Anomalous scattering of low-lying excitations in a spin-1 Bose-Einstein condensate

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Abstract – We study the scattering of the transverse spin wave by an impurity potential in the ferromagnetic spin-1 Bose-Einstein condensate (BEC) by using the mean-field theory. We present a simpler framework of anomalous tunneling effect, the perfect transmission in the low-energy limit through an external potential barrier. The transverse spin wave obeys a Schrödinger-type equation; yet, the effects of the potential barrier on its transmission coefficient and on its scattering cross-section vanish in the low-energy limit. The modulus of the order parameter alone determines its transmission coefficient. The momentum \( p \)-dependence of the scattering cross-section \( \sigma \) exhibits a Rayleigh scattering type (\( \sigma \propto p^4 \)). These properties are common to the transverse spin wave and the Bogoliubov mode, which belong to different types of Nambu-Goldstone modes.

Introduction. – Tunneling through a barrier without reflection is a curious phenomenon. Such anomalous behavior is observed in superfluids and superconductors. In the early 2000s, perfect transmission in the low-momentum limit against a potential barrier was predicted [1,2] for the low-lying excitation (the so-called Bogoliubov mode [3]) in a Bose-Einstein condensate (BEC). This perfect transmission was termed anomalous tunneling [2]. Subsequently, many interesting tunneling properties have been discovered in scalar BECs [4–11].

In quantum electrodynamics, it is known that a relativistic particle tunnels through a high and wide barrier, which is referred to as the Klein paradox [12], a recent subject of graphene [13]. This counterintuitive phenomenon is described by the Dirac equation for two-component wave functions (WFs) and its particle has linear dispersion. Since the Bogoliubov mode also has linear dispersion in the low-momentum limit and obeys the Bogoliubov equation for the two-component WFs, one might expect that the anomalous tunneling of the Bogoliubov mode is linked to the Klein paradox via some ways like the charge-conjugation symmetry for the Dirac equation. However, anomalous tunneling will be inherent in NG modes, because this tunneling phenomenon through a generic barrier was derived using a property of a NG mode in a BEC, the coincidence between the condensate WF and the WF of the Bogoliubov mode in the low-energy limit [5].

The ferromagnetic state of the spin-1 BEC [14,15] (a BEC composed of particles with spin-1 internal degrees of freedom) is suitable for investigating the anomalous tunneling of NG modes in various situations since both gauge and spin rotational symmetries are spontaneously broken and accordingly the system hosts two types of NG modes: the Bogoliubov mode (type-I) and the spin wave (type-II) [16]. Earlier studies have revealed that at small wave numbers \( k \), the Bogoliubov mode, which has phonon excitation and belongs to the type-I NG mode, exhibits Brewster’s law [7] as well as the Rayleigh scattering \( \sigma \propto k^4 \) [5]. The transverse spin wave in the ferromagnetic spin-1 BEC, which has a quadratic dispersion relation and belongs to the type-II NG mode, has been known to also show the anomalous tunneling phenomenon [17–20]. However, it is still unclear in the low-momentum limit, what determines the transmission coefficient for a junction of two ferromagnetic BECs, and whether this spin wave is
refractive. We also find ourselves confronted with the following question: Does the spin wave with a quadratic dispersion relation also exhibit anomalous scattering properties as the Rayleigh scattering? By comparing the scattering properties of the two NG modes, the Bogoliubov mode and the spin wave, we may gain a unified perspective and a deeper understanding of the scattering properties inherent to NG modes in broken-symmetry states.

With these backgrounds, we investigate the scattering properties of a transverse spin wave in a ferromagnetic spin-1 BEC within the mean-field theory at $T = 0$. We examine tunneling probability and the scattering cross-section $\sigma$ from a nonmagnetic external potential. Our main findings on the ferromagnetic spin wave are that anomalous tunneling occurs in the low-energy limit, and this spin wave is not refractive. The modulus of the order parameter, the magnetization, describes the transmission coefficient of the spin wave for a junction between BECs. Even though the ferromagnetic spin wave has a quadratic dispersion relation, the Rayleigh-scattering–type $k$-dependence $\sigma \propto k^4$ holds at small wave numbers. Comparison with the Bogoliubov mode and inspection of the derivation suggest that both findings are expected to hold for other NG modes regardless of the spectrum.

**Formulation.** – We start with a Hamiltonian for spin-1 bosons with an atomic mass $m$, given by [15]

$$\hat{H} = \int d\mathbf{r} \left[ \frac{\hbar^2}{2m} \nabla \hat{\Psi}^\dagger \nabla \hat{\Psi} + V_{\text{ext}} \hat{\Psi}^\dagger \hat{\Psi} + \frac{c_n}{2} (\hat{\Psi}^\dagger_0 \hat{\Psi}_j^2 + \hat{\Psi}_j^\dagger \hat{\Psi}_j^0 \hat{S}_{jj'} \hat{\Psi}_{j'})^2 \right],$$

(1)

where $\hat{\Psi}_j$ is a field annihilation operator for a spin-1 boson in a single hyperfine state $j$ ($j = \pm 1, 0$), $S_a$ ($a = x, y, z$) is a spin matrix, and repeated indices are summed. Two coupling constants are given by $c_0 \equiv 4\pi \hbar^2 (a_0 + 2a_2)/(3m)$ and $c_1 \equiv 4\pi \hbar^2 (a_2 - a_0)/(3m)$, where $a_0$ is the $x$-wave scattering length for the total spin $F$ channel. The external potential $V_{\text{ext}}$ is coupled only to the local density, so that $\hat{H}$ is $U(1) \times SO(3)$ invariant [15].

The ferromagnetic phase is realized for $c_1 < 0$ as a ground state, and its condensate WFs $\langle \hat{\Psi}_j \rangle \equiv \Phi^0 e^{-i\nu \phi} / \hbar$ are given by [15]

$$\langle \Phi^0_+ , \Phi^0_0 , \Phi^0_- \rangle = (\sqrt{n} \phi(\mathbf{r}) , 0 , 0) ,$$

(2)

where $\nu$ is the chemical potential and $n$ the condensate density. The WF $\phi$ obeys the Gross-Pitaevskii (GP) equation identical with that for a scalar BEC, $\hat{H}_{\text{GP}} \phi = 0$, where

$$\hat{H}_{\text{GP}} \equiv -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}} + c_+ n (\phi^2 - 1),$$

(3)

with $c_+ \equiv c_0 + c_1$. In this derivation, we may take $\phi$ as real.

This system offers three types of excitations [14,15]. The excitation of $j = +1$ is mathematically identical to the Bogoliubov mode of the scalar BEC; its tunneling properties are drawn from earlier studies [1,2,4–11]. The spectrum of the mode $j = -1$ is massive, $E = \hbar^2 k^2 / (2m) + 2|c_1|n$, the tunneling properties of which have been partially reported in [18,19]. The mode of $j = 0$ leads to the transverse spin wave, which we focus on in this paper. The equation of motion of this excitation mode is given by [15]

$$i\hbar \frac{\partial \phi_0}{\partial t} = \hat{H}_{\text{GP}} \phi_0 , \quad \text{with} \quad \phi_0 \equiv \phi_0 \exp (-iEt/\hbar).$$

(4)

Hereafter, we describe equations in dimensionless units $E \equiv E/(c_+ n)$, $V_{\text{ext}} \equiv V_{\text{ext}}/(c_+ n)$, $k \equiv \xi k$, and $\phi \equiv x/\xi$, where $\xi \equiv \hbar/\sqrt{mc_+ n}$, and omit the bar for simplicity.

**Transmission coefficient.** – We first study reflection and refraction of the transverse spin wave, taking the $x$-$y$ plane as an incident plane without loss of generality. We consider a potential $V_{\text{ext}}(\mathbf{r}) = V_{\text{ext}}(x)$ with an $x$-dependence alone, where the translational invariance holds in the $y$-axis, and the potential $V_{\text{ext}}(x)$ at $x \sim \ell$ acts as a potential wall. Here, we consider the shape of the barrier being arbitrary, but condensates must be weakly connected. In this sense, the barrier is not too high or wide so as not to completely divide a condensate. We further impose conditions $V_{\text{ext}}(x) = V_R$ at $x \gg 1$ as well as $V_{\text{ext}}(x) = V_L$ at $x \ll 1$. The bias potential $V_{LR}$ is less than unity in dimensionless units, or in the conventional units less than the chemical potential $\mu$, so that the condensates remain in uniform regions at very large $|x|$. For $|x| \to \infty$, the GP equation $\hat{H}_{\text{GP}} \phi = 0$ provides the result $\phi \to \sqrt{T - V_L}(\equiv A_L)$, where $A_L$ is the amplitude of the order parameter $|\Phi^+_{+1}(\equiv |\langle \Psi^+_{+1} \rangle)|$, and the subscript $\nu$ denotes L (or R) in the negative (or positive) side of the $x$-axis. The healing length is modified such that $L_{\text{eff}}(\nu\nu\nu) \equiv 1/|A_L(\nu\nu\nu)|$ in dimensionless form since condensate densities may be no longer unity owing to the bias potential $V_{LR}$. (See fig. 1(a).)

The incident wave is assumed to propagate at an angle $\theta_L$ with respect to the $x$-axis\footnote{See fig. 1 in [7] where $\phi_0$ denotes the incident angle.}. The energy spectrum in this tunneling problem is given by $E = \hbar^2 k^2 / (2m)$ since at very large $|x|$, the effective potential $V_{\text{eff}} \equiv V_{\text{ext}} + c_+ n (\phi^2 - 1)$ in (3) vanishes and $\hat{H}_{\text{GP}} \equiv -\hbar^2 \nabla^2 / (2m)$ holds. From this result, the spin wave is found to be not refractive. The energy conservation $\hbar^2 k_L^2 / (2m) = \hbar^2 k_R^2 / (2m)$ leads $k_L = k_R (\equiv k)$, where $k_{LR}$ is the modulus of the momentum in uniform regimes. Since the translational invariance holds along the $y$-axis, the $y$-component of the momentum is conserved, $k_L \sin \theta_L = k_R \sin \theta_L$, where $\theta_L$ is the angle of refraction. As a result, we have $\theta_L = \theta_R (\equiv \theta)$, so the spin wave is not refractive. This is to be contrasted with the behavior of the Bogoliubov mode, which is refractive and obeys Snell’s law [7]. The excitation WF may be then written in the form $\phi_0(\mathbf{r}) = X(x) \exp (iky \sin \theta)$.
and reflection coefficients are then given by
\[ r = \frac{(M_L - M_R)^2}{(M_L + M_R)^2} \quad \text{and} \quad T = \frac{4M_L M_R}{(M_L + M_R)^2} \]

(7)

Here, \( M_{L,R} \) denotes the modulus of the magnetization \( |M| \) in each uniform regime, which may be expressed as \( \mu_B n A_{x,R}^2 \) and given by \( 0 < |M| \leq \mu_B n \) in the conventional units. Here, \( \mu_B \) is the Bohr magneton. The result (7) satisfies \( T + R = 1 \), as well as the usual reciprocity relationship for tunneling. In the case \( V_L = V_R \), the spin wave transmits perfectly through a potential barrier in the low-momentum limit, \( T(k \to 0) = 1 \), regardless of the barrier shape and the barrier height, because the magnetizations in both sides are equal, \( M_L = M_R \). This is a simple example of anomalous tunneling; earlier anomalous tunneling was derived in a Bogoliubov equation including two-component WFs [1,2], while the anomalous tunneling of this spin wave is derived in a single-component Schrödinger equation. This phenomenon is nontrivial since a particle obeying a single-component Schrödinger equation generally undergoes total reflection at a potential barrier in the limit \( k \to 0 \).

Using an external potential of specific shape, we numerically confirm the validity of our result (7) (figs. 1(b) and (c)). The momentum dependence of the transmission coefficient \( T \) as well as that of the phase shift \( \varphi \) defined by \( t = \frac{t}{|t|} \exp(i \varphi) \) depend on details of an external potential. However, in the low-momentum limit, the transmission coefficient approaches the value of our analytic result (7). In the phase shift \( \varphi \), effects from a localized potential also vanish in the low-momentum limit.

Let us prove (7) without assuming a specific form of an external potential. We first seek the solution to (5) in the form of \( X(x) = \sum_{n=0}^{\infty} k^{2n} X^{(n)}(x) \). The function \( X^{(n)}(x) \) satisfies
\[ \hat{H}_{\text{GP}} X^{(n)}(x) = 0, \quad \hat{H}_{\text{GP}} X^{(n)}(x) = \frac{1}{2} X^{(n-1)}(x) \]

(8)

for \( n \geq 1 \). As a set of linearly independent solutions to the first equation of (8), we may choose
\[ X_0(x) \equiv \phi(x), \quad X_1(x) \equiv -\gamma \phi(x) + \phi(x) \int_0^x \frac{dx'}{\phi^2(x')} \]

(9)

with \( \gamma \equiv (\gamma_L + \gamma_R)/2 \), where
\[ \gamma_L,R \equiv \int_0^{a_{L,R}} dx \left[ \frac{1}{\phi^2(x)} - \frac{1}{A_{L,R}^2} \right] \]

(10)

and \((a_L, a_R) \equiv (-\infty, +\infty)\).
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We will show that in the anomalous tunneling problem, the WF in the low-momentum limit is given by one solution \( X_l(x) = \phi(x) \), and the other solution \( X_{II}(x) \) is absent. The solution to (5) is generally given by

\[
X(x; k_x) = C_l(k_x) X_l(x; k_x^2) + C_{II}(k_x) X_{II}(x; k_x^2)
\]

(11)

with \( X_\alpha(x; k_x^2) \equiv \sum_{n=0}^{\infty} k_x^{2n} X_\alpha^{(n)}(x) \) for \( \alpha = I, II \), where

\[
X^{(0)}_\alpha(x) = X_\alpha(x),
\]

(12)

\[
X^{(n)}_\alpha(x) = -\phi(x) \int_0^x \frac{dx'}{\phi'^2(x')} \int_0^{x'} dx'' \phi(x'') X^{(n-1)}_\alpha(x''),
\]

(13)

for \( n \geq 1 \). Here, \( C_{I,II}(k_x) \) are coefficients. In the regime \( x < -\xi_l (x > \xi_R) \), we have \( X_l(x) \sim A_L(k) \) as well as \( X_{II}(x) \sim x/4 A_{II}(x) + \text{sgn}(x) \gamma_{-} A_{II}(x) \) with \( \gamma_{-} = (\eta_{-} - \eta_{L})/2 \). The asymptotic behavior of \( X \) for small \( k_x \) then reads

\[
X = \left[ C_{I}^{(0)} + k_x C_{I}^{(1)} \right] A_L + \left[ C_{II}^{(0)} + k_x C_{II}^{(1)} \right] \frac{x}{A_L} \text{sgn}(x) \gamma_{-} A_L + \cdots.
\]

(14)

On the other hand, the plane-wave solution (6) in the low-\( k_x \) regime behaves as

\[
X = \begin{cases} 
1 + r^{(0)} + k_x \{ ix \{1 - r^{(0)} + r^{(1)}\} + \cdots \} (x \ll -\xi_L), \\
\ell^{(0)} + k_x \{ i\ell \{1 - \ell^{(0)} + \ell^{(1)}\} + \cdots \} (\xi_R \ll x).
\end{cases}
\]

(15)

We compare the coefficients of \( x^n k_x^m \) in (14) with those in (15), where \( m, n \in \mathbb{Z} \). For \( x^1 k_x^0 \), we have \( C_{II}^{(0)} = 0 \), which results in the absence of \( X_{II}(x) \) in the low-momentum limit. For \( x^0 k_x^0 \) as well as \( x^1 k_x^1 \), we reach

\[
\lim_{k \to 0} t(k) = \frac{2 A_{L} A_{R}}{A_{L}^2 + A_{R}^2}, \quad \lim_{k \to 0} r(k) = \frac{A_{L}^2 - A_{R}^2}{A_{L}^2 + A_{R}^2},
\]

(16)

which result in (7). The result (16) also provides \( C_{I}^{(0)} \neq 0 \), so the excitation WF in the low-momentum limit is proportional to the condensate WF, \( X(x) \propto X_l(x) = \phi(x) \).

In the derivation of (7), the following two points are crucial:

i) The WF of the excitation in the low-momentum limit is proportional to the condensate WF,

ii) The healing length of the condensate is finite so that the condensate WF recovers the value in the spatially uniform system from the depleted value around the potential over a finite transient region.

The property i) seems to be anomalous since a WF obeying a Schrödinger equation generally vanishes in the presence of an external potential barrier in the low-momentum limit. However, i) may be linked to the fact that the spin wave is a NG mode. The Hamiltonian (1) is invariant under gauge transformation as well as spin space rotation since \( V_{\text{ext}} \) is a symmetry-preserving potential coupled to local density alone. When continuous symmetry is spontaneously broken, a corresponding excitation WF in the low-energy limit may not vanish even in a spatially inhomogeneous system so that it describes the NG mode.

The finite healing length ii), which offers well-defined tunneling problems of excitations in BECs, is characteristic of an interacting Bose system. This interaction yields finite compressibility in a BEC. When the compressibility and the healing length are finite, the condensate WF persists with finite and nonzero amplitudes even in the presence of the potential barrier. According to i), the WF of the NG mode at the low-momentum limit also does. This property leads to the nonzero transmission of the NG mode in the low-momentum limit.

**Scattering cross-section.** – We investigate the scattering cross-section of the transverse spin wave from the spherical potential \( V_{\text{ext}}(r) = V(r) \) that satisfies \( \lim_{r \to 0} r^2 V(r) = 0 \) as well as \( \lim_{r \to \infty} V(r) = 0 \). In the ground state, the condensate WF is also spherically symmetric: \( \phi(r) = \phi(r) \). The GP equation in dimensionless form is reduced to \( \mathcal{H}_r \phi(r) = 0 \) with

\[
\mathcal{H}_r \equiv -\frac{1}{2} \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) + V_{\text{eff}}(r),
\]

(17)

where \( V_{\text{eff}}(r) \equiv V(r) + \phi^2(r) - 1 \). The WF \( \phi_0(r) \) is expanded with respect to the partial waves as

\[
\phi_0(r) = \sum_{l=0}^{\infty} P_l(\cos \theta) R_l(r; k)
\]

(18)

with the Legendre polynomial \( P_l(x) \) and a relative angle \( \theta \) between \( r \) and the incident momentum.

The radial WF \( R_l(r; k) \) satisfies

\[
\mathcal{H}_r + \frac{l(l+1)}{2r^2} R_l(r; k) = \frac{k^2}{2} R_l(r; k),
\]

(19)

with \( E = k^2/2 \), showing the form

\[
R_l(r; k) \propto \left\{ \begin{array}{ll}
\frac{r^l}{j_l(kr)} - \tan \delta_l(k) n_l(kr) & (r \to 0), \\
\frac{r^l}{j_l(kr)} & (r \to \infty).
\end{array} \right.
\]

(20)

Here, \( j_l(kr) \) and \( n_l(kr) \) denote the spherical Bessel and Neumann functions, respectively. The phase shift \( \delta_l(k) \) of the partial wave relates to the scattering cross-section

\[
\sigma(k) = 4\pi \frac{1}{k^2} \sum_{l=0}^{\infty} (2l + 1) \sin^2 \delta_l(k).
\]

(21)

We fix the overall normalization factor of \( R_0(r; k) \) such that

\[
R_0(r = 0; k) = \phi(r = 0).
\]

(22)

If \( V_{\text{ext}}(r) \) were a generic potential, the phase shift would behave as \( \delta_l(k) \propto k^{2l+1} \) for small \( k \) and the cross-section would be given by

\[
\sigma(k) \sim \frac{1}{k^2} \sin^2 \delta_0(k) \sim \text{const.(}> 0)
\]

(23)
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Table 1: Scattering properties of NG modes in the ferromagnetic phase of the spin-1 spinor BEC.

| NG mode   | type of NG mode | refraction | impedance | scattering cross-section |
|-----------|-----------------|------------|-----------|-------------------------|
| Bogoliubov | II              | Neill’s law | | $\sigma_0 \propto k^4$ |
| transverse spin | II              | $\times$ | $|\langle \Psi \rangle| / |M|$ | $\sigma_0 \propto k^4$ |

On the other hand, $j_0(kr)$ and $n_0(kr)$ at $kr \ll 1$ are given by $j_0(kr) \approx 1 - k^2r^2/6$ and $n_0(kr) \approx -1/(kr)$, respectively. We then obtain (24) from (20), (21), as well as (32), where the phase shift of the $s$-wave ($l = 0$) is given by

$$\tan \delta_0 \approx \frac{Ck^3}{1 + Bk^2} \propto k^3.$$  \hspace{1cm} (33)

The Rayleigh scattering type $\sigma(k) \propto k^4$ is characteristic of sound, light, as well as the Bogoliubov mode in the long-wavelength limit of a BEC [5,6], all of which have a linear dispersion relation $E \propto k$. As compared with (23), the result $\sigma(k) \propto k^4$ for low-energy scattering of the spin mode with a quadratic dispersion relation $E \propto k^2$ may be regarded as an anomalous power law. In ultracold gases, this scattering property would be observed by using a single ion trapped inside a BEC [22].

Discussion. – On the basis of these results, we suggest two hypotheses for NG modes in the presence of symmetry-preserving potentials:

a) In a system whose continuous symmetry is spontaneously broken, impedance of the corresponding NG mode in the transmission coefficient is given by the amplitude of its order parameter (e.g., $|\langle \Psi \rangle|$ and $|M|$).

b) The $s$-wave scattering cross-section shows an anomalous power law of the wave number $k$, such as a Rayleigh-scattering type proportional to the fourth power of $k$.

Table 1 summarizes the scattering properties of two NG modes in a BEC, the ferromagnetic spin wave and the Bogoliubov mode. In the Bogoliubov excitation, the transmission coefficient in the low-energy limit is given by [7]

$$\lim_{k \to 0} T = \frac{4A_L A_R \cos \theta_L \cos \theta_R}{(A_L \cos \theta_L + A_R \cos \theta_R)^2},$$  \hspace{1cm} (34)

and the scattering cross-section of the $s$-wave ($l = 0$) is a Rayleigh scattering type, $\sigma \propto k^4$. A unified perspective on (7) and (34) is that the modulus of magnetization $M_\nu$ and that of the condensate WF $A_\nu$ play the role of “impedance”. From (24) and results of refs. [5,6], the Rayleigh scattering is also common to both the spin wave and the Bogoliubov mode. These are derived from

\footnote{In ref. [7], the transmission coefficient $T$ is written in terms of the speed of sound $c_\nu$. Because the strength of the contact interaction is independent of space, this coupling constant is reduced from expression of $T$ and we may write $T$ in terms of the amplitude of the condensate WF $A_\nu$.}
the facts i) and ii), the derivation of which is straightforward and general without any other specific conditions. These properties have been derived in NG modes in BECs, so further investigations will be needed in other systems to examine the hypotheses a) and b).

We have discussed high tunneling probability through the nonmagnetic barrier, which stems from the degeneracy of the ground state induced by the spontaneous breaking of the continuous symmetry. Once a magnetic barrier is induced in a ferromagnetic or polar phase of a spin-1 BEC, the anomalous tunneling phenomenon of spin modes vanishes in the low-momentum limit [18,23]. (Also once a barrier that couples to the gauge is artificially induced, the anomalous tunneling of the Bogoliubov mode also vanishes in the low-momentum limit [18].) Since those tunneling properties for a barrier preserving/breaking symmetry are expected to be common to NG modes, a magnetic insulator will also offer a playground studying anomalous tunneling of the spin waves.

Indeed, in classical Heisenberg chain, the ferromagnetic spin wave in the low-energy limit transmits perfectly through a nonmagnetic inhomogeneity, while it undergoes total reflection at the local magnetic field that breaks the spin rotation symmetry [24]. A spin wave in antiferromagnets also shows similar tunneling properties [25]. Since the tunneling character is common to both the BEC and the classical Heisenberg chain, we may suggest the hypothesis that in the classical Heisenberg chain, the scattering cross-section of the spin wave shows an anomalous power law in the long-wavelength limit, such as the Rayleigh type (24), for a doped nonmagnetic impurity (e.g., a closed shell atom) or for a defect like a mechanical depression of the magnetization. On the other hand, for a doped magnetic impurity (e.g., an atom classified as transition metals or rare earth elements), a usual scattering (23) may be dominant. Based on this hypothesis, the propagation of the spin waves may be manipulated with impurities and defects adequately disposed. Without magnetic impurities, the scattering of spin waves from nonmagnetic impurities would be suppressed for lower momentum, and spin waves would be transported over longer distances at lower energy. Although it has been reported that spin waves in magnetic systems transmits a barrier induced by an inhomogeneous magnetic field [26,27], this high tunneling probability would be due to the nonlocal magnetic dipole interaction of spins [26], which is different from the anomalous tunneling phenomenon.

Conclusion. — The transverse spin wave in a ferromagnetic spin-1 Bose-Einstein condensate (BEC) obeys a Schrödinger-type equation; however, but surprisingly, the effects of an external potential on its transmission coefficient $T$ and on its scattering cross-section $\sigma$ vanish in the low-energy limit. Its transmission coefficient $T$ is described by the modulus of an order parameter. Its momentum $p$-dependence of $\sigma$ exhibits a Rayleigh scattering type ($\sigma \propto p^4$). These properties are common between two types of Nambu-Goldstone modes: this ferromagnetic spin wave and the Bogoliubov mode. These effects are proved for a localized potential of arbitrary shape. Interestingly, these scattering problems will connect to many other systems whose continuous symmetry is spontaneously broken.

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