A certain generalization of \(q\)-hypergeometric functions and their related monodromy preserving deformation

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We define a series \(F_{M,N}\) as a certain generalization of \(q\)-hypergeometric function. We study its duality and the system of \(q\)-difference nonlinear equations which admits particular solutions in terms of \(F_{1,M}\).

*Keywords*: generalized \(q\)-hypergeometric function; \(q\)-Garnier system; \(q\)-differences; a linear Pfaffian systems.

1 Introduction

Tsuda [1], [2] obtained a Hamiltonian system \(H_{N+1,M}\) which is an isomonodromy deformation of a certain \((N+1)\times(N+1)\) Fuchsian system on \(\mathbb{P}^1\) with \(M+3\) regular singularities, and which admits particular solutions in terms of a certain generalization of hypergeometric functions:

\[
F_{N+1,M}\left(\{\alpha_j\}, \{\beta_i\}; \{\gamma_j\}\right) = \sum_{m_i \geq 0} \prod_{j=1}^N (\alpha_j)^{m_j} \prod_{i=1}^M (\beta_i)^{m_i} \prod_{i=1}^M x_i^{m_i},
\]

where \(1 \leq i \leq M, 1 \leq j \leq N\), \((\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\) and, for a multi-index \(m = (m_1, \cdots, m_M)\), \(|m| = \sum_{i=1}^M m_i\).

Our goal is to obtain a \(q\)-difference analogue of the above result, and in this paper we give a partial result in this direction. Namely, we will obtain a \(q\)-difference analogue \(\mathcal{P}_{N,M}\) of Tsuda’s system \(\mathcal{H}_{N+1,M}\) in the case of \((M,N) = (M,1)\).

The construction of the system \(\mathcal{P}_{1,M}\) is as follows: Firstly, we define a series \(F_{M,N}\) as a generalization of the \(q\)-hypergeometric function \(2\phi_1\) [7] [8]. The series \(F_{M,N}\) has a duality relation (2.4), which gives an integral representation of \(F_{N,M}\). We derive a Pfaffian system from the integral representation of \(F_{1,M}\). Investigating the factorized structure of the coefficient matrices, we will interpret the Pfaffian system as a Lax pair \(T_z \overrightarrow{\Psi} = \overrightarrow{\Psi} A, T_t \overrightarrow{\Psi} = \overrightarrow{\Psi} B\), where accessory parameters are written in terms of the function
where \( (F) \). Finally, as suggested by this, we formulate the system \( P_{1,M} \) which admits particular solutions of \( F_{i,M} \).

**Remark 1.1.** As mentioned in next section, \( F_{i,M} \) is equal to the \( q \)-Appell-Lauricella function \( \varphi_D \). The system we will obtain in this paper can be identified with a system known as a \( q \)-difference analogue of the Garnier system \([9], [10]\). The \( q \)-Garnier system was first given by Sakai \([9]\), and he presented particular solutions of the system in terms of \( \varphi_D \) \([10]\). However, the present result is based on a different method from that.

This paper is organized as follows. A definition of the series \( F_{M,N} \) and its fundamental properties are given in section 2. A Pfaffian system which \( F_{1,M} \) satisfies is given in section 3. We formulate the system \( P_{1,M} \) in section 4. In the appendix we give a short summary of the result of Tsuda \([1], [2]\).

## 2 The series \( F_{N,M} \) as an extension of \( q \)-hypergeometric functions

In this section, we define the series \( F_{N,M} \) as an extension of \( q \)-hypergeometric functions and we show its duality relation and \( q \)-linear difference equations. The simplest case \( F_{1,1} \) corresponds to the Heine’s function \( 2\varphi_1 \) \([7], [8]\).

**Definition 2.1.** We define a series \( F_{N,M} \) as

\[
F_{N,M}\left(\left\{a_j\right\}, \left\{b_j\right\}; \left\{y_i\right\}\right) = \sum_{m_i \geq 0} \prod_{j=1}^{N} (a_j)_{m_j} \prod_{i=1}^{M} (b_i)_{m_i} \prod_{i=1}^{M} y_i^{m_i}, \tag{2.1}
\]

where \( (a)_n = \frac{(a)_\infty}{(q^n a)_\infty} \). Here and in what follows the symbol \( (a)_\infty \) means \( (a)_\infty = \prod_{i=0}^{\infty} (1 - q^i a) \). The series \( (2.1) \) converges in the region \( |y_i| < 1 \) and is continued analytically to \( |y_i| \geq 1 \).

When \( N = 1 \) or \( M = 1 \), the series \( (2.1) \) is equal to the \( q \)-Appell-Lauricella function \( \varphi_D \) or the generalized \( q \)-hypergeometric function \( _{N+1}\varphi_N \), respectively:

\[
F_{1,M}\left(\left\{a_i\right\}, \left\{b_i\right\}; \left\{y_i\right\}\right) = \sum_{m_i \geq 0} \prod_{i=1}^{M} (a_i)_{m_i} \prod_{i=1}^{M} (b_i)_{m_i} y_i^{m_i} = \varphi_D \left(\left\{a_i\right\}, \left\{b_i\right\}; \left\{y_i\right\}\right), \tag{2.2}
\]

\[
F_{N,1}\left(\left\{a_j\right\}, \left\{b_j\right\}; y\right) = \prod_{n \geq 0} \prod_{j=1}^{N} (a_j)_n (b_j)_n y^m = _{N+1}\varphi_N \left(\left\{a_j\right\}, \left\{b_j\right\}; y\right). \tag{2.3}
\]

There is a duality relation between the series \( F_{N,M} \) and \( F_{M,N} \) as follows:

**Proposition 2.1.** The series \( F_{N,M} \) satisfies the relation

\[
F_{N,M}\left(\left\{a_i\right\}, \left\{b_j\right\}; \left\{x_i\right\}\right) = \frac{\left(\{a_i\}\right)_\infty}{\left(\{b_j\}\right)_\infty} \frac{\left(\{x_i\}\right)_\infty}{\left(\{y_j\}\right)_\infty} F_{M,N}\left(\left\{a_i\right\}, \left\{b_j\right\}; \left\{y_j\right\}\right). \tag{2.4}
\]
Proof. We consider a double sum
\[
S = \sum_{m_i \geq 0} \sum_{n_j \geq 0} \prod_{i=1}^{M} (a_i)_{m_i} x_i^{m_i} \prod_{j=1}^{N} (b_j)_{n_j} y_j^{n_j} q^{m_i n_j}. \tag{2.5}
\]
Using the q-binomial theorem
\[
\sum_{n \geq 0} \frac{(b)^n}{(q)^n} z^n = \frac{(b z)^\infty}{(z)^\infty}, \tag{2.6}
\]
we can take the sum over \(n_j \geq 0\) in (2.5). Then the sum \(S\) is represented by the series \(\mathcal{F}_{N,M}\):
\[
S = \sum_{m_i \geq 0} \prod_{i=1}^{M} (a_i)_{m_i} x_i^{m_i} \prod_{j=1}^{N} (b_j y_j)_{\infty} = \prod_{j=1}^{N} \frac{(b_j y_j)_{\infty}}{(y_j)_{\infty}} \sum_{m_i \geq 0} \prod_{i=1}^{M} (a_i)_{m_i} x_i^{m_i} \prod_{j=1}^{N} \frac{(y_j)_{|m_j|}}{(b_j y_j)_{|m_j|}} \tag{2.7}
\]
\[
= \prod_{j=1}^{N} \frac{(b_j y_j)_{\infty}}{(y_j)_{\infty}} \mathcal{F}_{N,M} \left( \{y_j\}, \{a_i\}; \{x_i\} \right). \tag{2.8}
\]
Similarly, when we sum over \(m_i \geq 0\), the sum \(S\) is represented by the series \(\mathcal{F}_{M,N}\):
\[
S = \prod_{i=1}^{M} (a_i x_i)_{\infty} \mathcal{F}_{M,N} \left( \{x_i\}, \{b_j\}; \{a_i x_i\} \right). \tag{2.9}
\]
From (2.7), (2.8), we obtain the relation (2.4).
\[
\square
\]
Remark 2.1. When \(N = 1\) or \(M = 1\), the relation (2.4) is known in \([11]\) and \([12]\).

We can interpret the equation (2.4) as an integral representation of \(\mathcal{F}_{N,M}\) as follows.

Corollary 2.1. With \(y_j = q^{\gamma_j}\), the relation (2.4) can be rewritten as
\[
\mathcal{F}_{N,M} \left( \{q^{\gamma_j}\}, \{a_i\}; \{x_i\} \right) = \left( \{q^{\gamma_j}\}, \{b_j\}; q \right)_{\infty} \prod_{j=1}^{N} \int_0^c d_q t f(t) \prod_{i=1}^{M} \frac{(a_i x_i)_{|t_j|}}{(b_j y_j)_{|t_j|}} \prod_{j=1}^{N} (q t_j)_{\infty} t_j^{\gamma_j-1}, \tag{2.10}
\]
where the Jackson integral is defined as
\[
\int_0^c d_q t f(t) = c (1 - q) \sum_{n \geq 0} f(c q^n) q^n. \tag{2.11}
\]

Proof. The right-hand side of (2.4) can be written as
\[
(RHS) = \left( \{q^{\gamma_j}\}, \{a_i x_i\}; \{x_i\} \right) \sum_{n_j \geq 0} \prod_{i=1}^{M} (x_i)_{|n_j|} \prod_{j=1}^{N} \frac{(b_j)_{n_j}}{(q)_{n_j}} \prod_{i=1}^{M} (a_i x_i)_{|n_j|} \prod_{j=1}^{N} \frac{(b_j)_{n_j}}{(q)_{n_j}} (q^{\gamma_j})^{n_j}
\]
\[
= \left( \{q^{\gamma_j}\}, \{b_j\}; q \right)_{\infty} \prod_{n_j \geq 0} \prod_{i=1}^{M} (a_i x_i q^{n_j})_{|n_j|} \prod_{j=1}^{N} (q^{\gamma_j+1})_{\infty} q^{n_j \gamma_j}
\]
\[
= \left( \{q^{\gamma_j}\}, \{b_j\}; q \right)_{\infty} \prod_{j=1}^{N} \int_0^c d_q t f(t) \prod_{i=1}^{M} (a_i x_i)_{|t_j|} \prod_{j=1}^{N} (q t_j)_{\infty} t_j^{\gamma_j-1}. \tag{2.11}
\]
We set the integrand in (3.1) as

\[ \prod_{j=1}^{N} (1 - c_j q^{-1} T) \cdot (1 - T y_s) y_s - \prod_{j=1}^{N} (1 - a_j T) \cdot (1 - b_s T y_s) \]

\[ \mathcal{F} = 0 \quad (1 \leq s \leq M), \]

(2.12)

\[ \{y_s (1 - b_s T y_s) (1 - T y_s) - y_s (1 - b_s T y_s)(1 - T y_s)\} \mathcal{F} = 0 \quad (1 \leq r < s \leq M), \]

(2.13)

where \( T y_s \) is the \( q \)-shift operator for the variable \( y_s \) and \( T = T_{y_1} \cdots T_{y_M} \).

**Proof.** The coefficient of \( \prod_{i=1}^{M} y_i^{m_i} \cdot C(\{m_i\}) \) satisfies the following relation

\[ \prod_{j=1}^{N} (1 - q^{[m_j]} c_j) \cdot (1 - q^{m_s+1}) C(\{m_i + \delta_{i,s}\}) = \prod_{j=1}^{N} (1 - q^{[m_j]} a_j) \cdot (1 - q^{m_s} b_s) C(\{m_i\}). \]

(2.14)

Multiplying both sides of (2.14) by \( \prod_{i=1}^{M} y_i^{m_i} \) and summing them over \( m_i \geq 0 \), we obtain (2.12). The equation (2.13) is obtained similarly.

\[ \square \]

### 3 A Pfaffian system derived from \( \mathcal{F}_{1,M} \)

In this section, we derive a Pfaffian system from an integral representation of \( \mathcal{F}_{1,M} \), and then we derive a \( 2 \times 2 \) system by a certain specialization.

#### 3.1 A Pfaffian system

We derive a Pfaffian system which the integral representation of \( \mathcal{F}_{1,M} \) satisfies. In this subsection, we basically follow the method given in [13]. From Cor. 2.1, the integral representation of \( \mathcal{F}_{1,M} \) is given as

\[ \mathcal{F}_{1,M} \left( q^{11}, \{a_i\}; \{x_i\} \right) = \frac{(q^{11}, b_1)_{\infty}}{(b_1 q^{11}, q)_{\infty}} \int_0^1 \sum_{i=1}^M \frac{(a_i x_i u_1)_{\infty}}{(x_i u_1)_{\infty}} \frac{(q u_1)_{\infty}}{(b_1 u_1)_{\infty}} \frac{u_1^{11}}{1 - q}. \]

(3.1)

We set the integrand in (3.1) as

\[ \Phi(u_1) = u_1^{11} \prod_{i=0}^M \frac{(a_i x_i u_1)_{\infty}}{(x_i u_1)_{\infty}} \quad (x_0 = b_1, a_0 x_0 = q). \]

(3.2)

Following [13], we define functions \( \Psi_0, \Psi_{1,i} \) \((1 \leq i \leq M)\) as

\[ \Psi_0 = \langle \Phi p_0 \rangle, \]

(3.3)

\[ \Psi_{1,i} = \langle \Phi p_{1,i} \rangle \quad (1 \leq i \leq M), \]

(3.4)

where \( p_0 = 1, p_{1,i} = \frac{1 - x_{a+i}}{1 - a x_{a+i}} \prod_{k=1}^{i-1} \frac{1 - x_{a+k}}{1 - a x_{a+k}} \) and \( \langle \rangle \) means a kind of Jackson integral for \( u_1 \). Namely, \( \langle f(u_1) \rangle = \sum_{n \in \mathbb{Z}} f(q^n) \). We will see that \( \{\Psi_0, \Psi_{1,1}, \ldots, \Psi_{1,M}\} \) is a basis of a solution of the Pfaffian system.
We define an exchange operator $\sigma_i$ acting on a function $f$ of $\{x_i, a_i\}$ as
\[\sigma_i(f) = f|_{x_i \leftrightarrow x_{i+1}, a_i \leftrightarrow a_{i+1}}.\]  
(3.5)
We note that $\sigma_i(\Phi(u_1)) = \Phi(u_1)$. When the operator $\sigma_i$ acts on functions $p_0, p_{1,i}$ ($1 \leq i \leq M$), we have the following relations.

**Proposition 3.1.** We have
\[
\sigma_i\left[\begin{array}{c} p_{1,i} \\ p_{1,i+1} \end{array}\right] = \frac{1}{x_i - a_{i+1}x_{i+1}} \left[\begin{array}{c} (1 - a_i)x_i - a_ix_i - a_{i+1}x_{i+1} \\ x_i - x_{i+1} \end{array}\right] \left[\begin{array}{c} p_{1,i} \\ p_{1,i+1} \end{array}\right],
\](3.6)
\[
\sigma_i(p_{1,k}) = p_{1,k} \ (k \neq i, i+1).
\](3.7)

**Proof.** Consider the equations
\[
\sigma_i(p_{1,i}) = s_{11} p_{1,i} + s_{12} p_{1,i+1},
\](3.8)
\[
\sigma_i(p_{1,i+1}) = s_{21} p_{1,i} + s_{22} p_{1,i+1}.
\](3.9)

One can show that there exist unique coefficients $s_{11}, s_{12}, s_{21}, s_{22}$ independent of $u_1$ satisfying these relations. Hence we obtain the form (3.6). The equation (3.7) is obvious. \(\blacksquare\)

For the action of the $q$-shift operator $T_{x,M}$ on $\Psi_0, \cdots, \Psi_{1,M}$, we obtain the following equations.

**Proposition 3.2.** We have
\[
T_{x,M}(\Psi_0) = \frac{x_M - x_0}{a_M x_M - x_0} \rho(\Psi_0) + \frac{(a_M - 1)x_M}{a_M x_M - x_0} \rho(\Psi_{1,1}),
\](3.10)
\[
T_{x,M}(\Psi_{1,i}) = \rho(\Psi_{1,i+1}) \ (i \neq M),
\](3.11)
\[
T_{x,M}(\Psi_{1,M}) = q^{-\gamma_1} \left[\frac{1 - x_0}{a_M x_M - x_0} \rho(\Psi_0) + \frac{a_M x_M - 1}{1 - x_0} \rho(\Psi_{1,1})\right],
\](3.12)
where $\rho = \sigma_{M-1} \cdots \sigma_1$.

**Proof.** First, we make a shift by $T_{x,M}$ on $\Phi p_0, \Phi p_{1,i}$ ($1 \leq i \leq M$). We easily obtain the following equations
\[
T_{x,M}(\Phi p_{1,i}) = \Phi \rho(p_{1,i+1}),
\](3.13)
\[
T_{x,M}(\Phi p_0) = \Phi \frac{1 - x_M u_1}{1 - a_M x_M u_1},
\](3.14)
\[
T_{x,M} T_{u_1}^{-1}(\Phi p_{1,M}) = q^{-\gamma_1} \Phi \frac{1 - u_1}{1 - a_M x_M u_1}.
\](3.15)

The right-hand side of the equations (3.14), (3.15) can be rewritten as a linear combination of $\rho(\Phi p_0)$ and $\rho(\Phi p_{1,1})$ respectively, that is,
\[
T_{x,M}(\Phi p_0) = \frac{x_M - x_0}{a_M x_M - x_0} \rho(\Phi p_0) + \frac{(a_M - 1)x_M}{a_M x_M - x_0} \rho(\Phi p_{1,1}),
\](3.16)
\[
T_{x,M} T_{u_1}^{-1}(\Phi p_{1,M}) = q^{-\gamma_1} \left(\frac{1 - x_0}{a_M x_M - x_0} \rho(\Phi p_0) + \frac{a_M x_M - 1}{a_M x_M - x_0} \rho(\Phi p_{1,1})\right).
\](3.17)

Integrating the equations (3.13), (3.16) and (3.17) with respect to $u_1$, we obtain the equations (3.10)-(3.12). \(\blacksquare\)
Combining Prop. 3.1 and Prop. 3.2 we obtain the following theorem.

**Theorem 3.1.** The vector \( \vec{\Psi} = [\Psi_0, \Psi_1, \ldots, \Psi_M] \) satisfies the equation
\[
T_{x_M} \vec{\Psi} = \vec{\Psi} A_M = \vec{\Psi} R_{M-1} R_{M-2} \cdots R_1 Q,
\]
(3.18)
where the matrices \( R_i \) and \( Q \) are given as
\[
R_i = \begin{bmatrix}
E_i & \frac{(1-a_i)x_i}{x_i-a_M x_M} & \frac{x_i-x_M}{x_i-a_M x_M} \\
\end{bmatrix},
\]
(3.19)
\[
Q = \begin{bmatrix}
\frac{x_M-x_0}{a_M x_M-x_0} & \frac{q^{-\gamma_1}(1-x_0)}{a_M x_M-x_0} \\
\frac{a_M x_M-x_0}{(a_M-1)x_0} & \frac{q^{-\gamma_1}(a_M x_M-1)}{a_M x_M-x_0} \\
\end{bmatrix} E_{M-1}. 
\]
(3.20)

\( E_i \) is the unit matrix of dimension \( i \).

**Remark 3.1.** The equations for the other variables
\[
T_{x_i} \vec{\Psi} = \vec{\Psi} A_i,
\]
(3.21)
for \( i = 1, \ldots, M-1 \) can be derived from (3.18) and the action of \( \{\sigma_i\} \) in Prop. 3.1. The coefficient matrices in (3.18) and (3.21) satisfy a compatibility condition
\[
A_i(T_{x_i} A_j) = A_j(T_{x_i} A_i). 
\]
(3.22)

**Remark 3.2.** When \( M = 1 \), the equation is just solved by the \( 2\varphi_1 \) function. In fact we obtain, by Theorem 3.1, the equation
\[
T_{x_1} [\Psi_0, \Psi_{1,1}] = [\Psi_0, \Psi_{1,1}] \begin{bmatrix}
\frac{x_1-x_0}{a_1 x_1-x_0} & \frac{q^{-\gamma_1}(1-x_0)}{a_1 x_1-x_0} \\
\frac{a_1 x_1-x_0}{(a_1-1)x_0} & \frac{q^{-\gamma_1}(a_1 x_1-1)}{a_1 x_1-x_0} \\
\end{bmatrix}. 
\]
(3.23)

### 3.2 A reduction to \( 2 \times 2 \) form

In this section we reduce the equation (3.21) into \( 2 \times 2 \) form. To do this, we specialize the parameter \( a_M \) to be 1. Then the integrand (3.2), and hence \( \Psi_{1,i} \) (\( 1 \leq i \leq M-1 \)) and \( \Psi_0 \), become independent of \( x_M \). Therefore we can consider \( \Psi_{1,i} \) (\( 1 \leq i \leq M-1 \)) as \( \Psi_0 \) times \( r_i \), where \( r_i \) is a rational function in \( x_0, \ldots, x_{M-1} \). Based on this fact, we have the following.

**Theorem 3.2.** Specializing \( a_M = 1 \) and setting \( z = x_M, t = x_{M-1} \) and \( \Psi_{1,i} = r_i \Psi_0 \) (\( 1 \leq i \leq M-1 \)), the equations in Theorem 3.1 can be rewritten as
\[
\begin{cases}
T_z(\Psi_0, \Psi_{1,M}) = (\Psi_0, \Psi_{1,M}) \left( \prod_{i=1}^{M-1} \begin{bmatrix}
1/r_i & 1 \\
z & r_i x_i \\
\end{bmatrix}^{-1} \begin{bmatrix}
1/r_i & 1 \\
z & a_i r_i x_i \\
\end{bmatrix} \right) \begin{bmatrix}
1 & 1 \\
z & 0 \\
\end{bmatrix}^{-1} \begin{bmatrix}
1 & 1 \\
z & 0 \\
\end{bmatrix},

T_t(\Psi_0, \Psi_{1,M}) = (\Psi_0, \Psi_{1,M}) \begin{bmatrix}
u & 1 \\
z/q & a_{M-1} t/u \\
\end{bmatrix}^{-1} \begin{bmatrix}
u & 1 \\
z/q & t/u \\
\end{bmatrix},
\end{cases}
\]
(3.24)
where \( u \) is independent of \( z \).
Proof. When \( a_M = 1 \), obviously we have \( T_z(\Psi_0) = \Psi_0 \). We will compute \( T_z(\Psi_{1,M}) \). In equation (3.18), the coefficient matrices become

\[
R_i = \begin{bmatrix}
E_i & \frac{(1-a_i)x_i}{x_i-z} & 1 & 0 \\
\frac{a_i x_i - z}{x_i-z} & 0 & \vdots \\
E_{M-1} & & & \\
\end{bmatrix}, \quad Q = \begin{bmatrix}
1 & \frac{q^{-\gamma_1(1-x_0)}}{z-x_0} \\
0 & \frac{q^{-\gamma_1(z-1)}}{z-x_0} \\
\end{bmatrix}.
\] (3.25)

We rewrite the equation (3.18):

\[
\vec{\Psi}^{(1)} := \vec{\Psi} R_{M-1} = (\Psi_0, \cdots, \Psi_{1,M-2}, \Psi_{1,M-1}, \Psi_{1,M-1}), \\
\vec{\Psi}_{1,M-1} := a(M-1)\Psi_{1,M-1} + b(M-1)\Psi_{1,M},
\]

\[
\vec{\Psi}^{(2)} := \vec{\Psi}^{(1)} R_{M-2} = (\Psi_0, \cdots, \Psi_{1,M-2}, \Psi_{1,M-2}, \Psi_{1,M-1}), \\
\vec{\Psi}_{1,M-2} := a(M-2)\Psi_{1,M-2} + b(M-2)\Psi_{1,M-1},
\]

\[
\vdots
\]

\[
\vec{\Psi}^{(M-1)} := \vec{\Psi}^{(M-2)} R_1 = (\Psi_0, \Psi_{1,1}, \cdots, \Psi_{1,M-1}), \\
\vec{\Psi}_{1,1} := a(1)\Psi_{1,1} + b(1)\Psi_{1,2},
\]

\[
\vec{\Psi}^{(M)} := \vec{\Psi}^{(M-1)} Q = (\Psi_0, \Psi_{1,1}, \cdots, \Psi_{1,M-1}, \Psi_{1,M-1}), \\
\vec{\Psi}_{1,M-1} := a(M)\Psi_{1,1} + b(M)\Psi_{1,1},
\] (3.26)

where the coefficients \( a(i) \) and \( b(i) \) are as

\[
a(i) = \begin{cases} 
\frac{(1-a_i)x_i}{x_i-z} & (1 \leq i \leq M-1), \\
\frac{q^{-\gamma_1(1-x_0)}}{z-x_0} & (i = M),
\end{cases}
\]

\[
b(i) = \begin{cases} 
\frac{a_i x_i - z}{x_i-z} & (1 \leq i \leq M-1), \\
\frac{q^{-\gamma_1(1-z)}}{z-x_0} & (i = M).
\end{cases}
\] (3.27)

We obtain the following equation from (3.26) and (3.27):

\[
T_z(\Psi_{1,M}) = \Psi^{(M)}_{1,M-1}.
\] (3.28)

The equation (3.28) can be rewritten as

\[
\Psi^{(M)}_{1,M-1} = a(M)\Psi_0 + b(M) \sum_{i=1}^{M-1} a(i) \prod_{j=1}^{i-1} b(j) \Psi_{1,i} + b(M) \prod_{k=1}^{M-1} b(k) \Psi_{1,M},
\]

\[
= \left( a(M) + b(M) \sum_{i=1}^{M-1} r_i a(i) \prod_{j=1}^{i-1} b(j) \right) \Psi_0 + b(M) \prod_{k=1}^{M-1} b(k) \Psi_{1,M}.
\] (3.29)
From the above, we obtain the first equation of equation (3.24):

\[ T_z(\Psi_0, \Psi_{1,M}) = (\Psi_0, \Psi_{1,M}) \begin{bmatrix} 1 & a(M) + b(M) \left( \sum_{i=1}^{M-1} r_i a(i) \prod_{j=1}^{i-1} b(j) \right) \\ 0 & b(M) \left( \prod_{k=1}^{M-1} b(k) \right) \end{bmatrix} \]

\[ = (\Psi_0, \Psi_{1,M}) \left( \prod_{i=1}^{M-1} \begin{bmatrix} 1 & \frac{r_{M-i}}{a(M-i)} \\ 0 & b(M-i) \end{bmatrix} \right) \begin{bmatrix} 1 & a(M) \\ 0 & b(M) \end{bmatrix} \]

\[ = (\Psi_0, \Psi_{1,M}) \left( \prod_{i=1}^{M-1} \begin{bmatrix} 1/r_i & z \quad 1 \\ z - r_i x_i & a_i r_i x_i \end{bmatrix}^{-1} \right) \begin{bmatrix} 1 & 1 \\ z & x_0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ z & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ q^{-n} & 1 \end{bmatrix}. \]

The second equation \( T_t(\Psi_0, \Psi_{1,M}) \) is obtained similarly by considering \( T_t = \sigma_{M-1} T_z \sigma_{M-1}. \)

4 A monodromy preserving deformation \( \mathcal{P}_{1,M} \) related to \( \mathcal{F}_{1,M} \)

In this section, we interpret the equations (3.24) as a Lax pair for nonlinear equations associated with special solutions. As suggested by this, we obtain the system \( \mathcal{P}_{1,M} \) we are aiming at.

**Definition 4.1.** We consider the following Lax form for the unknown function \( \Psi(z) = [\Psi_1(z), \Psi_2(z)] \)

\[
\begin{align*}
\Psi(qz) &= \Psi A, \\
\Psi &= \Psi B,
\end{align*}
\]

where the coefficient matrices are

\[
\begin{align*}
A &= dX_1^{-1} X_2^{-1} \cdots X_{2M-1}^{-1} X_2, \\
B &= X_2 (z/q)^{-1} X_{2M-1} (z/q),
\end{align*}
\]

and \( d \) is a diagonal matrix and \( X_i = \begin{bmatrix} u_i & 1 \\ z & c_i/u_i \end{bmatrix} \). We denote by \( \pi \) a "time-evolution" defined as \( \{c_i \rightarrow q c_i, (i = 2M-1, 2M)\} \). Then we define a system of nonlinear \( q \)-difference equations \( \mathcal{P}_{1,M} \) for the unknown variables \( u_i \) (1 \( \leq \) \( i \) \( \leq \) \( 2M \)), through the compatibility conditions

\[ A(z) B(qz) = B(z) A(z). \]

**Proposition 4.1.** The equation (4.3) is satisfied if and only if the variables \( \{u_i\} \) and \( \{\overline{w_i}\} \) are related by a certain birational mapping.
Theorem 4.1. The system \( \mathcal{P}_{1,M} \) obtained from the equation (4.3) is equivalent to the \( q \)-Garnier system \([9], [14]\).

Remark 4.1. The existence of the birational mapping \( \{u_i\} \mapsto \{\pi_i\} \) which satisfies (4.3).

Proposition 4.2. The coefficient matrices of (3.24) have the forms in (4.2).

Proof. We will rewrite the coefficient matrix of the first equation of (3.24). We consider the relevant five factors of it. Then we obtain

\[
\begin{bmatrix}
1/r_{M-1} & 1 \\
z & r_{M-1}t
\end{bmatrix}^{-1}
\begin{bmatrix}
1/r_{M-1} & 1 \\
z & a_{M-1}r_{M-1}t
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
z & x_0
\end{bmatrix}^{-1}
\begin{bmatrix}
1 & 1 \\
z & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
z & q^{-1}n
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
p & 1 \\
z & x_0/p
\end{bmatrix}^{-1}
\begin{bmatrix}
p & 1 \\
z & 1/p
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
z & t/u
\end{bmatrix}^{-1}
\begin{bmatrix}
u & 1 \\
z & a_{M-1}t/u
\end{bmatrix}
\]

where

\[
p = q^{-1}n \frac{(1-a_{M-1})r_{M-1}+x_0-1}{(-1+(1-a_{M-1})r_{M-1})t+x_0}.\]

From this we see that the coefficient matrices of (3.24) take the form in (4.2).

As a corollary of Prop. 4.2 we have the following theorem.

Theorem 4.1. The system \( \mathcal{P}_{1,M} \) admits a particular solution in terms of \( \mathcal{F}_{1,M} \).

Remark 4.2. For the case of \( M = 2 \), the equation (4.3) is equivalent to \( q \)-Painlevé VI \([15]\). In fact, the Lax pair (4.1), (4.2) is

\[
A = \begin{bmatrix}
u_1 & 1 \\
z & c_1/u_1
\end{bmatrix}^{-1}
\begin{bmatrix}
u_2 & 1 \\
z & c_2/u_2
\end{bmatrix}
\begin{bmatrix}
u_3 & 1 \\
z & c_3/u_3
\end{bmatrix}^{-1}
\begin{bmatrix}
u_4 & 1 \\
z & c_4/u_4
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
u_4 & 1 \\
z/q & c_4/u_4
\end{bmatrix}^{-1}
\begin{bmatrix}
u_3 & 1 \\
z/q & c_3/u_3
\end{bmatrix}.
\]
Solving the compatibility condition
\[ A(z)B(qz) = B(z)\overline{A(z)}, \] (4.9)
we obtain the following \(q\)-PVI equation for \(f\), \(g\):
\[ \bar{f} f = -c_3(\bar{c}_2 - g)(ag - c_3q), \]
\[ \bar{g} g = \frac{c_3^2(\bar{c}_1 - \bar{f})(c_2 - \bar{f})q^2}{a^2(c_3q - \bar{f})}, \] (4.10)
where
\[ f = \frac{c_1c_4u_3^3 + c_2c_4u_1u_4^3 - c_1c_4u_2u_3^3 - c_1c_3u_2^3u_4}{c_4u_1u_2u_3 - c_3u_1u_2u_4 + c_2u_1u_3u_4 - c_1u_2u_3u_4}, \quad g = \frac{c_3(c_2u_1 - c_1u_2)u_4^2}{u_1^2(c_3u_4 - c_4u_3)}, \] (4.11)
and \(a = \frac{u_1u_3}{u_2u_4}\) is a constant.

**Acknowledgement**

The author would like to express her gratitude to Professor Yasuhiko Yamada for valuable suggestions and encouragement. She is also grateful to Professor Wayne Rossman for careful reading the manuscript and several improvements.

**Appendix**

In this appendix we give a short summary of the result of Tsuda [1], [2]. Tsuda derived a Hamiltonian system \(H_{N+1,M}\) which admits particular solutions in terms of a certain generalization of hypergeometric functions from the following Lax formalism [1]. Consider an \((N + 1) \times (N + 1)\) Fuchsian system
\[ \frac{\partial \Phi}{\partial z} = A\Phi = \sum_{i=0}^{M+1} A_i \frac{\Phi_i}{z - u_i}, \] (A.1)
with \(M + 3\) regular singularities \(\{u_0 = 1, u_1, \cdots, u_M, u_{M+1} = 0, u_{M+2} = \infty\} \subset \mathbb{P}^1\), of which the characteristic exponents at each singularity \(z = u_i\) are listed in the following table (Riemann scheme):

| Singularity | Exponents |
|-------------|-----------|
| \(u_i(0 \leq i \leq M)\) | \((-\theta_i, 0, \cdots, 0)\) |
| \(u_{M+1} = 0\) | \((e_0, e_1, \cdots, e_N)\) |
| \(u_{M+2} = \infty\) | \((\kappa_0 - e_0, \kappa_1 - e_1, \cdots, \kappa_N - e_N)\) |

We have Fuchs’ relation \(\sum_{n=0}^{N} \kappa_n = \sum_{i=0}^{M} \theta_i\). We can normalize the exponents as \(\sum_{n=0}^{N} e_n = \frac{N}{2}\) without loss of generality. Such Fuchsian systems as above then turn out
to constitute a $2MN$-dimensional family and can be written in terms of the accessory parameters $b_n^{(i)}$ and $c_n^{(i)}$ in the following way:

$$A_i = T(b_0^{(i)}, b_1^{(i)}, \ldots, b_N^{(i)}) \cdot (c_0^{(i)}, c_1^{(i)}, \ldots, c_N^{(i)}) \quad (0 \leq i \leq M),$$

$$A_{M+1} = \begin{bmatrix}
e_0 & w_{0,1} & \cdots & w_{0,N} \\
e_1 & \cdots & \cdots & \\
\vdots & \vdots & \ddots & \vdots \\
e_N & e_{N+1} & \cdots & e_{N+1}
\end{bmatrix},$$

(A.2)

where $c_0^{(i)} = 1$, $w_{m,n} = -\sum_{i=0}^{M} b_n^{(i)} c_n^{(i)}$, $(\text{tr} A_i =) \sum_{n=0}^{N} b_n^{(i)} c_n^{(i)} = -\theta_i$ and $\sum_{i=0}^{M} b_n^{(i)} c_n^{(i)} = -\kappa_n$. The essential number of the accessory parameters is confirmed to be $2MN$.

The isomonodromic family of Fuchsian systems of the form (A.1) is described by the integrability condition of the extended linear system, that is, (A.1) itself and its deformation equations

$$\frac{\partial \Phi}{\partial u_i} = B_i \Phi, \quad B_i = \frac{A_i}{u_i - z} - \frac{1}{u_i} \begin{bmatrix}
-\theta_i & & & \\
& \ddots & \ddots & \\
& & \ddots & \\
& & & -\theta_i
\end{bmatrix} \quad (1 \leq i \leq M).$$

(A.3)

In Theorem 7.2 in [1], it is proved that the integrability condition

$$\frac{\partial A}{\partial u_i} - \frac{\partial B_i}{\partial z} + [A, B_i] = 0,$$

(A.4)

of (A.1) and (A.3) is equivalent to the polynomial Hamiltonian system $\mathcal{H}_{N+1,M}$ via a certain change of variables.

The system in the case $(M, N) = (M, 1)$ coincides with the Garnier system in $M$ variables, [3], [4] and thus in the case $(M, N) = (1, 1)$ with the sixth Painlevé equation $P_{VI}$ [5], [6]. It was also shown that their solutions are expressed in terms of a generalization of hypergeometric functions $F_{N+1,M}$ given in (1.1). A linear Pfaffian system of rank $MN + 1$ which $F_{N+1,M}$ satisfies was derived. When $\kappa_0 - \sum_{i=1}^{M} \theta = 0$, the integrability condition (A.4) can be converted into the Pfaffian system for $F_{N+1,M}$. In this way the hypergeometric solution for the system (see Thm. 3.2 in [2] was obtained).

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