Essentials of Blackfold Dynamics

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Abstract

We develop and significantly generalize the effective worldvolume theory for higher-dimensional black holes recently proposed by the authors. The theory, which regards the black hole as a black brane curved into a submanifold of a background spacetime—a blackfold—, can be formulated in terms of an effective fluid that lives on a dynamical worldvolume. Thus the blackfold equations split into intrinsic (fluid-dynamical) equations, and extrinsic (generalized geodesic embedding) equations. The intrinsic equations can be easily solved for equilibrium configurations, thus providing an efficient formalism for the approximate construction of novel stationary black holes. Furthermore, it is possible to study time evolution. In particular, the long-wavelength component of the Gregory-Laflamme instability of black branes is obtained as a sound-mode instability of the effective fluid. We also discuss action principles, connections to black hole thermodynamics, and other consequences and possible extensions of the approach. Finally, we outline how the fluid/AdS-gravity correspondence is related to this formalism.
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References
1 Introduction

In recent work we have identified the origin of the rich variety of higher-dimensional black holes in the possibility of having horizons that are much longer along some directions than in others [1]. In such cases there are at least two different horizon length-scales, and we have proposed an effective theory that captures the long-distance physics when these scales are widely separate. Focusing on the simplest, universal case of neutral, vacuum black holes, the two length scales are associated with the mass and angular momentum $\ell_M \sim \left(\frac{GM}{\ell^3}\right)^{\frac{1}{D-3}}$, $\ell_J \sim \frac{J}{M}$.

In a four-dimensional black hole, by virtue of the Kerr bound $J \leq GM^2$ these lengths are always parametrically similar, but higher-dimensional black holes (including also black rings) with $\ell_J \gg \ell_M$ are known to exist in ultra-spinning regimes where the angular momentum for a given mass can be arbitrarily high [2, 3, 4, 5]. It appears that essentially all their novel features, compared to their four-dimensional cousins, arise from the ability to separate these two lengths. This suggests that higher-dimensional black holes must be organized according to a hierarchy of scales:

1. $\ell_J \ll \ell_M$: black holes behave qualitatively similarly to the four-dimensional Kerr black hole.
2. $\ell_J \approx \ell_M$: threshold of new black hole dynamics.
3. $\ell_J \gg \ell_M$: the separation of scales suggests an effective description of long-wavelength dynamics.

The first and second regimes fully involve the non-linearities of General Relativity, but in the first case we have no hints of qualitatively new properties of black holes compared to four-dimensional ones. In particular we conjecture that for $J < J_{\text{crit}} = \alpha D M (GM)^{\frac{1}{D-3}}$ (with $\alpha_D$ a yet-undetermined numerical constant of order one), the Myers-Perry black holes are dynamically stable and unique among solutions with connected regular horizons [2]. In contrast, as the two scales begin to diverge in the second regime, we have good evidence of the onset of new phenomena: horizon instabilities, inhomogeneous (‘pinched’) phases, non-spherical horizon topologies, and absence of uniqueness [4, 3, 6, 7]. This regime seems hard to investigate by means of exact analytical techniques, but on the other hand the presence of only one scale in the problem is actually convenient for numerical investigation, since one does not require high precision over widely different scales. In the third regime, which is the focus of this paper, the existence of a small parameter $\ell_M/\ell_J$ allows the

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1 $J = \left(\sum_i J_i^2\right)^{1/2}$ aggregates the effect of all possible angular momenta.
2 This extends a previous conjecture in [3] about stability of MP black holes to involve their uniqueness too. Here $J_{\text{crit}}$ refers to the minimum $J$ for which either instability or non-uniqueness first appear.
introduction of efficient approximate analytical methods. In this way, this organization in
scales provides an outline for a program to investigate higher-dimensional black holes.

We base the effective description of black holes in regime 3 on the idea that the limit \( \ell_M/\ell_J \rightarrow 0 \) of the black hole, keeping its horizon size finite, results in a flat, infinitely extended black brane, possibly boosted along some of its worldvolume directions. This has been observed in all exact solutions known so far: the horizon of Myers-Perry black holes pancakes along the plane of rotation and in the limit becomes a black brane \([3]\), and five-dimensional black rings become thin and locally approach the geometry of boosted black strings \([11, 12]\). This allows us to identify the variables for the effective description. Essentially, the black hole will be regarded as a black brane whose worldvolume spans a curved submanifold of a background spacetime — what we refer to as a blackfold. The simplest example is the characterization of a (thin) black ring as a circular boosted black string, which was worked out in detail in \([6]\) and expanded upon in \([13, 14]\). But, as shown in \([1]\), ultraspinning Myers-Perry black holes, as well as a number of other new black holes, can also be appropriately captured with this approach.

Thus, the effective theory of black holes when \( \ell_M/\ell_J \ll 1 \) is a theory that describes how to bend the worldvolume of a black brane in a background spacetime. In this regard we treat black branes in a manner similar to other familiar extended objects such as cosmic strings or D-branes. The main novelty is that black branes possess black hole horizons, so when their worldvolume is spatially compact we obtain a black hole with finite horizon area.

To leading order in the expansion in \( \ell_M/\ell_J \) the backreaction of the blackfold on the geometry is neglected — it is a ‘test’ blackfold. Corrections to the geometry are found by first computing the linearized gravitational backreaction on the background, which is of order \((\ell_M/\ell_J)^{D-3}\), then this correction induces a perturbation of the near-horizon geometry, and so forth. This can be systematically pursued in the form of a matched asymptotic expansion. For higher-dimensional black rings these first corrections have been computed \([9]\). In this paper, however, we remain at the ‘test blackfold’ level of approximation. This is enough to reveal new kinds of black holes, compute their physical properties, and also study time-dependent situations and stability.

Ref. \([4]\) gave a basic outline of the theory of blackfolds. Here we develop its conceptual basis and considerably improve and generalize the presentation. Its application to specific new classes of higher-dimensional black holes will be discussed in a forthcoming publication.

We draw heavily from the beautiful theory of classical brane dynamics developed by Carter in \([15]\). In this respect, we may say that a main part of our contribution is to

\[3\]The approach to extremality in a black hole introduces a long length scale transverse to the horizon and allows to decouple a different sector of the physics, namely the near-horizon region \([8]\).

\[4\]See \([9, 10]\) for brief reviews of higher-dimensional black holes and \([5]\) for a more extensive one.

\[5\]Our title deliberately highlights this.
apply this theory to black branes, and to interpret the results in the context of higher-dimensional black hole physics according to the general considerations above. But we also emphasize that the general classical brane dynamics, regarded as a long-wavelength effective theory, must take the form of the dynamics of a fluid that lives on a dynamical worldvolume. Black branes correspond to a specific type of fluid, with a certain equation of state and with specific values for transport coefficients. At the leading order that we work in this paper, the fluid is a perfect one and the brane equations are the Euler equations for the fluid \— intrinsic equations \— plus a generalization of the geodesic equation for the motion of a \( p \)-brane \— extrinsic equations for the worldvolume embedding. We believe that in principle it should be possible to incorporate higher-derivative corrections to these and compute transport coefficients by performing a derivative expansion of the underlying microscopic theory, in this case Einstein’s theory.

Closely related precedents of a mapping of black hole dynamics to fluid dynamics are the ‘membrane paradigm’ \[16\], and the more recent ‘fluid/AdS-gravity correspondence’ \[17\]. As we shall argue near the end, the fluid/AdS-gravity correspondence can be embedded within the approach we advocate here \— in fact it has been an important influence in developing it. The general arguments discussed above indicate that a fluid-dynamical description should indeed be expected to exist for any long-wavelength fluctuations around an equilibrium state. From this perspective, perhaps the main qualitative novelty of our approach is that the existence of a hierarchy of scales in higher-dimensional black holes makes this effective theory useful not only for studying fluctuations, but also for constructing and analyzing in a very general manner novel kinds of stationary (\textit{i.e.}, equilibrium) black holes, including vacuum solutions.

The outline of the paper is the following: Section \[2\] develops the conceptual basis underlying the blackfold approach as a worldvolume theory of the dynamics of black branes. Section \[3\] presents a main result of this paper: the \textit{blackfold equations}, a set of coupled non-linear differential equations for the collective coordinates of a neutral black brane. Section \[4\] focuses on the important case of stationary blackfolds, for which the intrinsic subset of these equations can be explicitly solved. Section \[5\] analyzes the issues raised by the possible presence of boundaries of the blackfold worldvolume. In section \[6\] we describe how to compute the physical magnitudes of a blackfold. Section \[7\] presents an action principle for stationary blackfolds. This is useful for practical calculations, but also admits a simple and appealing interpretation in terms of black hole thermodynamics. Section \[8\] discusses briefly the stability of blackfolds exhibiting how the approach can uncover in a remarkably simple way the Gregory-Laflamme instability of black branes. We close in Section \[9\] with a discussion of the relation of blackfolds to other effective theories of black hole dynamics, in particular the fluid/AdS-gravity correspondence. In the appendix we collect a number of technical results on the extrinsic geometry of submanifold embeddings.
Notation and terminology:

For clarity and later reference, we summarize here some of our notation.

For a blackfold of $p$ spatial dimensions in $D$-dimensional spacetime it is convenient to introduce

$$n = D - p - 3.$$  \hfill (1.2)

The codimension of the blackfold worldvolume is $n + 2$.

Spacetime (background) and worldvolume magnitudes are denoted and distinguished as follows:

- Spacetime coordinates (and embedding functions): $X^\mu, \mu, \nu \ldots = 0, \ldots, D - 1$.
  
  Background metric: $g_{\mu\nu}$.
  
  Background metric connection: $\Gamma^\sigma_{\mu\nu}$.
  
  Background covariant derivative: $\nabla_\mu$.

- Worldvolume coordinates: $\sigma^a, a, b \ldots = 0, \ldots, p$.
  
  (Induced) worldvolume metric: $\gamma_{ab}$.
  
  Worldvolume metric connection: $\{^{\ a}_{\ bc}\}$.
  
  Worldvolume covariant derivative: $D_a$.

Indices $\mu, \nu, \ldots$ are lowered and raised with $g_{\mu\nu}$, indices $a, b, \ldots$ with $\gamma_{ab}$.

We use the same letter for a background tensor tangent to the worldvolume, $t_{\mu\ldots\nu\ldots}$, and for its pullback onto the worldvolume, $t^{\ a\ldots\ b\ldots}$ (the only exception is the first fundamental form $h_{\mu\nu}$ and the induced metric $\gamma_{ab}$).

$\Omega_{(n)}$ denotes the volume of the unit $n$-sphere. $\Omega_i$ denotes the angular velocity of the blackfold in the $i$-th direction.

$V_{(p)}$ is the volume of a spatial section of the blackfold. $V_i$ is the spatial velocity field on the worldvolume, and $V = \sqrt{\sum_i V_i^2}$.

Note that we refer to ‘long-wavelength’ and not ‘low-energy’, effective theory, the reason being that in classical gravity, without $\hbar$, such notions are not equivalent (indeed large energies typically imply long distances in classical gravity).

2 Effective worldvolume theory

We present the effective theory of blackfolds trying to highlight the similarities with the field-theoretical effective description of other extended objects, such as cosmic strings or D-branes. The main differences with these are, first, that the short-distance degrees of freedom that are integrated out are not those of an Abelian Higgs model nor massive string modes, but rather purely gravitational degrees of freedom. Second, the extended objects—curved black branes—possess black hole horizons. We obtain the equations using general symmetry and conservation considerations, rather than doing a detailed derivation from first principles.
2.1 Collective coordinates for a black brane

Schematically, the degrees of freedom of General Relativity are split into long and short wavelength components,

$$g_{\mu\nu} = \{g^{(\text{long})}_{\mu\nu}, g^{(\text{short})}_{\mu\nu}\}.$$  \hfill (2.1)

The Einstein-Hilbert action is then approximated as

$$I_{EH} = \frac{1}{16\pi G} \int d^D x \sqrt{-g} R \approx \frac{1}{16\pi G} \int d^D x \sqrt{-g^{(\text{long})} R^{(\text{long})}} + I_{\text{eff}}[g^{(\text{long})}_{\mu\nu}, \phi],$$  \hfill (2.2)

where $I_{\text{eff}}[g^{(\text{long})}_{\mu\nu}, \phi]$ is an effective action obtained after integrating-out the short-wavelength gravitational degrees of freedom (precisely what we mean by this will be made clear in sec. 2.2). The coupling of these to the long-wavelength component of the gravitational field is captured through a set of ‘collective coordinates’ that we denote schematically by $\phi$. Our first task is to identify these effective field variables and the length scales that allow this splitting of degrees of freedom.

The main clue to the nature of the effective theory comes from the observation that the limit $\ell_M/\ell_J \to 0$ of known black holes, when it exists, results in flat black branes. Thus we shall take the effective theory to describe the collective dynamics of a black $p$-brane. Its geometry in $D = 3 + p + n$ spacetime dimensions is

$$ds^2_{p\text{-brane}} = -\left(1 - \frac{r_0}{r}\right) dt^2 + \sum_{i=1}^p (dz^i)^2 + \frac{dr^2}{1 - \frac{r_0}{r}} + r^2 d\Omega^2_{n+1}.$$  \hfill (2.3)

The coordinates $\sigma^a = (t, z^i)$ span the brane worldvolume. A more general form of the metric is obtained by boosting it along the worldvolume. If the velocity field is $u^a$, with $u^a u^b \eta_{ab} = -1$ then

$$ds^2_{p\text{-brane}} = \left(\eta_{ab} + \frac{r_0}{r} u_a u_b\right) d\sigma^a d\sigma^b + \frac{dr^2}{1 - \frac{r_0}{r}} + r^2 d\Omega^2_{n+1}.$$  \hfill (2.4)

The parameters of this black brane solution consist of the ‘horizon thickness’ $r_0$, the $p$ independent components of the velocity $u$ (say, its spatial components $u^i$), and the $D - p - 1$ coordinates that parametrize the position of the brane in directions transverse to the worldvolume, which we denote collectively by $X^\perp$. The $D$ collective coordinates of the black brane are

$$\phi(\sigma^a) = \{X^\perp(\sigma^a), r_0(\sigma^a), u^i(\sigma^a)\}$$  \hfill (2.5)

and in the long-wavelength effective theory one allows $\partial X^\perp$, $\ln r_0$ and $u_i$ to vary slowly along the worldvolume, $W_{p+1}$, over a length scale $R$ much longer than the size-scale of the black brane,

$$R \gg r_0.$$  \hfill (2.6)

\(^6\)For clarity of presentation, at this initial stage we consider that both long and short degrees of freedom obey vacuum gravity dynamics, $R_{\mu\nu} = 0$, but this can be easily generalized, see sec. 3.5 below.
Typically the scale $R$ is set by the smallest intrinsic or extrinsic curvature radius of the worldvolume. Observe that we require slow variations of $\partial X^\perp$, not of $X^\perp$. Like the longitudinal velocities $u^a$, the transverse ‘velocities’ $\partial X^\perp$ can be arbitrary.

In order to preserve manifest diffeomorphism invariance it is convenient to introduce some gauge redundancy and enlarge the set of embedding coordinates of the worldvolume of the black brane to include all the spacetime coordinates $X^\mu(\sigma^a)$. From this embedding we can compute an induced metric

$$\gamma_{ab} = g_{\mu\nu}^{(\text{long})}\partial_a X^\mu \partial_b X^\nu. \quad (2.7)$$

This is naturally interpreted as the geometry induced on the worldvolume of the brane. To understand what this means, regard the split between degrees of freedom as follows: the long-wavelength degrees of freedom live in a ‘far-zone’ $r \gg r_0$, and they describe the background geometry in which the (thin) brane lives. Then (2.7) is the metric induced on the brane worldvolume. The short-wavelength degrees of freedom live in the ‘near-zone’ $r \ll R$. In the strict limit where $R \to \infty$, the near-zone solution is (2.7), but when $R$ is large but finite, the collective coordinates depend on $\sigma$. Also, the long and short degrees of freedom interact together in the ‘overlap’ or ‘matching-zone’ $r_0 \ll r \ll R$, where the metrics $g_{\mu\nu}^{(\text{long})}$ and $g_{\mu\nu}^{(\text{short})}$ must match. Then the near-zone metric for the black brane must be of the form

$$ds^2_{(\text{short})} = \left(\gamma_{ab}(\sigma) + \frac{r_0^{n}(\sigma)}{r^n} u_a(\sigma) u_b(\sigma)\right) d\sigma^a d\sigma^b + \frac{dr^2}{1 - r_0^2(\sigma)/r^n} + r^2 d\Omega_{n+1}^2 + \ldots. \quad (2.8)$$

The dots here indicate that, without additional terms, in general this is not a solution to the Einstein equations. These equations contain terms with gradients of $\ln r_0$, $u^a$ and $\gamma_{ab}$. However these terms can be seen to come multiplied by powers of $r_0$ so they are small when $r_0/R \ll 1$. Then we can consider an expansion of the equations in derivatives and add a correction to (2.8) to find a solution to the Einstein equations to first order in the derivative expansion. A subset of the resulting Einstein equations can be rewritten as equations on the collective field variables $\phi(\sigma)$. An important requirement is that the perturbations preserve the regularity of the horizon, and to this effect working in a set of coordinates (Eddington-Finkelstein type) different than the ones above may be more appropriate.

The development of this line of argument, which can be regarded as a blend of the ideas for the effective descriptions of black hole dynamics in [18, 19] (and references therein), and in [17], produces a systematic derivation of the blackfold equations. This is however a technically involved approach that we hope to discuss elsewhere. Here we shall instead follow a less rigorous but quicker and physically well-motivated path, relying on general effective-theory-type of arguments that allow us to readily obtain the blackfold formalism valid to lowest order in the derivative expansion. As we will see, this is the ‘perfect fluid’ and ‘generalized geodesic’ approximation. The more systematic method outlined above
would be needed to go beyond these approximations and account for dissipation and effects of internal structure and gravitational self-force.

We have considered a black brane in (2.3), (2.4) that is not rotating along the transverse \((n + 1)\)-sphere, nor have we included any possible deformations of it. This is just a simplification and is not essential. Note first that ultraspins in this sphere, with rotation parameter \(a \propto J/M \gg r_0\), can indeed be considered in the blackfold approach, but then the starting point must be the black brane limit that results when \(a/r_0 \to \infty\). On the other hand small spins, with \(a \lesssim r_0\), can also be included easily to the order that we work in this paper. The reason is that the modifications to the blackfold dynamics introduced by these spins only enter at a higher order in the expansion in \(r_0/R\). This is familiar, for instance, in that spin effects on the worldline of a test particle enter through couplings to the background curvature tensor: they reflect the internal structure of the particle, which clearly is a higher-order correction. Thus the dynamics associated to internal spin and polarization effects of the spheres of size \(r_0\) is effectively integrated-over without affecting the lowest-order formalism. It must be noted, though, that higher-dimensional neutral black holes exhibit zero-mode deformations at discrete values of the spin when \(n \geq 3\) (when \(n = 1, 2\) such deformations appear to be always massive modes) \([7]\). For these spins, these deformations can be excited at arbitrarily low frequencies and therefore must be added to the collective coordinates of the black brane. Other than this, the internal spin can be treated just like a conserved charge in the worldvolume. As such, it must satisfy a continuity equation but, since the gravitational effects of this spin fall off faster at large \(r\) than those of the mass, the computation of the effective worldvolume stress tensor in the next subsection is unaffected. The blackfold equations also apply in the form below in the presence of internal spins.

### 2.2 Effective stress tensor

By the phrase ‘integrating out the short-distance dynamics’ we mean that the Einstein equations are solved at distances \(r \ll R\) and then the effects of the solution at distances \(r \gg r_0\) are encoded in a stress-energy tensor that depends only on the collective coordinates. The stress tensor is such that its effect on the long-wavelength field \(g^{(\text{long})}_{\mu\nu}\) is the same as that of the black brane at distances \(r \gg r_0\). For reasons that will become apparent as we proceed, it is both simpler and more convenient to work with an effective stress-energy tensor rather than with an effective action. In any case, nothing is lost since we work at the classical level.

The effective equations from (2.2) are

\[
R^{(\text{long})}_{\mu\nu} - \frac{1}{2} g^{(\text{long})}_{\mu\nu} R^{(\text{long})} = 8\pi G T^{\text{eff}}_{\mu\nu},
\]

\[\text{(2.9)}\]

\(^7\)Also, see \([20]\) for an approach to the calculation of these corrections (for zero-branes) in an effective-theory framework somewhat akin to the spirit in this paper.
where the effective worldvolume stress tensor is

\[ T_{\mu\nu}^{\text{eff}} = -\frac{2}{\sqrt{-g^{(\text{long})}\delta g^{\mu\nu}_{\text{long}}}} \delta I_{\text{eff}} \bigg|_{W_{p+1}}. \]  

(2.10)

We now argue that the appropriate notion for this effective stress-tensor that captures the coupling of the short-wavelength degrees of freedom to the long-wavelength ones, is the quasilocal stress-energy tensor introduced by Brown and York [21]. This is defined by considering a timelike hypersurface that lies away from the black brane and encloses it by extending along the worldvolume directions and the angular directions \( \Omega_{(n+1)} \), i.e., the hypersurface acts as a boundary. Actually, as we explained above, the angular directions are integrated over in our description (and to leading order they do not play any role), so we can simplify the discussion by focusing exclusively on the worldvolume directions of the boundary. If the boundary metric (along worldvolume directions) is \( \gamma_{ab} \) then the quasilocal stress tensor is

\[ T_{ab}^{(\text{quasilocal})} = -\frac{2}{\sqrt{-\gamma}} \frac{\delta I_{\text{cl}}}{\delta \gamma_{ab}}, \]  

(2.11)

where \( I_{\text{cl}} \) is the classical on-shell action of the solution. For our purposes, this is the action where the short-distance gravitational degrees of freedom, \( r \ll R \), are integrated and so it must be the same function of the collective variables as \( I_{\text{eff}} \). Together with the relation (2.7) this implies that we can identify (2.10) with (2.11). 

It is shown in [21] that the Einstein equations with an index orthogonal to the boundary are first-order equations equivalent to the equation of conservation of the quasilocal stress tensor,

\[ D_a T_{ab}^{(\text{quasilocal})} = 0, \]  

(2.12)

where \( D_a \) is the covariant derivative associated to the boundary metric \( \gamma_{ab} \). Hence, solving the equations (2.12) is equivalent to solving (a subset of) the Einstein equations.

Since we identify the stress tensors (2.10) and (2.11), henceforth we drop the superscripts from them. We also drop the superscript \( (\text{long}) \) from the background metric \( g_{\mu\nu} \).

The effective stress tensor is computed in the zone \( r_0 \ll r \ll R \), where the gravitational field is weak and the quasilocal stress tensor \( T_{ab} \) is, to leading order in \( r_0/R \), the same as the ADM stress tensor. For the boosted black \( p \)-brane (2.4) one can readily compute it and find

\[ T_{ab} = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n \left( nu^a u^b - \eta^{ab} \right). \]  

(2.13)

After introducing a slow variation of the collective coordinates the stress tensor becomes

\[ T_{ab}(\sigma) = \frac{\Omega_{(n+1)}}{16\pi G} r_0^n(\sigma) \left( nu^a(\sigma) u^b(\sigma) - \gamma^{ab}(\sigma) \right) + \ldots \]  

(2.14)

where the dots stand for terms with gradients of \( \ln r_0, u^a, \) and \( \gamma_{ab} \), which we are taking to be small and are neglected in this paper.
2.3 General branes: fluid perspective

On general grounds, the long-wavelength effective theory for any kind of brane will take the form of a derivative expansion for an effective stress-energy tensor that satisfies the conservation equations (2.12). This is the dynamics of an effective fluid that lives on the worldvolume spanned by the brane. If the worldvolume theory is isotropic, then to lowest derivative order the stress tensor is that of an isotropic perfect fluid,

\[ T^{ab} = (\varepsilon + P)u^a u^b + P \gamma^{ab}, \]  

(2.15)

with energy density \( \varepsilon \), pressure \( P \) and velocity \( u^a \) satisfying

\[ u^a u^b \gamma_{ab} = -1. \]  

(2.16)

Thermodynamics provides the universal macroscopic description of equilibrium configurations, and fluid dynamics is the general long-wavelength description of fluctuations under the assumption of local equilibrium. So in general there will be an equation of state, which we write in the form \( P(\varepsilon) \), and the system will obey locally the laws of thermodynamics

\[ d\varepsilon = T ds \]  

(2.17)

and Euler-Gibbs-Duhem relation

\[ \varepsilon + P = Ts \]  

(2.18)

where \( T \) is the local temperature and \( s \) the entropy density of the fluid in its rest frame. The fluid may also carry additional conserved charges, but we do not consider these in this paper.

For a black brane, (2.14) tells us that the effective fluid has

\[ \varepsilon = \frac{\Omega (n+1)}{16\pi G} (n+1) r_0^n, \quad P = -\frac{1}{n+1} \varepsilon. \]  

(2.19)

Moreover, in the rest frame of the fluid the Bekenstein-Hawking identification between horizon area and entropy

\[ s = \frac{\Omega (n+1)}{4G} r_0^{n+1} \]  

(2.20)

and between surface gravity and temperature

\[ T = \frac{n}{4\pi r_0} \]  

(2.21)

is well known to reproduce the correct thermodynamic relations (2.17), (2.18).

Going beyond the perfect fluid approximation (2.15), the stress tensor will acquire dissipative terms proportional to gradients of \( \ln r_0 \), \( u^a \), \( \gamma^{ab} \). As discussed above, these are neglected in this paper. At any rate their effects are absent for stationary configurations.
3 Blackfold dynamics

We have argued that the general effective theory of classical brane dynamics can be formulated as a theory of a fluid on a dynamical worldvolume. The fluid variables must satisfy the intrinsic equations (2.12), and they will be coupled to the ‘extrinsic’ equations for the dynamics of the worldvolume geometry, which we still have to determine. To this effect, in the next subsection we introduce a few notions about the geometry of worldvolume embeddings. More details and proofs are provided in the appendix.

3.1 Embedding and worldvolume geometry

Given the induced metric on $\mathcal{W}_{p+1}$, (2.7), the first fundamental form of the submanifold is

$$h^{\mu\nu} = \partial_a X^\mu \partial_b X^\nu \gamma^{ab}. \quad (3.1)$$

Indices $\mu, \nu$ are raised and lowered with $g_{\mu\nu}$, and $a, b$ with $\gamma_{ab}$. Defining

$$\perp_{\mu\nu} = g_{\mu\nu} - h_{\mu\nu} \quad (3.2)$$

it is easy to see that the tensor $h^{\mu\nu}$ acts as a projector onto $\mathcal{W}_{p+1}$, and $\perp^{\mu\nu}$ along directions orthogonal to $\mathcal{W}_{p+1}$.

Background tensors $t^{\mu\cdots\nu\cdots}$ with support on $\mathcal{W}_{p+1}$ can be converted into worldvolume tensors $t^{a\cdots b\cdots}$ and vice versa using $\partial_a X^\mu$. For instance, the velocity field

$$u^\mu = \partial_a X^\mu u^a, \quad (3.3)$$

preserves its negative-unit norm under this mapping.

The covariant differentiation of tensors that live in the worldvolume is well defined only along tangential directions, which we denote by an overbar,

$$\nabla_\mu = h_\mu^{\nu} \nabla_\nu. \quad (3.4)$$

Note that in general $\nabla_\rho t^{\mu\cdots\nu\cdots}$ has both orthogonal and tangential components. The tangentially projected part is essentially the same as the worldvolume covariant derivative $D_c t^{a\cdots b\cdots}$ for the metric $\gamma_{ab}$, both tensors being related via the pull-back map $\partial_a X^\mu$. In particular, the divergence of the stress-energy tensor

$$T^{\mu\nu} = \partial_a X^\mu \partial_b X^\nu T^{ab} \quad (3.5)$$

satisfies (see (A.21))

$$h^{\rho\nu} \nabla_\mu T^{\mu\nu} = \partial_b X^\rho D_a T^{ab}. \quad (3.6)$$

The extrinsic curvature tensor

$$K_{\mu\nu} = h^{\rho\sigma} \nabla_\nu h_\rho^{\sigma}. \quad (3.7)$$
is tangent to $W_{p+1}$ along its (symmetric) lower indices $\mu, \nu$, and orthogonal to $W_{p+1}$ along $\rho$. Its trace is the mean curvature vector

$$K^\rho = h^{\mu\nu} K_{\mu\nu}^\rho = \nabla_\mu h^{\mu\rho}.$$  \hspace{1cm} (3.8)

Explicit expressions for the extrinsic curvature tensor in terms of the embedding functions $X^\mu(\sigma^a)$ can be found in the appendix.

3.2 Blackfold equations

The general extrinsic dynamics of a brane has been analyzed by Carter in [15]. The equations are formulated in terms of a stress-energy tensor with support on the $p+1$-dimensional worldvolume $W_{p+1}$ satisfying the tangentiality condition

$$\nabla_\rho T^{\mu\nu} = 0 .$$  \hspace{1cm} (3.9)

The basic assumptions are that (i) this effective stress-energy tensor derives from an underlying conservative dynamics (in our case, General Relativity), even if the macroscopic (=long-wavelength) dynamics may be dissipative; and that (ii) spacetime diffeomorphism invariance holds, or equivalently, the worldvolume theory can be consistently coupled to the long-wavelength gravitational field $g_{\mu\nu}$. Under these assumptions, the stress tensor must obey the conservation equations

$$\nabla_\mu T^{\mu\rho} = 0.$$

(3.10)

These are in fact the generic equations of motion for the entire set of worldvolume field variables $\phi(\sigma^a)$, both intrinsic and extrinsic: we can decompose (3.10) along directions parallel and orthogonal to $W_{p+1}$ as

$$\nabla_\mu T^{\mu\rho} = \nabla_\mu (T^{\mu\nu} h_{\nu}^\rho) = T^{\mu\nu} \nabla_\mu h_{\nu}^\rho + h_{\nu}^\rho \nabla_\mu T^{\mu\nu}$$

$$= T^{\mu\nu} h_{\nu}^\sigma \nabla_\mu h_{\sigma}^\rho + h_{\nu}^\rho \nabla_\mu T^{\mu\nu}$$

$$= T^{\mu\nu} K_{\mu\nu}^\rho + \partial_\rho X^\rho D_a T^{ab}$$ \hspace{1cm} (3.11)

where in the last line we used (3.6) and (3.7). Thus the $D$ equations (3.10) separate into $D-p-1$ equations in directions orthogonal to $W_{p+1}$ and $p+1$ equations parallel to $W_{p+1}$,

$$T^{\mu\nu} K_{\mu\nu}^\rho = 0 \hspace{1cm} (extrinsic \ equations),$$  \hspace{1cm} (3.12)

$$D_a T^{ab} = 0 \hspace{1cm} (intrinsic \ equations).$$  \hspace{1cm} (3.13)

Let us now apply the equations (3.10) onto the generic stress-energy tensor of a perfect fluid on the worldvolume

$$T^{\mu\nu} = (\varepsilon + P) u^\mu u^\nu + P h^{\mu\nu}.$$  \hspace{1cm} (3.14)

We find

$$u^\mu u^\nu \nabla_\nu \varepsilon + (\varepsilon + P) (u^\mu + u^\nu \nabla_\nu u^\nu) + (h^{\mu\nu} + u^\mu u^\nu) \nabla_\nu P + PK^\mu = 0 ,$$  \hspace{1cm} (3.15)
where

\[ \dot{u} = u^\nu \nabla_\nu u \]  

(3.16)
is the acceleration of \( u^\mu \). These are the general equations, to leading order in the derivative expansion, for the dynamics of a classical brane with worldvolume spatial isotropy, possibly supplemented by conservation equations for charges, if present. A familiar example is a Nambu-Goto-Dirac brane, with \( T_{\mu\nu} = -|P| h_{\mu\nu} \) and \( \nabla P = \nabla \varepsilon = \nabla u = 0 \), for which the extrinsic equations, \( K^\rho = 0 \), require that the worldvolume be a minimal submanifold. But any classical brane will satisfy equations of this form.

As usual the projection of (3.15) onto \( u \) is the continuity equation for the energy of the fluid,

\[ u^\nu \nabla_\nu \varepsilon + (\varepsilon + P) \nabla_\mu u^\mu = 0 , \]  

(3.17)
while the projections orthogonal to \( u \)

\[ (\varepsilon + P) \dot{u}^\mu = -(h^\mu\nu + u^\mu u^\nu) \nabla_\nu P - PK^\mu \]  

(3.18)
say that the force that accelerates an element of the fluid is given along worldvolume directions by pressure gradients (Euler equation) and in directions transverse to the worldvolume by the extrinsic curvature.

For the specific stress tensor of a neutral black brane, (2.19), the equations (3.15) become, after a little manipulation,

\[ \dot{u}^\mu + \frac{1}{n+1} u^\mu u^\nu \nabla_\nu u^\nu = \frac{1}{n} K^\mu + \nabla^\mu \ln r_0 . \]  

(3.19)
These blackfold equations describe the general collective dynamics of a neutral black brane.

Again, we can decompose them into different projections. In directions orthogonal to the worldvolume we have

\[ K^\rho = n \nabla^\rho \dot{u}^\mu . \]  

(3.20)
The equivalence of this equation to (3.12) follows by using (A.12).

The equations parallel to the worldvolume are

\[ h_{\mu\nu} \dot{u}^\nu + \frac{1}{n+1} u^\mu \nabla_\nu u^\nu = \nabla_\mu \ln r_0 , \]  

(3.21)
which we can also write using worldvolume indices and derivatives,

\[ \dot{u}_a + \frac{1}{n+1} u_a D_b u^b = \partial_a \ln r_0 , \]  

(3.22)
with \( \dot{u}^b = u^c D_c u^b \). Thus the temporal and spatial worldvolume gradients of \( r_0 \) determine the worldvolume acceleration and expansion of \( u \), respectively.

Although we have emphasized the fluid-dynamical interpretation of the equations, it is interesting to observe that the extrinsic equations (3.12), when written explicitly in terms of the embedding \( X^\mu (\sigma^a) \) become

\[ T^{ab} \nabla^\sigma_{\rho} \left( \partial_a \partial_b X^\sigma + \Gamma^\sigma_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \right) = 0 , \]  

(3.23)
or alternatively
\[ T^{ab} \left( D_a \partial_b X^\mu + \Gamma^\rho_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \right) = 0 \] (3.24)

(see eqs. (A.28) and (A.30)). These can be regarded as generalizations to \( p \)-branes of the geodesic equation for free particles, or more simply, of “mass \times \text{acceleration}= 0”.

Blackfolds differ from other branes in that they represent objects with black hole horizons. In the long-distance effective theory we lose sight of the horizon, since its thickness is of the order of the scale \( r_0 \) that we integrate out. But the presence of the horizon is reflected in the effective theory in the existence of an entropy and in the local thermodynamic equilibrium of the effective fluid. Indeed, we shall assume that the regularity of the event horizon under long-wavelength perturbations —including those that bend the worldvolume away from the flat geometry or that excite the effective fluid away from equilibrium— is satisfied when the blackfold equations, which incorporate in particular local thermodynamic equilibrium, are satisfied. A proof of this statement requires the rigorous derivation, outlined at the end of sec. 2.1 of the blackfold equations in the derivative expansion of Einstein’s equations, which is outside the scope of this paper. However, there is already significant evidence that horizon regularity is preserved for blackfold solutions.

First, analyses of the perturbations of black strings that bend them into a circle [6] (and extensions thereof to branes curved into tori) show that the extrinsic equations (3.12) are equivalent to demanding absence of singularities on or outside the horizon. Second, the intrinsic, hydrodynamical perturbations of a black brane in AdS have been studied in detail in [17] and shown to be consistent with horizon regularity. Note however that the regular solution to higher orders may not preserve the same symmetries as the lowest-order solution. In particular, in some cases horizon regularity may require to abandon stationarity of the configuration at higher orders. In most instances it appears easy to decide from the physics of the problem whether such an effect is expected (specific examples will be discussed elsewhere), but it would be good if precise conditions could be stated in generality.

3.3 The metric at all length scales: Matched asymptotic expansion

Under the splitting in (2.2), the set of field variables in the system are the collective worldvolume fields, intrinsic and extrinsic, and the background gravitational field \( g_{\mu\nu} \). The complete set of equations are the extrinsic equations (3.12), intrinsic equations (3.13), and backreaction equations (2.9). Since they are a consequence of general symmetry and conservation principles, these equations retain their form at any perturbative order. The example studied in [22] appears to fall outside the remit of our approach since the blackfold (a small black hole) is not a small perturbation of the background spacetime.

This, however, is a somewhat formal statement due to the appearance of gravitational self-force divergences on the worldvolume that must be dealt with carefully [18]. Ref. [23] shows how the equation of stress tensor conservation can be used as the basis to obtain these corrections to particle motion.
specific form of the stress tensor, as well as the background metric, will in general be corrected at higher orders.

The only equations that one has to solve at the lowest order are those that suffice to ensure that $T_{\mu\nu}$ can be consistently coupled to the long-wavelength gravitational field. These are just the intrinsic and extrinsic equations, and backreaction is neglected. The explicit blackfold equations (3.19) that result are valid only for test branes.

From the point of view of effective field theory one is interested only in quantities that are measured in the long-wavelength regime. The short-wavelength dynamics enters only to determine the coefficients in the effective stress tensor, which can be computed by, e.g., matching the calculations of some observables [20]. However, in General Relativity one is often interested in also having an explicit solution, even if an approximate one, for the geometry at all scales, including near the horizon of the black hole. The systematic way to construct an explicit metric in an expansion in $r_0/R$ is through the method of matched asymptotic expansions [18]. In the context of blackfolds this was discussed in [6] (following [24, 25]), and explicitly applied to the construction of higher-dimensional black rings. We review it briefly here.

As described in sec. 2.1, the full geometry splits into near- and far-zones, that share a common overlap-zone. To zeroth order in $r_0/R$ the near-zone metric is (2.4) and the far-zone metric is the background metric $g_{\mu\nu}$. This is as far as we go in this paper in terms of providing explicit solutions to the Einstein equations: this is the test-brane approximation. The next order involves the gravitational backreaction of the brane: the equations (2.9) are linearized around the background and solved with the distributional worldvolume source $T_{\mu\nu}$. The blackfold equations would appear here among the Einstein equations as constraints (first-order equations), but we can assume they have already been solved at zeroth order. The solution of the linearized equations with appropriate asymptotics (e.g., asymptotic flatness) produces a corrected far-zone metric with corrections of order $(r_0/R)^n$. Its value in the overlap-zone provides new asymptotic conditions for the near-zone solution. The next step is to perturb the metric (2.4) linearly, with the boundary conditions that the horizon remains regular and that in the overlap zone the metric matches the corrected far-zone solution. In this manner we produce a new, corrected solution, at all scales. This process can then be iterated to higher orders.

3.4 Lumpy blackfolds

The formalism developed in this paper can easily be applied to other branes once their effective equation of state is known. In particular, there are other neutral black branes in vacuum gravity than the ‘smooth’ black branes of (2.3): refs. [26, 27] have shown that ‘lumpy’ black branes exist, branching off at the threshold of the Gregory-Laflamme (GL) instability [28]. Their horizons are inhomogeneous on a scale $\sim r_0$, so this small-scale inhomogeneity is averaged over in our effective description. However, there is an effect on the effective stress tensor measured at large distance from the brane, since the equation of
state (which is known only perturbatively near the GL threshold, or possibly numerically) is in general different than the one for smooth branes (2.19).

We can use these lumpy branes as the basis for the construction of lumpy blackfolds. It should be clear that their worldvolume is smooth, but the horizons of these blackfolds are inhomogeneous on the scale \( r_0 \). The simplest example would be a lumpy black ring, built by bending a lumpy black string into a circular shape. This example serves also to illustrate a feature of lumpy blackfolds: the lumps in a rotating lumpy black ring will emit gravitational radiation, so the ring will lose mass and angular momentum as it evolves in time. In more generality, if the lumps on a blackfold extend along a direction in which the fluid velocity is non-zero, and if this direction is not an isometry, then the lumps moving along these orbits will give rise to a varying quadrupole and hence to gravitational radiation.

Once this effect is taken into account, lumpy blackfolds are generically expected to evolve in time and not remain stationary. However, the time-scale for this evolution will be very long. The effect is only visible when the small scale is resolved, so it is suppressed by a power of \( r_0/R \). It will be further suppressed by the fact that gravitational radiation couples to a higher-multipole (quadrupole) and therefore is rather inefficient. In [29] this time scale was estimated for five-dimensional rings, and extending this estimate to black rings in \( D = 4 + n \) dimensions we find the time to be of order \( T_{gw} \sim R(R/r_0)^n \), longer by a factor \( (R/r_0)^{n+1} \) than the short time-scale \( r_0 \). Lumpy black branes may also be affected by GL-like instabilities, but these have not been investigated yet.

### 3.5 Generalization to non-vacuum theories

In (2.1) and (2.2) we assumed that the full dynamics at all wavelengths is described by vacuum General Relativity, i.e., the Einstein-Hilbert action with no matter nor cosmological constant. However, this is not actually necessary for our derivation of the equations of motion and the effective stress tensor. The only part of the field that is actually required to be governed by the Einstein-Hilbert action is the sector of short-wavelength degrees of freedom that we integrate out in order to obtain the effective stress tensor (2.13). For the long-wavelength components we only require diffeomorphism invariance, which implies the equations of motion (3.10). Thus, the blackfold equations (3.19) are enough to describe neutral blackfolds in any configuration that, at small distances, is dominated by the Einstein-Hilbert term. For instance, this will be the case for blackfolds in the presence of a cosmological constant as long as

\[
    r_0 \ll |\Lambda|^{-1/2} \tag{3.25}
\]

(see [13] for an explicit application), or for blackfolds in an external background gauge field as long as the typical length scale of the background field around the blackfold is much larger than \( r_0 \). No restriction on \( R \) other than \( R \gg r_0 \) needs to be imposed.
A different situation arises for charged blackfolds, since then the gauge field has short-wavelength components. The effective fluid is then charged, and additional current conservation equations must be added. This extension of our analysis will be discussed elsewhere.

4 Stationary blackfolds

Equilibrium configurations that remain stationary in time are of particular interest. For a blackfold, they correspond to stationary black holes. In this case it is possible to solve the blackfold equations explicitly for the worldvolume variables, namely the thickness $r_0$ and velocity $u$, so one is left only with the extrinsic equations for the worldvolume embedding $X^\mu(\sigma)$.

We employ a general result proven in [30] for stationary fluid configurations: if dissipative effects must be absent, then the fluid (intrinsic) equations require that the velocity field be proportional to a worldvolume Killing field $k^a \partial_a$. That is,

$$u = k/|k|$$  \hspace{1cm} (4.1)

where

$$|k| = \sqrt{-\gamma^{ab}k_a k_b}$$  \hspace{1cm} (4.2)

and $k$ satisfies the worldvolume Killing equation

$$D(\kappa k_b) = 0.$$  \hspace{1cm} (4.3)

This does not necessarily mean that $k$ is a Killing field of the background away from $W_{p+1}$, as the worldvolume could be at a locus of enhanced symmetry. However, this is not generic, and indeed when the blackfold thickness is small but non-zero, $k$ should satisfy the Killing equations on a finite region around the worldvolume. Thus we assume the existence of a timelike Killing vector $k^\mu \partial_\mu$ in the background,

$$\nabla(\kappa k_\nu) = 0,$$  \hspace{1cm} (4.4)

whose pull-back onto the worldvolume determines the velocity field as in (4.1). The existence of a timelike Killing vector field is in fact a necessary assumption if we intend to describe stationary black holes.

The contraction of the Killing equation (4.4) with $k^\mu k^\nu$ implies $k^\mu \partial_\mu |k| = 0$, and using this it follows easily that\footnote{For a stationary conformal fluid this is relaxed to the conformal Killing equation $D(\kappa k_b) = \lambda_\gamma k_b$. Conformal fluids do not dissipate the expansion of $u$ since the bulk viscosity vanishes.}

$$\dot{u}^\mu = \partial^\mu \ln |k|. \hspace{1cm} (4.5)$$

\footnote{In [30] it is shown that the worldvolume fluid equations directly imply $u^b D_b u^a = \partial^a \ln |k|$, which implies the worldvolume projection of (4.5), but not the orthogonal component of it, which will be used later below in eq. (4.11).}

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Since the expansion of \( u \) vanishes, the intrinsic blackfold equation (3.22) becomes
\[
\partial_a \ln |k| = \partial_a \ln r_0
\] (4.6)
so
\[
\frac{r_0}{|k|} = \text{constant}.
\] (4.7)
In order to fix the proportionality constant in (4.7), we turn to the fact that the stationary blackfold describes a black brane with a Killing horizon.

Above we have defined \( k \) as a Killing vector in the background. Its norm \( k^2 = -|k|^2 \) computed with the worldvolume metric \( \gamma_{ab} \) is negative, i.e., it is a timelike vector. But recall that we regard the background metric as only one part, the far-zone metric at \( r \gg r_0 \), of the full geometry. For \( r \ll R \) the geometry is instead well approximated by the near-zone metric (2.8), which matches with the far-zone metric in \( r_0 \ll r \ll R \). It is natural to extend \( k \) as a Killing vector to all the geometry, including the near-zone (2.8). Using the metric \( g^{(\text{short})}_{\mu\nu} \) in this region the norm of \( k \) is
\[
g^{(\text{short})}_{\mu\nu} k^\mu k^\nu = \left( \gamma_{ab} + \frac{r^0}{r^n} u_a u_b \right) k^a k^b = - \left( 1 - \frac{r_0}{r^n} \right) |k|^2
\] (4.8)
where we used (4.1) and (4.2). Thus \( k \) becomes null as we approach the horizon \( r \to r_0 \) and indeed it is the null Killing generator of the horizon. Its surface gravity is easily obtained as
\[
\kappa = \frac{n|k|}{2r_0}.
\] (4.9)
Equation (4.7) tells us that \( \kappa \) is a constant over the worldvolume of the blackfold. That is, the surface gravity is uniform not only over the \((n+1)\)-sphere but over the entire horizon.

Eqs. (4.1) and (4.9) provide the general solution to the intrinsic equations for stationary blackfolds. To give them a more explicit expression it is convenient to choose a preferred set of orthogonal commuting vectors of the background geometry, \( \xi \) timelike and \( \chi_i \) spacelike, such that the Killing vector \( k \) is a linear combination of them
\[
k = \xi + \sum_i \Omega_i \chi_i,
\] (4.10)
with constant \( \Omega_i \). Clearly \( i \) runs at most up to \( p \). A convenient (but not necessary) choice is to take for \( \xi \) the generator of asymptotic time translations, and for \( \chi_i \) the Cartan generators of asymptotic rotations with closed orbits of periodicity \( 2\pi \). They will often be Killing vectors themselves, but in principle only the linear combination in \( k \) need be so.

Introduce a set of worldvolume functions \( R_a(\sigma) \) via their norms on the worldvolume,
\[
R_0 = \sqrt{-\xi^2} \big|_{W_{p+1}}, \quad R_i = \sqrt{\chi_i^2} \big|_{W_{p+1}}.
\] (4.11)
The \( R_a \) must be regarded as part of the embedding coordinates \( X^\mu(\sigma) \). If \( \xi \) is the canonically-normalized generator of asymptotic time translations then \( R_0 \) is a redshift
factor between infinity and the blackfold worldvolume. The $R_i$ are the proper radii of the orbits generated by $\chi_i$ along the worldvolume. The $\Omega_i$ are the horizon angular velocities relative to observers that follow orbits of $\xi$.

The vectors
\[
\frac{\partial}{\partial t} = \frac{1}{R_0} \xi, \quad \frac{\partial}{\partial z_i} = \frac{1}{R_i} \chi_i
\]
(no sum in $i$) are orthonormal with respect to the metric $\gamma_{ab}$ on the worldvolume. It will also be useful to regard them as vectors that extend into all of $(2.8)$.

Let us introduce the worldvolume spatial velocity field
\[
V_i(\sigma) = \frac{u \cdot \partial_z i}{-u \cdot \partial_t} = \frac{\Omega_i R_i(\sigma)}{R_0(\sigma)}
\]
so that
\[
k = R_0 \left( \frac{\partial}{\partial t} + \sum_i V_i \frac{\partial}{\partial z_i} \right)
\]
and
\[
|k| = \left( -\xi^2 - \sum_i \Omega_i^2 \chi_i^2 \right)^{1/2}
= R_0 \sqrt{1 - V^2},
\]
where
\[
V^2 = \sum_i V_i^2 = \frac{1}{R_0^2} \sum_i \Omega_i^2 R_i^2.
\]
Thus $|k|$ can be regarded as the relativistic Lorentz factor at a point in $W_{p+1}$, with a possible local redshift, all relative to the reference frame of $\xi$-static observers. Plugging $(4.15)$ into $(4.9)$ we obtain that for given $\kappa$ and $\Omega_i$ the thickness $r_0$ is solved in terms of the $R_a$ as
\[
r_0(\sigma) = \frac{n R_0(\sigma)}{2\kappa} \sqrt{1 - V^2(\sigma)}.
\]

The conditions that $\kappa$ and $\Omega_i$ must remain uniform over the blackfold worldvolume were referred to in [1] as the blackness conditions and were imposed, invoking general theorems for stationary black holes — zeroth law of black hole mechanics and horizon rigidity [31, 32, 33] —, as necessary conditions for the regularity of the black hole horizon. Here instead we have derived them as general consequences of stationary fluid dynamics, where $\kappa$ and $\Omega_i$ appear as integration constants.

As a matter of fact it is also possible to work entirely within the framework of the effective theory and avoid any reference to the short-wavelength geometry of the horizon. Using only the fluid and thermodynamics equations one can derive, like in [30], that the variation of the local temperature $T$ (2.21) along the worldvolume is dictated by the local redshift
\[
T = \frac{T}{R_0 \sqrt{1 - V^2}}.
\]
The integration constant $T$ can then be interpreted, using the thermodynamic first law that we derive below, as the overall temperature of the black hole. As expected, $T = \kappa / 2\pi$. However, we think it is instructive to see how the concepts of black hole physics, like null horizon generators and surface gravity, are recovered in this scheme.

5 Blackfolds with boundaries

Typically branes (such as D-branes or cosmic strings) acquire boundaries when they intersect or end on other branes. This effect is accounted for by adding boundary terms to the effective stress tensor. However, black branes (and other fluid branes) may also have ‘free’ boundaries without any boundary stresses.

Consider the case where the worldvolume of the brane has a timelike boundary $\partial W_{p+1}$. Assume there is a smooth (but otherwise arbitrary) extension $\tilde{W}_{p+1}$ of $W_{p+1}$ across the boundary. Introduce a level-set function $f(\sigma^a)$ such that $f > 0$ in $W_{p+1}$ and $f < 0$ in $\tilde{W}_{p+1} - W_{p+1}$, with $f(\sigma^a) = 0$ on the boundary $\partial W_{p+1}$. Then $-\partial_a f$ is a one-form normal to $\partial W_{p+1}$ pointing away from the fluid. The stress tensor is

$$T_{ab} = [(\epsilon + P)u_a u_b + P \gamma_{ab}] \Theta(f),$$

where $\Theta$ is the step function.

If the fluid is to remain within its bounds then the boundary must be advected with the fluid, i.e., it must be Lie-dragged by $u$,

$$\mathcal{L}_u|_{\partial W_{p+1}} f = 0$$

or equivalently, the velocity must remain parallel to the boundary,

$$u^a \partial_a f|_{\partial W_{p+1}} = 0.$$  \hspace{1cm} (5.3)

At the boundary, the stress-energy conservation equation (3.13) becomes

$$((\epsilon + P) u_a u_b + P \gamma_{ab}) \partial^a f|_{\partial W_{p+1}} = 0.$$  \hspace{1cm} (5.4)

Imposing (5.3) we find that the pressure must approach zero at the boundary,

$$P|_{\partial W_{p+1}} = 0.$$  \hspace{1cm} (5.5)

This is simply the Young-Laplace equation for a bounded fluid when there is no surface tension that could balance the fluid pressure at the boundary. That is, we do not introduce any such boundary stresses, although, as we mentioned they are of interest when one studies brane intersections.

For a neutral blackfold, vanishing pressure at the boundary means

$$r_0|_{\partial W_{p+1}} = 0,$$  \hspace{1cm} (5.6)
which has a nice geometric interpretation: the thickness of the horizon must approach zero size at the boundary, so the horizon closes off at the edge of the blackfold.

If the blackfold is stationary, the condition \(|k| \to 0\) means that the fluid velocity becomes null at the boundary. This may happen either because the boundary is an infinite-redshift surface, \(R_0 \to 0\), or perhaps more commonly, because the fluid approaches the speed of light at the boundary,

\[
V^2 |_{\partial W_{p+1}} = 1. \tag{5.7}
\]

An example of this was presented in [1], where a rigidly-rotating blackfold disk was constructed and shown to reproduce accurately the properties of MP black holes with one ultraspin.

The extrinsic equations (3.20) must also be satisfied and they impose further constraints on the worldvolume variables. In the stationary case, using (4.5) and (4.7) we see that if the extrinsic curvature of the worldvolume remains finite at \(\partial W_{p+1}\), then not only \(r_0 \to 0\) but also \(\perp^{\mu} \partial_\mu r_0\) must vanish there at least as quickly as \(r_0\).

6 Horizon geometry, mass and angular momenta

The blackfold construction puts, on any point in the spatial section \(B_p\) of \(W_{p+1}\), a (small) transverse sphere \(s^{n+1}\) with Schwarzschild radius \(r_0(\sigma)\). Thus the blackfold represents a black hole with a horizon geometry that is a product of \(B_p\) and \(s^{n+1}\) — the product is warped since the radius of the \(s^{n+1}\) varies along \(B_p\). The null generators of the horizon are proportional to the velocity field \(u\).

If \(r_0\) is non-zero everywhere on \(B_p\) then the \(s^{n+1}\) are trivially fibered on \(B_p\) and the horizon topology is

\[
\text{(topology of } B_p) \times s^{n+1}. \tag{6.1}
\]

However, we have seen that if \(B_p\) has boundaries then \(r_0\) will shrink to zero size at them, resulting in a non-trivial fibration and different topology. A simple but very relevant instance of this happens when \(B_p\) is a topological p-ball. Then the horizon topology can easily be seen to be \(S^{p+n+1} = S^{D-2}\).

To analyze the horizon geometry we go to the metric (2.4) that locally describes the geometry of the blackfold to lowest order in \(r_0/R\) in the region \(r \ll R\). There we can choose a local orthonormal frame \((\partial_t, \partial_\sigma)\) on the worldvolume, such that \(\partial_t\) coincides in the overlap zone \(r_0 \ll r \ll R\) with the timelike unit normal \(n^\mu\) to \(B_p\),

\[
n^\mu = (\partial_t)^\mu. \tag{6.2}
\]

To lowest order in \(r_0/R\) the worldvolume metric is Minkowski and the spatial metric on the horizon at \(r = r_0\) is

\[
ds_H^2 = (\delta_{ij} + u_i u_j) dz^i dz^j + r_0^2 d\Omega_{(n+1)}^2 \tag{6.3}
\]
with \( u_i = u \cdot \partial z_i \). Then the local area density of the horizon \( a_H \) at a given point in \( B_p \) is \( \frac{\Omega_{aH}^{n+1}}{\sqrt{1 + \delta_{ij}u_iu_j}} \). 

For a stationary blackfold the choice for \( \partial_t \) and \( \partial_z \) was made in (4.12), so \( n^\mu = \frac{1}{R_0} \xi^\mu \), \n(6.5)\n
and \( u_i = \frac{\Omega_i R_i}{R_0 \sqrt{1 - V^2}} \) \n(6.6)\n
so the area density is \( \frac{\Omega_{(n+1)R_0^{n+1}}}{\sqrt{1 - V^2}} \) \n(6.7)\n
The total area of the horizon is then \( A_H = \int_{B_p} dV(p) a_H(\sigma^a) \), \( \frac{\Omega_{(n+1)R_0^{n+1}}}{\sqrt{1 - V^2}} \) \n(6.8)\n
where \( dV(p) \) denotes the volume form in \( B_p \).

Again, we could have avoided any reference to the geometry of the horizon and worked instead exclusively with quantities defined in the effective fluid theory. The entropy density of the blackfold fluid is given in (2.20) and after taking the relativistic Lorentz factor \( \sqrt{1 - V^2} \) into account, the total entropy is \( \frac{\Omega_{(n+1)R_0^{n+1}}}{4G} \int_{B_p} dV(p) R_0^{n+1}(\sigma^a)(1 - V^2(\sigma^a))^2 \) \n(6.9)\n
in agreement with the geometric area computed from (6.8). The geometric interpretation involves short-wavelength physics, but is useful for exhibiting how the blackfold construction gives precise information about the entire horizon geometry, including the size of the \( s^{n+1} \).

The mass and angular momenta are conjugate to the generators of asymptotic time translations and rotations, which we assume are the vectors \( \xi \) and \( \chi_i \) that we introduced in sec. 4. Then \n(6.10)\n
Plugging here (6.5) and the results from sec. 4 we obtain \( \frac{\Omega_{(n+1)R_0^{n+1}}}{16\pi G} \left( \frac{n}{2\kappa} \right)^n \int_{B_p} dV(p) R_0^{n+1}(1 - V^2)^{\frac{n-2}{2}} (n + 1 - V^2) \) \n(6.11)\n
\footnote{In a more general case the relation may involve linear combinations, but this, although straightforward, comes at the expense of more cumbersome expressions.}
and
\[ J_i = \frac{\Omega^{(n+1)}}{16\pi G} \left( \frac{n}{2\kappa} \right)^n n\Omega_i \int_{B_p} dV(p) R_0^{n-1} (1 - V^2)^{\frac{n-2}{2}} R_i^2. \] (6.12)

It is easy to extract some interesting consequences of these results. Let us assume that all length scales along \( B_p \) are \( \sim R \) and that the velocities and redshift are moderate (i.e., \( 1 - V^2 \) and \( R_0 \) of order one) over almost all the blackfold. Then the two black hole length scales introduced in (1.1) are
\[ \ell_M \sim (r_0^n R^p)^{\frac{1}{D-3}} \quad \ell_J \sim R, \] (6.13)
and the small expansion parameter for the effective theory is
\[ \left( \frac{\ell_M}{\ell_J} \right)^{D-3} \sim (\frac{r_0}{R})^n. \] (6.14)

It is interesting to compare the areas (i.e., entropies) of different blackfolds in a given dimension \( D \). In [6] we introduced the dimensionless angular momentum \( j \) and dimensionless area \( a \) for a given mass \( 13 \)
\[ j \sim \frac{J}{M(GM)^{1/(D-3)}} \sim \frac{\ell_J}{\ell_M}, \quad a \sim \frac{A_H}{\ell_M^{D-2}} . \] (6.15)
The blackfold approximation requires \( j \gg 1 \). Since \( A_H \sim r_0^{n+1} R^p \) we find that
\[ a(j) \sim j^{-\frac{n}{D-3-p}}. \] (6.16)
This shows that for a given number of large non-zero angular momenta, the blackfold with smallest \( p \) is entropically preferred at fixed mass. This is just like we observed for single-spin MP black holes and black rings in [6]: for a given mass, the smaller \( p \), the thicker the horizon, thus the cooler the black hole, and (since \( \kappa A_H \sim GM \) i.e., constant for fixed mass) the higher its entropy.

7 Action principle and first law of stationary blackfolds

7.1 Action for the embedding geometry

We have presented in sec. 4 the general solution to the intrinsic equations for a stationary blackfold. Given a Killing vector field \( \mathbf{k} \), eqs. (4.1), (4.15) and (4.17) allow to eliminate the variables \( r_0(\sigma) \) and \( u^a(\sigma) \) in terms of the embedding functions \( X^\mu(\sigma) \). The remaining extrinsic equation (3.20) that determines the embedding can be written, using (4.5), as
\[ K^\rho = \perp^{\rho\mu} \partial_\mu \ln |\mathbf{k}|^n. \] (7.1)

\footnote{One should not confuse the dimensionless total area for fixed mass \( a \) with the blackfold area density \( a_H \).}
Using the result \[ (A.40) \] from the appendix, this equation can be obtained from the action

\[
I[X^\mu(\sigma)] = \int_{\mathcal{W}_{p+1}} d^{p+1} \sigma \sqrt{-\gamma |k|^n}
\] (7.2)

by considering variations of \( X^\mu \) in directions transverse to the worldvolume.

We can write it in a form that is particularly practical for obtaining the blackfold equations in specific calculations. First observe that the asymptotic time, conjugate to the vector \( \xi \), is related to proper time \( t \) on the worldvolume by a factor of \( R_0 \). If we take the interval for the (trivial) integration over asymptotic time to have finite length \( \beta \), then

\[
\beta \int_{B_p} dV(p) R_0 |k|^n = \beta \int_{B_p} dV(p) R_0 |k|^n.
\] (7.3)

Using now (4.15) we find

\[
I[X^\mu(\sigma)] = \beta \int_{B_p} dV(p) R_0^{n+1} (1 - V^2)^{\frac{n}{2}}
\]

\[
= \beta \int_{B_p} dV(p) R_0(\sigma) \left( R_0^2(\sigma) - \sum_i \Omega_i^2 R_i^2(\sigma) \right)^{\frac{n}{2}}.
\] (7.4)

We emphasize again that \( R_\alpha \) are among the worldvolume field variables \( X^\mu(\sigma) \). Of course these enter as well through \( dV(p) \).

### 7.2 First law

Using eqs. (6.8), (6.11), (6.12), it is straightforward to check that the action (7.4) is

\[
I = \beta \left( M - \sum_i \Omega_i J_i - \frac{\kappa}{8\pi G} A_H \right).
\] (7.5)

This identity holds for any embedding, not necessarily a solution to the extrinsic equations. Thus, if we regard \( M \), \( J_i \) and \( A_H \) as functionals of \( X^\mu(\sigma) \), and consider variations at fixed surface gravity and angular velocities, we have

\[
\frac{\delta I}{\delta X^\mu} = 0 \quad \Leftrightarrow \quad \frac{\delta M}{\delta X^\mu} - \sum_i \Omega_i \frac{\delta J_i}{\delta X^\mu} - \frac{\kappa}{8\pi G} \frac{\delta A_H}{\delta X^\mu} = 0.
\] (7.6)

Hence, solutions of the blackfold equations satisfy the ‘equilibrium state’ version of the first law of black hole mechanics. Conversely, the blackfold equations for stationary configurations can be obtained as the requirement that the first law be satisfied. If we regard \( \kappa \) and \( \Omega_i \) as Lagrange multipliers we may also say that blackfolds extremize the horizon area for given mass and angular momenta.

\[^{14}\text{In [11] it was asserted that the blackfold equations can be derived from the action } \int_{\mathcal{W}_{p+1}} d^{p+1} \sigma \sqrt{-\gamma} T^{ab} \gamma_{ab}. \text{ This is not true in general, but is correct in the stationary case since } T^{ab} \gamma_{ab} \propto r_0^n \propto |k|^n.\]
In the Euclidean quantum gravity approach to black hole thermodynamics it is natural to take $\beta$ to be the period of Euclidean time, $\beta = 1/T$. Using $\kappa A_H/8\pi G = TS$ and eq. (7.5), we see that $I$ is equal, up to a factor, to the Gibbs free energy $G$, 

$$\beta^{-1} I = G = M - \sum_i \Omega_i J_i - TS. \quad (7.7)$$

Therefore (7.2) can be identified as the effective action that approximates, in the blackfold regime $r_0/R \ll 1$, the gravitational Euclidean action of the black hole [34].

It is also possible to find action functionals for general, possibly time-dependent blackfolds by adapting the action principles developed for perfect fluids [35]. However, the usefulness of these actions, which must deal with constraints such as $u^2 = -1$, appears to be somewhat limited so we do not dwell on them.

### 8 Gregory-Laflamme and correlated thermodynamic instability in blackfolds

The blackfold approach must capture the perturbative dynamics of a black hole when the perturbation wavelength $\lambda$ is long,

$$\lambda \gg r_0. \quad (8.1)$$

These perturbations can be either intrinsic variations in the thickness $r_0$ and local velocity $u$, or extrinsic variations in the worldvolume embedding geometry $X$. In general, these two kinds of perturbations are coupled. A detailed analysis of the perturbations of solutions to the blackfold equations and their stability will be presented elsewhere. Here we extract some simple but important consequences for perturbations with wavelength

$$r_0 \ll \lambda \ll R, \quad (8.2)$$

i.e., perturbations for which the worldvolume looks essentially flat, $K_{\mu\nu}\rho_0 \approx 0$. In this case it is easy to see that the intrinsic and extrinsic perturbations decouple.

It is instructive to perform the analysis for a general perfect fluid (2.15), and then particularize to the neutral blackfold fluid (2.19). For simplicity we consider a fluid initially at rest $u^a = (1, 0 \ldots)$, with uniform equilibrium energy density $\varepsilon$ and pressure $P$. The flat worldvolume metric is parametrized, in ‘static gauge’, using orthonormal coordinates $X^0 = t, X^i = z^i, i = 1, \ldots p$ and the transverse coordinates $X^m$ are held at constant values. Introduce small perturbations

$$\delta \varepsilon, \quad \delta P = \frac{dP}{d\varepsilon} \delta \varepsilon, \quad \delta u^a = (0, v^i), \quad \delta X^m = \xi^m, \quad (8.3)$$

and work to linearized order in them. The perturbed stress tensor is

$$T^{tt} = \varepsilon + \delta \varepsilon, \quad T^{ti} = (\varepsilon + P)v^i, \quad T^{ii} = P + \frac{dP}{d\varepsilon} \delta \varepsilon, \quad (8.4)$$
and the extrinsic curvature

\[ \delta K_{ab}^m = \partial_a \partial_b \xi^m. \]  

The extrinsic equations (3.12) then become

\[ (\varepsilon \partial_t^2 + P \partial_i^2) \xi^m = 0. \]  

Thus transverse, elastic oscillations of the brane propagate with speed

\[ c_T^2 = -\frac{P}{\varepsilon}. \]  

The intrinsic equations (3.13) are

\[ \partial_t T_{tt} + \partial_i T_{it} = 0, \quad \partial_t T_{ti} + \partial_j T_{ji} = 0, \]  

which can be combined into

\[ \partial_t^2 T_{tt} - \partial_{ij} T_{ij} = 0. \]  

For (8.4) we find

\[ \left( \partial_t^2 - \frac{dP}{d\varepsilon} \partial_i^2 \right) \delta \varepsilon = 0, \]  

so longitudinal, sound-mode oscillations of the fluid propagate with speed

\[ c_L^2 = \frac{dP}{d\varepsilon}. \]  

These derivations of eqs. (8.7) and (8.11) are hardly new: they are conventional ways to obtain the speeds of elastic and sound waves — in fact they have been obtained in [36, 37] for brane dynamics. They have a remarkable consequence: a brane with a worldvolume fluid equation of state such that

\[ \frac{P}{\varepsilon} \frac{dP}{d\varepsilon} > 0 \]  

has

\[ c_L^2 c_T^2 < 0 \]  

and so is unstable to either longitudinal or transverse oscillations with wavelengths in the range (8.2). For instance this happens in the simple case \( P = w\varepsilon \) with constant \( w \), where the interpretation is easy (we assume \( \varepsilon > 0 \)): positive tension is required for elastic stability, but positive pressure is needed to prevent that the fluid clumps under any density perturbation.

Neutral blackfolds have

\[ c_L^2 = -c_T^2 = -\frac{1}{n+1}, \]  

and therefore are generically unstable to longitudinal sound-mode oscillations and stable to elastic oscillations in the range of wavelengths (8.2).
This instability is not unexpected. Black branes suffer from the Gregory-Laflamme instability [28], which makes the horizon radius vary as

$$\delta r_0 \sim e^{\Omega t + ik_i z_i}.$$  \hspace{1cm} (8.15)

Here $\Omega$ is positive real and thus the frequency is imaginary. The threshold mode for the instability, with $\Omega = 0$ and $k = \sqrt{k_i^2 k_j^2} \neq 0$, has ‘small’ wavelength $\lambda = 2\pi/k \sim r_0$ and therefore cannot be seen in the blackfold approximation. But the GL instability extends to arbitrarily small $k$, i.e., arbitrarily long wavelengths, and when $k$ is very small it should be captured by the blackfold dynamics.

The sound-mode instability corresponds precisely to this long-wavelength part, $\Omega, k \rightarrow 0$, of the GL instability. Observe that sound waves in a blackfold produce $\delta \varepsilon \sim \delta P \sim \delta r_0$ i.e., variations in the horizon thickness. Eq. (8.14) tells us that these are unstable, of the form (8.15) with dispersion relation

$$\Omega = \frac{1}{\sqrt{n + 1}} k.$$  \hspace{1cm} (8.16)

A simple inspection of the slope at the origin in figure 1 of [28] shows good numerical agreement with (8.16). We leave a more precise derivation of this equation from a detailed GL-type analysis to future work. Note also that the dispersion relation (8.16) indicates that the collective coordinate $r_0$ is a ghost (i.e., its effective Lagrangian $-c_L^{-2}(\partial_t \ln r_0)^2 + (\partial_i \ln r_0)^2$ has the ‘wrong sign’ for the kinetic term).

Moreover, observe that the Gibbs-Duhem relation $dP = s dT$, from (2.17) and (2.18), implies in general that

$$\frac{dP}{d\varepsilon} = s \frac{dT}{d\varepsilon} = \frac{s}{c_v}$$  \hspace{1cm} (8.17)

where $c_v$ is the isovolumetric specific heat. Thus the black brane is dynamically unstable (to long-wavelength GL modes) if and only if it is locally thermodynamically unstable, $c_v < 0$. This is precisely the content of the ‘correlated stability conjecture’ of Gubser and Mitra [38]. In fact our method gives a quantitative expression for the unstable dynamical frequency in terms of local thermodynamics as

$$\Omega = \sqrt{\frac{s}{|c_v|}} k,$$  \hspace{1cm} (8.18)

which as far as we know is a new result.

The ordinary derivation of the GL instability involves a complicated analysis of linearized gravitational perturbations of a black brane and the numerical resolution of a boundary value problem for a differential equation (which is moreover compounded at small $k$ since larger grids are required to avoid finite-size problems). Here we have shown that the long-wavelength component of the instability, (8.16), can be obtained by an almost

\[15\] The GL mode at threshold is instead tachyonic, since its dispersion relation has imaginary mass.
trivial calculation of the sound speed in a fluid\textsuperscript{14}. In addition, the correlation between dynamical and thermodynamical stability follows as an elementary consequence of the thermodynamics of the effective fluid. In our opinion these results are striking evidence of the power of the blackfold approach.

9 Closing remarks

The formalism we have developed bears relation to two different earlier effective descriptions of black hole dynamics. The extrinsic part is a generalization to $p$-branes of the effective worldline formalism for small black holes, whose size $r_0$ is much smaller than the length scale $R = 1/(\text{acceleration})$ of their trajectories or the wavelength of the gravitational radiation they emit \textsuperscript{18} \textsuperscript{19} \textsuperscript{20}. The intrinsic part is similar to other fluid-dynamical formalisms for horizon fluctuations, such as the membrane paradigm and the fluid/AdS-gravity correspondence.

With respect to the first one, note that our formalism allows to consider time-dependent situations, which typically involve the emission of gravitational waves. This can be obtained by coupling the blackfold effective stress tensor to the quadrupole formula for gravitational radiation. This is also common in studies of gravitational wave emission from small black holes and from cosmic strings. However, accounting for the backreaction of this radiation on the blackfold requires going beyond the generalized-geodesic approximation and dealing with the notoriously subtle problem of the gravitational self-force \textsuperscript{39}.

To relate the fluid/AdS-gravity correspondence to our approach take, instead of a neutral black brane, a near-extremal D3-brane. The blackfold formalism can be applied to it, too\textsuperscript{17}. The small scale corresponds to the charge-radius $r_q$ of the D3-brane (which is much larger than the non-extremality length $r_0$), and in the overlap-zone $r_q \ll r \ll R$ where the blackfold effective stress tensor is computed, the metric is flat up to small corrections in $r_q/R$. The blackfold method here could be regarded as an extension of the DBI approach to describe thermally excited worldvolumes. The difference with respect to neutral branes is that there is a region near the horizon where excitations with long wavelengths $\gg r_q$ are localized. One can take a limit, Maldacena’s decoupling limit, to decouple all the far-zone effects from them. This region is asymptotic to $\text{AdS}_5 \times S^5$ with radius $r_q$, and far-zone effects are absent since they would give rise to non-normalizable modes in AdS and change the boundary geometry. Integrating the degrees of freedom in the asymptotically AdS region one gets only intrinsic, purely hydrodynamical collective modes. The effective stress tensor thus obtained is again of quasilocal type and in fact is the holographic stress tensor in AdS \textsuperscript{40}. This is in principle different

\textsuperscript{14}Observe that this is not a Jeans instability of the fluid (which has sometimes been suggested as possibly related to the GL instability) since gravitational forces within the fluid are entirely absent in our analysis.

\textsuperscript{17}Charged blackfolds will be discussed elsewhere.
than the one in the blackfold approach (which is computed in an asymptotically flat overlap-zone) but clearly they are related. For instance, at the perfect fluid level they are the same. The main point is that in the fluid/AdS-gravity correspondence there is no extrinsic worldvolume dynamics, nor far-zone backreaction. But the less general dynamics comes with a number of advantages: first, it makes much simpler to compute higher-derivative corrections to the perfect-fluid dynamics. Second, the charge near extremality eliminates the sound-mode instability. Finally, this specific correspondence can be argued to describe the hydrodynamical regime of a strongly-coupled quantum Yang-Mills theory. For a generic blackfold it is not known whether there is a useful dual interpretation of that sort.

Connections to the membrane paradigm are also suggestive but remain somewhat less precise. The membrane paradigm can be formulated for generic black holes, including vacuum ones, in terms of an effective fluid with a stress tensor of quasilocal type [41]. But in the membrane paradigm the boundary where the fluid lives is taken to lie right above the horizon, and not in an intermediate-asymptotic region (overlap zone) as in the blackfold approach. Perhaps the membrane-paradigm stress tensor and the one for blackfolds can be related through Komar integrals, at least in vacuum gravity. At any rate the membrane paradigm seems to capture only what we refer to as intrinsic fluid dynamics, and it is not clear to us how it could deal with the extrinsic embedding dynamics of a black brane.

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A Geometry of embedded submanifolds

We collect here several relevant definitions and results on the geometry of submanifold embeddings. Some aspects are more extensively discussed in [42].
A.1 Extrinsic curvature

Assume the submanifold $\mathcal{W}$ is embedded as $X^{\mu}(\sigma^a)$. The pull-back of the spacetime metric onto $\mathcal{W}$ is

$$\gamma_{ab} = g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu. \quad (A.1)$$

The first fundamental tensor of the surface is then

$$h^{\mu\nu} = \gamma_{ab} \partial_a X^\mu \partial_b X^\nu. \quad (A.2)$$

It follows easily that

$$h^\mu_\nu \partial_a X^\nu = \partial_a X^\mu, \quad (A.3)$$

and

$$h^\mu_\nu h^\nu_\rho = h^\mu_\rho \quad (A.4)$$

so $h^\mu_\nu$ projects tensors onto directions tangent to $\mathcal{W}$. Decomposing the metric as

$$g_{\mu\nu} = h_{\mu\nu} + \perp_{\mu\nu}, \quad (A.5)$$

we obtain the orthogonal projection tensor $\perp_{\mu\nu}$,

$$\perp_{\mu\nu} \partial_a X^\mu = 0, \quad \perp^\mu_\nu \perp^\nu_\rho = \perp^\mu_\rho. \quad (A.6)$$

The extrinsic curvature tensor can be defined as

$$K^{\mu\nu}_\rho = h^\lambda_\mu h^\rho_\sigma \nabla_\lambda h^\sigma_\nu = -h^\lambda_\mu h^\sigma_\nu \nabla_\lambda \perp^\sigma_\nu. \quad (A.7)$$

The tangentiality of the first two indices and orthogonality of the last,

$$\perp^\mu_\nu K^\sigma_\nu = \perp^\nu_\sigma K^\nu_\mu = h^\sigma_\mu K^\sigma_\nu = 0 \quad (A.8)$$

follows easily from this definition and the projector property (A.4).

Following [15], it is convenient to introduce the tangential covariant derivative

$$\nabla_\mu = h^\nu_\mu \nabla^\nu, \quad (A.9)$$

so

$$K^{\mu\nu}_\rho = h^\sigma_\nu \nabla_\mu h^\rho_\sigma. \quad (A.10)$$

Applying the tangential derivative on (A.4) one obtains

$$2K^{\mu(\nu\rho)} = \nabla_\mu h_{\nu\rho} = -\nabla_\mu \perp_{\nu\rho}. \quad (A.11)$$

Let $v$ be any vector tangent to $\mathcal{W}$. Then [42]

$$v^\mu v^\nu K^{\mu\nu}_\rho = -v^\mu v^\nu \nabla_\nu \perp^\rho_\mu = -v^\nu \nabla_\nu (v^\mu \perp^\rho_\mu) + \perp^\rho_\mu v^\nu \nabla_\nu v^\mu \quad (A.12)$$
where
\[ \dot{v}^\mu = v^\nu \nabla_\nu v^\mu. \]  
(A.13)

Now let \( N \) be any vector orthogonal to \( W \). Then
\[ N_\rho K_{\mu\nu}^\rho = N_\rho h_\nu^\sigma \nabla_\mu h_\sigma^\rho = -h_\nu^\rho \nabla_\mu N_\rho. \]  
(A.14)

The symmetry
\[ K_{[\mu\nu]}^\rho = 0 \]  
(A.15)
follows from the integrability of the subspaces orthogonal to \( \perp_{\mu\nu} \). To prove this, assume that the latter is true, namely that there is a submanifold \( W \) defined by a set of equations \( f_{(i)} (X) = 0 \) such that \( df_{(i)} \) are a basis of one-forms normal to the submanifold. Any one-form normal to \( W \) is a linear combination of them so the subspace orthogonal to it is also integrable. It is always possible to choose a one-form \( N \) orthogonal to this subspace such that
\[ N_\mu = \partial_\mu f(X) = \nabla_\mu f(X), \]  
(A.16)
so, using (A.14),
\[ N_\rho K_{\mu\nu}^\rho = -h_\nu^\rho h_\mu^\sigma \nabla_\sigma \nabla_\rho f, \]  
(A.17)
which is manifestly symmetric under \( \mu \leftrightarrow \nu \). The converse statement that (A.15) implies the integrability of the orthogonal subspace, can also be proven by a straightforward application of Frobenius’s theorem \[42\].

Background tensors \( t_{\mu_1 \mu_2 \ldots \nu_1 \nu_2 \ldots} \) can be pulled-back onto worldvolume tensors \( t_{a_1 a_2 \ldots b_1 b_2 \ldots} \) using \( \partial_a X^\mu \) as
\[ t_{a_1 a_2 \ldots b_1 b_2 \ldots} = \partial_a X^{\mu_1} \partial_{a_2} X^{\mu_2} \ldots \partial^{b_1} X_{\nu_1} \partial^{b_2} X_{\nu_2} \ldots t_{\mu_1 \mu_2 \ldots \nu_1 \nu_2 \ldots}, \]  
(A.18)
where
\[ \partial^{b} X_{\nu} = \gamma^{bc} h_{\nu\rho} \partial_c X^\rho. \]  
(A.19)

Observe that even when \( t_{\mu_1 \mu_2 \ldots \nu_1 \nu_2 \ldots} \) is a background tensor with indices parallel to \( W \), in general \( \nabla_\mu t_{\mu_1 \mu_2 \ldots \nu_1 \nu_2 \ldots} \) has both parallel and orthogonal components. The parallel projection along all indices is related to the worldvolume covariant derivative \( D_a t_{a_1 a_2 \ldots b_1 b_2 \ldots} \) as in (A.18). This can be shown by using the equation that relates the connection coefficients for each metric, \( \Gamma_{\mu\nu}^\rho \) and \( \{ c_{a b} \} \),
\[ \partial_a X^\nu \partial_b X^\nu h^\rho_\mu \Gamma_{\mu\nu}^\rho = \partial_c X^\sigma \{ c_{a b} \} - h^\sigma_\rho \partial_a \partial_b X^\rho, \]  
(A.20)
which can be proven by direct substitution of the definitions of each term involved.

In particular, the divergences of tensors are related as
\[ h^{\nu_1}_{\mu_1} \ldots \nabla_\rho t^{\rho \mu_1 \ldots} = \partial_a X^{\nu_1} \ldots D_c t^{ca_1 \ldots}. \]  
(A.21)
Such relations allow to dispense with the use of worldvolume coordinate tensors and derivatives in most formal manipulations. However, worldvolume coordinates are very practical.
for explicit calculations and also allow us to highlight the distinction between intrinsic (parallel to \( W \)) and extrinsic (orthogonal to \( W \)) equations.

Let us now consider the divergence of a totally antisymmetric tensor \( J \) (such as a current associated to a gauge form field) parallel to the worldvolume. It is easy to show that

\[
\perp_{\mu_1} \nabla_\mu J^{\mu_1 \ldots} = 0
\]  

(A.22)

holds as an identity. This implies that the conservation equation

\[
\nabla_\mu J^{\mu_1 \ldots} = 0
\]  

(A.23)

is equivalent to the worldvolume conservation equation

\[
D_a J^{a_1 \ldots} = 0,
\]  

(A.24)

i.e., the orthogonal component of the current conservation equation (A.23) does not yield any additional equation. In particular, for a 1-form current one has

\[
\nabla_\mu J^\mu = D_a J^a ,
\]  

(A.25)

and continuity of charge is only meaningful as an intrinsic equation. This is in contrast to the conservation of the stress energy tensor, where the orthogonal component gives independent extrinsic equations (3.12).

Let us now obtain more explicit expressions for the pull-back of the extrinsic curvature tensor onto \( W \) in terms of \( X^\mu(\sigma) \),

\[
K_{ab}^\rho = \partial_a X^\mu \partial_b X^\nu K_{\mu \nu}^\rho = -\partial_a X^\mu \partial_b X^\nu \nabla_\mu \perp_\nu \rho .
\]  

(A.26)

The property (A.6) implies

\[
0 = \partial_b X^\nu \nabla_\nu (\perp_\sigma \partial_a X^\sigma) = -K_{ab}^\rho + \perp_\sigma \partial_b X^\nu \nabla_\nu (\partial_a X^\sigma) .
\]  

(A.27)

Expanding the covariant derivative in the last term and using \( \partial_b X^\nu \partial_\nu = \partial_b \) we find

\[
K_{ab}^\rho = \perp_\sigma \rho \left( \partial_a \partial_b X^\sigma + \Gamma_{\mu \nu}^{\sigma} \partial_\mu X^\mu \partial_\nu X^\nu \right) ,
\]  

(A.28)

which is reminiscent of the expression for the acceleration (deviation from self-parallel transport) of a curve — indeed (A.12) makes this even more explicit. An alternative expression with this same feature can be obtained by performing some manipulations:

\[
\perp_\sigma \partial_a \partial_b X^\sigma = \partial_a \partial_b X^\rho - h^\rho \sigma \partial_a \partial_b X^\sigma
\]

\[
= \partial_a \partial_b X^\rho - \left\{ c \right\}_a^b \partial_a X^\rho + \partial_a X^\mu \partial_b X^\nu h^\rho \sigma \Gamma_{\mu \nu}^{\sigma}
\]

\[
= D_a \partial_b X^\rho + \partial_a X^\mu \partial_b X^\nu h^\rho \sigma \Gamma_{\mu \nu}^{\sigma} .
\]  

(A.29)

In the second line we have used (A.20). Inserting the last expression into (A.28) we find

\[
K_{ab}^\rho = D_a \partial_b X^\rho + \Gamma_{\mu \nu}^{\rho} \partial_\mu X^\mu \partial_\nu X^\nu .
\]  

(A.30)
A.2 Variational calculus

Consider a congruence of curves with tangent vector \( N \), that intersect \( W \) orthogonally

\[
N^\mu h_{\mu\nu} = 0, \quad N^\mu \perp_{\mu\nu} = N_{\nu},
\]
and Lie-drag \( W \) along these curves. The congruence is arbitrary, other than requiring it to be smooth in a finite neighbourhood of \( W \), so this realizes arbitrary smooth deformations of the worldvolume \( X^\mu \to X^\mu + N^\mu \).

Consider now the Lie derivative of \( h_{\mu\nu} \) along \( N \). In general,

\[
\mathcal{L}_N h_{\mu\nu} = N^\rho \nabla_\rho h_{\mu\nu} + h_{\mu\rho} \nabla_\mu N^\rho + h_{\nu\rho} \nabla_\nu N^\rho.
\]

Then

\[
h_{\mu} \lambda h_{\nu} \sigma \mathcal{L}_N h_{\lambda\sigma} = h_{\mu} \lambda h_{\nu} \sigma N^\rho \nabla_\rho h_{\lambda\sigma} + h_{\mu\rho} \nabla_\mu N^\rho + h_{\nu\rho} \nabla_\nu N^\rho.
\]

The first term in the rhs is zero:

\[
h_{\mu} \lambda h_{\nu} \sigma N^\rho \nabla_\rho h_{\lambda\sigma} = \frac{1}{2} h_{\mu} \lambda h_{\nu} \sigma \mathcal{L}_N h_{\lambda\sigma},
\]

and the other two terms can be rewritten using \( (A.14) \), so

\[
N_\rho K^{\rho} = -\frac{1}{2} h_{\mu} \lambda h_{\nu} \sigma \mathcal{L}_N h_{\lambda\sigma}.
\]

This implies

\[
N_\rho K^{\rho} = \frac{1}{2} h^{\mu\nu} \mathcal{L}_N h_{\mu\nu} = -\frac{1}{\sqrt{|h|}} \mathcal{L}_N \sqrt{|h|},
\]

where \( h = \det h_{\mu\nu} \). These equations generalize well-known expressions for the extrinsic curvature of a codimension-1 surface. The last one allows to derive the equation for a minimal-volume submanifold:

\[
Vol = \int_W \sqrt{|h|} \Rightarrow \delta_N Vol = -\sqrt{|h|} N_\rho K^{\rho}
\]

\( i.e., \) for variations along an arbitrary orthogonal direction \( N \), minimal (actually, extremal) volume requires \( K^{\rho} = 0 \). This is of course the variational principle for Nambu-Goto-Dirac branes.

Consider now a more general functional

\[
I = \int_W \sqrt{|h|} \Phi
\]

where \( \Phi \) is a worldvolume function. Then

\[
\delta_N I = \mathcal{L}_N \left( \sqrt{|h|} \Phi \right) = \sqrt{|h|} (\mathcal{L}_N K^{\rho} \Phi + N^{\rho} \partial_\rho \Phi).
\]

Since \( N \) is an arbitrary orthogonal vector we have

\[
\delta_N I = 0 \iff K^{\rho} = \perp_{\rho\mu} \partial_\mu \ln \Phi.
\]
References

[1] R. Emparan, T. Harmark, V. Niarchos and N. A. Obers, “World-volume effective theory for higher-dimensional black holes. (Blackfolds),” Phys. Rev. Lett. 102, 191301 (2009) [arXiv:0902.0427 [hep-th]].

[2] R. C. Myers and M. J. Perry, “Black Holes In Higher Dimensional Space-Times,” Annals Phys. 172 (1986) 304.

[3] R. Emparan and R. C. Myers, “Instability of ultra-spinning black holes,” JHEP 0309 (2003) 025 [arXiv:hep-th/0308056].

[4] R. Emparan and H. S. Reall, “A rotating black ring in five dimensions,” Phys. Rev. Lett. 88 (2002) 101101 [arXiv:hep-th/0110260].

[5] R. Emparan and H. S. Reall, “Black Holes in Higher Dimensions,” Living Rev. Rel. 11, 6 (2008) [arXiv:0801.3471 [hep-th]].

[6] R. Emparan, T. Harmark, V. Niarchos, N. A. Obers and M. J. Rodríguez, “The Phase Structure of Higher-Dimensional Black Rings and Black Holes,” JHEP 0710, 110 (2007) [arXiv:0708.2181 [hep-th]].

[7] O. J. C. Dias, P. Figueras, R. Monteiro, J. E. Santos and R. Emparan, “Instability and new phases of higher-dimensional rotating black holes,” arXiv:0907.2248 [hep-th].

[8] H. K. Kunduri, J. Lucietti and H. S. Reall, “Near-horizon symmetries of extremal black holes,” Class. Quant. Grav. 24, 4169 (2007) [arXiv:0705.4214 [hep-th]].

[9] N. A. Obers, “Black Holes in Higher-Dimensional Gravity,” Lect. Notes Phys. 769 (2009) 211 [arXiv:0802.0519 [hep-th]].

[10] V. Niarchos, “Phases of Higher Dimensional Black Holes,” Mod. Phys. Lett. A 23, 2625 (2008) [arXiv:0808.2776 [hep-th]].

[11] H. Elvang and R. Emparan, “Black rings, supertubes, and a stringy resolution of black hole non-uniqueness,” JHEP 0311, 035 (2003) [arXiv:hep-th/0310008].

[12] R. Emparan and H. S. Reall, “Black rings,” Class. Quant. Grav. 23 (2006) R169 [arXiv:hep-th/0608012].

[13] M. M. Caldarelli, R. Emparan and M. J. Rodríguez, “Black Rings in (Anti)-deSitter space,” JHEP 0811 (2008) 011 [arXiv:0806.1954 [hep-th]].

[14] J. Camps, R. Emparan, P. Figueras, S. Giusto and A. Saxena, “Black Rings in Taub-NUT and D0-D6 interactions,” JHEP 0902 (2009) 021 [arXiv:0811.2088 [hep-th]].
[15] B. Carter, “Essentials of classical brane dynamics,” Int. J. Theor. Phys. 40, 2099 (2001) [arXiv:gr-qc/0012036].

[16] R. S. Hanni and R. Ruffini, “Lines of Force of a Point Charge near a Schwarzschild Black Hole,” Phys. Rev. D 8, 3259 (1973).

T. Damour, “Black Hole Eddy Currents,” Phys. Rev. D 18, 3598 (1978).

R.L. Znajek, “The electric and magnetic conductivity of a Kerr hole,” Mon. Not. R. Astr. Soc. 185, 833 (1978).

T. Damour, Thèse de Doctorat dEtat, Université Pierre et Marie Curie, Paris VI, 1979; “Surface Effects in Black Hole Physics”, Proceedings of the Second Marcel Grossmann Meeting on General Relativity, (edited by R. Ruffini, North Holland, 1982) p. 587.

R. H. Price and K. S. Thorne, “Membrane viewpoint on black holes: Properties and evolution of the stretched horizon,” Phys. Rev. D 33 (1986) 915.

K. S. Thorne, R. H. Price and D. A. Macdonald, “Black Holes: The Membrane Paradigm”, Yale Univ. Press, New Haven, USA (1986).

[17] S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, “Nonlinear Fluid Dynamics from Gravity,” JHEP 0802, 045 (2008) [arXiv:0712.2456 [hep-th]].

[18] E. Poisson, “The motion of point particles in curved spacetime,” Living Rev. Rel. 7, 6 (2004) [arXiv:gr-qc/0306052].

[19] S. E. Gralla and R. M. Wald, “A Rigorous Derivation of Gravitational Self-force,” Class. Quant. Grav. 25 (2008) 205009 [arXiv:0806.3293 [gr-qc]].

[20] W. D. Goldberger, “Les Houches lectures on effective field theories and gravitational radiation,” arXiv:hep-ph/0701129.

W. D. Goldberger and I. Z. Rothstein, “An effective field theory of gravity for extended objects,” Phys. Rev. D 73, 104029 (2006) [arXiv:hep-th/0409156].

B. Kol and M. Smolkin, “Classical Effective Field Theory and Caged Black Holes,” Phys. Rev. D 77, 064033 (2008) [arXiv:0712.2822 [hep-th]].

[21] J. D. Brown and J. W. York, “Quasilocal energy and conserved charges derived from the gravitational action,” Phys. Rev. D 47, 1407 (1993) [arXiv:gr-qc/9209012].

[22] J. Le Witt and S. F. Ross, “Black holes and black strings in plane waves,” arXiv:0910.4332 [hep-th].

[23] Y. Mino, M. Sasaki and T. Tanaka, “Gravitational radiation reaction to a particle motion,” Phys. Rev. D 55, 3457 (1997) [arXiv:gr-qc/9606018].
[24] T. Harmark, “Small black holes on cylinders,” Phys. Rev. D 69 (2004) 104015 [arXiv:hep-th/0310259].

[25] D. Gorbonos and B. Kol, “A dialogue of multipoles: Matched asymptotic expansion for caged black holes,” JHEP 0406, 053 (2004) [arXiv:hep-th/0406002].

[26] S. S. Gubser, “On non-uniform black branes,” Class. Quant. Grav. 19 (2002) 4825 [arXiv:hep-th/0110193].

[27] T. Wiseman, “Static axisymmetric vacuum solutions and non-uniform black strings,” Class. Quant. Grav. 20 (2003) 1137 [arXiv:hep-th/0209051].

[28] R. Gregory and R. Laflamme, “Black strings and p-branes are unstable,” Phys. Rev. Lett. 70, 2837 (1993) [arXiv:hep-th/9301052].

For a review, see T. Harmark, V. Niarchos and N. A. Obers, “Instabilities of black strings and branes,” Class. Quant. Grav. 24 (2007) R1 [arXiv:hep-th/0701022].

[29] H. Elvang, R. Emparan and A. Virmani, “Dynamics and stability of black rings,” JHEP 0612, 074 (2006) [arXiv:hep-th/0608076].

[30] M. M. Caldarelli, O. J. C. Dias, R. Emparan and D. Klemm, “Black Holes as Lumps of Fluid,” JHEP 0904 (2009) 024 [arXiv:0811.2381 [hep-th]].

[31] S. W. Hawking, “Black holes in general relativity,” Commun. Math. Phys. 25, 152 (1972).

[32] B. S. Kay and R. M. Wald, “Theorems on the Uniqueness and Thermal Properties of Stationary, Nonsingular, Quasifree States on Space-Times with a Bifurcate Killing Horizon,” Phys. Rept. 207, 49 (1991).

[33] S. Hollands, A. Ishibashi and R. M. Wald, “A Higher Dimensional Stationary Rotating Black Hole Must be Axisymmetric,” Commun. Math. Phys. 271, 699 (2007) [arXiv:gr-qc/0605106].

[34] G. W. Gibbons and S. W. Hawking, “Action Integrals And Partition Functions In Quantum Gravity,” Phys. Rev. D 15 (1977) 2752.

[35] See e.g., J. D. Brown, “Action functionals for relativistic perfect fluids,” Class. Quant. Grav. 10, 1579 (1993) [arXiv:gr-qc/9304026], and references therein.

[36] B. Carter, “Stability And Characteristic Propagation Speeds In Superconducting Cosmic And Other String Models,” Phys. Lett. B 228 (1989) 466.

[37] B. Carter, “Perturbation Dynamics For Membranes And Strings Governed By Dirac Goto Nambu Action In Curved Space,” Phys. Rev. D 48 (1993) 4835.
[38] S. S. Gubser and I. Mitra, “The evolution of unstable black holes in anti-de Sitter space,” JHEP 0108 (2001) 018 [arXiv:hep-th/0011127].

[39] In addition to [18], some entries to the literature in the specific context of strings and branes are:

R. A. Battye and E. P. S. Shellard, “String radiative back reaction,” Phys. Rev. Lett. 75, 4354 (1995) [arXiv:astro-ph/9408078].

A. Buonanno and T. Damour, “Gravitational, dilatonic and axionic radiative damping of cosmic strings,” Phys. Rev. D 60 (1999) 023517 [arXiv:gr-qc/9801105].

R. A. Battye, B. Carter and A. Mennim, “Linearized self-forces for branes,” Phys. Rev. D 71 (2005) 104026 [arXiv:hep-th/0412053].

[40] V. Balasubramanian and P. Kraus, “A stress tensor for anti-de Sitter gravity,” Commun. Math. Phys. 208, 413 (1999) [arXiv:hep-th/9902121].

[41] See e.g., M. Parikh and F. Wilczek, “An action for black hole membranes,” Phys. Rev. D 58, 064011 (1998) [arXiv:gr-qc/9712077].

C. Eling and Y. Oz, “Relativistic CFT Hydrodynamics from the Membrane Paradigm,” arXiv:0906.4999 [hep-th].

[42] B. Carter, “Outer Curvature And Conformal Geometry Of An Imbedding,” J. Geom. Phys. 8, 53 (1992).