ARTIN PERVERSE SHEAVES

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Abstract. We study the possible properties of an integral perverse motivic t-structure, looking at an analog in the ℓ-adic setting. We show that the perverse t-structure induces a t-structure on the category $D^A(S, \mathbb{Z}_\ell)$ of Artin ℓ-adic complexes over excellent schemes of dimension less than 2 and provide a counter-example in dimension 3. Its heart $\text{Perv}^A(S, \mathbb{Z}_\ell)$ can be described explicitly in terms of representations in the case of a 1-dimensional excellent scheme.

Over schemes of finite type over a finite field and with coefficients $\mathbb{Q}_\ell$, we also construct a homotopy perverse t-structure and show that it is final among the t-structures such that the inclusion functor is right t-exact. We describe the simple objects of its heart $\text{Perv}^A(S, \mathbb{Q}_\ell)^\#$ and show that the Artin truncation functor $\omega^0$ is t-exact.

We also show that the weightless intersection complex $\text{EC}_S = \omega^0\text{IC}_S$ is a simple Artin perverse sheaf. If $S$ is a surface, it is also a perverse sheaf but it need not simple as a perverse sheaf.

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We have at our disposal several stable ∞-categories of mixed motives $DM(\_)$ endowed with Grothendieck’s six functors formalism and with ℓ-adic realization functors \cite{Voe92, VSF11, Ayo07, Ayo14, CD19, CD16}. Let $S$ be a noetherian finite-dimensional scheme. One of the major open problems regarding the category $DM(\_)$ is the existence of a motivic t-structure, whose heart would be the abelian category of mixed motivic sheaves that satisfies Beilinson’s conjectures \cite{Jan94}.

This problem seems completely out of reach at the moment. However, if $n = 0, 1$, an ordinary motivic t-structure has been constructed on the subcategory $DM^c_n(S, \mathbb{Q})$ of $n$-motives which is the subcategory generated by the cohomological motives of proper $S$-schemes of dimension less than $n$ \cite{Org04, Ayo11, ABV09, BVK16, Leh19b, Leh19a, Vai19}. We state the main result:

**Theorem.** (Voevodsky, Orgogozo, Ayoub, Barbieri-Viale, Kahn, Pépin Lehalleur, Vaiš) Let $S$ be a noetherian excellent finite dimensional scheme admitting resolution of singularities by alterations. Let $\ell$ be a prime number invertible on $S$. Let $n = 0, 1$.

Then, there exists a non-degenerate $t$-structure on $DM^n(S, \mathbb{Q})$ that restricts to the stable $\infty$-category $DM^c_n(S, \mathbb{Q})$ of constructible $n$-motives and such that the $\ell$-adic realization functor:

$$DM^c_n(S, \mathbb{Q}) \to Db^c(S, \mathbb{Q}_\ell)$$

is $t$-exact when the stable $\infty$-category $Db^c(S, \mathbb{Q}_\ell)$ of constructible $\ell$-adic complexes is endowed with its ordinary $t$-structure.

Working by analogy with $Db^c(S, \mathbb{Q}_\ell)$, there are two possible versions of the motivic $t$-structure: the perverse motivic $t$-structure and the ordinary motivic $t$-structure. As the category $Db(S, \mathbb{Q}_\ell)$ is the bounded derived category of the abelian category of perverse sheaves over $S$ \cite{Beb77b}, the category of mixed motivic sheaves is
conjectured to be the heart of perverse motivic t-structure. In [Bon15], Bondarko has shown that the perverse motivic t-structure on $\mathcal{DM}(S)$ can be recovered from the ordinary motivic t-structures on $\mathcal{DM}(K)$ (assuming it exists) when $K$ runs though the set of residue fields of $S$. However, his approach does not apply to the dimensional subcategories $\mathcal{DM}^n(S, \mathbb{Q})$.

On another note, with integral coefficients, it is not possible to construct a reasonable motivic t-structure on the Nisnevich model $\mathcal{DM}_{Nis}(S, \mathbb{Z})$ [VSF11, 4.3.8]. However, nothing is known about a t-structure on $\mathcal{DM}^n(S, \mathbb{Z})$ for the étale or the Nisnevitch model.

Hence, the above theorem leaves two open questions:

1. Is there a t-structure on $\mathcal{DM}^n(S, \mathbb{Z})$ that restricts to $\mathcal{DM}^n_c(S, \mathbb{Z})$ and such that the $\ell$-adic realization functor:

   $$\mathcal{DM}^n_c(S, \mathbb{Z}) \to \mathcal{D}^b_c(S, \mathbb{Z}_\ell)$$

   is t-exact when $\mathcal{D}^b(S, \mathbb{Z}_\ell)$ is endowed with its ordinary t-structure?

2. Let $R = \mathbb{Z}$ or $\mathbb{Q}$. Is there a t-structure on $\mathcal{DM}^n(S, R)$ that restricts to the stable $\infty$-category $\mathcal{DM}^n_c(S, R)$ and such that the $\ell$-adic realization functor:

   $$\mathcal{DM}^n_c(S, R) \to \mathcal{D}^b_c(S, R_\ell)$$

   is t-exact when $\mathcal{D}^b(S, R_\ell)$ is endowed with its perverse t-structure?

In this paper, we aim to determine what answer to this question could reasonably be expected in the case $n = 0$, looking at the analogous situation in the setting of $\ell$-adic complexes. Namely, we consider the category $\mathcal{DA}(S, \mathbb{Z}_\ell)$ of Artin $\ell$-adic complexes which is the stable $\infty$-subcategory of $\mathcal{D}^b_c(S, \mathbb{Z}_\ell)$ generated by the $f_*\mathbb{Z}_{\ell,X}$ for $f : X \to S$ finite, where $\mathbb{Z}_{\ell,X}$ is the constant sheaf of value $\mathbb{Z}_\ell$ on $X$. This category is the $\ell$-adic analog of the category $\mathcal{DM}^0_c(S)$ (also called category of constructible Artin motives).

To state our result regarding the ordinary t-structure, we need to introduce the abelian category $\text{Sh}^A(S, \mathbb{Z}_\ell)$ of Artin $\ell$-adic sheaves which is the smallest abelian full subcategory of the abelian category of $\ell$-adic sheaves which is stable under extensions and contains the $f_*\mathbb{Z}_{\ell,X}$, for $f : X \to S$ finite.

**Proposition.** (Proposition 2.6) Let $S$ be a noetherian finite dimensional scheme. Let $\ell$ be a prime number invertible on $S$. Then, the category $\mathcal{DA}(S, \mathbb{Z}_\ell)$ can be identified with the subcategory of those complexes in $\mathcal{D}^b_c(S, \mathbb{Z}_\ell)$ such that their ordinary cohomology sheaves are Artin $\ell$-adic sheaves. In particular the ordinary t-structure induces a t-structure on $\mathcal{DA}(S, \mathbb{Z}_\ell)$ whose heart is the category of Artin $\ell$-adic sheaves.

Our main result concerns the perverse t-structure:

**Theorem.** (Theorem 3.14 and Proposition 3.14) Let $S$ be a noetherian finite dimensional excellent scheme. Let $\ell$ be a prime number invertible on $S$. Assume that $S$ is of dimension $\leq 2$, or that $S$ is of dimension 3 and the residue field of the closed points of $S$ are separably closed or real close. Then, the perverse t-structure induces a t-structure on $\mathcal{DA}(S, \mathbb{Z}_\ell)$.

This result seems rather optimal: if $k$ is a finite field and $\ell$ is a prime number invertible on $k$, we prove (see Example 3.17) that the perverse t-structure of $\mathcal{D}^b_c(\mathbb{A}_k^3, \mathbb{Z}_\ell)$ does not induce a t-structure on $\mathcal{DA}(\mathbb{A}_k^3, \mathbb{Z}_\ell)$.
To describe the objects of the heart of the perverse $t$-structure on $\mathcal{D}^A(S,\mathbb{Z}_\ell)$ when it is defined, we need to introduce the stable $\infty$-subcategory $\mathcal{D}^{smA}(S,\mathbb{Z}_\ell)$ of $\mathcal{D}^A(S,\mathbb{Z}_\ell)$ of smooth Artin $\ell$-adic complexes which is the stable $\infty$-subcategories of $\mathcal{D}^b_c(S,\mathbb{Z}_\ell)$ generated by the $f_*\mathbb{Z}_\ell, X$, for $f : X \rightarrow S$ finite and étale. This category plays in our paper the role that the category of lisse $\ell$-adic sheaves plays in the theory of perverse sheaves.

If $\pi$ is a profinite group, we introduce the abelian category of representations of Artin origin of $\pi$ as the smallest full abelian subcategory of the abelian category of continuous representations of $\pi$ that contains all the Artin representations of $\pi$. If $\pi = \pi_\ell^\text{et}(S)$, we can view its objects as Artin $\ell$-adic sheaves on $S$. Then, $\mathcal{D}^{smA}(S,\mathbb{Z}_\ell)$ is the subcategory of those objects of $\mathcal{D}^b_c(S,\mathbb{Z}_\ell)$ whose ordinary cohomology sheaves are representations of $\pi_\ell^\text{et}(S)$ of Artin origin. This proves that the ordinary $t$-structure on $\mathcal{D}^b_c(S,\mathbb{Z}_\ell)$ induces a $t$-structure on $\mathcal{D}^{smA}(S,\mathbb{Z}_\ell)$ whose heart is the category of representations of Artin origin of $\pi_\ell^\text{et}(S)$. Now, assuming that the $t$-structure of $\mathcal{D}^b_c(S,\mathbb{Z}_\ell)$ induces a $t$-structure on $\mathcal{D}^A(S,\mathbb{Z}_\ell)$, we can describe an Artin perverse sheaf as the following data which is defined by induction on the dimension of $S$:

- A sequence $(U_0, \ldots, U_n)$ of locally closed subschemes of $S$ such that $U_k$ is regular, purely of dimension $\text{dim}(S) - k$ and is open in $F_{k-1} := S \setminus (U_0 \cup \cdots \cup U_{k-1})$; (and $F_{-1} = S$).
- On each $U_i$, a smooth Artin complex $M_i$ in perverse degree $[-i,0]$.
- A connection map $\phi_k : (i_k)_*M_{z,k}[-1] \rightarrow (j_k)_!M_k$ such that $\check{H}^0(\phi_k)$ is injective, where $i_k$ is the inclusion of $F_k$ in $F_{k-1}$, $j_k$ is the inclusion of $U_k$ in $F_{k-1}$ and $M_{z,k}$ is the perverse sheaf on $F_k$ given by the datum of $(M_{k+1}, \ldots, M_n)$ and the connection maps $\phi_k, \ldots, \phi_n$.

If $S$ is an excellent scheme of dimension 1, this description can be made more explicit: the category $\text{Perv}^A(S,\mathbb{Z}_\ell)$ is equivalent to the category $\text{P}(S,\mathbb{Z}_\ell)$ (Definition 5.10) defined completely in terms of representations and maps of representations.

Finally we introduce another notion of Artin perverse sheaves inspired by the approach of [Leh19b, Vai19, Ay07, 2.2.4] and [BD17]. More precisely, we define a homotopy perverse $t$-structure by generators on the ind-category $\mathcal{D}^A_{\text{ind}}(S,\mathbb{Z}_\ell)$ of $\mathcal{D}^A(S,\mathbb{Z}_\ell)$. When the perverse $t$-structure induces a $t$-structure on $\mathcal{D}^A(S,\mathbb{Z}_\ell)$, then so does the homotopy perverse $t$-structure and both induced $t$-structures coincide.

The properties of this $t$-structure are closely related to the properties of the right adjoint $\omega^0$ of the inclusion functor $\iota : \mathcal{D}^A_{\text{ind}}(S,\mathbb{Z}_\ell) \rightarrow \mathcal{D}^{\text{coh}}_{\text{ind}}(S,\mathbb{Z}_\ell)$ of Ind-Artin $\ell$-adic complexes into Ind-cohomological $\ell$-adic complexes. This functor was first introduced in [AZ12] in the motivic setting to study the relative Borel-Serre compactification of a symmetric variety.

We show the following results:

**Theorem.** (Propositions 4.13 and 4.18 and Remarks 4.12 and 4.27) Let $S$ be a scheme of finite type over $\mathbb{F}_p$. Let $\ell \neq p$ be a prime number.

1. As in the motivic setting (see [Vai16]), the functor $\omega^0 : \mathcal{D}^{\text{coh}}_{\text{ind}}(S,\mathbb{Q}_\ell) \rightarrow \mathcal{D}^A_{\text{ind}}(S,\mathbb{Q}_\ell)$ coincides with the weightless truncation functor of [NV15] over cohomological $\ell$-adic complexes. In particular, it sends cohomological $\ell$-adic complexes to Artin $\ell$-adic complexes.
The homotopy perverse t-structure induces a t-structure on $D^\lambda(S, Q_\ell)$. We denote its heart by $\text{Perv}^\lambda(S, Q_\ell)^\#$ and call it abelian category of Artin homotopy perverse sheaves.

When both categories are define, $\text{Perv}^\lambda(S, Q_\ell)^\#$ and $\text{Perv}^\text{coh}(S, Q_\ell)^\#$ coincide.

The functor $\omega^0$ is t-exact when $D^\text{coh}(S, Q_\ell)$ is endowed with the perverse t-structure. In particular, denoting $\text{Perv}^\text{coh}(S, Q_\ell)$ the category of cohomological perverse sheaves, $(\iota, \omega^0)$ induces an adjunction $\iota : \text{Perv}^\lambda(S, Q_\ell)^\# \rightleftarrows \text{Perv}^\text{coh}(S, Q_\ell)^\# : \omega^0$

such that $\omega^0$ is exact.

Furthermore, we show that $\text{Perv}^\lambda(S, Q_\ell)$ is similar to the category of perverse sheaves and contains the weightless complex of [NV15]:

**Proposition.** (Propositions 4.26 and 4.29 and Example 4.30) Let $S$ be a scheme of finite type over $\mathbb{F}_p$. Let $\ell \neq p$ be a prime number.

1. All the objects of $\text{Perv}^\lambda(S, Q_\ell)^\#$ are of finite length. Its simple objects can be described as $\omega^0 j_*(L[\dim(V)])$ where $j : V \to S$ is the inclusion of a regular subscheme and $L$ is an irreducible representation of Artin origin of $\pi^\text{et}_1(V)$.

2. Let $IC_S$ be the intersection complex of $S$. The weightless complex $EC_S = \omega^0 IC_S$ of [NV15] is a simple object of $\text{Perv}^\lambda(S, Q_\ell)^\#$.

3. If $S$ is a surface, then $EC_S$ is also a perverse sheaf, but it need not be simple as a perverse sheaf.

In an upcoming paper, we will study the motivic case and prove that we have similar results. One of the main interests of this paper is that the motivic constructions will realize into the constructions of this paper.

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1. **Preliminaries**

All schemes are assumed to be noetherian and of finite dimension; furthermore all smooth (and étale) morphisms are also implicitly assumed to be separated and of finite type.

In this text we will freely use the language of $(\infty, 1)$-categories of [Lur09, Lur17]. The term category will by default mean $(\infty, 1)$-category, and the term 2-category will mean $(\infty, 2)$-category. When we refer to derived categories, it will always be the ∞-categorical version. There is an exception to this rule: we use the term abelian category, to refer to what according to our conventions, we should call abelian 1-category.
There are several definitions of the derived category of constructible \( \ell \)-adic sheaves. The first definition was given in \cite{BBDG18} over schemes over a field. This definition was extended to all noetherian finite dimensional schemes in \cite{Ec07, CD16, BS15, LYF14}. We will use the theorems of \cite{BBDG18} over excellent schemes; in particular, to use the affine Lefschetz theorem \cite[4.1.1]{BBDG18} and its consequences, one has to replace \cite[XIV.3.1]{AGV73} in the proof with \cite[XV.1.1.2]{LYF14}.

We adopt the cohomological convention for t-structures (i.e. the convention of \cite[1.3.1]{BBDG18} and the opposite of \cite[1.2.1.1]{Lur17}).

Let \( i : Z \to X \) be a closed immersion and \( j : U \to X \) be the complementary open immersion. We call localization triangles the exact triangles of functors on \( \ell \)-adic sheaves over \( X \):

\[
(1.1) \quad j_!j^* \to \text{Id} \to i_*i^*.
\]

\[
(1.2) \quad i!i^! \to \text{Id} \to j_!j^*.
\]

Let \( S \) be a scheme. A stratification of \( S \) is a partition \( S \) of \( S \) into non-empty equidimensional locally closed subschemes called strata such that the topological closure of any stratum is a union strata.

If \( k \) is a field, we denote \( G_k \) its absolute Galois group.
If \( p \) is a prime number, we denote \( \mathbb{F}_p \) the field with \( p \) elements.

### 1.1. Notions of abelian and stable subcategories generated by a set of objects.
We will need the following definitions:

**Definition 1.1.** Let \( A \) be an abelian category and \( \mathcal{E} \) be a set of objects of \( A \). Then,

1. the thick abelian subcategory of \( A \) generated by \( \mathcal{E} \) is the smallest full subcategory of \( A \) which is abelian, contains \( \mathcal{E} \) and is stable under extensions, in other words it is the full subcategory of \( A \) spanned by those objects which are generated by extensions, kernels and cokernels from the objects of \( \mathcal{E} \).
2. the thick Grothendieck abelian subcategory of \( A \) generated by \( \mathcal{E} \) is the smallest full subcategory of \( A \) which is Grothendieck abelian, contains \( \mathcal{E} \) and is stable under extension, in other words it is the full subcategory of \( A \) spanned by those objects which are generated by arbitrary sums, extensions, kernels and cokernels from the objects of \( \mathcal{E} \).

**Definition 1.2.** Let \( C \) be a stable category and \( \mathcal{E} \) a set of objects. We call thick (resp. localizing) subcategory spanned by \( \mathcal{E} \) the full subcategory spanned by those objects which are generated from \( \mathcal{E} \) by suspensions, finite sums, cones and direct factors (resp. suspensions, arbitrary sums and cones) in the underlying triangulated category; it is a stable subcategory of \( C \).

**Definition 1.3.** Let \( D \) be a stable category endowed with a non-degenerated t-structure. Let \( A \) be a strictly full abelian subcategory of the heart of \( D \). We will denote \( D^A_\phi \) the subcategory of those bounded objects \( C \) of \( D \) such that for all integer \( n \), \( H^n(C) \) is in \( A \).

The following proposition establishes a link between those notions:

**Proposition 1.4.** Let \( D \) be a stable category endowed with a non-degenerated t-structure. Let \( A \) be the heart of \( D \) and \( \mathcal{E} \) be a set of objects of \( B \). Let \( A \) be the thick abelian subcategory of \( B \) generated by \( \mathcal{E} \), let \( \hat{A} \) be the thick Grothendieck...
abelian subcategory of $B$ generated by $E$ and let $D'$ be the localizing subcategory of $D$ generated by $E$. Denote by $E[0]$ the set of objects of $E$ seen as objects placed in degree 0 in $D$. Then,

1. The category $D^b_A$ is the thick subcategory of $D$ generated by $E[0]$.
2. The t-structure on $D$ induces a t-structure on $D^b_A$ whose heart is equivalent to $A$.
3. The t-structure on $D$ induces a t-structure on $D'$ whose heart is equivalent to $\hat{A}$.

Proof. First, notice that every object of $A$ is, as an object placed in degree 0, in the thick subcategory generated by $E$.

Let $C$ be a bounded object of $D$ such that for any integer $n$, $H^n(C)$ is in $A$. Then, by induction on $n$, $\tau_{[-n,n]}C$ is in the thick subcategory generated by $E$. Hence $C$ itself is in the thick subcategory generated by $E$.

Conversely, notice that $D^b_A$ is a thick subcategory of $D$ and contains the objects of $E[0]$. This proves the first point.

Now, the category $D^b_A$ is stable under the functor $\tau_{\geq 0}$ thus by [BBDG18, 1.3.19], the t-structure of $D$ induces a t-structure on $D^b_A$. The functor $H^0$ gives the equivalence of its heart with $A$.

Finally, to prove the last point, notice that every object $M$ of $D'$ is a small colimit of objects of $D^b_A$. Since $\tau_{\geq 0}$ is a left adjoint, it commutes with small colimits. Thus, $\tau_{\geq 0}M$ is a small colimit of objects of $D^b_A$ and is in $D'$. Hence, by [BBDG18, 1.3.19], the t-structure of $D$ induces a t-structure on $D'$.

The heart of this t-structure is an abelian subcategory of $B$. It contains $\hat{A}$ since every object of $A$ is, as an object placed in degree 0, in the localizing subcategory generated by $E$. Let us show that is also contained in $\hat{A}$. The full subcategory $D''$ of $D$ of those objects $C$ such that for all $n$, $H^n(C)$ is in $\hat{A}$ is localizing and contains $E$. Thus $D''$ contains $D'$, thus the heart of the induced t-structure on $D'$ is contained in $\hat{A}$.

\[\Box\]

1.2. A lemma on t-structures. The following lemma will allow us to restrict t-structure to thick subcategories:

Lemma 1.5. Let $C$ be a stable category, $E$ be a family of objects and $C_0$ be the thick stable subcategory of $C$ generated by $E$.

1. Assume that $C$ is endowed with a t-structure $t_C$ such that all the objects of $E$ are in the heart of this t-structure. Then, $t_C$ induces a t-structure on $C_0$.
2. If a t-structure on $C_0$ such that all the objects of $E$ are in its heart exists, then it is unique.

Proof. If $n$ is an integer, let $H^n$ be the $n$-th cohomology functor with respect to $t_C$. The full subcategory $C_1$ of $C_0$ of objects $M$ such that $M$ is bounded for $t_C$ and for all $n$, $H^n(M)$ is in $C_0$, is thick. Thus, if $C_1$ contains $E$, $C_1$ is equivalent to $C_0$. Hence, $C_0$ is stable under the functor $\tau_{\geq 0}$, thus by [BBDG18, 1.3.19], the t-structure of $C$ induces a t-structure on $C_0$.

Now, if $t$ and $t'$ are t-structures such that all the objects of $E$ are in the heart of $t$ and $t'$, then, the full subcategory $C_2$ of $C_0$ of objects $M$ such that for all $n$, the cohomology functors with respect to $t$ and $t'$ coincide, is thick and contains $E$. Thus $C_2$ and $C_0$ are equivalent. This yields the second statement of our lemma. \[\Box\]
1.3. Reminders on cdh-descent and Mayer-Vietoris for closed immersions.

**Definition 1.6.** We say following [Voe10, 2] and [CD19, 2.1.11] that a cartesian square of schemes:

\[
\begin{array}{ccc}
E & \rightarrow & F \\
\downarrow & & \downarrow i \\
Y & \rightarrow & X \\
\end{array}
\]

is cdh-distinguished if \( f \) is proper, \( i \) is a closed immersion and if \( U \) is the open complement, the induced map \( f^{-1}(U) \rightarrow U \) is an isomorphism.

The following proposition is a direct consequence of the fact that the category of étale motives satisfies cdh-descent [CD19, 3.3.10] and of the fact that the \( \ell \)-adic realization commutes with the six operations [CD16, 7.2.11].

**Proposition 1.7.** (cdh-descent) A cdh-distinguished square:

\[
\begin{array}{ccc}
E & \rightarrow & F \\
\downarrow & & \downarrow i \\
Y & \rightarrow & X \\
\end{array}
\]

induces an exact triangle:

\[
\mathbb{Z}_{\ell,X} \rightarrow f_*\mathbb{Z}_{\ell,Y} \oplus i_*\mathbb{Z}_{\ell,F} \rightarrow i_*p_*\mathbb{Z}_{\ell,E}.
\]

The following formula is a consequence of [DDØ21, 5.1.7]. It is closely linked to the Rapoport-Zink complex [RZ82, 2.5] and is a generalization of the Mayer-Vietoris distinguished triangle for closed immersions [Ayo07, 2.2.31].

**Proposition 1.8.** Let \( i : E \rightarrow Y \) be a closed immersion. Assume that \( E = \bigcup_{i \in I} E_i \) with \( I \) a finite set and \( (E_i)_{i \in I} \) a family of closed subschemes of \( E \). If \( J \subseteq I \), write \( E_J = \bigcup_{i \in J} E_i \). Then, \( i_*i^!\mathbb{Z}_{\ell,Y} \) is the homotopy colimit:

\[
\text{hocolim}_{p \in (\Delta^{[n]})^{op}} \left( \bigoplus_{J \subseteq I, |J| = p+1} (i_J)_*i_J^!\mathbb{Z}_{\ell,Y} \right),
\]

where \( \Delta^{[n]} \) is the category of finite ordered sets with morphisms the injective maps.

1.4. Artin representations. Recall the following definitions:

**Definition 1.9.** Let \( \pi \) be a topological group and let \( R \) be a ring. An \( R[\pi] \)-module \( M \) is discrete if the action of \( \pi \) on \( M \) is continuous when \( M \) is endowed with the discrete topology. We will denote \( \text{Mod}_{R,\pi} \) the Grothendieck abelian category of discrete \( R[\pi] \)-modules.

**Remark 1.10.** An \( R[\pi] \)-module is discrete if and only if the stabiliser of every point is open. Equivalently, this means that \( M \) is the union of all the \( M^U \) for \( U \) an open subgroup of \( \pi \).

If \( \pi \) is profinite, this also means that \( M \) is the union of all the \( M^H \) for \( H \) a normal subgroup of \( \pi \) of finite index.
**Definition 1.11.** Let \( \pi \) be a profinite group and let \( R \) be a ring. An Artin representation of \( \pi \) is a discrete \( R[\pi] \)-module which is of finite type as an \( R \)-module. We will denote \( \text{Rep}^A(\pi, R) \) the abelian category of Artin representations.

**Remark 1.12.** Keep the same notations. An \( R[\pi] \)-module is an Artin representation if and only if it is of finite type and the action of \( \pi \) factors through a finite quotient. Indeed, if \( M \) is an Artin representation, the stabiliser of a finite generating family is open and thus contains a normal subgroup \( N \) of finite index; the action of \( \pi \) factors through \( \pi/N \).

**Proposition 1.13.** Let \( \pi \) be a profinite group and let \( R \) be a noetherian ring. Then,

1. The category \( \text{Rep}^A(\pi, R) \) is the thick abelian subcategory of \( \text{Mod}_{\pi,R} \) generated by the \( E \) where \( E \) is a finite set endowed with a continuous action of \( \pi \).
2. The category \( \text{Mod}_{\pi,R} \) is the ind-category of \( \text{Rep}^A(\pi, R) \).
3. The category \( \text{Mod}_{\pi,R} \) is generated as a thick Grothendieck abelian subcategory of itself by the \( R[E] \) where \( E \) is a finite set endowed with a continuous action of \( \pi \).

**Proof.** To prove the first point, take an Artin representation \( M \) and let \( G \) be a finite quotient of \( \pi \) through which the action of \( \pi \) on \( M \) factors. Then, there exist integers \( n \) and \( m \) such that \( M \) is the cokernel of a map \( R[G]^n \to R[G]^m \).

To prove the second point, notice that the ind-category \( \mathcal{C} \) of \( \text{Rep}^A(\pi, R) \) is naturally embedded as a full subcategory of \( \text{Mod}_{\pi,R} \). Any object of \( \text{Mod}_{\pi,R} \) such that the action of \( \pi \) factors through a finite quotient is equivalent to an object of \( \mathcal{C} \) as it is the colimit of its submodules of finite type.

Now, let \( M \) be a discrete \( R[\pi] \)-module. Then, \( M \) is the union of the \( M_H \) where \( H \) is a normal subgroup of \( \pi \). The action of \( \pi \) on \( M_H \) factors through \( \pi/H \) which is finite. Thus, \( M \) is a colimit of Artin representations.

The third point follows from the first and second points. \( \Box \)

**Example 1.14.** \( R[\pi] \) is not a discrete \( R[\pi] \)-module.

**Example 1.15.** Let \( \pi = GL_n(\hat{\mathbb{Z}}) \). Then, \( \pi \) acts on \( \mathbb{Z}/N\mathbb{Z}^n \) for any \( N > 0 \) and the action factors through its finite quotient \( GL_n(\mathbb{Z}/N\mathbb{Z}) \).

Furthermore, if \( M, N > 0 \), \( \mathbb{Z}/N\mathbb{Z}^n \) is a sub-\( \pi \)-module of \( \mathbb{Z}/NM\mathbb{Z}^n \) (the map is the multiplication by \( M \)). This endows \( N/\mathbb{Z}^n \) with a structure of discrete \( \mathbb{Z}[\pi] \)-module.

If \( V \to S \) is a map of schemes and \( R \) is a ring, denote \( R_S(V) \) the \( R \)-linear étale sheaf on \( S \) represented by \( V \). We now explain the link with étale sheaves:

**Proposition 1.16.** Let \( S \) be a connected scheme and \( R \) be a noetherian ring. Let \( \xi \) be a geometric point of \( S \). The fiber functor \( \xi^* \) associated to \( \xi \) induces an equivalence of abelian monoidal categories between the abelian (resp. Grothendieck abelian) subcategory of \( \text{Sh}(S^{et}, R) \) generated by the \( R_S(V) \) with \( V \) finite étale over \( S \) and the category \( \text{Rep}^A(\pi^{et}_1(S, \xi), R) \) (resp. \( \text{Mod}_{\pi^{et}_1(S, \xi), R} \)).

**Proof.** It is a direct consequence of Galois-Grothendieck theory (see [GR71], VI). \( \Box \)
1.5. **Representations of Artin origin.** Let $\ell$ be a prime number. The topology on $\mathbb{Z}_\ell$ induces a topology on every $\mathbb{Z}_\ell$-module of finite type.

**Definition 1.17.** Let $\pi$ be a topological group and let $\ell$ be a prime number. A continuous representation of $\pi$ with coefficients in $\mathbb{Z}_\ell$ is a $\mathbb{Z}_\ell$-module of finite type endowed with a continuous action of $\pi$. We will denote $\text{Rep}(\pi, \mathbb{Z}_\ell)$ the abelian category of continuous representation of $\pi$ with coefficients in $\mathbb{Z}_\ell$.

Any Artin representation of a profinite group $\pi$ with coefficients in $\mathbb{Z}_\ell$ is a continuous representation. More precisely, $\text{Rep}^A(\pi, \mathbb{Z}_\ell)$ is a full abelian subcategory of $\text{Rep}(\pi, \mathbb{Z}_\ell)$.

In general, the category of Artin representations is not thick in the category of continuous representations. Indeed, consider an additive character $\chi : \pi \to \mathbb{Z}_\ell$ that does not factor through a finite quotient (for example the logarithm of the cyclotomic character if $\pi = \hat{\mathbb{Z}}$). Then, we have a morphism $\pi \to GL_2(\mathbb{Z}_\ell)$ given by

$$\begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix}$$

which defines a continuous representation of $\pi$ in $\mathbb{Z}_\ell^2$. The exact sequence:

$$0 \to \mathbb{Z}_\ell \to \mathbb{Z}_\ell^2 \to \mathbb{Z}_\ell \to 0$$

is an exact sequence of representations if $\pi$ acts trivially on both copies of $\mathbb{Z}_\ell$. Hence, this gives an extension of Artin representations which is not Artin.

This leads us to the following definitions:

**Definition 1.18.** Let $\pi$ be a profinite group and let $\ell$ be a prime number.

1. A continuous representation is of Artin origin if it is in the thick abelian subcategory of $\text{Rep}(\pi, \mathbb{Z}_\ell)$ generated by Artin representations. We denote $\text{Rep}^A(\pi, \mathbb{R})$ the category of representations of Artin origin.

2. We will say that an ind-object of $\text{Rep}(\pi, \mathbb{Z}_\ell)$ is of discrete origin if it is thick Grothendieck abelian subcategory of the category of ind-objects of $\text{Rep}(\pi, \mathbb{Z}_\ell)$ generated by discrete. We denote $\text{Mod}^{\ast}_{\pi, \mathbb{R}}$ the category of representations of discrete origin.

1.6. **Representations that are strongly of weight 0.** Artin representations are contained in an explicit thick abelian subcategory of $\text{Rep}(\pi, \mathbb{Z}_\ell)$ which we will call strongly of weight 0. In general, to prove that a representation is not of Artin origin, we will prove that it is not strongly of weight 0.

**Definition 1.19.** Let $\pi$ be a profinite group and $\ell$ be a prime number. Let $M$ be a continuous representation of $\pi$ with coefficients in $\mathbb{Z}_\ell$. We say that $M$ is strongly of weight 0 if for all $g \in \pi$, the eigenvalues of the matrix defined by the action of $g$ on $M \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell}$ are roots of unity. We will denote $\text{Rep}^{w=0}(\pi, \mathbb{Z}_\ell)$ the abelian category of representations that are strongly of weight 0.

**Remark 1.20.** Classically, if $k$ is a finite field and $\pi = G_k = \hat{\mathbb{Z}}$ is its absolute Galois group, an $\ell$-adic representation of $G_k$ is of weight 0 if the action of the topological generator $F$ of $G_k$ are algebraic numbers such that all their conjugates are of module 1 as complex numbers.

Hence, representations that are strongly of weight 0 of $G_k$ are contained in representations of weight 0 of $G_k$. 
**Proposition 1.21.** Let $\pi$ be a profinite group and $\ell$ be a prime number. The abelian category $\text{Rep}^{w=0}(\pi, \mathbb{Z}_\ell)$ is a thick abelian subcategory of $\text{Rep}(\pi, \mathbb{Z}_\ell)$ and contains the category of Artin representations.

**Proof.** First, any sub-representation and any quotient of a representation that is strongly of weight 0 is strongly of weight 0. Now, if

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of continuous representations where $M'$ and $M''$ are strongly of weight 0, the sequence

$$0 \to M' \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to M'' \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to 0$$

is still exact and if $g$ is an element of $\pi$, its action on $M$ is the action of a matrix of the form: $A = \begin{pmatrix} A' & * \\ 0 & A'' \end{pmatrix}$ where $A'$ (resp. $A''$) is the matrix of the action of $g$ on $M'$ (resp. $M''$). Hence, the set of eigenvalues of $A$ is the union of the set of eigenvalues of $A'$ and $A''$ and those are roots of unity. Thus, $\text{Rep}^{w=0}(\pi, \mathbb{Z}_\ell)$ is a thick abelian subcategory of $\text{Rep}(\pi, \mathbb{Z}_\ell)$.

Now, by Proposition 1.13 it suffices to show that if $E$ is a finite set endowed with a continuous action of $\pi$, then the eigenvalues of the action of any element of $\pi$ on $\mathbb{Z}_\ell[E]$ are roots of unity. But any element $g$ of $\pi$ permutes a basis of $\mathbb{Z}_\ell[E]$, hence, $g^{|E|}$ is the identity and the eigenvalues of $g$ are roots of unity. $\square$

## 2. Artin $\ell$-adic complexes

### 2.1. Artin $\ell$-adic sheaves

Let $S$ be a scheme. Let $\ell$ be a prime number which is invertible on $S$. By $\ell$-adic sheaf we mean an $\ell$-adic sheaf with coefficients in $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$ as in [Del77]. For simplicity, we only consider the case of $\mathbb{Z}_\ell$-coefficients in this paragraph, however, all the definition can be formulated with coefficients $\mathbb{Q}_\ell$ and all the propositions still hold.

We denote $\text{Sh}(S, \mathbb{Z}_\ell)$ the abelian category of $\ell$-adic sheaves with coefficients $\mathbb{Z}_\ell$ and we denote $\mathbb{Z}_\ell,S$ the constant $S$ of value $\mathbb{Z}_\ell$. Recall the following definitions:

**Definition 2.1.** Let $S$ be a scheme, $\ell$ be a prime number that is invertible on $S$ and $L = (L_n)_{n \geq 1}$ be an $\ell$-adic sheaf. Then,

1. $L$ is smooth of rank $r$ if all the $L_n$ are free sheaves of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules of rank $r$.
2. $L$ is constructible if there is a stratification $S$ of $S$ such that for all $T \in S$, $L|T$ is smooth.

The following proposition is classical. It is also a consequence of Galois-Grothendieck theory.

**Proposition 2.2.** ([Del77, 2.4]) Let $S$ be connected scheme. Let $\xi$ be a geometric point of $S$. The fiber functor $\xi^*$ associated to $\xi$ induces an equivalence of abelian monoidal categories between the abelian category of smooth $\ell$-adic sheaves with coefficients $\mathbb{Z}_\ell$ and the abelian category of continuous representations of $\pi_1^S(S, \xi)$ with coefficients in $\mathbb{Z}_\ell$ that are of finite type as $\mathbb{Z}_\ell$-modules.

**Definition 2.3.** Let $S$ be a scheme and $\ell$ a prime number that is invertible on $S$.

1. The category $\text{Sh}^\ell(S, \mathbb{Z}_\ell)$ of Artin $\ell$-adic sheaves over $S$ with coefficients $\mathbb{Z}_\ell$ is the abelian subcategory of $\text{Sh}(S, \mathbb{Z}_\ell)$ generated by the $f_*\mathbb{Z}_\ell,X$ with $f : X \to S$ finite.
(2) The category $\text{Sh}_{\text{smA}}(S, \mathbb{Z}_\ell)$ of smooth Artin $\ell$-adic sheaves over $S$ with coefficients $\mathbb{Z}_\ell$ is the abelian subcategory of $\text{Sh}(S, \mathbb{Z}_\ell)$ generated by the $f_*\mathbb{Z}_{\ell,X}$ with $f : X \to S$ étale and finite.

(3) The category $\text{Sh}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$ (resp. $\text{Sh}_{\text{Ind}}^{\text{smA}}(S, \mathbb{Z}_\ell)$) of Ind- (smooth) Artin $\ell$-adic sheaves is the Grothendieck abelian subcategory with the generators as in (1) (resp. (2)).

We have the following description of smooth Artin $\ell$-adic sheaves:

**Proposition 2.4.** Let $S$ be a connected scheme and $\ell$ a prime number that is invertible on $S$. Let $\xi$ be a geometric point of $S$. The fiber functor $\xi^*$ associated to $\xi$ induces an equivalence of abelian monoidal categories between the abelian category $\text{Sh}_{\text{smA}}^A(S, \mathbb{Z}_\ell)$ (resp. $\text{Sh}_{\text{Ind}}^{\text{smA}}(S, \mathbb{Z}_\ell)$) of smooth Artin $\ell$-adic sheaves over $S$ with coefficients $\mathbb{Z}_\ell$ and the abelian category $\text{Rep}^A\left(\pi_1^\text{\acute{e}t}(S, \xi), \mathbb{Z}_\ell\right)^*$ (resp. $\text{Mod}^*_\text{\acute{e}t}(S, \xi)$).

**Proof.** Using Propositions 1.10 and 2.2, it suffices to show that if $f : V \to S$ is étale and finite, the functor $\xi^*$ of Proposition 2.2 sends $f_*\mathbb{Z}_{\ell,V} = f_!\mathbb{Z}_{\ell,V}$ to the image of the étale sheaf $(\mathbb{Z}_\ell)_S(V)$ represented by $V$, by the functor $\xi^*$ of Proposition 1.10.

This is true since

$$(\mathbb{Z}_\ell)_S(V) = \lim (\mathbb{Z}/\ell^n\mathbb{Z})_S(V) = \lim f_!(\mathbb{Z}/\ell^n\mathbb{Z})(V).$$

\square

2.2. Artin $\ell$-adic complexes. Let $S$ be a scheme and $\ell$ a prime number which is invertible on $S$. We denote $\mathcal{D}_c^\ell(S, \mathbb{Z}_\ell)$ the stable category of constructible $\ell$-adic complexes. We also denote $\mathcal{D}(S, \mathbb{Z}_\ell)$ the ind-category of $\mathcal{D}_c^\ell(S, \mathbb{Z}_\ell)$.

**Definition 2.5.** Let $S$ be a scheme and $\ell$ a prime number which is invertible on $S$.

(1) The stable category $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ of Artin $\ell$-adic complexes is the thick stable subcategory of $\mathcal{D}_c^\ell(S, \mathbb{Z}_\ell)$ generated by the $f_*\mathbb{Z}_{\ell,X}$ with $f : X \to S$ finite.

(2) The stable category $\mathcal{D}_{\text{smA}}^A(S, \mathbb{Z}_\ell)$ of smooth Artin $\ell$-adic complexes over $S$ with coefficients $\mathbb{Z}_\ell$ is the thick stable subcategory of $\mathcal{D}_c^\ell(S, \mathbb{Z}_\ell)$ generated by the $f_*\mathbb{Z}_{\ell,X}$ with $f : X \to S$ étale and finite.

(3) The stable category $\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$ (resp. $\mathcal{D}_{\text{Ind}}^{\text{smA}}(S, \mathbb{Z}_\ell)$) is the localizing subcategory of $\mathcal{D}(S, \mathbb{Z}_\ell)$ with the generators as in (1) (resp. (2)).

The following proposition is a direct consequence of our definitions and of Proposition 1.4.

**Proposition 2.6.** Let $S$ be a scheme and $\ell$ a prime number which is invertible on $S$. Recall Definition 1.5.3. Then,

(1) The category $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ is the category $\mathcal{D}_c^\ell(S, \mathbb{Z}_\ell)_{\text{Sh}^A(S, \mathbb{Z}_\ell)}$. Hence, the canonical $t$-structure of $\mathcal{D}_c^\ell(S, \mathbb{Z}_\ell)$ induces a $t$-structure on $\mathcal{D}^A(S, \mathbb{Z}_\ell)$. The heart of this $t$-structure is equivalent to $\text{Sh}^A(S, \mathbb{Z}_\ell)$.

(2) The category $\mathcal{D}_{\text{smA}}^A(S, \mathbb{Z}_\ell)$ is the category $\mathcal{D}_c^\ell(S, \mathbb{Z}_\ell)_{\text{Sh}_{\text{smA}}(S, \mathbb{Z}_\ell)}$. Hence, the canonical $t$-structure of $\mathcal{D}_c^\ell(S, \mathbb{Z}_\ell)$ induces a $t$-structure on $\mathcal{D}_{\text{smA}}^A(S, \mathbb{Z}_\ell)$. The heart of this $t$-structure is equivalent to $\text{Sh}_{\text{smA}}^A(S, \mathbb{Z}_\ell)$.

(3) The canonical $t$-structure of $\mathcal{D}(S, \mathbb{Z}_\ell)$ induces a $t$-structure on $\mathcal{D}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$ (resp. $\mathcal{D}_{\text{Ind}}^{\text{smA}}(S, \mathbb{Z}_\ell)$). The heart of this $t$-structure is equivalent to $\text{Sh}_{\text{Ind}}^A(S, \mathbb{Z}_\ell)$ (resp. $\text{Sh}_{\text{Ind}}^{\text{smA}}(S, \mathbb{Z}_\ell)$).
Corollary 2.7. If $k$ is a field and $\ell$ is a prime number that is distinct of the characteristic of $k$, then, $D^A(k, \mathbb{Z}_\ell)$ is equivalent to the category of the objects $C$ of $D^b(\text{Rep}(G_k, \mathbb{Z}_\ell))$ such that $H^n(C)$ is an Artin representation for all $n$.

2.3. Six functors formalism. Artin $\ell$-adic sheaves are stable under some of the six functors on constructible $\ell$-adic sheaves. The results of this section could also have been formulated with the ind-categories. We will first need the following lemma that is due to Ayoub and Zucker in the motivic setting.

Proposition 2.8. Let $S$ be a scheme and $\ell$ a prime number which is invertible on $S$. The category $D^A(S, \mathbb{Z}_\ell)$ is the thick subcategory of $D^b(S, \mathbb{Z}_\ell)$ generated by any of the following families of objects:

1. $f_!\mathbb{Z}_{\ell,X}$ for $f : X \to S$ quasi-finite.
2. $f_*\mathbb{Z}_{\ell,X}$ for $f : X \to S$ finite.
3. $f_*\mathbb{Z}_{\ell,X}$ for $f : X \to S$ étale.

Proof. The proof is the same as [AZ12, 2.5].

Corollary 2.9. The six functors from $D(-, \mathbb{Z}_\ell)$ induce by restriction the following functors on $D^A(-, \mathbb{Z}_\ell)$:

1. $f^*$ for any morphism $f$,
2. $f_*$ for any quasi-finite morphism $f$,
3. A monoidal structure $\otimes$.

Remark 2.10. Notice that on the ind-category, we have formal right adjoint to the three functors that we have described.

In the case of smooth Artin $\ell$-adic sheaves, we also have functors $\otimes$, $f^*$ for any morphism $f$ and $g_*=g_!$ for any (pro)-finite étale morphism $g$. Since $f^*$ and $g_*$ are t-exact, they induce functors on the hearts. Furthermore, the monoidal structure on $D^{smA}(S, \mathbb{Z}_\ell)$ induces a monoidal structure on its heart: map two objects $M$ and $N$ to $H^0(M \otimes N)$. We can identify those functors with classical construction on categories of representations:

Proposition 2.11. Let $f : T \to S$ be a map between connected schemes. Then,

1. The map $f$ induces a group morphism $\pi^\text{étn}_{1}(f) : \pi^\text{étn}_1(T) \to \pi^\text{étn}_1(S)$; the functor $f^* : D^{smA}(S, \mathbb{Z}_\ell) \to D^{smA}(T, \mathbb{Z}_\ell)$ induces the forgetful functor $\pi^\text{étn}_1(f)^* : \text{Rep}^A(\pi^\text{étn}_1(S), \mathbb{Z}_\ell) \to \text{Rep}^A(\pi^\text{étn}_1(T), \mathbb{Z}_\ell)$ by restriction.
2. Furthermore, if $f$ is finite and étale, $\pi^\text{étn}_1(f)$ is injective and its image is open for the profinite topology. The map induced by $f_*$ by restriction is the induced representation functor $\text{Ind}_{\pi^\text{étn}_1(S)}^{\pi^\text{étn}_1(T)}$.
3. The monoidal structure of $D^{smA}(S, \mathbb{Z}_\ell)$ induces by restriction a monoidal structure on $\text{Rep}^A(\pi^\text{étn}_1(S), \mathbb{Z}_\ell)$ which is the usual monoidal structure.

2.4. Artin $\ell$-adic complexes as stratified complexes of Artin representations. In this section, we show that for any Artin $\ell$-adic complex, there is a stratification of the base scheme such that the restriction of the complex to any stratum is smooth Artin. We can assume the strata to be connected so that by Propositions 2.4 and 2.6, our complex is a glueing of complexes whose ordinary cohomology sheaves are Artin representations of the fundamental groups of the strata.
Proposition 2.12. Let \( S \) be a scheme, \( \ell \) be a prime number that is invertible on \( S \) and \( M \) be an Artin \( \ell \)-adic complex. Then, there exists a stratification \( S \) of \( S \) such that for all \( T \in S \), \( M|_T \) is a smooth Artin \( \ell \)-adic complex.

Proof.

Lemma 2.13. Under the same hypothesis, there is a dense open subscheme \( U \) of \( S \) such that \( M|_U \) is smooth Artin.

We start with a lemma:

Proof. Let \( C \) be the full subcategory of \( D^A(S, \mathbb{Z}_\ell) \) of those objects \( M \) such that there is a dense open subscheme \( U \) of \( S \) such that \( M|_U \) is smooth Artin. Then, \( C \) is thick.

Furthermore, if \( f : W \to S \) is étale, there is a dense open immersion \( j : U \to S \) such that the pull-back morphism \( g : V \to U \) is finite étale.

Therefore, \( f_!\mathbb{Z}_\ell|_U = j^*f_!\mathbb{Z}_\ell = g_!\mathbb{Z}_\ell \) is smooth Artin. Hence, by Proposition 2.8, \( C \) is equivalent to \( D^A(S, \mathbb{Z}_\ell) \).

To finish the proof, we proceed by noetherian induction on \( S \). Let \( M \) be an Artin \( \ell \)-adic sheaf over \( S \).

The lemma gives a dense open subscheme \( U \) of \( S \) such that \( M|_U \) is smooth Artin. Using the induction hypothesis, there is a stratification \( S_F \) of \( F = S \setminus U \) such that the restriction of \( M|_F \) to any stratum of \( S_F \) is smooth Artin. □

3. Perverse t-structure

From now on, all schemes are assumed to be endowed with a dimension function \( \delta \) (see for instance [BD17, 1.1.1]). Recall that all schemes are also assumed to be noetherian and of finite dimension. Finally, recall that if \( S \) is universally catenary and integral, \( \delta(x) = \dim(X) - \text{codim}_X(x) \) is a dimension function on \( S \).

The main goal of this paper is to understand when the perverse t-structure can be restricted to Artin \( \ell \)-adic complexes and to define Artin perverse sheaves when it is the case.

Recall that the auto-dual perversity \( p_{1/2} : S \mapsto -\delta(S) \) induces two t-structures \( [p_{1/2}] \) and \( [p^+_{1/2}] \) on \( D^b_c(S, \mathbb{Z}_\ell) \). In this paper, we only consider the t-structure \( [p_{1/2}] \). We refer the reader to [Gab08] for the definition of the perverse t-structure over a general base scheme.

If \( n \) is an integer, we will denote \( p^Hn \) the \( n \)-th cohomology functor with respect to the perverse t-structure.

3.1. The smooth Artin case. Recall that if \( S \) is regular, then \( \mathbb{Z}_{\ell,S} \) is in degree \( \delta(S) \) for the perverse t-structure. Thus, using Lemma 1.13, we have:

Proposition 3.1. Let \( S \) be a scheme and \( \ell \) a prime number which is invertible on \( S \). Then, the perverse t-structure of \( D^b_c(S, \mathbb{Z}_\ell) \) induces a t-structure on \( D^{smA}(S, \mathbb{Z}_\ell) \). This t-structure coincides with the canonical t-structure of \( D^{smA}(S, \mathbb{Z}_\ell) \) shifted by \( \delta(S) \). The heart of this t-structure is equivalent to \( \text{Rep}^A(\pi_1^\text{ét}(S), \mathbb{Z}_\ell)^* \).

In particular the case of fields is clear:
Proposition 3.2. Let $k$ be a field and $\ell$ a prime number which is invertible in $k$. Then, the perverse t-structure of $D^b(\text{Rep}(G_k, \mathbb{Z}_\ell))$ induces a t-structure on $D^A(k, \mathbb{Z}_\ell)$. This t-structure coincides with the canonical t-structure shifted by $\delta(k)$. The heart of this t-structure is equivalent to $\text{Rep}^A(G_k, \mathbb{Z}_\ell)^\ast$.

3.2. General properties. We have the following glueing property (see [BBDG18, 1.4]):

Proposition 3.3. Let $S$ be a scheme, $i : F \to S$ be a closed immersion and $j : U \to S$ be the open complementary immersion. Then, the perverse t-structure on $D^b_c(S, \mathbb{Z}_\ell)$ induces a t-structure on $D^A(S, \mathbb{Z}_\ell)$ if and only if the perverse t-structure on $D^b_c(U, \mathbb{Z}_\ell)$ induces a t-structure on $D^A(U, \mathbb{Z}_\ell)$ and the perverse t-structure on $D^b_c(F, \mathbb{Z}_\ell)$ induces a t-structure on $D^A(F, \mathbb{Z}_\ell)$.

In that case, the t-structure on $D^A(S, \mathbb{Z}_\ell)$ is obtained by glueing the t-structures on $D^A(U, \mathbb{Z}_\ell)$ and $D^A(F, \mathbb{Z}_\ell)$.

Definition 3.4. Let $S$ be a scheme. Assume that the perverse t-structure on $D^b_c(S, \mathbb{Z}_\ell)$ induces a t-structure on $D^A(S, \mathbb{Z}_\ell)$. Then, we define the category $\text{Perv}^A(S, \mathbb{Z}_\ell)$ to be the heart of this t-structure.

The following property are analogous to the classical properties of perverse sheaves:

Proposition 3.5. Assume that the perverse t-structure induces a t-structure on Artin $\ell$-adic sheaves on all schemes appearing below,

1. Let $f : T \to S$ be a quasi-finite morphism of schemes. Then, there is a right exact functor $\mathbb{P}H^0 f_! : \text{Perv}^A(T, \mathbb{Z}_\ell) \to \text{Perv}^A(S, \mathbb{Z}_\ell)$. Furthermore, if $f$ is finite, $\mathbb{P}H^0 f_! = f_!$ and this functor is exact.

2. Let $g : T \to S$ be a morphism of schemes. Assume that $\text{dim}(g) \leq d$. Then, there is a right exact functor $\mathbb{P}H^d f^* : \text{Perv}^A(S, \mathbb{Z}_\ell) \to \text{Perv}^A(T, \mathbb{Z}_\ell)$.

Furthermore, if $f$ is étale, $\mathbb{P}H^0 f^* = f^*$ and this functor is exact.

3. Consider a cartesian square of schemes

$$
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow & & \downarrow \\
T & \xrightarrow{f} & S
\end{array}
$$

such that $p$ is quasi-finite and $\text{dim}(f) \leq d$. Then we have a canonical equivalence:

$$
\mathbb{P}H^d f^* \mathbb{P}H^0 p_! \to \mathbb{P}H^0 q_! \mathbb{P}H^d g^*.
$$

4. Let $S$ be a scheme. The formula $M \otimes A N := H^0(M \otimes N)$ defines a monoidal product on $\text{Perv}^A(S, \mathbb{Z}_\ell)$.

5. Let $C$ be an excellent scheme of dimension 1. Let $i : F \to C$ be a closed immersion and $j : U \to C$ be the open complement. Then, we have an exact sequence of perverse Artin $\ell$-adic sheaves:

$$
0 \to i_* \mathbb{P}H^{-1} i^* M \to \mathbb{P}H^0 j_! j^* M \to M \to i_* \mathbb{P}H^0 i^* M \to 0.
$$
3.3. Construction of Artin perverse sheaves on schemes of dimension $\leq 2$.

Our first tool will be the sharpening of our set of generators:

**Proposition 3.6.** Let $S$ be a universally japanese scheme (e.g. an excellent scheme). Then, $\mathcal{D}^A(S,\mathbb{Z}_\ell)$ is generated by the $f_*\mathbb{Z}_{\ell,X}$ for $f : X \to S$ finite and $X$ normal.

**Proof.** By definition, $\mathcal{D}^A(S,\mathbb{Z}_\ell)$ is generated by the $f_*\mathbb{Z}_{\ell,X}$ for $f : X \to S$ finite. We proceed by induction on $\dim(X)$ to prove that $\mathcal{D}^A(S,\mathbb{Z}_\ell)$ is in the thick stable subcategory of $\mathcal{D}^b_c(S,\mathbb{Z}_\ell)$ generated by the $f_*\mathbb{Z}_{\ell,X}$ for $f : X \to S$ finite and $X$ normal.

The case where $S$ is a point is clear. Assume that $\dim(S) > 0$. Let $f : X \to S$ be finite. Let

$\nu : X^\nu \to X$

be the normalization of $X$. Since $X$ is japanese, $\nu$ is finite. Moreover, we have the cdh-distinguished square (see Definition 1.3):

$$
\begin{array}{ccc}
Z' & \xrightarrow{\nu_Z} & X^\nu \\
\downarrow{\nu_Z} & & \downarrow{\nu} \\
Z & \xrightarrow{i} & X
\end{array}
$$

where $Z$ is the (reduced) closed subscheme of $X$ complementary of the set where $\nu$ is an isomorphism.

By cdh-descent (see Proposition 1.7), we then have an exact triangle:

$$Z_{\ell,X} \to \nu_*Z_{\ell,X^\nu} \oplus i_*Z_{\ell,Z} \to (i \circ \nu)_*Z_{\ell,Z^\nu}.$$

Pushing forward to $S$, we have an exact triangle

$$f_*Z_{\ell,X} \to (f \circ \nu)_*Z_{\ell,X^\nu} \oplus (fi)_*Z_{\ell,Z} \to (fi \circ \nu)_*Z_{\ell,Z^\nu}.$$

Since $\dim(Z'), \dim(Z) < \dim(X)$ and since $X^\nu$ is normal, we conclude by our induction hypothesis.

**Corollary 3.7.** Assume that $C$ is an excellent scheme of dimension $\leq 1$, then, the perverse t-structure induces a t-structure on $\mathcal{D}^A(C,\mathbb{Z}_\ell)$.

**Proof.** Use the fact that a normal 1-dimensional scheme is regular, the fact that if $S$ is regular, then $Z_{\ell,S}$ is in degree $\delta(S)$ for the perverse t-structure and Lemma 1.5 and Proposition 3.6.

Finally, we have:

**Proposition 3.8.** Let $k$ be a field and $p : E \to \text{Spec}(k)$ be a proper morphism. Let

$$E \to \pi_0(E/k) \to \text{Spec}(k)$$

be the Stein factorization of $p$.

Then, $R^0p_*\mathbb{Z}_{\ell,X} = q_*\mathbb{Z}_{\ell,\pi_0(E/k)}$.

**Proof.** For any scheme $X$ and any ring of coefficients $R$,

$$H^0_{\ell}(X,R) = R^{\pi_0(X)}.$$

The proposition follows from this result applied to $R = \mathbb{Z}/\ell^n\mathbb{Z}$ for all $n$. 

Finally, we have:
Lemma 3.9. Let $S$ be an excellent scheme. Then, the perverse $t$-structure induces a $t$-structure on $\mathcal{D}^A(S, \mathbb{Z}_\ell)$ if and only if for all $f : X \to S$ finite with $X$ normal and all $n < \delta(X)$, $p^H_n(Z_{\ell}(X))$ is an Artin $\ell$-adic sheaf.

Proof. The only if part is a consequence of the $t$-exactness of $f^*$ for $f$ finite.

Let us prove the if part. Let $f : X \to S$ be finite with $X$ normal. Since $Z_{\ell}(X)$ is in degree $\leq \delta(X)$, this means that $p^H_n(Z_{\ell}(X))$ is an Artin $\ell$-adic complex for any integer $n$. Therefore, the $p^H_n(f_*Z_{\ell}(X))[-n]$ are in the heart of the perverse $t$-structure and generate $\mathcal{D}^A(S, \mathbb{Z}_\ell)$. We conclude by Lemma 1.5. □

This is the main result of our paper:

Theorem 3.10. Assume that $S$ is an excellent scheme of dimension $\leq 2$, then, the perverse $t$-structure induces a $t$-structure on $\mathcal{D}^A(S, \mathbb{Z}_\ell)$.

Proof. Take the convention that $\delta(S) = 2$, i.e. $\delta(x) = 2 - \text{codim}_X(x)$ for all $x \in X$ (since $X$ is normal and therefore universally catenary and integral). The bulk of the work is the following:

Lemma 3.11. Let $X$ be a normal scheme of dimension 2. Then, the perverse cohomology sheaves $p^H_0(Z_{\ell}(X))$ and $p^H_1(Z_{\ell}(X))$ are Artin $\ell$-adic complexes.

Assuming that this lemma is true, we conclude by Lemma 3.9. □

We now prove the lemma:

Proof. Step 1: Let $X$ be a normal surface. Let $F$ be the singular locus of $X$. Since $X$ is normal, $F$ is finite.

Now, recall that since $X$ is excellent, by [sta22, 54.14.5,54.15.2] there exists a resolution of singularities for $X$ such that the divisor over $F$ is a simple normal crossing divisor. Thus, we have a cdh-distinguished square:

$$
\begin{array}{ccc}
E & \xrightarrow{p} & F \\
\downarrow{i_E} & & \downarrow{i} \\
\widetilde{X} & \xrightarrow{f} & X 
\end{array}
$$

where $\widetilde{X}$ is regular and $E$ is a simple normal crossing divisor.

There is a finite set $I$ and regular 1-dimensional closed subschemes $E_i$ of $\widetilde{X}$ such that $E = \bigcup_{i \in I} E_i$. Write $E_{ij} = E_i \cap E_j$. Then, $E_{ij}$ is finite over $F$.

Using our cdh square, since $\ell$-adic cohomology satisfies cdh-descent, we have an exact triangle:

$$Z_{\ell,X} \to f_*Z_{\ell,\widetilde{X}} \oplus i_*Z_{\ell,F} \to i_*p_*Z_{\ell,E}.
$$

Step 2: In this step, we prove that $p^H_0(f_*Z_{\ell,\widetilde{X}})$ and $p^H_1(f_*Z_{\ell,\widetilde{X}})$ vanish.

Let $j : U \to X$ be open immersion in $X$ of its regular locus. Then, there is an open immersion $j' : U \to \widetilde{X}$ whose image is exactly the complement of $E$.

Thus, we have the localization triangle with respect to $i_E$ and $j'$:

$$(i_E)_*(i_E)^!Z_{\ell,\widetilde{X}} \to Z_{\ell,\widetilde{X}} \to j'_*Z_{\ell,U}
$$

Since $fj' = j$ and $fi_E = ip$, we get an exact triangle:
\[ i_*p_*(i_E)^!Z_{\ell,\bar{X}} \to f_*Z_{\ell,\bar{X}} \to j_*Z_{\ell,U}. \]

Since \( j \) is of relative dimension 0, by \[ \text{[BBDG18, 4.2.4]}, \] \( pH^0(j_!Z_{\ell,U}) \) and \( pH^1(j_!Z_{\ell,U}) \) vanish. As \( i \) is finite, \( i_* \) is t-exact for the perverse t-structure. Thus, if \( k = 0, 1, \)

\[ pH^k(f_!Z_{\ell,U}) = i_*pH^k(p_*(i_E)^!Z_{\ell,\bar{X}}). \]

Write \( i_E : E_i \to \bar{X} \) and \( u_{E_i} : E_i \to E \) the closed immersions. Chose a total order on the finite set \( I \). By Proposition 1.8, we have an exact triangle:

\[ \bigoplus_{i<j}(u_{E_{i,j}})_!((i_{E_{i,j}})^!Z_{\ell,\bar{X}}) \to \bigoplus_{i \in I}(u_{E_i})_!((i_{E_i})^!Z_{\ell,\bar{X}}) \to (i_E)^!Z_{\ell,\bar{X}}. \]

By absolute purity, we get:

\[ \bigoplus_{i<j}(u_{E_{i,j}})_!((i_{E_{i,j}})^!Z_{\ell,\bar{X}})^{-2} [-4] \to \bigoplus_{i \in I}(u_{E_i})_!((i_{E_i})^!Z_{\ell,\bar{X}})^{-1} [-2] \to (i_E)^!Z_{\ell,\bar{X}}. \]

Let \( p_i : E_i \to F \) be the natural map. Pushing forward this triangle to \( F \), we get an exact triangle:

\[ \bigoplus_{i<j}(p_{i,j})_!((p_{i,j})^!Z_{\ell,E_{i,j}})^{-2} [-4] \to \bigoplus_{i \in I}(p_i)_!((p_i)^!Z_{\ell,E_i})^{-1} [-2] \to p_*(i_E)^!Z_{\ell,\bar{X}}. \]

But the two first terms of this triangle are in degree bigger than 2 which concludes the second step.

**Step 3:** Using the exact triangle of step 1, and the result of step 2, we get an exact sequence:

\[ 0 \to pH^0(Z_{\ell,X}) \to i_*Z_{\ell,F} \to i_*pH^0(p_*Z_{\ell,E}) \to pH^1(Z_{\ell,X}) \to 0. \]

Now, for \( x \in F \), let \( E_x = p^{-1}(x) \) and \( p_x : E_x \to \text{Spec}(k(x)) \) be the canonical map. Then,

\[ pH^0(p_*Z_{\ell,E}) = \bigoplus_{x \in F} R^0(p_x)_*Z_{\ell,E_x} \]

Hence, by Proposition 3.8, \( pH^0(p_*Z_{\ell,E}) \) is an Artin \( \ell \)-adic sheaf. Therefore, \( pH^0(Z_{\ell,X}) \) and \( pH^1(Z_{\ell,X}) \) are also Artin \( \ell \)-adic sheaves. \( \square \)

### 3.4. The case 3-folds with real closed or separably closed closed points.

In the case of 3-folds, we will show that the perverse t-structure can be restricted to Artin \( \ell \)-adic complexes in some cases, and that it is impossible in other cases. Our main tools will be Lemma 3.9 and the following lemma, which is closely linked to the Rapoport-Zink spectral sequence \[ \text{[RZ82, 2.8]}. \]

**Lemma 3.12.** Consider a diagram:

\[ E \overset{p}{\longrightarrow} F \]
\[ \downarrow \]
\[ Y \]

where \( p \) is proper map, and \( i \) is a closed immersion of a simple normal crossing divisor \( E \) into a regular scheme \( Y \).
Write $E = \bigcup_{i \in I} E_i$ where $I$ is a finite set and for all $J \subseteq I$, the closed subscheme $E_J := \bigcap_{i \in J} E_i$ of $Y$ is a regular subscheme of codimension $|J|$.

For all $J \subseteq I$, let $p_J : E_J \to F$ be the natural map. Then, there is a spectral sequence such that:

$$E_1^{p,q} = \bigoplus_{J \subseteq I, |J| = p+1}^p H^{q-p-2} ((p_J)_* \mathbb{Z}_{\ell,E_J}) (-p-1) \implies p H^{p+q} (p_\ast \mathbb{Z}_{\ell,Y}).$$

**Proof.** Denote $i_J : E_J \to Y$ the inclusion. Proposition 1.8 asserts that $i_\ast i_!^! \mathbb{Z}_{\ell,Y}$ is the homotopy colimit:

$$\text{hocolim}_{p \in \Delta^{inj}} \left( \bigoplus_{J \subseteq I, |J| = p+1} (i_J)_\ast i_J^! \mathbb{Z}_{\ell,Y} \right).$$

where $\Delta^{inj}$ is the category of finite ordered sets with morphisms the injective maps.

We can truncate this diagram in each degree to write $i_\ast i_!^! \mathbb{Z}_{\ell,Y}$ as the colimit of a sequential diagram $(M_p)_{p \geq 0}$ such that the cofiber of the map $M_{p-1} \to M_p$ is $M(p-1,p) := \bigoplus_{J \subseteq I, |J| = p+1} (i_J)_\ast i_J^! \mathbb{Z}_{\ell,Y}.$

Pulling back to $E$, pushing forward to $F$ and using the absolute purity, we can write $p_\ast i_J^! \mathbb{Z}_{\ell,Y}$ as the colimit of a sequential diagram $(N_p)_{p \geq 0}$ such that the cofiber of the map $N_{p-1} \to N_p$ is $N(p-1,p) := \bigoplus_{J \subseteq I, |J| = p+1} (p_J)_\ast \mathbb{Z}_{\ell,E_J} (-p-1)[-2p-2].$

But by [Lur17, 1.2.2.14] such a sequential diagram gives rise to a spectral sequence such that

$$E_1^{p,q} = \bigoplus_{J \subseteq I, |J| = p+1}^p H^{p+q} (N(p-1,p)) \implies p H^{p+q} (p_\ast i_J^! \mathbb{Z}_{\ell,Y}).$$

Finally,

$$p H^{p+q} (N(p-1,p)) = p H^{q-p-2} ((p_J)_\ast \mathbb{Z}_{\ell,E_J}) (-p-1)$$

which finishes the proof. \hfill \square

The following consequence of this lemma will be useful:

**Corollary 3.13.** Keep the notations of the lemma. Take the convention that $\delta(F) = \dim(F)$. Then,

1. $p H^k (p_\ast i_J^! \mathbb{Z}_{\ell,Y})$ vanishes for $k = 0,1$.
2. Write $I_0 = \{ i \in I \mid p_i(E_i) \text{ is of dimension 0} \}$. If $i \in I_0$, let $E_i \to Z_i \to F$.
be the Stein factorization of $p_i$ so that the map $q_i$ is finite and its image is a finite subset of $F$. Then,

$$pH^2(p_* i^! \mathbb{Z}_{\ell,Y}) = \bigoplus_{i \in I_0} (q_i)_* \mathbb{Z}_{\ell,Z_i}(-1).$$

Proof. The only non-vanishing term of the spectral sequence of Lemma 3.12 such that $p + q \leq 2$ is the term

$$E_{1}^{0,2} = \bigoplus_{i \in I} pH^0((p_i)_* \mathbb{Z}_{\ell,E_i})(-1).$$

The corollary then follows from Proposition 3.8.

**Proposition 3.14.** Let $S$ be a quasi-excellent 3-fold such that all the closed points of $S$ have a separably closed or real closed residue field. Then, the perverse $t$-structure induces a $t$-structure on $D^A(S, \mathbb{Z}_\ell)$.

Proof. Assume that the following lemma is true. Then, the result is proved by Lemma 3.9.

**Lemma 3.15.** Let $X$ be a quasi-excellent normal scheme of dimension 3 such that all the closed points of $X$ have a separably closed or real closed residue field. Then, the perverse cohomology sheaves $pH^k(\mathbb{Z}_{\ell,X})$ for $k \leq 2$ are Artin $\ell$-adic complexes.

We now prove the lemma.

Proof. **Step 1:** Let $F$ be the singular locus of $X$ and $i : F \to X$ be the inclusion. The scheme $F$ is of codimension at least 2, hence $\delta(F) \leq 1$ and $F$ is of dimension $\leq 1$. By [CP19], there exists a resolution of singularities for $X$ such that the divisor over $F$ is a simple normal crossing divisor. Thus, we have a cdh-distinguished square:

$$
\begin{array}{ccc}
E & \xrightarrow{p} & F \\
\downarrow^{i_E} & & \downarrow^i \\
\tilde{X} & \xrightarrow{f} & X \\
\end{array}
$$

where $\tilde{X}$ is regular and $E$ is a simple normal crossing divisor. By cdh-descent, we have an exact triangle:

$$Z_{\ell,X} \to f_* Z_{\ell,\tilde{X}} \oplus i_* Z_{\ell,F} \to i_* p_* Z_{\ell,E}.$$  

**Step 2:** Like in step 2 of Lemma 3.11

$$pH^k(f_* Z_{\ell,\tilde{X}}) = i_* pH^k(p_* (i_E)_! Z_{\ell,E}).$$

Hence, by Corollary 3.13 $pH^k(f_* Z_{\ell,\tilde{X}})$ vanishes for $k = 0, 1$ and there is a 0-dimensional scheme $F_0$ and a finite map $k : F_0 \to F$ such that

$$pH^2(f_* Z_{\ell,\tilde{X}}) = i_* k_* Z_{\ell,F_0}(-1).$$

**Step 3:** Let $\pi : W \to T$ be proper. Assume that $\delta(T) \leq 1$ and that $T$ is of dimension $\leq 1$. We give an explicit computation of $pH^n(\pi_* Z_{\ell,W})$ if $n = 0, 1$. In the case where all the closed points of $T$ have separably closed or real closed residue fields, this computation will show that they are Artin $\ell$-adic complexes. We will use this computation in the case of the map $p : E \to F$. 


If $T' \to T$ is a map, we let $\pi_{T'} : W_{T'} \to T'$ be the pull-back of $\pi$ along this map. There is an affine open immersion $j : U \to T$ of 0-dimensional (reduced) complement $\iota : Z \to T$ such that

- All the points $x$ of $Z$ satisfy $\delta(x) = 0$ and all the generic points $\eta$ of $U$ satisfy $\delta(\eta) = 1$.
- If $n = 0, 1$,
  
  $j^* [p^H_n(\pi_*Z_\ell,W)]$

  is a lisse perverse sheaf.
- The reduced scheme associated to $U$ is regular.

Now, let $\Gamma$ be the set of generic points of $U$. Write $j_\eta : U^{(\eta)}$ the reduced closure of $\{\eta\}$ in $U$. Then, $U^{(\eta)}$ is connected and regular. Therefore,

$$p^H_n((\pi_U)_*Z_\ell,W_U) = \bigoplus_{\eta \in \Gamma} (j_\eta)_* M^n_{\eta}$$

where $M^n_{\eta}$ is a continuous representation of $\pi^{et}_1(U^{(\eta)})$.

By Proposition 3.8 if $\eta \in \Gamma$, $p^H_1((\pi_\eta)_*Z_\ell,W_\eta)$ is an Artin representation of the Galois group at $\eta$, $G_{k(\eta)}$ and $p^H_0((\pi_\eta)_*Z_\ell,W_\eta) = 0$.

Hence, if $\theta_\eta : \eta \to U^{(\eta)}$ is the inclusion, $\theta_\eta^* M^1_{\eta}$ is an Artin representation of $G_{k(\eta)}$ and $\theta_\eta^* M^1_{\eta} = 0$; thus, by [GR71, V.8.2], $M^1_{\eta}$ is an Artin representation of $\pi^{et}_1(U^{(\eta)})$ and $M^0_{\eta} = 0$.

Hence, $p^H_1((\pi_U)_*Z_\ell,W_U) = 0$ and $p^H_1((\pi_U)_*Z_\ell,W_U)$ is a smooth Artin $\ell$-adic sheaf that we will denote $M(U)$.

By Proposition 3.8 there is a finite map $\lambda : Z' \to Z$ such that

$$p^H_0((\pi_Z)_*Z_\ell,W_Z) = \lambda_* Z_\ell,Z'. $$

Now, since $j$ is affine, it is t-exact by [BBDG18, 4.1.3]. Hence, the localization triangle and the proper base change formula give a long exact sequence of perverse sheaves:

$$0 \to p^H_0(\pi_*Z_\ell,W) \to \mu_* \lambda_* Z_\ell,Z' \to j_! M(U) \to p^H_1(\pi_*Z_\ell,W) \to \ker(\Psi) = 0$$

Where $\Psi : \mu_* p^H_1((\pi_Z)_*Z_\ell,W_Z) \to j_! p^H_1((\pi_U)_*Z_\ell,W_U)$.

But by [BBDG18, 1.4.16(i), 1.4.16(ii)], $\Psi$ factors through

$$\Psi' = p^H_0(\mu_* \Psi : p^H_1((\pi_Z)_*Z_\ell,W_Z) \to p^H_0(\mu_* j_! p^H_1((\pi_U)_*Z_\ell,W_U))$$

and

$$\ker(\Psi) = \mu_* \ker(\Psi').$$

Now, assume that all the closed point of $T$ have separably closed or real closed residue fields. Then, any constructible $\ell$-adic complex on $Z$ is Artin, because $Z$ is a scheme of dimension 0 with only separably closed or real closed points. Hence $\ker(\Psi')$ is Artin and thus if $n = 0, 1$, $p^H_n(\pi_*Z_\ell,W)$ are Artin $\ell$-adic complexes.

**Step 4:** Using Step 1 and 2, we have a 0-dimensional scheme $F_0$ endowed with a finite map $k : F_0 \to F$ and a perverse sheaf $N$ on $S$ such that the following sequences are exact.

$$0 \to p^H_0(Z_\ell,X) \to i_* p^H_0(Z_\ell,F) \to i_* p^H_0(p_*Z_\ell,E)$$

$$i_* p^H_0(Z_\ell,F) \to i_* p^H_1(p_*Z_\ell,E) \to p^H_1(Z_\ell,X) \to i_* p^H_1(Z_\ell,F) \to i_* p^H_1(p_*Z_\ell,E)$$

$$i_* p^H_1(Z_\ell,F) \to i_* p^H_1(p_*Z_\ell,E) \to p^H_2(Z_\ell,X) \to i_* k_* Z_\ell,F_0(-1) \to N$$

currently unanswerable due to text-cutting issue
We show in the same fashion as in the end of Step 3, that the kernel of 
\[ i_* k_* \mathbb{Z}_{\ell,*}(−1) \rightarrow N \]
is an Artin \( \ell \)-adic complex. Thus, using Step 3, we are done. \( \square \)

3.5. A counter-example.

**Proposition 3.16.** Let \( X \) be a normal scheme of dimension 3 with singular locus a closed point \( x \). Assume that \( k(x) \) is finite. Let \( f : \overline{X} \rightarrow X \) be a resolution of singularities of \( X \) and assume that the exceptional divisor \( E \) is smooth over \( k(x) \) and has a non-zero first \( \ell \)-adic betti number.

Then, \( p H^2(\mathbb{Z}_{\ell,X}) \) is not an Artin \( \ell \)-adic sheaf over \( X \).

**Proof.** By Step 4 of the proof of Lemma 3.15 (in our case \( F = \{ x \} \) is of dimension 0), there is an exact sequence:

\[ 0 \rightarrow i_* p H^1(p_* \mathbb{Z}_{\ell,X}) \rightarrow p H^2(\mathbb{Z}_{\ell,X}) \rightarrow i_* M \rightarrow 0 \]

where \( M \) is an \( \ell \)-adic representation of \( G_{k(x)} \).

Hence there is an \( \ell \)-adic representation \( P \) such that \( p H^2(\mathbb{Z}_{\ell,X}) = i_* P \) and \( p H^1(p_* \mathbb{Z}_{\ell,X}) \) is a subobject of \( P \).

Thus, it suffices to show that \( p H^1(p_* \mathbb{Z}_{\ell,X}) \) is not a representation of \( G_{k(x)} \) of Artin origin. By Proposition 1.21 and Remark 1.20 it suffices to show that it is not of weight 0 as a perverse sheaf on \( \text{Spec}(k(x)) \).

But by [BBDG18, 5.1.14, 5.4.4], \( p H^1(p_* \mathbb{Z}_{\ell,X}) \) is pure of weight 1. Since the first \( \ell \)-adic Betti number \( b_1(E) \) is non-zero, \( p H^1(p_* \mathbb{Z}_{\ell,X}) \) is non-zero and thus, it is not of Artin origin. \( \square \)

**Example 3.17.** Let \( E \) be an elliptic curve over a finite field \( k \) and \( X \) be the affine cone over \( E \times \mathbb{P}^1_k \). \( X \) has a single singular point \( x \) and \( k^1 \times E \times \mathbb{P}^1_k \) is a resolution of singularities of \( X \) with exceptional divisor \( E \times \mathbb{P}^1_k \).

As \( b_1(E \times \mathbb{P}^1_k) = 1 \), the proposition applies and therefore, the perverse t-structure does not induce a t-structure on \( D^A(X, \mathbb{Z}_\ell) \).

Finally, notice that by Noether’s normalization lemma, there is a finite map \( X \rightarrow k^3 \). Thus, the perverse t-structure does not induce a t-structure on \( D^A(k^3, \mathbb{Z}_\ell) \). As \( k^3 \) is a closed subscheme of \( k^n \) for \( n \geq 3 \), the perverse t-structure does not induce a t-structure on \( D^A(k^n, \mathbb{Z}_\ell) \) if \( n \geq 3 \).

**Remark 3.18.** The condition that \( \dim(S) \leq 2 \) or \( \dim(S) \leq 3 \) and the residue field of closed points of \( S \) are separably closed or real closed seems rather optimal: loosely speaking, if \( X \) becomes more singular, the perverse cohomology sheaves of \( \mathbb{Z}_{\ell,X} \) should become more complicated. Here, the simplest possible singularity on a scheme of dimension 3 over a finite field already renders the cohomology sheaf not Artin.

4. HOMOTOPY PERVERSE T-STRUCTURE

Recall (see Proposition 4.2 below) that we can formally define t-structures on a presentable stable category by setting a certain set of objects to be the t-negative objects of our t-structure. Thus, an other attempt to define a t-structure on \( D^A(S, \mathbb{Z}_\ell) \) is to define it by generators on \( D^A_{\text{Ind}}(S, \mathbb{Z}_\ell) \) and then to see when this t-structure can be restricted to \( D^A(S, \mathbb{Z}_\ell) \).
4.1. Cohomological ℓ-adic complexes and the six functors for them.

**Definition 4.1.** Let $S$ be a scheme and $\ell$ be a prime number invertible on $S$. The category of cohomological (resp. ind-cohomological) $\ell$-adic complexes over $S$ is the thick (resp. localizing) subcategory $\mathcal{D}^\text{coh}(S, \mathbb{Z}_\ell)$ (resp. $\mathcal{D}^\text{coh}_{\text{Ind}}(S, \mathbb{Z}_\ell)$) of $\mathcal{D}(S, \mathbb{Z}_\ell)$ generated by the $f_!\mathbb{Z}_\ell,X$ for $f : X \to S$ proper.

In [Leh19b, 1.12], Pépin Lehalleur proved that (constructible) cohomological motives are stable under the six functors when the base schemes have resolution of singularities by alterations. In [CD16, 6.2], using Gabber’s method, Cisinski and Déglise showed that constructible étale motives are endowed with the six functors formalism when the base schemes are quasi-excellent. One can mimic their proof in the case of cohomological $\ell$-adic complexes. We outline how to do this:

1. The fibred subcategory $\mathcal{D}^\text{coh}(-, \mathbb{Z}_\ell)$ of $\mathcal{D}(-, \mathbb{Z}_\ell)$ is stable under tensor product, negative Tate twists, $f^*$ if $f$ is any morphism and $f_!$ if $f$ is separated and of finite type.

2. From the absolute purity property and the stability of $\mathcal{D}^\text{coh}$ under negative Tate twists and using the analog of [Ayo07, 2.2.31] for $\ell$-adic complexes, we deduce that if $i$ is the immersion of a simple normal crossing divisor with regular target $X$, $i^!\mathbb{Z}_\ell,X$ is cohomological.

3. Let $X$ be a scheme. $\mathcal{D}^\text{coh}(X, \mathbb{Z}_\ell)$ has the properties [CD16 6.2.9(b)(c)] replacing $\mathcal{D}M_b(X, \mathbb{R})$ with $\mathcal{D}(X, \mathbb{Z}_\ell)$. It does not necessarily have property [CD16 6.2.9(a)] however, in the proof of [CD16 6.2.7] we only need the following weaker version: $\mathcal{D}^\text{coh}(X, \mathbb{Z}_\ell)$ is a thick subcategory of $\mathcal{D}(X, \mathbb{Z}_\ell)$ which contains the object $\mathbb{Z}_\ell,X$. From this discussion, we deduce Gabber’s lemma for cohomological $h$-motives: if $X$ is quasi-excellent, then for any dense open immersion $U \to X$, the $\ell$-adic complex $j_*\mathbb{Z}_\ell,U$ is cohomological.

4. Follow the proofs of [CD16 6.2.13, 6.2.14] to prove that $\mathcal{D}^\text{coh}(-, \mathbb{Z}_\ell)$ is endowed with the six functors formalism over quasi-excellent schemes.

4.2. Definition on Ind-Artin $\ell$-adic complexes.

**Proposition 4.2.** Let $C$ be a presentable stable category which admits small co-products and push-outs. Given a family $E$ of objects, the smallest subcategory $E^{-}$ stable under extensions, coproducts and negative shifts is the set of non-positive objects of a t-structure.

**Proof.** [Lur17 1.2.1.16].

In the setting of Proposition 4.2 we will call this t-structure the t-structure generated by $E$. In the motivic setting, have been used to define the ordinary homotopy t-structure [Ayo07 2.2.3], the homotopy perverse t-structure [Ayo07 2.2.3] and the ordinary motivic t-structure on 0-motives and 1-motives [Leh19b]. Our perverse-homotopy t-structure is similar to a 'shifted' version of Pépin Lehalleur’s ordinary motivic t-structure.

**Definition 4.3.** Let $S$ be a scheme and $\ell$ be a prime number that is invertible on $S$. Then, the homotopy perverse t-structure on $\mathcal{D}^\text{Ind}_{\text{Ind}}(S, \mathbb{Z}_\ell)$ is the t-structure generated by the $f_!\mathbb{Z}_\ell,X[\delta(X)]$ for $f : X \to S$ finite.

Let $n$ be an integer. We denote $^hH^n$ the $n$-th cohomology functor with respect to this t-structure.
Using the same method as [AZ12, 2.5], it is also generated by the \( f_! \mathbb{Z}_\ell, X [\delta(X)] \) with \( X \) quasi-finite. By definition, an object \( M \) of \( \mathcal{D}^A_{\text{ind}}(S, \mathbb{Z}_\ell) \) is in \( \mathcal{D}^A_{\text{ind}}(S, \mathbb{Z}_\ell) \geq n \) if and only if for any finite map \( f : X \to S \) and any \( p > -n \),

\[
\text{Hom}_{\mathcal{D}(S, \mathbb{Z}_\ell)}(f_! \mathbb{Z}_\ell, X [\delta(X) + p], M) = 0.
\]

In other words, \( \text{Map}(f_! \mathbb{Z}_\ell, X, M) \) is \((n - \delta(X) - 1)\)-connected.

To study this t-structure, we need to introduce the Artin truncation functor:

**Definition 4.4.** Let \( S \) be a scheme and \( \ell \) be a prime number that is invertible on \( S \). The adjunction theorem [Lur09, 5.5.1.9] gives rise to a right adjoint functor to the inclusion of Ind-Artin \( \ell \)-adic complexes into Ind-cohomological complexes of \( \ell \)-adic sheaves:

\[
\omega^0 : \mathcal{D}^{\text{coh}}_{\text{ind}}(S, \mathbb{Z}_\ell) \to \mathcal{D}^A_{\text{ind}}(S, \mathbb{Z}_\ell).
\]

We call this functor Artin truncation

**Remark 4.5.** This functor was first introduced in [AZ12, 2.2] in the motivic setting. They predicted that their functor was an analog of a punctual weightless truncation functor on cohomological \( \ell \)-adic complexes. This punctual weightless truncation functor was introduced in [NV15] by Vaish and Nair. Vaish also introduced a punctual weightless truncation functor in the motivic setting in [Vai16] and showed that his functor coincides with Ayoub and Zucker’s over cohomological motives. We will show that our functor coincides with Vaish and Nair’s in the case of a scheme over a finite field (see Section 4.6).

**Proposition 4.6.** Let \( S \) be a scheme. Let \( f \) be a quasi-finite morphism and \( g \) be an essentially of finite type morphism. Then, for the homotopy perverse t-structure

1. The adjunction \((f^!, \omega^0 f^!\)]\) is a t-adjunction.
2. \( \otimes_S \) is right t-exact.
3. If \( \text{dim}(g) \leq d \), the adjunction \((g^* [d], \omega^0 g_* [-d])\) is a t-adjunction.

**Proof.** Use the description of generators of Proposition 2.8 \( \square \)

**Corollary 4.7.** Let \( S \) be a scheme. Let \( f \) be a morphism. Then, for the homotopy perverse t-structure

1. If \( f \) is étale, \( f^* = f^! = \omega^0 f^! \) is t-exact.
2. If \( f \) is finite, \( f_! = f_* = \omega^0 f_* \) is t-exact.

**4.3. The smooth Artin case.** Compare the following with Proposition 3.2.

**Proposition 4.8.** If \( S \) is regular and connected, the homotopy perverse t-structure induces a t-structure on \( \mathcal{D}^{\text{sm}}(S, \mathbb{Z}_\ell) \) which coincides with the t-structure induced by the perverse t-structure and with the canonical t-structure shifted by \( \delta(S) \).

**Proof.** By Lemma 1.1 and Corollary 4.7, it suffices to show that the object \( \mathbb{Z}_\ell(S) \) is in degree \( \delta(S) \) for the homotopy perverse t-structure. Hence, it suffices to prove that if \( f : X \to S \) is finite, \( \text{Map}(f_! \mathbb{Z}_\ell(X), \mathbb{Z}_\ell(S)) \) is \((\delta(S) - \delta(X) - 1)\)-connected.

But as \( \mathbb{Z}_\ell(S) \) is in degree \( \delta(S) \) and \( f_* \mathbb{Z}_\ell(X) \) is in degree \( \leq \delta(X) \) for the (classical) perverse t-structure, this is true. \( \square \)
4.4. Gluing formalism.

**Proposition 4.9.** Let $S$ be a scheme. Let $\ell$ be a prime number that is invertible on $S$. Let $i : F \to S$ be a closed immersion and $j : U \to S$ be the open complement. Then, the sequence

$$D_{\text{ind}}^A(F, \mathbb{Z}_\ell) \xrightarrow{i^*} D_{\text{ind}}^A(S, \mathbb{Z}_\ell) \xrightarrow{j^*} D_{\text{ind}}^A(U, \mathbb{Z}_\ell)$$

satisfies the axioms of the gluing formalism (see [BBDG18, 1.4.3]), i.e.

1. The functors $i_*$ and $j^*$ both have exact left and right adjoints $i^*$, $\omega^0 i_!, j_!$ and $\omega^0 j^*$.
2. $j^* i_* = 0$.
3. $i_*, j_*$ and $j_!$ are fully faithful.
4. We have exact triangles:

$$jj^* \to 1 \to i_* i^*$$

and

$$i_* \omega^0 j_! \to 1 \to \omega^0 j_* j^*.$$

Moreover, the homotopy perverse $t$-structure on $D_{\text{ind}}^A(S, \mathbb{Z}_\ell)$ is obtained by glueing the $t$-structures of $D_{\text{ind}}^A(U, \mathbb{Z}_\ell)$ and $D_{\text{ind}}^A(F, \mathbb{Z}_\ell)$, (see [BBDG18, 1.4.9]) i.e. for all object $M$ of $D^A(S, \mathbb{Z}_\ell)$

1. $M \geq 0$ if and only if $j^* M \geq 0$ and $\omega^0 i_! M \geq 0$.
2. $M \leq 0$ if and only if $j^* M \leq 0$ and $i^* M \leq 0$.

**Proof.** This follows from Proposition 4.6 Corollary 4.7 and the usual properties of the six functors. \qed

**Corollary 4.10.** Let $S$ be a scheme. Let $\ell$ be a prime number that is invertible on $S$. Let $M$ be an object of $D_{\text{ind}}^A(S, \mathbb{Z}_\ell)$. Then,

1. $M \geq 0$ if and only if there is a stratification $S$ of $S$ such that for all stratum $i : T \to S$, $\omega^0 i_! M \geq 0$.
2. $M \leq 0$ if and only if there is a stratification $S$ of $S$ such that for all stratum $i : T \to S$, $i^* M \leq 0$.

In particular if $S$ is excellent and $M$ is an object of $D^A(S, \mathbb{Z}_\ell)$. Then, $M \leq 0$ if and only if there is a stratification $S$ of $S$ with regular strata, such that for all stratum $i : T \to S$, $i^* M$ is a smooth Artin $\ell$-adic complex and $i^* M \leq 0$ for the perverse $t$-structure.

**Proof.** Recall that an excellent scheme admits a stratification with regular strata. Therefore, the last point follows from Proposition 2.12 and Proposition 4.8. \qed

4.5. The homotopy perverse and the perverse $t$-structure.

**Proposition 4.11.** Let $S$ be an excellent scheme and $\ell$ be a prime number that is invertible on $S$. Then,

1. Let $M$ be an object of $D^A(S, \mathbb{Z}_\ell)$, then $M$ is negative for the homotopy perverse $t$-structure if and only if it is negative for the perverse $t$-structure.
2. Let $M$ be an object of $D^A(S, \mathbb{Z}_\ell)$, then if $M$ is positive for the perverse $t$-structure, then it is positive for the homotopy perverse $t$-structure.
3. If the perverse $t$-structure on $D_{\text{ind}}^A(S, \mathbb{Z}_\ell)$ induces a $t$-structure on $D^A(S, \mathbb{Z}_\ell)$, then the homotopy perverse $t$-structure on $D_{\text{ind}}^A(S, \mathbb{Z}_\ell)$ induces a $t$-structure on $D^A(S, \mathbb{Z}_\ell)$. When this is the case, those two $t$-structures coincide.
Remark 4.12. We can reformulate the first point: when the homotopy perverse t-structure induces a t-structure on $D^A(S, \mathbb{Z}_\ell)$, it is final among those t-structures such that the inclusion $D^A(S, \mathbb{Z}_\ell) \to D^b_c(S, \mathbb{Z}_\ell)$ is right t-exact.

Proof. The first point follows from (2) of Corollary 4.10. Now assume that $M$ is positive for the perverse t-structure. Then, it is in the right orthogonal of non-positive objects for the perverse t-structure. Thus, it is in the right orthogonal of non-positive objects for the homotopy perverse t-structure that are Artin $\ell$-adic complexes; in particular for any finite map $f : X \to S$ and any $p > -n$, $M$ is right orthogonal to $f_*\mathbb{Z}_\ell[-(\delta(X) + p)]$ and hence $M$ is positive for the homotopy perverse t-structure.

If the perverse t-structure induces a t-structure on $D^A(S, \mathbb{Z}_\ell)$, the objects of $D = D^A_{ind}(S, \mathbb{Z}_\ell)^{\leq 0} \cap D^A(S, \mathbb{Z}_\ell)$ are the t-non-positive objects of a t-structure. Furthermore, the positive objects of this t-structure are positive for the homotopy perverse t-structure. But they contain the positive objects of the homotopy perverse t-structure since those objects are orthogonal to the elements of $D$. □

4.6. Comparison of $\omega^0$ and $w^{\leq 1d}$ for schemes of finite type over a finite field and with coefficients $\mathbb{Q}_p$. Let $S$ be a scheme of finite type over a finite field. Recall that Vaish and Nair introduced a functor $w^{\leq 1d} : D^b_c(S, \mathbb{Q}_\ell) \to D^b_c(S, \mathbb{Q}_\ell)$ in [NY15 3.1.1, 3.1.5]. It is a truncation functor with respect to a t-structure $(D^b_c(S, \mathbb{Q}_\ell)^{\leq 1d}, D^b_c(S, \mathbb{Q}_\ell)^{> 1d})$. We will still denote $w^{\leq 1d}$ its restriction to cohomological $\ell$-adic complexes (see Definition 4.1). This functor is the truncation functor with respect to the induced t-structure on $D^{coh}(S, \mathbb{Q}_\ell)$.

Proposition 4.13. Let $S$ be a scheme of finite type over $\mathbb{F}_p$. Let $\ell \neq p$ be a prime number. Then, the functors $\omega^0$ and $w^{\leq 1d}$ induce equivalent functors

$$D^{coh}(S, \mathbb{Q}_\ell) \to D^A(S, \mathbb{Q}_\ell).$$

Remark 4.14. A priori, $\omega^0$ sends cohomological $\ell$-adic complexes to ind-Artin $\ell$-adic complexes. This proposition proves that in fact, it sends cohomological $\ell$-adic complexes to Artin $\ell$-adic complexes.

Proof. Recall that in [CD16 5.1, 7.2.10], Cisinski and Déglise constructed a triangulated category $DM(S, \mathbb{Q})$ endowed with a realization functor $\rho : DM(S, \mathbb{Q}) \to D(S, \mathbb{Q}_\ell)$ that commutes with the six functors by [CD16 7.2.24]. We can promote their construction to a stable $\infty$-categorical one following [Kha16] and [Rob14].

Now, recall that Vaish constructed a functor (see [Va16 3.1]) $w^{mot}_{\leq 1d}$ from the category of cohomological motives to the category of Artin motives over $S$.

Lemma 4.15. Keep the same notations, then, $\rho_tw^{mot}_{\leq 1d} = w^{\leq 1d}\rho_t$

Proof. We follow the ideas of [Va16 4.2.4]. The proof of [Va16 4.2.4] tells us that $\rho_tDM(S, \mathbb{Q})^{\leq 1d} \subseteq D(S, \mathbb{Q}_\ell)^{\leq 1d}$ and that $\rho_tDM(S, \mathbb{Q})^{> 1d} \subseteq D(S, \mathbb{Q}_\ell)^{> 1d}$.

Now, if $X$ is a scheme, let $1_X$ be the constant object of $DM(X, \mathbb{Q})$. Let $f : X \to S$ be a proper map.

Applying $\rho_t$ to the exact triangle:

$$w^{mot}_{\leq 1d}f_*1_X \to f_*1_X \to w^{mot}_{> 1d}f_*1_X,$$

we get an exact triangle:

$$\rho_tw^{mot}_{\leq 1d}f_*1_X \to f_*\mathbb{Q}_\ell,X \to w^{mot}_{> 1d}f_*1_X.$$
But $\rho_\ell w^\mathrm{mot}_{\le\ell} f_* 1_X$ lies in $D(S, \mathbb{Q}_\ell)^{\le\ell}$ and $\rho_\ell w^\mathrm{mot}_{\ge\ell} f_* 1_X$ lies in $D(S, \mathbb{Q}_\ell)^{>\ell}$. Hence, $\rho_\ell w^\mathrm{mot}_{\le\ell} f_* 1_X$ must be the truncation of $f_* \mathbb{Q}_{\ell,X}$ with respect to the t-structure $(D^b_c(S, \mathbb{Q}_\ell)^{\le\ell}, D^b_c(S, \mathbb{Q}_\ell)^{>\ell})$. \qed

Using the lemma, we see that if $f : X \to S$ is proper, $w_{\le\ell} f_* \mathbb{Q}_{\ell,X}$ is an Artin $\ell$-adic complex. Hence, the functor $w_{\le\ell}$ has its essential image included in $D^A(S, \mathbb{Q}_\ell)$.

Recall that $w_{\le\ell} : D^\coho(S, \mathbb{Q}_\ell) \to D^\coho(S, \mathbb{Q}_\ell)^{\le\ell}$ is exact and is the right adjoint of the inclusion functor. Now, if $g : Y \to S$ is finite, $g_* \mathbb{Q}_{\ell,Y}$ is in $D^\coho(S, \mathbb{Q}_\ell)^{\le\ell}$ by [NV15, 3.1.8], hence, if $M$ is a cohomological $\ell$-adic complex, the adjunction map

$$\text{Map}(g_* \mathbb{Q}_{\ell,Y}, w_{\le\ell} M) \to \text{Map}(g_* \mathbb{Q}_{\ell,Y}, M)$$

is an equivalence. Therefore, if $N$ is an ind-Artin $\ell$-adic complex and $M$ is a cohomological $\ell$-adic complex, the adjunction map

$$\text{Map}(N, w_{\le\ell} M) \to \text{Map}(N, M)$$

is an equivalence. Thus, Proposition 4.13 follows. \qed

Using the Artin truncation functor $\omega^0$, we prove that there is a trace of the six functors formalism on Artin $\ell$-adic sheaves with coefficients $\mathbb{Q}_\ell$ over a finite field.

**Corollary 4.16.** Let $\ell \neq p$ be prime numbers. Then, on schemes of finite type over $\mathbb{F}_p$, the six functors from $D^\coho(-, \mathbb{Q}_\ell)$ (see Section 4.7) induce the following adjunctions on $D^A(-, \mathbb{Q}_\ell)$:

1. $(f^*, \omega^0 f_*)$ for any separated morphism of finite type $f$,
2. $(f^*, \omega^0 f^!)$ for any quasi-finite morphism $f$,
3. $(\otimes, \omega^0 \Hom)$.

**Remark 4.17.** These functors have quite remarkable features. Let $S$ be a scheme of finite type over $\mathbb{F}_p$. Let $\ell \neq p$ be a prime number. Let $X$ be a scheme of finite type over $k$.

First, Vaish and Nair used them to define the weightless complex $EC_X = \omega^0 j_* \mathbb{Q}_{\ell,U}$ with $j : U \to X$ the open immersion of a regular subscheme of $X$ (see [NV15, 3.1.15]). This is an analog of Ayoub and Zucker’s motive $E_X$ (see [AZ12, 2.20]).

In fact, if $f : X \to S$ is a resolution of singularities, then $EC_X = \omega^0 f_* \mathbb{Q}_{\ell,X}$ (see [NV15, 4.1.2]). The proof amounts to the following fact: if $i : F \to S$ is a closed immersion of dense complement, $\omega^0 i^!$ is zero on $D^\ginf(X, \mathbb{Q}_\ell)$ (see the proof of [NV15, 4.1.1]).

4.7. The homotopy perverse t-structure for schemes of finite type over a finite field and with coefficients $\mathbb{Q}_\ell$. We will now show that the homotopy perverse t-structure induces a t-structure on $D^A(S, \mathbb{Q}_\ell)$ when $S$ is of finite type over a finite field. To show this, we will construct a t-structure on $D^A(S, \mathbb{Q}_\ell)$ by glueing the perverse t-structure on smooth Artin objects along stratifications and show that it coincides with the perverse t-structure.

**Proposition 4.18.** Let $S$ be a scheme of finite type over $\mathbb{F}_p$. Let $\ell \neq p$ be a prime number. Then,

1. The homotopy perverse t-structure induces a t-structure on $D^A(S, \mathbb{Q}_\ell)$. 
(2) Let $i : F \to S$ be a closed immersion and $j : U \to S$ be the open complement. The sequence

$$D^A(F, \mathbb{Q}_\ell) \xrightarrow{i^*} D^A(S, \mathbb{Q}_\ell) \xrightarrow{j_!} D^A(U, \mathbb{Q}_\ell)$$

satisfies the axioms of the glueing formalism (see [BBDG18, 1.4.3] and Proposition 4.9) and the homotopy perverse t-structure on $D^A(S, \mathbb{Q}_\ell)$ is obtained by glueing the t-structures of $D^A(U, \mathbb{Q}_\ell)$ and $D^A(F, \mathbb{Q}_\ell)$, (see [BBDG18, 1.4.9]).

(3) Let $i : D^A(S, \mathbb{Q}_\ell) \to D^\text{coh}(S, \mathbb{Q}_\ell)$ be the inclusion. Then, with the homotopy perverse t-structure on the left hand side and the perverse t-structure on the right hand side, the adjunction $(i, \omega^0)$ is a t-adjunction and $\omega^0$ is t-exact.

Proof. The first part of (2) is a direct consequence of Proposition 4.9 and Corollary 4.10. To prove the rest of the proposition, we need the following:

Lemma 4.19. Let $S$ be a scheme of finite type over $\mathbb{F}_p$. Let $\ell \neq p$ be a prime number. Let $M$ be an object of $D^A(S, \mathbb{Z}_\ell)$. Then, there is a stratification $S$ of $S$ such that for all stratum $i : T \hookrightarrow S$, $\omega^0 i^! M$ and $i^* M$ are smooth Artin $\ell$-adic complexes.

Proof. We proceed by noetherian induction. By Proposition 2.12 there is a dense open immersion $j : U \to S$ such that $\omega^0 j^! M = j^* M$ is a smooth Artin $\ell$-adic sheaf. Now, if $i : F \to S$ is a closed complementary immersion, since $\omega^0 i^! M$ and $i^* M$ are Artin $\ell$-adic complexes by Corollary 4.10, there is a stratification $S'$ of $F$ such that for all stratum $k : T \hookrightarrow F$, $\omega^0 k^! \omega^0 i^! M$ and $k^* i^* M$ are smooth Artin $\ell$-adic complexes. The result follows.

We now finish the proof of the proposition. Let us start with (1). If $S$ is a stratification of $S$, we say that an object is $S$-constructible if for all stratum $i : T \hookrightarrow S$, $\omega^0 i^! M$ and $i^* M$ are smooth Artin $\ell$-adic complexes.

We can define a t-structure on $S$-constructible objects by glueing the perverse t-structure on the $D^m^{\text{et}}(T, \mathbb{Q}_\ell)$. By Corollary 4.10 the positive (resp. negative) objects of this t-structure are the positive (resp. negative) objects of the homotopy perverse t-structure that are $S$-constructible.

Thus, the perverse t-structure induces a t-structure on $S$-constructible objects. Therefore, $S$-constructible objects are stable under the truncation functors of the homotopy perverse t-structure. Hence $D^A(S, \mathbb{Q}_\ell)$ is stable under the truncation functors of the homotopy perverse t-structure. Therefore, the homotopy perverse t-structure induces a t-structure on $D^A(S, \mathbb{Q}_\ell)$.

Now, the second part (2) follows from Proposition 4.9.

Finally, Proposition 4.11 implies that $i$ is right t-exact. Thus, $\omega^0$ is left t-exact. But by Proposition 4.13 and [NV15, 3.2.6], $\omega^0$ is perverse right exact, i.e. it sends perverse-negative objects to perverse-negative objects. But any such object is negative with respect to the homotopy perverse t-structure using Proposition 4.11. Thus $\omega^0$ is right t-exact and therefore, $\omega^0$ is t-exact.

Definition 4.20. Let $S$ be a scheme of finite type over $\mathbb{F}_p$. Let $\ell \neq p$ be a prime number. We denote $\text{Perv}^A(S, \mathbb{Q}_\ell)^!$ and call abelian category of Artin homotopy perverse sheaves on $S$ the heart of the homotopy perverse t-structure on $D^A(S, \mathbb{Q}_\ell)$.
Remark 4.21. When both are defined, the abelian categories \( \text{Perv}^A(S, \mathbb{Q}_\ell) \) and \( \text{Perv}^A(S, \mathbb{Q}_\ell') \) coincide (see Definition 3.4). Recall that this is the case if \( S \) is of finite type over \( \mathbb{F}_p \) and \( \dim(S) \leq 2 \) according to our main theorem.

With the assumptions of the definition, there is a right exact functor

\[ \iota : \text{Perv}^A(S, \mathbb{Q}_\ell) \rightarrow \text{Perv}(S, \mathbb{Q}_\ell). \]

It is exact if and only if the perverse t-structure induces a t-structure on Artin \( \ell \)-adic sheaves.

Now, Lemma 1.5 allows us to prove that the perverse t-structure induces a t-structure on \( \text{D}^{\text{coh}}(S, \mathbb{Q}_\ell) \): using de Jong’s resolution of singularities we can prove that if \( X \rightarrow S \) is proper, the perverse cohomology sheaves of \( f_* \mathbb{Q}_{\ell,X} \) are cohomological by noetherian induction on \( X \); the case where \( X \) is smooth over \( k \) is clear because the perverse cohomology sheaves of \( f_* \mathbb{Q}_{\ell,X} \) are direct summands of \( f_* \mathbb{Q}_{\ell,X} \).

We let \( \text{Perv}^{\text{coh}}(S, \mathbb{Q}_\ell) \) be the heart of this t-structure.

The functor \( \iota \) factors through the abelian category \( \text{Perv}^{\text{coh}}(S, \mathbb{Q}_\ell) \). We still denote \( \iota \) the functor \( \text{Perv}^A(S, \mathbb{Q}_\ell) \rightarrow \text{Perv}^{\text{coh}}(S, \mathbb{Q}_\ell) \). The latter functor has an exact right adjoint:

Definition 4.22. Let \( S \) be a scheme of finite type over \( \mathbb{F}_p \). Let \( \ell \neq p \) be a prime number. We call weightless truncation the exact functor induced by \( \omega^0 : \)

\[ \omega^0 : \text{Perv}^{\text{coh}}(S, \mathbb{Q}_\ell) \rightarrow \text{Perv}^A(S, \mathbb{Q}_\ell). \]

Finally, Artin homotopy perverse sheaves have similar properties as those of Artin perverse sheaves: the analog of Proposition 3.5 holds.

4.8. Artin intermediate extension and simple Artin homotopy perverse sheaves. We now show that the abelian category of Artin homotopy perverse sheaves has similar features as the category of perverse sheaves. Recall that we denote \( \mathbf{h}^0 \mathcal{H}^* \) the cohomology functors with respect to the homotopy perverse t-structure.

Let \( j : U \rightarrow S \) be an open immersion of schemes of finite type over \( \mathbb{F}_p \). Recall (see [BBDG18, 1.4.22]) that the intermediate extension functor \( j_{!*} \) is defined as the image of the map \( \mathcal{p} \mathbf{H}^0 j_! \rightarrow \mathbf{p} \mathbf{H}^0 j_* \). In fact, whenever the glueing formalism (see [BBDG18, 1.4.3]) is satisfied, [BBDG18, 1.4.22] provides an intermediate extension functor. Hence, Proposition 14.18 allows us to define an Artin intermediate extension functor:

Definition 4.23. Let \( S \) be a scheme of finite type over \( \mathbb{F}_p \). Let \( \ell \neq p \) be a prime number. Let \( j : U \rightarrow S \) be an open immersion. The Artin intermediate extension functor \( j^A_* \) is the functor which associates to an object \( M \) in \( \mathcal{M}^A(S, \mathbb{Q}) \) the image of \( \mathbf{h}^0 j_! M \) in \( \mathbf{h}^0 \omega^0 j_* M \).

Since the weightless truncation functor is exact, the following holds:

Proposition 4.24. Let \( S \) be a scheme of finite type over \( \mathbb{F}_p \). Let \( \ell \neq p \) be a prime number. Let \( j : U \rightarrow S \) be an open immersion. Then, \( j^A_* = \omega^0 j_* \).

Using [BBDG18, 1.4.23], we have the following description of the intermediate extension functor (compare with [BBDG18, 2.1.9]):
Proposition 4.25. Let $S$ be a scheme of finite type over $\mathbb{F}_p$. Let $\ell \neq p$ be a prime number. Let $i : F \to S$ be a closed immersion and $j : U \to S$ be the open complementary immersion. Let $M$ be in $\text{Perv}^A(U, \mathbb{Q}_\ell)^\#$. Then, $j_!^A(M)$ is the only extension $P$ of $M$ to $S$ such that $i^* P < 0$ and $\omega^0 i^* P > 0$.

The following result is similar to [BBDG18, 4.3.1].

Proposition 4.26. Let $S$ be a scheme of finite type over $\mathbb{F}_p$. Let $\ell \neq p$ be a prime number.

(1) The abelian category of Artin homotopy perverse sheaves with coefficients $\mathbb{Q}_\ell$ on $S$ is artinian and noetherian: every object is of finite length.

(2) If $j : V \hookrightarrow S$ is the inclusion of a regular subscheme and if $L$ is an irreducible representation of Artin origin of $\pi_1(V)$, then $j_!^A(L[\delta(V)])$ is a simple Artin perverse sheaf over $S$. Every simple Artin perverse sheaf over $S$ is obtained in this way.

Proof. The proof is the same as in [BBDG18, 4.3.1] replacing [BBDG18, 4.3.2, 4.3.3] with the following lemmas.

Lemma 4.27. Let $V$ be a regular connected scheme. Let $L$ be an irreducible rational Artin representation of $\pi_1(V)$. Then, if $F = L[\delta(V)]$, for any open immersion $j : U \hookrightarrow V$, we have $F = j_!^A j^* F$.

Proof. We use Proposition 4.25. Let $i : F \to S$ be the reduced complementary closed immersion. Then, $i^* L$ is an Artin representation of $\pi_1(F)$ which is thus in degree $\delta(F)$ for the homotopy perverse t-structure. Therefore, $i^* F$ is in degree $\delta(F) - \delta(V) < 0$.

Finally, $\omega^0 i^* F$ is zero in $\mathcal{D}^{\text{sm}}(S, \mathbb{Q}_\ell)$ (see the proof of [NV15, 4.1.1]).

Lemma 4.28. If $L$ is an irreducible representation of Artin origin of $\pi_1(V)$, then $F = L[\delta(V)]$ is simple in $\text{Perv}^A(V, \mathbb{Q}_\ell)^\#$.

Proof. The proof is exactly the same as [BBDG18, 4.3.3].

Proposition 4.29. Let $X$ be a scheme of finite type over $\mathbb{F}_p$. Let $\ell \neq p$ be a prime number. Recall the construction of the weightless complex $EC_X$ of Vaish and Nair (see Remark 4.17). Let $j : U \to X$ be an open immersion with $U$ regular, then

1. $EC_X = j_!^A \mathbb{Q}_{\ell, U}$.
2. $EC_X$ is a simple Artin homotopy perverse sheaf.
3. If $X$ is a surface, $EC_X$ is an Artin perverse sheaf; in particular, it is a perverse sheaf.

Proof. Let $IC_X = j_! \mathbb{Q}_{\ell, U}$ be the intersection complex, then $EC_X = \omega^0 IC_X$ by [NV15, 3.1.13]. This proves (1) by Proposition 4.24; (2) follows by Proposition 4.26 and (3) follows by Remark 4.21.

However, when $X$ is a surface, the weightless complex $EC_X$ need not be simple in the category of perverse sheaves:

Example 4.30. Let $X$ be a normal surface with a single singular point $i : \{x\} \to X$. Let $f : Y \to X$ be a resolution of singularities such that the fiber of $f$ above $x$ is a simple normal crossing divisor with components of positive genus. Then, $EC_X$ is not simple.
Indeed, let \( i_F : E \to X \) be the pull-back of \( i \) along \( f \) and \( p : E \to \{ x \} \) be the pull-back of \( f \) along \( i \).

We have exact triangles:

\[
EC_X \to f_* \mathbb{Q}_{\ell,Y} \to w_{>\text{id}} f_* \mathbb{Q}_{\ell,Y}
\]

and

\[
w_{>\text{id}} j^q \mathbb{Q}_{\ell,U} \to w_{>\text{id}} f_* \mathbb{Q}_{\ell,Y} \to w_{>\text{id}} i_* p_* \mathbb{Q}_{\ell,E}.
\]

But as \( j^q \mathbb{Q}_{\ell,U} \) is Artin, \( w_{>\text{id}} j^q \mathbb{Q}_{\ell,U} = 0 \).

Now, by [a reference] 3.1.8, 3.1.10, \( i_* \) is t-exact with respect to the t-structure \((\mathcal{D}^b_{\leq 1}(\mathbb{Q}_{\ell}), \mathcal{D}^b_{> 1}(\mathbb{Q}_{\ell}))\) on \( \mathcal{D}^b_{\mathbb{C}}(\mathbb{Q}_{\ell}) \) and the t-structure \((\mathcal{D}^b_1(\mathbb{X}, \mathbb{Q}_{\ell}), \mathcal{D}^b_{>1}(\mathbb{X}, \mathbb{Q}_{\ell}))\) on \( \mathcal{D}^b_{\mathbb{C}}(\mathbb{X}, \mathbb{Q}_{\ell}) \). Thus, \( i_* \) commutes with \( w_{>\text{id}} \). But on \( \mathcal{D}^b_1(\mathbb{X}, \mathbb{Q}_{\ell}) \), \( w_{>\text{id}} \) is Morel’s weight truncation functor \( w_{\geq 1} \) (see [a reference] 4.1).

Now write \( E = \bigcup_{i \in I} E_i \) where \( E_i \) is regular. If \( J \subseteq I \), write \( E_J = \bigcap_{i \in J} E_i \) and \( p_J : E_J \to \{ x \} \) the structural morphism. By induction on \( |I| \), Proposition 1.7 gives an exact triangle

\[
p_i \mathbb{Q}_{\ell,E} \to \bigoplus_{i \in I} (p_i)_* \mathbb{Q}_{\ell,E_i} \to \bigoplus_{J \subseteq I, |J| = 2} (p_J)_* \mathbb{Q}_{\ell,E_J}.
\]

If \( |J| = 2 \), \( (p_J)_* \mathbb{Q}_{\ell,E_J} \) is pure of weight 0. Hence,

\[
w_{>\text{id}} i_* p_* \mathbb{Q}_{\ell,E} = \bigoplus_{i \in I} R^1 (p_i)_* \mathbb{Q}_{\ell,E_i} [-1] \oplus \bigoplus_{i \in I} R^2 (p_i)_* \mathbb{Q}_{\ell,E_i} [-2].
\]

Now, using [a reference] 1.8.1,

\[
f_* \mathbb{Q}_{\ell,Y} = IC_X \oplus i_* \left( \bigoplus_{i \in I} R^1 (p_i)_* \mathbb{Q}_{\ell,E_i} \right) [-2].
\]

Hence, we have an exact sequence of perverse sheaves:

\[
0 \to i_* \left( \bigoplus_{i \in I} R^1 (p_i)_* \mathbb{Q}_{\ell,E_i} \right) \to EC_X \to IC_X \to 0.
\]

Therefore, since one of the \( E_i \) has positive genus, \( i_* \left( \bigoplus_{i \in I} R^1 (p_i)_* \mathbb{Q}_{\ell,E_i} \right) \) is non-zero and thus \( EC_X \) is not simple.

5. Explicit description of Artin perverse sheaves and homotopy perverse sheaves

5.1. Description of perverse sheaves as gluing of representations. The material in this section is classical (see [a reference]) and figures here for expository purposes. Let \( M \) be a perverse sheaf on an excellent base scheme \( S \). Recall that there is a stratification \( S \) of \( S \) such that for all \( T \in S \), \( M|_T \) is a smooth perverse sheaf. We can recover \( M \) from the \( M|_T \) and additional data.

Indeed, let \( U \) be the (disjoint) union of the strata containing the generic points of \( S \). Without loss of generality, we can assume that \( U \) is a single stratum on which \( M \) is smooth. Shrinking \( U \), we can also assume that its complement \( F \) is a Cartier divisor, and in particular, the immersion \( j : U \to S \) is affine. Write \( i : F \to S \) the immersion. \( M|_U \) is a perverse sheaf and \( M|_F \) is a complex of \( \ell \)-adic sheaves in degrees \([-1, 0] \) by [a reference] 4.1.10.
By localization, we have an exact triangle:

\[ j_!M|_U \to M \to i_* M|_F \]

which gives rise to an exact sequence:

\[ 0 \to i_* p^H M|_F \to j_* M|_U \to M \to i_* p^H 0 M|_F. \]

Hence, we can recover \( M \) as the cone of a map: \( i_* M|_F[-1] \to j_* M|_U \) such that the induced map \( i_* p^H M|_F \to j_* M|_U \) is injective.

Conversely, a perverse sheaf \( M_U \) on \( U \), a complex \( M_F \) of \( \ell \)-adic sheaves in perverse degrees \([-1,0]\) on \( F \) and a connection map \( i_* M_F[-1] \to j_* M_U \) such that the induced map \( i_* p^H M_F \to j_* M_U \) is injective, give rise to a unique perverse sheaf \( M \) on \( S \).

Similarly, since \( S \) induces a stratification of \( F \), the datum of \( M_F \) is equivalent to the data of a complex \( M_1 \) of \( \ell \)-adic sheaves in perverse degrees \([-1,0]\) on an affine open subset \( U_1 \) of \( F \), a complex \( N_1 \) of \( \ell \)-adic sheaves in perverse degrees \([-2,0]\) on \( F_1 = F \setminus U_1 \) and a connection map with a similar condition.

This discussion provides the following proposition:

**Proposition 5.1.** Let \( S \) be an excellent scheme and \( \ell \) be a prime number invertible on \( S \). Then, a perverse sheaf on \( S \) is equivalent to the following data which is defined by induction on the dimension of \( S \):

- A sequence \((U_0, \ldots, U_n)\) of locally closed subschemes of \( S \) such that \( U_k \) is regular, purely of codimension \( k \) and is open in \( F_{k-1} := S \setminus (U_0 \cup \cdots \cup U_{k-1}) \); (and \( F_{-1} = S \)).
- A sequence \((M_0, \ldots, M_n)\) of smooth complexes of \( \ell \)-adic sheaves on \( U_i \) in perverse degree \([-i,0]\).
- A connection map \( \phi_k : (i_k)_* M_{\geq k}[-1] \to (j_k)_* M_k \) such that \( p^H(\phi_k) \) is injective, where \( i_k \) is the inclusion of \( F_k \) in \( F_{k-1} \), \( j_k \) is the inclusion of \( U_k \) in \( F_{k-1} \) and \( M_{\geq k} \) is the perverse sheaf on \( F_k \) given by the datum of \((M_{k+1}, \ldots, M_n)\) and the connection maps \((\phi_{k+1}, \ldots, \phi_n)\).

**5.2. Description of Artin perverse sheaves and Artin homotopy perverse sheaves.** Let \( S \) be a scheme such that the perverse t-structure induces a t-structure on Artin \( \ell \)-adic sheaves. Recall that if \( j : U \to S \) is an open immersion of closed complement \( i : F \to S \), the perverse t-structures on \( U \) and \( F \) induce t-structures on the categories of Artin \( \ell \)-adic sheaves which can be glued to obtain the global t-structure on \( S \).

Moreover, if \( j \) is affine, Artin perverse sheaves are stable under the functors \( j_! \), \( j^* \), \( i_* \) and \( i^* \). Hence, if the perverse t-structure induces a t-structure on Artin \( \ell \)-adic sheaves on \( S \), using the same ideas as in the previous paragraph, we get a complete description of Artin perverse sheaves on \( S \):

**Proposition 5.2.** Let \( S \) be an excellent scheme and \( \ell \) be a prime number invertible on \( S \). Assume that the perverse t-structure induces a t-structure on Artin perverse sheaves over \( S \). Then, an Artin perverse sheaf on \( S \) is equivalent to the following data which is defined by induction on the dimension of \( S \):

- A sequence \((U_0, \ldots, U_n)\) of locally closed subschemes of \( S \) such that \( U_k \) is regular, purely of codimension \( k \) and is open in \( F_{k-1} := S \setminus (U_0 \cup \cdots \cup U_{k-1}) \); (and \( F_{-1} = S \)).
• A sequence \((M_0, \ldots, M_n)\) of smooth Artin complexes of \(\ell\)-adic sheaves on \(U_i\) in perverse degree \([-i, 0]\).

• A connection map \(\phi_k : (i_k)_*M_{>k}[-1] \to (j_k)_!M_k\) such that \(pH^0(\phi_k)\) is injective, where \(i_k\) is the inclusion of \(F_k\) in \(F_{k-1}\), \(j_k\) is the inclusion of \(U_k\) in \(F_{k-1}\) and \(M_{>k}\) is the perverse sheaf on \(F_k\) given by the datum of \((M_{k+1}, \ldots, M_n)\) and the connection maps \((\phi_{k+1}, \ldots, \phi_n)\).

Similarly, we have:

**Proposition 5.3.** Let \(S\) be a scheme of finite type over \(\mathbb{F}_p\). Let \(\ell \neq p\) be a prime number. Then, an Artin homotopy perverse sheaf on \(S\) is equivalent to the following data which is defined by induction on the dimension of \(S\):

• A sequence \((U_0, \ldots, U_n)\) of locally closed subschemes of \(S\) such that \(U_k\) is smooth over \(\mathbb{F}_p\), purely of codimension \(k\) and is open in \(F_{k-1} := S \setminus (U_0 \cup \cdots \cup U_{k-1})\); (and \(F_{-1} = S\)).

• A sequence \((M_0, \ldots, M_n)\) of smooth Artin complexes of \(\ell\)-adic sheaves on \(U_i\) in perverse degree \([-i, 0]\).

• A connection map \(\phi_k : (i_k)_*M_{>k}[-1] \to (j_k)_!M_k\) such that \(pH^0(\phi_k)\) is injective, where \(i_k\) is the inclusion of \(F_k\) in \(F_{k-1}\), \(j_k\) is the inclusion of \(U_k\) in \(F_{k-1}\) and \(M_{>k}\) is the Artin homotopy perverse sheaf on \(F_k\) given by the datum of \((M_{k+1}, \ldots, M_n)\) and the connection maps \((\phi_{k+1}, \ldots, \phi_n)\).

### 5.3. Artin perverse sheaves on curves.

In the case of 1-dimensional schemes, this description can be made more explicit. Let \(C\) be a 1-dimensional excellent scheme. An Artin perverse sheaf on \(C\) is equivalent to the following data:

1. A dense open immersion \(j : U \to C\) of regular source. Let \(i : F \to U\) be the complementary closed immersion.

2. A smooth Artin \(\ell\)-adic sheaf \(M_U\) on \(U\).

3. A smooth Artin complex \(M_F\) of \(\ell\)-adic sheaves on \(F\) placed in degrees \([-1, 0]\).

4. A map \(\phi : i_*M_F[-1] \to j_!M_U\) such that \(pH^0(\phi)\) is injective.

Recall that we have an exact triangle:

\[j! \to j_* \to i_*i^*j_*\]

Moreover, \(j^*i_* = 0\). Therefore, \(\text{Hom}(i_*M_F[-1], j_!M_U) = \text{Hom}(M_F, i^*j_*M_U)\) and we have an exact sequence:

\[0 \to i_*\overline{pH}^{-1}(i^*j_*M_U) \to j_!M_U \to j_*M_U \to i_*pH^0(i^*j_*M_U) \to 0.\]

Therefore, if \(\phi : i_*M_F[-1] \to j_!M_U\), \(pH^0(\phi)\) is injective if and only if the induced map \(\overline{pH}^{-1}(M_F) \to pH^{-1}(i^*j_*M_U)\) is injective.

We can compute \(i^*j_*M_U\) more explicitly. The first case to consider is the case of a regular 1-dimensional scheme. We first need some notations:

**Notations 5.4.** Let \(C\) be a regular connected 1-dimensional scheme and \(K\) be its field of functions. Then, if \(U\) is an open subset of \(C\) and \(x\) is a closed point of \(C\), we let:

- \(K^U\) be the maximal extension of \(K\) which is unramified over \(U\).
- \(K_x\) be a completion of \(K\) with respect to the valuation defined by \(x\) on \(K\) and \(\phi_x : \text{Spec}(K_x) \to \text{Spec}(K)\) be the induced map.
be a separable closure of $K_x$ and $K_x^{nr}$ be the maximal extension of $K_x$ which is unramified at $x$.

- $G_0(x)$ be the absolute Galois group of $K_x^{nr}$.
- $K_x^U$ be the maximal extension of $K_x$ which is unramified on $U$.

The following lemma is classical (use [GR71 V.8.2,X.2.1]):

**Lemma 5.5.** Let $C$ be a regular connected curve, $U$ an open subset of $C$ and $x$ a closed point. Let $K$ be the field of regular functions on $C$. Then,

1. $K_x^U$ is a completion of $K^U$ with respect to the valuation defined by $x$.
2. If $x \in U$, $K_x^U = K_x^{nr}$.
3. If $x \not\in U$, $K_x^U = K_x$.
4. $\pi_1^{et}(U)$ can be identified with $\text{Gal}(K^U/K)$.
5. $G_{k(x)}$ can be identified with $\text{Gal}(K_x^{nr}/K_x) = G_{K_x}/G_0(x)$.
6. If $x \not\in U$, we get a diagram:

$$
\begin{array}{c}
G_{K_x} \longrightarrow \text{Gal}(K^U/K) = \pi_1^{et}(U) \\
\downarrow \\
\text{Gal}(K_x^C/K_x) = G_{k(x)}
\end{array}
$$

In this setting, we will denote $\Psi_x$ the map

$$
\text{Rep}(G_{K_x}, \mathbb{Z}_\ell) \rightarrow \text{Rep}(G_{k(x)}, \mathbb{Z}_\ell) \\
\quad M \mapsto M^{G_0(x)}
$$

7. The derived functor $R\Psi_x$ induces a map

$$
D^b_{\text{Rep}}(\text{Rep}(G_{K_x}, \mathbb{Z}_\ell)) \rightarrow D^b_{\text{Rep}}(\text{Rep}(G_{k(x)}, \mathbb{Z}_\ell)).
$$

**Proposition 5.6.** Let $C$ be a regular 1-dimensional connected scheme. Let $i : F \rightarrow C$ be a closed immersion and $j : U \rightarrow C$ be the open complement. Let $\ell$ be a prime number which is invertible on $S$.

Then, the functor

$$
i^* j_* : \text{Rep}^{\text{A}}(\pi_1^{et}(U), \mathbb{Z}_\ell) \rightarrow D^b_{\text{Rep}}(F, \mathbb{Z}_\ell)
$$

coincides with the functor $\bigoplus_{x \in F} (i_x)_*(R\Psi_x)\phi_x^*$, where $i_x : \{x\} \rightarrow F$ is the inclusion.

**Proof.** First, note that $i^* j_* = \bigoplus_{x \in F} (i_x)_* i_x^* j_*$. 

Let $x \in F$. Let $\overline{x} : \text{Spec}(k(x)) \rightarrow \{x\}$ be the associated geometric point. Let $j_0$ be the underived version of the functor $j_*$. Since the underived version of $i_x^* i_*$ is exact, it can be identified with its derived functor. Hence, $i_x^* i_* j_*$ is the derived functor of $i_x^* j_0$.

Now, if $F$ is an étale sheaf on $U$,

$$
i_x^* j_0^*(F) = F(U \times_S \text{Spec}(O^{sh}_{S,x}))
$$

where $O^{sh}_{S,x}$ is the strict henselization of the local ring $O_{S,x}$ with respect to $\overline{x}$.

If $V$ is étale over $U$ and connected, let $K(V)$ be its field of functions. By [Del77 2.4], the functor:

$$
\text{Rep}^{\text{A}}(\text{Gal}(K^U/K), \mathbb{Z}_\ell) \rightarrow \text{Sh}(U, \mathbb{Z}_\ell) \\
\quad M \mapsto \left(V \mapsto M^{\text{Gal}(K^U/K(V))}\right)
$$
is fully faithful.

When $W$ runs through the category of étale neighborhoods of $\mathfrak{S}$ in $S$, colim $K(U \times S W)$ is the fraction field of $\mathcal{O}_{\mathfrak{S}}^{\text{sh}}$, i.e. $K^{\text{nr}}$. Hence,

$$\lim \text{Gal}(K^U / K(U \times S W)) = \text{Gal}(K^U / K_x) = G_0(x).$$

Hence, identifying Artin étale sheaves on $U$ with representation of $\text{Gal}(K^U / K)$,

$$i_x^* j_x^0(M) = M^{G_0(x)}.$$

Thus, $i_x^* j_x^0 = \Psi_x \phi_x^*$ and the result follows. $\square$

We will now tackle the case of a general 1-dimensional scheme.

**Notations 5.7.** Let $C$ be an excellent scheme of dimension $\leq 1$. Let $\nu : \widetilde{C} \to C$ be the normalization of $C$. Let $\Gamma$ be the set of generic points of $\widetilde{C}$. Then, if $U$ is an open subset of $C$, $y$ is a closed point of $\widetilde{C}$ and $\eta \in \Gamma$, we let:

- $U_\eta = \{ \eta \} \cap \nu^{-1}(U) \subseteq \widetilde{C}$.
- $K_\eta$ be the field of regular functions on $C_\eta$.
- $K^U_\eta$ be the maximal extension of $K_\eta$ which is unramified on $U_\eta$.
- $\eta(y)$ be the unique element of $\Gamma$ such that $y \in C_{\eta(y)}$.
- $K_y$ be a completion of $K_{\eta(y)}$ with respect to the valuation defined by $y$ on $K^U_{\eta(y)}$ and $\phi_y : \text{Spec}(K_y) \to \text{Spec}(K_{\eta(y)})$ be the induced map.
- $\overline{K}_y$ be a separable closure of $K_y$ and $K^\text{nr}_y$ be the maximal extension of $K_y$ which is unramified at $y$.
- $G_0(y)$ be the absolute Galois group of $K^\text{nr}_y$.
- $K^U_y$ be the maximal extension of $K_y$ which is unramified on $U_{\eta(y)}$.

The following lemma follows readily from Lemma 5.5.

**Lemma 5.8.** Let $C$ be an excellent scheme of dimension $\leq 1$. Let $\nu : \widetilde{C} \to C$ be a normalization of $C$. Let $\Gamma$ be the set of generic points of $\widetilde{C}$. Let $U$ be a nil-regular open subset of $C$, $y$ closed point of $\widetilde{C}$ and $\eta \in \Gamma$. Then,

1. $K^U_y$ is a completion of $K^U_{\eta(y)}$ with respect to the valuation defined by $y$.
2. If $y \in U_{\eta(y)}$, $K^U_\eta = K^\text{nr}_\eta$.
3. If $y \notin U_{\eta(y)}$, $K^U_\eta = \overline{K}_y$.
4. $\pi_1^U(U_\eta)$ can be identified with $\text{Gal}(K^U_y / K_\eta)$.
5. $G_{k(y)}$ can be identified with $\text{Gal}(K^\text{nr}_y / K_y) = G_{K_\eta} / G_0(y)$.
6. If $y \notin U_{\eta(y)}$, we get a diagram:

$$G_{K_\eta} \longrightarrow \text{Gal}(K^U_{\eta(y)} / K_{\eta(y)}) = \pi_1^U(U_{\eta(y)})$$

7. A smooth Artin perverse sheaf on $U$ is the same as a direct sum $\bigoplus_{\eta \in \Gamma} M_\eta$

where each $M_\eta$ is an Artin representation of $\pi_1(U_\eta) = \text{Gal}(K^U_\eta / K_\eta)$.

In this setting, we will denote $\Psi_{\eta}$ the map

$$\begin{align*}
\text{Rep}(G_{K_\eta}, \mathbb{Z}_\ell) & \to \text{Rep}(G_{k(y)}, \mathbb{Z}_\ell) \\
M & \mapsto M^{G_0(y)}
\end{align*}$$
Corollary 5.9. Let $C$ be a 1-dimensional excellent scheme. Let $\ell$ be a prime number that is invertible on $C$. Let $\nu : \tilde{C} \to C$ be a normalization of $C$. Let $\Gamma$ be the set of generic points of $C$. Let $i : F \to C$ be a closed immersion and $j : U \to C$ be the open complement. Assume that $U$ is nil-regular.

Then, the functor $i^*j_* : \text{Sh}^A(U, \mathbb{Z}_\ell) \to \mathcal{D}^b_c(F, \mathbb{Z}_\ell)$ can be identified with the functor that sends $\otimes_{\eta \in \Gamma} M_\eta$ where each $M_\eta$ is an Artin representation of $G_{K_\eta}$ to

$$\bigoplus_{x \in F} (i_x)_* \bigoplus_{\nu(y) = x} \text{Ind}_{G_{K_\eta}}^{G_{K_{\nu(y)}}}(R\Psi_y \left[ \phi_y^* (M_\eta(y)) \right]).$$

Proof. Let $\nu_U : U_{\text{red}} \to U$ (resp. $\nu_F : \tilde{F} \to F$) be the inverse image of $\nu$ along $j$ (resp. $i$). Let $\tilde{i}$ (resp. $\tilde{r}$) be the inclusion of $U_{\text{red}}$ (resp. $F$) in $C$.

Recall that $\nu_U^*$ is an equivalence of categories. Thus,

$$i^* j_* = i^* j_* (\nu_U)^* \nu_U^* = i^* \nu_* j_* \nu_U^* = (\nu_F)_* \tilde{i}^* \tilde{j}_* \nu_U^*.$$

The result then follows from the proposition and Proposition 2.11.

This discussion leads us to the following definition:

Definition 5.10. Let $C$ be a 1-dimensional excellent scheme. Let $\ell$ be a prime number that is invertible on $C$. Let $\Gamma$ be the set of generic points of $C$. Let $\nu : \tilde{C} \to C$ be a normalization of $C$. We define an abelian category $\text{P}(C, \mathbb{Z}_\ell)$ as follows:

- An object of $\text{P}(C, \mathbb{Z}_\ell)$ is a quadruple $(U, (M_\eta)_{\eta \in \Gamma}, (M_x)_{x \in C \setminus U}, (f_x)_{x \in C \setminus U})$ where $U$ is a nil-regular open subset of $C$, for all $\eta \in \Gamma$, $M_\eta$ is an Artin $\ell$-adic representation of $\pi_1(U_\eta)$, and for all $x \in C \setminus U$, $M_x$ is a complex of Artin $\ell$-adic representations of $G_{k(x)}$ placed in degrees $[0, 1]$ and $f_x$ is an element of $\text{Hom}_{\mathcal{D}(\text{Rep}^A(G_{k(x)}, \mathbb{R})))} (M_x, \bigoplus_{\nu(y) = x} \text{Ind}_{G_{k(x)}}^{G_{k(\nu(y))}}(R\Psi_y \left[ \phi_y^* (M_\eta(y)) \right]))$ such that $\nu^* H^0(f_x)$ is injective.

- An element of $\text{Hom}_{\text{P}(C, \mathbb{Z}_\ell)}((U, (M_\eta), (M_x), (f_x)), (V, (N_\eta), (N_x), (g_x)))$ is a couple $((\Phi_\eta)_{\eta \in \Gamma}, (\Phi_x)_{x \in C \setminus U} \cap (C \setminus V))$ where $\Phi_\eta : M_\eta \to N_\eta$ is a map of representations of $G_{K_\eta}$, $\Phi_x : M_x \to N_x$ is a map of representations of $G_{k(x)}$ and the diagram:

$$\begin{array}{ccc}
M_x & \xrightarrow{f_x} & \bigoplus_{\nu(y) = x} \text{Ind}_{G_{k(x)}}^{G_{k(\nu(y))}}(R\Psi_y \left[ \phi_y^* (M_\eta(y)) \right]) \\
\downarrow \Phi_x & & \downarrow \bigoplus_{\nu(y) = x} \text{Ind}_{G_{k(x)}}^{G_{k(\nu(y))}}(R\Psi_y \left[ \phi_y^* (\Phi(\eta(y))) \right]) \\
N_x & \xrightarrow{g_x} & \bigoplus_{\nu(y) = x} \text{Ind}_{G_{k(x)}}^{G_{k(\nu(y))}}(R\Psi_y \left[ \phi_y^* (N_\eta(y)) \right])
\end{array}$$

is commutative.
Proposition 5.11. Let $C$ be an excellent 1-dimensional scheme and $\ell$ be a prime number invertible on $R$. Then, $\text{Perv}^A(C, \mathbb{Z}_\ell)$ is equivalent to $\text{P}(C, \mathbb{Z}_\ell)$.

Example 5.12. Let $k$ be an algebraically closed field of characteristic 0. A perverse Artin $\ell$-adic sheaf over $\mathbb{P}^1_k$ is a quadruple $(\mathbb{P}^1_k \setminus F, M, (M_x)_{x \in F}, (f_x)_{x \in F})$ where $F$ is a finite set of points of $\mathbb{P}^1_k$, $M$ is an Artin representation of $\pi^\ell_1(\mathbb{P}^1_k \setminus F)$, $M_x$ is a complex of $\mathbb{Z}_\ell$-modules of finite type placed in degrees $[0, 1]$ and $f_x$ is a map

$$M_x \rightarrow R\Psi_x[\phi_x^*(M)]$$

such that $^pH^0(f_x)$ is injective.

We can always assume that the point at infinity is in $F$. Write $F = \{\infty\} \sqcup F'$. Let $m = |F'|.$

Let $x \in F'$, then, $k((X - x))$ is the completion of the field $k(X)$ of regular functions on $\mathbb{P}^1_k$ with respect to the valuation defined by $x$.

On the other hand, $k((1/X))$ is the completion of $k(X)$ with respect to the valuation of the point at infinity.

Furthermore, recall that the fundamental group of $k((X))$ is $\hat{\mathbb{Z}}$.

The fundamental group of $\mathbb{P}^1_k \setminus F$ where $F$ is a finite set of closed points is trivial if $F$ is empty and is the completion of the free group over $F'$; we denote $g_x$ for $x \in F'$ its generators. The map

$$G_{k((x-x))} \rightarrow \pi^\ell_1(\mathbb{P}^1_k \setminus F)$$

is the only continuous map that sends the topological generator of $\hat{\mathbb{Z}}$ to $g_x$.

The map $G_{k((1/X))} \rightarrow \pi^\ell_1(\mathbb{P}^1_k \setminus F)$ is the map that sends the topological generator to a certain product $(g_{x_1}, \ldots, g_{x_m})^{-1}$ where $F' = \{x_1, \ldots, x_m\}$.

Now, the action of $\pi^\ell_1(\mathbb{P}^1_k \setminus F)$ on $M$ factors through a finite quotient $G$ and is therefore the same as a representation of a finite group with $m$ marked generators that are the images of the $g_x$ for $x \in F'$.

Let

$$\Psi : \text{Rep}(\hat{\mathbb{Z}}, \mathbb{Z}_\ell) \rightarrow \mathbb{Z}_\ell - \text{mod}$$

be the functor that sends a continuous $\mathbb{Z}_\ell[\hat{\mathbb{Z}}]$ module $M$ to the module of fixed points under the action of the topological generator.

Thus, a perverse Artin $\ell$-adic sheaf over $\mathbb{P}^1_k$ is equivalent to the following data:

- A finite number of distinct points $x_1, \ldots, x_m$ of $\mathbb{A}^1_k$ (i.e. of elements of $k$).
- A finite group $G$ generated by $m$ elements $g_1, \ldots, g_m$; let $\phi_i : \hat{\mathbb{Z}} \rightarrow G$ be the map given by $g_i$, and $\phi_\infty : \hat{\mathbb{Z}} \rightarrow G$ be the map given by $g_\infty = (g_1 \cdot \ldots \cdot g_m)^{-1}$.
- A $\mathbb{Z}_\ell$-linear representation $M$ of $G$.
- $m + 1$ complexes of $\mathbb{Z}_\ell$-modules of finite type $M_1, \ldots, M_m, M_\infty$ placed in degrees $[0, 1]$.
- $m + 1$ maps $f_1, \ldots, f_m, f_\infty$ with $f_i : M_i \rightarrow R\Psi(\phi_i^*M)$ such that $^pH^0(f_i) : ^pH^0(M_i) \rightarrow M^G$ is injective.

Example 5.13. Let $S = \text{Spec}(\mathbb{Z}_p)$. A perverse Artin $\ell$-adic sheaf over $S$ is given by a quadruple $(\text{Spec}(\mathbb{Q}_p), M, N, f)$ where $M$ is an Artin representation of $G_{\mathbb{Q}_p}, N$ is a complex of continuous representations of $\hat{\mathbb{Z}}$ placed in degrees $[0, 1]$ with Artin cohomology and $f : N \rightarrow R\Psi(M)$ (where $\Psi(M) = M^{G_0}$ and $G_0$ is the inertia group) is a map of complexes (in the derived category) such that $^pH^0(f)$ is injective.
Example 5.14. Let $k$ be an algebraically closed field of characteristic $0$. Let $A = k[X, Y]/(XY)$ be the localization of $k[X, Y]/(XY)$ at the ideal $(X, Y)$. It is the local ring at the intersection point of two lines in $\mathbb{P}^2_k$.

Let $S = \text{Spec}(A)$. Then, $S$ has two generic points of residue field $k(X)$ and the residue field at the closed point is $k$. A normalization of $S$ is given by $\text{Spec}(k[X] \{X\} \times k[Y] \{Y\})$.

Let $\Psi : \text{Rep}(\hat{\mathbb{Z}}, \mathbb{Z}_\ell) \to \mathbb{Z}_\ell \mod$ be the functor of fixed points and $\phi : \hat{\mathbb{Z}} = G_{k((X))} \to G_{k(X)}$ be the inclusion.

Now, a perverse Artin étale motive on $S$ is given by a quadruple $(M_1, M_2, N, f)$ where $M_1$ and $M_2$ are Artin representations of $k(X)$, $N$ is a complex of $\mathbb{Z}_\ell$-modules placed in degrees $[0, 1]$ and $f : N \to R\Psi(\phi^*(M_1 \oplus M_2))$ is such that $^p\text{H}^0(f)$ is injective.

Example 5.15. Let $S = \text{Spec}(\mathbb{Z}[\sqrt{5}]_{(2)})$. This scheme has two points: its generic point has residue field $\mathbb{Q}(\sqrt{5})$ and its closed point has residue field $\mathbb{F}_2$ (the field with two elements). The normalization of $S$ is $\text{Spec}(\mathbb{Z}[\frac{1+\sqrt{5}}{2}])$ which is a discrete valuation ring. The residue field of its closed point is the field $\mathbb{F}_4$ with 4 elements. The completion of $\mathbb{Q}(\sqrt{5})$ with respect to the valuation defined by 2 is the field $\mathbb{Q}_2(\sqrt{5})$.

Let $\Psi : M \in \text{Rep}(G_{\mathbb{Q}(\sqrt{5})}, \mathbb{Z}_\ell) \mapsto M^{G_0} \in \text{Rep}(\hat{\mathbb{Z}}, \mathbb{Z}_\ell)$ where $G_0$ is the inertia subgroup.

The map $\phi : \hat{\mathbb{Z}} \to \hat{\mathbb{Z}}$ that maps 1 to 2 induces a map $\text{Ind} : \text{Rep}(\hat{\mathbb{Z}}, \mathbb{Z}_\ell) \to \text{Rep}(\hat{\mathbb{Z}}, \mathbb{Z}_\ell)$ that is right adjoint to $\phi^*$.

Thus a perverse Artin $\ell$-adic sheaf on $S$ is the same as a triple: $(M, N, f)$ where $M$ is an Artin representation of $G_{\mathbb{Q}(\sqrt{5})}$, $N$ is a complex of continuous representations of $\hat{\mathbb{Z}}$ placed in degrees $[0, 1]$ with Artin cohomology and $f : N \to \text{Ind}(R\Psi(M))$ is such that $^p\text{H}^0(f)$ is injective.
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