RII NUMBER OF KNOT PROJECTIONS

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ABSTRACT. Every knot projection is simplified to the trivial spherical curve not increasing double points by using deformations of types 1, 2, and 3 which are analogies of Reidemeister moves of types 1, 2, and 3 on knot diagrams. We introduce RII number of a knot projection that is the minimum number of deformations of negative type 2 among such sequences. By definition, it is invariant under deformations of types 1 and 3. This is motivated by Östlund conjecture: Deformations of type 1 and 3 are sufficient to describe a homotopy from any generic immersion of a circle in a two dimensional plane to an embedding of the circle (2001), which implies RII number always would be zero. However, Hagge and Yazinski disproved the conjecture by showing the first counterexample with 16 double points, which implies that RII number is nontrivial. This paper shows that RII number can be any nonnegative number.

1. Introduction

A knot projection is the image of a generic immersion of a circle into the 2-sphere. Two knot projections are identified by ambient isotopy. Any self-intersection of a knot projection is a transverse double point and is simply called a double point. The trivial spherical curve is a knot projection with no double points. Deformations of types 1, 2, and 3 are local replacements defined in Fig. 1. This deformations are analogies of Reidemeister moves of knot diagrams. Every knot projection is related to the trivial spherical curve by a finite sequence of deformations of types 1, 2, and 3.

In 2001 [8], Östlund formulated a conjecture in the following:

Östlund Conjecture. Deformations of types 1 and 3 are sufficient to obtain a homotopy from any generic immersion $S^1 \to \mathbb{R}^2$ to an embedding.

In 2014, Hagge and Yazinski [2] disproved this conjecture as follows:

\[ \text{Date: Accepted August 1, 2019.} \]
\[ \text{Key words and phrases. knot projections; Östlund Conjecture; Reidemeister moves; spherical curves.} \]
\[ \text{MSC2020: 57K10, 57R42, 57K99.} \]
Hagge-Yazinski Theorem. For $P_{HY}$ that appears as Fig. 2, there is no finite deformations of types 1 and 3 from $P_{HY}$ to the trivial spherical curve up to ambient isotopies.

![Figure 2. Hagge-Yazinski’s example $P_{HY}$ having 16 double points](image)

In 2016, in [4], the authors obtain a generalization of the Hagge-Yazinski Theorem, that is, we show that there exists an infinite family of knot projections, which are counterexamples of the Östlund Conjecture for any $n$ double points ($n \geq 15$, for 15 double points, see Fig. 3, including $P(l,m,n)$ [4, Remark 1, Page 28, Fig. 13]. Throughout of this paper, let $P(m,n) = P(1,m,n)$ (Fig. 4).

![Figure 3. Our example having 15 double points](image)

We call a deformation of type 2 decreasing double points a deformation of negative type 2. Then the next fact is well known to the experts.

**Fact 1 ([7]).** Every knot projection is related to the trivial spherical curve by a finite sequence of deformations of types 1, negative 2, and 3.

In this paper, for a given knot projection $P$, we introduce RII number that is the minimum number of deformations of negative type 2 among sequences, each of which consists of deformations of type 1, negative type 2, and type 3. This number is denoted by $\text{RII}(P)$. By definition, Östlund conjecture implies RII number always would be zero. However, Hagge-Yazinski Theorem implies that RII number is nontrivial. In Section 4, we study 2 classes of knot projections called pretzel knot projections, and two-bridge knot projections, and show that every such knot projection $P$ satisfies $\text{RII}(P) = 0$ (Propositions 2 and 3). Theorem 1 implies that for any integer $m$ ($\geq 1$), there exists $P$ such that $\text{RII}(P) = m$.

Two knot projections are $(1, 3)$ homotopic if they are related by finite deformations of types 1 and 3 and ambient isotopies. The relation becomes an equivalence relation and is called $(1, 3)$ homotopy.
Proposition 1. Let $P$ be a knot projection. The number $\mathbb{RI}(P)$ is an invariant under $(1, 3)$ homotopy.

Theorem 1. For positive integers $m \geq 1$, and $n \geq 4$, let $P(m,n)$ be a knot projection, as in Fig. 4. Then, $\mathbb{RI}(P(m,n)) = m$ for any $(m,n)$.

Figure 4. $P(m,n)$. Note that $P(1,4) = P_{HY}$.

Corollary 1. There exists infinitely many $(1, 3)$ homotopy classes of knot projections.

For a proof of Theorem 1 see Section 3.

2. Preliminaries

Definition 1 (RI$^I$ number $\mathbb{RI}(P)$). Let $P$ be a knot projection. The RI$^I$ number is the minimum number of deformations of negative type 2 among sequences, each of which consists of deformations of type 1, negative type 2, and type 3. The number is denoted by $\mathbb{RI}(P)$.

Definition 2 ((3,3)-tangle). An unoriented (3,3)-tangle is the image of a generic immersion of 3 arcs into $[0,1] \times [0,1]$ such that:

- The boundary points of the arcs map bijectively to 6 points
  \[ \left\{ \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \right\} \times \{1\}, \left\{ \frac{1}{4}, \frac{2}{4}, \frac{3}{4} \right\} \times \{0\} \]

- Near the endpoints, the arcs are perpendicular to the boundary $[0,1]$.

In a (3,3)-tangle, we call an image of the map of a single arc a strand.

Definition 3. Let $P$ be a knot projection and $F$ the closure of a connected component in $S^3 \setminus P$. Let $n$ be a positive integer. Suppose that the double points of $P$ that lie on $\partial F$ are removed, the reminder consists of $n$ connected components, each of which is homeomorphic to an open interval. Then, $\partial F$ is called an $n$-gon. When we do not specify $n$, an $n$-gon is called a polygon.
Notation 1 ([1]). Let $P$ and $P'$ be two knot projections that are equivalent under deformations of type 1 and type 3, i.e., there exists a finite sequence of knot projections $P = P_0, P_1, \ldots, P_m = P'$, where $P_i$ is obtained from $P_{i-1}$ by a deformation of type 1 or type 3. Then, $Op_i$ denotes the deformation from $P_{i-1}$ to $P_i$, and the setting are expressed by using the notation:

$$P = P_0 \xrightarrow{Op_1} P_1 \xrightarrow{Op_2} \cdots \xrightarrow{Op_m} P_m = P'.$$

3. Proofs of Proposition 1 and Theorem 1

Before starting the proof, we explain our plan of the proof.

- We show that $RI(P)$ is invariant under $(1, 3)$ homotopy.
- We show that $P(m, n) \geq m$. Most of the part is obtained from a similar proof of Hagge-Yazinski Theorem [2] or [4]. Therefore, a reader who is familiar to [2] can skip this part except for Section 3.6. Since it is elementary to prove it, we prove it here. We use the terminology used in [2].
- We show that $P(m, n) \leq m$. We introduce new techniques and show this part.

**Proof of Proposition 1** Let $P$ and $P'$ be a pair of $(1, 3)$ homotopic knot projections. Let $m = RI(P)$, hence, there exists a sequence of deformations consisting of deformations of type 1, negative type 2, or type 3, and it contains exactly $m$ deformations of negative type 2. By combining the deformations from $P$ and $P'$, and $P'$ to the trivial spherical curve, we obtain a sequence from $P$ to the trivial spherical curve which contains exactly $m$ deformities of negative type 2. This shows that $RI(P) \leq RI(P')$. Since the argument is symmetric, we see that $RI(P) \geq RI(P')$ holds, too. Hence, $RI(P) = RI(P')$.

3.1. Proof of $P(m, n) \geq 1$. For any $P(m, n)$, there exist $2n$ boxes such that the intersections of $P(m, n)$ and $2n$ boxes are $(3, 3)$-tangles, and there are no double points outside the boxes (Figs. 4 and 5). Each box together with the portion of $P(m, n)$ contained in the box is equivalent to $[0, 1] \times [0, 1]$ corresponding to a $(3, 3)$-tangle. For example, for the case $m = 1$ and $n = 4$, 8 boxes are shown in Fig. 5. Each $P(m, n)$ satisfies the following two conditions:

![Figure 5](image)

(1) No double points are placed outside the boxes. Three arcs connect two adjacent boxes concentrically. There exist exactly two polygons, each of which has at least $n$ sides partially outside boxes.
(2) If we fix our gazing direction from infinity, which is selected as shown Figs. 4 and 5 we define the left-side and right-side of each box. In each box, strand 1 (2, resp.) is a strand that begins and ends on the left-side (right-side, resp.). In each box, strand 3 has one endpoint on the left-side and another endpoint the right-side. There exist 2n pairs of strand 1 and strand 2; for each pair, strand 1 and strand 2 intersect at exactly 2m double points. Strand 1 of a pair cannot intersect strand 2 belonging to a different pair.

Let m and n be positive integers (m ≥ 1, n ≥ 4). In general, for a knot projection Q, we say that Q satisfies (2m, 2n) box property if there exist 2n boxes B_i (1 ≤ i ≤ 2n) in S^2, as shown in Fig. 6, satisfying the following (A) and (B).

(A) There exist no double points of Q in S^2 \ (B_1 ∪ B_2 ∪ ··· ∪ B_{2n}), and Q ∩ (S^2 \ (B_1 ∪ B_2 ∪ ··· ∪ B_{2n})) consists of 3 × 2n simple arcs as in Fig. 6.

(B) For each i (1 ≤ i ≤ 2n), Q ∩ B_i is a (3, 3)-tangle that is the union of three immersed arcs. Then it satisfies the following condition.

- The endpoints of the three arcs are located on ∂B_i as in one of the figures of Fig. 7, where we name the immersed arcs 1, 2, and 3 as in Fig. 7. Further, there exist at least 2m double points formed by a subarc of 1, and a subarc of 2.

- Proof of RII(P(m, n)) ≥ 1. We consider an inductive proof with respect to the number of deformations, type 1 or 3, applied to P(m, n). This induction proves Claim 1 which implies that P(m, n) cannot be (1, 3) homotopic to the trivial spherical curve; that is, RII(P(m, n)) ≥ 1. Recall Notation 1.

Claim 1. Let Q be a knot projection satisfying a (2m, 2n) box property. Then, for each sequence of knot projections

\[ Q = Q_0 \xrightarrow{O_{P_1}} Q_1 \xrightarrow{O_{P_2}} \cdots \xrightarrow{O_{P_r}} Q_r \]

such that each O_{P_i} (1 ≤ i ≤ r) is of type 1 or 3, we have:

by retaking the boxes if necessary, each Q_i (1 ≤ i ≤ r) satisfies (2m, 2n) box property.

First, for Q_0, it is clear that Claim 1 holds. Second, we suppose that Claim 1 holds for Q_{r-1} and prove it Q_r.

3.2. On a deformation of type 1 or type 3 occurring within a box. Suppose that the O_{P_r} is a single deformation (type 1 or type 3) occurring entirely within a box. This deformation fixes the endpoints of the strands, and thus Claim 1 holds.
3.3. On a deformation of type 1 not occurring within a box and increasing double points. Suppose that $O_{p_r}$ is a single deformation of type 1 increasing double points. If the new 1-gon produced by $O_{p_r}$ is outside the boxes, by retaking the box by a sphere isotopy, as shown in Fig. 8, $O_{p_r}$ is entirely within a box. If the new 1-gon produced by $O_{p_r}$ is not completely outside the boxes, a similar modification works to retake the box.

3.4. On a deformation of type 1 not occurring within a box and decreasing double points. Suppose that the $O_{p_r}$ does not occur within a box. Since $O_{p_r}$ is a deformation of type 1 decreasing double points, there exists a 1-gon to be removed in $Q_{r-1}$. The two possibilities of appearing of a 1-gon are considered as follows.

- Suppose that the 1-gon contains a region having one side, as shown in Fig. 9. By the induction assumption of $P_{r-1}$, the region has at least four sides, which is a contradiction.
- Suppose that there exists the 1-gon containing a region having two sides outside the boxes, as shown in Fig. 10. If one of the two sides is directly connected to either strand 1 or 2, then the 1-gon has at least two double points, which is a contradiction (there is no 1-gon with two double points).
If one of the two sides is directly connected to strand 3 that is connected strand 1 or 2 in the adjacent box, then this also implies a contradiction which is similar to the case above.

3.5. On a deformation of type 3 not occurring within a box. If $O_{p_{r}}$ is a single deformation of type 3, we focus on the 3-gon $T_{r-1}$ in $Q_{r-1}$ with respect to the single deformation of type 3, as shown in Fig. 11.

- Suppose that $T_{r-1}$ contains a region having one side, as shown in Fig. 9. By the induction assumption of $Q_{r-1}$, the region has at least four sides, which is a contradiction.
- Suppose that $T_{r-1}$ contains a region having two sides outside the boxes, as shown in Fig. 12.
Figure 11. 3-gon $T_{r-1}$ in $Q_{r-1}$ with respect to a single deformation of type 3

Figure 12. A region surrounded by the 3-gon $T_{r-1}$ appearing in $Q_{r-1}$

(1) Suppose that no double point of $T_{r-1}$ is in a box, as shown in Fig. 13. However, this situation is prohibited by (B) of $(2m, 2n)$ box property.

Figure 13. Impossible case

(2) Suppose that $T_{r-1}$ has at least one double point in a box. By the induction assumption, there is no double point outside the boxes for $Q_{r-1}$. Thus, if $T_{r-1}$ is not inside a box, there are exactly two cases.

(a) Case 1. A double point of $T_{r-1}$ is inside a box and the other two double points of $T_{r-1}$ are in another box, as shown in the left figure of Fig. 14.

(b) Case 2. The three double points of $T_{r-1}$ are in three different boxes, respectively, as shown in the right figure of Fig. 14.
As a result, for Case 1 (Case 2, resp.), by retaking one box (two boxes, resp.), as shown in Fig. 15, $T_{r-1}$ is contained entirely within a box. Note that if $Q_{r-1}$ satisfies (B) of $(2m, 2n)$ box property, then $Q_r$ satisfies (B) of $(2m, 2n)$ box property since every connection preserves in each box and the positions of endpoints on $\partial B_i$ also preserve up to ambient isotopy under the deformation.

This completes the proof of Claim 1, which implies that $\text{RI}(P(m, n)) \geq 1$.

3.6. Proof of $\text{RI}(P(m, n)) \geq m$. Recall Claim 1. By the argument of the above proof of Claim 1 we have Lemma 1.

**Lemma 1.** Let $k$ be an integer $(0 \leq k \leq r)$. Let $Q_0 = P(m, n)$ and $Q_k$ be the knot projection obtained from $P(m, n)$ by $\text{Op}_k$ using Notation 1:

$$P(m, n) = Q_0 \xrightarrow{\text{Op}_1} Q_1 \xrightarrow{\text{Op}_2} \ldots \xrightarrow{\text{Op}_r} Q_r,$$

where the sequence consists of a single deformation of negative type 2 and deformations of type 1 and 3. If $\text{Op}_{k+1}$ is a deformation of negative type 2 within a box, then we have the following statements.

1. Each of $\{Q_0, Q_1, \ldots, Q_k\}$ preserves $(2m, 2n)$ box property.
2. $Q_{k+1}$ satisfies $(2m - 2, 2n)$ box property.
3. Each of $\{Q_{k+2}, Q_{k+3}, \ldots, Q_r\}$ preserves $(2m - 2, 2n)$ box property.
If a deformation $O_{p_{k+1}}$ of negative type 2 is not within a box, then, by retaking boxes, the case returns to the case that $O_{p_{k+1}}$ is a deformation of negative type 2 within a box.

Proof. Suppose that $O_{p_{k+1}}$ is a deformation of negative type 2 within a box.

1. (3), resp.) By using the argument of the proof of Claim 1, we may suppose that every $O_{p_j}$ $(j \neq k + 1)$ is a deformation of type 1 or 3 within a box by retaking boxes. Since $O_{p_j}$ of type 1 or 3 within a box preserve $(2m, 2n)$ $(2m − 2, 2n)$, resp.) box property, the statement (1) (3, resp.) holds.

2. The deformation of negative type 2 decreases or preserve double points formed by a subarc of 1 and a subarc of 2. It implies the statement (2).

Suppose that a deformation $O_{p_{k+1}}$ of negative type 2 is not within a box. Then, by retaking one box (two boxes, resp.), as shown in Fig. 16 the case returns to the case that $O_{p_{k+1}}$ is a deformation of negative type 2 within a box. □

Figure 16. A bigon to which will be applied a deformation of negative type 2 not occurring within a box

Lemma 1 immediately implies $R(P(m,n)) \geq m$.

3.7. Proof of $P(m,n) \leq m$. We prepare Lemma 2

Lemma 2. Let $k$ be a positive integer. Each of replacements $T(2k − 1)$ and $T(2k)$ as in Fig. 17 is always possible under (1, 3) homotopy.

Proof. We prove the statement of the lemma by the induction.

- Case $k = 1$: for $T(1)$, the statement is clear. $T(2)$ holds as in Fig. 18
- Case $k = i$: Suppose that $T(2i − 2)$ is a possible local replacement under (1, 3) homotopy. In this case, $T(2i − 1)$ holds, which also implies $T(2i)$. See Fig. 19

It completes the induction step. □

- $\text{Proof of } R(P(m,n)) \leq m$. Let $x_1$ $(x_k$, resp.) be the twisting part, including exactly $2m$ double points, corresponding to the first $(k\text{-th, resp.) box from the bottom of Fig. 20}$. First, we apply $m$ deformations of negative type 2 to the twisting part in a single box $x_1$, which erases $x_1$. Second, by applying $T(2m)$ to $x_3$ and the leftmost single double point of $x_2$, the number of double points in $x_2$ decreases by 1. Repeating this argument $2m − 1$ times, we complete erasing a single box $x_3$. By applying $T(2m)$ to $x_4$ and the rightmost single double point of $x_3$, the number of double points in $x_3$ decreases by 1. Repeating this argument $2m − 1$
Remark 1. The operation $T(n)$ ($n \in \mathbb{Z}_{>0}$) of Lemma 2 corresponds to a generalization of a type of an edge of a complex [1] or the operation $\alpha$ [3].

4. Applications

4.1. Every pretzel knot projection is (1, 3) homotopic to the trivial spherical curve.

Notation 2. A part consisting of $m$ double points of a knot projection as in Fig. 4 it is called a twist.

Proposition 2. Every pretzel knot projection as in Fig. 22 is (1, 3) homotopic to the trivial spherical curve.
Proof. Let $P(a_1, a_2, \ldots, a_n)$ be a knot projection as in Fig. 22 where each $a_i$ presents double points of a twist ($1 \leq i \leq n$).

It is sufficient to consider the two cases.
Case 1: for each \( i \), \( a_i \) are odd double points and \( n \) is an odd positive number.

Case 2: \( a_1 \) are even double points and for \( i \neq 1 \), \( a_i \) are odd double points.
It is easy to see that every \((2, 2p + 1)\)-torus knot projection \((p \in \mathbb{N})\), as shown in Fig. 23, is \((1, 3)\) homotopic to the trivial spherical curve by applying \(T(2p + 1)\) and applying deformations of type 1 decreasing double points.

For Case 1, we apply \(T(a_i)\) to each twist, and we have a \((2, 2p + 1)\)-torus knot projection where \(2p + 1 = \sum_{i=1}^{n} a_i\). For Case 2, since \(a_i \neq 1\) is an odd number, we apply \(T(a_i)\) to each twist corresponding to \(a_i\) double points \((i \neq 1)\), and apply \(T(a_1)\) \((\sum_{i=2}^{n} a_i\) times\) to the resulting knot projection, which implies a knot projection having exactly \(a_1\) double points. After that, deformations of type 1 decreasing double points obtain the trivial spherical curve.

\section*{4.2. Every two-bridge knot projection is \((1, 3)\) homotopic to the trivial spherical curve.}

\begin{proposition}
Every two-bridge knot projection as in Fig. 24 is \((1, 3)\) homotopic to the trivial spherical curve.
\end{proposition}

\begin{proof}
Let \(P(a_1, a_2, \ldots, a_n)\) be a knot projection as in Fig. 24 where each \(a_i\) presents double points of a twist \((1 \leq i \leq n)\).

- Case 1: Suppose that \(a_1\) is an odd number corresponding to \(a_1\) double points. By applying a single \(T(a_1)\), two twists \(a_1\) and \(a_2\) merge a twist consisting of \(a_1 + a_2\) double points.
- Case 2: Suppose that \(a_1\) is an even number corresponding to \(a_1\) double points. By applying \(T(a_1)\) \(a_2\) times, \(a_2\) double points are resolved, which implies a twist disappears.

By an induction of the number of twists, we resolve every twist.
\end{proof}

\begin{remark}
By using Lemma 2 for a knot projection \(P\) consisting of two-bridge knot projections via tangle sums or connected sums, it is elementary to show that such \(P\) is \((1, 3)\) homotopic to the trivial spherical curve by using Propositions 2 and 3.
\end{remark}

\section*{Acknowledgements}
We would like to explain the detail of the paper. First, Professor Tsuyoshi Kobayashi shared with a progress of the study of Ms. Sumika Kobayashi for her
Master thesis, on September 20, 2018. This work included $T(2k - 1)$. Then, we
sent a preprint (corresponding to an earlier version of this paper) with a slide of a
talk in a Colloquium of Department of Mathematics, Faculty of Education, Waseda
University on December 18, 2015, which includes Lemma 2. The motivations of the
two works, [5, 6] and this paper are different. The author would like to thank
Professor Tsuyoshi Kobayashi for encouraging us to write this paper and giving
us an information of [5, 6]. Ms. Sumika Kobayashi gave another application of
Lemma 2 by other motivation of hers [5, 6].

The authors also would like to thank the referee for the comments.

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