Abstract:
A basic modern picture of the universe is given here. The lectures start from the historical ideas of a static universe. Then I move on to Newtonian cosmology and derive the main cosmological equations in the framework of Newtonian mechanics for the sake of simplicity. With a qualitative description of general relativity, the expansion of the universe and elementary idea of the hot Big Bang models are introduced. The problems of the hot Big Bang, inflation and structure formation are explained here in simple language.
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A Hilary

Inflationary Universe

1. Introduction

What are we in? How big is it? How did it begin?
How will it end?

Some of the oldest questions in the world are surely those that ignited one of the world’s oldest subjects. The above questions are as such responsible for cosmology. Answers and explanations for these questions were sought from many concepts, cultures and beliefs. From ancient times until today our civilization has been gradually adding many small pieces of the jigsaw, either on mythological, philosophical or scientific aspects of our new picture of the universe. Different viewpoints of our universe vary from culture to culture. In science, cosmology started when we believed that our physical laws applied on earth can also be applied in any other parts of the universe. This is part of the thought that we are not special and we do not occupy a special place in the universe. Physics that works on earth must be the same physics that works everywhere in the universe.

2. The Newtonian Universe

It was not until the 1930s that Hubble began to realise that our Milky Way is not the only one main structure in the universe; in fact our Milky Way is just one of many galaxies. Modern cosmology was born. Historically Newton, without knowing that the sun is just one of billions of stars of the Milky Way, and without having any ideas about what lies beyond the Milky Way, attempted to apply his three laws of motion and his law of gravitation to work in all locations in the universe.

Newton realized that the universe can not have finite size since it would collapse quickly to the centre of mass. In the infinite universe there is no centre of the mass sphere and this fact allows the universe to be static agreeing with the beliefs of the people in the 17th century. At smaller scales gravity can bring about the formation of stars and smaller celestial objects. His belief in a static universe was expressed in the second edition of the Principia:

_The fixed stars, being equally spread out in all points of heavens, cancel out their mutual pulls by opposite attractions._
2.1 Olbers’ paradox and the Newton’s static and infinite universe

2.1.1 The paradox

Newton’s static infinite universe has a problem called **Olbers’ paradox** (Digges 1576 and Olbers 1826). The paradox is simply: _why is the night sky dark?_ This paradox can be seen in figure 1. In the figure we are the observers living at the centre of the imaginary sphere. \( r \) is the radial distance from us. Each shell with \( dr \) thickness occupies volume \( 4\pi r^2 dr \). The luminosity of stars \( L \) is proportional to the number of stars \( N \) and the number of stars is proportional to the volume of the shell. The flux light density is therefore

\[
\text{Flux} = \frac{L}{4\pi r^2} \propto \int \frac{4\pi r^2 dr}{4\pi r^2} \propto \int dr \tag{1}
\]

If the universe is infinite, \( \int dr \to \infty \) and the flux of night-sky’s light should be infinite. The other problem arose when the **Copernican principle**, which roughly says that we are not occupying a special place in the universe, was applied to this model (Halley 1721). This principle will be introduced formally later. The universe is isotropic about any point in the universe then the attraction force from stars in one side in figure 1 must be equal to the attraction force from stars in the opposite side. If there are infinite number of shells, the force from each side becomes stronger and stronger but the total force needs to be canceled out to zero. The universe is in balance with very high instability. This situation needs a very high degree of isotropy, otherwise this would unbalance the forces and pull
the matter toward one side, leading to collapse! Local forces that govern local motions of planets would disturb this low stability.

The problem of the Newtonian infinite universe is in fact what is called in mathematics a Dirichlet problem of a potential theory. To determine force, we need to know the boundary conditions. In a universe with a uniform, finite and bounded distribution of matter, we can determine the gravitational force at each point by boundary conditions. In Newton’s infinite, unbounded (edgeless) and centreless universe, gravitational potential diverges at infinite distance and the gravitational field depends on boundary conditions at infinity. Another problem is the high instability as mentioned earlier. To keep the Newtonian infinite universe static, we need to set one point in the universe to possess zero gravitational field but non-zero elsewhere. It does not sound reasonable. Then the problems can not be solved without modifying the dynamical law itself.

2.1.2 The way out of the paradox

Before the invention of General Relativity (GR) in 1915, gravity was thought to act simultaneously (action at a distance). This means matter particles can feel the force from other particles instantaneously without time elapsing. In GR, light has finite speed and so does the gravitational interaction. If we just add the idea of gravity propagating at the speed of light, in the finite age of the universe (approximately 10 billion years), we don’t feel force and don’t see any light from distances beyond 10 billion light years. Olbers’ paradox hence is solved since the light from beyond 10 billion light years has not reached us yet. Moreover, later when I introduce the expanding universe in which the galaxies are moving away from each other, creating a redshift in light and making distance between us and galaxies further and further away, we will have an even better resolution for Olbers’ paradox. More detailed discussion can be found in Refs. [1] and [2].

2.2 Cosmological principles

Modern cosmology relies on the basic important assumption called Cosmological principle which is a more general version of the Copernican principle. The Copernican principle states that the Earth is not the centre of the universe i.e. we are not living at a special location in the universe and the universe must be homogenous. Homogeneity of the universe means that the universe has the same property at any regions from point to point. The cosmological principle includes the Copernican principle together with isotropy of the universe. Isotropy of the universe means that the universe looks the same from all directions. In Figure 2, the vector field is homogenous but obviously it is not isotropic since when we observe it from a fixed point in the picture, it looks different from different directions. Figure 3 shows isotropy about one point but it fails to include homogeneity. These two properties are therefore separated and we need both of them to complete the
Homogeneity does not imply isotropy.

Figure 2: Homogenous vector field fails to be isotropy

cosmological principle. Mathematically, homogeneity and isotropy are respectively invariant properties under translation and rotation transformations.

We indeed do know that at small scales the universe is not homogenous and not isotropic otherwise any structures e.g. galaxies, stars, planets and humans would not even exist. However provided that we consider the universe on average on large scales, it looks approximately homogenous and isotropic. Strong evidence for this is the cosmic microwave background (CMB) observed in 1992 by the COBE mission to be very smooth to at least one part in $10^5$ [3].

3 The Expanding Universe

3.1 The non-static universe

Before 1915 cosmologists believed that the cosmos is static. From a Newtonian viewpoint, in addition the universe is also infinite. The solutions of GR in 1915 suggested that the universe should not be static. Einstein therefore added cosmological constant $\Lambda$ into his field equation in order to obtain a static solution and this was later admitted by him to be his greatest blunder. Many relativists (de Sitter 1917, Friedmann 1922 [4], Lemaître 1927) found solutions (either infinite or finite) in which universe is either expanding or collapsing. In 1929 the expansion of the universe was discovered by Hubble ([6],[7] and [8]). Later in 1934 Milne and McCrea discovered that the Friedman-Lemaître type model can be recovered by using Newtonian Mechanics. In 1936, Robertson and Walker gave the general form of the metric for a homogeneous and isotropic universe, called the Friedmann-Lemaître-Robertson-Walker (FLRW) metric.
Isotropy does not imply homogeneity.

Figure 3: Isotropy does not imply homogeneity

We will take advantage of Milne and McCrea’s work [5] by using the Newtonian treatment to obtain the main cosmological equations. We later discuss very briefly about GR. For a nice historical discussion, I refer readers to Refs. [1] and [9].

3.2 Hubble’s law

During the 1920s and 1930s, cosmologist began to realise that the distant nebulae are not only stardust but in fact other galaxies outside our Milky Way. The universe is much bigger than people at that time thought. In 1929, Hubble found that the universe does not stay static but is expanding [6]. This was noticed by observing that these distant galaxies’ spectra are redshifted and they are even more redshifted at further distance. Hubble found the empirical law

\[ v = H_0 R \]  

which was later dubbed after his name, Hubble’s law. This law implies that the further galaxy is, the faster it moves away from us. Here \( H_0 \) is the Hubble constant at the time \( t_0 \). The Hubble constant is in fact the proportionality constant of \( v \) and \( R \). The law is not exactly true since at the smaller scale the universe is neither completely homogenous nor isotropic and there is also the peculiar velocities from local gravitational forces. The law also breaks down at large distances, when it is no longer a good approximation.

The expansion of the universe leads us to the concept of expansion of space. Space itself is expanding and we need to introduce the so-called comoving coordinates \( x \). These coordinates are moving along with the expansion of space. The physical distance
Figure 4: Expanding comoving grids and comoving galaxy on a physical ruler.

$R$ is given by

$$ R = a(t) x $$  \hspace{1cm} (3) $$

where $a(t)$ is the scale factor. Here we assume homogeneity and isotropy of space. Figure 4 shows the expanding coordinates in one and two dimensions. The recession velocity of galaxies $v$ is in the same direction as the displacement from us $R(t)$ and is given by

$$ v = \frac{\dot{R}}{R} $$  \hspace{1cm} (4) $$

where dot denotes differentiation with respect to time e.g. $\dot{R} = \frac{dR(t)}{dt}$. Using equation (2), then

$$ v = \frac{\dot{a}}{a} R = H R $$  \hspace{1cm} (5) $$
where

\[ H = \frac{\dot{a}}{a} \]  

is the **Hubble parameter** which is time-dependent. The constant \( H_0 \) usually means \( H \) at the present age of the universe. There is a subtle question about why are not we expanding? Why does not the distance between the sun and the earth increase? The answer is that the expansion only has a dominant effect on large scales (inter-galactic scale). At scales smaller than this scale we hardly see any effect of expansion since it is dominated by local gravity.

Another doubt is something to do with Special Relativity (SR). As we see the distant galaxy moving away from us very fast, can it go away faster than the speed of light? Yes, it can. This expansion does not violate special relativity. Since special relativity governs the speed of objects that are just passing one another. That speed can not exceed the speed of light. Here in our circumstance, it is the distant cosmic expansion of space, not two objects passing each other.

### 3.3 Redshift

Redshift is the result of Doppler’s effect in physics. The expansion of the universe stretches the photon’s wavelength. Light spectra of elements which should be observed in some range of \( \lambda \) are shifted to the red end of the spectrum. The redshift \( z \) is defined by

\[ z = \frac{\lambda_{ob}}{\lambda_{emit}} - 1 = \frac{a_{ob}}{a_{emit}} - 1 = \frac{a_{ob}}{a_{emit}} - 1 \]

hence

\[ z + 1 = \frac{a_{ob}}{a_{emit}} \]

The subscript \( ob \) and \( emit \) denote the observed and emitted value of the wavelength.

### 3.4 Energy and matter species

Matter and energy are convertible according to \( E = mc^2 \). Matter or energy density in the universe can be classified into three species:

- dust
- radiation
- cosmological constant (\( \Lambda \)) and scalar fields (\( \phi \)).
Dust is the term for non-relativistic matter e.g. particles with small speed compared to the speed of light. Other species is radiation which is relativistic particles moving with the speed closed to that of light. Dust usually means baryons, electrons etc. and radiation here means photons, neutrinos. The cosmological constant is another form of energy that generates repulsive force to prevent the universe from collapsing due to gravity. This repulsive force can accelerate the expansion of the universe. We call types of energy that can accelerate the universe dark energy. Scalar fields are candidates for dark energy and they can be considered as a time-varying cosmological constant.

3.5 Newtonian gravity

In Newtonian mechanics

\[ F = \frac{GM}{r^2} m \hat{r} \]  \hspace{1cm} (9)

where \( F \), \( M \), \( m \), \( G \) denotes force, mass, test mass and the Newtonian gravitational constant. \( r \) is the distance between these two masses and \( \hat{r} \) is a unit vector pointing from \( m \) to \( M \). The force is always attractive and \( \frac{GM}{r^2} \hat{r} \) is the gravitational acceleration \( g \). The potential energy is given by

\[ P.E. = -\frac{GMm}{r} \]  \hspace{1cm} (10)

The potential function is

\[ \Phi = -\frac{GM}{r} \]  \hspace{1cm} (11)

For spherically symmetric and uniformly distributed mass (see figure 5, similar to what happens in electrostatics, the potential is constant within the sphere due to the symmetry property. Particles in this sphere feel no force according to

\[ g = -\nabla \Phi \]  \hspace{1cm} (12)

The mass outside the sphere feels a force as if all the mass of the sphere is located at the centre, and this force is independent of the radius of the sphere, \( R \).

In the picture used by Milne and McCrea the sphere of mass is expanding with a velocity \( v = \dot{R}(t) \). The spherical mass stays homogenous and isotropic (as viewed from centre of the sphere) during the expansion.

Let us look at the energy conservation law of a particle of mass \( m \) at the surface of this sphere. Here the total mass of our sphere is \( \rho V \) where \( \rho \), \( V \) are the mass density and the volume. \( V = \frac{4}{3}\pi R^3 \). Then the total energy of this particle is

\[ U = \frac{1}{2} m \dot{R}^2 + P.E. \]

\[ = \frac{1}{2} m \dot{R}^2 - \frac{4}{3} G \rho \pi R^2 m \]
Multiplying both sides by $2/(mR^2)$,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3} \rho + \frac{2U}{mR^2}$$

(13)

This equation is just another form of the energy conservation law.

### 4 Cosmological Equations

#### 4.1 Friedmann equation

Now we will include the idea of expansion in the energy conservation by putting $R(t) = a(t)x$ into equation (13). Hence

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho + \frac{2U/mx^2}{a^2}$$

(14)

The fraction $\frac{2U}{mx^2}$ is a constant. When space expands, the $x$ value remains the same since it comoves with the space. Total energy $U$ also remains constant. Let us define $kc^2$ as $-2U/(mx^2)$ where $c$ is the speed of light. $k$ is the curvature of space which describes geometry of the universe. If we add Einstein’s cosmological constant $\Lambda/3$ to the equation, it will be equivalent to adding a constant amount of energy to the universe. This does not
violate energy conservation. Now with $H = \dot{a}/a$ the equation becomes

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda}{3}$$

(15)

This equation is known as the **Friedmann equation**. According to the Copernican principle the equation we have here can be applied to any regions of the universe. The Friedmann equation is crucial in cosmology. It relates and constrains how the scale factor $a$ evolves given the total density $\rho$ (amount of energy and matter and hence geometry), $k$ and $\Lambda$ of the universe.

### 4.2 Fluid equation

We model the matter and radiation of the universe as a perfect fluid (fluid has no viscosity and no heat conduction in comoving coordinate). We shall start from look the first law of thermodynamics

$$dU = TdS + dW = TdS - pdV$$

(16)

where $S, T, W, p$ are entropy, temperature, work done and pressure respectively. This law is also an energy conservation law. Here we assume that the equation of state is of the form

$$p = p(\rho)$$

(17)

That is to say $p$ is an explicit function of $\rho$ only, and we assume that there are no other external forces. For dust $p = 0$ and radiation $p = \rho c^2 / 3$, but for $\Lambda$ or dark energy, $p < 0$ (negative pressure). $dS$ vanishes since reversible (adiabatic) expansion is assumed here. $dV$ is $d(\frac{4}{3}\pi R^3) = 4\pi R^2 dR$. Equation (16) finally becomes

$$dU = -p4\pi R^2 dR$$

(18)

The total relativistic energy of particles in a sphere is

$$U = Mc^2 = \rho Vc^2$$

$$dU = 4\pi R^2 \rho c^2 dR + \frac{4}{3}\pi R^3 c^2 d\rho$$

(19)

If we differentiate equation (18) with respect to $t$ and then combine it with equation (19), we get

$$-p4\pi R^2 \dot{R} = 4\pi R^2 \rho c^2 \dot{\rho} + \frac{4}{3}\pi R^3 c^2 \dot{\rho}$$

(20)
Rearranging terms in this equation and writing it in terms of the scale factor $a(t)$, we finally obtain the fluid equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + \frac{p}{c^2}) = 0$$

(21)

The equation of state (equation (17)) can be rewritten as

$$p = \rho c^2 w$$

(22)

where $w = 0$ and $1/3$ for dust and radiation, while $w < 0$ for $\Lambda$ or dark energy. Fluid equation is then simply

$$\dot{\rho} + 3H\rho(1 + w) = 0$$

(23)

Fluid equation, likes Friedmann equation, is in fact energy conservation law. The first term $\dot{\rho}$ tells us how fast density changes (e.g. dilutes) and the second term is the lost of kinetic energy from fluid into gravitational potential energy.

### 4.3 Acceleration equation

The acceleration equation tells us how rate of expansion of the universe changes i.e. slowing down or speeding up. The equation is in fact a mixture of Friedmann and fluid equations and these two equations are in fact energy conservation law in mechanics and thermodynamics respectively. After differentiating the Friedmann equation with respect to time and using the fluid equation, we finally obtain

$$\ddot{a} = \frac{-4\pi G}{3} \rho(1 + 3w) + \frac{\Lambda}{3}$$

(24)

The good feature of the acceleration equation is that it does not contain $k$ and we can use this equation regardless of the geometry of the universe. From the equation it seems that universe is decelerating. When we neglect the small value of the cosmological constant and the universe is dominated by $w < -1/3$ fluid (dark energy) with $p < -\rho/3$, it could make $\ddot{a}$ positive and will accelerate the universe. Indeed the recent observation from Type Ia Supernovae strongly supports that the universe now is accelerating [11]! From the viewpoint of high energy physics we can have dark energy in the form of scalar field that can yield negative pressure and hence accelerate the universe. These scalar fields are in general called Quintessence, the name of the fifth element in ancient Greek. For a good start to quintessence, Ref. [12] is recommended here.
4.4 Solutions of equations

4.4.1 Solution of the fluid equation

For simplicity, from now on we will work in the units where $c \equiv 1$. The equation of state of the cosmological perfect fluid is then just $p = w \rho$. The fluid equation (23) with the help of basic calculus can be rearranged as

$$\frac{d}{dt} \left( \rho a^{3(1+w)} \right) = 0$$

(25)

Obviously we see that $\rho a^{3(1+w)} = \text{constant}$, and therefore we get

$$\rho \propto a^{-3(1+w)}$$

(26)

An alternative way is to notice that equation (23) is separable differential equation

$$\frac{1}{\rho} \frac{d\rho}{dt} = -3 \frac{\dot{a}}{a} (1 + w)$$

$$\int \frac{1}{\rho} d\rho = -3(1 + w) \int \frac{1}{a} da$$

$$\ln \rho = -3(1 + w) \ln a + C$$

$$\rho = e^C a^{-3(1+w)}$$

(27)

where $e^C = \rho_0 a_0^{3(1+w)}$. Here we have assumed that $w$ has constant value. The subscript 0 denotes the value at present\(^1\). Here I introduce the e-folding number,

$$N(t) = \ln \frac{a(t_{\text{after}})}{a(t_{\text{initial}})}$$

(28)

for use in later sections. The e-folding number tells us the amount of expansion in log scale and is very useful when we consider inflationary expansion of the universe. Suppose that we consider the amount of expansion from the present era when the scale factor is $a_0$ to sometime $t$ in the future. Then $N(t) = \ln [a(t)/a_0]$ and

$$a(t) = a_0 e^{N(t)}$$

(29)

Using the e-folding number in the fluid equation (23), we have

$$\dot{a} \frac{d\rho}{da} = -3 \frac{\dot{a}}{a} \rho (1 + w)$$

(30)

\(^1\)For simplicity in many textbooks, authors usually re-scale the present size of scale factor to one ($a_0 \equiv 1$).
and
\[
\frac{d\rho}{da} = \frac{dp}{[a/a_0]^{-1}(a/a_0)} = \frac{dp}{d \ln (a/a_0)} = \frac{d\rho}{dN}
\] (31)

Therefore
\[
\frac{d\rho}{dN} = -3\rho(1 + w)
\] (32)

and equation (27) can be written as
\[
\rho = \rho_0 e^{-3(1+w)N}
\] (33)

4.4.2 Solution of the Friedmann equation

The exact solution of the Friedmann equation can be found easily when we assume that the universe is flat\(^2\) and \(\Lambda\) has negligible value. Using equation (27), the Friedmann equation reads
\[
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}
\] (34)

Then the solution is
\[
a = a_0 \left(\frac{t}{t_0}\right)^{2/[3(1+w)]}
\] (35)

where \(t_0\) is arbitrary constant. As the universe evolves, one particular component dominates the universe at some period of time. Following equation (35), domination of each type of fluid leads to a particular type of the expansion kinematics. For example in a universe with \(w < -1/3\), dark energy fluid dominates according to equation (24), and the expansion of the universe is therefore accelerating.

4.4.3 Evolution of density with time

Putting together solutions (27) and (35) we obtain
\[
\rho(t) = \rho_0 \left(\frac{t}{t_0}\right)^{-2}
\] (36)

regardless of the value of \(w\) (regardless of fluid types).

\(^2\)Inflation theory of universe predicts that universe has flat geometry \((k = 0)\). Moreover CMB detectors such as BOOMERanG, MAXIMA [13, 14] and Wilkinson Microwave Anisotropy Probe (WMAP) [16] have already released datasets confirming that the universe is very close to flat.
4.4.4 Evolution of Hubble parameter with time

Differentiating equation (35) with respect to time and dividing the result again by \( a \), the Hubble parameter evolves with time as

\[
H(t) = \left[ \frac{2}{3(1 + w)} \right] \frac{1}{t}
\]

(37)

The Hubble parameter at the present time is called the Hubble constant, \( H_0 \), which is given in terms of the dimensionless constant \( h \) as

\[
H_0 = 100h \text{ km/s/Mpc}
\]

The expansion increases 100\( h \) km/s for every distance increase of 1 Mpc. (The distance 1 pc \( \simeq 3.261 \) light years \( \simeq 3.086 \times 10^{16} \) m.) The value of the constant \( h \) is between 0.55 and 0.75. WMAP data (February 2003) leads to \( h = 0.71^{+0.04}_{-0.03} \) [17].

5 Simple Toy Models

5.1 The dust-filled universe

Dust (non-relativistic particles) is pressureless (\( p = 0 \) or \( w = 0 \)). In the dust-dominated universe solutions (27) and (35) become

\[
\rho(t) = \rho_0 \left( \frac{a(t)}{a_0} \right)^{-3}
\]

(38)

\[
a(t) = a_0 \left( \frac{t}{t_0} \right)^{2/3}
\]

(39)

respectively. The equation tells us that the density decreases with the expanding volume of the universe. In the dust universe, \( H \) evolves as

\[
H(t) = \frac{2}{3t}
\]

(40)

implying that the universe will stop expansion (\( H = 0 \)) when \( t \to \infty \). Let us assume that dust always dominates the universe. With \( h = 0.71 \) then we can find that

\[
H_0 \simeq 2.30 \times 10^{-18} \text{ s}^{-1}
\]

At the beginning of expansion \( t = 0 \) s, so that the age of the dust-filled universe today is

\[
t_0 = \frac{2}{3H_0} \simeq 2.90 \times 10^{17} \text{ s}
\]

or \( 9.20 \times 10^9 \) years! Dust has been dominating the universe longer than other type of fluids therefore this value is an approximate age of our universe.
5.2 The radiation-filled universe

Radiation (relativistic particles) has $p = \rho/3$ or $w = 1/3$. Carrying out the same procedure as we performed in the dust case, the solutions (27) and (35) become

$$\rho(t) = \rho_0 \left(\frac{a(t)}{a_0}\right)^{-4} \quad (41)$$

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{1/2} \quad (42)$$

respectively. The radiation-filled universe expands slower than the dust-filled universe. Density in both cases decreases with $t^2$ but the radiation density decreases with $a^4$ not $a^3$ as in dust case. The extra $a$ comes from the redshift effect on relativistic particle’s wavelength. We can obtain the relationship between redshift and scale factor from the Hubble law. At extra-galactic scale, for a small distance apart $dr$ the recession velocity differs by $dv$. According to the Hubble law,

$$dv = H dr = \frac{\dot{a}}{a} dr \quad (43)$$

$H$ is assumed constant within the variation of small distance $dr$. The Doppler effect stretches the radiation’s wavelength $\lambda$ by $d\lambda = \lambda_{ob} - \lambda_{emit}$. By Doppler’s law

$$\frac{d\lambda}{\lambda_{emit}} = \frac{dv}{c} \quad (44)$$

The time lapses in light traveling a distance $dr$ is $dt = dr/c$. Inserting equation (43) into (44) we get

$$\frac{d\lambda}{\lambda_{emit}} = \frac{(\dot{a}/a)dr}{c} = \frac{1}{a} \frac{da}{dt} dt = \frac{da}{a} \quad (45)$$

We can see that $\lambda \propto a$ therefore frequency $\nu \propto a^{-1}$. The energy density of radiation ($\rho_{\text{rad}}$) is

$$\epsilon = n h \nu = n h \frac{c}{\lambda} \propto \frac{1}{a} \quad (46)$$

where the quantum of energy is $E = h\nu$. Here $h$ is Planck’s constant and $n$ denotes the photon number density. Therefore redshift can add a factor $a^{-1}$ to the decreasing of $\rho_{\text{rad}}$, and therefore yields $\rho \propto a^{-4}$ instead of $a^{-3}$.

5.3 The dust and radiation-filled universe

In the dust-radiation universe we have contributions from both types of fluid. The Friedmann equation is now

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \left[\rho_{\text{dust}} + \rho_{\text{rad}}\right] \quad (47)$$

where $\rho_{\text{dust}}$ and $\rho_{\text{rad}}$ are given by (38) and (41) respectively. To get the exact solution for this mixed fluid equation is not easy. In the universe containing two components of
fluid, dust falls off slower than radiation, hence after sometime from the beginning, there must be a time when $\rho_{\text{rad}} = \rho_{\text{dust}}$, and it is called \textbf{matter-radiation equality}. After that, dust will start to dominate the universe. At early times, if we assume a Big Bang, it should be radiation that dominates the universe, and we can approximately use the solutions of the radiation-filled case. After the equality we then can approximately use all equations of the dust-dominated case. When dust becomes dominant, expansion becomes faster (compare equations (42) and (39)). Figure 6 illustrates schematically the evolution of dust and radiation density in the dust-radiation mixed universe. We can see from figure 6 that after dust becomes dominant at equality, both radiation and dust density fall off faster than before. This is because the scale factor in the dust case increases faster than in the case of radiation.

5.4 \textbf{Spatial curvature and fate of the universe}

The spatial curvature $k$ can play an important role in the evolution of the universe. We have already looked in detail at the $k = 0$ case. In the situation that $k \neq 0$, the term $k/a^2$
will dominate Friedmann equation quickly since it falls off much slower than both $\rho_{dust}$ and $\rho_{rad}$.

- For $k < 0$ after $k/a^2$ becomes dominant, the Friedmann equation reads

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{k}{a^2}$$

(48)

The solution of this equation is just

$$a \propto t$$

(49)

yielding that the universe will expand forever!

- For $k > 0$, the expansion will slow down because $-\frac{k}{a^2}$ reduces the Hubble rate. Now we can not neglect $\rho$ since then $H^2 < 0$. Eventually the curvature term balance the matter term

$$\frac{8\pi G}{3} \rho = \frac{k}{a^2}$$

(50)

so that the expansion stops

$$H = 0$$

(51)

From now the spatial curvature term becomes dominant and the universe starts to collapse. Notice that if we substitute $-t$ for $t$, the Friedmann equation remains unchanged [10]. This means that the equation is time-reversible and the universe will evolve reversely to where it first starts expanding. We can see the schematic evolution curves in figure 7.

6 The General Relativistic Universe and the Big Bang

The story of the hot Big Bang arose from revolution of physics ideas and astronomical observations in the early 20th century. After Einstein, in 1915, discovered his General Theory of Relativity (GR), in 1922 Friedmann found one solution of Einstein field equation that allows the non-static universe which can either expand or collapse [4]. In 1929, there was a great discovery by Hubble that distant nebulae are in fact other galaxies outside our Milky Way. By observing these distant galaxies redshifted, Hubble found that the universe does not stay static but it is expanding, following his empirical law, dubbed Hubble’s law, equation (2). In 1946, working on the theory of light-element abundances, Gamow proposed that the universe at early time should be very hot and dense [18]. Following Gamow’s work, Alpher and Herman [19] wrote a programming code predicting that the universe should be filled by microwave radiation with the black-body spectrum
at about 5 K.

All these ideas lead us to the picture that all galaxies should start from one infinitely dense and hot point and later expand, gradually cooling down while expanding. This concept is called the **hot Big Bang**.

### 6.1 Evidence supporting the hot big bang model

- **Redshift of Distant Galaxies**
  Galaxies’ redshift brought about the Hubble ’s law and is the first evidence of expanding universe.

- **Primodial Nucleosynthesis**
  The predictions of the abundances of the light elements from nucleosynthesis in the hot Big Bang have been proposed [20] and yield results that agree very well with astrophysical observations (for a review see [21]).

- **Detection of Cosmic Microwave Background Radiation (CMB)**
  Cosmic microwave background (CMB) was discovered accidentally in 1965 by Penzias and Wilson [22]. The spectrum of the background is very close to the theoretical
prediction since it has black-body (thermal equilibrium) shape and it seems to be very isotropic over the whole sky with the temperature about 2.7 K. This observed temperature is very close to Alpher and Herman’s prediction. Discovery of the CMB provides concrete evidence of the hot Big Bang theory, and against the steady state theory\(^3\).

The Big Bang cosmology relies on the cosmological principle and General Relativity (GR). The universe looks, on large scales, homogenous and isotropic but at smaller scales, it has structure and therefore is neither homogenous nor isotropic. In the early 1990s the results from the COBE mission revealed that there are small fluctuation ($\Delta T/T \sim 1/10^5$) [3] in the CMB temperature, and this could be the seeds for origin of structures we see today (see some standard textbooks, e.g. [23, 25, 26] for more discussion).

6.2 Einstein field equation and FLRW metric

The Einstein field equation in GR is

$$G_{\mu \nu} \equiv R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = 8\pi G T_{\mu \nu}$$

(52)

The field equations can be solved by introducing some assumptions on the metric (or line element) $g_{\mu \nu}$. The metric that satisfies homogeneity and isotropy is the Friedmann-Lemaître-Robertson-Walker metric (FLRW metric),

$$ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

(53)

where $t$ is proper time, $r$, $\theta$ and $\varphi$ are spherical coordinates, $a(t)$ is the scale factor and $k = \pm 1, 0$ is the spatial curvature mentioned earlier. With the FLRW metric, equation (52) leads to the the Friedmann and fluid equations.

6.3 Particle horizon

In the FLRW expanding universe, the particle horizon, i.e. the distance that light has travelled from the Big Bang $t = 0$ until the time $t = t_0$, is

$$d_H(t_0) \equiv a(t_0) \left[ \int_{t=0}^{t_0} \frac{c dt}{a(t)} \right]$$

(54)

The quantity $\int_{t_0}^{t_0} \frac{c dt}{a(t)}$ is called the comoving particle horizon.

Since in GR matter curves spacetime, the problems of Newtonian cosmology are solved.

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\(^3\)The steady state theory is not discussed here. For more detail, Ref. [27] is recommended.
6.4 Hubble time and Hubble length

We can notice from equation (37) that $H$ has a dimension of $t^{-1}$. We call $H^{-1}$ Hubble time. The Big Bang starts with $a = 0$, so the Hubble time can be used as the approximate age of the universe, as we used it to estimate the age of the dust-dominated universe before. The Hubble length at one particular time is the distance that light travels from Big Bang until that time. The Hubble length is defined simply as $cH^{-1}$.

6.5 The Einstein’s static universe

In the early 1900s the universe was thought to be static therefore Einstein, trying to match his theory with expectation, introduced the cosmological constant $\Lambda$ into his field equation. This ad hoc fine-tuned value of $\Lambda$ is just enough to hold the universe static. The field equation becomes

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\Lambda = 8\pi GT_{\mu\nu}$$

The discovery of the expansion of the universe ruled out the static model of the universe and the cosmological constant fell away. However recently, by observing type Ia supernova [11], the expansion of the universe was found to be accelerating and the idea of repulsive gravity e.g. cosmological constant, has come into favour again.  

6.6 Cosmological equations

The consistent fluids for the FLRW metric are the perfect fluids mentioned before (the fluids that have no viscosity and no heat conduction in comoving coordinate). We assume that the components of the universe are a mixture of perfect fluids. The energy momentum tensor for the perfect fluid is

$$T^\mu_\nu = diag(\rho, -p, -p, -p)$$

The field equation (55) in the FLRW, perfect fluid-filled universe yields the Friedmann and acceleration equations (15) and (24):

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) + \frac{\Lambda}{3}$$

The Einstein field equations imply the energy-momentum conservation equation,

$$\nabla^\nu T^\mu_\nu = 0$$

---

4The cosmological constant could be considered as the vacuum energy according to quantum field theory but there is one big problem: the value of vacuum energy from quantum field theory is about 124 order greater than the value needed in cosmology! This still remains a problem for cosmologists
For a FLRW universe, this gives the fluid equation (23)

\[ \dot{\rho} + 3H\rho(1 + w) = 0 \]  

(58)

### 6.7 Density parameter and geometry of the universe

We now rewrite the Friedmann equation (15) in another form

\[ -k = a^2H^2 \left[ 1 - \frac{8\pi G}{3H^2}\rho_{\text{tot}} \right] \]  

(59)

where \( \rho_{\text{tot}} \) is the total energy density of the universe

\[ \rho_{\text{tot}} = \rho_{\text{rad}} + \rho_{\text{dust}} + \rho_{\Lambda} \]  

(60)

where \( \Lambda \) here includes \( \Lambda \) or other dark energy. \( \rho_{\Lambda} \) is defined as \( \rho_{\Lambda} \equiv \Lambda/8\pi G \).

According to equation (59), the universe can be flat \((k = 0)\) only if

\[ \rho_{\text{tot}} = \frac{3H^2}{8\pi G} \equiv \rho_c \]  

(61)

This is how we define the critical energy density, \( \rho_c \), the energy density that is consistent with a flat universe. If \( \rho_{\text{tot}} > \rho_c \), then \( k > 0 \) and the universe has closed geometry which means that the expansion of the universe could halt at some point and then start to re-collapse. On the other hand if \( \rho_{\text{tot}} < \rho_c \) then \( k < 0 \) and the universe has open geometry. This means that universe will expand forever and in the far future, matter and radiation will be diluted away. Eventually the vacuum energy will dominate the universe and expansion enters the acceleration phase. The sum of the angles of a triangle is less than 180 degrees for open geometry. The sum is greater than 180 degrees for closed geometry and is equal to 180 degrees for the flat geometry.

The Friedmann equation can also be expressed in term of the total density parameter \( \Omega_{\text{tot}} \) as

\[ 1 - \Omega_{\text{tot}} = -\frac{k}{a^2H^2} \]  

(62)

where

\[ \Omega_{\text{tot}} \equiv \frac{\rho}{\rho_c} = \frac{8\pi G\rho}{3H^2} \]  

(63)

From equation (62) we can see that \( \Omega_{\text{tot}} \) determines conditions for spatial curvature of the universe

\[ \Omega_{\text{tot}} = 1 \Rightarrow k = 0 \]
\[ \Omega_{\text{tot}} < 1 \Rightarrow k = -1 \]
\[ \Omega_{\text{tot}} > 1 \Rightarrow k = 1 \]

In the year 2000, two CMB detector missions, BOOMERanG and MAXIMA [13, 14] confirmed that the universe’s geometry should be very close to flat. The BOOMERanG
result in 2001 gave $\Omega = 1.02^{+0.05}_{-0.05}$ [15]. The WMAP satellite’s data released in February 2003 was consistent with this: $\Omega = 1.02^{+0.02}_{-0.02}$ [16].

6.8 Cooling down with expansion

The temperature of the present universe is about 2.728 K. The background radiation of the universe has a black-body spectrum, therefore the relation between radiation energy density $\varepsilon$ and temperature $T$ of the universe is

$$\varepsilon = \rho \text{rad} c^2 = \alpha T^4$$  \hspace{1cm} (64)

where $\alpha \equiv \pi^2 k_B^4/15 h^3 c^3$, $k_B$ is Boltzmann’s constant and $h$ is the reduced Planck constant. Equation (41) for radiation implies that $\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-4}$. This leads to the relation

$$T \propto \frac{1}{a}$$  \hspace{1cm} (65)

and by equation (35) we have

$$T \propto \frac{1}{t^{2/3(1+w)}}$$  \hspace{1cm} (66)

This results imply that the universe with dust or radiation domination should be hotter and smaller at earlier time. This is the main concept of the standard hot Big Bang cosmology.

6.9 Problems of the hot big bang model

Despite the fact that the hot Big Bang theory can explain the redshift, abundance of primordial nucleus of light elements and the existence of the CMB, there are still some puzzles that can not be explained in the hot Big Bang framework. These problems are as follows.

6.9.1 Flatness problem

Using the Friedmann equation (62) to compare the value of $|\Omega_{\text{tot}}(t) - 1|$ in the early universe and today, we can show that

$$|1 - \Omega_{\text{tot}}(t)| \propto a^2 \propto t$$  \hspace{1cm} (67)

during the radiation-domination era and

$$|1 - \Omega_{\text{tot}}(t)| \propto a \propto t^{2/3}$$  \hspace{1cm} (68)

in the dust-domination era. These equations tell us how the density parameter evolves with time.
These equations show that if $\Omega_{\text{tot}}$ is not exactly 1, then its difference from 1 grows with expansion. The universe observed today is very flat, with $|1 - \Omega_{\text{tot},0}| \sim 0.05$, therefore the universe at early time should be very very close to flat. Working out the equation (67) and (68), we would require the universe to be extremely flat e.g. $|\Omega_{\text{tot}} - 1| \sim 10^{-16}$ at nucleosynthesis. If the initial value of $\Omega_{\text{tot}}$ was not very close to 1, the universe would either re-collapse or expand and dilute very quickly and will not evolve to our universe today. The hot Big Bang theory is not able to explain how and why the universe was almost perfectly flat in the first place.

6.9.2 Horizon problem

The CMB across the sky looks roughly isotropic. However there is something unacceptable about this fact. The CMB was emitted when the universe was about 300,000 years old at recombination and the horizon at that time was about 300,000 light years which is approximately one degree in the sky today. Any area in universe that has size greater than 1 degree therefore should not be able to be thermalized by photons and hence the smooth-looking thermal-equilibrium CMB sky should not be seen today. This can not be explained by the hot Big Bang picture.

6.9.3 Magnetic monopole problem

The theory of particle physics has predicted many of the exotic particles that could be created in the early universe. Some examples are magnetic monopoles, gravitinos, moduli fields and other higher dimensional objects from topological defects e.g. cosmic strings, domain walls and textures. Where are they today?

6.9.4 Origin of structure problem

The CMB anisotropies observed first by COBE in 1992 can not be explained by the hot Big Bang picture [29]. The scale of the anisotropies is too large to be produced during the time from the Big Bang to the time of decoupling, because it should be thermalized according to the Big Bang picture. This means that anisotropies must have been a part the initial condition which is very unlikely. Since the anisotropies in CMB are signatures telling us how structures in the universe were formed, the hot Big Bang theory fails to provide an answer for a theory of structure formation.

7 Inflationary Universe

Inflation proposed by Guth, Sato, Albrecht, Steinhardt and Linde in 1980s is the period of the early universe that undergoes an accelerating phase [28]. Inflation of the universe is equivalent to

$$\ddot{a} > 0$$

(69)
so that $\dot{a}$ increases during the inflation phase. As a result the comoving Hubble length $(aH)^{-1}$ must be decreasing in this phase, i.e.

$$\frac{d}{dt}\left(\frac{1}{aH}\right) < 0 \quad (70)$$

The acceleration equation (24) requires that

$$\rho + 3p < 0 \Rightarrow p < -\frac{\rho}{3} \quad (71)$$

Inflation is able to solve the problems of the hot Big Bang as will be explained. It does not substitute the hot Big Bang idea but instead it adds on some ideas and also modifies the hot Big Bang model.

### 7.1 Solving the hot big bang problems

- **Flatness Problem** The flatness problem can be solved directly with inflation. In equation (62), to make the difference $\Omega_{\text{tot}} - 1$ smaller, $aH$ must increase. The accelerating expansion yields directly an increase in $aH (= \dot{a})$. If enough increment in $aH$ has been made by inflation to drive $\Omega_{\text{tot}}$ very close to 1, we can explain why the final density parameter value is very close to one.

- **Horizon Problem** To obtain a universe that looks almost isotropic today, we need a universe that has causal contact over the whole sky before inflation. In this scenario, during inflation any small region of the universe will inflate to a very large region. This can explain why today sky looks almost the same in all directions.

- **Monopole Problem** The super-fast expansion of inflation can dilute away these relics predicted by particle physics. That is why we do not detect them today.

- **Origin of Structure Problem** The scalar field that drives inflation (see below) experiences quantum fluctuations. These fluctuations are stretched by inflation to scales where they cause ripples in the CMB temperature. These ripples are the seeds that lead to the formation of structures and galaxies.

### 7.2 What drives inflation?

To produce inflation, some matter that can yield this acceleration phase has to be dominant at that period. The candidate from particle physics responsible for driving inflation is a scalar field, which is called the *inflaton field*, $\phi(t)$. The pressure and energy of the inflaton field are given by

$$p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi) \quad (72)$$

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) \quad (73)$$
where $V(\phi)$ is the potential energy. During the inflation phase the inflaton field dominates the universe. The scalar field-dominated Friedmann equation reads

$$H^2 = \frac{8\pi G}{3} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) \right]$$

(74)

If we use the scalar field pressure and energy density in the fluid equation, we obtain the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi)$$

(75)

where $'$ denotes differentiation with respect to $\phi$. This equation governs the dynamics and energy conservation of the inflaton field. The condition for acceleration from equation (71) requires that

$$\dot{\phi}^2 < V$$

(76)

and it implies that inflation can be sustained when the field moves very slowly, i.e. the kinetic energy $\dot{\phi}^2$ is very small. This is called the slow-roll approximation. We can use this approximation as a criterion for inflation to happen. With very small $\dot{\phi}^2$, the Friedmann equation is approximately

$$H^2 \simeq \frac{8\pi G}{3} V$$

(77)

and the Klein-Gordon equation gives

$$\dot{\phi} \simeq -\frac{V'}{3H}$$

(78)

Since $|V'|$ is very small, $H$ is nearly constant, so that $a(t)$ grows nearly exponentially ($\frac{\dot{a}}{a} \simeq$ constant). The amount of inflation can be measured in term of the e-folding number, $N$ given by equation (28). Normally at least 50 e-foldings are needed to solve the hot Big Bang problems. Refs. [23, 24] are recommended for further reading on inflation.

7.3 After inflation

7.3.1 Reheating

During inflation the universe is supercooled by very rapid expansion. Inflation stops when the field begins to roll faster down the potential and the slow-roll approximation breaks down. After inflation, the inflaton field begins to decay, producing matter and radiation. The radiation produced by this decaying starts to reheat the universe, providing the standard hot Big Bang phase.

7.3.2 Structure formation

Predicting the seeds for structure formation is the true merit of inflationary theory. Inflation creates perturbations which are in the form of density (scalar) perturbations and
gravitational waves. Density perturbations are created from quantum vacuum fluctuations of the inflaton field, which leave an imprint as inhomogeneities in the radiation and matter at reheating [30]. The wavelength of the fluctuations is stretched by inflation. The fluctuations therefore become bigger in size and then provide seeds of gravitational instability for structure to form. However, another type of perturbation, the gravitational wave, does not contribute to structure formation. These perturbations instead cause ripples in the geometry of space-time. More detail of the structure formation, Ref. [25] is recommended.

Observational cosmologists have information on the present structures in the universe, and they try to match cosmic structure patterns with the information found in the CMB anisotropies which were created in the early universe.

8 Conclusion

I have outlined the historical background and derived the cosmological equations using Newtonian cosmology. Basic ideas of the hot big bang and relativistic cosmology are also given here. Finally I have introduced briefly inflation and explained how structure forms in an elementary language.

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