An estimation of level sets for non local KPP equations with delay

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Abstract
We study the large time asymptotic behavior of the solutions of the linear parabolic equation with delay (*): 
\[ u_t(t,x) = u_{xx}(t,x) - u(t,x) + \int_{\mathbb{R}} k(x-y) u(t-h,y) \, dy, \quad x \in \mathbb{R}, \ t > 0, \text{ and } 0 \leq k(x) \in L^1(\mathbb{R}). \] 
As an application, we obtain estimates on the measure of level sets of non local KPP-type equations with delay. For this type of nonlinear equation we prove that, in contrast to the classical case, the solution to the initial value problem with data of compact support may not be persistent.

Keywords: reaction–diffusion equation, spreading speed, level sets

Mathematics Subject Classification numbers: 35K57, 35R10, 35B40, 92D25

1. Introduction

In this paper, we consider the following delayed partial differential equation
\[
\begin{align*}
    u_t(t,x) &= u_{xx}(t,x) + mu_x(t,x) + pu(t,x) \\
    &\quad + \int_{\mathbb{R}} k(x-y) u(t-h,y) \, dy
\end{align*}
\]
for \( t > 0, \ x \in \mathbb{R}, \text{ and } h > 0. \) Here, \( m, p \in \mathbb{R} \) and \( 0 \leq k(x) \in L^1(\mathbb{R}). \)

Although the study of the solutions of (1) is interesting in itself (see, for example, [2, 7, 9, 20]), it has special relevance in the description of the dynamics of non-local reaction diffusion equations. In particular, the solutions of (1) are relevant in the study of the following nonlinear, nonlocal, evolution equation with delay,
\[
v_t(t,x) = v_{xx}(t,x) - v(t,x) + \int_{\mathbb{R}} k_0(y)g(v(t-h,x-y)) \, dy,
\]
for $t > 0$, $x \in \mathbb{R}$, and $h > 0$. In (2), $g$ is a Lipschitz function which has exactly two fixed points: 0 and $\kappa$. This equation models the dynamics of a broad class of populations [11, 25, 30, 35, 36, 39, 41, 42]. Here, $g$ stands for the birth rate of a given population, while $v(t,x)$ is the measure of sexually mature adults in that population at point $x$ at time $t$. Important in what follows is the parameter $h$, which is the total time spent by an individual from birth until becoming a sexually mature adult. Finally, the kernel $k_0$ takes into account the (non)local interaction between individuals. Of course, when $h = 0$, $g = 2u - u^2$ and $k_0(y) = \delta(y)$, the equation (2) reduces to the classical model of Kolmogorov et al [18].

Equation (2) was introduced by So et al [30] to describe the behavior of populations with age structure (see also, [35, section 4.1] and [11, section 5]). There are many situations related to population dynamics where the age structure of the population matters. One of the most studied models is the one by Nicholson, introduced in 1954 to describe the competition for food in laboratory populations of blowflies *Lucilia caprina*. In the Nicholson model, $g(u) = uw^{-u}$ (see, for example, [6, 15, 25, 41, 42] and references therein).

As is well known in classical reaction diffusion equations without delay, when the reaction term satisfies the so called KPP condition (i.e. $g(u) \leq g'(0)u$), the evolution of the disturbances is characterized by the dynamics of the linearized equation. The same situation occurs in our case. In fact, the linear equation (1) plays a crucial role in establishing the asymptotic behavior of perturbations for the so called semi-wavefronts (see, for example, [11, theorem 18]). These are nonnegative bounded solutions of the form $v(t,x) = \phi_c(x + ct)$, $\phi_c : \mathbb{R} \rightarrow \mathbb{R}^+$, which propagate with speed $c$, and such that either $\phi_c(-\infty) = 0$ or $\phi_c(\infty) = 0$. Moreover, if in the first case one additionally has $\phi(+\infty) = \kappa$ or in the second case $\phi(-\infty) = \kappa$, the solutions are called wavefronts. In the context of the population dynamics, wavefronts model the invasion, at constant speed, of one species over a given habitat. In fact, under the KPP condition, the perturbation of semi-wavefronts of (2) are approximated by the solutions to (1) — see [6, 15, 23, 25, 32].

The study of the asymptotic propagation speeds up and the existence of monotone wavefronts for (2) when $k_0$ is not symmetric (i.e. when $k_0(-x) \neq k_0(x)$) dates back to Weinberger [38]. Weinberger’s result was later extended in a more abstract setting by Liang and Zhao [22]. Recently, Yi and Zou [41], using lower and upper monotone semi-flows, extended the results of Weinberger to the case of non monotone $g$’s. For the results on the existence of the wavefronts of (2) under the KPP condition see, for example, [41, theorem 4.4] and [36, theorem 5], while for semi-wavefronts, possibly non wavefronts, see [36, theorem 4] and [11, theorem 18]. From [11, theorem 18], the existence of two critical speeds $c^*_+ \neq c^*_+$ follows. These values for speed are critical in the sense that if $c \geq c^*_+$ (or $c \leq c^*_-$) then there is a semi-wavefront with speed $c$ such that $\phi_c(+\infty) = 0$ ($\phi_c(-\infty) = 0$, respectively) and if $c \in (c^*_-, c^*_+)$ then there are no semi-wavefronts with speed $c$. Semi-wavefronts with critical speed $c^*_+$ or $c^*_-$ are called critical semi-wavefronts and they have the following asymptotic behavior $\lim_{t \rightarrow \pm \infty} \phi_c(z + ct + t')z e^{-\lambda^+_z t} = -1$ for some $\lambda^+_z = \lambda^+_z(h)$ satisfying $\pm \lambda^+ > 0$ and $t' \in \mathbb{R}$ (see [1, theorem 3]).

In the context of semi-wavefronts, we take $m = c$, $p = -1$ and $k(x) = g'(0)k_0(x - mh)$ in (1). One of the goals of this paper is to give an optimal rate of convergence for the approximation to semi-wavefronts given by equation (1). More precisely, the stability of wavefronts of (2) implies an approximation of the semi-wavefronts by solving (2) as an inhomogenous linear partial differential equation (PDE) on the time intervals $[0, h], [h, 2h], \ldots$ for an appropriate initial data. In order to carry out this iterative scheme, the knowledge of the asymptotic behavior of the solutions of (1) is crucial.
In this respect, recent investigations have mainly obtained results on the stability of wavefronts for small perturbations [6] to (2) and for global perturbations [31] when \( k_0 = \delta \). It is worth pointing out that by using this method, one can approximate non monotone wavefronts uniformly on the real line. Moreover, one can also asymptotically approximate periodic semi-wavefronts uniformly on any compact set of the real line (see, [31, theorem 3]). For \( g \) monotone and \( k_0 \) a heat kernel in (2), the authors in [25] proved that the approximation is \( O(t^{-1/2}) \) for critical wavefronts and \( O(e^{-\epsilon t}) \), some \( \epsilon > 0 \), for non-critical wavefronts. For more general \( g \) but \( k_0 = \delta \), numerical solutions are exhibited in [6] without providing an appropriate rate of convergence to the wavefront. Then, in [32] for \( g \) (not necessarily monotone) and \( k_0 \) general, the rate of convergence \( O(t^{-1/2}) \) is obtained for certain ‘hard perturbations’, and for ‘soft perturbations’, this rate can be exponentially improved. In our main result in section 3, we prove that the convergence rate must be \( o(t^{-1/2}) \) for critical semi-wavefronts.

In order to describe our main results concerning the solutions of (1), embodied in theorem 1.1 below, we need some preliminary definitions. Consider the functions,

\[
q_1(z) = -z^2 - mz - p \quad \text{and} \quad q_2(z) = \int k(y)e^{-\gamma y}dy,
\]

for which we assume the following hypothesis.

**K** The function \( q_2 \) is defined on a maximal open interval \((a, b) \supseteq 0\) for some real extended numbers \(a\) and \(b\), and

\[
0 < -p < \int_R k(y)dy.
\]

As is shown in [11, lemma 22], the condition (K) and the convexity of the function

\[
E_m(z) := -q_1(z) + q_2(z) = z^2 + mz + p + \tilde{q} e^{2mh} \int_R \tilde{k}(y)e^{-\gamma y}dy,
\]

where \( \tilde{q} := ||k||_{L^1(R)} \) and \( \tilde{k}(y) := k(y - mh)/\tilde{q} \) (following the structure of [11, lemma 22]), imply that there exist numbers \( m_- < m_+ \) such that for every \( m \in \mathcal{M} := (-\infty, m_-) \cup [m_+, +\infty) \) the functions \( q_1 \) and \( q_2 \) either have exactly two crossings or exactly one and if \( m \in (m_-, m_+) \) then \( q_1 \) and \( q_2 \) have no crossings.

Now, in order to state our main result, we need the following tangential property.

**T** For each \( m \in \mathcal{M} \) there is a number \( \gamma_m \) in such a way that \( q_1 - \gamma_m \) and \( e^{\gamma_m}q_2 \) are tangent at a point \( z = z_m \in (a, b) \).

Note that under the hypothesis (T), the functions \( q_1 \) and \( q_2 \) have at least one crossing, and therefore necessarily \( \gamma_m \geq 0 \). Also, we note that for each \( z \in (a, b) \), the function \( E^*(\epsilon) = -q_1(z) + \gamma_m - \epsilon + q_2(z)e^{(\gamma_m + \epsilon)h} \) is monotone so that the numbers \( \gamma_m \) and \( z_m \) can be computed as the only solution \((\gamma, z) = (\gamma_m, z_m)\) to the tangential equations

\[
\begin{align*}
\gamma^2 + mz + p + \gamma &= -e^{\gamma z} \int k(y)e^{-\gamma y}dy \\
2z + m &= e^{\gamma z} \int yk(y)e^{-\gamma y}dy.
\end{align*}
\]

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The usual assumptions on $k$ in the literature (see, for example, [16, 17, 36, 39, 42]) are that either

$$\int_{\mathbb{R}} e^{\lambda y} k(y) dy \quad \text{exists for all} \quad \lambda \in \mathbb{R}$$

or if $a$ (or $b$) is a real number then $q_2$ is defined for all $\lambda \in (a, b)$ and

$$\lim_{\lambda \to a^+} \int_{\mathbb{R}} e^{\lambda y} k(y) dy \quad \text{or} \quad \lim_{\lambda \to b^-} \int_{\mathbb{R}} e^{\lambda y} k(y) dy = \infty,$$

and, in both cases, the convex function $E_m(z)$ tend to $+\infty$ at $z = a$ and $z = b$ so that in these cases the assumption (T) is satisfied.

Moreover, in the context of semi-wavefronts, if $m_\pm = c_\pm^\pm$, then the minimality of $m_\pm$ for the existence of crossings between $q_1$ and $q_2$ [11, theorem 18] implies the condition (T) with $\gamma_m = 0$ and $z_m = \lambda_\pm^\pm$, so that, for critical semi-wavefronts (K) implies (T).

Finally, we introduce the function $k_\gamma(x) = k(x)e^{-\gamma x}$, $\gamma \in \mathbb{R}$, the number $k_\gamma^\pm := \int_{\mathbb{R}} y^2 k_\gamma(y) dy$ (6) and the Fourier transform of $u$ as

$$\hat{u}(z) = \int_{\mathbb{R}} e^{-izx} u(x) dx.$$

With all this notation we state our main result.

**Theorem 1.1.** Assume (K) and (T). If the initial data $u_0(s, \cdot) = u_0(\cdot)$, $s \in [-h, 0]$, to (1) is such that

$$\int_{\mathbb{R}} e^{-z_0 y} |u_0(y)| dy < \infty,$$

then for each $m \in \mathcal{M}$

$$\lim_{t \to \infty} \sqrt{t} e^{z_0 t} u(t, a(t, x)) = \frac{1}{2 \sqrt{\pi} \sigma_m} \int_{\mathbb{R}} u_0(y) dy \ e^{\sigma_m x} \quad \text{for all} \quad x \in \mathbb{R},$$

where

$$a(t, x) = o(\sqrt{t}) \quad \text{and} \quad \sigma_m = \frac{2 + k_\gamma^\pm e^{\gamma_m h}}{2(1 + he^{\gamma_m h} k_\gamma(0))}.$$

Moreover, without assuming (T), if for some $z_0 \in \mathbb{R}$ the initial data $u_0$ satisfies

$$\int_{\mathbb{R}} e^{-z_0 y} |u_0(y)| dy < \infty,$$

then $C > 0$ exists such that

$$|u(t, x)| \leq Ct^{-1/2} e^{-z_0 t} \ e^{\gamma_0 x} \sup_{x \in [-h, 0]} ||e^{-z_0(\cdot)} u_0(s, \cdot)||_{L^1(\mathbb{R})} \quad \text{for all} \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}$$

(9)
where $\gamma_0$ is defined as the only real solution to

$$-\gamma_0 + q_1(\gamma_0) = \hat{k}_0(0)e^{2\beta h}.$$  \hfill (10)

Therefore, for equations of type (1) associated with the stability of semi-wavefronts, i.e. for the equation

$$u_t(t,x) = \lambda u_x(t,x) - q u(t,x) + [k_0 * u(t-h,\cdot - mh)](x),$$  \hfill (11)

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, we have the following result on the optimality of the convergence rates where the speed set $\mathcal{M}_0$ is determined by $q_1(z) = -z^2 + mz + 1$ and $q_2 = g'(0)e^{-mh} \int_{\mathbb{R}} k_0(y)e^{-\gamma_0 y} dy$ as the numbers $m \in \mathcal{M}_0$ such that the curves $q_1$ and $q_2$ have at least one intersection.

**Corollary 1.2.** Assume (K) and (T). Let $m \in \mathcal{M}_0$ and the kernel $k_0$ be such that

$$k_m^*: = \int_{\mathbb{R}} y^2 k_0(y - mh)e^{-\gamma_0 y} dy < \infty.$$

If the initial data $u_0$ for (11) is such that $e^{-\gamma_0} u_0($) \in C([-h,0],L^1(\mathbb{R}))$ then,

$$\lim_{t \to \infty} \sqrt t e^{\gamma_0 t} u(t,x + a(t,x)) = e^{\gamma_0 x} \frac{1}{2\sqrt \pi \sigma_m} \int_{\mathbb{R}} u_0(y)dy \text{ for all } x \in \mathbb{R},$$  \hfill (12)

where $a(t,x) = o(\sqrt t)$ and $\sigma_m$ is given by (8).

It is instructive to compare our method with the approach used by Huang et al in [15, section 3] where the authors consider the stability of monotone wavefronts in non-local dispersive equations on the $n$-dimensional space where $k_0$ is a heat kernel. In [15], the estimate for the Fourier transform, in the $L^2$-sense, of the solutions is obtained by using the so-called delayed exponential function (see [19]), and the optimality of the convergence rates is stated with respect to the estimate function obtained in [15, proposition 1], i.e. the convergence rate in [15] for equation (1) is $O(t^{-n/2}e^{-\gamma_1 t})$, some $\gamma_1 > 0$, and $q_1(z_m) = q_2(z_m)$ implies $\gamma_1 = 0$. Our approach, which is developed for one-dimensional space, shows that in a suitable space, the rate of convergence is actually $o(t^{-1/2}e^{-\gamma_0 t})$ and $\gamma_0 = 0$ when $q_1(z_m) = q_2(z_m)$. Moreover, for the $n$-dimensional case we can also give the universal estimate $O(t^{-n/2}e^{-\gamma_0 t})$ (see remark 2.7) and it is interesting to see the $n$-dependence of our universal estimate function (for the optimality of this function, see remark 2.3) because the $L^1$-norm of this universal function turns out to be asymptotic to the Gamma function. More precisely, our approach strongly depends on the integrability, as $t \to +\infty$, of the function

$$E_n^h(t) := \frac{t^{n-1}}{1 + \frac{t^2}{2h^2} t^{1/h}},$$

and clearly $\lim_{t \to +\infty} ||E_n^h(t,\cdot)||_{L^1(\mathbb{R}_+)} = \Gamma(n/2)/2$ for all $h > 0$.

An important application that can be derived from the asymptotic behavior of the solutions of (1) is that we can prove a lower bound on the measure of the level sets of the solution $u(t,\cdot)$, for every $t$, of (2) when the initial data decays sufficiently fast. Known results on the stability of wavefronts (see, for example, [6, 15, 23, 25, 32]) provide information on the speed of propagation for initial data which are asymptotic to a semi-wavefront, i.e. for initial data that behave like $z^2 e^{\lambda z}$, $j = 1, 2$ for some $\lambda \in \mathbb{R}$. However, as far as we know, there are no studies in the case $h > 0$ that provide results on the asymptotic behavior of the solutions of (2) with
an initial data that decays faster than exponential. In this paper, we will provide information on the level sets of the solutions of (2) in that case.

In the local case without delay (i.e. when \( k_0 = \delta \) and \( h = 0 \)), after the seminal work of Kolmogorov et al [18], it was McKean [24], using probabilistic methods, who established an estimate depending on the logarithm of \( t \) for the distance between the level sets of wavefronts with minimal speed and the solutions of (2) with a Heaviside initial data. McKean’s estimate reads,

\[
D(t) := m(t) + ct \leq \frac{1}{2\lambda_*} \log(t) + B,
\]

where \( m(t) \) is such that \( u(t, m(t)) = 1/2 \) (here \( u(t, x) \) is the solution generated by the Heaviside initial datum), \( \lambda_* := \lambda_*^+(0) \) and \( B \in \mathbb{R} \). Later, Uchiyama [37, proposition 9.1], for the same equation, proved the same estimate as an immediate consequence of the fundamental solution of the heat equation, and, by using arguments from PDE, he obtained \( D(t) := \frac{3}{2\lambda_*} \log(t) + O(\log \log(t)) \). A more precise estimate, i.e. \( D(t) = \frac{3}{2\lambda_*} \log(t) + O(1) \) was obtained by Bramson [5] using a probabilistic approach. Then, Ebert and van Saarloos [8] gave, heuristically, a more refined representation for \( D(t) \), namely, \( D(t) = \frac{3}{2\lambda_*} \log(t) + x_0 - 3\sqrt{\pi}/\sqrt{t} + O(1/t) \). The heuristic result of Ebert and van Saarloos was confirmed, by probabilistic arguments, in a recent work by Nolen et al [27].

Noteworthily, Hamel et al [13] showed that \( D(t) = \frac{3}{2\lambda_*} \log(t) + O(1) \) by only using PDE arguments. Furthermore, in a recent work by Bouin et al [4], for a certain non-local version of the KPP equation, the authors show that, depending on \( k_0 \), the correction term can be \( O(t^{\beta}) \) for some \( \beta \in (0, 1) \). Due to the similarity given in (7) between the asymptotics for \( h = 0 \) and \( h > 0 \), we believe that, for local equations, the term \( \frac{3}{2\lambda_*} \log(t) \) persists when \( h > 0 \). Therefore we propose the following.

**Open problem:** prove that when \( k_0 = \delta \) in (2)

\[
D(t) = \frac{3}{2\lambda_*^+(h)} \log(t) + O(1),
\]

for all \( h > 0 \).

However, the main difficulty in the case with delay is how to obtain an explicit expression for the fundamental solution of (1). In spite of the fact that there are abstract expressions for the fundamental solutions of (1) (see, for example, [26] and [40, section 4.5]), in this case, their derivation seems quite intricate. In fact, Nakagiri in [26, theorem 4.1] gives an expression for the fundamental solution of (1) in terms of the fundamental solution of the heat equation but defined piecewise on the successive intervals \([0, h], [2h, 3h]\).... Following the theory of fundamental solutions in [10, chapter 1], we make the following definition.

**Definition 1.3.** A fundamental solution for (1) is a function \( \Gamma_h(t, x) \) defined for all \( x \in \mathbb{R} \) and \( t > 0 \) which satisfies the following conditions

(i) \( \Gamma_h(t, x) \) as a function of \( (t, x) \) satisfies (1) for each \( y \in \mathbb{R} \),

(ii) if \( \psi \in C([0, h], L^1(\mathbb{R})) \) then

\[
\lim_{t \to 0} \int_{\mathbb{R}} \Gamma_h(t, x - y) \psi(s, y) dy = \psi(s, x) \quad \text{for all} \quad (s, x) \in [-h, 0] \times \mathbb{R}.
\]
On the other hand, if we use Fourier transforms, the natural fundamental solution to (1) is the function

$$\Gamma_h(t, x) := \int_{\mathbb{R}} e^{i\nu e^{\lambda(t)r}} dy,$$

where $\lambda : \mathbb{R} \to \mathbb{C}$ is a function satisfying

$$\lambda(z) = -z^2 + imz + p + \hat{k}(z)e^{-h\lambda(t)}.$$  

(14)

The immediate problem is how to globally define a function $\lambda$ implicitly through (14) which, for each $z \in \mathbb{R}$ has infinitely many solutions. Even though, for our main result, we only need to define $\lambda$ locally in a neighborhood of 0, we discuss the situation in general, which is of independent interest. In the case when $\hat{k}(\cdot - mh)$ is positive (this condition is made, for example, in [3]), then, $\lambda(z) = \rho(z) + imz$ where $\rho : \mathbb{R} \to \mathbb{R}$ is the unique solution of

$$\rho(z) = -z^2 + p + \hat{k}(z - mh)e^{-h\rho(z)}.$$  

(15)

The main difficulty of this technique has to do with the integrability of the function $e^{\lambda(t)r}$. For example, in the local case, i.e. when formally $\hat{k}$ is equal to a positive constant $q$, by taking $m = 0$ and $p = -q$ in (1), the function $e^{\lambda(t)r}$ is no longer a Gaussian when $h > 0$, since $e^{\lambda(z)r} \sim (q/z^2)^t$ for $z \to \pm \infty$ for all $t > 0$ (see remark 2.3). Thus, even though $\Gamma_h$ is formally a fundamental solution to (1), by definition of $\lambda$ in (14), with $h > 0$, we cannot define $\Gamma_h(t, \cdot)$ for $t \in [0, h/2]$. This is the main difference between cases $h = 0$ and $h > 0$.

In the non local case, the situation can be different since a suitable convergence of $\hat{k}$ at $\pm \infty$ can be used to ensure the necessary integrability of $e^{\lambda(t)r}$ (see remark 2.3). In fact, if $\hat{k}(\cdot + mh)$ is positive and $\hat{k}(z) = O(e^{-h^2})$ then

$$\Gamma_h(t, x) = \int_{\mathbb{R}} e^{i(x+mt)r} e^{(\rho(y)+ym)t} dy,$$

with $\rho(y)$ given by (15), is actually a fundamental solution to (1) (see corollary 2.4 below).

The rest of the paper is organized as follows: in section 2 we prove theorem 1.1 and, in section 3, we apply this theorem to estimate the level sets of the evolution of the solutions of nonlocal nonlinear reaction–diffusion equations with delay.

2. Proof of theorem 1.1

We begin this section by proving a Halanay [12] type result.

**Lemma 2.1 (Halanay).** Let $\mu, k \in \mathbb{C}$ and let $X$ be a complex Banach space. If for all $h > 0$, $r \in C([\cdot - h, \infty), X)$ is a function satisfying

$$r_s(t) = \mu r(t) + kr(t-h),$$

almost everywhere, then,

$$|r(t)| \leq \left(\sup_{s \in [-h,0]} |r(s)|\right) e^{\min\{0,-\tau h\} \tau t},$$

(16)

for all $t > -h$. Here, $\tau$ is the only real root of the equation

$$\tau = \text{Re}(\mu) + |k| e^{-\tau h}.$$  

(17)
Moreover, we have that:

(i) \( \tau \leq 0 \) if and only if \( - \text{Re}(\mu) \geq |k| \), and

(ii) \( \tau = 0 \) if and only if \( - \text{Re}(\mu) = |k| \).

**Proof.** It is clear that

\[
\frac{d}{dt}(r(t)e^{-\mu t}) = ke^{-\mu t}r(t - h)
\]

almost everywhere, which implies

\[
x(t) = k \int_0^t e^{\mu(t-s)}x(s - h)ds + x(0),
\]

and from here we have that \( |r(t)| \) satisfies the following inequality

\[
x(t) \leq |k| \int_0^t e^{\text{Re}(\mu)(t-s)}x(s - h)ds + x(0),
\]

for all \( t > 0 \). Now notice that due to (17), for \( A \in \mathbb{R} \) the function \( e_A(t) = Ae^{\tau t} \) satisfies the equation \( \frac{d}{dt}(e_A(t)e^{-\text{Re}(\mu)t}) = |k|e_A(t - h)e^{-\text{Re}(\mu)t} \), and thus, \( e_A(t) \) satisfies (19) with equality. Hence, the function \( \delta(t) = |r(t)| - e_A \), with \( A = \sup_{s \in [-h, 0]} |r(s)|e^{\text{Re}(0, -\tau)h} \), for \( t \in [0, h] \) satisfies (19) and, thus, \( \delta(t) \leq 0 \) for all \( t \in [0, h] \). In an analogous way, we conclude that \( \delta(t) \leq 0 \) on the intervals \([h, 2h], [2h, 3h]... \) This proves (16).

Let us now prove (i). If \(-\text{Re}(\mu) \geq |k| \) then \( \tau \leq |k|(e^{-\tau h} - 1) \) which in turn implies that \( \tau \leq 0 \). On the other hand, if \( \tau \leq 0 \), we assume \(-\text{Re}(\mu) < |k| \). Hence, \( \tau > |k|(e^{-\tau h} - 1) \) which is a contradiction.

In order to prove (ii), notice that the derivative of \( a(\tau) := \tau - \text{Re}(\mu) - |k|e^{-\tau h} \) is always positive, hence \( a(\tau) \) has at most one zero. If \( \text{Re}(\mu) = -|k| \) then \( \tau = 0 \) is the unique zero. If \( \tau = 0 \), from (17) we conclude that \( a = b \).

For some \( z_0 \), let us define the function \( l : \mathbb{R} \rightarrow \mathbb{R} \) through the following equation:

\[
l(z) = -z^2 + \gamma_0 - q_1(z_0) + e^{\nu_0} \hat{k}_{z_0}(z) |e^{-h(z)}| \quad \text{for } z \in \mathbb{R}. \tag{20}
\]

Recalling that \( \hat{k}_{z}(x) = e^{-\alpha x}k(x) \) and that \( \gamma_0 \) is defined implicitly by

\[
q_1(z_0) - \gamma_0 = |\hat{k}_{z_0}(0)|e^{\nu_0} \tag{21}
\]

we can estimate the function \( l(z) \) in (20).

For \( \epsilon_h(z) := [1 + h|\hat{k}_{z_0}(z)|e^{\nu_0}]^{-1} \) we define the function

\[
\alpha_h(z) := -\frac{1}{h} \log(1 + \epsilon_h(z)z^2).
\]

Then we have:

**Lemma 2.2.** The function \( l(\zeta) \) satisfies the following estimate,

\[
-\epsilon_h(z) z^2 + e^{\nu_0} |\hat{k}_{z_0}(z)| - \hat{k}_{z_0}(0) | \leq l(z) \leq \alpha_h(z), \tag{22}
\]

for all \( z \in \mathbb{R} \) and
\[
\lim_{z \to \pm \infty} z^2 e^{h(z)} = 0. \tag{23}
\]

Moreover, if
\[
|\hat{k}(z)| e^{cz} \leq C, \tag{24}
\]
for some \(C > 0\) then
\[
l(z)/z^2 + 1 = O(z^{-2}). \tag{25}
\]

**Remark 2.3.** Note that for the local equation (i.e. when \(\hat{k} = q\), for some positive constant \(q\)), if we put \(m = 0\) and \(p = -q\) then the equation (1) is reduced to
\[
u_0(t, z) = u_\alpha(t, x) - q u(t, x) + q u(t - h, x),
\]
and the upper bound in (22) is sharp when \(h = 0\), i.e. \(\alpha_h(z) \sim -z^2\) as \(h \to 0\). However, when \(h > 0\), the asymptotic behavior for \(\alpha_h\) is different since applying the definitions of \(\gamma_0\) and \(z_0\) in (21) we have \(\gamma_0 = z_0 = 0\), then multiplying (20) by \(e^{h|z|}\) we get
\[
e^{h|z|} l(s) = -z^2 e^{h|z|} - q e^{h|z|} + q,
\]
while the upper estimate in (22) implies \(\lim_{|z| \to \infty} l(z) = -\infty\). So that, from (26), for each \(t > 0\) we obtain \(e^{l(z)} \sim (q/z^2)^t/h\) for \(z \to \pm \infty\).

For the non local case, by (24), if \(\hat{k} = O(e^{-hc^2})\) then by (25) we have \(e^{l(z)} \in L^1(\mathbb{R})\) for all \(t > 0\) and therefore we obtain the following result.

**Corollary 2.4 (Fundamental solutions).** If the kernel \(k\) satisfies \(\hat{k}(z - mh) \in \mathbb{R}_+\) for all \(z \in \mathbb{R}\) and \(\hat{k}(z) = O(e^{-hc^2})\) then \(\Gamma_h(t, x)\) defined by (13), with \(\lambda(z) = \rho(z) + \text{inz}\) and \(\rho\) given by (15), is a fundamental solution for (1).

**Proof.** If in (20) we take \(z_0 = 0\) we have \(-q_1(z_0) = p\) and \([\hat{k}_\rho(\cdot)] = [\hat{k}(\cdot)] = [e^{\lambda(z)}\hat{k}(\cdot)] = [\hat{k}(\cdot - mh)]\), therefore \(l = \rho\) and \(\gamma_0\) are given by (21), i.e. \(-p - \gamma_0 = [\hat{k}(0)]e^{\lambda(0)}\). Thus, by (25) we have \([e^{\lambda(z)}] = e^{\lambda(z)} \sim e^{-z^2}\) at \(z = \pm \infty\) for all \(t > 0\), therefore \(e^{l(z)} \in L^1(\mathbb{R})\) for all \(t > 0\), so that \(\Gamma_h\) is well defined for all \(t > 0\) and \(\Gamma_h(t, \cdot) \in L^\infty(\mathbb{R})\) for each \(t > 0\). Moreover, by the definition (14) of \(\lambda\) then \(\Gamma_h\) solves (1) for all \(t > 0\), therefore condition (i) in definition 1.3 is immediately satisfied. Otherwise, \(\Gamma_h(t, \cdot) u_0(s, x - \cdot) \in L^1(\mathbb{R})\) for all \((t, x) \in \mathbb{R}_+ \times \mathbb{R}\) and evaluating in \(x = 0\) we get
\[
\Gamma_h(t, \cdot) * u_0(s, \cdot)(0) = \int_{\mathbb{R}} \Gamma_0(t, y) u_0(s, -y) dy + \int_{\mathbb{R}} \Theta(t, y) u_0(s, -y) dy, \tag{27}
\]
where \(\Gamma_0(t, y) = \frac{e^{yt}}{2\sqrt{\pi t}} e^{-y^2/4t}\) and \(\Theta(t, y) = \Gamma(t, y) - \Gamma_0(t, y)\). Next, by the definition of \(\Gamma_h(t, y)\), for all \(t > 0\) we have
\[
\Theta(t, y) = e^{yt} \int_{\mathbb{R}} e^{it\tau} [e^{\rho(\tau)} - e^{-\tau^2}] d\tau
\]

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and using (25) there exists $C' > 0$ such that $|\tau^2 + \rho(\tau)| \leq C'$ for all $\tau \in \mathbb{R}$, therefore
\[
|\Theta(r, y)| \leq e^{rt} \int_{\mathbb{R}} e^{-\tau^2} |1 - e^{-(\rho(\tau)+\tau^2)}| d\tau \leq e^{C'(1 - e^{-C'})} e^{rt} \int_{\mathbb{R}} e^{-\tau^2} d\tau.
\]

However,
\[
(1 - e^{-C'}) \int_{\mathbb{R}} e^{-\tau^2} d\tau \leq \frac{1 - e^{C'}}{\sqrt{\pi}} \to 0,
\]
as $t \to 0$. Thus, by passing to the limit $t \to 0$ in (27) we obtain condition (ii) in definition 1.3 for $x = 0$. Finally, for each $x' \in \mathbb{R} \setminus \{0\}$ take the initial datum $w_0(s, x) := u_0(s, x + x')$ and note that $w(t, x) = u(t, x + x')$ for all $t > -h$ and $x \in \mathbb{R}$ which completes the proof.

**Proof of lemma 2.2.** Let us denote $Q(z) = |\dot{k}_0(z)e^{\gamma_0 h}|$, $P = \gamma_0 - q_1(\alpha_0)$, and $Q_0 = |\dot{k}_0(0)e^{\beta_0 h}|$. If we define $\beta(z) := -\alpha_h(z) + l(z)$ then
\[
\beta(z) = \frac{1}{h} \log(1 + h\epsilon_h(z)z^2) - z^2 + P + Q(z)(1 + h\epsilon_h(z)z^2)e^{-h\beta(z)}.
\]
From lemma 2.1 we have that $\beta(z) \leq 0$ if and only if:
\[
z^2 - \frac{1}{h} \log(1 + h\epsilon_h(z)z^2) - P \geq Q(z)(1 + h\epsilon_h(z)z^2). \tag{28}
\]
Now, using $\log(1 + x) \leq x$, for all $x \geq 0$, in order to obtain (28) it is enough to have
\[
z^2 - \epsilon_h(z)z^2 - P \geq Q_0 + hQ(z)\epsilon_h(z)z^2 \quad \text{for all} \quad z \in \mathbb{R}
\implies [1 - \epsilon_h(z) - Q(z)h\epsilon_h(z)]z^2 - P - Q_0 \geq 0 \quad \text{for all} \quad z \in \mathbb{R}
\implies -P \geq Q_0.
\]

Thus, as $P = -Q_0$ then $\beta(z) \leq 0$ for all $z \in \mathbb{R}$ which implies the right-hand side of (22). Otherwise, using (20) and the right-hand side of (22)
\[
l(z) \geq -z^2 - Q_0 + Q(z)(1 + h\epsilon_h(z)z^2) = -\epsilon_h(z)z^2 + Q(z) - Q_0.
\]
This proves the left-hand side of (22).

Finally, note that if $r_j$ is the only real solution for
\[
r_j = P + Q_j e^{-hr_j} \quad j = 1, 2 \tag{29}
\]
with $0 \leq Q_1 \leq Q_2$ then $r_1 \leq r_2$. Therefore, as $r(z) = l(z) + z^2$ satisfies (29) with $Q_j = Q(z)e^{h^2}$ we conclude that $P \leq r(z) \leq r_C$ for all $z \in \mathbb{R}$ where $r_C$ is the solution of (29) with $Q_j = C$, which implies (25).

**Remark 2.5.** Similarly, by taking $\beta(z) = l(z) - \alpha_h(z)$ but this time with $l, \alpha_h : \mathbb{R}^n \to \mathbb{R}$ defined by
\[
l(z) = -|z|^2 + P + Q(z)e^{-hl(z)}.
\]
and
\[ \alpha_h(z) := -\frac{1}{h} \log(1 + h\epsilon_h|z|^2) \]

it is an exercise completely analogous to show that the upper bound in (22) holds, so that
\[ l(z) \leq \alpha_h^*(z) := -\frac{1}{h} \log(1 + h\epsilon_h|z|^2), \tag{30} \]

where the number \( \epsilon_h := 1/[1 + hQ_0e^{-\gamma h}] \).

Let us consider the following characteristic equation,
\[ L(s) = -s^2 + i(2z_m + m)s - q_1(z_m) + k_{ca}(s)e^{-hL(s)}. \tag{31} \]

Since the linear transformation \( D : \mathbb{C} \to \mathbb{C} \), defined by \( D(s) = (1 + h\dot{k}_{ca}(0)e^{\gamma h})s \), is invertible, then there is a unique analytic function \( L : B_{\delta_0}(0) \subset \mathbb{C} \to \mathbb{C} \), with \( \delta_0 > 0 \), such that \( L(0) \in \mathbb{R} \). As \( L(0) \) satisfies (4) then \( L(0) = -\gamma_m \).

Then we have the following result.

**Lemma 2.6.** If we assume (K) is satisfied, then we have
\[ \lim_{s \to 0} (L(s) + \gamma_m)/s^2 = -\frac{k_m^*e^{\gamma h} + 2}{2(1 + he^{\gamma h}k_{ca}(0))} = -\sigma_m. \tag{32} \]

**Proof.** It follows from (31) that the tangency of the curves \( q_1 - \gamma_m \neq q_2e^{\gamma h} \) (by an appropriate choice of \( \gamma_m \) as we discussed above) and the hypothesis (K)
\[
\lim_{s \to 0} \frac{L(s) + \gamma_m}{s^2} = \frac{1}{2} \lim_{s \to 0} \frac{L'(s)}{s} = -\frac{1}{1 + he^{\gamma h}k_{ca}(0)} \left[ -1 + \frac{1}{2} \lim_{s \to 0} \frac{(2z_m + m)i + \dot{k}_{ca}(s)e^{-hL(s)}}{s} \right].
\]

However, \( \dot{k}_{ca}(s) = -i \int_\mathbb{R} ye^{-iyk_{ca}(y)}dy \) therefore (5) implies \( k_{ca}'(0) = -i(2z_m + m) \) so that
\[
\lim_{s \to 0} \frac{L(s) + \gamma_m}{s^2} = \frac{1}{1 + he^{\gamma h}k_{ca}(0)} \left[ -1 + \frac{1}{2} \lim_{s \to 0} (\dot{k}_{ca}'(s) - hL'(s)k_{ca}'(s)e^{-hL(s)}) \right] = -\frac{k_m^*e^{\gamma h} + 2}{2(1 + he^{\gamma h}k_{ca}(0))}.
\]

With all the previous results we are ready to prove the main result of this section.

**Proof of theorem 1.1.** Making the change of variable \( v(t, x) = e^{-z_m t}u(t, x) \) in (1) and \( v_0(s, x) = e^{-z_m s}v_0(s, x) \), and using, as before, that \( k_{ca}(x) = \hat{k}(x) \exp(-z_m x) \), we see that \( v \) satisfies,
\[ v(t, x) = v_{ss}(t, x) + (2z_m + m)v_s(t, x) - q_1(z_m)v(t, x) + k_{ca}(\cdot) \ast v(t - h, \cdot)(x). \tag{33} \]

Making the further change of variables \( v \to \alpha \) given by \( \alpha(t, x) := \exp(q_1(z_m)t)v(t, x - (2z_m + m)t) \) we see from (33) that \( \alpha(t, x) \) satisfies,
\[ \alpha_t(t, x) = \alpha_{\epsilon t}(t, x) + f(t, x) \]  
(34)

for \((t, x) \in [0, h] \times \mathbb{R}\). In (34), the second term of the right, given by

\[ f(t, x) := \int_{\mathbb{R}} e^{-q(t, x) h} k_m(x - y) \alpha(t - h, y - (2z_m + m) h) \, dy, \]

is the Lebesgue integrable for all \(t \in [0, h]\). Since the fundamental solution of (34), given by \(A(t, x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\), satisfies \(A(t, \cdot) \in C^\infty(\mathbb{R}) \cap W^{2,1}(\mathbb{R})\) for all \(t > 0\), using Duhamel’s formula we conclude that \(v(t, \cdot) \in C^2(\mathbb{R}) \cap L^1(\mathbb{R})\) for all \(t \in [0, h]\). Iterating this procedure successively on the intervals \([h, 2h], [2h, 3h], \ldots\) we conclude that \(v(t, \cdot) \in C^2(\mathbb{R}) \cap L^1(\mathbb{R})\) for all \(t > 0\). For convenience we introduce,

\[ A = \int_{\mathbb{R}} v_0(y) \, dy. \]

Given the smoothness of \(v(t, x)\) as a function of \(x\) for \(t > 0\), the Fourier inversion theorem implies that

\[ \sqrt{t} v(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega/\sqrt{t}} \hat{v}(t, y/\sqrt{t}) \, dy \]

(35)

for \(t > 0\) and \(x \in \mathbb{R}\). Moreover, the function \(e^{\omega t} \hat{v}(t, z)\) satisfies the equation

\[
\begin{align*}
\omega t \hat{v}(t, z) &= (-\omega^2 + i(2z_m + m)z + \gamma_m - q_1(z_m)) \hat{v}(t, z) \\
&+ \hat{k}_m(z) e^{\gamma_h t} \hat{w}(t - h, z)
\end{align*}
\]

(36)

for \(t > 0\) and all \(z \in \mathbb{R}\).

Next, we define \(I : \mathbb{R} \rightarrow \mathbb{R}\), implicitly, by the equation

\[ I(z) = -z^2 + \gamma_m - q_1(z_m) + |\hat{k}_m(z)| e^{\gamma_h t} e^{-I(z) h}. \]

(37)

Note that the pair \((\gamma_m, z_m)\) satisfies (21) and therefore by lemma 2.2 we have that \(I(z) \leq 0\) for all \(z \in \mathbb{R}\). Hence, by applying lemma 2.1 to (36) (with \(r = I(z)\)), we have,

\[ |e^{\omega t} \hat{v}(t, y/\sqrt{t})| \leq \sup_{t \in [-h, 0]} |e^{\omega t} \hat{v}(s, y/\sqrt{t})| e^{I(y/\sqrt{t})}. \]

(38)

Using (22), and the fact that \((1+x)^r \geq 1 + rx\), for all \(-1 \leq x < \infty\) and \(1 \leq r < \infty\) we have,

\[ |e^{\omega t} \hat{v}(t, y/\sqrt{t})| \leq \sup_{t \in [-h, 0]} \left| e^{\omega t} v(s, \cdot) \right|_{L^1(\mathbb{R})} \frac{1}{\left[ 1 + h e_h(y/\sqrt{t}) \right]^r} \]

(39)

\[ \leq \sup_{t \in [-h, 0]} \left| e^{\omega t} v(s, \cdot) \right|_{L^1(\mathbb{R})} \frac{1}{1 + e_h(y/\sqrt{t})^2}, \]

(40)

for all \(y \in \mathbb{R}\) and all \(t > h\).

Here, note that the last inequality (40) was obtained without assuming \((T)\) for the pair \((\gamma_m, z_m)\). In fact, we only use the fact that this pair satisfies (21) and lemma 2.1. Therefore (40) also holds for the pair \((\gamma_0, z_0)\) defined in (9), so that (40) and (35) imply (9).
Now, we use (T) in order to obtain the asymptotic behavior of $v(t, \cdot)$. By the Riemann–Lebesgue theorem, there exists $M > 0$ such that $\epsilon_b(z) > M$ for all $z \in \mathbb{R}$, so that (40) implies $e^{\gamma_0 t}|\hat{v}(t, y/\sqrt{t})|$ is dominated by an integrable function. Thus, to compute $\lim_{t \to \infty} e^{\gamma_0 t} \hat{v}(t, y/\sqrt{t})$ for all $y < \delta_0 \sqrt{t}$ we write,

$$|e^{\gamma_0 t}(t, y/\sqrt{t}) - Ae^{-\sigma_0 y^2}| \leq |A e^{[k(t/\sqrt{t})+\gamma_m] t} - Ae^{-\sigma_0 y^2}| + e^{\gamma_0 t}|A e^{k(t/\sqrt{t})} - \hat{v}(t, y/\sqrt{t})|. \quad (41)$$

Now set,

$$I_1(t) = |A e^{[k(t/\sqrt{t})+\gamma_m] t} - Ae^{-\sigma_0 y^2}|, \quad \text{and}$$

$$I_2(t) = |A e^{[k(t/\sqrt{t})+\gamma_m] t} - e^{\gamma_0 t}\hat{v}(t, y/\sqrt{t})|. \quad (32)$$

Then, because of (32), we have

$$\lim_{t \to \infty} I_1(t) = 0 \quad \text{for all } y \in \mathbb{R}. \quad (33)$$

On the other hand, due to the definition of $L$ in (31), $e^{[k(t)+\gamma_m] t}$ satisfies (36) for $z \in (-\delta_0, \delta_0)$ and for all $t \in \mathbb{R}$ Next, by applying lemma 2.1 to (36) with $r(t) = e^{\gamma_0 t} \hat{v}(t, y/\sqrt{t}) - Ae^{[k(t)+L(t/\sqrt{t})] t}$ and $\tau = I$, for $|y| < \delta \sqrt{t}$ and $t > h$ we get

$$I_2(t) = |e^{\gamma_0 t}\hat{v}(t, y/\sqrt{t}) - Ae^{\gamma_0 t}e^{[k(t/\sqrt{t})] t}| \leq \lim_{s \to -\infty} \sup_{y \in [-h, 0]} |e^{\gamma_0 t}\hat{v}(s, y/\sqrt{t}) - Ae^{[k(t/\sqrt{t})] t} e^{[k(t/\sqrt{t})] t}|$$

$$\leq \lim_{s \to -\infty} \sup_{y \in [-h, 0]} |e^{\gamma_0 t}\hat{v}(s, y/\sqrt{t}) - Ae^{[k(t/\sqrt{t})] t} e^{[k(t/\sqrt{t})] t}| \frac{1}{1 + \epsilon_1(y)^2}. \quad (34)$$

However, for each $s \in [-h, 0]$, $\lim_{t \to \infty} \hat{v}(s, y/\sqrt{t}) = A$ and $\lim_{t \to \infty} L(t/\sqrt{t}) = \gamma_m$, so that

$$\lim_{t \to \infty} I_2(t) = 0 \quad \text{for all } y \in \mathbb{R}. \quad (35)$$

Finally, (7) follows from (39) and (41), the dominated convergence theorem and (35) replacing $x$ by $a(t, x)$. \hfill \square

**Remark 2.7 (Multidimensional case).** Note that without assuming the tangential condition (T) for the equation (1) we can simply define the pair $(\gamma_0, z_0)$ by (9) with the initial datum $u_0(s, x) \in C([-h, 0], L^1(\mathbb{R}^n))$, for $z_0 \in \mathbb{R}^n$ and $n \in \mathbb{Z}_+$, and then by the same arguments of the proof of the theorem 1.1 we can obtain estimations for equation (1) when $x \in \mathbb{R}^n$. Indeed, similarly to (33), the function $v(t, x) = e^{-\gamma_0 t}u(t, x)$ satisfies the equation

$$v_t(t, x) = \Delta v(t, x) + (2z_0 + m) \cdot \nabla v(t, x) - q_1(z_0) v(t, x) + \int_{\mathbb{R}^n} k_{\gamma_0}(x-y) v(t-h, y) dy,$$

for $x \in \mathbb{R}^n$, $t > 0$; here the parameter $m \in \mathbb{R}^n$ and $q_1(z_0) = |z_0|^2 + m \cdot z + p$ for $p \in \mathbb{R}$. In this case, $e^{\gamma_0 t} \hat{v}(t, z)$ satisfies the following equation

$$w_t(t, z) = (-|z|^2 + i(2z_0 + m) \cdot z + \gamma_0 - q_1(z_0)) w(t, z) + \hat{k}_{\gamma_0}(z) e^{\gamma_0 t} w(t-h, z). \quad (42)$$
Next, by applying lemma 2.1 to (42) and remark 2.5
\[
\int_{\mathbb{R}^n} |e^{\gamma t} \hat{v}(t,y)| dy \leq \sup_{y \in [-h,0]} \|v_0(y, \cdot)\|_{L^1(\mathbb{R})} \int_{\mathbb{R}^n} \frac{dy}{1 + \epsilon h |y|^2} \leq \sup_{y \in [-h,0]} \|v_0(y, \cdot)\|_{L^1(\mathbb{R})} \int_0^{+\infty} \int_{\partial B(0,h)} \frac{dS}{1 + h |r|^2/\eta} dr ds
\]
which satisfies:
\[
\int_0^{+\infty} s^{-\frac{1}{2}} \frac{1}{1 + \epsilon h s^{\frac{1}{2}}/\eta} ds \leq \frac{1}{\epsilon^{n/2}} \int_0^{+\infty} s^{-\frac{1}{2}} e^{-s} ds = \frac{1}{\epsilon^{n/2}} \Gamma(n/2),
\]
so that by Fourier’s inversion formula
\[
e^{-\alpha s} |u(t,x)| \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{v}(t,y) dy \leq \frac{e^{-\alpha s}}{2 \pi^{n/2} s^{n/2}} \sup_{y \in [-h,0]} \|v_0(y, \cdot)\|_{L^1(\mathbb{R})}.
\]
Note that our estimation \(u(t,x) = e^{\alpha s} O(e^{-\alpha s} t^{-n/2})\) requires minimal conditions on the initial data \(u_0\). In particular, (9) is obtained with \(n = 1\).

3. Estimation of level set for non-local KPP equations

In this section, we study the level sets of the functions \(u : [-h, \infty) \times \mathbb{R} \to \mathbb{R}\) which satisfy
\[
u_t(t,x) - \nu_x(x) = u(t,x) + \int_{\mathbb{R}} k_0(y) g(u(t-h,x-y)) dy,
\]
for all \(t > 0, x \in \mathbb{R}\), and
\[
u(x) = u_0(x) \quad (x, x) \in [-h,0] \times \mathbb{R}.
\]
Here, \(u_0 \in C([-h,0], L^1(\mathbb{R}))\) and \(g\) satisfies:

\textbf{(M)} The function \(g : \mathbb{R}_+ \to \mathbb{R}_+\) is such that the equation \(g(x) = x\) has exactly two solutions: 0 and \(\kappa > 0\), and \(g(u) \leq g(0) u\) for all \(u \geq 0\). Moreover, \(g\) is \(C^1\)-smooth in some \(\delta_0\)-neighborhood of the equilibria where \(g'(0) > 1 > g'\). In addition, there are \(C > 0\), \(\theta \in (0,1]\), such that \(|g'(u) - g'(0)| + |g''(\kappa) - g''(\kappa - u)| \leq Cu^\theta\) for \(u \in (0, \delta_0]\).

The condition \(g(u) \leq g(0) u\), for \(u \in \mathbb{R}_+\) (i.e. the KPP condition in (M)) is satisfied in several models. For example, it holds in the Nicholson model where one has \(g(u) = pu e^{-aw}\) (with \(a, p > 0\), or in the Mackey–Glass model where \(g(u) = pu/[1 + au^q]\) (with \(a, p > 0\) and \(q > 1\)) (see, for example, [25]).

In order to continue with our discussion, we need to introduce the following definition.

\textbf{Definition 3.1.} For \(\beta > 0\) and a given initial data \(u_0\) to (43), we define the function \(m_\beta^\infty(t; u_0) : \mathbb{R}_+ \to \mathbb{R}\) as \(m_\beta^\infty(t; u_0) := \inf\{x \in \mathbb{R} : u(t,x) = \beta\}\). Here, \(u(t, \cdot)\) is the solution of (43) with the initial data \(u_0\) which attains the level \(\beta\) at some point in its do-
main. In case level $\beta$ is not attained, we set $m_{\beta}^{-}(t, u_0) = 0$. Analogously we define $m_{\beta}^{+}(t, u_0) := \sup\{x \in \mathbb{R} : u(t, x) = \beta\}$.

In the context of population dynamics, the functions $m_{\beta}^\pm(t, u_0)$ encode the information on the advance of the invading species, with initial population density $u_0$, over a resident species. In general, the behavior of $m_{\beta}^\pm(t, u_0)$ for (43) is unknown. However, many results have been obtained for (43) in the local case, i.e. for the equation,

$$w_t(t, x) = w_{xx}(t, x) - g(w(t, h, x)),$$

for $x \in \mathbb{R}$, and $t > 0$ (see, for example, [5, 8, 13, 14, 18, 24, 29]).

In this respect, the first result on the behavior of $m_{\beta}^\pm(t, u_0)$ was obtained in the classical work of Kolmogorov et al [18]. They considered (45) with $h = 0$ and $g(w) = w - w^2$ and proved that if $u_0$ is a Heaviside function, then,

$$\frac{d}{dt} \left[ m_{\beta}^\pm(t, u_0) \right] \to -c_* = -2.$$

Here, $c_*$ denotes the minimal speed for which monotone wavefronts exist. The actual study of the distance between $m_{\beta}^\pm(t, u_0)$ and $-c_*$ was initiated much later by McKean who proved a lower bound on $m_{\beta}^\pm(t) + c_* t$ [24] using probabilistic methods. Later, Uchiyama [37, theorem 9.1] was able to obtain McKean’s result using the maximum principle.

Still in the local case, for $h > 0$ and $g$ increasing, satisfying (M), it has been proven in [34, theorem 2] that for $c \geq c_*$, provided $u_0(x) \sim \phi_c(x)$ (i.e. $\lim_{t \to -\infty} u_0(x)/\phi_c(x) = 1$), where $\phi_c$ is a wavefront, then one has $|w(t, \cdot - ct) - \phi_c|/\phi_c \to 0$. That is, for all $\epsilon > 0$ there exists $T_\epsilon$ such that

$$(1 - \epsilon)\phi_c(x) \leq w(t, x - ct) \leq (1 + \epsilon)\phi_c(x) \quad \text{for all} \ (t, x) \in [T_\epsilon, \infty) \times \mathbb{R}.\tag{46}$$

Evaluating (46) at $x = ct + m_{\beta}^\pm(t, u_0)$ we conclude that $m_{\beta}^\pm(t, u_0) + ct$ is bounded for $\beta \in (0, \kappa)$ since, by taking $\epsilon > 0$ such that $(1 - \epsilon)\kappa > \beta$, if there exists a sequence $\{t_n\}$ such that $m_{\beta}^\pm(t_n, u_0) + ct_n \to -\infty$ then evaluating (46) at $x = m_{\beta}^\pm(t_n, u_0)$ we have $\beta \leq 0$, which is a contradiction. Similarly, if there exists a sequence $\{t_n\}$ such that $m_{\beta}^\pm(t_n, u_0) + ct_n \to +\infty$ in (46) we have $(1 - \epsilon)\kappa \leq \beta$ a contradiction.

Recall that under the KPP hypothesis (M), the asymptotic behavior of a wavefront $\phi_c$ with $c > c_*$ is $\phi_c(z + z') \sim e^{\lambda_c(z')}$ where $\lambda_c(1) > 0$ is the smallest solution of the characteristic equation $\lambda^2 - c\lambda - 1 + g'(0)e^{-\lambda c} = 0$ and $z' \in \mathbb{R}$; while if $c = c_*$ then the asymptotic behavior for the critical wavefront $\phi_c$ is $\phi_c(z + z') \sim -ze^{\lambda^* z}$ where $\lambda^* = \lambda_1(c_*)$ and $z' \in \mathbb{R}$. Thus, in this case, it only remains to establish what happens with $m_{\beta}^\pm(t, u_0) + ct$ for initial data decaying faster than $-ze^{\lambda^* z}$ for $z \to -\infty$. The next proposition sheds some light on this issue without assuming the monotonicity of $g$.

**Proposition 3.2.** Let $g$ be Lipschitz satisfying (M). Consider the level sets for the solutions of (45) for a non-negative initial data $u_0 \in C([-h, 0], L^\infty(\mathbb{R}))$ locally Holder continuous in $x \in \mathbb{R}$, uniformly with respect to $s \in [-h, 0]$, such that $\inf_{x \in [-h, 0]} \liminf_{t \to +\infty} u_0(s, x) > 0$.

If $c \geq c_*$, we have that:

(a) There exists a level $\beta_0(y) > 0$ such that if $u_0(s, x) \sim \phi_c(x)$, uniformly on $s \in [-h, 0]$, then $m_{\beta_0}(t, u_0) + ct$ is bounded for all $\beta \in (0, \beta_0]$.

(b) If $u_0(s, x) \leq Ae^{\lambda_* s}$, for some $A > 0$ and for all $(s, x) \in [-h, 0] \times \mathbb{R}$, then $m_{\beta}^\pm(t, u_0) + c_\ast t$ is not bounded for all $\beta > 0$. 


Proof.

(a) Let us introduce the monotone function $\bar{g}(w) = \max_{x \in [0, w]} g(x)$. It is simple to check that $\bar{g}$ satisfies (M) with a positive equilibrium $\bar{a} := \max_{w \in [0, x]} g(w)$ and $L_{\bar{g}} := \sup_{\bar{u} \in \mathbb{R}^+} \|\bar{g}(\bar{u}) - g(\bar{u})\|/|\bar{u} - v| = g'(0)$. Now, if we denote by $\bar{w}(t, x)$ the solution to (43) with initial data $\bar{u}_0$ and with $g = \bar{g}$, it follows from [31, lemma 16] that

$$w(t, x) \leq \bar{w}(t, x) \quad \text{for all} \ (t, x) \in [-h, \infty) \times \mathbb{R}.$$  

(47)

Denote by $\underline{a} = \min_{w \in [\bar{a}, \bar{a}]} g(w)$. Hence, it follows from the hypothesis (M) that one can find an increasing function $g$ satisfying (M) with a fixed point $\underline{a} > 0$ and $L_{\underline{g}} = L_{\bar{g}} = g'(0)$ such that $\underline{a} = g(\underline{a})$ for all $w \in [\bar{a}, \bar{a}]$ (for example, one can take $\bar{g}$ close to the function $g_0(w) := \min\{w, \bar{a}\}$ in the norm of $C^1(\mathbb{R}^+)$). Then, if $\bar{g}(t, x)$ denotes the solution of (45) with initial data $\bar{u}_0$ and with $g = \bar{g}$ then, using [31, lemma 16], we have

$$\underline{w}(t, x) \leq w(t, x) \quad \text{for all} \ (t, x) \in [-h, \infty) \times \mathbb{R}.$$  

(48)

Since $\bar{g}$ and $g$ are monotone functions satisfying the KPP condition, then the linear speed $c_\ast$ is the minimal speed for the existence of wavefronts of (45) for $g = \bar{g}$ and $g = g$, respectively. Consequently, for $\bar{g}$ and $g$ in (45) there exist monotone waves $\phi_\ast$ and $\phi_\ast$, respectively, such that $u_0(x, s) \sim \phi_\ast(x) \sim \phi_\ast(x)$ uniformly on $s \in [-h, 0]$. Therefore, by [34, theorem 2] (without restriction on the strict monotonicity of $g$) we conclude $\bar{w}(t, -ct) - \phi_\ast) / \phi_\ast \to 0$ and $\underline{w}(t, -ct) - \phi_\ast) / \phi_\ast \to 0$ for $t \to +\infty$, so that, if we set $\beta_0 := \kappa/2$, from (47), (48) we have,

$$\phi_\ast(x + ct) - \beta/2 \leq w(t, x) \leq \phi_\ast(x + ct) + \beta/2 \quad \text{for all} \ (t, x) \in [0, \infty) \times \mathbb{R},$$  

(49)

for some $T_0 = T_0(\beta, u_0)$. Now, notice that if we set $x = m^-_\beta(t; u_0)$ in (49) we have that $m^-_\beta(t; u_0) + ct$ must be bounded since if there exists a sequence $\{t_n\}$ such that $m^-_\beta(t_n; u_0) + ct_n \to -\infty$ then evaluating (49) at $x = m^-_\beta(t_n; u_0)$ we have $\beta \leq -\beta/2$ a contradiction. Similarly, if a sequence $\{t_n\}$ exists such that $m^-_\beta(t_n; u_0) + ct_n \to +\infty$ in (49) we have $\kappa - \beta/2 \leq \beta$ a contradiction.

(b) Assume there exists $C > 0$ such that $|m^-_\beta(t, u_0) + c(t)| < C$ for all $t > 0$. Then consider a wavefront $\phi_\ast(x) \sim -Ae^{\lambda x}$, and using the same notation as in (a), consider $b' \in \mathbb{R}$ such that: $(1 + \beta) \phi_\ast(C + b') < \beta$. Next, take $x_0 > -1$ such that $m_0(s, x) \leq Ae^{\lambda x_0}$ for all $(s, x) \in [h, 0] \times \mathbb{R}$ and denote by $\bar{w}$ the solution to (45) with this initial data

$$\tilde{u}(s, x) := \begin{cases} -Ae^{\lambda x}, & x < -1 \\ Ae^{\lambda x}, & -1 \leq x \leq x_0 \\ Ae^{\lambda x_0}, & x > x_0. \end{cases}$$

Clearly, $u_0(s, x) \leq \tilde{u}_0(s, x)$ for all $(s, x) \in [-h, 0] \times \mathbb{R}$, $\tilde{u}_0(s, x) \sim \phi_\ast$, uniformly on $s \in [-h, 0]$, $u_0(x, \cdot)$ is locally Holder continuous uniformly on $s \in [-h, 0]$ and

$$\inf_{s \in [-h, 0]} \liminf_{x \to +\infty} \tilde{u}_0(x, s) > 0.$$

Again, by using (47) and [34, corollary 1 (inequality 6)], we conclude that there exists $T_3 = T_3(\beta, b')$ such that,
From here, choosing \( x = m^\beta_T (t; u_0) \), we arrive at

\[
\beta \leq (1 + \beta) \tilde{\phi}_c (m^\beta_T (t; u_0) + c_s t + b') \leq (1 + \beta) \tilde{\phi}_c(C + b') \quad \text{for all } t > T_3,
\]

which contradicts the election of \( \beta \).

\( \Box \)

\textbf{Remark 3.3.} By similar arguments, using the stability of semi-wavefronts in the non-local case (recently obtained by one of us [32]) one can show that (a) and (b) also hold for equation (43).

Notice that in the proof of the proposition 3.2, the conclusions obtained depend strongly on the stability of the wavefronts. In fact, the main difficulty in the present case (in contrast with the situation without delay) is that the flow associated with (43) in general is not monotone if \( g \) is not increasing.

We are interested in obtaining information on the unboundness in case (b) of proposition 3.2 in the non-local case. In that case, the possible asymmetry of the kernel might give way to a different set of admissible speeds for the semi-wavefronts in comparison with the symmetric case. More precisely, it is well known that in the local case, there is a minimal speed \( c_* > 0 \) (i.e. \( c_* \) is the smallest positive real for which there exists a nonnegative bounded solution of (43) of the form \( u(t, x) = \phi_c(x + ct) \), \( \phi_c : \mathbb{R} \to \mathbb{R} \), satisfying \( \phi_c(\infty) = 0 \)). Moreover, for each \( c \geq c_* \), we can consider the solutions \( \psi_{-c}(x) := \phi_c(-x) \) as semi-wavefronts with speed \( -c \). In that way, we obtain a symmetric set of admissible speeds: \( (-\infty, -c_*] \cup [c_*, +\infty) \).

Now, if we take the kernel \( k_0 \) satisfying:

\( (K_0) \) The kernel \( k_0 \) satisfies \( k_0(\cdot) \geq 0 \), \( \int_R k_0(z)dz = 1 \) and, for given \( a < 0 < b \)

\[
\int_R k_0(z)e^{\lambda z}dz < \infty \quad \text{for all } \lambda \in (a, b).
\]

Then, by [11, theorem 18], the set of admissible speeds is given by \( (-\infty, c_*^-] \cup [c_*^+, +\infty) \), where the speeds \( c_*^- < c_*^+ \) are not necessarily opposed to each other (i.e. \( c_*^- \neq -c_*^+ \)).

Moreover, there exist values \( \lambda_*^+ < 0 < \lambda_*^- \) for which the curves

\[
f_1^\pm(z) = -z^2 + c_*^\pm z + 1 \quad \text{and} \quad f_2^\pm(z) = g'(0)e^{-\lambda_*^\pm z} \int_R k_0(y)e^{-\lambda_*^\pm y}dy
\]

are tangent at \( z = \lambda_*^\pm \) [11, lemma 22].

The next result is a modest generalization of the work of McKean [24].

\textbf{Theorem 3.4.} Let \( g \) satisfy (M), \( k_0 \) satisfy \( (K_0) \), and \( u_0 \) be an initial data for (43). Then we have:

(i) If the initial datum satisfies \( e^{\lambda_*^\pm z}u_0(\cdot) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) then there exists \( B \in \mathbb{R} \) such that
\[ m^\alpha_\beta(t,u_0) \geq \frac{1}{2\lambda_*^+} \log(t) - c^+_* t + B \quad \text{for all } t > 0. \] (51)

(ii) If the initial datum satisfies \( \mathbf{e}^{\lambda^-(\cdot)} u_0(\cdot) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \), then there exists \( B \in \mathbb{R} \) such that
\[ m^\beta_\alpha(t,u_0) \leq \frac{1}{2\lambda_*^-} \log(t) - c^-_* t + B \quad \text{for all } t > 0. \] (52)

**Remark 3.5 (Logarithmic term).** In case the kernel \( k_0 \) is non symmetric, it can be that \( c^+_* < 0 \). If \( u(t,\cdot) \) is asymptotically propagated with speed \(-c^+_*\) (like backward traveling fronts—see, for example, [11, p 16]) the logarithmic term in (51) increases the speed \( m^\alpha_\beta(t,u_0)/t \), for large \( t \), in contrast with the local case [5].

**Theorem 3.6.** For a kernel \( k_0 \) satisfying (K) and \( g \) satisfying (M):

(i) Let \( u(t,z) \) be the solution to (43) with initial data as in theorem 3.4 (i). If \( c \geq c^+_* \), then
\[ \lim_{t \to \infty, z \leq -c_* t} u(t,z) = 0. \]

(ii) Let \( u(t,z) \) be the solution to (43) with initial data as in theorem 3.4 (ii). If \( c \leq c^-_* \), then
\[ \lim_{t \to \infty, z \geq c_* t} u(t,z) = 0. \]

In particular, if \( c^+_* c^-_* > 0 \) and the initial data has compact support then,
\[ \lim_{t \to \infty} u(t,z) = 0 \quad \text{for all } z \in \mathbb{R}. \]

**Remark 3.7.** Thus, in the case \( c^-_* c^+_* > 0 \), the associated fronts extinguish in a different manner than in the local monotone case (but without assuming a KPP condition) (see [33, proposition 1.3]) or the symmetric non local monotone case (assuming the KPP condition) [35, theorem 4.1]. However, in a very abstract context, Liang and Zhao [21, theorem 3.4] obtained theorem 3.6 for a monotone \( g \) but for general \( k_0 \).

**Remark 3.8.** Note that if for some \( \sigma > 0 \) the initial datum \( u_0 \in BUC(\mathbb{R}) \) (i.e. is bounded and uniformly continuous) satisfies \( u_0 \geq \sigma \) on a ball of radius \( r_\sigma \), then, by using upper and lower nonlinearities given in [41, lemma 4.1] along with [41, theorem 3.2], we can conclude that for all \( c > c^+_* \) and \( c < c^-_* \)
\[ \lim_{t \to \infty} \min_{-c_* t \leq z \leq c_* t} u(t,z) \geq \epsilon_0, \] (53)
for some \( \epsilon_0 > 0 \). Moreover, the restriction on the radius \( r_\sigma \) can be dropped due to the fact that the KPP condition implies the sub homogenous condition assumed in [22, theorem 3.4 part (2)] and also, due to the comparison lemma A.1, the inequality (53) is valid for the class of exponentially bounded initial data. Finally, note that these propagation results admit a suitable interpretation in [41] when the minimal wave speeds exist, i.e. when \( g^2 \) has only one positive fix point and \( k \) satisfies the condition (K2) (see [41, theorem 4.4]) while this limitation is overcome in [11] by using the minimal semi-wavefront speeds (see [11, theorem 18]).

Theorems 3.4 and 3.6 follow from the following lemma.
Lemma 3.9. Let us consider (1) with
\[ m_{c_t^z} = 2\lambda_e^\pm - c_e^\pm, \quad q_{c_t^z} = -(\lambda_e^\pm)^2 + c_e^\pm \lambda_e^\pm + 1, \]
and
\[ k(x) = g'(0) e^{-\lambda_e^\pm c_e^\pm h} k_0(x + c_e^\pm h) e^{-\lambda_e^\pm x}. \]

If \( v^\pm(t, z) \) is the solution of (1), with this choice of coefficients, and with initial data \( v_0 \) we have the following:

(i) If the initial data \( v_0 \) is as in theorem 3.4 (i) then,
\[ u(t, z - c_t^+ t) \leq e^{\lambda_e^+ t} v^+(t, z) \quad \text{for all } t > 0, z \in \mathbb{R}. \] 
Moreover, \( v^+(t, z) \) satisfies (7) with \( \gamma_m = 0 \) and \( z_m = \lambda_e^+ \).

(ii) If the initial data \( v_0 \) is as in theorem 3.4 (ii) then,
\[ u(t, z - c_t^- t) \leq e^{\lambda_e^- t} v^-(t, z) \quad \text{for all } t > 0, z \in \mathbb{R}. \] 
Moreover, \( v^-(t, z) \) satisfies (7) with \( \gamma_m = 0 \) and \( z_m = \lambda_e^- \).

Proof.

(i) Setting \( w(t, z) := e^{-\lambda_e^+ t} u(t, z - c_t^+ t) \) then, for all \( (t, z) \in \mathbb{R}_+ \times \mathbb{R}, \) \( w \) satisfies,
\[ w_t(t, z) = w_{zz}(t, z) + m_z w(t, z) + e^{-\lambda_e^+ t} \int_\mathbb{R} k_0(z-y) g(e^{\lambda_e^+ t} e^{-c_t^+ h} w(t, y - c_t^+ h)) dy. \]

Hence (54) follows from lemma A.1.

Finally, since \( f^+_1(z) \) and \( f^+_2(z) \) are tangent at \( z = \lambda_e^+ \), then, by applying theorem 1.1, one obtains (7) with \( \gamma_m = 0 \) and \( z_m = \lambda_e^+ \).

(ii) This case is completely analogous to (i).

Proof of theorem 3.4.

(i) By using (54) and (9), we have
\[ u(t, z - c_t^+ t) \leq e^{\lambda_e^+ t} v^+(t, z) \leq C e^{\lambda_e^+ t - \frac{\log t}{2\lambda_e^+}} \quad \text{for all } t > 0, z \in \mathbb{R} \] 
and evaluating (56) at \( z = c_t^+ t + m^-_\beta(t) \) we obtain
\[ 0 < \beta \leq C e^{\lambda_e^+ M(t)} \quad \text{for all } t > 0 \] 
where \( M(t) = c_t^+ t + m^-_\beta(t) - \frac{\log t}{2\lambda_e^+} \).
But, if there exist \( \{t_n\} \) such that \( t_n \to +\infty \) and \( M(t_n) \to -\infty \) we obtain a contradiction in (57); therefore \( M(t) \) is bounded below which implies (51).

(ii) Similarly to (i), by using (57) since \( \lambda_e^- < 0 \), we conclude that \( M^+(t) := c_t^- t + m^+_\beta(t) - \frac{\log t}{2\lambda_e^-} \) must be bounded above.

We conclude this section with the proof of proposition 3.6.
Proof of theorem 3.6.

(i) By (56)

\[ u(t, z) \leq Ce^{\lambda t (t + e^z t)} \leq Ce^{(e^z - c)t} \quad \text{for all } t > 0, z \leq -ct \]  

(58)

which implies the assertion.

The proof of (ii) is analogous. \(\square\)

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Appendix

We consider the nonlinear equation

\[ u_i(t, x) = u_{\alpha_i}(t, x) + mu_{\alpha_i}(t, x) + pu(t, x) + \int_{\mathbb{R}} k_0(x - y) g(u(t - h, y)) dy. \]  

(A.1)

The following lemma can be obtained from [32, proposition 3.1 and lemma 5.6].

**Lemma A.1.** Let us consider \(k_0\) satisfying (K) and \(g\) satisfying the KPP condition \(g(u) \leq g'(0)u\) for all \(u \geq 0\). Denote by \(u(t, x)\) the solution to (A.1) generated by some initial data \(u_0\) and \(v(t, x)\) the solution to (A.1) with \(g(u) = g'(0)u\) generated by \(v_0\). If for some \(\lambda \in (a, b)\) and \(N > 0\)

\[ 0 \leq u_0(s, x) \leq N e^{\lambda x} \quad \text{for all } (s, x) \in [-h, 0] \times \mathbb{R}, \]  

(A.2)

then for each \(n \in \mathbb{Z}_{+}\)

\[ 0 \leq u(t, x) \leq v(t, x) \leq N' e^{\theta n x} \quad \text{for all } (t, x) \in [(n - 1)h, nh] \times \mathbb{R}, \]  

(A.3)

for some \(N' > 0\) and \(\theta > 1\).

**Proof.** We consider \(g\) only satisfying the KPP condition and \(\lambda \in \mathbb{R}\). Then, by making the change of variables \(\bar{u}(t, x) := u(t, x)e^{-\lambda x}\) the equation (A.1) is transformed to

\[ \bar{u}_i(t, x) = \bar{u}_{\alpha_i}(t, x) + (2\lambda + m)\bar{u}(t, x) - q_1(\lambda)\bar{u}(t, x) + \int_{\mathbb{R}} k'(x - y)d(t, y)\bar{u}(t - h, y)dy, \]  

(A.4)

where, \(k'(y) = k_0(y)e^{-\lambda y}\) and \(d(t, y) = g(u(t - h, y))/u(t - h, y)\). Next, by the change of variable \(\bar{u}(t, x) := \bar{u}(t, x - (2\lambda + m)t)e^{\theta(t + \lambda)t}\), the equation (A.4) is reduced to the inhomogeneous heat equation,

\[ \bar{u}_i(t, x) = \bar{u}_{\alpha_i}(t, x) + f(t, x) \quad \text{for all } (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d \]  

(A.5)
where
\[ f(t,x) = e^{t(\lambda)h} \int_{\mathbb{R}^d} k'(y) d(t,x-(2\lambda+m)t-y) \tilde{u}(t-h,x-y-(2\lambda+m)h)dy. \]

By (A.2) we obtain \( \tilde{u}(s,\cdot), \tilde{u}(s,\cdot) \in C([-h,0], L^\infty(\mathbb{R})) \) therefore \( f(t,\cdot) \in C([0,h], L^\infty(\mathbb{R})). \) Next, by denoting \( \Gamma_t \), the one-dimensional heat kernel, we have
\[ \tilde{u}(t) = \Gamma_t * \tilde{u}(0) + \int_0^t \Gamma_{t-s} * f(s,\cdot)ds, \tag{A.6} \]
so that, for \( t \in (0,h] \)
\[ \|\tilde{u}(t)\|_{L^\infty(\mathbb{R})} \leq \|u(0)\|_{L^\infty(\mathbb{R})} + h \sup_{s \in [-h,0]} \|f(s,\cdot)\|_{L^\infty(\mathbb{R})} \leq (1 + hg'(0)e^{(\lambda)h}\|k'\|_{L^2}) \|\tilde{u}_0\|_{C([-h,0],L^\infty(\mathbb{R}))}. \]
By defining \( \theta_0 := 1 + hg'^{(\lambda)h}\|d_2\|_{L^\infty} \|k'\|_{L^2} \) and repeating the argument with the initial data \( u(h+s,x), u(2h+s,x) \), we conclude
\[ \tilde{u}(nh+s,x) \leq \theta_0^n \|\tilde{u}_0\|_{C([-h,0],L^\infty)} \text{ for all } (s,z) \in [-h,0] \times \mathbb{R}, \]
which implies
\[ u(nh+s,x) \leq N' \theta^n e^{\lambda t} \text{ for all } (s,z) \in [-h,0] \times \mathbb{R}, \tag{A.7} \]
with \( N' = N e^{2\|g'\|_{L^2} h} \theta_0 \) and \( \theta = \theta_0 e^{(\lambda)h} \). Therefore, we conclude that \( u(t,z) \) and \( v(t,z) \) (for \( v \), we take \( g(v) = g(0)v \)) are exponentially bounded for each \( t \geq -h \) and uniformly exponentially bounded for \( t \) on any compacts, i.e. for each \( n \in \mathbb{Z}_+ \) we have
\[ u(t,x), v(t,x) \leq N' \theta^n e^{\lambda t} \text{ for all } (t,x) \in [(n-1)h,nh] \times \mathbb{R}. \]
Finally, by defining \( \delta(t,x) := u(t,x) - v(t,x) \) and
\[ (\mathcal{L}\delta)(t,x) = \delta_x(t,x) - \delta(t,x) + m\delta(t,x) + p\delta(t,x) \]
we have
\[ \mathcal{L}\delta(t,x) = \int_{\mathbb{R}} k(y)[g'(0)v(t-h,x-y) - g(u(t-h,x-y))]dy \geq \int_{\mathbb{R}} k(y)[g'(0)v(t-h,x-y) - g'(0)u(t-h,x-y)]dy \geq 0 \]
for all \( t \in [0,h], x \in \mathbb{R} \), and (A.2) implies \( \delta(0,x) \leq 0 \) for \( x \in \mathbb{R} \), so that, since by (A.7) the function \( \delta(t,x) \) is exponentially bounded, then the Phragmèn–Lindelöf principle from [28, chapter 3, theorem 10] implies \( \delta(t,x) \leq 0 \) for all \( t \in [0,h], x \in \mathbb{R} \). Then, by applying the same argument to \( \delta(h+s,x), \delta(2h+s,x) \ldots \) we conclude (A.3). \( \square \)
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