A New Signal Injection-based Method for Estimation of Position in Salient Permanent Magnet Synchronous Motors

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Abstract—Several heuristic procedures to estimate the rotor position of permanent magnet synchronous motors (PMSM) via signal injection have been reported in the literature. Using averaging theory, a framework to analyse such schemes has been recently proposed. However, to the best of our knowledge, no theoretical analysis of the performance of the conventional linear time invariant filtering methods, which are widely used as standard industrial practice, has been reported in the literature. The objective of this note, is to propose a new method that, on one hand, is amenable to a rigorous theoretical analysis and, on the other hand, ensures an improved accuracy in the position estimation. An additional advantage of the new method is that it relies on the use of linear operators, implementable with simple computations. The effectiveness of the proposed scheme is assessed by experiments on an interior PMSM platform driven by a 521 V DC bus with 5-kHz PWM.

Index Terms—Permanent magnet synchronous motors, signal injection, observer design, sensorless control.

NOMENCLATURE

Symbols

| Symbol | Description |
|--------|-------------|
| $L$   | Rotor flux angle [rad] |
| $\omega$ | Angular velocity [rad/s] |
| $\Phi$ | Magnetic flux [Wb] |
| $J$ | Drive inertia [kg-m²] |
| $f$ | Friction constant |
| $\theta$ | Rotor torque [N-m] |
| $R_s$ | Stator resistance [Ω] |
| $n_p$ | Number of pole pairs |
| $\omega_h$ | Angular frequency of injection signal [rad/s] |
| $\varepsilon$ | Period of injection signal ($\varepsilon = \frac{2\pi}{\omega}$) [s] |
| $V_i$ | Amplitude of injection signal [V] |
| $\text{HPF}$ | High-pass filter |
| $\text{LPF}$ | Low-pass filter |
| $\mathcal{H}$ | Bounded-input bounded-output (BIBO) operator acting on the input signal $u(t)$ to generate the output $y(t)$ |
| $y(t)$ | Virtual output |
| $i_{\alpha\beta}$ | $[i_{\alpha}, i_{\beta}]^T$ |
| $v_{\alpha\beta}$ | $[v_{\alpha}, v_{\beta}]^T$ |
| $I, J$ | Identity matrix on $\mathbb{R}^{2\times2}$ and $[0 \ -1 \ 1 \ 0]$ |

Superscripts

| Superscript | Description |
|-------------|-------------|
| $r$ | Actual reference frame |
| $\hat{r}$ | Estimated reference frame |
| $r^*$ | Reference value |
| $r^h$ | High-frequency component |
| $r^l$ | Low-frequency component |
| $v_{\alpha\beta}$ | Low frequency control input |

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I. INTRODUCTION

Permanent magnet synchronous motors (PMSMs) are widely used in industrial applications because of their superior power density and high efficiency. The sensorless control of PMSMs is increasingly attractive for its inherent advantages. Two different types of sensorless control methodologies are reported in the literature. The first type is the fundamental-model-based method, which is usually known as the back-emf or flux-linkage estimation, utilizing the fundamental components of electrical signals [1, 2, 14, 16]. The second one is saliency-tracking-based method, in which information is extracted from the high-frequency components of stator currents via high-frequency signal injections [5, 11].

It is well-known that, due to two facts, the fundamental-model-based method degrades or even fails under the conditions of low speeds and standstill: 1) in the low speed regions the magnitude of back-emf is relatively small—hampering its
extraction in the presence of measurement noises \[9\]; 2) the fundamental dynamical model loses observability at standstill \[18\]. However, the saliency tracking-based method, which utilizes the anisotropy due to rotor saliency and magnetic saturation, is favored in the low speed region. In this paper, we address the problem of position estimation for interior PMSMs at low speeds or standstill, thus with the second methodology adopted.

On one hand, the signal injection method is a widely-used technique-oriented method for electromechanical systems. With the notable exception of \[4\] \[5\] \[11\], no theoretical analysis can be found in the literature. On the other hand, in the last two decades, signal injection-based approaches were developed adopting the following approach\[1\] first, high frequency probing signals are injected into the motor terminal with the main driving power; then, extract the high-frequency components of the stator currents to get position estimates. Thus, the key problem is the signal processing of the measured stator currents, which is usually achieved via the combination of linear time invariant (LTI) high pass-filters (HPFs) and low-pass filters (LPFs) \[15\]. Several technique-oriented procedures have been reported in the literature to solve the signal processing problem, however, to the best of our knowledge, no theoretical analysis of these heuristic methods has been reported in the literature. An open problem is providing a theoretical interpretation to the existing technique-oriented methods, and establishing the connection between these methods and the methods based on averaging analysis, for instance, \[4\] \[21\]. The importance of disposing of rigorous analytic results can hardly be overestimated, since it allows, on one hand, to carry out a quantitative performance assessment while, on the other hand, it provides guidelines to make more systematic and simplify the parameter tuning procedure.

The main contributions of our paper are threefold.

- To utilize the new filtering stage, proposed in \[21\], to estimate the position from the measured currents. Since the proposed scheme also uses linear filters—but, in our case they are time varying—the increase in computational complexity is negligible.
- To apply the averaging technique to analyse the mechanism and the estimation accuracy of the conventional HPF/LPF procedure.
- To prove that the new proposed method, which was derived adopting parameter estimation based perspective, admits also an HPF/LPF interpretation. This is an important aspect, since it shows the connection of the new filtering method with the standard industrial practice, also providing a nice downwards-compatibility with the existing approaches.

The remainder of paper is organized as follows. In Section \[1\] we recall the mathematical model of interior PMSMs and formulate the problem of estimation of position using signal injection. Section \[III\] discusses the classical frequency-based accuracy analysis of the position estimators used for the conventional methods, and highlights their theoretical limitations.

To overcome these problems, we propose to use well-known and mathematically rigorous averaging techniques for this analysis. In Section \[IV\] the new method is proposed, and then some comparisons and similarities with the conventional methods are given in Section \[V\]. Simulation and experimental results are given in Section \[VI\] and the paper is wrapped-up with some concluding remarks in Section \[VII\]. To enhance the readability, in the body paragraphs we avoid the complicated language and notation. Therefore, in Appendix for interested readers.

II. MODEL AND PROBLEM FORMULATION

The voltage equations of the interior PMSM in the stationary frame are given by \[15\].

\[
v_{\alpha\beta} = \left[ R_s I + L(\theta) s - 2n_p \omega L_1 Q(\theta) \right] i_{\alpha\beta} + n_p \omega \Phi [\sin \theta \cos \theta],
\]

where we define the mappings

\[
L(\theta) := L_0 I + L_1 Q(\theta)
\]

\[
Q(\theta) := \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix},
\]

with the averaged inductance \(L_0\) and the inductance difference \(L_1\) defined as

\[
L_0 := \frac{1}{2}(L_d + L_q), \quad L_1 := \frac{1}{2}(L_d - L_q).
\]

The stationary model \[1\] together with the mechanical dynamics, can be expressed in the standard state-space form as follows.

\[
L(\theta) \frac{d}{dt} i_{\alpha\beta} = F(i_{\alpha\beta}, \theta, \omega) + v_{\alpha\beta}
\]

\[
\frac{d}{dt} \theta = n_p \omega
\]

\[
J \frac{d}{dt} \omega = n_p \Phi (i_{\beta} \cos \theta - i_{\alpha} \sin \theta) - f \omega - \tau_L,
\]

where we define the mapping

\[
F(i_{\alpha\beta}, \theta, \omega) := (2n_p \omega L_1 Q(\theta) J - R_s I) i_{\alpha\beta} + n_p \omega \Phi \begin{bmatrix} \sin \theta \\ -\cos \theta \end{bmatrix}.
\]

Position Estimation via Signal Injection Assume there is a stabilizing controller operator \(\Sigma_C\) measuring only \(i_{\alpha\beta}\), and define its output as

\[
v_{\alpha\beta}^C(t) := \Sigma_C [i_{\alpha\beta}(t)].
\]

Inject a high-frequency signal to one axis of the control voltage, say, the \(\alpha\)-axis, that is,

\[
v_{\alpha\beta} = v_{\alpha\beta}^C + \begin{bmatrix} V_h \sin \omega_h t \\ 0 \end{bmatrix},
\]

where \(\omega_h := \frac{2\pi}{\epsilon},\) with \(\epsilon > 0\) small, and \(V_h > 0\). The problem is to define an operator \(\Sigma_E:\ i_{\alpha\beta} \rightarrow \hat{\theta}\) such that

\[
\limsup_{t \rightarrow \infty} |\hat{\theta}(t) - \theta(t)| \leq O(\epsilon),
\]

\[1\] In this paper, the “classical heuristic” or “technique-oriented” procedures refer to this route.
where \( \mathcal{O} \) is the uniform big O symbol.\(^2\)

It is well-known that high frequency probing signals almost have no effects on the motor mechanical coordinates. However, due to the rotor saliency, it induces different high-frequency responses in the \( \alpha \)- and \( \beta \)-axes currents. This fact provides the possibility to recover the angle from the high-frequency components of stator currents.

III. FREQUENCY DECOMPOSITION OF \( i_{\alpha\beta} \) AND DEFINITION OF THE VIRTUAL OUTPUT

In this section, we give the analysis of frequency decomposition of the stator currents \( i_{\alpha\beta} \), which is instrumental for the design and analysis of position estimators.

A. Conventional Frequency Analysis

First we recall the conventional frequency decomposition in the technique-oriented literature, which relies on the ad-hoc application of the superposition law \([9, 15]\). That is, suppose the electrical states consist of high-frequency and low-frequency components as

\[
(\cdot)_{\alpha\beta} = (\cdot)^h_{\alpha\beta} + (\cdot)^l_{\alpha\beta}.
\]

If \( \omega \approx 0 \), the current responses can be separated as

\[
i^h_{\alpha\beta} + i^l_{\alpha\beta} = (R_s I + L(\theta)s)(i^h_{\alpha\beta} + i^l_{\alpha\beta}).
\]

(5)

For the approximative high-frequency model \( i^h_{\alpha\beta} \approx L(\theta)s^h_{\alpha\beta} \), neglecting the stator resistance, the angle \( \theta \) can be regarded as a constant, thus the high-frequency response contains the information of \( \theta \), namely, for the input \( \bar{I} \)

\[
i^h_{\alpha\beta} = \frac{V_h}{L_d L_q} [\sin \omega_h t] [L_0 - L_1 \cos 2\theta - L_1 \sin 2\theta].
\]

(6)

Substituting \( s = j\omega_h \), we approximately get the high-frequency components of the stator current as

\[
i^h_{\alpha\beta} = \frac{1}{\omega_h L_d L_q} [L_0 - L_1 \cos 2\theta - L_1 \sin 2\theta] (-V_h \cos \omega_h t).
\]

The derivation of the above high-frequency model is based on two assumptions, namely, the superposition law and the slow angular velocity \( \omega \approx 0 \), regarding which, the following remarks are in order.

- The dynamics \([16]\) is highly nonlinear. It is well-known that nonlinear systems “mix” the frequencies, making the superposition law not applicable. Although using the classical decomposition \([5]\) to estimate position may work in practice, it fails to reliably provide, neither a framework for a quantitative performance assessment, nor guidelines to tune parameters.
- The assumption \( \omega \approx 0 \) implies that the decomposition above is applicable only at standstill or very low speeds.

B. Frequency Analysis via Averaging

Averaging analysis provides a rigorous and elegant decomposition of the measured currents as follows. We refer the reader to \([4, 7, 19]\) for some basic knowledge on averaging analysis. Applying averaging analysis, it is shown that with \( \omega_h \) large enough

\[
i_{\alpha\beta} = \bar{i}_{\alpha\beta} + \varepsilon y_v S + O(\varepsilon^2),
\]

(7)

where, we defined the signal

\[
S(t) := \frac{V_h}{2\pi \cos \omega_h t},
\]

(8)

the (so-called) virtual output

\[
y_v := \frac{1}{L_d L_q} \left[ -L_1 \cos 2\theta + L_0 \right],
\]

(9)

and \( \bar{i}_{\alpha\beta} \) is the current of the closed-loop system with \( v_{\alpha\beta} = v^C_{\alpha\beta} \)—that is, without signal injection. From \((9)\) it is clear that

\[
\theta = \frac{1}{2} \arctan \left( \frac{y_{v2}}{y_{v1} - \frac{L_0}{L_d L_q}} \right).
\]

Hence the position estimation problem is translated into the estimation of \( y_v \). Towards this end, we notice that, from a frequency viewpoint, \( i_{\alpha\beta} \) contains fundamental frequency components \( \bar{i}_{\alpha\beta} \) and high frequency components \( \varepsilon y_v S \). It should be noticed that the high frequency term \( \varepsilon y_v S \) coincides with the one in \((9)\), but the averaging analysis characterizes all the components in \( i_{\alpha\beta} \) quantitatively.

It is natural, then, that to “reconstruct” \( y_v \)—out of measurements of \( i_{\alpha\beta} \)—we need to separate these components via some sort of HPF and LPF operations. This is the rationale underlying all existing position estimators reported in the literature, see \([15]\) for a recent review.

IV. PROPOSED ESTIMATION METHOD

Before presenting the new design, we define three BIBO-stable, linear operators,

- first, the delay operator \( D_d \), with parameter \( d > 0 \),

\[
D_d[u(t)] = u(t - d);
\]

(10)

- second, the weighted zero-order-hold operator \( \mathcal{Z}_w \), parameterized by \( w > 0 \), and defined as

\[
\hat{x}(t) = \chi(t) - \frac{1}{w} \left[ \chi(t) - \chi(t - w) \right];
\]

(11)

- third, the linear time-varying (LTV) operator \( \mathcal{G}_{\text{grad}} \) defined as

\[
\dot{x}(t) = -\gamma S^2(t) x(t) + \gamma S(t) u(t)
\]

(12)

where \( \gamma > 0 \) is a tuning gain.

These operators are instrumental for the following design. To construct the estimator, apply the first two operators to the currents as follows,

\[
\mathcal{Y}(t) := (D_d - \mathcal{Z}_d)[i_{\alpha\beta}(t)].
\]

(13)
We make the observation that, using the Laplace transform, the action of \( i_{\alpha} \) may be represented in the frequency domain as

\[
Y_f(s) = G_d(s)\sigma_{\alpha}(s),
\]

where we defined the transfer function

\[
G_d(s) := e^{-ds} + \frac{1}{2ds} \left( e^{-2ds} - 1 \right).
\] (14)

The description of the estimator is completed applying the third operator \( G_{rad} \) to \( Y_f \) to generate the estimate of \( y_v \), denoted \( \hat{y}_v \), that is,

\[
\hat{y}_v(t) = G_{rad}[Y_f(t)].
\] (15)

See Fig. 1.

![Fig. 1. Block diagram of the proposed estimation method](image)

Using the analysis reported in [21], with \( d = \varepsilon \) it is shown that the estimator \( i_{\alpha} \) verifies

\[
\lim_{t \to \infty} |\hat{y}_v(t) - y_v(t)| \leq O(\varepsilon).
\]

A rigorous statement of the result above is presented in Proposition 2 in Appendix B.

Thus, defining the angle estimate as

\[
\hat{\theta} = \frac{1}{2} \arctan \left( \frac{\hat{y}_{v2}}{\hat{y}_{v1} - \frac{L_m}{L_d L_q}} \right),
\] (16)

the required asymptotic accuracy \( \varepsilon \) is achieved.

V. COMPARISON WITH CONVENTIONAL METHODS

In [15] the position estimation method, for low rotation speeds, shown in Fig. 2 is proposed. In this section we, first, compare the performance of this classical estimator with the one proposed here. Then, we show the structural and functional similarities between the two methods.

![Fig. 2. Block diagram of the conventional signal injection method](image)

A. Estimation accuracy

To evaluate the performance of the classical method in Fig. 2 without loss of generality, select the LTI filters as

\[
\begin{align*}
\text{HPF}(s) &= \frac{2s^2}{(\lambda_\phi + s)^2} \\
\text{LPF}(s) &= \frac{\lambda_\theta}{\lambda_\theta + s},
\end{align*}
\] (17)

with parameters

\[
\lambda_\phi = \omega_h, \; \lambda_\theta = \max\{\sqrt{\omega_h \omega_s}, 1\}.
\] (18)

The Bode diagrams of two filters are given in Fig. 3 with \( \omega_h = 500, \omega_s = 1 \).

![Fig. 3. Bode Diagram of the HPF/LPF](image)

Applying averaging analysis at reduced speeds, and setting \( \phi = 0 \), we have\(^3\)

\[
\lim_{t \to \infty} \sup |\hat{\theta}(t) - \theta(t)| \leq O(\varepsilon^2),
\]

which compared with the accuracy achieved by the new estimator \( \hat{\theta} \), shows the performance enhancement\(^4\).

With the consideration that some readers are interested in the analysis of the conventional filtering method, we give it in a rigorous statement, Proposition 4 with its proof in Appendix A.

B. Frequency interpretation of new estimator

The new estimator proposed in this paper exactly coincides with the block diagram in Fig. 2 assigning \( \phi = \frac{\pi}{2} \) and the filters as follows

\[
\begin{align*}
\text{HPF} &= D_d - Z_{2d} \\
\text{LPF} &= \frac{1}{2} \left( \frac{V_h}{2\pi} \right)^2 H,
\end{align*}
\] (19)

where \( H \) is the single-input single-output LTV filter

\[
\dot{z}(t) = -\gamma S^2(t)z(t) + \gamma u(t) \\
H[u(t)] = z(t).
\] (20)

See Fig. 4.\(^5\)

\(^3\)Indeed, the saliency-tracking-based method has an angular ambiguity of \( \pi \) due to the \( \tan^{-1}(\cdot) \) operation. It is possible to utilize the saturation effect in \( d \)-axis of machine, as well as \( y_v \), to conduct the magnetic polarity identification. The problem is out of the scope of the paper, and we refer the readers to [10] for more details.

\(^4\)When the PMSM is working at standstill, the estimation accuracy at the steady stage becomes also \( O(\varepsilon) \).
proposed one, which are 0 and $\frac{\pi}{2}$, respectively. As shown in Figs. 3 and 5, this is caused by the different phase lags (+$\frac{\pi}{2}$ and $-2\pi$) of the second-order LTI filter (17) and the delayed LTI filter $G_d(s)$ at the frequency $\omega_h$.

C. Parameters Tuning

For the implementation of the proposed estimator, there are three tunable parameters, namely, $\gamma$, $\omega_h$ and $V_h$. We are in position to give some discussions on the parameter selection.

D1 A larger gain $\gamma$ yields a faster convergence speed of the estimation error, but at the price of being more sensitive to measurement noise. The performance assessment, including the transient and steady-state stages, of the virtual output estimates is instrumental for its tuning.

D2 For the frequency parameter $\omega_h$, there is a tradeoff between the estimation accuracy and the sensitivity to measurement noises [17]. On one hand, a higher frequency increases the accuracy. On the other hand, the measurement is

$$i_{\alpha\beta} = i_{\alpha\beta} + \varepsilon_y v + O(\varepsilon^2) + \nu,$$  \hspace{1cm} (24)

where $\nu$ is the unavoidable measurement noise. Thus, a higher frequency will decrease the signal-to-noise ratio.

D3 The amplitude $V_h$ shares some similar effects on the estimation performance with $\omega_h$, due to (24), since, a smaller amplitude $V_h$ yields a smaller signal-to-noise ratio. However, the parameter $V_h$ has limited effects on the estimation accuracy.

VI. SIMULATIONS AND EXPERIMENTS

A. Simulations

The proposed estimator is first tested by means of simulations in Matlab/Simulink. We use the parameters of Table I, the current-feedback controller $\Sigma_c$, given below, together with the proposed estimator.

1) Position estimator in Fig. 1 with (16).
2) Rotation between $\alpha\beta$-coordinates and misaligned $dq$-coordinates, namely,

$$i_{dq} = e^{-\theta} i_{\alpha\beta}, \hspace{0.5cm} v_{\alpha\beta} = e^{\theta} v_{dq}.$$

3) Speed regulation PI loops

$$i_{dq}^* = \left( K_p + K_i \frac{1}{n_p} \right) \left( \omega^* - \hat{\omega} \right),$$

where $\omega^*$ is the reference speed, and $\hat{\omega}$ is an estimate of the rotor speed obtained via the following PLL-type estimator.

$$\dot{\eta}_1 = K_p (\dot{\theta} - \eta_1) + K_i \eta_2,$$

$$\eta_2 = \dot{\theta} - \eta_1,$$

$$\dot{\omega}_p = K_p (\dot{\theta} - \eta_1) + K_i \eta_2,$$

$$\hat{\omega} = \frac{1}{n_p} \dot{\omega}_p.$$  \hspace{1cm} (25)
4) Current regulation loops

\[ v_d = \left( K_p + K_i \frac{1}{s} \right) (i_d^* - i_d^\ell) - L n_p \dot{\omega} i_q \]

\[ v_q = \left( K_p + K_i \frac{1}{s} \right) (i_q^* - i_q^\ell) + L n_p \dot{\omega} i_d + n_p \dot{\omega} \Phi, \]

where \( i_{dq}^\ell \) are filtered signals of \( i_{dq} \) by some LPFs.

**TABLE I**

| Parameters of the PMSM: Simulation (First Column) and Experiments (Second Column) |
|----------------------------------------------------------------------------------|
| Number of pole pairs (\( n_p \)) | 6 | 3 |
| PM flux linkage constant (\( \Phi \)) [Wb] | 0.11 | 0.39 |
| \( d \)-axis inductance (\( L_d \)) [mH] | 5.74 | 3.38 |
| \( q \)-axis inductance (\( L_q \)) [mH] | 8.68 | 5.07 |
| Stator resistance (\( R_s \)) [\( \Omega \)] | 0.43 | 0.47 |
| Drive inertia (\( J \)) [kg-m\(^2\)] | 0.01 \( \geq 0.01 \) |

We operate the motor at the slow speed of 30 rad/min with \( \tau_L = 0.5 \) N-m and the parameters \( \varepsilon = 10^{-3} \), \( \gamma = 10^{-4} \), \( V_h = 1 \), \( \omega^* = 0.5 \) and those in Table I. Fig. 7 shows the simulation results. In Fig. 7(a) we also give the position estimate obtained from the conventional LTI filters, denoted \( \hat{\theta}_{LTI} \). Considering the root-mean-square deviation (RMSD)

\[ \text{RMSD} = \sqrt{\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (\hat{\theta}(s) - \theta(s))^2 ds} \]

with \( \theta, \hat{\theta} \in S^1 \), we calculate the RMSDs for two methods in the interval [5, 10] s. They are 0.0872 and 0.1411 for the proposed design and the conventional LTI filtering method, respectively. We conclude that the new design outperforms the conventional LTI filtering method with a higher accuracy. It is also observed that the sensorless control law regulates the angular velocity at the desired value.

**B. Experiments**

**Losses and Compensations.** Before introducing the experimental results, let us say something about the loss of phase shift, which is unavoidable, as well as its compensations.

The excitation signal is injected into the modulation signal for the stationary \( \alpha \)-axis. The excitation in the \( \alpha \)-axis is, indeed, also affected by the inverter imperfections, for instance, the lockout time. Further on, the current \( i_{\alpha} \) responding to the excitation is phase-shifted by 90 degrees with respect to the voltage only in an ideal case, whereas the winding does not have any resistance. In practice, it should be a non-zero winding resistance, with the phase shift lower than 90 degrees. Additional impact is due to the excitation-frequency eddy-currents in the magnetic circuit. Acting as a short-circuited secondary winding in a transformer-like electromagnetic setup, the iron losses introduce an additional change of the phase shift. Thus, these factors, but not limited to them, cause the phase of high-frequency component in \( i_{\alpha} \), see the term \( S(\cdot) \) in (7), to be different from the one in the test signal.

**System Configuration.** The scheme developed in this paper was tested on an interior PMSM platform, shown in Fig. 8(b). The test IPMSM is a FAST PMSM, whose parameters are given in Table I. It has a 72 V line-to-line peak at 1000 RPM. The voltage of DC bus is 521 V, with the frequency of PWM 5 kHz.

The experimental setup comprises two synchronous motors with surface-mounted permanent magnets on the rotor. One of
them runs in the speed control mode, and it is used to maintain the speed at the desired level. The motors are coupled by means of a toothed belt, which also connects an inertial wheel. Experimental setup with two mechanically coupled, inverter supplied brushless dc motors: 1) main power supply unit comprising the line rectifier and two 3-phase PWM inverters with control circuits, 2) dc-bus support with dynamic breaking, 3) speed controlled motor, 4) torque controlled motor, 5) inertia coupled with both motors. The motor under the test is obtained by taking an industry-standard FAST motor and introducing changes into the rotor magnetic circuit so as to obtain the difference (2:3) between the d-axis and q-axis inductance. This motor runs in the torque-control mode. The speed and position are obtained through the high-speed digital serial link from standard industrial high-resolution sensors mounted on the shaft. The sampling time is $T_s = 300$ ns, and the acquisition time is set to cover at least two electrical periods. The three-phase currents, voltages and the rotor position were measured from the drive measurement system—a "Sincoder" shaft sensor.

In experiments, we only test the proposed estimator, thus the estimated signals are not used in the closed-loop system, whose bandwidth of the speed loop is larger than 100 Hz. The test signal is injected into the modulating signal in the $\alpha$-axis. The test motor is driven by another motor which kept the speed at 60 RPM. There is a large inertia involved, in excess of 0.01 kg m$^2$, thus keeping the speed rather constant. The motor under the test has zero current references ($i^\star_d$ and $i^\star_q$), and in this way we can increase the resolution of the measurement system.

**Experimental Results.** In the first experiment, the amplitude of the test signal is 2 V, with the frequency 400 Hz with zero reference currents. The parameters of the estimator are selected as $\gamma_\alpha = 1.25 \times 10^4$ and $\gamma_\beta = 2.5 \times 10^4$, with the compensation parameters $\ell_1 = 2.2$, $\ell_2 = 60$ and $\ell_3 = 1.7711^5$. For such a case, Fig. 9(a) shows the performance of the proposed position estimator. The test signal was only injected to the $\alpha$-axis, which is illustrated in Fig. 9(b) after Clarke transform.

We present some experimental results in Fig. 10. Figs. 10(a)-10(b) show the performance of the estimator with the injection frequencies 200 Hz and 100 Hz, respectively. We conclude that the performances degenerate for the frequencies from 400 Hz to 100 Hz. We also test the case with 800 Hz, but unfortunately, it does not work. A possible interpretation, as in D2, is that such a case is of a low signal-to-noise ratio.

Therefore, the injection frequency is suggested to be selected in [200, 400] Hz for the tested motor. In order to evaluate the proposed HPF, we consider the case with non-zero reference currents with the angular velocity at 60 rad/s and 40 rad/s, for which the corresponding results are illustrated in Figs. 10(c) and 10(d) with satisfactory performances. Fig. 10(e) shows the results when the motor is speeding up from 40 rad/s to 60 rad/s in around one second. A slight distortion can be observed when accelerating.

**VII. CONCLUSION**

This paper addresses the problem of position estimation of interior PMSMs at low speeds and standstill. Although the salience-tracking-based methods are effective and widely-studied, the theoretical analysis of the conventional methods, taking into account the nonlinear dynamics of PMSMs, was conspicuous by its absence. This paper attempts to fill in this gap analysing the stator current $i_{\alpha\beta}$ via the averaging method, with guaranteed error with respect to the injection frequency $\omega_h$. Also, with the key identity (7), we develop a new position estimator, which ensures an improved accuracy. Moreover, we establish the connection between the new method and the conventional one, showing that they can be
unified in the HPF/LPF framework from the perspective of signal processing.

The following extensions and issues are of interest to be further explored.

- For the sake of clarity, we only study the basic case of signal-injection methods for the PMSM model (1). The proposed method can also be extended to other motor models, for instance, saturated interior (or surface mounted) PMSMs.
- It is of interest to couple the proposed method with some model-based (non-invasive) techniques, for instance the gradient descent observer in [13, 16], in order to be able to operate the sensorless controller over a wide speed range. Such an approach has been pursued in [3].

**APPENDIX**

A. **ANALYSIS OF THE CONVENTIONAL METHOD**

In this section, we give the mathematical analysis of the conventional LTI filtering method. The main result is summarized as follows.

**Proposition 1.** For the IPMSM dynamics (1), suppose the control $v_{\alpha\beta}^C$ guarantees all the states bounded, with the speed

$$\|\omega, \dot{\omega}, v_{\alpha\beta}^C\| \leq \ell_\omega$$

for some constant $\ell_\omega$ independent of $\varepsilon$. If the filters are selected as (17)-(18), then the signal processing procedure in Fig. 2, namely,

$$y_h = \text{HPF}[i_{\alpha\beta}]$$

$$[Y_\alpha, Y_\beta]^\top = \frac{2\omega_h L_d L_q}{V_h} \cdot \text{LPF} \left[ y_h \times \sin(\omega_h t + \phi) \right]$$

$$\dot{\theta} = \frac{1}{2} \arctan \left( \frac{Y_\beta}{Y_\alpha - L_0} \right)$$

with $\phi = 0$, guarantees

$$\limsup_{t \to \infty} \left| \dot{\theta}(t) - \theta(t) \right| = n\pi + \mathcal{O}(\varepsilon^{\frac{1}{2}})$$

for $n \in \mathbb{Z}$, when $\omega_h \geq \omega_h^*$ for some $\omega_h^* \in \mathbb{R}_+$, with $\varepsilon = \frac{2\pi}{\omega_h}$.

**Proof.** Part A) **Filtered signal via HPF.**

Applying the operator HPF to (7), we have

$$\text{HPF}[i_{\alpha\beta}] = \text{HPF}[\bar{i}_{\alpha\beta}] + \frac{2\pi}{\omega_h} \text{HPF}[D(\theta)S] + \mathcal{O}(\varepsilon^2) + \epsilon_t,$$

with the definition

$$D(\theta) := L^{-1}(\theta) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For the first term of (27), we have

$$\text{HPF}[\bar{i}_{\alpha\beta}] = \frac{2\pi^2}{(w_h + s)^2} [\bar{i}_{\alpha\beta}]$$

$$= \frac{2\pi}{(w_h + s)^2} \left[ F(\bar{i}_{\alpha\beta}, \bar{\theta}, \bar{\omega}) + L^{-1}(\bar{\theta}) v_{\alpha\beta}^C \right]$$

$$= \frac{2}{(w_h + s)^2} \left[ \frac{\partial F}{\partial \bar{\alpha}} \cdot (F + L^{-1} v_{\alpha\beta}^C) + n_p \bar{\omega} \right]$$

$$\times \left( \frac{\partial F}{\partial \bar{\theta}} + \frac{\partial L^{-1}}{\partial \theta} v_{\alpha\beta}^C + \left( \frac{\partial F}{\partial \bar{\omega}} + L^{-1} \mathcal{O}(\bar{\omega}) \right) \right)$$

where we have used the assumption $|\bar{\omega}, \bar{\omega}, v_{\alpha\beta}^C| \leq \ell_\omega$ in the last term, with the definition

$$F(\bar{i}_{\alpha\beta}, \bar{\theta}, \bar{\omega}) := L^{-1}(\bar{\theta}) F(\bar{i}_{\alpha\beta}, \bar{\theta}, \bar{\omega}).$$

There always exists a constant $\omega_h^* \in \mathbb{R}_+$ such that for $\omega_h > \omega_h^*$

$$\text{HPF}[\bar{i}_{\alpha\beta}] = \frac{2}{\omega_h} \cdot \omega_h^2 [\text{O}(1)].$$

Consider a bounded signal $z(t) \in \mathbb{L}_\infty$. The signal $z(t)$ filtered by the operator $\frac{\omega_h^2}{(\omega_h + s)^2}$ is equivalent to that the signal $u(t)$ going through the following input-output dynamics (from $u$ to $y$)

$$\dot{r} = -\omega_h r + \omega_h u$$

$$\dot{y} = -\omega_h y + \omega_h r.$$

We get $y$ is bounded and $\frac{\omega_h^2}{(\omega_h + s)^2} [\text{O}(1)] = \mathcal{O}(1)$, thus

$$\text{HPF}[\bar{i}_{\alpha\beta}] = \mathcal{O}(\varepsilon^2).$$

For the second term in the right hand side of (27), we have

$$\frac{2\pi}{\omega_h} \text{HPF}[D(\theta)S]$$

$$= -\frac{V_h}{\omega_h} \text{HPF}[D(\theta) \cos(\omega_h t)]$$

$$= -\frac{2V_h}{\omega_h} \frac{s}{(\omega_h + s)^2} \left[ n_p \omega D'(\theta) \cos(\omega_h t) - \omega_h D(\theta) \sin(\omega_h t) \right]$$

$$= -\frac{2V_h}{\omega_h} \frac{1}{(\omega_h + s)^2} \left[ a_1(t) \cos(\omega_h t) + a_2(t) \omega_h \right.$$

$$\times \sin(\omega_h t) - a_3(t) \omega_h \sin(\omega_h t) - \omega_h^2 D(\theta) \cos(\omega_h t) \right].$$

with $a_1(t) := \frac{d}{dt} (n_p \omega D'(\theta))$, $a_2(t) := n_p \omega D'(\theta)$ and $a_3(t) = \omega_h D'(\theta) n_p \omega$, whose derivatives are bounded. When $\omega_h$ is large enough, we have

$$\frac{2\pi}{\omega_h} \text{HPF}[D(\theta)S] = \frac{1}{\omega_h} V_h D(\theta) \sin(\omega_h t) + \mathcal{O}(\varepsilon^2).$$

Therefore, the currents filtered by the HPFs become

$$y_h := \text{HPF}[i_{\alpha\beta}] = \frac{1}{\omega_h} V_h D(\theta) \sin(\omega_h t) + \mathcal{O}(\varepsilon^2).$$

Part B) **Filtered signal via LPF.**

Multiplying $\sin(\omega_h t + \phi)$ on both sides with $\phi = 0$, we get

$$\sin(\omega_h t) y_h = \frac{V_h}{2\omega_h} D(\theta) - \frac{V_h}{2\omega_h} D(\theta) \cos(2\omega_h t) + \mathcal{O}(\varepsilon^2).$$

\footnote{We omit the exponentially decaying term $\epsilon_t$ of filtered signals in the following analysis.}
where we have used the trigonometric identity
\[ \sin^2 \theta = \frac{1 - \cos 2\theta}{2}. \]

Applying the LPF to (29), for the first term we have
\[
\text{LPF} \left[ \frac{V_h}{2\omega_h} D(\theta) \right] = \frac{V_h}{2\omega_h} D(\theta) - \frac{V_h}{2\omega_h} \gamma_s + s \left[ D(\theta) \right] = \frac{V_h}{2\omega_h} D(\theta) + O(\varepsilon^2).
\]

For the second term, we have
\[
\text{LPF} \left[ \frac{V_h}{2\omega_h} D(\theta) \cos(2\omega_h t) \right] = O(\varepsilon^2),
\]

with straightforward calculations and the swapping lemma.

Therefore, the filtered signal satisfies
\[
y_{\alpha} := \text{LPF}[\sin(\omega_h t)y_{\alpha}] = \frac{V_h}{2\omega_h} D(\theta) + O(\varepsilon^2).
\]

Notice the exact form of \( D(\theta) \), thus we having
\[
\begin{bmatrix}
  Y_{\alpha} \\
  Y_{\beta}
\end{bmatrix} := \begin{bmatrix}
  2\omega_h L_d L_q & y_{\alpha} \\
  2\omega_h L_d L_q & y_{\beta} \\
\end{bmatrix} \begin{bmatrix}
  2\omega_h L_d L_q & y_{\alpha} \\
  2\omega_h L_d L_q & y_{\beta} \\
\end{bmatrix}^{-1} \begin{bmatrix}
  L_0 - L_1 \cos 2\theta \\
  -L_1 \sin 2\theta
\end{bmatrix} + O(\varepsilon^2).
\]

Therefore, when \( t \to \infty \) we have
\[
\tan 2\theta = \frac{Y_{\beta}}{Y_{\alpha} - L_0} + O(\varepsilon^2).
\]

Thus,
\[
\theta = \frac{1}{2} \tan^{-1} \left( \frac{Y_{\beta}}{Y_{\alpha} - L_0} \right) + n\pi + O(\varepsilon^2) + \epsilon_t,
\]

with \( n \in \mathbb{Z} \). It completes the proof.

A corollary at standstill is given as follows.

**Corollary 1.** For Proposition (1) with \( \omega \equiv 0 \), we have
\[
\limsup_{t \to \infty} |\dot{\theta}(t) - \theta(t)| = n\pi + O(\varepsilon)
\]

with \( n \in \mathbb{Z} \).

**Proof.** It follows clearly with \( \lambda_\ell = 1 \).

**B. ANALYSIS OF THE NEW ESTIMATOR**

**Proposition 2.** For the IPMSM dynamics (2), suppose the control \( v_{\alpha,\beta}^C \) guarantees all states bounded and the speed
\[
|y_v| \leq \ell_v
\]

for some constant \( \ell_v \), there exist constants \( \omega_h, \gamma^* > 0 \) such that for \( \omega_h > \omega_h^* \) and \( \gamma > \gamma^* \), the estimate satisfies
\[
\limsup_{t \to \infty} \left| \mathcal{G}_{\gamma} G_d(s)[\dot{v}_{\alpha,\beta}(t)] - y_v(t) \right| = O(\varepsilon)
\]

where \( y_v \) is defined in (9) with \( d = \varepsilon \).

**Proof.** The proof follows [21] directly. We give a brief outline here. With the Taylor expansion, we can obtain the time-varying regressor
\[
Y_f(t) = S(t)\theta_v(t - \delta) + O(\varepsilon^2),
\]

with
\[
Y_f(t) := G_d(s)[\dot{v}_{\alpha,\beta}(t)], \quad \theta_v(t) = \varepsilon y_v(t).
\]

Define the error signal
\[
\dot{\hat{\theta}} := \dot{\theta}_v - \varepsilon y_v.
\]

Invoking (31) and Assumption (30), we get
\[
\dot{\hat{\theta}} := -\gamma S^2(t)\dot{\hat{\theta}} + O(\varepsilon).
\]

Clearly, the signal \( S(t) \) is of persistent excitation, that is,
\[
\int_t^{t+\frac{\gamma}{2}} S^2(\tau) d\tau \geq S_0,
\]

for all \( t \geq 0 \) and some \( S_0 > 0 \). Invoking Krasovskii’s theorem and carrying-out some basic perturbation analysis, it completes the proof.

\[\Box\]

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