On optimizing interval data based critical path problem for the analysis of various fuzzy quantities using fuzzification and centroid based defuzzification techniques

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Abstract

In this paper, we propose two new views to optimize interval data based critical path problems (IBCPs). To optimize the IBCP problems, first it is converted to three different fuzzy critical path problems with activities being Trapezoidal, Hexagonal and Octagonal Fuzzy Numbers using proposed trisectional, pentasectional and heptasectional approaches of fuzzification. Secondly, the fuzzy numbers are converted to the crisp quantity in support of appropriate proposed method of defuzzification using centroid. Additionally, we have made an attempt to analyse the performance of two distinct algorithms for critical path identification and various fuzzy numbers and with help of numerical illustration through these views.

Keywords

Fuzzy Critical Path, Interval Data, Fuzzification, Ranking Function, Trapezoidal Fuzzy Number, Hexagonal Fuzzy Number, Octagonal Fuzzy Number.

AMS Subject Classification

03E72

1. Introduction

Critical path method is a network based method as well as project management technique which gives an effective scheduling to plan and execute the complicated projects in today’s competitive environment. This method provides the solutions when decision factors are crisp. Over the past years, Program Evaluation and Review Technique was applied to deal quantitatively with imprecise data. But in reality, if the data are vague, inexact and uncertain, the decision makers could not use the conventional approaches to solve the project problems. Consequently, Zadeh introduced the fuzzy set theory to tackle the uncertainty in 1965. In fuzzy critical path
problem, the decision makers consider the path lengths with fuzzy numbers. Nowadays, ranking of fuzzy numbers is an important aspect of decision making to find the critical path. So, we are solving the fuzzy critical path problems by using ranking techniques. Many authors have been proposing various ranking techniques to solve the critical path problems.

Yao and Lin in 2000 [13] used signed distance ranking of fuzzy numbers to find critical path. In 2008 [11], Shih-Pin Chen and Yi-Ju Hsueh formulated the fuzzy critical path problem as an LP problem and then used Yager’s ranking method to transform the LP as the crisp one. Shahsavari Poura, Kheramnnadb, Fallahc and Zeynali in 2011 [10] proposed a method to find the critical path in which processing times are trapezoidal fuzzy numbers and this method does not use any defuzzification technique to calculate the final processing time. Ravi Shankar, Sireesha, Srinivasa Rao and Vani introduced the metric distance ranking method for trapezoidal fuzzy numbers to find the critical path in 2010 [8]. Ravi Shankar, Pardha Saradhi and Suresh Babu in 2013 [9] proposed a ranking function using centroid of centroids of fuzzy numbers to find the critical path. Rajendran and Anantharayanan proposed fuzzy critical path using centroid based ranking function in which activities are considered as trapezoidal fuzzy numbers in 2015 [6]. In 2016, Naraynamoorthy and Maheswari [5] introduced a modified ranking method to determine the critical path in which each activity time is an octagonal fuzzy numbers. In 2018 [3], Dinesh and Pankaj Kumar Srivastava proposed two new thoughts to solve interval data based transportation problems. First, they converted the interval data into trapezoidal fuzzy number using trisectional method and then secondly, ranking method for conversion to crisp number. Stephen Dinagar and Rameshan [12] proposed a ranking function to find the critical path using octagonal fuzzy numbers in 2018. Rajendran and Anantharayanan [7] used magnitude based ranking function to find the critical path where all activity times are hexadecimal fuzzy numbers in 2018. In 2019 [1], Debrupa Sen, Debolina Roy and Samir Dey developed a parametric interval-valued functional form for trapezoidal fuzzy numbers to find the critical path. Dhanasekar, Hariharan, Manivannan and Umamaheswari [2] in 2019 proposed a wavelet based ranking namely, Haar ranking. In this, fuzzy numbers are converted into Haar timesteps using Haar wavelet technique and applied the critical path method to get the solution. In 2019, Leelavathy and Padmadevi [4] proposed a new critical path algorithm, to find an optimal duration of completion of a fuzzy network problem with activity duration represented as trapezoidal fuzzy number without converting the problem as a classical network problem.

This paper is organized as follows: Section 1 explains the literature and Section 2 briefly revises the basic definitions of fuzzy set. In Section 3, we introduced new trisectional, pentasectional and septasectional approaches of fuzzification for the interval data. In Section 4, new ranking techniques based on centroid of incenters has been proposed for trapezoidal, hexagonal and octagonal fuzzy numbers. Moreover, two algorithms are proposed to determine the critical path. In Section 5, numerical example is given to demonstrate the critical path. Finally results and discussions are presented in Section 6 and ended with conclusion.

## 2. Fuzzy Set Theory

### Definition 2.1
A non-empty subset A in a universal set X is called a fuzzy set if it is defined by the function µ which assigns a value within the range [0, 1] and this range indicates the membership grade of the element in the set A. It is defined by \( A = \{(x, \mu_A(x)) : x \in X, \mu_A(x) \in [0, 1]\} \).

### Definition 2.2
Non-normal trapezoidal fuzzy number \( \tilde{A} = (u_1, u_2, u_3, u_4; \omega) \) is defined with its membership function as

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
\frac{x - u_1}{u_2 - u_1}, & u_1 \leq x \leq u_2 \\
\frac{u_2 - x}{u_3 - u_2}, & u_2 \leq x \leq u_3 \\
\omega, & u_3 \leq x \leq u_4 \\
0, & \text{otherwise}
\end{cases}
\]

where \( \omega \) is any real number satisfying \( 0 < \omega \leq 1 \). It is normal if \( \omega = 1 \).

### Definition 2.3
Non-normal hexagonal fuzzy number \( \tilde{A} = (u_1, u_2, u_3, u_4, u_5, u_6, u_7; \omega) \) is non-normal hexagonal fuzzy number if it is defined with its membership function as

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
\frac{1}{2} \left( \frac{x - u_1}{u_2 - u_1} \right), & u_1 \leq x \leq u_2 \\
\frac{1}{2} \left( \frac{u_2 - x}{u_3 - u_2} \right), & u_2 \leq x \leq u_3 \\
\omega, & u_3 \leq x \leq u_4 \\
1 - \frac{1}{2} \left( \frac{u_4 - x}{u_5 - u_4} \right), & u_4 \leq x \leq u_5 \\
1 - \frac{1}{2} \left( \frac{u_6 - x}{u_7 - u_6} \right), & u_5 \leq x \leq u_6 \\
0, & \text{otherwise}
\end{cases}
\]

where \( \omega \) is any real number satisfying \( 0 < \omega \leq 1 \). It is called normal hexagonal fuzzy number if \( \omega = 1 \).

### Definition 2.4
Non-normal octagonal fuzzy number \( \tilde{A} = (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8; \omega) \) is non-normal octagonal fuzzy number if its membership function is

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
k \left( \frac{x - u_1}{u_2 - u_1} \right), & u_1 \leq x \leq u_2 \\
k, & u_2 \leq x \leq u_3 \\
k + (1 - k) \left( \frac{x - u_3}{u_4 - u_3} \right), & u_3 \leq x \leq u_4 \\
\omega, & u_4 \leq x \leq u_5 \\
k + (1 - k) \left( \frac{u_6 - x}{u_7 - u_6} \right), & u_5 \leq x \leq u_6 \\
k, & u_6 \leq x \leq u_7 \\
k \left( \frac{u_8 - x}{u_9 - u_8} \right), & u_7 \leq x \leq u_8 \\
0, & \text{otherwise}
\end{cases}
\]

where \( \omega \) is any real number satisfying \( 0 < \omega \leq 1 \). It is called normal if \( \omega = 1 \).
Definition 2.5. Let $\tilde{A} = (u_1, u_2, u_3, u_4; \omega_A)$ and $B = (v_1, v_2, v_3, v_4; \omega_B)$ be two trapezoidal fuzzy numbers. Then we define

1. $\tilde{A} + B = (u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4; \min(\omega_A, \omega_B))$
2. $\tilde{A} - B = (u_1 - v_1, u_2 - v_2, u_3 - v_3, u_4 - v_4; \min(\omega_A, \omega_B))$

Definition 2.6. Let $\tilde{A} = (u_1, u_2, u_3, u_4, u_5, u_6; \omega_A)$ and $B = (v_1, v_2, v_3, v_4, v_5, v_6; \omega_B)$ be two hexagonal fuzzy numbers. Then we define

1. $\tilde{A} + B = (u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4, u_5 + v_5, u_6 + v_6; \min(\omega_A, \omega_B))$
2. $\tilde{A} - B = (u_1 - v_1, u_2 - v_2, u_3 - v_3, u_4 - v_4, u_5 - v_5, u_6 - v_6; \min(\omega_A, \omega_B))$

Definition 2.7. Let $\tilde{A} = (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8; \omega_A)$ and $B = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8; \omega_B)$ be two octagonal fuzzy numbers. Then we define

1. $\tilde{A} + B = (u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4, u_5 + v_5, u_6 + v_6, u_7 + v_7, u_8 + v_8; \min(\omega_A, \omega_B))$
2. $\tilde{A} - B = (u_1 - v_1, u_2 - v_2, u_3 - v_3, u_4 - v_4, u_5 - v_5, u_6 - v_6, u_7 - v_7, u_8 - v_8; \min(\omega_A, \omega_B))$
The gravity point of any plane figure is centroid, so that the gravity point than the other center point of the hexagon.

The centroid of centroids triangles divided into three triangles their incenters are $I_1$, $I_2$, and $I_3$.

Centroid of incenters of hexagonal fuzzy number $\triangle$ they can form a triangle $\triangle I_1I_2I_3$.

Hence, the ranking function of $\tilde{\Delta}$ is

$$R(\tilde{\Delta}) = x_0 + (1 - \alpha) \bar{y}_0$$

(4.2)

4.2 Proposed Ranking Method of Hexagonal Fuzzy Numbers

The gravity point of any plane figure is centroid, so that the centroid of a hexagon is considered as the balancing point of the hexagon (Fig. 4.2). We divide the hexagon into two trapezoids AKHF and KLMH. The first trapezoid AKHF is divided into three triangles $\triangle$AEK, $\triangle$KEH and $\triangle$EFH and their incenters are $I_1$, $I_2$ and $I_3$. The centroid of $I_1$, $I_2$ and $I_3$ is $G_1$. The second trapezoid KLMH is divided into three triangles $\triangle KIL$, $\triangle LIM$ and $\triangle IHM$ and their incenters are $I_4$, $I_5$ and $I_6$. The centroid of incenters $I_4$, $I_5$ and $I_6$ is $G_2$. The centroid of centroids $G_1$ and $G_2$ is $G$. This point $G$ would be gravity point than the other center point of the hexagon.

The incenters of the triangles $\triangle$AEK, $\triangle$KEH and $\triangle$EFH are

$$I_1 = (x_{I_1}, y_{I_1}) = \left(\frac{a \alpha_1 + b \beta_1 + c \gamma_1}{\alpha_1 + \beta_1 + \gamma_1}, \frac{\omega \beta_1}{\alpha_1 + \beta_1 + \gamma_1}\right)$$

$$I_2 = (x_{I_2}, y_{I_2}) = \left(\frac{a \alpha_2 + c \beta_2}{\alpha_2 + \beta_2 + \gamma_2}, \frac{\omega \gamma_2}{\alpha_2 + \beta_2 + \gamma_2}\right)$$

$$I_3 = (x_{I_3}, y_{I_3}) = \left(\frac{e \alpha_3 + f \beta_3 + g \gamma_3}{\alpha_3 + \beta_3 + \gamma_3}, \frac{\omega \gamma_3}{\alpha_3 + \beta_3 + \gamma_3}\right)$$

where

$$\alpha_3 = \sqrt{(d - c)^2 + \omega^2}, \beta_3 = \omega, \gamma_3 = (d - c).$$

The points $I_1$, $I_2$ and $I_3$ are not lie on the same line. Therefore, they can form a triangle $\triangle I_1I_2I_3$.

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Figure 4.2. Centroid of Incenters of Hexagonal Fuzzy Number

\[\alpha = \frac{2}{3} + \omega, \beta = \frac{2}{3} + \omega, \gamma = \frac{2}{3} + \omega, \]
$\alpha_6 = \frac{1}{2} \sqrt{4(e - d)^2 + \omega^2}$, \( \beta_6 = \frac{\omega}{2} \), \( \gamma_6 = (e-d) \).

The centroid of incenters \( I_4, I_5 \) and \( I_6 \) is

$$G_2 = (x_2, y_2) = \left( \frac{x_4 + x_5 + x_6}{3}, \frac{y_4 + y_5 + y_6}{3} \right)$$

The centroid of centroids \( G_1 \) and \( G_2 \) of the generalized hexagonal fuzzy number \( \bar{A} = (a, b, c, d, e, f, \omega) \) is defined as

$$G = (\bar{x}, \bar{y}) = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \quad (4.3)$$

Hence, the ranking function of \( \bar{A} \) is

$$R(\bar{A}) = \bar{x} \alpha + (1 - \alpha) \bar{y} \quad (4.4)$$

### 4.3 Proposed Ranking Method of Octagonal Fuzzy Numbers

The gravity point of any plane figure is centroid, so that the centroid of an octagon is considered as the balancing point of the octagon (Fig. 4.3). Octagon is divided into four trapezoids AOND, NDEK, NPQK and EKJI. The first trapezoid AOND is divided into three triangles \( \triangle ACO, \triangle CNO \) and \( \triangle CND \) and their incenters are \( I_1, I_2 \) and \( I_3 \). The centroid of \( I_1, I_2 \) and \( I_3 \) is \( G_1 \). The second trapezoid NDEK is divided into three triangles \( \triangle KME, \triangle MDE \) and \( \triangle MDN \) and their incenters are \( I_4, I_5 \) and \( I_6 \). The centroid of incenters \( I_4, I_5 \) and \( I_6 \) is \( H_1 \). The third trapezoid NPQK is divided into three triangles \( \triangle NPL, \triangle PQM \) and \( \triangle QLM \) and the centroid of incenters \( I_7, I_8 \) and \( I_9 \) is \( H_2 \). The centroid of \( H_1 \) and \( H_2 \) is \( G_2 \). The fourth trapezoid EKJI is divided into three triangles \( \triangle EKH, \triangle KJI \) and \( \triangle HJI \) and their incenters are \( I_{10}, I_{11} \) and \( I_{12} \). The centroid of \( I_{10}, I_{11} \) and \( I_{12} \) is \( G_3 \). Now, the centroid of \( G_1, G_2 \) and \( G_3 \) is \( G \). This point \( G \) would be gravity point than the other center point of the octagon.

The incenters of the triangles \( \triangle ACO, \triangle CNO \) and \( \triangle CND \) are

- \( I_1 = (x_{11}, y_{11}) = \left( \frac{\alpha a_1 + c \beta_1 + b \gamma_1}{\alpha + \beta + \gamma}, \frac{\omega \beta_1}{\alpha + \beta + \gamma} \right) \)
- \( I_2 = (x_{22}, y_{22}) = \left( \frac{c (\alpha_2 + \beta_2) + b \gamma_2}{\alpha_2 + \beta_2 + \gamma}, \frac{\omega \beta_2}{\alpha_2 + \beta_2 + \gamma} \right) \)
- \( I_3 = (x_{33}, y_{33}) = \left( \frac{c (\alpha_3 + \beta_3) + d \gamma_3}{\alpha_3 + \beta_3 + \gamma}, \frac{\omega \beta_3}{\alpha_3 + \beta_3 + \gamma} \right) \)

where

- \( \alpha_1 = \frac{1}{2} \sqrt{4(c - b)^2 + \omega^2}, \beta_1 = \frac{1}{2} \sqrt{4(b - a)^2 + \omega^2}, \gamma_1 = (c-a) \)
- \( \alpha_2 = (c-b), \beta_2 = \frac{1}{2} \sqrt{4(c - b)^2 + \omega^2}, \gamma_2 = \frac{\omega}{2} \)
- \( \alpha_3 = \frac{1}{2} \sqrt{4(d - c)^2 + \omega^2}, \beta_3 = (d-c), \gamma_3 = \frac{\omega}{2} \)

The Centroid of incenters \( I_1, I_2 \) and \( I_3 \) is

$$G_1 = (x_{11}, y_{11}) = \left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

The incenters of the triangles \( \triangle KME, \triangle MDE \) and \( \triangle MDN \) are

- \( I_4 = (x_{44}, y_{44}) = \left( \frac{f \alpha_4 + d \beta_4 + e \gamma_4}{\alpha_4 + \beta_4 + \gamma_4}, \frac{\omega \beta_4}{\alpha_4 + \beta_4 + \gamma_4} \right) \)
- \( I_5 = (x_{55}, y_{55}) = \left( \frac{d (\alpha_5 + \beta_5) + e \gamma_5}{\alpha_5 + \beta_5 + \gamma_5}, \frac{\omega \beta_5}{\alpha_5 + \beta_5 + \gamma_5} \right) \)
- \( I_6 = (x_{66}, y_{66}) = \left( \frac{d (\alpha_6 + \beta_6) + e \gamma_6}{\alpha_6 + \beta_6 + \gamma_6}, \frac{\omega \beta_6}{\alpha_6 + \beta_6 + \gamma_6} \right) \)

where

- \( \alpha_4 = \frac{1}{2} \sqrt{4(e - d)^2 + \omega^2}, \beta_4 = \frac{1}{2} \sqrt{(f-e)^2 + \omega^2}, \gamma_4 = (f-d) \)
- \( \alpha_5 = (e-d), \beta_5 = \frac{1}{2} \sqrt{4(e - d)^2 + \omega^2}, \gamma_5 = \frac{\omega}{2} \)
- \( \alpha_6 = \frac{1}{2} \sqrt{4(d - c)^2 + \omega^2}, \beta_6 = (d-c), \gamma_6 = \frac{\omega}{2} \)

The centroid of incenters \( I_4, I_5 \) and \( I_6 \) is

$$H_1 = (x_{22}, y_{22}) = \left( \frac{x_4 + x_5 + x_6}{3}, \frac{y_4 + y_5 + y_6}{3} \right)$$

The incenters of the triangles \( \triangle NPL, \triangle PQM \) and \( \triangle QLM \) are

- \( I_7 = (x_{77}, y_{77}) = \left( \frac{c \alpha_7 + d \beta_7 + e \gamma_7}{\alpha_7 + \beta_7 + \gamma_7}, \frac{\omega \beta_7}{\alpha_7 + \beta_7 + \gamma_7} \right) \)
- \( I_8 = (x_{88}, y_{88}) = \left( \frac{d \alpha_8 + e \beta_8 + \gamma_8}{\alpha_8 + \beta_8 + \gamma_8}, \frac{\omega \beta_8}{\alpha_8 + \beta_8 + \gamma_8} \right) \)
- \( I_9 = (x_{99}, y_{99}) = \left( \frac{e \alpha_9 + f \gamma_9}{\alpha_9 + \beta_9 + \gamma_9}, \frac{\omega \beta_9}{\alpha_9 + \beta_9 + \gamma_9} \right) \)

where

- \( \alpha_7 = \frac{1}{2} \sqrt{4(e - d)^2 + \omega^2}, \beta_7 = (e-c) \)
- \( \gamma_7 = \frac{1}{2} \sqrt{(d-c)^2 + \omega^2} \)
- \( \alpha_8 = \frac{\omega}{2}, \beta_8 = \frac{1}{2} \sqrt{4(e - d)^2 + \omega^2}, \gamma_8 = (e-d) \)
- \( \alpha_9 = (f-e), \beta_9 = \frac{1}{2} \sqrt{(f-e)^2 + \omega^2}, \gamma_9 = \frac{\omega}{2} \)

The centroid of incenters \( I_7, I_8 \) and \( I_9 \) is

$$H_2 = (x_{33}, y_{33}) = \left( \frac{x_7 + x_8 + x_9}{3}, \frac{y_7 + y_8 + y_9}{3} \right)$$

The centroid of centroids \( H_1 \) and \( H_2 \) is defined as

$$G_2 = (x_{2}, y_{2}) = \left( \frac{x_7 + x_8 + x_9}{2}, \frac{y_7 + y_8 + y_9}{2} \right)$$

The incenters of the triangles \( \triangle EKH, \triangle KJI \) and \( \triangle HJI \) are

- \( I_{10} = (x_{10}, y_{10}) = \left( \frac{e \alpha_{10} + f \beta_{10} + h \gamma_{10}}{\alpha_{10} + \beta_{10} + \gamma_{10}}, \frac{\omega \beta_{10}}{\alpha_{10} + \beta_{10} + \gamma_{10}} \right) \)
The centroid of incenters

4.4 Algorithm for Finding the Critical Path Using

Hence, the ranking function of \( \tilde{T}_p \) is defined as

4.5 Algorithm for Finding the Critical Path Using Float Method

Step 2: Fuzzify the interval data \([L, H]\) using proposed fuzzification approach defined in (3.1) (or (3.3) or (3.5))

Step 3: Determine all the possible paths and their path lengths

Step 4: Apply the ranking function proposed in (4.2) (or (4.4) or (4.6)) for all the paths and select the path with the maximum score as the critical path

\[
I_{11} = (x_{I_{11}}, y_{I_{11}}) = \left( \frac{f \alpha_{I_{11}} + h (\beta_{I_{11}} + \gamma_{I_{11}})}{\alpha_{I_{11}} + \beta_{I_{11}} + \gamma_{I_{11}}} \right), \quad \left( \frac{\omega}{2} (\alpha_{I_{11}} + \beta_{I_{11}}) \right)
\]

\[
I_{12} = (x_{I_{12}}, y_{I_{12}}) = \left( \frac{h (\alpha_{I_{12}} + \beta_{I_{12}} + \gamma_{I_{12}})}{\alpha_{I_{12}} + \beta_{I_{12}} + \gamma_{I_{12}}} \right)
\]

where

\[
\alpha_{I_{10}} = \frac{1}{2} \sqrt{4(h-f)^2+\omega^2}, \quad \beta_{I_{10}} = (h-e),
\]

\[
\gamma_{I_{10}} = \frac{1}{\omega} \sqrt{4(f-e)^2+\omega^2},
\]

\[
\alpha_{I_{11}} = \frac{\omega}{2}, \quad \beta_{I_{11}} = \frac{1}{2} \sqrt{4(h-f)^2+\omega^2}, \quad \gamma_{I_{11}} = (h-f),
\]

\[
\alpha_{I_{12}} = \frac{1}{2} \sqrt{4(h-f)^2+\omega^2}, \quad \beta_{I_{12}} = (e-h), \quad \gamma_{I_{12}} = \frac{\omega}{2}
\]

The centroid of incenters \( I_{10}, I_{11} \) and \( I_{12} \) is

\[
G_{3} = (x_{g_{3}}, y_{g_{3}}) = \left( \frac{x_{I_{10}} + x_{I_{11}} + x_{I_{12}}}{3}, \frac{y_{I_{10}} + y_{I_{11}} + y_{I_{12}}}{3} \right)
\]

Hence, the centroid of \( G_{1}, G_{2} \) and \( G_{3} \) of the generalized octagonal fuzzy number \( \tilde{A} = (a, b, c, d, e, f, g, h; \omega) \) is defined as

\[
G = (x_{g}, y_{g}) = \left( \frac{x_{I_{10}} + x_{I_{11}} + x_{I_{12}}}{3}, \frac{y_{I_{10}} + y_{I_{11}} + y_{I_{12}}}{3} \right)
\]

(4.5)

Hence, the ranking function of \( \tilde{A} \) is

\[
R(\tilde{A}) = x_{g} \alpha + (1 - \alpha) y_{g}
\]

(4.6)

4.4 Algorithm for Finding the Critical Path Using

Direct Ranking Method

Step 1: In the project network, assign the interval data in the form \([L, H]\) for each activity

Step 2: Fuzzify the interval data \([L, H]\) using proposed fuzzification approach defined in (3.1) (or (3.3) or (3.5))

Step 3: Determine all the possible paths and their path lengths

Step 4: Apply the ranking function proposed in (4.2) (or (4.4) or (4.6)) for all the paths and select the path with the maximum score as the critical path

The total float fuzzy time of each path is calculated as

\[
L_{p}^{f} = \min \left\{ L_{q}^{f} + \tilde{t}_{q} \right\} \text{ and } L_{p}^{f} = L_{p}^{f} + \tilde{t}_{q}
\]

Step 5: Calculate total float fuzzy time \( \tilde{T}_{p}^{f} \) for each activity \((p,q)\) using

\[
\tilde{T}_{p}^{f} = L_{p}^{f} \Theta \tilde{E}_{p}^{f} \text{ or } \tilde{T}_{p}^{f} = L_{p}^{f} \Theta \tilde{E}_{p}^{f}
\]

Step 6: Determine all the possible paths in a project network and calculate the total float fuzzy time of each path.

Step 7: Convert the total float fuzzy time into standardized total float fuzzy time for each path

Figure 4.3. Centroid of Incenters of Octagonal Fuzzy Number
5. Numerical Example

The following road transport network represents two cities A and G with connecting five villages B, C, D, E and F by bus. Here A is located at the origin whereas G is at the end. This network contains links to represent the traffic time(in minutes) while travelling between two places which are given in the Fig. 5.1. There are 5 ways to reach the destination city G. If a person wants to go from A to G, which route would take a long time for him to reach the destiny G?

![Figure 5.1. Network Diagram for Critical Path](image)

**Solution**

**Finding Critical Path Using Direct Ranking Method**

**Case 1: Critical path based on trapezoidal fuzzy number**

Firstly, all interval data in the network project are fuzzified using equation (3.1) which is given in Table 5.1.

Here,

(i) A–D–G
(ii) A–D–E–G
(iii) A–C–E–G
(iv) A–B–C–E–G
(v) A–B–F–G

are the 5 possible paths and their path lengths are

\[ \hat{L}_1 = (9, 12.16667, 24.83333, 28) \]
\[ \hat{L}_2 = (15, 19.16667, 35.83333, 40) \]

Apply the ranking function in equation (4.2) for all the possible paths

For \( \alpha = 1 \),
\[ \hat{L}_1 = (14, 17.3, 23.9, 37.1, 43.7, 47) \]
\[ \hat{L}_2 = (21, 26.8, 38.4, 61.6, 73.2, 79) \]
\[ \hat{L}_3 = (19, 25.16667, 49.83333, 56) \]
\[ \hat{L}_4 = (21, 30.66667, 69.33333, 79) \]
\[ \hat{L}_5 = (14, 19.5, 41.5, 47) \]

Apply the ranking function in equation (4.2) for all the possible paths

For \( \alpha = 0 \),
\[ \hat{L}_1 = (15, 17.5, 22.5, 32.5, 37.5, 40) \]
\[ \hat{L}_2 = (19, 22.7, 30.1, 44.9, 52.3, 56) \]
\[ \hat{L}_3 = (14, 17.3, 23.9, 37.1, 43.7, 47) \]
\[ \hat{L}_4 = (14, 19.5, 41.5, 47) \]

**Case 2: Critical path based on hexagonal fuzzy number**

All interval data in the network project are fuzzified by using equation (3.3) which is given in Table 5.2.

Here,

(i) A–D–G
(ii) A–D–E–G
(iii) A–C–E–G
(iv) A–B–C–E–G
(v) A–B–F–G

are 5 possible paths and their path lengths are

\[ \bar{L}_1 = (9, 12.16667, 24.83333, 28) \]
\[ \bar{L}_2 = (15, 19.16667, 35.83333, 40) \]

Apply the ranking function in equation (4.4) for all the possible paths

For \( \alpha = 1 \),
\[ \bar{L}_1 = (20.61109, 30.29116, 41.6111, 56.44443, 34.16667) \]
\[ \bar{L}_2 = (56.44443, 34.16667) \]
\[ \bar{L}_3 = (41.6111, 34.16667) \]
\[ \bar{L}_4 = (21, 30.66667, 69.33333, 79) \]
\[ \bar{L}_5 = (14, 19.5, 41.5, 47) \]

Here maximum ranks are 56.44443, 28.46894 and 0.49345 respectively. So the critical path is A–B–C–E–G.
Table 5.3. Fuzzified Interval Data

| Activity | Interval data | Fuzzified Interval data (Octagonal fuzzy number) |
|----------|--------------|-----------------------------------------------|
| A–B      | [3,12]       | (3.3,64286,4.92857,6.21429,8.78571,10.07143,11.35714,12) |
| A–C      | [2,8]        | (2.2,42857,3.28571,4.14286,5.85714,6.71429,7.57143,8) |
| A–D      | [1,10]       | (1.1,64286,2.92857,4.21429,6.78571,8.07143,9.35714,10) |
| B–C      | [1,19]       | (1.2,28571,4.85714,7.42857,12.57143,15.14286,17.71429,19) |
| D–E      | [4,10]       | (4.4,42857,5.28571,6.14286,7.85714,8.71429,9.57143,10) |
| C–E      | [7,28]       | (7.8,5.11,5.14,5.20,5.23,5.26,5.28) |
| B–F      | [0, 15]      | (0.1,07143,3.21429,5.3714,9.64286,11.78571,13.92857,15) |
| D–G      | [8,18]       | (8.8,71429,1.0,14286,11.57143,14.42857,15.85714,17.28571,18) |
| E–G      | [10,20]      | (10.10,71429,1.2,14286,13.57143,16.42857,17.85714,19.28571,20) |
| F–G      | [11,20]      | (11.11,64286,1.2,92857,14.21429,16.78571,18.07143,19.35714,20) |

Table 5.4. Trapezoidal Total Float Fuzzy Time of Each Activity

| Activity | Trapezoidal Fuzzy Activity Time | Trapezoidal Total Float Fuzzy Time |
|----------|--------------------------------|-----------------------------------|
| A–B      | (3, 4.5, 10.5, 12)             | (0, 0, 0, 0)                      |
| A–C      | (2, 3, 7, 8)                   | (2, 5.5, 19.5, 23)                |
| A–D      | (1, 2.5, 8.5, 10)              | (6, 11.5, 33.5, 39)               |
| B–C      | (1, 4, 16, 19)                 | (0, 0, 0, 0)                      |
| D–E      | (4, 5, 9, 10)                  | (6, 11.5, 33.5, 39)               |
| C–E      | (7, 10.5, 24.5, 28)            | (0, 0, 0, 0)                      |
| B–F      | (0, 2.5, 12.5, 15)             | (7.11,1667, 27.8333, 32)          |
| D–G      | (8, 9,6667, 16.3333, 18)       | (12, 18.5, 44.5, 51)              |
| E–G      | (10, 11,6667, 18.3333, 20)     | (0, 0, 0, 0)                      |
| F–G      | (11, 12.5, 18.5, 20)           | (7.11,1667, 27.8333, 32)          |

Table 5.5. Trapezoidal Standardized Total Float Fuzzy Time for Each Path

| Paths   | Trapezoidal Total Float Fuzzy Time | Trapezoidal Standardized Total Float Fuzzy Time |
|---------|-----------------------------------|-----------------------------------------------|
| A–D–G  | (18, 30, 78, 90)                  | (0.2, 0.3333, 0.8667, 1)                      |
| A–D–E–G| (12, 23, 67, 78)                  | (0.1538, 0.2949, 0.859, 1)                    |
| A–C–E–G| (2, 5.5, 19.5, 23)                | (0.087, 0.2391, 0.8478, 1)                    |
| A–B–C–E–G| (0, 0, 0, 0)                   | (0, 0, 0, 0)                              |
| A–B–F–G| (14, 22.3334, 55.6666, 64)       | (0.2188, 0.349, 0.8698, 1)                   |

Table 5.6. Hexagonal Total Float Fuzzy Time of Each Activity

| Activity | Hexagonal Fuzzy Activity Time | Hexagonal Total Float Fuzzy Time |
|----------|------------------------------|--------------------------------|
| A–B      | (3, 3.9, 5.7, 9.3, 11.1, 12)| (0, 0, 0, 0, 0, 0)              |
| A–C      | (2, 2.6, 3.8, 6.2, 7.4, 8)   | (2, 4.1, 8.3, 16.7, 20.9, 23)   |
| A–D      | (1, 1.9, 3.7, 7.3, 9.1, 10)  | (6, 9.3, 15.9, 29.1, 35.7, 39)  |
| B–C      | (1, 2.8, 6.4, 13.6, 17.2, 19)| (0, 0, 0, 0, 0, 0)               |
| D–E      | (4, 4.6, 5.8, 8.2, 9.4, 10)   | (6, 9.3, 15.9, 29.1, 35.7, 39)   |
| C–E      | (7, 9.1, 13.3, 21.7, 25.9, 28)| (0, 0, 0, 0, 0, 0)               |
| B–F      | (0, 1.5, 4.5, 10.5, 13.5, 15)| (7, 9.5, 14.5, 24.5, 29.5, 32)   |
| D–G      | (8, 9, 11, 15, 17, 18)        | (12, 15.9, 23.7, 39.3, 47.1, 51)  |
| E–G      | (10, 11, 13, 17, 19, 20)      | (0, 0, 0, 0, 0)                   |
| F–G      | (11, 11.9, 13.7,17.3,19.1, 20)| (7, 9.5, 14.5, 24.5, 29.5, 32)   |
Case 3: Critical path based on octagonal fuzzy number

Here maximum ranks are 55.8, 28.14898 and 0.49796 for $\alpha = 0$, $\alpha = 0.5$ and $\alpha = 0$ respectively. So the critical path is A–B–C–E–G.

**Case 3: Critical path based on octagonal fuzzy number**

Fuzzification of all interval data using equation (3.5) is given in Table 5.3.

Here,  
(i) A–D–G  
(ii) A–D–E–G  
(iii) A–C–E–G  
(iv) A–B–C–E–G  
(v) A–B–F–G  
are 5 possible paths and their path lengths are

- $L_1 = (9, 10.35714, 13.07143, 15.78571, 21.21429, 23.92857, 26.64286, 28)$
- $L_2 = (15, 16.78571, 20.35714, 23.92857, 31.07143, 34.64286, 38.21429, 40)$
- $L_3 = (19, 21.64286, 26.92857, 32.21429,

For $\alpha = 0$,  
$R(L_1) = 0.49376, R(L_2) = 0.49525, R(L_3) = 0.4968$,  
$R(L_4) = 0.49796, R(L_5) = 0.49641$.  
Here maximum ranks are 55.8, 28.14898 and 0.49796 for $\alpha = 1, \alpha = 0.5$ and $\alpha = 0$ respectively. So the critical path is A–B–C–E–G.

**Finding Critical Path Using Float Method**

**Case 1: Critical path based on trapezoidal fuzzy number**
Case 2: Critical path based on hexagonal fuzzy number

Case 2: Critical path based on hexagonal fuzzy number

For all the fuzzified interval data given in Table 5.1, the trapezoidal total float fuzzy time using equation (4.7) is given in Table 5.4.

There are five paths and Trapezoidal Standardized Total Float Fuzzy Time for each path is given in Table 5.5.

Apply the ranking function in equation (4.2) for all the possible paths

For $\alpha = 1$, $R(L_1) = 0.6889, R(L_2) = 0.67095, R(L_3) = 0.64491, R(L_4) = 0, R(L_5) = 0.6962$.

For $\alpha = 0.5$, $R(L_1) = 0.52771, R(L_2) = 0.51946, R(L_3) = 0.50743, R(L_4) = 0.16667, R(L_5) = 0.53107$.

For $\alpha = 0$, $R(L_1) = 0.36652, R(L_2) = 0.36796, R(L_3) = 0.36995, R(L_4) = 0.33333, R(L_5) = 0.36594$.

Here minimum ranks are 0, 0.16667 and 0.33333 for $\alpha = 1$, $\alpha = 0.5$ and $\alpha = 0$ respectively. So the critical path is A–B–C–E–G.

Case 2: Critical path based on hexagonal fuzzy number

Compute the Hexagonal total float fuzzy time by using equation (4.7) for all the fuzzified interval data given in Table 5.2. The hexagonal total float fuzzy time is given in Table 5.6.

There are five path and Hexagonal Standardized Total Float Fuzzy Time for each path is given in the following Table 5.7.

Apply the ranking function in equation (4.4) for all the possible paths

For $\alpha = 1$, $R(L_1) = 0.67929, R(L_2) = 0.66156, R(L_3) = 0.6348, R(L_4) = 0, R(L_5) = 0.68751$.

For $\alpha = 0.5$, $R(L_1) = 0.56083, R(L_2) = 0.55242, R(L_3) = 0.53971, R(L_4) = 0.20834, R(L_5) = 0.56472$.

For $\alpha = 0$, $R(L_1) = 0.44236, R(L_2) = 0.44328, R(L_3) = 0.44461, R(L_4) = 0.41667, R(L_5) = 0.44193$.

Here minimum ranks are 0, 0.20834 and 0.41667 for $\alpha = 1$, $\alpha = 0.5$ and $\alpha = 0$ respectively. So the critical path is A–B–C–E–G.

Case 3: Critical path based on octagonal fuzzy number

For all the fuzzified interval data given in Table 5.3, calculate the octagonal total float fuzzy time using equation (4.7).

The octagonal total float fuzzy time is given in Table 5.8. There are five paths in the given network and Octagonal Standardized Total Float Fuzzy Time for each path is given in Table 5.9.

Apply the ranking function in equation (4.6) for all the possible paths

For $\alpha = 1$, $R(L_1) = 0.6127, R(L_2) = 0.59035, R(L_3) = 0.55797, R(L_4) = 0, R(L_5) = 0.62178$.

For $\alpha = 0.5$, $R(L_1) = 0.45299, R(L_2) = 0.44217, R(L_3) = 0.42648, R(L_4) = 0.13889, R(L_5) = 0.45739$.

For $\alpha = 0$, $R(L_1) = 0.29328, R(L_2) = 0.29399, R(L_3) = 0.29498, R(L_4) = 0.27778, R(L_5) = 0.29299$.

Here minimum ranks are 0, 0.13889 and 0.27778 for $\alpha = 1$, $\alpha = 0.5$ and $\alpha = 0$ respectively. So the critical path is A–B–C–E–G.

6. Results and Discussion

From the Tables 6.1 and 6.2, the path having maximum scoring is A–B–C–E–G and the path having minimum scoring is A–B–C–E–G. In both methods, critical path is A–B–C–E–G. It is observed that the proposed ranking methods perform better to determine the critical path in a road transport network.

![Figure 6.1. Comparison Among Three Fuzzy Numbers Using Direct Ranking Method for $\alpha = 0$](328/330)

7. Conclusion

Identifying critical path is an important problem in network analysis. In order to identify the critical path, the activities have to be observed from the field related to the problem
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Table 6.1. Comparison Among Three Fuzzy Numbers

| Path            | Critical Path Using Direct Ranking Method | Hexagonal Fuzzy Number | Octagonal Fuzzy Number |
|-----------------|------------------------------------------|------------------------|------------------------|
|                 | Trapezoidal Fuzzy Number | α = 0 | α = 0.5 | α = 1 | α = 0 | α = 0.5 | α = 1 | α = 0 | α = 0.5 | α = 1 |
| A–D–G           |                           | 0.49736 | 10.44688 | 20.4 | 0.33023 | 9.56591 | 18.80159 |
| A–D–E–G         |                           | 0.49525 | 15.24763 | 30  | 0.31105 | 14.11394 | 27.89683 |
| A–C–E–G         |                           | 0.49680 | 20.84840 | 41.2| 0.33185 | 19.20958 | 38.08730 |
| A–B–C–E–G       |                           | 0.49796 | 28.14898 | 55.8| 0.33242 | 15.67774 | 31.02381 |
| A–B–F–G         |                           | 0.49345 | 28.46894 | 56.44443 | 0.49796 | 28.14898 | 55.8 |
| A–B–C–E–G       |                           | 0.56083 | 0.67929 | 0.29328 | 0.45299 | 0.61270 |
| A–B–F–G         |                           | 0.53971 | 0.63480 | 0.29498 | 0.42648 | 0.55797 |

Figure 6.2. Comparison Among Three Fuzzy Numbers Using Direct Ranking Method for α = 0.5

Figure 6.3. Comparison Among Three Fuzzy Numbers Using Direct Ranking Method for α = 1

Figure 6.4. Comparison Among Three Fuzzy Numbers Using Float Method for α = 0

Table 6.2. Comparison Among Three Fuzzy Numbers

| Path            | Critical Path Using Float Method | Hexagonal Fuzzy Number | Octagonal Fuzzy Number |
|-----------------|---------------------------------|------------------------|------------------------|
|                 | Trapezoidal Fuzzy Number | α = 0 | α = 0.5 | α = 1 | α = 0 | α = 0.5 | α = 1 | α = 0 | α = 0.5 | α = 1 |
| A–D–G           |                               | 0.44236 | 0.56083 | 0.67929 | 0.29328 | 0.45299 | 0.61270 |
| A–D–E–G         |                               | 0.44328 | 0.55242 | 0.66156 | 0.29399 | 0.44217 | 0.59035 |
| A–C–E–G         |                               | 0.44461 | 0.53971 | 0.63480 | 0.29498 | 0.42648 | 0.55797 |
| A–B–C–E–G       |                               | 0.41667 | 0.20834 | 0  | 0.27778 | 0.13889 | 0 |
| A–B–F–G         |                               | 0.44193 | 0.56472 | 0.68751 | 0.29299 | 0.45739 | 0.62178 |

Figure 6.5. Comparison Among Three Fuzzy Numbers Using Float Method for α = 0.5

Figure 6.6. Comparison Among Three Fuzzy Numbers Using Float Method for α = 1

of network. But mostly in real life situations, information are observed in an non exact manner i.e., non exact quantity. So there is a confusion in the minds of the researchers in representing activity duration in a network problems under uncertainty. This confusion leads to the selection of best fuzzy quantity for representing activities in a network. In this paper, we have made an analysis of effectiveness of various fuzzy quantities in obtaining the best optimum solution for identifying critical path in a road transport network with the help of two different algorithms based on proposed fuzzification and defuzzification techniques. The results show that the trapezoidal fuzzy number is effective to represent activities criticality in determining the critical path than hexagonal and octagonal fuzzy numbers.

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