INVARIANT VECTOR FIELDS AND GROUPOIDS

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ABSTRACT. We use the notion of isomorphism between two invariant vector fields to shed new light on the issue of linearization of an invariant vector field near a relative equilibrium. We argue that the notion is useful in understanding the passage from the space of invariant vector fields in a tube around a group orbit to the space invariant vector fields on a slice to the orbit. The notion comes from Hepworth’s study of vector fields on stacks.

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1. INTRODUCTION

Dynamics and bifurcation theory of group-invariant vector fields is an old and well-established area of mathematics. The literature on the subject is vast, and we will not attempt to review it. The area draws on a number of fields that include representation theory, invariant theory, transformation groups, singularity theory, equivariant transversality and geometric theory of dynamical systems to name a few.

The goal of this paper is to add category theory to the arsenal of tools. More specifically we’d like to bring to the attention of the dynamics community the notion of isomorphism of invariant vector fields and to show that it is useful and natural. The source of the idea lies in Hepworth’s study of vector fields on stacks [4]. We do not assume any familiarity with stacks. In fact, except for the last section, stacks and groupoids will be kept out of sight. Instead most of the paper consists of revisiting Krupa’s work on relative equilibria [7].

Given a manifold $M$ with a proper action of a Lie group $G$, consider a $G$-invariant vector field $X \in \Gamma(TM)^G$. Recall that a point $x \in M$ is a relative equilibrium of $X$ if the vector $X(x)$ is tangent to the orbit $G \cdot x$. Let $H$ denote the stabilizer of $x$. Following Krupa choose a slice $S$ through $x$ to the action of $G$. Then the restriction of $X$ to the slice $S$ can be decomposed as

$$X|_S = X^h + X^S$$

where $X^S$ is an $H$-invariant vector field on the slice and $X^h$ is tangent to the $G$-orbits (see also Lemma 8.5.3 in [5]). Note that $X^S$ vanishes at $x$. One can then deduce a number of useful results about the dynamics of $X$ by studying the dynamics of $X^S$ in a neighborhood of its equilibrium $x$. Similarly, given a family of vector fields $X_\lambda$ one can analyze the bifurcations of a relative equilibrium of $X_\lambda$ in terms of the bifurcation of its projection $X_\lambda^S$ onto the slice. For all practical purposes one may think of the slice $S$ as a vector space with a linear action of the compact group $H$. We note that dynamics and bifurcation theory of invariant vector fields on representations have been studied intensely and extensively, and is well understood. So reducing the study of invariant vector fields near relative equilibria to the study of zeros of invariant vector fields in representations is a natural thing to do.
However, neither the slice $S$ nor the decomposition (1.1) for a given choice of a slice are unique. Nor is it clear that if a vector field $X$ is generic then its projection $X^S$ is generic and conversely. Thus given two different choices of slices $S, S'$ through $x$ and $x' \in G \cdot x$ respectively and two choices of splittings (1.1), it is far from clear how exactly the vector fields $X^S$ and $X^{S'}$ are related (if they are related at all). In particular there is no apparent relation between the spectra of the linearizations $DX^S(x)$ and $DX^{S'}(x')$. Consequently the notion of the spectrum of a vector field at a relative equilibrium doesn't seem to make sense (it is known the real part of the spectrum is well-defined; see [5, Lemma 8.5.2]). This is one instance where the notion of isomorphism of vector fields turns out to be useful.

Given an action of a Lie group $G$ on a manifold $M$, as before, consider the vector space
\[
C^\infty(M, \mathfrak{g})^G = \{\psi : M \to \mathfrak{g} \mid \psi(g \cdot m) = \text{Ad}(g)\psi(m) \quad \text{for all } m \in M, g \in G\}
\]
of Lie algebra valued equivariant maps. This vector space maps into the space of invariant vector fields: we have a linear map
\[
\partial : C^\infty(M, \mathfrak{g})^G \to \Gamma(TM)^G, \quad \partial(\psi) := \psi_M
\]
where the vector field $\psi_M$ is defined by
\[
\psi_M(m) := \frac{d}{dt} \bigg|_{t=0} \exp(t\psi(m)) \cdot m \quad \text{for all } m \in M.
\]

**Definition 1.5** (Isomorphic vector fields). Two invariant vector fields $X, Y \in \Gamma(TM)^G$ are isomorphic if there is an equivariant map $\psi \in C^\infty(M, \mathfrak{g})^G$ with
\[
X = Y + \psi_M,
\]
where $\psi_M$ is defined by (1.4) above.

The flows of two isomorphic vector fields are related by time-dependent “gauge transformation.” More precisely in Section 2 below we prove:

**Theorem 1.6.** Suppose two $G$-invariant vector fields $X$ and $Y$ on a manifold $M$ are isomorphic in the sense of Definition 1.5. Then there exists a family of maps $\{F_t : M \to G\}$ depending smoothly on $t$ so that flows $\Phi^X_t, \Phi^Y_t$ of $X$ and $Y$ respectively satisfy
\[
\Phi^X_t(m) = F_t(m) \cdot \Phi^Y_t(m)
\]
for all $(t, m) \in \mathbb{R} \times M$ for which $\Phi^X_t(m)$ is defined.

We then show that isomorphic vector fields occur naturally and that the notion of isomorphism of vector fields is useful.

**Theorem 1.7.** Let $X \in \Gamma(TM)^G$ be an invariant vector field, $x_1, x_2$ two points in $M$ on the same orbit, $S_1, S_2$ slices through the points $x_1, x_2$ respectively and $X^{S_i} \in \Gamma(TS_i)^{H_i}$ the components of $X$ tangent to the corresponding slices ($H_i$ is the stabilizer of $x_i$). Then, shrinking the slices if necessary, there exists an equivariant diffeomorphism $\varphi : S_1 \to S_2$ so that the vector fields $\varphi_* (X^{S_1})$ and $X^{S_2}$ are isomorphic.

**Remark 1.8.** The decomposition (1.1) implicitly used in the statement of Theorem 1.7 is different from the one defined by Krupa: instead of using an invariant Riemannian metric on $M$ we use a left invariant connections on the principal bundles $G \to G \cdot x_i, i = 1, 2$.

**Theorem 1.9.** Let $\rho : H \to GL(V)$ be a representation of a compact Lie group $H$ and $X, Y : V \to V$ two $H$-invariant vector fields that are isomorphic in the sense of Definition 1.5, i.e., differ by a vector field $\psi_V$ induced by an equivariant function $\psi : V \to \mathfrak{h}$. Suppose $X(0) = 0$. Then $Y(0) = 0$ and
\[
DX(0) = DY(0) + \delta \rho(\psi(0)).
\]
Here $\delta \rho : \mathfrak{h} \to \mathfrak{gl}(V)$ is the corresponding representation of the Lie algebra $\mathfrak{h}$ of $H$. 
Remark 1.10. Let $\xi = \psi(0)$. Since the linear maps $DY(0)$ and $\delta \rho(\xi)$ commute, $DY(0)$ and $\delta \rho(\xi)$ have the same eigenvectors. Hence $DX(0)$ and $DY(0)$ have the same eigenvectors as well. Moreover, the eigenvalues of $DX(0)$ are sums of the corresponding eigenvalues of $DY(0)$ and $\delta \rho(\xi)$. Since $H$ is compact the eigenvalues of $\delta \rho(\xi)$ are purely imaginary for all $\xi \in \mathfrak{h}$. Hence the real parts of the eigenvalues of $DX(0)$ and $DY(0)$ are the same. Combined with Theorem 1.1 this implies [5, Lemma 8.5.2].

In fact Theorem 1.9 tells us more. For instance, since $\psi$ is equivariant and $0 \in V$ is fixed by $H$, $\xi = \psi(0)$ is also fixed by $H$. It follows that the whole spectra of $DX(0)$ and $DY(0)$ have to be the same if the space of $H$ fixed vectors in $\mathfrak{h}$ is $0$, no matter what $\psi$ is. This happens, for instance, if the Lie algebra $\mathfrak{h}$ has no center. But it is also true for $H = O(2)$. Thus, depending on the stabilizer of a relative equilibrium, the whole spectrum of a vector field at a relative equilibrium may actually be well-defined; see Example 1.11 below.

Example 1.11. Consider a particle of mass $m$ in 3-space subject to a central force field $F$. Then the phase space of the system is $T\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$, the equations of motion are of the form

$$\begin{align*}
\dot{q} &= v \\
\dot{v} &= F(q, v),
\end{align*}$$

and the force $F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is $O(3)$-invariant:

$$F(Aq, Av) = F(q, v) \quad \text{for all } q, v \in \mathbb{R}^3, \text{ all } A \in O(3).$$

The action of $O(3)$ on $\mathbb{R}^3 \times \mathbb{R}^3$ has 3 orbit types. In more detail, $(0, 0)$ is the only fixed point. For $(q, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $q$ and $v$ linearly independent the stabilizer is isomorphic to $\{ \pm 1 \} = "O(1)"$. If $q, v$ are two linearly dependent vectors with $(q, v) \neq (0, 0)$, then the stabilizer of $(q, v)$ is isomorphic to $O(2)$. In all cases the space $\mathfrak{h}^H$ of fixed points in the Lie algebra $\mathfrak{h}$ of a stabilizer $H$ is zero. Consequently the spectrum of the “linearization” of the vector field

$$X = \sum_{i=1}^{3} \left( v_i \frac{\partial}{\partial q_i} + \frac{1}{m} F_i(q, v) \frac{\partial}{\partial v_i} \right)$$

at any relative equilibrium is well-defined — it doesn’t matter which slice through the relative equilibrium we pick, and it doesn’t matter which projection of the vector field $X$ onto the slice we choose in order to compute the linearization.

Remark 1.12. Theorem 1.7 holds for families of vector fields: if $\{X_\lambda\}_{\lambda \in \Lambda}$ is a family of invariant vector fields then the vector fields $\varphi_\ast(X_\lambda^{S_1})$ and $X_\lambda^{S_2}$ are isomorphic with the isomorphism depending smoothly on $\lambda$.

We next address the issue of genericity. Once again suppose we have a proper action of a Lie group $G$ on a manifold $M$, $H$ is the stabilizer of a point $x \in M$ and $S$ is a slice through $x$. We then have a canonical injective linear map

$$\Gamma(TS)^H \hookrightarrow \Gamma(T(G \cdot S))^G$$

from the space of $H$-invariant vector field on the slice $S$ to the space of $G$-invariant vector fields on the tube $G \cdot S$. It is clear that the image has infinite codimension. So it is difficult to compare the notion of being “generic” in these two spaces. A solution to the problem is provided by:

Theorem 1.14. The map (1.13) induces an invertible linear map

$$\Gamma(TS)^H/C^\infty(S, \mathfrak{h})^H \to \Gamma(T(G \cdot S))^G/C^\infty(G \cdot S, \mathfrak{g})^G$$

from the space of isomorphism classes of invariant vector fields on the slice to the space of isomorphism classes of invariant vector fields on the tube. Moreover if the spaces $\Gamma(TS)^H, C^\infty(S, \mathfrak{h})^H, \Gamma(T(G \cdot S))^G$ and $C^\infty(G \cdot S, \mathfrak{g})^G$ are given Whitney topologies, then the map (1.15) is a homeomorphism.
Remark 1.16. The space of sections of a vector bundle is not a topological vector space when given a Whitney topology: scalar multiplication is not continuous [6]. Hence the awkward circumlocution regarding the map (1.15): it is not an isomorphism of topological vector spaces. Rather it is an isomorphism of vector spaces and a homeomorphism.

The paper is organized as follows. In section 2 we prove Theorem 1.6. In section 3 we prove Theorem 1.7 and Theorem 1.9. In section 4 we prove Theorem 1.14. In the last section, section 5, we explain where the notion of isomorphic vector fields comes from and describe the connections with groupoids and stacks.

2. ISOMORPHIC VECTOR FIELDS AND THEIR FLOWS

The goal of this section is to prove Theorem 1.6. Throughout the section a Lie group $G$ with Lie algebra $\mathfrak{g}$ acts properly on a manifold $M$. For a point $m \in M$ we have the evaluation map $ev_m : G \to M$, $ev_m(a) = a \cdot m$ for all $a \in G$.

Dually, for any $a \in G$ we have a diffeomorphism $a_M : M \to M$, $a_M(m) = a \cdot m$ for all $m \in M$.

Recall that every $\xi \in \mathfrak{g}$ we have the induced vector field $\xi_M \in \Gamma(TM)$ defined by

$$\xi_M(m) = \frac{d}{dt} \bigg|_0 \exp(t\xi) \cdot m.$$  

Note that by the chain rule

$$\frac{d}{dt} \bigg|_0 \exp(t\xi) \cdot m = \frac{d}{dt} \bigg|_0 (ev_m(\exp t\xi)) = T(ev_m)_0 \xi.$$  

Thus

$$\xi_M(m) = T(ev_m)_0 \xi.$$  

Remark 2.3. We trust that the reader will have no difficulty distinguishing between the induced vector fields $\xi_M$ (2.1) and $\psi_M$ (1.4).

Proof of Theorem 1.6. Recall that we have a manifold $M$ with an action of a Lie group $G$, $X,Y \in \Gamma(TM)^G$ are two $G$-invariant vector fields and $\psi : M \to \mathfrak{g}$ is a $G$-equivariant function with $X = Y + \psi_M$.

Fix a point $z \in M$. Then

$$\gamma(t) := \Phi_t^Y(z)$$

is an integral curve of $Y$ through $z$. We’d like to prove the existence of a curve $g(t)$ in $G$ which depends smoothly on $z$ so that

$$\sigma(t) := g(t) \cdot \gamma(t)$$

is an integral curve of the vector field $X$ through $z$. It is well-known (see, for example, [3, Proposition 1.13.4]) that for any smooth curve $\tau : I \to \mathfrak{g}$ in the Lie algebra $\mathfrak{g}$ there is a unique smooth curve $g : I \to G$ defined on the same interval $I$ so that $g(0) = e$ and $g(t)$ solves the ODE

$$\dot{g}(t) = TR_{g(t)} \tau(t).$$

Here and elsewhere in the paper $R_a : G \to G$ denotes the right multiplication by $a \in G$. In our case we set

$$\tau(t) = \psi(\gamma(t)) = \psi(\Phi_t^Y(z))$$

We now check that the solutions $g(t)$ of the ODE

$$\dot{g}(t) = TR_{g(t)}\psi(\Phi_t^Y(z)), \quad g(0) = e,$$
which depend smoothly on \(z\) define the desired time-dependent family of maps \(\{F_t : M \to G\}\). For that it is enough to check that 

\[
\sigma(t) := g(t) \cdot \gamma(t)
\]

is the integral curve of \(X\) through \(z\). By the chain rule

\[
\frac{d}{dt} \bigg|_0 (g(t) \cdot \gamma(t)) = T(g(t)_M) \cdot \gamma(t) + T(e_{v_0(t)}g(t)) \cdot \gamma(t).
\]

By definition of \(\gamma(t)\),

\[
T(g(t)_M) \cdot \gamma(t) = T(g(t)_M) \cdot Y(\gamma(t)) = Y(g(t) \cdot \gamma(t)) = Y(\sigma(t)),
\]

where the next to last equality holds by \(G\)-invariance of the vector field \(Y\). On the other hand,

\[
T(e_{v_0(t)}g(t)) \cdot \gamma(t) = T(e_{v_0(t)}g(t)) \cdot T_{g(t)}\psi(\gamma(t)) \quad \text{by definition of } g(t)
\]

\[
= \frac{d}{ds} \bigg|_0 \left( g(t) \exp(s\psi(\gamma(t))) \right) \cdot \gamma(t)
\]

\[
= \frac{d}{ds} \bigg|_0 \left( g(t) \exp(s\psi(\gamma(t)))g(t)^{-1} \right) \cdot g(t) \cdot \gamma(t)
\]

\[
= \frac{d}{ds} \bigg|_0 \left( \exp(s\text{Ad}(g(t))\psi(\gamma(t))) \right) \cdot g(t) \cdot \gamma(t)
\]

\[
= \frac{d}{ds} \bigg|_0 \left( \exp(s\psi(g(t) \cdot \gamma(t))) \right) \cdot (g(t) \cdot \gamma(t)) \quad \text{by equivariance of } \psi
\]

\[
= \psi_M(g(t) \cdot \gamma(t)) = \psi_M(\sigma(t)).
\]

Therefore

\[
\frac{d}{dt} \bigg|_0 \sigma(t) = Y(\sigma(t)) + \psi_M(\sigma(t)) = X(\sigma(t))
\]

and we are done. \(\square\)

**Remark 2.6.** Something about the proof of the theorem may look vaguely familiar to some readers. Indeed it looks very much like the reconstruction argument in symplectic reduction theory. See for instance the discussion at the bottom of p. 304 in [1]. From the point of view of Theorem 1.6 the reconstruction argument says that the horizontal lift of the reduced vector field with respect to some connection and the original Hamiltonian vector field are isomorphic.

**Definition 2.7.** Since the orbit \(G \cdot x\) through a relative equilibrium \(x\) of a vector field \(X\) is preserved by the flow of \(X\) it makes sense to define the relative equilibrium \(x\) to be hyperbolic if the manifold \(G \cdot x\) is normally hyperbolic for the flow of \(X\).

**Corollary 2.8.** Suppose \(X,Y \in \Gamma(TM)^G\) are two isomorphic invariant vector fields. Then their flows induce the same flow on the space of orbits \(M/G\).

Consequently isomorphic vector fields have the same relative equilibria and relative periodic orbits. Moreover if \(x\) is a hyperbolic relative equilibrium for \(X\) it is hyperbolic for any vector field \(Y\) isomorphic to \(X\).

**Remark 2.9.** The first part of the corollary has a converse if the action of \(G\) on \(M\) is free and proper: if \(X\) and \(Y\) are two \(G\)-invariant vector fields inducing the same flow on the orbit space \(B := M/G\) then \(X\) and \(Y\) are isomorphic.

The argument proceeds as follows. Since the action of \(G\) is free and proper, the orbit space \(B := M/G\) is a manifold and the orbit map \(\pi : M \to B\) makes \(M\) into a principal \(G\)-bundle over \(M\). Then two \(G\)-invariant vector fields \(X\) and \(Y\) induce the same flow on \(B\) if and only if they are \(\pi\)-related to the same vector field on \(B\) if and only if \(T\pi(X - Y) = 0\) if and only if for every \(m \in M\) there is a vector
ψ(m) ∈ g so that X(m) − Y(m) = (ψ(m))M(m). It is also easy to see that the function ψ : M → g is smooth:

ψ(m) = αm (X(m) − Y(m)),

where α ∈ Ω1(M, g)G is a connection 1-form (any choice of α will do). Thus X and Y are isomorphic with the isomorphism provided by the G-equivariant map ψ defined above.

3. LOCAL NORMAL FORM FOR AN INVARIANT VECTOR FIELD

Once again let S denote a slice through a point x ∈ M for a proper action of a Lie group G on a manifold M and H denote the stabilizer of x. Then

G · S := \{g · y | g ∈ G, y ∈ S\}

is an open G-invariant neighborhood of the orbit G · x, which is often called a tube. We have a G-equivariant diffeomorphism ϕ from the associated bundle

G ×HS := (G × S)/H

to the tube G · S making the diagram

\[
\begin{array}{ccc}
G × HS & \xrightarrow{\varphi} & G · S \\
\downarrow & & \downarrow \pi \\
G/H & \xrightarrow{\pi} & G · x \\
\end{array}
\]

commute. Here the left vertical map is

[g, s] → gH,

the bottom horizontal map is

gH → g · x

and the right vertical map π : G · S → G · x is given by

π(g · y) = g · x;

it is well-defined.

Remark 3.1. The projection π : G · S → G · x very much depends on the choice of the slice S: the fiber of π above g · x ∈ G · x is the submanifold g · S, which is a slice through g · x. A choice of a different slice S′ through x with G · S′ = G · S defines a different submersion π′ : G · S′ → G · x, even though it is given by a seemingly identical formula:

π′(g · y′) = g · x

for all g ∈ G, y′ ∈ S′. In particular π and π′ have different fibers.

Lemma 3.2. Let M be a manifold with an action of a Lie group G, H the stabilizer of a point x ∈ M and S a slice through x for the action of G. A choice of an H-equivariant splitting

(3.3) \[ g = h ⊕ m \]

of the Lie algebra g of G into the Lie algebra h of H and a complement m gives rise to an isomorphism of vector spaces

\[ \Gamma(T(G · S))^G \rightarrow \Gamma(TS)^H \oplus C^\infty(S, m)^H. \]

Proof. It will be convenient for notational purposes to assume that G · S = M. Then, since S is a global slice, any G-invariant vector field X on M is uniquely determined by its restriction to S. Thus the restriction map

(3.4) \[ \Gamma(TM)^G \rightarrow \Gamma(TM)^G|_S, \quad X \rightarrow X|_S \]

is an isomorphism of vector spaces.
The splitting (3.3) defines a $G$-invariant connection on the principal $H$-bundle $G \to G \cdot x$ and consequently a $G$-invariant connection on the associated bundle $M \simeq G \times^H S \to G \cdot x$. Hence any $G$-invariant vector field $X$ on $M$ can be uniquely written as a sum
\begin{equation}
X = X^v + X^h
\end{equation}
of two $G$-invariant vector fields with $X^v$ being tangent to the fibers of $\pi : M \to G \cdot x$ and $X^h$ being tangent to the horizontal distribution $H \subset TM$ defined by the splitting (3.3). It follows that the restriction
\begin{equation}
X^S := X^v|_S
\end{equation}
is tangent to $S$. It is easy to see that $X^S$ is $H$-invariant.

The restriction $H|_S$ of the horizontal distribution to the slice is trivial: an isomorphism is given by
\begin{equation}
S \times m \to H|_S, \quad (y, \xi) \mapsto \xi_M(y).
\end{equation}
It follows that the space of $H$-equivariant sections of $H|_S \to S$ is isomorphic to the space of $H$-equivariant function $C^\infty(S, m)^H$:
\begin{equation}
\Gamma(H|_S)^H \simeq C^\infty(S, m)^H,
\end{equation}
and the result follows. \hfill \Box

Lemma 3.2 tells us that a choice of a slice $S$ and of a splitting (3.3) defines a surjective linear map
\begin{equation}
p : \Gamma(TG \cdot S) \to C^\infty(S, m)^H, \quad X \mapsto \psi^S_X.
\end{equation}
We note for future use

**Lemma 3.9.** The projection
\begin{equation}
p : \Gamma(TG \cdot S) \to C^\infty(S, m)^H, \quad X \mapsto \psi^S_X
\end{equation}
is continuous in Whitney topology. Consequently the projection
\begin{equation}
q = \Gamma(TG \cdot S)^G \to \Gamma(TS)^H, \quad q(X) = X|_S - (\psi^S_X)_S
\end{equation}
is continuous as well.

**Proof.** By definition of the space $m$ the map
\begin{equation}
T(ev_x)_0 : m \to T_xG \cdot x, \quad \xi \mapsto \xi_M(x) = T(ev_x)_0 \xi
\end{equation}
is an isomorphism. Denote its inverse by $\alpha$:
\begin{equation}
\alpha = (T(ev_x)_0)^{-1} : T_xG \cdot x \to m.
\end{equation}
Then
\begin{equation}
\psi^S_X(y) = \alpha(T\pi_y(X(y)));
\end{equation}
which depends continuously on $X$.

Note that the dependence on the choice of the slice $S$ is hidden in the definition of the map $\pi : G \cdot S \to G \cdot x$. The dependence of the function $\psi^S_X$ on the splitting (3.3) is suppressed in the notation. \hfill \Box

**Remark 3.12.** The function $\psi^S_X \in C^\infty(S, m)^H$ and the vector field $X^h$ in (3.5) are, of course, directly related. Indeed, define $\Psi^X \in C^\infty(G \cdot S), g)^G$ by
\begin{equation}
\Psi^X(g \cdot y) = Ad(g)\psi^S_X(y).
\end{equation}
Then
\begin{equation}
X^h = (\Psi^X)_M(m)
\end{equation}
for all $m \in G \cdot S$, and the decomposition (3.5) reads:
\begin{equation}
X(g \cdot y) = T(g_M)_y X^S(y) + (\Psi^X)_M(g \cdot y)
\end{equation}
for all $y \in S, g \in G$. 


Lemma 3.14. Let $S, S'$ be two different slices through the same point $x$ for a proper action of a Lie group $G$ on a manifold $M$. Let $H$ denote the stabilizer of $x$, as before. There exists an $H$-equivariant $G$-valued function $f$ defined on a neighborhood $U$ of $x$ in $S$ so that

$$
\varphi : U \to S', \quad \varphi(y) = f(y) \cdot y
$$

is an $H$-equivariant open embedding. Moreover we may assume $f$ takes values in $\exp m$, where $g = h \oplus m$ is an $H$-equivariant splitting.

Proof. We may assume that $M = G \times^H S$ and $S'$ is a slice through $[1, x] \in G \times^H S$, where $[1, x]$ is the $H$-orbit of $(1, x) \in G \times S$. It is a standard fact that an $H$-invariant splitting $g = m \oplus h$ defines an $H$-equivariant section $s$ of the principal $H$ bundle $G \to G/H$. Explicitly the section is given by the formula

$$
s(gH) = g
$$

for all $g \in \exp(m)$ sufficiently close to 1, say for $g$ in an $H$-invariant neighborhood $\mathcal{O}$ of 1 $\in \exp(m)$.

The section $s$ trivializes the associated bundle $\pi : G \times^H S \to G/H = G \cdot x$. Explicitly the trivialization is the map

$$
(3.15) \quad \mathcal{O} \times S \to \pi^{-1}(OG), \quad (g, y) \mapsto [g, y] = g [1, y].
$$

We may assume that $S' \subset \mathcal{O} \times S$ and that at $(1, x) \in \mathcal{O} \times S$ we have

$$
T_{(1, x)}(\mathcal{O} \times S) = T_{(1, x)} \mathcal{O} \oplus T_{(1, x)} S'.
$$

Consequently the differential of $pr_2|_{S'} : S' \to S$ at $(1, x)$ is an isomorphism, hence a diffeomorphism from a neighborhood $U'$ of $(1, x) \in S'$ to a neighborhood $U$ of $x$ in $S$. Since $pr_2$ is $H$-equivariant, $pr_2|_{S'}$ is $H$-equivariant as well, and we may take $U, U'$ to be $H$-invariant. We set

$$
\varphi = (pr_2|_{U'})^{-1} : U \to U'.
$$

It is the desired map, and it’s of the form

$$
\varphi(y) = (f(y), y)
$$

for some $H$-equivariant map $f : U \to \mathcal{O} \subset \exp(m)$.

When we identify $G \times^H S$ with $G \cdot S$ the map $\varphi$ takes the form

$$
\varphi(y) = f(y) \cdot y,
$$

as desired. \qed

Lemma 3.16. Let $\varphi : S \to S'$, $\varphi(y) = f(y) \cdot y$ be the equivariant map of Lemma 3.14. Denote the left multiplication by an element $a \in G$ by $La$. For any $y \in S$, $v \in T_y S$,

$$
(3.17) \quad T\varphi_y(v) = T(f(y)M)_y \left[ (TL_{f(y)}(Tf_y(v)))M(\varphi(y)) + v \right].
$$

Remark 3.18. In (3.17) $f(y) \in G$, $f(y)_M : M \to M$ is the corresponding diffeomorphism, and $T(f(y))_y : T_y M \to T_{f(y)} M$ is its differential. Similarly $Tf_y : T_y M \to T_{f(y)} G$ is the differential of $f$ at $y$, $TL_{f(y)} \left( Tf_y(v) \right) \in T_y G = \mathfrak{g}$, and $\left( TL_{f(y)} \left( Tf_y(v) \right) \right)_M(\varphi(y))$ is the value at $\varphi(y)$ of the vector field on $M$ induced by the vector $TL_{f(y)} \left( Tf_y(v) \right) \in \mathfrak{g}$.

Proof of Lemma 3.16. The derivative $T(\text{ev}_y)_g$ of the evaluation map

$$
\text{ev}_y : G \to M, \quad \text{ev}_y(g) = g \cdot y
$$

at a point $g \in G$ can be computed as follows: For $w \in T_g G$ set

$$
z = TL_{g^{-1}} w \in T_e G = \mathfrak{g}.
$$

Then $w = TL_g z$ and

$$
(3.19) \quad T(\text{ev}_y)w = \frac{d}{dt} \bigg|_0 ev_y \left( L_g(\exp tz) \right) = \frac{d}{dt} \bigg|_0 g(\exp tz)y = T(g_M)_y(TL_{g^{-1}} w)_M(y).
$$
Next choose a curve \( \gamma : I \rightarrow M \) with \( \gamma(0) = y, \quad \dot{\gamma}(0) = v \). Then
\[
\frac{d}{dt} \bigg|_0 \varphi(\gamma(t)) = \frac{d}{dt} \bigg|_0 (f(\gamma(t)) \cdot \gamma(t)) = T(\varphi_{\gamma(0)}) \frac{d}{dt} \bigg|_0 f(\gamma(t)) + T(f(\gamma(0))) M(\dot{\gamma}(0)) = T(f(y) M) ((TL_{g^{-1}} T f_y(v)) M(y) + v). \quad \text{by (3.19)}
\]

**Proof of Theorem 1.7.** We first address the dependence of the projection
\[ q : \Gamma(T(G \cdot S))^G \rightarrow \Gamma(TS)^H \]
on the choice of the splitting \( g = h \oplus m \).

**Lemma 3.20.** Let
\[ g = h \oplus m_1 = h \oplus m_2 \]
be two \( H \)-equivariant splittings and
\[ q_1, q_2 : \Gamma(T(G \cdot S))^G \rightarrow \Gamma(TS)^H \]
the two corresponding projections. Then for any \( X \in \Gamma(T(G \cdot S))^G \) the two vector fields
\[ X_i := q_i(X), \quad i = 1, 2 \]
are isomorphic in the sense of Definition 1.5: there is map \( \varphi \in C^\infty(S, h)^H \) with
\[ X_1 - X_2 = \varphi_S. \]

**Proof.** Since \( m_1, m_2 \) are both complementary to \( h \) in \( g \) there exists a linear map
\[ B : m_1 \rightarrow h \]
so that \( m_2 \) is the graph of \( B \):
\[ m_2 = \{ v + B(v) \in g \mid v \in m_1 \}. \]
Since the splittings (3.21) are \( H \)-equivariant, the map \( B \) is \( H \)-invariant. Define \( \psi_i \in C^\infty(S, m_i)^H \), \((i = 1, 2)\), by
\[ \psi_i(y) := (T(ev_x)_0|_{m_i})^{-1} (T\pi_y(X(y))). \]
Then
\[ \psi_1(y) + B(\psi_1(y)) \text{ is in } m_2 \text{ for all } y \in S. \]
Since \( B(\psi_1(y)) \in h \), we have
\[ T(ev_x)_0(B(\psi_1(y))) = 0 \quad \text{for all } y \in S. \]
Hence
\[ T(ev_x)_0(\psi_1(y) + B(\psi_1(y))) = T(ev_x)_0(\psi_1(y)) = T\pi_y(X(y)) = T(ev_x)_0(\psi_2(y)). \]
Since \( T((ev_x)_0|_{m_2} \) is 1-1, it follows that
\[ \psi_1(y) + B(\psi_1(y)) = \psi_2(y). \]
We define \( \varphi : S \rightarrow h \) by
\[ \varphi(y) = B(\psi_1(y)); \]
it is \( H \)-equivariant. Then
\[ \psi_2 - \psi_1 = \varphi. \]
Moreover
\[ (X_1 - X_2)(y) = (X(y) - (\psi_1)_M(y)) - (X(y) - (\psi_2)_M(y)) = (\psi_2 - \psi_1)_M(y) = \varphi_M(y), \]
which proves the lemma. \( \square \)
Next we address the dependence of the projection
\[ q: \Gamma(T(G\cdot S))^G \to \Gamma(TS)^H \]
on the point \( x \). For any \( g \in G \)
\[ S' := g_M(S) \quad (= g \cdot S) \]
is a slice through \( g \cdot x \), the stabilizer of \( g \cdot x \) is \( H' = gHg^{-1} \), and an \( H \)-equivariant splitting \( g = \mathfrak{h} \oplus \mathfrak{m} \) defines an \( H' \)-equivariant splitting
\[ g = Ad(g)\mathfrak{h} \oplus Ad(g)\mathfrak{m} = \mathfrak{h}' \oplus Ad(g)\mathfrak{m}. \]
Since \( g_M : S \to S' \) is an \( H - H' \)-equivariant diffeomorphism, it defines an isomorphism
\[ g_*: \Gamma(TS)^H \to \Gamma(TS')^{H'}, \]
given by
\[ (g_*Y)(y') = Tg_MY(g^{-1} \cdot y'), \quad y' \in S'. \]
Equation (3.13) now implies that the image \( X^{S'} \) of \( X \in \Gamma(T(G\cdot S))^G \) under the projection
\[ q': \Gamma(T(G\cdot S))^G \to \Gamma(TS')^{H'} \]
(which is defined by the slice \( S' \) and the splitting \( g = \mathfrak{h} \oplus Ad(g)\mathfrak{m} \) is exactly \( g_*X^S \). In other words, in this case
\[ X^{S'} \text{ and } g_*X^S \text{ are equal.} \]
Therefore, to finish our proof of Theorem 1.7 it is enough to show:

**Lemma 3.24.** Suppose \( S, S' \) are two slices through the same point \( x \). Let \( \varphi: S \to S' \), \( \varphi(y) = f(y) \cdot y \) be the equivariant diffeomorphism of Lemma 3.14. Then for any \( G \)-invariant vector field \( X \) on the original manifold \( M \) the vector fields \( X^{S'} \) and \( \varphi_*X^S \in \Gamma(TS')^H \) are isomorphic: there exists an \( H \)-equivariant map \( \nu: S' \to \mathfrak{h} \) with
\[ X^{S'} - \varphi_*X^S = \nu_S. \]
Here \( \nu_S(y') = \frac{d}{dt}|_0(\exp t\nu(y')) \cdot y' \) and \( \varphi_*X^S(y') = T\varphi_{\varphi^{-1}(y')}(X^{S}(\varphi^{-1}(y'))) \) for \( y' \in S' \).

**Proof.** As before fix an \( H \)-equivariant splitting \( g = \mathfrak{h} \oplus \mathfrak{m} \). Then by Remark 3.12 for any point \( y \) in the slice \( S \) we have
\[ X^S(y) = X(y) - (\Psi^S)_M(y) \]
with \( X^S \in \Gamma(TS)^H \) and \( \Psi^S \in C^\infty(G \cdot S, \mathfrak{g})^G \). Similarly
\[ X^{S'}(y') = X(y') - (\Psi^{S'})_M(y') \]
for \( y' \in S' \), \( X^{S'} \in \Gamma(TS')^H \) and \( \Psi^{S'} \in C^\infty(G \cdot S, \mathfrak{g})^G \). By Lemma 3.16, for any \( y \in S \),
\[ T\varphi_y(X^S(y)) = T(f(y)_M)_y(X^S(y) + \mu_M(y)), \]
where \( \mu \in C^\infty(G \cdot S, \mathfrak{g})^G \) is defined by
\[ \mu(y) = TL_{f(y)^{-1}}(Tf_y(X(y))) \]
for \( y \in S \) and \( \mu(g \cdot y) = Ad(g)\mu(y) \) for an arbitrary \( g \cdot y \in G \cdot S \). Recall that for any \( G \)-invariant vector field \( Z \) on \( G \cdot S \)
\[ T(g_M)Z(y) = Z(g \cdot y) \]
for all \( g \in M \) and \( y \in S \). Therefore, for \( y \in S \), \( y' = f(y) \cdot y \in S' \),
\[ X^{S'}(y') - \varphi_*X^S(y') = (X(y') - (\Psi^{S'})_M(y')) - T(f(y)_M)_y(X^S(y) + \mu_M(y)) \]
\[ = X(y') - (\Psi^{S'})_M(y') - T(f(y)_M)_y(X(y) - (\Psi^S)_M(y)) - \mu_M(y') \]
\[ = X(y') - (\Psi^{S'})_M(y') - X(y') + (\Psi^S)_M(y') - \mu_M(y') \]
\[ = (\Psi^S - \Psi^{S'})_M(y'). \]
Now define \( \nu(y') = (\Psi^S - \Psi^{S'} - \mu)(y') \) for all \( y' \in S' \). Since at points of the slice \( S' \) the induced vector field \( \nu_M \) is tangent to \( S' \), the function \( \nu \) takes values in \( \mathfrak{h} \).

This concludes our proof of Theorem 1.7.

We now turn out attention to Theorem 1.9.

**Proof of Theorem 1.9.** Since \( X - Y = \psi_\nu \), \( DX(0) - DY(0) = D(\psi_\nu)(0) \). Now, for any \( v \in V \),

\[
D(\psi_\nu)(0)v = \frac{\partial}{\partial s}\bigg|_{s=0} \psi_\nu(sv) = \frac{\partial^2}{\partial s \partial t}\bigg|_{(0,0)} \rho(\exp t\psi_\nu(sv))(sv)
\]

\[
= \frac{\partial^2}{\partial s \partial t}\bigg|_{(0,0)} e^{t\delta\rho(\psi_\nu(sv))} = \frac{\partial}{\partial s}\bigg|_{s=0} \delta\rho(\psi_\nu(sv))(sv)
\]

\[
= (\delta\rho(\psi(sv)|_{s=0})(\frac{\partial}{\partial s}\bigg|_{s=0})(sv) + (\frac{\partial}{\partial s}\bigg|_{s=0})(\delta\rho(\psi(sv)(sv)|_{s=0}))
\]

\[
= \delta\rho(\psi(0))v.
\]

\[ □ \]

4. **Topology on the space of the isomorphism classes of invariant vector fields**

The goal of this section is to prove Theorem 1.14. Given a manifold \( M \) with a proper action of a Lie group \( G \), think of the map (1.3) as a 2 term chain complex

\[
C^\infty(M, \mathfrak{g})^G \xrightarrow{\partial} \Gamma(TM)^G.
\]

We think of elements of \( C^\infty(M, \mathfrak{g})^G \) as having degree 1 and invariant vector fields in \( \Gamma(TM)^G \) as having degree 0. Then the quotient vector space \( \Gamma(TM)^G/\partial(C^\infty(M, \mathfrak{g})^G) \) is the 0th homology of the complex.

Next we briefly recall the notation: \( x \) is a point in \( M \), \( H \) is its stabilizer with Lie algebra \( \mathfrak{h} \) and \( S \) is a slice through \( x \). We fix an \( H \)-equivariant splitting \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \). Note that the splitting defines an \( H \)-equivariant projections

\[
A : \mathfrak{g} \to \mathfrak{h} \quad \text{and} \quad B : \mathfrak{g} \to \mathfrak{m}.
\]

Note also that for any vector \( \xi \in \mathfrak{g} \)

\[
(2.2) \quad B(\xi) = \alpha(\xi_M(x)),
\]

where \( \alpha : T_x G \cdot x \to \mathfrak{m} \) is the inverse of \( T(ev_x)_0|_{\mathfrak{m}} \); q.v. (3.11).

To prove Theorem 1.14 we produce two (continuous) maps of chain complexes

\[
(3.3) \quad K : \left( C^\infty(S, \mathfrak{h})^H \xrightarrow{\partial} \Gamma(TS)^H \right) \to \left( C^\infty(G \cdot S, \mathfrak{g})^G \xrightarrow{\partial} \Gamma(T(G \cdot S))^G \right)
\]

and

\[
(3.4) \quad L : \left( C^\infty(G \cdot S, \mathfrak{g})^G \xrightarrow{\partial} \Gamma(T(G \cdot S))^G \right) \to \left( C^\infty(S, \mathfrak{h})^H \xrightarrow{\partial} \Gamma(TS)^H \right)
\]

and show that \( K \circ L \) and \( L \circ K \) induce isomorphisms in 0th homology.\(^1\)

Given an equivariant map \( \psi \in C^\infty(S, \mathfrak{h})^H \) we define \( K(\psi) \in C^\infty(G \cdot S, \mathfrak{g})^G \) by

\[
K(\psi)(g \cdot y) := \text{Ad}(g)\psi(y)
\]

for all \( g \in G, y \in S \). Given an invariant vector field \( Y \in \Gamma(TS)^H \) we define the corresponding invariant vector field \( K(Y) \in \Gamma(T(G \cdot S))^G \) by

\[
K(Y)(g \cdot y) = T(g_M)_y Y(y)
\]

for all \( g \in G, y \in S \). It is not hard to check that \( K \) is a map of chain complexes. It amounts to showing that

\[
(K(\psi))_M(g \cdot y) = T(g_M)_y \psi_S(y),
\]

\(^1\)They also induce isomorphisms in the 1st homology but we won’t need this fact.
as the reader can readily verify.

Given a $\mathfrak{g}$-valued equivariant function $\Psi \in C^\infty(G\cdot S, \mathfrak{g})^G$ on the tube, we define an $\mathfrak{h}$-valued equivariant function $L(\Psi)$ on the slice by restricting it to the slice and projecting its values onto $\mathfrak{h}$: for all $y \in S$ we set

$$L(\Psi)(y) := A(\Psi(y)),$$

where $A : \mathfrak{g} \to \mathfrak{h}$ is the projection we chose earlier (q.v. (4.1)). We define $L : \Gamma(T(G\cdot S))^G \to \Gamma(TS)^H$ to be the map $q$ of Lemma 3.9:

$$(4.5) \quad L(X)(y) := X(y) - \frac{d}{dt} \bigg|_0 \exp(t\alpha(T\pi_y X(y))) \cdot y.$$ 

for all $X \in \Gamma(T(G\cdot S))$. To check that $L$ is a map of chain complexes, we compute. On the one hand, for any $G$-equivariant function $\Psi \in C^\infty(G\cdot S, \mathfrak{g})^G$ on the tube, for any $y \in S$,

$$(\partial \circ L(\Psi))(y) = \frac{d}{dt} \bigg|_0 \exp tA(\Psi(y)) \cdot y = (A \circ \Psi(y))_S(y).$$

On the other hand,

$$(L \circ \partial (\Psi))(y) = \Psi_M(y) - \frac{d}{dt} \bigg|_0 \exp(t\alpha(T\pi_y \Psi_M(y))) \cdot y.$$ 

Now

$$\alpha(T\pi_y \Psi_M(y)) = \frac{d}{dt} \bigg|_0 \alpha(\pi(\exp t\Psi(y)) \cdot y)$$

$$= \frac{d}{dt} \bigg|_0 \alpha(\exp t\Psi(y) \cdot \pi(y)) \quad (\text{since } \pi \text{ is equivariant})$$

$$= \frac{d}{dt} \bigg|_0 \alpha(\exp t\Psi(y) \cdot x)$$

$$= \alpha((\Psi(y))_M(x))$$

(here we think of $\Psi(y)$ as a vector in $\mathfrak{g}$ inducing the vector field $(\Psi(y))_M$)

$$= B(\Psi(y)) \quad \text{by (4.1)}.$$ 

It follows that

$$(L \circ \partial (\Psi))(y) = \Psi_M(y) - \frac{d}{dt} \bigg|_0 \exp(tB(\Psi(y))) \cdot y$$

$$= (\Psi(y))_M(y) - B(\Psi(y))_M(y)$$

$$= A(\Psi(y))_M(y) = (A(\Psi(y))_S(y) \quad (\text{Since } A(\Psi(y))_M(y) \text{ is tangent to the slice } S)$$

$$= (\partial \circ L(\Psi))(y).$$

Hence $L$ is a map of chain complexes.

We next check that $K \circ L$ and $L \circ K$ induce identity maps on the 0th homology. For an invariant vector field $Y$ on the slice $S$,

$$T\pi_y(K(Y))(y) = 0$$

for all $y \in S$. Hence

$$K(L(Y))(y) = Y(y) - \frac{d}{dt} \bigg|_0 \exp t\alpha(T\pi_y (KY(y))) \cdot y = Y(y).$$

In other words

$$L \circ K = id_{\Gamma(TS)^H}.$$ 

We now compute $K \circ L$. By Lemma 3.9 and its proof, for an invariant vector field $X$ on the tube $G\cdot S$ and a point $y \in S$

$$L(X)(y) = X(y) - (\psi_X^S)_S(y),$$
where \( \psi_X^S \in C^\infty(S, m)^{GH} \) depends continuously on \( X \). Hence by definition of the map \( K \), for any \( g \in G, y \in S \),
\[
K(L(X))(g \cdot y) = X(g \cdot y) - (\Psi_X)_M(y),
\]
where
\[
\Psi_X(g \cdot y) := \text{Ad}(g)\psi_X^S(y)
\]
is a function in \( C^\infty(G \cdot S, g)^G \). In other words, there is a continuous linear map
\[
a : \Gamma(T(G \cdot S))^G \rightarrow C^\infty(G \cdot S, g)^G, \quad a(X) := \Psi_X
\]
with
\[
(K \circ L)(X) = X - \partial(a(X))
\]
for all \( X \in \Gamma(T(G \cdot S))^G \). Therefore \( K \circ L \) induces the identity map on the quotient \( \Gamma(T(G \cdot S))^G / C^\infty(G \cdot S, g)^G \), the 0th homology of the chain complex \( \big(C^\infty(G \cdot S, g)^G \xrightarrow{\partial} \Gamma(T(G \cdot S))^G \big) \). This proves that the linear maps
\[
H_0(K) : \Gamma(TS)^H / C^\infty(S, h)^H \rightarrow \Gamma(T(G \cdot S))^G / C^\infty(G \cdot S, g)^G
\]
and
\[
H_0(L) : \Gamma(T(G \cdot S))^G / C^\infty(G \cdot S, g)^G \rightarrow \Gamma(TS)^H / C^\infty(S, h)^H
\]
are inverses of each other. In Lemma 3.9 we have shown that \( L : \Gamma(T(G \cdot S))^G \rightarrow \Gamma(TS)^H \) is continuous. It is easy to see that \( K : \Gamma(TS)^H \rightarrow \Gamma(T(G \cdot S))^G \) is continuous as well. Hence \( H_0(K), H_0(L) \) are continuous inverses of each other. This finishes our proof of Theorem 1.14.

5. Concluding remarks

Traditionally given a proper action of a Lie group \( G \) on a manifold \( M \) one thinks of a quotient \( M/G \) as a topological space with some additional structure. For instance one proves that \( M/G \) is a stratified space and that it is “smooth” in an appropriate sense. There is also another notion of a quotient that has its origins in the works of Grothendieck, Deligne, Mumford, Artin and their collaborators — this is the notion of a stack quotient \([M/G]\). Stack quotients are instances of geometric stacks and consequently come with atlases. A choice of an atlas for a stack defines a Lie groupoid that “represents” the stack. Two different choices of atlases give rise to Morita equivalent Lie groupoids. It turns out that the notion of a vector field on a stack does make sense, but instead of forming a vector space the collection of vector fields on a given stack forms a category. It was shown by Hepworth [4] that if a stack \( X \) is represented by a Lie groupoid \( \mathcal{G} \) then the category of vector fields on \( X \) is equivalent to the category of of the so called multiplicative vector fields on \( \mathcal{G} \).

Let me now sketch how the stacky point of view informs the paper. Given an action of a Lie group \( G \) on a manifold \( M \), we have an action Lie groupoid \( G \times M \rightrightarrows M \). Given a groupoid \( \mathcal{G} \) there is a tangent groupoid \( T\mathcal{G} \) with a canonical functor \( \pi : T\mathcal{G} \rightarrow \mathcal{G} \).

A multiplicative vector field \( X \) on a groupoid \( \mathcal{G} \) is a section of \( \pi : T\mathcal{G} \rightarrow \mathcal{G} \). In particular it is a functor. Since multiplicative vector fields are functors, it make sense to talk about natural transformations between them. Since vector fields take values in groupoids the natural transformations are natural isomorphisms. Thus multiplicative vector fields on a Lie groupoid \( \mathcal{G} \) form a category (in fact, a groupoid) which we denote by \( \mathcal{X}(\mathcal{G}) \).

We now return to an action of a group \( G \) on a manifold \( M \). On one hand we have the groupoid \( \mathcal{X}(G \times M \rightrightarrows M) \) of multiplicative vector fields. On the other hand we have the linear map
\[
A : C^\infty(M, g)^G \rightarrow \Gamma(TM)^G, \quad \psi \mapsto \psi_M
\]
which defines an action of \( C^\infty(M, g)^G \) on \( \Gamma(TM)^G \) by
\[
\psi \cdot X = A(\psi) + X = \psi_M + X.
\]
This this gives rise to an action groupoid
\[
\mathcal{X}(G \times M \rightrightarrows M) := C^\infty(M, g)^G \times \Gamma(TM)^G \rightrightarrows \Gamma(TM)^G,
\]
the groupoid of invariant vector fields. An invariant vector field \( X : M \to TM \) extends canonically to a multiplicative vector field \( G \times M \xrightarrow{(0,X)} TG \times TM \). Consequently the groupoid \( \mathcal{X}(G \times M \rightrightarrows M) \) is contained in the groupoid \( \mathcal{X}'(G \times M \rightrightarrows M) \). Usually the containment is strict. However, if \( G \) is compact, Hepworth shows that the inclusion is a fully faithful and essentially surjective functor. Hence the groupoids \( \mathcal{X}(G \times M \rightrightarrows M) \) and \( \mathcal{X}'(G \times M \rightrightarrows M) \) are equivalent. As was mentioned above the groupoid of multiplicative vector fields \( \mathcal{X}'(\mathcal{G}) \) on a groupoid \( \mathcal{G} \), in turn, is equivalent to the groupoid of vector fields on the stack quotient \( B\mathcal{G} \) of principal \( \mathcal{G} \) bundles. Thus if two Lie groupoids \( \mathcal{G} \) and \( \mathcal{H} \) are Morita equivalent, then the stacks \( B\mathcal{G} \) and \( B\mathcal{H} \) are isomorphic and consequently the groupoids of multiplicative vector fields \( \mathcal{X}'(\mathcal{G}) \) and \( \mathcal{X}'(\mathcal{H}) \) are equivalent as well.

On a manifold a vector field integrates to flow, which is, more or less, a one-parameter family of diffeomorphisms. Analogously on a Lie groupoid a multiplicative vector field integrates to a one-parameter family of \( C^\infty \) diffeomorphisms. Analogously on a Lie groupoid a multiplicative vector field integrates to a one-parameter family of \( C^\infty \) diffeomorphisms. Consequently the groupoids \( \mathcal{X}(\mathcal{G}) \) and \( \mathcal{X}'(\mathcal{G}) \) of invariant vector fields are Morita equivalent (at least when \( G \) is compact). So it is not surprising that the most natural functor

\[
\mathcal{X}(H \times S \rightrightarrows S) \to \mathcal{X}(G \times GS \rightrightarrows GS)
\]

one can write down is an equivalence of categories. Since such a functor is essentially surjective, any \( G \)-invariant vector field on the tube is isomorphic (but not necessarily equal) to a vector field tangent to the slice \( S \). This accounts for the decomposition of an invariant vector on the tube as a sum of a vector field tangent to the slice and another coming from the infinitesimal gauge transformation. Finally, equivalent groupoids have isomorphic orbit spaces. Therefore we should have a bijection

\[
\Gamma(TS)^H/C^\infty(S,\mathfrak{h})^H \to \Gamma(TM)^G/C^\infty(M,\mathfrak{g})^G,
\]

as we proved in Theorem 1.14.

We note that the groupoid of invariant vector fields is a 2-vector space in the sense of Baez and Crans [2]. But there is also more structure. As we have seen the space of \( G \)-invariant vector fields on a manifold \( M \) can be given topology. We chose Whitney \( C^\infty \) topology, but other choices are also reasonable. The space \( C^\infty(M,\mathfrak{g})^G \) of equivariant infinitesimal gauge transformation also carries a natural Whitney topology. We have shown that the map

\[
\partial : C^\infty(M,\mathfrak{g})^G \to \Gamma(TM)^G, \quad \psi \mapsto \psi_M
\]

is continuous. Consequently the 2-vector space \( \mathcal{X}(G \times M \rightrightarrows M) \) of invariant vector fields is a topological groupoid. Moreover we have shown that for a slice \( S \) and the tube \( GS \) the functor

\[
\mathcal{X}(H \times S \rightrightarrows S) \to \mathcal{X}(G \times GS \rightrightarrows GS)
\]

is an equivalence of topological groupoids. Consequently

\[
\Gamma(TS)^H/C^\infty(S,\mathfrak{h})^H \to \Gamma(TM)^G/C^\infty(M,\mathfrak{g})^G.
\]

is a homeomorphism. This is what allows us to talk about (isomorphism classes of) generic vector fields in such a way that the notions of genericity for the slice and the tube are the same.

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