Model for Optimization of Error and Uncertainty in the Generation of Calibration Charts for Horizontal Storage Tank

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Abstract. In this paper, we propose a sequence of steps for optimization of error and uncertainty in the generation of calibration charts in a horizontal tank. The calibration of the tank is carried out by means of the volumetric method, but the number of fillings and height measurements have been selected according to Chebyshev nodes and then the interpolation function of volume and height has been modeled according to the geometry of the tank. The proposed approach has been applied to 21 pairs of data, of which 11 pairs of data have been selected according before steps and the remaining pairs of data have been used to prove our propose, so that the errors in volumes have been less than maximum permissible errors (MPE) and the difference between remainder data of the volume and the interpolation data of the volume have been less than the uncertainty. The proposed approach has been calculated with two possible interpolation function of volume and height, but both have fullfilled the aims.

Keywords: Optimization, error, uncertainty, calibration charts, tank.

1. Introduction
In the chemical and food industry is common to use horizontal tanks for the storage of products. During inventory and transfer operations, the amount of liquid in the tanks must be known with accuracy and precision, since there are commercial and legal implications for any mistake that occurs during the buying and selling process of stored products. This determines that all storage tanks require an increment table of volume against the height of the liquid level, known as calibration charts.

This paper focuses on the optimization of the calibration process in the preparation of calibration charts, presenting an alternative to other methods used, a set of steps is proposed that determine the interpolation function for the calculation of the volume of a horizontal tank that solve the problems associated with compliance the maximum permissible errors (EMP) and expanded uncertainty (U).

2. Methodology

2.1. Number of fillings and height measurements
The calibration depends on the correct selection of the number of separate fillings and the capacity of the reference metering vessels, because the tank cross section changes with height. Therefore, the value of each fulling is selected using the Chebyshev node that a manner that minimize interpolation error. [1]
For an arbitrary interval \([h_o, h_f]\) can be used:

\[
h_k = \frac{1}{2}(h_o + h_f) + \frac{1}{2}(h_f - h_o)\cos\left(\frac{2k-1}{2N}\pi\right), \quad k = 1, \ldots, N
\]  

(1)

Where:
- \(h_o\): Minimum height.
- \(h_f\): Maximum height.
- \(h_k\): \(k\)-th height.
- \(2N\): Number of fillings.

| Number of fillings | Approximate value of one filling, in % of the tank capacity |
|--------------------|------------------------------------------------------------|
| 1                  | 5                                                          |
| 2                  | 10                                                         |
| 3                  | 15                                                         |
| 4                  | 30                                                         |
| 5                  | 40                                                         |
| 6                  | 50                                                         |
| 7                  | 60                                                         |
| 8                  | 70                                                         |
| 9                  | 90                                                         |
| 10                 | 95                                                         |
| 11                 | 100                                                        |

The table 1 can be used when take an approximate value of filling.

2.2. **Mathematical model of the relationship between volume and height of horizontal storage tank**

According to the cross-section diagram of the elliptical storage tank as shown in Figure 1, the elliptic equation:

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]  

(2)

The differential of volume according to figures 1 and 2, \(dV = 2Lx\,dy\), and substitute (2), which can obtain the following: \(dV = 2L\left(\frac{a}{b}\sqrt{y^2 + b^2} - y\right)\,dy\) and finally integrating the differential of volume, which can obtain the following:

\[
V(h') = abL\left(\frac{\pi}{2} + \arcsin\left(h'\right) + h'\sqrt{1 - h'^2}\right)
\]  

(3)

\[
h' = \frac{a}{b} - 1
\]  

(4)
According the equation (3) the appropriate basis functions for development the least squares method are: \( \varphi(h'_j) = \arcsin(h'_j) \) and \( \theta(h'_j) = h'_j \sqrt{1 - h'_j^2} \), where \( h'_j = \frac{h}{b_j} - 1 \) therefore we can take a polynomial equation according the basis functions:

\[
V(h'_j) = A_0 + \sum_{n=1}^{m} B_n \varphi^n(h'_j) + C_n \theta^n(h'_j)
\] (5)

\( A_0, B_n, C_n \) are constants and can be determined for the least squares method, using the following system of equations in matrix form:

\[
\begin{bmatrix}
\sum V \\
\sum V \varphi \\
\sum V \theta \\
\sum V \varphi^n \\
\sum V \theta^n
\end{bmatrix}
=
\begin{bmatrix}
N & \sum \varphi & \sum \varphi \theta & \sum \varphi^2 & \sum \varphi^2 \theta & \cdots & \sum \varphi^m & \sum \varphi \theta^m & \sum \varphi^m \theta & \cdots & \sum \varphi^m \theta^n \\
\sum \varphi & \sum \varphi^2 & \sum \varphi^3 & \sum \varphi^4 & \sum \varphi^5 & \cdots & \sum \varphi^m & \sum \varphi \theta^m & \sum \varphi^m \theta & \cdots & \sum \varphi^m \theta^n \\
\sum \varphi \theta & \sum \varphi \theta^2 & \sum \varphi \theta^3 & \sum \varphi \theta^4 & \sum \varphi \theta^5 & \cdots & \sum \varphi \theta^m & \sum \varphi \theta^m \theta & \sum \varphi \theta^m \theta^2 & \cdots & \sum \varphi \theta^m \theta^n \\
\sum \varphi^m & \sum \varphi^m \theta & \sum \varphi^m \theta^2 & \sum \varphi^m \theta^3 & \sum \varphi^m \theta^4 & \cdots & \sum \varphi^m \theta^m & \sum \varphi^m \theta^m \theta & \sum \varphi^m \theta^m \theta^2 & \cdots & \sum \varphi^m \theta^m \theta^n \\
\sum \varphi \theta^m & \sum \varphi \theta^m \theta & \sum \varphi \theta^m \theta^2 & \sum \varphi \theta^m \theta^3 & \sum \varphi \theta^m \theta^4 & \cdots & \sum \varphi \theta^m \theta^m & \sum \varphi \theta^m \theta^m \theta & \sum \varphi \theta^m \theta^m \theta^2 & \cdots & \sum \varphi \theta^m \theta^m \theta^n
\end{bmatrix}
\begin{bmatrix}
A_0 \\
B_1 \\
C_1 \\
B_m \\
C_m
\end{bmatrix}
\] (6)

Or in matrix notation

\[
[V] = [X][A]
\] (7)

For calculating \([A]\):

\[
[A] = [X]^{-1}[V]
\] (8)

2.3. Error and uncertainty of the adjustment curve

The uncertainty of the adjustment curve is determined based on the residual errors of the adjustment curve, evaluating the standard deviation of the residual errors, for all the interval: [2]

\[
up(x_i) = s_{\text{err}}([x][X]^{-1}[x]^T)^{1/2}
\] (9)

Where:

\[
s_{\text{err}}^2 = \frac{1}{N - m - 1} \sum_{i=1}^{N} (v_i - V(h'_j))^2
\] (10)

\[
x = \begin{bmatrix}
1 & \varphi_1 & \varphi_1 \varphi_1 & \cdots & \varphi_1 \varphi_1 \varphi_1 & \theta_1 \\
1 & \varphi_2 & \varphi_2 \varphi_2 & \cdots & \varphi_2 \varphi_2 \varphi_2 & \theta_2 \\
1 & \varphi_3 & \varphi_3 \varphi_3 & \cdots & \varphi_3 \varphi_3 \varphi_3 & \theta_3 \\
1 & \varphi_{N-1} & \varphi_{N-1} \varphi_{N-1} & \cdots & \varphi_{N-1} \varphi_{N-1} \varphi_{N-1} & \theta_{N-1} \\
1 & \varphi_N & \varphi_N \varphi_N & \cdots & \varphi_N \varphi_N \varphi_N & \theta_N
\end{bmatrix}
\] (11)

The function of optimization is:

\[
f(h'_j) \big|_{\text{min}} = \sum_{i=1}^{N} (v_i - V(h'_j(h, b^*)))^2 ; \quad \frac{h}{2} \leq b^* \leq H
\] (12)

We must find "b" by a numerical method where get a minimum for \( f(h'_j) \) and therefore \( V(h'_j(h, b^*)) \) is the optimal polynomial equation then calculated percentage measurement error (%E) where must fulfill:

\[
% E < \text{MPE}
\] (13)

**Table 2. Maximum permissible errors [3], [4].**

| Type                      | Maximum permissible errors (MPE) |
|----------------------------|----------------------------------|
| Static measuring system    | 0,50%                            |
| Transportable measuring    | 0,30%                            |
In the case of the generation of calibration charts, the expanded uncertainty is obtained by applying the following equation. [2], [6]

\[ U = k \sqrt{u_{\text{instrumental}}^2 + u_{\text{adjustment curve}}^2} \]  \hspace{1cm} (14)

Where:
- \( u_{\text{instrumental}} \): Instrumental uncertainty.
- \( u_{\text{adjustment curve}} \): Adjustment curve uncertainty.
- \( k \): Coverage factor.

Instrumental uncertainty in the process of elaboration of capacity tables, the volumetric method will be adopted, consisting of multiple discharges from a standard volumetric container [5]. Adjustment curve uncertainty is determined by evaluating the standard deviation of residual errors for the entire interval.

Considering the sufficient number of points to make a suitable adjustment, the intermediate point (not considered in the adjustment) will be consistent with the initial run when the difference between this point and the initial interpolation is less than the expanded uncertainty (yellow cell data for validation).

\[ |V_{\text{Teo}} - V| \leq U \]  \hspace{1cm} (15)

Where:
- \( V_{\text{Teo}} \): Volume for polynomial equation.
- \( V \): Volume transferred
- \( U \): Expanded uncertainty

2.4. Flowchart for mathematical model
The Mathematical Model for Optimization of Error and Uncertainty we call AMVA

Flowchart of approaching AMVA

![Flowchart AMVA](image-url)
3. Example of calibration [7]

Example of calibration of a horizontal tank of maximum capacity 4000 L (static tank)

Elements used:
- Capacity volume pattern equal to 200 L
- Digital thermometer, minimum division 0.1 °C

Table 3. Data for the calibration of a horizontal static tank.

| PATTERNS                      | C1  | C2  | C3  | C4  | C5  | C6  | C7  |
|-------------------------------|-----|-----|-----|-----|-----|-----|-----|
| Nominal volume pattern        | 200 | L   |     |     |     |     |     |
| Correction of volume pattern  | -0.02 | L   |     |     |     |     |     |
| Uncertainty of volume pattern | 0.01 | L   |     |     |     |     |     |
| Expansion coefficient vol. pattern | 5.16E-05 | 1/°C |     |     |     |     |     |
| Volume per division           | 0.1 | L   |     |     |     |     |     |
| Reference temperature of volume pattern | 20.0 | °C |     |     |     |     |     |
| Reading resolution            | 0.1 | L   |     |     |     |     |     |
| U enrase of pattern           | 0.05 | L   |     |     |     |     |     |
| TANK                          | 7   | 200.0 | 20.5 | 20.5 | 199.9 | 1399.0 | 511 |
| Expansion coefficient         | 5.16E-05 | 1/°C |     |     |     |     |     |
| Reference temperature         | 4.0 | °C   |     |     |     |     |     |
| Water cubic expansion coefficient | 2.10E-04 | 1/°C |     |     |     |     |     |
| ROD (DIP STICKS)              |     |     |     |     |     |     |     |
| The following data was considered: |     |     |     |     |     |     |     |
| Uncertainty pattern           | 0.1 | mm  |     |     |     |     |     |
| Reading resolution            | 1   | mm  |     |     |     |     |     |
| Where:                        |     |     |     |     |     |     |     |
| C1: Transfer number           | 16  | 200.0 | 20.7 | 20.5 | 199.9 | 2997.9 | 918 |
| C2: Obtained by volume pattern reading. | 17  | 200.0 | 20.6 | 20.5 | 199.9 | 3397.6 | 1024 |
| C3: Water temperature in volume pattern. | 18  | 200.0 | 20.3 | 20.5 | 199.9 | 3597.4 | 1081 |
| C4: Water temperature in cooler. | 19  | 200.0 | 20.0 | 20.5 | 199.9 | 3797.3 | 1141 |
| C5: Corrected volume of each transfer. | 20  | 200.0 | 19.8 | 20.5 | 199.8 | 3997.1 | 1206 |
| C6: Total volume transferred in cooler. | 21  | 200.0 | 20.0 | 20.4 | 199.8 | 4197.0 | 1282 |

In our example has been taken 11 data pairs from the table 3, according to Chebyshev nodes equation (1) and choose \( n = 1 \) and \( n = 2 \) of equation (5)
Table 4. Data according Chebyshev nodes.

| Number of data | h (mm) | Volumen Total (v) a 4°C (L) |
|----------------|--------|-----------------------------|
| 1              | 114    | 199,9                       |
| 2              | 206    | 399,7                       |
| 3              | 278    | 599,6                       |
| 4              | 343    | 799,4                       |
| 5              | 511    | 1399                        |
| 6              | 715    | 2198,4                      |
| 7              | 866    | 2798                        |
| 8              | 1024   | 3397,6                      |
| 9              | 1141   | 3797,3                      |
| 10             | 1206   | 3997,1                      |
| 11             | 1282   | 4197                        |

Table 5. Three-term and five-term polynomial.

| Constant | AMVA n = 1 | Polynomial of degree 2 | AMVA n = 2 | Polynomial of degree 4 |
|----------|------------|------------------------|------------|------------------------|
| b        | 733,63     | -                      | 769,88     | -                      |
| A        | 2272,6195  | -290,852757            | 2417,38071 | 27,0473526             |
| B        | 1347,4298  | 3,341666294            | 1305,85481 | 1,03280696             |
| C        | 1585,3203  | 0,000179798            | 1769,17843 | 0,00430372             |
| D        | -          | -                      | -92,806981 | -2,22E-06              |
| E        | -          | -                      | 0,29721553 | 1,66E-10               |

\[ V = A + Bx + Cx^2 (\text{Polynomial of degree 2}) \]  
\[ V = A + B. \arcsin(h') + C. h' \sqrt{1 - h'^2} (\text{AMVA n = 1; } b = 733,63) \]  
\[ V = A + Bx + Cx^2 + Dx^3 + Ex^4 (\text{Polynomial of degree 4}) \]  
\[ V = A + B. \arcsin(h') + C. h' \sqrt{1 - h'^2} + D. \arcsin(h')^2 + E. (h' \sqrt{1 - h'^2} )^2 (\text{AMVA n = 2; } b = 769,88) \]
4. Analysis of Results

Table 6. Comparison with AMVA $n = 1$ and normal polynomial of degree 2.

| $h$ [mm] | Total Volume (V) 4°C [L] | $V_{in}$ [L] | Error [E] [L] | E% | MPE | $U$ [L] $k=3.87$ | $V_{in}=V\leq U$ [L] | $V_{in}$ [L] | Error [E] [L] | E% | MPE | $U$ [L] $k=4.28$ | $V_{in}=V\leq U$ [L] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 114 | 199.9 | 200.5 | 0.56 | 0.28 | PASS | 4.08 | PASS | 92.4 | -107.47 | -53.76 | FAIL | 268 | PASS |
| 206 | 399.7 | 398.9 | -0.79 | -0.20 | PASS | 4.08 | PASS | 397.5 | -2.17 | -0.54 | FAIL | 268 | PASS |
| 278 | 599.6 | 598.0 | -1.57 | -0.26 | PASS | 4.08 | PASS | 638.1 | 38.53 | 6.43 | FAIL | 268 | PASS |
| 343 | 799.4 | 801.5 | 2.12 | 0.26 | PASS | 4.08 | PASS | 855.3 | 55.94 | 7.00 | FAIL | 268 | PASS |
| 401 | 999.3 | 997.9 | -1.41 | -0.14 | PASS | 4.08 | PASS | 1049.2 | 49.86 | 4.99 | FAIL | 268 | PASS |
| 458 | 1199.1 | 1201.7 | 2.56 | 0.21 | PASS | 4.08 | PASS | 1239.6 | 40.53 | 3.38 | FAIL | 268 | PASS |
| 511 | 1399.0 | 1398.8 | -0.23 | -0.02 | PASS | 4.08 | PASS | 1416.7 | 17.74 | 1.27 | FAIL | 268 | PASS |
| 563 | 1598.9 | 1597.7 | -1.17 | -0.07 | PASS | 4.08 | PASS | 1590.5 | -8.39 | -0.53 | FAIL | 268 | PASS |
| 614 | 1798.7 | 1796.9 | -1.84 | -0.10 | PASS | 4.08 | PASS | 1760.9 | -37.77 | -2.10 | FAIL | 268 | PASS |
| 665 | 1998.6 | 1998.7 | 0.13 | 0.01 | PASS | 4.08 | PASS | 1931.4 | -67.24 | -3.36 | FAIL | 268 | PASS |
| 715 | 2198.4 | 2198.2 | -0.25 | -0.01 | PASS | 4.08 | PASS | 2098.4 | -99.96 | -4.55 | FAIL | 268 | PASS |
| 765 | 2398.3 | 2398.0 | -0.32 | -0.01 | PASS | 4.08 | PASS | 2265.5 | -132.78 | -5.54 | FAIL | 268 | PASS |
| 815 | 2598.1 | 2597.1 | -0.97 | -0.04 | PASS | 4.08 | PASS | 2432.6 | -165.49 | -6.37 | FAIL | 268 | PASS |
| 866 | 2798.0 | 2798.4 | 0.42 | 0.02 | PASS | 4.08 | PASS | 2603.0 | -194.97 | -6.97 | FAIL | 268 | PASS |
| 918 | 2997.9 | 3000.5 | 2.64 | 0.09 | PASS | 4.08 | PASS | 2776.8 | -221.10 | -7.38 | FAIL | 268 | PASS |
| 970 | 3197.7 | 3198.2 | 0.48 | 0.01 | PASS | 4.08 | PASS | 2950.6 | -247.14 | -7.73 | FAIL | 268 | PASS |
| 1024 | 3397.6 | 3397.2 | -0.43 | -0.01 | PASS | 4.08 | PASS | 3131.0 | -266.39 | -7.85 | FAIL | 268 | PASS |
| 1081 | 3597.4 | 3598.4 | 1.01 | 0.03 | PASS | 4.08 | PASS | 3321.5 | -275.91 | -7.67 | FAIL | 268 | PASS |
| 1141 | 3797.3 | 3798.0 | 0.66 | 0.02 | PASS | 4.08 | PASS | 3522.0 | -275.31 | -7.25 | FAIL | 268 | FAIL |
| 1206 | 3997.1 | 3996.2 | -0.87 | -0.02 | PASS | 4.08 | PASS | 3739.2 | -257.90 | -6.45 | FAIL | 268 | PASS |
| 1282 | 4197.0 | 4197.4 | 0.37 | 0.01 | PASS | 4.08 | PASS | 3993.2 | -203.84 | -4.86 | FAIL | 268 | PASS |
Table 7. Comparison with AMVA $n = 2$ and normal polynomial of degree 4.

| $h$ [mm] | Volume (V) $4^\circ C$ [L] | $V_{in}$ [L] | Error(E) [L] | E% | MPE | $U$ [L] | $|V_{in}-V|$<U [L] | $V_{out}$ [L] | Error(E) [L] | E% | MPE | $U$ [L] | $|V_{out}-V|$<U [L] |
|----------|-----------------|----------|------------|---|-----|------|----------|--------|--------|---|-----|------|----------|
| 114      | 199,9           | 200,2    | 0,06       | 0,13 | PASS | 5,44 | PASS | 197,5   | -2,44  | -1,22 | FAIL | 18  | PASS | |
| 206      | 399,7           | 399,3    | -0,42      | -0,11 | PASS | 5,44 | PASS | 403,4   | 3,65  | 0,91  | FAIL | 18  | PASS | |
| 278      | 599,6           | 598,3    | -1,33      | -0,22 | PASS | 5,44 | PASS | 600,1   | 0,51  | 0,09  | PASS | 18  | PASS | |
| 343      | 799,4           | 801,5    | 2,13       | 0,27 | PASS | 5,44 | PASS | 800,4   | 1,02  | 0,13  | PASS | 18  | PASS | |
| 401      | 999,3           | 997,7    | -1,58      | -0,16 | PASS | 5,44 | PASS | 994,5   | -4,78 | -0,48 | PASS | 18  | PASS | |
| 458      | 1199,1          | 1201,4   | 2,26       | 0,19 | PASS | 5,44 | PASS | 1197    | -2,05 | -0,17 | PASS | 18  | PASS | |
| 511      | 1399,0          | 1398,4   | -0,60      | -0,04 | PASS | 5,44 | PASS | 1394    | -5,03 | -0,36 | PASS | 18  | PASS | |
| 563      | 1598,9          | 1597,3   | -1,56      | -0,10 | PASS | 5,44 | PASS | 1594    | -5,37 | -0,34 | PASS | 18  | PASS | |
| 614      | 1798,7          | 1796,5   | -2,20      | -0,12 | PASS | 5,44 | PASS | 1794    | -4,84 | -0,27 | PASS | 18  | PASS | |
| 665      | 1998,6          | 1998,4   | -0,16      | -0,01 | PASS | 5,44 | PASS | 1997    | -1,31 | -0,07 | PASS | 18  | PASS | |
| 715      | 2198,4          | 2198,0   | -0,45      | -0,02 | PASS | 5,44 | PASS | 2198    | -0,05 | 0,00  | PASS | 18  | PASS | |
| 765      | 2398,3          | 2397,9   | -0,41      | -0,02 | PASS | 5,44 | PASS | 2400    | 1,39  | 0,06  | PASS | 18  | PASS | |
| 815      | 2598,1          | 2597,1   | -0,95      | -0,04 | PASS | 5,44 | PASS | 2600    | 1,93  | 0,07  | PASS | 18  | PASS | |
| 866      | 2798,0          | 2798,5   | 0,55       | 0,02  | PASS | 5,44 | PASS | 2802    | 4,01  | 0,14  | PASS | 18  | PASS | |
| 918      | 2997,9          | 3000,7   | 2,85       | 0,09  | PASS | 5,44 | PASS | 3004    | 6,23  | 0,21  | PASS | 18  | PASS | |
| 970      | 3197,7          | 3198,4   | 0,73       | 0,02  | PASS | 5,44 | PASS | 3201    | 3,35  | 0,10  | PASS | 18  | PASS | |
| 1024     | 3397,6          | 3397,4   | -0,18      | -0,01 | PASS | 5,44 | PASS | 3399    | 0,99  | 0,03  | PASS | 18  | PASS | |
| 1081     | 3597,4          | 3598,6   | 1,18       | 0,03  | PASS | 5,44 | PASS | 3598    | 0,42  | 0,01  | PASS | 18  | PASS | |
| 1141     | 3797,3          | 3798,0   | 0,69       | 0,02  | PASS | 5,44 | PASS | 3795    | -1,89 | -0,05 | PASS | 18  | PASS | |
| 1206     | 3997,1          | 3996,1   | -1,00      | -0,03 | PASS | 5,44 | PASS | 3993    | -3,85 | -0,10 | PASS | 18  | PASS | |
| 1282     | 4197,0          | 4197,4   | 0,36       | 0,01  | PASS | 5,44 | PASS | 4200    | 3,09  | 0,07  | PASS | 18  | PASS | |

5. Conclusions

- According tables 6 and 7 our approach model (AMVA) uncertainty and error have been reduced and fulfill Maximum Permissible Errors (MPE).
- Our polynomial adjustment proposal converges faster (see tables 6 and 7) than normal polynomial adjustment, which minimizes uncertainty due to interpolation and error.
- Our proposal requires 11 points to obtain satisfactory results, minimizing the measurement time.

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