Stability Issues for $w < -1$ Dark Energy

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Abstract

Precision cosmological data hint that a dark energy with equation of state $w = P/\rho < -1$ and hence dubious stability is viable. Here we discuss for any $w$ nucleation from $\Lambda > 0$ to $\Lambda = 0$ in a first-order phase transition. The critical radius is argued to be at least of galactic size and the corresponding nucleation rate glacial, thus underwriting the dark energy’s stability and rendering remote any microscopic effect.
The equation of state for the dark energy component in cosmology has been the subject of much recent discussion [1–9]. Present data are consistent with a constant \( w(Z) = -1 \) corresponding to a cosmological constant. But the data allow a present value for \( w(Z = 0) \) in the range \(-1.38 < w(Z = 0) < -0.82 \) [1]. If one assumes, more generally, that \( w(Z) \) depends on \( Z \) then the allowed range for \( w(Z = 0) \) is approximately the same [8]. In the present article we shall forgo this greater generality as not relevant. Instead, in the present article we address the question of stability for a dark energy with constant \( w(Z) < -1 \).

**Interpretation as a limiting velocity**

Consider making a Lorentz boost along the 1-direction with velocity \( V \) (put \( c = 1 \)). Then the stress-energy tensor which in the dark energy rest frame has the form:

\[
T_{\mu\nu} = \Lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{pmatrix}
\]

is boosted to \( T'_{\mu\nu} \) given by

\[
T'_{\mu\nu} = \Lambda \begin{pmatrix} 1 & V & 0 & 0 \\ V & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{pmatrix} = \Lambda \begin{pmatrix} 1 + V^2w & V(1 + w) & 0 & 0 \\ V(1 + w) & V^2 + w & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{pmatrix}
\]

We learn several things by studying Eq.(2). First, consider the energy component \( T'_{00} = 1 + V^2w \). Since \( V < 1 \) we see that for \( w > -1 \) this is positive \( T'_{00} > 0 \) and the Weak Energy Condition (WEC) is respected [10]. For \( w = -1 \), \( T'_{00} \to 0 \) as \( V \to 1 \) and is still never negative. For \( w < -1 \), however, we see that \( T'_{00} < 0 \) if \( V^2 > -1/w \) and this violates...
the WEC and is the first sign that the case $w < -1$ must be studied with great care. Looking at the pressure component $T'_{11}$ we see the special role of the case $w = -1$ because $w = T'_{11}/T'_{00}$ remains Lorentz invariant as expected for a cosmological constant. Similarly the off-diagonal components $T'_{01}$ remain vanishing only in this case. The main concern is the negativity of $T'_{00} < 0$ which appears for $V^2 > -(1/w)$. One possibility is that it is impossible for $V^2 > -(1/w)$. The highest velocities known are those for the highest-energy cosmic rays which are protons with energy $\sim 10^{20}eV$. These have $\gamma = (1 - V^2)^{-1/2} \sim 10^{11}$ corresponding to $V \sim 1 - 10^{-22}$. This would imply that:

$$w > -1 - 10^{-22}$$

which is one possible conclusion.

First-Order Phase Transition and Nucleation Rate.

But let us suppose, as hinted at by [1–3] that more precise cosmological data reveals a dark matter which violates Eq.(3). Then, by boosting to an inertial frame with $V^2 > -(1/w)$, one arrives at $T'_{00} < 0$ and this would be a signal for vacuum instability [13]. If the cosmological background is a Friedmann-Robertson-Walker (FRW) metric the physics is Lorentz invariant and so one should be able to see evidence for the instability already in the preferred frame where $T_{\mu\nu}$ is given by Eq.(1). This goes back to work in the 1960’s and 1970’s where one compares the unstable vacuum to a superheated liquid. As an example, at one atmospheric pressure water can be heated carefully to above $100^0$C without boiling. The superheated water is metastable and attempts to nucleate bubbles containing steam. However, there is an energy balance for a three-dimensional bubble between the positive surface energy $\sim R^2$ and the negative latent heat energy of the interior $\sim R^3$ which leads to a critical radius below which the bubble shrinks away and above which the bubble expands and precipitates boiling [11,12]. For the vacuum the first idea in [13] was to treat the spacetime vacuum as a four-dimensional material medium just like superheated water. The second idea in
the same paper was to notice that a hyperspherical bubble expanding at the speed of light is the same to all inertial observers. This Lorentz invariance provided the mathematical relationship between the lifetime for unstable vacuum decay and the critical radius of the four-dimensional bubble or instanton.

In the rest frame, the energy density is

\[ T_{00} = \Lambda \sim (10^{-3} eV)^4 \sim (1\text{mm})^{-4} \]  

since \(10^{-3} eV \sim (1\text{mm})^{-1}\).

In order to make an estimate of the dark energy decay lifetime in the absence of a known potential, we can proceed by assuming (without motivation from observation) that there is a first-order phase transition possible between the \(\Lambda = (10^{-3} eV)^4\) “phase” and a \(\Lambda = 0\) “phase”. This hypothesized decay is the Lorentz invariant process of a hyperspherical bubble expanding at the speed of light, the same for all inertial observers. Let the radius of this hypersphere be \(R\), its energy density be \(\epsilon\) and its surface tension be \(S_1\). Then according to [13] the relevant instanton action is

\[ A = -\frac{1}{2} \pi^2 R^4 \epsilon + 2 \pi^2 R^3 S_1 \]  

where \(\epsilon\) and \(S_1\) are the volume and surface energy densities, respectively. The stationary value of this action is

\[ A_m = \frac{27}{2} \pi^2 S_1^4 / \epsilon^3 \]  

corresponding to the critical radius

\[ R_m = 3 S_1 / \epsilon \]  

We shall assume that the wall thickness is negligible compared to the bubble radius. The number of vacuum nucleations in the past lightcone is estimated as

\[ N = (V_u \Delta^4) exp(-A_m) \]
where $V_u$ is the 4-volume of the past and $\Delta$ is the mass scale relevant to the problem. This vacuum decay picture led to the proposals of inflation [14], for solving the horizon, flatness and monopole problems (only the horizon problem was generally known at the time of [13]). None of that work addressed why the true vacuum has zero energy. Now that the observed vacuum has non-zero energy density $+\epsilon \sim (10^{-3}eV)^4$ we may interpret it as a false vacuum lying above the true vacuum with $\epsilon = 0$. In order to use the full power of Eq.(8) taken from [13,15] and the requirement $N \ll 1$ we need to estimate the three mass-dimension parameters $\epsilon^{1/4}$, $S^{1/3}$ and $\Delta$ therein and so we discuss these three scales in turn.

The easiest of the three to select is $\epsilon$. If we imagine a tunneling through a barrier between a false vacuum with energy density $\epsilon$ to a true vacuum at energy density zero then the energy density inside the bubble will be $\epsilon = \Lambda = (10^{-3}eV)^4$. No other choice is reasonable. As for the typical mass scale $\Delta$ in the prefactor of Eq. (8), the value of $\Delta$ does not matter very much because it appears in a power rather than an exponential so let us put (the reader can check that the conclusions do not depend on this choice) $\Delta = \epsilon^{1/4} = (1mm)^{-1}$ whereupon the prefactor in Eq.(8) is $\sim 10^{116}$. The third and final scale to discuss is the surface tension, $S_1$. Here we appeal to comparison of spontaneous decay to stimulated decay. The former dictates $N \ll 1$ in Eq.(8): the latter requires further discussion.

Spontaneous dark energy decay brings us to the question of whether such decay can be initiated in an environment existing within our Universe. The question is analogous to one of electroweak phase transition in high energy particle collision. This was first raised in [13] and revisited for cosmic-ray collisions in [16]. That was in the context of the standard-model Higgs vacuum and the conclusion is that high-energy colliders are safe at all present and planned foreseeable energies because much more severe conditions have already occurred (without disaster) in cosmic-ray collisions within our galaxy. More recently, this issue has been addressed in connection with fears that the Relativistic Heavy Ion Collider (RHIC) might initiate a diastrous transition but according to careful analysis [17,18] there was no such danger.

The energy density involved for dark energy is some 58 orders of magnitude smaller
[(10^{-3}eV)^4 compared to (300GeV)^4] than for the electroweak case and so the nucleation scales are completely different. One is here led away from microscopic towards astronomical size scales.

The energy density of Eq.(4) is so readily exceeded that the critical radius cannot be microscopic. Think first of a macroscopic scale e.g. 1 meter and consider a magnetic field practically-attainable in bulk on Earth such as 10 Tesla. Its energy density is given by

$$\rho_{mag} = \frac{1}{\mu_0}B^2$$

(9)

Using the value $\mu_0 = 4\pi \times 10^{-7}NA^{-2}$ and $1T = 6.2 \times 10^{12}(MeV.s.m^{-2})$ leads to an energy density $\rho_{mag} = 2.5 \times 10^{17}eV/(mm)^3$, over 20 orders above the value of Eq.(4) for the interior of nucleation. Magnetic fields in bulk exist in galaxies with strength $\sim 1\mu G$ and the rescaling by $B^2$ then would give $\rho_{mag} \sim (2.5 \times 10^{-5}eV)(mm)^{-3}$, slightly below the value in Eq.(4). Assuming the dark energy can exchange energy with magnetic energy density the observed absence of stimulated decay would then imply a critical radius of at least galactic size, say, $\sim 10kpc$. Using Eq(7) then gives for the surface tension $S_1 > 10^{23}(mm)^{-3}$ and number of nucleations in Eq.(8) $N < exp(-10^{92})$. The spontaneous decay is thus glacial. Note that the dark energy has appeared only recently in cosmological time and has never interacted with background radiation of comparable energy density. Also, this nucleation argument does not require $w < -1$.

Discussion

As a first remark, since the critical radius $R_m$ for nucleation is astronomical, it appears that the instability cannot be triggered by any microscopic process. While it may be comforting to know that the dark energy is not such a doomsday phenomenon, it also implies at the same time the dreadful conclusion that dark energy may have no microscopic effect. If any such microscopic effect in a terrestrial experiment could be found, it would be crucial in investigating the dark energy phenomenon. We note that the present arguments are less model-dependent than those in [9].
In closing one may speculate how such stability arguments may evolve. One may expect most conservatively that the value \( w = -1 \) will eventually be established empirically in which case both quintessence and the “phantom menace” will be irrelevant. In that case, indeed for any \( w \), we may still hope that dark energy will provide the first connection between string theory and the real world as in \( e.g. [6] \). Even if precise data do establish \( w < -1 \), as in the “phantom menace” scenario, the dark energy stability issue is still under control.

Acknowledgements

We thank Stuart Raby and Tomo Takahashi for useful discussions. This work is supported in part by the US Department of Energy under Grant No. DE-FG02-97ER-41036.
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