ELIMINATION OF HAMILTON–JACOBI EQUATION IN EXTREME VARIATIONAL PROBLEMS

IGOR ORLOV

Taurida National V. Vernadsky University, Simferopol, Ukraine
E-mail address: old@crimea.edu

Abstract. It is shown that extreme problem for one–dimensional Euler–Lagrange variational functional in $C^1[a;b]$ under the strengthened Legendre condition can be solved without using Hamilton–Jacobi equation. In this case, exactly one of the two possible cases requires a restriction to a length of $[a;b]$, defined only by the form of integrand. The result is extended to the case of compact extremum in $H^1[a;b]$.

Key words and phrases: variational functional, Hamilton–Jacobi equation, Legendre condition, local extremum, compact extremum, Sobolev space.

INTRODUCTION

The classical scheme of the research to a local extremum for a one–dimensional Euler–Lagrange variational functional

$$\Phi(y) = \int_a^b f(x, y, y') dx \mapsto \text{extr} \quad (y \in C^1[a;b])$$

at an extremal point $y$ assumes [1], [2] checking the strengthened Legendre condition $f_{y'y'}(x, y, y') \neq 0$ and the Jacobi condition $U(x) \neq 0 \,(a < x \leq b)$ for the Hamilton–Jacobi equation:

$$-\frac{d}{dx} \left[ f_{y'y'}(x, y, y')U' \right] + \left[ -\frac{d}{dx} \left( f_{yy}(x, y, y') + f_{y'y}(x, y, y') \right) \right] U = 0 \quad (U(a) = 0, \ U'(a) = 1).$$

The second step is the most laborious, it requires to solve a complicated enough equation with a view to receive, really, a very small information about behavior of the solution $U(x)$.

Moreover, the initial conditions $U(a) = 0, U'(a) = 1$, have as a consequence automatical fulfilment of the Jacobi condition near $a$. The question is only — how much length has a suitable interval?

The aim of the present work is to show that the interval satisfying the Jacobi condition can be chosen depending only on the form of the integrand $f$ and not depending on a concrete extremal. More precisely, the main result (Theorem 1.1, 3.1) distinguishes two cases depending on the range of the coefficients in the Hamilton–Jacobi equation. For the first case, an extremum is guaranteed without any restriction to a length of $[a;b]$, for the second one, such a restriction is presented. The result above remains valid under the passage to the case of the research to a compact extremum in Sobolev space $H^1[a;b]$.

The first part of the work deals with elimination of the Hamilton–Jacobi equation in case of zero extremal in $C^1[a;b]$. The second part contains a quadratic estimate of tending $\Phi$ to a minimal value via norm of $y$ in $H^1[a;b]$. The third part determines a general form of $\Phi$ under the conditions of a local minimum and Legendre condition at zero. The fourth part
contains a passage to the case of an arbitrary $C^2$–smooth extremal in $C^1[a; b]$ and the last, fifth part contains a passage to the case of a compact minimum in $H^1[a; b]$.

1. Elimination of Jacobi condition: case of zero extremal

Let’s consider a classical Euler–Lagrange variational functional

$$
\Phi(y) = \int_a^b f(x, y, y')dx \quad (y \in C^1[a; b], y(a) = y(b) = 0, f \in C^2, f_{yz} \in C^1).
$$

(1)

We are going to show that, under fulfilment of Euler–Lagrange variational equation and the strengthened Legendre condition at zero, the functional (1) always attains a strong local extremum at zero. However, in addition, two different possible cases defined by form of the integrand $f$, arise: one of the cases assumes a restriction to a length of $[a; b]$, at the second case any restriction is absent.

So, let’s divide the integrand $f(x, y, z)$ into two terms:

$$
f_1(x, y, z) = f(x, y, z) - f(x, 0, 0) - [f_y(x, 0, 0) \cdot y + f_z(x, 0, 0) \cdot z] - 
\frac{1}{2} \left[ f_{yy}(x, 0, 0) \cdot y^2 + 2f_{yz}(x, 0, 0) \cdot yz + \lambda \cdot f_{zz}(x, 0, 0) \cdot z^2 \right]; \quad (0 < \lambda < 1)
$$

$$
f_2(x, y, z) = f(x, y, z) - f_1(x, y, z) = f(x, 0, 0) + [f_y(x, 0, 0) \cdot y + f_z(x, 0, 0) \cdot z] + 
\frac{1}{2} \left[ f_{yy}(x, 0, 0) \cdot y^2 + 2f_{yz}(x, 0, 0) \cdot yz + \lambda \cdot f_{zz}(x, 0, 0) \cdot z^2 \right].
$$

Let’s set, respectively,

$$
\Phi_i(y) = \int_a^b f_i(x, y, y')dx \quad (i = 1, 2); \quad \Phi(y) = \Phi_1(y) + \Phi_2(y).
$$

1) Let’s investigate $\Phi_1$ for a local extremum (minimum, for definiteness) at zero with the help of Euler–Lagrange, Legendre and Jacobi conditions.

(i) The Euler–Lagrange equation. Because

$$
(f_{1,y}(x, y, z) = f_y(x, y, z) - f_y(x, 0, 0) - f_{yy}(x, 0, 0) \cdot y - f_{yz}(x, 0, 0) \cdot z) \Rightarrow 
\Rightarrow (f_{1,y}(x, 0, 0) = 0);
$$

$$
(f_{1,z}(x, y, z) = f_z(x, y, z) - f_z(x, 0, 0) - f_{yz}(x, 0, 0) \cdot y - \lambda \cdot f_{zz}(x, 0, 0) \cdot z) \Rightarrow 
\Rightarrow (f_{1,z}(x, 0, 0) = 0);
$$

then the Euler–Lagrange equation for $\Phi_1$ at zero

$$
f_{1,y}(x, 0, 0) - \frac{d}{dx} [f_{1,z}(x, 0, 0)] = 0
$$

holds automatically, i.e. $y_0(x) \equiv 0$ is an extremal of the functional $\Phi_1$.

(ii) The strengthened Legendre condition. Because

$$
(f_{1,z}(x, y, z) = f_{zz}(x, y, z) - \lambda \cdot f_{zz}(x, 0, 0)) \Rightarrow (f_{1,z}(x, 0, 0) = (1 - \lambda) \cdot f_{zz}(x, 0, 0),
$$

then, under the additional requirement

$$
p(x) := f_{zz}(x, 0, 0) > 0, \quad (a \leq x \leq b)
$$

(2)

the strengthened Legendre condition for a strong minimum at zero holds.

(iii) The Hamilton–Jacobi equation and the Jacobi condition. Because

$$
(f_{1,yz}(x, y, z) = f_{yz}(x, y, z) - f_{yz}(x, 0, 0)) \Rightarrow (f_{1,yz}(x, 0, 0) = 0);
$$
Thus, under the condition (2), \( \Phi \) holds, i.e. the strengthened Jacobi condition at zero for a strong minimum of and consider at first the case of \( q \). From here, denoting by \( \Phi \) then the Hamilton–Jacobi equation for \( \Phi_1 \) at zero takes form of

\[
-\frac{d}{dx} \left[ (1 - \lambda) \cdot f_{x2}(x, 0, 0)U' \right] + \left[ -\frac{d}{dx} \left( f_{1,y2}(x, 0, 0) \right) + f_{1,y2}(x, 0, 0) \right] U = 0
\]

Hence, in view of condition (2), the required result

\[
\left( U(x) = p(a) \cdot \int_{a}^{x} \frac{dt}{p(t)} \right) \Rightarrow \left( U(x) \neq 0 \text{ for } a < x \leq b \right)
\]

holds, i.e. the strengthened Jacobi condition at zero for a strong minimum of \( \Phi_1 \) takes place. Thus, under the condition (2), \( \Phi_1 \) attains a strong local minimum at zero.

2) Let’s investigate now \( \Phi_2 \) for a local extremum at zero immediately. Note at first that \( \Phi_2(0) = \Phi(0) \).

(i) Suppose that the Euler–Lagrange equation for \( \Phi \) at zero

\[
(f_y(x, 0, 0) - f_{xz}(x, 0, 0) = 0 \quad (a \leq x \leq b)
\]

holds. Then integrating by parts gives us

\[
\Phi_2(y) = \int_{a}^{b} f(x, 0, 0)dx + \int_{a}^{b} \left[ f_y(x, 0, 0) \cdot y + f_z(x, 0, 0) \cdot y' \right] dx +
\]

\[
+ \int_{a}^{b} \left[ \frac{1}{2} f_{y2}(x, 0, 0) \cdot y^2 + f_{yz}(x, 0, 0) \cdot y'y' \right] dx + \frac{\lambda}{2} \cdot \int_{a}^{b} f_{z2}(x, 0, 0) \cdot y'^2 dx =
\]

\[
= \Phi_2(0) + \left[ \left( f_y - f_{xz} \right)(x, 0, 0)dx + f_z(x, 0, 0) \cdot y \right]_{a}^{b} +
\]

\[
+ \left[ \frac{1}{2} \int_{a}^{b} f_{y2} - f_{xy2} (x, 0, 0) \cdot y'^2 dx + \frac{1}{2} f_{yz}(x, 0, 0) \cdot y'^2 \right]_{a}^{b} + \lambda \int_{a}^{b} p(x) \cdot y'^2 dx.
\]

From here, denoting by

\[
q(x) := (f_{y2} - f_{xy2})(x, 0, 0),
\]

it follows

\[
\Phi_2(y) = \Phi_2(0) + \frac{1}{2} \int_{a}^{b} \left[ \lambda \cdot p(x) \cdot y'^2 + q(x) \cdot y'^2 \right] dx.
\]

(ii) Denote by

\[
p := \min_{a \leq x \leq b} p(x) > 0, \quad q := \min_{a \leq x \leq b} q(x),
\]

and consider at first the case of \( q \geq 0 \). Then

\[
\lambda p(x)y'^2 + q(x)y'^2 \geq \lambda p \cdot y'^2 + q \cdot y'^2 > 0 \quad \text{as} \quad y' \neq 0,
\]

whence, in view of (4), the inequality

\[
\Phi_2(y) > \Phi_2(0) \quad \text{as} \quad y(x) \neq 0
\]
follows. Thus, in this case $\Phi_2$ attains a strong absolute minimum at zero. Hence, in view of one was proved in i.1), $\Phi$ attains a strong local minimum at zero (without any restriction to a length of $[a; b]$).

(iii) Let’s consider now the case of $q < 0$. Then, using Friederichs inequality (see, e.g., [3], Ch. 18), it follows

$$\Phi_2(y) - \Phi_2(0) = \frac{1}{2} \int_a^b \left[ \lambda \cdot p(x) \cdot y'^2 + q(x) \cdot y^2 \right] dx \geq \frac{1}{2} \int_a^b \left[ \lambda \cdot p \cdot y'^2 - |q| \cdot y^2 \right] dx \geq \frac{1}{2} \int_a^b \left[ \lambda \cdot p \cdot y'^2 - \frac{16(b-a)^2}{\pi^2} |q| \cdot y^2 \right] dx = \frac{1}{2} \left( \lambda \cdot p - \frac{16(b-a)^2}{\pi^2} |q| \right) \cdot \int_a^b y'^2 dx. \quad (6)$$

Let’s require that the coefficient in front of the last integral in (6) will be strictly positive:

$$\left( \lambda \cdot p - \frac{16(b-a)^2}{\pi^2} |q| > 0 \right) \Leftrightarrow \left( b - a < \frac{\pi}{4} \sqrt{\frac{\lambda p}{|q|}} \right). \quad (7)$$

It follows from (6) and (7) that $\Phi_2(y) > \Phi_2(0)$ as $y \neq 0$, i.e. $\Phi_2$ attains a strong absolute minimum at zero and hence, by virtue of one was proved in i.1), $\Phi$ attains a strong local minimum at zero under the restriction (7) to a length of $[a; b]$.

Finally, passing to the limits in (7) as $\lambda \to 1 - 0$, the last statement can be extended to the case of the estimate of a length of $[a; b]$ not depending on $\lambda$:

$$b - a < \frac{\pi}{4} \sqrt{\frac{p}{|q|}}.$$

So, it is proved the following

**Theorem 1.1.** Let the variational functional (1) satisfies at zero the Euler–Lagrange equation (3) under the conditions $y(a) = y(b) = 0$. Then, under the notation of (5),

1) for $p > 0, q \geq 0$, $\Phi(y)$ attains a strong local minimum at zero (without any restriction to a length of $[a; b]$);

2) for $p > 0, q < 0$, under the restriction to a length of $[a; b]$:

$$b - a < \frac{\pi}{4} \sqrt{\frac{p}{|q|}}, \quad (8)$$

$\Phi(y)$ attains a strong local minimum at zero as well.

2. **Quadratic estimation from below of tending $\Phi$ to minimum at zero**

It’s easy to see that the estimate (8) at Theorem 1.1 is not optimal. For example, a generalized harmonic oscillator

$$\Phi(y) = \int_0^T (py'^2 - qy^2) dx \quad (p > 0, q > 0)$$

on zero extremal reduces to the Hamilton–Jacobi equation

$$pu'' + qU = 0 \quad (U(0) = 0, \ U'(0) = 1)$$

having the solution

$$U(x) = \sqrt{\frac{p}{q}} \sin \sqrt{\frac{q}{p}} x,$$
satisfying Jacobi condition $U(x) \neq 0$ as $0 < x < T$ for $T < \frac{\pi}{\sqrt{\frac{p}{q}}}$. At the same time, the estimate \( \square \) for given case leads to inequality $T < \frac{\pi}{4} \sqrt{\frac{p}{q}}$. However, as it’s easily can be seen, an advantage of the estimate \( \square \) consists of possibility to get a useful quadratic estimate from below for tending $\Phi(y)$ to the minimal value by means of norm of $y$ in the Sobolev space $H^1[a;b]$.

1) First, let’s consider a case of $p > 0, q > 0$. The equality (4) implies

$$\Phi(y) - \Phi(0) \geq \frac{1}{2} \min(p,q) \cdot \int_a^b (y'^2 + y^2)dx = \frac{1}{2} \min(p,q) \cdot \|y\|_{H^1[a;b]}^2.$$ 

Since $\Phi(y) - \Phi(0) \geq \Phi_2(y) - \Phi_2(0)$ in a small enough zero neighborhood, then given a zero neighborhood the inequality

$$\Phi(y) - \Phi(0) \geq \frac{1}{2} \min(p,q) \cdot \|y\|_{H^1[a;b]}^2$$ 

holds true.

2) Let’s pass to the case of $p > 0, q < 0$. The inequality (6) leads to the estimate

$$\Phi(y) - \Phi(0) \geq \frac{1}{2} \left[p - \frac{16(b-a)^2}{\pi^2} |q| \right] \cdot \int_a^b y'^2dx.$$ 

Since the Friederichs inequality implies

$$\int_a^b y'^2dx \geq \frac{\pi^2}{\pi^2 + 16(b-a)^2} \cdot \|y\|_{H^1[a;b]}^2,$$ 

then by combining of the last two inequalities for a small enough neighborhood of zero, under the conditions of inequality \( \square \), we get

$$\Phi(y) - \Phi(0) \geq \frac{\pi^2 p - 16(b-a)^2 |q|}{2(\pi^2 + 16(b-a)^2)} \cdot \|y\|_{H^1[a;b]}^2.$$ 

3) Note that the estimate \( \square \) can be applied as well in the case of $p > 0, q \geq 0$, whence the inequality

$$\Phi(y) - \Phi(0) \geq \frac{\pi^2 p}{2(\pi^2 + 16(b-a)^2)} \cdot \|y\|_{H^1[a;b]}^2$$ 

follows. So, it is proved the following

**Theorem 2.1.** Under the conditions and notation of Theorem 1.1, the following statements are valid:

1) in the case of $p > 0, q > 0$, in small enough zero neighborhood in $C^1[a;b]$ the estimate

$$\Phi(y) - \Phi(0) \geq \frac{1}{2} \min(p,q) \cdot \|y\|_{H^1[a;b]}^2$$ 

holds;

2) in the case of $p > 0, q \geq 0$, in small enough zero neighborhood in $C^1[a;b]$ the estimate

$$\Phi(y) - \Phi(0) \geq \frac{\pi^2 p}{2(\pi^2 + 16(b-a)^2)} \cdot \|y\|_{H^1[a;b]}^2$$ 

holds;
3) in the case of \( p > 0, \ q < 0, \) in small enough zero neighborhood in \( C^1[a; b], \) under the condition of estimate (8), the estimate

\[
\Phi(y) - \Phi(0) \geq \frac{\pi^2 p - 16(b - a)^2|q|}{2(\pi^2 + 16(b - a)^2)} \cdot \|y\|^2_{H^1[a; b]}
\]

holds.

3. APPLICATION: INVERSE EXTREME PROBLEM FOR VARIATIONAL FUNCTIONAL

Let’s set up a problem: to find a general form of the variational functional (1) possessing local minimum at zero under the strengthened Legendre condition.

1) Let’s shall find an integrand \( f \) of the functional (1) in the form of

\[
f(x, y, z) = P(x, y) + Q(x, y) \cdot z + \frac{1}{2}R(x, y, z) \cdot z^2.
\]

Then

\[
P(x, y) = f(x, y, 0), \quad Q(x, y) = f_z(x, y, 0), \quad R(x, y, 0) = f_{zz}(x, y, 0).
\]

Under this notation, the Euler–Lagrange equation on zero extremal (3) takes form of

\[
(Q_x - P_y)(x, 0) = 0 \quad (a \leq x \leq b);
\]

the strengthened Legendre condition on zero extremal (2) takes form of

\[
R(x, 0, 0) =: p(x) > 0 \quad (a \leq x \leq b).
\]

2) Let’s choose an arbitrary \( P(x, y) \in C^2. \) Then a general form of \( Q \) follows from (11):

\[
\left( Q_x(x, 0) = P_y(x, 0) \right) \Rightarrow \left( Q(x, 0) = C + \int_a^x P_y(t, 0)dt \right) \Rightarrow
\]

\[
\Rightarrow \left( Q(x, y) = C + \int_a^x P_y(t, 0)dt + \bar{Q}(x, y), \quad \text{where} \quad \bar{Q}(x, 0) = 0 \right) \Rightarrow
\]

\[
\Rightarrow \left( Q(x, y) = C + \int_a^x P_y(t, 0)dt + \left[ q(x, y) - q(x, 0) \right] \right),
\]

here \( C \in \mathbb{R} \) and \( q(x, y) \in C^2 \) can be chosen arbitrarily.

3) A general form of \( R \) easily follows from the condition (12):

\[
\left( R(x, 0, 0) = p(x) > 0 \right) \Rightarrow \left( R(x, y, z) = p(x) + [\rho(x, y, z) - \rho(x, 0, 0)] \right),
\]

where \( p(x) > 0, \ p \in C^2; \ \rho(x, y, z) \in C^2 \) can be chosen arbitrarily.

4) A general form of the integrand \( f \) follows now from (10), (13) and (14):

\[
f(x, y, z) = P(x, y) + \left( C + \int_a^x P_y(t, 0)dt + \left[ q(x, y) - q(x, 0) \right] \right) \cdot z +
\]

\[
+ \frac{1}{2} \left( p(x) + [\rho(x, y, z) - \rho(x, 0, 0)] \right) \cdot z^2,
\]

where \( C \in \mathbb{R}; \ q, p \in C^2 \ (p > 0) \) can be chosen arbitrarily. So, it is proved the following

**Theorem 3.1.** Let, under the conditions of Theorem 1.1, the functional (1) attains a local minimum at zero under the strengthened Legendre condition. Then the integrand \( f \) takes form of (15).
Remark 3.2. As it follows from Theorem 3.1, a general form of the variational functional (1) taking a local minimum at zero under the strengthened Legendre condition is

\[ \Phi(y) = \int_a^b \left( P(x, y) + \int_a^x P_y(t, 0)dt + q(x, y) - q(x, 0) \right) \cdot y' + \frac{1}{2} \left[ p(x) + \rho(x, y, y') - \rho(x, 0, 0) \right] \cdot y'^2 \]  \hspace{1cm} (16)

where \( P, q, p > 0, \rho \) are the arbitrary functions from \( C^2 \).

Thus, under the strengthened Legendre condition, the inverse extreme variational problem at zero is solved: all the functionals of type (1) taking a local minimum at zero are described.

4. CASE OF ARBITRARY \( C^2 \)--SMOOTH EXTREMAL IN \( C^1[a; b] \)

Let’s fix an arbitrary \( C^2 \)--smooth function \( y_0(x), a \leq x \leq b \), and consider a question on elimination of Jacobi condition for the local minimum of the variational functional (1) at the point \( y_0(\cdot) \) under the boundary conditions \( y(a) = y_0(a), y(b) = y_0(b) \).

To pass to the considered above (1.1) case of zero extremal, it suffices to consider an auxiliary variational functional:

\[ \tilde{\Phi}(y) = \Phi(y + y_0) = \int_a^b f(x, y + y_0(x), y' + y'_0(x))dx =: \int_a^b \tilde{f}(x, y, y')dx \]

\( (y(a) = y(b) = 0) \).

In this connection the condition \( y_0(\cdot) \in C^2 \) guarantees fulfilment of the condition from (1) for the auxiliary integrand \( \tilde{f} \) and permits to apply Theorem 1.1 to \( \tilde{\Phi} \). A not complicated calculation shows that it is valid the following

Theorem 4.1. Let variational functional (1) satisfies at a point \( y_0(\cdot) \in C^2[a; b] \) Euler–Lagrange equation

\[ f_y(x, y_0, y'_0) - \frac{d}{dx} \left[ f_z(x, y_0, y'_0) \right] = 0. \]  \hspace{1cm} (17)

Denote by

\[ p := \min_{a \leq x \leq b} f_z^2(x, y_0(x), y'_0(x)); \]

\[ q := \min_{a \leq x \leq b} \left[ f_y^2(x, y_0(x), y'_0(x)) - \frac{d}{dx} \left( f_{yz}(x, y_0(x), y'_0(x)) \right) \right]. \]

Then, under the boundary conditions \( y(a) = y_0(a), y(b) = y_0(b) \),

1) for \( p > 0, q \geq 0, \Phi(y) \) attains a strong local minimum at \( y_0(\cdot) \) (without any restriction to a length of \([a; b] \));

2) for \( p > 0, q < 0 \), and under the restriction

\[ b - a < \frac{\pi}{4} \sqrt{\frac{p}{|q|}}, \]  \hspace{1cm} (18)

to a length of \([a; b] \), \( \Phi(y) \) attains a strong local minimum at \( y_0(\cdot) \) as well.

Analogously, applying Theorem 2.1 to \( \tilde{\Phi} \) leads a general quadratic estimate for tending \( \Phi \) to a local minimum at \( y_0 \).

Theorem 4.2. Under the conditions and notation of Theorem 4.1:
1) for $p > 0$, $q > 0$, in some neighborhood of $y_0(\cdot)$ in $C^1[a; b]$ the estimate

$$\Phi(y) - \Phi(y_0) \geq \frac{1}{2} \min(p, q) \cdot \|y\|_{H^1[a, b]}^2$$

holds;

2) for $p > 0$, $q \geq 0$, in some neighborhood of $y_0(\cdot)$ in $C^1[a; b]$ the estimate

$$\Phi(y) - \Phi(y_0) \geq \frac{\pi^2 p}{2(\pi^2 + 16(b - a)^2)} \cdot \|y\|_{H^1[a, b]}^2$$

holds;

3) for $p > 0$, $q < 0$, under the restriction (18) to a length of $[a; b]$, in some neighborhood of $y_0(\cdot)$ in $C^1[a; b]$ the estimate

$$\Phi(y) - \Phi(y_0) \geq \frac{\pi^2 p - 16(b - a)^2|q|}{2(\pi^2 + 16(b - a)^2)} \cdot \|y\|_{H^1[a, b]}^2$$

holds.

At last, applying Theorem 3.1 to the auxiliary integrand $\tilde{f}$ leads to solution of the inverse extreme problem for $\Phi$ at an arbitrary point $y_0(\cdot) \in C^2[a; b]$.

**Theorem 4.3.** Let, under the conditions of Theorem 4.1, the variational functional (11) attains a local minimum at a point $y_0(\cdot) \in C^2[a; b]$ under the boundary conditions $y(a) = y_0(a)$, $y(b) = y_0(b)$ and the strengthened Legendre condition. Then the integrand $f$ takes form of

$$f(x, y, z) = P(x, y - y_0(x)) + \left( C + \int_a^x P_y(t, -y_0(t))dt + [q(x, y - y_0(x)) - q(x, -y_0(x))] \right) \cdot (z - y_0'(x)) + \frac{1}{2} \left( p(x) + [\rho(x, y - y_0(x), z - y_0'(x)) - \rho(x, -y_0(x), -y_0'(x))] \right) \cdot (z - y_0(x))^2,$$

where $C \in \mathbb{R}$; $P$, $q$, $p > 0$, $\rho \in C^2$ can be chosen arbitrarily.

From here a formula of the general form of the functional (11) taking a local minimum in $C^1[a; b]$ at a point $y_0(\cdot) \in C^2[a; b]$ under the strengthened Legendre condition:

$$\Phi(y) = \int_a^b \left( P(x, y - y_0(x)) + \int_a^x P_y(t, -y_0(t))dt + q(x, y - y_0(x)) - q(x, -y_0(x)) \right) \cdot (y' - y_0'(x)) + \frac{1}{2} \left[ p(x) + \rho(x, y - y_0(x), y' - y_0'(x)) - \rho(x, -y_0(x), -y_0'(x)) \right] \cdot (y' - y_0'(x))^2 dx,$$

where $C \in \mathbb{R}$; $P$, $q$, $p > 0$, $\rho \in C^2$ can be chosen arbitrarily, arises.

Thus, under the strengthened Legendre condition, the inverse extreme variational problem at an arbitrary point $y_0(\cdot) \in C^2$ is solved: the all functionals of the type (11), attaining a local minimum at a point $y_0(\cdot)$, are described.

**5. Case of compact extremum in $H^1[a; b]$**

In the Hilbert–Sobolev space $W^{1,2}[a; b] = H^1[a; b]$ equipped with the norm

$$\|y\|_{H^1[a,b]}^2 = \int_a^b (y^2 + y'^2)dx,$$

the estimate (19) is solved: the all functionals of the type (1), attaining a local minimum at a point $y_0(\cdot)$, are described.
as it is well known, by virtue of I.V. Skrypnik theorem ([4], Ch.11) the nonabsolute local extrema of the variational functionals practically absent. Note that in the present work the norm (24) was appeared above (Theorem 2.1) by natural way even for extreme problems in $C^1[a; b]$. In the our works [5, 7] and in the works by E.V. Bozhonok [8–10] a general notion of compact extremum (or $K$-extremum) of a functional was studied (see, also, [11]). It have been shown there that the classical, both necessary and sufficient conditions of a local extremum of variational functional in $C^1[a; b]$ can be extended to the case of $K$-extremum in $H^1[a; b]$. In this case, $K$-extrema inherit the important properties of the local extrema and can be considered as an analog of the ones in the case of variational functionals in $H^1[a; b]$. Let’s bring a relevant information.

**Definition 5.1.** Let a real functional $\Phi : H \to \mathbb{R}$ be defined in a Hilbert space $H$. Say that $\Phi$ has a compact minimum (or K-extremum) at a point $y_0 \in H$ if, for each absolutely convex (a.c.) compact set $C \subset H$, the restriction of $f$ to the subspace $(y_0 + \text{span } C)$ has a local minimum at $y_0$ respective to Banach norm $\| \cdot \|_C$ in span $C$ generated by $C$. In other words, for each a.c. compactum $C \subset H$ there exists such $\varepsilon = \varepsilon(C) > 0$ that $\varphi(y) \geq \varphi(y_0)$ as $y - y_0 \in \varepsilon \cdot C$.

The well posedness and the validity for the case of $K$ extremum of the variational functional (11) of the classical extreme conditions in $C^1$ (Euler–Lagrange equation, Legendre condition, Jacobi condition) require, as it was shown in [7], belonging coefficient $R(x, y, z)$ in the quadratic representation (10) of the integrand $f$:

$$f(x, y, z) = P(x, y) + Q(x, y) \cdot z + \frac{1}{2} R(x, y, z) \cdot z^2$$

to an appropriate dominated mixed smoothness space $C_{xy}^2$ (see [12, 13]). Namely, for the arbitrary compacta $C_x, C_y \subset \mathbb{R}$ the following property holds:

$$(x \in C_x, y \in C_y, -\infty < z < +\infty) \Rightarrow (R(x, y, z) \text{ is uniformly continuous and bounded, together with its first and second partial derivatives}).$$

Under the conditions above, the Euler–Lagrange equation, Legendre condition, strengthened Legendre condition and Jacobi condition for the Hamilton–Jacobi equation are extended to the case of $K$-extremum in an arbitrary $W^{2, 2}$-smooth point $y_0(\cdot) \in H^1[a; b]$. It allows to extend the results of i.4 to the case of $K$–minimum in $H^1[a; b]$. Let’s bring the corresponding formulations.

**Theorem 5.2.** Let the variational functional (11) at a $W^{2, 2}$-smooth point $y_0(\cdot) \in H^1[a; b]$ satisfies Euler–Lagrange equation (17), in addition $R(x, y, z) \in C_{xy}^2$. Then, under the conditions and notation of Theorem 2.1:

1) for $p > 0, q \geq 0$, $\Phi(y)$ attains a strong $K$-minimum at $y_0(\cdot)$ (without any restriction to a length of $[a; b]$);

2) for $p > 0, q < 0$, and under the restriction (18) to a length of $[a; b]$, $\Phi(y)$ attains a strong $K$-minimum at $y_0(\cdot)$ as well.

**Theorem 5.3.** Under the conditions and notation of Theorem 5.2:

1) for $p > 0, q > 0$, for each a.c. compactum $C \subset H^1[a; b]$ there exists such $\varepsilon = \varepsilon(C) > 0$ that inclusion $y - y_0 \in \varepsilon \cdot C$ implies estimate (19);

2) for $p > 0, q \geq 0$, for each a.c. compactum $C \subset H^1[a; b]$ there exists such $\varepsilon = \varepsilon(C) > 0$ that inclusion $y - y_0 \in \varepsilon \cdot C$ implies estimate (20);
3) for \( p > 0, q < 0 \), under the restriction (18) to a length of \([a; b]\), for each a.c. compactum \( C \subset H^1[a; b] \) there exists such \( \varepsilon = \varepsilon(C) > 0 \) that inclusion \( y - y_0 \in \varepsilon \cdot C \) implies estimate (21).

**Theorem 5.4.** Let, under the conditions and notation of Theorem 5.2, the variational functional (1) attains a K–minimum at a \( W^{2,2} \)–smooth point \( y_0(\cdot) \) from \( H^1[a; b] \) under the boundary conditions \( y(a) = y_0(a), y(b) = y_0(b) \) and under the strengthened Legendre condition. Then the integrand \( f \) takes form of (22), where \( C \in \mathbb{R}; P, q, p > 0 \) from \( C^2 \) and \( \rho \) from \( C^2_{xy} \) can be chosen arbitrarily.

From here the formula (23) of the general form of the functional (1) having a K–minimum at a \( W^{2,2} \)–smooth point \( y_0(\cdot) \) from \( H^1[a; b] \) under the strengthened Legendre condition, follows.

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