Skyrme Model with Different Mass Terms

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Abstract

We consider a one parameter family of mass terms for the Skyrme model that disfavours shell-like configurations for multi-baryon classical solutions. We argue that a model with such mass terms can provide a better description of nuclei as shell-like configurations are now less stable than in the traditional massive Skyrme model.

Initially proposed by Skyrme as a fundamental model for baryons [1], the Skyrme model has subsequently been shown [2] to be the low energy limit of QCD in the $1/N_c$ expansion. In dimensionless units, the Hamiltonian for the Skyrme model is given by

$$E = \frac{1}{12\pi^2} \int_{R^3} \left\{ -\frac{1}{2} \text{Tr} \left( \partial_i U U^{-1} \right)^2 + \frac{1}{16} \text{Tr} \left[ \partial_i U U^{-1}, \partial_j U U^{-1} \right]^2 \right\} d^3 \vec{x} + \frac{1}{12\pi^2} \int_{R^3} m^2 \text{Tr}(1 - U) d^3 \vec{x},$$

(1)

where $m$ denotes the pion mass.

Recently the model has been used also in the description of nuclei (as a semiclassical model describing their properties) [3]. The major problem with all the applications of the model is associated with the fact that the energy densities of its classical solutions [4] have shell-like configurations with a hole in the middle, even for a relatively large number of baryons, unless one takes for the pion mass $m$ a value several times larger than the physical one. To avoid this problem it has been suggested that one should consider the

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pion mass in the Lagrangian as a free parameter that must be fitted to the experimental data, thus somehow justifying the use of unphysical values of $m$.

Recently, we argued [5] that the mass term in the Skyrme model is not unique and that there exist a large family of such terms that have the correct asymptotic behaviour to describe pion fields. In general, the mass term, i.e., the last term in (1), which we call the potential, can be written as

$$V = \frac{1}{12\pi^2} Am^2_\pi \int_{\mathbb{R}^3} \text{Tr} \left[ 1 - \int_{-\infty}^{\infty} g(p) U^p dp \right] d^3 \vec{x},$$

(2)

where

$$\int_{-\infty}^{\infty} g(p) dp = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} g(p)p^2 dp = 1$$

(3)

and

$$A^{-1} = \int_{-\infty}^{\infty} g(p)p^2 dp.$$  

(4)

In [5] we looked in detail at the special case when $g(x) = \delta(x - p)$ for integer values of $p$ and showed that when $p$ is even and non-zero (note $p = 0$ is different), the classical solutions are very much like in the massless case and form hollow shells with a vanishing energy density at their centre. When $p$ is odd all shell-like configurations have a non-vanishing energy density at the centre which, if $m$ and $B$ are large enough, disfavour shell-like configurations. The energy density at the centre of the configuration is the largest when $p = 1$ which is the conventional mass term.

Of all the mass terms that we analysed in [5] the only term which disfavours shell-like configurations more than the traditional mass term is the special case $p = 0$, which, however, corresponds to a non-analytical $U$ potential term.

In this paper, we have decided to look at a one parameter family of mass terms which are a linear combination of three of the mass terms studied in [5] chosen in such a way that the shell like configurations are disfavoured.

To achieve this, we need a mass term that asymptotically leads to the usual expression for the pion fields in the linear limit, $m_\varphi |\vec{\pi}|^2$, but which is larger inside the shell in the baryon sector. We can obtain a family of such terms by taking (2) with

$$g(p) = \delta(p - 1) + D(\delta(p - 2) - \delta(p - 3)).$$

(5)

Notice that in this case we have

$$A^{-1} = 1 + D(4 - 9) = 1 - 5D$$

(6)
and
\[ \int_{-\infty}^{\infty} g(p) \, dp = 1 + D(1 - 1) = 1. \]  
(7)

So our potential term reduces to the explicit expression
\[ V = \frac{1}{12\pi^2} \frac{m_v^2}{1 - 5D} Tr(1 - U - D(U^2 - U^3)). \]  
(8)

Computing classical solutions of the Skyrme model when the baryon charge is larger than one is very difficult, but Houghton et al. [6] have presented the so called “rational map ansatz” which makes it possible to compute good approximations to shell-like solutions of the Skyrme model. The ansatz involves taking the fields \( U \) as
\[ U = \exp(\ii f(r)(2P - 1)), \]  
(9)

where \( f(r) \) is a radial profile function and \( P \) is a particular projector which depends only on the angular coordinates \( \theta \) and \( \varphi \). Defining \( z = \frac{1}{\ii} \tan \frac{\varphi}{2} \) we have \( P = \frac{v v^\dagger}{|v|^2} \) where \( v = (1, R(z)) \) is a two component holomorphic complex vector, \( \frac{\partial R}{\partial \bar{z}} = 0 \). Moreover, \( R(z) \) is a rational function of \( z \) and the degree of the rational map corresponds to the baryon number of the configuration. Both the radial profile and the rational map can be determined by inserting the ansatz (9) into (1) and minimising the obtained expression:
\[ H = \frac{1}{3\pi} \int \left[ f_r^2 + 2B \frac{\sin^2 f}{r^2} (1 + f_r^2) + \mathcal{I} \frac{\sin^4 f}{r^4} \\
+ 2 \frac{m_v^2}{1 - 5D} (1 - \cos(f) - D(\cos(2f) - \cos(3f))) \right] r^2 \, dr, \]  
(10)

where
\[ \mathcal{I} = \frac{1}{4\pi} \int \left( \frac{1 + |\xi|^2}{1 + |R|^2} \left| \frac{dR}{d\xi} \right|^4 \frac{2i}{1 + |\xi|^2} \right). \]  
(11)

The integral (11) depends only on the rational map \( R \) and so it must be minimised separately from the radial profile. This was done in [6] and [7].

Note that choosing a more general mass term like (2) does not change the angular dependence of the configuration within the rational map ansatz. It only changes the equation for the profile function \( f(r) \).

Before we attempt to minimise (10) we must check that the potential is positive definite. It easy to show this but only when \( D \) is in the range \([0, 0.2]\). Indeed, the roots of \( (1 - \cos(f) - D(\cos(2f) - \cos(3f))) \) are given by \( f = 0 \) and \( D = 1/(4\cos^2(f) + 2\cos(f) - 1) \). If we exclude the point \( f = 0 \), the energy density can only vanish if \( D \leq 0 \) or if \( D \geq 1/5 \). Notice that the case \( D = 0 \) corresponds to the standard mass term.
To compute the low energy configurations of our model that approximate its solutions we have solved the following equation for the radial profile $f(r)$

$$
f_{rr}(1 + 2B \frac{\sin^2 f}{r^2}) + 2 \frac{f_r}{r} + B \frac{\sin(2f)}{r^2} (f_r^2 - 1) - I \frac{\sin(2f) \sin^2 f}{r^4} - \frac{m^2}{1 - 5D} (\sin(f) + D(2\sin(2f) - 3\sin(3f))) = 0, \quad (12)
$$

taking from [7] the value for $I$.

The results are shown in figure 1 where we have plotted, as a function of $D$, the energy of several solutions divided by $B$ times the energy of the corresponding $B = 1$ solution. This relative energy describes how bound the skyrmions of the corresponding solution are.

We see clearly that, for a fixed value of the pion mass $m$, the relative energy per baryon increases with $D$. This shows that the potential we have chosen dislikes shell-like configurations more than the standard mass term. We also notice that for a relatively small value of the mass, $m = 0.4$ (Figure 1b), the curves for the relative energies of the $B = 17$ and $B = 22$ configurations both cross the curves for the $B = 4$ and $B = 8$ cases showing that these shell configurations could decay into smaller baryon configurations. Of course we expect the existence of solutions which are not shell like and which thus cannot be approximated by the rational map ansatz and we expect that the shell-like configurations would decay into these solutions instead of into the multiple shells. Even the configuration $B = 8$ is unstable with respect to the decay into two $B = 4$ configurations when $D > 0.16$. As the value of $m$ increases, we see that the instability of the shell-like configurations increases.

In figure 2 we present the radius, $r = \int_0^\infty \rho E(\rho) \rho^2 d\rho / \int_0^\infty E(\rho) \rho^2 d\rho$, of the energy density of the configuration, as a function of $D$, for different values of the mass $m$. We see that in each case, the radius decreases with $D$. This shows that the energy density at the center of the Skyrmion configuration increases with $D$, forcing the minimal energy configuration to have a smaller radius.

We have shown that if one takes (8) as the mass term for the Skyrme model, one obtains a model that disfavours shell-like configurations much more than the model with the traditional mass term, which corresponds to (8) with $D = 0$. This comes from the fact that the energy density at the centre of the shell increases with $D$ and, as a result, it is more favourable for the configuration to have a non-hollow shell-like shape (the fields do not take values close to $U = -1$ over a larger region than in the traditional model). Such non-shell like configurations have been constructed by Battye et al. [3][4] for the traditional mass term, but to obtain such configurations,
Battye et al. had to take a value of $m$ 6 or 7 times larger than the physical value of the pion mass.

To provide a semi-classical description of nuclei the Skyrme model must have solutions that are not hollow at their centres. While Battye et al. have shown that one can do this by taking a large value of the pion mass, we have shown that one can achieve the same effect with a more physical value of the pion mass by taking a different mass term for the Skyrme model \textit{i.e.} $(8)$.

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Figure 1: Relative baryon energies: $E(B)/BE(1)$ as a function of $D$ a) $m = 0.2$, b) $m = 0.4$, c) $m = 0.8$; d) $m = 1$. 
Figure 2: Shell radius: \( r = \int_0^\infty \rho E(\rho)\rho^2 d\rho / \int_0^\infty E(\rho)\rho^2 d\rho \) as a function of \( D \)

a) \( m = 0.2 \), b) \( m = 0.4 \), c) \( m = 0.8 \); d) \( m = 1 \).