Equilibrium points of the tilted perfect fluid Bianchi $VI_h$ state space

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Abstract

We present the full set of evolution equations for the spatially homogeneous cosmologies of type $VI_h$ filled with a tilted perfect fluid and we provide the corresponding equilibrium points of the resulting dynamical state space. It is found that only when the group parameter satisfies $h > -\frac{1}{9}$ a self-similar solution exists. In particular we show that for $h > -\frac{1}{9}$ there exists a self-similar equilibrium point provided that $\gamma \in \left(\frac{2(3+\sqrt{-h})}{5+3\sqrt{-h}}, \frac{3}{2}\right)$ whereas for $h < -\frac{1}{9}$ the state parameter belongs to the interval $\gamma \in \left(1, \frac{2(3+\sqrt{-h})}{5+3\sqrt{-h}}\right)$. This family of new exact self-similar solutions belongs to the subclass $n^\alpha_\alpha = 0$ having non-zero vorticity. In both cases the equilibrium points have a five dimensional stable manifold and may act as future attractors at least for the models satisfying $n^\alpha_\alpha = 0$. Also we give the exact form of the self-similar metrics in terms of the state and group parameters. As an illustrative example we provide the explicit form of the corresponding self-similar radiation model ($\gamma = \frac{4}{3}$), parametrised by the group parameter $h$. Finally we show that there are no tilted self-similar models of type $III$ and irrotational models of type $VI_h$.

KEY WORDS: Exact Solutions; Perfect Fluid Models; Self-Similarity.

1 Introduction

Although on a sufficiently large observational scale, the present state of the Universe is described by the Friedmann-Lemaître (FL) model which is isotropic and spatially homogeneous, there are potential problems mainly regarding the observed local structures of our “lumpy” Universe which cannot be explained within the class of FL models. Therefore more general cosmological models, which in some dynamical sense, are “close” to FL but not isotropic in local scale, can be used in order to answer many open and important questions. For example it is of interest to understand the presence, the form and the evolution of small (local) density and expansion anisotropies in the Universe or to investigate the constraints that measurements of temperature anisotropies are able to impose on the curvature of space-time. In addition it is important to classify all possible asymptotic states near the cosmological initial singularity (i.e. near the Planck time).

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and into the future that are permitted by the Einstein’s Field Equations (EFE) with a view to explaining how the real Universe may have evolved.

In this point of view the simplest (anisotropic) generalisation of the FL universes are the Spatially Homogeneous (SH) cosmologies admitting a $G_3$ group of isometries acting on 3-dimensional spacelike hypersurfaces $C$. From cosmological point of view, it is of importance to study the evolution of vacuum or models filled with a gamma-law perfect fluid matter source having an energy-momentum tensor of the form:

$$T_{ab} = (\tilde{\mu} + \tilde{p})u_au_b + \tilde{pg}_{ab}$$  

(1.1)

where $\tilde{\mu}, \tilde{p} = (\gamma - 1)\tilde{\mu}$ are the energy density and the pressure measured by the observers comoving with fluid velocity $u^a \ (u^au_a = -1)$. Since in SH models there is a preferred unit timelike congruence $n^a \ (n^an_a = -1)$ normal to the spatial foliations $C$ we can divide them into non-tilted [1] and tilted [2] models according to whether the fluid velocity $u^a$ is parallel or not to the timelike direction $n^a$.

In the last two decades, the study of SH models is heavily based on the qualitative analysis of the resulting system of the (induced) first order ordinary differential equations. Using the so called orthonormal frame formalism (pioneered by Ellis [1]) which is based on choosing a frame tetrad invariant under the group of isometries (thus ensuring the spatial independency of the kinematical and dynamical quantities of the models) and a set of expansion-normalized variables, the evolution and constraint equations, followed from the EFE, become an autonomous system of decoupled first order differential equations which can be studied with the aid of the well-established theory of dynamical systems [3]. The evolution of a specific model is studied in the so called dynamical state space which represents the set of all the physical states (at some instant of time) of the corresponding model [3]. Under this perspective of studying SH models, equilibrium points (i.e. fixed points) and their stability, of the resulting dynamical system, play an important role in the description of the asymptotic behavior (into the past/future as well as in the intermediate times of their evolution) since they may represent past or future attractors for more general models.

Vacuum and non-tilted perfect fluid SH models have been extensively studied in the literature [3, 4, 5, 6, 7, 8, 9] revealing new and important features of the SH models like e.g. the asymptotically self-similarity breaking and the divergence of the Weyl curvature at late times for type VII$_0$ models, providing a solid counterexample to the isotropisation conjecture (i.e. shear isotropisation implies a corresponding result for the Weyl curvature scalar). On the other hand it is natural to expect that the behavior of SH models will be modified accordingly by the presence of a tilted fluid velocity leading also to new interesting phenomena.

The first step of qualitative analyzing tilted models has been done for Bianchi type II models [10]. In particular it has been shown that $\gamma$–law tilted perfect fluid models, are future asymptotic to the Collins-Stewart non-tilted model [11] when $\frac{2}{3} < \gamma \leq \frac{10}{7}$, consequently these models do not isotropise and the angle of tilt becomes negligible at late times. At the value $\gamma = \frac{10}{7}$ the tilt destabilise the Collins-Stewart model and there is an exchange of stability with the self-similar equilibrium point in which $\gamma \in \left(\frac{10}{7}, \frac{14}{9}\right)$. Furthermore at the value $\gamma = \frac{14}{9}$ there is a second bifurcation between the equilibria $\left(\frac{10}{7}, \frac{14}{9}\right)$ and $\left(\frac{14}{9}, 2\right)$ and exhibits the property of the asymptotically extreme tilt for models where the state parameter $\gamma$ belongs to the interval $\left(\frac{14}{9}, 2\right)$.

Recently it was shown that the self-similar equilibrium points of Bianchi type VI$_0$ models...
play a similar role in the asymptotic behaviour of generic models. For example it was found \cite{12} that at the value $\gamma = \frac{6}{5}$ the tilt destabilise the Collins solution \cite{13} and a family of models satisfying $n^\alpha_\alpha = 0$ are future asymptotic to the Rosquist and Jantzen self-similar model \cite{12, 14} for $\gamma \in \left(\frac{5}{6}, \frac{2}{3}\right)$. However it has been shown that generic models (i.e. those satisfying $n^\alpha_\alpha \neq 0$) are not asymptotically self-similar \cite{12} and may be extreme tilted at late times for $\frac{6}{5} < \gamma < 2$ \cite{15}.

In the case of class B tilted models less information is available due to the increased complexity of the evolution equations. Recently the whole family of Bianchi class B models has been studied and some results concerning the stability of the non-tilted equilibrium points have been given \cite{16} (see also \cite{17, 18}).

Motivated from the above facts, the goal of this work is to present the full set of evolution equations, the vacuum, non-tilted and tilted equilibrium points for the important class of Bianchi type VI$_h$ models and interpret, for some of them, their geometric and dynamical properties.

An outline of the paper is as follows: section 2 reviews and presents the basic results concerning the set of equations which describes the dynamics of the tilted perfect fluid SH models. By specialising to the case of Bianchi type VI$_h$ models, we provide the complete set of the evolution equations and we identify the resulting dynamical state space. In section 3 we find the complete set of equilibrium points and for the case which are represented by self-similar models, we give all tilted perfect fluid models admitting a proper Homothetic Vector Field (HVF). Finally in section 4 we summarise and discuss the implications some of the obtained results.

Throughout the following conventions have been used: spatial frame indices are denoted by lower Greek letters $\alpha, \beta, ... = 1, 2, 3$, lower Latin letters denote space-time indices $a, b, ... = 0, 1, 2, 3$ and we use geometrised units such that $8\pi G = c = 1$.

## 2 Dynamical state space of tilted perfect fluid Bianchi type VI$_h$ models

In SH tilted perfect fluid models, the autonomous differential equation governing their evolution can be written in the form:

$$\frac{dx}{d\tau} = f(x)$$

(2.1)

where $x$ is the state vector representing the set of all the physical variables that describe the dynamics of the corresponding model, $f(x)$ is a polynomial function of the state vector and $\tau$ is the dimensionless time variable defined by:

$$\frac{dt}{d\tau} = \frac{1}{H}, \quad \frac{dH}{d\tau} = -(1 + q)H$$

(2.2)

where $q, H$ are the deceleration and Hubble parameter respectively.

In \cite{10} and using the orthonormal frame approach, the EFE are reformulated in terms of the components of the shear tensor of the normal timelike congruence $n^a$, the spatial curvature of the orbits of the $G_3$ isometry group and the spatial part of the tilted fluid velocity $u^a$. The evolution equations for the type VI$_h$ models can be found by specialising the set of equations given in \cite{10}:

$$\Sigma'_{\alpha\beta} = -(2 - q)\Sigma_{\alpha\beta} + 2\epsilon^{\mu\nu}_{(\alpha} \Sigma_{\beta)\mu} R_{\nu} - S_{\alpha\beta} + \Pi_{\alpha\beta}$$

(2.3)
\[ N'_{\alpha\beta} = q N_{\alpha\beta} + 2\Sigma_{(\alpha} \mu N_{\beta)\mu} + 2\epsilon^{\mu\nu}_{(\alpha} N_{\beta)\mu} R_{\nu} \]  
\[ A'_{\alpha} = q A_{\alpha} - \Sigma_{\alpha} \mu A_{\mu} + \epsilon^{\mu\nu\alpha}_{\mu} A_{\mu} R_{\nu} \]  
\[ \Omega' = \Omega G^{-1} \left[ 2Gq - (3\gamma - 2) - (2 - \gamma) v^2 - \gamma \Sigma_{\mu\nu} \epsilon^{\mu\nu} + 2\gamma A_{\mu} v^\mu \right] \]  
\[ v'_{\alpha} = \frac{v_{\alpha}}{1 - (\gamma - 1) v^2} \left\{ (3\gamma - 4) \left( 1 - v^2 \right) + (2 - \gamma) \Sigma_{\gamma\delta} v^\gamma v^\delta + \right. \]  
\[ \left. + \left[ (2 - \gamma) - (\gamma - 1) \left( 1 - v^2 \right) \right] A_{\beta} v^\beta \right\} - \Sigma_{\alpha} \beta v^\beta + \]  
\[ + \epsilon_{\alpha}^{\mu\nu} \left( -R_{\mu} + \Sigma_{\mu} \delta_{\beta} v^\delta \right) v_{\nu} - v^2 A_{\alpha} \]  
where a prime denotes derivative w.r.t. \( \tau \).

The above system is subjected to the algebraic constraints:

\[ \Omega = 1 - \Sigma^2 - K \]  
\[ 3\gamma G^{-1} \Omega v_{\alpha} = 3\Sigma_{\alpha} \beta A_{\beta} - \epsilon_{\alpha}^{\mu\nu} \Sigma_{\mu} \beta N_{\beta\nu} \]  

where we have set

\[ G = 1 + (\gamma - 1) v^2 \]  
and the deceleration parameter is given by the relation:

\[ q = 2\Sigma^2 + \frac{1}{2} G^{-1} \Omega \left[ (3\gamma - 2) \left( 1 - v^2 \right) + 2\gamma v^2 \right] = \]  
\[ = 2 \left( 1 - K \right) - \frac{1}{2} G^{-1} \Omega \left[ 3 \left( 2 - \gamma \right) \left( 1 - v^2 \right) + 2\gamma v^2 \right]. \]  

Using the freedom of a time-dependent spatial rotation, we may choose the orthonormal tetrad to be the eigenframe of \( N_{\alpha\beta} \) therefore the contracted form of Jacobi identities \( N_{\alpha\beta} A^{\beta} = 0 \) implies:

\[ N_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & N_2 & 0 \\ 0 & 0 & N_3 \end{pmatrix}, \quad A_{\alpha} = A_1 \delta_{\alpha}^1. \]  

The evolution equation of \( N_{\alpha\beta} \) can be used to express the angular velocity \( R_{\alpha} \) of the spatial frame in terms of the shear variables:

\[ R_1 = \frac{N_2 + N_3}{N_2 - N_3} \Sigma_{23}, \quad R_2 = \Sigma_{13}, \quad R_3 = -\Sigma_{12}. \]  

In addition equations (2.4) and (2.5) have a first integral which is used to express the component \( A_1 \) in the well known form:

\[ A_1^2 = h N_2 N_3. \]  

Following \[10\] we introduce the shear variables:

\[ \Sigma_+ = \frac{1}{2} \left( \Sigma_{22} + \Sigma_{33} \right), \quad \Sigma_- = \frac{1}{2\sqrt{3}} \left( \Sigma_{22} - \Sigma_{33} \right) \]
In the case of type VI \(h\) models we have \(h < 0\) and \(N_2 N_3 < 0\). With these identifications we obtain the following set of evolution equations for the basic expansion-normalised variables \(x = (\Sigma_+, \Sigma_-, \Sigma_1, \Sigma_3, \Sigma_{13}, N_2, N_3, v_\alpha)\):

\[
\Sigma'_+ = -(2 - q) \Sigma_+ - \frac{(N_2 - N_3)^2 - 18 (\Sigma_{13}^2 + \Sigma_3^2)}{6} - \frac{\Omega_{\gamma} (2v_1^2 - v_2^2 - v_3^2)}{2G}
\]

\[
\Sigma'_- = -(2 - q) \Sigma_- - \frac{\sqrt{3} (N_2^2 - N_3^2)}{6} + \frac{4\sqrt{3} N_3 \Sigma_1^2}{N_2 - N_3} + \frac{\sqrt{3} \Omega_{\gamma} (v_2^2 - v_3^2)}{2G}
\]

\[
\Sigma'_1 = - \left[ \frac{4\sqrt{3} N_3 \Sigma_-}{N_2 - N_3} - q + 2 \left( \sqrt{3} \Sigma_- + 1 \right) \right] \Sigma_1 + \frac{\sqrt{3} \sqrt{hN_2 N_3} (N_3 - N_2) + 6 \Sigma_{13} \Sigma_3}{3} + \frac{\sqrt{3} v_2 v_3 \Omega_{\gamma}}{G}
\]

\[
\Sigma'_3 = \left( q - \sqrt{3} \Sigma_- - 3 \Sigma_+ - 2 \right) \Sigma_3 + \frac{2\sqrt{3} N_3 \Sigma_{13} \Sigma_1}{N_2 - N_3} + \frac{\sqrt{3} v_1 v_3 \Omega_{\gamma}}{G}
\]

\[
\Sigma'_{13} = \left( q + \sqrt{3} \Sigma_- - 3 \Sigma_+ - 2 \right) \Sigma_{13} - \frac{2\sqrt{3} N_2 \Sigma_3 \Sigma_1}{N_2 - N_3} + \frac{\sqrt{3} v_1 v_3 \Omega_{\gamma}}{G}
\]

\[
N_2' = \left( q + 2\sqrt{3} \Sigma_- + 2 \Sigma_+ \right) N_2
\]

\[
N_3' = \left( q - 2\sqrt{3} \Sigma_- + 2 \Sigma_+ \right) N_3
\]

and the evolution equation (2.7) for the frame components of the tilted fluid velocity.

The algebraic constraint (2.9) reads:

\[
\sqrt{3} \Sigma_1 (N_3 - N_2) + 6 \sqrt{hN_2 N_3} \Sigma_+ + \frac{3v_1 \Omega_{\gamma}}{G} = 0
\]

\[
\sqrt{3} \Sigma_{13} N_3 + 3\sqrt{3} \sqrt{hN_2 N_3} \Sigma_3 - \frac{3v_2 \Omega_{\gamma}}{G} = 0
\]

\[
\sqrt{3} \Sigma_3 N_2 - 3\sqrt{3} \sqrt{hN_2 N_3} \Sigma_{13} + \frac{3v_3 \Omega_{\gamma}}{G} = 0
\]

We note that the shear scalar \(\Sigma^2 = \Sigma^{\alpha\beta} \Sigma_{\alpha\beta}/6\) and the spatial curvature \(K\) are:

\[
\Sigma^2 = \Sigma_+^2 + \Sigma_-^2 + \Sigma_1^2 + \Sigma_3^2 + \Sigma_{13}^2
\]

\[
K = hN_2 N_3 + \frac{(N_2 - N_3)^2}{12}
\]

therefore the inequality \(\Omega \geq 0\) and the constraint (2.8) imply that the state space \(\mathcal{D} \subset \mathbb{R}^7\) is bounded (we recall that \(N_2 N_3 < 0\)).
3 Determination of the equilibrium points

Equilibrium points of the autonomous differential equation (2.1) play an important role in the evolution of the SH models since they determine various stable and unstable invariant submanifolds of the state space $D$. These points can be found from the solution of the algebraic equations $f(x) = 0$ and (2.24)-(2.26) which we now list in the following subsections. We note that the case of type III models is included by setting (whenever is appropriate) $h = -1$.

3.1 Vacuum Equilibrium Points

1. Kasner Circle $K_{10}$

\[ N_2 = N_3 = 0, \quad v^\alpha v_\alpha = 0, \quad \Sigma^2 = 1 \]

\[ \Sigma_+^2 + \Sigma_-^2 = 1, \quad \Sigma_1 = \Sigma_{13} = \Sigma_3 = 0, \quad q = 2. \]

2. Kasner Line with tilt $K_{\text{tilt}}^{\pm}$

\[ N_2 = N_3 = 0, \quad v^\alpha v_\alpha = v_3^2 < 1, \quad v_1 = v_2 = 0, \quad \Sigma^2 = 1 \]

\[ \Sigma_+ = \sqrt{3}\Sigma_- + 3\gamma - 4, \quad \Sigma_- = -\frac{\sqrt{3}(3\gamma - 4) \pm \sqrt{3}(2 - \gamma)(3\gamma - 2)}{4} \]

\[ \Sigma_1 = \Sigma_3 = \Sigma_{13} = 0, \quad q = 2, \quad \frac{2}{3} \leq \gamma \leq 2. \]

3. Kasner Circle with extreme tilt $K_{\text{extreme}}^{10}$

\[ N_2 = N_3 = 0, \quad v^\alpha v_\alpha = v_3^2 = 1, \quad v_1 = v_2 = 0, \quad \Sigma^2 = 1 \]

\[ \Sigma_+^2 + \Sigma_-^2 = 1, \quad \Sigma_1 = \Sigma_3 = \Sigma_{13} = 0, \quad q = 2, \quad 0 < \gamma < 2. \]

4. Collins Vacuum Plane Wave Arc $L(VI_h)$ ($h \neq -\frac{1}{9}$)

\[ -12hN_2N_3 = 2\sqrt{3}\sqrt{- (N_2 - N_3)^2 + 3 + (N_2 - N_3)^2} - 6, \quad v^\alpha v_\alpha = 0, \]

\[ \Sigma^2 = \left[ \sqrt{3} + \sqrt{- (N_2 - N_3)^2 + 3} \right]^2 \left[ 12hN_2N_3 + (N_2 - N_3)^2 \right] \]

\[ \Sigma_+ = \frac{\sqrt{3} + \sqrt{-(N_2 - N_3)^2 + 3}}{6}, \quad \Sigma_1 = \frac{2\sqrt{3}hN_2N_3\Sigma_+}{(N_2 - N_3)}, \]

\[ \Sigma_- = \Sigma_{13} = \Sigma_3 = 0, \quad q = \frac{\sqrt{3}\left( \sqrt{- (N_2 - N_3)^2 + 3} + \sqrt{3} \right)}{3} \]
$0 < \gamma < 2$.

5. Vacuum plane wave with tilt\(^1\) $\mathcal{M}_{\text{tilt}}^\pm(VI_h)$ ($h \neq -\frac{1}{9}$)

$$N_2 = N_2, \quad N_3 = N_3, \quad v^\alpha v_\alpha = \left[ \sqrt{-(N_2 - N_3)^2 + 3 \mp \sqrt{3}(3\gamma - 5)} \right]^2 \cdot \frac{12hN_2N_3}{(\gamma - 1)^2},$$

$$\Sigma^2 = \left[ \sqrt{-(N_2 - N_3)^2 + 3 \mp \sqrt{3}} \right]^2 \cdot \left[ 12hN_2N_3 + (N_2 - N_3)^2 \right] \cdot \frac{12}{(N_2 - N_3)^2},$$

$$\Sigma_+ = \frac{\sqrt{3}}{6} \left[ \pm \sqrt{-(N_2 - N_3)^2 + 3 \mp \sqrt{3}} \right], \quad \Sigma_- = \Sigma_{13} = \Sigma_3 = v_2 = v_3 = 0$$

$$\Sigma_1 = \frac{6\Sigma_+ \sqrt{hN_2N_3}}{(N_2 - N_3) \sqrt{3}}$$

$$v_1 = \frac{\sqrt{3} \left[ \pm \sqrt{-(N_2 - N_3)^2 + 3 - \sqrt{3}(3\gamma - 5)} \right]}{6\sqrt{hN_2N_3(1 - \gamma)}},$$

$$q = \frac{\sqrt{3} \left[ \sqrt{3} \pm \sqrt{-(N_2 - N_3)^2 + 3} \right]}{3}$$

$$h = \frac{(N_2 - N_3)^4}{12N_2N_3 \left[ \pm 2\sqrt{3} \sqrt{-(N_2 - N_3)^2 + 3 - (N_2 - N_3)^2 + 6} \right]}.$$

We remark that the state parameter $\gamma$ is constrained via the inequality $1 - v^2 > 0$.

6. Vacuum plane wave with extreme tilt $\mathcal{M}_{\text{extreme}}(VI_h)$ ($h \neq -\frac{1}{9}$).

Same as the case $\mathcal{M}_{\text{tilt}}^\pm(VI_h)$. However the state parameter can take any value in the interval $(0, 2)$.

\(^1\)It appears that this form of the Collins type VI\(_h\) plane wave solution has been also given in [18] using, however, a different notation.
3.2 Non Vacuum Equilibrium Points

1. Flat Friedmann-Lemaître Equilibrium Point \( \mathcal{F} \) \([10]\)

\[
N_2 = N_3 = 0, \quad v^\alpha v_\alpha = 0, \quad \Sigma^2 = 0
\]

\[
\Sigma_+ = \Sigma_- = \Sigma_1 = \Sigma_{13} = \Sigma_3 = 0, \quad q = \frac{3\gamma - 2}{2}, \quad \Omega = 1
\]

\[0 < \gamma < 2.\]

2. Collins-Stewart type II non-tilted Equilibrium Point \( \text{CS(II)} \) \([3]\)

\[
N_2 = 0, N_3 = \sqrt{\frac{2(2-\gamma)(3\gamma - 2)}{4}}, \quad v^\alpha v_\alpha = 0, \quad \Sigma^2 = \frac{(3\gamma - 2)^2}{64}
\]

\[
\Sigma_+ = \frac{2 - 3\gamma}{16}, \quad \Sigma_- = \frac{\sqrt{3}(3\gamma - 2)}{16}, \quad \Sigma_1 = \Sigma_{13} = \Sigma_3 = 0
\]

\[
q = \frac{3\gamma - 2}{2}, \quad \frac{2}{3} < \gamma < 2, \quad \Omega = \frac{3(6 - \gamma)}{16}.
\]

3. Hewitt type II tilted Equilibrium Point \( \mathcal{P}_{\text{tilt}}(\text{II}) \) \([10]\)

\[
N_2 = 0, N_3 = 3 \sqrt{\frac{(\gamma - 2)(3\gamma - 4)(5\gamma - 4)}{18 - 17\gamma}}, \quad v^\alpha v_\alpha = \frac{(3\gamma - 4)(7\gamma - 10)}{(11\gamma - 10)(5\gamma - 4)},
\]

\[
\Sigma^2 = \frac{(3\gamma - 4)(9\gamma^2 - 20\gamma + 12)}{17\gamma - 18}, \quad \Sigma_+ = \frac{9\gamma - 14}{8}, \quad \Sigma_- = \frac{\sqrt{3}(5\gamma - 6)}{8},
\]

\[
\Sigma_{13} = \sqrt{\frac{3(\gamma - 2)(7\gamma - 10)(11\gamma - 10)}{16(18 - 17\gamma)}}, \quad \Sigma_1 = \Sigma_3 = 0,
\]

\[
\Omega = \frac{3(2 - \gamma)(21\gamma^2 - 24\gamma + 4)}{4(17\gamma - 18)}
\]

\[
v_2 = \sqrt{\frac{(3\gamma - 4)(7\gamma - 10)}{(11\gamma - 10)(5\gamma - 4)}}, \quad q = \frac{3\gamma - 2}{2}, \quad \frac{10}{7} < \gamma < 2.
\]

4. Type II Line of tilted Equilibrium Points \( \mathcal{L}_{\text{tilt}}(\text{II}) \) \([10]\)

\[
N_2 = \sqrt{\frac{2(27b^2 + 2)(17 - 54b^2)}{57}}, N_3 = 0, \quad v^\alpha v_\alpha = \frac{6(27b^2 + 1)(27b^2 + 2)}{(54b^2 - 17)(81b^2 - 32)},
\]

\[
\Sigma^2 = \frac{(2 - 3b^2)(27b^2 + 2)}{19}, \quad \Sigma_3 = -\frac{\sqrt{(27b^2 + 1)(32 - 81b^2)}}{3\sqrt{57}}, \quad \Sigma_{13} = b,
\]

\[
\Sigma_+ = \frac{2\sqrt{3}}{9}, \quad \Sigma_- = \Sigma_1 = 0, \quad v_3 = \sqrt{\frac{6(27b^2 + 1)(27b^2 + 2)}{(17 - 54b^2)(32 - 81b^2)}},
\]

\[
q = \frac{4}{3}, \quad \Omega = \frac{2916b^4 - 1215b^2 + 236}{342}, \quad |b| < \frac{2}{3\sqrt{3}}, \quad \gamma = \frac{14}{9}.
\]
5. Type II Extreme tilted Equilibrium Point $\mathcal{P}_{\text{extreme}}(II)$ \[10\]

\[ N_2 = \frac{6\sqrt{19}}{19}, \quad N_3 = 0, \quad v^\alpha v_\alpha = 1, \]
\[ \Sigma^2 = \frac{28}{57}, \quad \Sigma_3 = -\frac{10\sqrt{57}}{171}, \quad \Sigma_{13} = \frac{2\sqrt{3}}{3}, \]
\[ \Sigma_- = -\frac{2\sqrt{3}}{9}, \quad \Sigma_+ = \Sigma_1 = 0, \quad v_3 = 1, \]
\[ q = \frac{4}{3}, \quad \Omega = \frac{20}{57}, \quad 0 < \gamma < 2. \]

6. Collins Type VI$_h$ non-tilted Equilibrium Point $C^\pm(VI_h)$ \[3\]

\[ N_2 = \frac{3\sqrt{(2 - \gamma)(3\gamma - 2)}}{4}, \quad N_3 = -N_2, \quad v^\alpha v_\alpha = 0, \]
\[ \Sigma^2 = \frac{(3\gamma - 2)^2 (1 - 3h)}{16}, \quad \Sigma_3 = \Sigma_{13} = \Sigma_- = 0, \]
\[ \Sigma_+ = \frac{2 - 3\gamma}{4}, \quad \Sigma_1 = \frac{\pm \sqrt{3(2 - 3\gamma)\sqrt{-h}}}{4}, \]
\[ q = \frac{3\gamma - 2}{2}, \quad \Omega = \frac{3|h(3\gamma - 2) - \gamma + 2|}{4}, \quad \frac{2}{3} \leq \gamma \leq \frac{2(1 - h)}{(1 - 3h)}. \]

7. Type VI$_h$ tilted Equilibrium Point $C_{\text{tilt}}(VI_h)$ ($h = -h^2_1$ and $h_1 \neq -1/3$):

\[ -2h_1 \gamma \left[ -(3h_1 + 1) \right]^{1/2} \left\{ 3h^4_1 (4q\gamma - \beta) - 4h_1 [q(5\gamma - 4) + 2(\gamma - 2)] + 3\beta \right\}^{1/2} N_3 \]

\[ = \sqrt{3} \left\{ 6h^4_1 \gamma \left[ (q + 1)(2q\gamma - \beta)(6q(\gamma - 1) + 5\gamma - 6) \right] - h^3_1 \gamma^3 \left( 120q^3 + 16q^2 - 28q + 49 \right) + \right. \]
\[ \left. + (q + 1) \left( -2\gamma^2(144q^2 + 32q + 23) + 60\gamma(q + 1)(4q + 1) - 72(q + 1)^2 \right) \right\} + \]
\[ + h_1^2 \beta \gamma^2 (8q^2 - 66q - 41) + (q + 1)(4\gamma(q + 18) - 12(q + 1)) \]
\[ + h_1^2 \beta \left\{ 2q(4\gamma - 5) + 13\gamma - 10 \right\} - \beta^3 \right\}^{1/2} \]

\[ v^\alpha v_\alpha = \frac{3\beta^2 \left\{ 6q(\gamma - 1) + 5\gamma - 6 \right\} - 2h_1 [q(3\gamma - 2) + \gamma - 2] + \beta \right\}}{4h^2_1 N^2_3 \gamma^2 \left\{ 3h^4_1 (4q\gamma - \beta) - 4h_1 [q(5\gamma - 4) + 2(\gamma - 2)] + 3\beta \right\}} \]
\[ \Sigma_3 = -\sqrt{3} \left\{ 2h_1 N_3 \left[ \zeta + v^2_1 \gamma^2 (\gamma - 1) \right] + 2N_3 \zeta + v_1 \gamma^2 (\beta + 2 - q) \right\}, \]
\[\Sigma_{13} = \Sigma_3, \quad \Sigma_4 = \sqrt{3} \left[ h_1 (q + 1) (5\gamma - 6) - \beta \right] / 6h_1\gamma, \quad \Sigma_+ = -\frac{q}{2}, \quad \Sigma_- = 0\]

\[\Omega = -\frac{[3h_1^2 q\gamma + h_1 (q + 1) (6 - 5\gamma) + \beta] \left\{ 4h_1^2 N_3^2 \left[ 2\zeta (\gamma - 1) + \gamma^2 \right] + (\gamma - 1) \beta^2 \right\}}{6h_1^2 \gamma^3 \beta}\]

\[v_1 = -\frac{\beta}{2h_1 N_3\gamma}, \quad v_2 = v_3 = \zeta^{1/2} \gamma^{-1}\]

where we have set:

\[q = \frac{A + B}{\Lambda}\]

\[A = 3h_1\gamma |1 + 3h_1| |h_1 (3\gamma - 2) - 5\gamma + 6| \times \]
\[\times \{(\gamma - 1) [h_1^2 (\gamma - 1) (7\gamma - 6)^2 + 2h_1 (\gamma - 2) \left( 27\gamma^2 - 37\gamma + 6 \right) + (\gamma - 2)^2 (9\gamma - 1)] \}^{1/2}\]

\[B = 18h_1^4 \gamma^2 (\gamma - 1) (3\gamma - 2) + 3h_1^3 \left( 66\gamma^4 - 427\gamma^3 + 808\gamma^2 - 588\gamma + 144 \right) - \]
\[-3h_1^2 \left( 106\gamma^4 - 595\gamma^3 + 1200\gamma^2 - 1052\gamma + 336 \right) - \]
\[-3h_1 \left( 90\gamma^4 - 499\gamma^3 + 984\gamma^2 - 844\gamma + 272 \right) + (3\gamma - 2) (35\gamma - 36) (\gamma - 2)\]

\[\Lambda = 2[27h_1^4 \gamma (\gamma - 1)^2 (3\gamma - 2) - 18h_1^3 \left( 15\gamma^4 - 62\gamma^3 + 93\gamma^2 - 58\gamma + 12 \right) + \]
\[+ 3h_1^2 \left( 75\gamma^4 - 384\gamma^3 + 704\gamma^2 - 560\gamma + 168 \right) - \]
\[6h_1 \left( 30\gamma^3 - 127\gamma^2 + 166\gamma - 68 \right) + (35\gamma - 36) (\gamma - 2)\]

\[\beta = 2q - 3\gamma + 2,\]
\[ \zeta = \frac{\beta^2 (\gamma - 2) (1 + 3 h_1) (q + 1)}{4 h_1 N_3^2 \{3 h_1^2 (4 q \gamma - \beta) - 4 h_1 [q (5 \gamma - 4) + 2 (\gamma - 2)] + 3 \beta \}} \]

We note that the above solution is defined only when \( h_1 > -1 \) and belongs to the subclass \( n_\alpha^a = 0 \). In addition the state parameter satisfies:

\[
-1 < h_1 < -\frac{1}{3} \Rightarrow 1 < \gamma < \frac{2 (3 - h_1)}{5 - 3 h_1}
\]

\[
-\frac{1}{3} < h_1 < 0 \Rightarrow \frac{2 (3 - h_1)}{5 - 3 h_1} < \gamma < \frac{3}{2}
\]

It was not proved possible to study the stability properties of the solution in the full state space. However the simpler problem of the subclass of models satisfying \( n_\alpha^a = 0 \) can be treated. In particular a preliminary analysis indicates that this equilibrium point has a five dimensional stable manifold and may act as the future attractor at least for the models satisfying \( n_\alpha^a = 0 \).

8. Type VI\(_h\) Extreme tilted Set of Equilibrium Points \( C_{\text{extreme}}(VI_h) \) (\( h = -h_1^2 \) and \( h_1 \neq -1/3 \)):

\[ N_2 = N_2, N_3 = N_3, \quad v^\alpha v_\alpha = 1, \]

\[ \Sigma^2 = \frac{24 h_1 \sqrt{-N_2 N_3} - 12 h_1^2 N_2 N_3 + (N_2 - N_3)^2 + 12}{12}, \]

\[ \Sigma_{13} = \Sigma_3 = \Sigma_- = 0, \quad \Sigma_+ = -h_1 \sqrt{-N_2 N_3} - 1, \quad \Sigma_1 = \frac{\sqrt{3}}{6} (N_3 - N_2) \]

\[ v_1 = 1, \quad q = 2 \left( 1 + h_1 \sqrt{-N_2 N_3} \right), \]

\[ \Omega = \frac{-12 h_1 \sqrt{-N_2 N_3} + 12 h_1^2 N_2 N_3 - (N_2 - N_3)^2}{6}, \quad 0 < \gamma < 2 \]

where \( h_1 < 0 \).

3.3 The exact form of the self-similar equilibrium points

Apart from the cases with extreme tilt which are not representing by exact solutions of the EFE, the rest equilibrium points, whenever exist, correspond to self-similar models i.e. models admitting a proper Homothetic Vector Field (HVF) \( \mathbf{H} \) acting simply transitively on space-time:

\[ \mathcal{L}_\mathbf{H} g_{ab} = 2 \psi g_{ab} \quad (3.1) \]
where $\psi = \text{const.}$ is the homothetic factor which represents the (constant) scale transformation of the geometrical and dynamical variables.

For vacuum and non-tilted models, the corresponding self-similar solutions are all known (see e.g. Tables 9.1. and 9.2., pages 187-188 of [3]) whereas for tilted models the known self-similar solutions are of type II [10] and VI$_0$ [12, 14]. Taking into account the results of the previous section we conclude that only in type VI$_h$ (i.e. $h \neq -1$) there exists a self-similar model. In order to find the exact form of this new family of solutions we follow the procedure that was used in [19] (a complete study of the self-similar SH models of class B is reported elsewhere [20]).

Adopting the notation of [21], the KVFs $\{X_\alpha\}$ and the dual basis $\{\omega^\alpha\}$ are:

$$X_1 = \partial_y, \quad X_2 = \partial_z, \quad X_3 = \partial_x + y\partial_y + h_R z\partial_z.$$  

$$\omega^1 = e^{-x}dy, \quad \omega^2 = e^{-h_R x}dz, \quad \omega^3 = dx.$$  

where the group parameter $h_R$ is formally related with $h$ via ($h_1 < 0$):

$$h_R = \pm \frac{1 - h_1}{1 + h_1}. \quad (3.2)$$

It follows from the non-vanishing structure constants of the isometry group $C_{13}^1 = h_R^{-1}C_{23}^2 = 1$ and the Jacobi identities, that the remaining non-vanishing structure constants $C_{\beta_4}^\alpha$ of the simply transitive homothety group are:

$$C_{14}^1 = \frac{h_R (2 - p_1)}{p_2 - 2}, \quad C_{24}^2 = 1$$

and the HVF is:

$$H = \frac{h_R}{p_2 - 2} t\partial_t + \frac{h_R (p_1 - 2)}{2 - p_2} y\partial_y + z\partial_z. \quad (3.3)$$

In addition using the fact that in every perfect fluid model, the fluid velocity $u^a$ is conformally mapped by the HVF i.e. $\mathcal{L}_H u^a = \psi u^a$ in conjunction with the EFE, we write out explicitly the frame components of the four-velocity and the self-similar metric:

**Fluid velocity**

$$\Delta_1 = v_1 t^{p_1 - 1}, \quad \Delta_2 = 0, \quad \Delta_3 = v_3 t \quad (3.4)$$

**Metric**

$$g_{\alpha\beta} = \begin{pmatrix} c_{11} t^{2(p_1 - 1)} & 0 & c_{13} t^{p_1} \\ 0 & c_{22} t^{2(2h_R - p_1)/h_R} & 0 \\ c_{13} t^{p_1} & 0 & c_{33} t^2 \end{pmatrix} \quad (3.5)$$

where $u'_a = \Gamma (-\delta'_a + \Delta_a \omega^a)$ and $g = -dt^2 + g_{\alpha\beta} \omega^\alpha \omega^\beta dx^a dx^b$.

The various integration constants appearing in (3.4) and (3.5) are given by the following expressions:

$$\gamma = \frac{2}{2s + 1}, \quad p_1 = -\frac{2h_R^2 (s - 1) + 2h_R (2s - 1) + p_2 - 2}{h_R^2}.$$
\[
\frac{c_{33}}{c_{13}} = \frac{c_{11}(h_R + 2)(2s + 1) \left[ 2h_R^2 s + 2h_R (2s - 1) + p_2 - 2 \right] + 4\tilde{\mu}_0 h_R^2 v_1 \Gamma}{4c_{11} \tilde{\mu}_0 h_R^2 v_1 \Gamma}
\]

\[
\Gamma^2 = \frac{c_{11}(2s + 1) \left[ h_R^2 (p_2 + 2s - 2) + 2h_R (2s - 1) + p_2 - 2 \right]}{2\tilde{\mu}_0 h_R^2 (c_{11} v_3 - c_{13} v_1)}
\]

\[
v_3 = -\frac{h_R v_1^2 \left[ 2h_R^2 s - 2h_R (p_2 - 2s - 1) + p_2 - 2 \right]}{2c_{11} s (h_R + 2) [2h_R^2 s + 2h_R (2s - 1) + p_2 - 2]}
\]

\[
\tilde{\mu}_0 = (2s + 1) \{ 4h_R^6 s (2s - 1) (2p_2 + 2s - 3) - \\
-2h_R^4 \left[ p_2^2 (5s - 1) - p_2 \left( 24s^2 - 3s - 1 \right) - 2s \left( 20s^2 - 32s + 9 \right) \right] + \\
+ h_R^3 \left[ p_2^2 + 4p_2 \left( 12s^2 - 10s + 1 \right) + 4 \left( 32s^3 - 48s^2 + 20s - 1 \right) \right] + \\
+ h_R^2 \left[ p_2^2 (16s - 5) + 12p_2 \left( 4s^2 - 6s + 1 \right) + 4 \left( 16s^3 - 40s^2 + 24s - 1 \right) \right] + \\
+ 2h_R (p_2 - 2) \left[ p_2 (2s + 1) + 2 \left( 8s^2 - 6s - 1 \right) \right] + 4s (p_2 - 2)^2 \times \\
\times \left\{ 2h_R^3 (h_R + 2) (s - 1) \left[ 2h_R (p_2 - 1) - p_2 + 2 \right] \right\}^{-1}
\]

\[
v_1 = -\frac{c_{13} (h_R + 2) \left[ 2h_R (2s - 1) + p_2 - 2 \right] \left[ 2h_R^2 s + 2h_R (2s - 1) + p_2 - 2 \right]}{2h_R^2 [h_R^2 (p_2 + 2s - 2) + 2h_R (2s - 1) + p_2 - 2]}
\]

\[
c_{11} = \left[ 2h_R (2s - 1) + p_2 - 2 \right] \left[ 2h_R^2 s + 2h_R (2s - 1) + p_2 - 2 \right]^2 \times \\
\times \left\{ 4h_R^6 s (2s - 1) \left[ p_2 (s + 1) + 2s^2 - 2s - 1 \right] - \\
-2h_R^5 p_2^2 \left( 3s^2 + 1 \right) - p_2 \left( 10s^3 + 18s^2 - 13s + 5 \right) - \\
-2 \left( 24s^4 - 34s^3 + 4s^2 + 5s - 2 \right) \right\} - h_R^4 \left[ 2p_2^3 (s - 1) + \\
+ p_2^2 \left( 8s^2 - 24s + 15 \right) - 4p_2 \left( 2s^3 + 18s^2 - 28s + 11 \right) \right]
\]
\[-4 \left(48s^4 - 76s^3 + 24s^2 + 14s - 7\right) - h_R^3 [3p^3_2 (s - 1) +
+ p^2_2 \left(4s^2 - 38s + 23\right) - 4p_2 \left(12s^3 - 28s + 13\right) -
-4 \left(32s^4 - 88s^3 + 60s^2 - 5\right)] + 2h^2_R (2 - p_2) \times
\times [p_2 \left(6s^2 - 13s + 4\right) - 2 \left(24s^3 - 34s^2 + 13s - 2\right)] +
+h_R (p_2 - 2)^2 \left[2 \left(12s^2 - 13s + 3\right) - 3p_2 (s - 1)\right] +
2 (p_2 - 2)^3 (s - 1) \times
\times \{8h^5_R s (s - 1) [h^2_R (p_2 + 2s - 2) + 2h_R (2s - 1) + p_2 - 2]^3\}^{-1}

\[p_2 = 2 \{h^2_R [(h_R + 2) [h_R (2s - 1) - s]] \times
\times [(2s - 1) (h^2_R (s + 1)^2 (2s - 1) + 2h_R \left(4s^3 - 4s^2 + 5s - 1\right) +
+ 8s^3 - 28s^2 + 10s - 1)]^{1/2} + h^5_R (s + 1) (2s - 1)^2 -
-h_R^4 \left(2s^3 - 21s^2 + 17s - 4\right) - h_R^3 \left(8s^3 + 9s - 4\right) +
+h_R^2 \left(20s^3 - 16s^2 + 4s - 1\right) - 2h_R \left(4s^3 + 6s^2 - 8s + 1\right) + 4s^2\} \times
\times \{2h_R^4 \left(8s^2 - 5s + 1\right) - h_R^3 \left(16s^2 + 6s - 5\right) +
+h_R^2 \left(20s^2 - 14s + 1\right) - 2h_R \left(8s^2 - 8s + 1\right) + 4s^2\}^{-1}

From the expressions of the constants \(p_2, \Gamma\) and the positivity of the energy density it can be verified that the above family of self-similar tilted perfect fluid solutions of type VI \(h\) is defined for

\[\frac{1}{6} < s < \frac{h_R}{2h_R - 1} \Leftrightarrow \frac{2 (3 - h_1)}{5 - 3h_1} < \gamma < \frac{3}{2}\] (3.6)

when \(-1 < h_R < -\frac{1}{2}\) and

\[\frac{h_R}{2h_R - 1} < s < \frac{1}{2} \Leftrightarrow 1 < \gamma < \frac{2 (3 - h_1)}{5 - 3h_1}\] (3.7)
when $-\frac{1}{2} < h_R < 0$. We note that the inequalities involving the bounds of the group parameter $h_R$ are reduced to $-\frac{1}{3} < h_1 < 0$ and $-1 < h_1 < -\frac{1}{3}$ respectively as expected from the analysis in the previous section.

As a concrete and illustrative example of the general results given above, we present the self-similar metric and the associated energy density for the case of a radiation fluid i.e. $\gamma = \frac{4}{3} \Leftrightarrow s = \frac{1}{4}$:

$$p_1 = -\frac{|(h_R + 2)(2h_R + 1)| \sqrt{25h_R^2 - 4h_R + 4 - 12h_R^3 - 58h_R^3 + 7h_R^2 + 32h_R - 4}}{4(2h_R^4 + 10h_R^3 - 5h_R^2 + 4h_R + 1)}$$

$$p_2 = \frac{|(h_R + 2)(2h_R + 1)| \sqrt{25h_R^2 - 4h_R + 4 + 10h_R^5 + 33h_R^4 + 52h_R^3 - 22h_R^2 + 36h_R + 8}}{4h_R^2(2h_R^4 + 10h_R^3 - 5h_R^2 + 4h_R + 1)}$$

$$\tilde{\mu} = \{ -3h_R|h_R + 2| (6h_R^4 - 5h_R^3 + 6h_R^2 + 6h_R + 2) \sqrt{25h_R^2 - 4h_R + 4} +$$

$$+30h_R^5 - 29h_R^4 + 16h_R^3 - 30h_R^2 - 10h_R + 2 \} \times$$

$$\times \{ 4t^2 \left( 2h_R^4 + 10h_R^3 - 5h_R^2 + 4h_R + 1 \right) [-h_R(2h_R - 1) \sqrt{25h_R^2 - 4h_R + 4} +$$

$$+10h_R^3 - 5h_R^2 + 6h_R + 2] \}^{-1}.$$  

It turns out that the energy density is increasing as the group parameter varies in the interval $h_R \in (-1, -\frac{1}{2})$ or in terms of the “original” parameter $h \in (-\frac{1}{3}, 0)$.

4 Concluding Remarks

A long term goal of the qualitative study of SH tilted perfect fluid models rely on the determination of the equilibrium points of their dynamical state space in order to be able to make solid conjectures regarding the dynamics, at the asymptotic regimes, of the corresponding models. In the present paper, by exploiting the orthonormal frame formalism and the general form of evolution equations given in [10], we have found the complete set of equations describing the evolutionary behaviour of the important class of Bianchi type VI$_h$ tilted perfect fluid models.

As a result we have identified the dynamical state space to be a bounded region $D \subset R^7$ subjected to the constraints (2.24), (2.26) and we have given all the equilibrium points (both vacuum and non-vacuum). This study led us to a new family of exact solutions of EFE which are represented by the self-similar metrics (3.3). This family of rotational tilted perfect fluid models, belongs to the subclass $n_0^4 = 0$ and has some interesting stability properties.

In particular we have seen that this model is defined only for $-1 < h < 0$ and has a five dimensional stable manifold in the ranges $h_1 < -\frac{1}{3} \Rightarrow 1 < \gamma < \frac{2(3-h_1)}{5-3h_1}$ and $-\frac{1}{3} < h_1 < 0 \Rightarrow \frac{2(3-h_1)}{5-3h_1} < \gamma < \frac{2}{5}$. In conjunction with the fact that for $-1 < h < 0$ the Collins non-tilted solution $C^+(VI_h)$ is stable whenever $\frac{2}{3} < \gamma \leq \frac{2(3-h_1)}{5-3h_1}$ we conclude that, at the value$^2$ $\gamma = \frac{2(3-h_1)}{5-3h_1}$

$^2$We note that, in contrast with the case of VI$_0$ models in which a similar situation occurs concerning the value $\gamma = \frac{6}{5}$, there is no acceptable tilted solution for the value $\gamma = \frac{2(3-h_1)}{5-3h_1}$.
and when $1 + 3h_1 > 0$, there is an exchange of stability between the Collins model and the present solution. Moreover for $1 + 3h_1 < 0$ both models are stable whenever $1 < \gamma < \frac{2(3-h_1)}{5-3h_1}$ and they are also destabilised at $\gamma = \frac{2(3-h_1)}{5-3h_1}$. However, due to the complexity of the situation, there are no straightforward conclusions regarding the asymptotic behaviour of the models in the full state space. Perhaps a more sophisticated choice of the state variables is needed in order to efficiently answer many open questions. Nevertheless the case $h = -1$ appears to be more tractable, permitting a complete analysis of its dynamical properties. We also pointed out that the above procedure can be used to determine the equilibrium points for the rest of the Bianchi models and for the case of the exceptional models $h = -\frac{1}{5}$ as well. These matters will be the subject of a future work.

A quick glance of the equilibrium points in section 3 shows the lack of physically acceptable models for $h = -1$ which proves (this fact can be confirmed using the geometric results of [20]) the non-existence of tilted self-similar type III models. Furthermore in all (non extreme tilted) cases, the tilted fluid velocity is not parallel to $A_\alpha$. According to Theorem 3.2. of [2] this implies that self-similar type $VI_h$ models have necessary non-zero vorticity. In conclusion we have the following proposition:

**Proposition 1.** There are no type III and irrotational type $VI_h$ tilted perfect fluid models which admit a proper HVF.

We finally note that a stability analysis of the irrotational tilted perfect fluid models of type $VI_h$ shows that the equilibrium points (both non-tilted and tilted) found in section 3 are future unstable which, by means of Proposition 1, implies that irrotational type $VI_h$ models are not asymptotically self-similar but rather they are asymptotic to an extreme tilted model.

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**References**

[1] Ellis G. F. R., *Dynamics of pressure-free matter in general relativity*, 1967 J. Math. Phys. 8 1171-1194; Ellis G F R and MacCallum M A H, *A class of homogeneous cosmological models*, 1969 Commun. Math. Phys. 12 108-141.

[2] King A R and Ellis G F R, *Tilted Homogeneous Cosmological Models*, 1973 Commun. Math. Phys. 31 209-242.

[3] Wainwright J and Ellis G F R (Editors), *Dynamical Systems in Cosmology* (Cambridge University Press, Cambridge 1997); A. A. Coley, *Dynamical Systems and Cosmology*, (Kluwer, Academic Publishers 2003).
[4] Wainwright J, Hancock M J and Uggla C, *Asymptotic self-similarity breaking at late times in cosmology*, 1999 Class. Quantum Grav. **16** 2577-2598.

[5] Wainwright J, *Asymptotic self-similarity breaking in cosmology*, 2001 Gen. Relativ. Grav. **32** 1041-1054.

[6] Nilsson U S, Hancock M J and Wainwright J, *Non-tilted Bianchi VII\(_0\) models—the radiation fluid*, 2000 Class. Quantum Grav. **17** 3119-3134.

[7] Ringström H, *The future asymptotics of Bianchi VIII vacuum solutions*, 2001 Class. Quantum Grav. **18** 3791-3823.

[8] Horwood J T, Hancock M J, The D and Wainwright J, *Late-time asymptotic dynamics of Bianchi VIII cosmologies*, 2003 Class. Quantum Grav. **20** 1757-1777.

[9] Hewitt C G, Horwood J T and Wainwright J, *Asymptotic dynamics of the exceptional Bianchi cosmologies*, 2003 Class. Quantum Grav. **20** 1743-1756.

[10] Hewitt C G, Bridson R and Wainwright J, *The asymptotic regimes of tilted Bianchi II cosmologies*, 2001 Gen. Relativ. Grav. **33** 65-94.

[11] Collins C B and Stewart J M, *Qualitative cosmology*, 1971 Mon. Not. R. Astron. Soc. **153** 419-434.

[12] Apostolopoulos P S, *On tilted perfect fluid Bianchi type VI\(_0\) self-similar models*, 2004 Gen. Relativ. Grav. **36** 1939-1945 (see also Preprint [gr-qc/0310033](http://xxx.lanl.gov/abs/gr-qc/0310033)).

[13] Collins C B, *More qualitative cosmology*, 1971 Commun. Math. Phys. **23** 137-156.

[14] Rosquist K and Jantzen R T, *Exact power law solutions of the Einstein equations*, 1985 Phys. Lett. **107A** 29-32.

[15] Hervik S, *The asymptotic behaviour of tilted Bianchi type VI\(_0\) universes*, 2004 Class. Quantum Grav. **21** 2301-2317.

[16] Barrow J D and Hervik S, *The future of tilted Bianchi universes*, 2003 Class. Quantum Grav. **20** 2841-2854.

[17] Hervik S, van den Hoogen R, Coley A, *Future asymptotic behaviour of tilted Bianchi models of type IV and VII\(_h\)*, 2004 (Preprint [gr-qc/0409106](http://xxx.lanl.gov/abs/gr-qc/0409106)).

[18] Coley A A and Hervik S, *A dynamical systems approach to the tilted Bianchi models of solvable type*, 2004 (Preprint [gr-qc/0409100](http://xxx.lanl.gov/abs/gr-qc/0409100)).

[19] Apostolopoulos P S, *Self-similar Bianchi models: I. Class A models*, 2003 Class. Quantum Grav. **20** 3371-3384.

[20] Apostolopoulos P S, *Self-similar Bianchi models: II. Class B models*, 2005 Class. Quantum Grav. **22** 323-338.

[21] Ryan M P Jr and Shepley L C, *Homogeneous Relativistic Cosmologies*, (Princeton University Press, 1975).