Algebraic CMC hypersurfaces of order 3 in Euclidean spaces

Oscar Perdomo and Vladimir G. Tkachev

Abstract. Understanding and finding of general algebraic constant mean curvature surfaces in the Euclidean spaces is a hard open problem. The basic examples are the standard spheres and the round cylinders, all defined by a polynomial of degree 2. In this paper, we prove that there are no algebraic hypersurfaces of degree 3 in \( \mathbb{R}^n \), \( n \geq 3 \), with nonzero constant mean curvature.

Mathematics Subject Classification. 53C42, 53A10.
Keywords. Constant mean curvature, algebraic surfaces.

1. Introduction

There are plenty of examples of algebraic zero mean curvature (i.e. minimal) hypersurfaces: the classical Enneper degree 9 and Henneberg degree 15 minimal surfaces \([7]\) in \( \mathbb{R}^3 \), the minimal cones of degree 2 over the Clifford tori \( S^k \times S^m \) in \( \mathbb{R}^{n+m+1} \) and isoparametric cones of degrees 2, 3, 4, 6 in \( \mathbb{R}^n \), to name only few; see also \([1, 5, 8, 11–13]\) and references therein.

On the other hand, the problem of finding nonzero constant mean curvature (CMC) algebraic hypersurfaces in the Euclidean space is a hard open problem. The standard spheres \( S^n \) and the round cylinders \( S^k \times \mathbb{R}^{n-k} \) are the only algebraic hypersurfaces in the Euclidean space \( \mathbb{R}^{n+1} \) with constant \( (\neq 0) \) mean curvature known so far. All these hypersurfaces are given by a polynomial equation of degree 2.

Understanding of general algebraic CMC surfaces in the Euclidean spaces is a basic question that has very little progress. For the three dimensional space \( \mathbb{R}^3 \), the first author showed in \([9]\) that there are no degree 3 algebraic CMC surfaces. Also, for surfaces in the Euclidean space \( \mathbb{R}^3 \), Do Carmo and Barbosa proved \([2]\) that if \( M = f^{-1}(0) \) with \( f \) a polynomial and, \( \nabla f \) never vanishes
on \( M \), then \( M \) cannot have CMC unless \( M \) is a plane, a cylinder or a sphere. On the other hand, very little is known on the classification of all immersed algebraic surfaces in \( \mathbb{R}^3 \) or the classification of algebraic CMC hypersurfaces in \( \mathbb{R}^n \); see, however, a recent classification of semialgebraic CMC surfaces in \( \mathbb{R}^3 \) with isolated singularities in [10].

In this paper we will establish that the class of CMC hypersurfaces in \( \mathbb{R}^n \) given by a polynomial equation of degree 3 is empty. In [9] it was proven that if \( \phi : M \to \mathbb{R}^n \) is an immersion with constant mean curvature \( H \neq 0 \) and \( \phi(M) = f^{-1}(0) \) where \( f \) is an irreducible polynomial, and, for at least one point \( x_0 \in \phi(M) \), \( \nabla f(x_0) \) does not vanishes, then

\[
4(n-1)^2H^2|\nabla f|^6 - (2|\nabla f|^2\Delta f - \nabla f^t \nabla |\nabla f|^2)^2 = p f
\]

with \( p \) a polynomial.

We will say that an irreducible polynomial \( f \) defines an algebraic CMC hypersurface if \( f \) satisfies condition (1) with some \( H \neq 0 \).

Notice that \( f^{-1}(0) \) may have singular points. If \( x_0 \) is a regular point of \( f \) then (1) implies that \( f^{-1}(0) \) is a CMC hypersurface in a neighbourhood of \( x_0 \). With this definition in mind we show that there are not algebraic CMC hypersurfaces in the Euclidean \( n \)-dimensional space of degree 3.

**Theorem 1.1.** If \( f : \mathbb{R}^n \to \mathbb{R} \) is an irreducible polynomial of degree three, then, the zero level set of \( f \) cannot be an algebraic CMC hypersurface. That is, \( f \) cannot satisfy Eq. (1) with \( H \neq 0 \).

Some further remarks are worth making at this point. Recall that

\[
\Delta_p f := |\nabla f|^2 \Delta f + \frac{p-2}{2} \nabla f^t \nabla |\nabla f|^2 = 0
\]

is called the \( p \)-Laplace equation. Then for \( H = 0 \), the CMC Eq. (1) is closely related to the 1-Laplace equation, which appears very naturally in the context of minimal cones, see section 6 in [6]. The assumption \( H \neq 0 \) is crucial, because otherwise, when \( n > 3 \), there exists irreducible algebraic minimal \((H = 0)\) hypersurfaces of any arbitrarily higher degree, see for instance [13].

On the other hand, it is interesting to compare Theorem 1.1 with a similar situation for polynomial solutions of the general \( p \)-Laplacian equation, \( p \neq 2 \). It follows from [3] and the recent results in [4,14] that there are no homogeneous polynomial solutions to (2) in \( \mathbb{R}^n \) of degree \( d = 2, 3, 4 \) for any \( n \geq 2 \), of degree 5 in \( \mathbb{R}^3 \) and also of any degree in \( \mathbb{R}^2 \).

### 2. Proof of the main result

Before proving the Theorem 1.1, let us prove some lemmas.

**Lemma 2.1.** If \( g : \mathbb{R}^n \to \mathbb{R} \) is a polynomial of degree 3 and \( |\nabla g| \) vanishes anytime \( g \) vanishes, then \( g = l^3 \) where \( l \) is a linear function.
Proof. Lemma 2.5 in [13] states that if $g$ is a cubic irreducible polynomial then $g^{-1}(0)$ contains at least one regular point. Therefore, we conclude that our $g$ cannot be irreducible and then, for some polynomial $u$ of degree 2, we have that $g = uv$ where $v$ is linear. Since $\nabla v$ never vanishes and $\nabla g = u\nabla v + v\nabla u$ we conclude that $u(x) = 0$ whenever $v(x) = 0$. From the real nullstellensatz theorem we conclude that $u = wv$ for some polynomial $w$. Thus $g = wv^2$. Applying the same argument that we did above, using the polynomial $w$ instead of the polynomial $v$, we conclude that $v^2$ must be a multiple of $w$. Therefore the lemma follows. □

Proof of Theorem 1.1. Without loss of generality we can assume that the origin is in an element in the hypersurface, in other words, we assume that $f(0, \ldots, 0) = 0$. Let us write

$$f = f_3 + f_2 + f_1 \quad \text{and} \quad p = p_0 + \cdots + p_1 + p_0$$

where $f_i = f_i(x)$ and $p_i = p_i(x)$ are homogeneous polynomials of degree $i$. Comparing the degree 12 homogeneous part in both sides of equation (1), we conclude that

$$\tilde{H}^2|\nabla f_3|^6 = p_9 f_3,$$

where $\tilde{H} = 2(n-1)H \neq 0$. (3)

Using Lemma 2.1, we conclude that $f_3 = l^3$ where $l$ is linear. Since $f(0) = 0$, then $l(0) = 0$. By doing a rotation and a dilation of the coordinates, if necessary, we can assume that $l = x_1$. For notation sake, let us denote the coordinates in $\mathbb{R}^n$ as $x = x_1$ and $y = (y_1, \ldots, y_{n-1})^t = (x_2, \ldots, x_n)^t$. With this notation we have the for the homogenous parts in $f = f_3 + f_2 + f_1$:

$$f_3 = x^3$$
$$f_2 = y^tAy + k_0x^2 + y^trx$$
$$f_1 = k_1x + y^ts,$$

where $A$ is a symmetric $(n-1) \times (n-1)$ matrix, $r = (r_1, \ldots, r_{n-1})^t$ and $s = (s_1, \ldots, s_{n-1})^t$. (4)

At this moment we would like to point out that to finish the theorem it is enough to show that the matrix $A$ is zero. The reason for this is that if $A$ is the zero matrix and we relabel the axis so that the vectors $r$ and $s$ are in the plane spanned by the vectors $(1, 0, \ldots, 0)^t$ and $(0, 1, 0, \ldots, 0)^t$, then, the function $f$ would be a function that depends only on the three variables $x, y_1$ and $y_2$ and, by [9] we already know that there are not algebraic CMC surfaces in $\mathbb{R}^3$.

Let us continue the proof by showing that the matrix $A$ must be the zero matrix. A direct computation yields the following decomposition into homogeneous parts:

$$|\nabla f|^2 = h_0 + h_1 + h_2 + h_3 + h_4,$$

where $h_k$ is a homogeneous polynomial of degree $k$ given explicitly by

$$h_0 = k_1^2 + |s|^2$$
$$h_1 = 4s^tAy + 4k_0k_1x + 2k_1(r^ty) + 2x r^ts$$

$$h_2 = 2k_1^2x + 2k_1(r^ty) + 2x r^ts$$
$$h_3 = 2s^tAy + 2k_0k_1x + 2k_1(r^ty) + 2x r^ts$$

$$h_4 = 2k_1^2x + 2k_1(r^ty) + 2x r^ts$$

where $h_0, h_1, h_2, h_3, h_4$ are homogeneous polynomials of degree 0, 1, 2, 3, 4, respectively. (5)
Using this representation, we obtain

\[
\Delta_1 f := 2|\nabla f|\Delta f - \nabla f^t \nabla |\nabla f|^2
\]

\[
= 4|\nabla f|^2 \text{Tr} A + 16(k_0 + 3x)(s^t Ay + |Ay|^2) - 8r^t Ay (k_1 + r^t y - 3x^2)
\]

\[
- 8x s^t Ar - 4x^2 r^t Ar - 16x r^t A^2 y - 16s^t A^2 y - 4s^t A s - 16y^t A^3 y
\]

\[
- 4x(k_0 x + k_1 + r^t y)|r|^2 + 4(k_0 + 3x)|s|^2 - 4r^t s (k_1 + r^t y - 3x^2),
\]

which implies that

\[
\Delta_1 (f) \equiv 4 \text{Tr} A |\nabla f|^2 \mod \text{Pol}_3,
\]

where we follow an agreement to write

\[
A \equiv B \mod \text{Pol}_k
\]

if \(A - B\) is a polynomial of degree \(k\) or less.

It also follows from (5) that

\[
|\nabla f|^4 \equiv 81 x^8 \mod \text{Pol}_7
\]

Thus, using the made observations, equation (1) becomes

\[
(p_0 + p_1 + \cdots + p_9)(f_1 + f_2 + f_3) \equiv \tilde{H}^2 |\nabla f|^6 - 6^4 (\text{Tr} A)^2 x^8 \mod \text{Pol}_7.
\]

(6)

Our next goal is by using the decomposition for \(|\nabla f|^2\) in terms of the \(h_i\)'s and

and the expression for \((\Delta_1 f)^2 - 4(n - 1)^2 H^2 |\nabla f|^6\) in terms of the \(|\nabla f|^2\) up to order 8, deduce that the matrix \(A\) should be the zero matrix. To this end, we consider (6) as a polynomial identity with respect to a variable \(x\) over the ring \(\mathbb{R}[y]\).

A key observation is that since \(h_4 = Q_0 x^4\) and \(h_3 = Q_2 x^2\), where \(Q_i\) is a homogeneous polynomial of degree \(i\), one immediately obtains from (5) the following homogenous decomposition:

\[
\tilde{H}^2 |\nabla f|^6 - 6^4 (\text{Tr} A)^2 x^8 = \sum L_i x^{k_i} \mod \text{Pol}_7.
\]

(7)

Here \(L_i\) are homogeneous polynomials of degree \(i\). In particular,

\[
L_0 = 3^6 \tilde{H}^2 \neq 0.
\]

(8)

Next, identifying the homogeneous parts of degrees \(k, 8 \leq k \leq 12\), in both sides of (6) one obtains respectively

\[
p_9 f_3 = L_0 x^{12}
\]

\[
p_8 f_3 + p_9 f_2 = L_1 x^{10}
\]

\[
p_7 f_3 + p_8 f_2 + p_9 f_1 = L_2 x^8
\]
\[ p_6 f_3 + p_7 f_2 + p_8 f_1 = L_3 x^6 \]
\[ p_5 f_3 + p_6 f_2 + p_7 f_1 = L_4 x^4. \]  

(9)

The first two equations yield
\[ p_9 = L_0 x^9 \]

and
\[ p_8 = S_2 x^6, \]

where \( S_2 = L_1 x^2 - L_0 f_2 \). Arguing similarly, one easily finds that
\[ p_7 = S_4 x^3, \]
\[ p_6 = (L_3 - S_2 f_1) x^3 - S_4 f_2, \]

where \( S_4 = L_2 x^2 - L_0 f_1 S x^3 - S_2 f_2 \). Thus, evaluating the last identity of (9) for \( x = 0 \) and taking into account that \( f_3(0) = p_7(0) = 0 \) we obtain
\[ p_6(0)f_2(0) = 0. \]

Since
\[ p_6(0) = -S_4(0)f_2(0) = S_2(0)f_2^2(0) = -L_0 f_2^3(0), \]

it follows by (8) that \( f_2(0) = 0 \), therefore using (4) we obtain
\[ f_2(0) = y^t A y = 0 \text{ for any } y \in \mathbb{R}^{n-1}. \]

Since \( A \) is symmetric, we have \( A = 0 \). As explained before, after sowing that \( A \) vanishes we have that \( f \) essentially depends on three variable and therefore it cannot define an algebraic hypersurface with constant mean curvature. The theorem is proved.

\[ \square \]

**Acknowledgements**

This work was done while the first author visited Linköping University. He would like to thank the Mathematical Institution of Linköping University for hospitality. The second author acknowledges support from G D Magnusons Fond, MG2017-0101.

**Compliance with ethical standards**

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.
Open Access. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Hoppe, J., Linardopoulos, G., Turgut, O.T.: New minimal hypersurfaces in $\mathbb{R}^{(k+1)(2k+1)}$ and $\mathbb{S}^{2k^2+3k}$. Math. Nachr. 290(17–18), 2874–2878 (2017)

[2] Lucas, J., Barbosa, M., do Carmo, M.P.: On regular algebraic surfaces of $\mathbb{R}^3$ with constant mean curvature. J. Differ. Geom. 102(2), 173–178 (2016)

[3] Lewis, J.L.: Smoothness of certain degenerate elliptic equations. Proc. Am. Math. Soc. 80(2), 259–265 (1980)

[4] Lewis, J.L., Vogel, A.: On $p$-Laplace polynomial solutions. J. Anal. 24(1), 143–166 (2016)

[5] López, R.: Surfaces with constant mean curvature in Euclidean space. Int. Electron. J. Geom. 3(2), 67–101 (2010)

[6] Nadirashvili, N., Tkachev, V.G., Vlăduț, S.: Nonlinear Elliptic Equations and Nonassociative Algebras, vol. 200. Mathematical Surveys and Monographs AMS, Providence (2014)

[7] Nitsche, J.C.C.: Lectures on Minimal Surfaces, vol. 1. Cambridge University Press, Cambridge (1989). Introduction, fundamentals, geometry and basic boundary value problems. Translated from the German by Jerry M. Feinberg, With a German foreword

[8] Odehnal, B.: On algebraic minimal surfaces. KoG 20(20), 61–78 (2017)

[9] Perdomo, O.M.: Algebraic constant mean curvature surfaces in Euclidean space. Houst. J. Math. 39(1), 127–136 (2013)

[10] Sampaio, J.E.: Semialgebraic cmc surfaces in $\mathbb{R}^3$ with singularities. https://bird.bcamath.org/handle/20.500.11824/844 (2018) (preprint)

[11] Small, A.: Algebraic minimal surfaces in $\mathbb{R}^4$. Math. Scand. 94(1), 109–124 (2004)

[12] Small, A.: On algebraic minimal surfaces in $\mathbb{R}^3$ deriving from charge 2 monopole spectral curves. Int. J. Math. 16(2), 173–180 (2005)

[13] Tkachev, V.: Minimal cubic cones via Clifford algebras. Complex Anal. Oper. Theory 4(3), 685–700 (2010)

[14] Tkachev, V.G.: On the non-vanishing property for real analytic solutions of the $p$-Laplace equation. Proc. Am. Math. Soc. 144(6), 2375–2382 (2016)
Oscar Perdomo  
Department of Mathematics  
Central Connecticut State University  
New Britain  
CT 06050  
USA  
e-mail: perdomoosm@ccsu.edu

Vladimir G. Tkachev  
Department of Mathematics  
Linköping University  
58183 Linköping  
Sweden  
e-mail: vladimir.tkatjev@liu.se

Received: September 14, 2018.