A Study on Time-Varying Inertia Properties Estimation Using Dynamic Model Compensation

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1. Introduction

Since accurate inertia properties are necessary for precise attitude control, these properties are estimated by various methods. Zhao et al. applied the Kalman filter to estimate full inertia properties,1) Yang et al. presented a full inertia properties estimation method using the extended Kalman filter and recursive least squares algorithm.2) Kim et al. applied a combination method of the Savitzky-Golay filter, EKF and linear-least squares.3) Many researchers are utilizing rigid body rotational dynamics as the model to estimate inertia properties because it is assumed that inertia properties are not time-varying parameters. However, inertia properties are varied with respect to the time due to various reasons, such as fuel consumption, solar panel deployment, etc. In this case, the system model of the satellite will be different with those having a rigid body, and this difference will lead to degradation of the estimation performance in the model-based estimation approach. Therefore, it is necessary to research a method to obtain the reliable estimation of time-varying inertia properties in the model-based approach.

In this paper, a methodology to estimate time-varying inertia properties is presented. First, the rotational equations of motion, including the time-varying inertia properties, are presented. Next, a system model to estimate the time-varying inertia properties is constructed by applying dynamic model compensation (DMC) and the predicted variation information of the inertia properties. Last, the time-varying inertia properties are estimated using the unscented Kalman filter (UKF) based on the system model with DMC. To verify the availability of the method proposed, numerical simulation is demonstrated using a simple example.

2. Time-Varying Inertia Properties Estimation

2.1. System model with DMC

Assuming that the angular momentum \( I \) is equal to \( I\omega \), the time derivatives can be expressed as follows:

\[
\dot{\omega} = f_1(I, \omega, u) + \frac{1}{2} I \dot{\omega} + \frac{1}{2} \omega \times I \omega + \frac{1}{2} \omega \times (I \omega) + \frac{1}{2} \omega \times (I \omega)
\]

where, \( I \) is the inertia matrix, \( \omega \) is the angular velocity vector, \( u \) is the command torque vector, and \( \dot{a} \) means the time derivatives of \( a \).

In general, the function \( f_1(I, \omega, u) \) in Eq. (1) can be defined as \( f_1(I, \omega, u) = \Gamma^{-1}(u - \omega \times I \omega) \), which is known as the rigid body rotational dynamics. In the case of a rigid body, assuming that the time derivative of \( I \) is equal to zero, the function \( f_2(I, \dot{I}, u) \) is equal to zero. In a realistic situation, the function \( f_2(I, \dot{I}, u) \) may not be equal to zero or not be negligible due to variation in the inertia properties. To consider the variation in inertia properties in the function \( f_2(I, \dot{I}, u) \), time-varying modeling of the inertia properties is required. However, it is hard to conduct time-varying modeling of the inertia properties because it is calculated information such as the time-varying center of mass, mass change rate, etc.

To make up for this difficulty, DMC can be applied for substituting into the variation term \( f_2(I, \dot{I}, u) \). DMC is a supplementary technique to compensate the unmodeled terms in the equations of motion using the Gauss-Markov process.5,6) To apply DMC using this process, let the unmodeled term be defined as:

\[
\eta = T \eta + w
\]

where, \( \eta \) is term that substitutes for \( f_2(I, \dot{I}, u) \), \( T \) is the diagonal matrix consisting of the inverse of the time constant \((-1/\tau_c)\), and \( w \) is the normal random variable with mean zero and variance \( \sigma^2 \).

The variation information is required to apply model-based approach methods such as the various kinds of Kalman filters because the variables to be estimated are included in the state variables. This information can be obtained and predicted using the estimated inertia properties. The vector \( J \) consisting of the inertia matrix component \( I \) is defined as:

\[
J = [I_{xx} \ I_{yy} \ I_{zz} \ I_{xy} \ I_{xz} \ I_{yz}]^T
\]

The predicted variations of inertia properties can be expressed as:

\[
\dot{J}_p = g(J_p, p)
\]

where, \( g \) is a function of \( J_p \) and \( p \). \( J_p \) is the inertia properties estimated, and \( p \) is the parameter vector leading to change in
inertia properties. The change is caused by the fuel consumption, the solar panel deployment, etc.

From Eqs. (1), (2) and (4), the system model with DMC can be expressed as follows:

\[
\begin{bmatrix}
\dot{\mathbf{x}}_p \\
\dot{\mathbf{q}} \\
\dot{\mathbf{J}}_p
\end{bmatrix} = 
\begin{bmatrix}
f_1(J_{est}, \omega, u) + \eta \\
f_1(J_p, \omega, u) + \eta \\
g(J_p, p)
\end{bmatrix}
\]  

(5)

where \( J_{est} \) comprises the component of \( J_p \).

2.2. Inertia estimation using UKF

Let the state variables \( x \) be defined as

\[
\mathbf{x} = [\mathbf{q}^T \ \dot{\mathbf{q}}^T \ \mathbf{J}^T]^T
\]

(6)

where, \( \mathbf{q} \) is the quaternion.

Using Eqs. (5) and (6), the system model including the quaternion can be expressed as

\[
\begin{bmatrix}
\dot{\mathbf{q}} \\
\dot{\mathbf{J}}_p
\end{bmatrix} = 
\begin{bmatrix}
0.5S(\Omega)\mathbf{q} \\
0.5S(\Omega)\mathbf{J} + \mathbf{h}
\end{bmatrix}
\]

(7)

where, \( \Omega \) is the skew symmetric matrix whose elements are a satellite’s angular velocities.4)

From Eq. (7), the discrete-time system model can be written as

\[
\begin{align*}
\mathbf{x}_k &= \mathbf{F}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{p}_{k-1}) + \mathbf{e}_{k-1} \\
\mathbf{z}_k &= \mathbf{H}(\mathbf{x}_k) + \mathbf{v}_k
\end{align*}
\]

(8)

where, \( \mathbf{e} \) and \( \mathbf{v} \) are the process noise and measurement noise with mean zero and variances \( \mathbf{Q} \) and \( \mathbf{R} \), respectively.

The computation algorithm of UKF is presented below7):

- Initialization with:

\[
\hat{\mathbf{x}}_0 = \mathbf{E}[\mathbf{x}_0], \quad \mathbf{P}_0 = \mathbf{E}[(\mathbf{x}_0 - \hat{\mathbf{x}}_0)(\mathbf{x}_0 - \hat{\mathbf{x}}_0)^T]
\]

- Calculate sigma points:

\[
\begin{align*}
\chi_{k-1} &= \begin{bmatrix} \hat{\mathbf{x}}_{k-1} & \hat{\mathbf{x}}_{k-1} + \gamma \sqrt{\mathbf{P}_{k-1}} & \hat{\mathbf{x}}_{k-1} - \gamma \sqrt{\mathbf{P}_{k-1}} \end{bmatrix} \\
\end{align*}
\]

- Time update:

\[
\begin{align*}
\chi_{k|k-1} &= \mathbf{F}(\chi_{k-1}, \mathbf{u}_{k-1}, \mathbf{p}_{k-1}) \\
\hat{\mathbf{x}}_k &= \sum_{i=0}^{2L} \mathbf{W}_i^{(m)} \chi_{k|k-1}^{(i)} \\
\mathbf{P}_k &= \mathbf{Q} + \sum_{i=0}^{2L} \mathbf{W}_i^{(m)} [\chi_{k|k-1}^{(i)} - \hat{\mathbf{x}}_k][\chi_{k|k-1}^{(i)} - \hat{\mathbf{x}}_k]^T \\
\mathbf{Z}_{k|k-1} &= \mathbf{H}(\chi_{k|k-1}) \\
\hat{\mathbf{Z}}_k &= \sum_{i=0}^{2L} \mathbf{W}_i^{(m)} \mathbf{Z}_{k|k-1}^{(i)} \\
\end{align*}
\]

- Measurement update:

\[
\begin{align*}
\mathbf{P}_{z,k} &= \mathbf{R} + \sum_{i=0}^{2L} \mathbf{W}_i^{(m)} [\mathbf{Z}_{k|k-1}^{(i)} - \hat{\mathbf{Z}}_k][\mathbf{Z}_{k|k-1}^{(i)} - \hat{\mathbf{Z}}_k]^T \\
\end{align*}
\]

where, \( \mathbf{P}_{z,k} \) is the covariance matrix of \( \mathbf{z}_k \) and \( \mathbf{R} \) is the measurement noise covariance matrix.

- Inertia matrix update:

\[
\hat{\mathbf{j}}_k = \begin{bmatrix} \hat{I}_{xx} & \hat{I}_{xy} & \hat{I}_{xz} & \hat{I}_{yx} & \hat{I}_{yy} & \hat{I}_{yz} \end{bmatrix}^T
\]

where, \( \gamma = (L + \lambda), \lambda = \alpha^2(L + \kappa) - L, L \) is the dimension of the state variable, and both \( \alpha \) and \( \beta \) are the scaling parameters.

The weights \( \mathbf{W}^{(m)} \) and \( \mathbf{W}^{(c)} \) are represented as follows:

\[
\mathbf{W}^{(m)}_i = \begin{cases} 
\lambda/(L + \lambda) & (i = 0) \\
1/[2(L + \lambda)] & (i = 1, \ldots, 2L)
\end{cases}
\]

\[
\mathbf{W}^{(c)}_i = \begin{cases} 
\lambda/(L + \lambda) + (1 - \alpha^2 + \beta) & (i = 0) \\
1/[2(L + \lambda)] & (i = 1, \ldots, 2L)
\end{cases}
\]

3. Numerical Simulation

A numerical simulation is conducted under the following assumptions:

- The variation in inertia properties is only affected by the fuel consumption with respect to the command torque.
- The fuel tank’s center of mass is located at the center of mass of the satellite.
- The fuel is not sloshing in the fuel tank.
- The command torques act on three axes.
- It is a regulator problem simulation that the satellite rotates from \( \theta = [5\degree, 5\degree, 5\degree] \) to \( \theta = [0\degree, 0\degree, 0\degree] \).

In Eq. (7), the command torque \( \mathbf{u} \) can be calculated by

\[
\mathbf{u} = \mathbf{\omega} \times \mathbf{I} \mathbf{\omega} - D(\mathbf{\omega} - \omega_0) - K \mathbf{q}_e
\]

(9)

where, \( \mathbf{q}_e \) is the error quaternion, \( \omega_0 \) is the desired angular velocity, \( K \) is the proportional gain matrix, and \( D \) is the derivative gain matrix. Additionally, \( K \) and \( D \) are calculated as \( K = \mathbf{kI} \) and \( D = dl; \) where \( k \) and \( d \) are constants.

The variation in inertia properties is calculated by the magnitude of the command torque as follows8):

\[
\mathbf{J}_p = -A \| u \|
\]

(10)

where, \( A \) is the diagonal constant matrix expressing the proportional relationship between the fuel consumption and command torque.

The simulation conditions considering STSAT-3 as the simulation model are listed in Table 1. The noise reduction performance is shown in Figs. 1 and 2. The errors of the es-
The estimated quaternion are within approximately $8 \times 10^{-4}$. As shown in Fig. 2, the measurement noise in angular velocities is reduced approximately 66%. Table 2 and Fig. 3 show the estimation performance of the time-varying inertia properties. As can be seen in Fig. 3, inertia property estimates converge in approximately 15 s and the estimation error is within approximately 4.5%, as shown in Table 2. Figure 4 depicts the term $f_2$ in Eq. (1). As shown in Fig. 4, the $f_2$ term calculated using DMC is similar to the true value.

### 4. Conclusion

In this paper, the estimation method using DMC is suggested for determining reliable time-varying inertia properties. Since it is hard to model variations in inertia properties, DMC is applied to compensate the unmodeled term. UKF is applied to estimate the state variables and time-varying inertia properties because it can give more accurate estimates. To verify the estimation performance of the method proposed, a numerical simulation is conducted using a simple case. The simulation results reveal that combining DMC and UKF is an effective method for estimating time-varying inertia properties. In addition, the unmodeled term calculated using DMC can compensate equations of motion with the time-varying term.

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**Table 1. Simulation conditions.**

| Items          | Values                                                                 |
|----------------|------------------------------------------------------------------------|
| Simulation time| 30 s                                                                   |
| Initial quaternion | $[0.0416 0.0454 0.0416 0.9972]^T$                                        |
| Initial angular velocity | $[0 0 0]^T$ rad/s                                                       |
| Desired quaternion | $[0 0 0 1]^T$                                                          |
| Desired angular velocity | $[0 0 0]^T$ rad/s                                                       |
| Noise level     | 0.0039 rad/s                                                           |
| DMC time constant | $[1.5 1.5 1.5]^T$ s                                                  |
| DMC standard deviation | $[10^{-5} 10^{-5} 10^{-5}]^T$ rad/s²                                 |
| Initial inertia | $\begin{bmatrix} 15.62 & 0.0954 & 0.1493 \\ 0.0954 & 19.03 & 0.6618 \\ 0.1493 & 0.6618 & 22.33 \end{bmatrix}$ kg m² |

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**Table 2. Estimation result at 30 s.**

| Inertia | True (kg m²) | Estimates (kg m²) | Error (kg m²) | Error (%) |
|---------|--------------|-------------------|---------------|-----------|
| $I_{xx}$ | 13.9399     | 14.5181           | 0.5781        | 4.15      |
| $I_{yy}$ | 17.0399     | 17.1648           | 0.1249        | 0.73      |
| $I_{zz}$ | 20.0399     | 20.9322           | 0.8923        | 4.45      |
| $I_{xy}$ | 0.0815      | 0.0823            | 0.0008        | 0.95      |
| $I_{xz}$ | 0.1279      | 0.1256            | 0.0023        | 1.77      |
| $I_{yz}$ | 0.5886      | 0.5903            | 0.0017        | 0.29      |

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**Fig. 1. Quaternion.**

**Fig. 2. Error and 3σ boundary.**

**Fig. 3. Estimation results.**

**Fig. 4. $f_2$ term comparison results.**
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