classical solutions for the state path-dependent HJI equations.

Key words. stochastic zero-sum differential games, state and control paths-dependent PDEs, functional Itô calculus, viscosity solutions, dynamic programming principles.

AMS subject classifications. 49N70, 49L20, 49L25

1. Introduction. Since the seminal papers by Friedman [23] and Fleming and Souganidis [22], the study of two-player stochastic zero-sum differential games (SZSDGs) and nonzero-sum stochastic differential games (SDGs) has grown rapidly in various aspects; see [1, 2, 4, 5, 6, 7, 12, 17, 18, 24, 25, 30, 32, 33, 34, 35, 43, 44, 49] and the references therein. Specifically, Friedman in [23] considered SDGs with classical (or smooth) solutions of the associated partial differential equation (PDE) from dynamic programming to prove the existence of the Nash equilibrium and the game value. Fleming and Souganidis in [22] studied SZSDGs in the Markovian framework with nonanticipative strategies introduced in [19, 20]. They proved that the lower and upper value functions are unique viscosity solutions (in the sense of [10]) for lower and upper Hamilton-Jacobi-Isaacs (HJI) equations obtained from dynamic programming, which are nonlinear second-order partial differential equations (PDEs). They also showed the existence of the game value under the Isaacs condition. Later, the results of [22] were extended by Buckdahn and Li in [6], who defined the objective functional by the backward stochastic differential equation (BSDE). They used the backward semi-group associated with the BSDE introduced in [36] to obtain the generalized results of [22]. The weak formulation of SZSDGs and SDGs with random coefficients was considered in [17, 24, 25], where the existence of the open-loop type saddle-point equilibrium as well as the game value was established.

Recently, state path-dependent SZSDGs have been studied extensively in the literature to consider a general class of SZSDGs including SZSDGs with delayed systems

*Submitted to the editors DATE.

Funding: This research was supported in part by the National Research Foundation of Korea (NRF) Grant funded by the Ministry of Science and ICT, South Korea (NRF-2017R1E1A1A03070936, NRF-2017R1A5A1015311).

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in the state variable. This extends the results related to [6, 22] to the non-Markovian framework. Unlike [6, 22], for the path-dependent or non-Markovian case, the associated (lower and upper) HJI equations from the dynamic programming principle are the so-called (state) path-dependent PDEs (PPDEs) defined on a space of continuous functions, which is an infinite dimensional Banach space. Hence, the approach for the Hilbert space in [11, 31] cannot be applied to show the existence (and uniqueness) of viscosity solutions. In [38, 39], state path-dependent SZSDGs in weak formulation were studied, where the players are restricted to observe the state feedback information. They proved the existence and uniqueness of viscosity solutions in the sense of [14, 15, 16] for (lower and upper) state path-dependent HJI equations. Zhang in [50] considered path-dependent SZSDGs with strong formulation, where the existence of the game value was established via the approximating technique of the (lower and upper) state path-dependent HJI equations.

In this paper, we consider state and control paths-dependent stochastic zero-sum differential games (SZSDGs), where the dynamics and the running cost are dependent on both the state and control paths of the players. Note that the paper can be viewed as a generalization of [50] to the state and control paths-dependent case, and of [41] to the two-player SZSDG framework. We mention that the viscosity solution of state path-dependent HJI equations was not considered in [50]. In [41], the existence of classical (smooth) solutions of the state and control paths-dependent Hamilton-Jacobi-Bellman equation was assumed to establish the verification theorem for the state and control paths-dependent stochastic optimal control problem. Note also that the references [38, 39] do not deal with the control path-dependent case.

By using the notion of nonanticipative strategies, the lower and upper value functionals are defined, whereby these are functions of the initial state and control paths of the players. We prove that the (lower and upper) value functionals satisfy the dynamic programming principle. We also obtain the continuity property of the (lower and upper) value functionals in their arguments.

The associated lower and upper state and control paths-dependent Hamilton-Jacobi-Isaacs (PHJI) equations from the dynamic programming principle are state and control paths-dependent nonlinear second-order PDEs (PPDEs), whose structures are fundamentally different from those of PPDEs or state path-dependent HJI equations in [14, 15, 16, 37, 38, 39, 45, 52]. In particular, the time derivative term also depends on the control of the players, which is included in $\sup_{v \in V} \inf_{u \in U}$ of the lower PHJI equation and $\inf_{u \in U} \sup_{v \in V}$ of the upper PHJI equation (see (5.2) and (5.3)). We apply the functional Itô calculus introduced in [8, 9, 13] to prove that the lower and upper value functionals are viscosity solutions of the (lower and upper) PHJI equations, where the notion of viscosity solutions is defined on a compact subset of an $\kappa$-Hölder space introduced in [45, 52]. Specifically, the notion of viscosity solutions is defined on a compact set $\mathbb{C}^{\kappa, \mu, \mu_0}$, where $\kappa \in (0, \frac{1}{2})$ and $\mu, \mu_0 > 0$, which provides the precise estimate between the initial state path and its perturbed one. This initial state path perturbation is essential to prevent starting the dynamic programming at the boundary of $\mathbb{C}^{\kappa, \mu, \mu_0}$ (see [45, Remark 6]). Then using the functional Itô calculus and the dynamic programming principle, we show that the (lower and upper) value functionals are viscosity solutions of the corresponding PHJI equations. We mention that the uniqueness of viscosity solutions in our paper will be investigated in a future research study. Instead, we provide the uniqueness of classical solutions for the PHJI equations. The definition of viscosity solutions in this paper does not involve a nonlinear expectation included in the semi-jets in [14, 15, 16, 38, 39], which require additional assumptions to prove the existence and uniqueness of viscosity solutions.
For the state path-dependent case, we show the existence of the game value when the viscosity solution is unique and the Isaacs condition holds. This does not require the approximating technique of the (lower and upper) PHJI equations to the state dependent (not path-dependent) HJI equations studied in [50]. We provide the uniqueness result of classical solutions for the PHJI equations. In particular, under an additional assumption (see Assumption 5.9), we prove the comparison principle of classical sub- and super-solutions of the lower and upper PHJI equations, which further implies the uniqueness of classical solutions.

The paper is organized as follows. In Section 2, we provide notation and preliminary results of the functional Itô calculus introduced in [8, 9, 13]. The problem formulation of state and control paths-dependent SZSDGs is given in Section 3. In Section 4, we show that the lower and upper value functionals satisfy the dynamic programming principle and obtain their regularity. In Section 5, we introduce the lower and upper PHJI equations and prove that the value functionals are viscosity solutions of the corresponding PHJI equations. Several potential future research problems are also discussed at the end of Section 5.

2. Notation and Preliminaries. The n-dimensional Euclidean space is denoted by \( \mathbb{R}^n \), and the transpose of a vector \( x \in \mathbb{R}^n \) by \( x^\top \). The inner product of \( x, y \in \mathbb{R}^n \) is denoted by \( \langle x, y \rangle := x^\top y \), and the Euclidean norm of \( x \in \mathbb{R}^n \) by \( |x| := (x, x)^{\frac{1}{2}} \). Let \( \operatorname{Tr}(X) \) be the trace operator of a square matrix \( X \in \mathbb{R}^{n \times n} \). Let \( \mathbb{1} \) be the indicator function. Let \( \mathbb{S}^n \) be the set of \( n \times n \) symmetric matrices.

We introduce the calculus of path-dependent functionals in [8, 9, 13]; see also [14, 41, 45]. For a fixed \( T > 0 \) and \( t \in [0, T] \), let \( \Lambda^n_t := C([0, t], \mathbb{R}^n) \) be the set of \( \mathbb{R}^n \)-valued continuous functions on \( [0, t] \), and \( \hat{\Lambda}^n_t := D([0, t], \mathbb{R}^n) \) the set of \( \mathbb{R}^n \)-valued càdlàg functions on \( [0, t] \). Let \( \Lambda^n_{t,t} := C([0, t], E) \) and \( \Lambda^n_{t,t} := D([0, t], E) \) for \( E \subset \mathbb{R}^n \). Let \( \Lambda^n_{t,t} := \Lambda^n_{t,t} \times \Lambda^n_{t,t} \), \( \Lambda^n := \cup_{t \in [0, T]} \Lambda^n_t \), and \( \Lambda^n_{t,t} := \cup_{t \in [0, T]} \Lambda^n_{t,t} \). For any functions in \( \Lambda^n \), the capital letter stands for the path and the lowercase letter will denote the value of the function at a specific time. Specifically, for \( A \in \Lambda^n_t \), \( a_t \) stands for the value of \( A \) at \( t \in [0, T] \), and for \( t \in [0, T] \), we denote \( A_t := \{ a_r, r \in [0, t] \} \in \Lambda^n_t \) by the path of the corresponding function up to time \( t \in [0, T] \). A similar notation applies to \( \hat{\Lambda}^n \). Note that \( \Lambda^n \subset \hat{\Lambda}^n \).

For \( A \in \hat{\Lambda}^n \) and \( \delta > 0 \), we introduce the following notation:

\[
A_{t, \delta t}[s] := \begin{cases} a_s & \text{if } s \in [0, t) \\ a_t & \text{if } s \in [t, t + \delta t] \end{cases}, \quad A_t^{(h)}[s] := \begin{cases} a_s & \text{if } s \in [0, t) \\ a_t + h & \text{if } s = t. \end{cases}
\]

Note that \( A_{t, \delta t} \) is the flat extension, and \( A_t^{(h)} \) is the vertical extension of the path \( A \). The metric on \( \hat{\Lambda}^n \) is defined for \( A_t, B_{t'} \in \hat{\Lambda}^n \) with \( t, t' \in [0, T] \) and \( t \leq t' \),

\[
d_\infty(A_t, B_{t'}) := |t - t'| + \| A_t, t' - t - B_{t'} \|_\infty,
\]

where \( \| \cdot \|_\infty \) is the norm in \( \Lambda^n_{t,t} \) defined by

\[
\| B_t \|_\infty := \sup_{r \in [0, t]} |b_r|.
\]

Note that \((\hat{\Lambda}^n, d_\infty)\) is a complete metric space, and \((\hat{\Lambda}^n_{t,t}, \| \cdot \|_\infty)\) is a Banach space. The same results hold for \((\Lambda^n, d_\infty)\) and \((\Lambda^n_{t,t}, \| \cdot \|_\infty)\).

**Definition 2.1.** A functional is any function \( f : \hat{\Lambda}^n \to \mathbb{R} \). The functional \( f \) is said to be continuous at \( A_t \in \hat{\Lambda}^n \), if for each \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for
each $A'_t \in \hat{\Lambda}^n$, $d_\infty(A_t, A'_t) < \delta$ implies $|f(A_t) - f(A'_t)| < \epsilon$. Let $\mathcal{C}(\hat{\Lambda}^n)$ be the set of real-valued continuous functionals for every path $A_t \in \hat{\Lambda}^n$ under $d_\infty$. The set $\mathcal{C}(\hat{\Lambda}^n)$ is defined in a similar way.

Next, we introduce the concept of time and space derivatives of the functional $f$.

**Definition 2.2.** (i) Let $f : \hat{\Lambda}^n \to \mathbb{R}$ be the functional. The time derivative (or horizontal derivative) of $f$ at $A_t$ is defined by

$$\partial_t f(A_t) := \lim_{\delta t \downarrow 0} \frac{f(A_{t, \delta t}) - f(A_t)}{\delta t}.$$  

If the limit exists for all $A_t \in \hat{\Lambda}^n$, a functional $\partial_t f : \hat{\Lambda}^n \to \mathbb{R}$ is called the time derivative of $f$.

(ii) Let $f : \hat{\Lambda}^n \to \mathbb{R}$ be the functional. The space derivative (or vertical derivative) of $f$ at $A_t$ is defined by

$$\partial_x f(A_t) := [\partial^{(1)}_{x_1} f(A_t) \cdots \partial^{(n)}_{x_n} f(A_t)],$$

where for $e_i$, $i = 1, \ldots, n$, being a coordinate unit vector of $\mathbb{R}^n$,

$$\partial^{(i)}_{x_i} f(A_t) := \lim_{h \downarrow 0} \frac{f(A_{t, h e_i}) - f(A_t)}{h}.$$  

If the limit exists for all $A_t \in \hat{\Lambda}^n$ and $i = 1, \ldots, n$, a functional $\partial_x f : \hat{\Lambda}^n \to \mathbb{R}^n$ is called the space derivative of $f$. Note that the second-order space derivative (Hessian) $\partial_{xx} f$ can be defined in a similar way, where $\partial_{xx} f : \hat{\Lambda}^n \to \mathbb{S}^n$.

**Remark 2.3.** If a functional $f$ is differentiable in the sense of Definition 2.2 and depends only on a function (not its path), i.e., $f(A_t) = f(t, a_t)$, then the notion of derivatives in Definition 2.2 is equivalent to those for the classical ones.

From Definition 2.2, let $\mathcal{C}^{k,l}(\hat{\Lambda}^n)$ be the set of functionals such that for $f \in \mathcal{C}^{k,l}(\hat{\Lambda}^n)$, $f$ is $k$ times time differentiable and $l$ times space differentiable in $\hat{\Lambda}^n$, where all its derivatives are continuous in the sense of Definition 2.1. The set $\mathcal{C}^{k,l}(\Lambda^n)$ is defined similarly. We mention that these sets are well defined in view of [45, Definition 2.4 and Remark 2] (see also [14, Theorem 2.4] and [8, 9, 13]).

**Definition 2.4.** Let $A_t \in \Lambda^n$. For any $\kappa \in (0, 1]$, $A$ is an $\kappa$-Hölder continuous path if the following limit exists

$$[A_t]_\kappa := \sup_{0 \leq s \leq r \leq t} \frac{|a_s - a_r|}{|s - r|^{\kappa}} < \infty,$$

where we call $[A_t]_\kappa$ the $\kappa$-Hölder modulus of $A_t$. The $\kappa$-Hölder space is defined by $\mathcal{C}^\kappa(\Lambda^n) := \{ A_t \in \Lambda^n : [A_t]_\kappa < \infty \}$. The $\kappa$-Hölder space with $\mu > 0$ is defined by

$$\mathcal{C}^{\kappa,\mu}(\Lambda^n) := \{ A_t \in \Lambda^n : [A_t]_\kappa \leq \mu \}.$$  

The $\kappa$-Hölder space with $\mu > 0$ and $\mu_0 > 0$ is defined by

$$\mathcal{C}^{\kappa,\mu,\mu_0}(\Lambda^n) := \{ A_t \in \Lambda^n : [A_t]_\kappa \leq \mu, \| A_t \|_\infty \leq \mu_0 \}.$$  

We can easily see that $\mathcal{C}^\kappa(\Lambda^n) \subset \Lambda^n$. The spaces $\mathcal{C}^{\kappa,\mu}(\Lambda^n)$ and $\mathcal{C}^{\kappa,\mu,\mu_0}(\Lambda^n)$ have the following topological property [45, Proposition 1]:
Lemma 2.5. For \( \kappa \in (0, 1] \), \( C^{\kappa, \mu}(\Lambda^n) \) and \( C^{\kappa, \mu, \rho_0}(\Lambda^n) \) are compact subsets of \( (\Lambda^n, d_\infty) \).

Similar to Definition 2.4, we introduce the notion of the Hölder continuity for the functional \( f \).

Definition 2.6. Let \( f : \Lambda^n \to \mathbb{R} \) be the functional. For \( \kappa \in (0, 1] \), \( f \) is Hölder continuous if the following limit exists:

\[
[f]_{\kappa; \Lambda^n} := \sup_{A_t, A_t' \in \Lambda^n, A_t \neq A_t'} \frac{|f(A_t) - f(A_t')|}{d_\infty(A_t, A_t')}.
\]

Assume that \( f \in C^{1,2}(\Lambda^n) \). We define \( |f|_{\kappa; \Lambda^n} := \sup_{A_t \in \Lambda^n} |f(A_t)| + [f]_{\kappa; \Lambda^n} \) and

\[
|f|_{2, \kappa; \Lambda^n} := |f|_{\kappa; \Lambda^n} + |\partial_t f|_{\kappa; \Lambda^n} + |\partial_x f|_{\kappa; \Lambda^n} + |\partial_{xx} f|_{\kappa; \Lambda^n}.
\]

The set of functionals such that \( (2.1) \) is finite is denoted by \( C^{1,2}_0(\Lambda^n) \).

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete \( p \)-dimensional probability space satisfying the usual condition [27]. Let \( B \) be the standard \( p \)-dimensional Brownian motion defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \). Let \( \mathbb{F} = \{ \mathcal{F}_t, 0 \leq t \leq T \} \) be the standard natural filtration generated by the Brownian motion \( B \) augmented by all the \( \mathbb{P} \)-null sets of \( \mathcal{F} \). Let \( C^2(\Omega, \mathcal{F}_t, \mathbb{R}^n) \) be the set of \( \mathbb{R}^n \)-valued \( \mathcal{F}_t \)-measurable random vectors such that \( g \in C^2(\Omega, \mathcal{F}_t, \mathbb{R}^n) \) satisfies \( \mathbb{E}[|g|^2] < \infty \). Let \( C^2_f([t, T], \mathbb{R}^n) \) be the set of \( \mathbb{R}^n \)-valued \( \mathcal{F} \)-adapted stochastic processes such that \( g \in C^2_f([t, T], \mathbb{R}^n) \) satisfies \( \mathbb{E}[\sup_{s \in [t, T]} |g(s)|^2] < \infty \).

Let \( X \in \Lambda^n \) be the \( n \)-dimensional \( \mathcal{F} \)-adapted stochastic process, which is defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \). Note that \( X \) can be viewed as a mapping from \( \Omega \) to \( \Lambda^n \). Using the notation, \( x_t \) is the value of \( X \) at time \( t \in [0, T] \), and for \( t \in [0, T] \), \( X_t := \{ x_r, r \in [0, t] \} \in \Lambda^n \) is the corresponding path up to time \( t \in [0, T] \). We can see that for any functional \( f \in C(\Lambda^n) \), \( \{ f(X_t), t \in [0, T] \} \) is an \( \mathcal{F} \)-adapted stochastic process.

We now state the functional Itô formula in [8, 9, 13]

Lemma 2.7. Suppose that \( X \) is continuous semi-martingale, and \( f \in C^{1,2}(\Lambda^n) \). Then for any \( t \in [0, T] \),

\[
f(X_t) = f(X_0) + \int_0^t \partial_t f(X_r)dr + \int_0^t \partial_x f(X_r)dx_r + \frac{1}{2} \int_0^t \partial_{xx} f(X_r)d\langle x \rangle_r, \quad \mathbb{P}\text{-a.s.}
\]

3. Problem Formulation. This section provides the precise problem formulation of state and control paths-dependent SZSDGs.

Let \( U \) be the set of \( U \)-valued \( \mathcal{F}_t \)-progressively measurable and càdlàg processes, where \( U \subset \mathbb{R}^m \), which is the set of control processes for Player 1. The set of control processes for Player 2, \( V \), is defined similarly with \( V \subset \mathbb{R}^l \). It is assumed that \( U \) and \( V \) are compact metric spaces with the standard Euclidean norm. The precise definitions of \( U \) and \( V \) are given later.

The state and control paths-dependent stochastic differential equation (SDE) is given by

\[
\begin{align*}
dx_t^{X_1, A, U, V} & = f(X_t^{A, U, V}, U_s, V_s)ds + \sigma(X_t^{A, U, V}, U_s, V_s)dB_s, \quad s \in (t, \infty), \\
X_t^{A, U, V} & = A_t \in \Lambda^n_t,
\end{align*}
\]
where $X^{t,A;U,V}_s := \{ x^{t,A;U,V}_r : r \in [0,s] \} \in \Lambda^n$ is the whole path of the controlled state process from time 0 to $s$, and $U_s := \{ u_r \in U : r \in [0,s] \} \in \hat{\Lambda}^U \subset \hat{\Lambda}^n$ and $V_s := \{ v_r \in V : r \in [0,s] \} \in \hat{\Lambda}^V \subset \hat{\Lambda}^n$ are paths of the control processes of Players 1 and 2, respectively. In (3.1), $A_t \in \Lambda^n$ is the initial condition that is a continuous path starting from time $t = 0$. Let $\Lambda := \Lambda^n$ and $\hat{\Lambda} := \hat{\Lambda}^U \times \hat{\Lambda}^V$.

The state and control paths-dependent backward stochastic differential equation (BSDE) is given by

$$
\begin{cases}
    & dy^{t,A;U,V}_s = -l(X^{t,A;U,V}_s, y^{t,A;U,V}_s, q^{t,A;U,V}_s, U_s, V_s)ds \\
    & y^{t,A;U,V}_T = m(X^{t,A;U,V}_T),
\end{cases}
$$

(3.2)

where the pair $(y^{t,A;U,V}_s, q^{t,A;U,V}_s) \in \mathbb{R} \times \mathbb{R}^{1 \times p}$ is the solution of the BSDE. Note that the BSDE in (3.2) is coupled with the (forward) SDE in (3.1). Below, the BSDE in (3.2) is used to define the objective functional of Players 1 and 2.

We introduce the following assumption:

**Assumption 3.1.** In (3.1), the coefficients $f : \Lambda \times \hat{\Lambda} \to \mathbb{R}^n$ and $\sigma : \Lambda \times \hat{\Lambda} \to \mathbb{R}^{n \times p}$ are bounded. Furthermore, the running and terminal costs in (3.2), $l : \Lambda \times \mathbb{R} \times \mathbb{R}^{1 \times p} \times \Lambda \to \mathbb{R}$ and $m : \Lambda_T \to \mathbb{R}$, respectively, are bounded. There exists a constant $L > 0$ such that for $s_i \in [0,T]$ and $(X^{t_i}_i, U^{t_i}_i, V^{t_i}_i, y^{t_i}_i, q^{t_i}_i) \in \Lambda_T \times \hat{\Lambda} \times \mathbb{R} \times \mathbb{R}^{1 \times p}$, $i = 1, 2$, the following conditions hold:

$$
|f(X^{t_i}_1, U^{t_i}_1, V^{t_i}_1) - f(X^{t_i}_2, U^{t_i}_2, V^{t_i}_2)| \\
\leq L(d_\infty(X^{t_i}_1, X^{t_i}_2) + d_\infty(U^{t_i}_1, U^{t_i}_2) + d_\infty(V^{t_i}_1, V^{t_i}_2)) \\
|\sigma(X^{t_i}_1, U^{t_i}_1, V^{t_i}_1) - \sigma(X^{t_i}_2, U^{t_i}_2, V^{t_i}_2)| \\
\leq L(d_\infty(X^{t_i}_1, X^{t_i}_2) + d_\infty(U^{t_i}_1, U^{t_i}_2) + d_\infty(V^{t_i}_1, V^{t_i}_2)) \\
|l(X^{t_i}_1, y^{t_i}_1, q^{t_i}_1, U^{t_i}_1, V^{t_i}_1) - l(X^{t_i}_2, y^{t_i}_2, q^{t_i}_2, U^{t_i}_2, V^{t_i}_2)| \\
\leq L(d_\infty(X^{t_i}_1, X^{t_i}_2) + d_\infty(U^{t_i}_1, U^{t_i}_2) + d_\infty(V^{t_i}_1, V^{t_i}_2) + |y^{t_i}_1 - y^{t_i}_2| + |q^{t_i}_1 - q^{t_i}_2|) \\
|m(X^{t_i}_1) - m(X^{t_i}_2)| \leq L\|X^{t_i}_1 - X^{t_i}_2\|_\infty.
$$

Based on [6, 29, 45, 47, 50, 51], we have the following result:

**Lemma 3.2.** Suppose that Assumption 3.1 holds. Then, the following hold:

(i) For $t \in [0,T)$, $A_t \in \Lambda_t$ and $(U, V) \in \hat{\Lambda}$, the SDE in (3.1) and the BSDE in (3.2) admit unique strong solutions, $X^{t,A;U,V}$ with $E[\|X^{t,A;U,V}\|_\infty^2|F_t] < \infty$ and $(y^{t,A;U,V}, q^{t,A;U,V}) \in C_p([t,T], \mathbb{R}^n) \times L_\mathbb{F}^p([t,T], \mathbb{R}^{1 \times p})$, respectively.

(ii) For $t \in [0,T)$, $t_1, t_2 \in [t,T)$ with $t_2 \geq t_1$, $A^{t_1}_t \in \Lambda_t$, and $(U^{t_1}, V^{t_1}) \in \hat{\Lambda}$, $i = 1, 2$, there exists a constant $C > 0$, dependent on the Lipschitz constant $L$ in Assumption 3.1, such that

$$
E[\|X^{t,A;U,V}_s\|_\infty^2|F_t] \leq C(1 + \|A^1_t\|_\infty^2) \\
E[\|X^{t,A;U,V}_s - A^{t_1}_t\|_\infty^2|F_t] \leq C(1 + \|A^1_t\|_\infty^2)(t_2 - t_1) \\
E[\|X^{t,A;U,V}_s - X^{t,A;U,V}_s\|_\infty^2|F_t] \\
\leq C\|A^1_t - A^2_t\|_\infty + C\mathbb{E}\left[\int_t^T \|U^{t_1}_r - U^{t_2}_r\|_\infty^2 + \|V^{t_1}_r - V^{t_2}_r\|_\infty^2\]dr\right].
$$
(iii) For $t \in [0,T)$, $t_1, t_2 \in [t, T)$ with $t_2 \geq t_1$, $A_i^t \in \Lambda_i$, and $(U^i, V^i) \in \bar{\Lambda}$, $i = 1, 2$, there exists a constant $C > 0$, dependent on the Lipschitz constant $L$ in Assumption 3.1, such that

$$E\left[\sup_{s \in [t,T]} |y_s^{t, A_i^t, U^i, V^i}|^2 + \int_t^T |q_s^{t, A_i^t, U^i, V^i}|^2 \, dr \bigg| \mathcal{F}_t \right] \leq C(1 + \|A_i^t\|_\infty^2)$$

$$E\left[\sup_{s \in [t,t_2]} |y_s^{t_1, A_i^{t_1}, U^i, V^i} - y_{t_1}^{t_1, A_i^{t_1}, U^i, V^i}|^2 \bigg| \mathcal{F}_t \right] \leq C(1 + \|A_i^{t_1}\|_\infty^2)(t_2 - t_1)$$

$$E\left[\sup_{s \in [t,T]} |y_s^{t_1, A_i^{t_1}, U^i, V^i} - y_s^{t_2, A_i^{t_2}, U^i, V^i}|^2 \bigg| \mathcal{F}_t \right] \leq C\|A_i^{t_1} - A_i^{t_2}\|_\infty^2 + CE\left[\int_t^T [\|U_s^{t_1} - U_s^{t_2}\|_2^2 + \|V_s^{t_1} - V_s^{t_2}\|_2^2] \, dr \bigg| \mathcal{F}_t \right].$$

(iv) Suppose that $l^{(1)}$ and $l^{(2)}$ are coefficients of the BSDE in (3.2) satisfying Assumption 3.1, and $\eta^{(1)}$, $\eta^{(2)} \in L^2(\Omega, \mathcal{F}_T, \mathbb{R})$ are the corresponding terminal conditions. Let $(y^{(1)}, q^{(1)})$ and $(y^{(2)}, q^{(2)})$ be solutions of the BSDE in (3.2) with $(l^{(1)}, \eta^{(1)})$ and $(l^{(2)}, \eta^{(2)})$, respectively (note that $y_T^{(1)} = \eta^{(1)}$ and $y_T^{(2)} = \eta^{(2)}$). If $\eta^{(1)} \geq \eta^{(2)}$ and $l^{(1)} \geq l^{(2)}$, then $y^{(1)} \geq y^{(2)}$, a.s., for $s \in [t, T]$.

The objective functional of Players 1 and 2 is given by

$$J(t, A_i; U, V) = y_t^{t, A_i; U, V}, \quad t \in [0, T],$$

where $y$ is the first component of the BSDE in (3.2). Note that $J(T, A_i; U, V) = y_T^{t, A_i; U, V} = m(X_T^{t, A_i; U, V})$.

**Remark 3.3.** Suppose that $l$ in (3.2) is independent of $y$ and $q$. Then (3.3) becomes

$$J_t(t, A_i; U, V) = y_t^{t, A_i; U, V} = E\left[\int_t^T l(X_s^{t, A_i; U, V}, U_s, V_s) \, ds + m(X_T^{t, A_i; U, V}) \bigg| \mathcal{F}_t \right].$$

The admissible control of Players 1 and 2 is defined as follows:

**Definition 3.4.** For $t \in [0,T]$, the admissible control for Player 1 (respectively, Player 2) is defined such that $u := \{u_r \in U, \ r \in [t, T]\}$ (respectively, $v := \{v_r \in V, \ r \in [t, T]\}$) is an $U$-valued (respectively, $V$-valued) $\mathbb{F}$-progressively measurable and càdlàg process in $L^2([t,T], U)$ (respectively, $L^2([t,T], V)$). The set of admissible controls of Player 1 (respectively, Player 2) is denoted by $\mathcal{U}[t,T]$ (respectively, $\mathcal{V}[t,T]$).

We identify two admissible control processes of Player 1 (respectively, Player 2) $u$ and $\bar{u}$ in $\mathcal{U}[t,T]$ (respectively, $v$ and $\bar{v}$ in $\mathcal{V}[t,T]$) and write $u \equiv \bar{u}$ (respectively, $v \equiv \bar{v}$) on $[t,T]$, if $\mathbb{P}(u = \bar{u} \ a.e \text{ in } [t,T]) = 1$ (respectively, $\mathbb{P}(v = \bar{v} \ a.e \text{ in } [t,T]) = 1$).

Given the definition of the admissible controls of Players 1 and 2, we introduce the concept of nonanticipative strategies for Players 1 and 2.

**Definition 3.5.** For $t \in [0,T]$, a nonanticipative strategy for Player 1 (respectively, Player 2) is a mapping $\alpha : \mathcal{V}[t,T] \to \mathcal{U}[t,T]$ (respectively, $\beta : \mathcal{V}[t,T] \to \mathcal{U}[t,T]$) such that for any $\mathbb{F}$-stopping time $\tau : \Omega \to [t,T]$ and any $u^1, u^2 \in \mathcal{U}$ with $u^1 = u^2$ on $[t,\tau]$ (respectively, $v^1, v^2 \in \mathcal{V}$ with $v^1 = v^2$ on $[t,\tau]$), it holds that $\alpha(u^1) = \alpha(u^2)$ on $[t,\tau]$ (respectively, $\beta(u^1) = \beta(u^2)$ on $[t,\tau]$). The set of admissible strategies for Player 1 (respectively, Player 2) is denoted by $\mathcal{A}[t,T]$ (respectively, $\mathcal{B}[t,T]$).
The following notation captures control paths-dependent SZSDGs: for $t \in [0, T)$, 

$$
(Z_t \otimes u)[s] := \begin{cases} 
z_s, & \text{if } s \in [0, t) \\
u_s, & \text{if } s \in [t, T], 
\end{cases}
(W_t \otimes v)[s] := \begin{cases} 
w_s, & \text{if } s \in [0, t) \\
v_s, & \text{if } s \in [t, T],
\end{cases}
$$

where $Z_t := \{z_r, \ r \in [0, t]\} \in \hat{A}^U_t$, $u \in \mathcal{U}[t, T]$, $W_t := \{w_r, \ r \in [0, t]\} \in \hat{A}^V_t$, and $v \in \mathcal{V}[t, T]$. Note that $(Z_t \otimes u) \in \mathcal{U}[0, T]$ and $(W_t \otimes v) \in \mathcal{V}[0, T]$.

With the help of the notation in (3.4), the objective functional of (3.3) that includes path of the controls of Players 1 and 2 can be written as follows:

$$
J(t, A_t; Z_t \otimes u, W_t \otimes v) = y^{t, A_t; Z_t \otimes u, W_t \otimes v}.
$$

Then for $(t, A_t) \in [0, T] \times A_t$ and $(Z_t, W_t) \in \hat{A}_t$, the lower value functional of (3.5) for the state and control paths-dependent SZSDG can be defined by

$$
\mathcal{L}(A_t; Z_t, W_t) = \text{ess inf}_{\alpha \in A_t[t, T]} \text{ess sup}_{v \in \mathcal{V}[t, T]} J(t, A_t; Z_t \otimes \alpha(W_t \otimes v), W_t \otimes v)
$$

$$
= \text{ess inf}_{\alpha \in A_t[t, T]} \text{ess sup}_{v \in \mathcal{V}[t, T]} J(t, A_t; Z_t \otimes \alpha(v), W_t \otimes v),
$$

where the last equality follows from (3.4). Moreover, for $(t, A_t) \in [0, T] \times A_t$ and $(Z_t, W_t) \in \hat{A}_t$, the upper value functional of (3.5) is defined by

$$
\mathcal{U}(A_t; Z_t, W_t) = \text{ess sup}_{\beta \in \mathcal{B}[t, T]} \text{ess inf}_{u \in \mathcal{U}[t, T]} J(t, A_t; Z_t \otimes u, W_t \otimes \beta(Z_t \otimes u))
$$

$$
= \text{ess sup}_{\beta \in \mathcal{B}[t, T]} \text{ess inf}_{u \in \mathcal{U}[t, T]} J(t, A_t; Z_t \otimes u, W_t \otimes \beta(u)).
$$

Note that $\mathcal{L}(A_T; Z_T, W_T) = \mathcal{U}(A_T; Z_T, W_T) = m(A_T)$.

We state some remarks on various formulations of (paths-dependent) SZSDGs.

Remark 3.6. (1) One might formulate SZSDGs with control against control, in which the players can select admissible controls individually. Although this formulation is quite similar to stochastic optimal control and therefore can define the saddle-point equilibrium, the dynamic programming principle cannot be established and the value of the game may fail to exist; see [38, Appendix E] and [39, Example 2.1]. Note that under this formulation, the necessary condition for the existence of the saddle-point equilibrium in terms of the (stochastic) maximum principle was studied in [34, 48].

(2) The notion of nonanticipative strategies in Definition 3.5 is used in various zero-sum differential games; see [1, 2, 4, 5, 6, 22, 30, 33, 49, 50]. This is the strong formulation with strategy against control. Under this formulation, it is possible to establish the dynamic programming principle, to show the existence of viscosity solutions of Hamilton-Jacobi-Isaacs (HJI) equations, and to identify the existence of the game value under the Isaacs condition. We also note that instead of the strong formulation with strategy against control, one can use the notion of nonanticipative strategy with delay, which is still asymmetric information between the players that allows to show the existence of the (approximated) saddle-point equilibrium and the game value [5, 7, 50].

(3) Instead of the strong formulation with strategy against control, SZSDGs can be considered in weak formulation [17, 24, 25, 28, 38, 39]. Note that in [38, 39], the players are restricted to observe the state feedback information. Since the information is symmetric, it is convenient to define the saddle-point equilibrium and...
show the existence of the game value. The dynamic programming principle can also be obtained. However, the notion of viscosity solutions of the HJI equation requires the complex nonlinear expectation and some additional assumptions are required to show the existence of viscosity solutions [38, 39].

The next remark is on the (lower and upper) value functionals.

**Remark 3.7.** (1) We can see that the value functionals in (3.6) and (3.7) depend on initial paths of both state and control of the players. Consider the situation when the path-dependence is only in the state variable, i.e., \( f: \Lambda \times U \times V \to \mathbb{R}^n \), \( \sigma: \Lambda \times U \times V \to \mathbb{R}^{n \times p} \), and \( l: \Lambda \times \mathbb{R} \times \mathbb{R}^{1 \times p} \times U \times V \to \mathbb{R} \). Then, the value functionals can be written independent of \( Z \) and \( W \):

\[
L(A_t) = \text{ess inf}_{\alpha \in A(t,T)} \text{ess sup}_{v \in V(t,T]} J(t, A_t; \alpha(v), v)
\]

\[
U(A_t) = \text{ess sup}_{\beta \in B(t,T]} \text{ess inf}_{u \in U(t,T]} J(t, A_t; u, \beta(u)).
\]

This case was studied in [50]. In addition, for the state and control path-independent case, i.e., the SZSDG in the Markovian formulation, the value functionals are defined by

\[
L(t, x) = \text{ess inf}_{\alpha \in A(t,T)} \text{ess sup}_{V(t,T]} J(t, x; \alpha(v), v)
\]

\[
U(t, x) = \text{ess sup}_{\beta \in B(t,T]} \text{ess inf}_{U(t,T]} J(t, x; u, \beta(u)),
\]

for any initial state \( x \in \mathbb{R}^n \) and \( t \in [0, T] \). This has been considered in the various literature (see, e.g., [6, 22]).

(2) For the value functionals in (3.6) and (3.7), the essential supremum and the essential infimum are taken with respect to indexed family of random variables; see the precise notion in [26, Appendix A].

**4. Dynamic Programming Principle.** This section establishes the dynamic programming principle for the lower and upper value functionals.

We first state properties of the value functionals. The proof of the following result is similar to that in [6, Proposition 3.3] and [29, Proposition 3.3].

**Proposition 4.1.** Assume that Assumption 3.1 holds. The lower and upper value functionals \( L \) and \( U \) in (3.6) and (3.7), respectively, are \( \mathcal{F}_t \)-measurable random variables. In fact, they are deterministic functionals on \( \Lambda \times \hat{\Lambda} \).

In view of Assumption 3.1 the estimates in Lemma 3.2, and (3.4), the following result holds:

**Lemma 4.2.** Suppose that Assumption 3.1 holds. For any \( t \in [0, T] \), \( A_i^t \in \Lambda_t \) and \( (Z_i^1, W_i^1) \in \Lambda_t \), \( i = 1, 2 \), there exists a constant \( C > 0 \) such that the following estimates hold:

\[
|L(A_1^t; Z_1^1, W_1^1)| \leq C(1 + ||A_1^t||_{\infty})
\]

\[
|L(A_1^t; Z_1^1, W_1^1) - L(A_2^t; Z_2^1, W_2^1)| \leq C(||A_1^t - A_2^t||_{\infty} + ||Z_1^1 - Z_2^1||_{\infty} + ||W_1^1 - W_2^1||_{\infty})
\]

\[
|U(A_1^t; Z_1^1, W_1^1)| \leq C(1 + ||A_1^t||_{\infty})
\]

\[
|U(A_1^t; Z_1^1, W_1^1) - U(A_2^t; Z_2^1, W_2^1)| \leq C(||A_1^t - A_2^t||_{\infty} + ||Z_1^1 - Z_2^1||_{\infty} + ||W_1^1 - W_2^1||_{\infty}).
\]
Before stating the dynamic programming principle of the lower and upper value functionals, we introduce the backward semigroup associated with the BSDE in (3.2). For any $s \in [t, t+\tau]$ with $\tau \in [t, T-t)$ and $b \in L^2(\Omega, \mathcal{F}_{t+\tau}, \mathbb{R})$, we define
\[
\Pi^{t,t+\tau, A; U,V}_s[b] := y^{t,t+\tau, A; U,V}_s, \quad s \in [t, t+\tau],
\]
where $y$ is the first component of the pair $(y^{t,t+\tau, A; U,V}_s, q^{t,t+\tau, A; U,V}_s)$ that is the solution of the following BSDE:
\[
y^{t,t+\tau, A; U,V}_s = b + \int_s^{t+\tau} l(X^t_{r}, A; U,V, y^{t,t+\tau, A; U,V}_r, q^{t,t+\tau, A; U,V}_r) \, dr
- \int_s^{t+\tau} q^{t,t+\tau, A; U,V}_r \, dB_r, \quad s \in [t, t+\tau].
\]
Note that (4.1) can be regarded as a truncated BSDE in terms of the terminal time $t+\tau$ and the terminal condition. The superscripts $t$ and $t+\tau$ indicate the initial and terminal times, respectively. By definition, we have
\[
J(t; A_t; Z_t \otimes u, W_t \otimes v)
= \Pi^{t,T, A; Z_t \otimes u, W_t \otimes v}_t \left[ m(X^t_T; A; Z_t \otimes u, W_t \otimes v) \right]
= \Pi^{t,t+\tau, A; Z_t \otimes u, W_t \otimes v}_t [y^{t+\tau, A; Z_t \otimes u, W_t \otimes v}_{t+\tau}]
= \Pi^{t,t+\tau, A; Z_t \otimes u, W_t \otimes v}_t \left[ J(t+\tau, X^{t+\tau, A; Z_t \otimes u, W_t \otimes v}_t; (Z_t \otimes u)_{t+\tau}, (W_t \otimes v)_{t+\tau}) \right].
\]

Now, we state the dynamic programming principle of the lower and upper value functionals in (3.6) and (3.7).

**Theorem 4.3.** Suppose that Assumption 3.1 holds. Then for any $t, t+\tau \in [0, T]$ with $t < t+\tau$, and for any $A_t \in \mathcal{A}_t$ and $(Z_t, W_t) \in \bar{A}_t$, the lower and upper value functionals in (3.6) and (3.7), respectively, satisfy the following dynamic programming principles:
\[
\mathbb{L}(A_t; Z_t, W_t) = \inf_{\alpha \in \mathcal{A}_{[t,t+\tau]} \alpha \in \mathcal{V}_{[t,t+\tau]} \Pi^{t,t+\tau, A; Z_t \otimes \alpha(v), W_t \otimes v}_t \left[ \mathbb{L}(X^{t,A; Z_t \otimes \alpha(v), W_t \otimes v}_{t+\tau}; (Z_t \otimes \alpha(v))_{t+\tau}, (W_t \otimes v)_{t+\tau}) \right]
\]
\[
\mathbb{U}(A_t; Z_t, W_t) = \sup_{\beta \in \mathcal{B}_{[t,t+\tau]} \beta \in \mathcal{U}_{[t,t+\tau]} \Pi^{t,t+\tau, A; Z_t \otimes \beta(u), W_t \otimes \beta(u)}_t \left[ \mathbb{U}(X^{t,A; Z_t \otimes \beta(u), W_t \otimes \beta(u)}_{t+\tau}; (Z_t \otimes u)_{t+\tau}, (W_t \otimes \beta(u))_{t+\tau}) \right].
\]

**Proof.** We prove (4.3) only, as the proof for (4.4) is similar to that for (4.3).

Let us define
\[
\mathbb{L}'(A_t; Z_t, W_t) := \inf_{\alpha \in \mathcal{A}_{[t,t+\tau]} \alpha \in \mathcal{V}_{[t,t+\tau]} \Pi^{t,t+\tau, A; Z_t \otimes \alpha(v), W_t \otimes v}_t \left[ \mathbb{L}(X^{t,A; Z_t \otimes \alpha(v), W_t \otimes v}_{t+\tau}; (Z_t \otimes \alpha(v))_{t+\tau}, (W_t \otimes v)_{t+\tau}) \right].
\]
Below, we show $\mathbb{L}'(A_t; Z_t, W_t) \geq \mathbb{L}'(A_t; Z_t, W_t)$ and $\mathbb{L}(A_t; Z_t, W_t) \leq \mathbb{L}'(A_t; Z_t, W_t)$.

We modify the proof of [50] to the state and control paths-dependent case.
Part (i): \( L(A_t; Z_t, W_t) \leq L'(A_t; Z_t, W_t) \)

We first show that given \( A_t \in \Lambda_t \) and \( (Z_t, W_t) \in \hat{\Lambda}_t \), for any \( \epsilon > 0 \), there exists \( \alpha^\epsilon \in \mathcal{A}[t, T] \) such that

\[
L(A_t; Z_t, W_t) \geq \underset{v \in \mathcal{V}[t, T]}{\text{ess sup}} J(t, A_t; Z_t \otimes \alpha^\epsilon(v), W_t \otimes v) - \epsilon.
\]

(4.5)

Note that in view of [26, Theorem A.3] (see also [42]), there exists a sequence of nonanticipative strategies of Player 1, \( \{\alpha_k\} \) with \( \alpha_k \in \mathcal{A}[t, T] \), such that

\[
\limsup_{k \to \infty} \underset{v \in \mathcal{V}[t, T]}{\text{ess sup}} J(t, A_t; Z_t \otimes \alpha_k(v), W_t \otimes v) \leq L(A_t; Z_t, W_t).
\]

(4.6)

Let \( \hat{\Upsilon}_k := \{L(A_t; Z_t, W_t) \geq J(t, A_t; Z_t \otimes \alpha_k(v), W_t \otimes v) - \epsilon\}, k \geq 1 \). Note that \( \hat{\Upsilon}_k \)

is \( \mathcal{F}_t \)-measurable. To make the disjoint partition of \( \Omega \) with \( \{\hat{\Upsilon}_k\} \), let \( \hat{\Upsilon}_1 := \hat{\Upsilon}_1 \) and \( \hat{\Upsilon}_k := \hat{\Upsilon}_k \setminus \{\cup_{i=1}^{k-1} \hat{\Upsilon}_i\} \) for \( i \geq 2 \). Let \( \alpha^\epsilon := \sum_{k=1}^{\infty} 1_{\hat{\Upsilon}_k} \alpha_k \in \mathcal{A}[t, T] \). Then, in view of

the uniqueness of the solution to the BSDE and (4.6), we have

\[
L(A_t; Z_t, W_t) = \sum_{k=1}^{\infty} 1_{\hat{\Upsilon}_k} \text{ess sup}_{v \in \mathcal{V}[t, T]} J(t, A_t; Z_t \otimes \alpha_k(v), W_t \otimes v)
\]

\[
\geq \sum_{k=1}^{\infty} 1_{\hat{\Upsilon}_k} (J(t, A_t; Z_t \otimes \alpha^\epsilon(v), W_t \otimes v) - \epsilon)
\]

\[
= J(t, A_t; Z_t \otimes \alpha^\epsilon(v), W_t \otimes v) - \epsilon,
\]

which shows (4.5). In fact, to show the first equality in (4.7), for any \( k \geq 1 \), let \( \tilde{\alpha} := 1_{\hat{\Upsilon}_k} \alpha_k + 1_{\hat{\Upsilon}_k^C} \alpha_k^C \), where \( \alpha_k, \alpha_k^C \in \mathcal{A}[t, T] \), in which \( \alpha_k \) and \( \alpha_k^C \) correspond to \( \Upsilon_k \) and \( \Upsilon_k^C \), respectively. Based on this construction, we have

\[
\text{ess sup}_{v \in \mathcal{V}[t, T]} J(t, A_t; Z_t \otimes \tilde{\alpha}(v), W_t \otimes v)
\]

\[
\leq 1_{\hat{\Upsilon}_k} \text{ess sup}_{v \in \mathcal{V}[t, T]} J(t, A_t; Z_t \otimes \alpha_k(v), W_t \otimes v)
\]

\[
+ 1_{\hat{\Upsilon}_k^C} \text{ess sup}_{v \in \mathcal{V}[t, T]} J(t, A_t; Z_t \otimes \alpha^C_k(v), W_t \otimes v).
\]

On the other hand, from [26, Theorem A.3], there exist sequences of admissible controls of Player 2, \( \{v^t\} \) and \( \{v^C_t\} \) with \( v^t, v^C_t \in \mathcal{V}[t, T] \) such that

\[
\limsup_{l \to \infty} J(t, A_t; Z^t_l \otimes \alpha_k(v^t_l), Z^t_l \otimes v) = \text{ess sup}_{v \in \mathcal{V}[t, T]} J(t, A_t; Z^t_l \otimes \alpha_k(v), Z^t_l \otimes v)
\]

\[
\limsup_{l \to \infty} J(t, A_t; Z^C_l \otimes \alpha_k(v^C_l), Z^C_l \otimes v) = \text{ess sup}_{v \in \mathcal{V}[t, T]} J(t, A_t; Z^C_l \otimes \alpha^C_k(v), Z^C_l \otimes v).
\]

Hence, we have

\[
\text{ess sup}_{v \in \mathcal{V}[t, T]} J(t, A_t; Z_t \otimes \tilde{\alpha}(v), W_t \otimes v)
\]

\[
\geq \limsup_{l \to \infty} \{1_{\hat{\Upsilon}_k} J(t, A_t; Z_t \otimes \alpha_k(v^t_l), W_t \otimes v)
\]

\[
+ 1_{\hat{\Upsilon}_k^C} J(t, A_t; Z_t \otimes \alpha_k(v^C_l), W_t \otimes v)\}
\]

\[
= 1_{\hat{\Upsilon}_k} \text{ess sup}_{v \in \mathcal{V}[t, T]} J(t, A_t; Z_t \otimes \alpha_k(v), W_t \otimes v)
\]

\[
+ 1_{\hat{\Upsilon}_k^C} \text{ess sup}_{v \in \mathcal{V}[t, T]} J(t, A_t; Z_t \otimes \alpha^C_k(v), W_t \otimes v).
\]

(4.9)
Then, (4.8) and (4.9) imply (4.7); hence, (4.5) holds.

Now, given \( A_t \in \Lambda_t \), let
\[
\Psi_{t,t+\tau} := \{ \bar{A}_{t+\tau} \in \Lambda_{t+\tau} : \bar{a}_r = a_t, \forall r \in [t,t+\tau] \}.
\]
We note that \( \Psi \) is the set of continuous functions, which together with the metric \( \hat{d} \) induced by the norm \( \| \cdot \| \), implies that \( \Psi \) is a complete separable metric space.

Let \( d^* \) be the Skorokhod metric for \( \Lambda_t \) [3, Section 12]. Then in view of [3, Theorem 12.2], \( \Lambda_t \) is a complete separable metric space, and from [40], \( \Psi_{t,t+\tau}^{A_t; Z_t, W_t} := \Psi_{t,t+\tau}^{\hat{d}} \times \Lambda_t \) is a complete separable metric space with the metric \( \hat{d} := \hat{d} + d^* \). Therefore, there exists a countable dense subset, denoted by \( \{ \Psi_k \} \) [46], and for any \( \{ \bar{A}_{t+\tau}, Z_t, W_t \} \in \Psi_{t,t+\tau}^{A_t; Z_t, W_t} \) and \( \epsilon > 0 \), there exists \( (\bar{A}_{t+\tau}^{k}, \bar{Z}_t^{k}, \bar{W}_t^{k}) \in \Psi_k \), \( k \geq 1 \), such that we have
\[
\hat{d}((\bar{A}_{t+\tau}, Z_t, W_t), (\bar{A}_{t+\tau}^{k}, \bar{Z}_t^{k}, \bar{W}_t^{k})) < \epsilon.
\]
For \( (\bar{A}_{t+\tau}^{k}, \bar{Z}_t^{k}, \bar{W}_t^{k}) \in \Psi_k \), we define the set of neighborhood of \( \Psi_k \) by
\[
\Psi_k := \{ ((\bar{A}_{t+\tau}^{k}, Z_t, W_t) \in \Psi_{t,t+\tau}^{A_t; Z_t, W_t} : \hat{d}((\bar{A}_{t+\tau}, Z_t, W_t), (\bar{A}_{t+\tau}^{k}, \bar{Z}_t^{k}, \bar{W}_t^{k})) < \epsilon) \}.
\]
In view of this construction, \( \cup_{k=1}^{\infty} \Psi_k = \Psi_{t,t+\tau}^{A_t; Z_t, W_t} \), and by a slight abuse of notation, with \( \Psi_k := \Psi \setminus \cup_{k=1}^{\infty} \Psi_k \), \( k \geq 2 \), we still have \( \cup_{k=1}^{\infty} \Psi_k = \Psi_{t,t+\tau}^{A_t; Z_t, W_t} \), where \( \Psi_k \) is the disjoint partition of \( \Psi_{t,t+\tau}^{A_t; Z_t, W_t} \).

For any \( (A_t, Z_t, W_t) \in \Lambda_t \times \Lambda_t, \alpha' \in \mathcal{A}[t,t+\tau] \) and \( v' \in \mathcal{V}[t,t+\tau] \), with the above construction, together with Lemma 4.2, for each \( \epsilon > 0 \), there exists a constant \( C > 0 \) such that
\[
\begin{align*}
\mathbb{L}(X_{t+\tau}^{A_t; Z_t \otimes \alpha'(v')}, W_t \otimes v' : (Z_t \otimes \alpha'(v'))_{t+\tau}, (W_t \otimes v')_{t+\tau}) \\
= \sum_{k=1}^{\infty} \mathbb{I}(X_{t+\tau}^{A_t; Z_t \otimes \alpha'(v')}, W_t \otimes v', Z_t, W_t) \in \Psi_k' \\
\times \mathbb{L}(X_{t+\tau}^{A_t; Z_t \otimes \alpha'(v')}, W_t \otimes v' : (Z_t \otimes \alpha'(v'))_{t+\tau}, (W_t \otimes v')_{t+\tau}) \\
\geq \sum_{k=1}^{\infty} \mathbb{I}(X_{t+\tau}^{A_t; Z_t \otimes \alpha'(v')}, W_t \otimes v', Z_t, W_t) \in \Psi_k' \\
\times \mathbb{L}(\bar{A}_{t+\tau}^{k}; (\bar{Z}_t^{k} \otimes \alpha'(v'))_{t+\tau}, (\bar{W}_t^{k} \otimes v')_{t+\tau}) - C\epsilon.
\end{align*}
\]
Note that (4.5) implies that there exists \( \alpha'' \in \mathcal{A}[t+\tau, T] \) such that for \( k \geq 1 \),
\[
\begin{align*}
\mathbb{L}(\bar{A}_{t+\tau}^{k}; (\bar{Z}_t^{k} \otimes \alpha'(v'))_{t+\tau}, (\bar{W}_t^{k} \otimes v')_{t+\tau}) \\
\geq \sup_{v \in \mathcal{V}[t+\tau, T]} J(t + \tau, A_{t+\tau}; (\bar{Z}_t^{k} \otimes \alpha''(v))_{t+\tau}, (\bar{W}_t^{k} \otimes v)_{t+\tau}) - \epsilon.
\end{align*}
\]
Hence, from (4.10), for any \( v'' \in \mathcal{V}[t+\tau, T] \), we have
\[
\begin{align*}
\mathbb{L}(\bar{A}_{t+\tau}^{k}; (\bar{Z}_t^{k} \otimes \alpha'(v'))_{t+\tau}, (\bar{W}_t^{k} \otimes v')_{t+\tau}) \\
\geq \sum_{k=1}^{\infty} \mathbb{I}(X_{t+\tau}^{A_t; Z_t \otimes \alpha'(v')}, W_t \otimes v', Z_t, W_t) \in \Psi_k' \\
\times J(t + \tau, A_{t+\tau}; (\bar{Z}_t^{k} \otimes \alpha''(v''))_{t+\tau}, (\bar{W}_t^{k} \otimes v'')_{t+\tau}) - C\epsilon
\end{align*}
\]
where $\alpha''(v'') := \sum_{k=1}^{\infty} 1_{(A^{t,A_t;Z_t\otimes\alpha'(v')};W_t\otimes v')}(t,T)\alpha''_k(v'')$ and $v'' \in \mathcal{V}[t, T]$.

Let us define

$$v''' := 1_{s \in [t, t+\tau]} v'' + 1_{s \in (t+\tau, T]} v''
\alpha'''_s := 1_{s \in [t, t+\tau]} \alpha''_s + 1_{s \in (t+\tau, T]} \alpha''_s(v'').$$

Note that $v''' \in \mathcal{V}[t, T]$ and $\alpha''' \in \mathcal{A}[t, T]$. In view of (3.4), we have

$$(W_t \otimes v''')(s) = 1_{s \in [t, t+\tau]}(W_t \otimes v'')(s) + 1_{s \in (t+\tau, T]}(W_t \otimes v'')(s)
(Z_t \otimes \alpha'''(v'''))(s) = 1_{s \in [t, t+\tau]}(Z_t \otimes \alpha'(v'))(s) + 1_{s \in (t+\tau, T]}(Z_t \otimes \alpha''(v''))(s),$$

where it can be verified that $W_t \otimes v''' \in \mathcal{V}[0, T]$. Then from the comparison principle in (iv) of Lemma 3.2, (4.1) and (4.2), we have

$$\Pi_t^{t+\tau, A_t; Z_t \otimes \alpha'(v'), W_t \otimes v'}
\left[L(X^{t,A_t;Z_t\otimes\alpha'(v')};W_t\otimes v';(Z_t \otimes \alpha'(v'))_{t+\tau}, (W_t \otimes v')_{t+\tau})\right]
\geq \essinf_{\alpha \in [t, t+\tau]} \esssup_{v \in \mathcal{V}[t, T]}
\left[L(X^{t,A_t;Z_t\otimes\alpha'(v')};W_t\otimes v';(Z_t \otimes \alpha(v'))_{t+\tau}, (W_t \otimes v)_{t+\tau})\right]
\geq \essinf_{\alpha \in [t, t+\tau]} \esssup_{v \in \mathcal{V}[t, T]}
J(t, A_t; (Z_t \otimes \alpha(v'))_{t+\tau}, (W_t \otimes v)_{t+\tau}) - C\epsilon.$$

The arbitrariness of $v'''$ and $\alpha'''$, together with the definition of $\Pi$ and (4.2), yields

$$\essinf_{\alpha \in [t, t+\tau]} \esssup_{v \in \mathcal{V}[t, T]}
\left[L(X^{t,A_t;Z_t\otimes\alpha'(v')};W_t\otimes v';(Z_t \otimes \alpha(v'))_{t+\tau}, (W_t \otimes v)_{t+\tau})\right]
\geq \essinf_{\alpha \in [t, t+\tau]} \esssup_{v \in \mathcal{V}[t, T]}
J(t, A_t; (Z_t \otimes \alpha(v'))_{t+\tau}, (W_t \otimes v)_{t+\tau}) - C\epsilon.$$

By letting $\epsilon \downarrow 0$, we have the desired result.

Part (ii): $L(A_t; Z_t, W_t) \geq L'(A_t; Z_t, W_t)$

We first note that for any fixed $\alpha' \in \mathcal{A}[t, T]$ with $v \in \mathcal{V}[t, T]$, its restriction to $[t, t+\tau]$ is still nonanticipative independent of any special choice of $v \in \mathcal{V}[t, T]$, i.e., $\alpha'_{[t, t+\tau]} \in \mathcal{A}[t, t+\tau]$ for $v \in \mathcal{V}[t, t+\tau]$, due to the nonanticipative property of $\alpha'$. Recall the definition of $L'$; then with the restriction of $\alpha'$ to $[t, t+\tau]$, we have

$$L'(A_t; Z_t, W_t) \leq \esssup_{v \in \mathcal{V}[t, T]}
\left[L(X^{t,A_t;Z_t\otimes\alpha'(v')};W_t\otimes v';(Z_t \otimes \alpha'(v'))_{t+\tau}, (W_t \otimes v)_{t+\tau})\right].$$

Furthermore, similar to (4.6), there exist a sequence of admissible controls of Player 2, $\{v_k\}$, with $v_k \in \mathcal{V}[t, t+\tau]$, such that

$$\lim_{k \to \infty} \Pi_t^{t+\tau, A_t; Z_t \otimes \alpha'(v_k)} \leq \esssup_{v \in \mathcal{V}[t, T]}
\left[L(X^{t,A_t;Z_t\otimes\alpha'(v_k)};W_t\otimes v';(Z_t \otimes \alpha'(v_k))_{t+\tau}, (W_t \otimes v_k)_{t+\tau})\right].$$
Then by using (4.12) and the approach of (4.5), for each \( \epsilon \geq 0 \), there exists \( v' \in \mathcal{Y}[t, t + \tau] \) such that

\[
(4.13) \quad \mathbb{L}'(A_t; Z_t, W_t) \leq \Pi_{t}^{t, t + \tau, A_t; Z_t \otimes \alpha'(v'), W_t \otimes v'}
\]

\[
\left[ J(t + \tau, X_{t + \tau}^{t, A_t; Z_t \otimes \alpha'(v')}, W_t \otimes v'; (Z_t \otimes \alpha'(v'))_{t + \tau}, (W_t \otimes v')_{t + \tau} \right] + \epsilon.
\]

Similar to the argument and the notation introduced in (4.10) and (4.11), there exists \( v'' \in \mathcal{Y}[t + \tau, T], k \geq 1 \), such that

\[
\mathbb{L}(X_{t + \tau}^{t, A_t; Z_t \otimes \alpha'(v'), W_t \otimes v'}; (Z_t \otimes \alpha'(v'))_{t + \tau}, (W_t \otimes v')_{t + \tau})
\]

\[
= \sum_{k=1}^{\infty} \mathbb{I}(X_{t + \tau}^{t, A_t; Z_t \otimes \alpha'(v')}, W_t \otimes v'; Z_t, W_t) \in \Psi_k
\times \mathbb{L}(X_{t + \tau}^{t, A_t; Z_t \otimes \alpha'(v')}, W_t \otimes v'; (Z_t \otimes \alpha'(v'))_{t + \tau}, (W_t \otimes v')_{t + \tau})
\]

\[
\leq \sum_{k=1}^{\infty} \mathbb{I}(X_{t + \tau}^{t, A_t; Z_t \otimes \alpha'(v')}, W_t \otimes v'; Z_t, W_t) \in \Psi_k
\times J(t + \tau, A_{t + \tau}^k; (Z_{t + \tau}^k \otimes \alpha'(v''))_{t + \tau}, (W_{t + \tau}^k \otimes v''')_{t + \tau}) + C\epsilon
\]

\[
= J(t + \tau, X_{t + \tau}^{t, A_t; Z_t \otimes \alpha'(v'), W_t \otimes v'}; (Z_t \otimes \alpha'[t + \tau, T](v''))_{t + \tau}, (W_t \otimes v'')_{t + \tau}) + C\epsilon,
\]

where \( v''' = \sum_{k=1}^{\infty} \mathbb{I}(X_{t + \tau}^{t, A_t; Z_t \otimes \alpha'(v'), W_t \otimes v'}; Z_t, W_t) \in \Psi_k v''_k. \)

Let us define

\[
v'''_s := \mathbb{I}_{s \in [t, t + \tau]} v'_s + \mathbb{I}_{s \in (t + \tau, T]} v''_s, \quad \alpha'(v'') := \mathbb{I}_{s \in [t, t + \tau]} \alpha'_s(v'') + \mathbb{I}_{s \in (t + \tau, T]} \alpha'_s(v'').
\]

Note that \( v''' \in \mathcal{Y}[t, T] \) and \( \alpha' \in \mathcal{A}[t, T] \). Then, from (3.4),

\[
(W_t \otimes v''')|s] = \mathbb{I}_{s \in [t, t + \tau]} (W_t \otimes v')|s] + \mathbb{I}_{s \in (t + \tau, T]} (W_t \otimes v''')|s] \]

\[
(Z_t \otimes \alpha'(v'''))|s] = \mathbb{I}_{s \in [t, t + \tau]} (Z_t \otimes \alpha'(v'))|s] + \mathbb{I}_{s \in (t + \tau, T]} (Z_t \otimes \alpha'(v'''))|s],
\]

where \( W_t \otimes v''' \in \mathcal{Y}[0, T] \). From the comparison principle in (iv) of Lemma 3.2, (4.1), and (4.2), we have

\[
\mathbb{L}'(A_t; Z_t, W_t) \leq \Pi_{t}^{t, t + \tau, A_t; Z_t \otimes \alpha'(v'), W_t \otimes v'}
\]

\[
\left[ J(t + \tau, X_{t + \tau}^{t, A_t; Z_t \otimes \alpha'(v')}, W_t \otimes v'; (Z_t \otimes \alpha'(v''))_{t + \tau}, (W_t \otimes v'')_{t + \tau} \right] + C\epsilon
\]

\[
= J(t, A_t; Z_t \otimes \alpha'(v'''), W_t \otimes v') + C\epsilon.
\]

The arbitraryness of \( v''' \) and the definition of \( \Pi \) imply,

\[
\mathbb{L}'(A_t; Z_t, W_t) \leq \sup_{v \in \mathcal{Y}[t, T]} J(t, A_t; Z_t \otimes \alpha'(v), W_t \otimes v) + C\epsilon,
\]

and by taking \( \text{ess inf} \) with respect to \( \alpha \in \mathcal{A}[t, T] \) and then \( \epsilon \downarrow 0 \), we have the desired result. Hence, Parts (i) and (ii) show the dynamic programming principle of the lower value functional \( \mathbb{L} \) in (4.3). This completes the proof of the theorem. \( \square \)
Remark 4.4. By using standard localization and approximation techniques, Theorem 4.3 can be extended to stopping times.

From Lemma 4.2, the (lower and upper) value functionals are continuous with respect to the initial state and control paths. We next state the continuity of the (lower and upper) value functionals in \( t \in [0, T] \).

**Lemma 4.5.** Suppose that Assumption 3.1 holds. Then, the lower and upper value functionals are continuous in \( t \). In particular, there exists a constant \( C > 0 \) such that for any \( (A_T, Z_T, W_T) \in \Lambda_T \times \Lambda_T \) and \( t_1, t_2 \in [0, T] \) with \( t' := \max\{t_1, t_2\} \),

\[
\begin{align*}
|\mathbb{L}(A_{t_1}; Z_{t_1}, W_{t_1}) - \mathbb{L}(A_{t_2}; Z_{t_2}, W_{t_2})| &\leq C(1 + \|A_T\|_\infty)|t_1 - t_2|^{1/2}, \\
|\mathbb{U}(A_{t_1}; Z_{t_1}, W_{t_1}) - \mathbb{U}(A_{t_2}; Z_{t_2}, W_{t_2})| &\leq C(1 + \|A_T\|_\infty)|t_1 - t_2|^{1/2}.
\end{align*}
\]

**Proof.** We prove the case for the lower value functional only, since the proof for the upper value functional is similar. Without loss of generality, for any \( t_1 = t, t_2 = t + \tau \in [0, T] \) with \( t < t + \tau \), we need to prove

\[
-C(1 + \|A_{t+\tau}\|_\infty)^{\tau/2} \leq \mathbb{L}(A_t; Z_t, W_t) - \mathbb{L}(A_{t+\tau}; Z_{t+\tau}, W_{t+\tau}) \leq C(1 + \|A_{t+\tau}\|_\infty)^{\tau/2}.
\]

In view of the dynamic programming principle (4.3) in Theorem 4.3 and (4.5), for any \( \epsilon > 0 \), there exists \( \alpha^\epsilon \in A[t, t + \tau] \) such that for any \( v \in V[t, t + \tau] \),

\[
\mathbb{L}(A_t; Z_t, W_t) \geq \Pi_t^{t+\tau; A_t; Z_t; \alpha^\epsilon(v), W_t; v} \left[ \mathbb{L}(X_{t+\tau}^{A_t; Z_t; \alpha^\epsilon(v), W_t; v}, (Z_t \otimes \alpha^\epsilon(v))_{t+\tau}, (W_t \otimes v)_{t+\tau}) \right] - \epsilon.
\]

The definition of \( \Pi_t \) implies

\[
\mathbb{L}(A_t; Z_t, W_t) - \mathbb{L}(A_{t+\tau}; Z_{t+\tau}, W_{t+\tau}) \geq L^{(1)} + L^{(2)} - L^{(3)} - \epsilon,
\]

where

\[
\begin{align*}
L^{(1)} &:= \Pi_t^{t+\tau; A_t; Z_t; \alpha^\epsilon(v), W_t; v} \left[ \mathbb{L}(X_{t+\tau}^{A_t; Z_t; \alpha^\epsilon(v), W_t; v}, (Z_t \otimes \alpha^\epsilon(v))_{t+\tau}, (W_t \otimes v)_{t+\tau}) \right] \\
&\quad - \Pi_t^{t+\tau; A_t; Z_t; \alpha^\epsilon(v), W_t; v} \left[ \mathbb{L}(A_{t+\tau}; (Z_t \otimes \alpha^\epsilon(v))_{t+\tau}, (W_t \otimes v)_{t+\tau}) \right]
\end{align*}
\]

\[
\begin{align*}
L^{(2)} &:= \Pi_t^{t+\tau; A_t; Z_t; \alpha^\epsilon(v), W_t; v} \left[ \mathbb{L}(A_{t+\tau}; (Z_t \otimes \alpha^\epsilon(v))_{t+\tau}, (W_t \otimes v)_{t+\tau}) \right] \\
&\quad - \Pi_t^{t+\tau; A_t; Z_t; \alpha^\epsilon(v), W_t; v} \left[ \mathbb{L}(A_{t+\tau}; Z_{t+\tau}, W_{t+\tau}) \right]
\end{align*}
\]

\[
\begin{align*}
L^{(3)} &:= \Pi_t^{t+\tau; A_t; Z_t; \alpha^\epsilon(v), W_t; v} \left[ \Pi_t^{t+\tau; A_t; Z_t; \alpha^\epsilon(v), W_t; v} \left[ \mathbb{L}(A_{t+\tau}; Z_{t+\tau}, W_{t+\tau}) \right] \right] \\
&\quad - \mathbb{L}(A_{t+\tau}; Z_{t+\tau}, W_{t+\tau})
\end{align*}
\]

Note that \( L^{(i)} \geq -|L^{(i)}| \) for \( i = 1, 2, 3 \).

Now, Lemmas 3.2 and 4.2, the definition of \( \Pi_t \), and Jensen’s inequality imply that there exist a constant \( C > 0 \) such that

\[
|L^{(1)}| \leq CE \left[ \left| \mathbb{L}(X_{t+\tau}^{A_t; Z_t; \alpha^\epsilon(v), W_t; v}, (Z_t \otimes \alpha^\epsilon(v))_{t+\tau}, (W_t \otimes v)_{t+\tau}) \right| \right]^{1/2}
\]

\[
\leq CE \left[ \left| X_{t+\tau}^{A_t; Z_t; \alpha^\epsilon(v), W_t; v} - A_{t+\tau} \right|^2 \left| \mathcal{F}_t \right|^{1/2} \right] \leq C(1 + \|A_{t+\tau}\|_\infty)^{\tau/2}.
\]
Moreover, from the definition of Π and Proposition 4.1, $L^{(3)}$ is equivalent to
\[
L^{(3)} = E \left[ \int_t^{t+\tau} I(X_t^t, A_t; Z_t \otimes \alpha^*(v), W_t \otimes v), y_t^{t+\tau, A_t; Z_t \otimes \alpha^*(v), W_t \otimes v}, (Z_t \otimes \alpha^*(v))_r, (W_t \otimes v)_r \right] \mathcal{F}_t \right].
\]

Then Hölder inequality, Assumption 3.1, Lemma 3.2 imply that
\[
|L^{(3)}| \leq \tau^{1/2} E \left[ \int_t^{t+\tau} |J(t, X_t^t, A_t; Z_t \otimes \alpha^*(v), W_t \otimes v), y_t^{t+\tau, A_t; Z_t \otimes \alpha^*(v), W_t \otimes v}, W_t \otimes v), (Z_t \otimes \alpha^*(v))_r, (W_t \otimes v)_r| dr \mathcal{F}_t \right]^{1/2}
\]
\[
\leq C \tau^{1/2} E \left[ \int_t^{t+\tau} \left( 1 + \|X_t^t, A_t; Z_t \otimes \alpha^*(v), W_t \otimes v\|_\infty^2 + |y_t^{t+\tau, A_t; Z_t \otimes \alpha^*(v), W_t \otimes v}|^2 \right) dr \mathcal{F}_t \right]^{1/2}
\]
\[
+ |y_t^{t+\tau, A_t; Z_t \otimes \alpha^*(v), W_t \otimes v}|^2 | \mathcal{F}_t |^{1/2} \leq C (1 + \|A_{t+\tau}\|_\infty) \tau^{1/2}.
\]

Furthermore, in view of the definitions of the lower value functional in (3.6) and the objective functional in (3.5), we have
\[
\mathbb{L}(A_{t+\tau}; Z_{t+\tau}, W_{t+\tau})
= \text{ess inf}_{\alpha \in \mathcal{A}} \text{ess sup}_{v \in V} J(t, A_{t+\tau}; Z_{t+\tau} \otimes \alpha(v), W_{t+\tau} \otimes v)
\]
\[
\mathbb{L}(A_{t+\tau}; Z_t \otimes \alpha(v))_{t+\tau}, (W_t \otimes v)_{t+\tau})
= \text{ess inf}_{\alpha \in \mathcal{A}} \text{ess sup}_{v \in V} J(t, A_{t+\tau}; Z_t \otimes \alpha(v), W_t \otimes v).
\]

From (iii) of Lemmas 3.2, Lemma 4.2, (3.4), and the definition of Π, we have
\[
|L^{(2)}| \leq CE \left[ |\mathbb{L}(A_{t+\tau}; Z_t \otimes \alpha^*(v))_{t+\tau}, (W_t \otimes v)_{t+\tau})
- \mathbb{L}(A_{t+\tau}; Z_{t+\tau}, W_{t+\tau})|^2 | \mathcal{F}_t |^{1/2}
\right]
\[
\leq CE \left[ \int_t^{t+\tau} \left[ \| (Z_{t+\tau} \otimes \alpha^*(v))_r - (Z_t \otimes \alpha^*(v))_r \|_\infty^2 + \| (W_{t+\tau} \otimes v)_r - (W_t \otimes v)_r \|_\infty^2 \right] dr | \mathcal{F}_t |^{1/2} = 0.
\]

By substituting (4.16)-(4.18) into (4.15), we have
\[
\mathbb{L}(A_t; Z_t, W_t) - \mathbb{L}(A_{t+\tau}; Z_{t+\tau}, W_{t+\tau}) \geq -C (1 + \|A_{t+\tau}\|_\infty) \tau^{1/2} - \epsilon.
\]

Hence, the arbitrariness of ε implies the first inequality part in (4.14). The second inequality part in (4.14) can be shown in a similar way. We complete the proof. \[\square\]

5. State and Control Paths-Dependent Hamilton-Jacobi-Isaacs Equations and Viscosity Solutions. In this section, we first introduce the lower and upper state and control paths-dependent Hamilton-Jacobi-Isaacs (PHJI) equations that are state and control paths-dependent nonlinear second-order PDEs (PPDEs). We then show that the (lower and upper) value functionals in (3.6) and (3.7) are viscosity solutions of the corresponding PHJI equations.
The Hamiltonian, $\mathcal{H} : \Lambda \times \hat{\Lambda} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \rightarrow \mathbb{R}$, is defined by

\begin{equation}
\mathcal{H}(A_t, Z_t^u, W_t^v, y, p, P) = \langle f(A_t, Z_t^u, W_t^v), p \rangle + l(\mathcal{L}(A_t, Z_t^u, W_t^v), Z_t^u, W_t^v) + \frac{1}{2} \text{Tr}(P\sigma(A_t, Z_t^u, W_t^v)\sigma^\top(A_t, Z_t^u, W_t^v)),
\end{equation}

where

\[
Z_t^u[s] := \begin{cases} 
z_s, & \text{if } s \in [0,t) \\
u, & \text{if } s = t, 
\end{cases}
\quad \text{and} \quad
W_t^v[s] := \begin{cases} 
w_s, & \text{if } s \in [0,t) \\
v, & \text{if } s = t. 
\end{cases}
\]

With (5.1), we introduce the lower PHJI equation

\begin{equation}
\begin{aligned}
\mathbb{H}^-(A_t, Z_t, W_t, (\partial_t \mathbb{L}^{u,v}, \mathbb{L}, \partial_x \mathbb{L}, \partial_x \mathbb{L})(A_t; Z_t, W_t)) \\
:= \sup_{u \in U} \inf_{w \in V} \left\{ \partial_t \mathbb{L}(A_t; Z_t^u, W_t^v) \\
+ \mathcal{H}(A_t, Z_t^u, W_t^v, (\mathbb{L}, \partial_x \mathbb{L}, \partial_x \mathbb{L})(A_t; Z_t, W_t)) \right\} = 0,
\end{aligned}
\end{equation}

and the upper PHJI equation

\begin{equation}
\begin{aligned}
\mathbb{H}^+(A_t, Z_t, W_t, (\partial_t \mathbb{L}^{u,v}, \mathbb{U}, \partial_x \mathbb{U}, \partial_x \mathbb{U})(A_t; Z_t, W_t)) \\
:= \inf_{w \in V} \sup_{u \in U} \left\{ \partial_t \mathbb{U}(A_t; Z_t^u, W_t^v) \\
+ \mathcal{H}(A_t, Z_t^u, W_t^v, (\mathbb{U}, \partial_x \mathbb{U}, \partial_x \mathbb{U})(A_t; Z_t, W_t)) \right\} = 0,
\end{aligned}
\end{equation}

Remark 5.1. We note that the if the (lower and upper) value functionals are in $C^{1,2}(\Lambda; \hat{\Lambda})$ in the sense of Definition 2.2, i.e., $\mathbb{L}, \mathbb{U} \in C^{1,2}(\Lambda; \hat{\Lambda})$, then they are classical solutions to the corresponding PHJI equations. Specifically, the time derivative of $\mathbb{L}$ is with respect to $(A_t, Z_t, W_t)$, and can be written as

\[
\partial_t \mathbb{L}(A_t; Z_t^u, W_t^v) = \lim_{\delta t \downarrow 0} \frac{\mathbb{L}(A_t, Z_t, W_t, \delta t, \delta t) - \mathbb{L}(A_t; Z_t, W_t)}{\delta t},
\]

where $(Z_t^u, W_t^v)$ is induced due to the definition of $\otimes$ in (3.4) (see also [41]).

The space derivative of $\mathbb{L}$ with respect to $A_t$ is given as follows:

\[
\partial_h \mathbb{L}(A_t; Z_t, W_t) = \begin{bmatrix} \partial_1 \mathbb{L}(A_t; Z_t, W_t) & \cdots & \partial_n \mathbb{L}(A_t; Z_t, W_t) \end{bmatrix}
\]

\[
\partial_h \mathbb{L}(A_t; Z_t, W_t) = \lim_{h \downarrow 0} \frac{\mathbb{L}(A_t(\text{h.c.}); Z_t, W_t) - \mathbb{L}(A_t; Z_t, W_t)}{h}.
\]

Also, similar to [14, 37], the classical sub-solution (respectively, super-solution) of $\mathbb{L}$ is defined if the “= 0” in (5.2) is replaced by “$\geq 0$” (respectively, “$\leq 0$”). The same argument can be applied to the upper PHJI equation in (5.3).

The next remark discusses some special cases of the PHJI equations in view of (1) of Remark 3.7.
Remark 5.2. (1) For the state path-dependence case discussed in (1) of Remark 3.7, the PHJI equations are defined on \( \Lambda \); see (5.19) and (5.20).

(2) In addition, for the Markovian formulation (see (1) of Remark 3.7), the (lower and upper) PHJI equations are equivalent to those in [6, (4.1) and (4.2)].

We fix \( \kappa \in (0, \frac{1}{4}) \) in the \( \kappa \)-Hölder modulus. The notion of the viscosity solution is as follows. Similar definitions are also given in [45, 52], which are motivated by [31].

Definition 5.3. (i) A real-valued functional \( L \in C(\Lambda; \hat{\Lambda}) \) is said to be a viscosity sub-solution of the lower PHJI equation in (5.2) if for \((A_T, Z_T, W_T) \in \Lambda_T \times \hat{\Lambda}_T \) and \( \mu, \mu_0 > 0 \), \( L(A_T; Z_T, W_T) \leq m(A_T) \) and for all test functions \( \phi \in C^{1,2}_{\kappa}(\Lambda; \hat{\Lambda}) \) satisfying the predictable dependence in the sense of [8], i.e.,

\[
0 = (L - \phi)(\bar{A}_t; Z_t, W_t) = \sup_{A_s \in C^{1,2}_{\kappa} \mu, \mu_0} (\bar{L} - \phi)(A_s; Z_t, W_t),
\]

where \( \bar{A}_t \in C^{\kappa, \mu, \mu_0} \), the following inequality holds:

\[
\lim \inf_{\mu \to \infty} H^- (A_t, Z_t, W_t, (\partial_t \phi^{u,v}, \phi, \partial_x \phi, \partial_{xx} \phi)(\bar{A}_t; Z_t, W_t)) \geq 0.
\]

(ii) A real-valued functional \( L \in C(\Lambda; \hat{\Lambda}) \) is said to be a viscosity super-solution of the lower PHJI equation in (5.2) if for \((A_T, Z_T, W_T) \in \Lambda_T \times \hat{\Lambda}_T \) and \( \mu, \mu_0 > 0 \), \( L(A_T; Z_T, W_T) \geq m(A_T) \) and for all test functions \( \phi \in C^{1,2}_{\kappa}(\Lambda; \hat{\Lambda}) \) satisfying the predictable dependence in the sense of [8], i.e., \( \phi(A_t; Z_t, W_t) = \phi(A_t; Z_{t-}, W_{t-}) \) and

\[
0 = (L - \phi)(\bar{A}_t; Z_t, W_t) = \inf_{A_s \in C^{1,2}_{\kappa} \mu, \mu_0} (\bar{L} - \phi)(A_s; Z_t, W_t),
\]

where \( \bar{A}_t \in C^{\kappa, \mu, \mu_0} \), the following inequality holds:

\[
\lim \sup_{\mu \to \infty} H^- (A_t, Z_t, W_t, (\partial_t \phi^{u,v}, \phi, \partial_x \phi, \partial_{xx} \phi)(\bar{A}_t; Z_t, W_t)) \leq 0.
\]

(iii) A real-valued functional \( L \in C(\Lambda; \hat{\Lambda}) \) is said to be a viscosity solution if it is both a viscosity sub-solution and super-solution of (5.2).

(iv) The viscosity sub-solution, super-solution and solution of the upper PHJI equation in (5.3) are defined in similar ways.

Remark 5.4. (1) If the viscosity solutions of the PHJI equations in (5.2) and (5.3) are in \( C^{1,2}(\Lambda; \hat{\Lambda}) \), i.e., \( L, U \in C^{1,2}(\Lambda; \hat{\Lambda}) \), then they are also classical solutions of the PHJI equations discussed in Remark 5.1.

(2) For the Markovian case, the notion of viscosity solutions in Definition 5.3 is equivalent to that of the classical one in [6, 10, 21, 47].

(3) In Definition 5.3, due to the predictable dependence in the sense of [8], the derivative of \( \phi \) with respect to the second and third arguments is zero [8, Remark 4]; see also [41].

We state the main result of this section.

Theorem 5.5. Suppose that Assumption 3.1 holds. Then the lower value functional in (5.6) is the viscosity solution of the lower PHJI equation in (5.2). The upper value functional in (3.7) is the viscosity solution of the upper PHJI equation in (5.3).
Before proving the theorem, for $\mu > 0$, $\epsilon \in (0, \mu)$, $r \in [0, t]$ and $A_t \in \mathbb{C}^{\kappa, \mu, \mu_0}$, let $A^r_t$ be the perturbed version of $A_t$ defined by

$$a^r_t := \begin{cases} a_r, & \text{if } |a_r - a_t| \leq (\mu - \epsilon)|r - t|^{\kappa} \\ a + (\mu - \epsilon)(t - r)^{\kappa} \frac{a_r - a_t}{|a_r - a_t|}, & \text{if } |a_r - a_t| \geq (\mu - \epsilon)|r - t|^{\kappa}. \end{cases}$$

Note that $A^r_t := \{a^r_t, \; r \in [0, t]\}$. The perturbation is essential to prove Theorem 5.5; see [45, Remark 6].

We state the following lemma, whose proof is given in [45, Lemma 5.1].

**Lemma 5.6.** Let $\mu, \mu_0 > 0$. Assume that $[A_t]_\kappa \leq \mu$, $\|A_t\|_\infty \leq \mu_0$, i.e., $A_t \in \mathbb{C}^{\kappa, \mu, \mu_0}$, and $\epsilon \in (0, \frac{\mu}{2})$. Then we have

(i) $\|A^r_t - A_t\|_\kappa \leq 2\mu_0 \epsilon (\mu - \epsilon)^{-1} \leq 4\mu_0 \epsilon^{-1}$.

(ii) $\|A^r_t\|_\kappa \leq \mu$.

(iii) There exists a constant $C > 0$, independent of $\mu$, such that for any $d$ with $\frac{\mu}{2}(\frac{1}{2} - \kappa) > 1$ and $t + \tau < T$, $P(\|X^r_{t+\tau}; Z_t \otimes u, W_t \otimes v\|_\infty > \mu) \leq C T^{-d} \epsilon^{-d}.$

**Proof of Theorem 5.5.** We first prove that the lower value functional in (3.6) is the viscosity super-solution of the lower PHJI equation in (5.2). Note that in view of Lemmas 4.2 and 4.5, it is clear that $L \in C(A; \hat{A})$. Furthermore, from (3.6), we have $\mathbb{L}(A_T; Z_T, W_T) \geq m(A_T)$.

From the definition of the viscosity super-solution in (ii) of Definition 5.3 and Lemma 2.5, for $\phi \in \mathbb{C}^{1,2}(A; \hat{A})$, $\mu > 1$ and $\mu_0 > 0$,

$$0 = (L - \phi)(\bar{A}_t; Z_t, W_t) = \inf_{A_t \in \mathbb{C}^{\kappa, \mu, \mu_0}} (L - \phi)(A_t; Z_t, W_t),$$

where $A_t \in \mathbb{C}^{\kappa, \mu, \mu_0}$. Let $\delta := \mu_0 - \|\bar{A}_t\|_\infty > 0$.

For any $\epsilon \in (0, \frac{\mu}{2} \wedge \frac{\mu_0}{\|\bar{A}_t\|_\infty})$, in view of (i) and (ii) in Lemma 5.6, we have

$$\|\bar{A}^r_t - \bar{A}_t\|_\infty \leq 4\mu_0 \epsilon \mu^{-1} < \frac{\delta}{2}.$$ 

Consider the following $\mathbb{F}$-stopping time:

$$\xi^r := \inf \{r > t : \|X^r_{t+\tau}; Z_t \otimes \alpha(v), W_t \otimes v\|_\kappa > \mu \} \wedge \inf \{r > t : \|X^r_{t+\tau}; Z_t \otimes \alpha(v), W_t \otimes v\|_\infty > \mu_0 \}.$$ 

By definition, for any $s < \xi^r$, $X^r_s, A^r_t; Z_t, W_t \in \mathbb{C}^{\kappa, \mu, \mu_0}$ and for a small $\tau$ with $t + \tau \leq T$,

$$\{\xi^r \leq t + \tau\} \subset \{\|X^r_{t+\tau}; Z_t \otimes \alpha(v), W_t \otimes v\|_\kappa > \mu \} \cup \{\|X^r_{t+\tau}; Z_t \otimes \alpha(v), W_t \otimes v\|_\infty > \mu_0 \}.$$ 

Hence, from (iii) of Lemma 5.6, we have

$$P(\|X^r_{t+\tau}; Z_t \otimes \alpha(v), W_t \otimes v\|_\kappa > \mu) \leq C T^{-d} \epsilon^{-d},$$

and by (ii) of Lemma 3.2 and Markov inequality,

$$P(\|X^r_{t+\tau}; Z_t \otimes \alpha(v), W_t \otimes v\|_\infty > \mu_0) \leq P(\|X^r_{t+\tau}; Z_t \otimes \alpha(v), W_t \otimes v\|_\kappa - \bar{A}^r_{t; t+\tau} \|_\infty > \frac{\delta}{2}) \leq C(1 + \mu_0^6) \tau^3 / \delta^6.$$
This implies that
\[
\mathbb{P}(\xi \leq t + \tau) \leq C\tau^{d(\frac{1}{2} - \nu)} \epsilon^{-d} + C(1 + \mu_0^6)\tau^3 / \delta^6 \downarrow 0 \text{ as } \tau \downarrow 0.
\]

Now, from the dynamic programming principle in (4.3) of Theorem 4.3,
\[
\begin{aligned}
(5.5) & \quad \mathcal{L}(A_t^\nu; Z_t, W_t) - \phi(A_t^\nu; Z_t, W_t) \\
&= \text{ess inf}_{\alpha \in \mathcal{A}[t, t+\tau]} \text{ess sup}_{v \in \mathcal{V}[t, t+\tau]} \Pi_t^{\alpha; Z_t, W_t; v} \\
&\quad \left[ \mathcal{L}(X_{t+\tau}^{t, \alpha}; Z_t, W_t; v) ; (Z_t \otimes \alpha(v))_{t+\tau}, (W_t \otimes v)_{t+\tau} \right] - \phi(A_t^\nu; Z_t, W_t).
\end{aligned}
\]
Note also that
\[
\begin{aligned}
\mathcal{L}(A_t^\nu; Z_t, W_t) \geq \text{ess sup}_{v \in \mathcal{V}[t, t+\tau]} \text{ess inf}_{u \in \mathcal{U}[t, t+\tau]} \Pi_t^{u; \alpha; Z_t, W_t; v} \\
&\quad \left[ \mathcal{L}(X_{t+\tau}^{t, \alpha}; Z_t, W_t; v) ; (Z_t \otimes u)_{t+\tau}, (W_t \otimes v)_{t+\tau} \right].
\end{aligned}
\]
Then similar to (4.13), for any \(\epsilon > 0\), there exists \(u'' \in \mathcal{U}[t, t+\tau]\) such that for any \(v \in \mathcal{V}[t, t+\tau]\),
\[
(5.6) \quad \mathcal{L}(A_t^\nu; Z_t, W_t) \geq \Pi_t^{t+\tau, u''; Z_t, W_t; v} \\
&\quad \left[ \mathcal{L}(X_{t+\tau}^{t, u''}; Z_t, W_t; v) ; (Z_t \otimes u'')_{t+\tau}, (W_t \otimes v)_{t+\tau} \right] - \epsilon',
\]
where in view of the definition of \(\Pi, \Pi\) in the above expression can be rewritten as (superscript \(t + \tau\) is omitted)
\[
\begin{aligned}
\{ & d_s t_s A_s^{\nu; Z_t \otimes u''} ; W_t \otimes v, y_s t_s A_s^{\nu; Z_t \otimes u''} ; W_t \otimes v, \\
& q_s t_s A_s^{\nu; Z_t \otimes u''} ; W_t \otimes v, (Z_t \otimes u'')_s, (W_t \otimes v)_s ds \\
& + q_s t_s A_s^{\nu; Z_t \otimes u''} ; W_t \otimes v, d_B s, s \in [t, t+\tau] \}
\end{aligned}
\]
\[
(5.7)
\]
On the other hand, by using the functional Itô formula in Lemma 2.7, we have
\[
(5.8) \quad \phi(X_{t+\tau}^{t, A_t^{\nu}; Z_t \otimes u''}; W_t \otimes v; (Z_t \otimes u'')_{t+\tau}, (W_t \otimes v)_{t+\tau})
\]
\[
= \phi(A_t^{\nu}; Z_t, W_t) + \int_t^{t+\tau} F(X_r^{t, A_r^{\nu}; Z_t \otimes u''}; W_t \otimes v; (Z_t \otimes u'')_r, (W_t \otimes v)_r) dr \\
- \int_t^{t+\tau} l(X_r^{t, A_r^{\nu}; Z_t \otimes u''}; W_t \otimes v; \phi(X_r^{t, A_r^{\nu}; Z_t \otimes u''}; W_t \otimes v; (Z_t \otimes u'')_r, (W_t \otimes v)_r), (Z_t \otimes u'')_r, (W_t \otimes v)_r) dr \\
+ \int_t^{t+\tau} \sigma(X_r^{t, A_r^{\nu}; Z_t \otimes u''}; W_t \otimes v; (Z_t \otimes u'')_r, (W_t \otimes v)_r), (Z_t \otimes u'')_r, (W_t \otimes v)_r) dr \\
+ \int_t^{t+\tau} \partial_x \phi(X_r^{t, A_r^{\nu}; Z_t \otimes u''}; W_t \otimes v; (Z_t \otimes u'')_r, (W_t \otimes v)_r) d_B r,
\]
where
\[
F(A_t, Z_t^u, W_t^v) = \partial_t \phi(A_t; Z_t^u, W_t^v) + \frac{1}{2} \text{Tr} (\partial_{xx} \phi(A_t; Z_t, W_t) \sigma \sigma^T (A_t, Z_t^u, W_t^v)) \\
+ (\partial_x \phi(A_t; Z_t^u, W_t^v))) \quad (A_t, Z_t^u, W_t^v)
\]
\[
+ l(A_t, \phi(A_t; Z_t, W_t), (\partial_x \phi(A_t; Z_t, W_t), \sigma(A_t, Z_t^u, W_t^v)), Z_t^u, W_t^v)).
\]
where the derivative of \( \phi \) with respect to the control of the players is zero. This is similar to [8, 41] to make the functional dependence on the quadratic variation of \( X \); see also [8] for the predictable dependence of the functional.

Let

\[
\frac{d}{ds} \tilde{q}_s^{\overline{t}; Z_t \otimes u^r, W_t} = \phi(X_s^{\overline{t}; Z_t \otimes u^r, W_t}; (Z_t \otimes u^r)_s, (W_t \otimes v)_s)
\]

and

\[
\frac{d}{ds} q_s^{\overline{t}; Z_t \otimes u^r, W_t} := \frac{d}{ds} \phi(X_s^{\overline{t}; Z_t \otimes u^r, W_t}; (Z_t \otimes u^r)_s, (W_t \otimes v)_s)
\]

From (5.7) and (5.8),

\[
d \tilde{y}_s^{\overline{t}; Z_t \otimes u^r, W_t} = F(x_s^{\overline{t}; Z_t \otimes u^r, W_t}, (Z_t \otimes u^r)_s, (W_t \otimes v)_s) ds
\]

where \(|H| \leq C\) and \(|\tilde{H}| \leq C\) due to Assumption 3.1. We have from (5.9),

\[
\tilde{y}_s^{\overline{t}; Z_t \otimes u^r, W_t} = \phi(X_s^{\overline{t}; Z_t \otimes u^r, W_t}; (Z_t \otimes u^r)_s, (W_t \otimes v)_s)
\]

Notice that (5.11) is a linear BSDE; hence, it has a unique solution. In particular, by using the functional Itô formula in Lemma 2.7 and [51, Proposition 4.1.2], its explicit solution can be written as follows:

\[
\tilde{y}_s^{\overline{t}; Z_t \otimes u^r, W_t} = \mathbb{E}\left[\tilde{y}_{s+}\Phi_{s+}\Phi_{s+}\right]
\]

where \( \Phi \) is the scalar-valued state transition process given by

\[
\begin{cases}
d\Phi_r = \Phi_r H_r dr + \Phi_r \overline{H}_r dB_r, \quad r \in (t, t+i) \\
\Phi_t = 1.
\end{cases}
\]
From (5.5) and (5.6), together with (5.12) and the predictable dependence of \( \phi \),

\[
\mathbb{L}(\tilde{A}_t; Z_t, W_t) - \phi(\tilde{A}_t; Z_t, W_t) + \epsilon' \tau \geq \mathbb{E} \left[ -y_{t+\tau}^{t+\tau} \tilde{A}_t; Z_t \otimes u', W_t \otimes v \Phi_{t+\tau} \right. \\
+ \int_{t}^{t+\tau} \Phi_r F(X_r^{t+\tau}; Z_t \otimes u', W_t \otimes v, (Z_t \otimes u')_r, (W_t \otimes v)_r) \, dr \, |\mathcal{F}_t \right] \\
= \mathbb{E} \left[ \int_{t}^{t+\tau} F(\tilde{A}_t, (Z_t \otimes u')_r, (W_t \otimes v)_r) \, dr \, |\mathcal{F}_t \right] + L^{(1)} + L^{(2)} + L^{(3)},
\]

where

\[
L^{(1)} := -\mathbb{E} \left[ y_{t+\tau}^{t+\tau} \tilde{A}_t; Z_t \otimes u', W_t \otimes v \Phi_{t+\tau} |\mathcal{F}_t \right] \\
L^{(2)} := \mathbb{E} \left[ \int_{t}^{t+\tau} F(X_r^{t}; Z_t \otimes u', W_t \otimes v, (Z_t \otimes u')_r, (W_t \otimes v)_r) \, dr \, |\mathcal{F}_t \right] \\
- \int_{t}^{t+\tau} F(\tilde{A}_t, (Z_t \otimes u')_r, (W_t \otimes v)_r) \, dr \, |\mathcal{F}_t \right] \\
L^{(3)} := \mathbb{E} \left[ \int_{t}^{t+\tau} \Phi_r F(X_r^{t}; Z_t \otimes u', W_t \otimes v, (Z_t \otimes u')_r, (W_t \otimes v)_r) \, dr \, |\mathcal{F}_t \right] \\
- \int_{t}^{t+\tau} F(X_r^{t}; Z_t \otimes u', W_t \otimes v, (Z_t \otimes u')_r, (W_t \otimes v)_r) \, dr \, |\mathcal{F}_t \right].
\]

In view of (ii) in Lemma 3.2 and the fact that \( \Phi \) is the linear SDE, we have

\[
\mathbb{E} \left[ \|X_r^{t+\tau}; Z_t \otimes u', W_t \otimes v - \tilde{A}_r \|_\infty^2 |\mathcal{F}_t \right] \leq C \tau \\
\mathbb{E} \left[ \sup_{r \in [t, t+\tau]} |\Phi_r - 1|^2 |\mathcal{F}_t \right] \leq C \tau.
\]

Furthermore, due to the property of \( \phi \in \mathcal{C}^{1,2}_\alpha(A; \hat{A}) \) and Assumption 3.1, for \( t_1, t_2 \in [t, T] \) and \( A_{t_1}, A_{t_2} \in A \),

\[
|\phi(A_{t_1}; Z, W) - \phi(A_{t_2}; Z, W)| \leq C d_{\alpha}^{\infty}(A_{t_1}, A_{t_2}) \\
|F(A_{t_1}; Z, W) - F(A_{t_2}; Z, W)| \leq C d_{\alpha}^{\infty}(A_{t_1}, A_{t_2}).
\]

Then, from the definition of the viscosity super-solution (ii) in Definition 5.3, Lemmas 4.2 and 4.5, and the property of \( \phi \), we have

\[
\mathbb{L}(\tilde{A}_t; Z_t, W_t) - \phi(\tilde{A}_t; Z_t, W_t) \\
= \mathbb{L}(\tilde{A}_t; Z_t, W_t) - \phi(\tilde{A}_t; Z_t, W_t) + \mathbb{L}(\tilde{A}_t; Z_t, W_t) - \mathbb{L}(\tilde{A}_t; Z_t, W_t) \\
+ \phi(\tilde{A}_t; Z_t, W_t) - \phi(\tilde{A}_t; Z_t, W_t) \\
\leq C \|\tilde{A}_t - \tilde{A}_t\|_{\infty} + C \|\tilde{A}_t - \tilde{A}_t\|_{\infty}^\alpha \leq C (4 \mu_0 \epsilon \mu^{-1})^\alpha.
\]
Note that by (5.9), (5.14), Hölder inequality, Lemma 3.2 and (5.4),
\[
|L^{(1)}| \leq \mathbb{E}(\xi^c < t + \tau)^{\frac{1}{2}} \mathbb{E}\left[ \left| \phi(X_t^{\tilde{A}^c_t; Z_t \otimes u'}, W_t \otimes v) \right| \right. \\
- \phi(\tilde{A}^c_t, (Z_t \otimes u')_{t+\tau}, (W_t \otimes v)_{t+\tau}) \\
+ \phi(\tilde{A}^c_t, (Z_t \otimes u')_{t+\tau}, (W_t \otimes v)_{t+\tau}) \\
- \mathbb{L}(\tilde{A}^c_t, (Z_t \otimes u')_{t+\tau}, (W_t \otimes v)_{t+\tau}) \\
\left. + \mathbb{L}(\tilde{A}^c_t, (Z_t \otimes u')_{t+\tau}, (W_t \otimes v)_{t+\tau}) \right|^{2} \Phi_{t+\tau}^{2} |\mathcal{F}_{t}^{c}\right]^{\frac{1}{2}}
\]
(5.15) \]
\[
\leq C(1 + \mu_0^n) \left( \tau^{\frac{3}{4} (\frac{1}{2} - \kappa)} \epsilon^{-\frac{4}{5}} + \frac{\tau^{\frac{2}{3}}}{d^{\frac{2}{3}}} \right) \tau^{\frac{1}{2}} + C(4\mu_0 \epsilon \mu^{-1})^{\frac{2}{3}}.
\]
From (ii) of Lemma 3.2, we also have
\[
|L^{(2)}| \leq \mathbb{E}\int_{t}^{t+\tau} \left| F(\tilde{A}^c_t, (Z_t \otimes u')_{r}, (W_t \otimes v)_r) \\
- F(\tilde{A}^c_t, (Z_t \otimes u')_{r}, (W_t \otimes v)_r) \right| dr |\mathcal{F}_{t}^{c}|
\]
\[
+ \mathbb{E}\int_{t}^{t+\tau} \left| F(X_t^{\tilde{A}^c_t; Z_t \otimes u'}, W_t \otimes v, (Z_t \otimes u')_{r}, (W_t \otimes v)_r) \\
- F(\tilde{A}^c_t, (Z_t \otimes u')_{r}, (W_t \otimes v)_r) \right| dr |\mathcal{F}_{t}^{c}|
\]
\[
\leq C\tau \mathbb{E}\left[d_{\infty}^{c}(X_t^{\tilde{A}^c_t; Z_t \otimes u'}, W_t \otimes v, \tilde{A}^c_t)|\mathcal{F}_{t}^{c}\right] + C\tau d_{\infty}^{c}(\tilde{A}^c_t, \tilde{A}_t)
\]
(5.16)
\[
\leq C\tau^{1+\frac{3}{2}} + C\tau(4\mu_0 \epsilon \mu^{-1})^{\kappa},
\]
and
\[
|L^{(3)}| \leq \mathbb{E}\int_{t}^{t+\tau} \left( \Phi_{r} - 1 \right) dr |\mathcal{F}_{t}^{c} \right. \\
\leq C\tau \mathbb{E}\left[ \sup_{\tau \in [t, t+\tau]} \left| \Phi_{r} - 1 \right| |\mathcal{F}_{t}^{c} \right] \leq C\tau^{\frac{1}{2}}.
\]
Hence, by substituting (5.14)-(5.17) into (5.13), we have
\[
(4\mu_0 \epsilon \mu^{-1})^{\kappa} \frac{1}{\tau} + (1 + \mu_0^n) \left( \tau^{\frac{3}{4} (\frac{1}{2} - \kappa)} \epsilon^{-\frac{4}{5}} + \frac{\tau^{\frac{2}{3}}}{d^{\frac{2}{3}}} \right) \tau^{\frac{1}{2}} + C(4\mu_0 \epsilon \mu^{-1})^{\frac{2}{3}} + C\tau^{\frac{1}{2}} + \epsilon' \\
\geq \frac{1}{\tau} \mathbb{E}\left[ \int_{t}^{t+\tau} F(\tilde{A}_t, (Z_t \otimes u')_{r}, (W_t \otimes v)_r) dr |\mathcal{F}_{t}^{c} \right].
\]
Let \( \tau = \mu^{-\frac{2}{3}} \). Then the arbitrariness of \( v \) and \( \epsilon' \), and the definition of \( F \) imply that
\[
0 \geq \limsup_{\mu \to \infty} \mathbb{H}^{-}(\tilde{A}_t, Z_t, W_t, (\partial_t \phi^{\kappa, v}, \phi, \partial_x \phi, \partial_{xx} \phi)(\tilde{A}_t; Z_t, W_t)).
\]
This shows that the lower value functional in (3.6) is the viscosity super-solution of the lower PHJI equation in (5.2).
Next, we prove that (3.6) is the viscosity sub-solution of the lower PHJI equation in (5.2). From the definition of the viscosity sub-solution (i) in Definition 5.3 and
Lemma 2.5, for \( \phi \in \mathcal{C}^{1,2}(\Lambda; \hat{A}) \), \( \mu, \mu_0 > 0 \),
\[
0 = (L - \phi)(A_t; Z_t, W_t) = \sup_{A_t \in \mathcal{C}^{\mu,\mu_0}} (L - \phi)(A_t; Z_t, W_t),
\]
where \( \hat{A}_t \in \mathcal{C}^{\mu,\mu_0} \). This implies that \( L(A_t; Z_t, W_t) = \phi(A_t; Z_t, W_t) \), and for \( A_t \neq \hat{A}_t \),
\[
\phi(A_t; Z_t, W_t) \geq L(A_t; Z_t, W_t).
\]

In view of Lemmas 4.2 and 4.5, it is clear that \( L \in \mathcal{C}(\Lambda; \hat{A}) \), and due to the definition of the lower value functional, we have \( L(A_T; Z_T, W_T) \leq m(A_T) \). Then it is necessary to prove that
\[
\liminf_{\mu \to \infty} H^{-}(\hat{A}_t; Z_t, W_t, (\partial_t \phi^u, \phi, \partial_x \phi, \partial_{xx} \phi)(\hat{A}_t; Z_t, W_t)) \geq 0.
\]

Now, suppose that this is not true, i.e., there exists a finite \( \mu' > 0 \) such that for some \( \theta > 0 \),
\[
H^{-}(\hat{A}_t; Z_t, W_t, (\partial_t \phi^u, \phi, \partial_x \phi, \partial_{xx} \phi)(A_t; Z_t, W_t)) \leq -\theta < 0,
\]
where in view of the definition of \( F \),
\[
\sup_{v \in V} \inf_{u \in U} F(\hat{A}_t, Z_t^v, W_t^v) \leq -\theta < 0.
\]

Note that \( V \) and \( U \) are compact; hence, there exists a measurable function \( \gamma : V \to U \) such that for any \( v \in V \) with \( |r - t| \leq \tau_0 \),
\[
(5.18) \quad F(\hat{A}_r, Z_r^{\gamma(v)}, W_r^v) \leq -\frac{1}{2}\theta.
\]

On the other hand, in view of the dynamic programming principle in (4.3) of Theorem 4.3, we have
\[
\text{ess inf}_{\alpha \in \mathcal{A}[t,t+\tau]} \text{ess sup}_{v \in \mathcal{V}[t,t+\tau]} \Pi_t^{t+\tau,\hat{A}_t; Z_t; v + \alpha(v), W_t} \left[ L(X_t^{\alpha; v}, W_t^{v}; (A_t; Z_t \circ \alpha(v))_{t+\tau}, (W_t \circ v)_{t+\tau}) \right] = L(\hat{A}_t; Z_t, W_t) = 0.
\]

By defining \( \gamma_s(v) := \gamma(v_s(\omega)) \) for \( (s, \omega) \in [t,T] \times \Omega \), we have \( \gamma \in \mathcal{A}[t, t+\tau] \) and \( Z_t \circ \gamma \in \mathcal{A}[0,T] \). This, together with the definition of \( \Pi \) and the comparison principle in (iv) of Lemma 3.2, implies that
\[
\text{ess sup}_{v \in \mathcal{V}[t,t+\tau]} \Pi_t^{t+\tau,\hat{A}_t; Z_t; \gamma(v), W_t} \left[ \phi(X_t^{\alpha; v}, W_t^{v}; (Z_t \circ \gamma(v))_{t+\tau}, (W_t \circ v)_{t+\tau}) \right] - \phi(\hat{A}_t; Z_t, W_t) \geq 0.
\]

For each \( \epsilon' > 0 \), similar to (4.13), we can choose \( v' \in \mathcal{V}[t, t+\tau] \) such that
\[
\Pi_t^{t, t+\tau, \hat{A}_t; Z_t; \gamma(v'), W_t} \left[ \phi(X_t^{\alpha; v}, W_t^{v'}; (Z_t \circ \gamma(v'))_{t+\tau}, (W_t \circ v')_{t+\tau}) \right] - \phi(\hat{A}_t; Z_t, W_t) \geq -\epsilon' \tau.
\]

Note (5.9) and (5.10). Then, similar to (5.12), by using the functional Itô formula in Lemma 2.7, we have
\[
\frac{1}{\tau} \mathbb{E} \left[ \int_t^{t+\tau} \Phi_r F(X_t^{\hat{A}_r; Z_t \circ \gamma(v'), W_t \circ v'}; (Z_t \circ \gamma(v'))_r, (W_t \circ v')_r) d\mathcal{F}_r \right] \geq -\epsilon'.
\]
With the same technique as in the super-solution case and the definition of $\gamma$, by letting $\tau \downarrow 0$, the arbitrariness of $\epsilon'$ and (5.18) imply that
\[ 0 \leq F(A_t, Z_{\tau}^{(\nu)}, W_t) \leq -\frac{1}{2}\theta. \]
This induces $\theta \leq 0$, which leads to a contradiction. Hence, (3.6) is the viscosity sub-solution of the lower PHJI equation in (5.2).

The proof for the upper value functional $U$ being a viscosity solution to the upper PHJI equation in (5.3) is similar. We complete the proof of the theorem.

The remaining section provides the uniqueness result of classical solutions for the (lower and upper) PHJI equations, and shows the existence of the game value under the Isaacs condition for the state path-dependent case discussed in (1) of Remarks 3.7 and 5.2, i.e.,

\textbf{Assumption 5.7.} \( f : \Lambda \times U \times V \rightarrow \mathbb{R}^n, \sigma : \Lambda \times U \times V \rightarrow \mathbb{R}^{n \times p}, l : \Lambda \times \mathbb{R} \times \mathbb{R}^{1 \times p} \times U \times V \rightarrow \mathbb{R}. \)

For convenience, under Assumption 5.7, the lower and upper PHJI equations are given below (see (1) of Remark 5.2):
\[
\begin{cases}
\partial_t L(A_t) + \sup_{t \in \mathbb{V}} \inf_{u \in U} \mathcal{H}(A_t, u, v, (L, \partial_x L, \partial_{xx} L)(A_t)) = 0, \\
L(A_T) = m(A_T), \quad A_T \in \Lambda_T,
\end{cases}
\tag{5.19}
\]
and
\[
\begin{cases}
\partial_t U(A_t) + \inf_{u \in U} \sup_{t \in \mathbb{V}} \mathcal{H}(A_t, u, v, (U, \partial_x U, \partial_{xx} U)(A_t)) = 0, \\
U(A_T) = m(A_T), \quad A_T \in \Lambda_T.
\end{cases}
\tag{5.20}
\]
We introduce the state path-dependent Isaacs condition: for \((A_t, y, p, P) \in \Lambda \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S},\)
\[
\begin{align*}
\mathcal{H}'(A_t, y, p, P) &:= \tilde{\mathcal{H}}(A_t, y, p, P) = \tilde{\mathcal{H}}(A_t, y, p, P) \\
\mathcal{H}(A_t, y, p, P) &:= \sup_{u \in \mathbb{V}} \inf_{v \in \mathbb{U}} \mathcal{H}(A_t, u, v, y, p, P) \\
\tilde{\mathcal{H}}(A_t, y, p, P) &:= \inf_{u \in \mathbb{U}} \sup_{v \in \mathbb{V}} \mathcal{H}(A_t, u, v, y, p, P).
\end{align*}
\tag{5.21}
\]
We first state the result on the existence of the game value.

\textbf{Theorem 5.8.} Suppose that Assumptions 3.1 and 5.7, and the uniqueness of the viscosity solutions of (5.19) and (5.20) hold. Under the Isaacs condition in (5.21), the game has a value, i.e., \( L(A_t) = U(A_t) := G(A_t), \) where \( G \) is the unique viscosity solution of the PHJI equation:
\[
\begin{cases}
\partial_t G(A_t) + \mathcal{H}'(A_t, (G, \partial_x G, \partial_{xx} G)(A_t)) = 0, \quad t \in [0, T), \quad A_t \in \Lambda \\
G(A_T) = m(A_T), \quad A_T \in \Lambda_T.
\end{cases}
\tag{5.22}
\]
\[\]
\textbf{Proof.} In view of Theorem 5.5 and the uniqueness assumption, the lower value functional \( L \) and the upper value functional \( U \) are the unique viscosity solutions of (5.19) and (5.20), respectively. Then, the Isaacs condition in (5.21) implies \( L(A_t) = U(A_t) := G(A_t) \), which is the unique solution to the PHJI equation in (5.22). We complete the proof. ☐
Assumption 5.9. For any \((A_t, p) \in \Lambda \times \mathbb{R}, y_1, y_2 \in \mathbb{R}\) and \(P_1, P_2 \in \mathbb{S}^n\) with \(y_1 \geq y_2\) and \(P_1 \leq P_2\),
\[
\mathcal{H}(A_t, y_1, p, P_1) \leq \mathcal{H}(A_t, y_2, p, P_2), \quad \tilde{\mathcal{H}}(A_t, y_1, p, P_1) \leq \tilde{\mathcal{H}}(A_t, y_2, p, P_2).
\]

We now state the uniqueness result of classical solutions for (5.19) and (5.20) via the comparison principle.

**Theorem 5.10.** Assume that Assumptions 3.1, 5.7 and 5.9 hold. Suppose that \(\mathbb{L}_1\) and \(\mathbb{L}_2\) are classical sub- and super-solutions of the lower PHJI equation in (5.19), respectively. Then we have \(\mathbb{L}_1(A_t) \leq \mathbb{L}_2(A_t)\) for \(A_t \in \Lambda\). The same result holds for the upper PHJI equation in (5.20). Consequently, there is a unique classical solution of (5.19) and (5.20).

**Proof.** Let \(\tilde{\mathbb{L}}_1(A_t) := \mathbb{L}_1(A_t) - \frac{\delta}{t}\), where \(\delta > 0\). Then we can easily see that \(\tilde{\mathbb{L}}\) is a classical sub-solution of the following PDE (see Remark 5.1):
\[
\begin{cases}
\partial_t \tilde{\mathbb{L}}_1(A_t) + \tilde{\mathcal{H}}(A_t, (\tilde{\mathbb{L}}_1, \partial_x \tilde{\mathbb{L}}_1, \partial_{xx} \tilde{\mathbb{L}}_1)(A_t)) \geq \frac{\delta}{t}, & t \in [0, T), \ A_t \in \Lambda \\
\tilde{\mathbb{L}}_1(A_T) = m(A_T) - \frac{\delta}{T}, & A_T \in \Lambda.
\end{cases}
\]
Since \(\mathbb{L}_1 \leq \mathbb{L}_2\) follows from \(\tilde{\mathbb{L}}_1 \leq \tilde{\mathbb{L}}_2\) in the limit \(\delta \downarrow 0\), it suffices to prove the theorem with the following additional assumption:
\[
\partial_t \mathbb{L}_1(A_t) + \mathcal{H}(A_t, (\mathbb{L}_1, \partial_x \mathbb{L}_1, \partial_{xx} \mathbb{L}_1)(A_t)) \geq \nu > 0,
\]
where \(\nu := \frac{\delta}{T}\) and \(\lim_{t \to 0} \mathbb{L}_1(A_t) = -\infty\) uniformly on \([0, T]\).

Assume that this is not true, that is, there exists \(A_t' \in \Lambda\) with \(t' \in [0, T]\) such that \(k' := \mathbb{L}_1(A_t') - \mathbb{L}_2(A_t') > 0\). In view of Lemma 2.5, there exists \(\hat{A}_t \in \mathbb{C}^{\kappa, \mu, \mu_0}\) with \(\hat{t} \in [0, T]\) such that
\[
\mathbb{L}_1(A_t) - \mathbb{L}_2(A_t) = \sup_{A_t \in \mathbb{C}^{\kappa, \mu, \mu_0}} \mathbb{L}_1(A_t) - \mathbb{L}_2(A_t) \geq k'.
\]

Then from [37, Lemma 9], we have
\[
\partial_t (\mathbb{L}_1 - \mathbb{L}_2)(\hat{A}_t) \leq 0, \quad \partial_x (\mathbb{L}_1 - \mathbb{L}_2)(\hat{A}_t) = 0, \quad \partial_{xx} (\mathbb{L}_1 - \mathbb{L}_2)(\hat{A}_t) \leq 0.
\]
This, together with Assumption 5.9 and the fact that \(\mathbb{L}_2\) is the classical super-solution, implies that
\[
0 \geq \partial_t \mathbb{L}_2(\hat{A}_t) + \mathcal{H}(\hat{A}_t, (\mathbb{L}_2, \partial_x \mathbb{L}_2, \partial_{xx} \mathbb{L}_2)(\hat{A}_t))
\geq \partial_t \mathbb{L}_1(\hat{A}_t) + \mathcal{H}(\hat{A}_t, (\mathbb{L}_1, \partial_x \mathbb{L}_1, \partial_{xx} \mathbb{L}_1)(\hat{A}_t)) \geq \nu > 0,
\]
which induces a contradiction. Hence, \(\mathbb{L}_1(A_t) \leq \mathbb{L}_2(A_t)\) for \(A_t \in \Lambda\).

Now, suppose that \(\mathbb{L}_1\) and \(\mathbb{L}_2\) are classical solutions of (5.19). Then we have \(\mathbb{L}_1 \leq \mathbb{L}_2\) and \(\mathbb{L}_2 \geq \mathbb{L}_1\), which implies \(\mathbb{L} := \mathbb{L}_1 = \mathbb{L}_2\). Hence, the uniqueness follows. The same argument can be applied to the upper PHJI equation. This completes the proof. \(\square\)

**Remark 5.11.** As expected, the assumptions in Theorem 5.10 are less technical than those for the viscosity solution case in [38, Theorem 6.1].

Based on Theorems 5.8 and 5.10, we have the following corollary:

**Corollary 5.12.** Suppose that Assumptions 3.1, 5.7, 5.9 and the Isaacs condition in (5.21) hold, and that \(\mathbb{L}\) and \(\mathbb{U}\) are classical solutions of (5.19) and (5.20), respectively. Then the game has a value, i.e., \(\mathbb{L}(A_t) = \mathbb{U}(A_t) := \mathbb{G}(A_t)\), and \(\mathbb{G}\) is the unique classical solution of (5.22).
There are several interesting potential future research problems.

One important problem is the uniqueness of viscosity solutions of the (lower and upper) PHJI equations in (5.2) and (5.3). Under the different definitions of viscosity solutions, the uniqueness results of path-dependent PDEs are given in [16, 37, 38]. We expect to use the path-frozen PDE approach as in [16, 38] or the left-frozen approach in [37], where the latter approach might be useful, since our set $\mathbb{C}^{\kappa,\mu,\mu_0}$ is compact due to Lemma 2.5. However, these approaches require some strong technical assumptions.

Another problem is the existence of the (approximated) saddle-point equilibrium using the notion of nonanticipative strategies with delay as mentioned in (2) of Remark 3.6. With the additional Assumption 5.7 and the Isaacs condition in (5.21), this was shown in [50, Theorem 4.13], where the key step is approximating the PHJI equation in (5.19) and (5.20) to the state-dependent (not state path-dependent) HJI equations. Note that there is a unique viscosity solution of the approximated (lower and upper) state-dependent HJI equations in view of [6, Theorem 5.3]. Then, the existence of the (approximated) saddle-point equilibrium can be shown using the property of nonanticipative strategies with delay [5, Lemma 2.4]. We speculate that the approach of [50, Theorem 4.13] can be applied to the problem of this paper, that is, the case without Assumption 5.7.

We can consider the problem in weak formulation. As noted in (3) of Remark 3.6, one major feature of this formulation is the symmetric feedback information between the players, which is convenient to show the existence of the saddle-point equilibrium and the game value. However, in order to show the existence and uniqueness of viscosity solutions, additional technical assumptions are required [38, 39].

Finally, the forward-backward stochastic differential equation given in (3.1) and (3.2) is not fully coupled in the sense that the BSDE in (3.2) is not included in the (forward) SDE in (3.1). This can be extended to the fully-coupled FBSDE, where (3.1) is also dependent on (3.2). This can be viewed as a generalization of [29], where the major challenge is the case when the diffusion term of (3.1) depends on the second component of the solution of the BSDE, since the associated PHJI equation requires an additional algebraic equation.

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