EQUIVARIANT DERIVED CATEGORY OF A COMPLETE SYMMETRIC VARIETY

STÉPHANE GUILLERMOU

Abstract. Let $G$ be a complex algebraic semi-simple adjoint group and $X$ a smooth complete symmetric $G$-variety. Let $L = \bigoplus \alpha L_\alpha$ be the direct sum of all irreducible $G$-equivariant intersection cohomology complexes on $X$, and let $\mathcal{E} = \text{Ext}_{D^b_G(X)}(L, L)$ be the extension algebra of $L$, computed in the $G$-equivariant derived category of $X$. We considered $\mathcal{E}$ as a dg-algebra with differential $d_\mathcal{E} = 0$, and the $E_\alpha = \text{Ext}_{D^b_G(X)}(L, L_\alpha)$ as $E$-dg-modules. We show that the bounded equivariant derived category of sheaves of $C$-vector spaces on $X$ is equivalent to $D_E\langle E_\alpha \rangle$, the subcategory of the derived category of $E$-dg-modules generated by the $E_\alpha$.

1. Introduction

The aim of this paper is to give a description of the equivariant derived category of a smooth complete symmetric variety. Let $G$ be a complex algebraic semi-simple adjoint group, $\sigma$ an automorphism of $G$ of order 2 and $H = G^\sigma$. Let $X$ be a complete symmetric variety containing $G/H$, as defined in [7], section 5: this is a smooth compactification of $G/H$ with a $G$-action (extending the action on $G/H$) and with a $G$-equivariant morphism to the canonical compactification described in [6]. From [6] and [7] we have the following results on the $G$-orbits of $X$: $X \setminus (G/H)$ is the union of irreducible smooth $G$-stable divisors with normal crossings, say $D_1, \ldots, D_m$; any non-empty intersection $D_1 \cap \ldots \cap D_n$ is the closure of a single $G$-orbit, and, conversely, for any $G$-orbit $O$ in $X$, $O$ is the intersection of the $D_i$ containing $O$. We consider $X$ as a complex analytic variety, with its transcendental topology.

We denote by $D^b_G(X)$ the bounded equivariant derived category of sheaves of $C$-vector spaces on $X$; we let $D^b_{G,c}(X)$ be the subcategory formed by constructible objects (it is introduced in [5] – we recall the points we need in section 2A). Let $S$ be the set of $G$-orbits of $X$. For a $G$-orbit $O$, let $\tau_O$ be the group of components of the stabiliser. A representation $\rho$ of $\tau_O$ induces a $G$-equivariant local system on $O$ and we let $L_\rho^O$ be the corresponding intersection cohomology complex. Since $O \setminus O$ consists of normal crossings divisors, $L_\rho^O$ is in fact a sheaf. It is known that, for any orbit $O$, there exists $s$ with $\tau_O \simeq (\mathbb{Z}/2\mathbb{Z})^s$ (we recall this in section 4). In particular, for $\rho$ irreducible, the local system $L_\rho^O$ is of rank 1. The category $D^b_{G,c}(X)$ is generated by the $L_\rho^O$, for all $O$, $\rho$ as above. We set $L = \bigoplus L_\rho^O \in D^b_{G,c}(X)$, where $O$ runs over $S$ and $\rho$ runs over the irreducible representations of $\tau_O$.

We consider the graded ring $\mathcal{E} = \bigoplus_{i \in \mathbb{N}} \text{Ext}^i(L, L)$ (here $\text{Ext}^i(L, L)$ denotes $\text{Hom}_{D^b_{G,c}(X)}(L, L[i])$). We view $\mathcal{E}$ as a differential graded algebra (“dg-algebra”), with differential $d_\mathcal{E} = 0$, and we denote by $D_\mathcal{E}$ the derived category of $\mathcal{E}$-dg-modules, introduced in [5] (we recall its definition in section 2A). We let $D_\mathcal{E}(\mathcal{E}_O)$ be the
subcategory generated by the $\mathcal{E}$-modules $\mathcal{E}^0_\mathcal{O} = \bigoplus_{i \in \mathbb{Z}} \text{Ext}^i(L, L^0_\mathcal{O})$, for all $\mathcal{O}$ in $\mathcal{S}$ and all irreducible representations $\rho$ of $\tau_\mathcal{O}$.

On general grounds recalled below (a kind of derived version of the Freyd-Mitchell embedding theorem) there exists a dg-algebra $R$ with cohomology $H^*(R) \simeq \mathcal{E}$ such that the category $D^b_{G,c}(X)$ is equivalent to a subcategory of $D_R$. We will prove that we can actually take $R = \mathcal{E}$.

**Theorem 1.1.** With the above notations, the categories $D^b_{G,c}(X)$ and $D_{\mathcal{E}}(\mathcal{E}^0_\mathcal{O})$ are equivalent.

The statement of this theorem is inspired by questions of Soergel (see [14], [15]) where it is asked whether it holds for a Langlands parameter space, instead of a symmetric variety here. The case of general (possibly singular) toric varieties was done by Lunts in [13] and we follow the principle of his proof. Some difficulties appear: unlike the toric case, for a $G$-orbit $\mathcal{O}$, the smallest open $G$-stable set containing $\mathcal{O}$ is in general not homotopically equivalent to $\mathcal{O}$; moreover we may have non-connected isotropy groups.

Let us say a word about the algebra $\mathcal{E}$. For two $G$-orbits $\mathcal{O}$ and $\mathcal{O}'$, $\overline{\mathcal{O}} \cap \overline{\mathcal{O}'}$ is a (smooth) orbit closure, say $\overline{\mathcal{O}} \cap \overline{\mathcal{O}'} = \overline{\mathcal{O}''}$. Let $c$ be its complex codimension in $\overline{\mathcal{O}}$. If $\rho$ and $\rho'$ are the trivial representations of $\tau_\mathcal{O}$ and $\tau_{\mathcal{O}'}$, then $\text{Ext}(L^0_{\mathcal{O}'}, L^0_{\mathcal{O}'}) \simeq H^{+2c}(\overline{\mathcal{O}'})$ (in this paper for a group $G$ and a topological space $Y$ with a $G$-action, we denote by $H_G(Y) = H_G(Y; \mathbb{C})$ the $G$-equivariant cohomology of $Y$, with coefficient in $\mathbb{C}$). Hence, in the case where all $\tau_\mathcal{O}$ are trivial, the computation of $\mathcal{E}$ reduces to the computation of $H_G(\overline{\mathcal{O}})$, for all $G$-orbits $\mathcal{O}$ of $X$. But $\overline{\mathcal{O}}$ is a “regular embedding” in the sense of definition 5 of [2] (this means that $\overline{\mathcal{O}}$ has a finite number of $G$-orbits, that each $G$-orbit closure in $\overline{\mathcal{O}}$ is the transversal intersection of the codimension 1 orbits containing it, and that, for each $p \in \overline{\mathcal{O}}$, $G_p$ has a dense orbit in the normal space at $p$ to the $G$-orbit containing $p$). In [2] there is a description of the equivariant cohomology of a regular embedding, say $Y$, as a subalgebra of the product of the equivariant cohomology algebras of the $G$-orbits of $Y$. Note that, for a $G$-orbit $G \cdot p$, $p \in Y$, $H_G(G \cdot p) \simeq H_G(\{pt\})$ only depends on the stabiliser of $p$. For a symmetric variety, these stabilisers $G_p$ can be determined from the action of the involution $\theta$ on the root system of $G$ with respect to a suitable maximal torus. Hence we have a method to compute the $H_G(\overline{\mathcal{O}})$, for all $G$-orbits $\mathcal{O}$ of $X$. This could lead to a combinatorial description of the algebra $\mathcal{E}$ and the modules $\mathcal{E}^0_\mathcal{O}$, at least when the groups $\tau_\mathcal{O}$ are trivial. We note that in the case of toric varieties the computation of the equivariant intersection cohomology from combinatorial data has been carried out in [1] and [4].

The plan of the proof mostly follows that of Lunts in the toric case (see [13]). The principle is the following. We first show that $D^b_{G,c}(X)$ is equivalent to $D_{\mathcal{H}}(H^0_{\mathcal{O}})$, where $\mathcal{H}$ is a sheaf of dg-algebras on a finite set $I$, $D_{\mathcal{H}}$ denotes the derived category of sheaves of $\mathcal{H}$-dg-modules, and the $H^0_{\mathcal{O}}$ are $\mathcal{H}$-modules corresponding to the $L^0_{\mathcal{O}}$ ($I$ and $\mathcal{H}$ are described below). On a finite set, the category of sheaves has enough projectives, so that, for each $H^0_{\mathcal{O}}$, we may choose a projective resolution $P^0_{\mathcal{O}} \to H^0_{\mathcal{O}}$. We set $P = \bigoplus P^0_{\mathcal{O}}$, where the sum is over all $G$-orbits $\mathcal{O}$ and irreducible representations $\rho$ of $\tau_\mathcal{O}$. We consider the dg-algebra $R = \text{Hom}(P, P^0)$ and the left $R$-modules $R^0_{\mathcal{O}} = \text{Hom}(P, P^0)$. Since $D^b_{G,c}(X)$ and $D_{\mathcal{H}}(H^0_{\mathcal{O}})$ are equivalent, we have $H^*(R) \simeq \mathcal{E}$ and $H^*(R^0_{\mathcal{O}}) \simeq \mathcal{E}^0_{\mathcal{O}}$. We let $D_R$ be the derived category of left $R$-dg-modules. By general arguments, the functor $D_{\mathcal{H}}(H^0_{\mathcal{O}}) \to D_R(R^0_{\mathcal{O}})$, $F \mapsto$
Hom(\(P, F\)) is an equivalence of categories. Now, any quasi-isomorphism \(R \to R'\) between two dg-algebras induces an equivalence of categories between \(D_R\) and \(D_{R'}\), by restriction and extension of scalars. Hence the theorem is proved if we show that \(R\) is quasi-isomorphic to its cohomology (such dg-algebras are called "formal").

We give the details first assuming that all \(\tau_O\) are trivial.

1) We recall the following facts about \(D_{G,c}^b(X)\) (see section 12.1. Let \(E\) be a universal bundle for \(G\). By the construction of Bernstein-Lunts in [3], the category \(D_{G,c}^b(X)\) is the subcategory of \(D(E \times_G X)\) generated by the sheaves induced by \(G\)-equivariant constructible sheaves on \(X\). More precisely, in our situation it is generated by the sheaves \(E \times_G L^0_G\), which are equal to \(E \times_G C_{\sigma} = C_{E \times_G \sigma}\), since we assume that the \(\tau_O\) are trivial. However, there is no slice theorem for a \(G\)-action and we find it easier to work in the equivariant derived category for the action of a compact group. In fact, if \(K\) is a maximal compact subgroup of \(G\), the restriction functor \(D_G^b(X) \to D_K^b(X)\) is fully faithful. Since \(D_K^b(X)\) is itself a subcategory of \(D(E \times K X)\), this identifies \(D_{G,c}^b(X)\) with the subcategory of \(D(E \times K X)\) generated by the \(C_{E \times K \sigma}\).

2) Following [33], we obtain a category equivalent to \(D(E \times_G X)\) as follows. From now on, we assume that the maximal compact subgroup \(K\) is compatible with \(\sigma\) (i.e. \(\sigma\) commutes with the conjugation on \(G\) induced by \(K\)). We decompose \(X\) according to the \(K\)-orbit types, and take the connected components: \(X = \bigsqcup_{i \in I} X_i\). This is a stratification of \(X\), which is precisely described in [2]. We let \(\phi : X \to I\) be the map such that \(\phi(X_i) = \{i\}\) and endow \(I\) with the quotient topology. We also denote by \(\psi : E \times K X \to I\) the induced map. Since \(G\) is linear, we may take for \(E\) an increasing union of \(G\)-manifolds (e.g. Stiefel manifolds), \(E = \bigcup_{k \in \mathbb{N}} E_k\). Then the sheaves of \(C^\infty\)-forms of degree \(i\) on \(E_k \times K X\), \(\Omega^i_{E_k \times K X}\), form a projective system, and we define \(\Omega^i_{E \times K X}\) as the projective limit of the \(\Omega^i_{E_k \times K X}\). The complex \(\Omega^i_{E \times K X}\) has a natural structure of sheaf of dg-algebras and it gives a soft resolution of \(C_{E \times K X}\).

We set \(\mathcal{A} = \psi_* (\Omega_{E \times K X})\). This is a sheaf of dg-algebras on \(I\). We consider the direct image functor \(\gamma : D_{G,c}^b(X) \to D_{\mathcal{A}}, F \mapsto \psi_* (\Omega_{E \times K X} \otimes F)\). We prove that \(\gamma(C_{\sigma}) \simeq A_{\sigma(\sigma)}\), for any \(G\)-orbit \(\sigma\), and that \(\gamma\) gives an equivalence between \(D_{G,c}^b(X)\) and \(D_{\mathcal{A}}(A_{\sigma(\sigma)})\). This point uses the following property of our stratification. For \(j \in I\), we let \(V_j \subset X\) be the smallest open subset of \(X\) containing \(X_j\) and constructible with respect to the stratification \(X = \bigsqcup_{i \in I} X_i\). Then, there exists a \(K\)-equivariant homotopy contracting \(V_j\) to \(K \cdot x_j\), for some \(x_j \in X_j\).

3) Let \(\mathcal{H}\) be the cohomology sheaf of \(\mathcal{A}\), i.e. the sheaf on \(I\) associated to \(U \mapsto H^0(\mathcal{A}(U))\). We consider \(\mathcal{H}\) as a sheaf of dg-algebras with differential 0. We prove that there exists a sequence of quasi-isomorphisms \(\mathcal{A} \leftarrow \mathcal{A}' \leftarrow \mathcal{A}'' \leftarrow \cdots \leftarrow \mathcal{H}\) (actually there are 5 steps in this sequence). This implies that the categories \(D_{\mathcal{A}}(A_{\sigma(\sigma)})\) and \(D_{\mathcal{H}}(H_{\sigma(\sigma)})\) are equivalent.

Let us remark that the sheaf \(\mathcal{A}\) is determined by the stalks \(A_i\), \(i \in I\), since we are on a finite set. For a given \(i \in I\), the formality of the dg-algebra \(A_i\) is easy: we have \(A_i = A(\phi(V_i))\), with \(V_i\) as above, and \(H^0(\mathcal{A}_i)\) is the \(K\)-equivariant cohomology of \(K \cdot x_i\), since \(V_i\) has a retraction to \(K \cdot x_i\). Since \(H^0(K \cdot x_i) = H_{K,c_i}(\{pt\})\) is a polynomial algebra (because \(K_{c_i}\) is connected), any choice of representatives for its generators gives a quasi-isomorphism \(H_{K,c_i}(\{pt\}) \to A_i\). But, of course, to
obtain the formality of the sheaf of dg-algebras \( \mathcal{A} \), we need additionally that these quasi-isomorphisms commute with the restriction maps \( \mathcal{A}_i \to \mathcal{A}_j \), for \( i \leq j \).

The construction of the sequence of quasi-isomorphisms above makes use of the description of the stabilisers given in [2]. Let us briefly recall it. Let \( S \) be a maximal split torus of \( G \) and denote by \( x_0 \in X \) the class of \( 1_G \), \( x_0 \in G/H \subset X \). Then \( S \cdot x_0 \) is a smooth toric variety for the action of \( S' = S/(S \cap H) = S/\{ t \in S; t^2 = 1 \} \) and contains a toric subvariety \( Z \), with the following properties. Taking intersection with \( Z \) gives a bijection between the set of \( G \)-orbits of \( X \) and the set of \( S \)-orbits of \( Z \). Moreover the action of \( K \) in \( X \) has a fundamental domain \( C_X \subset Z \). We set \( S_c = S \cap K \); this is a maximal compact subgroup of \( S \). For \( x_i \in X_i \cap C_X \), we have, modulo a finite group, the decomposition \( K_{x_i} = S_{x_i}^c \times K_i \), where \( S_{x_i}^c = S_{x_i} \cap S_c \) and \( K_i = K_{x_i} \cap K_{\sigma} \), only depend on the stratum \( X_i \). Hence \( H_i = H_{K_{x_i}}(\{ pt \}) \cong H_{S_i}(\{ pt \}) \otimes H_{K_i}(\{ pt \}) \).

Now we build a morphism from \( H_{S_i}(\{ pt \}) \to \mathcal{A}_i \). Let \( D_v, v \in V \), be the irreducible \( G \)-stable divisors of \( X \) and \( O_v \) the \( G \)-orbits such that \( D_v = \overline{O_v} \). We denote by \( S_v^c \) the stabiliser in \( S^c \) of \( O_v \cap Z \). Then \( S_v^c = \prod_{v \in \Delta} S_v^c \), where \( \Delta = \{ v \in V; X_i \subset D_v \} \), and \( H_{S_v}(\{ pt \}) \cong \mathbb{C}[\Sigma_v; v \in \Delta] \), with deg \( \Sigma_v = 2 \). Let \( \delta_v \) be the \( G \)-equivariant fundamental class of \( D_v \) in \( X \) and \( \xi_v \in \Omega^2_{E \times K \times X} \) a representative of \( \delta_v \). For \( i \in I \), we define \( f_i : H_{S_i}(\{ pt \}) \to \mathcal{A}_i, \Sigma_v \mapsto \xi_v|_{E \times K \times V} \).

For the factor \( K_i \) in the decomposition, using the Cartan models for the \( K \)-equivariant cohomology, we prove that we have quasi-isomorphisms:

\[
\Gamma(E/K_i; \Omega_{E/K_i}) \sslash_{W_i} \to H_{K_i}(\{ pt \}),
\]

where the \( W_i \) are intermediate dg-algebras, which are subalgebras of the Weil algebra of the Lie algebra of \( K \). These quasi-isomorphisms only depend on the choice of a connection on \( E \). They are compatible with the natural maps from \( H_{K_i}(\{ pt \}) \) to \( H_{K_j}(\{ pt \}) \) induced by inclusions \( K_j \subset K_i \) (and the similar maps between the de Rham algebras). Finally, the maps \( f_i \otimes g_i \) and \( f_i \otimes h_i \) give compatible quasi-isomorphisms between the \( \mathcal{A}_i \) and the \( H_i \).

4) By steps 2 and 3, the categories \( \mathcal{D}^0_{G,c}(X) \) and \( \mathcal{D}(H_{(\mathcal{O}, \nu)}) \) are equivalent. Now we may apply the “principle” explained above to the category \( \mathcal{D}(H) \). Indeed, we can build explicit projective resolutions, using \( \mathbb{C} \)-coverings of \( I \). Let \( P_{\mathcal{O}} \to H_{(\mathcal{O}, \nu)} \) be such a resolution, \( P = \bigoplus P_{\mathcal{O}} \) and \( R = \text{Hom}(P, P) \). To conclude the proof of the theorem it is sufficient to show that \( R \) is quasi-isomorphic to its cohomology. The sheaf \( H \) itself has differential 0 and this fact can be used, as in [10], to endow \( R \) with a graduation different from the canonical one. With this graduation, we prove that \( H^i(R) \) vanishes for \( i \neq 0 \). Hence \( R \) is concentrated in degree 0, and quasi-isomorphic to its cohomology, by the natural morphisms \( R \to R \sim_{\tau \leq 0} R \to H(R) \).

This was the idea for the case where all isotropy groups are connected. In general, simply taking the direct image to \( J \) as above would send some local systems to objects quasi-isomorphic to 0 (for example, let \( H \) be a finite group, \( V \) a non-trivial irreducible representation of \( H \), \( L \) the \( H \)-equivariant local system on the point corresponding to \( V \), then we have \( H^0_{\mathcal{H}}(\{ pt \}; L) = 0 \) - recall that we work with coefficients in \( \mathbb{C} \)). So we modify step 2 as follows.

2.a) Let \( L_E \) be the sheaf induced by \( L = \bigoplus_{\mathcal{O}_c \in \delta} l^0_{\mathcal{O}_c} \) on \( E \times K X \). We will replace the dg-algebra \( \mathcal{A} = \psi_\bullet(\Omega_{E \times K X}) \), representing \( R\psi_\bullet(\mathcal{C}_{E \times K X}) \), by an algebra
representing $R\psi_*R\text{Hom}(L_E, L_E)$. Let us consider a variety $Y$ endowed with two sheaves $L, L'$ which are local systems on subvarieties $Z, Z'$ and 0 outside (here $Y = E_k \times_k V_i$, for some $k$ and $i$, and $L = E_k \times_k L^E_0|_{V_i}, L' = E_k \times_k L^E_0'|_{V_i}$). Here is how we represent $R\text{Hom}(L, L')$. We assume that $Z \cap Z'$ is a smooth subvariety of $Z'$. We consider tubular neighbourhoods: $T_1$ of $Z \cap Z'$ in $Y$, $T'$ of $Z \setminus T_1$ in $Y \setminus T_1, T'$ of $Z' \setminus T_1$ in $Y \setminus T_1$. We choose them so that $T \cap T' = 0$ and $T_1 \cap T'$ is a tubular neighbourhood of $T_1 \cap Z$ (and the same for $T'$). We extend $L$ to a local system $L_1$ on $T_1 \sqcup T$ and extend $L_1$ by 0 outside $T_1 \cap T$; we define $L'_1$ from $L'$ similarly. Then we show that $R\text{Hom}(L, L') \simeq R\text{Hom}(L_1, L'_1)$ and that the complex of sheaves $R\text{Hom}(L_1, L'_1)$ is actually a sheaf (concentrated in degree 0). Hence we may represent $R\text{Hom}(L, L')$ by the complex $\Gamma(Y; \Omega_Y \otimes \text{Hom}(L_1, L'_1))$.

We want to make this procedure work not only for two sheaves $L, L'$ as above, but for all pairs $L^p_0, L'^p_0$, simultaneously. For this, we prove that we can decompose $X$ into “tubes” $T_i$, such that $X = \bigsqcup_i T_i$, with the following properties. For a $G$-orbit $O$ and a representation $\rho$ of $T_O$, we set $Z^\rho_O = \{x \in X; (L^\rho_0)_x \neq 0\}$ and $T^\rho_O = \bigsqcup_{i: Y_i \subset Z^\rho_O} T_i$. Then $L^\rho_0$ has an extension, $L^\rho_0'$, to $T^\rho_O$ and we have, for any other pair $(O', \rho')$:

$$R\text{Hom}(L^\rho_0', L'^\rho_0') \simeq R\text{Hom}(L^\rho_0', L'^\rho_0')$$

and $H^i(R\text{Hom}(L^\rho_0', L'^\rho_0')) = 0$, for $i \neq 0$.

We set $L'_E = E \times_k \bigoplus_{(O, \rho)} L^\rho_0$ and define $\psi': E \times_k X \to J$ by $\psi'(E \times_k T_i) = \{i\}$. Then $A' = \psi'_* (\Omega_{E \times_k X} \otimes \text{Hom}(L'_E, L'_E))$ is a sheaf of dg-algebras on $I$, such that $H^i(A'_E) \simeq \text{Ext}_{D_k(V_i)}((L'_E|_{V_i}, L'_E|_{V_i}))$. We also have a direct image functor $\gamma : D^b_k(X) \to D_{A'}, F \mapsto \psi'_* (\Omega_{E \times_k X} \otimes \text{Hom}(L'_E, J))$, for an injective resolution $F \to J$ of $F$. Setting $M^\rho_O = \gamma(L^\rho_0)$, $\gamma$ gives an equivalence of categories between $D^b_{G, e}(X)$ and $D_{A'}(M^\rho_O)$.

2'.b) This procedure almost replaces step 2 above: we have built a sheaf of dg-algebras on $I$, whose derived category is equivalent to $D^b_{G, e}(X)$. But our sheaf $A'$ is not so easy to handle, because the “tubes” $T_i$ are not intrinsically related to the data. We first built a second sheaf $B$, quasi-isomorphic to $A'$ as follows. Let us consider again the variety $Y$ endowed with local systems $L, L'$ on subvarieties $Z, Z'$ extended to $L_1, L'_1$ on $T_1 \sqcup T, T_1 \sqcup T'$. Let us assume moreover that $Z \cap Z' \subset T_2$ is a homotopy equivalence. Hence we may extend $L_1$ and $L'_1$ to local systems, $L_2$ and $L'_2$ on $T_2$. For $c = \text{codim}^G_{Z'} Z \cap Z'$, we have a “Gysin isomorphism” $\text{Ext}(L|_{Z \cap Z'}, L'|_{Z \cap Z'}) \simeq \text{Ext}^+Z_{Z'}(L, L')$ given by the multiplication by the fundamental class, $\delta$, of $Z \cap Z'$ in $Z'$. Let $\xi \in \Gamma(Y; (\Omega^Z_{T_2})_{T_2})$ be a representative of $\delta$. Then the multiplication by $\xi$ gives a quasi-isomorphism $\Gamma(T_2; \Omega_{T_2} \otimes \text{Hom}(L_2, L'_2)) \to \Gamma(Y; \Omega_Y \otimes \text{Hom}(L_1, L'_1))[2c]$. Our sheaf $B$ is defined as follows. For $i \in I$, recall that the inclusion $X_i \subset V_i$ is a homotopy equivalence, so that the $L^\rho_0|_{X_i}$ extend as local systems, $L^\rho_0|_{V_i}$, to $V_i$. We set $L''_i = E \times_k \bigoplus_{(O, \rho)} L^\rho_{O,i}$ and $B_i = \Gamma(E \times_k V_i; \Omega_{E \times_k V_i} \otimes \text{Hom}(L''_i, L''_i))$. The above Gysin isomorphism implies that we have a quasi-isomorphism $B \to A'$. Now, as a sheaf, $B$ is intrinsically defined from the data of $X$, the local systems $L^\rho_0$ and the stratification. However we also have to understand what the multiplicative structure becomes through the Gysin isomorphism: for a third local system $L''_i$ on a subvariety $Z''_i$, we have a composition $\text{Ext}(L, L') \times \text{Ext}(L', L'') \to \text{Ext}(L, L'')$. Its counterpart on the extensions groups $\text{Ext}(L|_{Z \cap Z'}, L'|_{Z \cap Z'}) \to \text{Ext}(L, L')$. Is also given by the composition, but twisted by fundamental classes of some subvarieties obtained from intersections of $Z, Z', Z''$. 


More precisely, let us consider \( \Delta, \Delta', \Delta'' \subset V \), such that \( Z = \bigcap_{v \in \Delta} E \times_K D_v, Z', Z'' \) being obtained in the same way from \( \Delta', \Delta'' \). For \( W \subset V \), we set \( \delta(W) = \prod_{v \in W} \delta_v \).

The Gysin isomorphism, from \( \text{Ext}^r(L|_{Z \cap Z'}, L'|_{Z \cap Z'}) \) to \( \text{Ext}^{r+2c}(L, L') \) is then given by the multiplication by \( \delta(\Delta \setminus \Delta') \), and the product

\[
\text{Ext}^r(L|_{Z \cap Z'}, L'|_{Z \cap Z'}) \times \text{Ext}^r(L'|_{Z' \cap Z''}, L''|_{Z' \cap Z''}) \rightarrow \text{Ext}^r(L|_{Z \cap Z''}, L''|_{Z \cap Z''})
\]

is defined by the composition of the cup-product on \( Z \cap Z' \cap Z'' \) and the multiplication by \( \delta((\Delta' \setminus (\Delta \cup \Delta'')) \cup ((\Delta \cap \Delta'') \setminus \Delta')) \). The product in \( \mathcal{B} \) is defined by a similar formula, where the classes \( \delta_v \) are replaced by their representatives \( \xi_v \) already introduced above. We prove that we can choose a chain of quasi-isomorphisms as in step 3 compatible with the product. The final step 4 is the same as in the case of connected isotropy groups.

Here is the plan of the paper. In section 3 we recall some facts about equivariant derived categories, Weil algebras and constructible sheaves. In section 4 we construct the dg-algebras \( \mathcal{A}' \) and \( \mathcal{B} \) of steps 2.a and 2.b above. The main result of this section is proposition 3.7. In section 4 we recall some results of [2] on symmetric varieties and use them to prove that the hypothesis of proposition 3.7 are satisfied. Sections 5 and 6 are devoted to the proofs of steps 3 and 4.

Notations. Notations for functors on sheaves are taken from [10]. For a topological space \( X \), we denote by \( D(X) \) (resp. \( D^b(X) \)) the (resp. bounded) derived category of sheaves of \( \mathbb{C} \)-vector spaces on \( X \). If \( X \) is a real analytic manifold, we denote by \( D^b_{\text{R-an}}(X) \) (resp. \( D^b_{\text{R-c}}(X) \) the subcategory of \( D^b(X) \) formed by complexes with real constructible cohomology. The constant sheaf of group \( M \) on \( X \) is denoted \( M_X \). The direct and inverse images by a map \( i: X \rightarrow Y \) are denoted \( i_* \) and \( i^{-1} \). If \( X \) and \( Y \) are separated and locally compact, \( i* \) denotes the direct image with proper supports. If \( i \) is the embedding of a locally closed subset \( X \) of \( Y \), and \( F \in D(Y) \), we set \( F_X = i_!i^{-1}F \) and for a group \( M \), \( M_X = (M_Y)_X \). We will also use \( \Gamma_X(F) \), which is the subsheaf of \( F \) given by the sections with support in \( X \), when \( X \) is closed, and the sheaf \( U \rightarrow F(U \cap X) \) when \( X \) is open; in general we have \( \Gamma_X = \Gamma_X \Gamma_Y=F \). The homomorphisms sheaf is denoted \( \mathcal{H}om(\cdot, \cdot) \). We recall that \( R\Gamma_X(F) \simeq R\mathcal{H}om(C_X,F) \). For \( F \in D^b(X) \) we set \( F^* = R\mathcal{H}om(F,C_X) \). We will sometimes use the notation, for a subset \( Z \subset Y \) and \( F \in D(Y) \), \( R\Gamma(Z;F) = R\Gamma(Z;F|_Z) \).

For a triangulated category \( D \) and objects \( M, N \in D \), we denote by \( D(M, N) \) the triangulated subcategory generated by the \( M, N \), i.e. the smallest triangulated subcategory of \( D \) containing the \( M, N \). For a complex \( M' \), \( n \in \mathbb{Z}, M[n] \) denotes the complex \( M'[n] = M'[n] \) with differential \((-1)^n d_M \).

For a manifold \( X \), we denote by \( \Omega_X \) the de Rham complex of \( X \). If not specified, the cohomology of a space is taken with coefficients in \( \mathbb{C} \).

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2. Preliminaries

2.1. Equivariant derived categories. In this section we recall some results of [3] and [13] about equivariant derived categories and (sheaves of) dg-algebras. We consider, for the convenience of exposition, a linear Lie group \( G \) with finitely many connected components (in section 4 \( G \) will be complex semi-simple). We let \( K \) be...
a maximal compact subgroup of $G$. Hence the variety $G/K$ is isomorphic to an affine space $\mathbb{R}^k$.

We consider a sequence of embeddings of $G$-manifolds $E_i \subset E_{i+1}$ with free actions such that $H^k(E_i) = 0$ for $0 < k < i$ (since $G$ is a linear group, one may choose Stiefel varieties for the $E_i$). We set $E = \bigcup_{i \in \mathbb{N}} E_i$, endowed with the limit topology. The bounded equivariant derived category of $X$, $D^b_D(X)$, is the category formed by the triples $F = (F_X, \mathcal{F}, \beta)$ where $F_X \in D^b(D)$, $\mathcal{F} \in D^b(E \times_G X)$ and $\beta$ is an isomorphism between the inverse images of $F_X$ and $\mathcal{F}$ on $E \times X$. The morphisms from $F$ to $F' = (F'_X, \mathcal{F}', \beta')$ are the pairs of morphisms $(u_X, u)$, $u_X : F_X \to F'_X$, $\bar{u} : \mathcal{F} \to \mathcal{F}'$ commuting with $\beta$ and $\beta'$. It is shown in [3] that $D^b_D(X)$ is independent of the choice of $E$. (The reason to assume that $E$ is a limit of manifolds is to be able to define functors such as the proper direct image or the extraordinary inverse image.) If $X$ is a real analytic manifold, we denote by $D^b_{G,c}(X)$ the subcategory of $D^b_D(X)$ formed by the triples $F$ above such that $F_X$ has real constructible cohomology sheaves. By [3], lemma 2.9.2, the forgetful functor $F = (F_X, \mathcal{F}, \beta) \mapsto \mathcal{F}$ identifies $D^b_{G,c}(X)$ as a full subcategory of $D^b(E \times_G X)$.

Let $\bar{q} : E \times K X \to E \times_G X$ be the quotient map. The restriction functor $R_{K,G} : D^b_{G,c}(X) \to D^b_{K,c}(X)$, $(M_X, \mathcal{M}, \beta) \mapsto (M_X, \bar{q}^{-1}(\mathcal{M}), \beta)$, is fully faithful. Indeed for $F, F' \in D^b_{G,c}(X)$ we have an isomorphism $\text{Hom}_{D^b_{G,c}(X)}(F, F') \simeq H^0(E \times_G X; R\text{Hom}(\mathcal{F}, \mathcal{F}'))$, and the corresponding isomorphism in $D^b_{K,c}(X)$. Since the fibres of $\bar{q}$ are acyclic, we have, $\forall M \in D^b_{G,c}(X)$, $H^0(E \times_K X; \bar{q}^{-1}(\mathcal{M})) \simeq H^0(E \times_G X; \mathcal{M})$. This implies the claim because of the isomorphism $R_{K,G}(R\text{Hom}(\mathcal{F}, \mathcal{F}')) \simeq R\text{Hom}(R_{K,G}(\mathcal{F}), R_{K,G}(\mathcal{F}'))$.

For a $G$-orbit, $O \simeq G/H$, of $X$, we denote by $\tau_O = H/H^0$ the group of connected components of the isotropy group $H$. The $G$-equivariant sheaves with support $O$ are in correspondence with the representations of $\tau_O$. (Let us recall that the objects of $D^b_D(X)$ concentrated in degree 0 correspond to the $G$-equivariant sheaves on $X$.) For a representation $\rho$ of $\tau_O$, let $L^\rho_O$ denote the corresponding local system on $O$. Let us assume that $G$ has finitely many orbits in $X$ and that the $\tau_O$ are finite. Let $\iota_O : O \to X$ be the inclusion of the orbit $O$. For $F \in D^b_{G,c}(X)$, $i_O^* F$ decomposes as $i_O^* F \simeq \oplus_i L_i^\rho_O [d_i]$. We consider the morphisms $u : F \to R(\iota_O)_! i_O^* F$ and $v : \oplus_i (\iota_O)_! L_i^\rho_O [d_i] \to R(\iota_O)_! i_O^* F$ and denote by $F_u$ (resp. $F_v$) the third object of a distinguished triangle built on $u$ (resp. $v$). If $O$ is open in $supp F$, then $supp F_u$ and $supp F_v$ contain less orbits than $supp F$. We deduce that the category $D^b_{G,c}(X)$ is generated by the $\iota_O, L^\rho_O$, where $O$ runs over the $G$-orbits and $\rho$ over the irreducible representations of $\tau_O$. Hence the restriction functor identifies $D^b_{G,c}(X)$ with the subcategory of $D^b_{K,c}(X)$ generated by the $\iota_O, L^\rho_O$ (viewed as objects of $D^b_{K,c}(X)$).

As in the introduction, we let $\Omega^d_{E_i \times K X}$ denote the sheaf of $C^\infty$-forms of degree $d$ on $E_i \times K X$, and we set $\Omega^d_{E \times K X} = \lim_{\leftarrow i} \Omega^d_{E_i \times K X}$. The complex $\Omega^d_{E \times K X}$ is a soft resolution of $C_{E \times K X}$ by a sheaf of differential graded anti-commutative algebras.

We call dg-algebra a non-negatively graded algebra over $\mathbb{C}$, $\mathcal{A} = \oplus_{i \in \mathbb{N}} \mathcal{A}_i$, endowed with a differential $d$ of degree 1 such that, for any homogeneous elements $a, b$, $d(ab) = (da)b + (-1)^{\deg a}a db$. In [3] Lunts considers a sheaf of dg-algebras, $\mathcal{A}$, over a topological set $I$ with finitely many points and defines the derived category $D_{\mathcal{A}}$ as follows. We denote by $M_{\mathcal{A}}$ the category of sheaves of dg-modules over $\mathcal{A}$. A
morphism \( f : M \to M' \) is called a quasi-isomorphism if \( \forall i \in I, f_i : M_i \to M'_i \) is a quasi-isomorphism. We consider \( K_A \), the category with the same objects as \( A \) and with sets of morphisms quotiented by the null-homotopic morphisms. Then \( D_A \) is the localisation of \( K_A \) by the quasi-isomorphisms. There is a substitute for the notion of projective object in this framework: an object \( P \in M_A \) is said \( K \)-projective (see \[10\] and \[3\], p. 74) if \( \text{Hom}_{K_A}(P, -) = \text{Hom}_{D_A}(P, -) \).

Let \( A \) be the graded algebra underlying \( A \) and \( M_A \) the category of (non differential) graded \( A \)-modules. For \( M, N \in M_A \), we set \( \text{Hom}^n(M, N) = \text{Hom}_{M_A}(M, N[n]) \) and, for \( f \in \text{Hom}^n(M, N), df = d_N \circ f - (-1)^n f \circ d_M \). This turns \( \text{Hom}^n(M, N) \) into a complex, and we obtain a bifunctor from \( M_A^p \times M_A \) to the category of complexes of abelian groups. Denoting by \( \text{RHom} \) its derived functor, we have \( \text{Hom}_{D_A}(M, N) \approx \text{RHom}(M, N) \).

Here is how to obtain \( K \)-projectives in \( M_A \). For an open subset \( U \) of \( I \) and an \( A \)-module \( F \), let \( F_U \) be the extension by \( 0 \) of \( F|_U \). For \( M \in M_A \), we have \( \text{Hom}(A_U, M) = \Gamma(U; M) \). For a point \( i \in I \), we denote by \( U_i \) the smallest open subset of \( I \) containing \( i \). These fundamental open sets generate the topology of \( I \). For a sheaf \( F \) on \( I \) we have \( F_i = F(U_i) \). Hence the functor of sections over \( U_i \) is exact and the \( A \)-module \( A_{U_i} \) is \( K \)-projective. One may deduce that the category \( K_A \) has enough \( K \)-projectives and hence also enough \( K \)-flat objects (see \[13\], proposition 1.7.4). Let \( \phi : A \to B \) be a morphism of sheaves of dg-algebras on \( I \) such that \( \forall i \in I, H(\phi_i) : H(A_i) \to H(B_i) \) is an isomorphism. By \[13\], proposition 1.11.2, the functors of restriction and extension of scalars induce an equivalence of categories \( D_A \cong D_S \).

### 2.2. Formality of classifying spaces

An important point in the proof of theorem \[14\] is the fact that some de Rham algebras are formal, i.e. quasi-isomorphic to their cohomology algebras. We will use in particular the following consequence of the results of \[5\]: for a compact Lie group \( K \) with universal bundle \( E \) (given as above by a sequence of \( K \)-manifolds) the de Rham algebras \( \Gamma(E/H; \Omega_{E/H}) \) are formal in a compatible way for all subgroups \( H \subset K \) (see lemma \[23\] below).

Let us first recall the definition of the Weil algebra \( W(\mathfrak{k}) \) of a Lie algebra \( \mathfrak{k} \), as explained in \[5\] (see also \[9\]). As a graded \( C \)-algebra, \( W(\mathfrak{k}) = \Lambda(\mathfrak{k}^*) \otimes S(\mathfrak{k}^*) \), where \( \Lambda(\mathfrak{k}^*) \) denotes the exterior algebra of \( \mathfrak{k}^* \) and elements of \( \Lambda^1(\mathfrak{k}^*) \cong \mathfrak{k}^* \) have degree 1, and \( S(\mathfrak{k}^*) \) denotes the symmetric algebra of \( \mathfrak{k}^* \) and elements of \( S^1(\mathfrak{k}^*) \cong \mathfrak{k}^* \) have degree 2. The algebra \( W(\mathfrak{k}) \) is endowed with a differential, \( \delta \), of degree 1, and derivations, for any \( x \in \mathfrak{k}, i(x) \) of degree \(-1\), \( \theta(x) \) of degree 0. They satisfy the relations, for \( x, y \in \mathfrak{k} \):

\[
\begin{align*}
(1) & \quad \theta([x, y]) = \theta(x) \theta(y) - \theta(y) \theta(x), \\
(2) & \quad i([x, y]) = \theta(x) i(y) - i(y) \theta(x), \\
(3) & \quad \theta(x) = i(x) \delta + \delta i(x).
\end{align*}
\]

They are defined as follows. First we note that, for a connected Lie group \( K \), with Lie algebra \( \mathfrak{k} \), \( \Lambda(\mathfrak{k}^*) \) is identified with the left invariant subalgebra of \( \Omega(K) \) and inherits the differential \( d_A \), the contraction \( i_A(x) \) by the vector field associated to \( x \in \mathfrak{k} \), and the Lie derivative \( \theta_A(x) \). Explicitly, for \( (x, x') \in \mathfrak{k} \times \mathfrak{k}^* \), we have \( i_A(x)(x') = \langle x, x' \rangle \), \( \theta_A(x)(x') = -ad^*_x(x') \), and, for dual basis \( (x_i) \) of \( \mathfrak{k} \), \( (x'_i) \) of \( \mathfrak{k}^* \), we have \( d_A = \frac{1}{2} \sum_i x'_i \theta_A(x_i) \).
Now we define $i(x)$, $\theta(x)$ and $\delta$ on $W(\mathfrak{t})$. Since they are derivations, they are uniquely determined by their values on the generators of $W(\mathfrak{t})$. In the following formulas, $x \in \mathfrak{t}$, $x' \in \Lambda^1(\mathfrak{t}^*) \simeq \mathfrak{t}^*$, $\tilde{x}' \in S^1(\mathfrak{t}^*) \simeq \mathfrak{t}^*$ (recall that $\deg(x') = 1$, $\deg(\tilde{x}') = 2$). We let $\delta : \Lambda^1(\mathfrak{t}^*) \hookrightarrow S^1(\mathfrak{t}^*)$ be the natural isomorphism and we consider dual basis $(x_i)$ of $\mathfrak{t}$, $(x'_i)$ of $\mathfrak{t}^*$. With these notations we have:

$$
(i(x)(x' \otimes 1) = i_A(x)(x') = (x, x'), \quad i(x)(1 \otimes \tilde{x}') = 0,
$$

$$
\theta(x)(x' \otimes 1) = -ad^A_{x'}(x') \otimes 1, \quad \theta(x)(1 \otimes \tilde{x}') = -1 \otimes ad^g_{\tilde{x}'}(\tilde{x}'),
$$

$$
\delta(x' \otimes 1) = d_A(x') \otimes 1 + 1 \otimes h(x'), \quad \delta(1 \otimes \tilde{x}') = \sum_i x'_i \otimes \theta(x_i)(\tilde{x}').
$$

By [5], theorem 1, we have:

$$
H^0(W(\mathfrak{t}), \delta) = C, \quad \forall i > 0 \quad H^i(W(\mathfrak{t}), \delta) = 0.
$$

**Definition 2.1.** Let $(A, d_A)$ be a dg-algebra. One says that $\mathfrak{t}$ acts on $A$, if $A$ is endowed with two linear maps, $i, \theta : \mathfrak{t} \rightarrow \text{Der}(A)$, from $\mathfrak{t}$ to the space of derivations of $A$, such that, $\forall x \in \mathfrak{t}$, $i(x)$ is of degree $-1$, $\theta(x)$ of degree 0, $i(x)^2 = 0$ and $i, \theta$, $d_A$ satisfy the relations (1) to (3), with $d_A$ instead of $\delta$. In this case, the subspace of "$\mathfrak{t}$-basic" elements,

$$
A_{\mathfrak{t}-b} = \{a \in A; \forall x \in \mathfrak{t}, i(x)(a) = \theta(x)(a) = 0\},
$$

is a sub-dg-algebra.

We note that, if $\theta$ is given by differentiation of a $K$-action in $A$, and $K$ is connected, then the subalgebra of $K$-invariants is $A^K = \{a \in A; \forall x \in \mathfrak{t}, \theta(x)(a) = 0\}$ (in general $A^K$ is not stable by $d_A$). The main example is given by the de Rham algebra $\Omega(T)$ of the total space of a $K$-principal fibre bundle, $\tau : T \rightarrow B$: for $x \in \mathfrak{t}$, $i(x)$ and $\theta(x)$, are the usual contraction and Lie derivative associated to the vector field on $T$ induced by $x$. We have $\Omega(T)_{\mathfrak{t}-b} \simeq \Omega(B)$. For $W(\mathfrak{t})$, the elements annihilated by all contractions $i(x)$, $x \in \mathfrak{t}$, are the elements of $S(\mathfrak{t}^*)$; hence, if $K$ is connected, $W(\mathfrak{t})_{\mathfrak{t}-b} \simeq (S(\mathfrak{t}^*))^K$.

For a $K$-principal fibre bundle $T$ as above, recall that a connection on $T$ is the data of projections, $\forall P \in T$, $\phi_P : T_P T \rightarrow T_P(\tau^{-1}(\tau(P)))$, such that $\forall k \in K$, $\phi_{kP}$ is conjugate to $\phi_P$ by the action of $k$ (and the $\phi_P$ vary differentiably). Since the derivative of the $K$-action naturally identifies $\mathfrak{t}$ with $T_P(\tau^{-1}(\tau(P)))$, a connection corresponds to a morphism $f : \mathfrak{t}^* \rightarrow \Omega^1(T)$. More generally, for a dg-algebra $(A, d_A)$, with a $\mathfrak{t}$-action, a "connection" on $A$ is a linear morphism $f : \mathfrak{t}^* \rightarrow A^1$ satisfying:

$$
(5) \quad \forall x \in \mathfrak{t}, \forall x' \in \mathfrak{t}^*, \quad i(x)(f(x')) = (x, x'), \quad \theta(x)(f(x')) = f(-ad^A_{\tilde{x}'}(x')).
$$

We extend naturally $f$ to an algebras morphism, still denoted by $f$, from $A(\mathfrak{t}^*)$ to $A$. But in general, $f$ does not commute with the differential. The algebra $W(\mathfrak{t})$ has the following universal property: we may extend $f$ to an algebras morphism, $\tilde{f} : W(\mathfrak{t}) \rightarrow \Omega(T)$, with the following values on the generators:

$$
\tilde{f}(x' \otimes 1) = f(x'), \quad \tilde{f}(1 \otimes h(x')) = d_A(f(x')) - f(d_A(x')),
$$

commuting with the differentials, the "contractions", $i(x)$, and the "Lie derivatives", $\theta(x)$. In particular, for the $K$-principal fibre bundle $T$ above, we obtain a morphism $\tilde{f} : W(\mathfrak{t}) \rightarrow \Omega(T)$, and it induces a morphism on the basic sub-algebras $\tilde{f} : (S(\mathfrak{t}^*))^K \rightarrow \Omega(B)$.

The following result can be found in [5], though not explicitly stated. This is also a particular case of theorem 4.3.1 of [9].
Thm 2.2 ([3], [9]). Let $H$ be a connected compact Lie group with Lie algebra $\mathfrak{h}$, $A$ a dg-algebra with $\mathfrak{h}$-action and a connection. We assume that $H^i(A) = 0$ for $i > 0$ and $H^0(A) = \mathbb{C}$. Then $H^i(A_{\mathfrak{h}_b}) \simeq S(\mathfrak{h}^*H).

We return to the situation of a compact connected Lie group $K$, acting on a universal bundle $E$ which is an increasing union of $K$-manifolds, $E = \bigcup E_i$. We choose compatible connections on the $E_i$ (i.e. the connection on $E_i$ is the restriction of the one on $E_{i+1}$). This gives a connection, in the algebraic sense of [9], $f : \mathfrak{t}^* \to \Gamma(E; \Omega^1_E)$. It induces a dg-algebras morphism $\tilde{f} : W(\mathfrak{t}) \to \Gamma(E; \Omega^E)$, compatible with the contraction $i$ and the Lie derivative $\theta$. For a connected subgroup $H \subset K$, with Lie algebra $\mathfrak{h}$, the action of $\mathfrak{t}$ on $W(\mathfrak{t})$ obviously restricts to an action of $\mathfrak{h}$.

**Lemma 2.3.** With the notations $H$, $K$, $E$, $f$, introduced above, the space of $\mathfrak{h}$-basic elements of $W(\mathfrak{t})$ is $W(\mathfrak{t})_{\mathfrak{h}^b} \simeq (\Lambda(\mathfrak{h}^\perp) \otimes (\mathfrak{t}^*)^H)$, where $\mathfrak{h}^\perp \subset \mathfrak{t}^*$ denotes the orthogonal of $\mathfrak{h}$. The projection $W(\mathfrak{t})_{\mathfrak{h}^b} \to S(\mathfrak{h}^*H)$ is a quasi-isomorphism. The morphism induced by the connection, $W(\mathfrak{t})_{\mathfrak{h}^b} \to \Gamma(E; \Omega^E_{\mathfrak{h}^b}) \simeq \Gamma(E/H; \Omega^{E/H})$, also is a quasi-isomorphism.

The normaliser of $H$, $N_K(H)$, acts on $W(\mathfrak{t})_{\mathfrak{h}^b}$, $S(\mathfrak{t}^*)^H$ and $\Gamma(E/H; \Omega^{E/H})$, and the above morphisms are $N_K(H)$-equivariant. For another connected subgroup $H_1 \subset H \subset K$, with Lie algebra $\mathfrak{h}_1$, we have a commutative diagram

$$
\begin{array}{ccc}
\Gamma(E/H; \Omega^{E/H}) & \xleftarrow{\cong} & W(\mathfrak{t})_{\mathfrak{h}^b} \xrightarrow{\cong} S(\mathfrak{h}^*H) \\
\downarrow & & \downarrow \\
\Gamma(E/H_1; \Omega^{E/H_1}) & \xleftarrow{\cong} & W(\mathfrak{t})_{\mathfrak{h}_1^b} \xrightarrow{\cong} S(\mathfrak{h}_1^*H_1),
\end{array}
$$

where the horizontal arrows are quasi-isomorphisms.

**Proof.** The $\mathfrak{h}$-basic elements of $W(k)$ are the elements annihilated by all $i(x)$ and $\theta(x)$ for $x \in \mathfrak{h}$. Since $i(x)$ is a derivation and acts trivially on $S(\mathfrak{t}^*)$, the set of elements of $W(k)$ annihilated by all $i(x)$, $x \in \mathfrak{h}$, is $\Lambda(\mathfrak{h}^\perp) \otimes (\mathfrak{t}^*)$. Since $H$ is connected, the elements annihilated by the $\theta(x)$ are the $H$-invariants. Hence we have the description of $W(\mathfrak{t})_{\mathfrak{h}^b}$ given in the lemma. By this description, $W(\mathfrak{t})_{\mathfrak{h}^b}$ admits a projection to $S(\mathfrak{t}^*)^H$ and hence to $S(\mathfrak{h}^*H)$.

Let us choose an $H$-stable decomposition $\mathfrak{t} = \mathfrak{h} \oplus V$. It induces an $H$-equivariant splitting $g : \mathfrak{h}^* \to \mathfrak{t}^* \simeq W^1(\mathfrak{t})$. This is a connection on $W(\mathfrak{t})$, for the $\mathfrak{h}$-action, in the sense of [9]. Hence it gives a morphism of dg-algebras $\tilde{g} : W(\mathfrak{h}) \to W(\mathfrak{t})$. By [9] and theorem 2.2 the induced morphism $\tilde{g} : S(\mathfrak{h}^*H) \simeq W(\mathfrak{h})_{\mathfrak{h}^b} \to W(\mathfrak{t})_{\mathfrak{h}^b}$ is a quasi-isomorphism. We note that, by definition, $\tilde{g}$ also is a splitting of the projection $q : W(\mathfrak{t})_{\mathfrak{h}^b} \to S(\mathfrak{h}^*H)$, so that $g$ is a quasi-isomorphism too.

The composition $f_1 = f \circ g : \mathfrak{h}^* \to \Gamma(E; \Omega^E_{\mathfrak{h}})$ also is a connection on $\Gamma(E; \Omega^E_{\mathfrak{h}})$, for the $\mathfrak{h}$-action. Hence it gives a morphism $\tilde{f}_1 : W(\mathfrak{h}) \to \Gamma(E; \Omega^E_{\mathfrak{h}})$, and we have $\tilde{f}_1 = f \circ g$. By theorem 2.2 again, the induced morphism on the $\mathfrak{h}$-basic elements, $(\tilde{f}_1)_{\mathfrak{h}^b}$ is a quasi-isomorphism. Since $(\tilde{g})_{\mathfrak{h}^b}$ also is, $(\tilde{f}_1)_{\mathfrak{h}^b} : W(\mathfrak{t})_{\mathfrak{h}^b} \to \Gamma(E; \Omega^E_{\mathfrak{h}})_{\mathfrak{h}^b}$ is a quasi-isomorphism, as claimed.

The compatibility of the above morphisms with the $N_K(H)$-action and the commutativity of the diagram follows from the functoriality of the construction.

2.3. **Constructible sheaves.** Here we recall some results of [10] on constructible (complex of) sheaves on real analytic manifolds. Let $Y$ be a real analytic manifold. We say that a locally finite partition of $Y$ by locally closed real analytic manifolds,
\( Y = \bigsqcup_{i \in I} Y_i \), is a stratification if \( \forall i, j \in I, Y_i \cap Y_j \neq \emptyset \) implies \( Y_i \subset \overline{Y_j} \). For two closed subsets \( A, B \) of \( T^* Y \), which are conic, i.e. stably by the action of \( \mathbb{R}^{>0} \) in the fibres, we let \( A + B \) be the subset of \( T^* Y \) defined as follows (see [10] definition 6.2.3 and remark 6.2.8): in a local chart \( U \simeq \mathbb{R}^d \) of \( Y \), \( (x, \xi) \in T^* U \simeq \mathbb{R}^d \times \mathbb{R}^d \) belongs to \( A + B \) if and only if there exists sequences \( (x_n, \xi_n) \in A \), \( (y_n, \eta_n) \in B \) such that \( x_n \rightarrow x, y_n \rightarrow x, \xi_n + \eta_n \rightarrow \xi \) and \( |x_n - y_n||\xi_n| \rightarrow 0 \). We let \( \pi_Y : T^* Y \rightarrow Y \) be the projection and set \( T^*_Y Y = \{(y, \xi) \in T^* Y; y \in Y_i, (\xi, \tau_y Y_i) = 0\} \). We say that the stratification is a \( \mu \)-stratification if \( \forall i \neq j \in I \) such that \( Y_i \subset \overline{Y_j} \) we have \( (T^*_Y Y + T^*_Y Y) \cap \pi_Y^{-1}(Y_i) \subset T^*_Y Y \). We note that if \( Y = \bigsqcup_{i \in I} Y_i \) is a \( \mu \)-stratification then so is the product \( Y \times \mathbb{R}^d = \bigsqcup_{i \in I} Y_i \times \mathbb{R}^d \), and if \( S = \bigsqcup_{i \in I} S_i \) is a \( \mu \)-stratification of the \( d \)-sphere then so is the cone over it: \( \mathbb{R}^{d+1} = \{0\} \cup (\bigsqcup_{i \in I} \mathbb{R}^{>0} \cdot S_i) \) (the condition is trivial at the vertex and at other strata the stratification is diffeomorphic to a product).

Finally, if a submanifold \( Z \) of \( Y = \bigsqcup_{i \in I} Y_i \) intersects all strata transversally and the partition is a \( \mu \)-stratification, then so is the partition \( Z = \bigsqcup_{i \in I} Z \cap Y_i \).

For a complex of sheaves \( F \in \mathcal{D}^b(Y) \), we have the notion of micro-support, \( SS(F) \), which is a closed conic subset of \( T^* Y \). We refer to definition 5.1.1 of [10] and just recall that, if \( Y = \bigsqcup_{i \in I} Y_i \) is a \( \mu \)-stratification, and \( F \) is constructible with respect to this stratification, then \( SS(F) \subset \bigsqcup_{i \in I} T^*_Y Y \) (see proposition 8.4.1 of [10]). We denote by \( \mathcal{D}^b_{R-c}(Y) \) the subcategory of \( \mathcal{D}^b(Y) \) formed by complexes with real constructible cohomology (with respect to any stratification). We will use several times the following results of [10].

**Lemma 2.4** ([10], lemma 5.4.14). Let \( Y \) be a real analytic manifold, \( F \in \mathcal{D}^b_{R-c}(Y) \), \( G \in \mathcal{D}^b(Y) \) and assume that \( SS(F) \cap SS(G) \subset T^*_Y Y \). Then the natural morphism \( RHom(F, CV) \otimes G \rightarrow RHom(F, G) \) is an isomorphism.

**Lemma 2.5** ([10], lemma 8.4.7). Let \( Y \) be a real analytic manifold, \( F \in \mathcal{D}^b_{R-c}(Y) \), \( f : Y \rightarrow \mathbb{R} \) a real analytic function such that \( f|_{\text{supp} F} \) is proper. For \( \varepsilon > 0 \) we set \( Z = f^{-1}(0), Z_\varepsilon = f^{-1}([0, \varepsilon]), U_\varepsilon = f^{-1}([0, \varepsilon]) \). Then there exists \( \varepsilon_0 > 0 \) such that, \( \forall \varepsilon, 0 < \varepsilon < \varepsilon_0 \), we have the isomorphisms
\[
H^2_Z(Y; F) \xrightarrow{\sim} H^2_{Z_\varepsilon}(Y; F), \quad H^1(Z_\varepsilon; F) \xrightarrow{\sim} H^1(U_\varepsilon; F) \xrightarrow{\sim} H^1(Z; F).
\]

2.4. **Local systems outside normal crossings divisors.** We make here some easy remarks on local systems defined outside normal crossings divisors. Let \( Y \) be a smooth complex manifold and \( (D_v)_{v \in V} \) a finite family of smooth normal crossings divisors. We set \( U = Y \setminus \bigcup_{v \in V} D_v \). Local systems (over \( \mathbb{C} \)) on \( U \) are in bijective correspondence with complex representations of \( \pi_1(U) \). For such a representation, \( \rho \), we denote by \( L^\rho \) the associated local system.

We fix \( v \in V \) and set \( Y_v = Y \setminus \bigcup_{w \neq v} D_w, D'_v = D_v \cap Y_v \). We let \( T_v \) be a tubular neighbourhood of \( D'_v \) in \( Y_v \), homeomorphic to the normal bundle of \( D'_v \) in \( Y_v \), and with a projection \( \pi_v : T_v \rightarrow D'_v \). For \( x \in T_v \cap U = T_v \setminus D'_v \), the fibre \( \pi_v^{-1}(x) \cong \mathbb{C} \) is oriented, and we let \( \gamma_x \) be a loop in \( \pi_v^{-1}(x) \) with base-point \( x \) and turning +1 time around 0. Now, let \( b \in U \) be a base-point, \( \tau \) a path from \( b \) to \( x \). The conjugacy class of the image of \( \tau^{-1} \gamma_x \tau \) in \( \pi_1(U) \) is well-defined. We denote it by \( C_x \). If \( x' \) is another point of \( T_v \cap U \), and \( \gamma : [0, 1] \rightarrow T_v \cap U \) a path from \( x \) to \( x' \), then \( \gamma_x \) and \( \gamma^{-1} \gamma_x \gamma \) are homotopic. It follows that \( C_x \) is independent of \( x \). Hence the image of \( C_x \) by \( \rho \) is well-defined up to conjugacy. We call it the monodromy of \( L^\rho \) around \( D_v \). We quote the following facts for later use.

**Lemma 2.6.** In the above situation, let \( L \) be a local system of finite rank on \( U \).
(i) If the monodromy of $L$ around $D_v$ is $Id$, then $L$ extends as a local system, $L'$, to $Y_v$. For $w \neq v$, the monodromy of $L'$ around $D_w$ is the monodromy of $L$ around $D_w$.

(ii) Let $j : U \to Y$ be the inclusion. We assume that $\rho$ factors through a finite quotient of $\pi_1(U)$. If, for each $v \in V$, the monodromy of $L$ around $D_v$ has no eigenvalue equal to 1, then $Rj_*L \simeq j_*L \simeq j_!L$.

Proof. (i) Let $j_v : U \to Y_v$ be the inclusion. It follows from the definition that $L' = (j_v)_*L$ has the required properties.

(ii) The assertion is equivalent to $(Rj_*L)_x = 0$ for any $x \in Y \setminus U$. Since this is a local problem around $x$, we may assume that $Y = \mathbb{C}^n$, and we have coordinates $(x_1, \ldots, x_n)$ such that $x = (0, \ldots, 0)$, $D_v = \{x_v = 0\}$, $v = 1, \ldots, m$, $U = X \setminus \bigcup_{i=1,\ldots, m} D_v$. Then $(R^j j_*L)_x = \lim_{V} H^j(V \cap U; L)$, where $V$ runs over neighbourhoods of 0. We may assume $V$ of the type $V = \{(x); \forall i, |x_i| < \varepsilon\}$. Then $V \cap U$ decomposes as a product $V \cap U \simeq \mathbb{R}^{2n-m} \times (S^1)^m$ and $\pi_1(V \cap U)$ acts on $L|_{V \cap U}$ through a finite abelian group. Hence we may decompose $L|_{V \cap U}$ into a sum of irreducible components, $L_k$, which are local systems of rank 1. Then $L_k \simeq \mathbb{C}^{R^{2n-m} \times L_k^\times \cdots \times L_m^\times}$, for rank 1 local systems $L_k$ on $S^1$. The monodromy of $L_k$ around $D_v$ is the monodromy of $L_k^\times$ around $S^1$. By hypothesis, it is not 1, so that $L_k^\times$ is non trivial and we have $H^0(S^1; L_k^\times) = H^1(S^1; L_k) = 0$. The Künneth formula yields $\forall i, H^i(V \cap U; L) = 0$, as desired.

3. Categories of sheaves and dg-algebras

We consider a manifold $Y$ endowed with a finite stratification $Y = \bigsqcup_{i \in I} Y_i$ by locally closed submanifolds. We denote by $\phi : Y \to I$ the natural map and endow $I$ with the quotient topology. We consider sheaves, $(L_\alpha)_{\alpha \in A}$ on $Y$, constructible with respect to this stratification and which are local systems of finite rank on $Z_\alpha = \{x \in Y; (L_\alpha)_x \neq 0\}$. We will realize $D(Y)(L_\alpha)$ as a derived category of dg-modules over a sheaf of dg-algebras, $A$, on the finite set $I$ (see proposition 3.7 below). This sheaf $A$ will be quasi-isomorphic to $R\phi_*R\mathcal{H}om(\oplus L_\alpha, \oplus L_\alpha)$. We make the following hypothesis on the stratification and the $L_\alpha$.

Assumptions 3.1. Let $Y$ be a complex manifold endowed with a finite $\mu$-stratifica-
tion, $Y = \bigsqcup_{i \in I} Y_i$, by real analytic submanifolds. We assume that $Y$ is an analytic open subset of an analytic manifold, $X$, such that $\overline{Y}$ is compact and has a stratification, $\overline{Y} = \bigsqcup_{i \in I} Y_i$, satisfying: $\forall i \in I, Y_i = Y \cap Y'_i$ (note that $\overline{Y}$ has no additional stratum). For $i \in I$, we define, as in section 2.4, $U_i$ to be the smallest open subset of $I$ containing $i$:

$$U_i = \{j \in I; Y_i \subset \overline{Y}_j\}.$$

We consider a finite family of (complex) smooth, connected, normal crossings divisors, $(D_v)_{v \in V}$, on $Y$. We assume that the divisors are union of strata: $D_v = \bigsqcup_{i \in I_v} Y_i$, for some $I_v \subset I$. We define:

$$\text{for } \Delta \subset V, \quad Z_\Delta = \bigcap_{v \in \Delta} D_v, \quad S = \{\Delta \subset V; Z_\Delta \neq \emptyset\}.$$

We also consider a finite family of constructible sheaves, $(L_\alpha)_{\alpha \in A}$ on $Y$, and set

$$Z_\alpha = \{x \in Y; (L_\alpha)_x \neq 0\}.$$

We make the following hypothesis on these data:

(i) $\forall i, i' \in I, \exists j \in I, U_i \cap U_{i'} = U_j.$
Let \( A \) we set \( \tau \) a local system on \( O \) of order 2 elements and \( \phi \) cannot be in \( \{ i \} \). Hence the inclusions \( L_\alpha \cap D_v \) are one dimensional and the elements of \( \Delta_\alpha \) or \( \Gamma \). We have \( \alpha \in A \), \( L_\alpha|_{Z_\alpha} \) is a local system on \( Z_\alpha \) with monodromy \(-Id\) around each \( Z_\Delta \cap D_v \), for \( v \in \Delta_\alpha \) (see section 2.4).

**Example 3.2.** We will verify in section 2.4 that the decomposition of a symmetric variety given in (v) satisfies these assumptions. A more simple example is given by smooth toric varieties: let \( T = (\mathbb{C}^*)^l \) be a torus, \( D \subset T \) a subtorus consisting of order 2 elements and \( T' = T/D \). Let \( Y \) be a smooth \( T' \)-toric variety, with the action of \( T \) through \( T' \). Let \( (Y_i)_{i \in I} \) be the stratification given by the \( T' \)-orbits, \( (D_v)_{v \in V} \) be the set of \( T' \)-stable irreducible divisors. For a \( T' \)-orbit \( O \) and \( x \in O \), we set \( \tau_O = T_x/T_0 \), we have \( \tau_O \simeq \langle \mathbb{Z}/2\mathbb{Z} \rangle^c \), for some \( c \in \mathbb{N} \). The irreducible \( T \)-equivariant local systems on \( O \) correspond to irreducible representations of \( \tau_O \). Let \( A \) be the set of pairs \( \alpha = (O, \rho) \), where \( O \) is an orbit and \( \rho \) an irreducible representation of \( \tau_O \). We let \( \Delta_\alpha \subset S \) be such that \( \overline{O} = \bigcap_{v \in \Delta_\alpha} D_v \) and let \( L_\alpha \) be the local system on \( O \) given by \( \rho \). Since \( \tau_O \) is a 2-group, the irreducible representations are one dimensional and the elements of \( \tau_O \) act by 1 or \(-1\). In particular \( L_\alpha \) has monodromy \( Id \) or \(-Id\) around any divisor \( D_v \cap \overline{O} \) (for \( v \) such that \( O \not\subset D_v \) and \( \overline{O} \cap D_v \neq \emptyset \)). We let \( \Delta'_\alpha \) be the set of \( v \in V \) for which this monodromy is \(-Id\). Then \( L_\alpha \) extend as a local system to \( Z_\Delta \setminus \bigcup_{v \in \Delta'_\alpha} D_v \) and we let \( L_\alpha \) be the extension by 0 of this local system. Then the assumptions above are satisfied in this situation.

**Remarks 3.3.** 1) In fact we do not use the complex structure; only the geometry of the intersections of the \( D_v \) matters. In particular, the strata \( Y_i \) are not assumed to be complex.

2) In view of lemma 2.3 hypothesis (iv) and (v) have the following consequences: let \( j_\alpha : Z_\alpha \to Y \) be the inclusion. Then \( L_\alpha \simeq R(j_\alpha)_* (L_\alpha|_{Z_\alpha}) \simeq (j_\alpha)_*(L_\alpha|_{Z_\alpha}) \simeq (j_\alpha)_!(L_\alpha|_{Z_\alpha}) \) (or, with different notations, \( L_\alpha \simeq RG_{Z_\alpha}(L_\alpha) \simeq (L_\alpha)_{Z_\alpha} \)). For \( \alpha, \beta \in A \), we will give representatives for the complex \( R\text{Hom}(L_\alpha, L_\beta) \). We already note that

\[
R\text{Hom}(L_\alpha, L_\beta) \simeq R\text{Hom}((L_\alpha)_{Z_\alpha}, RG_{Z_\beta}(L_\beta)) \simeq R\text{Hom}(L_\alpha, RG_{Z_\alpha \cap Z_\beta}(L_\beta)).
\]

In particular, if \( Z_\alpha \cap Z_\beta = \emptyset \), then \( R\text{Hom}(L_\alpha, L_\beta) = 0 \).

3) We note that \( Y_i \) is closed in \( \phi^{-1}(U_i) \) (because the strata contained in \( Y_i \) cannot be in \( \phi^{-1}(U_i) \)) so that \( U_i \setminus \{ i \} \) is open in \( I \). In particular, if \( U_i = U_j \) then \( i = j \). Hence the \( j \) in hypothesis (ii) and (iii) are unique.

4) For any \( i \in I \) and any closed subset \( J \) of \( U_i \), the homotopy \( h \) of (ii), also contracts \( \phi^{-1}(J) \) to \( Y_i \). Hence the inclusions \( Y'_i \subset \phi^{-1}(J) \subset \phi^{-1}(U_i) \) are homotopy equivalences (i.e. induce isomorphisms on all homotopy groups). In particular the inclusion \( Y'_i \subset \phi^{-1}(U_i) \) is a homotopy equivalence. Hence, for any \( \alpha \in A \) and \( i \in I \) with \( Y_i \subset Z_\alpha \), the local system \( L_\alpha|_{Y_i} \) has a unique extension to a local system defined on \( \phi^{-1}(U_i) \). We denote this extension by \( L_{\alpha,i} \). We have \( L_{\alpha,i}|_{\phi^{-1}(U_i) \cap Z_\alpha} \simeq L_\alpha|_{\phi^{-1}(U_i) \cap Z_\alpha} \) and, for \( j \) such that \( Y_j \subset Z_\alpha \) and \( Y_j \subset Y_i \), we have an isomorphism \( u_{ij} : L_{\alpha,i}|_{\phi^{-1}(U_j)} \simeq L_{\alpha,j} \). Since \( u_{ij} \) is determined by its restriction to \( Y_j \), the \( u_{ij}, (i,j) \in I^2 \), satisfy the same relations as their restrictions to \( Z_\alpha \). In
particular they satisfy the cocycle condition which says that the $L_{a,i}$ glue together into a local system, say $L'$, on $\bigcup_{i, j \in \mathbb{Z}_0} U_i$.

5) Since the $U_i$ form a basis of the topology of $I$, a sheaf $F$ on $I$ is determined by its stalks $F_i = F(U_i)$, for all $i \in I$, and the restriction maps, $F_i \to F_j$, for all $i, j \in I$ with $i \in \{ j \}$. Conversely, the data of groups $F_i$ for all $i \in I$, and restriction maps, $f_{ji} : F_i \to F_j$, for all $i, j \in I$ with $i \in \{ j \}$, satisfying $f_{ij} \circ f_{jk} = f_{ik}$ (whenever it makes sense) define a sheaf on $I$.

**Notations 3.4.** We introduce the following notations, for $\alpha, \beta \in A$:

\[ Z_{a\beta} = Z_a \cap Z_{\beta}, \quad \delta_{a\beta} = \text{codim}_{Z_{\beta}} Z_{a\beta}, \quad I_{a\beta} = \phi(Z_{a\beta}), \]

\[ \Delta'_{a\beta} = (\Delta_a \setminus \Delta'_{\beta}) \cup (\Delta'_a \setminus \Delta'_a), \quad I'_{a\beta} = \phi(Z_{a\beta} \cup (\bigcup_{v \in \Delta_a} D_v)) \setminus I_{a\beta}, \]

and, for $i \in I_{a\beta}$, $L_{a,i}, L_{b,i}$ as in remark 3.3 (4), we introduce the following sheaf on $\phi^{-1}(U_i)$: $\Omega_{a\beta,i} = \Omega_{u^{-1}(U_i)} \otimes \text{Hom}(L_{a,i}, L_{b,i})$.

Let us prove the following facts:

(a) if $I_{a\beta} = \emptyset$ then $I'_{a\beta} = \emptyset$; in any case $I'_{a\beta} \subset \overline{I_{a\beta}}$.

(b) $I_{a\beta} \cup I'_{a\beta}$ is open in $\overline{I_{a\beta}}$.

(c) $\forall i \in I'_{a\beta}$ $\exists j \in I_{a\beta}$, such that $U_i \setminus \phi(\bigcup_{v \in \Delta_a \cup \Delta'_{a}} D_v) = U_j$.

For (a) we note that $I_{a\beta} = \emptyset$ means $(\bigcap_{v \in \Delta_a} D_v \setminus \bigcup_{w \in \Delta_a} D_w) \cap (\bigcap_{v \in \Delta_a} D_v \setminus \bigcup_{w \in \Delta_a \cup \Delta'_{a}} D_w) = \emptyset$, which is equivalent to $\bigcap_{v \in \Delta_a \cup \Delta'_{a}} D_v \subset \bigcup_{w \in \Delta_a \cup \Delta'_{a}} D_w$, or also to $(\Delta_a \cup \Delta_a) \cap (\Delta'_a \cup \Delta'_a) = \emptyset$. Since $\Delta_a \cap \Delta'_a = \emptyset$ and $\Delta_a \cap \Delta'_a = \emptyset$, this implies that $(\Delta_a \cap \Delta'_a) \setminus \Delta'_a = \emptyset$, and then $Z_{\Delta_a \cup \Delta_a \setminus \Delta'_a} = \emptyset$. In particular, $I'_{a\beta} = \emptyset$ as claimed, and $I'_{a\beta} \subset \overline{I_{a\beta}}$. Now we note that $Z_{a\beta}$ is open in $Z_{\Delta_a \cup \Delta_a}$. Arguing locally around each connected component of $Z_{\Delta_a \cup \Delta_a}$, we deduce that, if $I_{a\beta} \neq \emptyset$, we also have $\overline{I_{a\beta}} = \phi(Z_{\Delta_a \cup \Delta_a}) \supset I'_{a\beta}$.

For (b), we have $\overline{I_{a\beta} \cup I'_{a\beta}} = \phi(Z_{\Delta_a \cup \Delta_a} \cap (\bigcup_{v \in \Delta_a} D_v))$ and this is closed.

For (c), applying hypothesis (iii) of assumptions 3.4 several times, we know that there exists a unique $j \in I$ such that $U_i \setminus \phi(\bigcup_{v \in \Delta_a \cup \Delta'_{a}} D_v) = U_j$. We note that $U_i \setminus \overline{I_{a\beta}} \neq \emptyset$ and $\overline{I_{a\beta}} \not\subset \phi(\bigcup_{v \in \Delta_a \cup \Delta'_{a}} D_v)$ (remark that $I_{a\beta} \neq \emptyset$), hence $U_j \setminus \overline{I_{a\beta}} \neq \emptyset$. This implies $j \in I_{a\beta}$. But $i \not\in \overline{I_{a\beta} \cup I'_{a\beta}}$ which is closed, hence $j \not\in \overline{I_{a\beta} \cup I'_{a\beta}}$ as well. We thus obtain $j \in I_{a\beta} \cup I'_{a\beta}$. Since $j \not\in \phi(\bigcup_{v \in \Delta_a \cup \Delta'_{a}} D_v)$, this implies $j \in I_{a\beta}$.

**Definition 3.5.** For $i \in I'_{a\beta}$, we denote by $i(\alpha, \beta)$ the element $j \in I_{a\beta}$ given by assertion (c) above. We define a sheaf $A^{\alpha\beta}$ on $I$ by its stalks at $i \in I$:

\[ A^{\alpha\beta}_i = \begin{cases} 
\Gamma(\phi^{-1}(U_i); \Omega_{a\beta,i}) \otimes (-2d_{a\beta}) & \text{if } i \in I_{a\beta}, \\
\bigcup_{j = i}^{\bigcup_{j \in I_{a\beta} \setminus \bigcup_{j \in I_{a\beta}}} A^{\alpha\beta}_j & \text{if } i \in I'_{a\beta} \text{ and } j = i(\alpha, \beta), \\
0 & \text{if } i \not\in I_{a\beta} \cup I'_{a\beta},
\end{cases} \]

and the natural restriction maps. Let us check that this is indeed a sheaf. The first case ($i \in I_{a\beta}$) defines a sheaf, say $A' = \phi_u(\Omega_I \otimes \text{Hom}(L^1_{a}, L^1_{b}))$ (with $L^1_{a}, L^1_{b}$ as in remark 3.3 (4)), on $I_{a\beta}$. Then the second case defines a sheaf, say $A''$, on $I_{a\beta} \cup I'_{a\beta}$, as $u \ast A''$, where $u$ is the inclusion $u : I_{a\beta} \to I_{a\beta} \cup I'_{a\beta}$. Finally the third case defines $A^{\alpha\beta}$ as the extension by 0 of $A''$.

We also define $A = \bigoplus_{\alpha, \beta \in A} A^{\alpha\beta}$. 


Remarks 3.6. 1) The stalks $A^{i}_{\alpha \beta}$ are defined to be 0 when the stratum $Y_i$ is included in a divisor $D_w$ such that the local system $H^0(\mathcal{L}_\alpha, \mathcal{L}_\beta)$ (on $Z_{\alpha \beta}$) has monodromy $-\text{Id}$ around $D_w$. This definition is justified by Remark 3.8 (2) above.

2) For $A^{i}_{\alpha \beta} \neq 0$ (i.e., $i \in I_{\alpha \beta} \cup I'_{\alpha \beta}$) and $Y_i \subset D_v$, we have: $v \in \Delta'_\alpha \iff v \in \Delta'_\beta$. Hence:

(9) for $A^{i}_{\alpha \beta} \neq 0$, $U_i \cap \cup_{v \in \Delta'_\alpha} D_v = U_i \cap \cup_{v \in \Delta'_\beta} D_v = U_i \cap \cup_{v \in \Delta'_\alpha \cap \Delta'_\beta} D_v$.

We will introduce an algebra structure on $A$. For $v \in V$, $D_v$ has a fundamental class, $\delta_v \in H^2_{\mathbb{Q}}(Y; \mathcal{C}_Y)$. We choose representatives, $\xi_v \in \Gamma(Y; \Omega^2_Y)$, of the $\delta_v$. For $\Delta, \Delta', \Delta'' \in S$, we set

\begin{equation}
\nabla(\Delta, \Delta', \Delta'') = (\Delta' \setminus (\Delta \cup \Delta'')) \cup ((\Delta \cap \Delta'') \setminus \Delta')
\end{equation}

and for $\alpha, \beta, \gamma \in A$, $\eta_{\alpha \beta \gamma} = \prod_{v \in \Delta} \xi_v$, where $\nabla = \nabla(\Delta, \Delta, \Delta, \Delta)$. For $\alpha, \beta, \gamma \in A$, we define a morphism $m_{i}^{\alpha \beta \gamma} : A^{i \gamma} \otimes A^{i \beta} \to A^{i \alpha}$ as follows. For $i \in \phi(Z_{\alpha} \cap Z_{\beta} \cap Z_{\gamma})$, we define a sheaf morphism

\begin{equation}
n^{i}_{\alpha \beta \gamma} : \Omega_{\alpha \beta \gamma} \otimes \Omega_{\alpha \beta, i} \to \Omega_{\alpha \gamma, i}
\end{equation}

\begin{equation}
(\tau \otimes v) \otimes (\sigma \otimes u) \mapsto (\eta_{\alpha \beta \gamma} \tau \sigma) \otimes (v \circ u),
\end{equation}

where $\sigma, \tau$ are sections of $\Omega_{\phi^{-1}(U_i)}$ and $u, v$ sections of $\text{Hom}$ sheaves. We set $m_{i}^{\alpha \beta \gamma} = \Gamma(\phi^{-1}(U_i); n^{i}_{\alpha \beta \gamma})$. This definition extends to other $i \in I$, either by restriction to the case $i \in \phi(Z_{\alpha} \cap Z_{\beta} \cap Z_{\gamma})$ or, trivially, when one of the terms is 0. Indeed, if $i \in I \setminus \phi(Z_{\alpha} \cap Z_{\beta} \cap Z_{\gamma})$ satisfies $A^{i \gamma} \neq 0$, $A^{i \beta} \neq 0$ and $A^{i \alpha} \neq 0$, then we have, by (9),

\begin{equation}
U_i \setminus \phi(\cup_{v \in \Delta'_\beta \cap \Delta'_\gamma} D_v) = U_i \setminus \phi(\cup_{v \in \Delta'_\alpha \cap \Delta'_\beta} D_v) = U_i \setminus \phi(\cup_{v \in \Delta'_\alpha \cap \Delta'_\gamma} D_v).
\end{equation}

It follows that, for the same $j \in \phi(Z_{\alpha} \cap Z_{\beta} \cap Z_{\gamma})$, we have $A^{i \gamma} = A^{j \gamma}$, $A^{i \beta} = A^{j \beta}$, $A^{i \alpha} = A^{j \alpha}$. This allows one to define $m^{i}_{\alpha \beta \gamma}$. By definition, these morphisms $m_{i}^{\alpha \beta \gamma}$ commute with the restriction maps and we obtain a sheaves morphism $m_{i}^{\alpha \beta \gamma} : A^{i \gamma} \otimes A^{i \beta} \to A^{i \alpha}$, as claimed. (The justification for the definition of this product is given in section 3.8.2.)

Now we define a product $m$ on $A = \oplus_{\alpha, \beta \in A} A^{i \alpha}$ by $m = \oplus m^{i \alpha \beta}$. One checks that $m$ is an associative product using the straightforward identity:

\begin{equation}
\eta_{\alpha \beta \gamma} \eta_{\alpha \gamma \delta} = \eta_{\beta \gamma \delta} \eta_{\alpha \beta \delta}.
\end{equation}

Hence $A$ is a sheaf of dg-algebras on $I$. For $\alpha \in A$, $N_{\alpha} = \oplus_{\alpha' \in A} A^{i \alpha'}$ has a natural structure of $A$-module defined in the same way as the product of $A$. The result of this section is the following equivalence of categories.

Proposition 3.7. Let $Y = \bigcup_{i \in I} Y_i$ be a stratified complex analytic manifold, endowed with normal crossings divisors $D_v$, $v \in V$, and sheaves $L_{\alpha}$, $\alpha \in A$, satisfying assumptions 3.4. For a choice of forms $\xi_v \in \Gamma(Y; \Omega^2_Y)$, we define a sheaf of dg-algebras $A$ on $I$, and $A$-modules $N_{\alpha}$ as above.

Then, there exists a choice of $\xi_v$ such that we have an equivalence of categories between $\mathcal{D}(Y)(L_{\alpha})$ and $\mathcal{D}_{A}(N_{\alpha})$ sending $L_{\alpha}$ to $N_{\alpha}$.

The proof is given at the end of this section.

Remarks 3.8. 1) In fact one could prove that two choices of representatives $\xi_v$, $\xi'_v$ of the $\delta_v$ give quasi-isomorphic sheaves of dg-algebras, $A, A'$; with $\xi_v$ such that
\( \xi_n = \xi_n' = d\zeta_n \), and replacing \( Y \) by \( Y \times C \), endowed with \( \xi_n^+ = \xi_n + d(t(\zeta_n)) \) (\( t \) is the coordinate on \( C \)) and the data \( Y_i \times C, D_n \times C, L_\alpha \propto C_C \), we could build a third sheaf \( A^+ \) on \( I \), quasi-isomorphic to \( A \) and \( A' \). Hence the conclusion of the proposition is valid for any choice of \( \xi_n \), but we will not use this result.

2) The results of this section will be applied to \( Y = E \times K \), where \( X \) is a symmetric variety under the action of a semi-simple complex algebraic group \( G \), \( K \) a suitable maximal compact subgroup of \( G \) and \( E \) a universal bundle for \( K \). Of course, \( E \) is not a manifold, but we may assume that it is an increasing union of \( K \)-manifolds, \( E = \bigcup_k E_k \), and consider the de Rham algebra of \( Y \), \( \Omega_Y = \lim_k \Omega_{E_k \times K \times X} \).

The stratification \( Y = \bigsqcup_i Y_i \) and the divisors \( D_n \) will be given by a K-invariant stratification of \( X \) and \( K \)-invariant divisors. All constructions in this section can be made \( K \)-invariant (if we choose \( K \)-invariant functions \( f_i \) in notations \ref{eq:3.19} below, by averaging under the action of \( K \)) and transpose to \( Y = E \times_K X \).

3.1. System of tubes. Our first task in the proof of proposition \ref{prop:3.11} is to “compute” the global sections \( \Gamma_{\alpha\beta i} = R\Gamma(\phi^{-1}(U_i); R\text{Hom}(L_\alpha, L_\beta)) \), for \( i \in I, \alpha, \beta \in A \) (to compute just means to find suitable representatives). For this we replace the strata \( Y_i \) by a system of “tubes”, \( T_i \) (with \( T_i \) closed enough to \( Y_i \) so that the local systems \( L_\alpha|_{Y_i} \) extend to \( T_i \)) with the properties: (i) replacing \( L_\alpha \) by its extension, say \( L'_\alpha \), to the union of tubes \( \bigcup_{i: Y_i \subset Z_n} T_i \) doesn’t change the global sections \( \Gamma_{\alpha\beta i} \), (ii) the complexes \( R\text{Hom}(L'_\alpha, L'_\beta) \) are in fact sheaves. The precise statement is given in proposition \ref{prop:3.11} below. The first property implies that the category \( D(Y)\langle L_\alpha \rangle \) is equivalent to the \( D(Y)\langle L'_\alpha \rangle \) (see lemma \ref{lem:3.18} below). The second property will be used to define a sheaf of dg-algebras, \( B \), on \( I \), such that \( D(Y)\langle L'_\alpha \rangle \) is equivalent to a subcategory of \( D_B \) (definition \ref{def:3.18} and proposition \ref{prop:3.19} below). The proof of proposition \ref{prop:3.11} will then be achieved by showing that \( B \) and \( A \) are quasi-isomorphic.

Notations 3.9. First we assume that the finite set indexing the stratification is \( I = \{1, \ldots, n\} \), ordered such that \( \dim Y_i \leq \dim Y_{i+1} \), for \( i = 1, \ldots, n-1 \). Recall that \( Y \) is open in an analytic manifold \( X \) and \( \overline{Y} \) is compact, with a stratification \( \overline{Y} = \bigsqcup_i Y_i' \), satisfying: \( \forall i \in I, Y_i = Y \cap Y'_i \). For \( i = 1, \ldots, n-1 \), we choose a neighbourhood of \( Y_i \), \( \tilde{Y}_i \), whose closure is a neighbourhood of \( Y_i' \) in \( \overline{Y} \). We also choose real analytic functions \( f_i : \tilde{Y}_i \to \mathbb{R} \), such that \( f_i(\tilde{Y}_i) \subset \mathbb{R}_{\geq 0} \) and \( Y_i = f_i^{-1}(0) \cap \tilde{Y}_i \). For \( k < n \) and \( \varepsilon_1, \ldots, \varepsilon_k > 0 \), we define
\[
T_1(\varepsilon_1) = \{ y \in \tilde{Y}_1; f_1(y) \leq \varepsilon_1 \}, \ldots, T_k(\varepsilon_1, \ldots, \varepsilon_k) = \{ y \in \tilde{Y}_k; f_k(y) \leq \varepsilon_k \} \setminus \bigsqcup_{i=1}^k T_i.
\]

By abuse of notations we will write \( T_i(\varepsilon) = T_i(\varepsilon_1, \ldots, \varepsilon_i) \) for any \( \varepsilon \) of length greater than \( i \). We also set \( T_n(\varepsilon) = Y \setminus \bigsqcup_{i<n} T_i(\varepsilon) \). For a union of strata, \( Z \), we set:
\[
T_Z(\varepsilon) = \bigcup_{i \in J} T_i(\varepsilon), \quad \text{where } J \text{ satisfies } Z = \bigcup_{i \in J} Y_i.
\]

Definition 3.10. We call “set of bounds” a subset \( B \in [0, +\infty[^k \) such that \( \exists \varepsilon_i^0 > 0, \forall \varepsilon_1 < \varepsilon_1^0, \exists \varepsilon_2^0 > 0, \forall \varepsilon_2 < \varepsilon_2^0, \ldots, \exists \varepsilon_k^0 > 0, \forall \varepsilon_k < \varepsilon_k^0 \), \((\varepsilon_1, \ldots, \varepsilon_k) \in B \).

We note that a set of bounds is non-empty and that the intersection of two sets of bounds is a set of bounds. The aim of this paragraph is to prove the following result.

Proposition 3.11. Let \( Y = \bigsqcup_{i=1,\ldots,n} Y_i \) be a stratified analytic manifold as above.
Let \( Z_1, Z_2 \subset Y \) be locally closed subsets of \( Y \) which are unions of strata. Let \( L^1, L^2 \)
be local systems respectively defined on neighbourhoods of $Z_1$ and $Z_2$. Then there exists a set of bounds $B \subset [0, +\infty[^{n-1}$ such that $\forall \xi \in B$, setting $T_\xi = T_\xi(\xi)$, $\mathcal{L}_i$ is defined on $T_\xi$ and we have:

(i) There exist natural morphisms $\mathcal{L}_i^{T_\xi_1} \to \mathcal{L}_i^{T_\xi_2}$ and they induce isomorphisms

$$\text{RHom}(\mathcal{L}_i^{T_\xi_1}, \mathcal{L}_j^{T_\xi_2}) \simeq \text{RHom}(\mathcal{L}_i^{T_\xi_1}, \mathcal{L}_j^{T_\xi_2}) \simeq \text{RHom}(\mathcal{L}_i^{T_\xi_1}, \mathcal{L}_j^{T_\xi_2}).$$

(ii) For any locally closed union of strata $Z \subset Y$ such that $Z_1, Z_2 \subset Y \setminus \Xi$, we have an isomorphism

$$\text{R}^i(Y; (\text{RHom}(\mathcal{L}_i^{T_\xi_1}, \mathcal{L}_j^{T_\xi_2}))_{T_\xi}) \simeq \text{R}^i(Y; (\text{RHom}(\mathcal{L}_i^{T_\xi_1}, \mathcal{L}_j^{T_\xi_2}))_{Z}).$$

(iii) If $Z_1 \subset Z_2$, then $\text{RHom}(\mathcal{L}_i^{T_\xi_1}, \mathcal{L}_j^{T_\xi_2})$ is concentrated in degree 0.

(iv) Let us assume that $Z_2$ and $Z_1 \cap Z_2$ are smooth and let $\mu : Z_2 \to \Xi$, $\nu : Z_1 \cap Z_2 \to Z_1 \cap Z_2$ be the inclusions. We assume that $\text{R}_{\mu_*}(\mathcal{L}_i^2|_{Z_2}) = \mu_!(\mathcal{L}_i^2|_{Z_2})$ and $\text{R}_{\nu_*}(\mathcal{L}_2^2|_{Z_1 \cap Z_2}) = \nu_(\mathcal{L}_2^2|_{Z_1 \cap Z_2})$. Then, for any open union of strata $V \subset Y$,

$$\text{RHom}(\mathcal{L}_i^{T_\xi_1} \mid_V, \mathcal{L}_2^2 \mid_V) \simeq \text{R}^i(T_\nu; \text{Hom}(\mathcal{L}_i^{T_\xi_1} \mid_V, \mathcal{L}_2^2 \mid_V)).$$

The proof will be given at the end of the paragraph. We first deduce the following corollary, which gives a category equivalent to $D(Y)(L_\alpha)$.

We consider $Y$ and $L_\alpha$, $\alpha \in A$, as in assumptions 3.1. We choose neighbourhoods of the $Z_\alpha$ on which the local systems $L_\alpha$ may be extended to local systems $L_\alpha^+$. For $\alpha, \beta \in A$, we set $Z_1 = Z_\alpha$, $L_1 = L_\alpha^+$, $Z_2 = Z_\beta$, $L_2 = L_\beta$. Then, $Z_2$ and $Z_1 \cap Z_2$ are open subsets of intersections of some $D_\nu$, hence smooth. Moreover, by assumptions 3.1 (v), $L_2$ has monodromy $-Id$ around each irreducible divisor of $Z_2 \setminus Z_2$ and the similar property holds a fortiori for $L_2^2|_{Z_1 \cap Z_2}$. Hence, by lemma 3.10 the hypothesis of proposition 3.11 (iv), are verified.

**Notations 3.12.** We choose a set of bounds, $B$, such that the conclusions of proposition 3.11 hold for $Z_1 = Z_\alpha$, $L_1 = L_\alpha^+$, $Z_2 = Z_\beta$, $L_2 = L_\beta^+$, for any pair $(\alpha, \beta) \in A^2$. We fix $\xi \in B$ and set:

$$T_\alpha = T_\alpha(\xi), \quad T_\beta = T_{Z_\alpha}(\xi), \quad L_\alpha^+ = (L_\alpha^+)_T.$$

**Corollary 3.13.** The categories $D(Y)(L_\alpha)$ and $D(Y)(L_\alpha^+)$ are equivalent.

**Proof.** This is a consequence of the natural isomorphisms 3.11. By definition the category $D(Y)(L_\alpha)$ is the union of the full subcategories $D_\alpha$, $n \in N$, where $D_\alpha$ consists of the $L_\alpha[k], \alpha \in A$, $k \in Z$, and $D_{\alpha+1}$ is obtained from $D_\alpha$ by adding objects $H$ appearing in distinguished triangles $F \to G \to H^{-1} \to$, with $F, G \in D_\alpha$.

We write in the same way $D(Y)(L_\alpha^+)$ as $\bigcup D_n$. We assume by induction that we have an equivalence, $\delta_n$, between $D_n$ and $D_n'$, together with functorial morphisms $r_n(F) : \delta_n(F) \to F$ such that $\delta_n$ is given on the morphisms by composing isomorphisms

$$\text{Hom}_{D_n}(F, F') \simeq \text{Hom}_{D(Y)}(\delta_n(F), F') \simeq \text{Hom}_{D_n'}(\delta_n(F), \delta_n(F'))$$

induced by $r_n(F), r_n(F')$. (The first step is given by 3.11.) Let $F \to G \to H^{-1} \to$ be a distinguished triangle as above, $F' = \delta_n(F), G' = \delta_n(G), u' = \delta_n(u)$ and consider a distinguished triangle $F' \to G' \to H'^{-1} \to$. We extend the square built on $u, u', r_n(F)$ and $r_n(G)$ to a morphism of triangles:

$$\begin{array}{ccc}
F' & \to & G' & \to H'^{-1} \\
\downarrow & & \downarrow & & \downarrow \\
F & \to & G & \to H^{-1}
\end{array}$$
We set $H' = \delta_{n+1}(H)$, $r_{n+1}(H) = r$ and we have to define the images of the morphisms. First, for $X \in D_n$, we have long exact sequences of homomorphisms groups:

\[
\begin{array}{ccc}
\text{Hom}(X, F) & \longrightarrow & \text{Hom}(X, G) \\
\downarrow & & \downarrow \\
\text{Hom}(\delta_n(X), F) & \longrightarrow & \text{Hom}(\delta_n(X), G) \\
\downarrow & & \downarrow \\
\text{Hom}(\delta_n(X), \delta_n(F)) & \longrightarrow & \text{Hom}(\delta_n(X), \delta_n(G)) \\
\end{array}
\]

By the five lemma it gives an isomorphism $\text{Hom}(X, H) \cong \text{Hom}(\delta_n(X), \delta_n(H))$, which we use to define $\delta_n$ on $\text{Hom}(X, H)$. In the same way, we may define $\delta_n$ on $\text{Hom}(H, X)$, still for $X \in D_n$. Then we may assume $X \in D_{n+1}$ in the above diagram, and this defines $\delta_n$ on $\text{Hom}(X, H)$ for $X, H \in D_{n+1}$, satisfying the compatibility with $r_{n+1}$. □

Now we give some preliminary results before we prove proposition 3.11.

**Lemma 3.14.** Let $Y = \bigsqcup_{i=1}^{n} Y_i, f_i$, be as in assumptions 3.1 and notations 3.9. There exists a set of bounds $B \subset ]0, +\infty[^{n-1}$ such that $\forall \xi \in B$ and any union of strata $Z \subset Y$:

(i) if $Z$ is closed then so is $T_Z(\xi)$, and $Z \subset T_Z(\xi)$,

(ii) if $Z$ is open then so is $T_Z(\xi)$,

(iii) $\forall_{i_1} < \cdots < i_p < n$, and $y \in Y$ such that $f_{i_1}(y) = \varepsilon_{i_1}$, we have $d_{f_{i_1}} \ldots \wedge d_{f_{i_p}}(y) \neq 0$. In particular, locally around any point $y \in Y$, the partition $Y = \bigsqcup T_i(\xi)$ is homeomorphic to $\mathbb{R}^q = \{x_1 \leq 0\} \cup \{x_1 > 0, x_2 \leq 0\} \cup \{x_1 > 0, \ldots, x_q - 1 > 0, x_q \leq 0\} \cup \{x_1 > 0, \ldots, x_q > 0\}$, for some $q$.

**Proof.** Of course (ii) follows from (i) because $T_{Y \setminus Z}(\xi) = Y \setminus T_Z(\xi)$. We prove (i) by induction on $n$, the case $n = 1$ or 2 being obvious. Recall that $Y$ is open in a manifold $X$. For $i$ such that $Y_i \cap \overline{Y_i} = \emptyset$, we also have $\overline{Y_i} \cap \overline{Y_i} = \emptyset$, because otherwise the stratification of $\overline{Y_i}$ would have additional strata. Since $\overline{Y_i}$ is compact, we deduce $d(Y_i, Y_i) > 0$. Hence we may choose $r > 0$ smaller than $\min\{d(Y_i, Y_i); Y_i \cap \overline{Y_i} = \emptyset\}$ and $\sup\{d(Y_i, y); y \in Y_j\}$, for all $j$. Then, for $\varepsilon_{i_1}^0$ such that $T_i(\varepsilon_{i_1}^0) \subset \{y \in Y; d(Y_i, y) < r\}$ and for $0 < \varepsilon_1 < \varepsilon_{i_1}^0$, we have $T_i(\varepsilon_1) \cap \overline{Y_i} \neq \emptyset$ if and only if $Y_i \subset T_i(\varepsilon_1)$, and moreover $\forall j \neq 1$, $Y_j \not\subset T_i(\varepsilon_1)$.

The induction hypothesis applied to $Y' = Y \setminus T_i(\varepsilon_1)$ stratified by the $Y'_i = Y' \setminus Y_i$ gives a set of bounds $B'((\varepsilon_1)) \subset ]0, +\infty[^{n-2}$ for which (i) holds in $Y'$. For $i$ such that $Y_i \cap \overline{Y_i} = \emptyset$ we may choose $\varepsilon_i$, small enough so that $T_i(\varepsilon_1) \cap \{y \in Y; f_i(y) \leq \varepsilon_i\} = \emptyset$. In particular, restricting to a smaller set of bounds $B''(\varepsilon_1)$, we may assume that $T_i(\varepsilon_1) \cap \overline{T_i(\varepsilon_1)} = \emptyset$. Let $Z \subset Y$ be closed.

If $Y_1 \subset Z$ then $T_Z = T_1(\varepsilon_1) \cup T_{Z \setminus Y_1}'$. Since $T_{Z \setminus Y_1}'$ is closed in $Y'$, $T_Z$ is closed in $Y = T_1(\varepsilon_1) \cup Y'$. By induction we also have $Z \cap Y' \subset T_Z \cap Y'$ and this implies $Z \subset T_Z$.

If $Y_1 \not\subset Z$ then $Z$ only contains strata $Y_i$ such that $Y_1 \cap \overline{Y_i} = \emptyset$, so that $T_i(\varepsilon_1) \cap \overline{Y_i} = \emptyset$. It follows that $Z \subset Y'$ (and $Z$ is closed in $Y'$). This also gives $T_1(\varepsilon_1) \cap (T_2) Y' = \emptyset$, and since $T_2$ is closed in $Y'$, it is closed in $Y$ too. Finally $Z \subset T_Z$ since this is already true in $Y'$. In conclusion the set of bounds $B = \{(\varepsilon_1, \ldots, \varepsilon_{n-1}); 0 < \varepsilon_1 < \varepsilon_0, (\varepsilon_2, \ldots, \varepsilon_{n-1}) \in B''(\varepsilon_1)\}$ has the required property.
Now we prove by induction on $p$ that there exists a set of bounds $B_p$ such that the conclusion of (iii) holds for any $i_1 < \cdots < i_p < n$. For $p = 1$ this is a consequence of the curve selection lemma: by contradiction, if the closure of \{ $y \in Y_i; df_{i_1}(y) = 0$ \} intersects $Y_1$, then there exists a real analytic curve $\gamma \colon [0, 1] \to Y$ such that $\gamma(t) \in Y \setminus Y_1$ for $t \neq 0$ and $\gamma(0) \in Y_1$. But this implies $df_{i_1}(\gamma(t)) = 0$ so that $f_{i_1}(\gamma(t))$ is constant, which is impossible. Hence there exists $\varepsilon_{i_1}^0 > 0$ such that $0 < f_{i_1}(y) < \varepsilon_{i_1}^0$ implies $df_{i_1}(y) \neq 0$. We take $B_1 = \prod_i [0, \varepsilon_i^0]$.

Assuming (iii) holds for $p$, we consider, for $\underline{\varepsilon} \in B_p$ and $i_1 < \cdots < i_p$, the smooth subvariety of $Y$, $Y' = \{ y \in Y; f_{i_1}(y) = \varepsilon_{i_1}, \ldots, f_{i_p}(y) = \varepsilon_{i_p} \}$. For $i_{p+1} > i_p$, the function $f_{i_{p+1}}$ is not constant on $Y'$ and the proof of the first step gives $\varepsilon_{i_1, \ldots, i_p} \varepsilon_{i_1, \ldots, i_p} > 0$ such that the conclusion holds for $\varepsilon_{i_1, \ldots, i_{p+1}}$ with $\varepsilon_{i_1, \ldots, i_p, \varepsilon_{i_1, \ldots, i_p}} > 0$ and, for $k = p + 1, \ldots, n$, $B_p^k = \{ \underline{\varepsilon} \in B_p^k \mid \forall i_1 < \cdots < i_p < k, \varepsilon_k \in \varepsilon_{i_1, \ldots, i_p} \}$. Then $B_p^{k+1} = B_p^k$ is a set of bounds with the required property for step $p + 1$ and we take $B = B_{n-1}$.

Now, for $\varepsilon \in B$ and $y \in Y$, we let $i_1 < \cdots < i_q$ be the indices such that $f_{i_l}(y) = \varepsilon_{i_l}$. Since $df_{i_1} \wedge \ldots \wedge df_{i_q}(y) \neq 0$ the functions $x_l = f_{i_l} - \varepsilon_{i_l}, l = 1, \ldots, q$, may be extended to a coordinates system around $y$ and in any such system the description of the partition is the one given in the lemma. \hfill $\Box$

**Lemma 3.15.** Let $Y = \bigcup_{i=1, \ldots, n} Y_i$, $f_i$, be as in assumptions 3.1 and notations 3.9. Let $F_1, \ldots, F_m \in D_{R_{n-1}}(Y)$ (i.e. $F_j$ is constructible for some stratification of $Y$, not a priori $(Y_i)_{i \in I}$). Then there exists a set of bounds $B \subset [0, +\infty)^{n-1}$ such that $\forall \underline{\varepsilon} \in B$, setting for short $T_i = T_i(\underline{\varepsilon})$, we have isomorphisms $\forall i = 1, \ldots, n, \forall j = 1, \ldots, m$:

$$ H_{Y_j}(Y; F_j) \xrightarrow{\sim} H_{T_i}(Y; F_i), \quad H(Y; (F_j)_{T_i}) \xrightarrow{\sim} H(Y; (F_j)_{Y_i}). $$

**Proof.** We prove by induction on $k$ that there exists a set of bounds $B \subset [0, +\infty)^{k}$ such that the first isomorphism holds for any $j$ and for $i = 1, \ldots, k$. For $k = 1$, this is a consequence of lemma 2.35.

Let us assume the result is true for $k$, and apply it to $F_j$ and $F_j' = R\Gamma_{Y_k}(F_j)$, $j = 1, \ldots, m$. Let $B \subset [0, +\infty]^k$ be the set of bounds obtained and $(\varepsilon_1, \ldots, \varepsilon_k) \in B, T_i, i = 1, \ldots, k$ as in the lemma. Let us set $T = T_1 \sqcup \ldots \sqcup T_k$ and $U = Y \setminus T$. By lemma 2.35, again, applied to $f = f_{k+1}$ and the complexes $R\Gamma_U(F_j)$, there exists $\varepsilon_{k+1}^0 > 0$ such that $\forall 0 < \varepsilon_{k+1} < \varepsilon_{k+1}^0$ we have, setting $T_{k+1} = f_{k+1}^{-1}([0, \varepsilon_{k+1}]) \cap Y_{k+1} \cup U, H_{Y_{k+1}}(U; F_j) \xrightarrow{\sim} H_T(Y_{k+1}; F_j)$.

Since $T_{k+1} \subset U$, we have $H_{Y_{k+1}}(U; F_j) \cong H_{Y_{k+1}}(Y; F')$ and we have to prove that $H_{Y_{k+1}}(U; F_j) \cong H_{Y_{k+1}}(Y; F')$. Using an excision exact sequence, we are reduced to proving the vanishing of $A_j = H_{Y_{k+1}}(U; F_j)$. Now $A_j \cong H_T(Y; F_j')$ and, for $i \leq k$, we have

$$ H_{Y_j}(Y; F_j') \xrightarrow{\sim} H_{T_i}(Y; F_j') \quad \text{and} \quad H_{Y_j}(Y; F_j') \xrightarrow{\sim} H_{Y_j \cap Y_{k+1}}(Y; F_j) = 0. $$

Let us set $T_i' = T_1 \sqcup \ldots \sqcup T_i$; we have distinguished triangles $R\Gamma_{T_i-1}(-) \to R\Gamma_{T_i}(-) \to R\Gamma_{T_i'}(-)$. We deduce by induction on $i \leq k$ that $H_{T_i'}(Y; F_j') = 0$. For $i = k$ we obtain $A_j = 0$ as desired and this concludes the proof of the first isomorphism.

The proof of the second isomorphism is the same, again using lemma 2.35 (we just note that we apply the induction hypothesis to $F_j$ and $F_j'' = (F_j)_{Y_{k+1}}$, and lemma 2.35 to $(F_j)_{U'}$). \hfill $\Box$
We still consider $Y$ satisfying assumptions 3.1 and we keep notations 3.9. We consider moreover local systems $L^i$ defined on neighbourhoods of $Y_i$. The sheaves $L^i_{Y_i}$ are well-defined and, for $\varepsilon \in ]0, +\infty[^{n-1}$ small enough, the $L^i_{T_i(\varepsilon)}$ are also well-defined.

**Lemma 3.16.** Let $Y = \bigsqcup_{i=1, \ldots, n} Y_i$, $L^i$, be as above. Let $U \subset Y$ be an analytic open subset, $U'$ a neighbourhood of $Y$ and $f : U' \rightarrow \mathbb{R}$ an analytic function with 1 as regular value. We assume that the smooth hypersurface $S = \{ y \in U; f(y) = 1 \}$ meets the strata $U \cap Y_i$ transversally. Let $L$ be a sheaf on $Y$ which is a local system in a neighbourhood of $S$. We set $U_+ = \{ y; f(y) > 1 \}$. Then there exists a set of bounds $B \subset ]0, +\infty[^{n-1}$ such that $\forall \varepsilon \in B$, setting $T_i^\varepsilon = U \cap T_i(\varepsilon)$, we have an isomorphism $R\text{Hom}(L_S, L^i_{T_i^\varepsilon}|U) \simeq (L^* \otimes L^i|U)_{SS{T_i^\varepsilon}|[-1]}$ and the morphisms

$$R\text{Hom}(L_S, L^i_{T_i^\varepsilon}|U_+) \rightarrow R\text{Hom}(L_S, L^i_{T_i^\varepsilon}) \rightarrow R\text{Hom}(L_S, L^i_{Y_i})$$

are isomorphisms.

**Proof.** We have $R\text{Hom}(L_S, C_U) \simeq L^1_{SS[-1]}$ because $S$ is a smooth, relatively oriented, hypersurface. The micro-support of $L_S$ is $SS(L_S) = T_2^1 U$ and we also have the bound $SS(L^i_{Y_i}) \subset \bigcup_j T_j^1 U$. By the transversality hypothesis, we have $SS(C_S) \cap SS(L^i_{Y_i}) \subset T_1^1 U$. Hence, by lemma 2.4, we have isomorphisms

$$R\text{Hom}(L_S, L^i_{T_i^\varepsilon}|U) \simeq R\text{Hom}(L_S, C_U) \otimes L^i_{Y_i}|U \simeq (L^* \otimes L^i|U)_{SS{T_i^\varepsilon}|[-1]}.$$

For $\varepsilon_1$ small enough, $\partial T_i^\varepsilon$ also is transversal to $S$ and, since $SS(L^i_{T_i^\varepsilon})$ is the outer conormal of $\partial T_i^\varepsilon$ in $U$, we obtain similarly $R\text{Hom}(L_S, L^i_{T_i^\varepsilon}|U) \simeq (L^* \otimes L^i|U)_{SS{T_i^\varepsilon}|[-1]}$.

Hence, to show the last isomorphism, it is sufficient to find a set of bounds such that the morphisms $H(S; (L^* \otimes L^i|U)_{SS{T_i^\varepsilon}}) \rightarrow H(S; (L^* \otimes L^i|U)_{SSY})$ are isomorphisms. But this follows from lemma 3.10 applied to the stratification $S = \bigsqcup_i S \cap Y_i$.

Let us prove that the remaining morphism also is an isomorphism. Let us set $U_- = f^{-1}([-\infty, 1])$. We have $R\text{Hom}(C_{U_-}, C_U) \simeq C_{U_-}$ and $SS(C_{U_-})$ is the inner conormal to $U_-$. We deduce as above $R\text{Hom}(C_{U_-}, L^i_{T_i^\varepsilon}) \simeq L^i_{T_i^\varepsilon|U_-}$ and:

$$R\text{Hom}(L_S, L^i_{T_i^\varepsilon|U_-}) \simeq R\text{Hom}(L_S, R\text{Hom}(C_{U_-}, L^i_{T_i^\varepsilon})) \simeq R\text{Hom}(L_S, L^i_{T_i^\varepsilon}) \simeq 0.$$

Using the distinguished triangle $L^i_{T_i^\varepsilon|U_+} \rightarrow L^i_{T_i^\varepsilon} \rightarrow L^i_{T_i^\varepsilon|U_-}[1]$, we conclude that $R\text{Hom}(L_S, L^i_{T_i^\varepsilon|U_+}) \simeq R\text{Hom}(L_S, L^i_{T_i^\varepsilon})$ as desired. \hfill $\Box$

**Proposition 3.17.** Let $Y = \bigsqcup_{i=1, \ldots, n} Y_i$, $L^i$, be as in lemma 3.10. For $\varepsilon \in ]0, +\infty[^{n-1}$ and a union of strata $Z \subset Y$, we set:

$$T_i = T_i(\varepsilon), \quad T_Z = T_Z(\varepsilon), \quad \text{for } i < j, \quad T_{ij} = (T_i \cap T_j) \setminus (\bigcup_{1 \leq l < j} T_l).$$

There exists a set of bounds $B \subset ]0, +\infty[^{n-1}$ such that $\forall \varepsilon \in B$:

(i) for any $i < j$, the morphism $\alpha_{ij} : (L^{*i} \otimes L^j)|T_{ij}|[-1] \rightarrow R\text{Hom}(L^i_{T_{ij}}, L^j_{T_{ij}})$ is an isomorphism,

(ii) for any $i, j$, the natural morphisms

$$R\text{Hom}(L^i_{T_{ij}}, L^j_{T_{ij}}) \xrightarrow{\alpha_{ij}} R\text{Hom}(L^i_{T_{ij}}, L^j_{T_{ij}}) \xleftarrow{b_{ij}} R\text{Hom}(L^j_{Y_i}, L^j_{Y_j})$$

are isomorphisms,

(iii) for any $i, j, p$ with $i, j > p$, we have an isomorphism

$$R\Gamma(Y; (R\text{Hom}(L^i_{T_{ij}}, L^j_{T_{ij}}))_{T_{ip}}) \simeq R\Gamma(Y; (R\text{Hom}(L^i_{Y_i}, L^j_{Y_j}))_{Y_p}).$$
Proof: (i) and (ii). By lemma 3.13, we know that the $b_{ij}$ are isomorphisms for $\varepsilon$ in a suitable set of bounds. We prove by induction on $k$ that there exists a set of bounds $B \subset \{0, +\infty\}^{n-1}$ such that, for $i = 1, \ldots, k$ and $j = 1, \ldots, n$, the morphisms $a_{ij}$ and $\alpha_{ij}$ are isomorphisms. For $k = 1$, we set for short $K^j = L^1 \otimes L^j$ and we note that

$$\text{RHom}(L_1^j, L_1^j) \simeq \text{RHom}(L_1^j, L_1^j), \quad \text{RHom}(L_1^j, L_1^j) \simeq \text{RHom}(L_1^j, L_1^j),$$

Hence by lemma 2.10, we may choose $\varepsilon_0^j$ such that $\forall 0 < \varepsilon < \varepsilon_0^j, a_{11}$ is an isomorphism and

$$\text{RHom}(L_1^j, L_1^j) \simeq \text{RHom}(Y_1, K^1_2).$$

Let us prove that we may also find a set of bounds $B' \subset \{0, +\infty\}^{n-2}$ such that for $(\varepsilon_1, \ldots, \varepsilon_{n-1}) \in \{\varepsilon_1\} \times B'$ the $a_{ij}$ are isomorphisms. For $j \geq 2$, we have $\text{RHom}(Y, K^1_2) = 0$ because $Y_1 \cap Y_2 = \emptyset$. Hence the identity $16$ and the distinguished triangle $L_n^1(Y_1, K^1_2) \to L_1^1(Y_1, L_n^1) \to L_n^1(\partial T_1, K^1_2)$ imply $\text{RHom}(L_1^1, L_n^1) \simeq \text{RHom}(L_1^1, L_n^1)$. We may also assume since the beginning that $\varepsilon_0^1$ is small enough so that $\partial T_1$ is smooth. Then lemma 3.10, applied with $S = \partial T_1$, yields a set of bounds $B' \subset \{0, +\infty\}^{n-2}$ such that $\forall (\varepsilon_2, \ldots, \varepsilon_{n-1}) \in B'$, we have isomorphisms:

$$\text{RHom}(L_1^j, L_1^j) \simeq K_{T, j}^1[-1], \quad \text{RHom}(L_1^j, L_1^j) \simeq \text{RHom}(L_1^j, L_1^j).$$

(With the notations of lemma 3.10, we have $T_2 = T_2' \cap U$ and $T_1 = T_1')$. Now we just have to note that $\text{RHom}(L_1^j, L_1^j) = 0$, because $T_1 \cap \text{Int}(T_1) = \emptyset$, and use the same distinguished triangle as above to conclude.

Now let us assume the conclusion is true for $k$. Let $B$ be the set of bounds given in the statement and $(\varepsilon_0^1, \ldots, \varepsilon_0^j_{n-1}) \in B$. Let us set $T = T_1 \sqcup \ldots \sqcup T_k$, $U = Y \setminus T$. Arguing as in the case $k = 1$ on the open subset $U$, we find a set of bounds $B' \subset \{0, +\infty\}^{n-k-1}$ such that $\forall (\varepsilon_1, \ldots, \varepsilon_{n-1}) \in B \cap \{\varepsilon_0^1, \ldots, \varepsilon_0^j_{n-1}\} \times B'$ we have, with the “new” $T_j$ (but the $T_i$ for $i < k$ stay the same):

$$\text{RHom}(L_{T^{k+1}_j}^1|_U, L_{T^{k+1}_j}^1|_U) \simeq (L_{T^{k+1}_j}^1 \otimes L_{T^{k+1}_j}^1)_{T^{k+1}_j}[-1], \quad \text{RHom}(L_{T^{k+1}_j}^1|_U, L_{T^{k+1}_j}^1|_U) \simeq \text{RHom}(L_{T^{k+1}_j}^1|_U, L_{T^{k+1}_j}^1|_U)$$

where $T_{k+1} = U \cap \partial T_{k+1} \cap (\bigcup_{j<i<j} \overline{T_j})$. Since $T_{k+1} \subset U$, we have in fact $\text{RHom}(L_{T^{k+1}_j}^1|_U, L_{T^{k+1}_j}^1|_U) = \text{RHom}(L_{T^{k+1}_j}^1, L_{T^{k+1}_j}^1)$ and the same with $Y_j$ instead of $T_j$. Hence $a_{k+1,j}$ is an isomorphism, for $j \geq k+1$. Since we have chosen $(\varepsilon_1, \ldots, \varepsilon_{n-1}) \in B$, $a_{1,j}$ is an isomorphism for $i < k$ and any $j$. It just remains to check the case $i = k + 1$ and $j \leq k$. But $T_{k+1}$ is contained in the open set $U$ which does not meet $T_j$ and $Y_j$. Hence $\text{RHom}(L_{T^{k+1}_j}^1, L_{T^{k+1}_j}^1) = 0$ and the assertion is trivially true.

Now let us consider $a_{k+1,j}$. Let $u : U \to Y$ be the embedding. Since $T_{k+1} \subset U$, we have:

$$\text{RHom}(L_{T^{k+1}_j}^1, L_{T^{k+1}_j}^1) \simeq Ru_* \text{RHom}(L_{T^{k+1}_j}^1|_U, L_{T^{k+1}_j}^1|_U) \simeq Ru_*(L_{T^{k+1}_j}^1 \otimes L_{T^{k+1}_j}^1)_{T^{k+1}_j}[-1].$$

We may assume moreover, up to shrinking $B$, that the conclusions of lemma 3.13 hold. Hence we have local coordinates $(\xi_i)$ such that $U = \{\xi_i > 0; i = 1, \ldots, m\}$ and $C = T_{k+1,j}'$ is a convex locally closed subset of $U$. It follows that $Ru_* \mathcal{C}_C \simeq \mathcal{C}_D$.
for some locally closed subset $D \subset Y$. More precisely, if $C$ is closed in $U$, we check that $(Ru_x, C)_x = C$ if $x \in \overline{C}^Y$ and is 0 else. Hence $D = \overline{C}^Y$. If we write $C = \overline{C}_i \cap C_1$, with $C_1$ closed in $U$, we obtain, by the distinguished triangle $C_C \to C_{\overline{C}_i} \to C_{C_1} \to$, that $D = \overline{C}_i \setminus \overline{C}_1$. When applied to $T_{i+1}^j = \partial T_{i+1} \cap (\overline{T}_j \setminus \bigcup_{i<j} \overline{T}_i) = (T_{i+1} \cap T_j) \setminus \bigcup_{i<j} \overline{T}_i$, this formula for $D$ gives the expression of the proposition.

(iii) We first consider $p < i < j$, fixed. We set $F = R\mathcal{H}om(L^i_{Y_0}, L^j_{Y_0})$ and $G = R\mathcal{H}om(L^i_{Y_0}, L^j_{Y_0})$. By lemma 3.15 applied to $F$, there exists a set of bounds $B_0$, such that, $\forall (\varepsilon) \in B_0$, we have $\Gamma(Y; F_{T_y}) \simeq \Gamma(Y; F_{Y_0})$. Let us also consider the Verdier dual $DF = R\mathcal{H}om(F, C_Y(dy))$ (where $dy$ is the real dimension of $Y$).

In the proof of lemma 3.15 at the $p^{th}$ step, when $(\varepsilon_1, \ldots, \varepsilon_{p-1})$ are fixed, we can choose $\varepsilon_p$ so as to have moreover $\Gamma(\text{Int}(T_p); DF) \simeq \Gamma(\text{Int}(T_p) \cap Y_p; DF)$ (for this we apply lemma 2.2 to $DF$ restricted to $Y \setminus \bigcup_{k<p} T_k$). We also set $S_p = T_p \cap \text{Int}(T_p)$.

Let us prove that
\[ \Gamma(Y; F_{T_p}) \simeq \Gamma(\overline{S}_p; F_{S_p}). \]

We note that $\Gamma(Y; F_{T_p}) \simeq \Gamma(\overline{S}_p; F_{S_p})$. Since $F$ is constructible we get $(DF)_{T_p} \simeq D(\text{Int}(T_p); DF) = 0$, so that $\Gamma(\text{Int}(T_p); DF) \simeq \Gamma(\text{Int}(T_p) \cap Y_p; DF) = 0$. By Poincaré-Verdier duality, this gives the vanishing of the cohomology with compact supports, $\Gamma(\text{Int}(T_p); F) = 0$. Since $T_p$ is compact, this is equivalent to $\Gamma(Y; F_{\text{Int}(T_p)}) = 0$. By the distinguished triangle $F_{\text{Int}(T_p)} \to F_{T_p} \to F_{S_p} \to$, we deduce $\Gamma(Y; F_{T_p}) \simeq \Gamma(Y; F_{S_p})$. This yields (17) because $\Gamma(Y; F_{T_p}) \simeq \Gamma(Y; F_{T_p})$ and $\Gamma(Y; F_{T_p}) \simeq \Gamma(\overline{S}_p; F_{S_p})$.

On the other hand, $\text{supp} G \subset T_i \cap T_j$ and, for $k > p$ we have $T_k \cap T_p \subset S_p$, so that $\Gamma(Y; G_{T_p}) \simeq \Gamma(\overline{S}_p; G_{S_p})$. Together with (17), this shows that (iii) for our $p, i, j$, is equivalent to
\[ \forall S \in \mathcal{S}_{p, \text{min}}, \quad \Gamma(S; F) \simeq \Gamma(S; G). \]

We let $\mathcal{S}_p$ be the smallest set of subsets of $\overline{S}_p$ stable by taking intersections and complements and containing the $\overline{S}_p \cap T_k$, for $k < p$. This set $\mathcal{S}_p$ is finite and we denote by $\mathcal{S}_{p, \text{min}}$ the set of its minimal elements (for the inclusion relation); then any element of $\mathcal{S}_p$ is a union of elements of $\mathcal{S}_{p, \text{min}}$. By lemma 3.14 (iii), up to shrinking the set of bounds $B_0$, any $S \in \mathcal{S}_{p, \text{min}}$ is a locally closed submanifold of $Y$, transversal to every stratum $Y_i$ that it meets. Let us first prove:

\[ \forall S \in \mathcal{S}_{p, \text{min}}, \quad \Gamma(S; F) \simeq \Gamma(S; G). \]

By part (i) of the proposition, we have $G|_{S} \simeq (L^i \otimes L^j)_{T_i \cap S}[-1]$. Let us stratify $S$ by $S = \bigsqcup_{k > p} (S \cap Y_k)$ and define $T_k^S, T_k^S$, similarly as $T_k, T_{kl}$, with the functions $f_m|_{S}$. We have $\overline{T_k^S} = T_k \cap S$. Hence (i) applied to $S$ gives
\[ R\mathcal{H}om(L^i_{T_k^S}, L^j_{T_k^S}) \simeq (L^i \otimes L^j)_{T_k^S}[-1] \simeq (L^i \otimes L^j)_{T_{ij}}[S] [-1] \simeq G|_{S}. \]

Now, part (ii) of the proposition applied to $G$ gives, up to shrinking $B_0$ again:
\[ \Gamma(S; G) \simeq \Gamma(S; G) \simeq \Gamma(S; G) \simeq \Gamma(S; F), \]

where the last isomorphism follows from the transversality of $S$ and $Y_i, Y_j$. This is (19). We will deduce:

\[ \forall V \in S_p, \ V \text{ closed in } \overline{S}_p, \quad \Gamma(V; F) \simeq \Gamma(V; G). \]
Since $F$ is constructible with respect to the stratification $\bigsqcup Y_i$ and any $S \in S_{p,\min}$ is transversal to every stratum $Y_i$, we have the isomorphisms, $\forall S \in S_{p,\min}$, $u_S : R\Gamma(\mathcal{F}; F) \cong R\Gamma(S; F)$.

Recall that $G \simeq (L^i \otimes L^j)_{T_{ij}}[-1]$. For any $S \in S_{p,\min}$, the inclusion $(T_{ij} \cap S) \subset T_{ij} \cap S$ is an equivalence of homotopy (by lemma 3.14 (iii)), so that we also have an isomorphism $v_S : R\Gamma(\mathcal{F}; G) \cong R\Gamma(S; G)$.

Now let us prove (20). Let us write $V = V_1 \sqcup \ldots \sqcup V_r$, with $V_i \in S_{p,\min}$, and argue by induction on $r$. For $r = 1$, our assertion is (19). We may assume that $V_r$ is of maximal dimension (among the $V_i$), so that $V' = V \setminus V_r$ is closed and the induction hypothesis applies to $V'$. We have $\overline{V}_r = V$ or $\overline{V}'' = \overline{V}' \cap V'$ is a closed union of less than $r$ subsets $V_i$ and the induction hypothesis also applies to $V''$. By (19) and the isomorphisms $u_{V_i}$, $v_{V_r}$, we have $R\Gamma(\overline{V}_r; F) \cong R\Gamma(\overline{V}_r; G)$. We conclude by the Mayer-Vietoris distinguished triangle

$$R\Gamma(V; F) \to R\Gamma(\overline{V}_r; F) \oplus R\Gamma(V'; F) \to R\Gamma(V''; F) \xrightarrow{\pm 1},$$

and the similar one for $G$.

Now we can prove (19) (and thus (iii) for our $i,j,p$). We have an excision distinguished triangle,

$$R\Gamma(\overline{S}_p; F_{S_p}) \to R\Gamma(\overline{S}_p; F) \to R\Gamma(\overline{S}_p \setminus S_p; F) \xrightarrow{+1},$$

and a similar one for $G$. The last two terms of these distinguished triangles are isomorphic because of (20) applied to $V = \overline{S}_p$ and $V = \overline{S}_p \setminus S_p$. Hence the first terms are also isomorphic, as desired.

For $i = j$, the same proof works, replacing the isomorphism $G \simeq (L^i \otimes L^i)_{T_{ij}}[-1]$ by $G \simeq (L^i \otimes L^j)_{T_{ij}}$. We still have $R\Hom(L^j_{Y_i}, L^i_{Y_j}) \simeq G_S$ and $(T_{ij} \cap S) \subset T_{ij} \cap S$ is an equivalence of homotopy. For $j < i$, we have $F = R\Hom(L_{Y_j}, L^i_{Y_j}) = 0$ because $Y_j$ is open in $T_{ij}$ and $Y_j \subset (T_{ij} \setminus Y_i)$. In the same way $G = 0$ and (iii) is trivial.

We let $B_{i,j,p}$ be the subset of $B_0$ formed by the $i$ such that (iii) holds for $p, i,j$. This is a set of bounds and the intersection of all $B_{i,j,p}$, for $i,j > p$, gives us the desired set of bounds.

Proof of proposition 3.17. Let us set, for $i = 1, \ldots, n$, $k = 1, 2$, $L^k_i = \mathcal{L}^k_i$ if $Y_i \subset Z_k$ and $L^k_i = 0$ else. We set $L^i = L_i^1 \oplus L_i^2$; this is a local system defined in a neighbourhood of $Y_i$. We first choose a set of bounds $B$ such that the conclusions of lemma 3.13 and proposition 3.17 hold.

(i) Let us prove (19). We begin with the case where $Z_1 = Y_i$ is a single stratum. Let $Z' \subset Z_2$ be a closed subset of $Z_2$ (which is a union of strata). Set $W = Z_2 \setminus Z'$. We have $T_{Z_2} = T_{Z'} \cup T_W$ and $T_{Z'}$ is closed in $T_{Z_2}$. This gives a morphism of distinguished triangles:

$$\begin{array}{ccc}
\mathcal{C}_{W} & \to & \mathcal{C}_{Z_2} \\
\downarrow & & \downarrow \mathcal{C}_{Z_2} \to \mathcal{C}_{Z'} \\
\mathcal{C}_{W} & \to & \mathcal{C}_{Z_2} \to \mathcal{C}_{Z'} \to \mathcal{C}_{Z'} \\
& & \mathcal{C}_{Z_2} \to \mathcal{C}_{Z'} \\
end{array}$$

by $\mathcal{L}^2$ and applying the functor $R\Hom(L^i_{T_{ij}}, \cdot)$, where $Y_i$ is any stratum of $Y$, we
obtain the morphism of distinguished triangles:

\[
\begin{array}{ccc}
\text{RHom}(L_{T_1}^1, \mathcal{L}_{T_2}^2) & \longrightarrow & \text{RHom}(L_{T_1}^1, \mathcal{L}_{Z_2}^2) \\
\text{RHom}(L_{T_1}^2, \mathcal{L}_{T_2}^2) & \longrightarrow & \text{RHom}(L_{T_1}^2, \mathcal{L}_{Z_2}^2) \\
\text{RHom}(L_{T_1}^2, \mathcal{L}_{T_2}^2) & \longrightarrow & \text{RHom}(L_{T_1}^2, \mathcal{L}_{Z_2}^2) +1
\end{array}
\]

It follows that the morphism \( f_i \) in this diagram is an isomorphism, for any \( i = 1, \ldots, n \). Indeed this is true if \( Z_2 \) consists of one stratum, say \( Y_j \), by proposition \ref{prop:excision-distinguished-triangle}. We have here \( L^j = L^{ij} \oplus L^{2j} \), and \( f_i \) is the \( L^{2j} \)-component of isomorphism \( a_{ij} \) of this proposition. Then the above diagram allows an induction on the number of strata of \( Z_2 \).

The same reasoning, on the first argument of \( \text{RHom} \), gives the similar isomorphism with \( L_{T_1}^1 \) replaced by \( \mathcal{L}_{Z_2}^1 \). The same proof, using the isomorphisms \( b_{ij} \) of proposition \ref{prop:excision-distinguished-triangle} yields the second isomorphism of \ref{prop:excision-distinguished-triangle}.

(ii) We prove \ref{prop:excision-distinguished-triangle} first in the case where \( Z_1 = Y_i \) and \( Z_2 = Y_j \). Let us set \( F = R\text{Hom}(L_{1j}^1, \mathcal{L}_{1j}^2) \), \( G = R\text{Hom}(L_{1j}^2, \mathcal{L}_{1j}^2) \) and \( Z' = Z \cap Y_i \cap Y_j \). We see that \( F_Z = F_{Z'} \) and \( G_T = G_{T_{Z'}} \), hence we may assume that \( Z \) consists of strata \( Y_p \) with \( p < i, j \) (recall that \( Z_1, Z_2 \subset Y \setminus Z \), so that \( p \neq i, p \neq j \)). Let us argue by induction on the number of strata of \( Z \). By proposition \ref{prop:excision-distinguished-triangle} \ref{prop:excision-distinguished-triangle} is true when \( Z = Y_p \) with \( p < i, j \). Let us choose \( Y_p \subset Z \) such that it is open in \( Z \). We have an excision distinguished triangle \( R\Gamma(Y; F_{Y_p}) \to R\Gamma(Y; F_Z) \to R\Gamma(Y; G_{Z'}) +1 \), and a similar one with \( G_{T_p} \). By induction \( R\Gamma(Y; F_{Z'\setminus Y_p}) \simeq R\Gamma(Y; G_{T_{Z'\setminus Y_p}}) \), and by the first step \( R\Gamma(Y; F_{Y_p}) \simeq R\Gamma(Y; G_{T_p}) \). Hence we obtain \( R\Gamma(Y; F_Z) \simeq R\Gamma(Y; G_{T_Z}) \) as desired.

Going from a single stratum to arbitrary locally closed sets \( Z_1, Z_2 \), is the same as in the proof of \ref{prop:excision-distinguished-triangle}.

(iii) Let us first assume that \( Z_1 \) consists of a single stratum. Since the statement is local on \( Y \), we may take coordinates as in lemma \ref{prop:excision-distinguished-triangle} and assume \( \mathcal{L}^1 = \mathcal{L}^2 = \mathcal{C}_{R^d} \). We set \( C_l = \{ x_1 > 0, \ldots, x_{l-1} > 0, x_l \leq 0 \} \) and assume that \( Z_1 = C_l \). We set \( U = \{ x_1 > 0, \ldots, x_{l-1} > 0 \} \) and let \( u: U \to R^d \) be the inclusion. Since \( Z_2 \) is locally closed and \( Z_1 \subset Z_2 \), there exists \( l \geq i \) such that \( U \cap T_{Z_2} = C_l \sqcup \ldots \sqcup C_t \). Since \( C_l \subset U \), we have:

\[
\text{RHom}(\mathcal{C}_{C_l}, \mathcal{C}_{T_{Z_2}}) \simeq Ru_* \text{RHom}(\mathcal{C}_{C_l}, \mathcal{C}_{T_{Z_2}}|U) \simeq Ru_*(\mathcal{C}_{C_l \setminus C_i}),
\]

where \( C_i = C_{i+1} \sqcup \ldots \sqcup C_l \). Now \( Ru_*(\mathcal{C}_{C_l \setminus C_i}) \simeq \mathcal{C}_{C_l \setminus C_i} \) is concentrated in degree 0.

We deduce the result for an arbitrary locally closed subset \( Z_1 \subset Z_2 \) using an excision distinguished triangle.

(iv) We first note that we may assume \( V = Y \). Indeed, by lemma \ref{prop:excision-distinguished-triangle} applied to the complex \( G = R\text{Hom}(\mathcal{L}_{T_1}^1, \mathcal{L}_{T_2}^2) \), we may choose a set of bounds such that \( \text{RHom}(\mathcal{L}_{Z_1}^1|_V, \mathcal{L}_{Z_2}^2|_V) \simeq \text{RHom}(\mathcal{L}_{T_1}^1|_V, \mathcal{L}_{T_2}^2|_V) \). Hence, if \ref{prop:excision-distinguished-triangle} is true for global sections, applying it to \( Y = T_V \), with the induced stratification and local systems, gives the result for any open \( V \).

Let us set \( F = \text{Hom}(\mathcal{L}_{T_1}^1, \mathcal{L}_{T_2}^2), U = Y \setminus (\overline{T_1} \setminus Z_1) \) and show that we may replace \( Y \) by \( T_U \). Since \( T_{Z_2} \subset T_U \), we have \( F \simeq \Gamma_{T_U}(F) \). But \( F|_{T_U} \simeq (\mathcal{L}^1 \otimes \mathcal{L}^2)_{T_{Z_1 \cap Z_2}} \) and locally around any point of \( \overline{T_U} \), the inclusions \( T_{Z_1 \cap Z_2} \subset T_U \subset Y \) are homeomorphic.
to inclusions of convex subsets of $\mathbb{R}^d$. Hence $\Gamma_{T_U}(F) \simeq R\Gamma_{T_U}(F)$ and $R\Gamma(Y; F) \simeq R\Gamma(T_U; F)$. We also have $G \simeq R\Gamma_U(G)$, hence $R\text{Hom}(L^1_{Z_1}, L^2_{Z_2}) \simeq R\Gamma(U; G)$. By lemma 3.14 applied to $G$, we have, up to shrinking the set of bounds, $R\Gamma(U; G) \simeq R\Gamma(T_U; G) \simeq R\text{Hom}(L^1_{Z_1}|_{T_U}, L^2_{Z_2}|_{T_U})$. Hence we may replace $Y$ by $T_U$ and assume $Z_1$ is closed.

By assertion (iii) proved above, we have, setting $F' = R\text{Hom}(L^1_{T_{Z_1}}, L^2_{T_{Z_2}})$, $F_{T_{Z_2}} \simeq F_{T_{Z_2}}$. Since $Z_1$ is closed we also have an exact sequence $0 \to \Gamma_{T_{Z_2}}, L^2_{T_{Z_2}} \to L^2_{T'_{Z_2}}$, so that $F_x = 0$ for $x \notin T_{Z_2}$, and $F_{T_{Z_2}} \simeq F$. Hence $R\Gamma(Y; F) \simeq R\Gamma(Y; F')_{T_{Z_2}}$ and, setting $W = T_{Z_2} \setminus Z_2$ (which is closed), we have the distinguished triangles:

$$R\Gamma(Y; F')_{T_{Z_2}} \to R\Gamma(Y; F') \to R\Gamma(Y; F')_{+1},$$

$$R\text{Hom}(L^1_{T_{Z_1}, w}, L^2_{T_{Z_2}}) \to F' \to R\text{Hom}(L^1_{T_{Z_1}, (y \setminus w)}, L^2_{T_{Z_2}})_{+1}.$$  

By [13], $R\Gamma(Y; F') \simeq R\text{Hom}(L^1_{Z_1}, L^2_{Z_2})$, thus the first triangle implies that [13] is equivalent to the vanishing of $R\Gamma(Y; F')_{T_{Z_2}}$. Using the second triangle, this vanishing follows from the two sequences of isomorphisms below:

(21) $R\Gamma(Y; R\text{Hom}(L^1_{T_{Z_1}, w}, L^2_{T_{Z_2}})_{T_{Z_2}}) \simeq R\text{Hom}(L^1_{T_{Z_1}, w}, L^2_{T_{Z_2}})$

(22) $\simeq R\text{Hom}(L^1_{Z_1, w}, L^2_{Z_2})$

(23) $\simeq R\text{Hom}(L^1_{Z_1, w}, R\Gamma(Z_2, L^2_{Z_2})) = 0,$

(24) $R\Gamma(Y; R\text{Hom}(L^1_{T_{Z_1}, (y \setminus w)}, L^2_{T_{Z_2}})_{T_{Z_2}}) \simeq R\Gamma(Y; R\text{Hom}(L^1_{Z_1, (y \setminus w)}, L^2_{Z_2})_{w}) = 0.$

Let us explain these isomorphisms: (21) is true because the application of the functor $(\cdot)_{T_{Z_2}}$ does not change anything (since the support of the complex of sheaves is included in $T_{Z_2}$). (22) follows from (23), (22) comes from the hypothesis $R\mu_* (L^2_{Z_2}) = \mu_*(L^2_{Z_2})$ and the vanishing follows from $(Z_1 \cap W) \cap Z_2 = \emptyset$.

The first isomorphism in (24) follows from (23). The smoothness hypothesis gives $R\text{Hom}(L^1_{Z_1, (y \setminus w)}, L^2_{Z_2}) \simeq R\nu_*(L^1 \otimes L^2)|_{Z_1 \cap Z_2}$. Since $L^1$ is defined on $Z_1$, which is closed, we also have $R\nu_* ((L^1 \otimes L^2)|_{Z_1 \cap Z_2}) \simeq R\nu_* ((L^1 \otimes L^2)|_{Z_1 \cap Z_2})$; this implies the desired vanishing and concludes the proof of (13). □

3.2. Direct image to $I$. We consider a stratified analytic manifold $Y = \bigsqcup_{i \in I} Y_i$ endowed with normal crossings divisors $(D_\alpha)_{\alpha \in A}$ and sheaves $(L_\alpha)_{\alpha \in A}$, satisfying assumptions 3.1. As in notations 3.12 we fix $\xi$ in a set of bounds such that the conclusions of proposition 3.1 hold for any pair $(\alpha, \beta) \in A^2$. We keep the notations $T_i$, for $i \in I$, and $T_\alpha', L_\alpha'$, for $\alpha \in A$. We will give an equivalence between $D(Y)\langle L'_\alpha \rangle$ and a derived category of dg-modules on $I$ following the construction of [13].

**Definition 3.18.** We define a sheaf of dg-algebras on $Y$, $\Omega$, by

$$\Omega = \oplus_{(\alpha, \beta) \in A^2} \Omega_{\alpha \beta}, \quad \text{where, for } \alpha, \beta \in A, \quad \Omega_{\alpha \beta} = \Omega_Y \otimes \text{Hom}(L'_\alpha, L'_\beta).$$

The product $m_{\alpha \beta \gamma}: \Omega_{\alpha \beta} \otimes \Omega_{\beta \gamma} \to \Omega_{\alpha \gamma}$, for $\alpha, \beta, \gamma \in A$, is induced by the product of forms and the composition in the endomorphisms sheaves.

Considering the partition $Y = \bigsqcup_{i \in I} T_i$, we define $\phi': Y \to I$ (like $\phi: Y = \bigsqcup_{i \in I} Y_i \to I$) by $\phi'(T_i) = \{i\}$. We set:

$$B^{\alpha \beta} = \phi'_*(\Omega_{\alpha \beta}), \quad B = \phi'_*(\Omega) \simeq \oplus_{(\alpha, \beta) \in A^2} B^{\alpha \beta}.$$
These are sheaves of differential graded \( \mathbf{C} \)-vector spaces on \( I \), and \( \mathcal{B} \) is a sheaf of dg-algebras. We also define a direct image functor \( \gamma : \mathcal{D}^+(Y) \to \mathcal{D}_R \). For \( F \in \mathcal{D}^+(Y) \), we choose an injective resolution of \( F \), say \( F \to R_F \) (if \( R_F \) and the morphism \( F \to R_F \) depending functorially on \( F \)), and we set:

\[
\gamma(F) = \phi'_i(\sum_{\alpha \in A} \Omega_Y \otimes \text{Hom}(L'_\alpha, R_F)).
\]

We note that \( \gamma(F) \) has a natural structure of \( \mathcal{B} \)-dg-module, defined by multiplication of forms and composition of homomorphisms sheaves, like the multiplicative structure of \( \mathcal{B} \).

We also have the following description of the cohomology of the sections of \( \mathcal{B} \). Since \( \Omega_Y \) is a soft resolution of \( \mathcal{C}_Y \), \( \Omega_{\alpha\beta} \) is a soft resolution of \( \text{Hom}(L'_\alpha, L'_\beta) \). Let \( U \subset I \) be an open subset, \( V = \phi^{-1}(U) \), \( V' = \phi'^{-1}(U) = T_Y \), and let \( \alpha, \beta \in A \). By (iii) and (iv) of proposition 3.11, we obtain:

\[
\forall \phi \in \mathcal{B}, \exists \gamma \in \mathcal{B} \text{ such that } (\phi, \gamma) \text{ is a quasi-isomorphism of } \text{Hom}(L'_\alpha, L'_\beta) \text{ and } \text{Hom}(L'_\alpha, L'_\beta)
\]

Proposition 3.19. With the above notations, we set \( M_\alpha = \gamma(L'_\alpha) \), for \( \alpha \in A \). The functor \( \gamma \) induces an equivalence of categories between \( \mathcal{D}(Y)/L'_\alpha \) and \( \mathcal{D}_B(M_\alpha) \).

Proof. Since our categories are respectively generated by the \( L'_\alpha \) and the \( M_\alpha \), it is sufficient to prove that \( \gamma \) gives isomorphisms, \( \forall \alpha, \beta \in A, \forall p \in Z \):

\[
\text{Hom}_{\mathcal{D}(Y)}(L'_\alpha, L'_\beta[p]) \simeq \text{Hom}_{\mathcal{D}_B}(M_\alpha, M_\beta[p]).
\]

Indeed an inductive argument as in the proof of corollary 3.14 (but easier because, here, the functor giving the equivalence is a priori defined) implies that \( \gamma \) also gives a bijection between \( \text{Hom}_{\mathcal{D}(Y)}(L_1, L_2) \) and \( \text{Hom}_{\mathcal{D}_B}(\gamma(L_1), \gamma(L_2)) \) for any objects \( L_1, L_2 \) of \( \mathcal{D}(Y)/(L'_\alpha) \). Let us prove (26). We set \( L' = \bigoplus_{\alpha \in A} L'_\alpha \) and \( M = \gamma(L') = \bigoplus_{\alpha \in A} M_\alpha \). Then (26) is equivalent to:

\[
\forall p \in Z \quad \text{Hom}_{\mathcal{D}(Y)}(L', L'[p]) \simeq \text{Hom}_{\mathcal{D}_B}(M, M[p]).
\]

For \( \alpha \in A \), let \( L'_\alpha \to R_\beta \) be the chosen injective resolution of \( L'_\alpha \). It induces a morphism of differential graded sheaves, \( f^{\alpha\beta} \), from \( \mathcal{B}^{\alpha\beta} = \phi'_\alpha(\Omega_{\alpha\beta}) \) to \( \mathcal{B}^{\alpha\beta} = \phi'_\alpha(\Omega_Y \otimes \text{Hom}(L'_\alpha, R_\beta)) \). Since \( R_\beta \) is injective we have the isomorphisms above in \( \mathcal{D}(Y) \), which give the cohomology of sections of \( \mathcal{B}^{\alpha\beta} \), for an open set \( U \subset I \), and \( V' = \phi'^{-1}(U) \):

\[
\Omega_Y \otimes \text{Hom}(L'_\alpha, R_\beta) \simeq \text{Hom}(L'_\alpha, R_\beta) \simeq \mathcal{R}\text{Hom}(L'_\alpha, R_\beta),
\]

By (26) this means that \( f^{\alpha\beta} \) is a quasi-isomorphism of differential graded sheaves. Summing over all pairs \( (\alpha, \beta) \), we obtain a quasi-isomorphism of \( \mathcal{B} \)-modules between \( \mathcal{B} \) and \( \gamma(\bigoplus_{\beta \in A} L'_\beta) = M \). Hence we obtain:

\[
\text{Hom}_{\mathcal{D}_B}(M, M[p]) \simeq \text{Hom}_{\mathcal{D}_B}(\mathcal{B}, \mathcal{B}[p]).
\]

We have seen that \( \forall i \in I \), \( \mathcal{B}_{U_i} \) is \( K \)-projective. By (i) of assumptions 3.14 any intersection of open sets of the type \( U_i \) still is of this kind, hence \( \forall i_1, \ldots, i_n \in I \), \( \mathcal{B}_{U_{i_1} \cap \cdots \cap U_{i_n}} \) is \( K \)-projective. Let us put any total order on \( I \); we obtain a \( K \)-projective resolution of \( \mathcal{B} \) by taking the total complex of the following Čech-like complex of complexes:

\[
\cdots \to \bigoplus_{i_1 < i_2 < i_3} \mathcal{B}_{U_{i_1} \cap U_{i_2} \cap U_{i_3}} \to \bigoplus_{i_1 < i_2} \mathcal{B}_{U_{i_1} \cap U_{i_2}} \to \bigoplus_{i_1} \mathcal{B}_{U_{i_1}} \to 0.
\]
with the usual differential $(da)_{i_1,...,i_r} = \sum_{i_1<...<i_k<j<i_{k+1}<...<i_r}(-1)^ka_{i_1,...,j,...,i_r}$.
For an open set $U \subset I$, we have $\text{RHom}(\mathcal{B}_U, \mathcal{B}) \simeq \Gamma(U; \mathcal{B})$ and, for $U = U_1$, the functor $\Gamma(U_1; -) = (-)_1$ is exact. Hence:

$$\text{Hom}_D(B, \mathcal{B}[p]) \simeq H^0(\text{RHom}(\mathcal{B}_U, \mathcal{B})) \simeq H^0(\Gamma(U_1; \mathcal{B})) \simeq H^0(\Gamma(U_1; \mathcal{B})).$$

By definition $\Gamma(U_1; \mathcal{B}) = \Gamma(\phi^{-1}(U_1); \Omega)$ and we obtain that $\text{Hom}_D(M, M[p])$ is the $p$th cohomology group of the total complex of the double complex:

$$0 \to \bigoplus_{i_1 \in I} \Gamma(\phi^{-1}(U_{i_1}); \Omega) \to \bigoplus_{i_1 < i_2 \in I} \Gamma(\phi^{-1}(U_{i_1}) \cap \phi^{-1}(U_{i_2}); \Omega) \to \cdots$$

This is a Čech resolution of the complex $\Omega$, which is formed by soft sheaves. Hence $\text{Hom}_D(B, \mathcal{B}[p]) \simeq H^0(Y; \Omega)$, and, by (25), this last group is isomorphic to $\text{Ext}^p(L', L')$. In view of (28), this gives (27) and concludes the proof. 

3.3. Gysin isomorphism and product. In the previous paragraph, we have obtained a sheaf of dg-algebras, $\mathcal{B}$, on $I$ such that $D(Y)(L_a)$ is equivalent to a subcategory of $D_B$. In section 3.4 we will construct a sequence of quasi-isomorphisms between $\mathcal{B}$ and its cohomology. For this we will in particular replace sets like $U_{a,b} = \{x \in \phi^{-1}(U_i); \text{Hom}(L_a, L_b)_x \neq 0\}$ by homotopy equivalent ones. But before that, we will note that $\mathcal{B}$ is not immediately quasi-isomorphic to sections of $\Omega_{U_{a,b}} \otimes \text{Hom}(L'_a, L'_b)$. Indeed $U_{a,b}$ is not closed in $\phi^{-1}(U_i)$. The cohomology of $\mathcal{B}$ is $\text{Ext}(L_a|_{\phi^{-1}(U_i)}, L_b|_{\phi^{-1}(U_i)}) = \text{Ext}(L_a|_{\phi^{-1}(U_i)}, L_b|_{\phi^{-1}(U_i)})$, but the cohomology of $\Omega_{U_{a,b}} \otimes \text{Hom}(L'_a, L'_b)$ is $\text{Ext}(L'_a|_{U_{a,b}} \cap L'_b|_{U_{a,b}}) = \text{Ext}(L'_a|_{Y_{a,b}}, L'_b|_{Y_{a,b}})$. In our situation they are isomorphic under a “twisted” Gysin isomorphism. We build this isomorphism at the level of the de Rham algebras and describe the algebra structure. Then we use this description to obtain a quasi-isomorphism between $\mathcal{B}$ and the sheaf $\mathcal{A}$ defined in $\S 3.4$.

3.3.1. Gysin isomorphism. Let us first consider the usual Gysin isomorphism. If $M$ is an oriented manifold and $N$ a closed oriented submanifold of codimension $c$, we have an isomorphism $R\Gamma_N(C_M) \simeq C_N[-c]$. On the global sections it induces $H^*(N; C_N) \simeq H^{*+c}_N(M; C_M)$. Let us choose, by lemma (28), two open tubular neighbourhoods $U$, $V$ of $N$, such that $U \subset V$ and

$$H^*(V; C_V) \xrightarrow{\sim} H^*(U; C_U) \xrightarrow{\sim} H^*(N; C_N), \quad H^*(M; C_M) \xrightarrow{\sim} H^*(M; C_M).$$

If we assume moreover that the boundary of $U$ is smooth, we have $R\Gamma_{\overline{\Omega}_N} C_M \simeq C_U$, so that $H^*(M; C_M) \simeq H^*(M; C_U)$. Let $\delta(N, M) \in H^*_N(M; C_M)$ be the fundamental class of $N$ in $M$. We may choose a representative $\xi(N, M) \in \Gamma(M; \Omega_M)$ such that $\text{supp} \xi(N, M) \subset U$. Then, the multiplication by $\xi(N, M)$ induces a well-defined quasi-isomorphism of $\Gamma(N; \Omega_N)$-dg-modules between $\Gamma(V; \Omega_V)$ and $\Gamma(M; \Omega_M)[c]$. More generally, given local systems $L_1$ on $M$, and $L_2$ on $V$, we obtain isomorphisms between extension groups and a corresponding quasi-isomorphism between de Rham complexes:

\begin{align*}
(29) & \quad \delta(N, M) : \text{Ext}^i_{\text{D}(\mathcal{N})}(L_2|_N, L_1|_N) \xrightarrow{\sim} \text{Ext}^i_{\text{D}(\mathcal{M})}(L_2, L_1), \\
(30) & \quad \text{Ext}^i_{\text{D}(\mathcal{V})}(L_2, L_1|_V) \xrightarrow{\sim} \text{Ext}^i_{\text{D}(\mathcal{M})}(L_2, L_1), \\
(31) & \quad \xi(N, M) : \Gamma(V; \Omega_V \otimes \text{Hom}(L_2, L_1|_V)) \xrightarrow{\sim} \Gamma(M; \Omega_M \otimes \text{Hom}(\mathcal{L}_2, \mathcal{L}_1))[c].
\end{align*}

We will use (31) to build a second sheaf of dg-algebras on $I$, quasi-isomorphic to $\mathcal{B}$. For this we also need to describe the product structure: the composition
\( \text{Hom}(L_2, L_1) \otimes \text{Hom}(L_3, L_2) \to \text{Hom}(L_3, L_1) \) induces a product on the right hand side of the above isomorphisms and we want to understand the corresponding product on the left hand side.

3.3.2. Algebra structure. This paragraph mainly has an heuristic purpose, in order to justify the definition of the product \( m \) of the sheaf of dg-algebras \( A \) introduced in §3 above. We consider a complex manifold \( Y \), endowed with normal crossings divisors \( D_v \), \( v \in V \), and sheaves \( L_\alpha \), \( \alpha \in A \), satisfying assumptions §3.1. We keep the notations of §3.1 in particular for \( Z_\Delta, \Delta \in \mathcal{S}, Z_\alpha = Z_{\Delta_\alpha} \setminus \bigcup_{v \in \Delta_\alpha} D_v \), for \( \alpha \in A \). We set also:

\[
Z_{\alpha \beta} = Z_\alpha \cap Z_\beta, \quad Z_{\alpha \beta \gamma} = Z_\alpha \cap Z_\beta \cap Z_\gamma.
\]

We fix \( \alpha, \beta, \gamma \in A \) and, up to restricting ourselves to an open subset of \( Y \), we assume that \( Z_\alpha, Z_\beta, Z_\gamma \) are closed.

For \( v \in V \), \( D_v \) has a fundamental class \( \delta_v \in H^2_{D_v}(Y; C_Y) \). For \( \Delta \in \mathcal{S} \) the fundamental class of \( Z_\Delta \) in \( Y \) is \( \Delta = \prod_{v \in \Delta} \delta_v \). It belongs to \( H^2_{Z_\Delta}(Y; C_Y) \). For \( \Delta' \subset \Delta \), we have \( Z_{\Delta'} \subset Z_\Delta \) and \( Z_\Delta \cdot Z_{\Delta'} \) is the transversal intersection of \( Z_\Delta \) and \( Z_{\Delta'} \).

Hence the fundamental class \( \delta(Z_\Delta, Z_{\Delta'}) \in H^2_{Z_\Delta}(Z_{\Delta'}; C_{Z_{\Delta'}}) \), with \( d = |\Delta| - |\Delta'| \), is the image of \( \delta(Z_\Delta, Z_{\Delta'}) \in H^2_{Z_\Delta}(Z_{\Delta'}; C_{Z_{\Delta'}}) \) in \( H^2_{Z_{\Delta'}}(Z_{\Delta'}; C_{Z_{\Delta'}}) \). By abuse of notations we will write \( \delta_{\Delta_\alpha} \) for \( \delta(Z_\Delta, Z_{\Delta'}) \).

We define \( \varepsilon_{\alpha \beta} = \delta(Z_{\alpha \beta}, Z_{\beta}) \). We have:

\[
\varepsilon_{\alpha \beta} = \delta_{\Delta_{\alpha \beta}} \in H^2_{Z_{\alpha \beta}}(Z_{\beta}; C_{Z_{\beta}}), \quad \text{with} \quad \alpha \beta = |\Delta_{\alpha} \setminus \Delta_{\beta}|.
\]

We remark that \( \text{Ext}^1_{\mathcal{D}(Y)}(L_\alpha, L_\beta) \simeq \text{Ext}^1_{\mathcal{D}(Y)}((L_\alpha)_{Z_{\alpha \beta}}, L_\beta) \), so that, by (29), multiplication by \( \varepsilon_{\alpha \beta} \) gives an isomorphism:

\[
(32) \quad \varepsilon_{\alpha \beta} : \text{Ext}^1_{\mathcal{D}(Z_{\alpha \beta})}(L_\alpha|_{Z_{\alpha \beta}}, L_\beta|_{Z_{\alpha \beta}}) \cong \text{Ext}^1_{\mathcal{D}(Y)}(L_\alpha, L_\beta).
\]

From now on we forget the subscripts \( \mathcal{D}(Z_{\alpha}) \) if there is no ambiguity. We want to understand the product of two extension classes in \( \text{Ext}^0(L_\alpha, L_\beta) \) and \( \text{Ext}^0(L_\beta, L_\gamma) \) in terms of the corresponding classes on the left hand side of the above formula (see diagram (32) below). Let us introduce some notations:

\[
\begin{align*}
\Delta_1 &= \Delta_{\beta} \setminus (\Delta_{\gamma} \cup \Delta_{\alpha}), & d_1 &= |\Delta_1|, & \varepsilon_{\alpha \beta \gamma} &= \delta(Z_{\alpha \beta \gamma}, Z_{\alpha \gamma}) = \delta_{\Delta_1}, \\
\Delta_2 &= (\Delta_{\alpha} \cap \Delta_{\gamma}) \setminus \Delta_{\beta}, & d_2 &= |\Delta_2|, & \varepsilon_{\alpha \beta \gamma}^+ &= \delta_{\Delta_2}.
\end{align*}
\]

We first restrict the classes in \( \text{Ext}^0(L_\alpha|_{Z_{\alpha \beta \gamma}}, L_\beta|_{Z_{\alpha \beta \gamma}}) \) and \( \text{Ext}^0(L_\beta|_{Z_{\alpha \gamma}}, L_\gamma|_{Z_{\alpha \gamma}}) \) to \( Z_{\alpha \beta \gamma} \) and make the product, obtaining a morphism:

\[
r : \text{Ext}^0(L_\alpha|_{Z_{\alpha \beta}}, L_\beta|_{Z_{\alpha \beta}}) \otimes \text{Ext}^0(L_\beta|_{Z_{\alpha \beta}}, L_\gamma|_{Z_{\alpha \beta}}) \to \text{Ext}^{1}_{\mathcal{D}(Y)}(L_\alpha|_{Z_{\alpha \gamma}}, L_\gamma|_{Z_{\alpha \gamma}}).
\]

Now \( L_\alpha \) and \( L_\gamma \) both restrict to local systems on \( Z_{\alpha \beta \gamma} \) (and \( Z_{\alpha \beta \gamma} \)), so that we may identify \( \text{Ext}^0(L_\alpha|_{Z_{\alpha \beta}}, L_\gamma|_{Z_{\alpha \beta}}) \) with \( H^0(Z_{\alpha \gamma}; L_\alpha^* \otimes L_\gamma) \) (and the same on \( Z_{\alpha \beta \gamma} \)).

Hence multiplication by \( \varepsilon_{\alpha \beta \gamma} \) gives a morphism:

\[
\varepsilon_{\alpha \beta \gamma} : \text{Ext}^0(L_\alpha|_{Z_{\alpha \beta}}, L_\gamma|_{Z_{\alpha \beta}}) \to \text{Ext}^{2d_1}(L_\alpha|_{Z_{\alpha \gamma}}, L_\gamma|_{Z_{\alpha \gamma}}).
\]

The “correcting term” \( \varepsilon_{\alpha \beta \gamma}^+ \) is introduced to have the identity §5 below (in \( H^0_{Z_{\alpha \beta \gamma}}(Y; C_Y) \)). Multiplication by \( \varepsilon_{\alpha \beta \gamma}^+ \) gives a morphism:

\[
\varepsilon_{\alpha \beta \gamma} = \varepsilon_{\alpha \beta \gamma}^+ \cdot \varepsilon_{\alpha \beta \gamma} \cdot \varepsilon_{\alpha \gamma}.
\]

\[
(35) \quad \varepsilon_{\alpha \beta} \cdot \varepsilon_{\beta \gamma} = \varepsilon_{\alpha \beta \gamma}^+ \cdot \varepsilon_{\alpha \beta \gamma} \cdot \varepsilon_{\alpha \gamma},
\]

\[
(36) \quad \varepsilon_{\alpha \beta \gamma}^+ : \text{Ext}^0(L_\alpha|_{Z_{\alpha \beta}}, L_\gamma|_{Z_{\alpha \beta}}) \to \text{Ext}^{2d_2}(L_\alpha|_{Z_{\alpha \gamma}}, L_\gamma|_{Z_{\alpha \gamma}}).
\]
Because of (36), the composition of (35) with \( \varepsilon_{\alpha \beta \gamma} \circ r \) gives the commutative diagram
\[
\begin{array}{ccc}
\text{Ext}^i(L_\alpha | Z_{\alpha \beta}, L_\beta | Z_{\alpha \beta}) & \otimes & \text{Ext}^j(L_\beta | Z_{\beta \gamma}, L_\gamma | Z_{\beta \gamma}) \\
\varepsilon_{\alpha \beta} \otimes \varepsilon_{\beta \gamma} & \downarrow & \varepsilon_{\alpha \gamma} \\
\text{Ext}^{i+2d_{\alpha \beta}}(L_\alpha, L_\beta) & \otimes & \text{Ext}^{j+2d_{\beta \gamma}}(L_\beta, L_\gamma) \\
\end{array}
\]
(37)
where \( k = i + j + 2(d_1 + d_2) \), \( l = i + j + 2(d_{\alpha \beta} + d_{\beta \gamma}) \), \( p = (\varepsilon_{\alpha \beta}^+ \circ \varepsilon_{\alpha \beta}) \circ r \) and the bottom arrow is the composition of extension classes. This diagram gives us the composition of extension classes, through isomorphism (29). The definition of the product of \( \mathcal{A} \) in (53) is copied from the definition of \( p \) above.

3.4. **Proof of proposition 3.7**. Recall that quasi-isomorphic sheaves of dg-algebras have equivalent derived categories (see [13], proposition 1.11.2). Hence, by corollary 3.13 and proposition 3.19, the proof of proposition 3.7 will be achieved if we show that \( \mathcal{B} \) and \( \mathcal{A} \) are quasi-isomorphic.

We still consider \( Y = \bigsqcup_{i \in I} Y_i \), \( \{D_v \}_{v \in V} \), \( (L_\alpha)_{\alpha \in \mathcal{A}} \) satisfying assumptions 3.4. We keep the notation \( L_{\alpha,i} \) for the local system on \( \phi^{-1}(U_i) \) extending \( L_\alpha | Y_i \), and also notations 3.12 for \( T_i, T_\alpha, L_\alpha' \) as well as the notations of paragraph 3.3.2 for \( \delta_v, \delta_\Delta, \Delta_1, \Delta_2 \).

We choose representatives, \( \xi_v \in \Gamma(Y; \Omega^2_Y) \), of the \( \delta_v \), such that \( \text{supp} \xi_v \subset \text{Int}(T_{D_v}) \) (remember that \( T_{D_v} = \bigsqcup_{i, Y_i \subset D_v} T_i \)). For \( \Delta \subset V \) we define \( \xi_\Delta = \prod_{v \in \Delta} \xi_v \in \Gamma(Y; \Omega^2_Y) \); it is a representative of \( \delta_\Delta \) with \( \text{supp} \xi_\Delta \subset \text{Int}(T_{D_\Delta}) \). For \( \alpha, \beta, \gamma \in \mathcal{A} \) we define forms \( \eta_{\alpha \beta}, \eta_{\alpha \beta \gamma} \), representing the classes \( \varepsilon_{\alpha \beta}, (\varepsilon_{\alpha \beta \gamma} \cdot \varepsilon_{\alpha \beta}^+) \):
\[
\eta_{\alpha \beta} = \xi(\delta_\alpha \setminus \delta_\beta), \quad \eta_{\alpha \beta \gamma} = \xi(\Delta_1 \setminus \xi_\Delta).
\]
The forms \( \eta_{\alpha \beta \gamma} \) were already introduced when we defined the product of \( \mathcal{A} \). By lemma 3.20 below, the multiplications by the forms \( \eta_{\alpha \beta} \) give quasi-isomorphisms of sheaves \( g^{\alpha \beta} : \mathcal{A}^{\alpha \beta} \to \mathcal{B}^{\alpha \beta} \). By the definitions and the identity (similar to (36)) \( \eta_{\alpha \beta} \cdot \varepsilon_{\alpha \beta} = \eta_{\alpha \beta \gamma} \cdot \eta_{\alpha \beta} \), the morphism \( g = \bigoplus g^{\alpha \beta} : \mathcal{A} \to \mathcal{B} \) is a morphism of sheaves of dg-algebras.

We have thus obtained a quasi-isomorphism between \( \mathcal{B} \) and \( \mathcal{A} \), and hence an equivalence of categories between \( \text{D}(M_\alpha) \) and \( \text{D}(\mathcal{A} \otimes \mathcal{B} M_\alpha) \). We remark that \( M_\alpha \) is \( \mathcal{B} \)-flat because \( \mathcal{B} \simeq \bigoplus_{\alpha \in \mathcal{A}} M_\alpha \). It follows that \( \mathcal{A} \otimes \mathcal{B} M_\alpha \simeq N_\alpha \), for the \( \mathcal{A} \)-module \( N_\alpha = \bigoplus_{\alpha \in \mathcal{A}} \mathcal{A}^{\alpha \alpha} \). This concludes the proof of the proposition.

**Lemma 3.20.** Let us set, for \( i \in I \), \( V_i = \phi^{-1}(U_i) \), \( V'_i = \phi'^{-1}(U_i) = T_{V_i} \).

(i) For \( \alpha, \beta \in \mathcal{A} \), \( i \in \phi(Z_{\alpha \beta}) \), we have a well-defined morphism of sheaves on \( V'_i \):
\[
f^i_{\alpha \beta} : \Omega^2_{V'_i} \otimes \text{Hom}(L_{\alpha,i}, L_{\beta,i}) \to \Omega^2_{V'_i} \otimes \text{Hom}(L'_i, L'_i), \quad (\sigma \otimes u) \mapsto (\eta_{\alpha \beta} \sigma) \otimes u,
\]
where \( \sigma \in \Omega^2_{V'_i} \), \( u \in \text{Hom}(L'_i, L'_i) \). On the global sections, it induces a morphism \( g^{\alpha \beta} : \mathcal{A}^{\alpha \beta} \to \mathcal{B}^{\alpha \beta} \).

(ii) The morphisms \( g^{\alpha \beta}, i \in \phi(Z_{\alpha \beta}) \), extend to a morphism of differential graded sheaves on \( I \), \( g^{\alpha \beta} : \mathcal{A}^{\alpha \beta} \to \mathcal{B}^{\alpha \beta} \), which is a quasi-isomorphism.

**Proof.** Since \( U_i \) is open, we have, by lemma 3.14 (i), \( Y \setminus V_i \subset T_{Y \setminus V_i} = Y \setminus V'_i \). By definition of \( T_{Y \setminus V_i} \), this implies in fact \( Y \setminus V_i \subset \text{Int}(Y \setminus V'_i) \), or, as well, \( V'_i \subset V_i \).
such that $\text{supp}\, \omega^L \subset T\alpha \cap \beta \cap V_i'$. We first prove the isomorphism:

$$(38) \quad \text{Hom}(L'_\alpha, L'_\beta)|_{V_i'} \simeq \text{(Hom(L}_{\alpha,i}, L_{\beta,i}))_{T_{\alpha \beta}}.$$

We have by definition $L'_\alpha|_{V_i'} \simeq (L_{\alpha,i})_{T_{\alpha \beta}} \cap V_i'$. Since $i \in \phi(Z_{\alpha \beta})$, $Z_{\alpha} \cap V_i$ is closed in $V_i$ and, by lemma 3.14 (i), $(T_{\alpha \beta}) \cap V_i'$ is closed in $V_i'$. The same holds for $\beta$. Since $(38)$ can be checked locally, we may assume $L_{\alpha,i} = L_{\beta,i} = C_{V_i}$. Now, for two closed subsets $M, N$ of a manifold $X$, and $i_N$ the inclusion of $N$ in $X$, we have

$$\text{Hom}(C_M, C_N) \simeq (i_N) \text{Hom}(i_N^{-1} C_M, C_N) \simeq (i_N) \text{Hom}(C_{M \cap N}, C_N).$$

One checks that $\text{Hom}(C_{M \cap N}, C_N) \simeq C_U$, where $U$ is the interior of $M \cap N$ in $N$. This gives the formula for $T_{\alpha \beta}$.

We may write as well $T_{\alpha \beta} \cap (T_{\alpha \beta}) \cap V_i'$. A form $\omega \in \Gamma(V_i'; \Omega_Y)$ such that $\text{supp}\, \omega \cap (T_{\alpha \beta}) \cap V_i' = \emptyset$ belongs to $\Gamma(V_i'; \Omega_Y) \cap (T_{\alpha \beta}) \subset V_i'$. Since $Z_{\alpha} \cap T_{\beta}$ is closed in $V_i'$, we have a natural morphism from $(\Omega_Y)_{V_i'} \cap (T_{\alpha \beta}) \subset V_i'$ to $(\Omega_Y)_{V_i'} \cap (T_{\alpha \beta})$ such that $\omega$ induces an element of $\Gamma(V_i'; \Omega_Y) \cap (T_{\alpha \beta})$. This condition on the support is satisfied by $\eta_{\alpha \beta}$. Indeed, we have $\text{supp}\, \eta_{\alpha \beta} \subset \text{Int}(Z_{\alpha \beta} \cap \Delta_{\alpha})$, hence it is sufficient to check that $Z_{\Delta_{\alpha} \cap \Delta_{\beta}} \cap T_{\beta} \cap Z_{\alpha} = \emptyset$. This is equivalent to $Z_{\alpha \beta} \cap (Z_{\beta} \cap Z_{\alpha}) = \emptyset$, which is obvious. Hence the multiplication by $\eta_{\alpha \beta}$ sends $\Omega_{V_i'}$ into $(\Omega_{V_i'})_{T_{\alpha \beta}}[2d_{\alpha \beta}]$. In view of $(38)$, this gives the morphism $f_{i}^{\alpha \beta}$. Now, remember that:

$$B_{i}^{\alpha \beta} = \Gamma(V_i'; \Omega_Y \otimes \text{Hom}(L'_\alpha, L'_\beta)), \\
A_{i}^{\alpha \beta} = \Gamma(V_i; \Omega_Y \otimes \text{Hom}(L_{\alpha,i}, L_{\beta,i}))[-2d_{\alpha \beta}].$$

Hence the restriction from $V_i$ to $V_i'$, composed with $\Gamma(V_i'; f_{i}^{\alpha \beta})$ gives $g_{i}^{\alpha \beta}$.

(ii) Let us first see that the $g_{i}^{\alpha \beta}$ extend to $i \notin \phi(Z_{\alpha \beta})$. In view of the definition of $A$ (see 6.5), for $i \notin \phi(Z_{\alpha \beta}) \cup \phi(Z_{\Delta_{\alpha} \cap \Delta_{\beta}} \cap (\cup_{v \in \Delta_{\alpha} \cap \Delta_{\beta}} D_v))$, we have $A_{i}^{\alpha \beta} = 0$ and $g_{i}^{\alpha \beta}$ is trivially defined. So we assume $i \in \phi(Z_{\Delta_{\alpha} \cap \Delta_{\beta}} \cap (\cup_{v \in \Delta_{\alpha} \cap \Delta_{\beta}} D_v))$. We let $j$ be such that $V_i \cap (\cup_{v \in \Delta_{\alpha} \cap \Delta_{\beta}} D_v) = V_j$.

Let us first extend $(38)$ to this case:

$$(39) \quad \text{Hom}(L'_\alpha, L'_\beta)|_{V_i'} \simeq \text{(Hom(L}_{\alpha,i}, L_{\beta,i}))(U_{i}^{\alpha \beta}),$$

where $U_{i}^{\alpha \beta} = ((T_{\alpha \beta} \cap T_{\alpha \beta}) \setminus (T_{\beta} \setminus T_{\alpha})) \cap V_i'$. By definition, $L'_\alpha|_{T_{\beta} \cap T_{\alpha \beta}} = 0$ for $v \in \Delta_{\alpha}^{\prime}$, hence we have $L'_\alpha|_{V_i'} = (L'_\alpha)|_{V_i'}$. Denoting by $u : V_i' \to V_i'$ the inclusion, we deduce, in view of $(38)$:

$$(40) \quad \text{Hom}(L'_\alpha, L'_\beta)|_{V_i'} \simeq u_* (\text{Hom}(L'_\alpha, L'_\beta)|_{V_i'}) \simeq u_* (\text{Hom}(L_{\alpha,i}, L_{\beta,i}))(U_{i}^{\alpha \beta}).$$

By lemma 3.14 (iii), the inclusions $T_{\alpha \beta} \subset V_j \subset V_i'$ are locally homeomorphic to inclusions of convex subsets of $\mathbb{R}^d$, and $(40)$ follows.

Now we define $g_{i}^{\alpha \beta}$. Since $A_{i}^{\alpha \beta} = A_{j}^{\alpha \beta}$, we just have to check that $g_{j}^{\alpha \beta}$ factors through the restriction morphism $B_{i}^{\alpha \beta} \to B_{j}^{\alpha \beta}$. As in (i), formula $(39)$ implies the existence of a morphism

$$f_{i}^{\alpha \beta} : \Omega_{V_i'} \otimes \text{Hom}(L_{\alpha,i}, L_{\beta,i}) \to \Omega_{V_i'}^{+2d_{\alpha \beta}} \otimes \text{Hom}(L'_\alpha, L'_\beta).$$
We also note, by (11), that supp $\text{Hom}(L'_\alpha, L'_\beta)|_{V'_j} \subset \overline{V'_j}$. Since $\overline{V'_j} \subset V_j$, we obtain

$$B''_{i\alpha\beta} = \Gamma(V'_i \cap V'_j; \Omega_Y \otimes \text{Hom}(L'_\alpha, L'_\beta)).$$

Hence $\Gamma(V'_i \cap V'_j; f_{\alpha\beta}^i)$ yields a morphism from

$$B'_\alpha = \Gamma(V'_i \cap V'_j; \Omega_Y \otimes \text{Hom}(L_{\alpha,i}, L_{\beta,i})) [-2d_{\alpha\beta}]$$

to $B''_{i\alpha\beta}$. Composed with the restriction morphism from $A_j^{\alpha\beta}$ to $B'_\alpha$, it gives the required morphism $A_j^{\alpha\beta} \to B''_{i\alpha\beta}$.

Since $g_{i\alpha\beta}$ is defined by factorising $g_j^{\alpha\beta}$, it is clear that the $g^{\alpha\beta}$ commute with the restriction maps and define a morphism of sheaves, $g^{\alpha\beta}: A_j^{\alpha\beta} \to B''_{i\alpha\beta}$.

Now, let us check that $g^{\alpha\beta}$ is a quasi-isomorphism. We have to prove (with the notations 80):

(a) for $i \in I_{\alpha\beta}$, $H^0(g_{i\alpha\beta})$ is an isomorphism,

(b) for $i \in I'_{\alpha\beta}$ and $j \in I_{\alpha\beta}$ such that $V'_i \setminus (\bigcup_{i' \in \Delta_i \cap \Delta_j} D_{i'}) = V'_j$, we have

$$H^0(B''_{i\alpha\beta}) \simeq H^0(B'_{j\alpha\beta}),$$

(c) for $i \not\in I_{\alpha\beta} \cup I'_{\alpha\beta}$, $H^0(B''_{i\alpha\beta}) = 0$.

By the definition of $g_{i\alpha\beta}$, (a) follows directly from quasi-isomorphism 80. Let us verify (b). Since $L_{\alpha}|_{D_{i'}} = 0$ for $v \in \Delta_i'$, we have $L_{\alpha}|_{V_i} = (L_{\alpha})_{V_i}$. Hence, using (26), we have the required isomorphism:

$$H^0(B''_{i\alpha\beta}) \simeq \text{Ext}^0(L_{\alpha}|_{V_i}, L_{\beta}|_{V_i}) \simeq \text{Ext}^0(L_{\alpha}|_{V_j}, L_{\beta}|_{V_j}) \simeq H^0(B'_{j\alpha\beta}).$$

Let us prove (c). We first note that assumption 80(ii) implies, for $F \in D^b(Y)$, constructible with respect to the stratification $Y = \bigsqcup_{i \in I} Y_i$, $\forall i \in I$, $H^0(V_i; F) \simeq H^0(Y_i; F)$. By (26), it follows that:

$$H^0(B''_{i\alpha\beta}) \simeq \text{Ext}^0(L_{\alpha}|_{V_i}, L_{\beta}|_{V_i}) \simeq H^0(V_i; R\text{Hom}(L_{\alpha}, L_{\beta})) \simeq H^0(Y_i; R\text{Hom}(L_{\alpha}, L_{\beta})|_{Y_i}).$$

Now, we have either $i \not\in \phi(Z_{\Delta_i \cup \Delta_j} \cup \Delta')$ or there exists $v_0 \in \Delta_{i\alpha\beta} = (\Delta_i' \cup \Delta_j') \setminus (\Delta_i \cap \Delta_j')$ such that $V_i \subset D_{v_0}$. In the first case $V_i$ doesn’t meet supp($R\text{Hom}(L_{\alpha}, L_{\beta})$) and the vanishing of $H^0(B''_{i\alpha\beta})$ is clear. Let us assume we are in the second case. For $\alpha \in A$, we set $V_\alpha = Y \setminus \bigcup_{v \in \Delta_j} D_v$. By lemma 2.4 we have $L_{\alpha} \simeq (L_{\alpha})_{V_\alpha} \simeq \Gamma_{V_\alpha}(L_{\alpha})$. Let us set $V = V_\alpha \cap V_\beta$ and let $j : V \to Y$ be the inclusion. We obtain:

$$R\text{Hom}(L_{\alpha}, L_{\beta}) \simeq R\text{Hom}((L_{\alpha})_{V_\alpha}, R\Gamma_{V_\alpha}(L_{\beta})) \simeq R\Gamma_{V}R\text{Hom}(L_{\alpha}, L_{\beta}) \simeq Rj_*R\text{Hom}(L_{\alpha}|_V, L_{\beta}|_V).$$

Now $Z_\alpha \cap V$ and $Z_\beta \cap V$ are closed in $V$ and their intersection $Z_{\alpha\beta} \cap V$ is empty or a smooth submanifold of $Z_{\beta} \cap V$ of codimension $2d_{\alpha\beta}$. Hence $R\text{Hom}(L_{\alpha}|_V, L_{\beta}|_V)$ is isomorphic to $K_{\alpha\beta}(2d_{\alpha\beta})$, where $K_{\alpha\beta}$ is a local system on $Z_{\alpha\beta} \cap V$. By assumption 80(v), the monodromy of $K_{\alpha\beta}$ around $D_{v_0}$ is $-\text{Id}$. By lemma 2.4 we obtain $Rj_*K_{\alpha\beta}|_{D_{v_0}} = 0$ and, a fortiori, $R\text{Hom}(L_{\alpha}, L_{\beta})|_{V_i} = 0$ and $H^0(B''_{i\alpha\beta}) = 0$. □
4. Symmetric varieties

We recall some results of [2] on regular compactifications of homogeneous symmetric varieties, in particular the structure of the decomposition by the $K$-orbit types, for a suitable maximal compact subgroup $K$. Then we show that the hypothesis of proposition 5.7 are satisfied.

Let $G$ be a semi-simple algebraic group of adjoint type over $\mathbb{C}$, $\sigma$ an automorphism of order 2 of $G$ and $H = G^\sigma$. Let $T$ be a $\sigma$-stable maximal torus of $G$ containing a maximal $\sigma$-split torus $S$ (i.e. $\forall t \in S$, $\sigma(t) = t^{-1}$). The corresponding root system $\Phi = \Phi(G, T)$ decomposes as $\Phi = \Phi_0 \sqcup \Phi_1$, where $\Phi_0$ denotes the set of roots fixed by the action of $\sigma$. One may choose a basis of simple roots $\Sigma$ such that $\sigma$ exchanges the corresponding positive roots of $\Phi_1$ with the negative roots of $\Phi_1$. The non-fixed roots $\Phi_2$ induce a root system on $S$ with basis $\{\gamma_1, \ldots, \gamma_l\}$ given by the restriction of $\Sigma \cap \Phi_1$. The corresponding Weyl group is denoted by $W$. Set $D = H \cap S$, the subgroup of $S$ of elements of order 2, and $S' = S/D \simeq T/(T \cap H)$. The natural map $S \to S'$ gives an identification between the Lie algebras $\text{Lie}(S)$ and $\text{Lie}(S')$. Let $H^0$ be the identity component of $H$. By proposition 1 of [12] (see also proposition 7 of [17]), we have $H = D \cdot H^0$. In particular the group of components of $H$ is a quotient of $D$, hence of the type $H/H^0 \simeq (\mathbb{Z}/2\mathbb{Z})^a$, for some $a \in \mathbb{N}$.

Regular compactifications. Let $X$ be the canonical compactification of $G/H$ described in [6] and [7]. It can be defined as follows: let $G_{1, n}$ be the Grassmann variety of $n$-dimensional subspaces of $\text{Lie}(G)$, for $n = \dim(H)$, with the G-action induced by the adjoint action. Let $x \in G_{1, n}$ be the point associated to $\text{Lie}(H)$. One can show that $G \cdot x \simeq G/H$ and that $X$ is isomorphic to the closure of $G \cdot x$ in $G_{1, n}$. It is proved in [6] that $X$ is smooth, $X \setminus (G/H)$ is the union of $l$ smooth, normal crossings divisors, say $D_i$, $i = 1, \ldots, l$, which are closures of $G$-orbits, and any $G$-orbit closure is the intersection of the $D_i$ containing it. More precisely, the decomposition into $G$-orbits is identified with the decomposition of the toric variety $\mathbb{C}^l$ into orbits for the action of $(\mathbb{C}^*)^l$, as follows. The inclusion $S \subset G$ gives an embedding of $S'$ in $G/H$, and hence in $X$. The closure of $S'$ in $X$ is a $S'$-toric variety, whose fan can be identified with the subdivision of $\text{Lie}(S')$ into Weyl chambers under the action of $W$. We consider the affine space $\mathbb{C}^l$ associated to the negative Weyl chamber. Then the $G$-orbits in $X$ correspond bijectively (by taking the intersection with $\mathbb{C}^l$) to the $S'$-orbits in $\mathbb{C}^l$. In particular, there are $2^l$-orbits and one single closed orbit. Let $O \subset X$ be a $G$-orbit. Then its closure is fibred over a variety of parabolic subgroups of $G$, with fibre a symmetric variety. More precisely, there exist a parabolic subgroup $P \subset G$, and a $G$-equivariant fibration $\overline{O} \to G/P$, whose fibre, say $X_O$, has the following description. There exists a $\sigma$-stable Levi subgroup $L$ of $P$, such that, denoting by $L'$ the quotient $L/Z(L)$ ($Z(L)$ is the centre of $L$), $X_O$ is the canonical compactification of $L'/L'^\sigma$.

In [7] there is a description of the embeddings of $G/H$ over $X$. If $Y$ is such an embedding, with a map $\pi : Y \to X$, the closure of $S'$ in $Y$ gives a toric variety, say $Z'$, and $Z = Z' \cap \pi^{-1}(\mathbb{C}^l)$ is a toric variety over $\mathbb{C}^l$. It is shown in [7] that this gives a bijection between the embeddings of $G/H$ over $X$ and the toric varieties over $\mathbb{C}^l$. For $Z \to \mathbb{C}^l$ a morphism of toric varieties, let $\pi : X_Z \to X$ be the corresponding embedding of $G/H$. Then $Z = \pi^{-1}(\mathbb{C}^l)$ and the $G$-orbits in $X_Z$ correspond bijectively (by taking the intersection with $Z$) to the $S'$-orbits in $Z$. 
Moreover $X_Z$ is smooth if, and only if, $Z$ is smooth and $X_Z$ is complete if, and only if, $Z \to \mathbb{C}^l$ is proper. From now on we assume that $X_Z$ is smooth and complete. These are the symmetric varieties we consider here.

**Notations 4.1.** We denote by $V$ the set of irreducible $G$-stable divisors, $D_v, v \in V$. Any $G$-orbit closure is the intersection of the $D_v$ containing it. For $\Delta \subset V$ such that $\bigcap_{\Delta \subset \Delta} D_v \neq \emptyset$, we let $\mathcal{O}_\Delta$ be the $G$-orbit such that $\overline{\mathcal{O}_\Delta} = \bigcap_{\Delta \subset \Delta} D_v$. We denote by $\mathcal{S}$ the set of $G$-orbits; hence $\mathcal{S}$ is identified with a set of subsets of $V$.

**Fundamental domain.** In [2], we also have a description of the orbits of a suitable maximal compact subgroup of $G$. Let $K$ be a compact form of $G$ such that the Cartan involution of $G$ corresponding to $K$ commutes with $\sigma$ and such that $K \cap T$ is a maximal compact subgroup of $T$. We set $S^\sigma = K \cap S$. Let $t_i = t_i^{-2\gamma_i}, i = 1, \ldots, \ell$, be the characters of $S'$ associated to the simple roots $\gamma_i$, extended to a coordinates system on $\mathbb{C}^\ell$. We set $C = \{0, 1]^\ell \subset S' \subset G/H$. We consider the closure of $C$ in $X_Z$, $C_{X_Z} = \overline{C}$. In particular, $C_X = \{0, 1]^\ell$. Note that $(\pi|Z)^{-1}(\{0, 1]^\ell)$ is closed and contains $C$, so that $C_{X_Z}$ is in fact contained in $Z$. Hence $C_{X_Z}$ is mapped to $C_X$ by $\pi$. This is a fundamental domain for the action of $K$ on $X_Z$ (see [2] theorem 27).

**Stabilisers.** The stabiliser in $K$ of a point of $C_{X_Z}$ is described in [2], p. 27, as follows. For a point $q \in C_X = \{0, 1]^\ell$, we call its $J$-support the subset of $\{1, \ldots, \ell\}$ defined by $J(q) = \{i; q_i \neq 1\}$. Let $\tau : S \to S'$ denote the quotient map. For $q \in \{0, 1]^\ell \subset S'$, let $q \in \tau^{-1}(q)$ be in the connected component of $\tau^{-1}(\{0, 1]^\ell)$ containing 1. Then the centraliser of $q$ in $G$ only depends on $J = J(q)$; we set:

$$K_J = K^T \cap Z_G(q).$$

For $J \subset \{1, \ldots, \ell\}$, we denote by $C_{X_Z,J}$, or simply $C_J$ if there is no ambiguity, the subset of $C_{X_Z}$ formed by the $p$ such that $J(p) = J$. By definition this decomposition of $C_{X_Z}$ arises from the decomposition of $C_X = \{0, 1]^\ell$ according to the set of coordinates equal to 1: $C_{X_Z,J} = (\pi|C_{X_Z})^{-1}(C_X,J)$.

We also consider the partition of $C_{X_Z}$ given by the $G$-orbits: $C_{X_Z} = \bigsqcup_{J \subset \{1, \ldots, \ell\}} \mathcal{O}_J \cap C_{X_Z}$. All points $p \in \mathcal{O}_J \cap C_{X_Z}$ have the same stabiliser in $S^\sigma$. Indeed $C_{X_Z} \subset Z$, and $\mathcal{O}_J \cap Z$ is a single $S'$-orbit of the toric variety $Z$, so that all points of $\mathcal{O}_J \cap Z$ have the same stabiliser in $S'$. We set $S_{J}^\sigma = S_{J}^T$ for any $p \in \mathcal{O}_J \cap C_{X_Z}$.

Now we mix the two partitions above and set, for $\Delta \in \mathcal{S}$ and $J \subset \{1, \ldots, \ell\}$:

$$F_{\Delta,J} = \mathcal{O}_\Delta \cap C_J.$$

The $F_{\Delta,J}$ are called the “faces” of $C_{X_Z}$ (if $X_Z$ is projective, the moment map for the $K$-action identifies $C_{X_Z}$ with a polytope, and the “faces” are the usual faces of this polytope – see [2], p. 25). We have the following description of the stabilisers:

**Theorem 4.2 (theorem 32 and corollary 35 of [2]).** For $p \in C_{X_Z}$ let $F_{\Delta,J}$ be the face containing $p$, and let $K_p$ be the stabiliser of $p$ in $K$. Since $\mathcal{O}_\Delta$ is $K$-stable, $K_p$ acts on $N_p = \mathbb{T}_p X_Z/\mathbb{T}_p \mathcal{O}_\Delta$. We have:

\begin{align*}
(41) & \quad K_J = \ker(K_p \to \mathbb{GL}(N_p)), \\
(42) & \quad K_p = S_{\Delta}^\sigma \cdot K_J, \quad S_{\Delta}^\sigma \cap K_J = D.
\end{align*}

In particular $K_p$ only depends on the face to which $p$ belongs. For a face $F$, we set $K_F = K_p$, for any $p \in F$. Let us make some remarks on the decomposition of $C_{X_Z}$ into faces.
For $X_Z = X$, we have $Z = C^l$, $C_X = [0, 1]^l$. We may identify $V$ with \( \{1, \ldots, l\} \) so that, for $\Delta \subset V$, $O_\Delta \cap C^l = \{(t_1, \ldots, t_l) : t_i = 0 \ \text{iff} \ i \in \Delta \}$? Hence:

\[
 F_{\Delta, j} = \{(t_1, \ldots, t_l) \in [0, 1]^l : t_i = 0 \ \text{iff} \ i \in \Delta \ \text{and} \ t_i = 1 \ \text{iff} \ i \notin J \},
\]

The image of a $G$-orbit of $X_Z$, say $O_\Delta$, $\Delta \subset V$, by $\pi$ is a $G$-orbit of $X$, say $O_\Sigma$, $\Sigma \subset \{1, \ldots, l\}$. The restriction $\pi|_{O_\Delta} : O_\Delta \to O_\Sigma$ is a fibration, and gives a bijection between the faces $F_{\Delta, j}$ of $C_{X_Z} \cap O_\Delta$ and the faces $F_{\Sigma, j}$ of $C_X \cap O_\Sigma$. In particular $C_{X_Z} \cap O_\Delta$ has a unique closed face, $F_{\Delta, \Sigma}$, and a unique open face, $F_{\Delta, \{1, \ldots, l\}}$. We also deduce that $F_{\Delta, j} \subset F_{\Delta', j'}$ if and only if $J \subset J'$. We have similarly $F_{\Delta, j} \subset F_{\Delta', j'}$ if and only if $\Delta' \subset \Delta$. Indeed if $F_{\Delta, j} \subset F_{\Delta', j'}$, then certainly $O_\Delta \cap O_{\Delta'} \neq \emptyset$ and hence $\Delta' \subset \Delta$. Conversely, assume $\Delta' \subset \Delta$ and let $p \in F_{\Delta, j}$. Since $p \in O_\Delta$, we may write $p = \lim_n p_n$, with $p_n \in O_{\Delta'}$. Now each $p_n$ is itself a limit of points of $O_\emptyset$, which is identified with $(C^*)^l$ by $\pi$. Let us write $p_n = \lim_n q^*_n$, with $q^*_n = (t_{i_1}^n, \ldots, t_{i_d}^n)$. Since $p \in C_j$, for any $\varepsilon > 0$, there exists a neighbourhood $U$ of $p$, such that $q^*_n \in U$ implies $|t_{i_j}^n| - 1 < \varepsilon$, $\forall j \notin J$. Hence, up to restricting to a subsequence, we may assume that, for each $n$ great enough, $\forall j \notin J$, $\lim_n t_{i_j}^n = s_{i_j}$ for some $s_{i_j}$, with $|s_{i_j} - 1| \leq \varepsilon$. We set $s_{i_j}' = 1$ for $j \in J$. Then, for the element $g_n = (g^n) \in (C^*)^l$, the point $p'_{n_j} = g_{n_j}^{-1} \cdot p_n = \lim_n g_{n_j}^{-1} \cdot q^*_n$ is in $O_{\Delta'} \cap C_J$. The $p'_{n_j}$ converge to $p$, and we have $p \in F_{\Delta', j'}$, as required.

Combining both characterisations for the inclusions of closures of faces, we have: $F_{\Delta, j} \subset F_{\Delta', j'}$ is equivalent to $\Delta' \subset \Delta$ and $J \subset J'$. Now we summarise the notations and properties introduced so far and add some others.

Properties 4.3. For $\Delta \in S$, $J \subset \{1, \ldots, l\}$, we have $F_{\Delta, j} = O_\Delta \cap C_{X_Z, j}$. We let $F$ denote the set of faces, $F = \{F_{\Delta, j} : F_{\Delta, j} \neq \emptyset\}$. We let $\varphi : C_{X_Z} \to F$ be the natural map induced by this partition and endow $F$ with the quotient topology. For $\Delta \subset S$, $O_\Delta \cap C_{X_Z}$ has a unique closed face, which we denote by $F_{\Delta, \emptyset}$, $J \subset \{1, \ldots, l\}$. As in section 2.1 for a face $F$ of $C_{X_Z}$, we let $U_F$ be the smallest open subset of $F$ containing $F$. We set $U'_F = \varphi^{-1}(U_F)$; this is an open subset of $C_{X_Z}$ which contains $F$ as its unique closed face.

\[
 K_{F_{\Delta, j}} = S_{\Delta, j} \cdot K_j,
\]

\[
 \Delta' \subset \Delta \implies S_{\Delta, j} \subset S_{\Delta'} \quad \text{and} \quad J \subset J' \implies K_{\Delta' \cdot j} \subset K_{\Delta \cdot j},
\]

\[
 F_{\Delta, j} \subset F_{\Delta', j'} \iff \Delta' \subset \Delta \quad \text{and} \quad J \subset J',
\]

\[
 U_F = \{F' \in F : F \subset F'\}, \quad U'_F = \bigcup_{F \subset F'} F',
\]

\[
 U_{F_{\Delta, j}} = \bigcup_{\Delta' \subset \Delta, j' \subset J} F_{\Delta', j'} = (\bigcup_{\Delta' \subset \Delta} O_{\Delta'}) \cap (\bigcup_{J' \subset J} C_J),
\]

\[
 U_{F_{\Delta, j}} \cap U_{F_{\Delta', j'}} = U_{F_{\Delta \cap \Delta', j \cup j'}}.
\]

The first equality in (47) follows from (45). The second follows directly by applying the definition of the faces, and it implies (48).

4.1. Stratification by the faces.

Lemma 4.4. We keep the notations introduced above. Let $X'_Z = X_Z/K$ be the topological quotient, $p_Z : X_Z \to X'_Z$ the quotient map and $q_Z = p_Z|_{C_{X_Z}}$. We consider on $C_{X_Z}$ the topology induced by its inclusion in $X_Z$. We have:

(i) the map $q_Z$ is a homeomorphism,

(ii) the partition of $C_{X_Z}$ by the faces, $C_{X_Z} = \bigcup_{F \in F} F$, satisfies: if $F \cap \overline{F'} \neq \emptyset$ then $F \subset \overline{F'}$,
(iii) the induced partition of $X_Z$, $X_Z = \bigsqcup_{F \in \mathcal{F}} K \cdot F$, is a $\mu$-stratification; it satisfies the same inclusions relations for the closures of strata as the partition of $C_{X_Z}$ by the faces.

Proof. (i) Since $C_{X_Z}$ is a fundamental domain for the $K$-action, $q_Z$ is bijective. It is continuous by definition. For an open subset $U \subset C_{X_Z}$, $C_{X_Z} \setminus U$ is compact because $C_{X_Z}$ is. Hence $q_Z(C_{X_Z} \setminus U)$ is compact too, and $q_Z(U)$ is open. This proves that $q_Z^{-1}$ is continuous too.

(ii) Let $F_{\Delta, J}, F_{\Delta', J'}$ be two faces such that $F_{\Delta, J} \cap \overline{F_{\Delta', J'}} \neq \emptyset$. By definition of the faces, this implies $\mathcal{O}_\Delta \cap \overline{\mathcal{O}_{\Delta'}} \neq \emptyset$ and $C_J \cap C_{J'} \neq \emptyset$. Hence $\Delta' \subset \Delta$ and $J \subset J'$; we conclude by (31).

(iii) By (i), we have for any subset $C \subset C_{X_Z}$:

$$K \cdot C = p_Z^{-1}(\overline{p_Z(K \cdot C)}) = p_Z^{-1}(\overline{q_Z(C)}) = p_Z^{-1}(q_Z(\overline{C})) = K \cdot \overline{C}.$$  

This implies that our partition of $X_Z$ is a stratification and the last assertion. Let us verify conditions (i)–(iii). The first condition follows directly from (48). By (47), we verify the "$\mu$-condition" for two strata $\mathcal{O}_\Delta \subset H \cdot \lambda$. Since $\pi_1(E/H) \simeq H/H^0$, we have $\pi_1(E/K) \simeq E \times K$. We denote by $\psi : E \times K \to \mathcal{F}$ the universal bundle for $K$ which is an increasing union of manifolds, $E = \bigsqcup_k E_k$. Since $\pi_1(E/K) = 1$, for any subgroup $H \subset K$ we have $\pi_1(E/K) \simeq H/H^0$. We denote by $\phi : E \times K \to \mathcal{F}$ the map induced by $\phi$.

Notations 4.5. We stratify $X_Z$ as in the above lemma. We denote by $\phi : X_Z \to \mathcal{F}$ the continuous map defined by $\phi(K \cdot F) = F$, for $F \in \mathcal{F}$. We set $V_F = K \cdot U'_F = \phi^{-1}(U_F)$. From now on we denote by $E$ a universal bundle for $K$ which is an increasing union of manifolds, $E = \bigsqcup_k E_k$. Since $\pi_1(E/K) = 1$, for any subgroup $H \subset K$ we have $\pi_1(E/H) \simeq H/H^0$. We denote by $\psi : E \times K \to \mathcal{F}$ the map induced by $\phi$.

We are in the setting of assumption 3.3 with $Y = X_Z$ in a $K$-equivariant way, or $Y = E \times_K X_Z$ (see remark 3.31). $I = \mathcal{F}$, and a set of normal crossings divisors $D_v = \overline{\mathcal{O}_v}$, $v \in V$ (we will introduce local systems $L_\alpha$ in the next paragraph). Let us verify conditions (i)–(iii). The first condition follows directly from (31). By (31), we have, for $\Delta \in \mathcal{S}$ and $v \in \Delta$,

$$U'_{\mathcal{O}_v} \setminus D_v = (\bigsqcup_{\Delta' \subset \Delta} \mathcal{O}_{\Delta'}) \setminus D_v \cap (\bigsqcup_{J' \supset J} C_{J'}) = U'_{\mathcal{O}_v} \setminus (\bigsqcup_{J' \supset J} C_{J'})$$

and this gives condition (iii), the case $v \notin \Delta$ being trivial. Condition (ii) of 3.31 follows from the next lemma.

Lemma 4.6. Let $F$ be a face.

(i) There exists a homotopy $h : [0, 1] \times U'_F \to U'_F$, such that $h_1 = \text{id}$ and $h_0$ is the projection of $U'_F$ to a point of $F$, with the property that the closures of the faces of $U'_F$ are stable under $h_t$, $\forall t \in [0, 1]$.

(ii) It induces a $K$-equivariant homotopy $\tilde{h} : [0, 1] \times V_F \to V_F$ contracting $V_F$ to a $K$-orbit $K/K' \subset K \cdot F$ and preserving the closures of strata.
Proof: (i) By definition, the faces of $C_{XZ}$ are contained in $S'$-orbits of $Z$. Let $Y$ be the $S'$-orbit containing $F$ and let $U$ be the union of the $S'$-orbits whose closure contains $Y$. Then $U_F'$ is included in $U$. There exists a one-dimensional subtorus $i : C^* \hookrightarrow S'$ of $S'$ contracting $U$ to $Y$, i.e. $\forall y \in U \lim_{t \to 0} i(t) \cdot y \in Y$. Note that for $t \in [0,1]$, $x \in U \cap C_{XZ}$, $i(t) \cdot x$ still is in $U \cap C_{XZ}$ and belongs to the same face as $x$. Hence the homotopy $h_1 : [0,1] \times U'_F \to U'_F$, $(t,x) \mapsto i(t) \cdot x$ if $t \neq 0$, $(0,x) \mapsto \lim_{t \to 0} i(t) \cdot x$, contracts $U'_F$ to $U'_F \cap Y$ and preserves the closure of the faces.

Now $Y$ is fibred over $\pi(Y)$ with fibre $(\mathbb{C}^*)^k$, a quotient of $S'$. It follows that $C_{XZ} \cap Y$ is fibred over $C_X \cap \pi(Y)$, with fibre $W$, where $W$ is a subset of $\mathbb{R}_{\geq 0}^k$ stable by multiplication by $[0,1]^k$. In particular $W$ is contractible. Moreover this fibration is a bijection on the faces. Hence there exist sections $s : C_X \cap \pi(Y) \to C_{XZ} \cap Y$ of $\pi|C_{XZ} \cap Y'$, and for any such $s$, we may find a homotopy contracting $C_{XZ} \cap Y$ to $\text{im}(s)$, which preserves the faces and thus the faces.

Finally, $\text{im}(s) \simeq C_X \cap \pi(Y)$ is a union of standard faces in $\{0,1\}^I$, isomorphic to $[0,1]^m \times \{0\}^{I-m}$, up to a permutation of coordinates. Then $\text{im}(s) \cap F \simeq [0,1]^{n \times \{0\}^{I-m}}$ and $\text{im}(s) \cap U'_F \simeq [0,1]^{n \times \{0\}^{I-m}}$. Hence there exists a third homotopy contracting $\text{im}(s) \cap U'_F$ to a point of $F$ also preserving the closure of the faces.

(ii) By (14) and (15), for any faces $F_1$, $F_2$ with $F_1 \subset \overline{F_2}$, we have $K_{F_2} \subset K_{F_1}$. Since $\forall F' \in U_F$, $h([0,1] \times F') \subset \overline{F}$, we obtain $\forall (t,x) \in [0,1] \times U'_F$, $K_x \subset K_{h(t,x)}$. Hence it makes sense to define $h : [0,1] \times V_F \to V_F$, $(t,k \cdot x) \mapsto k \cdot h(t,x)$, for $t \in [0,1]$, $k \in K$, $x \in U'_F$. It is clear that $h_0 = \text{id}$ and $h_1$ is a $K$-equivariant projection to $K/K_F$. Let us verify that $h$ is continuous.

Let $(t_n,y_n)$ be any sequence in $[0,1] \times V_F$ converging to $(t,y)$. Let us see that a subsequence of $h(t_n,y_n)$ converges to $h(t,y)$. For each $n$ there exists a unique $x_n \in U'_F$ such that $y_n \in K \cdot x_n$. With the notations of lemma 4.6 we have $x_n = q^{-1}_Z(p_Z(y_n))$. Since these maps are continuous, the sequence $x_n$ converges to $x = q^{-1}_Z(p_Z(y))$. Let us write $y_n = k_n \cdot x_n$, $k_n \in K$. Since $K$ is compact, we may assume, up to restriction to a subsequence, that $k_n$ converges to $k \in K$. Then $h(t_n,y_n) = k_n \cdot h(t_n,x_n)$ converges to $h(t,y) = k \cdot h(t,x)$, as desired. \hfill \Box

Let us give some immediate consequences of this lemma. We consider the map $p_F : V_F \to K/K_F$, $k \cdot x \mapsto k$, where $k \in K$, $x \in U'_F$. This makes sense because $\forall x \in U'_F$, $K_x \subset K_F$ (recall that $U'_F = \bigsqcup_{F \in T} F'$). With the notations of lemma 4.6 let $x_0 = x_0 \in F$, be the point of $F$ such that $h(1,U'_F) = \{x_0\}$. By definition, we have in fact $p_F = h(1,\cdot)$ modulo the identification $K \cdot x_0 = K/K_F$. In particular $p_F$ is continuous. Since it is $K$-equivariant, it is a fibration over $K/K_F$, with fibre $K_F \cdot U'_F$. The homotopy $h$ also is $K$-equivariant and hence contracts each fibre $p_F^{-1}(y)$, $y \in K/K_F$ to the point $y$. We let

\begin{equation}
q_F : E \times_K V_F \to E/K_F
\end{equation}

be the map induced by $p_F$. This also is a fibration with contractible fibres.

For a $G$-orbit $\mathcal{O}_\Delta$, with closed face $F_{\Delta,J_{\Delta}}$, and a face $F' = F_{\Delta,J'}$ such that $F' \subset \mathcal{O}_\Delta$ (i.e. $J_{\Delta} \subset J'$), $\mathcal{O}_\Delta \cap V_{F'}$ is closed in $V_F$; indeed if a face $F_{\Delta,J_{\Delta}}$ is included in $\mathcal{O}_\Delta$ it satisfies $\Delta \subset \Delta_{\Delta}$, and if it is in $V_F$ it satisfies $\Delta_{1} \subset \Delta$; hence any face of $\mathcal{O}_\Delta \cap V_{F'}$ is in $\mathcal{O}_\Delta$. Since the homotopy $h$ of lemma 4.6 preserves the
closures of faces, it follows that \( h \) contracts \( \mathcal{O}_\Delta \cap V_{\mathcal{F}} \) to \( K/K_{\mathcal{P}} \). Hence the maps
\[
E \times_K (\mathcal{O}_\Delta \cap V_{\mathcal{F}'} \cap V_{\mathcal{F}'}) \to E \times_K V_{\mathcal{F}' \cap V_{\mathcal{F}'}} \to E/K_{\mathcal{P}'}
\]
are homotopy equivalences. In particular the fundamental groups are the same:
\[
(52) \quad \pi_1(E \times_K (\mathcal{O}_\Delta \cap V_{\mathcal{F}'})) = \pi_1(E/K_{\mathcal{P}'}) = K_{\mathcal{P}'}/K_{\mathcal{P}}^0.
\]

4.2. Equivariant local systems on \( X_S \). Remember that \( G \)-equivariant local systems on a homogeneous variety \( G/G' \) are in bijective correspondence with representations of the components group \( G'/G'_{\mathcal{O}} \) of \( G' \). For a \( G \)-orbit \( \mathcal{O} \), we denote by \( \tau_\mathcal{O} \) the group of components of a stabiliser, \( \tau_\mathcal{O} = G_p/G'_{\mathcal{O}} \), for \( p \in \mathcal{O} \). We introduce similar notations for the groups of components of the groups defined up to now:
\[
\tau_j = K_j/K_j^0, \quad \tau_{\Delta} = S_\Delta^0/K_\Delta^0 = D/D_\Delta, \quad \tau_{\mathcal{P}} = K_{\mathcal{P}}/K_{\mathcal{P}}^0.
\]
For a \( G \)-orbit \( \mathcal{O}_\Delta \), remember that \( F_{\Delta, \mathcal{O}_\Delta} \) denotes the unique closed face of \( \mathcal{O}_\Delta \cap C_{\mathcal{X}} \).

Let us fix a face \( F = F_{\Delta, \mathcal{O}_\Delta} \). By (11), \( K_j \) is a normal subgroup of \( K_F \). In fact the identity component \( S_\Delta^0 \) of \( S_\Delta \) centralises \( K_j \). Indeed, \( \forall s \in S_\Delta^0, \forall k \in K_j \), we have \( k' = s^k s^{-1} \in K_j \). Since \( K_j \subset K^\sigma \), we deduce \( k' = s^k s^{-1} = s^k s^{-2} \). Since any element of the torus \( S_\Delta^0 \) is a square, our claim follows. Then (12) gives the exact sequences (52), (in which \( S_\Delta^0 \times K_j \) is a direct product). We deduce the exact sequences (53).
\[
\begin{align*}
(53) & \quad 1 \to D \to S_\Delta^0 \times K_j \to K_F \to 1, & 1 \to D_\Delta \to S_\Delta^0 \times K_j \to K_F \to 1, \\
(54) & \quad D \to \tau_{\Delta} \times \tau_j \to \tau_{\mathcal{P}} \to 1, & D_\Delta \to \tau_{\Delta} \to \tau_{\mathcal{P}} \to 1.
\end{align*}
\]
In particular, the groups \( \tau_{\mathcal{P}} \) (and then \( \tau_\mathcal{O} \)) are quotients of the \( \tau_j \), hence of the type \((\mathbb{Z}/2\mathbb{Z})^a\), for some \( a \in \mathbb{N} \).

Let us consider more precisely the group \( \tau_\Delta \). Remember that \( S_\Delta^0 \) is the stabiliser of a point \( p \in C_{\mathcal{X}} \subset \mathbb{Z} \) in \( S^c = K \cap S \), which is the maximal compact subgroup of \( S \) (hence connected). Moreover \( D = \{ t \in S; t^2 = 1 \} \approx (\mathbb{Z}/2\mathbb{Z})^1 \), \( S \) acts on \( Z \) via \( S' = S/D \) and \( Z \) is toric for \( S' \), so that \( S_p^c \) is connected. Let \( S^c = S' \cap D \) be the maximal compact subgroup of \( S' \). Then \( S_p^{c} \) is connected too and the exact sequences
\[
1 \to D \to S_p^c \to S_p^{c} \to 1, \quad 1 \to (\mathbb{Z}/2\mathbb{Z})^{\nu} \to S_p^{c} \to S_p^{c} \to 1,
\]
where \( \nu = \dim S_p^{c} = |\Delta| \), show that \( \tau_{\Delta} \approx (\mathbb{Z}/2\mathbb{Z})^{\nu-|\Delta|} \) and \( D_\Delta \approx (\mathbb{Z}/2\mathbb{Z})^{|\Delta|} \). For \( \Delta_1 \in \mathcal{S} \), such that \( \Delta \subset \Delta_1 \), we have \( S_\Delta^0 \subset S_{\Delta_1}^0 \) and a natural morphism \( \tau_\Delta \to \tau_\Delta_1 \) which is surjective with kernel \((\mathbb{Z}/2\mathbb{Z})^{|\Delta_1| - |\Delta|}\).

**Lemma 4.7.** Let \( \mathcal{O}_\Delta \) be a \( G \)-orbit of \( X_S \) and \( L_{\rho} \) a \( G \)-equivariant local system on \( \mathcal{O}_\Delta \), corresponding to a representation \( \rho : \tau_\mathcal{O}_\Delta \to GL(V_p) \) of \( \tau_\mathcal{O}_\Delta \). We assume that \( L_{\rho} \) is irreducible. We set \( F = F_{\Delta, \mathcal{O}_\Delta} \) and let \( i_{\mathcal{O}} : \tau_{\Delta} \to \tau_{\mathcal{P}} = \tau_\mathcal{O}_\Delta \) be the morphism induced by (51). For \( v \in V \setminus \Delta \) such that \( \Delta_1 = \Delta \cup \{ v \} \in \mathcal{S} \), we have \( \ker(\tau_{\Delta} \to \tau_{\Delta_1}) \approx \mathbb{Z}/2\mathbb{Z} \). We let \( s_v \in \tau_{\Delta} \) be the generator of this kernel.
Then \( \rho(i_F(s_v)) = \pm Id_{V_v} \) and this is the monodromy of \( L_\rho \) around \( D_v \). If \( \rho(i_F(s_v)) = Id_{V_v} \) then \( \rho \) induces a representation, \( \rho_1 \), of \( \tau_{O_{\Delta_1}} \) and \( L_\rho \) extends to a local system, \( L_1 \), on \( O_\Delta \cup O_{\Delta_1} \), such that \( L_1|_{O_{\Delta_1}} \) corresponds to \( \rho_1 \).

**Proof.** Recall that the variety \( X_Z \) comes with a morphism \( \pi : X_Z \to X \). The \( G \)-orbits of \( X \) are parameterised by subsets of \( \{1, \ldots, l\} \) and \( J_\Delta \subseteq \{1, \ldots, l\} \) is determined by: \( \pi(O_{\Delta_1}) = O_{J_\Delta} \). The hypothesis gives \( \pi(O_{\Delta_1}) \subseteq \pi(O_\Delta) \), so that \( J_\Delta \subseteq J_{\Delta_1} \). We set \( F_1 = F_{\Delta_1,J_{\Delta_1}} \). By (53) we deduce that \( U_{F_1} \cap U_F = U_{F_2} \), where \( F_2 = F_{\Delta,J_{\Delta_1}} \). We obtain the following commutative diagram

\[
\begin{array}{cccc}
E \times_K V_{F_{\Delta_1,J_{\Delta_1}}} & \longrightarrow & E \times_K V_{F_{\Delta_1,J_{\Delta_1}}} & \longrightarrow \\
\cup & & \cup & \\
E \times_K O_{\Delta_1} & \longleftarrow & E \times_K (O_{\Delta_1} \cap V_{F_{\Delta_1,J_{\Delta_1}}}) & \longleftarrow \\
\downarrow & & \downarrow & \\
E/K_{F_{\Delta_1,J_{\Delta_1}}} & \longrightarrow & E/K_{F_{\Delta_1,J_{\Delta_1}}} & \longrightarrow \\
\end{array}
\]

where the vertical arrows are homotopy equivalences, by (51). Let us consider a “small” loop \( \gamma \) in \( E \times_K O_{\Delta_1} \) around \( E \times_K O_{\Delta_1} \), as in section 2.2. Since \( \gamma \) is small, we may assume that it is included in the neighbourhood \( E \times_K V_{F_1} \) of \( E \times_K O_{\Delta_1} \). Since \( \gamma \) doesn’t meet \( E \times_K D_v \) and \( V_{F_1} \setminus D_v = V_{F_2}, \) by (49), \( \gamma \) is in fact contained in \( E \times_K V_{F_2} \). Hence it represents a generator of the kernel of \( \pi_1(i) \), the map induced on the fundamental groups by the inclusion \( i \) of the above diagram.

Let \( j : \tau_{F_2} \to \tau_{F_1} \) be the morphism induced by \( \tau_\Delta \to \tau_{\Delta_1} \). By (49), \( j = \pi_1(i) \) and by (51) we have the commutative diagram:

\[
\begin{array}{cccc}
D & \longrightarrow & \tau_{\Delta_1} \ltimes \tau_{J_{\Delta_1}} & \longrightarrow \\
\downarrow & & \downarrow & \\
D & \longrightarrow & \tau_{\Delta_1} \ltimes \tau_{J_{\Delta_1}} & \longrightarrow \\
\end{array}
\]

We have already seen that \( \ker(\tau_\Delta \to \tau_{\Delta_1}) \simeq (\mathbb{Z}/2\mathbb{Z})^{\mid \Delta \mid - \mid \Delta_1 \mid} \simeq \mathbb{Z}/2\mathbb{Z} \). The above diagram implies that \( \ker(j) = 0 \) or \( \mathbb{Z}/2\mathbb{Z} \) and is generated by \( i_F(s_v) \). Since \( s_v^2 = 1 \) and \( V_v \) is irreducible it follows that \( \rho(i_F(s_v)) = \pm Id_{V_v} \).

We consider \( L_\rho \) as well as a \( G \)-equivariant local system on \( O_{\Delta} \) or as a local system on \( E \times_K O_{\Delta_1} \). Then \( L_\rho|_{E \times_K V_{F_1}} \) corresponds to the representation \( \rho_2 \) of \( \tau_{F_2} \) given by \( j \) and \( \rho \). The representation \( \rho_2 \) gives a representation \( \rho_1 \) of \( \tau_{F_1} = \tau_{O_{\Delta_1}} \) if and only if it sends \( \ker(j) \) to \( Id_{V_v} \), i.e. \( \rho(i_F(s_v)) = Id_{V_v} \). This also is equivalent to the fact that \( L_\rho \) extends to \( O_{\Delta} \cup O_{\Delta_1} \), with a restriction to \( O_{\Delta_1} \) corresponding to \( \rho_1 \).

**Definition 4.8.** We let \( A \) be the set of pairs \( \alpha = (O, \rho) \), where \( O \) is a \( G \)-orbit and \( \rho : \tau_O \to GL(V_\rho) \) an irreducible representation of \( \tau_O \). For \( \alpha = (O, \rho) \in A \), we let \( \Delta_\alpha \in S \) be such that \( O = O_{\Delta_\alpha} \) and, with the notations of the previous lemma, we set

\[
\Delta'_\alpha = \{ v \in V \setminus \Delta_\alpha \mid \Delta_\alpha \cup \{ v \} \in S \text{ and } \rho(i_F(s_v)) = -Id \}, \quad Z_\alpha = \overline{O \setminus \bigcup_{v \in \Delta'_\alpha} D_v}.
\]

By the lemma, the \( G \)-equivariant local system on \( O \) corresponding to \( \rho \) extends to a \( G \)-equivariant local system \( L^0_\alpha \) on \( Z_\alpha \). We extend it by 0 outside \( Z_\alpha \) (keeping the notation \( L^0_\alpha \) also for the extension). We denote by \( L_\alpha \) the corresponding local system on \( E \times_K X_Z \).
4.3. dg-algebras on the set of faces. By section 2.1 we have the equivalences of categories:

\[ D_{G,c}^b(X_Z) \simeq D_G(X_Z)\langle L^0, \alpha \in A \rangle \simeq D(E \times_K X_Z)\langle L, \alpha \in A \rangle. \]

We are in the situation of assumptions 3.1 with \( Y = E \times_K X_Z \), stratified by the set of faces \( \mathcal{F} \), the subspaces \( E \times_K D_v \) and the local systems \( L, \alpha \in A \). Conditions (i)–(iii) were verified in section 4.1 and (iv), (v) follow from definition 4.8. Hence we may apply proposition 3.7: the category \( D_G(X_Z)\langle L^0 \rangle \) is equivalent to a category of dg-modules over \( \mathcal{F}, D_R\langle N_\alpha \rangle \), where \( R \) is a sheaf of dg-algebras on \( \mathcal{F} \), whose description is recalled below, and the \( N_\alpha \) are \( R \)-modules.

First, for \( \alpha = (O, \rho), \beta = (O', \rho') \in A \), we define a sheaf \( R^{\alpha\beta} \) on \( \mathcal{F} \) by its stalks at any face \( F \). We have:

\[ \phi(O) = \{ F_{\Delta, J} \in \mathcal{F}; \Delta_\alpha \subset \Delta \}, \quad \phi(Z_\alpha) = \{ F_{\Delta, J} \in \mathcal{F}; \Delta_\alpha \subset \Delta \subset (V \setminus \Delta'_\alpha) \}. \]

We recall the notations 3.4 (with \( F_{\alpha\beta} \) instead of \( I_{\alpha\beta} \)):

\[ F_{\alpha\beta} = \phi(Z_\alpha \cap Z_\beta) = \{ F_{\Delta, J} \in \mathcal{F}; (\Delta_\alpha \cup \Delta_\beta) \subset \Delta \subset (V \setminus (\Delta'_\alpha \cup \Delta'_\beta)) \}; \]

\[ d_{\alpha\beta} = |\Delta_\alpha \setminus \Delta_\beta|, \quad \Delta'_{\alpha\beta} = (\Delta'_\alpha \setminus \Delta'_\beta) \cup (\Delta'_\beta \setminus \Delta'_\alpha), \]

\[ F'_{\alpha\beta} = \phi(Z_{\Delta_\alpha \cup \Delta_\beta} \setminus (\bigcup_{V_\in F_{\alpha\beta}} D_v)) \setminus F_{\alpha\beta} \]

\[ = \{ F_{\Delta, J} \in \mathcal{F}; (\Delta_\alpha \cup \Delta_\beta) \subset \Delta \subset (V \setminus \Delta'_{\alpha\beta}) \} \quad \text{and} \quad \Delta_\alpha \cap \Delta'_{\beta \cap \Delta} \neq \emptyset \}. \]

For \( F \in \phi(Z_\alpha) \), the restriction to \( E \times_K K \cdot F \) of the local system \( L_\alpha \) has an extension to \( E \times_K V_F \) (recall that \( V_F = \phi^{-1}(U_F) \) – notations 4.3); we denote it by \( L_{\alpha,F} \). According to the defining formula 4.3, we consider three cases: (i) \( F \in F_{\alpha\beta} \), (ii) \( F \in F'_{\alpha\beta} \), (iii) \( F \notin F_{\alpha\beta} \cup F'_{\alpha\beta} \). For \( F \in F_{\alpha\beta} \), we have

\[ R^{\alpha\beta}_F = \Gamma(E \times_K V_F; \Omega_{E \times_K V_F} \otimes \text{Hom}(L_{\alpha,F}, L_{\beta,F}))[-2d_{\alpha\beta}]. \]

Case (ii) is reduced to (i) as in 5.5 and in case (iii) we have \( R^{\alpha\beta}_F = 0 \). We set \( R = \oplus_{\alpha, \beta \in A} R^{\alpha\beta} \).

We denote by \( \delta_v \in H^2_{K,D_v}(X_Z; \mathbb{C}_{X_Z}) \) the K-equivariant fundamental class of \( D_v \) in \( X_Z \). We choose forms \( \xi_v \in \Gamma(E \times_K X_Z; \Omega^2_{E \times_K X_Z}) \) representing the \( \delta_v \), and use them to define a product on \( R \), as in 6.6, turning \( R \) into a sheaf of dg-algebras on \( \mathcal{F} \).

The \( R \)-dg-module \( N_\alpha \) is \( N_\alpha = \oplus_{\alpha', \alpha} R^{\alpha', \alpha} \), with a \( R \)-structure defined like the product of \( R \). Let us set \( L = \oplus_{\alpha \in A} L_\alpha \). By 25 and lemma 3.20 (ii), we have, for a face \( F \) in \( \mathcal{F} \), the isomorphism of algebras:

\[ H'(R_F) \simeq H'(\Gamma(U_F; R)) \simeq \text{Ext}^1_{D_G(X_Z)}(L|_{V_F}, L|_{V_F}). \]

We also introduce the sheaf \( \mathcal{H} \) on \( \mathcal{F} \) given by the cohomology of \( R \), i.e. the sheaf associated to the presheaf \( U \rightarrow H'(\Gamma(U; R)) \). This is a sheaf of dg-algebras on \( \mathcal{F} \), with differential 0. For a face \( F \in \mathcal{F} \) we have \( \Gamma(U_F; \mathcal{H}) = \mathcal{H}_F = H'(R_F) \). We define in the same way the \( \mathcal{H} \)-module \( \mathcal{H}_\alpha \) associated to \( U \rightarrow H'(\Gamma(U; N_\alpha)) \).

5. Formality of the de Rham algebra

We keep the notations introduced in the previous section. Our aim is to prove the following result.
Proposition 5.1. There exists a sequence of quasi-isomorphisms of sheaves of dg-algebras on $\mathcal{F}$, $\mathcal{R} \to \mathcal{R}A \leftarrow \mathcal{R}B \to \mathcal{R}C \leftarrow \mathcal{R}D \to \mathcal{R}E \simeq \mathcal{H}$, relating $\mathcal{R}$ and $\mathcal{H}$. It induces an equivalence of categories between $\mathcal{D}^b_{\mathcal{G},c}(X_Z)$ and $\mathcal{D}_H(\mathcal{H}_\alpha, \alpha \in \mathcal{A})$.

We know from the previous section that $\mathcal{D}^b_{\mathcal{G},c}(X_Z)$ is equivalent to $\mathcal{D}_R(\mathcal{N}_\alpha)$. Now a quasi-isomorphism between sheaves of dg-algebras induces an equivalence between their derived categories of dg-modules. Hence the first part of the proposition implies that $\mathcal{D}^b_{\mathcal{G},c}(X_Z)$ is equivalent to $\mathcal{D}_R(\mathcal{M}_\alpha)$, where $\mathcal{M}_\alpha$ is the image of $\mathcal{N}_\alpha$ by the chain of equivalences. The remainder of this section is devoted to the construction of a sequence of quasi-isomorphisms as in the proposition. It will follow from the construction that $\mathcal{M}_\alpha$ is indeed isomorphic to $\mathcal{H}_\alpha$.

5.1 Decomposition of the cohomology. By theorem [4,2] the isotropy group $K_F$ almost decomposes as a product, up to a finite subgroup. We deduce a decomposition for $H(\mathcal{R}^\alpha_{F,b})$.

For $F \in \mathcal{F}$, we have defined in (50) a fibration $q_F : E \times_K V_F \to E/K_F$, with contractible fibres $K_F : U'_F$. In particular $q_F$ gives an isomorphism between the fundamental groups. Hence, for $\alpha \in A$ such that $F \subset Z_\alpha$, the local system $L_{\alpha,F}$ on $E \times_K V_F$ is the inverse image of a local system $L'_{\alpha,F}$ on $E/K_F$. Setting $M = \mathcal{H}om(L'_{\alpha,F},L'_{\beta,F})$, for another $\beta \in A$ with $F \subset Z_\beta$, we obtain:

$$H(\mathcal{R}^\alpha_{F,b}) \simeq \mathcal{E}xt_{\mathcal{D}(E \times_K V_F)}(L_{\alpha,F},L_{\beta,F})$$

(57)

Now let us introduce some notations. For a face $F = F_{\Delta,J}$, we recall that $S^0_\Delta$ and $K_J$ commute and we consider the action of $S^0_\Delta \times K_J$ on $E^3$ by $(s,k) \cdot (e_1,e_2,e_3) = (sk \cdot e_1,s \cdot e_2,k \cdot e_3)$. Let also $a_F$ be the group morphism $S^0_\Delta \times K_J \to K_F$, $(s,k) \mapsto sk$. The first projection $E^3 \to E$ is $a_F$-equivariant and induces the morphism $r^1_F$ below. In view of (53), $r^1_F$ is a fibration with fibre $E^2/D_\Delta$, which is acyclic (i.e. $H^i(E^2/D_\Delta;\mathbb{C}) = \mathbb{C}$ and, for $i \neq 0$, $H^i(E^2/D_\Delta;\mathbb{C}) = 0$). The projection to the last two factors $E^3 \to E^2$ induces in the same way the morphism $r^2_F$ below, which is a fibration with acyclic fibre $E$.

$$E/K_F \xrightarrow{r_F^1} E^3/(S^0_\Delta \times K_J) \xrightarrow{r_F^2} (E/S^0_\Delta) \times (E/K_J).$$

Lemma 5.2. We consider a face $F = F_{\Delta,J} \in \mathcal{F}$, $\rho : \tau_F \to GL(V_\rho)$ a representation of $\tau_F$, and $M$ the local system on $E/K_F$ corresponding to $\rho$. We let $\rho_J : \tau_J \to GL(V_\rho)$ be the representation obtained from $\rho$ and the morphism $\tau_J \to \tau_F$. We let $M_J$ be the local system on $E/K_J$ corresponding to $\rho_J$. Then

$$H(E/K_F;M) \simeq C[X_v; v \in \Delta] \otimes H(E/K_J;M_J),$$

where the $X_v$ are indeterminates of degree 2.

Proof. Since $r^1_F$ is a fibration with acyclic fibres, we have $M \simeq R(r^1_F)_*(r^1_F)^{-1}M$. Hence $H(E/K_F;M) \simeq H(E^3/(S^0_\Delta \times K_J);(r^1_F)^{-1}M)$. We have $\pi_1(E/K_F) = \tau_F$, $\pi_1(E^3/(S^0_\Delta \times K_J)) = \tau_J$ and the morphism induced by $r^1_F$ on the fundamental groups is the morphism of the lemma $\tau_J \to \tau_F$. Hence $(r^1_F)^{-1}M$ is the local system corresponding to $\rho_J$.

Since $r^2_F$ has a contractible fibre, it gives an isomorphism on the fundamental groups. Hence $(r^1_F)^{-1}M \simeq (r^2_F)^{-1}(C_{E/S^0_\Delta} \boxtimes M_J)$. This also gives an isomorphism
on the cohomology groups of \((\tau_{F})^{-1}M\) and \(C_{E/S_{\Delta}^{0}} \boxtimes M_{J}\). We conclude by the Künneth formula and the fact that \(S_{\Delta}^{0}\) is the torus \((\mathbb{C}^{*})^{|\Delta|}\), so that \(H^{\ast}(E/S_{\Delta}^{0}; C) = H_{S_{\Delta}^{0}}(\{pt\}; C)\) is a polynomial algebra in \(|\Delta|\) variables. \(\square\)

We describe the local systems \(L_{\alpha,F}\) on \(E \times K V_{F}, L'_{\alpha,F}\) on \(E/K_{F}\) and \((L'_{\alpha,F})_{J}\) on \(E/K_{F}\), in terms of representations.

For \(\alpha = (\mathcal{O}, \rho) \in A\), let \(F_{\alpha} = F_{\Delta_{\alpha}} \cdot J_{\Delta_{\alpha}}\) be the closed face of \(\mathcal{O} \cap C X_{\Delta}\) and \(V_{\rho}\) be the representation space of \(\rho\). We have seen that \(\tau_{\mathcal{O}_{\alpha}} = \tau_{\rho_{\alpha}}\), so that we have a morphism \(\tau_{\rho_{\alpha}} \rightarrow \tau_{\mathcal{O}_{\alpha}}\). Let \(\mathcal{O}_{\Delta}\) be a \(G\)-orbit such that \(\mathcal{O}_{\Delta} \subset \overline{\mathcal{O}}\) and with closed face \(F_{\Delta_{\alpha}} \cdot J_{\Delta_{\alpha}}\). We recall that \(\pi(\mathcal{O}) = \mathcal{O}_{\Delta_{\alpha}}\) and \(\pi(\mathcal{O}_{\Delta}) = \mathcal{O}_{J_{\Delta}}\) (where \(\pi\) is the map from \(X_{\Delta}\) to \(X\)). Hence \(J_{\Delta_{\alpha}} \subset J_{\Delta}\). If \(\overline{F} = F_{\Delta_{\alpha}}\) is another face of \(\mathcal{O}_{\Delta}\), we have \(J_{\Delta_{\alpha}} \subset J_{\Delta}\). Finally, for any face \(F = F_{\Delta_{\alpha}}\) such that \(F \subset \overline{\mathcal{O}}\), we have \(J_{\Delta_{\alpha}} \subset J_{\Delta}\), so that \(K_{J} \subset K_{J_{\Delta_{\alpha}}}\) and we obtain a group morphism:

\[
(59) \quad \text{for } F_{\Delta_{\alpha}} \subset \overline{\mathcal{O}}, \quad t_{\alpha,F}^{J} : \tau_{J} \rightarrow \tau_{\mathcal{O}}.
\]

We let \(\rho_{J}\) be the representation of \(\tau_{J}\) given by \(V_{\rho}\) and \(t_{\alpha,F}^{J}\), and we let \(L_{\alpha,F}^{1}\) be the corresponding local system on \(E/K_{F}\).

Now we assume moreover that \(F \subset Z_{\alpha}\). This means, by lemma \(4.27\), that \(\rho\) induces a representation, say \(\rho'\), of \(\mathcal{O}_{\Delta_{\alpha}}\). Then the representation \(\tau_{\rho_{J}}\) of \(\tau_{J}\) is given by \(\rho'\) and the morphism \(\tau_{J} \rightarrow \tau_{\rho_{J}} \rightarrow \tau_{\mathcal{O}_{\Delta}}\). Since the morphism induced by \(\tau_{J}\) on the fundamental groups is \(\tau_{J} \rightarrow \tau_{F}\), we obtain the following relations:

\[
(60) \quad L_{\alpha,F} = q_{F}^{-1}(L'_{\alpha,F}), \quad (\rho_{J})^{-1}(L_{\alpha,F}) \simeq (\rho'_{J})^{-1}(C_{E/S_{\Delta}^{0}} \boxtimes L_{\alpha,F}).
\]

In lemma \(4.22\) we have used \(H_{S_{\Delta}^{0}}(\{pt\}; C) \simeq C[X_{v}; v \in \Delta]\). The choice of indexing the indeterminates by \(\Delta\) is not arbitrary, as explained in the next lemma.

For \(v \in V\), we have denoted by \(\delta_{v}\) the \(G\)- (or \(K\)-) equivariant fundamental class of \(D_{v}\) in \(X_{Z}\), \(\delta_{v} \in H_{K,D_{v}}^{2}(X_{Z}; C)\). Let us also denote by \(\delta_{v}\) its “restriction” to any \(K\) stable open subset of \(X_{Z}\). The following lemma describes the image of \(\delta_{v} \in H_{K}^{2}(V_{F}; C)\) by the isomorphism \(H_{K}(V_{F}; C) \simeq H(E/K_{F}; C)\) composed with the isomorphism of lemma \(4.22\).

Let us first recall the construction of the isomorphism

\[
(61) \quad H^{\ast}(E/S_{\Delta}^{0}; C) = H_{S_{\Delta}^{0}}(\{pt\}; C) \simeq \text{Sym}(\text{Lie}(S_{\Delta}^{0}),\mathbb{C})^{\ast},
\]

where the elements of \(\text{Lie}(S_{\Delta}^{0})^{\ast}\) have degree 2. A character \(\chi : S_{\Delta}^{0} \rightarrow \mathbb{C}^{\ast}\) gives an element \(d_{\chi} \in \text{Lie}(S_{\Delta}^{0})^{\ast}\) by differentiation. It also gives a one dimensional representation of \(S_{\Delta}^{0}, C_{\chi}\), and a line bundle \(l_{\chi} = E \times S_{\Delta}^{0} C_{\chi}\) over \(E/S_{\Delta}^{0}\). The above isomorphism sends \(d_{\chi}\) to the Chern class \(c_{2}(l_{\chi})\). We note that this Chern class is nothing but the \(S_{\Delta}^{0}\)-equivariant fundamental class of \(\{0\}\) in \(C_{\chi}\).

**Lemma 5.3.** Let \(v \in V\) and \(F = F_{\Delta_{\alpha}}\) be a face such that \(F \subset D_{v}\). For a point \(p \in F\), \(S_{\Delta}^{0}\) acts on \(T_{p}X_{Z}/T_{p}D_{v} \simeq C\). Let \(\chi_{v}\) be the corresponding character of \(S_{\Delta}^{0}\) and \(X_{v} \in H_{S_{\Delta}^{0}}^{2}(\{pt\}; C)\) the associated equivariant class. We have

\[
H_{K}(V_{F}; C) \simeq H_{K_{v}}(\{pt\}; C) \simeq H_{S_{\Delta}^{0}}(\{pt\}; C) \otimes H_{K_{J}}(\{pt\}; C)
\]

and this isomorphism sends \(\delta_{v}\) to \(X_{v} \otimes 1\). Moreover \(H_{S_{\Delta}^{0}}(\{pt\}; C) \simeq C[X_{v}; v \in \Delta]\).

**Proof.** We have seen that the first isomorphism follows from the homotopy equivalence \(q_{F}\) (see \(4.20\)). The second one is a special case of lemma \(4.22\) with \(M = C_{E/K_{F}}\).
We have an action of $K_F$ on $N_{p,v} = T_p X_{K} / T_p D_v$ and natural isomorphisms:

$$H_K^*(V_F; C) \cong H_{K_F}(N_{p,v}; C) \cong H_{K_F}^*(\{pt\}; C).$$

The class $\delta_v \in H_K^*(V_F; C)$ can be identified with the $K_F$-equivariant fundamental class of $\{0\}$ in $N_{p,v}$, $\delta_v \in H_{K_F}^*(\{0\}; N_{p,v}) \in H_{K_F}^*(\{pt\}; N_{p,v})$. Hence its image by the natural morphism from $H_{K_F}^*(\{pt\}; C)$ to $H_{K_F}^*(\{pt\}; C)$ is the $S^0_{\Delta}$-equivariant fundamental class of $\{0\}$ in $N_{p,v}$, i.e. $X_v$. By \cite{11}, $K_j$ acts trivially on $N_p$, so that the image of $\delta_v$ by $H_{K_j}^*(\{pt\}; C) \to H_{K_j}^*(\{pt\}; C)$ is 0. Since $H_{K_j}^*(V_F; C)$ only has the components $H_{K_j}^2(\{pt\}; C)$ and $H_{K_j}^2(\{pt\}; C)$, we deduce that $\delta_v$ is sent to $X_v \otimes 1$, as claimed.

Let us set $N_p = T_p X_{K} / T_p D_{\Delta}$. By theorem \ref{52} the kernel of $S^0_{\Delta} \to GL(N_p)$ is finite. Since $N_p \simeq \bigoplus_{v \in \Delta} N_{p,v}$, it follows that the characters $\chi_v, v \in \Delta$, are independent. Since $\dim S^0_{\Delta} = |\Delta|$ we obtain the last assertion.

\section{Decomposition of the dg-algebras.} We would like to decompose the de Rham complex $R_{\Delta}$ as we have decomposed its cohomology in \ref{52}. However in the sequence of fibrations,

$$E \times K V_F \xrightarrow{q_F} E/K_F \xrightarrow{r^1_F} E^3/(S^0_{\Delta} \times K_j) \xrightarrow{r^2_F} (E/S^0_{\Delta}) \times (E/K_j),$$

the morphism $r^1_F$ goes in the wrong direction, i.e. we have no natural map from $\Gamma(E/K_j; \Omega_{E/K_j})$ to $\Gamma(E \times K V_F; \Omega_{E \times K V_F})$. Hence we first replace $E \times K V_F$ by the fibre product built on $q_F$ and $r^1_F$.

\subsection{Pull-back to a fibre product.}

\begin{lemma}
We keep the notations $q_F$, $r^1_F$, $r^2_F$ defined in \ref{50} and \ref{58}. For a face $F = F_{\Delta,j} \in \mathcal{F}$, we set:

(i) For any face $F = F_{\Delta,j} \in \mathcal{F}$, we have fibrations

$$\nu_F : V^+_F \to E \times K V_F, \quad r_F : V^+_F \to (E/S^0_{\Delta}) \times (E/K_j),$$

$\nu_F$ has fibres homeomorphic to $E^2/D_{\Delta}$ and $r_F$ has contractible fibres.

(ii) For $F_i = F_{\Delta,j}, i = 1, 2$, with $F_1 \subset F_2$, we have a natural morphism $v_{F_1,F_2} : V^+_F \to V^+_F$, and a commutative diagram

\begin{equation}
\begin{array}{ccc}
E \times K V_F & \xrightarrow{\nu_{F_2}} & V^+_F \\
\downarrow & & \downarrow \\
E \times K V_{F_1} & \xrightarrow{v_{F_1}} & V^+_F \\
\end{array}
\end{equation}

(iii) For a third face $F_3$ with $F_1 \subset F_2 \subset F_3$, we have $v_{F_1,F_3} = v_{F_1,F_2} \circ v_{F_2,F_3}$.\end{lemma}

\begin{proof}
The proof is more or less tautological. By definition, $V^+_F$ comes with two fibrations, $r_F : V^+_F \to E \times K V_F$, with fibre $E^2/D_{\Delta}$, and $\mu_F : V^+_F \to E^3/(S^0_{\Delta} \times K_j)$, with contractible fibre $K_F \cdot U_{F}$. We set $r_F = r^2_F \circ \mu_F$; since $\mu_F$ and $r^2_F$ are fibrations with contractible fibres, so is $r_F$. This gives (i).

By \ref{52}, the kernel of $S^0_{\Delta} \to GL(N_p)$ is finite. Since $N_p \simeq \bigoplus_{v \in \Delta} N_{p,v}$, it follows that the characters $\chi_v, v \in \Delta$, are independent. Since $\dim S^0_{\Delta} = |\Delta|$ we obtain the last assertion.

\section{Decomposition of the dg-algebras.} We would like to decompose the de Rham complex $R_{\Delta}$ as we have decomposed its cohomology in \ref{52}. However in the sequence of fibrations,

$$E \times K V_F \xrightarrow{q_F} E/K_F \xrightarrow{r^1_F} E^3/(S^0_{\Delta} \times K_j) \xrightarrow{r^2_F} (E/S^0_{\Delta}) \times (E/K_j),$$

the morphism $r^1_F$ goes in the wrong direction, i.e. we have no natural map from $\Gamma(E/K_j; \Omega_{E/K_j})$ to $\Gamma(E \times K V_F; \Omega_{E \times K V_F})$. Hence we first replace $E \times K V_F$ by the fibre product built on $q_F$ and $r^1_F$.\end{proof}
For (ii), we have the inclusions \( K_{F_2} \subset K_F, \ K_{J_2} \subset K_{J_1}, \ S^0_{\Delta_2} \subset S^0_{\Delta_1} \), and they induce commutative squares of fibrations:

\[
\begin{array}{ccc}
E^3/(S^0_{\Delta_2} \times K_{J_2}) & \longrightarrow & E/K_{F_2} \\
\downarrow & & \downarrow \\
E^3/(S^0_{\Delta_1} \times K_{J_1}) & \longrightarrow & E/K_{F_1},
\end{array}
\]

(63)
and a similar square corresponding to \( r^2_F \). Now (ii) follows from these diagrams and the definitions. The proof of (iii) is similar.

**Definition of \( \mathcal{R}A \).** Now we pull back the construction of \( \mathcal{R} \) to the \( V^+_F \). For \( \alpha = (\mathcal{O}, \rho) \in A \), and \( F \in F \), we set \( L^+_{\alpha,F} = \nu_F^{-1} L_{\alpha,F} \). For \( v \in V \), we set \( \xi'_{v,F} = \nu'_F (\xi_v |_{E \times K V_F}) \in \Gamma(V^+_F; \Omega^2_{V^+_F}) \). For two faces \( F_1 \subset F_2 \), we have:

\[
\begin{array}{ccc}
v^{-1}_{F_1,F_2}(L^+_{\alpha,F_1}) = L^+_{\alpha,F_2}, & & \nu^*_F (\xi'_{v,F_1}) = \xi'_{v,F_2}.
\end{array}
\]

(64)
We introduce a sheaf \( \mathcal{R}A \) on \( F \), copying the definition of \( A \) in 3.5. For \( \alpha, \beta \in A \), we define the sheaf \( \mathcal{R}A^{\alpha\beta} \) by its stalks at a face \( F \in \mathcal{F}_\alpha \beta \):

\[
\mathcal{R}A^{\alpha\beta} = \Gamma(V^+_F; \Omega^2_{V^+_F} \otimes \text{Hom}(L^+_{\alpha,F}, L^+_{\beta,F})) [-2d_{\alpha\beta}],
\]
and we reduce the case \( F \notin \mathcal{F}_\alpha \beta \) to this one, as in definition 3.5. The only difference is that the restriction maps, say from \( \mathcal{R}A^{\alpha\beta}_F \) to \( \mathcal{R}A^{\alpha\rho}_E \), for \( F_1 \subset F_2 \), are induced by \( v^{-1}_{F_1,F_2} : V^+_F \rightarrow V^+_{F_1} \), instead of the inclusion \( E \times K V_{F_2} \subset E \times K V_F \). This gives a sheaf by (iii) of lemma 5.4.

We set \( \mathcal{R}A = \bigoplus_{(\alpha, \beta) \in A^2} \mathcal{R}A^{\alpha\beta} \) and endow it with an algebra structure as \( \mathcal{R} \): more precisely, we define the product on the stalks at a given face \( F \) by replacing the \( \xi_v \) in the definition of \( \mathcal{R} \) by the \( \xi'_{v,F} \); the compatibility of the product and the restriction maps follows from (64).

For a face \( F \), the inverse image by \( \nu_F \) induces a natural morphism \( \nu_F^* : \mathcal{R}_F \rightarrow \mathcal{R}A_F \). Since \( \xi'_{v,F} = \nu'_F \xi_v \), we see that the \( \nu_F^* \) are morphisms of dg-algebras. The commutative squares in lemma 5.4 imply that the \( \nu_F^* \) induce a morphism of sheaves on \( F \), say \( \nu^* : \mathcal{R} \rightarrow \mathcal{R}A \). Let us verify that it is a quasi-isomorphism. We have \( \text{Hom}(L^+_{\alpha,F}, L^+_{\beta,F}) \simeq \nu_F^{-1} \text{Hom}(L_{\alpha,F}, L_{\beta,F}) \). Since \( \nu_F \) is a fibration with fibre \( E^2/D_{\Delta_F} \), which is acyclic over \( C \), we have, for any sheaf \( L \) on \( E \times K V_F \), \( R(\nu_F)_* \nu_F^{-1} L \simeq L \). Hence \( H^*(\mathcal{R}A^\beta_F) = H^*(\mathcal{R}^\beta_F) \), as required.

5.2.2. **Decomposition.** For \( \alpha = (\mathcal{O}, \rho) \in A \), and \( F = F_{\Delta,J} \in F \) with \( F \subset Z_{\alpha} \), let \( L_{\alpha,F} \) be the local system on \( (E/K_J) \), corresponding to the representation of \( \tau_J \) given by \( \rho \) and \( t^0_J : \tau_J \rightarrow \tau_0 \) (see (50) and after). Then, by (60), \( L^+_{\alpha,F} \simeq r_F^{-1} (C_{E/S^0_{\Delta_F}} \boxtimes L^1_{\alpha,F}) \).

For \( \alpha, \beta \in A \), and \( F \in \mathcal{F}_{\alpha\beta} \), we have, by lemma 5.2:

\[
H^*(\mathcal{R}A^\beta_F) \simeq H^*(E/S^0_{\Delta_F}; C_{E/S^0_{\Delta_F}}) \otimes H^*(E/K_J, \text{Hom}(L^1_{\alpha,F}, L^1_{\beta,F})).
\]

This isomorphism corresponds to a quasi-isomorphism at the level of de Rham complexes. The product of forms, composed with the inverse image by \( r_F \) gives a quasi-isomorphism:

\[
\Gamma(E/S^0_{\Delta_F}; \Omega^*_{E/S^0_{\Delta_F}}) \otimes \Gamma(E/K_J, \Omega_{E/K_J} \otimes \text{Hom}(L^1_{\alpha,F}, L^1_{\beta,F})) \xrightarrow{\text{qis}} \mathcal{R}A^\beta_F.
\]
However the forms $\xi_{\alpha,F} \in \Omega^2_{V_F}$ do not have to be pull-backs of forms by $r_F$ and we have no natural algebra structure on the sum over $(\alpha, \beta) \in A^2$ of the groups appearing in the left hand side of (64). For this we will replace the factor $\Gamma(E/S^0_\Delta, \Omega_{E/S^0})$ by a free anti-commutative algebra quasi-isomorphic to it. We will define a sheaf $RB$ on $F$ (see (67) below) as the product of two sheaves: $RS$, quasi-isomorphic to the de Rham algebra of $E/S^0_\Delta$ appearing in (65), and $RK$, given by the twisted de Rham complex on $E/K_J$.

For $\Delta \subset V$, we introduce the dg-algebras $A(\Delta)$, $B(\Delta)$ below, which are free anti-commutative algebras, and a quasi-isomorphism, $b(\Delta)$, between them. (The reason for introducing $B(\Delta)$ is to be able to define morphism $f$ below, which would be impossible with $A(\Delta)$ instead of $B(\Delta)$.)

$$A(\Delta) = C[X_v; v \in \Delta], \quad B(\Delta) = C[X_v, Y_w; v \in V, w \in V \setminus \Delta],$$

$$b(\Delta) : B(\Delta) \to A(\Delta), \quad \forall v \in \Delta, X_v \mapsto X_v, \quad \forall w \in V \setminus \Delta, X_w \mapsto 0, Y_w \mapsto 0.$$

For $(\alpha, \beta) \in A^2$, and a face $F = F_{\Delta,j} \in F_{\alpha\beta}$ we set:

$$RS^\alpha_{\beta} = B(\Delta)[-2d_{\alpha\beta}].$$

For other faces, we reduce to the case $F \in F_{\alpha\beta}$, as in definition 5.5. The restriction maps are given by the inclusions $B(\Delta_1) \subset B(\Delta_2)$ if $\Delta_2 \subset \Delta_1$. We have a product similar to the product in $R$, as follows. For $\alpha, \beta, \gamma \in A$, we set $\varepsilon_{\alpha\beta\gamma} = \prod_{v \in \nabla} X_v$, where $\nabla$ is defined in (10). We define $m^{\alpha\beta\gamma}_{S} : RS^{\alpha_{\gamma}} \otimes RS^{\beta_{\gamma}} \to RS^{\alpha_{\beta}}$, $P \otimes Q \mapsto \varepsilon_{\alpha_{\beta_{\gamma}}PQ}$.

In a similar way, for $(\alpha, \beta) \in A^2$, and a face $F = F_{\Delta,j} \in F_{\alpha\beta}$ we set:

$$RK^{\alpha\beta}_{S} = \Gamma(E/K_J; \Omega_{E/K_J} \otimes \text{Hom}(L_{\alpha,F}, L_{\beta,F})).$$

For other faces, we reduce to the case $F \in F_{\alpha\beta}$, as in definition 5.5. The restriction maps are the following ones: for $U_{F_2} \subset U_F$, we have a map $r_2 : E/K_{J_2} \to E/K_J$, and $L_{\alpha,F_2} = r_2^{-1} L_{\alpha,F}$, hence an inverse image morphism $RK^{\alpha\beta}_{F_2} \to RK^{\alpha\beta}_{F}$. We also have an obvious product $m^{\alpha\beta\gamma}_{K} : RK^{\alpha\beta} \otimes RK^{\beta\gamma} \to RK^{\alpha\gamma}$ given by the product of forms and the composition of morphisms.

**Definition of $RB$.** Now we set:

$$\forall (\alpha, \beta) \in A^2, \quad RB^{\alpha\beta} = RS^{\alpha_{\beta}} \otimes RK^{\alpha\beta}, \quad RB = \oplus_{(\alpha, \beta) \in A^2} RB^{\alpha\beta}$$

We let $m^{\alpha\beta\gamma}_{S}$ be the tensor product of $m^{\alpha\beta}_{S}$ and $m^{\alpha\beta}_{K}$. Since, by definition, the $\varepsilon_{\alpha_{\beta_{\gamma}}}$ satisfy the same identity (11) as the $\eta_{\alpha_{\beta_{\gamma}}}$, we obtain a product on $RB$ defined by the sum of the $m^{\alpha\beta\gamma}_{S}$.

**Definition of $f : RB \to RA$.** For $F \in F$, we note that $RA_F$ is an algebra over $\Gamma(V_F^+, \Omega_{V_F^+})$. We define a morphism $f^{\alpha\beta}_{F} : RB^{\alpha\beta}_{F} \to RA^{\alpha\beta}_{F}$ as the product of $f^{\alpha\beta}_{S} : RS^{\alpha\beta}_{F} \to \Gamma(V_F^+; \Omega_{V_F^+})$ and $f^{\alpha\beta}_{K} : RK^{\alpha\beta}_{F} \to RA^{\alpha\beta}_{F}$, which are obtained as follows.

For a face $F$, let $s_F : V_F^+ \to E/K_J$ be the composition of $r_F$ and the projection to $E/K_J$. For a form $\sigma$ and a sheaves endomorphism $u$, we set $f^{\alpha\beta}_{F}(\sigma \otimes u) = s_F^*(\sigma) \otimes s_F^*(u)$.

Now we define $f^{\alpha\beta}_{S}$. For $v \in V$, the fundamental class, $\delta_v$ of $D_v$ in $X_Z$ restricts to 0 on $X_Z \setminus D_v$. Hence the restriction of $\xi_v$ on $E \times_K (X_Z \setminus D_v)$ is a boundary. Let us choose a form $\xi'_v \in \Gamma(E \times_K (X_Z \setminus D_v); \Omega_{E \times_K X_Z}^1)$ such that, on $E \times_K (X_Z \setminus D_v)$
we have $\xi_v = d\zeta_v$. We set $\zeta_{v,F} = \nu_F^p(\xi_v|_{E \times_K V_F})$ (we note that, for $E = F_{\Delta,J}$ and $v \in V$ such that $v \notin \Delta$, we have $V_F \cap D_v = \emptyset$, so that $\zeta_v$ is defined on $E \times_K V_F$).

We set:

$$\forall v \in V, \ f^S_{\alpha \beta}(X_v) = \xi_{v,F}, \ \forall w \in V \setminus \Delta, \ f^S_{\alpha \beta}(Y_w) = \zeta_{w,F}.$$  

Using lemma 5.3 one checks that $f^S_{\alpha \beta}$ and $f^K_{\alpha \beta}$ give morphisms of sheaves, say $f_{\alpha \beta}$ and $f_{\alpha \beta}$. We define $f_{\alpha \beta} = f^S_{\alpha \beta} \otimes f^K_{\alpha \beta}$ and $f = \otimes f_{\alpha \beta}$. In view of the definitions of the product and the differentials, $f$ is a morphism of dg-algebras. By lemmas 5.2 and 5.3 it is a quasi-isomorphism.

5.3. **Formality of the toric part.** We define a sheaf quasi-isomorphic to $\mathcal{R}S$ but with differential zero as follows. For $(\alpha, \beta) \in A^2$, and a face $F = F_{\Delta,J} \in \mathcal{F}_{\alpha \beta}$, we set:

$$f^\mathcal{R}_{\alpha \beta} = A(\Delta)[-2d_{\alpha \beta}],$$

with the following restriction maps. For two faces $F_i = F_{\Delta_i,J_i}$, $i = 1, 2$, such that $F_1 \subset F_2$ (i.e. $U_{F_2} \subset U_{F_1}$), we have $\Delta_2 \subset \Delta_1$, and the restriction $A(\Delta_1) \to A(\Delta_2)$ sends $X_v$ to $X_v$ for $v \in \Delta_2$ and to 0 for $v \in \Delta_1 \setminus \Delta_2$. As for $\mathcal{R}S$, other faces are reduced to this case.

We define a product, $m^\mathcal{R}_{\alpha \beta \gamma}$, similar to the product of $\mathcal{R}S$. For $\alpha, \beta, \gamma \in A$, we let $e_{\alpha \beta \gamma}$ be the section of $\mathcal{R}T^\alpha \otimes \mathcal{R}T^\beta \otimes \mathcal{R}T^\gamma$ defined by $(e_{\alpha \beta \gamma})_{F_{\Delta,J}} = b(\Delta)(e_{\alpha \beta \gamma})$. We set

$$m^\mathcal{R}_{\alpha \beta \gamma} : \mathcal{R}T^\alpha \otimes \mathcal{R}T^\beta \otimes \mathcal{R}T^\gamma \to \mathcal{R}T^\alpha, \ P \otimes Q \mapsto e_{\alpha \beta \gamma} PQ.$$

**Definition of $\mathcal{R}C$.** We set as in (63):

$$\forall (\alpha, \beta) \in A^2, \ \mathcal{R}C^\alpha \otimes \mathcal{R}C^\beta = \mathcal{R}C^\alpha \otimes \mathcal{R}C^\beta, \ \mathcal{R}C = \oplus_{(\alpha, \beta) \in A^2} \mathcal{R}C^\alpha \otimes \mathcal{R}C^\beta$$

We let $m^\mathcal{R}_{\alpha \beta \gamma}$ be the tensor product of $m^\mathcal{R}_{\alpha \beta \gamma}$ and $m^\mathcal{R}_{\alpha \beta \gamma}$. The sum $m^\mathcal{R}_C = \oplus m^\mathcal{R}_{\alpha \beta \gamma}$ defines a product on $\mathcal{R}C$. The $b(\Delta)$ defined in (68) give a quasi-isomorphism of sheaves of dg-algebras $\mathcal{R}S \to \mathcal{R}T$. This induces a quasi-isomorphism of sheaves of dg-algebras $\mathcal{R}B \to \mathcal{R}C$.

5.4. **Formality of $\mathcal{R}K$ and conclusion.** It remains to prove that the factor $\mathcal{R}K$ of $\mathcal{R}B$ also is quasi-isomorphic to its cohomology. For this we first give a more handy expression for $\mathcal{R}K^\alpha \otimes \beta$ (see formula (72) below). For $J \subset \{1, \ldots, l\}$, let $\pi_J : E/K_J \to E/K$ be the covering map, with group $\tau_J$, and set $A_J = (\pi_J)^*(C_{E/K})$. Then $A_J$ is a local system on $E/K_J$, considered as a right module over $C[\tau_J]$, locally free of rank one. It is also a sheaf of algebras. For $\alpha = (O, \rho) \in A$, $F = F_{\Delta,J} \in \mathcal{F}$, such that $F \subset Z_\alpha$, we have, by (59):

$$L^1_{\alpha,F} \simeq A_J \otimes C_{[\tau_J]} V_\rho.$$  

Hence, for another element $\beta = (O', \rho') \in A$, with $F \subset Z_\beta$, we have:

$$\mathcal{H}om(L^1_{\alpha,F}, L^1_{\beta,F}) \simeq \mathcal{H}om(A_J \otimes C_{[\tau_J]} V_\rho, A_J \otimes C_{[\tau_J]} V_\rho').$$

We want to “factorise” the local systems $A_J$ and the representation spaces $V_\rho, V_\rho'$ in this last formula. We will use the following definition (see 13).

**Definition 5.5.** For a group $W$ and a $C$-algebra $A$ with a right $W$-action by algebra automorphisms, we set $A^W = A \otimes C[W]$ with the product, for $a \in A$, $w \in W$, $(a \otimes w) \cdot (a' \otimes w') = (a(a' \cdot w)) \otimes (w'w)$. We have a natural embedding $C[W]^{op} \to A^W$, $w \mapsto a_w = 1 \otimes w$. It induces a structure of right $C[W] \otimes C[W]^{op}$-module on $A^W$: for $x \in A^W$, $w, w' \in W$, $x \cdot (w \otimes w') = a_w x a_{w'}$. 

We set $B_J = C[\tau_J] \otimes C[\tau_J]^\varrho$. We consider $\mathcal{H}om(A_J, A_J)$ as a sheaf of algebras, where the product is the composition, and right $B_J$-module by $(\phi \cdot (t \otimes t'))(a) = (\phi(a \cdot t')) \cdot t$, for $\phi$ a section of $\mathcal{H}om(A_J, A_J)$, $a$ a section of $A_J$ and $t, t' \in \tau_J$. With these definitions, one can check that the map
\[(70) \quad A^1_J[\tau_J] \to \mathcal{H}om(A_J, A_J), \quad a \otimes t \mapsto (\alpha \mapsto a(\alpha \cdot t)),\]
where $a, \alpha$ are sections of $A_J$ and $t \in \tau_J$, is an isomorphism of (sheaves of) algebras and right $B_J$-modules.

**Lemma 5.6.** For $J \subset \{1, \ldots, l\}$, $\alpha, \beta \in A$, $F = F_{\Delta,J} \in \mathcal{F}_{\alpha\beta}$, we have, with the above notations:
\[(71) \quad \mathcal{H}om(L^\alpha_{\alpha,F}, L^\beta_{\beta,F}) \simeq \mathcal{H}om(A_J, A_J) \otimes_{B_J} \mathcal{H}om(V^J_\rho, V^J_\rho'),\]
\[(72) \quad \mathcal{R}K^\alpha_{\beta,F} \simeq \Gamma(E/K_J^0; \Omega_{E/K_J}^1[\tau_J]) \otimes_{B_J} \mathcal{H}om(V^J_\rho, V^J_\rho').\]

**Proof.** Let us prove (71). We will use the following fact. Set $R = C[W]$ for a finite group $W$ and consider left and right $R$-modules, $M, N$, of finite ranks. We have a right $R$-module structure on $M^* = \mathcal{H}om_C(M, C)$ and a left one on $N^*$. Then the composition of (vector spaces) morphisms
\[(73) \quad (N \otimes_R M)^* \to (N \otimes_C M)^* \mapsto M^* \otimes_C N^* \to M^* \otimes_R N^*\]
is a canonical isomorphism. Indeed, it is compatible with direct sum, and any $R$-module is semi-simple. For $N, M$ irreducible with $N \neq M^*$, both $N \otimes_R M$ and $M^* \otimes_R N^*$ are 0. For $N = M^*$, both $N \otimes_R M$ and $M^* \otimes_R N^*$ are canonically identified with $C$ by the duality contractions $N \otimes M \to C$ and $M^* \otimes N^* \to C$.

Since the three morphisms in (73) commute with the duality contraction their composition corresponds to $id_C$. Using this we deduce canonical isomorphisms for left and right $R$-modules, $M_i, N_i, i = 1, 2$:
\[(74) \quad \mathcal{H}om_C(N_1 \otimes_R M_1, N_2 \otimes_R M_2) \simeq (N_1^* \otimes_R N_2^*) \otimes_C (N_2 \otimes_R M_2)\]
\[\simeq (N_1^* \otimes_C N_2) \otimes_{R \otimes R^\varrho} (M_1^* \otimes_C M_2)\]
\[\simeq \mathcal{H}om_C(N_1, N_2) \otimes_{R \otimes R^\varrho} \mathcal{H}om_C(M_1, M_2).\]

Since isomorphism (74) is canonical, it works as well for sheaves and we obtain (71).

Now we deduce the second isomorphism. We remark that $\Omega_{E/K_J} \otimes A_J$ is isomorphic to $(\tau_J)_* \Omega_{E/K_J}^0$. This isomorphism respects the $\tau_J$-action and we have, by isomorphisms (71) and (70):
\[\Omega_{E/K_J} \otimes \mathcal{H}om(L^\alpha_{\alpha,F}, L^\beta_{\beta,F}) \simeq ((\tau_J)_* \Omega_{E/K_J}^0)[\tau_J] \otimes_{B_J} \mathcal{H}om(V^J_\rho, V^J_\rho').\]

Since $\otimes_{B_J}$ is exact for $B_J$-modules, the constant sheaf $\mathcal{H}om(V^J_\rho, V^J_\rho')$ factors out when we take global sections, and we obtain (72). \(\square\)

**Remark 5.7.** On the right hand side of formula (72), the restriction maps are given as follows. For another face $F' = F_{\Delta,J'} \in \mathcal{F}_{\alpha\beta}$ such that $F \subset F'$, we have $K_{J'} \subset K_J$, hence a groups morphism $a : \tau_{J'} \to \tau_J$ and a quotient map $p : E/K_{J'}^0 \to E/K_J^0$. The inverse image of forms by $p$ is compatible with the action of $\tau_J$, $\tau_{J'}$ (via $a$) and we obtain a dg-algebras morphism:
\[b : \Gamma(E/K_J^0; \Omega_{E/K_J}^1[\tau_J]) \to \Gamma(E/K_{J'}^0; \Omega_{E/K_{J'}}^1[\tau_{J'}]) \otimes_{B_{J'}} B_J,\]
by $b(\omega \otimes t) = (p^*(\omega) \otimes 1) \otimes (1 \otimes t)$, for a form $\omega$ and $t \in \tau_J$. Tensorisation with $\mathcal{H}om(V^J_\rho, V^J_\rho')$ gives the desired restriction map from $\mathcal{R}K^\alpha_{\beta,F}$ to $\mathcal{R}K^\alpha_{\beta,F'}$. 

Let us explain how to recover the product $\mathcal{R}K^{\alpha\beta} \times \mathcal{R}K^{\beta\gamma} \to \mathcal{R}K^{\alpha\gamma}$ in the right hand side of (72). First we consider isomorphism (44). For $u : N_1 \to N_2$, $v : M_1 \to M_2$, let us denote by $u \cdot v : N_1 \otimes_R M_1 \to N_2 \otimes_R M_2$ the image of $u \otimes v$ by (44). For left and right $R$-modules, $M_3, N_3, u' : N_2 \to N_3, v' : M_2 \to M_3$, we see that $(u' \cdot v') \circ (u \cdot v) = (u' \circ u) \cdot (v' \circ v).

Hence the product in the right hand side of (71) is given by the composition in $\mathcal{R}Hom(A_J, A_J)$ and in $\mathcal{R}Hom(V, V')$. When we replace $\mathcal{R}Hom(A_J, A_J)$ by another algebra, say $R_1$, as in (72), we use the following lemma (with $R_2 = \mathcal{R}Hom(V, V), V$ being the sum of irreducible representations of $W$ and $\phi_2$ the action of $W$ on $V$).

**Lemma 5.8.** Let $W$ be a finite group, $R = \mathbb{C}[W]$ and $R_i, i = 1, 2$, algebras. Let $\phi_1 : R^{op} \to R_1, \phi_2 : R \to R_2$ be algebras morphisms. We consider the right $R \otimes R^{op}$-module structure on $R_1$ given by $a \cdot (w \otimes w') = \phi_1(w)a\phi_2(w')$, for $w, w' \in W, a \in R_1$, and the left structure on $R_2$, $(w \otimes w') \cdot a' = \phi_2(w)a\phi_2(w')$.

Then the formula $(a \otimes a', b \otimes b') \mapsto ab \otimes a'b'$, for $a, b \in R_1, a', b' \in R_2$, gives a well-defined product on $(R_1 \otimes_{R \otimes R^{op}} R_2)$.

**Proof.** By symmetry, it is sufficient to prove, for $a, b \in R_1, a', b' \in R_2, w,w' \in W$:

$$\phi_1(w)a\phi_2(w')b \otimes a'b' = ab \otimes (\phi_2(w)a\phi_2(w')b').$$

The tensor product is over $R \otimes R^{op}$, so that we are reduced to

$$(\phi_1(w)b) \otimes a'b' = ab \otimes (\phi_2(w)b').$$

Let us consider the subgroup $W' \subset W$ generated by $w'$, and $R' = \mathbb{C}[W']$. Then it is sufficient to prove that (75) holds with a tensor product over $R' \otimes R'^{op}$. Hence, replacing $W$ by $W'$, we may assume from the beginning that $W = \langle w' \rangle$ is commutative. We decompose $R_i$ under the action of $w'$: $R_i = \bigoplus_{\lambda, \mu} \mathbb{C} R_i^{\lambda \mu}$, where $\forall a \in R_i^{\lambda \mu}, \phi_1(w')a = \lambda a, \phi_2(w') = \mu a$. By additivity, we may assume that $a, b, a', b'$ are elements of some $R_i^{\lambda \mu}$. But $R_1^{\lambda \mu} \otimes_{R' \otimes R'^{op}} R_2^{\lambda' \nu'}$ is 0 unless $\lambda = \lambda'$ and $\mu = \mu'$. Similarly, for $a \in R_1^{\lambda \mu}, b \in R_2^{\lambda' \nu'}, \lambda', \mu' \in 1$. Hence we may assume $a \in R_1^{\lambda \mu}, b \in R_2^{\lambda' \nu'}, a' \in R_2^{\lambda \mu}, b' \in R_2^{\lambda' \nu'}$. In this case formula (75) is obvious.

**5.4.1. Formality of $\mathcal{R}K$.** Using (72) we will deduce the formality of $\mathcal{R}K$ from the formality of the de Rham algebras of $E/K_0$ obtained in lemma (21).

Let us denote by $\mathfrak{t}, \mathfrak{t}_j$, the Lie algebras of $K, K_j$, for $J \subset \{1, \ldots, l\}$. Let us set for short $H_0^J = (S(\mathfrak{t}_J))K_0^J$, viewed as a dg-algebra with differential 0; it is isomorphic to the $K_0^J$-equivariant cohomology algebra of the point, $H_0^{K_0^J} \langle \{pt\} \rangle$.

By lemma (21) the choice of a connection on $\Gamma(E; \Omega_E)$ (in the sense of (5)) gives functorial quasi-isomorphisms:

$$\Gamma(E/K_0^J; \Omega_{E/K_0^J}) \overset{f_J}{\longrightarrow} W(\mathfrak{t})_{\mathfrak{t}_J \rightarrow b} \overset{g_J}{\longrightarrow} H_0^J.$$

The normaliser $N_K(K_0^J)$ acts on the above dg-algebras and $K_0^J$ acts trivially, so that $\tau_J = K_j/K_0^J$ also acts. The morphisms $f_J$ and $g_J$ are $\tau_J$-equivariant and yield quasi-isomorphisms:

$$(76) \quad \Gamma(E/K_0^J; \Omega_{E/K_0^J})^{\tau_J} \overset{f_J}{\longleftarrow} (W(\mathfrak{t})_{\mathfrak{t}_J \rightarrow b})^{\tau_J} \overset{g_J}{\longrightarrow} (H_0^J)^{\tau_J}.$$

**Definition of $\mathcal{R}D$.** For $(\alpha, \beta) \in A^2$, we define $\mathcal{R}C^{\alpha\beta}$ similarly as $\mathcal{R}K^{\alpha\beta}$, replacing the forms over $E/K_0^J$ in expression (72) by a quasi-isomorphic dg-algebra. The
stalks at a face $F = F_{\Delta, j} \in \mathcal{F}_{\alpha \beta}$ are given by:
\[
\mathcal{RL}^\alpha_\beta = (W(\mathfrak{t})_{t_{j}-b})^j[\tau_j] \otimes_{B_j} \text{Hom}(V_\rho, V_{\rho'}),
\]
with restriction morphisms defined as in remark 5.7 and product as in lemma 5.8. As for $\mathcal{RK}$, other faces are reduced to this case. In view of the isomorphism $42$ and the quasi-isomorphism $f'_j$ of $(40)$, we have a quasi-isomorphism of sheaves $\mathcal{RL}^{\alpha \beta} \to \mathcal{RK}^{\alpha \beta}$. Setting
\[
\mathcal{RD}^{\alpha \beta} = \mathcal{RT}^{\alpha \beta} \otimes \mathcal{RL}^{\alpha \beta},
\]
with the product defined as the product of $\mathcal{RC}$, we deduce a quasi-isomorphism of sheaves of dg-algebras $\mathcal{RD} \to \mathcal{RC}$.

**Definition of $\mathcal{RE}$.** We define $\mathcal{RM}^{\alpha \beta}$ similarly as $\mathcal{RL}^{\alpha \beta}$, with stalks at a face $F = F_{\Delta, j} \in \mathcal{F}_{\alpha \beta}$:
\[
(77) \quad \mathcal{RM}^{\alpha \beta} = (H^0_j)^j[\tau_j] \otimes_{B_j} \text{Hom}(V_\rho, V_{\rho'}).
\]
Then the $g'_j$ induce a quasi-isomorphism $\mathcal{RL}^{\alpha \beta} \to \mathcal{RM}^{\alpha \beta}$ and, setting
\[
\mathcal{RE}^{\alpha \beta} = \mathcal{RT}^{\alpha \beta} \otimes \mathcal{RM}^{\alpha \beta},
\]
we obtain a quasi-isomorphism of sheaves of dg-algebras $\mathcal{RD} \to \mathcal{RE}$.

5.4.2. **End of proof.** Since the differential in $\mathcal{RE}$ is 0, $\mathcal{RE}$ coincides with its cohomology algebra $\mathcal{H}$. Finally we have built a sequence of quasi-isomorphic sheaves of dg-algebras $\mathcal{R} \to \mathcal{RA} \leftarrow \mathcal{RB} \to \cdots \to \mathcal{H}$, as required.

To conclude the proof of proposition $5.1$ we remark that in the above sequence of quasi-isomorphisms, each of the intermediate sheaves, say $\mathcal{A}$, is defined as a sum $\mathcal{A} = \oplus_{(\alpha, \beta) \in A^2} \mathcal{A}^{\alpha \beta}$, so that, for $\alpha \in A$, $\mathcal{A}_\alpha = \oplus_{\alpha_1} \mathcal{A}^{\alpha_1 \alpha}$ has a natural structure of $\mathcal{A}$-module. It follows that, for a quasi-isomorphism $\mathcal{A} \to \mathcal{A}'$ in the above sequence, the equivalence of categories between $D_{\mathcal{A}}$ and $D_{\mathcal{A}'}$ sends the $\mathcal{A}$-module $\mathcal{A}_\alpha$ to a $\mathcal{A}'$-module isomorphic to $\mathcal{A}'_{\alpha}$. This shows that $N_\alpha$ corresponds to $\mathcal{H}_\alpha$.

6. **PROOF OF THEOREM 1.1**

Now we prove theorem 1.1 using the equivalence of proposition 5.1 between $D_G^p(X_Z)$ and $D_\mathcal{H}(\mathcal{H}_\alpha, \alpha \in A)$. In view of this equivalence and the equality $\mathcal{H} = \oplus_{\alpha \in A} \mathcal{H}_\alpha$, the algebra $\mathcal{E}$ of the theorem is isomorphic to $\text{Ext}_{D_{\mathcal{H}}}(\mathcal{H}, \mathcal{H})$ and $\mathcal{E}_\alpha = \text{Ext}_{D_{\mathcal{H}}}(\mathcal{H}, \mathcal{H}_\alpha)$, for $\alpha = (\mathcal{O}, \rho)$. We have to prove that $D_{\mathcal{H}}(\mathcal{H}_\alpha)$ is equivalent to $D_{\mathcal{E}}(\mathcal{E}_\alpha)$. We recall the following construction:

1) Let $\mathcal{A}$ be a sheaf of dg-algebras on a finite set $I$ (hence $\mathcal{M}_\mathcal{A}$ has enough $K$-projectives). Let $M^1, \ldots, M^r$ be $\mathcal{A}$-modules, $P^i \to M^i$ a $K$-projective resolution of $M^i$ and $P = \oplus_i P^i$. The composition of morphisms induces a structure of dg-algebra on $R = \text{Hom}(P, P)$ (see section 23 for the definition of $\text{Hom}(\cdot, \cdot)$). We have a functor $F : \mathcal{M}_\mathcal{A} \to M_R$, $M \mapsto \text{Hom}(P, M)$, which sends quasi-isomorphisms to quasi-isomorphisms because $P$ is $K$-projective. Hence it induces $F : D_{\mathcal{A}} \to D_R$. We set $N^i = F(M^i)$. Then $F$ restricts to an equivalence of categories between $D_{\mathcal{A}}(M^i)$ and $D_R(N^i)$ (for example this is a very special case of $11$, theorem 4.3).

2) We apply this to $D_{\mathcal{H}}(\mathcal{H}_\alpha)$. Since $\mathcal{H}_\alpha$ is a direct summand of $\mathcal{H}$, $(\mathcal{H}_\alpha)_\rho$ is $K$-projective, for any $F \in \mathcal{F}$. Moreover, since $F$ satisfies (i) of assumptions $5.1$ we
see, as in the proof of proposition 4.19 that the Čech resolution (where we fix a total order on \( \mathcal{F} \)):

\[
(78) \quad P_\alpha = \cdots \to \bigoplus_{F_1, \ldots, F_k \in \mathcal{F}} (\mathcal{H}_\alpha)_{U_{F_1} \cap \cdots \cap U_{F_k}} \to \cdots \to \bigoplus_{F \in \mathcal{F}} (\mathcal{H}_\alpha)_{U_F} \to 0.
\]

is a \( K \)-projective resolution of \( \mathcal{H}_\alpha \). We set \( P = \bigoplus \alpha P_\alpha \) and \( R = \text{Hom}(P, P) \).

Following [13] we use the fact that the differential of \( \mathcal{H} \) is 0 and define \( \mathcal{H}' \) to be the sheaf of non-graded algebras underlying \( \mathcal{H} \). Replacing \( \mathcal{H} \) by \( \mathcal{H}' \) above, we define similarly \( \mathcal{H}'_\alpha, P'_\alpha, P', R' \) (we note that \( P'_\alpha \) still is a \( K \)-projective resolution of \( \mathcal{H}'_\alpha \)). The algebras \( R \) and \( R' \) are canonically isomorphic as differential algebras, but they do not have the same graduation. If we write \( P \) as a double complex \( P = \bigoplus_{i,j} P^{ij} \) and set \( Q^{ij}_\alpha = \text{Hom}_C(P^{ij}, P^{kl}) \), then \( R^d = R \cap (\bigoplus_{k+l=i+j+d} Q^{kl}_\alpha) \) and \( R'^d = R \cap (\bigoplus_{k+l=i+j+d} Q^{kl}_\alpha) \).

**Claim 6.1.** the dg-algebra \( R' \) is concentrated in degree 0.

We will prove this below. Let us see why it implies the theorem. We set \( R_0 = \tau_{<0} R' = \cdots \oplus R'^{-2} \oplus R'^{-1} \oplus \ker d_0 \). This is a differential sub-algebra of \( R' \) (or of \( R \) as well) and the claim implies that we have quasi-isomorphisms of differential algebras:

\[
R' \xrightarrow{\sim} R_0 \xrightarrow{v} H(R'), \quad R \xrightarrow{u} R_0 \xrightarrow{v'} H(R).
\]

In view of the decompositions of \( R \) and \( R' \) above, we have \( R'^d = \bigoplus \alpha (R^n \cap R'^d) \).

Hence we may endow \( R_0 \) with the graduation induced by the embedding \( u' \). Then \( v' \) is a graded morphism: indeed \( v' \) is the composition of the projection from \( R_0 \) to \( R'^0 \cap R_0 \) and the projection from \( \ker d_R \) to \( H(R) \), both morphisms are graded and so is \( v' \). Now we just remark that \( H(R) = \mathcal{E} \) by definition. Moreover, the functor from \( \mathcal{D}_R \) to \( \mathcal{D}_R \) sends \( P_\alpha \simeq \mathcal{H}_\alpha \) to \( \text{Hom}(P, P_\alpha) \). Since this last object is a summand of \( R \), we see that, in the equivalence from \( \mathcal{D}_R \) to \( \mathcal{D}_{H(R)} \), it is sent to its cohomology, \( \text{Ext}_{\mathcal{D}_{H}}(P, P_\alpha) \simeq \text{Ext}_{\mathcal{D}_{H}}(\mathcal{H}, \mathcal{H}_\alpha) = \mathcal{E}^0_\alpha \). Summing up we obtain an equivalence between \( \mathcal{D}_{H}(\mathcal{H}_\alpha) \) and \( \mathcal{D}_{\mathcal{E}}(\mathcal{E}^0_\alpha) \), as desired.

**Proof of claim 6.1.** We have to prove that \( \text{Hom}_\mathcal{D}_{H'}(\mathcal{H}', \mathcal{H}'[p]) = 0 \), for \( p \neq 0 \). We use the Čech resolution (13), with \( \mathcal{H}' \) instead of \( \mathcal{H}_\alpha \). Since \( \text{Hom}(\mathcal{H}'_{U_F}, \mathcal{H}') \simeq \Gamma(U_F; \mathcal{H}') \), we obtain

\[
\text{RHom}(\mathcal{H}', \mathcal{H}') \simeq 0 \to \bigoplus_{F \in \mathcal{F}} \Gamma(U_F; \mathcal{H}') \to \cdots.
\]

Since the open sets \( U_{F_1} \cap \cdots \cap U_{F_k} \) are fundamental open sets, the functor of sections over them is exact. Hence the above resolution computes \( H'(\mathcal{F}; \mathcal{H}') \) and the claim follows from the next lemma. \( \square \)

**Lemma 6.2.** For any \( G \)-stable open subset \( V \) of \( X_Z \), and \( U = \phi(V) \), we have \( H^i(U; \mathcal{H}') = 0 \) for \( i > 0 \), where \( \mathcal{H}' \) is the non-graded sheaf underlying \( \mathcal{H} \).

**Proof.** By proposition 6.1 we have an isomorphism \( \mathcal{H}' \simeq \mathcal{R}\mathcal{E} \), and, by definition, \( \mathcal{R}\mathcal{E} = \bigoplus_{(\alpha, \beta) \in A^2} \mathcal{R}\mathcal{E}^{\alpha\beta} \). Hence it is sufficient to prove the vanishing of \( H^i(U; \mathcal{R}\mathcal{E}^{\alpha\beta}) \).

We fix \( (\alpha, \beta) \in A^2 \) for the remainder of the proof and set for short \( \mathcal{A} = \mathcal{R}\mathcal{E}^{\alpha\beta} \).

1) Let us set \( V_{\alpha} = X_Z \setminus \bigcup_{\alpha} D_{\alpha}, V_{\beta} = X_Z \setminus \bigcup_{\beta} D_{\beta}, V_{\alpha\beta} = V_{\alpha} \cap V_{\beta}, V_{\alpha\beta} = \phi(V_{\alpha\beta}) \). We let \( j : U_{\alpha\beta} \to \mathcal{F} \) be the inclusion. We first prove that \( \mathcal{A} = j_*(\mathcal{A}|_{U_{\alpha\beta}}) \).

For a face \( F \in \mathcal{F} \), repeated applications of hypothesis (iii) of assumptions 5.1 show
that there exists a face \( F' \) such that \( U_F \cap U_{\alpha \beta} = U_{F'} \). Our claim is equivalent to the fact that for any \( F \), the restriction map \( A_F \to A_{F'} \) is an isomorphism. We have already seen, by definition of \( \Delta_1, \Delta_2 \) and lemma 2.3, that \( L_\alpha \simeq (L_\alpha)_c, \ L_\beta \simeq RT \gamma_\alpha (L_\beta) \). It follows that \( RH_\text{Hom}(L_\alpha, L_\beta) \simeq RT \gamma_\alpha \text{Hom}(L_\alpha, L_\beta) \). Using (56), this implies the desired result:

\[
A_F \simeq \text{Ext}_{DG(X_Z)}(L_\alpha|_{V_F}, L_\beta|_{V_F}) \simeq H_G(V_F; RH_\text{Hom}(L_\alpha, L_\beta)) \\
\simeq H_G(V_F \cap V_{\alpha \beta}; RH_\text{Hom}(L_\alpha, L_\beta)) \simeq A_{F'}.
\]

Now we remark that the functor \( j_* \) is exact, because, for any sheaf \( B \) on \( F \), any face \( F \), we have \( (j_*B)_F \simeq B_{F'} \), for \( F' \) satisfying \( U_F \cap U_{\alpha \beta} = U_{F'} \) as above. Hence we have \( H^i(U; A) \simeq H^i(U \cap U_{\alpha \beta}; A) \).

2) This means that we may assume \( V \subseteq V_{\alpha \beta} \). We prove the result by induction on the number of \( G \)-orbits in \( V \). If \( V \) consists of the open orbit of \( X_Z \), then \( U \) is the fundamental open set \( U_F \), with \( F = F_{0,0} \). Hence \( \Gamma(U; \cdot) = (\cdot)_F \) is exact and we are done.

Now let us assume that \( V = W \cap O_\Delta \), where \( W \subseteq V \) is a \( G \)-stable open subset of \( X_Z \). By induction the result is true for \( W \). Let \( F_{\Delta, J} \) be the closed face of \( O_\Delta \). Since \( U_{F_{\Delta, J}} \) contains \( O_\Delta \), we have \( \phi(V) = \phi(W) \cap U_{F_{\Delta, J}} \). Setting \( U' = \phi(W) \cap U_{F_{\Delta, J}} \) and using the Mayer-Vietoris sequence, it is sufficient to prove:

\[
\forall i > 0, \ H^i(U'; A) = 0, \quad \text{and} \quad H^0(U_{F_{\Delta, J}}; A) \to H^0(U'; A) \text{ is surjective.}
\]

3) We compute \( H^i(U'; A) \) with the help of a Čech covering. Remember that \( U_{F_{\Delta', J'}} \subset U_{F_{\Delta, J}} \) if and only if \( \Delta' \subset \Delta \) and \( J \subset J' \). For \( \Delta' \subset \Delta \), we have \( O_\Delta \cap U_{F_{\Delta', J'}} = \emptyset \) if and only if \( \Delta' \neq \Delta \); but this implicitly assume that \( F_{\Delta', J'} \) is a face, i.e. \( O_{\Delta'} \cap C_{J'} \) is non-empty. For any \( \Delta' \subset \Delta \), we have indeed \( F_{\Delta', J} \neq \emptyset \); this is easily seen if \( X_Z = X \), in which case \( F \) is the set of faces of \([0,1]^l\). This implies the general case because \( C_{X_Z, J_{\Delta}} = \pi^{-1}\pi(C_{X, J_{\Delta}}) \) (where \( \pi \) is the map \( X_Z \to X \)). It follows that

\[
U' = \bigcup_{\Delta' \subset \Delta} U_{F_{\Delta', J}}.
\]

By (15), this covering is stable by taking intersections. Since it consists of fundamental open sets, on which the functor of sections is exact, it can be used to compute \( H^i(U'; \cdot) \). For this, we have to know \( A(U_F) \). By definition, for any face \( F = F_{\Delta_1, J_1} \in F \):

\[
A(U_F) = A_F = \mathcal{R}T_{F}^{\alpha \beta} \otimes \mathcal{R}M_{F}^{\alpha \beta}.
\]

Let us describe more precisely the components of the tensor product. Recall that \( F \in U_{\alpha \beta} \); hence either either \( F \not\in F_{\alpha \beta} \cup F_{\alpha \beta} \) or \( F \in F_{\alpha \beta} \) (see (15)). In the first case we have \( \mathcal{R}M_{\alpha \beta} \otimes \mathcal{R}T_{F}^{\alpha \beta} = 0 \). In the second case, by definition (15), we have

\[
\mathcal{R}T_{F_{\Delta_1, J_1}}^{\alpha \beta} = C[X_v, v \in \Delta_1],
\]

and, by (17), \( \mathcal{R}M_{\alpha \beta} \) only depends on \( J_1 \), say \( \mathcal{R}M_{\alpha \beta} = M_{J_1} \). These descriptions of \( \mathcal{R}T_{F}^{\alpha \beta} \) and \( \mathcal{R}M_{\alpha \beta} \) assume that \( F_{\Delta_1, J_1} \) is a face. We have seen that this is the case for \( F_{\Delta', J} \) with \( \Delta' \subset \Delta \).

4) Since, in the covering (16), all faces have the same “\( J \)-index”, we obtain \( H^i(U'; A) = M(J_{\Delta}) \otimes H^i(C') \), where \( C' = C'(S_{\Delta}) \) is the following complex. We let
$S_\Delta$ be the set of subsets $\Delta'$ of $\Delta$ such that $(\Delta_{\alpha} \cup \Delta_{\beta}) \subset \Delta' \subseteq \Delta$ and we consider any total order on $S_\Delta$:

$$C(S_\Delta) = 0 \rightarrow \bigoplus_{\Delta' \in S_\Delta} C[X_v; \ v \in \Delta'_1] \rightarrow \bigoplus_{\Delta'_1 < \Delta'_2 \in S_\Delta} C[X_v; \ v \in \Delta'_1 \cap \Delta'_2] \rightarrow \cdots.$$  

We have a bijection between $S_\Delta$ and the set, $S_\Phi$, of strict subsets of $\Phi = \Delta \setminus (\Delta_{\alpha} \cup \Delta_{\beta})$. Setting $C_1 = C[X_v; \ v \in \Delta_{\alpha} \cup \Delta_{\beta}]$ and $C_2 = C(S_\Phi)$, we obtain $C' = C_1 \otimes C_2$.

Hence $H^i(U'; A) = M(J_\Delta) \otimes C_1 \otimes H^i(C_2)$. Since $H^0(U_{\Phi,\Delta}; A) \simeq M(J_\Delta) \otimes C[X_v; \ v \in \Delta]$. [\ref{E1}] will follow from

$$(81) \quad \forall i > 0, \ H^i(C_2) = 0, \text{ and } C[X_v; \ v \in \Phi] \rightarrow H^0(C_2) \text{ is surjective.}$$

5) We may interpret $C_2$ as another Čech complex: we consider the following sheaves on the quadrant $Q = \mathbb{R}_{>0}^2$.

$$M_1 = \oplus_{k \in \Phi} C_{\{x_k = 0\}}, \quad M_2 \text{ associated to } O \mapsto \text{Sym}(M_1(O)),$$

where $O \subset Q$ is open and Sym$(\cdot)$ denotes the symmetric algebra. We also consider the open subsets of $Q$, for $\Phi' \subset \Phi$, $U_{\Phi'} = \{x_k > 0; \ k \in \Phi \setminus \Phi'\}$. The $U_{\Phi'}$, for $\Phi' \subset \Phi$, give a covering of $Q \setminus \{0\}$, and

$$\Gamma(U_{\Phi'}; M_2) \simeq C[X_v; \ v \in \Phi'].$$

Hence $C_2$ is isomorphic to the Čech complex $C((U_{\Phi'})_{\Phi' \subset \Phi}; M_2)$. Now $M_2$ is constructible for the stratification of $Q$ by the strata $Q_{\Phi' = \{x_k = 0; k \in \Phi'; x_k > 0, k \notin \Phi'\}}$, $\Phi' \subset \Phi$. Each open $U_{\Phi'}$ is contractible to a point by a homotopy preserving the closures of strata; hence $H^i(U_{\Phi'}; M_2) = 0$, for $i > 0$. It follows that $\forall i$, $H^i(C_2) \simeq H^i(Q \setminus \{0\}; M_2)$.

For $\Phi_1 \subset \Phi$ we have of course $\otimes_{k \in \Phi_1} C_{\{x_k = 0\}} \simeq C_{\{x_k = 0; k \in \Phi_1\}}$. Hence $M_2$ is a sum of sheaves of the type $C_{\{x_k = 0; k \in \Phi_1\}}$. For each of them, say $N = C_{\{x_k = 0; k \in \Phi_1\}}$, we have $\forall i > 0, \ H^i(Q \setminus \{0\}; N) = 0$ and the map $H^0(Q; N) \rightarrow H^0(Q \setminus \{0\}; N)$ is surjective. By additivity, both assertions are true with $M_2$ instead of $N$. We also have $H^0(Q; M_2) \simeq C[X_v; \ v \in \Phi]$. Hence we obtain [\ref{E2}] and the lemma is proved. \hfill $\Box$

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Université de Grenoble I, Département de Mathématiques, Institut Fourier, UMR 5582 du CNRS, 38402 Saint-Martin d’Hères Cedex, France

E-mail address: Stephane.Guillermou@ujf-grenoble.fr