Implications of the Ganea Condition

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Abstract Suppose the spaces $X$ and $X \times A$ have the same Lusternik-Schnirelmann category: $\text{cat}(X \times A) = \text{cat}(X)$. Then there is a strict inequality $\text{cat}(X \times (A \times B)) < \text{cat}(X) + \text{cat}(A \times B)$ for every space $B$, provided the connectivity of $A$ is large enough (depending only on $X$). This is applied to give a partial verification of a conjecture of Iwase on the category of products of spaces with spheres.

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Introduction

The product formula $\text{cat}(X \times Y) \leq \text{cat}(X) + \text{cat}(Y)$ [1] is one of the most basic relations of Lusternik-Schnirelmann category. Taking $Y = S^r$, it implies that $\text{cat}(X \times S^r) \leq \text{cat}(X) + 1$ for any $r > 0$. In [5], Ganea asked whether the inequality can ever be strict in this special case. The study of the ‘Ganea condition’ $\text{cat}(X \times S^r) = \text{cat}(X) + 1$ has been, and remains, a formidable challenge to all techniques for the calculation of Lusternik-Schnirelmann category. In fact, it was only recently that techniques were developed which were powerful enough to identify a space which does not satisfy the Ganea condition [8] (see also [9, 12]). It is still not well understood exactly which spaces $X$ do not satisfy the Ganea condition, although it has been conjectured that they are precisely those spaces for which $\text{cat}(X)$ is not equal to the related invariant $\text{Qcat}(X)$ (see [14, 17]).

Since the failure of the Ganea condition appears to be a strange property for a space to have, it is reasonable to expect that such failure would have useful and interesting implications. In this paper we explore some of the implications of the equation $\text{cat}(X \times A) = \text{cat}(X)$ for general spaces $A$, and for $A = S^r$ in particular.
A brief look at the method of the paper \cite{8} will help to put our results into proper perspective. The new techniques begin with the following question: if $Y = X \cup_f e^{t+1}$, the cone on $f : S^t \to X$, then how can we tell if $\text{cat}(Y) > \text{cat}(X)$? It is shown (see \cite{9} Thm. 5.2 and \cite{12} Thm. 3.6) that, if $t \geq \dim(X)$, then $\text{cat}(Y) = \text{cat}(X) + 1$ if and only if a certain Hopf invariant $H_s(f)$ (which is a set of homotopy classes) does not contain the trivial map $\ast$. It is also shown \cite{9} Thm. 3.8 that if $\ast \in \Sigma^r H_s(f)$, then $\text{cat}(Y \times S^r) \leq \text{cat}(X) + 1$. Thus $Y$ does not satisfy Ganea’s condition if $\ast \notin H_s(f)$, but there is at least one $h \in H_s(f)$ such that $\Sigma^r h \simeq \ast$.

Of course, if $\Sigma^r h \simeq \ast$, then $\Sigma^{r+1} h \simeq \ast$ as well, and this suggests the following conjecture (formulated in \cite{8} Conj. 1.4):

**Conjecture** If $\text{cat}(X \times S^r) = \text{cat}(X)$, then $\text{cat}(X \times S^{r+1}) = \text{cat}(X)$.

In this paper we prove that this conjecture is true, provided $r$ is large enough.

**Theorem 1** Suppose $X$ is a $(c-1)$-connected space and let $r > \dim(X) - c \cdot \text{cat}(X) + 2$. If $\text{cat}(X \times S^r) = \text{cat}(X)$, then

$$\text{cat}(X \times S^t) = \text{cat}(X)$$

for all $t \geq r$.

The conjecture remains open for small values of $r$.

Our main result is much more general: it shows how the equation $\text{cat}(X \times A) = \text{cat}(X)$ governs the Lusternik-Schnirelmann category of products of $X$ with a vast collection of other spaces.

**Theorem 2** Let $X$ be a $(c-1)$-connected space and let $A$ be $(r-1)$-connected with $r > \dim(X) - c \cdot \text{cat}(X) + 2$. If $\text{cat}(X \times A) = \text{cat}(X)$ then

$$\text{cat}(X \times (A \times B)) < \text{cat}(X) + \text{cat}(A \times B)$$

for every space $B$.

Here $A \times B = (A \times B)/B$ is the half-smash product of $A$ with $B$. When $A$ is a suspension, the half-smash product decomposes as $A \times B \simeq A \vee (A \wedge B)$ (see, for example, \cite{12} Lem. 5.9), so we obtain the following.

**Corollary** Under the conditions of Theorem 2, if $A$ is a suspension, then

$$\text{cat}(X \times (A \wedge B)) = \text{cat}(X)$$
for every space $B$.

Our partial verification of the conjecture is an immediate consequence of this corollary: it the special case $A = S^r$ and $B = S^{s-r}$.

**Organization of the paper** In Section 1 we recall the necessary background information on homotopy pushouts, cone length and Lusternik-Schnirelmann category. We introduce an auxiliary space and establish its important properties in Section 2. The proof of Theorem 2 is presented in Section 3.

## 1 Preliminaries

In this paper all spaces are based and have the pointed homotopy type of CW complexes; maps and homotopies are also pointed. We denote by $\ast$ the one point space and any nullhomotopic map. Much of our exposition uses the language of homotopy pushouts; we refer to [9] for the definitions and basic properties.

### 1.1 Homotopy Pushouts

We begin by recalling some basic facts about homotopy pushout squares. We call a sequence $A \to B \to C$ a cofiber sequence if the associated square

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
\ast & \to & C
\end{array}
\]

is a homotopy pushout square. The space $C$ is called the cofiber of the map $f$. One special case that we use frequently is the half-smash product $A \sma B$, which is the cofiber of the inclusion $B \to A \times B$.

Finally, we recall the following result on products and homotopy pushouts.

**Proposition 3** Let $X$ be any space. Consider the squares

\[
\begin{array}{ccc}
A & \to & B \\
\downarrow & & \downarrow \\
C & \to & D \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X \times A & \to & X \times B \\
\downarrow & & \downarrow \\
X \times C & \to & X \times D
\end{array}
\]

If the first square is a homotopy pushout, then so is the second.

**Proof** This follows from Theorem 6.2 in [9].

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1.2 Cone Length and Category

A cone decomposition of a space \( Y \) is a diagram of the form

\[
\begin{array}{ccc}
L_0 & \longrightarrow & L_1 \\
\downarrow & & \downarrow \\
Y_0 & \longrightarrow & Y_1 \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
Y_{k-1} & \longrightarrow & Y_k
\end{array}
\]

in which \( Y_0 = * \), each sequence \( L_i \rightarrow Y_i \rightarrow Y_{i+1} \) is a cofiber sequence, and \( Y_k \simeq Y \); the displayed cone decomposition has length \( k \). The cone length of \( Y \), denoted \( \text{cl}(Y) \), is defined by

\[
\text{cl}(Y) = \begin{cases} 
0 & \text{if } Y \simeq * \\
\infty & \text{if } Y \text{ has no cone decomposition, and} \\
k & \text{if the shortest cone decomposition of } Y \text{ has length } k.
\end{cases}
\]

The Lusternik-Schnirelmann category of \( X \) may be defined in terms of the cone length of \( X \) by the formula

\[
\text{cat}(X) = \inf \{ \text{cl}(Y) \mid X \text{ is a homotopy retract of } Y \}.
\]

Berstein and Ganea proved this formula in [3, Prop. 1.7] with \( \text{cl}(Y) \) replaced by the strong category of \( Y \); the formula above follows from another result of Ganea — strong category is equal to cone length [7]. It follows directly from this definition that if \( X \) is a homotopy retract of \( Y \), then \( \text{cat}(X) \leq \text{cat}(Y) \). The reader may refer to [10] for a survey of Lusternik-Schnirelmann category.

The category of \( X \) can be defined in another way that is essential to our work. Begin by defining the \( n \)-th Ganea fibration sequence

\[
F_n(X) \longrightarrow G_n(X) \longrightarrow X,
\]

let \( \overline{G}_{n+1}(X) = G_n(X) \cup CF_n(X) \) be the cofiber of \( p_n \) and define \( \overline{p}_{n+1} : \overline{G}_{n+1}(X) \rightarrow X \) by sending the cone to the base point of \( X \). The \((n + 1)\)th Ganea fibration \( p_{n+1} : G_{n+1}(X) \rightarrow X \) results from converting the map \( \overline{p}_{n+1} \) to a fibration. The following result is due to Ganea (cf. Svarc).

**Theorem 4** For any space \( X \),

(a) \( \text{cl}(G_n(X)) \leq n \),

(b) the map \( p_n : G_n(X) \rightarrow X \) has a section if and only if \( \text{cat}(X) \leq n \), and
(c) $F_n(X) \simeq (\Omega(X))^\ast(n+1)$, the $(n+1)$-fold join of $\Omega X$ with itself.

**Proof** Assertion (a) follows immediately from the construction. For parts (b) and (c), see [6]; these results also appear, from a different point of view, in [16]. 

2 An Auxiliary Space

Let $\tilde{G}_n$ denote the homotopy pushout in the square

\[
\begin{array}{ccc}
G_{n-1}(X) & \xrightarrow{i_1} & G_{n-1}(X) \times A \\
\downarrow & & \downarrow \\
G_n(X) & \xrightarrow{p_n} & \tilde{G}_n.
\end{array}
\]

The maps $p_n : G_n(X) \to X$ and $1_A : A \to A$ piece together to give a map $\tilde{p}_n : \tilde{G}_n \to X \times A$. The space $\tilde{G}_n$ and the map $\tilde{p}_n$ play key roles in the forthcoming constructions; this section is devoted to establishing some of their properties.

2.1 Category Properties of $\tilde{G}_n$

We begin by estimating the category of $\tilde{G}_n$.

**Proposition 5** For any noncontractible $A$ and $n > 0$, $\text{cat}(\tilde{G}_n) < n + \text{cat}(A)$.

**Proof** For simplicity in this proof, we write $F_i$ for $F_i(X)$ and $G_i$ for $G_i(X)$. Let $A$ be a retract of another space $A'$ with $\text{cl}(A') = k$. Let $\tilde{G}_n = G_n \cup G_{n-1} \times A'$; clearly $\tilde{G}_n$ is a homotopy retract of $\tilde{G}_n'$ and so it suffices to show that $\text{cl}(\tilde{G}_n') < n + k$. Let

\[
\begin{array}{cccccc}
& L_0 & L_1 & \cdots & L_{k-1} & \\
A_0 & \xrightarrow{} & A_1 & \xrightarrow{} & \cdots & \xrightarrow{} & A_{k-1} & \xrightarrow{} & A_k
\end{array}
\]

be a cone decomposition of $A'$. We will also use the cone decomposition of $G_n$ given by the cofiber sequences $F_{i-1} \to G_{i-1} \to G_i$. According to a result of Baues [2] (see also [13, Prop. 2.9]), for each $i$ and $j$ there is a cofiber sequence

\[
F_{i-1} \ast L_{j-1} \to G_i \times A'_{i-1} \cup G_{i-1} \times A'_j \to G_i \times A'_j.
\]
Now define subspaces $W_s \subseteq \tilde{G}'_n$ by the formula

$$W_s = \begin{cases} 
\bigcup_{i+j=s} G_i \times A'_j & \text{if } s \leq n \\
G_n \times A'_0 \cup \left( \bigcup_{i+j=s, i<n} G_i \times A'_j \right) & \text{if } s > n
\end{cases}$$

with the understanding that $A'_j = A'_k$ for all $j \geq k$. The cofiber sequences guaranteed by Baues’ theorem can be pieced together with the given cone decompositions of $A'$ and $G_n$ to give the cofiber sequences

$$F_s \vee L_s \vee \left( \bigvee_{i<n} F_i \ast L_j \right) \rightarrow W_s \rightarrow W_{s+1}$$

for each $s < \min\{n, k\}$; when $s \geq n$ we alter the cobase of the cofiber sequence by removing the $F_s$ summand, and when $s \geq k$ we must remove the summand $L_s$. Since $\tilde{G}'_n = W_{n+k-1}$, we have the result.

Next, we show that the map $\tilde{p}_n : \tilde{G}_n \rightarrow X \times A$ has one of the category-detecting properties of $p_n : G_n(X \times A) \rightarrow X \times A$.

**Proposition 6** If $\text{cat}(X \times A) = \text{cat}(X) = n$, then $\tilde{p}_n$ has a homotopy section.

**Proof** We follow [4] (see also [3] Thm. 2.7) and define

$$\tilde{G}'_n(X \times A) = \bigcup_{i+j=n} G_i(X) \times G_j(A).$$

There is a natural map $h : \tilde{G}'_n(X \times A) \rightarrow X \times A$ induced by the Ganea fibrations over $X$ and $A$. According to [4] Thm. 2.3, $\text{cat}(X \times A) = n$ if and only if $h$ has a homotopy section.

Each map $G_i(X) \times G_j(A) \rightarrow X \times A$ (with $j > 0$) factors through $G_i(X) \times A$ and these factorizations are compatible because $p_{i+1}$ extends $p_i$. So $h$ factors as $\tilde{G}'_n(X \times A) \rightarrow G_n \rightarrow X \times A$. Therefore, if $\text{cat}(X \times A) = n$, then $h$, and hence $\tilde{p}_n$, has a section. \qed

### 2.2 Comparison of $\tilde{G}_n$ with $G_n(X) \times A$

Let $j : \tilde{G}_n \rightarrow G_n(X) \times A$ denote the natural inclusion map.

**Proposition 7** Assume that $X$ is $(c-1)$-connected and that $A$ is $(r-1)$-connected. Then the homotopy fiber $F$ of the map $j$ is $(nc+r-2)$-connected.
Proof There is a cofiber sequence
\[ \tilde{G}_n \to G_n(X) \times A \to \Sigma F_{n-1}(X) \wedge A. \]
Therefore the homotopy fiber of \( j \) has the same connectivity as the space \( \Omega(\Sigma F_{n-1}(X) \wedge A) \simeq \Omega(\Omega(X)^n * A) \), namely \( nc + r - 2 \). \( \square \)

Corollary 8 Assume \( \dim(Z) < nc + r - 2 \) and let \( f, g : Z \to \tilde{G}_n \). Then \( f \simeq g \) if and only if \( jf \simeq jg \).

The proof is standard, and we omit it.

2.3 New Sections from Old Ones

Suppose that \( \text{cat}(X) = \text{cat}(X \times A) = n \). By Proposition there is a section \( \sigma : X \times A \to \tilde{G}_n \) of the map \( \tilde{p}_n : \tilde{G}_n \to X \times A \). Define a new map \( \sigma' : X \to G_n(X) \) by the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\sigma'} & G_n(X) \\
\downarrow{\iota_1} & & \downarrow{\text{pr}_1} \\
X \times A & \xrightarrow{\sigma} & \tilde{G}_n \\
\end{array}
\]

\[
\begin{array}{ccc}
 & & \\
X \times A & \xrightarrow{j} & G_n(X) \times A \\
\end{array}
\]

We need the following basic properties of \( \sigma' \).

Proposition 9 If \( \text{cat}(X \times A) = \text{cat}(X) = n \), then

(a) \( \sigma' \) is a homotopy section of the projection \( p_n : G_n(X) \to X \), and

(b) if \( X \) is \((c - 1)\)-connected and \( A \) is \((r - 1)\)-connected with \( r > \dim(X) - nc + 2 \), then the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\sigma'} & G_n(X) \\
\downarrow{\iota_1} & & \downarrow{k} \\
X \times A & \xrightarrow{\sigma} & \tilde{G}_n \\
\end{array}
\]

commutes up to homotopy.
Proof First consider the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\sigma'} & G_n(X) \\
\downarrow{i_1} & & \downarrow{pr_1} \\
X \times A & \xrightarrow{\sigma} & \tilde{G}_n \\
\downarrow{1_{X \times A}} & & \downarrow{pr_1} \\
X \times A & \xrightarrow{\tilde{G}_n} & X \\
\end{array}
\]

The diagram of solid arrows is evidently commutative. Therefore, we have \( p_n \circ \sigma' \simeq pr_1 \circ 1_{X \times A} \circ i_1 \simeq 1_X \), proving (a).

To prove (b) we have to show that two maps \( X \to \tilde{G}_n \) are homotopic. Since \( \text{dim}(X) < n + r - 2 \), it suffices by Corollary \( \S \) to show that \( j \circ (\sigma \circ i_1) \simeq j \circ (k \circ \sigma') \). Since \( pr_2 \circ j \circ (\sigma \circ i_1) \simeq * \simeq pr_2 \circ j \circ (k \circ \sigma') \), it remains to show that \( pr_1 \circ j \circ (\sigma \circ i_1) \simeq pr_1 \circ j \circ (k \circ \sigma') \). But both of these maps are homotopic to \( \sigma' \).

3 Proof of the Main Theorem

Proof of Theorem \( \S \) We have \( n = \text{cat}(X) = \text{cat}(X \times A) \) by hypothesis. It follows from Proposition \( \S \) that there is a section \( \sigma : X \times A \to \tilde{G}_n \) of the map \( \tilde{p}_n : \tilde{G}_n \to X \times A \). We then get the section \( \sigma' : X \to G_n(X) \) that was constructed and studied in Section 2.3.

Consider the following diagram and the induced sequence of maps on the homotopy pushouts of the rows

\[
\begin{array}{ccc}
(X \times A) \times B & \xleftarrow{i_1 \times 1_B} & X \times B & \xrightarrow{pr_1} & X & \xrightarrow{pr_1} & X & \text{Y} \\
\downarrow{\sigma \times 1_B} & \simeq s & \downarrow{\sigma' \times 1_B} & \downarrow{\sigma'} & \text{homotopy pushout} & \text{P} \\
\tilde{G}_n \times B & \xrightarrow{k \times 1_B} & G_n(X) \times B & \xrightarrow{pr_1} & G_n(X) & \xrightarrow{pr_1} & G_n(X) & \text{Y} \\
\downarrow{\tilde{p}_n \times 1_B} & \downarrow{p_n \times 1_B} & \downarrow{p_n} & \downarrow{p_n} & \text{X} & \text{X} & \text{Y} & \text{Y} \\
(X \times A) \times B & \xrightarrow{i_1 \times 1_B} & X \times B & \xrightarrow{pr_1} & X & \text{X} & \text{X} & \text{X} & \text{X} \\
\end{array}
\]

Proposition \( \S \) implies that the upper left square commutes up to homotopy. Since \( i_1 \times 1_B \) is a cofibration, we can apply homotopy extension and replace the map \( \sigma \times 1_B : (X \times A) \times B \to \tilde{G}_n \times B \) with a homotopic map \( s \) which makes
that square strictly commute. All other squares are strictly commutative as they stand.

Since the composites \((\tilde{\rho}_n \times 1_B) \circ (\sigma' \times 1_B)\) and \(p_n \circ \sigma'\) are the identity maps and \((\tilde{\rho}_n \times 1_B) \circ s\) is a homotopy equivalence, each vertical composite in the modified diagram is a homotopy equivalence. Thus \(Y\) is a homotopy retract of \(P\), and consequently \(\text{cat}(Y) \leq \text{cat}(P)\).

The space \(Y\) is the homotopy pushout of the top row in the diagram, which is the product of the homotopy pushout diagram

\[
\begin{array}{ccc}
B & \longrightarrow & * \\
\downarrow & & \downarrow \\
A \times B & \longrightarrow & A \times B
\end{array}
\]

with the space \(X\). Therefore \(Y \simeq X \times (A \times B)\) by Proposition 3. Since \(Y\) is a homotopy retract of \(P\), it follows that

\[\text{cat}(X \times (A \times B)) \leq \text{cat}(P),\]

the proof will be complete once we establish that \(\text{cat}(P) < \text{cat}(X) + \text{cat}(A \times B)\). This is accomplished in Lemma 10 which is proved below.

**Lemma 10** The space \(P\) constructed in the proof of Theorem 2 satisfies

\[\text{cat}(P) \leq \text{cl}(P) < \text{cat}(X) + \text{cat}(A \times B).\]

**Proof** The space \(\tilde{G}_n\) is defined by the homotopy pushout square

\[
\begin{array}{ccc}
G_{n-1}(X) & \longrightarrow & G_n(X) \\
\downarrow & & \downarrow \\
G_{n-1}(X) \times A & \longrightarrow & \tilde{G}_n.
\end{array}
\]

Take the product of this square with the space \(B\) and adjoin the homotopy pushout square that defines \(P\) to obtain the diagram

\[
\begin{array}{ccc}
G_{n-1}(X) \times B & \longrightarrow & G_n(X) \times B & \longrightarrow & G_n(X) \\
\downarrow & & \downarrow & & \downarrow \\
G_{n-1}(X) \times A \times B & \longrightarrow & \tilde{G}_n \times B & \longrightarrow & P.
\end{array}
\]

By [11, Lem. 13], the outer square

\[
\begin{array}{ccc}
G_{n-1}(X) \times B & \longrightarrow & G_n(X) \\
\downarrow & & \downarrow \\
G_{n-1}(X) \times A \times B & \longrightarrow & P
\end{array}
\]

is a homotopy equivalence.
is also a homotopy pushout square. The top map is the composite
\[ G_{n-1}(X) \times B \xrightarrow{pr_1} G_{n-1}(X) \to G_n(X), \]
and so we have a new factorization into homotopy pushout squares:
\[ \begin{array}{ccc}
G_{n-1}(X) \times B & \xrightarrow{pr_1} & G_{n-1}(X) \\
\downarrow & & \downarrow \\
G_{n-1}(X) \times A \times B & \to & L \\
\downarrow & & \downarrow \\
G_{n-1}(X) \times (A \times B) & \to & P.
\end{array} \]

To identify the space \( L \), observe that the left square is simply the product of the space \( G_{n-1}(X) \) with the homotopy pushout square
\[ \begin{array}{ccc}
B & \to & * \\
\downarrow & & \downarrow \\
A \times B & \to & A \times B.
\end{array} \]

By Proposition 3, \( L \simeq G_{n-1}(X) \times (A \times B) \). Hence the right-hand square is the homotopy pushout square
\[ \begin{array}{ccc}
G_{n-1}(X) & \to & G_n(X) \\
\downarrow & & \downarrow \\
G_{n-1}(X) \times (A \times B) & \to & P.
\end{array} \]

Therefore \( \text{cl}(P) \leq \text{cat}(X) + \text{cat}(A \times B) \) by Proposition 5.

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