Further results on Functional Determinants of Laplacians in Simplicial Complexes

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Abstract

This paper is a sequel to a previous report (Aurell E. & Salomonson P., 1993), where we investigate the functional determinant of the laplacian on piece-wise flat two-dimensional surfaces, with conical singularities in the interior and/or corners on the boundary. Our results extend earlier investigations of the determinants on smooth surfaces with smooth boundaries. The differences to the smooth case are:
a) different “interaction energies” between pairs of conical singularities than one would expect from a naive extrapolation of the results for a smooth surface; and
b) “self-energies” of the singularities. In this paper we give the results for general simplicial complexes that are conformally related. Special attention is given to the case of disc topology with corners in the interior, and to the topology of a sphere, where we can compare with alternative computations in special cases where the spectra are known. We consider both Dirichlet and Neumann boundary conditions. In the limit when all corners are almost flat we recover the expressions for smooth surfaces with smooth boundaries.

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1 Introduction

Gaussian integrals appear in many branches in mathematical physics. When \( A \) is a symmetric real finite-dimensional matrix with positive eigenvalues, we have the elementary result

\[
\int \prod_{k=1}^{n} \frac{dx_k}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(\vec{x}, A\vec{x})\right) = (\det A)^{-\frac{1}{2}}. \tag{1}
\]

If \( A \) is an operator like the laplacian, both sides of (1) vanish. A regularization of the functional determinant of an operator attempts to give a meaning to (1) when the product of the eigenvalues of \( A \) diverges. We refer to [13, 15] for modern discussions on regularizations of functional integrals. One regularization proceeds from the zeta function of the operator

\[
Z_A(s) = \sum_{\nu : \lambda_\nu > 0} \lambda_\nu^{-s}. \tag{2}
\]

The sum in (2) goes over the positive eigenvalues of the operator, since we want to treat operators having zero eigenvalues, and \( s \) is a complex number with sufficiently large real part so that the sum converges. Then, motivated by the formal result \( \frac{dZ_A(s)}{ds}|_{s=0} \sim -\sum_n \log \lambda_n \), one defines

\[
\text{Regularized} (\det A) = \exp\left(-\frac{dZ_A(s)}{ds}|_{s=0}\right). \tag{3}
\]

We study here functional determinants of scalar laplacians in two-dimensional domains that are piece-wise flat with isolated conical singularities, and boundaries that are piece-wise straight with isolated corners. We refer to these surfaces as simplicial complexes.

Functional determinants of the scalar two-dimensional laplacian on smooth surfaces have been widely studied [14, 1, 15]. They appear in Polyakov’s approach to string theory as a double functional integral over embeddings in external space and metrics in internal space. The integral over embeddings is then just a Gaussian integral like (1) with \( A \) the scalar Laplace-Beltrami operator. The differences in functional determinants between two surfaces related by a conformal distortion can be written

\[
Z_D'(0) = Z_D'(0) + \frac{1}{4\pi} \int_{\partial D} d\hat{s} \hat{n} \cdot \partial \sigma + \frac{1}{6\pi} \int_{\partial D} d\hat{s} \hat{k} \sigma
\]

\[
+ \frac{1}{12\pi} \int_D d^2z \sqrt{\hat{g}} [\hat{g}^{ab} \partial_a \sigma \partial_b \sigma + \hat{R} \sigma]. \tag{4}
\]

In (3) \( \hat{R} \) is the Gaussian curvature of the reference surface \( D \), \( \hat{k} \) is the geodetic curvature of the boundary, and \( \sigma = \log |\frac{dz'}{dz}| \) (in local coordinates) is the conformal factor [14, 1, 13]. The – (+) sign refer to Dirichlet (Neumann) boundary conditions.

The kinetic energy term in (3) is infinite for a mapping that changes the opening angle of a conical singularity, although the quantity \( Z_D'(0) \) is known to be finite for many special cases (e.g. the equilateral triangle or any rectangle, see I or appendix [3] below). This kind of divergence looks at first sight similar to electrostatics. The interaction energy of a
collection of charges can be computed by multiplying a charge with the potential from the other charges, and then summing over the charges. One cannot include the potential from a charge on itself since that would give an infinite and meaningless result. One may also compute the interaction energy by integrating the square of the electric field over space, avoiding small regions around each charge (we assume for simplicity that the collection has zero total charge so that the integrals converge at infinity). That answer will then be divergent with the radii of the small excluded volumes, but if this divergence is discarded one gets the right finite result.

One might therefore assume that in the case of conical singularities, (4) is analogous to the integral over the square of the electric field, and should be regularized by taking away small regions around each singularity. Our initial motivation for considering simplicial complexes in detail was that this procedure does not work. The answer is nevertheless sufficiently close for the electrostatic picture to be useful. There will be a sum over interaction energies between charges, but the potential only reduces to the form of the standard electrostatic potential in the limit when all the conical singularities are almost flat. There will also be a finite self-energy determined locally at each singularity.

This paper is a sequel to [3], in the following referred to as I, where we investigated the functional determinants in simplicial complexes with disc topology and corners on the boundary, but without conical singularities in the interior, i.e. polygons. We extend here the results to general simplicial complexes that can be related by conformal distortions. Special attention is given to disc topology with conical singularities in the interior, and to spherical topology. In these cases there are a number of special integrable surfaces where the spectra can be determined exactly, and the functional determinants computed with methods from number theory. These special results fit our general expressions in all cases.

The paper is organized as follows: in section 2 we summarize the approach in I and the results obtained there. The modifications necessary when dealing with less simple topologies are included. In sections 3 and 4 we treat simplicial complexes of disc topology with Dirichlet and Neumann boundary conditions. In section 5 we treat simplicial complexes with the topology of a sphere. In section 6 we consider the general case, albeit in a less explicit form than for disc and spherical topology. In section 7 we show by an explicit example that it is not possible to obtain the functional determinants of a surface with conical singularities by rounding off the corners and extrapolating to the limit when the radius of the smoothing goes to zero. In section 8 we discuss the extension to more general variations, and possible application to the original problem of the Polyakov action. In appendix A we summarize relevant results from I. In appendix B we list the surfaces obtained by quoting a torus with a finite order symmetry.

### 2 Heat kernel and variational formula

We use the convention that the laplacian of a flat surface is is \(-(\partial_1^2 + \partial_2^2)\) such that the spectrum is non-negative. The heat kernel for the laplacian on a domain \(D\), and its trace,
can then be expressed in terms of its eigenvalues and normalised eigenfunctions as

\[ K_D(x, y, t) = \theta(t) \langle x|e^{-\Delta t}|y \rangle = \theta(t) \sum_\nu \Psi_\nu^*(x)\Psi_\nu(y)e^{-\lambda_\nu t}, \quad (5) \]

\[ K_D(t) = \theta(t) \text{Tr}(e^{-\Delta t}) = \theta(t) \sum_\nu e^{-\lambda_\nu t}. \quad (6) \]

As is well known[11, 12, 15], the trace of the heat kernel admits an expansion for short times, that for two-dimensional surfaces with a boundary goes as

\[ K_D(t) \sim \frac{c_1}{t} + \frac{c_{1/2}}{t^{1/2}} + c_0 + \mathcal{O}(t^{1/2}) + \ldots \quad (7) \]

The zeta function of the operator is the Mellin transform of the trace of the heat kernel, with the zero modes subtracted:

\[ Z_D(s) = \sum_{\nu: \lambda_\nu > 0} \lambda_\nu^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} \left[ \text{Tr}(e^{-\Delta t}) - \dim \ker \Delta \right]. \quad (8) \]

The integral in (8) converges from any finite time \( t \) to infinity. A term \( t^{-\gamma} \) in the asymptotic expansion for the heat kernel leads potentially to a pole in the zeta function at \( s = \gamma \) with residue \( c_\gamma/\Gamma(-\gamma) \). The exceptions are zero and the negative integers, where the prefactor \( 1/\Gamma(s) \) has a zero. Hence we have, e.g.,

\[ Z_D(0) = c_0 - \dim \ker \Delta. \quad (9) \]

One may continue (8) analytically around the poles at \( s = 1 \) and \( s = 1/2 \) to obtain for the derivative at the origin:

\[ Z'_D(0) = (3 + \gamma)(c_0 - \dim \ker \Delta) - 2 \int_0^\infty dt \log t \frac{\partial}{\partial t} \left[ t^{3/2} \frac{\partial}{\partial t} \left( t^{3/2} \frac{\partial}{\partial t} (K_D(t) - \dim \ker \Delta) \right) \right] \quad (10) \]

An alternative formula is obtained by taking the integral from \( \epsilon \) to infinity, where \( \epsilon \) is some small positive number. The integrand in (10) behaves as \( t^{-\frac{3}{2}} \log t \) for small \( t \) so the limit as \( \epsilon \) goes to zero is harmless. Partial integrations will however bring in divergent terms from the lower boundary. A short calculation gives

\[ Z'_D(0) = \gamma(c_0 - \dim \ker \Delta) + \text{Finite}_{\epsilon \to 0} \int_\epsilon^\infty \frac{dt}{t} (K_D(t) - \dim \ker \Delta) \quad (11) \]

where we understand that we keep the \( \epsilon \)-independent part of (11), and throw away the terms that diverge as \( \epsilon \) tends to zero. Equations (10) and (11) show that the derivative of the zeta function at the origin exists as a well-defined functional of the heat kernel, but are not of much practical interest, since they depend on the heat kernel for large times in a way that is difficult to handle.
Things are better if one considers variations of the laplacian. Then (11) turns into
\[ \delta[Z'_D(0)] = \gamma[\delta c_0] - \text{Finite}_{\epsilon \to 0} \sum_{\nu} ' (\delta \lambda_{r}) \epsilon^{-\epsilon \lambda_{r}} \frac{1}{\lambda_{r}}, \] (12)
where the prime on the sum indicates that we sum only over the non-zero eigenvalues. The operators \( e^{-\epsilon \Delta} \) and \( \frac{1}{\Delta} \) are diagonal in the basis of eigenfunctions of \( \Delta \). This allows us to rewrite the sum in (12) as
\[ \sum_{\nu} ' \sum_{\mu} \sum_{\kappa} \int d^2 x \int d^2 y \int d^2 z (\Psi_{\nu}^{*}(x) \delta \lambda_{\mu} \Psi_{\mu}(x)) (\Psi_{\mu}^{*}(y) e^{-\epsilon \lambda_{\kappa}} \Psi_{\kappa}(y)) (\Psi_{\kappa}^{*}(z) \frac{1}{\lambda_{\nu}} \Psi_{\nu}(z)), \] (13)
which can be rearranged to give
\[ \delta[Z'_D(0)] = \gamma[\delta c_0] - \text{Finite}_{\epsilon \to 0} \int d^2 y \int d^2 z [ (\delta \Delta)_{y} G_{D}(y, z)] K_{D}(z, y; \epsilon). \] (14)
In (14) \((\delta \Delta)_{y}\) is the variation of the laplacian, \( K_{D} \) is the heat kernel, and
\[ G_{D}(y, z) = \sum_{\nu} ' \frac{1}{\lambda_{\nu}} \Psi_{\nu}^{*}(y) \Psi_{\nu}(z) \] (15)
is the Green’s function. The heat kernel for short times only depend on the local metric properties of the surface. The nonlocal information in (14) is therefore only contained in the action of the varied laplacian on the Green’s function. We note that (14) can be interpreted as a regularization of the formal expression \( \delta(\log \det \Delta) = \text{Tr}[\delta \Delta / \Delta]. \)

In I and in this paper we only consider conformal variations. In conformal coordinates \((g_{ab} = e^{2\sigma(x)} \delta_{ab})\), the laplacian is written
\[ \Delta = -e^{-2\sigma(x)} (\partial_{1}^{2} + \partial_{2}^{2}). \] (16)
The action of the varied laplacian in (14) then has the simple form
\[ [ (\delta \Delta)_{y} G_{D}(y, z)] = (-2\delta \sigma(y)) [ \delta^{2}(y - z) - \sum_{\nu: \lambda_{\nu} = 0} \Psi_{\nu}^{*}(y) \Psi_{\nu}(z)], \] (17)
which gives
\[ \delta[Z'_D(0)] = \gamma[\delta c_0] + \text{Finite}_{\epsilon \to 0} \int d^2 x 2\delta \sigma(x) K_{D}(x, x; \epsilon) - \sum_{\nu: \lambda_{\nu} = 0} \int_{D} d^2 x 2\delta \sigma(x) |\Psi_{\nu}(x)|^2]. \] (18)

The scalar laplacian has in fact not more than one vanishing eigenvalue. The corresponding normalized eigenfunction is \( 1/\sqrt{\text{Area}} \), and (18) simplifies to
\[ \delta[Z'_D(0)] = \gamma[\delta c_0] + \text{Finite}_{\epsilon \to 0} \int d^2 x 2\delta \sigma(x) K_{D}(x, x; \epsilon) - \frac{\delta(\text{Area})}{\text{Area}}]. \] (19)
On surfaces with a boundary and Dirichlet boundary conditions the laplacian has no zero eigenvalue, and the correction term in (19) is absent.
Our calculation is now brought into the form of integrating a test function \((\delta \sigma)\) against the heat kernel for short times, and then extracting the time-independent piece. One convenient aspect of simplicial complexes is that there are qualitatively speaking only three different kinds of short-time behaviours that need to be considered: flat surface; close to a flat boundary, and close to a corner or a conical singularity. The flat surface does not contribute at all, and the flat boundary gives a term that only depends on the topology and may be absorbed in an undetermined constant of integration.

The entire contribution thus comes from the corners and conical singularities. The extraction of the finite piece in the short time asymptotics was done in I, with a convention for how the variations of the conical singularities are parametrized. Let us note that two conical singularities with opening angles \(\alpha\) and \(\alpha + \delta \alpha\) can be mapped from a flat surface by mappings that in local coordinates \((z\) for the first singularity, \(z'\) for the second and \(\omega\) for the flat surface) satisfy:

\[
\frac{dz}{d\omega} = e^{\lambda} \omega^{\alpha - 1} \quad \frac{dz'}{d\omega} = e^{\lambda + \delta \lambda} \omega^{\alpha + \delta \alpha - 1} \tag{20}
\]

The variation of the conformal factor of the mapping from \(z\) to \(z'\) then reads

\[
\delta \sigma(z) = \log \left| \frac{dz'}{dz} \right| = \left[ \delta \lambda - \frac{\delta \alpha}{\alpha} \lambda \right] + \frac{\delta \alpha}{\alpha} \log(\alpha z) \tag{21}
\]

The test function thus separates into one smooth part, and one that is logarithmically divergent.

When the test function is smooth we see that we may take its value at a corner outside of (18). The piece of the integral finite as \(\epsilon\) tends to zero will then give \(Z_\alpha(0)\), the contribution to \(c_0\) or to \(Z_D(0)\) (not the derivative), from the corner. This contribution has been computed long ago [11, 12, 6]: it is \(\frac{1}{24}(\frac{1}{\alpha} - \alpha)\) for a corner on the boundary with opening angle \(\pi \alpha\), and \(\frac{1}{12}(\frac{1}{\alpha} - \alpha)\) for a conical singularity in the interior with opening angle \(2\pi \alpha\). We thus have one contribution to \(\delta Z_D'(0)\)

\[
\left[ \delta \lambda - \frac{\delta \alpha}{\alpha} \lambda \right] Z_\alpha(0) \tag{22}
\]

which we call nonlocal, as the scale factors \(\lambda\) generally depend on the whole surface, and in particular on the positions and strengths of the other conical singularities.

The logarithmically divergent piece in (21), when inserted in (19) gives rise to a term which only depends on the local opening angle \(\alpha\). We can therefore combine it with \(\gamma c_0\), the other term in (14), and write it as a total variation \(\delta Z'_\alpha(0)\). The integrated quantity \(Z'_\alpha(0)\) for a corner on the boundary with Dirichlet boundary conditions was investigated in I. The changes when considering a Neumann boundary conditions or a conical singularity in the interior are small. These results are summarized in appendix A below.

### 3 Disc topology, Dirichlet boundary conditions

In this section we will establish convenient representations of surfaces with the topology of a disc, and then integrate \(Z'(0)\) using the formulae in section 2.
Let us say loosely that we map the unit disc (D, coordinate u) to M (coordinate z) by a transformation that satisfies
\[
\frac{dz(u)}{du} = e^{\lambda_0} \prod_{u_\mu \in \partial D} (u - u_\mu)^{-\beta_\mu} \prod_{u_i \in \text{int } D} (u - u_i)^{-\beta_i}(\bar{u}_i u - 1)^{-\beta_i}
\] (23)
which generalizes the Schwarz-Christoffel transform, corners on the boundary at positions z(u_\mu), to the case with conical singularities in the interior at positions z(u_i). An opening angle at the boundary is written \(\pi \alpha_\mu = \pi (1 - \beta_\mu)\), and an opening angle in the interior is written \(2 \pi \alpha_i = 2 \pi (1 - \beta_i)\). The image charges at positions \(1/u_i\) serve to make the boundary of M piece-wise straight. We will from now on use the convention that latin indices \((i,j,\ldots)\) refer to conical singularities in the interior and their image charges, while greek indices \((\mu,\nu,\ldots)\) refer to corners on the boundary. The exterior angles must satisfy \(\sum_i 2 \beta_i + \sum_\mu \beta_\mu = 2\).

It is clear that in general z in (23) is a coordinate defined on a branched surface over a part of the complex plane with cut lines properly identified. The simplicial complex M can be pictured as paper triangles glued together at the sides. One of the triangles can be put down on the plane. To construct the branched surface, one should put down the other triangles on the plane, and when doing so, one has to sever the triangles at the joints, if they make up less or more than a full turn around a vertex. Sides of triangles that have been unglued should be identified. With this understanding one can take z in (23) as a local variable everywhere on M and let (23) define the conformal factor \(\sigma(u) = \log |dz(u)|/du|\).

The normalized area of M is chosen
\[
A(M) = \int_D d^2 u \prod_M |u - u_\mu|^{-2\beta_\mu} |u - u_i|^{-2\beta_i} |\bar{u}_i u - 1|^{-2\beta_i}
\] (24)
such that the real and the normalized areas are related by
\[
\text{Area} = e^{2\lambda_0} A(M)
\] (25)
It is a convenient fact that we may use (23) to smoothly interpolate between one simplicial complex M at \(a = 0\), to another one M’ at \(a = 1\) by
\[
\frac{dz(u, a)}{du} = e^{(1-a)\lambda_0 + a \lambda_0'} \prod_M (u - u_\mu)^{-(1-a)\beta_\mu} (u - u_i)^{-(1-a)\beta_i} (\bar{u}_i u - 1)^{-(1-a)\beta_i}
\]
\[
\prod_{M'} (u - u_{\mu'})^{-(a\beta_\mu'} (u - u_{\nu'})^{-(a\beta_{\nu'})} (\bar{u}_{\nu'} u - 1)^{-a\beta_{\nu'}}
\] (26)
Each intermediate figure is then again a simplicial complex with the same topology. We may therefore without restriction consider variations of \(\lambda_0\) and the \(\beta\)'s, but leaving the branchpoints \(u_\mu\) and \(u_i\) fixed.

We now separate the conformal factor at a corner or a conical singularity into a local and a nonlocal part:
\[
\sigma(u \sim u_\mu) = -\beta_\mu \log |u - u_\mu| + \lambda_\mu + O(u - u_\mu)
\] (27)
\[ \lambda_\mu = \lambda_0 - \sum_{\nu \neq \mu} \beta_\nu \log |u_\mu - u_\nu| - \sum_j \beta_j \log |u_\mu - u_j| - \sum_j \beta_j \log |\bar{u}_j u_\mu - 1| \]

\[ \sigma(u \sim u_i) = -\beta_i \log |u - u_i| + \lambda_i + O(u - u_i) \]

\[ \lambda_i = \lambda_0 - \sum_{\nu} \beta_\nu \log |u_i - u_\nu| - \sum_j \beta_j \log |u_i - u_j| - \sum_j \beta_j \log |\bar{u}_j u_i - 1| \]

We notice that the nonlocal quantities in (29) may be rewritten as

We consider a variation of the simplex by varying \( \lambda_0 \) and the \( \beta \)'s. Locally, that means that we vary the opening angles \( \beta_i \) and \( \beta_\mu \) and the scale factors \( \lambda_i \) and \( \lambda_\mu \). We then have the variation of the conformal factor in the form of (21), and we can directly write

\[ \delta Z_\mathcal{M}(0) = \delta \left[ \sum_i (2Z_i' - \beta_i(0) + \frac{1}{2} \log(1 - \beta_i)) + \sum_\mu Z_\mu'(0) \right] + \]

\[ \sum_\mu \left( \frac{1}{12} \left( \frac{1}{\alpha_\mu} - \alpha_\mu \right) \right) \delta \lambda_\mu - \frac{\delta \alpha_\mu}{\alpha_\mu} \lambda_\mu + \]

\[ \sum_i \left( \frac{1}{6} \left( \frac{1}{\alpha_i} - \alpha_i \right) \right) \delta \lambda_i - \frac{\delta \alpha_i}{\alpha_i} \lambda_i \]

(29)

We notice that the nonlocal quantities in (29) may be rewritten as

\[ \delta \left[ \frac{1}{12} \sum_\mu \left( \frac{1}{\alpha_\mu} - \alpha_\mu \right) \lambda_\mu + \frac{1}{6} \sum_i \left( \frac{1}{\alpha_i} - \alpha_i \right) \lambda_i \right] + \frac{1}{6} \sum_\mu \delta \alpha_\mu \lambda_\mu + \frac{1}{3} \sum_i \delta \alpha_i \lambda_i. \]

(30)

and that the extra terms in (30) may be integrated to

\[ \sum_\mu \left( \frac{1}{12} \sum_{\nu \neq \mu} \beta_\mu \beta_\nu \log |u_\mu - u_\nu| + \frac{1}{3} \sum_j \beta_\mu \beta_j \log |u_\mu - u_j| \right) + \]

\[ \sum_i \left( \frac{1}{6} \sum_{j \neq i} \beta_i \beta_j \log |u_i - u_j| + \frac{1}{6} \sum_j \beta_i \beta_j \log |\bar{u}_j u_i - 1| \right) \]

(31)

Now we can combine (29), (30) and (31) and integrate the functional determinant of the Laplacian on a simplicial complex with disc topology and Dirichlet boundary conditions to:

\[ Z'_\mathcal{M}.\text{Area}(0) = \sum_i (2Z_i' - \beta_i(0) + \frac{1}{2} \log(1 - \beta_i)) + \sum_\mu Z_\mu'(0) + Z_\mathcal{M}(0) \log \frac{\text{Area}}{A(\mathcal{M})} \]

\[ -\frac{1}{12} \sum_\mu \beta_\mu \left[ \sum_{\nu \neq \mu} \beta_\nu \log |u_\mu - u_\nu| + \sum_j \beta_j \log |u_\mu - u_j| \right] + \]

\[ + \sum_j \beta_j \log |\bar{u}_j u_\mu - 1| \]

\[ -\frac{1}{6} \sum_i \beta_i \left[ \sum_{j \neq i} \beta_j \log |u_i - u_j| + \sum_\nu \beta_\nu \log |u_i - u_\nu| \right] + \]

\[ + \sum_j \beta_j \log |\bar{u}_j u_i - 1| \]

+ Integration constant  \hspace{1em} \text{(Disc topology, Dirichlet b.c.)} \]

(32)
The result (32) does not change if the branchpoints $u_\mu$ and $u_i$ are moved around by a Moebius transformation that leaves the unit disc invariant, although the normal area and the corner-corner interaction terms vary when taken separately. Formula (32) holds with or without conical singularities in the interior. In I the integration constant was determined to be zero.

In appendix B one can find two one-parameter families and seven more symmetric special surfaces of disc topology, for which the spectral functions can be computed directly. It can be checked that the general formula (32) reproduces the exact results in all cases.

In the limit where all exterior angles are small we have

$$Z',\text{Area}(0) \sim -2\frac{dz'(0)}{d\alpha} |_{\alpha=1} - \frac{1}{2} \sum_i \beta_i + \frac{1}{6} \log \frac{\text{Area}}{A(M)}$$

$$- \frac{1}{12} \sum_\mu \beta_\mu \left[ \sum_{\nu \neq \mu} \beta_\nu \log |u_\mu - u_\nu| + \sum_j \beta_j \log |u_\mu - u_j| \right]$$

$$+ \sum_j \beta_j \log |\bar{u}_j u_\mu - 1|$$

$$- \frac{1}{6} \sum_i \beta_i \left[ \sum_{j \neq i} \beta_j \log |u_i - u_j| + \sum_\nu \beta_\nu \log |u_i - u_\nu| \right]$$

$$+ \sum_j \beta_j \log |\bar{u}_j u_i - 1|$$

+ Integration constant (33)

Comparing (79) with the exact result for the functional determinant on a circular disc\[16\] we have

$$-2\frac{dz'(0)}{d\alpha} |_{\alpha=1} = Z'_{\text{disc}, \text{Dirichlet b.c.}}(0) = \frac{1}{6} \log 2 + \frac{1}{2} \log \pi + \frac{5}{12} + 2\zeta'(-1)$$

We may enclose every branchpoint $u_i$ in the interior of the disc in a small circular curve $C_i$ with radius $r_i$, and every branchpoint $u_\mu$ on the boundary in a small semicircle $C_\mu$. We may then approximate the interaction terms in (33) as

$$- \frac{1}{12\pi} \int_{C_i,C_\mu} [\hat{n} \cdot \partial\sigma](\sigma - \lambda_0)ds + \sum_i \frac{1}{6} \beta_i^2 \log r_i + \sum_\mu \frac{1}{12} \beta_\mu^2 \log r_\mu + O(r_i, r_\mu)$$

where $\hat{n}$ are the normals directed away from the points $u_i$ and $u_\mu$. We may close off the integration by including the circular segments between the branchpoints $u_\mu$. The extra integrals are evaluated by noting that by the Cauchy-Riemann equations

$$\hat{n} \cdot \partial\sigma|_{|u|=1} = -\frac{d}{d\phi} \text{Im} \log \frac{dz}{du} |_{u=e^{i\phi}}$$

(36)
and the right-hand of (36) is equal to 1 between the branchpoints. Then take the $\beta_i$’s, $\beta_\mu$’s, $r_i$’s and the $r_\mu$’s to zero such that correction terms vanishes, and after an integration by parts one finds:

$$\frac{1}{12\pi} \int_D (\partial \sigma)^2 d^2 u + \frac{1}{12\pi} \int_{\partial D} (\sigma - \lambda_0) ds$$

(37)

The line integral in (37) vanishes. We may identify

$$\frac{1}{3} \lambda_0 = \frac{1}{6\pi} \int_{\partial D} \sigma ds$$

and

$$-\frac{1}{2} \sum_i \beta_i = -\frac{1}{4\pi} \int_{\partial D} (\hat{n} \cdot \partial \sigma) ds.$$  

(38)

Including everything from (33), (34), (37), (38) and writing the integrals as over the disc with usual Cartesian volume element written as $\sqrt{g} d^2 z$, and the line element on the circle written $d \hat{s}$, we have

$$Z_{\mathcal{M}, \text{Area}}'(0) = Z_{\text{disc}}'(0) + \frac{1}{12\pi} \int_D d^2 z \sqrt{g} \hat{g}^{ab} \partial_a \sigma \partial_b \sigma$$

$$+ \frac{1}{6\pi} \int_{\partial D} \sigma d \hat{s} - \frac{1}{4\pi} \int_{\partial D} (\hat{n} \cdot \partial \sigma) d \hat{s}$$

$$+ \text{Integration constant} \quad \text{(Disc topology, Dirichlet b.c.)}$$  

(39)

Since the exact value of $Z_{\text{disc}}'(0)$ is included, we have determined again the integration constant in (33) to be zero.

4 Disc topology, Neumann boundary conditions

The changes when introducing Neumann boundary conditions are small. The mapping from the unit disc and the conformal factor are unchanged. The nonlocal terms in (32) are not sensitive to the kind of boundary conditions imposed. The boundaries change the local self-energies, and the zero-mode has to be deducted from diagonal element of the heat kernel according to (19).

We thus have the functional determinant of the Laplacian on a simplicial complex with disc topology and Neumann boundary conditions as:

$$Z_{\mathcal{M}, \text{Area}}'(0) = \sum_i (2Z'_{1-\beta_i}(0) + \frac{1}{2} \log(1 - \beta_i)) + \sum_\mu (Z'_{1-\beta_\mu}(0) + \frac{1}{2} \log(1 - \beta_\mu))$$

$$+ Z_{\mathcal{M}}(0) \log \frac{\text{Area}}{A(\mathcal{M})} - \log \text{Area}$$

$$- \frac{1}{12} \sum_\mu \frac{\beta_\mu}{1 - \beta_\mu} \left[ \sum_\nu \beta_\nu \log |u_\mu - u_\nu| + \sum_j \beta_j \log |u_\mu - u_j| \right]$$

$$+ \sum_j \beta_j \log |u_j u_\mu - 1|$$

10
\[-\frac{1}{6} \sum_i \frac{\beta_i}{1-\beta_i} \sum_{j \neq i} \beta_j \log |u_i - u_j| + \sum_{\nu} \beta_\nu \log |u_i - u_\nu| \]
\[+ \sum_j \beta_j \log |\bar{u}_j u_i - 1| \]
\[+ \text{Integration constant} \quad \text{(Disc topology, Neumann b.c.)} \quad (40)\]

Comparing with the exact result for the functional determinant on a circular disc\[16\]
we have
\[-2 \frac{dZ'_{\alpha}(0)}{d\alpha} |_{\alpha=1} = Z'_{\text{disc,Neumann}}(0) + (1 + \log \pi) \]
\[= \left( \frac{1}{6} \log 2 - \frac{1}{2} \log \pi - \frac{7}{12} + 2\zeta'(-1) \right) + (1 + \log \pi) \quad (41)\]

One sees that in the smooth limit
\[\frac{1}{2} \sum_i \log(1 - \beta_i) + \frac{1}{2} \sum_\mu \log(1 - \beta_\mu) \sim -1 + \frac{1}{2} \sum_i \beta_i \quad (42)\]

We can therefore write down the result in analogy to (39) as
\[Z'_{\mathcal{M},\text{Area}}(0) = Z'_{\text{disc}}(0) + \frac{1}{12\pi} \int_{\mathcal{D}} d^2 z \sqrt{\hat{g}} \hat{g}^{ab} \partial_a \sigma \partial_b \sigma \]
\[+ \frac{1}{6\pi} \int_{\partial \mathcal{D}} \sigma d\hat{s} + \frac{1}{4\pi} \int_{\partial \mathcal{D}} (\hat{n} \cdot \partial \sigma) d\hat{s} - \log \frac{\text{Area}}{\pi} \]
\[+ \text{Integration constant} \quad \text{(Disc topology, Neumann b.c.)} \quad (43)\]

Since the exact value of $Z'_{\text{disc}}(0)$ is included, we have determined the integration constant in (43) to be zero.

## 5 Spherical topology

In this section we will establish convenient representations of surfaces with the topology of a sphere, and then integrate $Z'(0)$ using the formulae in section 4. Visualize the sphere with unit radius as a ball lying on the plane. Take the point where the plane and the sphere touch to be the south pole of the sphere and the origin in the plane. By a stereographic projection, $r = 2 \tan \frac{\pi}{2} \theta$, we map the point $(\theta, \phi)$ in spherical coordinates on the sphere to $(r, \phi)$ in polar coordinates in the plane. The volume element on the sphere, $\sin \theta d\phi d\theta$, is in polar coordinates
\[\frac{1}{2} r dr d\phi.\]

Let the flat laplacian in the plane be $\hat{\Delta}$ and the laplacian on the sphere $\tilde{\Delta}$. Writing the laplacian on the sphere in coordinates $r$ and $\phi$, it is then related to the flat laplacian by
\[\tilde{\Delta} = (1 + (r/2)^2)^2 \hat{\Delta} \quad (44)\]
Call the conformal factor of the stereographic projection from the plane to the sphere $\tilde{\sigma}$. We have

$$\tilde{R} = e^{-2\tilde{\sigma}} (\hat{R} + 2 \hat{\Delta} \tilde{\sigma}) \quad (45)$$

where $\tilde{R}$ (equal to 2) is the Gaussian curvature of a point on the sphere and $\hat{R}$ (equal to zero) is the curvature of the plane. We can now map the plane (coordinate $\omega$) to a simplicial complex with the topology of a sphere ($\mathcal{M}$, coordinate $z$) by a transformation that satisfies

$$\frac{dz(\omega)}{d\omega} = e^{\lambda_0} \prod_i (\omega - \omega_i)^{-\beta_i} \quad (46)$$

The exterior angles must satisfy $\sum_i 2\beta_i = 4$. The normalised area of $\mathcal{M}$ is chosen

$$A(\mathcal{M}) = \int_C d^2 \omega \prod_M |\omega - \omega_i|^{-2\beta_i} \quad (47)$$

such that the real and the normalized areas are related by

$$\text{Area} = e^{2\lambda_0} A(\mathcal{M}) \quad (48)$$

We will call the conformal factor defined by (46) $\hat{\sigma}$. By combining (46) with a stereographic projection, we obtain a map from the sphere to $\mathcal{M}$, for which the conformal factor is

$$\sigma = \frac{1}{2} \log \left| \frac{\partial (z, \bar{z})}{\partial (\theta, \phi)} \right| = \hat{\sigma} - \tilde{\sigma} \quad (49)$$

We may now proceed in analogy with the case of a simplicial complex with topology of a disc, and write down the functional determinant for a simplicial complex of spherical topology:

$$Z'_{\mathcal{M}, \text{Area}}(0) = \sum_i (2Z'_{1-\beta_i}(0) + \frac{1}{2} \log (1 - \beta_i)) + Z_M(0) \log \frac{\text{Area}}{A(\mathcal{M})}$$

$$- \frac{1}{6} \sum_{i,j \neq i} \frac{\beta_i \beta_j}{1 - \beta_i} \log |\omega_i - \omega_j| - \log \text{Area} + \text{Integration constant} \quad \text{(Spherical topology)} \quad (50)$$

We determine below the integration constant in (50) to be $\log 2$.

In appendix B one can find one two-parameter family and five more symmetric special surfaces of disc topology, for which the spectral functions can be computed directly. It can be checked that the general formula (50) reproduces the exact results in all cases.

In the limit where all exterior angles are small we have

$$Z'_{\mathcal{M}, \text{Area}}(0) \sim -4 \frac{dZ'_\alpha(0)}{d\alpha} \bigg|_{\alpha = 1} - 1 + \frac{1}{3} \log \frac{\text{Area}}{A(\mathcal{M})} - \frac{1}{6} \sum_{i,j \neq i} \beta_i \beta_j \log |\omega_i - \omega_j|$$

$$- \log \text{Area} + \text{Integration constant} \quad (51)$$
Comparing (73) with the exact result for the functional determinant on a sphere [16], we have

\[-4 \frac{dZ'_\alpha(0)}{d\alpha} \big|_{\alpha=1} - 1 = Z'_{\text{sphere}}(0) - \frac{2}{3} \log 2 + \log 2\pi + \frac{1}{3}\]  

(52)

We may enclose every branchpoint \(\omega_i\) in a small circular curve \(C_i\) with radius \(r_i\) and approximate

\[-\frac{1}{6} \beta_i \sum_{j \neq i} \beta_j \log |\omega_i - \omega_j| = \frac{1}{6} \beta_i (\hat{\sigma} - \lambda_0 + \beta_i \log r_i) + O(r_i)\]

\[= -\frac{1}{12\pi} \int_{C_i} [\hat{n} \cdot \partial \hat{\sigma}] \hat{\sigma} ds - \frac{1}{6} \lambda_0 + \frac{1}{6} \beta_i^2 \log r_i + O(r_i)\]  

(53)

where \(\hat{n}\) is normal directed away from the point \(\omega_i\). Closing off with one circle at \(|\omega| = R\) far outside all the \(\omega_i\)'s, taking the \(\beta_i\)'s and the \(r_i\)'s to zero such that \(\frac{1}{6} \sum_i \beta_i^2 \log r_i\) vanishes and integrating by parts, we find:

\[-\frac{2}{3} \log R + \frac{1}{12\pi} \int_{|\omega|<R} (\partial \sigma)^2 d^2\omega + O(1/R)\]  

(54)

The integral in (54) can be rewritten as

\[\frac{1}{12\pi} \int_{|\omega|<R} (\partial \sigma)^2 + 2 \sigma \cdot \partial \sigma + (\partial \sigma)^2 d^2\omega\]  

(55)

The integral over \((\partial \sigma)^2\) is convergent at infinity and may be closed. The second term may be rewritten as

\[-\frac{2}{3} \lambda_0 + \frac{4}{3} \log 2 + \frac{1}{12\pi} \int_{|\omega|<R} \sigma \Re e^{2\hat{\sigma}} d^2\omega\]  

(56)

The integral in (56) is again convergent at infinity and may be closed. The third term in (54) is simply \(\frac{2}{3} \log R - \frac{4}{3} \log 2 - \frac{1}{3}\).

Including everything from (51), (52) and (54) and writing the integrals as over the sphere with usual volume element written as \(\sqrt{\tilde{g}} d^2z\) we have

\[Z'_{\mathcal{M}, \text{Area}}(0) = Z'_{\text{sphere}}(0) + \frac{1}{12\pi} \int_{S^2} d^2z \sqrt{\tilde{g}} [\tilde{g}^{ab} \partial_a \sigma \partial_b \sigma + \tilde{R} \sigma] - \log \text{Area} + \log 2\pi + \text{Integration constant}\]  

(57)

Since the area of the unit sphere is \(4\pi\), the integration constant in (57) must be \(\log 2\).

6 The general simplicial complex

When we extend the approach to functional determinants through simplicial approximations to higher genus surfaces, there is the problem of Teichmüller parameters,
which prevents us from going between two arbitrary surfaces with a conformal transformation. There is also a lack of integrable cases to fix an integration constant and use as checks on the calculations. We are not aware of any two surfaces with genus one or higher which are related by a conformal distortion and for which both spectra of the laplacian are known.

Here we will sketch how to compute the functional determinants under the restriction that all variations are conformal. For simplicity we will consider closed surfaces. The cases with Dirichlet and Neumann boundary conditions are easily obtained in analogy with sections 3 and 4. We may then say that we have two simplicial complexes $M_1$ and $M_2$, such that the second can be parametrized by the coordinates of the first, with a metric $g$ which is related to the piece-wise flat metric $\hat{g}$ on $M_1$ by $g_{ab} = e^{\sigma} \hat{g}_{ab}$.

The conformal factor $\sigma$ can be thought of as an electrostatic potential of a collection of charges on $M_1$ as follows: suppose that at the point labelled by coordinate $z_i^{(1)}$ on $M_1$, there is a conical singularity of $M_1$, or a conical singularity of $M_2$, or both. Then we can interpret the differences in exterior angles as charges of strength $(\beta_i^{(2)} - \beta_i^{(1)})$. We note that the sum of these charges is zero. The potential is determined up to an additive constant, which is fixed by comparing the areas of $M_1$ and $M_2$.

Now we may interpolate linearly between $M_1$ (at $a = 0$) and $M_2$ (at $a = 1$) by considering the family of electrostatic potentials generated by charges $a \cdot (\beta_i^{(2)} - \beta_i^{(1)})$, and an overall area changing term $a \cdot (\lambda_0^{(2)} - \lambda_0^{(1)})$. The electrostatic potential obeys an equation linear in the charges. By changing all charges by an overall factor, we will necessarily only change the electrostatic potential in the same proportion. Let us therefore write

$$\sigma(z; a) = a \cdot G(z; \{z_i^{(1)}\}; \{ (\beta_i^{(2)} - \beta_i^{(1)}) \}) + a \cdot (\lambda_0^{(2)} - \lambda_0^{(1)}),$$

where $G$ is the electrostatic potential at the point $z$ from the full charges. Close to a conical singularity we may separate out the logarithmically divergent term of the variation, and write

$$\delta \sigma(z \sim z_i; a) \sim \frac{\delta \alpha_i(a)}{\alpha_i(a)} \log |z - z_i| + [\delta \lambda_i(a) - \frac{\delta \alpha_i(a)}{\alpha_i(a)} \lambda_i(a)]$$

where

$$\lambda_i(a) = a \cdot \lambda_i(\{z_i^{(1)}\}, \{ (\beta_i^{(2)} - \beta_i^{(1)}) \}) + a \cdot (\lambda_0^{(2)} - \lambda_0^{(1)}).$$

contains the non-singular part of $G$. The variation (59) of the conformal factor can be integrated against the heat kernel around the conical singularity to give (29), with only a different definition of the $\lambda_i$’s. That gives the same self-energies as previously. The interactions between conical singularities may be obtained as in (30) by noting that both $\delta \alpha_i$ and $\lambda_i$ depend linearly on $a$ and on the angle differences $(\beta_i^{(2)} - \beta_i^{(1)})$. We can then write down the change of the functional determinant when going from one simplicial complex $M_1$ with area $\text{Area}_1$ to another simplicial complex $M_2$ with area $\text{Area}_2$. We write the normal area of $M_2$, considered as a conformal distortion of $M_1$, as a function also of $M_1$ and $\text{Area}_1$:
\[ Z_{M_2, \text{Area}_2}^\prime(0) = Z_{M_1, \text{Area}_1}^\prime(0) + \sum_i (2Z_{1-\beta_i^{(2)}}(0) + \frac{1}{2} \log(1 - \beta_i^{(2)})) \]
\[ - \sum_i (2Z_{1-\beta_i^{(1)}}(0) + \frac{1}{2} \log(1 - \beta_i^{(1)})) + Z_{M_2}(0) \log \frac{\text{Area}_2}{A(M_2; M_1, \text{Area}_1)} \]
\[ - \frac{1}{6} \sum_i \left( \frac{1}{\alpha_i^{(2)}} - \alpha_i^{(1)} \right) \lambda_i(\{z_i^{(1)}\}; \{2(\beta_i^{(2)} - \beta_i^{(1)})\}) - \log \frac{\text{Area}_2}{\text{Area}_1} \] (61)

In the limit when all the angle differences are small and all the angles are close to one, (61) tends to
\[ Z_{M_2, \text{Area}_2}^\prime(0) \sim Z_{M_1, \text{Area}_1}^\prime(0) + \frac{\chi}{6} \log \frac{\text{Area}_2}{A(M_2; M_1, \text{Area}_1)} \]
\[ - \frac{1}{6} \sum_i \left( (\beta_i^{(2)} - \beta_i^{(1)}) + 2(\beta_i^{(1)}) \right) \lambda_i(\{z_i^{(1)}\}; \{2(\beta_i^{(2)} - \beta_i^{(1)})\}) \]
\[ - \log \frac{\text{Area}_2}{\text{Area}_1} \] (62)

where \( \chi \) is the Euler characteristic of the surfaces.

We can rewrite the interaction term in (62) as line integrals along small circular curves around the branchpoints in the same way as in (53) for spherical topology, and we recover the general result (4). In addition we have the contribution from the zero mode \( - \log \frac{\text{Area}_2}{\text{Area}_1} \).

7 Comparison with smooth limits

We will here consider what happens when one tries to approximate the functional determinant of a simplicial complex, with the one on a smooth surface obtained by rounding off the corners with some characteristic radius \( \epsilon \).

This leads to problems with both the corner self-energies and the interaction energy. A smooth surface only feels the linear term in the expansion of \( Z_{\alpha}^\prime(0) \) around \( \alpha = 1 \). The interaction energies have a characteristic feature \( \frac{\beta_i \beta_j}{1-\beta_i} \), which is indistinguishable from \( \beta_i \beta_j \) when all the \( \beta \)'s are small, but different when they are not.

It is nice to look at a concrete example. Triangles with Dirichlet boundary conditions can be mapped from the unit disc with branchpoints that we may put without loss of generality at \( u_\mu = e^{2\pi i / 3} \), \( \mu = 1, 2, 3 \). Then the interaction term simplifies, and we find:
\[ Z_T^\prime(0) = \sum_{\mu=1,2,3} Z_{\alpha_\mu}^\prime(0) + Z_T(0) \log \frac{\text{Area}}{A(T)} \] (63)
if the normal area is chosen to be
\[ A(T) = \frac{\pi \Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma(\alpha_3)}{2 \Gamma(1 - \alpha_1) \Gamma(1 - \alpha_2) \Gamma(1 - \alpha_3)} \] (64)
which corresponds to the particular choice $\lambda_0 = \frac{1}{2} \log 3$ (see I). The relevant continuous action is \( (39) \). For two triangles we have two conformal factors

\[
\sigma_1(u) = \lambda_0 - \sum_{\mu} \beta_{\mu}^{(1)} \log |u - u_{\mu}| \quad \sigma_2(u) = \lambda_0 - \sum_{\mu} \beta_{\mu}^{(2)} \log |u - u_{\mu}|
\]  

(65)

The integral of the conformal factors over the boundary is equal to $\lambda_0/3$, and does not differ between two triangles. The integral of the normal derivative at the boundary is zero, because there are no singularities in the interior. The difference of the kinetic energy integrals in \( (39) \) can be written out as

\[
\sum_{\mu} \left( \frac{(\beta_{\mu}^{(2)})^2 - (\beta_{\mu}^{(1)})^2}{12\pi} \right) \int_{D} d^2u \frac{1}{|u - u_{\mu}|^2} + \sum_{\nu \neq \mu} \frac{\beta_{\mu}^{(2)}\beta_{\nu}^{(2)} - \beta_{\mu}^{(1)}\beta_{\nu}^{(1)}}{12\pi} \int_{D} d^2u \text{Re}[\frac{1}{u - u_{\mu}} \frac{1}{u - u_{\nu}}]
\]  

(66)

The diagonal terms are logarithmically divergent. Integrating over the disc except for small semi-discs of radius $\epsilon$ around each $u_{\mu}$ one obtains

\[
\sum_{\mu} \left( \frac{(\beta_{\mu}^{(2)})^2 - (\beta_{\mu}^{(1)})^2}{12} \right) \log \frac{1}{\epsilon} + \mathcal{O}(\epsilon)
\]  

(67)

The off-diagonal terms are finite and integrate to

\[
\sum_{\mu} \left( \frac{(\beta_{\mu}^{(2)})^2 - (\beta_{\mu}^{(1)})^2}{12} \right)(1 + \log 2),
\]  

(68)

if one eliminates the cross products using $\sum_{\mu} \beta_{\mu} = 2$. A comparison with \( (63) \) and \( (75, 76) \) shows that the simple result of \( (68) \) cannot give the complete difference of the functional determinants between two triangles.

### 8 Discussion

We have in this paper investigated functional determinants on simplicial complexes related by conformal distortions. The variation of the functional determinant is then given by the short-time behaviour of the heat kernel, weighted by the variation of the conformal factor \( (19) \).

A general pair of simplicial complexes of genus one or higher are not related by conformal distortions. The methods of this paper can then only give a partial answer. It would therefore be worthwhile to know how to compute the variations under small distortions that are not conformal. For simplicial complexes the most obvious candidate is linear shear. Even in the absence of integrable cases, that would reduce the undetermined integration constants to one for each topology, which is tolerable. The general variational formula for such a computation would be \( (14) \).
One ingredient in an analysis of the variation under linear $s$ would be the behaviour of the variation of the laplacian acting on the Green’s function $G(y, z)$ when the arguments $y$ and $z$ are close. Let us call the finite piece of this expression as $z$ tends to $y$ the point-splitting regularization of $(\delta \Delta / \Delta)$. For surfaces with disc topology it is proportional to the Schwarzian derivative of the mapping from the surface to the unit disc. Let us state in passing that the expression for the change of the functional determinant under linear distortions involving the Schwarzian derivative is also divergent at corners and conical singularities. One may attempt to regularize it by discarding small circles around the conical singularities and keeping the finite piece. For e.g. triangles that gives a different answer than we have computed for conformal $s$ in section 7, but one that is also incorrect if compared with the exact results.

It seems reasonable to expect that the finite in $\epsilon$ contribution in (14) will be the same as that for point-splitting, if we look at points far from the boundary and from conical singularities. The correct contributions from boundaries, corners and conical singularities remain to be computed. We hope to return to these questions in the future.

We now turn to the Polyakov action for random surfaces, expressed as a the double functional integral over $x^\mu$, embeddings in $d$–dimensional external space, and internal two–dimensional metric $g^{ab}$:

$$Z \sim \int D[g^{ab}] D[x^\mu] e^{-\frac{1}{2} \left[ \int \sqrt{g} g^{ab} \partial_a x^\mu \partial_b x^\mu \right]}.$$ (69)

If we perform the integration over $x^\mu$ first, we could formally write

$$Z \sim \int D[g^{ab}] e^{\frac{D}{2} Z_{\mathcal{M}(g)}(0)},$$ (70)

where $\mathcal{M}$ is the surface that corresponds to the metric $g$. In fact, the functional integral over metrics in (69) vastly overcounts the number of inequivalent surfaces, so (70) is meaningless as it stands. In the continuum theory this problem is solved by fixing a gauge for the reparametrization invariance, and computing the associated Fadeev-Popov determinants, which turn out to be proportional to the determinants of the laplacian. The basic problem of the resulting theory seems to be that if the embedding dimension is sufficiently large, these surfaces tend to degenerate into the phase of branched polymers.

If one considers from the beginning summation over simplicial complexes the redundancy of the metric is almost completely removed. If higher genus surfaces are related by a conformal distortion, they are generally so in just one way. There would then be no redundancy. For the disc and the sphere, the representations in terms of charges used in sections 3 and 4 are undetermined up to a group of Moebius transformations, which depends on six real parameters on the sphere, and on three real parameters on the disc. This invariance group is much smaller compared to the integral over metrics.

Because the elimination of the reparametrization invariance happens in a different way than in the continuum theory, and because the expressions of the functional determinants in simplicial complexes are not trivial extrapolations of the results from smooth surfaces,
it is interesting to see if one can say something about $-\frac{D}{2}Z'_M(0)$ as a possible action for random simplicial complexes. It is clear that the ensemble of simplicial complexes with a fixed number of corners and conical singularities, some of which may of course have zero exterior angle, can be considered as a gas of interacting charges. We are to sum over the positions and the strength of the charges. In the end we should then go to a canonical ensemble and sum over all possible number of charges.

According to (77) the self-energies asymptotically damp large opening angles but enhance small opening angles as

$$-\frac{D}{2}Z'_\alpha(0) \sim -0.095496\frac{D}{\alpha}. \tag{71}$$

If we look at how the signs in front of the logarithms go in (50) or (32), we see that like charges attract and unlike repel. In addition, positive charges attract strongly, and negative charges weakly. Taken together, the self-energy and the interaction energy enhance singularities with small opening angles, and such singularities attract each other, the stronger the smaller the opening angles are. For positive $D$ this alone would certainly favour the formation of many spikes in the surface, e.g. something qualitatively like a branched polymer.

There are however three more effects. First, for the topology of a sphere or a disc, there are the gauge fixing conditions. These should take away six real degrees of freedom of motion of the charges on the Riemann sphere, and three real degrees of freedom on the disc. We expect that they can therefore be written as ghost charges, three on the Riemann sphere, and for the disc one in the interior and one on the boundary. In all they compensate for the fact that the gauge group of Moebius transformations is not compact, but they should not matter much for the behaviour when two charges come close. A second effect is that we must choose an integration measure over the positions and the strengths of the charges. In the continuum theory the usual choice is

$$\int_M \sqrt{g} d^2\sigma = \int_M \sqrt{g} e^{2\sigma} (\delta\sigma)^2 d^2z. \tag{72}$$

If we translate (72) to e.g. Schwarz-Christoffel transformations of the plane to a simplicial complex with the topology of a sphere (see (46)) we could consider it to give one more piece of an effective action, depending on the exterior angles and positions of branchpoints.

Finally we should consider fixed area of the surface or fixed length of the boundary. One reason for doing so is that the action in (32) could in general contain also terms $\int_M \sqrt{g} d^2z$ and $\int_{\partial M} ds$, that give respectively the area and the length of the boundary. Alternatively we could say that the zeta function regularization of the functional determinants should be related to a discretization that introduces a cut-off scale. The intrinsic length scale of the surface in then no more arbitrary, and could be argued to be determined by the length scale of the surface embedded in external space [1]. The term from the normalized area $Z_M(0) \log \frac{1}{A_{(M)}}$ will in any case then effectively be one more interaction between the conical singularities.

We can argue qualitatively that this effective interaction favors smooth surfaces. The functional determinants in (31) or (32) are unchanged if the branchpoints are moved about
with Moebius transforms, since the changes in the interaction energies and the effective interaction from the normalized area cancel. If one has two singularities much closer to each other than to the others, one may bring them still closer by a Moebius transform. If the two that are close are positive and the others are negative, the dominant change of the interaction energy should then come from the two that are pushed close. But there is no change in the total energy. It therefore seems likely that the effective interaction of the normalized area evens out with the attractive interaction of like positive charges.

The normalized areas also counteracts the self-energies. Assume some local parameter space (|u| ≤ r) for a conical singularity, with some small radius r. This local parameter space would be the a small circular disc around a branchpoint for surfaces with disc topology. Conical singularities with different opening angles will be mapped from u by $z \sim \frac{1}{\alpha} u^{\alpha}$, and then have areas $A(\alpha, r) \sim r^{2\alpha}/\alpha$. This would give

$$- Z(0) \log A(\alpha, r) \sim - \frac{1}{12\alpha} \log \frac{1}{\alpha} - \frac{1}{12} \log r^2 \quad (\alpha \text{ small}) \quad (73)$$

$$- Z(0) \log A(\alpha, r) \sim - \frac{1}{12} \log \alpha + \frac{1}{12} \alpha^2 \log r^2 \quad (\alpha \text{ large}) \quad (74)$$

Hence here both large and small angle corners are damped, and small angles stronger damped than the enhancement from the self-energy. The argument from (73) is not conclusive of course. It assumes that one may take out one part of the normalized surface, while the effective interaction is proportional to the logarithm of the area of the surface as a whole. One could imagine a surface with a lot of both positive and negative charges such that the total area is around unity. It would then not be damped by the simple effect considered in (73).

Needless to say much work remains to be done to see whether $-\frac{D^2}{2} Z'_\alpha(0)$ is an interesting action for random simplicial complexes. Perhaps the question would have to be answered by a numerical investigation.

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A Summary of properties of $Z'_\alpha(0)$

In this appendix we summarize formulae from I concerning $Z'_\alpha(0)$, the corner self-energy contribution to the functional determinant. The formula written in this appendix are for a corner on the boundary with Dirichlet boundary conditions and opening angle $\pi \alpha$. For Neumann boundary conditions we get instead $Z'_\alpha(0) + \frac{1}{2} \log \alpha$. For a conical singularity in the interior with opening angle $2 \pi \alpha$ we get instead $2Z'_\alpha(0) + \frac{1}{2} \log \alpha$. $Z'_\alpha(0)$ can be written as an integral in different ways, of which one is

$$Z'_\alpha(0) = \frac{1}{12}(\frac{1}{\alpha} - \alpha)(\gamma - \log 2) - \frac{1}{12}(\frac{1}{\alpha} + 3 + \alpha) \log \alpha + \tilde{J}(\alpha), \quad (75)$$
\[ \tilde{J}(\alpha) = \int_0^\infty dy \frac{1}{e^y - 1} \left[ \frac{1}{2y} \left( \coth\left(\frac{y}{2\alpha}\right) - \alpha \coth\left(\frac{y}{2}\right) \right) - \frac{1}{12} \left( \frac{1}{\alpha} - \alpha \right) \right]. \]  

(76)

For small angles the following asymptotic expansion holds:

\[ Z'_\alpha(0) \sim_{\alpha \to 0} \frac{1}{\alpha} \left( 1 + 3 + \alpha \right) \log \alpha + \alpha \left( 1 + 3 + \alpha \right) \frac{\gamma + \log 2}{12} \]
\[ + \frac{1}{\alpha} \left( \frac{\gamma - \log 2}{12} \right) + \sum_{n=3}^{\infty} \frac{\zeta(n) B_{n+1}}{n(n+1)} \alpha^{-n}. \]  

(77)

The leading behaviour in (77) is \((0.190992 \ldots) / \alpha\). The asymptotic expansion for large angles reads:

\[ Z'_\alpha(0) \sim_{\alpha \to \infty} -\frac{1}{12} \left( \frac{1}{\alpha} + 3 + \alpha \right) \log(2) - \frac{1}{4} \log(2\pi) - \zeta'(-1) + \frac{1}{\alpha} \left( \frac{\gamma - \log 2}{12} \right) + \sum_{n=3}^{\infty} \frac{\zeta(n) B_{n+1}}{n(n+1)} \alpha^{-n}, \]  

(78)

for which the leading behaviour is \(-\frac{1}{12} \alpha \log \alpha\). We may also expand around a flat surface i.e. \(\alpha\) close to one:

\[ Z'_{1+\epsilon}(0) = \left( \frac{1}{6} \log 2 - \frac{5}{24} - \frac{1}{4} \log(2\pi) - \zeta'(-1) \right) \epsilon \]
\[ + \left( \frac{14}{72} + \frac{\gamma - \log 2}{12} \right) \epsilon^2 + \left( -\frac{29}{144} - \frac{\gamma - \log 2}{12} \right) \epsilon^3 + \mathcal{O}(\epsilon^4). \]  

(79)

For rational \(\alpha\) we evaluated in I in closed form \(Z'_\alpha(0)\) to be

\[ Z'_{p/q}(0) = \frac{q-p}{4q} \log(2\pi) + \frac{p^2 - q^2}{12pq} \log(2) - \frac{1}{q} \left( p - \frac{1}{p} \right) \zeta'(-1) - \]
\[ \frac{1}{12pq} \log(q) + \left( \frac{1}{4} + S(q,p) \right) \log\left( \frac{q}{p} \right) + \sum_{r=1}^{p-1} \left( \frac{1}{2} - \frac{r}{p} \right) \log(\Gamma\left( \frac{R(q,p)}{p} \right)) + \sum_{s=1}^{q-1} \left( \frac{1}{2} - \frac{s}{q} \right) \log(\Gamma\left( \frac{R(q,p)}{q} \right)), \]

where \(R(p,q)\) and \(S(p,q)\) are defined by:

\[ R(q,p) \equiv q - p \left[ \frac{q}{p} \right], \]  

(81)

\[ S(q,p) \equiv \frac{1}{p} \sum_{r=0}^{p-1} r \left( \frac{R(q,p)}{p} - \frac{1}{2} \right). \]  

(82)

For the special case \(p = 1\), (80) reduces to:

\[ Z'_{1/n}(0) = \frac{1}{4} \left( 1 - \frac{1}{n} \right) \log(\pi) - \left( \frac{n}{12} - \frac{1}{4} + \frac{1}{6n} \right) \log(2) + \]
\[ \left( \frac{1}{4} - \frac{1}{12n} \right) \log(n) + \sum_{\nu=1}^{n-1} \left( \frac{1}{2} - \frac{\nu}{n} \right) \log(\Gamma\left( \frac{\nu}{n} \right)). \]  

(83)
In the test with integrable domains we need $Z'_{1/n}(0)$ for $n = 2, 3, 4, 6$ which are therefore reproduced in the table below.

| $n$ | $Z'_{1/n}(0)$ |
|-----|--------------|
| 2   | $\log \frac{\pi}{2\pi}$ |
| 3   | $\log \frac{\pi^2 \cdot \pi^3 \cdot \pi^3}{1^4(4)}$ |
| 4   | $\log \frac{\pi \cdot \pi^2 \cdot \pi^4}{1^4(4)}$ |
| 6   | $\log \frac{\pi \cdot \pi^2 \cdot \pi^3 \cdot \pi^6}{1^4(4)}$ |

## B Exactly solvable spectra

In this appendix we list the orbifolds and planar polygons that can be obtained by quoting a torus with a symmetry. The zeta functions are computed from the spectrum with methods from number theory. See I for a computation of cases (121), (146) or (123) below, or [7] for a recent review.

**Rhomboidal lattice:**

A rhomboidal lattice $\Lambda^\tau$ is spanned by two basis vectors $\vec{e}_1$ and $\vec{e}_2$. We may without restriction take $\vec{e}_1$ to be the unit vector. The two-dimensional plane quoted by the lattice is then identified as the torus with modular parameter $\tau$ equal to $\vec{e}_2$, read as a complex number.

The eigenfunctions of the laplacian on the torus are

$$\Psi_{mn}^\tau = e^{2\pi i (m\vec{f}_1 + n\vec{f}_2) \cdot \vec{x}} \quad m \in \mathbb{Z}, n \in \mathbb{Z}$$

where $\vec{f}_1$ and $\vec{f}_2$ are the two dual basis vectors. The basis and the dual basis satisfy $\vec{e}_i \cdot \vec{f}_j = \delta_{ij}$. The eigenvalues of the laplacian and the spectral zeta function are

$$\lambda_{mn} = 4\pi^2 (m^2 |\vec{f}_1|^2 + 2mn \vec{f}_1 \cdot \vec{f}_2 + n^2 |\vec{f}_2|^2)$$

$$Z(\Lambda^\tau, s) = \sum_{(m,n) \neq (0,0)} (m^2 |\vec{f}_1|^2 + 2mn \vec{f}_1 \cdot \vec{f}_2 + n^2 |\vec{f}_2|^2)^{-s}$$

For simplicity we have in this appendix chosen to form the zeta function without the overall prefactor $4\pi^2$ in the eigenvalues.

The behaviour around the origin of (86) is [10]

$$Z(\Lambda^\tau, 0) = -1 \quad Z'(\Lambda^\tau, 0) = -\frac{1}{4} \log \text{Area} - \log \tau^4 \eta(q)$$

$$q = e^{2\pi i \tau} \quad \eta(q) = q^{\frac{1}{8}} \prod_{m=1}^{\infty} (1 - q^m)$$
where Area is the area of the torus, and $\eta$ is the modular form of Dedekind.

In general the only subgroup of the rhomboidal lattice is a rotation through $\pi$. Quoting out with that symmetry leads to rhombiodal envelope, visualized as two smaller copies of the torus with free sides, glued together at the four sides. The topology of the envelope is the sphere. The exterior angles at the corners are all $\pi$. The eigenvalues of the laplacian on the envelope are the same as on the torus, except that the degeneracy between states $(m, n)$ and $(-m, -n)$ have been divided out.

**Rhombic lattice:**

A rhombic lattice is generated by two lattice vectors of the same length. That means that the modular parameter is of absolute value one. In addition to rotation through $\pi$ there are now two reflexion symmetries, that we call $S_1$ and $S_2$. The reflexion symmetry lines must necessarily be normal to each other. When we quote with a reflexion symmetry, we must either add (for Neumann boundary conditions) or subtract (for Dirichlet boundary conditions) points on the reflexion line in the dual lattice. We therefore need two auxiliary number sequences:

\[
\begin{align*}
l_1 &= \{ |\vec{f}_1 + \vec{f}_2|^2 m^2 ; m > 0 \} \\
l_2 &= \{ |\vec{f}_1 - \vec{f}_2|^2 m^2 ; m > 0 \}
\end{align*}
\]  

(89) (90)

for which the spectral quantities are expressed in terms of the Riemann zeta function:

\[
\begin{align*}
Z(l_1, s) &= \sum_{m>0} |\vec{f}_1 + \vec{f}_2|^{-2s(m^2)^{-s}} = |\vec{f}_1 + \vec{f}_2|^{-2s}\zeta(2s) \\
Z(l_2, s) &= \sum_{m>0} |\vec{f}_1 - \vec{f}_2|^{-2s(m^2)^{-s}} = |\vec{f}_1 - \vec{f}_2|^{-2s}\zeta(2s)
\end{align*}
\]  

(91) (92)

The subgroups, their associated surfaces and spectral quantities are

1. $H = \{1\}$. This is rhombic torus, spectrum $\Lambda^r_1$.
2. $H = \{1, R_\pi\}$. This is a rhombic envelope, spectrum $\Lambda^r_\pi$.
3. $H = \{1, S_1\}$ or $H = \{1, S_2\}$ These are both Moebius bands with spectra respectively $\Lambda^r_2 \pm l_1$ and $\Lambda^r_2 \pm l_2$.
4. $H = \{1, S_1, S_2, R_\pi\}$. This is a cone with opening angle $\pi$. The base has two lines and two corners, both with opening angle $\pi/2$. The topology is that of a disc. The spectra are $\frac{1}{2} (\Lambda^r_2 \pm (l_1 + l_2))$.

**Rectangular lattice:**

A rectangular lattice is generated by two lattice vectors normal to each other. That
means that the modular parameter is purely imaginary. The symmetries are the reflections in the two lines parallel to the basis vectors and a rotation through $\pi$. It is again convenient to introduce number sequences $l_1$ and $l_2$, but this time defined as

$$l_1 = \{|\vec{f}_1|^2 m^2; m > 0\}$$  \hspace{1cm} (93)

$$l_2 = \{|\vec{f}_2|^2 m^2; m > 0\}$$  \hspace{1cm} (94)

The subgroups, their associated surfaces and spectral quantities are

1. $H = \{1\}$. This is a rectangular torus, spectrum $\Lambda_1^\tau$.

2. $H = \{1, R_\pi\}$. This is a rectangular envelope, spectrum $\Lambda_\frac{\pi}{2}^\tau$.

3. $H = \{1, S_0\}$ or $H = \{1, S_{\frac{\pi}{2}}\}$ These are rectangular bands. Spectra are $\Lambda_\frac{\pi}{2}^\tau \pm l_1$ and $\Lambda_\frac{\pi}{2}^\tau \pm l_2$.

4. $H = \{1, S_0, S_1, R_\pi\}$. This is a rectangle. Sidelength is half of the initial rectangular torus. Spectra are $\frac{1}{2}(\Lambda_\frac{\pi}{2}^\tau \pm (l_1 + l_2))$.

**Hexagonal lattice:**

The two-dimensional hexagonal lattice is spanned by the basis

$$\vec{e}_1 = (1, 0) \quad \vec{e}_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$  \hspace{1cm} (95)

The quotient of the plane with this lattice is identified with the torus with modular parameter $\tau = e^{2\pi i/3}$. The dual basis to (95) is

$$\vec{f}_1 = (1, \frac{1}{\sqrt{3}}) \quad \vec{f}_2 = (0, \frac{2}{\sqrt{3}})$$  \hspace{1cm} (96)

The eigenvalues of the laplacian are

$$\lambda_{mn} = 4\pi^2 \frac{4}{3} (m^2 + mn + n^2)$$  \hspace{1cm} (97)

To simplify we will in the following ignore the overall prefactor $4\pi^2 \frac{4}{3}$ in the eigenvalues.

The Weyl group of the hexagonal lattice consists of six rotations and six reflexions. Call $R_\alpha$ the rotation through angle an $\alpha$, which must be a multiple of $\pi/3$, and $S_\beta$ the reflexion through a line inclined an angle $\beta$ to the horizontal. By convention we take $\beta$’s equal to $\nu \pi/6$, with $\nu = 0, \ldots, 5$. The symmetry lines of even $\nu$ are parallel to vectors in the hexagonal lattice, while the lines of odd $\nu$ are parallel to vectors in the dual lattice. They hence transform the lattice differently. The subgroups of the Weyl group can contain either only rotations, or as many reflexions as rotations. Since reflexions with even and
odd $\nu$ act differently there are two inequivalent subgroups containing either one or three reflexions. In all we have ten inequivalent surfaces. Counting Neumann and Dirichlet and boundary conditions for the surfaces with a boundary (symmetric and anti-symmetric representations of the reflexions in the subgroup) we arrive at 16 different spectra.

It is convenient to first introduce a notation for some spectral sequences:

$$
\Lambda^e_{2\pi i} = \{ m^2 + mn + n^2; m \in \mathbb{Z}, n \in \mathbb{Z}; (m, n) \neq (0, 0) \} \quad (98)
$$

$$
\Lambda^e_{2\pi i} = \{ \Lambda^e_{2\pi i}; m > 0, \text{or } m = 0, n > 0 \} \quad (99)
$$

$$
\Lambda^e_{3\pi i} = \{ \Lambda^e_{2\pi i}; m > 0, n \leq 0 \} \quad (100)
$$

$$
\Lambda^e_{5\pi i} = \{ \Lambda^e_{2\pi i}; m > 0, n \geq 0 \} \quad (101)
$$

$$
l_1 = \{ m^2; m > 0 \} \quad (102)
$$

$$
l_1 = \{ 3m^2; m > 0 \} \quad (103)
$$

The meaning of these sequences is that $\Lambda^e_{k\pi i}$ is the lattice quoted by a subgroup of order $k$ containing only rotations. The spectral functions will be expressed in terms of the Riemann zeta function and

$$
Z(\Lambda^e_{k\pi i}, s) = \sum_{m > 0, n > 0} (m^2 + mn + n^2)^{-s} = L_3(s)\zeta(s) \quad (104)
$$

with

$$
L_3(s) = 1 - 2^{-s} + 4^{-s} - 5^{-s} + \cdots \quad (105)
$$

The behaviour around the origin of the Riemann zeta function and (104) is

$$
\zeta(0) = -\frac{1}{2} \quad \zeta'(0) = -\frac{1}{2} \log 2\pi \quad (106)
$$

$$
Z(\Lambda^e_{k\pi i}, 0) = -\frac{1}{6} \quad Z'(\Lambda^e_{k\pi i}, 0) = \frac{1}{2} \log \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} - \frac{1}{6} \log \frac{2\pi}{3} \quad (107)
$$

We now list the possible inequivalent subgroups, their associated surfaces and spectral quantities. For surfaces with boundary we use the convention that the result for Neumann boundary conditions is given first (generally with a plus sign), and then the result with Dirichlet boundary conditions (generally with a minus sign).

1. $H = \{1\}$. This is the torus itself, spectrum $\Lambda^e_{2\pi i}$.

$$
Z(\Lambda^e_{2\pi i}, 0) = -1 \quad Z'(\Lambda^e_{2\pi i}, 0) = 3\log \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} - \log \frac{2\pi}{3} \quad (108)
$$

2. $H = \{1, R_\pi\}$. This is the tetrahedron, spectrum $\Lambda^e_{3\pi i}$.

$$
Z(\Lambda^e_{3\pi i}, 0) = -\frac{1}{2} \quad Z'(\Lambda^e_{3\pi i}, 0) = \frac{3}{2} \log \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} - \frac{1}{2} \log \frac{2\pi}{3} \quad (109)
$$
3. $H = \{1, R_{\pm \frac{\pi}{3}}\}$. This is the equilateral triangle envelope, visualized as two equilateral triangles glued together at the three sides. Spectrum is $\Lambda_{\frac{2\pi i}{3}}$.

$$Z(\Lambda_{\frac{2\pi i}{3}}, 0) = -\frac{1}{3} \quad Z'(\Lambda_{\frac{2\pi i}{3}}, 0) = \log \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} - \frac{1}{3} \log \frac{2\pi}{3}$$  \hspace{1cm} (110)

4. $H = \{1, R_{\pm \frac{\pi}{4}}, R_{\pm \frac{\pi}{2}}, R_{\pi}\}$. This is bisected equilateral triangle envelope, visualized as two triangles with angles $\pi/2$, $\pi/3$ and $\pi/6$, glued together at the three sides. Spectrum is $\Lambda_{\frac{2\pi i}{6}}$.

$$Z(\Lambda_{\frac{2\pi i}{6}}, 0) = -\frac{1}{6} \quad Z'(\Lambda_{\frac{2\pi i}{6}}, 0) = \frac{1}{2} \log \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} - \frac{1}{6} \log \frac{2\pi}{3}$$  \hspace{1cm} (111)

5. $H = \{1, S_{\pm \frac{\pi}{3}}\}$. This is a Moebius band. The spectra are $\Lambda_{\frac{2\pi i}{2}} \pm l_{\frac{1}{\sqrt{3}}}$.

$$Z(\Lambda_{\frac{2\pi i}{2}} + l_{\frac{1}{\sqrt{3}}}, 0) = -1 \quad Z'(\Lambda_{\frac{2\pi i}{2}} + l_{\frac{1}{\sqrt{3}}}, 0) = \frac{3}{2} \log \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} + \log 3 - \frac{3}{2} \log 2\pi$$ \hspace{1cm} (112)

$$Z(\Lambda_{\frac{2\pi i}{2}} - l_{\frac{1}{\sqrt{3}}}, 0) = 0 \quad Z'(\Lambda_{\frac{2\pi i}{2}} - l_{\frac{1}{\sqrt{3}}}, 0) = \frac{3}{2} \log \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} + \frac{1}{2} \log 2\pi$$  \hspace{1cm} (113)

6. $H = \{1, S_{\pm \frac{\pi}{6}}\}$. This is another Moebius band. The spectra are $\Lambda_{\frac{2\pi i}{2}} \pm l_{1}$.

$$Z(\Lambda_{\frac{2\pi i}{2}} + l_{1}, 0) = -1 \quad Z'(\Lambda_{\frac{2\pi i}{2}} + l_{1}, 0) = \frac{3}{2} \log \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} + \frac{1}{2} \log 3 - \frac{3}{2} \log 2\pi$$ \hspace{1cm} (114)

$$Z(\Lambda_{\frac{2\pi i}{2}} - l_{1}, 0) = 0 \quad Z'(\Lambda_{\frac{2\pi i}{2}} - l_{1}, 0) = \frac{3}{2} \log \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} + \frac{1}{2} \log 3 + \frac{1}{2} \log 2\pi$$  \hspace{1cm} (115)

7. $H = \{1, S_{\pm \frac{\pi}{3}}, S_{\pm \frac{\pi}{2}}, R_{\pi}\}$. This is a cone with opening angle $\pi$. The base has one longer and one shorter side, and two corners with angles $\pi/2$. The spectra are $\frac{3}{2} \Lambda_{\frac{2\pi i}{6}} \pm (l_{1} + l_{\frac{1}{\sqrt{3}}})$.

$$Z\left(\frac{3}{2} \Lambda_{\frac{2\pi i}{6}} + l_{1} + l_{\frac{1}{\sqrt{3}}}, 0\right) = -\frac{3}{4} \quad Z'\left(\frac{3}{2} \Lambda_{\frac{2\pi i}{6}} + l_{1} + l_{\frac{1}{\sqrt{3}}}, 0\right) = \frac{3}{4} \log \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} + \frac{1}{2} \log 3 - \frac{3}{2} \log 2\pi$$ \hspace{1cm} (116)

$$Z\left(\frac{3}{2} \Lambda_{\frac{2\pi i}{6}} - l_{1} - l_{\frac{1}{\sqrt{3}}}, 0\right) = \frac{1}{4} \quad Z'\left(\frac{3}{2} \Lambda_{\frac{2\pi i}{6}} - l_{1} - l_{\frac{1}{\sqrt{3}}}, 0\right) = \frac{3}{4} \log \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} + \frac{1}{2} \log 2\pi$$  \hspace{1cm} (117)
8. \( H = \{1, S_0, S_x, S_y, R_{\pm \frac{2\pi}{3}}\} \). This is a cone with opening angle \(\frac{2\pi}{3}\). The base has one side and one corner with angle \(\frac{\pi}{3}\). The spectra are \(\Lambda_{\pm \frac{2T_k}{3}} \pm l\).

\[
Z(\Lambda_{\pm \frac{2T_k}{3}}, 0) = -\frac{2}{3} \quad Z'(\Lambda_{\pm \frac{2T_k}{3}}, 0) = \frac{1}{2} \log \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}
\]

\[
+ \frac{2}{3} \log 3 - \frac{7}{6} \log 2\pi
\]

(118)

\[
Z(\Lambda_{\pm \frac{2T_k}{3}}, 0) = \frac{1}{3} \quad Z'(\Lambda_{\pm \frac{2T_k}{3}}, 0) = \frac{1}{2} \log \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}
\]

\[
- \frac{1}{3} \log 3 + \frac{5}{6} \log 2\pi
\]

(119)

9. \( H = \{1, S_x, S_y, S_{\pm \frac{\pi}{3}}, S_{\pm \frac{2\pi}{3}}, \pm \frac{2\pi}{3}\} \). This is the equilateral triangle. The spectra are \(\Lambda_{\pm \frac{2T_k}{3}} \pm l_1\).

\[
Z(\Lambda_{\pm \frac{2T_k}{3}} + l_1, 0) = -\frac{2}{3} \quad Z'(\Lambda_{\pm \frac{2T_k}{3}} + l_1, 0) = \frac{1}{2} \log \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}
\]

\[
+ \frac{1}{6} \log 3 - \frac{7}{6} \log 2\pi
\]

(120)

\[
Z(\Lambda_{\pm \frac{2T_k}{3}} - l_1, 0) = \frac{1}{3} \quad Z'(\Lambda_{\pm \frac{2T_k}{3}} - l_1, 0) = \frac{1}{2} \log \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}
\]

\[
- \frac{1}{3} \log 3 + \frac{5}{6} \log 2\pi
\]

(121)

10. \( H = \) Entire Weyl group. This is the bisected equilateral triangle with angles \(\pi/2\), \(\pi/3\) and \(\pi/6\). The spectra are \(\frac{1}{2}(\Lambda_{\pm \frac{2T_k}{3}} \pm (l_1 + l_1\sqrt{3}))\).

\[
Z\left(\frac{1}{2}(\Lambda_{\pm \frac{2T_k}{3}} + l_1 + l_1\sqrt{3}), 0\right) = -\frac{7}{12} \quad Z'\left(\frac{1}{2}(\Lambda_{\pm \frac{2T_k}{3}} + l_1 + l_1\sqrt{3}), 0\right) = \frac{1}{4} \log \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}
\]

\[
+ \frac{7}{3} \log 3 - \frac{13}{12} \log 2\pi
\]

(122)

\[
Z\left(\frac{1}{2}(\Lambda_{\pm \frac{2T_k}{3}} - l_1 - l_1\sqrt{3}), 0\right) = \frac{5}{12} \quad Z'\left(\frac{1}{2}(\Lambda_{\pm \frac{2T_k}{3}} - l_1 - l_1\sqrt{3}), 0\right) = \frac{1}{4} \log \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}
\]

\[
- \frac{1}{6} \log 3 + \frac{11}{12} \log 2\pi
\]

(123)

Square lattice:
The eigenfunctions of the laplacian on the square torus are
\[ \Psi_{mn} = e^{2\pi i(mx_1 + nx_2)} \quad m \in \mathbb{Z}, n \in \mathbb{Z} \] (124)
with eigenvalues
\[ \lambda_{mn} = 4\pi^2(m^2 + n^2) \] (125)
The modular parameter is \( i \).

The Weyl group of the square lattice consists of four rotations and four reflexions. The reflexion symmetry lines parallel to the diagonals (\( S_{\pi/4} \) and \( S_{3\pi/4} \)) act differently than reflexions in the lines parallel to the lattice vectors (\( S_0 \) and \( S_{\pi/2} \)). We have thus 8 inequivalent subgroups, and counting Neumann and Dirichlet boundary conditions for the surfaces with a boundary, 13 different spectra.

It is again convenient to introduce a notation for some spectral sequences:
\[ \Lambda_{1}^i = \{m^2 + n^2; m \in \mathbb{Z}, n \in \mathbb{Z}; (m, n) \neq (0, 0)\} \] (126)
\[ \Lambda_{1/2}^i = \{\Lambda_1; m > 0, \text{ or } m = 0, n > 0\} \] (127)
\[ \Lambda_{1/4}^i = \{\Lambda_1; m > 0, n \geq 0\} \] (128)
\[ l_1 = \{m^2; m > 0\} \] (129)
\[ l_{1/2} = \{2m^2; m > 0\} \] (130)

We will need
\[ Z(\Lambda_{1/4}^i, s) = \sum_{m>0, n \geq 0} (m^2 + n^2)^{-s} = L_4(s)\zeta(s) \] (131)
with
\[ L_4(s) = 1 - 3^{-s} + 5^{-s} - 7^{-s} + \cdots \] (132)
The behaviour around the origin of (131) is
\[ Z(\Lambda_{1/4}^i, 0) = -\frac{1}{4} \quad Z'(\Lambda_{1/4}^i, 0) = \frac{1}{2} \log \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} + \frac{1}{4} \log \frac{1}{2\pi} + \frac{1}{2} \log 2 \] (133)

We now list the possible inequivalent subgroups, their associated surfaces and spectral quantities.

1. \( H = \{1\} \). This is the square torus, spectrum \( \Lambda_1 \).
\[ Z(\Lambda_1^i, 0) = -1 \quad Z'(\Lambda_1^i, 0) = 2 \log \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} + \log \frac{1}{2\pi} + 2 \log 2 \] (134)

2. \( H = \{1, R_{\pi}\} \). This is a square envelope, spectrum \( \Lambda_{1/2} \).
\[ Z(\Lambda_{1/2}^i, 0) = -\frac{1}{2} \quad Z'(\Lambda_{1/2}^i, 0) = \log \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} + \frac{1}{2} \log \frac{1}{2\pi} + \log 2 \] (135)
3. \( H = \{1, R_{\pm\frac{\pi}{2}}, R_\pi\} \). This is a right angle isosceles triangle envelope, visualized as two triangles with angles \( \frac{\pi}{2}, \frac{\pi}{4}, \) and \( \frac{\pi}{4} \) glued together at the sides. Spectrum is \( \Lambda_{\frac{\pi}{2}}^i \).

\[
Z(\Lambda_{\frac{\pi}{2}}^i, 0) = -\frac{1}{4} \quad Z'(\Lambda_{\frac{\pi}{2}}^i, 0) = \frac{1}{2} \log \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} + \frac{3}{4} \log \frac{1}{2\pi} + \frac{1}{2} \log 2 \quad (136)
\]

4. \( H = \{1, S_0\} \). This is a rectangular band. The spectra are \( \Lambda_{\frac{\pi}{2}}^i \pm l_1 \).

\[
Z(\Lambda_{\frac{\pi}{2}}^i + l_1, 0) = -1 \quad Z'(\Lambda_{\frac{\pi}{2}}^i + l_1, 0) = \log \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} + \frac{3}{2} \log \frac{1}{2\pi} + \log 2 \quad (137)
\]

\[
Z(\Lambda_{\frac{\pi}{2}}^i - l_1, 0) = 0 \quad Z'(\Lambda_{\frac{\pi}{2}}^i - l_1, 0) = \log \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} - \frac{1}{2} \log \frac{1}{2\pi} + \log 2 \quad (138)
\]

5. \( H = \{1, S_{\frac{\pi}{2}}\} \). This is Moebius band. The spectra are \( \Lambda_{\frac{\pi}{2}}^i \pm l_{\sqrt{2}} \).

\[
Z(\Lambda_{\frac{\pi}{2}}^i + l_{\frac{1}{\sqrt{2}}}, 0) = -1 \quad Z'(\Lambda_{\frac{\pi}{2}}^i + l_{\frac{1}{\sqrt{2}}}, 0) = \log \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} + \frac{3}{2} \log \frac{1}{2\pi} + \frac{3}{2} \log 2 \quad (139)
\]

\[
Z(\Lambda_{\frac{\pi}{2}}^i - l_{\frac{1}{\sqrt{2}}}, 0) = 0 \quad Z'(\Lambda_{\frac{\pi}{2}}^i - l_{\frac{1}{\sqrt{2}}}, 0) = \log \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} - \frac{1}{2} \log \frac{1}{2\pi} + \frac{1}{2} \log 2 \quad (140)
\]

6. \( H = \{1, S_0, S_{\frac{\pi}{2}}, R_\pi\} \). This is the square. The spectra, which could of course equally well have been written down directly, are \( \Lambda_{\frac{\pi}{4}}^i \pm l_1 \).

\[
Z(\Lambda_{\frac{\pi}{4}}^i + l_1, 0) = -\frac{3}{4} \quad Z'(\Lambda_{\frac{\pi}{4}}^i + l_1, 0) = \frac{1}{2} \log \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} + \frac{5}{4} \log \frac{1}{2\pi} + \frac{1}{2} \log 2 \quad (141)
\]

\[
Z(\Lambda_{\frac{\pi}{4}}^i - l_1, 0) = \frac{1}{4} \quad Z'(\Lambda_{\frac{\pi}{4}}^i - l_1, 0) = \frac{1}{2} \log \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} - \frac{3}{4} \log \frac{1}{2\pi} + \frac{1}{2} \log 2 \quad (142)
\]
7. $H = \{1, S_{\frac{\pi}{4}}, S_{\frac{3\pi}{4}}, R_{\pi}\}$. This is a cone with opening angle $\pi$. The base has two sides of equal length, and two corners with angles $\pi/2$. The spectra are $\Lambda^{i}_{\frac{3}{4}} \pm \frac{1}{\sqrt{2}}$.

\[
Z(\Lambda^{i}_{\frac{3}{4}} + \frac{l}{1/\sqrt{2}}), 0) = -\frac{3}{4}, \quad Z'(\Lambda^{i}_{\frac{3}{4}} + \frac{l}{1/\sqrt{2}}), 0) = \frac{1}{2} \log \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{1}{4})} + \frac{5}{4} \log \frac{1}{2\pi} + \log 2
\]

(143)

\[
Z(\Lambda^{i}_{\frac{3}{4}} - \frac{l}{1/\sqrt{2}}), 0) = \frac{1}{4}, \quad Z'(\Lambda^{i}_{\frac{3}{4}} - \frac{l}{1/\sqrt{2}}), 0) = \frac{1}{2} \log \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{1}{4})} - \frac{3}{4} \log \frac{1}{2\pi}
\]

(144)

8. $H = \text{Entire Weyl group}$. This is the right angles isosceles triangle, with angles $\pi/2$, $\pi/4$ and $\pi/4$. The spectra are $\frac{1}{2}(\Lambda^{i}_{\frac{3}{4}} \pm (l_1 + l_2))$.

\[
Z\left(\frac{1}{2}(\Lambda^{i}_{\frac{3}{4}} + l_1 + \frac{l}{1/\sqrt{2}}) , 0\right) = -\frac{5}{8}, \quad Z'\left(\frac{1}{2}(\Lambda^{i}_{\frac{3}{4}} + l_1 + \frac{l}{1/\sqrt{2}}) , 0\right) = \frac{1}{4} \log \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{1}{4})} + \frac{9}{8} \log \frac{1}{2\pi} + \frac{1}{2} \log 2
\]

(145)

\[
Z\left(\frac{1}{2}(\Lambda^{i}_{\frac{3}{4}} - l_1 + \frac{l}{1/\sqrt{2}}) , 0\right) = \frac{3}{8}, \quad Z'\left(\frac{1}{2}(\Lambda^{i}_{\frac{3}{4}} - l_1 + \frac{l}{1/\sqrt{2}}) , 0\right) = \frac{1}{4} \log \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{1}{4})} - \frac{7}{8} \log \frac{1}{2\pi}
\]

(146)

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