Research Article

Certain Class of Analytic Functions Connected with \( q \)-Analogue of the Bessel Function

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Received 20 February 2021; Revised 19 March 2021; Accepted 27 March 2021; Published 13 April 2021

Academic Editor: Ming-Sheng Liu

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The focus of this article is the introduction of a new subclass of analytic functions involving \( q \)-analogue of the Bessel function and obtained coefficient inequities, growth and distortion properties, radii of close-to-convexity, and starlikeness, as well as convex linear combination. Furthermore, we discussed partial sums, convolution, and neighborhood properties for this defined class.

1. Introduction

Let \( A \) specify the category of analytic functions and \( \eta \) represent on the unit disc \( \Delta = \{ w : |w| < 1 \} \) with normalization \( \eta(0) = 0 \) and \( \eta'(0) = 1 \) such that a function has the extension of the Taylor series on the origin in the form

\[
\eta(w) = w + \sum_{n=2}^{\infty} a_n w^n.
\]

(1)

Indicated by \( S \), the subclass of \( A \) is composed of functions that are univalent in \( \Delta \).

Then, a \( \eta(w) \) function of \( A \) is known as starlike and convex of order \( \theta \) if it delights the pursing

\[
\Re \left\{ \frac{w\eta'(w)}{\eta(w)} \right\} > \theta, \quad w \in \Delta,
\]

(2)

\[
\Re \left\{ 1 + \frac{w\eta''(w)}{\eta'(w)} \right\} > \theta, \quad w \in \Delta,
\]

for specific \( \theta(0 \leq \theta < 1) \), respectively, and we express by \( S^*(\theta) \) and \( K(\theta) \) the subclass of \( A \), which is expressed by the aforesaid functions, respectively. Also, indicated by \( T \), the subclass of \( A \) is made up of functions of this form

\[
\eta(w) = w - \sum_{n=2}^{\infty} a_n w^n, \quad a_n \geq 0, \quad w \in \Delta,
\]

(3)

and let \( T^*(\theta) = T \cap S^*(\theta), C(\theta) = T \cap K(\theta) \). There are interesting properties in the \( T^*(\theta) \) and \( C(\theta) \) classes which were thoroughly studied by Silverman [1] and Alessa et al. [2].

The intense devotion of scientists has recently fascinated the study of the \( q \)-calculus. The great focus in many fields of mathematics and physics is due to its benefits. In the analysis of many subclasses of analytic functions, the importance of the \( q \)-derivative operator \( D_q \) is very evident from its applications.

The concept of \( q \)-star functions was originally proposed by Ismail et al. [3] in the year 1990. However, in the sense of
One of the most significant special functions is the Bessel function. As a result, it is important for solving a wide range of problems in engineering, physics, and mathematics (see [14]). In recent years, several researchers have focused their efforts on forming different types of relationships. Many researchers have recently focused on determining the various conditions under which a Bessel function has geometric properties such as close-to-convexity, starlikeness, and convexity in the frame of a unit disc $\Delta$.

The first-order Bessel function $\varphi$ is defined by the infinite series [15]:

$$\mathcal{F}_\varphi(w) = \sum_{v=0}^{\infty} (-1)^v \frac{(w/2)^{2v+\varphi}}{\Gamma(v+\varphi+1)}, \quad w \in \mathbb{C}, \varphi \in \mathbb{R}, \quad (4)$$

where $\Gamma$ stands for a function of Gamma. Szasz and Kupan [16] and Thirupathi Reddy and Venkateswarlu [17] have recently explored the univalence of the first-kind normalization Bessel function $k_\varphi: \Delta \rightarrow \mathbb{C}$ defined by

$$k_\varphi(w) = 2^\varphi \Gamma(\varphi + 1) w^{1-(\varphi/2)} \mathcal{F}_\varphi(w^{1/2})$$

$$= w + \sum_{v=2}^{\infty} \frac{(-1)^{v-1} \Gamma(\varphi + 1)}{v-1! \Gamma(v+\varphi)} w^{v-1}, \quad w \in \Delta, \varphi \in \mathbb{R}. \quad (5)$$

For $0 < q < 1$, El-Deeb and Bulboaca [18] defined the $q$-derivative $k_\varphi$ operator as follows:

$$\partial_k k_\varphi(w) = \partial_k \left( w + \sum_{v=2}^{\infty} \frac{(-1)^{v-1} \Gamma(\varphi + 1)}{v-1! \Gamma(v+\varphi)} w^{v-1} \right) = \frac{k_\varphi(w)-k_\varphi(w)}{w(q-1)}$$

$$= 1 + \sum_{v=2}^{\infty} \frac{(-1)^{v-1} \Gamma(\varphi + 1)}{v-1! \Gamma(v+\varphi)} [v,q] w^{v-1}, \quad w \in \Delta, \quad (6)$$

where

$$[v,q] = \frac{1-q^v}{1-q} = 1 + \sum_{j=1}^{v-1} q^j, \quad [0,q] := 0. \quad (7)$$

Using (7), we are going to define two products in the text:

(1) The $q$-shifted factorial is given for any nonnegative integer $v$:

$$[v,q] = \begin{cases} 1, & \text{if } v = 0, \\ [1,q][2,q], \ldots, [k,q], & \text{if } v \in \mathbb{N}. \end{cases} \quad (8)$$

(2) The $q$-generalized Pochhammer symbol for any positive number $r$ is defined by

$$[r,q]_v = \begin{cases} 1, & \text{if } v = 0, \\ [r,q][r+1,q], \ldots, [r+k-1,q], & \text{if } v \in \mathbb{N}. \end{cases} \quad (9)$$

A simple computation shows that

$$\mathcal{F}_\varphi^{\ell}(w) \ast \mathcal{M}_{q,\ell+1}(w) = w \partial_k k_\varphi(w), \quad w \in \Delta, \quad (11)$$

where the function $\mathcal{M}_{q,\ell+1}(w)$ is supplied with the function

$$\mathcal{M}_{q,\ell+1}(w) = w + \sum_{v=2}^{\infty} \frac{[\ell+1,q]_{v-1}}{[v-1,q]_v} w^v, \quad w \in \Delta. \quad (12)$$

El-Deeb and Bulboaca [18] introduced the linear operator using the definition of $q$-derivative along with the idea of convolutions $\mathcal{H}^{\ell}_{\varphi,q}$: $\mathcal{A} \rightarrow \mathcal{A}$ defined by
\( \mathcal{N}_{\varphi, q} \eta (w) = \mathcal{F}_{\varphi, \ell} (w) \ast \eta (w) \)

\[ = w + \sum_{v=2}^{\infty} Y_{\varphi, q} (\varphi, \ell) a_v w^v, \quad \varphi > 0, \ell > -1, 0 < q < 1, w \in \Delta, \quad (13) \]

where \( Y_{\varphi, q} (\varphi, \ell) = \frac{(-1)^{\varphi - 1} \Gamma (\varphi + 1)}{4^{\varphi - 1} \Gamma (\varphi + \varphi) \Gamma (\ell + 1, q)} \frac{[\varphi, q]!}{\ell} \)

**Remark 1** (see [18]). We can easily verify from the definition relation (13) that the next relation holds for all \( \eta \in \mathcal{A} \):

\[ [\ell + 1, q] \mathcal{N}_{\varphi, q} \eta (w) = [\ell, q] \mathcal{N}_{\varphi, q} \eta (w) + q^2 w \partial_q ([\ell + 1, q] \mathcal{N}_{\varphi, q} \eta (w)), \quad w \in \Delta, \]

\[ \lim_{q \to 1} \mathcal{N}_{\varphi, q} \eta (w) = \mathcal{F}_{\varphi, \ell} \eta (w) = \mathcal{F}_{\varphi, \ell} \eta (w) \]

\[ = w + \sum_{v=2}^{\infty} \frac{(-1)^{\varphi - 1} \Gamma (\varphi + 1)}{4^{\varphi - 1} \Gamma (\varphi + \varphi) \Gamma (\ell + 1, q)} \frac{[\varphi, q]!}{\ell} \]

\[ \sum_{v=2}^{\infty} [\varphi + h \nu (\nu - 1) - 8] Y_{\varphi, q} (\varphi, \ell) |a_v| \leq 1 - \vartheta. \quad (16) \]

**Proof.** Since \( 0 \leq \vartheta < 1 \) and \( h \geq 0 \), now if we put

\[ q (w) = \frac{w (\mathcal{N}_{\varphi, q} \eta (w))' + h w^2 (\mathcal{N}_{\varphi, q} \eta (w))''}{\mathcal{N}_{\varphi, q} \eta (w)}, \quad w \in \Delta, \]

then it is a matter of proving it \( |q (w) - 1| < 1 - \vartheta, w \in \Delta \).

Indeed, if \( \eta (w) = w (w \in \Delta) \), then we have

\[ q (w) = w (w \in \Delta). \]

This implies (16) holds.

If \( \eta (w) \neq w (|w| = r < 1) \), then there exist a coefficient \( \Omega \), \( (\varphi, \ell) a_\nu > 0 \) for some \( \nu \geq 2 \). The consequence is that \( \sum_{\nu=2}^{\infty} Y_{\varphi, q} (\varphi, \ell) |a_\nu| > 0 \). Further note that

\[ \sum_{\nu=2}^{\infty} [\varphi + h \nu (\nu - 1) - 8] Y_{\varphi, q} (\varphi, \ell) |a_\nu| > (1 - \vartheta) \sum_{\nu=2}^{\infty} Y_{\varphi, q} (\varphi, \ell) |a_\nu| \]

\[ \implies \sum_{\nu=2}^{\infty} Y_{\varphi, q} (\varphi, \ell) |a_\nu| < 1. \]
By (16), we obtain

\[
[q(w) - 1] = \left| \frac{\sum_{n=2}^{\infty} [v + h\nu(v - 1) - 1]Y_{\nu q}(\psi, \ell)a_{r}w^{r-1}}{1 + \sum_{n=2}^{\infty} Y_{\nu q}(\psi, \ell)a_{r}w^{r-1}} \right|
\]

\[
\leq \frac{\sum_{n=2}^{\infty} [v + h\nu(v - 1) - 1]Y_{\nu q}(\psi, \ell)a_{r}}{1 - \sum_{n=2}^{\infty} Y_{\nu q}(\psi, \ell)a_{r}}
\]

\[
\leq \frac{\sum_{n=2}^{\infty} [v + h\nu(v - 1) - \delta]Y_{\nu q}(\psi, \ell)a_{r} - (1 - \delta)Y_{\nu q}(\psi, \ell)a_{r}}{1 - \sum_{n=2}^{\infty} Y_{\nu q}(\psi, \ell)a_{r}}
\]

\[
= 1 - \delta, \quad w \in \Delta. \tag{19}
\]

Hence, we obtain

\[
\mathcal{R}\left( \frac{w\left(A_{p,q}^{\nu} \eta(w)\right)' + hw^2\left(A_{p,q}^{\nu} \eta(w)\right)''}{A_{p,q}^{\nu} \eta(w)} \right) = \mathcal{R}(q(w)) > 1 - (1 - \delta) = \delta. \tag{20}
\]

Then, \(\eta \in \Phi_{p,q}^{\nu}(h, \delta)\).

\[\square\]

**Theorem 2.** Let \(\eta\) be given by (3). Then, the function

\[\eta \in T_{p,q}^{\nu}(h, \delta) \iff \sum_{n=2}^{\infty} [v + h\nu(v - 1) - \delta]Y_{\nu q}(\psi, \ell)a_{r} \leq 1 - \delta. \tag{21}\]

**Proof.** In view of Theorem 1, to examine it, \(\eta \in T_{p,q}^{\nu}(h, \delta)\) fulfills the coefficient inequality (16). If \(\eta \in T_{p,q}^{\nu}(h, \delta)\), then the function

\[\eta(w) = \frac{w\left(A_{p,q}^{\nu} \eta(w)\right)' + hw^2\left(A_{p,q}^{\nu} \eta(w)\right)''}{A_{p,q}^{\nu} \eta(w)}, \quad w \in \Delta, \tag{22}\]

satisfies \(\mathcal{R}(\eta(w)) > \delta\). This implies that

\[A_{p,q}^{\nu} \eta(w) = w - \sum_{n=2}^{\infty} Y_{\nu q}(\psi, \ell)a_{r}w \neq 0, \quad w \in \Delta \setminus \{0\}. \tag{23}\]

Noting that \(A_{p,q}^{\nu}(r)/r\) in the open interval \((0, 1)\), this is the real continuous function with \(\eta(0) = 1\), and we have

\[\frac{A_{p,q}^{\nu} \eta(r)}{r} = 1 - \sum_{n=2}^{\infty} Y_{\nu q}(\psi, \ell)a_{r}r^{r-1} > 0, \quad 0 < r < 1. \tag{24}\]

Now,

\[\delta < q(r) = \frac{1 - \sum_{n=2}^{\infty} [v + h\nu(v - 1) - \delta]Y_{\nu q}(\psi, \ell)a_{r}}{1 - \sum_{n=2}^{\infty} Y_{\nu q}(\psi, \ell)a_{r}}. \tag{25}\]

and consequently, by (24), we obtain

\[\sum_{n=2}^{\infty} [v + h\nu(v - 1) - \delta]Y_{\nu q}(\psi, \ell)a_{r}r^{r-1} \leq 1 - \delta. \tag{26}\]

Letting \(r \to 1\), we get \(\sum_{n=2}^{\infty} [v + h\nu(v - 1) - \delta]Y_{\nu q}(\psi, \ell)a_{r} \leq 1 - \delta\).

This proves the converse part. \(\square\)

**Remark 2.** If a function \(\eta\) of form (3) belongs to the class \(T_{p,q}^{\nu}(h, \delta)\), then

\[|a_{r}| \leq \frac{1 - \delta}{[v + h\nu(v - 1) - \delta]Y_{\nu q}(\psi, \ell)}, \quad \nu \geq 2. \tag{27}\]

The equality holds for the functions

\[\eta_{r}(w) = w - \frac{1 - \delta}{[v + h\nu(v - 1) - \delta]Y_{\nu q}(\psi, \ell)}w^{r}, \quad w \in \Delta, \nu \geq 2. \tag{28}\]
3. Distortion Theorem

In the section, the distortion limits of the functions are owned by the class $T_{p,q}^\ell(h,\vartheta)$.

Theorem 3. Let $\eta \in T_{p,q}^\ell(h,\vartheta)$ and $|w| = r < 1$. Then,

$$r - \frac{1 - \vartheta}{[2h - \vartheta + 2]Y_{2q}(\varphi,\ell)} r^2 \leq |\eta(w)| \leq r + \frac{1 - \vartheta}{[2h - \vartheta + 2]Y_{2q}(\varphi,\ell)} r^2,$$  \hspace{1cm} (29)

$$1 - \frac{1}{2} \frac{2(1 - \vartheta)}{[2h - \vartheta + 2]Y_{2q}(\varphi,\ell)} r \leq |\eta'(w)| \leq 1 + \frac{1}{2} \frac{2(1 - \vartheta)}{[2h - \vartheta + 2]Y_{2q}(\varphi,\ell)} r.$$  \hspace{1cm} (30)

The approximation is sharp, with the $\eta_2(w)$ extreme function indicated by (28).

Proof. Since $\eta \in T_{p,q}^\ell(h,\vartheta)$, we apply Theorem 2 to attain

$$[2h - \vartheta + 2]Y_{2q}(\varphi,\ell) \sum_{r=2}^\infty |a_r| \leq \sum_{r=2}^\infty [v + h(v - 1) - \vartheta]Y_{rq}(\varphi,\ell)|a_r| \leq 1 - \vartheta.$$  

Thus, $|\eta(w)| \leq |w| + |w|^2 \sum_{r=2}^\infty |a_r| \leq r + \frac{1 - \vartheta}{[2h - \vartheta + 2]Y_{2q}(\varphi,\ell)} r^2$.  \hspace{1cm} (31)

Also, we have $|\eta(w)| \leq |w| - |w|^2 \sum_{r=2}^\infty |a_r| \leq r - \frac{1 - \vartheta}{[2h - \vartheta + 2]Y_{2q}(\varphi,\ell)} r^2$.  \hspace{1cm} (32)

Equation (29) follows. In a similar way, for $\eta'$, the inequalities

$$|\eta'(w)| \leq 1 + \sum_{r=2}^\infty v|a_r||w|^{r-1} \leq 1 + |w| \sum_{r=2}^\infty v|a_r|,$$  \hspace{1cm} (33)

$$\sum_{r=2}^\infty v|a_r| \leq \frac{2(1 - \vartheta)}{[2h - \vartheta + 2]Y_{2q}(\varphi,\ell)}$$

are satisfied, which leads to (30).

4. Radii of Close-to-Convexity and Starlikeness

A close-to-convex and star-like radius of this class $T_{p,q}^\ell(h,\vartheta)$ is obtained in this section.

Theorem 4. Let $\eta$ be specified by (3) in $T_{p,q}^\ell(h,\vartheta)$. Then, $\eta$ is the order of close-to-convex $\ell (0 \leq \ell < 1)$ in the disc $|w| < t_1$, where

$$t_1 = \inf_{r \geq 2} \left[ \frac{1 - \ell}{v(1 - \vartheta)} \right]^{1/(r - 1)} \cdot \sum_{r=2}^\infty \frac{v|a_r|}{1 - \ell}.$$  \hspace{1cm} (34)

The estimate is sharp with the extremal function $\eta(w)$ is indicated by (28).

Proof. If $\eta \in T$ and $\eta$ is order of close-to-convex $\ell$, then we obtain

$$|\eta'(w)| \leq 1 - \ell.$$  

For the L.H.S of (34), we obtain

$$|\eta'(w)| \leq \sum_{r=2}^\infty v|a_r| |w|^{r-1} \leq 1 - \ell = \sum_{r=2}^\infty \frac{v|a_r| |w|^{r-1}}{1 - \ell}.$$  

We know that

$$\eta(w) \in T_{p,q}^\ell(h,\vartheta) \iff \sum_{r=2}^\infty \frac{v(\vartheta + h(v - 1) - \vartheta)Y_{rq}(\varphi,\ell)}{v(1 - \vartheta)} a_r \leq 1.$$  

Thus, (34) holds true if

Thus, (34) holds true if
Proof. We have $\eta \in T\phi_{p,q}^\ell \ (h, 0)$. Then, $\eta$ is order of starlike $\ell (0 \leq \ell < 1)$ in the disc $|w| < t_2$, where

$$t_2 = \inf_{\nu \in \mathbb{C}} \left[ \frac{(1 - \ell)(v + \nu h - 1 - \theta)Y_{\nu q}(p, \ell)}{(v - \ell)(1 - \theta)} \right]^{1/(\nu - 1)}.$$  

(38)

The estimate is sharp with the extremal function $\eta(w)$ indicated by (28).

Theorem 6.

Let $\eta \in T\phi_{p,q}^\ell \ (h, 0)$. Then, $\eta$ is order of starlike $\ell (0 \leq \ell < 1)$ in the disc $|w| < t_2$, where

$$t_2 = \inf_{\nu \in \mathbb{C}} \left[ \frac{(1 - \ell)(v + \nu h - 1 - \theta)Y_{\nu q}(p, \ell)}{(v - \ell)(1 - \theta)} \right]^{1/(\nu - 1)}.$$  

(39)

Thus, (39) is true if

$$\frac{v - \ell}{1 - \ell} |w|^\nu - 1 \leq \frac{[v + \nu h (v - 1) - \theta]Y_{\nu q}(p, \ell)}{(v - \ell)(1 - \theta)},$$  

(40)

and hence proved. □

For the L.H.S of (39), we have

$$\left| \frac{\omega \eta'(w)}{\eta(w)} - 1 \right| \leq \sum_{\nu = 1}^{\infty} \frac{(v - 1)a_{\nu}|w|^{\nu - 1}}{1 - \sum_{\nu = 0}^{\infty} a_{\nu}|w|^{\nu - 1}}.$$  

(41)

(1 - $\ell$) is bigger than the R.H.S of the left relation if

$$\sum_{\nu = 1}^{\infty} \frac{v - \ell}{1 - \ell} a_{\nu}|w|^{\nu - 1} < 1.$$  

(42)

We know that

$$\eta \in T\phi_{p,q}^\ell \ (h, 0) \iff \sum_{\nu = 1}^{\infty} \frac{[v + \nu h (v - 1) - \theta]Y_{\nu q}(p, \ell)}{(v - \ell)(1 - \theta)} a_{\nu} \leq 1.$$  

(42)

Thus, (39) is true if

$$\sum_{\nu = 1}^{\infty} \frac{[v + \nu h (v - 1) - \theta]Y_{\nu q}(p, \ell)}{(v - \ell)(1 - \theta)} a_{\nu} = 1.$$  

(42)

and $\sum_{\nu = 1}^{\infty} a_{\nu} = 1$.

5. Convex Linear Combinations

Theorem 7.

Let $\eta_1 (w) = w$ and

$$\eta_\nu (w) = w - \frac{1 - \theta}{[v + \nu h (v - 1) - \theta]Y_{\nu q}(p, \ell)} w^\nu, \quad w \in \Delta, \nu \geq 2.$$  

(44)

Then, $\eta \in T\phi_{p,q}^\ell \ (h, 0) \iff \eta$ in the way it can be expressed:

$$\eta(w) = \sum_{\nu = 1}^{\infty} \mu_{\nu} \eta_\nu (w), \quad \mu_{\nu} \geq 0,$$  

(45)

and $\sum_{\nu = 1}^{\infty} \mu_{\nu} = 1$.

Proof. If a function $\eta$ is of the form $\eta(w) = \sum_{\nu = 1}^{\infty} \mu_{\nu} \eta_\nu (w), \mu_{\nu} \geq 0$ and $\sum_{\nu = 1}^{\infty} \mu_{\nu} = 1$, then

$$\sum_{\nu = 1}^{\infty} \frac{[v + \nu h (v - 1) - \theta]Y_{\nu q}(p, \ell)}{(v - \ell)(1 - \theta)} a_{\nu} = 1.$$  

(42)

which provides (21), and hence $\eta \in T\phi_{p,q}^\ell \ (h, 0)$, by Theorem 2.

On the contrary, if $\eta$ is in the class $\eta \in T\phi_{p,q}^\ell \ (h, 0)$, then we may set

$$\sum_{\nu = 1}^{\infty} \frac{[v + \nu h (v - 1) - \theta]Y_{\nu q}(p, \ell)}{(v - \ell)(1 - \theta)} a_{\nu} < 1.$$  

(42)
\[ \mu_{v} = \frac{[\nu + h \nu (\nu - 1) - \theta]Y_{\nu d}(\nu, \theta)[a_{v}]}{1 - \theta}, \quad v \geq 2, \] \hspace{1cm} (47) \]

and \[ \mu_{1} = 1 - \sum_{v=2}^{\infty} \mu_{v}. \]

Then, the function \( \eta \) is of form (45).

\section*{6. Partial Sums}

Silverman [20] examined partial sums \( \eta \) for the function \( \eta \in A \) given by (1) and established through

\[ \eta_{m}(w) = w + \sum_{v=2}^{m} a_{v} w^{v}, \quad m = 2, 3, 4, \ldots \] \hspace{1cm} (48) \]

In this paragraph, in the class \( \Phi_{\nu, \theta}(h, \delta) \), partial function sums can be considered and sharp lower limits can be reached for the function. True component ratios are \( \eta \) to \( \eta_{m} \) and \( \eta' \) to \( \eta'_{m} \).

\textbf{Theorem 7.} Let \( \eta \in \Phi_{\nu, \theta}(h, \delta) \) fulfill (16). Then,

\[ R \left( \frac{\eta(w)}{\eta_{m}(w)} \right) \geq 1 - \frac{1}{d_{m+1}}, \quad w \in \Delta, m \in \mathbb{N}, \] \hspace{1cm} (49) \]

where

\[ d_{\nu} = \frac{[\nu + h \nu (\nu - 1) - \theta]}{1 - \theta} \] \hspace{1cm} (50) \]

\textbf{Proof.} Clearly, \( d_{m+1} > d_{\nu} > 1 \), \( \nu = 2, 3, 4, \ldots \)

Thus, by Theorem 1, we obtain

\[ \sum_{v=2}^{\infty} |a_{v}| + d_{m+1} \sum_{v=2}^{\infty} |a_{v}| \leq \sum_{v=2}^{\infty} d_{v} |a_{v}| \leq 1, \] \hspace{1cm} (57) \]

Setting \( h(w) = (1 + d_{m+1}) \frac{\eta_{m}(w)}{\eta(w)} - \left( \frac{d_{m+1}}{1 + d_{m+1}} \right) \),

\[ h(w) = 1 - \frac{(1 + d_{m+1}) \sum_{v=2}^{\infty} a_{v} w^{v-1}}{1 + \sum_{v=2}^{\infty} a_{v} w^{v-1}} \] \hspace{1cm} (58) \]

and hence proved.

\textbf{Theorem 8.} Let \( \eta \in T\Phi_{\nu, \theta}(h, \delta) \) fulfill (16). Then,

\[ R \left( \frac{\eta_{m}(w)}{\eta(w)} \right) \geq \frac{d_{m+1}}{1 + d_{m+1}}, \quad w \in \Delta, m \in \mathbb{N}, \] \hspace{1cm} (55) \]

where

\[ d_{\nu} = \frac{[\nu + h \nu (\nu - 1) - \theta]}{1 - \theta} \] \hspace{1cm} (56) \]

\textbf{Proof.} Clearly, \( d_{m+1} > d_{\nu} > 1 \), \( \nu = 2, 3, 4, \ldots \)

Thus, by Theorem 1, we obtain

\[ \sum_{v=2}^{\infty} |a_{v}| + d_{m+1} \sum_{v=2}^{\infty} |a_{v}| \leq \sum_{v=2}^{\infty} d_{v} |a_{v}| \leq 1, \] \hspace{1cm} (57) \]

Setting \( h(w) = (1 + d_{m+1}) \frac{\eta_{m}(w)}{\eta(w)} - \left( \frac{d_{m+1}}{1 + d_{m+1}} \right) \),

\[ h(w) = 1 - \frac{(1 + d_{m+1}) \sum_{v=2}^{\infty} a_{v} w^{v-1}}{1 + \sum_{v=2}^{\infty} a_{v} w^{v-1}} \] \hspace{1cm} (58) \]

and hence proved.

\textbf{Theorem 9.} Let \( \eta \in T\Phi_{\nu, \theta}(h, \delta) \) fulfill (16). Then,

\[ R \left( \frac{\eta'(w)}{\eta'(w)} \right) \geq 1 - \frac{m + 1}{m + d_{m+1}}, \quad w \in \Delta, m \in \mathbb{N}, \] \hspace{1cm} (61) \]

and hence proved.
the evidence is close to that of (51) and (52) theorems, so the specifics are omitted. □

7. Convolution Properties

We will prove in this section that the \( T\phi^{\ell}_{p,q}(h, \delta) \) class is closed by convolution.

**Theorem 10.** Let \( g(w) \) of the form,

\[
g(w) = w - \sum_{v=2}^{\infty} b_{v} w^{v},
\]

be regular in \( \Delta \). If \( \eta \in T\phi^{\ell}_{p,q}(h, \delta) \), then the function \( \eta^{*} g \) is in the class \( T\phi^{\ell}_{p,q}(h, \delta) \). Here, the symbol \( * \) denotes to the Hadamard product.

**Proof.** Since \( \eta \in T\phi^{\ell}_{p,q}(h, \delta) \), we have

\[
\sum_{v=2}^{\infty} \left[ v + h \nu (v - 1) - \delta \right] Y_{r,q}(\varphi, \ell) |a_{v}| \leq 1 - \delta.
\]  

Employing the last inequality and the fact that

\[
(\eta^{*} g)(w) = w - \sum_{v=2}^{\infty} a_{v} b_{v} w^{v},
\]

we obtain

\[
\sum_{v=2}^{\infty} \left[ v + h \nu (v - 1) - \delta \right] Y_{r,q}(\varphi, \ell) |a_{v}| |b_{v}|
\leq \sum_{v=2}^{\infty} \left[ v + h \nu (v - 1) - \delta \right] Y_{r,q}(\varphi, \ell) |a_{v}|
\leq 1 - \delta,
\]

and hence, in view of Theorem 1, the result follows. □

8. Neighborhood Property

Following [21, 22], we defined the \( \alpha \)-neighbourhood of the function \( \eta(w) \in T \) by

\[
N_{\alpha}(\eta) = \left\{ g \in T: g(w) = w - \sum_{v=2}^{\infty} b_{v} w^{v} \text{ and } \sum_{v=2}^{\infty} \nu |a_{v} - b_{v}| \leq \alpha, \right\}.
\]

**Definition 2.** The function \( \eta \in A \) is defined in the class \( T\phi^{\ell}_{p,q}(h, \delta) \) if the function \( h \in T\phi^{\ell}_{p,q}(h, \delta) \) occurs in such a way that the function is \( h \in T\phi^{\ell}_{p,q}(h, \delta) \):

\[
\left| \frac{\eta(w)}{h(w)} - 1 \right| < 1 - \gamma, \hspace{1em} \eta \in \Delta, 0 \leq \gamma < 1.
\]

**Theorem 11.** If \( h \in T\phi^{\ell}_{p,q}(h, \delta) \) and

\[
y = 1 - \frac{\alpha(2h - \delta + 2)Y_{2,q}(\varphi, \ell)}{(2h - \delta + 2)Y_{2,q}(\varphi, \ell) - (1 + \delta)}
\]

then \( N_{\alpha}(h) \subseteq T\phi^{\ell}_{p,q}(h, \delta) \).

**Proof.** Let \( \eta \in N_{\alpha}(h) \). We then find from

\[
\sum_{v=2}^{\infty} \nu |a_{v} - b_{v}| \leq \alpha
\]

which easily implies the coefficient inequality

\[
\sum_{v=2}^{\infty} |a_{v} - b_{v}| \leq \frac{\alpha}{\nu}
\]

Since \( h \in T\phi^{\ell}_{p,q}(h, \delta) \), we have from equation (16) that

\[
\sum_{v=2}^{\infty} |a_{v}| \leq \frac{1 - \delta}{(2h - \delta + 2)Y_{2,q}(\varphi, \ell)}
\]

\[
\left| \frac{\eta(w)}{h(w)} - 1 \right| < \frac{\sum_{v=2}^{\infty} \nu |a_{v} - b_{v}|}{1 - \sum_{v=2}^{\infty} b_{v}}
\]

\[
\leq \frac{\alpha}{\nu} \frac{2(2h - \delta + 2)Y_{2,q}(\varphi, \ell)}{2h - \delta + 2 - (1 + \delta)}
\]

\[
= 1 - \gamma,
\]

and hence proved. □

9. Conclusions

This research has introduced \( q \)-analogue of the Bessel function and studied some basic properties of geometric function theory. Accordingly, some results related to coefficient estimates, growth and distortion properties, convex linear combination, partial sums, radii of close-to-convexity
and starlikeness, convolution, and neighborhood properties have also been considered, inviting future research for this field of study.

**Data Availability**

No data were used to support the findings of the study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

All authors contributed equally to this work. And, all the authors have read and approved the final version manuscript.

**Acknowledgments**

This research was funded by the Deanship of Scientific Research at Princess Nourah Bint Abdulrahman University through the Fast-track Research Funding Program.

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