Abstract: A general method of regularisation of classical self interaction in strings is extended from the electromagnetic case (for which it was originally developed) to the gravitation case, for which the result can also be represented as a renormalisation.

1 Introduction

Concentration on supersymmetric quantum string theory has diverted attention from basic problems that remain to be dealt with in classical string theory. However even while laying the foundations of quantum electrodynamics, Dirac found time to obtain a regularised Lorentz covariant treatment of a classical electromagnetically self interacting point particle. Following this example, my purpose here is to treat the analogous problem of self interaction in classical relativistic string models.

I shall consider only the kinds of long range interaction that are experimentally familiar, namely electromagnetic interactions as governed by Maxwell’s equations for $A_\mu$, and weak gravitational interactions as governed by the linearised Einstein equations for a small amplitude but perhaps rapidly varying perturbation $h_{\mu\nu} = \delta g_{\mu\nu}$ of a slowly varying spacetime metric $g_{\mu\nu}$ characterising a 4-dimensional background with local coordinates $x^\mu$. In a suitable gauge, the field equations reduce to the standard forms

$$\nabla_\sigma \nabla^\sigma A^\mu = -4\pi \hat{j}^\mu, \tag{1}$$

and

$$\nabla_\sigma \nabla^\sigma h^{\mu\nu} = -8\pi G(2\hat{T}^{\mu\nu} - \hat{T}^\sigma g^{\mu\nu}), \tag{2}$$

in which $\hat{j}^\mu$ is the electromagnetic current density, $\hat{T}^{\mu\nu}$ is the stress momentum energy density tensor, and $G$ is Newton’s constant.
The problem of ultraviolet divergences for point particle or string models arises because, in these cases, the relevant source densities $\hat{j}^\mu$ and $\hat{T}^{\mu\nu}$ are not regular functions: they will be Dirac type distributions that vanish outside the relevant one or two dimensional worldsheets. In the case of a string with local worldsheet coordinates $\sigma^a$ ($a = 0, 1$) and induced metric $\gamma_{ab} = g_{\mu\nu}x^\mu_a x^{\nu}_b$, the latter will be expressible using the terminology of Dirac delta “functions” in the form

$$\hat{T}^{\mu\nu} = \|g\|^{-1/2} \int T^{\mu\nu} \delta^4(x - x(\sigma)) \|\gamma\|^{1/2} d^2\sigma,$$

(3)

and there will be a similar relation between $\hat{j}^\mu$ and $\hat{j}^\mu$, where the surface stress momentum energy density $T^{\mu\nu}$, and the surface current $J^\mu$ are regular tensorial functions on the worldsheet (but undefined off it). The string case is more awkward than that of a point particle, since it is beset by infrared as well as ultraviolet divergences. This complication has so far prevented the construction of a satisfactorily Lorentz covariant string analogue of Dirac’s (finite) radiation reaction formula for the point particle case.

2 Regularisation

What can be done for the string case is the analogue of the familiar treatment of the dominant (lowest differential order) contribution, which is divergent, and needs to be regularised by a cut-off procedure, following which – in the point particle case – it turns out to be absorbable by a simple mass renormalisation. It will be seen that an analogous, but not so simple, renormalisation is also possible in the string case. The basic force balance equation will be expressible in the form

$$\nabla_\nu T^{\mu\nu} = f_e^\mu + f_g^\mu,$$

(4)

in which the relevant tangentially projected differentiation operator is defined by $\nabla_\nu = \eta_{\nu\mu} \nabla_\mu$ where $\eta^{\mu\nu} = \gamma^{ab} x^\mu_a x^{\nu}_b$ is the fundamental tensor of the worldsheet, while the electromagnetic force density contribution will be given by the familiar formula

$$f_e^\mu = F^{\mu\nu} J_\nu, \quad F_{\mu\nu} = \nabla_{[\mu} A_{\nu]},$$

(5)

and, by the results of a recent gravitational perturbation analysis, the gravitational force contribution will be given in terms of the relevant worldsheet supported hyper-Cauchy tensor $\tilde{C}_{\mu\nu\rho\sigma}$ by an expression of the form

$$f_g^\mu = \frac{1}{2} T^{\sigma\nu} \nabla_\mu h_{\nu\sigma} - \nabla_\nu (T^{\nu\sigma} h_{\mu\sigma} + \tilde{C}_{\mu\nu\rho\sigma} h_{\rho\sigma}).$$

(6)

These force densities would evidently be well behaved if the fields $A_\mu$ and $h_{\mu\nu}$ were due just to passing radiation. However we are concerned with the case in which
they are obtained as the appropriate Dalembertian Green function solutions of the source equations (1), (2), which give values that (while finite outside) are divergent on the worldsheet where they are needed.

As in the point particle case, one can obtain realistically regularised values, $\hat{A}_\mu$ and $\hat{h}_{\mu\nu}$ say, on the string worldsheet itself, by taking account of the fact that the physical system one wishes to describe (a vacuum vortex defect in the cosmic string case) will not really be infinitely thin but will have a finite thickness that provides an appropriately microscopic “ultraviolet” cut-off length, $\delta_*$ say. In the string case it is also necessary to introduce a long range “infrared” cut-off length, $\Delta$ say, that might represent the macroscopic mean distance between neighbouring strings. The relevant Green function integrals will then be proportional to a logarithmic regularisation factor of the form

$$\hat{l} = \ln\left\{\frac{\Delta^2}{\delta_*^2}\right\}.$$  

More specifically (as pointed out in his original discussion of “superconducting” cosmic strings by Witten) the dominant contribution to the regularised electromagnetic self field $\hat{A}_\mu$ on the string will be obtained in the simple form

$$\hat{A}_\mu = \hat{l} \hat{J}_\mu.$$  

By comparing (1) and (2), it can thus be seen that the corresponding expression for the regularised gravitational self field, $\hat{h}_{\mu\nu}$ say, will have the form

$$\hat{h}_{\mu\nu} = 2G\hat{l}(2\bar{T}_{\mu\nu} - \bar{T}_\sigma^\sigma g_{\mu\nu}).$$  

(If the microscopic electromagnetic source distribution were very different from that of the gravitational source distribution, the natural cut-off $\delta_*$ that would be most appropriate for the former might be somewhat different from what would be most appropriate for the latter, but since the dependence on the cut off is only logarithmic there will not usually be any significant loss of accuracy in using the same regularisation factor $\hat{l}$ for both cases.)

For substitution in (5) and (6) – to get correspondingly regularised self force contributions $\hat{f}_e^\mu$ and $\hat{f}_g^\mu$ – knowledge of the simple regularised self fields $\hat{A}_\mu$ and $\hat{h}_{\mu\nu}$ is not sufficient. These regularised values are well defined only on the worldsheet and so do not provide what is needed for a direct evaluation of the gradients that are required: there is no difficulty for the terms involving just the tangentially projected gradient operator $\hat{\nabla}_\nu$, but there are also contributions from the unprojected gradient operator $\nabla_\nu$, which is directly meaningful only when acting on fields whose support extends off the worldsheet.

3 The regularised gradient operator

It fortunately turns out that this problem has a very simple and elegant solution. The appropriate regularisation of the gradients on the string worldsheet turns out to be obtainable simply by replacing the ill defined operator $\nabla_\nu$,
by a corresponding regularised gradient operator that is given in terms of the
worldsheet curvature vector, $K^\mu = \nabla_\nu \eta^{\mu \nu}$, by the formula

$$\hat{\nabla}_\nu = \nabla_\nu + \frac{1}{2} K_\nu.$$  \hspace{1cm} (9)

For the appropriately regularised electromagnetic field tensor on the string world-

sheet this gives

$$\hat{F}_{\mu \nu} = 2 \nabla_{[\mu} \hat{A}_{\nu]} = \hat{\ell} (2 \nabla_{[\mu} \hat{J}_{\nu]} + K_{[\mu} \hat{J}_{\nu]}),$$  \hspace{1cm} (10)

which, by the surface current conservation condition $\nabla_\nu \hat{J}^\nu = 0$, implies that the corresponding electromagnetic self force contribution in (3) will be expressible in the form

$$\hat{f}_{e \mu} = -\nabla_\nu \hat{T}_{e \nu} \mu \nu,$$  \hspace{1cm} (11)

where the relevant stress momentum energy density contribution from the electromagentic self interaction is

$$\hat{T}_{e \mu \nu} = \hat{\Lambda}^{\mu \nu} - \frac{1}{2} \hat{\Lambda}^{\rho \sigma} \eta^{\mu \nu}.$$  \hspace{1cm} (12)

It can be seen from (6) that the regularised gravitational self force contribution

will be similarly expressible in the form

$$\hat{f}_g \mu = -\nabla_\nu \hat{T}_g \nu \mu \nu,$$  \hspace{1cm} (13)

in which the relevant self gravitational stress momentum energy density contribu-
tion works out as

$$\hat{T}_g \mu \nu = \hat{\Lambda}_\sigma \mu \Tilde{T}^{\sigma \nu} - \frac{1}{4} \hat{\Lambda}_\rho \sigma \Tilde{T}^{\rho \sigma \eta^{\mu \nu} + \hat{\Lambda}_\rho \sigma \C^{\rho \sigma \mu \nu}.$$  \hspace{1cm} (14)

It evidently follows that the self force contributions will be absorbable by

a renormalisation whereby the original “bare” stress momentum energy density
tensor $T^{\mu \nu}$ is replaced by the “dressed” stress momentum energy density tensor

$$\Tilde{T}^{\mu \nu} = T^{\mu \nu} + \hat{f}_{e \mu} \nu \mu \nu + \hat{f}_g \mu \nu,$$  \hspace{1cm} (15)

so that – in the absence of radiation from outside – the basic force balance equation

$\nabla_\nu \tilde{T}^{\mu \nu} = 0$.

References

[1] Battye, R.A., Carter, B. (1995) Gravitational perturbations of relativistic
membranes and strings, Phys. Letters B357, pp 29-35 [hep-th/9507059]

[2] Carter, B. (1997) Electromagnetic self interaction in strings, Phys. Letters
B404, pp 246-252 [hep-th/9704210]