Twisted Sectors and Chern-Simons Terms in $M$-Theory Orbifolds

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Abstract

It is shown that the twisted sector spectrum, as well as the associated Chern-Simons interactions, can be determined on $M$-theory orbifold fixed planes that do not admit gravitational anomalies. This is demonstrated for the seven-planes arising within the context of an explicit $\mathbb{R}^6 \times S^1 / \mathbb{Z}_2 \times T^4 / \mathbb{Z}_2$ orbifold, although the results are completely general. Local anomaly cancellation in this context is shown to require fractional anomaly data that can only arise from a twisted sector on the seven-planes, thus determining the twisted spectrum up to a small ambiguity. These results open the door to the construction of arbitrary $M$-theory orbifolds, including those containing fixed four-planes which are of phenomenological interest.
1 Introduction

In their fundamental work [1, 2], Hořava and Witten discussed the eleven-dimensional realization of \( M \)-theory on the background spacetime \( \mathbb{R}^{10} \times S^1/\mathbb{Z}_2 \). A main thesis in that work is that local anomaly freedom fixes the “twisted sector” gauge field and matter spectrum on each of the two ten-dimensional orbifold fixed planes almost uniquely. This result is important in that, since the fundamental theory underlying \( M \)-theory remains unknown, there is no other way to determine this spectrum. It was subsequently demonstrated in a series of papers [3] that if the Hořava-Witten theory is compactified on elliptically fibered Calabi-Yau threefolds with holomorphic instantons, a quasi-realistic three family theory of particle physics with the standard gauge group emerges. Hence, \( M \)-theory orbifolds might provide a first principles foundation for low energy particle physics.

With this in mind, it is of interest to ask whether one can construct other orbifolds of \( M \)-theory beyond the \( S^1/\mathbb{Z}_2 \) example of [1, 2]. A first step in this direction was taken by Dasgupta and Muhki [4] and Witten [5] who discussed both local and global anomaly cancellation within the context of \( T^5/\mathbb{Z}_2 \) orbifolds. A major generalization of these results was presented in [6, 7, 8] and [9, 10] where all the \( M \)-theory orbifolds associated with the spacetime \( \mathbb{R}^6 \times K3 \) were constructed. In these papers, the complete anomaly polynomials for arbitrary gravity, gauge and matter supermultiplets were given for both ten and six-dimensional fixed planes. Local anomaly free solutions were specified on both ten-planes and six-planes and these were “woven” together to form orbifolds that were, in addition, free of global anomalies. Furthermore, it was shown in [4, 8] that many of these orbifold solutions were simply related to each other through the emission and absorption of \( M \)-five-branes at the six-planes, through the process of “small instanton” phase transitions [11].

In this paper we would like to address an essential physical feature of \( M \)-theory orbifolds, namely: how is it possible on orbifold fixed planes that do not admit a gravitational anomaly to determine the twisted sector spectrum? Recall that the main feature of the original Hořava-Witten theory was that there is a gravitational anomaly on each ten-dimensional orbifold fixed plane, and that this can, essentially, only be cancelled by adding an \( E_8 \) Yang-Mills \( N = 1 \) supermultiplet to each plane. That is, the existence of a gravitational anomaly dictated the structure of the twisted sector necessary to cancel it. This leads us to ask how one can determine the twisted sector spectrum on an orbifold fixed plane, such as a seven- or four-plane, that does not admit a gravitational anomaly. Unless this can be done, the associated \( M \)-theory orbifold cannot be constructed.
Here, we explicitly answer this question. We show, within the context of several simple but representative examples, how local anomaly cancellation on orbifold planes with gravitational anomalies determines the twisted sector spectrum on orbifold planes which intersect them, even if these planes have no gravitational anomalies themselves. This result follows from the details of anomaly cancellation which, we demonstrate, requires a chiral spectrum carrying fractional anomaly data. Such fractional data can only arise from a twisted sector on the intersecting fixed planes. In addition to determining the twisted sector on orbifold fixed planes without gravitational anomalies, local anomaly freedom also uniquely specifies additional Chern-Simons interactions that must appear on the worldvolumes of these planes.

In this paper, we work within the context of M-theory on a background $\mathbf{R}^6 \times S^1/\mathbb{Z}_2 \times T^4/\mathbb{Z}_2$ spacetime. In this case, our results will apply to the seven-dimensional fixed planes. This is, in some sense, the most illustrative example possible since seven-planes can admit no chiral anomalies, gravitational or gauge. However, it is clear that similar methods will apply to other M-theory orbifolds based on threefolds, including those with four-dimensional fixed planes and $N = 1$ supersymmetry. It is expected that one can now compute the twisted sector spectrum on these four-planes, opening the door for new phenomenological particle physics models. This work will be reported on elsewhere.

2 M-Theory on $S^1/\mathbb{Z}_2 \times T^4/\mathbb{Z}_2$ Orbifolds

In this paper, we will, for specificity, consider M-theory orbifolds on $S^1/\mathbb{Z}_2 \times T^4/\mathbb{Z}_2$. The spacetime has topology $\mathbf{R}^6 \times S^1 \times T^4$, where each of the five compact coordinates takes values on the interval $[-\pi, \pi]$ with the endpoints identified. Let $x^\mu$ parameterize the six non-compact dimensions, while $x^i$ and $x^{11}$ parameterize the $T^4$ and $S^1$ factors respectively. Then the $\mathbb{Z}_2$ action on $S^1$ is defined by

$$\alpha : (x^\mu, x^i, x^{11}) \longrightarrow (x^\mu, x^i, -x^{11})$$

whereas the $\mathbb{Z}_2$ action on $T^4$ is

$$\beta : (x^\mu, x^i, x^{11}) \longrightarrow (x^\mu, -x^i, x^{11}) .$$

The element $\alpha$ leaves invariant the two ten-planes defined by $x^{11} = 0$ and $x_{11} = \pi$, while $\beta$ leaves invariant the sixteen seven-planes defined when the four coordinates $x^i$ individually assume the values 0 or $\pi$. Finally, $\alpha \beta$ leaves invariant the thirty-two six-planes defined when all five compact coordinates individually assume the values 0 or $\pi$. The $\alpha \beta$ six-planes coincide with the intersections of the $\alpha$ ten-planes with the $\beta$ seven-planes. The global structure of this orbifold is shown in Figure 1.
Figure 1: The global structure of orbifold planes in the $S^1/Z_2 \times T^4/Z_2$ orbifold. The two horizontal lines represent the two ten-dimensional (“Hořava-Witten”) fixed planes associated with the $Z_2$ factor denoted $\alpha$, while the sixteen vertical lines represent the seven-dimensional fixed-planes associated with the $Z_2$ action denoted $\beta$. The thirty-two six-dimensional fixed planes associated with $\alpha\beta$ are represented by the solid dots. These coincide with the intersection of the $\alpha$ planes and the $\beta$ planes.

In ten-dimensional superstring theories, the spectrum of the theory on an orbifold background can be computed directly from the string equations. This is not possible in eleven-dimensional $M$-theory, since the fundamental underlying structure of this theory remains unknown. However, as was demonstrated in [1, 2] for the case of a one-dimensional $S^1/Z_2$ orbifold and generalized to higher dimensional orbifolds in [3, 4, 6, 7, 8, 10], the spectrum of $M$-theory orbifolds can be determined by exploiting the requirement that the theory be free of chiral anomalies. Since $S^1/Z_2$ appears as a subspace of the orbifold of interest in this paper, we begin our analysis by briefly reviewing the results of [1, 2].

3 The Ten-Plane Anomaly

A gravitational anomaly arises on each ten-plane due to the coupling of chiral projections of the bulk gravitino to currents localized on the fixed planes. Since the two ten-planes are indistinguishable aside from their position, this anomaly is identical on each of the two planes and can be computed by conventional means if proper care is used. The reason why extra care is needed is that each ten-plane anomaly arises from the coupling of eleven-dimensional fermions to ten-dimensional currents, whereas standard index theorem results only apply to ten-dimensional fermions coupled to ten-dimensional currents. If one notes
that the index theorem can be applied to the small radius limit where the two ten-planes coincide, then the gravitational anomaly on each individual ten-plane can be computed; it is simply one-half of the index theorem anomaly derived using the “untwisted” sector spectrum in ten-dimensions. By untwisted sector, we mean the $\mathbb{Z}_2$ projection of the eleven-dimensional bulk space supergravity multiplet onto each ten-dimension fixed plane. This untwisted spectrum forms the ten-dimensional $N = 1$ supergravity multiplet containing a graviton, a chiral gravitino, a two-form and a scalar dilaton. We denote by $R$ the ten-dimensional Riemann tensor, regarded as an $SO(9,1)$-valued form.

As pointed out in [1, 2], in addition to the untwisted spectrum, one must allow for the possibility of “twisted” sector $N = 1$ supermultiplets that live on each ten-dimensional orbifold plane only. For the case at hand, the twisted sector spectrum must fall into $N = 1$ Yang-Mills supermultiplets consisting of gauge fields and chiral gauginos. These will give rise to an additional contribution to the gravitational anomaly on each ten-plane, as well as to mixed and pure-gauge anomalies. However, since the twisted sector fields are ten-dimensional, these anomalies can be computed directly from the standard formulas, without multiplying by one-half. The twisted sector gauge group, the dimension of the gauge group and the gauge field strength on the $i$-th ten-plane are denoted by $G_i$, $n_i = \text{dim } G_i$ and $F_i$ respectively, for $i = 1, 2$.

The quantum mechanical one-loop local chiral anomaly on the $i$-th ten-plane is characterized by the twelve-form

$$I_{12(1\text{-loop})i} = \frac{1}{4} \left( I^{(3/2)}_{\text{GRAV}}(R) - I^{(1/2)}_{\text{GRAV}}(R) \right)$$

$$+ \frac{1}{2} \left( n_i I^{(1/2)}_{\text{GRAV}}(R) + I^{(1/2)}_{\text{MIXED}}(R, F_i) + I^{(1/2)}_{G\text{AUGE}}(F_i) \right)$$

(3.1)

from which the anomaly arises by descent. The constituent polynomials contributing to the pure gravitational anomaly due to the chiral spin 3/2 and chiral spin 1/2 fermions are

$$I^{(3/2)}_{\text{GRAV}}(R) = \frac{1}{(2\pi)^6} \left( \frac{55}{56} \text{tr } R^6 - \frac{75}{128} \text{tr } R^4 \wedge \text{tr } R^2 + \frac{35}{512} (\text{tr } R^2)^3 \right)$$

(3.2)

and

$$I^{(1/2)}_{\text{GRAV}}(R) = \frac{1}{(2\pi)^6} \left( -\frac{1}{504} \text{tr } R^6 - \frac{1}{384} \text{tr } R^4 \wedge \text{tr } R^2 - \frac{5}{4608} (\text{tr } R^2)^3 \right)$$

(3.3)

respectively, where tr is the trace of the $SO(9,1)$ indices. The polynomials contributing to the mixed and pure-gauge anomalies are due to chiral spin 1/2 fermions only and are given by

$$I^{(1/2)}_{\text{MIXED}}(R, F_i) = \frac{1}{(2\pi)^6} \left( \frac{1}{16} \text{tr } R^4 \wedge \text{Tr } F_i^2 + \frac{5}{64} (\text{tr } R^2)^2 \wedge \text{Tr } F_i^2 \right. \right.$$

$$- \frac{5}{8} \text{tr } R^2 \wedge \text{Tr } F_i^4 \right)$$

(3.4)
and

\[ I_{GAUGE}^{(1/2)}(F_i) = \frac{1}{(2\pi)^6!} \text{Tr} F_i^6. \]  

(3.5)

Here Tr is the trace over the adjoint representation of \( \mathcal{G}_i \). All the anomaly polynomials are computed using standard index theorems. Each term in (3.1) has a factor of 1/2 because the relevant fermions are Majorana-Weyl with half the degrees of freedom of Weyl fermions. The first two terms in (3.1) arise from untwisted sector fermions, whereas the last three terms are contributed by the twisted sector. It follows from the above discussion that the first two terms must have an additional factor of 1/2, accounting for the overall coefficient of 1/4, whereas the remaining three terms are given exactly by the index theorems.

The quantum anomaly (3.1) would spoil the consistency of the theory were it not to cancel against some sort of classical inflow anomaly. Hence, it is imperative to discern the presence of appropriate local classical counterterms to cancel against (3.1). One begins the analysis of anomaly cancellation by considering the pure \( \text{tr} R^6 \) term in (3.1) which is irreducible and must therefore identically vanish. It follows from the above that this term is

\[ -\frac{1}{2(2\pi)^6!} \frac{(n_i - 248)}{494} \text{tr} R^6. \]  

(3.6)

Therefore, the \( \text{tr} R^6 \) term will vanish if and only if each gauge group \( \mathcal{G}_i \) satisfies the constraint

\[ n_i = 248. \]  

(3.7)

Without yet specifying which 248-dimensional gauge group is permitted, we substitute \( n_i = 248 \) in (3.1) obtaining

\[ I_{12}^{(1\text{-loop})_i} = \frac{1}{2(2\pi)^6!} \left( \frac{-15}{16} \text{tr} R^4 \wedge \text{tr} R^2 - \frac{15}{64} (\text{tr} R^2)^3 + \frac{1}{16} \text{tr} R^4 \wedge \text{Tr} F_i^2 \right. \]

\[ + \frac{5}{64} (\text{tr} R^2)^2 \wedge \text{Tr} F_i^2 - \frac{5}{8} \text{tr} R^2 \wedge \text{Tr} F_i^4 + \text{Tr} F_i^6 \left. \right). \]  

(3.8)

Although non-vanishing, this part of the anomaly is reducible. It follows that it can be made to cancel as long as it can be factorized into the product of two terms, a four-form and an eight-form. A necessary requirement for this to be the case is that

\[ \text{Tr} F_i^6 = \frac{1}{24} \text{Tr} F_i^4 \wedge \text{Tr} F_i^2 - \frac{1}{3600} (\text{Tr} F_i^2)^3. \]  

(3.9)

There are two Lie groups with dimension 248 that satisfy this condition, the non-Abelian group \( E_8 \) and the Abelian group \( U(1)^{248} \). Both groups represent allowed twisted matter
gauge groups on each ten-plane. Hence, from anomaly considerations alone one can
determine the twisted sector on each ten-plane, albeit with a small ambiguity in the
allowed twisted sector gauge group. In this paper, we consider only the non-Abelian
gauge group $E_8$. Using (3.9) and several $E_8$ trace relations, the anomaly polynomial (3.8)
can be re-expressed as follows

$$I_{12}(1\text{-loop})_i = \frac{1}{3} \pi I^3_{4(i)} + X_8 \wedge I_{4(i)}$$

(3.10)

where $X_8$ is the eight-form

$$X_8 = \frac{1}{(2\pi)^34!} \left( \frac{1}{8} \text{tr} R^4 - \frac{1}{32} (\text{tr} R^2)^2 \right)$$

(3.11)

and $I_{4(i)}$ is the four-form given by

$$I_{4(i)} = \frac{1}{16\pi^2} \left( \frac{1}{30} \text{Tr} F_i^2 - \frac{1}{2} \text{tr} R^2 \right).$$

(3.12)

Once in this factorized form, the anomaly $I_{12}(1\text{-loop})_i$ can be cancelled as follows.

First, the Bianchi identity $dG = 0$, where $G$ is the field strength of the three-form $C$
in the eleven-dimensional supergravity multiplet, is modified to

$$dG = \sum_{i=1}^2 I_{4(i)} \wedge \delta^{(1)}_{M_{10}}$$

(3.13)

where $I_{4(i)}$ is the four-form given in (3.12) and $\delta^{(1)}_{M_{10}}$ is a one-form brane current with
support on the $i$-th ten-plane. Second, we note that the eleven-dimensional supergravity
action contains the terms

$$S = \cdots - \frac{\pi}{3} \int C \wedge G \wedge G + \int G \wedge X_7$$

(3.14)

where $X_7$ satisfies $dX_7 = X_8$. The $CGG$ interaction is required by the minimally-coupled
supergravity action, while the $GX_7$ term is an additional higher-derivative interaction
necessitated by five-brane anomaly cancellation. Using the modified Bianchi identity (3.13),
one can compute the variation of these two terms under Lorentz and gauge transforma-
tions. The result is that the $CGG$ and $GX_7$ terms have classical anomalies which descend
from the polynomials

$$I_{12}(CGG)_i = -\frac{\pi}{3} I^3_{4(i)}$$

(3.15)

and

$$I_{12}(GX_7)_i = -X_8 \wedge I_{4(i)}.$$  

(3.16)
respectively. It follows that

\[ I_{12}(1 - \text{loop})_i + I_{12}(CGG)_i + I_{12}(GX_7)_i = 0 \]  

(3.17)

and, hence, the total anomaly cancels exactly.

We conclude that the requirement of local anomaly cancellation on the each of the two \(S^1/\mathbb{Z}_2\) orbifold ten-planes specifies the twisted spectrum of the theory. This specification is almost, but not quite, unique, allowing \(N = 1\) vector supermultiplets with either gauge group \(E_8\) or \(U(1)^{248}\). An important ingredient in this analysis was the fact that the contribution to the anomaly on each ten-plane from the untwisted sector was a factor of \(1/2\) smaller than the index theorem result. This followed from the fact that the index theorem had to be spread over two equivalent ten-planes. A direct consequence of this is that the non-Abelian gauge group on each ten-plane is \(E_8\), not \(E_8 \times E_8\), and that the gauge group \(SO(32)\) is disallowed. Since \(S^1/\mathbb{Z}_2\) is a subspace of \(S^1/\mathbb{Z}_2 \times T^4/\mathbb{Z}_2\), the results of this section continue to hold on the larger orbifold. We now discuss the cancellation of local anomalies in the other factor space, \(T^4/\mathbb{Z}_2\).

4 The Seven-Plane Anomaly

The quantum anomalies on each of the sixteen indistinguishable seven-planes of the \(T^4/\mathbb{Z}_2\) orbifold are easy to analyze. In analogy with the ten-planes, an untwisted sector is induced on each seven-plane by the \(\mathbb{Z}_2\) projection of the eleven-dimensional supergravity multiplet. This untwisted spectrum forms the seven-dimensional \(N = 1\) supergravity multiplet consisting of a graviton, a gravitino, three vector fields, a two-form, a real scalar dilaton and a spin 1/2 dilitino. However, unlike the case of a ten-plane, gravitational anomalies cannot be supported on a seven-plane. In fact, since there are no chiral fermions in seven-dimensions, no chiral anomaly of any kind, gravitational or gauge, can arise. Hence, with no local chiral anomalies to cancel, it would appear to be impossible to compute the twisted sector spectrum of any seven-plane. As long as we focus on the seven-planes exclusively, this conclusion is correct. However, as we will see in the next section, the cancellation of the local anomalies on the thirty-two six-dimensional \(\alpha\beta\) orbifold planes, formed from the intersection of the \(\alpha\) ten-planes with the \(\beta\) seven-planes, will require a non-vanishing twisted sector spectrum on each seven-plane and dictate its structure. With this in mind, we now turn to the analysis of anomalies localized on the intersection six-planes in the full \(S^1/\mathbb{Z}_2 \times T^4/\mathbb{Z}_2\) orbifold.
5  The Six-Plane Anomaly and Seven-Plane Twisted Sector

As in the case for the ten-planes, a gravitational anomaly will arise on each six-plane due to the coupling of chiral projections of the bulk gravitino to currents localized on the thirty-two fixed planes. Since the thirty-two six-planes are indistinguishable, the anomaly is the same on each plane and can be computed by conventional means if proper care is taken. Noting that the standard index theorems can be applied to the small radius limit where the thirty-two six-planes coincide, it follows that the gravitational anomaly on each six-plane is simply one-thirty-second of the index theorem anomaly derived using the untwisted sector spectrum in six-dimensions. In this case, the untwisted sector spectrum is the $\mathbb{Z}_2 \times \mathbb{Z}_2$ projection of the eleven-dimensional bulk supergravity multiplet onto each six-dimensional fixed plane. This untwisted spectrum forms several $N = 1$ six-dimensional supermultiplets. Namely, the supergravity multiplet consisting of a graviton, a chiral gravitino and a self-dual two-form, four hypermultiplets each with four scalars and an anti-chiral hyperino, and one tensor multiplet with one anti-self-dual two-form, one scalar and an anti-chiral spin 1/2 fermion. A one-loop quantum gravitational anomaly then arises from one chiral spin 3/2 fermion, five anti-chiral spin 1/2 fermions and one each of self-dual and anti-self-dual tensors. However, the anomalies due to the tensors cancel each other. Noting that a chiral anomaly in six-dimensions is characterized by an eight-form, from which the anomaly arises by descent, we find, for the $i$-th six-plane, that

$$I_8(SG)_i = \frac{1}{32} \left( I_{GRAV}^{(3/2)}(R) - 5 I_{GRAV}^{(1/2)}(R) \right)$$

(5.1)

where

$$I_{GRAV}^{(3/2)}(R) = \frac{1}{(2\pi)^3 4!} \left( - \frac{49}{48} \text{tr} R^4 + \frac{43}{192} (\text{tr} R^2)^2 \right)$$

(5.2)

and

$$I_{GRAV}^{(1/2)}(R) = \frac{1}{(2\pi)^3 4!} \left( - \frac{1}{240} \text{tr} R^4 - \frac{1}{192} (\text{tr} R^2)^2 \right) ,$$

(5.3)

where $R$ is the six-dimensional Riemann tensor, regarded as an $SO(5,1)$-valued form. Note that the terms in brackets in (5.1) are the anomaly as computed by the index theorem. $I_8(SG)_i$ is obtained from that result by dividing by 32.

Noting that each six-plane is embedded in one of the two ten-dimensional planes, we see that there are additional “untwisted” sector fields on each six-plane. These arise from the $\beta$ $\mathbb{Z}_2$ projection of the $N = 1$ $E_8$ Yang-Mills supermultiplet on the associated ten-plane.
Such fields are untwisted from the point of view of the six-dimensional plane, although they arise from fields that were in the twisted sector of the ten-plane. In this paper, we will assume that the $\beta$ action on the ten-dimensional vector multiplets does not break the $E_8$ gauge group. A discussion of the case where $E_8$ is broken to a subgroup by the action of $\beta$ can be found in \cite{7, 8}. A ten-dimensional $N = 1$ vector supermultiplet decomposes in six-dimensions into an $N = 1$ vector multiplet and an $N = 1$ hypermultiplet. However, the action of $\beta$ projects out the hypermultiplet. Therefore, the ten-plane contribution to the untwisted sector of each six-plane is an $N = 1$ $E_8$ vector supermultiplet, which consists of gauge fields and chiral gauginos. The gauginos contribute to the gravitational anomaly on each six-plane, as well as adding mixed and $E_8$ gauge anomalies. Noting that the standard index theorems can be applied to the small radius limit, where each ten-plane shrinks to zero size and, hence, the sixteen six-planes it contains coincide, it follows that the anomaly is simply one-sixteenth of the index theorem result. We find that the one-loop quantum contribution of this $E_8$ supermultiplet to the gravitational, mixed and $E_8$ gauge anomalies on the $i$-th six-plane is

$$I_8(E_8)_i = \frac{1}{16} \left( 248 I^{(1/2)(R)}_{\text{GRAV}} + I^{(1/2)}_{\text{MIXED}}(R, F_i) + I^{(1/2)}_{\text{GAUGE}}(F_i) \right)$$

(5.4)

where

$$I^{(1/2)}_{\text{MIXED}}(R, F_i) = \frac{1}{(2\pi)^3 4!} \left( \frac{1}{4} \text{Tr} R^2 \wedge \text{Tr} F_i^2 \right)$$

(5.5)

and

$$I^{(1/2)}_{\text{GAUGE}}(F_i) = \frac{1}{(2\pi)^3 4!} \left( - \text{Tr} F_i^4 \right).$$

(5.6)

Here Tr is over the adjoint 248 representation of $E_8$. Note that the terms in brackets in (5.4) are the index theorem anomaly. $I_8(E_8)_i$ is obtained from that result by dividing by 16.

Are there other sources of untwisted sector anomalies on a six-plane? The answer is, potentially yes. We note that, in addition to being embedded in one of the two ten-planes, each six-plane is also embedded in one of the sixteen seven-dimensional orbifold planes. In analogy with the discussion above, if there were to be a non-vanishing twisted sector spectrum on each seven-plane, then this could descend under the $\alpha \mathbb{Z}_2$ projection as an addition to the untwisted spectrum on each six-plane. This additional untwisted spectrum could then contribute to the chiral anomalies on the six-plane. However, as noted above, a priori, there is no reason for one to believe that there is any twisted sector on a seven-dimensional orbifold plane. Therefore, for the time being, let us assume that there is no
such contribution to the six-dimensional anomaly. We will see below that this assumption
must be carefully revisited.

As for the ten-dimensional planes, one must allow for the possibility of twisted sector
$N = 1$ supermultiplets on each of the thirty-two six-planes. The most general allowed
spectrum on the $i$-th six-plane would be $n_{Vi}$ vector multiplets transforming in the adjoint
representation of some as yet unspecified gauge group $\mathcal{G}_i$, $n_{Hi}$ hypermultiplets transform-
ing under some representation (possibly reducible) $\mathcal{R}$ of $\mathcal{G}_i$, and $n_{Ti}$ gauge-singlet tensor
multiplets. We denote by $\mathcal{F}_i$ the gauge field strength. Since these fields are in the twisted
sector, their contribution to the chiral anomalies can be determined directly from the
index theorems without modification. We find that the one-loop quantum contribution
of the twisted spectrum to the gravitational, mixed and $\mathcal{G}_i$ gauge anomalies on the $i$-th
six-plane is

\[
I_8(\mathcal{G}_i) = (n_{Vi} - n_{Hi} - n_{Ti}) I_{GRAV}^{(1/2)}(R) - n_{Ti} I_{GRAV}^{(3\text{-form})}(R) + I_{\text{MIXED}}^{(1/2)}(R, \mathcal{F}_i) + I_{\text{GAUGE}}^{(1/2)}(\mathcal{F}_i)
\]

where $I_{GRAV}^{(1/2)}(R)$ is given in (5.3) and

\[
I_{GRAV}^{(3\text{-form})}(R) = \frac{1}{(2\pi)^3 4!} \left( - \frac{7}{60} \text{tr} R^4 + \frac{1}{24} \left( \text{tr} R^2 \right)^2 \right).
\]

Furthermore, the mixed and pure-gauge anomaly polynomials are modified to

\[
I_{\text{MIXED}}^{(1/2)}(R, \mathcal{F}_i) = \frac{1}{(2\pi)^3 4!} \left( \frac{1}{4} \text{tr} R^2 \wedge \text{trace} \mathcal{F}_i^2 \right)
\]

and

\[
I_{\text{GAUGE}}^{(1/2)}(\mathcal{F}_i) = \frac{1}{(2\pi)^3 4!} \left( - \text{trace} \mathcal{F}_i^4 \right),
\]

where

\[
\text{trace} \mathcal{F}_i^n = \text{Tr} \mathcal{F}_i^n - \sum_\alpha h_\alpha \text{tr}_\alpha \mathcal{F}_i^n.
\]

Here Tr is an adjoint trace, $h_\alpha$ is the number of hypermultiplets transforming in the
$\mathcal{R}_\alpha$ representation and $tr_\alpha$ is a trace over the $\mathcal{R}_\alpha$ representation. Note that the total
number of vector multiplets is $n_{Vi} = \dim(\mathcal{G}_i)$, while the total number of hypermultiplets
is $n_{Hi} = \sum_\alpha h_\alpha \times \dim(\mathcal{R}_\alpha)$. The relative minus sign in (5.11) reflects the anti-chirality
of the hyperinos.

Combining the contributions from the two untwisted sector sources and the twisted
sector, the total one-loop quantum anomaly on the $i$-th six-plane is the sum

\[
I_8(1\text{-loop})_i = I_8(SG)_i + I_8(E_8)_i + I_8(\mathcal{G}_i)
\]
where \( I_8(SG)_i \), \( I_8(E_8)_i \), and \( I_8(G_i) \) are given in (5.1), (5.4) and (5.7) respectively.

Unlike the case for the ten-dimensional planes, the classical anomaly associated with the \( GX_7 \) term in the eleven-dimensional action (3.14) can contribute to the irreducible curvature term which, in six-dimensions, is \( \text{tr} R^4 \). Therefore, our next step is to further modify the Bianchi identity for \( G = dC \) from expression (3.13) to

\[
dG = \sum_{i=1}^{2} I_4(i) \wedge \delta_{M^6_i}^{(1)} + \sum_{i=1}^{32} g_i \delta_{M^6_i}^{(5)}
\]

where \( \delta_{M^6_i}^{(5)} \) has support on the six-planes \( M^6_i \). As discussed in [6, 7], the magnetic charges \( g_i \) are required to take the values

\[
g_i = -3/4, -1/4, +1/4, ...
\]

Using the modified Bianchi identity (5.12), one can compute the variation of the \( GX_7 \) term under Lorentz and gauge transformations. The result is that this term gives rise to a classical anomaly that descends from the polynomial

\[
I_8(GX_7)_i = -g_i X_8
\]

where \( X_8 \) is presented in expression (3.11). The relevant anomaly is then

\[
I_8(1-\text{loop})_i + I_8(GX_7)_i
\]

where \( I_8(1-\text{loop})_i \) is given in (5.12). This anomaly spoils the consistency of the theory and, hence, must cancel. One begins the analysis of anomaly cancellation by considering the pure \( \text{tr} R^4 \) term in (5.16) which is irreducible and must identically vanish. It follows from the above that this term is

\[
- \frac{1}{(2\pi)^3 4! 240} (n_{V_i} - n_{H_i} - 29n_{T_i} + 30g_i + 23) \text{tr} R^4 .
\]

Therefore, the \( \text{tr} R^4 \) term will vanish if and only if on each orbifold plane the constraint

\[
n_{V_i} - n_{H_i} - 29n_{T_i} + 30g_i + 23 = 0
\]

is satisfied. Herein lies a problem, and the main point of this paper. Noting from (5.14) that \( g_i = c_i/4 \) where \( c_i = -3, -1, 1, 3, 5, ... \), we see that cancelling the \( \text{tr} R^4 \) term requires that we satisfy

\[
n_{V_i} - n_{H_i} - 29n_{T_i} = (-15c_i - 46)/2 .
\]

However, this is not possible since the left hand side of this expression is an integer and the right hand side always half integer. There is only one possible resolution of this problem,
which is to carefully review the only assumption that was made above, that is, that there is no twisted sector on a seven-plane and, hence, no contribution of the seven-planes by $\alpha Z_2$ projection to the untwisted anomaly on a six-plane. As we now show, this assumption is false.

Let us now allow for the possibility that there is a twisted sector of $N = 1$ supermultiplets on each of the sixteen seven-planes. The most general allowed spectrum on the $i$-th seven-plane would be $n_{7V_i}$ vector supermultiplets transforming in the adjoint representation of some as yet unspecified gauge group $G_{7i}$. Each seven-dimensional vector multiplet contains a gauge field, three scalars and a gaugino. With respect to six-dimensions, this vector multiplet decomposes into an $N = 1$ vector supermultiplet and a single hypermultiplet. Under the $\alpha Z_2$ projection to each of the two embedded six-planes, the gauge group $G_{7i}$ can be preserved or broken to a subgroup. In either case, we denote the six-dimensional gauge group arising in this manner as $\hat{G}_i$, define $\tilde{n}_{V_i} = \dim \hat{G}_i$ and write the associated gauge field strength as $\tilde{F}_i$. In this paper, for simplicity, we will assume that the gauge group is unbroken by the orbifold projection, that is, $\hat{G}_i = G_{7i}$. The more general case where it is broken to a subgroup is discussed in [7, 8]. Furthermore, the $\alpha$ action projects out either the six-dimensional vector supermultiplet, in which case the hypermultiplet descends to the six-dimensional untwisted sector, or the six-dimensional hypermultiplet, in which case the vector supermultiplet enters the six-dimensional untwisted sector. We denote by $\tilde{n}_{Hi}$ the number of hypermultiplets arising in the six-dimensional untwisted sector by projection from the seven-plane, and specify their (possibly reducible) representation under $\hat{G}_i$ as $\tilde{R}$. Since these fields are in the untwisted sector associated with a single seven-plane, and since there are two six-planes embedded in each seven-plane, their contribution to the quantum anomaly on each six-plane can be determined by taking $1/2$ of the index theorem result. We find that the one-loop quantum contribution of this part of the the untwisted spectrum to the gravitational, mixed and $\tilde{G}_i$ gauge anomalies on the $i$-th six-plane is

$$I_8(\hat{G}_i) = \frac{1}{2} \left( \tilde{n}_{V_i} - \tilde{n}_{Hi} \right) I^{(1/2)}_{\text{GRAV}}(R) + I^{(1/2)}_{\text{MIXED}}(R, \tilde{F}_i) + I^{(1/2)}_{\text{GAUGE}}(\tilde{F}_i) \right) \tag{5.20}$$

where $I^{(1/2)}_{\text{GRAV}}(R), I^{(1/2)}_{\text{MIXED}}(R, \tilde{F}_i)$ and $I^{(1/2)}_{\text{GAUGE}}(\tilde{F}_i)$ are given in (5.3), (5.9) and (5.10) respectively with the gauge and hypermultiplet quantities replaced by their “~” equivalents.

The total quantum anomaly on the $i$-th six-plane is now modified to

$$I_8(1\text{-loop})_i + I_8(\hat{G}_i) \tag{5.21}$$

where $I_8(1\text{-loop})_i$ and $I_8(\hat{G}_i)$ are given in (5.12) and (5.20) respectively. It follows that the relevant anomaly contributing to, among other things, the irreducible $\text{tr} R^4$ term is
modified to
\[ I_8(1\text{-loop})_i + I_8(\hat{G}_i) + I_8(GX_7)_i . \] (5.22)

This anomaly spoils the quantum consistency of the theory and, hence, must cancel. We again begin by considering the pure \( \text{tr} R^4 \) term in (5.22). This term is irreducible and must identically vanish. It follows from the above that this term is
\[ -\frac{1}{(2\pi)^3 4! 240} (n_{V_i} - n_{H_i} + \frac{1}{2} \tilde{n}_{V_i} - \frac{1}{2} \tilde{n}_{H_i} - 29n_{T_i} + 30g_i + 23 ) \text{tr} R^4 . \] (5.23)

Therefore, the \( \text{tr} R^4 \) term will vanish if and only if on each orbifold plane the constraint
\[ n_{V_i} - n_{H_i} + \frac{1}{2} \tilde{n}_{V_i} - \frac{1}{2} \tilde{n}_{H_i} - 29n_{T_i} + 30g_i + 23 = 0 \] (5.24)
is satisfied. Again, noting that \( g_i = c_i/4 \) where \( c_i = -3, -1, 1, 3, 5, ... \), we see that we must satisfy
\[ n_{V_i} - n_{H_i} + \frac{1}{2} \tilde{n}_{V_i} - \frac{1}{2} \tilde{n}_{H_i} - 29n_{T_i} = \frac{1}{2} (-15c_i - 46) . \] (5.25)

As above, the right hand side is always a half integer. Now, however, because of the addition of the untwisted spectrum arising from the seven-plane, the left hand side can also be chosen to be half integer. Hence, the pure \( \text{tr} R^4 \) term can be cancelled.

Having cancelled the irreducible \( \text{tr} R^4 \) term, we now compute the remaining terms in the anomaly eight-form. In addition to the contributions from (5.22), we must also take into account the classical anomaly associated with the \( \text{CGG} \) term in the eleven-dimensional action (3.14). Using the modified Bianchi identity (5.12), one can compute the variation of the \( \text{CGG} \) term under Lorentz and gauge transformations. The result is that this term gives rise to a classical anomaly that descends from the polynomial
\[ I_8(CGG)_i = -\pi g_i I^2_4(i) \] (5.26)
where \( I^2_4(i) \) is given in expression (3.12). Adding this anomaly to (5.22), and cancelling the \( \text{tr} R^4 \) term by imposing constraint (5.24), we can now determine the remaining terms in the anomaly eight-form.

Recall that, in this paper, we are assuming that the \( \beta \) action on the ten-dimensional vector supermultiplet does not break the \( E_8 \) gauge group. In this case, we can readily show that there can be no twisted sector vector multiplets on any six-plane. Rather than complicate the present discussion, we will simply assume here that gauge field strengths \( F_i \) do not appear. Furthermore, cancellation of the complete anomaly, in the case where
$E_8$ is unbroken, requires that $\tilde{G}_i$ be a product of $U(1)$ factors. Here, we will limit the discussion to the simplest case where

$$\tilde{G}_i = U(1) \quad (5.27)$$

The $\beta$ action on the seven-dimensional plane then either projects a single vector supermultiplet, or a single chargeless hypermultiplet, onto the untwisted sector of the six-plane. In either case, no $U(1)$ anomaly exists. Hence, the gauge field strengths $\tilde{F}_i$ also do not appear. With this in mind, we now compute the remaining terms in the anomaly eight-form. They are

$$\frac{1}{(2\pi)^3 4!} \frac{3}{16} \left( \frac{3}{4} (1 - 4 n_{T_i}) (\text{tr} R^2)^2 + \frac{1}{20} (5 + 8 g_i) \text{tr} R^2 \wedge \text{Tr} F_i^2 \right. $$

$$\left. - \frac{1}{100} (1 + \frac{4}{3} g_i) (\text{Tr} F_i^2)^2 \right) \quad (5.28)$$

where we have used the $E_8$ trace relation $\text{Tr} F^4 = \frac{1}{100} (\text{Tr} F^2)^2$. Note that, since $n_{T_i}$ is a non-negative integer and $g_i$ must satisfy (5.14), the first two terms of this expression term can never vanish. Furthermore, it is straightforward to show that (5.28) will factor into an exact square, and, hence, be potentially cancelled by a six-plane Green-Schwarz mechanism, if and only if

$$4 (4 n_{T_i} - 1) (3 + 4 g_i) = (5 + 8 g_i)^2 \quad (5.29)$$

Again, this equation has no solutions for the allowed values of $n_{T_i}$ and $g_i$. It follows that anomaly (5.28), as it presently stands, cannot be be made to identically vanish or cancel. The resolution of this problem was first described in [6], and consists of the realization that the existence of seven-planes in the theory necessitates the introduction of additional Chern-Simons interactions in the action, one for each seven-plane. The required terms are

$$S = \cdots + \sum_{i=1}^{16} \int \delta^{(4)}_{M_i} \wedge G \wedge Y_{03(i)} \quad (5.30)$$

where $dY_{03(i)} = Y_{4(i)}$ is a gauge-invariant four-form polynomial. $Y_{4(i)}$ arises from the curvature $R$ and also the field strength $\tilde{F}_i$ associated with the additional adjoint supergauge fields localized on the $i$-th seven-plane. It is given by

$$Y_{4(i)} = \frac{1}{4\pi} \left( - \frac{1}{32} \eta \text{tr} R^2 + \rho \text{tr} \tilde{F}_i \right) \quad (5.31)$$

where $\eta$ and $\rho$ are rational coefficients. Using the modified Bianchi identity (5.13), one can compute the variation of the $\delta^7 G Y_3$ terms under Lorentz and gauge transformations.
The result is that these give rise to a classical anomaly that descends from the polynomial

\[ I_8(\delta^7GY_3)_i = -I_{4(i)} \wedge Y_{4(i)} \]  

(5.32)

where \( I_{4(i)} \) is the four-form given in (3.12).

The total anomaly on the \( i \)-th six-plane is now modified to

\[ I_8(1-\text{loop})_i + I_8(\tilde{G}_i) + I_8(GX_7)_i + I_8(CGG)_i + I_8(\delta^7GY_3)_i \]  

(5.33)

where \( I_8(1-\text{loop})_i, I_8(\tilde{G}_i), I_8(GX_7)_i, I_8(CGG)_i \) and \( I_8(\delta^7GY_3)_i \) are given in (5.12), (5.20), (5.15), (5.26) and (5.32) respectively. Note that for the fixed plane intersection presently under discussion, the field strength \( \tilde{F}_i \) does not enter the anomaly eight-form (5.28). Therefore, within this context, we must take

\[ \rho = 0 . \]  

(5.34)

We will exhibit an example of non-vanishing \( \rho \) parameter at the end of this section. After cancelling the irreducible \( \text{tr } R^4 \) term, the remaining anomaly now becomes

\[ \frac{1}{(2\pi)^3 4! 16} \left( \frac{3}{4} \left(1 - 4n_{Ti} - \eta \right) (\text{tr } R^2)^2 \right. 
\[ + \frac{1}{20} (5+8g_i + \eta ) \text{tr } R^2 \wedge \text{Tr } F_i^2 - \frac{1}{100} (1 + \frac{4}{3} g_i ) (\text{Tr } F_i^2)^2 \) \]  

(5.35)

Depending on the number of untwisted hypermultiplets, \( n_{Ti} \), these terms can be made to cancel or to factor into the sum of exact squares. In this paper, we consider the \( n_{Ti} = 0, 1 \) cases only. As discussed in [7, 8], the solutions where \( n_{Ti} \geq 2 \) are related to the \( n_{Ti} = 0, 1 \) solutions by the absorption of one or more five-branes from the bulk space onto the \( i \)-th six-plane.

We first consider the case where

\[ n_{Ti} = 0 . \]  

(5.36)

In this case, no further Green-Schwarz type mechanism in six-dimensions is possible and the anomaly must vanish identically. We see from (5.33) that this is possible if and only if

\[ g_i = -3/4, \quad \eta = 1 . \]  

(5.37)

*It is important to note that this solution only exists for a non-vanishing value of parameter \( \eta \). Hence, the additional Chern-Simons interactions (5.30) are essential for the anomaly*
Figures 2a and 2b: The two solutions with unbroken ten-dimensional $E_8$ and $n_T = 0$ in which the seven-dimensional gauge group is $U(1)$. The left-hand figure depicts the case where only a chargeless hypermultiplet survives the $Z_2$ projection. The right-hand figure depicts the case where only a $U(1)$ vector multiplet survives. In the second case we also require an extra six-dimensional singlet hypermultiplet, which is depicted by the $1$ sitting below the vertex. In both cases, we require $\eta=1$, $\rho = 0$ and $g = -3/4$.

To vanish identically in the $n_{T_i} = 0$ case. Inserting these results into expression (5.24) for the vanishing of the irreducible $\text{tr} R^4$ term, and recalling that $n_{V_i} = 0$, we find that

$$-2n_{H_i} + \tilde{n}_{V_i} - \tilde{n}_{H_i} = -1.$$  \hspace{1cm} (5.38)

Equation (5.38) can be solved in several ways. Remembering that $\tilde{G}_i = U(1)$, the first solution then consists of allowing the $U(1)$ hypermultiplet to descend to the six-plane while projecting out the $U(1)$ vector multiplet. Equation (5.38) is then solved by taking the number of twisted hypermultiplets to vanish. That is, take

$$\tilde{n}_{H_i} = 1, \quad \tilde{n}_{V_i} = 0, \quad n_{H_i} = 0.$$  \hspace{1cm} (5.39)

The second solution follows by doing the reverse, that is, projecting out the $U(1)$ hypermultiplet and allowing the $U(1)$ vector multiplet to descend to the six-plane. In this case, equation (5.38) is solved by taking

$$\tilde{n}_{H_i} = 0, \quad \tilde{n}_{V_i} = 1, \quad n_{H_i} = 1.$$  \hspace{1cm} (5.40)

These two solutions are illustrated in Figure 2 (a) and (b) respectively.

Let us now consider the case where

$$n_{T_i} = 1.$$  \hspace{1cm} (5.41)

In this case, the anomaly (5.35) can be removed by a six-dimensional Green-Schwarz mechanism as long as it factors into an exact square. It is straightforward to show that this will be the case if and only if

$$4 \left( 3 + \eta \right) \left( 3 + 4 g_i \right) = (5 + 8 g_i + \eta)^2.$$  \hspace{1cm} (5.42)
This equation has two solutions

\[ g_i = -3/4, \quad \eta = 1 \quad (5.43) \]

and

\[ g_i = 1/4, \quad \eta = 1. \quad (5.44) \]

Again, note that these solutions require a non-vanishing value of the parameter \( \eta \). Hence, the additional Chern-Simons interactions \( (5.30) \) are also essential for anomaly factorization in the \( n_{T_i} = 1 \) case. Inserting these into the expression for the vanishing of the irreducible \( \text{tr} \ R^4 \) term, and recalling that \( n_{V_i} = 0 \), we find

\[ -2n_{H_i} + \tilde{n}_{V_i} - \tilde{n}_{H_i} = 57 \quad (5.45) \]

and

\[ -2n_{H_i} + \tilde{n}_{V_i} - \tilde{n}_{H_i} = -3. \quad (5.46) \]

The first equation \( (5.45) \) cannot be solved within the context of \( \tilde{G}_i = U(1) \), since \( \tilde{n}_{V_i} \leq 1 \). The second equation, however, has two solutions

\[ \tilde{n}_{H_i} = 1, \quad \tilde{n}_{V_i} = 0, \quad n_{H_i} = 1 \quad (5.47) \]

and

\[ \tilde{n}_{H_i} = 0, \quad \tilde{n}_{V_i} = 1, \quad n_{H_i} = 2. \quad (5.48) \]

These are illustrated in Figure 3 (a) and (b) respectively.

In either case, the anomaly \( (5.35) \) factors into an exact square given by

\[ -3 \frac{1}{(2\pi)^3 4! 16} \left( \text{tr} \ R^2 - \frac{1}{15} \text{Tr} \ F_i^2 \right)^2. \quad (5.49) \]

The anomaly can now be cancelled by a Green-Schwarz mechanism on the six-plane. First, one alters the Bianchi identity for the anti-self-dual tensor in the twisted sector tensor multiplet from \( dH_{T_i} = 0 \), where \( H_{T_i} \) is the tensor field strength three-form, to

\[ dH_{T_i} = \frac{1}{16\pi^2} \left( \text{tr} \ R^2 - \frac{1}{15} \text{Tr} \ F_i^2 \right). \quad (5.50) \]

Second, additional Chern-Simons terms are added to the action, one for each six-plane. The required terms are

\[ S = \cdots - \frac{1}{64\pi} \sum_{i=1}^{32} \int \delta_M^{(5)} \wedge B_{T_i} \wedge (\text{tr} \ R^2 - \frac{1}{15} \text{Tr} \ F_i^2), \quad (5.51) \]
Figures 3a and 3b: The two solutions with unbroken ten-dimensional $E_8$ and $n_T = 1$ in which the seven-dimensional gauge group is $U(1)$. These represent the cases where a fivebrane has wrapped the vertices depicted in Figure 2. In each case, we have $\eta = 1$, $\rho = 1$ and $g = +1/4$. The six-dimensional tensor multiplet is indicated by the $\times$ on the vertices.

where $B_{Ti}$ is the anti-self-dual tensor two-form on the $i$-th six-plane. Using Bianchi identity (5.50), one can compute the variation of each such term under Lorentz and gauge transformations. The result is a classical anomaly that descends from an eight-form that exactly cancels expression (5.49). The theory is now anomaly free.

Thus, we have demonstrated, within the context of an explicit orbifold fixed plane intersection where the $\beta \mathbb{Z}_2$ projection to the six-plane leaves $E_8$ unbroken, that all local anomalies can be cancelled. However, this cancellation requires that the intersecting seven-plane support a twisted sector consisting of a $U(1)$ $N = 1$ vector supermultiplet and an associated Chern-Simons term. This term is of the form (5.30) with $\eta = 1$ and $\rho = 0$. The fact that $\rho = 0$ in this context follows directly from the property that $E_8$ is unbroken by the $\beta$ projection.

We close this section by briefly presenting another possible orbifold fixed plane intersection where local anomaly freedom requires that the parameter $\rho$ in (5.30) be non-vanishing. For this to be the case, we must allow $E_8$ to be broken to a subgroup by the $\beta \mathbb{Z}_2$ projection to the six-plane. In this example, we take

$$E_8 \rightarrow E_7 \times SU(2).$$

(5.52)

This does not effect the $I_8(SG)_i$ contribution to the anomaly eight-form given in (5.1), but does alter the untwisted gauge anomaly from (5.4) to $I_8(E_7 \times SU(2))_i$. The analysis of the twisted sector spectrum on each six-plane remains identical to that discussed above. As in the previous example, we can show that there can be no twisted sector vector multiplets on any six-plane. That is, the field strengths $\mathcal{F}_i$ do not appear in the anomaly eight-form. Again, we will assume that $G_{7i} = \tilde{G}_i$ and limit our discussion to the simplest case which,
Figure 4: The solution with $n_T = 0$ where the ten-dimensional $E_8$ group is broken to $E_7 \times SU(2)$. This case requires a seven-dimensional gauge group $SU(2)$, which is identified with the $SU(2)$ factor on the ten-plane. This solution has $\eta = \rho = 1$ and $g = -1/4$.

in this example, is

$$\mathcal{G}_i = SU(2).$$

(5.53)

This seven-plane gauge group must be identified with the $SU(2)$ factor group in $E_7 \times SU(2)$. Analysis of the cancellation of all anomalies at this fixed plane intersection yields the following results. First of all, for there to be any solution, one must take the $\rho$ parameter in (5.31) to be

$$\rho = 1.\quad (5.54)$$

For the case where

$$n_{Ti} = 0\quad (5.55)$$

the complete anomaly will vanish identically if and only if

$$g_i = -1/4, \quad \eta = 1\quad (5.56)$$

and

$$\tilde{n}_{Hi} = 3, \quad \tilde{n}_{Vi} = 0, \quad n_{Hi} = 0\quad (5.57)$$

where the $\tilde{n}_{Hi}$ hypermultiplets transform in the $(1, 3)$ representation of $E_7 \times SU(2)$. This solution is illustrated in Figure 4.

For the case where

$$n_{Ti} = 1\quad (5.58)$$

the anomaly eight-form will factorize into a complete square, and, hence, be cancelled by a Green-Schwarz mechanism on the six-plane, if and only if

$$g_i = 3/4, \quad \eta = 1\quad (5.59)$$
Figure 5: The solution with $n_T = 1$ where the ten-dimensional $E_8$ group is broken to $E_7 \times SU(2)$. This represents the case where a fivebrane has wrapped the vertex depicted in Figure 4. The solution in Figure 5 has $\eta = \rho = 1$ and $g = -1/4$. The six-dimensional tensor multiplet is indicated by the $\times$ on the vertex.

and

$$\tilde{n}_{H_i} = 3, \quad \tilde{n}_{V_i} = 0, \quad n_{H_i} = 1$$

(5.60)

where the $\tilde{n}_{H_i}$ and $n_{H_i}$ hypermultiplets transform as $(1, 3)$ and $(1, 1)$ respectively under $E_7 \times SU(2)$. This solution is illustrated in Figure 5.

Therefore, within the context of a second orbifold fixed plane intersection where the $\beta Z_2$ projection breaks $E_8 \to E_7 \times SU(2)$, all local anomalies can be cancelled. This cancellation requires that the intersecting seven-plane support a twisted sector consisting of an $SU(2)$ adjoint representation of $N = 1$ vector supermultiplets. In addition, the seven-plane must support an associated Chern-Simons term of the form (5.30) with $\eta = \rho = 1$.

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