The Blow-Up Rate for Strongly Perturbed Semilinear Wave Equations

M. A. Hamza · O. Saidi

Received: 28 November 2013 / Revised: 23 March 2014 / Published online: 13 May 2014
© Springer Science+Business Media New York 2014

Abstract We consider in this paper a large class of perturbed semilinear wave equation with subconformal power nonlinearity. In particular, we allow log perturbations of the main source. We derive a Lyapunov functional in similarity variables and use it to derive the blow-up rate. Throughout this work, we use some techniques developed for the unperturbed case studied by Merle and Zaag (Int. Math. Res. Notices, 19(1):1127–1156, 2005) together with ideas introduced by Hamza and Zaag in (Nonlinearity, 25(9):2759–2773, 2012) for a class of perturbations.

Keywords Wave equation · Blow-up · Perturbations

Mathematics Subject Classification 35L05 · 35B44 · 35B20

1 Introduction

This paper is devoted to the study of blow-up solutions for the following semilinear wave equation:

\[
\begin{aligned}
\partial_t^2 u &= \Delta u + |u|^{p-1}u + f(u) + g(x, t, \nabla u, \partial_t u) \\
(u(x, 0), \partial_t u(x, 0)) &= (u_0(x), u_1(x))
\end{aligned}
\]  

(1.1)

where \( u(t) : x \in \mathbb{R}^N \rightarrow u(x, t) \in \mathbb{R} \), \( u_0(x) \in H^1_{loc, u} \) and \( u_1(x) \in L^2_{loc, u} \). The space \( L^2_{loc, u} \) is the set of all \( v \in L^2_{loc} \) such that
\[ \|v\|_{L^2_{loc,u}} \equiv \sup_{d \in \mathbb{R}^N} \left( \int_{|x-d|<1} |v(x)|^2 \, dx \right)^{\frac{1}{2}} < +\infty, \]

and the space \( H^1_{loc,u} = \{ v \mid v, |\nabla v| \in L^2_{loc,u} \} \).

We assume that the functions \( f \) and \( g \) are \( C^1 \), with \( f : \mathbb{R} \rightarrow \mathbb{R} \) and \( g : \mathbb{R}^{2N+2} \rightarrow \mathbb{R} \) globally lipschitz, satisfying the following conditions:

\[ (H_f) \quad |f(v)| \leq M \left( 1 + \frac{|v|^p}{(\log(2 + v^2))^a} \right), \quad \text{for all } v \in \mathbb{R} \text{ with } (M > 0, \ a > 1), \]

\[ (H_g) \quad |g(x, t, v, z)| \leq M(1 + |v| + |z|), \quad \text{for all } x, v \in \mathbb{R}^N, z \in \mathbb{R} \text{ with } (M > 0). \]

Finally, we assume that

\[ p > 1 \text{ and } p < p_c \equiv 1 + \frac{4}{N-2} \text{ if } N \geq 2. \]

The Cauchy problem of equation (1.1) is wellposed in \( H^1_{loc,u} \times L^2_{loc,u} \). This follows from the finite speed of propagation and the wellposdness in \( H^1 \times L^2 \), valid whenever \( 1 < p < p_S = 1 + \frac{4}{N-2} \). The existence of blow-up solutions \( u(t) \) of (1.1) follows from ODE techniques or the energy-based blow-up criterion of Levine [16] (see also Levine and Todorova [17] and Todorova [26]). More blow-up results can be founded in Caffarelli and Friedman [4,5], Kichenassamy and Littman [12,13], Killip and Visan [15].

If \( u(t) \) is a blow-up solution of (1.1), we define (see for example Alinhac [1,2]) \( \Gamma_1 \) as the graph of a function \( x \mapsto T(x) \) such that the domain of definition of \( u \) (also called the maximal influence domain)

\[ D_u = \{(x, t) | t < T(x) \}. \]

Moreover, from the finite speed of propagation, \( T \) is a 1-Lipschitz function. Let us first introduce the following non degeneracy condition for \( \Gamma \). If we introduce for all \( x \in \mathbb{R}^N, t \leq T(x) \) and \( \delta > 0 \), the cone

\[ C_{x,t,\delta} = \{(\xi, \tau) \neq (x, t) | 0 \leq \tau \leq t - \delta |\xi - x| \}, \]

then our non degeneracy condition is the following: \( x_0 \) is a non-characteristic point if

\[ \exists \delta_0 = \delta_0(x_0) \in (0, 1) \text{ such that } u \text{ is defined on } C_{x_0,T(x_0),\delta_0}. \]

In the case \( (f, g) \equiv (0, 0) \), Eq. (1.1) reduces to the semilinear wave equation:

\[ \partial_t^2 u = \Delta u + |u|^{p-1} u, \quad (x, t) \in \mathbb{R}^N \times [0, T). \]

It is interesting to recall that previously Merle and Zaag in [18,19] have proved, that if \( u \) a solution of (1.4) with blow-up graph \( \Gamma : \{ x \mapsto T(x) \} \) and \( x_0 \) is a non-characteristic point (in the sense (1.3)), then for all \( t \in \left[ \frac{3T(x_0)}{4}, T(x_0) \right] \).
\[ 0 < \varepsilon_0(N, p) \leq (T(x_0) - t)^{\frac{2}{p^2 - 1}} \frac{\|u(t)\|_{L^2(B(x_0, T(x_0) - t))}}{(T(x_0) - t)^{\frac{N}{2}}} + (T(x_0) - t)^{\frac{2}{p^2} + 1} \left( \frac{\|\partial_t u(t)\|_{L^2(B(x_0, T(x_0) - t))}}{(T(x_0) - t)^{\frac{N}{2}}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0, T(x_0) - t))}}{(T(x_0) - t)^{\frac{N}{2}}} \right) \leq K, \] 

where the constant \( K \) depends only on \( N, p \) and on an upper bound on \( T(x_0) \), \( \frac{1}{T(x_0)} \), \( \delta_0(x_0) \) and the initial data in \( H_{1,loc}^1(\mathbb{R}^N) \times L_{loc}^2(\mathbb{R}^N) \).

The unperturbed case (1.4) is considered in the mathematical community as a lab model for the development of efficient tools for the study of blow-up. Unfortunately, in more physical situations, the models are often more rich, hence more complicated, with dissipative terms (involving \( \partial_t u, \Delta(\partial_t u) \)) and other lower order source terms (for example if \( \frac{f(u)}{|u|^p} \longrightarrow 0 \) as \( u \longrightarrow \infty \) or the case where this term is in the form \( f = f(x, u) = V(x)|u|^p \) where \( V(x) \rightarrow 0 \) as \( x \rightarrow 0 \)). Therefore, it is completely meaningful for the mathematician to try to extend his methods and results to perturbations of the lab models, since the perturbed models are more encountered in the real-world models (see Whitham [27]).

Note that in [9,10], Hamza and Zaag consider a similar class of perturbed equations, with \( (H_f) \) and \( (H_g) \) replaced by a more restrictive conditions: \( |f(u)| \leq M(1 + |u|^q) \) and \( |g(u)| \leq M(1 + |u|) \) for some \( q < p, M > 0 \) and they proved a similar result as (1.5) valid when the exponent \( p \) is conformal or subconformal, i.e. \( 1 < p \leq p_c \).

The question of the perturbed nonlinear wave equation was later investigated by Killip et al. in [14] where they described the blow-up behavior for the following Klein–Gordon equation

\[ \partial^2_t u = \Delta u - u + |u|^{p-1}u, \] 

in spatial dimension \( N \geq 2 \) with \( 0 < p < 1 + \frac{4}{N-2} \). Concerning more specifically the Klein–Gordon equation (1.6), we refer the reader to the book by Nakanishi and Schlag [24]. We also mention the numerical study by Donninger and Schlag in [6] when \( N = 3 \) and \( p = p_c = 3 \).

Also, we would like to mention the remarkable result of Donninger and Schörkhuber in [7] who proved, in the subconformal range, the stability of the ODE solution \( u(t) = \kappa_0(p)(T - t)^{-\frac{2}{p^2 - 1}} \) among all radial solutions, with respect to small perturbations in initial data in the energy topology. Their approach is based in particular on a good understanding of the spectral properties of the linearized operator in self-similar variables, operator which is not self-adjoint. Similar results have also been obtained by the same authors [8] in the superconformal case and even in the Sobolev supercritical case (i.e for any \( p > p_c \)). They extend to this range the stability result obtained in the subconformal range in [7], though they need a topology stronger than needed by the energy.

In this paper, we consider the PDE case, and ask the question whether a strong perturbation like \( f(u) = \frac{|u|^p}{(\log(2+|u|^a))^a} \) may affect the blow-up dynamics. Focusing only on the blow-up rate, we show here that, like in the ODE case, it remains the same as in the unperturbed case (1.4) (we expect however some changes in the first-order terms, but this is beyond the scope of the present paper). Also, note that the method used here works without any problem in the case where \( f = f(x, u) = \frac{|u|^p}{(\log|\mathbf{x}^a|)^a} (a > 1) \) and we get a similar result. This paper can therefore be seen as an early understanding of the following equation
\[ \partial_t^2 u = \Delta u + V(x)|u|^{p-1}u, \quad (x, t) \in \mathbb{R}^N \times [0, T). \] \tag{1.7}

where \( V(x) \) is a smooth function satisfies \( V(x) - 1 \sim (-\log x)^{-a} \), as \( x \to 0 \).

Before handling the PDE, we first studied the associated ODE \( u'' = |u|^{p-1}u + f(u) \) and discovered that the perturbation doesn’t affect the main term and may show a different dynamic in the following term, (see Appendix for justification). In this paper, we show the blow-up rate remains unchanged, even under strong perturbation \( f(u) = \frac{|u|^p}{(\log(2+u^2))^{\alpha}} \), \( \alpha > 1 \).

Let us point out that logarithmic perturbations of pure power nonlinearities have been proved completely meaningful in other settings. This was the case in Tao’s contribution in [25] where the equation:

\[ \partial_t^2 u = \Delta u + |u|^{\frac{N+2}{N-2}} \log(2+u^2), \]

proved to be a good compromise, since it lays in the supercritical range and seems to be more tractable than the pure power supercritical case \( \partial_t^2 u = \Delta u + |u|^q \) with \( q > \frac{N+2}{N-2} \).

As we mentioned above, our aim in this paper is to extend the result of Hamza and Zaag [9] to Eq. (1.1) under the hypotheses \( (H_f) \) and \( (H_g) \). In order to keep our analysis clear, we may assume that \( f(u) = \frac{|u|^p}{(\log(2+u^2))^{\alpha}} \) and \( g \equiv 0 \), in the Eq. (1.1). The adaptation to the case \( g \neq 0 \) is straightforward from the techniques of [9].

As in [9,18–20], we want to write the solution \( v \) of the associate ordinary differential equation of (1.1). It is clear from Appendix that \( v \) is given by

\[ v'' = v^p + f(v), \quad v(T) = +\infty, \] \tag{1.8}

and satisfies: \( v(t) \sim \frac{T}{(T-t)^{\frac{1}{p-1}}} \) as \( t \to T \), where \( \kappa = \left( \frac{2p+2}{(p-1)^2} \right)^{\frac{1}{p-1}} \). For this reason, we define for all \( x_0 \in \mathbb{R}^N \), \( 0 < T_0 \leq T_0(x_0) \), the following similar transformation introduced in Antonini and Merle [3] and used in [9,10,18–20]:

\[ y = \frac{x - x_0}{T_0 - t} \quad s = -\log(T_0 - t), \quad \text{and} \quad w_{x_0} \to y(s) = (T_0 - t) \frac{2}{p-1} u(x, t). \] \tag{1.9}

The function \( w_{x_0} \to y(s) \) (we write \( w \) for simplicity) satisfies the following equation for all \( y \in B \) and \( s \geq -\log(T_0) \):

\[ \partial_s^2 w = \frac{1}{\rho} \partial y v(\rho \nabla w - \rho (y.\nabla w)y) - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1}w \]
\[ - \frac{p+3}{p-1} \partial_s w - 2(y.\nabla \partial_s w) + e^{-2\rho v} f(e^{-2\rho v}w), \] \tag{1.10}

where \( \rho(y) = (1 - |y|^2)^\alpha \) and \( \alpha = \frac{2}{p-1} - \frac{N-1}{2} > 0 \). In the new set of variable \( (y, s) \), the behavior of \( u \) as \( t \to T_0 \) is equivalent to the behavior of \( w \) as \( s \to +\infty \). The Eq. (1.10) will be studied in the space \( \mathcal{H} \)

\[ \mathcal{H} = \{(w_1, w_2)| \int_B (w_2^2 + |\nabla w_1|^2 (1 - |y|^2) + w_1^2) \rho dy < +\infty \}. \]

In the whole paper we denote

\[ F(u) = \int_0^u f(v)dv. \] \tag{1.11}
In the non-perturbed case, Antonini and Merle [3] proved that

\[ E_0(w(s)) = \int_B \left( \frac{1}{2} (\partial_s w)^2 + \frac{1}{2} |\nabla w|^2 - \frac{1}{2} (y \cdot \nabla w)^2 + \frac{p+1}{2(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho dy, \]

is a Lyapunov functional for Eq. (1.10). In our case we introduce

\[ H(w(s), s) = \exp \left( \frac{p+3}{(a-1)s^{b-1}} \right) E(w(s), s) + \theta e^{-\frac{(p+1)s}{p-1}}, \]

with \( b = \frac{a+1}{2} \) (1.13)

where \( \theta \) is a sufficiently large constant that will be determined later,

\[ E(w(s), s) = E_0(w(s)) + I(w(s), s) + J(w(s), s), \]

\[ I(w(s), s) = -e^{-\frac{2(p+1)s}{p-1}} \int_B F(e^{\frac{2s}{p-1}} w) \rho dy, \]

\[ J(w(s), s) = -\frac{1}{s^b} \int_B w \partial_s w \rho dy. \]

We now claim that the functional \( H(w(s), s) \) is a decreasing function of time for Eq. (1.10), provided that \( s \) is large enough.

Here we announce our main result.

**Theorem 1** Let \( N, p, a \) and \( M \) be fixed. There exists \( S_1 = S_1(N, p, a, M) \geq 1 \) such that, for all \( s_0 \in \mathbb{R} \) and \( w \) solution of equation (1.10) satisfying \( (w, \partial_s w) \in C([s_0, +\infty), \mathcal{H}) \), it holds that \( H(w(s), s) \) satisfies the following inequality, for all \( s_2 > s_1 \geq \max(S_1, s_0), \)

\[ H(w(s_2), s_2) - H(w(s_1), s_1) \leq -\alpha \int_{s_1}^{s_2} \left( \partial_s w \right)^2 \frac{\rho}{1 - |y|^2} dy ds. \]  

**Remark 1** (i) Our method breaks down in the conformal case when \( p \equiv p_c \), since in the energy estimates in similarity variables, the perturbations terms are integrated on the whole unit ball, hence, difficult to control with the dissipation of the non perturbed equation (1.4), which degenerates to the boundary of the unit ball.

(ii) The existence of this Lyapunov functional (and a blow-up criterion for Eq. (1.10) based in \( H(w(s), s) \), see Lemma 3.1 below) are a crucial step in the derivation of the blow-up rate for Eq. (1.1). Indeed, with the functional \( H(w(s), s) \) and some more work, as in [9,10] we are able to adapt the analysis performed in [19] for Eq. (1.4) and get the Theorem 2 below.

(iii) It is worth noticing that the method breaks down when \( a \leq 1 \) too, because with some analysis to the Lyapunov functional we find an equality of type \( \frac{d}{ds}(E(w(s)) \leq \frac{C}{s^\alpha} E(w(s)), \) \( E \) is upper bounded if \( a > 1 \) but if \( a \leq 1 \) we can not conclude.

**Theorem 2** Let \( N, p, a \) and \( M \) be fixed. There exists \( \hat{S}_0 = \hat{S}_0(N, p, a, M) \in \mathbb{R} \), and \( \epsilon_0 = \epsilon_0(N, p, a, M) \), such that if \( u \) is a solution of (1.1) with blow-up graph \( \Gamma : \{x \mapsto T(x)\} \) and \( x_0 \) is a non characteristic point, then

(i) For all \( s \geq \hat{S}_0(x_0) = \max(\hat{S}_0(N, p, a, M), -\log(t(x_0)/4)) \),

\[ 0 < \epsilon_0 \leq \|w_{x_0,T(x_0)}(s)\|_{H^1(B)} + \|\partial_s w_{x_0,T(x_0)}(s)\|_{L^2(B)} \leq K. \]
(ii) For all \( t \in [t_0(x_0), T(x_0)) \), where \( t_0(x_0) = \max(T(x_0) - e^{-\tilde{s}_0(x_0)}, \frac{3T(x_0)}{4}) \), we have

\[
0 < \varepsilon_0(N, p) \leq (T(x_0) - t)^{\frac{2}{p-1}} \frac{\|u(t)\|_{L^2(B(x_0, T(x_0) - t))}}{(T(x_0) - t)^{\frac{N}{2}}}
\]

\[
+ (T(x_0) - t)^{\frac{2}{p-1} + 1} \left( \frac{\|\partial_t u(t)\|_{L^2(B(x_0, T(x_0) - t))}}{(T(x_0) - t)^{\frac{N}{2}}} + \frac{\|\nabla u(t)\|_{L^2(B(x_0, T(x_0) - t))}}{(T(x_0) - t)^{\frac{N}{2}}} \right) \leq K,
\]

where \( K = K(N, p, a, \delta_0(x_0), \|u(t_0(x_0))\|, \|\partial_t u(t_0(x_0))\|_{L^1 \times L^2(B(x_0, e^{-\tilde{s}_0(x_0)}))} \) and \( \delta_0(x_0) \in (0, 1) \) is defined in (1.3).

**Remark 2** In a series of papers [18–23], Merle and Zaag give a full picture of the blow-up for solution of (1.4), in one space dimension and in dimension space \( N \geq 2 \).

The result of all this paper is extended by Hamza and Zaag for a class of perturbed problem in one space dimension or in higher dimension under radial symmetry outside origin in [11], or in dimension space \( N \geq 2 \) in [9,10], (blow-up, profile, regularity of the blow-up graph, existence of characteristic points, etc...). Once again, we believe that the key point in the analysis of blow-up for Eq. (1.1) is the derivation of a Lyapunov functional in similarity variables, which is the object of our paper.

As in [9–11,18–23], the proof of Theorem 2 relies on four ideas (the existence of a Lyapunov functional, interpolation in Sobolev spaces, some Gagliardo–Nirenberg estimates and a covering technique adapted to the geometric shape of the blow-up surface). It happens that adapting the proof of [18–20] given in the non-perturbed case (1.4) is straightforward, except for a key argument, where we bound the \( L^{p+1} \) space-time norm of \( w \). Therefore, we only present that argument, and refer to [18–20], for the rest of the proof.

This paper is divided in two sections, each of them devoted to the proof of a Theorem.

## 2 A Lyapunov Functional for Equation (1.10)

Throughout this section, we prove Theorem 1, we consider \((w, \partial_x w) \in C([s_0, +\infty), \mathcal{H})\) where \( w \) is a solution of (1.10) and \( s_0 \in \mathbb{R} \). We aim at proving that the functional \( H(w(s), s) \) defined in (1.13) is a Lyapunov functional for equation (1.10), provided that \( s \geq S_1 \), for some \( S_1 = S_1(N, p, M, a) \). We denote the unit ball of \( \mathbb{R}^N \) by \( B \). We denote by \( C \) a constant which depends only on \( N, p, a \) and on \( |B| \).

The starting point in our analysis is to prove the following lemma.

**Lemma 2.1** For all \( s \geq \max(s_0, 1) \),

\[
\frac{d}{ds} (E_0(w(s)) + I(w(s), s)) = -2\alpha \int_B (\partial_x w)^2 \frac{\rho}{1 - |y|^2} \, dy + \Sigma_0(s),
\]

where \( \Sigma_0(s) \) satisfies

\[
\Sigma_0(s) \leq Ce^{\frac{-p+1}{\rho-1}} + \frac{C}{s^d} \int_B |w|^{p+1} \rho \, dy.
\]
Proof Multiplying (1.10) by $\partial_s w\rho$ and integrating over the unit ball $B$, we obtain, for all $s \geq s_0$,
\[
\frac{d}{ds}(E_0(w(s)) + I(w(s), s)) = -2\alpha \int_B (\partial_s w)^2 \frac{\rho}{1 - |y|^2} dy + \Sigma_0^1(s) + \Sigma_0^2(s),
\]
where
\[
\Sigma_0^1(s) = \frac{2(p + 1)}{p - 1} e^{-\frac{2(p+1)s}{p-1}} \int_B F(e^{\frac{2s}{p-1}} w)\rho dy \quad \text{and} \quad \Sigma_0^2(s) = -\frac{2e^{\frac{2s}{p-1}}}{p - 1} \int_B f(e^{\frac{2s}{p-1}} w)\rho dy.
\]
(2.4)

It is clear that we obtain (2.1) with $\Sigma_0(s) = \Sigma_0^1(s) + \Sigma_0^2(s)$. Now, in order to obtain estimate (2.2), it is enough to control the terms $\Sigma_0^1(s)$ and $\Sigma_0^2(s)$.

Clearly the function $F$ defined in (1.11) satisfies the following estimate:
\[
|F(x)| + |xf(x)| \leq C(1 + \frac{|x|^{p+1}}{(\log(2 + x^2))^a}).
\]
(2.5)

Taking advantage of inequality (2.5), we see that,
\[
|\Sigma_0^1(s)| + |\Sigma_0^2(s)| \leq C e^{-\frac{2(p+1)s}{p-1}} + C \int_B \frac{|w|^{p+1}}{(\log(2 + e^{\frac{4s}{p-1}} w^2))^a} \rho dy.
\]
(2.6)

In order to prove (2.2), we divide the unit ball $B$ in two parts
\[
A_1(s) = \{y \in B \mid w^2(y, s) \leq e^{\frac{-2s}{p-1}}\} \quad \text{and} \quad A_2(s) = \{y \in B \mid w^2(y, s) > e^{\frac{-2s}{p-1}}\}.
\]

It follows then that
\[
\int_B \frac{|w|^{p+1}}{(\log(2 + e^{\frac{4s}{p-1}} w^2))^a} \rho dy = \int_{A_1(s)} \frac{|w|^{p+1}}{(\log(2 + e^{\frac{4s}{p-1}} w^2))^a} \rho dy + \int_{A_2(s)} \frac{|w|^{p+1}}{(\log(2 + e^{\frac{4s}{p-1}} w^2))^a} \rho dy.
\]
(2.7)

On the one hand, if $y \in A_1(s)$, we have
\[
\frac{|w|^{p+1}}{(\log(2 + e^{\frac{4s}{p-1}} w^2))^a} \leq \frac{e^{-\frac{(p+1)s}{p-1}}}{(\log(2))^a}.
\]

If we integrate over $A_1(s)$. Using the fact that $\int_{A_1(s)} \rho dy \leq |B|$ we see that,
\[
\int_{A_1(s)} \frac{|w|^{p+1}}{(\log(2 + e^{\frac{4s}{p-1}} w^2))^a} \rho dy \leq \frac{e^{-\frac{(p+1)s}{p-1}}}{(\log(2))^a} \int_{A_1(s)} \rho dy \leq Ce^{-\frac{(p+1)s}{p-1}}.
\]
(2.8)

On the other hand, if $y \in A_2(s)$, we have
\[
\log \left(2 + e^{\frac{4s}{p-1}} w^2\right) > \log \left(2 + e^{\frac{2s}{p-1}}\right) \geq \frac{2s}{p - 1},
\]
and for all $s \geq \max(s_0, 1)$, we write for all $y \in A_2(s),$
\[
\frac{|w|^{p+1}}{(\log(2 + e^{\frac{4s}{p-1}} w^2))^a} \leq C \frac{|w|^{p+1}}{s^a}.
\]
We integrate now over \( A_2(s) \), using the simple fact that \( A_2(s) \subset B \), we obtain for all \( s \geq \max(s_0, 1) \),

\[
\int_{A_2(s)} \frac{|w|^{p+1}}{(\log \left( 2 + e^{\frac{4s}{p-1} \rho} \right))^a} \rho \, dy \leq \frac{C}{s^a} \int_{A_2(s)} |w|^{p+1} \rho \, dy \leq \frac{C}{s^a} \int_{B} |w|^{p+1} \rho \, dy. \tag{2.9}
\]

To conclude, it suffices to combine (2.6), (2.7), (2.8) and (2.9), then write

\[
|\Sigma_0^1(s)| + |\Sigma_0^2(s)| \leq Ce^{-\frac{(p+1)s}{p-1}} + \frac{C}{s^a} \int_{B} |w|^{p+1} \rho \, dy,
\]

which ends the proof of Lemma 2.1. \qed

We are going now to prove the following estimate for the functional \( J(w(s), s) \):

**Lemma 2.2** For all \( s \geq \max(s_0, 1) \),

\[
\frac{d}{ds}(J(w(s), s)) \leq \alpha \int_{B} (\partial_s w)^2 \frac{\rho}{1 - |y|^2} \, dy + \frac{p + 3}{2s^b} E((w(s), s))
\]

\[
- \frac{p - 1}{4s^b} \int_{B} |\nabla w|^2 (1 - |y|^2) \rho \, dy - \frac{p + 1}{2(p - 1)s^b} \int_{B} w^2 \rho \, dy
\]

\[
- \frac{p - 1}{2(p + 1)s^b} \int_{B} |w|^{p+1} \rho \, dy + \Sigma_1(s), \tag{2.11}
\]

where \( \Sigma_1(s) \) satisfies the following inequality:

\[
\Sigma_1(s) \leq \frac{C}{s^{2b}} \int_{B} w^2 \rho \, dy + \frac{C}{s^{2b}} \int_{B} |\nabla w|^2 (1 - |y|^2) \rho \, dy
\]

\[
+ \frac{C}{s^{a+b}} \int_{B} |w|^{p+1} \rho \, dy + Ce^{-\frac{(p+1)s}{p-1}}. \tag{2.12}
\]

**Proof** Note that \( J(w(s), s) \) is a differentiable function for all \( s \geq s_0 \) and that

\[
\frac{d}{ds}(J(w(s), s)) = \frac{b}{s^{b+1}} \int_{B} w \partial_s w \rho \, dy - \frac{1}{s^b} \int_{B} (\partial_s w)^2 \rho \, dy - \frac{1}{s^b} \int_{B} w \partial_s^2 w \rho \, dy.
\]

By using the Eq. (1.10) and integrating by parts, we have

\[
\frac{d}{ds}(J(w(s), s)) = - \frac{1}{s^b} \int_{B} (\partial_s w)^2 \rho \, dy + \frac{1}{s^b} \int_{B} (|\nabla w|^2 - (y \cdot \nabla w)^2) \rho \, dy
\]

\[
+ \frac{2(p + 1)}{(p - 1)^2s^b} \int_{B} w^2 \rho \, dy - \frac{1}{s^b} \int_{B} |w|^{p+1} \rho \, dy
\]

\[
+ \Sigma_1^1(s) + \Sigma_2^2(s) + \Sigma_3^3(s) + \Sigma_4^4(s), \tag{2.13}
\]

\( \Box \)
where

\[ \Sigma_1^1(s) = \left( \frac{b}{s} + \frac{p + 3}{p - 1} - 2N \right) \frac{1}{s^b} \int_B w \partial_s w \rho dy \]
\[ \Sigma_1^2(s) = -\frac{2}{s^b} \int_B \partial_s w(y, \nabla w) \rho dy \]
\[ \Sigma_1^3(s) = -\frac{e^{-2\rho y}}{s^b} \int_B w f(e^{\frac{2y}{p-1}} w) \rho dy \]
\[ \Sigma_1^4(s) = -\frac{2}{s^b} \int_B w \partial_s w(y, \nabla \rho) dy. \]

By combining (1.12), (1.14), (2.13) and some straightforward computations we see that,

\[
\frac{d}{ds} (J(w(s), s)) = -\frac{p + 7}{4s^b} \int_B (\partial_s w)^2 \rho dy + \frac{p + 3}{2s^b} E(w(s), s) - \frac{p - 1}{4s^b} \int_B (|\nabla w|)^2
- (y, \nabla w)^2 \rho dy - \frac{p + 1}{2(p - 1)s^b} \int_B w^2 \rho dy - \frac{p - 1}{2(p + 1)s^b} \int_B |w|^{p+1} \rho dy
+ \Sigma_1^1(s) + \Sigma_1^2(s) + \Sigma_1^3(s) + \Sigma_1^4(s) + \Sigma_1^5(s) + \Sigma_1^6(s),
\]

(2.14)

where

\[ \Sigma_1^5(s) = \frac{p + 3}{2s^b} \int_B w \partial_s w \rho dy \]
\[ \Sigma_1^6(s) = \frac{p + 3}{2s^b} e^{-\frac{2(p+1)s}{p-1}} \int_B F(e^{\frac{2y}{p-1}} w) \rho dy. \]

We now study each of the last six terms. To estimate \( \Sigma_1^1(s) \) and \( \Sigma_1^5(s) \), using the fact that for all \( s \geq \max(s_0, 1) \),

\[
\left| \frac{b}{s} + \frac{p + 3}{p - 1} - 2N + \frac{p + 3}{2s^b} \right| \leq C,
\]

we get by virtue of Cauchy–Schwarz inequality

\[
|\Sigma_1^1(s)| + |\Sigma_1^5(s)| \leq C \frac{s^b}{s^b} \int_B |w \partial_s w| \rho dy \leq C \frac{s^b}{s^b} \int_B (\partial_s w)^2 \frac{\rho}{1 - |y|^2} dy + C \frac{s^b}{s^b} \int_B w^2 \rho dy.
\]

(2.15)

Using again Cauchy–Schwarz inequality, we obtain

\[
|\Sigma_1^2(s)| \leq \frac{\alpha}{3} \int_B (\partial_s w)^2 \frac{\rho}{1 - |y|^2} dy + C \frac{s^b}{s^b} \int_B |\nabla w|^2 \rho (1 - |y|^2) dy.
\]

(2.16)

Using (2.4), we write for all \( s \geq \max(s_0, 1) \)

\[ \Sigma_1^3(s) = \frac{(p - 1)}{2s^b} \Sigma_0^2(s), \quad \text{and} \quad \Sigma_1^6(s) = \frac{(p + 3)(p - 1)}{4(p + 1)s^b} \Sigma_0^1(s). \]
This easily leads to the following result
\[ |\Sigma_1^3(s)| + |\Sigma_1^6(s)| \leq \frac{C}{s^b}(|\Sigma_1^2(s)| + |\Sigma_0^2(s)|). \]

By exploiting inequality (2.10) and the fact that \( s \geq 1 \), we see that
\[ |\Sigma_1^3(s)| + |\Sigma_1^6(s)| \leq Ce^{-(p+1)\nu} + \frac{C}{s^{a+b}} \int_B |w|^{p+1}\rho \, dy. \] (2.17)

Now, we estimate the expression \( \Sigma_1^4(s) \). Since we know that \( y.\nabla \rho = -2\alpha \frac{|y|^2}{(1-|y|^2)} \rho \), we can use the Cauchy-Schwarz inequality to write
\[ |\Sigma_1^4(s)| \leq \frac{C}{s^b} \int_B |\partial_s w|(1 - |y|^2)^{\frac{a-1}{2}} |w| |y||(1 - |y|^2)^{\frac{a-1}{2}} \, dy, \]
\[ \leq \frac{\alpha}{3} \int_B (\partial_s w)^2 \frac{\rho}{1 - |y|^2} \, dy + \frac{C}{s^{2b}} \int_B w^2 \frac{|y|^2 \rho}{1 - |y|^2} \, dy. \] (2.18)

Since, we have the following Hardy type inequality for any \( w \in H^1_{loc, \alpha}(\mathbb{R}^N) \) (for more details on this subject, we refer the reader to Appendix B in [18]):
\[ \int_B w^2 \frac{|y|^2 \rho}{1 - |y|^2} \, dy \leq C \int_B |\nabla w|^2 \rho (1 - |y|^2) \, dy + C \int_B w^2 \rho \, dy, \] (2.19)
we get from (2.18) and (2.19)
\[ |\Sigma_1^4(s)| \leq \frac{\alpha}{3} \int_B (\partial_s w)^2 \frac{\rho}{1 - |y|^2} \, dy + \frac{C}{s^{2b}} \int_B w^2 \rho \, dy + \frac{C}{s^{2b}} \int_B |\nabla w|^2 \rho (1 - |y|^2) \, dy. \] (2.20)

Combining (2.14), (2.15), (2.16), (2.17) and (2.20), we write
\[ \frac{d}{ds}(J(w(s), s)) \leq \frac{p+3}{2s^b} E(s) - \frac{p-1}{4s^b} \int_B (|\nabla w|^2 - (y.\nabla w)^2) \rho \, dy - \frac{p+1}{2(p-1)s^b} \int_B w^2 \rho \, dy \]
\[ - \frac{p-1}{2(p+1)s^b} \int_B |w|^{p+1} \rho \, dy + \frac{C}{s^{2b}} \int_B w^2 \rho \, dy + \frac{C}{s^{2b}} \int_B |\nabla w|^2 (1 - |y|^2) \rho \, dy \]
\[ + \alpha \int_B (\partial_s w)^2 \frac{\rho}{1 - |y|^2} \, dy + C e^{-(p+1)\nu} + \frac{C}{s^{a+b}} \int_B |w|^{p+1} \rho \, dy. \] (2.21)

Since \( |y.\nabla w| \leq |y||\nabla w| \), it follows that
\[ \int_B |\nabla w|^2 (1 - |y|^2) \rho \, dy \leq \int_B (|\nabla w|^2 - (y.\nabla w)^2) \rho \, dy. \]

This leads finally to
\[ \frac{d}{ds}(J(w(s), s)) \leq \alpha \int_B (\partial_s w)^2 \frac{\rho}{1 - |y|^2} \, dy + \frac{p+3}{2s^b} E(s) - \frac{p-1}{4s^b} \int_B |\nabla w|^2 (1 - |y|^2) \rho \, dy \]
\[ - \frac{p+1}{2(p-1)s^b} \int_B w^2 \rho \, dy - \frac{p-1}{2(p+1)s^b} \int_B |w|^{p+1} \rho \, dy + \Sigma_1(s), \]
where $\Sigma_1(s)$ satisfies the following inequality

$$
\Sigma_1(s) \leq Ce^{-\frac{(p+1)s}{p-1}} + \frac{C}{s^{2b}} \int_B w^2 \rho dy + \frac{C}{s^{2b}} \int_B |\nabla w|^2 (1 - |y|^2) \rho dy + \frac{C}{s^{a+b}} \int_B |w|^{p+1} \rho dy.
$$

This ends the proof of Lemma 2.2. \qed

For the reader’s convenience we give the details of the proof of Theorem 1 in the following subsection.

2.1 Proof of Theorem 1

Proof Before going into the proof, let’s recall that from (1.14)

$$
E(w(s), s) = E_0(w(s)) + I(w(s), s) + J(w(s), s).
$$

Now, according to Lemmas 2.1 and 2.2, we have

$$
\frac{d}{ds}(E(w(s), s)) \leq Ce^{-\frac{(p+1)s}{p-1}} + \frac{p + 3}{2s^b} E(w(s), s) - \alpha \int_B (\partial_s w)^2 \frac{\rho}{1 - |y|^2} dy
$$

$$
\quad + \left( \frac{C}{s^b} - \frac{p - 1}{4} \right) \frac{1}{s^b} \int_B |\nabla w|^2 (1 - |y|^2) \rho dy
$$

$$
\quad + \left( \frac{C}{s^b} - \frac{p + 1}{2(p - 1)} \right) \frac{1}{s^b} \int_B w^2 \rho dy
$$

$$
\quad + \left( \frac{C}{s^{a-b}} + \frac{C}{s^a} - \frac{(p - 1)}{2(p + 1)} \right) \frac{1}{s^b} \int_B |w|^{p+1} \rho dy.
$$

Then, we consider $S_1 \geq 1$ such that, for all $s \geq \max(S_1, s_0)$, we have:

$$
\frac{C}{s^b} - \frac{p - 1}{4} \leq 0, \quad \frac{C}{s^b} - \frac{p + 1}{2(p - 1)} \leq 0, \quad \frac{C}{s^{a-b}} + \frac{C}{s^a} - \frac{(p - 1)}{2(p + 1)} \leq 0.
$$

Thus, implies that for all $s \geq \max(S_1, s_0)$,

$$
\frac{d}{ds}(E(w(s), s)) \leq Ce^{-\frac{(p+1)s}{p-1}} + \frac{p + 3}{2s^b} E(w(s), s) - \alpha \int_B (\partial_s w)^2 \frac{\rho}{1 - |y|^2} dy. \tag{2.22}
$$

Recalling that,

$$
H(w(s), s) = \exp\left( \frac{p + 3}{(a - 1)s^{b-1}} \right) E(w(s), s) + \theta e^{-\frac{(p+1)s}{p-1}},
$$

we get from straightforward computations

$$
\frac{d}{ds}(H(w(s), s)) = -\frac{p + 3}{2s^b} \exp\left( \frac{p + 3}{(a - 1)s^{b-1}} \right) E(w(s), s)
$$

$$
\quad + \exp\left( \frac{p + 3}{(a - 1)s^{b-1}} \right) \frac{d}{ds}(E(w(s), s)) - \theta (p + 1) e^{-\frac{(p+1)s}{p-1}}. \tag{2.23}
$$
Therefore, estimates (2.22) and (2.23) lead to the following crucial estimate:

\[
\frac{d}{ds}(H(w(s), s)) \leq \left( C \exp\left( \frac{p + 3}{(a-1)s^{b-1}} \right) - \theta \frac{(p+1)}{p-1} \right) e^{-\frac{(p+1)s}{p-1}} - \alpha \exp\left( \frac{p + 3}{(a-1)s^{b-1}} \right) \int_B (\partial_s w)^2 \frac{\rho}{1 - |y|^2} dy.
\]

Since, we have \(1 \leq \exp\left( \frac{p + 3}{(a-1)s^{b-1}} \right) \leq \exp\left( \frac{p + 3}{(a-1)} \right)\), we deduce for all \(s \geq \max(S_1, s_0)\),

\[
\frac{d}{ds}(H(w(s), s)) \leq \left( C - \theta \frac{(p+1)}{p-1} \right) e^{-\frac{(p+1)s}{p-1}} - \alpha \int_B (\partial_s w)^2 \frac{\rho}{1 - |y|^2} dy.
\]

We then choose \(\theta\) large enough, so that \(C - \theta \frac{(p+1)}{p-1} \leq 0\), which yields

\[
\frac{d}{ds}(H(w(s), s)) \leq -\alpha \int_B (\partial_s w)^2 \frac{\rho}{1 - |y|^2} dy.
\]

A simple integration between \(s_1\) and \(s_2\) ensures the result. This ends the proof of the Theorem 1. \(\square\)

3 Proof of Theorem 2

Throughout this section, we give a blow-up criterion in the \(w(y,s)\) variable and conclude the proof of Theorem 2.

3.1 A Blow-Up Criterion in the \(w(y,s)\) Variable

We now claim the following lemma:

**Lemma 3.1** There exists \(S_2 \geq S_1\), such that for all \(s_0 \in \mathbb{R}\) and \(w\) solution of equation (1.10) defined to the right of \(s_0\), such that \(\|w\|_{L^{p+1}(B)}\) is locally bounded, if \(H(w(s_3), s) < 0\) for some \(s_3 \geq \max(S_2, s_0)\), then \(w\) cannot be defined for all \((y, s) \in B \times [s_3 + 1, +\infty)\).

**Remark 3** Before going into the proof of Lemma 3.1, let’s remark that if \(w = w_{s_0}, t_0\) defined from a solution of (1.1) by (1.9) and \(x_0\) is a non characteristic point, then \(\|w\|_{H^1(B)}\) is locally bounded and so is \(\|w\|_{L^{p+1}(B)}\) by Sobolev’s embedding.

**Proof** The argument is the same as in the corresponding part in [3]. We sketch the proof for the reader’s convenience. Arguing by contradiction, we assume that there exists a solution \(w\) on \(B\), defined for all \((y, s) \in B \times [s_3 + 1, +\infty)\), with \(H(w(s_3), s_3) < 0\). Since the energy \(H(w(s_3), s_3))\) decreases in time, we have \(H(w(s_3 + 1), s_3 + 1) < 0\).

Consider now for \(\delta > 0\) the function \(w^\delta(y, s)\) for \((y, s) \in B \times [s_3 + 1, +\infty)\), defined for all \(s \geq s_3 + 1, y \in B\), by

\[
w^\delta(y, s) = \frac{1}{(1 + \delta e^s)^{\frac{p}{p-1}}} e^{\left( \frac{y}{1 + \delta e^s} - \log(\delta + e^{-s}) \right)},
\]

we have three observation:

- (A) Note that \(w^\delta\) is defined for all \((y, s) \in B \times [s_3 + 1, +\infty)\), whenever \(\delta > 0\) is small enough so that \(-\log(\delta + e^{-s_3 - 1}) \geq s_3\).
• (B) By construction, \( w^\delta \) is also a solution of (1.10) (indeed, let \( u \) be such that \( w = w_{0,0} \) in definition (1.9). Then \( u \) is a solution of (1.1) and \( w^\delta = w_{-\delta,0} \) is defined as in (1.9); so \( w^\delta \) is also a solution of (1.10)).

• (C) For \( \delta \) small enough, we have \( H(w^\delta(s_3 + 1), s_3 + 1) < 0 \) by continuity of the function \( \delta \mapsto H(w^\delta(s_3 + 1, s_3 + 1)) \).

Now, we fix \( \delta = \delta_0 > 0 \) such that (A), (B) and (C) hold. Let us note that we have

\[
- \frac{1}{s^b} \int_B w^{\delta_0}_1 \partial_s w^{\delta_0} \rho dy = - \frac{1}{4} \int_B (\partial_s w^{\delta_0})^2 \rho dy - \frac{1}{s^{2b}} \int_B (w^{\delta_0})^2 \rho dy. \tag{3.2}
\]

According to the inequality (2.10), we obtain

\[
- e^{-\frac{2(p+1)s}{p-1}} \int_B F(e^{\frac{2s}{p-1} w^{\delta_0}}) \rho dy \geq - C e^{-\frac{(p+1)s}{p-1}} - \frac{C}{s^a} \int_B |w^{\delta_0}|^{p+1} \rho dy. \tag{3.3}
\]

We recall that

\[
E(w^{\delta_0}(s), s) = E_0(w^{\delta_0}(s)) - e^{-\frac{2(p+1)s}{p-1}} \int_B F(e^{\frac{2s}{p-1} w^{\delta_0}}) \rho dy - \frac{1}{s^b} \int_B w^{\delta_0}_1 \partial_s w^{\delta_0} \rho dy.
\]

Plugging the estimates (3.2) and (3.3) together, we obtain

\[
E(w^{\delta_0}(s), s) \geq E_0(w^{\delta_0}(s)) - C e^{-\frac{(p+1)s}{p-1}} - \frac{C}{s^a} \int_B |w^{\delta_0}|^{p+1} \rho dy - \frac{1}{4} \int_B (\partial_s w^{\delta_0})^2 \rho dy
\]

\[- \frac{1}{s^{2b}} \int_B (w^{\delta_0})^2 \rho dy.
\]

By using the fact that

\[
E_0(w^{\delta_0}(s)) \geq \int_B \left( \frac{1}{2} (\partial_s w^{\delta_0})^2 + \frac{p+1}{(p-1)^2} (w^{\delta_0})^2 - \frac{1}{p+1} |w^{\delta_0}|^{p+1} \right) \rho dy,
\]

it follows that

\[
E(w^{\delta_0}(s), s) \geq \int_B \left( \frac{1}{2} (\partial_s w^{\delta_0})^2 + \frac{p+1}{(p-1)^2} (w^{\delta_0})^2 - \frac{1}{p+1} |w^{\delta_0}|^{p+1} \right) \rho dy - C e^{-\frac{(p+1)s}{p-1}}
\]

\[- \frac{C}{s^a} \int_B |w^{\delta_0}|^{p+1} \rho dy - \frac{1}{4} \int_B (\partial_s w^{\delta_0})^2 \rho dy - \frac{1}{s^{2b}} \int_B (w^{\delta_0})^2 \rho dy.
\]

Hence, for any \( s \geq s_3 + 1 \),

\[
E(w^{\delta_0}(s), s) \geq \frac{1}{4} \int_B (\partial_s w^{\delta_0})^2 \rho dy + \left( \frac{p+1}{(p-1)^2} - \frac{1}{s^{2b}} \right) \int_B (w^{\delta_0})^2 \rho dy
\]

\[- \left( \frac{1}{p+1} + \frac{C}{s^a} \right) \int_B |w^{\delta_0}|^{p+1} \rho dy - C e^{-\frac{(p+1)s}{p-1}}.
\]
We choose \( s_4 \geq s_3 \) large enough, so that we have \( \frac{p+1}{(p-1)^2} - \frac{1}{s_4^p} \geq 0 \). Then, we deduce, for all \( s \geq s_4 \),

\[
E(w^{\delta_0}) \geq - \left( \frac{1}{p+1} + \frac{C}{s^a} \right) \int_B |w^{\delta_0}|^{p+1} \rho dy - Ce^{-\frac{(p+1)s}{p-1}}.
\]

By using the construction of \( w^\delta \), we write

\[
E(w^{\delta_0}(s), s) \geq - \frac{1}{p+1} + \frac{C}{s^a} \int_B |w(y) - \log(\delta_0 + e^{-s})|^{p+1} \rho dy - Ce^{-\frac{(p+1)s}{p-1}}.
\]

Since \( \rho \leq 1 \), the change of variable \( z := \frac{y}{1 + \delta_0 e^s} \), yields

\[
E(w^{\delta_0}(s), s) \geq - \frac{1}{p+1} + \frac{C}{(1 + \delta_0 e^s)^{\frac{2(p+1)}{p-1}} - N} \int_B |w(z, - \log(\delta_0 + e^{-s}))|^{p+1} dz - Ce^{-\frac{(p+1)s}{p-1}}.
\]

It is clear that \( - \log(\delta_0 + e^{-s}) \to - \log(\delta_0) \) as \( s \to +\infty \) and since \( \|w\|_{L^{p+1}(B)} \) is locally bounded by hypothesis, by a continuity argument, it follows that the former integral remains bounded and

\[
E(w^{\delta_0}(s), s) \geq - \frac{C}{(1 + \delta_0 e^s)^\frac{4}{p-1} + 2 - N} - Ce^{-\frac{(p+1)s}{p-1}} \to 0,
\]

as \( s \to +\infty \) (this is due to the fact that \( \frac{4}{p-1} + 2 - N > 0 \) which follows from the fact that \( p < p_c \)). So, thanks to (1.13), it follows that

\[
\lim_{s \to +\infty} H(w^{\delta_0}(s), s) \geq 0. \tag{3.4}
\]

The inequality (3.4) contradicts the inequality \( H(w^{\delta_0}(s_3 + 1), s_3 + 1) < 0 \) (see item (C) above) and the fact that the energy \( H \) decreases in time for \( s \geq s_3 \), which leads to the result. This ends the proof of Lemma 3.1. \( \square \)

### 3.2 Boundedness of the Solution in Similarity Variables

We prove Theorem 2 here. Note that the lower bound follows from the finite speed of propagation and wellposedness in \( H^1 \times L^2 \). For a detailed argument in the similar case of equation (1.4), see Lemma 3.1 (p. 1136) in [19].

We consider \( u \) a solution of (1.1) which is defined under the graph of \( x \mapsto T(x) \) and \( x_0 \) is a non-characteristic point. Given some \( T_0 \in (0, T(x_0)) \), we introduce \( w_{x_0, T_0} \) defined in (1.9), and write \( w \) for simplicity, when there is no ambiguity. We aim at bounding \( \|w, \partial_s w\|_{H^1 \times L^2} \) for \( s \) large.

As in [18], by combining Theorem 1 and Lemma 3.1 (use in particular the remark after that lemma) we get the following bounds:

**Corollary 3.2** For all \( s \geq \hat{s}_3 = \hat{S}_3(T_0) = \max(S_3, -\log(T_0)) \), \( s_2 \geq s_1 \geq \hat{s}_3 \), it holds that

\[
-C \leq E(w(s), s) \leq M_0.
\]

\[
\int_{s_1}^{s_2} \int_B (\partial_s w)^2 \frac{\rho}{1 + |y|^2} dyds \leq M_0.
\]
Starting from these bounds, the proof of Theorem 2 is similar to the proof in [18–20]. To be more complete and in order to state our main result in a clear way, let us mention that the unique difference lays in the logarithmic term. In our opinion, handling these terms is straightforward in all the steps of the proof, except for the first step, where we bound the time averages of the $L^{p+1}_p(B)$ norm of $w$. For that reason, we only give that step and refer to [18–20], for the remaining steps in the proof of Theorem 2. This is the step we prove here (in the following, $K_3$ denotes a constant that depends on $p, N, C$).

**Proposition 3.3** For all $s \geq 1 + \delta_3$;

$$\int \int \frac{|w|^{p+1}}{\rho} dy \leq K_3.$$  

**Proof** The proof of Proposition 3.3 is the same as in Hamza and Zaag [9,10]. Exceptionally, the unique difference lays in the logarithmic term where we use the same technique as in the proof of Lemma 2.1 in Sect. 1. Since the derivation of Theorem 2 from Proposition 3.3 is the same as in the non-perturbed case treated in [18–20], (up to some very minor changes), this concludes the proof of Theorem 2. \hfill \Box

**Acknowledgments** The authors wish to thank Professor Hatem Zaag for many fruitful discussions. The authors are also grateful to the referee for his careful reading of the manuscript and for his valuable remarks. The first author is partially supported by the ERC Advanced Grant No. 291214, BLOWDISOL during his visit to LAGA, Univ P13 in 2013.

**Appendix: Blow-Up Dynamics for the Associated ODE**

In this appendix, we consider the following ODE:

$$u'' = |u|^{p-1}u + f(u),$$  \hspace{1cm} (4.1)

with either $f(u) \sim |u|^q$, $q < p$ as $u \to \infty$ or $f(u) \sim \frac{|u|^p}{(\log(2+u^2))^a}$, $a > 1$ as $u \to \infty$.

In this proposition, we give two terms in the solution’s expansion near blow-up

**Proposition 3.1** Let $u$ a solution of (4.1) that blows-up in some finite time $T$, the blow-up profile of $u$ near $T$ is given by the following quantities:

(i) If $f(u) \sim |u|^q$, then we have

$$u(t) - \frac{\kappa}{(T-t)^{\frac{s}{p-1}}} \sim \frac{A}{(T-t)^{\frac{s}{p-1}+\mu}}, \quad \text{as } t \to T^-, \hspace{1cm} (4.2)$$

where $\mu > 0$ and $A \in \mathbb{R}$.

(ii) If $f(u) \sim \frac{|u|^p}{(\log(2+u^2))^a}$, then we have

$$u(t) - \frac{\kappa}{(T-t)^{\frac{s}{p-1}}} \sim \frac{\kappa(p-1)^{a-1}}{4^a(T-t)^{\frac{s}{p-1}}(-\log(T-t))^a}, \quad \text{as } t \to T^-.$$  \hspace{1cm} (4.3)

**Remark 4** If $f(u) \equiv 0$, then we have

$$u(t) - \frac{\kappa}{(T-t)^{\frac{s}{p-1}}} \sim \frac{A}{(T-t)^{\frac{s}{p-1}+2}}, \quad \text{as } t \to T^-.$$
Proof  First it is clear that $u(t) \sim \frac{\kappa}{(T-t)^{\frac{p}{p-1}}}$, as $t \to T^-$, with $\kappa = \left( \frac{2p+2}{(p-1)^2} \right)^{\frac{1}{p-1}}$. By using the following change of variables:

$$s = -\log(T-t), \quad u(t) = \frac{1}{(T-t)^{\frac{p}{p-1}}} w(-\log(T-t)), \quad \forall \ t \in [0, T).$$

The function $w$ satisfies the following equation: $\forall \ s \geq -\log(T)$

$$w''(s) + \frac{p+3}{p-1} w'(s) + \frac{2p+2}{(p-1)^2} w(s) = |w(s)|^{p-1} w(s) + e^{-2ps} f(e^{-2s} w(s)). \quad (4.4)$$

By standard arguments, we easily study the asymptotic behaviour of equation (4.4) as $s \to \infty$ and we get (4.2) and (4.3). Which ends the proof of Proposition 3.1.  

References

1. Alinhac, S.: Blow up for nonlinear hyperbolic equations. In Volume 17 of Progress in Nonlinear Differential Equations and their Applications, pages Birkhäuser Boston Inc., Boston, MA (1995)
2. Alinhac, S.: A minicourse on global existence and blow-up of classical solutions to multidimensional quasilinear wave equations. In Journées “Équations aux Dérivées Partielles” (Forges-les-Eaux, 2002), pages Exp. No. I, 33. University of Nantes, Nantes (2002)
3. Antonini, C., Merle, F.: Optimal bounds on positive blow-up solutions for a semilinear wave equation. Int. Math. Res. Notices 21, 1141–1167 (2001)
4. Caffarelli, L.A., Friedman, A.: Differentiability of the blow-up curve for one dimensional nonlinear wave equations. Arch. Rational Mech. Anal. 91(1), 83–98 (1985)
5. Caffarelli, L.A., Friedman, A.: The blow-up boundary for nonlinear wave equations. Trans. Am. Math. Soc. 297(1), 223–241 (1986)
6. Donninger, R., Schlag, W.: Numerical study of the blow-up/global existence dichotomy for the focusing cubic nonlinear Klein–Gordon equation. Nonlinearity 24, 2547–2562 (2011)
7. Donninger, R., Schörkhuber, B.: Stable self-similar blow-up for energy subcritical wave equation. Dyn. Partial Differ. Equ. 9, 63–87 (2012)
8. Donninger, R., Schörkhuber, B.: Stable blow-up dynamics for energy supercritical wave equations. Trans. Am. Math. Soc. 366(4), 2167–2189 (2014)
9. Hamza, M.A., Zaag, H.: A Lyapunov functional and blow-up results for a class of perturbed semilinear wave equations. Bull. Sci. Math. 137, 1087–1109 (2013)
10. Kichenassamy, S., Littman, W.: Blow-up surfaces for nonlinear wave equations. I. Commun. Partial Differ. Equ. 18(3–4), 431–452 (1993)
11. Levine, H.A., Todorova, G.: Blow up of solutions of the Cauchy problem for a wave equation with nonlinear damping and source terms and positive initial energy. SIAM J. Math. Anal. 5(3), 793–805 (2001)
12. Merle, F., Zaag, H.: Blow-up rate near the blow-up surface for semilinear wave equation. Int. Math. Res. Notices 19(1), 1127–1156 (2005)
20. Merle, F., Zaag, H.: Determination of the blow-up rate for a critical semilinear wave equation. Math. Ann. 331(2), 395–416 (2005)
21. Merle, F., Zaag, H.: Existence and universality of the blow-up profile for the semilinear wave equation in one space dimension. J. Funct. Anal. 253(1), 43–121 (2007)
22. Merle, F., Zaag, H.: Openness of the set of non-characteristic points and regularity of the blow-up curve for the 1 D semilinear wave equation. Commun. Math. Phys. 282(1), 55–86 (2008)
23. Merle, F., Zaag, H.: Existence and classification of characteristic points at blow-up for a semilinear wave equation in one space dimension. Am. J. Math. 134(3), 581–648 (2012)
24. Nakanishi, K., Schlag, W.: Invariant Manifolds and Dispersive Hamiltonian Evolution Equations. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich (2011)
25. Tao, T.: Global regularity for a logarithmically supercritical defocusing nonlinear wave equation for spherically symmetric data. J. Hyperbolic Differ. Equ. 4(1), 259–265 (2007)
26. Todorova, G.: Cauchy problem for a non linear wave equation with non linear damping and source terms. Nonlinear Anal. 41, 891–905 (2000)
27. Whitham, G.B.: Linear and Nonlinear Waves, Pure and Applied Mathematics (New York). Wiley, New York (1999). Reprint of the 1974 original, A Wiley-Interscience Publication