A unified approach to determining forms for the 2D Navier–Stokes equations — the general interpolants case

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Abstract. It is shown that the long-time dynamics (the global attractor) of the 2D Navier–Stokes system is embedded in the long-time dynamics of an ordinary differential equation, called a determining form, in a space of trajectories which is isomorphic to $C^1_b(\mathbb{R}; \mathbb{R}^N)$ for sufficiently large $N$ depending on the physical parameters of the Navier–Stokes equations. A unified approach is presented, based on interpolant operators constructed from various determining parameters for the Navier–Stokes equations, namely, determining nodal values, Fourier modes, finite volume elements, finite elements, and so on. There are two immediate and interesting consequences of this unified approach. The first is that the constructed determining form has a Lyapunov function, and thus its solutions converge to the set of steady states of the determining form as the time goes to infinity. The second is that these steady states of the determining form can be uniquely identified with the trajectories in the global attractor of the Navier–Stokes system. It should be added that this unified approach is general enough that it applies, in an almost straightforward manner, to a whole class of dissipative dynamical systems.

Bibliography: 23 titles.

Keywords: Navier–Stokes equation, inertial manifold, determining forms, determining modes, dissipative dynamical systems.

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1. Introduction

The 2D system of Navier–Stokes equations (NSE), (2.1) and (2.2), in addition to being a fundamental component of many fluid models, is intriguing for several theoretical reasons. In featuring both a direct cascade of enstrophy and an inverse cascade of energy, it displays more complicated turbulence phenomena than does the 3D system of NSE [3], [19], [20]. Also unlike in 3D, the global existence theory for the 2D NSE is complete (see, for instance, [6], [22]). In fact, the long-time dynamics of the 2D NSE is entirely contained in the global attractor $\mathcal{A}$ (see (2.5)), a compact finite-dimensional subset of the infinite-dimensional phase space $H$ of solenoidal finite-energy vector fields, (see, for instance, [6], [10], [16], [22]). Sharp estimates of the dimension of the global attractor in terms of the relevant physical parameters were first established in [7] (see also [6], [21], [22] and references therein). If there were an inertial manifold $\mathcal{M}$ (that is, a Lipschitz, finite-dimensional, forward-invariant manifold which attracts each bounded set at an exponential rate), then $\mathcal{A} \subset \mathcal{M}$, and the dynamics on $\mathcal{A}$ would be captured by an ordinary differential equation (ODE), called an inertial form, in a finite-dimensional phase space $[6], [12], [13], [22]$. This can be attained through reduction of the original evolution equation to an equation on the inertial manifold $\mathcal{M}$. Yet the existence of an inertial manifold for the 2D NSE has been an open problem since the 1980s!!

This is very surprising since there are even stronger indicators of the finite-dimensional behavior of the 2D NSE. The solutions in $\mathcal{A}$ are determined by the asymptotic behavior of a sufficient (finite) number of determining parameters. If in the limit as $t \to \infty$ a sufficiently large number of low Fourier modes (or nodal values, or finite volume elements) for two solutions in $\mathcal{A}$ converge to each other, then the solutions coincide (see, for instance, [5] for a unified theory of determining parameters and projections). This is equivalent, at least in the case of Fourier modes, to the following:

If two complete trajectories in the global attractor coincide upon projection $P_m$ on a sufficient large number $m$ of low modes, then they are the same trajectory.

This notion of determining modes, which was introduced in [11] (see also, [18] for sharp estimates of the number of determining modes), was used in [8] to construct a system of ODEs in the Banach space $X = C_b(\mathbb{R}, P_m H)$ that govern the evolution
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of trajectories in the space X. We call the system of ODEs a determining form. Trajectories in the global attractor \( \mathcal{A} \) of the 2D NSE (2.2) are identified with travelling wave solutions of the determining form. There are conceptually two time variables in play for the determining form: the evolving time for the ODE and the original time variable of the NSE that now parameterizes complete trajectories in X. Though the determining form has an infinite-dimensional phase space, the vector field that governs the evolution is globally Lipschitz, so the determining form is an ODE in the true sense. The key to constructing the determining form in [8] is to extend to the whole space X the map \( W : P_m, \mathcal{A} \rightarrow (I - P_m), \mathcal{A} \) provided by (1.1) on the set \( \mathcal{A} \) of complete trajectories in \( \mathcal{A} \). The extended map is shown to be Lipschitz, and its image plays the role of recovering the higher modes, while the evolving trajectory in X represents the lower modes.

The determining form in this paper has an entirely different character. It is a system which possesses a Lyapunov function and whose steady states are precisely the trajectories in the global attractor of the 2D NSE. It is more general in that it can be used with a variety of determining parameters, including nodal values as well as Fourier modes. Furthermore, it provides a general framework and strategy that can be implemented for other dissipative systems. Like the determining form in [8], the key to its construction is the extension of a map \( W \) defined at first only for projections of trajectories in \( \mathcal{A} \). This is done by using the feedback control term added to the NSE as suggested in [1] and [2], and this involves an interpolating operator \( J_h \) approximating the identity map at the level \( h \) (for instance, \( J_h \) can be based on nodal values, where \( h \) represents the grid size). This construction and the statements of our main results are presented in §3. In §2 we provide some preliminary background material and useful inequalities concerning the Navier–Stokes equations. Details of the proofs of our main results are given in §§4 and 5.

2. Functional setting and the Navier–Stokes equations

We consider the two-dimensional incompressible Navier–Stokes equations

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p &= \Phi, \\
\text{div} u &= 0, \\
\int_{\Omega} u \, dx &= 0, \\
\int_{\Omega} \Phi \, dx &= 0, \\
u(0, x) &= u_0(x),
\end{align*}
\]

subject to periodic boundary conditions with the basic domain \( \Omega = [0, L]^2 \). The velocity field \( u \) and the pressure \( p \) are the unknown functions, while \( \Phi \) is a given forcing term, and \( \nu > 0 \) is a given constant viscosity.

Let

\[ \mathcal{V} = \left\{ \phi : \phi \text{ is an } \mathbb{R}^2 \text{-valued trigonometric polynomial, } \nabla \cdot \phi = 0, \text{ and } \int_{\Omega} \phi(x) \, dx = 0 \right\}. \]
For any subset $Z \subset L^1_{\text{per}}(\Omega)$ we define $\dot{Z} = \{ \phi \in Z : \int_{\Omega} \phi(x) \, dx = 0 \}$. We denote by $H$ and $V$ the closures of $\mathcal{V}$ in $(L^2_{\text{per}}(\Omega))^2$ and $(H^1_{\text{per}}(\Omega))^2$, respectively. The inner product and norm in the Hilbert spaces $(L^2_{\text{per}}(\Omega))^2$ and $H$ will be denoted by $(\cdot, \cdot)$ and $|\cdot|$, respectively, and the corresponding inner product and norm in the Hilbert spaces $(\dot{H}^1_{\text{per}}(\Omega))^2$ and $V$ will be denoted by $(\cdot, \cdot)$ and $\|\cdot\|$. Specifically, for every $u, v \in (\dot{H}^1_{\text{per}}(\Omega))^2$ we set

$$((u, v)) = \sum_{i,j=1}^{2} \int_{\Omega} \frac{\partial u_i(x)}{\partial x_j} \frac{\partial v_i(x)}{\partial x_j} \, dx.$$ 

Let $V'$ denote the dual space of the space $V$.

Using the above functional notation, we can write the Navier–Stokes equations as an evolution equation in the Hilbert space $H$ (cf. [6], [22]):

$$\frac{d}{dt}u(t) + \nu Au(t) + B(u(t), u(t)) = f, \quad t > 0,$$  

$$u(0) = u_0.$$  

(2.2)

The Stokes operator $A$, the bilinear operator $B$, and the force $f$ are defined as

$$A = -\mathcal{P} \Delta, \quad B(u, v) = \mathcal{P}(u \cdot \nabla)v, \quad f = \mathcal{P} \Phi,$$  

(2.3)

where $\mathcal{P}$ is the Helmholtz orthogonal projection from $(\dot{L}^2_{\text{per}}(\Omega))^2$ onto $H$, and where $u$ and $v$ are smooth enough that $B(u, v)$ makes sense. In this paper we will assume that $f \in V$.

We remark that $D(A) = (\dot{H}^2_{\text{per}}(\Omega))^2 \cap V$. The operator $A$ is self-adjoint, with compact inverse. Therefore, the space $H$ possesses an orthonormal basis $\{w_j\}_{j=1}^{\infty}$ of eigenfunctions of $A$, namely, $Aw_j = \lambda_j w_j$, with $0 < \lambda_1 = (2\pi/L)^2 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ (cf. [6], [22]). The powers $A^\alpha$ are given by

$$A^\alpha v = \sum_{j=1}^{\infty} \lambda_j^\alpha (v, w_j) w_j.$$ 

Note that all the powers of $A$ commute with $\mathcal{P}$. Note also that $V := D(A^{1/2})$ and that

$$\|u\| = \|A^{1/2}u\| = \left( \sum_{j=1}^{\infty} \lambda_j (u, w_j)^2 \right)^{1/2}.$$  

(2.4)

It is well known that the NSE (2.2) has a global attractor

$$\mathcal{A} = \left\{ u_0 \in H : \exists \text{ a solution } u(t, u_0) \text{ of (2.2)} \right\}$$ 

$$\forall t \in \mathbb{R}, \sup_t \|u(t)\| < \infty \};$$  

(2.5) 

that is, $\mathcal{A}$ is the maximal bounded invariant subset of $V$ under the NSE dynamics, or equivalently it is the minimal compact subset of $V$ which uniformly attracts all bounded sets in $V$ under the dynamics of (2.2). In particular, it is also known that

$$\mathcal{A} \subset \{ u \in V : \|u\| \leq G \nu \kappa_0 \}, \quad \text{where } G = \frac{|f|}{\nu^2 \kappa_0^2}. $$  

(2.6)
$G$ is the Grashof number, a dimensionless physical parameter, and $\kappa_0 = \lambda_1^{1/2} = 2\pi/L$. For the above properties see, for instance, [6], [10], [16], [22].

Next, we introduce a number of identities satisfied by the bilinear term. This includes the orthogonality relations

$$\langle B(u, v), w \rangle = -\langle B(u, w), v \rangle, \quad u, v, w \in V$$  \hspace{1cm} (2.7)

(where $\langle \cdot, \cdot \rangle$ denotes the dual action between $V'$ and $V$), and

$$(B(u, u), Au) = 0, \quad u \in D(A)$$  \hspace{1cm} (2.8)

(see, for instance, [6], [10], [22]). The relation (2.7) implies that (cf. [6], [22])

$$(B(v, v), Au) + (B(v, u), Av) + (B(u, v), Av) = 0, \quad u, v \in D(A).$$  \hspace{1cm} (2.9)

From now on, $c, c_A, c_B, c_L, c_S, c_T, c_1, c_2, \tilde{c}_1, \tilde{c}_2, \ldots$ will denote universal dimensionless positive constants. Our estimates for the non-linear term will involve Agmon’s inequality

$$\|u\|_\infty \leq c_A |u|^{1/2}|Au|^{1/2}, \quad u \in D(A),$$  \hspace{1cm} (2.10)

and the Sobolev and Ladyzhenskaya inequalities

$$\|u\|_{L^4(\Omega)} \leq c_S \|u\|_{H^{1/2}(\Omega)} \quad \text{for every } u \in (H^{1/2}(\Omega))^2,$$  \hspace{1cm} (2.11)

$$\|u\|_{H^{1/2}(\Omega)} \leq \tilde{c}_L \|u\|^{1/2}\|u\|^{1/2} \quad \text{for every } u \in (H^{1}_{\text{per}}(\Omega))^2,$$  \hspace{1cm} (2.12)

which imply that

$$\|u\|_{L^4(\Omega)} \leq c_L \|u\|^{1/2}\|u\|^{1/2} \quad \text{for every } u \in (H^{1}_{\text{per}}(\Omega))^2.$$  \hspace{1cm} (2.13)

We also use the versions of the Poincaré inequality

$$\kappa_0 |v| \leq \|v\|, \quad u \in V, \quad \text{and} \quad \kappa_0 \|v\| \leq |Av|, \quad u \in D(A),$$  \hspace{1cm} (2.14)

and Young’s inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad \text{for } a, b, p, q > 0 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} = 1.$$  \hspace{1cm} (2.15)

By (2.13)

$$|B(u, v), w| \leq c_S |u|^{1/2}\|u\|^{1/2}\|v\|^{1/2}|Av|^{1/2}|w| \quad \forall u \in V, \quad v \in D(A), \quad w \in H,$$  \hspace{1cm} (2.16)

and by (2.10)

$$|B(u, v), w| \leq c_A |u|^{1/2}|Au|^{1/2}\|v\| |w| \quad \forall u \in D(A), \quad v \in V, \quad w \in H.$$  \hspace{1cm} (2.17)

In addition,

$$|B(w, u), v| \leq c_T \|w\| \|u\| \left(\log \frac{|Au|}{\kappa_0 \|u\|}\right)^{1/2} |v| \quad \forall u \in D(A), \quad v \in H, \quad w \in V$$  \hspace{1cm} (2.18)
Using the Brézis-Gallouet inequality [4] (see also a different proof in [23]), we have
\[ |(B(w, u), v)| \leq c_B \|w\| \|u\| \left( \log \frac{e|Aw|}{\kappa_0 \|w\|} \right)^{1/2} |v| \quad \forall u \in V, \; v \in H, \; w \in D(A). \] (2.19)

We also use the following modified Gronwall inequality from [17] (see also [10]).

Lemma 2.1. Let \( \alpha \) and \( \beta \) be locally integrable real-valued functions on \( (0, \infty) \) which for some \( T \in (0, \infty) \) satisfy
\[
\lim_{t \to \infty} \frac{1}{T} \int_{t}^{t+T} \alpha(\tau) \, d\tau = \gamma > 0, \quad \lim_{t \to \infty} \frac{1}{T} \int_{t}^{t+T} \alpha^{-}(\tau) \, d\tau < \infty
\]
and
\[
\lim_{t \to \infty} \frac{1}{T} \int_{t}^{t+T} \beta^{+}(\tau) \, d\tau = 0,
\]
where \( \alpha^{-} = \max\{-\alpha, 0\} \) and \( \beta^{+} = \max\{\beta, 0\} \). Suppose that \( \xi \) is an absolutely continuous non-negative function on \( (0, \infty) \) such that
\[
\frac{d}{dt} \xi + \alpha \xi \leq \beta \quad \text{a.e. on } (0, \infty).
\]
Then \( \xi(t) \to 0 \) as \( t \to \infty \).

Lemma 2.1 will be combined later with the following estimates for averaged solutions (see [17], [18]).

Proposition 2.2. Let \( u \) be a solution of the NSE (2.2) and let \( T = (\nu \kappa_0^2)^{-1} \). Then
\[
\lim_{t \to \infty} \frac{1}{T} \int_{t}^{t+T} |Au(\tau)|^2 \, d\tau \leq 2\nu^2 \kappa_0^4 G^2.
\] (2.20)

If \( u \in \mathcal{A} \), then
\[
\lim_{t_0 \to -\infty} \frac{1}{t - t_0} \int_{t_0}^{t} |Au(\tau)|^2 \, d\tau \leq \nu^2 \kappa_0^4 G^2 \quad \text{for all } t \in \mathbb{R}. \] (2.21)

Moreover, it follows from the Cauchy–Schwarz inequality that
\[
\lim_{t_0 \to -\infty} \frac{1}{t - t_0} \int_{t_0}^{t} |Au(\tau)| \, d\tau \leq \nu \kappa_0^2 G \quad \text{for all } t \in \mathbb{R}. \] (2.22)

Proposition 2.3. Let \( u(t) \) be a solution of the NSE (2.2). Then
\[
\lim_{t \to \infty} \|u(t)\| \leq \nu \kappa_0 G \quad \text{and} \quad \lim_{t \to \infty} |Au(t)| \leq c \nu \kappa_0^2 G^3.
\]

In particular,
\[
\|u(t)\| \leq \nu \kappa_0 G \quad \text{and} \quad |Au| \leq c_0 \nu \kappa_0^2 G^3 \quad \forall u \in \mathcal{A}. \] (2.23)
Moreover, the solutions in the global attractor $\mathcal{A}$ are analytic with respect to the time variable in a strip about the real axis with width $\delta_{\text{time}} \geq c/(\nu \kappa_0^2 G^4)$. In addition, by the Cauchy formula the above estimates imply that

$$\sup_{t \in \mathbb{R}} \left| \frac{du}{dt}(t) \right| \leq c \nu^2 \kappa_0^3 G^5 \quad \text{and} \quad \sup_{t \in \mathbb{R}} \left| A \frac{du}{dt}(t) \right| \leq c \nu^2 \kappa_0^4 G^7.$$  

The ideas of the proof of the above proposition can be found in [6]. However, the new sharp estimates in Proposition 2.3 are obtained in [9].

We now derive two bounds for $A^{1/2}B(u, v)$.

**Proposition 2.4.** For all $u \in D(A^{3/4})$ and $v \in D(A)$

$$|A^{1/2}B(u, v)| \leq c \left( |A^{3/4}u| |A^{3/4}v| + |u|^{1/2} |Au|^{1/2} |Av| \right),$$  

(2.24)

and for all $u \in V$ and $v \in D(A^{3/2})$

$$|A^{1/2}B(u, v)| \leq c \left( |A^{1/2}u| |A^{1/2}v|^{1/2} |A^{3/2}v|^{1/2} + |A^{1/4}u| |A^{5/4}v| \right).$$  

(2.25)

**Proof.** First, we observe that

$$|A^{1/2} \mathcal{P} \phi| \leq |\nabla \phi| \quad \text{for every } \phi \in (\dot{H}^1_{\text{per}}(\Omega))^2.$$  

(2.26)

By virtue of (2.11) we have

$$((\partial_t u \cdot \nabla) v, w) \leq \|\partial_t u\|_{L^4} \|\nabla v\|_{L^4} |w| \leq \|\partial_t u\|_{H^{1/2}} \|\nabla v\|_{H^{1/2}} |w|. \tag{2.27}$$

It follows from (2.26), (2.27), (2.10), and (2.12) that

$$|A^{1/2}B(u, v)| \leq \sum_{i=1,2} |\partial_i(u \cdot \nabla)v| = \sum_{i=1,2} |(\partial_i u \cdot \nabla)v| + \sum_{i=1,2} |(u \cdot \nabla)\partial_i v|$$

$$\leq c \sum_{i=1,2} \left( \|\partial_i u\|_{H^{1/2}} \|\nabla v\|_{H^{1/2}} + \|u\|_{L^\infty} \|\nabla \partial_i v\| \right)$$

$$\leq c \left( |A^{3/4}u| |A^{3/4}v| + |u|^{1/2} |Au|^{1/2} |Av| \right)$$

and

$$|A^{1/2}B(u, v)| \leq \sum_{i=1,2} |\partial_i(u \cdot \nabla)v| = \sum_{i=1,2} |(\partial_i u \cdot \nabla)v| + \sum_{i=1,2} |(u \cdot \nabla)\partial_i v|$$

$$\leq c \sum_{i=1,2} \left( \|\partial_i u\| \|\nabla v\|_{L^\infty} + \|u\|_{H^{1/2}} \|\nabla \partial_i v\|_{H^{1/2}} \right)$$

$$\leq c \left( |A^{1/2}u| |A^{1/2}v|^{1/2} |A^{3/2}v|^{1/2} + |A^{1/4}u| |A^{5/4}v| \right). \quad \Box$$

Inspired by the proof of the Brézis–Gallouet inequality [4], we establish below a bound for the $L^\infty$-norm which we later use to optimize an estimate.

**Lemma 2.5.** Let $\phi \in \dot{H}^2_{\text{per}}(\Omega)$. Then for every $N \in \mathbb{R}^+$

$$\|\phi\|_{L^\infty} \leq \mathcal{L}_N |\nabla \phi| + (\sqrt{\pi} \kappa_0 N)^{-1} |\Delta \phi|,$$  

(2.28)

where $\mathcal{L}_N = (8 + 2\pi \log N)^{1/2} (2\pi)^{-1}$. 
Proof. First note that
\[
\sum_{1 \leq |k|^2 \leq N^2} \frac{1}{|k|^2} = 6 + \sum_{3 \leq |k|^2 \leq N^2} \frac{1}{|k|^2} \leq 6 + 4 \int_{2}^{N} \frac{dx}{x^2} + \int_{1}^{N} \int_{0}^{2\pi} \frac{1}{r^2} r \, d\theta \, dr \leq 8 + 2\pi \log N
\]
and for \( N \geq 3, \)
\[
\sum_{N+1 \leq |k|} \frac{1}{|k|^4} \leq 4 \left( \frac{1}{(N+1)^4} + 4 \int_{N+1}^{\infty} \frac{dx}{x^4} + \int_{N-1}^{\infty} \int_{0}^{2\pi} \frac{1}{r^4} r \, d\theta \, dr \right) \leq \frac{4\pi}{N^2}.
\]
Using the Cauchy–Schwarz inequality and Parseval’s identity, we have
\[
\|\phi\|_{L^\infty} \leq \sum_{1 \leq |k| \leq N} |\hat{\phi}_k| + \sum_{|k| \geq N+1} |\hat{\phi}_k| \leq \left[ \sum_{1 \leq |k| \leq N} \frac{1}{|k|^2} \right]^{1/2} \left[ \sum_{1 \leq |k| \leq N} |k|^2 |\hat{\phi}_k|^2 \right]^{1/2} + \left[ \sum_{|k| \geq N+1} \frac{1}{|k|^4} \right]^{1/2} \left[ \sum_{|k| \geq N+1} |k|^4 |\hat{\phi}_k|^2 \right]^{1/2} \leq (8 + 2\pi \log N)^{1/2} |\nabla \phi|_{2\pi} + 2\sqrt{\pi} \frac{|\Delta \phi|_{2\pi \kappa_0}}{N}. \quad \square
\]

3. Determining form and statements of main results

3.1. Interpolant operators. In this subsection we introduce a unified approach for using various determining parameters (modes, nodes, volume elements, and so on) by representing them in terms of interpolant operators that approximate the identity.

Let \( J = J_h : (H^2_{\text{per}}(\Omega))^2 \to (C^\infty_{\text{per}}(\Omega))^2 \) be a finite-rank linear operator approximating the identity in the following sense: for every \( \phi \in (H^2_{\text{per}}(\Omega))^2 \) we have \( J\phi \in (C^\infty_{\text{per}}(\Omega))^2, J\phi \) has zero spatial average, and

\[
|J\phi - \phi| \leq c_1 h|\nabla \phi| + c_2 h^2|\Delta \phi|, \quad (3.1)
\]

\[
|\nabla (J\phi - \phi)| \leq \bar{c}_1 |\nabla \phi| + \bar{c}_2 h|\Delta \phi|. \quad (3.2)
\]

Here \( h \) is a small parameter that determines the order of approximation. The rank of \( J_h \) is of order \( L/h \geq 1 \). For example, such interpolant polynomials are induced by the determining parameters of the NSE, such as determining modes, nodes, volume elements, projections of finite elements, and so on (see, for instance, [5], [11], [14], [15], [17], [18] and references therein). The most straightforward example of such interpolant operators is the projection operator \( J_h = P_m \) onto \( \text{span}\{w_1, w_2, \ldots, w_m\} \), where \( h = \lambda_m^{-1} \). Also, the appendix of [1] provides explicit examples of such interpolant operators that are based on nodal values and that satisfy (3.1) and (3.2). We remark that a general framework employing interpolant polynomials satisfying (3.1) was introduced in [5] for investigating the long-time dynamics of the NSE.
3.2. Determining form. In this subsection we present a determining form that is induced by the interpolant operators $J_h$. The connection between the long-time dynamics of the NSE (2.2) and the determining form is explained in the following result.

**Proposition 3.1.** Let $G \geq 1$ and let $u(s), s \in \mathbb{R}$, be a solution of the NSE (2.2) that lies in the global attractor $\mathcal{A}$. Suppose that $w \in C_b(\mathbb{R}; V) \cap L^2_{\text{loc}}(\mathbb{R}; D(A))$ with $dw/ds \in L^2_{\text{loc}}(\mathbb{R}; H)$ satisfies the equation

$$
\frac{dw}{ds} + \nu Aw + B(w, w) = f - \mu \nu \kappa_0^2 \mathcal{P} J(w - u).
$$

(3.3)

Then $w = u$ if

$$
\mu > 6c_T G \log(c_3 G), \quad \text{where} \quad c_3 = (2c_T c_0)^{1/3},
$$

(3.4)

and $h$ is small enough that

$$
2\mu \kappa_0^2 c_J h^2 \leq 1, \quad \text{where} \quad c_J = c_1 + \frac{c_2^2}{2},
$$

(3.5)

and $c_T, c_0, c_1, c_2$ are as in (2.18), (2.23), and (3.1).

The proof of Proposition 3.1 is given in §4. We remark that the existence of solutions of (3.3), as specified in Proposition 3.1, follows from Theorem 3.2 below.

Next, we introduce the phase space of the dynamics of our determining form. Let

$$
X = C_b^1(\mathbb{R}, J(\dot{H}^2_{\text{per}}(\Omega))^2),
$$

with the two norms

$$
\|v\|_X = \sup_{s \in \mathbb{R}} \frac{\|v(s)\|}{\nu \kappa_0} + \sup_{s \in \mathbb{R}} \frac{\|v'(s)\|}{\nu^2 \kappa_0^3},
$$

$$
\|v\|_{X,0} = \sup_{s \in \mathbb{R}} \frac{\|v(s)\|}{\nu \kappa_0}.
$$

Now we let $v$ be a given element of $X$ and consider the equation

$$
\frac{dw}{ds} + \nu Aw + B(w, w) = f - \mu \nu \kappa_0^2 \mathcal{P} (Jw - v).
$$

(3.6)

We show in the following result that under certain conditions on the parameters $\mu$ and $h$, which depend on $\|v\|_X$, (3.6) has a unique bounded global solution $w(s)$ for $s \in \mathbb{R}$.

**Theorem 3.2.** Let $f \in V$, and let $\mathcal{B}_X^\rho(0) = \{v \in X : \|v\|_X \leq \rho\}$ for some $\rho > 0$. Fix $K$ and $\mu$ so that

$$
K \geq \left(2\rho^2 + \frac{G^2}{\mu} + 1\right)^{1/2}
$$

(3.7)

and

$$
c_4 K^2 \log(c_5 K^2) < \mu < 2c_4 K^2 \log(c_5 K^2),
$$

(3.8)
where \( c_4 = 80(c_T + c_B + 1)^2 \) and \( c_5 = \sqrt{8} (c_T + c_B + 1) \). Choose \( h \) small enough so that
\[
2\mu h^2 \kappa_0^2 (c_1^2 + c_2) < \frac{1}{2}.
\]

Then for every \( v \in \mathcal{B}_X^0(0) \) equation (3.6) has a unique solution \( w(s) \) that exists globally for all \( s \in \mathbb{R} \) and has the following properties:

(i) \( \sup_{s \in \mathbb{R}} \| w(s) \| \leq \nu^2 \kappa_0^2 K^2 \),

(ii) \( \sup_{s \in \mathbb{R}} \| w'(s) \| \leq \nu^2 \kappa_0^3 C(K) \),

(iii) \( \sup_{s \in \mathbb{R}} |A w(s)| \leq \nu \kappa_0^2 C(K) \),

(iv) \( \sup_{s \in \mathbb{R}} |A^{3/2} w(s)| \leq \| f \| / \nu + \nu \kappa_0^3 C(K) \),

(v) \( \sup_{s \in \mathbb{R}} |A w'(s)| \leq \kappa_0 C(K) (\| f \| + \nu^2 \kappa_0^3) \).

Moreover, suppose that \( v_1, v_2 \in \mathcal{B}_X^0(0) \) and \( w_1 \) and \( w_2 \) are the corresponding solutions of (3.6). Let \( \gamma = v_1 - v_2 \) and \( \delta = w_1 - w_2 \). Then

(vi) \( \sup_{s \in \mathbb{R}} \| \delta(s) \| \leq 4 \nu \kappa_0 \| \gamma \| \),

(vii) \( \sup_{s \in \mathbb{R}} \| \delta'(s) \| \leq \nu^2 \kappa_0^3 C(K) \| \gamma \| X, \)

(viii) \( \sup_{s \in \mathbb{R}} |A \delta(s)| \leq \nu \kappa_0^2 C(K) \| \gamma \| X, \)

where \( C(K) = c \exp(cK^2 \log K) \) for some universal constant \( c > 0 \).

The proof of Theorem 3.2 will be presented in §5.

The following corollary is an immediate consequence of Theorem 3.2.

**Corollary 3.3.** Assume the conditions of Theorem 3.2. Then there exists a Lipschitz continuous map
\[
W : \mathcal{B}_X^0(0) \rightarrow C_0^1(\mathbb{R}; D(A))
\]

with the following properties:

(i) \( W(v)(s) = w(s) \) for every \( v \in \mathcal{B}_X^0(0) \) and for all \( s \in \mathbb{R} \), where \( w(s) \) is the unique solution of (3.6) corresponding to the input \( v(s) \);

(ii) for every \( v_1, v_2 \in \mathcal{B}_X^0(0) \)
\[
\frac{1}{\kappa_0} \sup_{s \in \mathbb{R}} |A(W(v_1)(s) - W(v_2)(s))| + \sup_{s \in \mathbb{R}} \| W(v_1)(s) - W(v_2)(s) \|
\]
\[
+ \frac{1}{\nu \kappa_0^2} \sup_{s \in \mathbb{R}} \left\| \frac{d}{ds} \left( W(v_1)(s) - W(v_2)(s) \right) \right\| \leq \nu \kappa_0 C(K) \| v_1 - v_2 \| X.
\]

This map \( W(v) \) plays a crucial role in the definition of our determining form. To be more specific, let \( u^* \) be a steady state of the NSE (2.2). Our determining form is the equation
\[
\frac{dv}{dt} = F(v) = -\| v - JW(v) \|_{X,0}^2 (v - Ju^*).
\]

The precise properties of (3.10) are stated in Theorem 3.5, below. But first we need the following result.
Proposition 3.4. Suppose that $G \geq 1$. Then for every $u \in \mathcal{A}$

$$
\|J(u)\| \leq c^* \nu \kappa_0 G^3 \quad \text{and} \quad \|J(u')\| \leq c^* \nu^2 \kappa_0^3 G^7.
$$

Consequently,

$$
\|J(u)\|_X \leq c^* G^7 =: R.
$$

Proof. Since $\|J(\phi)\| = |\nabla J(\phi)|$, we apply (3.2) and use the fact that by (3.9) we have $h \kappa_0 \leq 2\pi$, together with Proposition 2.3. □

Theorem 3.5. Let $G \geq 1$ and suppose that the conditions of Theorem 3.2 hold for $\rho = 4R$, where $R := c^* G^7$ as in Proposition 3.4. Then the following hold.

(i) The vector field in the determining form (3.10) is a Lipschitz map from the ball $\mathcal{B}_X^0(0) = \{v \in X : \|v\|_X < \rho\}$ into $X$. Hence, (3.10) is an ODE in $X$ which has short-time existence and uniqueness of a solution for initial data in $\mathcal{B}_X^0(0)$.

(ii) Moreover, the ball $\mathcal{B}_X^{3R}(J(u^*)) = \{v \in X : \|v - J(u^*)\|_X < 3R\} \subset \mathcal{B}_X^0(0)$ is forward invariant in time under the dynamics of the determining form (3.10). Consequently, (3.10) has global existence and uniqueness of a solution for all initial data in $\mathcal{B}_X^{3R}(J(u^*))$.

(iii) Furthermore, every solution of (3.10) with initial data in $\mathcal{B}_X^{3R}(J(u^*))$ converges to the set of steady states of (3.10) as $t \to \infty$.

(iv) All the steady states of the determining form (3.10) that are contained in $\mathcal{B}_X^0(0)$ have the form $v(s) = Ju(s)$ for all $s \in \mathbb{R}$, where $u(s)$ is a trajectory in the global attractor $\mathcal{A}$ of the NSE (2.2).

Proof. To show short-time existence it is sufficient to show that the vector field (3.10) is Lipschitz. Let $F(v) = -g^2(v)(v - u^*)$, where $g(v) = \|v - JW(v)\|_{X,0}$. Since

$$
\|F(v_1) - F(v_2)\|_X \leq g^2(v_1) - g^2(v_2)\|v_1 - u^*\|_X + g^2(v_2)\|v_1 - v_2\|_X,
$$

it suffices to show that the map $g: \mathcal{B}_X^0(0) \to \mathbb{R}$ is Lipschitz. Note that for $v_1, v_2 \in \mathcal{B}_X^0(0)$ we have

$$
\|v_1 - JW(v_1)\|_{X,0} - \|v_2 - JW(v_2)\|_{X,0} \leq \|v_1 - JW(v_1) - [v_2 - JW(v_2)]\|_{X,0} \leq \|v_1 - v_2\|_{X,0} + \|JW(v_1) - JW(v_2)\|_{X,0}.
$$

Next, by (3.2) and the triangle inequality,

$$
\|JW(v_1) - JW(v_2)\|_{X,0} \leq (c_1 + 1) \sup_{s \in \mathbb{R}} \|W(v_1)(s) - W(v_2)(s)\| + \tilde{c}_2 h \sup_{s \in \mathbb{R}} \|A(W(v_1)(s) - W(v_2)(s))\|.
$$

By virtue of Corollary 3.3 and the fact that $h \kappa_0 \leq 2\pi$,

$$
\|v_1 - JW(v_1)\|_{X,0} - \|v_2 - JW(v_2)\|_{X,0} \leq C(K)\|v_1 - v_2\|_X,
$$

where $K$ satisfies (3.7) with $\rho = 4R$. This completes the proof of (i).
By Proposition 3.4, $B^3_X(J(u^*)) \subset B^6_X(0)$. Thus, we have short-time existence of a solution of (3.10) with initial data in $B^3_X(J(u^*))$. The proof of (ii) follows from the dissipativity property of (3.10), namely: for every fixed $s \in \mathbb{R}$

$$\frac{d}{dt} \|v(t; s) - J(u^*)\|^2 = -2\|v - JW(v)\|^2_{\mathcal{X}, 0} v(t; s) - J(u^*)\|^2,$$

$$\frac{d}{dt} \|v'(t; s)\|^2 = -2\|v - JW(v)\|^2_{\mathcal{X}, 0} v'(t; s)\|^2,$$

where $' = d/ds$. This property implies that the ball $B^3_X(J(u^*))$ is forward invariant for all $t \geq 0$, which simultaneously proves (ii) and (iii) (see the justification below concerning the steady states of the determining form (3.10)).

To prove part (iv) we observe that the steady states of equation (3.10) in the ball $B^6_X(0)$ are either $v = Ju^*$ or $v \in B^6_X(0)$ such that $\|v - JW(v)\|_{\mathcal{X}, 0} = 0$. In the first case $u^* \in \mathcal{A}$, since $u^*$ is a steady state of the NSE (2.2). In the second case we have $v(s) = JW(v)(s)$, that is, $v(s) = J(w(s))$ for all $s \in \mathbb{R}$, where $w(s)$ is a solution of (3.6). In this case it follows from (3.6) that $w(s)$ is a bounded solution of the NSE (2.2). Therefore, one concludes from (2.6) that $w(\cdot)$ is a trajectory in the global attractor $\mathcal{A}$ of the NSE. Conversely, since $\rho = 4R$, it follows from Proposition 3.4 that $J(\mathcal{A}) \subset B^3_X(J(u^*)) \subset B^6_X(0)$. Thus, for every trajectory $u(\cdot) \subset \mathcal{A}$ it follows from Proposition 3.1 and (3.6) that $u(s) = W(Ju)(s)$ for all $s \in \mathbb{R}$. In particular, $Ju = JW(Ju)$, and consequently $Ju$ is a steady state of (3.10) in $\mathcal{B}^6_X(0)$. □

4. Proof of Proposition 3.1

In view of (2.2) and (3.3) the difference $\delta = w - u$ satisfies

$$\frac{d}{ds} \delta + \nu A\delta + B(\delta, u) + B(u, \delta) + B(\delta, \delta) = -\mu \nu \kappa^2_0 \mathcal{P} J\delta.$$

Suppose that $\delta(\bar{s}) \neq 0$ for some $\bar{s} \in \mathbb{R}$. Since $\delta(s)$ is a continuous function with values in $V$, there is a maximal interval $(s_0, s_1)$ containing $\bar{s}$ such that $\delta(s) \neq 0$ for all $s \in (s_0, s_1)$. Taking the scalar product with $A\delta$ and using (2.8), (2.9), (2.18), we find that for all $s \in (s_0, s_1)$

$$\frac{1}{2} \frac{d}{ds} \|\delta\|^2 + \nu \|A\delta\|^2 = -(B(\delta, u), A\delta) - (B(u, \delta), A\delta) - \mu \nu \kappa^2_0 \mathcal{P} J\delta,$$

$$= (B(\delta, \delta), Au) - \mu \nu \kappa^2_0 (J\delta - \delta, A\delta) - \mu \nu \kappa^2_0 \|\delta\|^2$$

$$\leq c_T \|\delta\|^2 \left(1 + \log \frac{|A\delta|^2}{\kappa^2_0 \|\delta\|^2}\right) |Au| + \mu \nu \kappa^2_0 c_1 h^2 |A\delta|^2$$

$$+ \mu \nu \kappa^2_0 c_2 h \|\delta\| |A\delta| - \mu \nu \kappa^2_0 \|\delta\|^2$$

$$\leq c_T \|\delta\|^2 \left(1 + \log \frac{|A\delta|^2}{\kappa^2_0 \|\delta\|^2}\right) |Au| + \mu \nu \kappa^2_0 c_1 h^2 |A\delta|^2 - \frac{1}{2} \mu \nu \kappa^2_0 \|\delta\|^2.$$

By (3.5) we have

$$\frac{d}{ds} \|\delta\|^2 + \nu \kappa^2_0 \left[\mu + \frac{|A\delta|^2}{\kappa^2_0 \|\delta\|^2} - \frac{2c_T |Au|}{\nu \kappa^2_0} \left(1 + \log \frac{|A\delta|^2}{\kappa^2_0 \|\delta\|^2}\right)\right] \|\delta\|^2 \leq 0,$$
that is,
\[
\frac{d}{ds} \|\delta\|^2 + \nu \kappa_0^2 \left[ \mu + \theta - \alpha (1 + \log \theta) \right] \|\delta\|^2 \leq 0, \tag{4.1}
\]
where
\[
\theta = \frac{|A\delta|^2}{\kappa_0^2 \|\delta\|^2} \geq 1, \quad \alpha = \frac{2c_T |Au|}{\nu \kappa_0^2}.
\]

We now seek a lower estimate of \(\psi(\theta) = \theta - \alpha (1 + \log \theta)\) for \(\theta \geq 1\). Note that

\[
\psi(1) = 1 - \alpha, \quad \lim_{\theta \to \infty} \psi(\theta) = \infty,
\]
and that \(\psi\) is decreasing for \(\theta < \alpha\) and increasing for \(\theta > \alpha\). Thus,

\[
\min_{\theta \geq 1} \psi(\theta) = \begin{cases} 
\psi(1) = 1 - \alpha \geq 0 & \text{if } 0 \leq \alpha \leq 1, \\
\psi(\alpha) = -\alpha \log \alpha \geq 0 & \text{if } \alpha \geq 1.
\end{cases}
\]

Now note that for \(\alpha \in (0, 1]\) we have \(1 - \alpha \geq -\alpha \log \alpha\). Indeed, it is easy to check that \(\eta(\alpha) = 1 - \alpha + \alpha \log \alpha\) satisfies \(\eta(0^+) = 1\), \(\eta(1) = 0\), and \(\eta'(\alpha) = \log \alpha \leq 0\). We conclude that

\[\min_{\theta \geq 1} \psi(\theta) \geq -\alpha \log \alpha. \tag{4.2}\]

Applying (4.2) and then (2.22) to (4.1), we find that since \(u \in \mathscr{A}\),

\[
\frac{d}{ds} \|\delta\|^2 + \nu \kappa_0^2 \left[ \mu - \frac{2c_T |Au|}{\nu \kappa_0^2} \log(2c_T c_0 G^3) \right] \|\delta\|^2 \leq 0.
\]

It follows that

\[
\|\delta(s)\|^2 \leq \exp \left\{ -\nu \kappa_0^2 \mu + \frac{6c_T \log(c_3 G)}{s - \sigma_0} \int_{\sigma_0}^s |Au| \, d\tau \right\} (s - \sigma_0)^\|\delta(\sigma_0)\|^2,
\]
where \(c_3 = (2c_T c_0)^{1/3}\) and \(s_0 < \sigma_0 < s < s_1\). If \(s_0 > -\infty\), then \(\delta(s_0) = 0\), so we take \(\sigma_0 \to s_0^-\) and conclude that \(\delta(s) = 0\) for \(s \in (s_0, s_1)\). Otherwise it follows from (2.22) that

\[
\|\delta(s)\|^2 \leq \exp \left\{ \nu \kappa_0^2 [-\mu + 6c_T \log(c_3 G)] (s - \sigma_0) \right\} \|\delta(\sigma_0)\|^2
\]

for large enough \(|\sigma_0|\). Letting \(\sigma_0 \to -\infty\), we find by (3.4) that \(\delta(s) = 0\). Since \(s \in (s_0, s_1)\) is arbitrary, \(\delta(s) = 0\) in particular, a contradiction proving the proposition.

5. Proof of Theorem 3.2

In this section we give a formal proof of each estimate stated in Theorem 3.2. However, we describe below how to give a rigorous justification for the existence of a solution \(w\) of (3.6) which satisfies, together with \(w'\), these estimates. First, one considers the following Galerkin approximation system for (3.6):

\[
\frac{d w_n}{ds} + \nu A w_n + P_n B(w_n, w_n) = P_n f - \mu \nu \kappa_0^2 P_n \mathcal{D}(J w_n - v), \tag{5.1}
\]

where \(P_n\) is the orthogonal projection from \(H\) onto \(H_n := \text{span}\{w_1, w_2, \ldots, w_n\}\), the space of the first \(n\) eigenfunctions of the Stokes operator \(A\).
Proposition 5.1. Equation (5.1) has a solution \( w_n(s) \) for all \( s \in \mathbb{R} \) which satisfies, together with \( (dw_n/ds)(s) \), all the estimates in Theorem 3.2.

Proof. For \( k \in \mathbb{N} \) we supplement (5.1) with an initial value \( w_n(-k\nu\kappa_0^2) = 0 \) in order to obtain a finite system of ODEs with a quadratic polynomial non-linearity. Therefore, (5.1) with this initial data possesses a unique solution \( w_{n,k} \) on a small time interval symmetric about the initial time \( s = -k\nu\kappa_0^2 \). Furthermore, on this small interval \( dw_{n,k}/ds \) is the unique solution of the Cauchy problem

\[
\frac{d\tilde{w}_n}{ds} + \nu A\tilde{w}_n + P_n B(\tilde{w}_n, w_n) + P_n B(w_n, \tilde{\nu}_n) = -\nu\kappa_0^2 P_n \mathcal{P}(J\tilde{w}_n - \nu'), \tag{5.2}
\]

For the interval \([-k\nu\kappa_0^2, \infty)\) one can follow the same steps as used below in establishing estimates of \( \|w\| \) to show that the same estimates are also valid for \( \|w_{n,k}(s)\| \) when \( s \geq -k\nu\kappa_0^2 \). Thus, \( w_{n,k}(s) \) remains bounded for \( s \geq -k\nu\kappa_0^2 \), and as a result it solves (5.1) for \( s \in [-k\nu\kappa_0^2, \infty) \). And since \( |A((dw_{n,k}/ds)(-k\nu\kappa_0^2))| = |A(P_nf + \nu\kappa_0^2 P_n \mathcal{P}v)| \) is finite and independent of \( k \), one can show, by following steps similar to those used below for estimating \( \|w'\| \) and \( |Aw'| \), that \( |A((dw_{n,k}/ds)(s))| \) is bounded uniformly independent of \( k \) for all \( s \in [-k\nu\kappa_0^2, \infty) \).

Now let \( j \in \mathbb{N} \). Then by employing the Arzelà–Ascoli compactness theorem one can extract a subsequence \( w_{n,k}(j) \) of \( w_{n,k} \) which converges as \( k(j) \to \infty \) to a solution \( w'_n \) of (5.1) on the interval \([-j\nu\kappa_0^2, j\nu\kappa_0^2] \). Moreover, \( w_n(j) \) and \( (dw_n(j)/ds)(j) \) satisfy the estimates in Theorem 3.2 for all \( s \in [-j\nu\kappa_0^2, j\nu\kappa_0^2] \). Now by the Cantor diagonal process one can show that \( w_{n,k}(k) \) converges to \( w_n \) as \( k \to \infty \), and that \( w_n \) has the properties stated in the proposition. \( \square \)

We continue with our proof of the theorem. Based on Proposition 5.1, we use the Aubin compactness theorem (see, for instance, [6] or [22]) to show that for every \( m \in \mathbb{N} \) there exists a subsequence \( w_{n(m)} \) of \( w_n \) which converges to \( w^{(m)} \) in the relevant spaces on the interval \([-\nu\kappa_0 m, \nu\kappa_0 m] \) as \( n(m) \to \infty \). Moreover, by passing to the limit and following arguments similar to those for the 2D NSE, one infers that \( w^{(m)} \) is a solution of (3.6) on the interval \([-\nu\kappa_0 m, \nu\kappa_0 m] \), and in addition, \( w^{(m)} \) and \( (dw^{(m)}/ds)(s) \) satisfy the estimates in Theorem 3.2 on the interval \([-\nu\kappa_0 m, \nu\kappa_0 m] \). Now we again use the Cantor diagonal process to show that the diagonal subsequence \( w_{n(n)} \) converges as \( n \to \infty \) to a solution \( w \) of (3.6). Moreover, \( w \) and \( w' \) satisfy the estimates in Theorem 3.2 for all \( s \in \mathbb{R} \). This in turn concludes the formal justification of the estimates that will be established below.

5.1. Bound for \( \|w\| \). Taking the inner product of (3.6) with \( Aw \) and using (2.8), we have

\[
\frac{1}{2} \frac{d}{ds} \|w\|^2 + \nu |Aw|^2 = (f, Aw) + \nu \kappa_0^2 (w, Aw) - \nu \kappa_0^2 (Jw - w, Aw) - \nu \kappa_0^2 \|w\|^2.
\]

From (3.1) we get that

\[
\frac{1}{2} \frac{d}{ds} \|w\|^2 + \nu \kappa_0^2 |Aw|^2 \leq \frac{|f|^2}{2\nu} + \frac{\nu}{2} |Aw|^2 + \nu \kappa_0^2 \|w\|^2 + \frac{\nu \kappa_0^2}{4} \|w\|^2 - \nu \kappa_0^2 \|w\|^2
\]

\[
+ \frac{\nu \kappa_0^2}{4} \|w\|^2 + \nu \kappa_0^2 c_2 h^2 |Aw|^2 + \nu \kappa_0^2 c_2 h^2 |Aw|^2,
\]
and thus
\[
\frac{d}{ds} \|w\|^2 + \mu \nu \kappa_0^2 \|w\|^2 + \nu(1 - 2\mu h^2 \kappa_0^2(c_1^2 + c_2)) |Aw|^2 \leq \frac{|f|^2}{\nu} + 2\mu \nu \kappa_0^2 \|v\|^2. \tag{5.3}
\]

Therefore, if we assume that \(h\) is small enough that
\[
2\mu h^2 \kappa_0^2(c_1^2 + c_2) < \frac{1}{2}, \tag{5.4}
\]
then by Gronwall’s inequality and the assumption that \(\|w(s)\|\) is bounded we have the estimate
\[
\|w(s)\|^2 \leq 2\nu^2 \kappa_0^2 \|v\|_X^2 + \frac{|f|^2}{\mu \nu^2 \kappa_0^2} \leq \nu^2 \kappa_0^2 \left(2 \rho^2 + \frac{G^2}{\mu}\right) \leq \nu^2 \kappa_0^2 K^2. \tag{5.5}
\]

Next, we consider the evolution equation for \(w' = dw/ds\):
\[
\frac{dw'}{ds} + \nu Aw' + B(w', w) + B(w, w') = -\mu \nu \kappa_0^2 \mathcal{D}(Jw' - v'). \tag{5.6}
\]

5.2. Bound for \(\|w'\|\). Taking the inner product of (5.6) with \(Aw'\) and using (2.9) and (3.1) (after following steps similar to those above), one obtains
\[
\frac{d}{ds} |w'|^2 + \mu \nu \kappa_0^2 |w'|^2 + 2\nu(1 - \mu h^2 \kappa_0^2(c_1^2 + c_2)) |Aw'|^2 \\
\leq 2 \left|B(w', w'), Aw\right| + 2\mu \nu \kappa_0^2 |v'|^2 \\
\leq 2 |w'|_{L^\infty} |w'| |Aw| + 2\mu \nu \kappa_0^2 |v'|^2. \tag{5.7}
\]

Now by applying (2.28) to \(\|w'\|_{L^\infty}\) we get that
\[
|w'|_{L^\infty} |w'| |Aw| \leq L_N |w'|^2 |Aw| + (\pi^{1/2} \kappa_0 N)^{-1} |w'| |Aw'| |Aw| \\
\leq L_N |w'|^2 |Aw| + \frac{1}{\nu \kappa_0^2 \pi N^2} |w'|^2 |Aw|^2 + \frac{\nu}{4} |Aw'|^2.
\]

Hence by (5.4)
\[
\frac{d}{ds} |w'|^2 + \alpha |w'|^2 \leq 2\mu \nu \kappa_0^2 |v'|^2 \leq \mu \nu^5 \kappa_0^8 K^2,
\]
where
\[
\alpha = \mu \nu \kappa_0^2 - 2L_N |Aw| - \frac{2}{\nu \kappa_0^2 \pi N^2} |Aw|^2.
\]

We get from (5.3) and (5.4) that
\[
\frac{\nu}{2} |Aw(s)|^2 \leq - \frac{d}{ds} |w(s)|^2 + \mu \nu^3 \kappa_0^4 K^2, \tag{5.8}
\]
and in view of (5.5) integration gives us
\[
\int_{s-1/(\nu \kappa_0^2)}^s |Aw(\sigma)|^2 d\sigma \leq 2(1 + \mu) \nu \kappa_0^2 K^2.
\]
Applying the Cauchy–Schwarz inequality, we have

$$\int_{s-1/(\nu\kappa_0^2)}^s |Aw| \, d\sigma \leq (2(1 + \mu))^{1/2} K,$$

and hence

$$\int_{s-1/(\nu\kappa_0^2)}^s \alpha \geq \mu - 2^{1/2} \left(8 + 2\pi \log N\right)^{1/2} \left(1 + \mu\right)^{1/2} K - \frac{4(1 + \mu)}{\pi N^2} K^2.$$

We want to make sure that

$$\int_{s-1/(\nu\kappa_0^2)}^s \alpha(\sigma) \, d\sigma \geq 1 \quad \text{for all} \quad s \in \mathbb{R}. \quad (5.9)$$

If we choose $N^2 = K(1 + \mu)^{1/2}$, then (5.9) follows from requiring that

$$\pi(\mu - 1) \geq 2^{1/2} \left[8 + \pi \log K + \frac{\pi}{2} \log(1 + \mu)\right]^{1/2} (1 + \mu)^{1/2} K + 4(1 + \mu)^{1/2} K,$$

which automatically holds if

$$\pi^2(\mu - 1)^2 \geq 4 \left[8 + \pi \log K + \frac{\pi}{2} \log(1 + \mu)\right](1 + \mu)K^2 + 32(1 + \mu)K^2,$$

which is equivalent to

$$\frac{(\mu - 1)^2}{1 + \mu} \geq \frac{64}{\pi^2} K^2 + \frac{4}{\pi} K^2 \log K + \frac{2}{\pi} K^2 \log(1 + \mu).$$

Since

$$\frac{(\mu - 1)^2}{1 + \mu} \geq \frac{1 + \mu}{4}, \quad \forall \mu \geq 3,$$

it suffices that

$$\frac{1 + \mu}{4} \left[1 - \frac{8}{\pi} K^2 \log(1 + \mu)\right] \geq \frac{64}{\pi^2} K^2 + \frac{4}{\pi} (\log K)K^2.$$

Since $a \geq 2b \log b$, $b \geq 9$, it follows that $a/\log a \geq b$, and we set $b = 16K^2/\pi$ and $a = 1 + \mu$, so that if

$$1 + \mu \geq \frac{32K^2}{\pi} \log \frac{16K^2}{\pi}, \quad (5.10)$$

then it suffices that

$$\frac{1 + \mu}{8} \geq \frac{64}{\pi^2} K^2 + \frac{4}{\pi} K^2 \log K.$$

Thus, to ensure that (5.9) holds, we take

$$\mu \geq 80K^2 \log K. \quad (5.11)$$
By a similar calculation, again taking $N^2 = K(1 + \mu)^{1/2}$, we get that for $0 \leq r \leq 1$
\[
\int_{s-r/(\nu \kappa_0^2)}^s \alpha(\tau) \, d\tau \geq r \mu - \left[ \frac{4}{\pi} - r^{1/2} 2^{1/2} \right] (1 + r \mu)^{1/2} \pi
\times \left( 8 + \pi \log K + \frac{\pi}{2} \log(1 + \mu) \right)^{1/2} (1 + r \mu)^{1/2} K.
\]
Therefore, if
\[\mu \leq c' K^2 \log(c' K) \quad (5.12)\]
then for some $c'$ we have
\[
\sup_{0 \leq r \leq 1} \exp \left( - \int_{s-r/(\nu \kappa_0^2)}^s \alpha \right) \leq \exp(cK^2 \log K) \quad (5.13)
\]
for some absolute constant $c$. Ultimately, we will choose $c'$ so that (5.12) is compatible with (5.11) (see (5.20)).

**Lemma 5.2.** Let $\beta \geq 0$ be a constant and let $y(s) \geq 0$ be an absolutely continuous bounded function satisfying the inequality $y' + \alpha y \leq \beta$ for all $s \in \mathbb{R}$. Suppose also that $\int_{s-1/(\nu \kappa_0^2)}^s \alpha(\sigma) \, d\sigma \geq 1$ for all $s \in \mathbb{R}$. Then
\[y(s) \leq \frac{2 \beta}{\nu \kappa_0^2} \sup_{0 \leq r \leq 1} \exp \left( - \int_{s-r/(\nu \kappa_0^2)}^s \alpha \right) \quad \text{for all } s \in \mathbb{R}.
\]

**Proof.** Note that since $\int_{s-1/(\nu \kappa_0^2)}^s \alpha(\sigma) \, d\sigma \geq 1$, we have $\int_{-\infty}^0 \alpha = +\infty$. Multiplying $y' + \alpha y \leq \beta$ by the integrating factor $\exp(\int_0^s \alpha(\sigma) \, d\sigma)$ and integrating from $-\infty$ to $s$, we find that
\[y(s) \leq \beta \int_{-\infty}^s \exp \left( - \int_{\sigma}^s \alpha \right) \, d\sigma
\]
\[= \beta \int_{-\infty}^s \exp \left( - \int_{\sigma}^s \frac{(s-\sigma) \nu \kappa_0^2}{\nu \kappa_0^2} \right) \prod_{k=1}^{\lfloor (s-\sigma) \nu \kappa_0^2 \rfloor} \exp \left( - \int_{\sigma+(k-1)/(\nu \kappa_0^2)}^{\sigma+k/(\nu \kappa_0^2)} \alpha \right) \, d\sigma
\]
\[\leq \beta \sup_{0 \leq r \leq 1} \exp \left( - \int_{s-r/(\nu \kappa_0^2)}^s \alpha \right) \int_{-\infty}^s \prod_{k=1}^{\lfloor (s-\sigma) \nu \kappa_0^2 \rfloor} e^{-1} \, d\sigma
\]
\[= \beta \sup_{0 \leq r \leq 1} \exp \left( - \int_{s-r/(\nu \kappa_0^2)}^s \alpha \right) \int_{-\infty}^s e^{-\lfloor (s-\sigma) \nu \kappa_0^2 \rfloor} \, d\sigma
\]
\[= \beta \sup_{0 \leq r \leq 1} \exp \left( - \int_{s-r/(\nu \kappa_0^2)}^s \alpha \right) \frac{1 + e^{-1} + e^{-2} + \ldots}{\nu \kappa_0^2}
\]
\[= \frac{2 \beta}{\nu \kappa_0^2} \sup_{0 \leq r \leq 1} \exp \left( - \int_{s-r/(\nu \kappa_0^2)}^s \alpha \right) \quad \blacksquare
\]

From Lemma 5.2, (5.9), and (5.13),
\[\|w'(s)\| \leq c v^2 \kappa_0^3 \exp(c K^2 \log K) \quad \text{for all } s \in \mathbb{R}, \text{ for some } c > 0. \quad (5.14)\]
5.3. Bounds for $|Aw|$ and $|A^{3/2}w|$. From (5.8) we get that
\[
\frac{\nu}{2}|Aw|^2 \leq 2\|w''\|\|w\| + \mu \nu^3 \kappa_0^4 K^2 \leq \frac{1}{\nu \kappa_0^2} \|w''\|^2 + \nu \kappa_0^2 \|w\|^2 + \mu \nu^3 \kappa_0^4 K^2,
\]
and hence by (5.14) and (5.5) we have
\[
\sup_{s \in \mathbb{R}} |Aw(s)| \leq c \nu \kappa_0^2 \exp(cK^2 \log K). \quad (5.15)
\]
From (3.6) we find that
\[
A^{1/2}w' + \nu A^{3/2}w + A^{1/2}B(w, w) = A^{1/2}f - \mu \nu \kappa_0^2 A^{1/2} \mathcal{P}Jw + \mu \nu \kappa_0^2 A^{1/2} \mathcal{P}v,
\]
and thus by (2.24)
\[
\nu |A^{3/2}w| \leq \|w''\| + \|f\| + \mu \nu \kappa_0^2 \|v\| + \mu \nu \kappa_0^2 \|\mathcal{P}Jw\| + c'(|A^{3/4}w|^2 + |w|^{1/2}|Aw|^{3/2}).
\]
Since $h \kappa_0 \leq c$, we get from (3.2) that
\[
\|\mathcal{P}Jw\| \leq c(\|w\| + h|Aw|) \leq c \nu \kappa_0 \exp(cK^2 \log K).
\]
Thus, by (5.12) and (5.15) we have
\[
\sup_{s \in \mathbb{R}} |A^{3/2}w(s)| \leq \frac{1}{\nu} \|f\| + \nu \kappa_0^3 c \exp(cK^2 \log K).
\]

5.4. Bound for $|Aw'|$. Taking the inner product of $A^2w'$ with the equation
\[
\frac{dw'}{ds} + \nu Aw' + B(w', w) + B(w, w') = -\mu \nu \kappa_0^2 \mathcal{P}(Jw' - v'),
\]
we get that
\[
\frac{1}{2} \frac{d}{ds} |Aw'|^2 + \nu |A^{3/2}w'|^2 \leq \|(B(w', w), A^2w')\| + \|(B(w, w'), A^2w')\| - \mu \nu \kappa_0^2 |Aw'|^2
\]
\[
+ \mu \nu \kappa_0^2 |Jw' - w'\| |A^{3/2}w'| + \mu \nu \kappa_0^2 \|v'\| |A^{3/2}w'|.
\]
We bound the first non-linear term using (2.15) and (2.25) to get that
\[
|(B(w', w), A^2w')| = |A^{1/2}B(w', w)| |A^{3/2}w'|
\]
\[
\leq \frac{1}{\nu} |A^{1/2}B(w', w)|^2 + \frac{\nu}{4} |A^{3/2}w'|^2
\]
\[
\leq \frac{c}{\nu} \left( |A^{1/2}w'| |A^{1/2}w|^{1/2} |A^{3/2}w|^{1/2} + |A^{1/4}w'| |A^{5/4}w| \right)^2 + \frac{\nu}{4} |A^{3/2}w'|^2.
\]
For the second non-linear term we integrate by parts, using the fact that $A = -\Delta$ under periodic boundary conditions, so that by (2.7), (2.8), (2.13), (2.11), and (2.15) we have

$$
|\langle B(w, w'), A^2 w' \rangle| = \left| \int_\Omega ((w \cdot \nabla)w') \cdot \Delta^2 w' \, dx \right|
\leq \left| \int_\Omega ((\Delta w \cdot \nabla)w') \cdot \Delta w' \, dx \right| + 2 \sum_{j=1}^{2} \left| \int_\Omega ((\partial_j w \cdot \nabla)\partial_j w') \cdot \Delta w' \, dx \right|
\leq c\|\Delta w\|_{L^4} \|\nabla w'\| \|\Delta w'\|_{L^4} + c \sum_{j,k=1}^{2} \|\nabla w\| \|\partial_j \partial_k w'\|_{L^4}^2
\leq c\|\nabla^2 w\|_{H^{1/2}} \|w'\| \|\Delta w'\|_{H^{1/2}} + c \sum_{j,k=1}^{2} \|\nabla w\| \|\partial_j \partial_k w'\|_{H^{1/2}}^2
\leq c\|Aw\|^{1/2} \|A^{3/2} w\|^{1/2} \|w'\| \|Aw'\|^{1/2} \|A^{3/2} w'\|^{1/2} + c\|w\| \|Aw'\| |A^{3/2} w'|
\leq \frac{c}{\nu} \|Aw\| \|A^{3/2} w\| \|w'\| \|Aw'\| + \frac{c}{\nu} \|w\|^2 \|Aw'\|^2 + \frac{\nu}{4} |A^{3/2} w'|^2
\leq \frac{c}{\nu^2 \kappa_0^2} \|Aw\|^2 \|A^{3/2} w\|^2 \|w'\|^4 + \nu \kappa_0^2 (1 + cK^2) |Aw'|^2 + \frac{\nu}{4} |A^{3/2} w'|^2.
$$

Using (2.25), (3.2), and the bounds for $\|w'\|$ and $|A^{3/2} w|$, we now obtain by Gronwall’s inequality the uniform bound

$$
|Aw'(s)| \leq \kappa_0 c \exp(cK^2 \log K)(\|f\| + \nu^2 \kappa_0^3) \quad \text{for all } s \in \mathbb{R}, \quad (5.16)
$$

provided that $\mu \geq 2(1 + K^2)$ (which, in turn follows from (5.11)).

### 5.5. Lipschitz property of $w$ and $w'$ in the $D(A)$ norm.

In this section we show that the bounded solutions of (3.6) are unique and depend continuously on the input trajectory $v \in X$, in a sense that will be specified below. In particular, these properties are instrumental for introducing a well-defined map $v \mapsto W(v)$ from the space $X$ to a space of trajectories, which is defined by $W(v)(s) = w(s)$ for all $s \in \mathbb{R}$.

To obtain these properties we consider the difference $\delta(s) = w_1(s) - w_2(s)$ between two trajectories $w_1(s)$ and $w_2(s)$ and establish estimates for $\delta$ and $\delta'$ similar to those for $w$ and $w'$. Indeed, by linearity the only complication is in the non-linear term. Let $\gamma = v_1 - v_2$, and $\bar{w} = w_1 + w_2$. Then

$$
\frac{d}{ds} \delta + \nu A \delta + \frac{1}{2} B(\bar{w}, \delta) + \frac{1}{2} B(\delta, \bar{w}) = -\mu \nu \kappa_0^2 \mathcal{P}(J \delta - \gamma). \quad (5.17)
$$
5.6. Bound for $\|\delta\|$. Taking the scalar product of (5.17) with $A\delta$, we find as in §5.1 that
\[
\frac{1}{2} \frac{d}{ds} \|\delta\|^2 + \nu |A\delta|^2 + \mu \nu \kappa_0^2 \|\delta\|^2 = \mu \nu \kappa_0^2 (\gamma, A\delta) - \mu \nu \kappa_0^2 (J\delta - \delta, A\delta)
\]
\[
- \frac{1}{2} (B(\bar{w}, \delta), A\delta) - \frac{1}{2} (B(\delta, \bar{w}), A\delta)
\]
\[
\leq \mu \nu \kappa_0^2 \|\gamma\|^2 + \frac{\mu \nu \kappa_0^2}{4} \|\delta\|^2 + (c_T + c_B) \|\delta\| \|\bar{w}\| \left( \log \frac{e |A\delta|}{\kappa_0 \|\delta\|} \right)^{1/2} |A\delta|
\]
\[
+ \frac{\mu \nu \kappa_0^2}{4} \|\delta\|^2 + \mu \nu \kappa_0^2 c_1^2 h^2 |A\delta|^2 + \mu \nu \kappa_0^2 c_2^2 h^2 |A\delta|^2,
\]
where for the non-linear terms we used (2.18) and (2.19). Applying Young’s inequality to the contribution from the non-linear terms, we find that if (3.5) holds, then by (5.5)
\[
\frac{d}{ds} \|\delta\|^2 + \nu \kappa_0^2 \left[ \mu + \frac{|A\delta|^2}{\kappa_0^2 \|\delta\|^2} - 8(c_T + c_B)^2 K^2 \left( 1 + \log \frac{|A\delta|^2}{\kappa_0^2 \|\delta\|^2} \right) \right] \|\delta\|^2 \leq 2 \mu \nu \kappa_0^2 \|\gamma\|^2.
\]
At this point we can use (4.2) with $\theta = |A\delta|^2 (\kappa_0^2 \|\delta\|^2)^{-1}$ and $\alpha = 8(c_T + c_B)^2 K^2$ to obtain
\[
\frac{d}{ds} \|\delta\|^2 + \nu \kappa_0^2 \left[ \mu - 8(c_T + c_B)^2 K^2 \log \left( 8(c_T + c_B)^2 K^2 \right) \right] \|\delta\|^2 \leq 2 \mu \nu \kappa_0^2 \|\gamma\|^2.
\]
Therefore, if
\[
\mu > 32(c_T + c_B)^2 K^2 \log (\sqrt{8} (c_T + c_B) K),
\]
then
\[
\sup_{s \in \mathbb{R}} \|\delta(s)\| \leq 4 \nu \kappa_0 \|\gamma\|_X.
\]
To ensure compatibility of (5.11) and (5.18) we take, as in (3.8),
\[
c_4 K^2 \log (c_5 K^2) < \mu < 2 c_4 K^2 \log (c_5 K^2),
\]
where $c_4 = 80(c_T + c_B + 1)^2$ and $c_5 = \sqrt{8} (c_T + c_B + 1)$.

5.7. Bound for $\|\delta'\|$. The equation for $\delta'$ is
\[
\frac{d}{ds} \delta' + \nu A\delta' + \frac{1}{2} B(\bar{w}, \delta') + \frac{1}{2} B(\delta', \bar{w}) + \frac{1}{2} B(\delta, \bar{w}') + \frac{1}{2} B(\delta, \bar{w}')
\]
\[
= -\mu \nu \kappa_0^2 (J\delta' - \gamma').
\]
Taking the scalar product with $A\delta'$, we get that
\[
\frac{1}{2} \frac{d}{ds} \|\delta'|^2 + \nu |A\delta'|^2 - \frac{1}{2} (B(\delta', \delta'), A\bar{w}) + \frac{1}{2} (B(\bar{w}', \delta), A\delta') + \frac{1}{2} (B(\delta, \bar{w}'), A\delta')
\]
\[
= -\mu \nu \kappa_0^2 \|\delta'|^2 + \mu \nu \kappa_0^2 (\gamma', A\delta') - \mu \nu \kappa_0^2 (J\delta' - \delta', A\delta').
\]
Note that the difference from (5.7) (when one changes $w'$ to $\delta'$ and $w$ to $\bar{w}$) is in the addition of the two terms $(B(\bar{w}', \delta), A\delta')$ and $(B(\delta, \bar{w}'), A\delta')$, so that if we can
show that they are bounded by a constant multiple of $\|\gamma\|_X$, then we obtain the Lipschitz property of $w'$ by using the same methods as in § 5.2.

We begin with

$$\frac{d}{ds}\|\delta'\|^2 + \mu \nu \kappa_0^2\|\delta'\|^2 + 2\nu[1 - \mu h^2 \kappa_0^2(c_1^2 + c_2^2)]|A\delta'|^2 \leq |(B(\delta', \delta'), A\tilde{w})| + |(B(\tilde{w}', \delta), A\delta')| + |(B(\delta, \tilde{w}'), A\delta')| + 2\mu \nu \kappa_0^2|\gamma'|^2.$$ 

Consequently, we have

$$|(B(\delta', \delta'), A\tilde{w})| \leq \|\delta'\|_{L^2} \|\delta'\| \|A\tilde{w}\| \leq \mathcal{L}_N \|\delta'\| \|A\tilde{w}\| + (\pi^{1/2} \kappa_0 N)^{-1}\|\delta'\| |A\delta'| |A\tilde{w}| \leq \mathcal{L}_N \|\delta'\| \|A\tilde{w}\| + \frac{\|\delta'\|^2 |A\tilde{w}|^2}{2\pi \nu \kappa_0^2 N^2} + \frac{\nu}{2}|A\delta'|^2$$

and (by (2.16))

$$|(B(\delta, \tilde{w}'), A\delta')| \leq c_L |\delta|^{1/2} |\delta|^{1/2} |\tilde{w}'|^{1/2} |A\tilde{w}'|^{1/2} |A\delta'| \leq \frac{\nu}{4}|A\delta'|^2 + c_L^2 \nu^{-1} \kappa_0^{-2} |\delta|^2 |A\tilde{w}'|^2,$$

along with (by (2.17))

$$|(B(\tilde{w}', \delta), A\delta')| \leq c_A |\tilde{w}'|^{1/2} |A\tilde{w}'|^{1/2} |\delta| |A\delta'| \leq \frac{\nu}{4}|A\delta'|^2 + c_A^2 \nu^{-1} \kappa_0^{-2} |\delta|^2 |A\tilde{w}'|^2.$$

Thus,

$$\frac{d}{ds}\|\delta'\|^2 + \mu \nu \kappa_0^2\|\delta'\|^2 + \nu[1 - 2\mu h^2 \kappa_0^2(c_1^2 + c_2^2)]|A\delta'|^2 \leq \mathcal{L}_N \|\delta'\| |A\tilde{w}| + \frac{\|\delta'\|^2 |A\tilde{w}|^2}{2\pi \nu \kappa_0^2 N^2} + 2\mu \nu \kappa_0^2|\gamma'|^2 + (c_A^2 + c_L^2) \nu^{-1} \kappa_0^{-2} |\delta|^2 |A\tilde{w}'|^2 \leq \mathcal{L}_N \|\delta'\| |A\tilde{w}| + \frac{\|\delta'\|^2 |A\tilde{w}|^2}{2\pi \nu \kappa_0^2 N^2} + 2\mu \nu \kappa_0^2|\gamma'|^2 + (c_A^2 + c_L^2) \nu^{-1} \kappa_0^{-2} |\delta|^2 |A\tilde{w}'|^2.$$

By (3.5) we can drop the term containing $|A\delta'|^2$. Then applying (5.16) to $|A\tilde{w}'| \leq |Aw'_1| + |Aw'_2|$ and using (5.19), we have

$$\frac{d}{ds}\|\delta'\|^2 + \alpha \|\delta'\|^2 \leq \beta,$$

where

$$\alpha = \mu \nu \kappa_0^2 - \mathcal{L}_N \|\delta'\| |A\tilde{w}| + \frac{|A\tilde{w}|^2}{2\pi \nu \kappa_0^2 N^2}$$

and

$$\beta = 2\mu \nu \kappa_0^2 |\gamma|^2 + 32 c \nu \kappa_0^2 (c_A^2 + c_L^2) \exp(c K^2 \log K)(\|f\| + \nu \kappa_0^2)^2 |\gamma|^2_X.$$
Proceeding as in §5.2, we get that

$$\sup_{s \in \mathbb{R}} \| \delta'(s) \| \leq (\| f \| + \nu^2 \kappa_0^3) C \| \gamma \| X,$$

where $C = c \exp(cK^2 \log K)$ for some universal constant $c$. Note that once again (5.9) suffices to ensure that (5.12) holds, so there is no need to modify the range for $\mu$ in (5.12).

5.8. **Bound for $|A\delta|$**. We find from (5.17) together with (2.16), (2.17), and (3.1) that

$$\nu |A\delta| \leq |\delta'| + \frac{cA + cL}{2\kappa_0} |A \tilde{w}| |\delta| + \mu \nu \kappa_0^2 |\gamma|$$

$$+ \mu \nu \kappa_0^2 |\delta| + \mu h \nu c_1 \kappa_0^2 |\delta| + \mu h^2 c_2 \nu \kappa_0^2 |A\delta|,$$

and hence

$$\nu (1 - \mu h^2 c_2 \kappa_0^2) |A\delta| \leq \kappa_0^{-1} |\delta'| + \mu \nu \kappa_0 |\gamma| \left[ \mu \nu \kappa_0 (1 + hc_1 \kappa_0) + \frac{cA + cL}{2\kappa_0} |A \tilde{w}| \right] |\delta|. $$

Therefore, if (3.5) holds, then by (5.15) and (5.19)

$$\sup_{s \in \mathbb{R}} |A\delta(s)| \leq \frac{C}{\nu \kappa_0} (\| f \| + \nu^2 \kappa_0^3) \| \gamma \| X,$$

with $C = c \exp(cK^2 \log K)$ for some universal constant $c > 0$.

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