ZETA FUNCTIONS ASSOCIATED TO ADMISSIBLE REPRESENTATIONS OF COMPACT $p$-ADIC LIE GROUPS

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Abstract. Let $G$ be a profinite group. A strongly admissible smooth representation $\rho$ of $G$ over $\mathbb{C}$ decomposes as a direct sum $\rho \cong \bigoplus_{\pi \in \text{Irr}(G)} m_{\pi}(\rho) \pi$ of irreducible representations with finite multiplicities $m_{\pi}(\rho)$ such that for every positive integer $n$ the number $r_n(\rho)$ of irreducible constituents of dimension $n$ is finite. Examples arise naturally in the representation theory of reductive groups over non-archimedean local fields. In this article we initiate an investigation of the Dirichlet generating function $\zeta_{\rho}(s) = \sum_{n=1}^{\infty} r_n(\rho)n^{-s} = \sum_{\pi \in \text{Irr}(G)} \frac{m_{\pi}(\rho)}{\dim \pi} s$ associated to such a representation $\rho$.

Our primary focus is on representations $\rho = \text{Ind}_H^G(\sigma)$ of compact $p$-adic Lie groups $G$ that arise from finite dimensional representations $\sigma$ of closed subgroups $H$ via the induction functor. In addition to a series of foundational results – including a description in terms of $p$-adic integrals – we establish rationality results and functional equations for zeta functions of globally defined families of induced representations of potent pro-$p$ groups. A key ingredient of our proof is Hironaka’s resolution of singularities, which yields formulae of Denef-type for the relevant zeta functions.

In some detail, we consider representations of open compact subgroups of reductive $p$-adic groups that are induced from parabolic subgroups. Explicit computations are carried out by means of complementing techniques: (i) geometric methods that are applicable via distance-transitive actions on spherically homogeneous rooted trees and (ii) the $p$-adic Kirillov orbit method. Approach (i) is closely related to the notion of Gelfand pairs and works equally well in positive defining characteristic.

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1. Introduction

In recent years the subject of representation growth has advanced with a primary focus on zeta functions enumerating (i) irreducible representations of arithmetic lattices and compact $p$-adic Lie groups, (ii) twist isoclasses of irreducible representations of finitely generated nilpotent groups; for instance, see \[38, 3, 4, 5, 1\] and \[52, 46, 21\], or the relevant surveys \[35, 55\]. The aim of this paper is to introduce and study a new, more general zeta function that can be associated to any ‘suitably tame’ infinite dimensional representation of a group. Our focus is on admissible smooth representations of compact $p$-adic Lie groups that arise from finite dimensional representations of closed subgroups via the induction functor. In addition to a series of foundational results that include a description in terms of $p$-adic integrals and provide a springboard for further investigations, we establish in Theorem D rationality results and functional equations for zeta functions of globally defined families of induced representations of potent pro-$p$ groups.

1.1. Background on representations zeta functions. A group $G$ is said to be representation rigid if, for each positive integer $n$, its number of (isomorphism classes of) irreducible complex representations of degree $n$, denoted $r_n(G)$, is finite. In the spirit of [27] the sequence $r_n(G), n \in \mathbb{N}$, is encoded in a Dirichlet generating function $\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G)n^{-s}$, yielding the conventional representation zeta function of $G$. If $G$ has polynomially bounded representation growth, then $\zeta_G(s)$ converges and defines a holomorphic function on a right half-plane $\{s \in \mathbb{C} \mid \Re(s) > \alpha(G)\}$, where the abscissa of convergence $\alpha(G)$ reflects the polynomial degree of representation growth. In favourable circumstances the function extends meromorphically to a larger domain, possibly the entire complex plane.

For a semisimple complex algebraic group $G = G(\mathbb{C})$ the representation zeta function $\zeta_G(s)$, encoding irreducible rational representations, is also known as the Witten zeta function; see [57]. The analytic properties of such zeta functions have been studied thoroughly, using the available classification of irreducible representations in terms of highest weights; e.g., see [37, 38]. We are concerned with groups where a similar classification is either impracticable or way out of reach.

Recent advances in [3, 4, 5] concern the representation zeta functions of arithmetic lattices in semisimple locally compact groups (mostly in characteristic 0) and their local Euler factors. The latter are, de facto, representation zeta functions enumerating irreducible continuous representations of compact $p$-adic Lie groups. The main focus has been on the following aspects: abscissae of convergence, possible meromorphic continuations and pole spectra. There are fundamental connections to Margulis super-rigidity and the classical Congruence Subgroup Problem; for instance, a quantitative refinement of the Congruence Subgroup Conjecture, due to Larsen and Lubotzky [38], has been reduced to the original conjecture in [5].
In these investigations the main tools are: Lie-theoretic techniques, the Kirillov orbit method, the character theory of finite groups and Clifford theory, all in parallel with algebro-geometric, model-theoretic and combinatorial methods from $p$-adic integration. This includes, among others, Deligne–Lusztig theory for representations of finite groups of Lie type, Hironaka’s resolution of singularities in characteristic 0 and aspects of the Weil conjectures regarding zeta functions of smooth projective varieties over finite fields.

Most relevant for our purposes are the results of Avni, Klopsch, Onn and Voll in [2, 3], including a ‘Denef formula’ for the representation zeta functions of principal congruence subgroups of compact $p$-adic Lie groups that arise from a global Lie lattice over the ring of integers of a number field; compare [18].

1.2. Zeta functions associated to admissible representations. Our motivation is to analyse more general enumeration problems than the one underlying past research on representation growth. Let $G$ be a profinite group; in due course we specialise to the case, where $G$ is a compact $p$-adic Lie group. Key examples are the completions $G = \hat{G}(\mathcal{O}_S)$ and $G = G(\mathcal{O}_p)$ of arithmetic groups $G(\mathcal{O}_S)$ with respect to the profinite topology or the $p$-adic topology associated to a non-archimedean prime $p$; here $G$ denotes a semisimple affine group scheme over the ring of $S$-integers $\mathcal{O}_S$ of a number field. In addition we are interested in the principal congruence subgroups of the groups $G(\mathcal{O}_p)$. In order to develop flexible methods for enumerating irreducible representations by their degrees according to natural weights, we shift emphasis and attach a zeta function to every ‘well behaved’ infinite dimensional representation of $G$. This point of view is motivated also by geometric applications, for instance, in the context of cohomology growth, where such representations are supported on direct limits of cohomology groups arising from systems of finite coverings of a fixed topological space; see [47]. In particular, in the realm of number theory the relevant profinite groups acting on such direct limits are completions of arithmetic groups as described above; e.g., see [28].

Technically, we consider admissible smooth representations of $G$ over $\mathbb{C}$, a concept which arises in number theory mainly through the study of automorphic representations, e.g., in the context of the Langlands program. Admissible smooth representations of reductive groups over $p$-adic fields have been studied thoroughly since the 1970s, starting from the work of Casselman as well as Bernstein and Zelevinsky; compare [12, 45]. Let $\text{Irr}(G)$ denote the set of all irreducible smooth representations of the profinite group $G$, up to isomorphism, and note that each $\pi \in \text{Irr}(G)$ is finite dimensional. An admissible smooth representation $\varrho$ of $G$ decomposes as a direct sum $\varrho \cong \bigoplus_{\pi \in \text{Irr}(G)} m_\pi(\varrho) \pi$ of irreducible representations with finite multiplicities $m_\pi(\varrho)$. We say that $\varrho$ is strongly admissible if, in addition, for every positive integer $n$ the number $r_n(\varrho)$ of irreducible constituents of dimension $n$ is finite; in this case we define the zeta function of the representation $\varrho$ as the Dirichlet generating function

$$
\zeta_{\varrho}(s) = \sum_{n=1}^{\infty} r_n(\varrho) n^{-s} = \sum_{\pi \in \text{Irr}(G)} \frac{m_\pi(\varrho)}{(\dim \pi)^s}, \quad (s \in \mathbb{C}).
$$
This raises the fundamental question which infinite dimensional \( \rho \) are moreover polynomially strongly admissible, i.e., for which \( \rho \) the zeta function \( \zeta_\rho \) has finite abscissa of convergence \( \alpha(\rho) \) and thus converges in the non-empty right half-plane \( \{ z \in \mathbb{C} \mid \text{Re}(z) > \alpha(\rho) \} \). Of course, the property can easily be formulated in terms of conditions on the multiplicities \( m_\pi(\rho) \); however, the latter are typically difficult to access. We remark that the definition gives a natural generalisation of the conventional representation zeta function of a profinite group \( G \): indeed, \( \zeta_G(s) = \zeta_{\rho_{\text{reg}}}(s+1) \) for the regular representation \( \rho_{\text{reg}} = \text{Ind}_\mathbb{1}^G(\mathbb{1}) \) of \( G \); see Example 2.5.

Generally, we are interested in relations between the algebraic properties of a polynomially strongly admissible representation \( \rho \) of the profinite group \( G \) and the analytic properties of the associated zeta function \( \zeta_\rho \). Depending on the context, different types of properties become relevant. Analytic properties of \( \zeta_\rho \) comprise, for instance: its abscissa of convergence, meromorphic extension, the location of zeros and poles, and possibly functional equations. Regarding algebraic properties of \( \rho \), the emphasis lies on conditions which do not refer directly to the multiplicities \( m_\pi(\rho) \) of the irreducible constituents.

The new zeta function defined in (1.1) transforms well with respect to basic operations on admissible representations, e.g., passing to the smooth dual, forming direct sums and tensor products. This suggests that other standard, but more sophisticated operations on representations transform the zeta functions in a controllable way. In this paper we study in detail representations \( \rho = \text{Ind}_H^G(\sigma) \) of \( G \) that are obtained by induction from representations \( \sigma \) of a closed subgroup \( H \leq_c G \). The induction functor is a central tool for producing infinite dimensional representations from finite dimensional ones.

1.3. Summary of main results. In dealing with profinite groups, it is often convenient to pass to open subgroups. In Section 3 we establish that various admissibility properties for a smooth representation \( \rho \) of a profinite group \( G \) are common properties of the twist similarity class of \( \rho \) defined there. In particular, if \( \rho \) is polynomially strongly admissible, the abscissa of convergence \( \alpha(\rho) \) is an invariant of the corresponding twist similarity class.

We recall that a finitely generated profinite group \( G \) is representation rigid if and only if it is \( \text{FAb} \), i.e., if every open subgroup \( K \leq_o G \) has finite abelianisation \( K/[K,K] \); see [7, Prop. 2]. We introduce the following refined notion: \( G \) is \( \text{FAb relative to} \) a closed subgroup \( H \leq_c G \) if \( K/(H \cap K)[K,K] \) is finite for every open subgroup \( K \leq_o G \). In Theorem 3.8 we establish the following natural characterisation.

**Theorem A.** Let \( H \leq_c G \) be a closed subgroup of a finitely generated profinite group \( G \). The following statements are equivalent.

(a) The group \( G \) is \( \text{FAb relative to} \) \( H \).

(b) The functor \( \text{Ind}^G_H \) preserves strong admissibility.

(c) The induced representation \( \text{Ind}^G_H(\mathbb{1}_H) \) is strongly admissible.

It is an open problem to characterise finitely generated profinite groups with polynomial representation growth. Theorem A leads to the following refined
question: Under what conditions on \( H \leq_c G \) does the induction functor \( \text{Ind}_H^G \) preserve polynomially strong admissibility? In Proposition 3.10, we provide a partial answer in the special setting, where \( G \) is a compact \( p \)-adic Lie group for some prime \( p \); our result can be regarded as a generalisation of [39] Prop. 2.7.

**Proposition B.** Let \( H \leq_c G \) be a closed subgroup of a compact \( p \)-adic Lie group \( G \). If \( G \) is FAb relative to \( H \) then \( \text{Ind}_H^G(\sigma) \) is polynomially strongly admissible for every finite dimensional smooth representation \( \sigma \) of \( H \).

For uniformly powerful – more generally, for finitely generated torsion-free potent – pro-\( p \) groups the Kirillov orbit method provides a powerful tool for handling the characters of smooth irreducible representations; see [23]. In Section 4 we generalise the approach used, for instance, in [2, 3, 4] to describe zeta functions of induced representations for potent pro-\( p \) groups. One of our key results is Proposition 4.3; it provides a formula for the relevant zeta function in terms of a \( p \)-adic integral. We refer to Sections 4.2 and 5.1 for an explanation of the notation \( || \cdot ||_p \) appearing in the integrands; the canonical \( p \)-adic measure is recalled in Remark 5.4.

**Proposition C.** Let \( \mathfrak{o} \) be a compact discrete valuation ring of characteristic 0 and residue characteristic \( p \), with uniformiser \( \pi \). Put \( \mathfrak{p} = \pi \mathfrak{o} \) and \( q = [\mathfrak{o}/\mathfrak{p}] \).

Let \( \mathfrak{g} \) be an \( \mathfrak{o} \)-Lie lattice and \( \mathfrak{h} \) an \( \mathfrak{o} \)-Lie sublattice of \( \mathfrak{g} \). Write \( m + 1 = \dim_{\mathfrak{o}} \mathfrak{g} - \dim_{\mathfrak{o}} \mathfrak{h} \). Let \( r \in \mathbb{N}_0 \) be such that \( G = \exp(\pi^r \mathfrak{g}) \) is a potent pro-\( p \) group with potent subgroup \( H = \exp(\pi^r \mathfrak{h}) \leq_c G \). Then the zeta function of \( \varrho = \text{Ind}_H^G(1_H) \) is given by the following integral formulae.

1. Writing \( \mathcal{W} = \{ w \in \text{Hom}_{\mathfrak{o}}(\mathfrak{g}, \mathfrak{f}) \mid w(\mathfrak{h}) \subseteq \mathfrak{o} \} \), where \( \mathfrak{f} \) is the fraction field of \( \mathfrak{o} \), we have

\[
\zeta_{\varrho}(s) = q^{r(m+1)} \int_{w \in \mathcal{W}} \left( \left\| \bigcup \{\text{Pfaff}_k(w) \mid 0 \leq k \leq [n/2]\} \right\|_p \right)^{-1-s} d\mu(w),
\]

where \( \mu \) denotes the normalised Haar measure satisfying \( \mu(\text{Hom}_{\mathfrak{o}}(\mathfrak{g}, \mathfrak{o})) = 1 \).

2. For simplicity, suppose further that the \( \mathfrak{o} \)-module \( \mathfrak{g} \) decomposes as a direct sum \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h} \). Interpreting \( \text{Hom}_{\mathfrak{o}}(\mathfrak{t}, \mathfrak{o}) \) as the \( \mathfrak{o} \)-points \( \mathcal{W}(\mathfrak{o}) \) of the \( m + 1 \)-dimensional affine space \( \mathcal{W} \) over \( \text{Spec}(\mathfrak{o}) \), let \( \mathfrak{X} \) denote the projectivisation \( \mathbb{P}\mathcal{W} \) over \( \text{Spec}(\mathfrak{o}) \). For \( 0 \leq k \leq [n/2] \), the map \( w \mapsto \text{Pfaff}_k(w) \) induces a sheaf of ideals \( \mathcal{I}_k \) on \( \mathfrak{X} \), and

\[
\zeta_{\varrho}(s) = (1 - q^{-1})q^{r(m+1)} \sum_{\ell \in \mathbb{Z}} q^{-\ell(m+1)} \int_{\mathfrak{X}(\mathfrak{o})} \left( \max_{0 \leq k \leq [n/2]} \left\| \pi^k \mathcal{I}_k \right\|_p \right)^{-1-s} d\mu_{\mathfrak{X}, \mathfrak{p}},
\]

where \( \mu_{\mathfrak{X}, \mathfrak{p}} \) denotes the canonical \( p \)-adic measure on \( \mathfrak{X}(\mathfrak{o}) \).

In Proposition 4.6 we obtain a concrete version of this formula, based on a particular choice of coordinates. We illustrate the usefulness of the explicit formula in Section 7 by computing the zeta functions of various representations induced from Borel or parabolic subgroups.

In Theorem 5.4 we obtain results on the rationality of globally induced representations, their abscissae of convergence and local functional equations. This
generalises similar results for conventional representation zeta functions, for instance, in [33] and [3]. Our result is a consequence of a general discussion of certain zeta functions which are obtained as an infinite series of Igusa integrals, similar to the one appearing in part (2) of Proposition C. We think that this approach is of independent interest, since it is very flexible and provides a new perspective on the abscissa of convergence.

Let $K_0$ be a number field with ring of integers $O_{K_0}$. Fix a finite set $S$ of maximal ideals of $O_{K_0}$ and let $O_{K_0,S} = \{ a \in K_0 \mid |a|_p \leq 1 \text{ for } p \notin S \}$ denote the ring of $S$-integers in $K_0$. We consider number fields $K$ that arise as finite extensions of $K_0$. Let $O_S = O_{K,S}$ denote the integral closure of $O_{K_0,S}$ in $K$. For a maximal ideal $p \subseteq O_S$ we write $\pi_p = O_S/p$ for the residue field and denote its cardinality by $q_p$. Furthermore, $O_p = O_{K,S,p}$ denotes the completion of $O_S$ with respect to $p$ and we fix a uniformiser $\pi_p$ so that $O_p$ has the valuation ideal $\pi_p O_p$.

Let $g$ be an $O_{K_0,S}$-Lie lattice. For every finite extension $K$ of $K_0$ and every maximal ideal $p \subseteq O_S$ we consider the $p$-Lie lattice $g_p = O_p \otimes_{O_{K_0,S}} g$ and, for $r \in \mathbb{N}_0$, its principal congruence sublattices $g_{p,r} = \pi_p^r g_p$. For any given $K$ and $p$, the Lie lattice $g_{p,r}$ is potent for all sufficiently large integers $r$ so that $G_{p,r} = \exp(g_{p,r})$ is a potent pro-$p$ group; we say that such $r$ are permissible for $g_p$.

**Theorem D.** As in the set-up described above, let $g$ be an $O_{K_0,S}$-Lie lattice and let $h \subseteq g$ be a Lie sublattice of codimension $m + 1 = \dim_{O_{K_0,S}} g - \dim_{O_{K_0,S}} h$. Suppose that $h$ is a direct summand as a submodule of the $O_{K_0,S}$-module $g$.

For maximal ideals $p \subseteq O_{K,S}$, where $K$ ranges over finite extensions of $K_0$, and for positive integers $r$ that are permissible for $g_p$ and $h_p$, we consider the induced representation

$$g_{p,r} = \text{Ind}_{h_{p,r}}^{G_{p,r}}(1_{h_{p,r}})$$

associated to the pro-$p$ groups $G_{p,r} = \exp(g_{p,r})$ and $H_{p,r} = \exp(h_{p,r})$.

1. For each $p$ there is a complex-valued function $Z_p$ of a complex variable $s$ that is rational in $q_p^{-s}$ with integer coefficients so that for all permissible $r$,

$$\zeta_{g_{p,r}}(s) = (1 - q_p^{-1}) q_p^{-m+1} Z_p(s).$$

2. The real parts of the poles of the functions $Z_p$, for all $p$, form a finite subset $P_{p,h} \subseteq \mathbb{Q}$.

3. There is a finite extension $K_1$ of $K_0$ such that, for all maximal ideals $p \subseteq O_{K,S}$, arising for extensions $K_1 \subseteq K$, and for all permissible $r$, the abscissa of convergence of the zeta function $\zeta_{g_{p,r}}$ satisfies

$$\alpha(\zeta_{g_{p,r}}) = \max P_{p,h}.$$

4. There are a rational function $F \in \mathbb{Q}(Y_1, Y_2, X_1, \ldots, X_p)$ and a finite set $T$ of maximal ideals of $O_{K_0}$ with $S \subseteq T$ such that the following holds:

For every maximal ideal $p_0 \subseteq O_{K_0}$ not contained in $T$ there are algebraic integers $\lambda_1 = \lambda_1(p_0), \ldots, \lambda_q = \lambda_q(p_0) \in \mathbb{C}^*$ so that for every finite extension $K$ of $K_0$ and every maximal ideal $p$ dividing $p_0$,

$$Z_p(s) = F(q^{-f_1}, q^{-f_q}, \lambda_1^{f_1}, \ldots, \lambda_q^{f_q}),$$
where \( q = q_{\mathfrak{p}_0} \) and \( f = [\kappa_{\mathfrak{p}} : \kappa_{\mathfrak{p}_0}] \) denotes the inertia degree of \( \mathfrak{p} \) over \( K_0 \).

Furthermore the following functional equation holds:

\[
\zeta_{\mathfrak{p},r}(s) \big|_{q \to q^{-1}} = q^{f(m+1)(1-2r)} \zeta_{\mathfrak{p},r}(s).
\]

The theorem is a consequence of our general discussion of certain zeta functions in Section 5; these functions are defined as infinite series of Igusa integrals, similar to the one in part (2) of Proposition \( \mathbf{C} \). This class of zeta functions is comparatively general and it is well suited to our applications. The proof of the rationality relies on Hironaka’s resolution of singularities, which yields a formula of Denef-type. Moreover, the functional equation is obtained along the lines of Denef and Meuser [18] and Voll [54]. The zeta functions treated here differ slightly from the ones considered before and our account provides a simplification of the still more general discussion in [54]. Concerning the abscissa of convergence we include a new approach which relies on the specific shape the zeta functions considered here. In Corollary 5.9 we deduce from statement (3) in Theorem \( \mathbf{D} \) that the abscissa of convergence is attained on a set of primes with positive Dirichlet density; indeed, statement (3) can be seen as a strengthening of [3, Theorem B].

In Theorem 6.3 we record an interesting class of examples, arising geometrically from distance-transitive actions of profinite groups on spherically homogeneous rooted trees. This result is based on ideas of Bekka, de la Harpe and Grigorchuk discussed in the appendix of [9].

**Theorem E.** Let \( T_m \) be a spherically homogeneous rooted tree with branching sequence \( m = (m_n)_{n \in \mathbb{N}} \) in \( \mathbb{N}_{\geq 2} \). Let \( G \leq \text{Aut}(T_m) \) act distance transitively on the boundary \( \partial T_m \), and let \( P = P_\xi \leq G \) be the stabiliser of a point \( \xi \in \partial T_m \).

Then \((G, P)\) is a Gelfand pair; more precisely, the induced representation \( \rho_\theta = \text{Ind}_G^P(1_P) \) decomposes as a direct sum of the trivial representation and a unique irreducible constituent \( \pi_n \) of dimension \( (m_n - 1) \prod_{j=1}^{n-1} m_j \) for each \( n \geq 1 \). In particular, the zeta function of \( \rho_\theta \) is

\[
\zeta_{\rho_\theta}(s) = 1 + \sum_{i=1}^{\infty} (m_i - 1)^{-s} \prod_{j=1}^{i-1} m_j^{-s}
\]

with abscissa of convergence \( \alpha(\rho_\theta) = 0 \).

Recall that \((G, P)\) is a Gelfand pair if the induced representation \( \text{Ind}_G^P(1_P) \) is multiplicity-free; for an introduction to Gelfand pairs we refer the reader to [26]. The zeta function of an induced representation of a Gelfand pair \((G, P)\) enumerates a subset of all irreducible representations of \( G \), which suggests that Gelfand pairs are of special interest in our setting. As an application of these new methods we obtain in Proposition 6.5 an explicit formula for the zeta functions associated to representations induced from maximal \((1, n)\)-parabolic subgroups to \( \text{GL}_{n+1}(\mathfrak{o}) \), where \( \mathfrak{o} \) is a compact discrete valuation ring. This approach works in arbitrary characteristic and complements the Kirillov orbit method described above.
1.4. Further discussion and open problems.

1.4.1. Our investigations are naturally related to the representation theory of reductive groups over $p$-adic fields or non-archimedean local fields in general; compare Remark 2.3. For simplicity, consider the reductive group $G = \text{GL}_n(\mathbb{Q}_p)$. We would like to gain a detailed understanding of the smooth irreducible representations of $G$, which match via the local Langlands correspondence, established by Harris–Taylor [29] and Henniart [31], with certain representations of the Weil–Deligne group of $\mathbb{Q}_p$. Consider a smooth irreducible representation $\rho$ of $G$ and its restriction $\rho|_K$ to the maximal compact subgroup $K = \text{GL}_n(\mathbb{Z}_p)$. The following question is fundamental: describe the decomposition of $\rho|_K$ into irreducible components. For $n = 2$, Casselman [14] obtain detailed results, but for larger degrees an answer seems to be unknown.

Already specific cases are of interest. For instance, the unramified principal series representations constitute a large class of smooth irreducible representations of $G$. These representations are induced to $G$ from 1-dimensional representations on a Borel subgroup $B$. As $KB = G$, the restriction of such a representation to $K$ is induced from $K \cap B$. The study of zeta functions of induced representations thus provides a quantitative approach to the decomposition problem for unramified principal series representations. The decomposition into irreducible $\text{GL}_3(\mathbb{Z}_p)$-constituents of unramified principal series representations of $\text{GL}_3(\mathbb{Q}_p)$ was determined by Onn and Singla [43], completing results of Campbell and Nevins [13]. They used direct representation-theoretic considerations and, as a consequence, deduced a formula for the associated zeta function. It is interesting to conduct similar investigations for supercuspidal representations of $G$; Nevins [42] has made steps in this direction.

1.4.2. In this article we consider, for simplicity, only smooth representations of profinite groups $G$ over the complex field $\mathbb{C}$. More generally, one could derive many of our results also for representations over fields $\mathbb{F}$ that are not necessarily algebraically closed and possibly have positive characteristic. Indeed, if the base field $\mathbb{F}$ is not algebraically closed, one can incorporate without much trouble the action of the absolute Galois group into the treatment; it is not so clear how to deal with Schur indices, but there is no extra complication in the important case, where $G$ is a pro-$p$ group for an odd prime $p$. Furthermore, if $\mathbb{F}$ has positive characteristic $\ell > 0$, there is no problem provided that $G$ has trivial pro-$\ell$-Sylow subgroup.

1.4.3. Dirichlet generating functions have also been employed to study the distribution of finite dimensional irreducible representations of finitely generated nilpotent groups. For such a group $\Gamma$ one defines and studies the zeta function enumerating twist-isoclasses of irreducible representations of $\Gamma$; for instance, see [52, 46, 21]. Many theorems on representation zeta functions, e.g., regarding rationality, pole spectra and functional equations, have analogues in the twist-isoclass setting; the Kirillov orbit method can be adjusted to take into
account twist-isoclasses and thus yields a basic tool. It is natural to investigate – in analogy to our approach in this paper – zeta functions \( \zeta_\psi \) associated to infinite dimensional twist-invariant representations \( \psi \) of a finitely generated nilpotent group \( \Gamma \) that are completely reducible into finite dimensional irreducible constituents: the Dirichlet series \( \zeta_\psi \) encodes the finite multiplicities of entire twist-isoclasses rather than individual constituents. Natural examples occur again in the form of arithmetic groups: the representation \( \psi \) spanned by all finite dimensional subrepresentations of the (co-)induced representation \( \text{Ind}_\Delta^\Gamma (1_\Delta) \), where \( \Gamma = G(O) \) for a unipotent affine group scheme \( G \) over the ring of integers \( O \) of a number field, and \( \Delta = H(O) \) for a subgroup \( H \subseteq [G, G] \).

1.4.4. Our results on induced representations of compact \( p \)-adic Lie groups can be seen as a starting point for further investigations. In particular, the general setting of induced representations provides many new tractable examples and significantly more flexibility to vary input parameters. To indicate possible future directions we formulate the following concrete open questions.

**Problem 1.1.** Jaikin-Zapirain [33, Section 7] computed the conventional representation zeta function of \( \text{SL}_2(R) \), where \( R \) is a compact discrete valuation ring of odd residue characteristic, and observed that it only depends on the residue field cardinality and not on the characteristic or isomorphism type of \( R \) itself. The same phenomenon occurs in our setting, for zeta functions associated to induced representations; see [43, Theorem 6.5] and Proposition 6.5 below. Find further examples or, possibly, counter-examples of this phenomenon in the context of induced representations. Give an explanation of the invariance of the zeta function where it holds.

**Problem 1.2.** One of the key invariants of a representation zeta function is its abscissa of convergence. The abscissa of the conventional representation zeta function is explicitly known for FAb compact \( p \)-adic Lie groups of small dimension and fully understood for norm-1 groups of \( p \)-adic division algebras; see [33, 3, 58] and [38, Theorem 7.1]. However, there is no general interpretation, even at the conjectural level. In our new setting, Propositions 7.1, 7.2 and Theorem 7.3 provide the abscissae of convergence of zeta functions associated to some families of induced representations.

Determine the abscissae for further families of induced representations. For instance, consider representations of open compact subgroups of reductive \( p \)-adic groups that are induced from (maximal) parabolic subgroups. Can the results be explained in terms of the root systems?

**Problem 1.3.** The conventional representation zeta functions of \( \text{SL}_3(\mathfrak{o}) \) and \( \text{SU}_3(\mathfrak{o}) \) over a compact discrete valuation ring \( \mathfrak{o} \) of characteristic 0 display an Ennola-type duality; see [3, 4]. The examples of induced representations in Propositions 7.1 and 7.2 do not suggest a similar form of duality, but form merely the beginning of a comprehensive analysis.
Study systematically the zeta functions of induced representations in groups of Type $A_2$ and determine when an Ennola-type duality holds. Find an explanation for this duality.

1.5. **Organisation.** In Section 2 we recall some standard notions from the representation theory of totally disconnected locally compact groups. Based on suitable refinements, we introduce zeta functions associated to strongly admissible smooth representations. In Section 3 we study the robustness of strong admissibility under the induction functor and introduce in this context the notion of relative $F_{\text{Ab}}$-ness. In Section 4 we develop a $p$-adic formalism to compute the zeta functions of induced representations between uniform pro-$p$ groups. Our main tool is the Kirillov orbit method, with special care taken for $p = 2$. In Section 5 we establish the rationality of the local factors of globally defined families of induced representations; we also study their abscissae of convergence and prove functional equations. In Section 6 we produce, with surprisingly little effort, instructive examples of zeta functions of induced representations, arising geometrically for groups acting on rooted trees. In Section 7 we give many concrete examples of zeta functions of induced representations of potent pro-$p$ groups.

2. **Basic concepts and preliminaries**

In this section we recall some standard notions from the representation theory of totally disconnected locally compact groups; see [12, 45]. Based on suitable refinements, we introduce zeta functions associated to strongly admissible smooth representations.

2.1. **Admissible smooth representations.** Let $G$ be a totally disconnected locally compact topological group; for instance, $G$ could be a reductive $p$-adic group such as $G = \text{GL}_n(\mathbb{Q}_p)$. Observe that $G$ is automatically Hausdorff, as $\{1\} \leq_c G$. A complex representation of $G$ is given by a homomorphism $\varrho: G \to \text{GL}(V_\varrho)$, where $V_\varrho$ is a vector space over $\mathbb{C}$. All representations we consider are over $\mathbb{C}$, and we usually drop the specification ‘complex’; compare Section 1.4.2.

Depending on the situation, it is convenient to denote a representation of $G$ either by the vector space $V_\varrho$ acted upon, or by the homomorphism $\varrho: G \to \text{GL}(V_\varrho)$, or by the pair $(\varrho, V_\varrho)$.

A vector $v \in V_\varrho$ is said to be smooth with respect to $\varrho$ if its stabiliser $\text{Stab}_G(v) = \{g \in G \mid \varrho(g).v = v\}$ is open in $G$. The vector subspace $V_\varrho^\infty = \{v \in V_\varrho \mid v \text{ is smooth with respect to } \varrho\}$ is $\varrho$-invariant and gives rise to a sub-representation $\varrho^\infty = (\varrho^\infty, V_\varrho^\infty)$. The representation $\varrho$ is said to be smooth if $V_\varrho = V_\varrho^\infty$; equivalently, $\varrho$ is smooth if the map $G \times V_\varrho \to V_\varrho$ is continuous when $G$ is equipped with its natural topology and $V_\varrho$ is equipped with the discrete topology.

We recall that the totally disconnected compact topological groups are precisely the profinite groups. The assertions of the following lemma are easy to prove and well known; see [12 Sec. 2.2].
Lemma 2.1. Let $G$ be a profinite group.

1. Every smooth representation of $G$ is semisimple, i.e., it decomposes as a direct sum of (smooth) irreducible constituents.

2. The smooth irreducible representations of $G$ are precisely the finite dimensional irreducible continuous representations of $G$; in particular, each of these factors through a finite continuous quotient of $G$.

Remark 2.2. The lemma has the following consequence for a totally disconnected locally compact topological group $G$. If $(\varrho, V_\varrho)$ is a smooth representation of $G$ and if $K \leq_o G$ is a compact open subgroup of $G$ then the restriction $\varrho|_K$ is semisimple and all its irreducible constituents are finite dimensional. If, in addition, $(\varrho, V_\varrho)$ is irreducible and $|G : K|$ countable then $V_\varrho$ is countable. A typical example of this situation is: $G = \GL_n(\Q_p)$ and $K = \GL_n(\Z_p)$.

A smooth representation $(\varrho, V_\varrho)$ of a totally disconnected locally compact topological group $G$ is said to be admissible if for every compact open subgroup $K \leq_o G$ the space of fixed vectors $V_\varrho^K = \{ v \in V_\varrho \mid \forall g \in K : \varrho(g).v = v \}$ has finite dimension.

Remark 2.3. According to a classical result in the representation theory of reductive $p$-adic groups, every smooth irreducible representation of a reductive $p$-adic group $G$ is admissible; see [53, Ch. II, 2.8]. It is worth noting that this result holds in great generality. Firstly, it applies to irreducible representations over any – not necessarily algebraically closed [10, Prop. 2] – field of characteristic different from $p$. Secondly, the statement remains valid, when $G$ is the group of rational points of a reductive group over a local field of positive characteristic. In view of Remark 2.2 this provides a wide range of interesting admissible smooth representations of profinite groups, where it is natural to study the multiplicities of irreducible components.

2.2. Polynomialsy strongly admissible representations. Let $G$ be a profinite group. We denote by $\Irr(G)$ the set of (isomorphism classes of) smooth irreducible – i.e. finite dimensional irreducible continuous – representations of $G$. For simplicity, we usually do not distinguish notationally between representations and isomorphism classes of representations. It is easy to see that a representation $(\varrho, V_\varrho)$ of $G$ is admissible smooth if and only if it decomposes as a direct sum

$$V_\varrho \cong \bigoplus_{\pi \in \Irr(G)} m(\pi, \varrho) \cdot V_\pi,$$

where the multiplicity $m(\pi, \varrho)$ of $\pi$ in $\varrho$ is finite for each $\pi \in \Irr(G)$.

We say that an admissible smooth representation $(\varrho, V_\varrho)$ of $G$ is strongly admissible if, for every $d \in \mathbb{N}$, the number

$$R_d(\varrho) = \sum_{\begin{subarray}{c} \pi \in \Irr(G) \\ \dim \pi \leq d \end{subarray}} m(\pi, \varrho)$$
of its irreducible constituents of dimension at most $d$ is finite. Furthermore, we define the zeta function of a strongly admissible smooth representation $(\varrho, V_\varrho)$ of $G$ to be the formal Dirichlet series

$$\zeta_\varrho(s) = \sum_{\pi \in \text{Irr}(G)} m(\pi, \varrho)(\dim \pi)^{-s},$$

where $s$ denotes a complex variable.

**Remark 2.4.** If $G$ is finitely generated as a profinite group, it is known that $G$ has only finitely many irreducible representations of any given finite dimension if and only if $G$ is $\text{FAb}$, meaning that every open subgroup $H \leq_o G$ has finite abelianisation $H/[H,H]$; compare [7]. Consequently, for a $\text{FAb}$ finitely generated profinite group every admissible smooth representation is strongly admissible.

We say that $(\varrho, V_\varrho)$ is polynomially strongly admissible if there exists $r \in \mathbb{R}_{>0}$ such that $R_d(\varrho) = O(d^r)$, i.e., if there are $r, C \in \mathbb{R}_{>0}$ such that for every $d \in \mathbb{N}$,

$$\sum_{\pi \in \text{Irr}(G)} m(\pi, \varrho) = R_d(\varrho) \leq C d^r.$$

Equivalently, $(\varrho, V_\varrho)$ is polynomially strongly admissible if its (polynomial) degree of irreducible constituent growth, defined as

$$\text{deg}(\varrho) = \inf\{r \in \mathbb{R}_{>0} \mid R_d(\varrho) = O(d^r)\} \in \mathbb{R}_{>0} \cup \{\infty\},$$

is finite.

For a polynomially strongly admissible smooth representation $(\varrho, V_\varrho)$ of $G$ the formal Dirichlet series $\zeta_\varrho(s)$ converges (absolutely) and defines an analytic function on a right half-plane $\{s \in \mathbb{C} \mid \text{Re}(s) > \alpha(\varrho)\}$, where $\alpha(\varrho) \in \mathbb{R} \cup \{-\infty\}$ denotes the abscissa of convergence. In fact, if $\varrho$ is infinite dimensional then $\alpha(\varrho) = \text{deg}(\varrho)$.

**Example 2.5.** The profinite group $G$ acts via right translations on the space $V = \mathcal{C}^\infty(G, \mathbb{C})$ of all locally constant functions, i.e. continuous functions when $\mathbb{C}$ is equipped with the discrete topology, from $G$ to $\mathbb{C}$:

$$(^g f)(x) = f(xg) \quad \text{for } g \in G, f \in \mathcal{C}^\infty(G, \mathbb{C}), x \in G.$$  

The resulting ‘regular’ representation $g_{\text{reg}}$ is admissible smooth if $G$ has only finitely many irreducible representations of any given finite dimension. As remarked above, this is for instance the case if $G$ is finitely generated and $\text{FAb}$. In this case the formal Dirichlet series

$$\zeta_{g_{\text{reg}}}(s) = \sum_{\pi \in \text{Irr}(G)} (\dim \pi)^{1-s}$$

is equal to $\zeta_G(s - 1)$, a shift of the conventional representation zeta function $\zeta_G(s)$ of $G$ that enumerates irreducible complex representations of $G$ and has been the sole focus of study until now; compare Section [II].
Finally, we take interest in yet another finiteness condition. An admissible smooth representation $\rho$ of the profinite group $G$ is said to have \textit{bounded multiplicities}, if there exists $M \in \mathbb{N}$ such that $m(\pi, \rho) \leq M$ for all $\pi \in \text{Irr}(G)$.

\textbf{Remark 2.6.} This concept arises naturally in two ways. Firstly, there are prominent examples, such as multiplicity-free representations associated to Gelfand pairs; compare Section \[. Secondly, the abscissa of convergence of the zeta function associated to any smooth representation $\rho$ of $G$ with bounded multiplicities yields a lower bound for the abscissa of convergence of the conventional representation zeta function of $G$.

\textbf{Definition 2.7 (Admissibility properties).} We refer to the properties of being admissible, strongly admissible, polynomially strongly admissible or having bounded multiplicities collectively as \textit{properties of type (A)}.

\section{2.3. Tensor products and induced representations.}

Let $G$ be a profinite group. The \textit{contragredient representation} $\rho^\vee$ of a smooth representation $\rho$ of $G$ is the smooth part $(\rho^*)^\infty$ of the abstract dual representation $\rho^*$. The \textit{tensor product} $(\rho \otimes \vartheta, V \otimes_C U)$ of two smooth representations $(\rho, V)$ and $(\vartheta, U)$ of $G$, defined via $(\rho \otimes \vartheta)(g)(v \otimes u) = \rho(g).v \otimes \vartheta(g).u$ for $g \in G$, $v \in V$, $u \in U$, is a smooth representation of $G$. By considering the tensor product $\rho \otimes \rho^\vee$ of an infinite dimensional admissible smooth representation $\rho$ with its contragredient representation $\rho^\vee$, one sees that admissibility need not be preserved under tensor products.

Let $H \leq_c G$ be a closed subgroup. The \textit{restriction} $\text{Res}_H^G(\vartheta)$ of a smooth representation $\vartheta$ of $G$ is a smooth representation of $H$, but, clearly, admissibility need not be preserved. Conversely, the \textit{induced representation} $\rho = \text{Ind}_H^G(\sigma)$ of a smooth representation $(\sigma, W)$ of $H$ is constructed as follows: $G$ acts on $V_\rho = \{f \in C^\infty(G, W) \mid \forall h \in H \forall x \in G: f(hx) = \sigma(h).f(x)\}$ via right translation $(\rho(g).f)(x) = (\sigma f)(x) = f(xg)$ for $g, x \in G, f \in V_\rho$.

In calling the representation ‘induced’ rather than ‘co-induced’, we follow \cite{49} I.\$2.5$ rather than \cite{18} VII.\$6$. We note that in the context of profinite groups the notions of induced, compactly induced and co-induced representations are in fact all the same. The next proposition is a consequence of Frobenius reciprocity; compare \cite{12} Sec. 2.4.

\textbf{Proposition 2.8.} Let $G$ be a profinite group and $H \leq_c G$ a closed subgroup. If $\sigma$ is an admissible smooth representation of $H$, then $\text{Ind}_H^G(\sigma)$ is an admissible smooth representation of $G$.

The induction functor is one of the key tools to construct new representations from known ones and, in particular, interesting infinite dimensional representations from finite dimensional ones.
3. Twist similarity classes and admissibility properties

Throughout this section $G$ denotes a profinite group and $H \leq_c G$ a closed subgroup. Given an admissible smooth representation $(\sigma, W)$ of $H$, we are keen to study the zeta function $\zeta_\varrho$ attached to the induced representation $\varrho = \Ind_H^G(\sigma)$. With this aim we develop conditions which ensure that $\varrho$ is (polynomially) strongly admissible. Furthermore, we establish that the abscissa of convergence $\alpha(\varrho)$ of $\zeta_\varrho$ is rather robust: it is, in fact, an invariant of the twist similarity class of $\sigma$, defined below.

3.1. Tensor products and twist similarity. Let $G$ be a profinite group. We say that two smooth representations $(\varrho, V)$ and $(\varrho', V')$ of $G$ are twist similar to one another if there are non-zero finite dimensional smooth representations $(\vartheta, U)$ and $(\vartheta', U')$ and a homomorphism $f : V \otimes_C U \to V' \otimes_C U'$ of $G$-representations with finite dimensional kernel and cokernel. Note that, in particular, all finite dimensional smooth representations of $G$ are twist similar to one another.

**Proposition 3.1.** Let $\varrho, \vartheta$ be a smooth representations of a profinite group $G$, and suppose that $1 \leq \dim \vartheta < \infty$. Then $\varrho$ has a property of type (A) if and only if $\varrho \otimes \vartheta$ has the same property.

Moreover, if $\varrho$ and $\varrho \otimes \vartheta$ are polynomially strongly admissible then $\zeta_\varrho$ and $\zeta_{\varrho \otimes \vartheta}$ have the same abscissa of convergence: $\alpha(\varrho) = \alpha(\varrho \otimes \vartheta)$.

**Proof.** Note that each property of type (A) is inherited by sub-representations. Denoting by $\vartheta^\vee$ the contragredient representation of $\vartheta$ acting on $V_{\vartheta} = V_{\vartheta}^\vee$, we observe that the trivial representation $\mathbb{1}_G$ occurs in $\vartheta \otimes \vartheta^\vee$. Hence $\varrho$ is a sub-representation of $\varrho \otimes \vartheta \otimes \vartheta^\vee$. Consequently, it suffices to show:

(i) if $\varrho$ has a property of type (A) then $\varrho \otimes \vartheta$ has the same property;
(ii) if $\varrho$ and $\varrho \otimes \vartheta$ are polynomially strongly admissible then $\alpha(\varrho \otimes \vartheta) \leq \alpha(\varrho)$.

For smooth irreducible representations $\sigma, \pi \in \Irr(G)$, we recall that

$$\text{Hom}_G(V_\sigma \otimes_C V_\vartheta, V_\pi) \cong \text{Hom}_G(V_\vartheta, \text{Hom}(V_\sigma, V_\pi)) \cong \text{Hom}_G(V_\vartheta, V_\sigma \otimes_C V_\pi)$$

as $\mathbb{C}$-vector spaces. Taking dimensions, we deduce from Schur’s lemma that the multiplicity of $\sigma$ in $\pi \otimes \vartheta$ and the multiplicity of $\pi$ in $\sigma \otimes \vartheta^\vee$ are the same:

$$m(\sigma, \pi \otimes \vartheta) = m(\pi, \sigma \otimes \vartheta^\vee). \hspace{1cm} (3.1)$$

Fix $\sigma \in \Irr(G)$ and, for $\pi \in \Irr(G)$, write $\pi \mid \sigma \otimes \vartheta^\vee$ to indicate that $\pi$ is a constituent of $\sigma \otimes \vartheta^\vee$, equivalently, that $m(\pi, \sigma \otimes \vartheta^\vee) > 0$. If $\varrho$ is admissible, then (2.1) and (3.1) yield:

$$m(\sigma, \varrho \otimes \vartheta) = \sum_{\pi \in \Irr(G) \atop \sigma | \pi \otimes \vartheta} m(\pi, \varrho) m(\sigma, \pi \otimes \vartheta) = \sum_{\pi \in \Irr(G) \atop \pi | \sigma \otimes \vartheta^\vee} m(\pi, \varrho) m(\sigma, \pi \otimes \vartheta). \hspace{1cm} (3.2)$$

Since $\sigma \otimes \vartheta^\vee$ and all $\pi \otimes \vartheta$ are finite dimensional, this shows: if $\varrho$ is admissible (respectively strongly admissible) then $\varrho \otimes \vartheta$ is admissible (respectively strongly admissible).
For all $\sigma, \pi \in \text{Irr}(G)$ with $\sigma \mid \pi \otimes \vartheta$ we infer from (3.1) that
\[(3.3) \quad \dim \sigma \leq (\dim \pi)(\dim \vartheta) \quad \text{and} \quad \dim \pi \leq (\dim \sigma)(\dim \vartheta).\]

If $\varrho$ has multiplicities bounded by $M$, then (3.2), (3.1) and the first inequality in (3.3) yield
\[m(\sigma, \varrho \otimes \vartheta) = \sum_{\pi \in \text{Irr}(G)} m(\pi, \varrho) m(\sigma, \pi \otimes \vartheta) \leq M \sum_{\pi \in \text{Irr}(G)} m(\pi, \varrho) \sum_{\sigma \in \text{Irr}(G)} m(\sigma, \pi \otimes \vartheta)(\dim \pi) \frac{\dim \vartheta}{\dim \sigma} \leq M(\dim \vartheta)^2,
\]
so that $\varrho \otimes \vartheta$ has multiplicities bounded by $M(\dim \vartheta)^2$.

Finally, suppose that $\varrho$ is polynomially strongly admissible, so that $\zeta_{\varrho}(s)$ converges absolutely for $\text{Re}(s) > \alpha(\varrho)$. Without loss of generality we may assume that $\varrho$ is infinite dimensional and consequently $\alpha(\varrho) \geq 0$. For all real $s > \alpha(\varrho)$, we use (3.2) and the second inequality in (3.3) to obtain
\[\zeta_{\varrho \otimes \vartheta}(s) = \sum_{\sigma \in \text{Irr}(G)} m(\sigma, \varrho \otimes \vartheta)(\dim \sigma)^{-s} = \sum_{\pi \in \text{Irr}(G)} m(\pi, \varrho) \sum_{\sigma \in \text{Irr}(G)} m(\sigma, \pi \otimes \vartheta)(\dim \sigma)^{-s} \leq \sum_{\pi \in \text{Irr}(G)} m(\pi, \varrho) \sum_{\sigma \in \text{Irr}(G)} m(\sigma, \pi \otimes \vartheta)(\dim \sigma) \frac{(\dim \vartheta)^{s+1}}{(\dim \pi)^{s+1}} = \sum_{\pi \in \text{Irr}(G)} m(\pi, \varrho) \frac{(\dim \vartheta)^{s+2}}{(\dim \pi)^{s}} = (\dim \vartheta)^{s+2} \zeta_{\varrho}(s).
\]

Consequently, $\varrho \otimes \vartheta$ is polynomially admissible and $\alpha(\varrho \otimes \vartheta) \leq \alpha(\varrho)$. \qed

**Corollary 3.2.** Let $\varrho$ be a smooth representation of a profinite group $G$. The properties of type (A) that hold for $\varrho$ are common properties of the twist similarity class of $\varrho$. If $\varrho$ is polynomially strongly admissible, then also the abscissa of convergence $\alpha(\varrho)$ is an invariant of the twist similarity class of $\varrho$.

### 3.2. Induction and restriction of smooth representations

Let $H \leq_G G$ be a closed subgroup of a profinite group $G$. Clearly, the restriction functor $\text{Res}^G_H$ preserves twist similarity classes. Hence, for any smooth representation $\varrho$ of $G$, the properties of type (A) of $\text{Res}^G_H(\varrho)$ only depend on the twist similarity class of $\varrho$. The next proposition shows that this conclusion also holds for the induction functor.
Proposition 3.3. Let $H \leq_c G$ be a closed subgroup of the profinite group $G$, and let $\sigma, \sigma'$ be twist similar non-zero smooth representations of $H$. If $\text{Ind}_H^G(\sigma)$ has a property of type (A) then $\text{Ind}_H^G(\sigma')$ has the same property.

Furthermore, if $\text{Ind}_H^G(\sigma)$ is polynomially strongly admissible then the abscissa of convergence $\alpha(\text{Ind}_H^G(\sigma))$ depends on $\sigma$ only up to twist similarity.

Proof. Properties of type (A) are inherited by sub-representations. Hence, by an argument similar to the one starting the proof of Proposition 3.1, it suffices to consider the special cases (i) $\sigma' = \sigma \otimes \tau$ and (ii) $\sigma' = \sigma \oplus \tau$ for a finite dimensional smooth representation $\tau$ of $H$; furthermore, it is enough to show that, if $\text{Ind}_H^G(\sigma)$ is polynomially strongly admissible in these cases, then $\alpha(\text{Ind}_H^G(\sigma')) \leq \alpha(\text{Ind}_H^G(\sigma))$.

First suppose that $\sigma' = \sigma \otimes \tau$. We choose a finite dimensional smooth representation $\vartheta$ of $G$ such that $\tau$ injects into the restriction $\text{Res}_H^G(\vartheta)$. Then $\text{Ind}_H^G(\sigma') = \text{Ind}_H^G(\sigma \otimes \tau)$ injects into $\text{Ind}_H^G(\sigma \otimes \text{Res}_H^G(\vartheta)) \cong \text{Ind}_H^G(\sigma) \otimes \vartheta$, and Proposition 3.1 implies that, if $\text{Ind}_H^G(\sigma)$ has a property of type (A), then $\text{Ind}_H^G(\sigma \otimes \tau)$ has the same property. Furthermore, if $\text{Ind}_H^G(\sigma)$ is polynomially strongly admissible then $\alpha(\text{Ind}_H^G(\sigma')) \leq \alpha(\text{Ind}_H^G(\sigma) \otimes \vartheta) = \alpha(\text{Ind}_H^G(\sigma))$.

Now suppose that $\sigma' = \sigma \oplus \tau$. Choose a smooth irreducible constituent $\pi$ of $\sigma$ and observe that $\tau$ is a sub-representation of $\sigma \otimes \pi^\vee \otimes \tau$. If $\text{Ind}_H^G(\sigma)$ has a property of type (A) then, by the argument above, the same property holds for $\text{Ind}_H^G(\sigma \otimes \pi^\vee \otimes \tau)$ and hence for the sub-representation $\text{Ind}_H^G(\tau)$. Furthermore, if $\text{Ind}_H^G(\sigma)$ is polynomially strongly admissible then we obtain $\alpha(\text{Ind}_H^G(\sigma')) = \alpha(\text{Ind}_H^G(\sigma) \oplus \text{Ind}_H^G(\tau)) = \max\{\alpha(\text{Ind}_H^G(\sigma)), \alpha(\text{Ind}_H^G(\tau))\} \leq \max\{\alpha(\text{Ind}_H^G(\sigma)), \alpha(\text{Ind}_H^G(\sigma \otimes \pi^\vee \otimes \tau))\} = \alpha(\text{Ind}_H^G(\sigma)).$ □

The following result generalises [39, Cor. 2.3].

Proposition 3.4. Let $H \leq_o G$ be an open subgroup of the profinite group $G$. The functors $\text{Ind}_H^G$ and $\text{Res}_H^G$ preserve all the properties of type (A). In addition, the functors preserve the abscissa of convergence for polynomially strongly admissible representations.

Proof. We only discuss the polynomially strongly admissible case; the other properties follow from similar arguments based on Frobenius reciprocity.

Let $\sigma \in \text{Irr}(H)$ and $\pi \in \text{Irr}(G)$ be smooth irreducible representations of $H$ and $G$ respectively. If $\pi$ occurs in $\text{Ind}_H^G(\sigma)$, or equivalently $\sigma$ occurs in $\text{Res}_H^G(\pi)$, then $\dim \sigma \leq \dim \pi \leq |G : H| \dim \sigma$.

For all $s \in \mathbb{R}_{\geq 0}$ this yields the inequalities

\begin{equation}
\frac{1}{(|G : H| \dim \sigma)^s} \leq \sum_{\hat{s} \in \text{Irr}(G)} \frac{m(\hat{\pi}, \text{Ind}_H^G(\sigma))}{(\dim \hat{\pi})^s} \leq \frac{|G : H|}{(\dim \sigma)^s},
\end{equation}

\begin{equation}
\frac{1}{(\dim \pi)^s} \leq \sum_{\hat{\sigma} \in \text{Irr}(H)} \frac{m(\hat{\sigma}, \text{Res}_H^G(\pi))}{(\dim \hat{\sigma})^s} \leq \frac{|G : H|^{s+1}}{(\dim \pi)^s}.
\end{equation}
Let \( \vartheta \) be a polynomially strongly admissible representation of \( H \). If \( \vartheta \) is finite dimensional so is \( \text{Ind}_H^G(\vartheta) \), and there is nothing further to show. Suppose that \( \vartheta \) has infinite dimension so that \( \alpha(\vartheta) \geq 0 \). Using (3.4), we deduce that

\[
|G : H|^{-s} \zeta_{\vartheta}(s) \leq \zeta_{\text{Ind}_H^G(\vartheta)}(s) \leq |G : H| \zeta_{\vartheta}(s)
\]

for all \( s \in \mathbb{R}_{\geq 0} \) so that \( \alpha(\vartheta) = \alpha(\text{Ind}_H^G(\vartheta)) \).

Similarly, the claim for the restriction functor follows from (3.5). \( \square \)

Example 3.5. While induction from and restriction to an open subgroup preserve properties of type (A), these functors can substantially alter algebraic properties of the associated zeta functions. Therefore our rationality results for zeta functions of induced representations of potent pro-\( p \)-groups in Section 5 (compare Remark 5.2) do not automatically extend to general compact pro-\( p \)-adic Lie groups; it remains an open problem to generalise the results of Jaikin-Zapirain [33].

To illustrate the underlying issue we construct, for every prime \( p \), a compact pro-\( p \)-adic Lie group \( G \), an open uniformly powerful pro-\( p \) subgroup \( H \leq \text{o}_G \) of index 2 and a polynomially strongly admissible representation \( \varrho \) of \( G \) such that \( \zeta_{\text{Res}_G^H(\varrho)}(s) \) is a rational function in \( \mathbb{Q}(p^{-s}) \), whereas \( \zeta_{\varrho}(s) \) cannot be expressed as a rational function in \( \mathbb{Q}\{n^{-s} \mid n \in \mathbb{N}\} \).

Let \( U \) be a uniformly powerful pro-\( p \) group which has an irreducible representation \( \eta_n \) of degree \( p^n \) for every integer \( n \geq 0 \). For instance, take \( U \) to be the 3-dimensional pro-\( p \)-adic Heisenberg group. Define \( H = U \times \mathbb{Z}_p \) and \( G = H \rtimes C_2 \), where the generator \( \tau \) of \( C_2 \) acts as the identity on \( U \) and by inversion on \( \mathbb{Z}_p \).

Fix a 1-dimensional irreducible representation \( \chi \) of \( \mathbb{Z}_p \) such that \( \chi(-t) \neq \chi(t) \) for some \( t \in \mathbb{Z}_p \).

Consider the irreducible \( p^n \)-dimensional representations \( \pi_n = \eta_n \otimes 1 \) and \( \sigma_n = \eta_n \otimes \chi \) of \( H \). Observe that the representation \( \pi_n \) is \( \tau \)-invariant, thus \( \pi_n \) extends to an irreducible representation \( \alpha_n \) of \( G \). However, the construction gives \( \sigma_n \neq \sigma_n^\tau \) and thus yields an irreducible representation \( \beta_n \) of \( G \) of degree \( 2p^n \) such that \( \text{Res}_G^H(\beta_n) = \sigma_n \oplus \sigma_n^\tau \).

Choose a sequence \( (m_n)_{n \in \mathbb{N}} \in \{0, 1\}^\mathbb{N} \) which is not eventually periodic. The smooth representation

\[
\varrho = \bigoplus_{n=0}^{\infty} 2(1 - m_n) \alpha_n \oplus \bigoplus_{n=0}^{\infty} m_n \beta_n
\]

of \( G \) is polynomially strongly admissible. By construction, we have

\[
\text{Res}_G^H(\varrho) = \bigoplus_{n=0}^{\infty} 2(1 - m_n) \pi_n \oplus \bigoplus_{n=0}^{\infty} m_n \sigma_n \oplus \bigoplus_{n=0}^{\infty} m_n \sigma_n^\tau,
\]

and a short calculation yields

\[
\zeta_{\text{Res}_G^H(\varrho)}(s) = \sum_{n=0}^{\infty} 2p^{-ns} = \frac{2}{1 - p^{-s}}.
\]
However, the Dirichlet series defining the zeta function of the representation $\varrho$ converges at 1 and evaluates to

$$\zeta_\varrho(1) = \sum_{n=0}^{\infty} (2 - 2m)p^{-n} + m_n(2p^n)^{-1} = \frac{2}{1 - p^{-1}} - \frac{3}{2} \sum_{n=0}^{\infty} m_n p^{-n}.$$

The sequence $(m_n)_{n \in \mathbb{N}}$ was chosen so that the number $\sum_{n=0}^{\infty} m_n p^{-n}$ is irrational, and we conclude that $\zeta_\varrho(s)$ cannot be expressed by as a rational function in $\{n^{-s} \mid n \in \mathbb{N}\}$ with coefficients in $\mathbb{Q}$.

3.3. **Strong admissibility of induced representations.** Let $H \leq c G$ be a closed subgroup of a profinite group $G$. As recorded in Proposition 2.8, the functor $\text{Ind}_G^H$ preserves admissibility. However, strong admissibility is in general not preserved by induction. For example, the regular representation of a finitely generated profinite group $G$ is strongly admissible if and only if $G$ is FAb. The latter means that every open subgroup $K \leq_o G$ has finite abelisation; see Section 2. We introduce a relative FAb-condition to deal with induced representations in general.

**Definition 3.6.** We say that $G$ is FAb relative to $H$ if for every open subgroup $K \leq_o G$ the abelian quotient $K/(H \cap K)[K, K]$ is finite.

**Remark 3.7.** (a) Note that $G$ is FAb if and only if it is FAb relative to the trivial subgroup. In general, $G$ is FAb relative to a closed normal subgroup $N \leq_c G$ exactly if the group $G/N$ is FAb.

(b) The group $G$ is FAb relative to $H$ if and only if for every open normal subgroup $K \leq_o G$ the abelian quotient $K/(H \cap K)[K, K]$ is finite.

(c) Suppose that $H_1 \leq_c H_2 \leq_c G$ are closed subgroups of $G$. Obviously, if $G$ is FAb relative to $H_1$ then $G$ is FAb relative to $H_2$. The converse holds if $H_1$ is open in $H_2$, because $|H_2[K, K] : H_1[K, K]| \leq |H_2 : H_1|$ for every $K \leq_o G$.

**Theorem 3.8.** Let $H \leq_c G$ be a closed subgroup of a finitely generated profinite group $G$. The following statements are equivalent.

(a) The group $G$ is FAb relative to $H$.

(b) The functor $\text{Ind}_G^H$ preserves strong admissibility.

(c) The induced representation $\text{Ind}_G^H(1_H)$ is strongly admissible.

**Proof.** In order to prove that (a) implies (b), we suppose that $\text{Ind}_G^H(1_H)$ is strongly admissible. All finite dimensional smooth representations of $H$ are twist similar to one another, thus Proposition 2.8 shows that $\text{Ind}_G^H(\sigma)$ is strongly admissible for every finite dimensional smooth representation $\sigma$ of $H$. Let $\vartheta$ be any strongly admissible representation of $H$. For every $d \in \mathbb{N}$ we obtain

$$R_d(\text{Ind}_G^H(\vartheta)) = \sum_{\sigma \in \text{Irr}(H)} m(\sigma, \vartheta) R_d(\text{Ind}_G^H(\sigma)) < \infty,$$

because $\vartheta$ and $\text{Ind}_G^H(\sigma)$ are strongly admissible and, furthermore, $\dim(\sigma) > d$ implies $R_d(\text{Ind}_G^H(\sigma)) = 0$.

We prove that (b) implies (a) by contraposition. Suppose that $G$ is not FAb relative to $H$. Take an open normal $K \leq_o G$ such that for $L = (H \cap K)[K, K]$
the abelian quotient $K/L$ is infinite. The representation $\text{Ind}_L^K(1_L)$ is infinite dimensional and decomposes into 1-dimensional representations of $K$, each occurring with multiplicity 1. In particular, using the embedding $\text{Ind}_L^K(1_L) \hookrightarrow \text{Ind}_{K\cap K}(1_{K\cap K})$, we conclude that $\text{Ind}_{K\cap K}(1_{K\cap K})$ contains an infinite number of distinct 1-dimensional representations. For the finite dimensional representation $\sigma = \text{Ind}_{K\cap K}(1_{K\cap K})$, we conclude that $\text{Ind}_G^K(\sigma) = \text{Ind}_{K\cap K}^G(1_{K\cap K})$ is not strongly admissible as it contains infinitely many irreducible constituents of dimension at most $|G : K|$

Finally, we prove that (m) implies (q), again by contraposition. Suppose that $\text{Ind}_H^K(1_H)$ is not strongly admissible. We find a positive integer $d$ and an infinite sequence $(\pi_i, V_i)_{i=1}^\infty$ of distinct $d$-dimensional smooth irreducible representations of $G$ which occur in $\text{Ind}_H^K(1_H)$. For every $i \in \mathbb{N}$ there is a non-zero vector $v_i$ in the space $V_i$ which is fixed by $H$. Setting $N_i = \ker(\pi_i)$, we observe that $G/N_i$ is a finite subgroup of $GL_d(\mathbb{C})$. By a classical theorem of Jordan (see [15] (36.13) or [19] Thm. 5.7) there exist $m = m(d) \in \mathbb{N}$ and open normal subgroups $A_i \unlhd G$ of index at most $m$ so that $N_i \subseteq A_i$ and $A_i/N_i$ is abelian for all $i \in \mathbb{N}$. As $G$ is finitely generated, it contains only a finite number of open subgroups of index at most $m$. Hence we find $A \leq_o G$ such that $I = \{ i \in \mathbb{N} \mid A = A_i \}$ is infinite. For each $i \in I$ the stabiliser of $v_i$ in $G$ contains the group $H[A, A]$. Therefore the distinct irreducible representations $(\pi_i)_{i \in I}$ all occur in the induced representation $\text{Ind}_{H[A, A]}^G(1_{H[A, A]})$. In particular, this representation is not finite dimensional and the index of $|G : H[A, A]|$ is infinite. We conclude that $G$ is not FAb relative to $H$. \hfill $\square$

3.4. Polynomiably strong admissibility and compact $p$-adic Lie groups. A profinite group $G$ has polynomial representation growth, as defined in [38], if and only if the regular representation of $G$ is polynomiably strongly admissible; see Example 2.5. As yet no simple characterisation of profinite groups of polynomial representation growth is known, not even at a conjectural level. The regular representation is obtained by inducing the trivial representation from the trivial subgroup; on that account we formulate the following more general problem.

**Problem 3.9.** Under what conditions on $H \leq_c G$ does the induction functor $\text{Ind}_H^G$ preserve polynomiably strong admissibility?

The next result and its proof generalise [39] Prop. 2.7. Let $p$ be a prime. We refer to [20] for the relevant structure theory of compact $p$-adic analytic groups.

**Proposition 3.10.** Let $H \leq_c G$ be a closed subgroup of a compact $p$-adic Lie group $G$. If $G$ is FAb relative to $H$ then $\text{Ind}_H^G(\sigma)$ is polynomiably strongly admissible for every finite dimensional smooth representation $\sigma$ of $H$.

**Proof.** We may assume that $H$ and $G$ are uniformly powerful pro-$p$ groups and that $\sigma = 1_H$ is the trivial representation. Indeed, there are open uniformly powerful pro-$p$ subgroups $H^* \leq_o H$ and $G^* \leq_o G$ such that $H^* \leq G^*$. By Proposition 3.3 it suffices to consider $\sigma = \text{Ind}_{H^*}^H(1_{H^*})$, thus we may assume
that \( H = H^* \) and \( \sigma = 1_H \). By Proposition 5.4 we may assume further that \( G = G^* \). Clearly, the property of being relatively FAb is also inherited.

Let \( g = \log(G) \) and \( h = \log(H) \) denote the powerful \( \mathbb{Z}_p \)-Lie lattices associated to \( G \) and \( H \). The lower \( p \)-series of \( G \), given by \( G_1 = G \) and \( G_n = (G_{n-1})^p[G_{n-1}, G] \) for \( n \geq 2 \), satisfies \( G_n = G^{p^{n-1}} = \exp(p^{n-1}g) \) for all \( n \in \mathbb{N} \). Since \( G \) is FAb relative to \( H \), we find \( r \in \mathbb{N}_0 \) such that \( G_{r+1} \subseteq H[G, G] \).

Observe that \([G, G]\) is powerfully embedded in \( G \), hence \( H[G, G] \) is uniformly powerful and \( \log(H[G, G]) = h + [g, g] \). This yields \( p^r g \subseteq h + [g, g] \). For \( n \in \mathbb{N} \) this implies that \( p^{2n+r} g \subseteq h + [p^n g, p^n g] \), equivalently \( G_{2n+r+1} \subseteq H[G_{n+1}, G_{n+1}] \), using again that \([G_{n+1}, G_{n+1}] \) is powerfully embedded in \( G \).

For \( n \in \mathbb{N} \) let \( \psi_n \) denote the \( G \)-sub-representation of \( \text{Ind}^G_H(1_H) \) spanned by all irreducible sub-representations of dimension at most \( p^n \). Let \((\eta, V_\eta) \in \text{Irr}(G) \) be an irreducible constituent of \( \psi_n \). As \( G \) is monomial, there are an open subgroup \( K \leq G \) and a linear character \( \chi : K \to \mathbb{C}^\times \) such that \( \eta = \text{Ind}^G_K(\chi) \). Note that \(|G : K| = \dim \eta \leq p^n\) and so \( G_{n+1} = G^{p^n} \subseteq K \).

The commutator group \([G_{n+1}, G_{n+1}] \) lies in the kernel of \( \eta \) and so every \( H \)-fixed vector in \( V_\eta \) is also fixed by \([G_{n+1}, G_{n+1}] \). We conclude that \( \eta \) occurs in \( \text{Ind}^G_H[G_{n+1}, G_{n+1}](1_H) \) with the same multiplicity as in \( \psi_n \). Denoting by \( \text{iso}_g(h) = g \cap (G_p \otimes_{\mathbb{Z}_p} h) \) the isolator of \( h \) in \( g \) (compare [23, §3]), we obtain in total

\[
R_{p^n}(\text{Ind}^G_H(\sigma)) \leq \dim \psi_n \leq |G : HG_{2n+r+1}|
\]

\[
= \dim [g : h + p^{2n+r} g] \leq C p^{(\dim(G) - \dim(H))n},
\]

where \( C = p^{(\dim(G) - \dim(H))} |\text{iso}_g(h) : h| \in \mathbb{R}_{>0} \) is independent of \( n \).

\[ \square \]

4. Induced representations of potent pro-\( p \) groups via the orbit method

Throughout this section \( p \) denotes a prime. Every \( p \)-adic Lie group contains a compact open pro-\( p \) group that is uniformly powerful; we refer to [20] for the general theory of \( p \)-adic analytic groups. A pro-\( p \) group \( G \) is called potent if \([G, G] \subseteq G^4\) for \( p = 2 \) and \( \gamma_{p-1}(G) \subseteq G^p \) for \( p > 2 \); there is an analogous definition for \( \mathbb{Z}_p \)-Lie lattices. Finitely generated torsion-free potent pro-\( p \) groups are a natural generalisation of uniformly powerful pro-\( p \) groups, and for all such groups the Kirillov orbit method provides a powerful tool for handling the characters of smooth irreducible representations; see [24, 25].

In this section we describe the representation zeta functions of induced representations for finitely generated torsion-free potent pro-\( p \) groups generalising the approach used, for instance, in [2, 3, 4]. There is a bijective correspondence between isomorphism classes of smooth irreducible representations of a profinite group \( G \) and the corresponding irreducible complex characters. We will interpret elements of \( \text{Irr}(G) \) in a flexible way as isomorphism classes of representations or characters, as befits the situation.

4.1. Potent pro-\( p \) groups. Let \( G \) be a finitely generated torsion-free potent pro-\( p \) group. Then \( G \) is saturable (in the sense of Lazard) and we denote by
\textbf{Lemma 4.1.} Let \( G \) be a finitely generated torsion-free potent pro-\( p \) group with associated potent \( \mathbb{Z}_p \)-Lie lattice \( \mathfrak{g} \). Then the logarithm map \( \log : G \to \mathfrak{g} \) transforms the multiplicative Haar measure on \( G \) to the additive Haar measure on \( \mathfrak{g} \).

\begin{proof}
It suffices to verify that the measures of cosets of open subgroups forming a base of neighbourhoods of 1 in \( G \) are preserved under the logarithm map. This follows from the fact that the multiplicative cosets \( xN \) of any open powerfully embedded normal subgroup \( N \trianglelefteq_o G \) are mapped to the additive cosets \( \log(x)+\mathfrak{n} \) of the associated \( \mathbb{Z}_p \)-Lie sublattice \( \mathfrak{n} \trianglelefteq \mathfrak{g} \); see the argument in [22, Cor. 6.38] or [22, Lem. 4.4]. \qed
\end{proof}

The adjoint action of \( G \) on \( \mathfrak{g} \) is related to conjugation in \( G \) via
\[
\mathfrak{g} X = \log(g \exp(X)g^{-1}) \quad \text{for } g \in G \text{ and } X \in \mathfrak{g}.
\]

We fix an isomorphism
\[
\mathbb{Q}_p/\mathbb{Z}_p \to \mu_\infty(C), \quad a + \mathbb{Z}_p \mapsto a_C,
\]
for instance, by decreeing that \( p^{-m}C = e^{2\pi ip^{-m}} \) for \( m \in \mathbb{N} \). In this way the Pontryagin dual of the additive compact group \( \mathfrak{g} \) can be realised as \( \mathfrak{g}^\vee = \mathfrak{g}_2^\vee = \text{Hom}(\mathfrak{g}, \mathbb{Q}_p/\mathbb{Z}_p) \). We denote the neutral element of \( \mathfrak{g}^\vee \), i.e. the zero map, by 0.

The co-adjoint action of \( G \) on \( \mathfrak{g}^\vee \) is given by
\[
(\mathfrak{g}^\vee)(X) = \omega(\mathfrak{g}^{-1}X) \quad \text{for } g \in G, \omega \in \mathfrak{g}^\vee \text{ and } X \in \mathfrak{g}.
\]

At the level of Lie lattices there is a corresponding co-adjoint action of \( \mathfrak{g} \) on \( \mathfrak{g}^\vee \) given by
\[
(Y.\omega)(X) = \omega([X,Y]) \quad \text{for } X,Y \in \mathfrak{g} \text{ and } \omega \in \mathfrak{g}^\vee.
\]

The Kirillov orbit method for \( p \)-adic analytic pro-\( p \) groups yields a bijective correspondence
\[
G/\mathfrak{g}^\vee \to \text{Irr}(G), \quad G.\omega \mapsto \chi_\omega
\]
between the collection \( G/\mathfrak{g}^\vee = \{G.\omega \mid \omega \in \mathfrak{g}^\vee \} \) of co-adjoint orbits and the set \( \text{Irr}(G) \) of irreducible complex characters of \( G \). If \( p \) is odd, the correspondence is canonical and can be made explicit via the formula
\[
(4.2) \quad \chi_\omega(x) = |G.\omega|^{-1/2} \sum_{\omega \in G.\omega} \overline{\omega}(\log(x))_C \quad \text{for } \omega \in \mathfrak{g}^\vee \text{ and } x \in G,
\]
where we make use of the isomorphism \( \text{(4.1)} \); see [23]. If \( p = 2 \), the expression \( \text{(4.2)} \) for \( \chi_\omega(x) \) remains valid whenever \( x \) lies in the open subgroup \( G^2 \), but does not hold in general; see [33, Thm. 2.12]. For \( \omega \in \mathfrak{g}^\vee \) we write \( \pi_\omega \) for a representation of \( G \) affording \( \chi_\omega \) so that
\[
\dim \pi_\omega = \chi_\omega(1) = |G.\omega|^{1/2} = |G : \text{Stab}_G(\omega)|^{1/2}.
\]
There is a useful Lie-theoretic description of the stabiliser \( \text{Stab}_G(\omega) \). The stabiliser of \( \omega \) in \( \mathfrak{g} \) under the co-adjoint action is
\[
\text{stab}_\mathfrak{g}(\omega) = \{Y \in \mathfrak{g} \mid Y.\omega = 0\},
\]
which can also be interpreted as the radical of an alternating bilinear form associated to \( \omega \). It is a fact that \( \log(\text{Stab}_G(\omega)) = \text{stab}_g(\omega) \) (see [33, Lem. 2.3]) and hence \( |G : \text{Stab}_G(\omega)| = |g : \text{stab}_g(\omega)| \).

Now consider a (finitely generated torsion-free) potent subgroup \( H \leq_c G \) and its associated potent \( \mathbb{Z}_p \)-Lie lattice \( g \leq g \). The inclusion map \( i_g : h \to g \) induces a surjective restriction map \( r_g : g^\vee \to h^\vee \). For \( \eta \in h^\vee \) we consider the \( r_g \)-fibres over elements of the co-adjoint orbit \( H.\eta \). For \( \omega \in g^\vee \) we define the intersection number

\[
\text{in}(\omega, \eta) = |G.\omega \cap (r_g)^{-1}(H.\eta)|.
\]

With this terminology we obtain the following application of the Kirillov orbit method to induced representations for all odd primes.

**Proposition 4.2.** Suppose that \( p > 2 \). Let \( G \) be a finitely generated torsion-free potent pro-\( p \) group, with associated \( \mathbb{Z}_p \)-Lie lattice \( g \), and \( H \leq_c G \) a potent subgroup, with associated \( \mathbb{Z}_p \)-Lie lattice \( h \). Let \( \omega \in g^\vee \) and \( \eta \in h^\vee \). Then the multiplicity of \( \pi_\omega \) in the induced representation \( \text{Ind}^G_H(\pi_\eta) \) is given by the formula

\[
m(\pi_\omega, \text{Ind}^G_H(\pi_\eta)) = \frac{\text{in}(\omega, \eta)}{|G.\omega|^{1/2}|H.\eta|^{1/2}}.
\]

**Proof.** Frobenius reciprocity, Lemma 4.1 and the orthogonality relations for irreducible characters of \( h \) yield

\[
m(\pi_\omega, \text{Ind}^G_H(\pi_\eta)) = \langle \chi_\omega, \text{Ind}^G_H(\chi_\eta) \rangle
= \langle \text{Res}_H^G(\chi_\omega), \chi_\eta \rangle
= \int_H \chi_\omega(y) \overline{\chi_\eta(y)} \, d\mu_H(y)
= |G.\omega|^{-1/2}|H.\eta|^{-1/2} \sum_{\omega \in G.\omega} \sum_{\eta \in H.\eta} \int_h \bar{\omega}(Y)_c \, \overline{\eta(Y)_c} \, d\mu_h(Y)
= |G.\omega|^{-1/2}|H.\eta|^{-1/2} \text{in}(\omega, \eta).
\]

We now specialise to the situation where we induce the trivial representation \( 1_H \) from a subgroup \( H \) to \( G \). We stress that the next result is valid also for \( p = 2 \).

**Proposition 4.3.** Let \( G \) be a finitely generated torsion-free potent pro-\( p \) group, with associated \( \mathbb{Z}_p \)-Lie lattice \( g \), and \( H \leq_c G \) a potent subgroup, with associated \( \mathbb{Z}_p \)-Lie lattice \( h \). The zeta function of \( g = \text{Ind}^G_H(1_H) \) is given by

\[
\zeta_g(s) = \sum_{\omega \in g^\vee \atop r_g^h(\omega) = 0} (|g : \text{stab}_g(\omega)|^{1/2})^{-1-s}.
\]
Proof. First suppose that \( p \) is an odd prime. Applying Proposition 4.2 to \( \eta = 0 \), hence \( \pi_\eta = \pi_0 = 1_H \) and \( H.\eta = \{0\} \), this is a simple computation:

\[
\zeta_\vartheta(s) = \sum_{\omega \in \mathfrak{g}^\vee} |G.\omega|^{-1} m(\pi_\omega, \text{Ind}_H^G(\pi_0)) |G.\omega|^{-s/2} = \sum_{\omega \in \mathfrak{g}^\vee} \frac{|G.\omega \cap (\mathfrak{g}_0^\vartheta)^{-1}(\{0\})|}{|G.\omega|} (|G.\omega|^{1/2})^{-1-s} = \sum_{\omega \in \mathfrak{g}^\vee} (|\mathfrak{g} : \text{stab}_\vartheta(\omega)|^{1/2})^{-1-s}.
\]

(4.3)

Now suppose that \( p = 2 \). In this case the Kirillov orbit method involves a choice, which makes it difficult to establish a possible analogue of Proposition 4.2. However, we obtain a canonical correspondence by considering suitable equivalence classes of irreducible characters.

We say that \( \vartheta, \tilde{\vartheta} \in \text{Irr}(G) \) are \( G^2 \)-equivalent if they restrict to the same character on \( G^2 \), i.e., \( \vartheta|_{G^2} = \tilde{\vartheta}|_{G^2} \). Denote by \( Z \) the group of linear characters of \( G \) that factor through the elementary abelian quotient \( G/G^2 \). The group \( Z \) acts on \( \text{Irr}(G) \) by multiplication and, as \( G/G^2 \) is abelian, the \( G^2 \)-equivalence classes are exactly the \( Z \)-orbits in \( \text{Irr}(G) \). For \( \omega \in \mathfrak{g}^\vee \) let \( \xi_\omega \) denote the sum of all \( \vartheta \in \text{Irr}(G) \) that are \( G^2 \)-equivalent to \( \chi_\omega \), i.e. the character associated to the \( G^2 \)-equivalence class of \( \chi_\omega \), and let \( \sigma_\omega \) denote a representation affording \( \xi_\omega \).

For \( \omega_1, \omega_2 \in \mathfrak{g}^\vee \) the characters \( \chi_{\omega_1} \) and \( \chi_{\omega_2} \) are \( G^2 \)-equivalent if and only if there are \( g \in G \) and \( \tau \in Z \) such that \( g \omega_1 \tau = \omega_2 \), equivalently if \( \omega_1, \omega_2 \) lie in the same orbit under the indicated action of the direct product \( G \times Z \) on \( \mathfrak{g}^\vee \). As explained in [33], the Kirillov orbit method yields a bijective correspondence

\[
(G \times Z).\omega \mapsto \xi_\omega
\]

between the \((G \times Z)\)-orbits in \( \mathfrak{g}^\vee \) and the characters associated to \( G^2 \)-equivalence classes in \( \text{Irr}(G) \). Moreover, the formula

\[
\xi_\omega(x) = |G.\omega|^{-1/2} \sum_{\tilde{\omega} \in (G \times Z).\omega} \tilde{\omega}(\log(x))_C
\]

holds for all \( x \in G \). Indeed, both functions are \( Z \)-invariant and hence vanish on \( G \setminus G^2 \); furthermore, by [12] the functions agree on \( G^2 \). Since all characters in a \( G^2 \)-equivalence class have the same degree, a slight modification of the argument in [13] suffices to complete the proof: replace the terms \( |G.\omega|^{-1}, \pi_\omega \) and \( \frac{|G.\omega \cap (\mathfrak{g}_0^\vartheta)^{-1}(\{0\})|}{|G.\omega|} \) by \( |(G \times Z).\omega|^{-1}, \sigma_\omega \) and \( \frac{|(G \times Z).\omega \cap (\mathfrak{g}_0^\vartheta)^{-1}(\{0\})|}{|(G \times Z).\omega|} \) respectively. \( \square \)

4.2. \( o \)-Lie lattices. Let \( o \) be a compact discrete valuation ring of characteristic 0, residue characteristic \( p \) and residue field cardinality \( q \). Fix a uniformiser \( \pi \) so that the valuation ideal of \( o \) takes the form \( p = \pi o \). Let \( \mathfrak{f} \) denote the fraction field of \( o \), a finite extension of \( Q_p \). We denote by \( e(\mathfrak{f}, Q_p) \) the absolute ramification index of \( \mathfrak{f} \).
Let $\mathfrak{g}$ be an $\mathfrak{o}$-Lie lattice. We take interest in the family of finitely generated torsion-free potent pro-$p$ groups $G$ that arise as $G = \exp(\pi^r \mathfrak{g})$ for all sufficiently large $r \in \mathbb{N}_0$.

The codifferent of $\mathfrak{f}$ over $\mathbb{Q}_p$ is the fractional ideal
\[ \{ x \in \mathfrak{f} \mid \forall y \in \mathfrak{o} : \text{Tr}_{\mathbb{Q}_p} (xy) \in \mathbb{Z}_p \} . \]

It lies in the kernel of the non-trivial character
\[ \mathfrak{f} \xrightarrow{\text{Tr}_{\mathbb{Q}_p}} \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \cong \mu_{p^n}(\mathbb{C}) \]
of the additive group $\mathfrak{f}$. The different $\mathcal{D}_{\mathfrak{f}/\mathbb{Q}_p}$ is the inverse of the codifferent and thus an ideal of $\mathfrak{o}$, say $\mathfrak{p}^\delta$, where $\delta = \delta(\mathfrak{f}) \in \mathbb{N}_0$ satisfies $\delta \geq e(\mathfrak{f}, \mathbb{Q}_p) - 1$ with equality if and only if $e(\mathfrak{f}, \mathbb{Q}_p)$ is prime to $p$. Using $[\pi^{-\delta} : \mathfrak{f} \to \mathfrak{f}, a \mapsto \pi^{-\delta} a]$, we obtain a map
\[ \text{Hom}_\mathfrak{o}(\mathfrak{g}, \mathfrak{f}) \to \text{Hom}(\mathfrak{g}, \mathbb{Q}_p), \quad \psi \mapsto \text{Tr}_{\mathbb{Q}_p} \circ [\pi^{-\delta}] \circ \psi \]
that induces a non-canonical isomorphism
\[ \mathfrak{g}^\vee = \mathfrak{g}^\vee_\mathfrak{o} = \text{Hom}_\mathfrak{o}(\mathfrak{g}, \mathfrak{f}/\mathfrak{o}) \cong \text{Hom}(\mathfrak{g}, \mathbb{Q}_p/\mathbb{Z}_p) . \]

Consider $\omega \in \mathfrak{g}^\vee$ and choose $\omega \in \text{Hom}_\mathfrak{o}(\mathfrak{g}, \mathfrak{f})$ such that $\omega(X) = w(X) + \mathfrak{o}$ for $X \in \mathfrak{g}$. The alternating $\mathfrak{o}$-bilinear form
\[ A_w : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{f}, \quad A_w(X, Y) = w([X, Y]) \]
induces the alternating $\mathfrak{o}$-bilinear form
\[ A_w : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{f}/\mathfrak{o}, \quad A_w(X, Y) = \omega([X, Y]) = A_w(X, Y) + \mathfrak{o} . \]

The $\mathfrak{o}$-submodule $\text{stab}_\mathfrak{g}(\omega)$ is equal to the radical of $A_w$, which can be described in terms of $A_w$ as
\[ \text{rad}(A_w) = \{ Y \in \mathfrak{g} \mid \forall X \in \mathfrak{g} : A_w(X, Y) \in \mathfrak{o} \} , \]
and the index $|\mathfrak{g} : \text{stab}_\mathfrak{g}(\omega)|$ can be expressed in terms of the invariant factors of the $\mathfrak{o}$-submodule $\text{rad}(A_w)$ of $\mathfrak{g}$. The latter are closely related to the Pfaffians of the alternating $\mathfrak{o}$-bilinear form $A_w$, which are defined as follows. Let $n = \dim \mathfrak{g}$.

For $0 \leq k \leq \lceil n/2 \rceil$, the degree-$k$ Pfaffian of $A_w$ is the fractional ideal $\text{Pfaff}_k(w)$ of $\mathfrak{f}$ generated by the elements
\[ \frac{1}{2^k k!} \sum_{\sigma \in \text{Sym}(2k)} \text{sgn}(\sigma) \prod_{j=1}^k A_w(Y_{(2j-1)} \sigma, Y_{(2j)} \sigma) = \sqrt{\det \left( (A_w(Y_i, Y_j))_{1 \leq i, j \leq 2k} \right)} , \]
where $Y_1, \ldots, Y_{2k}$ run through spanning sets of $\mathfrak{o}$-submodules of dimension $2k$ in $\mathfrak{g}$. As indicated the elements generating the degree-$k$ Pfaffian can be regarded as square roots of suitable determinants (up to an irrelevant choice of sign).

If $Y = (Y_1, \ldots, Y_n)$ is any $\mathfrak{o}$-basis of $\mathfrak{g}$, then the structure matrix $[A_w]_Y \in \text{Mat}_n(\mathfrak{f})$ of $A_w$ with respect to $Y$ is alternating. Furthermore, its elementary divisors $\pi^{\nu_1(w)}, \ldots, \pi^{\nu_n(w)}$, where $\nu_1(w), \ldots, \nu_n(w) \in \mathbb{Z} \cup \{ \infty \}$ with $\nu_1(w) \leq \ldots \leq \nu_n(w)$, come in pairs: $\nu_{2j-1}(w) = \nu_{2j}(w)$ for $1 \leq j \leq \lceil n/2 \rceil$. Finally, we see that
\[ \text{Pfaff}_k(w) = p^{\sum_{j=1}^k \nu_{2j-1}(w)} \quad \text{for } 1 \leq k \leq \lceil n/2 \rceil . \]
and this leads to
\[ |g : \text{stab}_g(\omega)|^{1/2} = q^{\frac{k}{2}} \sum_{i=1}^n \max \{-\nu_i(w), 0\} = \left\| \bigcup \{ \text{Pfaff}_k(w) \mid 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \} \right\|_p, \]
where \( \|S\|_p = \max \{|x|_p \mid x \in S\} \) for \( \emptyset \neq S \subseteq \mathfrak{g}. \)

As a consequence of Propositions 3.10 and 1.3 we obtain the following result.

**Proposition 4.4.** Let \( \mathfrak{g} \) be an \( \mathfrak{o}\)-Lie lattice with an \( \mathfrak{o}\)-Lie sublattice \( \mathfrak{h} \) such that \( |\mathfrak{g} : \mathfrak{h} + [\mathfrak{g}, \mathfrak{g}]| < \infty. \) Write \( m + 1 = \dim \mathfrak{g} - \dim \mathfrak{h}. \) Let \( r \in \mathbb{N}_0 \) be such that \( G = \exp(\pi^r \mathfrak{g}) \) is a finitely generated torsion-free potent pro-\( p \) group with potent subgroup \( H = \exp(\pi^r \mathfrak{h}) \leq_c G. \) Then \( G \) is FAb relative to \( H \) and the zeta function of \( \mathfrak{g} = \text{Ind}_H^G(\mathfrak{h}) \) is given by the following integral formulae.

1. Writing \( W = \{ w \in \text{Hom}_\mathfrak{o}(\mathfrak{g}, \mathfrak{f}) \mid w(\mathfrak{h}) \subseteq \mathfrak{o} \}, \) we have
\[ \zeta_q(s) = q^{r(m+1)} \int_{w \in W} \left\| \bigcup \{ \text{Pfaff}_k(w) \mid 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \} \right\|_p^{1-s} \, d\mu(w), \]
where \( \mu \) denotes the normalised Haar measure satisfying \( \mu(\text{Hom}_\mathfrak{o}(\mathfrak{g}, \mathfrak{o})) = 1 \)

2. For simplicity, suppose further that the \( \mathfrak{o}\)-module \( \mathfrak{g} \) decomposes as a direct sum \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h}. \) Interpreting \( \text{Hom}_\mathfrak{o}(\mathfrak{t}, \mathfrak{a}) \) as the \( \mathfrak{o}\)-points \( \mathcal{B}(\mathfrak{a}) \) of \( m+1 \)-dimensional affine space \( \mathcal{B} \) over \( \text{Spec}(\mathfrak{o}) \), let \( \mathcal{X} \) denote the projectivisation \( \mathcal{P}\mathcal{B} \) over \( \text{Spec}(\mathfrak{o}). \) For \( 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor, \) the map \( w \mapsto \text{Pfaff}_k(w) \) induces a sheaf of ideals \( \mathcal{I}_k \) on \( \mathcal{X}, \) and
\[ \zeta_q(s) = (1 - q^{-1})q^{r(m+1)} \sum_{\ell \in \mathbb{Z}} q^{-\ell(m+1)} \int_{\mathcal{X}(\mathfrak{a})} \left( \max_{0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor} \left\| \pi^{k\ell} \mathcal{I}_k \right\|_p \right)^{-1-s} \, d\mu_{\mathcal{X}, p}, \]
where the suggestive notation \( \| \pi^{k\ell} \mathcal{I}_k \|_p \) is employed in anticipation of a formal definition in \( 5.2 \) and \( \mu_{\mathcal{X}, p} \) denotes the canonical \( p \)-adic measure on \( \mathcal{X}(\mathfrak{a}); \) see Remark 5.4.

**Proof.** From \( |\mathfrak{g} : \mathfrak{h} + [\mathfrak{g}, \mathfrak{g}]| < \infty \) we infer that \( |G : H[G, G]| < \infty \) so that \( G \) is FAb relative to \( H. \) In the described set-up the formula for \( \zeta_q \) provided by Proposition 4.3 translates directly into the first integral formula. To derive the second formula write \( \text{Hom}_\mathfrak{o}(\mathfrak{t}, \mathfrak{f}) \) as a disjoint union of the sets \( \pi^\ell \text{Hom}_\mathfrak{o}(\mathfrak{t}, \mathfrak{a}) \setminus \pi^{\ell+1} \text{Hom}_\mathfrak{o}(\mathfrak{t}, \mathfrak{a}) \) and recall that the canonical \( p \)-adic measure \( \mu_{\mathcal{X}, p} \) is normalised so that \( \int_{\mathcal{X}(\mathfrak{a})} \, d\mu_{\mathcal{X}, p} = q^{-m |\mathcal{X}(\mathfrak{a}/p)|} = (1 - q^{-m-1})/(1 - q^{-1}) \) where we use that \( \mathcal{X} \) is the projective \( m \)-dimensional space.

**Remark 4.5.** By definition \( \text{Pfaff}_0(w) = \mathfrak{o} \) is the degree-0 Pfaffian of any form \( w \in \text{Hom}_\mathfrak{o}(\mathfrak{g}, \mathfrak{f}). \) This means, that the ideal sheaf \( \mathcal{I}_0 \) is the structure sheaf \( \mathcal{O}_X \) of \( \mathcal{X}. \) In addition, the assumption \( |\mathfrak{g} : \mathfrak{h} + [\mathfrak{g}, \mathfrak{g}]| < \infty \) implies that the image \( \text{pr}_\mathfrak{f}(\{\mathfrak{g}, \mathfrak{g}\}) \) of the projection of \( \{\mathfrak{g}, \mathfrak{g}\} \) to the complement \( \mathfrak{f} \) is of finite index; i.e., \( \pi^\ell \mathfrak{f} \subseteq \text{pr}_\mathfrak{f}(\{\mathfrak{g}, \mathfrak{g}\}) \) for some non-negative integer \( \ell \in \mathbb{Z}. \) Since \( \text{Pfaff}_1(w) = w(\text{pr}_\mathfrak{f}(\{\mathfrak{g}, \mathfrak{g}\})) \supseteq \pi^\ell w(\mathfrak{f}), \) we deduce that the ideal sheaf \( \mathcal{I}_1 \) contains \( \pi^\ell \mathcal{O}_X. \)

Next we formulate the two integral formulae more explicitly, subject to a choice of coordinates. For simplicity, we suppose that the \( \mathfrak{o}\)-module \( \mathfrak{g} \) decomposes as a direct sum \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h}. \) Fix an \( \mathfrak{o}\)-basis \( Y = (Y_1, \ldots, Y_n) \) of \( \mathfrak{g} \) such that
Proposition 4.6. Let \( g \) be an \( o \)-Lie lattice with an \( o \)-Lie sublattice \( h \) such that \([g : h + [g, g]] < \infty \). Suppose that the \( o \)-module \( g \) decomposes as a direct sum \( g = \mathfrak{k} \oplus h \), and put \( m + 1 = \dim_o \mathfrak{k} = \dim_o g - \dim_h h \). Let \( Y = (Y_1, \ldots, Y_n) \) be an \( o \)-basis of \( h \) such that \( Y_1, \ldots, Y_{m+1} \) (respectively \( Y_{m+2}, \ldots, Y_n \)) form an \( o \)-basis of \( \mathfrak{k} \) (respectively \( h \)), and let \( F_k \), depending on \( Y \), be defined as above.

Let \( r \in \mathbb{N}_0 \) such that \( G = \exp(\pi^r g) \) is a finitely generated torsion-free pro-\( p \) group with potent subgroup \( H = \exp(\pi^r h) \leq_o G \). Then \( G \) is FAb relative to \( H \) and the zeta function of \( q = \text{Ind}_H^G(1_H) \) is given by the following formulae:

\[
\zeta_q(s) = q^{r(m+1)} \int_{\mathbb{Q}_p} \left\| \bigcup_{0 \leq k \leq [n/2]} F_k(x) \right\|_p^{-1-s} \mu(x),
\]

where \( \mu \) denotes the normalised Haar measure satisfying \( \mu(o^{m+1}) = 1 \), and

\[
\zeta_q(s) = (1-q^{-1})q^{r(m+1)} \sum_{\ell \in \mathbb{Z}} q^{-\ell(m+1)} \int_{\mathbb{Q}_p} \left( \max_{0 \leq k \leq [n/2]} \left\| \pi^{k\ell} F_k(y) \right\|_p \right)^{-1-s} \mu_{\mathbb{Q}_p}(y),
\]

where \( \mu_{\mathbb{Q}_p} \) denotes the canonical \( p \)-adic measure on \( \mathbb{Q}_p \); see Remark 5.4.

Examples based on this concrete formula are discussed in Section 7 below.

5. A Denef-type formula for globally induced representations

Let \( K_0 \) be a number field with ring of integers \( O_{K_0} \), and fix a finite set of closed points \( S \subseteq \text{Spec}(O_{K_0}) \). Let \( O_{K_0, S} = \{ a \in K_0 \mid |a|_p \leq 1 \text{ for closed points } \mathfrak{p} \notin S \} \) denote the ring of \( S \)-integers in \( K_0 \).

We consider number fields \( K \) that arise as finite extensions of \( K_0 \). Let \( O_S = O_{K,S} \) denote the integral closure of \( O_{K_0,S} \) in \( K \). For a maximal ideal \( \mathfrak{p} \trianglelefteq O_S \) we write \( \kappa_\mathfrak{p} = O_S/\mathfrak{p} \) for the residue field and we denote its cardinality by \( q_\mathfrak{p} \).
Furthermore, $O_p = O_{K,S,p}$ denotes the completion of $O_S$ with respect to $p$ and we fix a uniformiser $\pi_p$ so that $O_p$ has the valuation ideal $\pi_p O_p$.

$$
\begin{align*}
O_K & \subseteq O_S = O_{K,S} \subseteq K \\
O_{K_0} & \subseteq O_{K_0,S} \subseteq K_0
\end{align*}
$$

$\kappa_p \cong O_p/\pi_p O_p \cong O_S/\pi_p \cong \mathbb{Q}_p$.

Let $\mathfrak{g}$ be an $O_{K_0,S}$-Lie lattice. For every finite extension $K$ of $K_0$ and every maximal ideal $p \triangleleft O_{S,S}$ we consider the $O_p$-Lie lattice $\mathfrak{g}_p = O_p \otimes_{O_{K_0,S}} \mathfrak{g}$ and, for $r \in \mathbb{N}_0$, its principal congruence sublattices $\mathfrak{g}_{p,r} = \pi_p^r \mathfrak{g}_p$. Observe that, for any given $K$ and $p$, the Lie lattice $\mathfrak{g}_{p,r}$ is potent for all sufficiently large integers $r$ so that $G_{p,r} = \exp(\mathfrak{g}_{p,r})$ is a finitely generated torsion-free potent pro-$p$ group; we say that such $r$ are permissible for $\mathfrak{g}_p$.

With these preparations we formulate the main result of this section.

**Theorem 5.1.** As in the set-up described above, let $\mathfrak{g}$ be an $O_{K_0,S}$-Lie lattice with an $O_{K_0,S}$-Lie sublattice $\mathfrak{h}$ such that $|\mathfrak{g} : \mathfrak{h} + [\mathfrak{g}, \mathfrak{g}]| < \infty$. Suppose that $\mathfrak{h}$ is a direct summand as a submodule of the $O_{K_0,S}$-module $\mathfrak{g}$, and put $m+1 = \dim_{O_{K_0,S}} \mathfrak{g} - \dim_{O_{K_0,S}} \mathfrak{h}$.

For closed points $p \in \text{Spec}(O_{K,S})$, where $K$ ranges over finite extensions of $K_0$, and for positive integers $r$ that are permissible for $\mathfrak{g}_p$ and $\mathfrak{h}_p$, we consider the induced representation

$$
\mathfrak{g}_{p,r} = \text{Ind}_{H_{p,r}}^{G_{p,r}} (\mathfrak{h}_{p,r})
$$

associated to the pro-$p$ groups $G_{p,r} = \exp(\mathfrak{g}_{p,r})$ and $H_{p,r} = \exp(\mathfrak{h}_{p,r})$.

1. For each $p$ there is a complex-valued function $Z_p$ of a complex variable $s$ that is rational in $q_p^{-s}$ with integer coefficients so that for all permissible $r$,

$$
\zeta_{\mathfrak{g}_{p,r}}(s) = (1 - q_p^{-1}) q_p^{-r(\mu+1)} Z_p(s).
$$

2. The real parts of the poles of the functions $Z_p$, for all $p$, form a finite subset $P_{p,h} \subseteq \mathbb{Q}$.

3. There is a finite extension $K_1$ of $K_0$ such that, for all closed points $p \in \text{Spec}(O_{K,S})$, arising for extensions $K \supseteq K_1$, and for all permissible $r$, the abscissa of convergence of the zeta function $\zeta_{\mathfrak{g}_{p,r}}$ satisfies

$$
\alpha(\zeta_{\mathfrak{g}_{p,r}}) = \max P_{p,h}.
$$

4. There are an open dense subscheme $\text{Spec}(O_{K_0,T}) \subseteq \text{Spec}(O_{K_0,S})$ and a rational function $F \in \mathbb{Q}(Y_1, Y_2, X_1, \ldots, X_g)$ such that the following holds:

For every closed point $p_0 \in \text{Spec}(O_{K_0,T})$ there are algebraic integers $\lambda_1 = \lambda_1(p_0), \ldots, \lambda_g = \lambda_g(p_0) \in \mathbb{C}^\times$ so that for every finite extension $K$ of $K_0$ and every closed point $p \in \text{Spec}(O_{K,T})$ lying above $p_0$,

$$
Z_p(s) = F(q^{-s} q^{-f s}, \lambda_1^f, \ldots, \lambda_g^f),
$$

where $q = q_{p_0}$ and $f = [\kappa_p : \kappa_{p_0}]$ denotes the inertia degree of $p$ over $K_0$.

Furthermore the following functional equation holds:

$$
F(q^f, q^{-fs}, \lambda_1^f, \ldots, \lambda_g^f) = -q^{-fm} F(q^{-f}, q^{-fs}, \lambda_1^f, \ldots, \lambda_g^f).
$$
This theorem is a consequence of Theorems 5.8 and 5.19 below, which hold for a larger class of integrals of Igusa type described in (5.3). Indeed, the zeta functions of induced representations are of the required form due to the formula given in Proposition 4.4. As discussed in the introduction, our proof proceeds along the lines of [18, 54]. Our approach is of independent interest, because we work on a different domain of integration and we modified the zeta functions with some infinite series, which is conceptually adapted to our applications.

Remark 5.2. The assertions in (1), (2) and (3) of Theorem 5.1 have local analogues for Lie lattices that are defined over the valuation ring of a \( p \)-adic field; i.e. a finite extension field of some \( \mathbb{Q}_p \). In fact, the proof of the underlying Theorem 5.8 given in Section 5.2.2 carries over directly if the base field \( K_0 \) is assumed to be a \( p \)-adic field (instead of a number field). Note that there are \( \mathbb{Z}_p \)-Lie lattices \( \mathfrak{g} \) which cannot be defined globally; this means, there is no Lie lattice \( \mathfrak{h} \) defined over the ring of \( S \)-integers \( \mathcal{O}_S \) of a number field and a prime ideal \( \mathfrak{p} \subseteq \mathcal{O}_S \) such that \( \mathcal{O}_\mathfrak{p} \cong \mathbb{Z}_p \) and \( \mathcal{O}_\mathfrak{p} \otimes \mathcal{O}_S \mathfrak{h} \cong \mathfrak{g} \).

5.1. \( p \)-Adic integrals. We continue to use the notation set up above and write

\[
O_S = O_{K,S}, \quad O_\mathfrak{p} = O_{K,S,\mathfrak{p}}, \quad \pi = \pi_\mathfrak{p}
\]

for short. Let \( \mathfrak{X} \) be a smooth integral projective scheme over \( \text{Spec}(O_{K_0,S}) \), and let \( m \) be the dimension of the generic fibre \( \mathfrak{X}_{K_0} \). Let \( d \in \mathbb{N} \) and let \( \mathcal{I} = (\mathcal{I}_j)_{j=0}^d \) and \( \mathcal{J} = (\mathcal{J}_j)_{j=0}^d \) be two collections of coherent sheaves of ideals on \( \mathfrak{X} \).

Assumption 5.3. We assume throughout that \( \mathcal{I}_0 = \mathcal{J}_0 = \mathcal{O}_\mathfrak{X} \) is the structure sheaf of \( \mathfrak{X} \), and \( \gamma \mathcal{O}_\mathfrak{X} \subseteq \mathcal{I}_{j_0} \) for some non-zero \( \gamma \in O_{K_0,S} \) and some \( j_0 \in \{1, \ldots, d\} \).

A sheaf of ideals \( \mathcal{I} \) on \( \mathfrak{X} \) defines a continuous function \( \|\mathcal{I}\|_\mathfrak{p} \) on the compact space \( \mathfrak{X}(O_\mathfrak{p}) \) by

\[
\|\mathcal{I}\|_\mathfrak{p}(x) = \max\{|f(x)|_\mathfrak{p} \mid f \in \mathcal{I}_x\},
\]

where \( \mathcal{I}_x = \mathcal{I} \cdot \mathcal{O}_{\mathfrak{X},x} \) denotes the stalk of \( \mathcal{I} \) at \( x \in \mathfrak{X}(O_\mathfrak{p}) \). We are interested in weighted combinations of such functions. For \( \ell \in \mathcal{I} \) we define

\[
\|\pi^{\ell}_j \mathcal{I}_j\|_{\mathfrak{p}} := \max_{0 \leq j \leq d} q_\mathfrak{p}^{-j} \|\mathcal{I}_j\|_\mathfrak{p}
\]

and likewise for \( \mathcal{J} \).

Remark 5.4. On the compact topological space \( \mathfrak{X}(O_\mathfrak{p}) \) we consider the canonical \( p \)-adic measure \( \mu_{\mathfrak{X},\mathfrak{p}} \) defined by Batyrev in [8, Def. 2.6]. This measure is locally defined in terms of differential forms, i.e. sections of the canonical bundle \( \omega_{\mathfrak{X}/O_\mathfrak{p}} \). It can be defined without assuming the existence of a global gauge form, on which Weil’s original approach [56] was based. Here we will mainly use the following property of the canonical \( p \)-adic measure: for every point \( x \in \mathfrak{X}(\kappa_\mathfrak{p}) \), the fibre \( Y \) over \( x \) in \( \mathfrak{X}(O_\mathfrak{p}) \) has measure \( \mu_{\mathfrak{X},\mathfrak{p}}(Y) = q_\mathfrak{p}^{-m} \); in particular, \( \mu_{\mathfrak{X},\mathfrak{p}}(\mathfrak{X}(O_\mathfrak{p})) = q_\mathfrak{p}^{-m} |\mathfrak{X}(\kappa_\mathfrak{p})| \) as shown in [8, Thm. 2.7].
Fix once and for all a positive integer \( c \in \mathbb{N} \). We consider the local zeta function
\[
Z_p(s) = Z_{\mathcal{X}, \mathcal{I}, \mathcal{J}, p}(s) = \sum_{\ell \in \mathbb{Z}} q_p^{-\ell c} \int_{\mathcal{X}(O_p)} \left\| (\pi^{j \ell} \mathcal{I})_{j=0}^{d} \right\|^{-s} \left\| (\pi^{j \ell} \mathcal{J})_{j=0}^{d} \right\|^{-1} \, d\mu_{\mathcal{X}, p}.
\]

For instance, in the setting of Proposition 4.4, we encounter such zeta functions for \( c = m + 1 \). In such applications the collections \( \mathcal{I} \) and \( \mathcal{J} \) may very well contain zero ideal sheaves; however, these do not contribute to the local zeta function. It follows from Remark 4.5 that Assumption 5.3 holds in this case. The next lemma shows that these assumptions guarantee the convergence of the infinite series in (5.3).

**Lemma 5.5.** Under Assumption 5.3 the series \( Z_p(s) \) converges absolutely for all \( s \in \mathbb{C} \) with \( \text{Re}(s) > \gamma_{j_0} \).

**Proof.** First we carry out the summation over all non-negative integers. For \( \ell \geq 0 \), the functions \( \| (\pi^{j \ell} \mathcal{I})_{j=0}^{d} \|_p \) and \( \| (\pi^{j \ell} \mathcal{J})_{j=0}^{d} \|_p \) have value 1 at every \( x \in \mathcal{X}(O_p) \), because \( \mathcal{I}_0 = \mathcal{J}_0 = \mathcal{O}_X \). We deduce that
\[
\sum_{\ell=0}^{\infty} q_p^{-\ell c} \int_{\mathcal{X}(O_p)} \left\| (\pi^{j \ell} \mathcal{I})_{j=0}^{d} \right\|^{-s} \left\| (\pi^{j \ell} \mathcal{J})_{j=0}^{d} \right\|^{-1} \, d\mu_{\mathcal{X}, p} = \sum_{\ell=0}^{\infty} q_p^{-\ell c} \mu_{\mathcal{X}, p}(\mathcal{X}(O_p)) \frac{1}{1 - q_p^{-c}} \text{ with absolute convergence independently of } s \in \mathbb{C}.
\]

It remains to sum over all negative integers. Using \( \gamma_{\mathcal{O}_X} \subseteq \mathcal{I}_{j_0} \) to justify the second inequality below, we observe that for all \( s \in \mathbb{C} \) with \( \text{Re}(s) > \gamma_{j_0} \),
\[
\sum_{\ell=1}^{\infty} \left| q_p^{\ell c} \int_{\mathcal{X}(O_p)} \left\| (\pi^{-j \ell} \mathcal{I})_{j=0}^{d} \right\|^{-s} \left\| (\pi^{-j \ell} \mathcal{J})_{j=0}^{d} \right\|^{-1} \, d\mu_{\mathcal{X}, p} \right| \
\leq \sum_{\ell=1}^{\infty} q_p^{\ell c} \int_{\mathcal{X}(O_p)} \left\| (\pi^{-j \ell} \mathcal{I})_{j=0}^{d} \right\|^{-\text{Re}(s)} \, d\mu_{\mathcal{X}, p} \leq \sum_{\ell=1}^{\infty} q_p^{\ell (c - \text{Re}(s))_{j_0}} |\gamma|_{p}^{-\text{Re}(s)} \mu_{\mathcal{X}, p}(\mathcal{X}(O_p)) = \frac{q_p^{c - \text{Re}(s)_{j_0}} |\gamma|_{p}^{-\text{Re}(s)}}{1 - q_p^{c - \text{Re}(s)_{j_0}}} \mu_{\mathcal{X}, p}(\mathcal{X}(O_p)).
\]

The infimum of all \( \sigma \in \mathbb{R} \) such that the series \( Z_p(\sigma) \) converges is called the abscissa of convergence \( \alpha(Z_p) \) of the local zeta function \( Z_p \); as seen above, the series \( Z_p(s) \) converges absolutely to an analytic function on the half-plane \( \{ s \in \mathbb{C} \mid \text{Re}(s) > \alpha(Z_p) \} \).

The computation of the integral defining the local zeta function \( Z_p \) is rather simple if the ideal sheaves in \( \mathcal{I} \) and \( \mathcal{J} \) are monomial in the following sense.

**Definition 5.6.** Let \( k \) be a field and let \( \mathfrak{Y} \) be a regular scheme over \( \text{Spec}(k) \). An ideal sheaf \( \mathcal{I} \) on \( \mathfrak{Y} \) is called monomial if for all \( y \in \mathfrak{Y} \) the ideal \( \mathcal{I}_y \subseteq \mathcal{O}_{\mathfrak{Y}, y} \)}
is of the form \( I_y = \prod_{i=1}^r z_i^{c_i} O_{Z,y} \), where \( z_1, \ldots, z_r \) are regular parameters at \( y \) and \( c_1, \ldots, c_r \in \mathbb{N}_0 \) are suitable exponents; cf. \cite{Mumford} (3.16).

Let \( X \) be a smooth scheme over \( \text{Spec}(O_S) \). Observe that for every \( a \in \text{Spec}(O_S) \), the fibre \( X_a \) is a regular scheme over \( \text{Spec}(k) \), where \( k = \kappa_p \) if \( a = p \) is a closed point and \( k = K \) otherwise. An ideal sheaf \( I \) on \( X \) is \textit{monomial}, if it is locally principal and for all \( a \in \text{Spec}(O_S) \) the ideal sheaf \( I \cdot O_{X_a} \) on the fibre \( X_a \) over \( a \) is monomial.

Many authors refer to a monomial sheaf of ideals as a \textit{sheaf of ideals of a divisor with simple normal crossings}. Hironaka’s famous resolution of singularities \cite{Hironaka} implies that over a field of characteristic 0 any ideal sheaf can be transformed into a monomial one using a suitable sequence of blow-ups. We will use the following version of this result, which is explained in \cite[Thm. 3.26]{Hironaka}. By a variety over a field \( k \) we mean an integral, separated scheme of finite type over \( \text{Spec}(k) \).

**Theorem 5.7** (Hironaka Monomialisation Theorem). Let \( X \) be a smooth variety over some field \( k \) of characteristic 0 and let \( I \) be an ideal sheaf on \( X \). There are a smooth variety \( Y \) over \( k \) and a projective morphism \( h : Y \rightarrow X \) such that the ideal sheaf \( h^*(I) \) is monomial and \( h \) restricts to an isomorphism between the complements of the closed subvarieties defined by \( h^*(I) \) and \( I \).

**5.2. The abscissa of convergence.** We continue to use the notation set up above. In this section we study the abscissa of convergence of the zeta functions \( Z_p = Z_{X,I,J,p} \), for finite extensions \( K \) of \( K_0 \) and closed points \( p \in \text{Spec}(O_S) \), as defined in \cite{Monodromy}. The main result is the following theorem.

**Theorem 5.8.** In the set-up described above, including Assumption \ref{assumption}, the following statements hold.

1. For every finite extension \( K \) of \( K_0 \) and every closed point \( p \in \text{Spec}(O_S) \), there is a rational function \( F_p \in \mathbb{Q}(X,Y) \) such that
   \[
   Z_p(s) = F_p(q_p, q_p^{-s}).
   \]

2. The real parts of the poles of the functions \( Z_p \), for all \( K \) and \( p \) as above, form a finite set \( P = P_{X,I,J} \subseteq \mathbb{Q} \) of rational numbers.

3. There is a finite extension \( K_1 \) of \( K_0 \) such that, for all \( K \) and \( p \) as above satisfying \( K_1 \subseteq K \), the abscissa of convergence of \( Z_p \) satisfies \( \alpha(Z_p) = \max P \).

The proof of the theorem is described in Section \ref{proof}. Here we briefly discuss an interesting consequence. Let \( K_1 \) be a finite extension of \( K_0 \) with the properties described in part (3) of the theorem. If a maximal ideal \( p_0 \) of \( O_{K_0,S} \) is unramified in \( K_1 \) and admits a prime divisor \( p_1 \leq O_{K_1,S} \) of inertia degree 1 over \( p_0 \), then the completions \( (K_0)_{p_0} \) and \( (K_1)_{p_1} \) coincide and hence \( Z_{p_0} = Z_{p_1} \), in particular \( \alpha(Z_{p_0}) = \alpha(Z_{p_1}) \). By the Chebotarev Density Theorem \cite[VII (13.6)]{Mumford} the set of such primes \( p_0 \leq O_{K_0,S} \) has Dirichlet density at least \( [K_1 : K_0]^{-1} \).
Corollary 5.9. The set of closed points \( p_0 \in \text{Spec}(O_{K_0,S}) \) satisfying 
\[
\alpha(Z_{p_0}) = \max P
\]
has positive Dirichlet density.

5.2.1. The abscissa of convergence of certain Dirichlet series. We discuss a general result on a family of power series. Let \( Q(Q)[[t]] \) be the ring of formal power series in \( t \) over the field of Laurent series \( Q(Q) \) in an indeterminate \( Q \). Fix integers \( d, u \in \mathbb{N} \) and two collections \( \lambda = (\lambda_j)_{j=0}^d, \beta = (\beta_j)_{j=0}^d \) of integral linear forms \( \lambda_j, \beta_j : \mathbb{Z}^{1+u} \to \mathbb{Z} \) for \( 0 \leq j \leq d \). We say that an integral linear form \( \lambda : \mathbb{Z}^{1+u} \to \mathbb{Z} \) is strictly negative if \( \lambda(v) < 0 \) for every \( v \in \mathbb{N}_0^{1+u} \setminus \{0\} \). It is convenient for us to write the elements of \( \mathbb{Z}^{1+u} \) as pairs \((\ell, n)\) where \( \ell \in \mathbb{Z} \) and \( n = (n_1, \ldots, n_u) \in \mathbb{Z}^u \). The following assumptions are analogous to those in Assumption 5.3.

Assumption 5.10. We assume throughout: (i) \( \lambda_0 \) is strictly negative, (ii) \( \beta_0 = 0 \), and (iii) there are an index \( j_0 \in \{1, \ldots, d\} \) and \( a \in \mathbb{N} \) such that \( \beta_{j_0}(\ell, n) = a\ell \) for all \((\ell, n) \in \mathbb{Z}^{1+u}\).

For every integer \( N \geq 0 \) and vectors \( \varepsilon = (\varepsilon_0, \ldots, \varepsilon_d), \delta = (\delta_0, \ldots, \delta_d) \in \mathbb{Z}^{d+1} \) with \( \delta_0 = 0 \) we define a power series
\[
\Xi_{N,\varepsilon,\delta}(Q, t) = \sum_{\ell \in \mathbb{Z}} \sum_{n(\mathbb{Z} \geq N)^u} Q^{-\min_{0 \leq j \leq d} \left( \lambda_j(\ell, n) + \varepsilon_j \right) - \min_{0 \leq j \leq d} (\beta_j(\ell, n) + \delta_j)}
\in Q(Q)[[t]].
\]
(5.4)

To simplify the notation, we write \( \Xi_{N,\varepsilon,\delta} \) in place of \( \Xi_{N,\varepsilon,\delta}^{\lambda,\beta} \). Let us verify that \( \Xi_{N,\varepsilon,\delta} \) defines an element of \( Q(Q)[[t]] \setminus Q(Q)[[t]] \). The variable \( t \) only occurs with non-negative exponents since \( \beta_0 = 0 \) and \( \delta_0 = 0 \). Consider summands contributing to the coefficient of \( t^e \), for some fixed exponent \( e \geq 0 \). By Assumption 5.10 (iii), the relevant summands only occur for \( \ell \geq -(\varepsilon + \delta_0)/a \). By Assumption 5.10 (i), \( \lambda_0 \) is strictly negative; hence every monomial \( Q^{k}t^e \) occurs only a finite number of times and, moreover, \( k \) is bounded below in terms of \( \ell \) and thus in terms of \( e \). Finally, we observe that the coefficients are all positive; as \( \ell \) decreases, we pick up non-zero terms of increasing degree in \( t \), thus \( \Xi_{N,\varepsilon,\delta} \in Q(Q)[[t]] \).

Lemma 5.11. In the above setting, including Assumption 5.10, the following statements hold.

1. The power series \( \Xi_{N,\varepsilon,\delta} \) is a rational function over \( Q \), i.e., \( \Xi_{N,\varepsilon,\delta} \in Q(Q, t) \).
2. There is a finite set \( \mathcal{P} \subseteq (\mathbb{Z} \times \mathbb{N}_0) \setminus \{(0, 0)\} \) such that, for all \( N, \varepsilon, \delta \), the rational function \( \Xi_{N,\varepsilon,\delta} \) can be written as the quotient of an element of \( Q(Q, Q^{-1}, t) \) by a power of \( \prod_{(A, B) \in \mathcal{P}}(1 - Q^{A}t^{B}) \).
3. The real parts of poles of \( \Xi_{N,\varepsilon,\delta}(q^{-1}, q^{-s}) \), for all \( N, \varepsilon, \delta \) and \( q, s \in \mathbb{R}_{>1} \), form a non-empty finite subset \( \mathcal{P}_{\lambda, \beta} \subseteq \mathbb{Q} \).
4. The maximum \( \max \Xi_{N,\varepsilon,\delta} \) is a pole for every \( \Xi_{N,\varepsilon,\delta}(q^{-1}, q^{-s}) \) as in \( \Xi_{N,\varepsilon,\delta} \).
5. The inversion property \( \Xi_{1,0,0}(Q^{-1}, t^{-1}) = (-1)^{u+1} \Xi_{0,0,0}(Q, t) \) holds.
Proof. We only sketch the proof, as the lemma is essentially a paraphrase of known results; see [51, Prop. 2.10] and [3, Prop. 4.5]. Furthermore, we may assume that \( N = 0 \) as \( \Xi_{\kappa,\lambda}(Q, t) = \Xi_{0,\kappa,\lambda}(Q, t) \) for \( \varepsilon'_{j} = \varepsilon_{j} + \lambda_{j}(0, N, \ldots, N) \) and \( \delta'_{j} = \delta_{j} + \beta_{j}(0, N, \ldots, N) \), where \( 0 \leq j \leq d \). For simplicity we drop \( N \) from the notation altogether.

Decompose \( Z \times \mathbb{N}_{0}^{d} \) into a finite number of disjoint rational cones \( C(\varepsilon, \delta)_{j,k}^{\pm} \), for \( 0 \leq j, k \leq d \), such that on each \( C(\varepsilon, \delta)_{j,k}^{\pm} \) the minimum in the exponent of \( Q \) is attained by \( \lambda_{j} + \varepsilon_{j} \), the minimum in the exponent of \( t \) is attained by \( \beta_{k} + \delta_{k} \), and \( \ell \) is positive or non-positive according to the attached sign. Introducing new variables we may describe each cone with the cone of positive solutions of a linear integral inhomogeneous system of equations. The generating function of such a cone is known to be rational (see [51, Ch. I]) and so we obtain (1).

Moreover, the denominator of the generating function of each cone is of the form \( \prod_{\alpha}(1 - X^{\varepsilon}) \) where the product runs over the finitely many completely fundamental solutions \( \gamma \) of the corresponding homogeneous system. Making the suitable substitutions we obtain a finite set \( \tilde{P} \) with the properties stated in (2). Observe that \( (0, 0) \notin \tilde{P} \) as \( \Xi(Q, t) \) is a well-defined power series, as explained just after the defining equation (2.4).

Statement (3) is a direct consequence of (2), as

\[
P_{A, B} \subseteq \left\{ -\frac{A}{B} \in \mathbb{Q} \mid (A, B) \in \tilde{P} \text{ and } B \neq 0 \right\},
\]

and the inversion property (5) follows from [51, Prop. 8.3].

It remains to justify (4). We give more details than provided in (3), where the corresponding discussion appears to be short if not incomplete. Fix \( q > 1 \).

As a first step we show that the abscissae of convergence of \( \Xi_{\kappa,\lambda}(q^{-1}, q^{-\sigma}) \) and \( \Xi_{0,0}(q^{-1}, q^{-\sigma}) \) agree. Define \( E = \max\{\varepsilon_{i} \mid 0 \leq i \leq d\} \) and \( \Delta = \max\{\delta_{i} \mid 0 \leq i \leq d\} \). Fix \((\ell, n) \in \mathbb{Z} \times \mathbb{N}_{0}^{d}\), and choose indices \( j, k \in \{0, \ldots, d\} \) such that

\[
\min_{0 \leq i \leq d} \beta_{i}(\ell, n) = \beta_{j}(\ell, n) \quad \text{and} \quad \min_{0 \leq i \leq d} \beta_{i}(\ell, n) + \delta_{i} = \beta_{k}(\ell, n) + \delta_{k}.
\]

Then \( \beta_{j}(\ell, n) \leq \beta_{k}(\ell, n) \), whereas \( \beta_{k}(\ell, n) + \delta_{k} \leq \beta_{j}(\ell, n) + \delta_{j} \), and we conclude

\[
-\Delta \leq \delta_{k} \leq (\beta_{k}(\ell, n) + \delta_{k}) - \beta_{j}(\ell, n) \leq \delta_{j} \leq \Delta.
\]

A similar inequality relates the minima of \( \lambda_{i}(\ell, n) \) and \( \lambda_{i}(\ell, n) + \varepsilon_{i} \). We conclude that, for every \( \sigma \in \mathbb{R} \) for which at least one of the two series \( \Xi_{\kappa,\lambda}(q^{-1}, q^{-\sigma}) \) and \( \Xi_{0,0}(q^{-1}, q^{-\sigma}) \) converges,

\[
q^{-\varepsilon_{i} + |\Delta|} \Xi_{0,0}(q^{-1}, q^{-\sigma}) \leq \Xi_{\kappa,\lambda}(q^{-1}, q^{-\sigma}) \leq q^{E + |\Delta|} \Xi_{0,0}(q^{-1}, q^{-\sigma}).
\]

Hence the abscissae of convergence of the two series are equal.

Consider the abscissa of convergence of \( \Xi_{0,0}(q^{-1}, q^{-\sigma}) \). Pick \((A, B) \in \tilde{P}\) with \( B \neq 0 \) such that \( -A/B \) is maximal among all the pairs \((A, B) \in \tilde{P}\). By (2), the series \( \Xi_{0,0}(q^{-1}, q^{-\sigma}) \) converges absolutely for \( \text{Re}(s) > -A/B \) (as for ordinary Dirichlet series this can be seen by looking at the Taylor series).
Let \( \sigma \in \mathbb{R} \) such that \( \Xi_{0,0}(q^{-1}, q^{-\sigma}) \) converges. The term \((1 - Q^{|t|B})\) corresponds to some extremal ray in some cone \( C(0, 0)_{j,k}^\pm \) in the decomposition chosen above. As \( \Xi_{0,0} \) is a series in \( Q \) and \( t \) with non-negative integral coefficients we may sum over the extremal ray to obtain \( \Xi_{0,0}(q^{-1}, q^{-\sigma}) \geq \sum_{i=0}^{\infty} q^{(-A-B)r_i} = \frac{1}{1-q^{-A-B}} \). This implies \( \sigma > -A/B \), and thus \(-A/B\) is the abscissa of convergence.

5.2.2. **Local abscissae of convergence and the proof of Theorem 5.3** Recall the notation set up in Section 5.1 in particular (5.1) and Assumption 5.3. Consider the generic fibre \( \mathfrak{X}_{K_0} \) of the scheme \( \mathfrak{X} \), equipped with the ideal sheaf \( \mathcal{L}_{K_0} \) which is the product of all non-zero ideals \( \mathcal{I}_{j,K_0} \) and \( \mathcal{J}_{j,K_0} \). To simplify the notation, we assume that all \( \mathcal{I}_{j,K_0} \) and \( \mathcal{J}_{j,K_0} \) are non-zero, so that \( \mathcal{L}_{K_0} = \prod_{j=0}^d \mathcal{I}_{j,K_0} \mathcal{J}_{j,K_0} \); in general, the ideal sheaves that are zero have to be removed. By Hironaka’s Theorem 5.7, there is a monomialisation

\[ h: Y \to \mathfrak{X}_{K_0} \]

of the ideal sheaf \( \mathcal{L}_{K_0} \). If a product of non-zero ideals is monomial, then each of the ideals in the product is monomial. For \( 0 \leq j \leq d \) we conclude that \( h^*(\mathcal{I}_{j,K_0}) \) (respectively \( h^*(\mathcal{J}_{j,K_0}) \)) is the sheaf of ideals of an effective divisor \( D_j \) (respectively \( C_j \)) with simple normal crossings; say

\[ D_j = \sum_{E \in T} N_E^{(j)} E \quad \text{and} \quad C_j = \sum_{E \in T} M_E^{(j)} E. \]

Here \( T \) denotes the finite set of smooth prime divisors, defined over \( K_0 \), occurring in the co-support of \( h^*(\mathcal{L}_{K_0}) \). Let \( D_{\text{disc}} \) denote the discrepancy divisor, that is, the divisor defined by the image of the canonical map \( h^*(\omega_{\mathfrak{X}_{K_0}}) \otimes \omega_X^\vee \to \mathcal{O}_Y \), where \( \omega_X \) denotes the canonical bundle on a variety \( X \). The support of \( D_{\text{disc}} \) is contained in \( T \), thus we may write

\[ D_{\text{disc}} = \sum_{E \in T} (\nu_E - 1) E \]

for certain \( \nu_E \in \mathbb{N} \). These parameters are relevant for our purposes, as they describe the pullback of the canonical \( p \)-adic measure; see Section 5.3. For any finite extension \( K \) of \( K_0 \) we obtain a monomialisation \( h_K: Y_K \to \mathfrak{X}_K \). As the divisors in \( T \) are smooth, they decompose into a disjoint union of prime divisors defined over \( K \). Let \( p \in \text{Spec}(\mathcal{O}_S) \) be a closed point. Since \( h \) is birational, we may pull back the integral defining \( Z_p \) to \( Y \) and we obtain

\[ Z_p(s) = \sum_{\ell \in \mathbb{Z}} q_{p}^{-\ell s} \int_{Y(K_p)} \left\| (\pi^{EF} h^{-1}(\mathcal{I}_j))_j \right\|_{\mathcal{O}_Y}^{-s} \left\| (\pi^{EF} h^{-1}(\mathcal{J}_j))_j \right\|_{\mathcal{O}_Y}^{-1} dh^*(\mu_{\mathfrak{X}_p}). \]

We point out two subtleties in connection with the formula (5.5). Firstly, the scheme \( \mathfrak{X} \) is projective and hence \( \mathfrak{X}(\mathcal{O}_p) = \mathfrak{X}(K_p) \). Secondly, the symbol \( h^{-1}(\mathcal{I}_j) \) denotes the sheaf-theoretic inverse image of the \( \mathcal{O}_X \)-module \( \mathcal{I}_j \), which is not an \( \mathcal{O}_Y \)-module; a similar remark applies to \( \mathcal{J}_j \).
Given an open subset \( V \subseteq_{o} Y(K_{p}) \), we define the restricted zeta function \( Z_{p}[V](s) \) by the formula obtained from (5.3) by restricting the domain of integration to \( V \). We denote by \( \alpha_{p}(V) \) the abscissa of convergence of \( Z_{p}[V] \). Observe that \( V \subseteq_{o} W \subseteq_{o} Y(K_{p}) \) implies \( \alpha_{p}(V) \leq \alpha_{p}(W) \).

**Definition 5.12.** Let \( K \) be a finite extension of \( K_{0} \) and let \( p \in \text{Spec}(O_{S}) \) be a closed point. We define the local abscissa of convergence at \( y \in Y(K_{p}) \) by

\[
\alpha_{p}(y) = \inf\{\alpha_{p}(V) \mid y \in V \subseteq_{o} Y(K_{p})\}.
\]

**Lemma 5.13.** In the above setting, for any open compact subset \( V \subseteq Y(K_{p}) \) the abscissa of convergence of \( Z_{p}[V] \) is

\[
\alpha_{p}(V) = \sup\{\alpha_{p}(y) \mid y \in V\}.
\]

**Proof.** Clearly, \( V \) is an open neighbourhood of all of the points that it contains, thus \( \alpha_{p}(V) \geq \sup\{\alpha_{p}(y) \mid y \in V\} \). Conversely, let \( \varepsilon > 0 \). For every \( y \in V \) we find an open neighbourhood \( W_{y} \) of \( y \) in \( V \) such that \( \alpha_{p}(W_{y}) < \alpha_{p}(y) + \varepsilon \). Since \( V \) is compact, we find a finite number of points \( y_{1}, \ldots, y_{n} \in V \) such that \( V = \bigcup\{W_{y_{i}} \mid 1 \leq i \leq n\} \). We deduce that

\[
\alpha_{p}(V) \leq \max\{\alpha_{p}(W_{y_{i}}) \mid 1 \leq i \leq n\} < \sup\{\alpha_{p}(y) \mid y \in V\} + \varepsilon.
\]

As \( \varepsilon \) tends to 0, we obtain \( \alpha_{p}(V) \leq \sup\{\alpha_{p}(y) \mid y \in V\} \). \( \square \)

Consider a point \( y \in Y(K_{p}) \). We use the monomialisation \( h : Y \rightarrow \mathfrak{x}_{K_{0}} \) to compute \( Z_{p}[V] \) for small coordinate neighbourhoods \( V \) of \( y \). Set \( T(y) = \{E \in T \mid y \in E\} \) and observe that \( u := \overline{|y|} \leq m = \dim(Y) \). Locally at \( y \), we can complement the collection of local equations \( g_{E} \) for \( E \in T(y) \) by \( m - u \) further elements to obtain a regular system of parameters.

Locally at \( x := h(y) \in \mathfrak{x}(O_{p}) \), the ideal \( I_{j} \), for \( j \in \{0, \ldots, d\} \), is finitely generated, since \( \mathfrak{x} \) is noetherian. Moreover, the sheaf \( h^{-1}(I_{j}) \) generates the monomial ideal \( h^{*}(I_{j,K_{0}}) \), therefore we obtain

\[
\|h^{-1}(I_{j})\|_{p} = |w|_{p} \prod_{E \in T(y)} \left| g_{E}^{N_{E}(j)} \right|_{p}
\]

locally at \( y \) for some regular function \( w \) with \( w(y) \neq 0 \). Similar considerations apply to the ideals \( J_{j} \). Therefore, if we choose \( V \cong p^{N}O_{p}^{m} \) to be a small coordinate neighbourhood, we obtain

\[
\left\| (\pi^{j}h^{-1}(I_{j}))^{d}_{j=0} \right\|_{p} = \left\| (\pi^{j+\delta_{j}} \prod_{E \in T(y)} g_{E}^{N_{E}(j)})^{d}_{j=0} \right\|_{p}
\]

\[
\left\| (\pi^{j}h^{-1}(J_{j}))^{d}_{j=0} \right\|_{p} = \left\| (\pi^{j+\varepsilon_{j}} \prod_{E \in T(y)} g_{E}^{M_{E}(j)})^{d}_{j=0} \right\|_{p}
\]

for certain integers \( \delta_{j}, \varepsilon_{j} \in \mathbb{Z} \). The canonical \( p \)-adic measure is locally defined by differential forms, thus the pullback \( h^{*}(\mu_{\mathfrak{x},p}) \) is transformed according to the discrepancy divisor. Writing \( T(y) = \{E_{1}, \ldots, E_{u}\} \), we deduce that the zeta
function $Z_p[V]$ is given by

$$Z_p[V](s) = \sum_{\ell \in \mathbb{Z}} q_p^{-\ell} \int_{p^{N_{O_p^n}}} \max_{0 \leq j \leq d} \left| x_1^{j+\delta_1} \prod_{i=1}^{u} x_i^{\sum_{i=1}^{u} N_{E_i}^{(i)} - s} \right|_p \cdot \max_{0 \leq j \leq d} \left| x_1^{j+\delta_2} \prod_{i=1}^{u} x_i^{\sum_{i=1}^{u} N_{E_i}^{(i)} - s} \right|_p \, d\mu(x)$$

(†)

$$= q_p^{N(u-m)-a} (1 - q_p^{-1})^u \Xi_{N,e,\delta}(q_p^{-1}, q_p^{-s}),$$

where $\mu$ denotes the normalised Haar measure satisfying $\mu(O_p^n) = 1$. Here $\Xi_{N,e,\delta}(Q, t)$ is equal to

$$\sum_{\ell \in \mathbb{Z}} \sum_{n \in \mathbb{Z}_{\geq 0}} Q^{-\min_{u \leq j \leq d} (\epsilon_j + (j-x) + \sum_{i=1}^{u} (M_{E_i}^{(i)} - \nu_{E_i} n_i))} - \min_{u \leq j \leq d} (\delta_j + j + \sum_{i=1}^{u} N_{E_i}^{(i)} n_i)} \Xi_{N,e,\delta}(Q, t)$$

and thus a power series of the form studied in Section 5.2.1 By our general assumptions we have $N_E^{(0)} = N_E^{j_0} = M_E^{(0)} = 0$ for all $E \in T$ and some $j_0 > 0$; so this series satisfies Assumption 5.10

**Proposition 5.14.** (1) Every point $y \in Y(K_p)$ has an open neighbourhood $V$ such that $\alpha_p(y) = \alpha_p(V)$.

(2) Let $K$ be a finite extension of $K_0$ and, for $i \in \{1, 2\}$, let $\iota_i : K \to K_{p_i}$ be a completion. For $y \in Y(K)$ the local abscissa does not depend on the completion, that is $\alpha_p(\iota_1(y)) = \alpha_p(\iota_2(y))$.

(3) The local abscissa at $y \in Y(K_p)$ depends only on $T(y) = \{E \in T \mid y \in E\}$.

**Proof.** The coordinate neighbourhoods on which the formula (†) holds form a neighbourhood base at $y$. By Lemma 5.11 the abscissa of convergence does not depend on $N$, hence (†) holds. Let $y'$ be another point in some completion $K_{p'}$ and assume that $T(y') = T(y)$. On small coordinate neighbourhoods of $y$ and $y'$, we have formulae of the form (†). In these formulae, different $N, e, \delta$ and $q$ may occur. However, by Lemma 5.11 these parameters do not change the abscissa of convergence and the assertions follow. □

**Proof of Theorem 5.8** For a subset $U \subseteq T$ we consider the closed smooth $K_0$-subscheme $E_U = \bigcap_{E \subseteq U} E \subseteq Y$ and the open subscheme $E_U^0 = E_U \setminus \bigcup_{E \not\subseteq U} E$ of $E_U$. The closed points $y \in E_U^0$ are concisely the ones which satisfy $T(y) = U$. By Proposition 5.11 the local abscissa is constant on all points of $E_U^0$ and we denote this abscissa by $\alpha(U)$.

Since $Y$ is projective, the space $Y(K_p)$ is compact. Therefore it can be covered by a finite number of disjoint open coordinate neighbourhoods on which the $\alpha_p$-function is given by a local formula as in (†). Now Lemma 5.11 implies that $Z_p(s)$ is rational in $q_p$ and $q_p^{-s}$. Moreover, the real parts of all poles of $Z_p$ are contained in a finite union of finite sets of the form $P_{\lambda, \beta}$ as in Lemma 5.11.

Let $p \in \text{Spec}(O_S)$ be a closed point. It follows from Lemma 5.13 that the abscissa of convergence of $Z_p$ is equal to

$$\alpha_p(Y(K_p)) = \max\{\alpha(U) \mid U \subseteq T \text{ and } E_U^0(K_p) \neq \emptyset\}.$$
Each of the schemes $E_U^0$ is of finite type over $K_0$, thus there is a finite extension $K_1$ of $K_0$ satisfying

$$E_U^0(K_1) \neq \emptyset$$

whenever $E_U^0$ is not the empty scheme. Suppose that $K_1 \subseteq K$. Then (5.6) shows that $\alpha_p(Y(K_p))$ is independent of $K$ and $p$; thus $\alpha(Z_p) = \max P$. \qed

5.3. The functional equation. In this section we prove part (4) of Theorem 5.1. We first consider the case $K = K_0$ and, to simplify the notation, it is convenient to use the short notation $O_S = O_{K,S}$ etc. introduced for $K$. Using this notation we construct in (3.7) the rational function $F$ featuring in Theorem 5.1. In a second step we go back to the original general notation and establish the functional equations for arbitrary extensions $K$ of $K_0$.

Consider the scheme $X$ over Spec($O_S$) as in Section 5.1. We would like to obtain a monomialisation of the ideal sheaves $\mathcal{I}_j$ and $\mathcal{J}_j$ over Spec($O_S$). For fields of positive characteristic a Monomialisation Theorem is not known and so we should not expect to achieve monomialisation globally. However, we can obtain a monomialisation on the generic fibre and expand it to some open set $U \subseteq_\circ \text{Spec}(O_S)$. It is straightforward to deduce the following result from Hironaka’s Monomialisation Theorem.

Theorem 5.15 (Generic Monomialisation). Let $X$ be a smooth integral projective scheme over Spec($O_S$) and let $\mathcal{I}$ be an ideal sheaf on $X$. There are a non-empty open subset $U \subseteq_\circ \text{Spec}(O_S)$, a smooth integral projective scheme $\mathfrak{X}$ over $U$ and a projective morphism $h: \mathfrak{X} \to X_U$ such that $h^*(\mathcal{I})$ is monomial and $h$ restricts to an isomorphism between the complements of the closed sets defined by $h^*(\mathcal{I})$ and $\mathcal{I}$.

We consider the sheaf of ideals $\mathcal{L}$ defined as the product over all $\mathcal{I}_j$ and $\mathcal{J}_j$ which are non-zero on $X$. As in Section 5.2.2 we assume that all $\mathcal{I}_j$ and $\mathcal{J}_j$ are non-zero, so that $\mathcal{L} = \prod_{j=0}^d \mathcal{I}_j \mathcal{J}_j$. We apply the generic Monomialisation Theorem to $\mathcal{L}$. Hence, after enlarging $S$, we may assume that there is a smooth integral projective scheme $\mathfrak{X}$ over Spec($O_S$) and a birational projective morphism $h: \mathfrak{X} \to X$ such that $h^*(\mathcal{L})$ is monomial. The structure morphism $\mathfrak{X} \to \text{Spec}(O_S)$ is a flat morphism of finite type between noetherian schemes and is hence open. Enlarging $S$ again if necessary, we may assume that the complement of the closed scheme $\mathfrak{Y}$ defined by $\mathcal{L}$ surjects onto Spec($O_S$).

The fact that $h$ is birational enables us to compute the zeta integral on $\mathfrak{Y}$ instead of $X$. Indeed, let $p \in \text{Spec}(O_S)$ be a closed point. The map $h: \mathfrak{Y}(O_p) \to X(O_p)$ is an homeomorphism, ignoring sets of measure zero, and consequently

$$Z_p(s) = \sum_{\ell \in \mathbb{Z}} q_p^{-\ell c} \int_{\mathfrak{Y}(O_p)} \left\| (\pi^{j \ell} h^*(\mathcal{I}_j))_{j=0}^d \right\|_{\mathfrak{Y}(O_p)}^{-s} \left\| (\pi^{j \ell} h^*(\mathcal{J}_j))_{j=0}^d \right\|_{\mathfrak{Y}(O_p)}^{-1} \left. dh^*(\mu_{\mathfrak{X},p}) \right.$$
These ideal sheaves are locally principal, hence they define effective Cartier divisors on \( Y \). For \( j \in \{0, \ldots, d\} \), let \( D_j \) (respectively \( C_j \)) denote the divisor defined by \( \tilde{J}_j \) (respectively \( \tilde{J}_j \)). Since \( Y \) is an integral noetherian separated regular scheme, there is, in fact, no difference between Cartier and Weil divisors on \( Y \); see [30, II.6.11]. Let \( T \) denote the set of prime divisors occurring in \( D = \sum_{j=0}^{d} C_j + D_j \). Hence we may write

\[
D_j = \sum_{E \in T} N_E^{(j)} E \quad \text{and} \quad C_j = \sum_{E \in T} M_E^{(j)} E, \quad \text{for } j \in \{0, \ldots, d\},
\]

where \( N_E^{(j)}, M_E^{(j)} \in \mathbb{Z} \) are certain non-negative integers. Note that \( M_E^{(0)} = N_E^{(0)} = 0 \) for all \( E \in T \). We enlarge \( S \) such that all the all prime divisors in \( T \) and all their intersections are smooth over \( \text{Spec}(O_S) \).

Now we consider the pullback \( h^*(\mu_{X,p}) \) of the canonical \( p \)-adic measure. The image of the canonical morphism of \( \mathcal{O}_Y \)-modules \( h^*\omega_X \otimes \omega_Y^p \to \mathcal{O}_Y \) is a locally principal ideal sheaf \( \mathcal{M} \), which defines the discrepancy divisor \( D^{\text{disc}} \), an effective Cartier divisor. Its support is contained in the support of \( D \) and so

\[
D^{\text{disc}} = \sum_{E \in T} (\nu_E - 1) E
\]

for certain multiplicities \( \nu_E - 1 \in \mathbb{N}_0 \). Note that \( \mathcal{M} \) is a monomial sheaf of ideals. The pullback of the canonical \( p \)-adic measure satisfies \( h^*(\mu_{X,p}) = \| M \|_{p} \mu_{Y,p} \).

For every set \( U \subseteq T \) of prime divisors, the intersection \( E_U = \bigcap_{E \in U} E \) is a smooth scheme over \( \text{Spec}(O_S) \). For every finite commutative \( O_S \)-algebra \( L \), we write \( b_U(L) = |E_U(L)| \) for the number of \( L \)-rational points of \( E_U \). We will consider the case where \( L \) is a finite extension of a residue field \( \kappa_p \) of \( O_p \).

**Proposition 5.16.** In the above situation for almost all closed points \( p \in \text{Spec}(O_S) \) the following formula holds:

\[
Z_p(s) = q_p^{-m} \sum_{U \subseteq W \subseteq T} (-1)^{|W \setminus U|} (q_p - 1)^{|U|} b_W(\kappa_p) \Xi_U(q_p^{-1}, q_p^{-s}),
\]

where the sum runs over all pairs \((U, W)\) of subsets \( U \subseteq W \subseteq T \) and

\[
\Xi_{U}(Q, t) = \sum_{(\ell, (n_E)_{E \in N_U})} Q^{-\left(\min_{E \in U} (\nu_E) + \sum_{E \in U} (M_E^{(j)} - \nu_E)n_E\right)} t^{-\left(\min_{E \in U} (j+1 + \sum_{E \in U} N_E^{(j)}n_E)\right)}.
\]

**Proof.** It suffices to establish the equation

\[
Z_p(s) = q_p^{-m} \sum_{U \subseteq T} (q_p - 1)^{|U|} c_U(\kappa_p) \Xi_U(q_p^{-1}, q_p^{-s}),
\]

where \( c_U(L) = |\{a \in \mathcal{Y}(L) \mid \forall E \in T: a \in E \iff E \in U\}| \). For then the proposition follows from the principle of inclusion and exclusion

\[
c_U(L) = \sum_{U \subseteq W \subseteq T} b_W(L)(-1)^{|W \setminus U|},
\]

where the sum runs over all sets \( W \) lying between \( U \) and \( T \).
The proof is very similar to the proof of [51, Thm. 2.1]. We only give a brief sketch. Take a \( \kappa_p \)-rational point \( a \in \mathcal{Y}(\kappa_p) \). Let \( T_a \) denote the subset of \( T \) consisting of those divisors passing through \( a \). Write \( T_a = \{ E_1, \ldots, E_r \} \) for \( r = |T_a| \), say. By construction the divisors have a simple normal crossing in \( a \), thus we find a regular local system of \( \mathcal{O}_{\mathcal{Y}, a} \) of the form \( \pi, g_1, g_2, \ldots, g_m \) such that \( g_i \) is the defining equation of \( E_i \) at \( a \), for \( i \in \{ 1, \ldots, r \} \). On the \( p \)-adic open set \( W_a = \{ y \in \mathcal{Y}(O_p) \mid y \equiv a \pmod{p} \} \), the following equalities hold:

\[
\| \widetilde{I}_j \|_p = \left| \prod_{i=1}^r g_i^{N_i^{(j)}} \right|_p, \quad \| \widetilde{J}_j \|_p = \left| \prod_{i=1}^r g_i^{M_i^{(j)}} \right|_p, \quad \| M \|_p = \left| \prod_{i=1}^r g_i^{\nu E_i} \right|_p.
\]

For any given \( \ell \in \mathbb{Z} \) this allows us to evaluate the integral over \( W_a \) as follows

\[
\int_{W_a} \left\| \left( \pi^{\ell} h^*(\mathcal{I}_j) \right) \right\|_p^d \left\| \left( \pi^{\ell} h^*(\mathcal{J}_j) \right) \right\|_p^d \ d\mathbf{h}^*(\mu_{X_p})
\]

\[
= \int_{W_a} \max_{0 \leq \ell \leq d} \left| \prod_{i=1}^r g_i^{N_i^{(j)}} \right|_p^{-s} \left| \prod_{i=1}^r g_i^{M_i^{(j)}} \right|_p^{-s} \ d\mu_{\mathcal{Y}_p}
\]

\[
= \int_{(\mathcal{Y}_p)^m} \max_{0 \leq \ell \leq d} \left| \prod_{i=1}^r x_i^{N_i^{(j)}} \right|_p^{-s} \left| \prod_{i=1}^r x_i^{M_i^{(j)}} \right|_p^{-s} \ d\mu_{\mathcal{Y}_p}
\]

\[
= \frac{(q_p - 1)^r}{q_p^m} \sum_{\mathbf{z} \in \mathbb{N}^r} q_p^m \left( \min_{0 \leq \ell \leq d} (\ell + \sum_{i=1}^r (M_i^{(j)} - \nu E_i) n_i) + s \min_{0 \leq \ell \leq d} (\ell + \sum_{i=1}^r N_i^{(j)} n_i) \right)
\]

Now the proposition follows by grouping the finitely many \( \kappa_p \)-rational points according to the prime divisors on which they lie. \( \square \)

Lemma 5.11 (1) and (5) imply that \( \Xi_U \) is rational and has the following inversion property.

**Lemma 5.17.** For every set \( U \subseteq T \) of prime divisors the formal sum \( \Xi_U(Q, t) \in Q((q))[[t]] \) is a rational function in \( Q(Q, t) \). The functions \( \Xi \) satisfy the inversion property

\[
\Xi_U(Q^{-1}, t^{-1}) = (-1)^{|U|+1} \sum_{V \subseteq U} \Xi_V(Q, t)
\]

for all \( U \subseteq T \).

We record a simple combinatorial lemma.

**Lemma 5.18.** Let \( V \subseteq W \) be finite sets and \( z \) some indeterminate. Then

\[
\sum_{V \subseteq U \subseteq W} (z - 1)^{|U|}(z - 1)^{|W \setminus U|} = (-1)^{|W \setminus V|}(z - 1)^{|V|}.
\]
Proof. Say \(|W| = n\) and \(|V| = r \leq n\) and write \(\Delta = W \setminus V\).

\[
(-1)^{n-r}(z-1)^r = ((z-1) - z)^{n-r}(z-1)^r
= \sum_{j=0}^{n-r} \binom{n-r}{j}(-1)^{n-r-j}(z-1)^{r+j}z^{n-r-j}
= \sum_{H \subseteq \Delta} (-1)^{n-r-|H|}(z-1)^{r+|H|}z^{n-r-|H|}
= \sum_{V \subseteq U \subseteq W} (-1)^{|W \setminus U|}(z-1)^{|U|}z^{|W \setminus U|}.
\]

\(\square\)

Let \(f\) be a positive integer, and let \(L_f\) denote the unique extension of \(\kappa_p\) of degree \(f\). For simplicity we write \(b_U(p, f) = b_U(L_f)\). We proceed along the lines of \([18]\); see \([16]\) for an overview of the relevant results on \(\ell\)-adic cohomology.

Write \(E_U = E_U \times \overline{\kappa_p}\), where \(\overline{\kappa_p}\) denotes the algebraic closure of \(\kappa_p\). Then, by the \(\ell\)-adic Lefschetz fixed point principle, we obtain

\[
b_U(p, f) = \sum_{i=0}^{2(m-|U|)} (-1)^i \text{Tr} (\varphi^f|H^i(E_U, \mathbb{Q}_\ell)) ,
\]

where \(\varphi\) denotes the endomorphism induced by the Frobenius automorphism. In particular, we get an expression

\[
b_U(p, f) = \sum_{i=0}^{2(m-|U|)} (-1)^i \sum_{j=1}^{g(U, i)} \lambda_{U, i, j}^f
\]

in terms of the eigenvalues \(\lambda_{U, i, j}^f\), for \(1 \leq j \leq g(U, i) := \dim H^i(E_U, \mathbb{Q}_\ell)\), of \(\varphi\) acting on the \(\ell\)-adic cohomology \(H^i(E_U, \mathbb{Q}_\ell)\). Poincaré duality in \(\ell\)-adic cohomology implies the functional equation of the Weil zeta function

\[
(\dagger) b_U(p, -f) = q_p^{-f(m-|U|)} b_U(p, f).
\]

By construction, the schemes \(E_U\) are smooth and projective (hence proper) over \(\text{Spec}(O_S)\); recall that \(S\) was enlarged to achieve this. By the smooth proper base change theorem (see \([16\), §6.8] or \([17\), Thm. (3.1), p. 62]) the \(i\)th \(\ell\)-adic Betti number is independent of the chosen place \(p\). In particular, the number of Frobenius eigenvalues is the same for all \(p\).

Let \(\Upsilon\) be the set of all triples \(v = (U, i, j)\) satisfying \(U \subseteq T\), \(0 \leq i \leq m-|U|\) and \(1 \leq j \leq g(U, i)\). Consider the polynomial ring

\[
\mathbb{Q}(Y_1, Y_2)[X] := \mathbb{Q}(Y_1, Y_2)[X_v \mid v \in \Upsilon]
\]

over the field of rational functions \(\mathbb{Q}(Y_1, Y_2)\), where \(X\) denotes the whole collection of independent variables \(X_v\). For every \(U \subseteq T\), we set

\[
b_U(X) = \sum_{i=0}^{2(m-|U|)} (-1)^i \sum_{j=1}^{g(U, i)} X_{U, i, j},
\]
and we define the rational function
\[ F = F(Y_1, Y_2, X) \]
\[ (5.7) \]
\[ = Y_1^m \sum_{U \subseteq W \subseteq T} (-1)^{|W \setminus U|} (Y_1^{-1} - 1)^{|U|} b_W(X) \Xi_U(Y_1, Y_2) \]
\[ \in \mathbb{Q}(Y_1, Y_2)[X]. \]

As explained above, we now revert to the original notation. It remains to establish the functional equations for arbitrary finite extension \( K \) of \( K_0 \). Let \( K \) be a finite extension of \( K_0 \) of inertia degree \( f \), and let \( p \in \text{Spec}(O_S) \) be a closed point over \( p_0 \in \text{Spec}(O_{K_0, S}) \). As in the statement of Theorem 5.1 set \( q = q_{p_0} \). The preceding discussion yields
\[ Z_p(s) = F(q^{-f}, t^{-f}, (\lambda_v^{-f})_{v \in \mathcal{Y}}), \]
where the \( \lambda_v \) are the corresponding Frobenius eigenvalues as above.

**Theorem 5.19.** In the above setting, the function \( F \) satisfies for every positive integer \( f \) the functional equation
\[ F(q^f, t^{-f}, (\lambda_v^{-f})_{v \in \mathcal{Y}}) = -q^{-fm} F(q^{-f}, t^f, (\lambda_v^f)_{v \in \mathcal{Y}}), \]
where \( t \) is any indeterminate.

**Proof.** The following calculation, based on [B], Lemma 5.17 and Lemma 5.18 as indicated, proves the functional equation:
\[ F(q^f, t^{-f}, (\lambda_v^{-f})_{v \in \mathcal{Y}}) \]
\[ = q^{-fm} \sum_{U \subseteq W \subseteq T} (-1)^{|W \setminus U|} (q^{-f} - 1)^{|U|} b_W(p, -f) \Xi_U(q^f, t^{-f}) \]
\[ = q^{-fm} \sum_{U \subseteq W \subseteq T} (-1)^{|W \setminus U|} (q^{-f} - 1)^{|U|} q^{-f(m-|W|)} b_W(p, f) \Xi_U(q^f, t^{-f}) \]
\[ = \sum_{V \subseteq U \subseteq W \subseteq T} (-1)^{|W \setminus U|} (q^f - 1)^{|U|} q^{f|W|} b_W(p, f) (-1)^{|U|+1} \Xi_V(q^{-f}, t^f) \]
\[ = - \sum_{V \subseteq U \subseteq W \subseteq T} (-1)^{|W \setminus U|} (q^f - 1)^{|U|} b_W(p, f) \Xi_V(q^{-f}, t^f) \]
\[ = - \sum_{W \subseteq T} (-1)^{|W|} (q^f - 1)^{||W|} b_W(p, f) \Xi_V(q^{-f}, t^f) \]
\[ = -q^{-fm} F(q^{-f}, t^f, (\lambda_v^f)_{v \in \mathcal{Y}}), \]
\[ \square \]

6. **Groups acting on rooted trees and multiplicity-free representations**

In this section we describe a geometric situation which leads to a transparent treatment of zeta functions associated to certain admissible representations. In particular, this yields a straightforward description of the zeta function associated to the induced representation \( \text{Ind}_{P_{k_1}}^{GL_{n+1}(O)}(1_{P_k}) \) from a maximal \((1, n)\)-parabolic subgroup \( P_k \) to the compact \( p \)-adic Lie group \( GL_{n+1}(O) \); see Proposition [5.5] for details.
Central to the geometric approach is a suitable notion of (weak) 2-transitivity.

**Definition 6.1.** The action of a group $G$ on a metric space $(X, d)$, from the right by isometries, is called distance transitive if for all $x_1, x_2, y_1, y_2 \in X$ with $d(x_1, x_2) = d(y_1, y_2)$ there is an element $g \in G$ with $x_ig = y_ig$ for $i \in \{1, 2\}$.

More specifically, if $G$ is a profinite group acting distance transitively on a rooted tree $T$ there is a geometric way to compute the zeta function of the representation of $G$ on the space $C^\infty(\partial T, \mathbb{C})$ of locally constant functions on the boundary $\partial T$. The corresponding method was used by Bartholdi and Grigorchuk [6] for specific groups. In [9], Bekka and de la Harpe discuss a general approach based on reproducing kernels; in collaboration with Grigorchuk, they treat groups acting on rooted trees in detail in an appendix. We briefly review the theoretic background and produce a formula for the zeta function. As an application we compute the zeta functions of induced representations of certain compact $p$-adic analytic groups that are not torsion-free pro-$p$.

### 6.1. Trees and distance transitive actions.

Let $m = (m_i)_{i \in \mathbb{N}}$ be an integer sequence with $m_i \geq 2$ for all $i \in \mathbb{N}$. We define the associated spherically homogeneous rooted tree $T_m$ as follows. The vertices of level $n \geq 0$ are finite sequences $w = (w_1, \ldots, w_n)$ of length $n$ with $w_i \in \{0, \ldots, m_i - 1\}$. We write $|w| = n$ to indicate that $w$ is of level $n$. The root of $T_m$ is the empty sequence. There is an edge between a vertex $v$ of level $n$ and a vertex $w$ of level $n + 1$ if and only if $v$ is a prefix of $w$. Let $L_m(n)$ denote the set of vertices of level $n$ in $T_m$. For any two vertices $v$ and $w$ we write $\text{pre}(v, w)$ for the longest common prefix of $v$ and $w$, regarded as a vertex of $T_m$.

The boundary $\partial T_m$ of $T_m$ is the space of geodesic rays in $T_m$ starting at the root, that is, the profinite topological space of infinite sequences $(\xi_i)_{i \in \mathbb{N}}$ with $\xi_i \in \{0, \ldots, m_i - 1\}$. For any $n \in \mathbb{N}_0$, we write $\xi(n)$ for the prefix of $\xi$ of length $n$ and we say that the geodesic ray $\xi$ passes through $\xi(n)$. The longest common prefix of $\xi, \eta \in \partial T_m$ is denoted $\text{pre}(\xi, \eta)$, which is a vertex of $T_m$ unless $\xi = \eta$.

There is a natural metric $d$ on $\partial T_m$ given by

$$d(\xi, \eta) = \frac{1}{1 + |\text{pre}(\xi, \eta)|} \quad \text{for all } \xi, \eta \in \partial T_m \text{ with } \xi \neq \eta.$$  

The same formula defines also a metric on $L_m(n)$.

Let $\text{Aut}(T_m)$ denote the profinite group of rooted automorphisms of $T_m$.

**Lemma 6.2.** A closed subgroup $G \leq_e \text{Aut}(T_m)$ acts distance transitively on $\partial T_m$ if and only if it acts distance transitively on each layer $L_m(n), n \in \mathbb{N}_0$.

**Proof.** First suppose that $G \leq_e \text{Aut}(T_m)$ acts distance transitively on the boundary $\partial T_m$. Let $n \in \mathbb{N}_0$ and, for $i \in \{1, 2\}$, let $x_i, y_i \in L_m(n)$ such that $d(x_1, x_2) = d(y_1, y_2)$. Choose geodesic rays $\xi_i$ passing through $x_i$ and $\eta_i$ passing to $y_i$ with $d(\xi_1, \xi_2) = d(\eta_1, \eta_2)$. Then there exists $g \in G$ such that $\xi_ig = \eta_i$ for $i \in \{1, 2\}$, and hence $x_ig = y_i$.

Conversely, suppose that $G$ acts distance transitively on each layer. For $i \in \{1, 2\}$, consider $\xi_i, \eta_i \in \partial T_m$ such that $d(\xi_1, \xi_2) = d(\eta_1, \eta_2)$. For each
Remark 6.4. Theorem 6.3 applies, in particular, to \( G = \text{Aut}(\mathcal{T}_m) \). Variation of the sequence \( m = (m_i)_{i \in \mathbb{N}} \) allows one to construct induced representations with various different zeta functions. For instance, if we apply the theorem to
the $d$-regular tree $T_m$ with $m_i = d$ for all $i$ and some fixed integer $d \geq 2$, then the zeta function of the representation on the boundary is $\zeta_{\eta_\gamma}(s) = 1 + \frac{(d-1)^{-s}}{1-d^{-s}}$ which admits a meromorphic continuation to the complex plane. By contrast, suppose that the sequence $(m_i)_{i \in \mathbb{N}}$ tends to infinity with $i$. In this case the zeta function $\zeta_{\eta_\gamma}(s)$ cannot be extended analytically and the vertical axis $\Re(s) = 0$ is the natural boundary; see [40 Thm. VI.2.2].

6.2. Induction from maximal $(1,n)$-parabolic subgroups to $GL_{n+1}(\Delta)$. In this section $\mathfrak{o}$ denotes a complete discrete valuation ring with maximal ideal $\mathfrak{p}$ and finite residue field $\kappa = \mathfrak{o}/\mathfrak{p}$ of cardinality $q$. Let $\pi \in \mathfrak{p}$ denote a uniformiser. We emphasise that here $\mathfrak{o}$ may have positive characteristic, e.g., $\mathfrak{o} = \mathbb{F}_p[[t]]$.

Consider a central division algebra $\mathfrak{d}$ of index $d$ over the fraction field $\mathfrak{f}$ of $\mathfrak{o}$ so that $\dim_{\mathfrak{f}} \mathfrak{d} = d^2$. It is known that $\mathfrak{d}$ contains a unique maximal $\mathfrak{o}$-order $\Delta$. Up to isomorphism, $\mathfrak{d}$ and $\Delta$ can be described explicitly, in terms of the index $d$ and a second invariant $r$ satisfying $1 \leq r \leq d$ and $\gcd(r,d) = 1$; see [44 §14] and the more explicit description given in Section 7.4. Here it suffices to fix a uniformiser $\Pi \in \mathfrak{d}$ so that $\Delta$ has maximal ideal $\mathfrak{P} = \Pi \Delta$. Recall that every element of $\Delta$ can be written as a converging power series $\sum_{k=0}^{\infty} c_k \Pi^k$ with coefficients coming from any set of representatives for the elements of the residue field $\mathfrak{f}/\mathfrak{P}$, which is a finite field of size $q^d$.

Fix $n \in \mathbb{N}$, and consider the general linear group $GL_{n+1}(\Delta)$ with a maximal parabolic subgroup $P_\xi$ which is the stabiliser of a point $\xi \in \mathbb{P}^n(\Delta)$ under the right $GL_{n+1}$-action. For instance, if $\xi_0 = (1:0:\ldots:0)$, then

$$P_{\xi_0} = \left( \begin{array}{c|c} \Delta & 0 \\ \hline \Delta & GL_n(\Delta) \end{array} \right) \leq GL_{n+1}(\Delta).$$

Proposition 6.5. For all $\xi \in \mathbb{P}^n(\Delta)$ the pair $(GL_{n+1}(\Delta), P_\xi)$ is a Gelfand pair and the induced representation $\varrho = Ind_{P_\xi}^{GL_{n+1}(\Delta)}(1_{P_\xi})$ has the zeta function

$$\zeta_{\varrho}(s) = 1 + \left( \frac{q^{dn} - 1}{q^d - 1} \right)^{-s} \left( q^{-ds} + \frac{(q^{dn+1} - 1)^{-s}}{1 - q^{-ds}} \right).$$

Proof. Consider the (left) projective $n$-space $\mathbb{P}^n(\Delta)$. Concretely, we describe elements $x$ in terms of homogeneous coordinates $x = (x_0 : \ldots : x_n)$, where $(x_0, \ldots, x_n) \in \Delta^n$ is primitive, i.e., has at least one entry not contained in $\mathfrak{P}$, and $(x_0 : \ldots : x_n) = (zx_0 : \ldots : zx_n)$ for all units $z \in \Delta^\times$.

We construct a rooted tree $T$ representing $\mathbb{P}^n(\Delta)$ as follows. The root $\varepsilon$ is the unique vertex of level 0 and, for each $k \in \mathbb{N}$, the vertices of level $k$ are the points of $\mathbb{P}^n(\Delta/\mathfrak{P}^k)$. There is an edge between the root and every vertex of level 1. Moreover, a vertex $x \in \mathbb{P}^n(\Delta/\mathfrak{P}^k)$ of level $k$ and a vertex $y \in \mathbb{P}^n(\Delta/\mathfrak{P}^{k+1})$ of level $k+1$ are connected by an edge if and only if $x \equiv y \mod \mathfrak{P}^k$. The boundary of $T$ is $\partial T \cong \mathbb{P}^n(\Delta)$ since $\Delta$ is complete. Note that $T$ is spherically homogeneous and $T \cong T_m$ for $m_1 = |\mathbb{P}^n(\Delta/\mathfrak{P})| = (q^{dn+1} - 1)/(q^d - 1)$ and $m_k = q^{dn}$ for $k \geq 2$. 


The group $\text{GL}_{n+1}(\Delta)$ acts on the tree $T$ from the right. We show below that the action on the boundary is distance transitive. Consequently, Theorem 6.3 implies that $(\text{GL}_{n+1}(\Delta), P^\Delta)$ is a Gelfand pair and

$$\zeta_0(s) = 1 + (|\mathbb{P}^n(\Delta/\mathbb{Q})| - 1)^{-s} + |\mathbb{P}^n(\Delta/\mathbb{Q})|^{-s}(q^d - 1)^{-s} \sum_{k=0}^{\infty} q^{-dnks}$$

$$= 1 + \left( \frac{q^{d(n+1)} - q^d}{q^d - 1} \right)^{-s} + \left( \frac{q^{d(n+1)} - 1}{q^d - 1} \right)^{-s} (q^d - 1)^{-s} \frac{1}{1 - q^{-dn}}$$

$$= 1 + \left( \frac{q^{dn} - 1}{q^d - 1} \right)^{-s} \left( q^{-dn} + \frac{(q^{dn+1} - 1)^{-s}}{1 - q^{-dn}} \right).$$

It remains to show that $\text{GL}_{n+1}(\Delta)$ acts distance transitively on $\mathbb{P}^n(\Delta)$. Since $\text{GL}_{n+1}(\Delta)$ acts transitively on $\mathbb{P}^n(\Delta)$, it suffices to show that the group $P_0 = P^{\xi_0}$ acts transitively on the spheres around $\xi_0 = (1 : 0 : \ldots : 0) \in \mathbb{P}^n(\Delta)$. Let $x, y \in \mathbb{P}^n(\Delta)$ lie on such a sphere, that is, $x \equiv \xi_0 \equiv y \pmod{\mathbb{P}}$ but $x \not\equiv \xi_0 \pmod{\mathbb{P}^{k+1}}$ and $y \not\equiv \xi_0 \pmod{\mathbb{P}^{k+1}}$ for some $k \geq 0$. Note that, for $k = 0$, the first congruence holds trivially for all $x, y$.

We need to find $g \in P_0$ such that $xg = y$. We may write

$$x = (x_0 : \Pi^k u_1 : \ldots : \Pi^k u_n) \quad \text{and} \quad y = (y_0 : \Pi^k v_1 : \ldots : \Pi^k v_n),$$

where $x_0 \equiv y_0 \equiv 1 \pmod{\mathbb{P}}$ and $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in \Delta^n$ are primitive. There is a matrix $h \in \text{GL}_n(\Delta)$ such that $uh = v$, since $\text{GL}_n(\Delta)$ acts transitively on primitive vectors. Furthermore, we find $a_1, \ldots, a_n \in \Delta$ such that $\Pi^k \sum_{i=1}^n u_i a_i = y_0 - x_0$. Thus

$$g = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ a_n & \cdots & 0 & h \end{pmatrix} \in P_0$$

has the property that $xg = y$. \hfill \Box

7. Examples of induced representations of potent pro-$p$ groups

Let $\mathfrak{o}$ be the ring of integers of a $p$-adic field $\mathfrak{f}$, with maximal ideal $p \leq \mathfrak{o}$ and finite residue field $\kappa = \mathfrak{o}/p$ of cardinality $q$. Let $\pi \in p$ denote a uniformiser.

7.1. Induction from a Borel subgroup to $\text{GL}_3(\mathfrak{o})$. We consider the general linear group $G = \text{GL}_3(\mathfrak{o})$ and, for $r \in \mathbb{N}$, its principal congruence subgroup $G^r = \ker(\text{GL}_3(\mathfrak{o}) \twoheadrightarrow \text{GL}_3(\mathfrak{o}/p^r))$.

Let $B \leq c G$ be the Borel subgroup consisting of upper triangular matrices, and set $B^r = B \cap G^r$. Our aim is to compute the zeta function of the induced representation $\text{Ind}_{B^r}^{G^r}(1_{B^r})$. We remark that the zeta function of the induced representation $\text{Ind}_{B^r}^{G^r}(1_{B^r})$ has already been computed by Omr and Singla [43 Thm. 6.5], using direct representation-theoretic considerations.

Let $\mathfrak{g} = \mathfrak{gl}_3(\mathfrak{o})$ denote the $\mathfrak{o}$-Lie lattice of $3 \times 3$-matrices and let $\mathfrak{b}$ denote the sublattice of all upper triangular matrices. The groups $G^r$ and $B^r$ are finitely
Proposition 7.1. In the set-up described above and subject to $r \geq 2e$ for $p = 2$ and $r \geq e(p-2)^{-1}$ for $p > 2$, the zeta function of the representation $\varrho = \text{Ind}_{B^r}(1_{B^r})$ is

$$\zeta_{\varrho}(s) = q^{3(r-1)}\frac{q^{1-3s}(q - q^{-s})(u(q^s) - qu(q^{-s}))}{(1-q^{-s})(1-q^{-6s})}$$

where $u(X) = X^2 - 2X + 1 - 2X^{-1} + X^{-2} - X^{-3}$.

The abscissa of convergence is $\alpha(\varrho) = 1/6$.

It would be interesting to find an interpretation of the value $1/6$ for the abscissa of convergence; compare the discussion in the introduction. Note that from the given formula it is easy to verify that $\zeta_{\varrho}$ satisfies a functional equation, in accordance with Theorem 5.1.

Proof of Proposition 7.1. We indicate how the result can be obtained, by applying the method described in Section 4.2. As most steps consist of simple computations we leave the verification of some details to the reader.

Consider the $\mathfrak{g}$-basis $(E_{i,j})_{i,j=1}^3$ of $\mathfrak{g}$ which consists of all elementary matrices; i.e., the only non-zero entry of $E_{i,j}$ is a 1 in position $(i,j)$. The subalgebra $\mathfrak{k}$ of strictly lower triangular matrices is a complement of $\mathfrak{b}$ in $\mathfrak{g}$, and the matrices $E_{i,j}$ with $j < i$ form a basis of $\mathfrak{k}$.

Using the commutator relations $[E_{i,j}, E_{s,t}] = \delta_{j,s}E_{i,t} - \delta_{i,t}E_{s,j}$, one can determine the commutator matrix $R(T) \in \text{Mat}_9(\mathbb{Z}[T_{i,j} \mid 1 \leq i,j \leq 3])$, introduced in Section 4.2, for a set of variables $T_{i,j}$ corresponding to the chosen basis of elementary matrices $E_{i,j}$. Substituting $T_{2,1} = x$, $T_{3,2} = y$, $T_{3,3} = z$ and $T_{i,j} = 0$ whenever $i \leq j$ we obtain the reduced commutator matrix

$$R(x,y,z) = \begin{pmatrix}
0 & 0 & 0 & z & -z & -x & y & 0 \\
0 & 0 & -z & x & -x & 0 & 0 & 0 \\
0 & z & 0 & 0 & y & -y & 0 & 0 \\
-z & -x & 0 & 0 & \cdots & \cdots & 0 \\
0 & x & -y & 0 & \cdots & \cdots & 0 \\
z & 0 & y & : & : & \ddots & : \\
x & 0 & 0 & : & : & \ddots & : \\
-y & 0 & 0 & : & : & \ddots & : \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0
\end{pmatrix}.$$  

An easy (but, if done by hand, lengthy) calculation yields the sets $F_k$ of degree-$k$ Pfaffians. Removing those Pfaffians, whose modulus cannot dominate the others, we obtain

$$\|F_0(x,y,z)\|_p = 1, \quad \|F_1(x,y,z)\|_p = \|x, y, z\|_p,$$

$$\|F_2(x,y,z)\|_p = \|x^2, y^2, z^2\|_p, \quad \|F_3(x,y,z)\|_p = \|x^2y, xy^2\|_p.$$
Hence Proposition 4.6 yields the formula

$$\zeta_\phi(s) = q^{3r} \int_{\mathbb{P}^3} \|1, x^2, y^2, z^2, x^2y, xy^2\|_{\mathbb{P}}^{-1-s} d\mu(x, y, z)$$

$$= q^{3r} + q^{3r}(1 - q^{-1}) \sum_{\ell=1}^{\infty} q^{3\ell} \int_{\mathbb{P}^2(\ell)}\|\pi^{-2\ell} : \pi^{-3\ell} \mathcal{I}_3\|_{\mathbb{P}}^{-1-s} d\mu_{\mathbb{P}^2,\ell},$$

where $\mathcal{I}_3$ is the sheaf of ideals on $\mathbb{P}^2/\mathfrak{o}$ generated by the image of $\mathcal{O}(3)^\vee = \mathcal{O}(-3)$ after pairing with the sections $x^2y$ and $xy^2$ of the 3rd twisting sheaf $\mathcal{O}(3)$.

The fibres of the reduction map $\mathbb{P}^2(\ell) \to \mathbb{P}^2(\kappa)$ admit global charts and via these charts the contribution to the integral (*) from any one fibre can be written as an integral over $\pi\mathfrak{o}^2$. Thus it remains to compute the integral over every fibre.

Let $(\bar{x} : \bar{y} : \bar{z}) \in \mathbb{P}^2(\kappa)$ be a point in projective coordinates. (a) There are $q(q - 1)$ many points satisfying $\bar{x}, \bar{y} \neq 0$. In these fibres the integral has constant value $q^{-2}q^{-3\ell - 3s}$ and working out the geometric series, we obtain the overall contribution

$$q^{3r}(1 - q^{-1}) \cdot q(q - 1) \cdot q^{-2} \sum_{\ell=1}^{\infty} q^{-3\ell s} = q^{3r}(1 - q^{-1})^2 \frac{q^{-3s}}{1 - q^{-3s}}$$

to $\zeta_\phi(s)$. (b) There are $q$ points satisfying $\bar{x} \neq 0$ and $\bar{y} = 0$. In these fibres one can compute the integral by distinction of the cases $|y|_p \geq q^{-\ell}$ and $|y|_p < q^{-\ell}$. Similarly, there are $q$ points satisfying $\bar{x} = 0$ and $\bar{y} \neq 0$; these points yield the same contributions, because the integral formula is symmetric in $x$ and $y$. We obtain the overall contribution

$$q^{3r}2(1 - q^{-1}) \frac{q^{-2s} - q^{-1-5s}}{(1 - q^{-3s})(1 - q^{-2s})}$$

to $\zeta_\phi(s)$. (c) Finally, the integral over the fibre of the remaining point $(0 : 0 : 1)$ is more intricate: the resulting 3-dimensional summation, over $\ell$ and the valuations of $x$ and $y$, can be calculated by choosing a suitable cone decomposition of the parameter space. For example, we first considered the case where $x$ and $y$ have the same valuation and then used the symmetry in $x$ and $y$ to treat the remaining cases. Skipping the details, we record the overall contribution

$$q^{3r}(1 - q^{-1})^3 \left( \frac{q^{1-2s} + q^{-4s} + q^{1-6s}}{(1-q^{-2})(1-q^{-6s})} + \frac{q^{1-9s}}{(1-q^{-3s})(1-q^{-6s})} \right)$$

$$+ \frac{2q^{2-2s}}{(1-q^{-2})(1-q^{-4})} + \frac{2(q^{1-4s} + q^{6s} + q^{1-8s})}{(1-q^{-2})(1-q^{-4})(1-q^{-8s})}$$

$$+ \frac{2q^{1-8s}}{(1-q^{-2})(1-q^{-4})(1-q^{-8s})}$$

to $\zeta_\phi(s)$. Addition and simplification yields the explicit formula. □

7.2. Induction from a Borel subgroup to $\text{U}_3(\mathfrak{o})$. Let $\mathfrak{e}$ be a quadratic extension field of $\mathfrak{f}$ and let $\mathfrak{o}_\mathfrak{e}$ denote the ring of integers of $\mathfrak{e}$. Denote the non-trivial Galois automorphism of $\mathfrak{e}$ over $\mathfrak{f}$ by $\sigma$. 

Let $U_3$ be the unitary group scheme over $\mathfrak{o}$ associated to the extension $\mathfrak{o}_\varepsilon$ and the non-degenerate hermitian matrix $W = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. We study the group

$$G = U_3(\mathfrak{o}) = \{ g \in \text{GL}_3(\mathfrak{o}) \mid \sigma(g)^{\text{tr}} W g = W \}$$

of $\mathfrak{o}$-rational points and its principal congruence subgroups $G^r = G \cap \text{GL}_3^r(\mathfrak{o})$ as in Section 7.1. From the given formula it is easy to verify that $\zeta$ form an $\mathfrak{o}$-Lie lattice associated to the extension $(\mathfrak{s}, \mathfrak{r})$. The groups $G^r$ and $B^r$ are finitely generated torsion-free pro-$p$ groups whenever $r \geq 2e$ for $p = 2$ and $r \geq e(p - 2)^{-1}$ for $p > 2$, where $e$ denotes the ramification index of $\mathfrak{f}$ over $\mathbb{Q}_p$. In this case $\pi^* u_3$, regarded as a $\mathbb{Z}_p$-Lie lattice, is the Lie lattice associated to $G^r$.

**Proposition 7.2.** Suppose that $p$ is odd and that $\varepsilon$ is unramified over $\mathfrak{f}$. In the set-up described above and subject to $r \geq e(p - 2)^{-1}$, the zeta function of the representation $\varrho = \text{Ind}_{B^r}^{G^r}(1_{B^r})$ is

$$\zeta_{\varrho}(s) = q^{3(r-1)} \frac{q^{1-2s}(q - q^{-s})(q^{-s}u(q^s) + qu(q^{-s}))}{(1 - q^{1-6s})}$$

where $u(X) = X^2 + X + X^{-2}$. The abscissa of convergence is $\alpha(\varrho) = 1/6$.

As before, it would be interesting to find an interpretation for the value $1/6$; it is highly suggestive that the abscissa agrees with the one found in Proposition 7.1. From the given formula it is easy to verify that $\zeta_{\varrho}$ satisfies a functional equation, in accordance with Theorem 5.1. A similar formula holds in case $\varepsilon$ is ramified over $\mathfrak{f}$, however the resulting zeta function does not admit a functional equation. In the ramified case the denominator is still $(1 - q^{1-6s})$, whereas the numerator is given by a more complicated formula.

**Proof of Proposition 7.2.** Since $p$ is odd, the canonical map $\mathfrak{o}_\varepsilon \otimes_{\mathfrak{o}} U_3 \to \text{gl}_3(\mathfrak{o})$ is an isomorphism of $\mathfrak{o}_\varepsilon$-Lie lattices. Since $\varepsilon$ is unramified over $\mathfrak{f}$, we find $\delta \in \mathfrak{d}^\perp \setminus \mathfrak{o}$ with $\delta^2 \in \mathfrak{o}^\varepsilon$. Then, $1, \delta$ form an $\mathfrak{o}$-basis for $\mathfrak{o}_\varepsilon$ and $\sigma(\delta) = -\delta$.

Let $\mathfrak{k} \leq \mathfrak{u}_3$ be the Lie sublattice of strictly lower triangular matrices; clearly $\mathfrak{k} \oplus \mathfrak{b} = \mathfrak{u}_3$, and the matrices

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

form an $\mathfrak{o}$-basis for $\mathfrak{k}$. Consider an $\mathfrak{o}$-linear map $\omega: U_3 \to \mathfrak{f}$ that vanishes on $\mathfrak{b}$, and put $u = \omega(A)$, $v = \omega(B)$ and $w = \omega(C)$. We need to find the symplectic minors of the symplectic form $\tilde{\omega}$ on $U_3$, given by $\tilde{\omega}(X,Y) = \omega([X,Y])$ for $X,Y \in U_3$.

The symplectic minors are invariant under base change. As $\mathfrak{o}_\varepsilon \otimes_{\mathfrak{o}} U_3 \cong \text{gl}_3(\mathfrak{o})$ we simply need to compare the basis $A, B, C$ with the basis $E_{2,1}, E_{3,2}, E_{3,1}$ used in the proof of Proposition 7.1. In the notation used there, we obtain $x = \frac{1}{2}(v + \delta^{-1}u)$, $y = \frac{1}{2}(\delta^{-1}u - v)$ and $z = \delta^{-1}w$. Observe that $|v + \delta^{-1}u|_p =$
7.3. **Induction from a maximal parabolic subgroup to** $\GL_n^0(\mathfrak{o})$. Let $n, t \in \mathbb{N}$ such that $n = t + t'$ with $1 \leq t \leq t' := n - t$. We consider the general linear group $G = \GL_n(\mathfrak{o})$ and, for $r \in \mathbb{N}$, its principal congruence subgroup $G^r = \ker(\GL_n(\mathfrak{o}) \to \GL_n(\mathfrak{o}/p^r))$. We are interested in the maximal parabolic subgroup of type $(t, n - t)$, defined by

$$ H = H_{n,t} = \{(g_{ij}) \in \GL_n(\mathfrak{o}) \mid g_{ij} = 0 \text{ for } i \leq t < j\} $$

and we set $H^r = H \cap G^r$. The groups $G^r$ and $H^r$ are finitely generated torsion-free potent pro-$p$ groups whenever $r \geq 2e$ for $p = 2$ and $r \geq e(p-2)^{-1}$ for $p > 2$, where $e$ denotes the ramification index of $\mathfrak{f}$ over $\mathbb{Q}_p$. Our aim is to compute the zeta function of the induced representation $\Ind_{H^r}^{G^r}(\mathbb{1}_{B^r})$ for such $r$.

For $m \in \mathbb{N}_0$, define $\mathcal{V}_q(m) = \prod_{j=1}^{m} (1 - q^{-j})$, viz. the volume of $\GL_m(\mathfrak{o})$ with respect to the additive Haar measure on $\Mat_{m,m}(\mathfrak{o})$. For a subset $J = \{x_1, \ldots, x_{|J|}\} \subseteq \{1, \ldots, t\}$ with $x_1 < \ldots < x_{|J|}$ we write

$$ V_{n,t}(J) = \frac{\mathcal{V}_q(t)\mathcal{V}_q(n-t)}{\mathcal{V}_q(t-x_{|J|})\mathcal{V}_q(n-t-x_{|J|})} \prod_{j=1}^{\left\lfloor \frac{|J|}{2} \right\rfloor} \mathcal{V}_q(x_j - x_{j-1})^{-1}, $$

with the convention $x_0 = 0$.

**Theorem 7.3.** In the set-up described above and subject to $r \geq 2e$ if $p = 2$ and $r \geq e(p-2)^{-1}$ if $p > 2$, the zeta function of the representation $\rho = \Ind_{H^r}^{G^r}(\mathbb{1}_{B^r})$ is

$$ \zeta_{\rho}(s) = q^{rt(n-t)} \sum_{J \subseteq \{1, \ldots, t\}} V_{n,t}(J) \prod_{j \in J} \frac{q^{-j(n-j)s}}{1 - q^{-j(n-j)s}}. $$

The abscissa of convergence is $\alpha(\rho) = 0$.

**Remark 7.4.** Even though the formula in (7.1) seems complicated, it is rather easy to evaluate the occurring terms for small values of $t$. For instance, for $t = 1$ we obtain

$$ \zeta_{\rho}(s) = q^{r(n-1)} \left( 1 + \left(1 - q^{-(n-1)}\right) \frac{q^{-(n-1)s}}{1 - q^{-(n-1)s}} \right) = q^{r(n-1)} \frac{1 - q^{-(n-1)-(n-1)s}}{1 - q^{-(n-1)s}}. $$
This formula will be generalised in Section 7.4. It should also be compared with the formula obtained in Proposition 6.3.

The proof of Theorem 7.3 requires some preparations. For $N, k \in \mathbb{N}_0$, a composition of length $k$ of $N$ is a $k$-tuple $\mathbf{u} = (u_1, \ldots, u_k)$ of positive integers such that $N = u_1 + \ldots + u_k$; we put $N(\mathbf{u}) = N$ and $\lambda(\mathbf{u}) = k$. In particular, the empty tuple is a composition of 0 of length 0. We denote by $\text{Comp}(N)$ the set of all compositions of $N$.

Set $\mathfrak{g} = \mathfrak{gl}_n(\mathfrak{o})$, and let

$$\mathfrak{h} = \{(X_{ij}) \in \mathfrak{g} \mid X_{ij} = 0 \text{ for } i \leq t < j\}$$

denote the parabolic $\mathfrak{o}$-Lie sublattice of type $(t, n-t)$, related to $H$. We identify the quotient $\mathfrak{g}/\mathfrak{h}$ with the $\mathfrak{o}$-lattice $\text{Mat}_{t,t'}(\mathfrak{o})$ of $t \times t'$-matrices, by projecting onto the upper right $t \times t'$-block. Furthermore, we identify the $\mathfrak{f}$-dual space $(\mathfrak{g}/\mathfrak{h})^* \cong \text{Hom}_c(\mathfrak{g}/\mathfrak{h}, \mathfrak{f})$ with the $\mathfrak{f}$-vector space $\text{Mat}_{t,t'}(\mathfrak{f})$ via the trace form

$$\text{Mat}_{t,t'}(\mathfrak{f}) \times \text{Mat}_{t,t'}(\mathfrak{o}) \to \mathfrak{f}, \quad (M, X) \mapsto \text{Tr}(MX),$$

and we identify the Pontryagin dual $(\mathfrak{g}/\mathfrak{h})^* \cong \text{Hom}_c(\mathfrak{g}/\mathfrak{h}, \mathfrak{f}/\mathfrak{o})$ with the space $\text{Mat}_{t,t'}(\mathfrak{f}/\mathfrak{o})$; compare Section 4.2. Indeed, writing $\overline{M} \in \text{Mat}_{t,t'}(\mathfrak{f}/\mathfrak{o})$ for the image of $M \in \text{Mat}_{t,t'}(\mathfrak{f})$, we obtain an isomorphism

$$\text{Mat}_{t,t'}(\mathfrak{f}/\mathfrak{o}) \to (\mathfrak{g}/\mathfrak{h})^*, \quad \overline{M} \mapsto \omega_{\overline{M}},$$

where $\omega_{\overline{M}}: \mathfrak{g}/\mathfrak{h} \to \mathfrak{f}/\mathfrak{o}$, $\omega_{\overline{M}}(X) = \text{Tr}(MX) + \mathfrak{o}$ for $M \in \text{Mat}_{t,t'}(\mathfrak{f})$.

In order to apply Proposition 4.3, we need to calculate $|\mathfrak{g} : \text{stab}_\mathfrak{g}(\omega)|$ for every form $\omega \in (\mathfrak{g}/\mathfrak{h})^*$, we make use of the Levi subgroup $L \leq_c H$. This is the group of block diagonal matrices

$$L = \{(g_{ij}) \in H \mid g_{ij} = 0 \text{ for } j \leq t < i\} \cong \text{GL}_t(\mathfrak{o}) \times \text{GL}_{n-t}(\mathfrak{o}),$$

and accordingly we write elements of $L$ as pairs $(g, h) \in \text{GL}_t(\mathfrak{o}) \times \text{GL}_{n-t}(\mathfrak{o})$. The action of $L$ on $(\mathfrak{g}/\mathfrak{h})^* \cong \text{Mat}_{t,t'}(\mathfrak{f}/\mathfrak{o})$ via the co-adjoint representation is described explicitly by

$$(g, h).\overline{M} = hMg^{-1} \quad \text{for } (g, h) \in L \text{ and } M \in \text{Mat}_{t,t'}(\mathfrak{f});$$

indeed, for $X \in \text{Mat}_{t,t'}(\mathfrak{o}) \cong \mathfrak{g}/\mathfrak{h}$ we observe that

$$(g, h).\omega_{\overline{M}}(X) = \text{Tr}(Mg^{-1}Xh) = \text{Tr}(hMg^{-1}X) = \omega_{hMg^{-1}}(X).$$

Next we determine an explicit set of orbit representatives for the $L$-orbits in $\text{Mat}_{t,t'}(\mathfrak{f}/\mathfrak{o})$. To this end we introduce the parameter set $\Xi = \Xi_{n,t}$ consisting of all pairs $\xi = (\mathbf{u}, \gamma)$, where $\mathbf{u}$ is a composition satisfying $N(\mathbf{u}) \leq t$ and $\gamma = (\gamma_1, \ldots, \gamma_{\lambda(\mathbf{u})})$ is a strictly increasing sequence of negative integers so that $\gamma_1 < \ldots < \gamma_{\lambda(\mathbf{u})} < 0$. For $\xi = (\mathbf{u}, \gamma) \in \Xi$ we define

$$\alpha_i(\xi) = \begin{cases} \gamma_k & \text{if } \sum_{j=1}^{k-1} u_j < i \leq \sum_{j=1}^{k} u_j, \\ 0 & \text{if } N(\mathbf{u}) < i, \end{cases} \quad \text{where } 1 \leq i \leq t,$$
and we associate to $\xi$ the matrices

$$M_\xi = \begin{pmatrix}
\pi^{\alpha_1(\xi)} \\
\pi^{\alpha_2(\xi)} \\
\vdots \\
0 & \cdots & 0
\end{pmatrix} \in \text{Mat}_{t',t}(f)$$

and $\overline{M}_\xi \in \text{Mat}_{t',t}(f/o)$. The elementary divisor theorem implies that $M = M_{\alpha,t} = \{M_\xi \mid \xi \in \Xi\}$ is a system of representatives of the $L$-orbits in $\text{Mat}_{t',t}(f/o)$.

Since $|g : \text{stab}_g(\omega)|$ is constant on the co-adjoint orbit of a form $\omega$ with regard to $L$, it suffices to determine, for $\xi \in \Xi$, the index $|g : \text{stab}_g(\omega M_\xi)|$ and the size of the $L$-orbit of $\overline{M}_\xi$.

**Lemma 7.5.** Let $\xi = (u, \gamma) \in \Xi$. The stabiliser of $\omega M_\xi$ in $g$ satisfies

$$|g : \text{stab}_g(\omega M_\xi)|^{1/2} = \prod_{i=1}^{\lambda(u)} q^{-u_i \gamma_i (n-u_i-2 \sum_{j=1}^{i-1} u_j)}.$$ 

**Lemma 7.6.** Let $\xi = (u, \gamma) \in \Xi$ and put $N = N(u)$. The $L$-orbit of $\overline{M}_\xi$ has the cardinality

$$|L,\overline{M}_\xi| = \frac{\mathcal{V}_q(t)\mathcal{V}_q(n-t)}{\mathcal{V}_q(t-N)\mathcal{V}_q((n-t)-N)} \prod_{i=1}^{\lambda(u)} \left(\mathcal{V}_q(u_i)^{-1} q^{-(n-N)\gamma_i u_i + \sum_{j=1}^{i-1} (\gamma_i - \gamma_j) u_i u_j}\right).$$

We postpone the proofs of the lemmata and explain first how Theorem 7.3 can be deduced.

**Proof of Theorem 7.3** For any composition $u$, we set

$$W_u = \frac{\mathcal{V}_q(t)\mathcal{V}_q(n-t)}{\mathcal{V}_q(t-N(u))\mathcal{V}_q((n-t)-N(u))} \prod_{i=1}^{\lambda(u)} \mathcal{V}_q(u_i)^{-1}.$$
It follows from Proposition 4.3 and from Lemmata 7.5 and 7.6 that \( \varrho = \text{Ind}_{G^p}^G(1_{B^p}) \) satisfies

\[
\zeta_\varrho(s) = q^{-rt(n-t)} \sum_{\omega \in (g/p)^{\gamma}} (|g : \text{stab}_g(\omega)|)^{1/2} - 1 - s
\]

\[
= q^{-rt(n-t)} \sum_{\xi = (u, \gamma) \in \Xi} L.M_{\xi} \prod_{i=1}^{\lambda(u)} q^{u_i \gamma_i (n-u_i-2 \sum_{j=1}^{i-1} u_j)(1+s)}
\]

\[
= q^{-rt(n-t)} \sum_{\xi = (u, \gamma) \in \Xi} W_u \prod_{i=1}^{\lambda(u)} q^{\gamma_i u_i (n-u_i-2 \sum_{j=1}^{i-1} u_j)s}
\]

\[
= q^{-rt(n-t)} \sum_{\xi = (u, \gamma) \in \Xi} W_u \prod_{i=1}^{\lambda(u)} q^{u_i \gamma_i (n-u_i-2 \sum_{j=1}^{i-1} u_j)s}
\]

Let \( \xi = (u, \gamma) \in \Xi \). Since \( \gamma_1 < \gamma_2 < \ldots < \gamma_{\lambda(u)} < 0 \) is strictly increasing, we may write \( \gamma_i = -\sum_{k=i}^{\lambda(u)} \beta_k \) for certain positive integers \( \beta_1, \ldots, \beta_{\lambda(u)} \in \mathbb{N} \). Using this reparametrisation, we obtain

\[
\zeta_\varrho(s) = q^{-rt(n-t)} \sum_{\xi = (u, \gamma) \in \Xi} W_u \prod_{i=1}^{\lambda(u)} q^{u_i \gamma_i (n-u_i-2 \sum_{j=1}^{i-1} u_j)s}
\]

\[
= q^{-rt(n-t)} \sum_{\xi = (u, \gamma) \in \Xi} W_u \prod_{i=1}^{\lambda(u)} q^{-u_i \sum_{k=1}^{\lambda(u)} \beta_k (n-u_i-2 \sum_{j<i} u_j)s}
\]

\[
= q^{-rt(n-t)} \sum_{\xi = (u, \gamma) \in \Xi} W_u \prod_{i=1}^{\lambda(u)} q^{-u_i \sum_{k=1}^{\lambda(u)} \beta_k (n-u_i-2 \sum_{j<i} u_j)s}
\]

where the last step is derived using the geometric series. Finally, we write \( x_k = \sum_{i=1}^{k} u_i \) for \( k \in \{1, \ldots, \lambda(u)\} \). Then every composition \( u \) yields a set

\[
J = J_u = \{x_1, x_2, \ldots, x_{\lambda(u)}\} \subseteq \{1, \ldots, t\}
\]

and every such set \( J \) corresponds uniquely to a composition of the number \( N = \text{max}(J) \). This proves the theorem since \( \sum_{i=1}^{k} u_i (n-u_i-2 \sum_{j=1}^{i-1} u_j) = x_k(n-x_k) \)

and \( W_u = V_{u,t}(J) \).

**Proof of Lemma 7.2.** Let \( \xi = (u, \gamma) \in \Xi \). As explained in Section 3.2, the index \( |g : \text{stab}_g(u_{M_{\xi}})| \) can be computed via the Pfaffians of the symplectic form

\[
A_\xi : g \times g \to \mathfrak{f}, \quad A_\xi(X, Y) = w_{M_{\xi}}([X, Y] + \mathfrak{h}) := \text{Tr}(M_{\xi}[X, Y]),
\]
where \( \overline{Z} \in \text{Mat}_{t,t'}(\mathfrak{o}) \) denotes the projection of \( Z \in \mathfrak{g} \) onto its upper right \( t \times t' \)-block. The elementary \( n \times n \)-matrices \( E_{i,j} \), with entry 1 in position \((i,j)\) and entries 0 elsewhere, form an \( \mathfrak{o} \)-basis of \( \mathfrak{g} \). A short calculation yields

\[
A_\xi(E_{i_1,j_1}, E_{i_2,j_2}) = \begin{cases} 
\pi_\alpha \xi(i) & \text{if } j_1 = i_2, j_2 = i_1 + t \text{ and } i_1 \leq t \\
-\pi_\alpha \xi(j) & \text{if } j_2 = i_1, j_1 = i_2 + t \text{ and } i_2 \leq t \\
0 & \text{otherwise}
\end{cases}
\]

Thus \( \mathfrak{g} \), equipped with the form \( A_\xi \), decomposes into an orthogonal direct sum of subspaces of three different types.

Type 1. For \( i > t \) and \( j \notin \{t+1,\ldots,2t\} \), the space \( \mathfrak{o}E_{i,j} \) is contained in the radical of \( A_\xi \), this means, \( A_\xi(E_{i,j}, X) = 0 \) for all \( X \in \mathfrak{g} \).

Type 2. For \( 1 \leq i \leq t \) and \( j > 2t \), the free \( \mathfrak{o} \)-module of rank 2 spanned by \( E_{i,j} \) and \( E_{j,i+t} \) is orthogonal to all other elementary matrices. The form \( A_\xi \) restricted to this space has the Pfaffian \( p_\alpha \xi(i) \). For each \( i \) with \( 1 \leq i \leq t \) there are \((n - 2t)\) possible choices for a corresponding index \( j \).

Type 3. For \( 1 \leq i,j \leq t \) the \( \mathfrak{o} \)-module of rank 3 spanned by \( E_{i,j}, E_{j,i+t} \) and \( E_{i+t,j+t} \) is orthogonal to all other elementary matrices. The form \( A_\xi \) restricted to this space has the degree-1 Pfaffian \( p_\min(\alpha_\xi(i),\alpha_\xi(j)) \).

The sequence of numbers \( \alpha_\xi(i), i \in \{1,\ldots,t\} \), is increasing. Thus we can determine the maximal Pfaffian of \( A_\xi \), and hence

\[
|\mathfrak{g} : \text{stab}_\mathfrak{g}(\omega_{\overline{M_\xi}})|^{1/2} = q^{-\sum_{i=1}^t \alpha_\xi(i)(n-2i+1)} = q^{-\sum_{i=1}^t \gamma_i u_i(n-u_i-2\sum_{j<i} u_j)}.
\]

**Proof of Lemma 1.3.** Let \( \xi = (\mathbf{u}, \gamma) \in \Xi \) and put \( N = N(\mathbf{u}) \). We compute the stabiliser \( \text{Stab}_L(M_\xi) \) of \( M_\xi \in \text{Mat}_{t,t'}(f/\mathfrak{o}) \) and its volume with respect to the normalised additive Haar measure on \( \text{Mat}_{t,t'}(\mathfrak{o}) \times \text{Mat}_{t',t'}(\mathfrak{o}) \). For simplicity we denote by \( \text{vol} \) the normalised additive Haar measure on any implicitly given \( \mathfrak{o} \)-lattice.

Suppose \((g, h) \in L = \text{GL}_t(\mathfrak{o}) \times \text{GL}_{t'}(\mathfrak{o}) \) lies in the stabiliser of \( \overline{M_\xi} \). Write

\[
g = \begin{pmatrix} A & C \\ U & V \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} B & X \\ D & Y \end{pmatrix}
\]

with

\[
A, B \in \text{Mat}_{N,N}(\mathfrak{o}), \quad C, U^\text{tr} \in \text{Mat}_{N,t-N}(\mathfrak{o}), \quad D, X^\text{tr} \in \text{Mat}_{t'-N,N}(\mathfrak{o}), \\
V \in \text{Mat}_{t-N,t-N}(\mathfrak{o}), \quad Y \in \text{Mat}_{t'-N,t'-N}(\mathfrak{o}).
\]

Let \( M_\xi^o \) denote the upper left \( N \times N \)-block of the matrix \( M_\xi \). The assumption \( \overline{hM_\xi} = \overline{M_\xi g} \) is equivalent to the following three conditions

(i) \( B M_\xi^o - M_\xi^o A \equiv_\mathfrak{o} 0 \),
(ii) \( D M_\xi^o \equiv_\mathfrak{o} 0 \),
(iii) \( M_\xi^o C \equiv_\mathfrak{o} 0 \).
Writing \( D = (d_{ij}) \), we observe that condition (iii) is equivalent to \( d_{ij} \in \mathfrak{p}^{-\alpha_j(\xi)} \) for \( 1 \leq i \leq t'-N \) and \( 1 \leq j \leq N \). Thus the set \( Z_2 \) of all matrices \( D \in \text{Mat}_{t'-N} (\mathfrak{g}) \) that satisfy condition (iii) has volume

\[
\text{vol}(Z_2) = \prod_{j=1}^{N} q^{\alpha_j(\xi)(t'-N)} = \prod_{i=1}^{\lambda(\mathfrak{u})} q^{\gamma_{u_i}(t'-N)}.
\]

Similarly, writing \( C = (c_{ij}) \), we see that condition (iii) means: \( c_{ij} \in \mathfrak{p}^{-\alpha_i(\xi)} \) for \( 1 \leq i \leq N \) and \( j \leq t - N \). Thus the volume of the set \( Z_3 \) of all matrices \( C \in \text{Mat}_{N,t-N} (\mathfrak{g}) \) that satisfy condition (iii) is

\[
\text{vol}(Z_3) = \prod_{i=1}^{N} q^{\alpha_i(\xi)(t-N)} = \prod_{i=1}^{\lambda(\mathfrak{u})} q^{\gamma_{u_i}(t-N)}.
\]

Since \( \alpha_i(\xi) < 0 \) for \( 1 \leq i \leq N \), the matrices \( C \) and \( D \) vanish modulo \( \pi \). Consequently, the square matrices \( A, B, V, Y \) are invertible over \( \mathfrak{g} \).

Now consider condition (i). Writing \( A = (a_{ij}) \) and \( B = (b_{ij}) \), we see that condition (i) is equivalent to

\[(7.2) \quad a_{ij} \pi^{\alpha_i(\xi)} \equiv_\mathfrak{p} b_{ij} \pi^{\alpha_j(\xi)} \quad \text{for } 1 \leq i, j \leq N.\]

Let \( Z_1 \) denote the set of all pairs \( (A, B) \in \text{GL}_N (\mathfrak{g}) \times \text{GL}_N (\mathfrak{g}) \) satisfying (7.2). We observe that, if \( (A, B) \in Z_1 \), then \( A \) is contained in the group

\[
G(\xi) = \{ (z_{ij}) \in \text{GL}_N (\mathfrak{g}) \mid z_{ij} \in \pi^{\alpha_j(\xi)-\alpha_i(\xi)} \mathfrak{g} \text{ for } i \leq j \}
\]

\[
= \left( \begin{array}{cccc}
\text{GL}_{u_1}(\mathfrak{g}) & \mathfrak{p}^{\gamma_{u_1}-\gamma_1} & \cdots & \mathfrak{p}^{\gamma_{u_1}-\gamma_1} \\
0 & \text{GL}_{u_2}(\mathfrak{g}) & \ddots & \vdots \\
0 & \ddots & \ddots & \mathfrak{p}^{\gamma_{u_N}-\gamma_{u_{N-1}}-\gamma_1} \\
0 & \cdots & 0 & \text{GL}_{u_{\lambda(\mathfrak{u})}}(\mathfrak{g})
\end{array} \right).
\]

Moreover, for any \( A \in G(\xi) \), there are matrices \( B \) such that \( (A, B) \in Z_1 \) and each entry \( b_{ij} \in \mathfrak{g} \) is uniquely determined modulo \( \pi^{-\alpha_j(\xi)} \). We deduce that

\[
\text{vol}(Z_1) = \text{vol}(G(\xi)) \prod_{k=1}^{N} q^{\alpha_k(\xi)N} = \prod_{i=1}^{\lambda(\mathfrak{u})} \left( \text{vol}(\text{GL}_{u_i}(\mathfrak{g})) q^{N\gamma_{u_i}} \prod_{j=i+1}^{\lambda(\mathfrak{u})} q^{(\gamma_{u_j}-\gamma_i)u_iu_j} \right).
\]

Recalling that \( \text{vol}(\text{GL}_m (\mathfrak{g})) = \mathcal{V}_q(m) \) for \( m \in \mathbb{N} \), we combine the three volume computations to conclude that

\[
\text{vol}(\text{Stab}_L (M_\xi)) = \text{vol}(Z_1) \text{vol}(Z_2) \text{vol}(Z_3) \text{vol}(\text{GL}_{t-N} (\mathfrak{g})) \text{vol}(\text{GL}_{t'-N} (\mathfrak{g}))
\]

\[
= \mathcal{V}_q(t-N) \mathcal{V}_q(t'-N) \prod_{i=1}^{\lambda(\mathfrak{u})} \left( \mathcal{V}_q(u_i) q^{(t+N)u_iu_j+\sum_{j=1}^{\lambda(\mathfrak{u})} (\gamma_{u_j}-\gamma_i)u_iu_j} \right).
\]

The claim now follows from the observation that the orbit length can be expressed in terms of the volume as follows:

\[
|L, M_\xi| = |L : \text{Stab}_L (M_\xi)| = \frac{\mathcal{V}_q(t)\mathcal{V}_q(n-t)}{\text{vol}(\text{Stab}_L (M_\xi))}.
\]

\( \square \)
7.4. Induction from a maximal $(1, n)$-parabolic subgroup to $GL_{n+1}^d(\Delta)$. Let $\mathfrak{o}$ be a compact discrete valuation ring of characteristic 0, residue characteristic $p$ and residue field cardinality $q$. Fix a uniformiser $\pi$ so that the valuation ideal of $\mathfrak{o}$ takes the form $\mathfrak{p} = \pi \mathfrak{o}$. Let $f$ denote the fraction field of $\mathfrak{o}$, a finite extension of $\mathbb{Q}_p$, and consider a central division algebra $\mathfrak{d}$ of index $d$ over $f$ so that $\dim_f \mathfrak{d} = d^2$. It is known that $\mathfrak{d}$ contains a unique maximal $\mathfrak{o}$-order $\Delta$. Up to isomorphism, $\mathfrak{d}$ and $\Delta$ can be described explicitly, in terms of the index $d$ and a second invariant $h$ satisfying $1 \leq h \leq d$ and $\gcd(h, d) = 1$; see [44, §14].

Fix $n \in \mathbb{N}$ and set $G = GL_{n+1}(\Delta)$. For $r \in \mathbb{N}$, we take interest in the principal congruence subgroup

$$G^dr = \ker(GL_{n+1}(\Delta) \to GL_{n+1}(\Delta/\pi^r \Delta)).$$

The maximal parabolic subgroup

$$H = \begin{pmatrix} GL_1(\Delta) & 0 & \cdots & 0 \\ \Delta & \vdots & \Delta \\ GL_n(\Delta) \end{pmatrix} \leq GL_{n+1}(\Delta)$$

gives rise to a subgroup $H^dr = H \cap G^dr \leq_c G^dr$. The groups $G^dr$ and $H^dr$ are finitely generated torsion-free potent pro-$p$ groups whenever $r \geq 2e$ for $p = 2$ and $r \geq e(p-2)^{-1}$ for $p > 2$, where $e$ denotes the ramification index of $f$ over $\mathbb{Q}_p$; compare [3, Prop. 2.3].

**Proposition 7.7.** In the set-up described above and subject to $r \geq 2e$ for $p = 2$ and $r \geq e(p-2)^{-1}$ for $p > 2$, the zeta function of the induced representation $g = \text{Ind}_{H^dr}^{G^dr}(\mathbb{1}_{H^dr})$ is

$$\zeta_g(s) = q^{rn^2} \frac{1 - q^{-dn(1+s)}}{1 - q^{-dns}}.$$  

**Proof.** The division algebra $\mathfrak{d}$ contains a splitting field $e$ that is unramified of degree $d$ over $f$. Thus $e = f(\xi)$, where $\xi$ is a primitive $(q^d - 1)$th root of unity, and $e|f$ is a cyclic Galois extension. The ring of integers in $e$ is $\mathfrak{o}[\xi]$. Furthermore there exists a uniformiser $\Pi \in \mathfrak{d}$ so that

$$\Pi^d = \pi \quad \text{and} \quad \Pi x = \sigma(x)\Pi \quad \text{for all} \ x \in e,$$

where $\sigma$ is the generator of $\text{Gal}(e|f)$ satisfying $\sigma(\xi) = \xi^{q^d}$. In this way we obtain an $f$-basis $(\xi^i\Pi^j)_{0 \leq i, j \leq d-1}$ for $\mathfrak{d}$ that is at the same time an $\mathfrak{o}$-basis for $\Delta$:

$$\Delta = \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \mathfrak{o} \xi^i\Pi^j.$$

The reduced trace $\text{tr}_{\mathfrak{o}|f}(x)$ of an element $x = \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} a_{i,j} \xi^i\Pi^j$ satisfies

$$\text{tr}_{\mathfrak{o}|f}(x) = \text{Tr}_{\mathfrak{d}|f} \left( \sum_{i=0}^{d-1} a_{i,0} \xi^i \right).$$

Observe that the symmetric $f$-bilinear trace form $\mathfrak{e} \times \mathfrak{e} \to f$, $(x, y) \mapsto \text{Tr}_{\mathfrak{d}|f}(xy)$ restricts to a symmetric $\mathfrak{o}$-bilinear form $\mathfrak{o}[\xi] \times \mathfrak{o}[\xi] \to \mathfrak{o}$ that is non-degenerate,
i.e. non-degenerate modulo \( p \) at the level of residue fields. Consequently, the \( \mathfrak{f} \)-bilinear pairing
\[
\Delta \times \mathfrak{d} \rightarrow \mathfrak{f}, \quad (x, y) \mapsto \text{tr}_{\mathfrak{d}|\mathfrak{f}}(xy)
\]
induces isomorphisms
\[
\text{Hom}_o(\Delta, \mathfrak{f}) \cong \mathfrak{d} \quad \text{and} \quad \text{Hom}_o(\Delta, \mathfrak{f}/\mathfrak{o}) \cong \mathfrak{d}/\mathfrak{d}^d \Delta.
\]
For \( m = m_0d + m_1 \) with \( m_0 \geq 0 \) and \( 0 \leq m_1 \leq d - 1 \), we take a closer look at the related \( \mathfrak{o} \)-bilinear form
\[
\beta_m: \Delta \times \Delta \rightarrow \mathfrak{f}, \quad (x, y) \mapsto \text{tr}_{\mathfrak{o}|\mathfrak{f}}(xy\mathfrak{d}^{-m}) = \pi^{-m_0} \text{tr}_{\mathfrak{o}|\mathfrak{f}}(xy\mathfrak{d}^{-m_1}).
\]
Its structure matrix with respect to the \( \mathfrak{o} \)-basis \((\xi^j\Pi^i)_{0 \leq i, j \leq d-1}\) of \( \Delta \), ordered as
\[
(1, \xi, \ldots, \xi^{d-1}, \Pi, \xi\Pi, \ldots, \xi^{d-1}\Pi, \ldots, \Pi^{d-1}, \xi^{d-1}\Pi^{d-1}),
\]
takes the shape
\[
B_m = \pi^{-m_0}
\begin{bmatrix}
A_0 & & & \\
& A_1 & & \\
& & \ddots & \\
& & & A_{m_1+1}
\end{bmatrix}
\in \text{Mat}_{d^2}(\mathfrak{o}),
\]
where \( A_i \in \text{Mat}_d(\mathfrak{o}) \) denotes the structure matrix of the \( \mathfrak{o} \)-bilinear form
\[
\mathfrak{o}[\xi] \times \mathfrak{o}[\xi] \rightarrow \mathfrak{o}, \quad (x, y) \mapsto \text{Tr}_{\mathfrak{o}|\mathfrak{f}}(x\sigma^i(y))
\]
with respect to the \( \mathfrak{o} \)-basis \((1, \xi, \ldots, \xi^{d-1})\). Since each of the latter forms is non-degenerate, we infer that the elementary divisors of \( B_m \) are \( \pi^{-m_0} \), with multiplicity \((m_1 + 1)d\), and \( \pi^{-m_1} \), with multiplicity \((d - m_1 - 1)d\).

We consider the \( \mathfrak{o} \)-Lie lattice \( \mathfrak{g} = \mathfrak{g}_{n+1}(\Delta) \). The elementary matrices \( E_{ij} \), with \( 1 \leq i, j \leq n + 1 \), form a \( \Delta \)-basis of the left \( \Delta \)-module \( \mathfrak{g} \). Let \( \mathfrak{h} \) denote the maximal parabolic \( \mathfrak{o} \)-Lie sublattice of \( \mathfrak{g} \) that is spanned as a \( \Delta \)-submodule by those \( E_{ij} \) satisfying \((i, j) = (1, 1)\) or \( i \geq 2 \). For \( m \in \mathbb{N}_0 \) with \( m \geq d - 1 \), we take interest in the \( \mathfrak{o} \)-linear form
\[
w_m: \mathfrak{g} \rightarrow \mathfrak{f}, \quad z = \sum_{1 \leq i, j \leq n+1} z_{ij} E_{ij} \mapsto \text{tr}_{\mathfrak{o}|\mathfrak{f}}(z_{1,n+1}\mathfrak{d}^{-m})
\]
and the induced \( \mathfrak{o} \)-linear form \( \omega_m: \mathfrak{g} \rightarrow \mathfrak{f}/\mathfrak{o}, \quad z \mapsto w_m(z) + \mathfrak{o} \). Clearly, \( w_m(\mathfrak{h}) = \{0\} \subseteq \mathfrak{o} \). Furthermore, the co-adjoint action of the Levi subgroup
\[
L = \begin{pmatrix}
\text{GL}_1(\Delta) & 0 & \cdots & 0 \\
0 & \ddots & \vdots & \vdots \\
0 & \cdots & \text{GL}_n(\Delta)
\end{pmatrix}
\leq \text{GL}_{n+1}(\Delta),
\]
associated with \( \mathfrak{h} \), on \( \text{Hom}_o(\mathfrak{g}, \mathfrak{f}) \) maps \( W = \{w \in \text{Hom}_o(\mathfrak{g}, \mathfrak{f}) \mid w(\mathfrak{h}) \subseteq \mathfrak{o}\} \) to itself. We observe that, modulo \( \text{Hom}_o(\mathfrak{g}, \mathfrak{o}) \), the \( L \)-orbits on \( W \) are parametrised
by the elements $w_m$, $m \geq d - 1$. Furthermore, the volume of $W_m = L.w_m + \text{Hom}_o(g, o)$ is equal to

$$
\mu(W_m) = \begin{cases} 
(q^{dn} - 1)q^{dn(m-d)} & \text{if } m \geq d, \\
1 & \text{if } m = d - 1,
\end{cases}
$$

where $\mu$ denotes the Haar measure normalised so that $\mu(\text{Hom}_o(g, o)) = 1$.

We choose a total order $\prec$ on index pairs $\{(k, l) \mid 1 \leq k, l \leq n + 1\}$ such that

$$(1, 1) \prec (1, n + 1) \prec (n + 1, n + 1) \prec (1, 2) \prec (2, n + 1) \prec (1, 3) \prec (3, n + 1) \prec \ldots \prec (1, n) \prec (n, n + 1) \prec \text{any } (k, l) \text{ with } k \neq n + 1 \text{ and } l \neq 1.$$

Let $Y$ denote the $o$-basis of $g$ consisting of the basis elements

$$\xi^i \Pi^j E_{kl}, \quad \text{where } 0 \leq i, j \leq d - 1 \text{ and } 1 \leq k, l \leq n + 1,$$

ordered according to

$$(i, j, k, l) < (i', j', k', l') \iff \begin{cases} 
(k, l) \prec (k', l') \text{, or} \\
(k, l) = (k', l') \text{ and } j < j', \text{ or} \\
(k, l) = (k', l') \text{ and } j = j' \text{ and } i < i'.
\end{cases}$$

Evaluating the commutator matrix $\mathcal{R}(T)$ of the $o$-Lie lattice $g$ with respect to $Y$ at the point $\underline{x}_m$ corresponding to $w_m$, we obtain

$$
\mathcal{R}(\underline{x}_m) = \begin{pmatrix} 
B_0 & B_1 & B_2 & \cdots & B_{n-1} \\
\end{pmatrix} \in \text{Mat}_{(n+1)^2d^2}(o),
$$

where

$$B_0 = \begin{pmatrix} 
0 & B_m & 0 \\
-B_m^\text{tr} & 0 & B_m \\
0 & -B_m^\text{tr} & 0 \\
\end{pmatrix} \in \text{Mat}_{3d^2}(o)$$

and

$$B_1 = \ldots = B_{n-1} = \begin{pmatrix} 
0 & B_m \\
-B_m^\text{tr} & 0 \\
\end{pmatrix} \in \text{Mat}_{2d^2}(o)$$

each have elementary divisors $\pi^{-m_0}$, with multiplicity $2(d_1 + 1)d$, and $\pi^{-m_0+1}$, with multiplicity $2(d - d_1 - 1)d$, the remaining $d^2$ elementary divisors of $B_0$.
being $\pi^\infty = 0$. This gives

$$ |g : \text{stab}_g(\omega_m)|^{1/2} = \left\| \bigcup \{ \text{Pfaff}_k(w_m) \mid 0 \leq k \leq \lfloor d^2(n+1)/2 \rfloor \} \right\|_p $$

$$ = \left\| \text{Pfaff}_{nd^2}(w) \right\|_p $$

$$ = \left| (\pi^{-m_0})(m_1+1)(\pi^{-m_0+1})(d-m_1-1)dn \right|_p $$

$$ = q^{nd(m-(d-1))}. $$

Thus, by Proposition 4.4, the zeta function of $\rho = \text{Ind}_{_{H_{dr}}}(\mathbb{1}_{H_{dr}})$ is given by

$$ \zeta_{\rho}(s) = q^{rn^{d^2}} \left( \sum_{m=d-1}^{\infty} \mu(W_m) \left\| \bigcup \{ \text{Pfaff}_k(w_m) \mid 0 \leq k \leq \lfloor d^2(n+1)/2 \rfloor \} \right\|^{-1-s}_p \right) $$

$$ = q^{rn^{d^2}} \left( 1 + \sum_{m=d}^{\infty} \left( (q^{dn} - 1) q^{dn(m-d)} \right) \left( q^{nd(m-(d-1))} \right)^{-1-s} \right) $$

$$ = q^{rn^{d^2}} \left( 1 + \frac{1 - q^{-dn}}{1 - q^{-dn}} \frac{q^{-dn(s+1)}}{1 - q^{-dn}} \right) $$

$$ = q^{rn^{d^2}} \frac{1 - q^{-dn(1+s)}}{1 - q^{-dn^2}}. \quad \square $$

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