Research article

On soft submaximal spaces

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ABSTRACT

In the present paper, detailed properties of soft submaximal topological spaces have been discussed. It starts by examining how well soft submaximal spaces behave when subjected to some operations, such as: taking soft subspaces, soft products, soft topological sum and images/preimages under certain soft mappings. Furthermore, several characterizations of soft submaximal spaces are obtained with respect to various types of soft subsets of soft topological spaces and their simple extensions.

1. Introduction

The area of topology which focuses on the fundamental set-theoretic concepts and procedures utilized in topology is known as general topology. It is the foundation of other branches of topology, including geometric topology, algebraic topology, and differential topology. The notion of submaximal spaces was introduced by Bourbaki [15] as a tool to study the topological spaces that do not admit a larger topology with the same semi-regularization. Such spaces were considered as a significant topic in both topological space and topological group, (see [10, 17]). Soft topology, particularly merges soft set theory and topology, is also a field of topology. It is driven by the basic assumptions of classic topological space and focuses on the set of all soft sets. A parameterized family of sets is known as a soft set. It was proposed by Molodtsov [28] in 1999 to concern with uncertainties. Shabir and Naz’s [30] contributions were especially important in establishing the area of soft topology. Later several subclasses of soft topological spaces were suggested, including soft separation axioms [1, 3, 14], soft separable spaces [14], soft connected [25], soft compact [13, 32], soft paracompact [25], soft extremally disconnected [11], soft J-spaces [27], and soft (nearly) Menger spaces [4, 24]. Furthermore, soft bioperators on soft topological spaces have been studied in [12]. Despite the fact that many studies followed their instructions and many notions developed in soft environments, substantial contributions can also be provided. The class of soft submaximal spaces was defined by Ilango and Ravindran [21], but not much study was done to further investigate soft submaximal spaces. The topic is crucial for considering the maximal element in the set of all soft topologies (see [21]) or soft topological groups on a common universe, and for extending many other different results done in [10]. Hence, we develop the properties of soft submaximal spaces and obtain several of their characterizations.

2. Preliminaries

Given a (discourse) set Z and an attribute set Λ. A soft set over (or in) Z is a pair \((Y, \Lambda) = \{(y, Y(e)) : e \in E\}\) such that \(Y : \Lambda \rightarrow \mathcal{P}(Z)\) is a set-valued mapping. Denote by \(P_{\Lambda}(Z)\) the set of all soft subsets in \(Z\). A soft set \((Y, \Lambda)\) in \(Z\) is called a soft element [29], denoted by \((z, \Lambda)\), if \(Y(e) = \{z\}\) for all \(e \in \Lambda\). A soft set \((Y, \Lambda)\) in Z is called soft point [7], denoted by \(z(e)\), if \(Y(e) = \{z\}\) and \(Y(e') = \emptyset\) for each \(e' \neq e, e' \in \Lambda\). The notation \(z(e) \in \mathcal{R}(Y, \Lambda)\) implies that \(z(e) \in Y(e)\). Two soft points \(z(e), y(e)\) are said to be distinct if either \(z \neq y\) or \(e \neq e'\). The soft sets \((Y, \Lambda), (X, \Lambda)\) over \(Z\) are said to be disjoint if \((Y, \Lambda) \cap (X, \Lambda) = \emptyset\). By a singleton soft set we mean set that includes one soft point. The complement of \((Y, \Lambda)\) is a soft set \((Y, \Lambda^c)\) such that \(Y^c : E \rightarrow \mathcal{P}(Z)\) is defined by \(Y^c(e) = Z \setminus Y(e)\) for all \(e \in \Lambda\) (we may refer to \((Z, \Lambda) \setminus (Y, \Lambda)\) as the complement of \((Y, \Lambda)\)). A null soft set \(\emptyset\) is a soft set \((Y, \Lambda)\) for which \(Y(e) = \emptyset\) for any \(e \in \Lambda\). An absolute soft set \(\mathcal{Z}\) is a soft set \((Y, \Lambda)\) for which \(Y(e) = Z\) for any \(e \in \Lambda\). Evidently, \(\mathcal{Z} = \mathcal{Z} = \emptyset = \emptyset\). A soft set \((X, \Lambda_1)\) is called a subset of \((Y, \Lambda_2)\) (denoted by \((X, \Lambda_1) \subseteq (Y, \Lambda_2)\), [26]) whenever \(\Lambda_1 \subseteq \Lambda_2\) and \(X(e) \subseteq Y(e)\) for all \(e \in \Lambda_1\). Clearly, \((X, \Lambda_1) \subseteq (Y, \Lambda_2)\) whenever \((X, \Lambda_1) \subseteq (Y, \Lambda_2)\) and \((Y, \Lambda_2) \subseteq (X, \Lambda_1)\). The union of soft sets \((R, \Lambda), (S, \Lambda)\) is defined by \((F, \Lambda) = (R, \Lambda) \cup (S, \Lambda)\), where \(F(e) = R(e) \cup S(e)\) for every \(e \in \Lambda\), and the intersection of soft sets \((R, \Lambda), (S, \Lambda)\) is defined by \((F, \Lambda) = (R, \Lambda) \cap (S, \Lambda)\), where \(F(e) = R(e) \cap S(e)\) for every \(e \in \Lambda\), (see, [6, 31]).
**Definition 2.1.** [18, 30] A soft topology on $Z$ is a collection $T$ of $P_A(Z)$ if $T$ meets the following conditions:

1. $\emptyset \subseteq T$;
2. $\{B_1, \ldots, B_n\} \in T$ for each $(B_1, \ldots, B_n) \in T$; and
3. $\bigcup_{\alpha \in I} (B, \Lambda) \in T$ for any $(B, \Lambda) : i \in I \subseteq T$.

Terminologically, we call $(Z, T, \Lambda)$ a soft topological space on $Z$. The members of $T$ are known as soft $T$-open sets (or shortly soft open sets when there is no confusion), and their complements are known as soft $T$-closed sets (or shortly soft closed sets).

In the next, we refer to a soft topological space as $(Z, T, \Lambda)$.

**Definition 2.2.** [16] Given a soft topology $T$. A soft base for $T$ is a subcollection $B \subseteq T$ such that elements of $T$ are unions of elements of $B$.

**Definition 2.3.** [30] Let $(Z, T, \Lambda)$ be a soft topological space and let $(Y, \Lambda) \in P_\Lambda(Z)$. A soft relative topology on $Y$ is defined by $T_Y := \{(G, \Lambda) : (G, \Lambda) \in T\}$.

The triple $(Y, T_Y, \Lambda)$ is called a soft subspace of $(Z, T, \Lambda)$.

**Definition 2.4.** [30] Let $(Z, T, \Lambda)$ be a soft topological space and let $(R, \Lambda) \in P_\Lambda(Z)$. The soft interior of $(R, \Lambda)$ is $\text{Int}_T((R, \Lambda)) := \bigcup \{(G, \Lambda) : (G, \Lambda) \in T, (G, \Lambda) \subseteq (R, \Lambda)\}$.

The soft closure of $(R, \Lambda)$ is $\text{Cl}_T((R, \Lambda)) := \bigcap \{(G, \Lambda) : (G, \Lambda) \in T, (G, \Lambda) \supseteq (R, \Lambda)\}$.

If $(X, T_X, \Lambda)$ is a soft subspace, the soft interior and soft closure of a soft $(R, \Lambda)$ in $(X, T_X, \Lambda)$ are denoted by $\text{Cl}_X((R, \Lambda))$ and $\text{Int}_X((R, \Lambda))$, respectively. The soft boundary of $(R, \Lambda)$ is defined by $\text{Bd}_I((R, \Lambda)) = \text{Cl}_I((R, \Lambda)) \cap \text{Cl}_I((R, \Lambda))'$.

**Definition 2.5.** [20] Let $(Z, T, \Lambda)$ be a soft topological space and let $(R, \Lambda) \in P_\Lambda(Z)$. A soft point $z \in \bar{Z}$ is called a soft limit point of $(R, \Lambda)$ if $(R, \Lambda) \cap (G, \Lambda) \ni z \neq \emptyset$ for each $(G, \Lambda) \in T$ with $z \notin \bar{G} = \text{Cl}_I((G, \Lambda))$.

**Lemma 2.6.** [20] Let $(Z, T, \Lambda)$ be a soft topological space and let $(R, \Lambda) \in P_\Lambda(Z)$. Then $(X, \Lambda) \in T$ if

1. soft regular open [19] if $(X, \Lambda) = \text{Int}_I((X, \Lambda))$,
2. soft dense [9] in $(Y, \Lambda)$ if $(Y, \Lambda) \subseteq \text{Cl}_I((X, \Lambda))$,
3. soft co-dense [9] if $\text{Int}_I((X, \Lambda)) = \emptyset$,
4. soft preopen [21] if $(X, \Lambda) \subseteq \text{Cl}_I((X, \Lambda))$, and
5. locally closed [23] if $(X, \Lambda) = (G, \Lambda) \cap \text{Cl}_I((F, \Lambda))$, where $(G, \Lambda) \in T$ and $(F, \Lambda) \in \bar{T}$.

**Definition 2.7.** Let $(Z, T, \Lambda)$ be a soft topological space and let $(X, \Lambda) \in P_\Lambda(Z)$. Then $(X, \Lambda)$ is called

1. soft regular open [19] if $(X, \Lambda) = \text{Int}_I((X, \Lambda))$,
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5. locally closed [23] if $(X, \Lambda) = (G, \Lambda) \cap \text{Cl}_I((F, \Lambda))$, where $(G, \Lambda) \in T$ and $(F, \Lambda) \in \bar{T}$.

**Definition 2.8.** [22, 33] Let $(R, \Lambda), (S, \Lambda')$ be soft sets, and let $g : R \rightarrow S, h : \Lambda \rightarrow \Lambda'$ be mappings. The image of a soft set $(B, \Lambda) \subseteq (S, \Lambda')$ under $f : (R, \Lambda) \rightarrow (S, \Lambda')$ is a soft subset $f(B, \Lambda) = (f(B), h(\Lambda))$ of $(S, \Lambda')$ which is given by $f(B, \Lambda) = \begin{cases} \bigcup_{e \in \text{Int}_I((B, \Lambda))} g(e), & \text{if } h^{-1}(e) \cap \Lambda \neq \emptyset; \\ \emptyset, & \text{otherwise,} \end{cases}$ for each $e \in \Lambda'$.

The inverse image of a soft set $(B, \Lambda') \subseteq (S, \Lambda')$ under $f$ is a soft subset $\text{f}^{-1}(B, \Lambda') = (f^{-1}(B), h^{-1}(\Lambda'))$ such that $\text{f}^{-1}(B, \Lambda') = \begin{cases} g^{-1}(f(h(e))), & h(e) \in \Lambda'; \\ \emptyset, & \text{otherwise,} \end{cases}$ for each $e \in \Lambda$.

If both mappings $g$ and $h$ are bijective, then $f$ is bijective.

**Definition 2.9.** [29] A soft mapping $f$ from a soft topology space $(Z, T, \Lambda)$ into a soft topology space $(Y, T', \Lambda')$ is called

1. soft continuous if $f^{-1}(H, \Lambda') \in T$ for each $(H, \Lambda') \in T'$,
2. soft open if $f((G, \Lambda)) \in T'$ for each $(G, \Lambda) \in T$,
3. soft homeomorphism if it is a soft open and a soft continuous bijection.

**Definition 2.10.** [13] Let $\{Z_i, T_i, \Lambda_i : i \in I\}$ be a collection of soft topological spaces with a fixed parametric set $\Lambda$. The product soft topology $T$ on $Z = \prod_{i \in I} Z_i$ is the initial soft topology on $Z$ generated by the family $\{p_i : i \in I\}$, where $p_i$ is the soft projection mapping from $Z$ to $Z_i$ for $i \in I$.

**Definition 2.11.** [5] Let $\{Z_i, T_i, \Lambda_i : i \in I\}$ be a collection of soft topological spaces such that $Z_i \cap Z_j = \emptyset$ for each $i \neq j$. The soft topology $T$ on $\bigcup_{i \in I} Z_i$ generated by the basis $B = \{(U, \Lambda) \subseteq \bigcup_{i \in I} Z_i : (U, \Lambda) \in T_i \text{ for some } i\}$ is called sum of soft topological spaces and denoted by $(\bigoplus_{i \in I} Z_i, T, \Lambda)$.

3. Some operations on soft submaximal spaces

**Definition 3.1.** [21] A soft submaximal space is soft topological space $(Z, T, \Lambda)$ in which each soft dense set over $Z$ is soft open.

We start by showing soft submaximality is a hereditary property.

**Proposition 3.2.** If $(Y, T_Y, \Lambda)$ is a sub space of a soft submaximal space $(Z, T, \Lambda)$, then $(Y, T_Y, \Lambda)$ is soft submaximal.

**Proof.** Let $(D, \Lambda)$ be soft dense over $Y$. Then $(Y, \Lambda) \subseteq \text{Cl}_I((D, \Lambda))$. But $(D, \Lambda) \subseteq \text{Cl}_I((\bar{D}, \Lambda))$ is soft dense over $Z$ and, by soft submaximality of $(Z, T, \Lambda)$, $(D, \Lambda) \subseteq \text{Cl}_I((\text{Cl}_I((D, \Lambda))))$ is a soft open set over $Z$. Now, we have $(D, \Lambda) \subseteq \text{Cl}_I((\text{Cl}_I((D, \Lambda)))) = (D, \Lambda)$ and so, by Theorem 2 in [30], $(D, \Lambda)$ is soft open in $(Y, \Lambda)$. Hence $(Y, T_Y, \Lambda)$ is soft submaximal.

**Proposition 3.3.** Let $(Z, T, \Lambda), (Z', T', \Lambda)$ be soft topological spaces and $T \subseteq T'$. If $T$ is a soft submaximal space, then $T'$ is soft submaximal.

**Proof.** If $(D, \Lambda)$ is a soft $T'$-dense set over $Z$, then $\bar{D} = \text{Cl}_I((\bar{D}, \Lambda)) \subseteq \text{Cl}_I((\text{Cl}_I((D, \Lambda))))$ and so $(D, \Lambda)$ is soft $T$-dense. But $(Z, T, \Lambda)$ is soft submaximal, then $(D, \Lambda)$ is soft $T'$-open. Since $T \subseteq T'$, so $(D, \Lambda)$ is soft $T'$-open and thus $(Z, T', \Lambda)$ is a soft submaximal space.

**Lemma 3.4.** If $f : (Z, T, \Lambda) \rightarrow (Y, T', \Lambda')$ is a soft open mapping, then $f^{-1}(\text{Cl}_I((B, \Lambda'))) \subseteq \text{Cl}_I(f^{-1}(B, \Lambda'))$ for each soft set $(B, \Lambda')$ over $Y$.

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Proof. Suppose $f$ is soft open. Let $(F, A') = \tilde{F}$ and let $z(e) \in f^{-1}(C_1(f, A'))$. Then $f(z(e)) \in \tilde{C}_1(f, A')$. Assume $(G, A)$ is any soft $T$-open set including $z(e)$. By soft openness of $f$, $f((G, A))$ is soft $T'$-open and so $(F, A') \cap f((G, A)) \neq \Phi$. Therefore $f^{-1}(C_1(f, A')) \cap (G, A) \neq \Phi$. Thus $z(e) \in \tilde{C}_1(f, A')$. The proof is finished.

**Proposition 3.5.** If $f : (Z, T, A) \to (Y, T', A')$ is a soft open surjection and $(Z, T, A)$ is a soft submaximal space, then $(Y, T', A')$ is soft submaximal.

**Proof.** Let $(H, A')$ be a soft $T'$-dense set over $Y$. By Lemma 3.4, $\tilde{Z} = f^{-1}(C_1(f, A')) \subseteq \tilde{C}_1(f^{-1}(H, A'))$ and thus $f^{-1}(H, A')$ is soft $T$-dense over $Z$. By soft submaximality of $(Z, T, A)$, $f^{-1}(H, A')$ is soft $T$-open. Since $f$ is soft open surjective, then $f(f^{-1}((H, A'))) = (H, A')$ soft open over $Y$. Hence $(Y, T', A')$ is soft submaximal.

One might conjecture that the inverse image of a soft submaximal space under soft continuous is soft submaximal, but this is generally false.

**Example 3.6.** Let $Z = \{1, 2\}$ and $\mathbb{R}$ be the set of reals. For any parametric set $A$, we respectively define soft topologies $T = \{\Phi, \{(1, 2)\}, \{(2, A\}, \{Z\}\}$ and $T' = \{\Phi, \{\mathbb{R}, \mathbb{R}\} \}$. Let $\mathbb{Q}$ be the set of rationals and $\mathbb{P}$ be the set of irrationals. The soft mapping $f : (R, T', A) \to (Z, T, A)$ defined by

$$f(r) = \begin{cases} 1, & \text{if } r \in \mathbb{Q}; \\ 2, & \text{if } r \notin \mathbb{P}, \end{cases}$$

is soft continuous and $(Z, T, A)$ is soft submaximal, while $(R, T', A)$ is not soft submaximal.

However, the following result is true:

**Corollary 3.7.** Let $f : (Z, T, A) \to (Y, T', A')$ be a soft homeomorphism. Then $(Z, T, A)$ is soft submaximal if and only if $(Y, T', A')$ is soft submaximal.

**Lemma 3.8.** [13, Theorem 3.8] The soft projection mapping $p_i : \prod Z_i \to (Z_i, T_i, A_i)$ is soft open for each $i$.

**Proposition 3.9.** If the product soft space $(\prod Z_i, \prod T_i, A_i)_{i \in I}$ is soft submaximal, then $(Z_i, T_i, A_i)$ is soft submaximal for each $i$.

**Proof.** Follows from Proposition 3.5 and Lemma 3.8.

The converse of the above proposition is not true.

**Example 3.10.** Let $Z = \{0, 1\}$ and let $A$ be any set parameters. Define a soft topology on $Z$ by $T = \{\Phi, \{(0, 1)\}, \{Z\}\}$. The soft submaximality of $(Z, T, A)$ can be easily followed, but not of $(Z \times Z, T \times T, A)$.

**Proposition 3.11.** The soft topological sum $(\bigoplus Z_i, T, A)$ is soft submaximal if and only if $(Z_i, T_i, A_i)$ is soft submaximal for each $i$.

**Proof.** The first direction follows from Proposition 3.2. Conversely, if $(D, A)$ is soft dense over $\bigoplus Z_i$, then $(D, A) \cap Z_i$ is soft dense over $Z_i$ for each $i$. Since $(Z_i, T_i, A_i)$ is soft submaximal, then $(D, A) \cap Z_i$ is soft open over $Z_i$ for each $i$, which implies that $(D, A) \cap Z_i$ is soft open over $\bigoplus Z_i$. Therefore $(D, A) = \bigcup Z_i((D, A) \cap Z_i)$ is a soft open set over $\bigoplus Z_i$. Thus $(\bigoplus Z_i, T, A)$ is soft submaximal.

**Proposition 3.12.** Let $(G, A)$ be a soft subset of a soft submaximal space $(Z, T, A)$. Then $(G, A)$ is soft open if and only if $(G, A) = (R, A)^c\cap (D, A)$ for some soft regular open $(R, A)$ and soft dense $(D, A)$ in $Z$.

**Proof.** If $(G, A)$ is soft open, then $(G, A)^c = (R, A)^c\cap (D, A)$ and so $(G, A) = (Cl((G, A)) \cap (D, A))$.

4. **Characterizations of soft submaximal spaces**

**Lemma 4.1.** Let $(Y, A)$ be a soft subset of $(Z, T, A)$. Then $(Y, A)$ is soft preopen if and only if $(Y, A) = (U, A)^c\cap (D, A)$, where $(U, A)$ is soft open and $(D, A)$ is soft dense over $Z$.

**Proof.** If $(Y, A) \subseteq Cl((Y, A))$, then $(U, A) = Int(Cl((Y, A)))$ is soft open. Set $(D, A) = Z \setminus (U, A) \setminus (Y, A)$. Now, $(Y, A) \subseteq Cl((Y, A))$, which concludes that $(D, A)$ is soft dense. Thus $(Y, A) = (U, A)^c\cap (D, A)$.

**Theorem 4.2.** The following statements are equivalent for a soft topological space $(Z, T, A)$:

1. $(Z, T, A)$ is a soft submaximal space;
2. Each soft preopen set over $Z$ is soft open.

**Proof.** $(1) \implies (2)$ Let $(U, A)$ be soft preopen. By Lemma 4.1, $(U, A) = (G, A)^c\cap (D, A)$, where $(G, A)$ is soft open and $(D, A)$ is soft dense over $Z$. Hence $(2)$.

$(2) \implies (1)$ Let $(D, A)$ be a soft dense set over $Z$. Then $(D, A)^c = Int(Z) = Int(Cl((D, A)))$. Therefore $(D, A)$ is soft preopen and by $(2)$ $(D, A)$ is soft open. Thus $(Z, T, A)$ is soft submaximal.

We assert that the above conclusion appeared in Theorem 16 [21], but our proof is different.

**Theorem 4.3.** The following statements are equivalent for a soft topological space $(Z, T, A)$:

1. $(Z, T, A)$ is soft submaximal;
2. Each soft subset in $Z$ is soft locally closed;
3. Each soft subset in $Z$ is soft co-locally closed.

**Proof.** $(1) \implies (2)$ Given a soft set $(Y, A)$ over $Z$. Since $Cl((Y, A))^c = Z$, by Proposition 3.2, $(Y, A)^c\subseteq Cl((Y, A))$ is soft submaximal. Since $(Y, A)$ is soft dense in $Cl((Y, A))$, so $(Y, A)$ is soft open in $Cl((Y, A))$. Therefore $(Y, A) = (G, A)^c\cap Cl((Y, A))$ for some soft open $(G, A)$ over $Z$. Hence $(Y, A)$ is a soft locally closed set.

$(2) \implies (1)$ Let $(D, A)$ be a soft dense in $Z$. By $(2) (D, A) = (G, A)^c\cap (F, A)$ for some soft open $(G, A)$ and soft closed $(F, A)$. We obtain $(D, A)^c = Cl((D, A))^c = Cl((F, A))^c$. This implies that $(F, A)^c = Z$ and so $(D, A) = (G, A)$ which is a soft open set over $Z$. This proves $(1)$.

$(2) \implies (3)$ They are complementary of each other.
Theorem 4.4. The following statements are equivalent for a soft topological space $(Z, T, \Lambda)$:

1. $(Z, T, \Lambda)$ is soft submaximal;
2. The soft boundary of any soft set over $Z$ is soft discrete;
3. The soft boundary of any soft set over $Z$ is soft closed;
4. Each soft co-closed set over $Z$ is soft closed.

Proof. $(1) \implies (2)$ Let $(Y, \Lambda)$ be a soft set. Let $z(e) \in Cl_T((Y, \Lambda)) \cap Cl_T((Y, \Lambda)^r)$. By Proposition 3.2, $Cl_T((Y, \Lambda)) \cap Cl_T((Y, \Lambda)^r)$ are soft submaximal subspaces of $Z$. Since $(Y, \Lambda)$ is soft dense in $Cl_T((Y, \Lambda))$, also $(Y, \Lambda) \cap Cl_T((Y, \Lambda)^r)$ is soft dense in $Cl_T((Y, \Lambda))$. By soft submaximality of $Cl_T((Y, \Lambda))$, there is soft open $(G, \Lambda)$ over $Z$ such that $(Y, \Lambda) \cap Cl_T((Y, \Lambda)^r) = Cl_T((Y, \Lambda)) \cap Cl_T((Y, \Lambda)^r)$. By the same reasoning, we can have $(Y, \Lambda) \cap Cl_T((Y, \Lambda)^r) \subseteq Cl_T((Y, \Lambda)) \cap Cl_T((Y, \Lambda)^r)$ for some soft open set $(H, \Lambda)$ over $Z$. Therefore $\{z(e)\} = (Y, \Lambda) \cap Cl_T((Y, \Lambda)^r) \subseteq Cl_T((Y, \Lambda)) \cap Cl_T((Y, \Lambda)^r) = Cl_T((Y, \Lambda)) \cap Cl_T((Y, \Lambda)^r)$. Hence $\{z(e)\}$ is a soft open subset of $Bd((Y, \Lambda))$ and so $Bd((Y, \Lambda))$ is soft discrete.

$(2) \implies (3)$ Let $(Y, \Lambda)$ be a soft set. By $(2) Bd((Y, \Lambda))$ is soft discrete, so $Bd((Y, \Lambda))$ has no soft limit point. Hence $Bd((Y, \Lambda))$ is soft closed.

$(3) \implies (4)$ Given a soft set $(Y, \Lambda)$, if $(Y, \Lambda)$ is soft co-dense, $(Y, \Lambda) \cap Cl((Y, \Lambda)) = Bd((Y, \Lambda))$. Thus $(Y, \Lambda)$ is soft closed.

$(4) \implies (1)$ The complement of $(4)$ is $(1)$. The result is proved. $\square$

Definition 4.5. [8] Let $(Z, T, \Lambda)$ be a soft topological space and let $\Phi \neq (Y, \Lambda) \subseteq T$. The family $T \subseteq (Y, \Lambda)$ generates a soft topology $T'$ on $Z$ called a simple extension of $T$ via $(Y, \Lambda)$. We shall denote $T' = T[T(Y, \Lambda)]$ (or briefly, $T' = T'[Y]$).

Lemma 4.6. [8, Remark 2.2] Let $T' = T[Y]$ be a simple extension of a soft topological space $(Z, T, \Lambda)$. Then $Cl_{T'}((X, \Lambda)) = Cl_T((X, \Lambda)) \cap \bigcap \{ (X, \Lambda) \cap Cl_T((X, \Lambda)) \}$ for each soft set $(X, \Lambda)$ over $Z$.

Lemma 4.7. Let $T' = T[Y]$ be a simple extension of a soft topological space $(Z, T, \Lambda)$. If $(Y, \Lambda)$ is a soft $T$-dense set, then it is soft $T'$-dense.

Proof. Apply Lemma 4.6. $\square$

Lemma 4.8. Let $T' = T[Y]$ be a simple extension of a soft topological space $(Z, T, \Lambda)$. If $(Y, \Lambda)$ is a soft $T$-dense set, then

1. If $(U, \Lambda) \subseteq T'$ is a soft $T'$-open set, then $Cl_{T'}((U, \Lambda)) = Cl_{T'}((U, \Lambda))$.

2. If $(F, \Lambda) \subseteq T'$ is a soft $T'$-closed set, then $Int_{T'}((F, \Lambda)) = Int_{T'}((F, \Lambda))$.

3. If $(U, \Lambda) \subseteq T'$ is a soft $T'$-open set, then $Int_{T'}((U, \Lambda)) = Int_{T'}((U, \Lambda))$.

4. If $(U, \Lambda) \subseteq T'$ is a soft $T'$-open set, then $(U, \Lambda)$ is a soft regular $T'$-open if and only if it is a soft regular $T'$-open.

Proof. $(1)$ Let $(U, \Lambda)$ be a soft $T'$-open set. Since $T' \subseteq T'$, we consider two cases: (a) If $(U, \Lambda) \subseteq T'$, clearly $Cl_{T'}((U, \Lambda)) = Cl_{T'}((U, \Lambda))$. On the other hand, suppose $z(e) \in Cl_{T'}((U, \Lambda))$. Let $(V, \Lambda)$ be any soft $T'$-open set containing $z(e)$. There exist soft $T'$-open sets $(G, \Lambda)$ and $(H, \Lambda)$ such that $(V, \Lambda) = (G, \Lambda) \cap (H, \Lambda)$ and $(G, \Lambda) \cap (H, \Lambda) \neq \emptyset$. Since $(H, \Lambda) \cap (H, \Lambda) \neq \emptyset$, we get $(G, \Lambda) \cap (H, \Lambda) \neq \emptyset$. Hence $Cl_{T'}((U, \Lambda)) = Cl_{T'}((U, \Lambda))$.

$(2)$ If $(F, \Lambda) \subseteq T'$ is a soft $T'$-closed set, then $Int_{T'}((F, \Lambda)) = Int_{T'}((F, \Lambda))$.

$(3)$ If $(U, \Lambda) \subseteq T'$ is a soft $T'$-open set, then $Int_{T'}((U, \Lambda)) = Int_{T'}((U, \Lambda))$.

$(4)$ If $(U, \Lambda) \subseteq T'$ is a soft $T'$-open set, then $(U, \Lambda)$ is a soft regular $T'$-open if and only if it is a soft regular $T'$-open.
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