\( \kappa \)-Minkowski star product in any dimension from symplectic realization

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Abstract

We derive an explicit expression for the star product reproducing the \( \kappa \)-Minkowski Lie algebra in any dimension \( n \). The result is obtained by suitably reducing the Wick–Voros star product defined on \( \mathcal{C}_d^\mathbb{R} \) with \( n = d + 1 \). It is thus shown that the new star product can be obtained from a Jordanian twist.

Keywords: star products, twist deformation, noncommutative spacetime

1. Introduction

Noncommutative space–time models represent an intermediate step in understanding quantum aspects of space–time in the search for a quantum theory of gravity. In analogy with quantum physics where quantum phase space ceases to be a pseudo-Riemannian manifold and classical observables (commuting functions defined on phase space) are replaced by operators, one expects that in general relativity space–time as a dynamical variable itself, becomes quantum at the Planck scale and space–time observables are replaced by noncommuting operators. In particular space–time coordinate functions are no longer classical variables but they belong to a noncommutative ‘coordinate algebra’.

One of the first models of quantum space–time was proposed in the 1990s [1], together with the fuzzy sphere [2, 3] (for a review see [37]), but already in 1986 space–time noncommutativity was found within string theory [4]. In the context of algebraic geometry, noncommutative geometry has been independently developed by Connes and collaborators [5, 6].

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Here we shall follow the star product approach, where the noncommutative algebra generated by coordinate functions is represented in terms of the original commutative algebra of functions, with the pointwise multiplication replaced by a noncommutative (star), associative product. There are many known procedures for introducing star products, for example using deformation quantization [7], which generalizes the canonical quantization approach of classical phase space, or, in connection with quantum groups, via twist deformation [8, 9]. Another possibility, which is the one pursued in the present article, is the symplectic realization proposed in [10], where noncommutative algebras are obtained as subalgebras of some ‘canonical’ algebra of Moyal type.

The star product formulation of space–time noncommutativity is especially relevant in physical applications, such as gravity [11, 12], field and gauge theories (see e.g [13] and refs. therein). Besides constant noncommutativity, which has been widely explored in the literature, there is a whole family of Lie algebra type noncommutativity which has interesting properties with respect to symmetries. For example in three-dimensions, there has recently been a renewed interest in noncommutative structures on $\mathbb{R}^3$ of Lie algebra type, in connection with their occurrence in three-dimensional quantum gravity models [14], where $\mathbb{R}^3$ is identified with the dual algebra of the local relativity group. In such a framework star products are mainly introduced through a group Fourier transform (see for example [15] for a review and comparison with other techniques), imposing compatibility with the group convolution.

We shall focus here on the $\kappa$-Minkowski space–time which was initially introduced in connection with the quantum (deformed) version of the Poincaré symmetry (the so called $\kappa$-Poincaré group) using the theory of quantum groups [16]. The $\kappa$-Minkowski space–time has been first introduced in [17, 18]. It has been further investigated by many authors [19–27, 45] in the framework of noncommutative quantum field theory and Planck scale physics.

As for the derivation of a star product associated to the $\kappa$-Minkowski space–time, many proposals are available in the literature. Among the others we quote [28] where an integral form for the star product is found for the three-dimensional case upon generalizing the deformation quantization derivation of the Moyal product; in [29] different expressions are found based on the correspondence between realizations and ordering prescriptions, whereas in [30, 31] the twist approach is pursued.

In this paper we shall derive yet another expression for the $\kappa$-Minkowski star product, which agrees with all the others, as it should, at the level of the star commutator of coordinate functions, but differs from the others already at the first order in the noncommutative parameter for the product of two generic functions. We start from the conjecture that most noncommutative spaces can be obtained by reduction of some ‘big enough’ noncommutative space with canonical noncommutativity\(^5\). By the word canonical we mean the quantization of the canonical symplectic form, which yields constant noncommutativity of the Moyal type. Let us recall here that the Moyal product [32] is just a representative of a whole equivalence class of translation invariant star products, all implementing the canonical star commutator among real coordinate functions on $\mathbb{R}^{2d}$

$$y_i \star y_j - y_j \star y_i = i\delta_{ij}, \ i,j = 1,\ldots, 2d$$

and differing among them by ordering prescriptions. In the present paper we shall use indeed the Wick–Voros product [33, 34] which is the one associated to normal ordering. For the case at hand, we will show that by considering $d$ copies of the noncommutative plane equipped

\(^5\) Let us notice however that symplectic realizations of noncommutative algebras do not require necessarily that the canonical noncommutative algebra to be larger than the one we wish to realize. Examples can be found in [35] where 3-d noncommutative spaces are realized in terms of the 2-d Moyal algebra.
with the Wick–Voros product we can obtain the $d + 1$ $\kappa$-Minkowski space, together with a star product, as a subalgebra of the large starting algebra. Additionally, we show that the star product obtained in this way can be obtained by a Jordanian twist, already known in the literature in a slightly modified form.

The paper is organized as follows. In section 2 we realize the $\kappa$-Minkowski algebra $\{x^\mu, \mu = 0, \ldots, n\}$ in terms of complex coordinate functions $\zeta^i, \zeta^i, i = 1, \ldots, d$ of the Wick–Voros plane $\mathbb{C}^d$, and derive an explicit expression of the star product for the subalgebra of functions $f(x^\mu)$. The content of this derivation constitutes the object of proposition 2.1 and represents the main result of the paper. In section 3 we show that our star product, obtained by symplectic realization, is actually obtainable by a Jordanian twist operator. This result is formalized in proposition 3.1. We finally discuss the issue of the existence of an integration measure with respect to which our star product would be cyclic and we conclude that no such measure exists, a result that we argue might be true for all Jordanian twists. We conclude with final remarks.

2. An explicit formula for the $\kappa$-Minkowski star product

Our starting point is the observation that the $\kappa$-Minkowski Lie algebra in $n = d + 1$ dimensions

\[
\begin{align*}
\{\hat{x}^0, \hat{x}^i\} &= \hat{x}^i, \quad i = 1, \ldots, d, \quad (2.1) \\
\{\hat{x}^i, \hat{x}^j\} &= 0, \quad i, j = 1, \ldots, d \quad (2.2)
\end{align*}
\]

may be realized in terms of $d$ families of creation and annihilation operators $a^i, a^i$, as follows

\[
\hat{x}^0 = \sum_{i=1}^d a^i a^i, \quad \hat{x}^i = a^i. \quad (2.3)
\]

This realization, which is far from being unique, is a generalization of the well known Jordan–Schwinger representation of all three-dimensional algebras in terms of two uncoupled harmonic oscillators. Note that the map, as it is written above works in any dimension and is applicable for arbitrary Lie algebra.

The observation may be extended to Poisson realizations. We consider $d$ copies of the complex plane $\mathbb{C}$ and the algebra of functions $\mathcal{F}(\mathbb{C}^d)$ equipped with the canonical Poisson bracket

\[
\begin{align*}
\{\zeta^i, \zeta^j\} &= i\delta^{ij}, \quad (2.4)
\end{align*}
\]

The subset of functions $f(x^\mu), \mu = 0, \ldots, d$, with

\[
x^0 = \sum_i \zeta^i, \quad x^i = \zeta^i, \quad i = 1, \ldots, d \quad (2.5)
\]

is a Poisson subalgebra of $\mathcal{F}(\mathbb{C}^d)$, with induced Poisson brackets

\[
\begin{align*}
\{x^0, x^i\} &= ix^i, \quad (2.6)
\{x^i, x^j\} &= 0 \quad (2.7)
\end{align*}
\]

\[^6\] Notice that for the moment all operators are dimensionless.
\[ \{ f(x), g(x) \} = \frac{\partial f}{\partial x^\mu} \{ x^\mu, x^\nu \} \frac{\partial g}{\partial x^\nu} = \partial_j f(x) \partial_j g(x) i \epsilon_{\mu \nu} x^\nu \] (2.8)

and \( \epsilon_{\mu \nu} \) are the structure constants of the \( \kappa \)-Minkowski algebra. We indicate such an algebra with the symbol \( \mathcal{A}^n \). The coordinate functions \( x^\mu, \mu = 0, \ldots, n - 1 \) polynomially generate \( \mathcal{A}^n \), with Poisson brackets of the \( \kappa \)-Minkowski type.

As a second step, the commutative algebra of functions \( \mathcal{F}(\mathbb{C}^d) \) is made into a noncommutative algebra replacing the pointwise product with the Moyal product

\[ f \ast_M g (z^i, \bar{z}^i) = f(z, \bar{z}) \exp \left( \frac{\theta}{2} \left( \bar{\partial}_i \partial_{\bar{z}^i} - \partial_i \bar{\partial}_{\bar{z}^i} \right) \right) g(z, \bar{z}), \quad i = 1, \ldots, d \] (2.9)

with \( \theta \) a constant, real parameter. Alternatively, the Wick–Voros product [33, 34] can be used

\[ f \ast_{WV} g (z^i, \bar{z}^i) = f(z, \bar{z}) \exp \left( \frac{\theta}{2} \bar{\partial}_i \partial_{\bar{z}^i} \right) g(z, \bar{z}), \quad i = 1, \ldots, d. \] (2.10)

As well known, they both belong to the same family of translation invariant star-products and are equivalent as they yield the same star-commutators for the coordinate functions

\[ [z^i, \bar{z}^j]_* = \theta \delta^{ij} \] (2.11)

only differing by symmetric terms.

In [10] the Moyal star product on \( \mathbb{R}^d \) has been used to induce, by different realizations, many nonequivalent star products on \( \mathcal{F}(\mathbb{R}^d) \), using the fact that all three-dimensional algebras could be realized as Poisson subalgebras of \( \mathcal{F}(\mathbb{R}^d) \), in the same spirit as above, and observing that those subalgebras are also Moyal subalgebras. It was already noticed that complex realizations are easier to deal with, by means of the identification \( \mathbb{R}^d \cong \mathbb{C}^2 \). It was however difficult to find a closed explicit form for such star products (only \( x^i \ast f(x) \) could be explicitly calculated), whereas it was observed in [36] (also see [38] for applications) that the Wick–Voros product (2.10) is easier to handle and closed expressions for star products on \( \mathbb{R}^3 \) could be obtained by reduction. An example is the noncommutative algebra \( \mathbb{R}^3_\lambda \), with the star product introduced in [36] which reproduces the \( \mathfrak{su}(2) \) Lie algebra commutation relations as a star-commutator of coordinate functions

\[ x^i \ast x^j - x^j \ast x^i = i \lambda \epsilon^{ijk} x^k, \quad i, j, k = 1, \ldots, 3 \] (2.12)

with

\[ f \ast g (x) = \exp \left( \frac{\lambda}{2} \left( \delta^j x^0 + i \epsilon^{ijk} x^k \right) \frac{\partial}{\partial \theta^j} \frac{\partial}{\partial \bar{\theta}^k} \right) f(u) g(v) \big|_{u=v=x} \] (2.13)

and \( \lambda \) the noncommutativity parameter, to be identified with \( \theta \) up to a constant. This was achieved on realizing the coordinates of \( \mathbb{R}^3 \) as quadratic-linear functions in \( \mathbb{R}^3, z^a, a = 1, 2 \)

\[ x^\mu = \frac{1}{2} z^a \sigma^\mu_{ab} z^b, \quad \mu = 0, \ldots, 3 \] (2.14)

with \( \sigma^i \) the Pauli matrices, \( \sigma^0 \) the identity matrix \( \mathbb{I}_2 \), and observing that the subalgebra polynomially generated by the coordinate functions \( x^\mu \) and properly completed, is closed with respect to the Wick–Voros star product\(^7\).

\(^7\) Notice that \( (x^0)^2 = \sum (x^i)^2 \). Therefore \( x^0 \) is functionally dependent from the other coordinates. It represents the radius in polar coordinates. This is not to be confused with the coordinate \( x^0 \) in the \( \kappa \)-Minkowski algebra which is an independent function.
The same procedure can be applied to the \( \kappa \)-Minkowski algebra, equations (2.1), (2.2), and extended to \( d \) dimensions. The outcome is contained in the following proposition, which represents our main result.

**Proposition 2.1.** Let us consider \( d \) copies of the noncommutative plane endowed with the Wick–Voros product, \( \mathcal{F}(\mathbb{C}^d), \ast_{\text{WV}} \) and the quadratic-linear functions defined in (2.5). Then, the Poisson subalgebra \( \mathcal{A}^{d+1} \ni f(x^0, x^i) \) is also a noncommutative subalgebra of \( \mathcal{F}(\mathbb{C}^d), \ast_{\text{WV}} \) with induced star product

\[
(f \ast g)(x) = \exp \left[ \theta \left( x^0 \frac{\partial}{\partial y^0} + x^i \frac{\partial}{\partial y^i} \right) \right] f(y) g(w) |_{y = w = x}.
\] (2.15)

**Proof.** This is obtained from equation (2.10) observing that

\[
\frac{\partial}{\partial z^i} \frac{\partial}{\partial z^j} \left( \frac{\partial}{\partial x^l} \frac{\partial}{\partial x^j} \right) = \left( \frac{\partial}{\partial x^l} \frac{\partial}{\partial x^j} \right) \left( \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \right) = z^l \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + z^j \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + z^i \frac{\partial}{\partial x^l} \frac{\partial}{\partial x^j} + z^j \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^l} + 2 z^l \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^l} \frac{\partial}{\partial x^j},
\] (2.16)

In order to exponentiate this result and to prove (2.15), the crucial observation is that the two summands in the rhs of (2.16) commute. At second order we have for instance

\[
\frac{\partial}{\partial z^i} \frac{\partial}{\partial z^j} = \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + 2 \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j},
\] (2.16)

where we have used

\[
\frac{\partial}{\partial z^i} \frac{\partial}{\partial z^j} = \frac{\partial}{\partial z^i} \left( \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^j} \right) = z^j \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + z^j \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + 2 z^j \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^j},
\] (2.17)

and this can be generalized to higher powers. \( \square \)

As a corollary, the star product (2.15) is associative, since the Wick–Voros product on \( \mathbb{C}^d \) is associative by construction, being the trace over coherent states of the operator product [33].

Notice that, had we started form the Moyal product (2.9) we should have considered powers of the sum of two differential operators, the first order in \( \theta \) being

\[
\theta \left[ \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^j} - \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^j} \right] \]
(2.18)
which should be understood in terms of (2.16), and its analogue with $z, \bar{z}$ exchanged, so to have

$$\theta \left[ \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^j} + \frac{\partial}{\partial \bar{z}^i} \frac{\partial}{\partial \bar{z}^j} \right] = \theta \left[ \frac{\partial}{\partial x^0} \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} + \frac{\partial}{\partial \bar{x}^0} \frac{\partial}{\partial \bar{x}^0} + \frac{\partial}{\partial \bar{x}_i} \frac{\partial}{\partial \bar{x}_i} \right]$$

$$= \theta \left[ \frac{\partial}{\partial x^0} \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} - \frac{\partial}{\partial \bar{x}^0} \frac{\partial}{\partial \bar{x}^0} - \frac{\partial}{\partial \bar{x}_i} \frac{\partial}{\partial \bar{x}_i} \right]. \quad (2.19)$$

This time the two summands do not commute, which signals a problem in expressing the star product in closed form. Indeed, considering higher orders in $\theta$ the Moyal product expansion develops mixed terms. For example at second order in $\theta$ we find mixed terms of the kind

$$\left[ z^i \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x_i} \right] \left[ x^0 \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^0} \right]$$

where, differently from (2.17), the first and second differential operator do not commute.

This problem was already noticed for the $su(2)$-like star product and led to the Wick–Voros based star product (2.13) as opposed to the Moyal based one [10].

In order to compare the $\kappa$-Minkowski star product with equation (2.13) we may rewrite it in terms of the structure constants of the $\kappa$-Minkowski algebra so to obtain

$$f * g(x) = \exp \left[ \theta \left( x^0 \delta^{i0} + x^i \epsilon^{i0} \right) \frac{\partial}{\partial y^0} \frac{\partial}{\partial w^\nu} \right] f(y) g(w) |_{y = w = x}. \quad (2.21)$$

Despite the similarity of this expression with equation (2.13), the two products are very different. The reason, already enunciated in the footnote 6 is that here $x^0$ is an independent coordinate function whereas in equation (2.13) it is functionally dependent from the other ones: the latter generates the non-trivial center of the algebra (which is related to the Casimir of $SU(2)$), while the $\kappa$-Minkowski algebra has no non-trivial center (the corresponding group has no Casimir).

The result of proposition 2.1 does not depend on space–time dimension and has the advantage of being given in terms of a closed expression unlike other known results in the literature.

For further developments in next sections it is useful to write the star product (2.15) (equiv. (2.21)) as a series expansion. We have

$$\exp \left[ \theta \left( x^0 \delta^{i0} + x^i \epsilon^{i0} \right) \frac{\partial}{\partial y^0} \frac{\partial}{\partial w^\nu} \right] = \exp \left[ \theta \frac{\partial}{\partial y^0} x^i \frac{\partial}{\partial w^\nu} \right]$$

$$= \sum_{n=0}^{\infty} \frac{\theta^n}{n!} \left( \frac{\partial}{\partial y^0} \right)^n \left( x^i \frac{\partial}{\partial w^\nu} \right)^n. \quad (2.22)$$

However, this is not a proof that a closed form for the Moyal based star-product does not exist. We will show in the next section that the WV based star product is related to a non-symmetric Jordanian twist. It would be interesting to investigate whether a symmetrized Jordanian twist might yield the Moyal based product.
We shall come back to this expression in section 3, where we will show that our star product can be obtained as a twisted product with Jordanian twist.

3. Twist operators and deformed symmetries

One can relate the star product formulation with the twist-deformation approach, but only for a certain type of quantum deformations. The deformations leading to noncommutative algebras (as noncommutative space–times) are related to the Hopf algebras formalism. Let us then recall a few important features of this approach.

Let us denote with $H$ a Hopf algebra, equipped with coproduct, counit and antipode maps $(H, \Delta, \epsilon, S)$. Hopf algebras generalize in noncommutative geometry the symmetries of space–time. The classical example of a Hopf algebra is provided by $U\mathfrak{g}$, the universal enveloping algebra of some Lie algebra $\mathfrak{g}$, with the Hopf algebra maps $(\Delta, \epsilon, S)$ appropriately defined.

Let us stick to the latter situation, with $\mathfrak{g}$ the Lie algebra of space–time symmetries. The Hopf algebra $U\mathfrak{g}$ can undergo a deformation procedure and the deformation may be obtained through a twist operator, $F$ [8, 9], which is an invertible element in the tensor product of Hopf algebras $H \otimes H$. One of the advantages of the twist deformation is that it provides straightforwardly the universal quantum $R$ matrix and an explicit formula for the star product on the Hopf module algebra, which is consistent with Hopf-algebra actions related to $H$.

Issues such as symmetries and invariances can thus be properly addressed.

The twist fulfills the cocycle condition:

\[(F \otimes 1)(\Delta \otimes 1)F = (1 \otimes F)(1 \otimes \Delta)F\] (3.1)

and the normalization condition

\[(1 \otimes \epsilon)F = (\epsilon \otimes 1)F = 1 \otimes 1.\] (3.2)

Under the twisted deformation the commutation relations of the Lie algebra sector of the Hopf algebra do not change but the coalgebra structure is suitably modified via relations:

\[\Delta_F(X) = F\Delta(X)F^{-1},\] (3.3)

\[\epsilon(X) = 0, \quad S_F(X) = f^{\alpha}\hat{S}(f_{\alpha})S(X)S(f_{\beta})\hat{f}_{\beta}.\] (3.4)

Where we used the notation $F = f^{\alpha} \otimes f_{\alpha}$, $F^{-1} = \hat{f}^{\alpha} \otimes \hat{f}_{\alpha}$. Such a deformation of the co-structures in the universal enveloping algebra of symmetries implies a simultaneous deformation in the space–time algebra. In this way the commutative multiplication in the algebra of functions $\mathcal{A}_{d+1}^{\mathbb{R}}$ is replaced by a new twisted one:

\[f \star_F g = \mu \circ F^{-1}(f \otimes g) = f^{\alpha}(f_{\alpha})\hat{f}_{\alpha}(g), \quad f, g \in \mathcal{A}_{d+1}^{\mathbb{R}},\] (3.5)

where $\mu$ is the usual point-wise multiplication. As a result one obtains a noncommutative algebra $(\mathcal{A}_{d+1}^{\mathbb{R}}, \star_F)$, which we identify with the noncommutative space–time as before. In the case of triangular deformations it is possible to associate with a given star product defined through a quantization–dequantization scheme the corresponding twisting element.

For example considering the Moyal product defined in (2.9) we can easily find the corresponding twist as:

9 For each value of $\alpha$, $f^{\alpha}$ and $f_{\alpha}$ (and similarly $\hat{f}^{\alpha}$ and $\hat{f}_{\alpha}$) are different elements of $H$.

10 Twist deformations can be introduced for general differentiable manifolds $M$, however here we are only interested in $M = \mathbb{R}^{d+1}$ to be consistent with the previous section.

11 For details on the quantization and dequantization operators see [39, 40] and references therein.
Such a twist is called Abelian\(^\text{12}\) because it has support in the Abelian (Lie) algebra generated by \(a = \text{span}\{\partial_x, \partial_{x'}\}, i = 1 \ldots d\). It is easy to define a Hopf algebra structure on its undeformed universal enveloping algebra \(Ua\) by defining all the maps on the generating elements as follows:

- coproduct \(\Delta(\partial_x) = \partial_x \otimes 1 + 1 \otimes \partial_x\);
- counit \(\epsilon(\partial_x) = 0\);
- antipode \(S(\partial_x) = -\partial_x\);

and then extending them to all the elements of \(Ua\).

Let us consider now the deformation of the Hopf algebra \(Ua\) associated with the Moyal twist (3.6). We can immediately see that the cocycle condition (3.1) is satisfied by \(FM\). To be more precise the twisted deformation of any Lie algebra \(g\) requires a topological extension of the corresponding enveloping algebra \(Ug\) into an algebra of formal power series \(Ug[[\theta]]\) in the formal parameter \(\theta\). We have \(U_{fg}a[[\theta]] = (U_{a[[\theta]]}, \Delta_{fg}, S_{fg}, \epsilon)\) with the twisted coproduct and the antipode maps obtained by (3.3) and (3.4). We notice that the coalgebra part stays undeformed under the action of this twist, i.e.:

\[
\Delta_{fg}(\partial_x) = \Delta_0(\partial_x); \quad \Delta_{fg}(\partial_{x'}) = \Delta_0(\partial_{x'}),
\]

\[
S_{fg}(\partial_x) = -\partial_x; \quad S_{fg}(\partial_{x'}) = -\partial_{x'}.
\]

The case we are interested in this paper, however, is related to a slightly more complicated kind of space–time noncommutativity, which is the \(\kappa\)-Minkowski star product defined in equation (2.15) or equivalently (2.22). We prove the following

**Proposition 3.1.** The star product defined in equation (2.22) is associated to a twist operator

\[
f \ast g = \mu \circ (F_J^{-1}(f \otimes g))
\]

with \(F_J\) the Jordanian twist [43]

\[
F_J = \exp(\sigma \otimes J); \quad \sigma = \ln(1 + \theta P_0).
\]

The twist \(F_J\) is an element of \((U\text{b} \otimes U\text{b})[[\theta]]\) with \(b = \text{span}\{P_0, J; [J, P_0] = P_0\}\) the two-dimensional Borel subalgebra of \(\mathfrak{gl}(2, \mathbb{C})\).

**Proof.** Using the representation for the generators \(J = -x^0\partial_0 - x^k\partial_k = -x^0\partial_0'\) and \(P_0 = \partial_0\) we can expand the inverse of the twist in the following way:

\[
F_J^{-1} = \exp\left(\ln\left(1 + \theta \partial_0\right) \otimes x^n \partial_n\right) = \left(1 + \alpha \partial_0\right)^{(x^n \partial_n)/(2^n)} = \sum_{n=0}^{\infty} \frac{1}{n!}(\theta \partial_0)^n \cdot (x^n \partial_n)^2,
\]

(3.11)

\(\text{Analogously we can define a twist operator for the Wick–Voros product [41], which is the one used in this paper, as } F_V = \exp\left(-\theta(\partial_x \otimes \partial_{x'})\right), \text{ with support in the same Abelian algebra. The two twists only differ by co-boundary terms [42].}
where \( y^2 = y(y - 1) \ldots (y - n + 1) \) and \( y^1 = y; \quad y^0 = 1 \). We can re-write the twist as:

\[
F^{-1}_J = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} \left( \partial_0 \right)^n_{(1)} \cdot \left( x^0 \partial_x \right)^n_{(2)} = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} \left( \partial_0 \right)^n_{(1)} \cdot \left( x^0 \partial_w \right)^n_{(2)},
\]

where

\[
\left( x^0 \partial_x \right)^n = x^0 \partial_x \left( x^0 \partial_x - 1 \right) \left( x^0 \partial_x - 2 \right) \ldots \left( x^0 \partial_x - n + 1 \right)
\]

and the notation \( x^0 \partial_w \) is a way to indicate that the differential operator \( \partial_w \) does not act on functions of \( x \) but only on functions of \( w \). Indeed we notice that

\[
\left( x^0 \partial_x \right)^n = \left( x^0 \partial_x \right)^n \partial_x = x^0 \partial_x \partial_x = x^0 x^2 \partial_x \partial_x = x^0 x^2 \partial_x \partial_x = \left( x^0 \partial_x \right)^n.
\]

and by induction we can prove that

\[
\left( x^0 \partial_x \right)^n = \left( x^0 x^2 \ldots x^n \right) \frac{\partial_x \partial_x \ldots \partial_x}{n} = \left( x^0 \partial_x \right)^n.
\]

Upon substituting this result into equation (3.9) we obtain

\[
f \ast g(x) = \left[ \mu \circ \left( F_J^{-1}(f \otimes g)(w) \right) \right]_{w=x} = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} \left( \partial_x \right)^n f(y) \left( x^0 \partial_x \right)^n g(w) |_{w=x}
\]

which agrees with equation (2.22).

The twist \( F_J \) (first introduced in [44], in a slightly different form) has support in the Borel algebra \( \mathfrak{b} \). However, if we are interested in the symmetries of the \( \kappa \)-Minkowski space–time, we use the fact that the above two-dimensional Borel algebra \( \mathfrak{b} \) is a subalgebra of the Poincaré–Weyl algebra (one generator extension of the Poincaré algebra) \( \mathfrak{p}\mathfrak{w} = \text{span} \{ M_{\mu \nu}, P_{\mu}, J, \} \), \( \mu, \nu = 0, 1, 2, 3 \) with the following commutation relations:

\[
\begin{align*}
\left[ M_{\mu \nu}, M_{\rho \lambda} \right] &= \eta_{\mu \rho} M_{\nu \lambda} + \eta_{\lambda \rho} M_{\nu \lambda} - \eta_{\lambda \nu} M_{\rho \lambda} - \eta_{\lambda \nu} M_{\rho \lambda}, \\
\left[ M_{\mu \nu}, P_{\rho} \right] &= \eta_{\nu \rho} P_{\mu} - \eta_{\nu \rho} P_{\mu}, \\
\left[ P_{\mu}, J \right] &= 0; \quad \left[ J, P_{\mu} \right] = P_{\mu}; \quad \left[ J, M_{\mu \nu} \right] = 0.
\end{align*}
\]

Again we can turn \( U \mathfrak{p}\mathfrak{w} \) into a Hopf algebra by defining the following maps on the generators:

\[
\Delta(J) = J \otimes 1 + 1 \otimes J, \quad \Delta(P_{\mu}) = P_{\mu} \otimes 1 + 1 \otimes P_{\mu}, \quad \Delta(M_{\mu \nu}) = M_{\mu \nu} \otimes 1 + 1 \otimes M_{\mu \nu},
\]

\[
e(\epsilon(J) = \epsilon(P_{\mu}) = \epsilon(M_{\mu \nu}) = 0, \quad S(J) = -J, \quad S(P_{\mu}) = -P_{\mu}, \quad S(M_{\mu \nu}) = -M_{\mu \nu} \quad \text{and then extending them to the whole } U \mathfrak{p}\mathfrak{w}.
\]

After twisting the coalgebra sector of \( U \mathfrak{p}\mathfrak{w} \) with \( F_J \) via equations (3.3) and (3.4) we get its twisted version \( U \mathfrak{p}\mathfrak{w}_{F_J}[[\theta]] \) (see [31]) with:

\[
\Delta_{F_J}(P_{\mu}) = P_{\mu} \otimes 1 + \epsilon^\sigma \otimes P_{\mu},
\]

\[
\Delta_{F_J}(M_{\mu \nu}) = M_{\mu \nu} \otimes 1 + 1 \otimes M_{\mu \nu} + \theta \left( \eta_{\sigma \nu} P_{\mu} - \eta_{\sigma \rho} P_{\nu} \right) e^{-\sigma} \otimes J,
\]

13 We are using anti-hermitean generators.
\[ \Delta_{\mathcal{F}_J}(J) = J \otimes 1 + 1 \otimes J - \theta \partial_0 e^{-\sigma} \otimes J = J \otimes 1 + e^{-\sigma} \otimes J, \]
\[ S_{\mathcal{F}_J}(P_\mu) = -P_\mu e^{-\sigma}; \quad S_{\mathcal{F}_J}(M_{\mu
u}) = -M_{\mu
u} + \theta \left( \eta_{\alpha\beta} P_\mu - \eta_{\beta\mu} P_\alpha \right) J; \quad S_{\mathcal{F}_J}(J) = -e^\sigma J. \]
(3.21)

The Poincaré–Weyl Lie algebra \( \mathfrak{po} \) is a subalgebra of a bigger one, the inhomogeneous general linear algebra \( i\mathfrak{gl}(n) \), which was investigated in the context of non-symmetric Jordanian twist in [31].

We may associate to a given twist deformation a realization of noncommuting coordinate functions in terms of differential operators, given by the following relations:

- **left-handed realization:**
  \[ \hat{x}_L^\mu = \left( \tilde{f}_\alpha \triangleright x^\mu \right) \cdot \tilde{f}_\alpha \quad (3.22) \]

- **right-handed realization:**
  \[ \hat{x}_R^\mu = \tilde{f}_\alpha \cdot \left( \tilde{f}_\alpha \triangleright x^\mu \right). \quad (3.23) \]

In the case of the Jordanian twist (3.10) we obtain these differential operators in the form:

\[ \hat{x}_L^\mu = x^\mu + \theta \delta_\mu \psi \partial_\nu, \quad \hat{x}_R^\mu = x^\mu \left( 1 + \theta \partial_0 \right). \quad (3.24) \]

Let us notice that such realizations are easily obtained from the explicit expression of the star product, equation (2.22), by star-multiplying a generic function \( f \) on the left (resp. on the right) with \( x^\mu \). Moreover, it is easy to check that using such realizations (left and right respectively) the \( \kappa \)-Minkowski relations are satisfied:

\[ \left[ \hat{x}_L^0, \hat{x}_R^0 \right] = \pm \theta \hat{x}_L^0 \hat{x}_R^0. \quad (3.25) \]

### 3.1. Integration measure

One of the advantages of having a star product obtained from a twist is that the differential calculus is completely determined by the twist operator [11, 12], (also see [46] for generalization to field theory). The guiding principle to obtain all the geometric structures related to the noncommutative algebra of functions is the following. Every time we have a bilinear map \( \mu : X \times Y \to Z \),

\[ \mu^\ast : X \times Y \to Z, \quad (3.26) \]

where \( X, Y, Z \) are modules over the algebra \( A \), with an action of the twist \( F \) on \( X \) and \( Y \), we combine the map with the action of the twist so to obtain a deformed map \( \mu^\ast \),

\[ \mu^\ast = \mu \circ F^{-1}. \]

In the particular case of the wedge product of elementary one-forms deformed via \( F_J \) (3.10) we have

\[ \text{d}x^\mu \wedge \ast \text{d}x^\nu = \wedge \circ F^{-1} \left( \text{d}x^\mu \otimes \text{d}x^\nu \right) = \tilde{f}_\alpha \left( \text{d}x^\mu \right) \wedge \tilde{f}_\alpha \left( \text{d}x^\nu \right) \]
\[ = \text{d}x^\mu \wedge \text{d}x^\nu + \theta \mathcal{L}_{\partial_0} \left( \text{d}x^\mu \right) \wedge \mathcal{L}_{\psi \partial_0} \left( \text{d}x^\nu \right) + O \left( \theta^2 \right) = \text{d}x^\mu \wedge \text{d}x^\nu \quad (3.27) \]
with $\mathcal{L}_X$ the Lie derivative with respect to $X$, so that the volume form on $\mathbb{R}^{d+1}$ is the undeformed one $\Omega = \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^d$. The star product (2.22) however, is not cyclic with respect to $[\Omega]$

$$\int \mathrm{d}^{d+1}x \ f \star g = \int \mathrm{d}^{d+1}x \ g \star f$$  \hspace{1cm} (3.28)

the reason being that one of the generators yielding the twist operator is the dilation, $J = -x^\mu \partial_\mu$, which does not preserve the volume form $\Omega$. For (2.22) to be cyclic, we could consider to modify the integral measure by the introduction of a measure function $h(x)$ such that:

$$\int h(x) \mathrm{d}^{d+1}x \ f \star g = \int h(x) \mathrm{d}^{d+1}x \ g \star f.$$  \hspace{1cm} (3.29)

On expanding the star product in powers of $\theta$ by means of (2.22) we obtain at first order in $\theta$

$$\left[ \int h(x)(f \star g - g \star f) \right]_\theta = \theta \int h(x) \mathrm{d}^k \left( \partial_\mu f \cdot \partial_\lambda g - \partial_\lambda f \cdot \partial_\mu g \right)$$  \hspace{1cm} (3.30)

thus, upon integrating by parts, we obtain that the cyclicity condition is satisfied at first order in $\theta$ if $h(x)$ satisfies the following equations:

$$\partial_\mu h(x) = 0, \quad x^k \partial_k h(x) = -d \ h(x)$$  \hspace{1cm} (3.31)

Examples of solutions are [48] in $d + 1$-dimensions:

$$r^{-d}, \text{ where } r = \sqrt{\sum_i x_i^2} \left( \prod_{i=1}^{d} x_i \right)^{-1} \left( \sum_{i=1}^{d} (x_i)^k \right)^{-\frac{d}{2}}.$$  \hspace{1cm} (3.32)

However, such choices break down already at second order. Indeed, after some calculation we obtain

$$\left[ \int \mathrm{d}^{d+1}x \ h(x)(f \star g - g \star f) \right]_\theta = \frac{\theta^2}{2} \int \mathrm{d}^{d+1}x \ h(x) \left\{ x^\lambda x^\mu \left( \partial_\mu f \cdot \partial_\lambda g - \partial_\lambda f \cdot \partial_\mu g \right) \right\}.$$  \hspace{1cm} (3.33)

On integrating by parts and on using the first order conditions (3.31), we get that the only solution is $h(x) = 0$.

Another possibility would be to consider a natural modification of the star product here introduced. Indeed, the star product in proposition 2.1 by reduction of the Wick–Voros product singles out just one of the possible twists reproducing $\kappa$-Minkowski relations, i.e. the non-symmetric Jordanian twist (3.10) [43, 44]. However there exists a possible symmetrization of this twist, introduced in [47] which we here briefly review.

For $F = f^a \otimes f_a$, a twisting two-tensor of the Hopf algebra $(U_\kappa, \Delta, S, \epsilon)$ and $\lambda = \sqrt{f^a S(f_a)}$ there exists a related twisting two-tensor $F^{(\lambda)}_r$,

$$F^{(\lambda)}_r \coloneqq \lambda^{-1} \otimes \lambda^{-1} F \Delta(\lambda)$$  \hspace{1cm} (3.34)

which is locally $r$-symmetric, with $r$ the classical $r$ matrix. One says that the twist is locally $r$-symmetric if it satisfies the cocycle (3.1) and normalization (3.2) conditions and if its
expansion in powers of the deformation parameter \( \theta \) has the form: 
\[
F_r (\theta) = 1 + c \theta r + O(\theta^2)
\]
where \( c \neq 0 \) is a numerical coefficient.

One can notice immediately that the Jordanian twist (3.10) is not \( r \)-symmetric, since the classical \( r \)-matrix associated with it is \( r = P_\theta \wedge J \).

The corresponding \( r \)-symmetric version of (3.10) is given by [47]
\[
F_{Jrs} = \exp \left( \frac{\theta}{2} (JP_0 \otimes 1 + 1 \otimes JP_0) \right) \exp \left( \ln (1 + \theta P_0) \otimes J \right)
\]
\[
\times \exp \left( -\frac{\theta}{2} (JP_0 \otimes 1 + J \otimes P_0 + P_0 \otimes J + 1 \otimes JP_0) \right).
\]
(3.35)

Its inverse can be re-written using the differential representation of the generators \( J \) and \( P_0 \):
\[
F_{Jrs}^{-1} = \exp \left( -\frac{\theta}{2} (x^\mu \partial_\mu \partial_0 \otimes 1 + x^\mu \partial_\mu \otimes \partial_0 + \partial_0 \otimes x^\mu \partial_\mu + 1 \otimes x^\mu \partial_\mu \partial_0) \right) \exp \left( \ln (1 + \theta \partial_0) \otimes x^\mu \partial_\mu \right)
\]
\[
\times \exp \left( \frac{\theta}{2} (x^\mu \partial_\mu \partial_0 \otimes 1 + 1 \otimes x^\mu \partial_\mu \partial_0) \right).
\]
(3.36)

For the \( * \)-product we can expand all three exponents in this symmetric twist to be able to check the cyclic property order by order:
\[
f_{Jrs} g = f \cdot g + \frac{\theta}{2} x^\mu \left( \partial_\mu f \cdot \partial_\mu g - \partial_\mu f \cdot \partial_0 g \right)
\]
\[
+ \left( \frac{\theta^2}{2} \right) \left\{ x^\mu x^\nu \left( \partial_\mu \partial_\nu \partial_\mu f \cdot g + f \cdot \partial_\mu \partial_\nu \partial_0 g + 2 \partial_\mu \partial_\nu \partial_\mu g \cdot \partial_0 f + 2 \partial_\mu \partial_\nu \partial_\mu f \cdot \partial_0 g + 2 \partial_\mu \partial_\nu \partial_\mu f \cdot \partial_\mu f + \partial_\nu \partial_0 g \cdot \partial_\mu f \right) \right.
\]
\[
+ 2 x^\mu \left( \partial_\mu \partial_\mu f \cdot \partial_0 g + \partial_\mu f \cdot \partial_\mu \partial_0 g + 3 \partial_\mu \partial_\mu f \cdot \partial_\mu g + \partial_\mu f \cdot \partial_\mu f \cdot \partial_0 g \right)
\]
\[
+ \partial_\mu \partial_\mu f \cdot \partial_0 g + \partial_\mu \partial_\mu \partial_0 g \left) \right) + O(\theta^3).
\]
(3.37)

One can notice that the first order is the usual \( \kappa \)-Minkowski \( * \)-product (see e.g. [20, 48, 49], which coincides for all the symmetric twists (Abelian and Jordanian) related to the \( \kappa \)-Minkowski algebra. Therefore we know that one has to modify the integral introducing the additional measure function \( h(x) \) such that:
\[
\int h(x) d^{d+1}x f *_{Jrs} g = \int h(x) d^{d+1}x \ g *_{Jrs} f. \]
(3.38)

At the first order we get:
\[
\left[ \int h(x) (f *_{Jrs} g - g *_{Jrs} f) \right] = \theta \int h(x) x^\nu \left( \partial_\nu f \cdot \partial_0 g - \partial_\nu f \cdot \partial_0 g \right)
\]
(3.39)

that is, the same expression as for the non-symmetric twist, equation (3.30). Therefore, after integrating by parts we obtain the same equations for the measure function \( h \), (3.31), as in the previous case, which again can be shown to be incompatible with the second order conditions except for the trivial solution \( h = 0 \). Therefore we conclude that we do not find any measure of the form \( d\mu = h(x) \Omega \) with respect to which \( \kappa \)-Minkowski star products deriving from Jordanian twists can be made cyclic.
4. Conclusions and outlook

In this paper we have derived a new ∗-product realizing the $κ$-Minkowski Lie algebra at the level of space–time coordinate functions, which is obtained as a symplectic realization in terms of quadratic linear functions on $\mathbb{C}^d$ with Wick–Voros noncommutativity. We have found that it corresponds to the ∗-product provided by a non-symmetric Jordanian twist constructed in terms of the generators of the two-dimensional Borel algebra $b = \text{span} \{ P_0, J; [J, P_0] = P_0 \}$. The new star product is not cyclic with respect to the natural integration volume on $\mathbb{R}^d$, $\Omega = dx^0dx^1 \ldots dx^d$ nor with respect to $h(x)[\Omega]$, with $h(x)$ any measure function. Moreover, the lack of cyclicity does not depend on the details of the Jordanian twist, because we have shown that the same negative result holds for symmetrized Jordanian twist.

The existence of a star product which is not only cyclic, but also closed with respect to the corresponding trace functional, which is known as the generalized Connes–Flato–Sternheimer conjecture, is proven\textsuperscript{14} in [50]. The problem is how to find it. A possible strategy, see e.g. [51, 52], which we plan to investigate elsewhere, is to start with some appropriate star product, for example the one we have derived in the present paper, and then use the gauge freedom [7] in the definition of the star product to obtain the desirable one. Indeed if ∗ and ∗′ are two different star products corresponding to the same Poisson bi-vector $x^k \partial_k$ (in the present case we have $\omega = x^k \partial_k$) they are related by a local transformation

$$T(f ∗ g) = (Tf ∗′ Tg),$$

where $T = 1 + O(\theta)$ is what we shall call the gauge operator. An instance of such a procedure can be found in [53], where a gauge operator $T$ was constructed, realizing the equivalence between the Gutt star product [54] and the Kontsevich one on the dual of Lie algebras. In [52] this approach is used to determine the gauge operator connecting the Weyl ordered star-product of $su(2)$ type on $\mathbb{R}^3$ to the closed one.

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\textsuperscript{14} For convenience we report the result theorem [50]. Let $\mathcal{M}$ be a Poisson manifold with the bivector field $\omega(x)$, and let $\Omega$ be any volume form on $\mathcal{M}$ such that $\text{div}_{\mathcal{M}} \omega = 0$. Then there exists a star product on $C^\infty(\mathcal{M})$ such that for any two functions $f$ and $g$ with compact support one has:

$$\int (f ∗ g) \cdot \Omega = \int f \cdot g \cdot \Omega.$$

(4.1)
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