MINIMAX BOUNDS FOR ESTIMATION OF NORMAL MIXTURES

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Abstract. This paper deals with minimax rates of convergence for estimation of density functions on the real line. The densities are assumed to be location mixtures of normals, a global regularity requirement that creates subtle difficulties for the application of standard minimax lower bound methods. Using novel Fourier and Hermite polynomial techniques, we determine the minimax optimal rate—slightly larger than the parametric rate—under the squared error loss. For the Hellinger loss, we provide a minimax lower bound using ideas modified from squared error loss case.

1. Introduction

This paper establishes the optimal minimax rate of convergence, under squared error loss, for densities that are normal mixtures. The analysis reveals a subtle difficulty in the application of Assouad’s Lemma to parameter spaces defined by indirect regularity conditions, which complicate the usual construction of subsets of the parameter space indexed by ‘hyper-rectangles’.

More precisely, we consider independent observations from probability distributions $P_f$ on the real line whose densities $f$ (with respect to Lebesgue measure) belong to the set of convolutions

$$\mathcal{F} = \left\{ f(x) = \phi \ast \Pi(x) = \int \phi(x-u)d\Pi(u) : \Pi \in \mathcal{P}(\mathbb{R}) \right\}$$

where $\phi$ denotes the standard normal N(0,1) density and $\mathcal{P}(\mathbb{R})$ denotes the set of all probability measures on the (Borel sigma-field of the) real line. The main result gives an asymptotic minimax lower bound for estimators $\hat{f}_n$ based on $n$ independent observations from a density $f$ in $\mathcal{F}$.

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Theorem 1.1. There exists a positive constant $c$ such that
\[
\sup_{f \in \mathcal{F}} \mathbb{E}_{n,f} \int_{-\infty}^{\infty} \left( \hat{f}_n(x) - f(x) \right)^2 \, dx \geq c \cdot \log n \cdot \frac{1}{n \sqrt{\log n}} := c\ell_n
\]
for every estimator sequence $\{\hat{f}_n\}$.

Zhang (1997) has established that the sinc kernel estimator attains the maximum expected loss as of order $O(l_n)$ over a class of normal location mixtures under the empirical Bayes setting. More precisely, under the assumption that $Y_i$ is a random variable with a density $f(y|\theta_i)$ given $\theta_i$ for $i = 1, \ldots, n$, Theorem 2 in Zhang (1997) proves that there exists constant $B_p$ depending on $p$ only, for which
\[
\left( \mathbb{E} \int \left( \hat{f}_n^{(s)}(x) - f_n^{(s)}(x) \right)^{2p} \, dx \right)^{1/p} \leq \frac{(B_{2p}^2 + o(1))(\sqrt{\log n})^{2s+1}}{\pi(2s+1)n}
\]
where $f_n^{(s)}(y) = \int \phi(y - u)dG_n(u)$ for which $s \geq 0$ is an integer and $G_n(x) := (1/n) \sum_{i=1}^{n} P(\theta_i \leq x)$ by using an estimator
\[
\hat{f}_n(x) = \frac{1}{n} \sum_{i=1}^{n} K_{\sqrt{\log n}}(Y_i - x)
\]
where $K_a(x) = \sin(ax)/\pi x$ for $x \neq 0$ and $K_a(x) = a/\pi$ for $x = 0$. Following the exact same proof, we can show the following,
\[
\sup_{f \in \mathcal{F}} \mathbb{E}_{n,f} \int_{-\infty}^{\infty} \left( \hat{f}_n(x) - f(x) \right)^2 \, dx = O(l_n)
\]
with the same sinc kernel estimator using an i.i.d random sample $X_1, \ldots, X_n$ from a density $f \in \mathcal{F}$.

Thus, Theorem 1.1 combined with this upper bound determines the optimal minimax rate for the problem of estimating a density in the class $\mathcal{F}$. The minimax rate $l_n$ reveals the difficulty of estimating $f$ in $\mathcal{F}$ in a sense that no estimator can achieve faster rate than $l_n$ in a worst case. That is, estimating a density in $\mathcal{F}$ under the $L_2$ loss is slightly more difficult than the parametric problem such as estimating a normal density with unknown mean. Actually, the same difficulty has been shown for larger classes of functions. For instance, Ibragimov (2001) proves the optimal minimax rate $l_n$ (in Theorem 4.1) for the class of analytic functions with a certain growth condition
\[
\left\{ f : \sup_{x} |f(x + iy)| \leq M \exp(c|y|^2) \right\}
\]
which includes the class $\mathcal{F}$ of normal location mixtures. Moreover, Efromovich (2008, Theorem 5.2) proves the sharp minimax risk as $(1/(\pi n))(\log n/(2\gamma))^{1/2}(1+o(1))$ for the infinitely differentiable class

$$\left\{ f : \frac{1}{\pi} \int_0^\infty |e^{\gamma u^2} h(u)|^2 du \leq Q, \ h(u) = \int_{-\infty}^\infty f(x)e^{iux}dx \right\},$$

which contains $\mathcal{F}$. This implies that a subclass $\mathcal{F}$ captures the difficulty in the class of analytic functions (or infinitely differentiable functions).

While minimax result under the $L_2$ loss presents most successful case, this loss is often criticized for giving too little weight to errors from the tails. As an alternative, we also consider the Hellinger loss. For the following class of normal location mixtures

$$\mathcal{F}_s := \left\{ f : f(x) = \phi * \Pi(x) = \int \phi(x-u) d\Pi(u), \ \Pi \in \mathcal{P}_s(\mathbb{R}) \right\}$$

where $\mathcal{P}_s(\mathbb{R})$ is a class of probability measures with sub-Gaussian tails,

$$\mathcal{P}_s(\mathbb{R}) := \{ \Pi, \ \Pi(|u| > t) \leq C \exp(-ct^2) \text{ for all real } t \}$$

with constants $c$ and $C$, Ghosal and van der Vaart (2001) provide a sieved maximum likelihood estimator whose convergence rate is $O((\log n)^2/n)$. However, as they pointed out, the optimal rate for $\mathcal{F}_s$ is still unknown (to the best of my knowledge, there is no lower bound proof under the Hellinger loss) and here we provide one possible lower bound.

**Theorem 1.2.** There exists a positive constant $c$ such that

$$\sup_{f \in \mathcal{F}_s} \mathbb{E}_{n,f} \int_{-\infty}^\infty \left( \sqrt{\hat{f}_n(x)} - \sqrt{f(x)} \right)^2 dx \geq c \cdot \log n \cdot \frac{1}{n}$$

for every estimator sequence $\{\hat{f}_n\}$.

It is interesting to notice that even if $\mathcal{F}_s$ is a subclass of $\mathcal{F}$, the optimal minimax rate under the Hellinger loss is larger than the optimal rate $l_n$ under the $L_2$ loss.

The proofs of theorems, which are given in Section 2, use a variation on Assouad’s lemma (cf. Van der Vaart, 1998, p.347). When specialized to density estimation, the Lemma can be cast into the following form. (Henceforth we omit the $\pm \infty$ terminals on the integrals when there is no ambiguity.) For completeness, we provide the proof in the appendix.
Lemma 1.3. Suppose \( \{f_{\alpha}, \, \alpha \in \{0,1\}^K\} \subseteq \mathcal{F} \) where \( K \) is a finite index set with a cardinality \( m \). For positive constants \( c_0 \) and \( c_1(<1) \), and for a nonnegative loss function \( W \) satisfying
\[
W(f,g_1) + W(f,g_2) \geq \zeta W(g_1,g_2)
\]
for a constant \( \zeta > 0 \), suppose
\[
W(f_{\alpha},f_{\beta}) \geq c_0 \epsilon^2 ||\alpha - \beta||_0 \quad \text{for all} \quad \alpha, \beta \in \{0,1\}^K
\]
and
\[
\int \frac{(f_{\alpha} - f_{\beta})^2}{f_{\alpha}} \leq \frac{c_1}{n} \quad \text{if} \quad ||\alpha - \beta||_0 = 1,
\]
where \( ||\alpha - \beta||_0 = \sum_{k \in K} \{\alpha_k \neq \beta_k\} \), which is the Hamming distance. Then, for every estimator \( \hat{f}_n \) based on \( n \) independent observations,
\[
\sup_{f \in \mathcal{F}} \mathbb{E}_{n,f} W(\hat{f}_n,f) \geq \frac{c_0 \zeta}{4} (1 - \sqrt{c_1}) m \epsilon^2.
\]

Remark 1.1. Assumption (3) regarding the \( \chi^2 \) distance is merely a convenient way to show that the testing affinity, \( ||P^n_{f_{\alpha}} \wedge P^n_{f_{\beta}}||_1 \), is greater than \( 1 - \sqrt{c_1} \), where \( P^n_f \) is a product probability measure under \( f \) and \( ||P \wedge Q||_1 \) is defined as \( \int \min(dP,dQ) \).

Remark 1.2. We try to obtain largest possible \( m \epsilon^2 \) for a better lower bound. While we construct the finite density class satisfying (2), we need to restrict the size \( \epsilon^2 \) and \( m \) so that two densities on the nearest edge should be reasonably close as in (3), and so that the constructed densities are truly in the parameter space \( \mathcal{F} \).

For the proof in Section 2, we construct \( f_{\alpha} \)'s of the form
\[
f_{\alpha}(x) = f_0(x) + \epsilon \sum_{k \in K} \alpha_k \Delta_k(x), \quad \alpha \in \{0,1\}^K
\]
where \( f_0 \) is the normal density function with a zero mean (and variance specified later) and \( K = \{1,3,\ldots,2m-1\} \), and \( m, \epsilon, \) and \( \Delta_k \) could change with \( n \). The main difficulty lies in choosing the (signed) perturbations \( \Delta_k \) so that each \( f_{\alpha} \) is a normal location mixture. The natural way around this problem is to construct the Assouad hyper-rectangle in the space of mixing distributions,
\[
f_{\alpha} = \phi \star \Pi_{\alpha} \quad \text{where} \quad \Pi_{\alpha}(u) = \Pi_0(u) + \epsilon \sum_{k \in K} \alpha_k V_k(u), \quad \alpha \in \{0,1\}^K
\]
where the signed measures $V_k$ must be chosen so that each $\Pi_\alpha$ is a probability measure. In contrast to this standard construction, the indirect from of $f_\alpha = \phi \ast \Pi_\alpha$ leads to an embedding condition like

$$(5) \quad W(\phi \ast \Pi_\alpha, \phi \ast \Pi_\beta) \geq \tau_n \sum_{k \in K} (\alpha_k - \beta_k)^2,$$

for some $\tau_n$. The right side of (5) is expressed with $\sum_{k \in K} (\alpha_k - \beta_k)^2$ instead of the Hamming distance, in order to emphasize orthogonal relation. Traditionally such a property is obtained by choosing the perturbations to be exactly orthogonal to each other, subject to various other regularity properties that define the parameter space. The smoothing effect of the convolution operation creates more complication to choose the $V_k$ to achieve such near-orthogonality. Nevertheless, we achieve (5) by choosing the perturbations so that their Fourier transforms are orthogonal as elements in $L_2(\phi^2)$, the space of complex-valued functions $g$ such that $\int \phi(x)^2|g(x)|^2 dx < \infty$ for the $L_2$ loss. Similarly, we achieve (5) under the Hellinger loss using the similar ideas under $L_2$ except that $\phi^2$ is replaced by other weight function.

2. The proof of Theorems

First, we introduce some notations used in this section. We let $\phi_{\sigma^2}$ be the normal density with mean zero and variance $\sigma^2$. Following [Rudin 1987, chap. 9], we define the Fourier transform $T$ by

$$Tf(t) := \tilde{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-ixt)f(x)dx$$

for $f \in L_1(\lambda)$ then extend from $L_1 \cap L_2$ to $L_2$ by the isometry.

For both theorems, we construct the signed measures $V_k$ to have (signed) densities $v_k$ with respect to Lebesgue measure $\lambda$:

$$(6) \quad \pi_\alpha(u) = \frac{d\Pi_\alpha}{d\lambda}(u) = \pi_0(u) + \epsilon \sum_{k \in K} \alpha_k v_k(u), \quad \alpha \in \{0, 1\}^K$$

where $\pi_0$ is the normal density with zero mean and each $v_k$ is a function for which $\int v_k = 0$ and

$$\pi_0(u) + \epsilon \sum_{k \in K} \alpha_k v_k(u) \geq 0 \quad \text{for all } u.$$

We then need to check the assumptions for Lemma 1.3.
2.1. Ideas in the proof of Theorem 1.1. Here we let \( W(f, g) := \|f - g\|_2^2 = \int (f - g)^2 \), then (1) is satisfied with \( \zeta = 1/2 \). The choice of the \( v_k \)'s is suggested by Fourier methods. By the Plancherel formula (and the fact that \( \hat{\phi} = \phi \)),

\[
\frac{1}{2\pi} \|f_\alpha - f_\beta\|_2^2 = \frac{1}{2\pi} \|f_\alpha - f_\beta\|_2^2 = \epsilon^2 \int_{-\infty}^{\infty} \left| \sum_{k \in K} (\alpha_k - \beta_k) \phi(t) \hat{v}_k(t) \right|^2 dt,
\]

which lets us write the desired property (2) of Lemma 1.3 as

\[
\int_{-\infty}^{\infty} \left| \sum_{k \in K} (\alpha_k - \beta_k) \phi(t) \hat{v}_k(t) \right|^2 dt \geq \frac{c_0}{2\pi} \sum_{k \in K} (\alpha_k - \beta_k)^2, \quad \forall \alpha, \beta \in \{0, 1\}^K,
\]

where we use \( \|\alpha - \beta\|_0 = \sum_{k \in K} (\alpha_k - \beta_k)^2 \). We might achieve such an inequality by choosing the \( v_k \)'s to make the functions \( \psi_k(t) := \phi(t) \hat{v}_k(t) \) orthogonal. Ignoring other requirements for the moment, we could even start from an orthonormal set \( \{\psi_k\} \) then try to define \( v_k \) as the (inverse) Fourier transform of \( \psi_k(t) / \phi(t) \), provided that the ratio is square integrable. This heuristic succeeds if we start from the normalized orthogonal functions (see Jackson, 2004, chap. 9),

\[
\psi_k(t) = Ci^{-k} \phi(t) \frac{H_k(2t)}{\sqrt{k!}} = i^{-k} \sqrt{2\phi(2t)} \frac{H_k(2t)}{\sqrt{k!}}
\]

for \( k \in K := \{1, 3, ..., 2m - 1\} \) where \( C = \sqrt{2(2\pi)^{3/4}} \) is chosen so that \( C\phi(t)^2 = \sqrt{2\phi(2t)} \) and \( H_k(t) \) is the Hermite polynomial of order \( k \), the polynomial for which \( \phi(t) \) has \( k^{th} \) derivative \( (-1)^k H_k(t) \phi(t) \).

Remark 2.1. \( \{H_k, k = 1, 2, ..., \} \) is sometimes called the “probabilists’ Hermite Polynomials” (denoted as ‘He’ in Gradshteyn and Ryzhik (2004)), as opposed to the “physicists’ Hermite Polynomials” \( H \). There is one-to-one relation between \( H \) and \( He \),

\[
H_k(t) = 2^{-k/2} He_k \left( \frac{t}{\sqrt{2}} \right).
\]

To calculate the Fourier inverse transform of \( \psi_k(t) / \phi(t) \), we provide the following lemma.

Lemma 2.1. For \( b > a > 0 \),

\[
\mathcal{T}^{-1} [\phi(at)H_k(bt)] (u) = Q_k \phi \left( \frac{u}{a} \right) H_k \left( \frac{b'u}{2} \right)
\]

where \( Q_k = (ic_{a,b})^k / a \) with \( c_{a,b} = \sqrt{b^2/a^2 - 1} \) and \( b' = b/(a^2c_{a,b}) \).
Remark 2.2. Lemma 2.1 illustrates a general form of the eigenvalue-eigenfunction relation for the Fourier transform of Hermite functions,

\[ T[\phi(t)H_k(\sqrt{2}t)](u) = (-i)^k \phi(u)H_k(\sqrt{2}u). \]

(See (7.376) in Gradshteyn and Ryzhik (2007), or for more details, see §4.11 in Kawata (1972)).

Proof of Theorem 1.1. By Lemma 2.1, this choice for the \( \psi_k \)'s in (7) leads to

(9) \[ v_k(u) = C \sqrt{\frac{3^k}{k!}} \phi(u)H_k \left( \frac{2}{\sqrt{3}} u \right) \] for \( k \in K \)

because \( T^{-1}[\phi(t)H_k(2t)](u) = i^k 3^{k/2} \phi(u)H_k(2u/\sqrt{3}) \). By restricting to odd values of \( k \) we make the \( v_k \)'s real-valued and odd, thereby ensuring that \( \int v_k d\lambda = 0 \) and \( \int \pi_\alpha d\lambda = 1 \) for each \( \alpha \) in \( \{0, 1\}^K \).

In summary, the choice of \( v_k \) as in (9) gives

(10) \[ \frac{1}{2\pi} \|f_\alpha - f_\beta\|^2 = e^2 \int_{-\infty}^{\infty} \left( \sum_{k \in K} (\alpha_k - \beta_k) \psi_k(t) \right)^2 dt = e^2 \sum_{k \in K} (\alpha_k - \beta_k)^2. \]

That is, the first condition (2) is satisfied with \( c_0 = 2\pi \).

We still need to check the second condition (3), and also show that \( \epsilon \) can be chosen small enough to make all the \( \pi_\alpha \)'s nonnegative. Actually, we first show that \( \pi_\alpha \geq \pi_0/2 > 0 \) by choosing

(11) \[ \epsilon \leq \frac{1}{16} 3^{-m+1/2} m^{-3/2}, \]

and by choosing \( \pi_0 \) as a normal density with zero mean and variance \( m \). Secondly, we find out the largest size \( m \) of these hypercubes while the two densities \( f_\alpha \) and \( f_\beta \) are close in terms of the \( \chi^2 \) distance as \( O(1/n) \) when there is only one different coordinate between \( \alpha \) and \( \beta \).

To control the denominator in (3), we first show that \( |v_k(u)| \leq C_k \sqrt{m} \pi_0(u) \) where \( C_k = 8 \cdot 3^{k/2} \). By Cramer’s inequality (Gradshteyn and Ryzhik, 2007, eqn. 8.954),

(12) \[ |H_k(u)| \leq \kappa \sqrt{k! \exp(u^2/4)} \] with \( \kappa \approx 1.086. \)
Applying this inequality to (9),

\[
|v_k(u)| \leq \kappa C \frac{3^{k/2}}{2\sqrt{2\pi}} \exp\left(-\frac{1}{6}u^2\right) \leq C_k \phi(u/\sqrt{3})
\]

\[
\leq C_k \phi(u/\sqrt{m}) = C_k \sqrt{m} \pi_0(u)
\]

Using (14), we have

\[
\pi_\alpha(u) = \pi_0(u) + \epsilon \sum_{k \in K} \alpha_k v_k(u) \geq \pi_0(u) - \epsilon \sum_{k \in K} C_k \sqrt{m} \pi_0(u)
\]

\[
\geq \pi_0(u) \left[1 - C_{2m-1} m^{3/2} \epsilon\right]
\]

\[
= \pi_0(u) \left[1 - 8 \cdot 3^{m-1/2} m^{3/2} \epsilon\right] \geq \frac{\pi_0(u)}{2}
\]

by the choice of \(\epsilon\) in (11).

Hence, under the condition (11), \(\phi \ast \Pi_\alpha := f_\alpha \geq f_0/2 := \phi \ast \Pi_0/2\), which implies that the second condition in Lemma 1.3 is rewritten as \(\int (f_\alpha - f_\beta)^2/f_0 \leq c_1/2n\) for \(\alpha\) and \(\beta\) having only one different coordinate. The denominator \(f_0 = \phi \ast \Pi_0\) is again normally distributed with mean zero and variance \(1 + m\) by the choice of \(d\Pi_0/d\lambda := \pi_0 = \mathcal{N}(0, m)\) density.

For convenience, we let \(\alpha_1 \neq \beta_1\) (all the other cases work the same way). By splitting the integral into two regions \(|x| \leq M\sqrt{m}\) and \(|x| > M\sqrt{m}\) with a constant \(M = 8 \log 9\),

\[
\int |f_\alpha - f_\beta|^2/f_0 = \int_{|x| \leq M\sqrt{m}} |f_\alpha - f_\beta|^2/f_0 + \epsilon^2 \int_{|x| > M\sqrt{m}} |f_\phi(x - u)v_1(u)d\lambda|^2/f_0
\]

For the first integral, the denominator is lower bounded under the interval \(|x| \leq M\sqrt{m}\), such as \(f_0(x)\{x| \leq M\sqrt{m} > \exp(-M^2/2)/(2\sqrt{2\pi}\sqrt{m}) := 1/(C^* \sqrt{m})\) where \(C^* := 2\sqrt{2\pi}\exp(M^2/2)\). Then, using the \(L_2\) loss calculation from (10),

\[
\int_{|x| \leq M\sqrt{m}} |f_\alpha(x) - f_\beta(x)|^2/f_0(x) dx \leq C^* \sqrt{m} ||f_\alpha - f_\beta||^2_2 = 2\pi C^* \sqrt{m} e^2.
\]

For the second integral, recall that for any \(k = 1, 3, \ldots, 2m-1\), \(|v_k(u)| \leq C_{2m-1}\phi(u/\sqrt{3}) := C_{2m-1}\sigma_0\phi_{\sigma_0^2}\) with \(\sigma_0 = \sqrt{3}\) as in (13). Using \(C_{2m-1} := 8 \cdot 3^{m-1/2}\) and \(\phi_{1+\sigma_0^2}(x) \leq \frac{1}{2}\).

\[
\int_{|x| > M\sqrt{m}} |f_\phi(x - u)v_1(u)d\lambda|^2/f_0 \leq C_{2m-1}\phi_{\sigma_0^2}(x) \leq \frac{1}{2}.
\]
\[ \sqrt{m} \phi_{1+m}(x), \] with a notation \( R(x) := \{|x| > M \sqrt{m}\}, \]

\[
e^2 \int_{R(x)} \left( \frac{\int (\phi(x-u)v_1(u)d\lambda)^2}{f_0(x)} \right) dx \leq e^2 C_{2m-1}^{2} \sigma_0^2 \int_{R(x)} \left( \frac{\int (\phi(x-u)\phi_0^2(u)d\lambda)^2}{\phi_{1+m}(x)} \right) dx
\]

\[
\leq \sqrt{m}e^2 C_{2m-1}^{2} \sigma_0^2 \int_{R(x)} \phi_{1+\sigma_0^2}(x) dx
\]

\[
= \left( \frac{64}{3} \sqrt{m}e^2 \right) \left( 3^{2m} \int_{R(x)} \phi_{1+\sigma_0^2}(x) dx \right)
\]

\[
\leq \frac{64}{3} \sqrt{m}e^2,
\]

where the last inequality is obtained by the Gaussian tail property with \( \sqrt{m} \gg \sigma_0 := \sqrt{3}, \)

\[
\int_{|x| > M \sqrt{m}} \phi_{1+\sigma_0^2}(x) dx \leq \exp \left( -\frac{1}{8} M^2 m \right) = 3^{-2m}
\]

by choosing \( M^2 = 8 \log 9. \)

Combining these two upper bounds for the integral, we obtain

\[
\int \frac{(f_\alpha - f_\beta)^2}{f_0} \leq \sqrt{me^2} (2\pi C_* + 64/3) = \frac{c_1}{2n}
\]

as long as

\[
\sqrt{me^2} \leq \frac{1}{n} \frac{c_1}{2(2\pi C_* + 64/3)}.
\]

As a consequence, the constructed mixing densities fulfill two requirements in Assouad’s lemma under conditions (11) and (15),

\[
e^2 \leq \min \left( \frac{1}{16^2 3^{-2m+1} m^{-3}}, \frac{1}{n \sqrt{m} 2(2\pi C_* + 64/3)} \frac{c_1}{n} \right).
\]

Following the simplified Assouad’s lemma 1.3, the lower bound is obtained as \( c e^2 m, \)

which is at most \( \min(3^{-2m} m^{-2}, \sqrt{m}/n) \) up to a constant. To find the largest \( me^2, \)

by equating \( 3^{-2m} m^{-2} = \sqrt{m}/n, \) we obtain \( m \) and \( e^2 \) as \( \log n \) and \( 1/(n \sqrt{\log n}) \)

respectively up to a constant, and hence the lower bound is obtained as \( \sqrt{\log n}/n \)

up to a constant. \( \square \)
2.2. Ideas in the proof of Theorem 1.2. Here we let $W(f, g) := ||\sqrt{f} - \sqrt{g}||^2 = \int (\sqrt{f} - \sqrt{g})^2$, then (11) is satisfied with $\zeta = 1$. First we relate the Hellinger distance and the $\chi^2$ distance. That is, suppose we can show $(1/2)\pi_0(u) \leq \pi_\alpha(u) \leq (3/2)\pi_0(u)$ which in turn says $(1/2)f_\alpha(x) \leq f_\alpha(x) \leq (3/2)f_\alpha(x)$ by convolving the standard normal density. Then, using the upper bound for $f_\alpha$ and $f_\beta$,

\[
\int \left( \sqrt{f_\alpha} - \sqrt{f_\beta} \right)^2 = \int \frac{(f_\alpha - f_\beta)^2}{(\sqrt{f_\alpha} + \sqrt{f_\beta})^2} \geq \frac{1}{6} \int \frac{(f_\alpha - f_\beta)^2}{f_\alpha}.
\]

Similarly, the lower bound for $f_\alpha$ would give an upper bound for the testing condition

\[
\int \frac{(f_\alpha - f_\beta)^2}{f_\alpha} \leq 2 \int \frac{(f_\alpha - f_\beta)^2}{f_\beta}.
\]

Thus it would be enough to work with the following quantity

\[
\int \frac{(f_\alpha - f_\beta)^2}{f_0} = \int \left( \frac{f_\alpha}{\sqrt{f_0}} - \frac{f_\beta}{\sqrt{f_0}} \right)^2 = \epsilon^2 \int \left( \sum_{k \in M} (\alpha_k - \beta_k) \phi * v_k \right)^2 \sqrt{f_0},
\]

where the second equality is given by (6).

At first glance, $\int (f_\alpha - f_\beta)^2/f_0$ does not look amenable to Fourier techniques. However, as Lemma 2.2 shows, $\phi * v_k/\sqrt{f_0}$ is expressed as convolution of normal (with a variance larger than 1) with a certain choice of the perturbation function $v_k$ and base function $\pi_0 = \phi_{\sigma^2}$.

**Lemma 2.2.** Consider the perturbation functions

\[v_k(u) = \frac{C_k}{\sqrt{k!}} \phi(\rho u) H_k(\gamma u), \quad \rho^2 \geq \frac{1}{\sigma^2} + \frac{\gamma^2}{2},\]

where $C_k$ is a constant depending on $k$ and $\gamma > 0$. Then

\[
\frac{[\phi * v_k(u)](x)}{\sqrt{\phi * \phi_{\sigma^2}(x)}} = \phi_{\tilde{\sigma}^2} * \tilde{v}_k
\]

where

\[
\tilde{v}_k(u) := \frac{\tilde{C}_k}{\sqrt{k!}} \phi(\tilde{\rho} u) H_k(\tilde{\gamma} u),
\]

with

\[
\tilde{\sigma}^2 = 1 + \frac{1}{2\sigma^2 + 1}, \quad \tilde{C}_k = C_k \left(4\pi \right)^{1/4}, \quad \tilde{\rho} = \sqrt{\rho^2 + \frac{1}{\tilde{\sigma}^2}}, \quad \tilde{\gamma} = \frac{\gamma}{\tilde{\sigma}}.
\]

By Lemma 2.2 the denominator effect can be incorporated into the normal convolution. Then we follow similar ideas used in the proof of Theorem 1.1.
Proof of Theorem 1.2. Again, the choice of $v_k$’s is suggested by Fourier methods. For convenience, we let $\pi_0 = \phi$, then $f_0 = \phi_2$ and $\sqrt{f_0} = 2^{3/4}\pi^{1/4}\phi_4$. Assuming $\tilde{v}_k$ in (19) are in $L_2$,

$$T \left[ \frac{f_0}{\sqrt{f_0}} \right] (t) = T \left[ \sqrt{f_0} \right] (t) + \epsilon \sum_{k \in K} \alpha_k T \left[ \frac{\phi \ast v_k}{\sqrt{f_0}} \right] (t)$$

$$= T[\sqrt{f_0}](t) + \epsilon \sum_{k \in K} \alpha_k T[\phi_4](t)T[\tilde{v}_k](t)$$

by Lemma 2.2.

By the Plancherel formula,

$$\frac{1}{2\pi} \left\| \frac{f_0}{\sqrt{f_0}} - \frac{f_0}{\sqrt{f_0}} \right\|^2 = \epsilon^2 \int \left\| \sum_{k \in M} (\alpha_k - \beta_k) T[\phi_4](t)T[\tilde{v}_k](t) \right\|^2 dt,$$

which lets us write the first condition (2) in Assouad’s Lemma 1.3 as

$$\int \left\| \sum_{k \in K} (\alpha_k - \beta_k) T[\phi_4](t)T[\tilde{v}_k](t) \right\|^2 dt \geq \frac{3c_0}{\pi} \sum_{k \in K} (\alpha_k - \beta_k)^2, \forall \alpha, \beta \in \{0, 1\}^K,$$

Similar to the case for the squared error loss, we might achieve even an equality with $c_0 = \pi/3$ by choosing $\tilde{v}_k$’s to make the functions $\psi_k(t) := T[\phi_4](t)T[\tilde{v}_k](t)$ orthonormal. Ignoring other requirements, we also start from the same orthonormal set (7), and then try to define $\tilde{v}_k$ as the inverse Fourier transform.

From the fact that $T[\phi_4](t) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{t^2}{2})$ and by definition of $\tilde{v}_k$ in (19), the requirement equals to

$$(21) \quad \psi_k(t) := \frac{\bar{C}_k}{\sqrt{2\pi k!}} \exp(-\frac{4t^2}{3}) T[\phi(\bar{\rho}u)H_k(\bar{\gamma}u)](t) = i^{-k}\sqrt{2\phi(2t)} H_k(2t)$$

If we find out all the parameters to make (21) true, we have the desired property for the loss separation condition, i.e. we have

$$(22) \quad \int \frac{(f_0 - f_0)^2}{f_0} = 2\pi \epsilon^2 \sum_{k \in K} (\alpha_k - \beta_k)^2.$$

Thus $\bar{\rho}, \bar{\gamma}$ and $\bar{C}_k$ are found satisfying (21). After some calculations, (21) equals to

$$T[\phi(\bar{\rho}u)H_k(\bar{\gamma}u)](t) = i^{-k}\sqrt{2(2\pi)^{3/4}} C_k \phi(t) \frac{\sqrt{2}}{\sqrt{3}} H_k(2t)$$

which leads to the following,

$$T^{-1} \left[ \phi(t) \frac{\sqrt{2}}{\sqrt{3}} H_k(2t) \right] (u) = i^k \frac{\tilde{C}_k}{\sqrt{2(2\pi)^{3/4}}} \phi(\bar{\rho}u)H_k(\bar{\gamma}u).$$
Recall Lemma 2.1, i.e. for \( b > a > 0 \),

\[
T^{-1}[\phi(at)H_k(bt)](u) = \frac{(ic_{a,b})^k}{a} \phi_a(u) H_k\left(\frac{bu}{a^2 c_{a,b}}\right), \quad c_{a,b} = \sqrt{\frac{b^2}{a^2} - 1}.
\]

Plugging in \( a = \sqrt{2/3} \) and \( b = 2 \) into the above expression, we have the following solutions,

\[
(23) \quad \tilde{C}_k = \left(2\pi\right)^{3/4} \sqrt{3} \sqrt{5^k}, \quad \tilde{\rho} = \sqrt{\frac{3}{2}}, \quad \tilde{\gamma} = \frac{3}{\sqrt{5}}.
\]

We need to ensure that the choice of \( \sigma^2 = 1 \) satisfies the inequality \( \rho^2 \geq \frac{1}{\sigma^2} + \frac{x^2}{2} \) needed for the Lemma 2.2. Comparing (20) and (23), we obtain \( \rho^2 = 3 \) and \( \gamma = \frac{4}{\sqrt{5}} \), which satisfy the condition. Also, \( C_k \) is obtained as \( C_k = \left(2^{5/4}/\pi\right) \sqrt{5^k} \).

Therefore, this choice for the \( \psi_k \)'s leads to

\[
(24) \quad v_k(u) = 2^{5/4} \sqrt{\pi} \sqrt{\frac{5^k}{k!}} \phi_0(\sqrt{3}u) H_k\left(\frac{4}{\sqrt{5}}u\right) \quad \text{for} \quad k \in K.
\]

By restricting to odd values of \( k \), we make the \( v_k \)'s real-valued and odd, thereby ensuring that \( \int v_k d\lambda = 0 \).

Using the exactly same idea in the previous section, if

\[
(25) \quad \epsilon \leq \frac{1}{2\kappa m C_{2m-1}} \quad \text{with} \quad \kappa \simeq 1.086,
\]

then

\[
\frac{1}{2} \pi_0(u) \leq \pi_\alpha(u) \leq \frac{3}{2} \pi_0(u) \quad \text{for all} \quad u \in \mathbb{R}, \quad \alpha \in \{0,1\}^K.
\]

Now the second testing condition can be treated straightforwardly. Indeed, once we choose an orthonormal function \( \psi_k \)'s, we obtain

\[
\int \frac{(f_\alpha - f_\beta)^2}{f_\alpha} \leq \int 2 \frac{(f_\alpha - f_\beta)^2}{f_0} = 4\pi \epsilon^2 \quad \text{for} \quad ||\alpha - \beta||_0 = 1 \quad \text{by} \quad (17) \text{and} \quad (22).
\]

Thus it is enough to choose \( \epsilon^2 < 1/(4\pi n) \). With our choice \( \epsilon = 1/(4\sqrt{m}) \), the testing condition is satisfied.

From the lower bound \( m \sigma^2 \), we want to choose \( m \) as large as possible. The condition in (25) restricts the size of \( m \),

\[
2\kappa m C_{2m-1} < 6m 5^m \leq 4\sqrt{n}.
\]

Thus, we have the upper bound for \( m \),

\[
m \lesssim \left(1/(2 \log 5)\right) \log n \simeq (0.31) \log n.
\]
Finally, we check these constructed $\pi_\alpha$’s are inside of the parameter space $\mathcal{P}_\alpha(\mathbb{R})$

From the fact that $\pi_\alpha(u) \leq (3/2)\pi_0(u)$ for all $u \in \mathbb{R}$ and $\alpha \in \{0,1\}^K$, it is clear that $\pi_0$ is in the space $\mathcal{P}_\alpha(\mathbb{R})$ from the tail property of normal density.

Consequently, the lower bound is obtained as $\log n/n$ up to a constant.

$\square$

3. Discussion

It has been claimed that the Fano’s method is more general in a sense (see Yu (1997, p. 428)). Indeed, using Varshamov-Gilbert’s lemma (e.g. Lemma 2.9 in Tsybakov (2009)), it is not very difficult to prove the same rate result for the class of normal location mixtures with similar types of the sub parameter space.

However, Assouad’s method seems more convenient in some cases. For instance, before knowing how to construct the subspace, it would be extremely difficult to figure out the right family of densities when there are only indirect regularity conditions as in this example. Assouad’s hyperrectangle method indicates that the problem can be solved if we can show the orthogonal relations between constructed densities. Specific constructions can cause another difficulty, but we at least have some clues to handle these problems.

On the other hand, in case we know the metric entropy (good packing and covering number bounds) results beforehand, the optimal minimax rates can be obtained almost automatically with predictive Bayes density estimator using the main theorems in Yang and Barron (1999). It will be interesting to see if we can calculate sharper metric entropy for $\mathcal{F}$ or $\mathcal{F}_s$ than one appeared in Ghosal and van der Vaart (2001).

4. Appendix

Proof of Lemma 1.3. Most of the proof is based on ideas borrowed from Le Cam (1973), Tsybakov (2009), and some unpublished notes by David Pollard. Denote $A = \{0,1\}^K$ and for convenience denote $E_\alpha$ for $E_{f_\alpha}$ and $P_\alpha$ for $P_{f_\alpha}$ where $P_{f_\alpha} = P_{f_\alpha}^n$.

For any density estimator $\hat{f}_n$ based on the observation $X_1,\ldots,X_n$, define an estimator

$$\hat{\alpha} = \arg \min_{\alpha \in A} W(\hat{f}_n, f_\alpha).$$
By restricting the parameter space and by the definition of \( \hat{\alpha} \),

\[
\sup_{f \in F} \mathbb{E}_f W(\hat{f}_n, f) \geq \max_{\alpha \in A} \mathbb{E}_\alpha W(\hat{f}_n, f_\alpha)
\]
\[
\geq \frac{1}{2} \max_{\alpha \in A} \mathbb{E}_\alpha \left( W(\hat{f}_n, f_\alpha) + W(\hat{f}_n, f_{\hat{\alpha}}) \right)
\]
\[
\geq \frac{\zeta}{2} \max_{\alpha \in A} \mathbb{E}_\alpha W(f_\alpha, f_{\hat{\alpha}})
\]

using pseudo distance property (1). Now, using the first condition (2) in the Lemma followed by the simple fact that the supremum is bounded by the average, the last equation can be lower bounded by

\[
\frac{c_0 \epsilon^2 \zeta}{2} \max_{\alpha \in A} \sum_{k=1}^{m} \mathbb{E}_\alpha 1\{\alpha_k \neq \hat{\alpha}_k\} \geq \frac{c_0 \epsilon^2 \zeta}{2} \frac{1}{2m} \sum_{\alpha \in A} \sum_{k=1}^{m} \mathbb{E}_\alpha 1\{\alpha_k \neq \hat{\alpha}_k\}. \]

Define

\[
\bar{P}_{0,k} = \frac{1}{2m-1} \sum_{\alpha \in A_{0,k}} \mathbb{P}_\alpha, \quad \bar{P}_{1,k} = \frac{1}{2m-1} \sum_{\alpha \in A_{1,k}} \mathbb{P}_\alpha, \quad k = 1, ..., m
\]

where \( A_{i,k} = \{\alpha \in A : \alpha_k = i\} \) for \( i = 0, 1 \).

Since \( \alpha_k, \hat{\alpha}_k \) can take only 0 and 1 values,

\[
\frac{1}{2m} \sum_{\alpha \in A} \sum_{k=1}^{m} \mathbb{E}_\alpha 1\{\alpha_k \neq \hat{\alpha}_k\} = \frac{1}{2m} \sum_{k=1}^{m} \left( \sum_{\alpha \in A_{0,k}} \mathbb{P}_\alpha 1\{\alpha_k \neq 0\} + \sum_{\alpha \in A_{1,k}} \mathbb{P}_\alpha 1\{\alpha_k \neq 1\} \right)
\]
\[
= \frac{1}{2} \sum_{k=1}^{m} \left( \bar{P}_{0,k} 1\{\hat{\alpha}_k \neq 0\} + \bar{P}_{1,k} 1\{\hat{\alpha}_k \neq 1\} \right),
\]

which gives us the following lower bound

\[
\sup_{f \in F} \mathbb{E}_f W(\hat{f}_n - f) \geq \frac{c_0 \epsilon^2}{8} \sum_{k=1}^{m} ||\bar{P}_{0,k} \wedge \bar{P}_{1,k}||_1
\]

by \( Ph + Q(1-h) \geq ||P \wedge Q||_1 \) for \( h \geq 0 \) with \( h = 1\{\hat{\alpha} \neq 0\} \).

For \( k = m \), each \( \alpha \) in \( A_{0,m} \) is of the form \( (\gamma, 0) \) with \( \gamma \in D := \{0, 1\}^{m-1} \). Similarly, each \( \alpha \) in \( A_{1,m} \) is of the form \( (\gamma, 1) \) with \( \gamma \in D \). The affinity between \( \bar{P}_{0,m} \) and \( \bar{P}_{1,m} \) equals

\[
\int \left( \frac{1}{2m-1} \sum_{\gamma \in D} p_{\gamma, 0} \right) \wedge \left( \frac{1}{2m-1} \sum_{\gamma \in D} p_{\gamma, 1} \right) \geq \int \frac{1}{2m-1} \sum_{\gamma \in D} (p_{\gamma, 0} \wedge p_{\gamma, 1})
\]
Note that \((\gamma, 0)\) and \((\gamma, 1)\) have only one different coordinate. By similar calculations for other \(k\)'s, we obtain

\[
\sup_{f \in \mathcal{F}} \mathbb{E}_f \int (\hat{f}_n - f)^2 \geq \frac{c_0 e^2 \xi}{2} m \min_{d(\alpha, \beta) = 1} ||\mathbb{P}_\alpha \wedge \mathbb{P}_\beta||_1.
\]

In general, it is difficult to calculate the testing affinity exactly. Fortunately, convenient lower bound can be used in terms of distances between marginals when \(\mathbb{P}_\alpha\) and \(\mathbb{P}_\beta\) are both product measures. For instance, when \(\mathbb{P}_\alpha = \mathbb{P}_\alpha^n\) for i.i.d. case, we can bound this using the chi-squared distance \(\chi^2\) by the following relation.

\[
(1 - ||\mathbb{P}_\alpha \wedge \mathbb{P}_\beta||_1)^2 \leq \frac{1}{n} \chi^2(\mathbb{P}_\alpha, \mathbb{P}_\beta) := \frac{1}{n} \int (\theta_\alpha - \theta_\beta)^2 \theta_\alpha.
\]

Thus the second condition (3) in the Lemma yields a lower bound

\[
\frac{c_0 e^2 \xi}{4} m (1 - \sqrt{c_1}).
\]

See Tsybakov (2009, Lemma 2.7 on page 90) or Le Cam (1973, Lemma 1 on page 40) for the derivation of facts about relations between distances.

\(\square\)

**Proof of Lemma 2.1.** For \(b > a > 0\),

\[
\phi(at) \exp(bt x - \frac{1}{2} x^2) = \phi(at) \sum_{k=0}^{\infty} \frac{H_k(bt)}{k!} x^k.
\]

Now, do the inverse Fourier transform of the left side of the above expression.

\[
\mathcal{T}^{-1} \left[ \phi(at) \exp(btx - \frac{1}{2} x^2) \right] (u) = \int_{-\infty}^{\infty} \exp(it u) \exp(-\frac{a^2 t^2}{2} + btx - \frac{1}{2} x^2) dt
\]

\[
= \frac{1}{a \sqrt{2\pi}} \exp (\frac{(bx + iu)^2}{2a^2} - \frac{1}{2} x^2)
\]

\[
= \frac{1}{a} \phi\left(\frac{u}{a}\right) \exp \left( \frac{bxu i}{a^2} - \frac{1}{2} (ixc_{a,b})^2 \right)
\]

\[
= \frac{1}{a} \phi\left(\frac{u}{a}\right) \sum_{k=0}^{\infty} \frac{H_k \left( \frac{b/(a^2 c_{a,b}) u}{k!} \right)}{k!} (ic_{a,b})^k x^k.
\]

The inverse Fourier transform of the right side is

\[
\sum_{k=0}^{\infty} \mathcal{T}^{-1} \left[ \phi(at) \frac{H_k(bt)}{k!} \right] (u) x^k
\]

By matching the coefficient for the \(k^{th}\) power of \(x\),

\[
\mathcal{T}^{-1} \left[ \phi(at) H_k(bt) \right] (u) = (ic_{a,b})^k \frac{1}{a} \phi\left(\frac{u}{a}\right) H_k \left( \frac{b}{a^2 c_{a,b}} u \right).
\]
which proves the claim. □

Proof of Lemma 2.2. Main ideas are just completing the square and change of variables. Note that $\phi \star \phi_{a^2} = \phi_{1+a^2}$. By definition of $v_k$, we have

$$\frac{[\phi \star v_k(u)](x)}{\sqrt{\phi_{1+a^2}(x)}} = \frac{C_k}{\sqrt{k!}} \frac{[\phi \star (pu)H_k(\gamma u)](x)}{\sqrt{\phi_{1+a^2}(x)}}$$

$$= \frac{C_k}{\sqrt{k!}} \int \frac{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x-u)^2\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\rho^2 u^2\right) H_k(\gamma u)}{(2\pi(1+\sigma^2))^{-1/4} \exp\left(-\frac{1}{4} \frac{x^2}{1+\sigma^2}\right)} du$$

$$= \frac{C_k}{\sqrt{k!}} (1 + \sigma^2)^{1/4} (2\pi)^{-3/4} \int E_{\sigma,\rho}(x,u)H_k(\gamma u)du$$

where $E_{\sigma,\rho}(x,u)$ is the exponential factor. By completing the square,

$$E_{\sigma,\rho}(x,u) := \exp \left( \left(-\frac{1}{2} + \frac{1}{4(1+\sigma^2)}\right)x^2 + xu - \left(\frac{1}{2} + \frac{1}{2}\rho^2\right)u^2 \right)$$

$$= \exp \left( -\frac{1}{2\sigma^2} x^2 + xu - \left(\frac{1}{2} + \frac{1}{2}\rho^2\right)u^2 \right) \text{ by def. of } \bar{\sigma}^2 \text{ in } (20)$$

$$= \exp \left( -\frac{1}{2\bar{\sigma}^2} (x - \bar{u})^2 \right) \exp \left( -\frac{1}{2} (1 + \rho^2 - \bar{\sigma}^2) \frac{\bar{u}^2}{\sigma^2} \right) \text{ by } \bar{u} := \sigma^2 u$$

$$= (2\pi\bar{\sigma}) \phi_{\bar{\sigma}^2}(x - \bar{u}) \phi \left( \frac{\sqrt{1 + \rho^2 - \bar{\sigma}^2}}{\bar{\sigma}^2} \bar{u} \right),$$

where the positive value for $(1 + \rho^2 - \bar{\sigma}^2)$ is guaranteed by the condition $\rho^2 > 1/\sigma^2 + \gamma^2/2 > 1/(1+2\sigma^2) := 1 - \bar{\sigma}^2$. By the change of variables,

$$\frac{[\phi \star v_k(u)](x)}{\sqrt{\phi_{1+a^2}(x)}} = \left( \frac{C_k}{\sqrt{k!}} \frac{2\pi(1+\sigma^2)^{1/4}}{\bar{\sigma}} \right) \phi_{\bar{\sigma}^2} \star \phi \left( \frac{\sqrt{1 + \rho^2 - \bar{\sigma}^2}}{\bar{\sigma}^2} \bar{u} \right) H_k \left( \frac{\gamma}{\bar{\sigma}^2} \bar{u} \right),$$

Using the definitions of each transformed variables (20), the proof is complete. □

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