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Latest measurements have revealed that the deviation from a maximal solar mixing angle is approximately the Cabibbo angle (i.e. QLC relation). We argue that it is not plausible that this deviation from maximality, be it a coincidence or not, comes from the charged lepton mixing. Consequently we have calculated the required corrections to the exactly bimaximal neutrino mass matrix ansatz necessary to account for the solar mass difference and the solar mixing angle. We point out that the relative size of these two corrections depends strongly on the hierarchy case under consideration. We find that the inverted hierarchy case with opposite CP parities, which is known to guarantee the RGE stability of the solar mixing angle, offers the most plausible scenario for a high energy origin of a QLC-corrected bimaximal neutrino mass matrix. This possibility may allow us to explain the QLC relation in connection with the origin of the charged fermion mass matrices.

I. INTRODUCTION

During the last year our knowledge of the leptonic mixing matrix has reached the precision level. The most recent 90% C.L. experimental results [1, 2, 3] and several global fits [1, 2, 4, 5] have improved our knowledge of the neutrino mass differences and indicate that the atmospheric mixing is almost maximal while the solar mixing deviates from maximality in a particular way. In the standard notation,

\[
\begin{align*}
\sin \theta_{12} & = 0.53 \pm 0.04, \\
\sin \theta_{23} & = 0.70 \pm 0.11, \\
\sin \theta_{13} & < 0.15,
\end{align*}
\]

\[
\Delta m_{\text{sun}}^2 = \Delta m_{21}^2 = (8.2 \pm 0.6) \times 10^{-5} \text{eV}^2, \\
|\Delta m_{\text{atm}}^2| = |\Delta m_{23}^2| = (2.45 \pm 0.55) \times 10^{-3} \text{eV}^2,
\]

(1)\hspace{1cm}(2)\hspace{1cm}(3)\hspace{1cm}(4)\hspace{1cm}(5)

We note that the mixing angle \(\theta_{13}\) is constrained to be \(\theta_{13} < 0.15\) by the non-observation of neutrino oscillations at the CHOOZ experiment [3] and a fit to the global data [5]. This substantial improvement has confirmed that the leptonic mixing matrix, hereafter called MNSP matrix [8], is nearly bimaximal [6, 10] and the deviation from bimaximality observed has revealed a surprising relation between the Cabibbo angle, \(\theta_C\) and the solar mixing angle [11],

\[
\theta_C + \theta_{12} = 45.1^\circ \pm 2.4^\circ (1\sigma),
\]

sometimes called the quark-lepton complementarity relation, heretofore referred to as QLC relation. There is a similar relation satisfied by the leptonic angle \(\theta_{23}\) and the corresponding angle in the quark sector, although the errors are somewhat larger. Based on the experimental data it is convenient to define the following parametrization [12] of the mixing angles,

\[
\begin{align*}
\sin \theta_{23} & = \frac{1}{\sqrt{2}} + \epsilon_A \lambda_v^2, \\
\sin \theta_{13} & = \frac{1}{\sqrt{2}} (1 - \lambda_v + \epsilon_S \lambda_v^2), \\
\sin \theta_{12} & = \epsilon_{CP} \lambda_v^2,
\end{align*}
\]

(6)\hspace{1cm}(7)\hspace{1cm}(8)

where \(s_{ij} = \sin \theta_{ij}\) and the coefficients \(\epsilon_A, \epsilon_S\) and \(\epsilon_{CP}\) are at most of order \(\sim 4\), as indicated by the experimental uncertainties. We note that we have defined the deviation from a maximal solar mixing angle as \(\lambda_v\) and not \(\lambda = \theta_C\) to emphasize that \(\lambda_v\) may not be exactly the Cabibbo angle. Therefore the MNSP matrix can be written to leading order in powers of \(\lambda_v\) as,

\[
\nu_{\text{MNSP}} = \begin{bmatrix}
\frac{1}{\sqrt{2}} (1 + \lambda_v) & -\frac{1}{\sqrt{2}} (1 - \lambda_v) & 0 \\
\frac{1}{\sqrt{2}} (1 - \lambda_v) & \frac{1}{\sqrt{2}} (1 + \lambda_v) & -\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} (1 - \lambda_v) & -\frac{1}{\sqrt{2}} (1 + \lambda_v) & \frac{1}{\sqrt{2}} \\
\end{bmatrix} + O(\lambda_v^2)
\]

(9)

The main implication of the QLC relation is fairly simple: the MNSP matrix is to first order bimaximal [8] and the deviation from the exact bimaximality is a correction of the order of the Cabibbo angle, i.e. around 20%. This resembles in certain way the situation in the quark sector, where it is known that to first order the CKM matrix is the unity matrix while the main correction is exactly the Cabibbo angle.

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Explaining the QLC relation is a real challenge that any future theory of flavor must address. Along with the extreme smallness of the neutrino masses, this is another feature which qualitatively distinguishes the neutrino sector from the charged fermion sector. The charged fermion spectra is very hierarchical, i.e. the third generation masses are much heavier than the first and second generation fermion masses. Therefore we expect that there is a basis, probably the flavor basis (also known as lagrangian or symmetry basis), where the charged fermion diagonalization matrices are approximately diagonal. On the other hand, it has been known for some time that the leptonic mixing matrix is nearly bimaximal. It was expected that this distinctive feature could be explained if the mechanism of neutrino mass generation is somehow disconnected from the mechanism generating the flavor structure in the charged fermion sector. This may explain why many people, surprised by the appearance of the Cabibbo angle in the leptonic mixing matrix, have proposed to explain the QLC relation as a contamination coming from the charged lepton mixing matrix.

In this paper we will analyze some generic implications of the QLC relation for models of neutrino masses. In Sec. II we argue that it is not plausible that the QLC relation is explained by effects arising from the charged lepton mixing sector. In Sec. III we analyze the form and relative size of the corrections to the bimaximal three neutrino mass matrix necessary to account for the QLC relation. In Sec. IV we analyze the effects of the neutrino mass hierarchy on the stability of the QLC relation and the implications for the scale of neutrino mass generation. In Sec. V we analyze the possibility that the solar mass difference being zero at a high energy scale is RGE generated, triggered by a high energy origin of the QLC relation. In Sec. VI we summarize the main results of this paper.

II. THE QLC RELATION CANNOT ARISE FROM CHARGED LEPTON MIXING

The MNSP mixing matrix is given by

$$V_{\text{MNSP}} = (V_L^f)^\dagger \nu_L$$

where $\nu_L$ is the neutrino diagonalization matrix and $V_L^f$ is the left handed charged lepton diagonalization matrix, $M_{\text{diag}} = (V_L^f)^\dagger M_{\text{fl}} V_L^f$. When trying to explain the QLC relation the first idea that comes to our mind is the possibility that the QLC relation may arise from the charged lepton mixing matrix. We will argue that this is not plausible if one wants to understand the well known empirical relations which connect the electron/muon mass ratio with the quark sector. There is an empirical relation which has been known for quite a long time,\footnote{12,14}{

$$|V_{us}| \approx \left( \frac{m_d}{m_s} \right)^{\frac{1}{2}} \approx 3 \left( \frac{m_e}{m_\mu} \right)^{\frac{1}{2}}$$

(11)}

This relation has been recently analyzed with precision by one of the authors who noted that indeed the relation surprisingly works at the level of $\pm 16\%$, as the following ratio shows (see Ref. 12 for details),

$$\left( \frac{m_d}{m_s} \right)^{1/2} : \left( \frac{m_e}{m_\mu} \right)^{1/2} = 3.06 \pm 0.48.$$  

(12)

The relation between the Cabibbo angle and the down-strange quark mass ratio can be simply explained, as known from the '70's,\footnote{12} if the down quark mass is generated from the mixing between the first and second families. Analogously, the relation between the Cabibbo angle and the electron-muon mass ratio can also be simply explained if the electron mass is generated from the mixing between the first and second lepton families. This implies that there is a leptonic basis where the charged lepton mass matrix is given to leading order by,

$$\hat{M}_L = \begin{bmatrix} 0 & \left( \frac{m_e m_\mu}{m_\tau^2} \right)^{\frac{1}{2}} & \mathcal{O}(\lambda^3) \\ \left( \frac{m_e m_\mu}{m_\tau^2} \right)^{\frac{1}{2}} & \mathcal{O}(\lambda^2) & \mathcal{O}(\lambda^3) \\ \mathcal{O}(\lambda^3) & \mathcal{O}(\lambda^2) & 1 \end{bmatrix}.$$  

(13)

Here $\lambda = \theta_C$. The order of magnitude in the coefficients ($\hat{M}_{113}$ and $\hat{M}_{223}$) can be obtained by requiring these entries not to affect the leading order terms for the charged lepton mass ratios. From the matrix in Eq. (13) and the empirical relation in Eq. (11) it follows that the charged lepton mixing matrix to leading order is given in this leptonic basis by,

$$V_L^f \approx \begin{bmatrix} 1 & \lambda/3 & \mathcal{O}(\lambda^3) \\ \lambda/3 & 1 & \mathcal{O}(\lambda^2) \\ \mathcal{O}(\lambda^3) & \mathcal{O}(\lambda^2) & 1 \end{bmatrix}.$$  

(14)

To sum up, Eq. (11) necessarily implies that there is a leptonic basis and a quark basis where the charged lepton mass matrix adopts the form given by Eq. (13) while the down-type quark mass matrix adopts a similar form with $m_\mu/m_\tau = 3m_s/m_b$. It is very plausible that this is the flavor basis in some underlying theory of flavor. For instance, this could be the basis where quarks and leptons unify in common representations of a Grand Unified group. It is known that some GUT models can explain the relation in Eq. (11)\textsuperscript{12,14} This could be achieved if the Higgs field giving mass to the charged leptons and down-type quarks transforms under particular representations of the GUT group: \textsuperscript{45} in the SU(5) case or \textsuperscript{126} in SO(10) models.

It has been recently proposed\textsuperscript{13,15} that, to explain the deviation from a maximal solar mixing angle, one could assume that the neutrino mixing matrix in the flavor basis is exactly or approximately bimaximal, i.e.,

$$V_{\nu} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$  

(15)
and that the QLC relation is generated from charged lepton mixing. We have pointed out above that most probably the flavor basis of the underlying theory of flavor is the basis where quarks and leptons unify in common representations. In this basis we expect the charged lepton diagonalization matrix to be given by Eq. 14. Nevertheless, if this was the case we would obtain that \( \theta_{12} = \pi + \frac{\Delta_{12}}{6} \) instead of the observed QLC relation, and this is quite inconsistent.

If one insists to fully generate the observed deviation from bimaximality in the MNSP matrix from the charged lepton mixing, assuming that the neutrino mixing matrix is approximately bimaximal in the flavor basis, the required mixing in the charged lepton sector would be very large and as a consequence the charged lepton mass matrix would adopt a very unnatural form in the flavor basis in order to reproduce the correct electron mass. This kind of scenarios do not provide a convincing explanation of the precise relation that connects the charged lepton spectra and the quark spectra, see Eq. 14.

Therefore, most probably the bulk of the difference between \( \theta_{12} \) and \( \frac{\pi}{4} \) is already present in the neutrino mass matrix in the flavor basis, or in other words the QLC relation must arise from the mechanism that generates the neutrino mass matrix and not from the charged lepton mixing.

### III. QLC CORRECTED BIMAXIMAL MASS MATRICES

The charged lepton mixing cannot account for the observed deviations from the bimaximal ansatz in the MNSP matrix. Therefore, it is interesting to study the generic corrections to the bimaximal neutrino mass matrix that can account for the QLC relation. The form and relative size of these corrections can give us some insight in the origin of the neutrino mass matrix. Let us denote the neutrino mass eigenstates by,

\[
\mathcal{M}_\nu^{\text{diag}} = (m_1, m_2, m_3) \tag{16}
\]

Neglecting the charged lepton mixing, which can only give a second order contribution to the QLC relation as we saw in the previous section, the reconstructed neutrino mass matrix is,

\[
\mathcal{M}_\nu = \mathcal{M}_\nu^{\text{MNSP}} \mathcal{M}_\nu^{\text{diag}} \mathcal{V}_\text{MNSP}^\dagger. \tag{17}
\]

This can be written as,

\[
\mathcal{M}_\nu = \mathcal{M}_\nu^{\text{BiMax}} + \mathcal{M}_\nu^{\text{QLC}}, \tag{18}
\]

where \( \mathcal{M}_\nu^{\text{BiMax}} \) is the well known bimaximal mass matrix whose general expression is given by Eq. 14.

\[
\mathcal{M}_\nu^{\text{BiMax}} = \left[ \begin{array}{ccc}
\frac{1}{2}m_{12} & \frac{1}{2}\Delta_{12} & \frac{1}{2}\sqrt{1 - \Delta_{12}^2} \\
\frac{1}{2}\sqrt{1 - \Delta_{12}^2} & m_2 + m_3 & \frac{1}{2}\sqrt{1 - \Delta_{12}^2} \\
\frac{1}{2}\Delta_{12} & \frac{1}{2}(m_2 + m_3) & m_1 + m_3 \\
\end{array} \right]. \tag{19}
\]

Here we have defined,

\[
m_{12} = \frac{1}{2}(m_1 + m_2), \quad \Delta_{12} = \frac{1}{2}(m_1 - m_2). \tag{20}
\]

The QLC correction, \( \lambda_\nu = \pi/4 - \theta_{12} \), to the bimaximal ansatz is generically given by,

\[
\mathcal{M}_\nu^{\text{QLC}} = \left[ \begin{array}{ccc}
2 & 0 & 0 \\
0 & -1 & -1 \\
0 & -1 & -1 \\
\end{array} \right] \lambda_\nu \Delta_{12}. \tag{21}
\]

We note that we used \( \lambda_\nu \) and not \( \lambda = \theta_{12} \) to emphasize that \( \lambda_\nu \) may not be exactly the Cabibbo angle. Additionally the bimaximal mass matrix can be separated into two pieces,

\[
\mathcal{M}_\nu^{\text{BiMax}} = \mathcal{M}_\nu^{\text{atm}} + \mathcal{M}_\nu^{\text{sol}}, \tag{22}
\]

The expressions for \( \mathcal{M}_\nu^{\text{atm}}, \mathcal{M}_\nu^{\text{sol}} \) and \( \mathcal{M}_\nu^{\text{QLC}} \) depend on the hierarchy case under consideration. The particular forms can be found in Table 1. Next we will comment on the main features of the different hierarchy cases.

| Normalized mass matrix \( \mathcal{M}_\nu \) | zero term \( \mathcal{M}_\nu^{\text{atm}} \) | solar mass correction \( \mathcal{M}_\nu^{\text{sol}} \) | QLC correction \( \mathcal{M}_\nu^{\text{QLC}} \) | Eigenvalues |
|---|---|---|---|---|
| normal hierarchy | \[ \begin{array}{ccc}
0 & 0 & 0 \\
\frac{1}{2} & -\frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2} \\
\end{array} \] | \[ \begin{array}{ccc}
\frac{1}{2} & -\frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2} \\
\end{array} \] | \[ \begin{array}{ccc}
-4\lambda_\nu & 0 & 0 \\
0 & \lambda_\nu & \lambda_\nu \\
0 & \lambda_\nu & \lambda_\nu \\
\end{array} \] | \( (0, \gamma, 1) \) |
| inverted hierarchy with same CP parities | \[ \begin{array}{ccc}
1 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2} \\
\end{array} \] | \[ \begin{array}{ccc}
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2} \\
\end{array} \] | \[ \begin{array}{ccc}
-2\lambda_\nu & 0 & 0 \\
0 & \lambda_\nu & \lambda_\nu \\
0 & \lambda_\nu & \lambda_\nu \\
\end{array} \] | \( (1, (1 + \gamma), 0) \) |
| inverted hierarchy with opposite CP parities | \[ \begin{array}{ccc}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\end{array} \] | \[ \begin{array}{ccc}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\end{array} \] | \[ \begin{array}{ccc}
2\lambda_\nu & 0 & 0 \\
0 & -\lambda_\nu & -\lambda_\nu \\
0 & -\lambda_\nu & -\lambda_\nu \\
\end{array} \] | \( (1, -(1 + \gamma), 0) \) |
A. Normal hierarchy case

In the normal hierarchy case we obtain the leading order term in the neutrino mass matrix assuming that \( m_1 = 0 \) and \( m_2 = 0 \),

\[
M_{\nu}^{\text{atm}} = \frac{m_3}{2} \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & -1 \\
0 & -1 & 1
\end{bmatrix} .
\] (23)

This matrix generates mass for one neutrino, \( \nu_3 \), which using the atmospheric mass difference, corresponds to,

\[
m_3 = \sqrt{\Delta m_{\text{atm}}^2} = (4.9 \pm 0.6) \times 10^{-2} \text{ eV} .
\] (24)

To generate the solar mass difference we need to give mass to the neutrino \( \nu_2 \). To this end we need to introduce a small perturbation of the previous matrix controlled by the parameter \( \gamma = m_2/m_3 \ll 1 \). To be consistent with bimaximal mixing we need the perturbation matrix to be of the form,

\[
M_{\nu}^{\text{sol}} = \frac{\gamma m_3}{2} \begin{bmatrix}
1 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix} .
\] (25)

In the normal hierarchy case \( \gamma \) is related to the neutrino mass differences by,

\[
\frac{(m_2^2 - m_1^2)}{(m_3^2 - m_2^2)} = \frac{\gamma^2}{(1 - \gamma^2)} \approx \gamma^2 .
\] (26)

Using experimental data \( \gamma \) is determined to be,

\[
\gamma \approx \left( \frac{\Delta m_{\text{sol}}^2}{\Delta m_{\text{atm}}^2} \right)^{\frac{1}{4}} \approx 0.18 \pm 0.03 .
\] (27)

We note that \( \gamma \) is curiously approximately the Cabibbo angle, \( \gamma \approx \lambda \), this was noticed earlier in Ref. [20]. Finally to generate a deviation from maximality in the solar mixing angle able to account for the QLC relation we need to introduce a second perturbation given by,

\[
M_{\nu}^{\text{QLC}} = \gamma \frac{m_3}{2} \lambda_{\nu} \begin{bmatrix}
-4 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1
\end{bmatrix} .
\] (28)

Therefore, in the normal hierarchy case, the correction to \( M_{\nu}^{\text{atm}} \) coming from the matrix \( M_{\nu}^{\text{sol}} \) is at most of order \( \gamma \approx \lambda \), i.e. approx 20\%, in the entry (11) and approx. \( \lambda^2/2 \) in the rest of entries of the matrix. The entries in the QLC correction, \( M_{\nu}^{\text{QLC}} \), are at most of order \( 4\gamma \lambda \nu \approx \lambda \) in the entry (11) and approx. \( \lambda^2 \) the rest. Therefore for the normal hierarchy case to reproduce the neutrino data we need the following hierarchy between the different corrections,

\[
M_{\nu}^{\text{QLC}} \simeq M_{\nu}^{\text{sol}} < M_{\nu}^{\text{atm}} .
\] (29)

B. Inverted hierarchy case with same CP parities

In the inverted hierarchy case with same CP-parities we obtain the leading order term in the neutrino mass matrix assuming that \( m_1 = m_2 \) and \( m_3 = 0 \),

\[
M_{\nu}^{\text{atm}} = m_1 \begin{bmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix} .
\] (30)

This matrix generates a degenerate mass for two neutrinos which corresponds roughly to the atmospheric mass scale,

\[
m_1 = m_2 = \sqrt{\Delta m_{\text{atm}}^2} .
\] (31)

In this case, to generate the solar mass difference we need to break the degeneracy between the masses of \( \nu_1 \) and \( \nu_2 \). To this end we introduce a small perturbation of the form \( m_2 = m_1 (1 + \gamma) \). To be consistent with bimaximal mixing we need the perturbation matrix to be given by,

\[
M_{\nu}^{\text{sol}} = \frac{\gamma m_1}{2} \begin{bmatrix}
1 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix} .
\] (32)

The solar mass difference is given by,

\[
\Delta m_{\text{sol}}^2 = m_1^2 \gamma (2 + \gamma) \approx 2m_1^2 \gamma .
\] (33)

In this case, \( \gamma \) can be determined from experimental data to be given by,

\[
\gamma \approx \frac{1}{2} \frac{\Delta m_{\text{sol}}^2}{\Delta m_{\text{atm}}^2} \approx \frac{1}{2} \lambda^2 \approx 0.024
\] (34)

Finally to generate a deviation from maximality in the solar mixing angle able to account for the QLC relation we need to introduce a second perturbation given by,

\[
M_{\nu}^{\text{QLC}} = \gamma m_1 \lambda_{\nu} \begin{bmatrix}
-1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{bmatrix} .
\] (35)

Therefore, in the inverted hierarchy case with same CP-parities, the correction to \( M_{\nu}^{\text{atm}} \) coming from the matrix \( M_{\nu}^{\text{sol}} \) is at most of order \( \gamma^2/2 \approx \lambda^3 \) in the entry (11) and \( \approx \lambda^3/2 \) the rest. The entries in the QLC correction, \( M_{\nu}^{\text{QLC}} \), are at most a correction of order \( \lambda^3/2 \) in the entry (11) and \( \approx \lambda^3 \) the rest. Therefore for the inverted hierarchy case with same CP-parities to reproduce the neutrino data we need the following hierarchy between the different corrections,

\[
M_{\nu}^{\text{QLC}} \simeq M_{\nu}^{\text{sol}} \ll M_{\nu}^{\text{atm}} .
\] (36)
C. Inverted hierarchy case with opposite CP-parities

In the inverted hierarchy case with opposite CP-parities we obtain the leading order term in the neutrino mass matrix assuming that \( m_2 = -m_1 \) and \( m_3 = 0 \),

\[
\mathcal{M}^\text{atm}_{\nu} = m_3 \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} .
\]  

(37)

As in the same parities case we need to break the degeneracy between the masses of \( \nu_1 \) and \( \nu_2 \) to generate the solar mass difference. To this end we introduce a small perturbation of the form \( m_2 = -m_1(1 + \gamma) \). To be consistent with bimaximal mixing we need the perturbation matrix to be given by,

\[
\mathcal{M}^\text{sol}_{\nu} = \gamma m_1 \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \sqrt{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \sqrt{2} & -\frac{1}{2} \end{bmatrix} .
\]  

(38)

The solar mass difference is again given by,

\[
\Delta m^2_{\text{sol}} = (m_2^2 - m_1^2) = m_1^2 \gamma (2 + \gamma) \approx 2 m_1^2 \gamma .
\]  

(39)

Therefore \( \gamma \approx \lambda^2 / 2 \). Finally to generate a deviation from maximality in the solar mixing angle able to account for the QLC relation we need to introduce a second perturbation given by,

\[
\mathcal{M}^\text{QLC}_{\nu} = \gamma m_1 \frac{1}{\sqrt{2}} \lambda \nu \begin{bmatrix} 2 \sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & -\sqrt{2} \\ 0 & -\sqrt{2} & -\sqrt{2} \end{bmatrix} .
\]  

(40)

Therefore, in the inverted hierarchy case with same CP-parities, the correction to \( \mathcal{M}^{\text{atm}}_{\nu} \) coming from the matrix \( \mathcal{M}^\text{QLC}_{\nu} \) is at most of order \( \gamma / 2 \sqrt{2} \approx \lambda^3 / \sqrt{2} \). Interestingly, the size of the entries to the QLC correction depends upon sign\( (m_2) \) and in the opposite CP-parities case under consideration we obtain that \( \mathcal{M}^\text{QLC}_{\nu} \) is between \( \sqrt{2} \lambda _\nu \) and \( 2 \sqrt{2} \lambda _\nu \approx 2 / 3 \), i.e. approximately between 30% and 60% of the leading term. Therefore for the inverted hierarchy case with opposite CP-parities to reproduce the neutrino data we need the following characteristic hierarchy between the different corrections,

\[
\mathcal{M}^\text{QLC}_{\nu} \ll \mathcal{M}^\text{sol}_{\nu} \ll \mathcal{M}^{\text{atm}}_{\nu} .
\]  

(41)

This is very different from the hierarchies required for the corrections generated in the normal hierarchy case and inverted hierarchy case with same CP-parities. In those two cases the QLC correction was of the same order or smaller than the solar correction respectively.

D. Generalization to the Dirac case

It is straightforward to extend the previous results to the case that neutrinos are Dirac fermions. We will assume again that the mixing in the charged lepton sector in the flavor basis is very small, as a consequence the MNSP matrix is very approximately the left-handed neutrino diagonalization matrix. We obtain,

\[
\mathcal{M}_\nu \mathcal{M}_\nu^\dagger = V_{\text{MNSP}} (\mathcal{M}^{\text{diag}}_{\nu})^2 V_{\text{MNSP}}^\dagger .
\]  

(42)

We can generalize the results of Secs. III A and III B for the normal and inverted hierarchy cases. In the first case we will introduce the same perturbation required to generate the solar mass difference, i.e. \( m_2 = \gamma m_1 \). The \( (\mathcal{M}_\nu \mathcal{M}_\nu^\dagger)^\text{sol} \) and \( (\mathcal{M}_\nu \mathcal{M}_\nu^\dagger)^\text{QLC} \) perturbations can be obtained from Eqs. 26 and 28 by implementing the substitution \( m_1 \rightarrow m_1^2 \) and \( \gamma \rightarrow \gamma^2 \). In the inverted hierarchy case we will now introduce the solar mass difference perturbation in the form, \( m_2^2 = m_1^2 (1 + \gamma^2) \). In doing so we can obtain the perturbations \( (\mathcal{M}_\nu \mathcal{M}_\nu^\dagger)^\text{sol} \) and \( (\mathcal{M}_\nu \mathcal{M}_\nu^\dagger)^\text{QLC} \) by implementing the same substitution, \( m_1 \rightarrow m_1^2 \) and \( \gamma \rightarrow \gamma^2 \), in Eqs. 29 and 33. Nevertheless, the perturbation parameter \( \gamma \) will be determined in this case by \( \gamma^2 \approx \Delta m^2_{\text{sol}} / \Delta m^2_{\text{atm}} \approx \lambda^2 \). Therefore we will obtain for the normal and inverted hierarchy cases corrections to the bimaximal ansatz similar to those in Eqs. 29 and 33 respectively.

IV. RADIATIVE STABILITY OF THE QLC RELATION

It has been known for some time that the RGE effects can considerably affect the neutrino mixing angles \( \theta_{12}, \theta_{13} \). These effects can be especially important in the context of SUSY SO(10) models, which are of especial interest for neutrino physics, since in this case all the third generation Yukawa couplings can be large \( \gtrsim 1 \). The RGE effects also depend crucially on the type of neutrino mass hierarchy under consideration.

In the normal hierarchy case the RGE effects are known to be very small and as a consequence they cannot account for a RGE generation of the QLC and or \( \Delta m^2_{\text{sol}} \), that, as we have seen in the previous section, must be of the same order of magnitude. Interestingly, in the inverted hierarchy case the RGE evolution of the solar mixing depends crucially on the neutrino CP-parities \( \theta_{12}, \theta_{13} \). The RGE equation for the solar mixing angle in this case adopts a simple form, which is valid for small \( \theta_{13} \), as experiments indicate, given by \( \sqrt{2} \lambda _\nu \).

\[
\frac{d \theta_{12}}{dt} = C \frac{1}{8 \pi^2} \frac{C_{12}^2}{\sin^2 \theta_{12}} \frac{m^2_{\text{atm}}}{m^2_{\text{sol}}} (1 + \cos(\phi_1 - \phi_2)) + C(\theta_{13}^2).
\]  

(43)

Here \( t = \ln(\mu / \mu_0) \), \( \mu \) is the renormalization scale and \( \phi_{1,2} \) are the neutrino CP-phases. We will assume that an exactly bimaximal neutrino mass matrix is generated at high energies, \( s_{12} = c_{12} = s_{23} = 1 / \sqrt{2} \), and that the solar and atmospheric neutrino mass differences are phenomenologically acceptable, i.e. that \( \Delta m^2_{\text{sol}} / \Delta m^2_{\text{atm}} \approx \lambda^2 \). We obtain for the RGE generated shift in the solar
mixing angle,
\[ \Delta \theta_{12} \approx \frac{C h^2}{32 \pi^2 \Lambda^2} \left(1 + \cos(\phi_1 - \phi_2)\right) \ln \left(\frac{\Lambda}{m_Z}\right). \]  
(44)

Here \( \Delta \theta_{12} = \theta_{12}(\Lambda) - \theta_{12}(m_Z) \). In the SM \( C = 3/2 \) and \( h^2 = m_Z^2/m_\tau^2 \approx 10^{-4} \) and assuming that \( \Lambda = 10^{16} \) GeV we obtain for the radiatively generated \( \Delta \theta_{12} \),
\[ \Delta \theta_{12}^{\text{max}}_{\text{SM}} \approx 3 \times 10^{-4} \left(1 + \cos(\phi_1 - \phi_2)\right) \]  
(45)

We note that to fit the experimental results we should obtain \( \Delta \theta_{12} \approx \lambda \). It has already been pointed out that in the SM this correction is very small and it cannot be the source of the QLC relation nor perturb a possible high energy origin of the QLC relation irrespective of the neutrino CP-parities.

In the MSSM the situation is more complicated. In this case \( C = -1 \) and \( h^2 \approx \tan^2 \beta m_\tau^2/m_t^2 \), where \( \tan \beta \) is the well known ratio of MSSM Higgs vacuum expectation values. This is relevant in the case of SUSY SO(10) models which require a large \( \tan \beta \). Assuming \( \tan \beta = 50 \) we obtain,
\[ \Delta \theta_{12}^{\text{max}}_{\text{MSSM}} \approx -\frac{1}{2} \left(1 + \cos(\phi_1 - \phi_2)\right) \]  
(46)

This shows that for the same CP-parities case the solar mixing angle would be unstable under RGE corrections as is well known. We cannot generate radiatively the QLC relation because the MSSM correction has a sign contrary to the required to fit the experimental data, \( \Delta \theta_{12} \approx \lambda \). On the other hand, Eq. (46) shows that the solar mixing angle in the case of an inverted neutrino spectra with a maximal CP-parity phase difference between the heaviest eigenvalues will be especially stable since in that case \( \cos(\phi_1 - \phi_2) = -1 \) and as a consequence \( d\theta_{12}/d\mu = 0 \). We note that the term \( O(\theta_{13}) \) which has not been included in the RGE for \( \theta_{13} \) also cancels for opposite CP-parities. This opens the possibility that the QLC relation is generated at a high energy scale, remaining stable all the way down to the electroweak scale.

V. A QLC TRIGGERED \( \Delta m^2_{\text{sol}} \)?

Let us assume that a QLC corrected bimaximal neutrino mass matrix is generated at some high energy scale. We have seen in the previous section that if there is an inverted neutrino hierarchy with opposite CP-parities, i.e., \( m_1 = -m_2 \), the QLC relation will remain stable under RGE evolution. It is interesting to study if an initial high-energy deviation from maximality in the solar mixing, like the one given by the QLC relation, can trigger the generation of the correct solar mass difference radiatively through RGE running. In some cases the solar mass difference, as pointed out some time ago, could be fully generated by RGE corrections. We will assume a limit case where at high energy \( \theta_{13} \) and \( \delta \), the Dirac CP-parities, are zero. The RGE for \( \Delta m^2_{\text{sol}} \) is given in this case by a simple expression
\[ 8\pi^2 \frac{d\Delta m^2_{\text{sol}}}{dt} = \alpha \Delta m^2_{\text{sol}} - CH^2 s^2 \left(\frac{m^2_{\mu}}{m^2_{\tau}} - \frac{m^2_{\tau}}{m^2_{\mu}}\right) + O(\theta_{13}) \]  
(47)

Assuming that at high energies \( \theta_{12} = \pi/4 - \lambda \) and \( \theta_{23} = \pi/4 \) we obtain for the radiatively generated solar mass difference,
\[ 8\pi^2 \frac{d\Delta m^2_{\text{sol}}}{dt} = \alpha \Delta m^2_{\text{sol}} - 2C \lambda^2 H^2 \Delta m^2_{\text{atm}}. \]  
(48)

This equation has a simple analytical solution. In the SM where \( C = 3/2 \) we obtain,
\[ \Delta m^2_{\text{sol}}(\mu)|_{\text{SM}} \approx \frac{3\lambda^2 m^2}{\alpha m^2_{\tau}} \Delta m^2_{\text{atm}}(1 - e^{-\frac{\mu}{M^2_{\text{SM}}}(\chi)}). \]  
(49)

Assuming that \( \Lambda = 10^{16} \) GeV and \( \mu = m_Z \) we obtain \( \Delta m^2_{\text{sol}}(m_Z)|_{\text{SM}} \approx 2.8 \times 10^{-5} \Delta m^2_{\text{atm}}, \) which is too small to account for the observed solar mass difference. On the other hand in the MSSM \( C = -1 \) and we obtain,
\[ \Delta m^2_{\text{sol}}(\mu)|_{\text{MSSM}} \approx -\frac{2\lambda^2 m^2}{\alpha m^2_{\tau}} \Delta m^2_{\text{atm}}(1 - e^{-\frac{\mu}{M^2_{\text{MSSM}}}(\chi)}). \]  
(50)

Assuming that \( \Lambda = 10^{16} \) GeV, \( \mu = m_Z \) and \( \tan \beta \) is large, \( \tan \beta = 50 \), we obtain \( \Delta m^2_{\text{sol}}(m_Z)|_{\text{MSSM}} \approx -2\lambda^2 \Delta m^2_{\text{atm}}. \) Therefore the radiatively generated \( \Delta m^2_{\text{sol}}(m_Z) \) is of the right magnitude but unfortunately of the wrong sign. The experimental data requires that \( \Delta m^2_{\text{sol}}(m_Z) \approx \lambda^2 \Delta m^2_{\text{atm}}. \) Therefore a RGE generation of \( \Delta m^2_{\text{sol}} \) triggered by a very high energy generation of the QLC perturbed bimaximal scenario, assuming and inverted hierarchy with opposite CP-parities, does not seem to be in agreement with the data.

VI. CONCLUSIONS

We have studied several model independent implications of the measured deviation from maximality in the solar mixing angle. We have pointed out that it is not plausible that this deviation is generated in the charged lepton mixing matrix. We have studied the generic low energy corrections to the exactly bimaximal ansatz necessary to account for both the solar mass difference and a non-maximal solar mixing angle. We pointed out that the relative size of these corrections depends strongly on the neutrino hierarchy under consideration. For the normal and inverted hierarchy with same CP parities it seems very difficult to understand the origin of the QLC relation independently from the origin of \( \Delta m^2_{\text{sol}} \) since the respective corrections are of the same order of magnitude. In that case the QLC relation is most probably a coincidence unless the neutrino mass matrix is generated at low energy scales.

On the other hand, for an inverted hierarchy with opposite CP parities the correction to the bimaximal ansatz
necessary to explain the QLC relation is of the same order but smaller than the leading term of the bimaximal matrix and both are much larger than the correction necessary to generate $\Delta m^2_{\text{sol}}$. Additionally the leading bimaximal term as well as the QLC perturbation could both have a high energy origin since the solar mixing angle is very stable under RGE effects. This raises the possibility to link the origin of the QLC relation with the origin of the charged fermion mass matrices. Although this setup does not allow us to radiatively generate $\Delta m^2_{\text{sol}}$ entirely by RGE corrections there are other possible explanations available in the literature for the origin of the measured $\Delta m^2_{\text{sol}}$. We believe, as our analysis indicates, that the inverted hierarchy case with opposite CP-parities may be the most interesting possibility from a model building point of view when searching for a non-coincidental, high-energy explanation of the QLC relation.

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