The Domination Number of Generalized Petersen Graphs with a Faulty Vertex

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Abstract. In this paper, we investigate the domination number of generalized Petersen graphs \(P(n, 2)\) when there is a faulty vertex. Denote by \(\gamma(P(n, 2))\) the domination number of \(P(n, 2)\) and \(\gamma(P_r(n, 2))\) the domination number of \(P(n, 2)\) with a faulty vertex \(u_r\). We show that \(\gamma(P_r(n, 2)) = \gamma(P(n, 2)) - 1\) when \(n = 5k + 1\) or \(5k + 2\) and \(\gamma(P_r(n, 2)) = \gamma(P(n, 2))\) for the other cases.

Keywords: Domination; Domination alternation; generalized Petersen Graph.

1 Introduction

A graph \(G\) is an ordered pair \((V(G), E(G))\) consisting of a set \(V(G)\) of vertices and a set \(E(G)\) of edges. When the context is clear, \(V(G)\) and \(E(G)\) are simply written as \(V\) and \(E\), respectively. The open neighborhood of vertex \(v \in V\) is the set \(N(v) = \{u \in V \mid uv \in E\}\). The closed neighborhood of vertex \(v \in V\) is the set \(N[v] = N(v) \cup \{v\}\). For a set \(S\) of vertices, \(N[S] = \bigcup_{v \in S} N[v]\). A set \(S \subseteq V\) is a dominating set of \(G\) if \(N[S] = V\) [3]. The domination number of \(G\), denoted by \(\gamma(G)\), is the cardinality of a minimum dominating set.

For two natural numbers \(n\) and \(k\) with \(n \geq 3\) and \(1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor\), the generalized Petersen graph \(P(n, k)\) is a graph on \(2n\) vertices with \(V(P(n, k)) = \{u_i, v_i \mid 1 \leq i \leq n\}\) and \(E(P(n, k)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} \mid 1 \leq i \leq n\}\) with subscripts modulo \(n\) [4,5,10]. Hereafter, all operations on the subscripts of vertices are taken modulo \(n\) unless stated otherwise.

In [2], Behzad, Behzad, and Praeger showed that \(\gamma(P(n, 2)) \leq \left\lfloor \frac{3n}{5} \right\rfloor\) for odd \(n \geq 3\) and conjectured that \(\left\lfloor \frac{3n}{5} \right\rfloor\) is exactly the domination number of \(P(n, 2)\).

In [6], Ebrahimi, Jahanbakht, and Mahmoodian (independently, Yan, Kang, and Xu [11] and Fu, Yang, and Jiang [8]) affirmed that \(\gamma(P(n, 2)) = \left\lfloor \frac{3n}{5} \right\rfloor\) for \(n \geq 3\).

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In this paper, we are concerned with \( \gamma(P_f(n, 2)) \) when there is a faulty vertex \( u_f \) in \( P(n, 2) \), where \( \gamma(P_f(n, 2)) \) denotes the domination number of \( P(n, 2) \) with faulty vertex \( u_f \), i.e., \( u_f \) is removed from \( P(n, 2) \). Thus a faulty vertex cannot be chosen as a vertex in the dominating set. We shall show that, for \( n \geq 3 \),

\[
\gamma(P_f(n, 2)) = \begin{cases} 
\gamma(P(n, 2)) - 1 & \text{if } n = 5k + 1 \text{ or } 5k + 2 \\
\gamma(P(n, 2)) & \text{otherwise.}
\end{cases}
\]

The alteration domination number of \( G \), denoted by \( \mu(G) \), is the minimum number of points whose removal increases or decreases the domination number of \( G \) [1]. The bondage number of \( G \), denoted by \( b(G) \), is the minimum number of edges whose removal from \( G \) results in a graph with larger domination number [7]. It can be regarded as the fault tolerance problem when removing vertices or edges from a graph. Fault tolerance is also an important issue on engineering [9]. This motivates us to study the domination number of \( P(n, 2) \) with a faulty vertex. By our result, we can find that the lower and upper bounds of \( \mu(P(n, 2)) \) are as follows: \( \mu(P(n, 2)) = 1 \) if \( n = 5k + 1 \) or \( 5k + 2 \); otherwise, \( \mu(P(n, 2)) \geq 2 \). Moreover, we can find that \( 2 \leq b(P(n, 2)) \leq 3 \) if \( n = 5k, 5k + 3 \) or \( 5k + 4 \).

This paper is organized as follows. In Section 2, we review some preliminaries of dominating sets in generalized Petersen graphs. In Section 3, some properties are introduced when there is a faulty vertex in \( P(n, 2) \). Section 4 contains our main results. We conclude in Section 5.

2 Preliminaries

In this paper, we follow the terminology of [6]. However, for clarity, we introduce some of them as follows.

Let \( P(n, 2) - u \) denote the resulting graph after \( u \) is removed from \( P(n, 2) \). In particular, \( P(n, 2) - u_f \) is denoted by \( P_f(n, 2) \) where \( u_f \), for some \( 1 \leq f \leq n \), is the faulty vertex in \( P(n, 2) \). We also use \( S + u \) and \( S - u \) to denote adding an element \( u \) to a set \( S \) and removing an element \( u \) from a set \( S \), respectively. In the rest of this paper, \( S \) always stands for a domination set of \( P_f(n, 2) \). A minimum dominating set of \( G \) is called a \( \gamma(G) \)-set. When the graph \( G \) is clear from the context, \( \gamma(G) \)-set is written as \( \gamma \)-set. A block of \( P(n, 2) \) is an induced subgraph of 5 consecutive pairs of vertices (see Figure 1). Denote by \( B_i \) if a block of \( P(n, 2) \) is centered at \( u_i \) and \( v_i \). When there is no possible ambiguity, \( B_i \) and \( V(B_i) \) are used interchangeably. The vertices of \( B_i - u_i \) can be partitioned into \( R_i = \{v_{i+1}, u_{i+2}, v_{i+3}\}, L_i = \{v_{i-1}, u_{i-2}, v_{i-3}\}, \) and \( M_i = \{u_{i-1}, v_i, u_{i+1}\} \). Let \( N^+(R_i) = N[R_i] \setminus B_i = \{v_{i+3}, u_{i+3}, v_{i+4}\}, N^+(L_i) = N[L_i] \setminus B_i = \{v_{i-3}, u_{i-3}, v_{i-4}\}, \) and \( \gamma_i(S) = |B_i \cap S| \). When the context is clear, \( \gamma_i(S) \) is written as \( \gamma_i \). Let \( F = \{f - 2, f - 1, f, f + 1, f + 2\} \) which contains the indices in \( B_f \).
Theorem 1 ([6,8,11]). For \( n \geq 3 \), \( \gamma(P(n, 2)) = \left\lceil \frac{3n}{5} \right\rceil \).

Corollary 1. For \( n \geq 3 \), \( \left\lceil \frac{3n}{5} \right\rceil - 1 \leq \gamma(P_f(n, 2)) \leq \left\lceil \frac{3n}{5} \right\rceil \).

Proof. Let \( S \) be a \( \gamma(P(n, 2)) \)-set. If \( u_f \not\in S \), then \( S \) is also a dominating set of \( P_f(n, 2) \) and \( \gamma(P_f(n, 2)) \leq \left\lceil \frac{3n}{5} \right\rceil \). For the case where \( u_f \in S \), since \( \gamma(P_f(n, 2)) \leq \left\lceil \frac{3n}{5} \right\rceil \), by symmetry, we can relabel the subscripts of the vertices in \( P(n, 2) \) but not \( S \) such that \( u_f \not\in S \). Thus, in this case, \( \gamma(P_f(n, 2)) \leq \left\lceil \frac{3n}{5} \right\rceil \).

To prove that \( \left\lceil \frac{3n}{5} \right\rceil - 1 \leq \gamma(P_f(n, 2)) \), suppose to the contrary that there exists a dominating set \( S \) of \( P_f(n, 2) \) with \( |S| = \left\lceil \frac{3n}{5} \right\rceil - 2 \). It is clear that \( S \cup \{u_f\} \) is also a dominating set of \( P(n, 2) \) whose cardinality is \( \left\lceil \frac{3n}{5} \right\rceil - 1 \), a contradiction. This completes the proof. \( \square \)

Lemma 1. Let \( S \) be a minimum dominating set in \( P_f(n, 2) \) and assume that the vertex \( u_f \) in the corresponding graph \( P(n, 2) \) has at least one neighbor in \( S \). Then \( |S| = \left\lceil \frac{3n}{5} \right\rceil \).

Proof. Since \( N(u_f) \cap S \neq \emptyset \), \( S \) is also a dominating set of \( P(n, 2) \). This implies that \( \gamma(P(n, 2)) \leq |S| \). By Theorem 1 there exists a \( \gamma(P(n, 2)) \)-set, say \( T \), with \( u_f \not\in T \). Clearly, \( T \) is also a dominating set of \( P_f(n, 2) \). Thus \( |S| \leq \gamma(P(n, 2)) \). This further implies that \( |S| = \gamma(P(n, 2)) \). By Theorem 1, the lemma follows. \( \square \)

3 Some properties when there is a faulty vertex

In this section, we introduce some properties of \( B_f \) in \( P(n, 2) \), where \( u_f \) is a faulty vertex. By Lemma 1, it remains to consider the case where \( N(u_f) \cap S = \emptyset \). Thus, in the rest of this paper, we assume that \( S \) is a minimum dominating set under the condition that \( N(u_f) \cap S = \emptyset \) unless stated otherwise. Thus, in this case, \( M_f \cap S = \emptyset \) which implies \( B_f \cap S \subseteq L_f \cup R_f \). Hereafter, when we say that a vertex \( x \) is dominated with respect to \( S \), then \( x \) is either in \( S \) or \( x \) is adjacent to some vertex in \( S \).
Proposition 1. Assume that $S$ is a dominating set of graph $G$ and $S' = S - x + y$, where $x \in S$ and $y \notin S$. If all vertices in $N[x]$ are dominated by $S'$, then $S'$ is also a dominating set of $G$ with $|S| = |S'|$.

Lemma 2. If there exists $u_i \notin S$ and $u_f \notin M_i$ for some $1 \leq i \leq n$, then $\gamma_1 \geq 3$.

Proof. Since $N[M_i] = B_i$, the vertices in $M_i$ can only be dominated by some vertices in $B_i$. Note that any two vertices in $M_i$ have no neighbor in common except $u_i$. However, $u_i \notin S$ and $u_f \notin M_i$. This results in $|N[M_i] \cap S| \geq 3$. Thus $\gamma_1 \geq 3$ and the lemma follows.

Lemma 3. Assume that there exists a minimum dominating set $S$ such that $N(u_f) \cap S = \emptyset$. Then there exists a minimum dominating set $S'$ such that $N(u_f) \cap S' = \emptyset$ and $\gamma_1 \geq 2$ for all $1 \leq i \leq n$.

Proof. If $\gamma_1 > 1$ for $1 \leq i \leq n$ in $S$, then we are done. Thus we assume that there exists $\gamma_1 = 1$ for some $i \neq f$. By Lemma 2, $u_i \notin S$; otherwise, $\gamma_1 \geq 2$. We may assume that none of $u_{i+2}$ and $u_{i+3}$ is $u_f$; otherwise, reverse $B_i$ so that $L_i$ and $R_i$ are interchanged. This further implies that all vertices in $R_i$ must be dominated by some vertices in $N^+(R_i)$. Since any two vertices in $R_i$ have no common neighbor in $N^+(R_i)$, all vertices in $N^+(R_i)$ must be in $S$ so that the vertices in $R_i$ can be dominated. By Proposition 1 the set $S' = S - u_{i+3} + u_{i+2}$ is also a minimum dominating set under the condition that $N(u_f) \cap S = \emptyset$ since the vertices in $N[u_{i+3}]$ are still dominated by the vertices in $S'$. Note that $\gamma_j(S') = \gamma_j(S)$ for $1 \leq j \leq n$ except $j \in \{i, i+3\}$. It is easy to verify that $\gamma_1(S') = 2$ and $\gamma_{i+3}(S') \geq 2$. Moreover, $S'$ has one less elements in $\{j | \gamma_j(S') = 1, 1 \leq j \leq n\}$ than that of $S$. By applying the above process repeatedly until the set $\{j | \gamma_j(S') = 1, 1 \leq j \leq n\}$ becomes empty, this results in a minimum dominating set with $\gamma_1 \geq 2$ for all $1 \leq i \leq n$. This completes the proof.

Definition 1. A minimum dominating set $S$ is called a Type I set if $\gamma_1 \geq 2$ for $1 \leq i \leq n$. For a Type I set $S$, the cardinality of the set $\{i | \gamma_i(S) = 2, u_i \notin B_i\}$ is called its couple number.

Proposition 2. Assume that $S$ is a Type I set. If $\gamma_f = 3$ and $N(u_f) \cap S = \emptyset$, then either $|L_f \cap S| = 1$ or $|R_f \cap S| = 1$.

Proof. Since $\gamma_f = 3$ and $N(u_f) \cap S = \emptyset$, $\gamma_f = |L_f \cap S| + |R_f \cap S| = 3$. To show that either $|L_f \cap S| = 1$ or $|R_f \cap S| = 1$, it is equivalent to showing that $|L_f \cap S| = 0$ or $|R_f \cap S| = 0$ is impossible. Suppose to the contrary that $L_f \cap S = \emptyset$ (or $R_f \cap S = \emptyset$). It can be found that vertex $u_{f-1}$ (or $u_{f+1}$) is not dominated, a contradiction.

By Proposition 2, in the rest of this paper, we only consider the case where $|L_f \cap S| = 1$ and $|R_f \cap S| = 2$ when $S$ is a Type I set with $\gamma_f = 3$. For the case where $|L_f \cap S| = 2$ and $|R_f \cap S| = 1$, we can reverse the generalized Petersen graph so that it yields $|L_f \cap S| = 1$ and $|R_f \cap S| = 2$.

In total, there are nine possible cases for the vertices in $B_f \cap S$ when $N(u_f) \cap S = \emptyset$, $\gamma_f = 3$, and $|L_f \cap S| = 1$. However, only four of them are feasible (see
Figures 2(a)-(d)). For example, if \(|L_f \cap S| = 1\) and \(v_{f-2} \in S\), then \(u_{f-1}\) is not dominated and it is an infeasible case.

For the case in Figure 2(b), by Proposition 1, \(S_1 = S - u_{f+2} + v_{f+1}\) is still a Type I set. Note that the pattern of \(S_1 \cap B_f\) is exactly the case in Figure 2(a). For the case in Figure 2(c), by Proposition 1, \(S_2 = S - v_{f-1} + u_{f-2}\) is also a Type I set. Furthermore, the pattern of \(S_2 \cap B_f\) is also exactly the case in Figure 2(a). For the case in Figure 2(d), by Proposition 1, \(S_3 = (S - u_{f-3} + u_{f-2})\) is also a Type I set while \(\gamma_f = 4\). We shall define a Type III set later for this case. Henceforth, if \(S\) is a Type I set with \(N(u_r) \cap S = \emptyset\) and \(\gamma_f = 3\), then we may assume that it is a Type II set which is defined as follows.

**Definition 2.** A Type I set \(S\) with \(\gamma_f = 3\) is called a Type II set if \(B_f \cap S = \{u_{f-2}, v_{f+1}, v_{f+2}\}\) (see Figure 2(a)).

Now we consider the case where \(N(u_r) \cap S = \emptyset\) and \(\gamma_f = 4\). By symmetry, we only need to consider the cases where \(|L_f \cap S| = 1\) and \(|L_f \cap S| = 2\). There are only seven feasible combinations (see Figures 3(a)-3(g)). Every minimum dominating set \(S\) in the cases of Figures 3(e)-3(g) can be transformed to a Type II set. That is, set \(S_1 = S - u_{f+2} + v_{f+3}\) in Figure 3(e), set \(S_2 = S - v_{f-2} + v_{f-4}\) in Figure 3(f), and set \(S_3 = S - v_{f+1} + v_{f+3}\) in Figure 3(g). Note that \(N[u_{f+2}], N[v_{f-2}],\) and \(N[v_{f+1}]\) are still dominated by the vertices in \(S_1, S_2,\) and \(S_3\), respectively. Thus, by Proposition 1, \(S_1, S_2,\) and \(S_3\) are Type II sets.

**Definition 3.** A Type I set \(S\) with \(\gamma_f = 4\) is called a Type III set if \(B_f \cap S\) is equal to one of the following four sets: \(\{v_{f-1}, v_{f+1}, u_{f+2}, v_{f+2}\}\), \(\{u_{f-2}, v_{f-2}, v_{f+2}, v_{f+2}\}\), \(\{v_{f-2}, v_{f-1}, v_{f+1}, v_{f+2}\}\), and \(\{u_{f-2}, v_{f-2}, v_{f+1}, u_{f+2}\}\) (see Figures 2(a), (b), (c), (d)), or precisely, are called Type III(a)-III(d) sets, respectively.

Henceforth, if \(S\) is a Type I set with \(\gamma_f = 4\) and \(N(u_r) \cap S = \emptyset\), then we may assume that it is a Type III set.
Fig. 3. All feasible $B_i \cap S$ when $N(u_i) \cap S = \emptyset$ and $\gamma_f = 4$.

**Lemma 4.** Assume that $S$ is a Type II (or III) set with couple number $c$. If there exists $\gamma_i = 2$ for $i \in F$ and $S \cap \{v_{i-1}, v_i, v_{i+1}\} = \emptyset$, then there exists a Type II (or III) set with couple number $c - 1$.

**Proof.** Clearly, by Lemma 2, $u_i \in S$. Since $S \cap \{v_{i-1}, v_i, v_{i+1}\} = \emptyset$, the other vertex of $B_i$ in $S$, say $y$, must be in $\{u_{i-2}, v_{i-2}, u_{i-1}\} \cup \{u_{i+2}, v_{i+2}, u_{i+1}\}$. We only consider the case where $y \in \{u_{i-2}, v_{i-2}, u_{i-1}\}$ (see Figure 4). The other case is similar. Since $R_i \cap S = \emptyset$, by using a similar argument as in Lemma 3, $N^+(R_i) \subset S$. Clearly, at least one vertex in $B_{i+5} \setminus N^+(R_i)$ must be in $S$ so that $u_{i+5}$ and $u_{i+6}$ are dominated. This results in $\gamma_{i+5} \geq 4$ no matter whether $i + 5$ or $i + 6$ is equal to $f$ or not. Now let $S' = S - u_{i+3} + u_{i+2}$. It is easy to check that $\gamma_j(S') = \gamma_j(S)$ for $1 \leq j \leq n$ except $j \in \{i, i + 5\}$. However, $\gamma_i(S') = 3$ and $\gamma_{i+5}(S') \geq 3$. Thus the couple number of $S'$ is one less than that of $S$. This completes the proof. \hfill \Box
Remark 1. Note that if \( i = f - 2 \) (respectively, \( i = f + 2 \)) in Lemma 4, then \( u_i \in R_i \) (respectively, \( u_i \in L_i \)). This yields \( u_{i+2} = u_f \) (respectively, \( u_{i-2} = u_f \)) which is removed under our assumption. Thus we cannot obtain a Type II (or III) set \( S' \) by setting \( S' = S - u_{i+3} + u_{i+2} \) (respectively, \( S' = S - u_{i-3} + u_{i-2} \)). For the case where \( i \in \{f - 1, f + 1\} \), \( u_i \) even might not be in \( S \) since \( u_f \) is either \( u_{i+1} \) or \( u_{i-1} \) which is removed.

Hereafter, we assume that \( S \) is with the smallest couple number if \( S \) is a Type II or III set.

Definition 4. Let \( S \) be a Type II (or III) set with the smallest couple number. A vertex \( u_i \) for \( i \not\in F \) is called a pseudo-couple vertex with respect to \( S \) if \( S \cap \{v_{i-1}, v_i, v_{i+1}\} = \emptyset \).

Notice that if \( u_i \) is a pseudo-couple vertex, then \( \gamma_i \geq 3 \) when \( S \) is a Type II (or III) set with the smallest couple number.

Lemma 5. Assume that \( S \) is a Type II or III set with the smallest couple number. If there exists \( \gamma_i = 2 \) for \( i \not\in F \), then either \( \gamma_{i+2} \geq 4 \) or \( \gamma_{i-2} \geq 4 \).

Proof. Since \( \gamma_i = 2 \) and \( S \) is with the smallest couple number, by Lemma 4, \( u_i \) cannot be a pseudo-couple vertex and a vertex \( x \in \{v_i, v_{i-1}, v_{i+1}\} \) must be in \( S \). We consider the following three cases.

Case 1. \( x = v_i \).

It is clear that \( v_{i+1} \) and \( u_{i+2} \) must be dominated by \( v_{i+3} \) and \( u_{i+3} \), respectively. Similarly, \( v_{i-1} \) and \( u_{i-2} \) must be dominated by \( v_{i-3} \) and \( u_{i-3} \), respectively (see Figure 5(a)). Thus both \( \gamma_{i+2} \) and \( \gamma_{i-2} \) are greater than or equal to 4.

Case 2. \( x = v_{l+1} \).

In this case, \( u_{i+3} \) and \( v_{i+4} \) must be in \( S \) so that \( u_{i+2} \) and \( v_{i+2} \) are dominated. Thus \( \gamma_{i+2} \geq 4 \) (see Figure 5(b)).

Case 3. \( x = v_{l-1} \).

By using a similar argument as in Case 2, the case holds. This completes the proof.

Lemma 6. Assume that \( S \) is a Type II or III set with the smallest couple number. If there exist \( \gamma_i = \gamma_j = 2 \) for distinct \( i, j \not\in F \), then \( |i - j| \neq 4 \).
Proof. Suppose to the contrary that $|i - j| = 4$. That is, $j$ is either equal to $i + 4$ or $i - 4$. We only consider the case where $j = i + 4$. The other case is similar. By Lemma 4, $u_i$ cannot be a pseudo-couple vertex and a vertex $x \in \{v_i, v_{i-1}, v_{i+1}\}$ must be in $S$. Analogous to Lemma 5, we consider the following three cases.

Case 1. $x = v_i$.

We can find that $u_{i+3}, v_{i+3} \in B_{i+4} \cap S$ (see Figure 5(a)). If $u_{i+4}$ is also in $S$, then $\gamma_j = \gamma_{i+4} \geq 3$, a contradiction. If $u_{i+4}$ is not in $S$, then, by Lemma 2, $\gamma_j \geq 3$, a contradiction too.

Case 2. $x = v_{i+1}$.

In this case, $u_{i+3}, v_{i+4} \in B_{i+4} \cap S$ (see Figure 5(b)). By using a similar argument as in Case 1, we can find that $\gamma_j \geq 3$. Thus this case is also impossible.

Case 3. $x = v_{i-1}$.

In this case, all vertices in $N^+(R_i)$ must be in $S$. This contradicts that $\gamma_j = 2$. This concludes the proof of this lemma. \hfill $\square$

4 Main results

By Lemma 2, $\gamma_f \geq 3$. In the following, we investigate the cardinalities of minimum dominating sets of $P_f(n, 2)$ under all possible values of $\gamma_f$.

By Lemma 5, if there exists $\gamma_i = 2$ for $i \notin F$, then either $\gamma_{i+2} \geq 4$ or $\gamma_{i-2} \geq 4$ must hold. This yields $(\gamma_i + \gamma_{i+2})/2 \geq 3$ or $(\gamma_i + \gamma_{i-2})/2 \geq 3$. By Lemma 6, if $\gamma_i = 2$, then both $\gamma_{i-4}$ and $\gamma_{i+4}$ are greater than or equal to 3. This means that no two distinct $\gamma_i = \gamma_j = 2$ use the same $\gamma_k$ to obtain the average number 3. This ensures that the average number of $\gamma_i$ is greater than or equal to 3 when $i \notin F$. To prove the lower bound of $|S|$, our main idea is to count the number of $\gamma_i = 2$ for $i \in F$, say $x$, which cannot gain support from any other $\gamma_j$ so that their average is greater than or equal to 3. This yields $5|S| = \sum_{i=1}^{n} \gamma_i \geq 3n - x$.

Lemma 7. If $S$ is a Type III set with the smallest couple number, then $|S| \geq \left\lceil \frac{3n}{5} \right\rceil$. 

8
Proof. First we consider the case where $S$ is a Type III(a), III(b), or III(c) set. By inspection (see Figure 3), it can be found that $\gamma_i \geq 3$ for $i \in F$ if $S$ is a Type III(a) or III(b) set. For the case where $S$ is a Type III(c) set, one of the elements in $\{v_{r-3}, u_{r-3}, u_{r-4}\}$ (respectively, $\{v_{r+3}, u_{r+3}, u_{r+4}\}$) must be in $S$ so that $u_{r-3}$ (respectively, $u_{r+3}$) is dominated. Thus $\gamma_i \geq 3$ for $i \in F$ if $S$ is a Type III(c) set. This also implies that, in those three types of dominating sets, if there exists $\gamma = 2$ in $S$, then, by Lemmas 5 and 6, either $(\gamma_i + \gamma_{i+2})/2 \geq 3$ or $(\gamma_i + \gamma_{i-2})/2 \geq 3$ must hold. As a consequence, $\sum_{i=1}^{n} \gamma_i = 5|S| \geq 3n$. This yields $|S| \geq \left\lceil \frac{3n}{5} \right\rceil$.

To complete the proof, it remains to consider the case where $S$ is a Type III(d) set. According to the possible values of $\gamma_f-2$, we consider the following two cases.

Case 1. $\gamma_f-2 = 2$.

Since $\gamma_f-2 = 2$, vertices $u_{r-5}$ and $v_{r-5}$ must be in $S$ so that $u_{r-4}$ and $v_{r-3}$ are dominated (see Figure 5(a)). This further implies that both $\gamma_{r-3}$ and $\gamma_{r-4}$ are greater than or equal to 4. Note that if $\gamma_{r-5} = 2$, then $u_{r-4}$ and $v_{r-8}$ must be in $S$ so that $u_{r-7}$ and $v_{r-6}$ are dominated. Accordingly, $\gamma_{r-7} \geq 4$. Thus $\gamma_{r-5}$ can gain support from $\gamma_{r-7}$ such that $(\gamma_{r-5} + \gamma_{r-7})/2 \geq 3$. We can find that the minimum values of $\gamma_{f-4}, \gamma_{f-3}, \ldots, \gamma_{f+2}$ are 4, 4, 2, 3, 4, 2, and 2, respectively. Thus every $\gamma_i = 2$ in $S$ can gain support from a vertex $\gamma_j = 4$. Thus $\sum_{i=1}^{n} \gamma_i = 5|S| \geq 3n$ which yields $|S| \geq \left\lceil \frac{3n}{5} \right\rceil$. Thus this case holds.

Case 2. $\gamma_f-2 \geq 3$.

If $u_{r-3} \in S$ or $v_{r-3} \in S$, then, after setting $S' = S - u_{r-2} + v_{r-1}$ and $S'' = S' - v_{r+1} + v_{r-3}, S''$ becomes a Type II set which will be considered in Lemma 9. We may assume that $u_{r-3}, v_{r-3} \notin S$ and either $u_{r-4}$ or $v_{r-4}$ is in $S$ (see Figure 5(b)). Note that, in this case, $v_{r-5}$ must be in $S$ so that $v_{r-3}$ is dominated. Thus the minimum values of $\gamma_{f-4}, \gamma_{f-3}, \ldots, \gamma_{f+2}$ are 4, 4, 3, 3, 4, 2, and 2, respectively. Clearly, their average is greater than or equal to 3. Note that one of the vertices in $N[u_{r-5}]$ must be in $S$ so that $u_{r-6}$ is dominated. Thus both $\gamma_{f-5}$ and $\gamma_{f-6}$ are greater than or equal to 3. This ensures that they will not gain support from $\gamma_{f-4}$ and $\gamma_{f-3}$ on computing the average value 3. Hence $\sum_{i=1}^{n} \gamma_i = 5|S| \geq 3n$ and $|S| \geq \left\lceil \frac{3n}{5} \right\rceil$. This completes the proof. \hfill \Box

Corollary 2. If $S$ is a minimum dominating set with the smallest couple number under the condition that $N(u_f) \cap S = \emptyset$ and $\gamma_f \geq 4$, then $|S| \geq \left\lceil \frac{3n}{5} \right\rceil$.

Proof. Since the case where $S$ is a Type III set with the smallest couple number is a special case of $\gamma_f \geq 4$, by Lemma 7, this corollary follows. \hfill \Box

It remains to investigate lower bounds for type II sets. By using a similar classification in [6], we consider the following five classes: $n = 5k, 5k + 1, 5k + 2, 5k + 3$, and $5k + 4$.

Lemma 8. For $n = 5k$ or $5k + 3$, if $S$ is a Type II set, then $|S| \geq \left\lceil \frac{3n}{5} \right\rceil$.

Proof. Let $S$ be a Type II set with the smallest couple number. By inspection on Figure 2(a), only the elements in $\{\gamma_{r-2}, \gamma_{r-1}, \gamma_{r+1}\}$ are possibly equal to 2 and
all other $\gamma_i \geq 3$ after gaining support. Note that $\gamma_{i+2} \geq 3$ since $N[u_{i+3}] \cap S \neq \emptyset$. This yields $\sum_{i=1}^n \gamma_i = 5|S| \geq 3n - 3$. For $n = 5k$, $|S| \geq \left\lceil \frac{3n-3}{5} \right\rceil = \left\lceil \frac{15k-3}{5} \right\rceil = 3k - \frac{3}{5} = 3k = \left\lceil \frac{3n}{5} \right\rceil$, and $|S| \geq \left\lceil 3k + \frac{6}{5} \right\rceil = 3k + 2 = \left\lceil \frac{3n}{5} \right\rceil$ for $n = 5k + 3$. This completes the proof. \hfill \square

**Definition 5.** Let $S$ be a dominating set of $P_f(n, 2)$. A block $B_i$ is called a self-contained block if $B_i \cap S = \{u_{i-2}, v_i, v_{i+1}\}$ (see Figure 7(a)).

**Proposition 3.** If both $B_i$ and $B_{i-5}$ are self-contained blocks, then $\gamma_x = 3$ for $i - 5 \leq x \leq i$.

**Proof.** By inspection (see Figure 7(b)), the proposition follows. \hfill \square

**Lemma 9.** For $n = 5k + 4$ and $\gamma_f = 3$, if $S$ is a Type II set and any two of $\gamma_{f-2}$, $\gamma_{f-1}$ and $\gamma_{f+1}$ are greater than 2, then $|S| \geq \left\lceil \frac{3n}{5} \right\rceil$.  

10
Proof. Analogous to Lemma 8, only the elements in \( \{ \gamma_{f-2}, \gamma_{f-1}, \gamma_{f+1} \} \) are possibly equal to 2 and all other \( \gamma_i \geq 3 \) after gaining support. If any two of \( \gamma_{f-2}, \gamma_{f-1} \) and \( \gamma_{f+1} \) are greater than 2, then \( \sum_{i=1}^{n} \gamma_i = 5|S| \geq 3n - 1 \). By replacing \( n \) by \( 5k + 4 \), this yields \( |S| \geq 3k + \frac{11}{5} \) = 3k + 3. Clearly, \( \left\lceil \frac{3n}{2} \right\rceil = 3k + \frac{12}{5} \) = 3k + 3 when \( n = 5k + 4 \). Thus \( |S| \geq \left\lceil \frac{3n}{2} \right\rceil \). This completes the proof. \( \square \)

Lemma 10. For \( n = 5k + 4 \), if \( S \) is a Type II set, then \( |S| \geq \left\lceil \frac{3n}{2} \right\rceil \).

Proof. Suppose to the contrary that \( |S| < \left\lceil \frac{3n}{2} \right\rceil \). We claim that exactly one of the vertices in \( \{ v_{f-3}, u_{f-3}, v_{f-4}, u_{f-4} \} \) is in \( S \). We argue the claim by contradiction and assume that there are two vertices in \( \{ v_{f-3}, u_{f-3}, v_{f-4}, u_{f-4} \} \cap S \). In this case, if one of \( v_{f-3} \) and \( u_{f-3} \) and one of \( v_{f-4} \) and \( u_{f-4} \) are in \( S \), then \( \gamma_{f-1} \geq 3 \) and \( \gamma_{f-2} \geq 3 \). By Lemma 9, \( |S| \geq \left\lceil \frac{3n}{2} \right\rceil \), a contradiction. If \( \{ v_{f-3}, u_{f-3}, v_{f-4}, u_{f-4} \} \cap S = \{ v_{f-4}, u_{f-4} \}, \) then \( \{ v_{f-5}, v_{f-3} \} \cap S \neq \emptyset \) so that \( v_{f-3} \) is dominated. This results in \( \gamma_{f-3}, \gamma_{f-4} \geq 4 \) and \( \gamma_{f-5} \geq 3 \). Thus \( \gamma_{f-1} \) can gain support from \( \gamma_{f-3} \), by Lemma 9, \( |S| \geq \left\lceil \frac{3n}{2} \right\rceil \), a contradiction. Thus the claim holds and \( \gamma_{f-2} = 2 \). Accordingly, we have the following four cases to consider.

Case 1. \( v_{f-3} \notin S \).

In this case, \( u_{f-5}, v_{f-6} \in S \) so that \( u_{f-4}, u_{f-5} \) are dominated (see Figure 8(a)). This results in \( \gamma_{f-1} \geq 3, \gamma_{f-4} \geq 4 \) and \( \gamma_{f-6} \geq 3 \) since \( N[v_{f-7}] \cap S \neq \emptyset \). The minimum values of \( \gamma_i \) for \( i = f - 6, f - 5, \ldots, f + 2 \) are 3, 3, 4, 2, 3, 3, 4, and 3, respectively, and \( \gamma_{f-2} \) can gain support from \( \gamma_{f-4} \). This leads to \( \sum_{i=1}^{n} \gamma_i = 5|S| \geq 3n - 1 \). By using a similar argument as in Lemma 9, \( |S| \geq \left\lceil \frac{3n}{2} \right\rceil \), a contradiction. Thus this case is impossible.

Case 2. \( u_{f-3} \notin S \).

Since \( \gamma_{f-2} = 2 \) and \( u_{f-4}, u_{f-5} \notin S \) (see Figure 8(b)). Furthermore, \( v_{f-6} \in S \) so that \( v_{f-4}, v_{f-5} \) is dominated. We claim that either \( u_{f-5} \) or \( v_{f-5} \) is in \( S \). If both \( u_{f-5} \) and \( v_{f-5} \) are not in \( S \), then \( \gamma_{f-3} = 2 \). However, \( u_{f-3} \) is a pseudo-couple vertex and, by Lemma 4, \( S \) does not have the smallest couple number, a contradiction. Thus this claim holds and \( \gamma_{f-3} \geq 3 \). Note that \( \gamma_{f-6} \geq 3 \). The reason is that if \( u_{f-6} \in S \), then \( \gamma_{f-6} \geq 3 \); otherwise, by Lemma 2, \( \gamma_{f-6} \geq 3 \). The minimum values of \( \gamma_i \) for \( i = f - 6, f - 5, \ldots, f + 2 \) are 3, 3, 4, 2, 3, 3, 2, and 3, respectively. Thus \( \gamma_{f-2} \) can gain support from \( \gamma_{f-4} \). Thus this case is also impossible.

Case 3. \( v_{f-4} \notin S \).

In this case, \( v_{f-5} \in S \) so that \( v_{f-3} \) is dominated (see Figure 8(c)). We claim that \( u_{f-5} \notin S \). Suppose to the contrary that \( u_{f-5} \in S \). This yields \( \gamma_{f-3}, \gamma_{f-4} \geq 4 \) and \( \gamma_{f-5}, \gamma_{f-6} \geq 3 \). This results in the minimum values of \( \gamma_i \) for \( i = f - 6, f - 5, \ldots, f + 2 \) to be 3, 3, 4, 2, 3, 3, 2, and 3, respectively, and \( \gamma_{f-2} \) can gain support from \( \gamma_{f-4} \) and \( \gamma_{f-3} \), respectively. Hence \( |S| \geq \left\lceil \frac{3n-1}{2} \right\rceil = \left\lceil \frac{3n}{2} \right\rceil \), a contradiction. Thus the claim holds. When \( u_{f-5} \notin S \), at least one vertex in \( \{ u_{f-6}, v_{f-6}, u_{f-7}, v_{f-7} \} \) must be in \( S \) so that \( u_{f-6} \) is dominated. If two vertices in \( \{ u_{f-6}, v_{f-6}, u_{f-7}, v_{f-7} \} \) are in \( S \), this results in \( \gamma_{f-5}, \gamma_{f-6} \geq 4 \) and \( \gamma_{f-3}, \gamma_{f-4}, \gamma_{f-7}, \gamma_{f-8} \geq 3 \). This further implies that \( |S| \geq \left\lceil \frac{3n-1}{2} \right\rceil = \left\lceil \frac{3n}{2} \right\rceil \), a contradiction. Thus at most one of \( u_{f-6}, v_{f-6}, \) and \( u_{f-7} \) can be in \( S \). We claim that \( v_{f-6} \) cannot be in \( S \) either. If \( v_{f-6} \in S \), then \( u_{f-8} \in S \) to ensure that \( u_{f-7} \)
dominated. Moreover, \( \gamma_{f-8} \geq 3 \) no matter whether \( u_{f-8} \) is a pseudo-couple vertex or not. This results in the minimum values of \( \gamma_i \) for \( i = f-6, f-5, \ldots, f+2 \) to be 4, 3, 4, 3, 2, 3, 2, and 3, respectively. Hence \( |S| \geq \lceil 3n-1 \rceil = \lceil 3n \rceil \), a contradiction. Thus the claim holds and only one of \( u_{f-6} \) and \( u_{f-7} \) can be in \( S \). If \( u_{f-6} \) is in \( S \), then, after replacing \( u_{f-6} \) by \( u_{f-7} \), all vertices in \( N[u_{f-6}] \) are also dominated. Thus we only consider the case where \( u_{f-7} \in S \). Note that, in this case, \( B_{f-5} \cap S \) is exactly a self-contained block. By repeating the above procedure on \( B_{f-5} \) for \( 2 \leq x \leq k \), the only possible result is that all \( B_{f-5} \) are also self-contained blocks and \( \{v_{5k+2}, v_{5k+3}\} \subset S \) (see Figure 8(d)). By Proposition 3, we have \( \gamma_{f-x} = 3 \) for \( 3 \leq x \leq 5k-2 \). Finally, we can find that at least one vertex in \( \{u_{5k+4}, v_{5k+4}, u_{5k+3}\} \) must be in \( S \) so that \( u_{5k+4} \) is dominated. This results in \( \gamma_x = 4 \) for \( x = 5k+2, 5k+3, 5k+4 \). Thus \( \gamma_{f-2}, \gamma_{f-1} \) and \( \gamma_{f+1} \) can gain support from those vertices. This yields \( \sum_{i=1}^{n} \gamma_i = |S| \geq 3n = \lceil \frac{3n}{2} \rceil \), a contradiction.

\( \text{Case 4.} \) \( u_{f-4} \in S \).

In this case, \( v_{f-5} \) and \( v_{f-6} \) are in \( S \) so that \( v_{f-3} \) and \( v_{f-4} \) are dominated. Since \( S' = S - u_{f-4} + v_{f-4} \) is still a a Type II set with the smallest couple number which is already considered in Case 3. Therefore, this case is also impossible. This concludes the proof of the lemma.

**Lemma 11.** For \( n = 5k+1 \) and \( n = 5k+2 \), if \( S \) is a Type II set, then \( |S| \leq \lceil \frac{3n}{2} \rceil - 1 \).

**Proof.** By using a similar argument as in Case 3 of Lemma 10 we can construct a dominating set \( S' \) of \( P(n, 2) \) when \( n = 5k+1 \) (see Figure 9(a)). Note that all \( \gamma_i(S') = 3 \) for \( 1 \leq i \leq n \) except \( \gamma_{f-1}(S') = \gamma_S(S') = 2 \). Thus \( \sum_{i=1}^{n} \gamma_i(S') = 5|S'| = 3n - 3 = 15k \) and \( |S'| = 3k \). However, \( \lceil \frac{3n}{2} \rceil = 3k + 1 \). This yields \( |S'| = \lceil \frac{3n}{2} \rceil - 1 \) and \( |S| \leq \lceil \frac{3n}{2} \rceil - 1 \).

Similarly, when \( n = 5k+2 \), let \( S'' = S' \cup \{v_{5k+2}\} \) (see Figure 9(b)). It can be verified easily that \( S'' \) is a dominating set of \( P(n, 2) - u_f \). Note that all \( \gamma_i(S'') = 3 \) for \( 1 \leq i \leq n \) except \( \gamma_{f-2}(S'') = \gamma_{f-1}(S'') = 2 \). Thus \( 5|S''| = 3n - 2 = 15k + 4 \) and \( |S''| = 3k + 1 \). However, \( \gamma(S(n, 2)) = \lceil \frac{3n}{2} \rceil = 3k + 2 \). This yields \( |S''| = \lceil \frac{3n}{2} \rceil - 1 \) and \( |S| \leq \lceil \frac{3n}{2} \rceil - 1 \). This completes the proof.

We summarize our results as the following theorem.

**Theorem 2.** Assume that \( u_f \) is a faulty vertex in \( P(n, 2) \). Then for \( n \geq 3 \)

\[
\gamma(P_f(n, 2)) = \begin{cases} 
\gamma(P(n, 2)) - 1 & \text{if } n = 5k + 1 \text{ or } 5k + 2 \\
\gamma(P(n, 2)) & \text{otherwise}
\end{cases}
\]

**Proof.** By Corollaries 1 and 2 and Lemmas 7 and 8, \( \gamma(P_f(n, 2)) = \gamma(P(n, 2)) \) when \( n = 5k \) and \( 5k + 3 \). By Corollaries 1 and 2 and Lemmas 7 and 10, \( \gamma(P_f(n, 2)) = \gamma(P(n, 2)) \) when \( 5k + 4 \). By Corollaries 1 and 2 and Lemmas 7 and 11, \( \lceil \frac{3n}{2} \rceil - 1 \) when \( n = 5k + 1 \) or \( 5k + 2 \). This completes the proof. \( \square \)
5 Concluding remarks

In this paper, we show that $\gamma(P_f(n, 2)) = \gamma(P(n, 2)) - 1$ if $n = 5k + 1$ or $5k + 2$; otherwise, $\gamma(P_f(n, 2)) = \gamma(P(n, 2))$. Our results can be applied to the alteration domination number of $P(n, 2)$. By Theorem 2, we can find the lower and upper bounds for $\mu(P(n, 2))$ as follows: $\mu(P(n, 2)) = 1$ if $n = 5k + 1$ or $5k + 2$; otherwise, $\mu(P(n, 2)) \geq 2$. As a further study, it is interesting to find out the exact value of $\mu(P(n, 2))$. On the bondage problem in $P(n, 2)$, it is clear that the domination number is still $\lceil \frac{3n}{2} \rceil$ after removing any edge from $P(n, 2)$. By Theorem 2, we can find that $2 \leq b(P(n, 2)) \leq 3$ if $n = 5k$, $5k + 3$ or $5k + 4$. It is also interesting to find out the exact value of $b(P(n, 2))$.

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