AN EXTENSION THEOREM IN SYMPLECTIC GEOMETRY

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ABSTRACT. We extend the “Extension after Restriction Principle” for symplectic embeddings of bounded starlike domains to a large class of symplectic embeddings of unbounded starlike domains.

1. INTRODUCTION

We endow each open subset $U$ of Euclidean space $\mathbb{R}^{2n}$ with the standard symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$ 

A smooth embedding $\varphi: U \hookrightarrow \mathbb{R}^{2n}$ is called symplectic if $\varphi^* \omega_0 = \omega_0$. In particular, every symplectic embedding preserves the volume form $\frac{1}{n!} \omega_0^n$ and hence the Lebesgue measure on $\mathbb{R}^{2n}$. Recall that a domain in $\mathbb{R}^{2n}$ is by definition a non-empty open connected subset of $\mathbb{R}^{2n}$.

1.1. Definition. Consider a symplectic embedding $\varphi: U \hookrightarrow \mathbb{R}^{2n}$ of a domain $U$ in $\mathbb{R}^{2n}$. We say that the pair $(U, \varphi)$ has the extension property if for each subset $A \subset U$ whose closure in $\mathbb{R}^{2n}$ is contained in $U$ there exists a symplectomorphism $\Phi_A$ of $\mathbb{R}^{2n}$ such that $\Phi_A|_A = \varphi|_A$.

Not every pair $(U, \varphi)$ as above has the extension property as the following example shows.

1.2. Example. For $0 \leq r_0 < r_1 < \infty$ we define the open annulus

$$A(r_0, r_1) = \{(x, y) \in \mathbb{R}^2 \mid r_0^2 < x^2 + y^2 < r_1^2\}.$$ 

Let $\varphi: A(0, 3) \to A(4, 5) \subset \mathbb{R}^2$ be the symplectic embedding which in polar coordinates is given by

$$\varphi(r, \vartheta) = \left(\sqrt{r^2 + 16}, \vartheta\right).$$

Any smooth extension of $\varphi|_{A(1, 2)}$ to $\mathbb{R}^2$ maps the disc of area $\pi$ to the disc of area $17\pi$ and hence cannot be symplectic.  

Date: November 9, 2018.

2000 Mathematics Subject Classification. Primary 53D35, Secondary 54C20.
On the other hand, the well-known “Extension after Restriction Principle” [2], which is reproved below, states that a pair \((U, \varphi)\) has the extension property whenever the geometry of \(U\) is simple enough. Recall that a subset \(U\) of \(\mathbb{R}^{2n}\) is said to be starlike if \(U\) contains a point \(p\) such that for every point \(x \in U\) the straight line between \(p\) and \(x\) is contained in \(U\).

### 1.3. Proposition. (Extension after Restriction Principle)
Assume that \(\varphi: U \hookrightarrow \mathbb{R}^{2n}\) is a symplectic embedding of a bounded starlike domain \(U \subset \mathbb{R}^{2n}\). Then the pair \((U, \varphi)\) has the extension property. In fact, for any subset \(A \subset U\) whose closure in \(\mathbb{R}^{2n}\) is contained in \(U\) there exists a compactly supported symplectomorphism \(\Phi_A\) of \(\mathbb{R}^{2n}\) such that \(\Phi_A|_A = \varphi|_A\).

The purpose of this paper is to prove the extension property for a large class of symplectic embeddings of unbounded starlike domains. The following example shows that it is not enough to assume that \(U\) is starlike.

### 1.4. Example. We let \(U \subset \mathbb{R}^2\) be the strip \([1, \infty[ \times ] -1, 1[\). Combining the methods used in Step 1 and Step 4 of Section 2.2 in [6] we find a symplectic embedding \(\varphi: U \hookrightarrow \mathbb{R}^{2n}\) such that \(\varphi(k, 0) = (\frac{1}{k}, 0)\), \(k = 2, 3, \ldots\). Then there does not exist any subset \(A \subset U\) containing the set \(\{(k, 0) \mid k = 2, 3, \ldots\}\) for which \(\varphi|_A\) extends to a diffeomorphism of \(\mathbb{R}^2\).

Observe that if \((U, \varphi)\) has the extension property, then \(\varphi\) is proper in the sense that each subset \(A \subset U\) whose closure in \(\mathbb{R}^{2n}\) is contained in \(U\) and whose image \(\varphi(A)\) is bounded is bounded. The map \(\varphi\) in Example 1.4 is not proper in this sense. However, the map \(\varphi\) in the following example is proper in this sense, and still \((U, \varphi)\) does not have the extension property.

### 1.5. Example. Let \(U \subset \mathbb{R}^2\) be the strip \([0, \frac{1}{2}]\) and let
\[
A = \{(x, y) \in U \mid \|y + \frac{1}{2}\| \leq f(x)\}
\]
where \(f: \mathbb{R} \to ]0, \frac{1}{2}[\) is a smooth function such that
\[
\int_{\mathbb{R}} \left(\frac{1}{2} - f(x)\right) dx < \infty,
\]
(1.1)
cf. Figure 11. Using the method used in Step 4 of Section 2.2 in [6] we find a symplectic embedding \(\varphi: U \hookrightarrow \mathbb{R}^{2n}\) such that
\[
\varphi(x, y) = (x, y) \text{ if } x \geq 1 \quad \text{and} \quad \varphi(x, y) = (-x, -y) \text{ if } x \leq -1,
\]
Figure 1. A pair \((U, \varphi)\) which does not have the extension property.

cf. Figure 1. In view of the estimate (1.1) the component \(C\) of \(\mathbb{R}^2 \setminus \varphi(A)\) which contains the point \((1, 0)\) has finite volume. Any symplectomorphism \(\Phi_A\) of \(\mathbb{R}^2\) such that \(\Phi_A|_A = \varphi|_A\) would map the “upper” component of \(\mathbb{R}^2 \setminus A\), which has infinite volume, to \(C\). This is impossible.

Example 1.5 shows that the assumption (1.2) on \(\varphi\) in Theorem 1.7 below cannot be omitted. For technical reasons in the proof of Theorem 1.7 we shall also impose a mild convexity condition on the starlike domain \(U\). The length of a smooth curve \(\gamma: [0, 1] \to \mathbb{R}^n\) is defined by

\[
\text{length}(\gamma) := \int_0^1 |\gamma'(s)| \, ds.
\]

On any domain \(U \subset \mathbb{R}^n\) we define a distance function \(d_U: U \times U \to \mathbb{R}\) by

\[
d_U(z, z') := \inf \{\text{length}(\gamma)\}
\]

where the infimum is taken over all smooth curves \(\gamma: [0, 1] \to U\) with \(\gamma(0) = z\) and \(\gamma(1) = z'\). Then \(|z - z'| \leq d_U(z, z')\) for all \(z, z' \in U\).
1.6. Definition. We say that a domain $U \subset \mathbb{R}^n$ is a Lipschitz domain if there exists a constant $\lambda > 0$ such that
\[ d_U(z, z') \leq \lambda |z - z'| \quad \text{for all } z, z' \in U. \]

Each convex domain $U \subset \mathbb{R}^n$ is a Lipschitz domain with Lipschitz constant $\lambda = 1$. It is not hard to see that there do exist starlike domains which are not Lipschitz domains. But we do not know of a starlike domain with smooth boundary which is not a Lipschitz domain.

1.7. Theorem. Assume that $\varphi : U \hookrightarrow \mathbb{R}^{2n}$ is a symplectic embedding of a starlike Lipschitz domain $U \subset \mathbb{R}^{2n}$ such that there exists a constant $L > 0$ satisfying
\[ |\varphi(z) - \varphi(z')| \geq L |z - z'| \quad \text{for all } z, z' \in U. \]

Then the pair $(U, \varphi)$ has the extension property.

Theorem 1.7 is applied in [6] to extend a symplectic vanishing theorem for bounded domains to certain unbounded domains.

Acknowledgements. I cordially thank Urs Lang, François Laudenbach and Edi Zehnder for useful discussions.

2. Proofs

We shall first proceed along the lines of [2] and then verify that our assumptions on $U$ and $\varphi$ are sufficient to push the arguments through.

Step 1. Reduction to a simpler case

We start with observing that we may assume that $U$ is starlike with respect to the origin and that $\varphi(0) = 0$ and $d\varphi(0) = id$. Indeed, suppose that Theorem 1.7 holds in this situation, that $A$ is a subset of $U$ whose closure in $\mathbb{R}^{2n}$ is contained in $U$, and that $U$ is starlike with respect to $p \neq 0$ or that $\varphi(p) \neq 0$ or that $D := d\varphi(p) \neq id$. For $w \in \mathbb{R}^{2n}$ we denote by $\tau_w$ the translation $z \mapsto z + w$. We define the symplectic embedding $\psi : (D \circ \tau_p)(U) \hookrightarrow \mathbb{R}^{2n}$ by
\[ \psi := \tau_{-\varphi(p)} \circ \varphi \circ \tau_p \circ D^{-1}. \]

Then $\psi(p) = 0$ and $d\psi(p) = id$. Since $U$ is starlike with respect to $p$ and $D$ is linear, the domain $(D \circ \tau_p)(U)$ is starlike with respect to the origin, and $(D \circ \tau_p)(A)$ is a subset of $(D \circ \tau_p)(U)$ whose closure in $\mathbb{R}^{2n}$ is contained in $(D \circ \tau_p)(U)$. Assume next that $U$ is a $\lambda$-Lipschitz domain. We fix $w, w' \in (D \circ \tau_p)(U)$ and set $z = D^{-1}(w) + p$, $z' = D^{-1}(w') + p$. Given any smooth path $\gamma : [0, 1] \rightarrow U$ with $\gamma(0) = z$, $\gamma(1) = z'$, the smooth path
\[ D \circ \tau_p \circ \gamma : [0, 1] \rightarrow (D \circ \tau_p)(U) \]
runs from $w$ to $w'$, and so
\[
\frac{d}{(D \circ \tau - p)(U)}(w, w') \leq \int_0^1 |D \gamma'(s)| \, ds \leq \|D\| \int_0^1 |\gamma'(s)| \, ds.
\]
It follows that
\[
\frac{d}{(D \circ \tau - p)(U)}(w, w') \leq \|D\| \|D^{-1}\| \lambda |w - w'|.
\]
Since $w, w' \in (D \circ \tau - p)(U)$ were arbitrary, we conclude that $(D \circ \tau - p)(U)$ is a $\|D\| \|D^{-1}\| \lambda$-Lipschitz domain. Finally, the assumption (1.2) on $\varphi$ yields
\[
|\psi(z) - \psi(z')| \geq \frac{L}{\|D\|} |D^{-1}(z - z')| \geq \frac{L}{\|D\|} |z - z'|
\]
for all $z, z' \in (D \circ \tau - p)(U)$. By assumption we therefore find a symplectomorphism $\Psi_{(D \circ \tau - p)(A)}$ of $\mathbb{R}^{2n}$ such that $\Psi_{(D \circ \tau - p)(A)}(\varphi_{(D \circ \tau - p)(A)}) = \psi_{(D \circ \tau - p)(A)}$. Define the symplectomorphism $\Phi_A$ of $\mathbb{R}^{2n}$ by
\[
\Phi_A := \tau_{\varphi(p)} \circ \Psi_{(D \circ \tau - p)(A)} \circ D \circ \tau_{-p}.
\]
Then $\Phi|_A = \varphi|_A$, as required.

**Step 2. The classical approach**

So assume that $U$ is starlike with respect to the origin and that $\varphi(0) = 0$ and $d\varphi(0) = id$. We denote the set of symplectic embeddings of $U$ into $\mathbb{R}^{2n}$ by $\text{Symp}(U, \mathbb{R}^{2n})$. Since $U$ is starlike with respect to the origin we can define a continuous path $\varphi_t \subset \text{Symp}(U, \mathbb{R}^{2n})$ by setting
\[
\varphi_t(z) := \begin{cases} z & \text{if } t = 0, \\ \frac{1}{t} \varphi(tz) & \text{if } t \in [0, 1]. \end{cases}
\]
The path $\varphi_t$ is smooth except possibly at $t = 0$. In order to smoothen $\varphi_t$, we define the diffeomorphism $\eta$ of $[0, 1]$ by
\[
\eta(t) := \begin{cases} 0 & \text{if } t = 0, \\ e^2 e^{-2/t} & \text{if } t \in [0, 1], \end{cases}
\]
where $e$ denotes the Euler number, and for $t \in [0, 1]$ and $z \in U$ we set
\[
\phi_t(z) := \varphi_{\eta(t)}(z).
\]
Then $\phi_t$ is a smooth path in $\text{Symp}(U, \mathbb{R}^{2n})$. We have $\phi_0 = id_U$ and $\phi_1 = \varphi$. 
Since $U$ is starlike, it is contractible, and so the same holds true for all the open sets $\phi_t(U)$, $t \in [0,1]$. We therefore find a smooth time-dependent Hamiltonian function

$$H: \bigcup_{t \in [0,1]} \{t\} \times \phi_t(U) \to \mathbb{R} \tag{2.4}$$

generating the path $\phi_t$, i.e., $\phi_t$ is the solution of the Hamiltonian system

$$\begin{aligned}
\frac{d}{dt} \phi_t(z) &= J\nabla H_t(\phi_t(z)), & z \in U, \ t \in [0,1], \\
\phi_0(z) &= z, & z \in U. \tag{2.5}
\end{aligned}$$

Here, $J$ denotes the standard complex structure defined by

$$\omega_0(z,w) = \langle Jz, w \rangle, \quad z, w \in \mathbb{R}^{2n}. \tag{2.6}$$

The function $H(z,t) = H_t(z)$ is determined by the first equation in (2.5) up to a smooth function $h(t): [0,1] \to \mathbb{R}$. Notice that $0 \in \phi_t(U)$ for all $t$. We choose $h(t)$ such that

$$H_t(0) = 0 \quad \text{for all } t \in [0,1].$$

**Step 3. Intermezzo: End of the proof of Proposition 1.3**

Before proceeding with the proof of Theorem 1.7 we shall prove Proposition 1.3. Fix a subset $A$ of $U$ whose closure $\overline{A}$ in $\mathbb{R}^{2n}$ is contained in $U$. Since $U$ is bounded, the set $\overline{A}$ is compact, and so the set

$$K = \bigcup_{t \in [0,1]} \{t\} \times \phi_t(\overline{A}) \subset [0,1] \times \mathbb{R}^{2n}$$

is also compact and hence bounded. We therefore find a bounded neighbourhood $V$ of $K$ which is open in $[0,1] \times \mathbb{R}^{2n}$ and is contained in the set $\bigcup_{t \in [0,1]} \{t\} \times \phi_t(U)$. By Whitney’s Theorem, there exists a smooth function $f$ on $[0,1] \times \mathbb{R}^{2n}$ which is equal to 1 on $K$ and vanishes outside $V$. Since $V$ is bounded, the function $fH: [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ has compact support, and so the Hamiltonian system associated with $fH$ can be solved for all $t \in [0,1]$. We define $\Phi_A$ to be the resulting time-1-map. Then $\Phi_A$ is a globally defined symplectomorphism of $\mathbb{R}^{2n}$ with compact support, and $\Phi_A|_A = \phi|_A$. The proof of Proposition 1.3 is thus complete.

**Step 4. End of the proof of Theorem 1.7**

If the set $U$ is not bounded, the subset $A \subset U$ does not need to be relatively compact, and so there might be no cut off $fH$ of $H$ whose Hamiltonian flow exists for all $t \in [0,1]$. We therefore need to extend the Hamiltonian $H$ more carefully. We shall first verify that our assumption (1.2) on $\varphi$ implies that $\nabla H$ is linearly bounded. Since we do
not know a direct way to extend a linearly bounded gradient field to a linearly bounded gradient field, we shall then pass to the function
\[ G(t, w) = \frac{H(t, w)}{g(|w|)} \]
where \( g(|w|) = |w| \) for \(|w|\) large. Our assumption that \( U \) is a Lipschitz domain will imply that \( G \) is Lipschitz continuous in \( w \) and can hence be extended to a Lipschitz continuous function \( \tilde{G} \) on \([0, 1] \times \mathbb{R}^{2n} \). After smoothing \( \tilde{G} \) in \( w \) to \( \tilde{G} \) we shall obtain an extension \( \tilde{H}(t, w) = g(|w|) \tilde{G}(t, w) \) whose gradient is linearly bounded.

2.1. Lemma. Let \( L > 0 \) be the constant guaranteed by assumption (1.2)\(^{1,2}\)

(i) \( |\phi_t(z) - \phi_t(z')| \geq L |z - z'| \) for all \( t \in [0, 1] \) and \( z, z' \in U \).

(ii) \( \|d\phi_t(z)\| \leq \frac{1}{L} \) for all \( t \in [0, 1] \) and \( z \in U \).

Proof. (i) In view of definitions (2.3) and (2.1) we have
\[ \phi_t(z) = \frac{1}{\eta(t)} \varphi(\eta(t)z) \]
for all \( t \in [0, 1] \) and \( z \in U \). Together with assumption (1.2) we find
\[ |\phi_t(z) - \phi_t(z')| = \frac{1}{\eta(t)} |\varphi(\eta(t)z) - \varphi(\eta(t)z')| \]
\[ \geq \frac{1}{\eta(t)} L |\eta(t)z - \eta(t)z'| \]
\[ = L |z - z'| . \]
Assertion (i) thus follows.

(ii) We fix \( t \in [0, 1] \) and \( z \in U \). Following the proof of Proposition 2.20 in [3] we decompose the linear symplectomorphism \( d\phi_t(z) \) as
\[ d\phi_t(z) = PQ \]
where both \( P \) and \( Q \) are symplectic and \( P \) is symmetric and positive definite and \( Q \) is orthogonal. According to [3] Lemma 2.18 the eigenvalues of \( P \) are of the form
\[ 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \lambda_n^{-1} \leq \ldots \leq \lambda_2^{-1} \leq \lambda_1^{-1} . \]
Since \( Q \) is orthogonal, we find
\[ \|d\phi_t(z)\| = \|P\| = \lambda_1^{-1} . \]
Let \( v_1 \) be an eigenvector of \( \lambda_1 \). In view of assertion (i) we have
\[ \lambda_1 |v_1| = |d\phi_t(z)v_1| \geq L |v_1| . \]
and so $\lambda^{-1} \leq L^{-1}$. This and the identity (2.8) yield $\|d\phi_t(z)\| \leq L^{-1}$, and so assertion (ii) follows.

For $r > 0$ we denote by $B(r)$ the closed $r$-ball around $0 \in \mathbb{R}^{2n}$. We choose $\epsilon > 0$ so small that $B(\epsilon) \subset U$. Finally, we abbreviate

$$U_t = U \cap B\left(\frac{\epsilon}{t} e^{1/t}\right), \quad t \in [0, 1].$$

2.2. Lemma. (i) There exists a constant $C_1 > 0$ such that

$$|\nabla H_t(w)| \leq \frac{C_1}{t^2} |w| \quad \text{for all } t \in [0, 1] \text{ and } w \in \phi_t(U).$$

(ii) There exists a constant $c_1 > 0$ such that

$$|\nabla H_t(w)| \leq \frac{c_1}{t^2} e^{-1/t} |w| \quad \text{for all } t \in [0, 1] \text{ and } w \in \phi_t(U_t).$$

Proof. (i) Fix $t \in [0, 1]$ and $w = \phi_t(z)$. Using the first line in (2.5) and the definitions (2.7) and (2.2) we compute

$$J \nabla H_t(w) = \frac{d}{dt} \phi_t(z) = \frac{d}{dt} \left(\frac{1}{\eta(t)} \varphi(\eta(t)z)\right) = \frac{\eta'(t)}{\eta(t)} \left(-\frac{1}{\eta(t)} \varphi(\eta(t)z) + d\varphi(\eta(t)z)z\right) = \frac{2}{t^2} \left(-w + d\varphi(\eta(t)z)z\right).$$

Lemma 2.1(ii) with $t = 1$ yields

$$\|d\varphi(z)\| \leq \frac{1}{L} \quad \text{for all } z \in U$$

and the identity $\phi_t(0) = \frac{1}{\eta(t)} \varphi(0) = 0$ and Lemma 2.1(i) with $z' = 0$ yield

$$|w| = |\phi_t(z)| \geq L|z|.$$

In view of the identity (2.11) and the estimates (2.12) and (2.13) we conclude

$$|\nabla H_t(w)| = |J \nabla H_t(w)| \leq \frac{2}{t^2} (|w| + \|d\varphi(\eta(t)z)\| |z|) \leq \frac{2}{t^2} \left(|w| + \frac{1}{L^2} |w|\right) = \frac{2}{t^2} \left(1 + \frac{1}{L^2}\right) |w|. $$
The constant $C_1 := 2 \left(1 + \frac{1}{L^2}\right)$ is as required.

(ii) By the choice of $\epsilon$, the smooth map $\varphi$ is $C^2$-bounded on $B(\epsilon)$, and so Taylor’s Theorem applied to $\varphi: B(\epsilon) \to \mathbb{R}^{2n}$ and $d\varphi: B(\epsilon) \to \mathcal{L}(\mathbb{R}^{2n})$ guarantees constants $M_1$ and $M_2$ such that for each $x \in B(\epsilon)$,

$$
\varphi(x) = \varphi(0) + d\varphi(0)x + r(x) \quad \text{with} \quad |r(x)| \leq M_1 |x|^2,
$$

$$
d\varphi(x) = d\varphi(0) + R(x) \quad \text{with} \quad \|R(x)\| \leq M_2 |x|,
$$

where $\|R(x)\|$ denotes the operator norm of the linear operator $R(x) \in \mathcal{L}(\mathbb{R}^{2n})$. Since $\varphi(0) = 0$ and $d\varphi(0) = \text{id}$, we conclude that

$$
|\varphi(x) - d\varphi(x)x| = |r(x) - R(x)x| \leq (M_1 + M_2)|x|^2 \quad \text{if} \quad |x| \leq \epsilon
$$

and so, with $x = \eta(t)z$,

$$
(2.14) \quad \left| \frac{1}{\eta(t)} \varphi(\eta(t)z) - d\varphi(\eta(t)z)z \right| \leq (M_1 + M_2)\eta(t)|z|^2 \quad \text{if} \quad \eta(t)|z| \leq \epsilon.
$$

Assume now $z \in U$. In view of the definition (2.9) of $U$ we then have

$$
\eta(t)|z| \leq e^2e^{-2/\epsilon} \frac{\epsilon}{e} e^{1/\epsilon} = ee^{-1/\epsilon} \leq \epsilon.
$$

Inserting the estimate (2.14) into (2.10) and using (2.13) we conclude that

$$
|\nabla H_t(w)| \leq \frac{2}{t^2}(M_1 + M_2)\eta(t)|z|^2
$$

$$
\leq \frac{2}{t^2}(M_1 + M_2)e(e^{-1/\epsilon})|z|
$$

$$
\leq \frac{2}{t^2}e^{-1/\epsilon}(M_1 + M_2)e\frac{1}{L}|w|.
$$

The constant $c_1 := 2(M_1 + M_2)e\frac{1}{L}$ is as required. \hfill \square

2.3. Lemma. (i) There exists a constant $C_2 > 0$ such that

$$
|H_t(w)| \leq \frac{C_2}{t^2}|w|^2 \quad \text{for all} \quad t \in [0,1] \text{ and } w \in \phi_t(U).
$$

(ii) There exists a constant $c_2 > 0$ such that

$$
|H_t(w)| \leq \frac{c_2}{t^2}e^{-1/\epsilon}|w|^2 \quad \text{for all} \quad t \in [0,1] \text{ and } w \in \phi_t(U_t).
$$

Proof. (i) Fix $t \in [0,1]$ and $w = \phi_t(z)$. The smooth path

$$
\gamma: [0,1] \to \phi_t(U), \quad \gamma(s) = \phi_t(sz)
$$

$$
|\nabla H_t(w)| \leq \frac{C_2}{t^2}|w|^2
$$

$$
\leq \frac{c_2}{t^2}e^{-1/\epsilon}|w|^2
$$

$$
\leq \frac{c_2}{t^2}e^{-1/\epsilon}(M_1 + M_2)e\frac{1}{L}|w|.
$$
joins 0 with \( w \). Since \( H_t(0) = 0 \) we find that

\[
H_t(w) = H_t(0) + \int_0^1 \langle \nabla H_t(\gamma(s)), \gamma'(s) \rangle \, ds
\]

(2.15)

\[
= \int_0^1 \langle \nabla H_t(\phi_t(sz)), d\phi_t(sz)z \rangle \, ds.
\]

The identity \( \phi_t(0) = 0 \), the mean value theorem and Lemma 2.1 (ii) yield

\[
|\phi_t(sz)| = |\phi_t(sz) - \phi_t(0)| \leq \frac{1}{L} |sz|.
\]

Using the identity (2.15), Lemma 2.2 (i), Lemma 2.1 (ii) and the estimates (2.16) and (2.13) we can estimate

\[
|H_t(w)| \leq \int_0^1 |\nabla H_t(\phi_t(sz))| |d\phi_t(sz)z| \, ds
\]

\[
\leq C_1 \frac{1}{t^2} \frac{1}{L} |sz| \int_0^1 |\phi_t(sz)| \, ds
\]

\[
\leq C_1 \frac{1}{t^2} \frac{1}{L^2} |sz|^2 \frac{1}{2}
\]

\[
\leq \frac{1}{2} C_1 \frac{1}{L^4 t^2} |w|^2.
\]

The constant \( C_2 := \frac{1}{2} C_1 \frac{1}{L^4 t} \) is as required.

(ii) Assume now \( z \in U_t \). Using Lemma 2.2 (ii) and estimating as above we obtain

\[
|H_t(w)| \leq \frac{1}{2} C_1 \frac{1}{L^4 t^2} e^{-1/t} |w|^2.
\]

The constant \( c_2 := \frac{1}{2} C_1 \frac{1}{L^4 t} \) is as required.

Choose a smooth function \( g: [0, \infty[ \rightarrow [1, \infty[ \) such that

\[
g(r) = \begin{cases} 
1 & \text{if } r \leq \frac{1}{2}, \\
r & \text{if } r \geq 2
\end{cases}
\]

(2.17)

and \( 0 \leq g'(r) \leq 1 \) for all \( r \).

We define the smooth function \( G: \bigcup_{t \in [0,1]} \{t\} \times \phi_t(U) \rightarrow \mathbb{R} \) by

\[
G(t, w) \equiv G_t(w) := \frac{H_t(w)}{g(|w|)}.
\]

2.4. Lemma. (i) There exists a constant \( C_3 > 0 \) such that

\[
|\nabla G_t(w)| \leq \frac{C_3}{t^2} \quad \text{for all } t \in ]0, 1] \text{ and } w \in \phi_t(U).
\]
(ii) There exists a constant $c_3 > 0$ such that

$$|\nabla G_t(w)| \leq \frac{c_3}{t^2} e^{-1/t} \quad \text{for all } t \in ]0, 1[ \text{ and } w \in \phi_t(U_t).$$

Proof. (i) We have that

$$\nabla \left( \frac{1}{g(|w|)} \right) = -\frac{g'(|w|)}{g(|w|)^2} \frac{w}{|w|}$$

and so

$$\nabla G_t(w) = -\frac{g'(|w|)}{g(|w|)^2} \frac{w}{|w|} H_t(w) + \frac{1}{g(|w|)} \nabla H_t(w).$$

Using $g'(t) \in [0, 1]$, Lemma 2.3(i) and Lemma 2.2(i) and $|w| \leq g(|w|)$ we therefore find that

$$|\nabla G_t(w)| \leq \frac{1}{g(|w|)^2} |H_t(w)| + \frac{1}{g(|w|)} |\nabla H_t(w)|$$

$$\leq \frac{1}{g(|w|)^2} \frac{C_2}{t^2} |w|^2 + \frac{1}{g(|w|)} \frac{C_1}{t^2} |w|$$

$$\leq \left( C_1 + C_2 \right) \frac{1}{t^2}. $$

The constant $C_3 := C_1 + C_2$ is as required.

(ii) Assume now $z \in U_t$. Using Lemma 2.3(ii) and Lemma 2.2(ii) and estimating as above we obtain

$$|\nabla G_t(w)| \leq (c_1 + c_2) \frac{1}{t^2} e^{-1/t}. $$

The constant $c_3 := c_1 + c_2$ is as required.  \qed
2.5. Lemma. (i) There exists a constant $C_4 > 0$ such that

$$|G_t(w) - G_t(w')| \leq \frac{C_4}{t^2} |w - w'| \quad \text{for all } t \in [0, 1] \text{ and } w, w' \in \phi_t(U).$$

(ii) There exists a constant $c_4 > 0$ such that

$$|G_t(w) - G_t(w')| \leq \frac{c_4}{t^2} e^{-1/t} |w - w'| \quad \text{for all } t \in [0, 1] \text{ and } w, w' \in \phi_t(U_t).$$

Proof. (i) Fix $t \in [0, 1]$ and $w = \phi_t(z)$, $w' = \phi_t(z')$, and assume that $U$ is a Lipschitz domain with Lipschitz constant $\lambda$. We then find a smooth path $\gamma: [0, 1] \to U$ such that $\gamma(0) = z$, $\gamma(1) = z'$ and such that

$$(2.19) \quad \text{length}(\gamma) = \int_0^1 |\gamma'(s)| \, ds \leq 2\lambda |z' - z|.$$

Using Lemma 2.4(i), Lemma 2.1(ii), the estimate (2.19) and Lemma 2.1(i) we can estimate

$$|G_t(w') - G_t(w)| = \left| \int_0^1 \langle \nabla G_t(\phi_t(\gamma(s))), d\phi_t(\gamma(s))\gamma'(s) \rangle \, ds \right|$$

$$\leq \frac{C_3}{t^2} \frac{1}{L} \int_0^1 |\gamma'(s)| \, ds$$

$$\leq \frac{C_3}{t^2} \frac{1}{L} 2\lambda |z' - z|$$

$$\leq \frac{C_3}{t^2} \frac{1}{L^2} 2\lambda |w' - w|.$$

The constant $C_4 := 2C_3 \frac{1}{L^2} \lambda$ is as required.

(ii) Assume now $z, z' \in U_t$. Since $U$ is starlike, we can assume that the path $\gamma$ chosen above is contained in $U_t$. Using Lemma 2.4(ii) and estimating as above we obtain

$$|G_t(w') - G_t(w)| \leq \frac{c_3}{t^2} e^{-1/t} \frac{1}{L^2} 2\lambda |w' - w|.$$

The constant $c_4 := 2c_3 \frac{1}{L^2} \lambda$ is as required. \qed

Our next goal is to extend the function $G$ on $\bigcup_{t \in [0, 1]} \{t\} \times \phi_t(U)$ to a continuous function $\hat{G}$ on $[0, 1] \times \mathbb{R}^{2n}$ having similar properties. We shall need two auxiliary lemmata.

2.6. Lemma. (McShane $\text{[5]}$) Consider a subset $W$ of the metric space $(X, d)$ and a function $f: W \to \mathbb{R}$ which is $\lambda$-Lipschitz continuous. Then the function $\overline{f}: X \to \mathbb{R}$ defined by

$$\overline{f}(x) := \inf \{ f(w) + \lambda d(x, w) \mid w \in W \}$$

I’m grateful to Urs Lang for pointing out to me this reference.
is a $\lambda$-Lipschitz continuous extension of $f$.

2.7. Lemma. Assume that $V$ is a subset of $\mathbb{R}^{2n}$ which contains the origin and that the function $h: V \cup B(2r) \to \mathbb{R}$ is $\lambda_V$-Lipschitz continuous on $V$ and $\lambda_B$-Lipschitz continuous on $B(2r)$. Then $h$ is $(2\lambda_V + \lambda_B)$-Lipschitz continuous on $V \cup B(r)$.

Proof. Fix $w, w' \in V \cup B(r)$. If $w, w' \in V$ or $w, w' \in B(2r)$ then by assumption

$$|h(w) - h(w')| \leq \max(\lambda_V, \lambda_B) |w - w'|.$$

So assume that $w \in V \setminus B(2r)$ and $w' \in B(r)$. Then $|w'| \leq r \leq \frac{|w|}{2}$ and so

$$\frac{|w|}{2} \leq |w| - |w'| \leq |w - w'|.$$

Since $0 \in V$ and $0 \in B(2r)$ we can now estimate

$$|h(w) - h(w')| \leq |h(w) - h(0)| + |h(w') - h(0)|$$

$$\leq \lambda_V |w| + \lambda_B |w'|$$

$$\leq (2\lambda_V + \lambda_B) \frac{|w|}{2}$$

$$\leq (2\lambda_V + \lambda_B) |w - w'|,$$

and so $h$ is $(2\lambda_V + \lambda_B)$-Lipschitz continuous on $V \cup B(r)$. \qed

2.8. Lemma. There exists a continuous function $\hat{G}: [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}$ with the following properties.

(i) $\hat{G}(t, w) = G(t, w)$ for all $t \in [0, 1]$ and $w \in \phi_t(U)$.

(ii) There exists a constant $C_5 > 0$ such that

$$|\hat{G}_t(w) - \hat{G}_t(w')| \leq C_5 \frac{|w - w'|}{t^2} \quad \text{for all } t \in [0, 1] \text{ and } w, w' \in \mathbb{R}^{2n}.$$

Proof. We shall first construct a function $\hat{G}: [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}$ meeting assertions (i) and (ii), and shall then verify that $\hat{G}$ is continuous.

We set $\hat{G}(0, w) = 0$ for all $w \in \mathbb{R}^{2n}$. Since $H: \bigcup_{t \in [0, 1]} \phi_t(U) \to \mathbb{R}$ is continuous, Lemma 2.3(ii) and the definition (2.18) of $U_t$ imply that $H(0, w) = 0$ for all $w \in U$. In view of definition (2.18) we therefore have $G(0, w) = 0$ for all $w \in U$, and so assertion (i) holds for $t = 0$.

We now fix $t \in [0, 1]$. We define the number $R_t$ by

$$R_t = \frac{L}{2} e^{1/t}. \quad (2.20)$$
Fix \( w = \phi_t(z) \in B(2R_t) \). In view of the estimate (2.13) and the definition (2.20) we have
\[
|z| \leq \frac{|w|}{L} \leq \frac{2}{L} R_t = \frac{\epsilon}{e^{1/t}},
\]
and so \( z \in U_t \) in view of definition (2.9). Lemma 2.3(ii) therefore implies that the function \( G_t \) is \( \frac{C_4}{t^2} e^{-1/t} \)-Lipschitz continuous on \( \phi_t(U) \cap B(2R_t) \). According to Lemma 2.6 the function \( \overline{G}_t : B(2R_t) \to \mathbb{R} \) defined by
\[
\overline{G}_t(x) := \inf \left\{ G_t(w) + \frac{C_4}{t^2} e^{-1/t} |x - w| \mid w \in \phi_t(U) \cap B(2R_t) \right\}
\]
is a \( \frac{C_5}{t^2} e^{-1/t} \)-Lipschitz extension of \( G_t \) to \( B(2R_t) \). In particular, the function \( \overline{G}_t : \phi_t(U) \cup B(2R_t) \to \mathbb{R} \),
\[
\overline{G}_t(x) := \begin{cases} G_t(x) & \text{if } x \in \phi_t(U), \\ G_t(x) & \text{if } x \in B(2R_t), \end{cases}
\]
is well-defined. According to Lemma 2.3(i), \( \overline{G}_t \) is \( \frac{C_4}{t^2} \)-Lipschitz continuous on \( \phi_t(U) \), and according to the above, \( \overline{G}_t \) is \( \frac{C_5}{t^2} \)-Lipschitz continuous on \( B(2R_t) \). According to Lemma 2.7 the restriction of \( \overline{G}_t \) to \( \phi_t(U) \cup B(R_t) \) is therefore \( \frac{C_5}{t^2} \)-Lipschitz continuous where we abbreviated
\[
C_5 := 2C_4 + c_4.
\]
Applying Lemma 2.6 once more, we find that the function \( \hat{G}_t : \mathbb{R}^{2n} \to \mathbb{R} \) defined by
\[
\hat{G}_t(x) := \inf \left\{ \overline{G}_t(w) + \frac{C_5}{t^2} |x - w| \mid w \in \phi_t(U) \cup B(R_t) \right\}
\]
is a \( \frac{C_5}{t^2} \)-Lipschitz extension of the restriction of \( \overline{G}_t \) to \( \phi_t(U) \cup B(R_t) \). In particular,
\[
\hat{G}(t,w) = G(t,w) \quad \text{for all } w \in \phi_t(U).
\]
The function \( \hat{G} : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R} \) thus defined therefore meets assertion (i) for \( t \in [0,1] \) and assertion (ii).

We are left with showing that the function \( \hat{G} : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R} \) constructed in the previous two steps is continuous. The definitions (2.20), (2.21), (2.22) and (2.23) show that the functions \( \hat{G}(\cdot,w) : [0,1] \to \mathbb{R} \) and \( \hat{G}(t,\cdot) : \mathbb{R}^{2n} \to \mathbb{R} \) are continuous. This and the fact that the functions \( \hat{G}(t,\cdot) \) are \( \frac{C_5}{t^2} \)-Lipschitz continuous imply that \( \hat{G} : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R} \)
is continuous. In order to show that $\hat{G}$ is also continuous at $(0, w)$ for each $w \in \mathbb{R}^{2n}$ we fix $w \in \mathbb{R}^{2n}$. We choose an open ball $B_w \subset \mathbb{R}^{2n}$ centered at $w$. In view of definition (2.20) we have $R_t \to \infty$ as $t \to 0^+$. We therefore find $t_0 > 0$ such that $B_w \subset B(R_t)$ for all $t \in [0, t_0]$. We fix $t \in [0, t_0]$ and $w' \in B_w$. Recalling the definition of $\hat{G}(t, w') \equiv \hat{G}_t(w')$ we see that
\[
\hat{G}_t(w') = \overline{G}_t(w') = \overline{G}_t(w) = \inf \left\{ G_t(v) + \frac{c_4}{t^2} e^{-1/t} |w' - v| \mid v \in \phi_t(U) \cap B(2R_t) \right\}.
\]

Since $0 = \phi_t(0) \in \phi_t(U) \cap B(2R_t)$ and $G_t(0) = H_t(0) = 0$ we conclude that
\[
\hat{G}_t(w') \leq \frac{c_4}{t^2} e^{-1/t} |w'|.
\]

Moreover, we recall from the beginning of the proof of Lemma 2.8 that $G_t$ is $\frac{c_4}{t^2} e^{-1/t}$-Lipschitz continuous on $\phi_t(U) \cap B(2R_t)$. This and $G_t(0) = 0$ yield
\[
|G_t(v)| \leq \frac{c_4}{t^2} e^{-1/t} |v| \quad \text{for all} \quad v \in \phi_t(U) \cap B(2R_t).
\]

Therefore,
\[
G_t(v) + \frac{c_4}{t^2} e^{-1/t} |w' - v| \geq -|G_t(v)| + \frac{c_4}{t^2} e^{-1/t} |w' - v| \geq \frac{c_4}{t^2} e^{-1/t} (-|v| + |w' - v|) \geq -\frac{c_4}{t^2} e^{-1/t} |w'|.
\]
for all $v \in \phi_t(U) \cap B(2R_t)$. We conclude that
\begin{equation}
\hat{G}_t(w') \geq -\frac{c_4}{t^2} e^{-1/t} |w'|.
\end{equation}
The estimates (2.24) and (2.25), which hold for all $t \in [0, t_0]$ and $w' \in B_w$, now imply that
\begin{equation}
\left| \hat{G}_t(w') \right| \leq \frac{c_4}{t^2} e^{-1/t} |w'| \quad \text{for all } t \in [0, t_0] \text{ and } w' \in B_w
\end{equation}
and so $\hat{G}$ is continuous at $(0, w)$. This completes the proof of Lemma 2.8. \hfill \Box

Let now $A$ be a subset of $U$ whose closure in $\mathbb{R}^{2n}$ is contained in $U$. Since also the origin is contained in $U$, we can assume that $A$ is closed and $0 \in A$. We abbreviate $\mathcal{A} := \bigcup_{t \in [0,1]} \{t\} \times \phi_t(A)$.

The next step is to smoothen $\hat{G}$ in the variable $w$ in such a way that the smoothened function $\tilde{G}$ coincides with $\hat{G}$ on $\mathcal{A}$. We shall first construct a smooth function $G^*$ which approximates $\hat{G}$ very well and shall then obtain $\tilde{G}$ by interpolating between $\hat{G}$ and $G^*$.

Since $\mathbb{R}^{2n}$ is a normal space, we find an open set $V$ in $\mathbb{R}^{2n}$ such that $A \subset V \subset \overline{V} \subset U$. Then
\begin{equation}
\phi_t(A) \subset \phi_t(V) \subset \phi_t(\overline{V}) = \phi_t(\overline{V}) \subset \phi_t(U) \quad \text{for all } t \in [0,1].
\end{equation}
We abbreviate $\mathcal{V} := \bigcup_{t \in [0,1]} \{t\} \times \phi_t(V)$.

Since $\mathcal{A}$ is closed and $\mathcal{V}$ is open in $[0,1] \times \mathbb{R}^{2n}$, we find a smooth function $f : [0,1] \times \mathbb{R}^{2n} \to [0,1]$ such that
\begin{equation}
f|_{\mathcal{A}} = 1 \quad \text{and} \quad f|_{[0,1] \times \mathbb{R}^{2n}\setminus \mathcal{V}} = 0.
\end{equation}
We say that a continuous function $F : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ is smooth in the variable $w \in \mathbb{R}^{2n}$ if all derivatives $D^k F_t(w)$ of $F$ with respect to $w$ exist and are continuous on $[0,1] \times \mathbb{R}^{2n}$.

**2.9. Lemma.** There exists a continuous function $G^* : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ which is smooth in the variable $w \in \mathbb{R}^{2n}$ and has the following properties.

(i) $|\nabla f_t(w)| |G^*_t(w) - \hat{G}_t(w)| \leq \frac{C_5}{t^2} \quad \text{for all } t \in [0,1] \text{ and } w \in \mathbb{R}^{2n}$. 
(ii) \(|\nabla G_i^*(w)| \leq \frac{2C_5}{t^2} \) for all \( t \in [0, 1] \) and \( w \in \mathbb{R}^{2n} \).

**Proof.** For each \( l \in \mathbb{N} \) we define the open subset \( V_l \) of \([0, 1] \times \mathbb{R}^{2n}\) by

\[
V_l := \{ (t, w) \in [0, 1] \times \mathbb{R}^{2n} \mid |\nabla f_t(w)| < l \}.
\]

Then there exists a smooth partition of unity \( \{ \theta_i \}_{i \in \mathbb{N}} \) on \([0, 1] \times \mathbb{R}^{2n}\) such that for each \( i \) the support \( \text{supp} \theta_i \) is compact and contained in some \( V_l \). We let \( l_i \) be a number such that \( \text{supp} \theta_i \subset V_{l_i} \). Since \( \{ \text{supp} \theta_i \} \) form a locally finite covering of \([0, 1] \times \mathbb{R}^{2n}\), the set

\[
\Theta_i := \{ j \in \mathbb{N} \mid \text{supp} \theta_i \cap \text{supp} \theta_j \neq \emptyset \}
\]

is finite; let its cardinality be \( m_i \). We set

\[
M_i := \max \{ m_j \mid j \in \Theta_i \}.
\]

Since the functions \( \theta_i \) have compact support, the numbers

\[
\mu_i := \max \{ |\nabla \theta_i(t, w)| \mid (t, w) \in [0, 1] \times \mathbb{R}^{2n} \} + 1
\]

are finite. We define positive numbers \( r_i \) by

\[
r_i := \frac{1}{l_i M_i \mu_i}.
\]

We next choose a smooth bump function \( K: \mathbb{R}^{2n} \to [0, \infty[ \) such that \( \text{supp} K \subset B(1) \) and \( \int_{\mathbb{R}^{2n}} K(v) \, dv = 1 \). We abbreviate

\[
\kappa := \max \{ |\nabla K(v)| \mid v \in \mathbb{R}^{2n} \}.
\]

For each \( i \) we define a smooth function \( K_i: \mathbb{R}^{2n} \to [0, \infty[ \) by

\[
K_i(w) := \frac{1}{r_{i}^{2n+1}} K \left( \frac{w}{r_i} \right).
\]

Then \( \text{supp} K_i \subset B(r_i) \) and \( \int_{\mathbb{R}^{2n}} K_i(v) \, dv = 1 \), and

\[
|\nabla K_i(w)| \leq \frac{1}{r_{i}^{2n+1}} \kappa \quad \text{for all } w \in \mathbb{R}^{2n}.
\]

Let \( \hat{G} \) be the function guaranteed by Lemma 2.8. For each \( i \) we define the function \( G_i^* : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R} \) as the convolution

\[
G_i^*(t, w) := \left( \hat{G}_t * K_i \right)(w) \equiv \int_{\mathbb{R}^{2n}} \hat{G}_t(v) K_i(w - v) \, dv.
\]

Since for each \( t \) the function \( \hat{G}_t \) is continuous and since \( K_i \) is smooth, the function \( w \mapsto G_i^*(t, w) \) is smooth and

\[
D^k G_i^*(t, w) = \int_{\mathbb{R}^{2n}} \hat{G}_t(v) D^k K_i(w - v) \, dv, \quad k = 0, 1, 2, \ldots
\]
The function $\hat{G}$ is continuous, and $D^k K_i$ is continuous and has compact support and is thus uniformly continuous. Formula (2.34) therefore shows that $D^k G_i^*$ is continuous, $k = 0, 1, 2, \ldots$, and so $G_i^*$ is continuous and smooth in $w$. It follows that the function $G^* : [0, 1] \times \mathbb{R}^{2^n} \to \mathbb{R}$ defined by

$$G^*(t, w) := \sum_i \theta_i(t, w) G_i^*(t, w)$$

is continuous and smooth in $w$. In order to prove assertions (i) and (ii) we fix $t \in [0, 1]$ and abbreviate

$$\theta_i(w) = \theta_i(t, w), \quad \hat{G}(w) = \hat{G}(t, w), \quad G_i^*(w) = G_i^*(t, w), \quad G^*(w) = G^*(t, w).$$

Proof of (i). Using the definition (2.33) of the function $G_i^*$ and the identity $\int_{\mathbb{R}^{2^n}} K_j(v) \, dv = 1$ we find

$$G_j^*(w) - \hat{G}(w) = \int_{\mathbb{R}^{2^n}} \left( \hat{G}(v) - \hat{G}(w) \right) K_j(w - v) \, dv$$

and so, together with Lemma 2.8 (ii),

$$\left| G_j^*(w) - \hat{G}(w) \right| \leq \int_{\mathbb{R}^{2^n}} \left| \hat{G}(w - v) - \hat{G}(w) \right| K_j(v) \, dv$$

$$\leq \int_{B(r_j)} \frac{C_5}{t^2} |v| K_j(v) \, dv$$

$$\leq \frac{C_5}{t^2} r_j \int_{B(r_j)} K_j(v) \, dv$$

$$= \frac{C_5}{t^2} r_j.$$

If $\nabla f_t(w) = 0$, assertion (i) is obvious. So assume $|\nabla f_t(w)| > 0$. Recall from the definitions (2.29) and (2.30) that $M_j \geq 1$ and $\mu_j \geq 1$. This, the definition (2.31) of $r_j$, the inclusion supp$\theta_j \subset V_j$, and the definition (2.28) of $V_j$ yield

$$r_j = \frac{1}{i_j M_j \mu_j} \leq \frac{1}{i_j} \leq \frac{1}{|\nabla f_t(w)|} \quad \text{for all } w \in \text{supp } \theta_j.$$
The definition (2.35) of $G^*$ and the estimates (2.36) and (2.37) now yield

$$
\left| G^*(w) - \hat{G}(w) \right| = \left| \sum_j \theta_j(w) \left( G_j^*(w) - \hat{G}(w) \right) \right|
\leq \sum_j \theta_j(w) \frac{C_5}{t^2} \frac{1}{|\nabla f_i(w)|}
= \frac{C_5}{t^2} \frac{1}{|\nabla f_i(w)|}
$$

and so assertion (i) follows.

**Proof of (ii).** Using the definition (2.35) of $G^*$ and the identities

$$
\sum_j \theta_j(w) = \sum_j \theta_j(w') = 1
$$
we compute that for all $w, w' \in \mathbb{R}^{2n},$

$$
G^*(w') - G^*(w) = \sum_j \theta_j(w') G_j^*(w') - \sum_j \theta_j(w) G_j^*(w) = \sum_j \left( \theta_j(w') - \theta_j(w) \right) \left( G_j^*(w') - \hat{G}(w') \right)
\quad = \sum_j \theta_j(w) \left( G_j^*(w) - G_j^*(w') \right).
\quad (2.38)
$$

Fix now $w$. We choose $i$ such that $\theta_i(w) > 0$, and we choose an open ball $B_w \subset \mathbb{R}^{2n}$ centered at $w$ such that $B_w \subset \text{supp } \theta_i$. Fix $w' \in B_w$. In view of the mean value theorem and the definition (2.30) of $\mu_j$ we find that

$$
|\theta_j(w') - \theta_j(w)| \leq \max_{v \in B_w} |\nabla \theta_j(v)||w' - w| \leq \mu_j |w' - w|
\quad (2.39)
$$
and the estimate (2.36) with $w$ replaced by $w'$ yields

$$
\left| G_j^*(w') - \hat{G}(w') \right| \leq \frac{C_5}{t^2} r_j.
\quad (2.40)
$$
The definition (2.29) of $M_j$ implies that $M_j \geq m_i$ whenever $j \in \Theta_i$, and so

$$
\sum_{j \in \Theta_i} \frac{1}{M_j} \leq \sum_{j \in \Theta_i} \frac{1}{m_i} = 1
\quad (2.41)
$$
in view of the definition of $m_i$. The definition (2.31) of $r_j$ and the inequalities $l_j \geq 1$ and (2.41) yield

$$
\sum_{j \in \Theta_i} \mu_j r_j = \sum_{j \in \Theta_i} \mu_j \frac{1}{l_j M_j \mu_j} \leq \sum_{j \in \Theta_i} \frac{1}{M_j} \leq 1.
\quad (2.42)
$$
Since \( w, w' \in B_w \subset \text{supp } \theta \), we have \( \theta_j(w') - \theta_j(w) = 0 \) if \( j \notin \Theta_i \). This and the estimates (2.39), (2.40) and (2.42) now show that

\[
\left| \sum_j (\theta_j(w') - \theta_j(w)) \left( G_j^*(w') - \tilde{G}(w') \right) \right| \leq \sum_{j \in \Theta_i} \mu_j |w' - w| \frac{C_5}{t^2} r_j
\]

(2.43)

\[
\leq \frac{C_5}{t^2} |w' - w|.
\]

Next, the definition (2.33) of \( G_j^* \) and the identity \( \int_{\mathbb{R}^{2n}} K_j(v) \, dv = 1 \) yield

\[
G_j^*(w') - G_j^*(w) = \int_{\mathbb{R}^{2n}} \tilde{G}(v) (K_j(w' - v) - K_j(w - v)) \, dv
\]

\[
= \int_{\mathbb{R}^{2n}} \left( \tilde{G}(w' - v) - \tilde{G}(w - v) \right) K_j(v) \, dv.
\]

Together with Lemma 2.8 (ii) we obtain

\[
\left| G_j^*(w') - G_j^*(w) \right| \leq \frac{C_5}{t^2} \int_{\mathbb{R}^{2n}} |w' - w| K_j(v) \, dv = \frac{C_5}{t^2} |w' - w|
\]

and so

(2.44) \[
\left| \sum_j \theta_j(w) (G_j^*(w') - G_j^*(w)) \right| \leq \frac{C_5}{t^2} |w' - w|.
\]

The identity (2.38) and the estimates (2.43) and (2.44) now imply

\[
|G^*(w') - G^*(w)| \leq \frac{2C_5}{t^2} |w' - w|.
\]

Since \( w' \in B_w \) was arbitrary, we conclude that

\[
|\nabla G^*(w)| \leq \frac{2C_5}{t^2}
\]

and so assertion (ii) follows. The proof of Lemma 2.9 is complete. \( \Box \)

2.10. Lemma. There exists a continuous function \( \tilde{G} : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R} \) which is smooth in the variable \( w \in \mathbb{R}^{2n} \) and has the following properties.

(i) \( \tilde{G}(t, w) = G(t, w) \) for all \( t \in [0, 1] \) and \( w \in \phi_t(A) \).

(ii) There exists a constant \( C_6 > 0 \) such that

\[
|\nabla \tilde{G}_t(w)| \leq \frac{C_6}{t^2} \text{ for all } t \in [0, 1] \text{ and } w \in \mathbb{R}^{2n}.
\]
Proof. Let \( f : [0, 1] \times \mathbb{R}^{2n} \to [0, 1] \) be the smooth function chosen before Lemma 2.9 and let \( \tilde{G} \) and \( G^* \) be the continuous functions on \([0, 1] \times \mathbb{R}^{2n}\) guaranteed by Lemma 2.8 and Lemma 2.9.

We define a continuous function \( \tilde{G} : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R} \) by

\[
\tilde{G}(t, w) := f(t, w)\hat{G}(t, w) + (1 - f(t, w))G^*(t, w).
\]

The inclusions (2.26) and the identities (2.27), Lemma 2.8(i) and the fact that \( G^* \) is smooth in \( w \) imply that \( \tilde{G} \) is smooth in \( w \) and that assertion (i) holds true. In order to verify assertion (ii) we fix \( t \in [0, 1] \).

We first assume \( w \in \phi_t(U) \). On \( \phi_t(U) \) we have \( \hat{G}_t = G_t \), and so

\[
\nabla \tilde{G}_t = \nabla f_t \left( \tilde{G}_t - G_t^* \right) + f_t \nabla G_t + (1 - f_t) \nabla G_t^*.
\]

In view of Lemma 2.9(i), Lemma 2.4(i) and Lemma 2.9(ii) we can therefore estimate

\[
\left| \nabla \tilde{G}_t \right| \leq |\nabla f_t| \left| G_t^* - \tilde{G}_t \right| + |\nabla G_t| + |\nabla G_t^*| \leq \frac{C_5}{t^2} + \frac{C_3}{t^2} + \frac{2C_5}{t^2}.
\]

We next assume \( w \in \mathbb{R}^{2n} \setminus \phi_t(V) \). On \( \mathbb{R}^{2n} \setminus \phi_t(V) \) we have \( f_t \equiv 0 \), and so

\[
\left| \nabla \tilde{G}_t \right| = |\nabla G_t^*| \leq \frac{2C_5}{t^2}.
\]

Setting \( C_6 := C_3 + 3C_5 \) assertion (ii) follows. The proof of Lemma 2.10 is complete.

We are now in a position to define the desired extension \( \tilde{H} \) of \( H \). Let \( g \) be the function chosen in (2.17) and let \( \tilde{G} \) be the function guaranteed by Lemma 2.10.

2.11. Lemma. The continuous function \( \tilde{H} : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R} \) defined by

\[
\tilde{H}(t, w) \equiv \tilde{H}_t(w) := g(|w|)\tilde{G}_t(w).
\]

is smooth in the variable \( w \in \mathbb{R}^{2n} \) and has the following properties.

(i) \( \tilde{H}(t, w) = H(t, w) \) for all \( t \in [0, 1] \) and \( w \in \phi_t(A) \).

(ii) There exists a constant \( C > 0 \) such that

\[
\left| \nabla \tilde{H}_t \right| \leq \frac{C}{t^2} \left( |w| + 1 \right) \text{ for all } t \in [0, 1] \text{ and } w \in \mathbb{R}^{2n}.
\]
Proof. Since \( g: [0, \infty[ \to [1, \infty[ \) is smooth and \( \tilde{G}: [0,1] \times \mathbb{R}^{2n} \to \mathbb{R} \) is continuous, the function \( \tilde{H} \) is indeed continuous, and since \( g(r) = 1 \) if \( r \leq \frac{1}{2} \) and \( \tilde{G} \) is smooth in \( w \), the function \( \tilde{H} \) is smooth in \( w \). Assertion (i) follows from the definition (2.45) of \( \tilde{H} \), from Lemma 2.10 (i) and from the definition (2.18) of \( G \). In order to verify assertion (ii) we fix \( t \in ]0,1] \) and \( w \in \mathbb{R}^{2n} \). Using definition (2.45) we compute

\[
\nabla \tilde{H}_t(w) = g'(|w|) \frac{w}{|w|} \tilde{G}_t(w) + g(|w|) \nabla \tilde{G}_t(w).
\]

Since \( 0 \in A \) and \( \phi_t(0) = 0 \) we have, together with equation (2.6), \( \tilde{G}_t(0) = G_t(0) = H_t(0) = 0 \).

This, the mean value theorem and Lemma 2.10 (ii) yield

\[
|\tilde{G}_t(w)| \leq \frac{C_6}{t^2} |w| \quad \text{and} \quad |\nabla \tilde{G}_t(w)| \leq \frac{C_6}{t^2}.
\]

Using the identity (2.47), the estimates (2.48) and the estimates \(|g'(r)| \leq 1 \) and \( g(r) \leq r + 2 \) holding for all \( r \geq 0 \) we can estimate

\[
|\nabla \tilde{H}_t(w)| \leq \frac{C_6}{t^2} |w| + (|w| + 2) \frac{C_6}{t^2} = 2 \frac{C_6}{t^2} (|w| + 1).
\]

Setting \( C := 2C_6 \) assertion (ii) follows. The proof of Lemma 2.11 is complete.

Theorem 1.7 is a consequence of Lemma 2.11. The time-dependent vector field \( \nabla \tilde{H}_t(w) \) on \([0,1] \times \mathbb{R}^{2n} \) is continuous, and since it is smooth in \( w \), it is locally Lipschitz continuous in \( w \). This and assertion (ii) of Lemma 2.11 imply that the Hamiltonian system associated with \( \tilde{H} \) can be solved for all \( t \in [0,1] \). We define \( \Phi_A \) to be the resulting time-1-map. Since \( \nabla \tilde{H}_t(w) \) is continuous and smooth in \( w \), the map \( \Phi_A \) is smooth (see [1, Proposition 9.4]), and so \( \Phi_A \) is a globally defined symplectomorphism of \( \mathbb{R}^{2n} \). Moreover, Lemma 2.11 (i) shows that \( \Phi_A|_A = \varphi|_A \). The proof of Theorem 1.7 is finally complete.

\[\Box\]

2.12. Remark. Proceeding as in Step 2 we obtain a smooth Hamiltonian

\[
H_A: [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}
\]

which generates the symplectomorphism \( \Phi_A \) and is such that \( H_A|_A = \tilde{H}|_A \). However, \( H_A \) might not be \( C^0 \)-close to \( \tilde{H} \), and \( \nabla H_A \) might not be linearly bounded.
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