QUANTITATIVE STATISTICAL STABILITY FOR EQUILIBRIUM STATES OF PIECEWISE PARTIALLY HYPERBOLIC MAPS.

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Abstract. We consider a class of endomorphisms that contains a set of piecewise partially hyperbolic dynamics semi-conjugated to non-uniformly expanding maps. Our goal is to study a class of endomorphisms that preserve a foliation that is almost everywhere uniformly contracted, with possible discontinuity sets parallel to the contracting direction. We apply the spectral gap property and the $\zeta$-Hölder regularity of the disintegration of its equilibrium states to prove a quantitative statistical stability statement. More precisely, under deterministic perturbations of the system of size $\delta$, we show that the $F$-invariant measure varies continuously with respect to a suitable anisotropic norm. Furthermore, we establish that certain interesting classes of perturbations exhibit a modulus of continuity estimated by $D_2 \delta^\zeta \log \delta$, where $D_2$ is a constant.

1. Introduction

Understanding how statistical properties change when a system is perturbed is of significant interest in both pure and applied mathematics. When a statistical property of a system varies continuously after deterministic or even stochastic variations, we say it is statistically stable. The study of these properties is motivated by the desire to understand how uncertainty impacts the quantitative and qualitative measurements of systems.

An important ergodic object of a dynamical system that makes interesting the investigation of its stability is the invariant measure, given that it is key in understanding the long-term behavior of the dynamics. To do this, consider a one-parameter family of dynamics $\{F_\delta\}_{\delta \in [0,1)}$ as a perturbation of a system $F = F_0$. Suppose that $\{F_\delta\}_{\delta \in [0,1)}$ admits a one-parameter family of invariant measures $\{\mu_\delta\}_{\delta \in [0,1)}$, i.e., $\mu_\delta$ is a $F_\delta$-invariant probability measure for all $\delta \in [0,1)$. We say that $\mu_0$ is statistically stable if the function $\delta \mapsto \mu_\delta$ is continuous at 0 in a suitable topology. In this paper, our aim is to prove the continuity and to estimate the modulus of continuity of the function $\delta \mapsto \mu_\delta$ at 0.

To prove our results, we aim to construct a suitable vector space of signed measures that satisfy three main properties. This space includes the family

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\{\mu_\delta\}_{\delta \in [0,1)}$, where \(\mu_\delta\) is the unique \(F_\delta\)-invariant measure inside this space for all \(\delta \in [0,1)\). Additionally, the function \(\delta \mapsto \mu_\delta\) is continuous at 0, with a modulus of continuity of the order of \(\delta^\zeta \log \delta\) (where \(\zeta\) is a constant that depends on \(F\)). We use the functional analytic approach of \cite{5} to study the transfer operator of \(F_*\), which is the linear operator \(F_*\) that associates each signed measure \(\mu\) with the signed measure \(F_* \mu\) defined by \(F_* \mu(A) = \mu(F^{-1}(A))\). Positive fixed points of \(F_*\) are \(F\)-invariant measures, so the problem is reduced to understanding how the eigenvectors of the induced family of transfer operators \(\{F_\delta_*\}_{\delta \in [0,1)}\) associated with unitary eigenvalues (invariant measures) vary when the system changes.

In \cite{5}, the authors studied this property for Lorenz-like systems, \(F = (f, G)\). In this case, the quotient the quotient map \(f\) is a piecewise expanding map and the fiber function \(G\) is Lipschitz in the first variable on each element of a finite family of vertical strips. This family covers a full measure set of the ambient space. Moreover, they defined anisotropic spaces where the action of the transfer operator had a spectral gap, which provided a quantitative stability statement, estimating the modulus of continuity of the function \(\delta \mapsto \mu_\delta\) at 0. It is worth mentioning that the Bounded Variation regularity of the disintegration of the invariant measure was a crucial ingredient used in \cite{5} to obtain the stability result.

The dynamical system under consideration in this work is a skew-product of the form \(F = (f, G)\), where the quotient map \(f\) is a non-uniformly expanding system. Moreover, the fiber function \(G\) is H"older in the first variable on each element of a finite family of vertical strips. As in \cite{5}, this family covers a full measure set of the ambient space. To handle this system, we utilize the H"older regularity of the disintegration of the invariant measure established in \cite{6}. In addition, to overcome the challenges posed by the new hypotheses, we introduce several new definitions, including the concepts of an admissible \(R(\delta)\)-perturbation and a \((R(\delta), \zeta)\)-family of operators (see the next paragraphs for these definitions). Some of these definitions generalize the ones given in \cite{5}.

Although the uniform hyperbolic scenario is well understood, our understanding of partially hyperbolic systems, especially those that are non-invertible or have discontinuities, is far from complete. For further information on this topic, interested readers can refer to \cite{4} and \cite{2}. While the former deals with systems that allow discontinuities, the latter is restricted to smooth invertible systems.

In the approach presented here, a finite number of sets of discontinuities (lines) parallel to the contracting direction are allowed. In comparison with the works cited above, \cite{5} requires a uniform expansion (piecewise) on the base map \(f\) despite allowing discontinuities. On the other hand, \cite{4} obtains quantitative estimates for statistical stability for piecewise constant toral extensions \(F = (f, G)\) with a uniform expanding quotient map \(f\) and uses norms similar to those employed in our work.
In this paper, we study skew-product maps $F : \Sigma \rightarrow \Sigma$, where $F(x, y) = (f(x), G(x, y))$, $\Sigma = M \times K$ is a product space, and $M$ is a compact and connected Riemannian manifold equipped with a Riemannian metric $d_1$, while $K$ is a compact metric space equipped with a metric $d_2$. The space $\Sigma$ is endowed with the metric $d_1 + d_2$. For simplicity, we assume that $\text{diam}(M) = 1$, which is not restrictive but will avoid multiplicative constants.

We also assume that $F$ contracts almost every vertical fiber $\gamma = \{x\} \times K$ and its quotient map $f : M \rightarrow M$ is a non-uniformly expanding system. More precisely, $f : M \rightarrow M$ is a local diffeomorphism, and there exists a continuous function $L : M \rightarrow \mathbb{R}$ such that for every $x \in M$, there exists a neighborhood $U_x$ of $x$ such that $f_x := f|_{U_x} : U_x \rightarrow f(U_x)$ is invertible and satisfies

$$d_1(f^{-1}_x(y), f^{-1}_x(z)) \leq L(x)d_1(y, z)$$

for all $y, z \in f(U_x)$. In particular, $\#f^{-1}(x)$ is constant for all $x \in M$, and we set $\deg(f) := \#f^{-1}(x)$ as the degree of $f$.

Define the function $\rho : M \rightarrow \mathbb{R}$ by

$$\rho(x) := \frac{1}{|\text{det} Df(x)|},$$

where $\text{det} Df$ is the Jacobian of $f$ with respect to a fixed probability $m_1$ on $M$ (see [9] for definitions and basic results on the Jacobian). We assume that $m_1$ is an equilibrium state for the potential $\phi = \log \frac{1}{|\text{det} Df|}$. That is, $m_1$ satisfies

$$\int \log \frac{1}{|\text{det} Df|}dm_1 + h_{m_1}(f) = \sup_{f*\mu=\mu} \left\{ \int \log \frac{1}{|\text{det} Df|}d\mu + h_{\mu}(f) \right\}, \quad (1)$$

where $h_{\mu}(f)$ denotes the entropy of the system $(f, \mu)$. By [3] and [11] a measure $m_1$ which satisfies (1) exists. Note that $\rho$ is defined $m_1$-a.e. $x \in M$.

Suppose that there exists an open region $A \subset M$ and constants $\sigma > 1$ and $L_1 \geq 1$ such that the following conditions hold:

(f1) $L(x) \leq L_1$ for every $x \in A$ and $L(x) < \sigma^{-1}$ for every $x \in A^c$.
Moreover, the constant $L_1$ satisfies the inequality given by equation [4] that will be presented ahead.

(f2) There exists a finite covering $\mathcal{U}$ of $M$ by open domains of injectivity of $f$, such that $A$ can be covered by $q < \deg(f)$ of these domains.

Let $H_\zeta (\zeta \leq 1)$ represent the set of $\zeta$-Hölder functions $h : M \rightarrow \mathbb{R}$. In other words, defining

$$H_\zeta (h) := \sup_{x \neq y} \frac{|h(x) - h(y)|}{d_1(x, y)^\zeta},$$

we have

$$H_\zeta := \{ h : M \rightarrow \mathbb{R} : H_\zeta(h) < \infty \}. $$

Next, we require that condition (f3) holds, which is an open condition with respect to the Hölder norm. Equation [5], presented below, specifies that $\rho$
is $\zeta$-Holder and belongs to a small cone of Hölder continuous functions (see [3]). For examples of non-uniformly expanding transformations that satisfy (f1), (f2) and (f3), the reader may refer to [3] and [11]. In this paper, we also explore such maps in Section 2 and provide Example 2.1, where (f1), (f2), and (f3) have been explicitly demonstrated.

(f3) There exists a small enough $\epsilon_\rho > 0$ such that
\[
\sup \log(\rho) - \inf \log(\rho) < \epsilon_\rho; \tag{2}
\]
and
\[
H_\zeta(\rho) < \epsilon_\rho \inf \rho. \tag{3}
\]
Precisely, we assume that the constants $\epsilon_\rho$ and $L_1$ satisfy the condition:
\[
e^{\epsilon_\rho} \left( \frac{(\deg(f) - q)\sigma^{-\zeta} + qL_1^{\zeta}[1 + (L_1 - 1)^{\zeta}]}{\deg(f)} \right) < 1. \tag{4}
\]
We assume that the fiber map $G : \Sigma \to K$ satisfies:

(G1) $G$ is uniformly contracting on $m_1$-a.e. vertical fiber $\gamma_x := x \times K$.

Precisely, there exists $0 < \alpha < 1$ such that for $m_1$-a.e. $x \in M$, it holds that
\[
d_2(G(x, z_1), G(x, z_2)) \leq \alpha d_2(z_1, z_2), \quad \forall z_1, z_2 \in K. \tag{5}
\]
We denote the set of all vertical fibers $\gamma_x$ by $\mathcal{F}^s$:
\[
\mathcal{F}^s := \{\gamma_x := \{x\} \times K; x \in M\}.
\]
When no confusion is present, the elements of $\mathcal{F}^s$ will be denoted simply by $\gamma$ instead of $\gamma_x$.

(G2) Let $P_1, \ldots, P_{\deg(f)}$ be the partition of $M$ given in Remark 3.10 and let $\zeta \leq 1$. Suppose that
\[
|G_i|_\zeta := \sup_{y} \sup_{x_1, x_2 \in P_i} \frac{d_2(G(x_1, y), G(x_2, y))}{d_1(x_1, x_2)^{\zeta}} < \infty.
\]
Denote by $|G|_\zeta$ the following constant:
\[
|G|_\zeta := \max_{i=1, \ldots, \deg(f)} \{ |G_i|_\zeta \}. \tag{6}
\]

**Remark 1.1.** The condition (G2) implies that $G$ may be discontinuous on the sets $\partial P_i \times K$ for all $i = 1, \ldots, \deg(f)$, where $\partial P_i$ denotes the boundary of $P_i$.

**Remark 1.2.** Since there is a bijective correspondence between the elements $x \in M$ and the fibers $\gamma = \{x\} \times K$, from now on we also use $\gamma$ to denote the elements of $M$.

For the system $F$ under consideration (see [6]), the transfer operator $F_* \mu$ has a spectral gap on a space of signed measures, $\mu$, such that its projection ($\pi_1(x, y) = x$ for all $x \in M$ and $y \in K$) onto the first coordinate, $\pi_{1*}\mu$ (the pushforward), is absolutely continuous with respect to $m_1$ and
its density satisfies \( \frac{d\rho_1}{dm} \in H_{\zeta} \). We denote this space by \( S^\infty \) (see definition 3.5). It is worth mentioning that \( S^\infty \) is a suitable anisotropic space of disintegrated measures. For these maps, we prove quantitative results on the statistical stability of the unique equilibrium state of \( S \) in \( S^\infty \) under a class of deterministic perturbations of the system, \( \{F_\delta\}_{\delta \in \{0, 1\}}, F_\delta = (f_\delta, G_\delta), \)

\( f_\delta : M \rightarrow M, G_\delta : M \times K \rightarrow K \) for all \( \delta \in [0, 1) \) and \( F_0 = F \).

For such a perturbation, we suppose the following conditions:

(U1) There exists a small enough \( \delta_1 \) such that for all \( \delta \in (0, \delta_1) \), it holds

\[
\deg(f_\delta) = \deg(f),
\]

for all \( \delta \in (0, \delta_1) \).

(U2) For every \( \gamma \in M \) and for all \( i = 1, \ldots, \deg(f) \) denote by \( \gamma_{\delta, i} \) the \( i \)-th pre-image of \( \gamma \) by \( f_\delta \) (see Remark 1.2). Suppose there exists a real-valued function \( \delta \rightarrow R(\delta) \in \mathbb{R}^+ \) such that

\[
\lim_{\delta \rightarrow 0^+} R(\delta) \log(\delta) = 0
\]

and the following three conditions hold:

(U2.1) \[
\sum_{i=1}^{\deg(f)} \left| \frac{1}{\det Df_\delta(\gamma_{\delta, i})} - \frac{1}{\det Df_0(\gamma_{0, i})} \right| \leq R(\delta);
\]

(U2.2) \[
\sup_{\gamma} \max_{i=1, \ldots, \deg(f)} d_1(\gamma_{0, i}, \gamma_{\delta, i}) \leq R(\delta);
\]

(U2.3) \[
G_\delta \text{ and } G_0 \text{ are } R(\delta)-\text{close in the sup norm: for all } \delta,
\]

\[
d_2(G_0(x, y), G_\delta(x, y)) \leq R(\delta) \quad \forall (x, y) \in M \times K;
\]

(U3) For all \( \delta \in (0, \delta_1) \), \( f_\delta \) has an equilibrium state \( m_{1, \delta} \), and \( m_{1, \delta} \) is equivalent to \( m_1 \) for all \( \delta \in [0, \delta_1) \). This implies that \( m_1 \ll m_{1, \delta} \) and \( m_{1, \delta} \ll m_1 \) for all \( \delta \in [0, \delta_1) \).

Remark 1.3. By (U3), note that (see Remark 1.2) \[
\sum_{i=1}^{\deg(f)} \frac{1}{\det Df_\delta(\gamma_{\delta, i})} = 1 \quad m_1\text{-a.e.}, \text{ since } m_1 \ll m_{1, \delta} \text{ for all } \delta \text{ and } m_{1, \delta} \text{ is } f_\delta\text{-invariant}.
\]

(A1) There exist constants \( D > 0 \) and \( 0 < \lambda < 1 \) such that for all \( g \in H_{\zeta} \), all \( \delta \in [0, 1) \), and all \( n \geq 1 \), the following inequality holds:

\[
|P^n_{f_\delta} g|_{\zeta} \leq D \lambda^n |g|_{\zeta} + D|g|_{\infty},
\]

where \( |g|_{\zeta} = H_{\zeta}(g) + |g|_{\infty} \) and \( P_{f_\delta} \) is the Perron-Frobenius operator of \( f_\delta \). That is, for all \( \delta \in [0, 1) \), \( P_{f_\delta} \) is the unique linear operator \( P_{f_\delta} : L^1_{m_1} \rightarrow L^1_{m_1} \) such that for all \( \psi \in L^\infty_{m_1} \) and all \( \phi \in L^1_{m_1} \) it holds that

\[
\int \psi P_{f_\delta}(\phi)dm_1 = \int (\psi \circ f_\delta)\phi dm_1.
\]

(A2) For all \( \delta \in [0, 1) \), let \( \alpha_\delta, L_1, \delta \) and \( |G_\delta|_{\zeta} \) be the contraction rate \( \alpha \) given by Equation (5) for \( G_\delta \), the constant \( L_1 \) given by (f1) for \( f_\delta \),
and the constant $|G|_\zeta$ defined by Equation (6), respectively. Set $eta_\delta := (\alpha_\delta L_{1,\delta})^\zeta$ and $D_{2,\delta} := \{\epsilon_{\rho,\delta}L_{1,\delta}^\zeta + |G|_\zeta L_{1,\delta}^\zeta\}$. Suppose that,

$$\sup_\delta \beta_\delta < 1$$

and

$$\sup_\delta D_{2,\delta} < \infty.$$ 

We define an admissible $R(\delta)$-perturbation as a family $\{F_\delta\}_{\delta \in [0,1)}$, where $F_\delta$ satisfies conditions (U1), (U2), (U3), (A1), (A2), as well as (f1), (f2), (f3), (G1), and (G2) for all $\delta$.

**Statements of the Main Results.** In this section, we present the main results of this article and provide an explanation of how Theorems A, B, and Corollary 1.4 are proven.

The first result guarantees the existence and uniqueness of an invariant measure for $F$ in the space $S^\infty$, which is an equilibrium state if $F$ is continuous. In particular, all admissible $R(\delta)$-perturbation, $\{F_\delta\}_{\delta \in [0,1)}$, has a family of $F_\delta$-invariant measures, $\{\mu_\delta\}_{\delta \in [0,1)}$.

**Theorem A.** The system $F$ has a unique invariant probability, $\mu_0 \in S^\infty$. If $F$ is continuous, then $\mu_0$ is an equilibrium state.

The next Theorem gives a relation between an admissible $R(\delta)$-perturbation, $\{F_\delta\}_{\delta \in [0,1)}$, and the variation of the induced family of invariant measures, $\{\mu_\delta\}_{\delta \in [0,1)}$. Moreover, it estimates the modulus of continuity on 0 of the induced function $\delta \mapsto \mu_\delta$, given by

$$\delta \mapsto F_\delta \mapsto \mu_\delta, \quad \delta \in [0,1),$$

with respect to the norm $|| \cdot ||_\infty$ defined by

$$||\mu||_\infty := \sup_{\gamma, \delta} \left\{ \int g d\mu |_{\gamma} \right\},$$

where $\gamma \in M$, $g$ ranges over the set $H_\zeta$ satisfying $H_\zeta(g) \leq 1$, $|g|_\infty \leq 1$ and $\mu |_{\gamma}$ is defined from the conditional measure $\mu_\gamma$ of the disintegration of $\mu$ along $F^s$ (see Definition 3.5).

**Theorem B** (Quantitative stability for deterministic perturbations). Let $\{F_\delta\}_{\delta \in [0,1)}$ be an admissible $R(\delta)$-perturbation. Denote by $\mu_\delta$ the invariant measure of $F_\delta$ in $S^\infty$, for all $\delta$. Then, there exist constants $D_2 < 0$ and $\delta_1 \in (0, \delta_0)$ such that for all $\delta \in [0, \delta_1)$, it holds

$$||\mu_\delta - \mu_0||_\infty \leq D_2 R(\delta)^\zeta \log \delta. \quad (7)$$

Many interesting perturbations of $F$ ensure the existence of a linear $R(\delta)$. For instance, perturbations with respect to topologies defined in the set of the skew-products, induced by the $C^r$ topologies. Therefore, if the function $R(\delta)$ is of the type,

$$R(\delta) = K_6 \delta,$$
for all $\delta$ and a constant $K_6$ we immediately get the following corollary.

**Corollary 1.4 (Quantitative stability for deterministic perturbations with a linear $R(\delta)$).** Let $\{F_\delta\}_{\delta \in [0,1)}$ be an admissible $R(\delta)$-perturbation, where $R(\delta)$ is defined by $R(\delta) = K_5 \delta$. Denote by $\mu_\delta$ the unique invariant probability of $F_\delta$ in $S^\infty$, for all $\delta$. Then, there exist constants $D_2 < 0$ and $\delta_1 \in (0, \delta_0)$ such that for all $\delta \in [0, \delta_1)$, it holds:

$$||\mu_\delta - \mu_0||_\infty \leq D_2 \delta^\zeta \log \delta.$$ 

**Plan of the paper.** The paper is structured as follows:

- **Section 1**: in this section, we introduce the type of systems we consider in the paper. Essentially, it is a class of systems that comprises a set of piecewise partially hyperbolic dynamics $F(x, y) = (f(x), G(x, y))$. The system $F$ has a non-uniformly expanding basis map $f$, and a fiber map $G$ that uniformly contracts $m_1$-a.e. vertical fiber $\gamma \in M$, where $m_1$ is an $f$-invariant equilibrium state. Additionally, in this section, we state the main results and definitions. For instance, we introduce the concept of *admissible $R(\delta)$-perturbations*;
- **Section 2**: we present some examples;
- **Section 3**: we present some tools and preliminary results, some of which have already been published in the literature. Most of them are from [5] and [6]. We use these tools to introduce the functional spaces discussed in the previous paragraphs;
- **Section 4**: we prove some basic results satisfied by *admissible $R(\delta)$-perturbations* which are important to obtain Theorem [3] and Lemma [4.2];
- **Section 5**: we introduce the definition of ($R(\delta), \zeta$)-family of operators and present results relating this family to *admissible $R(\delta)$-perturbations*;
- **Section 6**: we prove Theorem [A];
- **Section 7**: we prove Theorems [B] and Corollary [1.4];

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\[1\] A question to be answered is: is $O(\delta^\zeta \log \delta)$ an optimal modulus of continuity?
2. Examples

In what follows, we present some examples which satisfy the assumptions described in the previous section.

Example 2.1. Let \( f_0 : M \to M \) be a map defined by \( f_0(x, y) = (\text{id}(x), 3y \mod 1) \), where \( M := [0, 1]^2 \) is endowed with the \( T^2 \) topology and \( \text{id} \) is the identity map on \([0, 1]\). On \([0, 1]^2\), we consider the metric \( d_1((x_0, y_0), (x_1, y_1)) = \max d(x_0, y_0), d(x_1, y_1) \), where \( d \) is the metric of \([0, 1]\). This system has \((0, 0) = (1, 1)\) and all points of the horizontal segment \([0, 1] \times \{1/2\}\) as fixed points.

Consider the partition \( P_0 = [0, 1/3] \times [0, 1], P_1 = [1/3, 2/3] \times [0, 1], \) and \( P_2 = [2/3, 1] \times [0, 1] \). In particular, the fixed point \( p_0 = (1/2, 1/2) \in A := \text{int} \ P_1 \) (where \( \text{int} \ P_1 \) means the interior of \( P_1 \)).

For a given \( \delta > 0 \), consider a perturbation \( f \) of \( f_0 \), given by \( f(x, y) = (g(x), 3y \mod 1) \) such that \( g(1/2) = 1/2, 0 < g'(1/2) < 1 \) and \( g \) is \( \delta \)-close to \( \text{id}(x) = x \) in the \( C^2 \) topology. Moreover, suppose that \( |g'(x)| \geq k_0 > 1 \) for all \( x \in P_0 \cup P_2 \). In particular, without loss of generality, suppose that \( 1-\delta < g'(1/2) < 1+\delta < 3 \) and \( \deg(g) = 1 \). Below, the reader can see the graph of such a function \( g \).

![Figure 1. The graph of the perturbed map \( g \).](image)

Thus, we have

\[
Df(1/2, 1/2) = \begin{pmatrix} g'(1/2) & 0 \\ 0 & 3 \end{pmatrix}.
\]

And since \( p_0 = (1/2, 1/2) \) is still a fixed for \( f \) we have that \( p_0 \) becomes a saddle point (for \( f \)) as in the next Example 2.2. Moreover, since \( \deg(g) = 1 \), we have that \( \deg(f_0) = \deg(f) = 3, q = 1, \sigma = 3, L(x, y) := 1/g'(x), \forall (x, y) \in \)
In general, we have the following expression for the derivative of \( f \):

\[
Df(x, y) = \begin{pmatrix}
g'(x) & 0 \\
0 & 3
\end{pmatrix},
\]

for all \((x, y) \in [0, 1]^2\). This expression ensures that \( \rho \) is \( \zeta \)-Holder for \( 0 < \zeta \leq 1 \). Therefore, for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( L(x, y) \in (1 - \epsilon, 1 + \epsilon) \) for all \((x, y) \in [0, 1]^2\), so that \( L_1 \) can be defined as \( L_1 := 1 + \epsilon \).

Since \( g : [0, 1] \to [0, 1] \) is \( \delta \)-close to \( \text{id} : [0, 1] \to [0, 1] \), we have that
\[
1 - \delta < g'(x) < 1 + \delta \quad \text{for all} \quad x \in [0, 1],
\]
and
\[
1 - 3\delta < \det Df(x, y) < 3(1 + \delta) \quad \text{for all} \quad (x, y) \in [0, 1]^2.
\]

Thus,
\[
\sup_{(x, y) \in [0, 1]^2} \log \frac{1}{\det Df(x, y)} - \inf_{(x, y) \in [0, 1]^2} \log \frac{1}{\det Df(x, y)} \leq \log \frac{1}{3(1 - \delta)} - \log \frac{1}{3(1 + \delta)} = \log \frac{(1 + \delta)}{(1 - \delta)}.
\]

Therefore, it holds
\[
\sup_{(x, y) \in [0, 1]^2} \log \frac{1}{\det Df(x, y)} - \inf_{(x, y) \in [0, 1]^2} \log \frac{1}{\det Df(x, y)} \leq \log \frac{1 + \delta}{1 - \delta}. \tag{8}
\]

Note that, since \( e^{\rho} \approx 1, L_1 \approx 1, 0 < \zeta \leq 1 \) and \( q(L_1^\zeta[1 + (L_1 - 1)^\zeta]) \approx 1 \) we have that
\[
e^{\rho} \left( \frac{(\deg(f) - q)\zeta^{-\zeta} + qL_1^\zeta[1 + (L_1 - 1)^\zeta]}{\deg(f)} \right) \approx \frac{2(3 - \zeta) + 1}{3} < 1. \tag{9}
\]

The above relation shows that the system satisfies (4).

Now we will prove that this system satisfies (3) of (f3). Note that,
\[
\rho(x, y) := \frac{1}{|\det Df(x, y)|} = \frac{1}{3g'(x)}. \tag{10}
\]

Besides that, since \( g \) is \( \delta \)-close to \( \text{id} \) in the \( C^2 \) topology, we have
\[
-\delta \leq g(x) - x \leq \delta, \tag{11}
\]
\[
1 - \delta \leq g'(x) \leq 1 + \delta \tag{12}
\]
and
\[
-\delta \leq g''(x) \leq \delta. \tag{13}
\]

In what follows, the point \( x_2 \) is obtained by an application of the Mean Value Theorem. Then, we have
\[
\frac{1}{|\det Df(x_0, y_0)|} - \frac{1}{|\det Df(x_1, y_1)|} = \frac{1}{|3g'(x_0)|} - \frac{1}{|3g'(x_1)|} \leq \frac{1}{\max\{d_1(x_0, x_1), d_1(y_0, y_1)\}} \cdot \delta
\]

\[
\leq \frac{1}{3} \frac{|g'(x_1) - g'(x_0)|}{|g'(x_1)g'(x_0)|}
\]

\[
\leq \frac{1}{3} \frac{|g''(x_2)|}{|g'(x_1)g'(x_0)|} \leq \frac{1}{3} \delta \frac{1}{(1 - \delta)^2}
\]

\[
\leq \frac{\sqrt{\delta}}{(1 - \delta)^2} \inf_{x \in [0,1]} \frac{1}{3g'(x)}.
\]

Thus,

\[
H_\delta(\rho) \leq \frac{\sqrt{\delta}}{(1 - \delta)^2} \inf_{x \in [0,1]} \rho.
\] (14)

If \(\delta\) is small enough, by equations (8) and (14), the perturbed system satisfies (2) and (3) of (f3).

We emphasize that this example satisfies the hypotheses of both articles, [3] and [11]. More precisely, it satisfies (H1), (H2) and (P) of [3] and (H1), (H2) and (P) of [11]. In fact, by the variational principle, we have that \(h(f) > 0\).

**Example 2.2.** Now we present a general idea to generate examples by perturbing the identity or an expanding map close to the identity.

Let \(f_0 : \mathbb{T}^d \rightarrow \mathbb{T}^d\) be an expanding map. Choose a covering \(\mathcal{P}\) and an atom \(P_1 \in \mathcal{P}\) that contains a periodic point (possibly a fixed point) \(p\). Next, consider a perturbation \(f\), of \(f_0\), within \(P_1\) using a pitchfork bifurcation in such a way that \(p\) becomes a saddle point for \(f\). Consequently, \(f\) coincides with \(f_0\) in \(P_1^c\), where we have uniform expansion. The perturbation can be designed to satisfy condition (f1). This ensures that \(f\) is never overly contracting in \(P_1\), and it remains topologically mixing. However, it is important to note that a small perturbation with these properties may not always exist. If such a perturbation does exist, then condition (f3) can be satisfied. In this case, \(m_1\) is absolutely continuous with respect to the Lebesgue measure, which is an expanding conformal and positive measure on open sets. Consequently, there can be no periodic attractors.

**Example 2.3.** In the previous example, assume that \(f_0\) is diagonalizable, with eigenvalues \(1 < 1 + a < \lambda\), associated with \(e_1, e_2\), respectively, where
where $S$ is again from the previous Example 2.1. Consider a real number $0 < \alpha_2 < 1$ and $\zeta$-Hölder functions $h_1 : [0, \frac{1}{2}] \to [0, 1], h_2 : [0, 1] \to [0, 1]$.

Example 2.4. (Discontinuous Maps) Let $F = (f, G)$ be the measurable map, where $f$ is from the previous Example 2.4. Consider the real numbers $\alpha_1$ and $\alpha_2$ s.t $0 \leq \alpha_1 < \alpha_2 < 1$. Let $G : [0, 1] \times [0, 1] \to [0, 1]$ be the function defined by

$$G(x, y) = \begin{cases} \alpha_1y & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \alpha_2y & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

It is easy to see that $G$ is discontinuous on the set $\{\frac{1}{2}\} \times [0, 1)$. Moreover, $G$ satisfies (H2) since $|G|_{\zeta} = 0$ (see equation (6)), for all $\zeta$. Thus, $G$ is a $\alpha_3$-contraction, where $\alpha_3 = \max\{\alpha_1, \alpha_2\}$. Since $L_1 = 1$ we have that $(\alpha_3 L_1)^{\zeta} < 1$, for all $\zeta$. Therefore, $F$ satisfies all hypothesis (f1), (f2), (f3), (G1), (G2) and $(\alpha_3 L_1)^{\zeta} < 1$.

Example 2.5. (Discontinuous Maps) Let $F = (f, G)$ be the measurable map, where $f$ is again from the previous Example 2.5. Consider a real number $0 \leq \alpha_2 < 1$ and $\zeta$-Hölder functions $h_1 : [0, \frac{1}{2}] \to [0, 1], h_2 :...
$[\frac{1}{2}, 1] \rightarrow [0, 1]$ such that $h_1(\frac{1}{2}) \neq h_2(\frac{1}{2})$ and $0 \leq h_1, h_2 < \alpha_2 < 1$. Let $G : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be the function defined by

$$G(x, y) = \begin{cases} h_1(x)y & \text{if } 0 \leq x \leq \frac{1}{2}, \\ h_2(x)y & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

It is easy to see that $G$ is discontinuous on the set $\{\frac{1}{2}\} \times [0, 1]$. Moreover, $G$ satisfies (H2) since $|G|_\zeta \leq \max\{|h_1|_\zeta, |h_2|_\zeta\}$. Thus, $G$ is a $\alpha_2$-contraction.

While Theorem 1.4 deals with a linear $R(\delta)$, this function can take other forms, as shown in the next example.

**Example 2.6.** Let us consider $F : M \times [0, 1] \rightarrow M \times [0, 1]$ such that $F(x, y) = (f(x), G(x, y))$, where $G(x, y) = \lambda y$ for all $(x, y) \in M \times [0, 1]$ and $0 < \lambda < 1$. Suppose that $\delta_0$ is small enough in a way that $0 < \lambda + \sqrt{\delta} < 1$ for all $\delta \in (0, \delta_0]$. Define $\{F_\delta\}_{\delta \in [0, 1]}$ by $f_\delta := f$ for all $\delta \in (0, 1)$, $G_\delta(x, y) = \lambda y$ for all $(x, y) \in M \times [0, 1]$ if $\delta > \delta_0$ and $G_\delta(x, y) = (\sqrt{\delta} + \lambda)y$ for all $(x, y) \in M \times [0, 1]$ if $\delta \in (0, \delta_0]$.

We have that $\{F_\delta\}_{\delta \in [0, 1]}$ is an $R(\delta)$-perturbation with $R(\delta) := \sqrt{\delta}$. Indeed,

$$|G(x, y) - G_\delta(x, y)| = |\lambda y - (\sqrt{\delta} + \lambda)y| \\ \leq |\sqrt{\delta}y| \\ \leq \sqrt{\delta},$$

for all $\delta \in [0, 1)$. Thus (U2.3) is satisfied. The other conditions are straightforward to check.

### 3. Preliminary Results

In this section, we present some preliminary and well-established results and introduce the functional analytic framework suitable for our approach. Some of these results are taken from [6] and [5].

#### 3.1. Weak and Strong Spaces

**3.1.1. $L^\infty$-like spaces.** In this subsection, we define the vector spaces of signed measures that we will be working with. Specifically, we define the norm of the left-hand side of equation (7). To do this, we need to briefly review some facts about the disintegration of measures, state Rokhlin’s Disintegration Theorem, and establish certain notations.
Rokhlin’s Disintegration Theorem. Consider a probability space \((\Sigma, \mathcal{B}, \mu)\) and a partition \(\Gamma\) of \(\Sigma\) into measurable sets \(\gamma \in \mathcal{B}\). Denote by \(\pi : \Sigma \rightarrow \Gamma\) the projection that associates to each point \(x \in M\) the element \(\gamma_x\) of \(\Gamma\) that contains \(x\). That is, \(\pi(x) = \gamma_x\). Let \(\hat{\mathcal{B}}\) be the \(\sigma\)-algebra of \(\Gamma\) provided by \(\pi\). Precisely, a subset \(Q \subset \Gamma\) is measurable if, and only if, \(\pi^{-1}(Q) \in \mathcal{B}\). We define the quotient measure \(\mu_{\pi}\) on \(\Gamma\) by \(\mu_{\pi}(Q) = \mu(\pi^{-1}(Q))\).

The proof of the following theorem can be found in [9, Theorem 5.1.11 (items a), b) and c)) and Proposition 5.1.7 (item d)).

**Theorem 3.1.** (Rokhlin’s Disintegration Theorem) Suppose that \(\Sigma\) is a complete and separable metric space, \(\Gamma\) is a measurable partition of \(\Sigma\) and \(\mu\) is a probability on \(\Sigma\). Then, \(\mu\) admits a disintegration relative to \(\Gamma\). That is, there exists a family \(\{\mu_\gamma\}_{\gamma \in \Gamma}\) of probabilities on \(\Sigma\) and a quotient measure \(\mu_{\pi}\), such that:

(a) \(\mu_\gamma(\gamma) = 1\) for \(\mu_{\pi}\)-a.e. \(\gamma \in \Gamma\);
(b) for all measurable set \(E \subset \Sigma\) the function \(\Gamma \rightarrow \mathbb{R}\) defined by \(\gamma \mapsto \mu_\gamma(E)\), is measurable;
(c) for all measurable set \(E \subset \Sigma\), it holds \(\mu(E) = \int \mu_\gamma(E)d\mu_{\pi}(\gamma)\).
(d) If the \(\sigma\)-algebra \(\mathcal{B}\) on \(\Sigma\) has a countable generator, then the disintegration is unique in the following sense. If \(\{\mu_\gamma'\}_{\gamma \in \Gamma}, \mu_{\pi}\) is another disintegration of the measure \(\mu\) relative to \(\Gamma\), then \(\mu_\gamma = \mu_\gamma'\), for \(\mu_{\pi}\)-almost every \(\gamma \in \Gamma\).

### 3.1.2. The \(L^\infty\) and \(S^\infty\) spaces.

Let \(SB(\Sigma)\) be the space of Borel signed measures on \(\Sigma := M \times K\). Given \(\mu \in SB(\Sigma)\), denote by \(\mu^+\) and \(\mu^-\) the positive and the negative parts of its Jordan decomposition, \(\mu = \mu^+ - \mu^-\) (see remark 3.3). Let \(\pi_1 : \Sigma \rightarrow M\) be the projection defined by \(\pi_1(x, y) = x\), denote by \(\pi_{1*} : SB(\Sigma) \rightarrow SB(M)\) the pushforward map associated to \(\pi_1\). Denote by \(AB\) the set of signed measures \(\mu \in SB(\Sigma)\) such that its associated positive and negative marginal measures, \(\pi_{1*}\mu^+\) and \(\pi_{1*}\mu^-\), are absolutely continuous with respect to \(m_1\). That is,

\[
AB = \{\mu \in SB(\Sigma) : \pi_{1*}\mu^+ << m_1 \text{ and } \pi_{1*}\mu^- << m_1\}.
\]

Given a probability measure \(\mu \in AB\) on \(\Sigma\), Theorem 3.1 describes a disintegration \(\{\mu_\gamma\}_{\gamma, \mu_x}\) along \(F^s\) by a family of probability measures \(\{\mu_\gamma\}_{\gamma}\), defined on the stable leaves. Moreover, since \(\mu \in AB\), \(\mu_x\) can be identified with a non-negative marginal density \(\phi_1 : M \rightarrow \mathbb{R}\), defined almost everywhere, where \(|\phi_1| = 1\). For a non-normalized positive measure \(\mu \in AB\) we can define its disintegration following the same idea. In this case, \(\{\mu_\gamma\}\) is still a family of probability measures, \(\phi_1\) is still defined and \(|\phi_1| = \mu(\Sigma)\).

**Definition 3.2.** Let \(\pi_2 : \Sigma \rightarrow K\) be the projection defined by \(\pi_2(x, y) = y\). Consider \(\pi_{\gamma, 2} : \gamma \rightarrow K\), the restriction of the map \(\pi_2\) to the vertical leaf \(\gamma\), and the associated pushforward map \(\pi_{\gamma, 2*}\). Given a positive measure \(\mu \in AB\) and its disintegration along the stable leaves \(F^s\), \(\{\mu_\gamma\}_{\gamma, \mu_x = \phi_1 m_1}\), we define the **restriction of \(\mu\) on \(\gamma\)** and denote it by \(\mu|_{\gamma}\) as the positive measure.
on $K$ (not on the leaf $\gamma$) defined, for all measurable set $A \subset K$, as
$$\mu|_{\gamma}(A) = \pi_{\gamma_{2}}'(\phi_{1}(\gamma)\mu_{\gamma})(A).$$

For a given signed measure $\mu \in AB$ and its Jordan decomposition $\mu = \mu^{+} - \mu^{-}$, define the restriction of $\mu$ on $\gamma$ by
$$\mu|_{\gamma} = \mu^{+}|_{\gamma} - \mu^{-}|_{\gamma}.$$

**Remark 3.3.** As proved in Appendix 2 of [5], restriction $\mu|_{\gamma}$ does not depend on decomposition. Precisely, if $\mu = \mu_{1} - \mu_{2}$, where $\mu_{1}$ and $\mu_{2}$ are any positive measures, then $\mu|_{\gamma} = \mu_{1}|_{\gamma} - \mu_{2}|_{\gamma}$ m$_{1}$-a.e. $\gamma \in M$.

Let $(X,d)$ be a compact metric space, $g : X \rightarrow \mathbb{R}$ be a $\zeta$-Hölder function, and $H_{\zeta}(g)$ be its best $\zeta$-Hölder’s constant. That is,
$$H_{\zeta}(g) = \sup_{x,y \in X, x \neq y} \left\{ \frac{|g(x) - g(y)|}{d(x,y)^{\zeta}} \right\}. \quad (15)$$

In what follows, we present a generalization of the Wasserstein-Kantorovich-like metric given in [5] and [1].

**Definition 3.4.** Given two signed measures, $\mu$ and $\nu$ on $X$, we define the Wasserstein-Kantorovich-like distance between $\mu$ and $\nu$ by
$$W_{1}^{\zeta}(\mu, \nu) := \sup_{H_{\zeta}(g) \leq 1, |g|_{\infty} \leq 1} \left| \int gd\mu - \int gd\nu \right|.$$

Since $\zeta$ is a constant, we denote
$$||\mu||_{W} := W_{1}^{\zeta}(0, \mu), \quad (16)$$
and observe that $||\cdot||_{W}$ defines a norm on the vector space of signed measures defined on a compact metric space. It is worth remarking that this norm is equivalent to the standard norm of the dual space of $\zeta$-Hölder functions.

**Definition 3.5.** Let $\mathcal{L}^{\infty} \subseteq AB(\Sigma)$ be the set of signed measures defined as
$$\mathcal{L}^{\infty} = \left\{ \mu \in AB : \text{ess sup}(W_{1}^{\zeta}(\mu^{+}|_{\gamma}, \mu^{-}|_{\gamma})) < \infty \right\},$$
where the essential supremum is taken over $M$ with respect to $m_{1}$. Define the function $||\cdot||_{\infty} : \mathcal{L}^{\infty} \rightarrow \mathbb{R}$ by
$$||\mu||_{\infty} = \text{ess sup}(W_{1}^{\zeta}(\mu^{+}|_{\gamma}, \mu^{-}|_{\gamma})).$$

Finally, consider the following set of signed measures on $\Sigma$
$$\mathcal{S}^{\infty} = \{ \mu \in \mathcal{L}^{\infty} ; \phi_{1} \in H_{\zeta} \}, \quad (17)$$
and the function, $||\cdot||_{\mathcal{S}^{\infty}} : \mathcal{S}^{\infty} \rightarrow \mathbb{R}$, defined by
$$||\mu||_{\mathcal{S}^{\infty}} = |\phi_{1}|_{\zeta} + ||\mu||_{\infty}.$$
Remark 3.6. A straightforward computation yields $\| \cdot \|_{\infty} \leq \| \cdot \|_{W}$. Then, supposing that $\{ F_\delta \}_{\delta \in [0,1]}$ satisfies Theorem (B), it holds

$$\| \mu_\delta - \mu_0 \|_{W} \leq A R(\delta)^\zeta \log \delta,$$

for some $A > 0$. Therefore, for all $\zeta$-Holder function $g : \Sigma \rightarrow \mathbb{R}$, the following estimate holds

$$\left| \int g d\mu_\delta - \int g d\mu_0 \right| \leq A \| g \|_\zeta R(\delta)^\zeta \log \delta,$$

where $\| g \|_\zeta = \| g \|_\infty + H_\zeta(g)$ (see equation (15), for the definition of $H_\zeta(g)$).

Thus, for all $\zeta$-Holder function, $g : \Sigma \rightarrow \mathbb{R}$, the limit $\lim_{\delta \rightarrow 0} \int g d\mu_\delta = \int g d\mu_0$ holds, with a rate of convergence smaller than or equal to $R(\delta)^\zeta \log \delta$.

The proof of the next proposition is straightforward and can be found in [8].

Proposition 3.7. $(\mathcal{L}^\infty, \| \cdot \|_\infty)$ and $(\mathcal{S}^\infty, \| \cdot \|_{S^\infty})$ are normed vector spaces.

3.2. The transfer operator associated to $F$. In this section, we examine the transfer operator associated with skew-product maps, $F = (f, G)$, as defined in Section 1. We analyze its action on our disintegrated measure spaces, $\mathcal{L}^\infty$ and $\mathcal{S}^\infty$, which were introduced in Section 3.1.2. For the transfer operator applied to measures, a type of Perron-Frobenius formula holds (see Corollary 3.11). This formula bears some resemblance to the one that applies to one-dimensional maps.

Consider the pushforward map (also known as the ”transfer operator”) $F_* \mu$ associated with $F$, defined by

$$[F_* \mu](E) = \mu(F^{-1}(E)),$$

for each signed measure $\mu \in \mathcal{SB}(\Sigma)$ and for all measurable set $E \subset \Sigma$, where $\Sigma := M \times K$.

The reader can find the proofs of the following three results in Lemma 4.1, Proposition 2, and Corollary 2 of [9], respectively.

Lemma 3.8. For every probability $\mu \in AB$ disintegrated by $\{ \mu_\gamma \}_{\gamma}, \phi_1$, the disintegration $\{(F_* \mu)_\gamma \}_{\gamma}, (F_* \mu)_x$ of the pushforward $F_* \mu$ satisfies the following relations

$$(F_* \mu)_x = P_f(\phi_1)m_1 \quad (18)$$

and

$$(F_* \mu)_\gamma = \nu_\gamma := \frac{1}{P_f(\phi_1)(\gamma)} \sum_{i=1}^{\deg(f)} \frac{\phi_1}{\det Df_i} \circ f_i^{-1}(\gamma) \cdot \chi_{f_i(P_i)(\gamma)} \cdot F_* \mu_{f_i^{-1}(\gamma)} \quad (19)$$
when $P_f(\phi_2)(\gamma) \neq 0$. Otherwise, if $P_f(\phi_1)(\gamma) = 0$, then $\nu_\gamma$ is the Lebesgue measure on $\gamma$ (the expression $\frac{\phi_1}{|\det Df_i|} \circ f_i^{-1}(\gamma) \cdot \chi_{f_i(P_i)}(\gamma) \cdot F_{\gamma_i} \mu_{f_i^{-1}(\gamma)}$ is understood to be zero outside $f_i(P_i)$ for all $i = 1, \ldots, \deg(f)$). Here and above, $\chi_A$ is the characteristic function of the set $A$.

**Proposition 3.9.** Let $\gamma \in F^s$ be a stable leaf. Let us define the map $F_\gamma : K \to K$ by

$$F_\gamma = \pi_2 \circ F \circ \pi_{\gamma,2}^{-1}. \quad (20)$$

Then, for each $\mu \in L^\infty$ and for almost all $\gamma \in M$ it holds

$$(F_* \mu)|\gamma = \sum_{i=1}^{\deg(f)} F_{\gamma_i,*} \mu|_{\gamma_i} \rho_i(\gamma_i) \chi_{f_i(P_i)}(\gamma) \ m_1-a.e. \ \gamma \in M \quad (21)$$

where $F_{\gamma_i,*}$ is the pushforward map associated to $F_{\gamma_i}$, $\gamma_i = f_i^{-1}(\gamma)$ when $\gamma \in f_i(P_i)$ and $\rho_i(\gamma) = \frac{1}{|\det(f_i'(\gamma))|}$, where $f_i = f|_{P_i}$.

**Remark 3.10.** By (f2), (see [6]) there exists a disjoint finite family, $\mathcal{P}$, of open sets, $P_1, \ldots, P_{\deg(f)}$, s.t. $\bigcup_{i=1}^{\deg(f)} P_i = M$ $m_1$-a.e., and $f|_{P_i} : P_i \to f(P_i)$ is a diffeomorphism for all $i = 1, \ldots, \deg(f)$. Moreover, $f(P_i) = M$ $m_1$-a.e., for all $i = 1, \ldots, \deg(f)$. Therefore, it holds that

$$P_f(\varphi)(x) = \sum_{i=1}^{\deg(f)} \varphi(x_i)\rho(x_i),$$

for $m_1$-a.e. $x \in M$, where

$$\rho_i(\gamma) := \frac{1}{|\det(f_i'(\gamma))|}$$

and $f_i = f|_{P_i}$. This expression will be used later on.

Sometimes it will be convenient to use the following expression for $(F_* \mu)|\gamma$, which is a consequence of Remark 3.10 and Proposition 3.9.

**Corollary 3.11.** For each $\mu \in L^\infty$ it holds

$$(F_* \mu)|\gamma = \sum_{i=1}^{\deg(f)} F_{\gamma_i,*} \mu|_{\gamma_i} \rho_i(\gamma_i) \ m_1-a.e. \ \gamma \in M, \quad (22)$$

where $\gamma_i$ is the $i$-th pre image of $\gamma$, $i = 1, \ldots, \deg(f)$.

---

2There is nothing special about the Lebesgue measure here. We could replace it with any other positive measure.
3.3. Basic properties of the norms and convergence to equilibrium.

In this part, we list the properties of the norms and their behavior concerning the action of the transfer operator.

According to [3] and [11], a map \( f : M \to M \) satisfying (f1), (f2), and (f3) has an invariant probability measure \( m_1 \) of maximal entropy. The Perron-Frobenius operator of \( f \), denoted as \( P_f : L^1_{m_1} \to L^1_{m_1} \), satisfies the following two results, the proofs of which can be found in [6].

**Theorem 3.12.** There exist \( 0 < r < 1 \) and \( D > 0 \) such that for all \( \varphi \in H_\zeta \), and \( \int \varphi dm_1 = 0 \), it holds

\[
|P^n f(\varphi)|_\zeta \leq Dr^n|\varphi|_\zeta \quad \forall \ n \geq 1,
\]
where \( |\varphi|_\zeta := H_\zeta(\varphi) + |\varphi|_\infty \).

**Theorem 3.13.** (Lasota-Yorke inequality) There exist \( k \in \mathbb{N} \), \( 0 < \beta_0 < 1 \) and \( C > 0 \) such that, for all \( g \in H_\zeta \), it holds

\[
|P^k f g|_\zeta \leq \beta_0|g|_\zeta + C|g|_\infty,
\]
(23)
where \( |g|_\zeta := H_\zeta(g) + |g|_\infty \).

**Corollary 3.14.** There exist constants \( B_3 > 0 \), \( C_2 > 0 \) and \( 0 < \lambda < 1 \) such that for all \( g \in H_\zeta \), and all \( n \geq 1 \), it holds

\[
|P^n f g|_\zeta \leq B_3\lambda^n|g|_\zeta + C_2|g|_\infty.
\]
(24)

In the following, item (1) demonstrates the continuity and weak contraction of the transfer operator, \( F_* \), with respect to the norm \( || \cdot ||_\infty \). Items (2) and (3) provide Lasota-Yorke inequalities for the norms \( || \cdot ||_\infty \) and \( || \cdot ||_{S^\infty} \), showing a regularizing property of the transfer operator for these norms. These inequalities are also commonly referred to as Doeblin-Fortet inequalities. The proofs of equations (25), (26), (27), and (28) can be found in Proposition 3, Proposition 4, Corollary 3, and Lemma 5.2 of [6], respectively.

1. (Weak Contraction for \( || \cdot ||_\infty \)) If \( \mu \in \mathcal{L}_\infty \), then

\[
||F_* \mu||_\infty \leq ||\mu||_\infty;
\]
(25)

2. (Lasota-Yorke inequality for \( S^\infty \)) There exist \( A, B_2 > 0 \) and \( \lambda < 1 \) (\( \lambda \) of Corollary 3.14) such that, for all \( \mu \in S^\infty \), it holds

\[
||F^n_* \mu||_{S^\infty} \leq A\lambda^n||\mu||_{S^\infty} + B_2||\mu||_\infty, \quad \forall n \geq 1;
\]
(26)

3. For every signed measure \( \mu \in \mathcal{L}_\infty \), it holds

\[
||F^n_* \mu||_\infty \leq (\alpha^{\zeta})^n||\mu||_\infty + \overline{\alpha}||\phi_1||_\infty,
\]
(27)
where \( \overline{\alpha} = \frac{1}{1-\alpha^{\zeta}} \);

4. For every signed measure \( \mu \) on \( K \), such that \( \mu(K) = 0 \) it holds

\[
||F_\gamma \mu||_W \leq \alpha^{\zeta}||\mu||_W,
\]
(28)
where \( F_\gamma \) is defined in equation 20.
3.4. **Convergence to equilibrium.** Let $X$ be a compact metric space. Consider the space $SB(X)$ of Borel signed measures on $X$ and two normed vectors subspaces, $(B_s, ||\cdot||_s) \subseteq (B_w, ||\cdot||_w) \subseteq SB(X)$ with norms satisfying

$$||\cdot||_w \leq ||\cdot||_s.$$  

We say that a Markov operator $L : B_w \rightarrow B_w$

has convergence to the equilibrium with a speed of at least $\Phi$ for the norms $||\cdot||_s$ and $||\cdot||_w$, if for each $\mu \in V_s$, where

$$V_s = \{ \mu \in B_s, \mu(X) = 0 \}$$  

is the space of zero-average measures, it holds

$$||L^n(\mu)||_w \leq \Phi(n)||\mu||_s,$$

where $\Phi(n) \rightarrow 0$ as $n \rightarrow \infty$.

Let us consider the set of zero average measures in $S^\infty$ defined by

$$V_s = \{ \mu \in S^\infty : \mu(\Sigma) = 0 \}. \quad (30)$$

The proof of the next proposition can be found in [6, Proposition 6].

**Theorem 3.15** (Exponential convergence to equilibrium). There exist $D_2 \in \mathbb{R}$ and $0 < \beta_1 < 1$ such that for every signed measure $\mu \in V_s$, it holds

$$||F_n^\mu||_\infty \leq D_2 \beta_1^n||\mu||_{s^\infty},$$

for all $n \geq 1$, where $\beta_1 = \max\{\sqrt{r}, \sqrt{\alpha \zeta}\}$ and $D_2 = (\sqrt{\alpha \zeta^{-1}} + \pi D \sqrt{r^{-1}})$.

3.5. **Hölder-Measures.** In this section, we introduce in Definition 3.17 the concept of Hölder’s constant of a signed measure on $\Sigma$. We also make use of the hypotheses (G2) for the first time. Moreover, apart from satisfying equation (4), the constant $L_1$ mentioned in (f1) and (f3) is also required to be sufficiently close to 1 such that $(\alpha \cdot L_1) \zeta < 1$ (or $\alpha$ is close enough to 0). This condition is satisfied by the examples of Section 2.

We have observed that a positive measure on $M \times K$ can be disintegrated along the stable leaves $F^s$ in such a way that we can regard it as a family of positive measures on $M$, denoted by $\{\mu_\gamma\}_{\gamma \in F^s}$. Since there exists a one-to-one correspondence between $F^s$ and $M$, this defines a path in the metric space of positive measures ($SB(K)$) defined on $K$, represented by $M \mapsto SB(K)$, where $SB(K)$ is equipped with the Wasserstein-Kantorovich-like metric (see Definition 3.4).

It will be convenient to use functional notation and denote such a path by $\Gamma_\mu : M \mapsto SB(K)$, defined almost everywhere by $\Gamma_\mu(\gamma) = \mu_\gamma$, where $(\{\mu_\gamma\}_{\gamma \in M}, \phi_1)$ is some disintegration of $\mu$. However, since this disintegration is defined $\mu$-a.e., $\gamma \in M$, the path $\Gamma_\mu$ is not unique. For this reason, we define $\Gamma_\mu$ as the class of almost everywhere equivalent paths corresponding to $\mu$. 
Definition 3.16. Consider a positive Borelean measure $\mu$ on $M \times K$, and a disintegration $\omega = \{\{\mu_\gamma\}_{\gamma \in M}, \phi_1\}$, where $\{\mu_\gamma\}_{\gamma \in M}$ is a family of probabilities on $M \times K$ defined $\mu$-a.e. $\gamma \in M$ (where $\hat{\mu} := \pi_1^*\mu = \phi_1 m_1$) and $\phi_1 : M \to \mathbb{R}$ is a non-negative marginal density. Denote by $\Gamma_\mu$ the class of equivalent paths associated to $\mu$

\[ \Gamma_\mu = \{\Gamma_\omega\}_{\omega} \]

where $\omega$ ranges on all the possible disintegrations of $\mu$ and $\Gamma_\omega : M \to SB(K)$ is the map associated to a given disintegration, $\omega$:

\[ \Gamma_\omega(\gamma) = \mu_\gamma = \pi_{\gamma,2}^* \phi_1(\gamma) \mu_\gamma. \]

Let us call the set on which $\Gamma_\omega$ is defined by $I_{\Gamma_\omega} \subset M$.

Definition 3.17. For a given $0 < \zeta < 1$, a disintegration $\omega$ of $\mu$, and its functional representation $\Gamma_\omega$, we define the $\zeta$-Hölder constant of $\mu$ associated to $\omega$ by

\begin{equation}
|\mu|^\omega_\zeta := \text{ess sup}_{\gamma_1,\gamma_2 \in I_{\Gamma_\omega}} \left\{ \frac{||\mu|_{\gamma_1} - \mu|_{\gamma_2}||_W}{d_1(\gamma_1, \gamma_2)^\zeta} \right\}.
\end{equation}

Finally, we define the $\zeta$-Hölder constant of the positive measure $\mu$ by

\begin{equation}
|\mu|_\zeta := \inf_{\Gamma_\omega \in \Gamma_\mu} \{|\mu|^\omega_\zeta\}.
\end{equation}

Remark 3.18. When no confusion is possible, to simplify the notation, we denote $\Gamma_\omega(\gamma)$ just by $\mu|_\gamma$.

Definition 3.19. From the Definition 3.17 we define the set of the $\zeta$-Hölder's positive measures $\mathcal{H}_\zeta^+$ as

\begin{equation}
\mathcal{H}_\zeta^+ = \{\mu \in AB : \mu \geq 0, |\mu|_\zeta < \infty\}.
\end{equation}

For the next lemma, for a given path $\Gamma_\mu$ which represents the measure $\mu$, we define for each $\gamma \in I_{\Gamma_\mu} \subset M$, the map

\begin{equation}
\mu_F(\gamma) := F_\gamma^* \mu|_\gamma,
\end{equation}

where $F_\gamma : K \to K$ is defined as

\begin{equation}
F_\gamma(y) = \pi_2 \circ F \circ (\pi_2|_\gamma)^{-1}(y)
\end{equation}

and $\pi_2 : M \times K \to K$ is the second coordinate projection $\pi_2(x, y) = y$.

The proofs of Lemma 3.20, Proposition 3.23, and Corollary 3.24 can be found in Lemma 7.4, Proposition 9, and Corollary 4 of [6].

Lemma 3.20. Suppose that $F : \Sigma \to \Sigma$ satisfies (G1) and (G2). Then, for all $\mu \in \mathcal{H}_\zeta^+$ which satisfy $\phi_1 = 1 m_1$-a.e., it holds

\[ ||F(x) \mu|_x - F(y) \mu|_y||_W \leq \alpha^\zeta|\mu|_\zeta d_1(x, y)^\zeta + |G|_\zeta d_1(x, y)^\zeta||\mu||_\infty, \]

for all $x, y \in P_i$ and all $i = 1, \ldots, \deg(f)$. 

Corollary 3.21. Let \( \{F_\delta\}_{\delta \in [0,1)} \) an admissible \( R(\delta) \)-perturbation and \( \gamma_{\delta,i} \) the \( i \)-th pre-image of \( \gamma \in \mathcal{M} \) by \( f_\delta \), \( i = 1, \cdots, \deg(f_\delta) \). Then, for all \( \mu \in H_\delta^+ \) which satisfy \( \phi_1 = 1 \) \( m_1 \)-a.e., the following inequality holds:

\[
\left\| (F_{0, \gamma_{0,i}} * - F_{0, \gamma_{\delta,i}} *) \mu \right\|_{\gamma_{0,i}, W} \leq R(\delta)^{\zeta} (2\alpha^\zeta |\mu|_\zeta + |G|_\zeta |\mu|_\infty), \forall i = 1, \cdots, \deg(f),
\]

where \( F_{\delta, \gamma_{\delta,i}} \) is defined by equation (35), for all \( \delta \in [0,1) \).

Proof. To simplify the notation, we denote \( F := F_0 \) and \( \gamma := \gamma_{0,i} \). Thus, we have

\[
\left\| (F_{0, \gamma_{0,i}} * - F_{0, \gamma_{\delta,i}} *) \mu \right\|_{\gamma_{0,i}, W} = \left\| (F_{\gamma} * - F_{\gamma_{\delta,i}} *) \mu \right\|_{\gamma_{0,i}, W}
\]

\[
= \left\| F_{\gamma} * \mu \gamma - F_{\gamma_{\delta,i}} * \mu \gamma \right\|_{\gamma_{0,i}, W}
\]

\[
\leq \left\| F_{\gamma} * \mu \gamma - F_{\gamma_{\delta,i}} * \mu \gamma_{\delta,i} \right\|_{\gamma_{0,i}, W} + \left\| F_{\gamma_{\delta,i}} * (\mu |_{\gamma_{\delta,i}} - \mu |_\gamma) \right\|_{\gamma_{0,i}, W}
\]

Since \( \phi_1 = 1 \) \( m_1 \)-a.e., \( \mu |_{\gamma_{\delta,i}} - \mu |_\gamma \) has zero average. Therefore, by Lemma 3.20, equation (28), (U2.2) and definition (3.17) applied on \( \mu \), we get

\[
\left\| (F_{0, \gamma_{0,i}} * - F_{0, \gamma_{\delta,i}} *) \mu \right\|_{\gamma_{0,i}, W} \leq \alpha^\zeta |\mu|_\zeta d_1 (\gamma_{\delta,i}, \gamma) + |G|_\zeta d_1 (\gamma_{\delta,i}, \gamma) |\mu|_\infty
\]

\[
\leq R(\delta)^{\zeta} (2\alpha^\zeta |\mu|_\zeta + |G|_\zeta |\mu|_\infty).
\]

\( \square \)

Lemma 3.22. Let \( \{F_\delta\}_{\delta \in [0,1)} \) an admissible \( R(\delta) \)-perturbation and \( \gamma_{\delta,i} \) the \( i \)-th pre-image of \( \gamma \in \mathcal{M} \) by \( f_\delta \), \( i = 1, \cdots, \deg(f_\delta) \). Then, the following inequality holds:

\[
\left\| (F_{0, \gamma_{\delta,i}} * - F_{0, \gamma_{\delta,i}} *) \mu \right\|_{\gamma_{0,i}, W} \leq ||\mu|_{\gamma_{0,i}}|| R(\delta)^{\zeta}, \forall i = 1, \cdots, \deg(f),
\]

where \( F_{\delta, \gamma_{\delta,i}} \) is defined by equation (35), for all \( \delta \in [0,1) \).

Proof. To simplify the notation, we denote \( \gamma := \gamma_{\delta,i} \). Thus, by definition (3.4) and (U2.3), we have
\begin{align*}
\left| (F_{0,\gamma} * F_{\delta,\gamma} )\mu|_{\gamma_0,i} \right|_W &= \left| (F_{0,\gamma} * F_{\delta,\gamma} )\mu|_{\gamma_0,i} \right|_W \\
&= \sup_{H_\epsilon(g) \leq 1, |g| \leq 1} \left| \int g d(F_{0,\gamma} * \mu|_{\gamma_0,i} - F_{\delta,\gamma} * \mu|_{\gamma_0,i}) \right| \\
&= \sup_{H_\epsilon(g) \leq 1, |g| \leq 1} \left| \int g(G_0(\gamma, y)) - g(G_\delta(\gamma, y)) d\mu|_{\gamma_0,i} \right| \\
&\leq \sup_{H_\epsilon(g) \leq 1, |g| \leq 1} \int |g(G_0(\gamma, y)) - g(G_\delta(\gamma, y))| d\mu|_{\gamma_0,i} \\
&\leq \int d_2(G_0(\gamma, y), G_\delta(\gamma, y)) d\mu|_{\gamma_0,i} \\
&\leq R(\delta)^c |\int 1 d\mu|_{\gamma_0,i} | \\
&\leq R(\delta)^c ||\mu||_{\gamma_0,i} ||_W.
\end{align*}

For the next, proposition and henceforth, for a given path \( \Gamma_\omega \in \Gamma_\mu \) (associated with the disintegration \( \omega = \{ \mu_\gamma \}_\gamma, \phi_1 \), of \( \mu \), unless written otherwise, we consider the particular path \( \Gamma_\omega F_* \mu \in \Gamma_{F_* \mu} \) defined by Corollary 3.11 by the expression

\[
\Gamma_{F_* \mu}^\omega (\gamma) = \sum_{i=1}^{\deg(f)} \Gamma_{\gamma_i} F_* \mu (\gamma_i) \rho_1(\gamma_i) \quad m_1\text{-a.e. } \gamma \in M. \tag{36}
\]

Recall that \( \Gamma_{\mu}^\omega (\gamma) = \mu|_\gamma := \pi_{2*}(\phi_1(\gamma)\mu_\gamma) \) and in particular \( \Gamma_{F_* \mu}^\omega (\gamma) = (F_* \mu)|_\gamma = \pi_{2*}(P F \phi_1(\gamma)\mu_\gamma) \), where \( \phi_1 = \frac{d\pi_{1*} \mu}{dm_1} \) and \( P \) is the Perron-Frobenius operator of \( f \).

**Proposition 3.23.** If \( F : \Sigma \longrightarrow \Sigma \) satisfies (f1), (f2), (f3), (G1), (G2) and \((\alpha \cdot L_1)^c < 1\), then there exist \( 0 < \beta < 1 \) and \( D > 0 \), such that for all \( \mu \in \mathcal{H}_\zeta^+ \) which satisfy \( \phi_1 = 1 \) \( m_1\)-a.e. and for all \( \Gamma_\mu^\omega \in \Gamma_\mu \), it holds

\[
|\Gamma_{F_* \mu}^\omega|_\zeta \leq \beta |\Gamma_{\mu}^\omega|_\zeta + D_2 ||\mu||_\infty,
\]

for \( \beta := (\alpha L_1)^c \) and \( D_2 := \{ \epsilon \rho L_1^2 + |G|_\zeta L_1^2 \} \).

**Corollary 3.24.** Suppose that \( F : \Sigma \longrightarrow \Sigma \) satisfies (f1), (f2), (f3), (G1), (G2) and \((\alpha \cdot L_1)^c < 1\). Then, for all \( \mu \in \mathcal{H}_\zeta^+ \) which satisfy \( \phi_1 = 1 \) \( m_1\)-a.e. and \( ||F_* \mu||_\infty \leq ||\mu||_\infty \), it holds

\[
|\Gamma_{F_* \mu}^\omega|_\zeta \leq \beta^n |\Gamma_{\mu}^\omega|_\zeta + \frac{D_2}{1 - \beta} ||\mu||_\infty, \tag{37}
\]

for all \( n \geq 1 \), where \( \beta \) and \( D_2 \) are from Proposition 3.23.
Remark 3.25. Taking the infimum over all paths $\Gamma^\omega_\mu \in \Gamma_\mu$ and all $\Gamma^\omega_{F^* n \mu} \in \Gamma_{F^* n \mu}$ on both sides of inequality (37), we get

$$|F^* n \mu|_\zeta \leq \beta^n |\mu|_\zeta + \frac{D_2}{1-\beta}||\mu||_\infty.$$  (38)

The above Equation (38) will provide a uniform bound (see the proof of Theorem 3.26) for the H"older’s constant of the measure $F^* n m$, for all $n$ where $m$ is defined, as the product $m = m_1 \times \nu$, for a fixed probability measure $\nu$ on $K$. The uniform bound will be useful later on.

Remark 3.26. Consider the probability measure $m$ defined in Remark 3.25, i.e., $m = m_1 \times \nu$, where $\nu$ is a given probability measure on $K$ and $m_1$ is the $f$-invariant measure fixed in Section 1. Besides that, consider its trivial disintegration $\omega_0 = \{m_\gamma\}_{\gamma}$, given by $m_\gamma = \pi^{-1/2}_\gamma \nu$, for all $\gamma$ and $\phi_1 \equiv 1$. According to this definition, it holds that $m|_\gamma = \nu$, $\forall \gamma$.

In other words, the path $\Gamma^\omega_\mu$ is constant: $\Gamma^\omega_\mu(\gamma) = \nu$ for all $\gamma$. Moreover, for each $n \in \mathbb{N}$, let $\omega_n$ be the particular disintegration of the measure $F^* n m$ defined from $\omega_0$ as an application of Lemma 3.8 and consider the path $\Gamma^\omega_{F^* n m}$ associated with this disintegration. By Proposition 3.9, we have

$$\Gamma^\omega_{F^* n m}(\gamma) = \sum_{i=1}^s \frac{F^* n (\gamma)_i}{|\det Df^n \circ f^{-n}(\gamma)|} \chi_{f^n(P_i)}(\gamma) \ m_1 \ a.e \ \gamma \in M, \quad (39)$$

where $P_i$, $i = 1, \cdots, s = s(n)$, ranges over the partition $P^{(n)}$ defined in the following way: for all $n \geq 1$, let $P^{(n)}(x) = P^{(n)}(y)$ if and only if $P^{(1)}(f^n(x)) = P^{(1)}(f^n(y))$ for all $j = 0, \cdots, n-1$, where $P^{(1)} = P$ (see remark 3.10). This path will be used in the next section 4.

The following result is an estimate for the regularity of the invariant measure of $F$ and its proof can be found in Theorem 7.5 of [6]. This sort of result has many applications and can also be found in [5] and [7], wherein [7] the authors reach an analogous result for random dynamical systems.

Theorem 3.27. Suppose that $F : \Sigma \rightarrow \Sigma$ satisfies (f1), (f2), (f3), (G1), (G2) and $(\alpha \cdot L_1)^\zeta < 1$ and consider the unique $F$-invariant probability $\mu_0 \in S^\infty$. Then $\mu_0 \in H^+_\zeta$ and

$$|\mu_0|_\zeta \leq \frac{D_2}{1-\beta},$$

where $D_2$ and $\beta$ are from Proposition 3.23.

4. Properties of Admissible $R(\delta)$-Perturbations

In this section, we will prove some properties about admissible $R(\delta)$-perturbations perturbations. These properties will be used in the following sections, specifically to prove Theorem 3.
Lemma 4.1. Let \( \{ F_\delta \}_{\delta \in [0,1)} \) be an admissible \( R(\delta) \)-perturbation. Denote by \( F_\delta \) their transfer operators, and by \( \mu_\delta \) their fixed points (probabilities) in \( S^\infty \). Suppose that the family \( \{ \mu_\delta \}_{\delta \in [0,1)} \) satisfies

\[
|\mu_\delta|_\zeta \leq B_u,
\]

for all \( \delta \in [0, \delta_1) \). Then, there is a constant \( C_1 \) such that, it holds

\[
|| (F_0 - F_\delta) \mu_\delta ||_\infty \leq C_1 R(\delta)^\zeta,
\]

for all \( \delta \in [0, \delta_1) \), where \( C_1 := |G_0|_\zeta + 3B_u + 2 \).

Proof. Let us estimate

\[
|| (F_0 - F_\delta) \mu_\delta ||_\infty = \text{ess sup}_M || (F_0 - F_\delta) \mu_\delta ||_\gamma.
\]

Denote by \( f_{\delta,i} \), with \( 1 \leq i \leq \text{deg}(f) \), the branches of \( f_\delta \) defined in the sets \( P_i \in \mathcal{P} \) (where \( \mathcal{P} \) depends on \( \delta \)), \( f_{\delta,i} = f_\delta|_{P_i} \). Moreover, remember that we denote \( \gamma_{\delta,i} := f_{\delta,i}^{-1}(\gamma) \) for all \( \gamma \in M \), and by (U2.2) there exists \( R(\delta) \) such that

\[
d_1(\gamma_{0,i}, \gamma_{\delta,i}) \leq R(\delta) \quad \forall i = 1 \cdots \text{deg}(f).
\]

We also recall that by (U1) \( \text{deg}(f_\delta) = \text{deg}(f) \) for all \( \delta \in [0, \delta_1) \).

Thus, denoting \( F_{\delta,\gamma_{\delta,i}} := F_{\delta,f_{\delta,i}^{-1}(\gamma)} \) and \( \mu := \mu_\delta \), we get

\[
(F_0 \mu - F_\delta \mu) |_{\gamma} = \sum_{i=1}^{\text{deg}(f)} \frac{F_0 |_{\gamma_{0,i}} \mu |_{\gamma_{0,i}}}{\det Df_0(\gamma_{0,i})} - \sum_{i=1}^{\text{deg}(f)} \frac{F_{\delta,\gamma_{\delta,i}} \mu |_{\gamma_{\delta,i}}}{\det Df_{\delta}(\gamma_{\delta,i})}, \quad \mu_x - a.e. \quad \gamma \in M.
\]

Then, we have

\[
|| (F_0 - F_\delta) \mu ||_\infty \leq I + II,
\]

where

\[
I := \text{ess sup}_M \left\| \sum_{i=1}^{\text{deg}(f)} \frac{F_0 |_{\gamma_{0,i}} \mu |_{\gamma_{0,i}}}{\det Df_0(\gamma_{0,i})} - \sum_{i=1}^{\text{deg}(f)} \frac{F_{\delta,\gamma_{\delta,i}} \mu |_{\gamma_{\delta,i}}}{\det Df_{\delta}(\gamma_{\delta,i})} \right\|_W
\]

and

\[
II := \text{ess sup}_M \left\| \sum_{i=1}^{\text{deg}(f)} \frac{F_{\delta,\gamma_{\delta,i}} \mu |_{\gamma_{\delta,i}}}{\det Df_{\delta}(\gamma_{\delta,i})} - \sum_{i=1}^{\text{deg}(f)} \frac{F_{\delta,\gamma_{\delta,i}} \mu |_{\gamma_{\delta,i}}}{\det Df_{\delta}(\gamma_{\delta,i})} \right\|_W.
\]

Let us estimate \( I \) of equation (42). An analogous application of the triangular inequality, we have

\[
I \leq \text{ess sup}_M I_a(\gamma) + \text{ess sup}_M I_b(\gamma),
\]

where
\[ I_a(\gamma) := \left\| \sum_{i=1}^{\deg(f)} \frac{F_{0,\gamma_0,i} \ast \mu|_{\gamma_0,i}}{\det Df_0(\gamma_0,i)} - \sum_{i=1}^{\deg(f)} \frac{F_{\delta,\gamma_\delta,i} \ast \mu|_{\gamma_0,i}}{\det Df_0(\gamma_\delta,i)} \right\|_W \]  

and

\[ I_b(\gamma) := \left\| \sum_{i=1}^{\deg(f)} \frac{F_{\delta,\gamma_\delta,i} \ast \mu|_{\gamma_0,i}}{\det Df_0(\gamma_0,i)} - \sum_{i=1}^{\deg(f)} \frac{F_{\delta,\gamma_\delta,i} \ast \mu|_{\gamma_0,i}}{\det Df_\delta(\gamma_\delta,i)} \right\|_W . \]  

The summands will be treated separately.

For \( I_a \), we note that

\[ I_a(\gamma) \leq \left\| \sum_{i=1}^{\deg(f)} \frac{F_{0,\gamma_0,i} \ast \mu|_{\gamma_0,i}}{\det Df_0(\gamma_0,i)} - \sum_{i=1}^{\deg(f)} \frac{F_{\delta,\gamma_\delta,i} \ast \mu|_{\gamma_0,i}}{\det Df_0(\gamma_\delta,i)} \right\|_W \]

\[ \leq \sum_{i=1}^{\deg(f)} \frac{\|F_{0,\gamma_0,i} \ast \mu|_{\gamma_0,i}\|_W}{\det Df_0(\gamma_0,i)} \]

\[ \leq \sum_{i=1}^{\deg(f)} \frac{\|F_{0,\gamma_0,i} \ast F_{\delta,\gamma_\delta,i} \ast \mu|_{\gamma_0,i}\|_W}{\det Df_0(\gamma_0,i)} + \sum_{i=1}^{\deg(f)} \frac{\|F_{\delta,\gamma_\delta,i} \ast \mu|_{\gamma_0,i}\|_W}{\det Df_0(\gamma_\delta,i)} . \]

Now we note that \( \mu \), satisfy \( \phi_1 \equiv 1 \). By Remark 1.3, Corollary 3.21 and Lemma 3.22 applied on the last inequality above, we have

\[ I_a(\gamma) \leq \left( \sum_{i=1}^{\deg(f)} \frac{1}{\det Df_0(\gamma_0,i)} \right) R(\delta)^\zeta (2B_u + |G_0|_\zeta + 1) . \]

For \( I_b(\gamma) \), by (U2.1) we have

\[ I_b(\gamma) \leq \left( \sum_{i=1}^{\deg(f)} \frac{1}{\det Df_0(\gamma_0,i)} \right) R(\delta)^\zeta \left( |\mu|_{\gamma_0,i} \right)_W . \]
Let us estimate II. By (Remark 1.3), note that
\[ \sum_{i=1}^{\deg(f)} \left| \frac{1}{\det Df_{\delta_i}(\gamma_{\delta,i})} \right| = 1 \]
m_{1\text{-a.e.}}. Thus, we have
\[ \text{II} \leq \text{ess sup}_M \sum_{i=1}^{\deg(f)} \left| \frac{1}{\det Df_{\delta}(\gamma_{\delta,i})} \left( \left| F_{\delta,\gamma_{\delta,i}} * (\mu|_{\gamma_{0,i}} - \mu|_{\gamma_{\delta,i}}) \right| \right) \right|_W \]
\[ \leq \text{ess sup}_M \sum_{i=1}^{\deg(f)} \left| \frac{1}{\det Df_{\delta}(\gamma_{\delta,i})} \left| \left| d_1(\gamma_{\delta,i}, \gamma_{0,i}) \right| \right|_W \right| \]
\[ \leq \text{ess sup}_M \sum_{i=1}^{\deg(f)} \left| \frac{1}{\det Df_{\delta}(\gamma_{\delta,i})} \right| R(\delta)^\xi |\mu|_\zeta \]
\[ \leq R(\delta)^\xi B_u. \]

Since \( \zeta < 1 \), then \( \delta \leq \delta^\xi \). Thus, all these facts yield
\[ \left| |(F_0 - F_{\delta})\mu_\delta|_\infty \right| \leq I + \text{II} \]
\[ \leq I_{1a} + I_{1b} + \text{II} \]
\[ \leq R(\delta)^\xi (2B_u + |G_0|_\zeta + 1) + R(\delta)^\xi + R(\delta)^\xi B_u \]
\[ \leq C_1 R(\delta)^\xi, \]
where \( C_1 := |G_0|_\zeta + 3B_u + 2. \)

The following result is an important tool to reach Theorem B. It states that the function
\[ \delta \mapsto |\mu_\delta|_\zeta \]
(see Definition 3.17) is uniformly bounded, where \( \{\mu_\delta\}_{\delta \in [0,1]} \) is the family of \( F_\delta \)-invariant probabilities of an admissible perturbation \( \{F_\delta\}_{\delta \in [0,1]} \) of \( F (= F_0) \).

**Lemma 4.2.** Let \( \{F_\delta\}_{\delta \in [0,1]} \) be an admissible \( R(\delta) \)-perturbation and \( \mu_\delta \) be the unique \( F_\delta \)-invariant probability in \( S^\infty \), for all \( \delta \in [0,1] \). Then, there exists \( B_u > 0 \) such that
\[ |\mu_\delta|_\zeta \leq B_u, \]
for all \( \delta \in [0,1] \).

First, we need a preliminary sublemma.

**Sublemma 4.1.** If \( \{F_\delta\}_{\delta \in [0,1]} \) is an admissible \( R(\delta) \)-perturbation. Then, there exist uniform constants \( 0 < \beta_u < 1 \) and \( D_{2,u} > 0 \) such that for every
\( \mu \in \mathcal{H}_\xi^t \) which satisfies \( \phi_1 = 1 \) \( m_1 \)-a.e., it holds

\[
|\Gamma_{F_{\delta}^n}^\omega \mu|_\zeta \leq \beta_u |\mu|_\zeta + \frac{D_{2,u}}{1 - \beta_\delta} ||\mu||_\infty,
\]

for all \( \delta \in [0,1) \) and all \( n \geq 0 \).

Proof. We apply Corollary 3.24 to each \( F_\delta \) and obtain,

\[
|F_\delta^* \mu|_\zeta \leq \beta_\delta |\mu|_\zeta + D_{2,\delta} ||\mu||_\infty, \ \forall \delta \in [0,1),
\]

where \( \beta_\delta := (\alpha_\delta L_{1,\delta})^\zeta \) and \( D_{2,\delta} := \{ \epsilon_{\rho, \delta} L_{1,\delta}^\zeta + |G_{\delta}| L_{1,\delta}^\zeta \} \).

By A2, we define \( \beta_u := \sup_{\delta} \beta_\delta \) and \( D_{2,u} := \sup_{\delta} D_{2,\delta} \), and the result is established.

\[\square\]

Proof. (of Lemma 3.2)

Consider path \( \Gamma_{F_{\delta}^n}^\omega m \), defined in Remark 3.26 which represents the measure \( F_{\delta}^n m \).

According to Theorem A, let \( \mu_\delta \in S^\infty \) be the unique \( F_\delta \)-invariant probability measure in \( S^\infty \). Consider the measure \( m \), defined in Remark 3.26 and its iterates \( F_{\delta}^n(m) \). By Theorem 3.15, these iterates converge to \( \mu_\delta \) in \( L^\infty \).

It implies that the sequence \( \{ \Gamma_{F_{\delta}^n}^\omega m \} \) converges \( m_1 \)-a.e. to \( \Gamma_\delta^\omega \) in \( S^\infty \).

By A2, we define \( \beta_u := \sup_{\delta} \beta_\delta \) and \( D_{2,u} := \sup_{\delta} D_{2,\delta} \), and the result is established.

\[\square\]

Proof. (of Lemma 3.2)

Consider path \( \Gamma_{F_{\delta}^n}^\omega m \), defined in Remark 3.26 which represents the measure \( F_{\delta}^n m \).

According to Theorem A, let \( \mu_\delta \in S^\infty \) be the unique \( F_\delta \)-invariant probability measure in \( S^\infty \). Consider the measure \( m \), defined in Remark 3.26 and its iterates \( F_{\delta}^n(m) \). By Theorem 3.15, these iterates converge to \( \mu_\delta \) in \( L^\infty \).

It implies that the sequence \( \{ \Gamma_{F_{\delta}^n}^\omega m \} \) converges \( m_1 \)-a.e. to \( \Gamma_{\mu_\delta}^\omega \) in \( S^\infty \).

By A2, we define \( \beta_u := \sup_{\delta} \beta_\delta \) and \( D_{2,u} := \sup_{\delta} D_{2,\delta} \), and the result is established.

\[\square\]
On the other hand, by Lemma 4.2, the argument of the left-hand side is bounded by $|\Gamma_{n,\delta}|_\zeta \leq D_u \frac{1}{1-\beta_u}$ for all $n \geq 1$. Then,

$$\frac{||\Gamma_\delta(x) - \Gamma_\delta(y)||_w}{d_1(x,y)\zeta} \leq D_u \frac{1}{1-\beta_u}.$$ 

Thus $|\Gamma_\omega_{\mu_\delta}|_\zeta \leq D_u \frac{1}{1-\beta_u}$, and taking the infimum we get $|\mu_\delta|_\zeta \leq D_u \frac{1}{1-\beta_u}$. We finish the proof defining $B_u := D_u \frac{1}{1-\beta_u}$. □

5. Perturbation of Operators

The main results of this article (Theorems B and Corollary 1.4) are proven by demonstrating that an admissible $R(\delta)$-perturbation induces a family of transfer operators, $T_\delta : B_w \rightarrow B_w$, referred to as the $(R(\delta), \zeta)$-family of operators, which is defined in the following paragraph. The main tool used to establish this is Lemma 5.2, which is stated and proved in this section.

Definition 5.1. Suppose there are vector spaces $(B_w, ||\cdot||_w)$ and $(B_s, ||\cdot||_s)$, satisfying $B_s \subset B_w$ and $||\cdot||_s \geq ||\cdot||_w$, where the actions $T_\delta : B_w \rightarrow B_w$, $T_\delta : B_s \rightarrow B_s$ are well defined and, for each $\delta \in [0,1)$, $\mu_\delta \in B_s$ is a fixed point for $T_\delta$. Moreover, suppose that:

(P1) There are $C \in \mathbb{R}^+$ and a real-valued function $\delta \mapsto R(\delta) \in \mathbb{R}^+$, defined on $[0,1)$, such that

$$\lim_{\delta \to 0^+} R(\delta) \log(\delta) = 0$$

and

$$||(T_0 - T_\delta)\mu_\delta||_w \leq R(\delta)^\zeta C \forall \delta \in [0,1);$$

(P2) Suppose there is $M > 0$ such that for all $\delta \in [0,1)$, it holds

$$||\mu_\delta||_s \leq M;$$

(P3) $T_0$ has exponential convergence to equilibrium with respect to the norms $||\cdot||_s$ and $||\cdot||_w$: there exists $0 < \rho_2 < 1$ and $C_2 > 0$ such that

$$\forall \mu \in V_s := \{\mu \in B_s : \mu(\Sigma) = 0\}$$

it holds

$$||T_0^n \mu||_w \leq \rho_2^n C_2 ||\mu||_s;$$

(P4) The iterates of the operators are uniformly bounded for the weak norm: there exists $M_2 > 0$ such that for all $\delta \in [0,1)$, all $n \in \mathbb{N}$, and all $\nu \in B_s$, it holds $||T_\delta^n \nu||_w \leq M_2 ||\nu||_w$.

A family of operators that satisfies (P1), (P2), (P3) and (P4) is called a $(R(\delta), \zeta)$-family of operators.
The following Lemma 5.2 establishes a general and quantitative relation between the variation of the fixed points, \( \{ \mu_\delta \}_{\delta \in [0,1)} \), of a \((R(\delta), \zeta)\)-family of operators concerning the parameter \( \delta \). It states that the function \( \delta \mapsto \mu_\delta \), given by

\[
\delta \mapsto T_\delta \mapsto \mu_\delta, \quad \delta \in [0,1)
\]
vary continuously at 0, with respect to the norm \( \| \cdot \|_w \), and provides an explicit bound for its modulus of continuity: \( D_1 R(\delta)^C \log \delta, \) where \( D_1 \geq 0 \).

**Lemma 5.2** (Quantitative stability for fixed points of operators). Suppose \( \{ T_\delta \}_{\delta \in [0,1)} \) is a \((R(\delta), \zeta)\)-family of operators, where \( \mu_0 \) is the unique fixed point of \( T_0 \) in \( B_w \) and \( \mu_\delta \) is a fixed point of \( T_\delta \). Then, there exist constants \( D_1 < 0 \) and \( \delta_0 \in (0,1) \) such that for all \( \delta \in [0, \delta_0) \), it holds

\[
\| \mu_\delta - \mu_0 \|_w \leq D_1 R(\delta)^C \log \delta.
\]

To prove Lemma 5.2, we state a general result on the stability of fixed points. We will omit its proof, but the reader can find it for instance in [5], Lemma 12.1.

Consider two operators \( T_0 \) and \( T_\delta \) preserving a normed space of signed measures \( B \subseteq SB(X) \) with norm \( \| \cdot \|_B \). Suppose that \( f_0, f_\delta \in B \) are fixed points of \( T_0 \) and \( T_\delta \), respectively.

**Sublemma 5.1.** Suppose that:

a) \( \| T_\delta f_\delta - T_0 f_\delta \|_B < \infty \);

b) For all \( i \geq 1 \), \( T_0^i \) is continuous on \( B \): for each \( i \geq 1 \), \( \exists C_i \) s.t. \( \forall g \in B, \| T_0^i g \|_B \leq C_i \| g \|_B \).

Then, for each \( N \geq 1 \), it holds

\[
\| f_\delta - f_0 \|_B \leq \| T_0^N (f_\delta - f_0) \|_B + \| T_\delta f_\delta - T_0 f_\delta \|_B \sum_{i \in [0, N-1]} C_i.
\]

**Proof.** (of Lemma 5.2)

First, note that if \( \delta \geq 0 \) is small enough, then \( \delta \leq -\delta \log \delta \). Moreover, \( x - 1 \leq \lfloor x \rfloor \), for all \( x \in \mathbb{R} \).

By P1,

\[
\| T_\delta \mu_\delta - T_0 \mu_\delta \|_w \leq R(\delta)^C \]

(see Lemma 5.1, item a) ) and P4 yields \( C_i \leq M_2 \).

Hence, by Lemma 5.1 we have

\[
\| \mu_\delta - \mu_0 \|_w \leq R(\delta)^C M_2 N + \| T_0^N (\mu_0 - \mu_\delta) \|_w.
\]

By the exponential convergence to equilibrium of \( T_0 \) (P3), there exists \( 0 < \rho_2 < 1 \) and \( C_2 > 0 \) such that (recalling that by P2 \( \| (\mu_\delta - \mu_0) \|_s \leq 2M) \)

\[
\| T_0^N (\mu_\delta - \mu_0) \|_w \leq C_2 \rho_2^N \| (\mu_\delta - \mu_0) \|_s \leq 2C_2 \rho_2^N M
\]

hence

\[
\| \mu_\delta - \mu_0 \|_w \leq R(\delta)^C M_2 N + 2C_2 \rho_2^N M.
\]
Choosing $N = \left\lfloor \frac{\log \delta}{\log \rho_2} \right\rfloor$, we have

$$
||\mu_\delta - \mu_0||_w \leq R(\delta)\zeta C M_2\left\lfloor \frac{\log \delta}{\log \rho_2} \right\rfloor + 2C_2 \rho_2 \frac{\log \delta}{\log \rho_2} + 1 M
$$

$$
\leq R(\delta)\zeta \log C M_2 \frac{1}{\log \rho_2} + 2C_2 \rho_2 \frac{\log \delta}{\log \rho_2} + 1 M
$$

$$
\leq R(\delta)\zeta \log C M_2 \frac{1}{\log \rho_2} + 2C_2 \frac{\log \delta}{\log \rho_2} M
$$

$$
\leq R(\delta)\zeta \log C M_2 \frac{1}{\log \rho_2} + 2C_2 \delta M
$$

$$
\leq R(\delta)\zeta \log C M_2 \frac{1}{\log \rho_2} - 2C_2 \delta \log M
$$

$$
\leq R(\delta)\zeta \left( \frac{CM_2}{\log \rho_2} - \frac{2C_2 M}{\rho_2} \right).
$$

We finish the proof by setting, $D_1 = \frac{CM_2}{\log \rho_2} - \frac{2C_2 M}{\rho_2}$. □

6. Proof of Theorem A

First, let us prove the existence and uniqueness of an $F$-invariant measure in $S^\infty$.

The following lemma 6.1 ensures the existence and uniqueness of an $F$-invariant measure that projects onto $m_1$. Since its proof is based on standard arguments (see [10], for instance), we will omit it here.

**Lemma 6.1.** There exists a unique measure $\mu_0$ on $M \times K$ such that for every continuous function $\psi \in C^0(M \times K)$, it holds

$$
\psi^+ = \psi^- = \int \psi d\mu_0,
$$

(47)

where

$$
\psi^- := \lim_{n \to \infty} \int_{(\gamma,y) \in \gamma \times K} \inf \psi \circ F^n(\gamma,y) d m_1(\gamma)
$$

and

$$
\psi^+ := \lim_{n \to \infty} \int_{(\gamma,y) \in \gamma \times K} \sup \psi \circ F^n(\gamma,y) d m_1(\gamma).
$$

Moreover, the measure $\mu_0$ is $F$-invariant and $\pi_{1*}\mu_0 = m_1$.

Let $\mu_0$ be the $F$-invariant measure such that $\pi_{1*}\mu_0 = m_1$ (which exists by Lemma 6.1), where 1 is the unique $f$-invariant density in $H_\zeta$. Suppose that $g : K \to \mathbb{R}$ is a $\zeta$-Hölder function such that $|g|_\infty \leq 1$ and $H_\zeta(g) \leq 1$. Then, it holds $|\int g d(\mu_0)| \leq |g|_\infty \leq 1$. Hence, $\mu_0 \in L^\infty$. Since $\frac{\pi_{1*}\mu_0}{dm_1} \equiv 1$, we have $\mu_0 \in S^\infty$.

The uniqueness follows directly from Theorem 3.15 since the difference between two probabilities $(\mu_1 - \mu_0)$ is a zero-average signed measure.
where $M$ and $K$, respectively. For $\varphi \in C^0(M \times K, \mathbb{R})$ (space of continuous functions), define the \textit{topological pressure} of $\varphi$ by

$$
P_t(F, \varphi) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \inf_{E \in \Sigma} \left( \sum_{(x,y) \in E} e^{S_n \varphi(x,y)} \right)
$$

where $S_n(\varphi)(x,y) := \sum_{j=0}^{n-1} \varphi(F^j(x,y)) = \sum_{j=0}^{n-1} \varphi(f^j(x), G^j(y))$, and the infimum is taken over all $(n, \varepsilon)-$spanning subsets $E$ of $\Sigma.$

It is known that the variational principle holds. That is,

$$
P_t(F, \varphi) = \sup_{\mu \in \mathcal{M}^1(F \times K)} \left\{ h_\mu(F) + \int \varphi d\mu \right\}
$$

(48)

where $\mathcal{M}^1(F \times K)$ is the set of measures $\mu$ that are invariant by $F$ ($\mu \circ F^{-1} = \mu$). On the other hand, for a given $\varphi^* \in C^0(M, \mathbb{R})$ define the function

$$
\varphi : M \times K \longrightarrow \mathbb{R}
$$

$$
(x, y) \longmapsto \varphi(x, y) := \varphi^*(x).
$$

We have that $\varphi \in C^0(M \times K, \mathbb{R})$. Now, let $\mathcal{M}^1_m(M \times K)$ be the set of all probability measures $\mu$ on $M \times K$ such that

$$
\pi_1 \ast \mu = \mu \circ \pi_1^{-1} = m_1.
$$

Where $\pi_1 : M \times K \to M$ stands for the first projection ($\pi_1(x,y) = x$). \textit{Theorem 3.1} (Rokhli’s disintegration theorem) describes a disintegration $\left( \{ \mu_\gamma \}_\gamma, m_1 \right)$ of $\mu.$ So that

$$
\int_{M \times K} \varphi d\mu = \int_M \int_K \varphi(\gamma, y) d\mu_\gamma(y) dm_1(\gamma)
$$

$$
= \int_M \int_K \varphi^*(\gamma) d\mu_\gamma(y) dm_1(\gamma)
$$

$$
= \int_M \varphi^*(\gamma) dm_1(\gamma) < \infty.
$$

If we consider an $(n, \varepsilon)$-spanning set $E \subset M \times K$, then by the metric $d$, $E^* = \{ x \in M : (x,y) \in E \}$ is an $(n, \varepsilon)$-spanning set for the system.
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Let $f : M \to M$. Hence, by definition of topological pressure, we get

$$P_t(f, \varphi^*) \leq P_t(F, \varphi). \quad (49)$$

For the other inequality, we will use the following result (see [12]).

\textbf{Theorem 6.3 (Ledrappier-Walters Formula).} Let $\hat{X}, X$ be compact metric spaces and let $\hat{T} : \hat{X} \to \hat{X}, T : X \to X$ and $\hat{\pi} : \hat{X} \to X$ be continuous maps such that $\hat{\pi}$ is surjective and $\hat{\pi} \circ T = T \circ \hat{\pi}$. Then

$$\sup_{\hat{\nu} : \hat{\pi}^* \hat{\nu} = \nu} h_{\hat{\nu}}(\hat{T}) = h_{\nu}(T) + \int h_{\text{top}}(\hat{T}, \hat{\pi}^{-1}(y)) \, d\nu(y).$$

Since $G(x, \cdot) : K \to K$ is a uniform contraction, for every $x \in M$, we have $h_{\text{top}}(F, \pi_1^{-1}(x)) = 0$ for every $x \in M$. Then, by Theorem 6.3, we obtain

$$h_\mu(F) = h_{m_1}(f) \quad (50)$$

for every $m_1 \in \mathcal{M}_f(M)$ and $\mu \in \mathcal{M}_F(M \times K)$ such that $\pi_1 \ast \mu = m_1$. Therefore, by (48) and (50) we get

$$P_t(F, \varphi) \leq P_t(f, \varphi^*). \quad (51)$$

Combining (49) and (51) we get

$$P_t(F, \varphi) = P_t(f, \varphi^*). \quad (52)$$

\textbf{Proposition 6.4.} The measure $m_1 \in \mathcal{M}_f(M)$ is an equilibrium state for $(f, \varphi^*)$, if and only if, $\mu \in \mathcal{M}_F(M \times K)$ such that $m_1 = \pi_1 \ast \mu$, is an equilibrium state for $(F, \varphi)$. Moreover, if $m_1$ is the unique equilibrium state, then $\mu$ is unique.

\textbf{Proof.} (of Theorem A) The proof of the theorem follows from (50) and (52). For the second part, it is a consequence of Lemma 6.1. \qed

7. PROOF OF THEOREM B AND COROLLARY 1.4

Before to establish Theorem B we need to prove the following Lemma

\textbf{Lemma 7.1.} Let $\{F_\delta\}_{\delta \in [0, 1)}$ be an admissible $R(\delta)$-perturbation and let $\{F_\delta^\ast\}_{\delta \in [0, 1)}$ be the induced family of transfer operators. Then, $\{F_\delta^\ast\}_{\delta \in [0, 1)}$ is an $(R(\delta), \zeta)$-family of operators with weak space ($L^\infty, \| \cdot \|_\infty$) and strong space ($S^\infty, \| \cdot \|_{S^\infty}$).

\textbf{Proof.} We need to prove that $\{F_\delta^\ast\}_{\delta \in [0, 1)}$ satisfies P1, P2, P3 and P4. To prove P2, note that, by (A1) and equation (25) we have

$$\| F_\delta^\ast n \mu_\delta \|_{S^\infty} = \| P_{f_\delta}^n \phi_1 \|_\zeta + \| F_\delta^\ast n \mu \|_\infty \leq D \lambda^n \| \phi_1 \|_\zeta + D \| \phi_1 \|_\infty + \| \mu \|_\infty \leq D \lambda^n \| \mu \|_{S^\infty} + (D + 1) \| \mu \|_\infty.$$  

Therefore, if $\mu_\delta$ is a fixed probability measure for the operator $F_\delta^\ast$, by the above inequality, we get P2 with $M = D + 1$. 

A direct application of Theorem 4.2 and Lemma 4.1 gives P1. The items P3 and P4 follow, respectively, from proposition 3.15, equation (25) applied to each $F_{\delta}$.

**Proof.** (of Theorem [B] and Corollary [1.4])

We directly apply the above results together with Theorem 5.2, and the proof of Theorem [B] is completed. The proof of Corollary [1.4] is straightforward.

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