Real Jacobian mates

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Abstract

Let $p$ be a real polynomial in two variables. We say that a polynomial $q$ is a real Jacobian mate of $p$ if the Jacobian determinant of the mapping $(p, q) : \mathbb{R}^2 \to \mathbb{R}^2$ is everywhere positive. We present a class of polynomials that do not have real Jacobian mates.

1 Introduction

This note is inspired by [2] where Braun and dos Santos Filho proved that every polynomial mapping $(p, q) : \mathbb{R}^2 \to \mathbb{R}^2$ with everywhere positive Jacobian determinant and such that $\deg p \leq 3$ is a global diffeomorphism.

A pair of polynomials $p, q \in \mathbb{R}[x, y]$ such that the mapping $(p, q) : \mathbb{R}^2 \to \mathbb{R}^2$ has everywhere positive Jacobian will be called real Jacobian mates. The key role in [2] plays the result that $p = x(1 + xy)$ does not have a real Jacobian mate.

This result is a special case of Theorem 1. In Theorem 2 a wide class of polynomials that do not have real Jacobian mates is characterized. In particular every polynomial such that its Newton polygon has an edge described in Corollary 1 belong to this class. This gives a new proof of [3, Theorem 5.5] that polynomials of degree 4 with at least one disconnected level set do not have real Jacobian mates.

2 Glacial tongues

Theorem 1 Let $p$ be a real polynomial in two variables and let $B \subseteq A$, be the subsets of the real plane such that:

(i) the set $B$ is bounded,

(ii) for every $t \in \mathbb{R}$ the set $p^{-1}(t) \cap A$ is either empty, or is contained in $B$, or is homeomorphic to a segment and its endpoints belong to $B$,

(iii) the border of $A$ contains a half-line.

Then for every $q \in \mathbb{R}[x, y]$ there exists $v \in \mathbb{R}^2$, such that $\text{Jac}(p, q)(v) = 0$. 

Proof. Suppose that there exists \( q \in \mathbb{R}[x, y] \) such that \( \text{Jac}(p, q) \) vanishes nowhere. Under this assumption the mapping \( \Phi = (p, q) : \mathbb{R}^2 \to \mathbb{R}^2 \) is a local diffeomorphism.

Take any \( t \in \mathbb{R} \) such that the set \( A_t = p^{-1}(t) \cap A \) is nonempty. If \( A_t \subset B \) then \( \Phi(A_t) \subset \Phi(B) \). If \( A_t \) is homeomorphic to a segment with endpoints in \( B \) then the restriction of \( \Phi \) to \( A_t \) is a locally injective continuous mapping from the source \( A_t \) which is homeomorphic to a segment to a vertical line \( \{t\} \times \mathbb{R} \) which is homeomorphic to \( \mathbb{R} \). By the extreme value theorem and the mean value theorem such a mapping is either increasing or decreasing. Hence, the image \( \Phi(A_t) \) is a vertical segment with endpoints that belong to \( \Phi(B) \).

Since \( A \) is the union of the sets \( A_t \) and \( \Phi(B) \) is bounded, so it is \( \Phi(A) \).

Let \( L \) be a half-line contained in the border of \( A \). Because the mapping \( \Phi \) is bounded on \( A \) it is also bounded on \( L \). Consequently the polynomials \( p \) and \( q \) restricted to \( L \) are constant (because they behave on \( L \) like polynomials in one variable). We arrived to a contradiction with the condition that \( \Phi \) is locally injective. ☐

Every set \( A \) satisfying assumptions of Theorem 1 will be called a glacial tongue with a straight border.

Example 1 Let \( p = x(1 + xy) \). In [2] it is checked (Lemma 4.1 and Remark 1) that \( A = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, -\frac{1}{2} < y \leq -1\} \) is a glacial tongue with a straight border for the polynomial \( p \). Hence, \( p \) does not have a Jacobian mate.

3 Newton polygon and branches at infinity

Let \( p = \sum a_{i,j}x^iy^j \) be a nonzero polynomial. By definition the Newton polygon \( \Delta(p) \) is the convex hull of the set \( \{(i, j) \in \mathbb{N}^2 : a_{i,j} \neq 0\} \). An edge \( S \) of the Newton polygon \( \Delta(p) \) will be called an outer edge if it has a normal vector \( \vec{v} = (v_1, v_2) \) directed outwards \( \Delta(p) \) such that \( v_1 > 0 \) or \( v_2 > 0 \). If \( v_1 > 0 \) then \( S \) will be called a right outer edge. With every right outer edge \( S \) we associate a rational number \( \theta(S) = v_2/v_1 \) and call this number the slope of \( S \).

Example 2 The Newton polygon of \( p = x + x^2 + x^3y + y^2 + x^3y^2 + xy^4 \) has 4 outer edges. Three of them are right outer edges with slopes \(-1, 0, \) and \( 2 \).

\(^1\) Suppose that a continuous and locally injective function \( f : [a, b] \to \mathbb{R} \) in neither increasing nor decreasing. Then there exist \( x_1, x_2, x_3, a \leq x_1 < x_2 < x_3 \leq b \) such that \( f(x_1) \leq f(x_2) \geq f(x_3) \) or \( f(x_1) \geq f(x_2) \leq f(x_3) \). By the extreme value theorem \( f \) restricted to \([x_1, x_3]\) has a maximal or a minimal value at some point \( c \) inside the interval \([x_1, x_3]\). Shrinking \([x_1, x_3]\), if necessary, we may assume that \( f \) restricted to \([x_1, x_3]\) is injective. By the mean value theorem \( f(x_3) \in f([x_1, c]) \) or \( f(x_1) \in f([c, x_3]) \) which gives a contradiction.
The objective of this section is to describe branches at infinity of a curve \( p(x, y) = 0 \) and associate with each branch a certain outer edge of the Newton polygon of \( p \).

Let \( V = \{(x, y) \in \mathbb{R}^2 : p(x, y) = 0\} \). Assume that the curve \( V \) is unbounded and consider a one-point algebraic compactification \( \tilde{\mathbb{R}}^2 = \mathbb{R}^2 \cup \{\infty\} \) of the real plane (see [1] Definition 3.6.12). Then \( \infty \) belongs to the Zariski closure of \( V \) in \( \tilde{\mathbb{R}}^2 \). By [1] Lemma 3.3] in a suitably chosen neighborhood of \( \infty \) the curve \( V \cup \{\infty\} \) is the union of finitely many branches which intersect only at \( \infty \). Each branch is homeomorphic to an open interval under an analytic homeomorphism \( p : (\epsilon, \epsilon) \to V \cup \{\infty\}, p(0) = \infty \).

It follows from the above that after passing to coordinates \( x \) and \( y \) in \( \mathbb{R}^2 \) and substituting \( s = t^{-1} \) in \( p \) we get the following characterization of branches at infinity.

**Lemma 1** Assume that \( V = \{(x, y) \in \mathbb{R}^2 : p(x, y) = 0\} \) is an unbounded polynomial curve. Then in a suitably chosen neighborhood of infinity in \( \mathbb{R}^2 \) the curve \( V \) is the union of finitely many pairwise disjoint “branches at infinity”. Each branch at infinity is homeomorphic to a union of two open intervals \( (\epsilon, -\epsilon) \cup (\epsilon, +\epsilon) \) under a homeomorphism \( (x, y) \to (\hat{x}(t), \hat{y}(t)) \) which is given by Laurent power series

\[
\hat{x}(t) = a_k t^k + a_{k-1} t^{k-1} + \cdots \\
\hat{y}(t) = b_l t^l + b_{l-1} t^{l-1} + \cdots
\]

convergent for \(|x| > R\).

**Lemma 2** Keep the assumptions an notations of Lemma 1. If \( a_k \neq 0, b_l \neq 0 \) then \((k, l)\) is a normal vector to some outer edge of the Newton polygon of \( p \).

**Proof.** Let \( d = \max\{ki + lj : (i, j) \in \Delta(p)\} \). The polynomial \( p \) can be written as a sum \( p = \sum_{ki+lj \leq d} c_{i,j} x^i y^j \). Substituting \((x, y) \to (\hat{x}(t), \hat{y}(t))\) to \( p \) and collecting together the terms of the highest degree we get

\[
0 = p(\hat{x}(t), \hat{y}(t)) = \left( \sum_{ki+lj = d} c_{i,j} a_k^i b_l^j \right) t^d + \text{terms of lower degrees}.
\]

The necessary condition for this identity is a cancellation of terms in the sum in parenthesis. If there are at least two distinct coefficients \( c_{i_1,j_1} \neq 0, c_{i_2,j_2} \neq 0 \), satisfying \( ki_1 + lj_1 = ki_2 + lj_2 = d \) then the straight line \( \{(i, j) \in \mathbb{R}^2 : ki + lj = d\} \) touches \( \Delta(p) \) at least two points, hence along the edge.

Since \((x, y) \to (\hat{x}(t), \hat{y}(t))\) is a Laurent parametrization of a branch at infinity, we have \(|(\hat{x}(t), \hat{y}(t))| \to \infty \) as \( t \to +\infty \) which proves that at least one of exponents \( k, l \) is positive and shows that \( \Delta(p) \cap \{(i, j) \in \mathbb{R}^2 : ki + lj = d\} \) is an outer edge. \(\blacksquare\)

Using Lemma 2 we may associate to every branch at infinity of the curve \( p = 0 \) the unique outer edge of the Newton polygon of \( p \). In the next lemma we
will show that the slope of the associated edge characterizes the asymptotic of the branch at infinity.

For two real valued functions $g, h$ defined in some interval $(R, \infty)$ we will write $g(x) \sim h(x)$ if there exist constants $c > 0$, $C > 0$, and $r > 0$ such that $c|h(x)| \leq |g(x)| \leq C|h(x)|$ for all $x > r$.

**Lemma 3** Let $p(x, y)$ be a nonzero real polynomial such that for every $x_0$ the set $X = \{(x, y) \in \mathbb{R}^2 : x > x_0, y > 0, p(x, y) = 0\}$ is nonempty. Then for sufficiently large $x_0$ there exists a finite collection of continuous semialgebraic functions $f_k : (x_0, +\infty) \to \mathbb{R}, k = 1, \ldots, s$ such that

(i) $0 < f_1(x) < \ldots < f_s(x)$ for $x > x_0$,

(ii) $X$ is the union of graphs $\{(x, y) \in \mathbb{R}^2 : y = f_k(x), x > x_0\}, k = 1, \ldots, s$,

(iii) for every $f_k$ there exists a right outer edge $S_k$ of the Newton polygon of $p(x,y)$ such that $f_k(x) \sim x^{\theta(S_k)}$.

**Proof.** Part (i) and (ii) follow from the Cylindrical Decomposition Theorem for semialgebraic sets (see for example [1, Theorem 2.2.1]).

To prove (iii) observe that the graph of $f_k$ is unbounded and homeomorphic to an open interval. Thus, we may assume, increasing $x_0$ if necessary, that this graph is a half-branch at infinity. By Lemma 1 there exists a homeomorphism of an open interval $(R, +\infty)$ to the graph given by Laurent power series (1), (2) with $a_k \neq 0, b_l \neq 0$. Since $\tilde{x}(t) \to +\infty$ for $t \to +\infty$, the leading term of $\tilde{x}(t)$ has a positive exponent $k$. By estimations $\tilde{x}(t) \sim t^k$, $\tilde{y}(t) \sim t^l$ and identity $f_k(\tilde{x}(t)) = \tilde{y}(t)$ we get $f_k(x) \sim x^{l/k}$. Finally, by Lemma 2 there exists a right outer edge $S$ of the Newton polygon of $p$ such that $l/k = \theta(S)$. ■

4 Main result

**Theorem 2** Assume that the Newton polygon of $p \in \mathbb{R}[x,y]$ has a right outer edge $S$ with endpoint $(0, 1)$ and positive inclination and that the curve $p = 0$ has a real branch at infinity associated with the edge $S$. Then $p$ has a glacial tongue with a straight border.

**Proof.** Without loss of generality we may assume, changing signs of variables if necessary, that one of half-branches associated with the edge $S$ lies in the positive quadrant $x > 0, y > 0$.

Then, under notation of Lemma 3 this half-branch at infinity is a graph $y = f(x)$ where $f$ is one of the functions $f_k, k = 1, \ldots, s$. Comparing the asymptotic of these functions we see that $\theta(S_1) \leq \theta(S_2) \leq \ldots \leq \theta(S_s)$. Since $S$ has the smallest slope among all right outer edges of the Newton polygon $\Delta(p)$, we have $S = S_1$ and we may assume that $f(x) = f_1(x)$.

Let $V = \{(x,y) \in \mathbb{R}^2 : x > x_0, 0 < y < f(x)\}$. 

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The polynomial \( p \) vanishes nowhere on \( V \), hence without loss of generality we may assume that \( p \) is positive on this set.

**Claim 1.** For every \( t \neq 0 \) the set \( p^{-1}(t) \cap V \) is bounded.

**Proof of 1.** If not, then by the Curve Selection Lemma there exists a half-branch at infinity of a curve \( p(x,y) = t \) contained in \( V \). Let \( y = g(x) \) be the graph of this half-branch at infinity. By Lemma 3 \( g(x) \sim x^{\theta(S_1)} \), where \( S_1 \) is one of the right outer edges of the Newton polygon \( \Delta(p - t) \). By inequalities \( 0 < g(x) < f(x) \) we get \( \theta(S_1) \leq \theta(S) \). This is impossible because all right outer edges of \( \Delta(p - t) \) have slopes bigger than the slope of \( S \).

**Claim 2.** For \( x_0 \) sufficiently large, \( V \) does not contain any critical point of \( p \).

**Proof of 2.** If the intersection of \( V \) with the set of critical points is bounded then it is enough to enlarge \( x_0 \). If this intersection is unbounded then by the Curve Selection Lemma it contains an unbounded semi-algebraic arc \( \Gamma \subset V \). It follows that \( p \) restricted to \( \Gamma \) is constant and nonzero – contrary to Claim 1.

Further, we will assume that \( V \) satisfies assumptions of Claim 2. Then every level set \( p^{-1}(t) \) intersected with \( V \) is a one-dimensional smooth semialgebraic manifold. By Poincare-Bendixon Theorem \( V_t = p^{-1}(t) \cap V \) has a finite number of connected components, each homeomorphic to a circle or to an open interval.

**Claim 3.** There is no connected component of \( V_t \) homeomorphic to a circle.

**Proof of 3.** Suppose there is. Then by Jordan’s Theorem it cuts the set \( V \) to two open regions. One of these regions is bounded. Since the function \( p \) is constant on the boundary of this region, it attains an extreme value at some point inside. This is impossible because \( p \) has no critical points in the set \( V \).

Let \( h(y) = p(x_0, y) \) be the restriction of \( p \) to the vertical line \( \{x_0\} \times \mathbb{R} \). A function \( h \) vanishes at the endpoints of the interval \([0, f(x_0)]\) and is positive inside. It is easy to find \( t_0 > 0 \) and two points \( a < b \) inside the interval \([0, f(x_0)]\) such that:

\[
\begin{align*}
&h'(y) \neq 0 \text{ for } y \in (0, a) \cup [b, f(x_0)), \\
h &\text{ increases from } 0 \text{ to } t_0 \text{ in the interval } [0, a], \\
h(y) &> t_0 \text{ for } a < y < b, \\
h &\text{ decreases from } t_0 \text{ to } 0 \text{ in the interval } [b, f(x_0)].
\end{align*}
\]

**Claim 4.** For every \( t \) such that \( 0 < t \leq t_0 \) the set \( V_t = p^{-1}(t) \cap V \) is connected and homeomorphic to an open interval. The topological closure of \( V_t \) intersects the vertical segment \( \{x_0\} \times (0, f(x_0)) \) at two points.

**Proof of 4.** By the discussion proceeding Claim 4 the polynomial \( p \) attains value \( t \) precisely at two points of the boundary of \( V \). These are \((x_0, y_1)\) and \((x_0, y_2)\), where \( 0 < y_1 \leq a \) and \( b \leq y_2 < f(x_0) \). Moreover \( \partial p / \partial y \) does not vanish at these points.

By Claim 2 and Claim 3 the set \( V_t \) is a one-dimensional smooth manifold having a finite number of connected component; each component is semialgebraic and homeomorphic to an open interval. Thus, the closure of \( V_t \) is a graph with vertexes \((x_0, y_1), (x_0, y_2)\) and edges which are connected components of \( V_t \).
By the Implicit Function Theorem the closure of $V_t$ has in a small neighborhood of $(x_0, y_1)$, where $i = 1, 2$ a topological type of an interval $[0, 1)$ which shows that there is exactly one edge which connects $(x_0, y_1)$ and $(x_0, y_2)$.

By Claim 4 the closure of $V_{t_0}$ is a line with two endpoints: $(x_0, a)$ and $(x_0, b)$. Joining them by a vertical segment we get a non-smooth oval. By Jordan’s Theorem this oval cuts the plane into two open regions. Let $B_0$ be the bounded region, let $B = B_0 \cup \{x_0\} \times (0, f(x_0))$ and let $A = V \cup \{x_0\} \times (0, f(x_0))$.

If $t \leq 0$ then $A_t$ is empty. If $0 < t \leq t_0$ then $A_t$ is homeomorphic to a line with endpoints at $\{x_0\} \times (0, f(x_0))$. If $t > t_0$ then either $A_t$ is empty or the closure of every connected component of $A_t$ intersects the border of $A$ along $x_0 \times (a, b)$. In this case $A_t \subset B$. 

**Corollary 1** Assume that the Newton polygon of a polynomial $p \in \mathbb{R}[x,y]$ has a right outer edge that: begins at $(0,1)$, has a positive inclination, and its only lattice points are the endpoints. Then $p$ does not have a real Jacobian mate.

**Proof.** It is enough to prove that there exists a branch at infinity of the curve $p = 0$ associated with the edge $S$ satisfying assumptions of Corollary 1. Let $(0,1), (a,b)$ be the endpoints of $S$. Then the polynomial $p$ has two terms $Ax^ay^b$ and $By$ corresponding to the lattice points of $S$. Multiplying $x, y$ and $p$ by nonzero constants, we may reduce our considerations to $A = 1$ and $B = -1$. Substituting $(x(t), y(t)) = (ct^{b-1}, t^{-a})$ we get $p(x(t), y(t)) = (c^a - 1)t^{-a} + \text{terms of lower degrees}$. Hence, the sign of the polynomial $p$ on the curve $(x(t), y(t))$ for large $t$ depends on the sign of $c^a - 1$. The curve $(x(t), y(t))$ for $t > 0$ is a graph of a function. By the appropriate choice of $c$ we can find two functions $f_1, f_2$ such that $0 < f_1 < f_2$, $f_1(x) \sim f_2(x) \sim x^{\theta(S)}$, $p$ has negative values on the graph of $f_1$ and has positive values on the graph of $f_2$. By Lemma 3 this can happen if and only if there is a half-branch at infinity of the curve $p = 0$ which is a graph of a function $g$ with $g(x) \sim x^{\theta(S)}$. 

**Remark.** Using toric modifications of the real plane one can present a shorter proof of Corollary 1.

**Example 3** Every polynomial from the list: $p_1 = y + xy^2 + y^4$, $p_2 = y + a y^2 + xy^3$, $p_3 = y + x^2 y^2$, $p_4 = y + ay^2 + y^3 + x^2 y^2$ satisfies assumptions of Corollary 1. The Newton polygons of these polynomials are drawn below.

The polynomials in the above example are taken from [3]. Theorem 1.3 in the cited paper states that these polynomials are canonical forms, up to affine substitution of polynomials of degree 4 without critical points and with at least
one disconnected level set. Theorem 5.5 says that none of these polynomials has a real Jacobian mate. The method of its proof uses an integration based on Green’s formula and requires an analysis of each case separately.

References

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