A modified walk-on-sphere method for high dimensional fractional Poisson equation

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Abstract
We develop walk-on-sphere method for fractional Poisson equations with Dirichlet boundary conditions in high dimensions. The walk-on-sphere method is based on probabilistic representation of the fractional Poisson equation. We propose efficient quadrature rules to evaluate integral representation in the ball and apply rejection sampling method to drawing from the computed probabilities in general domains. Moreover, we provide an estimate of the number of walks in the mean value for the method when the domain is a ball. We show that the number of walks is increasing in the fractional order and the distance of the starting point to the origin. We also give the relationship between the Green function of fractional Laplace equation and that of the classical Laplace equation. Numerical results for problems in 2–10 dimensions verify our theory and the efficiency of the modified walk-on-sphere method.

KEYWORDS
fractional Laplacian, inexact sampling, modified walk-on-sphere method, walk on spheres

1 | INTRODUCTION
The fractional Laplacian, \((-\Delta)^s\), is a prototypical operator for modeling nonlocal and anomalous phenomenon which incorporates long range interactions [1–5]. It arises in many areas of applications, including models for turbulent flows, porous media flows, pollutant transport, quantum mechanics, stochastic dynamics, and finance [6–9].

Numerical methods for fractional Laplacian operator and differential equations with fractional Laplacian operator have been investigated in dozens of few papers, such as in finite difference
methods [10–12], spectral methods [13–15], finite element methods [16–18] and probabilistic methods [19, 20]. See review papers [21–23] for more details. All these methods are nonlocal and thus expensive in high dimensions, except the probabilistic methods. While the most economical method is with quasi-linear complexity in number of physical nodes [17] using finite element methods in 2D, no numerical results are reported for Poisson equation with fractional Laplacian over general bounded domain in high dimensions such as in three or much higher dimensions.

Probabilistic methods (usually implemented with Monte Carlo methods, say, e.g., [24, 25]) for partial differential equations with/without fractional Laplacian are based on the probabilistic representation of the Laplacian/fractional Laplacian, see, for example, [26]. These methods do not require any discretization in space. In one of such methods, walk-on-sphere method (e.g., [27]) does not even require discretization in time or even the diffusion trajectory of the stochastic process. Such probabilistic methods are particularly advantageous when the geometry domain $\Omega$ is very complex or if the solution of the partial differential equation is required at a relatively small number of points.

Though Monte Carlo methods need $\mathcal{O}(M^2)$ walks to achieve $\mathcal{O}(M^{-1})$, it is a reliable method in arbitrarily high dimensions. In addition, they can be efficiently implemented on massively parallel computers.

In this work, we develop efficient probabilistic methods in high dimensions for the following fractional Poisson equation on an open bounded domain with an extended Dirichlet boundary value condition (see, e.g., in [28]):

$$\begin{cases}
(-\Delta)^s u(x) = f(x), & x \in \Omega, \\
u(x) = g(x), & x \in \mathbb{R}^n \setminus \Omega,
\end{cases} \quad (1.1)$$

where $s \in (0, 1)$, $n \geq 1$ and we use the integral definition defined by a singular integral [29] which coincides with Riesz derivative definition on the whole space [30],

$$(-\Delta)^s u(x) = C(n, s) \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy. \quad (1.2)$$

Here P.V. stands for the Cauchy principal value and the constant $C(n, s)$ is given by [30]

$$C(n, s) = \left( \int_{\mathbb{R}^n} \frac{1 - \cos \zeta_1}{|\zeta|^{n+2s}} \, d\zeta \right)^{-1} = \frac{s^{2n} \Gamma \left( \frac{n}{2} + s \right)}{\pi^{n/2} \Gamma(1-s)} \quad (1.3)$$

with $\zeta_1$ being the first component of $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{R}^n$ and $\Gamma$ representing the Gamma function.

We develop our probabilistic method along the line of walk on spheres developed in [20, 27, 31–34]. We use the modified walk-on-sphere method based on Poisson kernel and Green function to solve Equation (1.1), which is also called “one point random estimation” (OPRE) method [31]. Specifically, every jump of one particle can be simulated with a certain probability until this particle is out of domain $\Omega$ (see Figure 1) and all of the particles’ processes compose the approximate solution.

The difficulty in the implementation of this method is the computation of the probability in high dimensions, which has not been addressed in literature. We will introduce some quadrature methods in Sections 2–4 and use rejection sampling method to draw samples whenever the exact sampling is not feasible. Another practical issue when implementing this approach is to estimate the average number of steps for it. When $s \to 1$, the Green function of fractional Laplacian equation is reduced to that of integer-order case. This issue is discussed in Section 4.
The contributions of this work are summarized as follows.

- We apply quadrature rules to the singular representation of walk on spheres and then numerically solve Equation (1.1) in $n$-dimensional ball. We present some convergence analysis of the proposed approach. Compared with [20], we give the deterministic numerical method to solve equation in $n$-dimensional ball.

- We provide the modified walk-on-sphere method to numerically solve Equation (1.1) on general domains in high dimensions. We find approximate probabilities of walk on spheres and draw samples from these probabilities. Compared with [20], the current work can be applied to arbitrary dimensions by rejection sampling method and reduces the computational time because of OPRE method. We illustrate the efficiency of the proposed approach using a numerical example of fractional Poisson equation in 10 dimensions.

- We give an upper bound of the number of walks in the modified walk-on-sphere method for fractional Poisson equation in a ball. The upper bound is a function increasing with respect to the fractional order $s$ and the distance of the starting point $x_0$ to the origin. When $s \to 1$, Green function for the fractional Laplacian equation degenerates into that of the classical Laplace equation.

The rest of this paper is outlined as follows. In Section 2, we present the probabilistic representation for the homogeneous problem (1.1) where the domain $\Omega$ is a ball. We also present quadrature rules to approximate the integrals in the representation.

In Section 3, we present the modified walk-on-sphere algorithm for the Equation (1.1) on an open bounded domain $\Omega$ in one dimension and high dimensions. For high dimensional problems, we present a simple and efficient rejection sampling method based on the truncated Gaussian distribution to draw samples from the probabilities of random walks.

In Section 4, we derive an estimate of number of walks for the method when the problem is considered on a ball. We show that the number of walks is increasing with respect to the fractional order and the distance of the starting point to the origin. We also give the relationship of Green functions between the fractional Laplacian and the classical Laplace equation.

In Section 5, numerical experiments are performed to confirm the convergent order of the proposed numerical evaluations in Section 2 and the efficiency of the modified walk-on-sphere method in Section 3. Finally, we summarize our work in the last section.
2 | PROBABILISTIC REPRESENTATION FOR FRACTIONAL POISSON EQUATION

In this section, we give an integral representation of $u(x)$ in (1.1) where $\Omega$ is a ball centered at $x \in \mathbb{R}^n$. The representation formula of the homogeneous equation is discretized by using quadrature formula and the corresponding error estimates are derived for $n$ dimensional case.

2.1 Fractional Poisson equation on a ball

We start from Equation (1.1) where $\Omega$ is a ball centered at the origin with radius $r > 0$, that is,

$$\begin{cases}
(−Δ)^s u(x) = f(x), & x \in \mathbb{B}_r, \\
u(x) = g(x), & x \in \mathbb{R}^n \setminus \mathbb{B}_r.
\end{cases}$$

(2.1)

To give the integral representation for $u(x)$, we introduce the following definitions.

**Definition 2.1** ([35]) Let $r > 0$ be fixed. For any $x \in \mathbb{B}_r$ and any $y \in \mathbb{R}^n \setminus \mathbb{B}_r$, the Poisson kernel $P_r$ is defined by

$$P_r(x, y) = \frac{\alpha(n, s)}{|x - y|^n},$$

(2.2)

where the constant $\alpha(n, s)$ is given by

$$\alpha(n, s) = \frac{\Gamma\left(\frac{n}{2}\right) \sin(\pi s)}{\pi^{\frac{n+1}{2}}},$$

(2.3)

**Definition 2.2** ([35]) Let $r > 0$ be fixed. For any $x, y \in \mathbb{B}_r$ and $x \neq y$, Green function $G$ is defined by

$$G(x, y) = \begin{cases}
\kappa(1, \frac{1}{2}) \log \left(\frac{r^2 - |x|^2 + \sqrt{(r^2 - x^2)(r^2 - y^2)}}{|x - y|}\right), & n = 1, \\
\kappa(n, s) |x - y|^{2-n} \int_0^{r(x, y)} \frac{t^{n-1}}{(t+1)^{\frac{n}{2}}} dt, & n \geq 2,
\end{cases}$$

(2.4)

where

$$r^s(x, y) = \frac{(r^2 - |x|^2)(r^2 - |y|^2)}{r^2 |x - y|^2},$$

(2.5)

and $\kappa(n, s)$ denotes a normalization constant

$$\kappa(n, s) = \begin{cases}
\frac{1}{\pi}, & n = 1, \\
\frac{\Gamma\left(\frac{n}{2}\right)}{2^{\frac{n}{2}} \pi^{\frac{n}{2}} \Gamma(s)}, & n \geq 2.
\end{cases}$$

(2.6)

Then the representation formula for (2.1) is stated in the following theorem.

**Theorem 2.1** ([35]) Let $r > 0$, $f \in C^{2+\epsilon}(\mathbb{B}_r) \cap C(\overline{\mathbb{B}_r})$ for sufficiently small $\epsilon > 0$ and $g \in L^1_1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Then there exists a unique continuous solution to (2.1) which is given by

$$u(x) = \int_{\mathbb{R}^n \setminus \mathbb{B}_r} P_r(x, y)g(y)dy + \int_{\mathbb{B}_r} G(x, y)f(y)dy.$$  

(2.7)

From Theorem 2.1, we can derive the representation formula for problem (1.1) with $\Omega$ being an arbitrary ball, centered at $x_0 \in \mathbb{R}^n$, namely
\[
\begin{aligned}
(-\Delta)^n u(x) &= f(x), & x &\in B_r(x_0), \\
u(x) &= g(x), & x &\in \mathbb{R}^n \setminus B_r(x_0).
\end{aligned}
\]

(2.8)

Through variable translation and replacement, we obtain

\[
u(x) = \int_{\mathbb{R}^n \setminus B_r(x_0)} P_r(x-x_0, y-x_0) g(y) dy + \int_{B_r(x_0)} G(x-x_0, y-x_0) f(y) dy.
\]

(2.9)

2.2 | Numerical method for (2.1) using the Poisson kernel

In this subsection, we first derive the numerical method for the following Dirichlet problem,

\[
\begin{aligned}
(-\Delta)^n u(x) &= 0, & x &\in B_r, \\
u(x) &= g(x), & x &\in \mathbb{R}^n \setminus B_r.
\end{aligned}
\]

(2.10)

From Theorem 2.1,

\[
u(x) = \int_{\mathbb{R}^n \setminus B_r} P_r(x, y) g(y) dy,
\]

(2.11)

provided that \( g(x) \) satisfies the condition in Theorem 2.1.

To compute this integral in the above formula, we use change of variables by utilizing the hyperspherical coordinates with radius \( \rho > r \), angles \( \varphi_1, \varphi_2, \ldots, \varphi_{n-2} \in [0, \pi] \), and \( \theta \in [0, 2\pi] \). Then, it holds for \( n \geq 3 \) that

\[
\begin{aligned}
y_1 &= \rho \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{n-2} \sin \theta, \\
y_2 &= \rho \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{n-2} \cos \theta, \\
y_3 &= \rho \sin \varphi_1 \sin \varphi_2 \cdots \cos \varphi_{n-2}, \\
\vdots \\
y_{n-1} &= \rho \sin \varphi_1 \cos \varphi_2, \\
y_n &= \rho \cos \varphi_1.
\end{aligned}
\]

(2.12)

The Jacobian of the transformation is given by \( \rho^{n-1} \sin^{n-2} (\varphi_1) \sin^{n-3} (\varphi_2) \cdots \sin (\varphi_{n-2}) \). Here we discuss the case with \( n \geq 3 \). Two dimensional case can be derived similarly so is omitted here or is left for the interested reader. Without loss of generality and up to rotations, we assume \( x = e_n = |x| \begin{pmatrix} 0 \\ 0 \\ \ldots \\ 1 \end{pmatrix} \), so

\[|x-y|^2 = \rho^2 + |x|^2 - 2 \rho |x| \cos \varphi_1, \quad n \geq 3,\]

(2.13)

see Figure 2.
Now we have

\[
    u(x) = \alpha(n,s)(r^2 - |x|^2)^s \int_0^\pi \int_0^{2\pi} \int_0^\infty \frac{g(\rho, \theta, \varphi_1, \ldots, \varphi_{n-2})}{(\rho^2 + |x|^2 - 2|\rho x| \cos \varphi_1)^{3/2}} \rho^{n-1} \sin^{n-2}(\varphi_1) \sin^{n-3}(\varphi_2) \cdot \cdot \cdot \sin (\varphi_{n-2}) \, d\rho \, d\theta \, d\varphi_1 \cdot \cdot \cdot d\varphi_{n-2}. \tag{2.14}
\]

To compute this integral, we perform the substitution \( \rho = \frac{r}{\rho'} \) and rename \( \rho' \) as \( \rho \),

\[
    u(x) = \alpha(n,s)(r^2 - |x|^2)^s r^{n-2s} \int_0^\pi \int_0^{2\pi} \int_0^1 \rho^{2s-1} (1 - \rho^2)^{-s} \frac{g\left(\frac{r}{\rho}, \theta, \varphi_1, \ldots, \varphi_{n-2}\right)}{(r^2 + \rho^2|x|^2 - 2r\rho|x| \cos \varphi_1)^{3/2}} \times \sin^{n-2}(\varphi_1) \sin^{n-3}(\varphi_2) \cdot \cdot \cdot \sin (\varphi_{n-2}) \, d\rho \, d\theta \, d\varphi_1 \cdot \cdot \cdot d\varphi_{n-2}. \tag{2.15}
\]

When \( s \in \left(0, \frac{1}{2}\right) \), we separate integral into two parts as follows

\[
    u(x) = \alpha(n,s)(r^2 - |x|^2)^s r^{n-2s} \int_0^\pi \int_0^{2\pi} \int_0^1 \rho^{2s-1} (1 - \rho^2)^{-s} \frac{g\left(\frac{r}{\rho}, \theta, \varphi_1, \ldots, \varphi_{n-2}\right)}{(r^2 + \rho^2|x|^2 - 2r\rho|x| \cos \varphi_1)^{3/2}} \times \sin^{n-2}(\varphi_1) \sin^{n-3}(\varphi_2) \cdot \cdot \cdot \sin (\varphi_{n-2}) \, d\rho \, d\theta \, d\varphi_1 \cdot \cdot \cdot d\varphi_{n-2} = c(n,s)(r^2 - |x|^2)^s r^{n-2s} [I_1(x) + I_2(x)]. \tag{2.16}
\]

Through change of variable \( \rho = \frac{\rho'}{2} \) and \( \theta = 2\pi \theta' \), \( \varphi_1 = \pi \varphi_1', \ldots, \varphi_{n-2} = \pi \varphi_{n-2}' \), \( I_1(x) \) can be rewritten as

\[
    I_1(x) = \pi^{n-1} \int_0^1 \int_0^1 \int_0^1 \left( \frac{\rho'}{2} \right)^{2s-1} \left[ 1 - \left( \frac{\rho'}{2} \right)^2 \right]^{-s} \frac{g\left(\frac{2\rho'}{r}, 2\pi \theta', \pi \varphi_1', \ldots, \pi \varphi_{n-2}'\right)}{(r^2 + \left(\frac{\rho'}{2}\right)^2 |x|^2 - r\rho'|x| \cos \left(\varphi_1'\right))^{3/2}} \times \sin^{n-2}(\pi \varphi_1') \sin^{n-3}(\pi \varphi_2') \cdot \cdot \cdot \sin (\pi \varphi_{n-2}') \, d\rho' \, d\theta' \, d\varphi_1' \cdot \cdot \cdot d\varphi_{n-2}' \tag{2.17}
\]

where

\[
    \omega_1(x, \rho, \theta, \varphi_1, \ldots, \varphi_{n-2}) = \left[ 1 - \left( \frac{\rho}{2} \right)^2 \right]^{-s} \frac{g\left(\frac{2\rho}{r}, 2\pi \theta, \pi \varphi_1, \ldots, \pi \varphi_{n-2}\right)}{(r^2 + \left(\frac{\rho}{2}\right)^2 |x|^2 - r\rho|x| \cos \pi \varphi_1)^{3/2}} \times \sin^{n-2}(\pi \varphi_1) \sin^{n-3}(\pi \varphi_2) \cdot \cdot \cdot \sin (\pi \varphi_{n-2}). \tag{2.18}
\]

By the affine transformations \( \rho = \frac{1}{2} \rho' + \frac{1}{2} \) and \( \theta = 2\pi \theta' \), \( \varphi_1 = \pi \varphi_1' \), \ldots, \( \varphi_{n-2} = \pi \varphi_{n-2}' \), \( I_2(x) \) is given by
\[ I_2(x) = \pi^{n-1} \int_0^1 \cdots \int_0^1 \left[ 1 - \left( \frac{\rho' + 1}{2} \right)^2 \right]^{-s} g \left( \frac{2s}{\rho' + 1}, 2\pi \theta', \pi \varphi', \ldots, \pi \varphi'_{n-2} \right) \frac{1}{\left( r^2 + \left( \frac{\rho' + 1}{2} \right)^2 \right)^{|x|} - r(\rho' + 1)|x| \cos \pi \varphi'_1} \] 
\[ \times \left( \frac{\rho' + 1}{2} \right)^{2s-1} \sin^{n-2}(\pi \varphi'_1) \sin^{n-3}(\pi \varphi'_2) \cdots \sin(\pi \varphi'_{n-2}) \right) d\rho' d\theta' d\varphi' \cdots d\varphi'_{n-2} \]
\[ = \pi^{n-1} \int_0^1 \cdots \int_0^1 \left( \frac{1 - \rho}{2} \right)^{-s} \omega_2(x, \rho, \theta, \varphi_1, \ldots, \varphi_{n-2}) d\rho d\theta d\varphi_1 \cdots d\varphi_{n-2}, \quad (2.19) \]
where
\[ \omega_2(x, \rho, \theta, \varphi_1, \ldots, \varphi_{n-2}) = \left( \frac{3 + \rho}{2} \right)^{-s} \left( \rho + \frac{1}{2} \right)^{2s-1} g \left( \frac{2s}{\rho + 1}, 2\pi \theta, \pi \varphi_1, \ldots, \pi \varphi_{n-2} \right) \frac{1}{\left( r^2 + \left( \frac{\rho' + 1}{2} \right)^2 \right)^{|x|} - r(\rho + 1)|x| \cos \pi \varphi_1} \] 
\[ \times \sin^{n-2}(\pi \varphi_1) \sin^{n-3}(\pi \varphi_2) \cdots \sin(\pi \varphi_{n-2}). \quad (2.20) \]

We set the uniform grid \( \{ \rho_i \}_{j=1}^N \) for \( \rho \), \( \{ \theta_j \}_{j=1}^M \) for \( \theta \), \( \{(\varphi_m)_{j_k} \}_{j_k=1}^{M_m} \) for \( \varphi_m \), \( m = 1, 2, \ldots, n - 2 \), and \( \{ t_k \}_{k=1}^L \) for \( t \). Here \( \rho_i = \rho t_i, \theta_j = \theta_t, (\varphi_m)_{j_k} = k_m h_m \), and \( t_k = t h_t \) with \( h_{\rho} = \frac{1}{N}, h_{\theta} = \frac{1}{M}, h_m = \frac{1}{M_m}, \) and \( h_t = \frac{1}{L} \). We also define the interpolation operator by recursive formula,
\[ I_{[\rho_{i-1}, \rho_i]}^{[\theta_{j-1}, \theta_j]}(\varphi, \rho, \theta, \varphi_1, \ldots, \varphi_{n-2}) \]
\[ = \frac{1}{h_{\rho}} \left[ (\rho - \rho) v(\rho_{i-1}, \theta, \varphi_1, \ldots, \varphi_{n-2}) + (\rho - \rho_{i-1}) v(\rho_i, \theta, \varphi_1, \ldots, \varphi_{n-2}) \right], \]
\[ I_{[\rho_{i-1}, \rho_i]}^{[\theta_{j-1}, \theta_j]}(\varphi, \rho, \theta, \varphi_1, \ldots, \varphi_{n-2}) = I_{[\rho_{i-1}, \rho_i]}^{[\theta_{j-1}, \theta_j]}(\varphi, \rho, \theta, \varphi_1, \ldots, \varphi_{n-2}), \]
\[ \ldots \]
\[ I_{[\rho_{i-1}, \rho_i]}^{[\theta_{j-1}, \theta_j]}(\varphi, \rho, \theta, \varphi_1, \ldots, \varphi_{n-2}) = I_{[\rho_{i-1}, \rho_i]}^{[\theta_{j-1}, \theta_j]}(\varphi, \rho, \theta, \varphi_1, \ldots, \varphi_{n-2}), \]
\[ = \left( \frac{\pi}{2} \right)^{-n} h_{\rho} \prod_{m=1}^{n-2} h_{m} \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{k_1=1}^{M_1} \cdots \sum_{k_{n-2}=1}^{M_{n-2}} \left[ \omega_1(x, \rho, \theta, \varphi_1, \ldots, \varphi_{n-2}) \right] \right) d\rho d\theta d\varphi_1 \cdots d\varphi_{n-2} \]
\[ = \left( \frac{\pi}{2} \right)^{-n} h_{\rho} \prod_{m=1}^{n-2} h_{m} \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{k_1=1}^{M_1} \cdots \sum_{k_{n-2}=1}^{M_{n-2}} \left[ \omega_1(x, \rho_1, \theta_j, \varphi_{1_{k_1}}, \ldots, \varphi_{n-2_{k_{n-2}}}) \right] + A_2(i) \omega_1(x, \rho, \theta_j, \varphi_{1_{k_1}}, \ldots, \varphi_{n-2_{k_{n-2}}}) \]
\[ \triangleq I_1^h(x), \quad (2.22) \]

where
\[ \begin{align*}
A_1(i) &= 2^{2s-2s} \left[ \rho_i^{2s} - \rho_{i-1}^{2s} \right] - 2^{2s-2s} \left[ \rho_i^{2s+1} - \rho_{i-1}^{2s+1} \right], \\
A_2(i) &= 2^{1-2s} \left[ \frac{1}{2s+1} \left( \rho_i^{2s+1} - \rho_{i-1}^{2s+1} \right) + \frac{2s+1}{2s} \left( \rho_i^{2s} - \rho_{i-1}^{2s} \right) \right].
\end{align*} \quad (2.23)
Similarly, $I_2(x)$ can be approximated as follows,

$$I_2(x) \approx \pi^{n-1} \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{k_i=1}^{M_{i-1}} \cdots \sum_{k_{n-2}=1}^{M_{n-2}} \int_{(\varphi_{n-2})_{k_{n-2}}}^{(\varphi_{n-2})_{k_{n-1}}} \cdots \int_{(\varphi_{n-2})_{k_{i+1}}}^{(\varphi_{n-2})_{k_i}} \left[ \omega_2(x, \rho, \theta, \varphi_1, \ldots, \varphi_{n-2}) \right] d\rho d\theta d\varphi_1 \cdots d\varphi_{n-2}$$

$$= \left( \frac{\pi}{2} \right)^{n-1} h_B \prod_{m=1}^{n-2} \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{k_i=1}^{M_{i-1}} \cdots \sum_{k_{n-2}=1}^{M_{n-2}} \left\{ \sum_{j'=j-1}^{j} \sum_{k_i'=k_i-1}^{k_i} \cdots \sum_{k_{n-2}'=k_{n-2}-1}^{k_{n-2}} \left[ B_1(i) \omega_2 \left( x, \rho_{j-1}, \theta_{j'}, (\varphi_1)_{k_i'}, \ldots, (\varphi_{n-2})_{k_{n-2}'} \right) + B_2(i) \omega_2 \left( x, \rho_{j-1}, \theta_{j'}, (\varphi_1)_{k_i'}, \ldots, (\varphi_{n-2})_{k_{n-2}'} \right) \right] \right\}$$

$$\triangleq I_2^h(x), \quad (2.24)$$

where

$$B_1(i) = 2^n \left\{ \frac{1-\rho_i}{1-s} \left[ (1-\rho_i)^{1-s} - (1-\rho_{i-1})^{1-s} \right] + \frac{1}{2-s} \left[ (1-\rho_{i-1})^{2-s} - (1-\rho_i)^{2-s} \right] \right\},$$

$$B_2(i) = 2^n \left\{ \frac{1}{2-s} \left[ (1-\rho_i)^{2-s} - (1-\rho_{i-1})^{2-s} \right] + \frac{1-\rho_i}{1-s} \left[ (1-\rho_{i-1})^{1-s} - (1-\rho_i)^{1-s} \right] \right\}. \quad (2.25)$$

Then we derive Scheme I for the approximation of the solution $u(x)$ when $s \in \left( 0, \frac{1}{2} \right)$,

$$u(x) \approx c(n, s) \left( r^2 - |x|^2 \right)^s \rho^{n-2s} \left[ I_1^h(x) + I_2^h(x) \right] \triangleq u_h(x). \quad (2.26)$$

When $s \in \left[ \frac{1}{2}, 1 \right)$, Scheme I is given as follows,

$$u(x) \approx \left( \frac{\pi}{2} \right)^{n-1} h_B \prod_{m=1}^{n-2} h_m 2^{-s} \alpha(n, s) \left( r^2 - |x|^2 \right)^s \rho^{n-2s} \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{k_i=1}^{M_{i-1}} \cdots \sum_{k_{n-2}=1}^{M_{n-2}}$$

$$\left\{ \sum_{j'=j-1}^{j} \sum_{k_i'=k_i-1}^{k_i} \cdots \sum_{k_{n-2}'=k_{n-2}-1}^{k_{n-2}} B_1(i) \left[ \omega_2 \left( x, 2\rho_{j-1} - 1, \theta_{j'}, (\varphi_1)_{k_i'}, \ldots, (\varphi_{n-2})_{k_{n-2}'} \right) \right] + B_2(i) \left[ \omega_2 \left( x, 2\rho_{j-1} - 1, \theta_{j'}, (\varphi_1)_{k_i'}, \ldots, (\varphi_{n-2})_{k_{n-2}'} \right) \right] \right\}, \quad (2.27)$$

where $B_1(i)$ and $B_2(i)$ are defined in Equation (2.25).

**Remark 2.1** The complexity of the quadrature rule is $O\left( N M \prod_{m=1}^{n-2} M_m \right)$. Specially, when $h = h_B = h_0 = h_1 = \ldots = h_{n-2} = \frac{1}{N}$, the complexity is $O(N^n)$.

When the equation has nonconstant source term with homogeneous boundary value, that is, $f(x) \neq 0$ and $g(x) = 0$ in Equation (2.1), we take the two dimensional case as an example. It follows from

$$u(x) = \int_{B_x} G(x, y) f(y) dy = \kappa(n, s) \rho^{n-2s} \left( r^2 - |x|^2 \right)^s$$

$$\times \int_0^1 \left( \int_{B_y \setminus S_0} + \int_{S_0} \right) \left( r^2 - |y|^2 \right)^s \left[ (r^2 - |x|^2) \left( r^2 - |y|^2 \right) t + r^2 |x - y|^2 \right]^{-1} t^{-1} f(y) dy \quad (2.28)$$

in two dimensions that for $x = (|x|, 0) \neq (0, 0)$
where \( S_h \) is a square with width \( h \) centered at \( x \), \( \phi = \arctan \frac{h}{|x|} \) and \( \omega_{31}, \omega_{32}, \omega_{33} \) are integrands in the corresponding fields of Figure 3,

\[
\begin{align*}
\omega_{31}(\rho, \theta, t) &= (r^2 - \rho^2)^2 \frac{f(\rho \cos \theta, \rho \sin \theta)}{(r^2 - |x|^2) (r^2 - \rho^2) t + r^2 [(\rho \cos \theta - |x|)^2 + (\rho \sin \theta)^2]}, \\
\omega_{32}(\rho, \theta, t) &= (r^2 - \rho^2)^2 \frac{f(\rho \cos \theta + |x| - h, \rho \sin \theta + h)}{(r^2 - |x|^2) (r^2 - \rho^2) t + r^2 [(\rho \cos \theta - h)^2 + (\rho \sin \theta + h)^2]}, \\
\omega_{33}(\rho, \theta, t) &= (r^2 - \rho^2)^2 \frac{f(\rho \cos \theta - |x| + h, \rho \sin \theta - h)}{(r^2 - |x|^2) (r^2 - \rho^2) t + r^2 [(\rho \cos \theta + h)^2 + (\rho \sin \theta - h)^2]}. 
\end{align*}
\]

Then we have

\[
u_0(x) = \kappa(2, s)r^{-2L}(r^2 - |x|^2)^5 \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{k=1}^{L} C(k)((2\pi - 2\phi)\rho \omega_{31} \left((r_{l-\frac{1}{2}}, (2\pi - 2\phi)\theta_{j-\frac{1}{2}}, t_{k-\frac{1}{2}}) \right) \\
+ \frac{2\phi(|x| - h)}{\cos(2\phi j_{l-\frac{1}{2}} - \phi)} \omega_{31} \left(\frac{|x| - h}{\cos(2\phi j_{l-\frac{1}{2}} - \phi)}, 2\phi \theta_{j-\frac{1}{2}} - \phi, t_{k-\frac{1}{2}}\right) + 2\phi \left(r - \frac{|x| + h}{\cos(2\phi j_{l-\frac{1}{2}} - \phi)}\right) \\
\times \omega_{31} \left(\frac{|x| - h}{\cos(2\phi j_{l-\frac{1}{2}} - \phi)}, 2\phi \theta_{j-\frac{1}{2}} - \phi, t_{k-\frac{1}{2}}\right) + 2\phi \left(r - \frac{|x| + h}{\cos(2\phi j_{l-\frac{1}{2}} - \phi)}\right) \\
+ \frac{2h\phi}{\cos(\phi j_{l-\frac{1}{2}} - \phi)} \omega_{32} \left(\frac{2h\rho_{l-\frac{1}{2}}}{\cos(\phi j_{l-\frac{1}{2}} - \phi)}, \phi \theta_{j-\frac{1}{2}}, t_{k-\frac{1}{2}}\right) \\
+ \frac{2h\phi}{\cos(\phi j_{l-\frac{1}{2}} - \phi)} \omega_{33} \left(\frac{2h\rho_{l-\frac{1}{2}}}{\cos(\phi j_{l-\frac{1}{2}} - \phi)}, \phi \theta_{j-\frac{1}{2}} - \phi, t_{k-\frac{1}{2}}\right), \quad (2.31)
\]

where \( C(k) = \frac{1}{4} (t_{k}^e - t_{k-1}^e) \).

**FIGURE 3** Two dimensional case
When \( x = (0, 0) \), we have

\[
    u_h(x) = 2\pi(r-h)\kappa(2, s) \sum_{i=1}^{N} \sum_{j=1}^{M} \sum_{k=1}^{L} C(k)\omega_4 \left( (r-h)p_{i-\frac{1}{2}}, 2\pi\theta_{j-\frac{1}{2}}, t_{k-\frac{1}{2}} \right),
\]

where

\[
    \omega_4(\rho, \theta, t) = \left( r^2 - \rho^2 \right)^{2s} \frac{f(\rho, \theta)}{\left( r^2 - \rho^2 \right) t + \rho^2}.
\]

For the higher-dimensional case, the numerical scheme can be similarly derived so is omitted here.

### 2.3 Error estimates of the quadrature rules

In this subsection, we provide error estimates for our numerical method in discretizing the representation formula of the solution \( u(x) \). Firstly we have the following lemma which can be readily derived.

**Lemma 2.1** Let \( I \) be the interpolation operator defined in (2.21) on the domain \( K = [0, \varepsilon_1] \times [0, \varepsilon_2] \times \cdots \times [0, \varepsilon_n] \). Then for any functions \( v \in C^2_b(K) \), we have the error estimate,

\[
    \| v - I v \|_{L^n} \leq \frac{1}{8} \left( \sum_{i=1}^{n} \varepsilon_i \left\| \frac{\partial^2 v}{\partial x_i^2} \right\|_{L^n} \right).
\]

In \( n (n \geq 2) \) dimensional cases, suppose \( g(x) \in C^2_b\left( \mathbb{R}^n \setminus \mathbb{B}_r \right) \). Then when \( s \in \left( 0, \frac{1}{2} \right) \), we obtain

\[
    |u(x) - u_h(x)| \leq c(n, s) \left( r^2 - |x|^2 \right)^s r^{n-2s} |I_1(x) - I^h_1(x)| + |I_2(x) - I^h_2(x)|.
\]

From Lemma 2.1 and the fact \( x \neq y \), we derive

\[
    |I_1(x) - I^h_1(x)| = \left| \pi^{n-1} \sum_{i=1}^{n} \sum_{j=1}^{M} \sum_{k_{n-1}=1}^{M_{n-1}} \int_{\rho_{n-1}}^{\rho} \int_{\theta_{n-1}}^{\theta} \int_{\phi_{n-2}}^{\phi} \omega_1(\rho, \theta, \phi_1, \ldots, \phi_{n-2}) \right|
\]

\[
    \times \left( \frac{B^2}{2} \right)^{2s-1} \rho \, d\rho \, d\theta \, d\phi \cdots d\phi_{n-2}
\]

\[
    \leq \frac{\pi}{s^{2s+3}} \left( h^2_r \left\| \frac{\partial^2 \omega_1}{\partial \rho^2} \right\|_{L^n} + h_\theta^2 \left\| \frac{\partial^2 \omega_1}{\partial \theta^2} \right\|_{L^n} + \sum_{r=1}^{n-2} h^2_r \left\| \frac{\partial^2 \omega_1}{\partial \phi_r^2} \right\|_{L^n} \right)
\]

\[
    \leq C \left( h^2_r + h_\theta^2 + \sum_{r=1}^{n-2} h^2_r \right). \tag{2.36}
\]

Similarly,

\[
    |I_2(x) - I^h_2(x)| \leq C \left( h^2_r + h_\theta^2 + \sum_{r=1}^{n-2} h^2_r \right). \tag{2.37}
\]

When \( s \in \left[ \frac{1}{2}, 1 \right) \), the truncated errors is still \( \Theta \left( h^2_r + h_\theta^2 + \sum_{r=1}^{n-2} h^2_r \right) \). Thus we have the theorem as follows.

**Theorem 2.2** Let \( r > 0, g \in C^2_b\left( \mathbb{R}^n \setminus \mathbb{B}_r \right) \) and \( s \in (0, 1) \). Then it holds that

\[
    |u(x) - u_h(x)| \leq C \left( h^2_r + h_\theta^2 + \sum_{r=1}^{n-2} h^2_r \right), \tag{2.38}
\]

where \( C \) is a positive constant.
When \( f(x) \in C^2_b(\mathbb{B}_1 \setminus S_h) \), it is easily to derive the error estimates for Equation (2.31), \( O(h^{2s} + h_r + h_0^2 + h_0^2) \).

Obviously, the above quadrature for \( n \)-dimensional fractional Laplacian is often difficult to be implemented in computer simulations if \( n > 3 \). So the Monte Carlo method is very likely the best choice for numerical experiments. In the following, we introduce modified walk-on-sphere method, which is one of Monte Carlo methods.

### 3 Modified Walk-on-Sphere Method

In this section, we utilize the modified walk-on-sphere method to solve Equation (1.1) with \( \Omega \) being an arbitrary domain instead of a ball. Assume \( f(x) \) and \( g(x) \) satisfy the condition in Theorem 2.1. Then it is known from Section 2 that the solution to (1.1), \( u(x_0) \), \( x_0 \in \Omega \), has the integral representation

\[
u(x_0) = \int_{\mathbb{R}^n \setminus \mathbb{B}_{r_0}(x_0)} p_{1,r_0}(x, x_0) u(x) dx + a(x_0) \int_{\mathbb{B}_{r_0}(x_0)} p_{2,r_0}(y, x_0) f(y) dy,
\]

where \( \mathbb{B}_{r_0}(x_0) \) is the largest ball contained in \( \Omega \) with radius \( r_0 = \sup \{ r : \mathbb{B}(x_0, r) \subset \Omega \} \), centered at \( x_0 \),

\[
\begin{align*}
p_{1,r_0}(x, x_0) &= P_{r_0}(0, x - x_0), \\
p_{2,r_0}(y, x_0) &= \frac{1}{a(x_0)} G(0, y - x_0),
\end{align*}
\]

and

\[
a(x_0) = \int_{\mathbb{B}_{r_0}(x_0)} G(0, y - x_0) dy = \begin{cases} \\
\kappa(n, s) \int_{\mathbb{B}_{r_0}(x_0)} |y - x_0|^{2s-n} \int_0^{|y-x_0|^2} \frac{r^{-1}}{(t+1)^n} dtdy & \\
\omega_{n-1}(n) \int_0^{r_0^2} \rho^{2s-n} \int_0^{\frac{r_0^2}{(t+1)^n}} \frac{r^{-1}}{(t+1)^n} drdt & \\
\omega_{n-1}(n) \int_0^\infty \int_0^{\frac{r_0^2}{(t+1)^n}} \frac{r^{-1}}{(t+1)^n} drdt & \\
\kappa(n, s) B \left( s, \frac{n}{2} \right) \frac{a_{n-1}}{2s} r_0^{2s}.
\end{cases}
\]

Here \( a(x_0) \) is the normalizing constant such that \( \int_{\mathbb{B}_{r_0}(x_0)} p_{2,r_0}(y, x_0) dy = 1 \). And \( \omega_{n-1} \) denotes the \((n-1)\)-dimensional measure of the unit sphere \( S^{n-1} \). We recall from [35] that

\[
\int_{\mathbb{R}^n \setminus \mathbb{B}_{r_0}(x_0)} p_{1,r_0}(x, x_0) dx = 1.
\]

Both \( p_{1,r_0}(x, x_0) \) and \( p_{2,r_0}(y, x_0) \) can be viewed as probability density function for the random variables \( X \) outside the ball \( \mathbb{B}_{r_0}(x_0) \) and \( Y \) inside the ball, respectively. So we can suppose \( X \) and \( Y \) denote random variables outside the ball with density \( p_{1,r_0}(x, x_0) \) and inside the ball with \( p_{2,r_0}(y, x_0) \), accordingly.

Then the integral representation (3.1) can be rewritten as

\[
u(x_0) = \mathbb{E}u(X) + a(x_0) \mathbb{E}f(Y).
\]

Here \( \mathbb{E} \) indicates expected operation. The first term describes a mean value with respect to \( p_{1,r_0}(y, x_0) \) outside the ball and represents the average score upon exiting the ball \( \mathbb{B}_{r_0}(x_0) \). The second term is a weighted average taken with respect to density \( p_{2,r_0}(x, x_0) \) and represents the expected contribution from sources inside the ball.
Both $p_{1, r_0} (x, x_0)$ and $p_{2, r_0} (y, x_0)$ can be used to construct transition probabilities for a Markov chain. The transition from an initial point $X_0 = x_0$ is performed by selecting a point $X_1$ outside the ball $B_{r_0} (x_0)$ with density $p_{1, r_0} (x, x_0)$ and by generating a random variable $Y_1$ with density $p_{2, r_0} (y, x_0)$. Given the position $X_k = x_k$ at the $k$th step, the transition to the $(k + 1)$-th step is carried out by choosing $X_{k+1}$ with $p_{1, r_k} (x, x_k)$, $r_k = \sup \{ r : B_r (x_k, r) \subset \Omega \}$, outside the ball $B_{r_k} (x_k)$ and by selecting $Y_{k+1}$ according to the density $p_{2, r_k} (y, x_k) (X_{k+1} \text{ and } Y_{k+1} \text{ are independent of each other})$. The walk on spheres is simulated by repeating this procedure until the particle exits the domain $\Omega$. For the point $X_k$, the solution $u(x)$ must satisfy (3.5). We obtain

$$ u (x_k) = \mathbb{E} [u (X_{k+1}) | X_k] + a (X_k) \mathbb{E} [f (Y_{k+1}) | X_k] . $$

(3.6)

Here the conditional expectations are used as the densities of $X_{k+1}$ and $Y_{k+1}$ are determined by the position of $X_k$.

Generally speaking, the connection between the solution to the fractional Laplacian Dirichlet problem and the random process follows from the telescoping summation

$$ u (x_0) = \mathbb{E} u (X_0) = \mathbb{E} u (X_i) + \mathbb{E} \left\{ \sum_{k=0}^{l-1} [u (X_k) - u (X_{k+1})] \right\} $$

$$ = \mathbb{E} u (X_i) + \mathbb{E} \left\{ \sum_{k=0}^{l-1} (u (X_k) - \mathbb{E} [u (X_{k+1}) | X_k]) \right\} , $$

(3.7)

where $X_0 = x_0$. In the last equality, we have used the fact that

$$ \mathbb{E} u (X_{k+1}) = \mathbb{E} \{ \mathbb{E} [u (X_{k+1}) | X_k] \} . $$

(3.8)

Applying identity (3.6) yields

$$ u (x_0) = \mathbb{E} u (X_i) + \mathbb{E} \left\{ \sum_{k=0}^{l-1} a (X_k) \mathbb{E} [f (Y_{k+1}) | X_k] \right\} $$

$$ = \mathbb{E} u (X_i) + \mathbb{E} \left\{ \sum_{k=0}^{l-1} a (X_k) f (Y_{k+1}) \right\} . $$

(3.9)

Now suppose the process jumps out of the boundary on the $l$th step. Then all of the terms on the right-hand side of Equation (3.9) would be known and $u (X_i)$ can be replaced by $g (X_i)$. This suggests that $u (x_0)$ be the mean value of the exit points plus a weighted average from internal contribution.

Monte Carlo method makes use of the preceding observation to estimate $u (x_0)$. According to the density $p_{1, r_j} (x, x_k)$, $X_{k+1}$ will jump out of the ball $B_{r_j} (x_k)$ so that the particle will exit the domain in a finite number of steps. At the conclusion of each walk, we compute the random sample

$$ Z_i = g (X_i^j) + \sum_{k=0}^{l-1} a (X_k^j) f (Y_{k+1}^j) , $$

(3.10)

where $i$ denotes the $i$-th experiment. By identity (3.10), we have $u (x_0) = \mathbb{E} Z_i$. An estimate for the mean of $Z_i$ is given by the statistic

$$ S_N = \frac{1}{N} \sum_{i=1}^{N} Z_i , $$

(3.11)

where $N$ is the number of trials. By the law of large numbers,

$$ \lim_{N \to \infty} S_N = u (x_0) . $$

(3.12)

The central limit theorem gives $O(1/N)$ upper bounds on the variance of the $N$-term sum, which serves as a numerical error estimate.
3.1 Modifying the walk-on-sphere method via approximate sampling methods

For the Monte Carlo sampling, we need to sample $X_{k+1}$ and $Y_{k+1}$ according to their probability density functions. Given the position $X_{k} = x_{k} = (x_{k,1}, x_{k,2}, \ldots, x_{k,n})$, we obtain probability measure for $X_{k+1}$,

$$
\mathbb{P}_{x_{k}}(X_{k+1} \in dx) = p_{1,r_{k}}(x_{k}) = P_{r_{k}}(0, x - x_{k}) = x \in \mathbb{R}^{n} \backslash \mathcal{B}_{r_{k}}(x_{k})
$$

$$
= c(n, s) \left( \frac{r_{k}^{2} - r_{k}^{2}}{|x - x_{k}|^{2}} \right)^{s} \frac{1}{|x - x_{k}|^{n}} dx.
$$

(3.13)

We change variables by using the hyperspherical coordinates with radius $\rho > r_{k}$, angles $\varphi_{1}, \varphi_{2}, \ldots, \varphi_{n-2} \in [0, \pi]$, and $\theta \in [0, 2\pi]$. In this case, it holds that

$$
\begin{align*}
&x_{1} = x_{k,1} + \rho \sin \varphi_{1} \sin \varphi_{2} \cdots \sin \varphi_{n-2} \sin \theta, \\
&x_{2} = x_{k,2} + \rho \sin \varphi_{1} \sin \varphi_{2} \cdots \sin \varphi_{n-2} \cos \theta, \\
&x_{3} = x_{k,3} + \rho \sin \varphi_{1} \sin \varphi_{2} \cdots \cos \varphi_{n-2}, \\
&\cdots \\
&x_{n-1} = x_{k,n-1} + \rho \sin \varphi_{1} \cos \varphi_{2}, \\
&x_{n} = x_{k,n} + \rho \cos \varphi_{1}.
\end{align*}
$$

(3.14)

The Jacobian of the transformation is given by $\rho^{n-1} \sin^{n-2} \varphi_{1} \sin^{n-3} \varphi_{2} \cdots \sin \varphi_{n-2}$. Then we derive

$$
\mathbb{P}_{x_{k}}(X_{k+1} \in dx) = 2 \frac{\pi^{n/2}}{\Gamma(n/2)} c(n, s) r_{k}^{2s} \left( \rho^{2} - r_{k}^{2} \right)^{-\frac{n}{2}} \frac{\rho \sin \varphi_{1} \sin \varphi_{2} \cdots \sin \varphi_{n-2}}{2\pi} \frac{d\theta}{I_{n-2}} \frac{\sin^{n-2} \varphi_{1}}{I_{1}} \frac{1}{r_{k}^{2}} \frac{d\varphi_{2}}{I_{1}} \cdots \frac{\sin \varphi_{n-2}}{I_{1}} d\varphi_{n-2},
$$

(3.15)

where $\rho \geq r_{k}$,

$$
I_{n} = \int_{0}^{\frac{\pi}{2}} \sin^{n} \varphi d\varphi = \left\{ \begin{array}{ll} \frac{n-3}{n-2} \cdots \frac{1}{2} I_{0}, & n \text{ is even}, \\
\frac{n-3}{n-2} \cdots \frac{1}{2} I_{1}, & n \text{ is odd}, \end{array} \right.
$$

(3.16)

with $I_{0} = \pi$, $I_{1} = 2$, and

$$
\prod_{k=1}^{n-2} \int_{0}^{\frac{\pi}{2}} \sin^{k} \varphi d\varphi = I_{1} I_{2} \cdots I_{n-2} = \frac{\pi^{n/2-1}}{\Gamma(n/2)}.
$$

(3.17)

From (3.15), we observe that $\theta$ is sampled uniformly on $[0, 2\pi]$, whereas we can sample the radius $\rho$ via the inverse transform sampling method [20]. For $U \sim ([0, 1])$,

$$
\rho = F^{-1}(U) = r_{k} \left( I^{-1}(1 - U; s, 1 - s) \right)^{-1/2},
$$

(3.18)

where $I(x; z, w)$ is the incomplete beta function with $z$ and $w$ being positive

$$
I(x; z, w) = \frac{1}{B(z, w)} \int_{0}^{x} u^{z-1} (1 - u)^{w-1} du, \quad x \in [0, 1].
$$

(3.19)

For $\varphi_{1}, \varphi_{2}, \cdots, \varphi_{n-2}$, we use again the inverse transform method to simulate them. Denote $I_{n}^{*}(\varphi) = \int_{0}^{\varphi} \sin^{n} \varphi d\varphi$ and $I_{n}(\varphi) = \int_{1}^{\varphi} I_{n}^{*}(\varphi), \varphi \in [0, \pi]$. Then we have

$$
I_{n}(\varphi) = \frac{1}{I_{n}} \left\{ \begin{array}{ll}
\frac{1}{n} \sin^{n-1} \varphi \cos \varphi + \sum_{i=1}^{n/2-1} \frac{1}{n-2i} \sin^{n-2i-1} \varphi \cos \varphi \prod_{j=0}^{i-1} \frac{n-2j}{n-2j}, & n \text{ is even}, \\
\frac{1}{n} \sin^{n-1} \varphi \cos \varphi + \sum_{i=1}^{(n-1)/2} \frac{1}{n-2i} \sin^{n-2i-1} \varphi \cos \varphi \prod_{j=0}^{i-1} \frac{n-2j}{n-2j}, & n \text{ is odd},
\end{array} \right.
$$

(3.20)
where \( I_0^p(\phi) = \phi, I_1^p(\phi) = 1 - \cos \phi \). For random number \( U \sim ([0, 1]) \), we have
\[
\varphi_i = I_{n-i-1}^{-1}(U). \tag{3.21}
\]

We can readily get \( \varphi_{n-2} = \arccos(1 - 2U), U \sim ([0, 1]) \). When \( i \neq n - 2 \), it is complicated to get the inverse density function and we use the rejection sampling method to generate samples \( \varphi_{n-m-1} \) from a target PDF \( \frac{1}{l_m} p(x) = \frac{1}{l_m} \sin^m x, m = 2, \ldots, n - 2 \). The standard RS algorithm \([36]\) allows us to draw samples exactly from the target PDF \( p_0(x) \).

**Algorithm 3.1** ([36]). *Step 1. Choose an alternative simpler proposal PDF \( q_0(x) \).*
*Step 2. Draw \( x' \sim q_0(x) \) and \( \omega' \sim U([0, 1]) \).*
*Step 3. If \( \omega' \leq \frac{p_0(x')}{K q_0(x')} \), then \( x' \) is accepted. Otherwise, \( x' \) is discarded.*
*Step 4. Repeated steps 2–3 until the desired number of samples has been obtained.*

Since the target PDF for \( \varphi_{n-m-1} \) \( p_0(x) = \frac{1}{l_m} p_m(x) = \frac{1}{l_m} \sin^m x, m = 2, \ldots, n - 2 \) is unimodal and symmetric, the proposal PDF is given by truncated Gaussian density,
\[
q_0(x) = C_{q,m} q_m(x) = C_{q,m} \beta \exp \left( -\alpha_m (x - \mu)^2 \right), \quad x \in (0, \pi), \tag{3.22}
\]
where the determination of parameters \( \alpha_m, \beta, \mu \) and the normalizing constant \( C_{q,m} \) is obtained via a proposal function used in the rejection sampling method, and \( K = I_m / C_{q,m} \), such that
\[
q_m(x) \geq p_m(x) \tag{3.23}
\]
It is easy to get \( \beta = 1, \mu = \frac{\pi}{2} \). For parameter \( \alpha_m \), noting that Equation (3.23) implies
\[
\ln q_m(x) = -\alpha_m \left( x - \frac{\pi}{2} \right)^2 \geq m \ln \sin(x), \tag{3.24}
\]
one has
\[
\alpha_m \leq \frac{-m \ln \sin(x)}{\left( x - \frac{\pi}{2} \right)^2} \quad x \in (0, \pi). \tag{3.25}
\]
In order to obtain best positive fit between the proposal and the target, we must set \( \alpha_m = \lim_{x \to \frac{\pi}{2}} \frac{-m \ln \sin(x)}{(x - \frac{\pi}{2})^2} = \frac{m}{2} \); see Figure 4 for details. Thus, samples must be drawn from the selected proposal PDF, \( C_{q,m} e^{-\frac{\beta}{2} \left( x - \frac{\pi}{2} \right)^2} \), \( x \in (0, \pi) \). For the truncated Gaussians, the technique available in the literature \([37]\) allows us to draw samples efficiently. Finally, the acceptance rate which is the key performance measure for rejection sampling method is
\[
\eta = \int_0^\pi \frac{p_m(x)}{q_m(x)} q_0(x) dx = \frac{I_m}{\sqrt{2\pi m} \text{erf} \left( \sqrt{\frac{2m \pi}{4}} \right)}, \tag{3.26}
\]
where \( \text{erf}(x) \) denotes the error function. It is obvious that \( \eta \) is a monotonically increasing function converging to 1 with respect to \( m \) and the acceptance rate is more than 91% (see Figure 4).

Based on the simulation of \( \rho, \theta, \varphi_1, \varphi_2, \cdots, \varphi_{n-2} \), we derive \( x \) so that the random variable \( X_{k+1} \) can be simulated by \( X_{k+1} = x \).

For random variable \( Y_{k+1} \), it is complicated to sample \( Y_{k+1} \) by using density \( p_{2,r_k}(y, x_k) \). Thus, we rewrite the second part in (3.1) in the form of
\[
\int_{B_{r_0}(x_0)} f(y) G(0, y - x_0) dy = \kappa(n, s) \int_{B_{r_0}(x_0)} f(y) |y - x_0|^{2-s} \int_0^{\frac{\pi}{2}} \frac{\gamma-|y-x_0|^2}{(t+1)^{\frac{3}{2}}} dt \tag{3.27}
\]
Performing substitution \( t = \frac{1 - \rho'}{r'} \) yields that

\[
\int_{B_n(x_0)} f(y) G(0, y - x_0) \, dy = \kappa(n, s) \int_{B_n(x_0)} f(y) |y - x_0|^{2s-n} \int_{|y-x_0|^2/\rho_0}^1 (1-t')^{s-1} \, dt' \, dy
\]

\[
= b(x_0) \mathbb{E} \left\{ 1 - I \left( \frac{|Y - x_0|^2}{\rho^2} ; \frac{n}{2} - s, s \right) \right\} f(Y) , \tag{3.28}
\]

where

\[
b(x_0) = \kappa(n, s) B \left( \frac{n}{2} - s, s \right) \int_{B_n(x_0)} |y - x_0|^{2s-n} \, dy = \kappa(n, s) B \left( \frac{n}{2} - s, s \right) \frac{r_n^s \pi^{n/2}}{s \Gamma(n/2)} \tag{3.29}
\]

and the probability density function for \( Y \) is

\[
p_{2r_n}^s(y, x_0) = \frac{s \Gamma(n/2)}{r_n^s \pi^{n/2}} |y - x_0|^{2s-n}. \tag{3.30}
\]

Thus, the Equation (3.9) becomes

\[
u(x_0) = \mathbb{E} \nu(X_i) + \sum_{k=1}^{l-1} \mathbb{E} \left\{ b(X_k) \left[ 1 - I \left( \frac{|Y_{k+1} - x_k|^2}{r_k^2} ; \frac{n}{2} - s, 1 - s \right) \right] f(Y_{k+1}) \right\} , \tag{3.31}
\]

where the random variable \( Y_{k+1} \) obeys the density \( p_{2r_n}^s(y, x_k) \) for the given position \( X_k = x_k \).

Using the hyperspherical coordinates (3.14) for PDF and replacing \( x \) with \( y \) yield

\[
P_{X_k}(Y_{k+1} \in dy) = \frac{2s}{r_k^s} \rho^{2s-1} \, d\rho \times \frac{d\theta}{2\pi} \times \frac{\sin^{n-2} \varphi_1 \, d\varphi_1 \times \cdots \times \sin \varphi_{n-2} \, d\varphi_{n-2}}{I_{n-2} \cdots I_1}, \quad \rho < r_k. \tag{3.32}
\]

Notice that \( \theta \sim U([0, 2\pi]) \), \( \rho = r_k R^{1/2} \), and \( R \sim U([0, 1]) \) enable us to simulate \( \theta \) and \( \rho \). Moreover, we sample \( \varphi_i \) by inverse sample method and \( \varphi_i, i = 2, 3, \ldots, n-2 \), by rejection sampling method given before. Then we can simulate \( Y_{k+1} = y \).

3.2 | One-dimensional fractional Poisson equation

When \( n = 1 \), the integral \( I \left( \frac{|Y_{k+1} - x_k|^2}{r_k^2} ; \frac{1}{2} - s, s \right) \) in (3.27) is infinite if \( \frac{1}{2} < s < 1 \). When \( 0 < s < \frac{1}{2} \) and \( n = 1 \), we can still use the method (3.31) with the probability of the moving point’s direction
\( \theta \) is \( \mathbb{P}(\theta = +1) = \mathbb{P}(\theta = -1) = \frac{1}{2} \). To deal with this troublesome integral, we rewrite (3.27) as follows:

\[
\int_{x_0 - r_0}^{x_0 + r_0} f(y) G(0, y - x_0) \, dy = \kappa(1, s) \int_{x_0 - r_0}^{x_0 + r_0} f(y) \, dy \int_0^{\frac{x_0 + r_0 - y}{2b - \alpha}} \frac{t^{s-1}}{(t + 1)^{\frac{1}{2}}} \, dt \rho^{2s-1} \int_0^{\frac{x_0 + r_0 - y}{2b - \alpha}} \frac{\rho^{s-1}}{(t + 1)^{\frac{1}{2}}} \, d\rho, \quad (3.33)
\]

where we perform substitution \( \rho = \frac{|y - x_0|}{2b} \) and let \( \theta \) denote the random variable with \( \mathbb{P}(\theta = +1) = \mathbb{P}(\theta = -1) = \frac{1}{2} \). Via a change of variable \( t' = \rho^2t \) for the inner integral, we obtain

\[
\int_0^{\frac{x_0 + r_0 - y}{2b - \alpha}} \left( \frac{t'}{\rho^2} \right)^{s-1} \left( \frac{t'}{\rho^2} + 1 \right)^{-\frac{1}{2}} \, dt' = \rho^{1-2s} \int_0^{\frac{x_0 + r_0 - y}{2b - \alpha}} t^{s-1}(1 + \rho^2)^{-\frac{1}{2}} \, dt. \quad (3.34)
\]

It follows from (3.33) that

\[
2\kappa(1, s) \int_0^{x_0 + r_0} \mathbb{E}_\theta \left[ f(x_0 + \rho \theta) \right] \int_0^{\frac{x_0 + r_0 - y}{2b - \alpha}} t^{s-1}(1 + \rho^2)^{-\frac{1}{2}} \, dt \rho^{2s-1} \int_0^{\frac{x_0 + r_0 - y}{2b - \alpha}} \frac{\rho^{s-1}}{(t + 1)^{\frac{1}{2}}} \, d\rho
\]

\[
= 2r_0 \kappa(1, s) \mathbb{E}_\rho \left\{ \mathbb{E}_\theta \left[ f(x_0 + \rho \theta) \right] \int_0^{\frac{x_0 + r_0 - y}{2b - \alpha}} t^{s-1}(1 + \rho^2)^{-\frac{1}{2}} \, dt \right\}
\]

\[
= 2r_0 \kappa(1, s) \mathbb{E}_Y \left[ f(Y) \int_0^{\frac{x_0 + r_0 - y}{2b - \alpha}} t^{s-1}(1 + \rho^2)^{-\frac{1}{2}} \, dt \right] \quad (3.35)
\]

where \( Y = x_0 + \rho \theta \) with \( \rho \sim U([0, r_0]) \) and \( \mathbb{P}(\theta = +1) = \mathbb{P}(\theta = -1) = \frac{1}{2} \). Then we show that the integral in the expectation can be represented by the hypergeometric function. For simplicity, let \( a = |Y - x_0|^2 \) and \( b = r_0^2 - a \). We obtain

\[
\int_0^b t^{s-1}(1 + a)^{-\frac{1}{2}} \, dt = \int_0^1 (bt)^{s-1}(a + bt)^{-\frac{1}{2}} \, dbt
\]

\[
= a^{-\frac{1}{2}} b^s \int_0^1 t^{s-1} \left( 1 + \frac{bt}{a} \right)^{-\frac{1}{2}} \, dt
\]

\[
= a^{-\frac{1}{2}} b^s \left[ \int_0^1 t^{s-1} \left( 1 + \frac{bt}{a} \right)^{-\frac{1}{2}} \, dt - b \int_0^1 t^{s-1} \left( 1 + \frac{b}{a} \right)^{-\frac{1}{2}} \, dt \right]
\]

\[
= \frac{a^{-\frac{1}{2}} b^s}{s(s + 1)} \left[ a(s + 1) F_1 \left( -0.5, s; s + 1; -\frac{b}{a} \right) - bs_2 F_1 \left( 0.5, s + 1; s + 2; -\frac{b}{a} \right) \right], \quad (3.36)
\]

where \( F_1(b, c; d; z) \) is the hypergeometric function given by the analytic continuation

\[
F_1(a, b, c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - tx)^{-a} \, dt. \quad (3.37)
\]
Finally, Equation (3.9) can be changed into

$$u(x_0) = \mathbb{E}u(X_l) + 2\kappa(1, s) \sum_{k=1}^{l-1} \mathbb{E} \left\{ r_k f(Y_{k+1}) \int_0^{r_k^2 - |Y_{k+1} - X_k|^2} \frac{1}{t^2} \left( t + |Y_{k+1} - X_k|^2 \right)^{-\frac{3}{2}} dt \right\}$$

$$= \mathbb{E}u(X_l) + 2\kappa(1, s) \sum_{k=1}^{l-1} \mathbb{E} \left\{ r_k f(Y_{k+1}) \frac{1}{|Y_{k+1} - X_k|^3(s+1)} \right\}$$

$$\times \left[ |Y_{k+1} - X_k|^2(s+1) F_1 \left( -0.5, s+1; 2; \frac{r_k^2 - |Y_{k+1} - X_k|^2}{|Y_{k+1} - X_k|^2} \right) \right]$$

$$- \left( r_k^2 - |Y_{k+1} - X_k|^2 \right) s F_1 \left( 0.5, s+1; 2; \frac{r_k^2 - |Y_{k+1} - X_k|^2}{|Y_{k+1} - X_k|^2} \right) \right\}.$$  (3.38)

When \( n = 2s \), that is, \( s = \frac{1}{2} \), we have

$$u(x_0) = \mathbb{E}u(X_l) + 2\kappa \left( 1, \frac{1}{2} \right) \sum_{k=1}^{l-1} \mathbb{E} \left\{ r_k f(Y_{k+1}) \log \left( r_k + \sqrt{r_k^2 - |Y_{k+1} - X_k|^2} \right) \right\}.$$  (3.39)

Here, \( Y_{k+1} = x_k + \rho \theta \) with \( \rho \sim U(0, r_k) \) and \( \mathbb{P}\{\theta = 1\} = \mathbb{P}\{\theta = -1\} = \frac{1}{2} \). And the random variable \( X_k, k = 1, 2, \ldots, l \) can be derived by the method introduced in high dimension which will undergo a long jump.

### 3.3 | Summary of the modified walk-one-spheres method

We summarize the method in the following algorithm for Equation (1.1), \( n \geq 2 \):

**Algorithm 3.2** Assign fractional order \( s \), the domain \( \Omega \), the point \( x_0 \in \mathbb{R}^n \), and the number of samples \( N \).

1. **Step 1.** Sample \( X_1 \) and \( Y_1 \) obeying probability density functions \( p_{1,r_0}(x, x_0) \) in (3.13) and \( p_{2,r_0}^s(y, x_0) \) in (3.30) based on \( X_0 = x_0 \), respectively.
2. **Step 2.** If the latest \( X_k \) is out of \( \Omega \), go to Step 4. Otherwise, go to Step 3.
3. **Step 3.** Sample \( X_{k+1} \) and \( Y_{k+1} \) obeying probability density functions \( p_{1,r_k}(x, x_k) \) in (3.13) and \( p_{2,r_k}^s(y, x_k) \) in (3.30) based on \( X_k = x_k \), respectively and go back to Step 2.
4. **Step 4.** Calculate \( u(x) = g(X_n) + \sum_{k=0}^{l-1} \mathbb{E} \left\{ b(X_k) \left[ 1 - I \left( \frac{|Y_{k+1} - X_k|^2}{r_k^2}, \frac{n}{2} - s, 1 - s \right) \right] f(Y_{k+1}) \right\} \).
5. **Step 5.** Implement Steps 1–4 for \( N \) times. Then calculate \( u(x) \approx \frac{1}{N} \sum_{i=1}^N u^i(x) \).

### 4 | BOUNDS ON EXPECTED STEPS OF WALKS ON SPHERES

In the section, we focus on the problem (2.1) when the dimensionality \( n \geq 2 \). We give the upper bound on the number of steps \( l \) in expectation as follows.

**Theorem 4.1** Consider the problem (2.1) when \( n \geq 2 \). For random walks originating at \( x_0 \in \mathbb{B}_r \), the expectation of the number of steps \( l \) in Algorithm 3.2 is
where $s_1 = s$, $s \in \left(0, \frac{1}{3}\right]$ and $s_1 = \frac{1-s}{2}$, $s \in \left(\frac{1}{3}, 1\right)$. The right-hand-side function is increasing with respect to $s$ and $|x_0|$, respectively.

Before the proof of Theorem 4.1, we need to introduce the following lemmas. Observe that the number of walks $l$ depends only on the domain $\Omega$ and is independent of $f(x)$ and $g(x)$. Thus we may consider the problem, for fixed $r > 0$, given by

$$
\begin{cases}
(-\Delta)^s u(x) = f(x), & x \in B_r, \\
u(x) = 0, & x \in \mathbb{R}^n \setminus B_r,
\end{cases}
$$

where $s \in (0,1)$ and $f(x) = (d(x))^{-2s}$ and $d(x) = r - |x|$ denotes the minimum distance from $x$ to the boundary $\partial B_r$. For any $x_k \in B_r$, $d(x_k) = r_k$.

Recall that the Green function $G_B(x, y)$ in Section 1 is given by

$$G_B(x, y) = 2^{-2s} \pi^{-n/2} \Gamma(s)^{-2} \Gamma \left(\frac{n}{2}\right) |x - y|^{2s-n} \int_0^{r(x, y)} \frac{r^{t-1}}{(t+1)^{n/2}} dr. \tag{4.3}\label{4.3}
$$

**Lemma 4.1** ([38]) For any ball centered at the origin, $B_r \subset \mathbb{R}^n$, we have

$$G_B(x, y) \leq C_1(n, s) \frac{(d(x))^s (d(y))^s}{|x - y|^n}, \quad x, y \in B_r, \tag{4.4}\label{4.4}
$$

where $C_1(n, s) = \pi^{-n/2} \Gamma(s)^{-1} \Gamma(s + 1)^{-1} \Gamma \left(\frac{n}{2}\right)$.

**Lemma 4.2** ([38]) For any ball centered at the origin, $B_r \subset \mathbb{R}^n$, we have

$$G_B(x, y) \leq C_3(n, s) \frac{(d(x))^s}{(d(y))^s |x - y|^{n-2s}}, \quad x, y \in B_r, \tag{4.5}\label{4.5}
$$

where $C_3(n, s) = 2^{2s} \max \{C_1(n, s), C_2(n, s)\}$ and $C_2(n, s) = 2^{-2s} \pi^{-n/2} \Gamma \left(\frac{n}{2} - s\right) \Gamma(s)^{-1}$.

Then we have the result in the following.

**Lemma 4.3** For any ball centered at the origin, $B_r \subset \mathbb{R}^n$, and $s_1 \in (0, 2s),

$$G_B(x, y) \leq C_4(n, s) (d(x))^s (d(y))^{s_1} \frac{(d(y))^{s_1}}{|x - y|^{n-s_1}}, \quad x, y \in B_r, \tag{4.6}\label{4.6}
$$

where $C_4(n, s) = \pi^{-n/2} \Gamma(s)^{-1} \max \left\{\frac{2^{2s} \Gamma \left(\frac{n}{2}\right)}{\Gamma(1+s), \Gamma \left(\frac{n}{2} - s\right)}\right\}$.
Then we have the definition of $E$

By using Equation (3.9), we derive where $a$ is finite and is the solution to problem (4.2), that is,

$$
G_{\mathbb{B}_r}(x, y) \leq \left[ C_1(n, s) \frac{(d(x))^s(d(y))^s}{|x - y|^n} \right] \land \left[ C_2(n, s) \frac{(d(x))^s}{(d(y))^s|x - y|^{n-2s}} \right]
$$

$$
\leq \max \left\{ C_1(n, s), C_3(n, s) \right\} (d(x))^s \left( \frac{(d(y))^{s-s_1}}{|x - y|^{n-s_1}} \right) \left( \frac{|x - y|}{d(y)} \right)^{2s-s_1}
$$

$$
\leq C_4(n, s)(d(x))^s \frac{(d(y))^{s-s_1}}{|x - y|^{n-s_1}},
$$

(4.7)

where $a \land b = \min\{a, b\}$ and

$$
C_4(n, s) = \max \left\{ C_1(n, s), 2^{2s} \max \left\{ C_1(n, s)C_2(n, s) \right\} \right\} = 2^{2s} \max \left\{ C_1(n, s), C_2(n, s) \right\}
$$

$$
= \pi^{-n/2} \Gamma(s)^{-1} \max \left\{ 2^{2s} \Gamma \left( \frac{n}{2} \right), \Gamma \left( \frac{n}{2} - s \right) \right\}. \quad (4.8)
$$

Proof of Theorem 4.1 From Lemma 4.3 we know that $\int_{\mathbb{B}_r} f(y)G(x_0, y) \, dy$ with $x_0 \in \mathbb{B}_r$ is finite and is the solution to problem (4.2), that is,

$$
u(x_0) = \int_{\mathbb{B}_r} f(y)G(x_0, y) \, dy, \quad x_0 \in \mathbb{B}_r. \quad (4.9)
$$

By using Equation (3.9), we derive

$$
u(x_0) = \int_{\mathbb{B}_r} f(y)G(x_0, y) \, dy = \mathbb{E} g(X_t) + \mathbb{E} \left[ \sum_{k=0}^{l-1} a(X_k)f(Y_{k+1}) \right]
$$

$$
= \mathbb{E} \left[ \sum_{k=0}^{l-1} a(X_k)f(Y_{k+1}) \right]. \quad (4.10)
$$

where $\mathbb{E}u(X_t) = \mathbb{E}g(X_t) = 0$ have been used since $X_t$ is in the outside of the ball. For $k = 0, 1, \ldots, l - 1$, recalling

$$
a(X_k) = \kappa(n, s)B \left( s, \frac{n}{2} \right) \frac{\omega_{n-1}}{2s} d(X_k)^{2s}, \quad (4.11)
$$

the definition of $f(x) = (d(x))^{-2s}$ in Equation (4.2) and the fact that $d(Y_{k+1}) \leq 2d(X_k)$ (see Figure 5) lead to the lower bound

$$
a(X_k)f(Y_{k+1}) = \kappa(n, s)B \left( s, \frac{n}{2} \right) \frac{\omega_{n-1}}{2s} \left[ \frac{d(X_k)}{d(Y_{k+1})} \right]^{2s} \geq \kappa(n, s)B \left( s, \frac{n}{2} \right) \frac{\omega_{n-1}}{2s} \left( \frac{1}{2} \right)^{2s}. \quad (4.12)
$$

Then we have

$$
\kappa(n, s)B \left( s, \frac{n}{2} \right) \frac{\omega_{n-1}}{2s} \left( \frac{1}{2} \right)^{2s} \mathbb{E}\{l\} \leq \int_{\mathbb{B}_r} f(y)G(x_0, y) \, dy. \quad (4.13)
$$

For the right hand side of the inequality, we utilize Lemma 4.3 and partition domain $B_r$ into two parts,

$$
\int_{\mathbb{B}_r} f(y)G(x_0, y) \, dy = \kappa(n, s) \int_{\mathbb{B}_r} f(y)|y - x_0|^{2s-n} \int_0^{r(x_0, y)} \frac{r^{s-1}}{(t + 1)^{\frac{n}{2}}} \, dr \, dy
$$

$$
\leq C_4(n, s)(d(x_0))^s \int_{\mathbb{B}_r} f(y) \frac{(d(y))^{s-s_1}}{|x_0 - y|^{n-s_1}} \, dy
$$
\[ X_n = C_4(n, s)(d(x_0))^s \left[ \int_{\mathbb{B}_h(x_0)} + \int_{\mathbb{B}_r \setminus \mathbb{B}_h(x_0)} \right] f(y) \frac{(d(y))^{s-s_1}}{|x_0 - y|^{n-s_1}} \, dy \]

\[ = C_4(n, s)(d(x_0))^s [I_1 + I_2], \quad (4.14) \]

where we set \( h = \frac{r - |x_0|}{2}, s_1 = s, s \in \left(0, \frac{1}{3}\right) \) and \( s_1 = \frac{1-s}{2}, s \in \left(\frac{1}{3}, 1\right) \). For \( I_1 \), we obtain

\[ I_1 = \int_{\mathbb{B}_h(x_0)} (d(y))^{s-s_1} |y - x_0|^{s_1-n} \, dy \]

\[ \leq (r - |x_0| - h)^{-s-s_1} \int_{\mathbb{B}_h(x_0)} |y - x_0|^{s_1-n} \, dy \]

\[ = (r - |x_0| - h)^{-s-s_1} \frac{\omega_{n-1} h^{s_1}}{s_1} \]

\[ = \frac{2^n \omega_{n-1}}{s_1} (r - |x_0|)^{-s}. \quad (4.15) \]

For \( I_2 \), we have

\[ I_2 = \int_{\mathbb{B}_r \setminus \mathbb{B}_h(x_0)} d(y)^{-s-s_1} |x_0 - y|^{s_1-n} \, dy \]

\[ \leq \left( \int_{\mathbb{B}_r \setminus \mathbb{B}_h(x_0)} [d(y)^{-s-s_1}]^{\frac{1+n_1}{1+n_2}} dy \right)^{\frac{1+n_2}{2}} \left( \int_{\mathbb{B}_r \setminus \mathbb{B}_h(x_0)} (|x_0 - y|^{s_1-n})^{\frac{1+n_1}{1+n_2}} dy \right)^{\frac{1+n_1}{2}} \]

\[ = (I_{2,1})^{\frac{2(n_1+n_2)}{1+n_1+n_2}} (I_{2,2})^{\frac{n_1+n_2}{1+n_1+n_2}}. \quad (4.16) \]

For \( I_{2,1} \), we use the polar coordinates, hence

\[ I_{2,1} \leq \omega_{n-1} \int_0^r (r - \rho)^{-\frac{n_1+n_2}{2}-1} \, d\rho \]

\[ = \omega_{n-1} r^{n-\frac{1+n_1+n_2}{2}} \int_0^{\frac{1}{2}} (1 - \rho)^{-\frac{n_1+n_2}{2}-1} \, d\rho \]

\[ = \omega_{n-1} r^{n-\frac{1+n_1+n_2}{2}} B\left(1 - \frac{1+s_1+s}{2}, n \right). \quad (4.17) \]
For $I_{2,2}$, we use polar coordinates $(\rho, \theta)$ with $x_0$ being treated as the origin. Let us consider a ray that originates from $x_0$ and has angle $\theta$, which intersect $\partial \mathbb{B}_r$ on $z$ (see Figure 5). Then we define $r(\theta) = |z - x_0|$ and the integral $I_{2,2}$ can be rewritten as

$$I_{2,2} = \int_{\mathbb{B}_r \setminus \mathbb{B}_s(x_0)} |x_0 - y|^{\frac{1}{1-s_1-s}} dy$$

$$= \omega_{n-1} \int_0^{r(\theta)} \rho^{\frac{1}{1-s_1-s}} \sqrt{n-1} d\rho$$

$$\leq \frac{\omega_{n-1}}{(1+s_1+s)(1-s)} + n \left[ (r + |x_0|) \right]^{\frac{1}{1-s_1-s}} + n - \left( \frac{r - |x_0|}{2} \right)^{\frac{1}{1-s_1-s}} + n \right] (4.18)$$

Bringing (4.14), (4.15), and (4.16) into inequality (4.13) and noticing $\kappa(n, s) > 0$ yield that

$$\kappa(n, s)B \left( s, \frac{n}{2} \right) \frac{\omega_{n-1}}{2s} (\frac{1}{2}) 2^s \mathbb{E}(l)$$

$$\leq C_4(n, s)(r - |x_0|)^s \left[ \frac{2^{s-1} \omega_{n-1}}{s} (r - |x_0|)^s + \left[ \omega_{n-1} \rho^s \frac{1}{1+s_1+s} \right] \left( 1 - \frac{1 + s_1 + s}{2}, n \right) \right]$$

$$\times \left[ (r + |x_0|) \right]^{\frac{1}{1-s_1-s}} + n - \left( \frac{r - |x_0|}{2} \right)^{\frac{1}{1-s_1-s}} + n \right] \right].$$

Thus, we arrive at the following estimate:

$$\mathbb{E}(l) \leq 2^{4s+1} \pi^2 \frac{\Gamma \left( s + \frac{n}{2} \right) \Gamma(s + 1)}{\Gamma^2 \left( \frac{n}{2} \right)} C_4(n, s) \left\{ \frac{2^s}{s_1} + (r - |x_0|)^s \right\}$$

$$\times \left[ \rho^{\frac{1}{1+s_1+s}} B \left( 1 - \frac{1 + s_1 + s}{2}, n \right) \right]^{\frac{2(s+1)}{1+s_1+s}} \left( \frac{1}{1-s_1-s} \right)$$

$$\times \left[ \left( r + |x_0| \right)^{\frac{1}{1-s_1-s}} + n - \left( \frac{r - |x_0|}{2} \right)^{\frac{1}{1-s_1-s}} + n \right] \right].$$

(4.19)

Here $s_1 = s$, for $s \in \left( 0, \frac{1}{3} \right)$ and $s_1 = \frac{1-s}{2}$ for $s \in \left( \frac{1}{3}, 1 \right)$. So inequality (4.1) is shown.

We are now in position to bound the expected number of steps $l$ before stopping. Let $a(s) = -\left[ \frac{s^3 + 4n_{s+3} + 6n_{s+3} - 3}{2(1-s)} \right]$ and $\rho = \frac{|x_0|}{r}$. Then, for $s \in \left( \frac{1}{3}, 1 \right)$, we have

$$\mathbb{E}(l) \leq \max \left\{ C_5(n, s), C_6(n, s) \right\} \left\{ \frac{2^s+1}{1-s} + (1 - \rho)^s \right\} \left[ B \left( 1 - \frac{s - s}{4}, 2 \right) \right]^{2s+2}$$

$$\times \left( \frac{1}{a(s)} \right) \left[ (1 + \rho)^{-a(s)} - 2^{a(s)}(1 - \rho)^{-a(s)} \right] \right\},$$

(4.20)

where

$$C_5(n, s) = \frac{2^{s+1}}{B \left( s, \frac{n}{2} \right)}.$$
Let \( v(s, \rho) \) be the right hand side of Equation (4.20). Since we have
\[
(1 - \rho)^2 \left( -\frac{1}{a(s)} \left[ (1 + \rho)^{-a(s)} - 2^a(s)(1 - \rho)^{-a(s)} \right] \right)
= \left( \frac{1}{a(s)} \right) \left[ 2^a(s)(1 - \rho)^{-a(s)} - (1 + \rho)^{-a(s)}(1 - \rho)^{-a(s)} \right]
\]
and \( \frac{a(3+a)}{1-a} - a(s) < 0 \), it can be readily checked that \( v(s, \rho) \) is a monotonically increasing function with respect to \( \rho \) so that we discuss monotonicity for \( s \). For \( C_5(n, s) \), it is easily obtain \( C_5(n, s) \) monotonically increases with respect to \( s \). Observe that \( C_6(n, s) \) can be written as follows.
\[
C_6(n, s) = 2^{4s+1} B \left( \frac{n}{2}, \frac{n}{2} - s \right) \frac{\Gamma(n)}{\Gamma^2 \left( \frac{n}{2} \right)}.
\]

Differentiating Beta function with respect to \( s \), we obtain
\[
B' \left( \frac{n}{2} + s, \frac{n}{2} - s \right) = \int_0^1 x^{\frac{n}{2} - s - 1}(1 - x)^{\frac{n}{2} - s - 1} \ln \left( \frac{x}{1-x} \right) dx
= \int_0^\frac{1}{2} x^s - 1(1 - x)^{\frac{n}{2} - s - 1} \left[ 2^s \ln \left( \frac{x}{1-x} \right) + (1 - x)^{2s} \ln \left( \frac{1-x}{x} \right) \right] dx.
\]

Since the integrand in the square bracket is nonnegative, \( B \left( \frac{n}{2} + s, \frac{n}{2} - s \right) \) increases monotonically such that both \( C_6(n, s) \) and max \( \{ C_5(n, s), C_6(n, s) \} \) are monotonically increasing coefficients. Then we discuss the second term in the brace of the Equation (4.20), which is denoted by \( v_1(s, \rho) \). Taking logarithm of \( v_1(s, \rho) \) yields
\[
\ln(v_1(s, \rho)) = s \ln(1 - \rho) + \frac{2 + 2\rho}{3 + s} \ln \left[ B \left( \frac{1-s}{4}, 2 \right) \right]
+ \frac{1-s}{3+s} \ln \left( \frac{1}{a(s)} \left[ 2^a(s)(1 - \rho)^{-a(s)} - (1 + \rho)^{-a(s)} \right] \right)
= v_{1,1}(s, \rho) + v_{1,2}(s, \rho).
\]

where
\[
v_{1,1}(s, \rho) = \frac{1-s}{3+s} \ln \left( \frac{1}{a(s)} \left[ 2^a(s)(1 + \rho)^{a(s)} - (1 - \rho)^{-a(s)} \right] \right)
\]
\[
v_{1,2}(s, \rho) = \left( s - a(s) \frac{1-s}{3+s} \right) \ln(1 - \rho) + \frac{2 + 2\rho}{3 + s} \ln \left[ B \left( \frac{1-s}{4}, 2 \right) \right] - a(s) \frac{1-s}{3+s} \ln(1 + \rho).
\]

After careful calculations, we obtain the derivative of \( v_{1,1}(s, \rho) \) with respect to \( s \)
\[
(v_{1,1}(s, \rho))' = -\frac{4}{(s+3)^2} \ln \left( \frac{2(1-s)}{s^2 + (4n+2)s + 4n - 3} \right)
- \frac{4}{(s+3)^2} \ln \left[ (2 + 2\rho)^{a(s)} - (1 - \rho)^{a(s)} \right] + \frac{s^2 - 2s - 8n + 1}{(s+3)(s^2 + (4n+2)s + 4n - 3)}.
\]
When $s$ is near $1$, we have
\[
\frac{\nu_{1,1}(s, \rho)}{\nu_{1,2}(s, \rho)} = \left( 1 - a(s) \frac{1 - s}{3 + s} \right) \ln(1 - \rho) + \frac{2 + 2s}{3 + s} \left( \ln \Gamma(n) + \ln \prod_{k=0}^{n-1} \frac{1}{1 - s + 4k} \right)
\]
\[
+ \left( \frac{2 + 2s}{3 + s} \ln 4^n - a(s) \frac{1 - s}{s + 3} \ln(1 + \rho) \right)
\]
\[
:= A_1 + A_2 + A_3.
\]

It is easy to know that $A_1$ and $A_2$ are increasing with respect to $s$. A simple calculation gives
\[
A_3 = \left[ \frac{8ns^3 + 8n}{s + 3} - a(s) \frac{1 - s}{s + 3} \right] \ln 2 + a(s) \frac{1 - s}{s + 3} \ln 2 - \ln(1 + \rho)
\]
\[
(4.29)
\]
Since $\left[ \frac{8ns^3 + 8n}{2(s + 3)} - a(s) \frac{1 - s}{s + 3} \right] = \frac{s^2 - 6s + 9}{2(s + 3)^2} \geq 0$ and $\left[ a(s) \frac{1 - s}{s + 3} \right] = \frac{s^2 + 6s + 8n + 9}{2(s + 3)^2} > 0$, $A_3$ is increasing with respect to $s$. Combining the monotonicity of $\nu_{1,1}(s, \rho)$ and $\nu_{1,2}(s, \rho)$ with respect to $s$, $\nu(s, \rho)$ will then grow for $s > \frac{1}{3}$. Similarly, we can derive the same result for $s \leq \frac{1}{3}$. Thus we obtain the desired result. \hfill \blacksquare

When $s \to 1$, the upper bounds for the expected stopping steps cannot work, since fractional Laplacian degenerates into the classical Laplace operator so that the Lévy flight becomes Brownian motion. Though the Brownian motion originated at $x$ will reach boundary $B_x$ in the probability sense, the expected stopping steps are infinite. We also have the following theorem.

**Theorem 4.2** When $s \to 1$, $G(x, y)$ in (2.4) is the Green function for the classical Laplace equation with ball boundary.

**Proof.** When $n = 2$, we have
\[
G(x, y) = \kappa(2, 1) \int_0^{r(x, y)} \frac{1}{1 + r} dr = \kappa(2, 1) \log \left( \frac{(r^2 - |x|^2) (r^2 - |y|^2) + r^2 |x - y|^2}{r^2 |x - y|^2} \right)
\]
\[
= \frac{1}{4\pi} \log \left( \frac{r^4 + |x|^2 |y|^2 - 2r^2 |x| |y| \cos(x, y)}{r^2 |x - y|^2} \right)
\]
\[
(4.30)
\]
When \( n \geq 3 \), we have

\[
G(x, y) = \kappa(n, 1) |x - y|^{2-n} \int_0^{r^*(x, y)} \frac{1}{(1 + t)\frac{n}{2}} \, dt
\]

\[
= \frac{2\kappa(n, 1)}{n-2} |x - y|^{2-n} \left[ 1 - \left( \frac{r^2 - |x|^2}{r^2 |x - y|^2} \right) \right]^{-\frac{n-2}{2}}
\]

\[
= \frac{2\kappa(n, 1)}{n-2} \left[ \frac{1}{\left( |x|^2 + |y|^2 - 2|x||y|\cos\langle x, y \rangle \right)^{\frac{n-2}{2}}} - \left( r^4 + |x|^2 |y|^2 \right) - 2r^2 \cos \langle x, y \rangle \right]
\]

(4.31)

\[\blacksquare\]

Remark 4.1 When \( f(x) = (d(x))^{-2} \) which does not satisfy the condition in Theorem 2.1, we know that \( u(x) \) in (4.9) is still the solution of the problem (4.2) in the sense of mild solution, while \( u(x) \) may not satisfy the regularity in Theorem 2.1.

5 | NUMERICAL EXAMPLES

In this section, numerical examples are carried out by using Scheme I (quadrature method 2.26 and 2.31) discussed in Section 2 and Algorithm 3.2 introduced in Section 3 on an i5-8250 U CPU.

In the experiments, we consider two special cases of Equation (1.1): the homogeneous equation with inhomogeneous boundary value condition, and nonconstant source term with homogeneous boundary value condition.

We set step sizes \( h = h_{\rho} = h_{\theta} = h_t = h_r, r = 1, 2, \ldots, n-2 \). In addition, \( E(h) = |u_{2h} - u_h| \) denotes posteriori error estimates in Scheme I and \( E \) denotes the absolute error in the modified walk-on-sphere method. Then the convergence order is given by

\[ \text{rate} = \log_2 \frac{E(2h)}{E(h)}. \]

Example 5.1 Let \( \Omega \) be a unit ball in \( \mathbb{R}^n \) centered at the origin

\[
\begin{cases}
(-\Delta)^s u(x) = 0, & x \in \Omega, \\
u(x) = g(x), & x \in \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

(5.1)

where \( g(x) = \exp\left(-|x - x'|^2\right) \).

In this example, we take \( s = 0.25, 0.5, 0.75 \) and set different step sizes \( \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}, \frac{1}{512} \) in two spacial dimensions and \( \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128} \) in three spacial dimensions for Scheme I. For the modified walk-on-sphere method, the number of samples are set by 1000, 10000, 100000. We evaluate \( u(0.6,0.6) \) with \( x' = (3, 0) \) in two spacial dimensions. The numerical results of Scheme I and modified walk-on-sphere method are presented in Tables 1 and 2, respectively.

Table 1 shows the convergent order coincides with the theoretical analysis. It can be seen from Table 2 that the simulation results by the Monte Carlo method are close to the approximations in Table 1.

Next, we evaluate \( u(0.5,0.5,0.5) \) with \( x' = (3, 0, 0) \) in three spacial dimensions. Numerical results are given in Tables 3 and 4. Table 3 shows that although Scheme I achieves same convergent order while the CPU time in three spacial dimensions grows a lot. Compared with the Scheme I, modified walk-on-sphere method presented in Table 4 saves much more time.
### Table 1
**Numerical results of Example 5.1 using Scheme I in 2D**

| $s$ | $\frac{1}{h}$ | Approximation | $E(h)$ | Rate | CPU time (s) |
|-----|----------------|---------------|--------|------|--------------|
| $s = 0.25$ | 32 | 0.0234077 | 5.6370E-06 | – | 0.0631 |
|       | 64 | 0.0234021 | 8.9306E-07 | 2.6581 | 0.1039 |
|       | 128 | 0.0234012 | 2.2166E-07 | 2.0104 | 0.3229 |
|       | 256 | 0.0234009 | 5.5441E-08 | 1.9994 | 1.1362 |
|       | 512 | 0.0234009 | 1.3863E-08 | 1.9996 | 4.4059 |
| $s = 0.50$ | 32 | 0.0187671 | 8.2695E-06 | – | 0.0595 |
|       | 64 | 0.0187558 | 4.6391E-07 | 4.1559 | 0.1041 |
|       | 128 | 0.0187583 | 1.1117E-07 | 2.0612 | 0.3031 |
|       | 256 | 0.0187582 | 2.8056E-08 | 1.9863 | 1.0778 |
|       | 512 | 0.0187582 | 7.0603E-09 | 1.9905 | 4.1187 |
| $s = 0.75$ | 32 | 0.0099238 | 1.5558E-05 | – | 0.0599 |
|       | 64 | 0.0099082 | 3.7635E-07 | 5.3694 | 0.1407 |
|       | 128 | 0.0099079 | 8.3237E-08 | 2.1768 | 0.7417 |
|       | 256 | 0.0099078 | 2.1912E-08 | 1.9255 | 1.4891 |
|       | 512 | 0.0099077 | 5.7075E-09 | 1.9407 | 5.5057 |

### Table 2
**Numerical results of Example 5.1 using modified walk-on-sphere method in 2D**

| $s$ | Samples | Approximation | Average no. steps | Variance | CPU time (s) |
|-----|---------|---------------|--------------------|----------|--------------|
| $s = 0.25$ | 1000   | 0.0230409 | 1.7470            | 9.4958E-03 | 0.0078 |
|       | 10000  | 0.0187558 | 2.9460            | 1.0672E-02 | 0.0171 |
|       | 100000 | 0.0187583 | 3.0142            | 5.7382E-03 | 1.1908 |
| $s = 0.50$ | 1000   | 0.0185969 | 1.7515            | 9.0476E-03 | 0.0835 |
|       | 10000  | 0.0188054 | 3.0495            | 5.8763E-03 | 0.1179 |
|       | 100000 | 0.0187276 | 3.0142            | 5.7382E-03 | 1.1908 |
| $s = 0.75$ | 1000   | 0.0099238 | 1.7543            | 8.4807E-03 | 0.6804 |
|       | 10000  | 0.0099082 | 3.0142            | 5.8763E-03 | 0.2601 |
|       | 100000 | 0.0099079 | 3.0142            | 5.7382E-03 | 2.5103 |

### Example 5.2
Consider Equation (5.1) with $\Omega$ being a unit ball in $\mathbb{R}^n$ centered at the origin,

\[
g(x) = \begin{cases} 
\frac{1}{\pi} \log |x - x'|, & n = 1, \\
\frac{n}{2} a(n, s)|x - x'|^{-n+2s}, & n \geq 2,
\end{cases}
\]

and

\[
a(n, s) = \frac{\Gamma\left(\frac{n}{2} - s\right)}{2^{\frac{n}{2}} \pi^{\frac{n}{2}} \Gamma(s)}.
\]

Here $g(x)$ is just Green’s function and $x' \in \mathbb{R}^n \setminus \Omega$, where $x'$ can be any points out of the unit ball. The exact solution to (5.1) is given below [35],

\[
u(x) = \begin{cases} 
\frac{1}{\pi} \log |x - x'|, & n = 1, \\
\frac{n}{2} a(n, s)|x - x'|^{-n+2s}, & n \geq 2.
\end{cases}
\]
### Table 3: Numerical results of Example 5.1 using Scheme I in 3D

| $s$ | $\frac{1}{h}$ | Approximation | $E(h)$ | Rate | CPU time (s) |
|-----|---------------|---------------|--------|------|--------------|
| $s = 0.25$ | 8              | 0.0084161     | 7.3967E-05 | --   | 0.0983       |
|      | 16             | 0.0079807     | 9.5757E-05 | 3.4423 | 0.2698       |
|      | 32             | 0.0080208     | 2.1302E-05 | 2.1448 | 1.3501       |
|      | 64             | 0.0080298     | 4.9834E-06 | 2.0779 | 14.830       |
|      | 128            | 0.0080320     | 1.2291E-06 | 2.0155 | 103.99       |
| $s = 0.50$ | 8              | 0.0066376     | 7.3967E-05 | --   | 0.0752       |
|      | 16             | 0.0065636     | 9.5757E-06 | -0.3725 | 0.2861       |
|      | 32             | 0.0066594     | 2.1377E-05 | 2.1684 | 1.2785       |
|      | 64             | 0.0066807     | 4.6231E-06 | 2.0957 | 9.4483       |
|      | 128            | 0.0066856     | 1.0635E-06 | 2.0195 | 78.617       |
| $s = 0.75$ | 8              | 0.0033824     | 2.7559E-04 | --   | 0.0792       |
|      | 16             | 0.0036580     | 1.5625E-04 | 0.8185 | 0.2571       |
|      | 32             | 0.0038143     | 3.4898E-05 | 2.1627 | 1.3584       |
|      | 64             | 0.0038491     | 8.0415E-06 | 2.1176 | 10.167       |
|      | 128            | 0.0038572     | 1.9762E-06 | 2.0247 | 85.237       |

### Table 4: Numerical results of Example 5.1 for modified walk-on-sphere method in 3D

| $s$ | Samples | Approximation | Average no. steps | Variance | CPU time (s) |
|-----|---------|---------------|-------------------|----------|--------------|
| $s = 0.25$ | 1000    | 0.0083326     | 1.8890            | 2.2310E-03 | 0.0340       |
|      | 10000   | 0.0080532     | 1.9248            | 1.9605E-03 | 0.1520       |
|      | 100000  | 0.0080475     | 1.9259            | 1.8473E-03 | 1.3697       |
| $s = 0.50$ | 1000    | 0.0068445     | 3.9280            | 3.1425E-03 | 0.0476       |
|      | 10000   | 0.0067445     | 3.8514            | 1.3806E-03 | 0.3632       |
|      | 100000  | 0.0066647     | 3.8748            | 1.2729E-03 | 2.7069       |
| $s = 0.75$ | 1000    | 0.0039063     | 10.327            | 1.8671E-03 | 0.0936       |
|      | 10000   | 0.0037649     | 10.010            | 5.4232E-04 | 0.7213       |
|      | 100000  | 0.0038088     | 10.110            | 4.1431E-04 | 7.4822       |

### Table 5: Numerical results of Example 5.2 using modified walk-on-sphere method in 1D

| $s$ | Samples | $E$ | Average no. steps | Variance | CPU time (s) |
|-----|---------|-----|-------------------|----------|--------------|
| $s = 0.25$ | 1000    | 8.8457E-03 | 1.2910           | 3.1650E-01 | 0.0129       |
|      | 10000   | 2.2407E-03 | 1.2929           | 2.4775E-01 | 0.0912       |
|      | 100000  | 8.4281E-04 | 1.2915           | 2.3637E-01 | 0.8159       |
| $s = 0.50$ | 1000    | 6.9838E-03 | 1.5000           | 2.8124E-01 | 0.0150       |
|      | 10000   | 3.7475E-03 | 1.5290           | 2.7348E-01 | 0.0982       |
|      | 100000  | 4.8992E-04 | 1.5246           | 2.6451E-01 | 0.8901       |
| $s = 0.75$ | 1000    | 7.8929E-03 | 1.7200           | 3.7211E-01 | 0.0253       |
|      | 10000   | 3.1675E-03 | 1.7069           | 3.7141E-01 | 0.1236       |
|      | 100000  | 2.8304E-05 | 1.6879           | 3.5764E-01 | 1.1568       |
### Table 6
Numerical results of Example 5.2 using modified walk-on-sphere method in 2D

| $s$  | Samples | $E$       | Average no. steps | Variance | CPU time (s) |
|------|---------|-----------|-------------------|----------|--------------|
|     | 1000    | 7.7764E-03| 1.7220            | 1.5125E-01| 0.0191       |
|     | 10000   | 3.3265E-03| 1.7338            | 3.3816E-02| 0.1893       |
|     | 100000  | 1.7565E-03| 1.7338            | 1.2221E-02| 1.6108       |
|     | 1000    | 2.3412E-03| 3.0770            | 1.4768E-02| 0.0292       |
|     | 10000   | 1.3937E-04| 3.0016            | 8.2579E-03| 0.2636       |
|     | 100000  | 7.8162E-05| 3.0004            | 6.7842E-03| 2.5823       |
|     | 1000    | 9.5021E-03| 6.5530            | 4.0156E-02| 0.0813       |
|     | 10000   | 1.0741E-04| 6.2859            | 3.6001E-02| 0.5657       |
|     | 100000  | 2.2905E-05| 6.2344            | 2.9957E-02| 5.5934       |

**Figure 6** Result for Example 5.1 with the walk-one-spheres method based on $10^5$ samples in two dimensions. When $s$ becomes bigger, the error will be smaller whereas the average number of steps will increase.

We use the modified walk-on-sphere method to simulate the solution. The number of samples are set by 1000, 10000, and 100000 for modified walk-on-sphere method. Though $g(x)$ does not satisfy the condition in Theorem 2.1, modified walk-on-sphere method still takes effect since the representation formula is finite. The value of $u \left( \frac{1}{2} \right)$ with $x' = 2$ in one spacial dimension is showed in Table 5. We also evaluate $u(0.6,0.6)$ with $x' = \left( \sqrt{2}, \sqrt{2} \right)$ in two spacial dimensions. Table 6 gives the numerical results. The average number of step in Table 6 is basically the same as that in Table 2, indicating that the average number of step is not related to $g(x)$. When $10^5$ samples are used in modified walk-on-sphere method, Figure 6 shows that the larger the $s$ is, the smaller the errors will be, which is caused by the singularity of $g(x)$. And the average number of steps will increase when $s$ tends to 1, which explains why the CPU time will become longer when $s$ grows.

Next, we evaluate $u(0.5,0.5,0.5)$ with $x' = \left( \frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3} \right)$ in three spacial dimensions. The numerical results are given in Table 7. The consuming time does not grow too much as the dimension increases. Figure 7 also indicates the relation between the average number of step and index $s$ remains, which coincides with theoretical analysis.

For higher dimensional cases, we first evaluate $u(x)$ with $x = \frac{1}{4} \times \text{ones}(4)$ and $x' = \text{ones}(4)$ in four spacial dimensions and $u(x)$ with $x = \frac{1}{5} \times \text{ones}(5)$ and $x' = \frac{2\sqrt{5}}{5} \times \text{ones}(5)$ in five spacial dimensions,
where \( \text{ones}(n) \) is an \( n \)-dimensional vector with all ones. The number of samples are set by \( 10^5 \). Since when \( s \in \left( 0, \frac{1}{2} \right) \), \( g(x) \) has singularity, we mainly give the numerical results for \( s \in \left( \frac{1}{2}, 1 \right) \) in Table 8.

We then evaluate \( u(x) \) with \( x = \frac{1}{10} \times \text{ones}(10) \) and \( x' = \frac{\sqrt{10}}{5} \times \text{ones}(10) \) in 10 spacial dimensions. The number of samples are set by \( 10^5 \) and the numerical results is given in Table 9. Compared with the computational time in lower dimensions, the time in 10 dimensions only increases in multiple, which shows the efficiency of the algorithm.

**Example 5.3** Consider the following fractional Poisson equation with vanishing Dirichlet boundary condition

\[
\begin{cases}
(-\Delta)^s u(x) = f(x), & x \in \Omega, \\
u(x) = 0, & x \in \mathbb{R}^n \setminus \Omega,
\end{cases}
\tag{5.5}
\]

where \( \Omega \) is a unit ball in \( \mathbb{R}^n \) and

\[
f(x) = \begin{cases}
\Gamma \left( \frac{n}{2} + 2 \right) x, & n = 1, \\
2^s \Gamma(2 + s) \Gamma \left( \frac{n}{2} + s \right) \Gamma \left( \frac{n}{2} \right)^{-1} \left( 1 - \left( 1 + \frac{2s}{n} \right) |x|^2 \right), & n \geq 2.
\end{cases}
\tag{5.6}
\]
TABLE 8  Numerical results of Example 5.2 for modified walk-on-sphere method in 4D and 5D by $10^5$ samples

| s   | n   | E       | Average no. steps | Variance   | CPU time (s) |
|-----|-----|---------|-------------------|------------|--------------|
| 0.25| 4   | 1.1203E-02 | 1.5387           | 2.1507E-01 | 5.8991       |
| 0.50| 4   | 2.0822E-03 | 3.3463           | 8.8047E-02 | 12.082       |
| 0.60| 4   | 1.4179E-03 | 5.0610           | 4.8773E-02 | 19.741       |
| 0.70| 4   | 6.4099E-04 | 8.2784           | 1.9234E-02 | 32.903       |
| 0.80| 4   | 3.4514E-04 | 15.663           | 8.9541E-03 | 60.322       |
| 0.90| 4   | 2.8155E-04 | 38.629           | 2.9072E-03 | 208.21       |

| s   | n   | E       | Average no. steps | Variance   | CPU time (s) |
|-----|-----|---------|-------------------|------------|--------------|
| 0.25| 5   | 1.7871E-03 | 1.5178           | 4.6222E-02 | 8.7034       |
| 0.50| 5   | 1.6622E-03 | 3.4818           | 2.8613E-02 | 19.241       |
| 0.60| 5   | 8.2384E-04 | 5.4826           | 1.6533E-02 | 30.330       |
| 0.70| 5   | 6.3443E-04 | 9.4176           | 4.8255E-03 | 52.100       |
| 0.80| 5   | 3.4322E-04 | 18.598           | 2.2152E-03 | 100.84       |
| 0.90| 5   | 2.9972E-04 | 48.296           | 1.0710E-03 | 271.01       |

TABLE 9  Numerical results of Example 5.2 for modified walk-on-sphere method in 10D by $10^5$ samples

| s   | n   | E       | Average no. steps | Variance   | CPU time (s) |
|-----|-----|---------|-------------------|------------|--------------|
| 0.25| 10  | 1.1617E-03 | 1.4501           | 2.4291E-01 | 23.991       |
| 0.50| 10  | 6.8564E-04 | 3.6944           | 7.5413E-02 | 52.489       |
| 0.60| 10  | 5.3812E-04 | 6.4110           | 5.2138E-04 | 94.023       |
| 0.70| 10  | 2.9153E-04 | 12.266           | 1.6533E-02 | 178.82       |
| 0.80| 10  | 2.4108E-04 | 27.366           | 6.1517E-05 | 395.73       |
| 0.90| 10  | 1.2341E-04 | 80.842           | 5.4785E-05 | 1168.0       |

The exact solution to (5.8) is given in [39]

\[
\begin{align*}
    u(x) = \begin{cases} 
    x(1-x^2)^s, & n = 1, \\
    (1-|x|^2)^{1+s}, & n \geq 2.
    \end{cases}
\end{align*}
\]

(5.7)

We use the modified walk-on-sphere method to simulate the solution and the number of samples are set by 1000, 10000, and 100000 for modified walk-on-sphere method. We evaluated \( u \left( \frac{1}{2} \right) \) in one spacial dimension, which is showed in Table 10. Since we need to approximate the integral or the hypergeometric function when \( \frac{1}{2} < s < 1 \), the computational time is a bit longer. \( u(0.6,0.6) \) is also evaluated in two spacial dimensions by both Scheme I (2.31) and the modified walk-on-sphere method. The numerical results are presented in Tables 11 and 12. It is obvious that Scheme I has bigger errors and costs more computational time. Comparing Tables 2 and 6, it is obvious that the average number of step is independent of \( f(x) \) and \( g(x) \). Figure 8 shows that there is no obvious trend in absolute error when \( s \) changes. As expected, we again observe that when \( x \) is closed to the origin, the average number of steps will become smaller. In particular, when \( x \) is at the origin, the number of steps is one.

Next, we evaluate \( u(0.5,0.5,0.5) \) in three spacial dimensions. Table 13 shows that the CPU time of modified walk-on-sphere method does not increase too much in three dimensions compared with the time in two dimensions. Figure 9 gives the same result derived in Figure 8. When \( x \) approaches the origin, the average number of steps will be small, which coincides with the theoretical analysis.
**TABLE 10** Numerical results of Example 5.3 using modified walk-on-sphere method in 1D

| $s$ | Samples | $E$     | Average no. steps | Variance | CPU time (s) |
|-----|---------|---------|-------------------|----------|--------------|
| 0.25 | 1000    | 9.9759E-03 | 1.2540           | 1.1209E-01 | 0.0159       |
|     | 10000   | 8.7500E-04 | 1.2952           | 1.1191E-01 | 0.1131       |
|     | 100000  | 4.6390E-04 | 1.2879           | 1.0927E-01 | 0.8675       |
| 0.50 | 1000    | 9.4393E-03 | 1.5410           | 1.7148E-01 | 0.0178       |
|     | 10000   | 2.9952E-03 | 1.5316           | 1.6840E-01 | 0.1139       |
|     | 100000  | 2.0324E-04 | 1.5281           | 1.6630E-01 | 0.9490       |
| 0.75 | 1000    | 3.5671E-03 | 1.7000           | 8.6924E-01 | 0.8696       |
|     | 10000   | 1.6581E-03 | 1.6945           | 2.3160E-01 | 6.3171       |
|     | 100000  | 4.2174E-04 | 1.6838           | 2.7423E-01 | 45.534       |

**TABLE 11** Numerical results of Example 5.3 for modified walk-on-sphere method in 2D

| $s$ | Samples | $E$     | Average no. steps | Variance | CPU time (s) |
|-----|---------|---------|-------------------|----------|--------------|
| 0.25 | 1000    | 6.9086E-03 | 1.7620           | 1.1539E-01 | 0.0216       |
|     | 10000   | 3.1608E-03 | 1.7716           | 1.0973E-01 | 0.1662       |
|     | 100000  | 1.3496E-04 | 1.7606           | 8.7965E-02 | 1.5673       |
| 0.50 | 1000    | 6.4914E-03 | 2.8210           | 1.9263E-01 | 0.0332       |
|     | 10000   | 2.6209E-03 | 2.9709           | 1.7505E-01 | 0.2636       |
|     | 100000  | 1.3063E-04 | 2.9997           | 1.7088E-01 | 2.8465       |
| 0.75 | 1000    | 9.5021E-03 | 5.9050           | 2.2955E-01 | 0.0538       |
|     | 10000   | 1.0741E-04 | 6.1569           | 2.1898E-01 | 0.5109       |
|     | 100000  | 2.2905E-05 | 6.1818           | 1.8771E-01 | 5.6996       |

**TABLE 12** Numerical results of Example 5.1 using Scheme I (2.31) in 2D

| $s$ | $\frac{1}{h}$ | $E(h)$     | Rate | CPU time (s) |
|-----|----------------|------------|------|--------------|
| 0.25 | 32             | 3.4047E-02 | –    | 0.9227       |
|     | 64             | 2.5291E-02 | 0.5159 | 10.797       |
|     | 128            | 1.9142E-02 | 0.5099 | 53.474       |
|     | 256            | 1.4927E-02 | 0.5462 | 246.99       |
|     | 512            | 1.2011E-02 | 0.5317 | 1922.1       |
| 0.50 | 32             | 8.6860E-03 | –    | 0.6604       |
|     | 64             | 4.7663E-03 | 0.4627 | 4.2421       |
|     | 128            | 2.7036E-03 | 0.9262 | 36.764       |
|     | 256            | 1.6377E-03 | 0.9524 | 335.90       |
|     | 512            | 1.0602E-03 | 0.8392 | 1712.6       |
| 0.75 | 32             | 1.1589E-02 | –    | 0.7423       |
|     | 64             | 4.6710E-03 | 0.9681 | 5.3990       |
|     | 128            | 1.8928E-03 | 1.3157 | 39.301       |
|     | 256            | 8.2243E-04 | 1.3769 | 441.43       |
|     | 512            | 3.9633E-04 | 1.3286 | 2169.6       |
TABLE 13  Numerical results of Example 5.3 for modified walk-on-sphere method in 3D

| $s$     | Samples | $E$          | Average no. steps | Variance   | CPU time (s) |
|---------|---------|--------------|-------------------|------------|--------------|
| $s = 0.25$ | 1000    | 4.2582E-03   | 1.8890            | 7.4618E-02 | 0.0327       |
|         | 10000   | 8.6731E-04   | 1.9480            | 7.2944E-02 | 0.1464       |
|         | 100000  | 1.5456E-04   | 1.9233            | 7.2383E-02 | 1.4511       |
| $s = 0.50$ | 1000    | 7.2182E-03   | 3.9970            | 1.093E-01  | 0.0348       |
|         | 10000   | 1.7504E-04   | 3.8960            | 1.0408E-01 | 0.3011       |
|         | 100000  | 1.3108E-04   | 3.9187            | 1.0043E-01 | 2.9787       |
| $s = 0.75$ | 1000    | 9.3257E-04   | 10.357            | 1.2448E-01 | 0.0810       |
|         | 10000   | 5.2534E-04   | 10.067            | 1.1665E-01 | 0.7787       |
|         | 100000  | 2.5671E-04   | 10.132            | 1.1586E-01 | 7.7002       |
TABLE 14  Numerical results of Example 5.3 for modified walk-on-sphere method in 4D and 5D by $10^5$ samples

| $s$  | $n$  | $E$    | Average no. steps | Variance  | CPU time (s) |
|------|------|--------|-------------------|-----------|--------------|
| 0.20 | 4    | 1.0203E-03 | 1.3759           | 7.0657E-02| 10.931       |
| 0.40 | 4    | 3.8298E-04 | 2.3480           | 1.3780E-01| 18.279       |
| 0.60 | 4    | 5.9519E-04 | 5.0809           | 1.9744E-01| 40.125       |
| 0.80 | 4    | 6.2239E-04 | 15.702           | 2.5873E-01| 121.50       |
| 0.20 | 5    | 9.3904E-04 | 1.3557           | 5.8018E-02| 16.349       |
| 0.40 | 5    | 5.6123E-04 | 2.3758           | 1.1718E-01| 28.644       |
| 0.60 | 5    | 9.7228E-04 | 5.4498           | 1.6952E-01| 65.567       |
| 0.80 | 5    | 2.5286E-03 | 18.684           | 2.2059E-01| 221.95       |

TABLE 15  Numerical results of Example 5.3 for modified walk-on-sphere method in 10D by $10^5$ samples

| $s$  | $n$  | $E$    | Average no. steps | Variance  | CPU time (s) |
|------|------|--------|-------------------|-----------|--------------|
| 0.10 | 10   | 2.1003E-04 | 1.0953           | 1.9213E-02| 34.905       |
| 0.30 | 10   | 1.6807E-03 | 1.6611           | 5.9872E-02| 49.659       |
| 0.50 | 10   | 6.3033E-03 | 3.6920           | 9.9123E-02| 110.85       |
| 0.70 | 10   | 2.8531E-03 | 12.230           | 1.3642E-01| 344.63       |
| 0.90 | 10   | 1.8440E-03 | 80.954           | 1.6415E-01| 2955.7       |

For higher dimensional cases, we still first evaluate $u(x)$ with $x = \frac{1}{4} \times \text{ones}(4)$ in four spatial dimensions and $u(x)$ with $x = \frac{1}{5} \times \text{ones}(5)$ in five spatial dimensions. The number of samples are set by $10^5$ and the numerical results is given in Table 14. It is noticed that the average number of steps will increase when the fractional order $s$ increases.

We then evaluate $u(x)$ with $x = \frac{1}{10} \times \text{ones}(10)$ in 10 spatial dimensions. The number of samples are set by $10^5$. Unlike the homogeneous equation in Examples 5.1 and 5.2, we need to sample $Y$ in every step so that it will cost more computational time. However, based on the numerical results given in Table 15 the algorithm is still fast and efficient.

Remark 5.1  Since the numerical experiments for modified walk-on-sphere method contain randomness, the variance sometimes does not converge (e.g., Table 15).

As discussed in Section 4, the average number of steps depends only on the domain $\Omega$, the point $x$ at which the solution we want to arrive, and $s$. Combining Figures 6–9, we conclude that when $\Omega$ is a ball, the steps will increase if $x$ is far from the center of the sphere or $s$ becomes larger.

Next, we take an example of the fractional Poisson equation defined on a square.

Example 5.4  Consider the following fractional Poisson equation with vanishing Dirichlet boundary condition

$$
\begin{cases}
(-\Delta)^s u(x) = f(x), & x \in [0, 1]^n, \\
u(x) = 0, & x \in \mathbb{R}^n \setminus [0, 1]^n,
\end{cases}
$$

where $f(x) = 1$. 
We evaluate $u(x)$ at points $x^{(1)} = \frac{1}{1000} \times \text{ones}(10)$ and $x^{(2)} = \frac{1}{10} \times \text{ones}(10)$ in 10 spacial dimensions, respectively. The number of samples are set by $10^5$. Average number of steps will increase when $s$ grows. From the numerical experiments in Table 16, it seems that the average number of steps has no relation to the location point $x$.

## 6 | CONCLUSION

We propose a modified walk-on-sphere method for the fractional Laplacian problem on general domains in high dimensions. Based on the probabilistic representation of the problem, we carefully compute the probabilities of the random walks, using proper quadrature rules and the modified walk-on-sphere method to sample from the probabilities. We show that the quadrature rules are of second-order convergent when the boundary data $g(x) \in C^2_b(\mathbb{R}^n \setminus B_r)$ and the forcing $f = 0$. When $f(x) \in C^2_b(B_r \setminus S_h)$ and $g(x) = 0$, we derive the numerical method in two dimensions, while the convergent order is only $O(h^{2s/1})$ because of the poor property of Green function and it will cost more computational time. So, it is necessary to propose much more efficient method for the problem. Thus, for problems in higher dimensions, we apply an efficient rejection sampling method based on truncated Gaussian distribution. Also, we estimate the mean of the number of walks for the problem in a ball in $n (n \geq 2)$ dimensions and $s \in (0, 1)$ and show that the mean of the number of walks is increasing in $s$ and the distance of the initial point to the origin. Numerical results verify the theoretical analysis and show the efficiency of the proposed method. Extensions to fractional advection–diffusion equations are currently ongoing.

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## DATA AVAILABILITY STATEMENT

The datasets generated during the current study are available from the corresponding author upon reasonable request.
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