Closed form Solutions to Some Nonlinear equations by a Generalized Cole-Hopf Transformation

Mayer Humi
Department of Mathematical Sciences,
Worcester Polytechnic Institute,
100 Institute Road,
Worcester, MA 01609

Abstract

In the first part of this paper we linearize and solve the Van der Pol and Lienard equations with some additional nonlinear terms by the application of a generalized form of Cole-Hopf transformation. We then show that the same transformation can be used to linearize Painlevé III equation for certain combinations of its parameters. Finally we linearize new forms of Burger’s and related convective equations with higher order nonlinearities.
1 Background

While there is no existing general theory for integrating nonlinear ordinary and partial differential equations, the derivation of exact (closed form) solutions for these equations is a non-trivial and important problem. Some methods to this end are Lie, and related symmetry methods [8]. Many other ad-hoc methods for solving nonlinear differential equations were independently suggested during the last decades [18]. Among these methods Cole-Hopf transformation [9,10,11,12] has been used originally to linearize the Burger’s equation [13]

\[
\frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial x} = \nu \frac{\partial^2 \psi}{\partial x^2}, \quad \nu = \text{constant.} \tag{1.1}
\]

This equation contains a convective term and serves as a prototype for turbulence modeling, gas dynamics and traffic flow. Such equations with convective terms appear in various applications in applied mathematics and theoretical physics.

Using Cole-Hopf transformation [9,10]

\[
\psi = \frac{\partial \phi}{\partial x}. \tag{1.2}
\]

it was found that (1.1) can be linearized and reduced to the linear Heat equation

\[
\frac{\partial \phi}{\partial t} = k \frac{\partial^2 \phi}{\partial x^2}. \tag{1.3}
\]

Since this discovery many attempts were made in the literature to generalize this result to larger class of equations or to relate other nonlinear equations to this equation[11,12].

Another important nonlinear equation is the Van der Pol equation [1] which without forcing is

\[
\psi(x)'' = \mu(\beta - \psi(x)^2)\psi(x)' - \alpha \psi(x), \tag{1.4}
\]

where \(\alpha\) and \(\beta\) and \(\mu\) are arbitrary constants. This equation was derived originally to model electrical circuits in vacuum tubes. However since then it has been used to model phenomena in both the physical [2-6] and biological sciences [7]. The solutions of this equation were investigated extensively (both numerically and analytically) in the literature [2-6,16,17].

Recently however, we introduced [14] a generalization of Cole-Hopf transformation and used it to linearize and solve various nonlinear ordinary differential equations e.g Duffing
equation. We showed also that many of the special functions of mathematical physics are exact solutions for a class of nonlinear equations. In this paper we apply in Sec. 2 this generalization of Cole-Hopf transformation to linearize the Van der Pol equation with additional quadratic and cubic nonlinear terms. These terms can be considered as perturbations to the original equation. Next in Sec. 3 we present explicit solutions (in closed analytic form) to this equation and related (polynomial) Lienard equations with and without external forcing.

In Sec 4 we consider Painleve III equation and show that it can be linearized by the same transformation for certain combinations of its parameters.

In Sec. 5 we consider a class of Burger’s equations with additional quadratic nonlinear terms which can be solved by the same transformation. Next in Sec. 6 we consider second nonlinear ordinary differential equations (ODEs) with convective term and derive conditions under which they can be linearized. These type of equations represent steady state convection in one dimension [15].We end up in Sec. 7 with a summary.

We wish to point out that we do not include in this paper direct physical applications to the solutions of the equations treated in this paper. However there exist an extensive research literature devoted to these solutions and their applications. Accordingly the method presented in this paper is generic to the class of methods in mathematical physics.

2 The Perturbed Van der Pol Equation

In this section we consider Van der Pol equation (1.4) with additional nonlinear and forcing terms

$$\psi(x)'' = \mu(\beta - \psi(x)^2)\psi(x)' - \alpha\psi(x) + v(x)\psi(x)^2 + h(x)\psi(x)^3 + g(x)\psi(x)^4 + f(x). \quad (2.1)$$

We shall say that the solutions of (2.1) and

$$\phi(x)'' = U(x)\phi(x) \quad (2.2)$$

are related by a generalized Cole-Hopf transformation if we can find a function $P(x)$ so that

$$\psi(x) = P(x) + \frac{\phi(x)'}{\phi(x)}. \quad (2.3)$$
To classify those nonlinear equations of the form (2.1) whose solutions can be obtained from those of the linear equation (2.2) we differentiate (2.3) twice and in each step replace the second order derivative of $\phi(x)$ using (2.2). Substituting these results in (2.1) leads to the following equation

$$a_4(x)\left(\frac{\phi(x)'}{\phi(x)}\right)^4 + a_3(x)\left(\frac{\phi(x)'}{\phi(x)}\right)^3 + a_2(x)\left(\frac{\phi(x)'}{\phi(x)}\right)^2 + a_1(x)\frac{\phi(x)'}{\phi(x)} + a_0(x) = 0 \quad (2.4)$$

where

$$a_4(x) = -\mu - g(x) \quad (2.5)$$
$$a_3(x) = -[2\mu + 4g(x)]P(x) - h(x) + 2 \quad (2.6)$$
$$a_2(x) = \mu P(x)' - (6g(x) + \mu)P(x)^2 - 3h(x)P(x) - v(x) + \mu U(x) + \mu \beta \quad (2.7)$$
$$a_1(x) = -4g(x)P(x)^3 - 3h(x)P(x)^2 + [2\mu P(x)' - 2v(x) + 2\mu U(x)]P(x) - 2U(x) + \alpha \quad (2.8)$$

$$a_0(x) = P(x)'' + \mu [P(x)^2 - \beta]P(x)' + U(x)' - g(x)P(x)^4 - h(x)P(x)^3 + [\mu U(x) - v(x)]P(x)^2 + \alpha P(x) - \mu \beta U(x) - f(x) \quad (2.9)$$

To satisfy (2.4) it is sufficient to let $a_i(x) = 0$, $i = 1, 2, 3, 4$. From $a_4 = 0$ we obtain

$$g(x) = -\mu. \quad (2.10)$$

Using this result and $a_3(x) = 0$ we solve for $h(x)$ (in terms of $P(x)$)

$$h(x) = 2(\mu P(x) + 1). \quad (2.11)$$

The condition $a_2(x) = 0$ can be solved then for $v(x)$

$$v(x) = \mu P(x)' + \mu U(x) - [\mu P(x) + 6]P(x) + \mu \beta. \quad (2.12)$$

substituting (2.10)- (2.12) in $a_1 = 0$ we solve for $U(x)$ in terms of $P(x)$

$$U(x) = 3P(x)^2 - \mu \beta P(x) + \frac{\alpha}{2} \quad (2.13)$$

Finally using the expressions derived in (2.10)- (2.13) in $a_0 = 0$ we have

$$f(x) = P(x)'' - 2\mu \beta P(x)' + [6P(x)' + \alpha + \mu^2 \beta^2]P(x) + 4[P(x) - \mu \beta]P(x)^2 - \frac{\mu \beta \alpha}{2} \quad (2.14)$$
This equation can be solved (in principle) for $P(x)$ for a given $f(x)$ or it can be used to determine $f(x)$ for apriori choice of $P(x)$. Eqs. (2.11)-(2.13) can be solved then (in reverse order) to compute the functions $U(x), v(x)$ and $h(x)$.

Using this algorithm we provide in the next section explicit solutions to (2.1) under various conditions.

### 3 Solutions to the Van der Pol Equation

#### 3.1 Van der Pol Equation with No External Forcing

When $f(x) = 0$ the general solution of (2.14) for $P(x)$ is

$$P(x) = \frac{2C_1\mu\beta e^{\mu\beta x} + C_2(\mu\beta + k)e^{kx} + (\mu\beta - k)e^{-kx}}{4[C_1e^{\mu\beta x} + C_2e^{kx} + e^{-kx}]} \quad (3.1)$$

where $k^2 = \mu^2\beta^2 - 4\alpha$ and $C_1, C_2$ are arbitrary constants. Similar (but more cumbersome) expressions can be obtained for $P(x)$ when $f(x) = c$ where $c$ is a non zero constant.

It is now only a matter of simple algebra to compute the general form of the functions $U(x), v(x)$ and $h(x)$ and then derive solutions to (2.1) by solving the linear equation (2.2).

We consider some simple cases.

**Case 1: $C_1 = C_2 = 0$.** In this case (3.1) reduces to

$$P(x) = \frac{\mu\beta - k}{4}. \quad (3.2)$$

The resulting expressions for $U(x), v(x)$ and $h(x)$ are

$$U(x) = \frac{\alpha}{2} + \frac{(3k + \mu\beta)(k - \mu\beta)}{16} \quad (3.3)$$

$$v(x) = -\frac{\mu^3\beta^2}{8} + \frac{\mu}{2} \left( \alpha - \beta + \frac{k^2}{4} \right) + \frac{3k}{2} \quad (3.4)$$

$$h(x) = \frac{\mu}{2}(\mu\beta - k) + 2. \quad (3.5)$$

With these settings the general solution of (2.2) is

$$\phi(x) = C_3 \cos(\omega x) + C_4 \sin(\omega x) \quad (3.6)$$
where \( \omega = \frac{1}{4} \sqrt{\mu^2 \beta^2 + 2 \mu \beta k - 3k^2 - 8\alpha} \). These solutions are related to the solutions of (2.1) by the transformation (2.3).

**Case 2:** Assume \( k = 0 \) and \( C_1 = 0 \).

In this case

\[
P(x) = \frac{\mu \beta}{4}, \quad U(x) = \frac{\alpha}{2} - \frac{\mu^2 \beta^2}{16}, \quad h(x) = \frac{\mu^2 \beta}{2} + 2, \quad v(x) = \frac{\mu}{2} \left( \alpha - \beta - \frac{\mu^2 \beta^2}{4} \right)
\]

(3.7)

The solution for \( \phi(x) \) is in the same form as in (3.6) with \( \omega = \frac{1}{4} \sqrt{\mu^2 \beta^2 - 8\alpha} \).

**Case 3:** Assume \( \alpha = 0 \).

When \( \alpha = 0 \), \( k = \pm \mu \beta \). (In the following we consider only the plus sign). Under this constraint (3.1) reduces to

\[
P(x) = \frac{(C_1 + C_2) \mu \beta e^{\frac{\mu \beta x}{2}}}{(C_1 + C_2 + e^{-\mu \beta x})}
\]

(3.8)

Using (2.13) yields

\[
U(x) = -\frac{c \mu^2 \beta^2 (e^{-\mu \beta x} - c)}{e^{-\mu \beta x} + c^2}
\]

(3.9)

where \( c = C_1 + C_2 \) Similarly (2.12) and (2.13) lead to

\[
v(x) = \frac{\mu \beta (e^{-\mu \beta x} - 2c)}{e^{-\mu \beta x} + c}, \quad h(x) = 2 + \frac{\mu^2 \beta c}{e^{-\mu \beta x} + c}
\]

(3.10)

With this data the general solution for (2.2) is

\[
\phi(x) = \frac{C_3 + C_4 (ce^{\mu \beta x} + \mu \beta x)}{\sqrt{1 + ce^{\mu \beta x}}}
\]

(3.11)

### 3.2 Van der Pol Equation with External forcing

When the forcing function \( f(x) \) is not zero or a constant (2.14) can not be solved in closed form for \( P(x) \). It is expedient in this case to reverse the process and classify those equations of the form (2.1) which can be linearized for a given \( P(x) \).

To make a proper choice of \( P(x) \) we observe that in order to obtain closed form solutions for \( \psi(x) \) the function \( U(x) \) has to be in a form which enables the solution of (2.2) to be expressed in "simple form".

To carry this out it is easy to see from (2.13) that when

\[
U(x) = 3g(x)^2 - \mu \beta g(x) + \frac{\alpha}{2}
\]
where g(x) is an arbitrary smooth function) then

\[ P(x) = g(x), \text{ or } P(x) = -g(x) + \frac{\mu \beta}{3} \]

The computation of the functions \( f(x), v(x) \) and \( h(x) \) can be carried out then by substitution.

We present some examples.

**Example 1:** Let \( g(x) = ax \) where \( a \) is a constant. We then have

\[ U(x) = 3a^2 x^2 + a \mu \beta x - \frac{\alpha}{2}, \]
\[ f(x) = -4a^3 x^3 + a \left( 6a - \alpha + \frac{\mu^2 \beta^2}{3} \right) x - \frac{\mu \beta \alpha}{6} + \frac{\mu^3 \beta^3}{27} \]

The general solution for \( \phi(x) \) can be expressed then in terms of Hypergeometric functions.

**Example 2:** Let \( g(x) = \tan(x) \). In this case we obtain an oscillatory forcing function and the solutions of (2.2) for \( \phi(x) \) can be expressed again in terms of Hypergeometric functions.

### 3.3 Polynomial Lienard Equations

The general form of Lienard equations is

\[
\frac{d^2 \psi}{dt^2} + f(\psi) \frac{d\psi}{dt} + g(\psi) = 0 \quad (3.12)
\]

where \( f(\psi) \) and \( g(\psi) \) are smooth functions.

Since Van der Pol equation with no external forcing belongs to this class of equations it is appropriate to ask if the same method used to solve (2.1) can be applied to this more general class of equations. In this section we explore this application to (3.12) when \( f(x) \) and \( g(x) \) are respectively a second and fourth order polynomials

\[ f(x) = \sum_{k=0}^{2} c_k \psi(x)^k, \quad g(\psi) = \sum_{k=0}^{4} b_k \psi(x)^k \]

Applying the transformation (2.3) with \( \phi(x) \) satisfying (2.2) to (3.12) leads to an equation similar to (2.4), which can be satisfied if we set \( a_i = 0 \) for \( i = 0, \ldots, 4 \). We solve this set of equations for \( b_i \) in terms of \( c_i, P(x) \) and \( U(x) \). We obtain the following relations:

\[ b_4(x) = c_2(x), \quad b_3(x) = c_1(x) - 2c_2(x) P(x) + 2, \quad (3.13) \]
\[ b_2(x) = c_2(x)P(x)^2 - 2(3 - c_1(x))P(x) - c_2(x)(P(x)' - U(x)) + c_0(x) \]  
\[ b_1(x) = (6 + c_1(x))P(x)^2 - 2c_0(x)P(x) - c_1(x)P(x)' - (c_1(x) + 2)U(x), \]  
\[ b_0(x) = P(x)'' - c_0(x)P(x)' + U(x)' - 2P(x)^3 + 2P(x)U(x) + c_0(x)(P(x)^2 - U(x)) \]  

For apriori choice of the functions \( c_i(x) \) and \( U(x) \), these equations can be solved for the \( b_i(x) \). When these relations hold we obtain an equation of the form (3.12) whose solutions are related to those of (2.2) by the transformation (2.5).

An interesting case arises when \( b_0 = 0 \). Under this condition (3.16) will be satisfied if
\[ U(x) = P(x)^2 - P(x)' \]
which is a Riccati equation for \( P(x) \) and \( c_0(x) \) remains a free parameter.

### 4 Painleve III Equation

The general form of Painleve III equation is
\[ \psi'' = \frac{(\psi')^2}{\psi} - \frac{\psi'}{x} + \frac{\alpha \psi^2 + \beta}{x} + \frac{\gamma \psi^3 + \delta}{\psi} \]
where \( \alpha, \beta, \gamma, \delta \) are parameters.

We shall say that we can linearize this equation (for certain values of its parameters) if we can find functions \( P(x), Q(x) \) so that the functions \( \psi(x), \phi(x) \) are related by eq. (2.3) and \( \phi(x) \) satisfies the linear equation
\[ \phi(x)'' = K(x)\phi(x)' + U(x)\phi(x). \]
Following the same steps as in Sec 2 we obtain an equation of the form (2.4) where
\[ a_4(x) = Q(x)^2 - \gamma Q(x)^4 \]
\[ a_3(x) = \frac{Q(x)}{x} [2P(x)x + \alpha Q(x)^2 \alpha + Q(x)K(x)x + Q(x)] \]
\[ a_2(x) = \frac{1}{x} [2xQ(x)P(x)' + Q(x)Q(x)' - 3\alpha P(x)Q(x)^2 - x(Q(x))'^2 - 3P(x)Q(x)K(x) - P(x)Q(x) + xQ(x)Q(x)'' + xQ(x)^2 K(x) - 6xP(x)^2 + Q(x)^2 K(x) - 2xP(x)Q(x)'] \]
\[ a_1(x) = \frac{1}{xQ(x)} \{ [-U(x) - xU'(x) + xK(x)U(x)]Q(x)^3 + [-K(x)P(x) - xP(x)K(x)'] - xP(x)K(x)^2 + 2xP(x)'K(x) + \beta - P(x)' - xP(x)'' + 3\alpha P(x)^2 + 2xP(x)U(x)]Q(x)^2 + [-xP(x)Q(x)'' - P(x)Q(x)' + 2xP(x)Q(x)'] - 2xP(x)K(x)Q(x)']Q(x) + 4xP(x)^3 \} \]

\[ a_0(x) = - \frac{P(x)^3(xP(x) + \alpha Q(x)^2)}{xQ(x)^2} + \]  
\[ \{ P(x)'' - \frac{\beta}{x} + [K(x)U(x) + U(x)' + \frac{U(x)}{x}]Q(x) + 2Q(x)'U(x) + \frac{P(x)'}{x}P(x) - Q(x)^2U(x)^2 - 2P(x)'Q(x)U(x) - (P(x)')^2 - \delta \]

To satisfy (4.4) it is sufficient to let \[ a_i(x) = 0, \quad i = 1, 2, 3, 4. \] From \[ a_4 = 0 \] it follows that \[ Q(x)^{-2} = \gamma \] i.e. \[ Q(x) \] a constant. (Hence, in the following, we refer to it as \[ Q \]). Using this result and solving (4.4) for \[ K(x) \] we have

\[ K(x) = - \frac{2xP(x) + \alpha Q^2 + Q}{xQ} \]

Inserting these results in (4.3) we find that it being satisfied automatically (no further restrictions). Eq (4.6) reduces then to a first order linear equation for \[ U(x) \] in terms of \[ P(x) \] whose general solution is

\[ U(x) = - \frac{P(x)'}{Q} - \frac{P(x)^2}{Q^2} - \frac{\alpha Q + 1}{xQ} P(x) + \frac{\beta}{Q(\alpha Q + 2)} + Cx^{-(\alpha Q + 2)} \]

where \[ C \] is an arbitrary constant. Substituting all these results in (4.7) we find that it reduces to

\[ \delta + \frac{\beta^2}{(\alpha Q + 2)^2} + CQ[\frac{2\beta}{(\alpha Q + 2)xQ + 2}] + \frac{CQ}{x^{2\alpha Q + 4}} = 0 \]

(viz. the coefficient of \[ P(x) \] is zero and therefore \[ P(x) \] remains as a free parameter function). Eq. (4.10) is satisfied if we let \[ C = 0 \] and

\[ \delta = - \frac{\beta^2}{(\alpha Q + 2)^2} \]

When (4.11) is satisfied Painleve III equation can be linearized. We can summarize these results by the following lemma:
Lemma Let the parameters in eq. (4.1) satisfy eq. (4.11) (with $Q^2 = 1/\gamma$). For any (smooth) choice of the function $P(x)$ the solutions of eq. (4.2) where $K(x), U(x)$ are given by eqs. (4.8), (4.9), provide solutions to eq. (4.1) through eq. (2.3).

**Example 1**: If we let the parameters in eq. (4.1) be chosen as

$$\alpha = -1, \beta = 1, \gamma = 1 (Q = 1) \text{ and } \delta = -1$$

then eq. (4.11) is satisfied. For

$$P(x) = ax + bx^2 + cx^3 + d$$

we have

$$K(x) = -2P(x), \ U(x) = -P(x)^2 - P(x)\prime + 1.$$

The general solution for $\phi$ is

$$\phi(x) = C_1 \exp\left\{-\frac{x}{2} \left[\frac{cx^3}{2} + \frac{2bx^2}{3} + ax + 2d - 2\right]\right\} + C_2 \exp\left\{-\frac{x}{2} \left[\frac{cx^3}{2} + \frac{2bx^2}{3} + ax + 2d + 2\right]\right\}$$

The solution $\psi(x)$ to eq. (4.1) is given then by eq. (2.2) for any values of the integration constants $C_1, C_2$ and this can verified by direct substitution. (This demonstrates also that the superposition principle holds for these solutions).

**Example 2**: With the same parameters for eq. (4.1) as in example 1 we let

$$P(x) = \sin x$$

Computing the corresponding expressions for $K(x)$ and $U(x)$ we find that the solution of eq. (4.2) is

$$\phi(x) = C_1 e^{\cos x} \sinh x + C_2 e^{\cos x} \cosh x.$$

Once again the solution $\psi(x)$ to eq. (4.1) is given then by eq. (2.2).

**Example 3**: With the same parameters for eq. (4.1) as in example 1 we let

$$P(x) = xe^x$$

The solution for $\phi(x)$ is

$$\phi(x) = C_1 \exp((1 - x)e^x) \sinh x + C_2 \exp((1 - x)e^x) \cosh x$$

and $\psi(x)$ is given by (2.2).
5 Generalized Burger Equation

In this section we consider equations of the form

\[
\frac{\partial \psi}{\partial t} - M(x) \frac{\partial^2 \psi}{\partial x^2} = H(x)\psi \frac{\partial \psi}{\partial x} + V(x)\psi + W(x)\psi^2
\]  

(5.1)

where \( \psi = \psi(x,t) \). In this case we shall say that the solutions of this equation are related to the solution of the linear equation

\[
\frac{\partial \phi}{\partial t} - M(x) \frac{\partial^2 \phi}{\partial x^2} = 0
\]  

(5.2)

if we can find functions \( P(x), Q(x) \) so that

\[
\psi = P(x) + Q(x) \frac{\partial \phi}{\partial x}
\]  

(5.3)

To classify those equations of the form (5.1) which can be paired to the linear equation (5.2) we substitute (5.3) in (5.1). After some algebra we find that the following equation has to be satisfied

\[
\begin{align*}
a_3 \left( \frac{\partial \phi}{\partial x} \right)^2 + a_2 \left( \frac{\partial \phi}{\partial x} \right)^2 + a_{21} \frac{\partial \phi}{\partial x} + a_1 \frac{\partial \phi}{\partial \phi(x)} + a_{11} + a_0 &= 0, \\
\end{align*}
\]  

(5.4)

where

\[
\begin{align*}
a_{21} &= -Q(x) \left[ (Q(x)H(x) - 3M(x)) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial t} \right], \\
a_{11} &= -(Q(x)H(x)P(x) + 2M(x)Q(x)) \frac{\partial^2 \phi}{\partial x^2} + Q(x) \frac{\partial^2 \phi}{\partial \phi} - M(x)Q(x) \frac{\partial^3 \phi}{\partial x^2}, \\
a_3 &= 2M(x) - Q(x)H(x), \\
a_2 &= H(x)Q(x)P(x) + (2M(x) - H(x)Q(x))Q(x)' - W(x)Q(x)^2, \\
a_1 &= -H(x)Q(x)P(x)' - H(x)P(x)Q(x)' - M(x)Q(x)'' - V(x)Q(x) - 2W(x)P(x)Q(x), \\
a_0 &= -M(x)P(x)'' - H(x)P(x)P(x)' - W(x)P(x)^2 - V(x)P(x),
\end{align*}
\]  

(5.5-5.10)

where primes denote differentiation with respect to \( x \). To satisfy (5.4) it is sufficient to let \( a_{21} = a_{11} = 0 \) and \( a_3 = a_2 = a_1 = a_0 = 0 \).

From \( a_3 = 0 \) it follows that

\[
Q(x) = \frac{2M(x)}{H(x)}
\]  

(5.11)
In view of this relationship $a_{21} = 0$ is satisfied in virtue of (5.2). To satisfy $a_{11} = 0$ we add and subtract $M(x)\frac{\partial^2 \phi}{\partial x^2}$ and rewrite this equation in the form

$$a_{11} = -(Q(x)H(x)P(x)+2M(x)Q(x)'-Q(x)M(x)')\frac{\partial^2 \phi}{\partial x^2}+Q(x)\frac{\partial \phi}{\partial t}\frac{\partial}{\partial x} - Q(x)\frac{\partial}{\partial x} \left( M\frac{\partial \phi}{\partial x^2} + Q(x) \right) = 0.$$  
(5.12)

Letting

$$P(x) = \frac{2M(x)H(x)'}{H(x)^2} - \frac{M(x)'}{H(x)}$$  
(5.13)

we infer that (5.12) is satisfied using (5.2).

We now use $a_2 = a_1 = 0$ to express $W(x)$ and $V(x)$ in terms of $M(x)$, $H(x)$. Substituting the expressions for $Q(x)$ and $P(x)$ in $a_{22} = 0$ we obtain

$$W(x) = -\frac{H(x)M(x)'}{2M(x)} + H(x)'$$  
(5.14)

Similarly $a_1 = 0$ yields

$$V(x) = -\frac{M(x)H(x)'}{H(x)}$$  
(5.15)

With these results $a_0 = 0$ yield a differential equation relating $M(x)$ and $H(x)$

$$\left[ \frac{H(x)M(x)'}{2M(x)} - H(x)' \right] P(x)^2 + \left[ \frac{M(x)H(x)''}{H(x)} - H(x)P(x)' \right] P(x) - M(x)P(x)'' = 0$$  
(5.16)

where for brevity we did not substitute for $P(x)$.

When $M(x) = A$ where $A$ is a constant (5.16) becomes

$$H(x)^2H(x)''' - 5H(x)H(x)'H(x)'' + 4(H(x)')^3 = 0.$$  
(5.17)

To solve this equation we introduce $z(x) = \frac{1}{H(x)}$. The resulting equation can be written as

$$\frac{d}{dx} \left( \frac{1}{z(x)} \frac{d^2 z(x)}{dx^2} \right) = 0.$$  
(5.18)

Hence either

$$H(x) = \frac{1}{ax+b}$$  
(5.19)

where $a, b$ are constants or

$$H(x) = \frac{B}{\cos(\omega x + \beta)}, \quad B, \omega, \beta, \text{ constants},$$  
(5.20)
or

\[ H(x) = Ce^{\alpha x}, \ C, \alpha, \ constants, \] \hspace{1cm} (5.21)

When \( H(x) = 1 \) eq. (5.16) becomes

\[ \frac{M(x)'}{2M(x)} - M(x)'M(x)'' + M(x)M(x)''' = 0. \] \hspace{1cm} (5.22)

Substituting \( M(x) = w(x)^2 \) this equation reduces to

\[ \frac{d}{dx} \left( w(x) \frac{d^2 w}{dx^2} \right) = 0. \] \hspace{1cm} (5.23)

Hence

\[ w(x) \frac{d^2 w}{dx^2} = c \]

where \( c \) is a constant. When \( c = 0 \) it follows that

\[ M(x) = (a_1 x + b_1)^2 \] \hspace{1cm} (5.24)

where \( a_1, b_1 \) are constants. When \( c \neq 0 \) we can find an implicit expression \( w(x) \)

\[ x = C_2 \pm \int_{0}^{w(x)} \frac{ds}{\sqrt{2c \ln(s) - C_1 c}}, \]

where \( C_1, C_2 \) are integration constants.

To our best knowledge the linearization of the ”modified Burger’s equations” represented
by (5.19)-(5.21) and (5.24) did not appear in the literature so far.

Example: For \( M(x) = 1, H(x) = Ce^{\alpha x} \) we obtain from (5.14) (5.15) respectively that

\[ W(x) = C\alpha e^{\alpha x}, \ V(x) = -\alpha^2. \]

Hence the solutions of (5.1) with these coefficients is related to the solutions of the Heat
equation (5.2) by the transformation (5.3) with

\[ Q(x) = \frac{2e^{-\alpha x}}{C}, \ P(x) = \frac{2\alpha e^{-\alpha x}}{C} \]

and this fact can be verified by direct substitution.
6 Second Order Convective ODEs

We shall say that the solutions of the equations

$$\psi''(x) = S(x) + [V(x) + F(x)\psi(x)']\psi(x) + W(x)\psi(x)^2$$  \hspace{1cm} (6.1)

and

$$\phi''(x) = U(x)\phi(x)$$  \hspace{1cm} (6.2)

are related if we can find functions $P(x)$ and $Q(x)$ so that

$$\psi(x) = P(x) + Q(x)\frac{\phi(x)'}{\phi(x)}.$$  \hspace{1cm} (6.3)

Furthermore we observe that (6.1) can take the more general form

$$\psi''(x) = S(x) + (V(x) + F(x)\psi(x)')\psi(x) + V_1(x)\psi(x)' + W(x)\psi(x)^2.$$  \hspace{1cm} (6.4)

In this case we can find $p(x)$ so that $V_1(x) = -2\frac{p(x)'}{p(x)}$. Introducing $\xi(x) = p(x)\psi(x)$, (6.4) becomes

$$\xi''(x) = p(x)S(x) + \left[ V(x) + \frac{p(x)''}{p(x)} + \frac{F(x)}{p(x)}\xi(x)' \right] \xi(x) + \left[ \frac{W(x)}{p(x)} - \frac{F(x)p(x)'}{p(x)^2} \right] \xi(x)^2.$$  \hspace{1cm} (6.5)

which has the same form as (6.1).

To classify those nonlinear equations (6.1) which can be "paired" with a linear equation of the form (6.2) we follow the same steps as in the previous section and find that the following equation must hold;

$$a_3(x) \left( \frac{\phi(x)'}{\phi(x)} \right)^3 + a_2(x) \left( \frac{\phi(x)'}{\phi(x)} \right)^2 + a_1(x) \frac{\phi(x)'}{\phi(x)} + a_0(x) = 0$$  \hspace{1cm} (6.6)

where

$$a_3(x) = (Q(x)F(x)+2), \hspace{0.5cm} a_2(x) = Q(x)(F(x)P(x)-W(x)Q(x))-(2+F(x)Q(x))Q(x)', \hspace{0.5cm} (6.7)$$

$$a_1(x) = Q(x)'' - F(x)P(x)Q(x)' - U(x)F(x)Q(x)^2 - (2U(x) + V(x) + F(x)P(x)' + 2W(x)P(x))Q(x),$$  \hspace{1cm} (6.8)
\[ a_0(x) = P(x)'' - W(x)P(x)^2 - (F(x)Q(x)U(x) + V(x) + F(x)P(x)')P(x) + Q(x)U(x) + 2U(x)Q(x)' - S(x). \] (6.9)

To satisfy (6.6) it is therefore sufficient to let \( a_i(x) = 0, \ i = 0, 1, 2, 3. \)

As in previous sections one can use these conditions in two ways. The first is to assume that a nonlinear equation (6.1) is given and try to determine the appropriate \( P, Q, U \) (if they exist) that relates it to (6.3). Otherwise one may fix the functions \( P, Q, U \) and classify those nonlinear equations of the form (6.1) which are related to (6.3) by the transformation (6.2).

In the following we provide separate solutions for these two possibilities.

Assuming that one starts from (6.1) i.e the functions \( V(x), W(x), F(x) \) and \( S(x) \) (with \( F(x) \neq 0 \)) are given it follows from (6.6) that

\[ Q(x) = -\frac{2}{F(x)}. \] (6.10)

Substituting this result in (6.7) and solving for \( P(x) \) it follows that

\[ P(x) = -\frac{2W(x)}{F(x)^2}. \] (6.11)

Using (6.10), (6.11) and (6.9) we obtain a linear first order differential equation for \( U(x) \)

\[
U(x)'' + \frac{F(x)S(x)}{2} + \frac{W(x)'' - V(x)W(x) + (2W(x) - 2F(x)')U(x)}{F(x)}
\]

\[
\frac{1}{2}(2W(x) - 4F(x)')W(x)' - 2W(x)F(x)'' + \frac{2W(x)(W(x)^2 + 3(F(x))' - 2W(x)F(x)'^2 - 4W(x)^2}{F(x)^2} = 0.
\] (6.12)

Finally eq. (6.8) provides (after using (6.10) and (6.11)) an intrinsic constraint on the functions \( V(x), W(x), F(x) \) and \( S(x) \) which have to be satisfied for the relationship between (6.1) and (6.3) to exist.

\[ V(x) + \frac{F(x)'' - 2W(x)' - 6W(x)F(x)' - 2(F(x)')^2 - 4W^2}{F(x)^2} = 0. \] (6.13)

We observe that when \( F(x) = 0 \) the algorithm can be implemented in the same way by adding a term \( R(x)\psi(x)^3 \) to eq. (6.1).

For the reverse procedure where one elects the functions \( U(x), P(x) \) and \( Q(x) \) and attempts to evaluate the corresponding nonlinear equation (6.1). We have from (6.9)

\[ F(x) = -\frac{2}{Q(x)}, \quad W(x) = -\frac{2P(x)}{Q(x)^2}. \] (6.14)
Substituting these results in (6.9) and solving for \( V(x) \) yields

\[
V(x) = \frac{Q(x)Q'' + 4P(x)^2 + 2(P(x)Q(x))'}{Q(x)^2}. \tag{6.15}
\]

Finally from (6.10) we derive an expression for \( S(x) \)

\[
S(x) = P(x)'' + Q(x)U(x)' + 2(Q(x)' + P(x))U(x) - \frac{P(x)Q(x)''}{Q(x)}(x) - \frac{2P(x)^2(Q(x)' + P(x))}{Q(x)^2}. \tag{6.16}
\]

**Example:** In the differential equation

\[
\psi'' = [4a^2 + \psi(x)']\psi(x) + a\psi(x)^2, \tag{6.17}
\]

where \( a \) is a constant we have \( F(x) = 1, W(x) = a, V(x) = 4a^2 \) and \( S(x) = 0 \). The equation satisfies the constraint (6.13) and using (6.10)-(6.12) we find that

\[
Q(x) = -2, \ P(x) = -2a. \]

From (6.12) we find that the general solution for \( U(x) \) is

\[
U(x) = Ce^{-2ax} + a^2. \]

With this \( U(x) \) the general solution of (6.2) is

\[
\phi(x) = C_1BesselI(1, z) + C_2BesselY(1, iz)
\]

where \( z = \sqrt{Ce^{-ax}} \) and \( BesselI, BesselY \) are the modified Bessel functions of the first and second kind. Letting \( C_2 = 0 \) (real solution) it is straightforward to verify that

\[
\psi(x) = -2a - 2\frac{\phi(x)'}{\phi(x)}
\]

is a solution of (6.17).

### 7 Summary

We demonstrated in this paper that a generalized form of Cole-Hopf transformation (2.5) can be used to find solutions of the perturbed Van der Pol equation without forcing and for Painleve III equation. In addition we applied this transformation to linearize a generalized form of Burger’s equation and second order nonlinear ODEs with convective terms.
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