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Abstract. Rota–Baxter operators for groups were recently introduced by L. Guo, H. Lang, and Y. Sheng.

V. G. Bardakov and V. Gubarev showed that with each Rota–Baxter operator one can associate a skew brace. Skew braces on a group $G$ can be characterised in terms of functions $\gamma$ from $G$ to its automorphism group $\text{Aut}(G)$, which satisfy a certain functional equation. For the skew braces obtained from a Rota–Baxter operator the corresponding functions $\gamma$ takes values in the inner automorphism group $\text{Inn}(G)$ of $G$.

In this paper, we give a characterisation of the gamma functions on a group $G$, with values in $\text{Inn}(G)$, that come from a Rota–Baxter operator, in terms of the vanishing of a certain element in a suitable second cohomology group.

Exploiting this characterisation, we are able to exhibit an example of a skew brace whose corresponding function $\gamma$ takes value in the inner automorphism group, but cannot be obtained from a Rota–Baxter operator.

1. Introduction

Rota–Baxter operators for various kinds of algebras have been studied by several authors since G. Baxter introduced them for commutative algebras [Bax60] in 1960.

Recently, L. Guo, H. Lang, and Y. Sheng introduced Rota–Baxter operators for groups [GLS21]. These were studied further by V. G. Bardakov and V. Gubarev [BG21b, BG21a]. In particular, in [BG21a], it is showed how to associate a skew brace with a Rota–Baxter operator.

Recall that a skew (left) brace, defined in [GV17], is a triple $(G, \cdot, \circ)$ such that $(G, \cdot)$ and $(G, \circ)$ are groups and the two operations are related,
by the identity
\[ g \circ (h \cdot k) = (g \circ h) \cdot g^{-1} \cdot (g \circ k), \]
for all \( g, h, k \in G \), where the inverse refers to the operation \(^-1\).

The interplay between set-theoretic solutions of the Yang–Baxter equation of Mathematical Physics, (skew) braces, regular subgroups, and Hopf–Galois structures has spawned a considerable body of literature in recent years; see, for example, [Rum07, GV17, CJO14, GP87, Byo96, SV18].

Given a group \((G, \cdot)\), the group operations \(\circ\) such that \((G, \cdot, \circ)\) is a skew brace can be characterised in the form
\[ g \circ h = g \cdot \gamma(g)h, \quad (1.1) \]
where \(\gamma: G \to \text{Aut}(G)\) is a function, which we write as an exponent, satisfying the identity
\[ \gamma(g \cdot \gamma(g)h) = \gamma(g)\gamma(h) \]
for all \( g, h \in G \). We call such a function a \textit{gamma function}. These functions are usually referred to as \(\lambda\) in the literature of skew braces. (See for instance [CCDC20, CS21b] for this context.)

A \textit{Rota–Baxter operator} on the group \((G, \cdot)\) is a function \(B: G \to G\) which satisfies the functional equation
\[ B(g \cdot B(g) \cdot h \cdot B(g)^{-1}) = B(g) \cdot B(h) \]
for all \( g, h \in G \).

Consider the morphism from the group \(G\) onto the group of its inner automorphisms
\[ \iota: G \to \text{Inn}(G) \]
\[ g \mapsto (x \mapsto g \cdot x \cdot g^{-1}), \]
whose kernel is the centre \(Z(G)\) of \(G\). If \(B\) is a Rota–Baxter operator on the group \(G\), then the function
\[ \gamma(g) = \iota(B(g)) \]
is immediately seen to be a gamma function on \(G\), with values in the inner automorphisms group \(\text{Inn}(G)\). This function yields the skew brace \(G(B)\) introduced in [BG21a], where
\[ g \circ h = g \cdot \iota(B(g))h = g \cdot B(g) \cdot h \cdot B(g)^{-1} \]
for all \( g, h \in G \).

We address here the question whether the converse holds.

\textbf{Question 1.1.} Let \(\gamma\) be a gamma function on the group \((G, \cdot)\), which takes values in \(\text{Inn}(G)\).

Can the skew brace \((G, \cdot, \circ)\), where \(g \circ h = g \cdot \gamma(g)h\) for all \( g, h \in G \), be obtained from a Rota–Baxter operator \(B: G \to G\)?
Equivalently, does there exist a Rota–Baxter operator $B$ on $G$ such that
\[ \gamma(g) = \iota(B(g)) \] holds for $g \in G$?

Remark 1.2. It might be noted that there are skew braces for which the values of the corresponding gamma function do not lie in the group of inner automorphisms.

For instance, if $G$ is a cyclic group of order $pq$, with $p, q$ primes such that $q \mid p - 1$, then the group morphism $\gamma: G \to \text{Aut}(G)$ whose image has order $q$ is of this kind. (See [CCDC20 Section 2.9].)

Question 1.1 has a positive answer when $G$ is a centreless group. In fact, in this case the map $\iota$ is an isomorphism, so if $\gamma: G \to \text{Inn}(G)$ is a gamma function, then the composition of $\gamma$ with $\iota^{-1}$ yields a Rota–Baxter operator $B$ for which (1.2) holds. (Compare with [BG21a Proposition 3.13].)

In Section 3 of this paper we show that the question has a negative answer in general, by exhibiting as a counterexample a finite $p$-group $G$ of order $p^5$, for $p$ an odd prime; this group $G$ is a simplified version of an example of [CS21a].

Our example relies on a cohomological characterisation (Theorem 2.1 of Section 2) of the gamma functions that come from a Rota–Baxter operators, in terms of the vanishing of a certain cocycle in a suitable second cohomology group. Our approach to the cohomological setting is rather concrete, as it exploits the standard connection between group extensions and the second cohomology group, which we recall briefly in Section 3.

2. Cohomology

The arguments in this section are basically those relating group extensions with a suitable second cohomology group. Since the context is slightly different from the standard one, though, we go briefly through the details for the convenience of the reader.

Let $\gamma$ be a gamma function on a group $(G, \cdot)$ whose values are inner automorphisms. In what follows, write $Z(G)$ for $Z(G, \cdot)$. As usual, for all $g, h \in G$, let $g \circ h = g \cdot \gamma(g^{-1})h$.

Choose a function $C: G \to G$ such that $\gamma(g) = \iota(C(g))$. This implies that
\[ g \circ h = g \cdot C(g) \cdot h \cdot C(g)^{-1}. \]

We have
\[ \gamma(g) \gamma(h) = \iota(C(g) \cdot C(h)) \]
and
\[ \gamma(g \cdot \gamma(g^{-1})h) = \gamma(g \cdot \iota(C(g))h) = \iota(C(g \cdot \iota(C(g))h)) = \iota(C(g \circ h)), \]
so that there is a function \( \kappa : G \times G \to Z(G) \) such that
\[
C(g) \cdot C(h) = C(g \circ h) \cdot \kappa(g, h)
\] (2.1)
for all \( g, h \in G \).

We claim that \( \kappa \) is a 2-cocycle on the group \((G, \circ)\) with values in the trivial module \( Z(G) \), that is, for all \( x, y, z \in G \),
\[
\kappa(y, z) \cdot \kappa(x, y \circ z) = \kappa(x, y) \cdot \kappa(x \circ y, z).
\]
We get this by computing \( C(x \circ y \circ z) \) first as
\[
C(x \circ (y \circ z)) = C(x) \cdot C(y \circ z) \cdot \kappa(x, y \circ z)^{-1}
\]
and then as
\[
C((x \circ y) \circ z) = C(x \circ y) \cdot C(z) \cdot \kappa(x \circ y, z)^{-1}
\]
\[
= C(x) \cdot C(y) \cdot C(z) \cdot \kappa(y, z)^{-1} \cdot \kappa(x, y \circ z)^{-1},
\]
If we choose another function \( C' : G \to G \) such that \( \iota(C'(g)) = \gamma(g) = \iota(C(g)) \) for all \( g \in G \), then we have \( C'(g) = C(g) \cdot \sigma(g) \), for some function \( \sigma : G \to Z(G) \). Equation (2.1) yields
\[
C'(g) \cdot C'(h) = C(g) \cdot \sigma(g) \cdot C(h) \cdot \sigma(h)
\]
\[
= C(g \circ h) \cdot \kappa(g, h) \cdot \sigma(g) \cdot \sigma(h)
\]
\[
= C'(g \circ h) \cdot \kappa(g, h) \cdot \sigma(g) \cdot \sigma(h) \cdot \sigma(g \circ h)^{-1},
\]
that is, the 2-cocycle corresponding to \( C' \) is
\[
\kappa'(g, h) = \kappa(g, h) \cdot \sigma(g) \cdot \sigma(h) \cdot \sigma(g \circ h)^{-1},
\]
and thus it differs from \( \kappa \) by a 2-coboundary. We have obtained the first statement in the following result.

**Theorem 2.1.** Let \( \gamma \) be a gamma function on the group \( G \), which takes values in \( \text{Inn}(G) \). Let \( \kappa : G \times G \to Z(G) \) be defined as in (2.1). The following hold.

\begin{itemize}
  \item[(1)] \( \kappa \) is a 2-cocycle, which well-defines a cohomology class in
  \[
  H^2((G, \circ), Z(G)),
  \]
  where \( Z(G) \) is a trivial \((G, \circ)\)-module.
  \item[(2)] The following are equivalent:
    \item[(a)] There is a Rota–Baxter operator \( B \) such that \( \gamma(g) = \iota(B(g)) \) for all \( g \in G \).
    \item[(b)] The cohomology class of \( \kappa \) is trivial.
\end{itemize}

For the second statement, choose arbitrarily a function \( C : G \to G \) such that \( \gamma(G) = \iota(C(g)) \), for all \( g \in G \), and define as above the cocycle \( \kappa \). It follows from (2.2) that there is a Rota–Baxter operator \( B \) such that \( \gamma(g) = \iota(B(g)) \) if and only if \( \kappa \) is a coboundary.
3. The example

This is a simplified version of [CS21a, Example 5.3].

Let $p$ be an odd prime, and let $H$ be the Heisenberg group modulo $p$:

$$H = \langle u, v, k : u^p, v^p, k^p, [u, v] = k, [u, k], [v, k] \rangle.$$  

Let $(G, \cdot) = \langle x, y \rangle \times H$, where $\langle x, y \rangle$ is elementary abelian of order $p^2$, so that $G$ has order $p^5$. Write $Z(G) = Z(G, \cdot)$, and write $K = \langle k \rangle = Z(H) \leq Z(G)$. Since $H/K = \langle uK, vK \rangle$ is elementary abelian of order $p^2$, the assignment

$$\psi(xK) = uK, \quad \psi(yK) = vK, \quad \psi(hK) = K$$

for all $h \in H$ uniquely defines an endomorphism $\psi$ of $G/K$, whose kernel and image are $H/K$.

Let $C : G \to G$ be any function such that

$$C(g)K = \psi(gK)$$

for all $g \in G$. (Such a function is denoted by $\psi^\uparrow$ in [CS21a].) Note that we do not assume $C$ to be constant on the cosets of $K$. Also, note that $C(G) \subseteq H$ and $C(H) \subseteq K$.

It was proved in [CS21a], and it is immediate to see, that the function $\gamma : G \to \text{Inn}(G)$ defined by $\gamma(g) = \iota(C(g))$ depends only on $\psi$, and not on the particular choice of $C$. It was also proved in [CS21a, Theorem 3.2] that $\gamma$ is a gamma function, which thus defines a skew brace $(G, \cdot, \circ)$, where “$\cdot$” is the original group operation on $G$ and “$\circ$” is the group operation of (1.1), namely

$$a \circ b = a \cdot C(a) \cdot b \cdot C(a)^{-1}$$

for all $a, b \in G$. (Actually, gamma functions are not mentioned explicitly in [CS21a], but they are used implicitly via (1.1).)

Note that $[a, b] = 1$ for all $a \in \langle x, y \rangle$ and $b \in C(G) \subseteq H$; therefore the operations “$\cdot$” and “$\circ$” coincide on $\langle x, y \rangle$, which is thus a subgroup of $(G, \circ)$.

Similarly, as $C(h) \in K \leq Z(G)$ for all $h \in H$, the operations “$\cdot$” and “$\circ$” coincide also on $H$, which is thus also a subgroup of $(G, \circ)$.

Moreover, $(H, \circ)$ is normal in $(G, \circ)$. This follows from the general formula for conjugation in $(G, \circ)$ (see [CCDC20, Lemma 2.10]):

$$g \circ h \circ \overline{g} = g \cdot \gamma(g) h \cdot \gamma(h) \cdot \gamma(g)^{-1}(g^{-1}),$$

where $\overline{g}$ is the inverse of $g$ in $(G, \circ)$. In fact here we have, for all $g \in G$ and $h \in H$,

$$g \circ h \circ \overline{g} = g \cdot \gamma(g) h \cdot \gamma(h) \cdot \gamma(g)^{-1}(g^{-1}) = g \cdot C(g) \cdot h \cdot C(g)^{-1} \cdot g^{-1} \in H,$$

as $\gamma(h) = 1$ and $(H, \cdot)$ is normal in $(G, \cdot)$.

We now compute the cohomology class of the 2-cocycle $\kappa$ associated to $\gamma$, as per Section 2. To compute this class, we are free to choose any
of the functions $C$ satisfying (3.1). Our choice is the map $C : G \to G$ defined by
\[ C(x^i y^j c) = u^i v^j \in H \]
for all $0 \leq i, j < p$ and $c \in H$. Let us compute the relevant 2-cocycle. We have, for all $0 \leq i, j, m, n < p$ and $c, d \in H$,
\[ C((x^i y^j c) \cdot C(x^i y^j c) \cdot (x^m y^n d) \cdot C(x^i y^j c)^{-1}) = C(x^{i+m} y^{j+n} e) = u^{i+m} v^{j+n}, \]
for some $e \in H$. On the other hand,
\[ C(x^i y^j c) C(x^m y^n d) = u^i v^j u^m v^n \]
\[ = u^{i+m} v^{j+n} [v^{-j}, u^{-m}] \]
\[ = u^{i+m} v^{j+n} k^{-jm}. \]
So the relevant 2-cocycle here is
\[ \kappa(x^i y^j c, x^m y^n d) = k^{-jm}, \quad (3.2) \]
with image in $K = Z(H) \leq Z(G) = \langle x, y \rangle \times K$.

The skew brace $(G, \cdot, \circ)$ provides a negative answer to Question 1.1. This will follow from Theorem 2.1 and the following lemma.

**Lemma 3.1.** The cocycle $\kappa$ of (3.2) yields a non-trivial class in the 2-cohomology group $H^2((G, \circ), Z(G))$.

We will give two proofs of this lemma. Both proofs rely on the standard connection between central extensions and a suitable cohomology group (see [Rob82, Section 11.4], [RZ10, Section 6.8]), which we briefly recall in the following.

Let $Q$ and $G$ be groups, and let
\[ 1 \to Q \to D \xrightarrow{\pi} G \to 1. \quad (3.3) \]
be a central extension of $G$ by $Q$, that is, an exact sequence of groups where the image of $Q$ is contained in the centre of $D$. Let $s : G \to D$ be a section, that is, a set-theoretic map $G \to D$ such that $\pi(s(x)) = x$ for all $x \in G$. There will be a function $\vartheta : G \times G \to Q$ such that for all $a, b \in G$,
\[ s(a)s(b) = s(ab)\vartheta(a, b). \quad (3.4) \]
Then, regarding $Q$ as a trivial $G$-module, we have the following result, whose proof is basically the argument we gave in Section 2.

**Proposition 3.2.**

1. $\vartheta$ is a 2-cocycle, whose class in $H^2(G, Q)$ does not depend on the particular section we have chosen.
2. Every element of $H^2(G, Q)$ occurs in this form.
3. The following are equivalent:
   a. The extension $D$ of $G$ by $Q$ splits.
   b. The cocycle $\vartheta$ yields the trivial class in $H^2(G, Q)$. 
First proof of Lemma 3.1. Consider $Z(G)$ as trivial module for both $(G, \circ)$ and its subgroup $S = \langle x, y \rangle$. Equation (3.2) shows that the restriction map

$$\text{res}^{(G, \circ)}_{(S, \circ)} : H^2((G, \circ), Z(G)) \to H^2((S, \circ), Z(G))$$

maps the cohomology class of the 2-cocycle $\kappa$ to the cohomology class of the 2-cocycle $\kappa'$ defined by

$$\kappa'(x^iy^j, x^my^n) = k^{-jm}.$$

We now show that $\kappa'$ is non-trivial in cohomology. It will follow that the same holds for $\kappa$, thereby concluding the proof of the lemma.

The surjective homomorphism

$$\pi : S \times H = G \to S$$

$$x^my^nu^iv^j \mapsto x^iy^j,$$

with $0 \leq i, j, m, n < p$ and $c \in K$, has kernel $S \times K$, so it yields the central extension

$$1 \to S \times K \hookrightarrow G \xrightarrow{\pi} S \to 1.$$

This extension is clearly non-split, as $S$ is abelian, $S \times K = Z(G)$, and $G$ is non-abelian. We now show that the cohomology class associated to this central extension is that of $\kappa'$. According to Proposition 3.2, this will yield that $\kappa'$ is non-trivial in cohomology.

Consider the following section of $\pi$:

$$s : S \to G$$

$$x^iy^j \mapsto u^iv^j,$$

where $0 \leq i, j < p$. As

$$s(x^iy^j)s(x^my^n) = u^iv^jumvn = u^{i+m}v^{j+n}[u, v]^{-jm} = u^{i+m}v^{j+n}k^{-jm}$$

and

$$s((x^iy^j)(x^my^n)) = s(x^{i+m}v^{j+n}) = u^{i+m}v^{j+n},$$

the cocycle we are looking for is

$$(x^iy^j, x^my^n) \mapsto k^{-jm},$$

as claimed. □

We now give an alternative proof of Lemma 3.1, which avoids the use of the restriction map.

Second proof of Lemma 3.1. Let $\tilde{K} = \langle \tilde{k} \rangle$ be a copy of $K$, and let $\tilde{S}$ be a copy of $S = \langle x, y \rangle$. Regard $\tilde{S} \times \tilde{K}$ as a trivial $(G, \circ)$-module, and $\kappa$ as a 2-cocycle $\tilde{\kappa} : (G, \circ) \times (G, \circ) \to \tilde{S} \times \tilde{K}$. Let

$$1 \to \tilde{S} \times \tilde{K} \hookrightarrow D \to (G, \circ) \to 1$$

(3.5)

be the central extension associated to $\tilde{\kappa}$. 

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Let \( s : (G, \circ) \rightarrow D \) be a section. It follows from equation (3.4) with \( \vartheta = \tilde{\kappa} \) and the formula (3.2) for \( \kappa \) that

\[
[s(x), s(y)] = s(x)s(y)(s(y)s(x))^{-1}
= s(xy)\tilde{\kappa}(x, y)(s(yx)\tilde{\kappa}(y, x))^{-1}
= \kappa(y, x)^{-1} = \tilde{\kappa},
\]

so that \( \tilde{K} \) is contained in the derived subgroup \([D, D]\) of \( D \).

Assume by way of contradiction that the sequence (3.5) split \( s \), and let \( T \) be a complement to \( \tilde{S} \times \tilde{K} \) in \( D \). Then \( M = \tilde{S}T \) is a maximal subgroup of the finite \( p \)-group \( D \). Now in a finite \( p \)-group every maximal subgroup is normal, with cyclic quotient, and thus contains the derived subgroup; but here \( M \) does not contain \( \tilde{K} \), a non-trivial subgroup of \([D, D]\).

This contradiction shows that (3.5) does not split, so that by Proposition 3.2 \( \kappa \) is non-trivial in \( H^2((G, \circ), Z(G)) \). □

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