On some conjectures concerning critical independent sets of a graph

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Abstract

Let $G$ be a simple graph with vertex set $V(G)$. A set $S \subseteq V(G)$ is independent if no two vertices from $S$ are adjacent. For $X \subseteq V(G)$, the difference of $X$ is $d(X) = |X| - |N(X)|$ and an independent set $A$ is critical if $d(A) = \max\{d(X) : X \subseteq V(G)\text{ is an independent set}\}$ (possibly $A = \emptyset$). Let nucleus($G$) and diadem($G$) be the intersection and union, respectively, of all maximum size critical independent sets in $G$. In this paper, we will give two new characterizations of König-Egerváry graphs involving nucleus($G$) and diadem($G$). We also prove a related lower bound for the independence number of a graph. This work answers several conjectures posed by Jarden, Levit, and Man-drescu.

Keywords: maximum independent set, maximum critical independent set, König-Egerváry graph, maximum matching, core, corona, ker, diadem, nucleus.

1 Introduction

In this paper $G$ is a simple graph with vertex set $V(G)$, $|V(G)| = n$, and edge set $E(G)$. The set of neighbors of a vertex $v$ is $N_G(v)$ or simply $N(v)$ if there is no possibility of ambiguity. If $X \subseteq V(G)$, then the set of neighbors of $X$ is $N(X) = \cup_{u \in X} N(u)$, $G[X]$ is the subgraph induced by $X$, and $X^c$ is the complement of the subset $X$. For sets $A, B \subseteq V(G)$, we use $A \setminus B$ to denote the vertices belonging to $A$ but not $B$. For such disjoint $A$ and $B$ we let $(A, B)$ denote the set of edges such that each edge is incident to both a vertex in $A$ and a vertex in $B$.

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A matching $M$ is a set of pairwise non-incident edges of $G$. A matching of maximum cardinality is a maximum matching and $\mu(G)$ is the cardinality of such a maximum matching. For a set $A \subseteq V(G)$ and matching $M$, we say $A$ is saturated by $M$ if every vertex of $A$ is incident to an edge in $M$. For two disjoint sets $A, B \subseteq V(G)$, we say there is a matching $M$ of $A$ into $B$ if $M$ is a matching of $G$ such that every edge of $M$ belongs to $(A, B)$ and each vertex of $A$ is saturated. An $M$-alternating path is a path that alternates between edges in $M$ and those not in $M$. An $M$-augmenting path is an $M$-alternating path which begins and ends with an edge not in $M$.

A set $S \subseteq V(G)$ is independent if no two vertices from $S$ are adjacent. An independent set of maximum cardinality is a maximum independent set and $\alpha(G)$ is the cardinality of such a maximum independent set. For a graph $G$, let $\Omega(G)$ denote the family of all its maximum independent sets, let $\text{core}(G) = \bigcap\{S : S \in \Omega(G)\}$, and $\text{corona}(G) = \bigcup\{S : S \in \Omega(G)\}$.

See [1, 9, 14] for background and properties of $\text{core}(G)$ and $\text{corona}(G)$.

For a graph $G$ and a set $X \subseteq V(G)$, the difference of $X$ is $d(X) = |X| - |N(X)|$ and the critical difference $d(G)$ is $\max\{d(X) : X \subseteq V(G)\}$. Zhang [16] showed that $\max\{d(X) : X \subseteq V(G)\} = \max\{d(S) : S \subseteq V(G) \text{ is an independent set}\}$. The set $X$ is a critical set if $d(X) = d(G)$. The set $S \subseteq V(G)$ a critical independent set if $S$ is both a critical set and independent. A critical independent set of maximum cardinality is called a maximum critical independent set. Note that for some graphs the empty set is the only critical independent set, for example odd cycles or complete graphs. See [2, 7, 8, 16] for more background and properties of critical independent sets.

Finding a maximum independent set is a well-known NP-hard problem. Zhang [16] first showed that a critical independent set can be found in polynomial time. Butenko and Trukhanov [2] showed that every critical independent set is contained in a maximum independent set, thereby directly connecting the problem of finding a critical independent set to that of finding a maximum independent set.

For a graph $G$ the inequality $\alpha(G) + \mu(G) \leq n$ always holds. A graph $G$ is a König-Egerváry graph if $\alpha(G) + \mu(G) = n$. All bipartite graphs are König-Egerváry but there are non-bipartite graphs which are König-Egerváry as well, see Figure [2] for an example. We adopt the convention that the empty graph $K_0$, without vertices, is a König-Egerváry graph. In [7] it was shown that König-Egerváry graphs are closely related to critical independent sets.
Theorem 1.1. [7] A graph $G$ is König-Egerváry if, and only if, every maximum independent set in $G$ is critical.

Theorem 1.2. [7] For any graph $G$, there is a unique set $X \subseteq V(G)$ such that all of the following hold:

(i) $\alpha(G) = \alpha(G[X]) + \alpha(G[X^c])$,
(ii) $G[X]$ is a König-Egerváry graph,
(iii) for every non-empty independent set $S$ in $G[X^c]$, $|N(S)| \geq |S|$, and
(iv) for every maximum critical independent set $I$ of $G$, $X = I \cup N(I)$.

Larson in [8] showed that a maximum critical independent set can be found in polynomial time. So the decomposition in Theorem 1.2 of a graph $G$ into $X$ and $X^c$ is also computable in polynomial time. Figure 1 gives an example of this decomposition, where both the sets $X$ and $X^c$ are non-empty. Recall, for some graphs the empty set is the only critical independent set, so for such graphs the set $X$ would be empty. If a graph $G$ is a König-Egerváry graph, then the set $X^c$ would be empty. We adopt the convention that if $K_0$ is empty graph, then $\alpha(K_0) = 0$.

![Figure 1](image_url)

Figure 1: $G$ has maximum critical independent set $I = \{a, b, c\}$. Theorem 1.2 gives that $X = \{a, b, c, d, e\}$ and $X^c = \{f, g, h, i, j\}$.

In [5][11] the following concepts were introduced: for a graph $G$,

$\ker(G) = \bigcap \{S : S \text{ is a critical independent set in } G\}$,
$\diadem(G) = \bigcup \{S : S \text{ is a critical independent set in } G\}$, and
$\nucleus(G) = \bigcap \{S : S \text{ is a maximum critical independent set in } G\}$.

However, the following result due to Larson allows us to use a more suitable definition for $\diadem(G)$.

Theorem 1.3. [8] Each critical independent set is contained in some maximum critical independent set.
For the remainder of this paper we define
\[ \text{diadem}(G) = \bigcup \{ S : S \text{ is a maximum critical independent set in } G \}. \]

Note that if \( G \) is a graph where the empty set is the only critical independent set (including the case \( G = K_0 \), the empty graph), then \( \text{ker}(G) \), \( \text{diadem}(G) \), and \( \text{nucleus}(G) \) are all empty. See Figure 2 for examples of the sets \( \text{ker}(G) \), \( \text{diadem}(G) \), and \( \text{nucleus}(G) \).

Figure 2: \( G_1 \) is a König-Egerváry graph with \( \text{ker}(G_1) = \{a, b\} \subsetneq \text{core}(G_1) = \{a, b, d\} \) and \( \text{diadem}(G_1) = \text{corona}(G_1) = \{a, b, c, d, f\} \). \( G_2 \) is not a König-Egerváry graph and has \( \text{ker}(G_2) = \text{core}(G_2) = \{a, b\} \subsetneq \text{nucleus}(G_2) = \{a, b, d\} \) and \( \text{diadem}(G_2) = \{a, b, c, d, f\} \subsetneq \text{corona}(G) = \{a, b, c, d, f, g, h, i, j\} \).

In [4, 5], the following necessary conditions for König-Egerváry graphs were given:

**Theorem 1.4.** [4] If \( G \) is a König-Egerváry graph, then
\[ (i) \quad \text{diadem}(G) = \text{corona}(G), \quad \text{and} \]
\[ (ii) \quad |\text{ker}(G)| + |\text{diadem}(G)| \leq 2\alpha(G). \]

**Theorem 1.5.** [5] If \( G \) is a König-Egerváry graph, then \[ |\text{nucleus}(G)| + |\text{diadem}(G)| = 2\alpha(G). \]

In [4] it was conjectured that condition (i) of Theorem 1.4 is sufficient for König-Egerváry graphs and in [5] it was conjectured the necessary condition in Theorem 1.5 is also sufficient. The purpose of this paper is to affirm these conjectures by proving the following new characterizations of König-Egerváry graphs.
Theorem 1.6. For a graph $G$, the following are equivalent:

(i) $G$ is a König-Egerváry graph,
(ii) $\text{diadem}(G) = \text{corona}(G)$, and
(iii) $|\text{diadem}(G)| + |\text{nucleus}(G)| = 2\alpha(G)$.

The paper [4] gives an upper bound for $\alpha(G)$ in terms of unions and intersections of maximum independent sets, proving

$$2\alpha(G) \leq |\text{core}(G)| + |\text{corona}(G)|$$

for any graph $G$. It is natural to ask whether a similar lower bound for $\alpha(G)$ can be formulated in terms of unions and intersections of critical independent sets. Jarden, Levit, and Mandrescu in [4] conjectured that for any graph $G$, the inequality $|\ker(G)| + |\text{diadem}(G)| \leq 2\alpha(G)$ always holds. We will prove a slightly stronger statement. By Theorem 1.3 we see that $\ker(G) \subseteq \text{nucleus}(G)$ holds implying that $|\ker(G)| + |\text{diadem}(G)| \leq |\text{nucleus}(G)| + |\text{diadem}(G)|$. In section 4 we will prove the following statement, resolving the cited conjecture:

Theorem 1.7. For any graph $G$,

$$|\text{nucleus}(G)| + |\text{diadem}(G)| \leq 2\alpha(G).$$

It would be interesting to know whether the sets $\text{nucleus}(G)$ and $\text{diadem}(G)$, or their sizes, can be computed in polynomial time.

2 Some structural lemmas

Here we prove several crucial lemmas which will be needed in our proofs. Our results hinge upon the structure of the set $X$ as described in Theorem 1.2.

Lemma 2.1. Let $I$ be a maximum critical independent set in $G$ and set $X = I \cup N(I)$. Then $\text{diadem}(G) \cup N(\text{diadem}(G)) = X$.

Proof. By Theorem 1.2 the set $X$ is unique in $G$, that is, for any maximum critical independent set $S$, $X = S \cup N(S)$. Then $\text{diadem}(G) = X$ follows by definition.

Lemma 2.2. Let $I$ be a maximum critical independent set in $G$ and set $X = I \cup N(I)$. Then $\text{diadem}(G) \subseteq \text{diadem}(G[X])$ and $\text{nucleus}(G[X]) \subseteq \text{nucleus}(G)$.
Proof. Let $S$ be a maximum critical independent set in $G$. Using Theorem 1.2 we see that $S$ is a maximum independent set in $G[X]$ and also $G[X]$ is a König-Egerváry graph. Then Theorem 1.1 gives that $S$ must also be critical in $G[X]$, which implies that $\text{diadem}(G) \subseteq \text{diadem}(G[X])$.

Now let $v \in \text{nucleus}(G[X])$. Then $v$ belongs to every maximum critical independent set in $G[X]$. As remarked above, since every maximum critical independent set in $G$ is also a maximum critical independent set in $G[X]$, then $v$ belongs to every maximum critical independent set in $G$. This shows that $v \in \text{nucleus}(G)$ and $\text{nucleus}(G[X]) \subseteq \text{nucleus}(G)$ follows.

Lemma 2.3. Suppose $I$ is a non-empty maximum critical independent set in $G$, set $X = I \cup N(I)$, let $A = \text{nucleus}(G) \setminus \text{nucleus}(G[X])$, and let $S$ be a maximum independent set in $G[X]$. For $S' \subseteq S \cap N(A)$, if there exists $A' \subseteq A$ such that $N(A') \cap S \subseteq S'$, then $|S'| \geq |A'|$.

Proof. For $S' \subseteq S \cap N(A)$ suppose such an $A'$ exists. For sake of contradiction, suppose that $|S'| < |A'|$. Since $A' \subseteq \text{nucleus}(G)$, then $A'$ is an independent set. Also since $A' \subseteq \text{nucleus}(G) \subseteq \text{diadem}(G)$, by Lemma 2.1 we have $A' \subseteq X$. Furthermore, since $N(A') \cap S \subseteq S'$ then $A' \cup (S \setminus S')$ is an independent set in $G[X]$. Now by assumption $|S'| < |A'|$, so $A' \cup (S \setminus S')$ is an independent set in $G[X]$ larger than $S$, which cannot happen. Therefore we must have $|S'| \geq |A'|$ as desired.

Lemma 2.4. Let $I$ be a maximum critical independent set in $G$ and set $X = I \cup N(I)$. Then

$$|\text{nucleus}(G)| + |\text{diadem}(G)| \leq |\text{nucleus}(G[X])| + |\text{diadem}(G[X])|.$$ 

Proof. First note that if the set $X$ is empty, then by Lemma 2.1 both sides of the inequality are zero. So let us assume that $X$ is non-empty. Now consider the set $A = \text{nucleus}(G) \setminus \text{nucleus}(G[X])$. If this independent set is empty, then $\text{nucleus}(G) = \text{nucleus}(G[X])$ and there is nothing to prove since $\text{diadem}(G) \subseteq \text{diadem}(G[X])$ holds by Lemma 2.2. If $A$ is non-empty, for each $v \in A$ there is some maximum independent set $S$ of $G[X]$ which doesn’t contain $v$. Since $S$ is a maximum independent set there exists $u \in N(v) \cap S$. Since $v \in \text{nucleus}(G)$, then $u$ does not belong to any maximum critical independent set in $G$. Recall by Theorem 1.2 (ii) $G[X]$ is a König-Egerváry graph, so Theorem 1.1 gives that $S$ is a maximum critical independent set in $G[X]$. It follows that $u \in \text{diadem}(G[X]) \setminus \text{diadem}(G)$, which shows each vertex in $A$ is adjacent to at least one vertex in $\text{diadem}(G[X]) \setminus \text{diadem}(G)$.
Now we will show there is a maximum matching from $A$ into $\text{diadem}(G[X]) \setminus \text{diadem}(G)$ with size $|A|$. For sake of contradiction, suppose such a matching $M$ has less than $|A|$ edges. Then there exists some vertex $v \in A$ not saturated by $M$. By the above, $v$ is adjacent to some vertex $u \in \text{diadem}(G[X]) \setminus \text{diadem}(G)$. Since $M$ is maximum, $u$ is matched to some vertex $w \in A$ under $M$. Now let $S$ be a maximum independent set of $G[X]$ containing $u$. We now restrict ourselves to the subgraph induced by the edges $(A \cap N(S), S \cap N(A))$, noting this subgraph is bipartite since both $A \cap N(S)$ and $S \cap N(A)$ are independent. In this subgraph, consider the set $\mathcal{P}$ of all $M$-alternating paths starting with the edge $vu$. Note that all such paths must start with the vertices $v, u$, then $w$. Also, such paths must end at either a matched vertex in $A \cap N(S)$ or an unmatched vertex in $S \cap N(A)$.

We wish to show that there is some alternating path ending at an unmatched vertex in $S \cap N(A)$. For sake of contradiction, suppose all alternating paths end at a matched vertex in $A \cap N(S)$ and let $V(\mathcal{P})$ denote the union of all vertices belonging to such an alternating path. We aim to show this scenario contradicts Lemma 2.3. Now clearly we must have $N(V(\mathcal{P}) \cap A) \cap S \subseteq V(\mathcal{P}) \cap S$, else we could extend an alternating path to any vertex in $(N(V(\mathcal{P}) \cap A) \cap S) \setminus (V(\mathcal{P}) \cap S)$. Also, since all paths in $\mathcal{P}$ end at a matched vertex in $A \cap N(S)$, then every vertex of $V(\mathcal{P}) \cap S$ is matched under $M$, and such a situation should look as in Figure 3.

![Figure 3](image_url)

Figure 3: What the $M$-alternating paths could look like between $V(\mathcal{P}) \cap A$ and $V(\mathcal{P}) \cap S$, where solid lines represent matched edges in $M$ and dotted lines represent the unmatched edges.

From this it follows that $|V(\mathcal{P}) \cap S| < |V(\mathcal{P}) \cap A|$. The previous statements exactly contradict Lemma 2.3 so there is some alternating path $P$ ending at
an unmatched vertex \( x \in S \cap N(A) \). This means that \( P \) is an \( M \)-augmenting path. A well-known theorem in graph theory states that a matching is maximum in \( G \) if, and only if, there is no augmenting path \([15]\). So \( P \) being an \( M \)-augmenting path contradicts our assumption that \( M \) is a maximum matching.

Therefore there is a matching \( M \) from \( A \) into \( \text{diadem}(G[X]) \setminus \text{diadem}(G) \). This matching implies that \( | \text{nucleus}(G) \setminus \text{nucleus}(G[X]) | \leq | \text{diadem}(G[X]) \setminus \text{diadem}(G) | \). Since both \( \text{nucleus}(G[X]) \subseteq \text{nucleus}(G) \) and \( \text{diadem}(G) \subseteq \text{diadem}(G[X]) \) by Lemma 2.2, the lemma follows.

3 New characterizations of König-Egerváry graphs

Proof (of Theorem 1.6). First we prove \((ii) \Rightarrow (i)\). Suppose that \( \text{diadem}(G) = \text{corona}(G) \) holds and let \( I \) be a maximum critical independent set with \( X = I \cup N(I) \). We will use the decomposition in Theorem 1.2 to show that \( X^c \) must be empty and hence, \( G = G[X] \) is a König-Egerváry graph. By Lemma 2.1 we have \( \text{corona}(G) = \text{diadem}(G) \subseteq X \), in other words every maximum independent set in \( G \) is contained in \( X \). This implies that \( |I| = \alpha(G[X]) = \alpha(G) \). Now by Theorem 1.2 \((i)\), \( \alpha(G) = \alpha(G[X]) + \alpha(G[X^c]) \) showing that we must have \( \alpha(G[X^c]) = 0 \). Now clearly the result follows, since \( \alpha(G[X^c]) = 0 \) implies that \( X^c \) must be empty.

To prove \((iii) \Rightarrow (i)\), again we will use the decomposition in Theorem 1.2 to show that \( X^c \) must be empty and hence, \( G \) is a König-Egerváry graph. So suppose that \( | \text{diadem}(G) \setminus \text{nucleus}(G) | = 2\alpha(G) \) and let \( I \) be a maximum critical independent set in \( G \) with \( X = I \cup N(I) \). Lemma 2.1 implies that

\[
2\alpha(G) = | \text{diadem}(G) | + | \text{nucleus}(G) | \leq | \text{diadem}(G[X]) | + | \text{nucleus}(G[X]) | .
\]

Theorem 1.2 \((ii)\) gives that \( G[X] \) is König-Egerváry, so by Corollary 1.5 we have \( | \text{diadem}(G[X]) | + | \text{nucleus}(G[X]) | = 2\alpha(G[X]) \) implying that \( \alpha(G) \leq \alpha(G[X]) \). It follows by Theorem 1.2 \((i)\) we must have \( \alpha(G) = \alpha(G[X]) \), so again we know that \( \alpha(G[X^c]) = 0 \) which finishes this part of the proof.

The implications \((i) \Rightarrow (ii)\) and \((i) \Rightarrow (iii)\) are given in Theorem 1.4 and in Theorem 1.5.

4 A bound on \( \alpha(G) \)

Proof (of Theorem 1.7). Let \( I \) be a maximum critical independent set in \( G \) and \( X = I \cup N(I) \). By Theorem 1.2 \((ii)\), \( G[X] \) is a König-Egerváry graph.
so by Theorem 1.5 we have

\[ |\text{nucleus}(G[X])| + |\text{diadem}(G[X])| = 2\alpha(G[X]) \leq 2\alpha(G). \]

Now by Lemma 2.4 we must have

\[ |\text{nucleus}(G)| + |\text{diadem}(G)| \leq |\text{nucleus}(G[X])| + |\text{diadem}(G[X])| \]

and the theorem follows. \qed

Combining Theorem 1.7 and the inequality $2\alpha(G) \leq |\text{core}(G)| + |\text{corona}(G)|$ proven in [4], the following corollary is immediate.

**Corollary 4.1.** For any graph $G$,

\[ |\text{nucleus}(G)| + |\text{diadem}(G)| \leq 2\alpha(G) \leq |\text{core}(G)| + |\text{corona}(G)|. \]

These upper and lower bounds are quite interesting. The fact that every critical independent set is contained in a maximum independent set implies that $\text{diadem}(G) \subseteq \text{corona}(G)$ for all graphs $G$. However, the graph $G_2$ in Figure 2 has $\text{core}(G_2) \subsetneq \text{nucleus}(G_2)$ while the graph $G$ in Figure 1 has $\text{nucleus}(G) = \{a, b, c\} \subsetneq \text{core}(G) = \{a, b, c, h\}$.

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