HYPERGEOMETRIC FUNCTION AND MODULAR CURVATURE. II. CONNES-MOSCOVICI
FUNCTIONAL RELATION AFTER LESCH’S WORK

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ABSTRACT. As the second part of the sequel, we investigate the variation of rearrangement operators
(more precisely, the spectral functions behind) arising in the study of modular geometry on noncom-
mutative (two) tori. We initiate a systematic approach by introducing transformations corresponding
to basic operations in calculus, like differentiation and integration by parts. As for applications, we
extend, in a uniform way, the Connes-Moscovici’s functional relations on noncommutative two tori
attached to the variation of second heat coefficients to noncommutative tori of arbitrary dimension.
Moreover, those transformations lead to more internal relations among the hypergeometric family
obtained in part I of the sequel, which allows us to obtain, the first time, a computer-aid free verification
of those Connes-Moscovici type functional relations.

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1. Introduction

As a continuation of [23], the paper concerns the variational aspect of the $a_2$-coefficient of the heat trace asymptotic on noncommutative tori and toric noncommutative manifolds\(^1\). The integration of the $a_2$-coefficients allows us to establish the analogue of Gauss-Bonnet theorem on noncommutative two tori [3] and [11] and the full $a_2$-coefficients gives rise to the notion of scalar curvature in conformal geometry on noncommutative two tori [7], [20], also [12] [13] and later on toric noncommutative manifolds [22] [24]. The complexity of the calculation increases dramatically for the next coefficient, the $a_4$-term. An experimental computation has been achieved in [4] which is in need of more conceptual understanding and further simplification. To this end, we have accomplished in [23] the first step suggested by Connes, namely, to find a suitable basis of functions to record the spectral functions arising from the rearrangement process. It means that, for instance, one can replace the explicit expressions of the functions $K_1$ to $K_{20}$ of the $a_4$-term in [4] by combinations of the hypergeometric family $H_{\alpha}$, $\alpha \in \mathbb{Z}_{>0}$ obtained [23] \(^2\) similar to the form shown in Eqs. (4.2) and (4.3). As we can see from [4] §9 and Appendix C], their original form generated from a CAS (computer algebra system) takes pages to record. The next step, which is one of the primary motivations of the paper, is to search for simplifications for the functional relations cf. [4] §7. Back to the $a_2$-term, there are only two functions and the relation (2.12) \(^3\) simply reads

$$-rac{1}{2} \tilde{H}(s_1, s_2) = \frac{\tilde{K}(s_2) - \tilde{K}(s_1)}{s_1 + s_2} + \frac{\tilde{K}(s_1 + s_2) - \tilde{K}(s_2)}{s_1} - \frac{\tilde{K}(s_1 + s_2) - \tilde{K}(s_1)}{s_2}.$$  

(1.1)

It reflects the variational nature of modular Gaussian curvature $K_{\varphi}$ (7 Eq. (4.34)) and similar results in the paper are often referred as Connes-Moscovici type functional relations.

Even for the $a_2$-term, when performing similar calculus on higher dimensional examples, it is not obvious, at least to the author, that one can obtain such a simple equation as Eq. (1.1). After repeating the variational computation several times, the author finally realized that the sum of [22] Eq. (4.5)-(4.8) \(^3\) indeed admits reduction to a similar form of Eq. (1.1). The first contribution \(^4\) of the paper consists of a systematic framework (built upon the work of Lešč [19]) for the variational calculus for a noncommutative variable. The fundamental obstruction is the fact that the variable and its derivatives do not commute. There is a rearrangement process that compresses the Ansatz to some rearrangement operators (or equivalently, to the underlying spectral functions, cf. §2.1).

To keep track of the change of those functions under basic operations in variational calculus, we introduce the following transformations

- with respect to the log-Weyl factor $h$: $\{t, \tau_j, \eta_{0,j}, \eta^+, \eta^-\}$,
- with respect to the Weyl factor $k = e^h$: $\{\eta, \sigma_j, \eta_{0,j}, \eta^+, \sigma^-\}$.

The transformations $\{\eta_{0,j}, \eta^+, \eta^-\}$ arising from differentiation are generated by divided difference, thanks to [19] §3.5. The cyclic operator $\tau_j$ (resp. $\sigma_j$) resembles integration by parts with respect to the derivation $[\cdot, h]$ and the volume weight $\varphi_j$ in Eq. (2.10). A noteworthy new observation (Proposition 2.11) is a set of internal relations, which explains the cancellation behind [22] Eq. (4.5)-(4.8) mentioned above. As a consequence, the whole variational calculus can be generated by only three transformations $\{t, \tau_j, \eta^+\}$, $\{\eta, \sigma_j, \eta^+\}$. In contrast to classical calculus, formulas under the simple the change of variable $h \rightarrow k = e^h$ could be quite subtle, again due to the noncommutativity between $h$ and its derivatives. We keep parallel discussions on purpose.

For instance, the main result has two versions Theorem 2.15 and Theorem 2.17 \(^4\) in terms of $h$ and $k$ respectively, which generalizes [7] Thm. 4.10. Later, in Prop. 4.10 we carried out a direct verification of the equivalence of the two versions.

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\(^1\)a.k.a Connes-Landi deformations or $\theta$-deformations, cf. [8] [5], [24], also [29].

\(^2\)The author has finished the computation but the result will be postponed to future publications.

\(^3\)The relations have appeared before in the proof of Theorem 5.1 in [10].
The abstract discussion in §2 is designed for gradient computation in the modular geometry on noncommutative tori explained in §3. We first extend the geometric functional attached to the $a_2$-coefficient from dimension two (cf. [17] §4) two to higher dimensions. Their functional gradients can be reached via two routes [4]. One relies on variation on the heat trace $\text{Tr}(e^{-t\Delta})$ or the spectral zeta function $\zeta_{\Delta}(s)$. The results are recorded in §3.3 and 3.3. The other approach (§4.2 and 4.3) makes use of Theorems 2.15 and 2.17, which brings in the functional relations. By equating the two methods, we see that the two coeﬃcients (Eqs. (4.2) and (4.3)) of $R_{\Delta}$ are subject to the corresponding functional relations inherited from the gradient $\text{grad}_x F$ (resp. $\text{grad}_k F$). A crucial technical result that is only quoted without proof is Prop. 4.1 the explicit form of $R_{\Delta}$, as it is the main focus of part I of the sequel [23].

If one is willing to think of the lengthy computation of the heat coeﬃcients as a physics theory, then the functional relations (like Eq. (1.1)) derived from variation provides, so far, the most robust experiment for testing its validation. For the $a_2$-coefficient [4], similar verification is indeed the most sophisticated part hidden behind the paper. The reader is encouraged to go through the Mathematica notebook files attached to §2 and §4 to see how the tests were actually achieved. With the better understanding of the variational calculus, we are able to provide in §5 the first time, a computer-aid free veriﬁcation of the functional relations. The new input is the compatibility between transformations in §2 and the hypergeometric family $H_\alpha$ arising in pseudo-diﬀerential calculus. For instance, the cyclic operator $\sigma_j$ indeed corresponds to cyclic permutation of the index $\alpha$, see Prop. 5.1. Furthermore, such “by hand” calculation reveals more information. For example, all the functional relations share the same root: they can be reduced to same the relations among some initial values of the hypergeometric family, see Props. 5.4, 5.6 and 5.8. Another interesting observation, which is hard to achieve without using the hypergeometric family, is that the two functions given in Prop. 4.1 form a continuous family (i.e. for $m \in [2, \infty)$, $m$ is the dimension) of solutions to the Connes-Moscovici type functional relations.

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2. Variation on the Rearrangement Operators

2.1. Schwartz functional calculus on $A_{\otimes n+1}$. We follow [19] §3 to set up notations for the rearrangement process. Let $A$ be a $C^*$-algebra. At algebraic level, a toy model of the rearrangement is given by the contraction map

$$\cdots : A_{\otimes n+1} \times A_{\otimes n} \rightarrow A$$

$$(a_0, \ldots, a_n) \cdot (\rho_1, \ldots, \rho_n) \mapsto a_0 \rho_1 a_1 \cdots \rho_n a_n.$$

(2.1)

When looking from right to the left, we factor out $a_i$’s as an operator $(a_0, \ldots, a_n) \in L(A_{\otimes n}, A)$. To extend the construction, we consider the following set of generators of those rearrangement operators. For the scope of the paper, we shall focus only one element $h \in A$ and its associated multiplication operators (at the $l$-th slot), that is, when acting on $A_{\otimes n}$, we have

$$h^{(l)} = (1, \ldots, h, \ldots, 1) \in L(A_{\otimes n}, A), \ h \text{ appears at the } l\text{-th slot}.$$
labeled by a superscript $l = 0, 1, \ldots, n$. For instance, when $n = 1$, $h(0)$ and $h(1)$ are simply the left and the right multiplication respectively. We fix such a self-adjoint $h = h^* \in A$ with its exponential $k = e^h$ and denote by bold font letters
\begin{equation}
\mathbf{y} = \text{Ad}_k = k^{-1}(\cdot)k, \quad \mathbf{x} = \log \mathbf{y} = -\text{ad}_h = [\cdot, h]
\end{equation}
the corresponding conjugation and commutator operators. In a similar way, each of them yields $n$ operators $\mathbf{y}_l, \mathbf{x}_l \in L(A^{\otimes n}, A)$ whose subscript $l = 1, \ldots, n$ indicates the operator only acts on the $l$-th factor of elementary tensors, that is,
\begin{equation}
\mathbf{x}_l = -h(l-1) + h(l), \quad \mathbf{y}_l = e^{-h(l-1)} e^{-h(l)} = (k^{(l-1)})^{-1} k^{(l)}, \quad l = 1, \ldots, n,
\end{equation}
and vice versa:
\begin{equation}
k^{(l)} = (k^{(0)})^{-1} \mathbf{y}_1 \cdots \mathbf{y}_l, \quad h(l) = h(0) + \mathbf{x}_1 + \cdots + \mathbf{x}_l.
\end{equation}

The Schwartz functional calculus can be think of a way to evaluate Schwartz functions $f(x_1, \ldots, x_n)$ at the tuple of operators $(x_1, \ldots, x_n)$ so that it obeys similar arithmetics as in classical multi-variable calculus. To be precise, it is an algebra homomorphism:
\[ C^\infty(U^n) \rightarrow L(A^{\otimes n}, A) : f \mapsto f(x_1, \ldots, x_n), \]
where $U := U_h \subset \mathbb{R}$ is a bounded open subset containing the spectrum of $x$. To define the operator, we first pick an extension, still denote by $f$\footnote{The functional calculus on depends only the restriction of $f$ on $U^n$, (more precisely, on $(\text{spec}(x))^n \subset \mathbb{R}^n$), cf. [19] §3.2.} that belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, so that the Fourier transform $\hat{f}$ exists with the normalization that
\[ f(x_1, \ldots, x_n) = \int_{\mathbb{R}^n} \hat{f}(\xi_1, \ldots, \xi_n) e^{i(x_1 \xi_1 + \cdots + x_n \xi_n)} d\xi_1 \cdots d\xi_n, \]
then define
\begin{equation}
f(x_1, \ldots, x_n) = \int_{\mathbb{R}^n} \hat{f}(\xi_1, \ldots, \xi_n) e^{i(x_1 \xi_1 + \cdots + x_n \xi_n)} d\xi_1 \cdots d\xi_n,
\end{equation}
that is, when acting on elementary tensors $(\rho_1, \ldots, \rho_n) \in A^{\otimes n},$
\[ f(x_1, \ldots, x_n)(\rho_1 \otimes \cdots \otimes \rho_n) \]
\[ = \int_{\mathbb{R}^n} f(\xi_1, \ldots, \xi_n) e^{i\xi_1 x_1(\rho_1)} \cdots e^{i\xi_n x_n(\rho_n)} d\xi 
\]
\[ = \int_{\mathbb{R}^n} f(\xi_1, \ldots, \xi_n)(y_1^{i\xi_1}(\rho_1) \cdots (y_1^{i\xi_n}(\rho_n)) d\xi. \]
The last line above also gives a definition of the functional calculus $f(y_1, \ldots, y_n)$ for the modular operators $y_l = e^{x_l}$, namely, by the substitution
\[ f(y_1, \ldots, y_n) := f_{\text{exp}}(x_1, \ldots, x_n) := f(e^{x_1}, \ldots, e^{x_n}). \]

**Definition 2.1.** For any $n \in \mathbb{N}$, we will denote by $C_{\mathcal{S}, h}(\mathbb{R}^n)$ (resp. $C_{\mathcal{S}, h}(\mathbb{R}^n)$), or simply $C_{\mathcal{S}}(\mathbb{R}^n)$ and $C_{\mathcal{S}}(\mathbb{R}^n)$ when $h$ is fixed, the collection of all the $n$-variable spectral functions $f(x_1, \ldots, x_n)$ such that the functional calculus
\[ f(x_1, \ldots, x_n) \] (resp. $f(y_1, \ldots, y_n)$)
is well-defined.
2.2. Divided differences. The crucial role of divided differences in the variational calculus was pointed out by Lesch [19]. For a one-variable function $f(z)$, the divided difference can be defined inductively as below:

$$f[x_0] := f(x_0);$$
$$f[x_0, x_1, \ldots, x_n] := \left( f[x_0, x_1, \ldots, x_{n-1}] - f[x_1, \ldots, x_n] \right) / (x_0 - x_n)$$

For example,

$$f[x_0, x_2] = (f(x_0) - f(x_1)) / (x_0 - x_1).$$

By induction, one derives in general:

$$f[x_0, x_1, \ldots, x_n] = \sum_{i=0}^{n} f(x_i) \prod_{s=0, s \neq i}^{n} (x_i - x_s)^{-1}. \quad (2.6)$$

In particular, the function $f[x_0, x_1, \ldots, x_n]$ is symmetric in all the arguments. This fact will be quoted several times in later discussions. For multivariable functions, we shall use a subscript to indicate on which variable the divided difference acts, for example:

$$f(z_1, z_2, z_3)[x_1, \ldots, x_3]_{z_2}$$

means the divided difference is taken with respect to the function $f(z_1, \bullet, z_3)$. Through out the paper, we fix the variable $z$ as the default choice for the divided difference operator: $[\bullet, \ldots, \bullet] := [\bullet, \ldots, \bullet]_z$.

The following basic properties will also be needed:

1. Leibniz rule:

$$f(g)(x_0, \ldots, x_n) = f(x_0)g(x_0, \ldots, x_n) + f(x_0, \ldots, x_n)g(x_n) \quad (2.7)$$

2. Composition rule:

$$(f[y_0, \ldots, y_p, z])[x_0, \ldots, x_p]_z = f[y_0, \ldots, y_p, x_1, \ldots, x_p]_z \quad (2.8)$$

3. The confluent case: suppose there are $\alpha + 1$ copies of $x$ in the arguments of the divided difference, then:

$$f[y, x, \ldots, x] = \frac{1}{\alpha!} \partial^\alpha_x f[y, x].$$

2.3. The cyclic transformations. Let us keep the notations in Eqs. (2.2) to (2.4) and further assume that there exists a tracial functional $\varphi_0 : \mathcal{A} \to \mathbb{C}$ on the algebra $\mathcal{A}$ that plays the role of integration (volume form functional). Trace property of $\varphi_0$ leads to the integration by parts formula with respect to the derivation $x$:

$$\varphi_0(x(a) \cdot b) = \varphi_0(a \cdot [(-x)(b)]), \quad \forall a, b \in \mathcal{A}. \quad (2.9)$$

Let $j \in \mathbb{R}$ be a real parameter, denote by

$$\varphi_j(a) := \varphi_0(k^j a) = \varphi_0(e^{jx} a), \quad \forall a \in \mathcal{A} \quad (2.10)$$

the rescaled volume functional by the factor $k^j = e^{jx}$. We push Eq. (2.9) further:

$$\varphi_j(x_n(\rho_1 \otimes \cdots \otimes \rho_n)) = \varphi_0(e^{jx}(\rho_1 \otimes \cdots \otimes \rho_{n-1}) \cdot x(\rho_n))$$

$$= \varphi_0(-x(\rho_1 \cdots \rho_{n-1}) \rho_n) = \varphi_0(e^{jx}(-x)(\rho_1 \cdots \rho_{n-1}) \cdot \rho_n)$$

$$= \varphi_j((-x_1 - \cdots - x_{n-1})(\rho_1 \otimes \cdots \otimes \rho_{n-1}) \cdot \rho_n).$$
The computation suggests that one can reduced the number of arguments of the rearrangement operator \( f(x_1, \ldots, x_n) \) defined in Eq. (2.5) when the integration \( \varphi_j \) is applied. More precisely, we have transformations:

\[
\iota: C_\mathcal{Y}(\mathbb{R}^n) \to C_\mathcal{Y}(\mathbb{R}^{n-1}), \quad \eta: C_\mathcal{Y}(\mathbb{R}^n) \to C_\mathcal{Y}(\mathbb{R}^{n-1}), \quad n = 1, 2, \ldots
\]

**Lemma 2.1.** Let \( f \in C_\mathcal{Y}(\mathbb{R}^n) \) be a spectral function with \( n \)-arguments and \( \varphi_j \) is the rescaled weight in (2.10) with \( j \in \mathbb{R} \).

\[
(2.11) \quad \varphi_j(f(x_1, \ldots, x_n)(\rho_1 \otimes \cdots \otimes \rho_n)) = \varphi_j(\iota(f)(x_1, \ldots, x_{n-1})(\rho_1 \otimes \cdots \otimes \rho_{n-1})\rho_n)
\]

where the operator \( \iota: C_\mathcal{Y}(\mathbb{R}^n) \to C_\mathcal{Y}(\mathbb{R}^{n-1}), \quad n = 1, 2, \ldots \), reduces the number of variable by one by restricting spectral functions onto a hyperplane:

\[
(2.12) \quad \iota(f)(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, -x_1 - \cdots - x_{n-1}), \quad n > 1,
\]

\[
\iota(f)(x) = f(0), \quad n = 1.
\]

**Lemma 2.2.** Let \( f \in C_\mathcal{Y}(\mathbb{R}_+^n) \) be a spectral function with \( n \)-arguments and \( \varphi_j \) is the rescaled weight in (2.10) with \( j \in \mathbb{R} \).

\[
(2.13) \quad \varphi_j(f(y_1, \ldots, y_n)(\rho_1 \otimes \cdots \otimes \rho_n)) = \varphi_j(\eta(y_1, \ldots, y_{n-1})(\rho_1 \otimes \cdots \otimes \rho_{n-1})\rho_n)
\]

where the operator \( \eta: C_\mathcal{Y}(\mathbb{R}_+^n) \to C_\mathcal{Y}(\mathbb{R}_+^{n-1}), \quad n = 1, 2, \ldots \), reduces the number of variable by one: for

\[
(2.14) \quad \eta(f)(y_1, \ldots, y_{n-1}) = f(y_1, \ldots, y_{n-1}, (y_1 \cdots y_n)^{-1}), \quad \text{when } n > 1,
\]

\[
\eta(f)(y) = f(1), \quad \text{when } n = 1.
\]

Notice that Eq. (2.9) is the infinitesimal version of the KMS-property of the weight \( \varphi_j \):

\[
(2.15) \quad \varphi_j(a \cdot b) = \varphi_j(\iota^l(b) \cdot a).
\]

It leads to cyclic transformations \( \tau_j \) and \( \sigma_j \) that generate all cyclic permutations on \( \rho_1, \ldots, \rho_n \) in Eq. (2.16) and Eq. (2.19) respectively.

**Lemma 2.3.** Let \( f(x_1, \ldots, x_n) \) be a function with \( n \)-arguments and \( \varphi_j \) is the rescaled weight in (2.10) with \( j \in \mathbb{R} \).

\[
(2.16) \quad \varphi_j(f(x_1, \ldots, x_n)(\rho_1 \otimes \cdots \otimes \rho_n) \cdot \rho_{n+1}) = \varphi_j(\tau_j(f)(x_1, \ldots, x_n)(\rho_2 \otimes \cdots \otimes \rho_{n+1}) \cdot \rho_1)
\]

where

\[
(2.17) \quad \tau_j(f)(x_1, \ldots, x_n) = \left( e^{-i\pi n} f(x_1, \ldots, x_n) \right) M_{\text{cyc}}^{(n)},
\]

where the notation \( f(\vec{x}) M := f(M \cdot \vec{x}) \) means applying the linear transformation \( M \) onto the arguments (as a column vector). The \( n \times n \) matrix is given by:

\[
(2.18) \quad M_{\text{cyc}}^{(n)} = \begin{bmatrix}
-1 & \cdots & -1 & -1 \\
1 & 0 & \ddots & \\
& \ddots & \ddots & 0 \\
& & 1 & 0
\end{bmatrix},
\]

which implements the substitutions:

\[
x_1 \mapsto -x_1 - \cdots - x_n, \quad x_2 \mapsto x_1, \quad x_3 \mapsto x_2, \quad \ldots, \quad x_n \mapsto x_{n-1}.
\]

**Remark.** Let us abbreviate \( M := M_{\text{cyc}}^{(n)} \). We compute all \( \tau_j^l \), for \( 1 \leq l \leq n \):

\[
\tau_j^l(f)(x_1, \ldots, x_n) = (e^{-i\pi n} f(M) \cdots (e^{-i\pi n} f(M)^l f(\vec{x}) M^l)
\]

\[
= (e^{-i\pi n} f(M + M^2 + \cdots + M^l)(f(\vec{x}) M^l),
\]
where $\vec{x} = (x_1, \ldots, x_n)$ and
\[ e^{-jx_1}(M + M^2 + \cdots + M^j) = e^{j(x_1 + \cdots + x_{n-1})}. \]

They implement all cyclic permutations on $\{\rho_1, \ldots, \rho_{n+1}\}$.

**Proof.** Let $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{z} = (z_1, \ldots, z_n)$. We first look at the case in which $j = 0$ and the function $f(\vec{x}) = e^{jz}$ Since $\varphi_0$ is a trace, or using Eq. (2.9) instead, we compute:
\[
\varphi_0\left( e^{jz}(\rho_1 \otimes \cdots \otimes \rho_n) \cdot \rho_{n+1} \right) = \varphi_0\left( e^{jz}(\rho_1 \otimes \cdots \otimes \rho_n) \cdot \rho_{n+1} \right) \\
= \varphi_0\left( e^{jz}(\rho_1 \otimes \cdots \otimes \rho_n) \cdot \rho_{n+1} \right) \\
= \varphi_0\left( e^{jz}(\rho_1 \otimes \cdots \otimes \rho_n) \cdot \rho_{n+1} \right) \\
= \varphi_0\left( e^{jz}(\rho_1 \otimes \cdots \otimes \rho_n) \cdot \rho_{n+1} \right) \\
= \varphi_0\left( e^{jz}(\rho_1 \otimes \cdots \otimes \rho_n) \cdot \rho_{n+1} \right),
\]
where the matrix $M := M^{(n)}_\text{cy}$ is defined in Eq. (2.18). The general case follows from the Schwartz functional calculus in Eq. (2.5):
\[
\varphi_0\left( f(\vec{x})(\rho_1 \otimes \cdots \otimes \rho_n) \cdot \rho_{n+1} \right) = \int_{\mathbb{R}^n} f(\vec{z}) \varphi_0\left( e^{j\vec{z}}(\rho_1 \otimes \cdots \otimes \rho_n) \cdot \rho_{n+1} \right) d\vec{z} \\
= \int_{\mathbb{R}^n} f(\vec{z}) \varphi_0\left( e^{j\vec{z}}(\rho_1 \otimes \cdots \otimes \rho_n) \cdot \rho_{n+1} \right) d\vec{z} \\
= \varphi_0\left( \left( \int_{\mathbb{R}^n} f(\vec{z}) e^{j\vec{z}} d\vec{z} \right)(\rho_1 \otimes \cdots \otimes \rho_n) \cdot \rho_{n+1} \right) \\
= \varphi_0\left( f(M \cdot \vec{x})(\rho_1 \otimes \cdots \otimes \rho_n) \cdot \rho_{n+1} \right),
\]
So far, we have proved (2.17) for the tracial weight $\varphi_0$, that is for $\tau_0$. For $j \neq 0$,
\[
\varphi_j\left( f(\vec{x})(\rho_1 \otimes \cdots \otimes \rho_n) \cdot \rho_{n+1} \right) = \varphi_0\left( e^{j\vec{x}} f(\vec{x})(\rho_1 \otimes \cdots \otimes \rho_n) \cdot \rho_{n+1} \right) \\
= \varphi_0\left( f(\vec{x})(\rho_1 \cdots \rho_n) \cdot (\rho_{n+1} e^{j\vec{x}}) \right) \\
= \varphi_0\left( f(M \cdot \vec{x})(\rho_2 \otimes \cdots \otimes (\rho_{n+1} e^{j\vec{x}})) \cdot \rho_{n+1} \right) \\
= \varphi_0\left( e^{j\vec{x}} f(\vec{x})(\rho_2 \otimes \cdots \otimes (\rho_{n+1} e^{j\vec{x}})) \cdot \rho_{n+1} \right),
\]
where is spectral function $e^{j(x_1 + \cdots + x_n)} f(M \cdot \vec{x})$ agrees with the right hand side of (2.17).

After the substitution $y = e^x$, we obtain the parallel version in terms of the Weyl factor $k$ and the modular operator $\gamma$.

**Lemma 2.4.** Let $f \in C_\varphi(\mathbb{R}_+^n)$ and $\varphi_j$, $j \in \mathbb{R}$, be the rescaled volume weight. For all $\rho_1, \ldots, \rho_{n+1} \in \mathcal{A}$,
\[
\varphi_j\left( f(y_1, \ldots, y_n)(\rho_1 \otimes \cdots \otimes \rho_n) \cdot \rho_{n+1} \right) = \varphi_0\left( \varphi_j(f)(y_1, \ldots, y_n)(\rho_2 \otimes \cdots \otimes \rho_{n+1}) \cdot \rho_{n+1} \right)
\]
where
\[
\varphi_j(f)(y_1, \ldots, y_n) = (y_1 \cdots y_n)^{-j} f((y_1 \cdots y_n)^{-1}, y_1, \ldots, y_{n-1}).
\]

**2.4. Taylor expansion and the first variation.** Recall the exponential expansion:
\[
e^{a+b} = e^a + \sum_{n=1}^{\infty} \int_{0 \leq s_n \leq s_{n-1} \leq 1} e^{(1-s_n)a} \cdot b \cdot e^{(s_{n-1}-s_n)a} \cdot b \cdots b \cdot e^{s_1a} ds.
\]
The integrand of summands above can be rewritten (using notations in §2.1) as:
\[
\exp\left( (1-s_1)a^{(0)} + (s_1-s_2)a^{(1)} + \cdots + s_{n-1}a^{(n-1)} \right) \cdot (b \otimes \cdots \otimes b).
\]
The Genocchi–Hermite formula turns integration over the standard $n$-simplex into a divided difference:

$$
\int_{0 \leq s_n \leq \cdots \leq s_t \leq 1} e^{(1-s_t)a \cdot b \cdot e^{(s_1-s_2)a \cdot b \cdots b} \cdot e^{s_n a} ds = e^z[a^{(0)}, \ldots, a^{(n)}]z}.
$$

For any self-adjoint elements $a, b \in \mathcal{A}$ and a function $f$ whose Schwartz functional calculus (cf. Eq. [2.5]) makes sense, we have the noncommutative Taylor expansion (cf. [19] Prop. 3.7)

$$
(2.21) \quad f(a+b) \sim_{b \to 0} \sum_{n=0}^{\infty} f[a^{(0)}, \ldots, a^{(n)}] \cdot (b \otimes \cdots \otimes b).
$$

Let $\delta : \mathcal{A} \to \mathcal{A}$ be a derivation with the associated one-parameter group of automorphisms $\alpha_t : \mathcal{A} \to \mathcal{A}$, $t \in \mathbb{R}$:

$$
(2.22) \quad \delta(a) = \left. \frac{d}{dt} \right|_{t=0} \alpha_t(a), \ a \in \mathcal{A}.
$$

As a warm up, we compute the derivatives of the exponential function. Put $a = h$ and $b = \alpha_t(h)$ in the exponential expansion, we obtain the Duhamel’s formula in terms of divided difference:

$$
\delta(e^h) = \int_0^1 e^{(1-u)h} \delta(h) e^{uh} du = e^h \int_0^1 e^{uh} du \delta(h) = e^h e^{u_x 1}_{x=0} (\delta(h)) = e^h \exp[0, x](\delta(h)).
$$

**Lemma 2.5.** Let $j \in \mathbb{R}$ and $h = h^* \in \mathcal{A}$ be a log-Weyl factor and $\delta$ be a derivation as in eq. (2.22). The first derivative of the exponential is given by:

$$
(2.23) \quad \delta(e^{jh}) = e^{jh} G^{(1)}_{\exp}(x; j) (\delta(h)).
$$

In terms of the Weyl factor $k = e^h$:

$$
(2.24) \quad \delta(k^j) = k^{j-1} G^{(1)}_{\pow}(y; j) (\delta(k)).
$$

The spectral functions are given in terms of the divided differences of the exponential and power functions:

$$
(2.25) \quad G^{(1)}_{\exp}(x; j) = e^{jz} [0, x]_z, \ G^{(1)}_{\pow}(y; j) = z^j [1, y]_z.
$$

**Remark.** With $y = e^z$, we observe that

$$
(2.26) \quad G^{(1)}_{\pow}(y; j) = G^{(1)}_{\exp}(x; j) (G^{(1)}_{\exp}(x; 1))^{-1} = e^{jz} [0, x]_z (\exp[0, x])^{-1}.
$$

**Proof.** Eq. (2.23) is simply the Duhamel’s formula proved above with $h \to jh$, see also [19] Example 3.9. As a consequence, we can solve for $\delta(h)$ (with $j = 1$) in terms of $\delta(k)$:

$$
(2.27) \quad \delta(h) = k^{-1}(\exp[0, x])^{-1} (\delta(k)).
$$

Eq. (2.24) follows quickly: for $j \in \mathbb{R}$,

$$
\delta(k^j) = \delta(e^{jh}) = e^{jh} (e^{jz} [0, x]_z (\delta(h))) = k^j e^{jz} [0, x]_z (k^{-1}(\exp[0, x])^{-1}(\delta(k)))
$$

$$
= k^{j-1} (e^{jz} [0, x]_z \exp[0, x])^{-1} (\delta(k)) = k^{j-1} (z^j [0, y]_z (\delta(k))).
$$

Recall that for fixed $h$, the associated commutator and conjugation $x, y$ belong to $\mathcal{A}^{t\otimes 2} \subset L(\mathcal{A}, \mathcal{A})$ according to Eq. (2.3). One can derive the Taylor expansion below by applying Eq. (2.21) to $\mathcal{A}^{t\otimes 2}$.
Proposition 2.6 ([19], Prop. 3.11). Let \( h = h^* \in \mathcal{A} \) be a log-Weyl factor and \( \mathbf{x} = -\text{ad}_h = [\cdot, h] \) be the corresponding modular derivation. Let \( f(x) \in C(\mathbb{R}) \), given a self-adjoint perturbation: \( h \to h + b \) with \( b = b^* \), we have the Taylor expansion for the modular action \( f(x_{h+b}) \) up to the first order:

\[
(2.28) \quad f(x_{h+b})(\rho) = f(x)(\rho) - f([x_1 + x_2, x_2])\rho + o(b),
\]
as \( b \to 0 \) and \( \forall \rho \in \mathcal{A} \).

Proof. See [19] §3.5.

The first variation of the rearrangement operator \( f(x) \) follows immediately from the Taylor expansion Eq. (2.28).

Corollary 2.7. For \( f(x) \in C_\gamma(\mathbb{R}), \rho \in \mathcal{A}, \) and a derivation \( \delta \) given in Eq. (2.22), we have

\[
\delta(f(x))(\rho) = \delta(f(x)(\rho)) = \delta(f(x)(\rho)) - f([x_1 + x_2, x_2])\rho \delta(h) - f([x_1 + x_2, x_2])\rho \delta(h) \rho.
\]

(2.29)

With new notations: \( \bigtriangleup^\pm : C_\gamma(\mathbb{R}) \to C_\gamma(\mathbb{R}^2) \)

\[\bigtriangleup^+(f)(x_1, x_2) = f(x_1, x_1 + x_2), \quad \bigtriangleup^-(f)(x_1, x_2) = f(x_2, x_1 + x_2),\]

Eq. (2.29) can be rewritten as

\[\delta(f(x)) = [\delta, f(x)] = \bigtriangleup^+(f)(x_1, x_2)\delta(h)^{0(1)} - \bigtriangleup^-(f)(x_1, x_2)\delta(h)^{0(0)}.\]

We also need a similar result for \([\delta, e^{ih} f(x)]\), which has an extra component coming from \(\delta(e^{ih}) = e^{ih}G^{(1)}_{\exp}(x; f)(\delta(h))\).

Corollary 2.8. For \( j \in \mathbb{R}, f(x) \in C_\gamma(\mathbb{R}) \), we have

\[
(2.30) \quad \delta(e^{ih} f(x)) = [\delta, e^{ih} f(x)] = e^{ih\rho(0)}(\bigtriangleup^0_{0,j}(f) - \bigtriangleup^0(f)(x_1, x_2)\delta(h)^{0(0)} + e^{ih\rho(0)}(\bigtriangleup^0(f)(x_1, x_2)\delta(h)^{0(1)},
\]

where the operator \( \bigtriangleup^0_{0,j} : C(\mathbb{R}) \to C(\mathbb{R}^2) \):

\[
(2.31) \quad \bigtriangleup^0_{0,j}(f)(x_1, x_2) = G^{(1)}_{\exp}(x_1; j)f(x_2),
\]

where \( G^{(1)}_{\exp}(x; j) \) is defined in Eq. (2.25).

Proof. For any \( \rho \in \mathcal{A}, \)

\[
[\delta, e^{ih} f(x)] = [\delta, e^{ih} f(x)] + e^{ih}[\delta, f(x)]
\]

where \([\delta, f(x)]\) have been computed in the previous result.

Let us move on to the multiplicative version (regarding to the change of variable \( h \to k = e^h \)):

\[\bigtriangleup^0_{0,j}, \bigtriangleup^+, \bigtriangleup^- : C_\gamma(\mathbb{R}^+) \to C_\gamma(\mathbb{R}^2)\]

of the operators \( \{\bigtriangleup^0_{0,j}, \bigtriangleup^+, \bigtriangleup^-, \bigtriangleup^-\} \). They all increase the number of arguments by one via divided difference.
Lemma 2.9. Let \( f \in C_{\mathcal{V}}(\mathbb{R}_+) \) and \( \delta \) be a derivation on \( A \),

\[
\delta(k^j f(y)) = [\delta, k^j f(y)]
\]

(2.32) 

\[
= (k^{j-1})^0 \left( \delta^+_0(f) - \delta^-(f) \right)(y_1, y_2) + (k^{j-1})^0 \delta^+(f)(y_1, y_2) \delta^k(1),
\]

where

\[
\delta^+_0(f)(y_1, y_2) = f(y_2)(z_1^0, y_1),
\]

(2.33) 

\[
\delta^+(f)(y_1, y_2) = f(y_1, y_1 y_2), \quad \delta^-(f)(y_1, y_2) = y_2(f[y_2, y_1 y_2]).
\]

Proof. For \( f \in C_{\mathcal{V}}(\mathbb{R}_+) \), denote \( f_{\exp} = f \circ \exp \in C_{\mathcal{V}}(\mathbb{R}) \). For any \( \rho \in A \), we apply (2.30):

\[
[\delta, k^j f(y)](\rho) = [\delta, e^{ih} f_{\exp}(x)](\rho)
\]

\[
= f_{\exp}(1, x, \ldots, x) = f_{\exp}(x_1, x_2) (\delta(h) \otimes \rho) + e^{ih} f_{\exp}(x_1, x_2) (\rho \otimes \delta(h)).
\]

It remains to replace all \( \delta(h) \) by \( \delta(k) \) via Eq. (2.26). Let us do it one by one:

\[
\delta^+(f)(y_1, y_2) = k^{-1} z_1^0 f_{\exp}(x_1, x_2) (\delta(k) \otimes \rho)
\]

\[
= k^{-1} z_1^0 f_{\exp}(x_1, x_2) (\rho \otimes \delta(k)),
\]

where we have used Eq. (2.26). For the second term,

\[
\delta^{-}(f_{\exp})(x_1, x_2) (\delta(h) \otimes \rho) = k^{-1} f_{\exp}[x_2, x_1 + x_2] (\exp[0, x_1])^{-1}(\delta(k) \otimes \rho)
\]

\[
= k^{-1} y_2(f[y_2, y_1 y_2]) (\delta(k) \otimes \rho)
\]

\[
= k^{-1} \delta^-(f)(y_1, y_2) (\rho \otimes \delta(k)).
\]

At last,

\[
\delta^+(f_{\exp})(x_1, x_2) (\rho \delta(h)) = k^{-1} f_{\exp}[x_1, x_1 + x_2] \exp[0, x_2] (\delta(k) \otimes \rho)
\]

\[
= k^{-1} f[y_1, y_1 y_2] (\rho \otimes \delta(k))
\]

\[
= k^{-1} \delta^+(f)(y_1, y_2) (\rho \otimes \delta(k)).
\]

So far, we have obtained Lemma 2.3, Corollary 2.7 and 2.8 respectively in Lemma 2.4 and 2.9 two sets of transformations:

\[
\{ \tau_j, \delta^+_0, \delta^+, \delta^- \} \quad \text{vs.} \quad \{ \sigma_j, \delta^+_0, \delta^+, \delta^- \}
\]

linked via the change of coordinate \( y = e^x \). For \( f \in C_{\mathcal{V}}(\mathbb{R}_+) \), we denote the change of variable by \( f_{\exp} = f \circ \exp \), that is

\[
f_{\exp}(x_1, \ldots, x_n) := f(e^{x_1}, \ldots, e^{x_n}) = f(y_1, \ldots, y_n),
\]

where \( y_l = e^{x_l}, l = 1, \ldots, n \).

Proposition 2.10. Keep notations as above. For \( f \in C_{\mathcal{V}}(\mathbb{R}_+) \), \( y = e^x \), \( y_l = e^{x_l} \) with \( l = 1, 2 \), we have

\[
\delta^+_0(f)(y_1, y_2) = \delta^+_0(f_{\exp})(x_1, x_2)(\exp[0, x_1])^{-1},
\]

(2.34) 

\[
\delta^-(f)(y_1, y_2) = \delta^-(f_{\exp})(x_1, x_2)(\exp[0, x_1])^{-1},
\]

\[
\delta^+(f)(y_1, y_2) = \delta^+(f_{\exp})(x_1, x_2)(e^{x_1} \exp[0, x_2])^{-1}.
\]

For \( f \in C_{\mathcal{V}}(\mathbb{R}_+) \) and \( \tilde{f} \in C_{\mathcal{V}}(\mathbb{R}_+^2) \),

\[
\sigma_j(f)(y) = \tau_j(f_{\exp})(x), \quad \sigma_j(\tilde{f})(y_1, y_2) = \tau_j(\tilde{f}_{\exp})(x_1, x_2).
\]

\[
\tau_j(f)(y) = \sigma_j(f_{\exp})(x), \quad \tau_j(\tilde{f})(y_1, y_2) = \sigma_j(\tilde{f}_{\exp})(x_1, x_2).
\]
2.5. Reduction relations. We now have arrived at the first key result of the paper. An important takeaway from the relations described below is the fact that the variational calculus on the rearrangement operators is generated by two transformations: \{\tau_j, \bullet^+\} (resp. \{\sigma_j, \bullet^{-}\}).

**Proposition 2.11.** As operators on $C^\infty_{\mathcal{A}}(\mathbb{R})$, we have:

\begin{equation}
(\bullet_0^+ - \bullet^-)(f)(x_1, x_2) = (\tau_j \cdot \bullet^+ \cdot \tau_j)(f)(x_1, x_2).
\end{equation}

The multiplicative version reads: $\forall f \in C^\infty_{\mathcal{A}}(\mathbb{R}_+)$:

\begin{equation}
(\bullet_0^+ - \bullet^-)(f)(y_1, y_2) = (\sigma_{j-1} \cdot \bullet^+ \cdot \sigma_j)(f)(y_1, y_2).
\end{equation}

**Proof.** We only prove \((2.37)\) as an example. Recall that on $C^\infty_{\mathcal{A}}(\mathbb{R}_+)$ and $C^\infty_{\mathcal{A}}(\mathbb{R}_+^2)$, the cyclic operator $\sigma_j$ is given by:

$$\sigma_j(f)(y) = y^j f(y^{-1}), \quad \sigma_j(f)(y_1, y_2) = (y_1 y_2)^j f((y_1 y_2)^{-1}, y_1).$$

For any $f \in C^\infty_{\mathcal{A}}(\mathbb{R}_+)$,

\begin{align*}
(\sigma_j \cdot \bullet^+ \cdot \sigma_j)(f)(y_1, y_2) &= (y_1 y_2)^j (\bullet^+ \cdot \sigma_j)(f)((y_1 y_2)^{-1}, y_1) = (y_1 y_2)^j \sigma_j(f)((y_1 y_2)^{-1}, y_2^{-1}) \\
&= (y_1 y_2)^j \sigma_j(f)((y_1 y_2)^{-1} - \sigma_j(f)(y_2^{-1}) = \frac{\sigma_j^2(f)(y_1 y_2) - y_1^j f(y_1)}{(y_1 y_2)^{-1} - y_2^{-1}} = (y_1 y_2) \frac{f(y_1 y_2) - y_1^j f(y_2)}{1 - y_1},
\end{align*}

here we have used the fact that $\sigma_j$ is of order two when acting on one-variable functions $C^\infty_{\mathcal{A}}(\mathbb{R}_+)$. Observe that $\sigma_j = (y_1 y_2) \sigma_{j-1}$ on $C^\infty_{\mathcal{A}}(\mathbb{R}_+^2)$, thus:

\begin{equation}
(\sigma_{j-1} \cdot \bullet^+ \cdot \sigma_j)(f)(y_1, y_2) = \frac{f(y_1 y_2) - y_1^j f(y_2)}{1 - y_1},
\end{equation}

On the other hand,

\begin{equation}
(\bullet_0^+ - \bullet^-)(f)(y_1, y_2) = \frac{y_1^j - 1}{y_1 - 1} f(y_2) - \frac{f(y_1 y_2) - f(y_2)}{y_1 - 1} = \frac{y_1^j f(y_2) - f(y_1 y_2)}{y_1 - 1},
\end{equation}

which agrees with $(\sigma_{j-1} \cdot \bullet^+ \cdot \sigma_j)(f)$ obtained above. The proof of \((2.37)\) is complete. \(\square\)

In fact, \((2.37)\) and \((2.36)\) are equivalent due to the correspondence in Prop. 2.10. For instance, let us assume \((2.36)\) and would like to derive \((2.37)\). Denote $y_l = e^{x_l}$ for $l = 1, 2$. We compute $(\bullet^+ \cdot \sigma_j)(f)$ in terms of $x_1, x_2$ following Prop. 2.10:

\begin{equation}
(\bullet^+ \cdot \sigma_j)(f)(y_1, y_2) = (\tau_j \cdot \bullet^+ \cdot \tau_j)(f_{\exp}(x_1, x_2)[e^{x_1 \exp[0, x_2]}])^{-1}
\end{equation}

Now apply $\sigma_j$ on both sides:

\begin{equation}
(\sigma_j \cdot \bullet^+ \cdot \sigma_j)(f)(y_1, y_2) = (\tau_j \cdot \bullet^+ \cdot \tau_j)(f_{\exp}(x_1, x_2)\sigma_j[\sigma_j[\tau_j \cdot \bullet^+ \cdot \tau_j](e^{x_1 \exp[0, x_2]}))]^{-1},
\end{equation}

Proof. The first set of comparison Eq. \((2.34)\) is a byproduct of the proof of Lemma 2.9. The verification of \((2.35)\) is straightforward. \(\square\)
here we have used the fact that $\tau_j(f f') = \tau_0(f) \tau_j(f') = \tau_0(f') \tau_j(f)$. To continue, we now make use of Eq. (2.36):

$$
(\tau_j \cdot \Delta^- \cdot \tau_j)(f_{\exp})(x_1, x_2) = (\Delta^+_{0,j} - \Delta^-)(f_{\exp})(x_1, x_2) = \exp[0, x_1][\Delta^+_{0,j} - \Delta^-](f)(y_1, y_2)
$$

Finally, we have reached (2.37) by multiplying the two terms together:

$$
(\tau_j \cdot \Delta^- \cdot \tau_j)(f_{\exp})(x_1, x_2) = e^{x_1 + x_2}(\Delta^+_{0,j} - \Delta^-)(f)(y_1, y_2) = (y_1 y_2)(\Delta^+_{0,j} - \Delta^-)(f)(y_1, y_2).
$$

**Corollary 2.12.** Let $x = [\cdot, h]$ and $y = e^{\cdot h}(\cdot)^h$ be the modular derivation and modular operator of a self-adjoint $h \in A$, and $f \in \mathcal{C}(\mathbb{R})$, $\tilde{f} \in \mathcal{C}_{\mathcal{R}}(\mathbb{R})$ and $j \in \mathbb{R}$. For a derivation $\delta : A \to A$, we have

$$
\delta(e^{j h} f(x)) = (e^{j h})^0(\tau_j \cdot \Delta^- \cdot \tau_j)(f)(x_1, x_2) + (e^{j h})^0 \Delta^+(f)(x_1, x_2) ((\delta h)^{[1]}),
$$

$$
\delta(k \tilde{f}(y)) = (k^{-1})^0(\sigma_{j-1} \cdot \Delta^- \cdot \sigma_j)(\tilde{f})(y_1, y_2) + (k^{-1})^0 \Delta^+(\tilde{f})(y_1, y_2) ((\delta k)^{[1]}).
$$

As an example, we apply the result onto Eq. (2.23) and Eq. (2.24) to compute the second derivative of $e^{j h} = k^j$, $j \in \mathbb{R}$, in terms of $h$ and $k$ respectively, by taking advantage of the following identities:

$$
\tau_j[G^{(1)}_{\exp}(x; j)] = G^{(1)}_{\exp}(x; j), \quad (\tau_j \cdot \Delta^+ \cdot \tau_j)(G^{(1)}_{\exp}(x; j)) = \Delta^+(G^{(1)}_{\exp}(x; j))
$$

and

$$
\sigma_{j-1}[G^{(1)}_{\pow}(y; j)] = G^{(1)}_{\pow}(y; j), \quad (\sigma_{j-2} \cdot \Delta^- \cdot \sigma_j)(G^{(1)}_{\pow}(y; j)) = \Delta^+[G^{(1)}_{\pow}(y; j)],
$$

where $G^{(1)}_{\exp}$ and $G^{(1)}_{\pow}$ are given in Eq. (2.25). To see the identities, we first observe that the functions $e^{j z}$ and $z^j$ are multiplicative, thus

$$
e^{j x_0}(e^{j z}[x_1, \ldots, x_n] z) = e^{j z}[x_1 + x_0, \ldots, x_n + x_0] z
$$

$$
y_0^{-j+n+1}(z^j[y_1, \ldots, y_n]) z = z^j[y_0 y_1, \ldots, y_0 y_n].
$$

In particular,

$$
\tau_j(e^{j z}[0, x]) = e^{j x}(e^{j z}[0, -x]) = e^{j z}[0, x]
$$

$$
\tau_j(e^{j z}[0, x_1, x_2]) = e^{j (x_1 + x_2)}(e^{j z}[0, -x_1 - x_2]) = e^{j z}[x_1 + x_2, 0, x_1].
$$

For the function $z^j$,

$$
\sigma_{j-1}(z^j[1, y]) = y^{j-1} z^j[1, y^{-1}] = z^j[1, y],
$$

$$
\sigma_{j-2}(z^j[1, y_1, y_2]) = (y_1 y_2)^{j-2} z^j[1, (y_1 y_2)^{-1}, y_2^{-1}] = z^j[y_1 y_2, 1, y_1].
$$

Since divided differences are symmetric in their arguments, we have verified (2.40) and (2.41).

**Lemma 2.13.** Let $\delta_1, \delta_2$ be derivations on $A$ and $j \in \mathbb{R}$. We have

$$
\delta_1(\delta_2(e^{j h})) = e^{j h} G^{(2)}_{\exp}(x; j)(\delta_1(\delta_2 h))
$$

$$
+ G^{(1)}_{\exp}(x_1, x_2; j)(\delta_1(h) \otimes \delta_2(h) + \delta_2(h) \otimes \delta_1(h)).
$$

In terms of $k = e^h$,

$$
\delta_1(\delta_2(k^j)) = k^{-1} G^{(2)}_{\pow}(y; j)(\delta_1(\delta_2 k))
$$

$$
+ k^{-j} G^{(1)}_{\pow}(y_1, y_2; j)(\delta_1(k) \otimes \delta_2(k) + \delta_2(k) \otimes \delta_1(k)),
$$

where the spectral functions are given by:

$$
G^{(2)}_{\exp}(x; j) = G^{(1)}_{\exp}(x; j) = e^{j z}[0, x] z, \quad G^{(2)}_{\pow}(y; j) = G^{(1)}_{\pow}(y; j) = z^j[1, y] z,
$$

$$
\delta_1(\delta_2 h) = \delta_2(h) \otimes \delta_1(h), \quad \delta_1(h) \otimes \delta_2(h) + \delta_2(h) \otimes \delta_1(h) = \delta_1(h) \otimes \delta_2(h) + \delta_2(h) \otimes \delta_1(h).
$$
and
\begin{align}
G^{(1,1)}_{\exp}(x_1, x_2; j) &= \Delta^+(G^{[2]}_{\exp}(x; j)) = \left( e^{i z[0, x, x_1, x_2]} \right) x_1, x_1 + x_2 = e^{i z[0, x_1, x_1 + x_2]}\, z,
\end{align}
\begin{align}
G^{(1,1)}_{\mathrm{pow}}(y_1, y_2; j) &= \Pi^+(G^{(1)}_{\mathrm{pow}}(y; j)) = \left( z^{[1, y, y_1, y_2]} \right) z = z^{[1, y_1, y_1 y_2]}.
\end{align}

Remark. One usually computes higher derivatives of \( e^{i z} \) by the exponential expansion (cf. [7, §6.1]). Our argument is very similar to [19, Example 3.9], but the new notations seem to reveal some hierarchy behind the noncommutative analogue of Taylor coefficients, such as \( G^{\mu}_{\exp} \) and \( G^{\mu}_{\mathrm{pow}} \), \( \mu \in \{2\} \), of the functions \( e^{i z} \) and \( z^j \). We leave the exploration to future papers.

Proof. According to Eq. (2.39):
\begin{align}
\delta_1(\delta_2(k)) &= \delta_1 \left( k^{j-1} G^{(1)}_{\mathrm{pow}}(y; j) \delta_2(k) \right)
= k^{j-1} \left( \sigma_{j-2} \cdot \Pi^+ \cdot \sigma_{j-1} \right) G^{(1)}_{\mathrm{pow}, j}(y_1, y_2) \delta_1 \delta_2(k) + k^{j-1} \Pi^+(G^{(1)}_{\mathrm{pow}, j})(y_1, y_2) \delta_1 \delta_2(k) \, 
+ k^{j-1} G^{(1)}_{\mathrm{pow}, j}(y; j) \delta_1 \delta_2(k).
\end{align}

To reach (2.43), it suffices to show \( \Pi^+(G^{(1)}_{\mathrm{pow}, j}) = (\sigma_{j-2} \cdot \Pi^+ \cdot \sigma_{j-1})(G^{(1)}_{\mathrm{pow}, j}) \), which follows immediately from Eq. (2.41). Similarly, (2.42) follows from (2.40). \( \square \)

2.6. Variation on Local Expressions. For a fixed Weyl factor \( k = e^h \), \( h = h^* \in A \). A differential calculus is usually generated by a family of derivations. We consider only one derivation \( \delta \) (on \( A \)) to simplify the notation. Similar to the commutative setting, differential expressions like \( L = L(h, \delta(h), \delta^2(h), \ldots) \) are generated by \( h \) and its derivatives. The new ingredients are rearrangement operators of the form: with \( f \in C^\omega(x^n) \), \( \tilde{f} \in C^\omega(y^n) \), \( x = [1, h] \) and \( y = k^{-1}(x)k \):
\[ f(x_1, \ldots, x_n), \tilde{f}(y_1, \ldots, y_n) : A^{\otimes n} \rightarrow A, \]
appearing as "coefficients " of differential operators. We shall refer the corresponding integrations \( \varphi_\circ \) as local expressions, where \( \varphi_0 : A \rightarrow C \) is a tracial functional.

Let us start with a change of coordinate \( k \rightarrow \log k \) formula for differential expressions consisting of (up to) second derivatives of \( k \).

Lemma 2.14. Let \( \delta \) be a derivation on \( A \) and \( j \in \mathbb{R} \). Consider the following element \( R \in A \)
\[ R = k^{j-1} K(y) \delta^2(k) + k^{j-2} H(y_1, y_2) \delta(k) \otimes \delta(k). \]
The change of variable \( k \rightarrow h = \log k \) is given by
\[ R = e^{i h} \left( K(x) \delta^2(h) + H(x_1, x_2) \delta(h) \otimes \delta(h) \right), \]
where the spectral functions are transformed as below:
\[ K(x) = K(x) G^{(1)}_{\exp}(x) = K(x) \exp[0, x], \]
\[ H(x_1, x_2) = 2K(e^{x_1}, e^{x_2}) + \exp[0, x_1, x_2] \delta(h) \otimes \delta(h) \],
\begin{align}
&= 2K(e^{x_1}, e^{x_2}) \exp[0, x_1, x_2] + H(e^{x_1}, e^{x_2}) \exp[0, x_1, x_2] + H(e, e^{x_1}) \exp[0, x_1, x_2].
\end{align}

Proof. We apply Eq. (2.42) (with \( j = 1 \))
\[ K(y) \delta^2(k) = kK(y) \left[ \exp[0, x] \delta^2(h) + \exp[0, x_1, x_1 + x_2] \delta(h) \otimes \delta(h) \right] \]
\[ = kK(e^x) \exp[0, x] \delta^2(h) + kK(e^{x_1}, e^{x_2}) \exp[0, x_1, x_2] \delta(h) \otimes \delta(h). \]
To get \( H \), we start with Eq. (2.23):
\[ H(y_1, y_2) \delta(k) = H(y_1, y_2) \delta(h) \exp[0, x] \delta(h) \otimes \delta(h) \]
\[ = k^2 H(e^{x_1}, e^{x_2}) \exp[0, x] \delta(h) \otimes \delta(h). \]
Notice that the rearrangement operator \( e^{x_1} \) in the second line moves the second \( k \) (in the first line) to the very left. Since the exponential function is multiplicative, we have \( e^{x_1} \exp[0, x_2] = \exp[x_1, x_1 + x_2] \). The proof is complete.

For a real parameter \( j \in \mathbb{R} \), consider functional of the form:

\[
(2.48) 
F(h) = \varphi_0(e^{jh} f(x)(\delta h) \cdot \delta h), \quad f \in C_c(Q(\mathbb{R}))
\]

For any self-adjoint \( a \in A \), denote by \( \delta_a \) the variation along \( a \):

\[
(2.49) 
h \rightarrow h + \varepsilon a, \quad \text{and} \quad \delta_a := \frac{d}{d\varepsilon} \Big|_{\varepsilon=0}.
\]

**Definition 2.2.** Let \( F(h) \) be the functional on self-adjoint elements given in Eq. (2.48). The functional gradient \( \operatorname{grad}_h F \in A \) at \( h \) with respect to the inner product given by \( \varphi_0 \) is the unique element determined by the equation:

\[
(2.50) 
\delta_a F(h) = \varphi_0(\delta_a(h) \operatorname{grad}_h F), \quad \forall a = a^* \in A.
\]

**Theorem 2.15.** Keep the notations as above. For the functional \( F(h) \) given in eq. (2.48), the gradient defined in (2.50) has the following explicit formula:

\[
\operatorname{grad}_h F = e^{jh} \left( K_f(x)(\delta^2 h) + H_f(x_1, x_2)(\delta(h) \otimes \delta(h)) \right),
\]

where the one-variable spectral function is the average of \( f \) with respect to the cyclic operator (of order two) \( \tau_j \):

\[
(2.51) 
K_f = -(1 + \tau_j)(f).
\]

The two-variable function \( H_f \) is determined by \( K_f \) in terms of the following Connes-Moscovici type functional relation:

\[
(2.52) 
H_f = ((1 + \tau_j - \tau_j^2) \cdot \Delta^+)(K_f)
\]

**Remark.** After substituting the definitions of \( \tau_j \) and \( \Delta^+ \) in the previous section, we have explicit relations:

\[
K_f(x) = f(x) + e^{jx} f(-x)
\]

and for \( H_f \):

\[
\Delta^+(K_f)(x_1, x_2) = K_f[x_1, x_1 + x_2], \quad (\tau_j^2 \cdot \Delta^+(K_f))(x_1, x_2) = e^{jx_1} K_f[x_2, -x_1],
\]

\[
(\tau_j \cdot \Delta^+(K_f)(x_1, x_2) = e^{-j(x_1 + x_2)} K_f[-x_1 - x_2, -x_2].
\]

**Proof.** According to the Leibniz property, the variation splits into three terms:

\[
\delta_a F(h) = \varphi_0(\delta_a(e^{jh} f(x)(\delta h) \cdot (\delta h)))
\]

\[
+ \varphi_0(e^{jh} f(x) \delta(\delta_a(h)) \cdot (\delta h)) + \varphi_0(e^{jh} f(x)(\delta h) \cdot (\delta(\delta_a(h)))
\]

where we have used the fact that \( \tau_a \) and \( \delta \) commute. We postpone the computation of the first term to Lemma 2.16 which gives rise to a contribution \( \operatorname{Grad}(f, \delta h, \delta h) \) (cf. Eq. (2.54))

\[
\operatorname{Grad}(f, \delta h, \delta h) = -e^{jh}(\tau_j^2 \cdot \Delta^+)(K_f)(x_1, x_2)(\delta(h) \otimes \delta(h)).
\]

The last two terms are of the same form and can be handled together:

\[
\varphi_0(e^{jh} f(x) \delta(\delta_a(h)) \cdot (\delta h)) + \varphi_0(e^{jh} f(x)(\delta h) \cdot (\delta(\delta_a(h)))
\]

\[
= \varphi_0(e^{jh}(1 + \tau_j)(f)(x)(\delta h) \cdot (\delta(\delta_a(h)))) = \varphi_0((\delta_a(h) \delta(e^{jh} K_f(x)(\delta h)))).
\]

Note that, in both equations above, \( K_f \) appears through \( K_f = -(1 + \tau_j)(f) \) (cf. Eq. (2.51)). So far, we have reached:

\[
\operatorname{grad}_h F = \operatorname{Grad}(f, \delta h, \delta h) + \delta(e^{jh} K_f(x)(\delta h)).
\]
It remains to compute:

\[
\delta \left( e^{ih} K_f(x)(\delta h) \right) = e^{ih} K_f(x)(\delta^2 h) + \delta(e^{ih} K_f(x))(\delta h)
\]

\[
= e^{ih} K_f(x)(\delta^2 h) + e^{ih}(\tau_j \cdot \Delta^+ \cdot \tau_j + \Delta^+)(K_f)(x_1, x_2)(\delta(h) \otimes \delta(h))
\]

\[
= e^{ih} K_f(x)(\delta^2 h) + e^{ih}((\tau_j + 1) \cdot \Delta^+)(K_f)(x_1, x_2)(\delta(h) \otimes \delta(h)).
\]

To see the second = sign, we expand \( \delta(e^{ih} K_f(x)) \) via Corollary 2.12 with \( \rho = \delta h \). To reach the third = sign, we need the fact that \( K_f \) is \( \tau_j \)-invariant because \( \tau_j \) is of order two when acting on one-variable functions.

\[\square\]

**Lemma 2.16.** Keep notations. For any \( \rho_1, \rho_2 \in A \) and self-adjoint \( a \in A \):

\[
(2.53) \quad \varphi_0(\delta_a(e^{ih} f(x))(\rho_1 \cdot \rho_2)) = \varphi_0(\delta_a(h) \text{Grad}_1(h, \rho_1, \rho_2))
\]

where

\[
(2.54) \quad \text{Grad}(h, \rho_1, \rho_2) = a^{ih}(\tau_j^2 \cdot \Delta^+ \cdot (1 + \tau_j))(f)(x_1, x_2)(\rho_1 \cdot \rho_2).
\]

**Proof.** We first expand \( \delta_a(e^{ih} f(x))(\rho_1) \) using Corollary 2.12 and then apply the \( \tau_j \) operation (see, Lemma 2.3) to move \( \delta_a(h) \) to the very left as indicated in \(2.53\):

\[
\varphi_0(\delta_a(e^{ih} f(x))(\rho_1 \cdot \rho_2))
\]

\[
= \varphi_0(e^{ih}(\tau_j \cdot \Delta^+ \cdot \tau_j)(f)(x_1, x_2)(\delta_a(h) \otimes \rho_1 \cdot \rho_2)) + \varphi_0(e^{ih}(\Delta^+)(f)(x_1, x_2)(\rho_1 \otimes \delta_a(h)) \cdot \rho_2)
\]

\[
= \varphi_0(e^{ih}(\tau_j^2 \cdot \Delta^+ \cdot \tau_j)(f)(x_1, x_2)(\rho_1 \otimes \rho_2) \cdot \delta_a(h)) + \varphi_0(e^{ih}(\tau_j^2 \cdot \Delta^+)(f)(x_1, x_2)(\rho_1 \otimes \rho_2) \cdot \delta_a(h))
\]

\[
= \varphi_0(\delta_a(h)(e^{ih}(\tau_j^2 \cdot \Delta^+ \cdot (1 + \tau_j))(f)(x_1, x_2)(\rho_1 \otimes \rho_2))
\]

By definition, \( \text{Grad}(h, \rho_1, \rho_2) \) is given by the expression enclosed by the curly brackets. \(\square\)

We perform the parallel computation with respect to the Weyl factor \( k = e^h \) itself. Same functional \( F(h) \) in Eq. \(2.48\) can be written as:

\[
(2.55) \quad F(k) = \varphi_0(k^l \tilde{f}(y) \delta k \cdot \delta k), \quad \tilde{f} \in C_{\varphi}(\mathbb{R}_+)
\]

The variation is identical: for any self-adjoint \( a \in A \), we perturb the its logarithm along \( a \):

\[
k \mapsto \exp(\log k + \varepsilon a), \quad \delta_a := \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0}
\]

The functional gradient \( \text{grad}_k F \) is defined in a slightly different way:

\[
(2.56) \quad \delta_a F(k) = \varphi_0(\delta_a(k) \text{grad}_k F), \quad \forall a \in A.
\]

Observe that \( \delta_a(h) = a \) and

\[
\varphi_0(\text{grad}_h F) = \varphi_0(\delta_a(k) \text{grad}_k F) = \varphi_0(k e^z [0, x](\delta_a(h)) \text{grad}_k F)
\]

\[
= \varphi_0(k e^z [0, x](\delta_a(h))(\text{grad}_k F) = \varphi_0(\delta_a(k) \text{grad}_k F)
\]

\[
= \varphi_0(a k e^z e^z [0, x](\text{grad}_k F)) \varphi_0(a k e^z [0, x](\text{grad}_k F))
\]

that is:

\[
(2.57) \quad \text{grad}_h F = k e^z [0, x] \text{grad}_k F.
\]

**Theorem 2.17.** Consider the functional \( F \) given in Eq. \(2.55\). The functional gradient \( \text{grad}_k F \) defined in Eq. \(2.56\) is of the form:

\[
\text{grad}_k F = k^j \tilde{K}_f(y)(\delta^2 k) + k^{j-1} \tilde{H}_f(y_1, y_2)(\delta(k) \otimes \delta(k)),
\]
where
\[ K_f(y) = -(1 + \sigma_j)(\bar{f})(y), \]
\[ H_f(y_1, y_2) = (1 + \sigma_{j-1} - \sigma_j^2) \cdot \Box^+ (K_f)(y_1, y_2). \]

**Remark.** We have the explicit form for \( K_f \):
\[ -K_f(y) = \bar{f}(y) + y^j \bar{f}(y^{-1}). \]

and for \( H_f \):
\[ \Box^+(K_f)(y_1, y_2) = K_f[y_1, y_1 y_2], \]
\[ (\sigma_{j-1} \cdot \Box^+)(K_f)(y_1, y_2) = (y_1 y_2)^{j-1} K_f[(y_1 y_2)^{-1}, y_2^{-1}] \]
\[ (\sigma_j^2 \cdot \Box^+)(K_f)(y_1, y_2) = (y_1 y_2)^{j-1} K_f[y_2, y_1^{-1}]. \]

**Proof.** Let us start with the Leibniz property:
\[ \delta_a F(k) = \varphi_0(\delta_a(k^j \bar{f}(y)(\delta(k)) \cdot \delta(k))) \]
\[ + \varphi_0(k^j \bar{f}(y)(\delta(\delta_a(k))) \cdot \delta(k)) + \varphi_0(k^j \bar{f}(y)\delta(k) \cdot (\delta(\delta_a(k))))). \]

To combine the two terms in the second line, we first use the cyclic operator \( \sigma_j \) to move \( \delta_a(k) \) to the very left and then add up the spectral functions:
\[ \varphi_0(k^j \bar{f}(y)(\delta(\delta_a(k))) \cdot \delta(k)) + \varphi_0(k^j \bar{f}(y)\delta(k) \cdot (\delta(\delta_a(k)))) = \varphi_0(\delta_a(k) \delta(k^j K_f(y) \delta(k))). \]

Again, Eq. (2.58) brings in \( K_f \). The first line is computed in Lemma 2.18 below. As a result,
\[ \text{grad}_k F = k^{-1} \text{Grad}_{\exp}(\bar{f}, \delta(k), \delta(k)) + \delta(k^j K_f(y) \delta(k)), \]
where
\[ \text{Grad}_{\exp}(\bar{f}, \delta(k), \delta(k)) = (\sigma_{j-1}^2 \cdot \Box^+)(1 + \sigma_j)(\bar{f})(y_1, y_2)(\delta(k) \delta(k)) \]
\[ = -(\sigma_{j-1}^2 \cdot \Box^+)(K_f)(y_1, y_2)(\delta(k) \delta(k)). \]

The second term is given by Eq. (2.39) in Corollary 2.12
\[ \delta(k^j K_f(y) \delta(k)) = k^{-1}((\sigma_{j-1} \cdot \Box^+ \cdot \sigma_j + \Box^+)(K_f)(y_1, y_2)(\delta(k) \delta(k)) + k^j K_f(y) \delta^2(k)) \]
\[ = k^{-1}((1 + \sigma_{j-1} \cdot \Box^+)(K_f)(y_1, y_2)(\delta(k) \delta(k)) + k^j K_f(y) \delta^2(k)). \]

We complete the proof of Eq. (2.59) by adding up the corresponding terms. \( \square \)

**Lemma 2.18.** Keep notations. For any \( \rho_1, \rho_2 \in A \), and \( a = a^* \in A \), we have
\[ \varphi_0(\delta_a(k^j \bar{f}(y))(\rho_1) \cdot \rho_2) = \varphi_0(\delta_a(k^{j-1}) \text{Grad}_{\exp}(\bar{f}, \rho_1, \rho_2)), \]
with
\[ \text{Grad}_{\exp}(\bar{f}, \rho_1, \rho_2) = (\sigma_{j-1}^2 \cdot \Box^+ \cdot (1 + \sigma_j))(\bar{f})(y_1, y_2)(\rho_1 \rho_2). \]

**Proof.** We first apply Corollary 2.12 to expand \( \delta_a(k^j \bar{f}(y)) \) and then use \( \sigma_j \) to move \( \delta_a(k) \) to the vary right in the local expression (cf. Lemma 2.24):
\[ \varphi_0(\delta_a(k^j \bar{f}(y))(\rho_1) \cdot \rho_2) = \varphi_0(k^{j-1}((\sigma_{j-1} \cdot \Box^+ \cdot \sigma_j)(\bar{f})(y_1, y_2)\delta_a(k) \delta(k)) \delta(k) \rho_1) \cdot \rho_2) \]
\[ + \varphi_0(k^{j-1}(\Box^+)(\bar{f})(y_1, y_2)(\rho_1 \delta_a(k)) \cdot \rho_2) \]
\[ = \varphi_0(k^{j-1}((\sigma_{j-1} \cdot \Box^+ \cdot (1 + \sigma_j))(\bar{f})(y_1, y_2)\rho_2 \delta_a(k)) \rho_2 \delta_a(k)) \]
\[ = \varphi_0(\delta_a(k) \{ k^{j-1}((\sigma_{j-1} \cdot \Box^+ \cdot (1 + \sigma_j))(\bar{f})(y_1, y_2)\rho_2 \delta_a(k) \}). \]
By definition, \( \text{Grad}_e(f, \rho_1, \rho_2) \) is equal to the part between curly brackets.

\[ \square \]

**Corollary 2.19.** The functional \( F(k) \) in Eq. (2.55) is the zero functional if and only if the spectral function satisfies \( (1 + \sigma_j)(f) = 0 \).

**Proof.** As shown in Eq. (2.58) and Eq. (2.59), \( \tilde{K}_f = 0 \) implies \( H_f = 0 \). In particular, \( \text{grad}_k F = 0 \) if and only if \( \tilde{K}_f = 0 \), that is \( (1 + \sigma_j)(f) = 0 \).

\[ \square \]

In fact, the relation \( (1 + \sigma_j)(f) = 0 \) implies directly that \( F(k) = F(e^h) = 0 \) for all self-adjoint \( h \) without using Eq. (2.58) and Eq. (2.59): with the help of \( \sigma_j \) (cf. Lemma 2.3), we get

\[ F(k) = \varphi_j(f(y)(\delta k) \cdot (\delta k)) = \varphi_j(\sigma_j(f)(y)(\delta k) \cdot (\delta k)). \]

The argument was first used in the first proof of Gauss-Bonnet on \( T^2_\theta \), cf. [8 §3.3].

### 3. Modular Curvature as a Functional Gradient

#### 3.1. Notations for \( T^m_\theta \)

Let us quickly review the basic notations of the differential calculus on noncommutative \( m \)-tori \( T^m_\theta \) from deformation point of view (along a \( m \)-torus action). The smooth structure of the noncommutative manifold \( T^m_\theta \) is represented by the deformed algebra \( C^\infty(T^m_\theta) = (C^\infty(T^m), \times_\theta) \) which takes the underlying topological vector space from its commutative counterpart and the multiplication \( \times_\theta \) is deformed along a torus action and parametrized by \( m \) by \( m \) skew-symmetric matrices \( \theta \): for any \( f, g \in C^\infty(T^m_\theta) \),

\[ f \times_\theta g := \sum_{r, l \in \mathbb{Z}^m} \exp(2\pi i \langle \theta r, l \rangle) f_r g_l. \]

where \( f = \sum_{r \in \mathbb{Z}^m} f_r \) and \( g = \sum_{l \in \mathbb{Z}^m} g_l \) are the isotypical decomposition with respect to the \( T^m_\theta \)-action. Any translation invariant (i.e. \( T^m \)-invariant) measure \( d\mu \) on \( T^m \) can be deform to a tracial functional:

\[ \varphi_0 : C^\infty(T^m_\theta) \to \mathbb{C} : f \mapsto \int_{T^m} f d\mu, \]

where we further assume that \( \varphi_0 \) is normalized \( \varphi_0(1) = 1 \). The Hilbert space of \( L^2 \)-functions can be recovered by the standard GNS construction with respect to \( \varphi_0 \):

\[ \mathcal{H} := L^2(C^\infty(T^m_\theta), \varphi_0) \]

whose the inner product is given by

\[ (a, b) := (a, b)_{\varphi_0} = \varphi_0(b^* a), \quad \forall a, b \in C^\infty(T^m_\theta). \]

The coordinate system on \( \mathbb{R}^m : (x_1, \ldots, x_m) \) induces one on the quotient \( T^m = \mathbb{R}^m / 2\pi i \mathbb{Z}^m \). We use the standard Euclidean metric and denote by \( \nabla \) the metric connection. The associated covariant differentials

\[ \nabla_l := \nabla_{\delta_l}, \quad l = 1, \ldots, m, \]

generate the algebra of differential operators. In particular, the flat Laplacian reads:

\[ \Delta = -(\nabla_1^2 + \cdots + \nabla_m^2). \]

We will also feel free to use the classical notations [7 §1.3] (for \( T^2_\theta \)) and [28 §6] (for general \( T^m_\theta \)). The volume functional is denoted by \( \varphi_0 \) in all the references and the basic derivations \( \delta_l \) in [28 §6] are slightly different from the covariant differential above, that is, \( \delta_l = -i\nabla_l, \ l = 1, \ldots, m \).
3.2. Modular Gaussian Curvature on \( T^2_{\theta} \). Modeled on spectral geometry of Riemannian manifolds (cf. [18] Ch. 4), local invariants to be investigated are derived from spectral asymptotic of some Laplacian type operators attached to the underlying Riemannian metric. On noncommutative two tori \( T^2_{\theta}, \theta \in \mathbb{R} \setminus \mathbb{Q} \), we consider conformal change of the flat metric, starting with rescaling the flat volume functional, i.e., the canonical trace \( \varphi_0 \), by a Weyl factor \( k = e^h \) where \( h = h^* \in C^\infty(T^2_{\theta}) \):

\[
\varphi(a) = \varphi_0(a e^{-h}) = \varphi_0(a k^{-1}).
\]

The associated Dolbeault Laplacian is changed from the flat one \( \Delta = \tilde{\partial}^* \tilde{\partial} \) in Eq. \( \text{(3.5)} \) to \( \tilde{\partial}^* \tilde{\partial} \), where the new adjoint is taken with respect to the inner product: \( \langle a, b \rangle_\varphi := \varphi(b^* a), \forall a, b \in C^\infty(T^2_{\theta}) \). To keep the same Hilbert space \( \mathcal{H} \) (or the same inner product in Eq. \( \text{(3.3)} \)) of \( L^2 \)-functions defined in Eq. \( \text{(3.2)} \), we shall work with

\[
\Delta_\varphi = k^{1/2} \Delta k^{1/2} : \mathcal{H} \rightarrow \mathcal{H}
\]

which is anti-unitary equivalent to \( \tilde{\partial}^* \tilde{\partial} \), see [7] §1.5.

The analogue of the Gaussian curvature is based on the following spectral realization on Riemann surfaces, known as the conformal anomaly, due to Polyakov [27, 26], of the functional assigning each metric \( g \) to the Ray-Singer determinant of its scalar Laplacian \( \Delta_g; g \rightarrow -\log \det \Delta_g := \gamma_{\Delta_g}(0) \). In [25], a slightly modified version was introduced whose gradient flow recovers Hamilton’s Ricci flow.

**Definition 3.1** (Modular Gaussian Curvature). In [17], the precise analogue of the OPS-functional (Osgood-Phillips-Sarnak) on \( T^2_{\theta} \) is defined to be

\[
F_{\text{OPS}}(h) = \gamma'_{\Delta_\varphi}(0) + \log \varphi_0(e^{-h})
\]

as a function on self-adjoint elements in \( C^\infty(T^2_{\theta}) \). The functional gradient in the sense of Eq. \( \text{(2.50)} \) yields the notion of modular Gaussian curvature: \( K_\varphi := \nabla_{\text{grad}} F_{\text{OPS}} \).

The analytic continuation of the spectral zeta function is often proved by means of the heat trace asymptotic. On general noncommutative tori \( T^m_{\theta} \), we can establish the small time asymptotic by, for instance, Connes pseudo-differential calculus\(^6\)

\[
\text{Tr}(f e^{-t \Delta_\varphi}) \sim \sum_{j=0}^{\infty} V_j(f, \Delta_\varphi) t^{(j-m)/2}, \forall f \in C^\infty(T^m_{\theta}).
\]

Each \( V_j(\cdot, \Delta_\varphi) \) is a linear functional in \( f \) and is absolutely continuous with respect to the volume functional \( \varphi_0 \) with a smooth functional density in \( C^\infty(T^m_{\theta}) \). For instance, the density \( R_{\Delta_\varphi} \in C^\infty(T^m_{\theta}) \) of the \( V_2 \)-term is determined by the property:

\[
V_2(f, \Delta_\varphi) = \varphi_0(f R_{\Delta_\varphi}), \forall f \in C^\infty(T^m_{\theta}).
\]

Back to dimension \( m = 2 \), let \( P_\varphi \) be the orthogonal projection onto \( \ker \Delta_\varphi \). For any \( f \in C^\infty(T^2_{\theta}) \), consider the zeta function \( \zeta_{\Delta_\varphi}(z) := \zeta_{\Delta_\varphi}(1; z) \), where

\[
\zeta_{\Delta_\varphi}(f; z) = \text{Tr}(f \Delta_{\varphi}^{-z}(1 - P_\varphi)), \Re z > 2,
\]

in which we define the inverse restricted on the kernel \( \Delta_{\kappa}^{-1}|_{\ker \Delta_\kappa} = 1 \). It is related to the heat trace by Mellin transform:

\[
\zeta_{\Delta_\varphi}(f; z) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \left( \text{Tr}(f e^{-t \Delta_\varphi}(1 - P_\varphi)) \right) dt
\]

---

\(^6\)We also mention two other approaches: [18] using Duhamel type perturbation series of the exponential function and [21] using harmonic analysis on \( T^2_{\theta} \).
and asymptotic in (3.8) implies that the right hand side above is a meromorphic function in \( z \). Moreover, \( z = 0 \) is a regular point (cf. \( [\blacksquare] \) Eq. (3.11)) and:

\[
(3.12) \quad \zeta_{\Delta_{\varphi}}(f ; 0) = V_2(f, \Delta_{\varphi}) - \text{Tr}(f \, P_{\varphi}) = V_2(f, \Delta_{\varphi}) - \frac{\varphi_0(f \, k^{-1})}{\varphi_0(k^{-1})}.
\]

In particular,

\[
\zeta_{\Delta_{\varphi}}(1 ; 0) = V_2(1, \Delta_{\varphi}) - 1.
\]

One has the following variational formulas for \( \zeta_{\Delta_{\varphi}}(0) \) and \( \zeta_{\Delta_{\varphi}'}(0) \).

**Lemma 3.1.** Consider the two types of variation: dilation \( h_s = s \, h \) by \( s \in \mathbb{R} \) so that \( k_s := k^s \) and \( \Delta_{\varphi_s} = k^{s/2} \Delta k^{s/2} \), and a small perturbation along a self-adjoint \( a = a^* \in C^\infty(\mathbb{T}_0^2) \): \( h_s = h + \varepsilon a \) and \( \Delta_{\varphi_s} = e^{(h+\varepsilon a)/2} \Delta e^{(h+\varepsilon a)/2} \). We have

\[
(3.13) \quad \frac{d}{ds} \zeta_{\Delta_{\varphi_s}}(z) = -z \zeta_{\Delta_{\varphi_s}}(\log k; z), \quad \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \zeta_{\Delta_{\varphi_s}}(z) = -\frac{z}{2} \zeta_{\Delta_{\varphi_s}} \left( e^{-\varepsilon/2} e^{\varepsilon} [x, 0]_z(a); z \right).
\]

Applying \( d / dz \big|_{z=0} \) on both sides above yields:

\[
(3.14) \quad -\frac{d}{ds} \zeta_{\Delta_{\varphi_s}}'(0) = \zeta_{\Delta_{\varphi_s}}(\log k; 0), \quad -\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \zeta_{\Delta_{\varphi_s}}'(z) = \zeta_{\Delta_{\varphi_s}} \left( e^{-\varepsilon/2} e^{\varepsilon} [x, 0]_z(a); z \right).
\]

**Remark.** The first type of the variation is a special case of the second type with \( a = h \).

**Proof.** See \( [\blacksquare] \) §4.1, 4.2, in which \( e^{-\varepsilon/2} e^{\varepsilon} [x, 0]_z(a) \) is written as \( \frac{1}{2} \int_{-1}^{1} e^{\frac{u}{2}} a e^{-\frac{u}{2}} \, du \).

After substituting Eq. (3.12) to the right hand sides of Eq. (3.14), we have arrived at:

**Proposition 3.2.** The OPS-functional in Eq. (3.7) is determined by the second heat coefficient in the following way.

\[
(3.15) \quad R_{\text{OPS}}(k) = \int_0^1 V_2(\log k, \Delta_{\varphi_s}) ds + \zeta_{\varphi}'(0),
\]

where \( \Delta_{\varphi_s} = k^{s/2} \Delta k^{s/2} \) with \( s \in \mathbb{R} \).

**Proof.** Again, we refer the details to \( [\blacksquare] \) §4.1, 4.2.

**Proposition 3.3.** For any self-adjoint \( a \in C^\infty(\mathbb{T}_0^2) \), the variation of the OPS-functional (see Eq. (3.7)) is given by:

\[
(3.16) \quad \varphi_0(a \, \text{grad}_h \, R_{\text{OPS}}) = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} R_{\text{OPS}}(e^{h+\varepsilon a}) = -V_2 \left( e^{-\varepsilon/2} e^{\varepsilon} [x, 0]_z(a), \Delta_{\varphi} \right).
\]

In particular,

\[
(3.17) \quad \text{grad}_h \, R_{\text{OPS}} = -e^{\varepsilon/2} e^{\varepsilon} [-x, 0]_z(R_{\text{OPS}}).
\]

We also work with the operator

\[
(3.18) \quad \Delta_k := k \Delta = k^{1/2} (k^{1/2} \Delta k^{1/2}) k^{-1/2} = k^{1/2} \Delta k^{-1/2},
\]

whose complete symbol \( \sigma(\Delta_k)(\xi) = k |\xi|^2 \) has only one term, the leading part so that the heat asymptotic is much easier to compute. Their heat traces are related in a similar way: \( \text{Tr}(f \, e^{-t \Delta_k}) = \text{Tr}(k^{1/2} f k^{-1/2} e^{-t \Delta_k}) \) for all \( f \in C^\infty(\mathbb{T}_0^2) \). Therefore, \( V_l(f, \Delta_{\varphi}) = V_l(y^{1/2}(f), \Delta_k) \), \( l \in \mathbb{N} \). For the second coefficient, we have \( \varphi_0(f \, R_{\Delta_k}) = \varphi_0(y^{1/2}(f)R_{\Delta_k}) \), that is

\[
(3.19) \quad R_{\Delta_k} = y^{1/2}(R_{\Delta_k}).
\]

**Corollary 3.4.** In terms of \( R_{\Delta_k} \) defined above, the two versions of gradient (cf. Eq. (2.50) and Eq. (2.57)) can be rewritten as below:

\[
(3.20) \quad \text{grad}_k \, R_{\text{OPS}} = -k^{-1} R_{\Delta_k}, \quad \text{grad}_h \, R_{\text{OPS}} = -\exp[0, x](R_{\Delta_k}).
\]
Remark. The OPS-functional gives rise to a renormalization of the coefficient $(2 - m)/2$ of the EH-functional in Eq. (3.24) in later section.

Proof. We apply Eq. (3.19) to replace $R_{\Delta_\varphi}$ in Eq. (3.17):

$$\text{grad}_h F_{\text{OPS}} = -e^{\chi/2} e^z[-x,0]z(R_{\Delta_\varphi}) = -\left( e^{\chi/2} e^z[-x,0]z e^{\chi/2} \right) (R_{\Delta_k}) = -\exp[0,x](R_{\Delta_k}),$$

where we have used the multiplicative property of the exponential function:

$$e^x e^z[-x,0]z = e^z[-x + x, x]z = e^z[0, x]z.$$

The formula for $\text{grad}_k F_{\text{OPS}}$ follows immediately from the relation in Eq. (2.57). \hfill $\square$

### 3.3. Modular curvature on higher dimensional $T^m_\theta$.

Among known examples of noncommutative manifolds, not even the conformal geometry has been fully explored. The author did a case study on a larger class toric noncommutative manifold in [22], and on noncommutative tori, some variations related to exploration of scalar curvature\footnote{A more complete list of references can be found in surveys [21][14].} can be found in [1], [9] and [15]. To go beyond conformal geometry on $T^m_\theta$, Ponge and Ha [17] proposed a construction of Laplace-Beltrami operators for more general Riemannian metrics (influenced by [30]), but no explicit forms of local invariants are available at the current stage.

In the paper, we take the setting in dimension two $\Delta \mapsto \Delta_\varphi = e^{h/2} \Delta e^{h/2}$, where $h = h^* \in C^\infty(T^m_\theta)$, as a simplified model of the conformal change of metric $g \mapsto e^{-h} g$ for general noncommutative tori $T^m_\theta$. What is crucial to our discussion is some variational properties like Proposition 3.5 which shall hold for so-called conformally covariant operators in the sense of [3] Eq. (1.1)]. The squared spinorial Dirac operator on toric noncommutative manifold in [22] $\mathcal{D}^2_h$, where $\mathcal{D}_h = e^h \mathcal{D} e^{-h}$, is another set of examples in noncommutative geometry.

It well-known that, on Riemannian manifolds $(M, g)$, the EH (Einstein–Hilbert) action $g \mapsto \int_M S_g \, dg$ has a spectral realization as the second heat coefficient of the scalar Laplacian $\Delta_g$:

$$f \mapsto V_2(f, \Delta_g) = c_m \int_M (f S_g)/6 \, dg$$

for some universal constant $c_m$ (depending on the dimension). Moreover, when $\dim M \geq 3$, that is without the restriction of Gauss–Bonnet, the EH-action has non-trivial variation inside a conformal class of metrics, and the functional gradient gives the scalar curvature $S_g$ a variational interpretation. Consequently, on noncommutative tori $T^m_\theta$, $m \geq 2$, we define the modular curvature to be the functional gradient of the following Riemannian functional.

**Definition 3.2.** On $T^m_\theta$, we view $\Delta \mapsto \Delta_\varphi = e^{h/2} \Delta e^{h/2}$ as a conformal change of the flat metric represented by $\Delta$ in Eq. (3.5). The Riemannian functional $F$ is defined on the conformal class of metrics parametrized by Weyl factors $k = e^h$, whose tangent space consists of self-adjoint elements $h = h^* \in C^\infty(T^m_\theta)$:

$$F(k) := F(h) = \begin{cases} F_{\text{OPS}}(h) = \zeta'_{\Delta_\varphi}(0) + \log \varphi_0(k^{-1}), & \text{if } m = 2, \\ F_{\text{EH}}(h) = V_2(1, \Delta_\varphi), & \text{if } m > 2, \end{cases}$$

where $V_2(\cdot, \Delta_\varphi)$ is the second heat coefficient in Eq. (3.8).

**Definition 3.3** (Modular Curvature). We call the gradients of $F$ above (in terms of $k$ or $h$),

$$\text{grad}_k F \text{ or } \text{grad}_h F \in C^\infty(T^m_\theta)$$

the modular curvature of the metric associated with the Laplacian $\Delta_\varphi$.\footnote{A more complete list of references can be found in surveys [21][14].}
Again, the gradients are defined by means of Gâteaux differential using the inner product associated to the canonical trace $\varphi_0$ (cf. Definition $\text{(2.2)}$ and Eq. $\text{(2.57)}$).

The variation of $F$ relies on the variation of the heat coefficient $V_2(\cdot, \Delta_k)$, where $\Delta_k$ and $\Delta_\varphi$ are related in Eq. $\text{(3.18)}$ and Eq. $\text{(3.19)}$.

**Proposition 3.5.** As before, consider the variation along a self-adjoint direction $a \in C^\infty(T^m_\theta)$: $h \mapsto h + \varepsilon a$ and set $\delta_a := d/d\varepsilon|_{\varepsilon=0}$. Then for heat coefficients in in Eq. $\text{(3.9)}$, we have for $l = 0, 1, 2, \ldots$

$$
\delta_a V_l(1, \Delta_k) = \frac{l-m}{2} V_l(\delta_a(k)k^{-1}, \Delta_k) = \frac{l-m}{2} V_l(\exp[0, -x]([\delta_a(h)], \Delta_k)).
$$

**Proof.** Start with $\delta_a(\Delta_k) = \delta_a(k) = \delta_a(k)k^{-1}\Delta_k$. Apply the Duhamel’s formula, for $t > 0$,

$$
\delta_a \Tr(e^{-t\Delta_k}) = -t \Tr(\delta_a(\Delta_k) e^{-t\Delta_k}) = \Tr(\delta_a(k)k^{-1}\Delta_k e^{-t\Delta_k})
$$

$$
= t \frac{d}{dt} \Tr(\delta_a(k)k^{-1} e^{-t\Delta_k}).
$$

Since both $\delta_a$ and $d/dt$ pass through the asymptotic expansion, we can continue:

$$
\sum_{l=0}^\infty \delta_a V_l(1, \Delta_k) t^{(l-m)/2} = \sum_{l=0}^\infty V_l(\delta_a(k)k^{-1}, \Delta_k) t \frac{d}{dt} t^{(l-m)/2}
$$

$$
= \sum_{l=0}^\infty \frac{l-m}{2} V_l(\delta_a(k)k^{-1}, \Delta_k) t^{(l-m)/2}.
$$

The first $\text{sign}$ in Eq. $\text{(3.22)}$ follows from comparing the coefficients of $t^{(l-m)/2}$. To see the second equal sign, we need Lemma $\text{(2.5)}$

$$
\delta_a(k)^{-1} = e^\frac{x}{y} \delta_a(h)k^{-1} = y^{-1} \left(\frac{e^x-1}{y}\right)(\delta_a(h)) = \frac{1-e^{-x}}{y}(\delta_a(h)).
$$

**Corollary 3.6.** When the dimension $m \geq 2$, we have

$$
\text{grad}_k F_{\text{EH}} = \frac{2-m}{2} k^{-1} R_{\Delta_k}, \quad \text{grad}_h F_{\text{EH}} = \frac{2-m}{2} \exp[0, x](R_{\Delta_k}).
$$

**Remark.** When $m = 2$, it follows immediately that $\text{grad}_k F_{\text{EH}} = \text{grad}_h F_{\text{EH}} = 0$. The result is known as Connes-Moscovici’s variational proof of Gauss-Bonnet theorem on noncommutative two tori $T^2_\theta$.

**Proof.** For the EH-action $F_{\text{EH}}$:

$$
\delta_a F_{\text{EH}}(k) = \delta_a V_2(1, \Delta_k) = \frac{2-m}{2} V_2(\delta_a(k)k^{-1}, \Delta_k) = \frac{2-m}{2} \varphi_0(\delta_a(k)(k^{-1} R_{\Delta_k})).
$$

Similarly,

$$
\delta_a F_{\text{EH}}(h) = \delta_a V_2(1, \Delta_k) = \frac{2-m}{2} V_2(\exp[0, -x]([\delta_a(h)], \Delta_k))
$$

$$
= \frac{2-m}{2} \varphi_0(\exp[0, -x]([\delta_a(h)] R_{\Delta_k})) = \frac{2-m}{2} \varphi_0(\delta_a(h) \exp[0, x](R_{\Delta_k})).
$$

**4. Local Expressions of the Geometric Functional**

4.1. **Functional density of $V_2(\cdot, \Delta_k)$.** The full expression of $R_{\Delta_k}$ on $T^2_\theta$ was first achieved in $[7]$ and further confirmed via independent calculation in $[12]$. In this section, we shall start with the following version obtained in part I of the sequel $[23]$ §4 which works for arbitrary dimension $T^m_\theta$. We will not repeat the computation here.
**Proposition 4.1.** For the perturbed Laplacian $\Delta_k = k\Delta$ acting on $C^\infty(T^m_\theta)$ with $m \geq 2$, the corresponding functional density of the second heat coefficient $R_{\Delta_k}$ is given by, upto a constant factor $\text{Vol}(S^{m-2})/2$,

\begin{equation}
R_{\Delta_k} = \sum_{a=1}^{m} k^{-m/2} K_{\Delta_k}(y)(\nabla_a^2 k; m) + k^{-m/2-1} H_{\Delta_k}(y_1,y_2;m)(\nabla_a k \otimes \nabla_a k).
\end{equation}

The one-variable function is given by

\begin{equation}
K_{\Delta_k}(y;m) = \frac{4}{m} H_{3,1}(z;m) - H_{2,1}(z;m), \quad z = 1 - y.
\end{equation}

Similarly, we have:

\begin{equation}
H_{\Delta_k}(y_1,y_2;m) = \frac{4}{m} H_{2,1}(z_1,z_2;m) - \frac{4(1-z_1)H_{2,2,1}(z_1,z_2;m)}{m}
\end{equation}

where $z_1 = 1 - y_1$ and $z_2 = 1 - y_1 y_2$. The precise definition of the hypergeometric families $\{H_{a,b}, H_{a,b,c}\}$ will be need in the next section, see Eqs. (5.1) and (5.2).

**Remark.** According to Eq. (3.19), we see that $R_{\Delta_y}$ is also of the form Eq. (4.1), with spectral functions:

\begin{equation}
K_{\Delta_y}(y;m) = \sqrt{y} K_{\Delta_k}(y;m),
\end{equation}

\begin{equation}
H_{\Delta_y}(y_1,y_2;m) = \sqrt{y_1 y_2} H_{\Delta_k}(y_1,y_2;m).
\end{equation}

**Proposition 4.2.** In terms of $h = \log k \in C^\infty(T^m_\theta)$, the functional density $R_{\Delta_k}$ can be rewritten as:

\begin{equation}
R_{\Delta_k} = \sum_{a=1}^{m} e^{-m/2+1} \left( \tilde{K}_{\Delta_k}(x)(\nabla_a^2 h) + \tilde{H}_{\Delta_k}(x_1,x_2)(\nabla_a h \otimes \nabla_a h) \right),
\end{equation}

with

\begin{equation}
\tilde{K}_{\Delta_k}(x) = \exp[0,x] K_{\Delta_k}(e^x)
\end{equation}

\begin{equation}
\tilde{H}_{\Delta_k}(x_1,x_2) = e^{x_1} \exp[0,x_1] \exp[0,x_2] H_{\Delta_k}(e^{x_1},e^{x_2}) + 2 K_{\Delta_k}(e^{x_1+x_2}) \exp[0,x_1+x_2].
\end{equation}

**Proof.** One simply follows Lemma 2.14 to carry out the change of variable $k \mapsto h = \log k$ and $y \mapsto x$. The details are left to the reader. \hfill \Box

4.2. **The EH (Einstein-Hilbert) action.** Recall that

\[ F_{\text{EH}}(k) := V_2(1,\Delta_y) = V_2(1,\Delta_k) = \varphi_0(R_{\Delta_k}), \]

where $\varphi_0(R_{\Delta_k})$ can be further simplified using the operators $\mathbf{t} : C_{\mathcal{G}}(\mathbb{R}^n) \to C_{\mathcal{G}}(\mathbb{R}^{n-1})$ and $\eta : C_{\mathcal{G}}(\mathbb{R}^n) \to C_{\mathcal{G}}(\mathbb{R}^{n-1})$ in Lemma 2.1 and 2.2 respectively.

**Proposition 4.3.** On $T^m_\theta$, the Einstein-Hilbert action defined above admits the following closed formula. In terms of the Weyl factor $k$:

\begin{equation}
F_{\text{EH}}(k) = \sum_{a=1}^{m} \varphi_0 \left( k^{-m/2-1} T_{\Delta_k}(y;m)(\nabla_a k) \cdot (\nabla_a k) \right).
\end{equation}

In terms of $h = \log k$:

\begin{equation}
F_{\text{EH}}(h) = \sum_{a=1}^{m} \varphi_0 \left( e^{-(m/2+1)h} \tilde{T}_{\Delta_k}(x;m)(\nabla_a h) \cdot (\nabla_a h) \right).
\end{equation}
The function $T_{\Delta_k}$ and $\tilde{T}_{\Delta_k}$ are determined by the spectral functions in Eq. (4.10) and (4.6) respectively:

\[ T_{\Delta_k}(y; m) = -\eta(K_{\Delta_k})z^{-m/2}[1, y] + \eta(H_{\Delta_k})(y) \]
\[ = -K_{\Delta_k}(1; m)\frac{y^{-m/2}-1}{y-1} + H_{\Delta_k}(y, y^{-1}, m). \]

Also,

\[ \tilde{T}_{\Delta_k}(x; m) = -\iota(K_{\Delta_k})e^{(-m/2+1)z}[0, x]z + \iota(H_{\Delta_k})(x) \]
\[ = -\tilde{K}_{\Delta_k}(0; m)\frac{e^{-mx/2}-1}{x} + \tilde{H}_{\Delta_k}(x, -x; m). \]

In term of hypergeometric functions:

\[ T_{\Delta_k}(y; m) = \left( \frac{4}{m} + 2 \right)H_{2,1}(1 - y; m) - \frac{4y}{m}H_{2,2}(1 - y; m) - \frac{8}{m}H_{4,1}(1 - y; m) \]
\[ - \eta(K_{\Delta_k})z^{-m/2}[1, y]. \]

**Proof.** Let us apply the volume functional $\varphi_0$ in Eq. (4.1). The second term is handled by Lemma (2.2):

\[ \varphi_0 \left( k^{-m/2-1}H_{\Delta_k}(y_1, y_2)(\nabla_a k \odot \nabla_a k) \right) = \left( k^{-m/2-1} \eta(H_{\Delta_k})(y)(\nabla_a k) \cdot (\nabla_a k) \right), \]

and the first term involves the classical integration by parts regarding to $\nabla_a$

\[ \varphi_0 \left( k^{-m/2}K_{\Delta_k}(y)(\nabla_a k) \right) = \eta(K_{\Delta_k})\varphi_0 \left( k^{-m/2}\nabla_a k \right) = -\eta(K_{\Delta_k})\varphi_0 \left( \nabla_a(k^{-m/2})\nabla_a k \right) \]
\[ = -\eta(K_{\Delta_k})\varphi_0 \left( k^{-m/2-1}(z^{-m/2}[0, y]z)(\nabla_a k) \cdot (\nabla_a k) \right), \]

here we have used Eq. (2.24): $\nabla_a k^{-m/2} = k^{-m/2-1}(z^{-m/2}[0, y]z)(\nabla_a k)$. By adding up the two terms, we have finished the proof of Eqs. (4.8) and (4.10). We leave the parallel verification of Eqs. (4.9) and (4.11) to the reader.

Last but not least, Eq. (4.12) is obtained by expressing $\eta(H_{\Delta_k})$ in terms of the Gauss hypergeometric functions $H_{a,b}$ according to the relation in Eq. (4.6). \hfill \Box

Now, we are ready to apply Theorem (2.17) (resp. Theorem (2.15) to Eq. (4.8) (resp. Eq. (4.9)) with the parameter $j = -m/2 - 1$ (resp. $j = -m/2 + 1$).

**Proposition 4.4.** Let $T_{\Delta_k}$ and $\tilde{T}_{\Delta_k}$ be the spectral functions defined in Prop. (4.3) with the dimension $m \geq 2$. The functional gradient at metric $k$ is given by:

\[ \mathrm{grad}_k F_{\mathrm{EH}} = \sum_{a=1}^{m} k^{-m/2-1}K_{\mathrm{EH}}(y)(\nabla_a k) + k^{-m/2-2}H_{\mathrm{EH}}(y_1, y_2)(\nabla_a k \cdot \nabla_a k), \]

with

\[ K_{\mathrm{EH}} = -(1 + \sigma_{-m/2-1})(T_{\Delta_k}), \]
\[ H_{\mathrm{EH}} = (1 + \sigma_{-m/2-2} - \sigma_{-m/2-2}) \cdot \bigtriangleup^{+}(K_{\mathrm{EH}}). \]

In terms of the log-Weyl factor $h$,

\[ \mathrm{grad}_h F_{\mathrm{EH}} = e^{(-m/2+1)h}\left( \sum_{a=1}^{m} \tilde{K}_{\mathrm{EH}}(x)(\nabla_a h) + \tilde{H}_{\mathrm{EH}}(x_1, x_2)(\nabla_a h \cdot \nabla_a h) \right), \]

with

\[ \tilde{K}_{\mathrm{EH}} = -(1 + \tau_{-m/2+1})(\tilde{T}_{\Delta_k}), \]
\[ \tilde{H}_{\mathrm{EH}} = (1 + \tau_{-m/2+1} - \tau_{-m/2+1}) \cdot \bigtriangleup^{+}(\tilde{K}_{\mathrm{EH}}). \]
In dimension two, we have \( \text{grad}_k F_{\text{EH}} = 0 \), which is nothing but another version of the Gauss-Bonnet theorem first achieved in \([8]\). In fact, the very first nontrivial discovery on the \( a_2 \)-term is the observation that the function \( \tilde{T}_{\Delta_k}(x) \) (denoted by \( K(x) \) in \([8]\) Lemma 3.3)) is odd and the vanishing of the functional \( F_{\text{EH}} \) follows immediately from a one-line argument cf. \([8]\) §3.3), in particular, no need to look at the gradient. The Proposition above confirms the vanishing of the gradient. Indeed, \( \tilde{T}_{\Delta_k}(x) \) is odd means that

\[
(1+\tau_0)\tilde{T}_{\Delta_k}(x) = \tilde{T}_{\Delta_k}(x) + \tilde{T}_{\Delta_k}(-x) = 0,
\]

so that \( K_{\text{EH}} = \tilde{K}_{\text{EH}} = 0 \) and then \( \text{grad}_k F_{\text{EH}} = 0 \) according to Eqs. (4.17) and (4.18). We will provide another verification of the oddness in Prop. 5.4 to illustrate the power of the hypergeometric family \( H_\alpha \) obtained in \([23]\).

### 4.3. Osgood-Phillips-Sarnak (OPS) functional

Let us specialized to noncommutative tori \( m = 2 \) so that \( \text{grad}_k F_{\text{EH}} = 0 \), which leads to another functional \( F_{\text{OPS}} \). For \( s \in \mathbb{R} \), put \( \Delta_{k_i} = k^i \Delta \). With the relation Eq. (3.19) in mind and the fact that \( [h,k] = 0 \), we see that \( V_2(h,\Delta_{k_i}) = V_2(h,\Delta_{\varphi_i}) \) and then Eq. (3.15) becomes:

\[
F_{\text{OPS}}(k) = -\int_0^1 V_2(h,\Delta_{k_i})d s + \zeta_{\Delta}^\prime(0).
\]

The proposition below is another version of \([7]\) Thm 4.6] in which the computation was carried out in terms of \( h \). We make a few comparisons before proceeding to the statement and the proof:

1. In terms of \( h \), the factor \( e^{-m/2+1} \) in Eq. (4.6) disappears when \( m = 2 \), while Eq. (4.1) reads

\[
R_{\Delta_k} = \sum_{a=1}^{2} k^{-1} K(y)(\nabla_a k) + k^{-2} H(y_1,y_2)(\nabla_a k \nabla_a k).
\]

The factors \( k^{-1} \) and \( k^{-2} \) indeed bring in extra work which makes the appearance of \( T_{\text{OPS}} \) in Eq. (4.21) more intimidating than its counterpart \( K_2 \) in \([7]\) Lemma 4.4.

2. We intentionally perform calculation in terms of \( k \) to illustrate subtleties behind the change of variable \( h \rightarrow k = e^h \), by comparing with the parallel computation in \([7]\) §4.

3. The Gauss-Bonnet functional equation \( (1+\sigma_2)(T) = 0 \), with \( T \) defined in Eq. (4.23), plays a key role in the computation, which should be compared with \([7]\) Lemma 4.3.

**Proposition 4.5.** On \( T^2_6 \), the OPS-functional has the following local expression:

\[
F_{\text{OPS}}(k) = \sum_{a=1}^{2} \varphi_0(k^{-2} T_{\text{OPS}}(y)(\nabla_a k \cdot \nabla_a k) + \zeta_{\Delta}^\prime(0),
\]

where the spectral function is given by

\[
T_{\text{OPS}}(y) = T_{\text{OPS}}(y) + H_{\text{OPS}}(y)
\]

\[
= \eta(K) \ln[1, y] \int_0^1 \sigma_1 \left( z^4[1, y] \right) d s + \frac{1}{2} \int_0^1 (z^4[1, y])^2 T(y^4; 2) \ln y d s.
\]

where \( K, H \) and \( T \) are the abbreviations listed in Eq. (4.22) and \( \eta(K) = K(1) = 1/6 \).

**Remark.** The validation of Eq. (4.21) will be further confirmed in §5.4 by explicit verification of the functional relation Eq. (5.33).

**Proof.** We shall abbreviate the spectral functions Eqs. (4.2), (4.3) and (4.10) related to \( R_{\Delta_k} \) by dropping the subscript:

\[
K(y) := K_{\Delta_k}(y; 2), \ T(y) := T_{\Delta_k}(y; 2), \ H(y_1, y_2) := H_{\Delta_k}(y_1, y_2; 2),
\]
so that for $m = 2$, Eq. (4.10) reads:

$$T(y) = \eta(K)y^{-1} + \eta(H)(y).$$

(4.23)

The $R_{\Delta_i}$ in Eq. (4.19) yields:

$$V_2(h, \Delta_i) = \varphi_0(hR_{\Delta_i}) = \eta(K)\varphi_0(hk^{-2}\nabla_a^2 k^s) + \varphi_0(hk^{-2s}\eta(H)(y^s)(\nabla_a k^s)(\nabla_a k^s)),$$

where Lemma 2.2 has been applied. For the first term:

$$\varphi_0(hk^{-2}\nabla_a^2 k^s) = -\varphi_0((\nabla_a h)k^{-2}\nabla_a k^s) - \varphi_0(h(\nabla_a k^{-2})\nabla_a k^s)$$

$$= -\varphi_0((\nabla_a h)k^{-2}\nabla_a k^s) + \varphi_0(\eta h^{-2s}y^{-s}(\nabla_a k^s)(\nabla_a k^s)),$$

where we need $\nabla_a k^{-2} = -k^{-2}(\nabla_a k^s)k^{-2} = -y^{-s}(\nabla a k^s)$. To continue, with Eq. (4.23) in mind:

$$V_2(h, \Delta_i)$$

(4.24)

$$= -\eta(K)\varphi_0((\nabla_a h)k^{-2}\nabla_a k^s) + \varphi_0(\eta h^{-2s}y^{-s}(\nabla_a k^s)(\nabla_a k^s))$$

$$= -\eta(K)\varphi_0((\nabla_a h)k^{-2}\nabla_a k^s) + \varphi_0(\eta h^{-2s}T(y^s)(\nabla_a k^s)(\nabla_a k^s)).$$

We would like to convert the two terms in the RHS of (4.24) into the desired form in Eq. (4.20). The calculation of the first term is postponed to Lemma 4.6 which leads to the first part of (4.21):

$$I_{\Gamma_v}(y) = \eta(K)\ln[1, y]\int_0^1 \sigma_{-1}(z^s[1, y])ds.$$

(4.25)

The $-1$ in front of $\eta(K)$ in (4.24) disappears because of the minus sign in the definition of $F_{\text{OPS}}$. There is an extra factor $h$ in the second term of Eq. (4.24), which can be turned into a modular derivation provided the Gauss-Bonnet functional equation $(1 + \sigma_{-2})(T) = 0$:

$$\varphi_0(k^{-2s}T(y^s)(\nabla_a k^s)h(\nabla_a k^s)) = \varphi_0(k^{-2s}\sigma_{-2s}(T(y^s))(\nabla_a k^s))$$

$$= -\varphi_0(hk^{-2s}T(y^s)(\nabla_a k^s)(\nabla_a k^s)).$$

It follows that

$$\varphi_0(hk^{-2s}T(y^s)(\nabla_a k^s)(\nabla_a k^s)) = \frac{1}{2}\varphi_0(\text{ad}_h[k^{-2s}T(y^s)(\nabla_a k^s)](\nabla_a k^s))$$

$$= -\frac{1}{2}\varphi_0(k^{-2s}\ln(yT(y^s))(\nabla_a k^s)(\nabla_a k^s)).$$

Before integrating in $s$, we further move the parameter to the modular operator.

$$(\nabla_a k^s)(\nabla_a k^s) = (k^{-s}z^s[1, y])z(\nabla_a k)$$

$$= k^{2(s-1)}(y_1)^{s-1}(z^s[1, y])z(\nabla_a k)$$

in which the spectral function becomes:

$$\eta(y_1^{s-1}(z^s[1, y])z(\nabla a k^s))(y) = (z^s[1, y])$$

after applying $\varphi_0$, that is

$$\varphi_0(hk^{-2s}T(y^s)(\nabla_a k^s)(\nabla_a k^s)) = -\frac{1}{2}\varphi_0(k^{-2s}(\ln(yT(y^s))(z^s[1, y]^2))(\nabla_a k)(\nabla_a k)),$$

which, after integration $\int_0^1$, gives rise exactly to the second part $\Pi\Gamma_v(y)$ in Eq. (4.21). The proof is complete.

Here is the last piece needed for Eq. (4.25).
Lemma 4.6. Keep notations. For \( \alpha = 1, 2 \), we have
\[
\int_0^1 \varphi_0((\nabla_y h)k^{-s}\nabla_y k^s)\, ds = \varphi_0\left(k^{-2}L(y)(\nabla_y k)(\nabla_y k)\right)
\]
with
\[
L(y) = \ln[1, y] \int_0^1 \sigma_{-1}(z^s[1, y])\, ds = \frac{-y + y \ln y + 1}{(y - 1)^2 y}.
\]

Proof. We start with Lemma 2.5:
\[
\nabla_y h = k^{-1}(\exp[0, x])^{-1}(\nabla_y k) = k^{-1}\ln[1, y](\nabla_y k), \quad \nabla_y k^s = k^{-s-1}z^s[1, y].
\]
Hence
\[
\varphi_0\left((\nabla_y h)k^{-s}\nabla_y k^s\right)
= \varphi_0\left(k^{-1}\left(\ln[1, y](\nabla_y k)\right)(k^{-s-1}(z^s[1, y])(\nabla_y k))\right)
= \varphi_0\left(k^{-2}\eta\left(y_1^{-1}\ln[1, y_1]z^s[1, y_2]\right)\right|_{y = e^y}(\nabla_y k)\cdot(\nabla_y k),
\]
where \( \eta \) is defined in (2.14):
\[
\eta\left(y_1^{-1}\ln[1, y_1]z^s[1, y_2]\right) = y^{-1}\ln[1, y]z^s[1, y^{-1}] = \ln[1, y]\sigma_{-1}(z^s[1, y]).
\]
The result follows from integrating the spectral function in \( s \) from 0 to 1.

Again, up to the constant \( \epsilon'_{\Delta}(0) \), the local expression of OPS-functional Eq. (4.20) is indeed landed in the setup of Theorem 2.17 with \( f = -2 \). As a consequence, the gradient is given as below.

Proposition 4.7. For a Weyl factor \( k \in C^\infty(\mathbb{T}_g^2) \), the gradient of the OPS-functional is given by:
\[
\text{(4.26)} \quad \text{grad}_k F_{\text{OPS}} = \sum_{\alpha=1}^2 k^{-2}K_{\text{OPS}}(y)(\nabla_y^2 k) + k^{-3}H_{\text{OPS}}(y_1, y_2)(\nabla_y k \otimes \nabla_y k),
\]
where
\[
\text{(4.27)} \quad K_{\text{OPS}} = (1 + \sigma_{-2})(T_{\text{OPS}}),
\]
\[
\text{(4.28)} \quad H_{\text{OPS}} = (1 + \sigma_{-3} - \sigma_{-3}^2) \cdot \mathbf{i}(K_{\text{OPS}}).
\]

4.4. Functional relations and hypergeometric functions. To sum up, we have given two approaches to \( \text{grad}_k F \) by performing variation: on heat traces and zeta functions in §3 and on the local expression of \( F \) in §4. To put the pieces together, we recall from §3.2 [3.3]
\[
\text{grad}_k F_{\text{EH}} = \frac{2 - m}{2}k^{-1}R_{\Delta_k} \quad \text{grad}_h F_{\text{EH}} = \frac{2 - m}{2}\exp[0, x](R_{\Delta_k}).
\]
When \( m = 2 \):
\[
\text{grad}_k F_{\text{OPS}} = -k^{-1}R_{\Delta_k}, \quad \text{grad}_h F_{\text{OPS}} = -\exp[0, x](R_{\Delta_k}).
\]
By comparing the spectral functions of the rearrangement operators, we have
\[
K_{\text{EH}}(y; m) = \frac{2 - m}{2}K_{\Delta_k}(y; m), \quad H_{\text{EH}}(y; m) = \frac{2 - m}{2}H_{\Delta_k}(y_1, y_2; m),
\]
\[
K_{\text{OPS}}(x; m) = \frac{2 - m}{2}\exp[0, x]K_{\Delta_k}(x; m), \quad \tilde{H}_{\text{OPS}}(x; m) = \frac{2 - m}{2}\exp[0, x_1 + x_2]H_{\Delta_k}(x_1, x_2; m).
\]
In dimension two,
\[
K_{\text{OPS}}(y) = -K_{\Delta_k}(y; 2), \quad H_{\text{OPS}}(y) = -H_{\Delta_k}(y_1, y_2; 2),
\]
\[
K_{\text{OPS}}(x) = \exp[0, x]K_{\Delta_k}(x; 2), \quad \tilde{H}_{\text{OPS}}(x) = \exp[0, x_1 + x_2]H_{\Delta_k}(x_1, x_2; 2).
\]
Therefore functional relations obtained in Propositions 4.7 and 4.4 for \( K \) and \( H \) on the left hand sides, can be transferred to functions \( K_{\Delta_k} \) and \( H_{\Delta_k} \) on the right hand sides.

**Theorem 4.8.** For one-variable functions, we have for \( m \geq 2 \)

\[
\frac{m-2}{2} K_{\Delta_k}(y; m) = (1 + \sigma_{-m/2-1}) T_{\Delta_k}(y; m).
\]

When \( m = 2 \),

\[
K_{\Delta_k}(y; 2) = (1 + \sigma_{-2}) T_{\text{OPS}}(y; 2),
\]

where \( T_{\text{OPS}} \) is defined in Prop. 4.5.

**Theorem 4.9.** For all \( m \geq 2 \), the spectral function defined in Eq. (4.2) and (4.3) in terms of hypergeometric geometric functions fulfill the functional relation:

\[
(4.29) \quad H_{\Delta_k}(y; m) = (1 + \sigma_{-m/2-2} - \sigma^2_{m/2-2}) \cdot \mathbf{\Pi}^+(K_{\Delta_k}(y; m)).
\]

For the log-Weyl factor side, recall \( \tilde{K}_{\Delta_k} \) and \( \tilde{H}_{\Delta_k} \) defined in Eq. (4.7), denote by

\[
(4.30) \quad \tilde{K}_{\Delta_k}(x; m) = \exp[0, x] \tilde{K}_{\Delta_k}(x; m), \quad \tilde{H}_{\Delta_k}(x_1, x_2; m) = \exp[0, x_1 + x_2] \tilde{H}_{\Delta_k}(x_1, x_2; m),
\]

then

\[
(4.31) \quad \tilde{H}_{\Delta_k}(x_1, x_2; m) = (1 + \tau_{-m/2+1} - \tau^2_{-m/2+1}) \cdot \mathbf{\Pi}^+(\tilde{K}_{\Delta_k}(x; m))
\]

**Remark.** In dimension \( m = 2 \), (4.31) reads:

\[
\tilde{H}_{\Delta_k}(x_1, x_2) = (1 + \tau_0 - \tau^2_0) \cdot \mathbf{\Pi}^+(\tilde{K}_{\Delta_k}(x)) = K_{\Delta_k}[x_1, x_1 + x_2] + K_{\Delta_k}[-x_1, -x_1 - x_2] - K_{\Delta_k}[-x_1, x_2] +
\]

\[
= \frac{K_{\Delta_k}(x_1 + x_2) - K_{\Delta_k}(x_1) + K_{\Delta_k}(-x_1 - x_2) - K_{\Delta_k}(-x_1)}{x_1 + x_2}.
\]

Provided the fact that \( \tau_0(K_{\Delta_k}) = K_{\Delta_k} \) (that is \( K_{\Delta_k} \) is an even function), the right hand side above recovers exactly the original version Eq. (1.1) of Connes-Moscovici.

The two versions of the Connes-Moscovici type relation Eq. (4.29) and Eq. (4.31) represent the same equation with respect to the change of variable \( k = h = \log k \). However, the verification of the equivalence are much less straightforward.

**Proposition 4.10.** Let us simply define the functions \( \tilde{K}_{\Delta_k} \) and \( \tilde{H}_{\Delta_k} \) in terms of \( K_{\Delta_k} \) and \( H_{\Delta_k} \) according to Eqs. (4.7) and (4.30) without any geometric setting. The functional relations Eq. (4.29) implies Eq. (4.31) provided the fact \( \sigma_{-m/2-1}(K_{\Delta_k}) = K_{\Delta_k} \).

**Proof.** Because of the cyclic property: \( \tau_3^{m/2+1} = 1 \) and \( \sigma_3^{m/2-1} = 1 \), Eq. (4.29) and (4.31) are equivalent to

\[
(4.32) \quad (1 + \sigma_{-m/2-2})(H_{\Delta_k}) = 2 \mathbf{\Pi}^+(K_{\Delta_k}), \text{ resp. } (1 + \tau_{-m/2+1})(\tilde{H}_{\Delta_k}) = 2 \mathbf{\Pi}^+(\tilde{K}_{\Delta_k}).
\]

We now assume the first relation in (4.32) and would like to prove the second one. Starting with

\[
\mathbf{\Pi}^+(\tilde{K}_{\Delta_k})(x_1, x_2) = (\exp[0, z])^2 K_{\Delta_k}(z)[x_1, x_1 + x_2] = (Q^I + Q^{II} + Q^{III})(x_1, x_2),
\]

\( ^8 \)It follows from the relations in Theorem 4.8.
where \( Q^I \) to \( Q^{III} \) are obtained by applying the Leibniz property of divided differences:

\[
Q^I(x_1, x_2) = \exp[0, x_1, x_1 + x_2] \exp[0, x_1 + x_2] K_{\Delta}(e^{x_1 + x_2}),
\]

\[
Q^{II}(x_1, x_2) = \exp[0, x_1] \exp[0, x_1, x_1 + x_2] K(e^{x_1}),
\]

\[
Q^{III}(x_1, x_2) = \exp[0, x_1] \exp[0, x_1 + x_2] K(e^z)[x_1, x_1 + x_2],
\]

here we have used the composition rule of divided differences: \( \exp[0, z][x_1, x_1 + x_2] = \exp[0, x_1, x_1 + x_2] \).

Also, we will freely use the following fact due to the multiplicative nature of the exponential function:

\[
e^\nu(\exp[u_1, \ldots, u_n]) = \exp[u_1 + v, \ldots, u_n + v], \quad u_1, \ldots, u_n, v \in \mathbb{R}.
\]

On the other side, with Eq. (4.7), we see that

\[
\hat{\mathcal{H}}_\Delta(x_1, x_2) = \exp[0, x_1 + x_2] \hat{\mathcal{H}}_\Delta(x_1, x_2) = f_H(x_1, x_2) H_\Delta(e^{x_1}, e^{x_2}) + 2Q^I(x_1, x_2),
\]

with

\[
f_H(x_1, x_2) = \exp[0, x_1] \exp[0, x_1 + x_2] \exp[0, x_1 + x_2].
\]

Now we claim that

\[
(4.35) \quad f_H = \tau^2(f_H), \quad Q^{II} = \tau^2_{-m/2+1}(Q^I).
\]

If true, we have \((1 + \tau_{-m/2+1})(Q^I) = Q^{II} + Q^I\) and

\[
\tau^2_{-m/2+1}(f_H) = \tau^2_{\Delta}(f_H) \cdot \tau^2_{-m/2+1}(H_\Delta^0) = f_H \cdot \tau^2_{-m/2+1}(H_\Delta^0),
\]

where \( H_\Delta^0(x_1, x_2) := H_\Delta(e^{x_1}, e^{x_2}) \). Then

\[
(1 + \tau_{-m/2-1})(f_H) = f_H(x_1, x_2) \cdot (1 + \tau_{-m/2})(H_\Delta^0)(x_1, x_2) = f_H(x_1, x_2) \cdot (1 + \tau_{-m/2-1})(H_\Delta^0)(x_1, x_2)
\]

\[
= 2 f_H(x_1, x_2) \exp[0, x_1, x_1 + x_2] K_{\Delta}(e^z)[x_1, x_1 + x_2] = 2Q^{III}(x_1, x_2),
\]

which concludes the proof of the second relation in Eq. (4.32). In the calculation above, we have used our assumption (first equation in Eq. (4.32)) and also Prop. 2.10 to carefully switch the variational operators from \( \{\sigma_{-m/2-1}, \tau^+\} \) to \( \{\tau_{-m/2+1}, \tau^+\} \).

Let us check the claim (4.35). Recall for \( j \in \mathbb{R} \), \( \tau_j^2 \) is implemented by the substitutions

\[
x_1 \rightarrow x_2, \quad x_2 \rightarrow -x_1 - x_2, \quad x_1 + x_2 \rightarrow -x_1
\]

followed by multiplying \( e^{jx_1} \). Therefore

\[
\tau^2_{x_1}(f_H)(x_1, x_2) = e^{3x_1} \exp[0, x_2] \exp[x_2, x_2, -x_1] \exp[0, -x_1]
\]

\[
= \exp[x_1, x_1 + x_2] \exp[0, x_1 + x_2] \exp[0, x_1] = f_H(x_1, x_2),
\]

and

\[
\tau_{-m/2+1}(Q^I)(x_1, x_2) = e^{-(m/2+1)x_1} \exp[0, x_2, -x_1] \exp[0, -x_1] K_{\Delta}(e^{-x_1})
\]

\[
= e^{2x_1} \exp[0, x_2, x_2, -x_1] \exp[0, -x_1] e^{-(m/2-1)x_1} K_{\Delta}(e^{-x_1})
\]

\[
= \exp[x_1, x_2 + x_1, 0] \exp[0, x_1] \left( y_1^{m/2-1} K_{\Delta}(y_1^{-1}) \right)
\]

notice that we have used the property \( \sigma_{-m/2-1}(K_{\Delta}) = K_{\Delta} \) to complete the argument. \( \square \)
5. Verification of the Functional Relations

It is highly nontrivial to see why the explicit functions $K_{\Delta_k}$ and $H_{\Delta_k}$ in Eqs. (4.2) and (4.3) fulfill all the functional relations derived in the previous section. In the last part of the paper, we give a computer-aid free verification, to further illustrate the power of hypergeometric family $H_a, a \in \mathbb{Z}^n_>$ and the variational operators discussed in §2.

5.1. Hypergeometric functions appeared in the rearrangement process. Since only the second heat coefficient is concerned, we only need those of one (Gauss hypergeometric functions $\,_2F_1$) and two variable (Appell’s $F_1$ functions). For $a, b, c \in \mathbb{Z}_+$, we denote:

\[
H_{a,b}(z; m) = \frac{\Gamma(d_m)}{\Gamma(a)\Gamma(b)} \int_0^1 (1-t)^{a-1} t^{b-1} (1-z t)^{-d_m} d t
\]

(5.1)

where $m$ is the dimension and $d_m = a + b + m/2 - 2$. In a like manner,

\[
H_{a,b,c}(z_1, z_2; m) = \frac{\Gamma(d_m)}{\Gamma(a+b+c)} \int_0^1 \int_0^{1-t} (1-t-u)^{a-1} u^{b-1} (1-z_1 t - z_2 u)^{-d_m} d u d t
\]

(5.2)

where $d_m = a + b + c + m/2 - 2$. They fulfill the following differential and divided difference relations (cf. [23] Theorem 3.3):

\[
H_{a,c}(z; m) = \frac{1}{(c-1)!} \frac{d^{c-1}}{d z} H_{a,1}(z; m),
\]

(5.3)

\[
H_{a,b,c}(z_1, z_2; m) = \left( \frac{b}{b-1} \right) \left( \frac{c}{c-1} \right) ! H_{a,1,1}(z_1, z_2; m),
\]

(5.4)

\[
H_{a,1,1}(z_1, z_2; m) = (z H_{a+1,1}(z; m)) [z_1, z_2]_z.
\]

(5.5)

Since $F_1(\alpha; \beta, \beta'; \gamma; z_1, z_2) = F_1(\alpha; \beta', \beta; \gamma; z_2, z_1)$, we have

\[
H_{a,b,c}(z_1, z_2; m) = H_{a,c,b}(z_2, z_1; m).
\]

(5.6)

An interesting takeaway of the relations above is the fact that one can reach any $H_{a,b}$ or $H_{a,b,c}$ from $H_{a,1}(z; m)$ via basic algebraic manipulations and differentiations, while the Gauss hypergeometric functions $H_{a,1}(z; m)$ admits fast evaluation, as functions in $a, m$ and $z$, in many CASs (computer algebra systems), such Mathematica. The observations allows us to achieve symbolic verification of the functional relations for all dimensions $m = 2, 3, \ldots$, in part I [23].

5.2. Further recurrence relations. We now discuss how the operators $\{\eta, \sigma_j, \blacksquare^i\}$ act on the $H$-family. Since the precise functions appeared in the rearrangement lemma are $H_{a,b}(1-y; m)$ and $H_{a,b,c}(1-y_1, 1-y_1 y_2; m)$, we introduce the change of variables

\[
z = 1 - y, \quad z_1 = 1 - y_1, \quad z_2 = 1 - y_1 y_2
\]

(5.7)

and freely use abbreviations like

\[
H_{a,b} := H_{a,b}(z; m), \quad H_{a,b,c} := H_{a,b,c}(z_1, z_2; m)
\]

when no confusion arises. For example, for $f \in C(\mathbb{R}_+)$, we have

\[
\blacksquare^i(f)(y_1, y_2) = f(y_1, y_1 y_2) = f[1 - z_1, 1 - z_2] = -f[z_1, z_2]
\]
and Eq. (5.5) becomes:
\[ H_{a,1,1} = -\eta^+(z H_{a+1,1,1}). \]

The contraction map \( \eta \) in Eq. (2.14) is obtained by setting \( y_2 \to y_1^{-1} \), that is \( y_1 y_2 \to 1 \) or \( z_2 \to 0 \) according to Eq. (5.7). It is not difficult to see \( H_{a,b,c}(z; 0; m) = H_{a+c,b}(z; m) \), that is,
\[ \eta(H_{a,b,c}) = H_{a+c,b}. \]

For the cyclic permutation \( \sigma_0 \), the transformations are \( y \to y^{-1} \) for \( H_{a,b} \) and \( y_1 \to (y_1 y_2)^{-1} \) and \( y_2 \to y_1 \) for \( H_{a,b,c} \), which are linear fractional transformations in terms of \( z, z_1 \) and \( z_2 \):
\[
z \to \frac{-z}{1-z}, \quad z_1 \to \frac{z_2}{z_2-1}, \quad z_2 \to \frac{z_2-z_1}{z_2-1}.
\]

**Proposition 5.1.** The cyclic operator \( \sigma_0 \) indeed permutes the indices \( \{a, b, c\} \) in the \( H \)-family:
\[
\sigma_0(H_{a,b,c}(z; m))(z) := H_{a,b,c}
\left(
\frac{1}{1-z} ; m \right) = (1-z)^{a+b+m/2-2} H_{b,a}(z; m).
\]

For two variable functions,
\[
\sigma_0(H_{a,b,c}(z_1,z_2); m)(z_1,z_2) := H_{a,b,c}
\left(
\frac{z_2}{z_2-1} ; \frac{z_2-z_1}{z_2-1} ; m \right)
= (1-z)^{a+b+c+m/2-2} H_{b,c,a}(z_1,z_2;m).
\]

In particular, we have:
\[
H_{a,1} = (1-z)^{m/2-1} \sigma_0(H_{1,a}), \quad H_{a,1,1} = (1-z_2)^{m/2} \sigma_0(H_{1,a,1}),
\]
where \( H_{1,a} \) and \( H_{1,a,1} \) are derivatives of \( H_{1,1} \) and \( H_{1,1,1} \), see Eq. (5.3) and (5.4). As a consequence, we have obtained an algorithm to write any \( H_{a,b} \) (resp. \( H_{a,b,c} \)) as derivatives of \( H_{1,1} \) (resp. \( H_{1,1,1} \)).

**Corollary 5.2.** When \( a, b, c \) and \( a+b+m/2-2 \) are all non-zero, we have:
\[
b H_{b+1,a} = (a+b+m/2-2)H_{a,b} - \alpha(1-z)H_{b,a+1},
\]
\[
b H_{b+1,c,a} = (a+b+c+m/2-2)H_{b,c,a} - c(1-z_1)H_{b,c+1,a} - \alpha(1-z_2)H_{b,c,a+1}.
\]

**Proof.** They are obtained by computing \( d/dz(\sigma_0(H_{a,b})) \) and \( \partial_{z_2}(\sigma_0(H_{a,b,c})) \) in two ways respectively. We will proof the two-variable case and left the rest to the reader. Denote
\[
J_{\sigma_0}(z_1,z_2) := (J_1(z_1,z_2), J_2(z_1,z_2)) := \left( \frac{z_2}{z_2-1}, \frac{z_2-z_1}{z_2-1} \right).
\]
Then
\[
\partial_{z_2}(\sigma_0(H_{a,b,c}(z_1,z_2)) = \partial_{z_2}
\left( H_{a,b,c}
\left( J_{\sigma_0}(z_1,z_2) \right) \right)
= \left( \partial_1 H_{a,b,c} \right)(J_{\sigma_0}(z_1,z_2)) \partial_{z_2} J_1 + \left( \partial_2 H_{a,b,c} \right)(J_{\sigma_0}(z_1,z_2)) \partial_{z_2} J_2
= \frac{1}{(z_2-1)^2} b \sigma_0(H_{a,b+1,c})(z_1,z_2) + \frac{z_1-1}{(z_2-1)^2} c \sigma_0(H_{a,b,c+1})(z_1,z_2)
= -(1-z_2)^{a+b+c+m/2-3} \left( b H_{b+1,c,a} + c(1-z_1)H_{b,c+1,a} \right)
\]
where we have used the differential relations from Eq. (5.3): \( \partial_1 H_{a,b,c} = b H_{a,b+1,c} \) and \( \partial_2 H_{a,b,c} = c H_{a,b,c+1} \) and Eq. (5.10) to remove the \( \sigma_0 \) in the third line.

On the other hand, we can first apply Eq. (5.10) and then carry out the differentiation:
\[
\partial_{z_2}(\sigma_0(H_{a,b,c}(z_1,z_2)) = \partial_{z_2}
\left( (1-z_2)^{a+b+c+m/2-2} H_{b,c,a}(z_1,z_2) \right)
= (1-z_2)^{a+b+c+m/2-3} \left( (a+b+c+m/2-2)H_{b,c,a} + (1-z_2)a H_{b,c,a+1} \right).
\]
We complete the proof by equating Eq. (5.12) and (5.13).
On the other hand, the differential equations attached to \( _2F_1 \) and \( F_1 \) leads to the following recurrence relations of the \( H \)-family. The hypergeometric ODEs of \( _2F_1 \) are transformed into

\[
(5.14) \quad B_2 H_{a,b+2} + B_1 H_{a,b+1} + B_0 H_{a,b} = 0
\]

with

\[
B_2 = b(b+1)(1-z)z, \quad B_1 = b(-z(a+2b+m/2-2)+a+b), \quad B_0 = -b(a+b+m/2-2).
\]

The PDE system of Appell's \( F_1 \) reads:

\[
(5.15) \quad C_{2,0} H_{a,b+2,c} + C_{1,1} H_{a,b+1,c+1} + C_{1,0} H_{a,b+1,c} + C_{0,1} H_{a,b,c+1} + C_{0,0} H_{a,b,c} = 0
\]

\[
(5.16) \quad C_{0,2} H_{a,b,c+2} + \tilde{C}_{1,1} H_{a,b+1,c+1} + \tilde{C}_{1,0} H_{a,b+1,c} + \tilde{C}_{0,1} H_{a,b,c+1} + \tilde{C}_{0,0} H_{a,b,c} = 0
\]

where the coefficients are given as below:

\[
C_{2,0} = b(b+1)(1-z_1)z_1, \quad C_{1,1} = b c (1-z_1)z_2, \quad C_{0,1} = -b c z_2, \\
C_{1,0} = b (-z_1(a+2b+c+m/2-1)+a+b+c), \quad C_{0,0} = -b (a+b+c+m/2-2),
\]

and

\[
C_{0,2} = c (c+1)(1-z_2)z_2, \quad \tilde{C}_{1,1} = b c (1-z_2)z_1, \quad \tilde{C}_{0,1} = -b c z_1, \\
C_{0,1} = c (-z_1(a+b+2c+m/2-1)+a+b+c), \quad C_{0,0} = -c (a+b+c+m/2-2).
\]

Some remarks:

1. Eq. (5.15) and (5.16) are equivalent provided the fact that \( H_{a,b,c}(z_1,z_2;m) = H_{a,c,b}(z_2,z_1;m) \).
2. By applying \( \sigma_0 \) onto Eq. (5.14) and (5.15), one obtains another set of relations among \( \{H_{a,b}, H_{a+1,b}, H_{a+2,b}\} \) and \( \{H_{a,b,c}, H_{a+1,b,c}, H_{a+2,b,c}\} \) which provide new routes for the reduction of \( H_{a,1} \) (resp. \( H_{1,1,1} \)) to \( H_{1,1} \) (resp. \( H_{1,1,1} \)).

The recurrence relations allow us to express \( H_{a,b} \) (resp. \( H_{a,b,c} \)) as linear combinations of \( H_{1,1} \) and \( H_{1,2} \) (resp. \( H_{1,1,1}, H_{1,2,1}, H_{1,1,2} \) and \( H_{1,2,2} \)) with rational function coefficients. For the two variable functions, one can further remove \( H_{1,2,2} \) using the fact that \( H_{a,1,1} = (z H_{a+1,1})(z_1,z_2) \) is a divided difference. In fact,

\[
H_{a,2,1} = \partial_{z_1} (z H_{a+1,1}) = (z H_{a+1,1})(z_1,z_1,z_2), \\
H_{a,1,2} = \partial_{z_2} (z H_{a+1,1}) = (z H_{a+1,1})(z_1,z_2,z_2).
\]

Thus

\[
H_{a,2,2} = \partial_{z_1} \partial_{z_2} ([z H_{a+1,1}]) = (z H_{a+1,1})(z_1,z_1,z_2) \\
= \frac{(z H_{a+1,1})(z_1,z_1,z_2) - (z H_{a+1,1})(z_1,z_2,z_2)}{z_1-z_2} = \frac{H_{a,2,1} - H_{a,1,2}}{z_1-z_2}.
\]

As for the dependence on the dimension \( m \) (assume \( m \geq 2 \)), we recall from [22] that:

\[
H_{a,b}(z;m+2) = a H_{a+1,b} + b H_{a,b+1}, \\
H_{a,b,c}(z_1,z_2;m+2) = a H_{a+1,b,c} + b H_{a,b+1,c} + c H_{a,b,c+1},
\]

where \( H_{a,b} := H_{a,b}(z;m) \) and \( H_{a,b,c} := H_{a,b,c}(z_1,z_2;m) \).

5.3. Initial values and relations. Last but not least, we discuss initial values and their relations:

\[
H_{1,1}(z;m) = \frac{\Gamma(m/2)}{\Gamma(2)} \int_0^1 (1-z t)^{-m/2} d t \\
= \begin{cases} 
-\log(1-u) [0,z], & m=2 \\
\Gamma(m/2-1)(1-u)^{-m/2+1} [0,z], & m \neq 2
\end{cases}
\]

\[
(5.18)
\]
Notice that, for \( m \neq 2 \), one can use \( z_{F_1} \) to make sense of \( H_{0,1}(z; m) = (1 - z)^{1-\frac{m}{2}} \Gamma\left(\frac{m}{2} - 1\right) \) even though the integral representation in Eq. (5.1) diverges and then \( H_{1,1} = H_{0,1}[0, z] \). In other words, if we define

\[
H_{0,1}(z; m) = \begin{cases} 
-\log(1 - z), & m = 2 \\
\Gamma(m/2 - 1)(1 - z)^{-m/2 + 1}, & m \neq 2
\end{cases}
\]

(5.19)

\[
H_{0,2}(z; m) := \frac{d}{dz} H_{0,1}(z; m) = \Gamma(m/2)(1 - z)^{-\frac{m}{2}}.
\]

(5.20)

then

\[
H_{1,1} = H_{0,1}[0, z],
\]

(5.21)

and for \( H_{1,2} := H_{1,2}(z; m) \):

\[
H_{1,2} = \frac{d}{dz} H_{1,1} = \frac{d}{dz} H_{0,1}[0, z] = H_{0,1}[0, z, z] = \frac{H_{0,1}[z, z] - H_{0,1}[0, z]}{z - 0}
\]

(5.22)

We shall need the following initial relation later: for \( m \neq 2 \)

\[
m H_{1,1} + 2z H_{1,2} - 2H_{0,2}[0, z] = (m - 2) H_{1,1} + 2H_{0,2} - 2H_{0,2}[0, z] = 0.
\]

(5.23)

Too see the vanishing of the right hand side, note that for \( m \neq 2 \), \((1 - z)H_{0,2} = (-m/2 + 1)H_{0,1}\), applying the divided difference \([0, z]\) on both sides yields

\[
-H_{0,2} + H_{0,2}[0, z] = \frac{2 - m}{2} H_{0,1}[0, z] = \frac{2 - m}{2} H_{1,1}.
\]

5.4. Verification of Theorem 4.8

**Lemma 5.3.** Let us abbreviate \( H_{a,b} := H_{a,b}(z; m) \). All the required recurrence relations for the one-variable family are listed as below.

\[
H_{2,1} = \frac{1}{2} m H_{1,1} + (z - 1) H_{1,2}, \quad H_{3,1} = \frac{1}{4} ((m + 2) H_{2,1} + 2(z - 1) H_{2,2}),
\]

(5.24)

\[
H_{4,1} = \frac{1}{6} ((m + 4) H_{3,1} + 2(z - 1) H_{3,2}).
\]

\[
H_{1,3} = -\frac{(m + 4)z - 4}{4(z - 1)} H_{1,2} + m H_{1,1}, \quad H_{1,4} = -\frac{(m + 8)z - 6}{6(z - 1)} H_{1,3} + (m + 2) H_{1,2}
\]

(5.25)

\[
H_{2,3} = -\frac{(m + 6)z - 6}{4(z - 1)} H_{2,2} + (m + 2) H_{2,1}
\]

(5.26)

\[
H_{2,2} = (m/2 + 1) H_{1,2} + 2(z - 1) H_{1,3}, \quad H_{3,2} = (m/4 + 1) H_{2,2} + (z - 1) H_{2,3}.
\]

*Proof.* The first set (5.24) is a special case of Lemma Corollary 5.2. The second set (5.25) comes from the differential equation (5.14). At last, we obtain (5.26) by differentiating the relations of \( H_{2,1} \) and \( H_{3,1} \) in (5.24).

At \( y = 1 \), that is \( z = 1 - y = 0 \), we have:

\[
H_{a,b}(0; m) = \frac{\Gamma(a + b + m/2 - 1)}{\Gamma(a + b + 1)},
\]

then

\[
K_{\Delta_1}(1; m) = -H_{2,1}(0; m) + \frac{4}{m} H_{3,1}(0; m) = \Gamma(m/2)(4 - m)/12
\]
where the vanishing of right hand side was checked in (5.23) (with the assumption □ where the vanishing is due to Eq. (5.22).

Consider the spectral functions \( K_{\Delta_k} \) and \( T_{\Delta_k} \) given in Eq. (5.27), then for \( m > 2 \), the relation

\[
\frac{m-2}{2} \Delta_{\Delta_k} = (1 + \sigma_{-m/2-1})(T_{\Delta_k})
\]

derived in Theorem 4.8 can be reduced to the following one among the initial values of the hypergeometric family, cf. Eq. (5.23):

\[
m H_{1,1} + 2 z H_{1,2} - 2 H_{0,2}[0, z] = 0.
\]

For \( m = 2 \), Eq. (5.28) becomes \((1 + \sigma_{-2})(T_{\Delta_k}) = 0\), which is reduced to Eq. (5.22):

\[
H_{0,2} - H_{1,1} - z H_{1,2} = 0.
\]

**Proof.** According to Prop. 5.1 \( \sigma_{-m/2-1}(T_{\Delta_k}) \) is given by:

\[
\sigma_{-m/2-1}(T_{\Delta_k}) = \frac{1}{12} m H_{0,2}[0, z] + \frac{2}{m} (1 - z) \left( (m + 2) H_{1,3} + 4 (z - 1) H_{1,4} - 2 H_{2,3} \right),
\]

here we have also used the fact that \( \sigma_{-m/2-1}(H_{0,2}[0, z]) = H_{0,2}[0, z] \). By repeating the substitutions given in Lemma 5.3 one can replace all the \( H_{a,b} \) with \( a, b > 0 \) Eqs. (5.27) and (5.29) by \( H_{1,1} \) and \( H_{1,2} \), the result reads as follows:

\[
(1 + \sigma_{-m/2-1})(T_{\Delta_k}) - (m - 2) \Delta_{\Delta_k}/2 = \frac{m-4}{12} \left( m H_{1,1} + 2 z H_{1,2} - 2 H_{0,2}[0, z] \right) = 0,
\]

where the vanishing of right hand side was checked in (5.23) (with the assumption \( m \neq 2 \)). When \( m = 2 \), Eq. (5.28) reduces to \((1 + \sigma_{-2})(T_{\Delta_k}) = 0\). Indeed, again using the reductions relations in Lemma 5.3 we obtain

\[
(1 + \sigma_{-2})(T_{\Delta_k}) = \frac{1}{3} \left( H_{0,2} - H_{1,1} - z H_{1,2} \right) = 0,
\]

where the vanishing is due to Eq. (5.22). \( \square \)

In dimension \( m = 2 \), the corresponding functional of (5.28) is derived by the variation of the OPS-functional and the function \( T_{\Delta_k} \) is replaced by \( T_{\text{OPS}} \), which we recall from (1.21):

\[
T_{\text{OPS}}(y) = I_{T_r}(y) + \Pi_{T_r}(y)
\]

\[
= \eta(K) \ln[1, y] \int_0^1 \sigma_{-1}(z^2[1, y]) \, ds + \frac{1}{2} \int_0^1 (z^2[1, y])^2 T_{\Delta_k}(y^2; 2) \ln y \, ds.
\]

**Lemma 5.5.** For the given function \( T_{\text{OPS}} \) as above, we have

\[
(1 + \sigma_{-2})(T_{\text{OPS}})(y) = \eta(K) y^{-1} + (y - 1)^{-2} \int_1^y (u - 1)^2 T(u) u^{-1} \, du,
\]

where \( T(u) = T_{\Delta_k}(u; 2) \).
Proof. One checks that:

\[ \sigma_{-1}(\ln[1, y]) = \ln[1, y], \quad \sigma_s(z^s[1, y]_z) = z^s[1, y]_z. \]  

Also, since \( \sigma_{-2}(T) = -T \), we see that

\[ \sigma_{2s}(T(y^s)) = -T(y^s), \quad \sigma_0(\ln(y)) = -\ln(y). \]

Therefore, \( \Pi_T \) is invariant under \( \sigma_{-2} \):

\[ \sigma_{-2}\left((z^s[1, y])^2 T(y^s) \ln y\right) = (\sigma_s(z^s[1, y]))^2 \sigma_{-2}(T(y^s)) \sigma_0(\ln y) \]

Thus

\[ (1 + \sigma_{-2})(\Pi_T)(y) = 2\Pi_T(y) = \int_0^1 (z^s[1, y])^2 T(y^s) \ln y \, ds \]

\[ = (y - 1)^{-2} \int_1^y (u - 1)^2 T(u) u^{-1} \, du. \]

Meanwhile, to compute \( \sigma_{-2}(I_T) \), note that \( \sigma_{-1}^2 = 1 \):

\[ \sigma_{-2}[\ln(1, y) \sigma_{-1}(z^s[1, y])] = \sigma_{-1}(\ln(1, y)) \sigma_{-1}^2(z^s[1, y]) = \ln(1, y) z^s[1, y]. \]

It follows that

\[ (1 + \sigma_{-2})(I_T)(y) = \eta(K) \ln[1, y] \int_0^1 (1 + \sigma_{-1})(z^s[1, y]) ds = \eta(K) y^{-1}. \]

\[ \Box \]

Proposition 5.6. Keep notations as above. The functional relations

\[ K_{\Delta_k} = (1 + \sigma_{-2})(T_{OPS}) \]

is can be reduced to the initial relation defined in Eq. (5.22):

\[ (5.33) \]

\[ z H_{2,1} = H_{0,2} - H_{1,1}, \quad H_{0,2} = (1 - z)^{-1}. \]

Proof. Set \( K := K_{\Delta_k}(; 2) \) and \( T := T_{\Delta_k}(; 2) \), we have simplified the right hand side of Eq. (5.33):

\[ (1 + \sigma_{-2})(T_{OPS})(y) = \eta(K)y^{-1} + (y - 1)^{-2} \int_1^y (u - 1)^2 T(u) u^{-1} \, du. \]

In particular, we see that \((1 + \sigma_{-2})(T_{OPS})(1) = \eta(K) = K(1)\), thus it suffices to check their derivatives are equal. We group Eq. (5.33) in the following way:

\[ (y - 1)^2(K - \eta(K)y^{-1}) = \int_1^y (u - 1)^2 T(u) u^{-1} \, du \]

and then apply \( d/dy \) to both sides:

\[ (2 + (y - 1)d/dy)\left(K(y) - \eta(K)y^{-1}\right) = y^{-1}(y - 1)T(z). \]

\[ -(1 - z)^{-1}(2 + z d/dz)\left[K(1 - z) - \eta(K)(1 - z)^{-1}\right] = T(z), \]

where \( z = 1 - y \). With the help of \( d/dz(H_{0,b}) = b H_{0,b+1} \), we write the difference of two sides

\[ T(z) + (1 - z)^{-1}(2 + z d/dz)\left[K(1 - z) - \eta(K)(1 - z)^{-1}\right] \]

\[ = \frac{-6(z - 1)H_{2,1} - 3(z - 1)z H_{1,1} + 12 z H_{4,1} - 12 H_{3,1} + 1}{3z} \]

where the last line is obtained by applying the relations in Lemma 5.3. Moreover, the vanishing of the last line is equivalent to the conditions in (5.34). \( \Box \)
5.5. **Verification of Theorem 4.9**

**Lemma 5.7.** Keep the abbreviation: \(H_{a,b,c} := H_{a,b,c}(z_1, z_2; m)\). The recurrence relations will be needed in later computation are listed as below:

\[
H_{3,1,1} = \frac{1}{4} ((m + 4)H_{2,1,1} + 2(z_2 - 1)H_{2,1,2} + 2(z_1 - 1)H_{2,2,1}),
\]

\[
(5.36)
\]

\[
H_{2,1,2} = \frac{1}{2} (m + 4)H_{1,1,2} + 2(z_2 - 1)H_{1,1,3} + (z_1 - 1)H_{1,2,2},
\]

\[
H_{2,1,1} = \frac{1}{2} (m + 2)H_{1,1,1} + (z_2 - 1)H_{1,1,2} + (z_1 - 1)H_{1,2,1},
\]

and

\[
H_{1,1,3} = -\frac{(m + 6)z_2 - 6)H_{1,1,2} + (m + 2)H_{1,1,1} + 2z_1((z_2 - 1)H_{1,2,2} + H_{1,2,1})}{4(z_2 - 1)z_2},
\]

\[
(5.37)
\]

\[
H_{1,3,1} = -\frac{(m + 6)z_1 - 6)H_{1,2,1} + (m + 2)H_{1,1,1} + 2z_2((z_1 - 1)H_{1,2,2} + H_{1,1,2})}{4(z_1 - 1)z_1},
\]

and

\[
(5.38)
\]

\[
H_{1,2,2} = \frac{H_{1,2,1} - H_{1,1,2}}{z_1 - z_2}.
\]

**Proposition 5.8.** Let \(H_{\Delta_k}\) and \(K_{\Delta_k}\) be the spectral functions given in (4.3) and (4.2) respectively. The verification of functional relation

\[
H_{\Delta_k} = (1 + \sigma_{-m/2-2} + \sigma_{-m/2-2}^2)\cdot \Delta^+(K_{\Delta_k})
\]

can be reduced to the following initial relations: for \(m > 2\), we use Eq. (5.22) and (5.23):

\[
(5.39)
\]

\[
zH_{1,2} = H_{0,2} - H_{1,1}, \quad mH_{1,1} + 2zH_{1,2} - 2H_{0,2}[0, z] = 0,
\]

While in dimension \(m = 2\), we need (5.22) and (5.20):

\[
(5.40)
\]

\[
zH_{1,2} = H_{0,2} - H_{1,1}, \quad H_{0,2} = (1 - z)^{-1}.
\]

**Proof.** Due to the cyclic property \(\sigma_{-m/2-2}^3 = 1\), we shall prove the equivalent functional relation

\[
\frac{1}{2}(1 + \sigma_{-m/2-2})(H_{\Delta_k}) = \Delta^+(K_{\Delta_k}).
\]

Start with the left hand side,

\[
\sigma_{-m/2-1}(H_{\Delta_k}) = \frac{2(m + 2)H_{1,1,2} + 8(z_2 - 1)H_{1,1,3} - 4H_{2,1,2}}{m}
\]

and then

\[
(1 + \sigma_{-m/2-1})(H_{\Delta_k}) = \frac{2}{m} \left[ (m + 2)H_{1,2,1} + (m + 2)H_{2,1,1} + 2(z_2 - 1)H_{1,2,2} + 4(z_1 - 1)H_{1,3,1} + 2(z_1 - 1)H_{2,2,1} - 4H_{3,1,1} \right],
\]

which admits further simplification due to the recurrence relations listed in Lemma 5.7

\[
-\frac{2((m + 2)(z_1 + z_2)H_{1,1,1} + 2(z_2 - 1)(2z_1 + z_2)H_{1,1,2} + 2(z_1 - 1)(z_1 + 2z_2)H_{1,2,1})}{m z_1 z_2}.
\]

Notice that

\[
H_{1,1,1} = (zH_{2,1})[z_1, z_2]z, \quad H_{1,2,1} = \partial_{z_1}H_{1,1,1}, \quad H_{1,1,2} = \partial_{z_2}H_{1,1,1}.
\]
and apply again the recurrence relations in Lemma 5.3 for one-variable functions, we can express the difference, which we would like to prove to be zero, in terms of $H_{1,1}$ and $H_{1,2}$:

\[
\frac{1}{2} (1 + \sigma_{-m/2-\frac{3}{2}})(H_{\Delta}) - \sigma^+ (K_{\Delta}) = \frac{z_1 + z_2}{2z_1 z_2} \left( mz H_{1,1} + 2z^2 H_{1,2} - 2z H_{1,2} - 2H_{1,1} \right) [z_1, z_2].
\]

It remains to show that $m z H_{1,1} + 2z^2 H_{1,2} - 2z H_{1,2} - 2H_{1,1}$ is a constant function (in $z$), therefore becomes zero after applying the divided difference. Indeed, with the initial relations in Eq. (5.39) for dimension $m > 2$, one sees that

\[
(m z H_{1,1} + 2z^2 H_{1,2}) - (2z H_{1,2} + 2H_{1,1}) = 2H_{0,2}[0, z] - 2H_{0,2} = -2H_{0,2}[0; m],
\]

and when $m = 2$, the initial relations in Eq. (5.40) implies that

\[
m z H_{1,1} + 2z^2 H_{1,2} - 2z H_{1,2} - 2H_{1,1} = 2(z_1 - 1)(z_1 H_{1,2} + H_{1,1}) = 2(z_1 - 1) H_{0,2} = -2.
\]

\[
\square
\]

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