CONNECTED COMPONENTS OF PARTITION PRESERVING DIFFEOMORPHISMS

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Abstract. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial and $\mathcal{S}(f)$ be the group of diffeomorphisms $h$ of $\mathbb{R}^2$ preserving $f$, i.e. $f \circ h = f$. Denote by $\mathcal{S}_{id}(f)^r$, $(0 \leq r \leq \infty)$, the identity component of $\mathcal{S}(f)$ with respect to the weak Whitney $C^r_W$-topology. We prove that $\mathcal{S}_{id}(f)\infty = \cdots = \mathcal{S}_{id}(f)^1$ for all $f$ and that $\mathcal{S}_{id}(f)^1 \neq \mathcal{S}_{id}(f)^0$ if and only if $f$ is a product of at least two distinct irreducible over $\mathbb{R}$ quadratic forms.

1. Introduction

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial of degree $p \geq 1$. Thus up to sign we can write

\begin{equation}
\quad f(x, y) = \pm \prod_{i=1}^l L_i^{\alpha_i}(x, y) \cdot \prod_{j=1}^k Q_j^{\beta_j}(x, y),
\end{equation}

where every $L_i$ is a linear function, $Q_j$ is a positive definite quadratic form, $\alpha_i, \beta_j \geq 1$, and

$$\frac{L_i}{L_i'} \neq \text{const for } i \neq i', \quad \frac{Q_j}{Q_{j'}} \neq \text{const for } j \neq j'. $$

Denote by $\mathcal{S}(f) = \{h \in \mathcal{D}(\mathbb{R}^2) : f \circ h = f\}$ the stabilizer of $f$ with respect to the right action of the group $\mathcal{D}(\mathbb{R}^2)$ of $C^\infty$-diffeomorphisms of $\mathbb{R}^2$ on the space $C^\infty(\mathbb{R}^2, \mathbb{R})$. It consists of diffeomorphisms of $\mathbb{R}^2$ preserving every level-set $f^{-1}(c)$ of $f$, $c \in \mathbb{R}$.

Let $\mathcal{S}_{id}(f)^r$, $0 \leq r \leq \infty$, be the identity component of $\mathcal{S}(f)$ with respect to the weak Whitney $C^r_W$-topology. Thus $\mathcal{S}_{id}(f)^r$ consists of diffeomorphisms $h \in \mathcal{S}(f)$ isotopic in $\mathcal{S}(f)$ to $\text{id}_{\mathbb{R}^2}$ via an $f$-preserving isotopy $H : \mathbb{R}^2 \times I \to \mathbb{R}^2$ whose partial derivatives in $(x, y) \in \mathbb{R}^2$ up to order $r$ continuously depend on $(x, y, t)$, see Section 2 for a precise definition. Then it is easy to see that

$$\mathcal{S}_{id}(f)\infty \subset \cdots \mathcal{S}_{id}(f)^r \subset \cdots \subset \mathcal{S}_{id}(f)^1 \subset \mathcal{S}_{id}(f)^0.$$ 

It follows from results in [12, 13] that $\mathcal{S}_{id}(f)\infty = \mathcal{S}_{id}(f)^1$. Moreover, it is actually proved in [9] that $\mathcal{S}_{id}(f)\infty = \mathcal{S}_{id}(f)^p$ for $p \leq 2$, see also [11]. The aim of this note is to prove the following theorem describing the relation between $\mathcal{S}_{id}(f)^r$ for all $p \geq 1$.

**Theorem 1.1.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial of degree $p \geq 1$. Then $\mathcal{S}_{id}(f)\infty = \cdots = \mathcal{S}_{id}(f)^1$. Moreover, $\mathcal{S}_{id}(f)^1 \neq \mathcal{S}_{id}(f)^0$ if and only if $f$ is a product of at least two distinct irreducible quadratic forms, i.e. $f = Q_1^{\alpha_1} \cdots Q_k^{\alpha_k}$ for $k \geq 2$.

This theorem is based on a rather general result about partition preserving diffeomorphisms, see Theorem 4.7. The applications of Theorem 1.1 will be given in another paper concerning smooth functions on surfaces with isolated singularities.

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Structure of the paper. In Section 2 we describe homotopies which induce continuous paths into functional spaces with the weak Whitney $C^r_\mathrm{w}$-topologies. Section 3 introduces the so-called singular partitions of manifolds being the main object of the paper. Section 4 contains the main result, Theorem 4.7, about invariant contractions of singular partitions. In Section 5 an application of this theorem to local extremes of smooth functions is given. Section 6 contains a description of the group of linear symmetries of $f$. Finally in Section 7 we prove Theorem 1.1.

2. $r$-homotopies

Denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\overline{\mathbb{N}}_0 = \mathbb{N} \cup \{0, \infty\}$.

Let $M, N$ be two smooth manifolds of dimensions $m$ and $n$ respectively. Then for every $r \in \overline{\mathbb{N}}_0$ the space $C^r(M, N)$ admits the so-called weak Whitney topology denoted by $C^r_\mathrm{w}$, see e.g. [8, 5].

Recall, e.g. [7, § 44.IV] that there exists a homeomorphism

$$C^0(I, C^0(M, N)) \approx C^0(M \times I, N)$$

with respect to the corresponding $C^r_\mathrm{w}$-topologies (also called compact open ones) associating to every (continuous) path $w : I \to C^0(M, N)$ a homotopy $H : M \times I \to N$ defined by $H(x, t) = w(t)(x)$.

We will now describe homotopies inducing continuous paths $w : I \to C^r(M, N)$ with respect to $C^r_\mathrm{w}$-topologies.

**Definition 2.1.** Let $H : M \times I \to N$ be a homotopy and $r \in \overline{\mathbb{N}}_0$. We say that $H$ is an $r$-homotopy if

1. $H_t : M \to N$ is $C^r$ for every $t \in I$;
2. partial derivatives of $H(x, t)$ with respect to $x$, up to order $r$, continuously depend on $(x, t)$.

More precisely, let $z \in M \times I$. Then in some local coordinates at $z$ we can regard $H$ as a map

$$H = (H_1, \ldots, H_n) : \mathbb{R}^m \times I \to \mathbb{R}^n$$

such that for every fixed $t$ and $i$ the function $H_i(x, t)$ is $C^r$. Condition (2) requires that for every $i = 1, \ldots, n$ and every non-negative integer vector $k = (k_1, \ldots, k_m)$ of norm $|k| = \sum_{j=1}^m k_j \leq r$ the function

$$\frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_m^{k_m}} H_i(x_1, \ldots, x_m, t)$$

continuously depend on $(x, t)$.

Thus a $0$-homotopy $H$ is just the usual homotopy.

It easily follows from the definition of $C^r_\mathrm{w}$-topologies that a path $w : I \to C^r(M, N)$ is continuous in the standard topology of $I$ and $C^r_\mathrm{w}$-topology of $C^r(M, N)$ if and only if the corresponding homotopy $H : M \times I \to N$ is an $r$-homotopy.

We can also define a $C^r$-homotopy as a $C^r$-map $M \times I \to N$. Evidently, every $C^r$-homotopy is an $r$-homotopy as well, but the converse is not true.

**Example 2.2.** Let $H : \mathbb{R} \times I \to \mathbb{R}$ be given by

$$H(x, t) = \begin{cases} t \ln(x^2 + t^2), & (x, t) \neq (0, 0), \\ 0, & (x, t) = (0, 0). \end{cases}$$

Then $H$ is continuous, while $\frac{\partial H}{\partial x} = \frac{2tx}{x^2 + y^2}$ is $C^\infty$ for every fixed $t$ as a function in $x$ but discontinuous at $(0, 0)$ as a function in $(x, t)$. In other words $H$ is a 0-homotopy but not a 1-homotopy.
Moreover, define \( G : \mathbb{R} \times I \to \mathbb{R} \) by \( G(x,t) = \int_0^x H(y,t) \, dy \). Then \( G \) a 1-homotopy but not a 2-homotopy.

3. SINGULAR PARTITIONS OF MANIFOLDS

Let \( M \) be a smooth manifold equipped with a partition \( \mathcal{P} = \{ \omega_i \}_{i \in \Lambda} \), i.e., a family of subsets \( \omega_i \) such that
\[
M = \bigcup_{i \in \Lambda} \omega_i, \quad \omega_i \cap \omega_j = \emptyset \quad (i \neq j).
\]

In general \( \Lambda \) may be even uncountable and \( \omega_i \) are not necessarily closed in \( M \). Let also \( \Lambda' \) be a (possibly empty) subset of \( \Lambda \) and \( \Sigma = \{ \omega_i \}_{i \in \Lambda'} \) be a subfamily of \( \mathcal{P} \) thought of as a set of “singular” elements. Then the pair \( \Theta = (\mathcal{P}, \Sigma) \) will be called a singular partition of \( M \).

**Example 3.1.** Let \( F \) be a vector field on \( M \), \( \mathcal{P}_F \) the set of orbits of \( F \), and \( \Sigma_F \) the set of singular points of \( F \). Then the pair \( \Theta_F = (\mathcal{P}_F, \Sigma_F) \) will be called the singular partition of \( F \).

**Example 3.2.** Let \( f : \mathbb{R}^m \to \mathbb{R}^n \) be a smooth map, \( x \in \mathbb{R}^m \) a point, and \( J(f, x) \) the Jacobi matrix of \( f \) at \( x \). Then \( x \in \mathbb{R}^m \) is called critical for \( f \) if \( \text{rank} \, J(f, x) < \min\{m, n\} \). Otherwise \( x \) is regular. This definition naturally extends to maps between manifolds.

Let \( M, N \) be smooth manifolds and \( f : M \to N \) a smooth map. Denote by \( \Sigma_f \) the set of critical points of \( f \). Consider the following partition \( \mathcal{P}_f \) of \( M \): a subset \( \omega \subset M \) belongs to \( \mathcal{P}_f \) iff \( \omega \) is either a critical point of \( f \), or a connected component of the set of the form \( f^{-1}(y) \setminus \Sigma_f \) for some \( y \in N \). Then the pair \( \Theta_f = (\mathcal{P}_f, \Sigma_f) \) will be called the singular partition of \( f \). Evidently, every \( \omega \in \mathcal{P}_f \setminus \Sigma_f \) is a submanifold of \( M \).

**Example 3.3.** Assume that, in Example 3.2, \( \dim M = \dim N + 1 \) and both \( M \) and \( N \) are orientable. Then every element of \( \mathcal{P}_f \setminus \Sigma_f \) is one-dimensional and orientations of \( M \) and \( N \) allow to coherently orient all the elements of \( \mathcal{P}_f \setminus \Sigma_f \). Moreover, it is even possible to construct a vector field \( F \) on \( M \) such that the singular partitions \( \Theta_f \) and \( \Theta_F \) coincide.

In particular, let \( M \) be an orientable surface and \( f : M \to \mathbb{R} \) a smooth function. Then \( M \) admits a symplectic structure, and in this case we can assume that \( F \) is the corresponding Hamiltonian vector field of \( f \).

**Example 3.4.** Let \( \mathcal{F} \) be a foliation on \( M \) with singular leaves, \( \mathcal{P} \) be the set of leaves of \( \mathcal{F} \), and \( \Sigma \) be the set of its singular leaves (having non-maximal dimension). Then the pair \( \Theta_{\mathcal{F}} = (\mathcal{P}_{\mathcal{F}}, \Sigma_{\mathcal{F}}) \) will be called the singular partition of \( \mathcal{F} \). This example generalizes all previous ones.

Let \( \Theta = (\mathcal{P}, \Sigma) \) be a singular partition on \( M \). For every open subset \( V \subset M \) denote by \( \mathcal{E}(\Theta, V) \) the subset of \( C^\infty(V, M) \) consisting of maps \( f : V \to M \) such that
\[
(1) \quad f(\omega_i \cap V) \subset \omega_i \quad \text{for all} \quad \omega_i \in \mathcal{P} \quad \text{and}
(2) \quad f \text{ is a local diffeomorphism at every point } z \text{ belonging to some singular element} \omega \in \Sigma.
\]

Let also \( \mathcal{D}(\Theta, V) \) be the subset of \( \mathcal{E}(\Theta, V) \) consisting of immersions, i.e., local diffeomorphisms. For \( V = M \) we abbreviate
\[
\mathcal{E}(\Theta) = \mathcal{E}(\Theta, M), \quad \mathcal{D}(\Theta) = \mathcal{D}(\Theta, M).
\]

For every \( r \in \mathbb{N}_0 \) denote by \( \mathcal{E}_{id}(\Theta, V)^r \), resp. \( \mathcal{D}_{id}(\Theta, V)^r \), the path-component of the identity inclusion \( i_V : V \subset M \) in \( \mathcal{E}(\Theta, V) \), resp. in \( \mathcal{D}(\Theta, V) \), with respect to the induced \( C^r_W \)-topology, see Section 2.

Thus \( \mathcal{E}_{id}(\Theta, V)^r \) (resp. \( \mathcal{D}_{id}(\Theta, V)^r \)) consists of maps (resp. immersions) \( V \subset M \) which are \( r \)-homotopic (\( r \)-isotopic) to \( i_V : V \subset M \) in \( \mathcal{E}(\Theta, V) \) (resp. in \( \mathcal{D}(\Theta, V) \)).
Lemma 3.6. \( \alpha \) be the local flow of \( C \) where \( \Gamma \)

\[ (3.1) \quad \mathcal{E}_{id}(\Theta, V) \subset C^{\infty}C(\Theta, V)^{-1} \subset \mathcal{E}_{id}(\Theta, V)^0, \]

and similar relations hold for \( \mathcal{D}_{id}(\Theta, V)^{r} \).

The following notion turns out to be useful for studying singular partitions of vector fields.

3.5. **Shift-map of a vector field.** Let \( F \) be a vector field on \( M \) and \( \Phi : M \times \mathbb{R} \supset \text{dom}(\Phi) \to M \)

be the local flow of \( F \) defined on some open neighbourhood \( \text{dom}(\Phi) \) of \( M \times 0 \) in \( M \times \mathbb{R} \).

For every open subset \( V \subset M \) let also

\[ \text{func}(\Phi, V) = \{ \alpha \in C^{\infty}(V, \mathbb{R}) : \Gamma_{\alpha} \subset \text{dom}(\Phi) \}, \]

where \( \Gamma_{\alpha} = \{ (x, \alpha(x)) \setminus x \in V \} \subset M \times \mathbb{R} \) is the graph of \( \alpha \). Then \( \text{func}(\Phi, V) \) is the largest subset of \( C^{\infty}(V, \mathbb{R}) \) on which the following shift-map is defined:

\[ \varphi_{V} : \text{func}(\Phi, V) \to C^{\infty}(V, M), \quad \varphi_{V}(\alpha)(x) = \Phi(x, \alpha(x)), \]

for \( \alpha \in \text{func}(\Phi, V), x \in V \).

**Lemma 3.6.** Let \( \Theta_{F} \) be the singular partition of \( M \) by orbits of \( F \). Then

\[ (3.2) \quad \text{im}(\varphi_{V}) \subset \mathcal{E}_{id}(\Theta_{F}, V)^{\infty}. \]

Moreover, if \( \mathcal{D}_{id}(\Theta_{F}, V)^{r} \subset \text{im}(\varphi) \) for some \( r \in \mathbb{N}_{0} \), then

\[ \mathcal{D}_{id}(\Theta_{F}, V)^{\infty} = \cdots = \mathcal{D}_{id}(\Theta_{F}, V)^{r+1} = \mathcal{D}_{id}(\Theta_{F}, V)^{r}. \]

**Proof.** Let \( \alpha \in \text{func}(\Phi, V) \) and \( f = \varphi(\alpha) \), i.e., \( f(x) = \Phi(x, \alpha(x)) \). Then \( f(\omega \cap V) \subset \omega \) for every orbit of \( F \). Moreover by [9, Lemma 20] \( f \) is a local diffeomorphism at a point \( x \in V \) iff \( da(F)(x) \neq -1 \), where \( da(F)(x) \) is the Lie derivative of \( \alpha \) along \( F \) at \( x \). Hence \( f \) is so at every singular point \( z \) of \( F \), since \( da(F)(z) = 0 \neq -1 \), [9, Corollary 21]. Therefore \( f \in \mathcal{E}(\Theta_{F}, V) \). Moreover an \( \infty \)-homotopy of \( f \) to \( \text{i}_{V} : V \subset M \in \mathcal{E}(\Theta_{F}, V) \) can be given by \( f_{t}(x) = \Phi(x, t\alpha(x)) \). Thus \( f \in \mathcal{E}_{id}(\Theta_{F}, V)^{\infty} \).

Finally, suppose that \( f \in \mathcal{D}_{id}(\Theta_{F}, V)^{r} \). Then the restriction of \( f \) to any non-constant orbit \( \omega \) of \( F \) is an orientation preserving local diffeomorphism. Therefore \( da(F)(z) > -1 \) on all of \( V \). Hence \( d(t\alpha)(F)(z) > -1 \) for all \( t \in I \) as well, i.e., \( f_{t} \in \mathcal{D}(\Theta_{F}, V) \). This implies that \( f \in \mathcal{D}_{id}(\Theta_{F}, V)^{\infty} \).

**Example 3.7.** Let \( A \) be a real non-zero \((m \times m)\)-matrix, \( F(x) = Ax \) be the corresponding linear vector field on \( \mathbb{R}^{m} \), and \( V \) be a neighbourhood of the origin \( 0 \). Then the shift-map \( \varphi_{V} \) is given by

\[ \varphi(\alpha)(x) = \Phi(x, \alpha(x)) = e^{A\alpha(x)x}. \]

It is shown in [9] that in this case \( \text{im}(\varphi_{V}) = \mathcal{E}_{id}(\Theta_{F}, V)^{0} \). Hence for all \( r \in \mathbb{N}_{0} \) we have

\[ \text{im}(\varphi_{V}) = \mathcal{E}_{id}(\Theta_{F}, V)^{\infty} = \cdots = \mathcal{E}_{id}(\Theta_{F}, V)^{r}, \]

\[ \mathcal{D}_{id}(\Theta_{F}, V)^{\infty} = \cdots = \mathcal{D}_{id}(\Theta_{F}, V)^{r}. \]

4. **Invariant contractions**

Let \( \Theta = (\mathcal{P}, \Sigma) \) be a singular partition on a manifold \( M \). We will say that a subset \( V \subset M \) is \( \Theta \)-invariant, if it consists of full elements of \( \Theta \), i.e., if \( \omega \in \mathcal{P} \) and \( \omega \cap V \neq \emptyset \), then \( \omega \subset V \).

**Definition 4.1.** Let \( Z \subset M \) be a closed subset such that every point \( z \in Z \) is a singular element of \( \Theta \), i.e., \( \{ z \} \in \Sigma \). Say that \( \Theta \) has an invariant \( r \)-contraction to \( Z \) if there exists a closed \( \Theta \)-invariant neighbourhood \( V \) of \( Z \) being a smooth submanifold of \( M \) and a homotopy \( r : V \times I \to V \) such that:

(i) \( r_{1} = \text{id}_{V} \);
(ii) \( r_0 \) is a proper retraction of \( V \) to \( Z \), i.e., \( r_0(V) = Z \), \( r_0(z) = z \) for \( z \in Z \), and \( r_0^{-1}(K) \) is compact for every compact \( K \subset Z \);

(iii) for every \( t \in (0, 1] \) the map \( r_t \) is a closed \( C^r \)-embedding of \( V \) into \( V \) such that for each \( \omega \in \mathcal{P} \) (resp. \( \omega \in \Sigma \)) its image \( r_t(\omega) \) is also an element of \( \mathcal{P} \) (resp. \( \Sigma \));

(iv) for each \( z \in Z \) the set \( V_z = r_0^{-1}(z) \) is \( \Theta \)-invariant, and \( r_t(V_z) \subset V_z \) for all \( t \in I \).

Since \( V \) is \( \Theta \)-invariant, it follows from (iii) that so is its image \( r_t(V) \).

**Example 4.2.** Define \( f : \mathbb{R}^m \to \mathbb{R} \) by \( f(x_1, \ldots, x_m) = \sum_{i=1}^m x_i^2 \). Evidently, the singular partition \( \Theta_f \) consists of the origin 0 and concentric spheres centered at 0. For every \( s > 0 \) let \( V_s = f^{-1}[0, s^2] \) be a closed \( m \)-disk of radius \( s \). Then \( V_s \) is \( \Theta_f \)-invariant and its invariant contraction of \( V_s \) to \( Z = \{0\} \) can be given by \( r(x, t) = tx \).

**Example 4.3.** The previous example can be parametrized as follows. Let \( p : M \to Z \) be an \( m \)-dimensional vector bundle over a connected, smooth manifold \( Z \). We will identify \( Z \) with the image \( Z \subset M \) of the corresponding zero-section of \( p \). Suppose that we are given a norm \( \| \cdot \| \) on fibers such that the following function \( f : M \to \mathbb{R} \) is smooth:

\[
f(\xi, z) = \|\xi\|^2, \quad z \in Z, \quad \xi \in p^{-1}(z).
\]

Define the following singular partition \( \Theta = (\mathcal{P}, \Sigma) \) on \( M \), where \( \mathcal{P} \) consists of the subsets \( \omega_{s,z} = f^{-1}(s) \cap p^{-1}(z) \) for \( s \geq 0 \) and \( z \in Z \), and \( \Sigma = \{\omega_{0,z} = \{z\} : z \in Z\} \) consists of points of \( Z \). Thus every fiber \( p^{-1}(z) \) is \( \Theta \)-invariant, and the restriction of \( \Theta \) to \( p^{-1}(z) \) is the same as the one in Example 4.2.

Fix \( s > 0 \) and put \( V = f^{-1}[0, s] \). Then \( V \) is \( \Theta \)-invariant and a \( \Theta \)-invariant contraction of \( V \) to \( Z \) can be given by \( r(\xi, z, t) = (t\xi, z) \).

We will now generalize these examples. Let \( f : \mathbb{R}^m \to \mathbb{R} \) be a smooth function and suppose that there exists a neighbourhood \( V \) of \( 0 \) and smooth functions \( \alpha_1, \ldots, \alpha_m : V \to \mathbb{R} \) such that

\[
f = \alpha_1 \cdot f'_{x_1} + \cdots + \alpha_m \cdot f'_{x_m}.
\]

Equivalently, let \( \Delta(f, 0) \) be the Jacobian ideal of \( f \) in \( C^\infty(V, \mathbb{R}) \) generated by partial derivatives of \( f \). Then (4.1) means that \( f \in \Delta(f, 0) \).

For instance, let \( f \) be quasi-homogeneous of degree \( d \) with weights \( s_1, \ldots, s_m \), i.e., \( f(t^{s_1} x_1, \ldots, t^{s_m} x_m) = t^d f(x_1, \ldots, x_m) \) for \( t > 0 \), see e.g. [1, §12]. Equivalently, we may require that the function

\[
g(x_1, \ldots, x_m) = f(x_1^{s_1}, \ldots, x_m^{s_m})
\]

be homogeneous of degree \( d \). Then the following Euler identity holds true:

\[
f = \frac{x_1}{s_1} f'_{x_1} + \cdots + \frac{x_m}{s_m} f'_{x_m}.
\]

In particular, \( f \) satisfies (4.1). Moreover in the complex analytical case\(^1\) the identity (4.1) characterizes quasi-homogeneous functions, see [21].

**Lemma 4.4.** Let \( f : M \to \mathbb{R} \) be a smooth function and \( z \in M \) be an isolated local minimum of \( f \). Suppose that \( f \) satisfies condition (4.1) at \( z \), i.e., \( f \in \Delta(f, z) \). Then the singular partition \( \Theta_f \) admits an invariant contraction to \( z \).

**Proof.** Since the situation is local, we may assume that \( M = \mathbb{R}^m \), \( z = 0 \) is a unique critical point of \( f \) being its global minimum, \( f(0) = 0 \), and there exists an \( \varepsilon > 0 \) such that \( V = f^{-1}[0, \varepsilon] \) is a smooth compact \( m \)-dimensional manifold with boundary \( L = f^{-1}(\varepsilon) \).

First we give a precise description of the partition \( \mathcal{P}_f \) on \( V \). Let \( F \) be any gradient like vector field on \( V \) for \( f \), i.e., \( df(F)(x) > 0 \) for \( x \neq 0 \). Then following [15, Th. 3.1] we can construct a diffeomorphism

\[
\eta : V \setminus \{0\} \to L \times (0, \varepsilon]
\]

\(^1\)I would like to thank V. A. Vasilyev for referring me to the paper [21] by K. Saito.
such that \( f \circ \eta^{-1}(y, t) = t \) for all \((y, t) \in L \times (0, \varepsilon)\), see Figure 4.1.

Let \( CL = L \times [0, \varepsilon]/\{L \times 0\} \) be the cone over \(L\) and \(L_t = f^{-1}(t)\) for \(t \in (0, \varepsilon]\). Since \(\text{diam}(L_t) \to 0\) when \(t \to 0\), we obtain that \(\eta\) extends to a homeomorphism \(\eta : V \to CL\) by \(\eta(0) = \{L \times 0\}\).

For every \(t \in (0, \varepsilon]\) put \(L_t = f^{-1}(t)\). Then \(\eta\) diffeomorphically maps \(L_t\) onto \(L \times \{t\}\).

![Figure 4.1](image.png)

**Lemma 4.5.** \(L\) is homotopy equivalent to \(S^{m-1}\).

We will prove this lemma below. Then it will follow from the generalized Poincaré conjecture that \(L\) is homeomorphic with the sphere \(S^{m-1}\), and even diffeomorphic to \(S^{m-1}\) if \(k = m - 1 \neq 4\). For \(k = 1, 2\) this statement is rather elementary, for \(k = 3\) this follows from a recent work of G. Perelman [18, 19], for \(k = 4\) from M. Freedman [3], and for \(k \geq 5\) from S. Smale [23], see also [14].

In particular, every \(L_t\) is connected, whence the partition \(P_f\) on \(V\) consists of a unique singular element \(\{0\} \in \Sigma_f\) and sets \(L_t, t \in (0, \varepsilon]\).

Let us recall the definition of \(\eta\). Notice that every orbit of \(F\) starts at 0 and transversely intersects every \(L_t\). For each \(x \in V \setminus \{0\}\) denote by \(q(x)\) a unique point of the intersection of the orbit of \(x\) with \(L = L_\varepsilon = \partial V\). Then \(\eta : V \setminus \{0\} \to L \times (0, \varepsilon]\) can be given by the following formula:

\[
\eta(x) = (q(x), f(x)).
\]

Also notice that if \(\phi : [0, \varepsilon] \to [0, \varepsilon]\) is a (not necessarily surjective) \(C^\infty\) embedding such that \(\phi(0) = 0\), then we can define the embedding

\[
w_\phi : CL \to CL, \quad w_\phi(y, t) = (y, \phi(t))
\]

and therefore the embedding \(r_\phi = \eta \circ w_\phi \circ \eta^{-1} : V \to V\). Then \(r_\phi\) is \(C^\infty\) on \(V \setminus 0\) and diffeomorphically maps \(L_t\) onto \(L_{q(t)}\) for all \(t \in (0, \varepsilon]\). Moreover \(r_\phi(0) = 0\), but in general \(r_\phi\) is not even smooth at 0.

Suppose now that \(f \in \Delta(f, 0)\), i.e., we have a presentation (4.1). Consider the following vector field

\[
F = \alpha_1 \frac{\partial}{\partial x_1} + \cdots + \alpha_m \frac{\partial}{\partial x_m}.
\]

Then (4.1) means that \(f = df(F)\). Since \(f(x) > 0\) for \(x \neq 0\), it follows that \(F\) is a gradient like vector field for \(f\). Therefore we can construct a homeomorphism \(\eta : V \to CL\) using \(F\) as above. It follows from [10] that in our case this \(\eta\) has the following feature:

- if \(\phi : [0, \varepsilon] \to [0, \varepsilon]\) is a \(C^\infty\) embedding such that \(\phi(0) = 0\), then the corresponding embedding \(r_\phi : V \to V\) is a diffeomorphism onto its image. Moreover, if \(\phi_s, (s \in I),\) is a \(C^\infty\) isotopy, then so is \(r_\phi_s : V \to V\).

In particular, consider the following homotopy

\[
\phi : [0, \varepsilon] \times I \to [0, \varepsilon], \quad \phi(t, s) = t(1 - s)
\]

which contracts \([0, \varepsilon]\) to a point and being an isotopy for \(t > 0\). Then the induced homotopy \(r : V \times I \to V\) is an invariant \(\infty\)-contraction of \(\Theta_f\) to 0. \(\square\)
Proof of Lemma 4.5. It suffices to establish that

\[
\pi_k L \cong \pi_k S^{m-1} = \begin{cases} 0, & k = 0, \ldots, m-2, \\ \mathbb{Z}, & k = m-1. \end{cases}
\]

Then the generator \( \mu : S^{m-1} \to L \) of \( \pi_{m-1} L \cong \mathbb{Z} \) will yield isomorphisms of the homotopy groups \( \pi_k S^{m-1} \cong \pi_k L \) for all \( k \leq m-1 = \dim S^{m-1} = \dim L \). Now by the well-known Whitehead’s theorem \( \mu \) will be a homotopy equivalence between \( S^{m-1} \) and \( L \).

For the calculation of homotopy groups of \( L \) consider the exact sequence of homotopy groups of the pair \( (V, L) \):

\[
\cdots \to \pi_{k+1}(V, L) \to \pi_k L \to \pi_k V \to \cdots
\]

Since \( V \) is homeomorphic with the cone \( CL \), \( V \) is contractible, whence \( \pi_k V = 0 \) for all \( k \geq 0 \).

Moreover, \( \pi_{k+1}(V, L) = 0 \) for all \( k = 0, \ldots, m-2 \). Indeed, let

\[ \xi : (D^{k+1}, S^k) \to (V, L) \]

be a continuous map. We have to show that \( \xi \) is homotopic (as a map of pairs) to a map into \( L \). Since \( k + 1 \leq m - 1 < \dim V \), \( \xi \) is homotopic to a map into \( V \setminus \{0\} \). But \( L \) is a deformation retract of \( V \setminus \{0\} \), therefore \( \xi \) is homotopic to a map into \( L \).

Now it follows from (4.3) that \( \pi_k L = 0 \) for \( k = 0, \ldots, m-2 \). Hence from Hurewicz’s theorem we obtain that \( \pi_{m-1} L \cong H_{m-1}(L, \mathbb{Z}) \). It remains to note that \( L \) is a connected closed orientable \((m - 1)\)-manifold, whence (by Poincaré duality) \( H_{m-1}(L, \mathbb{Z}) \cong \mathbb{Z} \). This proves (4.2).

\[ \square \]

Remark 4.6. If \( f \) is a quasi-homogeneous function of degree \( d \) with weights \( s_1, \ldots, s_m \), then we can define an invariant contraction by

\[ r(x_1, \ldots, x_m, t) = (t^{s_1} x_1, \ldots, t^{s_m} x_m). \]

If \( f \) is homogeneous, then we can even put \( r(x, t) = tx \).

Theorem 4.7. Let \( \Theta = (\mathcal{P}, \Sigma) \) be a singular partition on a manifold \( M \), and \( Z \subset M \) be a closed subset such that every \( z \in Z \) is a singular element of \( \mathcal{P} \), i.e., \( \{z\} \in \Sigma \). Suppose that \( \Theta \) has an invariant \( \infty \)-contraction to \( Z \) defined on a \( \Theta \)-invariant neighbourhood \( V \) of \( Z \). Let also \( h \in \mathcal{E}(\Theta, V) \) be a map fixed outside some neighbourhood \( U \) of \( z \) such that \( \bar{U} \subset \text{Int} V \). Then \( h \in \mathcal{E}_{id}(\Theta, V)^0 \), though \( h \) not necessarily belongs to \( \mathcal{E}_{id}(\Theta, V)^r \) for some \( r \geq 1 \).

Remark 4.8. Let \( \Theta_f \) be the singular partition of \( f(x) = \|x\|^2 \) as in Example 4.2, \( V \) be the unit \( n \)-disk centered at \( 0 \) and \( r : V \times I \to V \) be the invariant contraction of \( \Theta_f \) to a point \( Z = \{0\} \) defined by \( r(x, t) = tx \). Then a 0-homotopy between \( h \) and \( \text{id}_V \) can be defined by

\[ H_h(x) \begin{cases} th(\xi), & t > 0, \\ 0, & t = 0. \end{cases} \]

c.f. [5, Ch. 4, Theorems 5.3 & 6.7]. Theorem 4.7 generalizes this example.

Proof of Theorem 4.7. Let \( r : V \times I \to V \) be an invariant \( \infty \)-contraction of \( \Theta_F \) to \( Z \). Define the following map \( H : V \times I \to M \) by

\[ H(x, t) = \begin{cases} r_t \circ h \circ r_t^{-1}(x), & \text{if } t > 0 \text{ and } x \in r_t(V), \\ x, & \text{otherwise}. \end{cases} \]

We claim that \( H \) is a 0-homotopy (i.e. just a homotopy) between \( h \) and the identity inclusion \( i_V : V \subset M \) in \( \mathcal{E}(\Theta, V) \). To make this more obvious we rewrite the formulas for \( H \) in another way.
The homotopy \( r \) can be regarded as the composition

\[
    r = p_1 \circ \tilde{r} : V \times I \longrightarrow V \times I \longrightarrow V,
\]

where \( \tilde{r} \) is the following level-preserving map

\[
    \tilde{r} : V \times I \rightarrow V \times I, \quad \tilde{r}(x, t) = (r(x, t), t),
\]

and \( p_1 : V \times I \rightarrow V \) is the projection to the first coordinate. It follows from the definition

\[
\begin{align*}
    &V \times I \\
    &\downarrow \quad \downarrow \tilde{r} \\
    &V \times 0
\end{align*}
\]

![Diagram](image.png)

**Figure 4.2**

that \( r \) yields a level-preserving embedding \( V \times (0, 1] \) to \( V \times I \), see Figure 4.2. Denote

\[
    R' = \tilde{r}(V \times (0, 1]), \quad R = \tilde{r}(V \times I).
\]

Then \( R \setminus R' = Z \times 0 \). Define also the following map

\[
    \tilde{h} : V \times I \rightarrow V \times I, \quad \tilde{h}(x, t) = (h(x), t).
\]

In these terms, the homotopy \( H \) is defined by

\[
    H = p_1 \circ \tilde{H} : V \times I \longrightarrow V \times I \longrightarrow V,
\]

where \( \tilde{H} : V \times I \rightarrow V \times I \) is a level-preserving map given by

\[
    \tilde{H}(x, t) = \begin{cases} 
    \tilde{r} \circ \tilde{h} \circ \tilde{r}^{-1}(x, t), & (x, t) \in R', \\
    (x, t), & (x, t) \in (V \times I) \setminus R'.
\end{cases}
\]

Now we can prove that \( H \) has the desired properties.

Since \( r_1 = \text{id}_V \), we have \( H_1 = h \). Moreover \( H_0 = \text{id}_V \).

1. **Continuity of \( \tilde{H} \) on \( V \times (0, 1] \).** Notice that \( \tilde{r} \circ \tilde{h} \circ \tilde{r}^{-1} \) is well-defined and continuous on \( R' \). Moreover, since \( h \) is fixed on \( V \setminus U \), it follows that \( \tilde{h} \) is fixed on \( (V \setminus U) \times I \), whence \( \tilde{r} \circ \tilde{h} \circ \tilde{r}^{-1} \) is fixed on the subset \( \tilde{r}((V \setminus U) \times (0, 1]) \subset R' \). This implies that \( \tilde{H} \) is continuous on \( V \times (0, 1] \).

2. **Continuity of \( \tilde{H} \) when \( t \to 0 \).** Let \( z \in V \). Then \( \tilde{H}(z, 0) = (z, 0) \).

Suppose that \( z \in V \setminus Z \). Since \( Z \) is closed in \( V \), \( \tilde{H} \) is also fixed and therefore continuous on some neighbourhood of \( (z, 0) \) in \( (V \times I) \setminus R \).

Let \( z \in Z \) and let \( W \) be a neighbourhood of \( (z, 0) \) in \( V \times I \). We have to find another neighbourhood \( W' \) of \( (z, 0) \) such that \( \tilde{H}(W') \subset W \).

Recall that for every \( y \in Z \) we denoted \( V_y = r_0^{-1}(y) \). Then \( V_y \) is compact and \( \Theta \)-invariant.

**Claim.** There exist \( \varepsilon > 0 \) and an open neighbourhood \( N \) of \( z \) in \( V \) such that \( \overline{N} \times [0, \varepsilon] \subset W \) and

\[
    \tilde{r}(V_y \times [0, \varepsilon]) \subset W, \quad (y \in \overline{N} \cap Z).
\]

**Proof.** Let \( N \) be an open neighbourhood of \( z \) such that \( \overline{N} \) is compact and \( \overline{N} \times 0 \subset W \). Denote

\[
    Q = r_0^{-1}(\overline{N} \cap Z) = \bigcup_{y \in \overline{N} \cap Z} r_0^{-1}(y) = \bigcup_{y \in \overline{N} \cap Z} V_y.
\]
Then $Q$ is a compact subset of $V$, and $\tilde{r}^{-1}(W)$ is an open neighbourhood of $Q \times 0$ in $V \times I$. Hence there exists $\varepsilon > 0$ such that $Q \times [0, \varepsilon] \subset \tilde{r}^{-1}(W)$. This implies (4.4). Decreasing $\varepsilon$ if necessary we can also assume that $\nabla \times [0, \varepsilon] \subset W$ as well.

Denote $W' = N \times [0, \varepsilon]$. We claim that $\tilde{H}(W') \subset W$.

Let $(x, t) \in W'$. If either $(x, t) \in W' \setminus W'$ or $t = 0$, then $\tilde{H}(x, t) = (x, t) \in W' \subset W$.

Suppose that $(x, t) \in W' \cap W$. Then $t > 0$. Let also $y = r_0(x) \in \Sigma$. Then $\tilde{r}^{-1}(x, t) \in V_y \times t$ for all $t \in I$. Hence

$$\tilde{r} \circ \tilde{h} \circ \tilde{r}^{-1}(x, t) \in \tilde{r} \circ \tilde{h}(V_y \times t) \subset \tilde{r}(V_y \times t) \subset W.$$  

In the second inclusion we have used a $\Theta$-invariance of $V_y$ and the assumption that $h \in \mathcal{E}(\Theta, V)$.

3. Proof that $H_t \in \mathcal{E}(\Theta, V)$ for $t \in I$. We have to show that (i) for every $t \in I$ the mapping $H_t$ is $C^\infty$, (ii) $H_t(\omega) \subset \omega$ for every element $\omega \in \mathcal{P}$ included in $V$, and (iii) $H_t$ is a local diffeomorphism at every point $z$ belonging to some $\omega \in \Sigma$.

(i) Since $r_t$, $t > 0$, is $C^\infty$ and $h$ is fixed on $V \setminus U$, it follows that $H_t$ is $C^\infty$ as well.

(ii) Let $\omega \in V$ be an element of $\mathcal{P}$ (resp. $\Sigma$).

If $\omega \in V \setminus r_t(V)$, then $H_t$ is fixed on $\omega$, whence $H_t(\omega) = \omega \in \mathcal{P}$ (resp. $\Sigma$).

Suppose that $\omega \subset r_t(V)$. Since $r_t(V)$ is $\Theta$-invariant, $\omega = r_t(\omega')$ for some another element $\omega' \in \mathcal{P}$ (resp. $\Sigma$). Then $h(\omega') \subset \omega'$, whence

$$H_t(\omega) = r_t \circ h \circ r_t^{-1}(\omega) = r_t \circ h(\omega') \subset r_t(\omega') = \omega.$$  

(iii) Suppose that $\omega \in \Sigma$ and let $x \in \omega$.

If $x \in V \setminus r_t(U)$, then $H_t$ is fixed in a neighbourhood of $x$, and therefore it is a local diffeomorphism at $x$.

Suppose that $x = r_t(x') \in r_t(U)$ for some $x' \in U$ and let $\omega' \in \Sigma$ be the element containing $x'$. Then $h$ is a local diffeomorphism at $x'$, whence $H_t = r_t \circ h \circ r_t^{-1}$ is a local diffeomorphism at $x$.

5. Stabilizers of smooth functions

Let $B^m \subset \mathbb{R}^m$ be the unit disk centered at the origin $0$, $S^{m-1} = \partial B^m$ be its boundary sphere, $f : B^m \to \mathbb{R}$ be a $C^\infty$ function, and $\Theta_f$ be the singular partition of $f$.

Let $\mathcal{S}(f) = \{ h \in \mathcal{D}(B^m) : f \circ h = f \}$ be the stabilizer of $f$ with respect to the right action of the group $\mathcal{D}(B^m)$ of diffeomorphisms of $B^m$ on the space $C^\infty(B^m, \mathbb{R})$. Denote by $\mathcal{S}^+(f)$ the subgroup of $\mathcal{S}(f)$ consisting of orientation preserving diffeomorphisms. For $r \in \nabla$, let also $\mathcal{S}_d(f)^r$ be the identity component of $\mathcal{S}(f)$ with respect to the $C^r$-topology. Then

$$\mathcal{S}_d(f)^\infty \subset \cdots \subset \mathcal{S}_d(f)^{r+1} \subset \mathcal{S}_d(f)^r \subset \cdots \subset \mathcal{S}_d(f)^0 \subset \mathcal{S}^+(f).$$  

**Theorem 5.1.** Let $m \geq 2$, $f : B^m \to [0, 1]$ be a $C^\infty$ function such that $0$ is a unique critical point of $f$ being its global minimum, $f(0) = 0$, and $f(S^{m-1}) = 1$. Denote by $\mathcal{S}$ the subgroup of $\mathcal{S}(f)$ consisting of diffeomorphisms $h$ such that $h|_{S^{m-1}} : S^{m-1} \to S^{m-1}$ is $C^\infty$-isotopic to id$_{S^{m-1}}$. Suppose also that the singular partition $\Theta_f$ of $f$ has an invariant $\infty$-contraction to $0$. Then $\mathcal{S} \subset \mathcal{S}_d(f)^0$.

If $m = 2, 3, 4$, then $\mathcal{S} = \mathcal{S}_d(f)^0 = \mathcal{S}^+(f)$.

For the proof we need the following two simple standard statements concerning smoothing homotopies at the beginning and at the end, see e.g. [20, pp. 74 & 118] and [16, p. 205]. Let $M$ be a closed smooth manifold.

**Claim 5.2.** Let $a, b, c \in \mathbb{R}$ be numbers such that $0 < a < b < c$, and $N = M \times (0, c]$. Then we have a foliation on $N$ by submanifolds $M \times t$, $t \in (0, c]$. Let also $h : N \to N$ be a $C^\infty$ leaf preserving diffeomorphism, i.e., $h(x, t) = (\phi(x, t), t)$ for some $C^\infty$ map $\phi : M \times (0, c] \to M$ such that for every $t \in (0, c]$ the map $\phi_t : M \to M$ is a diffeomorphism.
Then there exists a leaf preserving isotopy relatively to \( M \times [0, a] \) of \( h \) to a diffeomorphism \( \hat{h}(x, t) = (\hat{\phi}(x, t), t) \) such that \( \hat{\phi}_t = \phi_c \) for all \( t \in [b, c] \).

**Proof.** Let \( \mu : (0, c) \to (0, c) \) be a \( C^\infty \) function such that \( \mu(t) = t \) for \( t \in (0, a) \) and \( \mu(t) = c \) for \( t \in [b, c] \), see Figure 5.1a. Define the following leaf preserving isotopy \( H : N \times I \to N \) by

\[
H_s(x, t) = (\phi(x, (1-s)t + s \mu(t)), t).
\]

Then it easy to see that \( H_0 = \text{id}_N \), \( H_1 = h \) on \( M \times (0, a] \) and \( \hat{h} = H_1 \) satisfies conditions of our claim.  

**Claim 5.3.** Let \( d < c \in (0, 1) \) and \( G : M \times I \to M \) be a \( C^\infty \) homotopy (isotopy). Then there exists another \( C^\infty \) homotopy (isotopy) \( G' : M \times I \to M \) such that \( G_t = G_0 \) for \( t \in [0, d] \) and \( \hat{G}_t = G_1 \) for \( t \in [e, 1] \).

**Proof.** Take any \( C^\infty \) function \( \nu : I \to I \) such that \( \nu(0, d] = 0 \) and \( \nu(e, 1] = 1 \), and put \( \hat{G}_t = G_\nu(t) \), see Figure 5.1b).

![Figure 5.1](attachment:image.png)

**Proof of Theorem 5.1.** Since 0 is a unique critical point of \( f \) and \( f \) is constant on \( S^{m-1} \), it follows from the arguments of the proof of Lemma 4.4 that there exists a diffeomorphism \( \eta : D^n \setminus 0 \to S^{m-1} \times (0, 1] \) such that \( f \circ \eta^{-1}(y, t) = t \). In particular, for every \( t \in [0, 1] \) the set \( f^{-1}(t) \) is diffeomorphic with \( S^{m-1} \). Since \( S^{m-1} \) is connected, we obtain that \( D(\Theta_f) = \mathcal{S}(f) \).

Therefore \( D_{\text{id}}(\Theta_f)^r = \mathcal{S}_{\text{id}}(f)^r \) for all \( r \in \mathbb{N}_0 \).

By assumption there exists an invariant \( \infty \)-contraction of \( \Theta_f \) to 0 defined on some \( \Theta_f \)-invariant neighbourhood \( V \) of 0. Therefore we can assume that \( V = f^{-1}[0, 2c] \) for some \( c \in (0, \frac{1}{2}) \).

**Lemma 5.4.** There exists a \( C^\infty \)-isotopy of \( h \) in \( S \) to a diffeomorphism \( \hat{h} \) fixed on \( f^{-1}[c, 1] \). Then is follows from Theorem 4.7 that \( \hat{h} \) and therefore \( h \) belong to \( D_{\text{id}}(\Theta_f)^0 = \mathcal{S}_{\text{id}}(f)^0 \).

**Proof.** Since \( h \) preserves \( f \), it follows that the following diffeomorphism

\[
g = \eta \circ h \circ \eta^{-1} : S^{m-1} \times (0, 1] \to S^{m-1} \times (0, 1]
\]

is leaf preserving, i.e., \( h(S^{m-1} \times t) = S^{m-1} \times t \) for all \( t \in (0, 1] \). Then by Claim 5.2 we can assume that \( g|_{S^{m-1} \times t} = h|_{S^{m-1}} \) for all \( t \in [0, 1] \).

Take any \( a \in (0, c) \). It suffices to find a leaf preserving isotopy relatively to \( S^{m-1} \times (0, a] \) of \( g \) to a diffeomorphism \( \hat{g} \) fixed on \( S^{m-1} \times [c, 1] \). This isotopy will yield an isotopy relatively to \( f^{-1}[0, a] \) of \( h \) in \( S \) to a diffeomorphism \( \hat{h} \) which is fixed on \( f^{-1}[c, 1] \).

By assumptions of our theorem there exists a \( C^\infty \) isotopy \( G : S^{m-1} \times [1, 2] \to S^{m-1} \) such that \( G_1 = h|_{S^{m-1}} \) and \( G_2 = \text{id}_{S^{m-1}} \). By Claim 5.3 we can assume that \( G_t = h|_{S^{m-1}} \).

**Figure 5.1**

![Figure 5.1](attachment:image.png)
for all $t \in [1, 1.5]$. Hence $g$ and $G$ yield the following $C^\infty$ leaf preserving diffeomorphism:

$$T : S^{m-1} \times (0,2] \to S^{m-1} \times (0,2], \quad T(y,s) = \begin{cases} g(y,s), & s \in (0,1], \\ G(y,s), & s \in [1,2]. \end{cases}$$

Notice that $T(y,2) = G(y,2) = y$. Then, by Claim 5.3, $T$ is isotopic via a leaf preserving isotopy relatively $S^{m-1} \times (0,a]$ to a diffeomorphism $\hat{T}$ which is fixed on $S^{m-1} \times (c,2]$. Denote $\hat{g} = \hat{T}|_{S^{m-1} \times (0,1]}$. The restriction of this isotopy to $S^{m-1} \times (0,1]$ gives a leaf preserving isotopy relatively $S^{m-1} \times (0,a]$ of $g$ to a diffeomorphism $\hat{g}$ with desired properties. The construction of homotopy is schematically presented in Figure 5.2. □

Figure 5.2

Suppose now that $m = 2, 3, 4$. Then every orientation preserving diffeomorphism of $S^{m-1}$ is $C^\infty$-isotopic to $\text{id}_{S^{m-1}}$, whence $\mathcal{S} = \mathcal{S}^+(f)$, and therefore $\mathcal{S} = S_{\text{id}}(f)^0 = \mathcal{S}^+(f)$. For $m = 2$ this is rather trivial, for $m = 3$ is proved by S. Smale [22], and for $m = 4$ by A. Hatcher [4].

If $m \geq 5$, then $\mathcal{D}^+(S^{m-1})$ is not connected in general, see e.g. [17], and therefore Theorem 4.7 is not applicable. □

6. Linear symmetries of homogeneous polynomials

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial of degree $p \geq 2$ given by (1.1),

$$f(x,y) = \pm \prod_{i=1}^{l} L^n_i(x,y) \cdot \prod_{j=1}^{k} Q^m_j(x,y).$$

Denote

$$\mathcal{LS}(f) = \mathcal{S}^+(f) \cap \text{GL}^+(2,\mathbb{R}).$$

Thus $\mathcal{LS}(f)$ consists of preserving orientation linear automorphisms $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f \circ h = f$. Also notice that $\mathcal{LS}(f)$ is a closed subgroup of $\text{GL}^+(2,\mathbb{R})$, and therefore it is a Lie group. Denote by $\mathcal{LS}(f)_0$ the connected component of the unit matrix $\text{id}_{\mathbb{R}^2}$ in $\mathcal{LS}(f)$.

In this section we recall the structure of $\mathcal{LS}(f)$. Notice that we may make linear changes of coordinates to reduce $f$ to a convenient form. Then $\mathcal{LS}(f)$ will change to a conjugate subgroup in $\text{GL}^+(2,\mathbb{R})$.

The following statement is evident.

**Lemma 6.1.** If $\deg f$ is even, then $f(-z) \equiv f(z)$, i.e., $-\text{id}_{\mathbb{R}^2} \in \mathcal{LS}(f)$. Therefore in this case $\mathcal{LS}(f)$ is a non-trivial group.
We will distinguish the following five cases of $f$.

(A) $l = 1$, $k = 0$, $f = L_1^\alpha$. By a linear change of coordinates we can assume that $L_1(x, y) = y$ and thus $f(x, y) = y^{\alpha_1}$. Then

$$\mathcal{L}_0(y^{\alpha_1}) = \left\{ \left( \begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix} \right) : a > 0 \right\}.$$  

If $\alpha_1$ is odd then $\mathcal{L}_0 = \mathcal{L}(f)$, otherwise, $\mathcal{L}(f)$ consists of two connected components $\mathcal{L}_0$ and $-\mathcal{L}_0$.

(B) $l = 2$, $k = 0$, $f = L_1^\alpha L_2^\beta$. By a linear change of coordinates we can assume that $L_1(x, y) = x$, $L_2(x, y) = y$ and thus $f(x, y) = x^{\alpha_1} y^{\alpha_2}$. Then

$$\mathcal{L}_0(x^{\alpha_1} y^{\alpha_2}) = \left\{ \left( \begin{smallmatrix} e^{a_1 t} & 0 \\ 0 & e^{-a_1 t} \end{smallmatrix} \right) : t \in \mathbb{R} \right\}.$$  

Moreover, $\mathcal{L}(f)/\mathcal{L}_0$ is isomorphic with some subgroup of $\mathbb{Z}_4$ generated by the rotation of $\mathbb{R}^2$ by $\pi/2$.

(C) $l = 0$, $k = 1$, $f = Q^\beta_1$. By a linear change of coordinates we can assume that $Q_1(x, y) = x^2 + y^2$, whence $f(x, y) = (x^2 + y^2)^{\beta_1}$. Then

$$\mathcal{L}(x^2 + y^2) = \text{SO}(2, \mathbb{R}) = \left\{ \left( \begin{smallmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{smallmatrix} \right) : t \in [0, 2\pi) \right\}.$$  

The above statements are elementary and we left them to the reader. Notice also that, in the cases (A)-(C), $l + 2k \leq 2$. The remaining two cases are the following:

(D) $l = 0$, $k \geq 2$, $f = Q^\beta_1 \cdots Q^\beta_k$. In this case deg $f$ is even, whence $\mathcal{L}(f)$ is non-trivial.

(E) $l \geq 1$, $l + 2k \geq 3$.

**Lemma 6.2.** In the cases (D) and (E), $\mathcal{L}(f)$ is a finite cyclic subgroup of $\text{GL}^+(2, \mathbb{R})$. Moreover, in the case (E), $\mathcal{L}(f)$ is a subgroup of $\mathbb{Z}_2^l$.

**Proof.** In fact cyclicity of $\mathcal{L}(f)$ for the case $l + 2k - 1 \geq 2$ can be extracted from the paper of W. C. Huffman, [6], where the symmetries of complex binary forms are classified. Regard $f : \mathbb{C}^2 \to \mathbb{C}$ as a complex polynomial with real coefficients. Then by [6] the subgroup $\mathcal{L}_C(f)$ of $\text{GL}(2, \mathbb{C})$ consisting of complex symmetries of $f$ turned out to be of one of the following types: cyclic, dihedral, tetrahedral, octahedral, and icosahedral. Notice $\mathcal{L}(f)$ is the subgroup of $\mathcal{L}_C(f)$ consisting of preserving orientation real symmetries of $f$, i.e., automorphisms which also leave invariant the 2-plane $\mathbb{R}^2 \subset \mathbb{C}^2$ of real coordinates and preserve its orientation. Then it follows from the structure of symmetries of regular polyhedrons that $\mathcal{L}(f)$ must be cyclic.

Nevertheless, since we need a very particular case of [6] and for the sake of completeness, we will present a short elementary proof. It suffices to show that $\mathcal{L}(f)$ is finite, see 6.6. This will imply that $\mathcal{L}(f)$ is isomorphic to a finite subgroup of $\text{SO}(2)$, and therefore is cyclic. Also notice that the fact that $\mathcal{L}(f)$ is discrete also follows from [12]. First we establish the following three statements.

**Claim 6.3.** Let $h \in \text{GL}^+(2, \mathbb{R})$ and $Q$ be a positive definite quadratic form such that $Q \circ h = tQ$ for some $t > 0$. Then $t = \det(h)$.

**Proof.** By a linear change of coordinates we can assume that $Q(z) = |z|^2$. Then $h(z) = \sqrt{t} e^{i\psi} z$ for some $\psi \in \mathbb{R}$, hence $\det(h) = t$. \hfill $\Box$

**Claim 6.4.** Let $Q_1, Q_2$ be a positive definite quadratic form such that $Q_1/Q_2 \neq \text{const}$. Let also $h \in \text{GL}^+(2, \mathbb{R})$ be such that $Q_i \circ h = tQ_i$ for $i = 1, 2$, where $t = \det(h)$. Then $h(z) = \pm \sqrt{t} z$.

**Proof.** We can assume that $Q_1(x, y) = x^2 + y^2$ and $Q_2(x, y) = ax^2 + by^2$, where $a, b > 0$ and either $a \neq 1$ or $b \neq 1$. Denote $g(z) = h(z)/\sqrt{t}$. Then $Q_1 \circ g = Q_1$, i.e., $g$ preserves every circle $x^2 + y^2 = \text{const}$ and every ellipse $ax^2 + by^2 = \text{const}$. Therefore $g = \pm \text{id}_{\mathbb{R}^2}$, and $h(z) = \pm \sqrt{t} z$. \hfill $\Box$
Claim 6.5. If \( t \cdot \text{id}_{\mathbb{R}^2} \in \mathcal{LS}(f) \) for some \( t \in \mathbb{R} \), then \( t = \pm 1 \).

Proof. Let \( z \in \mathbb{R}^2 \) be such that \( f(z) \neq 0 \). Since \( f \) is homogeneous, we have \( f(tz) = t^{\deg f} f(z) \), whence \( t = \pm 1 \).

Let \( h \in \mathcal{LS}(f) \). Since \( L_i \) and \( Q_j \) are irreducible over \( \mathbb{R} \), so are \( L_i \circ h \) and \( Q_j \circ h \). Therefore the identity \( f \circ h = f \) implies that “\( h \) permutes \( L_i \) and \( Q_j \) up to non-zero multiples”. This means that for every \( i \) there exist \( i' \) and \( s_i \in \mathbb{R} \setminus \{0\} \), and for every \( j \) there exist \( j' \) and \( t_j > 0 \) such that
\[
L_i(h(z)) = s_i L_{i'}(z), \quad Q_j(h(z)) = t_j Q_{j'}(z).
\]

Denote by \( \text{Sym}(r) \) the group of permutations of \( r \) symbols. Then we have a well-defined homomorphism
\[
\mu : \mathcal{LS}(f) \rightarrow \text{Sym}(l) \times \text{Sym}(k)
\]
associating to every \( h \in \mathcal{LS} \) its permutations of \( L_i \) and \( Q_j \).

Claim 6.6. If \( l + 2k \geq 3 \), then \( \ker \mu \subset \{ \pm \text{id}_{\mathbb{R}^2} \} \), whence \( \mathcal{LS}(f) \) is a finite group.

Proof. Let \( h \in \ker \mu \). Thus \( L_i \circ h = s_i L_i \) and \( Q_j \circ h = t_j Q_j \) for all \( i, j \). We will show that \( h = t \cdot \text{id}_{\mathbb{R}^2} \) for some \( t \neq 0 \). Then it will follow from Claim 6.5 that \( h = \pm \text{id} \).

Notice that \( h \) preserves every line \( \{ L_i = 0 \} \) and thus has \( l \) distinct eigen directions.

a) Therefore if \( l \geq 3 \), then \( h = t \cdot \text{id}_{\mathbb{R}^2} \) for some \( t \in \mathbb{R} \).

b) Moreover, if \( k \geq 2 \), then by Claim 6.4, we also have \( h = \pm \text{id}_{\mathbb{R}^2} \).

c) Suppose that \( 1 \leq l \leq 2 \) and \( k = 1 \). We can assume that \( Q(z) = |z|^2 \) and that \( h(z) = t e^{i\psi} z \) for some \( t > 0 \) and \( \psi \in \mathbb{R} \). Since \( h \) has \( l \geq 1 \) eigen directions, we obtain that \( h(z) = \pm \sqrt{t} z \).

Thus \( \mathcal{LS}(f) \approx \mathbb{Z}_n \) for some \( n \in \mathbb{N} \). Let \( h \) be a generator of \( \mathcal{LS}(f) \). Then we can assume that \( h(z) = e^{2\pi i/n} z \). It remains to prove the latter statement.

Claim 6.7. Suppose that \( l \geq 1 \). Then \( n \) divides \( 2l \), whence \( \mathcal{LS}(f) \) is isomorphic to a subgroup of \( \mathbb{Z}_{2l} \).

Proof. Since \( f \circ h = f \), it follows that \( h(f^{-1}(0)) = f^{-1}(0) \). By assumption \( l \geq 1 \), whence \( f^{-1}(0) = \bigcup_{i=1}^l \{ L_i = 0 \} \) is the union of \( l \) lines passing through the origin. This set can be viewed as the union of \( 2l \) rays starting at the origin, and these rays are cyclically shifted by \( h \). Moreover, if \( h^l \) preserves at least one of these rays, then \( h^l = \text{id}_{\mathbb{R}^2} \). Therefore \( n \) divides \( 2l \).

Lemma 6.2 is completed.

7. Proof of Theorem 1.1.

Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a homogeneous polynomial of degree \( p \geq 1 \) given by (1.1)
\[
f(x, y) = \pm \prod_{i=1}^l L_i^{\alpha_i}(x, y) \cdot \prod_{j=1}^k Q_j^{\beta_j}(x, y).
\]

We will refer to the cases (A)–(E) of \( f \) considered in the previous section. We have to show that \( \mathcal{S}_{\text{id}}(f)^\infty = \cdots = \mathcal{S}_{\text{id}}(f)^1 \) and that \( \mathcal{S}_{\text{id}}(f)^{1} \neq \mathcal{S}_{\text{id}}(f)^0 \) iff \( f \) is of the case (D).

Our first aim is to identify \( \mathcal{S}_{\text{id}}(f)^r \) with the group \( \mathcal{D}_{\text{id}}(\Theta_G)^r \) for some vector field \( G \) on \( \mathbb{R}^2 \), see Lemma 7.1. Then we will use the shift map of \( G \). Denote
\[
D = \pm \prod_{i=1}^l L_i^{\alpha_i-1} \cdot \prod_{j=1}^k Q_j^{\beta_j-1}.
\]

Then
\[
f = L_1 \cdots L_l \cdot Q_1 \cdots Q_q \cdot D
\]
and it is easy to see that $D$ is the greatest common divisor of $f_x'$ and $f_y'$ in the ring $\mathbb{R}[x, y]$.

Let $F = -f_y' \frac{\partial}{\partial x} + f_x' \frac{\partial}{\partial x}$ be the Hamiltonian vector field of $f$ on $\mathbb{R}^2$ and

$$G = F/D = -(f_y'/D) \frac{\partial}{\partial x} + (f_x'/D) \frac{\partial}{\partial x}.$$ 

We will call $G$ the reduced Hamiltonian vector field of $f$. Notice that

$$\deg G = l + 2k - 1$$

and the coordinate functions of $G$ are relatively prime in $\mathbb{R}[x, y]$.

As noted in Example 3.3 the singular partitions $\Theta_f$ and $\Theta_F$ coincide. Let us describe the singular partition $\Theta_G = (\mathcal{P}_G, \Sigma_G)$. Recall that elements of $\mathcal{P}_G$ are the orbits of $G$ and $\Sigma_G$ consists of zeros of $G$. Consider the following cases, see Figure 7.1.

(A) $f = y^{\alpha_1}$. Then $D = y^{\alpha_1 - 1}$ and $F(x, y) = \alpha_1 y^{\alpha_1 - 1} \frac{\partial}{\partial y}$. Hence $G(x, y) = \alpha_1 \frac{\partial}{\partial y}$ is a constant vector field and the partition $\Theta_G$ consists of horizontal lines $\{y = \text{const}\}$ being non-singular elements of $\Theta_G$.

(C) and (D) $f = Q_1^{\beta_1} \cdots Q_k^{\beta_k}$. In this case $\Theta_F = \Theta_G$. The origin is a unique singular element of $\Theta_G$. All other elements of $\Theta_G$ are level-sets $f^{-1}(c)$ of $f$ for $c > 0$.

(B) and (E) either $l = 2$ and $k = 0$ or $l \geq 1$ and $k \geq 1$. In both cases the set of singular points of $\Theta_F$ consist of the origin and the set $D^{-1}(0) = \bigcup_{i: \alpha_i \geq 2} \{L_i = 0\}$ of zeros of $D$ being the union of those lines $\{L_i = 0\}$ for which $L_i$ is a multiple factor of $f$. Since after division of $F$ by $D$ the coordinate functions of $G = F/D$ are relatively prime, it follows that $0$ is a unique singular element of $\Theta_G$. Hence non-singular elements of $\Theta_G$ are the connected components $f^{-1}(c)$ for $c \neq 0$ and the half-lines in $f^{-1}(0) \setminus 0$, see Figure 7.2.

\begin{figure}[h]
\centering
\begin{tabular}{cccc}
\includegraphics[width=0.2\textwidth]{case_a.png} & \includegraphics[width=0.2\textwidth]{case_c.png} & \includegraphics[width=0.2\textwidth]{case_d.png} & \includegraphics[width=0.2\textwidth]{case_be.png} \\
Case (A) & Case (C) & Cases (D) & Cases (B) and (E)
\end{tabular}
\caption{Figure 7.1}
\end{figure}

\begin{figure}[h]
\centering
\begin{tabular}{ll}
a) $F(x, y) = -2xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$ & b) $G(x, y) = -2xy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$
\end{tabular}
\caption{Figure 7.2. Case (B). Hamiltonian and reduced Hamiltonian vector fields for $f(x, y) = xy^2$.}
\end{figure}

Since $f$ is constant along orbits of $F$ and $G$, it follows that

\begin{equation}
D(\Theta_F) \subset D(\Theta_G) \subset S(f).
\end{equation}

Also notice that $D(\Theta_F)$ consists of those $h \in D(\Theta_G)$ which fixes every critical point of $f$.

Lemma 7.1. $D_{id}(\Theta_G)^r = S_{id}(f)^r$ for all $r \in \overline{\mathbb{N}_0}$.
Proof. It follows from (7.1) that \( \mathcal{D}_{id}(\Theta_G)^r \subset \mathcal{S}_d(f)^r \).

Conversely, let \( h \in \mathcal{S}_d(f)^r \); then there exists an \( r \)-isotopy \( h_t : \mathbb{R}^2 \to \mathbb{R}^2 \) between \( h_0 = id_{\mathbb{R}^2} \) and \( h_1 = h \) in \( \mathcal{S}(f) \), i.e.,

\[
(7.2) \quad f \circ h_t = f, \quad t \in I.
\]

We claim that every \( h_t \in \mathcal{D}(\Theta_G) \), i.e., \( h_t(\omega) = \omega \) for every element \( \omega \) of \( \Theta_G \). This will mean that \( \{h_t\} \) is an \( r \)-isotopy in \( \mathcal{D}(\Theta_G) \), whence \( h \in \mathcal{D}_{id}(\Theta_G)^r \).

It follows from (7.2) that \( h_t(f^{-1}(c)) = f^{-1}(c) \) for every \( c \in \mathbb{R} \) and \( h_t(\Sigma_f) = \Sigma_f \). Since \( h_0 = id_{\mathbb{R}^2} \) preserves every connected component \( \omega \) of \( f^{-1}(c) \setminus \Sigma_f \), so does \( h_t \), \( t \in I \). If either \( c \neq 0 \), or \( c = 0 \) but \( f \) is of either the cases (A), (C), or (D), then by definition every such \( \omega \) is an element of \( \Theta_G \).

Let \( c = 0 \). We claim that in the cases (B) and (E) \( h_t(0) = 0 \) for all \( t \in I \). Indeed, in these cases the origin is “the most degenerate point among all other points of \( f^{-1}(0) \)”. This means the following.

For every \( z \in f^{-1}(0) \) denote by \( p_z \) the least number such that \( p_z \)-jet of \( f \) at \( z \) does not vanish, i.e., \( j^{p_z-1}(f, z) = 0 \) while \( j^{p_z}(f, z) \neq 0 \). In other words, the Taylor series of \( f \) at \( z \) starts with terms of order \( p_z \). It is easy to see that for the origin \( p_0 = \deg f \), while for all other points \( z \in f^{-1}(0) \) we have that \( p_z < \deg f \). Also notice that this number \( p_z \) is preserved by any diffeomorphism \( h \in \mathcal{S}(f) \), i.e., \( p_{h(z)} = p_z \). It follows that \( h_t(0) = 0 \).

It remains to note that by continuity every \( h_t \) preserves connected components of \( D^{-1}(0) \setminus \{0\} \). Hence \( h_t \in \mathcal{D}(\Theta_G) \) for all \( t \in I \). \( \square \)

Now we can complete Theorem 1.1. Let \( \Phi \) be the local flow on \( \mathbb{R}^2 \) generated by \( G \), and \( \varphi \) be the shift map of \( G \), see Section 3.5. The following statement was established in [9].

Lemma 7.2. In the cases (A)-(C) (i.e., when \( \deg G \leq 1 \)) for every \( h \in \mathcal{E}_{id}(\Theta_G)^0 \) there exists a smooth function \( \sigma : \mathbb{R}^2 \to \mathbb{R} \) such that \( h(z) = \Phi(x, \sigma(x)) \), i.e., \( \im(\varphi) = \mathcal{E}_{id}(\Theta_G)^0 \).

It follows from Lemmas 3.6 and 7.1 that, in the cases (A)-(C),

\[
(7.3) \quad \mathcal{S}_{id}(f)^\infty = \mathcal{D}_{id}(\Theta_G)^\infty = \cdots = \mathcal{D}_{id}(\Theta_G)^0 = \mathcal{S}_{id}(f)^0.
\]

The following lemma is a consequence of results of [12, 13].

Lemma 7.3. In the cases (D) and (E) (i.e., when \( \deg G \geq 2 \)) \( \im(\varphi) \) consists of all \( h \in \mathcal{E}(\Theta_G) \) whose tangent map \( T_0 h : T_0 \mathbb{R}^2 \to T_0 \mathbb{R}^2 \) at \( 0 \) is the identity.

Corollary 7.4. \( \mathcal{E}_{id}(\Theta_G)^1 \subset \im(\varphi) \), whence similarly to (7.3) we get \( \mathcal{S}_{id}(f)^\infty = \cdots = \mathcal{S}_{id}(f)^1 \).

Proof of Corollary. Let \( h \in \mathcal{E}_{id}(\Theta_G)^1 \). Then there exists a 1-homotopy \( h_t \) between \( h_0 = id_{\mathbb{R}^2} \) and \( h_1 = h \) in \( \mathcal{E}(\Theta_G) \). In particular, \( T_0 h_t \) is continuous in \( t \).

Since \( f \) is homogeneous, it follows from [9, Lemma 36], that \( T_0 h_t \) regarded as a linear automorphism of \( \mathbb{R}^2 \) also must preserve \( f \), i.e., \( T_0 h_t \in \mathcal{L}(f) \). Therefore the family of maps \( T_0 h_t \) can be regarded as a homotopy in \( \mathcal{L}(f) \). But by Lemma 6.2 in the cases (D) and (E) the group \( \mathcal{L}(f) \) is discrete, whence all \( T_0 h_t \) coincide with the identity map \( T_0 h_0 = id_{\mathbb{R}^2} \). In particular, \( T_0 h = id_{\mathbb{R}^2} \), whence by Lemma 7.3 \( h \in \im(\varphi) \). \( \square \)

It remains to show that \( \mathcal{S}_{id}(f)^1 \) and \( \mathcal{S}_{id}(f)^0 \) coincide in the case (E) and are distinct in the case (D).

(D) Let \( f \) be a product of at least two distinct definite quadratic forms. Then by Theorem 5.1 \( \mathcal{S}_{id}(f)^0 = \mathcal{S}^+(f) \). Moreover, since \( \deg f \) is even, we have that \( -id_{\mathbb{R}^2} \in \mathcal{S}^+(f) = \mathcal{S}_{id}(f)^0 \), see Lemma 6.1. On the other hand by Lemma 7.3 for every \( h \in \mathcal{S}_{id}(f)^1 \) its tangent map \( T_0 h = id_{\mathbb{R}^2} \neq -id_{\mathbb{R}^2} \). Hence \( \mathcal{S}_{id}(f)^1 \neq \mathcal{S}_{id}(f)^0 \).

(E) In this case \( f^{-1}(0) \) is a union of \( l \geq 1 \) straight lines \( \{L_i = 0\} \) passing through the origin. Let \( h \in \mathcal{D}_{id}(\Theta_G)^0 \). Since there exists a homotopy between \( h \) and \( id_{\mathbb{R}^2} \) in
it follows that \( h \) preserves every half-line of \( f^{-1}(0) \setminus \{0\} \). Therefore so does \( T_0 h \in \mathcal{LS}(f) \). Then it follows from Claim 6.7 that \( T_0 h = \text{id}_{\mathbb{R}^2} \), whence by Lemma 7.3 \( h \in \text{im}(\varphi) \). Thus \( \mathcal{D}_{\text{id}}(\Theta_G)^0 \subset \text{im}(\varphi) \).

Now we get from Lemma 3.6 that
\[
\mathcal{S}_{\text{id}}(f) = \mathcal{D}_{\text{id}}(\Theta_G) = \mathcal{D}_{\text{id}}(\Theta_G)^0 = \mathcal{S}_{\text{id}}(f).
\]

Theorem 1.1 is completed.

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