Generation and application of hypocycloid and astroid

Bei Wang*, Yuejie Geng and Jixun Chu

School of Mathematics and Physics, University of Science & Technology Beijing, Beijing, 100083, China

*Corresponding author’s email: 18811120586@163.com

Abstract. As a kind of plane curve, hypocycloid can be defined as the trajectory of a moving circular point in a fixed circle. In this paper, we start with the generation process of hypocycloid and explore its parametric equation. Then astroid and its related properties are introduced. Meanwhile, we discuss the application of the astroid on the bus door. When the door opens, the envelope of the moving straight line projected on the ground is an astroid, and it can effectively save space. Finally, we use MATLAB to simulate its dynamic opening process.

1. Introduction

A curve can be regarded as a collection of moving points that some meet certain criteria or a trajectory of a point moving according to some certain conditions. Hypotrochoid is a trajectory of a fixed point on the circumference of a moving circle when it rolls around the inside of a fixed circle. This paper first introduces the formation of the hypocycloid and then discusses a special form of hypocycloid: astroid. Finally, the application of astroid on the bus door is introduced, and the simulation is carried out by the MATLAB program.

2. Hypocycloid

A hypotrochoid is a curve traced by a point $P$ attached to a circle of radius $r$ rolling around the inside of a fixed circle of radius $R$ ($R > r$), where $P$ has a distance $h$ from the center of the interior circle(Figure 1). When $h = r$, the curve is called hypocycloid[1].

2.1. Function

To obtain the equation of the hypocycloid, establish a Cartesian coordinate system with the point $O$ as the origin. denote that $\angle PQT = \theta$ and $\angle TOT = t$ (Figure 1). When the circle $Q$ rolls around the inside of the fixed circle $O$, it can be seen that:

$$PT_2 = T_2T \rightarrow Rt = r\theta \rightarrow \theta = \frac{R}{r} t.$$
So, the coordinates of \( P(x, y) \) are as follows:

\[
\begin{align*}
x &= (R - r) \cos \theta - r \cos (\theta - t) \\
y &= (R - r) \sin \theta - r \sin (\theta - t)
\end{align*}
\]

(1)

As shown in Figure 2, different curves are obtained when radius ratio \( k \) changes. The hypocycloid has many unique characteristics that are different from other curves, and many of its features have been widely used in various industries of machinery \(^{[2,3]}\). For example, many scholars have studied the design of different cycloidal cycloid pumps with different \( k \) values, and some have been used in engineering practice \(^{[4]}\).

2.2. Algorithm Process and Simulation Results

Taking the center of the circle \( O \) as the origin, the coordinates of any point \( M \) on the circle \( O \) can be expressed as:

\[
M(x_r, y_r) = (R \cos \alpha, R \sin \alpha) \quad \alpha \in [0, 2\pi]
\]

(2)

where \( \alpha \) denotes the angle between the radius \( OM \) and \( X \) positive half-axis.

The coordinates of the center of the small circle \( Q \) are:

\[
Q(x_0, y_0) = ((R - r) \cos t, (R - r) \sin t)
\]

(3)

The coordinates of the any point \( N \) on the small circle \( Q \) can be expressed as:

\[
N(x_r, y_r) : \begin{align*}
x_r &= x_0 + r \cos \beta \\
y_r &= y_0 + r \sin \beta
\end{align*} \quad \Rightarrow \begin{align*}
x_r &= (R - r) \cos t + r \cos \beta \\
y_r &= (R - r) \sin t + r \sin \beta
\end{align*} \quad \beta \in [0, 2\pi]
\]

(4)

Where \( \beta \) denotes the angle between the radius \( QN \) and \( X \) positive half-axis.

The coordinates of the rotation point \( P \) are:

\[
P(x, y) = \left( (R - r) \cos t + r \cos \left( \frac{R - r}{r} t \right), (R - r) \sin t - r \sin \left( \frac{R - r}{r} t \right) \right)
\]

(5)

Based on the above calculation, the following algorithm is designed by MATLAB to realize the dynamic generation of the hypocycloid.
a) Parameter initialization: radius ratio $k$, the radius of the big circle $O: R$, the range of $\angle T_1OT_2: t \in [0, T]$

b) At a fixed $t$, calculate the coordinates of the center of the small circle $Q$, the point $N$ on the small circle $Q$, and the rotation point $P$ according to the formula (2)(3)(4)(5).

c) For $t=0...T$
For $\alpha(\beta) = 0...2\pi$
Plot $(R \cos \alpha, R \sin \alpha)$
Plot $((R - r)\cos t + r \cos \beta, (R - r)\sin t + r \sin \beta)$
End for
Plot $\left( (R - r)\cos t + r \cos \left( \frac{R - r}{r}t \right), (R - r)\sin t - r \sin \left( \frac{R - r}{r}t \right) \right)$
End for

When $k = 3$ the results are as shown in Figure 3.

![Figure 3 Hypocycloid when $k = 3$](image)

### 3. Astroid

Astroid, also known as tetracuspid, is a kind of hypocycloid when $k = 4$ and have a wide range of applications in graphic design and engineering\(^5\).

#### 3.1. Function

According to Formula 1, the parametric equation of astroid is as follows\(^6\).

\[
\begin{align*}
    x &= \frac{3R}{4} \cos t + \frac{R}{4} \cos(3t) \\
    y &= \frac{3R}{4} \sin t - \frac{R}{4} \sin(3t)
\end{align*}
\]

Note that $\cos(3t) = -3 \cos t + 4 \cos^3 t$, $\sin(3t) = 3 \sin t - 4 \sin^3 t$, the equations can be simplified to:

\[
    x = R \cos^3 t, \quad y = R \sin^3 t
\]

Eliminate the parameter $t$:

\[
    x^\frac{2}{3} + y^\frac{2}{3} = R^\frac{2}{3}
\]

#### 3.2. Property

1) The length of the line segment cut by the x-axis y-axis of the curve tangent is constant.

Denote that the tangent of any point on the astroid intersects the x-axis and y-axis at points $A$ and $B$, respectively.
The tangent equation is:

\[ y - R \sin^3 t = -\tan(t)(x - R \cos^3 t) \Rightarrow \frac{x}{R \cos t} + \frac{y}{R \sin t} = 1 \]

So

\[ A = (R \cos t, 0), \quad B = (0, R \sin t) \rightarrow |AB| = \sqrt{R^2 \sin^2 t + R^2 \cos^2 t} = R \]

2) Astroid can also be defined as the envelope of the moving straight line whose length is \( R \) by the x-axis and y-axis.

The moving straight line whose length is \( R \) by the x-axis y-axis can be expressed as

\[ \frac{x}{a \cos t} + \frac{y}{a \sin t} = 1 \]

Define that \( F(x, y, t) = \frac{x}{a \cos t} + \frac{y}{a \sin t} - 1 \)

Find the equation of the envelope:

\[
\begin{cases}
F(x, y, t) = 0 \\
\frac{\partial F}{\partial t} = 0
\end{cases}
\Rightarrow
\begin{cases}
\frac{x}{a \cos t} + \frac{y}{a \sin t} - 1 = 0 \\
\sin t \cdot x - \cos t \cdot y = 0
\end{cases}
\]

Solve the equation: \( x = a \cos^3 t, y = a \sin^3 t \), which is the parametric equation of the astroid.

### 4. Application

In this section, we introduce the application of the astroid in the bus door.

#### 4.1. Mathematical principle

The schematic diagram of the two types of doors is as follows:

Figure 5 shows the opening method of the ordinary door. When the door is opened, MA and NA rotate around points M and N respectively, \( AM = BN = R \), the area swept by each door is a quarter of a circle. The area required is:

\[
S_1 = 2 \times \frac{1}{4} \times \pi R^2 = \frac{1}{2} \pi R^2
\]
Figure 6 shows the opening method of the bus door. When the door is opened, \( AB = CD = R \) the four points A, B, C, and D slide along the chute. The area swept by each door is a quarter of the astroid. The area required is:

\[
S_2 = 2 \int_0^{\pi/2} R \sin^3 t \, dt (R \cos^3 t) = -6R^2 \int_0^{\pi/2} \sin^4 t - \sin^6 t \, dt = \frac{3}{16} \pi R^2
\]

According to the design of the bus door, the required area is \( \frac{3}{8} \) times the area required for the ordinary door design. Therefore, the space required can be greatly reduced.

4.2. Algorithm Process and Simulation Results

In this section, we simulate the opening process of the ordinary door and the bus door. The width of both sides of the door is \( R \), the height of the door is \( h \) (Figure 7).

For the ordinary door:

The coordinates of the point on the right and left circle are:

\[
(R - R \sin \alpha, R - R \cos \alpha, 0), (R \sin \alpha - R, R - R \cos \alpha, 0), \alpha \in \left[0, \frac{\pi}{2}\right]
\]

The coordinates of the four apex angles on the two sides of the door are:

\[
M(-R, R, 0) A(R \sin t - R, R - R \cos t, 0) M_{\beta}(-R, R, h) A_{\beta}(R \sin t - R, R - R \cos t, h),
\]

\[
N(R, R, 0) B(R - R \sin t, R - R \cos t, 0) N_{\beta}(R, R, h) B_{\beta}(R - R \sin t, R - R \cos t, h), t \in \left[0, \frac{\pi}{2}\right]
\]

The projection of the armrest of the door on the \( xoy \) coordinate plane is on the line \( MA \) and \( NB \). The heights of the two points are \( h \), and the coordinates of the two points on the right armrest are:

\[
x_{H_{1i}} = \frac{1}{2} (x_M + x_A) \\
y_{H_{1i}} = \frac{1}{2} (y_M + y_A) \Rightarrow H_{1i}(R - \frac{R}{2} \sin t, R - \frac{R}{2} \cos t, 0.6h) \\
z_{H_{1i}} = 0.6h
\]

\[
x_{H_{2i}} = \frac{1}{3} x_M + \frac{2}{3} x_A \\
y_{H_{2i}} = \frac{1}{3} y_M + \frac{2}{3} y_A \Rightarrow H_{2i}(R - \frac{R}{3} \sin t, R - \frac{2R}{3} \cos t, 0.4h) \\
z_{H_{2i}} = 0.4h
\]

Similarly, the coordinates of the two points on the left armrest are:

\[
H_{1i}(\frac{R}{2} \sin t - R, -\frac{R}{2} \cos t, 0.6h) \quad H_{2i}(\frac{R}{3} \sin t - R, -\frac{2R}{3} \cos t, 0.4h)
\]

For the bus door:

The coordinates of the point on the right and left circle are:

\[
(R - R \sin^3 \beta, R - R \cos^3 \beta, 0), (R \sin^3 \beta - R, R - R \cos^3 \beta), \beta \in \left[0, \frac{\pi}{2}\right]
\]

The coordinates of the four apex angles on the two sides of the door are:

\[
A(-R, R - R \cos t, 0) B(R \sin t - R, R, 0) A_{\beta}(-R, R - R \cos t, h) B_{\beta}(R \sin t - R, R, h), t \in \left[0, \frac{\pi}{2}\right]
\]

\[
C(R, R - R \cos t, 0) D(R - R \sin t, R, 0) C_{\beta}(R, R - R \cos t, h) D_{\beta}(R - R \sin t, R, h)
\]

Similarly, the coordinates of the two points on the armrest are:
Based on the above calculation, the following algorithm is designed by MATLAB to realize the dynamic simulation of the process of the door opening.

a) Parameter initialization: height of the door $h$, weight of the door $R$.

b) At a fixed $t$, calculate the coordinates of the four points on the door and the two points of the handrail.

c) For $t=0,\ldots,T$

   Draw the rectangle of the door and fill it.
   Draw the armrest.
   Draw the line projected by the door plane on the xoy coordinate plane.

End for

d) Draw the astroid.

The results are as shown in Figure 7.

\[
H_{r1}(R - \frac{R}{2}\sin t, R - \frac{R}{2}\cos t, 0.6h) \quad H_{r2}(R - \frac{R}{3}\sin t, R - \frac{2R}{3}\cos t, 0.4h)
\]

\[
H_{l1}(\frac{R}{2}\sin t - R, R - \frac{R}{2}\cos t, 0.6h) \quad H_{l2}(\frac{R}{3}\sin t - R, R - \frac{2R}{3}\cos t, 0.4h)
\]

5. **Summary**

As a special type of plane curve, the hypocycloid and the astroid have important applications in the engineering field. This paper introduces the formation process of hypocycloid and astroid, and then discusses the application of the astroid on the bus door. Finally, the MATLAB program is used to simulate the door opening process.

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