TESTING ANALYTICITY ON CIRCLES

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Let \( \Omega \) be a domain in complex plane and let \( C_t \) be a continuous one parameter family of Jordan curves such that \( \cup C_t = \bar{\Omega} \). Let \( f \) be a continuous function in \( \bar{\Omega} \) such that the restrictions \( f|_{C_t} \) extend holomorphically inside \( C_t \). When does this imply that \( f \) is holomorphic in \( \Omega \)? This question has been known for a long time and is related to inverse problems for PDE and integral geometry, see [Z,E1].

Globevnik [G] answered the question in the affirmative for rotation invariant families. Agranovsky and Globevnik [AG] resolved the question for families of circles when \( f \) is a rational or real-analytic functions of two real variables. Ehrenpreis [E2] independently found solution for the circles of constant radii with centers on a line and real-analytic functions. In [T] we obtain the affirmative answer for the family of radius one circles \( C_t \) with centers at \( t \in \mathbb{R} \), where \( |t| < 2 + \epsilon, \epsilon > 0 \) and for two more special families. Ehrenpreis [E1] has independently obtained the main result of [T].

In this paper we extend the result of [T] to fairly general families of circles with variable radii. In particular, the conclusion holds for radius one circles with centers at \( t \in \mathbb{R}, |t| < 1 + \epsilon \), improving the result of [T]. However for curves other than the circles the question is largely open.

Despite the one variable nature of the problem, our method involves analysis of several complex variables, in particular the extendibility of CR functions. Surprisingly, the proof is simpler than in our previous paper. The main tool comes form the original proof of the classical H. Lewy [L] extension theorem.

The author thanks Mark Agranovsky, Leon Ehrenpreis, and Josip Globevnik for useful discussions.

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Let \( \{C_t : \alpha \leq t \leq \beta\} \), be a continuous one parameter family of circles in complex plane \( \mathbb{C} \) with centers at \( c(t) \in \mathbb{C} \) and radii \( r(t) > 0 \). Let \( D_t \) denote the disc bounded by \( C_t \). Suppose the following hold.
(a) \( \bar{D}_\alpha \cap \bar{D}_\beta = \emptyset \), that is \( |c(\alpha) - c(\beta)| > r(\alpha) + r(\beta) \).

(b) The functions \( c(t) \) and \( r(t) \) are piecewise \( C^3 \) smooth. The curve \( t \mapsto c(t) \) is injective and regular, that is \( c'(t) \neq 0 \).

(c) No circle \( C_t \) is contained in the closed disc \( \bar{D}_s \) for \( t \neq s \), that is \( |c(t) - c(s)| > |r(t) - r(s)| \).

(d) \( |c'(t)| > |r'(t)| \) whenever defined.

**Theorem.** Let the family \( \{ C_t : \alpha \leq t \leq \beta \} \) satisfy (a)-(d). Let \( \Omega = \bigcup D_t \). Let \( f : \bar{\Omega} \to \mathbb{C} \) be a continuous function. Suppose for every \( \alpha \leq t \leq \beta \) the restriction \( f|_{C_t} \) extends holomorphically to \( D_t \). Then \( f \) is holomorphic in \( \Omega \).

**Remarks.** The condition (a) is crucial, the others being added for simplicity and convenience of the proof. The smoothness in excess of \( C^1 \) is used only to deal with triple intersections of the circles. The condition that the circles can’t lie inside one another is natural because otherwise the values of \( f \) on them are unrelated. We assume the slightly stronger property (c) that they can’t even touch. The condition (d) is the infinitesimal version of (c). In fact (c) implies (d) with possible equality, but for simplicity we assume the strict inequality.

Turning to the proof, we define

\[
\Sigma = \{(z, w) \in \mathbb{C}^2 : w = \bar{z}\}
\]

\[
X_t = \{(z, w) \in \mathbb{C}^2 : (z - c(t))(w - \bar{c}(t)) = r(t)^2, |z - c(t)| \leq r(t)\}.
\]

The complex curve \( X_t \) can be considered (a part of) the complexification of \( C_t \). Indeed, let \( \tilde{C}_t = \{(z, w) \in \mathbb{C}^2 : z \in C_t, w = \bar{z}\} \). Then \( \partial X_t = \tilde{C}_t \) because \( (z, w) \in X_t \) and \( |z - c(t)| = r(t) \) imply \( w - \bar{c}(t) = \frac{r(t)^2}{z - c(t)} = \bar{z} - \bar{c}(t) \), whence \( w = \bar{z} \).

**Lemma 1.** The condition (c) implies \( X_t \cap X_s = \tilde{C}_t \cap \tilde{C}_s \subset \Sigma \) for \( t \neq s \).

**Proof.** By (c) we have the following possibilities.

Case 1: \( \bar{D}_t \cap \bar{D}_s = \emptyset \). Then \( X_t \cap X_s = \emptyset \) because \( (z, w) \in X_t \cap X_s \) implies \( z \in \bar{D}_t \cap \bar{D}_s \).

Case 2: \( C_t \cap C_s \neq \emptyset \). By eliminating \( w \) from the equations of \( X_t \) and \( X_s \), we get a quadratic equation in \( z \). Hence \( X_t \cap X_s \) contains no more points than \( \tilde{C}_t \cap \tilde{C}_s \). The lemma is proved.

Define \( M = \bigcup X_t \). Then by Lemma 1, \( M \setminus \Sigma \) is a piecewise smooth Levi-flat hypersurface in \( \mathbb{C}^2 \). Let \( f_t \) denote the holomorphic extension of \( f \) to \( D_t \). For \( (z, w) \in X_t \) we define
\( F(z, w) = f_t(z) \). Then \( F \) is a continuous CR function on \( M \) because \( F \) is holomorphic on the fibers \( X_t \).

We plan to prove that \( F \) actually is independent of \( w \). That would mean that all the extensions \( f_t(z) \) match at \( z \), and \( f \) is holomorphic.

Let \( \Omega' = \Omega \setminus (\overline{D}_\alpha \cup \overline{D}_\beta) \). For \( z \in \Omega' \) put \( I_z = \{ \alpha \leq t \leq \beta : |z - c(t)| \leq r(t) \} \subset \mathbb{R} \). Since \( z \in \Omega' \), then \( \alpha \) and \( \beta \) are not in \( I_z \). Let \( \mathbb{C} \) be the Riemann sphere \( \mathbb{C} \cup \{ \infty \} \).

Put \( \Gamma_z = \{ w \in \mathbb{C} : (z, w) \in M \} \). If \( \Gamma_z \) is unbounded, then we will consider \( \infty \in \Gamma_z \). It will suffice to consider the case in which \( I_z \) consists of finitely many closed disjoint intervals \( I_z = I_1 \cup \ldots \cup I_k \), where \( k \) depends on \( z \). Then \( \Gamma_z \) is a parametrized curve \( I_z \to \mathbb{C} \), \( t \mapsto w(t) = \bar{c}(t) + \frac{r(t)^2}{z - c(t)} \).

If \( t \) is an end point of one of the intervals \( I_j \), then \( |z - c(t)| = r(t) \) and as we noticed before, \( w(t) = \bar{z} \). By Lemma 1 the mapping \( t \mapsto w(t) \) is injective except that all the end points of the intervals \( I_j \) are mapped to the same point \( \bar{z} \).

Hence the curve \( \Gamma_z \) consists of \( k \) simple closed loops \( \Gamma_z = \Gamma_1 + \ldots + \Gamma_k \) corresponding to the partition \( I_z = I_1 \cup \ldots \cup I_k \).

Let \( C = \{ c(t) : \alpha \leq t \leq \beta \} \). If \( z \in C \), then exactly one loop of the above passes through the point at infinity.

We first explain the idea of the proof in the case when the circles have no triple intersections. Then the curve \( \Gamma_z \) consists of just one loop. By the argument of the classical work of H. Lewy \([L]\), the CR function \( F \) holomorphically extends inside \( \Gamma_z \) for \( z \in \Omega' \setminus C \). When \( z \) crosses the centers line \( C \), the interior and exterior of the loop \( \Gamma_z \) interchange. This implies that \( F(z, w) \) extends to the whole Riemann sphere for fixed \( z \), hence \( F(z, w) \) is independent of \( z \), and \( f \) is holomorphic.

In the general case we examine the behavior of \( \Gamma_z \) as a function of \( z \). Consider the natural map \( Z : (\alpha, \beta) \times \mathbb{R} \to \mathbb{C} \), \( Z(t, \theta) = c(t) + r(t)e^{i\theta} \). By the implicit function theorem, if \( z_0 \) is not a critical value of \( Z \), then for \( z \) close to \( z_0 \), \( \Gamma_z \) varies continuously with \( z \), in particular the number of loops in \( \Gamma_z \) is constant.

Let \( P \) denote the set of all critical values of \( Z \). They are found by solving the equation \( \frac{\partial Z}{\partial t} = \lambda \frac{\partial Z}{\partial \theta} \), \( \lambda \in \mathbb{R} \). This equation means that the velocity of the point of \( C_t \) is tangential to \( C_t \), so we call such point sliding points. Straightforward calculations show that every circle has exactly two sliding points

\[
p(t) = c(t) - \frac{r'(t) \pm i \sqrt{|c'(t)|^2 - r'(t)^2}}{c'(t)}.
\]
Hence \( P \) consists of finitely many \( C^2 \)-smooth curves possibly with singular points, in which \( p'(t) = 0 \). Regular pieces of \( P \) are the osculating curves for the family \( \{C_t\} \), that is \( C_t \) is tangent to \( P \) at \( p(t) \).

Let \( P' \) be the set of regular points of \( P \). Without loss of generality \( P' \) is “simple”, that is if \( p(t) \) and \( p(s) \) are regular points for \( t \neq s \), then \( p(t) \neq p(s) \). Indeed, by (c) the circles \( C_t \) and \( C_s \) cannot touch in interior fashion. Exterior tangency can be eliminated by shrinking the interval \((\alpha, \beta)\) and passing to a subfamily.

We describe the qualitative behavior of \( \Gamma_z \) as \( z \) crosses the critical set \( P \) at \( p(t) \). Let \( p(t) \in P' \). Let \( \rho(t) \) denote the radius of curvature of \( P \) at \( p(t) \). We distinguish between the interior and exterior tangency of \( P \) and \( C_t \) at \( p(t) \) if \( \rho(t) < \infty \). Only the following cases may occur.

1. Interior tangency and \( \rho(t) > r(t) \) or exterior tangency.
2. Interior tangency and \( \rho(t) < r(t) \).
3. Interior tangency and \( \rho(t) = r(t) \).

The last case (3) actually can’t occur because it implies \( c'(t) = 0 \) in violation of (b). In case (1), if \( z \) crosses to the side of \( P \) where \( C_t \) approaches \( p(t) \), then a new small loop of \( \Gamma_z \) is created. In case (2), one loop of \( \Gamma_z \) splits into two loops.

As a tool for proving that \( F(z, w) \) is constant on \( \Gamma_z \), we use the following simple

**Lemma 2.** Let \( G \subset \mathbb{C} \) be a closed piecewise smooth curve and let \( f \in L^\infty(G) \). Suppose \( \int_G f(\zeta)(\zeta - w)^{-2} \, d\zeta = 0 \) for all \( w \in \mathbb{C} \setminus G \). Then \( f \) is constant on every segment of \( G \) with no points of self-intersection of \( G \).

**Proof.** Consider the Cauchy type integral

\[
F(w) = \frac{1}{2\pi i} \oint_G \frac{f(\zeta) \, d\zeta}{\zeta - w}, \quad w \in \mathbb{C} \setminus G.
\]

The hypotheses imply that \( F'(w) \equiv 0 \), whence \( F \) is locally constant. The lemma now follows by the Plemelj-Sokhotsky jump formula.

In view of the last lemma, we put

\[
\Phi(z, w) = \int_{G_z} F(z, \zeta)(\zeta - w)^{-2} \, d\zeta,
\]

where \( G_z \) is a curve that consists of some of the loops of \( \Gamma_z \) and depends continuously on \( z \) in an open set \( U \subset \Omega' \), and \( w \in \mathbb{C} \setminus G_z \). The second power in the denominator is convenient to avoid convergence problems if \( G_z \) passes through the infinity.

**Lemma 3.** \( \Phi(z, w) \) is holomorphic in \( z \in \Omega' \setminus (P \cup C) \), \( w \in \mathbb{C} \setminus G_z \).
Proof. Obviously, $\Phi$ is holomorphic in $w$. To show that $\Phi$ is holomorphic in $z$, we follow the original proof of the H. Lewy [L] extension theorem. Put $\Phi(z) = \Phi(z, w)$, $H(z, \zeta) = F(z, \zeta)(\zeta - w)^{-2}$, then $\Phi(z) = \int_{G_z} H(z, \zeta) d\zeta$. By the Morera theorem, it suffices to show that $\int_\gamma \Phi(z) dz = 0$ for every small loop $\gamma$ in $U$.

Without loss of generality, $G_z$ consists of just one simple loop. Consider the “torus” $T = \{(z, \zeta) \in M : z \in \gamma, \zeta \in G_z\}$. Then $T$ bounds a “solid torus” $S \subset M$ obtained by filling the loop $\gamma$. Then by Stokes’ formula

$$\int_\gamma \Phi(z) dz = \int_T H(z, \zeta) d\zeta \wedge dz = \int_S dH(z, \zeta) \wedge d\zeta \wedge dz = 0$$

because $H$ is a CR function. The lemma is proved.

Let $L_0$ be a straight line separating the circles $C_\alpha$ and $C_\beta$. Let $L$ be a small perturbation of $L_0$ such that

(i) $L \cap (P \setminus P') = \emptyset$;

(ii) $L \cap (P \cup C)$ is finite;

(iii) $L$ has only transverse intersections with $P$ and $C$.

Such a curve $L$ exists because $P$ and $C$ are smooth and the singular set $P \setminus P'$ has length zero.

Consider the set $\mathcal{A}$ of all loops of $\Gamma_z$ for all $z \in L$. For $G_1, G_2 \in \mathcal{A}$, $G_1 \subset \Gamma_{z_1}$, $G_2 \subset \Gamma_{z_2}$, $z_1, z_2 \in L$, we put $G_1 \sim G_2$ if $G_1$ deforms into $G_2$ as $z$ runs from $z_1$ to $z_2$ in $L$. If a single loop $G \in \mathcal{A}$ splits into two loops $G_1$ and $G_2$ as $z \in L$ crosses the critical set $P$, then we also put $G_1 \sim G_2$. We define an equivalence relation $\sim$ on $\mathcal{A}$ using the above two basic equivalences.

Since the centers line $C$ comes from one side of $L$ to the other and all intersections $L \cap C$ are transverse, then $L \cap C$ consists of odd number of points. Since no more than one loop of $\Gamma_z$ passes through the infinity, then there exists an equivalence class $A \subset \mathcal{A}$ that includes odd number of infinite loops.

Define $G_z$ as a curve that consists of all loops of $\Gamma_z$ that are in $A$.

Let $U_0$ be a small neighborhood of $L$. Then $U_0$ is an infinite thin band separated into sub-bands by segments of the singular set $P$ and centers line $C$. We continuously extend $G_z$ from $z \in L$ to $z \in U_0$. We restrict to the points $z$ for which $G_z$ is nontrivial, that is we put $U = \{z \in U_0 : G_z \neq \emptyset, G_z \neq \{z\}\}$.

By Lemma 3 the function $\Phi(z, w)$ defined above is holomorphic with respect to both $z$ and $w$ for $z \in U \setminus (P \cup C)$, $w \in \mathcal{C} \setminus G_z$. Since $G_z$ varies continuously and the integrand
in the formula for $\Phi$ decays as $\zeta^{-2}$ at infinity, then $\Phi$ is continuous on $V = \{(z, w): z \in U, w \in \mathfrak{C} \setminus G_z\}$. Hence $\Phi$ is holomorphic on $V$.

Let $G$ be an oriented piecewise smooth curve in $\mathfrak{C}$. Define $G^+$ and $G^-$ as the sets of all points in $\mathfrak{C}$ of index respectively 1 and 0 with respect to $G$. We call $G$ quasi-simple if $G^+ \cup G^- \cup G = \mathfrak{C}$.

By the nature of the equivalence relation $\sim$, all the curves $G_z$ are quasi-simple (which might not be the case for $\Gamma_z$), so $V = V^+ \cup V^-$, where $V^\pm = \{(z, w): z \in U, w \in G_z^\pm\}$. Also, it follows that the sets $V^\pm$ are connected.

The set $U$ is a thin band with two short edges $K^\pm \subset P$. If $z \in U$ is close to $K^\pm$, then $G_z$ consists of a single small loop contracting into $\bar{z}$ as $z$ approaches $K^\pm$.

Here is the key point of the proof. As $z \in U$ runs from $K^+$ to $K^-$, it crosses the centers line $\mathcal{C}$ odd number of times so $G_z$ crosses through the $\infty$ odd number of times. If say $\infty \in G_z^+$ for $z \in U$ close to $K^+$, then $\infty \in G_z^-$ for $z \in U$ close to $K^-$. For definiteness choose the signs in the notation $K^\pm$ so this is the case.

As $z$ approaches $K^\pm$, the loop $G_z$ contracts into a point, hence $\Phi(z, w) \to 0$ as $z \to z_0 \in K^\pm$. By the uniqueness theorem, $\Phi \equiv 0$ in the connected set $V^\pm$. By Lemma 2 the function $F$ is constant on every loop of $G_z$. This implies that $f$ is holomorphic in $U$ and this property propagates over the whole set $\Omega$. The theorem is now proved.
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