The asymptotics of the Struve function $H_\nu(z)$ for large complex order and argument

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Abstract

We re-examine the asymptotic expansion of the Struve function $H_\nu(z)$ for large complex values of $\nu$ and $z$ satisfying $|\arg \nu| \leq \frac{1}{2}\pi$ and $|\arg z| < \frac{1}{2}\pi$. Watson's analysis [4, §10.43] covers only the case of $\nu$ and $z$ of the same phase with $\nu/z$ in the intervals $(0, 1)$ and $(1, \infty)$. The domains in the complex $\nu/z$-plane where the expansion takes on different forms are obtained.

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1. Introduction

The Struve function $H_\nu(z)$ is a particular solution of the inhomogeneous Bessel equation

$$\frac{d^2 w(z)}{dz^2} + \frac{1}{z} \frac{dw(z)}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) w(z) = \frac{(\frac{1}{2}z)^{\nu-1}}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})}$$

which possesses the series expansion

$$H_\nu(z) = (\frac{1}{2}z)^{\nu+1} \sum_{n=0}^{\infty} \frac{(-)^n (\frac{1}{2}z)^{2n}}{\Gamma(n + \frac{3}{2}) \Gamma(n + \nu + \frac{3}{2})}$$

valid for all finite $z$.

An integral representation, valid when $\Re(\nu) > -\frac{1}{2}$, is given by [4, p. 330] as

$$J_\nu(z) \pm iH_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^1 e^{\pm izt} (1 - t^2)^{\nu-\frac{1}{2}} dt,$$

where $J_\nu(z)$ is the usual Bessel function. Upon replacement of the variable $t$ by $\pm iu$, we obtain

$$H_\nu(z) \pm iJ_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^{\pm i} e^{-zu} (1 + u^2)^{\nu-\frac{1}{2}} du \quad (\Re(\nu) > -\frac{1}{2}).$$

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The integration path corresponding to the upper sign in (1.2) can be deformed to pass along the positive real axis to +∞ and back to the point i along the parallel path i + u (0 ≤ u ≤ ∞). The contribution from the path (i + ∞, i] is equal to iHν(2)(z), where Hν is the Hankel function; see [4, p. 166]. Thus we find the alternative representation [2, p. 292]

$$H_\nu(z) - Y_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{-zu}(1 + u^2)^{\nu - \frac{1}{2}} du$$

valid for unrestricted ν and | arg z | < $\frac{1}{2}$π, where $Y_\nu(z)$ denotes the Bessel function of the second kind.

Here we shall consider the asymptotic expansion of $H_\nu(z)$ for large complex values of ν and z satisfying | arg ν | ≤ $\frac{1}{2}$π and | arg z | < $\frac{1}{2}$π. Values of arg z outside this range can be dealt with by means of the continuation formula

$$H_\nu(ze^{\pi mi}) = e^{\pi mi(\nu + 1)} H_\nu(z), \quad m = \pm 1, \pm 2 \ldots$$

obtained from (1.1).

2. Asymptotic expansion when $z > 0$

We set

$$q := \nu/z = \alpha + i\beta, \quad \theta := \text{arg } z, \quad \omega := \text{arg } q.$$ 

In view of (1.2) and (1.3), we are led to the consideration of the integral

$$\int_C e^{-|z|\tau} \frac{du}{\sqrt{1 + u^2}} \tau := e^{i\theta}\{u - q \log(1 + u^2)\},$$

where C is a suitably chosen path in the $u$-plane.

Saddle points are situated at $d\tau/du = 0$; that is, at the points

$$u_\pm = q \pm \sqrt{q^2 - 1}.$$ 

We shall refer to these saddles as $S_1$ (upper sign) and $S_2$ (lower sign). Inversion of (2.1) in the form $u = \sum_{k=1}^\infty a_k(\tau e^{-i\theta})^k$, where $a_0 = 1$, shows that

$$\frac{1}{\sqrt{1 + u^2}} \frac{du}{d\tau} = e^{-i\theta} \sum_{k=0}^\infty c_k(q)(\tau e^{-i\theta})^k$$

valid in a disc centered at $\tau = 0$ of radius determined by the nearest singularity corresponding to the saddles $S_1$ or $S_2$ (or both). The values of the coefficients $c_k(q)$ (0 ≤ k ≤ 10) are listed in Table 1; see also [1, p. 203].

Watson [4, §10.43] has considered the two cases (i) $q \in [1, \infty)$ and (ii) $q \in (0, 1)$ when $z > 0$ ($\theta = 0$). The steepest descent paths emanating from the origin in the complex $u$-plane in these two cases are shown in Fig. 1; branch cuts have been taken along the segments of the imaginary axis [±i, ±∞i). In case (i), the desired path C consists of the real axis between the origin and the saddle $S_2$ and then either along the arc to the branch point at $u = i$ or along the arc to the branch point at $u = -i$. In case (ii), the path C from the origin coincides with the positive real axis and passes to $+\infty$. In both cases $\tau$ increases monotonically from 0 to $+\infty$ as we traverse these paths.
Asymptotics of the Struve function

Figure 1: The steepest paths when $\theta = 0$: (a) when $q \in (1, \infty)$ and (b) when $q \in (0, 1)$. The heavy dots indicate the saddle points and the heavy lines denote the branch cuts.

Table 1: The coefficients $c_k(q)$ for $0 \leq k \leq 10$.

| $k$ | $c_k(q)$ |
|-----|----------|
| 0   | 1        |
| 1   | $2q$     |
| 2   | $6q^2 - \frac{1}{2}$ |
| 3   | $20q^3 - 4q$ |
| 4   | $70q^4 - \frac{45}{2}q^2 + \frac{3}{8}$ |
| 5   | $252q^5 - 112q^3 + \frac{23}{2}q$ |
| 6   | $924q^6 - 525q^4 + \frac{301}{6}q^2 - \frac{5}{16}$ |
| 7   | $3432q^7 - 2376q^5 + 345q^3 - \frac{22}{3}q$ |
| 8   | $12870q^8 - \frac{21021}{2}q^6 + \frac{16665}{8}q^4 - \frac{1425}{16}q^2 + \frac{35}{128}$ |
| 9   | $48620q^9 - 45760q^7 + \frac{139139}{12}q^5 - \frac{1595}{2}q^3 + \frac{563}{64}q$ |
| 10  | $184756q^{10} - 196911q^8 + 61061q^6 - \frac{287287}{48}q^4 + \frac{133529}{960}q^2 - \frac{63}{256}$ |

Then in case (i) we find

$$\int_0^{\pm i} e^{-zu}(1 + u^2)^{\nu - \frac{1}{2}} du = \int_0^{\infty} e^{-z\tau} \left( \frac{1}{\sqrt{1 + u^2}} \frac{du}{d\tau} \right) d\tau \sim \sum_{k=0}^{\infty} c_k(q)\Gamma(k + 1) \frac{1}{z^{k+1}}$$

for $z \to +\infty$. Hence, for large real $\nu$ and $z$ with $\nu/z \in [1, \infty)$ (when the deformed path

\begin{footnote}
1Suitable rotation of the integration path through an acute angle enables the validity of (1.3) to be extended to the wider sector $|\arg z| < \pi$; see [4, p. 331].

2When $q = 1$, the saddles $S_1$ and $S_2$ form a double saddle at $u = 1$. In this case, the path $C$ consists of the real axis $0 \leq u \leq 1$ followed by similar arcs to the points $u = \pm i$.
\end{footnote}
both saddles corresponds to the case when the steepest descent path from the origin connects with

\[ (\frac{1}{2}z)^{\nu-1} \sum_{k=0}^{\infty} \frac{c_k(q)\Gamma(k+1)}{z^k}, \]  

(2.2)

respectively. Similarly, for \( \nu/z \in (0,1) \) (when the path \( C \) passes to \( +\infty \) along the real axis), we have from (1.3)

\[ H_\nu(z) - Y_\nu(z) \sim \frac{(\frac{1}{2}z)^{\nu-1}}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \sum_{k=0}^{\infty} \frac{c_k(q)\Gamma(k+1)}{z^k}. \]  

(2.3)

These are the results given in [4, §10.43]; see also the discussion in Section 3.

When \( \nu \) is allowed to take on complex values with \( z > 0 \), the steepest descent paths in Fig. 1 undergo a progressive change. Recalling that \( q = \alpha + i\beta \), we find that as \( \beta \) increases from zero when \( \alpha \in (0,1) \) the steepest descent path from the origin \( \Im \tau = 0 \) becomes increasingly deformed in the upper-half plane, until at a critical value \( \beta = \beta^* \) this path connects with the saddle \( S_1 \). For example, when \( \alpha = 0.80 \) the critical value is \( \beta^* \approx 0.143900 \). Then, the path \( \Im \tau = 0 \) passes to infinity when \( \beta < \beta^* \), connects with \( S_1 \) when \( \beta = \beta^* \) and approaches the branch point at \( u = i \) (possibly spiralling onto different Riemann sheets) when \( \beta > \beta^* \). An analogous transition occurs when \( \beta < 0 \) at \( \beta = -\beta^* \), with the saddle \( S_1 \) replaced by \( S_2 \). When \( \alpha > 1 \), the steepest path \( \Im \tau = 0 \) passes to \( u = i \) when \( \beta > 0 \), and to \( u = -i \) when \( \beta < 0 \), without undergoing any transition as \( \beta \) increases.

The transitions that occur when \( z > 0 \) and \( |\arg \nu| \leq \frac{1}{2}\pi \) are summarised in Fig. 2(a). This shows the three curves in the complex \( q \)-plane, on which a transition takes place, that emanate from the point \( P \) (corresponding to \( q = 1 \)). The curves in the upper and lower half-planes are conjugate curves with the third being the segment \([1, \infty)\) of the real \( q \)-axis. In the domain numbered 1 (between the conjugate curves and the imaginary \( q \)-axis), the path \( C \) passes to \( \infty \) and the expansion (2.3) applies. In the domain numbered 2, the path \( C \) terminates at \( u = +i \) and the expansion (2.2) applies with the upper sign; in the domain numbered 3, the terminal point is \( u = -i \) and the expansion (2.2) applies with the lower sign. For \( q \) situated on these curves the transition is associated with a Stokes phenomenon; see below.

### 3. Asymptotic expansion for complex \( z \)

When \( z \) is complex (\( \theta \neq 0 \)) the transition curves in the sector of the \( q \)-plane given by\(^3\) \((-\frac{1}{2}\pi - \theta, \frac{1}{2}\pi - \theta)\) are \( \theta \)-dependent. In Fig.2(b)–(d) we show these curves for \( \theta/\pi = 0.10, 0.20 \) and 0.30. The curves for \( \theta < 0 \) are the conjugate of those for \( \theta > 0 \). The point \( P \) corresponds to the case when the steepest descent path from the origin connects with both saddles \( S_1 \) and \( S_2 \). The point labelled \( Q \) is the intercept of the lower curve with the positive \( q \)-axis. Values of \( q \) at \( P \) and \( Q \) are presented in Table 2 for different \( \theta \).

As in the case \( \theta = 0 \) in Fig. 2(a), for \( q \)-values in domain 1 the endpoint of the steepest descent path from the origin terminates at infinity, whereas those situated in domains 2 and 3 pass to the branch points (possibly spiralling onto adjacent Riemann surfaces) at \( u = \pm i \), respectively. As one crosses one of these curves, say from domain 1 to domain

\(^3\)This sector corresponds to \( |\arg \nu| \leq \frac{1}{2}\pi \) and \( |\arg z| < \frac{1}{2}\pi \).
Table 2: The coordinates of the triple point $P$ and the intercept $Q$ on the real $q$-axis as a function of $\theta$.

| $\theta/\pi$ | $P$     | $Q$     | $\theta/\pi$ | $P$     | $Q$     |
|--------------|---------|---------|--------------|---------|---------|
| 0            | 1       | 1       | 0.30         | 1.08553 + 1.38238$i$ | 0.27561 |
| 0.05         | 0.96385 + 0.08606$i$ | 0.83360 | 0.35         | 1.36479 + 2.60425$i$ | 0.18575 |
| 0.10         | 0.93778 + 0.18745$i$ | 0.70952 | 0.40         | 2.36238 + 7.23955$i$ | 0.10710 |
| 0.20         | 0.93437 + 0.53249$i$ | 0.48057 | 0.42         | 3.72266 + 14.4826$i$ | 0.07942 |
| 0.25         | 0.97678 + 0.84047$i$ | 0.37449 | 0.45         | 16.4886 + 104.102$i$ | 0.04275 |

Figure 2: The domains in the sector of the $q$-plane bounded by $-\frac{1}{2}\pi - \theta < \omega < \frac{1}{2}\pi - \theta$ showing the termination points of the steepest descent path from the origin: (a) $\theta = 0$, (b) $\theta = 0.10\pi$, (c) $\theta = 0.20\pi$ and (d) $\theta = 0.30\pi$. The termination point in domain 1 is at infinity and that in domains 2 and 3 is at $\pm i$, respectively.

2, there is a change in the endpoint via a Stokes phenomenon. Examples of the steepest descent paths when $\theta = 0.10\pi$ on the three curves labelled $PA$, $PB$ and $PC$ in Fig. 2(b), and at $P$, are shown in Fig. 3 demonstrating that on each curve the change of endpoint is associated with a Stokes phenomenon.
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Figure 3: The steepest paths the through the saddles when \( \theta = 0.10\pi \): (a) on \( PA \) with \( q = 0.60 + 0.95307i \), (b) on \( PB \) with \( q = 1.40 + 0.39474i \), (c) on \( PC \) with \( q = 0.40 - 0.42914i \) and (d) at \( P \) with \( q = 0.93778 + 0.18745i \). The heavy lines denote the branch cuts.

4. Numerical results

To verify these assertions, we carry out calculations using (2.2) and (2.3) for a series of values of \( q \equiv \nu/z \) situated in different domains in Fig. 2. The results are presented in Table 3 which shows the absolute relative error in the computation of \( H_\nu(z) \). The values of the Bessel functions \( J_\nu(z) \) and \( Y_\nu(z) \) were evaluated with the in-built codes in Mathematica. In each case, the asymptotic series on the right-hand sides of (2.2) and (2.3) is optimally truncated; that is, at or just before the least term.

Table 3: The absolute relative error in the computation of \( H_\nu(z) \) from (2.2) and (2.3) when \( z = 40e^{i\theta} \).

| \( q = \nu/z \) | \( \theta = 0 \) | \( \theta = 0.10\pi \) |
|----------------|----------------|-----------------|
|                | Error         | Endpoint        | Error           | Endpoint        |
| 0.60           | 7.764 \times 10^{-9} | \infty          | 9.556 \times 10^{-9} | \infty          |
| 1.00           | 1.041 \times 10^{-4} | \pm i           | 2.751 \times 10^{-5} | -i              |
| 1.25           | 8.835 \times 10^{-4} | \pm i           | 4.830 \times 10^{-4} | -i              |
| 0.60 + 0.40i   | 2.355 \times 10^{-6} | \infty          | 3.280 \times 10^{-6} | \infty          |
| 1.00 + 0.60i   | 2.783 \times 10^{-4} | +i              | 4.136 \times 10^{-3} | +i              |
| 1.00 - 0.30i   | 7.342 \times 10^{-5} | -i              | 5.000 \times 10^{-5} | -i              |

In [4, §10.43], Watson claims that (2.3) and (2.2) hold for \( q \in (0, 1) \) and \( q \in [1, \infty) \),
respectively, when $|\arg z| < \frac{1}{2} \pi$. Our calculations have shown that when $\arg z \neq 0$ with $q = \nu/z > 0$ (that is, when $\nu$ and $z$ have the same phase), the expansion (2.3) holds for $q \in (0, Q)$ and the expansion (2.2) holds for $q \in [Q, \infty)$, where $Q \equiv Q(\theta)$ is tabulated in Table 2.

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