THE MASS GAP AND GLUON CONFINEMENT

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In our previous publications [1, 2, 3] it has been proven that the general iteration solution of the Schwinger-Dyson equation for the full gluon propagator (i.e., when the skeleton loop integrals, contributing into the gluon self-energy, have to be iterated, which means that no any truncations/approximations have been made) can be algebraically (i.e., exactly) decomposed as the sum of the two principally different terms. The first term is the Laurent expansion in integer powers of severe (i.e., more singular than $1/q^2$) infrared singularities accompanied by the corresponding powers of the mass gap and multiplied by the corresponding residues. The standard second term is always as much singular as $1/q^2$ and otherwise remaining undetermined. Here it is explicitly shown that the infrared renormalization of the mass gap only is needed to render theory free of all severe infrared singularities in the gluon sector. Moreover, this leads to the gluon confinement criterion in a gauge-invariant way. As a result of the infrared renormalization of the mass gap in the initial Laurent expansion, that is dimensionally regularized, the simplest severe infrared singularity $(q^2)^{-2}$ survives only. It is multiplied by the mass gap squared, which is the scale responsible for the large scale structure of the true QCD vacuum. The $\delta$-type regularization of the simplest severe infrared singularity (and its generalization for the multi-loop skeleton integrals) is provided by the dimensional regularization method correctly implemented into the theory of distributions. This makes it possible to formulate exactly and explicitly the full gluon propagator (up to its unimportant perturbative part).

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I. INTRODUCTION

The propagation of gluons is one of the main dynamical effects in the true QCD vacuum. The gluon Green's function is (Euclidean signature here and everywhere below)

$$D_{\mu\nu}(q) = i \{ T_{\mu\nu}(q)d(q^2, \xi) + \xi L_{\mu\nu}(q) \} \frac{1}{q^2}, \tag{1.1}$$

where $\xi$ is the gauge fixing parameter ($\xi = 0$ - Landau gauge and $\xi = 1$ - Feynman gauge) and $T_{\mu\nu}(q) = \delta_{\mu\nu} - q_{\mu}q_{\nu}/q^2 = \delta_{\mu\nu} - L_{\mu\nu}(q)$. Evidently, $T_{\mu\nu}(q)$ is the transverse (“physical”) component of the full gluon propagator, while $L_{\mu\nu}(q)$ is its longitudinal (unphysical) one. The free gluon propagator is obtained by setting simply the full gluon form factor $d(q^2, \xi) = 1$ in Eq. (1.1), i.e.,

$$D^0_{\mu\nu}(q) = i \{ T_{\mu\nu}(q) + \xi L_{\mu\nu}(q) \} \frac{1}{q^2}. \tag{1.2}$$

The dynamical equation of motion for the full gluon propagator (1.1) is known as the gluon Schwinger-Dyson (SD) one (see below Sect. 3), and its solutions reflect the quantum-dynamical structure of the true QCD ground state.

In our previous works [1, 2, 3] we have investigated the general iteration solution of the above-mentioned gluon SD equation for the full gluon propagator (1.1). No any truncations/approximations have been made since the corresponding skeleton loop integrals, contributing into the gluon self-energy, have been iterated. It has been exactly proven that the full gluon propagator is to be algebraically decomposed into the two principally different terms as follows:

$$D_{\mu\nu}(q) = D^{LP}_{\mu\nu}(q) + O_{\mu\nu}(1/q^2), \tag{1.3}$$
where

$$D_{\mu\nu}^{\text{INP}}(q, \Delta^2) = iT_{\mu\nu}(q)\frac{\Delta^2}{(q^2)^2}\sum_{k=0}^{\infty}(\Delta^2/q^2)^k \phi_k(\lambda, \nu, \xi, g^2), \quad (1.4)$$

and $O_{\mu\nu}(1/q^2)$ denotes the terms which are the terms of the order $1/q^2$ at small $q^2$ (see also section IV below). Here the superscript "INP" stands for the intrinsically nonperturbative (NP) part of the full gluon propagator. It is nothing but the Laurent expansion in integer powers of severe (i.e., more singular than $1/q^2$) infrared (IR) singularities accompanied by the corresponding powers of the mass gap $\Delta^2$ and multiplied by the corresponding residues (which are dimensionless, of course). They are the sum of all iterations, namely

$$\phi_k(\lambda, \nu, \xi, g^2) = \sum_{m=0}^{\infty} \phi_{k,m}(\lambda, \nu, \xi, g^2), \quad (1.5)$$

and this clearly shows that an infinite number of iterations (each iteration) of the skeleton loop integrals invokes each severe IR singularity and hence the mass gap in the full gluon propagator. Evidently, the mass gap $\Delta^2$ determines the deviation of the full gluon propagator from the free one in the deep IR limit ($q^2 \to 0$). It is worth emphasizing that it has not been introduced by hand. It is hidden in the skeleton loop contributions into the gluon self-energy. It explicitly shows up in the terms dominating the IR structure of the full gluon propagator (the first term in Eq. (1.3)). The appropriate regularization procedure have been applied to make the existence of the mass gap perfectly clear [1]. Let us also recall that here $\lambda$ and $\nu$ are the dimensionless UV cut-off and the renormalization point, respectively, while $\xi$ and $g^2$ are the gauge fixing parameter and the dimensionless coupling constant squared, respectively. As emphasized in Ref. [1] is all that matters within our approach is the dependence of the residues on their arguments and not their concrete values (see below as well). The perturbative (PT) part of the full gluon propagator $O_{\mu\nu}(1/q^2)$ remains undetermined, which, however, is not important as well within our approach (for more detail discussion of these results see our previous papers [1, 2, 3]).

The main purpose of this paper is to complete the previous work by investigating the IR renormalization properties of the theory, in particular of the mass gap. We will show that the gluon confinement criterion is the gauge-invariant IR renormalization of the mass gap only effect within our approach. Moreover, we will establish uniquely and exactly the INP part of the full gluon propagator by correctly implementing the dimensional regularization (DR) method [4] into the theory of distributions [5].

II. IR RENORMALIZATION OF THE GLUON SD EQUATION

The power-type severe (or equivalently NP) IR singularities represent a rather broad and important class of functions with algebraic singularities. They regularization should be done within the theory of distributions [5], complemented by the DR method [4]. The crucial observation is that the regularization of these singularities does not depend on their powers [5], namely

$$(q^2)^{2-k} = \frac{1}{\epsilon} \left[ a(k) [\delta^4(q)]^{(k)} + O(\epsilon) \right], \quad \epsilon \to 0^+, \quad (2.1)$$

where $a(k)$ is a finite constant depending only on $k$ and $[\delta^4(q)]^{(k)}$ represents the $k$th derivative of the $\delta$ function. Here $\epsilon$ is the IR regularization parameter, introduced within the DR method [4], and which should go to zero at the end of the computations. We point out that after introducing this expansion everywhere one can fix the number of dimensions to four without any further problems. The important observation is that each NP IR singularity scales as $1/\epsilon$ as $\epsilon$ goes to zero, and the difference between them appears only in the residues. Just this plays a crucial role in the IR renormalization of the theory within our approach (see below). This regularization expansion takes place only in four-dimensional QCD with Euclidean signature. In other dimensions and signature it is more complicated [5, 6].

In the presence of such severe IR singularities all the quantities which appear in the gluon SD equation should depend in principle on $\epsilon$. Thus the general IR multiplicative renormalization (IRMR) program is needed in order to express all the above-mentioned quantities in terms of their IR renormalized versions. Fortunately, the gluon SD equation (see below) does not contain unknown scattering amplitudes, which usually are determined by an infinite series of the multi-loop skeleton diagrams. It is a closed system in the sense that there is a dependence only on the pure gluon vertices, quark- and ghost-gluon vertices and on the corresponding propagators [5, 6, 7, 8]. Its IR renormalization
can be carried out on general ground. Symbolically (however, this is sufficient to perform the IRMR program) it can be written down as follows:

\[
D(q) = D^0(q) - D^0(q)T_2(q)D(q) - D^0(q)T_{gh}(q)D(q) + D^0(q)\frac{1}{2}T_1D(q) + D^0(q)\frac{1}{2}T_2D(q) + D^0(q)\frac{1}{6}T_3D(q),
\]

where all the skeleton loop integrals are shown explicitly below

\[
T_q(q) = -g^2\int \frac{id^4p}{(2\pi)^4}Tr[\gamma_\mu S(p - q)\Gamma_\mu(p - q, q)S(p)],
\]

\[
T_{gh}(q) = g^2\int \frac{id^4k}{(2\pi)^4}\kappa_\nu G(k)G(k - q)G_\mu(k - q, q),
\]

\[
T_t = g^2\int \frac{id^4q_1}{(2\pi)^4}T_4^0D(q_1),
\]

\[
T_1(q) = g^2\int \frac{id^4q_1}{(2\pi)^4}T_4^0(q, -q, q_1 - q)T_3(-q, q, q - q_1)D(q_1)D(q - q_1),
\]

\[
T_2(q) = g^4\int \frac{id^4q_1}{(2\pi)^4}\int \frac{id^4q_2}{(2\pi)^4}T_4^0(q_2, q_3, q_2)T_3(q, q_3)D(q_1)D(q_2)D(q_3)D(q_3 - q_2),
\]

\[
T_2(q) = g^4\int \frac{id^4q_1}{(2\pi)^4}\int \frac{id^4q_2}{(2\pi)^4}T_4^0(q, q_1, -q_2, q_3)D(q_1)D(q_2)D(q_3),
\]

and in the last two skeleton loop integrals \(q - q_1 + q_2 - q_3 = 0\) as usual. It is instructive to complete the IRMR program for the full gluon SD equation (i.e., including the quark and ghost degrees of freedom) and not only for its Yang-Mills (YM) part.

### A. Multiplicative Renormalizability

The next step is to introduce the IR renormalized quantities. As noted above, in the presence of such severe IR singularities (2.1), all the quantities should in principle depend on \(\epsilon\) as well, i.e., they become IR regularized. So, one has to put

\[
g^2 = X(\epsilon)\bar{g}^2, \quad G(k) = \tilde{Z}_2(\epsilon)\tilde{G}(k), \quad S(p) = Z_2(\epsilon)\tilde{S}(p),
\]

\[
G_\mu(k, q) = \tilde{Z}_1(\epsilon)\tilde{G}_\mu(k, q), \quad \Gamma_\mu(p, q) = Z_1^{-1}(\epsilon)\tilde{\Gamma}_\mu(p, q),
\]

\[
D(q) = Z_D(\epsilon)\tilde{D}(q),
\]

\[
T_3(q, q_1) = Z_3(\epsilon)\tilde{T}_3(q, q_1),
\]

\[
T_4(q, q_1, q_2) = Z_4(\epsilon)\tilde{T}_4(q, q_1, q_2).
\]

In all these relations the quantities with bar are, by definition, IR renormalized, i.e., they exist as \(\epsilon\) goes to zero. In both quantities, the IR regularized (without bar) and the IR renormalized (with bar), the dependence on \(\epsilon\) is assumed but not shown explicitly, for simplicity. In the corresponding IRMR constants this dependence is not omitted in order to distinguish them clearly from the corresponding UVMR constants. Since we are interested in the IR renormalization of the SD equation for the full gluon propagator, it is convenient not to distinguish between the IR renormalization of its INP and PT parts at this stage.
Substituting these relations into the gluon SD equation (2.2), and on account of the explicit expressions for the corresponding skeleton loop integrals given in Eqs. (2.3)-(2.8), one obtains

\[ \bar{N}_1 D(q) = D^0(q) - \bar{N}_6 D^0(q) T_q(q) D(q) - \bar{N}_7 D^0(q) T_{gh}(q) \bar{D}(q) + \bar{N}_2 D^0(q) \frac{1}{2} \bar{T}_1 D(q) + \bar{N}_3 D^0(q) \frac{1}{2} \bar{T}_1(q) \bar{D}(q) + \bar{N}_4 D^0(q) \frac{1}{2} \bar{T}_2(q) \bar{D}(q) + \bar{N}_5 D^0(q) \frac{1}{6} \bar{T}_2(q), \]

if and only if the so-called following IR convergence conditions hold:

\[ Z_D(\epsilon) = \bar{N}_1(\epsilon), \quad X(\epsilon) Z_D^2(\epsilon) = \bar{N}_2(\epsilon), \]

\[ X(\epsilon) Z_D^2(\epsilon) Z_3(\epsilon) = \bar{N}_3(\epsilon), \quad X^2(\epsilon) Z_D^2(\epsilon) Z_3^2(\epsilon) = \bar{N}_4(\epsilon), \quad X^2(\epsilon) Z_D^2(\epsilon) Z_4^2(\epsilon) = \bar{N}_5(\epsilon), \]

\[ X(\epsilon) Z_D^2(\epsilon) Z_3^2(\epsilon) Z_D(\epsilon) = \bar{N}_6(\epsilon), \quad X(\epsilon) Z_D^2(\epsilon) Z_3^2(\epsilon) Z_D(\epsilon) = \bar{N}_7(\epsilon), \]

(2.10)

where all the quantities \( \bar{N}_i(\epsilon), \quad i = 1, 2, 3, 4, 5, 6, 7 \) exist as \( \epsilon \) goes to zero, so these quantities are simply arbitrary but finite numbers, i.e., \( \bar{N}_i(\epsilon) \equiv \bar{N}_i \). Evidently, by imposing these IR convergence conditions one requires that all the terms entering the gluon SD equation (2.2) should survive in the \( \epsilon \to 0^+ \) limit, maintaining thus its full dynamical structure. This makes it possible not to lose even one bit of the information on the true QCD vacuum, the dynamical and topological structures of which are supposed to be reflected by the solutions of this equation. The last two IR convergence conditions are known as quark and ghost self-energy IR convergence conditions, respectively.

Let us show now that all the finite but arbitrary and different constants \( N_i \) appeared in the last gluon SD equation (2.10) as well as in the IR convergence conditions (2.11) can be put to one not losing generality. Moreover, this is general feature of our approach. In all the IR convergence conditions, all the finite, but arbitrary numbers can be put to one, by simply redefining the corresponding IRMR constants as well as the corresponding IR renormalized quantities. In order to show this explicitly, let us introduce indeed new IRMR constants as follows:

\[ Z_D(\epsilon) = \bar{N}_1 Z_D(\epsilon), \quad X(\epsilon) = \bar{N}_2^{-1} N_2 X'(\epsilon), \]

\[ Z_3(\epsilon) = \bar{N}_3^{-1} N_3^{-1} Z_3(\epsilon), \quad \bar{N}_3^{-1} N_3^{-1} Z_3^{-1} = 1, \quad Z_4(\epsilon) = \bar{N}_5 N_5^{-2} Z_4'(\epsilon), \]

\[ \bar{N}_1^{-1} N_2^{-1} Z_2(\epsilon) Z_3^{-1}(\epsilon) = Z_2^2(\epsilon) Z_1(\epsilon), \quad \bar{N}_2^{-1} N_2^{-1} Z_2(\epsilon) Z_3^{-1}(\epsilon) = Z_2^2(\epsilon) Z_1(\epsilon), \quad \bar{N}_1^{-1} N_2^{-1} Z_2(\epsilon) Z_3^{-1}(\epsilon) = Z_2^2(\epsilon) Z_1(\epsilon),\]

(2.12)

then the IR convergence conditions (2.11) in terms of new IRMR constants become

\[ Z_D'(\epsilon) = 1, \quad X'(\epsilon) Z_D^2(\epsilon) = 1, \]

\[ X'(\epsilon) Z_D^2(\epsilon) Z_3(\epsilon) = 1, \quad X'(\epsilon) Z_D^2(\epsilon) Z_3^2(\epsilon) = 1, \quad X'(\epsilon) Z_D^2(\epsilon) Z_4(\epsilon) = 1, \]

(2.13)

while the gluon SD equation (2.10) in terms of new IR renormalized quantities is

\[ \bar{D}'(q) = D^0(q) - D^0(q) T_q(q) D'(q) - D^0(q) T_{gh}(q) \bar{D}'(q) + D^0(q) \frac{1}{2} T_1 D'(q) + D^0(q) \frac{1}{2} T_1(q) \bar{D}'(q) + D^0(q) \frac{1}{2} T_2(q) \bar{D}'(q) + D^0(q) \frac{1}{6} T_2'(q) \bar{D}'(q). \]

(2.14)

This simply means that all the arbitrary but finite constants \( N_i \) in the gluon SD equation (2.10) and in the IR convergence conditions (2.12) can be put to one, i.e., \( N_i = 1 \), indeed. Returning then to the previous notations, the gluon SD equation (2.10) becomes

\[ \bar{D}(q) = D^0(q) - D^0(q) T_q(q) D(q) - D^0(q) T_{gh}(q) \bar{D}(q) + D^0(q) \frac{1}{2} T_1 D(q) + D^0(q) \frac{1}{2} T_1(q) \bar{D}(q) + D^0(q) \frac{1}{2} T_2(q) \bar{D}(q) + D^0(q) \frac{1}{6} T_2'(q), \]

(2.15)
while the corresponding IR convergence conditions (2.11) become

\[ Z_D(\epsilon) = 1, \quad X(\epsilon)Z_D^2(\epsilon) = 1, \]
\[ X(\epsilon)Z_D^2(\epsilon)Z_3(\epsilon) = 1, \quad X^2(\epsilon)Z_D^3(\epsilon)Z_4(\epsilon) = 1, \]
\[ X(\epsilon)Z_D^2(\epsilon)Z_1^{-1}(\epsilon)Z_D(\epsilon) = 1, \quad X(\epsilon)\bar{Z}_2^2(\epsilon)\bar{Z}_1(\epsilon)Z_D(\epsilon) = 1. \]  

(2.16)

Evidently, the solutions of these relations are

\[ Z_D(\epsilon) = X(\epsilon) = 1, \quad Z_3(\epsilon) = Z_4(\epsilon) = 1, \quad Z_2^2(\epsilon)Z_1^{-1}(\epsilon) = 1, \quad \bar{Z}_2^2(\epsilon)\bar{Z}_1(\epsilon) = 1. \] 

(2.17)

Thus the IRMR constants of quark and ghost degrees of freedom remain undetermined at this stage. They will be determined elsewhere via the corresponding ST identities, which relate them to each other (see, for example, Refs. \[ \text{and references therein.} \] Evidently, one can start from any place in the SD system of equations, for example, to start from the quark and ghost sectors, ST identities, etc. However, finally the system of the corresponding IR convergence conditions will have the same solutions (2.17), of course.

From the solutions (2.17) and definitions (2.9) it clearly follows that

\[ g^2 = \bar{g}^2, \quad D(q) = \bar{D}(q) \Rightarrow (\xi = \bar{\xi}), \quad T_3(q_1,q_1) = \bar{T}_3(q_1,q_1), \quad T_4(q_1,q_1,q_2) = \bar{T}_4(q_1,q_1,q_2), \] 

(2.18)

i.e., all these quantities are IR renormalized from the very beginning since all their corresponding IRMR constants are equal to one. Thus the IR regularized (without bar) quantities in Eq. (2.18) coincide with their IR renormalized (with bar) counterparts. Moreover, from the fact that the full gluon propagator is IR renormalized from the very beginning, it is easy to understand (see Eq. (1.1)) that the gauge fixing parameter is IR renormalized from the very beginning either (i.e., \( \xi = \bar{\xi} \)), which is explicitly shown in Eq. (2.18). So the dependence all of these quantities on \( \epsilon \) in the \( \epsilon \rightarrow 0^+ \) limit can be neglected.

### III. IR RENORMALIZATION OF THE MASS GAP

In order to investigate the IR renormalization properties of the INP part of the full gluon propagator, it is convenient to rewrite Eq. (1.4) as follows:

\[ D^{\text{INP}}(q,\Delta^2) = \sum_{k=0}^{\infty} (\Delta^2)^{k+1}(q^2)^{-2-k}\phi_k(\lambda,\nu,\xi,g^2), \] 

(3.1)

where we suppress the tensor \( iT_{\mu\nu}(q) \), for simplicity. Thus the IR renormalization properties of \( D^{\text{INP}}(q,\Delta^2) \) depend on the mass gap \( \Delta^2 \) and the corresponding residues \( \phi_k(\lambda,\nu,\xi,g^2) \) only.

In our previous works \[ \text{and references therein.} \] we have derived the gluon confinement criterion in the most general form, i.e., considering the above-mentioned both quantities as depending on the IR regularization parameter \( \epsilon \). Our aim here is to specify the dependence of the residues \( \phi_k(\lambda,\nu,\xi,g^2) \) on \( \epsilon \) via their arguments. As a function of \( \epsilon \) beside the quantities pointed out above it may depend on the full gluon propagator, the triple and quartic full gluon vertices (see Eqs. (2.5)-(2.8) above), so that one has

\[ \phi_k(\lambda,\nu,\xi,g^2) \equiv \phi_k(\lambda,\nu;g^2,D,T_3,T_4), \] 

(3.2)

where instead the dependence on the gauge fixing parameter \( \xi \) we show explicitly the equivalent dependence on the full gluon propagator \( D \). As mentioned above, in the presence of such severe IR singularities (shown in Eq. (3.1)) all the quantities, which appear in the residues, should depend in principle on \( \epsilon \). Thus, one obtains

\[ \phi_k(\lambda,\nu;g^2,D,T_3,T_4) \equiv \phi_k(\lambda(\epsilon),\nu(\epsilon);g^2(\epsilon),D(\epsilon),T_3(\epsilon),T_4(\epsilon)). \] 

(3.3)

However, from the solutions (2.18) it follows that none of the coupling constant squared, the full gluon propagator, the triple and quartic full gluon vertices depend on \( \epsilon \) as it goes to zero, so we can write

\[ \phi_k(\lambda(\epsilon),\nu(\epsilon);g^2(\epsilon),D(\epsilon),T_3(\epsilon),T_4(\epsilon)) \equiv \phi_k(\lambda(\epsilon),\nu(\epsilon);g^2,D,T_3,T_4). \] 

(3.4)
On the other hand, the dimensionless UV cut-off $\lambda$ and the dimensionless renormalization point $\nu$ might be functions of the coupling constant squared $g^2$ and the gauge fixing parameter $\xi$ since they have been introduced by hand, i.e., $\lambda = \lambda(\xi, g^2)$ and $\nu = \nu(\xi, g^2)$. In its turn, this means that $\lambda(\epsilon) = \lambda(\epsilon(\xi), g^2(\epsilon))$ and $\nu(\epsilon) = \nu(\epsilon(\xi), g^2(\epsilon))$. However, again from the solutions (2.18) it follows that we can neglect the dependence on $\epsilon$ in the coupling constant squared as well as in the gauge fixing parameter. Hence the previous equation is to be present as follows:

$$
\phi_k(\lambda(\epsilon), \nu(\epsilon); g^2(\epsilon), D(\epsilon), T_3(\epsilon), T_4(\epsilon)) \equiv \phi_k(\bar{\lambda}, \bar{\nu}; \bar{g}^2, \bar{D}, \bar{T}_3, \bar{T}_4),
$$

(3.5)
since from $\lambda = \lambda(\xi, g^2) = \lambda(\bar{\xi}, \bar{g}^2)$ and $\nu = \nu(\xi, g^2) = \nu(\bar{\xi}, \bar{g}^2)$ it follows that $\lambda = \bar{\lambda}$ and $\nu = \bar{\nu}$, where $\lambda = \lambda(\xi, \bar{g}^2)$ and $\bar{\nu} = \nu(\xi, \bar{g}^2)$, by definition.

That’s the dimensionless UV cut-off $\lambda$ and the dimensionless renormalization point $\nu$ are IR renormalized from the very beginning can be in addition proven in the following way. Let us recall that the above-mentioned quantities are the ratios between the corresponding mass squared scale parameters and the mass gap. As underlined above, all these squared masses depend in general on the IR regularization parameter $\epsilon$, namely $\Lambda^2 \equiv \Lambda^2(\epsilon)$, $\mu^2 \equiv \mu^2(\epsilon)$ and $\Delta^2 \equiv \Delta^2(\epsilon)$. So on general ground, one puts

$$
\Lambda^2(\epsilon) = \alpha(\epsilon)\Delta^2_1(\epsilon) = \lambda\alpha_1(\epsilon)\Delta^2_1(\epsilon) = \lambda\Delta^2(\epsilon),
$$

$$
\mu^2(\epsilon) = \beta(\epsilon)\Delta^2_2(\epsilon) = \nu\beta_2(\epsilon)\Delta^2_2(\epsilon) = \nu\Delta^2(\epsilon),
$$

(3.6)
which assumes

$$
\alpha_1(\epsilon)\Delta^2_1(\epsilon) = \beta_2(\epsilon)\Delta^2_2(\epsilon) = \Delta^2(\epsilon),
$$

(3.7)
and which can be always satisfied, of course. Evidently, in these relations we introduce auxiliary intermediate masses squared $\Delta^2_1(\epsilon)$ and $\Delta^2_2(\epsilon)$. Also, the dimensionless numbers $\lambda$, $\nu$ do not depend on $\epsilon$, by derivation (more precisely they have finite limit as $\epsilon$ goes to zero, by definition, i.e., $\lambda \equiv \bar{\lambda}$ and $\nu \equiv \bar{\nu}$). Since nothing depends on the auxiliary intermediate masses squared in our approach, all the ratios between the initial masses squared $\Lambda^2(\epsilon)$, $\mu^2(\epsilon)$ and the mass gap $\Delta^2(\epsilon)$ itself are $\epsilon$-independent. In other words, by introducing the above-mentioned auxiliary intermediate masses squared, we pass the dependence on $\epsilon$ from the initial masses squared on the mass gap, leaving the corresponding dimensionless numbers not depending on it (evidently, this is true for any arbitrary mass squared $M^2 \equiv M^2(\epsilon)$ by assuming the introduction of the corresponding auxiliary intermediate mass squared). This is in complete agreement with the above-derived result (that’s none of $\lambda$ and $\nu$ depends on $\epsilon$ in the $\epsilon \to 0^+$ limit) which was obtained by the IR renormalization of the gluon SD equation itself.

Going back to Eq. (3.5), it thus becomes

$$
\phi_k(\lambda, \nu; g^2, D, T_3, T_4) \equiv \phi_k(\bar{\lambda}, \bar{\nu}; \bar{g}^2, \bar{D}, \bar{T}_3, \bar{T}_4),
$$

(3.8)
since we can neglect the dependence on $\epsilon$ in all arguments of the residues. This means that all the residues in the initial equation (3.1) are IR renormalized from the very beginning, namely

$$
\phi_k(\lambda, \nu, \xi, g^2) = \phi_k(\bar{\lambda}, \bar{\nu}, \bar{\xi}, \bar{g}^2).
$$

(3.9)
Let us recall that the corresponding IRMR constants for the residues (see paper [1]) are equal to one, i.e., $Z_k(\epsilon) = 1$ in this case. Evidently, from here on we are going back to the same arguments in the residues as in the starting Eq. (3.1).

Then in Eq. (3.1) the only quantity which should be IR renormalized remains the mass gap itself. Let us introduce further the following relation:

$$
\Delta^2 = X(\epsilon)\tilde{\Delta}^2,
$$

(3.10)
where the mass gap with bar is IR renormalized, i.e., it exists as $\epsilon$ goes to zero, by definition, while the mass gap without bar is IR regularized. In complete analogy with the relations (2.9) in both quantities the dependence on $\epsilon$ is assumed. Here $X(\epsilon)$ is the corresponding IRMR constant. Substituting further this relation into the Laurent expansion (3.1), in terms of the IR renormalized quantities, it then becomes

$$
D^{\text{INP}}(\bar{q}, \bar{\Delta}^2) = \sum_{k=0}^{\infty} (\bar{\Delta}^2)^{k+1}(\bar{g}^2)^{-2-k}\tilde{\phi}_k(\bar{\lambda}, \bar{\nu}, \bar{\xi}, \bar{g}^2)X^{k+1}(\epsilon).
$$

(3.11)
A. Gluon confinement

Due to the distribution nature of severe IR singularities, which appear in the full gluon propagator, the two different cases should be distinguished.

I. If there is an explicit integration over the gluon momentum, then from the dimensional regularization (2.1) and Eq. (3.11), it follows

$$D^{\text{INP}}(q, \Delta^2) = \sum_{k=0}^{\infty} (\Delta^2)^{k+1} a(k)|\delta^4(q)|^{(k)} \tilde{\phi}_k(\lambda, \bar{\nu}, \xi, \bar{g}^2)\tilde{B}_k(\epsilon),$$

provided the INP part will not depend on \( \epsilon \) at all as it goes to zero. For this we should put

$$X^{k+1}(\epsilon) = \epsilon \tilde{B}_k(\epsilon), \quad k = 0, 1, 2, 3..., \quad \epsilon \to 0^+, \quad (3.12)$$

then the cancellation with respect to \( \epsilon \) will be guaranteed term by term (each NP IR singularity is completely independent distribution) in the Laurent skeleton loop expansion (3.11), that is dimensionally regularized and IR renormalized in Eq. (3.12). Here \( \tilde{B}_k(\epsilon) \) exists as \( \epsilon \) goes to zero, by definition. It is easy to show that the unique solution of the IR convergence condition (3.13) is

$$X(\epsilon) = \epsilon, \quad \Delta^2 = \epsilon \bar{\Delta}^2, \quad \tilde{B}_k(\epsilon) = \epsilon^k, \quad k = 0, 1, 2, 3..., \quad \epsilon \to 0^+, \quad (3.14)$$

where we put \( \tilde{B}_0(\epsilon) \equiv \tilde{B}_0 = 1 \) not loosing generality.

II. If there is no explicit integration over the gluon momentum, then the functions \((q^2)^{-2-k}\) in the Laurent skeleton loops expansion (3.11) cannot be treated as the distributions, i.e., there is no scaling as \( 1/\epsilon \). The INP part of the full gluon propagator, expressed in the IR renormalized terms, in this case disappears as \( \epsilon \), namely

$$D^{\text{INP}}(q, \Delta^2) = \epsilon \sum_{k=0}^{\infty} (\Delta^2)^{k+1} (q^2)^{-2-k} \tilde{\phi}_k(\lambda, \bar{\nu}, \xi, \bar{g}^2)\tilde{B}_k(\epsilon) \sim \epsilon, \quad \epsilon \to 0^+. \quad (3.15)$$

This means that any amplitude for any number of soft-gluon emissions (no integration over their momenta) will vanish in the IR limit in our picture. In other words, there are no transverse gluons in the IR (let us remind that the INP part of the full gluon propagator responsible for confinement of gluons within our approach depends only on the transverse degrees of freedom of gauge bosons), i.e., at large distances (small momenta, \( q^2 \to 0 \)) there is no possibility to observe physical gluons experimentally as free particles. Let us emphasize that the PT part of the full gluon propagator is to be totally neglected in comparison with its INP one in the \( q^2 \to 0 \) limit, i.e., the full gluon propagator \( D \) is reduced to its INP part in this limit before taking the \( \epsilon \to 0^+ \) limit. So color gluons can never be isolated. This behavior can be treated as the gluon confinement criterion. It does not depend explicitly on a gauge choice in the full gluon propagator, i.e., it is a gauge-invariant. It is also general one, since even going beyond the gluon sector nothing can invalidate it. For the first time it has been derived in Ref. [2] (see Ref. [3] as well). Evidently, it coincides with the general criterion of gluon confinement derived in our previous works [11, 12] when the IR renormalization properties of the residues via their arguments have not been specified.

IV. EXACT STRUCTURE OF THE FULL GLUON PROPAGATOR

Our quantum-dynamical approach to the true QCD ground state is based on the existence and the importance of such kind of the NP excitations and fluctuations of virtual gluon fields which are mainly due to the nonlinear (NL) interactions between massless gluons without explicitly involving some extra degrees of freedom. They are to be summarized (accumulated) into the purely transverse part of the full gluon propagator, and are to be effectively correctly described by its severely singular structure in the deep IR domain. We will call them the purely transverse singular gluon fields, for simplicity. In other words, they represent the purely transverse quantum virtual fields with the enhanced low-frequency components/large scale amplitudes due to the NL dynamics of the massless gluon modes.

At this stage, it is difficult to identify actually which type of gauge field configurations can be finally formed by the purely transverse singular gluon fields in the QCD ground state, i.e., to identify relevant field configurations: chromomagnetic, self-dual, stochastic, etc. However, if these gauge field configurations can be absorbed into the gluon
propagator (i.e., if they can be considered as solutions to the corresponding SD equation), then its severe IR singular behavior is a common feature for all of them. Being thus a general phenomenon, the existence and the importance of quantum excitations and fluctuations of severely singular IR degrees of freedom inevitably lead to the general zero momentum modes enhancement (ZMME) effect in the QCD ground state.

Our approach to the true QCD ground state, based on the general ZMME phenomenon there, can be thus analytically formulated in terms of the exact decomposition of the full gluon propagator. In order to define correctly the NP phase in comparison with the PT one in QCD, let us introduce (following our paper [3]) the exact decomposition of the full gluon form factor (which in principle can be treated as the effective charge) as follows:

\[ d(q^2, \xi) = d(q^2, \xi) - d^{PT}(q^2, \xi) + d^{PT}(q^2, \xi) = d^{NP}(q^2, \xi) + d^{PT}(q^2, \xi). \] (4.1)

Evidently, \( d(q^2, \xi) \) being the NP effective charge, nevertheless, is contaminated by the PT contributions, while \( d^{NP}(q^2, \xi) \) is the truly NP one since it is free of them, by construction. Substituting now this decomposition into the full gluon propagator (1.1), one obtains

\[ D_{\mu\nu}(q) = D_{\mu\nu}^{INP}(q) + D_{\mu\nu}^{PT}(q), \] (4.2)

where

\[ D_{\mu\nu}^{INP}(q) = iT_{\mu\nu}(q)d^{NP}(q^2, \xi) \frac{1}{q^2} = iT_{\mu\nu}(q)d^{INP}(q^2, \xi), \] (4.3)

and

\[ D_{\mu\nu}^{PT}(q) = i \left[ T_{\mu\nu}(q)d^{PT}(q^2, \xi) + \xi L_{\mu\nu}(q) \right] \frac{1}{q^2}. \] (4.4)

As mentioned above, the PT part, denoted in Eq. (1.3) as \( O_{\mu\nu}(1/q^2) \), remains undetermined. At the same time, the INP part (representing the above-mentioned ZMME effect) in terms of the IR renormalized quantities is

\[ D_{\mu\nu}^{INP}(q, \Delta^2) = iT_{\mu\nu}(q) \times \epsilon \sum_{k=0}^{\infty} (\Delta^2)^{k+1}(q^2)^{-2-k} \phi_k(\lambda, \bar{\nu}, \xi, g^2) \tilde{B}_k(\epsilon). \] (4.5)

However, it is perfectly clear now that due to solutions (3.14) only the simplest NP IR singularity \( (q^2)^{-2} \) will survive in the \( \epsilon \to 0^+ \) limit, namely

\[ D_{\mu\nu}^{INP}(q, \Delta^2) = iT_{\mu\nu}(q) \times \epsilon \Delta^2(q^2)^{-2} \phi_0(\lambda, \bar{\nu}, \xi, g^2), \] (4.6)

since all other terms in the expansion (4.5) become terms of the order \( \epsilon^2 \), at least, in the \( \epsilon \to 0^+ \) limit. Here

\[ \phi_0(\lambda, \bar{\nu}, \xi, g^2) = \sum_{m=0}^{\infty} \phi_{0,m}(\lambda, \bar{\nu}, \xi, g^2). \] (4.7)

Thus an infinite number of iterations of the relevant skeleton loops gives rise to a simplest NP IR singularity (and hence to the mass gap).

Again, if there is an explicit integration over the gluon momentum, then on account of the regularization relation (2.1) for \( k = 0 \) and \( a(0) = \pi^2 \), one finally gets

\[ D_{\mu\nu}^{INP}(q, \Delta^2) = iT_{\mu\nu}(q) \times \Delta_R^2 \delta^4(q). \] (4.8)

The \( \delta \)-type regularization of the simplest NP IR singularity \( (q^2)^{-2} \) is valid even for the multi-loop skeleton diagrams, where the number of independent loops is equal to the number of the gluon propagators. In the multi-loop skeleton diagrams, where these numbers do not coincide (for example, in the diagrams containing three or four-gluon proper
vertices), the general regularization (2.1) is to be used, i.e., the derivatives of the \( \delta \) functions, which should be understood in the sense of the theory of distributions (for detail prescription how to correctly proceed in this case see our paper [6]). In Eq. (4.8) we introduce the UV renormalized (which has been already the IR renormalized) mass gap as follows:

\[
\bar{\Delta}_r^2 = Z_\Delta(\bar{\lambda}, \bar{\nu}, \bar{\xi}, \bar{g}^2) \bar{\Delta}^2(\bar{\lambda}, \bar{\nu}, \bar{\xi}, \bar{g}^2),
\]

(4.9)

and

\[
Z_\Delta(\bar{\lambda}, \bar{\nu}, \bar{\xi}, \bar{g}^2) = \left[ \pi^2 \sum_{m=0}^{\infty} \bar{\phi}_{0,m}(\bar{\lambda}, \bar{\nu}, \bar{\xi}, \bar{g}^2) \right],
\]

(4.10)

where \( Z_\Delta(\bar{\lambda}, \bar{\nu}, \bar{\xi}, \bar{g}^2) \) is the corresponding UVMR constant, and \( \bar{\Delta}_r^2 \) exists in the \( \bar{\lambda} \to \infty \) limit, by definition. It is also a gauge-invariant since nothing in its definition depends explicitly on the gauge fixing parameter. Precisely this quantity should be considered as the physical mass gap, which must be strictly positive, by definition. Let us emphasize that it survives after summing up an infinite number of the relevant contributions (skeleton loops expansion) and performing the IR renormalization program. This is similar to \( \Lambda_{QCD}^2 \), which also appears after summing up an infinite number of the relevant contributions by solving the renormalization group equations for the effective coupling in the weak coupling regime and taking the \( \lambda \to 0^+ \) limit. At the same time, the dependence of the mass gap on the coupling constant squared \( g^2 \) is completely arbitrary because of an infinite summation of the relevant skeleton loop integrals, so its all orders contribute into the mass gap. In other words, it plays no role in the presence of the mass gap. Let us also emphasize that the UVMR constant \( Z_\Delta(\bar{\lambda}, \bar{\nu}, \bar{\xi}, \bar{g}^2) \), introduced in Eq. (4.9) and defined in Eq. (4.10), itself is an infinite sum over all NL iterations of the relevant dimensionless skeleton loop integrals, and hence it cannot be calculated perturbatively. It is essentially NP UVMR constant, by its nature.

II. Again, if there is no explicit integration over the gluon momentum, then the criterion of gluon confinement (3.15) remains valid, of course. Now it looks like

\[
D^{\text{INP}}_{\mu\nu}(q, \bar{\Delta}^2) = \epsilon \times iT_{\mu\nu}(q)\bar{\Delta}^2(q^2)^{-2}\bar{\phi}_0(\bar{\lambda}, \bar{\nu}, \bar{\xi}, \bar{g}^2) \sim \epsilon, \quad \epsilon \to 0^+,
\]

(4.11)
in complete agreement with Eq. (4.6). Let us emphasize once more that it takes place at any gauge, and thus is gauge-invariant.

So, we have established the exact structure of the INP part of the full gluon propagator which is responsible for color confinement of gluons within our approach. In all loop integrals for the independent loop variable it is explicitly given in Eq. (4.8). In all other cases the derivatives of the \( \delta \) function are in order as mentioned above. All this has been achieved at the expense of the PT part of the full gluon propagator which remains undetermined. However, this is not important within our approach (see discussion below in section V).

The ZMME mechanism of quark confinement is nothing but the well forgotten IR slavery (IRS) one, which can be equivalently referred to as a strong coupling regime [1, 11]. Indeed, at the very beginning of QCD it was expressed a general idea [11, 12, 13, 14, 15, 16] that the quantum excitations of the IR degrees of freedom, because of self-interaction of massless gluons in the QCD vacuum, made it only possible to understand confinement, dynamical (spontaneous) breakdown of chiral symmetry and other NP effects. In other words, the importance of the deep IR structure of the true QCD vacuum has been emphasized as well as its relevance to the above-mentioned NP effects and the other way around. This development was stopped by the wide-spread wrong opinion that severe IR singularities cannot be put under control. We have explicitly shown (see our recent papers [1, 2, 3, 6] and references therein) that the correct mathematical theory of quantum YM physical theory is the theory of distributions (the theory of generalized functions) [1], complemented by the DR method [4]. They provide a correct treatment of these severe IR singularities without any problems. Thus, we come back to the old idea but on a new basis that is why it becomes new ("new is well forgotten old"). In other words, we put the IRS mechanism of quark confinement on a firm mathematical ground provided by the distribution theory. Moreover, we also emphasize the role of the purely transverse singular gauge fields in this mechanism.

Working always in the momentum space, we are speaking about the purely transverse singular gluon fields responsible for color confinement in our approach. Discussing the relevant field configurations, we always will mean the functional (configuration) space. Speaking about relevant field configurations (chromomagnetic, self-dual, stochastic, etc), we mean all the low-frequency modes of these virtual transverse fields. Only large scale amplitudes of these fields ("large transverse gluon fields") are to be taken into account by the INP part of the full gluon propagators. All other frequencies are to be taken into account by corresponding PT part of the gluon propagators. Apparently,
to speak about specific field configurations that are solely responsible for color confinement is not the case, indeed. The low-frequency components/large scale amplitudes of all the possible in the QCD vacuum the purely transverse virtual fields are important for the dynamical and topological formation of such gluon field configurations which are responsible for color confinement and other NP effects within our approach to low-energy QCD. For convenience, we will call them the purely transverse severely singular gluon field configurations as mentioned above.

A. A few technical remarks

The exact separation of the full gluon propagator into the two principally different parts, shown in Eq. (4.2), does not, of course, contradict to the full gluon propagator being IR renormalized from the very beginning \((D = \bar{D})\) and hence \(Z_D(\epsilon) = 1\). If there is no explicit integration over the gluon momentum \(q\), then \(D\) is reduced to \(D^{PT}\) which implies \(D = D^{PT} = \bar{D}^{PT}\), indeed. On the other hand, if there is an explicit integration over the gluon momentum \(q\), then, nevertheless, the INP part has a finite limit as \(\epsilon\) goes to zero (see Eq. (3.12)). So \(D = \bar{D}\) will be again satisfied. Also, there is no doubt that our solution for the full gluon propagator, obtained at the expense of remaining unknown its PT part, nevertheless, satisfies the gluon SD equation (2.15) apart from the quark and ghost skeleton loops since it has been obtained by the direct iteration solution of this equation. To show this explicitly by substituting it back into the gluon SD equation (2.15) is not a simple task, and this is to be done elsewhere (for preliminary procedure see our paper \([6]\)). The problem is that the decomposition of the full gluon propagator into the INP and PT parts by regrouping the so-called mixed up terms in Ref. \([1]\) (see also Refs. \([2, 3]\)) was a well defined procedure (there was an exact criterion how to distinguish between these two terms in a single \(D\)). However, to do the same at the level of the gluon SD equation itself, which is nonlinear in \(D\), is not so obvious.

Fortunately, there exists a rather simple method as how to show explicitly that the INP part of the full gluon propagator can be completely decoupled from the rest of the gluon SD equation in the \(\epsilon \to 0^+\) limit. In other words, let us consider it as a function of \(\epsilon\) rather than as a function of its momentum. For this purpose, let us present explicitly the gluon SD equation which was the starting point for the general iteration solution for the full gluon propagator in Ref. \([1]\), namely

\[
D(q) = D^0(q) + D^0(q)T_g[D]D(q) + D^0(q)O(q^2; D)D(q). \tag{4.12}
\]

From its iteration solution we already know that on general ground the block \(T_g[D]\) can be represented as follows:

\[
T_g[D] = \Delta^2 \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} b_{k,m}(\lambda, \nu, \xi, g^2), \tag{4.13}
\]

where the \(q^2\)-independent factors \(b_{k,m}(\lambda, \nu, \xi, g^2)\) may only depend on the same arguments as the real residues \(\phi_{k,m}(\lambda, \nu, \xi, g^2)\) in Eq. (3.2). From the IRMR program formulated and developed here (see sections II and III) it follows that just as the real residues the \(q^2\)-independent factors do not depend on \(\epsilon\) in the \(\epsilon \to 0^+\) limit. Taking this into account, and going to the IR renormalized quantities in Eq. (4.13), one obtains

\[
T_g[\bar{D}] = \epsilon \Delta^2 \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \bar{b}_{k,m}(\bar{\lambda}, \bar{\nu}, \bar{\xi}, \bar{g}^2) \sim \epsilon, \quad \epsilon \to 0^+, \tag{4.14}
\]

where, evidently, we equivalently replace all quantities with their IR renormalized counterparts since \(D = \bar{D}\) and so on (let us emphasize that after doing so here and everywhere, only then we can go to the \(\epsilon \to 0^+\) limit). In other words, this block disappears as \(\epsilon\) in the \(\epsilon \to 0^+\) limit (at any \(k\)), so Eq. (4.12) becomes

\[
\bar{D}(q) = D^0(q) + D^0(q)O(q^2; \bar{D})\bar{D}(q). \tag{4.15}
\]

Now we can distinguish between the INP and PT parts of the full gluon propagator shown in Eq. (4.2). Since there is no explicit integration over the gluon momentum \(q\), the INP part of the full gluon propagator \(D\) vanishes as \(\epsilon\) in the \(\epsilon \to 0^+\) limit (see Eq. (4.11)). So from Eq. (4.15) it finally follows

\[
\bar{D}^{PT}(q) = D^0(q) + D^0(q)O(q^2; \bar{D})\bar{D}^{PT}(q). \tag{4.16}
\]
Let us underline that this equation will produce only the PT-type of the IR singularities within its iteration solution since the block $O(q^2, \bar{D})$ is always of the order $q^2$ whatever $\bar{D}$ is. Thus the INP part of the full gluon propagator automatically satisfies the gluon SD equation. The part of the gluon SD equation responsible for the NP IR singularities in its general iteration solution vanishes on account of the solution for its INP part. Just in this sense should be understood in general terms the solution of the gluon SD equation within our approach since it leaves the PT part of the solution undetermined. To show this explicitly by treating the INP part of the full gluon propagator as a function of its momentum and substituting the decomposition (4.2) back to the nonlinear gluon SD equation (4.12) is far more complicate case as mentioned above.

For the calculations of such NP quantities as the gluon condensate or the truly NP vacuum energy density (the Bag constant apart from the sign, by definition), the effective coupling constant (effective charge) should be used from the very beginning. From the fact that within our approach the simplest NP IR singularity $(q^2)^{-2}$ survives only and in our notations (see Eq. (4.3)) it then follows

$$\alpha_s(q^2) = q^2 d^{INP}(q^2) = \Lambda_{NP}^2 / q^2,$$

(4.17)

where $\Lambda_{NP}^2$ is identified with $\bar{\Delta}_R^2$, for simplicity (a possible difference between them is not important). The fact that the effective charge (4.17) has a PT IR singularity from the very beginning makes it formally possible to put $\Delta^2 = \bar{\Delta}_R^2$ (up to some mentioned above unimportant finite constant), i.e., to omit the dependence on $\epsilon$ in Eq. (4.6). Though the gluon condensate is not directly measured quantity, it enters many physical relations, that is why it should depend on $\Lambda_{NP}^2$. For its correct calculations (free from the PT "contaminations") see, for example, papers [17, 18]. The renormalization group equation which determines the corresponding $\beta$ function and its solution for this effective charge is

$$q^2 d\alpha_s(q^2) = \beta(\alpha_s(q^2)) = -\alpha_s(q^2),$$

(4.18)

so that the $\beta$ function as a function of the effective charge is always in the domain of attraction (i.e., negative) as it is required for the confining theory [7]. Also, in order to calculate the linear rising potential between heavy quarks the INP gluon form factor shown in Eq. (4.17) is to be used.

V. DISCUSSION AND CONCLUSIONS

A few years ago Jaffe and Witten (JW) have formulated the following theorem [19]:

**Yang-Mills Existence And Mass Gap:** Prove that for any compact simple gauge group $G$, quantum Yang-Mills on $\mathbb{R}^4$ exists and has a mass gap $\Delta > 0$.

Of course, at present to prove the existence of the YM theory with compact simple gauge group $G$ is a formidable task yet. It is rather mathematical than physical problem. However, the general result of our investigation in Refs. [1, 2, 3] and here can be formulated similar to the above-mentioned JW theorem as follows:

**Yang-Mills Existence, Mass Gap And Gluon Confinement:** If quantum Yang-Mills with compact simple gauge group $G = SU(3)$ exists on $\mathbb{R}^4$, then it exhibits a mass gap and confines gluons.

Though our mass gap (4.9) reproduces many features of the JW mass gap [19], nevertheless, the latter is more general conception, at least at this stage (see below). The symbolic relation between our mass gap ($\bar{\Delta}_R \equiv \Lambda_{NP}$), the JW one ($\Delta \equiv \Delta_{JW}$) and $\Lambda_{QCD} \equiv \Lambda_{PT}$ is

$$\Lambda_{NP} \leftarrow_{\Lambda_{JW} \rightarrow \Lambda_{PT}} \Delta_{JW} \leftarrow_{\Lambda_{JW} \rightarrow \Lambda_{PT}} M_{UV} \rightarrow \Lambda_{PT},$$

(5.1)

where $\alpha_s$ is obviously the fine structure coupling constant of strong interactions, while $M_{UV}$ and $M_{IR}$ are the UV and IR cut-offs, respectively. The right-hand-side limit is well known as the weak coupling regime, while the left-hand-side can be regarded as the strong coupling regime. We know how to take the former [1, 2, 3] and how to deal with the latter one, not solving the gluon SD equation directly, which is formidable task, anyway. However, there is no doubt that the final goal of this limit, namely, the mass gap
\( \Lambda_{NP} \) exists, and should be the renormalization group invariant in the same way as \( \Lambda_{QCD} \). It is solely responsible for the large scale structure of the true QCD ground state. It is important to emphasize once more that it has not been introduced by hand. We have explicitly shown that it was hidden in the skeleton loop integrals contributing into the gluon self-energy due to the NL interaction of massless gluon modes (Eqs. (2.5)-(2.8)). The mass gap shows explicitly up when the gluon momentum goes to zero. The appropriate regularization procedure has been applied to make the existence of the mass gap perfectly clear. Moreover, it survives an infinite summation of the corresponding skeleton loop contributions (skeleton loop expansion) after completing the general IRMR program.

Let us continue our discussion recalling that many important quantities in QCD, such as the gluon and quark condensates, the topological susceptibility, the Bag constant, etc., are defined beyond the PT only \( \pi \). This means that they are determined by such S-matrix elements (correlation functions) from which all types of the PT contributions should be, by definition, subtracted. Anyway, our theory for low-energy QCD which we call INP QCD will be precisely defined by the subtraction of all types of the PT contributions. At the fundamental (microscopic) gluon propagator level the first subtraction is provided by the exact decomposition (4.1). The second one is to omit the PT part of the full gluon propagator \( D^{PT} \). Then one obtains the full gluon propagator free from all types of the PT "contamination" (that is why the PT part is not important within our approach as underlined above). Only after these subtractions one can identify our mass gap \( \Lambda_{NP} \) with the JW mass gap \( \Delta_{JW} \) (for more detailed discussion of the necessary subtractions see Ref. \( \pi \)). At the same time, having made these subtractions, we thus know the full gluon propagator (4.8) and its generalizations (the derivatives of the \( \delta \) function) which can be used for the solutions of the quark SD equation, quark-gluon ST identity, etc., \( \pi \). This opens the possibilities to calculate physical observables from first principles (for preliminary calculations see our papers \( \pi, \pi \)).

Color confinement of gluons is the IR renormalization of the mass gap gauge-invariant effect within our approach. An infinite number of iterations of the relevant skeleton loops (skeleton loops expansion) has to be made in order to invoke the mass gap. No any truncations/approximations have been made as well in such obtained general iteration solution of the gluon SD equation for the full gluon propagator. The important feature of our investigation, that's, the existence of the mass gap assumes certainly confinement of gluons, somehow has been missed from the discussion in Ref. \( \pi \). However, it is worth emphasizing that our mass gap and the JW mass gap cannot be interpreted as the gluon mass, i.e., they always remain massless. Our gluon propagator, described in the previous section IV, takes into account the importance of the quantum excitations of severely singular IR degrees of freedom in the true QCD vacuum. They lead to the formation of the purely transverse severely singular gluon field configurations there. Just these configurations are primary responsible for the NP effects, such as color confinement, dynamical chiral symmetry breaking, etc., within our approach.

As mentioned above, we have explicitly shown that in the initial Laurent loops expansion (3.1), that is dimensionally regularized in Eq. (3.12), the simplest severe IR singularity \((a^2)^{-2}\) survives only (see Eq. (4.6)). So, we have confirmed and thus revitalized the previous investigations \( \pi, \pi, \pi, \pi, \pi, \pi, \pi, \pi \) (and references therein), in which this behavior has been obtained as asymptotic solution to the gluon SD equation in different gauges. Let us emphasize, however, that our result is exact. We have already shown that it leads to quark confinement and spontaneous (dynamical) breakdown of chiral symmetry \( \pi, \pi \) (and references therein). In this connection let us note in advance that quark and ghost degrees of freedom play no any significant role in the dynamical generation of the mass gap in our approach. As explicitly shown here and in our previous works \( \pi, \pi, \pi \), the NL interaction of massless gluon modes is only important. At the same time, the quark and ghost skeleton loop contributions into the gluon self-energy (see Eqs. (2.3) and (2.4)) do not depend on the full gluon propagator. As a result their contributions can be summed up into the geometrical progression series within the corresponding linear iteration procedure. All this will complicate the IRMR program from a technical point of view only, and it is left to be done elsewhere (see also Ref. \( \pi \)).

The smooth in the IR gluon propagator is also possible depending on different truncations/approximations used \( \pi \) (see papers \( \pi, \pi \) and references therein as well) since the gluon SD equation is highly nonlinear one. The number of solutions for such kind of systems is not fixed \( a \) \( priori \). The singular and smooth in the IR solutions for the gluon propagator are independent from each other, and thus should be considered on equal footing. However, the smooth gluon propagator is rather difficult to relate to color confinement in a gauge-invariant way, in particular to gluon confinement, while severely IR singular one is directly related to it as explicitly demonstrated here and in our previous works \( \pi, \pi, \pi \) (and references therein).

It is interesting to note that Gribov \( \pi \) by differentiating twice the quark SD equation in fact arrived at the same \( \delta \)-type potential (4.8) as well. However, the principal feature of our approach – the mass gap – was missing in this procedure.

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