Quantum entanglement distribution using a magnetic field sensor

M Schaffry\textsuperscript{1}, S C Benjamin\textsuperscript{1,2} and Y Matsuzaki\textsuperscript{1,3}

\textsuperscript{1} Department of Materials, University of Oxford, Parks Road, Oxford OX1 3PH, UK
\textsuperscript{2} Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543, Singapore
E-mail: matsuzaki.yuichiro@lab.ntt.co.jp

\textit{New Journal of Physics} 14 (2012) 023046 (16pp)
Received 3 September 2011
Published 21 February 2012
Online at \url{http://www.njp.org/}
doi:10.1088/1367-2630/14/2/023046

Abstract. Sensors based on crystal defects, especially nitrogen vacancy (NV) centres in nanodiamond, can achieve detection of single magnetic moments. Here, we show that this exquisite control can be utilized to entangle remote electronic spins for applications in quantum computing; the mobile sensor provides a ‘flying’ qubit while the act of sensing the local field constitutes a two-qubit projective measurement. Thus, the NV centre mediates entanglement between an array of well-separated (and thus well-controlled) qubits. Our calculations establish that such a device would be remarkably robust against realistic issues such as dephasing, inaccurate timing and both positioning errors and multimodal vibrations in the sensor tip. Interestingly, the fact that this form of flying qubit is readily measurable allows one to convert certain classes of unknown errors into heralded failures, which are relatively easy to deal with using established quantum information processing techniques. We also provide calculations establishing the feasibility of performing a demonstrator experiment with a fixed sensor in the immediate future.

\textsuperscript{3} Author to whom any correspondence should be addressed.
1. Introduction

One possible architecture for a quantum computer is based on the idea of distributed quantum information processing (QIP) [1]. Individual qubits (or small groups of qubits) are physically well separated from each other, thereby affording ease of control. The challenge then is to accomplish entanglement; it is known that suitable optically active qubits can be entangled by measurements on emitted photons [2–4]. In this paper, however, we show how one can use the dipole–dipole interaction between electronic spins in conjunction with optical detected magnetic resonance (ODMR) to create entanglement between remote spin qubits that need not be optically active. The nitrogen vacancy (NV) centre defects in diamond are very suitable for ODMR and QIP as these possess a long-lived spin triplet electronic ground state with the levels $|0\rangle$ and $|\pm 1\rangle$ that can be easily initialized with a laser, manipulated with microwave pulses and read out optically [5, 6]. This exquisite control enables the observation of their coupling to adjacent nuclear spins [7, 8] and the measurement of a nearby nuclear spin [9]. A very promising application of NV centres is their capability to detect the strength of very small magnetic fields through an induced Zeeman splitting [10–14]. We will show how this sensitivity can be used to entangle remote electronic spins in a remarkably robust fashion. Indeed, because of the measurement-based nature of the operation, certain classes of experimental imperfection including finite decoherence time in the NV centre and both timing and positioning errors can be probabilistically mapped to a far higher fidelity operation (with the other outcome being a heralded outright failure). Since it has been established that QIP is possible with very high heralded error rates exceeding 90% [15], this is a very powerful approach.

Ultimately, our proposal is to move an NV centre sensor between remote spins to entangle them. We begin by outlining a simpler experimental scenario, which could be tested in the immediate future. Suppose that we are given two electronic spin qubits and can measure the field of these two spins by using a crystal defect in a nanodiamond, which is placed in the middle of the two qubits (see figure 1(a)). If each qubit produces a field of strength $b_z$ at the site of the NV centre, then the NV centre experiences either $-2b_z$, 0 or $2b_z$, depending on the spin orientation of the qubits: the field is 0 if the two spins are in the state $|\uparrow\downarrow\rangle$ or $|\downarrow\uparrow\rangle$, and $\pm 2b_z$ if the two spins are in the state $|\downarrow\downarrow\rangle$ or $|\uparrow\uparrow\rangle$. Given a sufficiently large external magnetic field,
typically of the order of tens of millitesla, the $|{-1}\rangle$-level of the NV centre triplet splits away and we can prepare the system in the state $|+\rangle_{NV} = 1/\sqrt{2}(|0\rangle + |1\rangle)$ using green light pumping. Over time $t$, this state collects either a phase of $\pm 2b_2\mu_{NV}t/\hbar \equiv \omega_2t$ or 0 depending on the qubits, where $\mu_{NV}$ denotes the magnetic moment of the NV centre. Measuring this phase will implement a projective two-qubit measurement. For example, suppose that the two qubits are each prepared in the state $|+\rangle_1/2 = 1/\sqrt{2}(|\downarrow\rangle + |\uparrow\rangle)$ and we let the NV centre precess for the time $\tau = \pi/\omega$, before measuring it in the basis $|\pm\rangle_{NV} = 1/\sqrt{2}(|\downarrow\rangle \mp |\uparrow\rangle)$. If the measurement results in $|+\rangle_{NV}$, then the two qubits will be in the Bell state $1/\sqrt{2}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$. Similarly, the outcome $|-\rangle_{NV}$ leaves us with $1/\sqrt{2}(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle)$. Thus, a deterministic parity projection is realized, which is very robust to errors as we will show in this paper. Moreover, we will show that for other times $\tau$ the procedure still achieves a probabilistic parity projection, known to be sufficient to implement the entanglement needed for full QIP [2].

2. The basic scheme

We now show how robust this general idea is with respect to the translations and vibrations of the NV centre. First, we look at a configuration where the two qubits are located at the origin and at $z = 2\Delta$ while the NV centre, which is oriented along the $z$-direction, is placed in the middle of the two qubits (see figure 1(a)). The interaction between the three particles is given by the dipole–dipole coupling:

$$ H_{D_{ij}} = C_{i,j}d_{i,j}^{-3}(3(\hat{\mathbf{x}}_{i,j} \cdot \mathbf{S}_i)(\hat{\mathbf{x}}_{i,j} \cdot \mathbf{S}_j) - \mathbf{S}_i \cdot \mathbf{S}_j), $$

where $\mathbf{S}_i$ is the spin-operator of the particle $i$, $\hat{\mathbf{x}}_{i,j}$ is a unit vector pointing from spin $i$ to spin $j$, $d_{i,j}$ is the distance between the spins $i$ and $j$ and $C_{i,j} = -\mu_0/4\pi \mu_i \mu_j = C$ is a constant [16]. Here, $\mu_0$ is the magnetic constant, $\mu_{1/2,NV} = 2\mu_B$ are the magnetic moments of the spins and $\mu_B$ denotes the Bohr magneton. In an external magnetic field $B$ in the $z$-direction, the whole system can be described by the following Hamiltonian:

$$ H_{\text{static}} = H_0 + H_{\text{DIP}} \quad \text{with} $$

$$ H_0 = -\mu_{NV}BS_z,_{NV} + D_{NV}S^2_{z,NV} - \mu_1BS_{z,1} - \mu_2BS_{z,2}, $$

$$ H_{\text{DIP}} = H_{D_{NV1}} + H_{D_{NV2}} + H_{D_{12}}, $$

[New Journal of Physics 14 (2012) 023046 (http://www.njp.org/)]
where, $D_{\text{NV}} = 2.87 \text{ GHz}$ is the zero-field splitting (ZFS). We transform this Hamiltonian into a rotating frame with respect to $\exp(iH_0 t)$ and neglect fast oscillating terms originating from a sufficiently large external field and the ZFS (rotating wave approximation). This gives us

$$\tilde{H}_{\text{static}, \text{RWA}} = W_{\text{NV}, 1} + W_{\text{NV}, 2} + H_{D, 12} \quad \text{with}$$

$$W_{i, j} = 2C d_{i, j}^3 S_{x, i} S_{x, j}. \quad (6)$$

As with many solid-state systems, in NV centres dephasing is by far the most severe form of decoherence; spin relaxation is typically orders of magnitudes slower and therefore we can neglect it. Using a standard master equation [17] as follows, we now evaluate the effect of dephasing on our basic proposal:

$$\rho'(t) = -i[\tilde{H}_{\text{static}, \text{RWA}}, \rho(t)] + \sum_{j=1, 2, \text{NV}} \frac{2}{T_{2, j}} \left( S_{j, z} \rho(t) S_{j, z}^\dagger - \frac{1}{2} \{ \rho(t), S_{j, z} S_{j, z}^\dagger \} \right). \quad (7)$$

We assume the three particles to be initially in the state

$$|\psi_i \rangle = |\psi(0) \rangle = |+\rangle_1 |+\rangle_{\text{NV}} |+\rangle_2. \quad (8)$$

After time $t$, we measure the NV centre in the $|\pm\rangle_{\text{NV}}$-basis. The readout can be implemented in several ways. At low temperature the readout of the NV centre spin can be done resonantly in one shot [18]. Although one cannot use such resonant measurements at higher temperature, a single-shot measurement has been demonstrated at room temperature with the help of a nuclear spin in diamond [9]. This experiment involved transfer of information from the electron onto the nuclear spin via a controlled NOT (CNOT)-gate and employed projective measurements by using a quantum nondemolition scheme. Alternatively, at room temperature, it is possible to utilize a nonresonant technique. Here, due to the limited detection efficiency in the current technology, a single-shot measurement is not possible and therefore the entangling operation needs to be repeated many times [10]. The statistics of the repetitive entangling operations will eventually determine which parity between the static qubits is present. Fortunately, the nature of the operation we seek to perform, i.e. a parity projection, is such that an arbitrary number of repetitions will not change the state after the first application. Regardless of the implementation of the measurement, the qubits are after the measurement, depending on the outcome, either in the state $\rho_+$ or $\rho_-$. This holds for both with the probability

$$p_\pm = \left( 1 \pm \exp(-t/T_{2, \text{NV}}) \cos^2(\alpha t/2) \right)/2, \quad (9)$$

where $\alpha = 2C \Delta^{-3}$. Within the limits of infinite dephasing times, $\rho_\pm = |\psi_\pm \rangle \langle \psi_\pm |$ are pure, where

$$|\psi_+ \rangle = n_+ \left( e^{-i\alpha t} |\downarrow\downarrow \rangle + |\uparrow\uparrow \rangle + \frac{\sqrt{2} - e^{-i\alpha t}}{1 + e^{-i\alpha t}} ( |\downarrow\uparrow \rangle + |\uparrow\downarrow \rangle ) \right),$$

$$|\psi_- \rangle = \frac{1}{\sqrt{2}} ( -e^{-i\alpha t} |\downarrow\downarrow \rangle + |\uparrow\uparrow \rangle )$$

and where $n_+$ is a normalization factor. We find that the entanglement of formation (EF) [19] is given by

$$E_F(\rho_-) = 1 \quad \text{and} \quad E_F(\rho_+) = H \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - \eta} \right), \quad (10)$$
Figure 2. Static NV centre: mean EF with respect to the measurement time. (a) For the nonsolid curves: $T_{2,\text{NV}} = 2 \text{ ms}$. Note that, in [20], a nanodiamond with an embedded NV centre of 5 nm diameter has been synthesized; however, the coherence time of this nanocrystal is currently unknown. Assuming that it is of the order of microseconds, we use in (b) $T_{2,\text{NV}} = 20 \mu\text{s}$. In both plots the distance between the NV centre and the qubits is $\Delta = 10 \text{ nm}$.

where

$$\eta = \frac{5 - 4 \cos \left( \frac{3}{16} \alpha t \right) (1 + \cos(\alpha t)) + \cos(\alpha t)(2 + \cos(\alpha t))}{(3 + \cos(\alpha t))^2}.$$  \hspace{1cm} (11)

Hence, whenever we measure the outcome $|\rightarrow\rangle$, the qubits are maximally entangled. Note that, for QIP, we could simply discard $\rho_\rightarrow$, retaining $\rho_\leftarrow$, which represents a perfect parity projection if we know $t$.

In order to quantify the amount of entanglement between the qubits that we obtain after measuring the NV centre at time $t$, we calculate the mean EF, i.e.

$$\langle E_F(t) \rangle := p_\leftarrow E_F(\rho_\leftarrow) + p_\rightarrow E_F(\rho_\rightarrow),$$  \hspace{1cm} (12)

which we then plot with respect to the time of the measurement in figure 2. This figure displays the expected oscillatory behaviour due to the phase that the NV centre collects. In addition, one can show that the distortion of the oscillation is a result of the qubit–qubit coupling in equation (5) and as these terms are relatively small, they have a negligible effect on the first maximum of the mean EF. Finally, we conclude that the effect of dephasing for the first maximum of the mean EF is also small given large coherence times. At room temperature, in bulk diamond, coherence times of about 2 ms have been demonstrated [21, 22]. However, currently, due to impurities in and on the surface of nanodiamonds, these values are degraded to the order of microseconds [11, 23] and further investigations need to be carried out to improve these values. In this context, decoupling techniques like in [22, 24] are very valuable, and these can be analogously applied to our spins. Another way of improving these values is to employ an amplified NV centre sensor scheme [14]. Here, an amplifier spin is attached to the surface of the nanodiamond so that the NV centre can be effectively brought closer to the static qubits. Thus, the NV centre can be placed in a larger nanocrystal, which can increase the coherence time of the NV centre.
3. Tolerating poor nitrogen vacancy (NV) decoherence times

In addition, for short dephasing times of the NV centre, we can simply repeat the entangling operation in order to increase the amount of entanglement. Since we need to obtain measurement results of the same parity at each entangling operation, such repetitions tend to decrease the success probability. However, such repetitions drastically increase the fidelity of the entanglement under the effect of the dephasing of the NV centre. We quantify this idea by considering two qubits with infinite dephasing times and an NV centre with a short dephasing time. We repetitively apply our entangling operation, where each operation lasts for the time $t = \pi / |\alpha|$. If we measure the NV centre $k$ times and consecutively obtain the outcome $|\rangle$, then we call the state of the two qubits $\rho_{-k}$. The success probability for such an event to happen also depends on the dephasing time of the NV centre. In figure 3, we plot the EF for this state $\rho_{-k}$ and the success probability with respect to the dephasing time of the NV centre. We see that the repetitions substantially increase the amount of entanglement.

The static qubits can be embodied by various types of spins. For example, at low temperature, electron spins bound to donors in silicon are suitable, as these possess a coherence time of several seconds [25]. It is remarkable that the NV centre spin can also be used to initialize and read out the qubits acting analogously to the read and write head in a classical Turing machine. If room temperature operation is considered highly desirable, shallow implanted NV centres in bulk diamond could provide the static qubits. However, in this case it is necessary to ensure that the illumination of the flying NV centre does not affect the static qubits. This can be done for example by using focused light beams [26] or by storing the information in the nuclear spins of the NV centres, which remain coherent under optical illumination, and by only using the NV centre spins for the entangling operation [4].
4. Implications of tip vibration

In a complete QIP system, the nanocrystal bearing our NV sensor would be mounted on an atomic force microscopy (AFM)-tip, thus enabling it to move between various sites with atomic resolution [27]. This mobility comes at the price of limited controllable vibrational modes of the AFM-tip, with typical frequencies varying between 50 and 500 kHz [28] and amplitudes up to 1.5 nm [27]. To characterize the consequences of these vibrations for our entangling operation, we consider first the effect of the NV centre oscillating between the two qubits in a single mode, i.e. $d_{NV,1/2}(t) = \Delta \pm \delta \cos(\omega t + \phi)$, where $\delta$ denotes the amplitude, $\omega$ the frequency and $\phi$ the phase of the oscillation\(^4\). As above, we derive a rotating wave approximation Hamiltonian and end up with the time-dependent Hamiltonian

$$\tilde{H}_{vib,RWA}(t) = W_{NV,1}(t) + W_{NV,2}(t) + H_{D,12}. \quad (13)$$

We can consider this vibrational scenario as a perturbed static case and thus we expect a maximum of the mean EF at around $\pi/|\omega|$ (similar to figure 2). Note that, for larger times, the qubit–qubit coupling and the dephasing become relevant. For this reason we define the maximal achievable mean EF, $M$, as

$$M = M(\delta, \omega, \phi) = \max_{0 \leq t \leq 2\pi/|\omega|} \langle E_F(t) \rangle. \quad (14)$$

We are interested in knowing how the introduction of many modes with various parameters ($\delta, \omega, \phi$) affect this maximal achievable mean EF and, in particular, for which parameter regime $M$ is close to 1. For this purpose, we analyse the effect of many modes that share the common parameter $p$ on $M$ and define

$$\tilde{D}(p) = \int_{-\infty}^{\infty} D(p) |\psi(t, p)\rangle \langle \psi(t, p)| \, dp$$

as the mean density matrix with respect to the distribution $D(p)$ of the parameter $p$, where $|\psi(t, p)\rangle$ is the solution of the Schrödinger equation for $\tilde{H}_{vib,RWA}(t)$. We find an analytic approximation of this solution as follows. First, we neglect the qubit–qubit coupling in $\tilde{H}_{vib,RWA}(t)$

$$\tilde{H}_{vib,RWA,app}(t) = W_{NV,1}(t) + W_{NV,2}(t). \quad (16)$$

We then transform this Hamiltonian into an interaction picture and expand it in a power series with respect to $\delta(t) = \delta \cos(\omega t + \phi)$. This gives

$$H_1(t) = \exp(i\tilde{H}_{static,RWA}t) \tilde{H}_{vib,RWA,app} \exp(-i\tilde{H}_{static,RWA}t) = \sum_{j=1,2} \frac{2C}{\Delta^3} \left( (-1)^j \frac{\delta(t)}{\Delta} + 6 \frac{\delta(t)^2}{\Delta^2} + O(\delta(t)^3) \right) S_{z,0} S_{z,j}. \quad (17)$$

Finally, we apply time-dependent perturbation theory:

$$\rho(t) = \rho(t) - i \int_{0}^{t} \left[ H_1(t'), \rho_1(0) \right] \, dt' + (-i)^2 \int_{0}^{t} \int_{0}^{t'} \left[ H_1(t'), [H_1(t''), \rho_1(0)] \right] \, dt'' \, dt', \quad (18)$$

\(^4\) For a given amplitude $\delta$, vibrations perpendicular to the axis connecting the qubits would change $d_{NV,1/2}(t)$ less dramatically than the on-axis vibrations. Taking all vibrations to lie along the axis connecting the qubits is therefore adequate to capture the worst case; moreover, this assumption then admits analytic treatment.
Figure 4. (a) Maximal achievable mean EF for vibration modes where the vibration frequencies (amplitudes) are distributed according to a truncated normal distribution on $[0, \infty)$; with mean $\omega(\delta)$ and standard deviation $0.01\omega$ ($0.01\delta$), i.e. $\overline{\rho}^\omega$ ($\overline{\rho}^\delta$) with respect to the mean (common) vibration frequency $\omega/2\pi$ for different common (mean) vibration amplitudes $\delta$ and $\phi = \frac{\pi}{2}$.

(b) Maximal achievable mean EF for vibration modes where the phase is uniformly distributed, i.e. $\overline{\rho}^\phi$ with respect to their common vibration amplitude $\delta$ for different common vibration frequencies. In both plots the solid lines are calculated from the numerical solution of the von Neumann equation for $\tilde{H}_{\text{vib,RWA}}(t)$ and the dashed lines are calculated by using the approximations discussed in the text. Here $\Delta = 10$ nm.

where $\rho_1(0) = |\psi_i\rangle\langle\psi_i|$ and get an analytic approximation of the solution of the Schrödinger equation for equation (13). The first-order approximation in $\delta$ reads

$$\rho_1(t) = \rho_1(0) + \frac{6C}{\Delta^4} \delta \sin(\omega t + \phi) - \sin(\phi) \overline{S_{z,NV} S_{z,1} - S_{z,NV} S_{z,2}} \rho_1(0).$$  \hspace{1cm} (19)$$

This approximation allows us to understand the effect of many modes. First, we analyse the effect of many vibrational modes with different frequencies. In figure 4(a), we compare our approximation with the exact solution for (13) by plotting the maximal achievable mean EF for $\overline{\rho}^\omega$ (in both cases) with respect to the vibration frequency $\omega$ for different amplitudes and for a particular phase. For the average $\overline{\rho}^\omega$ in equation (15), we use a truncated normal distribution on $[0, \infty)$ with mean $\omega$ and standard deviation $\sigma = 0.01\omega$. In the static case, we have seen that the mean EF is maximal when we wait about time $\pi/|\alpha|$ before we measure the NV centre. Firstly, assume that the NV centre is initially in the middle of the two qubits ($\phi = \pi/2$). If the NV centre oscillates for $1/2 + k$ periods ($k$ periods) before it is measured, where $k$ is a non-negative integer, then the asymmetry in the system is maximal (minimal) and we get a low (high) mean EF. Obviously for a large $k$, this effect is less pronounced. The minus sign in the commutator in equation (19) describes exactly this asymmetry which decreases for large frequencies and thus the perturbative solution (19) explains the characteristics of figure 4(a). We note that analogous plots where we increase and decrease the width of the frequency distribution are not very sensitive to the width of the distribution, which is in agreement with the rate in equation (19).

Secondly, we can repeat this analysis by averaging over the amplitude instead of the frequency, i.e. by considering $\overline{\rho}^\delta$. Again we see from equation (15) that averaging over a
reasonably narrow truncated normal distribution on $[0, \infty)$ has little effect, and hence, we get the same plot as in the analysis before.

Thirdly, we analyse the effect of many vibrational modes with different phases. Thus, we assume a uniform distribution of phases on $[0, 2\pi]$. With equation (18), we obtain in second order in $\delta$ the following density matrix:

$$
\bar{\rho}_\phi(t) = \rho_I(0) - i\frac{C}{\Delta^3} t^2 [S_{z,NV} S_{z,1} + S_{z,NV} S_{z,2}, \rho_I(0)] - i6 \frac{C^2}{\Delta^8} t^2 \frac{1 - \cos(\omega t)}{\omega^2} S_z S_{z,NV} S_z, \rho_I(0)].
$$

Again we compare this approximation with the exact solution for (13) by plotting in figure 4(b) the maximal achievable mean EF with respect to the vibration amplitude for different frequencies and find good agreement between the two solutions, especially for small amplitudes. We see that the last term in equation (20) can be seen as a (time-dependent) decoherence rate and explains why higher frequencies and smaller amplitudes are less disturbing to our entangling operation.

In summary, we conclude that oscillations of the NV centre do not cause a substantial adverse effect for our entangling operation provided that they are relatively fast and of low amplitude.

5. Moving the NV centre between remote locations

Having established that the measurement of a static NV centre can entangle two nearby spins and having further determined that the realistic issues of dephasing and oscillation do not present fundamental difficulties, we can now characterize our full protocol with two remote spins and a moving, or ‘flying’, NV centre. Therefore, we consider a distance $D \gg \Delta$ between the two qubits and propose that the NV centre should now fly from one qubit to the other, i.e., from $z = z_0$ to $z = D - z_0$ with velocity $v$. Hence, the NV centre interacts first with qubit 1 and then with qubit 2 before it is measured at $t_F = (D - 2z_0)/v$ (see figure 1(b)).

As above, we derive with the time-dependent distances $d_{NV,1}(t) = tv + z_0$ and $d_{NV,2}(t) = D - tv - z_0$ the following rotating frame approximation Hamiltonian:

$$
\tilde{H}_{flying,RWA}(t) = W_{NV,1}(t) + W_{NV,2}(t),
$$

where we neglect the qubit–qubit coupling due to the large distance between the qubits.

Again, we initialize the three spins in the state $|\Psi_1\rangle = |\psi_1\rangle$ (see equation (8)). The Schrödinger equation determines the state $|\Psi(t_F)\rangle$ when the NV centre travelled from one qubit to the other. The measurement of the NV centre in the $|\pm\rangle_{NV}$-basis projects the qubits to one of the states

$$
|\Psi_-\rangle = (-e^{i\beta} |\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle)/\sqrt{2} \quad \text{or}
$$

$$
|\Psi_+\rangle = N_+ \left( e^{i\beta} |\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle + \frac{2e^{i\beta}}{1 + e^{i\beta}} (|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) \right)
$$

with the probability

$$
p_{By,-} = \sin^2(\beta/2)/2 \quad \text{and} \quad p_{By,+} = (3 + \cos(\beta))/4.
$$
Figure 5. Flying NV centre: (a) EFs, mean EF and success probabilities as a function of the inverse velocity $v^{-1}$ when the NV centre has moved from one qubit to the other ($D = 100$ nm). Note that, for $D \gg z_0$, the parameter $\beta$ is almost independent of $D$. Hence, plots like the one shown in this figure with $D$ larger than 100 nm almost look identical. (b) EF for the two static qubits as a function of their separation $D$, after $k$ applications of the protocol (assuming that state $|\!\!-\rangle$ is measured on each occasion). In each entanglement operation the NV centre flies with velocity $v$ such that $\beta(v) = \pi$ from one qubit ($z = z_0$) to the other ($z = D - z_0$), where it is finally measured. We assume that the NV centre is affected by dephasing with a coherence time of $T_{2,NV} = 20 \mu s$. In both plots $z_0 = 5$ nm.

where $\beta = C \left( (D - z_0)^{-2} - z_0^{-2} \right) / v$ and $N_+$ is a normalization factor. The EF of the first state is 1 and that of the second state is given by the binary entropy function $H$

$$E_F(\rho_+) = H \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - \xi^2} \right) \quad \text{with} \quad \xi = \frac{1 - \cos(\beta)}{3 + \cos(\beta)}. \quad (25)$$

Hence, whenever $\beta = \pi \mod 2\pi$, the entangling operation is maximally entangling. Again, for very high velocities, we could discard $\rho_+$ retaining $\rho_-$ and still achieve a (probabilistic) maximally entangling parity projection. In figure 5(a), we plot the EFs and probabilities around the velocity for which $\beta(v) = \pi$.

6. Comparison of entanglement protocols

One of the advantages of the entangling operation presented here is that it entangles remote qubits. For this reason there is a physical distance between the spins that makes it easier to control the qubits. In particular, if the qubits are implemented by NV centres, a separation of hundreds of nanometres enables the use of sub-wavelength techniques such as STED [26] or spin-RESOLFT [29] to optically read out the spin states of the qubits.

Currently, remote entanglement can be achieved in about three different ways [30]. The first method makes use of a mediator system, which can, for example, be an optical excitation [31, 32] or two nuclei with an effective interaction mediated by their partner electron spins [33]. Here, the physical characteristics limit the distance between the qubits. In [33], a distance between the qubits of about 10 nm is considered. A second class of schemes is given by spin chains connecting the two qubits [34–36]. These proposals allow one to bridge a distance of
up to 1 µm. However, a rigorous analysis of the decoherence occurring in these many-body systems still needs to be done. An even larger distance is possible with optical means. In this third class, the qubits are required to possess an optical degree of freedom that is harnessed in a path-erasure experiment for the creation of entanglement [2, 37]. Here, the challenge is that the photons emitted from the qubits need to be indistinguishable.

Our proposal adds another method for entanglement creation to this list. We will now consider the performance characteristics of our scheme and discuss several strategies for maximizing the range of the remote operation. Again, we model the dynamics with a standard master equation as follows:

$$\dot{\rho}(t) = -i[H_{\text{flying,RWA}}(t), \rho(t)] + \frac{2}{T_{2,\text{NV}}} \left( S_{\text{NV},z} \rho(t) S_{\text{NV},z}^\dagger - \frac{1}{2} \{ \rho(t), S_{\text{NV},z}^\dagger S_{\text{NV},z} \} \right).$$

(26)

If we choose the velocity $v$ such that $\beta(v) = \pi$ and determine the EF of $\rho_\pm$, when the NV centre has travelled from one qubit to the other, i.e. when $t = t_1$, then the EF will decrease with distance between the qubits (see figure 5(b)) because the entangling operation takes longer. As in section 3, the fact that the entangling operation is a projection can be used to improve the amount of entanglement obtained by simply repeating the operation. As shown in figure 5(b), for a poor coherence time of 20 µs, the EF can be significantly increased if the distance between the qubits is close to 100 nm. However, for large distances and short coherence times no noticeable improvement is achieved as the coherence of the NV centre is completely lost.

A way to preserve the coherence of the NV centre spin is to transfer the spin state onto a spin with a much longer coherence time. In general, nuclear spins possess coherence times that are significantly longer, typically three orders of magnitude, than electron spins. As the NV centre also possesses a nuclear spin, we could in principle shelve the electronic coherence on the nuclear spin, while the NV centre is not interacting with the qubits. We estimate the duration of this SWAP operation to be less than 2 µs as it is essentially determined by the hyperfine coupling, which is about 3 MHz [36].

In figure 6, we model this shelving technique for two variants. In the first variant, we consider as above a nanocrystal that flies with constant velocity $v$, such that $\beta(v) = \pi$, from one qubit to the other. As soon as the interaction strength between the NV centre and the first qubit drops to 1% of its initial strength, we transfer the electron spin state onto the nuclear spin of the NV centre. For a conservative estimate we can model the spin state of the NV centre spin with the master equation (26). While the coherence is stored on the nuclear spin, we can assume a much longer coherence time (in figure 6 we use $T_{2,\text{NV,nuc}} = 1000 T_{2,\text{NV,el}}$). Before the flying NV centre gets close to the second qubit (we use the same criteria as for the first qubit), we transfer the nuclear spin state back onto the electron spin so that the NV centre can interact with the second qubit.

In the second variant, we assume that the flying NV centre is at first static and interacts with one qubit at $z = z_0$. Once the right amount of phase is acquired, i.e. $t = \frac{\pi}{2|\alpha|}$ (see section 2), we shelve the electronic spin state of the NV centre on its nuclear spin and accelerate the crystal towards the second qubit with a constant acceleration $a$. Currently, the maximum speed (acceleration) of an AFM tip is about 0.5 m s$^{-1}$ (30 m s$^{-2}$) [38]. When the nanodiamond is situated midway, we decelerate the nanodiamond with the acceleration $-a$ until it is situated at the second qubit, i.e. when $z = D - z_0$. While the crystal is in motion, the nuclear spin dephases much more slowly than the electron spin would. For this reason much longer distances can be
Figure 6. The EF $E_F(\rho_{-k})$ of the two-qubit state that is obtained after the entangling operation is performed $k$ times in a row and each time the NV centre is found to be in the state $|-\rangle$. The plot includes the success probability $p_{-k}$ for such an event to happen. (a) The nanodiamond moves with a constant velocity $v$ such that $\beta(v) = \pi$ (the first variant as described in the text). (b) The nanodiamond is accelerated and decelerated with $a = \pm 30 \text{m s}^{-2}$ (the second variant as described in the text). Parameters for both plots: $T_{2,\text{NV,elc}} = 20 \mu\text{s}$, $T_{2,\text{NV,nuc}} = 20 \text{ms}$, $z_0 = 5 \text{nm}$ and the duration of the SWAP operation is $2 \mu\text{s}$.

covered. Eventually, we swap the spin states again and the NV centre interacts with the second qubit for time $t = \pi/|\alpha|$, before it is measured.

In figure 6, we see that the shelving technique enables a distance between the qubits of 500 nm and more. For much larger distances the dephasing of the static qubits would become relevant. Additionally, the EF stays virtually independent for a distance of $D = 100–500$ nm. Finally, already after three repetitions of the second variant, an almost perfectly entangled state is obtained with a success probability of about 35%.

7. Robustness—imperfections in the phase acquisition

We proceed in this paper by discussing a very remarkable robustness property, originating from the fact that we measure the NV centre. For this analysis, we consider the special case in which we aim to implement a parity projection that is deterministic, i.e. $t = \pi/(2|\alpha|)$ in the static case and $\beta = \pi \mod 2\pi$ for the flying NV centre scenario. The interaction part of the entangling operation can effectively be described by two unitaries acting on the system. These unitaries are

\[
U_{1/2} = \exp\left(-i\frac{\pi}{2} (1 + \delta_{1/2}) S_{\text{NV},z} S_{1/2,z}\right),
\]

where $\delta_{1/2}$ denotes the (relative) imperfection in the phase acquisition. A typical example of an error, for which $\delta_{1/2} \neq 0$, is given by timing errors due to imprecise knowledge of the physical distances in the system. Another example is an uncertainty in the speed of the flying NV centre.

As we have seen before, the measurement of the NV centre implements a parity projection on the qubits. It is determined by the general measurement operators

\[
M_{\pm}(\delta_1, \delta_2) = \langle \pm |_{\text{NV}} U_2 U_1 |+\rangle_{\text{NV}}.
\]
Figure 7. (a) EF of the two qubits, after the entanglement operation described in the text projected the NV centre onto the $|\!-\rangle$ state, with respect to the relative errors $\delta_{1/2}$ in phase acquisition. (b) EF of the two qubits, after two successive applications of the entanglement operation described in the text projected the NV centre each time onto the $|\!-\rangle$ state, with respect to the relative errors $\delta_{1/2}$ in phase acquisition.

In the computational basis of the qubits, these operators have the following representation:

$$M_\pm = \frac{1}{2} \begin{pmatrix} \pm 1 - e^{-(1/2)i\pi(\delta_1+\delta_2)} & 0 & 0 & 0 \\ 0 & \pm 1 + e^{-(1/2)i\pi(\delta_1-\delta_2)} & 0 & 0 \\ 0 & 0 & \pm 1 + e^{(1/2)i\pi(\delta_1-\delta_2)} & 0 \\ 0 & 0 & 0 & \pm 1 - e^{(1/2)i\pi(\delta_1+\delta_2)} \end{pmatrix}.$$  \hspace{1cm} (29)

Again, we assume that the two qubits are initially in the $|+\rangle |+\rangle$ state, and the measurement of the NV centre yields the two-qubit state $\rho_\pm$. In figure 7(a), we plot the EF of the state $\rho_-\pm$ with respect to the relative errors $\delta_{1/2}$. We see that even for errors of several per cent, the resulting state possesses an EF that is greater than 0.99, and therefore our method is very robust with respect to errors.

Even more worthy of attention is that, in case the errors are systematic (i.e. the same in consecutive applications of the entangling operation), a simple repetition of the entangling operation significantly increases the EF if the measurement outcome of the second round is the same as in the first round. Note that the probability to measure $|\pm\rangle$ in the first round is about 1/2 and the probability to measure $|\pm\rangle$ again in the second round is close to unity as long as the dephasing of the NV centre is negligible. In figure 7(b), we consider the case where the outcomes of two successive measurements is $\{|\!-\rangle, |\!-\rangle\}$, and we plot the EF of the resulting state $\rho_{-2}$ with respect to the two relative errors $\delta_{1/2}$. We see that the second measurement drastically increases the robustness of our scheme, and therefore, it easily tolerates errors of more than 10%.
Figure 8. EF of the two qubits after our entanglement operation has been applied to a system where the NV centre is initially in a mixed state (described by the parameter $\lambda$) and the measurement of the NV centre is imperfect (described by the parameter $\gamma$).

8. Polarized initial state and imperfect readout

In addition to the specific problems that we have considered so far, in a real system, of course, there will be numerous other imperfections giving rise to finite infidelities in both preparation and measurement. For example, a subtle effect of the unwanted vibration may be to alter the orientation of the NV centre with respect to the static field, thus impairing the initialization process. Here, we describe the general effects of such infidelities. Firstly, we analyse the case of a partially polarized initial state and, secondly, we consider the case of an imperfect measurement of the NV centre. Initially, the NV centre shall be in a mixed state that we parameterize by $\lambda$ as follows:

$$\rho_{\text{pol}} = \frac{1 + \lambda}{2} |+\rangle \langle + | + \frac{1 - \lambda}{2} |-\rangle \langle - | \quad \text{with} \quad 0 \leq \lambda \leq 1. \quad (30)$$

For clarity we assume that the qubits are in the pure state $|+\rangle$. Again, we describe the interaction of the spins by the unitaries $U_{1/2}$ defined in equation (27). Here, however, we assume that there are no imperfections in the phase acquisition, i.e. $\delta_{1/2} = 0$. If a measurement of the NV centre yields $|\pm\rangle$, then the qubits are projected onto the state

$$\rho_{\pm} = \frac{1}{4} \begin{pmatrix}
1 \mp \lambda & 0 & 0 & 1 \mp \lambda \\
0 & 1 \pm \lambda & 1 \pm \lambda & 0 \\
0 & 1 \pm \lambda & 1 \pm \lambda & 0 \\
1 \mp \lambda & 0 & 0 & 1 \mp \lambda
\end{pmatrix}. \quad (31)$$

Further, we can model an imperfect measurement of $\rho_{\pm}$ by considering the following mixture:

$$\tilde{\rho}_{\pm} = \frac{1 + \gamma}{2} \rho_{\pm} + \frac{1 - \gamma}{2} \rho_{\mp} \quad \text{with} \quad -1 \leq \gamma \leq 1. \quad (32)$$

as our measurement outcome. Here, $\gamma = 1$ constitutes a perfect measurement, and $\gamma = -1$ is a measurement that always reports the wrong outcome. The EF for this state is given by the
binary entropy function

\[ E_F(\tilde{\rho}_\pm) = H \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - \gamma^2 \lambda^2} \right) \]  

and is plotted in figure 8.

9. Conclusion

In summary, for this paper we present an entanglement operation that is based on sensing small magnetic fields. Our calculations establish that this operation can be performed within the coherence time of the qubits. Repetitions of the entangling operation compensate for the effect of dephasing of the NV centre. Moreover, imperfections in the scheme caused by vibrations of the NV centre are tiny for high frequencies and low amplitudes. A particular strength of the scheme is its enormous robustness with respect to timing errors. In particular, repetitions of the entangling operation mitigate the effect of systematic errors. Eventually, remote qubits can be entangled by physically moving the NV centre mounted, for example, on an AFM-tip between the two qubits—making the entangling operation of high value for a distributed quantum computer architecture.

Acknowledgments

We thank Erik Gauger, Brendon Lovett, Brian Patton and Jason Smith for discussions. This work was supported by the DAAD (German Academic Exchange Service), Linacre College, the National Research Foundation and the Ministry of Education, Singapore, the John Templeton Foundation and the Japanese Ministry of Education, Culture, Sports, Science and Technology.

References

[1] Cirac J I, Ekert A K, Huelga S F and Macchiavello C 1999 Phys. Rev. A 59 4249–54
[2] Barrett S D and Kok P 2005 Phys. Rev. A 71 060310
[3] Lim Y L, Beige A and Kwek L C 2005 Phys. Rev. Lett. 95 030505
[4] Benjamin S C, Browne D E, Fitzsimons J and Morton J J L 2006 New J. Phys. 8 141
[5] Wrachtrup J, Kilin S Y and Nizovtsev A P 2001 Opt. Spectrosc. 91 429–37
[6] Jelezko F, Gaebel T, Popa I, Gruber A and Wrachtrup J 2004 Phys. Rev. Lett. 92 076401
[7] Gaebel T et al 2006 Nature Phys. 2 408–13
[8] Dutt M V G, Childress L, Jiang L, Togan E, Maze J, Jelezko F, Zibrov A S, Hemmer P R and Lukin M D 2007 Science 316 1312
[9] Neumann P, Beck J, Steiner M, Rempp F, Fedder H, Hemmer P R, Wrachtrup J and Jelezko F 2010 Science 329 542–44
[10] Taylor J M, Cappellaro P, Childress L, Jiang L, Budker D, Hemmer P R, Yacoby A, Walsworth R and Lukin M D 2008 Nature Phys. 4 810–6
[11] Maze J R et al 2008 Nature 455 644–7
[12] Balasubramanian G et al 2008 Nature 455 648–51
[13] Degen C L 2008 Appl. Phys. Lett. 92 243111
[14] Schaffry M, Gauger E M, Morton J J L and Benjamin S C 2011 Phys. Rev. Lett. 107 207210
[15] Li Y, Barrett S D, Stace T M and Benjamin S C 2010 Phys. Rev. Lett. 105 250502
[16] Levitt M H 2008 Spin Dynamics: Basics of Nuclear Magnetic Resonance 2nd edn (New York: Wiley)
[17] Breuer H P and Petruccione F 2002 The Theory of Open Quantum Systems (New York: Oxford University Press)
[18] Wrachtrup J and Jelezko F 2006 J. Phys.: Condens. Matter 18 S807
[19] Wootters W K 1998 Phys. Rev. Lett. 80 2245–8
[20] Bradac C, Gaebel T, Naidoo N, Sellars M J, Twamley J, Brown L J, Barnard A S, Plakhotnik T, Zvyagin A V and Rabeau J R 2010 Nature Nanotechnol. 5 345–9
[21] Balasubramanian G et al 2009 Nature Mater. 8 383–7
[22] Naydenov B, Dolde F, Hall L T, Shin C, Fedder H, Hollenberg L C L, Jelezko F and Wrachtrup J 2011 Phys. Rev. B 83 081201
[23] Tisler J et al 2009 ACS Nano 3 1959–65
[24] Facchi P, Tasaki S, Pascazio S, Nakazato H, Tokuse A and Lidar D A 2005 Phys. Rev. A 71 022302
[25] Tyryshkin A M et al 2011 Nature Mat. at press
[26] Rittweger E, Han K Y, Irvine S E, Eggeling C and Hell S W 2009 Nature Photonics 3 144–7
[27] Herz M, Giessibl F J and Mannhart J 2003 Phys. Rev. B 68 045301
[28] Yun K, Park S, Pyo H, Kim S and Lee S 1999 Biotechnol. Bioprocess Eng. 4 72–7
[29] Maurer P C et al 2010 Nature Phys. 6 912–8
[30] Benjamin S C and Smith J M 2011 Physics 4 78
[31] Schaffry M et al 2010 Phys. Rev. Lett. 104 200501
[32] Schaffry M, Gauger E M, Morton J J L, Fitzsimons J, Benjamin S C and Lovett B W 2010 Phys. Rev. A 82 042114
[33] Bermudez A, Jelezko F, Plenio M B and Retzker A 2011 Phys. Rev. Lett. 107 150503
[34] Epstein R J, Mendoza F M, Kato Y K and Awschalom D D 2005 Nature Phys. 1 94–8
[35] Yao N Y, Jiang L, Gorshkov A V, Gong Z X, Zhai A, Duan L M and Lukin M D 2011 Phys. Rev. Lett. 106 040505
[36] Yao N Y, Jiang L, Gorshkov A V, Maurer P C, Giedke G, Cirac J I and Lukin M D 2010 Scalable architecture for a room temperature solid-state quantum information processor arXiv:1012.2864
[37] Matsuzaki Y, Benjamin S C and Fitzsimons J 2010 Phys. Rev. Lett. 104 050501
[38] Shakir H 2007 Control strategies and motion planning for nanopositioning applications with multi-axis magnetic-levitation instruments PhD Thesis Texas A&M University, http://hdl.handle.net/1969.1/5942