Conductivity of 2D HgTe layer near the critical width. The role of developed edge states network and random mixture of p- and n-domains

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The conductivity of 2D HgTe quantum well with a width $\sim 6.3\text{nm}$, close to the transition from ordinary to topological insulating phases, is studied. The Fermi level is supposed to get to the overall energy gap. The consideration is based on the percolation theory. We have found that the width fluctuations convert the system to a random mixture of domains with positive and negative energy gaps with internal edge states formed near zero gap lines. In the case with no potential fluctuations, the conductance of a finite sample is provided by a random network of the edge states. The zero temperature conductivity of an infinite sample is determined by the free motion of electrons along the zero-gap lines and tunneling between them.

The conductance of a single p-n junction, which is crossed by the edge state, is found. The result is applied when potential fluctuations transform the system to a mixture of p- and n-domains. It is stated that the tunneling across p-n junctions forbids the low-temperature conductivity of a random system, but the latter is restored due to the random edge states crossing the junctions.

I. INTRODUCTION

Topological insulators (TI) attracted much attention in the last decade\cite{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15} (more references see in reviews\cite{16,17,18}). Unlike ordinary insulators (OI), TIs have the energy gap between occupied valence and empty conduction bands. However, unlike ordinary insulators, they have a negative energy gap sign. Due to topological reasons, this inevitably leads to conductive TI borders. The edge states energies cover the entire energy gap.

The idea of TI has a similarity to the classification of electron states in the quantum Hall effect\cite{19,20,21} based on the topological order. The edge states in the quantum Hall effect are characterized by the topological phase that produces the gapless boundary modes insensitive to smooth changes in material parameters. In TIs, the role of magnetic field is transferred to the spin-orbit interaction.

The most known representative of TI is a 2D HgTe quantum layer\cite{22,23,24}. In two dimensions, the backscattering processes in the edge states are strongly forbidden by the time reversibility. In such case, the electron transport should be one-dimensional spin-conserving and non-local. However, the experimental observations show that many aspects of this picture contradict this ideal picture\cite{25}. In particular, this concerns the absence of the 2D transport and the backscattering on the edge states.

Often, the edge states in a 2D TI are considered based on the 6-band Bernevig-Hughes-Zhang\cite{26} model with zero boundary conditions on the external border. Less known is the minimal Dirac-like 2-band model by Volkov-Pankratov (VP)\cite{27} first invented for the 3D case and used for the 2D case later\cite{28,29,30,31,32,33}.

In the VP model, the gap in the system continuously changes its value passing through zero, and it provides the electron states localized near the zero-gap line (ZGL). Different aspects of this model have been studied recently. The VP model specificity is the linearity of the edge-state spectrum which is symmetric around the gap center. The linearity leads to the suppression of electron-electron interaction\cite{34,35,36}. The edge states on the curved edges were studied also\cite{37}. It was found that the microwave absorption in the insulating phase is a result of the transitions from the edge-state to 2D states or between the edge states with the opposite direction of motion with 2D states virtual participation\cite{38}.

The 2D TI with a potential disorder was investigated in\cite{39}. Unlike\cite{40}, we study the system with a disordered energy gap. The difference between the approaches of the present paper and\cite{39} is schematically demonstrated in Fig. 1.

FIG. 1: (Color online) (left) The narrow-gap semiconductor with developed potential fluctuations. The energy gap is marked in light-red. The Fermi level is situated in the mean gap center. If the hybridization of states in electron and hole lakes is weak, the system is insulating. The strong hybridization of these states converts the system to a metal. (right) The narrow-gap semiconductor with developed gap fluctuations. The edge states appear near ZGLs. The density of electrons on the edge states is plotted on top of the figure.

The purpose of the present paper is the study of 2D low-temperature stationary electron conductivity in a 2D TI. Our specific interest is focused on the system with...
a near-critical thickness $w \sim 6.3$ nm, where the energy gap changes its sign. In this situation the gap relief randomness leads to the edge states network formation (see Figs. 1 (right), 2, 3).

![Image](image1.png)

**FIG. 2**: (Color online) Relief of random function $\Delta(r)$ near the critical width. The domains of $\Delta(r) < 0$ are blue and $\Delta(r) < 0$ are white.

![Image](image2.png)

**FIG. 3**: (Color online) The lines of $\Delta(r) = 0$, corresponding to Fig. (2), along which the edge states are located. The electron motion direction at a fixed spin projection is shown by arrows. In a square sample there is one ZGL AA, percolating in the $x$-direction and no percolation in the $y$-direction. In sample elongated in the $y$-direction (separated by a dashed line from the square), there are 2 edge lines crossing the sample in the $x$-direction (AA and BB) and no such ZGLs in the $y$-direction.

### II. RANDOM VOLKOV-PANKRATOV MODEL

A basic assumption of this paper is that the HgTe layer experiences the transition between ordinary (with positive gap) and inverted (with negative gap) insulating phases (OI and TI), accordingly, when the mean well width $\overline{w}$ changes from zero to infinity at $\overline{w} = w_0 = 6.3$ nm. We suppose that the energy gap $2\Delta$ has the linear dependence on the well width $w(r)$ ($r = (x, y)$): $\Delta(w) = \alpha(w - w_0)$, $\alpha = \delta \Delta(w) / w_0$. Using the data from we obtain $\alpha \approx -0.09$ meV $\cdot$ nm$^{-1}$.

Near the critical width $w_0$, the inevitable fluctuation of $w(r)$ leads to the separation of a sample to OI and TI domains. The borders between them, where the gap $2\Delta(r) = 0$, should form the edge states.

Electrons are described by the 2D Volkov-Pankratov Hamiltonian (VP)

$$H = \begin{pmatrix} \Delta(r) & v\sigma p \\ v\sigma p & V(r) - \Delta(r) \end{pmatrix},$$

(1)

where $p = (p_x, p_y)$ is the 2D momentum operator, $\sigma_i$ are the Pauli matrices.

If $V(r)$ and $\Delta(r)$ are constant, the Hamiltonian (1) has the energy spectrum $\pm \sqrt{\Delta^2 + v^2 p^2}$ with the gap $2\Delta$.

If the system has one straight edge, $V(r) = 0$, $\Delta(r) = \Delta(y)$, $\Delta(y < 0) > 0$, $\Delta(y > 0) < 0$, the VP Hamiltonian yields the edge states with a wave function

$$\Psi_x = \begin{pmatrix} 1 + \sigma \\ 1 - \sigma \\ \sigma - 1 \\ 1 + \sigma \end{pmatrix} \exp \left( i px + \frac{1}{v} \int_0^y \Delta(y') dy' \right),$$

(2)

where $\sigma = \pm 1$ is a spin quantum number (here and in what follows, $\hbar = 1$; in the final equations we restore the dimensionalities).

These edge states energy spectrum is linear:

$$\epsilon = \sigma v p.$$

(3)

Below we deal with a smooth dependence of $\Delta(r)$ on $r$. In a particular case of $\Delta(y)$, assuming that $\Delta(0) = 0$ and expanding $\Delta(y) \approx -\varepsilon y$, $\varepsilon = -d\Delta(y)/dy|_{y=0}$, we get to $\psi = \exp(-\varepsilon y^2/2v)$ that, at $\varepsilon > 0$, yields the wave function localized near $y = 0$ with the edge-state half-width $l_{ed} = \sqrt{8\varepsilon / \varepsilon}$.

The edge-states spectrum overlaps the bandgap of the infinite system.

The paper deals with $V(r)$ and $\Delta(r)$ randomly depending on both coordinates. We will study the edge states in a quasiclassical system, where the characteristic planar sizes of potential $b$ and gap $a$ are large, as compared with $l_{ed}$ (for $\Delta(r)$ depending on both coordinates $l_{ed} \sim \sqrt{8\varepsilon / |\nabla \Delta|}$). The functions $V(r) \pm \Delta(r)$ represent the conduction band bottom and the valence band top, correspondingly. Below we assume that $V(r)$ and $\Delta(r)$ are independent random functions with Gaussian distributions. The first one results from the random distribution of charge impurities and the second one results from quantum well width fluctuations.

The quantity $\Delta(r)$ is characterized by its mean value $\overline{\Delta(r)} = \alpha(\overline{w} - w_0)$ and mean-squared fluctuations $\delta^2 \Delta^2 = \alpha^2 \delta w^2$; the overline stands for the spatial mean one. The mean-spatial sizes of $\Delta(r)$ and $V(r)$, $a$ and $b$, are: $a^2 = b\delta w^2 / (\overline{w} w)^2$, $b^2 = \delta V^2 / (\overline{V} V)^2$, where $\delta w^2 = \overline{w^2} - \overline{w}^2$.
In the quasiclassical approximation, the ZGLs $\Delta(r) = 0$ can be considered as locally straight. In such case one can apply Eq. 4.

An additional simplification can be done if to assume that wavelength $h/p$ is less than the ZGL characteristic length. This permits a quasiclassical description of the electron motion along the edges. Such particles have the Hamiltonian function $\sigma p + V(r(u))$, where $u$ is a coordinate along the ZGL and $p$ is a conjugated momentum. In accordance with this Hamiltonian, independently from the potential, electrons move along the edge with constant velocity $\sigma v$. If necessary, the electron motion along the edge can be quantized.

III. PROBLEM FORMULATION

First, we shall describe the edge states in the framework of percolation theory. After that, we shall study the conductance of p-n junction with a crossing edge state. This state produces a channel short-circuiting the insulating p-n junction.

Then, the conductance of the finite sample with the fluctuating gap sign and without potential fluctuations will be studied. The random network of edge states is formed in such system. The problem can be reformulated as a motion with a constant velocity of particles along ZGLs of a random function $\Delta(r)$. In such approach the problem of electron transport converts to the study of geometrical properties of ZGLs. Near the threshold width the conductance experiences strong fluctuations and depends on the sample shape. We shall find the conductance of wide or long samples. The theory shows that the conductivity of infinite sample tends to zero, unless the interedge transitions are taken into account. The problem solution is given by taking into account the finiteness of the edge state width that provides the interedge transitions resulting in the 2D conductivity at zero temperature. The other studied approach is the conductivity fluctuations in the system, one can define the order parameter $\xi$ of potential distributions for conduction and valence bands:

$$\xi = \frac{1}{\sqrt{2\pi} \Delta} \int_{w_0}^{\infty} e^{-\frac{(w-w_0)^2}{2\Delta^2}} dw = \frac{1}{2} \operatorname{erfc} \frac{w_0-w}{\sqrt{2}\Delta},$$

(4)

where $\operatorname{erfc}(x)$ is a complementary error function.

Analogically, if there are potential fluctuations with no width fluctuations in the system, one can define the order parameters $\xi_{c,h}$ of potential distributions for conduction and valence bands:

$$\xi_{c,h} = \frac{1}{\sqrt{2\pi} \delta V} \int_{\Delta}^{\infty} e^{-\frac{\delta V^2}{2\mu^2}} dV = \frac{1}{2} \operatorname{erfc} \frac{\Delta + \mu}{\sqrt{2}\delta V},$$

(5)

where $\mu$ is the chemical potential. If one takes into account $\delta w$, the quantity $\delta V^2$ in Eq. 5 should be replaced by the mean-squared fluctuations of the conduction (valence) band edges $\delta E^2 = \delta V^2 + \delta \Delta^2$.

The order parameters $\xi_{c,h}$ determine the n- and p-domains connectivity (in infinite systems) by conditions $\xi_{c,h} > \xi_c = 0.5$, correspondingly.

IV. ORDER PARAMETERS

As the properties of our system are determined by potential and quantum well width fluctuations, we should introduce their distribution and order parameters. For the Gaussian distribution of widths, the corresponding order parameter is

$$\xi = \frac{1}{\sqrt{2\pi} \Delta} \int_{0}^{\infty} e^{-\frac{w^2}{2\Delta^2}} dw = \frac{1}{2} \operatorname{erfc} \frac{w}{\sqrt{2}\Delta}.$$  

(6)

V. CONDUCTANCE OF A PLANAR P-N JUNCTION WITH A CROSSING EDGE STATE

Let us consider a potential in the planar p-n junction with the impurity charge density distribution as $en \tanh(x/b)$, where $e$ is the electron charge, $n$ is the carrier density at infinity. Planar charge carriers screen this distribution. That determines the potential across the p-n junction (see Fig. 1). The potential caused by the 2D charge distribution is

$$V = \int d^2r e[n(r') - n_e(r')] \frac{r - r'}{r^2}.$$  

(6)
where, $\kappa$ is the dielectric constant. In the other form,

$$V = \frac{1}{\kappa} \int dx' e(n(x') - n_e(x')) \ln |x - x'|.$$  \hfill (7)

If the 2D screening length is small enough, the p and n domains are neutral, and the potential is formed by the charge in the insulating domain $-l_d/2 < x < l_d/2$. This gives the potential difference

$$V = \frac{1}{\kappa} \int_{-l_d/2}^{l_d/2} dx' e(n(x') - n_e(x')) \ln |x - x'|$$  \hfill (8)

and the boundary condition for the insulating domain width

$$2\Delta_0 = \frac{e^2 n}{\kappa} \int_{-l_d/2}^{l_d/2} dx \tanh \left( \frac{x}{b} \right) \ln |x|.$$  \hfill (9)

Here $2\Delta_0$ is the gap on infinity. If $b \ll l_d$,

$$l_d \approx 2\Delta_0 \kappa / e^2 \ln(\Delta_0 \kappa / e^2 b).$$

The characteristic decrement for the tunneling is $\Delta_0 / v$ and if $l_d\Delta_0 / v \gg 1$, the p-n junction is impermeable. The parameter $l_d\Delta_0 / v$ can be rewritten as $l_d\Delta_0 / v \sim (\Delta_0/E_F)(v_F/e^2)(v/v_F)$, where $E_F$ and $v_F$ are the Fermi energy and velocity. All 3 factors in the right part are greater than 1: $(\Delta_0/E_F) \gg 1$ if electrons (holes) fill the extrema of the spectrum, which is an ordinary situation, $v_F/e^2 \gg 1$ for a weakly interacting electronic liquid, and $v_F > v$ always.

Then we consider the edge state that crosses the straight p-n junction. Let the potential profile have the form of Eq. (7), the gap be $2\Delta(y) = 2\Delta_0 \tanh(y/a)$, the potential $V(x) = \Delta_0 \tanh(x/l_d)$ and the parameters satisfy inequalities $\Delta_0 - V_0 < \mu < V_0 - \Delta_0$ (see Fig. 4). The gap goes to constants $\pm \Delta_0$ at $y \to \mp \infty$. The potential goes to $\pm \Delta_0$ at $x \to \mp \infty$. In such system, the $x \to \mp \infty$ domains belong to n and p, correspondingly. In that case the electron gas in p and n domains is degenerate. Apart from the $y = 0$ line, the conductivity across the edge at a low temperature vanishes. The line $y = 0$, where $\Delta(y) = 0$ crosses the p-n junction. Just this place is responsible for the p-n junction conductance.

Let there be a single edge state along this line. Independently from the potential, the edge state conductance is $e^2 / h$, where $h = 2\pi \hbar$. Hence, this value will determine the total conductance of the p-n junction $\Sigma_{junc}$. The edge $y = 0$ plays the role of the p-n junction short-circuit.

Now, consider the case when width $a$ is large enough. Then, electrons are not quantized by the conducting channel width. Considering the tunneling in any point in an independent manner we can find the probability of tunneling across the junction at a coordinate $y$

$$P_t(y) = \exp \left( -\frac{2}{v} \int \sqrt{\Delta^2 - V^2(x)} dx \right) = e^{-\frac{2\pi e}{\hbar} \Delta_0 |\tanh \frac{y}{a}|}.$$  \hfill (10)

The ballistic motion channels are determined by $P_t \sim 1$. This yields the characteristic width $l_t \sim v_0/l_d\Delta_0$. The number of quantum channels for ballistic tunneling is determined by this width ratio to the edge state width: $l_t / l_d = \sqrt{v_0/8\Delta_0}$. The p-n junction conductance $\Sigma$ is $l_t / l_d \cdot e^2 / h$. This approach is valid if $l_t \gg l_d$; otherwise the approach with a single channel holds. Summarizing this, we have

$$\Sigma_{junc} = \frac{e^2}{\hbar} f \left( \frac{l_t}{l_d} \right),$$

where $f(0) = 1$, $f(x) = x$ at $x \to \infty$.

Note that the conductance does not change if to apply a finite voltage to the p-n junction (less than $2\Delta_0$).

Hence, the conductance across the p-n junction is determined by the edge states crossing it. The results of the present part are applicable to the case when the p-n junction width is less than a characteristic spatial size of gap $a$ or has the same order.

The p-n barriers can block the overall conductivity in a system with a random potential. This conductivity is restored by the edge states inclusion.

## VI. Edge-states conductance without potential fluctuations

### A. Quantized conductance

First, we should concretize what we mean by the word "sample". The source and drain contacts are metallic. The edge state electrons coming into these contacts are quickly mixed with the electron sea due to elastic scattering. The most simple is the case of a sample without side boundaries. For example, one can consider the ring sample (the "Corbino" disk) with metallic inner and outer
parts. Another example is an infinite strip across the current direction.

The other situation is with the lateral faces of the rectangular sample. One can not use the limitation due to the potential: edge state electrons freely come through this p-n junction (see previous section). Instead of this, the limitation of the electron motion in the side direction can be done by the contact with a large-gap OI situated outside the sample. However, this procedure inevitably produces the edge states on the side border.

We shall assume that the sample sizes exceed a. Let the Fermi level in a ring sample fall inside the forbidden band. At the same time, fluctuations in the gap sign create the edge states on ZGLs. Depending on the percolation threshold proximity, the ZGLs turn out to be closed at \( |\xi - \xi_c| \approx 1/2 \), or open lines appear at \( |\xi - \xi_c| \ll 1/2 \). In the first case, the conductance of a large sample vanishes, with the exception of external edges contribution. In the case \( |\xi - \xi_c| \ll 1/2 \), the conductance appears at zero temperature (see Fig. 3).

If they also exceed the correlation length, the edge channels at \( \xi \neq \xi_c \) are the close lines which do not cross all the sample. If \( \xi \to \xi_c \), the ZGLs begin to cross the sample; that provides the conductance \( \Sigma = \frac{N_0 e^2}{h} \), where \( 2N_0 \) is the number of the lines \( \Delta (r) = 0 \) crossing the sample in the field direction (radius for the ring case), \( 2e^2/h \) is the conductance quantum. As the ZGLs has no branching, the presence of conductance in the radial direction means no conductance in the angular direction and vice versa. So, the conductance fluctuates between 0 and the value of the order of \( e^2/h \).

The conductance along the external edges \( e^2/h \) appears in a square sample. Besides, the additional inner ZGLs crossing the sample in the field direction can exist. So, \( \Sigma = (N_0 + 1)e^2/h \).

**B. Consideration based on the percolation theory**

Here we shall consider a finite rectangular sample based on the percolation theory. We shall start from a square sample. Assume, that the ZGL is absolutely random and starts from the right edge of a square. Then it has approximately equal opportunities to cross any other square edges, and the probability to cross the sample in the field direction is 1/4.

In a sample elongated in the applied voltage direction, with width \( L_2 \) and length \( L_1 \gg L_2 \), the percolation probability should qualitatively be a product of the probabilities of percolation through the square samples from which it is composed (Fig. 3). So, for an elongated sample, the percolation probability should be proportional to \( 1/4^{L_1/L_2} \). Hence, the mean conductance of such sample decays exponentially with its length.

The situation is different for a wide sample \( (L_2 > L_1) \). Blocks \( L_1 \times L_1 \) are independent from each other. Hence, the conductances are added in a parallel manner.

The previous consideration was too qualitative. To be more accurate, let us include the percolation theory. For the conductance finiteness, ZGLs should connect the opposite sides of the sample \( x = 0 \) and \( x = L_1 \). The probability of one of the phases in connecting the borders can be found using the correlation function \( G(r) \). In the percolation threshold vicinity,

\[
G(r) = \left( \frac{d}{r} \right)^{\eta} e^{-r/L_c},
\]

where \( L_c = a \xi \xi_c^{-\nu} \), \( \eta = 2 - \frac{2}{\nu} \approx 0.22 \), \( \gamma \approx 2.38 \), \( \nu \approx 1.34 \). Near the percolation threshold, the probability, for the edge, of crossing the sample in points \( r_1 = (0, y_1) \), \( r_2 = (L_1, y_2) \), \( 0 < y_1 \) and \( y_2 < L_2 \) has the same order as the probability of \( r_1 \) and \( r_2 \) points to belong to the same cluster, namely \( G(|r_1 - r_2|) \xi_1^2 dy_1 dy_2 \). The fractal dimension of the large cluster hull is close to 1, see Fig. 2.

This follows from the fact that, on the threshold, exactly half of the neighboring sites of OI belongs to an OI or a TI subset. So, the probability of that a site belongs to a cluster border has the order of 1 for example, in the site problem on a square lattice, this probability is 15/16).

As a result, we can replace the correlation function of edge points by the correlation function of OI or TI sites.

Now, let us consider a \( L \times L \) square sample. The probability of percolation along the edge in the \( x \)-direction is collected from the probability for two points on the sample borders \( x = 0 \) and \( x = L \) of belonging to the same cluster \( G(|r_1 - r_2|) \xi_1^2 dy_1 dy_2 \) and the probability \( \exp (-Z) \) of that no one of other points of the rectangle border \( \Gamma \) to be connected with the starting point. We should find the number of the ZGLs, which cross the opposite edges \( x = 0 \) and \( x = L \), with the limitation that these lines remain inside the rectangle. It is the product of the number of points \( r_1 \) and \( r_2 \) belonging to the same cluster

\[
N_c = \int_0^L \int_0^L G(|r_1 - r_2|) \xi_1^2 dy_1 dy_2,
\]

where \( r_1 = (0, y_1) \), \( r_2 = (L, y_2) \), and the ZGL probability of not crossing all borders in other points is given by a product

\[
\prod_{r \in \Gamma} \left( 1 - G(|r_1 - r_2|) \xi_1^2 dt_2 \right) = e^{-Z}, \quad (11)
\]

\[
Z = \int_0^L \int_0^L G(|r_1 - r_2|) \xi_1^2 dt_1 dt_2 L a. \quad (12)
\]

Here \( t_2 \) is the length along \( \Gamma \).

The system conductance \( \Sigma(L) \) is the product of the half conductance quantum and the probability of connection of two opposite sample sides:

\[
\Sigma(L) \sim \frac{e^2}{h} \xi_c \exp (-Z). \quad (13)
\]

In the limiting cases \( L \ll L_c \)

\[
\Sigma(L) = \xi_c \xi_1 \left( \frac{1}{a} \right)^{2-\eta} \xi_1 e^{-\xi_1 \Gamma (\xi)}^{1-\eta}, \quad (14)
\]
and at $L \gg L_c$

$$\Sigma(L) = e^2 \frac{c_3}{h} \left( \frac{L}{a} \right)^{\frac{3}{2} - \eta} \left( \frac{L_c}{a} \right)^{\frac{3}{2}} e^{-\frac{L}{L_c} - c_4 \left( \frac{L}{L_c} \right)^{1-\eta}}. \quad (15)$$

Here $c_3 = \sqrt{\pi/2} \approx 1.25$; at $\eta = 0.22$ quantities $c_4 \approx 0.97$, $c_2 \approx 4.56$ and $c_4 \approx 1.19$. If $L \ll L_c$, $\Sigma(L)$, experiences a power-like drop with $L$; if $L \gg L_c$, $\Sigma(L)$ exponentially drops.

Let us consider the conductance of a rectangular sample, when $L_2 \gg L_1$ or $L_1 \gg L_2$. Like the approximate consideration, the conductance of samples with $L_2 \gg L_1$ and $\Sigma(L_1, L_2)$ is the sum of squares conductances:

$$\Sigma(L_1, L_2) = \Sigma(L_1) L_2 / L_1. \quad (16)$$

This is valid in a very wide sample or in a ring sample. In a sample of finite width, one should add $e^2 / h$ to this value.

In the case of $L_1 \gg L_2$ the percolation probability exponentially decays with the length $L_1$ at $L_1 \gg L_c$, while the external edge conductance remains the same. This means that

$$\Sigma(L_1, L_2) = e^2 / h + o \left( \exp \left( -L_1 / L_c \right) \right). \quad (17)$$

We should emphasize that Eqs. (13)-(15) give estimations only. Besides, the conductance is strongly fluctuating for the samples with $L_2 \lesssim L_1$, while, at $L_2 \gg L_1$, the conductance is self-averaging.

There are different origins of the obtained results inaccuracy. First, we replaced the edge sites correlation function by the TI phase correlation function (when the OI phase prevails) or the OI phase in the opposite case. In principle, the behavior of both correlation functions should be similar, but the corresponding exponents can differ. We hope that, in accordance with the abovementioned, this numerical difference is not strong. The other inaccuracy consists in using the percolation correlation functions for the unbounded sample to describe the strip. This inaccuracy manifests itself in the difference of the percolation probability along the strip obtained by ZGL in non-correlated and correlated models (different probability logarithm dependence on the strip length). Again, this difference is inessential in the sufficiently rough approimation which is used here.

This consideration neglects the tunneling, which is considered in the next section, between ZGLs.

VII. 2D EDGE CONDUCTIVITY AT ZERO TEMPERATURE. FINITE EDGE-STATE WIDTH

Here we discuss the consequences of the edge-state width finiteness. It is obvious that the exaggerated picture of the transport along ZGLs is limited by the edge width. If the width becomes comparable with the distance between different ZGLs or different parts of the same line, the intensive tunneling will destroy this picture. Instead of strict following ZGLs, an electron can jump from one place to another (see Fig. 5).

The free jump distance is determined by the edge width $l_{ed}$, which can be estimated as $\sqrt{a \pi / \Delta}$. At $\xi_\Delta \ll 1$ or $1 - \xi_\Delta \ll 1$, we suppose the equality of the densities of clusters internal and edge points. Hence, for all $\xi_\Delta$, we can set the edge points density $N_e = \xi_\Delta (1 - \xi_\Delta) / a^2$. Neglecting the edge points correlation, one can estimate the threshold $\xi_{\Delta, e}$ when the transport along ZGLs converts to the two-dimensional traveling due to jumps as $\pi N_e l_{ed}^2 = B_c$. Here $B_c$ is some number calculated in the percolation theory. $B_c$ runs from 3.2 to 4.5 in different 2D percolation models. As a result, $\xi_{\Delta, e} (1 - \xi_{\Delta, e}) = B_c a \Delta / 2 \pi v$. If $\xi_\Delta (1 - \xi_\Delta) > \xi_{\Delta, e} (1 - \xi_{\Delta, e})$, the edge traveling prevails; else two-dimensional jumps occur and that means the 2D delocalization, and, in such case, the conductivity should be determined by the Drude-like expression.

Note that we neglected the quantization of electron motion along edges. The longitudinal quantization results in a minimal energy needed to jump from one close edge state to another with a characteristic quantum of the order of $\xi_\Delta (1 - \xi_\Delta) v / a$. This quantum should be less than the characteristic hopping amplitude $\Delta \exp \left( -\Delta (\bar{h} v) / N_e \right)$, where the characteristic hopping distance $l_{hop} = 1 / \sqrt{N_e}$. Obviously, this is impossible and the quantization is inessential.

The developed ZGLs near the threshold $\xi_\Delta \to \xi_c$ are long. There is a finite probability that, somewhere, such lines will approach the other ZGL to the distance comparable with the edge state width. That results in the electron possibility of jumping from one ZGL to another and the 2D diffusion will be established.

We shall find the diffusion coefficient and the conductivity based on the model of random $\Delta(r)$ with the Gaussian distribution. Traveling along a ZGL, an electron meets a different relief of random $\Delta(r)$. The quasiclassical edge state width depends on the coordinate along the edge as $l_{ed} \sim \sqrt{4 \pi v / |\nabla \Delta(r)|}$. (The quasiclassical approach is valid if $a \gg l_{ed}$.)

The edge-to-edge transition occurs in the places where two ZGLs are closer to each other than $l_{ed}$. Let us intro-
duce two curvilinear coordinates: \( t \) - along ZGL \( \Delta(t) = 0 \) and \( \rho \) - across it. In these coordinates \( \Delta(t, \rho) = 0 \). Besides, in the vicinity of point \( t_0 \) of a minimal distance between two ZGLs, one can expand \( \Delta(t, \rho) \) as

\[
\Delta(t, \rho) = \partial_\rho \Delta(t_0, 0) \rho + \frac{\partial_\rho^2 \Delta(t_0, 0)}{2} \rho^2 + \frac{\partial_{t, \rho}^2 \Delta(t_0, 0)}{2} (t - t_0)^2 \rho^2.
\] (18)

Eq. (18) takes into account the need to have two close solutions for the ZGLs:

\[
\rho = 0, \quad \rho = \rho_0 + a_1 (t - t_0)^2,
\]

\[
\rho_0 = -2 \frac{\partial_\rho \Delta(t_0, 0)}{\partial_\rho^2 \Delta(t_0, 0),} \quad a_1 = - \frac{\partial_{t, \rho}^3 \Delta(t_0, 0)}{\partial_\rho^3 \Delta(t_0, 0),}.
\] (19)

The quantity \( \rho_0 \) is the minimal distance between the ZGLs reached at point \( t_0 \). The condition for a jump between edges is \( \rho_0 < l_d \). In other terms,

\[
2h \nu (\partial_\rho^2 \Delta(t_0, 0))^2 - |\partial_\rho \Delta(t_0, 0)|^3 > 0.
\] (20)

Eq. (20) can be estimated as \( h \nu \Delta^2 / a^4 > \Delta^3 / a^3 \). Let us divide the ZGL into cuts of minimal length \( a \). The portion of configurations, when Eq. (20) is valid, is determined by the ratio of value \( \Delta \) from this inequality to the mean fluctuation of \( \Delta \). Hence, the portion of cuts, where the jumps can occur, is \( h \nu / a \sqrt{\Delta^2 / h \nu} \), and the mean free path for a jump is \( l_p = a^2 \sqrt{\Delta^2 / h \nu} \).

Now we should express the Cartesian distance of travel \( L \) via \( l_p \). This relation is given by the hull fractal dimension:

\[
l_p = a (L/a) D_h, \quad D_h \approx 1.74 \Delta.
\]

Thus, an electron randomly shifts at distance \( L \) at the mean free time \( l_p / v \). This yields the 2D conductivity \( \sigma \) expressed via the diffusion coefficient \( D = L^2 v / l_p = a v \left( \sqrt{\Delta^2 / h \nu} \right)^{2/D_h} \). The diffusion coefficient is connected with the conductivity and the density of edge states \( g \). For a linear spectrum, \( g \) does not depend on the energy and is determined by the number of edge nodes per unite area \( 1/2a \):

\[
\sigma = \frac{e^2 D}{2 \pi \hbar v a} = \frac{e^2}{h} \left( \sqrt{\Delta^2 a / h \nu} \right)^{2/D_h}.
\]

That explains how the 2D diffusion and 2D conductivity appear.

The threshold for the appearance of 2D conductivity is determined by the requirement that the gap fluctuations correlation length \( L_c \) exceeds \( L \): \( L_c (\xi_c) = a (\xi_c - \xi_c) - \nu \sim L = a (L/a)^{1/D_h}, \) or \( \xi_c - \xi_c \sim (h \nu / a \sqrt{\Delta^2})^{1/1/D_h} = (h \nu / a |\xi| \sqrt{\Delta^2})^{1/1/D_h} \).

The 2D diffusion picture is realized if \( |\xi_c - \xi_c| < |\xi_c - \xi_c| \). Otherwise, the 2D diffusion and conductivity vanish.

Recounting for the HgTe width, we find that the 2D conductivity appears if \( |\xi_c - \xi_c| < |\xi| \Delta \) and, vice versa, the low-temperature conductivity is absent. This shows the extremely unusual behavior of the conductivity with \( w \): it exists in a narrow window of widths near \( w_0 \). The origin of this phenomenon is the growth of the number of contacts between ZGLs with the edge length growth at approaching the critical width.

Note that, in this approximation, the transition does not depend on the Fermi level inside the gap. This is explained by the spectrum linearity because all physical properties of such system do not depend on the electron energy.

### VIII. Percolation in the System with Gap and Potential Fluctuations Near P- and N-Type Conductivity Thresholds

In a pure OI or TI, the conductivity appears when the Fermi level comes in electron or hole permitted bands. If the chaos is weak, the band edges smear. The conductivity near thresholds is realized through the electron or hole seas. The inner part of p- and n-regions has the Drude \( \sigma_{e,h} \sim (e^2 / h) \cdot E_{F,e,h} \tau_{e,h} \) (for \( \delta E_{F,e,h} \gg 1 \)) or minimal metallic \( e^2 / h \) (for \( \delta E_{F,e,h} \gg 1 \)) conductivity, where \( E_{F,e,h} \) are the Fermi energies of electrons and holes, \( \tau_{e,h} \) are their mean free times.

It is known that, when the Fermi level \( \mu \) approaches the bands edge, the conductivity has the power-law behavior \( \sigma_{e,h} \propto (|\xi_{e,h} - \xi_c|)^{t} \) with \( t \approx 1.38 \). Thus, in the case of \( \delta E \tau \sim 1 \), we have

\[
\sigma_{e,h} = \frac{e^2}{h} (|\xi_{e,h} - \xi_c|)^{t},
\] (21)

where the potential distribution order parameter is

\[
\xi_{e,h} = \left\{ \begin{array}{ll}
\text{erf} \left( \frac{\mu - \overline{E}}{\sqrt{2 \delta E}} \right), & |\mu| > |\overline{E}| \\
0, & |\mu| < |\overline{E}|.
\end{array} \right.
\] (22)

The conductivity versus the Fermi level for different average HgTe layer thicknesses is shown in Fig. 6.

![FIG. 6: Conductivity (in units of e^2/h) dependence on the Fermi level at \( \delta E = 0.5 \overline{E} \), in accordance with Eqs. (21,22).](image)
IX. DEVELOPED POTENTIAL FLUCTUATIONS IN THE PRESENCE OF RANDOM EDGE STATES

In the absence of edge states, the conductivity of an infinite system inside the bandgap with a fluctuating potential vanishes. Let the potential fluctuations \( \Delta V \) be large as compared with the mean gap \( \overline{\Delta} \). Then the system will consist of large n and p metallic lakes (with size \( b \)) separated by narrow p-n junctions. The characteristic p-n junction width is \( b \overline{\Delta}/\Delta V \ll b \). The p-n junction isolates these regions from each other. The ZGLs \( \Delta(r) \gg \Delta \) (larger than the conductance quantum \( e^2/2h \)) when the number of ZGLs crossing each junction is large enough, that provides the 2D metallic conductivity (Fig. 7). According to the weak localization theory, for delocalization, the characteristic length \( L \) that the p-n junction has a width \( b \) is \( \sim b \overline{\Delta}/\Delta V \). Let the potential fluctuations (blue, light-orange and light-yellow colors correspond to n-, i- and p-domains, correspondingly). Red lines represent the ZGLs short-circuiting p-n junctions. Right: equivalent circuit. Squares stand for p and n lakes; lines replace the edge states.

The simplest estimation can be done if to assume that the p-n junction has a width \( L_1 \rightarrow b \overline{\Delta}/\Delta V \) and length \( L_2 \rightarrow b \). Let \( |\xi_\Delta - \xi_e| \) be not much less than 0.5. The number of TI phase inclusions to OI (or OI to TI) with a size \( a \) in this rectangle is \( N_{\text{cross}} = \overline{\Delta}^2/(\Delta V)|\xi_\Delta(1 - \xi_\Delta)|/a^2 \). Let \( a \gg b \overline{\Delta}/\Delta V \) and \( N_{\text{cross}} \gg 1 \). Then the conductance between two metallic lakes \( N_{\text{cross}}e^2/h \) exceeds \( e^2/h \). Thus, the characteristic conductance, apart from the threshold in \( \xi_\Delta = \xi_e \), is \( \Sigma_{\text{junc}} \sim (e^2/h)(b/a)^2(\overline{\Delta}/\Delta V)|\xi_\Delta(1 - \xi_\Delta)| \).

In fact, the factor \( |\xi_\Delta - \xi_e| = (\overline{\Delta}/\Delta V) \) is the proximity of the potential to the threshold. The smallness of this quantity leads to the lake edges fractality. The mean lake size grows like \( b|\xi_\Delta - \xi_e|^{-\gamma} \). Replacing \( L_2 \) by this cluster perimeter, we have a more accurate estimation for \( \Sigma_{\text{junc}} \sim (e^2/h)(b/a)^2(\overline{\Delta}/\Delta V)^{1-\gamma}|\xi_\Delta(1 - \xi_\Delta)| \).

The total conductivity is formed upon the series connection of \( \Sigma_{\text{junc}} \) and the conductivity of p and n domains \( \sigma_{e,h} \); \( \sigma^{-1} = \Sigma_{\text{junc}}^{-1} + \sigma_{e,h}^{-1} \) (see Fig. 7, right). This yields the resulting effective 2D conductivity.

X. CONCLUSIONS

In conclusions, we have qualitatively studied the low-temperature conductivity of 2D narrow-gap semiconductor with gap and potential fluctuations. If gap fluctuations do not change its sign, they result in the shift of the thresholds for n- and p-type conductivities. In the system, where the gap fluctuations change their sign, an internal edge states network appears. The global conductivity is absent without the potential fluctuations and interedge tunneling, unless in the case when the gap fluctuations order parameter gets just to the threshold value. In the last case, the conductance of the square sample is unstable and fluctuates from zero to the conductance quantum. In a sample, elongated along the electric field, the conductance exponentially drops with the length, while, in the widened sample, it is proportional to the sample width. If the sample possesses the developed potential fluctuations that exceed the mean gap, the sample composed of p and n domains is separated by p-n junctions. If the latter are tunneling impenetrable ones, the conductivity between conducting p and n lakes is provided by the edge states crossing p-n junctions. This picture was reduced to the conductance of the single p-n junction with short-circuiting edge states.

Note that narrow-gap semiconductors with strong short-periodic potential fluctuations (see Fig. 11) can experience the Anderson-Mott transition which closes the energy gap.\(^{24}\) When the spatial length is large enough, the energy gap can disappear, but the states inside the mean gap are localized; vice versa, in the layers with adiabatically fluctuating width and gap considered here, the edge states are formed near the zero gap lines covering the entire sample.

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