HOM-LIE-YAMAGUTI SUPERALGEBRAS

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Abstract. (Multiplicative) Hom-Lie-Yamaguti superalgebras which generalize Hom-Lie supertriple systems (and subsequently ternary multiplicative Hom-Nambu superalgebras) and Hom-Lie superalgebras in the same way as Lie-Yamaguti superalgebras [12] generalize Lie supertriple systems and Lie superalgebras are defined. We show that the category of (multiplicative) Hom-Lie-Yamaguti superalgebras is closed under twisting by self-morphisms. Construction of some examples of Hom-Lie-Yamaguti superalgebras is given. The notion of an n th -derived (binary) Hom-superalgebras is extended to the one of an n th -derived binary-ternary Hom-superalgebra and it is shown that the category of Hom-Lie-Yamaguti superalgebras is closed under the process of taking n th -derived Hom-superalgebras.

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1. INTRODUCTION

[12] A Lie-Yamaguti superalgebra is a triple (L, *, { , , }) where L = L 0 ⊕ L 1 is K-vector superspace i.e Z 2 -graded vector space, " * " a bilinear map (the binary superoperation on L) and " { , , } " a trilinear map (the ternary superoperation on L) such that

(SLY1) x * y = −(−1) |x||y| y * x,
(SLY2) {x, y, z} = (−1) |x||y| {y, x, z},
(SLY3) {x, y, z} = (−1) |x||y||z| [y, z, x] + (−1) |x||y||z| [z, x, y] + J(x, y, z) = 0,
(SLY4) {x * y, z, u} + (−1) |x||y||z| [y * z, x, u] + (−1) |x||y||z| [z * x, y, u] = 0,
(SLY5) {x, y, u * v} = {x, y, u} * v + (−1) |u| |v| u * {x, y, v},
(SLY6) {x, y, {u, v, w}} = {{x, y, u}, v, w} + (−1) |u| |v| |w| {u, {x, y, v}, w} + (−1) |x| |y| |v| |w| {u, v, {x, y, w}},

for all x, y, z, u, v, w in L and

(1) J(x, y, z) = (x * y) * z + (−1) |x||y||z| [y * z] * x + (−1) |x||y||z| [z * x] * y.

The relation (1) is called super-Jacobian. Observe that if x * y = 0, for all x, y in L, then a Lie-Yamaguti superalgebra (L, *, { , , }) reduces to a Lie supertriple system (L, *, { , , }) as defined in [17] and if {x, y, z} = 0 for all x, y, z in L, then (L, *, { , , }) is a Lie superalgebra (L, *) [17]. Recall that a Lie supertriple system [17] is a pair (T, { , , }) where T = T 0 ⊕ T 1 is a K-vector superspace and
"\((, , )\)" a trilinear map (the ternary superoperation on \(T\)) such that

\[
\begin{align*}
(i) \quad \{x, y, z\} &= (-1)^{|x||y|}\{y, x, z\} \\
(ii) \quad \{x, y, z\} + (-1)^{|x|(|y|+|z|)}\{y, z, x\} + (-1)^{|z|(|x|+|y|)}\{z, x, y\} = 0 \\
(iii) \quad \{x, y, \{u, v, w\}\} &= \{(x, y, u), v, v\} + (-1)^{|(x|+|y|)|(|u|+|v|)}\{u, \{x, y, v\}, w\}
\end{align*}
\]

for all \(x, y, z\) in \(T\) and a Lie superalgebra \(\mathfrak{a}\) is a pair \((A, *)\) where \(A = A_0 \oplus A_1\) is a \(K\)–vector superspace with "*" a bilinear map (the binary superoperation on \(A\)) such that

\[
\begin{align*}
(i) \quad x * y &= (-1)^{|x||y|}y * x \\
(ii) \quad J(x, y, z) &= 0
\end{align*}
\]

for all \(x, y, z\) in \(A\).

A Hom-type generalization of a kind of algebra is obtained by certain twisting of the defining identities by a linear self-map, called the twisting map, in such way that when the twisting map is identity map, then one recovers the original kind of algebra. In this scheme, e.g., associative algebras and Leibniz algebras are twisted into Hom-associative algebras and Hom-Leibniz algebras respectively \([10]\) and, likewise, Hom-type analogues of Novikov algebras, alternative algebras, Jordan algebras and Leibniz algebras are twisted into Hom-algebras and Hom-Leibniz algebras respectively \([15], [19], [20]\). In the same way, this generalization of binary or ternary algebras has been extended on the binary-ternary algebras. Indeed, Hom-Akivis algebras \([10]\), Hom-Lie-Yamaguti algebras \([8]\) and Hom-Bol algebras \([6]\) which generalize Akivis algebras, Lie-Yamaguti algebras and Bol algebras respectively are introduced. One could say that the theory of Hom-algebras originated in \([9]\) (see also \([13], [14]\)) in the study of deformations of Witt and Virasoro algebras (in fact, some \(q\)–deformations of Witt and Virasoro algebras have a structures of a Hom-Lie algebra \([9]\)). Some algebraic abstractions of this study are given in \([16], [18]\) For further more information on other Hom-type algebras, one may refer to, e.g., \([5], [7], [15], [19], [20], [21]\).

In \([11]\), the authors introduce Hom-associative superalgebras and Hom-Lie superalgebras which generalize associative superalgebras \([17]\) and Lie superalgebras \([17]\) respectively. They provide a way for constructing Hom-Lie superalgebras from Hom-associative superalgebras which extend the fundamental construction of Lie superalgebras from associative superalgebras via supercommutator bracket (see the Proposition 1.1. in \([17]\)). Indeed, they show also that the supercommutator bracket defined using the multiplication in a Hom-associative superalgebra leads naturally to Hom-Lie superalgebras. In \([2]\) the authors introduce the Hom-alternative, Hom-Malcev and Hom-Jordan superalgebras which are the generalization of Alternative, Malcev and Jordan superalgebras respectively.

Our present study extends the Hom-type generalization of binary superalgebras to the one of ternary superalgebras or binary-ternary superalgebras. The purpose of this paper is to introduced Hom-type generalization of Lie-Yamaguti superalgebras \([12]\), called Hom-Lie-Yamaguti superalgebras. It is also to extend the notion of an \(n^{th}\)–derived (binary) Hom-superalgebras \([2]\) to the one of an \(n^{th}\)–derived ternary or binary-ternary Hom-superalgebras and we shown that ternary or binary-ternary superalgebras are closed under the process of taking \(n^{th}\)–derived Hom-superalgebras.

The rest of this paper is organized as follows. In Section two, we recall basic definitions in Hom-superalgebras theory and useful results about Hom-associative superalgebras and Hom-Lie superalgebras. In \([11]\), the authors show that the supercommutator bracket defined using the multiplication in a
Hom-associative superalgebra leads naturally to Hom-Lie superalgebra. Also, we recall the notion of \(n^{th}\)-derived (binary) Hom-superalgebra introduced in [1] and as example, we show that Hom-Lie superalgebras are closed under the process of taking \(n^{th}\)-derived (binary) Hom-superalgebras (see the Proposition 2.5). In the third Section, we introduce ternary and binary-ternary Hom-superalgebras. In particular, Hom-Lie-Yamaguti superalgebras which are binary-ternary Hom-superalgebras are defined. Hom-Lie-Yamaguti superalgebras generalize Hom-Lie triple supersystems (and subsequently ternary multiplicative Hom-Nambu superalgebras) and Hom-Lie superalgebras. We provide that any non-Hom-superassociative Hom-superalgebra is a Hom-supertriple system. We also extend a Yau’s ternary multiplicative Hom-Nambu superalgebras) and Hom-Lie superalgebras. We provide that any any defined. Hom-Lie-Yamaguti superalgebras generalize Hom-Lie triple supersystems (and subsequently Proposition 2.5). In the third Section, we introduce ternary and binary-ternary Hom-superalgebras. We recall some basic facts about Hom-superalgebras, including Hom-associative and Hom-Lie superalgebras [1]. Also, we recall the notion of (binary) \(n^{th}\)-derived Hom-superalgebras. As example, we show that the category of Hom-superalgebras is closed under the process of taking \(n^{th}\)-derived Hom-superalgebras [1].

2. SOME BASICS ON SUPERALGEBRAS

We recall some basic facts about Hom-superalgebras, including Hom-associative and Hom-Lie superalgebras [1]. Also, we recall the notion of (binary) \(n^{th}\)-derived Hom-superalgebras. As example, we show that the category of Hom-superalgebras is closed under the process of taking \(n^{th}\)-derived Hom-superalgebras [1].

\textbf{Definition 2.1.} (i) Let \( f : (A, *, \alpha) \to (A', *, \alpha') \) be a map, where \( A = A_0 \oplus A_1 \) and \( A' = A'_0 \oplus A'_1 \) are \( Z_2 \)-graded vector spaces. The map \( f \) is called an even (resp. odd) map if \( f(A_i) \subset A'_i \) (resp. \( f(A_i) \subset A'_{i+1} \)), for \( i = 0, 1 \).

(ii) A Hom-superalgebra is a triple \((A, *, \alpha)\) in which \( A = A_0 \oplus A_1 \) is a \( K \)-super-module, \(* : A \times A \to A\) is an even bilinear map, and \( \alpha : A \to A\) is an even linear map such that \( \alpha(x * y) = \alpha(x) * \alpha(y) \) (multiplicativity).

\textbf{Remark 2.2.} For convenience, we assume throughout this paper that all Hom-superalgebras are multiplicative.

\textbf{Definition 2.3.} Let \((A, *, \alpha)\) be a Hom-superalgebra, that is a \( K \)-vector superspace \( A \) together with a multiplication "\(*" and an even linear self-map \( \alpha \).

(i) \[ a_{\alpha}(x, y, z) = (x * y) * \alpha(z) - \alpha(x) * (y * z). \]

(ii) \[ a_{\alpha}(x, y, z) = 0, \quad \forall x, y, z \in A \]

(iii) \[ J_{\alpha}(x, y, z) = (x * y) * \alpha(z) + (-1)^{|x||y|+|z|} (y * z) * \alpha(x) + (-1)^{|x||z|+|y|} (z * x) * \alpha(y) \]

(iv) \[ J_{\alpha}(x, y, z) = 0 \text{ (the Hom-super-Jacobi identity)} \]

for all \( x, y, z \in A \). If \( \alpha = Id \), a Hom-Lie superalgebra reduces to a usual Lie superalgebra.
Recall that the supercommutator bracket defined using the multiplication in any Hom-associative superalgebra leads naturally to a Hom-Lie superalgebra (see the Proposition 2.6 in [1]). Others Hom-superalgebras can be constructed from Lie superalgebras using the Theorem 2.7 in [1]. In the following, we recall the notion of (binary) \( n \)-th derived Hom-superalgebra.

**Definition 2.4.** [2] Let \((A, *, \alpha)\) be a Hom-superalgebra and \(n \geq 0\) an integer (”*” is the binary operation on \(A\)). The Hom-superalgebra \(A^n\) defined by

\[
A^n := (A, *^{(n)}, \alpha^{2^n}), \quad \text{where } (x *^{(n)} y) := \alpha^{2^n-1}(x * y), \forall x, y \in A
\]

is called the \(n\)-th derived Hom-superalgebra of \(A\).

For simplicity of exposition, \(*^{(n)}\) is written as \(*^{\alpha} = \alpha^{2^n-1} \circ \alpha^n\). Then notes that \(A^0 = (A, *, \alpha), A^1 = (A, *^{(1)} = \alpha \circ \alpha, \alpha^2), \) and \(A^{n+1} = (A^n)^1\).

**Proposition 2.5.** Let \((A, [\cdot, \cdot], \alpha)\) be a multiplicative Hom-Lie superalgebra. Then the \(n\)-th derived Hom-superalgebra \(A^n = (A, [\cdot, \cdot]^{(n)}, \alpha^{2^n})\) is also a multiplicative Hom-Lie superalgebra for each \(n \geq 0\).

**Proof:** Observe that \([x, y]^{(n)} = -(\alpha^{n})[x][y][x, y]^{(n)}. Indeed,

\[
[x, y]^{(n)} = \alpha^{2^n-1}([x, y]) = \alpha^{2^n-1}((\alpha^{n})[x][y][x, y]) = (\alpha^{n})[x, y]^{(n)} = (\alpha^{n})\alpha^{2^n-1}([y, x]) = (\alpha^{n})\alpha^{2^n-1}([x, y])
\]

Then, the superantisymmetry of \([x, y]^{(n)}\) holds in \(A^n\). Next, we have

\[
\begin{align*}
&\left([x, y]^{(n)}, \alpha^{2^n}(z)\right)^{(n)} + \left(1\right)^{2^n-1}([y][z][y, z]) [\alpha^{2^n}(x)]^{(n)} + (1)^{2^n-1}([y][z][y, z]) [\alpha^{2^n}(y)]^{(n)} \\
&= \alpha^{2^n-1}(\alpha^{n}([x, y, \alpha^{2^n}(z)]) + (1)^{2^n-1}([y][z][y, z]) \alpha^{2^n-1}(\alpha^{n}([y, y, \alpha^{2^n}(z)]) + (1)^{2^n-1}([y][z][y, z]) \alpha^{2^n-1}(\alpha^{n}([y, z, \alpha^{2^n}(y)]) + (1)^{2^n-1}([y][z][y, z]) \alpha^{2^n-1}(\alpha^{n}([x, y, \alpha^{2^n}(y)]) \\
&= (\alpha^{n})^{2^n-1}(\alpha^{n}([x, y, \alpha^{2^n}(z)]) + (1)^{2^n-1}([y][z][y, z]) \alpha^{2^n-1}(\alpha^{n}([y, y, \alpha^{2^n}(z)]) + (1)^{2^n-1}([y][z][y, z]) \alpha^{2^n-1}(\alpha^{n}([y, z, \alpha^{2^n}(y)]) + (1)^{2^n-1}([y][z][y, z]) \alpha^{2^n-1}(\alpha^{n}([x, y, \alpha^{2^n}(y)]) \\
&= (\alpha^{n})^{2^n-1}(0) \quad \text{(by (3))} \\
&= 0
\end{align*}
\]

and so the Hom-superalgebra in \(A^{(n)}\) holds in \(A^n\). Thus, we conclude that \(A^{(n)}\) is a (multiplicative) Hom-Lie superalgebra. In section 4, we extend this notion of \(n\)-th one derived (binary) Hom-superalgebra to the case of ternary and binary-ternary Hom-superalgebras.

### 3. Ternary and binary-ternary Hom-superalgebras

In this section, we introduce Hom-Lie supertriple systems and consequently ternary multiplicative Hom-Nambu superalgebras. Hom-Lie-Yamaguti superalgebras which are binary-ternary Hom-superalgebras are defined. Hom-Lie-Yamaguti superalgebras reduce to Lie-Yamaguti superalgebras [12] when the twisting map is identity map. Note also that Hom-Lie-Yamaguti superalgebras generalize Hom-Lie supertriple systems (and subsequently ternary multiplicative Hom-Nambu superalgebras) and Hom-Lie superalgebras. We finish this section by giving some examples of Hom-Lie-Yamaguti superalgebras constructed by using Theorem 3.10 via Corollary 3.11.

#### 3.1. Definitions and Theorem of construction

In this subsection, we give some definitions of ternary Hom-superalgebras and we show that any non-Hom-superassociative Hom-superalgebra has a natural structure of Hom-superalgebraic system. In the Theorem 3.10 we show that Hom-Lie-Yamaguti superalgebras are closed under twisting by self-morphisms respectively.

**Definition 3.1.** A ternary Hom-superalgebra is triple \((T, \{\cdot, \cdot, \cdot\}, \alpha = (\alpha_1, \alpha_2))\) constituted by a vector \(K\)-superspace ”\(T = T_0 \oplus T_1\)”, a trilinear map \(\{\cdot, \cdot, \cdot\} : T \times T \times T \rightarrow T\), and even linear maps \(\alpha_i : T \rightarrow T, i = 1, 2\), called the twisting maps. The algebra \((T, \{\cdot, \cdot, \cdot\}, \alpha = (\alpha_1, \alpha_2))\) is said
any non-Hom-associative superalgebra is a Hom-supertriple system.

**Proposition 3.5.** relationships between nonassociative superalgebras and supertriple systems (see Proposition 3.5).

Our definition here is motivated by the concern of giving a Hom-type analogue of the **Remark 3.4.**

\[
(1) = (x, y, z) = 0 \forall x, y, z \in T.
\]

**Definition 3.2.** A (multiplicative) ternary Hom-Nambu superalgebra is a (multiplicative) ternary Hom-superalgebra \((T, \{\cdot, \cdot, \cdot\}, \alpha)\) satisfying

\[
\{\alpha(x), \alpha(y), \{u, v, w\}\} = \{\alpha(x), \alpha(y), \alpha(w)\}
\]

\[
+ (-1)^{|u+(x+y)}}\{\alpha(u), \{x, y, v\}, \alpha(w)\}
\]

\[
+ (-1)^{|w+(x+y)}}\{\alpha(u), \alpha(v), \{x, y, w\}\}
\]

(4)

for all \(u, v, w, x, y, z \in T\). The condition (4) is called the ternary Hom-Nambu superidentity. In general, the ternary Hom-Nambu superidentity reads:

\[
\{\alpha_1(x), \alpha_2(y), \{u, v, w\}\} = \{\alpha_1(x), \alpha_2(y), \alpha_2(w)\}
\]

\[
+ (-1)^{|u+(x+y)}}\{\alpha_1(u), \{x, y, v\}, \alpha_2(w)\}
\]

\[
+ (-1)^{|w+(x+y)}}\{\alpha_1(u), \alpha_2(v), \{x, y, w\}\}
\]

for all \(u, v, w, x, y, z \in T\), where \(\alpha_1\) and \(\alpha_2\) are linear self-maps of \(T\).

**Definition 3.3.** A (multiplicative) Hom-supertriple system is a (multiplicative) ternary Hom-superalgebra \((T, \{\cdot, \cdot, \cdot\}, \alpha)\) such that

1. \(\{x, y, z\} = -(-1)^{|x||y|}\{y, x, z\},\)
2. \(\{x, y, z\} + (-1)^{|x||y|+|z|}\{y, z, x\} + (-1)^{|z||x|+|y|}\{z, x, y\} = 0 \forall x, y, z \in T.\)

**Remark 3.4.** Our definition here is motivated by the concern of giving a Hom-type analogue of the relationships between nonassociative superalgebras and supertriple systems (see Proposition 3.5).

**Proposition 3.5.** Any non-Hom-associative superalgebra is a Hom-supertriple system.

**Proof:** Let \((A, \ast, \alpha)\) be a non-Hom-associative superalgebra. Define the supercommutator by \([x, y] := x \ast y - (-1)^{|x||y|}y \ast x\) and the ternary superoperation by \(\{x, y, z\} := [[x, y], \alpha(z)] - as_\alpha(x, y, z) + (-1)^{|x||y|}as_\alpha(y, x, z)\) for all homogeneous elements \(x, y, z \in A\) and where the superassociator \(as(x, y, z)\) is given by (2). Then we have \(\{x, y, z\} = -(-1)^{|x||y|}\{y, x, z\}\) and \(\circ_{(x,y,z)}(-1)^{|x||z|}\{x, y, z\} = 0.\) Thus \((A, \{\cdot, \cdot, \cdot\}, \alpha)\) is a Hom-supertriple system. \(\square\)

**Remark 3.6.** For \(\alpha = Id\) (the identity map), we recover the supertriple system with ternary superoperation \(\{x, y, z\} = [[x, y], z] - as(x, y, z) + (-1)^{|x||y|}as(y, x, z)\) that is associated to each nonassociative superalgebra and where \([x, y]\) and \(as(x, y, z)\) are supercommutator and superassociator respectively for all homogeneous elements \(x, y, z\).

**Definition 3.7.** A **Hom-Lie supertriple system** is a Hom-supertriple system \((A, \{\cdot, \cdot, \cdot\}, \alpha)\) satisfying the ternary Hom-Nambu superidentity (4). When \(\alpha = Id\), a Hom-Lie supertriple system reduces to a Lie supertriple system.

One can note that Hom-Bol superalgebras introduced in [11] may be viewed as some generalization of Hom-supertriple systems.

In the following, we now give the definition of the basic object of this paper and we consider construction methods for Hom-LY superalgebras. These methods allow to find examples of Hom-LY superalgebras starting from ordinary LY superalgebras or even from Malcev superalgebras. Recall that from a Malcev superalgebra \((A, \ast)\) and consider on \(A\) the ternary superoperation

\[
\{x, y, z\} := x \ast (y \ast z) - (-1)^{|x||y|}y \ast (x \ast z) + (x \ast y) \ast z, \forall x, y, z \in A,
\]

(5)
Theorem 3.10. Let $(\beta x, y, z)^\ast \{u, v, w, x, y, z\}$ for all $u, v, w, x, y, z$.

Definition 3.8. A Hom-Lie Yamaguti superalgebra (Hom-LY superalgebra for short) is a quadruple $(L, \ast, \{,\}, \alpha)$ in which $L$ is $\mathbb{K}$-vector superspace, “$\ast$” a binary superoperation and “$\{,\}$” a ternary superoperation on $L$, and $\alpha : L \to L$ an even linear map such that

\begin{align*}
(SHLY1) \quad & \alpha(x \ast y) = \alpha(x) \ast \alpha(y), \\
(SHLY2) \quad & \alpha(x, y, z) = \{\alpha(x), \alpha(y), \alpha(z)\}, \\
(SHLY3) \quad & x \ast y = -(-1)^{|x||y|}y \ast x, \\
(SHLY4) \quad & \{x, y, z\} = -(-1)^{|x||y|}\{y, x, z\}, \\
(SHLY5) \quad & \circ_{(x,y,z)} (1)^{|x||y|}[(x \ast y) \ast \alpha(z) + \{x, y, z\}] = 0, \\
(SHLY6) \quad & \circ_{(x,y,z)} (1)^{|x||y|}\{x, y, \alpha(z), \alpha(u)\} = 0, \\
(SHLY7) \quad & \{\alpha(x), \alpha(y), u \ast v\} = \{x, y, u\} \ast \alpha^2(v) + (1)^{|u|(|x|+|y|)}\alpha^2(u) \ast \{x, y, v\}, \\
(SHLY8) \quad & \{\alpha^2(x), \alpha^2(y), u, v, w\} = \{x, y, u\}, \alpha^2(v), \alpha^2(w)\\
& \quad + (1)^{|u|(|x|+|y|)}\alpha^2(u), \{x, y, v\}, \alpha^2(w)\\
& \quad + (1)^{|u|(|x|+|y|)}\alpha^2(u), \alpha^2(v), \{x, y, w\},
\end{align*}

for all $u, v, w, x, y, z \in L$ and where $\circ_{(x,y,z)}$ denotes the sum over cyclic permutation of $x, y, z$.

Note that the conditions $(SHLY1)$ and $(SHLY2)$ mean the multiplicativity of $(L, \ast, \{,\}, \alpha)$.

Remark 3.9. 
(1) If $\alpha = Id$, then the Hom-LY superalgebra $(L, \ast, \{,\}, \alpha)$ reduces to a LY superalgebra $(L, \ast, \{,\})$ (see $(SLY1) - (SLY6)$).

(2) If $x \ast y = 0$, for all $x, y \in L$, then $(L, \ast, \{,\}, \alpha)$ becomes a Hom-Lie supertriple system $(L, \{,\}, \alpha^2)$ and, subsequently, a ternary Hom-Nambu superalgebra (since, by Definition 3.7, any Hom-Lie supertriple system is automatically a ternary Hom-Nambu superalgebra).

(3) If $\{x, y, z\} = 0$ for all $x, y, z \in L$, then the Hom-LY superalgebra $(L, \ast, \{,\}, \alpha)$ becomes a Hom-Lie superalgebra $(L, \ast, \alpha)$.

Theorem 3.10. Let $A_\alpha := (A, \ast, \{,\}, \alpha)$ be a Hom-LY superalgebra and let ”$\beta$” be an even endomorphism of the superalgebra $(A, \ast, \{,\})$ such that $\beta \alpha = \alpha \beta$. Let $\beta^0 = Id$ and, for any $n \geq 1$ $\beta^n := \beta \circ \beta^{n-1}$. Define on $A$ the superoperations

\begin{align*}
& x \ast y := \beta^n(x \ast y), \\
& \{x, y, z\}_\beta := \beta^{2n}(\{x, y, z\}),
\end{align*}

for all $x, y, z \in A$. Then, $A_{\beta^n} := (A, \ast, \{,\}, \beta^n \alpha)$ are Hom-LY superalgebras, with $n \geq 1$.

Proof. First, we observe that the condition $\beta \alpha = \alpha \beta$ implies $\beta^n \alpha = \alpha \beta^n$, $n \geq 1$. Next we have

\begin{align*}
& (\beta^n \alpha)(x \ast y) = (\beta \alpha)(\beta^n(x) \ast \beta^n(y)) = \beta^n((\alpha \beta^n)(x) \ast (\alpha \beta^n)(y)) \\
& = (\alpha \beta^n)(x) \ast (\alpha \beta^n)(y) = (\beta \alpha)(x) \ast (\beta \alpha)(y) \quad \text{and we get} \quad (SHLY1) \text{ for } A_{\beta^n}. \text{ Likewise, the condition } \beta \alpha = \alpha \beta \text{ implies } (SHLY2). \text{ The identities } (SHLY3) \text{ and } (SHLY4) \text{ for } A_{\beta^n} \text{ follow from the skew-supersymmetry of } "\ast" \text{ and } "\{,\}" \text{ respectively.}
Consider now $\mathcal{O}_{x,y,z} [(x \ast y) \ast_{\beta} (\beta^{n}\alpha)(z) + \{x, y, z\}_{\beta}]$. Then

\[
(x \ast y) \ast_{\beta} (\beta^{n}\alpha)(z) + (-1)^{|x||y|+|z|} (y \ast z) \ast_{\beta} (\beta^{n}\alpha)(x) + (-1)^{|x||y|+|z|} (z \ast x) \ast_{\beta} (\beta^{n}\alpha)(y) + \{x, y, z\}_{\beta} + (-1)^{|x||y|+|z|} \{x, y, z\}_{\beta} + (-1)^{|x||y|+|z|} \{z, x, y\}_{\beta} = \beta^{n}(\beta^{n}(x \ast y) \ast_{\beta} (\beta^{n}\alpha)(z)) + (-1)^{|x||y|+|z|}\beta^{n}(\beta^{n}(y \ast z) \ast_{\beta} (\beta^{n}\alpha)(x)) + (-1)^{|x||y|+|z|}\beta^{n}(\beta^{n}(z \ast x) \ast_{\beta} (\beta^{n}\alpha)(y)) + \beta^{2n}([x, y, z]) + (-1)^{|x||y|+|z|}\beta^{2n}([y, z, x]) + (-1)^{|x||y|+|z|}\beta^{2n}([z, x, y]) = \beta^{2n}((x \ast y) \ast_{\beta} (\beta^{n}\alpha)(z)) + (-1)^{|x||y|+|z|}\beta^{2n}((y \ast z) \ast_{\beta} (\beta^{n}\alpha)(x)) + (-1)^{|x||y|+|z|}\beta^{2n}((z \ast x) \ast_{\beta} (\beta^{n}\alpha)(y)) + \beta^{2n}([x, y, z]) + (-1)^{|x||y|+|z|}\beta^{2n}([y, z, x]) + (-1)^{|x||y|+|z|}\beta^{2n}([z, x, y]) = \beta^{2n}((x \ast y) \ast_{\beta} (\beta^{n}\alpha)(z)) + (-1)^{|x||y|+|z|}\beta^{2n}((y \ast z) \ast_{\beta} (\beta^{n}\alpha)(x)) + (-1)^{|x||y|+|z|}\beta^{2n}((z \ast x) \ast_{\beta} (\beta^{n}\alpha)(y)) + \{x, y, z\}_{\beta} + (-1)^{|x||y|+|z|} \{x, y, z\}_{\beta} + (-1)^{|x||y|+|z|} \{y, z, x\}_{\beta} + (-1)^{|x||y|+|z|} \{z, x, y\}_{\beta} = \beta^{2n}(0) \text{ (by } SHLY5 \text{ for } A_{\alpha})
\]

and thus we get \((SHLY5)\) for $A_{\beta^{n}}$. Next,

\[
\{x \ast y, (\beta^{n}\alpha)(z), (\beta^{n}\alpha)(u)\}_{\beta} = \{\beta^{3n}(x \ast y), \beta^{3n}(\alpha(z)), \beta^{3n}(\alpha(u))\} = \beta^{3n}(\{x \ast y, \alpha(z), \alpha(u)\}).
\]

Therefore

\[
\mathcal{O}_{x,y,z} (-1)^{|x||y|}\{x \ast y, (\beta^{n}\alpha)(z), (\beta^{n}\alpha)(u)\}_{\beta} = \mathcal{O}_{x,y,z} (-1)^{|x||y|}\{\beta^{3n}(x \ast y), \beta^{3n}(\alpha(z)), \beta^{3n}(\alpha(u))\} = \beta^{3n}(\{x \ast y, \alpha(z), \alpha(u)\}) = \beta^{3n}(0) \text{ (by } HLY6 \text{ for } A_{\alpha})
\]

So that we get \((SHLY6)\) for $A_{\beta^{n}}$. Further, using \((SHLY7)\) for $A_{\alpha}$ and condition $\alpha\beta = \beta\alpha$, we compute

\[
\{\beta^{n}(\alpha)(x), (\beta^{n}\alpha)(y), u \ast_{\beta} v\}_{\beta} = \beta^{3n}(\{\alpha(x), \alpha(y), u \ast v\})
\]

Thus \((SHLY7)\) holds for $A_{\beta^{n}}$. Using repeatedly the condition $\alpha\beta = \beta\alpha$ and the identity \((SHLY8)\) for $A_{\alpha}$, the verification for \((SHLY8)\) for $A_{\beta^{n}}$ is as follows.
\{ (\beta^n \alpha)^2(x), (\beta^n \alpha)^2(y), \{ u, v, w \}\}_{\beta} \\
= \beta^{2n}\{ (\beta^{2n} \alpha^2)(x), (\beta^{2n} \alpha^2)(y), \beta^{2n}\{ u, v, w \}\}\}
= \beta^{4n}\{ \alpha^2(x), \alpha^2(y), \{ u, v, w \}\}\}
= \beta^{4n}\{ \{ x, y, u \}, \alpha^2(v), \alpha^2(w)\}\}
+ \beta^{4n}\{ (\beta^{n} \alpha)^2(u), \{ x, y, v \}, \alpha^2(w)\}\}
+ \beta^{4n}\{ (\beta^{n} \alpha)^2(u), \{ x, y, w \}, \alpha^2(v)\}\}
= \beta^{2n}\{ \beta^{2n}\{ x, y, u \}, (\beta^{n} \alpha^2)(v), (\beta^{n} \alpha^2)(w)\}\}
+ (\beta^{n} \alpha)^2(u), \{ x, y, v \}, (\beta^{n} \alpha)^2(w)\}\}
+ (\beta^{n} \alpha)^2(u), \{ x, y, w \}, (\beta^{n} \alpha)^2(v)\}\}
= \{ \{ x, y, u \}, (\beta^{n} \alpha)^2(v), (\beta^{n} \alpha)^2(w)\}\}
+ (\beta^{n} \alpha)^2(u), \{ x, y, v \}, (\beta^{n} \alpha)^2(w)\}\}
+ (\beta^{n} \alpha)^2(u), \{ x, y, w \}, (\beta^{n} \alpha)^2(v)\}\}

Thus \( (SHLY8) \) holds for \( A_{\beta^n} \). Therefore, we get that \( A_{\beta} \) is a Hom-LY superalgebra. This finishes the proof. \( \square \)

In \cite{18}, D. Yau established a general method of construction of Hom-algebras from their corresponding untwisted algebras. From Theorem 3.1 we have the following method construction of Hom-LY superalgebras from LY superalgebras(this yields examples of Hom-Ly superalgebras). This method is an extension to binary-ternary superalgebras of D. Yau’s result \cite{18}, Theorem 2.3. Such an extension to binary-ternary algebras is first mentioned in \cite{10}, Corollary 4.5.

**Corollary 3.11.** Let \( (A, \ast, [, ,]) \) be a LY algebra and \( \beta \) an endomorphism of \( (A, \ast, [, ,]) \). If define on \( A \) a binary operation \( ^{\beta}\ast \) and a ternary operation \( \{ , , \} \) by
\[ x \ast y := \beta(x \ast y), \]
\[ \{ x, y, z \} := \beta^2([x, y, z]), \]
then \( (A, ^{\beta}\ast, \{ , , \}, \beta) \) is a Hom-LY algebra.

**Proof.** The proof follows if observe that Corollary 3.11 is Theorem 3.10 when \( \alpha = Id \) and \( n = 1 \). \( \square \)

**Proposition 3.12.** Let \( (L, [, \alpha], \alpha) \) be a (multiplicative) Hom-Lie superalgebra. Define on \( (L, [, \alpha], \alpha) \) a ternary superoperation by
\[ (6) \quad \{ x, y, z \} := [[x, y], \alpha(z)]. \]
Then, \( (L, [, \alpha], \{ , , \}, \alpha) \) is a Hom-Lie-Yamaguti Superalgebras.

**Proof:** Straightforward calculations by verification of \( (SHLY1 – SHLY8) \) identities of Definition 3.2.

3.2. **Examples of Hom-Lie-Yamaguti superalgebras.** In the following, we give some examples of Hom-Lie-Yamaguti superalgebras which are constructed firstly from the example 2.8 in \cite{11} and using Proposition 3.12. In the second from the example 3.2 given in \cite{3} (see also in \cite{2}) and using the Theorem 3.10 via Corollary 3.11. Then, we obtain some family of Hom-Lie-Yamaguti algebras of dimension 5 and dimension 3 respectively. Indeed,
Example 3.13. Consider the family of Hom-Lie superalgebra \( \text{osp}(1,2)_\lambda = (\text{osp}(1,2), [\cdot]_{\alpha_\lambda}, \alpha_\lambda) \) given in the example 2.8 in [1]. The Hom-Lie superalgebra bracket \([\cdot]_{\alpha_\lambda}\) on the basis elements is given, for \( \lambda \neq 0 \), by:

\[
[H, X]_{\alpha_\lambda} = 2\lambda^2 X, \quad [H, Y]_{\alpha_\lambda} = -\frac{2}{\lambda} Y, \quad [X, Y]_{\alpha_\lambda} = H, [Y, G]_{\alpha_\lambda} = \frac{1}{\lambda} F, \quad [X, F]_{\alpha_\lambda} = \lambda G,
\]

\[
[H, F]_{\alpha_\lambda} = -\frac{1}{\lambda} F, \quad [H, G]_{\alpha_\lambda} = \lambda G, \quad [G, F]_{\alpha_\lambda} = H, \quad [G, G]_{\alpha_\lambda} = -2\lambda^2 X, \quad [F, F]_{\alpha_\lambda} = \frac{2}{\lambda^2} Y,
\]

where \( \alpha_\lambda : \text{osp}(1,2) \to \text{osp}(1,2) \) is a linear map defined by

\[
\alpha_\lambda(X) = \lambda^2 X, \quad \alpha_\lambda(Y) = \frac{1}{\lambda^2} Y, \quad \alpha_\lambda(H) = H, \quad \alpha_\lambda(F) = \frac{1}{\lambda} F, \quad \alpha_\lambda(G) = \lambda G,
\]

and \( \text{osp}(1,2) = V_0 \oplus V_1 \) is a Lie superalgebra where \( V_0 \) is generated by:

\[
H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

and \( V_1 \) is generated by:

\[
F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.
\]

such that

\[
[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H, \quad [Y, G] = F, \quad [X, F] = G,
\]

\[
[H, F] = -F, \quad [H, G] = G, \quad [G, F] = H, \quad [G, G] = -2X, \quad [F, F] = 2Y,
\]

Now, we define on \( \text{osp}(1,2)_\lambda = (\text{osp}(1,2), [\cdot]_{\alpha_\lambda}, \{\cdot, \cdot\}_{\alpha_\lambda}) \) a ternary superoperation by [9]. Then, \( sLY(1,2)_\lambda = (\text{osp}(1,2), [\cdot]_{\alpha_\lambda}, \{\cdot, \cdot\}_{\alpha_\lambda}, \alpha_\lambda) \) is a family of Hom-Lie-Yamaguti superalgebras where the supercommutator \([\cdot]_{\alpha_\lambda}\) is defined as above in this example 3.13 and the ternary superoperation \( \{\cdot, \cdot\}_{\alpha_\lambda} \) (we give only the ones with non zero values in the left hand side and using the identity (SHLY4) of Definition 3.13 one can deduce the others values which are non zero) is defined by

\[
2H = \{H, X, Y\}_{\alpha_\lambda} = \{H, Y, X\}_{\alpha_\lambda} = 2\{H, F, G\}_{\alpha_\lambda} = 2\{H, G, F\}_{\alpha_\lambda} = 2\{X, F, F\}_{\alpha_\lambda} = -2\{Y, G, G\}_{\alpha_\lambda} = -\{F, F, X\}_{\alpha_\lambda}
\]

\[
-2\lambda^4 X = \frac{1}{2}\{H, X, H\}_{\alpha_\lambda} = -\{X, X, Y\}_{\alpha_\lambda} = \{F, G, X\}_{\alpha_\lambda} = \{H, G, G\}_{\alpha_\lambda} = \{X, F, G\}_{\alpha_\lambda}
\]

\[
\frac{4}{\lambda^4} Y = \{H, Y, H\}_{\alpha_\lambda} = \{2, Y, X, Y\}_{\alpha_\lambda} = \{2, F, G, Y\}_{\alpha_\lambda} = -2\{H, F, F\}_{\alpha_\lambda} = 2\{Y, G, F\}_{\alpha_\lambda}
\]

\[
\frac{-2}{\lambda^2} F = \{2, H, F, H\}_{\alpha_\lambda} = \{H, Y, G\}_{\alpha_\lambda} = \{2, H, G, Y\}_{\alpha_\lambda} = 2\{X, Y, F\}_{\alpha_\lambda} = -\{F, F, G\}_{\alpha_\lambda}
\]

\[
-\lambda^2 G = \{2, H, G, H\}_{\alpha_\lambda} = \lambda^2 \{H, X, F\}_{\alpha_\lambda} = -\{H, F, X\}_{\alpha_\lambda} = -\{X, Y, G\}_{\alpha_\lambda} = \{X, F, H\}_{\alpha_\lambda}
\]

These Hom-Lie-Yamaguti superalgebras are not Lie-Yamaguti superalgebras for \( \lambda \neq \pm 1 \). Indeed, the left hand side of the identity (SLY3), for \( \alpha = \text{Id} \), leads to

\[
\sum (H, Y, X) (-1)^{|H| |X|} ([H, Y], X) + \{H, Y, X\} = 2\left(1 - \frac{\lambda^4}{\lambda^2}\right) H
\]
and also the left hand side of the identity \((SLY 4)\), for \(\alpha = \text{Id}\), leads to

\[ \bigcap_{(H,Y,X)} (-1)^{[H][X]}\{[H,Y], X,G\} = 2 (1 - \lambda^2) G. \]

Then, they not vanish for \(\lambda \neq \pm 1\).

**Example 3.14.** From the example \(M^3(3,1)\) of a non-Lie Malcev superalgebra given in \([3]\) (see also in \([2]\)) and defining on it a ternary superoperation by \([3]\), we obtain a Lie-Yamaguti superalgebra \((SLY(3,1), [\cdot, \cdot, \cdot])\) of dimension 4. \(\text{def}\) with respect to a basis \((e_1,e_2,e_3,e_4)\), where \((SLY(3,1))_0 = \text{span}(e_1,e_2,e_3)\) and \((SLY(3,1))_1 = \text{span}(e_4)\), by the following multiplication table

\[
\begin{align*}
[e_1,e_3] &= -e_1 \quad (= -[e_3,e_1]), \quad [e_2,e_3] = 2e_2 \quad (= -[e_3,e_2]), \quad [e_3,e_4] = -e_4 \quad (= -[e_4,e_3]), \\
[e_4,e_3] &= e_1 + e_2, \\
\{e_1,e_3,e_3\} &= 2e_1 \quad (= -\{e_3,e_1,e_3\}), \quad \{e_2,e_3,e_3\} = 8e_2 \quad (= -\{e_3,e_2,e_3\}), \\
\{e_3,e_4,e_3\} &= 2e_4 \quad (= -\{e_4,e_3,e_3\}), \quad \{e_3,e_4,e_4\} = e_1 - 2e_2 \quad (= -\{e_4,e_3,e_4\}) \\
\{e_4,e_4,e_3\} &= -e_1 - 4e_2.
\end{align*}
\]

Consider the even superalgebra endomorphism \(\alpha_1 : sLY(3,1) \to sLY(3,1)\) with respect to the same basis define by

\[
\alpha_1(e_1) = a^2 e_1, \quad \alpha_1(e_2) = a^2 e_2, \quad \alpha_1(e_3) = be_1 + ce_2 + e_3, \quad \alpha_1(e_4) = a^2 e_4 \quad \forall \ a,b,c \in \mathbb{K}.
\]

For each such even superalgebra endomorphism \(\alpha_1\) and using the Theorem 3.10 via Corollary 3.11, there is a Hom-Lie superalgebra \((sLY(3,1), [\cdot, \cdot, \cdot])_{\alpha_1}\) such that

\[
\begin{align*}
[e_1,e_3]_{\alpha_1} &= -a^2 e_1 \quad (= -[e_3,e_1]_{\alpha_1}), \quad [e_2,e_3]_{\alpha_1} = 2a^2 e_2 \quad (= -[e_3,e_2]_{\alpha_1}), \\
[e_3,e_4]_{\alpha_1} &= -ae_4 \quad (= -[e_4,e_3]_{\alpha_1}), \quad [e_4,e_4]_{\alpha_1} = a^2(e_1 + e_2), \\
\{e_1,e_3,e_3\}_{\alpha_1} &= 2a^4 e_1 \quad (= -\{e_3,e_1,e_3\}_{\alpha_1}), \quad \{e_2,e_3,e_3\}_{\alpha_1} = 8a^4 e_2 \quad (= -\{e_3,e_2,e_3\}_{\alpha_1}), \\
\{e_3,e_4,e_3\}_{\alpha_1} &= -2a^5 e_4 \quad (= -\{e_4,e_3,e_3\}_{\alpha_1}), \quad \{e_3,e_4,e_4\}_{\alpha_1} = a^4(e_1 - 2e_2) \quad (= -\{e_4,e_3,e_4\}_{\alpha_1}) \\
\{e_4,e_4,e_3\}_{\alpha_1} &= a^4(2a - 1)e_1 + 2(a + 1)e_2.
\end{align*}
\]

While for example

\[
\bigcap_{(e_3,e_4,e_4)} (-1)^{e_3+e_4}\{([e_3,e_4], e_4) + \{e_3,e_4,e_4\}\} = a^4(2a - 1)e_1 + (2a + 3)e_2,
\]

then for \(a \neq 0\), these Hom-Lie-Yamaguti superalgebras are not Lie-Yamaguti superalgebras.

In the same way, when we consider an even superalgebra endomorphism \(\alpha_2 : sLY(3,1) \to sLY(3,1)\) with respect to the same basis define by \(\alpha_2(e_3) = be_1 + ce_2 + \frac{3}{2}e_3, \quad \alpha_2(e_4) = de_4, \quad \forall \ a,b,c,d \in \mathbb{K}\) and using the Theorem 3.10 via Corollary 3.11, we obtain a twisting of \(sLY(3,1)\) into a family of Hom-Lie superalgebras define by \([e_3,e_4] = -de_2 \quad (= -[e_4,e_3])\) which are also Hom-Lie-Yamaguti superalgebras in particular case where the ternary superoperation is zero. Observe that they are also Lie superalgebras and consequently they are Lie-Yamaguti superalgebras where the ternary superoperation is zero.

4. \(n^{th}\)-DERIVED BINARY-TERNARY HOM-SUPERALGEBRAS

In this section, we extend the notion of \(n^{th}\)-derived (binary) Hom-superalgebra to the case of ternary and binary-ternary Hom-superalgebras. In particular, we introduce \(n^{th}\)-derived of Hom-Lie-Yamaguti superalgebra and we shown that the category of Hom-Lie-Yamaguti superalgebras is closed under the process of taking \(n^{th}\)-derived Hom-superalgebras.
Definition 4.1. Let \( A := (A, \{, , \}, \alpha) \) be a ternary Hom-superalgebra and \( n \geq 0 \) an integer. Define on \( A \) the \( n^{th} \)-derived ternary operation \( \{, , \}^{(n)} \) by

\[
\{x, y, z\}^{(n)} := \alpha^{2n+1-2}(\{x, y, z\}), \forall \; x, y, z \in A.
\]

Then \( A^{(n)} := (A, \{, , \}^{(n)}, \alpha^{2n}) \) will be called the \( n^{th} \)-one derived ternary Hom-superalgebra of \( A \). Now denote \( \{x, y, z\}^{(n)} = \alpha^{2n+1-2}(\{x, y, z\}) \). Then we note that \( A^0 = (A, \{, , \}, \alpha), A^1 = (A, \{x, y, z\}^{(1)} = \alpha^2 \circ \{x, y, z\}, \alpha^2) \), and \( A^{n+1} = (A^n)^1 \).

Definition 4.2. Let \( A := (A, *; \{, , \}, \alpha) \) be a binary-ternary Hom-superalgebra and \( n \geq 0 \) an integer. Define on \( A \) the \( n^{th} \)-derived binary and the \( n^{th} \)-derived ternary operation \( \{, , \}, \{, , \}^{(n)} \) by

\[
\alpha^{2n-1}(x * y) = \{x, y, z\}^{(n)}.
\]

Then \( A^{(n)} := (A, *^{(n)}, \{, , \}^{(n)}, \alpha^{2n}) \) will be called the \( n^{th} \)-one derived (binary-ternary) Hom-superalgebra of \( A \). Denote \( *^{(n)} = \alpha^{2n-1} \circ * \) and \( \{x, y, z\}^{(n)} = \alpha^{2n+1-2}(\{x, y, z\}) \). Then we note that \( A^0 = (A, *; \{, , \}, \alpha), A^1 = (A, *^{(1)} = \alpha \circ *; \{x, y, z\}^{(1)} = \alpha^2 \circ \{x, y, z\}, \alpha^2) \), and \( A^{n+1} = (A^n)^1 \).

One observes that, from Definition 4.2 if set \( \{x, y, z\} = 0, \forall \; x, y, z \in A \), we recover the \( n^{th} \)-derived (binary) Hom-superalgebra of Definition 2.4 and if \( x * y = 0 \; \forall \; x, y \in A \), then we have an \( n^{th} \)-derived (ternary) Hom-superalgebra (see Definition 4.1).

Proposition 4.3. Let \( A := (A, \{, \}, \alpha) \) be a multiplicative Hom-Lie supertriple system. Then for each \( n \geq 0 \), the \( n^{th} \)-derived Hom-superalgebra \( A^{(n)} = (A, \{, \}^{(n)} = \alpha^{2n+1-2} \circ \{, \}, \alpha^{2n}) \) of \( A \) is a multiplicative Hom-Lie supertriple system.

Proof: The proof of this Proposition 4.3 can be constituted throughout the proof of Theorem 4.4 below if the binary superoperation is zero.

In the following result, we show that the category of Hom-Lie-Yamaguti superalgebras is closed under the process of taking \( n^{th} \)-derived Hom-superalgebras.

Theorem 4.4. Let \( A_\alpha := (A, [ , ], \alpha) \) be a Hom-Lie-Yamaguti. Then, for each \( n \geq 0 \) \( n^{th} \) derived Hom-algebra

\[
A^{(n)} := (A, [ , ]^{(n)} = \alpha^{2n-1} \circ [ , ]^{(n)} = \alpha^{2n+1-2} \circ [ , ], \alpha^{2n})
\]

is a Hom-Lie-Yamaguti algebra.

Proof: The identities \((SHLY)_1\) \(- (SHLY)_4\) for \( A^{(n)} \) are obvious. The checking of \((SHLY)_5\) for \( A^{(n)} \) is as follows. Indeed, consider \( \bigcirc_{x,y,z} \left( (1)_{x,y}^{||y||}(\alpha(y))^{n}((\alpha^{2n}(z))^{(n)} + (1)_{x,y}^{||y||}(x, y, z)\right) \). Then

\[
\bigcirc_{x,y,z} \left( (1)_{x,y}^{||y||}(\alpha(y))^{n}((\alpha^{2n}(z))^{(n)} + (1)_{x,y}^{||y||}(x, y, z)\right)
\]

\[
= \bigcirc_{x,y,z} \left( (1)_{x,y}^{||y||}(\alpha^{2n-1}(\alpha^{2n-1}(x, y))^{(n)} + (1)_{x,y}^{||y||}(x, y, z)\right)
\]

\[
= \alpha^{2n+1-2}(\bigcirc_{x,y,z} \left( (1)_{x,y}^{||y||}(\alpha(y))^{n}((\alpha^{2n}(z))^{(n)} + (1)_{x,y}^{||y||}(x, y, z)\right)
\]

\[
= \alpha^{2n+1-2}(\bigcirc_{x,y,z} \left( (1)_{x,y}^{||y||}(\alpha(y))^{n}((\alpha^{2n}(z))^{(n)} + (1)_{x,y}^{||y||}(x, y, z)\right)
\]

\[
= \alpha^{2n+1-2}(0) \; (by \; (SHLY)_5 \; for \; A_\alpha)
\]

\[
= 0
\]

and thus we get \((SHLY)_5\) for \( A^{(n)} \).
Next,
\[
\{ [x, y]^{(n)}, (\alpha^2)^{(n)}(z), (\alpha^2)^{(n)}(u) \}^{(n)} = \{(\alpha^{2n-1}([x, y]), (\alpha^{2n})(z), (\alpha^{2n})(u))^{(n)}
\]
\[
= \alpha^{2n+1-2}\{(\alpha^{2n-1}([x, y]), (\alpha^{2n})(z), (\alpha^{2n})(u))\}
\]
\[
= \alpha^{2n+1-2}\alpha^{2n-1}\{(x, y, \alpha(z), \alpha(u))\}
\]
\[
= \alpha^{3(2n-1)}\{(x, y, \alpha(z), \alpha(u))\}.
\]

Therefore
\[
\bigcup\big((x, y, z) \big| (|x|+|y|)\big)\{(x, y)^{(n)}, (\alpha^2)^{(n)}(z), (\alpha^2)^{(n)}(u)\}^{(n)}
\]
\[
= \bigcup\big((x, y, z) \big| (|x|+|y|)\big)\{(x, y)^{(n)}, (\alpha^2)^{(n)}(z), (\alpha^2)^{(n)}(u)\}^{(n)}
\]
\[
= \alpha^{3(2n-1)}\bigcup\big((x, y, z) \big| (|x|+|y|)\big)\{(x, y)^{(n)}, (\alpha^2)^{(n)}(z), (\alpha^2)^{(n)}(u)\}^{(n)}
\]
\[
= \alpha^{3(2n-1)}(0) \text{ (by (SHLY6) for } A_{\alpha})
\]
\[
= 0
\]

So that we get (SHLY6) for \(A^{(n)}\). Further, using (SHLY7) for \(A_{\alpha}\) and condition of multiplicativity linearity of \(\alpha\), we compute
\[
\{(\alpha^{2n})(x), (\alpha^{2n})(y), [u, v]^{(n)}\}^{(n)}
\]
\[
= \alpha^{2n+1-2}\{(\alpha^{2n})(x), (\alpha^{2n})(y), (\alpha^{2n-1}([u, v]))\}
\]
\[
= \alpha^{2n+1-2}\alpha^{2n-1}\{(\alpha(x), \alpha(y), [u, v])\}
\]
\[
= \alpha^{2n+1-2}\alpha^{2n-1}\{(x, y, u, \alpha^2(v)) + (-1)^{|u||(|x|+|y|)}[\alpha^2(u), \{x, y, v\}]\}
\]
\[
= \alpha^{2n-1}\{(x, y, u)^{(n)}, (\alpha^2)^{(n)}(v)\} + (-1)^{|u||(|x|+|y|)}[(\alpha^2)^{(n)}(u), \{x, y, v\}^{(n)}]
\]
\[
= \{(x, y, u)^{(n)}, (\alpha^2)^{(n)}(v)\} + (-1)^{|u||(|x|+|y|)}[(\alpha^2)^{(n)}(u), \{x, y, v\}^{(n)}].
\]

Thus (SHLY7) holds for \(A^{(n)}\). Using repeatedly the condition of multiplicativity and the identity (SHLY8) for \(A_{\alpha}\), the verification for (SHLY8) for \(A^{(n)}\) is as follows.
\[
\{(\alpha^{2n})(x), (\alpha^{2n})(y), [u, v]^{(n)}\}^{(n)}
\]
\[
= \alpha^{2n+1-2}\{(\alpha^{2n})(x), (\alpha^{2n})(y), (\alpha^{2n-1}([u, v]))\}
\]
\[
= \alpha^{2n+1-2}\{(\alpha^{2n+1})(x), (\alpha^{2n+1})(y), (\alpha^{2n-1}([u, v]))\}
\]
\[
= \alpha^{2n+1-2}\alpha^{2n-1}\{(x, y, u, \alpha^2(v))\}
\]
\[
= \alpha^{2n+1-2}\alpha^{2n-1}\{(x, y, u, \alpha^2(v))\}
\]
\[
= \{(x, y, u)^{(n)}, (\alpha^2)^{(n)}(v)\} + (-1)^{|u||(|x|+|y|)}[(\alpha^2)^{(n)}(u), \{x, y, v\}^{(n)}].
\]
Thus $\text{(SHLY)8}$ holds for $A^{(n)}$. Therefore, we get that $A^{(n)}$ is a Hom-LY superalgebra. This finishes the proof. □

Remark 4.5. (1) If $[x, y]^{(n)} = 0$, for all $x, y \in L$, then $n^{th}$-derived of Hom-Lie-Yamaguti superalgebras becomes an $n^{th}$-derived of Hom-Lie supertriple system and, subsequently, $n^{th}$-derived of ternary Hom-Nambu superalgebra (since, by Definition 3.7, any Hom-Lie supertriple system is automatically a ternary Hom-Nambu superalgebra).

(2) If $\{x, y, z\}^{(n)} = 0$, for all $x, y, z \in L$, then the $n^{th}$-derived of Hom-LY superalgebra becomes a $n^{th}$-derived of Hom-Lie superalgebra.

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