MODULI OF THETA-CHARACTERISTICS VIA NIKULIN SURFACES

GAVRIL FARKAS AND ALESSANDRO VERRA

The importance of the locus \( K_g := \{ [C] \in M_g : C \text{ lies on a K3 surface} \} \) has been recognized for some time. Fundamental results in the theory of algebraic curves like the Brill-Noether Theorem [Laz], or Green’s Conjecture for generic curves [Vo] have been proved by specialization to a general point \([C] \in K_g\). The variety \( K_g \) viewed as a subvariety of \( \overline{M}_g \) serves as an obstruction for effective divisors on \( \overline{M}_g \) to having small slope [FP] and thus plays a significant role in determining the cone of effective divisors on \( \overline{M}_g \).

The first aim of this paper is to show that at the level of the Prym moduli space \( R_g \) classifying étale double covers of curves of genus \( g \), the locus of curves lying on a Nikulin K3 surface plays a similar role. The analogy is far-reaching: Nikulin surfaces furnish an explicit unirational parametrization of \( R_g \) in small genus, see Theorem 0.2, just like ordinary K3 surfaces do the same for \( M_g \); numerous results involving curves on K3 surfaces have a Prym-Nikulin analogue, see Theorem 0.4, and even exceptions to uniform statements concerning curves on K3 surfaces carry over in this analogy!

Our other aim is to complete the birational classification of the moduli space \( S_g^+ \) of even spin curves of genus \( g \). It is known [F] that \( S_g^+ \) is of general type when \( g \geq 9 \). Using Nikulin surfaces we show that \( S_g^+ \) is uniruled for \( g \leq 7 \), see Theorem 0.7 which leaves \( S_8^+ \) as the only case missing from the classification. We prove the following:

**Theorem 0.1.** The Kodaira dimension of \( S_g^+ \) is equal to zero.

Theorems 0.1 and 0.7 highlight the fact that the birational type of \( S_g^+ \) is entirely governed by the world of K3 surfaces, in the sense that \( S_g^+ \) is uniruled precisely when a general even spin curve of genus \( g \) moves on a special K3 surface. This is in contrast to \( \overline{M}_g \) which is known to be uniruled at least for \( g \leq 16 \), whereas the general curve of genus \( g \geq 12 \) does not lie on a K3 surface.

A Nikulin surface [Ni] is a K3 surface \( S \) endowed with a non-trivial double cover

\[ f : \tilde{S} \to S \]

with a branch divisor \( N := N_1 + \cdots + N_8 \) consisting of 8 disjoint smooth rational curves \( N_i \subset S \). Blowing down the \((-1)\)-curves \( E_i := f^{-1}(N_i) \subset \tilde{S} \), one obtains a minimal K3 surface \( \sigma : \tilde{S} \to Y \), together with an involution \( \iota \in \text{Aut}(Y) \) having 8 fixed points corresponding to the images \( \sigma(E_i) \) of the exceptional divisors. The class \( O_S(N) \) is divisible by 2 in Pic\( (S) \) and we set \( e := \frac{1}{2}O_S(N_1 + \cdots + N_8) \in \text{Pic}(S) \). Assume that \( C \subset S \) is a smooth curve of genus \( g \) such that \( C \cdot N_i = 0 \) for \( i = 1, \ldots, 8 \). We say that the triple \((S, e, O_S(C))\) is a polarized Nikulin surface of genus \( g \) and denote by \( F_g^{31} \) the 11-dimensional moduli space of such objects. Over \( F_g^{31} \) we consider the \( \mathbb{P}^g \)-bundle

\[ F_g^{31} := \{ (S, e, C) : C \subset S \text{ is a smooth curve such that } [S, e, O_S(C)] \in F_g^{31} \}, \]
which comes equipped with two maps

\[ p_g : \mathcal{P}_g^{\mathfrak{n}} \rightarrow \mathcal{F}_g \rightarrow \mathcal{R}_g \]

where \( p_g([S, e, C]) := [S, e, \mathcal{O}_S(C)] \) and \( \chi_g([S, e, C]) := [C, eC := e \otimes \mathcal{O}_C] \). Since \( C \cdot N = 0 \), it follows that \( eC \otimes \mathcal{O}_C = \mathcal{O}_C \). The étale double cover induced by \( eC \) is precisely the restriction \( f_C := f_{\tilde{C}} : \tilde{C} \rightarrow C \), where \( \tilde{C} := f^{-1}(C) \). Note that \( \dim(\mathcal{P}_g^{\mathfrak{n}}) = 11 + g \) and it is natural to ask when is \( \chi_g \) dominant and induces a uniruled parametrization of \( \mathcal{R}_g \).

**Theorem 0.2.** The general Prym curve \([C, eC] \in \mathcal{R}_g \) lies on a Nikulin surface if and only if \( g \leq 7 \) and \( g \neq 6 \), that is, the morphism \( \chi_g : \mathcal{P}_g^{\mathfrak{n}} \rightarrow \mathcal{R}_g \) is dominant precisely in this range.

In contrast, the general Prym curve \([C, eC] \in \mathcal{R}_6 \) lies on an Enriques surface [V1] but not on a Nikulin surface. Since \( \mathcal{P}_g^{\mathfrak{n}} \) is a uniruled variety being a \( \mathbb{P}^g \)-bundle over \( \mathcal{F}_g^{\mathfrak{n}} \), we derive from Theorem [0.2] the following immediate consequence:

**Corollary 0.3.** The Prym moduli space \( \mathcal{R}_g \) is uniruled for \( g \leq 7 \).

The discussion in Sections 2 and 3 implies the stronger result that \( \mathcal{F}_g^{\mathfrak{n}} \) (and thus \( \mathcal{N}_g := \text{Im}(\chi_g) \)) is unirational for \( g \leq 6 \). It was known that \( \mathcal{R}_g \) is rational for \( g \leq 4 \), see [Do2], [Ca], and unirational for \( g = 5, 6 \), see [Do2], [ILS], [V1], [V2]. Apart from the result in genus 7 which is new, the significance of Corollary 0.3 is that Nikulin surfaces provide an explicit uniform parametrization of \( \mathcal{R}_g \) that works for all genera \( g \leq 7 \).

Before going into a more detailed explanation of our results on \( \mathcal{F}_g^{\mathfrak{n}} \), it is instructive to recall Mukai’s work on the moduli space \( \mathcal{F}_g \) of polarized \( K3 \) surfaces of genus \( g \):

**Mukai’s results [M1], [M2], [M3]:**

1. A general curve \([C] \in \mathcal{M}_g \) lies on a \( K3 \) surface if and only if \( g \leq 11 \) and \( g \neq 10 \), that is, the equality \( \mathcal{K}_g = \mathcal{M}_g \) holds precisely in this range.

2. \( \mathcal{M}_{11} \) is birationally isomorphic to the tautological \( \mathbb{P}^{11} \)-bundle \( \mathcal{P}_{11} \) over the moduli space \( \mathcal{F}_{11} \) of polarized \( K3 \) surfaces of genus 11. There is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}_{11} & \xrightarrow{\approx} & \mathcal{P}_{11} \\
\downarrow{q_{11}} & & \downarrow{p_{11}} \\
\mathcal{F}_{11} & \xrightarrow{\ } & \\
\end{array}
\]

with \( q_{11}^{-1}([C]) = [S, C] \), where \( S \) is the unique \( K3 \) surface containing a general \([C] \in \mathcal{M}_{11} \).

3. The locus \( \mathcal{K}_{10} \) is a divisor on \( \mathcal{M}_{10} \) which has the following set-theoretic incarnation:

\[
\mathcal{K}_{10} = \{ [C] \in \mathcal{M}_{10} \mid \exists L \in W_{12}^l(C) \text{ such that } \mu_0(L) : \text{Sym}^2 H^0(C, L) \xrightarrow{\not\sim} H^0(C, L^{-2}) \}.
\]

4. There exists a rational variety \( X \subset \mathbb{P}^{13} \) with \( K_X = \mathcal{O}_X(-3) \) and \( \dim(X) = 5 \), such that the general \( K3 \) surface of genus 10 appears as a 2-dimensional linear section of \( X \). Such a realization is unique up to the action of \( \text{Aut}(X) \) and one has birational isomorphisms:

\[
\mathcal{F}_{10} \cong G(\mathbb{P}^{10}, \mathbb{P}^{13})^{ss} // \text{Aut}(X) \quad \text{and} \quad \mathcal{K}_{10} \cong G(\mathbb{P}^9, \mathbb{P}^{13})^{ss} // \text{Aut}(X).
\]
To this list of well-known results, one could add the following statement from \[FP\]:

(5) The closure $\overline{\mathcal{K}}_{10}$ of $\mathcal{K}_{10}$ inside $\overline{\mathcal{M}}_{10}$ is an extremal point in the effective cone $\text{Eff}(\overline{\mathcal{M}}_{10})$; its class $\overline{\mathcal{K}}_{10} \equiv 7\lambda - 5\delta - 9\delta_2 - 12\delta_3 - 14\delta_4 - \cdots \in \text{Pic}(\overline{\mathcal{M}}_{10})$ has minimal slope among all effective divisors on $\overline{\mathcal{M}}_{10}$ and provides a counterexample to the Slope Conjecture [HMo].

Quite remarkably, each of the statements (1)-(5) has a precise Prym-Nikulin analogue. Theorem 0.2 is the analogue of (1). For the highest genus when the Prym-Nikulin condition is generic, the moduli space acquires a surprising Mori fibre space structure:

**Theorem 0.4.** The moduli space $\mathcal{R}_7$ is birationally isomorphic to the tautological $\mathbb{P}^7$-bundle $\mathcal{P}_{7}^{\mathbb{P}^7}$ and there is a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{R}_7 & \cong & \mathcal{P}_{7}^{\mathbb{P}^7} \\
\downarrow \chi_7 & & \downarrow \chi_7 \\
\mathcal{F}_{7}^{\mathbb{P}^7} & \rightarrow & \mathbb{P}^7
\end{array}
$$

Furthermore, $\chi_7^{-1}([C, \eta]) = [S, C]$, where the unique Nikulin surface $S$ containing $C$ is given by the base locus of the net of quadrics containing the Prym-canonical embedding $\phi_{K_C \otimes \eta} : C \rightarrow \mathbb{P}^5$.

Just like in Mukai’s work, the genus next to maximal from the point of view of Prym-Nikulin theory, behaves exotically.

**Theorem 0.5.** The Prym-Nikulin locus $\mathcal{N}_6 := \text{Im}(\chi_6)$ is a divisor on $\mathcal{R}_6$ which can be identified with the ramification locus of the Prym map $Pr_6 : \mathcal{R}_6 \rightarrow \mathcal{A}_5$:

$$
\mathcal{N}_6 = \{(C, \eta) \in \mathcal{R}_6 : \mu_0(K_C \otimes \eta) : \text{Sym}^2 H^0(C, K_C \otimes \eta) \twoheadrightarrow H^0(C, K_C^{\otimes 2})\}.
$$

Observe that both divisors $\mathcal{K}_{10}$ and $\mathcal{N}_6$ share the same Koszul-theoretic description. Furthermore, they are both extremal points in their respective effective cones, cf. Proposition 3.6. Is there a Prym analogue of the genus 10 Mukai $G_2$-variety $X := G_2/P \subset \mathbb{P}^{13}$? The answer to this question is in the affirmative and we outline the construction of a Grassmannian model for $\mathcal{F}_{6}^{\mathbb{P}^7}$ while referring to Section 3 for details.

Set $V := \mathbb{C}^5$ and $U := \mathbb{C}^4$ and view $\mathbb{P}^3 = \mathbb{P}(U^\vee)$ as the space of planes inside $\mathbb{P}(U^\vee)$. Let us choose a smooth quadric $Q \subset \mathbb{P}(V)$. The quadratic line complex $W_Q \subset G(2, V) \subset \mathbb{P}(\wedge^2 V)$ consisting of tangent lines to $Q$ is singular along the codimension 2 subvariety $V_Q$ of lines contained in $Q$. One can identify $V_Q$ with the Veronese 3-fold

$$
\nu_2(\mathbb{P}^3) \subset \mathbb{P}(\text{Sym}^2(U)) = \mathbb{P}(\wedge^2 V) = \mathbb{P}^9.
$$

The projective tangent bundle $\mathbb{P}_Q$ of $Q$, viewed as the blow-up of $W_Q$ along $V_Q$, is endowed with a double cover branched along $V_Q$ and induced by the map

$$
\mathbb{P}^3 \times \mathbb{P}^3 \overset{2:1}{\rightarrow} \mathbb{P}(\text{Sym}^2(U)), \quad (H_1, H_2) \mapsto H_1 + H_2.
$$

We show in Theorem 0.4 that codimension 3 linear sections of $W_Q$ are Nikulin surfaces of genus 6 with general moduli. Moreover there is a birational isomorphism

$$
\mathcal{F}_{6}^{\mathbb{P}^7} \cong G(7, \wedge^2 V)^{ss} \slash \text{Aut}(Q).
$$

Taking codimension 4 linear sections of $W_Q$ one obtains a similar realization of $\mathcal{N}_6$, which should be viewed as the Prym counterpart of Mukai’s construction of $\mathcal{K}_{10}$. 
The subvariety $K_g \subset M_g$ is intrinsic in moduli, that is, its generic point $[C]$ admits characterizations that involve $C$ alone and the $K$ surface containing $C$ is a result of some peculiarity of the canonical curve. For instance [BM], if $[C] \in K_g$ then the Wahl map

$$\psi_{KC} : \lambda^2 H^0(C, KC) \rightarrow H^0(C, K_C^{\otimes 3}),$$

is not surjective. It is natural to ask for similar intrinsic characterizations of the Prym-Nikulin locus $N_g \subset R_g$ in terms of Prym curves alone, without making reference to Nikulin surfaces. In this direction, we prove in Section 1 the following result: 

**Theorem 0.6.** Set $g := 2i + 6$. Then $K_{i,2}(C, KC \otimes \eta) \neq 0$ for any $[C, \eta] \in N_g$, that is, the Prym-canonical curve $C \mapsto p^g_{\eta} = K_{i,2}(C, KC \otimes \eta) P^g$ of a Prym-Nikulin section fails to satisfy property $(N_i)$.

It is the content of the Prym-Green Conjecture [FL] that $K_{i,2}(C, KC \otimes \eta) = 0$ for a general Prym curve $[C, \eta] \in R_{2i+6}$. This suggests that curves on Nikulin surfaces can be recognized by extra syzygies of their Prym-canonical embedding.

Our initial motivation for considering Nikulin surfaces was to use them for the birational classification of moduli spaces of even theta-characteristics and we propose to turn our attention to the moduli space $S^+_g$ of even spin curves classifying pairs $[C, \eta]$, where $[C] \in M_g$ is a smooth curve of genus $g$ and $\eta \in Pic^{g-1}(C)$ is an even theta-characteristic. Let $\overline{S}^+_g$ be the coarse moduli space associated to the Deligne-Mumford stack of even stable spin curves of genus $g$, cf. [Cor]. The projection $\pi : S^+_g \rightarrow M_g$ extends to a finite covering $\pi : \overline{S}^+_g \rightarrow \overline{M}_g$ branched along the boundary divisor $\Delta_0$ of $\overline{M}_g$. It is shown in [FL] that $\overline{S}^+_g$ is a variety of general type as soon as $g \geq 9$.

The existence of the dominant morphism $\chi_g : P^g_{\eta} \rightarrow R_g$ when $g \leq 7$ and $g \neq 6$, leads to a straightforward uniruled parametrization of $\overline{S}^+_g$, which we briefly describe. Let us start with a general even spin curve $[C, \eta] \in S^+_g$ and a non-trivial point of order two $e_C \in Pic^0(C)$ in the Jacobian, such that $h^0(C, e_C \otimes \eta) \geq 1$. Since the curve $[C] \in M_g$ is general, it follows that $h^0(C, e_C \otimes \eta) = 1$ and $Z := supp(e_C \otimes \eta)$ consists of $g - 1$ distinct points. Applying Theorem [CL] if $g \neq 6$ there exists a Nikulin $K$ surface $(S, e)$ containing $C$ such that $e_C = e \otimes O_C$. When $g = 6$, there exists an Enriques surface $(S, e)$ satisfying the same property, see [VII], and the construction described below goes through in that case as well. In the embedding $\varphi_{[O_S(C)]} : S \rightarrow P^g$, the span $\langle Z \rangle \subset P^g$ is a codimension $2$ linear subspace and $h^0(S, I_Z/S(1)) = 2$. Let

$$P := PH^0(S, I_Z/S(1)) \subset |O_S(C)|$$

be the corresponding pencil of curves on $S$. Each curve $D \in P$ is endowed with the odd theta-characteristic $O_D(Z)$. Twisting this line bundle with $e \otimes O_D \in Pic^0(D)$, we obtain an even theta-characteristic on $D$. This procedure induces a rational curve in moduli

$$m : P \rightarrow \overline{S}^+_g, \quad P \ni D \mapsto [D, e \otimes O_D(Z)],$$

which passes through the general point $[C, \eta] \in \overline{S}^+_g$. This proves the following result:

**Theorem 0.7.** The moduli space $\overline{S}^+_g$ is uniruled for $g \leq 7$.

It is known [FL] that $\overline{S}^+_g$ is of general type when $g \geq 9$. We complete the birational classification of $\overline{S}^+_g$ and wish to highlight the following result, see Theorem [0.11].
where divisor effective representative for the canonical divisor explicit problem. Settling these outstanding cases is expected to require genuinely new ideas.

there exists a smooth \(K \subset R\) space \(\pi\) boundary divisors Proposition 0.8. classes of moduli spaces is not complete. The Kodaira dimension of \(K\) is unknown for \(17 \leq g \leq 21\), see [HM], [EH1], the birational type of \(R\) is not understood in the range \(8 \leq g \leq 13\), whereas finding the Kodaira dimension of \(A_6\) is a notorious open problem. Setting these outstanding cases is expected to require genuinely new ideas.

The proof of Theorem 0.1 relies on two main ideas: Following [F], one finds an explicit effective representative for the canonical divisor \(K_{s_8^+}\) as a \(\mathbb{Q}\)-combination of the divisor \(\Omega_{\text{null}} \subset S_{s_8^+}\) of vanishing theta-nulls, the pull-back \(\pi^*(\overline{\mathcal{M}}_{s_8,7}^2)\) of the Brill-Noether divisor \(\overline{\mathcal{M}}_{s_8,7}\) on \(\overline{\mathcal{M}}_{s_8}\) of curves with a \(g_{s_8^2}\), and boundary divisor classes corresponding to spin curves whose underlying stable model is of compact type. This already implies the inequality \(\kappa(S_{s_8^+}) \geq 0\). Each irreducible component of this particular representative of \(K_{s_8^+}\) is rigid (see Section 3), and the goal is to show that \(K_{s_8^+}\) is rigid as well. To that end, we use the existence of a birational model \(\mathcal{M}_{s_8}\) of \(\overline{\mathcal{M}}_{s_8}\) inspired by Mukai’s work [M2]. The space \(\mathcal{M}_{s_8}\) is realized as the following GIT quotient

\[\mathcal{M}_{s_8} := G(8, \wedge^2 V)^{ss//SL(V)},\]

where \(V = \mathbb{C}^6\). We note that \(\rho(\mathcal{M}_{s_8}) = 1\) and there exists a birational morphism

\[f : \overline{\mathcal{M}}_{s_8} \dashrightarrow \mathcal{M}_{s_8},\]

which contracts all the boundary divisors \(\Delta_1, \ldots, \Delta_4\) as well as \(\overline{\mathcal{M}}_{s_8,7}\). Using the geometric description of \(f\), we establish a geometric characterization of points inside \(\Omega_{\text{null}}\):

**Proposition 0.8.** Let \(C\) be a smooth curve of genus 8 without a \(g_{s_8^2}\). The following are equivalent:

- There exists a vanishing theta-null \(L\) on \(C\), that is, \([C, L] \in \Omega_{\text{null}}\).
- There exists a smooth K3 surface \(S\) together with elliptic pencils \(|F_1|\) and \(|F_2|\) on \(S\), such that \(C \in |F_1 + F_2|\) and \(L = \mathcal{O}_C(F_1) = \mathcal{O}_C(F_2)\).

The existence of such a doubly elliptic K3 surface \(S\) is equivalent to stating that there exists a smooth K3 extension \(S \subset \mathbb{P}^8\) of the canonical curve \(C \subset \mathbb{P}^7\), such that the rank three quadric \(C \subset Q \subset \mathbb{P}^7\) which induces the theta-null \(L\), lifts to a rank 4 quadric \(S \subset Q_S \subset \mathbb{P}^8\). Having produced \(S\), the pencils \(|F_1|\) and \(|F_2|\) define a product map

\[\phi : S \to \mathbb{P}^1 \times \mathbb{P}^1,\]

such that each smooth member \(D \in I := |\phi^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)|\) is a canonical curve contained in a rank 3 quadric. A general pencil in \(I\) passing through \(C\) induces a rational curve \(R \subset S_{s_8^+}\), and after intersection theoretic calculations on the stack \(S_{s_8^+}\), we prove the following:

**Proposition 0.9.** The theta-null divisor \(\Omega_{\text{null}} \subset S_{s_8^+}\) is uniruled and swept by rational curves \(R \subset S_{s_8^+}\) such that \(R : \Omega_{\text{null}} < 0\) and \(R : \pi^*(\overline{\mathcal{M}}_{s_8,7}^2) = 0\). Furthermore \(R\) is disjoint from all boundary divisors \(\pi^*(\Delta_i)\) for \(i = 1, \ldots, 4\).
Proposition \[14\] implies that $K_{\overline{S}_8^+}$, expressed as a weighted sum of $\overline{\Theta_{\text{null}}}$, the pull-back $\pi^*(\overline{M}_{8,7}^2)$ and boundary divisors $\pi^*(\Delta_i)$ for $i = 1, \ldots, 4$, is rigid as well. Equivalently, $\kappa(\overline{S}_8^+) = 0$. Note that since $K_{\overline{S}_8^+}$ consists of 10 uniruled base components which can be blown-down, the variety $\overline{S}_8^+$ is not minimal and there exists a birational model $S$ of $\overline{S}_8^+$ which is a genuine Calabi-Yau variety in the sense that $K_S = 0$. Finding an explicit modular interpretation of this Calabi-Yau 21-fold (or perhaps even its equations!) is a very interesting question.

1. Prym-canonical curves on Nikulin surfaces

Let us start with a smooth $K3$ surface $Y$. A Nikulin involution on $Y$ is an automorphism $\iota \in \text{Aut}(Y)$ of order 2 which is symplectic, that is, $\iota^*(\omega) = \omega$, for all $\omega \in H^{2,0}(Y)$. A Nikulin involution has 8 fixed points, see [Ni] Lemma 3, and the quotient $\bar{Y} := Y/\langle \iota \rangle$ has 8 ordinary double point singularities. Let $\sigma : \bar{S} \to \bar{Y}$ be the blow-up of the 8 fixed points and denote by $E_1, \ldots, E_8 \subset \bar{S}$ the exceptional divisors and by $\iota \in \text{Aut}(\bar{S})$ the automorphism induced by $\iota$. Then $S := \bar{S}/\langle \iota \rangle$ is a smooth $K3$ surface and if $f : \tilde{S} \to S$ is the projection, then $N_i := f(E_i)$ are $(-2)$-curves on $S$. The branch divisor of $f$ is equal to $N := \sum_{i=1}^8 N_i$. We summarize the situation in the following diagram:

$$
\begin{array}{c}
\tilde{S} \\
\downarrow \sigma \\
S \\
\downarrow \\
\bar{Y}
\end{array}
$$

(1)

Sometimes we shall refer to the pair $(Y, \iota)$ as a Nikulin surface, while keeping the previous diagram in mind. We refer to [Mo], [vGS] for a lattice-theoretic study on the action of the Nikulin involution on the cohomology $H^2(Y, \mathbb{Z}) = U^3 \oplus E_8(-1) \oplus E_8(-1)$, where $U$ is the standard rank 2 hyperbolic lattice and $E_8$ is the unique even, negative-definite unimodular lattice of rank 8. It follows from [Mo] Theorem 5.7 that the orthogonal complement $E_8(-2) \cong (H^2(Y, \mathbb{Z}^4)^\perp$ is contained in $\text{Pic}(Y)$, hence $Y$ has Picard number at least 9. The class $O_S(N_1 + \cdots + N_8)$ is divisible by 2, and we denote by $e \in \text{Pic}(S)$ the class such that $e^{\otimes 2} = O_S(N_1 + \cdots + N_8)$.

**Definition 1.1.** The Nikulin lattice is an even lattice $\mathfrak{H}$ of rank 8 generated by elements $\{n_i\}_{i=1}^8$ and $e := \frac{1}{2} \sum_{i=1}^8 n_i$, with the bilinear form induced by $n_i^2 = -2$ for $i = 1, \ldots, 8$ and $n_i \cdot n_j = 0$ for $i \neq j$.

Note that $\mathfrak{H}$ is the minimal primitive sublattice of $H^2(S, \mathbb{Z})$ containing the classes $N_1, \ldots, N_8$ and $e$. For any Nikulin surface one has an embedding $\mathfrak{H} \subset \text{Pic}(S)$. Assuming that $(Y, \iota)$ defines a general point in an irreducible component of the moduli space of Nikulin involutions, both $Y$ and $S$ have Picard number 9 and there is a decomposition $\text{Pic}(S) = \mathbb{Z} \cdot [C] \oplus \mathfrak{H}$, where $C$ is an integral curve of genus $g \geq 2$. According to [vGS] Proposition 2.2, only two cases are possible: either $C \cdot e = 0$ so that the previous decomposition is an orthogonal sum, or else, $C \cdot e \neq 0$, this second case being possible only when $g$ is odd. In this paper we consider only Nikulin surfaces of the first kind.

We fix an integer $g \geq 2$ and consider the lattice $\Lambda_g := \mathbb{Z} \cdot e \oplus \mathfrak{H}$, where $e \cdot e = 2g - 2$.

**Definition 1.2.** A Nikulin surface of genus $g$ is a $K3$ surface $S$ together with a primitive embedding of lattices $j : \Lambda_g \hookrightarrow \text{Pic}(S)$ such that $C := j(e)$ is a nef class.
The coarse moduli space $\mathcal{F}_g$ of Nikulin surfaces of genus $g$ is the quotient of the 11-dimensional domain

$$
\mathcal{D}_g := \{ \omega \in \mathbb{P} (\Lambda_g \otimes \mathbb{Z} \mathbb{C}) : \omega^2 = 0, \ \omega \cdot \bar{\omega} > 0 \}
$$

by an arithmetic subgroup of $O(\Lambda_g)$. Its existence follows e.g. from [Do1] Section 3.

We now consider a Nikulin surface $f: \hat{S} \to S$, together with a smooth curve $C \subset S$ of genus $g$ such that $C \cdot N = 0$. If $\hat{C} := f^{-1}(C)$, then $f_C := f|_{\hat{C}} : \hat{C} \to C$ is an étale double covering. By the Hodge index theorem, $\hat{C}$ cannot split in two disjoint connected components, hence $f_C$ is non-trivial and $e_C := \mathcal{O}_C(e) \in \text{Pic}^0(C)$ is the non trivial 2-torsion element defining the covering $f_C$. We set $H \equiv C - e \in NS(S)$, hence $H^2 = 2g - 6$ and $H \cdot C = 2g - 2$. For further reference we collect a few easy facts:

**Lemma 1.3.** Let $[S, e, \mathcal{O}_S(C)] \in \mathcal{F}_g$ be a Nikulin surface such that $\text{Pic}(S) = \Lambda_g$. The following statements hold:

1. $H^i(S, e) = 0$ for all $i \geq 0$.
2. $\text{Cliff}(C) = \left[ \frac{g-1}{2} \right]$.
3. The line bundle $\mathcal{O}_S(H)$ is ample for $g \geq 4$ and very ample for $g \geq 6$. In this range, it defines an embedding $\phi_H : S \to \mathbb{P}^{g-2}$ such that the images $\phi_H(N_i)$ are lines for all $i = 1, \ldots, 8$.
4. If $g \geq 7$, the ideal of the surface $\Phi_H(S) \subset \mathbb{P}^{g-2}$ is cut out by quadrics.

**Proof.** Recalling that $e^\otimes 2 = \mathcal{O}_S(N_1 + \cdots + N_8)$ and that the curves $\{N_i\}_{i=1}^8$ are pairwise disjoint, it follows that $H^0(S, e) = 0$ and clearly $H^2(S, e) = 0$. Since $e^2 = -4$, by Riemann-Roch one finds that $H^1(S, e) = 0$ as well.

In order to prove (ii) we assume that $\text{Cliff}(C) < \left[ \frac{g-1}{2} \right]$. From [GL2] it follows that there exists a divisor $D \in \text{Pic}(S)$ such that $h^i(S, \mathcal{O}_C(D)) \geq 2$ for $i = 0, 1$ and $C \cdot D \leq g - 1$, such that $\mathcal{O}_C(D)$ computes the Clifford index of $C$, that is, $\text{Cliff}(C) = \text{Cliff}(\mathcal{O}_C(D))$. But $C \cdot \ell \equiv 0 \bmod 2g - 2$ for every class $\ell \in \text{Pic}(S)$, hence no such divisor $D$ can exist.

Moving to (iii), the ampleness (respectively very ampleness) of $\mathcal{O}_S(H)$ is proved in [GS] Proposition 3.2 (respectively Lemma 3.1). From the exact sequence

$$
0 \to \mathcal{O}_S(-H) \to \mathcal{O}_S(e) \to \mathcal{O}_C(e) \to 0,
$$

one finds that $h^1(S, \mathcal{O}_S(H)) = 0$ and then $\dim |H| = g - 2$. Furthermore $H \cdot N_i = 1$ for $i = 1, \ldots, 8$ and the claim follows.

To prove (iv), following [SD] Theorem 7.2, it suffices to show that there exists no irreducible curve $\Gamma \subset S$ with $\Gamma^2 = 0$ and $H \cdot \Gamma = 3$. Assume by contradiction that $\Gamma \equiv aC - b_1N_1 - \cdots - b_8N_8$ is such a curve, where necessarily $a, b_i \in \mathbb{Z}_{\leq 0}$. Then $\sum_{i=1}^8 b_i = 2ag - 2a - 3$ and $\sum_{i=1}^8 b_i^2 = a^2(g - 1)$. Applying the Cauchy-Schwarz inequality $(\sum_{i=1}^8 b_i)^2 \leq 8(\sum_{i=1}^8 b_i^2)$, we obtain an immediate contradiction. \hfill $\Box$

We consider the $\mathbb{P}^g$-bundle $p_g : \mathcal{P}^\mathcal{R}_g \to \mathcal{F}^\mathcal{R}_g$, as well as the map

$$
\chi_g : \mathcal{P}^\mathcal{R}_g \to \mathcal{R}_g, \quad \chi_g([S, e, C]) := [C, e_C := e \otimes \mathcal{O}_C]
$$

defined in the introduction. We fix a Nikulin surface $[S, e, \mathcal{O}_S(C)] \in \mathcal{P}^\mathcal{R}_g$. A Lefschetz pencil of curves $\{C_\lambda\}_{\lambda \in \mathbb{P}^1}$ inside $|\mathcal{O}_S(C)|$ induces a rational curve

$$
\Xi_g := \{ [C_\lambda, e_{C_\lambda} := e \otimes \mathcal{O}_{C_\lambda}] : \lambda \in \mathbb{P}^1 \} \subset \mathcal{R}_g.
$$
In the range where \( x_g \) is a dominant map, \( \Xi_g \) is a rational curve passing through a general point of \( \overline{\mathcal{R}}_g \), and it is of some interest to compute its numerical characters. If \( \pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g \) denotes the projection map, we recall the formula [FL] Example 1.4

\[
(2) \quad \pi^* (\delta_0) = \delta'_0 + \delta''_0 + 2\delta^\text{ram},
\]

where \( \delta'_0 := [\Delta'_0], \delta''_0 := [\Delta''_0] \) and \( \delta^\text{ram}_0 := [\Delta^\text{ram}_0] \) are boundary divisor classes on \( \overline{\mathcal{R}}_g \) whose meaning we recall. Let us fix a general point \( [C_{xy}] \in \Delta_0 \) induced by a 2-pointed curve \( [C, x, y] \in \mathcal{M}_{g-1,2} \), and the normalization map \( \nu : C \to C_{xy} \), where \( \nu(x) = \nu(y) \).

A general point of \( \Delta_0 \) (respectively of \( \Delta''_0 \)) corresponds to a stable Prym curve \( [C_{xy}, \eta] \), where \( \eta \in \text{Pic}^0(C_{xy})[2] \) and \( \nu^*(\eta) \in \text{Pic}^0(C) \) is non-trivial (respectively, \( \nu^*(\eta) = \mathcal{O}_C \)). A general point of \( \Delta^\text{ram}_0 \) is of the form \([X, \eta]\), where \( X := C \cup \{x, y\} \mathbb{P}^1 \) is a quasi-stable curve, whereas \( \eta \in \text{Pic}^0(X) \) is characterized by \( \eta_{p_1} = \mathcal{O}_{p_1}(1) \) and \( \eta_{C^2} = \mathcal{O}_C(-x - y) \).

**Proposition 1.4.** If \( \Xi_g \subset \overline{\mathcal{R}}_g \) is the curve induced by a pencil on a Nikulin surface, then

\[
\Xi_g \cdot \lambda = g + 1, \quad \Xi_g \cdot \delta'_0 = 6g + 2, \quad \Xi_g \cdot \delta''_0 = 0 \quad \text{and} \quad \Xi_g \cdot \delta^\text{ram}_0 = 8.
\]

It follows that \( \Xi_g \cdot K_{\overline{\mathcal{R}}_g} = g - 15 \).

**Proof.** We use [FL] Lemma 2.4 to find that \( \Xi_g \cdot \lambda = \pi_*(\Xi_g) \cdot \lambda = g + 1 \) and \( \Xi_g \cdot \pi^*(\delta_0) = \pi_*(\Xi_g) \cdot \delta_0 = 6g + 18 \), as well as \( \Xi_g \cdot \pi^*(\delta_i) = 0 \) for \( 1 \leq i \leq [g/2] \). For each \( 1 \leq i \leq 8 \), the sublinear system \( P H^0(\mathcal{O}_S(C - N_i)) \subset P H^0(\mathcal{O}_S(C)) \) intersects \( \Xi_g \) transversally in one point which corresponds to a curve \( N_i + C_i \in |\mathcal{O}_S(C)| \), where \( N_i \cdot C_i = -N_i^2 = 2 \) and \( C_i \equiv C - N_i \). Furthermore \( e \otimes \mathcal{O}_{N_i} = \mathcal{O}_{N_i}(1) \) and \( e_{C_i}^\text{ram} = \mathcal{O}_{C_i}(-N_i \cdot C_i) \). Each of these points lie in the intersection \( \Xi_g \cap \Delta^\text{ram}_0 \). All remaining curves in \( \Xi_g \) are irreducible, hence \( \Xi_g \cdot \delta^\text{ram}_0 = 8 \). Since \( \Xi_g \cdot \delta''_0 = 0 \), from (2) we find that \( \Xi_g \cdot \delta'_0 = 6g + 2 \). Finally, according to [FL] Theorem 1.5 the formula \( K_{\overline{\mathcal{R}}_g} \equiv 13\lambda - 2(\delta'_0 + \delta''_0) - 3\delta^\text{ram}_0 - \cdots \in \text{Pic}(\overline{\mathcal{R}}_g) \) holds, therefore putting everything together, \( \Xi_g \cdot K_{\overline{\mathcal{R}}_g} = g - 15 \).

The calculations in Proposition 1.4 are applied now to show that syzygies of Prym-canonical curves on Nikulin surfaces are exceptional when compared to those of general Prym-canonical curves. To make this statement precise, let us recall the **Prym-Green Conjecture**, see [FL] Conjecture 0.7: If \( g := 2i + 6 \) with \( i \geq 0 \), then the locus

\[
\mathcal{U}_{g,i} := \{ [C, \eta] \in \mathcal{R}_{2i+6} : K_{i,2}(C, K_C \otimes \eta) \neq 0 \}
\]

is a virtual divisor, that is, the degeneracy locus of two vector bundles of the same rank defined over \( \mathcal{R}_{2i+6} \). The statement of the Prym-Green Conjecture is that this vector bundle morphism is generically non-degenerate:

**Prym-Green Conjecture:** \( K_{i,2}(C, K_C \otimes \eta) = 0 \) for a general Prym curve \( [C, \eta] \in \mathcal{R}_{2i+6} \).

The conjecture is known to hold in bounded genus and has been used in [FL] to show that \( \Xi_g \) is of general type when \( g \geq 14 \) is even.

**Theorem 1.5.** For each \( [S, e, C] \in \mathcal{P}_g^{\mathbb{P}^1} \mathcal{M}_{2i+6} \) one has \( K_{i,2}(C, K_C \otimes e_C) \neq 0 \). In particular, the Prym-Green Conjecture fails along the locus \( N_{2i+6} \).

**Proof.** If the non-vanishing \( K_{i,2}(C, K_C \otimes \eta) \neq 0 \) holds for a general point \( [C, \eta] \in \mathcal{R}_g \), then there is nothing to prove, hence we may assume that \( \mathcal{U}_{g,i} \) is a genuine divisor on \( \mathcal{R}_g \). The
Lemma 2.1. For a general $C$ the Prym-canonical image

$$\overline{U}_{g,i} \equiv \left( \binom{2i+2}{i} \right) \left( \frac{3(2i + 7)}{i + 3} \lambda - \frac{3}{2} b_{0}^{\text{ram}} - b_{0}' - b_{0}'' - \cdots \right) \in \text{Pic}(\mathbb{P}_{2i+6}).$$

From Proposition [1.4] by direct calculation one finds that $\Xi_{g} \cdot \overline{U}_{g,i} = -(\frac{2i+3}{2}) < 0$, thus $\Xi_{g} \subset \overline{U}_{g,i}$. By varying $\Xi_{g}$ inside $\overline{R}_{g}$, we obtain that $N_{g} \subset \overline{U}_{g,i}$, which ends the proof. □

Remark 1.6. A geometric proof of Theorem 1.5 using the Lefschetz hyperplane principle for Koszul cohomology is given in [AF] Theorem 3.5. The indirect proof presented here is however shorter and illustrates how cohomology calculations on $\overline{R}_{g}$ can be used to derive geometric consequences for individual Prym curves.

Remark 1.7. One might ask whether similar applications to $\overline{R}_{g}$ can be obtained using Enriques surfaces. There is a major difference between Prym curves $[C, \eta] \in \overline{R}_{g}$ lying on a Nikulin surface and those lying on an Enriques surface. For instance, if $C \subset S$ is a curve of genus $g$ lying on an Enriques surface $S$, then from [CD] Corollary 2.7.1

$$\text{gon}(C) \leq 2 \inf \{ F \cdot C : F \in \text{Pic}(S), F^{2} = 0, F \not\equiv 0 \} \leq 2\sqrt{2g-2}.$$

In particular, for $g$ sufficiently high, $C$ is far from being Brill-Noether general. On the other hand, we have seen that for $[S, e, C] \in \mathcal{P}_{g}^{\text{m}}$ such that $\text{Pic}(S) = \Lambda_{g}$, one has that $\text{gon}(C) = \left[ \frac{g + 3}{2} \right]$. For this reason, the Prym-Nikulin locus $N_{g} := \text{Im}(\chi_{g}) \subset \overline{R}_{g}$ appears as a more promising and less constrained locus than the Prym-Enriques locus in $\overline{R}_{g}$, being transversal to stratifications of $\overline{R}_{g}$ coming from Brill-Noether theory.

2. The Prym-Nikulin locus in $\overline{R}_{g}$ for $g \leq 7$

In this section we give constructive proofs of Theorems 0.2 and 0.4. Comparing the dimensions $\text{dim}(\mathcal{P}_{g}^{\text{m}}) = 11 + g$ and $\text{dim}(\overline{R}_{g}) = 3g - 3$, one may inquire whether the morphism $\chi_{g} : \mathcal{P}_{g}^{\text{m}} \rightarrow \overline{R}_{g}$ is dominant when $g \leq 7$. The similar question for ordinary $K3$ surfaces has been answered by Mukai [M1]. Let $\mathcal{F}_{g}$ denote the 19-dimensional moduli space of polarized $K3$ surfaces of genus $g$ and consider the associated $\mathbf{P}^{9}$-bundle

$$\mathcal{P}_{g} := \{ [S, e, C] : C \subset S \text{ is a smooth curve such that } [S, \mathcal{O}_{S}(C)] \in \mathcal{F}_{g} \}.$$

The map $q_{g} : \mathcal{P}_{g} \rightarrow \mathcal{M}_{g}$ forgetting the $K3$ surface is dominant if and only if $g \leq 11$ and $g \neq 10$. The result for $g = 10$ is contrary to untutored expectation since the general fibre of $q_{10}$ is 3-dimensional, hence $\text{dim}(\text{Im}(q_{10})) = \text{dim}(\mathcal{P}_{10}) - 3 = 26$. A strikingly similar picture emerges for Nikulin surfaces and Prym curves. The morphism $\chi_{g} : \mathcal{P}_{g}^{\text{m}} \rightarrow \overline{R}_{g}$ is dominant when $g \leq 7$ and $g \neq 6$. For each genus we describe a geometric construction that furnishes a Nikulin surface in the fibre $\chi_{g}^{-1}(\{C, \eta\})$ over a general point $[C, \eta] \in \overline{R}_{g}$.

2.1. Nikulin surfaces of genus 7. We start with a general element $[C, \eta] \in \overline{R}_{7}$ and construct a Nikulin surface containing $C$. One may assume that $\text{gon}(C) = 5$ and that the line bundle $\eta$ does not lie in the difference variety $C_{2} - C_{2} \subset \text{Pic}^{0}(C)$, or equivalently, the linear series $L := K_{C} \otimes \eta \in W_{12}^{0}(C)$ is very ample. It is a consequence of [GL1] Theorem 2.1 that the Prym-canonical image $C \rightarrow L_{|C/\mathbf{P}^{5}}$ is quadratically normal, that is, $h^{0}(\mathbf{P}^{5}, \mathcal{I}_{C/\mathbf{P}^{5}}(2)) = 3$.

Lemma 2.1. For a general $[C, \eta] \in \overline{R}_{7}$, the base locus of $|\mathcal{I}_{C/\mathbf{P}^{5}}(2)|$ is a smooth $K3$ surface.
\textbf{Proof.} The property that the base locus of \([\mathcal{I}_C/p^5(2)]\) is smooth, is open in \(\mathcal{R}_7\) and it suffices to exhibit a single Prym-canonical curve \([C, \eta] \in \mathcal{R}_7\) satisfying it. Let us fix an element \((S, e, C) \in \mathcal{P}_7^{\mathbb{N}}\) such that \(\text{Pic}(S) = \Lambda_7\) and set \(H = C - e\). Then according to Lemma \ref{lemma1}, \(\phi_H : S \to \mathbb{P}^5\) is an embedding whose image \(\phi_H(S)\) is ideal-theoretically cut out by quadrics. Moreover \(\text{gon}(C) = 5\), hence \(K_C \otimes e_C \in W_{12}^5(C)\) is quadratically normal. This implies that \(H^0(S, \mathcal{O}_S(2H - C)) = H^1(S, \mathcal{O}_S(2H - C)) = 0\), and then \(H^0(\mathbb{P}^5, \mathcal{I}_S/p^5(2)) \cong H^0(\mathbb{P}^5, \mathcal{I}_C/p^5(2))\) cut out precisely the surface \(S\).

\[\square\]

\textbf{Remark 2.2.} This proof shows that if \([S, e, C] \in \mathcal{P}_7^{\mathbb{N}}\) is general then \(\chi_7^{-1}([C, e_C]) = [S, e, C]\) and in particular the fibre \(\chi_7^{-1}([C, e_C])\) is reduced. Indeed, let \([S', e', C] \in \mathcal{P}_7^{\mathbb{N}}\) be an arbitrary Nikulin surface containing \(C\). Set \(H' = C - e' \in NS(S')\). We may assume that \(\text{Pic}(S') = \Lambda_7\), therefore the map \(\phi_{H'} : S' \to \mathbb{P}^5\) is an embedding whose image is cut out by quadrics. Since \(\text{Cliff}(C) = 3\), from Lemma \ref{lemma1} we find that \(K_C \otimes e_C\) is quadratically normal and then \(S'\) is cut out by the quadrics contained in Prym-canonical embedding of \(C \subset \mathbb{P}^5\).

Since both \(\mathcal{P}_7^{\mathbb{N}}\) and \(\mathcal{R}_7\) are irreducible varieties of dimension 18, Remark \ref{remark2.2} shows that \(\chi_7 : \mathcal{P}_7^{\mathbb{N}} \to \mathcal{R}_7\) is a birational morphism and we now describe \(\chi_7^{-1}\).

\textbf{Proposition 2.3.} For a general \([C, \eta] \in \mathcal{R}_7\), the surface \(S := \text{bs } \mathcal{I}_C/p^5(2)\) is a polarized Nikulin surface of genus 7.

\textbf{Proof.} We show that \(\text{Pic}(S) \supset \mathbb{Z} \cdot C \oplus \mathfrak{N}\). Denote by \(H \subset S\) the hyperplane class and let \(N = 2(C - H)\), thus \(N^2 = -16, N \cdot H = 8\) and \(N \cdot C = 0\). We aim to prove that \(N\) is linearly equivalent to a sum of 8 pairwise disjoint integral (-2) curves on \(S\). We consider the following exact sequence

\[0 \to \mathcal{O}_S(N - C) \to \mathcal{O}_S(N) \to \mathcal{O}_C(N) \to 0.\]

Note that \(\mathcal{O}_C(N)\) is trivial because \(e_C = \mathcal{O}_C(C - H)\) and that \(h^1(S, \mathcal{O}_S(N - C)) = h^1(S, \mathcal{O}_S(C - 2H)) = 0\), because \(C \subset \mathbb{P}^5\) is quadratically normal. Passing to the long exact sequence, it follows that \(h^0(S, \mathcal{O}_S(N)) = 1\). Using Remark \ref{remark2.2} it follows that \(N \equiv N_1 + \ldots + N_8\), where \(N_i \cdot N_j = -2\delta_{ij}\). Finally, to conclude that \([S, \mathbb{Z} \cdot C \oplus \mathfrak{N}] \in \mathcal{I}_C^{\mathbb{N}}\) we must show that there is a primitive embedding \(\mathbb{Z} \cdot C \oplus \mathfrak{N} \hookrightarrow \text{Pic}(S)\). We apply \cite{vGS} Proposition 2.7. Since \(H^0(S, \mathcal{O}_S(\tilde{C})) = H^0(S, \mathcal{O}_S(\tilde{C})) \oplus H^0(S, \mathcal{O}_S(C) \otimes e^\vee)\) and sections in the second summand vanish on the exceptional divisor of the morphism \(\sigma : \tilde{S} \to Y\), it follows that this is precisely the decomposition of \(H^0(Y, \mathcal{O}_{\tilde{Y}}(\tilde{C}))\) into \(\iota_1^*\)-eigenspaces. Invoking loc. cit., we finish the proof. \[\square\]

\subsection{2.2. The symmetric determinantal cubic hypersurface and Prym curves.} We provide a general set-up that allows us to reconstruct a Nikulin surface from a Prym curve of genus \(g \leq 5\). Let us start with a curve \([C, \eta] \in \mathcal{R}_g\) inducing an étale double cover \(f : \tilde{C} \to C\) together with an involution \(\iota : \tilde{C} \to \tilde{C}\) such that \(f \circ \iota = f\). For each integer \(r \geq 1\), the Prym-Brill-Noether locus is defined as the locus

\[V^r(C, \eta) := \{L \in \text{Pic}^{2g-2}(\tilde{C}) : \text{Nm}_f(L) = K_C, h^0(L) \geq r + 1\} \text{ and } h^0(L) \equiv r + 1 \text{ mod } 2\}.

Note that \(V^{-1}(C, \eta) = \text{Pr}(C, \eta)\). For each line bundle \(L \in V^r(C, \eta)\), the Petri map

\[\mu_0(L) : H^0(\tilde{C}, L) \otimes H^0(\tilde{C}, K_{\tilde{C}} \otimes L^\vee) \to H^0(\tilde{C}, K_{\tilde{C}})\]
splits into an $\iota$-anti-invariant part
$$\mu_0^-(L) : \Lambda^2 H^0(\tilde{C}, L) \to H^0(C, K_C \otimes \eta), \quad s \wedge t \mapsto s \cdot \iota^*(t) - t \cdot \iota^*(s),$$
and an $\iota$-invariant part respectively
$$\mu_0^+(L) : \text{Sym}^2 H^0(\tilde{C}, L) \to H^0(C, K_C), \quad s \otimes t + t \otimes s \mapsto s \cdot \iota^*(t) + t \cdot \iota^*(s).$$
For a general $[C, \eta] \in \mathcal{R}_g$, the Prym-Petri map $\mu_0^-(L)$ is injective for every $L \in V^r(C, \eta)$ and $V^r(C, \eta)$ is equidimensional of dimension $g - 1 - \binom{r + 1}{2}$, see [We]. We introduce the universal Prym-Brill-Noether variety
$$\mathcal{R}_g^r : = \left\{ ([C, \eta], L) : [C, \eta] \in \mathcal{R}_g, \ L \in V^r(C, \eta) \right\}.$$
When $g - 1 - \binom{r + 1}{2} \geq 0$, the variety $\mathcal{R}_g^r$ is irreducible of dimension $4g - 4 - \binom{r + 1}{2}$. We propose to focus on the case $r = 2$ and $g \geq 4$ and choose a general triple $(f : \tilde{C} \to C, L) \in \mathcal{R}_g^2$, such that $L$ is base point free and $h^0(\tilde{C}, L) = 3$.

Setting $\mathbb{P}^2 := \mathbb{P}(H^0(L)^\vee)$, we consider the quasi-étale double cover $q : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^5$ obtained by projecting via the Segre embedding to the space of symmetric tensors. Note that $q$ is ramified along the diagonal $\Delta \subset \mathbb{P}^2 \times \mathbb{P}^2$ and $V_4 := q(\Delta) \subset \mathbb{P}^5$ is the Veronese surface. Moreover $\Sigma := \text{Im}(q)$ is the determinantal symmetric cubic hypersurface isomorphic to the secant variety of $V_4$. We have the following commutative diagram:

$$
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{(L, \iota^* L)} & \mathbb{P}^2 \times \mathbb{P}^2 \\
\downarrow f & & \downarrow q \\
\mathbb{C} & \xrightarrow{\mu_0^-(L)} & \mathbb{P}^5 = \mathbb{P}(\text{Sym}^2 H^0(L)^\vee)
\end{array}
$$

Observe that the involution $\iota : \mathbb{P}^8 \to \mathbb{P}^8$ given by $\iota[v \otimes w] := [w \otimes v]$ where $v, w \in H^0(L)$, is compatible with $\iota : \tilde{C} \to \tilde{C}$. To summarize, giving a point $(\tilde{C} \to C, L) \in \mathcal{R}_g^2$ is equivalent to specifying a symmetric determinantal cubic hypersurface $\Sigma \subset H^0(\mathbb{P}^{g-1}, \mathcal{I}_C/\mathbb{P}^{g-1}(3))$ containing the canonical curve.

2.3. A birational model of $\mathcal{F}_4^3$. As a warm-up, we indicate how the set-up described above is a generalization of the construction that Catanese [Ca] used to prove that $\mathcal{R}_4$ is rational. For a general point $[C, \eta] \in \mathcal{R}_4$ we find that $V^2(C, \eta) = \{ \text{Id}, \iota^* \}$, that is, the pair $(L, \iota^* L)$ is uniquely determined. The map $\mu_0(L)$ has corank 2 and $\mathbb{P}^8_C := \text{Im} \, \mu_0(L) \subset \mathbb{P}^8$ has codimension 2. The intersection $\tilde{T} := (\mathbb{P}^2 \times \mathbb{P}^2) \cap \mathbb{P}^6_C$ is a del Pezzo surface of degree 6, whereas $T := \Sigma \cap \mathbb{P}^4_+ \subset \mathbb{P}^5$ is a 4-nodal Cayley cubic. Here we set $\mathbb{P}^3_+ := \mathbb{P}(H^0(K_C)^\vee)$. The double cover $q : \tilde{T} \to T$ is ramified at the singular points of $T$.

To obtain a Nikulin surface containing $[C, \eta]$, we reverse this construction and start with a quartic rational normal curve $R \subset \mathbb{P}^4$ and denote by $\mathcal{Y} := \text{Sec}(R) \subset \mathbb{P}^4$ its secant variety, which we view as a hyperplane section of $\Sigma \subset \mathbb{P}^5$. Retaining the notation of diagram (1), for a general quadric $Q \in |\mathcal{O}_{\mathbb{P}^4}(2)|$, the intersection $\tilde{Y} := \mathcal{Y} \cap Q$ is a $K3$ surface with 8 rational double points at $R \cap Q$. There exists a cover $q : Y \xrightarrow{\mathbb{P}^4} \mathcal{Y}$ ramified at the singular points of $Y$, induced by restriction from the map $q : \mathbb{P}^2 \times \mathbb{P}^2 \to \Sigma$. Clearly
q : Y → Y is a Nikulin covering, and a hyperplane section in \( |\mathcal{O}_Y(1)| \) induces a Prym curve \([C, \eta] \in \mathcal{R}_4\) having general moduli. Moreover we have a birational isomorphism

\[
\mathcal{F}_3^{\mathfrak{pr}} \cong \mathbf{P} \left( H^0(\mathcal{O}_{\mathbf{P}^4}(2)) \right)^{ss} // SL_2,
\]

where \(PGL_2 = \text{Aut}(R) \subset PGL_5\). An immediate consequence is that \(\mathcal{F}_3^{\mathfrak{pr}}\) is unirational.

2.4. Nikulin surfaces of genus 3. We prove that \( \chi_3 : \mathcal{P}_3^{\mathfrak{pr}} \to \mathcal{R}_3\) is dominant and fix a complete intersection of 3 quadrics \( Y \subset \mathbf{P}^5\) invariant with respect to an involution fixing a line \( L \subset \mathbf{P}^5\) and a 3-dimensional linear subspace \( \Lambda \subset \mathbf{P}^5\). The projection \( \pi_L : \mathbf{P}^5 \dashrightarrow \Lambda\) induces a quartic \( \bar{Y} := \pi_L(Y)\) with 8 nodes, which is a Nikulin surface. We check that a general Prym curve \([C, \eta] \in \mathcal{R}_3\) corresponding to an étale cover \( f : \bar{C} \to C\) embeds in such a surface.

Indeed, the canonical model \(\bar{C} \subset \mathbf{P}^4\) is a complete intersection of 3 quadrics. Fixing projective coordinates on \( \mathbf{P}^4\), we can assume that the involution \( \iota : \bar{C} \to \bar{C}\) is induced by the projective involution \([x : y : u : v : t] \mapsto [-x : -y : u : v : t]\). Note that the \( \iota\) anti-invariant quadratic forms are vectors \( q = ax + by \), where \( a, b \) are linear forms in \( u, v, t.\) Since \(\bar{C}\) is complete intersection of 3 quadrics, no non-zero quadric \( q = ax + by \) vanishes on \(\bar{C}\), for not, \(\bar{C}\) would intersect the plane \([x = y = 0]\) and then \(\iota\) would have fixed points. Thus \(\iota\) acts as the identity on the space \( H^0(\mathbf{P}^4, \mathcal{I}_{\bar{C}/\mathbf{P}^4}(2))\). Hence it follows \(\bar{C} = \{a_1 + b_1 = a_2 + b_2 = a_3 + b_3 = 0\}\), where \( a_i, b_i \) are quadratic forms in \( x, y \) and \( u, v, t.\) Passing to \(\mathbf{P}^5\) by adding one coordinate \( h\), we can choose quadratic forms \( a_i + b_i + hl_i \), where \( l_i \) is a general linear form in \( h, u, v, t.\) Consider the surface \( Y \subset \mathbf{P}^5\) defined by the latter 3 equations. Then \([x : y : h : u : v : t] \leftrightarrow [-x : -y : h : u : v : t]\) induces a Nikulin involution on \( Y\). Let \( \pi_L : Y \to \mathbf{P}^3\) be the projection of center \( L = \{h = u = v = t = 0\}\). Then \(\bar{Y} := \pi_L(Y)\) is a quartic Nikulin surface and \( \bar{C} = \pi_L(\bar{C})\) is a plane section of it.

2.5. Nikulin surfaces of genus 5. To describe the morphism \( \chi_5 : \mathcal{P}_5^{\mathfrak{pr}} \to \mathcal{R}_5\) more geometrically, we use the set-up introduced in Subsection 2.2. If \([C, \eta] \in \mathcal{R}_5\) is general, then \( \dim V^2(C, \eta) = 1\), the \( \iota\)-invariant Petri map \( \mu_0^-(L)\) is injective, \( \mu_0^+(L)\) surjective, thus \( \dim(\text{Coker} \mu_0(L)) = 1\). We consider the hyperplane

\[
\mathbf{P}_C^7 := \mathbf{P}(\text{Im}(\mu_0(L)) \subset \mathbf{P}(H^0(L)^\vee \otimes H^0(L)^\vee)
\]

and also set \( \mathbf{P}_+^4 := \mathbf{P}(H^0(K_C)^\vee) \subset \mathbf{P}^5\). Then we further denote

\[
\mathcal{T} := (\mathbf{P}^2 \times \mathbf{P}^2) \cap \mathbf{P}_C^7, \quad \text{and} \quad T := \Sigma \cap \mathbf{P}_+^4.
\]

Note that \(\mathcal{T}\) is a degree 6 threefold in \(\mathbf{P}_C^7\). Since the hyperplane \(\mathbf{P}_C^7\) is \( \iota\)-invariant, it follows \(\mathcal{T}\) is also endowed with the involution \(\iota_{\mathcal{T}} \in \text{Aut}(\mathcal{T})\) such that \(\text{Fix}(\iota_{\mathcal{T}}) = \Delta \cap \mathcal{T}\) is a rational quartic curve in \(\mathbf{P}_+^4\). Furthermore \( T \subset \mathbf{P}_+^4 \) is the secant variety of \( R\).

Proposition 2.4. For a general point \([C, \eta, L] \in \mathcal{R}_5^2\) the following statements hold:

(i) The threefold \( \mathcal{T} \subset \mathbf{P}^2 \times \mathbf{P}^2\) is smooth, while \(T \subset \mathbf{P}_+^4\) is singular precisely along \( R.\)

(ii) \( h^0(\mathcal{T}, \mathcal{I}_{\mathcal{C}/\mathcal{T}}(2)) = 3.\) Moreover \( H^i(\mathcal{T}, \mathcal{I}_{\mathcal{C}/\mathcal{T}}(2)) = 0\) for \( i = 1, 2.\)

(iii) Every quadratic section in the linear system \( |\mathcal{I}_{\mathcal{C}/\mathcal{T}}(2)|\) is \( \iota\)-invariant, that is,

\[
H^0(\mathcal{T}, \mathcal{I}_{\mathcal{C}/\mathcal{T}}(2)) = q^* H^0(\mathcal{T}, \mathcal{I}_{\mathcal{C}/\mathcal{T}}(2)).
\]
(iv) A general quadratic section \( Y \in |\mathcal{I}_{\tilde{C}/\tilde{T}}(2)| \) is a smooth K3 surface endowed with an involution \( \nu_Y \) with fixed points precisely at the 8 points in the intersection \( R \cap Y \).

**Proof.** We take cohomology in the following exact sequence
\[
0 \to \mathcal{I}_{\tilde{C}/\mathbb{P}^2 \times \mathbb{P}^2}(2) \to \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(2) \to K_{\tilde{C}}^\otimes 2 \to 0,
\]
to note that \( h^0(\mathcal{I}_{\tilde{C}/\tilde{T}}(2)) = 3(\iff h^1(\mathcal{I}_{\tilde{C}/\mathbb{P}^2 \times \mathbb{P}^2}(2)) = 0) \), if and only if the composed map
\[
\text{Sym}^2 H^0(\tilde{C}, L) \otimes \text{Sym}^2 H^0(\tilde{C}, \nu^* L) \to H^0(\tilde{C}, L^\otimes 2) \otimes H^0(\tilde{C}, \nu^*(L^\otimes 2)) \to H^0(\tilde{C}, K_{\tilde{C}}^\otimes 2)
\]
is surjective. This is an open condition and a triple \((\tilde{C} \overset{\iota}{\to} C, L) \in \mathcal{R}_5 \) satisfying it, and for which moreover \( \tilde{T} \subset \mathbb{P}^2 \times \mathbb{P}^2 \) is smooth, has been constructed in [V2] Section 4. Finally, from the exact sequence
\[
0 \to \mathcal{I}_T/\mathbb{P}^4(2) \to \mathcal{I}_C/\mathbb{P}^4(2) \to \mathcal{I}_{C/T}(2) \to 0,
\]
we compute that \( h^0(T, \mathcal{I}_{C/T}(2)) = 3 \), therefore \( q^* : H^0(T, \mathcal{I}_{C/T}(2)) \to H^0(\tilde{T}, \mathcal{I}_{\tilde{C}/\tilde{T}}(2)) \) is an isomorphism, based on dimension count. Part (iv) is a consequence of (i)-(iii). Assume that \( \tilde{Y} = T \cap Q \), where \( Q \in H^0(\mathcal{I}_{C/\mathbb{P}^4}(2)) \). Then \( Y = \tilde{T} \cap q^*(Q) \) and \( \tilde{Y} \) is the quotient of \( Y \) by the involution \( \nu_Y \) obtained by restriction from \( \iota \in \text{Aut}(\mathbb{P}^2 \times \mathbb{P}^2) \). It follows that the covering \( q : Y \to \tilde{Y} \) is a Nikulin surface such that \( C \subset Y \subset \mathbb{P}^1 \). To conclude, we must check that for a general choice of \( Y \in |\mathcal{I}_{\tilde{C}/\tilde{T}}(2)| \), the point \([Y, \nu_Y]\) gives rise to an element of \( \mathcal{F}_5^\eta \), that is, using the notation of diagram (1), that \( \text{Pic}(S) = \Lambda_5 \). Proposition 2.7 from [vGS] picks out two possibilities for \( \text{Pic}(Y) \) (or equivalently for \( \text{Pic}(S) \)), and we must check that \( \mathbb{Z} \cdot \mathcal{O}_Y(\tilde{C}) \oplus E_8(-2) \) has index 2 in \( \text{Pic}(Y) \), see also [GS Corollary 2.2.]

This is achieved by finding the decomposition of \( H^0(\mathcal{O}_Y(\tilde{C})) \) into \( \nu_Y^* \)-eigenspaces. In the course of the proof of [V2] Proposition 5.2 an example of a smooth quadratic section \( Y \in |\mathcal{I}_{\tilde{C}/\tilde{T}}(2)| \) is constructed such that
\[
H^0(Y, \mathcal{O}_Y(\tilde{C}))^+ = q^* H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(\tilde{C})).
\]
In particular the \((+1)\)-eigenspace of \( H^0(Y, \mathcal{O}_Y(\tilde{C})) \) is 6-dimensional and invoking once more [vGS] Proposition 2.7, we conclude that \([Y, \nu_Y] \in \mathcal{F}_5^\eta \). \( \square \)

We close this subsection with an amusing result on a geometric divisor on \( \mathcal{R}_5 \). For a Prym curve \([C, \eta] \in \mathcal{R}_5 \) and \( L := K_C \otimes \eta \in W_3(C) \), we observe that the vector spaces entering the multiplication map \( \nu_3(L) : \text{Sym}^3 H^0(C, L) \to H^0(C, L^\otimes 3) \) have the same dimension. The condition that \( \nu_3(L) \) be not an isomorphism is divisorial in \( \mathcal{R}_5 \). We have not been able to find a direct proof of the following equality of cycles on \( \mathcal{R}_5 \), even though one inclusion is straightforward:

**Theorem 2.5.** Let \([C, \eta] \in \mathcal{R}_5 \) be a Prym curve such that the Prym-canonical line bundle \( K_C \otimes \eta \) is very ample. Then \( \phi_{K_C \otimes \eta} : C \to \mathbb{P}^4 \) lies on a cubic surface if and only if \( C \) is trigonal.

**Proof.** Let \( \mathcal{D}_1 \) be the locus of Prym curves whose Prym-canonical model lies on a cubic
\[
\mathcal{D}_1 := \{[C, \eta] \in \overline{\mathcal{R}_5} : \nu_3(\omega_C \otimes \eta) : \text{Sym}^3 H^0(C, \omega_C \otimes \eta) \overset{\mathbb{Z}}{\to} H^0(C, \omega_C^\otimes 3 \otimes \eta^\otimes 3)\},
\]

We are grateful to the referee for raising this point that we have initially overlooked.
and $\mathcal{D}_2$ the closure inside $\overline{\mathcal{R}}_5$ of the divisor $\{ [C, \eta] : C \in \mathcal{C}_2 - \mathcal{C}_3 \}$ of smooth Prym curves for which $L := K_C \otimes \eta \in W^3_{12}(C)$ is not very ample. Obviously, $\mathcal{D}_1 - \mathcal{D}_2 \geq 0$, for if $L$ is not very ample, then the multiplication map $\nu_3(L) : \text{Sym}^3 H^0(C, L) \to H^0(C, L^{\otimes 3})$ cannot be an isomorphism. The class of $\mathcal{D}_2$ can be read off [FL] Theorem 5.2:

$$\mathcal{D}_2 \equiv 14\lambda - 2(\delta_0' + \delta_0'') - \frac{5}{2}\delta_0^{\text{ram}} - \cdots \in \operatorname{Pic}(\overline{\mathcal{R}}_5).$$

For $i \geq 1$, let $E_i$ be the vector bundle over $\overline{\mathcal{R}}_5$ with fibre $E_i[C, \eta] = H^0(C, \omega_C^{\otimes i} \otimes \eta^{\otimes i})$ for every $[C, \eta] \in \overline{\mathcal{R}}_5$. One has the following formulas from [FL] Proposition 1.7:

$$c_1(E_i) = \left( \frac{i}{2} \right)(12\lambda - \delta_0' - \delta_0'' - 2\delta_0^{\text{ram}}) + \lambda - \frac{i^2}{4}\delta_0^{\text{ram}} \in \operatorname{Pic}(\overline{\mathcal{R}}_5).$$

As a consequence, $\mathcal{D}_1 \equiv c_1(E_3) - c_1(\text{Sym}^3 E_1) \equiv 37\lambda - 3(\delta_0' + \delta_0'') - \frac{33}{4}\delta_0^{\text{ram}} - \cdots \in \operatorname{Pic}(\overline{\mathcal{R}}_5)$, therefore $\mathcal{D}_1 - \mathcal{D}_2 \equiv 8\lambda - (\delta_0' + \delta_0'') - 2\delta_0^{\text{ram}} - \cdots \equiv \pi^*(8\lambda - \delta_0 - \cdots) \geq 0$, where the terms left out are combinations of boundary divisors $\pi^*(\delta_i)$ with $i \geq 1$, corresponding to reducible curves. The only effective divisors $D \equiv a\lambda - b_0\delta_0 - b_1\delta_1 - b_2\delta_2$ on $\overline{\mathcal{M}}_5$ such that $\frac{a}{b_0} \leq 8$ and satisfying $\Delta_i \not\subseteq \text{supp}(D)$ for $i = 1, 2$, are multiples of the trigonal locus $\overline{\mathcal{M}}_{5,3}^1$ (the proof is identical to that of Proposition 5.1). This proves that if $[C, \eta] \in \mathcal{D}_1 - \mathcal{D}_2$, with $C$ being a smooth curve, then necessarily $[C] \in \mathcal{M}_{5,3}^1$, which finishes the proof. \square

3. A SINGULAR QUADRATIC COMPLEX AND A BIRATIONAL MODEL FOR $\mathcal{F}_5^{\text{PT}}$

Let us set $V := \mathbb{C}^{n+1}$ and denote by $G := G(2, V) \subset \mathbb{P}(\wedge^2 V)$ the Grassmannian of lines in $\mathbb{P}(V)$. We fix once and for all a smooth quadric $Q \subset \mathbb{P}(V)$. The projective tangent bundle $\mathbb{P}_Q := \mathbb{P}(T_Q)$ can be realized as the incidence correspondence

$$\mathbb{P}_Q = \{ (x, \ell) \in Q \times G : x \in \ell \subset \mathbb{P}(T_x Q) \}.$$

For each point $x \in Q$, the fibre $\mathbb{P}_Q(x)$ is the space of lines tangent to $Q$ at $x$. We introduce the projections $p : \mathbb{P}_Q \to G$ and $q : \mathbb{P}_Q \to Q$, then set

$$W_Q := p(\mathbb{P}_Q) = \{ \ell \in G : \ell \text{ is tangent to the quadric } Q \}.$$

Note that $W_Q$ contains the Hilbert scheme of lines in $Q$, which we denote by $V_Q \subset W_Q$. It is well-known that $V_Q$ is smooth, irreducible and $\dim(V_Q) = 2n - 5$. The restriction $p|_{\nu^*Q - V_Q}$ is an isomorphism and $E_Q := p^{-1}(V_Q) \subset \mathbb{P}_Q$ is the exceptional divisor of $p$.

**Proposition 3.1.** The variety $W_Q$ is a quadratic complex of lines in $G$. Its singular locus is equal to $V_Q$ and each point of $V_Q$ is an ordinary double point of $W_Q$.

**Proof.** Let $Q : V \to \mathbb{C}$ be the quadratic form whose zero locus is the quadric hypersurface also denoted by $Q \subset \mathbb{P}(V)$, and $\tilde{Q} : V \times V \to \mathbb{C}$ the associated bilinear map. We define the bilinear map $\nu_2(\tilde{Q}) : \wedge^2 V \times \wedge^2 V \to \mathbb{C}$ by the formula

$$\nu_2(\tilde{Q})(u \wedge v, s \wedge t) := \tilde{Q}(u, s)\tilde{Q}(v, t) - \tilde{Q}(u, v)\tilde{Q}(s, t)$$

for $u, v, s, t \in V$, and denote by $\nu_2(Q) : \wedge^2 V \to \mathbb{C}$ the induced quadratic form.

For fixed points $x = [u] \in Q$ and $y = [v] \in \mathbb{P}(V)$, we observe that the line $\ell = \langle x, y \rangle$ is tangent to $Q$ if and only if $Q(u, v) = 0 \Leftrightarrow \nu_2(Q)(u \wedge v) = 0$. Therefore $W_Q = G \cap \nu_2(Q)$ is a quadratic line complex in $G$, being the vanishing locus of $\nu_2(Q)$.

Keeping the same notation, a point $\ell = [u \wedge v] \in W_Q$ is a singular point, if and only if the linear form $\nu_2(\tilde{Q})(u \wedge v, -)$ vanishes along $\mathbb{P}(T_\ell G)$. Since $\mathbb{P}(T_\ell G)$ is spanned by
the Schubert cycle \( \{ m \in G : m \cap \ell \neq \emptyset \} \), any tangent vector in \( T_\ell(G) \) has a representative of the form \( u \wedge a - v \wedge b \), where \( a, b \in V \). We obtain that \( [u \wedge v] \in \text{Sing}(W_Q) \) if and only if \( Q(v, v) = 0 \), that is, \( \ell = [u \wedge v] \in V_Q \). Since \( W_Q \) is a quadratic complex, each point \( \ell \in V_Q \) has multiplicity 2.

The map \( p : P_Q \to W_Q \) appears as a desingularization of the quadratic complex \( W_Q \). We shall compute the class of the exceptional divisor \( E_Q \) of \( P_Q \). Let \( H := p^*(O_G(1)) \) be the class of the family of tangent lines to \( Q \) intersecting a fixed \( (n - 2) \)-plane in \( P(V) \) and \( B := q^*(O_1(1)) \in \text{Pic}(P_Q) \). Furthermore, we consider the class \( h \in N_S(1)(P_Q) \) of the pencil of tangent lines to \( Q \) with center a given point \( x \in Q \). It is clear that

\[
h \cdot H = 1 \quad \text{and} \quad h \cdot B = 0.
\]

If \( \ell \in V_Q \) is a fixed line, let \( s \in NS(1)(P_Q) \) be the class of the family \( \{(x, \ell) : x \in \ell \} \). Then

\[
s \cdot H = 0 \quad \text{and} \quad s \cdot B = 1.\]

**Lemma 3.2.** The linear equivalence \( E_Q \equiv 2H - 2B \) in \( \text{Pic}(P_Q) \) holds. In particular, the class \( E_Q \) is divisible by 2 and it is the branch divisor of a double cover

\[
f : \tilde{P}_Q \to P_Q.
\]

**Proof.** To compute the class of \( E_Q \) it suffices to compute \( h \cdot E_Q \) and \( s \cdot E_Q \). First we note that \( h \cdot E_Q = 2 \). Indeed a pencil of tangent lines to \( Q \) through a fixed point \( x \in Q \) has two elements which are in \( Q \). Finally, recalling that \( V_Q = \text{Sing}(W_Q) \) consists of ordinary double points, we obtain that \( s \cdot E_Q = -2 \), since \( p^{-1}(\ell) \) is a conic inside \( P(N_{V_Q/G}(\ell)) \). \( \square \)

3.1. A birational model for \( F_{6}^{\mathfrak{r}} \). Let us now specialize to the case \( n = 4 \), that is,

\[
Q \subset P^4, \quad G = G(2, 5) \subset P^9 \text{ and } \dim(W_Q) = 5.
\]

The class of \( V_Q \) equals \( 4\sigma_{2, 1} \in H^0(G, \mathbb{Z}) \) see [HP] p. 366, therefore \( \deg(V_Q) = 4\sigma_{2, 1} \cdot \sigma_{1}^3 = 8 \). This can also be seen by recalling that \( V_Q \) is isomorphic to the Veronese 3-fold \( \nu_2(P^3) \subset P^9 \).

The double covering \( f : \tilde{P}_Q \to P_Q \) constructed above has a transparent projective interpretation. For \( (x, \ell) \in P_Q \), we denote by \( \Pi_\ell \subset G(3, V) \) the polar space of \( \ell \) defined as the base locus of the pencil of polar hyperplanes \( \{ z \in P(V) : Q(y, z) = 0 \} \) \( y \in \ell \). Clearly \( x \in \Pi_\ell \subset P(T_xQ) \) and \( Q \cap \Pi_\ell \) is a conic of rank at most 2 in \( \Pi_\ell \). When \( \ell \in W_Q - V_Q \), the quadric has rank exactly 2 which corresponds to a pair of lines \( \ell_1 + \ell_2 \) with \( \ell_1, \ell_2 \in V_Q \). The double cover is induced by the map from the parameter space of the lines themselves.

In the next statement we shall keep in mind the notation of diagram (1):

**Proposition 3.3.** A general codimension 3 linear section \( \tilde{Y} := \Lambda \cap W_Q \) of the quadratic complex \( W_Q \) where \( \Lambda \in G(7, \wedge^2 V) \), is a 8-nodal \( K3 \) surface with desingularization

\[
p : S := p^{-1}(\tilde{Y}) \to \tilde{Y}.
\]

The triple \( [S, O_S(H - B), O_S(H)] \in F_{6}^{\mathfrak{r}} \) is a Nikulin surface of genus 6 and the induced double cover is the restriction \( f : \tilde{S} := f^{-1}(S) \to S \).

**Proof.** We fix a general 6-plane \( \Lambda \in G(7, \wedge^2 V) \). Since \( K_{W_Q} = O_{W_Q}(-3H) \), by adjunction we obtain that \( \tilde{Y} := \Lambda \cap W_Q \) is a \( K3 \) surface. From Bertini’s theorem, \( \tilde{Y} \) has ordinary double points at the 8 points of intersection \( \Lambda \cap V_Q \). General hyperplane sections of \( C \in \{ O_S(H) \} \), viewed as codimension 4 linear sections of \( W_Q \), are canonical curves of
genus 6, endowed with a line bundle of order 2 given by \( \mathcal{O}_C(H - B) \). The remaining statements are immediate. \( \square \)

It turns out that the general Nikulin surface of genus 6 arises in this way:

**Theorem 3.4.** Let \( V := \mathbb{C}^5 \) and \( Q \subset \mathbb{P}(V) \) be a smooth quadric. One has a dominant map

\[
\varphi : G(7, \wedge^2 V)_{\text{ss}} \to \mathcal{F}_6^{31},
\]

given by \( \varphi(\Lambda) := [S := p^{-1}(\Lambda \cap W_Q), \mathcal{O}_S(H - B), \mathcal{O}_S(H)] \).

**Proof.** Via the embedding \( \text{Aut}(Q) \subset \text{PGL}(V) \hookrightarrow \text{PGL}(\wedge^2 V) \), we observe that every automorphism of \( Q \) induces an automorphism of \( \mathbb{P}(\wedge^2 V) \) that fixes both \( W_Q \) and \( V_Q \). Since (i) the moduli space \( \mathcal{F}_6^{31} \) is irreducible and (ii) polarized Nikulin surfaces have finite automorphism groups, it suffices to observe that \( \dim G(7, \wedge^2 V)_{\text{ss}} / \text{Aut}(Q) = 21 - 10 = 11 \) and \( \dim(\mathcal{F}_6^{31}) = 11 \) as well. \( \square \)

**Corollary 3.5.** The Prym-Nikulin locus \( \mathcal{N}_6 \subset \mathcal{R}_6 \) is an irreducible unirational divisor, which is set-theoretically equal to the ramification locus of the Prym map \( \text{Pr} : \mathcal{R}_6 \to \mathcal{A}_5 \)

\[
\mathcal{U}_{6,0} = \{ [C, \eta] \in \mathcal{R}_6 : K_{0,2}(C, K_C \otimes \eta) \neq 0 \}.
\]

Furthermore, there exists a dominant rational map \( G(6, \wedge^2 V)_{\text{ss}} / \text{Aut}(Q) \to \mathcal{N}_6 \).

**Proof.** Just observe that \( \langle C \rangle = \mathbb{P}^5 \) and that this has codimension 4 in \( \mathbb{P}(\wedge^2 V) \), hence there is a \( \mathbb{P}^3 \) of Nikulin sections of \( W_Q \) containing \( C \). \( \square \)

The divisor \( \overline{\mathcal{N}}_{10} \subset \overline{\mathcal{M}}_{10} \) of sections of K3 surfaces is known to be an extremal point of the effective cone \( \text{Eff}(\mathcal{M}_{10}) \). An analogous result holds for the closure of \( \mathcal{N}_6 \):

**Proposition 3.6.** The Prym-Nikulin divisor \( \overline{\mathcal{N}}_6 \) is extremal in the effective cone \( \text{Eff}(\overline{\mathcal{R}}_6) \):

**Proof.** It follows from [FL] Theorem 0.6 that \( \mathcal{N}_6 \equiv 7\lambda - 3\delta_0^{\text{ram}} - (\delta_0' + \delta_0'') - \cdots \in \text{Pic}(\overline{\mathcal{R}}_6) \). The divisor \( \overline{\mathcal{N}}_6 \) is filled-up by the rational curves \( \Xi_6 \subset \overline{\mathcal{R}}_6 \) constructed in the course of proving Theorem 1.4. We compute that \( \Xi_6 \cdot \overline{\mathcal{N}}_6 = -1 \), which completes the proof. \( \square \)

4. Spin curves and the divisor \( \overline{\mathcal{G}}_{\text{null}} \)

We turn our attention to the moduli space of spin curves and begin by setting notation and terminology. If \( \mathcal{M} \) is a Deligne-Mumford stack, we denote by \( \mathcal{M} \) its associated coarse moduli space. A Q-Weil divisor \( D \) on a normal \( \mathbb{Q} \)-factorial projective variety \( X \) is said to be movable if \( \text{codim}(\bigcap_{m \in \mathbb{N}} \text{Bs}[mD], X) \geq 2 \), where the intersection is taken over all \( m \) which are sufficiently large and divisible. We say that \( D \) is rigid if \( [mD] = \{ mD \} \), for all \( m \geq 1 \) such that \( mD \) is an integral Cartier divisor. The Kodaira-Iitaka dimension of a divisor \( D \) on \( X \) is denoted by \( \kappa(X, D) \).

If \( D = m_1D_1 + \cdots + m_sD_s \) is an effective \( \mathbb{Q} \)-divisor on \( X \), with irreducible components \( D_i \subset X \) and \( m_i > 0 \) for \( i = 1, \ldots, s \), a (trivial) way of showing that \( \kappa(X, D) = 0 \) is by exhibiting for each \( 1 \leq i \leq s \), an irreducible curve \( \Gamma_i \subset X \) passing through a general point of \( D_i \), such that \( \Gamma_i \cdot D_i < 0 \) and \( \Gamma_i \cdot D_j = 0 \) for \( i \neq j \).

We recall basic facts about the moduli space \( \overline{\mathcal{F}}_g \) and refer to [Cor], [F] for details.
Definition 4.1. An even spin curve of genus \(g\) consists of a triple \((X, \eta, \beta)\), where \(X\) is a genus \(g\) quasi-stable curve, \(\eta \in \text{Pic}^{g-1}(X)\) is a line bundle of degree \(g - 1\) such that \(\eta_E = \mathcal{O}_E(1)\) for every rational component \(E \subset X\) with \(|E \cap (X - E)| = 2\) (such a component is called exceptional), \(h^0(X, \eta) \equiv 0 \pmod{2}\), and \(\beta : \eta^{\otimes 2} \to \omega_X\) is a morphism of sheaves which is generically non-zero along each non-exceptional component of \(X\).

Even spin curves of genus \(g\) form a smooth Deligne-Mumford stack \(\pi : \mathcal{S}^+_g \to \overline{M}_g\). At the level of coarse moduli schemes, the morphism \(\pi : \mathcal{S}^+_g \to \overline{M}_g\) is the stabilization map \(\pi(\{[X, \eta, \beta]\}) := \text{st}(X)\), which associates to a quasi-stable curve its stable model.

We explain the boundary structure of \(\mathcal{S}^+_g\): If \([X, \eta, \beta] \in \pi^{-1}(\{C \cup y D\})\), where \([C, y] \in \mathcal{M}_{g-i,1}, [D, y] \in \mathcal{M}_{g-i,1}\) and \(1 \leq i \leq \lfloor g/2 \rfloor\), then necessarily \(X = C \cup_{y_1} E \cup_{y_2} D\), where \(E\) is an exceptional component such that \(C \cap E = \{y_1\}\) and \(D \cap E = \{y_2\}\). Moreover \(\eta = (\eta_C, \eta_D, \eta_E = \mathcal{O}_E(1)) \in \text{Pic}^{g-1}(X)\), where \(\eta_C^{\otimes 2} = K_C, \eta_D^{\otimes 2} = K_D\). The condition \(h^0(X, \eta) \equiv 0 \pmod{2}\) implies that the theta-characteristics \(\eta_C\) and \(\eta_D\) have the same parity.

We denote by \(A_i \subset \mathcal{S}^+_g\) the closure of the locus corresponding to pairs
\[
([C, y], D, y) \in S^+_i \times S^+_{g-i,1}
\]
and by \(B_i \subset \mathcal{S}^+_g\) the closure of the locus corresponding to pairs
\[
([C, y], D, y) \in S^-_i \times S^-_{g-i,1}.
\]

We set \(\alpha_i := [A_i] \in \text{Pic}(\mathcal{S}^+_g), \beta_i := [B_i] \in \text{Pic}(\mathcal{S}^+_g)\), and then one has the relation
\[
\pi^*(\delta_i) = \alpha_i + \beta_i.
\]

We recall the description of the ramification divisor of the covering \(\pi : \mathcal{S}^+_g \to \overline{M}_g\).

For a point \([X, \eta, \beta] \in \mathcal{S}^+_g\) corresponding to a stable model \(\text{st}(X) = C_{yy} := C/y \sim q\), with \([C, y, q] \in \mathcal{M}_{g-1,2}\), there are two possibilities depending on whether \(X\) possesses an exceptional component or not. If \(X = C_{yy}\) (i.e. \(X\) has no exceptional component) and \(\eta_C := \nu^*(\eta)\) where \(\nu : C \to X\) denotes the normalization map, then \(\eta_C^{\otimes 2} = K_C\). For each choice of \(\eta_C \in \text{Pic}^{g-1}(C)\) as above, there is precisely one choice of gluing the fibres \(\eta_C(y)\) and \(\eta_C(q)\) such that \(h^0(X, \eta) \equiv 0 \pmod{2}\). We denote by \(A_0\) the closure in \(\mathcal{S}^+_g\) of the locus of spin curves \([C_{yy}, \eta_C] = \sqrt{K_C(y + q)}\) as above.

If \(X = C \cup_{(y, q)} E\), where \(E\) is an exceptional component, then \(\eta_C := \eta \otimes \mathcal{O}_C\) is a theta-characteristic on \(C\). Since \(H^0(X, \omega) \cong H^0(C, \omega_C)\), it follows that \([C, \eta_C] \in S^+_g\). We denote by \(B_0 \subset \mathcal{S}^+_g\) the closure of the locus of spin curves
\[
[C \cup_{(y, q)} E, E \cong \mathbb{P}^1], \eta_C \in \sqrt{K_C}, \eta_E = \mathcal{O}_E(1)] \in S^+_g.
\]

If \(\alpha_0 := [A_0], \beta_0 := [B_0] \in \text{Pic}(\mathcal{S}^+_g)\), we have the relation, see \([\text{Cor}]:\)
\[
\pi^*(\delta_0) = \alpha_0 + 2\beta_0.
\]

In particular, \(B_0\) is the ramification divisor of \(\pi\). An important effective divisor on \(\mathcal{S}^+_g\) is the locus of vanishing theta-nulls
\[
\Theta_{\text{null}} := \{[C, \eta] \in S^+_g : H^0(C, \eta) \neq 0\}.
\]
The class of its compactification inside $\overline{S}_g^+$ is given by the formula, cf. [F]:

$$\overline{\Theta}_{\text{null}} \equiv \frac{1}{4} \lambda - \frac{1}{16} \alpha_0 - \frac{1}{2} \sum_{i=1}^{[g/2]} \beta_i \in \text{Pic}(\overline{S}_g^+) .$$

It is also useful to recall the formula for the canonical class of $\overline{S}_g^+$:

$$K_{\overline{S}_g^+} \equiv \pi^*(K_{\overline{M}_g}) + \beta_0 \equiv 13 \lambda - 2 \alpha_0 - 3 \beta_0 - 2 \sum_{i=1}^{[g/2]} (\alpha_i + \beta_i) - (\alpha_1 + \beta_1) .$$

An argument involving spin curves on certain singular canonical surfaces in $\mathbb{P}^6$, implies that for $g \leq 9$, the divisor $\overline{\Theta}_{\text{null}}$ is uniruled and a rigid point in the cone of effective divisors $\text{Eff}(\overline{S}_g^+)$:

**Theorem 4.2.** For $g \leq 9$ the divisor $\overline{\Theta}_{\text{null}} \subset \overline{S}_g^+$ is uniruled and rigid. Precisely, through a general point of $\overline{\Theta}_{\text{null}}$ there passes a rational curve $\Gamma \subset \overline{S}_g^+$ such that $\Gamma \cdot \overline{\Theta}_{\text{null}} < 0$. In particular, if $D$ is an effective divisor on $\overline{S}_g^+$ with $D \equiv n \overline{\Theta}_{\text{null}}$ for some $n \geq 1$, then $D = n \overline{\Theta}_{\text{null}}$.

**Proof.** A general point $[C, \eta_C] \in \overline{\Theta}_{\text{null}}$ corresponds to a canonical curve $C \hookrightarrow \mathbb{P}^{g-1}$ lying on a rank 3 quadric $Q \subset \mathbb{P}^{g-1}$ such that $C \cap \text{Sing}(Q) = \emptyset$. The pencil $\eta_C$ is recovered from the ruling of $Q$. We construct the pencil $\Gamma \subset \overline{S}_g^+$ by representing $C$ as a section of a nodal canonical surface $S \subset Q$ and noting that $\dim \mid O_S(C) \mid = 1$. The construction of $S$ depends on the genus and we describe the various cases separately.

**(i)** $7 \leq g \leq 9$. We choose $V \in G(7, H^0(C, K_C))$ such that if $\pi_V : \mathbb{P}^{g-1} \dashrightarrow \mathbb{P}(V^\vee)$ denotes the projection, then $\hat{Q} := \pi_V(Q)$ is a quadric of rank 3. Let $C' := \pi_V(C) \subset \mathbb{P}(V^\vee)$ be the projection of the canonical curve $C$. By counting dimensions we find that

$$\dim \left\{ \text{Sym}^2(V) \to H^0(C, K_C^{\otimes 2}) \right\} \geq 31 - 3g \geq 4,$$

that is, the embedded curve $C' \subset \mathbb{P}^6$ lies on at least 4 independent quadrics, namely the rank 3 quadric $\hat{Q}$ and $Q_1, Q_2, Q_3 \in |I_{C'/\mathbb{P}(V^\vee)}(2)|$. By choosing $V$ sufficiently general we make sure that $S := \hat{Q} \cap Q_1 \cap Q_2 \cap Q_3$ is a canonical surface in $\mathbb{P}(V^\vee)$ with 8 nodes corresponding to the intersection $\bigcap_{i=1}^3 Q_i \cap \text{Sing}(\hat{Q})$ (This transversality statement can also be checked with Macaulay by representing $C$ as a section of the corresponding Mukai variety). From the exact sequence on $S$,

$$0 \to O_S \to O_S(C) \to O_C(C) \to 0,$$

coupled with the adjunction formula $O_C(C) = K_C \otimes K_S^{\vee} = O_C$, as well as the fact $H^1(S, O_S) = 0$, it follows that $\dim |C| = 1$, that is, $C \subset S$ moves in its linear system. In particular, $\overline{\Theta}_{\text{null}}$ is a uniruled divisor for $g \leq 9$.

We determine the numerical parameters of the family $\Gamma \subset \overline{S}_g^+$ induced by varying $C \subset S$. Since $C^2 = 0$, the pencil $|C|$ is base point free and gives rise to a fibration $f : \overline{S} \to \mathbb{P}^1$, where $\overline{S} := \text{Bl}_8(S)$ is the blow-up of the nodes of $S$. This in turn induces a moduli map $m : \mathbb{P}^1 \to \overline{S}_g^+$ and $\Gamma =: m(\mathbb{P}^1)$. We have the formulas

$$\Gamma \cdot \lambda = m^*(\lambda) = \chi(S, O_S) + g - 1 = 8 + g - 1 = g + 7,$$
and
\[ \Gamma \cdot \alpha_0 + 2 \Gamma \cdot \beta_0 = m^*(\pi^*(\delta_0)) = m^*(\alpha_0) + 2m^*(\beta_0) = c_2(\tilde{S}) + 4(g - 1). \]
Noether’s formula gives that \( c_2(\tilde{S}) = 12\chi(\tilde{S}, \mathcal{O}_\tilde{S}) - K_\tilde{S}^2 = 12\chi(S, \mathcal{O}_S) - K_S^2 = 80 \), hence \( m^*(\alpha_0) + 2m^*(\beta_0) = 4g + 76 \). The singular fibres corresponding to spin curves lying in \( B_0 \) are those in the fibres over the blown-up nodes and all contribute with multiplicity 1, that is, \( \Gamma \cdot \beta_0 = 8 \) and then \( \Gamma \cdot \alpha_0 = 4g + 60 \). It follows that \( \Gamma \cdot \mathcal{C}_{\text{null}} = -2 < 0 \) (independent of \( g \)), which finishes the proof.

(ii) \( g = 5 \). In the case \( C \subset Q \subset \mathbb{P}^4 \) and we choose a general quartic \( X \in H^0(\mathbb{P}^4, \mathcal{I}_C/\mathcal{I}_C^2(4)) \) and set \( S := Q \cap X \). Then \( S \) is a canonical surface with nodes at the 4 points \( X \cap \text{Sing}(Q) \). As in the previous case \( \dim |C| = 1 \), and the numerical characters of the induced family \( \Gamma \subset \mathcal{S}_5^{\perp} \) can be readily computed:
\[ \Gamma \cdot \lambda = g + 5 = 10, \quad \Gamma \cdot \beta_0 = |\text{Sing}(S)| = 4, \quad \text{and} \quad \Gamma \cdot \alpha_0 = 4g + 52, \]
where the last equality is a consequence of Noether’s formula \( \Gamma \cdot (\alpha_0 + 2\beta_0) = 12\chi(S, \mathcal{O}_S) - K_S^2 + 4(g - 1) = 4g + 60 \). By direct calculation, we obtain once more that \( \Gamma \cdot \mathcal{C}_{\text{null}} = -2 \). The case \( g = 6 \) is similar, except that the canonical surface \( S \) is a \((2, 2, 3)\) complete intersection in \( \mathbb{P}^5 \), where one of the quadrics is the rank 3 quadric \( Q \).

(iii) \( g = 4 \). In this last case we proceed slightly differently and denote by \( S = \mathbb{F}_2 \) the blow-up of the vertex of a cone \( Q \subset \mathbb{P}^3 \) over a conic in \( \mathbb{P}^3 \) and write \( \text{Pic}(S) = \mathbb{Z} \cdot F + \mathbb{Z} \cdot C_0 \), where \( F^2 = 0, C_0^2 = -2 \) and \( C_0 \cdot F = 1 \). We choose a Lefschetz pencil of genus 4 curves in the linear system \( |3(C_0 + 2F)| \). By blowing-up the 18 = 9\((C_0 + 2F)^2\) base points, we obtain a fibration \( f: \tilde{S} := \mathbb{B}_{18}(S) \to \mathbb{P}^1 \) which induces a family of spin curves \( m: \mathbb{P}^1 \to \mathcal{S}_4^{\perp} \) given by \( m(t) := [f^{-1}(t), \mathcal{O}_{f^{-1}(t)}(F)] \). We have the formulas
\[ m^*(\lambda) = \chi(\tilde{S}, \mathcal{O}_\tilde{S}) + g - 1 = 4, \quad \text{and} \]
\[ m^*(\pi^*(\delta_0)) = m^*(\alpha_0) + 2m^*(\beta_0) = c_2(\tilde{S}) + 4(g - 1) = 34. \]
The singular fibres lying in \( B_0 \) correspond to curves in the Lefschetz pencil on \( Q \) passing through the vertex of the cone, that is, when \( f^{-1}(t_0) \) splits as \( C_0 + D \), where \( D \subset \tilde{S} \) is the residual curve. Since \( C_0 \cdot D = 2 \) and \( \mathcal{O}_{C_0}(F) = \mathcal{O}_{C_0}(1) \), it follows that \( m(t_0) \equiv B_0 \). One finds that \( m^*(\beta_0) = 1 \), hence \( m^*(\alpha_0) = 32 \) and \( m^*(\mathcal{C}_{\text{null}}) = -1 \). Since \( \Gamma := m(\mathbb{P}^1) \) fills-up the divisor \( \mathcal{C}_{\text{null}} \), we obtain that \( [\mathcal{C}_{\text{null}}] \in \text{Eff}(\mathcal{S}_4^{\perp}) \) is rigid.

5. Spin curves of genus 8

The moduli space \( \mathcal{M}_8 \) carries one Brill-Noether divisor, the locus of plane septics
\[ \mathcal{M}_{8,7}^2 := \{ [C] \in \mathcal{M}_8 : G^2_7(C) \neq \emptyset \}. \]
The locus \( \overline{\mathcal{M}}_{8,7}^2 \) is irreducible and for a known constant \( c_{8,7}^2 \in \mathbb{Z}_{>0} \), one has, cf. [EH1],
\[ b_{8,7} := \frac{1}{c_{8,7}^2} \overline{\mathcal{M}}_{8,7}^2 \equiv 22\lambda - 3\delta_0 - 14\delta_1 - 24\delta_2 - 30\delta_3 - 32\delta_4 \in \text{Pic}(\overline{\mathcal{M}}_8). \]
In particular, \( s(\overline{\mathcal{M}}_{8,7}^2) = 6 + 12/(g + 1) \) and this is the minimal slope of an effective divisor on \( \overline{\mathcal{M}}_8 \). The following fact is probably well-known:

**Proposition 5.1.** Through a general point of \( \overline{\mathcal{M}}_{8,7}^2 \) there passes a rational curve \( R \subset \overline{\mathcal{M}}_8 \) such that \( R \cdot \overline{\mathcal{M}}_{8,7}^2 < 0 \). In particular, the class \([\overline{\mathcal{M}}_{8,7}^2] \in \text{Eff}(\overline{\mathcal{M}}_8) \) is rigid.
Proof. One takes a Lefschetz pencil of nodal plane septic curves with 7 assigned nodes in general position (and 21 unassigned base points). After blowing up the 21 unassigned base points as well as the 7 nodes, we obtain a fibration \( f : S := \text{Bl}_{28}(\mathbb{P}^2) \rightarrow \mathbb{P}^1 \), and the corresponding moduli map \( m : \mathbb{P}^1 \rightarrow \overline{M}_8 \) is a covering curve for the irreducible divisor \( \overline{M}_{8,7}^2 \). The numerical invariants of this pencil are

\[
m^*(\lambda) = \chi(S, \mathcal{O}_S) + g - 1 = 8 \quad \text{and} \quad m^*(\delta_0) = c_2(S) + 4(g - 1) = 59,
\]

while \( m^*(\delta_i) = 0 \) for \( i = 1, \ldots, 4 \). We find \( m^*(\overline{M}_{8,7}^2) = c_2^2(S, 8 \cdot 22 - 3 \cdot 59) = -c_{8,7}^2 < 0 \).  

Using (5) we find the following explicit representative for the canonical class \( K_{\overline{S}^+_8}^\dagger \):

\[
K_{\overline{S}^+_8}^\dagger = \frac{1}{2} \pi^*(bn_8) + 8\Theta_{\text{null}} + \sum_{i=1}^4 (a_i \alpha_i + b_i \beta_i),
\]

where \( a_i, b_i > 0 \) for \( i = 1, \ldots, 4 \). The multiples of each irreducible component appearing in (6) are rigid divisors on \( \overline{S}^+_8 \), but in principle, their sum could still be a movable class. Assuming for a moment Proposition 0.9, we explain how this implies Theorem 0.1.  

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Proof of Theorem 0.1. The covering curve \( R \subset \Theta_{\text{null}} \) constructed in Proposition 0.9 satisfies \( R \cdot \Theta_{\text{null}} < 0 \) as well as \( R \cdot \pi^*(\overline{M}_{8,7}^2) = 0 \) and \( R \cdot \alpha_i = R \cdot \beta_i = 0 \) for \( i = 1, \ldots, 4 \). It follows from (6) that for each \( n \geq 1 \), one has an equality of linear series on \( \overline{S}^+_8 \):

\[
|nK_{\overline{S}^+_8}| = 8n\Theta_{\text{null}} + |n(K_{\overline{S}^+_8} - 8\Theta_{\text{null}})|.
\]

Furthermore, from (6) one finds constants \( a_i' > 0 \) for \( i = 1, \ldots, 4 \), such that if

\[
D \equiv 22\lambda - 3\delta_0 - \sum_{i=1}^4 a_i' \delta_i \in \text{Pic}(\overline{M}_8),
\]

then the difference \( \frac{1}{2} \pi^*(D) - (K_{\overline{S}^+_8} - 8\Theta_{\text{null}}) \) is still effective on \( \overline{S}^+_8 \). We can thus write

\[
0 \leq \kappa(\overline{S}^+_8, K_{\overline{S}^+_8} - 8\Theta_{\text{null}}) \leq \kappa(\overline{S}^+_8, \frac{1}{2} \pi^*(D)) = \kappa(\overline{S}^+_8, \pi^*(D)).
\]

We claim that \( \kappa(\overline{S}^+_8, \pi^*(D)) = 0 \). Indeed, in the course of the proof of Proposition 5.1 we have constructed a covering family \( B \subset \overline{M}_8 \) for the divisor \( \overline{M}_{8,7}^2 \) such that \( B \cdot \overline{M}_{8,7}^2 < 0 \) and \( B \cdot \delta_i = 0 \) for \( i = 1, \ldots, 4 \). We lift \( B \) to a family \( R \subset \overline{S}^+_8 \) of spin curves by taking

\[
\tilde{B} := B \times_{\overline{M}_8} \overline{S}^+_8 = \{ [C_t, \eta_{C_t}] \in \overline{S}^+_8 : [C_t] \in B, \eta_{C_t} \in \text{Pic}^2(C_t), t \in \mathbb{P}^1 \} \subset \overline{S}^+_8.
\]

One notes that \( \tilde{B} \) is disjoint from the boundary divisors \( A_i, B_i \subset \overline{S}^+_8 \) for \( i = 1, \ldots, 4 \), hence \( \tilde{B} \cdot \pi^*(D) = 2^{d-1}(2^d + 1)(B \cdot \overline{M}_{8,7}^2)_{\overline{M}_8} < 0 \). Thus we write that

\[
\kappa(\overline{S}^+_8, \pi^*(D)) = \kappa(\overline{S}^+_8, \pi^*(D - (22\lambda - 3\delta_0))) = \kappa(\overline{S}^+_8, \sum_{i=1}^4 a_i'(\alpha_i + \beta_i)) = 0.
\]
6. A FAMILY OF SPIN CURVES $R \subset \overline{S}_8$ WITH $R \cdot \pi^*(\overline{M}_{8,7}^2) = 0$ AND $R \cdot \overline{C}_{\text{null}} < 0$

The aim of this section is to prove Proposition 0.9 which is the key ingredient in the proof of Theorem 0.1. We begin by reviewing facts about the geometry of $\overline{M}_8$, in particular the construction of general curves of genus 8 as complete intersections in a rational homogeneous variety, see [M2].

We fix $V := \mathbb{C}^6$ and denote by $G := G(2, V) \subset \mathbb{P}(\Lambda^2 V)$ the Grassmannian of lines. Noting that smooth codimension 7 linear sections of $G$ are canonical curves of genus 8, one is led to consider the Mukai model of the moduli space of curves of genus 8

$$\mathcal{M}_8 := G(8, \Lambda^2 V)^\text{st}/\!/SL(V).$$

There is a birational map $f : \overline{M}_8 \twoheadrightarrow \mathcal{M}_8$, whose inverse is given by $f^{-1}(H) := G \cap H,$ for a general $H \in G(8, \Lambda^2 V)$. The map $f$ is constructed as follows: Starting with a curve $[C] \in M_8 - M_{8,7}^2$, one notes that $C$ has a finite number of pencils $g_1$. We choose $A \in W_3^2(C)$ and set $L := K_C \otimes A^\vee \in W_3^2(C)$. There exists a unique rank 2 vector bundle $E \in SU_C(2, K_C)$ (independent of $A$!), sitting in an extension

$$0 \to A \to E \to L \to 0,$$

such that $h^0(E) = h^0(A) + h^0(L) = 6$. Since $E$ is globally generated, we define the map

$$\phi_E : C \to G(2, H^0(E)^\vee), \quad \phi_E(p) := E(p)^\vee (\to H^0(E)^\vee),$$

and let $\varphi : G(2, H^0(E)^\vee) \to \mathbb{P}(\Lambda^2 H^0(E)^\vee)$ be the Plücker embedding. The determinant map $u : \Lambda^2 H^0(E) \to H^0(K_C)$ is surjective and we can view $H^0(K_C)^\vee \in G(8, \Lambda^2 H^0(E)^\vee)$, see [M2] Theorem C. We set

$$f([C]) := H^0(K_C)^\vee \mod SL(H^0(E)^\vee) \in \mathcal{M}_8,$$

that is, we assign to $C$ its linear span $\langle C \rangle$ under the Plücker map $\varphi \circ \phi_E : C \to \mathbb{P}(\Lambda^2 H^0(E)^\vee)$.

Even though this is not strictly needed for our proof, it follows from [M2] that the exceptional divisors of $f$ are the Brill-Noether locus $\overline{M}_{8,7}^2$ and the boundary divisors $\Delta_1, \ldots, \Delta_4$. The map $f^{-1}$ does not contract any divisors.

Inside the moduli space $\mathcal{F}_8$ of polarized K3 surfaces $[S, h]$ of degree $h^2 = 14$, we consider the following Noether-Lefschetz divisor

$$\mathfrak{N}_L := \{[S, \mathcal{O}_S(C_1 + C_2)] \in \mathcal{F}_8 : \text{Pic}(S) \supset \mathbb{Z} \cdot C_1 \oplus \mathbb{Z} \cdot C_2, \quad C_1^2 = C_2^2 = 0, \quad C_1 \cdot C_2 = 7\},$$

of doubly-elliptic K3 surfaces. For a general element $[S, \mathcal{O}_S(C)] \in \mathfrak{N}_L$, the embedded surface $\phi_{\mathcal{O}_S(C)} : S \hookrightarrow \mathbb{P}^5$ lies on a rank 4 quadric whose rulings induce the elliptic pencils $|C_1|$ and $|C_2|$ on $S$.

Let $\mathcal{U} \to \mathfrak{N}_L$ be the space classifying pairs $([S, \mathcal{O}_S(C_1 + C_2)], C \subset S)$, where $C \in |H^0(S, \mathcal{O}_S(C_1)) \otimes H^0(S, \mathcal{O}_S(C_2))| \subset |H^0(S, \mathcal{O}_S(C_1 + C_2))|.$

An element of $\mathcal{U}$ corresponds to a hyperplane section $C \subset S \subset \mathbb{P}^5$ of a doubly-elliptic K3 surface, such that the intersection of $\langle C \rangle$ with the rank 4 quadric induced by the elliptic pencils, has rank at most 3. There exists a rational map

$$q : \mathcal{U} \dashrightarrow \overline{\mathcal{C}}_{\text{null}}, \quad q([S, \mathcal{O}_S(C_1 + C_2)], C) := [C, \mathcal{O}_C(C_1) = \mathcal{O}_C(C_2)].$$

Since $\mathcal{U}$ is birational to a $\mathbb{P}^3$-bundle over an open subvariety of $\mathfrak{N}_L$, we obtain that $\mathcal{U}$ is irreducible and $\dim(\mathcal{U}) = 21 (= 3 + \dim(\mathfrak{N}_L))$. We shall show that the morphism $q$ is dominant (see Corollary 8.3) and begin with some preparations.
We fix a general point \([C, \eta] \in \overline{\Theta}_{\text{null}} \subset \overline{\Theta}^8_5\), with \(\eta\) a vanishing theta-null. Then
\[
C \subset Q \subset P^7 := P(H^0(C, K_C)^\vee),
\]
where \(Q \in H^0(P^7, I_{C/P^7}(2))\) is the rank 3 quadric such that the ruling of \(Q\) cuts out on \(C\) precisely \(\eta\). As explained, there exists a linear embedding \(P^7 \subset P^{14} := P(\wedge^2 H^0(E)^\vee)\) such that \(P^7 \cap G = C\). The restriction map yields an isomorphism between spaces of quadrics, cf. \([M2]\),
\[
\text{res}_C : H^0(G, I_{G/P^{14}}(2)) \xrightarrow{\cong} H^0(P^7, I_{C/P^7}(2)).
\]
In particular there is a unique quadric \(G \subset \tilde{Q} \subset P^{14}\) such that \(\tilde{Q} \cap P^7 = Q\).

There are three possibilities for the rank of any quadric \(\tilde{Q} \in H^0(P^{14}, I_{G/P^{14}}(2))\): (a) \(\text{rk}(\tilde{Q}) = 15\), (b) \(\text{rk}(\tilde{Q}) = 6\) and then \(\tilde{Q}\) is a Plücker quadric, or (c) \(\text{rk}(\tilde{Q}) = 10\), in which case \(\tilde{Q}\) is a sum of two Plücker quadrics, see \([M2]\) Proposition 1.4.

**Proposition 6.1.** For a general \([C, \eta] \in \overline{\Theta}_{\text{null}}\) the quadric \(\tilde{Q}\) is smooth, that is, \(\text{rk}(\tilde{Q}) = 15\).

**Proof.** We may assume that \(\dim G^1_2(C) = 0\) (in particular \(C\) has no \(g^1_5\)'s), and \(G^2_2(C) = 0\). The space \(P(\text{Ker}(u)) \subset P(\wedge^2 H^0(E))\) is identified with the space of hyperplanes \(H \in (P^{14})^\vee\) containing the canonical space \(P^7\).

**Claim:** If \(\text{rk}(\tilde{Q}) < 15\), there exists a pencil of 8-dimensional planes \(P^7 \subset \Xi \subset P^{14}\), such that \(S := G \cap \Xi\) is a \(K^3\) surface containing \(C\) as a hyperplane section, and
\[
\text{rk}\{Q_\Xi := \tilde{Q} \cap H^0(\Xi, I_{S/\Xi}(2))\} = 3.
\]

The conclusion of the claim contradicts the assumption that \([C, \eta] \in \overline{\Theta}_{\text{null}}\) is general. Indeed, we pick such an 8-plane \(\Xi\) and corresponding \(K^3\) surface \(S\). Since \(\text{Sing}(Q) \cap C = \emptyset\), where \(Q_\Xi \cap P^7 = Q\), it follows that \(S \cap \text{Sing}(Q_\Xi)\) is finite. The ruling of \(Q_\Xi\) cuts out an elliptic pencil \(|E|\) on \(S\). Furthermore, \(S\) has nodes at the points \(S \cap \text{Sing}(Q_\Xi)\). For numerical reasons, \(|\text{Sing}(S)| = 7\), and then on the surface \(\tilde{S}\) obtained from \(S\) by resolving the 7 nodes, one has the linear equivalence
\[
C \equiv 2E + \Gamma_1 + \cdots + \Gamma_7,
\]
where \(\Gamma_i = -2\), \(\Gamma_i \cdot E = 1\) for \(i = 1, \ldots, 7\) and \(\Gamma_i \cdot \Gamma_j = 0\) for \(i \neq j\). In particular \(\text{rk}(\text{Pic}(\tilde{S})) \geq 8\). A standard parameter count, see e.g. \([Dol]\), shows that
\[
\dim \{ (S, C) : C \in |O_S(2E + \Gamma_1 + \cdots + \Gamma_7)| \} \leq 19 - 7 + \dim |O_S(C)| = 20.
\]
Since \(\dim(\overline{\Theta}_{\text{null}}) = 20\) and a general curve \([C] \in \overline{\Theta}_{\text{null}}\) lies on infinitely many such \(K^3\) surfaces \(S\), one obtains a contradiction.

We are left with proving the claim made in the course of the proof. The key point is to describe the intersection \(P(\text{Ker}(u)) \cap \tilde{Q}^\vee\), where we recall that the linear span \(\langle \tilde{Q}^\vee \rangle\) classifies hyperplanes \(H \in (P^{14})^\vee\) such that \(\text{rk}(\tilde{Q} \cap H) \leq \text{rk}(\tilde{Q}) - 1\). Note also that \(\dim \langle \tilde{Q} \rangle = \text{rk}(\tilde{Q}) - 2\).

If \(\text{rk}(\tilde{Q}) = 6\), then \(\tilde{Q}^\vee\) is contained in the dual Grassmannian \(G^\vee := G(2, H^0(E))\), cf. \([M2]\) Proposition 1.8. Points in the intersection \(P(\text{Ker}(u)) \cap G^\vee\) correspond to decomposable tensors \(s_1 \wedge s_2\), with \(s_1, s_2 \in H^0(C, E)\), such that \(u(s_1 \wedge s_2) = 0\). The image of the morphism \(\mathcal{O}_C^{\oplus s_1} \to E\) is thus a subbundle \(g^3_5\) of \(E\) and there is a bijection
\[
P(\text{Ker}(u)) \cap G(2, H^0(E)) \cong W^1_5(C).
\]
It follows, there are at most finitely many tangent hyperplanes to \( \tilde{Q} \) containing the space \( P^7 = \langle C \rangle \), and consequently, \( \dim \left( P(\text{Ker}(u)) \cap \langle \tilde{Q} \rangle \right) \leq 1 \). Then there exists a codimension 2 linear space \( W^{12} \subset P^{14} \) such that \( \text{rk}(\tilde{Q} \cap W) = 3 \), which proves the claim (and much more), in the case \( \text{rk}(\tilde{Q}) = 6 \).

When \( \text{rk}(\tilde{Q}) = 10 \), using the explicit description of the dual quadric \( \tilde{Q}^\vee \) provided in [M2] Proposition 1.8, one finds that \( \dim \left( P(\text{Ker}(u)) \cap \langle \tilde{Q} \rangle \right) \leq 4 \). Thus there exists a codimension 5 linear section \( W^9 \subset P^{14} \) such that \( \text{rk}(\tilde{Q} \cap W) = 3 \), which implies the claim when \( \text{rk}(\tilde{Q}) = 10 \) as well.

We consider an 8-dimensional linear extension \( P^7 \subset \Lambda^8 \subset P^{14} \) of the canonical space \( P^7 = \langle C \rangle \), such that \( S_\Lambda := \Lambda \cap G \) is a smooth K3 surface. The restriction map
\[
\text{res}_{C/S_\Lambda} : H^0(\Lambda, I_{S_\Lambda/\Lambda}(2)) \to H^0(P^7, I_{C/P^7}(2))
\]
is an isomorphism, see [SD]. Thus there exists a unique quadric \( S_\Lambda \subset Q_\Lambda \subset \Lambda \) with \( Q_\Lambda \cap P^7 = Q \). Since \( \text{rk}(Q) = 3 \), it follows that \( 3 \leq \text{rk}(Q_\Lambda) \leq 5 \) and it is easy to see that for a general \( \Lambda \), the corresponding quadric \( Q_\Lambda \subset \Lambda \) is of rank 5. We show however, that one can find \( K \)-extensions of the canonical curve \( C \), which lie on quadrics of rank 4:

**Proposition 6.2.** For a general \( [C, \eta] \in \overline{\Theta}_{null} \), there exists a pencil of 8-dimensional extensions
\[
P(H^0(C, K_C)^\vee) \subset \Lambda \subset P^{14}
\]
such that \( \text{rk}(Q_\Lambda) = 4 \). It follows that there exists a smooth K3 surface \( S_\Lambda \subset \Lambda \) containing \( C \) as a transversal hyperplane section, such that \( \text{rk}(Q_\Lambda) = 4 \).

**Proof.** We pass from projective to vector spaces and view the rank 15 quadric
\[
\tilde{Q} : \wedge^2 H^0(C, E)^\vee \to \wedge^2 H^0(C, E)
\]
as an isomorphism, which by restriction to \( H^0(C, K_C)^\vee \subset \wedge^2 H^0(C, E)^\vee \), induces the rank 3 quadric \( Q : H^0(C, K_C)^\vee \to H^0(C, K_C) \). The map \( u \circ \tilde{Q} : \wedge^2 H^0(E)^\vee \to H^0(K_C) \) being surjective, its kernel \( \text{Ker}(u \circ \tilde{Q}) \) is a 7-dimensional vector space containing the 5-dimensional subspace \( \text{Ker}(Q) \). We choose an arbitrary element
\[
[u := v + \text{Ker}(Q)] \in P\left( \frac{\text{Ker}(u \circ \tilde{Q})}{\text{Ker}(Q)} \right) = P^1,
\]
inducing a subspace \( H^0(C, K_C)^\vee \subset \Lambda := H^0(C, K_C)^\vee + Cv \subset \wedge^2 H^0(C, E)^\vee \), with the property that \( \text{Ker}(Q_\Lambda) = \text{Ker}(Q) \), where \( Q_\Lambda : \Lambda \to \Lambda^\vee \) is induced from \( \tilde{Q} \) by restriction and projection. It follows that \( \text{rk}(Q_\Lambda) = 4 \) and there is a pencil of 8-planes \( \Lambda \supset P^7 \) with this property.

Let \( C \subset Q \subset P^7 \) be a general canonical curve endowed with a vanishing theta-null, where \( Q \in H^0(P^7, I_{C/P^7}(2)) \) is the corresponding rank 3 quadric. We choose a general 8-plane \( P^7 \subset \Lambda \subset P^{14} \) such that \( S := \Lambda \cap G \) is a smooth K3 surface, and the lift of \( Q \) to \( \Lambda \)
\[
Q_\Lambda \in H^0(\Lambda, I_{S/\Lambda}(2))
\]
has rank 4 (cf. Proposition 6.2). Moreover, we can assume that \( S \cap \text{Sing}(Q_\Lambda) = \emptyset \). The linear projection \( f_\Lambda : \Lambda \to P^3 \) with center \( \text{Sing}(Q_\Lambda) \), induces a regular map \( f : S \to P^3 \) with image the smooth quadric \( Q_0 \subset P^3 \). Then \( S \) is endowed with two elliptic pencils
The rational morphism where 
\[
\{ P_i \} = \{ \Pi_i \} \cap \{ \Lambda \}
\]
follows that for genus \( c \) precisely, if \( S \) the rank \( \text{rk}(Q_{\Lambda} \cap P^7) = \text{rk}(Q_{\Lambda}) - 1 \), implies that the hyperplane \( P_i^7 \in (\Lambda)^{\prime} \) is the pull-back of a hyperplane from \( P^7 \), that is, \( P_i^7 = f_\Lambda^{-1}(\Pi_0) \), where \( \Pi_0 \in (P^3)^{\prime} \). This proves the following:

**Corollary 6.3.** The rational morphism \( q : \mathcal{U} \rightarrow \mathcal{S}_{\text{null}} \) is dominant.

**Proof.** Keeping the notation from above, if \( [C] \in \mathcal{S}_{\text{null}} \) is a general point corresponding to the rank 3 quadric \( Q \in H^0(P^7, I_{C_{\Lambda}/P^7}(2)) \), then \( [S, \mathcal{O}_S(C_1 + C_2), C] \in q^{-1}([C]) \). □

We begin the proof of Proposition [0.9] while retaining the set-up described above. Let us choose a general line \( l_0 \subset \Pi_0 \) and denote by \( \{ q_1, q_2 \} := l_0 \cap Q_0 \). We consider the pencil \( \{ \Pi_i \}_{i \in \mathbb{P}^1} \subset (P^3)^{\prime} \) of planes through \( l_0 \) as well as the induced pencil of curves of genus 8

\[
\{ C_t := f_\Lambda^{-1}(\Pi_i) \subset S \}_{i \in \mathbb{P}^1},
\]
each endowed with a vanishing theta-null induced by the pencil \( f_t : C_t \rightarrow Q_0 \cap \Pi_i \).

This pencil contains precisely two reducible curves, corresponding to the planes \( \Pi_1, \Pi_2 \) in \( P^3 \) spanned by the rulings of \( Q_0 \) passing through \( q_1 \) and \( q_2 \) respectively. Precisely, if \( l_i, m_i \subset Q_0 \) are the rulings passing through \( q_i \) such that \( l_1 \cdot l_2 = m_1 \cdot m_2 = 0 \), then it follows that for \( \Pi_1 = \{ l_1, m_2 \}, \Pi_2 = \{ l_2, m_1 \} \), the fibres \( f_\Lambda^{-1}(\Pi_1) \) and \( f_\Lambda^{-1}(\Pi_2) \) split into two elliptic curves \( f_\Lambda^{-1}(l_i) \) and \( f_\Lambda^{-1}(m_j) \) meeting transversally in 7 points. The half-canonical \( g^1_7 \) specializes to a degree 7 admissible covering

\[
f_\Lambda^{-1}(l_i) \cup f_\Lambda^{-1}(m_j) \xrightarrow{f} l_i \cup m_j, \ i \neq j,
\]
such that the 7 points in \( f_\Lambda^{-1}(l_i) \cap f_\Lambda^{-1}(m_j) \) map to \( l_i \cap m_j \). To determine the point in \( \mathcal{S}^+_{8} \) corresponding to the admissible covering \( (f_\Lambda^{-1}(l_i) \cup f_\Lambda^{-1}(m_j), f_\Lambda^{-1}(l_i) \cup f_\Lambda^{-1}(m_j)) \), one must insert 7 exceptional components at all the points of intersection of the two components. We denote by \( R \subset \mathcal{S}_{\text{null}} \subset \mathcal{S}^+_{8} \) the pencil of spin curves obtained via this construction.

**Lemma 6.4.** Each member \( C_t \subset S \) in the above constructed pencil is nodal. Moreover, each curve \( C_t \) different from \( f^{-1}(l_1) \cup f^{-1}(m_2) \) and \( f^{-1}(l_2) \cup f^{-1}(m_1) \) is irreducible. It follows that \( R \cdot \alpha_i = R \cdot \beta_i = 0 \) for \( i = 1, \ldots, 4 \).

**Proof.** This follows since \( f : S \rightarrow Q_0 \) is a regular morphism and the base line \( l_0 \subset H_0 \) of the pencil \( \{ \Pi_i \}_{i \in \mathbb{P}^1} \) is chosen to be general. □

**Lemma 6.5.** \( R \cdot \pi^*(\mathcal{M}^2_{7,8}) = 0 \).

**Proof.** We show instead that \( \pi_*(R) \cdot \mathcal{M}^2_{8,7} = 0 \). From Lemma [6.4] the curve \( R \) is disjoint from the divisors \( A_i, B_i \) for \( i = 1, \ldots, 4 \), hence \( \pi_*(R) \) has the numerical characteristics of a Lefschetz pencil of curves of genus 8 on a fixed K3 surface.

In particular, \( \pi_*(R) \cdot \delta/\pi_*(R) \cdot \lambda = 6 + 12/(g+1) = s(\mathcal{M}^2_{8,7}) \) and \( \pi_*(R) \cdot \delta_i = 0 \) for \( i = 1, \ldots, 4 \). This implies the statement. □

**Lemma 6.6.** \( R \cdot \mathcal{S}_{\text{null}} = -1 \).
Proof. We have already determined that $R \cdot \lambda = \pi \lambda = \chi(S, \mathcal{O}_S) + g - 1 = 9$, where $\tilde{S} := \text{Bl}_{2g-2}(S)$ is the blow-up of $S$ at the points $f^{-1}(q_1) \cup f^{-1}(q_2)$. Moreover,

$$R \cdot \alpha_0 + 2R \cdot \beta_0 = \pi \cdot \delta_0 = c_2(\tilde{X}) + 4(g - 1) = 38 + 28 = 66. \tag{7}$$

To determine $R \cdot \beta_0$ we study the local structure of $\mathcal{S}_g^+$ in a neighbourhood of one of the two points, say $t^* \in R$ corresponding to a reducible curve, say $f^{-1}(l_1) \cup f^{-1}(m_2)$, the situation for $f^{-1}(l_1) \cup f^{-1}(m_1)$ being of course identical. We set $\{p\} := l_1 \cap m_2 \in Q_0$ and $\{x_1, \ldots, x_7\} := f^{-1}(p) \subset S$. We insert exceptional components $E_1, \ldots, E_7$ at the nodes $x_1, \ldots, x_7$ of $f^{-1}(l_1) \cup f^{-1}(m_2)$ and denote by $X$ the resulting quasi-stable curve. If

$$\mu : f^{-1}(l_1) \cup f^{-1}(m_2) \cup E_1 \cup \ldots \cup E_7 \rightarrow f^{-1}(l_1) \cup f^{-1}(m_2)$$

is the stabilization morphism, we set $\{y_i, z_i\} := \mu^{-1}(x_i)$, where $y_i \in E_i \cap f^{-1}(l_1)$ and $z_i \in E_i \cap f^{-1}(m_2)$ for $i = 1, \ldots, 7$. If $t^* = [X, \eta, \beta]$, then $\eta_{f^{-1}(l_1)} = \mathcal{O}_{f^{-1}(l_1)}$, $\eta_{f^{-1}(m_2)} = \mathcal{O}_{f^{-1}(m_2)}$, and of course $\eta_{E_i} = \mathcal{O}_{E_i}(1)$. Moreover, one computes that $\text{Aut}(X, \eta, \beta) = \mathbb{Z}_2$, see [Cor] Lemma 2.2, while clearly $\text{Aut}(f^{-1}(l_1) \cup f^{-1}(m_2)) = \{\text{Id}\}$.

If $\mathbb{C}^{3g-3}_\tau$ denotes the versal deformation space of $[X, \eta, \beta] \in \mathcal{S}_g^+$, then there are local parameters $(\tau_1, \ldots, \tau_{3g-3})$, such that for $i = 1, \ldots, 7$, the locus $(\tau_i = 0) \subset \mathbb{C}^{3g-3}$ parameterizes spin curves for which the exceptional component $E_i$ persists. It particular, the pull-back $\mathbb{C}^{3g-3}_\tau \times \mathcal{S}_g^+ B_0$ of the boundary divisor $B_0 \subset \mathcal{S}_g^+$ is given by the equation $(\tau_1 \cdots \tau_7 = 0) \subset \mathbb{C}^{3g-3}_\tau$. The group $\text{Aut}(X, \eta, \beta)$ acts on $\mathbb{C}^{3g-3}_\tau$ by

$$(\tau_1, \ldots, \tau_7, \tau_8, \ldots, \tau_{3g-3}) \mapsto (-\tau_1, \ldots, -\tau_7, \tau_8, \ldots, \tau_{3g-3}),$$

and since an étale neighbourhood of $t^* \in \mathcal{S}_g^+$ is isomorphic to $\mathbb{C}^{3g-3}_\tau/\text{Aut}(X, \eta, \beta)$, we find that the Weil divisor $B_0$ is not Cartier around $t^*$ (though $2B_0$ is Cartier). It follows that the intersection multiplicity of $R \times \mathcal{S}_g^+ \mathbb{C}^{3g-3}_\tau$ with the locus $(\tau_1 \cdots \tau_7 = 0)$ equals 7, that is, the intersection multiplicity of $R \cap \beta_0$ at the point $t^*$ equals $7/2$, hence

$$R \cdot \beta_0 = (R \cdot \beta_0)_{f^{-1}(l_1) \cup f^{-1}(m_2)} + (R \cdot \beta_0)_{f^{-1}(l_2) \cup f^{-1}(m_1)} = \frac{7}{2} + \frac{7}{2} = 7.$$

Then using (7) we find that $R \cdot \alpha_0 = 66 - 14 = 52$, and finally

$$R \cdot \Theta_{\text{null}} = \frac{1}{4} R \cdot \lambda - \frac{1}{16} R \cdot \alpha_0 = \frac{9}{4} - \frac{52}{16} = -1.$$

\[\square\]

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HUMBOLDT-UNIVERSITÄT ZU BERLIN, INSTITUT FÜR MATHEMATIK, UNTER DEN LINDEN 6 10099 BERLIN, GERMANY  
E-mail address: farkas@math.hu-berlin.de

UNIVERSITÀ ROMA TRE, DIPARTIMENTO DI MATEMATICA, LARGO SAN LEONARDO MURIALDO 1-00146 ROMA, ITALY  
E-mail address:verra@mat.uniroma3.it