Somewhat smooth numbers in short intervals

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Abstract
We use exponent pairs to establish the existence of many \( x^a \)-smooth numbers in short intervals \([x - x^b, x]\), when \( a > 1/2 \). In particular, \( b = 1 - a - a(1 - a)^3 \) is admissible. Assuming the exponent-pairs conjecture, one can take \( b = (1 - a)/2 + \epsilon \).
As an application, we show that \([x - x^{0.4872}, x]\) contains many practical numbers when \( x \) is large.

Keywords Smooth numbers · Short intervals · Exponent pairs

Mathematics Subject Classification 11N25

1 Introduction
We say that a natural number \( n \) is \( y \)-smooth if all of its prime factors are \( \leq y \). Let \( \Psi(x, y) \) be the number of such \( n \leq x \). Improving on many earlier efforts by a number of different authors, Matomäki and Radziwiłł[12] established the existence of many \( x^\epsilon \)-smooth numbers in intervals of the form \([x, x + c(\epsilon)\sqrt{x}]\), for every \( \epsilon > 0 \). Harman [6] showed that intervals around \( x \) of length \( x^{0.45...} \) contain many \( x^{0.27...} \)-smooth numbers.

We are interested in the existence of \( x^a \)-smooth numbers in much shorter intervals, when \( a > 1/2 \). More precisely, given \( a \in (1/2, 1) \), how small can we take \( b \) such that
\[
\Psi(x, x^a) - \Psi(x - x^b, x^a) \gg x^{b-\epsilon}
\]
for every \( \epsilon > 0 \)? In that direction, Friedlander and Lagarias [3] showed that there exists a constant \( c > 0 \) such that \( b = 1 - a - ca(1 - a)^3 \) is admissible, even with \( \epsilon = 0 \), but without providing any numerical estimate for \( c \). We will use exponent pairs...
(see [4]) to find explicit values of $b < 1 - a$. In particular, $b = 1 - a - a(1 - a)^3$ is admissible for every $a \in (1/2, 1)$.

Let $\psi(x) = x - [x] - 1/2 = \{x\} - 1/2$. The method used by Friedlander and Lagarias [3] starts with Chebyshev’s identity and requires estimates for sums of $\psi(x/p) \log p$, where $p$ runs over primes. Our approach involves sums of $\psi(x/n)$ over all integers $n$ from an interval. We use the estimate

$$\sum_{N \leq n \leq 2N} \psi(x/n) \ll \min(x^\theta, x^{k/(k+1)} N^{(l-k)/(k+1)}) \quad (1 \leq N \leq \sqrt{x}), \quad (1)$$

where $(k, l)$ is any exponent pair. The two most recent records for $\theta$ are $\theta = \frac{131}{146} + \epsilon = 0.3149...$ by Huxley [11, Thm. 4] and $\theta = \frac{517}{1648} + \epsilon = 0.3137...$ by Bourgain and Watt [2, Eq. (7.4)]. For the second estimate in (1), see Graham and Kolesnik [4, Lemma 4.3].

Let $v = 2.9882...$ be the minimum value of $(2^u - 1)/(u - 1)$ for $u > 1$.

**Theorem 1** Let $(k, l)$ be an exponent pair and $\theta$ as in (1). There is a constant $K$ such that

$$\Psi(x, y) - \Psi(x - z, y) \gg \frac{z}{(\log x)^v},$$

provided $x \geq y \geq \sqrt{2x}$ and $x \geq z \geq K \min(x^\theta, x^{l/(k+1)} y^{(k-l)/(k+1)})$.

Define

$$b = b(a, k, l) = \frac{l + a(k - l)}{k + 1}. \quad (2)$$

**Corollary 1** Let $(k, l)$ be an exponent pair, $\theta$ as in (1) and $1/2 < a \leq 1$. There is a constant $K$ such that for $x \geq z \geq K x^{\min(\theta, b)}$,

$$\Psi(x, x^a) - \Psi(x - z, x^a) \gg \frac{z}{(\log x)^v}.$$

If $a = 1/2$, the conclusion holds if $x^a$ is replaced by $\sqrt{2x}$.

Starting with the exponent pair $(\kappa, \lambda) = (13/84 + \epsilon, 55/84 + \epsilon)$ of Bourgain [1, Thm. 6], and possibly applying van der Corput’s processes A or B, we find a sequence of linear functions in $a$, shown in Table 1. When $a$ is close to $1/2$, then $\theta$ is smaller than any $b$ obtained from known exponent pairs. When $a$ is close to 1, we rely on exponent pairs $(k, l)$ with small $k$. Heath-Brown [8, Thm. 2] found that for integers $m \geq 3$ and every $\epsilon > 0$,

$$k_m = \frac{2}{(m - 1)^2 (m + 2)}, \quad l_m = 1 - \frac{3m - 2}{m(m - 1)(m + 2)} + \epsilon \quad (3)$$

is an exponent pair. This enables us to prove the following result.

**Corollary 2** For each $a \in [1/2, 1)$, the conclusion of Corollary 1 holds for some $b < 1 - a - a(1 - a)^3 - 4.32 a(1 - a)^5$.  

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Table 1 Admissible values of $b$, depending on $a$

| $b$                  | Interval for $a$ | Exponent Pair |
|----------------------|------------------|---------------|
| $517/1648 + \epsilon$ | [0.500..., 0.579...] | $BA(\kappa, \lambda)$ |
| $(110 - 55a)/249 + \epsilon$ | [0.579..., 0.590...] | $(\kappa, \lambda)$ |
| $(55 - 42a)/97 + \epsilon$ | [0.590..., 0.701...] | $A(\kappa, \lambda)$ |
| $(152 - 139a)/207 + \epsilon$ | [0.701..., 0.766...] | $AA(\kappa, \lambda)$ |
| $(359 - 346a)/427 + \epsilon$ | [0.766..., 0.796...] |              |
| $b(a, k_m, l_m)$     | $[a_{m-1}, a_m]$, $m \geq 5$ |              |

The value of $a$, for which $b(a, k_m, l_m) = b(a, k_{m+1}, l_{m+1})$, is given by

$$a_m := 1 - \frac{2}{m} + \frac{2 - m^{-1}}{m^3 + m^2 + 2m - 1}.$$

If $a > 0.796...$ and $a \in [a_{m-1}, a_m]$, then $b$ is minimized by $b(a, k_m, l_m)$. This yields slightly smaller values of $b$ than Corollary 2.

The values $a = 1 - 1/m$, where $m \geq 2$ is an integer, may be of particular interest. Here, we have $a_{m-1} < a = 1 - 1/m < a_m$ and

$$b = b(1 - 1/m, k_m, l_m) = \frac{(m - 1) (m^3 + m^2 - 3m + 2)}{m^2 (m^3 - 3m + 4)} + \epsilon.$$

The exponent-pairs conjecture states that $(k, l) = (\epsilon, 1/2 + \epsilon)$ is an exponent pair for every $\epsilon > 0$.

![Fig. 1 Admissible values of $b$ based on Table 1 (solid); $b = 1 - a - a(1 - a)^3 - 4.32 a(1 - a)^5$ (dotted) from Cor. 2; $b = 1 - a$ (dashed) from the exponent pair $(k, l) = (0, 1)$; $b = \frac{1}{2} (1 - a)$ (dot-dashed) from the exponent-pairs conjecture](image-url)
Corollary 3 If \((\epsilon, 1/2 + \epsilon)\) is an exponent pair, then the conclusion of Corollary 1 holds with \(b = (1 - a)/2 + \epsilon\) for each \(a \in [1/2, 1]\) (Fig. 1).

If one is only concerned with the existence of a single \(y\)-smooth number in short intervals, then a construction due to Friedlander and Lagarias [3] (consider integers of the form \(m^2 - h^2 = (m - h)(m + h)\), where \(m = \lceil \sqrt{x} \rceil\) and \(h = 0, 1, 2, \ldots\)) and an easy exercise (aided by a computer to deal with small values of \(x\)) lead to the explicit estimate

\[
\Psi(x, \sqrt{2}x) - \Psi(x - 3x^{1/4}, \sqrt{2}x) \geq 1 \quad (x \geq 1).
\]

From Table 1, we find that our intervals are wider than \(3x^{1/4}\) when \(a < 401/556 = 0.721\ldots\), but are shorter when \(a > 401/556\).

2 Proofs

Let \(\tau(n)\) be the number of positive divisors of \(n\). The following estimate is a special case of Theorem 2 of Shiu [18].

Lemma 1 Let \(\epsilon > 0\) and \(u \in \mathbb{R}\) be fixed. For \(x \geq 2\) and \(x^\epsilon \leq z \leq x\), we have

\[
\sum_{x-z \leq n \leq x} (\tau(n))^u \ll z(\log x)^{2u-1}.
\]

Proof of Theorem 1 Let \(P(n)\) denote the largest prime factor of \(n\). Note that the result holds if \(z > x/2\), so we may assume \(z \leq x/2\). Define

\[
S := \sum_{x/y \leq d \leq 2x/y} \sum_{x-z \leq n \leq x \atop n \equiv 0 \mod d} 1. \quad (4)
\]

We have

\[
S = \sum_{x/y \leq d \leq 2x/y} ([x/d] - [(x-z)/d])
\]

\[
= \sum_{x/y \leq d \leq 2x/y} z/d - \sum_{x/y \leq d \leq 2x/y} \psi(x/d) + \sum_{x/y \leq d \leq 2x/y} \psi((x-z)/d)
\]

\[
\geq z/3 + O(\min(x^\theta, x^{j/(k+1)}y^{(k-1)/(k+1)}))
\]

\[
\geq z/4,
\]

by (1) and the assumptions of Theorem 1.

Note that \(y \geq \sqrt{2x}\) implies \(2x/y \leq y\). Every \(n\) counted in the inner sum of (4) has a divisor \(d \in [x/y, 2x/y] \subseteq [x/y, y]\). Since \(d \leq y\) and \(n/d \leq x/(x/y) = y\), we have \(P(n) \leq y\), i.e. \(n\) is \(y\)-smooth. Moreover, each \(n\) is counted at most \(\tau(n)\) times,
once for each divisor $d$ of $n$ with $d \in [x/y, 2x/y]$. Thus,

$$S \leq \sum_{x-z<n \leq x, P(n) \leq y} \tau(n).$$

For real numbers $t, u > 1$ with $1/t + 1/u = 1$, Hölder’s inequality yields

$$S \leq \left( \sum_{x-z<n \leq x, P(n) \leq y} 1 \right)^{1/t} \left( \sum_{x-z<n \leq x} \tau(n)^u \right)^{1/u} \ll (\Psi(x, y) - \Psi(x - z, y))^{1/t} \frac{z^{1/u}}{(\log x)^{(2u-1)/u}},$$

by Lemma 1. Since $S \geq z/4$, we get

$$\Psi(x, y) - \Psi(x - z, y) \gg \frac{z}{(\log x)^{(2u-1)/(u-1)}}.$$

The last exponent has a minimum value of $\nu = 2.9882...$ at $u = 2.1080...$ \hspace{1em} \Box

**Proof of Corollary 2** For $m \geq 3$ and $a \in [a_{m-1}, a_m]$, we want to show that $b(a, k_m, l_m) < f(a)$, where $f(a) = 1 - a - a(1-a)^3 - 4.32a(1-a)^5$. Since $f''(a) < 0$ for $1/2 < a < 1$ and $b(a, k_m, l_m)$ is a linear function in $a$ for each $m$, it suffices to verify the inequality at the endpoints $a = a_m$. That is, we need to show that $b(a_m, k_m, l_m) < f(a_m)$ for $m \geq 2$. We find that $f(a_m) - b(a_m, k_m, l_m)$ is a rational function in $m$ that is positive for every $m \geq 1$. This proves the claim for $a \geq a_2 = 3/5$. If $1/2 \leq a < 3/5$, the result follows from Table 1. \hspace{1em} \Box

### 3 Application to practical numbers

Let $\mathcal{A}$ be the set of positive integers containing $n = 1$ and all those $n \geq 2$ with prime factorization $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $p_1 < p_2 < \ldots < p_k$, which satisfy $p_1 = 2$ and

$$p_i \leq p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}} \quad (2 \leq i \leq k).$$

The significance of the set $\mathcal{A}$ is that it is a subset of several notable integer sequences, including the practical numbers (i.e. integers $n$ such that every natural number $m \leq n$ can be expressed as a sum of distinct positive divisors of $n$ [16,17,19,20]), the $t$-dense numbers, for every $t \geq 2$, (i.e. the ratios of consecutive divisors of $n$ are at most $t$, [16,17,19,20]), and the $\varphi$-practical numbers (i.e. $x^n - 1$ has a divisor in $\mathbb{Z}[x]$ of every degree up to $n$, [14]).

Let $v = 2.9882...$ be as in Theorem 1, $C = (1 - e^{-\gamma})^{-1} = 2.280...$, where $\gamma = 0.5772...$ is Euler’s constant, and

$$\mu_0 := 2v + 2 + C \log 2 = 9.5569...$$

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Theorem 2 Let \((k, l)\) be an exponent pair, \(\beta = \frac{5k + l + 2}{6(k + 1)}\) and \(\mu > \mu_0\). There exists a constant \(K\) such that for \(x \geq z \geq K x^\beta\), the interval \([x - z, x]\) contains \(\gg z (\log x)^{-\mu}\) members of \(\mathcal{A}\).

The exponent pair \((k, l) = (13/194 + \epsilon, 76/97 + \epsilon) = (\kappa, \lambda)\) yields:

**Corollary 4** For every \(\beta > 605/1242 = 0.4871\ldots\) and \(\mu > \mu_0\), the conclusion of Theorem 2 holds. Assuming the exponent-pairs conjecture, it holds for every \(\beta > 5/12 = 0.4166\ldots\).

**Corollary 5** The interval \([x - x^{0.4872}, x]\) contains at least \(x^{0.4872}(\log x)^{-9.557}\) members of \(\mathcal{A}\), for all sufficiently large \(x\).

A quick search on a computer suggests that Corollary 5 probably holds for all \(x \geq 504\).

It is clear that Theorem 2 and its corollaries remain valid if \(\mathcal{A}\) is replaced by any superset of \(\mathcal{A}\). In the case of practical numbers, Corollary 5 improves on two earlier results: Hausman and Shapiro [7] found that the interval \([x^2, (x + 1)^2]\) contains a practical number for every \(x \geq 1\), in analogy with Legendre’s conjecture for primes. Melfi [13, Thm. 9] sharpened this by showing that the interval \([x, x + K \sqrt{x/\log \log x}]\) contains a practical number for all large \(x\) and some constant \(K\).

Granville [5, Conj. 4.4.2] states the conjecture that for every fixed \(\epsilon > 0\), the interval \([x - x^\epsilon, x]\) contains a \(x^\epsilon\)-smooth number for all \(x \geq x_0(\epsilon)\). Pomerance [15] points out that this would imply the existence of a practical number (or member of \(\mathcal{A}\)) in every interval \([x - x^\epsilon, x]\) for large \(x\).

The following observation follows at once from the definition of the set \(\mathcal{A}\).

**Lemma 2** If \(n \in \mathcal{A}\) and \(P(m) \leq n\), then \(mn \in \mathcal{A}\).

**Proof of Theorem 2** If \(z > x/2\), the result follows from Theorem 1.2 of [20], so we may assume \(z \leq x/2\). Let \(a = 3/4\). We have \(b = \frac{3k + l}{4(k + 1)} > 0\), according to (2), and \(\beta = 1/3 + (2/3)b > 1/3\).

Theorem 1.2 of [20] shows that the number of \(n \in \mathcal{A} \cap (2x^{1/3}, 3x^{1/3}]\) is \(\sim cx^{1/3} / \log x\) for some positive constant \(c\). Let \(\epsilon > 0\) and \(C = (1 - e^{-\gamma})^{-1} = 2.280\ldots\).

By Corollary 1 of [21], the number of these \(n\) with \(\Omega(n) > (C + \epsilon) \log \log n\) is \(o( x^{1/3} / \log x)\), so we may exclude such \(n\) and assume \(\Omega(n) \leq (C + \epsilon) \log \log n\).

Since \(n \in (2x^{1/3}, 3x^{1/3}]\), the condition \(z \geq 3Kx^\beta\) implies \(z/n \geq K(x/n)^b\).

By Corollary 1, for each of these \(n\), the interval \(I_n := [x/n - z/n, x/n]\) contains \(\gg (z/n)(\log x/n)^{-\epsilon} \gg z x^{-1/3}(\log x)^{-\epsilon}\) integers \(m\) that are \((x/n)^{3/4}\)-smooth. Note that \(mn \in [x - z, x]\) for \(m \in I_n\).

We will show that for each of these pairs \((n, m)\) as described above, we have \(mn \in \mathcal{A}\). Let \(p = P(m)\). Since \(n \geq 2x^{1/3}\), \(p \leq (x/n)^{3/4} \leq x^{1/2} 2^{-3/4}\). If \(p \leq x^{1/3}\), then \(mn \in \mathcal{A}\), by Lemma 2. If \(p > x^{1/3}\), write \(m = pr\) and note that \(r = m/p \leq x/(np) < x^{1/3}\). Thus, \(rn \in \mathcal{A}\) by Lemma 2. Since \(p \leq x^{1/2} 2^{-3/4}\), we have \(p^2 \leq x^{-3/2} < mn = prn\) and hence \(p < rn\). Thus, \(mn = prn \in \mathcal{A}\) also holds in this case, by Lemma 2.
The number of pairs \((m, n)\) is \(\gg (\log x)^{-1-v}\), but several pairs may lead to the same product \(mn\). We have \(\tau(n) \leq 2\Omega(n) \leq (\log x)^C \log^{2+\varepsilon} x\). By Lemma 1, we have \(\sum_{m \in I_n} \tau(m) \ll (z/n) \log x\). Since the number of \(m \in I_n\) that are \((x/n)^{3/4}\)-smooth is \(\gg (z/n)(\log x)^{-v}\), we have \(\tau(m) \ll (\log x)^{v+1}\) for a positive proportion of them. Thus, we may assume \(\tau(m) \ll (\log x)^{v+1}\), and therefore \(\tau(mn) \ll (\log x)^{v+1+C \log 2+\varepsilon}\). It follows that the number of distinct products \(mn\) is

\[
\gg \frac{z(\log x)^{-1-v}}{(\log x)^{v+1+C \log 2+\varepsilon}} = \frac{z}{(\log x)^{\mu_0+\varepsilon}}.
\]

\(\square\)

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