Abstract. We show that the Khovanov-Jacobsson number of an embedded torus in $\mathbb{R}^4$ is always $\pm 2$.

1. Introduction

In his original paper describing his Jones polynomial homology for knots [5] — now known as the Khovanov homology — Khovanov described how his construction could be used to define an invariant of embedded surfaces in $\mathbb{R}^4$. More precisely, let $S \subset \mathbb{R}^3 \times [0,1]$ be a smoothly embedded, orientable cobordism between knots $K_0$ and $K_1$. Given a decomposition of $S$ into elementary cobordisms, Khovanov defined a graded map $\phi_S : Kh(K_0) \to Kh(K_1)$ and conjectured that up to sign, $\phi_S$ was an invariant of $S$. This conjecture was subsequently proved by Jacobsson [4] and Khovanov [6], and later in a more general form by Bar-Natan [1].

The Khovanov homology of the empty link is $\mathbb{Z}$, so this construction associates a homomorphism $\phi_{\Sigma} : \mathbb{Z} \to \mathbb{Z}$ (well defined up to sign) to a closed, orientable, smoothly embedded surface $\Sigma \subset \mathbb{R}^4$. Defining $n_{\Sigma} = |\phi_{\Sigma}(1)|$, we obtain an invariant of $\Sigma$, which is generally referred to as the Khovanov-Jacobsson number. Since $\phi_{\Sigma}$ is a graded map of degree $\chi(\Sigma)$, $n_{\Sigma} = 0$ unless $\chi(\Sigma) = 0$. Thus, the simplest case in which one might have a nonzero invariant is when $\Sigma$ is a torus. In [3], Carter, Saito, and Satoh showed that $n_{\Sigma} = 2$ for a large class of embedded tori, namely those which can be unknotted by a double point move. In fact, this relation holds for all embedded tori in $\mathbb{R}^4$:

**Theorem.** If $\Sigma \subset \mathbb{R}^4$ is a smoothly embedded torus, then $n_{\Sigma} = 2$.

The same result has also been obtained by Tanaka [9] using a slightly different method.

The proof of the theorem is straightforward extension of the techniques of [8]. We briefly recall the setup of that paper, and refer the reader to it for more details. Let $D$ be a planar diagram of a link $L$. Associated to $D$, there is a graded chain complex $CKh(D)$ whose homology is $Kh(L)$. In [7], Lee defined a related filtered complex $CKh'(D)$. The filtration gives rise to a spectral sequence whose $E_1$ term is the complex $CKh(D)$. This spectral sequence converges to the homology group $Kh'(L)$, which she explicitly calculated.

To be precise, if $L$ has $l$ components, then $Kh'(L)$ has rank $2^l$ over any field with characteristic not equal to 2. Moreover, Lee gave an explicit correspondence between orientations on $L$ and generators of $Kh'(L)$. In [5], it was observed that these generators are "projectively canonical," in the sense that they are preserved up to scalar multiplication by the isomorphisms induced by Reidemeister moves. As we explain in section 3 it turns out that after a slight modification, Lee’s generators are actually canonical up to sign. This enables us to give a more precise calculation of the maps induced by cobordisms in Lee’s theory. We then explain how this calculation can be used to prove the theorem above.

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2. Cobordisms

We begin with some generalities on cobordisms, orientations, and induced maps. Let $S \subset \mathbb{R}^3 \times [0, 1]$ be a smoothly, properly embedded, orientable surface with boundary $L_1 \times 0 \cup L_2 \times 1$. We say that $S$ is a cobordism from $L_1$ to $L_2$, and write $S : L_1 \to L_2$. Next, suppose that $o_1$ and $o_2$ are orientations on $L_1$ and $L_2$. We say that $o_1$ and $o_2$ are compatible if there is an orientation $o$ on $S$ which induces $o_1$ on $L_1$ and the reverse of $o_2$ on $L_2$. (Thus if $S : L_1 \to L_1$ is the identity cobordism, $o_1$ is compatible with itself.)

2.1. Induced Maps. If $S_1 : L_1 \to L_2$ and $S_2 : L_2 \to L_3$ are cobordisms, we can compose them to get a cobordism $S : L_1 \to L_3$. Conversely, it is well known \[\text{[2]}\] that any cobordism is isotopic to a composition of certain elementary cobordisms which may be represented by fixed local moves on planar diagrams. The necessary local moves are given by the Reidemeister moves, their inverses, and the Morse moves (addition of a 0 or 2-handle.)

Following Khovanov [3], we want to assign to a cobordism $S : L_1 \to L_2$ an induced map $\phi_S : Kh(L_1) \to Kh(L_2)$ and $\phi'_S : Kh'(L_1) \to Kh'(L_2)$. (Note that this notation differs from that of \[\text{[8]}\], where the induced map on $Kh'$ was denoted by $\phi_S$.) Since we would like this assignment to be functorial, it suffices to define the induced maps associated to the various elementary cobordisms.

First, suppose $S : L \to L$ is a Reidemeister move relating two diagrams $D$ and $D'$ of $L$. In \[\text{[3]}\], Khovanov defined maps $\rho_i : CKh(D) \to CKh(D')$ and $\rho'_i : CKh'(D) \to CKh'(D')$ which induce isomorphisms on homology. In this case, we define $\phi_S$ to be $\rho_i \rho'_i$. Likewise, if $S : L \to L$ is the inverse of a Reidemeister move, $\phi_S = \rho_i^{-1} \rho'_i^{-1}$.

Now suppose $S : D_1 \to D_2$ is an elementary cobordism associated to a Morse move. To define $\phi_S$ in this case, we recall that the chain complex $CKh(D_1)$ is defined using a certain 1 + 1-dimensional TQFT $A$. $CKh(D_1)$ is generated by elements of the form $v \in A(D_{1,v})$, where $D_{1,v}$ is a complete resolution of the diagram $D_1$. $D_{1,v}$ naturally determines a complete resolution $D_{2,v}$ of $D_2$, and the cobordism $S$ induces a cobordism $S_v : D_{1,v} \to D_{2,v}$. We define a map $\phi : CKh(D_1) \to CKh(D_2)$ by setting $\phi(v) = A(S_v)(v)$ for $v \in D_{1,v}$. $\phi_S$ is defined to be the induced map on homology. With these definitions, Jacobsson \[\text{[4]}\] and Khovanov [3] showed that $\phi_S$ is well defined up to sign, regardless of how $S$ was decomposed into elementary cobordisms.

Exactly the same procedure can be used to define an induced map $\phi'_S : Kh'(L_1) \to Kh'(L_2)$ in Lee’s theory. Indeed, Lee’s theory is formally identical to Khovanov’s, but with the TQFT $A$ replaced by a new TQFT $A'$. (See [1] for a realization of both theories as specializations of a more general, geometric theory, and for a universal proof that maps induced by cobordisms are well defined.)

2.2. Comparison of $\phi_S$ and $\phi'_S$. Recall from \[\text{[2]}\] that $Kh(L)$ and $Kh'(L)$ are connected by a spectral sequence. Indeed, the TQFT $A$ used to define $Kh$ is a graded TQFT, and this naturally gives rise to a grading (known as the $q$-grading) on $CKh(D)$. Moreover, the TQFT $A'$ used to define $CKh'(D)$ is a perturbation of $D$: on the level of groups, the two theories are isomorphic, and the maps induced by cobordisms agree to lowest order in the $q$-grading. As a consequence, the $q$ grading defines a filtration on $CKh'(D)$, which gives a spectral sequence $E^i(D)$ converging to $Kh'(L)$. Moreover, we have $E^1(D) \cong Kh(L)$. This spectral sequence behaves functorially with respect to cobordisms. To be precise, we have

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Lemma 2.1. Let $S: L_1 \to L_2$ be a cobordism with a fixed decomposition into elementary cobordisms. Then for $i \geq 1$, $S$ induces a morphism of spectral sequences $\phi_1^i: E^i(L_1) \to E^i(L_2)$ converging to $\phi_1^\infty$. If $\phi_S^1: Kh(L_1) \to Kh(L_2)$ is the induced map on filtered gradeds, then $\phi_S^1 = \phi_S$.

Proof. This follows from standard properties of spectral sequences. Indeed, if $f: A \to B$ is a map of filtered complexes, then there is an induced morphism of spectral sequences $f^*: A^i \to B^i$ which converges to $f_\ast$. Moreover, if $\overline{f}^i: \overline{A}^i \to \overline{B}^i$ is the induced chain map on filtered gradeds, it is not difficult to see that $\overline{f}^i = \overline{f}_1^i$. In the case when $S$ is an elementary cobordism induced by a Morse move, we are in precisely this situation with the map $\phi : CKh(D_1) \to CKh(D_2)$. The argument is very similar when $S$ is an elementary cobordism associated to a Reidemeister move. (See the proof of Theorem 1 in [8] for more details). Finally, the result for a general cobordism $S$ follows by functoriality. \hfill $\square$

3. Canonical generators

Let $L$ be a link represented by a planar diagram $D$. Given an orientation $o$ of $L$, Lee associates to it a state $s_o \in CKh'(D)$ and shows that $Kh'(L)$ is freely generated by $\{|s_o| \mid o \text{ is an orientation of } L\}$. It turns out that there is a somewhat more natural way to normalize the $s_o$’s:

Definition 3.1. Let $w(o)$ be the writhe of the diagram $D$ endowed with the orientation $o$, and let $k(o)$ be the number of circles in its oriented resolution. We define the rescaled canonical generator associated to the orientation $o$ by

$$\overline{s}_o = 2^{w(o) - k(o)/2}s_o$$

To see why this choice of generators is a good one, we consider the behavior of $\overline{s}_o$ under the map induced by a Morse cobordism.

Proposition 3.2. Let $S: L_1 \to L_2$ be a cobordism with no closed components. If $o$ is an orientation on $L_1$, then

$$\phi_1(S)(\overline{s}_o) = 2^{-\chi(S)}\sum_I \pm \overline{s}_I$$

where the sum runs over all orientations on $L_2$ compatible with $o$.

Proof. We work our way up to the proof in stages, starting with the easiest possible case.

Lemma 3.3. Equation 1 holds in the case where $S: L_1 \to L_2$ is an elementary Morse cobordism.

The proof is an easy calculation along the lines of the proof of Proposition 4.1 in [8]. We leave its verification to the reader.

Lemma 3.4. Equation 1 holds in the case where $S: L_1 \to L_2$ is a composition of elementary Morse cobordisms with no closed component.

Proof. We induct on the number of elementary cobordisms in the composition. The base case of one cobordism is covered by the lemma; if there is more than one cobordism, we decompose $S$ into the composition of two cobordisms $S_1: L_1 \to L_{1,5}$ and $S_2: L_{1,5} \to L_2$ for which the statement is known to hold. Then we compute

$$\phi_1(S)(\overline{s}_o) = 2^{-\chi(S_1) - \chi(S_2)}\sum_{(o_1,5,0_2)} \pm \overline{s}_{o_2} = 2^{-\chi(S)}\sum_{(o_1,5,0_2)} \pm \overline{s}_{o_2}$$
where the sum runs over pairs \((o_1, o_2)\) such that \(o\) is compatible with \(o_{1,5}\) and \(o_{1,5}\) is compatible with \(o_2\). If this is the case, \(o_2\) is clearly compatible with \(o_1\). Conversely, given \(o_2\) on \(L_2\) compatible with \(o\), the fact that \(S\) has no closed components implies that there is a unique orientation on \(S\) compatible with \(o\) and \(o_2\). Thus \(o_2\) is associated with a unique compatible pair \((o_{1,5}, o_2)\).

To prove the proposition in general, we must check that the generators \(\mathfrak{F}_o\) behave well under the isomorphisms associated to Reidemeister moves.

**Lemma 3.5.** Suppose \(R_i : D \to \tilde{D}\) is a Reidemeister move relating \(D\) to another diagram \(\tilde{D}\) of \(L\), and let \(\rho_i' : \text{Kh}'(D) \to \text{Kh}'(\tilde{D})\) be the corresponding induced map. Let \(o\) be an orientation on \(D\) and \(\tilde{o}\) be the corresponding orientation on \(\tilde{D}\). Then \(\rho_i'([\mathfrak{F}_o]) = \pm [\mathfrak{F}_{\tilde{o}}]\).

**Proof.** Essentially, this follows from the fact that the \(\rho_i\) are defined in terms of maps induced by elementary cobordisms. Below, we give a more detailed argument for each Reidemeister move.

**Reidemeister I Move:** Let \(\tilde{D}\) be a diagram obtained from \(D\) by adding a left-hand curl, and let \(\tilde{D}(\ast 0)\) be the diagram obtained by giving this crossing the 0 resolution. Then the map \(\rho_1 : \text{Kh}'(D_1) \to \text{Kh}'(\tilde{D}(\ast 0)) \subset \text{CKh}'(\tilde{D})\) is given by \(\rho_1 = \phi_{S_1}^\prime - \phi_{S_2}^\prime\), where \(S_1\) is the obvious 1-handle cobordism from \(D\) to \(\tilde{D}(\ast 0)\), and \(S_2\) is the product cobordism connect summed with a trivial torus, followed by the addition of a zero handle. Both \(S_1\) and \(S_2\) have \(\chi = -1\). Using Lemma 3.4 (with a little direct computation to get the signs right), we see that

\[
\rho_1([\mathfrak{F}_o]) = 2^{1/2} [\mathfrak{F}_{o_1} - ([\mathfrak{F}_{o_1} - [\mathfrak{F}_{o_2}])]
= 2^{1/2} \mathfrak{F}_{o_2}
= \mathfrak{F}_{\tilde{o}}.
\]

Here \(o_1\) and \(o_2\) are the two orientations on \(\tilde{D}(\ast 0)\) compatible with \(o\) under \(S_2\). The last equality follows from the fact that \(w(\tilde{D}) = w(\tilde{D}(\ast 0)) + 1\).
Reidemeister II Move: Let \( \tilde{D} \) be obtained from \( D \) by adding a pair of cancelling intersections. The complex \( \CKh'(\tilde{D}) \) is illustrated in Figure 4. The part labeled \( \tilde{D}(01) \) is naturally identified with \( \CKh'(D) \), and for \( x \in \CKh'(D) \), we have \( \rho'_2(x) = x + \alpha(x) \), where the map \( \alpha : \CKh'(\tilde{D}(01)) \to \CKh'(\tilde{D}(10)) \) is induced by a cobordism \( S : \tilde{D}(01) \to \tilde{D}(10) \). \( S \) is the composition of two elementary Morse cobordisms: from \( \tilde{D}(01) \) to \( \tilde{D}(11) \) by addition of a 1-handle, and then from \( \tilde{D}(11) \) to \( \tilde{D}(10) \) by addition of a 0-handle.

Fix an orientation \( o \) on \( D \). We consider two cases. First, suppose that the two strands involved in the move have parallel orientations, so that the oriented resolution of \( \tilde{D}(01) \). There is no orientation on \( \tilde{D}(11) \) compatible with \( o \), so \( \alpha(\mathfrak{s}_o) = 0 \). Since \( w(o) = w(\bar{o}) \), it follows that \( \rho'_2(\mathfrak{s}_o) = \mathfrak{s}_{\bar{o}} \).

Now suppose that the two strands involved in the move have opposite orientations, so that the oriented resolution of \( \tilde{D} \) is the oriented resolution of \( \tilde{D}(01) \). In this case, there are two orientations \( o_1 \) and \( o_2 \) on \( \tilde{D}(01) \) compatible with \( o \). Since \( \chi(S) = 0 \), Corollary 3.4 implies that \( \alpha(\mathfrak{s}_o) = \pm \mathfrak{s}_{o_1} \pm \mathfrak{s}_{o_2} \). Thus we have \( \rho'_2(\mathfrak{s}_o) = \mathfrak{s}_o \pm \mathfrak{s}_{o_1} \pm \mathfrak{s}_{o_2} \). Now one of \( \mathfrak{s}_{o_1} \) or \( \mathfrak{s}_{o_2} \) is equal to \( \mathfrak{s}_{\bar{o}} \) — without loss of generality, let us assume it is \( o_1 \). Then, as was observed in the proof of Proposition 2.3 of \( \mathbb{R} \), the remaining term \( \mathfrak{s}_o \pm \mathfrak{s}_{o_2} \) is exact in \( \CKh'(\tilde{D}) \). Thus we have \( \rho'_2(\mathfrak{s}_o) = \pm \mathfrak{s}_o \), and the claim holds in this case as well.

Reidemeister III Move: Let \( D \) and \( \tilde{D} \) be as shown in Figure 2. In this case, we decompose the complexes as shown in the figure. If \( \mathfrak{s}_{\bar{o}} \) is contained in \( \tilde{D}(01) \), the argument is easy. Indeed, there is a natural identification \( \iota : D(1) \to \tilde{D}(1) \), and we have \( \rho_3(\mathfrak{s}_{\bar{o}}) = \iota(\mathfrak{s}_{\bar{o}}) = \mathfrak{s}_o \).

On the other hand, if \( \mathfrak{s}_o \) is contained in \( \tilde{D}(0) \), the argument proceeds much as in the case of the Reidemeister II move. Given \( \mathfrak{s}_o \), we first find a homologous element of the form \( x + \alpha(x) \), where \( x \in \tilde{D}(100) \). Under \( \rho_3 \), this element is mapped to \( x + \bar{\alpha}(x) \) via the natural identification \( \tilde{D}(100) = \tilde{D}(010) \). As with the Reidemeister II move, there are two cases to consider. Either \( \mathfrak{s}_o \in \tilde{D}(101) \), in which case \( \alpha(\mathfrak{s}_o) = \bar{\alpha}(\mathfrak{s}_o) = 0 \), or \( \mathfrak{s}_o \in \tilde{D}(010) \), in which case it is homologous to \( \pm \mathfrak{s}_{o_1} \pm \mathfrak{s}_{o_2} = \pm \mathfrak{s}_{o_1} + \alpha(\pm \mathfrak{s}_{o_1}) \). This in turn, maps to \( \pm \mathfrak{s}_{o_1} + \bar{\alpha}(\pm \mathfrak{s}_{o_1}) \), which is homologous to \( \pm \mathfrak{s}_{\bar{o}} \).

Proposition 3.2 now follows by combining Lemma 3.4 with the proof of Lemma 3.4.

We now specialize to the case where \( S : K_1 \to K_2 \) is a connected cobordism between two knots. In this case, there are two orientations \( o \) and \( o' \) on \( K_1 \). When there is no risk of confusion, we denote the corresponding compatible orientations on \( K_2 \) by \( o \) and \( o' \) as well. Recall from section 3 of \( \mathbb{R} \) that the \( q \)-grading on \( Kh'(K_1) \) is well-defined mod 4, and that \( \mathfrak{s}_o + \mathfrak{s}_{o'} \) and \( \mathfrak{s}_o - \mathfrak{s}_{o'} \) are homogenous elements whose \( q \)-gradings differ by 2 mod 4. From the proposition, it follows that \( \phi'_S(\mathfrak{s}_o - \mathfrak{s}_{o'}) \) is a multiple of either \( \mathfrak{s}_o - \mathfrak{s}_{o'} \) or \( \mathfrak{s}_o + \mathfrak{s}_{o'} \); which one is determined by the mod 4 \( q \)-grading of \( \phi'_S \). We have thus arrived at

Corollary 3.6. Let \( S : K_1 \to K_2 \) be a connected cobordism between two knots. Then if \( \chi(S) \equiv 0 \mod 4 \),

\[ \phi'_S(\mathfrak{s}_o - \mathfrak{s}_{o'}) = \pm 2^{-\chi(S)}(\mathfrak{s}_o - \mathfrak{s}_{o'}) \]

while if \( \chi(S) \equiv 2 \mod 4 \),

\[ \phi'_S(\mathfrak{s}_o - \mathfrak{s}_{o'}) = \pm 2^{-\chi(S)}(\mathfrak{s}_o + \mathfrak{s}_{o'}). \]

It is now easy to assemble the proof of the theorem stated in the introduction. Indeed, suppose \( \Sigma \subset \mathbb{R}^4 \) is a closed surface of genus one. Then by choosing an appropriate Morse function on \( \mathbb{R}^4 \), we can decompose \( \Sigma \) into the union of a 0-handle, a cobordism \( S : U \to U \), and a 2-handle. It follows that \( n_\Sigma = |\epsilon(\phi_S(v_+))| \), where \( \epsilon(v_-) = 1 \).
We claim that in this case, we have $\phi_S(v_+) = \phi'_S(v_+)$. Indeed, by Lemma 2.1, we know that $\phi_S$ is the map on filtered gradeds associated to a morphism of spectral sequences which converges to $\phi'_S$. Since the knots in question are both the unknot, the spectral sequences
converge at the $E^1$ term. Thus we can write

$$
\phi'_S(v_+) = \phi_S(v_+) + \psi(v_+)
$$

where $q(\psi(v_+)) \geq q(v_+) + \chi(S) + 4 = 3$. Since $Kh(U)$ is trivial in $q$-grading 3 and higher, we must have $\psi(v_+) = 0$. This proves the claim.

We can now calculate

$$
\phi'_S(v_+) = \phi'_S\left(\frac{1}{2}(s_o - s_{o'})\right)
$$

$$
= \phi'_S\left(2^{-1/2}(s_o - s_{o'})\right)
$$

$$
= \pm 2^{-1/2}\chi(S)/2\left[2^{-1/2}(s_o + s_{o'})\right]
$$

$$
= \pm 2\left[\frac{1}{2}(s_o - s_{o'})\right]
$$

$$
= \pm 2v_-
$$

which completes the proof. □

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Princeton University Dept. of Mathematics, Princeton, NJ 08544

E-mail address: jrasmus@math.princeton.edu