Squashed entanglement: order parameter for topological superconductors

Alfonso Maiellaro,1 Antonio Marino,1 and Fabrizio Illuminati1,2, ∗

1Dipartimento di Ingegneria Industriale, Università di Salerno,
Via Giovanni Paolo II, 132, I-84084 Fisciano (SA), Italy
2INFN, Sezione di Napoli, Gruppo collegato di Salerno, Italy

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Identifying entanglement-based order parameters characterizing topological systems has remained a major challenge for the physics of quantum matter in the last two decades. Here we show that squashed entanglement between the system edges, defined in terms of the edge-edge quantum conditional mutual information, is the natural order parameter for topological superconductors and for systems with edge modes. In the topological phase, the long-distance edge-edge squashed entanglement is quantized to $\log(2)/2$, half the maximal Bell-state entanglement, and vanishes in the trivial phase. Such topological squashed entanglement exhibits the correct scaling at the quantum phase transition, discriminates between topological order and ordered phases associated to broken symmetries, counts the number of Majorana excitations for systems of increasing geometrical complexity, and is robust against the effects of disorder and local perturbations. In fact, it turns out to be a valid signature of topological order even for systems with weakened bulk-edge correspondence. Squashed entanglement is defined for all quantum states, reducing to the von Neumann entanglement entropy on pure states. As such, it provides a unified framework for the characterization of topological order in any dimension and, moreover, can be generalized to any setting, including finite temperature, multipartite correlations, and nonequilibrium. Squashed entanglement and quantum conditional mutual information are defined in terms of linear combinations of reduced entropies and can thus be probed using currently available experimental setups and techniques.

I. INTRODUCTION

Condensed matter physics is witnessing, among others, two groundbreaking and concurring developments, respectively the application of concepts and methods of quantum information science and the investigation of topological phases of quantum matter.

Via the identification of entanglement boundary laws and their violations, bipartite block entanglement (block von Neumann entropy) has become a central tool for the characterization and the diagnostics of large classes of phenomena in quantum many-body physics [1–5].

At the same time, by featuring perfectly conducting edge modes, patterns of long-distance entanglement and robust ground-state degeneracy without symmetry breaking, three traits that generalize the concept of ordered phase and phase transition beyond the Ginzburg-Landau paradigm, topological states of matter have attracted increasing attention both for their fundamental interest and their potentiality for applications [1–3, 6–9].

In particular, topological superconductors hosting Majorana edge zero modes (MZEMs) [10] have been proposed as the working principle of various disruptive quantum technologies, including fault-tolerant topological quantum computation [11–13].

Topological order in two-dimensional systems is identified and detected by the sub-leading contribution to the bipartite bulk-boundary von Neumann entanglement entropy, the so-called topological entanglement entropy (TEE) [14–16]. This is a great success of entanglement theory applied to the investigation of quantum matter.

Yet, the approach based on block entropies and simple bipartitions is limited and cannot be applied to various important instances, thus showing the need for more general and advanced methods of entanglement theory. Indeed, in one dimension at zero temperature, no measure of bipartite entanglement based on simple bipartitions, including the entanglement spectrum, can discriminate topologically ordered phases from those with symmetry-breaking order [17].

Moreover, irrespective of the spatial dimension, being TEE the subleading correction to the leading boundary-law contribution to the ground-state block von Neumann entanglement entropy, it cannot be generalized straightforwardly to finite-temperature and nonequilibrium processes since the von Neumann entropy, when defined on mixed states, includes contributions both from quantum and classical correlations, thus ceasing to be a bona fide measure of entanglement and nonclassicality.

A further limitation of the approach based on the elementary bipartition of the system in two halves is that it is not suitable for the characterization and quantification of the nonlocal correlations that are in place between arbitrary subsystems; in particular, block entanglement entropy provides no information on the physics of edge correlations and cannot characterize and quantify the long-distance entanglement between edge modes. Therefore, going beyond the approach based on simple bipartitions by introducing bipartite entanglement measures defined on multipartitions would allow to identify richer conceptual structures and investigate broader classes of complex physical phenomena.
Thus motivated, in the present work we discuss a multipartition-based measure of bipartite entanglement, the squashed entanglement (SE), previously introduced in the general context of quantum information theory, and we apply it to the study of topological superconductors with model Hamiltonians supporting Majorana zero energy modes (MZEMs) at the system edges. We show that the long-distance squashed entanglement SE between the edges, that we name topological squashed entanglement (TSE), is the nonvanishing order parameter that 1) characterizes unambiguously the topologically ordered phases of topological superconductors, 2) discriminates topological order from Ginzburg-Landau order associated to spontaneous symmetry breaking and nonvanishing local order parameters, and 3) distinguishes between different types of topologically ordered phases.

As we will show, one can define two basic forms of upper bounds on bipartite SE, respectively in terms of tripartitions and quadripartitions. This general property of SE allows to introduce two forms of TSE, the first one based on edge-bulk-edge tripartitions, the second one based on edge-bulk-bulk-edge quadripartitions. We will then show how the latter identifies unambiguously topological order in the Kitaev chain, discriminates it from standard Ginzburg-Landau order in the Ising chain, is stabilized in the presence of interactions, and is robust against the effects of disorder. Further, we will discuss how the TSE discriminates between different topological phases in models defined on geometries of higher complexity such as two-leg Kitaev ladders. Finally, we will consider models with long-range hopping and consequently suppressed bulk-boundary correspondence, such as the Kitaev tie, and we will show that even in this case the TSE is a good identifier of topological order.

The paper is organized as follows. In Section II we introduce the SE, review its main properties, and introduce the two fundamental forms of upper bounds. In Section III we review the basic models of topological superconductivity in different lattice geometries and define the different forms of TSE corresponding to tripartitions and quadripartitions. In Section IV we discuss in detail the TSE of the 1D Kitaev model for topological superconductivity, compare it to the end-to-end SE in the Ising chain, and discuss the effects of interactions and disorder. In Section V we study the TSE of the two-leg Kitaev ladder and show how it distinguishes between the different topological phases of the model, while in Section VI we carry out the same investigation for the case of the Kitaev tie with long-range hopping and attenuated bulk-boundary correspondence. Finally, in Section VII we discuss our results and further possible applications of SE in the study of quantum matter.

II. SQUASHED ENTANGLEMENT

A. Definition and fundamental properties

We are looking for a measure of bipartite entanglement that has the following properties:

1. It is defined in any spatial dimension, at any temperature, and on all quantum states (pure or mixed).
2. It is bipartite but generically defined in terms of non-overlapping, distinguishable multipartitions, and it reduces to the bipartite von Neumann entanglement entropy on pure states of simple bipartitions.
3. It is a true, bona fide measure of bipartite entanglement, i.e. a convex entanglement monotone that in addition is also asymptotically continuous, monogamous, and additive on tensor products [18, 19].

In fact, such a measure exists and is the so-called squashed entanglement (SE) [20, 21]. Given a quantum state $\rho_{AB}$ of a bipartite quantum system $AB$, the SE $E_{sq}(\rho_{AB})$ between subsystems $A$ and $B$ in state $\rho_{AB}$ is defined as:

$$E_{sq}(\rho_{AB}) = \inf_{\rho_{ABC}} \left\{ \frac{1}{2} I(A : B|C) \right\},$$

where the infimum is taken over all the quantum-state extensions of arbitrary size $\rho_{ABC}$ such that $\rho_{AB} = Tr_C(\rho_{ABC})$, and $I(A : B|C)$ is the quantum conditional mutual information (QCMI) between $A$ and $B$ conditioned by $C$ [22, 23]:

$$I(A : B|C) = S(\rho_{AC}) + S(\rho_{BC}) - S(\rho_{AC}) - S(\rho_{ABC}),$$

where $\rho_{AC} = Tr_B(\rho_{ABC})$, $\rho_{BC} = Tr_A(\rho_{ABC})$, $\rho_{C} = Tr_{ABC}(\rho_{ABC})$, and $S(\rho)$ denotes the von Neumann entropy of a given quantum state $\rho$ [19].

SE owes its name by the construction Eqs. 1–2 that “squashes” out the classical correlations to leave only the quantum contributions to the mutual information between parties $A$ and $B$ conditioned by party $C$ (that in informative terms can be seen as the “conditioning environment”). SE is a lower bound on the entanglement of formation and an upper bound on the distillable secret key or distillable entanglement of a quantum state or a quantum channel [21, 24], and reduces to the von Neumann entanglement entropy on pure states of a bipartite system $AB$. Indeed, if $\rho_{AB}$ is pure, then $\rho_{ABC} = \rho_{AB} \otimes \rho_{C}$, and thus $E_{sq}(\rho_{AB}) = [S(\rho_{A}) + S(\rho_{B})]/2 = S(\rho_{A}) = S(\rho_{B})$.

SE is unique in that it is the only known entanglement quantifier that enjoys all the desirable axiomatic properties required for a bona fide measure of entanglement [19], SE is a full entanglement monotone, i.e. non increasing under local operations and classical communication (LOCC), and convex [21]: $E_{sq}(\lambda\rho + (1 - \lambda)\sigma) \leq$
\( \lambda E_{sq}(\rho) + (1 - \lambda) E_{sq}(\sigma) \), with \( \lambda \in [0, 1] \). Moreover, SE enjoys important additional properties that promote it to a full entanglement measure.

First of all, SE is additive on tensor products [21]:
\[ E_{sq}(\rho \otimes \sigma) = E_{sq}(\rho) + E_{sq}(\sigma) \]. SE is also continuous [25]: if two sequences of states converge in trace norm: \( \lim_{m \to \infty} \| \rho_m - \sigma_m \|_1 = 0 \), then \( \lim_{m \to \infty} E_{sq}(\rho_m) - E_{sq}(\sigma_m) = 0 \). Furthermore, SE is faithful, that is, \( E_{sq}(\rho_{AB}) \geq 0 \), and \( E_{sq}(\rho_{AB}) = 0 \) if and only if \( \rho_{AB} \) is separable [26].

Finally, SE is monogamous [27]: given three parties \( A, B, \) and \( C \), one has:
\[ E_{sq}(\rho_{|ABC}) \geq E_{sq}(\rho_{AB}) + E_{sq}(\rho_{AC}) \].

This property has important consequences on the SE between edge modes that we will study later on in this work.

SE is an entanglement measure with very important operational meaning. In quantum communication theory it is defined in terms of the communication cost in the distribution of quantum states among multiple parties [28] and it is a tight upper bound on the length of a secret key shared by two parties holding many copies of a quantum state [24, 29]. Moreover, it allows for multipartite extensions with operational meaning [30]. Finally, SE plays a fundamental role in channel theory, as the SE of a quantum channel is the optimal upper bound on the quantum communication capacity of any channel assisted by unlimited classical communication [31]. In particular, SE provides the tightest known bound for this type of capacity (also called two-way assisted quantum capacity) for the case of the amplitude damping channel [32].

Although computing SE is NP-complete [33], it has been calculated analytically for some nontrivial classes of states [34, 35]; moreover, it enjoys a set of very useful lower bounds in terms of the reduced von Neumann entropies, the relative entropy of entanglement and the relative 2-Rényi entropy [29, 36–38].

### B. From bipartitions and two blocks to multipartitions and any pair of subsystems

Given a physical system \( G \) in a pure state \( \rho_G = |G\rangle\langle G| \), typically a ground state in condensed matter physics and quantum statistical mechanics or a vacuum state in quantum field theory, one typically focuses on simple bipartitions \( G = AB \) of system \( G \) in two blocks \( A \) and \( B \), and considers then the pure-state bipartite entanglement between the two blocks as quantified by the von Neumann block entanglement entropy \( S(\rho_A) = S(\rho_B) \) of the reduced local states \( \rho_A = Tr_{BG} \rho_G \) or, equivalently, \( \rho_B = Tr_{AG} \rho_G \).

Squashed entanglement greatly extends this picture to include all possible different forms of bipartite entanglement. To set the stage, consider a quantum system \( G \) in an arbitrary state \( \rho_G \). Next, consider partitioning the global system \( G \) in any two subsystems \( A \) and \( B \) plus a reminder \( C: G = ABC \). When \( C \) is the empty set and \( \rho_G \) is a pure state, we recover the standard two-block bipartition and SE reduces to the von Neumann block entanglement entropy.

We now wish to determine the SE \( E_{sq}(\rho_{AB}) \) existing between subsystem \( A \) and subsystem \( B \) in the reduced state \( \rho_{AB} \). The first important observation in order is that there are always two, and only two, equivalent ways to obtain the same reduced state \( \rho_{AB} \) from the global state \( \rho_G \) that however give rise to two, and only two, different expressions for the QCMI of Eq. 2 prior to the extremization procedure in Eq. 1 that defines the SE. These two different expressions of the QCMI define two fundamental upper bounds on the true SE that may in principle be different. The two expressions obviously give rise to the same unique SE once extremization is performed in Eq. 1.

Indeed, besides the immediate tripartite form \( \rho_{ABC} = \rho_{ABC} \) such that \( \rho_{AB} = Tr_G(\rho_{ABC}) \), one can consider a further bipartite splitting of the reminder: \( C = C_1 C_2 \) and obtain the corresponding quadrupartite form \( \rho_G = \rho_{ABC} = \rho_{ABC} C_2 \) such that \( \rho_{AB} = Tr_G(\rho_{ABC} C_1) \) (or, equivalently, \( \rho_{AB} = Tr_G(\rho_{ABC} C_2) \)), where in turn \( \rho_{ABC} C_1 = Tr_C(\rho_{ABC} C_1) \) (or, equivalently, \( \rho_{ABC} C_2 = Tr_C(\rho_{ABC} C_2) \)). We can thus introduce in Eq. 2 two different expressions for the QCMI:\n
\[ I_{3}(A:B\mid C) = S(\rho_{AC}) + S(\rho_{BC}) - S(\rho_{ABC}) - S(\rho_C), \quad (3) \]
\[ I_{4}(A:B\mid C_1) = S(\rho_{AC_1}) + S(\rho_{BC_1}) - S(\rho_{ABC_1}) - S(\rho_{C_1}), \quad (4) \]

where \( S(\rho_{ABC}) = 0 \) if \( \rho_{ABC} = \rho_G \) is pure.

The QCMI’s \( I_{3} \) and \( I_{4} \) define the two natural upper bounds, respectively in terms of tripartitions and quadrupartitions, to the SE \( E_{sq}(\rho_{AB}) \). It is immediate to verify that any further splitting of the reminder \( C \) in multipartitions of higher order \( C = C_1 C_2 C_3 \ldots C_n \) is redundant and trivially reproduces the QCMI \( I_{4} \). Taking into account the subadditivity properties of the von Neumann entropy and the triangle inequality, one has \( I_{3} \geq I_{4} \), so that the chain of inequalities reads

\[ I_{3}(A:B\mid C) \geq I_{4}(A:B\mid C_1) \geq E_{sq}(\rho_{AB}). \quad (5) \]

The two fundamental constructions, respectively via the tripartition \( ABC \) and the quadrpartition \( ABC_1 C_2 \), are summarized in panels (a) and (b) in Fig. 1.

The exact \( A-B \) bipartite SE in state \( \rho_{AB} \) is obtained by computing the infimum of the two expressions Eqs. 3–4 over all extensions of unbounded dimension \( \rho_{ABC} \) and \( \rho_{ABC_1} \), respectively. This is in general an exceedingly hard task; on the other hand, the true SE is readily obtained whenever there are lower bounds available that coincide with some upper bounds, for instance like the ones provided by Eqs. 3–4.
Summarizing what has been discussed so far, we have shown how by resorting to the QCMI and the SE one moves from the elementary paradigm of bipartite pure-state block entanglement entropy defined over minimal, irreducible bipartitions of the global system to a general framework of bipartite entanglement between any two subsystems defined over arbitrary multipartitions of the global system. Moreover, we have introduced two classes of upper bounds to the exact SE defined over tripartitions and quadripartitions of the global system. The transition from bipartitions to multipartitions and from block entanglement to SE of generic subsystems is pictured in panels (c) and (d) of Fig. 1, where we illustrate schematically how such transition allows in principle to discriminate systems with broken symmetries and short-ranged entanglement from systems with a different type of global order and long-distance boundary entanglement. In the next sections we investigate in detail some significant consequences of this paradigm shift in the study of topological quantum matter.

III. SQUASHED ENTANGLEMENT AND TOPOLOGICAL SUPERCONDUCTORS

A. The quest for order parameters

The archetypal model of topological p-wave superconductivity supporting Majorana edge zero modes (MZEMs) is the Kitaev Hamiltonian of spinless fermions on a 1D lattice [39]. Proposals for realistic implementations of the Kitaev chain consider heterostructures made of semiconducting nanowires coupled to s-wave superconducting substrates [40–43].

Experimental evidence of MZEMs localized at the system edges has been obtained in the study of the tunnel conductance of an InAs nanowire proximized by an s-wave superconductor [44] and in the scanning tunneling microscopy of iron-atom chains deposited on lead substrates [45].

The one-dimensional Kitaev model can be generalized to geometries of significantly increasing complexity, e.g. via Kitaev ladders and ties, to describe coupled superconducting nanowires with a phase diagram hosting a rich variety of different topological phases [46–51].

In view of its conceptual significance and potential realizability in realistic systems of condensed matter physics, the Kitaev model has become a central paradigm in the study of topologically ordered phases of matter hosting MZEMs and, more generically, edge modes and edge states.

The problem of identifying unambiguous signatures of topologically ordered phases, such as topological invariants and/or nonlocal order parameters, has turned out to be a highly nontrivial task addressed in a number of different approaches. One can look to momentum-space properties [52], when, after imposing periodic boundary conditions, the translational invariance is restored. This meaningful relation between the edges and the bulk of a system is also known as bulk-edge correspondence [52] and leads to the definitions of geometric indices $Q$ which signal the presence and the number of topological zero-energy states of matter.

Unfortunately, for disordered systems or, in general, when translational invariance cannot be restored, or also in the presence of interactions, the topological invariants $Q$ are no longer valid signatures and indicators of topological phase transitions and topologically ordered phases [53–56].

In a more fundamental approach one tries then to identify topological invariants based on the entanglement properties of the system, such as the entanglement spectrum [17, 57, 58] or the Rényi entropies of disconnected and partially overlapping partitions [59].

Indeed, the entanglement spectrum becomes two-fold degenerate when a quantum system undergoes a topological phase transition [60–62]; however, this is not a discriminating signature of topological order: exactly the same degeneration is featured by any system with the same Hamiltonian symmetries that can support symmetry-breaking order [17, 63]. As a consequence, block entanglement based on simple bipartitions and two blocks does not provide the information necessary to identify and discriminate unambiguously topological order associated to edge modes and edge states.

Suitable combinations of Rényi entropies of disconnected and partially overlapping multipartitions actually allow to define quantized topological invariants able to characterize topologically ordered phases in one-dimensional topological superconductors [59]. This is a significant step forward. Unfortunately, these invariants are not entanglement monotones between distinguishable subsystems, let alone entanglement measures, and thus are devoid of a clear physical meaning in terms of nonlocal quantum correlations.

Even leaving aside their lack of conceptual significance, it is difficult from a purely pragmatic point of view to understand how to develop such invariants systematically for the characterization of different types of topologically ordered phases or to extend them to general situations such as finite temperature and/or complex geometries and higher dimensions.

In the present work, we discuss SE in the context of topological quantum matter featuring edge modes and edge states. We show that the edge-to-edge, long-distance bipartite SE is an order parameter, properly quantized, featuring the correct scaling behavior at the critical point, characterizing the ordered phases of topological superconductors in various geometries, and discriminating them unambiguously from other classes of ordered phases of matter.

Shifting focus from block entanglement entropy to SE between subsystems finds a basic physical motivation by observing that systems with topological order display bulk band gaps like those of ordinary insulators and conducting surface states that are topologically protected...
by some symmetries. This naturally prompts to look for nonlocal correlations between subsystems (e.g., the system edges) rather than the block entanglement between two halves of the same system. In turn, this implies replacing bipartitions with multipartitions, and resorting to SE as the prospective quantity able to:

- Quantify the *prima facie* bipartite edge-edge, long-distance entanglement in the presence of edge modes.

- Discriminate unambiguously topological order from symmetry-breaking order and distinguish between different classes of topologically ordered phases.

In the following we will consider systems hosting MZEMs and for such systems we will investigate the two bipartite edge-to-edge QCMI, \( I_{(3)}(A : B|C) \) and \( I_{(4)}(A : B|C_1) \) that realize two upper bounds on the true SE \( E_{sq}(\rho_{AB}) \) between edge \( A \) and edge \( B \). In particular, we will show that the QCMI \( I_{(4)} \) discriminates the one-dimensional Kitaev model featuring edge modes and long-distance entanglement between the edges from the Ising chain, its symmetry-breaking counterpart featuring no edge modes and a short-distance entanglement structure.

For the Kitaev chain we will show that \( I_{(4)} \) satisfies all the requirements for a genuine topological order parameter, including: quantization as topological invariant at the exact topological ground-state degeneracy point and throughout the topologically ordered phase, scaling with the system size at the phase transition point, and robustness against localized disorder and imperfections.

Moreover, for the Kitaev chain the topological invariant in fact coincides with a lower bound to the SE, and therefore the QCMI \( I_{(4)} \) indeed coincides with the edge SE \( E_{sq} \) at the exact topological degeneracy point and throughout the entire topologically ordered phase. The transition from simple bipartitions and block entanglement entropy to multipartitions and edge to edge long-distance entanglement is illustrated pictorially in panels (c) and (d) of Fig. 1.

We will then generalize the method to consider models
of topological superconductors defined on geometries of higher complexity, the Kitaev ladder [50] and the Kitaev tie [51]. The Kitaev ladder model is obtained by coupling two Kitaev chains by means of transverse hopping and pairing terms.

The Kitaev chain and the Kitaev ladder belong to the two different topological classes D and BDI which are characterized by two different topological bulk invariants, respectively the Pfaffian invariant [64] and the winding number [50]. In the case of the Kitaev tie, the long-range hopping term added to the Hamiltonian of the Kitaev chain define a knotted–ring geometry which can be rearranged in the form of a tie. The model is then the simplest realization of geometric frustration with no associated bulk [65].

For all three models SE identifies and characterizes the different topological behaviors of the systems, well characterized by the band topology for the Kitaev chain and the Kitaev ladder and by the Majorana polarization and topological transfer matrix for the Kitaev tie [65]. We will show that $I_4$ identifies the topological phase transitions in the Kitaev ladder and distinguishes between the different topologically ordered phases of the model corresponding to different numbers of Majorana excitations.

Remarkably, in the case of the Kitaev tie, a system which lacks a clearly identifiable bulk, the SE is still very efficient in characterizing the topological phases of the model even if, as expected, perfect quantization is partially blurred due to the less pronounced physical bulk–boundary separation.

### B. Topological squashed entanglement

For the model systems defined in the following we consider tripartitions and quadripartitions as sketched in Fig. 1 and we proceed to compute the upper bounds $I_4(A : B|C)$ and $I_4(A : B|C_1)$ on the SE $\rho_{AB}$ between the edges $A$ and $B$ given a bulk $C = C_1C_2$.

In a condensed matter setting, the two QCMIs are defined in terms of the many body ground state, i.e. they are the quantum mutual information of the edges $A$ and $B$ conditioned, respectively, on the existence of the total bulk $C$ and the partial bulk $C_1$. When going to sufficiently large system sizes such that the edges are far from each other and decoupled from the bulk, a non-vanishing $\rho_{AB}$ establishes the existence of a topological squashed entanglement (TSE), i.e. a nontrivial long-distance quantum correlation between the edges.

Although the physical mechanisms are rather different, the edge-edge fermionic TSE is reminiscent of other forms of long-distance entanglement (LDE) that are established by entanglement monogamy between the end points of dimerized and quasi-dimerized spin chains with alternate patterns of strong and weak nearest-neighbor couplings or patterns of competing finite-range interactions [66–68]. In turn, the latter can be seen as particular instances of more general patterns of modular entanglement [69] and surface entanglement on networks [70].

In panels (a)–(c) of Fig. 2 we provide a sketch of the three model geometries considered (Kitaev chain, Kitaev ladder, Kitaev tie), while in panels (d) and (e) we draw a synthetic scheme of how the multipartitions reported in Fig. 1 are applied to these three explicit cases. As detailed in the next sections, both the Kitaev ladder and the Kitaev tie can be obtained, respectively, from two coupled Kitaev chains with superconducting and hopping terms. For a Kitaev chain of length $L$, the Kitaev tie is obtained by adding a long-range hopping term coupling a site at position $d$ with the symmetric one at the position $L - d + 1$. The basic tripartition and quadripartition are reported, respectively, in panels (d) and (e). In both panels the edges are denoted by $A$ and $B$ (red color); in panel (d) the total bulk $C = C_1C_2$ is reported in blue; in panel (e) the bulk portions $C_1$ and $C_2$ are reported in blue and in green, respectively.

The diamonds are generic and can correspond to a single fermionic site when referred to a chain or a tie and to two fermionic sites when referred to a ladder. In the quadripartition, when not otherwise specified, the length $L_{C_1}$ of the bulk portion $C_1$ is fixed at $L_{C_1} = 1$.

FIG. 2. Schematic of a Kitaev chain (a), a two-leg Kitaev ladder (b) and a Kitaev tie (c). The Kitaev ladder is obtained coupling two Kitaev chains by superconducting and hopping terms. A Kitaev chain of length $L$, the Kitaev tie is obtained by adding a long-range hopping term coupling a site at position $d$ with the symmetric one at the position $L - d + 1$. The basic tripartition and quadripartition are reported, respectively, in panels (d) and (e). In both panels the edges are denoted by $A$ and $B$ (red color); in panel (d) the total bulk $C = C_1C_2$ is reported in blue; in panel (e) the bulk portions $C_1$ and $C_2$ are reported in blue and in green, respectively. The diamonds are generic and can correspond to a single fermionic site when referred to a chain or a tie and to two fermionic sites when referred to a ladder. In the quadripartition, when not otherwise specified, the length $L_{C_1}$ of the bulk portion $C_1$ is fixed at $L_{C_1} = 1$. 

Moreover, as explained later on (see Fig. 5, panel (a)),
our results will be independent of the relative size of the two sub-bulks, so that without loss of generality we will take \( L_{C_1} = 1 \) throughout, as reported in panel (e) of Fig. 2.

A crucial step in order to compute the TSE consists in diagonalizing the reduced density matrices of the various subsystems, once the many body ground state density matrix \( \rho = |\Psi\rangle_G \langle \Psi| \) is assigned.

When interactions are neglected, the Kitaev-type Hamiltonians are quadratic in the fermionic degrees of freedoms and can be diagonalized by a Bogoliubov transformation. In this case, an appropriate approach to compute the von Neumann entropies of the subsystems has been introduced by Peschel [71].

The Bogoliubov transformation ensures a direct access to fermionic correlations on the many body ground state \( |\Psi\rangle_G \) and the reduced density matrix of a given subsystem \( \alpha \) can be recast in the form of a thermal density matrix, \( \rho_\alpha = (1/Z)e^{-H_\alpha} \), where \( H_\alpha = \sum_{l=1}^{L_\alpha} \epsilon l \hat{T}_l \) is the effective Hamiltonian of the reduced system [72–75], and the constant factor \( Z \) (the "partition function") ensures the correct normalization \( Tr(\rho_\alpha) = 1 \).

Given the lattice fermionic creation and annihilation operators \( \{ c^\dagger_l, c^\dagger_j \} \), the spectrum of the entanglement Hamiltonian can be evaluated numerically by solving the eigenvalue problem:

\[
(2T - 1 - 2D)(2T - 1 + 2D)\phi_l = \tanh^2 \left( \frac{\epsilon_l}{2} \right) \phi_l,
\]

where \( T_{ij} = \langle \Psi | c^\dagger_i c^\dagger_j | \Psi \rangle \) and \( D_{ij} = \langle \Psi | c^\dagger_i c^\dagger_j c^\dagger_j c^\dagger_i | \Psi \rangle \) are the matrix elements of the fermionic two-point correlations in a quantum state \( |\Psi\rangle \) (for instance, the ground state \( |\Psi\rangle_G \)), and 1 is the identity matrix. The index \( l \) runs over all the lattice sites belonging to subsystem \( \alpha \).

Due to the form of \( \rho_\alpha \), the von Neumann entropy of the reduced state \( S(\rho_\alpha) \) can be obtained in terms of the eigenvalues of entanglement Hamiltonian:

\[
S(\rho_\alpha) = \sum_l \ln(1 + e^{-\epsilon_l}) + \sum_l \frac{\epsilon_l}{e^{\epsilon_l} + 1}.
\]

The Peschel algorithm for free fermionic systems reduces the computational efforts and allows to access the entanglement properties for systems of very large sizes.

When interactions are included in the models, the mapping to the effective entanglement Hamiltonian and the Gaussian thermal structure of the reduced density matrix are no longer applicable and one must resort to numerical techniques, e.g. the density matrix renormalization group (DMRG) method, particularly well suited to treat one-dimensional systems [76–78].

Complementary to numerical methods, one can resort to direct numerical matrix diagonalization in the parameter region that allows for exact analytic expressions of the state vectors, such as along the exact factorization lines of interacting spin models and Kitaev–type fermionic systems [79–82], as we will discuss in the following.

IV. TOPOLOGICAL SQUASHED ENTANGLEMENT OF THE KITAEV CHAIN

A. The order parameter

Here and in the following we investigate the paradigmatic case of the fermionic Kitaev chain [39], considering first the non-interacting model:

\[
H_K = \sum_{j=1}^{L} -\mu c^\dagger_j c^\dagger_j + \sum_{j=1}^{L-1} (\Delta c^\dagger_{j+1} c^\dagger_j - tc^\dagger_j c^\dagger_{j+1} + h.c.).
\]

The Hamiltonian \( H_K \) in Eq. 8 describes a system of spinless fermions confined to a one dimensional lattice of length \( L \) with on site creation and annihilation operators \( c^\dagger_j, c_j (j = 1, \ldots, L) \), subject to a p-wave superconducting coupling. The coefficients \( \Delta, t \) and \( \mu \) are respectively the strength of the superconducting pairing, the nearest-neighbor hopping amplitude and the on-site energy offset.

This model features a topological phase for \( \mu < 2t \) and \( \Delta \neq 0 \) with two robust Majorana zero-energy modes localized at the two edges. The energy \( E_M \) of such MZEms scales exponentially to zero with the system’s size and is expected to vanish asymptotically in the thermodynamic limit.

In the limit of small on site energy offset (chemical potential), precisely at the analytically solvable point \( \mu = 0 \) and \( t = \Delta \), the model features exact topological two-fold ground state degeneracy and two MZEms with exactly \( E_M = 0 \), independently of the chain length. The remaining non-topological modes of the spectrum are gapped and form a band. In the opposite regime of large on site energy offset \( \mu > 2t \) the Kitaev wire is a band insulator.

By construction the Hamiltonian satisfies the particle-hole symmetry and belongs to the class \( D \) of the ten-fold classification [83], with topological invariant given by the Pfaffian invariant [64].

For the non-interacting Kitaev chain, we have computed the two QCMIs \( I(3) \) and \( I(4) \) and verified that \( I(3) = I(4) \) throughout the entire phase diagram, so that in the reminder of the present subsection we will focus only on \( I(4) \). The physical origin of the two QCMIs’ equality in the topological case will become clear in the following.

In panel (a) of Fig. 3 we report the phase diagram obtained by means of the ratio between the QCFM \( I(4) \) and the constant quantity \( E_M^0 = \log(2)/2 \). One finds that throughout the entire topological phase the ratio \( I(4)/E_M^0 = 1 \) so that \( I(4) \) is quantized exactly at the value \( I(4) = E_M^0 = \log(2)/2 \). Moreover, one has that \( I(4) = 0 \) throughout the entire trivial phase.

Finite size effects are clearly visible near the phase transition line at \( \mu = 2t \), as also shown in panel (b) of Fig. 3, where the ratio \( I(4)/E_M^0 \) is plotted as a function of the length \( L \) of the chain for \( t = \Delta = 1 \) and three different values of the chemical potential \( \mu \), respectively close to the exact topological point, in the topological phase close to the critical point, and well into the trivial phase.
In particular, \(I(4)\) scales exponentially to the quantized value \(E^0_{sq}\) inside the topological phase (red curve) and it coincides with \(E^0_{sq}\) independently of the size of the chain for values of \(\mu\) sufficiently close to the analytic point (blue curve). The QCMI identically vanishes at any value of \(\mu\) in the trivial phase (green curve).

Finally, in panel (c) of Fig. 3 we report the behavior of the QCMI as a function of \(\mu\) for different values of the length of the chain \(L\), showing the correct approach to quantization inside the topological phase and the correct scaling behavior approaching the critical point. These results show that the QCMI \(I(4)\) detects and characterizes the non-trivial regime of the Kitaev chain.

In fact, by applying the lower-bound inequalities holding on SE [29, 36–38], it turns out that the constant value \(E^0_{sq} = \log(2)/2\) is a lower bound to the true SE \(E_{sq}(\rho_{AB})\) between the edges at the exact topological point \(\mu = 0\). By continuity, the bound extends to neighboring values of \(\mu\). Since the upper bound \(I(4)\) and the lower bound \(E^0_{sq}\) coincide in this region, both coincide with the true long-distance SE \(E_{sq}(\rho_{AB})\) between edges \(A\) and \(B\).

In conclusion, collecting all of the above results, the quantized TSE

\[
E_{sq}(\rho_{AB}) = E^0_{sq} = \frac{\log(2)}{2}
\]

is the order parameter for the the Kitaev model of topological superconductivity.
The physical origin of a quantized non-vanishing long-distance entanglement between the system edges in the topological phase arises from the interplay between the exponential bulk-edge separation due to the topological nature of the system and the fundamental monogamy property of nonlocal quantum correlations [27].

As the two edges progressively decouple from the bulk their mutual information and correlation are enhanced by the monogamy constraint on the shared information and on the amount of shareable entanglement among the different subsystems. When the system moves away from the topological phase, correlations become once again short-ranged and the long-distance entanglement between the two extremes of the chain collapses.

The quantized limit of the edge-edge TSE is \( \log(2)/2 \) rather than the maximal Bell-pair entanglement \( \log(2) \) because of the non-vanishing entanglement pattern existing between the internal degrees of freedom inside each edge.

**B. SE and TSE: discriminating between symmetry breaking and topological order**

A core question concerns the ability of a proposed order parameter to discriminate systems with topological order from systems with spontaneous symmetry breaking and Ginzburg-Landau type of order, addressing their different entanglement properties and patterns.

We focus our attention on the Ising spin-1/2 chain and the fermionic Kitaev chain. Setting \( t = \Delta = 1 \) and applying the global Jordan-Wigner transformations [63], the Hamiltonian in Eq. (8) can be mapped into that of the 1D Ising model in transverse field:

\[
H_I = J \sum_{j=1}^{L-1} \sigma_j^x \sigma_{j+1}^x + h \sum_{j=1}^{L} \sigma_j^z, \tag{10}
\]

with \( J = -2t \) and \( h = \mu/2 \). Both models share a two-fold degenerate ground state and the parity symmetry \( \mathbb{Z}_2 \), but the physics of the order displayed is radically different. While in the Ising model the \( \mathbb{Z}_2 \) spin reflection symmetry is spontaneously broken with the appearance of a nonvanishing local order parameter, the topological degeneracy in the Kitaev chain is realized with perfect conservation of the Hamiltonian symmetry and no local order parameters.

Despite the different type of order that they support, the two models share the same bipartite entanglement spectrum and bipartite von Neumann block entanglement entropy. This is not surprising, since the nonlocal Jordan-Wigner transformation (a particular case of the general Klein transformation in quantum field theory) does not affect the short-distance entanglement structure across the separation between the system’s halves for Hamiltonians sums of local (nearest-neighbor or short-ranged) interaction terms.

The landscape is quite different when dealing with long-distance entanglement structures between the system’s boundaries: clearly, they are expected to be significantly affected under the action of global transformations on the entire system.

In panels (a) and (b) of Fig. 4 we report the behavior of the end-to-end QCMI \( I(3) \) for the Ising and the Kitaev chains for two lengths \( L = 6 \) and \( L = 9 \). The two curves intersect at the critical points \( h = 1 \) and \( \mu = 2 \). In the thermodynamic limit, the following behaviors are expected: \( I(3)_{\text{Ising}}/E_{sq,\text{Ising}}^0 = 1 \) for \( 0 < h < 1 \) (\( 0 < \mu < 2 \)) and \( I(3)_{\text{Kitaev}}/E_{sq,\text{Kitaev}}^0 = 0 \) for \( h > 1 \) (\( \mu > 2 \)). This analysis shows that the QCMI \( I(3) \) detects and characterizes both topological and symmetry breaking orders but it is not capable of distinguishing between them.

In panels (c) and (d) of Fig. 4 we report the behavior of the end-to-end QCMI \( I(4) \) for the Ising and Kitaev chains; we see that \( I(4) \) distinguishes topological order from ordered phases with spontaneously broken symmetries.

For the Ising chain, we see that removing any finite part \( C_2 \) of the bulk \( C = C_1C_2 \) before performing the partial trace with respect to the reminder \( C_1 \) returns a strongly suppressed end-to-end QCMI \( I(4) \) that is expected to vanish in the thermodynamic limit.

At variance with the symmetry-breaking case, the topology of the Kitaev chain is essentially concentrated on the boundaries, so that in the computation of the end-to-end QCMIIs performing the partial trace with respect to the entire bulk \( C \) is entirely equivalent to removing a part of the bulk \( (C_2) \) and then performing the partial trace with respect to the remaining part of the bulk \( (C_1) \). As a consequence, \( I(3) = I(4) \) for a topological system, while in general \( I(3) \neq I(4) \) for a system with symmetry-breaking order.

Ultimately, these results depend on and highlight the different global properties of the two systems. Indeed, the bulk of the Kitaev chain is insulating and exponentially decoupled from the boundaries, forcing (by monogamy) the onset of the long-distance correlation between the ends of the chain.

Viceversa, the bulk of the Ising chain is conductive, so that cutting one part of it before performing the partial trace on the reminder abruptly interrupts the information flow and destroys the correlation structure between the two ends of the chain. These physical differences determine the different structure of the end-to-end QCMIIs and of the end-to-end SE.

Concerning quadripartitions, an interesting, although not surprising, feature is the independence of the results on the relative sizes of the two parts of the bulk \( C_2 \) (the cut) and \( C_1 \) (the reminder), so that one can always adopt the most convenient choice \( C_1 = 1 \) in the actual calculations.

In Fig. 5 panel (a) we report the behavior of the difference \( I(4) - E_{sq}^{qu} \) as a function of \( L_C \), with \( \mu = 0.5 \), \( t = 1 \) and \( L = 60 \), and for \( \Delta = t \) (green curve) and \( \Delta = 0.1t \).
FIG. 4. Ratios of the end-to-end QCMIs $I(3)$ and $I(4)$ to the TSE $E_{SQ}^0 = \log(2)/2$ for the Ising chain (panels (a) and (c)) as functions of the transverse field $h$, and for the Kitaev chain (panels (b) and (d)) as functions of the chemical potential $\mu$. While for the Kitaev chain the equality $I(3) = I(4)$ always holds, for the Ising chain $I(3) \neq I(4)$ and $I(4)$ tends to vanish as the system size increases. The QMI $I(4)$ discriminates between topological and symmetry-breaking systems. Throughout, the Hamiltonian parameters are $t = \Delta = 1$ for the Kitaev chain and $J = -1$ for the Ising chain.

(red curve). We see that $I(4) - E_{SQ}^0$ is constant in both cases, regardless of the length $L_{C_1}$.

In the balanced hopping-pairing case $t = \Delta$, the SE is fully quantized at the fundamental unit $E_{SQ}^0 = \log(2)/2$ already for an intermediate chain length of $L = 60$ (green curve), while in the strongly unbalanced case $\Delta = 0.1t$ one can observe (red curve) a small finite-size effect on $I(4) - E_{SQ}^0$.

These behaviors show that the balanced case $t = \Delta$ is way more insensitive to finite size effects than the unbalanced case $t \neq \Delta$. On the other hand, as the inset in panel (a) of Fig. 5 shows, for the unbalanced case $t \neq \Delta$ the difference $I(4) - E_{SQ}^0$ decreases with increasing system’s sizes, tending to vanish, as expected, in the thermodynamic limit.

The different robustness to finite size effects between the balanced and unbalanced configurations is illustrated in panels (b) and (c) of Fig. 5, where we draw sketches of the Kitaev chain in the basis of Majorana fermions $a_j = c_j + c_j^\dagger$ and $b_j = -i(c_j - c_j^\dagger)$ for the two aforementioned cases.

The balanced setting $t = \Delta$ illustrated in panel (c) results in turning off the coupling $(t - \Delta) a_j b_{j+1}$ between third-neighboring Majorana fermions. In this configuration the Majorana fermions are thus more weakly coupled than in the unbalanced case $t \neq \Delta$ illustrated in panel (b).

On the other hand, the setting $t = \Delta$ and $\mu = 0$ corresponds to the exact analytic, topological degeneracy point of the model, where the edge modes $a_1$ and $b_L$ decouple from the lattice and become exact zero energy modes.

C. TSE in the presence of interactions

The interacting Kitaev chain is obtained by adding a nearest-neighbor density-density interaction term to the free Kitaev Hamiltonian:

$$H_{IK} = H_K + U \sum_{j=1}^{L-1} n_j n_{j+1},$$

where $U$ is the interaction coupling constant and $n_j = c_j^\dagger c_j$ is the on site fermion number operator. The intervals $U > 0$ and $U < 0$ correspond, respectively, to a repulsive and to an attractive interaction. The phase diagram of the model has been investigated in depth by a variety of
FIG. 5. Panel (a): Behavior of the QCMI $I_{\text{all}}$ as a function of the size $L_{c_1}$ of the reminder of the bulk in a Kitaev chain of $L = 60$ sites. We have considered both the balanced case $t = \Delta = 1$ (green curve) and the unbalanced case $t = 1$, $\Delta = 0.1$ (red curve). Different choices of $L_{c_1}$ have no effect on the evaluation of $I_{\text{all}}$. The inset shows the scaling of $I_{\text{all}}$ towards the quantized value $E_{\text{SO}}^{\text{3D}} = \log(2)/2$ of the TSE in the balanced regime as the size $L$ of the chain is increased. Panel (b): Schematic representation of the chain in the Majorana basis $\{a_j, b_j\}$ for $t \neq \Delta$. Panel (c): Same, with $t = \Delta$.

numerical and approximate analytical methods [59, 82, 84, 85] and is reported schematically in Fig. 6.

We see that, although very strong repulsive interactions eventually force the system into a Mott insulating phase, a topological superconducting state is established in between the trivial band insulator phase and the Mott localization even for fairly strong repulsive interactions. The black dashed line inside the topological phase is the factorization line of equation $\mu = \mu^*$ described in the following.

The interacting Kitaev chain can be mapped exactly onto the interacting $XYZ$ spin-$1/2$ model in an external magnetic field

$$H_{xyz} = \sum_{j=1}^{L-1} [J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z] + \mu \sum_{j=1}^{L} \sigma_j^z,$$

where $J_x = - (\Delta + t)/2$, $J_y = (\Delta - t)/2$, $J_z = -U$, and $\mu = \mu^*/2$.

The model admits exact ground state solutions factorized in the product of single-spin (single-site) wave functions along the so-called factorization line $h = h^* = z_d \sqrt{(J_x + J_z)(J_y + J_z)}$ in any dimension [79, 80, 86], with $z_d$ the coordination number (so that, e.g., $z_d = 2$ in the 1D case).

For the Kitaev chain, this corresponds to the exact solution for the many-body ground states along the factorization line [82]:

$$\mu = \mu^* = 4\sqrt{U^2 + t U + (t^2 - \Delta^2)/4},$$

which reduces to $\mu^* = 4\sqrt{U(U+t)}$ for $t = \Delta$.

In general, along any factorization line (provided it exists) an interacting model Hamiltonian $H$ becomes frustration–free and can be written as the sum of commuting local terms $H_j$: $H = \sum_j H_j$ with $[H_i, H_j] = 0$ [81].

The exact ground states of definite symmetry of the interacting Kitaev chain for $\mu = \mu^*$ are the following entangled linear combinations [82]:

$$|\Psi^{\text{even}}\rangle = \frac{1}{\sqrt{2}} (|\Psi^+\rangle + |\Psi^-\rangle),$$

$$|\Psi^{\text{odd}}\rangle = \frac{1}{\sqrt{2}} (|\Psi^+\rangle - |\Psi^-\rangle),$$

where the non-orthogonal states $|\Psi^+\rangle$ and $|\Psi^-\rangle$ are fully factorized in the product of single-site wave functions and read [82]:

$$|\Psi^{\pm}\rangle = \frac{1}{(\alpha + 1)^{L/2}} e^{\pm \alpha c_i^\dagger} \ldots e^{\pm \alpha c_L^\dagger} |\text{vac}\rangle,$$

where

$$\alpha = \sqrt{\cot(\theta^*/2)}, \quad \theta^* = \arctan(2\Delta/\mu^*),$$

with $|\text{vac}\rangle$ denoting the lattice vacuum state.

The ground states $|\Psi^{\text{even}}\rangle$ and $|\Psi^{\text{odd}}\rangle$ are orthogonal and are the only possible ground state of $H_{IK}$ for $\mu = \mu^*$.
Ground state factorization holds in the topological phase with a repulsive interaction $U \geq 0$, and the even and odd ground states are degenerate with energy $E_0 = -(L - 1)(U + t)$.

The behavior of the QCMI $I_{(4)}^*$ normalized to the quantized value of the SE $E_{sq}^0 = \log(2)/2$ along the Illuminati-Katsura factorization line $\mu = \mu^*$ is reported in Fig. 7 panel (a) as a function of the chain length $L$ for different values of the interaction strength, and in panel (b) as a function of the interaction strength for different values of the chain length $L$.

The QCMI $I_{(4)}^*$ exhibits the correct scaling, saturating to the quantized value $\log(2)/2$ of the SE for sufficiently large values of the chain size $L$, and vanishing asymptotically for very large values of the interaction strength $U$ as the factorization line pinches the boundary of the topological phase and the system enters in the trivial phase.

In conclusion, we see that the TSE $E_{sq}^0$ identifies the correct quantized order parameter for the topological phase of the fermionic Kitaev chain, either free or interacting.

D. Robustness of TSE against disorder

Topological phases of the Kitaev chain are immune to the detrimental effects of disorder and local perturbations [53, 87–89]. Such immunity is a fundamental property that is expected to hold whenever topological order is involved. This observation suggests that $I_{(4)}$ should also be as immune to disorder as the other topological signatures previously defined in terms of the spectral properties [90] or related to the transport properties [91] of the systems involved.

We study two distinct classes of configurations, each characterized by a different source of disorder with a clear tracking of its physical origin. Since one of the way to realize a topological Kitaev chain is by proximity effect between a semiconducting nanowire and a conventional superconductor [40, 41], we consider two distinct sources of disorder: random site-dependent hopping amplitudes $t_i$ and random site-dependent pairing potentials $\Delta_i$. The former models an effective mass gradient and random doping along the nanowire, originating from the growth process of the nanowire itself, while the latter emulates unwanted spatial variations along the wire which can affect the nanowire-superconducting coupling and therefore the induced superconducting gap.

In Fig. 8 we report the behavior of the QCMI to TSE ratio $I_{(4)}^*/E_{sq}^0$ of the Kitaev chain Eq. 8 as a function of the chemical potential $\mu$ for the aforementioned two different sources of disorder. In panels (a) and (b) we consider respectively random hopping integrals $t_i = t + \tau_i^{dis}$ and random pairing potentials $\Delta_i = \Delta + \delta_i^{dis}$ with the random strengths of disorder $\tau_i^{dis}$ and $\delta_i^{dis}$ uniformly distributed in the intervals $(-\tau, \tau)$ and $(-\delta, \delta)$.

In Fig. 8, for the sake of completeness disorder effects are investigated spanning the spectrum of noise strengths from the perturbative regime up to the strongly disordered configurations. It is worth remarking that the maximum noise amplitudes considered correspond to extreme levels of disorder, which are not even expected in realistic experimental conditions.

Fig. 8 shows that within the typical regimes of weak to moderate disorder the TSE order parameter is strongly resilient up to the transition point, as it should be expected.

TSE and topological order are completely destroyed for very strong levels of noise on the hopping integrals (see panel (a)), while they are preserved in all regimes of disorder on the pairing amplitudes (see panel (b)).
is the simplest model which takes into account multi-mode topological phases, which usually arise in models of significant complexity, often involving two-dimensional platforms [95–97].

The Hamiltonian $H_{KL}$ of the two-leg Kitaev ladder can be expressed as follows:

$$H_{KL} = H_{K_1} + H_{K_2} + H_{K_{12}},$$

(18)

where $H_{K_j}$ $(j = 1, 2)$ is the Hamiltonian of a single Kitaev chain and

$$H_{K_{12}} = \sum_{i=1}^{L} [-t_1 c_{i,1}^\dagger c_{i,2} + \Delta_1 c_{i,1} c_{i,2} + h.c.]$$

(19)

describes the coupling between the two chains by means of a transverse hopping with amplitude $t_1$ and a transverse pairing with amplitude $\Delta_1$. The model has been introduced and discussed at length in Ref. [50]. The Kitaev ladder satisfies particle-hole, time reversal and chiral symmetry, belonging to the BDI class of the ten-fold classification.

The phase diagram of the model can be investigated by means of the winding number:

$$W = Tr \int_{-\pi}^{\pi} \frac{dk}{2\pi i} A_k^{-1} \partial_k A_k =$$

$$- \int_{-\pi}^{\pi} \frac{dk}{2\pi i} \partial_k \ln Det A_k$$

(20)

where $A_k$ can be expressed in terms of the model parameters once periodic boundary conditions have been imposed and the bulk-edge correspondence is invoked:

$$A_k = \begin{pmatrix}
\epsilon_k - \Delta_k & -t_1 - \Delta_1 \\
-t_1 + \Delta_1 & \epsilon_k - \Delta_k
\end{pmatrix},$$

(21)

where $\epsilon_k = 2t \cos k + \mu$, $\Delta_k = 2i \Delta \sin k$ and $k \in [-\pi, \pi]$.

In panels (a), (c), and (b) of Fig. 9 we report the phase diagrams of the ladder obtained respectively by means of the winding number (panels (a) and (c)) and by means of the QCMI $I_{(4)}$ normalized by the quantized edge-edge TSE $E_{sq}^0$ (panel (b)). The diagram in panel (b) is obtained for a ladder of $L = 50$ sites per chain, with $L_e = 12$ and via interpolation on a grid of $25 \times 25$ points.

The digits drawn on the phase diagrams in panels (a) and (c) count the number of Majorana zero energy modes per ladder edge. Indeed, the system possesses a richer phase diagram compared to the one of the single Kitaev chain. Denoting by $N_m$ the number of Majorana zero energy modes per edge, the two-leg Kitaev ladder features two alternating topological phases, respectively endowed with $N_m = 1$ and $N_m = 2$, and a trivial phase with no edge modes.

Despite the fact that the phase diagram in panels (a) and (c) is defined in terms of bulk properties, while the phase diagram in panel (b) is obtained in terms of the

![Image](79x613.png)

FIG. 8. The QCMI to TSE ratio $I_{(4)}/E_{sq}^0$ as a function of the chemical potential $\mu$ for a chain of length $L = 70$ and two different types of disorder. Panel (a): behavior of $I_{(4)}/E_{sq}^0$ in the presence of random hopping amplitudes $t_i$. Panel (b): behavior of $I_{(4)}/E_{sq}^0$ in the presence of random pairing potentials $\Delta_i$. The random realizations are generated by a continuous probability distribution defined in the interval $t_i \in (t - \tau, t + \tau)$, $\Delta_i \in (\Delta - \delta, \Delta + \delta)$. The reference values of the parameters have been fixed as $t = 1$, $\Delta = 0.1$. The different curves are parameterized by disorder of increasing strength.

showing that noise on the hopping is more effective in perturbing the topological phase. In particular, in order to even allow a superconducting phase the maximum $\delta$ can never exceed $\Delta$.

In conclusion, we have found that realistic values of the noise strengths do not perturb TSE significantly, thus indicating that the latter enjoys resilience against disorder, a crucially necessary feature of any candidate topological order parameter.

V. SYSTEMS WITH MULTIPLE EDGE MODES: TSE OF THE KITAEV LADDER

In this section we generalize the previous investigations to consider systems enjoying multiple topological phases and multiple MZEMs, i.e. more than one Majorana zero mode per edge.

Various such generalizations of the 1D Kitaev model have been introduced [46–49, 92, 93], but the simplest instance can be realized by coupling multiple Kitaev chains with transverse hopping and pairing terms to form $n$-legs Kitaev ladders [50, 94]. The two-leg Kitaev ladder
FIG. 9. Panels (a) and (c): phase diagram of the two-leg Kitaev ladder determined by means of the winding number in $k$-space. Panel (a): phase diagram with parameters fixed at $t = 1$, $\Delta = 0.8$, $\Delta_1 = 0.8$. Panel (c): phase diagram with parameters fixed at $t = 1$, $\Delta = 0.8$, $\Delta_1 = 0.09$. Panel (b): the same phase diagram determined by means of the QCMI to TSE ratio $I_{(4)}/E_{sq}^0$, obtained on a grid of $25 \times 25$ points with the order of the interpolating polynomial equal to 1. Panels (d) and (e): QCMI to TSE ratio $I_{(4)}/E_{sq}^0$, reported respectively along the horizontal and the vertical red cuts of panel (c). The single chain and edge lengths are fixed, respectively, at $L = 50$ and $L_e = 12$.

In panels (d) and (e) we report the behavior of $I_{(4)}/E_{sq}^0$ along the horizontal and vertical red cuts of panel (c). In both cases, we observe the three plateaux $I_{(4)}/E_{sq}^0 = 2, 1, 0$ corresponding, respectively, to the two topological phases and to the trivial phase. Remarkably, TSE not only discriminates topological phases from trivial ones but also distinguishes different topological phases by counting the corresponding number of Majorana edge modes $N_m$:

$$\frac{I_{(4)}}{E_{sq}^0} = N_m.$$  \hspace{1cm} (22)

If the above relation has general validity, then TSE can apply to systems belonging to arbitrary topological classes. Besides its conceptual importance, this property bears a clear practical advantage, since while the definition of the TSE is unique and system-independent, different invariants have to be defined each time, following the ten-fold classification, depending on the symmetries obeyed by each specific system under investigation.

VI. SYSTEMS WITH SUPPRESSED BULK EDGE CORRESPONDENCE: TSE OF THE KITAEV TIE

In this section we study by means of the TSE the effects of geometrical frustration induced by adding an hopping term of arbitrary spatial range to a Kitaev chain. The model has been introduced and investigated in Refs. [51, 65]. The tight-binding Hamiltonian of the model can be written as

$$H_T = H_K + H_d,$$ \hspace{1cm} (23)

where $H_K$ denotes the Kitaev chain Hamiltonian and

$$H_d = -t_d(c_d^\dagger c_{L-d+1} + h.c.)$$ \hspace{1cm} (24)

is an extra long-range hopping term, connecting two symmetrical sites of the chain $d$ and $L - d + 1$ with an hopping amplitude $t_d$. The long-range hopping, identified by the parameter $d$, can vary along the length of the chain, playing the role of a movable knot which induces a geometric frustration on the original chain Hamiltonian. This results in a legged-ring system with no clearly identifiable bulk, and consequently a suppressed bulk-edge correspondence, referred to as a Kitaev tie.

The phase diagram of the system in the $\mu$-$d$ plane shows an interstitial character with topological phases nucleating inside trivial regions as the knot position $d$ is varied while keeping $\mu < 2t$. This phenomenon is referred to as topological frustration.

For small values of $d$, the interstitial character of the non-trivial phases is more evident, since the system resembles a ring with very short legs. On the other hand, for large values of $d$ there is a significant growth of the topological phase domains, since the ring is reduced and the system approaches the limiting regime of a perturbed Kitaev chain.
The Kitaev tie is realizable in single-walled carbon nanotubes [51, 65], flexible ballistic conductors [98] where superconducting proximity effect can be easily implemented [99–101].

In Fig. 10, we report the behavior of the QCMI $I_{(4)}$ as a function of $\mu$ for a Kitaev tie of $L = 121$ sites (blue curves) and we compare it with the behavior of the spectral energy function of the system (orange curves). Specifically, denoting by $I_{\text{max}}$ the maximum value achieved by the QCMI $I_{(4)}$ and by $E$ the lowest energy eigenvalue, in panel (a) we compare $E$ and the complement $1 - I_{(4)}/I_{\text{max}}$ of the QCMI normalized to its maximum value as functions of $\mu$, while in panel (b) we compare $E$ and the complement of the QCMI to TSE ratio $1 - I_{(4)}/E^0_{\text{sq}}$. We consider two different positions of the long-range hopping, respectively $d = 30$ (panel (a)) and $d = 50$ (panel (b)).

The presence of unpaired Majorana modes is related, respectively, to minima or vanishing values of the energy eigenvalue and to the minima of the TSE-related estimators. All plots clearly show signatures of the topological frustration, pinpointed by an oscillating behavior of the various quantities as functions of the chemical potential. For systems with an associated bulk, the QCMI $I_{(4)}$ converges to the quantized TSE $E_{\text{sq}}^0$ already for moderate lattice sizes, as shown in the discussion of the Kitaev chain and of the Kitaev ladder. For the Kitaev tie, the absence of an associated bulk is signaled by the fact that the complement of the QCMI to TSE ratio does not vanish, reaching instead non-zero minima, even for lattices of relatively large sizes ($L = 121$).

On the other hand, the perfect correspondence of the positions of both the minima and the maxima of the different functions shows that the TSE captures the same physics as the energy. Although the energy eigenvalue provides a stronger marker of the presence of MZEMs, remarkably the QCMI $I_{(4)}$ and the TSE $E_{\text{sq}}^0$ are reliable quantifiers of the emerging topological features even for systems, like the Kitaev tie, that do not allow for a clear identification of the bulk geometry.

Fig. 10 shows that the actual value of the absolute maximum $I_{\text{max}}$ reached by the QCMI $I_{(4)}$ strongly depends on whether the scope $d$ of the long-range hopping falls inside the edge extension, i.e. $d < L_e$, as in panel (a), or outside the edge extension, i.e. $d > L_e$, as in panel (b). Interestingly, $I_{(4)}$ exceeds $E_{\text{sq}}^0 (I_{\text{max}} \approx 0.62)$ when $d < L_e$, while when $d > L_e$, the QCMI converges to the TSE $E_{\text{sq}}^0$, as in the cases of the Kitaev chains and ladders. In the case of the Kitaev tie the inapplicability of the bulk-edge correspondence is responsible for the partial loss of quantization of the QCMI.

VII. DISCUSSION AND OUTLOOK

In the present work we have discussed the application of squashed entanglement to the study of topological quantum matter. Squashed entanglement is a measure of bipartite entanglement defined on multipartitions that generalizes the von Neumann entanglement entropy; it allows to quantify the bipartite entanglement between any two subsystems in any quantum state, either pure or mixed, and reduces to the von Neumann entanglement entropy on pure states. Introducing generic tripartitions and quadripartitions of a quantum system, we have identified two general classes of upper bounds on the bipartite squashed entanglement in terms of quantum conditional mutual information between subsystems.

For models of topological superconductors that admit Majorana zero energy modes at the system edges, we have shown that in the topological phase the upper bound associated to the system quadripartition in fact coincides with the squashed entanglement between the system edges in the ground state at the exact topological point. The squashed entanglement in the topological phase is quantized to $\log(2)/2$, half the value of the maximal Bell entanglement, counting the Majorana splitting...
of the Dirac fermions.

The long-distance topological squashed entanglement between the edges is constant throughout the topological phase, exhibits the correct scaling when approaching the critical point, and vanishes identically in the trivial phase. Moreover, it is robust in the presence of interactions, resilient to the effects of disorder and local perturbations, and discriminates topological order from orders associated to spontaneous symmetry breaking. It thus realizes the desired, and long-sought for, entanglement-based order parameter for symmetry-protected topological superconductivity.

For systems defined on geometries of higher complexity with respect to the 1D Kitaev chain, featuring multiple topological phases and more than one Majorana zero energy mode per edge, like e.g. the Kitaev ladder, topological squashed entanglement distinguishes between the multiple phases by plateaus that count the number of Majorana edge modes. For systems with weakened bulk-edge correspondence, like e.g. the Kitaev tie, topological squashed entanglement, though ceasing to be perfectly quantized, is anyway able to identify the interstitial topological phases and discriminate them from the trivial ones.

Given that on pure quantum states, e.g. ground states of many-body systems, the squashed entanglement reduces exactly to the von Neumann block entanglement entropy defined on bipartitions of a quantum system into two blocks, the subleading contribution to the block entropy, the block topological squashed entanglement, trivially coincides with the topological entropy in the case of true topological order in two-dimensional systems. Therefore squashed entanglement provides the general framework that unifies all classes of topological order in any dimension.

Concerning the experimental accessibility of topological squashed entanglement, the problem boils down to that of measuring quantum entropies of a set of reduced states in quantum many-body systems. A recent proposal relies on the thermodynamic study of the entanglement Hamiltonian for the direct experimental probing of von Neumann entropies via quantum quenches [102]. Another possibility, specific for systems featuring topological order, consists in identifying minimum entropy states and then experimentally simulating the behavior of the associated von Neumann entropies via the classical microwave analogs of such states. In this way it is in principle possible to simulate various non-trivial instances of reduced entropies and topological order, as shown explicitly for the transition from a trivial phase to a $Z_2$-symmetric topological phase [103]. Since highly informative bounds on von Neumann entropies, quantum conditional mutual information, and squashed entanglement can be constructed in terms of Rényi entropies [37, 38], a further possibility is to adapt to fermionic systems [104] the schemes previously proposed for the experimental measurement of Rényi entropies in bosonic and spin systems [105–107] and the corresponding experimental techniques that lead to the first experimental measurement of the 2-Rényi entropy in a many-body system [108].

The generality of the concept of squashed entanglement between subsystems on multipartitions allows for several possible future research directions. One line of investigation of current interest is the application of squashed entanglement to the study of topological quantum matter at finite temperature and out of equilibrium [109]. In fact, even for non topological matter it would be interesting to study finite temperature quantum criticality resorting to squashed entanglement and compare it systematically with other measures of nonclassicality such as, e.g., the entropic discord [110] or Bell nonlocality [111]. In this perspective, hierarchies of quantum complexity according to different layers of quantumness, ranging from discord and coherence to entanglement and nonlocality, could realize the scaffold of a unified research framework applicable to a wide spectrum of problems, ranging from quantum information and quantum matter to elementary particle physics and quantum gravity.

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