The coloring problem for classes with two small obstructions

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Abstract

The coloring problem is studied in the paper for graph classes defined by two small forbidden induced subgraphs. We prove some sufficient conditions for effective solvability of the problem in such classes. As their corollary we determine the computational complexity for all sets of two connected forbidden induced subgraphs with at most five vertices except 13 explicitly enumerated cases.

Keywords: vertex coloring, computational complexity, polynomial-time algorithm

1 Introduction

The coloring problem is one of classical problems on graphs. Its formulation is as follows. A coloring is an arbitrary mapping of colors to vertices of

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some graph. A graph coloring is called proper if any neighbors are colored in different colors. The chromatic number of a graph $G$ (denoted by $\chi(G)$) is the minimal number of colors in proper colorings of $G$. The coloring problem for a given graph is to find its chromatic number. The vertex $k$-colorability problem is to verify whether vertices of a given graph can be colored with at most $k$ colors. The edge $k$-colorability problem is formulated by analogy.

A graph $H$ is called an induced subgraph of $G$ if $H$ is obtained from $G$ by deletions of its vertices. The induced subgraph relation is denoted by $\subseteq_i$. In other words, $H \subseteq_i G$ if $H$ is an induced subgraph of $G$. A class is a set of simple unlabeled graphs. A class of graphs is called hereditary if it is closed under deletions of vertices. It is well known that any hereditary (and only hereditary) graph class $\mathcal{X}$ can be defined by a set of its forbidden induced subgraphs $\mathcal{S}$. We write $\mathcal{X} = \text{Free}(\mathcal{S})$ in this case. If a hereditary class can be defined by a finite set of forbidden induced subgraphs, then it is called finitely defined.

The coloring problem for $G$-free graphs is polynomial-time solvable iff $G \subseteq_i P_4$ or $G \subseteq_i P_3 \oplus K_1$ (Kral’ et al. 2002). A study of forbidden pairs was also initialized in the paper. When we forbid two induced subgraphs, the situation becomes more difficult than in the monogenic case. Here only partial results are known (Kral’ et al. 2002; Brandstät et al. 2002; Brandstät et al. 2006; Schindl 2005; Golovach, Paulusma and Song 2011; Dabrowski et al. 2012; Golovach and Paulusma 2013). The next statement is a survey of such achievements (Golovach and Paulusma 2013).

**Theorem 1** Let $G_1$ and $G_2$ be two fixed graphs. The coloring problem is NP-complete for Free($\{G_1, G_2\}$) if:

- $C_p \subseteq_i G_1$ for some $p \geq 3$ and $C_q \subseteq_i G_2$ for some $q \geq 3$
- $K_{1,3} \subseteq_i G_1$ and $K_{1,3} \subseteq_i G_2$
- $K_{1,3} \subseteq_i G_1$ and either $K_4 \subseteq_i G_2$ or $K_4 - e \subseteq_i G_2$ (or vice versa)
- $K_{1,3} \subseteq_i G_1$ and $C_p \subseteq_i G_2$ for some $p \geq 4$ (or vice versa)
- $G_1$ and $G_2$ contain a spanning subgraph of $2K_2$ as an induced subgraph
- $C_3 \subseteq_i G_1$ and $K_{1,p} \subseteq_i G_2$ for some $p \geq 5$ (or vice versa)
- $C_3 \subseteq_i G_1$ and $P_{164} \subseteq_i G_2$ (or vice versa)
• $C_p \subseteq G_1$ for $p \geq 5$ and $G_2$ contains a spanning subgraph of $2K_2$ as an induced subgraph (or vice versa)

• either $C_p \oplus K_1 \subseteq G_1$ for $p \in \{3, 4\}$ or $C_q \subseteq G_1$ for $q \geq 6$ and $G_2$ contains a spanning subgraph of $2K_2$ as an induced subgraph (or vice versa)

It is polynomial-time solvable for $\text{Free}(\{G_1, G_2\})$ if:

• $G_1$ and $G_2$ are induced subgraphs of $P_4$ or $P_3 \oplus K_1$

• $G_1 \subseteq K_{1,3}$ and $G_2 \subseteq C_3 \oplus K_1$ (or vice versa)

• $G_1 \subseteq \text{paw}$ and $G_2 \neq K_{1,5}$ is a forest with at most six vertices (or vice versa)

• $G_1 \subseteq \text{paw}$ and either $G_2 \subseteq pK_2$ or $G_2 \subseteq P_5 \oplus pK_1$ for some $p \geq 1$ (or vice versa)

• $G_1 \subseteq K_p$ for $p \geq 3$ and either $G_2 \subseteq qK_2$ or $G_2 \subseteq P_5 \oplus qK_1$ for some $q \geq 1$ (or vice versa)

• $G_1 \subseteq \text{gem}$ and either $G_2 \subseteq P_4 \oplus K_1$ or $G_2 \subseteq P_5$ (or vice versa)

• $G_1 \subseteq \overline{P_5}$ and either $G_2 \subseteq P_4 \oplus K_1$ or $G_2 \subseteq 2K_2$ (or vice versa)

In the present article we prove some sufficient conditions for NP-completeness and polynomial-time solvability of the coloring problem for $\{G_1, G_2\}$-free graphs. They add new information about its complexity for some cases that Theorem 1 does not cover. For instance, the problem is appeared to be NP-complete for $\{K_{1,4}, \text{bull}\}$-free graphs, but it is polynomial-time solvable for $\text{Free}(\{K_{1,3}, P_3\}), \text{Free}(\{K_{1,3}, \text{hammer}\}), \text{Free}(\{P_5, C_4\})$. The complexity was earlier open for these four cases. As a corollary of the conditions we determine the complexity for all sets $\{G_1, G_2\}$ of connected graphs with at most five vertices except 13 listed cases.

2 Notation

As usual, $P_n, C_n, K_n, O_n$ and $K_{p,q}$ stand respectively for the simple path with $n$ vertices, the chordless cycle with $n$ vertices, the complete graph with $n$ vertices, the empty graph with $n$ vertices and the complete bipartite graph with
vertices in the first part and \( q \) vertices in second. The graph \( K_n - e \) is obtained by deleting an arbitrary edge in \( K_n \). The graph \textit{paw} is obtained from \( K_{1,3} \) by adding a new edge incident to its vertices of degree two. The graphs \textit{fork}, \textit{gem}, \textit{hammer}, \textit{bull}, \textit{butterfly} have the vertex set \( \{x_1, x_2, x_3, x_4, x_5\} \). The edge set for \textit{fork} is \( \{(x_1, x_2), (x_1, x_3), (x_1, x_4), (x_4, x_5)\} \), for \textit{gem} is \( \{(x_1, x_2), (x_1, x_3), (x_1, x_4), (x_2, x_3), (x_3, x_4), (x_4, x_5)\} \), for \textit{hammer} is \( \{(x_1, x_2), (x_1, x_3), (x_2, x_3), (x_1, x_4), (x_2, x_5)\} \), for \textit{bull} is \( \{(x_1, x_2), (x_1, x_3), (x_2, x_3), (x_1, x_4), (x_2, x_5)\} \), for \textit{butterfly} is \( \{(x_1, x_2), (x_1, x_3), (x_2, x_3), (x_1, x_4), (x_1, x_5), (x_4, x_5)\} \).

The \textit{complemental graph} of \( G \) (denoted by \( \overline{G} \)) is a graph on the same set of vertices and two vertices of \( \overline{G} \) are adjacent if and only if they are not adjacent in \( G \). The sum \( G_1 \oplus G_2 \) is the disjoint union of \( G_1 \) and \( G_2 \). The disjoint union of \( k \) copies of a graph \( G \) is denoted by \( kG \). For a graph \( G \) and a set \( V \subseteq V(G) \) the formula \( G \setminus V \) denotes the subgraph of \( G \) obtained by deleting all vertices in \( V \).

All graph notions and properties that are not formulated in this paper can be found in the textbooks (Bondy and Murty 2008; Distel 2010).

3 Boundary graph classes

The notion of a boundary graph class is a helpful tool for analysis of the computational complexity of graph problems in the family of hereditary graph classes. This notion was originally introduced by Alekseev for the independent set problem (Alekseev 2004). It was applied for the dominating set problem later (Alekseev, Korobitsyn and Lozin 2004). A study of boundary graph classes for some graph problems was extended in the paper (Alekseev et al. 2007), where the notion was formulated in its most general form. Let us give the necessary definitions.

Let \( \Pi \) be an NP-complete graph problem. A hereditary graph class is called \( \Pi \)-\textit{easy} if \( \Pi \) is polynomial-time solvable for its graphs. If the problem \( \Pi \) is NP-complete for graphs in a hereditary class, then this class is called \( \Pi \)-\textit{hard}. A class of graphs is said to be \( \Pi \)-\textit{limit} if this class is the limit of an infinite monotonously decreasing chain of \( \Pi \)-hard classes. In other words, \( \mathcal{X} \) is \( \Pi \)-limit if there is an infinite sequence \( \mathcal{X}_1 \supseteq \mathcal{X}_1 \supseteq \ldots \) of \( \Pi \)-hard classes, such that \( \mathcal{X} = \bigcap_{k=1}^{\infty} \mathcal{X}_k \). A minimal under inclusion \( \Pi \)-limit class is called \( \Pi \)-\textit{boundary}.

The following theorem certifies the significance of the boundary class
Theorem 2 A finitely defined class is Π-hard iff it contains some Π-boundary class.

The theorem shows that knowledge of all Π-boundary classes leads to a complete classification of finitely defined graph classes with respect to the complexity of Π. Two concrete classes of graphs are known to be boundary for several graph problems. First of them is $S$. It is constituted by all forests with at most three leaves in each connected component. The second one is $T$, which is a set of the line graphs of graphs in $S$. The paper (Alekseev et al. 2007) is a good survey about graph problems, for which either $S$ or $T$ is boundary.

Some classes are known to be limit and boundary for the coloring problem. The set of all forests (denoted by $F$) and the set of line graphs of forests with degrees at most three are limit classes for it (Lozin and Kaminski 2007). The last class we will denote by $T'$. The set $co(T) = \{ G : \overline{G} \in T \}$ is a boundary class for the problem (Malyshev 2012(a)). The set of boundary graph classes for the coloring problem is continuous (Korpeilainen et al. 2011). Some continuous sets of boundary classes for the vertex $k$-colorability and the edge $k$-colorability problems are known for any fixed $k \geq 3$ (Malyshev 2012(a); Malyshev 2012(b)).

4 NP-completeness of the coloring problem for $\{K_{1,4}, bull\}$-free graphs

The listed above results on limit and boundary classes for the coloring problem together with Theorem 1 allow to prove NP-completeness of the problem for some finitely defined classes. Namely, if $\mathcal{Y}$ is a finite set of graphs and no one graph in $\mathcal{Y}$ belongs to a class in $\{F, T', co(T)\}$, then the problem is NP-complete for $Free(\mathcal{Y})$. But this idea can not be applied to $Free(\{K_{1,4}, bull\})$, because $K_{1,4} \in F, bull \in T', bull \in co(T)$. Nevertheless, the coloring problem is NP-complete for it. To show this we use the operation with a graph called the diamond implantation.

Let $G$ be a graph and $x$ be its nonpendant vertex. Applying the diamond implantation to $x$ implies:
• an arbitrary splitting of the neighborhood of $x$ into two nonempty parts $A$ and $B$
• deletion of $x$ and addition of new vertices $y_1, y_2, y_3, y_4$
• addition of all edges of the kind $(y_1, a), a \in A$ and of the kind $(y_4, b), b \in B$
• addition of the edges $(y_1, y_2), (y_1, y_3), (y_2, y_3), (y_2, y_4), (y_3, y_4)$

Clearly that for any graph and any its nonleaf vertex applying the diamond implantation preserves vertex 3-colorability. This property and the paper (Kochol, Lozin and Randerath 2003) give the key idea of the proof of Lemma 1.

**Lemma 1** The vertex 3-colorability problem (hence, the coloring problem) is NP-complete for the class $\text{Free}(\{K_{1,4}, \text{bull}\})$.

**Proof.** The vertex 3-colorability problem is known to be NP-complete for triangle-free graphs with degrees at most four (Maffray and Preissmann 1996). Let us consider connected such a graph with at least two vertices. We will sequentially apply the described above operation to its vertices with edgeless neighborhoods. In other words, if $H$ is a current graph, then it is applied to an arbitrary vertex of $H$ that does not belong to any triangle. The sets $A$ and $B$ are arbitrarily formed with the condition $|A| - |B| \leq 1$. The whole process is finite, because the number of its steps is no more than the quantity of vertices in the initial graph. It is easy to see that the resulted graph belongs to $\text{Free}(\{K_{1,4}, \text{bull}\})$. Thus, the vertex 3-colorability problem for triangle-free graphs with degrees at most four is polynomially reduced to the same problem for graphs in $\text{Free}(\{K_{1,4}, \text{bull}\})$. Hence, it is NP-complete for $\text{Free}(\{K_{1,4}, \text{bull}\})$. ■

**5 Some structural results on graphs in some classes defined by small obstructions**

For a hereditary class $\mathcal{X}$ and a number $k$ the formula $[\mathcal{X}]_k$ is a set of graphs, for which one can delete at most $k$ vertices that a result belongs to $\mathcal{X}$. 
Lemma 2 If some connected graph \( G \in \text{Free}(\{K_{1,3}, P_5\}) \) contains an induced cycle \( C \) of length at least four, then \( G \in [\text{Free}(\{O_3\})]_5 \).

Proof. Length of \( C \) is equal to either four or five. We will show first that \( C \) dominates all vertices of \( G \). Let there is a vertex of \( G \) that does not belong to \( C \) and adjacent to no one vertex of the cycle. Then due to the connectivity of \( G \) there are vertices \( x, y \in V(G) \setminus V(C) \), such that \( (x, y) \in E(G) \), \( x \) is not adjacent to any vertex of \( C \) and \( y \) is adjacent to at least one vertex of \( C \). Since \( G \in \text{Free}(\{K_{1,3}\}) \), then \( y \) is adjacent to exactly two vertices of \( C \). The vertices \( x, y \) and some three consecutive vertices of the cycle (one of which is adjacent to \( y \)) induce the subgraph \( P_5 \). Thus, \( C \) really dominates all vertices of \( G \).

We will show that the graph \( G \setminus V(C) \) does not contain three pairwise nonadjacent vertices. This fact implies the validity of Lemma 2. Assume that \( G \setminus V(C) \) has a set \( V \) of three pairwise nonadjacent vertices. Since \( G \) is \( K_{1,3} \)-free, then the intersection of the neighborhood of each vertex in \( V \) with \( V(C) \) is a set of at least two (three for \( C = C_5 \)) consecutive vertices of \( C \).

Let us consider the case \( C = C_5 \). No one vertex of \( V \) can be adjacent to all vertices of \( C \), since otherwise some vertex of \( C \) is adjacent to all vertices of \( V \) (and \( G \) contains \( K_{1,3} \) as an induced subgraph). One can assume that no one vertex in \( V \) is adjacent to all vertices of the cycle \( C \), since in this case the graph \( G \) contains the induced cycle \( C_4 \) and the case \( C = C_4 \) will be considered later. Therefore, we can consider only the situation, where each vertex of \( V \) is adjacent to three consecutive vertices of \( C \) and the corresponding sets of three consecutive vertices are distinct (otherwise \( G \) contains \( K_{1,3} \) as an induced subgraph). Then, some two vertices of \( V \) and some three vertices of \( C \) induce \( P_5 \). So, if \( C = C_5 \), then we have a contradiction.

Now we consider the case \( C = C_4 \). It is easy to verify that avoiding induced \( K_{1,3} \) in \( G \) leads to only the following situations:

- one vertex of \( V \) is adjacent to all vertices of \( C \) and the other two vertices of \( V \) are adjacent to disjoint pairs of its consecutive vertices

- one vertex of \( V \) is adjacent to two consecutive vertices of \( C \) and each of the other two vertices is adjacent to three consecutive vertices of \( C \), they have two common neighbors in \( C \) and the first vertex has only one common neighbor in \( C \) with each of them
• each of two vertices of $V$ is adjacent to two consecutive vertices of $C$, the third one is adjacent to three consecutive vertices of $C$ and any two vertices of $V$ have only one common neighbor in $C$

The graph $G$ contains $P_5$ as an induced subgraph in all three cases. We come to a contradiction. Thus, the initial assumption was false. ■

**Lemma 3** If some connected graph $G \in \text{Free}([K_{1,3}, \text{hammer}])$ contains an induced cycle $C_n$ ($n \geq 7$), then $G$ is isomorphic to $C_n$.

**Proof.** Assume opposite, i.e. there is a vertex $x \in V(G) \setminus V(C_n)$. One can easily show that the vertex $x$ is adjacent to at least one vertex of $C_n$. It is easy to verify that the set of $x$'s neighbors in $C_n$ is constituted either by two, three or four consecutive vertices or by two pairs of consecutive vertices (otherwise $G \not\in \text{Free}([K_{1,3}])$). In both situations the graph $G$ contains hammer as an induced subgraph. Hence, the assumption was false. ■

**Lemma 4** If some connected graph $G \in \text{Free}([K_{1,3}, \text{hammer}])$ contains $C_6$ as an induced subgraph, then $G \setminus V(C_6)$ is the disjoint union of at most three cliques.

**Proof.** Let us consider the set $V = V(G) \setminus V(C_6)$. It is easy to verify that the intersection of the neighborhood of each vertex in $V$ with $C_6$ induces in $G$ the subgraph $2K_2$. Let us consider now two arbitrary vertices in $V$. If they are adjacent, then they have in $C_6$ the same sets of neighbors and if they are not adjacent, then the mentioned sets are distinct. This implies that $V$ does not contain four pairwise nonadjacent vertices. Thus, $G \setminus V(C_6)$ is the disjoint union of at most three cliques. ■

**Lemma 5** For any connected graph $G \in \text{Free}([K_{1,3}, \text{hammer}])$ at least one of the following properties is true:

• $G$ is a simple cycle
• $G$ contains the induced subgraph $C_6$
• $G$ has a pendant vertex
• $G$ belongs to the class $\text{Free}([P_5])$
• $G$ belongs to the class $[\text{Free}([O_3])]_5$

Proof. Assume that $G \notin \text{Free}([P_3])$. Let us consider an induced path $P_n$ of $G$ having the maximal length. Clearly, $n \geq 5$. Let us consider an arbitrary end of this path. One can assume that it is adjacent to some vertex $x \in V(G) \setminus V(P_n)$, otherwise $G$ contains a pendant vertex. By the maximality of $P_n$ the vertex $x$ is adjacent to at least two vertices of the path. One can consider that $x$ is adjacent to at least one interior vertex of $P_n$, otherwise $G$ is a simple cycle (by Lemma 3) or it contains $C_6$ as an induced subgraph.

Let $n > 5$. To avoid induced $K_{1,3}$ the vertex $x$ must be adjacent to three or four consecutive vertices of $P_n$ or to two end its vertices or to three vertices of $P_n$ that induce the subgraph $K_2 \oplus K_1$ in $G$ or to four vertices inducing $2K_2$. The graph $G$ contains hammer as an induced subgraph in all these situations.

Let $n = 5$ now. One can assume that the graph $G \setminus V(P_5)$ has three pairwise nonadjacent vertices (otherwise $G \in [\text{Free}([O_3])]_5$). It is easy to check that any of these three vertices must be adjacent to either three central vertices of $P_5$ or to all its vertices, except central or to the first, the third and the fourth vertices of $P_5$ (counting from some of the $P_5$’s ends) or to the first and the last its vertices. The graph $G$ contains $C_6$ as an induced subgraph in the last case. Hence, we can consider that no one among the three vertices is adjacent to only the ends of $P_5$. If one of the three vertices is adjacent to the first, the third and the fourth vertices of $P_5$ and other of these vertices is adjacent to the second, the third and the fifth ones, then $G$ contains induced $C_6$. Therefore, one can assume that there are no such two vertices. Either the second or the fourth vertex of $P_5$ is adjacent to the three vertices and, hence, $G$ is not $K_{1,3}$-free. Thus, the initial assumption was false.

6 On formulae connecting the chromatic numbers of a graph and of its induced subgraphs

The following statement is obvious.

Lemma 6 If $G$ is a connected graph with at most three vertices and a pendant vertex $v$, then $\chi(G \setminus \{v\}) = \chi(G)$.
Lemma 7  Let $G$ be a connected graph in $Free(\{P_5, C_4\})$ that contains induced $C_5$. Let $V_1$ be the set its vertices that adjacent to all vertices of $C_5$, $V_2$ be the set of vertices in $G$ that have three neighbors in $C_5$, $G_1$ and $G_2$ be the subgraphs of $G$, induced by $V(G) \setminus (V_1 \cup V_2 \cup V(C_5))$ and $V_1 \cup V_2 \cup V(C_5)$ correspondingly. Then, $G_2$ is $O_3$-free and the relation $\chi(G) = \max(\chi(G_1), \chi(G_2))$ holds.

Proof. Any vertex outside $C_5$ that adjacent to at least one vertex of the cycle must be adjacent to either all vertices of the cycle or to three consecutive its vertices. It is easy to verify taking into account that $G$ is $\{P_5, C_4\}$-free. Therefore, any such a vertex belongs to either $V_1$ or $V_2$. Each vertex in $V_2$ has no a neighbor outside $V(C_5) \cup V_1 \cup V_2$ (since $G \in Free(\{P_5\})$). As $G$ is $C_4$-free, then any vertex in $V_1$ is adjacent to every vertex in $V_1 \cup V_2 \cup V(C_5)$ except itself. It is easy to verify that $G_2$ is $O_3$-free.

The inequality $\chi(G) \geq \max(\chi(G_1), \chi(G_2))$ is obvious. We will show that $G$ can be colored with $\max(\chi(G_1), \chi(G_2))$ colors. Let $c_1$ and $c_2$ be optimal colorings of $G_1$ and $G_2$ correspondingly. If $\chi(G_1) \geq \chi(G_2)$, then $c_1$ has $\chi(G_1) - |V_1| \geq \chi(G_2) - |V_1| \geq 0$ colors that do not appear in $V_1$. Hence, $c_1$ can be extended to a proper coloring of $G$ with $\chi(G_1)$ colors by coloring $G_2 \setminus V_1$ with $\chi(G_2) - |V_1|$ colors of the mentioned type. By the same reasons $c_2$ is extendable to a proper coloring of $G$ with $\chi(G_2)$ colors when $\chi(G_2) \geq \chi(G_1)$.

7 Some results on polynomial-time solvability of the coloring problem

Lemma 8  Let $\mathcal{X}$ be an easy case for the coloring problem and for some number $p$ the inclusion $\mathcal{X} \subseteq Free(\{O_p\})$ holds. Then, for any fixed $q$ this problem is polynomial-time solvable in the class $[\mathcal{X}]_q$.

Proof. Let $G$ be a graph in $[\mathcal{X}]_q$. Deleting some set $V (|V| \leq q)$ of its vertices leads to a graph in $\mathcal{X}$. We will consider all partial proper colorings of $G$ with at most $|V|$ color classes, in which every vertex of $V$ is colored. Obviously, any such a coloring has at most $(p - 1)q$ colored vertices. Hence, the quantity of the colorings is bounded by a polynomial on $|V(G)|$. For any considered partial coloring deleting all colored vertices leads to a graph in $\mathcal{X}$ and its chromatic number is computed in polynomial time. For every our
partial coloring we will find the sum of the number of used colors and the chromatic number of the subgraph induced by the set of uncolored vertices. Minimal among these sums is equal to $\chi(G)$. Thus, $\chi(G)$ is computed in polynomial time. □

A graph is called chordal if it does not contain induced cycles with four and more vertices. The coloring problem is known to be polynomial-time solvable for chordal graphs (Golumbic 1980).

**Lemma 9** The classes $\text{Free}(\{K_{1,3}, P_5\})$, $\text{Free}(\{K_{1,3}, \text{hammer}\})$, $\text{Free}(\{P_5, C_4\})$ are easy for the coloring problem.

**Proof.** We will show that for every considered class the coloring problem is polynomially reduced to the same problem for chordal graphs. This fact implies the lemma. The problem is polynomial-time solvable in $\text{Free}(\{O_3\})$, since it is equivalent to the matching problem. The reduction for $\{K_{1,3}, P_3\}$-free graphs follows from this observation, Lemma 2 and Lemma 8.

Let $G$ be a graph in $\text{Free}(\{K_{1,3}, \text{hammer}\})$ containing the induced subgraph $C_6$. By Lemma 4, deleting vertices of this cycle leads to a chordal $O_4$-free graph. Hence, by Lemma 8, $\chi(G)$ is computed in polynomial time. Thus, by Lemma 5 and Lemma 6 the coloring problem for the class is polynomially reduced to the same problem for graphs in $\text{Free}(\{K_{1,3}, P_5\}) \cup \text{Free}(\{O_3\})^5$. Hence, it is reduced to chordal graphs.

Let $G$ be a connected graph in $\text{Free}(\{P_5, C_4\})$ that is not chordal. Hence, $G$ contains induced $C_5$. The graphs $G_1$ and $G_2$ defined in the formulation of Lemma 7 are constructed in polynomial time. Moreover, $G_2$ is $O_3$-free and $|V(G)| - |V(G_1)| \geq 5$. Therefore, by Lemma 7 the considered problem for $\{P_5, C_4\}$-free graphs is also polynomially reduced to the same problem for chordal graphs. □

8 The main result and its corollaries

The following theorem is the main result of the paper.

**Theorem 3** Let $H_1$ and $H_2$ be some graphs. If there is a class $\mathcal{Y} \in \{\mathcal{F}, \mathcal{T}', \text{co}(\mathcal{T})\}$ with either $H_1, H_2 \notin \mathcal{Y}$ or $K_{1,4} \subseteq_i H_1$ and bull $\subseteq_i H_2$ (or vice versa), then the coloring problem is NP-complete for $\text{Free}(\{H_1, H_2\})$. It is polynomial-time solvable in the class if at least one of the following properties holds:
\[ H_1 \subseteq_i P_4 \text{ or } H_2 \subseteq_i P_4 \]
\[ H_1 \subseteq_i P_5 \text{ or } H_2 \subseteq_i K_5 \text{ (or vice versa)} \]
\[ H_1 \subseteq_i P_5 \text{ or } H_2 \subseteq_i \text{gem} \text{ (or vice versa)} \]
\[ H_1 \subseteq_i P_5 \text{ or } H_2 \subseteq_i C_4 \text{ (or vice versa)} \]
\[ H_1 \subseteq_i P_5 \text{ or } H_2 \subseteq_i K_{1,3} \text{ (or vice versa)} \]
\[ H_1 \subseteq_i K_{1,4} \text{ or } H_2 \subseteq_i \text{paw} \text{ (or vice versa)} \]
\[ H_1 \subseteq_i \text{fork} \text{ or } H_2 \subseteq_i \text{paw} \text{ (or vice versa)} \]
\[ H_1 \subseteq_i K_{1,3} \text{ or } H_2 \subseteq_i \text{hammer} \text{ (or vice versa)} \]

**Proof.** Let us recall that the classes \(F, T', \text{co}(T)\) are limit for the coloring problem. This fact, Theorem 1 and Lemma 1 imply the first part of the statement. The set of \(P_4\)-free graphs is well known to be an easy case for the coloring problem (Courcelle and Olariu 2000). The classes \(\text{Free}(\{P_5, \text{gem}\})\) and \(\text{Free}(\{P_5, K_5\})\) are easy for the problem (Brandstädt et al. 2002; Golovach and Paulusma 2013). The same is true for \(\text{Free}(\{\text{fork, paw}\})\) (Golovach and Paulusma 2013) and \(\text{Free}(\{K_{1,4}, \text{paw}\})\) (Kral’ et al. 2001). These facts and Lemma 9 imply the second part of the theorem.

Both parts of Theorem 3 add new information about the complexity of the coloring problem for some classes. For example, its complexity status for the classes \(\text{Free}(\{K_{1,3}, \text{bull}\}), \text{Free}(\{K_{1,3}, P_5\}), \text{Free}(\{K_{1,3}, \text{hammer}\}), \text{Free}(\{P_5, C_4\})\) was open.

Theorem 3 gives the following criterion in the case of connected \(H_1\) and \(H_2\) with at most four vertices.

**Corollary 1** If \(H_1\) and \(H_2\) are connected graphs with at most four vertices, then the coloring problem is polynomial-time solvable for \(\{H_1, H_2\}\)-free graphs iff either \(H_1 \subseteq_i P_4\) or \(H_2 \subseteq_i P_4\) or \(\{H_1, H_2\} = \{K_{1,3}, \text{paw}\}\) or \(\{H_1, H_2\} = \{K_{1,3}, C_3\}\).

Theorem 3 can not be applied to some pairs of connected graphs with at most five vertices. If \(\{H_1, H_2\}\) is such a set, then either \(H_1\) or \(H_2\) belongs to \(\{K_{1,3}, \text{fork}, K_{1,4}, P_5\}\). This observation helps to enumerate all connected cases with at most five vertices that the theorem does not cover.
Corollary 2 Theorem 3 does not give the complexity status of the coloring problem for the following sets of forbidden induced connected subgraphs (a number in the brackets shows the quantity of such kind sets):

- \(\{K_{1,3}, G\}\), where \(G \in \{\text{bull, butterfly}\}\) (2)
- \(\{\text{fork, bull}\}\) (1)
- \(\{P_5, G\}\), where \(G \notin \{K_5, \text{gem}\}\) is an arbitrary connected five-vertex graph in \(\text{co}(T)\) (10)

Determining the complexity of the problem for any of the listed above 13 cases is a challenging research problem.

References

[1] Alekseev V (2004) On easy and hard hereditary classes of graphs with respect to the independent set problem. Discrete Applied Mathematics 132:17–26.

[2] Alekseev V, Boliac R, Korobitsyn D and Lozin V (2007) NP-hard graph problems and boundary classes of graphs. Theoretical Computer Science 389:219–236.

[3] Alekseev V, Korobitsyn D and Lozin V (2004) Boundary classes of graphs for the dominating set problem. Discrete Mathematics 285:1–6.

[4] Brandstädt A, Dragan F, Le H and Mosca R (2002) New graph classes of bounded clique-width. Lecture Notes in Computer Science 2573:57–67.

[5] Brandstädt A, Engelfriet J, Le H and Lozin V (2006) Clique-width for 4-vertex forbidden subgraphs. Theory Comput. Syst. 39:561–590.

[6] Bondy A and Murty U (2008) Graph theory. Springer-Verlag, Graduate texts in mathematics.

[7] Courcelle B and Olariu S (2000) Upper bounds to the clique width of graphs. Discrete Applied Mathematics 101(1–3):77–144.

[8] Dabrowski K, Lozin V, Raman R and Ries B (2012) Colouring vertices of triangle-free graphs without forests. Discrete Mathematics 312:1372–1385.
[9] Diestel R (2010) Graph theory. Springer-Verlag, Graduate texts in mathematics.

[10] Golovach P, Paulusma D and Song J (2011) Coloring graphs without short cycles and long induced paths. Lecture Notes in Computer Science 6914:193–204.

[11] Golovach P and Paulusma D (2013) List coloring in the absence of two subgraphs. CIAC 288–299.

[12] Golumbic M. Algorithmic graph theory and perfect graphs (1980) Academic Press, New York.

[13] Kochol M, Lozin V and Randerath B (2003) The 3-colorability problem on graphs with maximum degree four. SIAM J. Computing 32:1128-1139.

[14] Korpeilainen N, Lozin V, Malyshev D and Tiskin A (2011) Boundary properties of graphs for algorithmic graph problems. Theoretical Computer Science 412:3544–3554.

[15] Kral’ D, Kratochvil J, Tuza Z and Woeginger G (2001) Complexity of coloring graphs without forbidden induced subgraphs. Lecture Notes in Computer Science 2204:254–262.

[16] Lozin V and Kaminski M (2007) Coloring edges and vertices of graphs without short or long cycles. Contributions to Discrete Mathematics 2(1).

[17] Schindl D (2005) Some new hereditary classes where graph coloring remains NP-hard. Discrete Math. 295:197–202.

[18] Maffray F and Preissmann M (1996) On the NP-completeness of the \( k \)-colorability problem for triangle-free graphs. Discrete Mathematics 162(1–3): 313–317.

[19] Malyshev D (2009) Continual sets of boundary classes of graphs for colorability problems. Discrete Analysis and Operations Research 16(5):41–51 (in Russian).

[20] Malyshev D (2012) On intersection and symmetric difference of families of boundary classes in the problems of colouring and on the chromatic number. Discrete Mathematics 24(2):75–78 (in Russian). English translation in Discrete Mathematics and Applications (2011) 22(5–6):645–649.
[21] Malyshev D (2012) A study of boundary graph classes for colorability problems. Discrete Analysis and Operations Research 19(6):37–48 (in Russian). English translation in Journal of Applied and Industrial Mathematics (2013) 7(2):221–228.