A CLASSIFICATION OF TAUT, STEIN SURFACES WITH A PROPER $\mathbb{R}$-ACTION

A. IANNUZZI AND S. TRAPANI

ABSTRACT. We present a classification of 2-dimensional, taut, Stein manifolds with a proper $\mathbb{R}$-action. For such manifolds the globalization with respect to the induced local $\mathbb{C}$-action turns out to be Stein. As an application we determine all 2-dimensional taut, non-complete, Hartogs domains over a Riemann surface.

1. Introduction

Let the group $(\mathbb{R}, +)$ act on a complex manifold $X$ by biholomorphism. Then, by integrating the associated vector field one obtains a local action of $(\mathbb{C}, +)$. For taut, Stein manifolds, the universal globalization with respect to such a local action is Hausdorff ([Ian]). That is, there exists a complex $\mathbb{C}$-manifold $X^*$ containing $X$ as an $\mathbb{R}$-invariant domain such that every $\mathbb{R}$-equivariant holomorphic map from $X$ onto a complex $\mathbb{C}$-manifold extends $\mathbb{C}$-equivariantly on $X^*$. Recently C. Miebach and K. Oeljeklaus have shown that if $X$ is 2-dimensional and the $\mathbb{R}$-action is proper, then the $\mathbb{C}$-action on $X^*$ is also proper, implying that the globalization $X^*$ can be regarded as a holomorphic principal $\mathbb{C}$-bundle over the Riemann surface $S := X^*/\mathbb{C}$ ([MiOe]).

Our main goal here is to present a classification of all such $X$, up to $\mathbb{R}$-equivariant biholomorphism. We first exploit the above bundle structure in order to give a more precise description of $X^*$. In the case when $S$ is non compact, $X^*$ is $\mathbb{C}$-equivariantly biholomorphic to $\mathbb{C} \times S$, where $\mathbb{C}$ acts by translations on the first factor. If the base $S$ is compact, then it is hyperbolic and $X^*$ turns out to be $\mathbb{C}$-equivariantly biholomorphic to a certain twisted bundle $\mathbb{C} \times \Delta/\Gamma$, where $\Delta$ is the unit disk in $\mathbb{C}$ and $\Gamma$ is the group of deck transformations of the universal covering $\Delta \to S$. Then, by using a result of T. Ueda ([Ued]) as the main ingredient, we prove the following

Theorem. Let $X$ be a 2-dimensional, taut, Stein manifold with a proper $\mathbb{R}$-action. Then its universal globalization $X^*$ is Stein.

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Note that in the more general context of Stein $\mathbb{R}$-manifolds it is an open problem to determine whether $X^*$ is always Stein or at least Hausdorff (cf. [HeIa], [CIT], [IST]). Once $X^*$ is understood, we look at the realization of $X$ as an $\mathbb{R}$-invariant domain of $X^*$ and the following question turns out to be crucial. Given an upper semicontinuous function $a : \mathbb{C} \to \{ -\infty \} \cup \mathbb{R}$, consider the ($\mathbb{R}$-invariant) subdomain of $\mathbb{C}^2$ defined by

$$\Omega_a := \{(z, w) \in \mathbb{C}^2 : a(w) < \Im z \}.$$ 

Under which conditions on $a$ is $\Omega_a$ taut? Since taut domains in $\mathbb{C}^n$ are Stein, the function $a$ is necessarily subharmonic. Moreover $\Omega_a$ cannot contain complex lines, therefore $a(w) > -\infty$ for all $w \in \mathbb{C}$.

Partial answers to this problem can be found, e.g., in [Yu] and [Gau]. Here the following necessary and sufficient condition is obtained by using tools of potential theory (Thm. 3.4).

**Theorem.** The domain $\Omega_a$ is taut if and only if $a$ is real valued, subharmonic, non-harmonic and continuous.

This result put us in the position of showing that the 2-dimensional manifolds listed below are all taut and Stein.

**Type CH** If $S$ is compact hyperbolic, say $S = \Delta/\Gamma$, the models are certain twisted bundles $H \times \Delta/\Gamma$, with $H$ a proper, $\mathbb{R}$-invariant, connected strip in $\mathbb{C}$.

**Type NCH** If $S$ is non compact hyperbolic, the models are

$$\{(z, p) \in \mathbb{C} \times S : a(p) < \Im z < -b(p) \} ,$$

where $a$ and $b$ are subharmonic, continuous functions on $S$ such that $a + b < 0$ and $\max\{a(p), b(p)\} > -\infty$ for all $p \in S$.

**Type NCNH** If $S = \mathbb{C}$ or $S = \mathbb{C}^*$, the models are

$$\{(z, p) \in \mathbb{C} \times S : a(p) < \Im z \} \text{ or } \{(z, p) \in \mathbb{C} \times S : \Im z < -b(p) \} ,$$

with $a, b$ subharmonic, non-harmonic, real valued, continuous functions on $S$.

On each such manifold let $\mathbb{R}$ act by translations on the first factor. Then the classification follows by proving that a 2-dimensional, taut, Stein manifold $X$ with a proper $\mathbb{R}$-action is $\mathbb{R}$-equivariantly biholomorphic to a model as above and its type depends on compactness and hyperbolicity of the base $S$ (cf. Thm. 6.1). We recall that in the non compact, simply connected case, a partial result is obtained in [MiOe], Theorem 6.3.

It is worth noting that $X$ turns out to be homotopically equivalent to its base $S$. As a consequence, the corresponding type is strongly related to the topology of $X$. For instance, $X$ is of type CH if and only if $H^2(X, \mathbb{Z}) \neq 0$ (cf. Sect. 6).

We also wish to recall that every taut manifold is Kobayashi hyperbolic, therefore its automorphism group is a Lie group acting properly on $X$ (see
[Kob], Thm. 5.4.2). It follows that there exists a proper $\mathbb{R}$-action on $X$ if and only if the connected component of the identity in $\text{Aut}(X)$ is non compact (cf. [Hoc] p. 180, [MiOe] Lemma 6.3).

As an application of the above classification we determine all 2-dimensional, taut, non-complete Hartogs domains over a Riemann surface (Prop. 7.1). For a characterization of complete Hartogs domains see [ThDu], [Par].

The paper is organized as follows. In Section 2 we point out a characterization of taut manifolds and collect those results which are used in the sequel.

In Section 3 we characterize those domains of the form $\Omega_a$ which are taut (Thm. 3.4).

In Section 4 we study models of type CH and show that their globalization is Stein. We also prove that if the base $S$ is compact, then $X$ is $\mathbb{R}$-equivariantly biholomorphic to one of these models.

In Section 5 the analogous results are proved for models of type NCH and NCNH.

In Section 6 we point out that in most cases the type of $X$ is determined by the topology of $X$ (Cor. 6.3 and Rem. 6.4).

In Section 7 we classify 2-dimensional, taut, non-complete Hartogs domains over a Riemann surface.

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2. Preliminaries

By definition a complex manifold is taut if and only if every sequence of holomorphic maps $f_n : \Delta \rightarrow X$ admits a subsequence which is either converging uniformly on compact subsets or compactly divergent. If $X$ is taut, then it is hyperbolic ([Kob], Thm. 5.1.3). We first recall a result of M. Abate and give a characterization of taut manifolds.

**Theorem 2.1.** ([Aba], Thm. 1.3) Let $X$ be a complex manifold and $X \cup \{\infty\}$ its Alexandroff compactification. Then $X$ is hyperbolic if and only if $\text{Hol}(\Delta, X)$ is relatively compact in $C(\Delta, X \cup \{\infty\})$ with respect to the compact-open topology.

Note that since $X \cup \{\infty\}$ is metrizable, the compact open topology of $C(\Delta, X \cup \{\infty\})$ coincides with the topology of uniform convergence on compact subsets.
Proposition 2.2. For a complex manifold \(X\) the following conditions are equivalent.

(i) \(X\) is taut,

(ii) for every sequence of holomorphic maps \(f_n : \Delta \to X\) such that \(f_n(\zeta_0) \to x_0\) for some \(\zeta_0 \in \Delta\) and \(x_0 \in X\), there exists a subsequence converging uniformly on compact sets of \(\Delta\).

Proof. Condition (ii) is clearly satisfied if \(X\) is taut. Assume that (ii) holds true. We first show that \(X\) is hyperbolic. Let \(K_X\) denote the Kobayashi infinitesimal pseudo metric and assume by contradiction that there exists \(x_0\) in \(X\) and a non zero vector \(v\) in the tangent space \(T_{x_0}X\) such that \(K_X(v) = 0\). Then, by definition of \(K_X\) there exists a sequence \(f_n : \Delta \to X\) of holomorphic maps, such that \(f_n(0) = x_0\) and \(\|f'_n(0)\| \to +\infty\), where \(\| \|\) denotes a chosen norm on \(T_{x_0}X\). However, by assumption up to subsequence \(f_n\) converges uniformly on compact subsets to a holomorphic map from \(\Delta\) to \(X\), giving a contradiction. Thus \(X\) is Kobayashi hyperbolic.

Finally, let \(X \cup \{\infty\}\) be the Alexandroff compactification of \(X\). As a consequence of Theorem 2.1 up to subsequence every sequence of holomorphic maps \(f_n : \Delta \to X\) converges uniformly on compact subsets either to the constant map of value \(\infty\) or there exists \(\zeta_0 \in \Delta\) and \(x_0 \in X\) such that \(f_n(\zeta_0) \to x_0\). In the latter case (ii) implies that there exists a subsequence converging uniformly on compact subsets of \(\Delta\). Hence \(X\) is taut. \(\square\)

As a corollary to Proposition 2.2, one has

Corollary 2.3. Let \(\alpha\) be a plurisubharmonic, continuous function on a taut manifold \(X\). Then the sublevel sets of \(\alpha\) are taut.

Proof. For \(C \in \mathbb{R}\) consider the sublevel set \(O_C = \{x \in X : \alpha(x) < C\}\) and let \(f_n : \Delta \to O_C\) be a sequence of holomorphic maps such that \(f_n(\zeta_0) \to x_0\), for some \(\zeta_0 \in \Delta\) and \(x_0 \in O_C\). Since \(X\) is taut, Proposition 2.2 applies to show that up to subsequence \(f_n\) converges uniformly on compact subsets to a holomorphic map \(f : \Delta \to X\). Note that \(\alpha \circ f(\zeta_0) < C\) and by continuity \(\alpha \circ f \leq C\) on \(\Delta\). Then the maximum principle for plurisubharmonic functions implies that \(\alpha \circ f < C\) on \(\Delta\), i.e. \(f(\Delta) \subset O_C\). Finally the statement follows from Proposition 2.2. \(\square\)

Next we recall two results due to D.D. Thai and N. L. Huong. (\cite{ThHu}, Lemma 3 and Cor. 4). For analogous statements where tautness is replaced by hyperbolicity or complete hyperbolicity, see \cite{Kob}, Thm 3.2.8.
Proposition 2.4. Let $X$ and $Y$ be complex manifolds and $F : X \to Y$ a holomorphic map. If $Y$ is taut and admits an open covering $\{U_j\}$ such that $F^{-1}(U_j)$ is taut for all $j$, then $X$ is taut.

Proposition 2.5. Let $X$ and $Y$ be complex manifolds and $F : X \to Y$ a holomorphic covering. Then $Y$ is taut if and only if so is $X$.

For later use we also collect the following well-known facts.

Lemma 2.6. Let $\theta$ be a real, positive and closed $(1,1)$-current on a complex manifold $X$.

(i) If $H^1(X, \mathcal{O}) = H^2(X, \mathbb{R}) = 0$, then there exists a plurisubharmonic function $\tau$ on $X$ such that $\theta = i\partial\bar{} \partial \tau$.

(ii) If $X$ is compact Kähler and $\theta$ is exact then $\theta = 0$.

(iii) If $H^1(X, \mathbb{R}) = 0$ and $\tau$ is a pluriharmonic function on $X$, then there exists a holomorphic function $f : X \to \mathbb{C}$ such that $\text{Im } f = \tau$.

Proof. (i) follows from the proof of Prop. III 1.19 in [Dem]. For (ii) note that (i) implies that there exist a locally finite open covering $\{U_j\}$ of $X$ and plurisubharmonic functions $\tau_j$ on $U_j$ such that $\theta|_{U_j} = i\partial\bar{} \partial \tau_j$. Let $\psi_j$ be a partition of unity associated to $\{U_j\}$, define $T := \sum_j \psi_j \tau_j$ and $\Theta := \theta - i\partial\bar{} \partial T$. Then for $j_0$ fixed one has $\Theta|_{U_{j_0}} = (\theta - i\partial\bar{} \partial T)|_{U_{j_0}} = i\partial\bar{} \partial \sum_j \psi_j (\tau_{j_0} - \tau_j)$. Since $\tau_{j_0} - \tau_j$ is pluriharmonic on $U_{j_0} \cap U_j$, it follows that $\Theta$ is a smooth, exact, real $(1,1)$-form on $X$. Then, the classical $\partial\bar{}$-Lemma for compact Kähler manifolds (see e.g. [GrHa], Lemma 1.2, p. 148) implies that there exists a smooth function $Q$ on $X$ such that $\Theta = i\partial\bar{} \partial Q$. Hence $\theta = i\partial\bar{} (Q + T)$ and Thm. I 3.31 in [Dem] implies that $Q + T$ is plurisubharmonic on $X$. Since $X$ is compact, $Q + T$ is constant and consequently $\theta$ is zero. For (iii) see [Dem] Theorem I 5.16. □

Let us briefly recall the notion of globalization in the context of $\mathbb{R}$-manifolds. For further details and generalizations we refer to [Pal], [Helm], [CIT] and [MiOe]. An $\mathbb{R}$-action by biholomorphisms on a complex manifold $X$ induces a local holomorphic $\mathbb{C}$-action by integration of the associated holomorphic vector field. This is given by an open neighborhood $\Sigma$ of the neutral section $\{e\} \times X$ in $\mathbb{C} \times X$ and a holomorphic map $\Phi : \Sigma \to X$, $(\lambda, x) \to \lambda \cdot x$, such that

(i) the set $\{\lambda \in \mathbb{C} : (\lambda, x) \in \Sigma\}$ is connected for all $x \in X$,

(ii) for all $x \in X$ one has $0 \cdot x = x$,

(iii) if $(\mu + \lambda, x) \in \Sigma$, $(\lambda, x) \in \Sigma$ and $(\mu, \lambda \cdot x) \in \Sigma$, then $(\mu + \lambda) \cdot x = \mu \cdot (\lambda \cdot x)$. 

A possibly non-Hausdorff complex manifold with a global $\mathbb{C}$-action containing $X$ as an $\mathbb{R}$-invariant domain is called a globalization of the local $\mathbb{C}$-action. By [HeIa], if $X$ is holomorphically separable there exists a (unique) universal globalization $X^*$. That is, a globalization with the following universal property: for any $\mathbb{R}$-equivariant holomorphic map $f : X \to Y$ into a $\mathbb{C}$-manifold there exists a $\mathbb{C}$-equivariant holomorphic extension $f^* : X^* \to Y$.

For $x$ in $X^*$ let $\Sigma_x = \{ \lambda \in \mathbb{C} : \lambda \cdot x \in X \}$. Then $\Sigma_x$ is $\mathbb{R}$-invariant, connected and there exist upper semicontinuous functions $\alpha, \beta : X^* \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\Sigma_x = \{ \lambda \in \mathbb{C} : \alpha(x) < \Im \lambda < -\beta(x) \}.$$  

Note that $\alpha$ and $\beta$ are $\mathbb{R}$-invariant and $\alpha + \beta < 0$. Moreover, an element $x$ of $X^*$ belongs to $X$ if and only if $\alpha(x) < 0 < -\beta(x)$. Thus

$$X = \{ x \in X^* : \alpha(x) < 0 \text{ and } \beta(x) < 0 \}.$$  

We recall the basic properties of $\alpha$ and $\beta$ in the case when $X$ is a taut, Stein manifold.

**Lemma 2.7.** Let $X$ be a taut, Stein $\mathbb{R}$-manifold. Then

(i) the functions $\alpha$ and $\beta$ are continuous and plurisubharmonic,

(ii) for $\lambda \in \mathbb{C}$ and $x \in X^*$ one has

$$\alpha(\lambda \cdot x) = -\Im (\lambda) + \alpha(x) \quad \beta(\lambda \cdot x) = \Im (\lambda) + \beta(x).$$

(iii) the sum $\alpha + \beta$ is a negative, $\mathbb{C}$-invariant, plurisubharmonic, continuous function,

(iv) if the $\mathbb{R}$-action is proper, then $\max(\alpha(x), \beta(x)) > -\infty$ for all $x$ in $X^*$.

**Proof.** (i) Plurisubharmonicity of $\alpha$ and $\beta$ in the case where $X$ is a Stein $\mathbb{R}$-manifold is proved in [For]. Since $X$ is also taut, such functions are continuous ([Ian], [MiOe], Prop. 3.2). (ii) is a direct consequence of the definition and (iii) follows from (i) and (ii). For (iv) note that properness of the $\mathbb{R}$-action implies that there are no fixed points. Therefore if $\alpha(x) = \beta(x) = -\infty$ for some $x$ in $X$, the (local) $\mathbb{C}$-orbit through $x$ is biholomorphic either to $\mathbb{C}$ or to $\mathbb{C}^*$. Since $X$ is taut, this gives a contradiction. Recalling that $X^* = \mathbb{C} \cdot X$, the result follows from (ii). \qed

Finally we recall the following result of C. Miebach and K. Oeljeklaus (see [MiOe], Thm. 4.4) which is often used in the sequel.
Theorem 2.8. Let $X$ be a 2-dimensional, taut, Stein manifold with a proper $\mathbb{R}$-action. Then the $\mathbb{C}$-action on $X^*$ is proper, i.e. $X^*$ can be regarded as a holomorphic principal $\mathbb{C}$-bundle over the Riemann surface $S := X^*/\mathbb{C}$. In particular if $S$ is non compact, then $X^*$ is $\mathbb{C}$-equivariantly biholomorphic to $\mathbb{C} \times S$.

Note that last part of the statement follows directly from the fact that on a non compact Riemann surface $S$ the cohomology group $H^1(S, \mathcal{O})$ vanishes.

3. Distinguished $\mathbb{R}$-invariant domains in $\mathbb{C}^2$

Consider the domains of $\mathbb{C}^2$ of the form $\Omega_a = \{ (z, w) \in \mathbb{C}^2 : a(w) < \text{Im} \, z \}$, with $a : \mathbb{C} \to (-\infty) \cup \mathbb{R}$ an upper semicontinuous function. Note that $\mathbb{R}$ acts properly on $\Omega_a$ by translations on the first factor. The main result of this section is Theorem 3.4 where we determine necessary and sufficient conditions for $\Omega_a$ to be taut.

We already noted in the introduction that if $\Omega_a$ is taut, then $a$ is real valued and subharmonic. Moreover $\mathbb{C}^2$ is the universal globalization of $\Omega_a$, therefore by (i) of Lemma 2.7 the function $\alpha : \mathbb{C}^2 \to (-\infty) \cup \mathbb{R}$, given by $(z, w) \to a(w) - \text{Im} \, z$, is continuous. As a consequence $a$ necessarily belongs to

$$C := \{ \text{subharmonic, real valued, continuous functions on } \mathbb{C} \}.$$

Also note that if $\Omega_a$ is taut, then for all positive $\tau \in \mathbb{R}$ the domain $\Omega_{\tau a}$ is also taut, since the biholomorphism $\mathbb{C}^2 \to \mathbb{C}^2$, defined by $(z, w) \to (\tau z, w)$, maps $\Omega_a$ onto $\Omega_{\tau a}$. Thus the set of interest $\mathcal{F} := \{ a \in C : \Omega_a \text{ is taut} \}$ is a cone. Here we show that $\mathcal{F}$ coincides with $\{ a \in C : a \text{ is not harmonic} \}$. We need some preliminary lemmata.

Lemma 3.1. Let $a \in C$ and $(f_n, g_n) : \Delta \to \Omega_a$ be a sequence of holomorphic maps such that

(i) $f_n(\zeta_0) \to z_0$, for some $\zeta_0 \in \Delta$ and $z_0 \in \mathbb{C}$,

(ii) $g_n$ converges uniformly on compact subsets of $\Delta$ to a holomorphic map $g : \Delta \to \mathbb{C}$ such that $(z_0, g(\zeta_0)) \in \Omega_a$.

Then there exists a subsequence of $f_n$ converging uniformly on compact subsets of $\Delta$ to a holomorphic map $f : \Delta \to \mathbb{C}$ such that $(f, g)(\Delta) \subset \Omega_a$.

Proof. Let $U_1$ be a relatively compact disk of $\Delta$ containing $\zeta_0$. By condition (i) the sequence $g_n$ converges uniformly to $g$ on the closure $\overline{U_1}$ of $U_1$. Then, for $n$ large enough $f_n(U_1)$ is contained in the set

$$S_1 := \{ z \in \mathbb{C} : \text{Im} \, z > \min_{w \in U_1} \{ a(g(w)) \} - 1 \},$$
which is biholomorphic to the unit disc of \( \mathbb{C} \). In particular \( S_1 \) is taut, therefore there exists a subsequence \( f_{n,1} \) of \( f_n \) converging uniformly on compact subsets of \( U_1 \) to a holomorphic map \( f_1 : U_1 \to S_1 \).

Complete \( U_1 \) to an increasing sequence of simply connected domains \( \{U_k\}_{k \in \mathbb{N}} \) which exhaust \( \Delta \). By iterating the above argument, for each \( k \in \mathbb{N} \) one obtains subsequences \( \{f_{n,k}\}_{n \in \mathbb{N}} \) converging uniformly on compact subsets of \( U_k \) to holomorphic maps \( f_k : U_k \to S_k \). Then the diagonal sequence \( \{f_{j,j}\}_{j \in \mathbb{N}} \) converges uniformly on compact subsets of \( \Delta \).

Finally note that \( a \circ g(\zeta_0) - \text{Im} f(\zeta_0) < 0 \) and by continuity \( a \circ g - \text{Im} f \leq 0 \) on \( \Delta \). Then, by the maximum principle for subharmonic functions, \( a \circ g - \text{Im} f < 0 \) on \( \Delta \), i.e. \( (f, g)(\Delta) \subset \Omega_a \).

Given a subharmonic function \( a \) on \( \mathbb{C} \), denote by \( M_{\zeta_0,r}(a) \) its mean value
\[
\frac{1}{2\pi} \int_0^{2\pi} a(\zeta_0 + re^{i\theta})d\theta.
\]

**Lemma 3.2.** For \( a \) in \( \mathcal{C} \) the following conditions are equivalent.

(i) \( a \in \mathcal{F} \),

(ii) for any sequence of holomorphic functions \( g_n : \Delta \to \mathbb{C} \) satisfying

(a) \( g_n(\zeta_0) \to w_0 \) for some \( \zeta_0 \in \Delta \) and \( w_0 \in \mathbb{C} \),

(b) for every \( 0 < r < 1 - |\zeta_0| \) there exists \( M_r \in \mathbb{R} \) such that
\[
M_{\zeta_0,r}(a \circ g_n) < M_r \text{ for all } n \in \mathbb{N},
\]

there exists a subsequence converging uniformly on compact subsets of \( \Delta_{1-|\zeta_0|}(\zeta_0) \).

**Proof.** Assume that \( \Omega_a \) is taut and let \( g_n \) be a sequence as in (ii). For \( n \in \mathbb{N} \) and \( 0 < r < 1 - |\zeta_0| \), denote by \( h_n \) the harmonic function on \( \Delta_{r}(\zeta_0) \) which coincides with \( a \circ g_n \) on the boundary of \( \Delta_{r}(\zeta_0) \). Then \( h_n(\zeta_0) = M_{\zeta_0,r}(a \circ g_n) \) and consequently, for \( n \) large enough, one has
\[
a(w_0) - 1 < a(g_n(\zeta_0)) \leq h_n(\zeta_0) < M_r.
\]

As a consequence, up to subsequence \( h_n(\zeta_0) \) converges to a real number \( y \). Let \( f_n : \Delta_{r}(\zeta_0) \to \mathbb{C} \) be the sequence of holomorphic functions defined by \( \text{Im} f_n = h_n + 1 \) and \( \text{Re} f_n(\zeta_0) = 0 \). Since \( \text{Im} f_n = h_n + 1 \geq a \circ g_n + 1 > a \circ g_n \), it follows that \( (f_n, g_n) \) defines a sequence of holomorphic maps from \( \Delta_{r}(\zeta_0) \) to \( \Omega_a \).

Moreover \( (f_n, g_n)(\zeta_0) \to (i(y+1), w_0) \in \Omega_a \) and \( \Omega_a \) is taut. Then by Lemma \ref{lemma:taut} there exists a subsequence \( (f_n, g_n) \) converging uniformly on compact subsets of \( \Delta_{r}(\zeta_0) \).

Let \( r_k \) be an increasing sequence of positive numbers converging to \( 1 - |\zeta_0| \) such that \( r_1 = r \). The analogous argument as above shows that there exist subsequences \( (f_{n,k}, g_{n,k}) \) converging uniformly on compact subsets of \( \Delta_{r_k}(\zeta_0) \). Then
the diagonal subsequence \((f_{n,n}, g_{n,n})\) converges uniformly on compact subsets of \(\Delta_{1-|\zeta_0|}^{(\zeta_0)}\) and so does \(g_{n,n}\). This implies (ii).

Conversely assume (ii) and let \((f_n, g_n) : \Delta \to \Omega_a\) be a sequence of holomorphic maps such that \((f_n, g_n)(\zeta_0) \to (z_0, w_0)\), for some \(\zeta_0\) in \(\Delta\) and \((z_0, w_0)\) in \(\Omega_a\). By Lemma 2.2, it is enough to show that, up to subsequence, \((f_n, g_n)\) converges uniformly on compact subsets of \(\Delta\) to some \((f, g)\) with \((f, g)(\Delta) \subset \Omega_a\). Note that for \(0 < r < 1 - |z_0|\) and \(n\) large enough one has

\[
\text{Im } z_0 + 1 > \text{Im } f_n(\zeta_0) = M_{\zeta_0,r}(\text{Im } f_n) > M_{\zeta_0,r}(a \circ g_n). 
\]

Thus, by assumption, up to subsequence \(g_n\) converges uniformly on compact sets of the disk \(\Delta_{1-|z_0|}(z_0)\) and, by Lemma 3.1, so does \(f_n\). Therefore for every point \(\zeta \in \Delta_{1-|z_0|}(z_0)\) there exists a subsequence of \((f_n, g_n)\) converging at \(\zeta\) to an element of \(\Omega_a\). Then by constructing a finite chain of disks one shows that, up to subsequence, \((f_n, g_n)\) converges at 0 to an element of \(\Omega_a\). Finally the analogous argument as above implies that, up to subsequence \((f_n, g_n)\), converges uniformly on compact subsets of \(\Delta\) to some \((f, g)\) with \((f, g)(\Delta) \subset \Omega_a\). \(\square\)

**Lemma 3.3.** The cone \(F\) has the following properties.

(i) Harmonic functions do not belong to \(F\).

(ii) If \(a \in C\) is non constant and bounded from below, then \(a \in F\).

(iii) If \(b \in C\) and \(c \in F\) then \(b + c \in F\).

*Proof.* (i) If \(a\) is harmonic, then \(a = \text{Im } f\) for some holomorphic \(f : \mathbb{C} \to \mathbb{C}\). Then the biholomorphism of \(\mathbb{C}^2\) defined by \((z, w) \to (z - f(w), w)\) maps \(\Omega_a\) onto \(\{ (z, w) \in \mathbb{C}^2 : \text{Im } z > 0 \}\), which is not taut. Thus \(\Omega_a\) is not taut.

For (ii) consider the restriction to \(\Omega_a\) of the projection from \(\mathbb{C}^2\) onto the first factor given by

\[
p|_{\Omega_a} : \Omega_a \to p(\Omega_a), \quad (z, w) \to z.
\]

Since \(a\) is bounded from below, the image \(p(\Omega_a)\) is contained in the half plane \(\{ \text{Im } z > \inf_C a \}\), which is taut. Then, by Lemma 2.4, in order to prove that \(\Omega_a\) is taut it is enough to show that \((p|_{\Omega_a})^{-1}(U)\) is taut for every relatively compact open subset \(U\) in \(p(\Omega_a)\).

For this, let \(M\) be the maximum of \(\text{Im } z\) on the closure of \(U\) and note that \((p|_{\Omega_a})^{-1}(U)\) is contained in \(U \times \{ a < M \}\). Since \(a\) is not constant, it is not bounded. As a consequence \(\{ a < M \}\) is a hyperbolic domain of \(\mathbb{C}\). Thus it is taut and so is \(U \times \{ a < M \}\). Finally, the image \((p|_{\Omega_a})^{-1}(U)\) is the zero sublevel set in \(U \times \{ a < M \}\) of the subharmonic, continuous function \((z, w) \to a(w) - \text{Im } z\). Thus it is taut by Corollary 2.3, concluding (ii).

For (iii) let \(g_n : \Delta \to \mathbb{C}\) be a sequence of holomorphic maps such that \(g_n(\zeta_0) \to w_0\) for some \(\zeta_0 \in \Delta\), \(w_0 \in \mathbb{C}\) and for every \(0 < r < 1 - |\zeta_0|\) there exists a real number \(M_r\) such that \(M_{\zeta_0,r}((b + c) \circ g_n) < M_r\) for all \(n \in \mathbb{N}\).
Then by Lemma 3.2 in order to show that \( b + c \) belongs to \( \mathcal{F} \), it is enough to find a subsequence of \( g_n \) converging uniformly on compact subsets of \( \Delta \). For 

\[
0 < r < 1 - |\zeta_0| \quad \text{and} \quad n \ \text{large enough one has}
\]

\[
M_r > M_{\zeta_0,r}((b + c) \circ g_n) \geq b(g_n(\zeta_0)) + M_{\zeta_0,r}(c \circ g_n) > b(w_0) - 1 + M_{\zeta_0,r}(c \circ g_n).
\]

Hence

\[
M_{\zeta_0,r}(c \circ g_n) < M_r - b(w_0) + 1.
\]

Since \( c \in \mathcal{F} \), Lemma 3.2 implies that there exists a subsequence of \( g_n \) converging uniformly on compact subsets of \( \Delta \), as wished. \( \square \)

**Theorem 3.4.** Let \( a : \mathbb{C} \to \{-\infty\} \cup \mathbb{R} \) be an upper semicontinuous function. Then \( \Omega_a := \{(z,w) \in \mathbb{C}^2 : a(w) < \text{Im} \, z\} \) is taut if and only if \( a \) is a real valued, subharmonic, non-harmonic, continuous function.

**Proof.** We already noted at the beginning of the section that if \( \Omega_a \) is taut, then \( a \) belongs to \( \mathcal{C} \). Moreover, by (i) of the above lemma \( a \) is not harmonic, giving one implication.

Conversely, given \( a \in \mathcal{C} \) non-harmonic we want to show that \( a \in \mathcal{F} \). By (ii) and (iii) of the above lemma, it is enough to show that \( a = b + c \), with \( b, \ c \in \mathcal{C} \) and \( c \) non constant and bounded from below.

For this consider the positive measure \( \mu = L(a) \), where \( L(a) \) denotes the laplacian of \( a \), and choose \( r \) big enough such that \( \mu \) is non zero on \( \Delta_r(0) \). Let \( \chi_{\Delta_r(0)} \) be the characteristic function of \( \Delta_r(0) \) and define \( \mu_1 = (1 - \chi_{\Delta_r(0)})\mu \) and \( \mu_2 = \chi_{\Delta_r(0)}\mu \), so that \( \mu = \mu_1 + \mu_2 \) gives a decomposition of \( \mu \) as a sum of positive measure on \( \mathbb{C} \). Note that \( \mu_2 \) is non zero with compact support and consider the potential \( c : \mathbb{C} \to \mathbb{R} \cup \{-\infty\} \) associated to \( \mu_2 \) defined by

\[
c(w) := \frac{1}{2\pi} \int_{\mathbb{C}} \log(|w - \xi|)d\mu_2(\xi) = \frac{1}{2\pi} \int_{\Delta_r(0)} \log(|w - \xi|)d\mu_2(\xi).
\]

Then the laplacian \( L(c) \) of \( c \) coincides with \( \mu_2 \) (see e.g. [KH, Prop. 4.1.2]), therefore \( c \) is non constant and subharmonic.

Furthermore, the real \((1,1)\)-current \( \mu_1 d\xi d\bar{\xi} \) is closed and positive on \( \mathbb{C} \), hence by (i) of Lemma 2.6 there exists a subharmonic function \( \tilde{b} : \mathbb{C} \to \mathbb{R} \cup \{-\infty\} \) such that \( L(\tilde{b}) = \mu_2 \). It follows that \( L(\tilde{b} + c) = L(a) \) and consequently \( a = \tilde{b} + c + h \), with \( h \) harmonic on \( \mathbb{C} \). This implies that \( \tilde{b} + c \) is continuous and real valued. Since \( \tilde{b} \) and \( c \) are everywhere smaller than \( +\infty \), they are also real valued. Moreover \( c \) is upper semicontinuous, \( -\tilde{b} \) is lower semicontinuous and \( c = -\tilde{b} + a - h \), with \( a - h \) continuous. Thus \( \tilde{b} \) and \( c \) are continuous subharmonic functions, i.e. they belong to \( \mathcal{C} \), and \( b := \tilde{b} + h \in \mathcal{C} \).

Finally note that the non constant function \( c \) is bounded from below. Indeed by definition of \( c \), if \( w \) is not in \( \Delta_{r+1}(0) \) then \( c(w) \geq 0 \). Since \( c \) is continuous,
this implies that \( c \geq \min\{0, m\} \), with \( m := \min_{w \in \Delta_{r+1}(0)} \{c(w)\} \). Then \( a = b + c \) gives the desired decomposition. \( \square \)

4. MODELS WITH COMPACT BASE

Let \( S \) be a compact hyperbolic Riemann surface, say \( S = \Delta/\Gamma \), with \( \Gamma \) the subgroup in \( \text{Aut}(\Delta) \) of deck transformations of the universal covering \( \Delta \to S \). Choose a non-trivial group homomorphism \( \Psi : \Gamma \to \mathbb{R} \) and let \( \Gamma \) act on \( \mathbb{C} \times \Delta \) by \( \gamma \cdot (z, w) := (z + \Psi(\gamma), \gamma \cdot w) \). Endow the quotient \( \mathbb{C} \times \Delta/\Gamma \) with the \( \mathbb{R} \)-action defined by \( t \cdot (z, w) := (z + t, w) \). We introduce the first class of models as \( \mathbb{R} \)-invariant subdomains of \( \mathbb{C} \times \Delta/\Gamma \).

**Type CH** A model of type CH with compact hyperbolic base \( S = \Delta/\Gamma \) is given by

\[ H \times \Delta/\Gamma, \]

where \( H \) is a proper, \( \mathbb{R} \)-invariant, connected strip of \( \mathbb{C} \). Up to \( \mathbb{R} \)-equivariant biholomorphism, we may assume that \( H \) is one of the strips \( \{0 < \text{Im } z\} \), \( \{\text{Im } z < 0\} \) or \( \{0 < \text{Im } z < C\} \), for some real positive \( C \).

**Proposition 4.1.** Let \( X \) be a model of type CH with base \( S = \Delta/\Gamma \). Then

(i) the universal globalization of \( X \) is \( \mathbb{C} \times \Delta/\Gamma \), which is Stein.

(ii) \( X \) is a taut, Stein manifold with a proper \( \mathbb{R} \)-action.

Before proving the above proposition we need a preparatory lemma. Given a rank two holomorphic vector bundle \( E \) over a compact Riemann surface \( S \), denote by \( P \) its (fiberwise) projectification and let \( p : E \setminus S \to P \) be the canonical projection. Here \( S \) is identified with the zero section in \( E \). Let \( \sigma : S \to P \) be a holomorphic section of \( P \) and consider its image \( C := \sigma(S) \). Recall that the normal bundle \( N \) of the curve \( C \) is given by \( TP|_C/TC \) and it can be identified with the line bundle \( \sigma^*(N) \) over \( S \).

Regard the tautological line bundle \( \mathcal{O}(-1) \) as a subbundle in \( \pi^*(E) \), where \( \pi : P \to S \) is the bundle projection. Then the holomorphic line bundle associated to \( \sigma \) is \( L := \sigma^*(\mathcal{O}(-1)) \) and can be identified with the subbundle of \( E \) given by \( p^{-1}(C) \cup S \).

**Lemma 4.2.** The normal bundle \( \sigma^*(N) \) is isomorphic to \( (E/L) \otimes L^* \).
Proof. Consider the relative tangent bundle $T_{P/S} := \ker d\pi$. We first note that $N$ is isomorphic to the restriction $T_{P/S}|_C$ of such a bundle to $C$, since one has the short exact sequence of vector bundles over $C$

$$0 \to TC \to TP|_{C} \to T_{P/S}|_{C} \to 0,$$

where the third map is defined by $v \mapsto v - d\sigma \circ d\pi(v)$.

We first assume that $L$ is trivial, i.e. it admits a non zero holomorphic section $\tau$. Then one has the commutative diagram

$$\begin{array}{ccc}
E \setminus S & \xrightarrow{p} & P \\
\downarrow \tau & & \downarrow \sigma \\
S & & 
\end{array}$$

and an exact sequence of vector bundles over $\tau(S)$

$$0 \to T_{L/S}|_{\tau(S)} \to T_{E/S}|_{\tau(S)} \to p^*(T_{P/S}|_C) \to 0,$$

where the third map is given by $v \mapsto dp(v)$. Since $p \circ \tau = \sigma$, by applying $\tau^*$ one obtains the following exact sequence of vector bundles over $S$

$$0 \to L \to E \to \sigma^*(T_{P/S}|_C) \to 0,$$

where we use the natural identification $\tau^*(T_{E/S}|_{\tau(S)}) \cong F$ for any vector sub-bundle $F$ of $E$. Moreover, by recalling that $N$ is isomorphic to $T_{P/S}|_C$, one obtains that $\sigma^*(N)$ is isomorphic to $E/L$, as wished.

Finally, if $L$ is non trivial note that $P$ can be regarded as the projectification of $E \otimes L^*$ and in this realization $\sigma(S)$ is the projectification of the trivial line bundle $L \otimes L^*$. Then an analogous argument as above implies that $\sigma^*(N)$ is isomorphic to $E \otimes L^*/L \otimes L^*$ and by the exactness of the sequence of vector bundles over $S$

$$0 \to L \otimes L^* \to E \otimes L^* \to (E/L) \otimes L^* \to 0,$$

one has $E \otimes L^*/L \otimes L^* \cong (E/L) \otimes L^*$. □

Proof of Proposition 4.1 (i) Note that $X$ is orbit-connected in $\mathbb{C} \times \Delta/\Gamma$. Then Lemma 1.5 in [CTT] implies that $X^* := \mathbb{C} \times \Delta/\Gamma$ is the universal globalization of $X$. Consider the $\mathbb{P}^1$-bundle $P := \mathbb{P}^1 \times \Delta/\Gamma$, where $\Gamma$ act on $\mathbb{P}^1 \times \Delta$ by $\gamma \cdot ([z_1 : z_2], w) := ([z_1 + \Psi(\gamma)z_2 : z_2], \gamma \cdot w)$. Then $X^*$ is embedded in $P$ via the map

$$[z, w] \mapsto [[z : 1], w].$$

and the union of points at infinity defines the complex curve $C := \{[[1 : 0], w] \in P : w \in \Delta\}$ which is biholomorphic to $S$. Indeed it can be regarded as the holomorphic section $\sigma : S \to P$, defined by $[w] \mapsto [[[1 : 0], w]$. We wish to apply Theorem 1, p. 590 in [Ued] in order to obtain a suitable strictly plurisubharmonic function on $V_0 \setminus C$, for some open neighborhood $V_0$ of
$C$ in $P$. For this we first check that the normal bundle of $C$ is trivial. Consider the rank two vector bundle over $S$ defined by $E := \mathbb{C}^2 \times \Delta / \Gamma$, where $\Gamma$ acts on $\mathbb{C}^2 \times \Delta$ by $\gamma \cdot ((z_1, z_2), w) := ((z_1 + \Psi(\gamma)z_2, z_2), \gamma \cdot w)$. Note that the line subbundle $L := \{ ([z_1, z_2], w) \in E : z_2 = 0 \}$ associated to the section $\sigma$ is trivial. Indeed it admits the global section $[w] \to [(1, 0), w]$. Since $P$ is the projectification of $E$, by Lemma 4.2 this implies that the normal bundle of $C := \sigma(S)$ is isomorphic to $E/L$. Moreover one has the short exact sequence of vector bundles over $S$

$$0 \to L \to E \to \mathbb{C} \times S \to 0,$$

where the third map is defined by $[(z_1, z_2), w] \to (z_2, [w])$. Therefore $E/L$ is trivial and so is the normal bundle of $C$.

Next we check that the curve $C$ is of type 1, in the sense of Definition p. 589 in [Ued]. For this choose an open covering $\{ U_j \}$ of $S$ such that there exist injective, local sections $s_j : U_j \to \Delta$ of the universal covering $\Delta \to S$. Define local trivializations of $P$ by

$$\mathbb{P}^1 \times U_j \to P, \quad (z_1 : z_2, p) \to [[z_1 : z_2], s_j(p)].$$

Note that the curve $C$ is locally defined by $\{ z_2 = 0 \}$ and in a neighborhood of $C$ the intersection of two trivializations associated to the sections $s_j$ and $s_k$ is given by

$$[[1 : z_2], s_k(p)] = [[1 : z_2], s_j(p)].$$

This implies that there exists $\gamma \in \Gamma$ such that $s_j(p) = \gamma \cdot s_k(p)$ and consequently

$$[[1 : z_2], s_j(p)] = [[1 : z_2], \gamma \cdot s_k(p)] = [[1 - \Psi(\gamma)z_2', z_2', s_k(p)]].$$

Since $\Gamma$ acts freely on $\Delta$, it follows that $z_2 = z_2'/(1 - \Psi(\gamma)z_2')$ and

$$z_2 - z_2' = z_2'\left(\frac{1}{1 - \Psi(\gamma)z_2'} - 1\right) = (z_2')^2 \frac{\Psi(\gamma)}{1 - \Psi(\gamma)z_2'} = (z_2')^2(\Psi(\gamma) + o(z_2')).$$

In our setting the normal bundle of $C$ is holomorphically trivial, therefore the locally constant maps $f_{jk} : U_j \cap U_k \to \mathbb{C}$, given by $p \to \Psi(\gamma)$, define a cocycle in $H^1(S, \mathcal{O})$ (cf. [Ued], p. 588).

**Claim.** The cocycle $f_{jk}$ is cohomologous to zero if and only if $\Psi$ is trivial.

**Proof of Claim.** By using the above defined sections $s_j : U_j \to \Delta$ one has local trivializations of $X^*$ given by

$$\mathbb{C} \times U_j \to X^*, \quad (z, p) \to [z, s_j(p)].$$

It follows that $f_{jk}$ is the cocycle defining $X^*$ as a holomorphic principal $\mathbb{C}$-bundle over $S$. Assume that there exists a holomorphic ($\mathbb{C}$-equivariant) trivialization $F : X^* \to \mathbb{C} \times S$. We can choose a ($\mathbb{C}$-equivariant) lifting $\bar{F} : \mathbb{C} \times \Delta \to \mathbb{C} \times \Delta$ to the universal coverings such that $\bar{F}(z, w) = (z + \bar{f}(w), w)$, with $\bar{f} : \mathbb{C} \to \mathbb{C}$ holomorphic. Moreover for every $\gamma \in \Gamma$ one has

$$\bar{F}(\gamma \cdot (z, w)) = \gamma \cdot \bar{F}(z, w) = (z + \bar{f}(w), \gamma \cdot w).$$
implying that $\tilde{f}(\gamma \cdot w) + \Psi(\gamma) = \tilde{f}(w)$. In particular

$$\tilde{f}(w) - \tilde{f}(\gamma \cdot w) = \Psi(\gamma) \in \mathbb{R}. $$

Hence $\text{Im } \tilde{f}$ is $\Gamma$-invariant, therefore it pushes down to a harmonic function on $S := \Delta/\Gamma$. Then the compactness of $S$ implies that $\text{Im } \tilde{f}$ is constant and consequently $\tilde{f}$ is constant. Hence $\Psi(\gamma) = 0$ for all $\gamma \in \Gamma$, proving the claim.

Since $\Psi$ is non-trivial by assumption, the cocycle $f_{jk}$ is not cohomologous to zero, i.e. the curve $C$ is of type 1. Then, by Theorem 1, p. 590 in [Ued], there exists an open neighborhood $V_0$ of $C$ in $P$ and a smooth, strictly plurisubharmonic function $\rho$ defined on $V_0 \setminus C$ such that $\lim \rho(p) = \infty$ for $p$ approaching $C$. In particular we may assume that $\rho$ is positive.

Fix $N$ large enough such that the domain $X_N := \{ [z, w] \in \mathbb{C} \times \Delta/\Gamma : \text{Im } z > N \}$ is contained in $V_0$. Note that $X_N$ is Stein, since it admits the smooth, strictly plurisubharmonic exhaustion $\rho + \frac{1}{\text{Im } z - N}$. Moreover for all $n \in \mathbb{N}$ the domains $X_{N-n}$ are also Stein, being biholomorphic to $X_N$ via a translation in the first factor. Furthermore $X_{N-n}$ can be regarded as a sublevel set of the plurisubharmonic function $\text{Im } z$, therefore it is Runge in $X_{N-1}$.

(i) Note that $X$ is an $\mathbb{R}$-invariant, locally Stein domain in the Stein, principal $\mathbb{C}$-bundle $X^* = \mathbb{C} \times \Delta/\Gamma$ over $S$. Thus the $\mathbb{R}$-action on $X$ is proper and $X$ is Stein by [DoGr]. Finally the universal covering of $X$ is given by $H \times \Delta$, which is taut. Thus $X$ is taut by Proposition 2.5.

**Remark 4.3.** It was pointed out to us by Christian Miebach that a similar strategy as above applies to show that every non trivial principal $\mathbb{C}$-bundle over a compact Riemann surface is Stein.

**Remark 4.4.** Let $F : H \times \Delta/\Gamma \to H' \times \Delta/\Gamma'$ be an $\mathbb{R}$-equivariant biholomorphism between two models of type CH and consider a holomorphic lifting $\tilde{F} : H \times \Delta \to H' \times \Delta$ to the universal covering spaces. We claim that $\tilde{F}(z, w) = (z + f(w), \tilde{\varphi}(w))$, where $r \in \mathbb{R}$ and $\tilde{\varphi} \in \text{Aut}(\Delta)$. In particular $H = H'$.

In order to prove this, note that $\tilde{F}$ is also $\mathbb{R}$-equivariant. Therefore it induces a biholomorphism $\varphi : \Delta/\Gamma \to \Delta/\Gamma'$. As a consequence $\tilde{F}(z, w) = (z + f(w), \tilde{\varphi}(w))$, with $f : \Delta \to \mathbb{C}$ holomorphic and $\tilde{\varphi} \in \text{Aut}(\Delta)$ a lifting of $\varphi$ with $\Gamma' = \tilde{\varphi} \Gamma \tilde{\varphi}^{-1}$. Since the actions of $\Gamma$ and $\Gamma'$ on $\mathbb{C} \times \Delta$ are given respectively by $\gamma \cdot (z, w) = (z + \Psi(\gamma), \gamma(w))$ and $\gamma' \cdot (z, w) = (z + \Psi'(\gamma'), \gamma'(w))$, for $\gamma \in \Gamma$ one has

$$f \circ \gamma - f = \Psi'(\tilde{\varphi} \gamma \tilde{\varphi}^{-1}) - \Psi(\gamma) \in \mathbb{R}. $$

Hence $\text{Im } f$ is $\Gamma$-invariant. Then the analogous argument as in the claim in the proof of Proposition 2.4 implies that $f \equiv r$, with $r \in \mathbb{C}$. In particular $\Psi'(\tilde{\varphi} \gamma \tilde{\varphi}^{-1}) = \Psi(\gamma)$ and $H, H'$ are either both of finite width or of infinite
TAUT STEIN SURFACES

width. Assume that, e.g. $H = \{0 < \text{Im } z < C\}$ and $H' = \{0 < \text{Im } z < C'\}$. By applying $\tilde{F}$ to any $(z, w) \in H \times \Delta$ one sees that $0 < \text{Im } z$ if and only if $0 < \text{Im } z + \text{Im } r$. This implies that $\text{Im } r = 0$, i.e. that $r$ is a real number and consequently $C = C'$. An analogous argument applies to the case when $H$ has infinite width.

Let $X$ be a 2-dimensional, taut, Stein manifold with a proper $\mathbb{R}$-action. By Theorem 2.8 the $\mathbb{C}$-action on $X^*$ is proper and one can consider the associated holomorphic principal $\mathbb{C}$-bundle

$\Pi : X^* \longrightarrow S := X^*/\mathbb{C}$.

If $S$ is compact, we show that $X$ is $\mathbb{R}$-equivariantly biholomorphic to a model of type CH. Then Proposition 4.1 implies that the globalization $X^*$ is Stein. We need a preliminary result. Let the functions $\alpha$, $\beta$ be defined as in Lemma 2.7.

Lemma 4.5. If $S$ is compact then $\alpha$, respectively $\beta$, is either pluriharmonic or constantly equal to $-\infty$.

Proof. Assume that $\alpha$ is not constantly equal to $-\infty$. Since by (ii) of Lemma 2.7 one has $\alpha(\lambda \cdot x) = -\text{Im } (\lambda) + \alpha(x)$ for all $x \in X^*$ and $\lambda \in \mathbb{C}$, the real, positive $(1, 1)$-current $i\partial \bar{\partial} \alpha$ is $\mathbb{C}$-invariant. Therefore it pushes down to a $(1, 1)$-current $\theta$ on $S$ such that $\Pi^* (\theta) = i\partial \bar{\partial} \alpha$. Note that $\theta$ is also positive.

Recall that all cohomology groups with values in the sheaf of smooth functions on $S$ vanish. Thus $X^*$ is trivial as a differentiable principal $\mathbb{C}$-bundle and the maps induced by $\Pi$ in cohomology are isomorphisms. Since $i\partial \bar{\partial} \alpha$ is an exact current, this implies that $\theta$ is also an exact current. Then, from (ii) of Lemma 2.6 it follows that $\theta = 0$ and consequently $i\partial \bar{\partial} \alpha = \Pi^* (\theta) = 0$. Hence $\alpha$ is pluriharmonic. An analogous argument applies to show that if the function $\beta$ is not constantly equal to $-\infty$, then it is is pluriharmonic.

Proposition 4.6. Let $X$ be a 2-dimensional, taut, Stein manifold with a proper $\mathbb{R}$-action and assume that $S := X^*/\mathbb{C}$ is compact. Then $S$ is hyperbolic and $X$ is $\mathbb{R}$-equivariantly biholomorphic to a model of type CH. In particular $X^*$ is Stein.

Proof. First note that $S$ can not be biholomorphic to the Riemann sphere. Indeed $H^1(\mathbb{P}^1(\mathbb{C}), O) = 0$, thus if $S = \mathbb{P}^1(\mathbb{C})$ then $X^* = \mathbb{C} \times \mathbb{P}^1(\mathbb{C})$. Moreover the functions $\mathbb{P}^1(\mathbb{C}) \to \mathbb{R} \cup \{\infty\}$, defined by $p \to \alpha(0, p)$ and $p \to \beta(0, p)$, are constant, being subharmonic on $\mathbb{P}^1(\mathbb{C})$. Since (ii) of Lemma 2.7 implies that $X = \{(z, p) \in \mathbb{C} \times \mathbb{P}^1(\mathbb{C}) : \alpha(0, p) < \text{Im } z < -\beta(0, p)\}$, it follows that $X$ is the product of a strip in $\mathbb{C}$ and $\mathbb{P}^1(\mathbb{C})$. However $X$ is Stein, therefore this is impossible.
Now let us show that $S$ is hyperbolic. Consider the universal covering space $\pi: \tilde{X}^* \to X^*$ of $X^*$ with deck transformation group $\Gamma$. The proper $C$-action on $X^*$ lifts to a proper $C$-action on $\tilde{X}^*$, therefore $\tilde{X}^*$ is a principal $C$-bundle over $\tilde{S} \cong \tilde{X}^*/C$. Note that $X^*$ and $\tilde{X}^*$ are trivial as differentiable principal $C$-bundles over $S$ and $\tilde{S}$, respectively. This implies that the Riemann surface $\tilde{S}$ is simply connected, therefore it is non compact and consequently $\tilde{X}^*$ is $C$-equivariantly biholomorphic to $\mathbb{C} \times \tilde{S}$ (cf. Thm. 2.8). One has a commutative diagram of holomorphic maps

$$\tilde{X}^* = \mathbb{C} \times \tilde{S} \xrightarrow{\pi} X^* = \mathbb{C} \times \tilde{S}/\Gamma \xrightarrow{\tilde{\pi}} \tilde{S} = S = \tilde{S}/\Gamma,$$

where $\tilde{\pi}$ is the universal covering of $S$ with deck transformation group $\Gamma$.

By (iii) of Lemma 2.7, the sum $\alpha + \beta$ is $C$-invariant, thus it can be regarded as a subharmonic function on $S$. Since $S$ is compact, $\alpha + \beta$ is constant. Moreover polar sets have zero measure, therefore if $\alpha + \beta \equiv -\infty$ then either $\alpha \equiv -\infty$ or $\beta \equiv -\infty$. As a consequence $X = \{ x \in X^* : \alpha(x) < 0 \}$ or $X = \{ x \in X^* : \beta(x) < 0 \}$. On the other hand, if $\alpha + \beta = -C$ for some positive real number $C$, one has $X = \{ x \in X^* : \alpha(x) < 0 < -\beta(x) \} = \{ x \in X^* : -C < \alpha(x) < 0 \}$.

First consider the case when $X = \{ x \in X^* : \alpha(x) < 0 \}$. Set $\tilde{\alpha} = \alpha \circ \pi$ and let $\tilde{X} := \pi^{-1}(X) = \{(z, p) \in \mathbb{C} \times \tilde{S} : \tilde{\alpha}(z, p) < 0 \}$. Recall that $\alpha$ is pluriharmonic by Lemma 4.5, therefore so is $\tilde{\alpha}$. Since $\tilde{X}^*$ is simply connected, (iii) of Lemma 2.6 implies that there exists a holomorphic function $f : \tilde{X}^* \to \mathbb{C}$ such that $\text{Im}(f) = \tilde{\alpha}$. Moreover, for all $(z, p) \in \tilde{X}^*$ one has

$$\tilde{\alpha}(z, p) = \alpha \circ \pi(z \cdot (0, p)) = \alpha(z \cdot \pi(0, p)) = \tilde{\alpha}(0, p) - \text{Im} z,$$

therefore $f(z, p) = \alpha(0, p) = f(0, p) - z$. Then the map defined by

$$(z, p) \mapsto (-f(z, p), p) = (z - f(0, p), p)$$

gives a $C$-equivariant biholomorphism of $\mathbb{C} \times \tilde{S}$ and its restriction to $\tilde{X}$ defines an $\mathbb{R}$-equivariant biholomorphism onto $\{(z, p) \in \mathbb{C} \times \tilde{S} : 0 < \text{Im} z \}$, which is simply connected. Thus $\tilde{X}$ can be regarded as the universal covering of $X$ and since $\tilde{X}$ is taut by Proposition 2.5 this implies that $\tilde{S} \cong \Delta$, i.e. that $S$ is hyperbolic.

An analogous argument applies to the cases when $X = \{ x \in X^* : \beta(x) < 0 \}$ and $X = \{ x \in X^* : \beta(x) < 0 \}$, showing that $S$ is hyperbolic and that $\tilde{X}$ is $\mathbb{R}$-equivariantly biholomorphic to $H \times \Delta$, where $H$ is given by $\{0 < \text{Im} z\}$, $\{\text{Im} z < 0\}$ or $\{0 < \text{Im} z < C\}$, for some positive real $C$.

Identify the universal covering $\tilde{X}$ with $H \times \Delta$ and note that it is $\Gamma$-invariant in $X^* \cong \mathbb{C} \times \Delta$. In order to describe the $\Gamma$-action, observe that every $\gamma$ in $\Gamma$ is $\mathbb{C}$-equivariant, therefore there exists a holomorphic map $F_\gamma : \Delta \to \mathbb{C}$ such that

$$\gamma \cdot (z, w) = (z + F_\gamma(w), \gamma \cdot w),$$
for all \((z, w) \in \mathbb{C} \times \Delta\). Since \(\gamma(H \times \Delta) = H \times \Delta\), it follows that \(\text{Im} \, F_\gamma \equiv 0\) and consequently the holomorphic function \(F_\gamma\) is a real constant. Thus the \(\Gamma\)-action on \(\mathbb{C} \times \Delta\) is given by \(\gamma \cdot (z, w) = (z + \Psi(\gamma), \gamma \cdot w)\), where the group homomorphism \(\Psi : \Gamma \to \mathbb{R}\) is defined by \(\gamma \to F_\gamma\).

Finally note that \(\text{Ker} \, \Psi \neq \Gamma\). Otherwise one has \(X = H \times \Delta/\Gamma = H \times (\Delta/\Gamma) = H \times S\). Since \(X\) is Stein and \(S\) is compact, this gives a contradiction. Thus \(X\) is \(\mathbb{R}\)-equivariantly biholomorphic to a model of type CH and \(X^*\) is Stein by (i) of Proposition 4.1. \(\square\)

**Remark 4.7.** Note that two models of type CH, one of the form \(H \times \Delta/\Gamma\), with \(H\) of finite width, and one of the form \(H' \times \Delta/\Gamma'\), with \(H'\) of infinite width, cannot be biholomorphic. Let \(\Gamma\) act on \(H \times \Delta\) by \(\gamma \cdot (z, w) = (z + \Psi(\gamma), \gamma \cdot w)\) and let \(\Gamma'\) act on \(H' \times \Delta\) by \(\gamma' \cdot (z, w) = (z + \Psi'(\gamma'), \gamma' \cdot w)\), where \(\Psi : \Gamma \to \mathbb{R}\) and \(\Psi' : \Gamma' \to \mathbb{R}\) are non trivial homomorphisms. Recall that the elements of \(\Gamma\) and \(\Gamma'\) are all hyperbolic, i.e. they have two fixed point on the boundary of \(\Delta\) (see [FaKr], Cor. 2, p. 216).

In particular every element of \(\Gamma\) which does not belong to \(\text{Ker} \, \Psi\) has 4 fixed points on the boundary of the universal covering \(H \times \Delta\) of \(X\), while an element of \(\Gamma'\) has either infinite or 2 fixed points on the boundary of \(H' \times \Delta\). \(\square\)

### 5. MODELS WITH NON COMPACT BASE

Here we consider the models with base a non compact Riemann surface. Let us start with the hyperbolic case.

**Type NCH** Let \(S\) be a non compact hyperbolic Riemann surface. A model of type NCH with base \(S\) is given by
\[
\{(z, p) \in \mathbb{C} \times S : a(p) < \text{Im} \, z < -b(p)\},
\]
where \(a\) and \(b\) are subharmonic, continuous functions on \(S\) such that \(a + b < 0\) and \(\max\{a(p), b(p)\} > -\infty\) for all \(p \in S\).

**Type NCNH** A model of type NCNH with base \(S = \mathbb{C}\) or \(S = \mathbb{C}^*\) is given by
\[
\{(z, p) \in \mathbb{C} \times S : a(p) < \text{Im} \, z\} \quad \text{or} \quad \{(z, p) \in \mathbb{C} \times S : \text{Im} \, z < -b(p)\},
\]
with \(a, b\) subharmonic, non-harmonic, real valued, continuous functions on \(S\).

On each manifold as above let \(\mathbb{R}\) act by translations on the first factor.

**Proposition 5.1.** Let \(X\) be a model of type NCH or NCNH with base \(S\). Then (i) the universal globalization of \(X\) is \(\mathbb{C} \times S\), which is Stein,
(ii) \( X \) is a taut, Stein manifold with a proper \( \mathbb{R} \)-action.

**Proof.** (i) Note that \( X \) is orbit-connected in \( \mathbb{C} \times S \). Then Lemma 1.5 in \cite{CIT} implies that \( \mathbb{C} \times S \) is the (Stein) universal globalization of \( X \).

(ii) Since \( X \) is an \( \mathbb{R} \)-invariant submanifold in \( \mathbb{C} \times S \), the \( \mathbb{R} \)-action on \( X \) is proper. Moreover, \( X \) is given as the sublevel set of plurisubharmonic functions defined on the product \( \mathbb{C} \times S \), which is Stein. Thus \( X \) is Stein.

Finally we show that \( X \) is taut. For \( X \) a model of type NCH consider the projection \( \Pi|_X : X \rightarrow S \), \( (z,p) \rightarrow p \), onto the second factor. By Lemma 2.4 it is sufficient to prove that for every \( p \) in \( S \) there exists a neighborhood \( U \) of \( p \) in \( S \) such that \( (\Pi|_X)^{-1}(U) \) is taut. Since \( \max\{a(p),b(p)\} > -\infty \) we may assume that, e.g. \( a(p) > -\infty \). By continuity \( a > M \) on a neighbourhood \( U \) of \( p \), for some real constant \( M \). Then \( (\Pi|_X)^{-1}(U) \) is contained in \( H \times S \), with \( H = \{ z \in \mathbb{C} : M < \text{Im} \ z \} \). Moreover the inverse image \( (\Pi|_X)^{-1}(U) \) it is defined as a sublevel set of continuous plurisubharmonic functions, therefore it is taut by Corollary 2.3.

Assume now that \( X \) is a model of type NCNH. Note that if \( S = \mathbb{C}^* \), then the universal covering \( \tilde{X} \) of \( X \) is contained in \( \mathbb{C}^2 \) and it is also of type NCNH. Moreover, by Proposition 2.5 the manifold \( X \) is taut if and only if so is \( \tilde{X} \). Thus we may assume that \( S = \mathbb{C} \). Since a domain of the form \( \{ (z,w) \in \mathbb{C}^2 : \text{Im} \ z < -b(w) \} \) is biholomorphic to \( \Omega_b \) via the biholomorphism of \( \mathbb{C}^2 \) defined by \( (z,w) \rightarrow (-z,w) \), all such models are taut by Theorem 3.4. \( \square \)

**Remark 5.2.** Let \( F \) be a \( \mathbb{R} \)-equivariant biholomorphism between two models of type NCH defined by \( \{ (z,p) \in \mathbb{C} \times S : a(p) < \text{Im} \ z < -b(p) \} \) and \( \{ (z,p') \in \mathbb{C} \times S' : a'(p') < \text{Im} \ z < -b'(p') \} \). Then \( F(z,p) = (z + f(p),\varphi(p)) \) where \( f : S \rightarrow \mathbb{C} \) is holomorphic and \( \varphi : S \rightarrow S' \) is a biholomorphism such that \( a = (a' \circ \varphi - \text{Im} \ f) \) and \( b = (b' \circ \varphi + \text{Im} \ f) \). An analogous statement holds for models of type NCNH. \( \square \)

**Proposition 5.3.** Let \( X \) be a 2-dimensional, taut, Stein manifold with a proper \( \mathbb{R} \)-action and assume that the Riemann surface \( S := X^*/\mathbb{C} \) is non compact. Then \( X^* \) is \( \mathbb{C} \)-equivariantly biholomorphic to \( \mathbb{C} \times S \), which is Stein. Moreover, depending on hyperbolicity of \( S \) the manifold \( X \) is \( \mathbb{R} \)-equivariantly biholomorphic either to a model of type NCH or NCNH.

**Proof.** Since \( S \) is non compact by assumption, the principal \( \mathbb{C} \)-bundle \( X^* \) is trivial (cf. Thm. 2.8), implying the first statement. Regard \( X \) as \( \{ (z,p) \in \mathbb{C} \times S : \alpha(z,p) < 0 < -\beta(z,p) \} \) and define \( a(p) := \alpha(0,p) \) and \( b(p) := \beta(0,p) \). Since from (ii) of Lemma 2.7 it follows that \( \alpha(z,p) = -\text{Im} \ z + \alpha(0,p) \) and \( \beta(z,p) = \text{Im} \ z + \beta(0,p) \), one has

\[
X = \{ (z,p) \in \mathbb{C} \times S : a(p) < \text{Im} \ z < -b(p) \}.
\]
Moreover the same lemma implies that $a$ and $b$ are subharmonic, continuous functions, $a + b < 0$ and $\max\{a(p), b(p)\} > -\infty$ for all $p \in S$. This concludes the case when $S$ is hyperbolic.

For $S = \mathbb{C}$ or $S = \mathbb{C}^*$ we first note that $a + b$ is constant, being a subharmonic, negative function on $S$. We claim that $a + b \equiv -\infty$. Assume by contradiction that $a + b = -C$ for some positive $C$. Then $a = -b - C$ is harmonic and $X = \{(z, p) \in \mathbb{C} \times S : a(p) < \text{Im} \ z < a(p) + C\}$. In the case when $S = \mathbb{C}$, there exists a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that $\text{Im} \ f = a$. Then the map $\zeta \to (f(\zeta) + IC/2, \zeta)$ is a non constant holomorphic map from $\mathbb{C}$ into $X$. Since $X$ is taut, this gives a contradiction. If $S = \mathbb{C}^*$, one can show that $a + b \equiv -\infty$ by applying the analogous argument to the universal covering of $X$, which is taut by Proposition 2.5.

Thus $a + b \equiv -\infty$ and since the sets $\{a = -\infty\}$ and $\{b = -\infty\}$ have zero measure, either $a$ or $b$ are constantly equal to $-\infty$. Assume, e.g. that $b \equiv -\infty$. Since $X$ is taut, $a$ is necessarily real valued and the above argument also proves that $a$ can not be harmonic. Thus $X$ is $\mathbb{R}$-equivariantly biholomorphic to a model of type NCNH.

Corollary 5.4. Let $S$ be a non compact Riemann surface and consider the subdomain of $\mathbb{C} \times S$ defined by

$$\Omega := \{(z, p) \in \mathbb{C} \times S : a(p) < \text{Im} \ z < b(p)\},$$

where $a, b : S \to \{-\infty\} \cup \mathbb{R}$ are upper semicontinuous functions. Then $\Omega$ is taut if and only if it is a model of type NCH or NCNH.

Proof. First note that if $\Omega$ is taut, then it is Stein. For this consider the universal covering $Id \times \pi : \mathbb{C} \times \tilde{S} \to \mathbb{C} \times S$, where $\tilde{S} = \mathbb{C}$ or $\tilde{S} = \Delta$. Then Proposition 2.5 applies to show that the inverse image $(Id \times \pi)^{-1}(\Omega)$ is a taut domain of $\mathbb{C}^2$. Thus it is Stein by Thm. 5.4.1 in [Kob] and consequently it is locally Stein in $\mathbb{C} \times \tilde{S}$. It follows that $\Omega$ is locally Stein in $\mathbb{C} \times S$, which is Stein. Thus $\Omega$ is Stein by [DoGr].

Then an analogous argument as in the above proof applies to prove that $\Omega$ is a model of type NCH or NCNH.

6. Homotopy of the models

Let us summarize the main results of the previous sections as follows (see Prop. 4.1, 4.6, 5.1 and 5.3).
Theorem 6.1. Every model of type CH, NCH or NCNH is taut and Stein. Moreover a 2-dimensional, taut, Stein manifold with a proper $\mathbb{R}$-action is $\mathbb{R}$-equivariantly biholomorphic to one of them. In particular its universal globalization is Stein.

Here we show that in most cases, but not all of them, the type of a 2-dimensional, taut, Stein manifold with a proper $\mathbb{R}$-action is uniquely determined by its topology.

Proposition 6.2. Let $X$ be a 2-dimensional, taut, Stein manifold with a proper $\mathbb{R}$-action. Then $X$ is homotopically equivalent to $S$.

Proof. We first find a smooth global section of the restriction $\Pi|_X : X \to S$ of $\Pi$ to $X$. In a given smooth, trivialization $\mathbb{C} \times S$ of $X^*$ one has $X = \{(z, p) \in \mathbb{C} \times S : a(p) < \mathrm{Im}z < -b(p)\}$, with $a$ and $b$ continuous functions on $S$ (maybe no longer subharmonic) with values in $\{-\infty\} \cup \mathbb{R}$. Moreover, by continuity of $a$ and $b$ one can choose a locally finite covering $\{U_j\}$ of $S$ and real constants $M_j$ such that $a < M_j < -b$ on each $U_j$. Thus the constant functions $iM_j$ can be regarded as smooth local sections of $\Pi|_X$. Choose a smooth partition of unity $\{\psi_j\}$ subordinated to $\{U_j\}$. Then $\theta = \sum_j iM_j \psi_j$ defines a smooth global section of $\Pi|_X$, since $a(p) < \mathrm{Im} \theta < -b(p)$ on $S$.

Finally note that the map $X \times [0, 1] \to X$ defined by $((z, p), t) \to (z + t(\theta(p) - z), p)$ is a homotopy equivalence, showing that $S$ is a strong deformation retract of $X$. \hfill \Box

Corollary 6.3. Let $X$ be a 2-dimensional, taut Stein manifold with a proper $\mathbb{R}$-action. Then

(i) $X$ is $\mathbb{R}$-equivariantly biholomorphic to a model of type CH if and only if $H^2(X, \mathbb{Z}) \neq 0$;

(ii) if $H^2(X, \mathbb{Z}) = 0$ and $\pi_1(X)$ is neither trivial, nor isomorphic to $\mathbb{Z}$, then $X$ is $\mathbb{R}$-equivariantly biholomorphic to a model of type NCH,

(iii) if $\pi_1(X) = 0$ or $\pi_1(X) = \mathbb{Z}$, then $X$ is $\mathbb{R}$-equivariantly biholomorphic to a model of type NCH (type NCNH) if and only if it (does not) admits a non constant, bounded holomorphic $\mathbb{R}$-invariant function.

Proof. (i) and (ii) are direct consequences of the above proposition. For (iii) note that an $\mathbb{R}$-invariant holomorphic function on $X$ pushes down to a holomorphic function on $S$. Moreover, the assumption on the fundamental group implies that $S$ is biholomorphic to one of the following domains $\mathbb{C}$, $\mathbb{C}^*$, $\Delta$, $\Delta^*$ or an annulus. This implies the statement. \hfill \Box
Remark 6.4. Let $X$ be a taut, Stein surface such that either $\pi_1(X) = 0$ or $\pi_1(X) = \mathbb{Z}$. Then, for different proper $\mathbb{R}$-actions, the manifold $X$ may be $\mathbb{R}$-equivariantly biholomorphic to models of different types.

As an example consider the unbounded realization of the unit ball of $\mathbb{C}^2$ given by $X = \{(u, v) \in \mathbb{C}^2 : |v|^2 < \text{Im } u\}$ and the two different $\mathbb{R}$-actions on $X$ defined by $$t \circ (u, v) := (u + t, v), \quad t \ast (u, v) := (u - 2tv + it^2, v - it).$$

Such actions appear in [FaIa] as normal forms of parabolic elements in the automorphism group of $X$. It is clear that the globalization with respect to the first action is $\mathbb{C}^2$ and its $\mathbb{C}$-quotient is $\mathbb{C}$.

Note that the second $\mathbb{R}$-action extends to a $\mathbb{C}$-action on $\mathbb{C}^2$ and a simple computation shows that $\mathbb{C} \ast X = \{(u, v) \in \mathbb{C}^2 : \text{Im } u > (\text{Im } v)^2 - (\text{Re } v)^2\}$. Moreover, one checks that $X$ is orbit-connected in $\mathbb{C} \ast X$. Then Lemma 1.5 in [CIT] implies that $\mathbb{C} \ast X$ is the universal globalization with respect to the local $\mathbb{C}$-action on $X$ induced by the second $\mathbb{R}$-action. Let $\mathbb{H} = \{z \in \mathbb{C} : 0 < \text{Im } z\}$. One has a $\mathbb{C}$-equivariant biholomorphism $\Psi : \mathbb{C} \times \mathbb{H} \to \mathbb{C} \ast X$, $(\lambda, u) \to \lambda \ast (u, 0) = (u + i\lambda^2, -i\lambda)$.

Therefore the $\mathbb{C}$-quotient of $\mathbb{C} \ast X$ is biholomorphic to $\mathbb{H}$. This completes the example in the case when the fundamental group is trivial.

A similar example with fundamental group isomorphic to $\mathbb{Z}$ is as follows. Since $\Psi$ is a biholomorphism, $X' := \Psi^{-1}(X) = \{(u, v) \in \mathbb{C}^2 : \text{Im } v > 2(\text{Im } u)^2\}$ is also a model for the unit ball of $\mathbb{C}^2$. On this model the above actions look like $$t \circ (u, v) := (u, v + t), \quad t \ast (u, v) := (u + t, v).$$

Then the $\mathbb{Z}$-action on $X'$ defined by $n \cdot (u, v) := (u + n, v + n)$ commutes with both the $\mathbb{R}$-actions. Thus such $\mathbb{R}$-actions push down to proper $\mathbb{R}$-actions on the quotient $X'/\mathbb{Z}$, whose fundamental group is $\mathbb{Z}$. Observe that $X'/\mathbb{Z}$ is taut by Prop. 2.5 and it is Stein by [FaIa]

Finally note that the restrictions $X' \to \mathbb{C}$ and $X' \to \mathbb{H}$ to $X'$ of the projections of the associated holomorphic principal $\mathbb{C}$-bundles are $\mathbb{Z}$ equivariant. Thus they factorize to the restrictions $X'/\mathbb{Z} \to \mathbb{C}^*$ and $X'/\mathbb{Z} \to \Delta^* = \mathbb{H}/\mathbb{Z}$ to $X'/\mathbb{Z}$ of the projections of the holomorphic principal $\mathbb{C}$-bundles associated to the pushed down $\mathbb{R}$-actions on $X'/\mathbb{Z}$. Since the bases of this bundles are $\mathbb{C}^*$ and $\Delta^*$, this shows that also in this case the type of $X$ depends on the chosen $\mathbb{R}$-action. \qed
7. TAUT HARTOGS DOMAINS

As an application of the given classification, we give necessary and sufficient conditions for tautness of (non-complete) Hartogs domains over a non compact Riemann surface $S$. A complete Hartogs domain over $S$ is given by

$$\{(u, p) \in \mathbb{C} \times S : |u| < e^{-b(p)}\},$$

with $b : S \to \mathbb{R} \cup \{-\infty\}$ an upper semicontinuous function. A non-complete Hartogs domain over $S$ is given by

$$\{(u, p) \in \mathbb{C} \times S : e^{a(p)} < |u| < e^{-b(p)}\},$$

where $a, b : S \to \mathbb{R} \cup \{-\infty\}$ are upper semicontinuous functions with $a + b < 0$.

We wish to determine under which conditions on $a$ and $b$ such domains are taut. A result of Thai-Duc ([ThDu]), which applies in a more general context, implies that a complete Hartogs domain is taut if and only if $S$ is hyperbolic and $b$ is a real valued, subharmonic, continuous function. The following proposition gives a characterization of non-complete Hartogs domains (cf. [Par] for related results).

**Proposition 7.1.** Let $\Omega = \{(u, p) \in \mathbb{C} \times S : e^{a(p)} < |u| < e^{-b(p)}\}$ be a non-complete Hartogs domain over a non compact Riemann surface. Then $\Omega$ is taut if and only if either

(i) the Riemann surface $S$ is hyperbolic, the functions $a$, $b$ are continuous subharmonic and $\max(a(p), b(p)) > -\infty$ for every $p \in S$, or

(ii) the Riemann surface $S$ is not hyperbolic, $b \equiv -\infty$ (respectively $a \equiv -\infty$) and $a$ (respectively $b$) is a subharmonic, non-harmonic, real-valued, continuous function.

**Proof.** Consider the covering map $F : \mathbb{C} \times S \to \mathbb{C}^* \times S$ given by $(z, p) \to (e^{-iz}, p)$. Since the restriction of $F$ to $F^{-1}(\Omega)$ is a covering, from proposition 2.5 it follows that $F^{-1}(\Omega)$ is taut if and only if so is $\Omega$. Note that $\Omega$ is invariant under the $S^1$-action defined by $e^{i\theta} \cdot (u, p) := (e^{i\theta}u, p)$. As a consequence $\mathbb{R}$ acts properly on $F^{-1}(\Omega)$ by $t \cdot (z, p) := (z + t, p)$. Then Corollary 5.4 applies to show that $F^{-1}(\Omega)$ is taut if and only if it is a model of type NCH or NCNH, depending on hyperbolicity of $S$. This implies the statement. \qed
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Andrea Iannuzzi and Stefano Trapani: Dip. di Matematica, Università di Roma “Tor Vergata”, Vía della Ricerca Scientifica, I-00133 Roma, Italy.

E-mail address: iannuzzi@mat.uniroma2.it, trapani@mat.uniroma2.it