A Sharp Analysis of Covariate Adjusted Precision Matrix Estimation via Alternating Gradient Descent with Hard Thresholding

Xiao Lv*, Wei Cui* and Yulong Liu†

*School of Information and Electronics, Beijing Institute of Technology
Email: {xiaolv, cuiwei}@bit.edu.cn
†School of Physics, Beijing Institute of Technology
Email: yulongliu@bit.edu.cn

Abstract

In this paper, we present a sharp analysis for an alternating gradient descent algorithm which is used to solve the covariate adjusted precision matrix estimation problem in the high dimensional setting. Without the resampling assumption, we demonstrate that this algorithm not only enjoys a linear rate of convergence, but also attains the optimal statistical rate (i.e., minimax rate). Moreover, our analysis also characterizes the time-data tradeoffs in the covariate adjusted precision matrix estimation problem. Numerical experiments are provided to verify our theoretical results.

I. INTRODUCTION

Multivariate regression problems and their variants have received a lot of attention for their diverse applications. In this paper, we consider one of their variants, the covariate adjusted precision matrix estimation problem [1], [2]. The traditional multivariate regression model is

\[ y_i = \Gamma_i^T x_i + \epsilon_i, \]  

(1)

where \( \Gamma_i \in \mathbb{R}^{d \times m} \). \( \{\epsilon_i\}_{i=1}^n \) are independent vectors following \( N(0, \Sigma_\epsilon) \). We could write the model in the matrix form

\[ Y = X \Gamma_\star + E, \]  

(2)

where \( X = [x_1, \ldots, x_n]^T \), \( Y = [y_1, \ldots, y_n]^T \) and \( E = [\epsilon_1, \ldots, \epsilon_n]^T \).

The objective of the covariate adjusted precision matrix estimation problem is to estimate the regression parameter \( \Gamma_\star \) and the precision matrix \( \Omega_\star = \Sigma_\epsilon^{-1} \) simultaneously. This model has been explored in various fields. In Graph Theory, \( \Gamma_\star \) and \( \Omega_\star \) represent the directed graph and the undirected graph respectively. The edges of directed graphs indicate casual relationships and those of undirected graphs reveal conditional dependency relationships [3], [4]. Both of them provide insights for exploring the interaction among data, especially in the high dimensional setting.

Estimating precision matrices is the objective of graphical models. Gaussian graphical models [5] are wildly applied to infer the precision matrix. They have achieved a great success in interpreting the conditional independence between genes at the transcriptional level [6]. The Graphical Lasso [7] and the Nodewise Regression [8] are the popular methods to estimate undirected graphs with sparsity. On the other hand, regression parameters could be estimated by the traditional multivariate regression model.

Despite the respective success, considering regression parameters and precision matrices jointly could lead to a better result in many application scenarios. When applying the Gaussian graphical model to gene expression data, the introduction of genetic variants as the regression parameter would benefit the interpretation of gene regulation relationships [9], [10]. In [4], the influence from the key macroeconomic indicators to the returns of financial assets is modeled as regression parameters and the co-dependency relationships between the economic variables and the returns could be viewed as precision matrices in the layered network structures.

Compared with the diverse applications, the theoretical guarantee for the covariate adjusted precision matrix estimation is still to be explored. Rothman et al. [11] use the multivariate regression with covariance estimation (MRCE) method to estimate the regression parameters with the incorporation of the covariance information. [12] and [13] also consider the simultaneous estimation of the regression matrix and the precision matrix. Compared with the asymptotic analysis before, Cai et al. [1] provide the nonasymptotic analysis for the statistical error of a two-stage algorithm without the optimization analysis. Recently, Chen et al. [2] first provide the non-asymptotic optimization performance guarantee of the alternating gradient descent method with hard thresholding for the covariate-adjusted precision matrix estimation and illustrate the method converges linearly. However, their analysis is based on the resampling assumption and there is an additional logarithmic factor compared with the minimax rate.
Resampling is a technology to simplify the analysis by eliminating the dependency among iterations used in [14] and [15]. It requires independent data for each iteration, which oversimplifies the analysis for the method in practice [2], [16]. For the resampling assumption would cause the loss of effective measurements, [16], [17] have removed this assumption in their own application scenarios with sophisticated analysis technologies.

To estimate $\Gamma$, and $\Omega$, jointly, we consider the maximum likelihood estimator according to the Gaussian mapping. The corresponding conditional negative log-likelihood function is (neglect the constants)

$$f_n(\Gamma, \Omega) = -\log |\Omega| + \frac{1}{n} \| (Y - X\Gamma)\Omega^{\frac{1}{2}} \|_F^2$$

$$= -\log |\Omega| + \frac{1}{n} \text{tr}((Y - X\Gamma)\Omega(Y - X\Gamma)^T).$$

In the high dimensional and underdetermined case, we need to refer to the structure information of parameters to guarantee the performance of estimation. The sparsity priors of $\Gamma$ and $\Omega$, are considered in [1], [4], [12]. In this paper, we follow the line of [2] and consider the model

$$\min_{\Gamma, \Sigma} -\log |\Omega| + \frac{1}{n} \text{tr}((Y - X\Gamma)\Omega(Y - X\Gamma)^T)$$

s.t. $\| \text{vec}(\Gamma^T) \|_0 \leq s_{\Gamma}, \| \text{vec}(\Omega^T) \|_0 \leq s_{\Omega}.$

The key challenge to analyze the model (4) is that the function $f_n(\Gamma, \Omega)$ is not jointly convex about $\Gamma$ and $\Omega$. There is a line of research [18]–[21] adopting a different parameterization which makes the objective function convex. The difference and comparison of these two models are provided in [4] and [2].

For the bi-convexity of the loss function $f_n(\Gamma, \Omega)$, the alternating method is a natural choice. Alternating methods have been widely used to solve joint estimation problems, latent variable models and matrix factorization problems, such as [14]–[17], [22]. However, their methods could not be adopted to our model directly.

The other topic discussed in this paper is the time-data tradeoffs. Interpreting the relationships among data complexity (the number of measurements), structural complexity (the structure of unknown signals) and time complexity (the convergence rate of the algorithm) is crucial in the high dimensional data science. For linear inverse problems, [23] and [24] have illustrated the tradeoff between data and structural complexity by the phase transition curve, which characterizes the requirement of measurements to guarantee the successful recovery of structure signals. Recently, there are several literatures considering the time-data tradeoffs for linear inverse problems that more data would accelerate the rate of convergence, such as [25]–[27]. As the model (4) could be viewed as a multivariate regression problem with an unknown precision matrix, it is natural to conjecture the method solving (4) also has a similar time-data tradeoff.

In this paper, we present a sharp analysis for the alternating gradient descent with hard thresholding applied to the covariate adjusted precision matrix estimation problem. To the best of our knowledge, our analysis provides the first non-asymptotic optimization performance guarantee for this model without resampling. We illustrate the algorithm not only converges linearly, but also attains the minimax rate. Last but not the least, we demonstrate the time-data tradeoffs also exist for the covariate adjusted precision matrix estimation problem.

II. ALGORITHM

For the bi-convex property of (4), we apply the alternating gradient descent with hard thresholding (Algorithm 1) to jointly estimate $\Gamma_*$ and $\Omega_*$, where the hard thresholding operator $\mathcal{H}_T(\Gamma, s)$ only remains the top $s$ entries of $\Gamma$ in terms of magnitude [28].

**Algorithm 1:** Alternating Gradient Descent with Hard Thresholding

```
Input: Iteration number $T$, step size $\eta_\Gamma, \eta_\Omega$, sparsity $s_\Gamma, s_\Omega$.  
for $t = 0$ to $T - 1$ do  
    $\Gamma_{t+1} = \mathcal{H}_T(\Gamma_t - \eta_\Gamma \nabla_\Gamma f_n(\Gamma_t, \Omega_t), s_\Gamma)$  
    $\Omega_{t+1} = \mathcal{H}_T(\Omega_t - \eta_\Omega \nabla_\Omega f_n(\Gamma_t, \Omega_t), s_\Omega)$  
end for  
Output: $\Gamma_T, \Omega_T$
```

Considering the non-convexity of the objective function of (4), a good initialization is required to guarantee the estimation performance. Compared with [2], we adopt a different initialization algorithm (Algorithm 2), which avoids the use of the unknown parameters $\lambda_{\Gamma}$ and $\lambda_{\Omega}$.
Algorithm 2: Initialization

Input: Sparsity $s_T$, $s_\Omega$.

- $\Gamma_{ini} = \arg\min_{\|\text{vec}(\Gamma)\|_0 \leq s_T} \frac{1}{2} \|X - \text{vec}(\Gamma)\|_F^2$
- $S = \frac{1}{n} (Y - X\Gamma_{ini})^T (Y - X\Gamma_{ini})$
- $\Omega_{ini} = \mathcal{H}(S^{-1}, s_\Omega)$

Output: $\Gamma_{ini}$, $\Omega_{ini}$

### III. Main Theory

Before diving into the main theoretical results, we first present two assumptions required by our analysis.

**Assumption 1.** The rows of $E$ are independent with the distribution $\mathcal{N}(0, \Omega^*_s)$. We suppose the eigenvalues of $\Omega_s$ satisfy

$$\frac{1}{\tau} \leq \lambda_{\min}(\Omega_s) \leq \lambda_{\max}(\Omega_s) \leq \nu.$$  

(5)

This assumption is also declared in [1], [2], [13].

**Assumption 2.** Suppose $X$ is independent with $E$ and the rows of $X$ are independent following the distribution $\mathcal{N}(0, \Sigma_X)$. Further, the eigenvalues of $\Sigma_X$ satisfy

$$\frac{1}{\tau} \leq \lambda_{\min}(\Sigma_X) \leq \lambda_{\max}(\Sigma_X) \leq \tau.$$  

(6)

The Gaussian assumption about $X$ guarantees $\text{vec}(X^T)$ to satisfy the Hanson-Wright inequality [29]. This assumption could be extended to the case where $\text{vec}(X^T)$ satisfies the convex concentration property [30]. In [2], the authors also require $\|\Omega_s\|_\infty \leq M$, where $\|\Omega_s\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^m (\Omega_{ij})_j$. Our analysis does not rely on this condition.

Then we introduce some notations that are useful for our analysis.

**Definition 1.** The Gaussian width is a simple way to quantify the size of a set $C$

$$\omega(C) := \mathbb{E}\sup_{x \in C} \langle g, x \rangle, \text{ where } g \sim \mathcal{N}(0, I).$$

We define $C_{2s_T} = \{ \Gamma \in \mathbb{R}^{d \times m} \mid \|\text{vec}(\Gamma^T)\|_0 \leq 2s_T \}$ which is the set composing of matrices with at most $2s_T$ nonzero entries. Similarly, we set $C_{2s_\Omega} = \{ \Omega \in \mathbb{R}^{m \times m} \mid \|\text{vec}(\Omega^T)\|_0 \leq 2s_\Omega \}$. For simplicity, we write $\omega_T = \omega(C_{2s_T} \cap S^{d \times m-1})$ and $\omega_\Omega = \omega(C_{2s_\Omega} \cap S^{m \times m-1})$.

We use $C$ and $\epsilon$ to denote positive constants which might change from line to line throughout the paper.

**Theorem 1.** Under Assumption 1 and 2, suppose $R = \min(1/(\nu^2), 1/(\tau^2), 1/(2\nu))$, $s_T \geq (1 + 4(1/\rho_{pop} - 1)^2)s_T^i$, and $s_\Omega \geq (1 + 4(1/\rho_{sam} - 1)^2)s_\Omega^i$. Starting from $\Gamma_0$ and $\Omega_0$ satisfying $\max(\|\Gamma_0 - \Gamma_s\|_F, \|\Omega_0 - \Omega_s\|_F) \leq R$ with the step sizes $\eta_T = \nu \tau/(\nu^2 + 1)$ and $\eta_\Omega = 8\nu^2/(16\nu^4 + 1)$, if the number of measurements satisfies

$$n \geq C \frac{\max(1, \nu^2)}{\rho_{pop} (1 - \sqrt{\rho_{pop}^2 - R^2})} \left( \omega_T + \omega_\Omega + u \right)^2,$$

(7)

the alternating gradient descent with hard thresholding would converge linearly and its iterations obey

$$\Delta_{t+1} \leq \rho^{t+1} \Delta_0 + \frac{\epsilon}{1 - \rho},$$

(8)

with probability $1 - 14 \exp(-u^2)$. Here $\rho = \sqrt{\rho_{pop} + 1/\sqrt{\rho_{pop}}} \rho_{sam}$, $\Delta_t = \max(\|\Gamma_t - \Gamma_s\|_F, \|\Omega_t - \Omega_s\|_F)$, $\Delta_{t+1}$ and $\Delta_0$ are defined correspondingly. We set

$$\rho_{pop} = \max\left(1, \frac{2 - 2\nu^2 R}{\nu^2 + 1}, 1 - \frac{2 - 8\nu^2 R}{16\nu^4 + 1}\right)$$

(9)

$$\rho_{sam} = C \frac{\omega_T + \omega_\Omega + u}{\sqrt{n}}$$

(10)

$$\epsilon = \frac{C}{\sqrt{\rho_{pop}}} \max\left(1, \frac{\omega_T + u}{\nu}, 1, \frac{\omega_\Omega + u}{\nu}\right).$$

(11)

**Remark 1.** When analyzing this model, Chen et al. [2] impose the resampling assumption, that a fresh piece of data is used for every iteration. Under this assumption, $\Gamma_t$ and $\Omega_t$ are viewed as fixed matrices at each iteration, which simplifies the analysis. However, this assumption neither coincides with the practical algorithm nor produces a tight result (an additional logarithmic factor). To remove this assumption, we use the generic chaining [31] and the structure information of iteration points to take an union bound, which lead to a sharp estimation error.
Remark 2. When the Gaussian width $\omega_\Omega$ is dominant, the estimation error of $\Omega_*$ in [2] is in the order of $O(\sqrt{\log n} \sqrt{s_\Omega^2 \log(m)}/\sqrt{n})$. Considering the concrete bound of $\omega_r$ in [32], our estimation about $\Omega_*$ is in the order of $O(\sqrt{s_\Omega^2 \log(m)}/\sqrt{n})$ matching the minimax lower bound [33].

Remark 3. Different from the result in [2], the total convergence rate $\rho$ is composed of two parts, the population part $\rho_{\text{pop}}$ and the sample part $\rho_{\text{sam}}$. Particularly, the sample part $\rho_{\text{sam}}$ indicates the time-data tradeoffs that more data would accelerate the convergence rate.

Theorem 2. Under Assumption 1 and 2, suppose $s_\Gamma \geq s_r^*$ and $s_\Omega \geq s_r^* \Gamma$ and $R = \min(1/(\nu_r^2), 1/(\nu_r^2), 1/(2\nu))$. If the number of measurements satisfies

$$n \geq C \frac{\nu^4 \nu^4}{R^2} \frac{R^2}{R^2} \left( m + \omega_r + u \right)^2,$$

then we have

$$\max(\|\Gamma_{\text{ini}} - \Gamma_*\|_F, \|\Omega_{\text{ini}} - \Omega_*\|_F) \leq R,$$

with probability at least $1 - 18 \exp(-u^2)$.

Remark 4. We adopt a different initialization algorithm from [2] to avoid the selection of the parameters $\lambda_\Gamma$ and $\lambda_\Omega$.

Remark 5. In Theorem 4.7 of [2], the requirement of measurements contains the coefficient $\nu^2$, which is of the same order as $n^2$ in most situations. We also note that if we adopt the Graphical Lasso estimator, we could expect tighter sample complexity with the price of higher computation complexity.

When $\Omega_*$ is known, the model (4) degrades to the multivariate regression and the alternating method reduces to the iterative hard thresholding method (IHT). Our analysis naturally adapts to this condition.

$$\min_{\Gamma} \frac{1}{2n} \text{tr}( (Y - X\Gamma)\Omega_*(Y - X\Gamma)^T )$$

s.t. $\|\text{vec}(\Gamma^T)\|_F \leq s_\Gamma$.

Corollary 1. Under Assumption 1 and 2, we apply the IHT starting from $\Gamma_0 = 0$ with the step size $\eta_r = 2\nu/(\nu_r^2 \nu^2 + 1)$ and $s_\Gamma \geq (1 + 4(1/\rho_{\text{pop}} - 1)^2) s_r^*$. When the number of measurements satisfies

$$n \geq C \frac{\nu^4 \nu^4}{\rho_{\text{pop}} (1 - \sqrt{\rho_{\text{pop}}})},$$

we have

$$\|\Gamma_{t+1} - \Gamma_*\|_F \leq \rho \|\Gamma_t - \Gamma_*\|_F + \epsilon$$

with probability at least $1 - 4 \exp(-u^2)$. Here $\rho = \sqrt{\rho_{\text{pop}}} + \frac{1}{\sqrt{\rho_{\text{pop}}}} \rho_{\text{sam}}$, $\rho_{\text{pop}} = \frac{\nu^2 \nu^2 - 1}{\nu^2 \nu^2 + 1}$, $\rho_{\text{sam}} = C \frac{\omega_r + u}{\sqrt{n}}$ and

$$\epsilon = C \frac{1}{\sqrt{\rho_{\text{pop}}} \sqrt{\nu_r^2 \nu^2} \sqrt{n}}.$$

Remark 6. Compared with the traditional analysis of IHT in [28], our analysis reveals the time-data tradeoffs directly. The result in [25] could be applied to the analysis of IHT. However, its analysis is based on the linear regression problem, which is a special case of the multivariate regression problem we consider in this paper.

IV. EXPERIMENTS

In this section, we verify our theoretical results with numerical simulations.

Through the experiments, the support of $\Gamma_*$ is selected at random and its entries have i.i.d $\mathcal{N}(0,1)$ values. The predictor matrix $X$ is generated under the Gaussian mapping with i.i.d. standard Gaussian entries as in [1]. The precision matrix follows a band graph, where $\Omega_{ii} = 1$, $\Omega_{i,i+1} = \Omega_{i+1,i} = 0.4$ and $\Omega_{ij} = 0$, $|i - j| > 1$.

A. Comparison between different initialization algorithms

Compared with [2], we adopt a different initialization algorithm. To verify the performance of the initialization, we consider $n = 500$, $m = 100$, $d = 100$ and $n = 500$, $m = 200$, $d = 200$ two scenarios. In our initialization algorithm, we perform 2 hard thresholding iterations. We set $s_\Gamma = 300$ and record the time of 100 experiments and their average estimation errors.

From Table 1 we could derive a preciser estimation of $\Omega_*$ and a similar estimation of $\Gamma_*$ with less input parameters and moderate calculation burden. This is because the initialization algorithm itself converges linearly to a rough region, which means we could expect a fast convergence.
TABLE I
COMPARISON BETWEEN DIFFERENT INITIALIZATION ALGORITHMS.

| Methods         | $\|\Gamma_{in} - \Gamma^*\|_F$ | $\|\Omega_{in} - \Omega^*\|_F$ | Time |
|-----------------|-------------------------------|-------------------------------|------|
| [2]             | 9.57                         | 11.48                        | 0.44 |
| Ours            | 15.63                        | 6.75                         | 0.89 |

| Methods         | $\|\Gamma_{in} - \Gamma^*\|_F$ | $\|\Omega_{in} - \Omega^*\|_F$ | Time |
|-----------------|-------------------------------|-------------------------------|------|
| [2]             | 9.91                         | 16.24                        | 1.51 |
| Ours            | 16.95                        | 9.38                         | 2.99 |

B. Time-data tradeoffs

To verify the time-data tradeoffs of the optimization problem, we perform the algorithm under different numbers of measurements. We set $d = m = 100$, $s^*_\Gamma = 400$. We conduct the simulations under three scenarios $n_1 = 3000$, $n_2 = 4000$, $n_3 = 5000$ and each scenario is repeated for 50 trials. In the initialization algorithm, two hard thresholding operations are conducted.

![Fig. 1](image)

Fig. 1. (a) Convergence of $\|\Gamma_t - \Gamma^*\|_F / \|\Gamma^*\|_F$ under different numbers of measurements. (b) Convergence of $\|\Omega_t - \Omega^*\|_F / \|\Omega^*\|_F$ under different numbers of measurements.

In Figure 1(a) and 1(b), we present the convergence results of the alternating gradient descent for $\|\Gamma_t - \Gamma^*\|_F / \|\Gamma^*\|_F$ and $\|\Omega_t - \Omega^*\|_F / \|\Omega^*\|_F$. From the figures we could illustrate that more data would lead to faster convergence rates and smaller estimation errors, which support the theoretical result in Theorem 1.

C. Statistical estimation error

In this part, we verify the scaling of the statistical estimation error in Theorem 1. We consider two different scenarios, $\Gamma^*$-sparsity dominated case and $\Omega^*$-sparsity dominated case. For the $\Gamma^*$-sparsity dominated case, we set $d = m = 50$ and consider $s^*_\Gamma = 500, 550, 600$ three subcases. The number of measurements varies from 1200 to 3000. For the $\Omega^*$-sparsity dominated case, we set $d = 50$ and consider $m = 60, 70, 80$ three subcases corresponding to $s^*_\Omega = 178, 208, 238$. Each subcase is repeated 200 trials.

The scaling of the estimation error about $\Gamma^*$ and $\Omega^*$ is presented in Figure 2. The two diagrams illustrate the estimation error is proportion to $\omega_{\Gamma} / \sqrt{n}$ and $\omega_{\Omega} / \sqrt{n}$ without any logarithmic factor, which verifies our theoretical result in Theorem 1.

V. DISCUSSION

In this paper, we provide a sharp analysis of the alternating gradient descent method for the covariate adjusted precision matrix estimation problem. Though the whole optimization problem is highly nonconvex, we illustrate the discussed method
converges linearly and attains the minimax rate under the Gaussian mapping. At the same time, our analysis doesn’t refer to the resampling assumption. By introducing the generic chaining into the analysis, we could reveal this method possesses the phenomenon of time-data tradeoffs. As the generalization of the ordinary IHT for linear regression models, our analysis indicates there is also a time-data tradeoff for the ordinary IHT to solve sparse signals recovery problems.

Moreover, we believe our analysis is not restricted to this model. The stochastic processes we deal with are common in the multivariate regression problem and its variants. In fact, our result refer to the Gaussian width, which is suitable for all structure signals. Considering the diverse applications of the covariate adjusted precision matrix estimation problem in time series samples and general structure priors, such as group sparse and low rank matrices, we would extend our analysis to these models in the future works.

In this supplementary, we present the complete proof for the theoretical results in the paper. We use $C$ and $c$ to denote positive constants which might change from line to line throughout the paper.

**Appendix A**

**Preliminaries**

The core of our analysis is the sample-based analysis about the iteration process. The following two lemmas make it convenient for us to analyze the items like $\langle U, X^T X \rangle$ and $\langle U, X^T E \rangle$, which would appear many times in the remained part.

**Lemma 1.** Suppose $\text{vec}(X^T)$ follows the distribution $\mathcal{N}(0, \Upsilon_X)$. We have the tail bound

$$P(\mid \text{tr}(XUX^T) - \mathbb{E}\text{tr}(XUX^T)\mid > u) \leq 2\exp(-c\min(\frac{u^2}{n\|\Upsilon_X\|^2\|U\|^2_F}, \frac{u}{\|\Upsilon_X\|\|U\|^2_F})), \tag{19}$$

where $c$ is a constant.

**Lemma 2.** Suppose $X$ is independent with $E$ and $\text{vec}(X^T) \sim \mathcal{N}(0, \Upsilon_X)$, $\text{vec}(E^T) \sim \mathcal{N}(0, \Upsilon_E)$. Then

$$P(\mid \text{tr}(EUX^T)\mid > u) \leq 2\exp(-c\min(\frac{u^2}{n\|\Upsilon_E\|^2\|\Upsilon_X\|^2\|U\|^2_F}, \frac{u}{\|\Upsilon_E\|\|\Upsilon_X\|\|U\|^2_F})). \tag{20}$$

The following lemma is the fundamental tool to analyze the suprema of random processes with a mixed tail, which is based on the generic chaining [31] itself.

**Lemma 3.** [37] Suppose the random process $(X_t)_{t \in T}$ has a mixed tail

$$P(\mid X_t - X_s \mid > u) \leq 2\exp(-u^2\min\left(\frac{1}{d_2(t, s)^2}, \frac{1}{d_1(t, s)}\right)), \tag{21}$$

then we could derive

$$P\left(\sup_{t \in T}|X_t - X_{t_0}| > C(\gamma_2(T, d_2) + \gamma_1(T, d_1) + u\Delta_2(T) + u^2\Delta_1(T))\right) \leq 2\exp(-u^2), \tag{22}$$

where $\Delta_2(T)$ ($\Delta_1(T)$) is the diameter of $T$ with respect to $d_2$ ($d_1$).
APPENDIX B
MODEL

The corresponding negative log-likelihood function is
\[
 f_n(\Gamma, \Omega) = -\log |\Omega| + \frac{1}{n} \text{tr}(Y - X\Gamma)^T \Omega (Y - X\Gamma)^T \\
= -\log |\Omega| + \frac{1}{n} \text{tr}(\Gamma - \Gamma_x)^T X^T X (\Gamma - \Gamma_x) \Omega - 2E^T X (\Gamma - \Gamma_x) \Omega + E^T E \Omega, 
\]
where \( X \in \mathbb{R}^{n \times d}, Y \in \mathbb{R}^{n \times m}, \Gamma \in \mathbb{R}^{d \times m}, \Omega \in \mathbb{R}^{m \times m} \). Without the generality, we suppose \( \Omega \) is symmetric. The population function is
\[
 f(\Gamma, \Omega) = -\log |\Omega| + \text{tr}((\Gamma - \Gamma_x)^T \Sigma_X (\Gamma - \Gamma_x) \Omega + \Omega^{-1} x). 
\]
(24)

For the convenience of analysis, we collect the corresponding gradients and Hessian matrices here
\[
 \nabla_\Gamma f_n(\Gamma, \Omega) = \frac{2}{n} X^T X (\Gamma - \Gamma_x) \Omega - \frac{2}{n} X^T E \Omega \\
 \nabla_{\Omega} f_n(\Gamma, \Omega) = -\Omega^{-1} + \frac{1}{n} (\Gamma - \Gamma_x)^T X^T X (\Gamma - \Gamma_x) - \frac{2}{n} (\Gamma - \Gamma_x)^T X^T E + \frac{1}{n} E^T E \\
 \nabla_\Gamma f(\Gamma, \Omega) = 2\Sigma_X (\Gamma - \Gamma_x) \Omega \\
 \nabla_{\Omega} f(\Gamma, \Omega) = -\Omega^{-1} + (\Gamma - \Gamma_x)^T \Sigma_X (\Gamma - \Gamma_x) + \Omega^{-1} \\
 \nabla^2_\Gamma f(\Gamma, \Omega) = \Omega \otimes 2\Sigma_X \\
 \nabla^2_{\Omega} f(\Gamma, \Omega) = \Omega^{-1} \otimes \Omega^{-1}.
\]
(25) (26) (27) (28) (29) (30)

We set the step sizes as
\[
 \eta_\Gamma = \frac{\nu T}{\nu^2 + \nu^2} \quad \text{and} \quad \eta_\Omega = \frac{8\nu^2}{16\nu^2 + 1}. 
\]
(31)

In [2], the authors introduce the following local properties of the population function \( f(\Gamma, \Omega) \) required by the analysis.

**Lemma 4.** Under Assumption 1 and 2, for any \( \Gamma, \Gamma' \in B_F(\Gamma_x; R) \), we have
\[
 \frac{1}{\nu T} \| \Gamma' - \Gamma \|^2_F \leq f(\Gamma', \Omega_x) - f(\Gamma, \Omega_x) - \langle \nabla_\Gamma f(\Gamma, \Omega), \Gamma' - \Gamma \rangle \leq \nu T \| \Gamma' - \Gamma \|^2_F. 
\]
(32)

**Lemma 5.** Under Assumption 1 and 2, for any \( \Omega, \Omega' \in B_F(\Omega_x; R) \) where \( R \leq \frac{1}{\nu T} \), we have
\[
 \frac{1}{8\nu^2} \| \Omega' - \Omega \|^2_F \leq f(\Gamma_x, \Omega') - f(\Gamma_x, \Omega) - \langle \nabla_\Omega f(\Gamma_x, \Omega), \Omega' - \Omega \rangle \leq 2\nu^2 \| \Omega' - \Omega \|^2_F. 
\]
(33)

**Lemma 6.** Under Assumption 1 and Assumption 2, for any \( \Omega \in B_F(\Omega_x; R) \), we could derive
\[
 \| \nabla_\Gamma f(\Gamma, \Omega_x) - \nabla_\Gamma f(\Gamma, \Omega) \|_F \leq 2\tau R \| \Omega - \Omega_x \|_F. 
\]
(34)

For any \( \Gamma \in B_F(\Gamma_x; R) \), we could derive
\[
 \| \nabla_\Omega f(\Gamma, \Omega) - \nabla_\Omega f(\Gamma, \Omega) \|_F \leq \tau R \| \Gamma - \Gamma_x \|_F. 
\]
(35)

APPENDIX C
ANALYSIS FOR THE ALTERNATING GRADIENT DESCENT WITH HARD THRESHOLDING

Our analysis is based on the facts \( \Gamma_t \in B_F(\Gamma_x, R) \) and \( \Omega_t \in B_F(\Omega_x, R) \).

A. Analysis for the iteration about \( \Gamma \)

We write \( \mathcal{I} = \mathcal{I}_{t+1} \cup \mathcal{I}_s \), where \( \mathcal{I}_{t+1} \) and \( \mathcal{I}_s \) are the support sets of \( \Gamma_{t+1} \) and \( \Gamma_s \), respectively.

**Lemma 7.** [34] Suppose \( \mathbf{x}^* \) is a sparse vector satisfying \( \| \mathbf{x}^* \|_0 \leq s_\star \). \( \mathcal{HT}(\cdot, s) \) is the hard thresholding operator with \( s \geq s_\star \). Then we could bound the difference \( \| \mathcal{HT}(\mathbf{x}, s) - \mathbf{x}^* \|^2_2 \) for any \( \mathbf{x} \) by
\[
 \| \mathcal{HT}(\mathbf{x}, s) - \mathbf{x}^* \|^2_2 \leq (1 + \frac{2\sqrt{s_\star}}{\sqrt{s} - s_\star}) \| \mathbf{x} - \mathbf{x}^* \|^2_2. 
\]
(36)

**Lemma 8.** [35] Suppose \( f(\mathbf{x}) \) is \( \mu \)-strongly convex and \( L \)-smooth. With the step size \( \eta = 2/(L + \mu) \), the gradient descent iteration would contract as
\[
 \| (\mathbf{x} - \eta \nabla f(\mathbf{x}) - \mathbf{x}^*) \|_2 \leq \frac{L - \mu}{L + \mu} \| \mathbf{x} - \mathbf{x}^* \|_2. 
\]
(37)
where $x^*$ is the optimal point.

First, for $\mathcal{I}$ contains $\mathcal{I}_{t+1}$ and $\mathcal{I}_*$, we could rewrite $\|\nabla \Gamma_{t+1} - \Gamma_*\|_F$ as

$$\|\nabla \Gamma_{t+1} - \Gamma_*\|_F = \|\nabla \Gamma(T_{t+1} \in (1 - \eta_t \nabla \Gamma f_n(\Gamma_t, \Omega_t), s_{t+1}) - \Gamma_*\|_F \\
\leq \sqrt{1 + \frac{2\sqrt{\rho}}{\sqrt{s_t - s_{t+1}}}} \|\Gamma_t - \eta_t \nabla \Gamma f_n(\Gamma_t, \Omega_t), s_{t+1}) - \Gamma_*\|_F \\
\leq \sqrt{1 + \frac{2\sqrt{\rho}}{\sqrt{s_t - s_{t+1}}}} \|\Gamma_t - \eta_t \nabla \Gamma f(\Gamma_t, \Omega_t) - \Gamma_*\|_F + \eta_t \|\nabla \Gamma f(\Gamma_t, \Omega_t) - \nabla \Gamma f_n(\Gamma_t, \Omega_t)\|_F,$$

where the first inequality is from Lemma 7.

The first term of (40) could be bounded by the strong convexity and the smoothness of the population function $f(\Gamma, \Omega)$ about $\Gamma$ in Lemma 4 and the corresponding convergence result in Lemma 8

$$\|\nabla \Gamma f(\Gamma_t, \Omega_t) - \Gamma_*\|_F \leq \frac{R(\sqrt{\nu + u})}{\sqrt{n}} \|\Gamma_0 - \Gamma_*\|_F + \frac{\eta}{\sqrt{\nu + u}} \|\nabla \Gamma f(\Gamma, \Omega) - \nabla \Gamma f_n(\Gamma, \Omega)\|_F.$$

The second term of (40) could be rewritten as

$$\|\nabla \Gamma f(\Gamma_t, \Omega_t) - \nabla \Gamma f_n(\Gamma_t, \Omega_t)\|_F \leq \|\nabla \Gamma f(\Gamma, \Omega) - \nabla \Gamma f_n(\Gamma, \Omega)\|_F + \|\nabla \Gamma f_n(\Gamma, \Omega) - \nabla \Gamma f_n(\Gamma_t, \Omega_t)\|_F.$$

The first item could be bounded by the Lipschitz property of $\nabla \Gamma f(\Gamma, .)$ about $\Omega$ around $\Omega_*$ in Lemma 6

$$\|\nabla \Gamma f_n(\Gamma, \Omega) - \nabla \Gamma f_n(\Gamma_t, \Omega_t)\|_F \leq 2\sqrt{R(\sqrt{\nu + u})},$$

and the sample-based analysis.

**Lemma 9.** Under Assumption 1 and 2, we set $\eta_t = \frac{\sqrt{\nu + u}}{\sqrt{n} \nu + u}$ and $\eta_t = \frac{8\nu^2}{16\nu + 8}$ for any $\Gamma_t \in \mathcal{B}_F(\Gamma, R)$ and $\Omega_t \in \mathcal{B}_F(\Omega, R),$ the difference $\nabla \Gamma f(\Gamma_t, \Omega_t) - \nabla \Gamma f_n(\Gamma_t, \Omega_t)$ could be bounded by

$$\|\nabla \Gamma f_n(\Gamma_t, \Omega_t) - \nabla \Gamma f_n(\Gamma_t, \Omega_t)\|_F \leq \frac{C}{\sqrt{\nu + u}} \|\Omega_* - \Gamma_*\|_F + \frac{\nu + u}{\sqrt{n}} \|\Gamma_t - \Gamma_*\|_F + \frac{\rho}{\sqrt{\nu + u}} \|\Omega_* - \Gamma_*\|_F.$$

with probability at least $1 - 8 \exp(-u^2),$ when $n \geq (\omega + \omega + u)^2.$

**B. Analysis for the iteration about $\Omega$**

Similarly, we write $\mathcal{T} = \mathcal{T}_{t+1} \cup \mathcal{T}_*$, where $\mathcal{T}_{t+1}$ and $\mathcal{T}_*$ are the support sets of $\Omega_{t+1}$ and $\Omega_*$, respectively. For $\mathcal{T}$ contains $\mathcal{T}_{t+1}$ and $\mathcal{T}_*$, we could rearrange $\|\Omega_{t+1} - \Omega_*\|_F$ as

$$\|\Omega_{t+1} - \Omega_*\|_F = \|\nabla \Omega(T_{t+1} \in (1 - \eta_t \nabla \Omega f_n(\Gamma_t, \Omega_t), s_{t+1}) - \Omega_*\|_F \\
\leq \sqrt{1 + \frac{2\sqrt{\rho}}{\sqrt{s_t - s_{t+1}}}} \|\Omega_t - \eta_t \nabla \Omega f_n(\Gamma_t, \Omega_t) - \Omega_*\|_F \\
\leq \sqrt{1 + \frac{2\sqrt{\rho}}{\sqrt{s_t - s_{t+1}}}} \|\Omega_t - \eta_t \nabla \Omega f(\Gamma_t, \Omega_t) - \Omega_*\|_F + \eta_t \|\nabla \Omega f(\Gamma_t, \Omega_t) - \nabla \Omega f_n(\Gamma_t, \Omega_t)\|_F,$$

The first term of (47) could be bounded by the strong convexity and the smoothness of the population function $f(\Gamma, \Omega)$ about $\Omega$ in Lemma 5 and the corresponding convergence result in Lemma 8

$$\|\Omega_t - \eta_t \nabla \Omega f(\Gamma_t, \Omega_t) - \Omega_*\|_F \leq \frac{16\nu^4 - 1}{16\nu^4 + 1} \|\Omega_t - \Omega_*\|_F.$$
The first item could be bounded by the Lipschitz property of $\nabla_{\Omega} f(\cdot, \Omega)$ about $\Gamma$ around $\Gamma_*$ in Lemma 6

$$\|\nabla_{\Omega} f(\Gamma_*, \Omega_t) - \nabla_{\Omega} f(\Gamma_t, \Omega_t)\|_F \leq \tau R \|\Gamma_t - \Gamma_*\|_F.$$  

(50)

The second item is associated with the sample loss function $f_n(\Gamma, \Omega)$ and needs the sample-based analysis.

**Lemma 10.** Under the same condition as Lemma 9. For any $\Gamma_t \in B_F(\Gamma_*, R)$ and $\Omega_t \in B_F(\Omega_*, R)$, the difference $(\nabla_{\Omega} f(\Gamma_t, \Omega_t) - \nabla_{\Omega} f_n(\Gamma_t, \Omega_t))_{\Omega}$ could be bounded by

$$\eta_2 \|\nabla_{\Omega} f(\Gamma_t, \Omega_t) - \nabla_{\Omega} f_n(\Gamma_t, \Omega_t)\|_F \leq C \frac{8 \nu^4}{16 \nu^4 + 1} \frac{(\tau R \omega_t + \omega_t + n)}{\sqrt{n}} \|\Gamma_t - \Gamma_*\|_F + \frac{\sqrt{\tau R} \omega_t + \omega_t + u}{\sqrt{\nu}} + \frac{1}{\nu} \frac{\omega_t + u}{\sqrt{n}},$$

(51)

with probability at least $1 - 6 \exp(-u^2)$, when $n \geq (\omega_t + \omega_t + u)^2$.

**C. The whole convergence result (Proof of Theorem 1)**

We define the convergence parameter $\rho_{\text{pop}}$ associated with the population loss function as

$$\rho_{\text{pop}} = \max\left(\frac{\nu^2 \tau^2 - 1}{\nu^2 \tau^2 + 1} + \frac{2 \nu \tau^2 R}{\nu^2 \tau^2 + 1}, \frac{16 \nu^4 - 1}{16 \nu^4 + 1} + \frac{8 \nu^2 R}{16 \nu^4 + 1}\right),$$

(52)

where $R \leq \min\left(\frac{1}{\nu^2 \tau^2}, \frac{1}{\nu^2 \tau R}\right)$ from the requirement $\rho_{\text{pop}} < 1$.

By the assumptions $s_t \geq (1 + 4(1/\rho_{\text{pop}} - 1)^2) s_t^2$ and $s_t \geq (1 + 4(1/\rho_{\text{pop}} - 1)^2) s_t^2$, we could bound the two parameters associated with the hard thresholding operation by

$$\max\left(1 + \frac{2 \sqrt{s_t^2}}{\sqrt{s_t} - s_t}, 1 + \frac{2 \sqrt{s_t^2}}{\sqrt{s_t^2} - s_t}\right) \leq \frac{1}{\sqrt{\rho_{\text{pop}}}}$$

(53)

Then, we consider all components of $\|\Gamma_{t+1} - \Gamma_*\|_F$. Taking (41), (43) and Lemma 9 into (40), we could derive

$$\|\Gamma_{t+1} - \Gamma_*\|_F \leq \left(\sqrt{\rho_{\text{pop}}} + \frac{1}{\sqrt{\rho_{\text{pop}}} \rho_{T_{\text{sam}}}}\right) \max(\|\Gamma_t - \Gamma_*\|_F, \|\Omega_t - \Omega_*\|_F) + \frac{1}{\sqrt{\rho_{\text{pop}}}} \epsilon_T,$$

(54)

where

$$\rho_{T_{\text{pop}}} = \frac{\nu^2 \tau^2 - 1}{\nu^2 \tau^2 + 1} + \frac{2 \nu \tau^2 R}{\nu^2 \tau^2 + 1} = 1 - 2 \frac{\nu \tau^2 R}{\nu^2 \tau^2 + 1},$$

(55)

$$\rho_{T_{\text{sam}}} = C \frac{\nu^2 \tau^2}{\nu^2 \tau^2 + 1} \left(\frac{R \omega_t + \omega_t + u}{\sqrt{n}} + \frac{\omega_t + u}{\sqrt{n}}\right),$$

(56)

$$\epsilon_T = C \frac{\nu^2 \tau^2}{\nu^2 \tau^2 + 1} \left(\frac{R \omega_t + \omega_t + u}{\sqrt{n}} + \frac{\omega_t + u}{\sqrt{n}}\right).$$

(57)

If we want $\|\Gamma_{t+1} - \Gamma_*\|_F \leq R$, we need to guarantee

$$\left(\sqrt{\rho_{\text{pop}}} + \frac{1}{\sqrt{\rho_{\text{pop}}} \rho_{T_{\text{sam}}}}\right) R + \frac{1}{\sqrt{\rho_{\text{pop}}}} \epsilon_T \leq R,$$

(58)

or

$$\epsilon_T + \rho_{T_{\text{sam}}} R \leq \sqrt{\rho_{\text{pop}}}(1 - \sqrt{\rho_{\text{pop}}}) R.$$  

(59)

When $R \leq \min\left(\frac{1}{\nu^2 \tau^2}, \frac{1}{\nu^2 \tau R}, \frac{1}{\nu^2 \tau^2}\right)$, we could derive

$$\epsilon_T + \rho_{T_{\text{sam}}} R \leq C \frac{\nu^2 \tau^2}{\nu^2 \tau^2 + 1} \left(\frac{R \omega_t + \omega_t + u}{\sqrt{n}} + \frac{1}{\sqrt{n}} \frac{\omega_t + u}{\sqrt{n}}\right) + \left(\frac{R \omega_t + \omega_t + u}{\sqrt{n}} + \frac{\omega_t + u}{\sqrt{n}}\right) R,$$

$$\leq C \frac{1}{\sqrt{n}} \left(\frac{\omega_t + \omega_t + u}{\sqrt{n}}\right).$$
When the number of measurements satisfies
\[ n \geq C \frac{1}{\rho_{\text{pop}}(1 - \sqrt{\rho_{\text{pop}}})} \frac{(\omega_T + \omega_{\Omega} + u)^2}{R^2}, \]  

(60)
we could guarantee \( \| \Gamma_{t+1} - \Gamma_* \|_F \leq R. \)

Next, we consider all components of \( \| \Omega_{t+1} - \Omega_* \|_F. \) Taking (48), (50) and Lemma 10 into (47), we could derive
\[
\| \Omega_{t+1} - \Omega_* \|_F \leq \sqrt{1 + 2\frac{s_{\Omega}^2}{s_{\Omega}^2 - s_{\Omega}^2}} (16\nu^4 - 1) \| \Omega_t - \Omega_* \|_F + \frac{8\nu^2 R}{16\nu^4 + 1} \| \Gamma_t - \Gamma_* \|_F + \epsilon_{\Omega}(\sqrt{\nu} f_t(\Gamma_t, \Omega_t) - \sqrt{\nu} f_t(\Gamma_t, \Omega_0)) T \|_F
\]
\[
\leq \sqrt{1 + 2\frac{s_{\Omega}^2}{s_{\Omega}^2 - s_{\Omega}^2}} \rho_{\text{pop}} + \rho_{\text{sam}} \max(\| \Gamma_t - \Gamma_* \|_F, \| \Omega_t - \Omega_* \|_F) + \sqrt{1 + 2\frac{s_{\Omega}^2}{s_{\Omega}^2 - s_{\Omega}^2}} \epsilon_{\Omega}
\]
\[
\leq (\sqrt{\rho_{\text{pop}} + \frac{1}{\rho_{\text{pop}}} \rho_{\text{sam}}}) \max(\| \Gamma_t - \Gamma_* \|_F, \| \Omega_t - \Omega_* \|_F) + \frac{1}{\rho_{\text{pop}}} \epsilon_{\Omega},
\]

(61)
where
\[
\rho_{\text{pop}} = \frac{16\nu^4 - 1}{16\nu^4 + 1} + \frac{8\nu^2 R}{16\nu^4 + 1} = 1 - \frac{2 - 8\nu^2 R}{16\nu^4 + 1}
\]
\[
\rho_{\text{sam}} = C \frac{8\nu^4}{16\nu^4 + 1} \left( \frac{\tau R \omega_T + \omega_{\Omega} + u}{\sqrt{n}} \right)
\]
\[
\epsilon_{\Omega} = C \frac{8\nu^4}{16\nu^4 + 1} \left( \frac{\sqrt{\nu} R \omega_T + \omega_{\Omega} + u}{\sqrt{n}} + \frac{1}{\nu} \omega_{\Omega} + u \right).
\]

If we want \( \| \Omega_{t+1} - \Omega_* \|_F \leq R, \) we need to guarantee
\[
(\sqrt{\rho_{\text{pop}} + \frac{1}{\rho_{\text{pop}}} \rho_{\text{sam}}}) R + \frac{1}{\rho_{\text{pop}}} \epsilon_{\Omega} \leq R
\]
or
\[
\epsilon_{\Omega} + \rho_{\text{sam}} R \leq \sqrt{\rho_{\text{pop}}(1 - \sqrt{\rho_{\text{pop}}})} R.
\]

(65)
(66)
When \( R = \min \left( \frac{1}{\sqrt{\nu}}, \frac{1}{\nu}, \frac{1}{2\nu} \right), \) we could derive
\[
\epsilon_{\Omega} + \rho_{\text{sam}} R \leq C \frac{8\nu^4}{16\nu^4 + 1} \left( \frac{\sqrt{\nu} R \omega_T + \omega_{\Omega} + u}{\sqrt{n}} + \frac{1}{\nu} \omega_{\Omega} + u \right) + \left( \frac{\tau R \omega_T + \omega_{\Omega} + u}{\sqrt{n}} \right) R
\]
\[
\leq C \frac{1}{\nu} \omega_T + \omega_{\Omega} + u.
\]

When the number of measurements satisfies
\[
n \geq C \frac{1}{\rho_{\text{pop}}(1 - \sqrt{\rho_{\text{pop}}})} \frac{(\omega_T + \omega_{\Omega} + u)^2}{R^2},
\]

(67)
we could guarantee \( \| \Omega_{t+1} - \Omega_* \|_F \leq R. \)

Finally, we consider \( \| \Gamma_{t+1} - \Gamma_* \|_F \) and \( \| \Omega_{t+1} - \Omega_* \|_F \) as a whole and derive
\[
\max(\| \Gamma_{t+1} - \Gamma_* \|_F, \| \Omega_{t+1} - \Omega_* \|_F) \leq (\sqrt{\rho_{\text{pop}} + \frac{1}{\rho_{\text{pop}}} \max(\rho_{\text{sam}}, \rho_{\text{sam}})}) \max(\| \Gamma_t - \Gamma_* \|_F, \| \Omega_t - \Omega_* \|_F) + \frac{1}{\rho_{\text{pop}}} \max(\epsilon_T, \epsilon_{\Omega}).
\]

(68)

D. Initialization (Proof of Theorem 2)

The initialization of \( \Gamma \) is derived from the following optimization problem
\[
\min_{\Gamma} \frac{1}{2} \| Y - X\Gamma \|_F^2
\]
\[
s.t. \| \text{vec}(\Gamma^T) \|_0 \leq s_T.
\]

(69)
The initialization of \( \Omega \) is derived from the following optimization problem
\[
\min_{\Omega} \frac{1}{2} \| \Omega - S^{-1} \|_F^2
\]
\[
s.t. \| \text{vec}(\Omega^T) \|_0 \leq s_{\Omega},
\]

(70)
where $S = (Y - X\Gamma_{in})^T(Y - X\Gamma_{in})/n$.  

The error $\|\Gamma_{ni} - \Gamma_*\|_F$ is analyzed as the Lasso.

**Lemma 11.** When $n \geq C r_1^2 (\omega_T + u)^2$, we could derive

$$\|\Gamma_{ni} - \Gamma_*\|_F \leq C \frac{\omega_T + u}{\sqrt n} \nu \frac{1}{\sqrt n},$$

(71)

with probability at least $1 - 4 \exp(-u^2)$.

When $n \geq C r_1^2 (\omega_T + u)^2/\nu^2$, we could derive

$$\|\Gamma_{ni} - \Gamma_*\|_F \leq R.$$

(72)

The analysis of $\|\Omega_{mi} - \Omega_*\|_F$ is more complicated.

**Lemma 12.** When $n > \tau_1^4 \nu^4 (m + \omega_T + u)^2$, we could derive

$$\|\Omega_{mi} - \Omega_*\|_F \leq C \tau_1^2 \nu^3 \frac{m + \omega_T + u}{\sqrt n},$$

(73)

with probability at least $1 - 18 \exp(-u^2)$.

When $n > C \tau_1^4 \nu^4 (m + \omega_T + u)^2$, we could derive $\|\Omega_{mi} - \Omega_*\|_F \leq R$.

**E. Ordinary Iterative hard thresholding (Proof of Corollary 3)**

In this condition, the loss function becomes

$$f_n(\Gamma) = \frac{1}{2n} tr((Y - X\Gamma)\Omega_*(Y - X\Gamma)^T).$$

(74)

The corresponding gradients and Hessian matrix are

$$\nabla f_n(\Gamma) = \frac{1}{n} X^T X (\Gamma - \Gamma_*) \Omega_* - \frac{1}{n} X^T E \Omega_*$$

(75)

$$\nabla^2 f(\Gamma) = \Omega_* \otimes \Sigma_X.$$  

(76)

(77)

We set the step sizes as

$$\eta_T = \frac{\nu_T}{\nu^2 r^2 + 1}.$$  

(78)

We write $I = I_{t+1} \cup I_*$, where $I_{t+1}$ and $I_*$ are the support sets of $\Gamma_{t+1}$ and $\Gamma_*$, respectively.

We could write the hard thresholding iteration as

$$\|\Gamma_{t+1} - \Gamma_*\|_F = \|\mathcal{H}(I_T) (\Gamma_T - \eta_T \nabla f_n (\Gamma_T))_I_2 - \Gamma_*\|_F$$

$$\leq \frac{1 + \frac{2\sqrt{s_I}}{\sqrt{s_I} - s_T}}{\nabla f(\Gamma_{t})} \|\Gamma_T - \Gamma_*\|_F - \eta_T \nabla f_n (\Gamma_T)_{I_2} - \Gamma_*\|_F$$

$$\leq 1 + \frac{2\sqrt{s_I}}{\sqrt{s_I} - s_T} \|\Gamma_T - \Gamma_* - \eta_T \nabla f(\Gamma_T) + \eta_T \nabla f_n (\Gamma_T)_{I_2} - \Gamma_*\|_F$$

$$\leq 1 + \frac{2\sqrt{s_I}}{\sqrt{s_I} - s_T} \|\Gamma_T - \Gamma_* - \eta_T \nabla f(\Gamma_T)\|_F + \|\eta_T \nabla f_n (\Gamma_T) - \Gamma_*\|_F$$

$$\leq 1 + \frac{2\sqrt{s_I}}{\sqrt{s_I} - s_T} \|\Gamma_T - \Gamma_* - \eta_T \nabla f(\Gamma_T)\|_F + 1 + \frac{2\sqrt{s_I}}{\sqrt{s_I} - s_T} \|\eta_T \nabla f_n (\Gamma_T) - \Gamma_*\|_F,$$

where the first inequality is from Lemma 7.

The first item could be bounded by the strong convexity and the smoothness of $f(\Gamma)$, which could be derived from the Hessian matrix $\nabla^2 f(\Gamma)$ and Assumption 1, 2. With Lemma 8, we have

$$\|\Gamma_* - \eta_T \nabla f(\Gamma_T) - \Gamma_*\|_F \leq \frac{\nu_T^2 \tau^2}{\nu^2 r^2 + 1} \|\Gamma_T - \Gamma_*\|_F.$$  

(79)

The second item could be rewritten as

$$\|\eta_T \nabla f_n (\Gamma_T) - \eta_T \nabla f_n (\Gamma_T)_{I_2}\|_F = \eta_T \| \Sigma_X - \frac{1}{n} X^T X (\Gamma_T - \Gamma_*) \Omega_* + \frac{1}{n} X^T E \Omega_*\|_F$$

$$\leq \eta_T \| \Sigma_X - \frac{1}{n} X^T X (\Gamma_T - \Gamma_*) \Omega_*\|_F + \eta_T \| \frac{1}{n} X^T E \Omega_*\|_F.$$  

(80)
These two parts have been analyzed in Lemma 9. The next two lemmas follow the same procedures as Lemma 18 and Lemma 20.

**Lemma 13.** Under the condition of \( n \geq (\omega_T + u)^2 \), we could derive

\[
P(\sup_{U, V \in \mathcal{C}_{21 - \epsilon}} \langle V, (\Sigma_X - \frac{X^TX}{n})U\Omega_s \rangle > C\|\Sigma_X\|\|\Omega_s\|\left(\frac{\omega_T + u}{\sqrt{n}}\right)) \leq 2\exp(-u^2).
\]  

\((81)\)

**Lemma 14.** Under the condition of \( n \geq (\omega_T + u)^2 \), we could derive

\[
P(\sup_{V \in \mathcal{C}_{21 - \epsilon}} \langle V, \frac{1}{n}X^T E \Omega_s \rangle > C\|\Sigma_X\|\|\Omega_s\|\left(\frac{\omega_T + u}{\sqrt{n}}\right)) \leq 2\exp(-u^2).
\]  

We set

\[
\rho_{pop} = \frac{\nu^2\tau^2 - 1}{\nu^2\tau^2 + 1}.
\]  

\((83)\)

By the assumption \( s_T \geq (1 + 4(1 - \rho)^2)s^2_T \), we could derive

\[
\sqrt{1 + \frac{2\sqrt{s_T^2}}{s_{T} - s^2_{T}} \leq \frac{1}{\sqrt{\rho_{pop}}}
\]  

\((84)\)

When \( n \geq (\omega_T + u)^2 \), we could derive

\[
\|\Gamma_{t+1} - \Gamma_s\|_F \leq \left(\sqrt{\rho_{pop}} + C \frac{1}{\sqrt{\rho_{pop}}\nu^2\tau^2 + 1} \left(\frac{\omega_T + u}{\sqrt{n}}\right)\right)\|\Gamma_t - \Gamma_s\|_F + C \frac{1}{\sqrt{\rho_{pop}}\nu^2\tau^2 + 1} \left(\frac{\omega_T + u}{\sqrt{n}}\right)
\]  

\[
\leq \left(\sqrt{\rho_{pop}} + C \frac{1}{\sqrt{\rho_{pop}}\nu^2\tau^2 + 1} \left(\frac{\omega_T + u}{\sqrt{n}}\right)\right)\|\Gamma_t - \Gamma_s\|_F + C \frac{1}{\sqrt{\rho_{pop}}\nu^2\tau^2 + 1} \left(\frac{\omega_T + u}{\sqrt{n}}\right),
\]

with probability at least \( 1 - 4\exp(-u^2) \).

**APPENDIX D**

**PROOF OF TECHNICAL LEMMAS**

A. **Proof of Lemma 1**

This lemma is a direct proposition of the Hanson-Wright inequality

**Lemma 15 (Hanson-Wright inequality [29]).** Suppose \( x \) is a random vector with independent sub-Gaussian components \( x_i \) satisfying \( \mathbb{E}[x_i] = 0 \) and \( \|x_i\|_{\psi_2} \leq K \). \( A \in \mathbb{R}^{n \times n} \) is a fixed matrix. For \( u > 0 \), we could get

\[
P(\|x^T Ax - \mathbb{E}x^T Ax\| > u) \leq 2\exp(-c \min\left(-\frac{u^2}{K^2\|A\|_F^2}, \frac{u}{K^2\|A\|}\right)),
\]  

\((85)\)

where \( c > 0 \) is a constant.

We could rearrange

\[
\text{tr}(XUX^T) = \text{vec}(X^T)(I_n \otimes U)\text{vec}(X^T) = \text{vec}(X^T)^T Y^\frac{1}{2}_X Y^\frac{1}{2}_X (I_n \otimes U) Y^\frac{1}{2}_X Y^\frac{1}{2}_X \text{vec}(X^T).
\]  

\((86)\)

In this way, \( Y^\frac{1}{2}_X \text{vec}(X^T) \) becomes an isotropic Gaussian vector. Combining the rotation invariance of Gaussian vectors, we could derive

\[
P(\text{tr}(XUX^T) - \mathbb{E}\text{tr}(XUX^T) > u) = P(\|g^T Y^\frac{1}{2}_X (I_n \otimes U) Y^\frac{1}{2}_X g - \mathbb{E}g^T Y^\frac{1}{2}_X (I_n \otimes U) Y^\frac{1}{2}_X g\| > u)
\]  

\[
\leq 2\exp(-c \min\left(-\frac{u^2}{\|Y^\frac{1}{2}_X (I_n \otimes U) Y^\frac{1}{2}_X\|_F^2}, \frac{u}{\|Y^\frac{1}{2}_X (I_n \otimes U) Y^\frac{1}{2}_X\|}\right))
\]  

\[
\leq 2\exp(-c \min\left(-\frac{u^2}{n\|Y^\frac{1}{2}_X \|_F^2 \|U\|_F^2}, \frac{u}{\|Y^\frac{1}{2}_X \|_F}\right)),
\]

where \( g \) is a vector with independent standard Gaussian entries. Here we use \( \|AB\|_F \leq \|A\|\|B\|_F \), \( \|AB\| \leq \|A\|\|B\| \) and \( \|A\| \leq \|A\|_F \) for two matrices \( A \) and \( B \).
B. Proof of Lemma 2

This lemma could be viewed as a proposition of the Bernstein’s inequality (Theorem 2.8.1 in [32]). And we note the proof is similar to the one for the Hanson-Wright inequality after decoupling (Theorem 6.2.1 in [32]).

From the independence between $X$ and $E$ and the rotation invariance of Gaussian vectors, we could derive

$$P(|\text{tr}(EU_X^T)| > u) = P(|\text{vec}(X^T)^T(I_n \otimes U^T)\text{vec}(E^T)| > u) = P(|g_X^T \mathbf{Y}_X^T(I_n \otimes U^T)\mathbf{Y}_E^T g_E| > u),$$

where $g_E$ and $g_X$ are two independent vectors with independent standard Gaussian entries.

We set $Q = \mathbf{Y}_X^T(I_n \otimes U^T)\mathbf{Y}_E^T$ with the singular value decomposition $U_Q \Sigma_Q V_Q$, where $U_Q$ and $V_Q$ are two unitary matrices.

We adopt the rotation invariance of Gaussian vectors again and derive

$$P(|g_X^T \mathbf{Y}_X^T(I_n \otimes U^T)\mathbf{Y}_E^T| > u) = P(|g_X^T U_Q \Sigma_Q V_Q g_E| > u)$$

$$\leq 2 \exp(-c \min(u^2, u/\|\Sigma_Q\|_F^2)$$

$$\leq 2 \exp(-c \min(\frac{u^2}{\|\mathbf{Y}_X^T\|^2 \|\mathbf{Y}_E^T\|^2}, \frac{u}{\|\mathbf{Y}_X^T\|_F \|\mathbf{Y}_E^T\|_F \|U\|_F^2})),$$

where we use the Bernstein’s inequality for the sum of the product of independent Gaussian variables. We also use $\|AB\|_F \leq \|A\|_F \|B\|_F$, $\|AB\| \leq \|A\| \|B\|$ and $\|A\| \leq \|A\|_F$ for two matrices $A$ and $B$.

C. Proof of Lemma 9

We first rewrite $\nabla_{\Gamma} f(\Gamma_t, \Omega_t) \rightarrow \nabla_{\Gamma} f_n(\Gamma_t, \Omega_t)$ as

$$\nabla_{\Gamma} f(\Gamma_t, \Omega_t) \rightarrow \nabla_{\Gamma} f_n(\Gamma_t, \Omega_t)$$

$$= 2 \Sigma_X (\Gamma_t - \Gamma_s) \Omega_t - \frac{2}{n} X^T X (\Gamma_t - \Gamma_s) \Omega_t - \frac{2}{n} X^T E \Omega_t$$

$$= 2 (\Sigma_X - \frac{X^T X}{n})(\Gamma_t - \Gamma_s)(\Omega_t - \Omega_s) + 2 (\Sigma_X - \frac{X^T X}{n})(\Gamma_t - \Gamma_s)(\Omega_t - \Omega_s) - \frac{2}{n} X^T E (\Omega_t - \Omega_s) - \frac{2}{n} X^T E \Omega_s.$$

With the definition of $C_{2\eta r}$, we could derive

$$\|\nabla_{\Gamma} f(\Gamma_t, \Omega_t) \rightarrow \nabla_{\Gamma} f_n(\Gamma_t, \Omega_t)\|_F \leq \sup_{V \in C_{2\eta r} \cap S^{d \times m-1}} \langle V, \nabla_{\Gamma} f(\Gamma_t, \Omega_t) \rightarrow \nabla_{\Gamma} f_n(\Gamma_t, \Omega_t)\rangle,$$

where we use the fact $\text{Card}(Z) \leq 2s_f$.

In this way, to bound $\|\nabla_{\Gamma} f(\Gamma_t, \Omega_t) \rightarrow \nabla_{\Gamma} f_n(\Gamma_t, \Omega_t)\|_F$, we need to deal with the suprema of random processes.

The supreme of the random process associated with the first term of (87) could be bounded by Lemma 3. We need to verify it has a mixed type. We rewrite the random process as

$$\langle V, 2 (\Sigma_X - \frac{X^T X}{n})(\Gamma_t - \Gamma_s)(\Omega_t - \Omega_s) \rangle = \langle V, 2 (\Sigma_X - \frac{X^T X}{n})PU\rangle \|\Gamma_t - \Gamma_s\|_F \|\Omega_t - \Omega_s\|_F,$$

where $P, V \in C_{2\eta r} \cap S^{d \times m-1}$ and $U \in C_{2\eta r} \cap S^{m \times m-1}$.

Then we could rearrange the increment as

$$X_{U,V,P} \rightarrow X_{W,Z,Q}$$

$$= \langle V, 2 (\Sigma_X - \frac{X^T X}{n})(PU) \rangle$$

$$= \text{E}[\frac{2}{n} \text{vec}(X)^T (I_n \otimes (PU^T - QWZ^T))\text{vec}(X)] - \frac{1}{n} \Sigma_X (I_n \otimes (PU^T - QWZ^T))\text{vec}(X).$$

We could further rearrange $PUV^T - QWZ^T$ as

$$PUV^T - QWZ^T = \frac{1}{2} P(U - W)(V + Z)^T + \frac{1}{2} P(U + W)(V - Z)^T + (P - Q)WZ^T.$$

Its Frobenius norm could be bounded as

$$\|PUV^T - QWZ^T\|_F^2 \leq 4\|U - W\|_F^2 + 4\|V - Z\|_F^2 + 2\|P - Q\|_F^2 \leq 4\|\frac{U}{P} - \frac{W}{Q}\|_F^2.$$
Combining Lemma 1 with $X_{U,V,P} - X_{W,Z,Q}$, we could derive the mixed tail 

$$P(\|V, 2(\Sigma_X - \frac{X^TX}{n})P\| - \|Z, 2(\Sigma_X - \frac{X^TX}{n})QW\| > u) \leq 2 \exp(-\min(\frac{u^2}{\n^2}\|\Sigma_X\|^2, \frac{u}{\n^2}\|\Sigma_X\|\|\frac{X^TX}{n}\|)),$$

(92)

where we use $\|\Sigma_X\| = \|\Sigma\|$ under Assumption 2.

This means the increment has a mixed tail with $d_2 = 4\|\Sigma_X\|\cdot \n/\sqrt{n}$ and $d_1 = 4\|\Sigma_X\|\cdot \n/\n$.

With Lemma 3, we could derive the event

$$\sup_{P, V \in C_{2n}, c \in S^{d \times m - 1}} \|V, 2(\Sigma_X - \frac{X^TX}{n})P\| > C(\gamma_2(T, d_2) + \gamma_1(T, d_1) + u\Delta_2(T) + u^2\Delta_1(T))$$

(94)

holds with probability at most $2 \exp(-u^2)$. Here $T = C_{2n} \cap S^{d \times m - 1} \times C_{2n} \cap S^{d \times m - 1} \times C_{2n} \cap S^{d \times m - 1}$.

We adopt the following lemma to transfer the $\gamma_1$-functional to the $\gamma_2$-functional and deal with the coefficients of metrics.

**Lemma 16.** [38] For $\gamma_\alpha$-functional, we have

$$\gamma_1(S, \|.\|_2) = \frac{1}{2} \left( \gamma_2(S, \|.\|_2) + \gamma_\alpha(S, d) \right)$$

(95)

$$\gamma_\alpha(S, cd) = c\gamma_\alpha(S, d)$$

(96)

where $\alpha > 0$, $c > 0$.

Combining with the Talagrand’s majorizing measure theorem [31], we could bound the $\gamma_2$-functional by the Gaussian width

$$\gamma_2(T, \|.\|_F) \leq C(\omega(T, \{S, d\}) + \omega(T, \{S, d\}))$$

(97)

where the Frobenius norm for a matrix is equivalent to the $l_2$ norm for a vector.

Then we could rearrange (94) further and derive the event

$$\sup_{P, V \in C_{2n}, c \in S^{d \times m - 1}} \|V, 2(\Sigma_X - \frac{X^TX}{n})P\|$$

$$> C(\|\Sigma_X\|\frac{\omega_T + \omega_{1T}}{\n} + 4\|\Sigma_X\|(\frac{\omega_T + \omega_{1T}}{\n})^2 + 4\|\Sigma_X\|\frac{u}{\n}\sqrt{\Delta_F(T)} + 4\|\Sigma_X\|\frac{u^2}{\n}\Delta_F(T))$$

holds with probability at most $2 \exp(-u^2)$.

From the facts $(\omega_T + \omega_{1T})^2 + u^2 \leq (\omega_T + \omega_{1T} + u)^2$ and $\Delta_F(T) \leq 6$, we could derive the following lemma when the item $(\omega_T + \omega_{1T} + u)/\sqrt{\n}$ is dominant.

**Lemma 17.** Under the condition of $n \geq (\omega_T + \omega_{1T} + u)^2$, we have

$$P(\sup_{P, V \in C_{2n}, c \in S^{d \times m - 1}} \|V, 2(\Sigma_X - \frac{X^TX}{n})P\| > C\|\Sigma_X\|(\frac{\omega_T + \omega_{1T} + u}{\n}) \leq 2 \exp(-u^2).$$

(98)

The random process associated with the second item of (87) could be written as

$$\|2(\Sigma_X - \frac{X^TX}{n})(\Gamma_t - \Gamma_*\Omega_s)\|_F \leq \sup_{U, V \in C_{2n}, c \in S^{d \times m - 1}} \langle V, 2(\Sigma_X - \frac{X^TX}{n})U\Omega_s\rangle \|\Gamma_t - \Gamma_*\|_F.$$  

(99)

We rearrange the random process $X_{U,V} - X_{Z,W}$ as

$$X_{U,V} - X_{Z,W} = \langle V, 2(\Sigma_X - \frac{X^TX}{n})U\Omega_s\rangle - \langle W, 2(\Sigma_X - \frac{X^TX}{n})Z\Omega_s\rangle$$

$$= E[\frac{u}{n}\text{vec}(X^T)^T(I_n \otimes (U\Omega_sV^T - Z\Omega_sW^T))\text{vec}(X^T)] - \frac{2}{n}\text{vec}(X^T)^T(I_n \otimes (U\Omega_sV^T - Z\Omega_sW^T))\text{vec}(X^T).$$

(100)

From the facts

$$U\Omega_sV^T - Z\Omega_sW^T = (U - Z)\Omega_sV^T + Z\Omega_s(V - W)^T$$

(101)

and

$$\|U\Omega_sV^T - Z\Omega_sW^T\|_F^2 \leq 2\|\Omega_s\|^2\|U - Z\|_F^2 + 2\|\Omega_s\|^2\|V - W\|_F^2,$$

(102)
we could derive the mixed tail according to Lemma 1
\[
P( \| (V, 2(\Sigma - \frac{X^TX}{n})U\Omega_\ast ) - (W, 2(\Sigma - \frac{X^TX}{n})Z\Omega_\ast ) \| > u )
\leq 2 \exp(-c \min( \frac{u^2}{\| \Sigma \| \| \Omega_\ast \| \| (\frac{\sqrt{\omega T} + u}{\sqrt{n}}) \|_F^2 }, \frac{2u^2}{\| \Sigma \| \| \Omega_\ast \| \| (\frac{\sqrt{\omega T} + u}{\sqrt{n}}) \|_F} )).
\] (103)
Combining with Lemma 3, we have the following lemma.

**Lemma 18.** When \( n \geq (\omega T + u)^2 \), we could derive
\[
P( \sup_{U, V \in \mathbb{C}^{d \times m - 1}} \| (V, 2(\Sigma - \frac{X^TX}{n})U\Omega_\ast ) \| > C\| \Sigma \| \| \Omega_\ast \| (\frac{\omega T + u}{\sqrt{n}}) \|_F \leq 2 \exp(-u^2). \] (104)
The random process associated with the third item of (87) could be written as
\[
\| \frac{2}{n} X^T E(\Omega_\ast - \Omega_\ast) \|_F \leq \sup_{V \in \mathbb{C}^{d \times m - 1}} \langle V, \frac{2}{n} X^T EP \rangle \| \Omega_\ast - \Omega_\ast \|_F \] (105)
The random process \( X_{V, P} - X_{Z, Q} \) could be rearranged as
\[
X_{V, P} - X_{Z, Q} = \langle V, \frac{2}{n} X^T EP \rangle - \langle Z, \frac{2}{n} X^T EQ \rangle = \frac{2}{n} \text{vec}(E^T)(I_n \otimes (PV^T - QZ^T))\text{vec}(X^T).
\] (106)
From the facts
\[
PV^T - QZ^T = (P - Q)V^T + Q(V - Z)^T
\] (107)
and
\[
\| PV^T - QZ^T \|_F^2 = 2\| P - Q \|_F^2 + 2\| V - Z \|_F^2,
\] (108)
we could derive the mixed tail according to Lemma 2
\[
P( \| (V, \frac{2}{n} X^T EP) - (Z, \frac{2}{n} X^T EQ) \| > t )
\leq 2 \exp(-c \min( \frac{t^2}{\| \Omega_\ast \| \| \Omega_\ast \| \| (\frac{\omega T + u}{\sqrt{n}}) \|_F^2 }, \frac{2t^2}{\| \Omega_\ast \| \| \Omega_\ast \| \| (\frac{\omega T + u}{\sqrt{n}}) \|_F} )).
\] (109)
Combining with Lemma 3, we have the following lemma.

**Lemma 19.** Under the condition of \( n \geq (\omega T + \omega \Omega + u)^2 \), we could derive
\[
P( \sup_{V \in \mathbb{C}^{d \times m - 1}} \| (V, \frac{2}{n} X^T EP) \| > C\| \Omega_\ast \| \| (\frac{\omega T + \omega \Omega + u}{\sqrt{n}}) \|_F \leq 2 \exp(-u^2). \] (110)
The random process associated with the fourth item of (87) could be written as
\[
\| \frac{2}{n} X^T E\Omega_\ast \|_F \leq \sup_{V \in \mathbb{C}^{d \times m - 1}} \langle V, \frac{2}{n} X^T EP \rangle \| \Omega_\ast \|_F \] (111)
We arrange the random process \( X_V - X_Z \) as
\[
X_V - X_Z = \langle V, \frac{2}{n} X^T EP \rangle - \langle Z, \frac{2}{n} X^T EQ \rangle = \frac{2}{n} \text{vec}(E^T)(I_n \otimes (\Omega_\ast V^T - \Omega_\ast Z^T))\text{vec}(X^T).
\] (112)
Then we could derive the mixed tail according to Lemma 2
\[
P( \| (V, \frac{2}{n} X^T EP) - (Z, \frac{2}{n} X^T EQ) \| > u )
\leq 2 \exp(-c \min( \frac{u^2}{\| \Omega_\ast \| \| \Omega_\ast \| \| (\frac{\omega T + \omega \Omega + u}{\sqrt{n}}) \|_F^2 }, \frac{2u^2}{\| \Omega_\ast \| \| \Omega_\ast \| \| (\frac{\omega T + \omega \Omega + u}{\sqrt{n}}) \|_F} )),
\] (113)
where we use the fact \( \text{vec}(E\Omega_\ast^T) \sim \mathcal{N}(0, I_n \otimes \Omega_\ast) \) under Assumption 1.
Combining with Lemma 3, we have the following lemma.

**Lemma 20.** Under the condition of \( n \geq (\omega T + u)^2 \), we could derive
\[
P( \sup_{V \in \mathbb{C}^{d \times m - 1}} \| (V, \frac{2}{n} X^T EP) \| > C\| \Omega_\ast \| \| (\frac{\omega T + u}{\sqrt{n}}) \|_F \leq 2 \exp(-u^2). \] (114)
Taking Lemma 17, 18, 19 and 20 into consideration, we could derive the event

$$\|\nabla_f(\Gamma_t, \Omega_t) - \nabla_f n(\Gamma_t, \Omega_t)\|_F \leq C(\|X\|_{\omega} + \|\Omega\|_{\omega} + ||X||_T) + \|\Omega_{\star}||_F + \|\Sigma_{\star}||_F (\|\nabla_{\Gamma} + u\|_F) + \|\Sigma_{\star}||_F + (\|\Omega_{\star}||_F + ||\Sigma_{\star}||_F)$$

$$\leq C(\tau R \frac{\omega_{\star} + \|\Omega\|_{\omega} + u}{\sqrt{\tau}}) ||\Omega_t - \Omega_{\star}||_F + \tau R \frac{\omega_{\star} + \|\Omega\|_{\omega} + u}{\sqrt{\tau}} + \tau \frac{3}{n} \|\Omega_{\star}||_F (\|\nabla_{\Gamma} + u\|_F) + \tau \frac{3}{n} \|\Omega_{\star}||_F (\|\nabla_{\Gamma} + u\|_F)$$

holds with probability at least $1 - 8 \exp(-u^2)$, when $n \geq (\omega_{\star} + \|\Omega\|_{\omega} + u)^2$. Here we use Assumption 1, 2 and max$(||\Gamma_t - \Gamma_{\star}||_F, ||\Omega_t - \Omega_{\star}||_F) \leq R$.

**D. Proof of Lemma 10**

We first rewrite $\nabla f(\Gamma_t, \Omega_t) - \nabla f n(\Gamma_t, \Omega_t)$ as

$$\nabla f(\Gamma_t, \Omega_t) - \nabla f n(\Gamma_t, \Omega_t) = (\Gamma_t - \Gamma_{\star})^T (\Sigma X - \frac{X^T X}{n})(\Gamma_t - \Gamma_{\star}) + \frac{2}{n} (\Gamma - \Gamma_{\star})^T X T E + (\Omega_{\star}^{-1} - \frac{1}{n} E T E).$$

(115)

With the definition of $C_{2n}$, we could derive

$$\|\nabla f(\Gamma_t, \Omega_t) - \nabla f n(\Gamma_t, \Omega_t)\|_F \leq \sup_{V \subset C_{2n} \cap S_{m \times m - 1}} \|V, \nabla f(\Gamma_t, \Omega_t) - \nabla f n(\Gamma_t, \Omega_t)\|_F,$$

(116)

where we use the fact Card$(T) \leq 2s$.

In this way, to bound $\|\nabla f(\Gamma_t, \Omega_t) - \nabla f n(\Gamma_t, \Omega_t)\|_F$, we need to deal with the suprema of random processes.

The random process associated with the first item of (115) could be written as

$$\|(\Gamma_t - \Gamma_{\star})^T (\Sigma X - \frac{X^T X}{n})(\Gamma_t - \Gamma_{\star})\|_F \leq \sup_{U \subset C_{2n} \cap S_{d \times d - 1}} \|V, U^T (\Sigma X - \frac{X^T X}{n})U\|_F.$$
The random process $X_U, V - X_{W, Z}$ could be rearranged as

$$X_U, V - X_{W, Z} = \text{vec}(X^T)^T(I_n \otimes (UV^T - WZ^T))\text{vec}(E^T).$$

From the fact

$$UV^T - WZ^T = (U - W)V^T + W(V - Z)^T,$$

we could derive the mixed tail according to Lemma 2

$$P(|(V, 2n^{-1}E^TXU) - (Z, 2n^{-1}E^TXW)| > u) \leq 2\exp(-c\min(\frac{n^2}{2}\|\Omega_x^\perp\|^2\|\Sigma_X^\perp\|\|V\|_F^2 - \frac{u}{\sqrt{n}}\|\Omega_x^\perp\|^2\|\Sigma_X^\perp\|\|V\|_F - (\frac{W}{Z})_F^2)).$$

Combining with Lemma 3, we have the following lemma.

**Lemma 22.** When $n \geq (\omega_T + \omega_\Omega + u)^2$, we could derive

$$P\left(\sup_{U \in C_{2m^2} \cap S^{m \times n-1}} |\langle V, 2n^{-1}E^TXU \rangle| > C\|\Sigma_x^\perp\|\|\Omega_x^\perp\|\|\Omega_x^\perp\|^2\|\Sigma_x^\perp\|\langle \omega_T + \omega_\Omega + u \rangle \right) \leq 2\exp(-u^2).$$

The random process associated with the third item of (115) could be written as

$$\|\langle \Omega_x^\perp - \frac{1}{n}E^T E \rangle \|_F \leq \sup_{V \in C_{2m^2} \cap S^{m \times n-1}} \langle V, \Omega_x^\perp - \frac{1}{n}E^T E \rangle.$$

The random process $X_U - X_Z$ could be rearranged as

$$X_U - X_Z = E\left(\frac{1}{n}\text{vec}(E^T)^T(I_n \otimes (V^T - Z^T))\text{vec}(E^T)\right) - \frac{1}{n}\text{vec}(E^T)^T(I_n \otimes (V^T - Z^T))\text{vec}(E^T).$$

We could derive the mixed tail according to Lemma 1

$$P(|(V - Z, \Omega_x^\perp - \frac{1}{n}E^T E)| > u) \leq 2\exp(-c\min(\frac{n^2}{2}\|\Omega_x^\perp\|^2\|\Sigma_x^\perp\|\|V - Z\|_F^2 - \frac{u}{\sqrt{n}}\|\Omega_x^\perp\|^2\|\Sigma_x^\perp\|\|V - Z\|_F^2)).$$

Combining with Lemma 3, we have the following lemma.

**Lemma 23.** When $n \geq (\omega_T + \omega_\Omega + u)^2$, we could derive

$$P\left(\sup_{V \in C_{2m^2} \cap S^{m \times n-1}} |\langle V, \Omega_x^\perp - \frac{1}{n}E^T E \rangle| > C\|\Sigma_x^\perp\|\|\Omega_x^\perp\|^2\|\Sigma_x^\perp\|\langle \omega_T + \omega_\Omega + u \rangle \right) \leq 2\exp(-u^2).$$

Taking Lemma 21, 22 and 23 into consideration, we could derive the event

$$\|\langle \Sigma_x^\perp \Omega_x^\perp - \frac{1}{n}\text{vec}(E^T)^T(I_n \otimes (V^T - Z^T))\text{vec}(E^T) \rangle \|_F \leq C(\|\Sigma_x^\perp\|\|\Omega_x^\perp\|^2\|\Sigma_x^\perp\|\|\Omega_x^\perp\|\|\Omega_x^\perp\|^2\|\Sigma_x^\perp\|\langle \omega_T + \omega_\Omega + u \rangle \|\|\Omega_x^\perp\|^2\|\Sigma_x^\perp\|\|\Omega_x^\perp\|\|\Omega_x^\perp\|^2\|\Sigma_x^\perp\|\langle \omega_T + \omega_\Omega + u \rangle \langle \omega_T + \omega_\Omega + u \rangle)$$

$$\leq C(\tau R(\|\Sigma_x^\perp\|\|\Omega_x^\perp\|^2\|\Sigma_x^\perp\|\|\Omega_x^\perp\|\|\Omega_x^\perp\|^2\|\Sigma_x^\perp\|\langle \omega_T + \omega_\Omega + u \rangle \|\|\Omega_x^\perp\|^2\|\Sigma_x^\perp\|\|\Omega_x^\perp\|\|\Omega_x^\perp\|^2\|\Sigma_x^\perp\|\langle \omega_T + \omega_\Omega + u \rangle \langle \omega_T + \omega_\Omega + u \rangle))$$

hold with probability at least $1 - 6\exp(-u^2)$, when $n \geq (\omega_T + \omega_\Omega + u)^2$. Here we use Assumption 1, 2 and $\|\Gamma_t - \Gamma_s\|_F \leq R$.

**E. Proof of Lemma 11**

From the optimality of $\Gamma_{in}$, we could derive

$$\frac{1}{2}\|Y - X\Gamma_{in}\|_F^2 \leq \frac{1}{2}\|Y - X\Gamma_s\|_F^2$$

$$\frac{1}{2n}\|X(\Gamma_{in} - \Gamma_s)\|_F^2 \leq \frac{1}{n}(E, X(\Gamma_{in} - \Gamma_s)).$$

The left hand of (132) could be rewritten as

$$\frac{1}{2n}\|X(\Gamma_{in} - \Gamma_s)\|_F^2 = \frac{1}{2n}\langle U, X^T XU \rangle\|\Gamma_{in} - \Gamma_s\|_F^2,$$

where $U \in C_{2m^2} \cap S^{m \times n-1}$. Here we use the fact $\|\text{vec}(\Gamma_{in} - \Gamma_s)^T\|_0 \leq 2s^G$. 
Then we illustrate the random process $X_U = \langle U, (\Sigma_X - \frac{X^T X}{n}) U \rangle$ has a mixed tail.

\[
X_U - X_W = \mathbb{E}[\frac{1}{n} \text{vec}(X^T)^T (I_n \otimes (UU^T - WW^T)) \text{vec}(X^T)] - \frac{1}{n} \text{vec}(X^T)^T (I_n \otimes (UU^T - WW^T)) \text{vec}(X^T).
\]

From the fact

\[
UU^T - WW^T = \frac{1}{2}(U + W)(U - W)^T + \frac{1}{2}(U - W)(U + W)^T,
\]

we could derive

\[
P(\sup_{U \in C_{2n} \cap S^{d \times m-1}} |\langle U, (\Sigma_X - \frac{X^T X}{n}) U \rangle - \langle W, (\Sigma_X - \frac{X^T X}{n}) W \rangle| > u) \leq 2 \exp(-c \min(\frac{u^2}{\frac{1}{n} \|\Sigma_X\|^2 \|U - W\|^2_F}, \frac{u}{\frac{1}{n} \|\Sigma_X\| \|U - W\|^2_F}))
\]

where we use Lemma 1. Then we could derive the following statement by Lemma 3.

**Lemma 24.** When $n \geq (\omega n + u)^2$, we could derive

\[
P(\sup_{U \in C_{2n} \cap S^{d \times m-1}} \|\langle U, (\Sigma_X - \frac{X^T X}{n}) U \rangle - \langle W, (\Sigma_X - \frac{X^T X}{n}) W \rangle\| \|\Sigma_X\| \|U - W\|^2_F) \leq 2 \exp(-u^2).
\]

From the above lemma we could derive

\[
\frac{1}{2n} \|X(\Gamma_{mi} - \Gamma_*)\|^2_\mathbb{F} \geq \frac{1}{2}(\lambda_{\min}(\Sigma_X) - C\lambda_{\max}(\Sigma_X)\omega n + u(\Sigma_X)^{1/2})\|\Gamma_{mi} - \Gamma_*\|^2_\mathbb{F}
\]

with probability at least $1 - 2 \exp(-u^2)$.

The right hand of (132) could be rewritten as

\[
\frac{1}{n} \langle E, X(\Gamma_{mi} - \Gamma_*) \rangle = \frac{1}{n} \langle V, X^T E \rangle \|\Gamma_{mi} - \Gamma_*\|^2_\mathbb{F}
\]

where $V \in C_{2n} \cap S^{d \times m-1}$. Then we illustrate the random process $X_V = \frac{1}{n} \langle V, X^T E \rangle$ has a mixed tail.

\[
X_V - X_Z = \frac{1}{n} \langle V, X^T E \rangle - \frac{1}{n} \langle Z, X^T E \rangle = \frac{1}{n} \text{vec}(E^T)^T (I_n \otimes (V^T - Z^T)) \text{vec}(X^T).
\]

With Lemma 2 and Lemma 3, we could derive

\[
P(\sup_{V \in C_{2n} \cap S^{d \times m-1}} \|\langle V, \frac{1}{n} X^T E \rangle\| \|\Sigma_X\| \|\Sigma_X^{-1/2}\| \|\Sigma_X^{1/2}\| \|U - W\|_F) \leq 2 \exp(-u^2).
\]

and the following lemma.

**Lemma 25.** Under the condition of $n \geq (\omega n + u)^2$, we could derive

\[
P(\sup_{V \in C_{2n} \cap S^{d \times m-1}} |\langle V, \frac{1}{n} X^T E \rangle| \|\Sigma_X\| \|\Sigma_X^{-1/2}\| \|\Sigma_X^{1/2}\| \|U - W\|_F) \leq 2 \exp(-u^2).
\]

Taking the two processes into consideration, we could derive

\[
\|\Gamma_{mi} - \Gamma_*\|_\mathbb{F} \leq C\lambda_{\max}(\Sigma_X^{1/2}) \|\Sigma_X\| \|\Sigma_X^{-1/2}\| \|\Sigma_X^{1/2}\| \|U - W\|_F
\]

\[
\leq C\lambda_{\max}(\Sigma_X^{1/2}) \|\Sigma_X\| \|\Sigma_X^{-1/2}\| \|\Sigma_X^{1/2}\| \|U - W\|_F
\]

\[
\leq C\lambda_{\min}(\Sigma_X) \|\Sigma_X^{-1/2}\| \|U - W\|_F
\]

\[
\leq C\omega n + u \sqrt{\frac{\|\Gamma_{mi} - \Gamma_*\|_\mathbb{F}}{\|\Sigma_X\|}},
\]

with probability at least $1 - 4 \exp(-u^2)$, when $n \geq C\tau^4(\omega n + u)^2$. Here, we use Assumption 1 and 2.
Lemma 26. The event
\[ \| \Omega_\ast^\prime - S \|_F \leq C \tau^2 \sqrt{m + \omega T + u} \]  
holds with probability at least \( 1 - 12 \exp(-u^2) \), when \( n \geq C \tau^4 (m + \omega T + u)^2 \).

Our method to bound \( \| S^{-1} \| \) is inspired by [17]. To upper bound \( \| S^{-1} \| \), we need to lower bound \( \lambda_{\min}(S) \).

Lemma 27. The event
\[ \lambda_{\min}(S) \geq c \nu \]  
holds with probability \( 1 - 10 \exp(-u^2) \), when \( n \geq C \tau^4 \nu^2 (\sqrt{m} + \omega T + u)^2 \).

Then we could derive \( \| S^{-1} \| \leq C \nu \).

Considering the two above lemmas, we derive the final result.

G. Proof of Lemma 26

The item \( \| \Omega_\ast^\prime - S \|_F \) could be rewritten as
\[
\| \Omega_\ast^\prime - S \|_F = \| (\Gamma_\ast - \Gamma_\ast') (\Sigma X (\Gamma_\ast - \Gamma_\ast') \Gamma_\ast - \Gamma_\ast') \|_F \\
+ \| (\Gamma_\ast - \Gamma_\ast') \Sigma X (\Gamma_\ast - \Gamma_\ast') \|_F \\
= \sup_{V \in S^{m \times m-1}} \left[ \langle V, (\Gamma_\ast - \Gamma_\ast') (\Sigma X (\Gamma_\ast - \Gamma_\ast') \Gamma_\ast - \Gamma_\ast') \|_F \right] \\
+ \| (\Gamma_\ast - \Gamma_\ast') \Sigma X (\Gamma_\ast - \Gamma_\ast') \|_F.
\]

We still bound these items by the generic chaining.

From the facts
\[
UV^T U^T - WZ^T W^T = \frac{1}{2} (U + W) V^T (U - W) + \frac{1}{2} (U - W) V^T (U + W) + W (V - Z)^T W^T
\]
and
\[
\| UV^T U^T - WZ^T W^T \|_F^2 \leq 8 \| U - W \|_F^2 + 2 \| V - Z \|_F^2,
\]
we could derive the mixed tail according to Lemma 1
\[
P(\| (V, U^T X^T W^T U) - \mathbb{E}(V, U^T X^T W^T U) \| > u) \leq 2 \exp(-c \min(\frac{u^2}{\nu^2 \| \Sigma \|_F^2}, \frac{u^2}{\omega T + u} \| \Sigma X \|_F, \| \Sigma \|_F^2)) \)
\]
Then the supremum of the random process could be bounded as
\[
P(\sup_{U \in S^{m \times m-1}} \| (V, U^T X^T W^T U) - \mathbb{E}(V, U^T X^T W^T U) \| > C \| \Sigma \|_F (\frac{m + \omega T + u}{\sqrt{n}})) \leq 2 \exp(-u^2),
\]
when \( n \geq (m + \omega T + u)^2 \), according to Lemma 3.
Following the procedure of Lemma 22, the second and third items could be bounded as
\[ P\left( \sup_{U \in C_{2^m} \cap S^{d \times m-1}} \sup_{V \in S^{m \times m-1}} |\langle V, U^T X^T E \rangle| > C \| \Omega_{\alpha}^{-1} \| \Sigma X^2 \frac{m + \omega + u}{\sqrt{n}} \right) \leq 2 \exp(-u^2), \]  
when \( n \geq (m + \omega + u)^2 \).

Following the procedure of Lemma 23, the fourth item could be bounded as
\[ P\left( \sup_{V \in S^{m \times m-1}} |\langle V, \frac{E^T E}{n} - \Omega_{\alpha}^{-1} \rangle| > C \| \Omega_{\alpha}^{-1} \| \Sigma X \frac{m + u}{\sqrt{n}} \right) \leq 2 \exp(-u^2), \]
when \( n \geq (m + u)^2 \).

The last determined item could be bounded as
\[ \| (\Gamma_{\alpha} - \Gamma_{\alpha}^*)^T \Sigma X (\Gamma_{\alpha} - \Gamma_{\alpha}^*) \|_F \leq \| \Sigma X \| \| \Gamma_{\alpha} - \Gamma_{\alpha}^* \|_F^2. \]

Taking all items into consideration, we could derive the event
\[ \| \Omega_{\alpha}^{-1} - S \|_F \leq C(\| \Sigma X \| (1 + \frac{m + \omega + u}{\sqrt{n}}) \| \Gamma_{\alpha} - \Gamma_{\alpha}^* \|_F + \| \Omega_{\alpha}^{-1} \| \Sigma X \frac{m + \omega + u}{\sqrt{n}} \| \Gamma_{\alpha} - \Gamma_{\alpha}^* \|_F) \]
\[ \leq C(\tau^4 \nu (1 + \frac{m + \omega + u}{\sqrt{n}})(\omega + u)^2 + \tau^2 \nu \frac{m + \omega + u + \omega + u}{\sqrt{n}} + \nu \frac{m + u}{\sqrt{n}}) \]
\[ \leq C\tau^2 \nu \frac{m + \omega + u}{\sqrt{n}} \]
holds with probability at least \( 1 - 12 \exp(-u^2) \), when \( n \geq C \tau^4 (m + \omega + u)^2 \).

\[ \text{H. Proof of Lemma 27} \]

We could rewrite \( v^T S v \) as
\[ v^T S v = v^T (\Gamma_{\alpha} - \Gamma_{\alpha}^*)^T X^T X (\Gamma_{\alpha} - \Gamma_{\alpha}^*) - 2(\Gamma_{\alpha} - \Gamma_{\alpha}^*)^T X^T E \frac{E^T E}{n} \]
\[ + (\Gamma_{\alpha} - \Gamma_{\alpha}^*)^T \Sigma X (\Gamma_{\alpha} - \Gamma_{\alpha}^*) \]
\[ \geq v^T (\Gamma_{\alpha} - \Gamma_{\alpha}^*)^T X^T X (\Gamma_{\alpha} - \Gamma_{\alpha}^*) - 2(\Gamma_{\alpha} - \Gamma_{\alpha}^*)^T X^T E \frac{E^T E}{n} - \Omega_{\alpha}^{-1} \]
where we use the fact that \((\Gamma_{\alpha} - \Gamma_{\alpha}^*)^T \Sigma X (\Gamma_{\alpha} - \Gamma_{\alpha}^*)\) is positive semidefinite.

We need to deal with three random processes. The first item is bound by the following lemma.

**Lemma 28.** The event
\[ \inf_{U \in C_{2^m} \cap S^{d \times m-1}} \sup_{V \in S^{m \times m-1}} v^T U U^T \frac{X^T X}{n} - \Sigma X U v \| \Gamma_{\alpha} - \Gamma_{\alpha}^* \|_F^2 \geq -C \| \Sigma X \| \frac{\sqrt{m + \omega + u}}{\sqrt{n}} \| \Gamma_{\alpha} - \Gamma_{\alpha}^* \|_F^2 \]
holds with probability \( 1 - 2 \exp(-u^2) \), when \( n > (\sqrt{m} + \omega + u)^2 \).

The second item could be rewritten as
\[ v^T (\Gamma_{\alpha} - \Gamma_{\alpha}^*)^T X^T E \frac{E^T E}{n} v = v^T U U^T \frac{X^T E}{n} v \| \Gamma_{\alpha} - \Gamma_{\alpha}^* \|_F^2, \]
where \( U \in C_{2^m} \cap S^{d \times m-1} \).

We could rearrange \( X_{U, v} - X_{W, z} \) as
\[ X_{U, v} - X_{W, z} = v^T U U^T X^T E \frac{E^T E}{n} v - z^T W T \frac{X^T E}{n} z = \frac{1}{n} \text{vec}(E^T)(I_n \otimes (v v^T U^T - z z^T W^T)) \text{vec}(X^T). \]
From the facts
\[ v v^T U^T - z z^T W^T = \frac{1}{2}(v + z)(v - z) U^T + \frac{1}{2}(v - z)(v + z) U^T + z z^T (U - W) \]
(153)}
and
\[ \| vv^T U^T - zz^T W^T \|_F^2 \leq 8 \| v - z \|_2^2 + 2 \| U - W \|_F^2 \leq 8 \| ( U^T ) - ( W^T ) \|_F^2, \] (154)
we could derive the mixed tail according to Lemma 2
\[ P( \frac{1}{n} \vec(E^T) ( I_n \otimes ( vv^T U^T - zz^T W^T )) \vec(X^T) ) > u ) \]
\[ \leq 2 \exp(- \min( \frac{u^2}{n \| \Sigma X \|_F^2 \| \Omega_* \|_2^2 \| \Omega_* \|_2^2 \| ( U^T ) - ( W^T ) \|_F^2, \frac{2 \| z \|_2 \| \Sigma X \| \Omega_* \|_2^2 \| ( U^T ) - ( W^T ) \|_F^2 }{n \| \Sigma X \|_F^2 \| \Omega_* \|_2^2 \| ( U^T ) - ( W^T ) \|_F^2 ) } ). \] (155)

Then we could derive from Lemma 3
\[ P( \sup_{ U \in C_{21}, v \in S^{d \times m - 1} } \| vv^T U^T X^T E \|_n > C \| \Sigma X \| \Omega_* \|_2^2 \| \sqrt{m + \omega_T + u} \|_n ) \leq 2 \exp(-u^2), \] (156)
when \( n > ( \sqrt{m} + \omega_T + u )^2 \).

Now we deal with the third item. From the facts
\[ vv^T - zz^T = \frac{1}{2} ( v + z ) ( v - z )^T + \frac{1}{2} ( v - z ) ( v + z )^T \] (157)
and
\[ \| vv^T - zz^T \|_F^2 \leq 4 \| v - z \|_2^2, \] (158)
we could get the mixed tail according to Lemma 1
\[ P( \| vv^T ( \frac{1}{n} E^T E - \Omega_*^{-1} ) v + z^T ( \frac{1}{n} E^T E - \Omega_*^{-1} ) z \| > u ) \leq 2 \exp(- \min( \frac{u^2}{n \| \Sigma X \|_F^2 \| \Omega_* \|_2^2 \| \Omega_* \|_2^2 \| ( U^T ) - ( W^T ) \|_F^2 }{n \| \Sigma X \|_F^2 \| \Omega_* \|_2^2 \| ( U^T ) - ( W^T ) \|_F^2 ) } ). \] (159)
Then we could derive
\[ P( \sup_{ v \in S^{d - 1} } \| vv^T ( \frac{1}{n} E^T E - \Omega_*^{-1} ) v \| > C \| \Omega_*^{-1} \| \sqrt{m + u} \|_n ) \leq 2 \exp(-u^2), \] (160)
when \( n > ( \sqrt{m} + u )^2 \), according to Lemma 3.

Taking all parts into consideration, we could derive
\[ vv^T S v \]
\[ \geq - C \| \Sigma X \|_F \| \Gamma_{i+1} - \Gamma_* \|_F^2 - 2 C \| \frac{ \sqrt{m + \omega_T + u} \| \Sigma X \|_F \| \Omega_* \|_2 \| \Gamma_{i+1} - \Gamma_* \|_F - C \| \Omega_*^{-1} \| \sqrt{m + u} \|_n \]
\[ + \lambda_\min( \Omega_*^{-1} ) \] (161)
\[ \geq - C T^4 \| \frac{ \sqrt{m + \omega_T + u} \|_2^2 }{ n } - C T^2 \| \frac{ \sqrt{m + \omega_T + u} \|_2^2 }{ n } \]
\[ \geq C \nu, \]
when \( n > C T^4 \nu^4 ( \sqrt{m} + \omega_T + u )^2 \), where we use Lemma 11.

I. Proof of Lemma 28

We could rewrite the item as
\[ \lambda_\min( ( \Gamma_{i+1} - \Gamma_*)^T \frac{ X^T X }{ n } - \Sigma X ) ( \Gamma_{i+1} - \Gamma_* ) ) = \inf_{ v \in S^{d - 1} } v^T U^T \frac{ X^T X }{ n } - \Sigma X U v \| \Gamma_{i+1} - \Gamma_* \|_F^2. \] (161)

From the facts
\[ U \vec{v} v^T U^T - W z z^T W^T = ( U - W ) \vec{v} v^T U^T + \frac{ W ( v + z ) ( v - z )^T U^T }{ 2 } + \frac{ W ( v - z ) ( v + z )^T U^T }{ 2 } + W z z^T ( U - W )^T \] (162)
and
\[ \| U \vec{v} v^T U^T - W z z^T W^T \|_F^2 \leq 6 \| U - W \|_F^2 + 16 \| v - z \|_2^2, \] (163)
we could derive the mixed tail according to Lemma 1
\[ P( \vec{v} ( X^T T ( U \vec{v} v^T U^T - W z z^T W^T ) vec(X) ) - E \vec{v} ( X^T T ( U \vec{v} v^T U^T - W z z^T W^T ) vec(X) ) ) > u ) \]
\[ \leq 2 \exp(- \min( \frac{ u^2 }{ 4 \| \Sigma X \|_F^2 \| ( U^T ) - ( W^T ) \|_F^2 + \frac{ u }{ 4 \| \Sigma X \| \| ( U^T ) - ( W^T ) \|_F^2 ) } ), \] (164)
Then we could derive
\[
P\left( \sup_{U \in \mathbb{C}_{m \times 1}, \Sigma \in \mathbb{S}_{m \times m}^{+}} |v^T U^T X \frac{X^T X}{n} U - E[v^T U^T X^T X U]| > C \Sigma \frac{\sqrt{m + \omega_r + u}}{\sqrt{n}} \right) \leq 2 \exp(-u^2),
\]
when \( n > (\sqrt{m + \omega_r + u})^2 \), according to Lemma 3.

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