ON THE PERIODICITY PROBLEM OF RESIDUAL $r$-FUBINI SEQUENCES

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Abstract. For any positive integer $r$, the $r$-Fubini number with parameter $n$, denoted by $F_{n,r}$, is equal to the number of ways that the elements of a set with $n + r$ elements can be weak ordered such that the $r$ least elements are in distinct orders. In this article we focus on the sequence of residues of the $r$-Fubini numbers modulo a positive integer $s$ and show that this sequence is periodic and then, exhibit how to calculate its period length. As an extra result, an explicit formula for the $r$-Stirling numbers is obtained which is frequently used in calculations.

1. Introduction

The Fubini numbers (also known as the Ordered Bell numbers) form an integer sequence in which the $n$th term counts the number of weak orderings of a set with $n$ elements. Weak ordering means that the elements can be ordered, allowing ties. A. Cayley studied the Fubini numbers as the number of a certain kind of trees with $n + 1$ terminal nodes [2]. The Fubini numbers can also be defined as the summation of the Stirling numbers of the second kind. The Stirling number of the second kind which is denoted by $\{n\}_k$, counts the number of partitions of $n$ elements into $k$ non-empty subsets. The sequence of residues of the Fubini numbers modulo a positive integer $s$ is pointed out by Bjorn Poonen. He showed that this sequence is periodic and calculated the period length for each positive integer $s$ [5].

The $r$-Stirling numbers of the second kind are defined as an extension to the Stirling numbers of the second kind, and similarly, an $r$-Fubini number is defined as the number of ways which the elements of a set with $n + r$ elements can be weak ordered such that the first $r$ elements are in distinct places. One can study the same problem of periodicity of residual sequence in case of the $r$-Fubini numbers. I. Mezo investigated this problem for $s = 10$ [4]. In this article, $\omega(A_{r,s})$, the period of the $r$-Fubini numbers modulo any positive integer $s \in \mathbb{N}$ is computed. Based on the Fundamental Theorem of Arithmetic, $\omega(A_{r,p})$ is calculated for powers of odd primes $p^m$. The cases $s = 2^m$ are studied separately. Therefore if $s = 2^mp_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$ is the prime factorization, then the $\omega(A_{r,s})$ is equal to the least common multiple (LCM) of $\omega(A_{r,p_i^m})$’s and $\omega(A_{r,2^m})$, for $i = 1, 2, \ldots, k$.

The preliminaries are presented in Section 2 and in Sections 3 and 4 the length of the periods are computed in the case of odd prime powers and the $2$ powers, respectively. The last section contains the final theorem which presents the conclusion of this article.

2. Basic Concepts

The Stirling number of the second kind with the parameters $n$ and $k$ counts the number of ways that the set $A = \{1, 2, \ldots, n\}$ with $n$ elements can be partitioned into $k$ non-empty subsets. If we want that the first $r$ elements of $A$ are in distinct subsets, the number of ways to do so is the $r$-Stirling number of the second kind with parameters
n and k, which is denoted by \( \{ \binom{n}{k} \}_r \) (so it is clear that \( n \geq k \geq r \)). Fubini numbers are defined as follows \[4\]

\[ F_n = \sum_{k=0}^{n} k! \binom{n}{k}. \]

In a similar way we can define the \( r \)-Fubini numbers as the number of ways which the elements of \( A \) can be weak ordered such that the elements \( \{1, 2, \ldots, r\} \) are in distinct ranks. These numbers are denoted by \( F_{n,r} \) and are evaluated by

\[ F_{n,r} = \sum_{k=0}^{n} (k + r)! \binom{n + r}{k + r}. \]

There are simple relations and formulae about \( \{ \binom{n}{k} \}_r \) which are listed below. One can find a proof of them in \[4\], \[1\] and \[3, \S 4\].

\begin{align*}
\{ \binom{n}{m} \}_r &= \{ \binom{n}{m} \}_{r-1} - (r - 1) \{ \binom{n-1}{m} \}_{r-1}, 1 \leq r \leq n \\
\{ \binom{n}{m} \}_1 &= \{ \binom{n}{m} \} \\
\{ \binom{n + r}{r} \}_r &= r^n \\
\{ \binom{n + r}{r + 1} \}_r &= (r + 1)^n - r^n \\
\{ \binom{n}{m} \} &= \frac{1}{m!} \sum_{j=1}^{m} (-1)^{m-j} \binom{m}{j} j^n.
\end{align*}

In addition to above recurrence relations of the \( r \)-Stirling numbers of the second kind, a direct way to compute these numbers is given in the next theorem.

**Theorem 2.1.** For \( n, m \in \mathbb{N} \) and \( r \leq m \leq n \), the \( r \)-Stirling number of the second kind with the parameters \( n \) and \( m \) is

\[ \{ \binom{n}{m} \}_r = \frac{1}{m!} \sum_{j=r}^{m} (-1)^{m-j} \binom{m}{j} j^{n-(r-1)} \left( \frac{(j-1)!}{(j-r)!} \right). \]

**Proof.** We prove it by the induction on \( r \). For \( r = 2 \), the relations (1) and (5) result

\[ \{ \binom{n}{m} \}_2 = \{ \binom{n}{m} \} - \{ \binom{n-1}{m} \} = \frac{1}{m!} \left( \sum_{j=1}^{m} (-1)^{m-j} \binom{m}{j} j^n - \sum_{j=1}^{m} (-1)^{m-j} \binom{m}{j} j^{n-1} \right) = \frac{1}{m!} \sum_{j=2}^{m} (-1)^{m-j} \binom{m}{j} j^{n-1} (j - 1) \]

Assume that

\[ \{ \binom{n}{m} \}_r = \frac{1}{m!} \sum_{j=r}^{m} (-1)^{m-j} \binom{m}{j} j^{n-(r-1)} \left( \frac{(j-1)!}{(j-r)!} \right). \]

For \( \{ \binom{n}{m} \}_{r+1} \), use (1) to conclude that
\[
\begin{aligned}
\left\{ \frac{n}{m} \right\}_{r+1} &= \left\{ \frac{n}{m} \right\}_r - r \left\{ \frac{n-1}{m} \right\}_r \\
&= \frac{1}{m!} \sum_{j=r}^{m} (-1)^{m-j} \binom{m}{j} j^{n-(r-1)} \frac{(j-1)!}{(j-r)!} \\
- r \left( \frac{1}{m!} \sum_{j=r}^{m} (-1)^{m-j} \binom{m}{j} j^{n-1-(r-1)} \frac{(j-1)!}{(j-r)!} \right) \\
&= \frac{1}{m!} \sum_{j=r+1}^{m} (-1)^{m-j} \binom{m}{j} j^{n-r} \frac{(j-1)!}{(j-r+1)!}.
\end{aligned}
\]

\[\square\]

By \(\varphi(n)\) we indicate the number of integer numbers less than \(n\) and co-prime to it. It is known as Euler’s totient function. The value of \(\varphi(n)\) can be computed via the following relation [3]

\[\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)\]

3. The \(r\)-Fubini residues modulo prime powers

Let \(p\) be a prime number greater than 2 and \(m\) be a positive integer. If \(\{F_{n,r}\}\) denotes the \(r\)-Fubini numbers for a fixed positive integer \(r\), we indicate by \(A_{r,q} = \{F_{n,r} \mod q\}\), for \(n \in \mathbb{N}\), the sequence of residues of the \(r\)-Fubini numbers modulo the positive integer \(q\). In this section we try to compute the period length of the sequence \(A_{r,q}\) when \(q = p^m\). We denote this length by \(\omega(A_{r,q})\).

**Proposition 3.1.** Let \(p\) be an odd prime and let \(q = p^m\), \(m \in \mathbb{N}\). If \(q \leq r\), then \(\omega(A_{r,q}) = 1\)

**Proof.** The proof is very simple. Since \(p \leq r\), we can deduce that \(p | (k+r)!\), for \(k \geq 0\), and by the relation \(F_{n,r} = \sum_{k=0}^{n} (k+r)! \left\{ \begin{array}{c} n+r \\ k \end{array} \right\}_r\), we have \(p | F_{n,r}\). Therefore \(\omega(A_{r,p}) = 1\). \(\square\)

As pointed out in above proposition, the cases in which the prime power factor of \(s\) is less than or equal to \(r\) have the period length 1, so it is sufficient to investigate the period length in the cases of \(q > r\).

**Lemma 3.2.** Let \(p\) be an odd prime and \(r, m \in \mathbb{N}\) with \(p \geq r + 1\). Then

\[p^m - r \geq m.\]

**Proof.** For \(m = 1\) the result is obvious. Suppose the inequality holds for any \(m \geq 2\). Since \(p(p+m) > 2(p+m) > 2p + m\), we have

\[(7) \quad p^2 + pm - p \geq p + m.\]

Since \(p - 1 \geq r\), the induction hypothesis can be reformulated to \(p^m \geq p - 1 + m\). Multiplication by \(p\) results \(p^{m+1} \geq p^2 + pm - p\). By (7) we have \(p^{m+1} \geq p + (m+1) - 1\), Q.E.D. \(\square\)

**Theorem 3.3.** Let \(p\) be an odd prime and \(q = p^m\). After the \((m-1)\)th term the sequence \(A_{r,q}\) has a period with length \(\omega(A_{r,q}) = \varphi(q)\).
Proof. If \( n \geq q - r - 1 \) we can write
\[
F_{n+\phi(q),r} - F_{n,r} = \sum_{k=0}^{n+\phi(q)} \binom{n+\phi(q)+r}{k+r} - \sum_{k=0}^{n} \binom{n+r}{k+r} \equiv q-r-1 \sum_{k=0}^{q-r-1} \binom{n+\phi(q)+r}{k+r} - \sum_{k=0}^{n+r} \binom{n+r}{k+r} \quad (\text{mod } q)
\]
\[
\equiv q-r-1 \sum_{k=0}^{q-r-1} \binom{n+\phi(q)+r}{k+r} - \sum_{k=0}^{n+r} \binom{n+r}{k+r} \quad (\text{mod } q)
\]

If \( j = cp, c \in \mathbb{N} \), then \( j^{n+1} = (cp)^{q-r+h} \), for some \( h \geq 0 \), so from Lemma 3.2 it follows that \( j^{n+1} \equiv 0 \pmod{q} \). If \( (j, q) = 1 \), by Euler’s Theorem \( j^\phi(q) - 1 \equiv 0 \pmod{q} \), so the right hand side of the above congruence relation vanished and we have
\[
(8) \quad F_{n+\phi(q),r} \equiv F_{n,r} \quad (\text{mod } q), \quad \text{for } n \geq q - r - 1.
\]

If \( m-1 \leq n < q-r-1 \) then
\[
F_{n+\phi(q),r} - F_{n,r} \equiv q-r-1 \sum_{k=0}^{q-r-1} \binom{n+\phi(q)+r}{k+r} - \sum_{k=0}^{n+r} \binom{n+r}{k+r} \quad (\text{mod } q)
\]
\[
\equiv q-r-1 \sum_{k=0}^{q-r-1} \binom{n+\phi(q)+r}{k+r} - \sum_{k=0}^{n+r} \binom{n+r}{k+r} \quad (\text{mod } q)
\]

Since \( n \geq m - 1 \), in the indices where \( j = cp, c \in \mathbb{N} \), we have \( j^{n+1} = (cp)^{n+r} \), for some \( h \geq 0 \), and it deduced that \( j^{n+1} \equiv 0 \pmod{q} \). When \( (j, q) = 1 \), again by Euler’s Theorem \( j^{\phi(q)} - 1 \equiv 0 \pmod{q} \). In the sums \( \sum_{k=n+1}^{q-r-1} \binom{n+r}{k+r} \) and \( \sum_{k=n+\phi(q)+1}^{q-r-1} \binom{n+\phi(q)+r}{k+r} \), the upper parameter of the \( r \)-Stirling number is less than the lower one and therefore these two sums are equal to zero. So
\[
F_{n+\phi(q),r} - F_{n,r} = \sum_{k=0}^{q-r-1} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} \frac{(j-1)!}{(j-r)!} (j^{\phi(q)} - 1)
\]

and therefore
\[
(9) \quad F_{n+\phi(q),r} \equiv F_{n,r} \quad (\text{mod } q), \quad \text{for } m-1 \leq n < q-r-1.
\]

Combining results (8) and (9) gives \( F_{n+\phi(q),r} \equiv F_{n,r} \pmod{q} \), for \( n \geq m - 1 \). \( \square \)
4. The $r$-Fubini residues modulo powers of 2

Similar to many computations in number theory, the case of $p = 2$ has its own difficulties which needs special manipulations. In the case of powers of 2, initially we calculate the residues of 2-Fubini numbers and then use the results in the case of the $r$-Fubini numbers. We classify the sequence of remainders of 2-Fubini numbers modulo $2^n$, $m \geq 7$, in Theorem 4.1 and then, work on remainders of the $r$-Fubini numbers modulo $2^n$, $m \geq 7$ in Theorem 4.4. The special cases will be proved in Theorems 4.1, 4.5 and 4.6. The trivial cases in which $2^m \leq r$ with period length 1 are omitted.

**Theorem 4.1.** If $3 \leq m \leq 6$, then after the $(m-1)$th term the sequence $A_{2, 2^m}$ has a period with length $\omega(A_{2, 2^m}) = 2$.

*Proof.* By using the formula $F_{n, 2} = \sum_{k=0}^{n} (k+2)! \left\{\frac{n+1}{k+2}\right\}^2$ we prove that $F_{n+2, 2} - F_{n, 2} \equiv 0 \pmod{2^6}$ then it is concluded that $F_{n+2, 2} - F_{n, 2} \equiv 0 \pmod{2^n}$, $3 \leq m \leq 5$.

$$F_{n+2, 2} - F_{n, 2} = \sum_{k=0}^{n+2} (k+2)! \left\{\frac{n+1}{k+2}\right\}^2 - \sum_{k=0}^{n} (k+2)! \left\{\frac{n+2}{k+2}\right\}^2$$

$$\equiv \sum_{k=0}^{5} (k+2)! \left(\left\{\frac{n+1}{k+2}\right\}^2 - \left\{\frac{n+2}{k+2}\right\}^2\right) \pmod{2^6}$$

$$\equiv \sum_{k=0}^{5} \sum_{j=2}^{k+2} (-1)^{k+2-j} \binom{k+2}{j} j^{n+1}(j^2 - 1)(j-1) \pmod{2^6}.$$

$m = 6$ implies that $n \geq 5$, so if $j$ is even, then $j^{n+1} = (2c)^{6+h}$, for some $h \geq 0$ and therefore $64 \mid j^{n+1}$. For odd $j$'s, $(j, 64) = 1$ so by Euler's Theorem we have $j^{32} \equiv 1 \pmod{64}$ and therefore $j^{n+1+32} \equiv j^{n+1} \pmod{64}$. This implies

$$F_{n+2, 2} - F_{n, 2} \equiv 5 \sum_{k=0}^{5} \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+2}{2l+1} (2l+1)^{n+1}$$

$$\times \left(\frac{(2l+1)^2 - 1)}{2l} \right) \times 2l \pmod{64}$$

$$\equiv 16 \sum_{k=0}^{5} (-1)^{k+1} \sum_{l=1}^{\lfloor (k+1)/2 \rfloor} \binom{k+2}{2l+1} (2l+1)^{n+1} \frac{l(l+1)}{2} l \pmod{64}.$$

Enumerating the last summation for $2 \leq n \leq 33$ shows that it is divisible by 64 and because of periodicity of remainders of $j^{n+1}$ modulo 64, the result follows. \qed

Now we present the following lemma analogues to Lemma 3.2 in previous section.

**Lemma 4.2.** If $m \in \mathbb{N}$ and $m > 1$, then $2^m - 2 \geq m$.

*Proof.* For $m=2$ the result is obvious. If we assume that

(10)

$$2^m - 2 \geq m$$

then multiplication by 2 gives $2^{m+1} - 4 \geq 2m$. Since $2^{m+1} \geq 2m + 4 > m + 3$, we have $2^{m+1} \geq m + 3$ and so $2^{m+1} - 2 \geq m + 1$. \qed

The following lemma provides a simple but essential relation used in the next theorem. Its proof is provided in Appendix A.

**Lemma 4.3.** For $m \geq 7$ and $5 \leq i \leq 2^{m-6}$ we have $2^{m-6} - i \mid 2^{i-5} (2^{m-6} - 1)$.

**Theorem 4.4.** If $m \geq 7$, after the $(m-1)$th term, the sequence $A_{2, 2^m}$ has a period with length $\omega(A_{2, 2^m}) = 2^{m-6}$. 

Proof. In the case of $n \geq 2^m - 3$, from Lemma 4.2 we can deduce that $n \geq 2^m - 3 \geq m - 1$. So we have

\[
F_{n+2^m-2} - F_{n,2} = \sum_{k=0}^{2^m-3} (k+2)! \left\{ \frac{n + 2^m - 2}{k + 2} \right\}_2 - \sum_{k=0}^{n} (k+2)! \left\{ \frac{n + 2}{k + 2} \right\}_2
\]

\[
\equiv \sum_{k=0}^{2^m-3} (k+2)! \left\{ \frac{n + 2^m - 2}{k + 2} \right\}_2 - \left\{ \frac{n + 2}{k + 2} \right\}_2 \pmod {2^m}
\]

\[
\equiv \sum_{k=0}^{2^m-3} k+2 \sum_{j=2}^{n} (-1)^{k+2-j} \left( \frac{k+2}{j} \right) j^{n+1} (j^{2^m-6} - 1)(j - 1) \pmod {2^m}.
\]

When $j$ is even, then $j^{n+1} = (2c)^{2^m-2+h}$, for some $h \geq 0$. So by Lemma 4.2 $2^m \mid j^{n+1}$. For odd $j$’s we have

\[
F_{n+2^m-2} - F_{n,2} \equiv \sum_{k=0}^{2^m-3} \left( \frac{(k+1/2)}{2^{m-6}} \right) \sum_{l=1}^{2^m-6} (-1)^{k+1} \left( \frac{k+2}{2l+1} \right) (2l + 1)^n + 1
\]

\[
\times \left( \frac{(2l + 1)^{2^m-6} - 1}{2^{m-5}} \right) l \pmod {2^m}
\]

\[
\equiv 2^m-4 \sum_{k=0}^{2^m-3} (-1)^{k+1} \left( \frac{k+2}{2l+1} \right) (2l + 1)^n + 1
\]

\[
\times \sum_{i=1}^{2^m-6} l^i 2^{i-1} \left( \frac{(2^m-1)!}{i!(2^m-1)!} \right) \pmod {2^m}.
\]

Last expression contains $m - 4$ factors of 2, so it is sufficient to prove the last summation is divisible by 16. We denote this summation by $S$. Simplify the summation $\sum_{i=1}^{2^m-6} l^i 2^{i-1} \left( \frac{(2^m-1)!}{i!(2^m-1)!} \right)$ and use the Lemma 4.3 gives

\[
\sum_{i=1}^{2^m-6} l^i 2^{i-1} \left( \frac{(2^m-1)!}{i!(2^m-1)!} \right) \equiv \sum_{i=1}^{2^m-6} l^i 2^{i-1} \left( \frac{(2^m-1)!}{i!(2^m-1)!} \right) \pmod {16}
\]

\[
\equiv l + l^2 (2^m - 1) + l^3 \times 2(2^m - 1)(2^m - 2)
\]

\[
+ l^4 (2^m - 1)(2^m - 2)(2^m - 3) \pmod {16}.
\]

Assume $m \geq 10$ (the case $7 \leq m \leq 9$ is proceeded at last). So $16 \mid 2^m - 6$. Let $3a = 2(2^m - 1)(2^m - 2)$ and $3b = (2^m - 1)(2^m - 2)(2^m - 3)$. Then $3a \equiv 4 \pmod {16}$ and $3b \equiv -6 \pmod {16}$. Therefore $a \equiv -4 \pmod {16}$ and $b \equiv -2 \pmod {16}$. So the proof continues as follows

\[
S_{16} \equiv \sum_{k=0}^{2^m-3} (-1)^{k+1} \left( \frac{(2l+1)!}{2^{m-6}} \right) \left( \frac{k+2}{2l+1} \right) (2l + 1)^n + 1 l - l^2 - 4l^3 - 2l^4)
\]

\[
S_8 \equiv \sum_{k=0}^{2^m-3} (-1)^{k+1} \sum_{l=1}^{2^m-6} \left( \frac{k+2}{2l+1} \right) (2l + 1)^n + 1 \left( \frac{l(l+1)}{2} \right) (-2l^2 - 2l + 1).
\]
Let $P(l)$ and $A(k, r, n)$ be the remainder of $\frac{1}{2}(2l + 1)^{n+1}(l(l + 1))(-2l^2 - 2l + 1)l$ and $\sum_{l=-\infty}^{l=\infty} \binom{k+2}{2l+r} P(l)$ divided by 8, respectively. By Pascal identity, we have $\binom{k+2}{2l+r} = \binom{k+1}{2l} + \binom{k+1}{2l+1}$ and therefore
\[
\sum_{l=-\infty}^{\infty} \binom{k+2}{2l+r} P(l) = \sum_{l=-\infty}^{\infty} \binom{k+1}{2l+r} P(l) + \sum_{l=-\infty}^{\infty} \binom{k+1}{2l+r-1} P(l),
\]
so
\[(11) \quad A(k, r, n) = A(k-1, r, n) + A(k-1, r-1, n).
\]
We can write
\[
A(k, r + 32, n) \equiv \sum_{l=-\infty}^{\infty} \binom{k+2}{2l+r+32} P(l) \pmod{8}
\]
The sequence $(P(l))_{l=-\infty}^{\infty}$ has period 16, so $P(l + 16) = P(l)$. Set $l' = l + 16$, then
\[(12) \quad A(k, r + 32, n) \equiv \sum_{l'=-\infty}^{\infty} \binom{k+2}{2l'+r} P(l') \equiv A(k, r, n) \pmod{8}.
\]
Since $(2l + 1, 16) = 1$, the Euler’s Theorem implies $(2l + 1)^8 \equiv 1 \pmod{16}$ and therefore $(2l + 1)^{n+1+8} \equiv (2l + 1)^{n+1} \pmod{16}$. $A(6, r, n)$ vanishes for $1 \leq r \leq 32$ and $9 \leq n \leq 24$, by enumeration, then by (11) and (12), we deduce that
\[(13) \quad A(k, r, n) = 0, \text{ for } k \geq 6.
\]
Therefore
\[
A(k, 1, n) \equiv \sum_{l=-\infty}^{\infty} \binom{k+2}{2l+1} (2l + 1)^{n+1} \left( \frac{l(l+1)}{2} \right) (-2l^2 - 2l + 1)l \pmod{8}
\]
\[
\equiv \sum_{l=1}^{[(k+1)/2]} \binom{k+2}{2l+1} (2l + 1)^{n+1} \left( \frac{l(l+1)}{2} \right) (-2l^2 - 2l + 1)l
\]
\[
\equiv 0 \pmod{8},
\]
for $k \geq 6$. If $1 \leq k \leq 5$, $9 \leq n \leq 24$ and $1 \leq r \leq 32$ we have $\sum_{k=1}^{5} (-1)^{k+1} A(k, r, n) \equiv 0 \pmod{8}$. The period length of $A(k, r, n)$ with respect to $r$ and $n$ implies that $\sum_{k=1}^{5} (-1)^{k+1} A(k, 1, n) \equiv 0 \pmod{8}$, for $n \geq 9$. Combining this with (13) we have
\[
S \equiv \sum_{k=1}^{2^m-3} (-1)^{k+1} A(k, 1, n) \equiv 0 \pmod{8}, \text{ for } n \geq 0.
\]
So the result follows in the case of $n \geq 2^m - 3$. If $m - 1 \leq n < 2^m - 3$ we can write
\[
F_{n+2^m-6, 2} - F_{n, 2} = \sum_{k=0}^{2^m-6} \frac{(k+2)!}{2} \left\{ \frac{n+2^{m-6}+2}{k+2} \right\} - \sum_{k=0}^{n} \frac{(k+2)!}{2} \left\{ \frac{n+2}{k+2} \right\}
\]
\[
= \sum_{k=0}^{2^m-3} \frac{(k+2)!}{2} \left( \left\{ \frac{n+2^{m-6}+2}{k+2} \right\} - \left\{ \frac{n+2}{k+2} \right\} \right)
\]
\[
- \sum_{k=n+2^{m-6}+1}^{2^m-3} (k+2)! \left\{ \frac{n+2^{m-6}+2}{k+2} \right\} + \sum_{k=n+1}^{2^m-3} (k+2)! \left\{ \frac{n+2}{k+2} \right\}
\]
\[
\equiv \sum_{j=1}^{2^m-3} \sum_{j=1}^{2^m-3} (-1)^{k+2-j} \binom{k+2}{j} j^{n+1} (j^{2^{m-6}} - 1) (j - 1) \pmod{2^m}
\]
When \( j \) is even, then \( j^{n+1} = (2c)^{m+h} \), for some \( h \geq 0 \), so \( 2^m \mid j^{n+1} \). Since \( m \geq 10 \), for odd \( j \)'s we have

\[
\sum_{k=0}^{2^m-3} \sum_{j=1}^{k+2} (-1)^{k+2-j} \binom{k+2}{j} j^{n+1} (j^{2^m-6} - 1)(j-1)
\]

\[
\equiv 2^{m-4} \sum_{k=0}^{2^m-3} (-1)^{k+1} \sum_{l=1}^{(k+1)/2} \binom{k+2}{2l+1} (2l+1)^{n+1} (l^2 - 4l^3 - 2l^4)l \pmod{2^m}.
\]

The last summation is exactly the \( S \) and the proof will be similar as above. Combine with the previous case we have the following congruence relation

(14) \[ F_{n+2^m-6,2} \equiv F_{n,2} \pmod{2^m}, \text{ for } m \geq 10. \]

In the case where \( 7 \leq m \leq 9 \), the remainder of the sum \( \sum_{k=0}^{2^m-3} (-1)^{k+1} \times \sum_{l=1}^{(k+1)/2} \binom{k+2}{2l+1} (2l+1)^{n+1} \pmod{2^m} \) modulo 16 is computed for \( m - 1 \leq n \leq m + 14 \). Divisibility of all these values by 16 implies that the recent sum is divisible by 16 therefore

(15) \[ F_{n+2^m-6,2} \equiv F_{n,2} \pmod{2^m}, \text{ for } 7 \leq m \leq 9. \]

Summing up the congruence relations (14) and (15) gives

\[ \omega(A_{2,2^n}) = 2^{m-6}, \text{ for } m \geq 7. \]

\[ \square \]

**Theorem 4.5.** For \( m = 1 \) and \( m = 2 \), the sequence \( A_r,2^n \) is periodic from the first term and the period length is \( \omega(A_{r,2^n}) = 1 \).

**Proof.** The proof of this theorem is divided into three cases. For \( r = 2 \) we have

\[
F_{n+1,2} - F_{n,2} = \sum_{k=0}^{n+1} (k+2)! \left\{ \binom{n+3}{k+2} - \sum_{k=0}^{n} (k+2)! \binom{n+2}{k+2} \right\} \equiv 2 \left( \binom{n+3}{2} - \binom{n+2}{2} \right) + 6 \left( \binom{n+3}{3} - \binom{n+2}{3} \right) \pmod{4}
\]

\[
= 2 (2^{n+1} - 2^n) + 6 (3^{n+1} - 2^{n+1} - 3^n - 2^n) = 2^{n+1} + 6 (2 \times 3^n - 2^n) = 4 (2^{n-1} + 3^{n+1} - 3 \times 2^{n-1})
\]

\[
= 4 (3^{n+1} - 2^n) \equiv 0 \pmod{4}.
\]

So we can deduce that \( \omega(A_{2,4}) = 1 \) and obviously \( \omega(A_{2,2}) = 1 \).

For \( r = 3 \) we can write

\[
F_{n+1,3} - F_{n,3} = \sum_{k=0}^{n+1} (k+3)! \left\{ \binom{n+4}{k+3} - \sum_{k=0}^{n} (k+3)! \binom{n+3}{k+3} \right\} \equiv 6 \left( \binom{n+4}{3} - \binom{n+3}{3} \right) \pmod{4}
\]

\[
= 6 (3^{n+1} - 3^n) = 6 \times 2 \times 3^n = 4 \times 3^{n+1} \equiv 0 \pmod{4}.
\]

Therefore we have \( \omega(A_{3,4}) = 1 \) and \( \omega(A_{3,2}) = 1 \).

Finally if \( r \geq 4 \), let \( r = 4 + h \), for some \( h \geq 0 \), then

\[
F_{n+1,r} - F_{n,r} = \sum_{k=0}^{n+1} (k+r)! \left\{ \binom{n+1+r}{k+r} - \sum_{k=0}^{n} (k+r)! \binom{n+r}{k+r} \right\}.
\]

Since \( 4 \mid (k+r)! \), for all \( k \geq 0 \), we can write \( F_{n+1,r} - F_{n,r} \equiv 0 \pmod{4} \). Therefore \( \omega(A_{r,4}) = 1 \) and \( \omega(A_{r,2}) = 1 \). \[ \square \]
Theorem 4.6. If $3 \leq m \leq 6$, after the $(m-1)$th term, the sequence $A_{r,2^m}$ has a period with length $\omega(A_{r,2^m}) = 2$.

Proof. The proof of this theorem is similar to the proof of Theorem 4.1. Proving for $m = 6$ deduces the result for $m = 3, 4, 5$. Let $m = 6$, so $n \geq 5$; because $n \geq m - 1$. For $3 \leq r \leq 7$ we have

$$F_{n+2,r} - F_{n,r} = \sum_{k=0}^{n+2} (k+r)! \binom{n+2+r}{k+r} - \sum_{k=0}^{n} (k+r)! \binom{n+r}{k+r}$$

$$= \sum_{k=0}^{7-r} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} (j^2-1) \left( \frac{(j-1)!}{(j-r)!} \right) \pmod{64}$$

$$= \sum_{k=0}^{7-r} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} (j^2-1)(j-1) \left( \frac{(j-2)!}{(j-r)!} \right) \pmod{64}$$

When $j$ is even, then $j^{n+1} = (2e)^{n+h}$, for some $h \geq 0$, and so $64 \mid j^{n+1}$. For odd $j$’s we have $(j, 64) = 1$ and the Euler’s Theorem gives $j^{32} \equiv 1 \pmod{64}$. Therefore $j^{n+1+32} \equiv j^{n+1} \pmod{64}$ and we can write

$$\sum_{k=0}^{7-r} \sum_{j=r}^{k+r} (-1)^{k+r-j} \binom{k+r}{j} j^{n+1} (j^2-1)(j-1) \left( \frac{(j-2)!}{(j-r)!} \right) \pmod{64}$$

$$= 16 \sum_{k=0}^{7-r} (-1)^{k+r-1} \binom{k+r}{2l+1} (2l+1)^{n+1} \left( \frac{l+l+1}{2} \right) \pmod{64}$$

By computation we see that the recent summation is divisible by 4, for $2 \leq n \leq 33$. So the proof for $3 \leq r \leq 7$ is completed.

If $r \geq 8$, since $64 \mid 8!$, then $64 \mid (k+r)!$, and

$$F_{n+2,r} - F_{n,r} = \sum_{k=0}^{n+2} (k+r)! \binom{n+2+r}{k+r} - \sum_{k=0}^{n} (k+r)! \binom{n+r}{k+r}$$

$$\equiv 0 \pmod{64},$$

so $\omega(A_{r,2^k}) = 2$, for $r \geq 8$, and the proof is completed. \[\square\]

Theorem 4.7. If $m \geq 7$, after the $(m-1)$th term, the sequence $A_{r,2^m}$ has a period with length $\omega(A_{r,2^m}) = 2^{m-6}$.
Proof. The proof of this theorem is similar to the proof of Theorem \[ \text{4.3} \] In the case of \( n \geq 2^m - r - 1 \) and \( r \geq 8 \) we have

\[
F_{n+2^{m-6},r} - F_{n,r} = \sum_{k=0}^{n+2^{m-6}} (k+r)! \left\{ \frac{n+2^{m-6}+r}{k+r} \right\}_r - \sum_{k=0}^{n} (k+r)! \left\{ \frac{n+r}{k+r} \right\}_r
\]

\[
\equiv \sum_{k=0}^{2^{m-r-1}} (k+r)! \left( \left\{ \frac{n+2^{m-6}+r}{k+r} \right\}_r - \left\{ \frac{n+r}{k+r} \right\}_r \right) \pmod{2^m}
\]

\[
\equiv \sum_{k=0}^{2^{m-r-1}} \sum_{j=r}^{k+r} (-1)^{k-r} \left( \begin{array}{c} k+r \\ j \end{array} \right) j^{n+1}(j^{2^{m-6}}-1) \left( \frac{(j-1)!}{(j-r)!} \right) \pmod{2^m}
\]

In the case of \( 2^m > r > 2^m - m \), since \( m \geq 7 \) this implies that \( r > 2^m - m \geq 2^{m-1} \) and so

\[
2^m \mid (2^{m-1})! \mid (k+r)!, \text{ for each } k \geq 0.
\]

Therefore both summations in the above first equation are zero modulo \( 2^m \) and in this case \( \omega(A_{r,2^{m-6}}) = 2^{m-6} \). When \( r \leq 2^m - m \), if \( j \) is even then \( j^{n+1} = (2e)^{2^m-r+h} \), for some \( h \geq 0 \). So \( 2^m \mid j^{n+1} \). For odd \( j \)'s, \( (j, j^{2^{m-5}}) = 1 \) and by Euler’s Theorem \( 2^{m-5} \mid j^{2^{m-6}}-1 \). Since \( r \geq 8 \) we can write \( \frac{(j-1)!}{(j-r)!} = \left( \frac{(j-8)!}{(j-r)!} \right) \prod_{i=1}^{7} (j-i) \). Therefore \( 32 \mid \frac{(j-1)!}{(j-r)!} \) and \( 2^m \mid (j^{2^{m-6}}-1) \left( \frac{(j-1)!}{(j-r)!} \right) \).

In the case of \( m - 1 \leq n < 2^m - r - 1 \) and \( r \geq 8 \) we have

\[
F_{n+2^{m-6},r} - F_{n,r} = \sum_{k=0}^{n+2^{m-6}} (k+r)! \left\{ \frac{n+2^{m-6}+r}{k+r} \right\}_r - \sum_{k=0}^{n} (k+r)! \left\{ \frac{n+r}{k+r} \right\}_r
\]

\[
\equiv \sum_{k=0}^{2^{m-r-1}} (k+r)! \left( \left\{ \frac{n+2^{m-6}+r}{k+r} \right\}_r - \left\{ \frac{n+r}{k+r} \right\}_r \right)
\]

\[
- \sum_{k=n+2^{m-6}+1}^{2^{m-r-1}} (k+r)! \left\{ \frac{n+2^{m-6}+r}{k+r} \right\}_r
\]

\[
+ \sum_{k=n+1}^{2^{m-r-1}} (k+r)! \left\{ \frac{n+r}{k+r} \right\}_r \pmod{2^m}
\]

\[
= \sum_{k=0}^{2^{m-r-1}} (k+r)! \left( \left\{ \frac{n+2^{m-6}+r}{k+r} \right\}_r - \left\{ \frac{n+r}{k+r} \right\}_r \right) + 0,
\]

and the proof proceeds as the previous case. In the case of \( 3 \leq r \leq 7 \) one can deduce similar to the proof of Theorem \[ \text{4.3} \] that

\[
F_{n+2^{m-6},r} - F_{n,r} \equiv \sum_{k=0}^{2^{m-r-1}k+r} \sum_{j=r}^{k+r} (-1)^{k-r-j} \left( \begin{array}{c} k+r \\ j \end{array} \right) j^{n+1}(j^{2^{m-6}}-1)
\]

\[
\times \left( \frac{(j-1)!}{(j-r)!} \right) \pmod{2^m}.
\]
Exactly the same as Theorem 4.4, the even j’s run out. Therefore only the terms with odd j remain. So we have

\[ F_{n+2^m-6}, r - F_{n,r} \]

\[ \equiv 2^{m-r-1} \sum_{k=0}^{2^{m-r-1}} (k+r)^{2l+1} \sum_{l=[r/2]}^{2^{m-r-1}} (-1)^{k+r-2l+1} \left( \frac{k+r}{2l+1} \right) \left( (2l+1)^{n+1}((2l+1)^{2^{m-r}} - 1) \right) \]

\[ \times \left( \frac{((2l+1)-1)!}{((2l+1)-r)!} \right) \pmod{2^m} \]

\[ \equiv 2^{m-5} \sum_{k=0}^{2^{m-5}} (-1)^{k+r-2l+1} \sum_{l=[r/2]}^{2^{m-5}} \left( \frac{k+r}{2l+1} \right) \left( (2l+1)^{n+1} \right) \]

\[ \times \left( \sum_{i=1}^{2^{m-6}} i \cdot 2^{i-1} \left( \frac{(2^{m-6}-1)!}{(2l+1)^{i}} \right) \right) \left( \frac{(2l)!}{(2l-r+1)!} \right) \pmod{2^m}. \]

Since \((2l+1, 16) = 1\), Euler’s Theorem shows that \((2l+1)^{n+1} \equiv (2l+1)^{n+1} \pmod{16}\).

If \(m \geq 1\), we have

\[ F_{n+2^m-6}, r - F_{n,r} \equiv 2^{m-4} \sum_{k=0}^{2^{m-4}} (-1)^{k+r-2l+1} \sum_{l=[r/2]}^{2^{m-4}} \left( \frac{k+r}{2l+1} \right) \left( (2l+1)^{n+1} \right) \]

\[ \times \left( \frac{(l+1)}{2} \right) (-2l^2 - 2l + 1) \left( \frac{(2l)!}{(2l-r+1)!} \right) \pmod{2^m}. \]

Therefore it is sufficient to compute the above summation (without factor \(2^{m-4}\)) for \(3 \leq r \leq 7\) and \(9 \leq n \leq 16\) to show that it is divisible by 16.

For \(7 \leq m \leq 9\) we evaluate the summation

\[ \sum_{k=0}^{2^{m-1}} (-1)^{k+r-2l+1} \sum_{l=[r/2]}^{2^{m-1}} \left( \frac{k+r}{2l+1} \right) \left( (2l+1)^{n+1} \right) \]

\[ \times \left( \sum_{i=1}^{2^{m-6}} i \cdot 2^{i-1} \left( \frac{(2^{m-6}-1)!}{(2l+1)^{i}} \right) \right) \left( \frac{(2l)!}{(2l-r+1)!} \right) \]

for \(m-1 \leq n \leq m+6\) to show that it is divisible by 32. Then it follows that \(\omega(A_{r,2^m}) = 2^{m-6}\), for all \(m \geq 7\).

\[ \square \]

5. The Conclussion

We have the final theorem which shows how to compute \(\omega(A_{r,s})\) for any \(s \in \mathbb{N}\).

**Theorem 5.1.** Let \(s \in \mathbb{N}\) and \(s > 1\) with the prime factorization \(s = 2^m p_1^{m_1} p_2^{m_2} \ldots p_k^{m_k}\) and let \(D = \{p_i^{m_i} \mid p_i^{m_i} > r, 1 \leq i \leq k\}\). Define \(E = \{m_i - 1 \mid p_i^{m_i} \in D\}\), \(F = \{\varphi(p_i^{m_i}) \mid p_i^{m_i} \in D\}\) and \(a = \max(E \cup \{m-1\})\) and let \(b\) be the Least Common Multiple (LCM) of the elements of \(F\). Then

\[ \omega(A_{r,s}) = \begin{cases} b, & \text{if } 0 \leq m \leq 2 \text{ or } 2^m \leq r; \\ \text{LCM}(2, b), & \text{if } 3 \leq m \leq 6 \text{ and } 2^m > r; \\ \text{LCM}(2^{m-6}, b), & \text{if } m \geq 7 \text{ and } 2^m > r; \end{cases} \]

and periodicity of the sequence \(A_{r,s}\) is seen after the \(a\)th term.

**Proof.** Let \(l\) be the right hand side of \((16)\). For each \(d \in D \cup \{2^m\}, \omega(A_{r,d}) \mid l\) and for each \(p_j^{m_j} \notin D\) that \(1 \leq j \leq k, 1 = \omega(A_{r,p_j^{m_j}}) \mid l\), so

\[ F_{n+l,r} \equiv F_{n,r} \pmod{2^m}, \]

\[ F_{n+l,r} \equiv F_{n,r} \pmod{p_i^{m_i}}, \text{ for } i = 1, 2, \ldots, k. \]

Since \((2^m, p_1^{m_1}, p_2^{m_2}, \ldots, p_k^{m_k}) = 1\), the multiplication of all above congruence relations gives the required result. \[ \square \]
Appendix A. Proof of Lemma 4.3

Simplify the lemma’s relation we have
\[ \frac{2^{i-5}(\binom{2^m-6}{i})}{2^m-5} = \frac{2^{i-5}(2^{m-6} - 1)(2^{m-6} - 2) \cdots (2^{m-6} - i + 1)}{i!}. \]  

(17)

It is sufficient to show that the right hand side of (17) is integer. We know that \( \binom{2^m-6}{i} \in \mathbb{N}, \) i.e.,
\[ i! \mid 2^{m-6}(2^{m-6} - 1) \cdots (2^{m-6} - i + 1). \]

If \( O_i \) denotes the product of the odd factors of \( i! \), since \( (O_i, 2^{m-6}) = 1 \), then \( O_i \mid (2^{m-6} - 1) \cdots (2^{m-6} - i + 1). \) So in (17) we only want to prove that the powers of 2 in \( 2^{i-5}(2^{m-6} - 1)(2^{m-6} - 2) \cdots (2^{m-6} - i + 1) \) is greater than or equal to the powers of 2 in \( i! \). Let \( A \) and \( B \) be the greatest integers such that
\[ 2^A \mid (2^{m-6} - 1)(2^{m-6} - 2) \cdots (2^{m-6} - i + 1), \]
\[ 2^B \mid i!. \]

Let \( e \) be the unique integer such that \( 2^e \leq i < 2^{e+1} \). So
\[ A = \sum_{k=1}^{e} \left\lfloor \frac{i-1}{2^k} \right\rfloor, \quad B = \sum_{k=1}^{e} \left\lfloor \frac{i}{2^k} \right\rfloor. \]

(18)

If we show that
\[ B - A \leq e \]

then the lemma is concluded if it is proved that
\[ i + A \geq B + 5. \]

(20)

Determine (20) by induction. For \( e = 2 \), three cases are exist as shown in the following table.

| \( i \) | \( A \) | \( B \) |
|---|---|---|
| 5 | 3 | 3 |
| 6 | 3 | 4 |
| 7 | 4 | 4 |

For \( e \geq 3 \) one can deduce by simple induction that
\[ 2^e \geq e + 5, \]
so \( i \geq 2^e \geq e + 5 \). Add \( B - e \) to these inequalities and use (18) demonstrates (20) for \( i \geq 8 \).

To prove the equality part of (19), consider \( i = 2^e \), for \( e \geq 3 \). In the case where \( 2^e < i < 2^{e+1} \), consider \( 1 \leq k < e \). By The Division Algorithm, \( i = 2^h + r \) where \( 0 \leq r < 2^k \), therefore \( \left\lfloor \frac{i}{2^k} \right\rfloor = \left\lfloor \frac{2^h + r}{2^k} \right\rfloor = h \). Furthermore we have
\[ \left\lfloor \frac{i-1}{2^k} \right\rfloor = \left\lfloor \frac{h + \frac{r-1}{2^k}}{2^k} \right\rfloor = h + \left\lfloor \frac{r-1}{2^k} \right\rfloor. \]

(22)
Dividing the inequality $-1 \leq r - 1 < 2^k - 1$ by $2^k$ gives $-1 < \frac{r - 1}{2^k} < \frac{2^k - 1}{2^k} < 1$ and therefore $-1 \leq \lfloor \frac{r - 1}{2^k} \rfloor \leq 0$. It means that the right hand side of (22) is equal to $h$ or $h - 1$. Therefore

$$0 \leq \lfloor \frac{i}{2^k} \rfloor - \lfloor \frac{i - 1}{2^k} \rfloor \leq 1, \text{ for } 1 \leq k < e.$$ (23)

If $k = e$, we have $i = 2^e + r'$, $r' > 0$, and $i - 1 = 2^e + r' - 1$. So $\lfloor \frac{i}{2^e} \rfloor = \lfloor 1 + \frac{r'}{2^e} \rfloor = 1$.

Since $0 \leq r' - 1 < 2^e$, then $0 \leq \frac{r'-1}{2^e} < 1$ and so $\lfloor \frac{r'-1}{2^e} \rfloor = 0$. Now we can deduce that $\lfloor \frac{i}{2^e} \rfloor = \lfloor 1 + \frac{r'-1}{2^e} \rfloor = 1$. Using (23) and $\lfloor \frac{i}{2^e} \rfloor - \lfloor \frac{i - 1}{2^e} \rfloor = 0$ we have

$$\sum_{k=1}^{e} \lfloor \frac{i}{2^k} \rfloor - \lfloor \frac{i - 1}{2^k} \rfloor \leq e - 1 < e.$$ (24)

Substituting these results in the definitions of $A$ and $B$ in (18) gives (19).