Three charged particles in the continuum. Astrophysical examples.

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Abstract

We suggest a new adiabatic approach for description of three charged particles in the continuum. This approach is based on the Coulomb-Fourier transformation (CFT) of three body Hamiltonian, which allows to develop a scheme, alternative to Born-Oppenheimer one. The approach appears as an expansion of the kernels of corresponding integral transformations in terms of small mass-ratio parameter. To be specific, the results are presented for the system $ppe$ in the continuum. The wave function of a such system is compared with that one which is used for estimation of the rate for triple reaction $p + p + e \rightarrow d + \nu$, which take place as a step of $pp$-cycle in the center of the Sun. The problem of microscopic screening for this particular reaction is discussed.

1 Introduction.

An accurate treatment of three charged particles in the continuum at low relative energies represents till now a very difficult problem which is actual in many areas of physics. Indeed, only some special cases with specific properties of the system has been considered in
the literature. In this respect one should mention papers [1, 2] where
the asymptotic solution for the three-body wave function has been
obtained for configurations when all interparticle distances are much
larger of the characteristic size of the system. An alternative limiting
case considered in [3] corresponds to configurations where one Jacobi
coordinate is much larger than the other one. The near threshold
breakup of hydrogen by proton (or electron) studied in [4] is another
example of approximative solution of three body Schrödinger equation
obtained so far for three charged particles in the continuum.

The purpose of this paper is to develop a new adiabatic expansion
for three-body Hamiltonian for the system consisting of one light and
two heavy particles. The asymptotic behavior of the wave-function
with respect to the coordinate of light particle will be presented in
the framework of this new adiabatic expansion. We use the Coulomb-
Fourier transform formalism proposed in [5] to make a unitary trans-
formation of the Hamiltonian, which leads to a convenient representa-
tion, where, for example, one long-range interactions is eliminated.
The known explicit form of eigenfunctions allows us to construct the
useful integral representation of interaction potentials for transformed
Hamiltonian. The important feature of this representation for poten-
tials is the appearance of universal integral with the integrand contain-
ing the exponential factor, proportional to square root of mass ratio
$\tau \sim \sqrt{m_e/m_p}$ of light (electron) and heavy (proton) particles. The
natural power series in $\tau$ of this integral then generate power series
representation of the transformed Hamiltonian what is the basis for
our new variant of adiabatic representation of the problem. As the
first stage, in this paper we study the solution of the problem taking
into account terms up to $O(\tau^2)$ order.

The paper is organized as following. Section 2 plays the central
role and includes formulation of the problem, description of necessary
portions of Coulomb-Fourier transformation. Here we give as exact
formulas for CF transformed Hamiltonian for $ppe$ system as well as
derive the expansion of matrix elements of the Hamiltonian and wave
function as power series in $\tau^2$. Some technical details concerning eval-
uation of singular integrals from the main tex is placed in Appendix.
Third section contains application of approximation for three-body
wave function obtained in Section 2 to description of some reactions
of $pp$ cycle. Short conclusion summarizes the paper.

In the paper we use units such that $\hbar = c = 1$, for the unit electric
charge the symbol $e$ is used. Three-dimensional vectors are denoted by $x, y, k, p$... and its modules and respective unite vectors by $\hat{x}, \hat{y}, \hat{k}, \hat{p}$... Sometimes we combine pairs of three-dimensional vectors in six-dimensional ones as $X = \{x, y\}, P = \{k, p\}$. The Hilbert space of functions depending on vectors $X$ which in our paper play the role of configuration space vectors will be denoted as $\mathcal{H}$ and the Hilbert space $\hat{\mathcal{H}}$ will be associated with function depending on momentum variables $P$.

2 Adiabatic expansion for three-body Hamiltonian and solution

The $ppe$ system where $p$ is proton and $e$ is electron with masses $m_p$ and $m_e$ ($m_p \gg m_e$) respectively is considered. Number 1 we assign to electron whereas 2 and 3 to protons. The Hamiltonian of the system in the center of mass frame using mass-renormalized Jacobi coordinates can be written in the form

$$H = -\Delta x_1 - \Delta y_1 + v_s(x_1) + n_1/x_1 + n_2/x_2 + n_3/x_3.$$  

Here $V_s(x_1)$ is a short-range potential describing strong $pp$ interaction. Mass-renormalized charge factors $n_i$ are defined by the formulas $n_i = e_j e_k \sqrt{2 \mu_{jk}}$, where $e_1 = -e$, $e_2 = e_3 = e$ are electron and protons charges and $\mu_{ij}$ stands for reduced mass of a pair of particles $ij$, i.e. $\mu_{ij} = m_i m_j / (m_i + m_j)$. Introducing proton and electron masses into this formula we get $\mu_{23} = m_p/2$, $\mu_{31} = m_e m_p / (m_e + m_p)$, $\mu_{12} = \mu_{31}$.

Before proceed further, let us make three clarifying comments. First, throughout the paper we systematically use a convention that indices of any pair of particles $ij$ are considered as a part of triad $ijk$ which itself is a cyclic permutation of 1, 2, 3. Second, let us recall that mass-renormalized Jacobi set $x_i, y_i$ is defined in such a way that the vector $x_i$ up to the factor $\sqrt{2 \mu_{jk}}$ is proportional to the relative position vector of particles $j$ and $k$ and the vector $y_i$ is the position vector of particle $i$ with respect to the center of mass of corresponding two-body subsystem. There are three possible sets $x_i, y_i, i = 1, 2, 3$ and different sets are related to each other by kinematic rotation relations

$$x_i = c_{ij} x_j + s_{ij} y_j$$
$$y_i = -s_{ij} x_i + c_{ij} y_j$$

(1)
with coefficients being defined in terms of particle masses by the formulas

\[ c_{ij} = -\sqrt{\frac{m_im_j}{(m_i + m_k)(m_j + m_k)}} \]

(2)

\[ s_{ij} = \epsilon_{ijk}\sqrt{1 - c_{ij}^2}, \]

where \( \epsilon_{ijk} \) is fully antisymmetric tensor normalized as \( \epsilon_{123} = 1 \). Third, in all mass factors we keep general situation, i.e. not neglecting \( m_e \) with respect to \( m_p \), making sometimes simplifications for illustrative purposes, as for example \( \mu_{31} = m_em_p/(m_e + m_p) = m_e(1 + O(m_e/m_p)) \).

Now let us turn to the solution of the Schrödinger equation

\[ H\Psi = E\Psi \]  

(3)

for three particles in the continuum \((E > 0)\). To this end we will construct a special representation. As the basis of this representation we take the eigenfunctions \( \Psi_{e0}(X, P) \), i.e. \( H_{e0}\Psi_{e0} = P^2\Psi_{e0} \), of the operator

\[ H_{e0} = -\Delta_{x1} - \Delta_{y1} + n_1/x_1 \]

with repulsive \((n_1 > 0)\) Coulomb potential. It is obvious that this eigenfunctions have the form

\[ \Psi_{e0}(X, P) = \psi_{k_1}^c(x_1)\psi_{p_1}^0(y_1). \]

Here \( \psi_{p_1}^0(y_1) = \frac{1}{(2\pi)^{3/2}} e^{i(p_1\cdot y_1)} \) is normalized plane wave and

\[ \psi_{k_1}^c(x_1) = \frac{1}{(2\pi)^{3/2}} e^{i(k_1\cdot x_1)} e^{-\pi\gamma_1/2}\Gamma(1 + i\gamma_1)\Phi(-i\gamma_1, 1, ik_1\xi_1) \]

(4)

is the normalized Coulomb wave function. The standard notations for Sommerfeld parameter \( \gamma_1 = n_1/2k_1 \), parabolic coordinate \( \xi_1 = x_1 - (x_1, k_1) \), Gamma function \( \Gamma(z) \) and Confluent Hypergeometric function \( \Phi(a,c,z) \) have been used.

The representation which we call Coulomb-Fourier (CF) one is generated by the transform [5]

\[ G(X) = F_c\hat{G} = \int dP\, \Psi_{e0}(X, P)\hat{G}(P). \]

(5)

The integral operator \( F_c \) transforms the Hilbert space \( \hat{H} \) of functions depending on momentum variables \( P \) into Hilbert space \( H \) of functions
depending on coordinates. Note that in the limiting case \( n_1 = 0 \) the \( \mathcal{F}_c \) operator is reduced to the standard inverse Fourier transform which connects conventional momentum space and configuration space representations. The Hilbert space adjoint \( \mathcal{F}_c^\dagger \) acts from \( \mathcal{H} \) to \( \hat{\mathcal{H}} \) and for the pair \( \mathcal{F}_c \) and \( \mathcal{F}_c^\dagger \) the unitarity properties hold

\[
\mathcal{F}_c^\dagger \mathcal{F}_c = I_{\hat{\mathcal{H}}}, \quad \mathcal{F}_c \mathcal{F}_c^\dagger = I_{\mathcal{H}},
\]

which are just the operator form of the orthogonality and completeness of eigenfunctions of the Hamiltonian \( H_{c0} \).

In the Schrödinger equation in CF representation described above

\[
\hat{H} \Psi = E \Psi, \quad (6)
\]

the CF transformed Hamiltonian \( \hat{H} = \mathcal{F}_c^\dagger H \mathcal{F}_c \) appears now as integral operator with the kernel (matrix elements)

\[
\hat{H}(P, P') = \langle \psi_0^c \psi_{k_1}^c | H | \psi_0^c \psi_{k_1'}^c \rangle = (7)
\]

\[
(k_1^2 + p_1^2) \delta(k_1 - k_1') \delta(p_1 - p_1') + \hat{v}_s(k_1, k_1') \delta(p_1 - p_1') + W_2(P, P') + W_3(P, P')
\]

operating on CF-transformed wave function \( \Psi(P) \). Here the first term corresponds to kinetic energy operator \( \hat{H}_0 = k_1^2 + p_1^2 \), in the second term \( \hat{v}_s \) stands for the CF-transformed short-range pp interaction potential

\[
\hat{v}_s(k_1, k_1') = \langle \psi_{k_1}^c | v_s | \psi_{k_1'}^c \rangle = \int d\mathbf{x}_1 \psi_{k_1}^c(\mathbf{x}_1) v_s(\mathbf{x}_1) \psi_{k_1'}^c(\mathbf{x}_1)
\]

where * means the complex conjugation, and \( W_j \) are Coulomb potentials \( n_j/x_j \) in CF representation. Let us notice, that the contribution from \( n_1/x_1 \) potential has been eliminated by CF transform. The functions \( W_j(P, P') \), \( j = 2, 3 \) have the following structure

\[
W_j(P, P') = |s_{j1}|^{-3} \hat{v}_j^c(s_{j1}^{-1}(P - P')) \mathcal{L}_j(P, P'),
\]

where

\[
\hat{v}_j^c(q) = \frac{1}{2\pi} \frac{n_j}{|q|^2}
\]

is the familiar Fourier transform of Coulomb potential \( n_j/x_j \) and the functions \( \mathcal{L}_j(P, P') \), \( j = 2, 3 \) are given by the integrals

\[
\mathcal{L}_j(P, P') = \lim_{\lambda \to +0} \int d\mathbf{x}_1 e^{i\mathbf{r}_j(\mathbf{x}_1 - \mathbf{P} - \mathbf{P}')} \lambda |\mathbf{x}_1| \psi_{k_1}^c(\mathbf{x}_1) \psi_{k_1'}^c(\mathbf{x}_1). \quad (8)
\]
The parameters $\tau_j$, $j = 2, 3$, have the kinematical origin and are represented in terms of kinematic rotation matrix elements (1) as

$$\tau_j = c_{j1}/s_{j1}.$$

Noting that $\tau_3 = -\tau_2$ which is the consequence of equality of heavy particles (protons) masses and using definitions (2) and the fact that $m_e \ll m_p$ we get

$$\tau_2 = \sqrt{m_e/2m_p(1 + O(m_e/m_p))},$$

what shows that $\tau_j$ are small. This allows us to expand the exponential factor in the integrand of (8) and obtain in general case the expression

$$L_j(P, P') = \delta(k_1 - k'_1) +$$

$$\frac{i\tau_j}{1!}L^{(1)}(P, P') + \frac{(i\tau_j)^2}{2!}L^{(2)}(P, P') + \frac{(i\tau_j)^3}{3!}L^{(3)}(P, P') + ...$$

Here $L^{(l)}(P, P')$ are integrals

$$L^{(l)}(P, P') = \lim_{\lambda \to +0} \int dx_1 e^{-\lambda|x_1|} \psi^c_{k_1}(x_1) \langle x_1, p_1 - p'_1 \rangle \psi^c_{k'_1}(x_1)$$

which are independent on $j$. This last fact and the property $\tau_3 = -\tau_2$ leads us to the following expansion for the sum of the CF transformed Coulomb potentials $W_2 + W_3$, which contains only even power terms

$$W_2(P, P') + W_3(P, P') = \hat{v}_{eff}(p_1, p'_1) \times$$

$$\left\{ \delta(k_1 - k'_1) - \frac{\tau^2}{2!} L^{(2)}(P, P') + \frac{\tau^4}{4!} L^{(4)}(P, P') - ... \right\},$$

where we have introduced parameter $\tau = |\tau_2|$. The quantity $\hat{v}_{eff}(p_1, p'_1)$ stands for the Coulomb potential corresponding to interaction between electron and effective particle with the charge $2e$ and the mass $2m_p$ and has the form

$$\hat{v}_{eff}(p_1, p'_1) = \frac{1}{2\pi^2} \frac{n_{eff}}{|p - p'|^2}$$

with $n_{eff} = -2e^2 \sqrt{2m_e} \sim -2e^2 \sqrt{2 \frac{m_e 2m_p}{m_e + 2m_p}}.$
Inserting formula (11) into (7) we arrive at the representation of the CF-transformed Hamiltonian $\hat{H}$ which plays a central role in the solution of the problem

$$\hat{H} = \hat{H}_0 + \hat{V}_s + \hat{V}_{\text{eff}} + \tau^2 \hat{W}.$$  \hspace{1cm} (13)

The kernels of operators involved in (13) read

$$\hat{H}_0(P, P') = \left( k_1^2 + p_1^2 \right) \delta(P - P'),$$

$$\hat{V}_s(P, P') = \hat{v}_s(k_1, k_1')\delta(p_1 - p_1'),$$

$$\hat{V}_{\text{eff}}(P, P') = \hat{v}_{\text{eff}}(p_1, p_1')\delta(k_1 - k_1')$$

and

$$\hat{W}(P, P') = \hat{W}^{(0)}(P, P') - \tau^2 \hat{W}^{(2)}(P, P') + \tau^4 \hat{W}^{(4)}(P, P') - ..., \hspace{1cm} (14)$$

$$\hat{W}^{(l)}(P, P') = \hat{v}_{\text{eff}}(p_1, p_1') \frac{1}{l!} L^{(l)}(P, P')$$  \hspace{1cm} (15)

where in the last case we have factored out the small parameter $\tau^2$ to show explicitly that the last term in (13) is as small as $\tau^2$.

The structure of the Hamiltonian (13) suggests now the natural perturbative scheme for solution of Schrödinger equation (6). Let us represent the wave-function $\hat{\Psi}$ as power series in $\tau^2$

$$\hat{\Psi} = \hat{\Psi}_0 + \tau^2 \hat{\Psi}_2 + \tau^4 \hat{\Psi}_4 + ...$$  \hspace{1cm} (16)

then inserting (16) into (6) one immediately gets the recursive set of equations for $\hat{\Psi}_k$, i.e.

$$\left( \hat{H}_0 + \hat{V}_s + \hat{V}_{\text{eff}} \right) \hat{\Psi}_0 = E\hat{\Psi}_0,$$  \hspace{1cm} (17)

$$\left( \hat{H}_0 + \hat{V}_s + \hat{V}_{\text{eff}} \right) \hat{\Psi}_{2l} = E\hat{\Psi}_{2l} - \sum_{s=0}^{l-1} (-1)^{l-s} \hat{W}^{(2l-2s)} \hat{\Psi}_{2s}, \hspace{0.5cm} l = 1, 2, 3, ...$$  \hspace{1cm} (18)

The scheme (17, 18) has a remarkable property, namely, the solution of the three-body problem in framework of this scheme can be obtained in terms of solutions of two-body problems. Indeed, equation (17) allows the separation of variables, so that its solution is reduced to the solution of respective two-body equations, at the same time the solution
of inhomogeneous equations (18) can be obtained in terms of Green’s function of the operator $\hat{H}_0 + \hat{V}_s + \hat{V}_{eff}^c$ which can be constructed from two-body Green’s functions due to separability of variables.

For the specific case of three particles in the continuum we are considering the solution scheme outlined above yields the following results. The solution of the first equation (17) reads

$$\hat{\Psi}_0(P, P^{in}) = \hat{\psi}^+_k(k_1) \hat{\psi}_{p_1}^{ce}(p_1),$$

where initial state momentums $k_1^{in}$ and $p_1^{in}$ are related to the energy $E$ by the formula $P_{in}^2 = k_1^{in2} + p_1^{in2} = E$. The function $\hat{\psi}_{p_1}^{ce}(p_1)$ is the momentum space Coulomb wave function corresponding to the potential $\hat{V}_{eff}^c(p_1, p_1')$. The inverse Fourier transform of $\hat{\psi}_{p_1}^{ce}(p_1)$ we will denote by $\hat{\psi}_{p_1}^{ce}(y_1)$ which explicit form can be obtained from (4) when $n = n_{eff}$ and replacing $k_1^{in}, x_1$ by $p_1^{in}, y_1$, respectively. The term $\hat{\psi}^+_k(k_1)$ is the scattering solution to the two-body Schrödinger equation with the potential $\hat{V}_s(k_1, k_1')$ and is conventionally represented as the solution of the Lippmann-Schwinger integral equation

$$\hat{\psi}^+_k(k_1) = \delta(k - k_1^{in}) - \frac{1}{k^2 - k_1^{in2} - i0} \int dq \hat{\psi}(k, q) \hat{\psi}^+_k(q).$$

The solutions of inhomogeneous equations (18) are given by recursive formulas starting from $\hat{\Psi}_0(P, P^{in})$

$$\hat{\Psi}_{2l} = -\hat{G}_{s,eff}(E + i0) \sum_{s=0}^{l-1} (-1)^{l-s} \hat{\Psi}_{2l-2s}.$$  

Here the kernel of the operator $\hat{G}_{s,eff}(z) = (\hat{H}_0 + \hat{V}_s + \hat{V}_{eff}^c - z)^{-1}$ is represented through two-body Green’s functions $\hat{g}_s$ for potential $\hat{V}_s$ and $\hat{g}_{eff}^c$ for potential $\hat{V}^c_{eff}$ by the counter integral

$$\hat{G}_{s,eff}(P, P', z) = \frac{1}{2i\pi} \oint_C d\zeta \hat{g}_s(k_1, k_1', \zeta) \hat{g}_{eff}^c(p_1, p_1', z - \zeta)$$

with counter $C$ encircling the cut of $\hat{g}_s$ in anticlockwise direction.

So that, we have constructed the formal solution to the CF transformed Schrödinger equation (6) for the system $ppe$ in the continuum.
The configuration space wave function which obeys (3) can be obtained from $\hat{\Psi}$ by CF transform (5)

$$\Psi(X, P_{in}) = \int dP \Psi_{c0}(X, P) \hat{\Psi}(P, P_{in}).$$  \hfill (22)

Now one can see, that the structure of our solution (16) and respective series in $\tau^2$ for $\Psi(X, P_{in})$ generated from (22) by (16) and the structure of the representation for the Hamiltonian (13) outline the framework of our approach as an alternative to Born-Oppenheimer one. It is worth mentioning here, that the formalism given above is rather general and with minor evident modifications is applicable for the three charged particle systems for the case of different masses when the mass of one particle is significantly smaller than the masses of others.

Before proceed further, let us give some explicit formulas for approximation to the wave function $\Psi(X, P_{in})$ generated by our complete formal solution which we will use in the next section discussing some astrophysical reactions. Introducing (19) and (21) into (22) we get

$$\Psi(X, P_{in}) = \psi_{\vec{k}_{1n}}^+(x_1) \psi_{\vec{p}_{1n}}^{ce}(y_1) + \tau^2 \Psi_2(X, P_{in}) + O(\tau^4)$$ \hfill (23)

where

$$\psi_{\vec{k}_{1n}}^+(x_1) = \int d\vec{k}_1 \psi_{\vec{k}_1}^c(x_1) \hat{\psi}_{\vec{k}_{1n}}^+(\vec{k}_1)$$

and $\Psi_2(X, P_{in})$ is given by transform (22) of $\hat{\Psi}_2$ calculated through $\hat{\Psi}_0$ from (19) by formula

$$\hat{\Psi}_2 = -\hat{G}_{e,eff}(E + i0)\hat{W}^{(2)}\hat{\Psi}_0.$$ \hfill (24)

The complete investigation of properties of the solutions to (17,18) is out of the scope of this paper and will be made elsewhere. Below in this section, we consider two points which plays the key role for the formalism, namely the singular structure of operator $\hat{W}$ and the structure of correction term $\hat{\Psi}_2$ (and consequently $\Psi_2$) which possesses the most important properties specific for all correction terms $\hat{\Psi}_{2l}$.

The kernels of the operators $\hat{W}^{(l)}$ in (15) are given in terms of singular integrals (10). These integrals obviously can be computed in terms of distributions (generalized functions) as it could be seen from (10) in the trivial case $l = 0$ which yields $L^{(0)}(P, P') = \delta(k_1 - k_1')$. 

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The general case of arbitrary \( l \) in (10) is considered in the Appendix where it is shown that the most singular part of the integral \( L^{(l)}(\mathbf{P}, \mathbf{P}') \) has the form

\[
L^{(l)}(\mathbf{P}, \mathbf{P}') = \delta(\mathbf{k}, \mathbf{k}') \delta^{(l)}(k - k') (\mathbf{k}, \mathbf{p} - \mathbf{p}') L^{(l)}(k, k').
\]  

(25)

Here and in what follows we omit subscript 1 from definition of momentums and coordinates using, for instance, \( k \) instead of \( k_1 \) and so on. Delta-function \( \delta(\mathbf{k}, \mathbf{k}') \) on unit sphere \( S^2 = \{ \mathbf{k} : k = 1 \} \) and \( l \)-th derivative of delta-function \( \delta^{(l)}(k - k') \) are defined by

\[
\int_{S^2} d\hat{k} \delta(\hat{k}, \hat{k}')g(\hat{k}') = f(\hat{k}),
\]

\[
\int_{-\infty}^{\infty} dk' \delta^{(l)}(k - k')g(k) = (-1)^l g^{(l)}(k).
\]

The function \( L^{(l)}(k, k') \) being a smooth function of \( k \) and \( k' \) for \( l \)-even has the form

\[
L^{(l)}(k, k') = \frac{1}{kk'} e^{-i(\sigma(k) - \sigma(k'))} \frac{\sinh(\frac{\pi(\gamma - \gamma')}{2})}{\pi(\gamma - \gamma')} \times \\
\Re \left[ e^{i(\sigma_0(k) - \sigma_0(k')) + \frac{i\pi}{2} (2k - i\gamma)(2k' - i\gamma') \Gamma(1 - i(\gamma - \gamma'))} \right].
\]

Above formulas for \( L^{(l)}(\mathbf{P}, \mathbf{P}') \) can be used to compute the action of \( \hat{W}^{(2)} \) operator on \( \hat{\Psi}_0 \)

\[
\hat{W}^{(2)} \hat{\Psi}_0(\mathbf{P}, \mathbf{P}^{in}) = \frac{1}{2} I_1(\mathbf{k}, \mathbf{k}^{in}) I_2(\mathbf{p}, \mathbf{p}^{in})
\]

(26)

where

\[
I_1(\mathbf{k}, \mathbf{k}^{in}) = \int d\mathbf{k}' \delta(\hat{k}, \hat{k}') \delta^{(2)}(k - k') L^{(2)}(k, k') \hat{\psi}^+_k(\mathbf{k}')
\]

and

\[
I_2(\mathbf{p}, \mathbf{p}^{in}) = \int d\mathbf{p}' \hat{\psi}^{ce}_{\mathbf{p}'}(\mathbf{p}, \mathbf{p}') (\mathbf{k}, \mathbf{p} - \mathbf{p}')^2 \hat{\psi}^{ce}_{\mathbf{p}^{in}}(\mathbf{p}').
\]

Both integrals \( I_j \) are singular distributions. To make its structure more transparent let us find its most singular parts. For the first integral it means leaving in the integrand the most singular part of \( \hat{\psi}^+_k(\mathbf{k}') \) from (20), i.e. \( \delta(\mathbf{k} - \mathbf{k}^{in}) \) which yields

\[
I_1(\mathbf{k}, \mathbf{k}^{in}) = \delta(\hat{k}, \hat{k}^{in}) \delta^{(2)}(k - k^{in}) L^{(2)}(k, k^{in}).
\]

(27)
For the second integral, it is useful to make a linear change of variables to get
\[ I_2(p, p^{in}) = \frac{n_{eff}}{2\pi^2} \int dq \hat{\psi}_{p^{in}}(q + p)^2 \hat{\psi}_{p^{in}}(q) \]
and then using Fourier transform for \( \hat{\psi}_{p^{in}}(p) \) rewrite this integral as
\[ I_2(p, p^{in}) = \frac{n_{eff}}{2\pi^2} \int dy \, D(y, \hat{k}) e^{-i(p\cdot y)} \hat{\psi}_{p^{in}}(y). \tag{28} \]
Here the function \( D(y, \hat{k}) \) is given by
\[ D(y, \hat{k}) = \lim_{\lambda \to +0} \frac{1}{(2\pi)^{3/2}} \int dq \, e^{-i(q\cdot y) - \lambda q \langle \hat{k}, \hat{q} \rangle^2}. \]
In the Appendix it is shown that the main singular part of \( D(y, \hat{k}) \) is proportional to delta-function, i.e.
\[ D(y, \hat{k}) = \frac{1}{3(2\pi)^{3/2}} \delta(y). \]
The latter gives for the integral \( I_2(p, p^{in}) \)
\[ I_2(p, p^{in}) = N_{eff} \hat{\psi}_{p^{in}}(0), \tag{29} \]
\[ N_{eff} = \frac{n_{eff}}{2\pi^2} \frac{1}{3(2\pi)^{3/2}}. \]
Now, inserting formulas given above in (24) we can represent the correction term \( \hat{\Psi}_2 \) as the integral
\[ \hat{\Psi}_2(P, P^{in}) = -\frac{1}{4\pi i} \int dP' \int_0 d\zeta \, \hat{g}_s(k, k', \zeta) \hat{g}_{eff}(p, p', E - \zeta + i0) \times \]
\[ I_1(k', k^{in}) I_2(p', p^{in}). \tag{30} \]
This general formula can be simplified if we take instead of full Green’s function \( \hat{g}_s(k, k', \zeta) \) its main singular part which is Green’s function of the two-body kinetic energy operator \( \delta(k - k')(k^2 - \zeta)^{-1} \). This case in fact has the particular physical sense, since taking into account that we left only delta-function for \( \hat{\psi}_s^+ \) the resulting approximation is exactly equivalent to the neglect of the short-range potential \( V_s \) from the very beginning. Formula (30) is transformed in this case to
\[ \hat{\Psi}_2(P, P^{in}) = \tag{31} \]
\[-\frac{1}{2} N_{\text{eff}} I_1(k, k^{\text{in}}) \int dp' \hat{g}^{\text{ce}}_{\text{eff}}(p, p', E - k^2 + i0) \psi_{\text{p}^{\text{in}}}^{\text{ce}}(0). \]

The configuration space representation for \( \Psi_2 \) which can be obtained from formula (22) is reduced now to the integral

\[ \Psi_2(X, P^{\text{in}}) = \]

\[-\frac{1}{2} N_{\text{eff}} \psi^{\text{ce}}_{\text{p}^{\text{in}}}(0) \int dk \psi^{\text{ce}}_k(x) I_1(k, k^{\text{in}}) g^{\text{ce}}_{\text{eff}}(y, 0, E - k^2 + i0). \]

Final form for this integral follows immediately from delta-functional factors of \( I_1 \) and reads

\[ \Psi_2(X, P^{\text{in}}) = -\frac{1}{2} N_{\text{eff}} \psi^{\text{ce}}_{\text{p}^{\text{in}}}(0) \times \]

\[ \frac{\partial^2}{\partial t^2} \left[ t^2 \mathcal{L}^{(2)}(t, k^{\text{in}}) \psi^{\text{ce}}_{k^{\text{in}}}(x) g^{\text{ce}}_{\text{eff}}(y, 0, E - t^2 + i0) \right]_{t = k^{\text{in}}}. \]

This formula describes the correction term \( \Psi_2 \) for the \( ppe \) system when strong \( pp \) interaction is neglected and at the same time is approximation to the term \( \Psi_2 \) in the general case.

The formula (32) is useful for constructing the coordinate asymptotics of \( \Psi_2(X, P^{\text{in}}) \) as \( y \to \infty \). One needs to use well known coordinate asymptotics of Coulomb Green’s function as \( y \to \infty \) and \( y' \) is bound

\[ g^{\text{ce}}_{\text{eff}}(y, y', s^2 + i0) \sim \frac{\exp \left\{ isy - i \frac{N_{\text{eff}}}{2s} \log 2sy \right\}}{4\pi y} \psi^{\text{ce}}_{\text{p}^{\text{in}}}(y') \]

which can be found for instance in [7]. This asymptotic formula gives the following asymptotics of \( \Psi_2(X, P^{\text{in}}) \)
\[
\Psi_2(X, P^{in}) \sim \left( \frac{\alpha}{4\pi y} \right) \exp \left\{ ip^{in} y - \frac{i\text{eff}}{2p^{in}} \log 2p^{in} y \right\} \left( 1 + O \left( \frac{y}{p^{in}} \right) \right)
\]

where the amplitude \( A \) has the explicit form

\[
A(x, k^{in}, p^{in}, \hat{y}) = -\frac{1}{2} N_{eff} \psi_{\pi}^{(0)} \times \\
\frac{\partial^2}{\partial t^2} \left[ t^2 \mathcal{L}^{(2)}(t, k^{in}) \psi_{\pi}^{(0)}(x) \psi^{(0)}_{\pi} - \sqrt{E - t^2} \psi^{(0)}(0) \right] \bigg|_{t=k^{in}}
\]

Here by \( O \left( \frac{y}{p^{in}} \right) \) we have denoted terms corresponding to derivatives of exponential factor in (32). The order of this terms shows the range of validity of the asymptotics (33), i.e. \( \frac{y}{p^{in}} \) has to be small, what in terms of masses must be equivalent to the fact that \( y\tau^2 \) has to be small. Let us emphasize that all treatment above devoted to consideration of three body charged particles at kinetic energies, comparable with corresponding potential energy, which means that Coulomb interactions are essential. In this situation for the systems, consisting from heavy and light particles, one can develop the adiabatic description, which actually means smallness of momentums of heavy and light particles ratio \( \frac{k^{in}}{p^{in}} \). This smallness obviously appear due to the small mass ratio parameter \( \tau \), introduced above.

Therefore in each fixed order of expansion in small parameter \( \tau \) one should neglect also by all terms, proportional to the ratio \( \frac{k^{in}}{p^{in}} \).

### 3 Astrophysical Examples

Now let us discuss ways of describing some reactions of the pp-cycle on the Sun, which can be done on the ground of 3-body wave function given by (23). In other words, we will consider situations when in the initial state the system consists of three charged particles in the continuum and the mass of one of them is much smaller than other masses.

The first example gives the reaction

\[
p + p + e \rightarrow d + \nu
\]
considered in [6]. As it follows from the form of the main term in the right hand side of (23), with very good accuracy we have separation of the Jacobi coordinates in the wave function of the initial state for the reaction (34). This means that the rate of three-body process (34) can be expressed in terms of a binary process

\[ p + p \rightarrow d + e^+ + \nu \]  

(35)

This is just the main result of paper [6]. Now it becomes clear that the physical background of the above result from the point of view of the few-body theory consists in two points:

a) the system has two heavy and one light particle such that the parameter \( \frac{m}{M} \ll 1 \) and therefore one can neglect the second term in the right-hand side of (23).

b) heavy particles are slow enough to neglect higher partial waves in their relative motion. One should emphasize that free ”effective charge of the initial nuclear system” \( Z \), introduced in [6], can now be fixed to value \( Z = 2 \) which is supported by the structure of (23).

Let us consider another example of 3-body initial state

\[ p + ^7 Be + e, \]  

(36)

which can produce \(^8 Be\) or \(^7 Li\) nuclei via the following reactions

\[ p + ^7 Be + e \quad \rightarrow \quad ^8 Be + e (\gamma) \]

\[ \quad \searrow \quad ^7 Li + \nu + p \]

First, from the previous discussion one can see that due to different masses of heavy particles in this case the contribution from the linear term \( \tau \) is nonzero in contrast to the \( p + p + e \) system, and this contribution should be estimated.

If the electron in state (36) is in the continuum, then again due to the separation of Jacobi coordinate in the first term of (23) the rate of the proton capture from the initial three-body state (36) can be expressed via the rate of the binary reaction \( p + ^7 Be \rightarrow ^8 B + \gamma \).

However, the rate of the electron capture from the initial three-body state (36), as it follows from (23), (modified for the state (36)), will be defined by the Coulomb wave function of the electron moving in the Coulomb field with the charge \( Z = 5 \) instead of \( Z = 4 \) for the
capture from the two-body state $^7\text{Be}+e$. This means that the production rate of $^7\text{Li}$ from the three-body state (36) cannot be expressed via the binary $(e+^7\text{Be} \rightarrow ^7\text{Li}+\nu)$ reaction rate. Roughly speaking, the ratio of these rates will be proportional to the ratio of the corresponding electron Coulomb functions at energy in the center of the Sun $E_s$. In other words,

$$\frac{w_3}{w_2} \sim \frac{|\psi_c(0, E_s, Z = 5)|^2}{|\psi_c(0, E_s, Z = 4)|} \sim \frac{5}{4}.$$

Now let us discuss the problem of screening of the Coulomb interaction between two protons for the system $p+p+e$. We restrict ourselves by lowest order in the ratio $m_e/m_p$ for the three-body wave function, i.e. consider only first term in the (23). It is evident, that the screening effect in this approximation appears due to the electronic wave function $\psi_c(p, y)$, where $y = \sqrt{\frac{4m_e m_p}{m_e + 2m_p}} \left( \frac{R}{2} + r \right)$, $R$ being the distance between protons and $r$ being the distance between electron and one of the protons. Taking the asymptotics of this function in the region where $R \gg r$, one can see that the Coulomb phases of $pp$ wave function and electronic wave function can cancel each other for the specific configurations of initial momentums $k$ and $p$ of three-body system under consideration. Hence the resulting motion of two protons in this configuration would be described by plane wave, which means the total screening effect.

## 4 Conclusion

In conclusion we emphasize, that the CF-transformed three-body Hamiltonian (13) for the system of two heavy and one light particles can be used for realization of adiabatic expansion which is alternative to the Born-Oppenheimer one. This approach allows to treat screening effects on the microscopic level. In astrophysical examples it was shown, that in the lowest order of small parameter $\tau$, it is possible for some reactions only express rates of 3-body processes in terms of binary ones.
5 Appendix

In this section we give a brief description of evaluation stages of integrals
$L^{(l)}(P, P')$ and $D(y, \hat{k})$ which are defined as

$$L^{(l)}(P, P', \lambda) = \lim_{\lambda \to +0} L^{(l)}(P, P', \lambda),$$

and

$$D(y, \hat{k}) = \lim_{\lambda \to +0} D(y, \hat{k}, \lambda),$$

and

$$D(y, \hat{k}, \lambda) = \frac{1}{(2\pi)^{3/2}} \int dq e^{-\lambda q - i(q,y) \langle \hat{k}, \hat{q} \rangle^2}. \quad (38)$$

The technical tool we use for calculations of prelimiting integrals (37, 38) is so
called weak asymptotics. For Coulomb wave function $\psi^c_k(x)$ as $y \to \infty$ [7] it reads

$$\psi^c_k(x) \sim \frac{i}{(2\pi)^{1/2} k^2} \left\{ \delta(\hat{x}, -\hat{k}) \frac{\exp (-i k x - i w_0(x, k))}{x} - s_c(k, \hat{x}, \hat{k}) \frac{\exp (i k x + i w_0(x, k))}{x} \right\} \quad (39)$$

where $w_0(x, k) = -\gamma \log 2kx$, $\gamma = n/2k$ and $s_c(k, \hat{x}, \hat{k})$ is the Coulomb
s-matrix

$$s_c(k, \hat{x}, \hat{k}) = \frac{2^{1+2i\gamma} \gamma e^{2i\sigma_0}}{2i\pi |\hat{x} - \hat{k}|^{2+i2\gamma}},$$

$\sigma_0(k) = \arg \Gamma(1 + i\gamma)$. If $n = 0$ this formula is reduced to the weak
asymptotics of plane wave [9]

$$\frac{1}{(2\pi)^{3/2}} e^{i(k,x)} \sim \frac{i}{(2\pi)^{1/2} k^2} \left\{ \delta(\hat{x}, -\hat{k}) \frac{\exp (-i k x)}{x} - \delta(\hat{x}, \hat{k}) \frac{\exp (i k x)}{x} \right\} \quad (40)$$

The last key formula we need to compute above integrals is

$$\int d\hat{k} s_c(k, \hat{x}, \hat{k}) g(\hat{k}) = e^{2i\sigma_0} g(\hat{x}) + \frac{2^{i\gamma} \gamma e^{2i\sigma_0}}{2i\pi} \int_0^2 dt \frac{G(t) - G(0)}{t^{1+i\gamma}},$$

with

$$G(0) = 2\pi g(\hat{x}), \quad G(t) = \int_0^{2\pi} d\phi g(t, \phi).$$
In these equations, $\theta$ and $\phi$ are spherical angles and $t = 1 - \cos \theta$. In the sense of distribution it means that

$$s_c(k, \hat{x}, \hat{k}) = e^{2i\sigma_0} \delta(x, k) + ... \quad (41)$$

The integral from (37) for $\lambda \neq 0$ can be calculated by following procedure, inserting (39) into the integral, leaving the main singular part of $s_c$ from (41), and using the formula [10]

$$\int_0^\infty dx \ x^{ia} e^{\pm i tx - \lambda x} = e^{\pm i \pi / 2} \Gamma(1 + ia) \Gamma(1 + i\lambda)$$

one arrives at

$$L^{(l)}(P, P', \lambda) = \frac{1}{2\pi kk'} \delta(\hat{k}, \hat{k}') e^{-i\sigma_0(k) + \sigma_0(k')} \times$$

$$\left[ (-\hat{k}, p - p')^l B_l(k, k' \lambda) + (\hat{k}, p - p')^l B^*_l(k, k' \lambda) \right],$$

where

$$B_l(k, k' \lambda) = e^{i\sigma_0(k) - i\sigma_0(k')} (2k)^{-i\gamma} (2k')^{-i\gamma} e^{ix(l+1) / (k - k' + i\lambda)} \Gamma(1 + l - i(\gamma - \gamma')).$$

It remains to evaluate the $\lambda \to 0$ limit, which can be done by means of the following representation which one can verify by straightforward calculations

$$\lim_{\lambda \to 0} \int_{-\infty}^{\infty} \frac{dt \ g(t)}{(t - i\lambda)^{n+1+\mu}} = \frac{\Gamma(1 + i\mu)}{\Gamma(n + i\mu)} \left\{ i\mu^{-1} g^{(n-1)}(0) \left[ 1 - e^{\pm\pi\mu} \right] + \int_{-1}^{1} \frac{dt \ g^{(n-1)}(t)}{(t + i\mu)^{n+1+\mu}} \right\}.$$ 

Using symbolic notations and leaving explicitly only main singular part

$$\frac{1}{(t + i\mu)^{n+1+\mu}} = \frac{\Gamma(1 + i\mu)}{\Gamma(n + i\mu)} i\mu^{-1} (-1)^{n-1} \delta^{(n-1)}(t) \left[ 1 - e^{\pm\pi\mu} \right] + ...$$

we arrive finally at the formula (25) given in the main text.

By very similar way the integral (38) can be evaluated, i.e. usage of (40) which is at the same time the asymptotics of plane wave as $q \to \infty$
helps to calculate the integral over $\mathbf{q}$ and then the radial integral over $q$ gives

$$D(y, \mathbf{k}, \lambda) = \frac{1}{(2\pi)^{3/2}} \frac{8\pi \lambda \langle \hat{y}, \hat{k} \rangle^2}{(y^2 + \lambda^2)^2}.$$ 

The $\lambda \to +0$ limit is now straightforward

$$\lim_{\lambda \to +0} D(y, \mathbf{k}, \lambda) = \frac{1}{3(2\pi)^{3/2}} \delta(y).$$

References

[1] R.K. Peterkop, Zh. Eksp. Teor. Fiz, 43, 616 (1962) (in russian) [Sov. Phys. JETP 14, 1377 (1962)].

[2] S.P. Merkuriev, Theor. Math. Phys., 32, 680 (1977), M. Brauner, J.S. Briggs and H.J. Klar, J. Phys. B, 22, 2265 (1989).

[3] E.O. Alt, A.M. Mukhamedzhanov, JETP Lett., 56, 435 (1992), Phys. Rev. A, 47, 2004 (1993); Y.E. Kim, A.L. Zubarev, Phys. Rev. A, 56, 521 (1997).

[4] J.H. Macek, S.Yu. Ovchinnikov, Phys. Rev. A, 54, 1 (1996); M.Yu.Kuchiev and V.N.Ostrovsky, Phys. Rev. A, 58, 321 (1998).

[5] E.O. Alt, S.B. Levin and S.L. Yakovlev, Coulomb-Fourier transformation: a novel approach to three-body scattering with charged particles. (submitted to Phys. Rev. C); E.O. Alt, S.B. Levin and S.L. Yakovlev, Few-Body Systems Suppl. 14, 221 (2003); E.O.Alt, Few-Body Systems Suppl. 14, 179 (2003); E.O. Alt, S.B. Levin and S.L. Yakovlev, Book of Abstracts, Few Body 17, Durham 2003, p. 283, 287; Belyaev V.B., Levin S.L., Yakovlev S.L., ibid, p. 346.

[6] J.N. Bahcall, R.M. May, Astrophysical Journal, 155, 501, (1969).

[7] L.D. Faddeev, S.P. Merkuriev, Quantum Scattering Theory for Several Particle Systems, (Kluwer, Dordrecht, 1993).

[8] S.B. Levin, E.O. Alt and S.L. Yakovlev, Integral Representation for the Two-Body Coulomb Wave Function, in: Selected topics in theoretical physics and astrophysics: collection of papers dedicated to Vladimir B. Belyaev on the occasion of his birthday. - Dubna: JINR, 2003, 167 p.; ISBN 5-9530-0022-7.
[9] R.K. Peterkop, *Theory of Ionization of Atoms by Electron Impact* (Colorado Associated University Press, Boulder, 1977).

[10] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, San Diego, 1980).