Maximum Likelihood Estimation in the Beta Model

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Abstract

We study maximum likelihood estimation for the statistical model for both directed and undirected random graph models in which the degree sequences are minimal sufficient statistics. In the undirected case, the model is known as the beta model. We derive necessary and sufficient conditions for the existence of the MLE that are based on the polytope of degree sequences. We characterize in a combinatorial fashion sample points leading to a nonexistent MLE, and non-estimability of the probability parameters under a nonexistent MLE. We formulate conditions that guarantee that the MLE exists with probability tending to one as the number nodes increases. We illustrate our approach on other random graph models for networks, such as the Rasch model, the Bradley-Terry model and the more general $p_1$ model of Holland and Leinhardt (1981).

Keywords: beta model, polytope of degree sequences, random graphs, Rasch model, $p_1$ model

1 Introduction

Many statistical models for the representation and analysis of network data rely on information contained in the degree sequence, the vector of node degrees of the observed graph. Node degrees not only quantify the overall connectivity of the network, but also reveal other potentially more refined features of interest. The study of the degree sequences and, in particular, of the degree distributions of real networks is a classic topic in network analysis, which has received extensive treatment in the statistical literature (see, e.g., Holland and Leinhardt, 1981; Fienberg and Wasserman, 1981a; Fienberg et al., 1985), the physics literature (see, e.g., Newman et al., 2001; Albert and Barabási, 2002; Newman, 2003; Park and Newman, 2004; Newman et al., 2006; Foster et al., 2007; Willinger et al., 2009) as well as in the social network literature (see, e.g., Robins et al., 2008; Goodreau, 2007; Handcock and Morris, 2007, and references therein). See also the monograph by Goldenberg et al. (2010) and the books by Kolaczyk (2009), Cohen and Havlin (2010) and Neman (2010).

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The simplest instance of a statistical network model based exclusively on the node degrees is the exponential family of probability distributions for undirected random graphs with the degree sequence as its natural sufficient statistic. This is in fact a simpler, undirected version of the broader class of statistical models for directed networks known as the $p_1$-models, introduced by Holland and Leinhardt (1981). We will refer to this model as the beta model (henceforth the $\beta$-model), a name recently coined by Chatterjee et al. (2011), and refer to Blitzstein and Diaconis (2009) for details and extensive references.

Despite its apparent simplicity and popularity, the $\beta$-model, much like most network models, exhibit non-standard statistical features, since its complexity, measured by the dimension of the parameter space, increases with the size of the graph. Lauritzen (2003, 2008) characterized $\beta$-models as the natural models for representing exchangeable binary arrays that are weakly summarized, i.e., random arrays whose distribution only depends on the row and column totals. More recently, Chatterjee et al. (2011) conducted an analysis of the asymptotic properties of the $\beta$-model, including existence and consistency of the maximum likelihood estimator (MLE) as the dimension of the network increases, and provided a simple algorithm for estimating the natural parameters. They also characterized the graph limits, or graphons (see Lovász and Szegedy, 2006), corresponding to a sequence of $\beta$-models with given degree sequences (for a connection between the theory of graphons and exchangeable arrays see Diaconis and Janson, 2007). Concurrently, Barvinok and Hartigan (2010) explored the asymptotic behavior of sequences of random graphs with given degree sequences, and studied a different mode of stochastic convergence. Among other things, they show that, as the size of the network increases and under a “tameness” condition, the number of edges of a uniform graph with given degree sequence converges in probability to the number of edges of a random graph drawn from a $\beta$-model parametrized by the MLE corresponding to degree sequence.

Subsequently to the submission of our article, Yan and Xu (2012) and Yan et al. (2012) derived asymptotic conditions for uniform consistency and asymptotic normality of the MLE of the $\beta$-model, and asymptotic normality of the likelihood ratio test for homogeneity of the model parameters. Perry and Wolfe (2012) consider a general class of models for network data parametrized by node-specific parameters, of which the $\beta$-model is a special case. The authors show that, under suitable conditions, the MLEs of model parameters exist and can be well approximated by simple estimators.

In an attempt to avoid the reliance on asymptotic methods, whose applicability to network models remains largely unclear (see, e.g., Haberman, 1981), several researchers have turned to exact inference for the $\beta$-model, which hinges upon the non-trivial task of sampling from the set of graphs with a given degree sequence. Blitzstein and Diaconis (2009) developed and analyzed a sequential importance sampling algorithm for generating a random graph with the prescribed degree sequence (see also Viger and Latapay, 2005, for a different algorithm). Hara and Takemura (2010) and Ogawa et al. (2011) tackled the same task using more abstract algebraic methods and Petrović et al. (2010) studied Markov bases for the more general $p_1$-model.

In this article we study the existence of the MLE for the parameters of the beta model under a more general sampling scheme in which each edge is observed a fixed number of times (instead of just once, like in previous works) and for increasing network sizes. The reasons of our focus on the issue of existence of the MLE, which we view as a natural measure of the intrinsic statistical difficulty of the beta model, is twofold. First, existence of the MLE is a natural minimum requirement for feasibility of statistical inference in discrete exponential families, such as the beta model: nonexistence of the MLE is in fact equivalent to non-estimability of the model parameters, as illustrated in Fienberg and Rinaldo (2011). Thus, establishing conditions for existence of the MLE amounts to specifying the conditions under which statistical inference for these models is fully possible. Secondly, under the asymptotic scenario of growing network sizes, existence of the MLE will provide a natural measure of sample complexity of the beta model, and will indicate the asymptotic scaling of the model parameters for which statistical inference is viable. In fact, our results from Section 4 will prove that the parameters of the beta model can be estimated consistently even when the edge probabilities approach 0 or 1, provided the network size is sufficiently large and under appropriate conditions.

Though prior studies of the beta model by Chatterjee et al. (2011) and Barvinok and Hartigan (2010)\footnote{In the analysis of Barvinok and Hartigan (2010), the maximum entropy matrix associated to a degree sequence is in fact exactly the MLE corresponding to the observed degree sequence. This is a well-known property of linear exponential families: see, e.g.,}
also revolve around the very same issue of existence of the MLE, our method of analysis is significantly
different from existing contributions in that it is rooted in the statistical theory of discrete linear exponential
families and relies in a fundamental way on the geometric properties of these families (see, in particular,
Rinaldo et al., 2009; Geyer, 2009). Our contributions are as follows.

• We provide explicit necessary and sufficient conditions for existence of the MLE for the beta model that
are based on the polytope of degree sequences, a well-studied polytope arising in the study of threshold
graphs (see Mahadev and Peled, 1996). In contrast, the conditions of Chatterjee et al. (2011)
are only sufficient. We then show that non-existence of the MLE is brought on by certain forbidden
patterns of extremal network configurations, which we fully characterize in a combinatorial way. Furthermore, when the MLE does not exist, we can identify exactly which probability parameters are estimable. To illustrate our findings, we rely on the computational geometry software polymake (see Gawrilow and Joswig, 2000) to compute the forbidden configurations leading to nonexistence of the MLE for some simple beta models.

• We use the properties of the polytope of degree sequences to formulate geometric conditions that
allow us to derive finite sample bounds on the probability that the MLE does not exist. In particular,
our results imply that the MLE exists with overwhelming probability even when the edge probabilities
tend to zero or one as the network grows, a case unaccounted for by existing literature. Our asymptotic
results improve analogous results of Chatterjee et al. (2011) and our proof is both simpler and more
direct. Furthermore, we show that the tameness condition of Barvinok and Hartigan (2010) is stronger
than our conditions for existence of the MLE.

• Our analysis is not specific to the beta model but, in fact, follows a principled way for detecting nonex-
istence of the MLE and identifying non-estimable parameters that is based on polyhedral geometry and
applies more generally to discrete models. We illustrate this point by analyzing other network models
that are variations or generalizations of the beta model: the Rasch model, the Bradley-Terry model and
the p1 model.

Finally, we remark that our results arise as non-trivial applications of the geometric and combinatorial
properties of log-linear models under general sampling schemes, as thoroughly described in the companion
paper Fienberg and Rinaldo (2011), to which the reader is referred for further details as well as for practical
algorithms.

The paper is organized as follows. In section 2 we introduce a generalized version of the beta model in
which we observe the edges of a graph a fixed number of times, possibly larger than one, and we express
it as a natural exponential family with linear sufficient statistics. We obtain the beta model as a special
case in which we observe edges only once. In section 3 we introduce the polytope of degree sequences and
use it to derive necessary and sufficient conditions for the existence of the MLE. In particular, we charac-
terize the patterns of edge counts for which the MLE does not exist, called co-facial sets. In section 3.1 we
show a number of examples of co-facial sets, obtained using polymake. Furthermore, we use a result from
Mahadev and Peled (1996) to show in section 3.2 how to construct virtually any example of random graphs
for which the MLE of the beta parameters does not exist. In section 4 we once again use the polytope of
degree sequences to obtain finite sample bounds on the probability that the MLE does not exist. As the
number of objects to be compared increases, the MLE exists with probability approaching one. In section
5 and in the appendix we describe an algorithm for computing and identifying facial sets. In section 6, we
apply our theory and algorithm to a variety of other, related models: the Rasch model, a generalized beta
model with no sampling restriction on the number of observed edges, the Bradley-Terry model and the p1
model for directed networks of Holland and Leinhardt (1981).

Cover and Thomas (1991, Chapter 11)
Notation

For vectors $x$ and $y$ in the Euclidean space $\mathbb{R}^n$, we will denote with $x_i$ the value of $x$ at its $i$-th coordinate and with $\langle x, y \rangle := x^T y = \sum_i x_i y_i$ their standard inner product. Operations on vectors will be performed element-wise. For a matrix $A$, $\text{convhull}(A)$ and $\text{cone}(A)$ denote the set of all convex and conic combinations of the columns of $A$, respectively. For a polyhedron $P$, we denote with $ri(P)$ its relative interior. We will assume throughout some familiarity with basic concepts from polyhedral geometry (see, e.g., Schrijver, 1998) and the theory of exponential families (see, e.g., Barndorff-Nielsen, 1978; Brown, 1986).

2 The (Generalized) Beta Model

In this section we describe a simple generalization of the beta model and introduce the exponential family parametrization we will be using throughout the article. Though our analysis applies to the generalized beta model and recovers the original beta model as described in Chatterjee et al. (2011) as a special case, for simplicity and with slight abuse of notation, we will refer to our more general setting as the beta model as well.

The beta model is concerned with the occurrence of edges in a simple undirected random graph, with the nodes labeled $\{1, \ldots, n\}$ for convenience. The associated statistical experiment consists of recording, for each pair of nodes $(i,j)$ with $i < j$, the number of edges appearing in $N_{i,j}$ distinct observations, where the integers $\{N_{i,j}, i < j\}$ are deterministic and positive (both the non-randomness and positivity assumptions can in fact be relaxed). For $i < j$, we denote with $x_{i,j}$, the number of times the edge $(i, j)$ was observed and, accordingly, with $x_{j,i}$ the number of times object edge $(i, j)$ was missing. Thus, for all $(i, j)$,

$$x_{i,j} + x_{j,i} = N_{i,j}.$$

This is the natural heterogenous version of the well-known Erdős-Rényi random graph model (Erdős and Rényi, 1959). For a discussion of this model and its generalizations see Goldenberg et al. (2010). The observed edge counts $\{x_{i,j}, i < j\}$ are modeled as draws from mutually independent binomial distributions, with $x_{i,j} \sim \text{Bin}(N_{i,j}, p_{i,j})$, where $p_{i,j} \in (0, 1)$ for each $i < j$. Accordingly, $x_{j,i} = N_{i,j} - x_{i,j}$ has a $\text{Bin}(N_{i,j}, p_{j,i})$ distribution, where $p_{j,i} = 1 - p_{i,j}$.

Data arising from such an experiment can be naturally represented through a $n \times n$ contingency table with empty diagonal cells and whose $(i,j)$-th cell contains the count $x_{i,j}$, $i \neq j$. For modeling purposes, however, it is enough to consider the upper-triangular part of this contingency table. Indeed, since, given $x_{i,j}$, the value of $x_{j,i}$ is determined by $N_{i,j} - x_{i,j}$, the set of all possible outcomes can be represented more parsimoniously as the following subset of $\mathbb{N}^2$:

$$S_n := \{x_{i,j} : i < j \text{ and } x_{i,j} \in \{0,1,\ldots,N_{i,j}\}\}.$$

Throughout the article, we index the coordinates $\{(i,j) : i < j\}$ of any point $x$ in the sample space $S_n$ lexicographically.

In the beta model, the $\binom{n}{2}$ edge probabilities are parametrized by points $\beta \in \mathbb{R}^n$ as follows. For each $\beta \in \mathbb{R}^n$, the probability parameters are uniquely determined as

$$p_{i,j} = \frac{e^{\beta_i + \beta_j}}{1 + e^{\beta_i + \beta_j}} \quad \text{and} \quad p_{j,i} = 1 - p_{i,j} = \frac{1}{1 + e^{\beta_i + \beta_j}}, \quad \forall i \neq j,$$

or, equivalently, in term of odds ratios,

$$\log \frac{p_{i,j}}{p_{j,i}} = \beta_i + \beta_j, \forall i \neq j.$$

Therefore, for a given choice of $\beta$, the probability of observing the vector of edge counts $x \in S_n$ is

$$p_\beta(x) = \prod_{i < j} \binom{N_{i,j}}{x_{i,j}} p_{i,j}^{x_{i,j}} (1 - p_{i,j})^{N_{i,j} - x_{i,j}}.$$
with the probability values \( p_{i,j} \) satisfying (1). Simple algebra shows that this expression can be written in exponential family form as

\[
p_{\beta}(x) = \exp \left\{ \sum_{i=1}^{n} d_i \beta_i - \psi(\beta) \right\} \prod_{i<j} \left( \frac{N_{i,j}}{x_{i,j}} \right),
\]

where the coordinates of the vector of minimal sufficient statistics \( d = d(x) \in \mathbb{N}^n \) are

\[
d_i = \sum_{j<i} x_{j,i} + \sum_{j>i} x_{i,j}, \quad i = 1, \ldots, n,
\]

and the log-partition function \( \psi: \mathbb{R}^n \to \mathbb{R} \) is given by \( \beta \mapsto \sum_{i<j} N_{i,j} \log (1 + e^{\beta_i + \beta_j}) \). Note that \( e^{\psi(\beta)} < \infty \) for all \( \beta \in \mathbb{R}^n \), so \( \mathbb{R}^n \) is the natural parameter space of the full and steep exponential family with support \( S_n \) (see, e.g. Barndorff-Nielsen, 1978) and densities given by the exponential term in (4). We take note that in the beta model parametrization the probability of an undirected simple graph with possibly multiple edges is determined only by the \( n \) natural parameters in \( \beta \) instead of the \( \binom{n}{2} \) edge probability parameters \( \{p_{i,j}, i < j\} \).

### Random graphs with fixed degree sequence

In the special case in which \( N_{i,j} = 1 \) for all \( (i, j) \), the support \( S_n \) reduces to the set \( G_n := \{0, 1\}^{\binom{n}{2}} \), which encodes all undirected simple graphs on \( n \) nodes: for any \( x \in G_n \), the corresponding graph has an edge between nodes \( i \) and \( j \), with \( i < j \), if and only if \( x_{i,j} = 1 \). In this case the beta model yields a class of distributions for random undirected simple graphs on \( n \) nodes, where the edges are mutually independent Bernoulli random variables with probabilities of success \( \{p_{i,j}, i < j\} \) satisfying (1). Then, by (5), the \( i \)-th minimal sufficient statistic \( d_i \) is the degree of node \( i \), i.e. the number of nodes adjacent to \( i \), and the vector \( d(x) \) of sufficient statistics is the degree sequence of the observed graph \( x \). This precisely the version of the beta model studied by Chatterjee et al. (2011).

#### The Rasch model

The Rasch model (see, e.g., Rasch, 1960; Andersen, 1980) is concerned with modeling the joint probabilities that \( k \) subjects provide correct answers to a set of \( l \) items, and is one of the most popular statistical models used in item response theory and in educational tests. This model can be recast as a random bipartite graph model in which, without loss of generality, the bipartition of the nodes consists of the sets \( I := \{1, \ldots, k\} \) and \( J := \{k+1, n-1, n\} \), with \( k \geq 2 \) and \( l := n-k \geq 2 \). The set \( I \) represents the subjects and the set \( J \) the items, and edges can only be of the form \((i, j)\), with \( i \in I \) and \( j \in J \). In particular, the presence of an edge \((i, j)\) indicates that the \( i \)-th subject has responded correctly to the \( j \)-th item. The sample space is given by the set \( \mathcal{R}_n = \{0, 1\}^{kl} \), and the vector \( x \in \{x_{i,j}, i \in I, j \in J\} \in \mathcal{R}_n \) encodes the bipartite graphs in which the edge \((i, j)\) is present if and only if \( x_{i,j} = 1 \), i.e. if and only if subject \( i \) answered correctly to item \( j \).

The Rasch model (see, e.g. Rasch, 1960) is then formulated by assuming that the edge probabilities satisfy equation (1), for some \( \beta \in \mathbb{R}^n \). Then, it follows directly from our discussion above that the Rasch model is a beta model for bipartite graphs, and, in particular, that the degree sequence provides the sufficient statistics.

### 3 Existence of the MLE for the Beta Model

In this section we derive a necessary and sufficient condition for the existence of the MLE of the natural parameter \( \beta \in \mathbb{R}^n \) of the beta model or, equivalently, of the probability parameters \( \{p_{i,j}, i < j\} \) as defined in (1). For a given \( x \in S_n \), we say that the MLE does not exist when

\[
\{\beta^* : p_{\beta^*}(x) = \sup_{\beta \in \mathbb{R}^n} p_{\beta}(x)\} = \emptyset,
\]
where \( p_{ij}(x) \) is given in (4). Notice that nonexistence of the MLE entails, in the case of the natural parameters, that the supremum of the likelihood function (4) cannot be attained by any finite vector in \( \mathbb{R}^n \), and, in the case of the probability parameters, that the supremum of (3) cannot be attained by any set of probability values bounded away from 0 and 1, and satisfying the equations (1).

We will formulate conditions for the existence of the MLE for the beta model based on a geometric object that will play a key role throughout the rest of the paper: the polytope of degree sequences. To this end, note that, for each \( x \in \mathcal{S}_n \), the vector of sufficient statistics \( d(x) \) for the beta model can be obtained as

\[
d(x) = Ax
\]

where \( A \) is the \( n \times \binom{n}{2} \) design matrix consisting of the node-edge incidence matrix of a complete graph on \( n \) nodes. Specifically, the rows of \( A \) are indexed by the node labels \( i \in \{1, \ldots, n\} \), and the columns are indexed by the set of all pairs \((i, j)\) with \( i < j \), ordered lexicographically. The entries of \( A \) are ones along the coordinates \((i, (i, j))\), when \( i < j \) and \((i, (j, i))\) when \( j < i \), and zeros otherwise. For instance, when \( n = 4 \)

\[
A = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix},
\]

where the columns are indexed lexicographically by the pairs \((1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\). In particular, as pointed out above, for any undirected simple graph \( x \in \mathcal{G}_n \), \( Ax \) is the associated degree sequence. The polytope of degree sequences \( P_n \) is the convex hull of all possible degree sequences, i.e.

\[
P_n := \text{convhull}\left(\{Ax, x \in \mathcal{G}_n\}\right).
\]

The integral polytope \( P_n \) is a well-studied object: see Chapter 3 in Mahadev and Peled (1996). In the language of algebraic statistics, \( P_n \) is called the model polytope (see Sturmfels and Welker, 2011). In particular, when \( n = 2 \), \( P_n \) is just a line segment in \( \mathbb{R}^2 \) connecting the points \((0, 0)\) and \((1, 1)\), while, for all \( n \geq 3 \), \( \dim(P_n) = n \).

The main result in this section is to show that existence of the MLE for the beta model can be fully characterized using the polytope of degree sequences in the following fashion. For any \( x \in \mathcal{S}_n \), let

\[
\tilde{p}_{i,j} := \frac{x_{i,j}}{N_{i,j}}, \quad i < j,
\]

and set \( \tilde{d} = \tilde{d}(x) \in \mathbb{R}^n \) to be the vector with coordinates

\[
\tilde{d}_i := \sum_{j < i} \tilde{p}_{j,i} + \sum_{j > i} \tilde{p}_{i,j}, \quad i = 1, \ldots, n.
\]

Notice that, \( \tilde{d} \) is a just a rescaled version of the sufficient statistics (5), normalized by the number of observations. It is also clear that, for the random graph model, \( \tilde{d} = d \).

**Theorem 3.1.** Let \( x \in \mathcal{S}_n \) be the observed vector of edge counts. The MLE exists if and only if \( \tilde{d}(x) \in \text{int}(P_n) \).

**Remark**

Theorem 3.1 verifies the conjecture contained in Addenda A in Chatterjee et al. (2011): for the random graph model, the MLE exists if and only if the degree sequence belongs to the interior of \( P_n \). This result follows from the standard properties of exponential families: see Theorem 9.13 in Barndorff-Nielsen (1978) or Theorem 5.5 in Brown (1986). The theorem also confirms the observation made by Chatterjee et al. (2011) that the MLE never exists if \( n = 3 \): indeed, since \( P_3 \) has exactly 8 vertices, as many as the possible graphs on 3 nodes, no degree sequence can be inside \( P_3 \).

The geometric nature of Theorem 3.1 has important consequences. First, it provides the algorithmic basis for detecting existence of the MLE, as discussed in the appendix. Secondly, and quite importantly, it allows to indentify the patterns of observed edge counts that cause nonexistence of the MLE, i.e. the sample points for which the MLE is undefined. This is done in the next result.
Lemma 3.2. A point \( y \) belongs to the interior of some face \( F \) of \( P_n \) if and only if there exists a set \( F \subset \{(i, j), i < j\} \) such that
\[
y = Ap,
\]
where \( p = \{p_{i,j}: i < j, p_{i,j} \in [0, 1]\} \in \mathbb{R}^2 \) is such that \( p_{i,j} \in \{0, 1\} \) if \( (i, j) \notin F \) and \( p_{i,j} \in (0, 1) \) if \( (i, j) \in F \). The set \( F \) is uniquely determined by the face \( F \) and is the maximal set for which (7) holds.

Following Geiger et al. (2006) and Fienberg and Rinaldo (2011), we call any such set \( F \) a facial set of \( S_n \) and its complement, \( F^c = \{(i, j): i < j \} \setminus F \), a co-facial set. Facial sets form a lattice that is isomorphic to the face lattice of \( P_n \) as shown by Fienberg and Rinaldo (2011, Lemma 3.4). This means that the faces of \( S_n \) are in one-to-one correspondence with the facial sets of \( S_n \) and, for any pair of faces \( F \) and \( F' \) of \( S_n \) with associated facial sets \( F \) and \( F' \), \( F \cap F' \) if and only if \( F \subseteq F' = \emptyset \) if and only if \( F \subseteq F' \). In particular, for a point \( x \in S_n \), \( d(x) = Ax \) belongs to the interior of a face \( F \)'s \( P_n \) if and only if there exists a non-negative \( p \) such that \( d(x) = Ap \), where \( F = \{(i, j): p_{i,j} > 0\} \) is the facial set corresponding to \( F \). By the same token, \( y \in \text{int}(P_n) \) if and only if \( y = Ap \) for a vector \( p \) whose coordinates are strictly between 0 and 1.

Facial sets are combinatorial objects that have statistical relevance for two reasons. First, non-existence of the MLE can be described combinatorially in terms of co-facial sets, i.e. patterns of edge counts that are either 0 or \( N_{i,j} \). In particular, the MLE does not exist if and only if the set \( \{(i, j): i < j, x_{i,j} = 0 \text{ or } N_{i,j}\} \) contains a co-facial set. Secondly, apart from exhausting all possible patterns of forbidden entries in the table leading to a non-existent MLE, facial sets specify which probability parameters are estimable. In fact, inspection of the likelihood function (3) reveals that, for any observable set of counts \( \{x_{i,j}: i < j\} \), there always exists a unique set of maximizers \( \tilde{p} = \{\tilde{p}_{i,j}, i < j\} \) which, by strict concavity, are uniquely determined by the first order optimality conditions
\[
d(x) = A\tilde{p},
\]
also known as the moment equations. Existence of the MLE is then equivalent to \( 0 < \tilde{p}_{i,j} < 1 \) for all \( i < j \). When the MLE does not exist, i.e. when \( d \) is on the boundary of \( P_n \), the moment equations still hold, but the entries of the optimizer \( \{\tilde{p}_{i,j}, i < j\} \), known as the extended MLE, are no longer strictly between 0 and 1. Instead, by Lemma (3.2), the extended MLE is such that \( \tilde{p}_{i,j} = \tilde{p}_{i,j} \in \{0, 1\} \) for all \( (i, j) \in F^c \). Furthermore, it is possible to show (see, e.g., Morton, 2008) that \( \tilde{p}_{i,j} \in (0, 1) \) for all \( (i, j) \in F \). Therefore, when the MLE does not exist, only the probabilities \( \{p_{i,j}, (i, j) \in F\} \) are estimable.

Therefore, while co-facial sets encode the patterns of table entries leading to a non-existent MLE, facial sets indicate which probability parameters are estimable. A similar, though more involved interpretation holds for the estimability of the natural parameters, for which the reader is referred to Fienberg and Rinaldo (2011).

Below, we further investigate the properties of \( P_n \) and provide several examples of co-facial sets associated to the facets of \( P_n \).

### 3.1 The Co-facial Sets of \( P_n \)

Theorem 3.1 and Lemma 3.2 both show that the boundary of the polytope \( P_n \) plays a fundamental role in determining the existence of the MLE for beta models and in specifying which parameters are estimable.

Mahadev and Peled (1996) have fully characterized the boundary of \( P_n \) and derived the facet-defining inequalities of \( P_n \), for all \( n \geq 4 \) (when \( n \leq 3 \) the problem is of little interest). For the reader’s convenience,

| × | 0 |
|---|---|
| N₁₂ | × |
| × | N₄₄ |
| 0 | × |

Table 1: Example of a co-facial set leading to a nonexistent MLE.
we describe some of our computations. which the MLE is nonexistent, most of which seem to be unaccounted for in the statistical literature. Below we report this result below. Let \( P \) be the set of all pairs \((S, T)\) of disjoint non-empty subsets of \(\{1, \ldots, n\}\), such that \(|S \cup T| \in \{2, \ldots, n-3, n\}\). For any \((S, T) \in P\) and \(y \in P_n\), let
\[
g(S, T, y, n) := |S|(n - 1 - |T|) - \sum_{i \in S} y_i + \sum_{i \in T} y_i.
\]

Table 2: Left: data exhibiting the pattern reported in Table 1, when \(N_{i,j} = 3\) for all \(i \neq j\). Right: table of the extended MLE of the estimated probabilities. Under the natural parametrization, the supremum of the log-likelihood is achieved in the limit for any sequence of natural parameters \(\beta^{(k)}\) of the form \(\beta^{(k)} = (-c_k, -c_k, c_k, c_k)\), where \(c_k \to \infty\) as \(k \to \infty\).

Table 3: Left: same data as in Table 2, but with the values for the cells (1, 2) and (2, 3) switched with the values in the cells (2, 1) and (3, 2), respectively. Right: table of probabilities at which the log-likelihood is optimal. The MLE of the natural parameters are \(\beta = (-0.237, -1.002, -0.237, 1.205)\).

The combinatorial complexity of the face lattice of an \(n\)-dimensional polytope can be summarized by its \(f\)-vector, the vector of length \(n + 1\) whose \(i\)-th entry contains the number of \(i\)-dimensional faces, \(i = 0, \ldots, n\). Stanley (1991) studies the number faces of the polytope of degree sequences \(P_n\) and derives an expression for computing the entries of the \(f\)-vector of \(P_n\). For example, the \(f\)-vector of \(P_8\) is the 9-dimensional vector
\[
(334982, 1726648, 3529344, 3679872, 2074660, 610288, 81144, 3322, 1),
\]
so \(P_8\) is an 8-dimensional polytope with 334982 vertices, 1726648 edges, and so on, up to 3322 facets. Also, according to Stanley's formula, the number of facets of \(P_4\), \(P_5\), \(P_6\) and \(P_7\) are 22, 60, 224 and 882, respectively. These numbers correspond to the numbers we obtained with \texttt{polymake}, using the methods described in the appendix.

Despite the fact that much is known about \(P_n\), the number of facet-defining inequalities appears to be exponential in \(n\) and, consequently, the tasks of identifying points on the boundary of \(P_n\) and the associated facial set remain computationally challenging. In the appendix, we discuss these difficulties and propose an algorithm for detecting boundary points and the associated facial sets that is based on a log-linear model reparametrization. Using the methods described there, we were able to identify a few interesting cases in which the MLE is nonexistent, most of which seem to be unaccounted for in the statistical literature. Below we describe some of our computations.
Table 4: Example of a co-facial set leading to a nonexistent MLE. In this case $\tilde{d}_2 = 0$.

|   | N_{1,2} |   |
|---|---|---|
| × | 0 | 0 |
| N_{3,2} | × |   |
| N_{4,2} | × |   |

Table 5: Example of a co-facial set leading to a nonexistent MLE. In this case the second row sum is 0.

|   | N_{1,2} |   |
|---|---|---|
| × | 0 | 0 |
| N_{3,2} | × |   |
| N_{4,2} | × |   |

Table 6: Example of a co-facial set leading to a nonexistent MLE.

Recall that the data can be represented as a $n \times n$ table of counts, in which the diagonal elements are expunged and where the $(i,j)$ - th entry of the table indicates the number of times, out of $N_{i,j}$, in which the edges $(i,j)$ was observed. In our examples, empty cells correspond to facial set and may contain any count values, in contrast to the cells in the co-facial sets that contain either a zero value or a maximal value, namely $N_{i,j}$. As we say in Lemma 3.2, extreme count values of this nature are precisely what leads to a nonexistent MLE.

Table 1 provides an instance of a co-facial set, which corresponds to a facet of $P_1$. Assume for simplicity that each of the empty cells contain counts bounded away from 0 and $N_{i,j}$. Then the sufficient statistics $\tilde{d}$ are also bounded away from 0 and $n-1$ and, so are the row and column sums of the normalized counts $\{\frac{x_{i,j}}{N_{i,j}}; i \neq j\}$, yet the MLE does not exist. This is further illustrated in Table 2, which shows, on the left, an instance of data with $N_{i,j} = 3$ for all $i \neq j$, satisfying the pattern indicated in Table 1 and, on the right, the probability values maximizing the log-likelihood function. Since the MLE does not exist, some of these probability values are 0 and 1. The order of the pattern is crucial. Indeed, Table 3 shows, on the left, data containing precisely the same counts as in Table 2, but with the values in cells (1, 2) and (2, 3) switched with the values in cell (2, 1) and (3, 2), respectively. On the left of Table 3 the MLE of the cell probabilities are shown; as the MLE exists, they are bounded away from 0 and 1.

In Table 4 we show another example of a co-facial set that is easy to detect, since it corresponds to a value of 0 for the normalized sufficient statistic $\tilde{d}_2$. Indeed, from cases (i) and (ii) of Theorem 3.3, the MLE does not exist if $\tilde{d}_i = 0$ or $\tilde{d}_i = n-1$, for some $i$. Table 5 shows yet one more example of a co-facial set that is easy to detect, as it leads to a zero row margin for the second row. Finally, Table 6 provides one more example of a co-facial set, which unlike the ones in Tables 4 and 5, has normalized row sums and the normalized sufficient statistics bounded away from 0 and $n-1$. In Table 7 we list all 22 co-facial sets associated with the facets of $P_n$, including the cases already shown in Tables 1, 4, 5 and 6.

In general, there are $2n$ facets of $P_n$ that are determined by $\tilde{d}_i$ equal to 0 or $n-1$, and $2n$ other facets associated to values of the normalized row sums equal to 0 or $n-1$. Thus, just by inspecting the row sums or the observed sufficient statistics, one can detect $4n$ co-facial sets associated to as many facets of $P_n$. However, comparing this number to the entries of the $f$-vector calculated in Stanley (1991) and as our computations
confirm, most of the facets of $P_n$ do not yield co-facial sets of this form. Since the number of facets appear to grow exponentially in $n$, we conclude that most of the co-facial sets do not appear to arise in this fashion.

### 3.2 Random Graphs with Nonexistent MLEs

When dealing with the special case of $N_{i,j} = 1$ for all $i < j$, which we showed to be equivalent to a model for random undirected graphs, points on the boundary of $P_n$ are, by construction, degree sequences and have a
Table 8: Patterns of zeros and ones yielding random graphs with non-existent MLE (empty cells indicate that the entry could be a 0 or a 1).

direct graph-theoretical interpretation, as shown in the next result.

**Lemma 3.4** (Lemma 3.3.13 in Mahadev and Peled (1996)). Let $d$ be a degree sequence of a graph $G$ that lies on the boundary of $P_n$. Then either $d_i = 0$, or $d_i = n - 1$ for some $i$, or there exist non-empty and disjoint subsets $S$ and $T$ of $\{1, \ldots, n\}$ such that

1. $S$ is clique of $G$;
2. $T$ is a stable set of $G$;
3. every vertex in $S$ is adjacent to every vertex in $(S \cup T)^c$ in $G$;
4. no vertex of $T$ is adjacent to any vertex of $(S \cup T)^c$ in $G$.

A direct consequence of lemma 3.4 is that the MLE does not exist if the observed network is a split graph, i.e. a graph whose node sets can be partitioned into a clique $S$ and a stable set $T$. More generally, Lemma 3.4 can be used to create virtually any example of random graphs with fixed degree sequences for which the MLE does not exist. Notice that, in particular, having node degrees bounded away from 0 and $n - 1$ is not a sufficient condition for the existence of the MLE (though its violation implies nonexistence of the MLE). We point out that, in order to detect boundary points and the associated co-facial sets, Lemma 3.4 is, however, of little help.

Below, we provide some examples of co-facial sets for random graphs with fixed degree sequences for which the MLE does not exist, yet the node degrees are bounded from 0 and $n - 1$.

For the case $n = 4$, our computations show that there are 14 distinct co-facial sets associated to the facets of $P_n$. Eight of them correspond to degree sequences containing a 0 or a 3, and the remaining six are shown in Table 8, which we computed numerically using the procedure described in the appendix. Notice that the three tables on the second row are obtained from the first three tables by switching zeros with ones. Furthermore, the number of the co-facial sets we found is smaller than the number of facets of $P_n$, which is 22, as shown in Table 7. This is a consequence of the fact that the only observed counts in the random graph model are 0’s or 1’s: it is in fact easy to see in Table 7 that any co-facial set containing three zero counts and three maximal counts $N_{i,j}$ is equivalent, in the random graph case, to a node having degree zero or 3. However, as soon as $N_{i,j} \geq 2$, the number of possible co-facial sets matches the number of faces of $P_n$.

Table 9 shows an observed graph with degrees all larger than 0 and less than 3 but for which the MLE does not exist. Notice that the co-facial set corresponds to the one shown in the upper left corner of Table 8. Finally, Tables 9 and 10 show two more examples of random graphs on $n = 5$ and $n = 6$ nodes, respectively, for which the MLE does not exist (by Lemma 3.4), and yet the degrees are such that $0 < d_i < n - 1$ for all $i$. 

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Table 9: Random graph with node degrees larger than 0 and smaller than 3 exhibiting the same co-facial set show in the upper left corner of Table 8. In this case, lemma 3.4 applies with $S = \{3, 4\}$ and $T = \{1, 2\}$.

Table 10: Network with $n = 5$ for which the MLE does not exist and the degrees are bounded away from 0 and 4. In this case, lemma 3.4 applies with $S = \{2, 3, 4\}$ and $T = \{1, 5\}$.

Table 11: Network with $n = 6$ for which the MLE does not exist and the degrees are bounded away from 0 and 5. In this case, lemma 3.4 applies with $S = \{1, 2, 6\}$ and $T = \{3, 4, 5\}$.

4 Existence of the MLE: Asymptotics

In this section we derive sufficient conditions that imply existence of the MLE with large probability as the size of the network $n$ grows. We will make the simplifying assumption that $N_{i,j} = N_n$, for all $i$ and $j$, where $N_n \geq 1$ could itself depend on $n$.

Recall the random vector $\tilde{d}$, whose coordinate are given in (6) and set $\tilde{d} = \mathbb{E}[\tilde{d}] \in \mathbb{R}^n$. Then

$$d_i = \sum_{j<i} p_{j,i} + \sum_{j>i} p_{i,j}, \quad i = 1, \ldots, n.$$  

We formulate sufficient conditions for the existence of the MLE in terms of the entries of the vector $\tilde{d}$.

**Theorem 4.1.** Assume that, for all $n \geq \max\{4, 2\sqrt{c n \log n} + 1\}$, the vector $\tilde{d}$ satisfies the conditions

(i) $\min_i \min \{\tilde{d}_i, n - 1 - \tilde{d}_i\} \geq 2 \sqrt{c n \log n} + C$,

(ii) $\min_{(S,T) \in \mathcal{P}} g(S, T, \tilde{d}, n) > |S \cup T| \sqrt{c n \log n} + C$,

where $c > 1/2$ and $C \in \left(0, \frac{n-1}{2} - \sqrt{c n \log n}\right)$. Then, with probability at least $1 - \frac{2}{n^{c-1}}$, the MLE exists.

When $N_n$ is constant, for instance when $N_n = 1$, as in the random graph case, the conditions of Theorem 4.1 can be relaxed by requiring condition (ii) to hold only over subsets $S$ and $T$ of cardinality of order...
Corollary 4.2. Let \( n \geq \max\{N, 4, 2\sqrt{cn \log n} + 1\}, \) \( c > 1 \) and \( C \in \left(0, \frac{2}{d^2} - \sqrt{cn \log n}\right)\). Assume the vector \( \overrightarrow{d} = E[d] \in \mathbb{R}^n \) satisfies the conditions

(i) \( \min_i \min \{d_i, n - 1 - d_i\} \geq 2 \sqrt{cn \log n} + C; \)

(ii) \( \min_{(S,T) \in \mathcal{P}_n} g(S, T, \overrightarrow{d}, n) > |S \cup T| \sqrt{cn \log n} + C, \)

where

\[
\mathcal{P}_n := \{(S, T) \in \mathcal{P} : \min\{|S|, |T|\} > \sqrt{cn \log n} + C\},
\]

where the set \( \mathcal{P} \) was defined before Theorem 3.3. Then, the MLE exists with probability at least \( 1 - \frac{2}{n^{c/2}} \). If \( N = 1 \), it is enough to have \( c > 1/2 \), and the MLE exists with probability larger than \( 1 - \frac{2}{n^{c/2}} \).

4.1 Discussion and Comparisons with Previous Works

It is clear that, asymptotically, the value of the constant \( C \) in both Theorem 4.1 and Corollary 4.2 becomes irrelevant, as the constraints on its range will be satisfied by any positive \( C \), for all \( n \) large enough.

Since \( |S \cup T| \leq n \), one could replace assumption (ii) of Theorem 4.1 with the simpler but stronger condition

\[
\min_{(S,T) \in \mathcal{P}_n} g(S, T, \overrightarrow{d}, n) > n^{3/2} \sqrt{c \log n} + C_n.
\]

Then, assuming for simplicity that \( N_n \) is a constant, as in Corollary 4.2, the MLE exists with probability tending to one at a rate that is polynomial in \( n \) whenever

\[
\min_i \min \{d_i, n - 1 - d_i\} = \Omega\left(\sqrt{n \log n}\right)
\]

and, for all pairs \( (S,T) \in \mathcal{P}, \)

\[
g(S, T, \overrightarrow{d}, n) > \Omega\left(n^{3/2} \sqrt{\log n}\right).
\]

For the case \( N_n = 1 \), Corollary 4.2 should be compared with Theorem 3.1 in Chatterjee et al. (2011), which also provides sufficient conditions for the existence of the MLE with probability no smaller than \( 1 - \frac{1}{n^{c/2}} \) (for all \( n \) large enough), but appear to be stronger than ours. In detail, their conditions require that, for some constant \( c_1, c_2 \) and \( c_3 \) in \((0,1), c_1(n-1) < d_i < c_2(n-1)\) for all \( i \) and

\[
|S|(|S| - 1) - \sum_{i \in S} d_i + \sum_{i \notin S} \min\{d_i, |S|\} > c_3 n^2,
\]

for all sets \( S \) such that \( |S| > (c_1)^2 n^2 \). It is easy to see that, for any non-empty subsets \( S \subset \{1, \ldots, n\} \) and \( T \subset \{1, \ldots, n\} \setminus S, \)

\[
\sum_{i \notin S} \min\{d_i, |S|\} \leq \sum_{i \in T} d_i + |S|(|S \cup T)^c|,
\]

which implies that

\[
|S|(n - 1 - |T|) - \sum_{i \in S} d_i + \sum_{i \notin T} d_i > |S|(|S| - 1) - \sum_{i \in S} d_i + \sum_{i \notin S} \min\{d_i, |S|\},
\]

where we have used the equality \( n = |S| + |T| + |(S \cup T)^c| \). Thus if (9) holds for some non-empty \( S \subset \{1, \ldots, n\} \), it satisfies the facet conditions implied by all the pairs \( (S,T), \) for any non-empty set \( T \subset \{1, \ldots, n\} \setminus S. \) As a result, for any subset \( S, \) (9) is a stronger condition than any of the facet conditions of \( \mathcal{P}_n \) specified by \( S. \) In addition, we weakened significantly their requirements that \( c_1(n-1) < d_i < c_2(n-1) \)
for all $i$ to $\min_i \min \{ \overline{d}_i, n - 1 - \overline{d}_i \} \geq 2\sqrt{cn \log n} + C$. As a direct consequence of this weakening, in our analysis we only need $|S| > \sqrt{cn \log n} + C$ as opposed to $|S| > (c_1/n)^2$. Overall, in our setting, the vector of expected degrees of the sequence of networks is allowed to lie much closer to the boundary of $P_n$. As we explain next, such weakening is significant, since the setting of Chatterjee et al. (2011) only allows to estimate an increasing number of probability parameter (the edge probabilities) that are uniformly bounded away from 0 and 1, while our assumptions allow for these probabilities to become degenerate as the network size grows.

**The non-degenerate case**

We now briefly discuss the case of sequences of networks for which $N_n = 1$ and the edge probabilities are uniformly bounded away from 0 and 1, i.e.

$$
\delta < p_{i,j} < 1 - \delta, \quad \forall i,j,
$$

for some $\delta \in (0, 1)$ independent of $n$. In this scenario, the number of probability parameters to be estimated grows with $n$, but their values are guaranteed to be non-degenerate. It immediately follows from the non-degenerate assumption (10) that $\overline{d} \in \text{int}(P_n)$ and

$$
\delta(n - 1) < \overline{d}_i < (1 - \delta)(n - 1), \quad i = 1, \ldots, n.
$$

Then, the same arguments used in the proof of corollary 4.2 imply that the MLE exists with high probability. We only provide a sketch of the proof. First, we note that, with high probability, $g(S, T, \overline{d}, n) \geq g(S, T, \overline{d}, n) - |S \cup T|/\Omega(\sqrt{n \log n})$, for each pair $(S, T) \in P$. Furthermore, because of (11), it is enough to consider only pairs $(S, T)$ of disjoint subsets of $\{1, \ldots, n\}$ of sizes of order $\Omega(n)$. For each such pair, the condition on $\overline{d}_i$ further yields that $g(S, T, \overline{d}, n)$ is of order $\Omega(n^2)$, and, by Theorem 8 the MLE exists with high probability.

In fact, the boundedness assumption of Chatterjee et al. (2011) that $\|\beta\|_\infty < L$, with $L$ independent of $n$, is equivalent to the non-degenerate assumption (10), as it can be easily seen from equation (1). Unlike the analysis of Chatterjee et al. (2011), which focusses on the non-degenerate case, our results hold under weaker scaling, as we only require for instance that $\overline{d}_i$ be of order $\Omega(\sqrt{n \log n})$ for all $i$.

Finally, we note that the tameness condition of Barvinok and Hartigan (2010) is equivalent to $\delta < \tilde{p}_{i,j} < 1 - \delta$ for all $i$ and $j$ and a fixed $\delta \in (0, 1)$, where $\tilde{p}_{i,j}$ is the MLE of $p_{i,j}$. Therefore, the tameness condition is stronger than existence of the MLE. In fact, using again Theorem 1.3 in Chatterjee et al. (2011), for all $n$ sufficiently large, the tameness condition is equivalent to the boundedness condition of Chatterjee et al. (2011).

We conclude this section with two final remarks. First, Theorem 1.3 in Chatterjee et al. (2011) shows that, when the MLE exists, $\max_i |\tilde{\beta}_i - \beta_i| = O(\sqrt{\log n / n})$, with probability at least $1 - 2^{-n}$. Combined with our Corollary 4.2, this implies that the MLE is a consistent estimator under a growing network size and with edge probabilities approaching the degenerate values of 0 and 1. Secondly, after the submission of this article we learned about the interesting asymptotic results of Yan and Xu (2012); Yan et al. (2012), who claim that, based on a modification of the arguments of Chatterjee et al. (2011), it is possible to show the MLE of the $\beta$-model exists and is uniformly consistent if $L = o(\log n)$ and $L = o(\log \log n)$, respectively, where $L = \max_i |\tilde{\beta}_i|$.

## 5 Computations

The main difficulty in applying the theory presented so far is that the polytope of degree sequences $P_n$ is in general difficult to handle algorithmically. Indeed, $P_n$ arises a Minkowski sum and, even though the system of defining inequalities is given explicitly, its combinatorial complexity grows exponentially in $n$. Furthermore, the vertices of $P_n$ are not known explicitly. Algorithms for obtaining the vertices of $P_n$, such as $\text{minksum}$ (see Weibel, 2005), are computationally expensive and require generating all the points $\{Ax, x \in \mathcal{G}_n\}$, where
\(|G_n| = 2^{\binom{n}{2}}\). In general, when \(n\) is as small as 10, this is not feasible. See for instance, Table 6.4 below. Thus, deciding whether a given degree sequence is a point in the interior of \(P_n\) and identifying the facial set corresponding to an observed degree sequence on the boundary of \(P_n\).

Our strategy to overcome these problems entails re-expressing the beta model as a log-linear model with \(\binom{n}{2}\) product-multinomial sampling constraints. This approach is not new, and it harks back to the earlier re-expression of the Holland-Leinhardt \(p_1\) model and its natural generalizations as log-linear models (Fienberg and Wasserman, 1981a,b; Fienberg et al., 1985). Though this re-parametrization increases the dimensionality of the problem, it nonetheless has the crucial computational advantage of reducing the determination of the facial sets of \(P_n\) to the determination of the facial sets of a pointed polyhedral cone spanned by \(n(n-1)\) vectors, which is a much simpler object to analyze, both theoretically and algorithmically. This procedure is known as the Cayley embedding in polyhedral geometry, and its use in the analysis of log-linear models is described in Fienberg and Rinaldo (2011). The advantages of this re-parametrization are two-fold. First, it allows us to use the highly optimized algorithms available in polymake for listing explicitly all the facial sets of \(P_n\). This is how we computed the facial sets in all the examples presented in this article. Secondly, the general algorithms for detecting nonexistence of the MLE and identifying facial sets proposed in Fieberg and Rinaldo (2011), which can handle larger dimensional models, can be directly applied to this problem. This reference is also relevant for dealing with inference under a non-existent MLE.

The appendix describes the details of our computations and the associated algorithms.

6 Applications and Extensions

The main arguments that we have used to explore nonexistence of the MLE and parameter estimability in the beta model are rather general, as they pertain to all log-linear models (see, e.g., Fienberg and Rinaldo, 2011). In this section we extend them to different models for networks.

6.1 The Rasch model

Just like in Section 3.2, necessary and sufficient conditions for the existence of the MLE of the Rasch model parameters can also be formulated in geometric terms based on the polytope of degree sequences. In detail, for a bipartition of the \(n\) nodes of the form \(I = \{1, \ldots, k\}\) and \(J = \{k + 1, n - 1, n\}\), where \(l = n - k\), let \(P_{k,l} \subset \mathbb{R}^n\) denote the associated polytope of bipartite degree sequences, i.e. the convex hull of all degree sequences of bipartite undirected simple graphs on \(n\) nodes, with the bipartition specified by \(I\) and \(J\). Let \(d(x)\) denote the degree sequence associated with the observed bipartite graph \(x \in \mathbb{R}_n\). Then, a straightforward application of Theorem 9.13 in Barndorff-Nielsen (1978) yields the following result.

**Theorem 6.1.** The MLE of the Rasch model parameters exists if and only if \(d(x) \in \text{ri}(P_{p,q})\).

The polytope of bipartite degree sequences was introduced by Hammer et al. (1990). We briefly recall its properties (see Mahdev and Peled, 1996, Section 3.4 for more details). Let

\[F_{I,J} := \{y \in P_n : g(y, I, J, n) = 0\}\]

be the facet of \(P_n\) specified by \(I\) and \(J\), where \(g\) is given in (8) (the sets \(I\) and \(J\) can be interchanged). Also, let \(c \in \mathbb{R}^n\) be the vector with coordinates

\[c_i = \begin{cases} k - 1 & i = 1, \ldots, k \\ 0 & i = k + 1, \ldots, n. \end{cases}\]

The polytope of bipartite degree sequences \(P_{k,l}\) is just the translate by \(c\) of the facet \(F_{I,J}\), which implies, in particular, that \(\dim(P_{p,q}) = n - 1\) (this explains why, in Theorem 6.1, we used the correct notation \(\text{ri}(P_{k,l})\) instead of \(\text{int}(P_{p,q})\)).

**Theorem 6.2** (Theorem 3.4.4 in Mahdev and Peled (1996)). \(P_{k,l} = \{y - c, y \in F_{I,J}\}\).
The previous result is rather useful: in order to determine whether the MLE fails to exist, i.e. whether the degree sequence of the observed bipartite graph is on the relative boundary of $P_{k,t}$, one can use Lemma 3.4 as follows. First add an edge between each pair of nodes in $I$ (so, the graph is no longer bipartite). Then, check whether there is a pair of sets $S$ and $T$, different from $I$ and $J$, for which the conditions of Lemma 3.4 apply. Thus, the MLE does not exists if and only if there exists a partition of the nodes into three non-empty sets $S$, $T$ and $(S \cup T)^c$, such that, with respect to this enlarged graph,

1. $S \subseteq I$ (hence $S$ is complete);
2. $T \subseteq J$ (hence, $T$ is stable);
3. every vertex of $S$ is adjacent to every vertex in $(S \cup T)^c$;
4. no vertex in $T$ is adjacent to any vertex in $(S \cup T)^c$.

In fact, the above conditions are equivalent to the conditions for existence of the MLE in the Rasch model found independently by Haberman (1977) and Fischer (1981). Indeed, recall that Haberman’s conditions are as follows: the MLE does not exists if there there exists sets $A$, $B$, $C$ and $D$ such that

1. $A \cup B = I$ and $C \cup D = J$, with $A \cap B \cap C \cap D = \emptyset$;
2. $A \neq \emptyset$ and $C \neq \emptyset$ or $B \neq \emptyset$ and $D \neq \emptyset$;
3. $x_{i,j} = 0$ for all $i \in A$ and $j \in C$;
4. $x_{i,j} = 10$ for all $i \in B$ and $j \in D$,

were $x \in \mathbb{R}_n$ is the observed graph. Then, to see the equivalence, take $S = B$, $T = C$ and $(S \cup T)^c = A \cup D$.

### 6.2 Removing the Sampling Constraint in the Beta Model

In this section we analyze the behavior the generalized beta model when the number of recorded edges for each pair of nodes is also random. Specifically, we assume the number of observed edges $\{x_{i,j}: i \neq j\}$ are realizations of $n(n-1)$ independent Poisson random variables with means $\{m_{i,j}: i \neq j\}$. As a result, the quantities $\{N_{i,j}, i \neq j\}$ are now random and can be zero with positive probabilities. Unlike the beta model described in Section 2, in this more general case $x_{i,j}$ is not determined by $x_{i,j}$, thus we need to account for all possible quantities $\{x_{i,j}\}_{i \neq j}$. We index the points of this enlarged set of $n(n-1)$ numbers as pairs $\{(x_{i,j}, x_{j,i}): i < j\} \subset \mathbb{N}^{n(n-1)}$, with the pairs ordered lexicographically based on $(i,j)$.

In this setting, natural generalization of the beta model is to consider a parametrization of the mean edge counts by points $\alpha \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}^n$ so that

$$
\log m_{i,j} = \alpha_i + \gamma_j, \quad \forall i \neq j.
$$

(12)

Some algebra then shows that the probability of observing any point $x \in \mathbb{N}^{n(n-1)}$ is

$$
p_{\alpha,\gamma}(x) = \exp \left\{ \sum_i \alpha_i d_i^{\text{out}} + \sum_j \gamma_j d_j^{\text{in}} - \phi(\alpha, \gamma) \right\} \prod_{i \neq j} \frac{1}{x_{i,j}}. \tag{13}
$$

where the coordinates of the vectors of minimal sufficient statistics $d^{\text{out}} = d^{\text{out}}(x)$ and $d^{\text{in}} = d^{\text{in}}(x)$ are

$$
d_i^{\text{out}} := \sum_{j \neq i} x_{i,j} \quad \text{and} \quad d_j^{\text{in}} := \sum_{i \neq j} x_{i,j}, \quad i = 1, \ldots, n,
$$

respectively, and the log-partition function $\phi: \mathbb{R}^{2n} \to \mathbb{R}$ is given by $(\alpha, \gamma) \mapsto \sum_{i \neq j} \exp(\alpha_i + \gamma_j)$. The sufficient statistics $d = d(x)$ can be obtained as

$$
d = \left( \begin{array}{c} d_i^{\text{out}} \\ d_j^{\text{in}} \end{array} \right) = Ax,
$$

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\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\hline
\end{tabular}
\caption{Co-facial sets of the second kind, as specified in theorem 6.3, for the case $n = 4$. Empty cells refer to arbitrary entries.}
\end{table}

where $A$ is a $2n \times n(n-1)$ matrix whose columns are indexed by the points in the sample space, and whose rows are indexed by the parameters $\{\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_n\}$. The entries of the row corresponding to $\alpha_i$ are all zeros, except for the coordinates corresponding the columns $(i, j)$ with $i < j$ and $(j, i)$ with $i > j$, which are ones. Similarly, the rows corresponding to $\gamma_j$ are all zeros, except for the coordinates corresponding the columns $(j, i)$ with $i < j$ and $(i, j)$ with $i > j$, which are ones. For instance, when $n = 4$,

$$A = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
\end{bmatrix},$$

We remark that $A$ is rank-deficient, as its rank is $2n - 1$, which reflects the fact that the parametrization in (12) is non-identifiable (this can be easily fixed by imposing, for instance, the constraint $\sum_i \alpha_i = 0$).

Notice that if the entries of $x \in \{0, 1\}^{n(n-1)}$ are all zeros and ones, then $x$ encodes a directed graph on $n$ nodes, with an arrow going from node $i$ to node $j$ if and only if $x_{i,j} = 1$ (thus, there may be two edges connecting any pair of nodes, directed in opposite ways). In this case, the sufficient statistics $d_{\text{out}}$ and $d_{\text{in}}$ correspond to the in-degrees and out-degrees of the nodes.

Below we provide necessary and sufficient conditions for the existence of the MLE of $(\alpha, \gamma)$ or, equivalently, of $\{m_{i,j} : i \neq j\}$ satisfying equation (12). To this end, let $C_n$ denotes the polyhedral cone spanned by the columns of $A$.

**Theorem 6.3.** Let $x \in \mathbb{R}^{n(n-1)}$ be the vector of observed edge counts. Then, the MLE exists if $d(x) \in \text{int}(C_n)$. The polyhedral cone $C_n$ has $3n$ facets. The co-facial sets corresponding to the facets of $C_n$ can be classified as follows:

1. the $2n$ support sets of the columns of $A$, each corresponding to a zero entry in the vectors of in-degree or out-degree statistics;

2. $n$ co-facial sets of the form $\{(i, j) : i \neq j \neq k\}$, one for each $k = 1, \ldots, n$.

For instance, when $n = 4$, there are 12 facial sets, 8 of them associated to a zero value in the 8 dimensional vector of sufficient statistics. The remaining 4 co-facial sets are shown in Table 12.

The previous Theorem implies that the number of facets of $C_n$ grows only linearly in $n$, unlike the number of facets of the polytope of degree sequences $P_n$. Thus, for this model, nonexistence of the MLE is a much less frequent phenomenon, at least combinatorially. Note in particular, that the MLE exists even if $x_{i,j} + x_{j,i} = 0$ for some (in fact many) pairs. Theorem 6.3 can be used to easily show that the MLE exists with probability tending to one as $n$ increases. Indeed, the probability of a nonexistent MLE is no larger than

$$\sum_{i=1}^{n} e^{-\sum_{j \neq i} m_{i,j}} + \sum_{j=1}^{n} e^{-\sum_{i \neq j} m_{i,j}} + \sum_{k=1}^{n} e^{-\sum_{i \neq j \neq k} m_{i,j}}.$$
Then, letting \( m^* := \min_{i \neq j} m_{i,j} \), the first two terms in equation (14) are each smaller than \( n e^{-(n-1)m^*} \), while the last term is bounded from above by
\[
ne^{-(\frac{1}{2} + 2(n-1)m^*)} \leq ne^{-(n-1)m^*},
\]
where the last inequality is due to the fact that \( \binom{n}{2} - 2(n-1) \geq n - 1 \) for all \( n \geq 7 \). Thus, (14) is bounded from above by \( 3ne^{-(n-1)m^*} \), which implies that, if \( m^* = m^*(n) = \frac{c \log n}{n-1} \), the MLE exists with probability at least \( 1 - \frac{3}{n} \). This simple calculation then shows that the MLE exists with overwhelming probability even if the expected edge counts all tend to zero, as long as these values decay at a rate \( \Omega \left( \frac{\log n}{n} \right) \).

The results just obtained can be specialized to the Rasch model, in which the nodes are partitioned into two sets \( I \) and \( J \) of cardinality \( k \) and \( l = n - k \), and edges can only occur between a node \( i \in I \) and a node \( j \in J \), though the number of edges between any pair of nodes \( (i, j) \) is random. The observed set edge counts takes the form of a \( k \times l \) contingency table and the sufficient statistics are the \( k \) row sums and the \( l \) column sums. As noted by Haberman (1977), in this case the MLE exists if and only if the row and column sums are all positive.

**A simpler Poisson model**

When the Poisson model for directed graphs described in (12) is specialized to the case of undirected graph, we obtain a simpler model in which the number of edges between nodes \( i \) and \( j \), with \( i \neq j \), has a Poisson distribution with mean
\[
\log m_{i,j} = \theta_i + \theta_j.
\]
The previous display should be compared to (2).

In this setting, the design matrix \( A \) is the same design matrix described in Section 3, of dimension \( n \times \binom{n}{2} \). The associated convex support is the pointed polyhedral cone \( C_n' \) spanned by its columns. The proof of Theorem 6.3 can then be easily adapted to derive the co-facial sets of \( C_n' \), which we describe in the next result.

**Corollary 6.4.** The cone \( C_n' \) has \( 2n \) facets. The corresponding facial sets are as follows:

1. the \( n \) support sets of the columns of \( A \)
2. \( n \) co-facial sets of the form \( \{(i, j) : i \neq j \neq k, i < j \} \), one for each \( k = 1, \ldots, n \).

Following the same arguments above, we see that the MLE exists with probability tending to one as long as the expected edge counts are of order \( \Omega \left( \frac{\log n}{n} \right) \).

**6.3 The Bradley-Terry Model**

We can specialize the model described in Section 6.2 to a directed graph without multiple edges, thus obtaining the Bradley-Terry model for pairwise comparisons. See Bradley and Terry (1952), David (1988), Hunter (2004) and references therein. In detail, let \( p_{i,j} \) denote the probability of a directed edge from \( i \) to \( j \) and \( p_{j,i} \) the probability of a directed edge from \( j \) to \( i \). According to the Bradley-Terry model, the probabilities of directed edges can be parametrized by vectors \( \beta \in \mathbb{R}^n \) so that
\[
p_{i,j} = \frac{e^{\beta_i}}{e^{\beta_i} + e^{\beta_j}}, \quad \forall i \neq j,
\]
or, equivalently, in terms of log-odd ratios, \( \log \frac{p_{i,j}}{p_{j,i}} = \beta_i - \beta_j \), \( \forall i < j \). Notice that this parametrization is redundant, and identifiability is typically enforced by requiring that \( \sum_{i=1}^n e^{\beta_i} = 1 \). Data are obtained by recording, for each pair of nodes \( (i, j) \) the outcomes of \( N_{i,j} \) pairwise comparisons, where \( N_{i,j} \) are fixed positive integers, resulting in \( x_{i,j} \) instances of node \( i \) being preferred to node \( j \) and \( x_{j,i} \) instances of node \( j \)
being preferred to node \( i \), with \( x_{i,j} + x_{j,i} = N_{i,j} \). The outcomes of the pairwise comparison are assumed mutually independent. Thus, for \( i < j \), the Bradley-Terry model treats the \( n(n-1) \) observed counts \( \{x_{i,j} : i \neq j\} \) as a realization of mutually independent Bin\( (N_{i,j}, p_{i,j}) \) distributions, where the probability parameters \( \{p_{i,j} : i \neq j\} \) satisfy \((15)\).

Despite the apparent similarity between equations \((1)\) and \((15)\), the beta model and the Bradley-Terry model are radically different. Indeed, for the Bradley-Terry model, it is well known that the minimal sufficient statistics are the row sums (or the column sums) of the observed table, which corresponds to the vector of out-degrees (or in-degrees, respectively) of the network. Indeed, this model can be alternatively prescribed as a model of quasi-symmetry and quasi-independence (see, e.g. Fienberg and Larntz, 1976). Necessary and sufficient conditions for the existence of the MLE are due to Petrović et al. (1981). We remark that the asymptotic properties of \( p_1 \) models are largely unknown and, as discussed by Haber (2010), such an analysis appears to be rather daunting.

Just like in the other network models considered thus far, in \( p_1 \) models the occurrence of a random edge between any pair of nodes \( i \) and \( j \), or dyad, is modeled independently from all the others edges. We keep track of four possible edge configurations within each dyad: node \( i \) has an outgoing edge into node \( j \) \((i \rightarrow j)\); node \( i \) as an incoming edge originating from node \( j \) \((i \leftarrow j)\); nodes \( i \) and \( j \) are linked by a bi-directed edge \((i \leftrightarrow j)\); and node \( i \) and \( j \) are not adjacent in the network. Following the notation we established in Petrović et al. (2010), which is slightly different than the original notation of Holland and Leinhardt (1981), for every pair of nodes \((i,j)\) we define the probability vector

\[
\begin{align*}
p_{i,j} = (p_{i,j}(0,0), p_{i,j}(1,0), p_{i,j}(0,1), p_{i,j}(1,1)) & \in \Delta_3
\end{align*}
\]

containing the probabilities of the four possible edge types, where \( \Delta_3 \) is the standard simplex in \( \mathbb{R}^4 \). The numbers \( p_{i,j}(1,0), p_{i,j}(0,1) \) and \( p_{i,j}(1,1) \) denote the probabilities of the edge configurations \( i \rightarrow j \), \( i \leftarrow j \) and \( i \leftrightarrow j \), respectively, and \( p_{i,j}(0,0) \) is the probability that there is no edge between \( i \) and \( j \) (thus, 1 denotes the outgoing side of the edge). Notice that, by symmetry \( p_{i,j}(a,b) = p_{j,i}(b,a) \), for all \( a,b \in \{0,1\} \) and that \( p_{i,j}(0,0) + p_{i,j}(1,0) + p_{i,j}(0,1) + p_{i,j}(1,1) = 1 \).

In \( p_1 \) models, the \( \binom{n}{2} \) dyads are modeled as mutually independent draws from multinomial distributions with class probabilities \( p_{i,j}, i < j \). Specifically, the Holland-Leinhardt \( p_1 \) model specifies the multinomial
probabilities of each dyad \((i, j)\) in logarithmic form as follows (see Holland and Leinhardt, 1981):

\[
\begin{align*}
\log p_{i,j}(0, 0) &= \lambda_{ij} \\
\log p_{i,j}(1, 0) &= \lambda_{ij} + \alpha_i + \beta_j + \theta \\
\log p_{i,j}(0, 1) &= \lambda_{ij} + \alpha_i + \beta_i + \theta \\
\log p_{i,j}(1, 1) &= \lambda_{ij} + \alpha_i + \beta_i + \alpha_j + \beta_i + 2\theta + \rho_{i,j}.
\end{align*}
\]

(18)

The parameter \(\alpha_i\) quantifies the effect of an outgoing edge from node \(i\), the parameter \(\beta_j\) instead measures the effect of an incoming edge into node \(j\), while \(\rho_{i,j}\) controls the added effect of reciprocated edges (in both directions). The parameter \(\theta\) measures the propensity of the network to have edges and, therefore, controls the “density” of the graph. The parameters \(\{\lambda_{i,j} : i < j\}\) are normalizing constants to ensure that (17) holds for each each dyad \((i, j)\) and need not be estimated. Note that, in order for the model to be identifiable, additional linear constraints need to be imposed on its parameters. We refer the interested readers to the original paper on \(p_1\) model by Holland and Leinhardt (1981) for an extensive interpretation of the model parameters.

As noted in Fienberg and Wasserman (1981a,b), different variants of the \(p_1\) model can be obtained by constraining the model parameters. In Petrović et al. (2010) we consider three special cases of the basic \(p_1\) model, which differ in the way the reciprocity parameter is modeled:

1. \(\rho_{i,j} = 0\), no reciprocal effect;
2. \(\rho_{i,j} = \rho\), constant reciprocation;
3. \(\rho_{i,j} = \rho + \rho_i + \rho_j\), edge-dependent reciprocation.

As it is often the case with network data, we assume that data become available in the form of one observed network. Thus, each dyad \((i, j)\) is observed in only one of its four possible states and this one observation is a random vector in \(\mathbb{R}^4\) with a Multinomial\((1, p_{i,j})\) distribution. As a result, data are sparse and, even though the dyadic probabilities are strictly positive according to the defining equations (18), only some of the model parameters may be estimated from the data. Extension to the case in which the dyads are observed multiple times are straightforward.

For a network on \(n\) nodes, we represent the vector of \(2n(n-1)\) dyadic probabilities as

\[ p = (p_{12}, p_{13}, \ldots, p_{n-1,n}) \in \mathbb{R}^{2n(n-1)}, \]

where, for each \(i < j\), \(p_{ij}\) is given as in (16). The \(p_1\) model is the set of all probability distributions that satisfy the Holland-Leinhardt equations (18). The design matrix associated with a given \(p_1\) model can be constructed as follows (this construction is by no means unique and leads to rank-deficient matrices, though it is rather simple). The columns of \(\Lambda\) are indexed by the entries of the vectors \(p_{i,j}, i < j\), where the \(p_{i,j}\)’s are ordered lexicographically, and its rows by the model parameters, ordered arbitrarily. The \((r, c)\) entry of \(\Lambda\) is equal to the coefficient of the \(c\)-th parameter in the logarithmic expansion of the \(r\)-the probability as indicated in (18). In particular, notice that the entries of \(\Lambda\) can only be 0, 1 or 2. For example, in the case \(\rho_{ij} = \rho + \rho_i + \rho_j\), the matrix \(\Lambda\) has \(\binom{n}{2} + 3n + 2\) rows. When \(n = 3\), the design matrix corresponding to this model is
Let $S_n = \{x_{i,j}, i \neq j\} \subset \{0, 1\}^{2n(n-1)}$ denote the sample space, i.e. the set of all observable networks on $n$ nodes. Then, every point $x$ in the sample space $\mathcal{X}$ can be written as

$$x = (x_{1,2}, x_{1,3}, \ldots, x_{n-1,n}),$$

where each of the $\binom{n}{2}$ subvectors $x_{i,j}$ is a vertex of $\Delta_3$. Notice that $|\mathcal{X}_n| = 4^n(n-1)$. This way of representing a network on $n$ nodes with a highly-constrained $0/1$ vector of dimension $2n(n-1)$ may appear cumbersome and redundant. Indeed, as in Holland and Leinhardt (1981), we could more naturally represent an $n$-node network using the $n \times n$ incidence matrix with $0/1$ off-diagonal entries, where the $(i,j)$ entry is 1 if there is an edge from $i$ to $j$ and 0 otherwise. While this representation is more intuitive and parsimonious (as it only requires $n(n-1)$ bits), whenever $\rho \neq 0$, the sufficient statistics for the reciprocity parameter are not linear functions of the observed network. As a consequence, the adjacency matrix representation does not lead directly to a linear exponential family.

The convex support for this family is the polytope obtained as the Minkowski sum

$$P_A := \sum_{i<j} A_{i,j},$$

where $A_{i,j}$ is the sub-matrix of $A$ comprised by the four columns referring to the dyad $(i,j)$. Given an observed network $x \in S_n$ the MLE of the parameters exists if and only of $Ax \in \text{ri}(S_n)$ and, when the MLE does not exist, the associated facial set provides the non-estimable probability parameters. Like with the polytope of degree sequences for the beta model, the combinatorial complexity of this object is quite high and increases very rapidly with $n$ (though, unlike the beta model, the convex supports for these models do not appear to be a known or well studied polytopes). See table 6.4 and the discussion below.

The arguments and results of Section 3 and the Cayley trick described in the appendix apply to the case of $p_1$ models as well, and yield the following result.

**Theorem 6.5.** For any $p_1$ model with associated design matrix $A$, the MLE exists if and only if $Ax \in \text{ri}(C_A)$, where $C_A = \text{cone}(A)$, and the facial sets of $P_A$ are also facial sets of $C_A$.

As shown in the appendix and further illustrated in Table 6.4, it is algorithmically much simpler do deal with the cone $C_A$ than with the polytope $P_A$.

**Numerical Experiments**

We conclude this section by describing some numerical experiments illustrating the reduction in complexity associated to the Cayley trick described in the appendix for the general $p_1$ model. Table 6.4 displays the
Table 13: Number of vertices for the polytopes $P_A$ for different specifications of the $p_1$ model and different network sizes. Computations carried out using minksum Weibel (2005). The last column indicates the number of columns of the design matrix $A$, which correspond to the number of generators of $C_A$.

| $n$ | $\rho_{i,j} = 0$ | $\rho_{i,j} = \rho$ | $\rho = \rho_i + \rho_j$ | $2n(n - 1)$ |
|-----|-----------------|-----------------|-----------------|-------------|
| 3   | 62              | 62              | 62              | 12          |
| 4   | 1,862           | 2,415           | 3,086           | 24          |
| 5   | 88,232          | 158,072         | 347,032         | 40          |

Table 14: Number of facets, dimensions and ambient dimensions of the cones $C_A$ for different specifications of the $p_1$ model and different network sizes. The number of facets of $C_A$ is equal to the number of facets of $P_A$ plus $\binom{n}{2}$, these additional facets corresponding to the sampling constraints of one observation per dyad.

The number of vertices of the polytopes $P_A$ for the three $p_1$ model specifications we consider and various networks sizes. The last column of the table contains the number of columns of the design matrix $A$, which is also the number of extreme rays of the marginal cone $C_A$. In comparison, the number of vertices of $P_A$, whose determination is computationally very hard, is very large and grows extremely fast with $n$.

In Table 6.4 we report the number of facets, dimensions and ambient dimensions of the cones $C_A$ for different values of $n$ and for the three specification of the reciprocity parameters $\rho_{i,j}$ we consider here. Though this only provides and indirect measure of the complexity of these models and of the non-zero patterns in extended MLEs, it does show how quickly the complexity of $p_1$ models may scale with the network size $n$.

Another point of interest is the assessment of how often the existence of the MLE arises. In fact, because of the product Multinomial sampling constraint, nonexistence of the MLE is quite severe, especially for smaller networks. Below we report our findings, which are necessarily restricted to networks of small sizes.

The case $n = 3$. The sample space consists of $4^3 = 64$ possible networks. When $\rho_{i,j} = 0$ for all $i$ and $j$, there are 63 different observable sufficient statistics, only one of which belongs to $\text{ri}(P_A)$. Thus, only one of the 63 observable sufficient statistics leads to the existence of the MLE. This sufficient statistic corresponds to the two nextworks

$$
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
$$

In both cases, the associated MLE is the 12-dimensional vector with entries all equal to 0.25. Incidentally, the polytope $P_A$ has 62 vertices and 30 facets. When $\rho_{i,j} = \rho \neq 0$ or $\rho_{i,j} = \rho_i + \rho_j$ the MLE never exists.

The case $n = 4$. The sample space contains 4096 observable networks. If $\rho_{i,j} = 0$, there are 2,656 different observable sufficient statistics, only 64 of which yield existent MLEs. Overall, out of the 4096 possible networks, only 426 have MLEs. When $\rho_{i,j} = \rho \neq 0$, there are 3,150 different observable sufficient statistics, only 48 of which yield existent MLEs. Overall, out of the 4,096 possible networks, only 96 have MLEs. When $\rho_{i,j} = \rho_i + \rho_j$, there are 3,150 different observable sufficient statistics and the MLE never exists.
The case \( n = 5 \). The sample space consists of \( 4^{10} = 1,048,576 \) different networks. If \( \rho_{i,j} = 0 \), there are 225,025 different sufficient statistics, and the MLE exists for 7,983. If \( \rho_{i,j} = \rho \neq 0 \) the number of distinct possible sufficient statistics is 349,500, and the MLE exists in 12,684 cases. Finally, when \( \rho_{i,j} = \rho_1 + \rho_2 \), the number of different sufficient statistics is 583,346 and the MLE never exists.

7 Discussion and extensions

We have used polyhedral geometry to analyze the conditions for existence of the MLE of a generalized version of the \( \beta \)-model and to derive finite sample bounds for the probability associated with the existence of the MLE. Our results offer a novel and explicit characterization of the patterns of edge counts leading to non-existent MLEs. The problem of nonexistence occurs in numbers and with a complexity that was not previously known. Our results allow us to sharpen conditions for existence of the MLE. Our analysis in particular highlights the fact that requiring node degrees equal to 0 and \( n - 1 \) is only a sufficient condition for nonexistence of the MLE and non-estimability of the edge probabilities. We show that we need to account for many more edge patterns. We note that the use of polyhedral geometry in statistical models for discrete data is a hallmark of the theory of exponential families, but its considerable potential for use and applications in the analysis of log-linear and network models has only recently begun to be investigated (see Fienberg and Rinaldo, 2011; Rinaldo et al., 2009).

Our generalization of the \( \beta \)-model allows for Poisson and binomial, not simply Bernoulli distributions for edges. Email databases and others involving repeated transactions among pairs of parties provides the simplest examples of situations for networks where edges can occur multiple times. These are often analyzed as weighted networks but that may not necessarily make as much sense as using a Poisson for random numbers of occurrences.

As our results indicate, the nonexistence of the MLE is equivalent to non-estimability of a subset of the parameters of the model, but by no means does it imply that no statistical inference can take place. In fact, when the MLE does not exist, there always exists a “restricted” \( \beta \)-model that is fully specified by the appropriate facial set, and for which all parameters are fully estimable. Thus, for such smaller model, traditional statistical tasks such as hypothesis testing and assessment of parameter uncertainty are possible, even though it becomes necessary to adjust the number of degrees of freedom for the non-estimable parameters. A complete description of this approach, which is rooted in the theory of extended exponential families, is beyond the scope of the article. See Fienberg and Rinaldo (2011) for details.

We can extend our study of the \( \beta \)-model in a number of ways. In the full original version of the article we considered various generalizations of the \( \beta \)-model setting, including the \( \beta \)-model with random numbers of edges, the Rasch model from item response theory, the Bradley-Terry paired comparisons model and the \( p_1 \) network model. For most of these models we were able to carry out a fairly explicit analysis based on the underlying geometry, but for the full \( p_1 \) model the complexity of the model polytope appears to make such a direct analysis very difficult (this is reflected in the high complexity of the Markov basis for \( p_1 \) model, of which we give full account in Petrović et al., 2010). Another interesting extension of our results of Section 4 would be to translate our conditions, which are formulated in terms of expected degree sequences, into conditions on the \( p_{i,j} \)‘s themselves, for instance by establishing appropriate bounds for \( \min_{i<j} p_{i,j} \), \( \max_{i<j} p_{i,j} \), or \( \max_{i\neq j} \frac{p_{i,j}}{1-p_{i,j}} \).

We conclude with some remarks on the computational aspects of our analysis, which constitute a non-trivial component of our work and is of key importance for detecting the nonexistence of the MLE and identifying estimable parameters. The main difficulty in applying our results is that the polytope of degree sequences \( P_n \) is difficult to handle algorithmically in general. Indeed, \( P_n \) arises a Minkowskï suma sum and, even though the system of defining inequalities is given explicitly, its combinatorial complexity grows exponentially in \( n \). More importantly, the vertices of \( P_n \) are not known explicitly. Algorithms for obtaining the vertices of \( P_n \), such as \( \text{minksum} \) (see Weibel, 2005), are computationally expensive and require generating all the points \( \{Ax, x \in G_n\} \), where \( |G_n| = 2^{|S|} \), a task that, even for \( n \) as small as 10, is impractical. See for instance, our analysis of the \( p_1 \) model in the full version of the article. Thus, deciding whether a given
degree sequence is a point in the interior of $P_n$ and identifying the facial set corresponding to an observed degree sequence on its boundary is highly non-trivial. Our strategy to overcome these problems entails re-expressing the $\beta$-model as a log-linear model with $\binom{n}{2}$ product-multinomial sampling constraints. This approach is not new, and it harks back to the earlier re-expression of the Holland-Leinhardt $p_1$ model and its natural generalizations as log-linear models (Fienberg and Wasserman, 1981a,b; Fienberg et al., 1985; Meyer, 1983). Though this re-parametrization increases the dimensionality of the problem, it nonetheless has the crucial computational advantage of reducing the determination of the facial sets of $P_n$ to the determination of the facial sets of a pointed polyhedral cone spanned by $n(n-1)$ vectors, which is a much simpler object to analyze, both theoretically and algorithmically. This procedure is known as the Cayley embedding in polyhedral geometry, and Fienberg and Rinaldo (2011) describe its use in the analysis of log-linear models. The advantages of this re-parametrization are two-fold. First, it allows us to use the highly optimized algorithms available in polymake for listing explicitly all the facial sets of $P_n$. This is how we computed the facial sets in all the examples presented in this article. Secondly, the general algorithms for detecting non-existence of the MLE and identifying facial sets proposed in Fienberg and Rinaldo (2011), which can handle larger dimensional models (with $n$ in the order of hundreds), can be directly applied to this problem. This reference is also relevant for dealing with inference under a non-existent MLE.

Software
The R routines used to carry out the computations for the results presented in the paper and for creating the input files for polymake are available at http://www.stat.cmu.edu/~arinaldo/Rinaldo_Petrovic_Fienberg_Rcode.txt

8 Acknowledgments
This research was supported in part by grant FA9550-12-1-0392 from the U.S. Air Force Office of Scientific Research (AFOSR) and the Defense Advanced Research Projects Agency (DARPA), NSF grant DMS-0631589, and by a grant from the Singapore National Research Foundation (NRF) under the Interactive & Digital Media Programme Office to the Living Analytics Research Centre (LARC). The authors would like to thank Christian Haase and Josephine Yu for suggesting the proof of Theorem 6.3. This manuscript was completed while the second author was in residence at Institut Mittag-Leffler.

9 Proofs

Proof of Theorem 3.1. Throughout the proof, we will use standard results and terminology from the theory of exponential families, for which standard references are Brown (1986) and Barndorff-Nielsen (1978). The polytope

$$S_n := \text{convhull}\{Ax, x \in S_n\}$$

is the convex support for the sufficient statistics of the natural exponential family described in Section 2. Furthermore, by a fundamental result in the theory of exponential families (see, e.g., Theorem 9.13 in Barndorff-Nielsen, 1978), the MLE of the natural parameter $\beta \in \mathbb{R}^n$ (or, equivalently of the set probabilities \{p_{i,j}, i < j\} $\in \mathbb{R}^{\binom{n}{2}}$ satisfying (1)) exists if and only if $d \in \text{int}(S_n)$. Thus, it is sufficient to show that $d \in \text{int}(S_n)$ if and only if $\tilde{d} \in \text{int}(P_n)$.

Denote with $a_{i,j}$ the column of $A$ corresponding to the ordered pair $(i, j)$, with $i < j$, and set

$$P_{i,j} = \text{convhull}\{0, a_{i,j}\} \subset \mathbb{R}^n.$$ (19)

Each $P_{i,j}$ is a line segment between its vertices 0 and $a_{i,j}$. Then, $P_n$ can be expressed as the zonotope obtained as the Minkowski sum of the line segments $P_{i,j}$:

$$P_n = \sum_{i<j} P_{i,j},$$ (20)

24
This identity can be established as follows. On one hand, \( P_n \) is the convex hull of vectors that are Boolean combinations of the columns of \( \Lambda \). Since all such combinations are in \( \sum_{i<j} P_{i,j} \), and both \( P_n \) and \( \sum_{i<j} P_{i,j} \) are closed sets, we obtain \( P_n \subseteq \sum_{i<j} P_{i,j} \). On the other hand, the vertices of \( \sum_{i<j} P_{i,j} \) are also Boolean combinations of the columns of \( \Lambda \) (see, e.g., corollary 2.2 in Fukuda, 2004), and, therefore, \( \sum_{i<j} P_{i,j} \subseteq P_n \).

Equation (20) shows, in particular, that \( \tilde{d} \in P_n \). Furthermore, using the same arguments, we see that, similarly to \( P_n \), \( S_n \) too can be expressed as a Minkowski sum:

\[
S_n = \sum_{i<j} S_{i,j},
\]

where

\[
S_{i,j} := P_{i,j} N_{i,j} = \{ x N_{i,j} : x \in P_{i,j} \}
\]

is the rescaling of \( P_{i,j} \) by a factor of \( N_{i,j} \). In fact, we will prove that \( S_n \) and \( P_n \) are combinatorially equivalent.

For a polytope \( P \) and a vector \( c \), we set \( F(P; c) := \{ x \in P : x^\top c \geq y^\top c, \forall y \in P \} \). Any face \( F \) of \( P \) can be written in this way, where is \( c \) is any vector in the interior of the normal cone to \( F \). By Proposition 2.1 in Fukuda (2004), \( F \) is a face of \( P_n \) with \( F = F(P_n, c) \) if and only if it can be written uniquely as

\[
F(P_n, c) = \sum_{i<j} F(P_{i,j}, c),
\]

for any \( c \) in the interior of the normal cone to \( F \). It is immediate to see that \( F(P_{i,j}, c) \) is a face of \( P_{i,j} \) if and only if \( F(S_{i,j}, c) \) is a face of \( S_{i,j} \), and that \( F(S_{i,j}, c) = N_{i,j} F(P_{i,j}, c) \); in fact, \( P_{i,j} \) and \( S_{i,j} \) are combinatorially equivalent. Therefore, invoking again Proposition 2.1 in Fukuda (2004), we conclude that \( F(P_{i,j}, c) \) is a face of \( P_n \) if and only if

\[
\sum_{i<j} N_{i,j} F(P_{i,j}, c)
\]

is a face of \( S_n \) (and this representation is unique). From this, we see that \( P_n \) and \( S_n \) have the same normal fan and, therefore, are combinatorially equivalent.

**Proof of Lemma 3.2.** By Proposition 2.1 in Fukuda (2004),

\[
F = F(P_n, c) = \sum_{i<j} F(P_{i,j}, c),
\]

for any \( c \) in the interior of the normal cone to \( F \), where the above representation is unique. Since \( P_{i,j} \) is a line segment (see (19)), its only proper faces are the vertices 0 and \( a_{i,j} \). Let the set \( F \) be the complement of the set of pairs \((i, j)\) with \( i < j \) such that \( F(P_{i,j}, c) \) is either the vector 0 or \( a_{i,j} \). By the uniqueness of the representation (21), \( F \) is unique as well and, in particular, maximal. Furthermore, as it depends on \( F \) only through the interior of its normal cone and since the interiors of the normal cones of \( P_n \) are disjoint, different faces will be associated with different facial sets.

**Proof of Theorem 4.1.** Let \( \tilde{d} = (\tilde{d}_1, \ldots, \tilde{d}_n) \) be the random vector defined in (6). We will show that, under the stated assumptions, \( \tilde{d} \in \text{int}(P_n) \) with probability no smaller than \( 1 - \frac{2}{n^2} \).

Since \( N_n \) is constant, we can conveniently re-express the random vector \( \tilde{d} \) as an average of independent and identically distributed graphical degree sequences. In details, we can write

\[
\tilde{d} = \frac{1}{N} \sum_{k=1}^{N} d^{(k)},
\]

where each \( d^{(k)} \) is the degree sequence arising from an independent realization of random graph with edge probabilities \( \{ p_{i,j} : i < j \} \), for \( k = 1, \ldots, N \).
Thus, each $\tilde{d}_i$ is the sum of $N(n-1)$ independent random variables taking values in $\{0, \frac{1}{N}\}$. Then, an application of Hoeffding’s inequality and of the union bound yields that the event

$$O_n := \left\{ \max_i |\tilde{d}_i - \bar{d}_i| \leq \sqrt{\frac{c n \log n}{N}} \right\} \tag{23}$$

occurs with probability at least $1 - \frac{2}{n^{2c-2}}$. Throughout the rest of the proof we will assume that the event $O_n$ holds.

By assumption (i), for each $i$,

$$0 < C + \sqrt{c \frac{n \log n}{N}} \leq \tilde{d}_i - \sqrt{c \frac{n \log n}{N}} \leq \tilde{d}_i \leq \tilde{d}_i + \sqrt{c \frac{n \log n}{N}} \leq n - 1 - C - \sqrt{c \frac{n \log n}{N}} < n - 1,$$

so that

$$0 < \tilde{d}_i < n - 1, \quad i = 1, \ldots, n. \tag{24}$$

Notice that the assumed constraint on the range of $C$ guarantees the above inequalities are well defined.

Next, for each pair $(S, T) \in \mathcal{P}$,

$$|g(S, T, \tilde{d}, n) - g(S, T, \bar{d}, n)| \leq |S \cup T| \max_i |\tilde{d}_i - \bar{d}_i|,$$

which yields

$$g(S, T, \tilde{d}, n) \geq g(S, T, \bar{d}, n) - |S \cup T| \sqrt{c \frac{n \log n}{N}}. \tag{27}$$

Using assumption (ii), the previous displays implies that

$$\min_{(S, T) \in \mathcal{P}} g(S, T, \tilde{d}, n) > C > 0. \tag{25}$$

Thus, we have shown that (24) and (25) hold, provided that the event $O_n$ is true and assuming (i) and (ii). Therefore, by Theorem 3.3 the MLE exists.

Proof of Corollary 4.2. Using the same setting and notation of Theorem 4.1, we will assume throughout the proof that the event

$$O'_n := \left\{ \max_k \max_i |d^{(k)}_i - \bar{d}_i| \leq \sqrt{c n \log n} \right\}$$

holds true. Note that by Hoeffding’s inequality and the union bound,

$$\mathbb{P}(O'_n) \leq 2 \exp \left\{ -2c \log n + \log n + \log N \right\} \leq \frac{2}{n^{2c-2}},$$

where we have used the inequality $\log N \leq \log n$. A simple calculation shows that, when $O'_n$ is satisfied, we also have

$$\left\{ \max_i |d_i - \bar{d}_i| \leq \sqrt{c n \log n} \right\}.$$

Then, by the same arguments used in the proof of Theorem 4.1, assumption (i) yields that

$$0 < d_i < n - 1, \quad i = 1, \ldots, n. \tag{26}$$

and, for each pair $(S, T) \in \mathcal{P}$,

$$g(S, T, d, n) \geq g(S, T, \bar{d}, n) - |S \cup T| \sqrt{c n \log n}. \tag{27}$$
Now, it is easy to see that, on the event $O_n^*$, assumption (i) also yields
\[
\min_k \min_i \min \left\{ d^{(k)}_i, n - 1 - d^{(k)}_i \right\} \ge \sqrt{cn \log n} + C. \tag{28}
\]

We now show that, when (26) and the previous equation are satisfied, the MLE exists if
\[
\min_{(S,T) \in P_n^*} g(S,T,d,n) > C > 0. \tag{29}
\]

Indeed, suppose that (26) is true and that $d$ belongs to the boundary of $P_n$. Then, by the integrality of the polytope $P_n$, there exist non-empty and disjoint subsets $T$ and $S$ of $\{1, \ldots, n\}$ satisfying the conditions of lemma 3.4 for each of the degree sequences $d^{(1)}, \ldots, d^{(k)}$. If $\min_k \min_i d_i^{(k)} > \sqrt{cn \log n} + C$, then, necessarily, $|S| > \sqrt{cn \log n} + C$, because $|S|$ is the maximal degree of every node $i \in T$. Similarly, since each $i \in S$ has degree at least $|S| - 1 + |(S \cup T)^c|$, if $\max_k \max_i d_i^{(k)} < n - 1 - \sqrt{cn \log n} - C$, the inequality
\[
|S| - 1 + |(S \cup T)^c| < n - 1 - \sqrt{cn \log n} - C
\]
must hold, implying that $|T| = n - |S| - |(S \cup T)^c| > \sqrt{cn \log n} + C$. Thus, we have shown that, if (26) and (28) hold, and $d$ belongs to the boundary of $P_n$, the cardinalities of the sets $S$ and $T$ defining the facet of $P_n$ to which $d$ belongs cannot be smaller than $\sqrt{cn \log n} + C$. By Theorem 3.3, when (26) and (28) hold, (29) implies that $d \in \text{int}(P_n^*)$, so the MLE exists. However, equation (27) and assumption (ii) implies (29), so the proof is complete.

\textbf{Proof of Theorem 6.3.} The result about existence of the MLE follows from a direct application of Theorem 9.13 in Barndorff-Nielsen (1978) or Theorem 5.5 in Brown (1986), since $C_n$ is the convex support for the exponential family of equation 13.

As for the claims regarding the facets of $C_n$, since the row span of $A$ contains the constant vectors, we study the facets of the polytope $P := \text{conv}(B) \subset \mathbb{R}^n \times \mathbb{R}^n$. Denote by $x_i$ and $x'_j$ the coordinates of the two spaces, and by $e_i$ and $e'_j$ the corresponding standard unit vectors in $\mathbb{R}^n$. The polytope $P$ is contained in the product of simplices $\Delta_{n-1} \times \Delta_{n-1} := \text{conv}\{e_i \times e'_j : 1 \leq i, j \leq n\}$, where, for two vectors $x$ and $x'$ in $\mathbb{R}^n$,
\[
x \times x' := \begin{pmatrix} x \\ x' \end{pmatrix} \in \mathbb{R}^{2n}.
\]

The point $e_i \times e'_j$ corresponds to the $(i,j)$-entry of the $n \times n$ incidence table of the network. $P$ is obtained from the product of simplices by removing the $n$ vertices $\{e_i \times e'_j : i = 1, \ldots, n\}$. To show that $P$ has $3n$ facets, we will use the fact that $\Delta_{n-1} \times \Delta_{n-1}$ has $2n$ facets whose defining inequalities are $x_i \geq 0, x'_j \geq 0$, for $i = 1, \ldots, n$. Note that these facets correspond to zero margins in the incidence table: for example, $x_i = 0$ refers to the zero margin corresponding to the $i$-th row and $x'_j = 0$ to the zero margin for the $(i+n)$-th row.

Define a new polytope, $P'$, cut out by the following $3n$ inequalities:
\[
P' := \{ x_i \geq 0, x'_j \geq 0, x_i + x'_j \leq 1, \text{ for all } i \}.
\]

We need to show that $P = P'$ and that the defining inequalities are all facets. For the first claim, we already see that $P \subseteq P'$. Since $\Delta_{n-1} \times \Delta_{n-1}$ is simple, every vertex has dimension many neighbors. Thus, removing the vertex $e_i \times e'_j$ introduces one new facet, namely, $x_i + x'_j \leq 1$. Since we are removing $n$ non-adjacent vertices, $P = P'$. Next, our arguments so far already imply that the $n$ new inequalities $\{x_i + x'_j \leq 1 : i = 1, \ldots, n\}$ define facets, so we need to show that other $2n$ inequalities, corresponding to zero row margins, define facets as well. But this follows from the fact that the support sets of each of the rows of $A$ are facial sets of $P$ and that they are incomparable, in the sense that none of them is contained in any of the others. Thus, since the lattice of facial sets of $P$ is isomorphic to the face lattice of $P$, the $2n$ null margins each specifies a different facet of $P$. 

10 Appendix: Computations

In this appendix, we provide details on how to determine whether a given degree sequence belongs to the interior of the polytope of degree sequences $P_n$ and on how to compute the facial set corresponding to a degree sequence on the boundary of $P_n$. We will only deal with the polytope $P_n$, even though the arguments below are general and extend, for instance, to the Rasch model, the Bradley-Terry model and $p_1$ models.

Below, we describe the procedure we used to compute the facial sets of $P_n$. The main difficulties with working directly with $P_n$ is that this polytope arises a Minkowksi sum and, even though the system of defining inequalities is given explicitly, its combinatorial complexity grows exponentially in $n$. Furthermore, we do not have available a set of vertices for $P_n$. Algorithms for obtaining the vertices of $P_n$, such as \texttt{minksum} (see Weibel, 2005), are computationally expensive and require generating all the points $\{Ax, x \in G_n\}$, where $|G_n| = 2^{\binom{n}{2}}$. In general, when $n$ is as small as 10, this is not feasible.

Our basic strategy to overcome these problems is quite simple, and entails representing the beta model as a log-linear model with \binom{n}{2} product-multinomial sampling constraints. Though this re-parametrization increases the dimensionality of the problem, it nonetheless has the crucial computational advantage of reducing the determination of the facial sets of $P_n$ to the determination of the facial sets of a pointed polyhedral cone spanned by $n(n-1)$ vectors, which is a much simpler object to analyze, both theoretically and algorithmically. This procedure is known as the Cayley embedding in polyhedral geometry, and its use in the analysis of log-linear models is described in Fienberg and Rinaldo (2011). The advantages of this re-parametrization are two-fold. First, it allows us to use the highly optimized algorithms available in \texttt{polyomke} for listing explicitly all the facial sets of $P_n$, which is the strategy we used. Secondly, the general algorithms for detecting nonexistence of the MLE and identifying facial sets proposed in Fienberg and Rinaldo (2011), which can handle larger dimensional models, can be directly applied to this problem. This reference is also relevant for dealing with inference under a non-existent MLE.

In the interest of space, we do not provide all the details, and instead only sketch the two main steps of our procedure.

- **Step 1: Enlarging the space**

  In the first step, we switch to a redundant representation of the data by considering all the observed counts $\{x_{i,j}, i \neq j\}$ and not just $\{x_{i,j}, i < j\}$. We index the points of this enlarged set of $n(n-1)$ numbers as pairs $S'_n = \{(x_{i,j}, x_{j,i}) : i < j\} \subset \mathbb{N}^{n(n-1)}$, with the pairs ordered lexicographically based on $(i, j)$. For instance, when $n = 4$, any point $x' \in S'_4$ has coordinates indexed by

  $$(1, 2), (2, 1), (1, 3), (3, 1), (1, 4), (4, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4, 3).$$

  It is clear that the sets $S_n$ and $S'_n$ are in one-to-one correspondence with each other and that, for each corresponding pair $x \in S_n$ and $x' \in S'_n$, $x'_{i,j} = x_{i,j}$ for all $i < j$ and $x'_{j,i} = N_{i,j} - x_{i,j}$ for all $j > i$.

  In this new setting, we construct a new polytope $P'_n \subset \mathbb{R}^{2n}$ that is combinatorially equivalent to $P_n$ but whose facial sets are easier to interpret. This is achieved by first constructing a new design matrix $B$ of dimension $(2n) \times n(n-1)$, with the columns indexed according to the order described above. The matrix $B$ has the form

  $$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

  where both $B_1$ and $B_2$ have $n$ rows. For all $i < j$, the columns of $B_1$ corresponding to the coordinate $(i, j)$ and the columns of $B_2$ corresponding to the coordinate $(j, i)$ are both equal to $a_{i,j}$, and all the
other columns are zeros. For instance, when \( n = 4 \),

\[
B = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
\end{bmatrix}.
\]

By construction, \( d = Ax = B_1 x' \) for any corresponding pair \( x \in S_n \) and \( x' \in S'_n \). Furthermore, if we let \( d' = B_2 x' \), it is easy to see that \( d' \) and \( d \) are in one-to-one correspondence with each other. Indeed, recalling that \( N_{i,j} = N_{j,i} \),

\[
d'_i = \sum_{j<i} x'_{i,j} + \sum_{j>i} x'_{j,i} \\
= \sum_{j<i} (N_{i,j} - x_{j,i}) + \sum_{j>i} (N_{i,j} - x_{i,j}) \\
= \sum_{j\neq i} N_{i,j} - \left( \sum_{j<i} x_{j,i} + \sum_{j>i} x_{i,j} \right) \\
= \sum_{j\neq i} N_{i,j} - d_i,
\]

where we used equation (5) in the last step. Thus, \( Bx' \) is also a sufficient statistic, though highly redundant due to linear dependencies. Next, for any \( i < j \), let

\[
B_{i,j} = \text{convhull}(\{b_{i,j}, b_{j,i}\})
\]

where \( b_{i,j} \) is the column of \( B \) indexed by \((i, j)\), and set

\[
P'_n = \sum_{i<j} B_{i,j}.
\]

The polytopes \( P_n \) and \( P'_n \) are combinatorially equivalent, even though their ambient dimensions are different. In fact, using arguments similar to the ones used in the proof of Lemma 33.2, one can characterize the facial sets of \( P'_n \) as follows.

**Lemma 10.1.** A point \( y' \) belongs to the interior of some face \( F' \) of \( P'_n \) if and only if there exists a set \( F' \subset \{(i, j), i \neq j\} \) such that

\[
y' = B p', \tag{31}
\]

where \( p' = \{p'_{i,j}; i \neq j, p'_{i,j} \in [0,1], p'_{i,j} = 1 - p'_{j,i}\} \) is such that \( p'_{i,j} = 0 \) for all \((i, j) \notin F'\) and \( p'_{i,j} > 0 \) for all \((i, j) \in F'\). The set \( F' \) is uniquely determined by the face \( F \) and is a maximal set for which (31) holds.

Because \( P_n \) and \( P'_n \) are combinatorially equivalent, their co-facial sets are also in one-to-one correspondence. The advantage of using \( P'_n \) instead of \( P_n \) is that its co-facial sets arise by entries of \( p' \) that are all zeros, as opposed to the more complicated co-facial sets of \( P_n \), which are obtained from entries of \( p = \{p_{i,j}; i < j\} \) which are both ones and zeros. For instance, the co-facial set of \( P_n \) corresponding to the counts reported in Table 1 is \( \{(1,2), (3,4)\} \) with \( p_{1,2} = 0 \) and \( p_{3,4} = 1 \). In contrast, the corresponding co-facial set for \( P'_n \) is \( \{(1,2), (4,3)\} \), with \( p'_{1,2} = 0 \) and \( p'_{4,3} = 0 \). Clearly, they convey the same information.

- **Step 2: Lifting**

As we saw, the advantage of the larger polytope \( P'_n \) derived in the first step is that, when searching for co-facial sets, it is enough to consider points of the form \( p' = \{p'_{i,j}; i \neq j, p'_{i,j} \in [0,1]\} \) with zero coordinates only. However, \( P'_n \) is still a hard object to deal with computationally, since it is prescribed as a Minkowski sum of \( \binom{n}{2} \) polytopes. In this second step, we lift \( P_n \) to a polyhedral cone.

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in dimension \(2n + \binom{n}{2}\) which is simpler to analyze (in fact, as remarked below, this polyhedral cone has smaller dimension: \(n + \binom{n}{2}\)). This cone is spanned by the columns of a matrix \(C\) of dimension \((2n + \binom{n}{2}) \times n(n-1)\) which has the form

\[
C = \begin{pmatrix}
C_1 \\
B
\end{pmatrix},
\]

where the rows of \(C_1\) are indexed by the pairs \((i, j): i < j\) ordered lexicographically. Each row \((i, j)\) of \(C_1\) contains all zeroes, except for two ones in the coordinates \((i, j)\) and \((j, i)\). In fact for any \(x' \in S_n\), the vector \(C_1 x'\) is constant, and its \((i, j)\)-the entry is

\[
x'_{i,j} + x'_{j,i} = N_{i,j}.
\]

For instance, when \(n = 4\),

\[
C = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

Let \(D_n = \text{cone}(C)\) be the polyhedral cone of spanned by the columns of \(C\). The facial sets of \(D_n\) are defined as follows (see, e.g., Geiger et al., 2006). The subset \(\mathcal{F} \subset \{(i, j): i \neq j\}\) is a facial set of \(D_n\) when there exists a \(v \in \mathbb{R}^{2n+\binom{n}{2}}\) such that

\[
\langle v, c_{i,j}\rangle = 0, \quad \forall i \in \mathcal{F} \quad \text{and} \quad \langle v, c_{i,j}\rangle < 0, \quad \forall i \notin \mathcal{F},
\]

where \(c_{i,j}\) indicates the column of \(C\) indexed by the pair \((i, j)\). It follows that \(F\) is face of \(D_n\) if and only if \(F = \text{cone}\{c_i: i \in \mathcal{F}\}\), for some facial set \(\mathcal{F}\) of \(D_n\), and that there is a one-to-one correspondence between the facial sets and the faces of \(D_n\). Thus, as before, facial sets form a lattice isomorphic to the face lattice of \(D_n\). Following Eriksson et al. (2006), we will call \(D_n\) the marginal cone.

The following result shows how one can obtain the facial sets of \(P_n\) from the facial set of \(D_n\) through the facial sets of \(P_n'\) (see also Section 3 in Fienberg and Rinaldo (2011)).

**Theorem 10.2.** Let \(p' = \{p'_{i,j}: i \neq j, p'_{i,j} \in [0, 1], p'_{i,j} = 1 - p'_{j,i}\}\). Then \(Bp' \in \text{ri}(P_n')\) if and only if \(Cp' \in \text{ri}(D_n)\). Furthermore, if \(\mathcal{F}'\) is a facial set of \(P_n'\), then \(\mathcal{F}'\) is a facial set of \(D_n\).

**Proof.** We first define a new polytope \(Q_n \subset \mathbb{R}^{2n+\binom{n}{2}}\) which is combinatorially equivalent to \(P_n'\) and, therefore, to the polytope of degree sequences \(P_n\). Let \(c_{i,j}\) be the column of \(C\) index by the pair \((i, j)\) and, for each \(i < j\), set

\[
C_{i,j} := \text{convhull}\{c_{i,j}, c_{j,i}\}
\]

and

\[
Q_n := \sum_{i<j} C_{i,j}.
\]
By construction, $w \in P'_n$ if and only if
\[
\begin{pmatrix}
1 \\
\end{pmatrix} \in Q_n,
\]
where $1 \in \mathbb{R}^{\binom{n}{2}}$ is a vector of all ones, which shows that $P'_n$ and $Q_n$ are combinatorially equivalent, so they have the same facial sets. We make a simple but useful observation: because the first $\binom{n}{2}$ coordinates of any point in $Q_n$ are all ones, and given the pattern of non-zero entries in the first $\binom{n}{2}$ rows of $C$, it must be that if $y \in Q_n$ and $y = Cp'$, the vector $p'$ is of the form $\{p'_{i,j}; i \neq j, p'_{i,j} \in [0,1], p'_{i,j} = 1 - p'_{j,i}\}$.

Since $Q_n \subset D_n$ and both sets are closed, $y \in \text{ri}(Q_n)$ implies that $y \in \text{ri}(D_n)$. As for the converse statement, suppose $y$ belongs to the interior of a proper face of $Q_n$ with facial set $F'$. Then, by Proposition 2.1 in Fukuda (2004), $y$ can be uniquely expressed as
\[
y = y_{1,2} + y_{1,3} + \ldots + y_{n-1,n}
\]
where $y_{i,j} \in \text{ri}(C_{i,j})$ if and only if $(i, j)$ and $(j, i)$ are in $F'$. Equivalently, $y_{i,j} = c_{i,j}$ or $y_{i,j} = c_{j,i}$ if and only if $(i, j) \notin F'$ or $(j, i) \notin F'$, respectively. Arguing by contradiction, suppose that $y \in \text{ri}(D_n)$. Then, there exists a point $p^* = \{p^*_{i,j}; i \neq j\}$ with strictly positive entries such that $y = Cp^*$. By the observation above, it must be that $p^*_{i,j} \in (0,1)$ and $p^*_{i,j} = 1 - p^*_{j,i}$, for all $i < j$. In turn, this implies that, in equation (32), $y_{i,j} \in \text{ri}(C_{i,j})$ for all $i < j$, i.e. $y_{i,j} \notin \{c_{i,j}, c_{j,i}\}$ for all $i < j$. Then, using again Proposition 2.1 in Fukuda (2004), $y \in \text{ri}(Q_n)$, a contradiction.

To prove the second claim, notice that, the arguments so far yield that, for every proper face $F$ of $Q_n$, there exists one face $G$ of $D_n$ such that $\text{ri}(F) \subset \text{ri}(G)$, so that $F' \subseteq G$, where $F'$ and $G$ are the facial sets associated with $F$ and $G$, respectively. We now show that $F' = G$. To see this, let $y \in \text{ri}(F)$ for some face $F$ of $Q_n$ with facial set $F'$, so that
\[
y = Cp'
\]
for some $p' = \{p'_{i,j}; i \neq j, p'_{i,j} \in [0,1], p'_{i,j} = 1 - p'_{j,i}\}$ such that $p'_{i,j} > 0$ if and only if $(i, j) \in F'$. On the other hand, since $y \in \text{ri}(G)$,
\[
y = Cp^*,
\]
where $p^* = \{p^*_{i,j}; p^*_{i,j} \geq 0\}$ is such that $p^*_{i,j} > 0$ if and only if $(i, j) \in G$. However, using the observation above, it must be that $p^*_{i,j} \in [0,1]$ and $p^*_{i,j} = 1 - p^*_{j,i}$, for all $i < j$. By maximality of the facial sets, $F' = G$, as claimed.

Thus, we have shown that if $F'$ is a facial set of $Q_n$ and hence of $P'_n$, it is also a facial set of $D_n$. ■

In particular, the only facial sets of $D_n$ that are not facial sets of $P'_n$ are the ones corresponding to the supports of the first $\binom{n}{2}$ rows of $C$, so that $D_n$ has $\binom{n}{2}$ more facets than $P_n$ (and $P'_n$). Since, by construction $x'_{i,j} + x_{j,i} = N_{i,j}, Cx'$ will never be a point in the interior of the first $\binom{n}{2}$ facets of $D$ whose facial sets are the supports of the first $\binom{n}{2}$ rows of $C$.

Theorem 10.2 can be used as follows. The MLE exists if and only if $Cx' \in \text{ri}(D_n)$. When the MLE does not exist, the corresponding facial set of $D_n$ gives the required facial set for $P'_n$ and, therefore, for $P_n$.

Finally, it is clear to see that $C$ is rank-deficient due to linear dependencies among the rows, so one could instead consider the marginal cone spanned by the columns of the matrix
\[
\begin{pmatrix}
C_1 \\
B_1
\end{pmatrix},
\]
which has full dimension $\binom{n}{2} + n$ and is combinatorially equivalent to $D_n$. 31
The final result of the two-step procedure just outlined is a reparametrization of the beta model in the form of a log-linear model with full-rank design matrix given in (33) and Poisson sampling scheme. The constrains on the number of observed edges translate into \( \binom{n}{2} \) product-multinomial sampling restrictions for this log-linear model. However, it is well known that the conditions for existence of the MLE are the same under Poisson and product-multinomial scheme, so whether we incorporate these constraints or not has no bearing on parameter estimability. See Haberman (1974, Chapter 2) and Fienberg and Rinaldo (2011, Section 3.4).

The examples of co-facial sets were obtained by first computing the matrix (33) and then using polymake to compute the facial sets of the resulting marginal cone. For a detailed description of the connection with log-linear models, and for algorithms to compute the facial sets of this cone that can be used in higher dimensions, the reader is referred to Fienberg and Rinaldo (2011).

Finally, to deal with the Rasch model, the procedure can be trivially modified by eliminating the columns to be as in (33) and only if \( Cx' \) belongs to the relative interior of \( D_n \), where for convenience the matrix \( C \) can be taken to be as in (33) (so it has dimension \( n + \binom{n}{2} \times n(n-1) \) and is of full rank). In turn, we can decide this by solving the following simple linear program

\[
\begin{align*}
\max & \quad s \\
\text{s.t.} & \quad Cx' = t \\
& \quad x'_{i,j} - s \geq 0 \\
& \quad s \geq 0,
\end{align*}
\]

where the scalar \( s \) and vector \( x' = \{x'_{i,j}, i \neq j\} \in \mathbb{R}^{n(n-1)} \) are the variables. At the optimum \( (s^*, x^*) \), the MLE exists if and only if \( s^* > 0 \). Though very simple, the previous algorithm may not be sufficient to compute the support of \( \hat{\theta} \) if the MLE does not exist. To this end, we need to resort to a more sophisticated algorithm. Consider the following \( n(n-1) \) programs, one for each column of \( C \):

\[
\begin{align*}
\max & \quad \langle c_{i,j}, y \rangle \\
\text{s.t.} & \quad y^\top t = 0 \\
& \quad C^\top y \geq 0 \\
& \quad -1 \leq y \leq 1,
\end{align*}
\]

where the last inequalities are taken element-wise. Let \( y^*_{i,j} \in \mathbb{R}^{n+\binom{n}{2}} \) denote the solution to the linear program corresponding to the \( (i,j) \)-th column of \( C \).

**Lemma 10.3.** The MLE does not exist if and only if \( \langle c_{i,j}, y^*_{i,j} \rangle > 0 \) for some \( (i,j) \), in which case the co-facial set associated with \( t \) is given by

\[\{(i,j) : \langle c_{i,j}, y^*_{i,j} \rangle > 0\}\].

**Proof.** Let \( \bar{F} = \{(i,j) : \langle c_{i,j}, y^*_{i,j} \rangle = 0\} \). If \( \bar{F} = \{1, \ldots, n\} \), then there does not exist any vector \( v \in \mathbb{R}^{n+\binom{n}{2}} \) such that \( \langle v, c_{i,j} \rangle \geq 0 \) with strict inequality for some \( (i,j) \). Thus, the normal cone at \( t \) is the zero vector, so \( t \in ri(D_n) \), and the MLE exists by Theorem 10.2. We now show that the if the MLE does not exist, then \( \bar{F} = F \), where \( F \) is the facial set associated with the face of \( D_n \) whose relative interior contains \( t \). To see this, let \( \hat{v} = \sum_{(i,j) \in \bar{F}} y^*_{i,j} \). It is clear that \( \bar{F} \subseteq F \), for otherwise the vector \( \hat{v} \) would produce a strictly larger
facial set, which violates the maximality of $\mathcal{F}$. On the other hand, if $(i,j) \in \mathcal{F} \setminus \tilde{\mathcal{F}}$, then there does not exist any vector $y^*_{i,j}$ in the feasible set of the $(i,j)$-th program such that $\langle y^*_{i,j}, c_{i,j} \rangle = 0$. However, the vector $v$ specifying $\mathcal{F}$ is clearly in that feasible set and, by definition, $\langle v, c_{i,j} \rangle = 0$, which gives a contradiction. Thus $\tilde{\mathcal{F}} = \mathcal{F}$, as claimed.

See Fienberg and Rinaldo (2011, Section 4.1) for a more refined and efficient implementation of the above algorithms.

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