The generalized Euler–Poinso rigid body equations: explicit elliptic solutions

Yuri N Fedorov, Andrzej J Maciejewski and Maria Przybylska

1 Department de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Barcelona, E-08028, Spain
2 Kepler Institute of Astronomy, University of Zielona Gòra, Licealna 9, PL-65-407, Zielona Gòra, Poland
3 Institute of Physics, University of Zielona Gòra, Licealna 9, PL-65-407, Zielona Gòra, Poland

E-mail: Yuri.Fedorov@upc.edu, maciejka@astro.ia.uz.zgora.pl and M.Przybylska@proton.if.uz.zgora.pl

Received 12 May 2013, in final form 2 September 2013
Published 27 September 2013
Online at stacks.iop.org/JPhysA/46/415201

Abstract

The classical Euler–Poinso case of the rigid body dynamics admits a class of simple but non-trivial integrable generalizations, which modify the Poisson equations describing the motion of the body in space. These generalizations possess first integrals which are polynomial in the angular momenta. We consider the modified Poisson equations as a system of linear equations with elliptic coefficients and show that all the solutions of it are single-valued. By using the vector generalization of the Picard theorem, we derive the solutions explicitly in terms of sigma-functions of the corresponding elliptic curve. The solutions are accompanied by a numerical example. We also compare the generalized Poisson equations with the classical third order Halphen equation.

PACS numbers: 45.40.−f, 02.30.Ik, 02.30.Gp
Mathematics Subject Classification: 70E40, 70H06, 37J35, 33E05

1. Introduction

As in [1], we consider the following system

\[ \dot{\omega} = J \omega \times \omega, \quad \dot{\gamma} = \gamma \times B \omega, \quad \omega, \gamma \in \mathbb{R}^3, \]

(1.1)

which is a certain limit of the Kirchhoff equations describing the motion of a rigid body in an ideal fluid. Here \( \omega \) is the angular velocity of the body, \( \gamma \) is the linear momentum; \( 3 \times 3 \) matrices \( J \), and \( B \) are tensors of adjoint masses. The first vector equation in (1.1) represents the well-known Euler equations describing the free motion of the body with the inertia tensor \( J \).

In what follows, \( J \) and \( B \) are assumed to be arbitrary diagonal matrices. In the special case \( B = \text{Id}_3 \), the system (1.1) becomes the classical integrable Euler–Poinso case of the rigid body motion, when \( \gamma \) is interpreted as a vector fixed in space. Then, the equation for \( \gamma \) is
called the Poisson equation. Its three independent solutions are elliptic functions and elliptic functions of the second kind, see e.g., [8, 12].

Setting $M = J\omega$, the system (1.1) can be rewritten in the form

$$M = M \times aM, \quad \dot{\gamma} = \gamma \times bM,$$

where

$$a = \text{diag}(a_1, a_2, a_3) := J^{-1}, \quad b = \text{diag}(b_1, b_2, b_3) := B J^{-1}.$$

(1.2)

It has three independent polynomial first integrals

$$H_1 := \langle M, aM \rangle, \quad H_2 := \langle M, M \rangle, \quad H_3 := \langle \gamma, \gamma \rangle.$$

(1.4)

Here and below $(x, y)$ denotes the scalar product of vectors $x, y \in \mathbb{R}^3$. As the system (1.2) is divergence free, according to the Euler–Jacobi theorem, see e.g. [9], for its integrability only one additional first integral is required.

In [1], the authors applied the Kovalevskaya–Painlevé method to search for integrable cases of the considered system. It was shown that if all the solutions of the system (1.2) are meromorphic, or single-valued, then

$$k^2 a_{12} a_{13} a_{21} + b_1^2 a_{12} + b_2^2 a_{13} + b_3^2 a_{21} = 0, \quad a_{ij} = a_i - a_j,$$

(1.5)

where $k$ is an odd integer. Geometrically, the above condition describes a quadric in $\mathbb{R}^3$ with coordinates $(b_1, b_2, b_3)$. In particular this condition is fulfilled for $b = ka$. In this case the equations (1.2) become what can be called the modified Euler–Poinsot system

$$M = M \times aM, \quad \dot{\gamma} = k \gamma \times aM.$$

(1.6)

As was shown in [1], if the condition (1.5) is satisfied, then for odd positive $k$ the system (1.2) possesses an additional first integral $H_4$, which is algebraically independent with (1.4). It is linear in $\gamma$, and of degree $k$ in $M$, and can be written in the following form

$$H_4 = \langle P(M), \gamma \rangle,$$

(1.7)

where the vector $P(M)$ is given by

$$P(M) = \text{diag}(M_1, M_2, M_3) \Phi_k(M) T,$$

(1.8)

$$\Phi_k(M) := (A_1^{-1} K) \cdot (A_3^{-1} K) \cdots (A_{k-2}^{-1} K).$$

(1.9)

The matrices $K, A_n$ are defined as follows

$$K = \text{diag} \left( M_1^2, M_2^2, M_3^2 \right), \quad A_n = \begin{pmatrix} -na_{32} & b_2 & -b_2 \\ -b_3 & -na_{13} & b_3 \\ b_2 & -b_1 & -na_{21} \end{pmatrix}, \quad n \in \mathbb{N}.$$ 

The constant vector $T \in \mathbb{R}^3$ in formula (1.8) spans the kernel of the matrix $A_k$.

Notice that

$$\det A_n = -n(n^2 a_{32} a_{13} a_{21} + b_1^2 a_{32} + b_2^2 a_{13} + b_3^2 a_{21}),$$

hence, under (1.5), it vanishes only when $n = k$.

As will be shown in section 8, the case of negative odd $k$ can be reduced to the one above.

In the simplest non-trivial case $b = ka$ with $k = 3$, we have

$$P(M) = P^{(3)} := \left( P_1^{(3)}, P_2^{(3)}, P_3^{(3)} \right)^T.$$ 

(1.10)
with
\[
P_1^{(3)} = M_1 \left[ (a_1a_2 + 8a_1^2 - a_3a_1 + a_3a_1)M_1^2 + (3a_3a_1 - 3a_3a_1 + 9a_1a_2)M_2^2 + (9a_3a_1 - 3a_1a_2 + 3a_3a_2)M_3^2 \right],
\]
\[
P_2^{(3)} = M_2 \left[ (-3a_3a_2 + 3a_3a_1 + 9a_1a_2)M_1^2 + (a_3a_2 - a_3a_1 + 8a_2^2 + a_1a_2)M_2^2 + (3a_3a_1 - 3a_1a_2 + 9a_3a_2)M_3^2 \right],
\]
\[
P_3^{(3)} = M_3 \left[ (9a_3a_1 + 3a_1a_2 - 3a_3a_2)M_1^2 + (-3a_3a_1 + 3a_1a_2 + 9a_3a_2)M_2^2 + (a_3a_1 + 8a_2^2 - a_1a_2 + a_3a_2)M_3^2 \right] .
\]

In [1] it was also shown that the vector \( P(t) := P(M(t)) \), where \( M(t) \) is a solution of the Euler equations in (1.2), is itself a meromorphic solution of the Poisson equations in (1.2). Since the solution \( M(t) \) in terms of elliptic or hyperbolic functions is well-known, the solution \( P(t) \) can be found using (1.8).

In what follows we will regard the Poisson equations in (1.2) as a separate system of linear equations
\[
\gamma' = A(t)\gamma, \quad A = \begin{pmatrix} 0 & b_1M_3 & -b_2M_2 \\ -b_1M_3 & 0 & b_1M_1 \\ b_2M_2 & -b_1M_1 & 0 \end{pmatrix} \in \text{so}(3)
\]
with the coefficients given by the elliptic functions \( M(t) \). (We will not consider the special cases when \( M(t) \) are hyperbolic functions describing asymptotic motions of the Euler top.)

In the present paper we show that, under the condition (1.5), all the solutions of (1.11) are meromorphic. Next, for the case \( b = ka \), we derive explicitly its three independent complex solutions in terms of elliptic functions and elliptic functions of the second kind (sigma-functions and exponents), as presented in theorems 4 and 6 below.

Additionally, in theorem 7, we give expressions for the components of the associated real orthogonal rotation matrix \( R(t) \) whose columns satisfy the Poisson equations.

These equations give rise to a third order ODE for one of the components of the vector \( \gamma \). In the final part, we compare this ODE with the best known integrable ODE with elliptic coefficients, namely the Halphen equation, and show that, in general, they cannot be transformed into each other.

2. General properties of the solutions

As was already mentioned, the Kovalevskaya–Painlevé analysis made in [1] shows that for all the solutions of the system (1.2) (or the Poisson equations (1.11)) to be single-valued, the condition (1.5) must hold and \( k \) must be an odd integer. We show that these conditions are also sufficient.

**Lemma 1.** For an arbitrary solution \( M_1(t) \), \( M_2(t) \), \( M_3(t) \) of the Euler equations, all solutions of the generalized Poisson equations (1.11) are single-valued if and only if \( k \) is an odd integer and condition (1.5) is fulfilled.

**Proof.** An elliptic or a hyperbolic solution \( M(t) \) of the Euler equation has four simple poles in the fundamental region. Hence, all singular points of equation (1.11) on \( \mathbb{C} \) are regular. Since the equation is linear, branching of its solutions can happen only at the singular points, see [7].

\footnote{In fact, this was already stated in [1], but without a proof.}
If \( k \) is an odd integer and condition (1.5) is satisfied, then all exponents at each singular point are integers. However a branching can still occur if the local series solution in a neighborhood of a singular point has logarithmic terms. We will show that this never happens due to the presence of the first integral (1.7). Namely, the integral implies that the equations (1.11) have a time-dependent first integral \( I_4(\gamma, \gamma) := \langle P(t), \gamma \rangle \), which is polynomial of degree \( k \) in \( \gamma \), and \( P(t) \) is the corresponding elliptic solution of (1.11). Assume that \( P(t) \) is normalized: \( \langle P(t), P(t) \rangle = 1 \).

Now take \( t_0 = \mathbb{C} \) which does not coincide with a pole of \( M(t) \), and a loop
\[
s \mapsto \tau(s) \in \mathbb{C}, \quad s \in [0, 1], \quad \tau(0) = \tau(1) = t_0,
\]
which encircles once counterclockwise a pole \( t^* \) of \( M(t) \). Let \( \Gamma(t) \) be a fundamental matrix of (1.11) with the first column proportional to \( P(t) \) and let \( \Gamma(t_0) \in SO(3, \mathbb{C}) \). Then \( \Gamma(t) \in SO(3, \mathbb{C}) \) for all \( t \) where it is defined.

A continuation along the loop \( \tau \) gives a monodromy matrix \( \mathcal{M}_\tau \in SO(3, \mathbb{C}) \):
\[
\Gamma(\tau(s + 1)) = \Gamma(\tau(s))\mathcal{M}_\tau.
\]
For any solution \( \gamma(t) = \Gamma(t)\vec{v} \), \( \vec{v} = \text{const} \in \mathbb{C}^3 \), the integral \( I_4(t, \gamma) \) implies
\[
\langle P(t_0), \gamma(t_0) \rangle = \langle P(t_0), \Gamma(t_0)\vec{v} \rangle = \langle P(t_0), \Gamma(t_0)\mathcal{M}_\tau\vec{v} \rangle.
\]
Since \( \vec{v} \) is arbitrary, this yields \( P^T(t_0)\Gamma(t_0) = P^T(t_0)\Gamma(t_0)\mathcal{M}_\tau \) and, due to the orthogonality of \( \Gamma(t) \) and the normalization of \( P(t) \),
\[
(1, 0, 0) = (1, 0, 0)\mathcal{M}_\tau.
\]
Then, since \( \mathcal{M}_\tau \) is also orthogonal, it must have the block structure
\[
\mathcal{M}_\tau = \begin{pmatrix}
1 & 0 & 0 \\
0 & \theta & \vartheta \\
0 & -\vartheta & \theta
\end{pmatrix}, \quad \theta, \vartheta \in \mathbb{C}, \quad \theta^2 + \vartheta^2 = 1. \tag{2.1}
\]
We now recall that the formal series solution in a neighborhood of the singular point \( t^* \) has logarithmic terms if and only if the monodromy matrix \( \mathcal{M}_\tau \) is not diagonalizable (see, e.g., [6]). However (2.1) is diagonalizable for any \( \theta, \vartheta \) satisfying the above condition. \( \square \)

One of the main tools of our subsequent analysis will be a vector extension of the known Picard theorem formulated, in particular, in [3, 4]. For our purposes we adopt it in the following form.

**Theorem 2.** Let \( T_1 \) and \( T_2 \) be the common, real and, respectively, imaginary periods of the elliptic solutions \( M_1(t), M_2(t), M_3(t) \) of the Euler equations. If all the solutions of (1.11) are meromorphic, then, apart from the elliptic vector solution \( \gamma(t) = P(M(t)) \) of (1.11), there exist two elliptic solutions of the second kind \( \gamma(t) = G^{(1)}(t) \) and \( \gamma(t) = G^{(2)}(t) \), which satisfy
\[
G^{(1)}(t + T_j) = S_jG^{(1)}(t), \quad G^{(2)}(t + T_j) = S_j^{-1}G^{(2)}(t), \quad j = 1, 2, \tag{2.2}
\]
where \( S_1, S_2 \in \mathbb{C} \), and, moreover, \(|S_1| = 1\).

**Proof.** The existence of at least one vector solution of the second kind, \( G(t) \), follows from the vector extension of the Picard theorem mentioned above. Let \( s_1, s_2 \) be its monodromy factors with respect to the periods \( T_1, T_2 \).
Let $\tilde{G}(t)$ be another solution of (1.11), and $\Gamma(t) = (G(t), \tilde{G}(t), P(t))$ be a fundamental matrix. The monodromy matrices $M_1$, and $M_2$, corresponding to the periods $T_1$, $T_2$, respectively, are given by

$$\Gamma(t + T_j) = (s_j G, \chi_j G + \tilde{\chi}_j \tilde{G} + \rho_j P, P) = \Gamma(t)M_j,$$

with

$$M_j = \begin{pmatrix} s_j & \chi_j & 0 \\ 0 & \tilde{\chi}_j & 0 \\ 0 & \rho_j & 1 \end{pmatrix}, \quad j = 1, 2,$$

where $\chi_j$, $\tilde{\chi}_j$, $\rho_j$ are certain constants. Observe that, regardless of the values of the constants, both monodromy matrices $M_1$, and $M_2$ are diagonalizable.

Next, since, by the assumption, all the solutions of (1.11) are meromorphic, the monodromy group must be trivial. Therefore, $M_1$, and $M_2$ commute, and are diagonalizable in the same basis. As a result, there exist two independent solutions of the second kind $\mathcal{G}^{(1)}(t), \mathcal{G}^{(2)}(t)$ forming the fundamental matrix $(\mathcal{G}^{(1)}(t), \mathcal{G}^{(2)}(t), P(t))$. Following the general Floquet theory, the corresponding monodromy matrices $\tilde{M}_1$, $\tilde{M}_2$ must satisfy

$$\det \tilde{M}_j = \exp \left( \int_{0}^{T_j} \text{Tr} A(t) \, dt \right) = 1, \quad j = 1, 2.$$

(Here we used the property $A(t) \in \text{so}(3, \mathbb{C})$.) Hence, since the monodromy of the elliptic solution $P(t)$ is trivial, the monodromy factors of $\mathcal{G}^{(1)}(t)$, and $\mathcal{G}^{(2)}(t)$ must be reciprocal, and this implies (2.2).

Further, for certain constants $v_1, v_2 \in \mathbb{C}$ let

$$\gamma(t) = v_1 \mathcal{G}^{(1)}(t) + v_2 \mathcal{G}^{(2)}(t), \quad t \in \mathbb{R}$$

be a real vector solution of the Poisson equations. This means that, for $i = 1, 2, 3$, the numbers $v_1 \mathcal{G}_{i}^{(1)}(t)$ and $v_2 \mathcal{G}_{i}^{(2)}(t)$ are complex conjugates. Then, for the real period $T_1$, the vector $\gamma(t + T_1)$ is also a real solution. On the other hand, from the above and from the monodromy (2.2), we deduce that

$$\gamma(t + T_1) = S_1 v_1 \mathcal{G}^{(1)}(t + T_1) + S_1^{-1} v_2 \mathcal{G}^{(2)}(t + T_1),$$

which is real if and only if $|S_1| = 1$. □

3. Algebraic parametrization and the elliptic sigma-function solution for $M$ and $P(M)$

We first recall how generic solutions of the Euler equation in (1.2) can be expressed in terms of the Weierstrass sigma-functions. We need this fact to derive the general solution of the Poisson equations.

Let us fix a common level of first integrals (1.4)

$$(M, aM) = 1, \quad (M, M) = m^2, \quad (\gamma, \gamma) = 1 \quad (3.1)$$

For generic values of $\chi$ and $m$, the solutions $M(t)$ of the Euler equations are elliptic functions related to the elliptic curve $E$ given by

$$E = \{ \mu^2 = U_4(\lambda) \}, \quad U_4(\lambda) := -(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - c),$$

where $c := 1/m^2$. Here and below we assume that $c \neq a_1, a_2, a_3$. This curve, compactified and regularized, has two infinite points $\infty_{\pm}$.
A ‘rational’ parametrization of the momenta $M_\alpha$ in terms of the coordinate $\lambda$, see, e.g., [2], has the following form

$$M_\alpha = m \left( \lambda - a_\alpha \right) \left( \lambda - a_\beta \right) \left( \lambda - a_\gamma \right) \left( \lambda - a_\delta \right) \sqrt{\lambda - c}.$$  \hfill (3.3)

That is, for any $\lambda \in \mathbb{C}$, the right-hand sides of (3.3) satisfy (3.1). Then, from the Euler equations in (1.2), we easily deduce that the evolution of $\lambda$ is given by the equation

$$\dot{\lambda} = 2m \sqrt{-(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - c)}.$$  \hfill (3.4)

For a real motion, i.e., for real values of $l$, $m$, and $t$, if $a_1 < a_2 < a_3$, then one has $c \in (a_1, a_3)$, $c \neq a_2$. Moreover,

$$\lambda \in \begin{cases} [a_2, a_3], & \text{if } a_1 < c < a_2, \\ [a_1, a_2], & \text{if } a_2 < c < a_3. \end{cases}$$

Next, the birational map $(\lambda, \mu) \rightarrow (z, w)$, given by

$$z = \frac{1}{3} \left( \frac{t_1 - 2ct_2 + 3\lambda^2}{\lambda - c} + 2ct_2 + c^2 t_1 - 3t_3 \right), \quad w = \frac{\mu}{(\lambda - c)^2},$$

transforms the elliptic curve $E$ to its canonical Weierstrass form $E = \{ w^2 = U_3(z) \}$, $U_3(z) := 4(z - e_1)(z - e_2)(z - e_3) = 4z - g_2z^2 - g_3$, \hfill (3.5)

where

$$3e_\alpha = t_2 + c t_1 - 3(a_\alpha a_\beta + c a_\delta), \quad e_1 + e_2 + e_3 = 0,$$
$$t_1 = a_1 + a_2 + a_3, \quad t_2 = a_1 a_2 + a_2 a_3 + a_3 a_1, \quad t_3 = a_1 a_2 a_3. $$  \hfill (3.7)

The above map sends $\lambda = c$ to $z = \infty$, and $a_i$ to $e_i$, respectively. Then there is the following relation between the holomorphic differentials on $E$ and $\mathcal{E}$:

$$\int_c^p \frac{d\lambda}{2\sqrt{U_4(\lambda)}} = \int_c^p \frac{dz}{2\sqrt{4(z-e_1)(z-e_2)(z-e_3)}}.$$  \hfill (3.8)

Let us introduce the Abel map $u = i \int_c^p \frac{d\lambda}{2\sqrt{U_4(\lambda)}}$, where $p = (\lambda, \mu) \in E$. \hfill (3.8)

The integrals

$$\Omega_\alpha := i \int_c^{\alpha \Omega} \frac{d\lambda}{2\sqrt{U_4(\lambda)}}, \quad \alpha = 1, 2, 3$$

are the half-periods of the curve $E$. We choose the sign of the root $U_4(\lambda)$ to ensure $\Omega_1 + \Omega_2 + \Omega_3 = 0$.

According to (3.8), in the case $a_1 < c < a_2 < a_3$ the half-period $\Omega_1$ is imaginary and $\Omega_2$ is real, whereas for $a_1 < a_2 < c < a_3$ the half-period $\Omega_3$ is imaginary and $\Omega_2$ is real. In both cases, comparing (3.8) with (3.4), we get

$$u = im(t - t_0) + \Omega_2. $$  \hfill (3.9)

Using the Weierstrass sigma-function $\sigma(u) = \sigma(u|2\Omega_1, 2\Omega_3)$, one can write

$$\frac{\lambda - c}{\lambda - \rho} = \text{const} \cdot \frac{\sigma^2(u)}{\sigma(u - h)\sigma(u + h)}, \quad h = \int_c^\infty \frac{d\lambda}{\sqrt{U_4(\lambda)}},$$  \hfill (3.10)

$$\frac{\lambda - \rho}{\lambda - c} = \text{const} \cdot \frac{\sigma(u - \beta)\sigma(u + \beta)}{\sigma^2(u)}, \quad \beta = \int_c^p \frac{d\lambda}{\sqrt{U_4(\lambda)}}.$$  \hfill (3.11)
Moreover, we also have
\[
\sqrt{\frac{\lambda - a_\alpha}{\lambda - c}} = C_\alpha \frac{\sigma_\alpha(u)}{\sigma(u)}, \quad \alpha = 1, 2, 3,
\] (3.12)
where \(C_\alpha\) are certain constants and \(\sigma_\alpha(u)\) are the sigma-functions obtained from \(\sigma(u)\) by shift of \(u\), and by multiplication by an exponent:
\[
\sigma_\alpha(u) := e^{\rho_{\alpha}u} \frac{\sigma(\Omega_\alpha - u)}{\sigma(\Omega_\alpha)}, \quad \eta_\alpha = \zeta(\Omega_\alpha), \quad \zeta(u) = \frac{\sigma(u)}{\sigma(\Omega_\alpha)}.
\] (3.13)

Note that we have
\[
\sigma(u) = u - \frac{g_3}{240} u^5 - \frac{g_1}{840} u^7 + \cdots, \quad \text{and} \quad \sigma_1(0) = \sigma_2(0) = \sigma_3(0) = 1,
\] (3.14)
see, e.g., [5] or [10]. From (3.3), (3.12), it follows that the solutions of the Euler equations have the form
\[
M_\alpha = h_\alpha \frac{\sigma_\alpha(u)}{\sigma(u)}, \quad u = im t + \Omega_\alpha, \quad \alpha = 1, 2, 3,
\] (3.15)
with certain constants \(h_\alpha\) which we determine below. In view of (3.13), the sigma-quotients have the quasiperiodic property
\[
\frac{\sigma_\alpha(u + 2\Omega_\alpha)}{\sigma(u + 2\Omega_\alpha)} = (-1)^{1-h_{\beta}/h_\alpha} \frac{\sigma_\alpha(u)}{\sigma(u)},
\] (3.16)
where \(\delta_{\alpha, \beta}\) is the Kronecker symbol. Hence, the coefficients \(M_\alpha(u)\) of the Poisson equations (1.11) have common periods \(4\Omega_1, 4\Omega_2\).

Next, using the parametrization (3.3), and the expressions (1.8), for each odd \(k\) we get the following parametrization for the elliptic solution \(P(M) = (P_1, P_2, P_3)^T\) in terms of \(\lambda\):
\[
P_\alpha = \sqrt{\frac{(a_\alpha - c)(a_\beta - c)(a_\gamma - c)}{(a_\alpha - a_\beta)(a_\beta - a_\gamma)(a_\gamma - a_\alpha)}} \sqrt{\frac{\lambda - a_\alpha}{\lambda - c}},
\] (3.17)
Here \(F_{i,\alpha}(\lambda) = \rho_{i,\alpha} \prod_{r=1}^{s} (\lambda - \rho_{r,\alpha})\) is a polynomial of degree \(s = (k - 1)/2\), which is obtained by substituting (3.3) into the vector \(\Phi_4 T\) in (1.8) and taking the numerator. The sum \(\Delta_k = P_1^2(\lambda) + P_2^2(\lambda) + P_3^2(\lambda)\) is a constant depending on \(a_\alpha\) and \(c\) only.

In particular, for \(k = 3\), and \(h_\alpha = 3a_\alpha\), by using (1.10), we have
\[
F_{1,1}(\lambda) = [3\tau_2 + 4c(c - \tau_1 - 2a_1) - 2a_2 a_3] \lambda + c(\tau_2 + 2a_2 a_3) - 3(\tau_3 + 8c^2 a_1),
\]
\[
F_{1,2}(\lambda) = [3\tau_2 + 4c(c - \tau_1 - 2a_2) - 2a_3 a_1] \lambda + c(\tau_2 + 2a_3 a_1) - 3(\tau_3 + 8c^2 a_2),
\]
\[
F_{1,3}(\lambda) = [3\tau_2 + 4c(c - \tau_1 - 2a_3) - 2a_1 a_2] \lambda + c(\tau_2 + 2a_1 a_2) - 3(\tau_3 + 8c^2 a_3)
\] (3.18)
and \(\Delta_3 = \tau_2^2 - 4\tau_1 \tau_3 + 36c^2 \tau_3 - 48c^2 \tau_2 + 64c^3 \tau_1\).

Now, applying expressions (3.11), (3.12) to (3.17), we get
\[
P_\alpha = c_\alpha \frac{\sigma_\alpha(u)}{\sigma(u)} \prod_{r=1}^{s} \frac{\sigma(u + \nu_{r,\alpha}) \sigma(u - \nu_{r,\alpha})}{\sigma^2(u)},
\] (3.19)
where
\[
\nu_{r,\alpha} = \pm i \int_{c}^{\rho_{r,\alpha}} \frac{d\lambda}{2 \sqrt{U_3(\lambda)}}, \quad u = im t + \Omega_\alpha,
\]
for \(r = 1, \ldots, s\). Thus, the components \(P_\alpha(u)\) have a pole of order \(k\) at \(u = 0\) and, like \(M_\alpha(u)\), they are doubly periodic with common periods \(4\Omega_1, 4\Omega_2\). We finally have
Proposition 3. The momentum vector $M$ and the elliptic vector solution $P$ of the Poisson equations (1.11) for $b = ka$ can be written as

$$M_\alpha = me_\alpha \frac{\sigma_\alpha (u)}{\sigma (u)}, \quad P_\alpha = e_\alpha \frac{\sigma_\alpha (u)}{\sigma (u)} \prod_{r=1}^s \frac{\sigma (u + v_{r,\alpha}) \sigma (u - v_{r,\alpha})}{\sigma^2 (v_{r,\alpha}) \sigma^2 (u)},$$

(3.20)

$$e_\alpha = \frac{1}{\sqrt{(a_\alpha - a_\beta)(a_\alpha - a_\gamma)}}, \quad (\alpha, \beta, \gamma) = (1, 2, 3),$$

(3.21)

where signs of $e_\alpha$ are chosen according to the condition

$$\frac{1}{\epsilon_1 \epsilon_2 \epsilon_3} = -(a_1 - a_2)(a_2 - a_3)(a_3 - a_1),$$

and $u$ depends on time $t$ via (3.9).

Then also

$$P^i_\alpha (u) = e_\alpha (\varphi (u) - \varphi (\Omega_\alpha)) \prod_{l=1}^s \left( \varphi (u) - \varphi (v_{l,\alpha}) \right)^2,$$

(3.22)

where $\varphi (u) = \varphi (u|g_2, g_3)$ is the Weierstrass P-function.

Here, for any $u \in \mathbb{C}$

$$M^2_\alpha (u) + M^2_\beta (u) + M^2_\gamma (u) = m^2, \quad P^2_\alpha (u) + P^2_\beta (u) + P^2_\gamma (u) = \Pi,$$

(3.23)

$$\Pi = \epsilon_\beta^2 (\varphi (\Omega_\alpha) - \varphi (\Omega_\beta)) \prod_{l=1}^s \left( \varphi (\Omega_\alpha) - \varphi (v_{l,\beta}) \right)^2$$

$$+ \epsilon_\gamma^2 (\varphi (\Omega_\alpha) - \varphi (\Omega_\gamma)) \prod_{l=1}^s \left( \varphi (\Omega_\alpha) - \varphi (v_{l,\gamma}) \right)^2,$$

(3.24)

for any permutation $(\alpha, \beta, \gamma) = (1, 2, 3)$.

Remark. According to the rule (3.16), the shift $u \rightarrow u + 2\Omega_\alpha$ in the solutions (3.20) is equivalent to flipping the signs of some of the constants $\epsilon_i$ in such a way that the above condition is satisfied.

Proof of proposition 3. To calculate the constants $h_\alpha, c_\alpha$ in the elliptic solutions (3.15), (3.19), we compare the leading terms of the Laurent expansions of $M_\alpha (t), P_\alpha (t)$ near the poles and the expansions of the sigma-functions. Namely, let $t_0 \in \mathbb{C}$ be a pole of the functions $M(t), P(t)$, and $\delta t = t - t_0$. Substituting

$$M = \frac{1}{\delta t} (M^{(0)} + M^{(1)} \delta t + \cdots), \quad P = \frac{1}{(\delta t)^s} (P^{(0)} + P^{(1)} \delta t + \cdots)$$

into the system (1.6) for $M$ and $\gamma$, for any $k \in \mathbb{N}$ one gets

$$M^{(k)}(u) \in \{ (\epsilon_1, \epsilon_2, \epsilon_3)^T, \ i(-\epsilon_1, -\epsilon_2, -\epsilon_3)^T, \ i(\epsilon_1, -\epsilon_2, -\epsilon_3)^T, \ i(\epsilon_1, -\epsilon_2, -\epsilon_3)^T \},$$

(3.25)

with $e_\alpha$ given by (3.21), and $P^{(0)}$ is just proportional to $M^{(0)}$.

On the other hand, in view of (3.14), near $u = 0$ we have the expansions

$$\frac{\sigma_\alpha (u)}{\sigma (u)} = \frac{1}{u} + O(1), \quad \prod_{r=1}^s \frac{\sigma (u + v_{r,\alpha}) \sigma (u - v_{r,\alpha})}{\sigma^2 (v_{r,\alpha}) \sigma^2 (u)} = -\frac{\sigma^2 (v_{1,\alpha}) \cdots \sigma^2 (v_{s,\alpha})}{u^{k-1}} + O(1),$$

with $s = (k - 1)/2$. Since in the above expansions $u = im \cdot \delta t$, comparing them, we obtain

$$h_\alpha = me_\alpha, \quad c_\alpha = \frac{e_\alpha}{\sigma^2 (v_1) \cdots \sigma^2 (v_s)}.$$

Substituting this into (3.15), (3.19), we get (3.20).

5 In fact, one can write $h_\alpha, c_\alpha$ only in terms of sigma-constants and $\sigma (v_i)$, as was written for the Euler top (the case $k = 1$) (see [8, 12]), but this process is tedious and requires more calculations.
The latter, in view of the known relations (see, e.g., [5, 10])
\[
\frac{\sigma^2(u)}{\sigma^2(u)} = \varphi(u) - \varphi(\Omega u), \quad \frac{\sigma(u + \beta)\sigma(u - \beta)}{\sigma^2(\beta)\sigma^2(u)} = \varphi(u) - \varphi(\beta).
\]
implies (3.22).

Finally, since \( P(u) \) is a solution of the Poisson equations, it satisfies the integral (3.23). Setting there \( u = \Omega u \) and using (3.22) one obtains (3.24).

**Remark.** As follows from the formal Laurent solution for \( M(t) \) with the coefficients (3.25), near a pole \( t = t_0 \) any vector solution \( \gamma(t) = (\gamma_1, \gamma_2, \gamma_3)^T \) of the Poisson equations (1.11) for \( b = ka \) has the expansion
\[
\gamma_a(t) = \text{const} \frac{(1)}{(\delta t)^k} (\epsilon_a + O(\delta t)), \quad a = 1, 2, 3.
\]

(3.26)

**4. Algebraic structure of elliptic solutions of the second kind**

Using the algebraic parameterizations (3.3) and (3.17), we obtain

**Theorem 4.**

(1) If \( k \) is a positive odd integer and \( k \geq 3 \), then, apart from the solution \( P(\lambda) \) in (3.17), the Poisson equations (1.11) for \( b = ka \) have two independent solutions
\[
\gamma^{(j)} = G^{(j)} = (G_1^{(j)}, G_2^{(j)}, G_3^{(j)})^T, \quad j = 1, 2
\]
which can be represented as the following algebraic functions of the parameter \( \lambda \) in (3.3), (3.4):
\[
G^{(1)}_a = c_{1,a} \frac{\sqrt{Q_{1,a}(\lambda)}}{\sqrt{(\lambda - c)^k}} \exp \left( \frac{1}{2} \int W_a \right),
\]
\[
G^{(2)}_a = c_{2,a} \frac{\sqrt{Q_{1,a}(\lambda)}}{(\lambda - c)^k} \exp \left( -\frac{1}{2} \int W_a \right).
\]

Here \( c_{1,a}, c_{2,a} \) are certain constants to be specified below, and
\[
W_a = \frac{q_{i+1,a}(\lambda) \cdot (\lambda - c)^k}{Q_{i,a}(\lambda)} \frac{d\lambda}{\sqrt{U_{i+1}(\lambda)}},
\]

\[
Q_{k,a}(\lambda) = r_{0,a} \prod_{i=1}^k (\lambda - r_{i,a}) = \text{const} \cdot (P_{\beta}^2(\lambda) + P_\gamma^2(\lambda)) \cdot (\lambda - c)^k
\]
\[
= (a_\beta - c)(a_\gamma - a_\beta)(\lambda - a_\beta)F_{\beta}^2(\lambda) + (a_\gamma - c)(a_\beta - a_\gamma)(\lambda - a_\gamma)F_\gamma^2(\lambda),
\]
\[(\alpha, \beta, \gamma) = (1, 2, 3),
\]

where the polynomials \( F_{\beta,\gamma}(\lambda) \) of degree \( s = (k - 1)/2 \) are specified in (3.17).

The above formula implies that the zeros of the polynomials \( Q_{k,a}(\lambda) \) coincide with the zeros of \( P_{\beta}^2(\lambda) + P_\gamma^2(\lambda) \).

(2) The differential \( W_a \) is a meromorphic differential of the third kind on the curve \( E \) having pairs of only simple poles at the points \( T_{i,a}^\pm = (r_{i,a}, \pm \sqrt{U_{i}(r_{i,a})}) \in \mathcal{E}, i = 1, \ldots, k \) with residues \( \pm 1 \) respectively:
\[
\text{Res}_{T_{i,a}^\pm} W_a = \pm 1, \quad i = 1, \ldots, k.
\]

Finally, \( q_{i+1,a}(\lambda) \) in (4.2) are polynomials of degree \( s + 1 = (k + 1)/2 \) completely defined by the conditions (4.4).

The algebraic solutions in the classical case \( k = 1 \) will be described separately in section 6.
Remark. The polynomials $F_{a,1}$, $F_{a,2}$, $F_{a,3}$ and $Q_{a,1}$, $Q_{a,2}$, $Q_{a,3}$ are obtained by the corresponding permutation of $a_1$, $a_2$, $a_3$. Note that their Abel images ($u$-coordinates) of their roots $\tilde{\rho}_{1,a}, \tilde{\rho}_{2,a}$ are not obtained from each other by the translations by the half-periods $\Omega_j$ of the elliptic curve $E$.

Proof of theorem 4

(1) According to the kinematic interpretation, the Poisson equations in (1.2) describe the evolution of a fixed in the space vector $\gamma$ in a frame rotating with the angular velocity $\hat{\omega} = bM$ (also taken in the body frame). Now choose an orthonormal frame $\{O, e_1, e_2, e_3\}$ fixed in space. Let $\theta, \psi, \phi$ be the Euler nutation, precession, and rotation angles associated to this frame so that the corresponding rotation matrix is

$$\mathbf{R} \equiv \begin{pmatrix} \cos \theta \cos \psi & \cos \phi \sin \psi \cos \theta + \cos \theta \sin \phi \sin \psi & \sin \phi \sin \psi \\ -\sin \theta \sin \psi & \cos \theta \cos \psi \cos \phi - \sin \theta \cos \phi \sin \psi & \cos \phi \sin \psi \\ \sin \theta \cos \psi & -\sin \phi \sin \psi \cos \theta - \cos \phi \cos \theta \sin \psi & \cos \theta \cos \phi \sin \psi \end{pmatrix}. $$

Let $P(t) = (P_1, P_2, P_3)^T$ be a solution of (1.6) describing the motion of the vector $|P|e_3$. Then, in view of the structure of $\mathbf{R}$, and from the Euler kinematic equations, one has

$$P_1 = |P| \sin \phi \sin \theta, \quad P_2 = |P| \cos \psi \sin \theta, \quad P_3 = |P| \cos \phi,$$

and

$$\dot{\psi} = |P| \frac{\dot{\omega}_1 P_1 + \dot{\omega}_2 P_2}{P_1^2 + P_2^2},$$

see e.g., [12]. Hence the third components of the other two independent solutions $G^{(1)}$, and $G^{(2)}$ of (1.11) can be written in the complex form

$$G_3^{(j)} = \tilde{c}_j \sin \theta \exp(\pm i \psi), \quad \text{or} \quad G_3^{(j)} = c_j \sqrt{P_1^2 + P_2^2} \exp \left( \pm i \int \dot{\psi} \, dt \right),$$

where $\tilde{c}_j$, and $c_j$ are certain constants, and $j = 1, 2$. Using the parametrization (3.17) for $P_d(\lambda)$, as well as relations (3.3) and (3.4), and setting $b = k\alpha$, we get

$$i \dot{\psi} \, dt = i\sqrt{\Delta_k} \frac{a_1 M_1(\lambda) P_2(\lambda) + a_2 M_2(\lambda) P_1(\lambda)}{P_1^2(\lambda) + P_2^2(\lambda)} \frac{d\lambda}{2m\sqrt{U_{3}(\lambda)}} \equiv \frac{1}{2} W_3.$$  

After simplifications this takes the form

$$i \dot{\psi} \, dt = \frac{1}{2} \frac{Q_{1,3}(\lambda) \cdot (\lambda - c)^r}{\sqrt{U_{3}(\lambda)}} \frac{d\lambda}{\sqrt{\Delta_k}}$$

with

$$Q_{1,3}(\lambda) = \frac{i\sqrt{\Delta_k}}{m} \frac{[(a_2 - c)(a_3 - a_2)b_1 \cdot (\lambda - a_1)F_{1,1}(\lambda)}{+ (a_1 - c)(a_1 - a_3)b_2 \cdot (\lambda - a_2)F_{2,2}(\lambda)],$$

$$Q_{2,3}(\lambda) = ((a_2 - c)(a_2 - a_3) \cdot (\lambda - a_1)F_{2,1}^2(\lambda) + (a_1 - c)(a_1 - a_3) \cdot (\lambda - a_2)F_{1,2}^2(\lambda),$$

$$\sqrt{P_1^2 + P_2^2} = \text{const} \frac{\sqrt{Q_{1,3}(\lambda)}}{\sqrt{(\lambda - c)^r}}.$$

The above implies the formulas (4.1)–(4.3) for $\alpha = 3$. Repeating the same geometric argumentation for $\alpha = 1, 2$, we get the whole set of formulas of theorem 4.
Therefore the residue $/kappa_1$ Proposition 5. Following formula. In order to convert the algebraic solutions of theorem 4 to analytic ones, we shall need the 5. Sigma-function solutions of the second kind

$\Gamma_1$ the solution

in $t$ or $u$ and, locally, in $\lambda$. Namely, let $\tau = \lambda - r_{i,a}$ be a local coordinate on $E$ near the root $r_{i,a}$ and the meromorphic differentials have the expansion

$$W_\alpha = \left( \frac{K}{\tau} + O(1) \right) \,dt.$$  

Assume $\kappa > 0$. Then, as follows from (4.6) for $\alpha = 3$, the leading term of the expansion of the solution $\Gamma_1$ has the form

$$\text{const} \cdot \sqrt{\tau} \exp \left( \frac{K}{2} \ln \tau \right) = \text{const} \cdot \sqrt{\tau}^{\kappa/2}.$$  

Hence, $\kappa$ must be 1 or 3, 5, . . . . Since $\Gamma_1^{(1,2)}$ is an elliptic function of the second kind, the total number of its zeros on $E$ must be equal to that of its poles (with multiplicity), that is, $k$, therefore the residue $\kappa$ must be 1. The same argument for $\alpha = 1, 2$ completes the proof. $\Box$

5. Sigma-function solutions of the second kind

In order to convert the algebraic solutions of theorem 4 to analytic ones, we shall need the following formula.

Proposition 5. Let $K_3(\lambda)$ be a polynomial of odd degree $k$, and

$$W = \frac{K_3(\lambda)}{Q_3(\lambda)} \frac{d\lambda}{2\sqrt{U_3(\lambda)}}, \qquad Q_3(\lambda) = r_0(\lambda - r_1) \cdots (\lambda - r_k)$$  

be a differential of the third kind on the degree 4 curve $E$ with simple poles at the points $P_j^\pm = (r_j, \pm \sqrt{U_3(r_j)})$, $j = 1, \ldots, k$ with residues $\pm 1$ respectively. Let the point $(\lambda, \mu) \in E$ and $w \in \mathbb{C}$ be related by the Abel map (3.8). Then

$$\int_{(c,0)}^{(\lambda,\mu)} W = \log \frac{\sigma(u - w_1) \cdots \sigma(u - w_k)}{\sigma(u + w_1) \cdots \sigma(u + w_k)} + \frac{2}{\pi} \left[ \sum_{s} \zeta(w_s) \right] \mu + \delta u - \pi i, \quad (5.1)$$  

$$\sqrt{(\lambda - r_1) \cdots (\lambda - r_k)} \frac{d\lambda}{\sqrt{(\lambda - c)^k}} = \text{const} \sqrt{\sigma(u - w_1) \cdots \sigma(u - w_k)} \sigma(u + w_1) \cdots \sigma(u + w_k) \sigma^k(u), \quad (5.2)$$  

where, as above, $\zeta(u)$ is the Weierstrass zeta-function, $\delta = K(c)/Q(c)$, and, up to a sign,

$$w_j = i \int_{(c,0)}^{P_j} \frac{d\lambda}{2\sqrt{U_3(\lambda)}}, \quad (5.3)$$  

The signs of $w_j$ can be determined from the conditions

$$\frac{K_3(r_j)}{(r_j - c)Q_3(r_j)} = -\frac{3i}{(a_1 - c)(a_2 - c)(a_3 - c)} \psi^3(w_j). \quad (5.4)$$  

The proposition is a reformulation of known relations in the theory of elliptic functions; its proof is purely technical and is given in appendix B.

Note that if the polynomial $K_3(\lambda)$ contains the factor $(\lambda - c)^s$, $s \geq 1$, the constant $\delta$ in (5.1) is zero.

Theorem 4 and proposition 5 allow us to formulate the following theorem.
Theorem 6.

(1) The two complex vector elliptic solutions of the second kind of the Poisson equations (1.11) with \( b = ka \) are

\[
G^{(1)}(u) = \left( G^{(1)}_1, G^{(1)}_2, G^{(1)}_3 \right)^T, \quad G^{(2)}(u) = \left( G^{(2)}_1, G^{(2)}_2, G^{(2)}_3 \right)^T,
\]

(5.5)

\[
G^{(1)}(u) = \varepsilon_a e^{\Theta_a u} \prod_{l=1}^{k} \frac{\sigma(u - w_{l,a})}{\sigma(u) \sigma(-w_{l,a})}, \quad G^{(2)}(u) = \varepsilon_a e^{-\Theta_a u} \prod_{l=1}^{k} \frac{\sigma(u + w_{l,a})}{\sigma(u) \sigma(w_{l,a})},
\]

\[w_{l,a} = i \int_{c}^{\infty} \frac{\lambda d\lambda}{2 \sqrt{U_{\lambda}(\lambda)}}, \quad l = 1, \ldots, k, \quad \alpha = 1, 2, 3,
\]

where \( r_{l,a} \) are the roots of the polynomials \( Q_{k,a}(\lambda) \) in (4.3) and the signs of \( w_{l,a} \) are defined according to (5.4), namely,

\[
\frac{g_{k+a}(r_{l,a})}{Q_{k,a}(r_{l,a})} = -\frac{3i}{(a_1 - c)(a_2 - c)(a_3 - c)} g^3(w_{l,a}).
\]

Next, \( \varepsilon_a \) are specified in (3.21), \( u = i\text{mt} + 2\Omega_2 \), and

\[
\Theta_1 = \sum_{j=1}^{k} \zeta(w_{j,1}), \quad \Theta_2 = \sum_{j=1}^{k} \zeta(w_{j,2}), \quad \Theta_3 = \sum_{j=1}^{k} \zeta(w_{j,3}).
\]

Together with (3.20), (3.21), the expressions (5.5) form a complete basis of independent solutions of the equations (1.11).

(2) Let also

\[
\Sigma_1 = \sum_{j=1}^{k} w_{j,1}, \quad \Sigma_2 = \sum_{j=1}^{k} w_{j,2}, \quad \Sigma_3 = \sum_{j=1}^{k} w_{j,3}.
\]

The solutions (5.5) have the following quasi-periodicity property

\[
G^{(1)}(u + 2\Omega_j) = (-1)^{s_j} G^{(1)}(u), \quad G^{(2)}(u + 2\Omega_j) = (-1)^{s_j} G^{(2)}(u),
\]

(5.6)

\[j, \alpha = 1, 2, 3,
\]

and imply the vector monodromy

\[
G^{(1)}(u + 4\Omega_j) = s_j^2 G^{(1)}(u), \quad G^{(2)}(u + 4\Omega_j) = s_j^{-2} G^{(2)}(u),
\]

(5.7)

where

\[
s_1 = -\exp(2\Theta_1 \Omega_2 - 2\Sigma_1 \eta_1) = \exp(2\Theta_2 \Omega_2 - 2\Sigma_2 \eta_1) = \exp(2\Theta_3 \Omega_2 - 2\Sigma_3 \eta_1),
\]

\[
s_2 = -\exp(2\Theta_2 \Omega_3 - 2\Sigma_2 \eta_2) = \exp(2\Theta_1 \Omega_3 - 2\Sigma_1 \eta_2) = \exp(2\Theta_3 \Omega_3 - 2\Sigma_3 \eta_2),
\]

\[
s_3 = -\exp(2\Theta_3 \Omega_1 - 2\Sigma_3 \eta_3) = \exp(2\Theta_2 \Omega_1 - 2\Sigma_2 \eta_3) = \exp(2\Theta_1 \Omega_1 - 2\Sigma_1 \eta_3).
\]

(5.8)

(3) If \( \Omega_j \) is the imaginary half-period, then \(|s_j| = 1\). For the real half-period \( \Omega_2 \) one has \(|s_j| \neq 1\). Moreover,

\[
\Sigma_\alpha - \Sigma_\beta = \Omega_\gamma \mod \{ 2\Omega_1 \mathbb{Z} + 2\Omega_2 \mathbb{Z} \}, \quad \Theta_\alpha - \Theta_\beta = \eta_\gamma \mod \{ 2\eta_1 \mathbb{Z} + 2\eta_2 \mathbb{Z} \}
\]

(5.9)

for \((\alpha, \beta, \gamma) = (1, 2, 3)\).
\( G^{(1)}(u), G^{(2)}(u) \) = 0, \( G_a^{(1)}(u)G_a^{(2)}(u) \) = \(-\epsilon_a^2 \prod_{r=1}^{k} [\varphi(u) - \varphi(w_{r,a})] \). \( (5.10) \)

and

\[ \left[ G_a^{(1)}(\Omega_j) \right] ^2 = s_j (-1)^{1-h_j} \epsilon_i^2 \prod_{r=1}^{k} (\varphi(\Omega_j) - \varphi(w_{r,a})), \]
\[ \left[ G_a^{(2)}(\Omega_j) \right] ^2 = s_j^{-1} (-1)^{1-h_j} \epsilon_i^2 \prod_{r=1}^{k} (\varphi(\Omega_j) - \varphi(w_{r,a})). \] \( (5.11) \)

for any \( \alpha, j = 1, 2, 3 \). Moreover, for any \( u \in \mathbb{C} \),

\[ \langle G^{(1)}(u), G^{(2)}(u) \rangle = -2 \epsilon_i^2 \prod_{l=1}^{k} (\varphi(\Omega_1) - \varphi(w_{l,1})) = -2 \epsilon_i^2 \prod_{l=1}^{k} (\varphi(\Omega_2) - \varphi(w_{l,2})) \]
\[ = -2 \epsilon_i^2 \prod_{l=1}^{k} (\varphi(\Omega_3) - \varphi(w_{l,3})) = 2\Pi, \] \( (5.13) \)

where the constant \( \Pi \) is defined in (3.23), (3.24).

The proof of the theorem is given in appendix B.

Remark. One can easily recognize that, for each index \( \alpha \), the components \( G_a^{(1)}(u), G_a^{(2)}(u) \) have the same structure as solutions of the Lamé equation

\[ \frac{d^2 \Lambda}{du^2} = (n(n + 1)\varphi(u) + B)\Lambda, \quad n \in \mathbb{N}, \quad B = \text{const}, \]

with \( n = k \) (see, e.g., [13]), namely,

\[ \Lambda_1 = \prod_{r=1}^{k} \left( \frac{\sigma(u - h_r)}{\sigma(u)\sigma(h_r)} \exp(\zeta(h_r)u) \right), \quad \Lambda_2 = \prod_{r=1}^{k} \left( \frac{\sigma(u + h_r)}{\sigma(u)\sigma(h_r)} \exp(-\zeta(h_r)u) \right), \]

where the zeros \( h_1, \ldots, h_k \) satisfy various conditions, in particular,

\[ \varphi(h_1) + \cdots + \varphi(h_k) = kB. \]

However, as numerical tests show, the zeros \( w_{1,a}, \ldots, w_{k,a} \) of the solutions (5.5) do not satisfy all the conditions on \( h_1, \ldots, h_k \). Hence \( G_a^{(1)}(u), G_a^{(2)}(u) \) cannot be solutions of the Lamé equation.

Thus, if the relation between the Poisson equations (1.11) and the Lamé equation (or some of its generalizations) exists, it should be a rather non-trivial one.

6. The classical case \( k = 1 \)

The Poisson equations in this case were first integrated by Jacobi [8], who used previous results of Legendre (see, e.g., [12]). The case does not fit completely into theorems 4 and 6 because the corresponding meromorphic differentials (4.2) do not contain the factor \((\lambda - c)^s\), and the solutions do not have precisely the structure of (5.5).
Namely, for \( k = 1 \) the elliptic solution \( P(u) \) is just \( M(u) \) given by (3.20), and the algebraic solutions (4.1) reread
\[
G_a^{(1,2)} = c_a \sqrt{M_1^2 + M_2^2} \exp \left( \pm \frac{1}{2} \int W_\alpha \right), \quad (\alpha, \beta, \gamma) = (1, 2, 3).
\]
Set, for concreteness, \( \alpha = 3 \). Using the algebraic parameterization (3.3) for \( M(\lambda) \), from (4.7) we get
\[
W_3 = \frac{2 \pi i a_1 M_1^2(\lambda) + a_2 M_2^2(\lambda)}{M_1^2(\lambda) + M_2^2(\lambda)} \frac{d\lambda}{2(\lambda - \lambda_\alpha^2 - c_{\alpha 2}^2)\lambda - (ca_1 - a_1 a_2)}
\]
\[
= \frac{2(\lambda_1^2 - a_1 a_2)\lambda + c(a_1 a_2 - a_1 a_3 - a_2 a_3) - a_1 a_2 a_3}{(c + a_3 - a_1 - a_2)\lambda - (ca_1 - a_1 a_2)\lambda} \frac{d\lambda}{2\sqrt{U(\lambda)}}.
\]
This is a differential of the third kind having a pair of simple poles \( (\lambda^*, \pm \sqrt{U(\lambda^*)}) \) on \( E \) with
\[
\lambda^* = \frac{a_1 a_2 - ca_3}{c + a_3 - a_1 - a_2}.
\]
Observe that in (6.1)
\[
2 \frac{(ca_3 - a_1 a_2)\lambda + c(a_1 a_2 - a_1 a_3 - a_2 a_3) - a_1 a_2 a_3}{(c + a_3 - a_1 - a_2)\lambda - (ca_1 - a_1 a_2)\lambda} \bigg|_{\lambda = \lambda_3} = 2a_3.
\]
Then, according to proposition 5,
\[
\int_{(c,0)}^{(\lambda, \mu)} W_3 = \log \frac{\sigma(u - w_3)}{\sigma(u + w_3)} + 2[\xi(w_3) + a_3]u - \pi i,
\]
\[
\sqrt{M_1^2 + M_2^2} = \text{const} \frac{\sqrt{\sigma(u - w_3)\sigma(u + w_3)}}{\sigma(u)},
\]
where \( w_3 \) is the Abel image of the pole \( (\lambda^*, \mu) \) of \( W_3 \) with residue \(-1\), namely
\[
w_3 = \int_{(c,0)}^{(\lambda^*, 0)} \frac{i d\lambda}{2\sqrt{U(\lambda)}}.
\]
Repeating the above calculations for the differentials \( W_1, W_2 \), we conclude that, up to multiplication by \(-1\), the elliptic solutions of the second kind are
\[
G_a^{(1)}(u) = e_a \frac{\sigma(u - w_\alpha)}{\sigma(u) \sigma(-w_\alpha)} \exp \left( (\zeta(w_\alpha) + a_\alpha)u \right),
\]
\[
G_a^{(2)}(u) = e_a \frac{\sigma(u - w_\alpha)}{\sigma(u) \sigma(-w_\alpha)} \exp \left( -(\zeta(w_\alpha) + a_\alpha)u \right)
\]
(compare with (5.5)), where \( w_\alpha \) denotes the Abel image of the pole of the differential \( W_\alpha \) with the residue \(-1\), and, as above, \( u = i m t + \Omega_2 \).

As follows from item 3 of theorem 6, here
\[
w_\alpha - w_\beta \equiv \Omega_\gamma \mod \{2\Omega_1 \mathbb{Z} + 2\Omega_2 \mathbb{Z} \},
\]
\[
\zeta(w_\alpha) - \zeta(w_\beta) = \eta_\gamma = \zeta(\Omega_\gamma) \mod \{2\eta_1 \mathbb{Z} + 2\eta_2 \mathbb{Z} \},
\]
\[
(\alpha, \beta, \gamma) = (1, 2, 3).
\]
Then, introducing
\[
w^* = w_\alpha - \Omega_\alpha, \quad \theta^* = \zeta(w_\alpha) - \eta_\alpha = \zeta(w^*) \quad \text{for any} \ \alpha
\]
and using the definition (3.13) of the sigma-functions with indices, one can represent the complex solutions (6.3) in the following form
\[
G_a^{(1)}(u) = e_a \frac{\sigma_\alpha(u - w^*)}{\sigma(u) \sigma_\alpha(-w^*)} \exp \left( (\theta^* + a_\alpha)u \right),
\]
\[
G_a^{(2)}(u) = e_a \frac{\sigma_\alpha(u + w^*)}{\sigma(u) \sigma_\alpha(w^*)} \exp \left( -(\theta^* + a_\alpha)u \right).
\]
By transforming the integrals defining $w_\eta$ (in particular, (6.2)), one can show that

$$w^* = \int_{\mathbb{C}(\lambda)}^{\infty} \frac{i d\lambda}{2\sqrt{U_4(\lambda)}},$$

where $\infty$ stands for one of the two infinite points on $E$.

Finally notice that being rewritten in terms of theta-functions with characteristics, the expressions (6.4) coincide with the complex elliptic solutions of the second kind presented by Jacobi (see also [12]).

7. Real normalized vector solutions

As was shown in section 2, the elliptic solution $P(u)$ in (3.20), (3.21) for $u = imt + \Omega_2$, $t \in \mathbb{R}$, is real. Then, by their construction (see (4.6)), the basis elliptic second kind solutions $G_a(1)(u), G_a(2)(u)$ have opposite arguments:

$$\text{Arg}(G_a(1)(u)) = \text{Arg}(G_a(2)(u)), \quad \alpha = 1, 2, 3. \quad (7.1)$$

Then two independent non-normalized real vector solutions of the Poisson equations can be written in the form

$$\gamma^{(1)}(t) = v_1 G^{(1)}(imt + \Omega_2) + v_2 G^{(2)}(imt + \Omega_2),$$

$$\gamma^{(2)}(t) = \frac{1}{2} \left[ v_1 G^{(1)}(imt + \Omega_2) - v_2 G^{(2)}(imt + \Omega_2) \right] \quad (7.2)$$

with some appropriate constants $v_1, v_2$. In view of (7.1), it is sufficient to set

$$v_1 = \chi [G_a(1)(imt + \Omega_2)]^{-1}, \quad v_2 = \chi [G_a(2)(imt + \Omega_2)]^{-1}, \quad (7.3)$$

for any fixed real $t^*$, any real nonzero $\chi$, and any $\alpha \in \{1, 2, 3\}$. Then we arrive at

**Theorem 7.** A real orthogonal rotation matrix formed by the three independent unit vector solutions of the Poisson equations (1.11) for $b = \kappa a$ has the form

$$R(t) = \frac{1}{\sqrt{\Pi}} \left( \tilde{\gamma}^{(1)}(t), \tilde{\gamma}^{(2)}(t), \tilde{P}(t) \right), \quad \tilde{P}(t) = P(imt + \Omega_2),$$

$$\tilde{\gamma}^{(1)}(t) = \frac{1}{2} \left[ \frac{1}{\sqrt{s_2}} G^{(1)}(imt + \Omega_2) + \sqrt{s_2} G^{(2)}(imt + \Omega_2) \right],$$

$$\tilde{\gamma}^{(2)}(t) = \frac{1}{2} \left[ \frac{1}{\sqrt{-s_2}} G^{(1)}(imt + \Omega_2) + \sqrt{-s_2} G^{(2)}(imt + \Omega_2) \right]. \quad (7.4)$$

where $P(u)$ is the elliptic solution (3.20), (3.21), and $G^{(1)}(u), G^{(2)}(u)$ are the elliptic solutions of the second kind described in (5.5) and theorem 6. Next, $s_2$ is real and is specified in (5.8), whereas the constant $\Pi$ is defined in (3.23), (3.24).

Note that the columns of $R(t)$ form a left- or right-oriented orthonormal basis.

**Proof of theorem 7** Setting in (7.3) $t^* = 0, \alpha = 2$, and

$$\chi = \frac{\sigma_2}{2} \sqrt{(\phi(\Omega_2) - \phi(\alpha_{1,2})) \cdots (\phi(\Omega_2) - \phi(\alpha_{k,2}))},$$

in view of (5.12), we get $v_1 = s_2^{-1/2}, v_2 = s_2^{1/2}$. Then (7.2) gives

$$\gamma^{(1)}(0) = \frac{1}{2} \left[ \frac{1}{\sqrt{s_2}} G^{(1)}(\Omega_2) + \sqrt{s_2} G^{(2)}(\Omega_2) \right],$$

$$\gamma^{(2)}(0) = \frac{1}{2} \left[ \frac{1}{\sqrt{-s_2}} G^{(1)}(\Omega_2) + \sqrt{-s_2} G^{(2)}(\Omega_2) \right].$$
Then, according to (5.10) and (5.13),
\[
3 \sum_{a=1}^{A} [Y_a^{(1)}(0)]^2 = \sum_{a=1}^{A} [Y_a^{(2)}(0)]^2 = \frac{1}{2} \langle G^{(1)}(\Omega_2), G^{(2)}(\Omega_2) \rangle = \Pi. \tag{7.5}
\]

As a result, the real vectors \( \vec{\gamma}^{(1)}(t), \vec{\gamma}^{(2)}(t), \vec{P}(t) \) all have the same length. By their construction, they are all orthogonal. Hence, we obtain the matrix \( R(t) \) in (7.4). \( \square \)

8. The case of negative odd \( k \)

We first note that the case of negative odd \( k \) cannot be reduced to the already considered case \( k > 0 \) by the trivial substitution \( t = -T \) in the Poisson equations in (1.6). Indeed, this change gives
\[
\frac{d\gamma}{dT} = -k\gamma \times aM(-T).
\]
The elliptic vector solution \( M(-T) \) given by (3.20) with \( u = -imT + \Omega_2 \) is neither odd nor even, hence one cannot write \( M(-T) = M(T) \), and the above equation cannot be transformed to the form
\[
\frac{d\gamma}{dT} = -k\gamma \times aM(T).
\]

Nevertheless, the analysis for positive \( k \) is sufficient to cover all the cases. Indeed, upon introducing new moments of inertia
\[
\Lambda_i = (J_i + J_2 - J_3)J_i, \quad (i, j, k) = (1, 2, 3), \tag{8.1}
\]
the system (1.6) can be rewritten as
\[
\dot{\Lambda}\omega = -\Lambda\omega \times \omega, \quad \dot{\gamma} = k\gamma \times \omega, \tag{8.2}
\]
which, under the change \( t = -T \), gives
\[
\dot{\Lambda}\omega' = \Lambda\omega \times \omega, \quad \dot{\gamma}' = -k\gamma \times \omega, \quad (') = \frac{d}{dT}. \tag{8.3}
\]
The Euler equations here have the integrals
\[
(\omega, \Lambda\omega) = L, \quad (\omega, \Lambda^2\omega) = M^2
\]
with integration constants \( L, M \). Then, applying to these equations the procedure of section 3, we express the solutions \( \omega(T) \) in terms elliptic functions of the curve
\[
E' = \{W^2 = -(Z - A_1)(Z - A_2)(Z - A_3)(Z - C)\}
\]
with the parameters
\[
A_i = 1/\Lambda_i = \frac{a_i^2a_ia_k}{a_ia_j + a_ia_k - a_ja_k},
\]
\[
C = \frac{L}{M^2} = \frac{(\tau_2c - 2\tau_3)\tau_3}{\tau_2^2 - 4\tau_1\tau_3 + 4\tau_3^2}, \quad M^2 = \langle \omega, \Lambda^2\omega \rangle = \frac{\tau_2^2 - 4\tau_1\tau_3 + 4\tau_3^2}{\tau_3^2}m^2, \quad \tau_1 = a_1 + a_2 + a_3, \quad \tau_2 = a_1a_2 + a_2a_3 + a_3a_1, \quad \tau_3 = a_1a_2a_3.
\]
Here, as above, in (1.3),
\[
a_i = 1/J_i, \quad c = 1/m^2, \quad m^2 = \langle M, M \rangle = \langle \omega, J^2\omega \rangle.
Since \( \omega(T) \), like \( M(t) \), must also be elliptic functions of the original curve \( E \), we get the following relation

\[
d\tau = \frac{d\lambda}{2m\sqrt{-(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - c)}} = -\frac{dZ}{2M\sqrt{-(\lambda - A_1)(\lambda - A_2)(\lambda - A_3)(\lambda - C)}}. \tag{8.4}
\]

**Remark.** As one may expect, the elliptic curves \( E, E' \) with the parameters \( a_i, c \) and \( A_i, C \) are birationally equivalent. Indeed, \( E' \) is transformed to \( E \) by the substitution

\[
Z = \frac{(\tau_2\lambda - 2\tau_3)\tau_3}{\tau_2^2 - 4\tau_1\tau_3 + 4\tau_3\lambda},
\]

\[
W = \mu \frac{(a_1a_2 - a_2a_3 - a_1a_3)(a_1a_3 - a_2a_3 - a_1a_2)(a_2a_3 - a_1a_3 - a_1a_2)}{(\tau_2^2 - 4\tau_1\tau_3 + 4\tau_3\lambda)^2 (\tau_2^2 - 4\tau_1\tau_3 + 4\tau_3c)}.
\]

We stress that the half-periods of \( E \) and \( E' \)

\[
\Omega'_\alpha = \int_c^{\omega} \frac{i d\lambda}{\sqrt{U_4(\lambda)}}, \quad \Omega_\alpha = \int_c^{\lambda} \frac{i dZ}{2\sqrt{-(\lambda - A_1)(\lambda - A_2)(\lambda - A_3)(\lambda - C)}},
\]

\( \alpha = 1, 2, 3 \)

in general, do not coincide, but are only proportional to each other: in view of (8.4),

\[
\Omega'_\alpha = \frac{M}{m} \Omega_\alpha, \quad \alpha = 1, 2, 3.
\]

Now comparing (1.6) or (8.2) with (8.3), we arrive at the following observation.

**Proposition 8.** Let \( k \) be an odd negative integer and the vector

\[
\gamma(T) = \gamma(T|A_1, A_2, A_3, C, M) \text{ be a solution of the Poisson equations in (8.3) with the elliptic coefficients } \omega(T), \text{ related to the parameters } A_i, C, M. \text{ Then } \gamma(t) = \gamma(-T) \text{ is a solution of the Poisson equations (1.11) with the elliptic coefficients } M_\alpha(t), \text{ related to the parameters } a_i, c, m, \text{ and vice versa.}
\]

In other words, the solutions \( \gamma(t) \) of (1.11) with an odd negative \( k \) and the parameters \( a_1, a_2, a_3, c, m \) are given by \( \gamma(-t|A_1, A_2, A_3, C, M) \). The latter are described by the formulas of theorems 4 and 6 corresponding to \|k\|, the parameters \( A_1, A_2, A_3, C, M \) and the corresponding roots of the polynomials \( F_{t,\alpha}(\lambda), Q_{k,\alpha}(\lambda) \).

We stress that although the elliptic vector functions \( \omega(t) \) for \( a_1, a_2, a_3, c, m \) and \( \omega(-t) \) for \( A_1, A_2, A_3, C, M \) coincide, this is no longer true for the solutions \( \gamma(t|a_1, a_2, a_3, c, m) \) and \( \gamma(-t|A_1, A_2, A_3, C, M) \).

**9. Comparison with the Halphen equation**

By using the algebraic parametrization (3.3), the generalized Poisson equations (1.11) can be rewritten as third order ODE for one of the components of the vector \( \gamma \), say \( \gamma_1 \), with the independent variable \( \lambda \in \mathbb{C} \). For general \( a_i, k \) the explicit expressions for the coefficients of the ODE are very long. We give an example for \( a_1 = 1, a_2 = 2, a_3 = 3 \) and \( k = 3 \) (\( c \) is arbitrary):

\[
\frac{d^3}{d\lambda^3}\gamma_1 + g_2(\lambda) \frac{d^2}{d\lambda^2}\gamma_1 + g_1(\lambda) \frac{d}{d\lambda}\gamma_1 + g_0(\lambda)\gamma_1 = 0, \tag{9.1}
\]
with
\[
g_2 = \frac{3}{2(\lambda - 2)} + \frac{3}{2(\lambda - 3)} + \frac{3}{\lambda - c} - \frac{c + 10}{10\lambda - 15c + 6 + c\lambda} + \frac{1}{\lambda - 1},
\]
\[
g_1 = -\frac{52(c - 2)(\lambda - 2)}{3(138c - 37)} + \frac{24(c - 3)(\lambda - 3)}{82c - 119} - \frac{4(c - 2)(c - 3)(c - 1)(c - \lambda)}{(10 + c)^2(10c^2 + 2267c - 2666)} \frac{1}{4(\lambda - 1)^2} + \frac{156(c - 2)(-3 + c)(7c - 8)(-15c + 10\lambda + 6 + c\lambda)}{5 \cdot 34c^2 - 276c + 263} - \frac{4(c - \lambda)^2}{5(7c - 8)(c - 1)(\lambda - 1)}.
\]
\[
g_0 = -\frac{27(c - 1)(23c - 30)}{104(\lambda - 2)(c - 2)^2} - \frac{3(\lambda - 1)(17c - 22)}{9(5c - 6)} + \frac{32(c - 3)(c - 3)^2}{9(\lambda - 1)} + \frac{9(4c^3 - 81c^2 + 171c - 98)}{9(109c^3 - 235c^2 + 106c + 24)}.
\]

That is, the coefficients have poles at \( \lambda = a_1, a_2, a_3, c, \) and at an extra pole defined by the condition \(-15c + 6 + (10 + c)\lambda = 0\).

A natural question is how the above third order equation is related to known linear equations with elliptic coefficients admitting elliptic solutions of the second kind. The best known example is the Halphen equation
\[
\frac{d^3}{du^3} \Psi + (1 - n^2)\varphi(u) \frac{d}{du} \Psi + \varphi'(u) \frac{1 - n^2}{2} \Psi = h \Psi, \quad \Psi = \Psi(u),
\tag{9.2}
\]
where \( n \) is integer and \( h \) is an arbitrary parameter. As above, \( \varphi(u) \) is the Weierstrass function. For any such \( n \) the three independent solutions \( \Psi_1(u), \Psi_2(u), \Psi_3(u) \) are elliptic functions of the second kind with poles of order \( g = n - 1 \) at \( u = 0 \):
\[
\Psi_{\alpha}(u) = \frac{\sigma(u - w^{(\alpha)}_1(h)) \ldots \sigma(u - w^{(\alpha)}_{n-1}(h))}{\sigma(u)^{n-1}} \exp[(\zeta(w^{(\alpha)}_1) + \ldots + \zeta(w^{(\alpha)}_{n-1}))u],
\]
\( \alpha = 1, 2, 3. \)

The structure of the solutions generalizes that of solutions (5.5) of our equation (9.1). So, one might suppose that the equation (9.1) is a special case of the Halphen equation (for \( h = 0 \)), when one of its solutions is elliptic. For a general discussion of the Halphen equation see [11].

However, written in the algebraic form with the independent variable \( z \) such that
\[
\Psi(u) = \frac{d}{dz} \int_{\Gamma} \frac{dz}{2\sqrt{(z - e_1)(z - e_2)(z - e_3)}},
\]
the Halphen equation with \( h = 0 \) is
\[
4(z - e_1)(z - e_2)(z - e_3) \frac{d^3}{dz^3} \Psi + g_2(z) \frac{d^2}{dz^2} \Psi + g_1(z) \frac{d}{dz} \Psi_1 + g_0(z) \Psi = 0,
\]
\[
g_2 = 36\lambda^2 + 18\lambda + 81, \quad g_1 = 12\lambda - 1 - n^2, \quad g_0 = -(1 - n^2)^2/2.
\]

Hence, the coefficients of the normalized equation have finite poles only at \( z = e_1, e_2, e_3 \). Taking into account that the equation (9.1) has five poles (which can be reduced to four finite poles), it cannot be identified with a special case of the Halphen equation.
Acknowledgments

The work of YF was supported by the Spanish MINECO-FEDER grants MTM2009-06973, MTM2009-09676 and MTM2012-31714. AJM and MP acknowledge the support of grant DEC-2011/02/A/ST1/00208 of the National Science Centre of Poland. The authors are grateful to V Enolski for discussions, and to anonymous referees for valuable remarks and comments that improved the article.

Appendix A. A numerical example

This example was made with Maple by using the functions \( \text{WeierstrassP}(u, g[2], g[3]) \), \( \text{WeierstrassSigma}(u, g[2], g[3]) \), \( \text{WeierstrassZeta}(u, g[2], g[3]) \). Consider the simplest nontrivial case \( \kappa = 3 \) and \( b = ka \). Choose the inertia tensor \( J = \text{diag}(1, 1/2, 1/3) \), i.e., \( a = \text{diag}(1, 2, 3) \) and \( l/n^2 = c = 5/2 \in [a_2, a_3], n = 1 \). Therefore, in the real case the parameter \( \lambda \in [a_1, a_2] = [1; 2] \). The elliptic curve \( E \) has the form

\[
\mu^2 = 2U_4(\lambda) = -2(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 5/2).
\]

The birational transformation \((3.5)\), namely,

\[
z = -\frac{1}{12} \frac{\lambda + 2}{\lambda - 5/2}, \quad \lambda = \frac{15z - 1}{12z + 1},
\]

takes it to the Weierstrass form

\[
w^2 = 4z^3 - \frac{8}{7} z + \frac{10}{27} = 4(z + 5/6)(z - 1/6)(z - 2/3),
\]

with \( z(\lambda = 1) = 1/6, \quad z(2/3) = 2/3, \quad z(3) = -5/6. \)

Hence the parameters of the Weierstrass functions of \( E \) are \( g_2 = 7/3, \quad g_3 = -10/27 \). The half-periods are\(^6\):

\[
\begin{align*}
\Omega_1 &= i \int_{\epsilon}^{a_1} \frac{d\lambda}{\mu} = \int_{\infty}^{1/6} \frac{dz}{w} = -1.656638 - 1.415737 \cdot i, \\
\Omega_2 &= \int_{\infty}^{2/3} \frac{dz}{w} = 1.656638, \quad \Omega_3 = \int_{\infty}^{1/6} \frac{dz}{w} = 1.415737 \cdot i.
\end{align*}
\]

The corresponding constants \( \eta_i = \xi(\Omega_i) \) in \((3.13)\) are

\[
\eta_1 = -0.440206 + 0.571999 \cdot i, \quad \eta_2 = 0.4402056, \quad \eta_3 = -0.571999 \cdot i.
\]

This allows us to calculate

\[
\sigma_\alpha(u) = \exp(\eta_\alpha u) \frac{\sigma(\Omega_\alpha - u|g_2, g_3)}{\sigma(\Omega_\alpha|g_2, g_3)}, \quad \alpha = 1, 2, 3.
\]

Next,

\[
\epsilon_1 = \frac{1}{\sqrt{(a_1 - a_2)(a_1 - a_3)}} = \frac{\sqrt{2}}{2}, \quad \epsilon_2 = -i, \quad \epsilon_3 = \frac{\sqrt{2}}{2}.
\]

From \((3.18)\) we have

\[
F_{11}(\lambda) = -34\lambda + 167/2, \quad F_{12} = -48\lambda + 237/2, \quad F_{13} = -66\lambda + 327/2.
\]

The Abel images of their zeros on the complex plane \( u \) are

\[
u_1 = \pm 0.344972, \quad v_2 = \pm 0.289812, \quad v_3 = \pm 0.246863.
\]

\(^6\) In this example most of the floating point numbers are indicated to six decimal places.
Then the real elliptic solutions for the Euler and the Poisson equations are given by

\[ M_\alpha = \epsilon_\alpha \frac{\sigma_\alpha(u)}{\sigma(u)}, \quad P_\alpha = \epsilon_\alpha \frac{\sigma_\alpha(u - v)\sigma(u + v)}{\sigma^3(u)}, \]

\[ \alpha = 1, 2, 3, \quad u = it + \Omega_2, \quad t \in \mathbb{R}, \]

with the above-indicated values of the parameters. Note that here \(|M| = 1\) and \(|P|^2 = \Pi = 201.062507.

Next, the meromorphic differentials in (4.2) are

\[ W_1 = i \frac{3\sqrt{3217}(2y - 5)(260y^3 - 1295y + 1613)}{2(3984y^3 - 29608y^2 + 73363y - 60607)\sqrt{\Omega_4(\lambda)}}, \]

\[ W_2 = i \frac{3\sqrt{3217}(2y - 5)(560y^3 - 3136y + 4331)}{2(23824y^3 - 194232y^2 + 523569y - 467236)\sqrt{\Omega_4(\lambda)}}, \]

\[ W_3 = i \frac{3\sqrt{3217}(2y - 5)(1084y^3 - 4913y + 5512)}{2(50672y^3 - 356280y^2 + 832461y - 646139)\sqrt{\Omega_4(\lambda)}}. \]

Then, up to constant factors, the polynomials \(Q_{3,\alpha}\) in theorem 4 are

\[ Q_{3,1}(\lambda) = 3167/2\lambda^3 - 44553/4\lambda^2 + 832461/32\lambda - 646139/32, \]

\[ Q_{3,2}(\lambda) = 1489\lambda^3 - 24279/2\lambda^2 + 523569/16\lambda - 116809/4, \]

\[ Q_{3,3}(\lambda) = 6723/2\lambda^2 - 99927/4\lambda^2 + 1980801/32\lambda - 1636389/32. \]

The Abel images of their zeros on the complex \(u\)-plane are

\[ w_{1,3} = \pm 0.21963966, \quad w_{2,3} = \pm 0.309571742, \quad w_{3,3} = \pm 1.232049297, \]

\[ w_{1,2} = \pm (\Omega_1 + 0.40885), \quad w_{2,2} = \pm (0.272475 \pm 0.041941 \cdot i), \]

\[ w_{1,1} = -0.26888528, \quad w_{2,1} = \pm (0.3424574955 \pm 0.2549408658 \cdot i). \]

Applying the condition (5.4), we choose the signs

\[ w_{1,3} = -0.21963966, \quad w_{2,3} = -0.309571742, \quad w_{3,3} = 1.232049297, \]

\[ w_{1,2} = -0.40885 + \Omega_3, \quad w_{2,2} = -0.272475 + \Omega_3, \]

\[ w_{3,2} = -0.272475 - \Omega_3, \]

\[ w_{1,1} = -0.26888528, \quad w_{2,1} = -0.3424574955 - 0.2549408658 \cdot i, \]

\[ w_{3,1} = -0.3424574955 + 0.2549408658 \cdot i. \]

Then

\[ \Sigma_3 = \sum_{j=1}^{3} w_{j,3} = 0.7028378946, \quad \Sigma_2 = -0.95380028 + \Omega_3, \quad \Sigma_1 = -0.95380028 \]

and

\[ \Theta_3 = \sum_{j=1}^{3} \zeta(w_{j,3}) = -7.03775, \quad \Theta_2 = -7.477955 - 0.572 \cdot i, \quad \Theta_1 = -7.477955. \]

Notice that

\[ \Sigma_3 - \Sigma_2 = 1.656638 - 1.41573 \cdot i = i = \Omega_1 \equiv \Omega_1, \quad \Sigma_2 - \Sigma_1 = \Omega_3, \quad \Sigma_3 - \Sigma_1 = \Omega_2 \]

(A.3)

and

\[ \Theta_3 - \Theta_2 = \zeta(\Omega_3), \quad \Theta_2 - \Theta_1 = \zeta(\Omega_3) = \eta_3, \quad \Theta_3 - \Theta_1 = \zeta(\Omega_2) = \eta_2. \]

(A.4)
That is, the sums of zeros of $Q_j(\lambda)$ on the complex $u$-plane differ by the half-periods of the curve $E$, as predicted by item 3 of theorem 6. Note that for arbitrary values of $w_{l,j}$, the relations (A.3) do not imply (A.4).

The basis of the complex vector elliptic solutions of the second kind in (5.5) is specified by

$$
G^{(1)}_a(u) = e_a \frac{\sigma(u - w_{1,a})\sigma(u - w_{2,a})\sigma(u - w_{3,a})}{\sigma^3(u)\sigma(-w_{1,a})\sigma(-w_{2,a})\sigma(-w_{3,a})} e^{\theta_{a,u}},
$$

$$
G^{(2)}_a(u) = e_a \frac{\sigma(u + w_{1,a})\sigma(u + w_{2,a})\sigma(u + w_{3,a})}{\sigma^3(u)\sigma(w_{1,a})\sigma(w_{2,a})\sigma(w_{3,a})} e^{-\theta_{a,u}},
$$

for any $\alpha = 1, 2, 3$, with $w_{l,a}$ as in (A.2). Next,

$$
\begin{pmatrix}
G^{(1)}_1(u + 2\Omega_3) \\
G^{(1)}_2(u + 2\Omega_3) \\
G^{(1)}_3(u + 2\Omega_3)
\end{pmatrix} = s_3
\begin{pmatrix}
G^{(1)}_1 \\
G^{(1)}_2 \\
G^{(1)}_3
\end{pmatrix}(u),
$$

$$
s_3 = \exp(2[\Theta_1\Omega_3 - \Sigma_1\eta_3]) = -0.962799 + 0.2702178 \cdot i, \quad |s_3| = 1,
$$

$$
\begin{pmatrix}
G^{(1)}_1(u + 2\Omega_2) \\
G^{(1)}_2(u + 2\Omega_2) \\
G^{(1)}_3(u + 2\Omega_2)
\end{pmatrix} = s_2
\begin{pmatrix}
G^{(1)}_1 \\
G^{(1)}_2 \\
G^{(1)}_3
\end{pmatrix}(u), \quad s_2 = \exp(2[\Theta_1\Omega_2 - \Sigma_1\eta_2]) = 0.402144 \cdot 10^{-10}.
$$

The monodromy of the solutions $G^{(2)}_a(u)$ is inverse to the one above.

Finally, the orthogonal matrix of real solutions is

$$
\mathcal{R}(t) = \frac{1}{\sqrt{\Pi}} \begin{pmatrix}
\tilde{\gamma}^{(1)}(t) & \tilde{\gamma}^{(2)}(t) & \tilde{P}(t) \\
\end{pmatrix},
\tilde{P}(t) = P(u + \Omega_2),
$$

$$
\tilde{\gamma}^{(1)}(t) = \frac{1}{2} \left[ \frac{1}{\sqrt{s_2}} G^{(1)}(u + \Omega_2) + \sqrt{s_2} G^{(2)}(u + \Omega_2) \right],
$$

$$
\tilde{\gamma}^{(2)}(t) = \frac{1}{2} \left[ \frac{1}{\sqrt{s_2}} G^{(1)}(u + \Omega_2) + \sqrt{s_2} G^{(2)}(u + \Omega_2) \right],
$$

with $\sqrt{s_2} = 0.634148 \cdot 10^{-5}$ and $\Pi = 201.0625$.

Appendix B. Proofs of proposition 5 and theorem 6

Proposition 5 is a reformulation of known relations of the theory of elliptic functions. Consider first an elliptic curve $E$ in the canonical Weierstrass form (3.6),

$$
E = \{ w^2 = P_3(z) \equiv 4(z - e_1)(z - e_2)(z - e_3), \quad e_1 + e_2 + e_3 = 0, \}
$$

a point $P = (z, w) = (z, \sqrt{P_3(z)})$ on it, and the Abel map

$$
u = \int^P \frac{dz}{2\sqrt{(z - e_1)(z - e_2)(z - e_3)}},
$$

which gives $z = \varphi(u)$.

Now let

$$
\tilde{W} = \frac{\tilde{q}_k(z)}{2\sqrt{P_3(z)} \tilde{Q}_k(z)} \, dz, \quad \tilde{Q}_k(z) = (z - z_1) \cdots (z - z_k)
$$

be a meromorphic differential of the third kind having pairs of only simple poles at the finite points $P_i^\pm = (z_i, \pm 2\sqrt{R_i(z_i)})$, $i = 1, \ldots, k$ with residua $\pm 1$, respectively. Here $q_k(z) = b_kz^k + \cdots + b_0$ is a polynomial of degree at most $k$. 

21
Theorem 9. If $u$ and $P \in \mathcal{E}$ are related by the map (B.1), then, up to an additive constant,
\[
\int_{\infty}^{P} W = \log \frac{\sigma(u - w_1) \cdots \sigma(u - w_k)}{\sigma(u + w_1) \cdots \sigma(u + w_k)} + 2[\zeta(w_1) + \cdots + \zeta(w_k)]u + \kappa u, \tag{B.3}
\]
where $w_i$ are uniquely determined from the conditions
\[
g(\pm w_i) = z_i, \quad \frac{\tilde{q}_k(z_i)}{\tilde{Q}_k(z_i)} = -g'(w_i). \tag{B.4}
\]
Here, $\zeta(u)$ is the Weierstrass zeta-function, $g(u)$ is the derivative of the Weierstrass $P$-function, and $\kappa$ is the first coefficient in the expansion of $W$ at the infinite point $\infty \in \mathcal{E}$:
\[
W = (\kappa + O(u)) u, \text{ that is,}
\]
\[
\kappa = \lim_{z \to \infty} \frac{\tilde{q}_k(z)}{\tilde{Q}_k(z)} = b_k.
\]

Proof. Since $2\sqrt{K_1(z)} = g'(w_i)$, the condition $\text{Res}_{P-\tilde{P}} \tilde{W} = -1$ is equivalent to (B.4).

It is known ([5]) that the above integral has the form
\[
\int_{\infty}^{P} W = \log \frac{\sigma(u - w_1) \cdots \sigma(u - w_k)}{\sigma(u + w_1) \cdots \sigma(u + w_k)} + C_1 u + C_0, \quad C_1, C_0 = \text{const.}
\]
So, it remains to calculate $C_1$ for the differential (B.2). Differentiate both parts of (B.3) by $u$ and evaluate the result at $u = 0 (z = \infty)$. Then the right-hand side gives
\[
\sum_{i=1}^{k} \left[ \frac{\sigma'(u - w_i)\sigma(u + w_i) - \sigma'(u + w_i)\sigma(u - w_i)}{\sigma(u - w_i)\sigma(u + w_i)} + 2\zeta(w_i) \right]_{\sigma = 0} + \kappa
\]
\[
= \sum_{i=1}^{k} [\zeta(u - w_i) - \zeta(u + w_i) + 2\zeta(w_i)]_{\sigma = 0} + \kappa = \kappa.
\]

Differentiation of the left-hand side of (B.3) gives
\[
\left( \frac{d}{dz} \int_{\infty}^{P(z, w)} W \, dz \right) \left. \frac{dz}{du} \right|_{\sigma = 0} = \lim_{z \to \infty} \frac{\tilde{q}_k(z)}{\tilde{Q}_k(z)},
\]
which is precisely $b_k$.

Proof. Under a birational transformation $(z, u) \to (\lambda, \mu)$, which sends $z = \infty$ to $\lambda = c$ and converts $\mathcal{E}$ to the even order curve (3.2),
\[
\mu^2 = U_4(\lambda) = -(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)(\lambda - c),
\]
the differential (B.2) takes the form
\[
W = \frac{K_k(\lambda)}{\tilde{Q}_k(\lambda)} \frac{d\lambda}{\sqrt{U_4(\lambda)}}
\]
with certain degree $k$ polynomials $K_k(\lambda)$ and $Q_k(\lambda) = r_0(\lambda - r_1) \cdots (\lambda - r_n)$. Then theorem 9 implies
\[
\int_{(\lambda, \mu)}^{P(\lambda, \mu)} q_k(\lambda) \frac{d\lambda}{\tilde{Q}_k(\lambda) \sqrt{U_4(\lambda)}} = \log \frac{\sigma(u - w_1) \cdots \sigma(u - w_k)}{\sigma(u + w_1) \cdots \sigma(u + w_k)} + 2[\zeta(w_1) + \cdots + \zeta(w_k)]u + \kappa u,
\]
\[
\text{B.5}
\]

7 Here we used $\zeta(u) = \sigma'(u)/\sigma(u)$ and the oddness of $\zeta(u)$.
\[ w_k = \int_{c}^{(\lambda, \sqrt{U_{\lambda}(\lambda)})} \frac{d\lambda}{\sqrt{U_{\lambda}(\lambda)}}, \quad \delta = \frac{K_\delta(c)}{Q_\delta(c)}, \quad (B.6) \]

which is the expression (5.1) in proposition 5. Under the birational transformation (3.5), the condition (B.4) takes the form (5.4).

Proof of theorem 6

1. The structure of the solutions (5.5) follows from theorem 4 and proposition 5. Namely, substituting the sigma-function expressions (5.2), (5.1) for each \( F_{\lambda u}(\lambda) \) and \( W_\alpha \) into (4.1), one obtains these solutions.

Now notice that, in view of the leading behavior (3.14), the Laurent expansions of (5.5) near \( u = 0 \) and \( t = t_0 \) are

\[ G_\alpha^{(1,2)}(u + 2\Omega_1) = \pm s_j G_\alpha^{(1)}(u), \quad G_\alpha^{(2)}(u + 2\Omega_2) = \pm s_j^{-1} G_\alpha^{(2)}(u). \]

On the other hand, from the quasi-periodicity law of \( \sigma(u) \) we have, in particular,

\[ G_\alpha^{(1)}(u + 2\Omega_1) = s_1 G_\alpha^{(1)}(u), \quad G_\alpha^{(2)}(u + 2\Omega_2) = s_1^{-1} G_\alpha^{(2)}(u), \]

\[ s_1 = \exp(2\Theta_1 \Omega_1 - 2\Sigma \eta_1). \quad (B.7) \]

By the construction of the vectors \( G_\alpha^{(1,2)} \) (see (4.6)), for any \( u \in \mathbb{C} \)

\[ G_\alpha^{(1)}(u)P_3(u) + G_\alpha^{(2)}(u)P_2(u) + G_\alpha^{(1)}(u)P_3(u) \equiv 0. \quad (B.8) \]

Here, from (3.16) and (3.20), we have

\[ P(u + 2\Omega_1) = (P_1(u), -P_2(u), -P_3(u))^T, \]
\[ P(u + 2\Omega_2) = (-P_1(u), P_2(u), -P_3(u))^T, \]
\[ P(u + 2\Omega_3) = (-P_1(u), -P_2(u), P_3(u))^T. \]

This, together with the integral (B.8) and (B.7), implies

\[ G_\alpha^{(1)}(u + 2\Omega_1) = (s_1 G_\alpha^{(1)}(u), -s_1 G_\alpha^{(1)}(u), -s_1 G_\alpha^{(1)}(u))^T. \]

Repeating the argument for other \( \Omega_j \), we obtain the behavior (5.6), (5.8).

2. Let the half-period \( \Omega_j \) be imaginary and \( \Omega_2 \) be real. Then item (2) of theorem 2 implies that \( |s_j^2| = 1 \) if we identify the periods \( T_1, T_2 \) with some of the full periods \( 4\Omega_1, 4\Omega_2, 4\Omega_3 \) of \( M_1(u) \). Hence, also \( |s_j| = 1 \). By (5.8), \( s_j = \pm \exp(2\Theta_1 \Omega_1 - 2\Sigma \eta_j) \), therefore the argument \( 2\Theta_1 \Omega_1 - 2\Sigma \eta_j \) is imaginary. Since \( \eta_j \) is imaginary and \( \eta_2 \) is real, this means that \( 2\Theta_1 \Omega_1 - 2\Sigma \eta_2 \) is real, and \( s_j \) is real.

Next, from (5.6), we obtain, for example,

\[ \frac{G_\alpha^{(1)}(u + 2\Omega_1)}{G_\alpha^{(1)}(u)} = \frac{G_\alpha^{(1)}(u)}{G_\alpha^{(1)}(u + 2\Omega_2)} = \frac{G_\alpha^{(1)}(u)}{G_\alpha^{(1)}(u)}. \quad (B.9) \]
Hence, $G_1^{(1)}(u)/G_3^{(1)}(u)$ is an elliptic function with the periods $4\Omega_1, 2\Omega_2$. Its zeros and poles in the parallelogram of periods are

$$\{ w_{1,1}, \ldots, w_{k,1}, w_{1,1} + 2\Omega_1, \ldots, w_{k,1} + 2\Omega_1 \}$$

and, respectively,

$$\{ w_{1,3}, \ldots, w_{k,3}, w_{1,3} + 2\Omega_1, \ldots, w_{k,3} + 2\Omega_1 \}.$$  

Then, according to the Abel theorem, the difference of their sums must be zero modulo the lattice $\{4\Omega_1\mathbb{Z} + 2\Omega_2\mathbb{Z}\}$:

$$2\Sigma_1 - 2\Sigma_3 \in \{4\Omega_1\mathbb{Z} + 2\Omega_2\mathbb{Z}\},$$

therefore, $\Sigma_1 - \Sigma_3 \in \{2\Omega_1\mathbb{Z} + \Omega_2\mathbb{Z}\}$. Further, the case $\Sigma_1 - \Sigma_3 \in \{2\Omega_1\mathbb{Z} + 2\Omega_2\mathbb{Z}\}$ is not possible (otherwise $G_1^{(1)}(u)/G_3^{(1)}(u)$ would have been a product of an elliptic function with the period lattice $\{2\Omega_1\mathbb{Z} + 2\Omega_2\mathbb{Z}\}$ and an exponent, i.e., not doubly-periodic itself). Hence,

$$\Sigma_1 - \Sigma_3 \equiv \Omega_2 \mod \{2\Omega_1\mathbb{Z} + 2\Omega_2\mathbb{Z}\}.$$  

Applying the same argument to the quotients $G_2^{(1)}(u)/G_3^{(1)}(u), G_1^{(1)}(u)/G_2^{(1)}(u)$, we arrive at the first part of relations (5.9).

Next, the quasi-periodicity of $\sigma(u)$ implies

$$G_1^{(1)}(u + 2\Omega_1) = \exp \{2(\Sigma_1 - \Sigma_3)\Omega_1 - 2(\Theta_1 - \Theta_3)\eta_1 \} G_1^{(1)}(u),$$

$$G_3^{(1)}(u + 2\Omega_2) = \exp \{2(\Sigma_1 - \Sigma_3)\Omega_2 - 2(\Theta_1 - \Theta_3)\eta_2 \} G_3^{(1)}(u).$$

Comparing this with (B.9) and using $\Sigma_1 - \Sigma_3 \equiv \Omega_2$, as well as the known Legendre relations

$$\eta_1\Omega_2 - \eta_3\Omega_1 = \eta_1\Omega_1 - \eta_3\Omega_3 = \eta_1\Omega_2 - \eta_2\Omega_1 = \frac{1}{2}\pi i,$$

we get the second half of (5.9).

(3) Since $G_1^{(1)}(u)$ is a solution of the Poisson equations, it must satisfy the integral

$$G_1^{(1)2}(u) + G_2^{(1)2}(u) + G_3^{(1)2}(u) = M.$$  

Then

$$G_1^{(1)2}(u + 2n\Omega_2) + G_2^{(1)2}(u + 2n\Omega_2) + G_3^{(1)2}(u + 2n\Omega_2)$$

$$= e^{2\pi i 2n} [G_1^{(1)2}(u) + G_2^{(1)2}(u) + G_3^{(1)2}(u)] = M \quad \forall n \in \mathbb{Z}.$$  

Since $|s_2| \neq 1$, letting above $n \to \infty$ or $n \to -\infty$, we conclude that $M$ must be zero. The same argumentation applied to $G_3^{(2)}(u)$ gives the second identity in (5.10).

Let now $\Omega$ be a half-period of $E$, $\eta = \xi(\Omega)$ and $w \in \mathbb{C}$ an arbitrary number. From the identity

$$\frac{\sigma(\Omega + w)\sigma(\Omega - w)}{\sigma^2(\Omega)\sigma^2(w)} = \varphi(\Omega) - \varphi(w),$$

using the quasi-periodicity of $\sigma(u)$, we get

$$- e^{2\pi i} \frac{\sigma^2(\Omega - w)}{\sigma^2(\Omega)\sigma^2(w)} = \varphi(\Omega) - \varphi(w). \quad \text{(B.10)}$$
Applying this to the solutions \((5.5)\) in the case \(\Omega = \Omega_i, \ w = w_{i,1}\) and using \((5.8)\) gives us
\[
\left[ G^{(1)}_\alpha(\Omega_j) \right]^2 = \epsilon^2 \prod_{i=1}^k e^{2u(w_{i,1})} \frac{\sigma^2(\Omega_j - w_{i,1})}{\sigma^2(\Omega_j)\sigma^2(w_{i,1})} = (-1)^{k+1} \epsilon^2 \frac{\sigma^2(\Omega_j - w_{i,1})}{\sigma^2(\Omega_j)\sigma^2(w_{i,1})}
\]
\[
= (-1)^{k+1} \epsilon^2 \pi^j \prod_{i=1}^k e^{2u(w_{i,1})} \frac{\sigma^2(\Omega_j - w_{i,1})}{\sigma^2(\Omega_j)\sigma^2(w_{i,1})}
\]
i.e., the expressions \((5.12)\) for \(G^{(1)}_j(\Omega_j)\). The expressions for \(G^{(2)}_j(\Omega_j)\) are obtained in the same way.

Next, the structure of the solutions \((5.5)\) gives
\[
\sum_{\alpha=1}^3 G^{(1)}_\alpha(u)G^{(2)}_\alpha(u) = -\sum_{\alpha=1}^3 \epsilon^2 \prod_{i=1}^k (\phi(u) - \phi(w_{i,1})). \tag{B.11}
\]
On the other hand, as follows from \((5.10)\) and \((5.12)\), for any \(\alpha = 1, 2, 3\)
\[
\epsilon^2 \pi^1 \prod_{i=1}^k (\phi(\Omega_a) - \phi(w_{i,1})) = \epsilon^2 \prod_{i=1}^k (\phi(\Omega_\alpha) - \phi(w_{i,1})) + \epsilon^2 \prod_{i=1}^k (\phi(w_{i,1}) - \phi(w_{i,1})),
\]
\((\alpha, \beta, \gamma) = (1, 2, 3)\).
Then, for \(u = \Omega_j, j = 1, 2, 3\), relation \((B.11)\) yields
\[
\sum_{\alpha=1}^3 G^{(1)}_\alpha(\Omega_j)G^{(2)}_\alpha(\Omega_j) = -2\epsilon^2 \sum_{i=1}^k (\phi(\Omega_j) - \phi(w_{i,1})). \tag{B.12}
\]

Since \((G^{(1)}(u), G^{(2)}(u))\) is a first integral, we get the relations \((5.13)\).

It remains to prove that \((B.12)\) equals \(2\Pi\). According to theorem 4 and relation \((4.3)\),
\[
Q_{k,\alpha}(\lambda) = r_{k,\alpha} \prod_{i=1}^k (\lambda - r_{i,\alpha}) = \text{const}(P^2_{\beta}(\lambda) + P^2_{\gamma}(\lambda)) \cdot (\lambda - c)^k
\]
for \((\alpha, \beta, \gamma) = (1, 2, 3)\). This implies, in particular,
\[
C \prod_{i=1}^k (\phi(u) - \phi(w_{i,1})) = P^2_{\beta}(u) + P^2_{\gamma}(u) \tag{B.13}
\]
with a certain constant \(C\). On the other hand, in view of the solutions \((3.22)\), one obtains
\[
P^2_{\beta}(u) + P^2_{\gamma}(u) = \epsilon^2 \prod_{i=1}^k (\phi(u) - \phi(w_{i,1})) \prod_{i=1}^k (\phi(u) - \phi(w_{i,1})).
\]
Letting in the above two relations \(u \to 0\), taking into account the expansion \(\phi(u) = 1/u^2 + O(1)\), and comparing the leading coefficients of \(1/u^2\), we find
\[
C = \epsilon^2 + \epsilon^2 = \frac{1}{(a_2 - a_1)(a_2 - a_3)} + \frac{1}{(a_3 - a_1)(a_3 - a_1)} = -\epsilon^2.
\]
Since \(P_3(u) = \Omega_1 = 0\), we have \(\Pi = P^2_1(u) + P^2_2(u) + P^2_3(u) = P_2(\Omega_1) + P_3(\Omega_1)\). Comparing this with \((B.13)\) for \(u = \Omega_1\), we get
\[
\Pi = -\epsilon^2 \sum_{i=1}^k (\phi(\Omega_i) - \phi(w_{i,1})),
\]
and similar expressions when \(\Omega_1\) is replaced by \(\Omega_2, \Omega_3\). This, together with \((B.12)\), gives the last equality in \((5.13)\).
References

[1] Borisov A V and Tsygvintsev A V 1997 Kovalevskaya’s method in the dynamics of a rigid body Prikl. Mat. Mekh. 61 30–36 (in Russian)
Borisov A V and Tsygvintsev A V 1997 Kovalevskaya’s method in the dynamics of a rigid body J. Appl. Math. Mech. 61 27–32 (Engl. transl.)

[2] Fedorov Yu 1999 Classical integrable systems and billiards related to generalized Jacobians Acta Appl. Math. 55 251–301

[3] Fedoryuk M V 1987 Lamé wave functions in the Jacobi form: I Differentsialnye Uravneniya 23 1715–24 (in Russian)
Fedoryuk M V 1987 Lamé wave functions in the Jacobi form: II Differentsialnye Uravneniya 23 1913–22 (in Russian)

[4] Gesztesy F and Sticka W 1998 On a theorem of Picard Proc. Am. Math. Soc. 126 1089–99

[5] Hurwitz A and Courant R 1964 Vorlesungen uber Algemeine Funktiontheorie und Elliptische Funktionen (Berlin: Springer)

[6] Ilyashenko Yu and Yakovenko S 2008 Lectures on Analytic Differential Equations (Graduate Studies in Mathematics vol 86) (Providence, RI: American Mathematical Society)

[7] Ince E 1956 Ordinary Differential Equations (New York: Dover)

[8] Jacobi C G J 1882 Sur la rotation d’un corps Gesammelte Werke 2 289–352

[9] Arnold V I, Kozlov V V and Neishtadt A I 2006 Mathematical aspects of classical and celestial mechanics Dynamical Systems (Encyclopedia of Mathematical Sciences vol 3) 3rd edn (Berlin: Springer) p 518 (translated from the Russian)

[10] Lawden D 1989 Elliptic Functions and Applications (Berlin: Springer)

[11] Unterkofler K 2001 On the solutions of the Halphen’s equation Differ. Integral Equns 14 1025–50

[12] Whittaker E and Watson G 1927 A Course of Modern Analysis 4th edn (Cambridge: Cambridge University Press)

[13] Whittaker E and Watson G 1927 A Course of Modern Analysis 4th edn (Cambridge: Cambridge University Press)