Vertex-type thermal correction to the one-photon transition rates

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Abstract
Thermal corrections to the one-photon spontaneous and induced transition probabilities for hydrogen and hydrogen-like ions are evaluated. The found thermal corrections are given by the vertex Feynman graph, where the vertex represents the thermal interaction between the bound electron and the nucleus. All derivations of thermal corrections to bound–bound transitions for an atom exposed to blackbody radiation are made in a fully relativistic approach within the framework of the adiabatic S-matrix formalism. It is found that the vertex-type radiative corrections to the transition rates can be at the level of a few percent to corresponding spontaneous rates for highly excited states in the hydrogen atom. A comprehensive analysis of the vertex-type thermal corrections for hydrogen-like atomic systems is presented.

Keywords: transition rates, radiative QED corrections, hydrogen, hydrogen like ions, blackbody effects

1. Introduction

Since the early days of quantum mechanics (QM), the study of absorption and emission of photons by atomic systems played a key role in the development of modern quantum field theory and its practical application in various fields of physics, chemistry, engineering and etc. Further development of QM led to the creation of a quantum electrodynamical description (QED) of light interaction with matter, within which all processes are described in terms of scattering cross-sections and transition rates. Subsequent experimental observations and their growing accuracy have required taking into accounting more complex effects, in particular, radiative QED corrections. In this regard, a detailed theoretical analysis of various radiative corrections providing a versatile verification of fundamental physics is needed. The precise values of the transition rates in various atomic systems are also of interest for studying the processes of an atomic collision or interpreting spectra from astrophysical sources [1, 2]. Moreover, accurate calculations of the transition rates can serve for the verification of basic parts in more complicated processes, such as, e.g., the parity violation amplitudes in heavy ions and atoms [3, 4].

To study the above examples, it becomes extremely important to determine accurately the lifetimes and decay rates of different atomic systems. Since real physical processes always occur in the presence of certain external fields, it is necessary to study theoretically their influence on the spectral characteristics of atoms. One of these fields is represented by a black-body radiation (BBR). In astrophysics, an external field with a mostly Planck spectrum (cosmic microwave background, radiation from powerful sources, nebulae, etc.) affects the population of atomic states, which leads to significant changes in the dynamic of astrophysical processes. Under laboratory conditions, the radiation of BBR also has impact on the physics of the studied processes and, consequently, requires detailed study [5–9]. A rigorous analysis of these effects is possible within the framework of the quantum electrodynamics theory at finite temperatures (TQED) for bound states, see, for example, [10] and references therein.

Concentrating on the electric and magnetic dipole transitions in the H-like ions, we extend the approach developed in [10–12] to the study of the lowest-order thermal radiative corrections to decay rates given by Feynman diagram figure 2. Below we demonstrate that these corrections can be significant for the measurements of lifetimes of highly excited (Rydberg) states in these atomic systems since their relative
magnitude can reach a level of several percent at room temperature. The interaction potential, figure 1, which shifts atomic energy levels and corrects wave function of the bound electron, was recently derived in [10]. It was found that thermal potential, figure 1, produces the dominant thermal frequency shift arising in a heat bath and could exceed well-known BBR-induced Stark shift [13].

The paper is organized as follows. In appendix A we start from a brief description of the Gell–Mann and low adiabatic formalism used to evaluate the one-photon transition rates and radiative corrections. The derivation of master equations for vertex-type correction to the transition rate is given in section 2. In section 2 the nonrelativistic limit of derived equations is also presented. All the derivations are performed within the framework of rigorous quantum electrodynamics theory at finite temperatures and are applicable for the H-like ions. The results of numerical calculations of the thermal vertex-type corrections for electric and magnetic dipole transitions in H-like ions are discussed in section 5. The relativistic units $\hbar = m_e = c = 1$ ($m_e$ is the electron rest mass, $c$ is the speed of light and $\hbar$ is the reduced Planck constant) are used throughout the paper.

2. One-photon emission

Evaluation of the natural decay width, $\Gamma^{(1)}_A$, for the arbitrary excited state $A$ can be performed in a conventional manner by the formula equation (A11):

$$
\Gamma^{(1)}_A = \lim_{\eta \to 0} \eta \sum_{\beta \neq A} \langle F | \hat{S}^{(1)}_{\eta} | A \rangle^2.
$$

First, we evaluate the matrix element $\langle A' | \hat{S}^{(1)}_{\eta} | A \rangle$ of $\langle A' | \vec{A}_\eta \vec{e} \rangle$ for the emission of the photon with momentum $\vec{k}$ and polarization $\vec{e}$. The corresponding adiabatic $S_{\eta}$-matrix element reads

$$
\langle A' | \hat{S}^{(1)}_{\eta} | A \rangle = -i e \int d^3x \psi_{A'}(x) \gamma_\mu A^\mu_{\eta}(x) \psi_A(x) e^{-i\eta|\vec{k}|}.
$$

Here $\psi_A(x) = \psi_{A}(\vec{r}) e^{-iE_A t}$, $\psi_{A'}(\vec{r})$ is the solution of the Dirac equation for the atomic electron, $E_A$ is the Dirac energy, $\psi_{A'}^\dagger = \psi_{A'}^\dagger(\vec{r})$ is the Dirac conjugated wave function with $\psi_{A'}^\dagger$ being its Hermitian conjugate and $\gamma_\mu = (\gamma_0, \vec{\gamma})$ are the Dirac matrices and $A_{\eta}(x)$ is the photon wave function (electromagnetic field potential)

$$
A_{\eta}(x) = \sqrt{\frac{2\pi}{\omega}} e^{i\gamma_\mu \phi_{\mu\nu}}.
$$

Here $\phi_{\mu\nu}$ are the components of the photon polarization four-vector, $x_\mu$ is the space-time four-vector, $k_\mu$ is the photon momentum four-vector with the space vector $\vec{k}$ and photon frequency $\omega = |\vec{k}|$. Using the transversality condition $\gamma_\mu \phi^{\mu\nu} = e^{\nu\alpha}$ ($\vec{e}$ is a transverse space vector of the photon polarization), the wave function for the emitted/absorbed real photon takes the form:

$$
\tilde{A}(\vec{x}) = \sqrt{\frac{2\pi}{\omega}} e^{i(k_x x - \omega t)} = \sqrt{\frac{2\pi}{\omega}} e^{-i\omega t} \Lambda(\vec{k}, \vec{r}).
$$

Now the integration over the time variable in equation (2) yields essentially a representation of the $\delta$-function

$$
\int_{-\infty}^{\infty} d\tau e^{i(E_{\eta} - E_{A'} + \omega - \eta)|\vec{k}|} = \frac{2\pi}{\omega^{2} + \eta^{2}}
$$

which yields the following relation for the frequency $\omega = |\vec{k}|$

$$
\int_{0}^{\infty} \frac{\omega d\omega}{(\omega - \omega_{AA'})^2 + \eta^2} = \frac{\pi}{4\eta^3} + \frac{1}{\eta^2} + \frac{\omega_{AA'}}{2\eta^3} \arctg \left[ \frac{\omega_{AA'}}{\eta} \right].
$$

Having in mind the factor $\lim_{\eta \to 0} \eta$ we can replace equation (6) by

$$
\frac{\pi}{4\eta^3} + \frac{1}{\eta^2} + \frac{\omega_{AA'}}{2\eta^3} = \frac{2\pi \omega_{AA'}}{\eta}.
$$

Multiplying the square modulus of equation (2) by the factor $d^3(2\pi)^3$ (the phase volume) with the account for (6) and with the summation over the electron states, lower by energy than the state $A$, we arrive at

$$
\Gamma^{(1)}_A = \frac{\pi}{2} \sum_{A'} \frac{\omega_{AA'}}{\eta} \int d^3\vec{e} \left| \hat{\alpha} A' \right| A'^\dagger_A^2,
$$

where $\hat{\alpha}$ is the Dirac matrix. In the derivation above the manipulations with $\Delta$-functions, like in ‘normal’ $S$-matrix formalism, are avoided. Multiplying the result by the adiabatic parameter $\eta$ in equation (A11) plays the same role as dividing the result by the time $T$: the adiabatic factor $\eta$ has the dimensionality $s^{-1}$. Note, that in this approach the automatic exclusion of the transitions to the states higher than $A$ in the summation over $F$ in equation (A11) does not occur and we have to refer to the energy conservation law.
In the nonrelativistic limit and electric dipole approximation equation (8) after the integration over photon emission direction, summation over photon polarizations, summation over projections of final states and averaging over projections of initial state equation (8) takes the form:

$$\Gamma_{\text{A}}^{(1)} = \sum_{A'} W_{AA'},$$

where $W_{AA'}$ is the partial transition rate

$$W_{AA'} = \frac{e^2}{2\hbar + 1} \sum_{n_{\text{e}},n_{\text{n}}'} \frac{4}{3} \omega_{AA'}^2 |\langle A' | \vec{p} | A \rangle|^2.$$  

Below we will be also interested in BBR-induced one-photon transition rate which is given by

$$W_{AA'}^{\text{ind}} = \frac{e^2}{2\hbar + 1} \sum_{n_{\text{e}},n_{\text{n}}'} \frac{4}{3} \omega_{AA'}^2 |\langle A' | \vec{p} | A \rangle|^2 n_{\beta}(\omega_{AA'}).$$

where $n_{\beta}(\omega) = (e^{\beta} - 1)^{-1}$, $\beta = (k_{\text{B}}T)^{-1}$, $k_{\text{B}}$ is the Boltzmann constant and $T$ is the radiation temperature in kelvin.

In the next sections we will evaluate the one-photon and one-photon plus thermal interaction correction decay widths.

3. One-photon emission with vertex correction

The radiative correction to the natural level width (see computations in appendix A and [14–19]) of the order of $e^4$ is

$$\Gamma_{\text{A}}^{\text{rad}} = \lim_{\eta \to 0} \eta \sum_{F \neq A} 2 \text{Re} \langle A | \tilde{S}^{(1)} | F \rangle \langle F | \tilde{S}^{(3)} | A \rangle.$$  

The first matrix element in equation (12) corresponds to the amplitude of the spontaneous emission process, while the second matrix element contains additional interaction. In our case the interaction with an external field is considered according to the Feynman graph in figure 1.

Thus, the expression (12) is the cross product of the one-photon emission amplitude and one-photon emission amplitude modified by the interaction of a bound electron with an external field.

The vertex diagram in figure 1 can be interpreted as a thermal potential, which, in principle, can be considered as the electron–nucleus potential modified by the BBR [10]. In contrast to the traditional relativistic QM description [20], the QED approach shows that the thermal interaction determined by the multipolar method [21] corresponds to an accurate evaluation of the thermal self-energy correction of a bound electron [22]. In turn, this description can be associated with the advanced and retarded interactions of a bound electron coupled with the BBR, and does not reflect the thermal potential of two interacting charges [10].

Recently, the QED approach at finite temperature has been used to describe the Feynman diagram in figure 1 corresponding to the photon exchange between a bound electron and a nucleus [10, 13]. As a consequence, a thermal interaction potential was found that corresponds to the Coulomb part of the thermal photon propagator. The lowest-order thermal correction to the level energies arising from the series expansion of this potential at room temperature is of the order of accuracy in modern experiments [23]. Here we consider this thermal correction to the one-photon emission process. The inclusion of thermal interaction in the process of one-photon emission is schematically depicted by the Feynman graphs in figure 2.

The corresponding $S_{\mu\nu}$-matrix element for the graph a) reads

$$S_{\mu\nu}^{(3)} = (-ie)^2 iZ \int d^4x_1 d^4x_2 d^4x_3 \psi_f(x_1) \gamma^\nu A_\mu(x_1) \times \psi_0(x_2) e^{-i\varphi} D^3_{\mu\nu}(x_2, x_3) f^3(x_3) \psi_0(x_2),$$

where $Z$ is the external charge (nucleus), $\gamma^\nu$ is the electron propagator and $f^3(x)$ is the external nuclear current which can be written using the Fourier transform as

$$f^3(x) = \frac{1}{(2\pi)^3} \int \frac{dq}{(2\pi)^3} \psi(q) j^3(q).$$

The eigenmode decomposition of $S(x_1, x_2)$ with respect to one-electron eigenstates is

$$S(x_1, x_2) = \frac{i}{2\pi} \int d\Omega e^{-i\varphi} \sum_n \psi_n(\vec{r}_1) \psi_n(\vec{r}_2) \Omega - E_n(1 - \rho_n),$$

where summation runs over the entire Dirac spectrum.

The thermal interaction shown by the dashed line in figure 2 corresponds to the zero component of the thermal photon propagator, $D^0_{\mu\nu}(x_2, x_3)$. In the case of an infinitely heavy and static external charge (nucleus), it is convenient to use the contour integral representation found in [10]:

$$D^0_{\mu\nu}(x, x') = -4\pi g_{\mu\nu} \int \frac{d^4k}{c_1} \frac{e^{i(k-x)\rho}}{(2\pi)^4} \frac{1}{k^2} n_{\beta}(k),$$

where $g_{\mu\nu}$ is the metric tensor, $k^2 \equiv k_0^2 - |\vec{k}|^2$.

Analytical evaluation of equation (13) is given in appendix B. The final result for the radiative correction
equation (12) is
\[
\Delta W_{\text{rad}}^{\text{rel}} = \frac{8Ze^4}{3} (\int \varphi \hat{a} \hat{A} \varphi d) \left[ \omega \sum_{n} \langle f|\varphi \hat{a} \hat{A} \varphi \rangle \langle n|V^2(\vec{r})|i \rangle \frac{1}{E_f - E_n} \right.
\]
\[+ \omega \sum_{n} \langle f|V^2(\vec{r})|n \rangle \langle n|\varphi \hat{A} \varphi \rangle \frac{1}{E_f - E_n} + \frac{1}{2} \langle f|\varphi \hat{A} \varphi \rangle \langle \vec{r}^2 |i \rangle \right] d\vec{r},
\]
(17)

where \( \omega \) is the gauge parameter (the 'length form') and \( \lambda \) is the photon angular momentum and its projection. Terms with \( \lambda = 0 \) are electric multipoles and terms with \( \lambda = 1 \) are magnetic multipoles. Then the reduction of matrix elements can be performed with the use of relation
\[
\langle n' f | m' | n | m \rangle = (-1)^{f - m'} \times \left( \begin{array}{c} L + 1 \lambda + 1 \lambda - 1 \lambda - 1/2 \\ L + 1 \lambda - 1 \lambda + 1 \lambda - 1/2 \end{array} \right) \mathcal{M}_{n'f,n}^{(L,\lambda)},
\]
(20)

where
\[
\mathcal{M}_{n'f,n}^{(L,\lambda)} \equiv \frac{2L + 1}{(L + 1)^{1/2}} \begin{pmatrix} \kappa' - \kappa \end{pmatrix}^{1/2} (L + 1) I_{L+1}^\kappa - (L + 1) I_{L-1}^{\kappa} + (L + 1) I_{L+1}^\kappa \right) \times \frac{4\pi r}{(L + 1)^{1/2}} \begin{pmatrix} L + 1 \lambda - 1 \lambda + 1 \lambda - 1/2 \end{pmatrix} \mathcal{M}_{n'f,n}^{(L,\lambda)}.
\]

Now we turn to the thermal interaction potential \( V^3(\vec{r}) \). In [10] it was found that the matrix element \( \langle A|V^3(\vec{r})|A \rangle \) diverges. To avoid infrared divergence of this type, a regularization procedure can be applied, see [10]. However, this procedure is redundant in this case (without affecting the result). As it was found in [11, 12] for the thermal self-energy radiative corrections to one- and two-photon decay rates, all infrared divergences vanish for the complete set of the same order diagrams. For vertex-type corrections, this divergence can also be singled out explicitly.

Integration of \( V^3 \) over angles in momentum space yields
\[
V^3(r) = \frac{1}{2\pi^2} \int_0^\infty dq \frac{\sin qr}{qr} \approx \frac{1}{2\pi^2} \int_0^\infty dq \frac{q^2 r^2}{e^{\hbar qr} - 1}.
\]
(18)

Here we have used that the Planck distribution function cuts off the high-frequency range at relevant temperatures and \( r \) is limited by the atomic radius.

The first term of equation (18) represents a divergent contribution which, however, turns to be zero in view of the orthogonality of wave functions in the irreducible part of equation (29). For the reducible contributions (last two terms in equation (29)) divergent part of equation (18) cancels out due to the opposite signs. Thus, there is no divergence in equation (29), and the dominant contribution can be reduced to the expression:
\[
\Delta W_{\text{rad}}^{\text{rel}} = \frac{4Ze^4\zeta(3)}{9\pi^2\beta^3} (\int \varphi \hat{a} \hat{A} \varphi d) \left[ \omega \sum_{n} \langle f|\varphi \hat{a} \hat{A} \varphi \rangle \langle n|\vec{r}^2 |i \rangle \frac{1}{E_f - E_n} \right.
\]
\[+ \omega \sum_{n} \langle f|\vec{r}^2 |n \rangle \langle n|\varphi \hat{A} \varphi \rangle \frac{1}{E_f - E_n} + \frac{1}{2} \langle f|\varphi \hat{A} \varphi \rangle \langle \vec{r}^2 |i \rangle \right] d\vec{r},
\]
(19)

where the second term in equation (18) was integrated over \( q \) and \( \zeta \) is the Riemann zeta function. It is important to note that the exact cancellation of infrared divergences in the equations was also found in [11, 12].
Table 1. Numerical values of electric dipole decay rates and thermal correction for 2p_{1/2} → 1s_{1/2} + γ(E1) (upper line) and 5p_{1/2} → 1s_{1/2} + γ(E1) (lower line) transitions in H-like ions at room temperature. The first column shows the nuclear charge Z. All values are given in s⁻¹.

| Z   | W_{ij}  | ΔW_{ij}^{rad}, equation (28) |
|-----|---------|-----------------------------|
| 1   | 6.26835 × 10⁸ | 1.463 × 10⁻² |
| 2   | 3.43942 × 10⁷  | 2.821 × 10⁻³ |
| 5   | 1.00295 × 10¹⁰ | 2.925 × 10⁻⁵ |
| 20  | 5.50276 × 10⁸  | 5.642 × 10⁻⁴ |
| 35  | 3.91828 × 10¹¹ | 7.305 × 10⁻⁸ |
| 50  | 2.14864 × 10¹⁰ | 1.409 × 10⁻³ |
| 70  | 6.27219 × 10¹²  | 1.455 × 10⁻⁴ |
| 100 | 3.43281 × 10¹¹ | 2.808 × 10⁻⁵ |
| 200 | 1.00545 × 10¹⁴ | 2.859 × 10⁻⁸ |
| 350 | 5.46004 × 10¹⁴ | 5.533 × 10⁻⁸ |
| 500 | 9.47919 × 10¹⁴ | 4.764 × 10⁻⁸ |
| 700 | 5.03405 × 10¹³ | 9.281 × 10⁻⁸ |
| 1000| 3.98002 × 10¹⁵ | 6.280 × 10⁻⁸ |
| 1200| 2.03662 × 10¹⁴ | 1.236 × 10⁻⁸ |
| 1500| 1.55211 × 10¹⁶  | 7.419 × 10⁻⁸ |
| 2000| 7.32772 × 10¹⁴ | 1.486 × 10⁻² |
| 2500| 4.72597 × 10¹⁶  | 7.122 × 10⁻² |
| 3000| 1.90898 × 10¹⁵  | 1.422 × 10⁻² |

Integration over angles and summation over polarizations can be performed with the orthogonality condition. Then the total radiative correction to the one-photon transition, after summation over photon polarization and integration over photon emission direction, summation over the projections of the final state and averaging over projection of the initial state, is

\[
\Delta W_{ij}^{rad} = \frac{1}{2} \frac{4Ze^4c^3}{9\pi^3\beta^3} \sum_{\ell m \ell'M} \langle i|\tilde{a}^{(\ell)\ell'}(\mathbf{r}|f) \rangle \left[ \frac{\omega_f}{E_f - E_n} + \frac{\omega_i}{E_i - E_n} \right] \\
\times \left[ \sum_n \frac{\langle f|\tilde{a}^{(\ell)\ell}(\mathbf{r}|n)\rangle \langle n|\tilde{r}|i\rangle}{E_f - E_n} + \frac{1}{2} \frac{\langle f|\tilde{a}^{(\ell)\ell'}(\mathbf{r}|n)\rangle \langle n|\tilde{r}|i\rangle}{E_f - E_n} \right] \\
- \frac{1}{2} \frac{\langle f|\tilde{r}|f\rangle \langle \tilde{r}|i\rangle}{\beta^2} \, d\beta. \tag{28}
\]

The results of evaluation of thermal correction, equation (28), to 2p_{1/2} → 1s_{1/2} + γ(E1) and 5p_{1/2} → 1s_{1/2} + γ(E1) transitions in H-like ions with different nuclear charge Z are given in table 1.

From table 1 it follows that the thermal correction equation (28) is much less than the corresponding one-photon electric dipole correction. Nonetheless, according to the parametric estimation ~ α²Z(kᵩT)³ for the hydrogen atom in atomic units, it can be compared with two-photon E1M1 and E1E2 transitions, see [26], since such transitions obey the (αZ)⁶. In particular, the sum of the two-photon transition rates for hydrogen is \( W_{2p_{1/2}}^{E1M1+E1E2} = 1.627 × 10^{-5} \) s⁻¹, while the vertex-type thermal correction is 1.462 × 10⁻⁵ s⁻¹. In turn, for an H-like ion with Z = 2, the total two-photon transition rate is \( W_{2p_{1/2}}^{E1M1+E1E2} = 4.164 × 10^{-3} \) s⁻¹ [26], and the thermal correction is 2.925 × 10⁻⁵ s⁻¹. This tendency persists at high values of the nuclear charge Z. Notwithstanding, the rise in temperature should lead to a cubic increase of thermal correction, which makes it essential for LHe transitions in light one-electron ions.

The thermal correction, equation (28), calculated for the magnetic dipole 2S_{1/2} → 1S_{1/2} + γ(M1) transition in different H-like ions is collected in table 2.

Within the framework of the dipole approximation and the nonrelativistic limit kr ≪ 1, the expression (28) can be significantly simplified. In this case, the emission operator \( \tilde{a}\tilde{g} \), see equation (4), is reduced to \( \langle \tilde{a}\tilde{g} \rangle \). Then the summation over photon polarization with integration over the photon emission direction, the summation over the projections of the final state, and averaging over projection of the initial state, gives

\[
\Delta W_{ij}^{rad} = - \frac{1}{2i} \frac{25Ze^4c^3}{27\pi^3\beta^3} \sum_{m,m'} \langle i|\tilde{a}\tilde{g}|f\rangle \left[ \sum_n \frac{\omega_f}{E_f - E_n} + \frac{\omega_i}{E_i - E_n} \right] \\
\times \sum_n \frac{\langle f|\tilde{a}\tilde{g}(\mathbf{r}|n)\rangle \langle n|\tilde{r}|i\rangle}{E_f - E_n} + \frac{1}{2} \left[ \frac{\langle f|\tilde{a}\tilde{g}(\mathbf{r}|n)\rangle \langle n|\tilde{r}|i\rangle}{E_f - E_n} \right] \\
+ \frac{1}{2} \left\langle \tilde{a}\tilde{g}(\mathbf{r}|i)\langle i|\tilde{r}|i\rangle - \frac{1}{2} \langle f|\tilde{r}|f\rangle \langle \tilde{r}|i\rangle \right\rangle, \tag{29}
\]

where the relation \( \langle a|\tilde{p}|b\rangle = i\omega_{ab} \langle a|\tilde{r}|b\rangle \) was used. The numerical results of Equation (29) for various dipole one-photon \( i \rightarrow f \) transitions in the hydrogen atom at room temperature are collected in table 3. Evaluation at other temperatures can be easily obtained by multiplying by the coefficient \( (T/300)^3 \).

4. Vertex contribution to the transition frequency

As a next step it is necessary to evaluate thermal radiative corrections to vertex type to the energy levels of final and initial states and their contribution to the transition rate. This correction corresponds to the Feynman graph depicted in figure 1.

In [10] it was found that the thermal potential \( V^{\gamma}(r) \) can be derived within the framework of the rigorous QED theory. One of the advantages of the description in [10] is the ability to introduce thermal gauges in an obvious manner. Moreover, the regularization procedure proposed in [10], which has
eliminated divergences discussed above, led to obtaining the thermal Coulomb potential in closed form:

\[
V^q_v(r) = \frac{4}{\pi} \left[ \frac{-\gamma}{\beta} + \frac{i}{2\pi} \ln \frac{\Gamma(1 + \frac{4\beta}{\gamma})}{\Gamma(1 - \frac{4\beta}{\gamma})} \right],
\]

where \( \Gamma(z) \) is the gamma function. At room temperature and the low-lying states equation (30) could be approximated with sufficient accuracy by the regular term of equation (18).

Then, the thermal shift corresponding to the diagram in figure 1 for the hydrogen atom in the nonrelativistic limit and the point-nucleus approximation, can be found as

\[
\Delta E_A^\beta = (A|V^q_v(r)|A)
\]

\[
= -\frac{4Z^2e^2\xi(3)}{3\pi\beta^3} n^2_A \left[ 5n^2_A + 1 - 3\lambda_A(l_A + 1) \right] a_0^2,
\]

where \( n_A \) is the principal quantum number of the hydrogenic state \( A \), \( \lambda_A \) is the corresponding angular momentum, and \( a_0 \) is the Bohr radius. It should be noted that at room temperature this correction to the transition frequencies is at the level of the measurement accuracy in hydrogen [27] and in helium [28] atoms, see [10, 13].

This approximation, however, is violated for the Rydberg states. Numerical comparison of energy shifts \( \Delta E_A^\beta = (A|V^q_v(r)|A) \) for different atomic states \( A \) calculated for both equations (30) and (31) is given in table 4.

In particular, from table 4 it follows that equation (18) is a good approximation for low lying states at room temperatures, while for Rydberg states the accuracy and temperature behavior of equation (18) fall out. Thus, the complete form, equation (30), should be used to evaluate the decay rates of highly excited states.

The energy shift defined by equation (31) results in a corresponding correction to the transition frequency and, consequently, to the transition rate. The latter can be written as the difference between equation (10) calculated with the zero-order transition frequency \( \omega_{i\ell} = E_i - E_f \) and equation (10) calculated with the corrected transition frequency \( \tilde{\omega}_{i\ell} = E_i + \Delta E_A^\beta - E_f - \Delta E_f^\beta \), see [12, 29]:

\[
\Delta W_{i\ell}^\beta = \frac{e^2}{2\hbar} \sum_{n, m, \ell'} \left\langle \omega_{i\ell}^3 - \tilde{\omega}_{i\ell}^3 \right\rangle |f|\langle \ell |\ell'\rangle|^2.
\]  

Table 3. Numerical values of the thermal corrections \( \Delta W_{i\ell, \beta}^{\text{rad}} \) and \( \Delta W_{i\ell, \beta}^{\text{ind}} \) for different one-photon transitions at room temperature \( T = 300 \) K in the hydrogen atom. The first column shows the considered one-photon transitions. In the second column the natural one-photon transition rates \( W_{i\ell} \) are calculated within the dipole and nonrelativistic approximations. The third and fourth columns contain vertex-type thermal correction \( \Delta W_{i\ell, \beta}^{\text{rad}} \) and \( \Delta W_{i\ell, \beta}^{\text{ind}} \), respectively. In the fifth column, the total contribution \( \Delta W_{i\ell, \beta}^{\text{total}} = \Delta W_{i\ell, \beta}^{\text{rad}} + \Delta W_{i\ell, \beta}^{\text{ind}} \) is presented for the corresponding transitions. All values are in s\(^{-1}\).

| Transition | \( W_{i\ell, \beta} \), equation (10) | \( \Delta W_{i\ell, \beta}^{\text{rad}}, \) equation (29) | \( \Delta W_{i\ell, \beta}^{\text{ind}}, \) equation (32) | \( \Delta W_{i\ell, \beta}^{\text{total}} \) |
|------------|------------------|------------------|------------------|------------------|
| 2p–1s      | 6.286 × 10\(^8\) | 1.463 × 10\(^{-5}\) | 2.285 × 10\(^{-5}\) | 3.748 × 10\(^{-5}\) |
| 3p–1s      | 1.673 × 10\(^8\) | 5.562 × 10\(^{-5}\) | 3.394 × 10\(^{-5}\) | 8.956 × 10\(^{-5}\) |
| 3p–2s      | 2.246 × 10\(^7\) | 6.147 × 10\(^{-6}\) | 2.277 × 10\(^{-5}\) | 2.892 × 10\(^{-5}\) |
| 4p–1s      | 6.822 × 10\(^7\) | 1.405 × 10\(^{-5}\) | 4.429 × 10\(^{-5}\) | 1.847 × 10\(^{-4}\) |
| 4p–2s      | 9.673 × 10\(^6\) | 1.595 × 10\(^{-5}\) | 2.937 × 10\(^{-5}\) | 4.532 × 10\(^{-5}\) |
| 4p–3s      | 3.067 × 10\(^6\) | 7.475 × 10\(^{-6}\) | 2.529 × 10\(^{-5}\) | 3.277 × 10\(^{-5}\) |
| 5p–1s      | 3.439 × 10\(^7\) | 2.821 × 10\(^{-4}\) | 5.469 × 10\(^{-5}\) | 3.368 × 10\(^{-4}\) |
| 5p–2s      | 4.951 × 10\(^6\) | 3.536 × 10\(^{-5}\) | 3.506 × 10\(^{-5}\) | 7.042 × 10\(^{-5}\) |
| 5p–3s      | 1.639 × 10\(^6\) | 8.787 × 10\(^{-6}\) | 3.039 × 10\(^{-5}\) | 3.918 × 10\(^{-5}\) |
| 5p–4s      | 7.376 × 10\(^5\) | 7.340 × 10\(^{-6}\) | 2.849 × 10\(^{-5}\) | 3.583 × 10\(^{-5}\) |

Table 4. Energy shifts determined by the approximate expression (31) (second column) and the potential (30) (third column) for different atomic states \( A \) at temperatures \( T = 300 \) K (upper line) and \( T = 3000 \) K (lower line) in hydrogen atom. The first column shows the considered state (\( n_A, \lambda_A \)). All values are given in Hz.

| \( n_A, \lambda_A \) | \( \Delta E_{n_A\lambda_A}^\beta, \) equation (18) | \( \Delta E_{n_A\lambda_A}^\beta, \) equation (30) |
|------------------|------------------|------------------|
| (1, 0)            | -3.36            | -3.36            |
| (2, 0)            | -46.98           | -46.98           |
| (10, 0)           | -4.7 × 10\(^4\)  | -4.68 × 10\(^4\) |
| (10, 9)           | -2.80 × 10\(^4\) | -2.76 × 10\(^4\) |
| (20, 0)           | -2.80 × 10\(^4\) | -2.76 × 10\(^4\) |
| (20, 19)          | -1.93 × 10\(^5\) | -1.87 × 10\(^5\) |
| (100, 0)          | -1.93 × 10\(^8\) | -1.64 × 10\(^7\) |
| (100, 99)         | -2.80 × 10\(^8\) | -1.79 × 10\(^6\) |
| (100, 99)         | -1.14 × 10\(^8\) | -1.78 × 10\(^6\) |

The calculated values of the correction equation (32) are collected in table 3 showing that the corrections \( \Delta W_{i\ell, \beta}^{\text{rad}} \) and \( \Delta W_{i\ell, \beta}^{\text{ind}} \) are of the same order.

In addition to the vertex correction to a spontaneous transition, \( \Delta W_{i\ell, \beta}^{\text{ind}} \), the corresponding correction to the induced transition rate, \( \Delta W_{i\ell, \beta}^{\text{ind}} \), should be also evaluated, when equation (32) is multiplied by the factor \( n_\beta(\omega) \) taken at the appropriate frequency. The difference between induced decay rates, equation (11) with the energies of 'zero-order' and
In the recent decades, photon emission processes have become of high interest in fundamental investigations on field theories, astrophysics, laboratory experiments, constructing of atomic clocks [2, 12, 30–33]. The one-photon transitions play a special role in experiments pursuing the goal of precision determination of the fundamental physical constants [34], and forbidden (magnetic dipole) one-photon transitions have found their application in atomic clocks, see, for example, [35]. To increase accuracy in all such experiments, the influence of the thermal environment should be taken into account.

The most known phenomenon that affects the transition rate, is the BBR induced decays [5, 36]. Basically, BBR-induced transitions are calculated in the framework of the quantum mechanical approach, while, as was recently shown in [22], the application of the QED theory is more appropriate for detecting obscure effects arising in emission processes (an accurate accounting for the finite lifetimes of the excited states, for example). The QED approach allows the revealing (an accurate accounting for the finite lifetimes of the excited states, for example). The QED approach allows the revealing of the thermal effects which correspond to the known Feynman graphs with the replacement of the ordinary photon propagator by the thermal one [10]. For example, it has recently been shown that thermal self-energy radiative corrections to spontaneous one- and two-photon transition rates in the hydrogen atom, evaluated within the framework of rigorous QED theory at finite temperature [11, 12], are of particular importance in this field.

In particular, it was demonstrated in [12] that the thermal radiative corrections to the spontaneous Lyα decay rate can dominate over an ordinary induced transition rate up to temperatures \( T < 6000 \text{ K} \). As a result of calculations in [12], a contribution \( 2.42 \times 10^{-3} \text{ s}^{-1} \) at room temperature was found. From table 3 it follows that the total vertex-type thermal correction to the Lyα decay rate is \( 3.748 \times 10^{-5} \text{ s}^{-1} \). At other temperatures the total thermal correction for low-lying states could be easily obtained by the multiplying by the factor \( T^3/300^3 \). As a consequence, the vertex-type thermal correction reaches the value \( 3.748 \times 10^{-5} \text{ s}^{-1} \) at \( T = 3000 \text{ K} \) and \( 1.388 \times 10^{-5} \text{ s}^{-1} \) at \( 10^4 \text{ K} \) for the Lyα transition in the hydrogen atom, while the thermal self-energy correction found in [12] is \( 2.911 \times 10^{-5} \text{ s}^{-1} \) and \( 3.35 \times 10^{-5} \text{ s}^{-1} \), respectively. Thus, we can conclude that vertex-type thermal correction can also exceed the induced transition rates for certain transitions and, therefore, is important in the astrophysical context.

Vertex-type thermal corrections can be compared to ‘zero-vacuum’ QED corrections (including the finite size and mass of the nucleus), see, for example, [37], disclosing a relatively small contribution of thermal effects under room conditions for low lying states. However, the results collected in table 5 demonstrate that the vertex-type thermal correction to electric dipole transitions becomes important and has a relative order of magnitude of about \( 10^{-2} \) (see the last line and column in the table), whereas the QED corrections [37] remain minor for highly excited states in hydrogen.

Thermal corrections of the vertex type can also be compared with the two-photon E1E2 and E1M1 transitions [26]. For the hydrogen atom and the Lyα transition probability, the total contribution of these corrections exceeds the aggregated contribution of the corresponding two-photon transitions. On the other hand, calculations for H-like ions show that with increasing nuclear charge \( Z \), the vertex-type corrections become less important and even negligible for high \( Z \), see table 1. The same conclusion follows for the thermal corrections to the forbidden M1 transitions collected in table 2. However, the most interesting results arise for transitions between highly excited (Rydberg) states.

The relatively long lifetimes of Rydberg states make them suitable candidates for the implementation of quantum computers [38–40]. Thus, studying the accompanying effects can be important for their development. In particular, from table 5 it follows that the vertex-type correction has the greatest effect on highly excited states. This conclusion seems obvious in conjunction with the result equation (31), which shows an increase in thermal correction with the principal quantum number of the excited state. However, the results of numerical calculations of the thermal correction equation (31), listed

| \( n_i, l_i \) | \( n_f, l_f \) | \( W_{if} \) | \( \Delta W_{if}^{\text{ind}} \) | \( \Delta W_{if}^{\gamma} \) | \( \Delta W_{if}^{\text{v,ind}} \) |
|-------------|-------------|-------|----------------|----------------|----------------|
| (10, 9)     | (9, 8)      | 1.320 \times 10^4 | 5.419 \times 10^3 | 2.190 \times 10^{-5} | 3.772 \times 10^{-6} |
| (50, 1)     | (49, 0)     | 2.682 | 3.077 \times 10^2 | 6.780 \times 10^{-6} | 5.173 \times 10^{-4} |
| (50, 49)    | (49, 48)    | 7.137 \times 10^{-1} | 81.861 | 3.406 \times 10^{-6} | 2.590 \times 10^{-4} |
| (70, 1)     | (69, 0)     | 4.840 \times 10^{-1} | 1.541 \times 10^2 | -5.779 \times 10^{-6} | -1.226 \times 10^{-3} |
| (70, 69)    | (69, 68)    | 9.369 \times 10^{-2} | 29.830 | 1.633 \times 10^{-6} | 3.464 \times 10^{-4} |
| (100, 1)    | (99, 0)     | 7.953 \times 10^{-2} | 74.387 | 3.917 \times 10^{-4} | 2.444 \times 10^{-1} |
| (100, 99)   | (99, 98)    | 1.093 \times 10^{-2} | 10.221 | -1.972 \times 10^{-7} | -1.229 \times 10^{-4} |
in table 4, demonstrate the need to use the potential as a whole (without the series expansion over small parameter $r/\beta$) for highly excited states. Moreover, the numerical calculations of contributions corresponding to figure 2 are nontrivial for such states within the B-splines method used in this work. The problem arises due to the summation over the entire spectrum and extremely large basis set for the qualitative approximation of final, initial, and intermediate states. Over methods such as Coulomb Green functions will also face this problem.

The total thermal corrections to the transition rate should also include radiative self-energy corrections considered in [12], which have double summation and more complicated analytical form in respect to equation (28). Thus, the calculations of thermal corrections to the decay rates of highly excited states represent a separate task requiring the application of special methodology. However, the vertex-type correction to the transition rate arising due to the thermal shift of energy levels has a simpler representation, see equations (32) and (33), in the nonrelativistic limit. The corresponding contribution is the same order as the correction to wave function given by figure 1. Therefore, the vertex-type thermal correction to the transition rates between Rydberg states can be roughly estimated in the nonrelativistic limit. The corresponding contribution is of the form (up to terms $e^2$) [15]

\[
\Delta E_A = \lim_{\eta \to 0} \frac{1}{2} \eta \left[ \frac{1}{\gamma} \delta_{A} A \right] \left( \delta_{A} A \right) = \lim_{\eta \to 0} \frac{1}{2} \eta \left[ A \hat{S}_{\eta}^{(2)} | A \right] . \tag{A1}
\]

The adiabatic $S$-matrix $\hat{S}_{\eta}$ in equation (A1) differs from the ordinary $S$-matrix by the presence of the exponential factor $e^{-i\eta t}$ in each (interaction) vertex. It refers to the concept of adiabatic switching on and off the interaction introduced formally by the replacement $\hat{H}_{\text{int}}(t) \rightarrow \hat{H}_{\text{int}}^{\eta}(t) = e^{-i\eta t} \hat{H}_{\text{int}}(t)$. The symmetric version of the adiabatic formula containing $\hat{S}_{\eta}(\infty, -\infty)$, which is more convenient for the QED calculations, was proposed by Sucher [16]. The first application of the formula (A1) to calculations within bound-state QED was made in [46]. In [46] it was shown how to deal with the adiabatic exponential factor when evaluating the real part of corrections to the energy levels equation (A1) (see also [15]). In this section, we employ the same methods for evaluating the imaginary part of equation (A1), see [19].

For a free atom (or ion) in the state $|A\rangle$ interacting with the photon vacuum $|0\rangle$ (i.e. $\langle A | 0 \rangle = |A\rangle |0\rangle$ in the absence of external fields) the complex energy correction contains only diagonal $S$-matrix elements of even order, since $\langle 0 \rangle | \hat{S}_{\eta}^{(2)} | 0 \rangle = \langle 0 \rangle | \hat{S}_{\eta}^{(4)} | 0 \rangle = 0$ etc. For the separation of the imaginary part of the energy shift $\Delta E_{\eta}^{(2)}$ of a given order 2i, it is more convenient to represent equation (A1) in terms of a perturbation series of the form (up to terms $e^2$) [15]

\[
\Delta E_A = \lim_{\eta \to 0} i \eta \left[ \left( A \hat{S}_{\eta}^{(2)} | A \right) + \left( 2 \langle A | \hat{S}_{\eta}^{(4)} | A \rangle - \langle A | \hat{S}_{\eta}^{(2)} | A \rangle \right)^2 \right] + \ldots . \tag{A2}
\]

For the adiabatic $\hat{S}_{\eta}$ matrix the standard expansion in powers of the interaction constant $e$ was used

\[
\hat{S}_{\eta}(\infty, -\infty) = 1 + \sum_{i=1}^{\infty} \hat{S}_{\eta}^{(2i)}(\infty, -\infty). \tag{A3}
\]

To separate real and imaginary parts of the matrix elements at any given order of perturbation theory, one can write

\[
\langle A | \hat{S}_{\eta}^{(2i)} | A \rangle = \text{Re} \langle A | \hat{S}_{\eta}^{(2i)} | A \rangle + i \text{Im} \langle A | \hat{S}_{\eta}^{(2i)} | A \rangle . \tag{A4}
\]

The only one second-order term describes the pure one-photon decay width

\[
\text{Im} \Delta E_{\eta}^{(2)} = \lim_{\eta \to 0} \eta \text{Re} \langle A | \hat{S}_{\eta}^{(2)} | A \rangle . \tag{A5}
\]

Arranging all the terms of fourth order, which describe the pure two-photon decay width including a part of the radiative (one-loop) corrections to the one-photon width, one obtains

\[
\text{Im} \Delta E_{\eta}^{(4)} = \lim_{\eta \to 0} \eta \left[ 2 \text{Re} \langle A | \hat{S}_{\eta}^{(4)} | A \rangle \right] \left[ \langle A | \hat{S}_{\eta}^{(2)} | A \rangle \right]^2 - 2 \left( \text{Re} \langle A | \hat{S}_{\eta}^{(2)} | A \rangle \right)^2 , \tag{A6}
\]

where the last two terms result from the expression $\langle A | \hat{S}_{\eta}^{(2)} | A \rangle^2$. 

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**Appendix A. Adiabatic S-matrix formalism**

For the description of radiative QED correction we start from the basic formulas of the Gell–Mann and low adiabatic formalism [45]. Within this approach the initial formula for the energy shift $\Delta E_A$ of an excited atomic state $A$ is


The total width $\Gamma_A$ of an excited electron state $A$ (specifying
the initial state as $|A, O_i \rangle \equiv |A\rangle$) should follow (by definition)
from the imaginary part of the total energy-shift:

$$\Gamma_A = -2\Im \Delta E_A. \quad \text{(A7)}$$

Respectively, after expansion of $\Delta E_A$ (up to order $\epsilon^4$) as

$$\Gamma_A = -\lim_{\eta \to 0} 2\eta \left[ 2\text{Re}\langle A|S^{(2)}_\eta|A\rangle + 2\text{Re}\langle A|S^{(4)}_\eta|A\rangle + \left| \langle A|S^{(2)}_\eta|A\rangle \right|^2 - 2\left( \text{Re}\langle A|S^{(2)}_\eta|A\rangle \right)^2 \right]. \quad \text{(A8)}$$

As indicated above the adiabatic $S$-matrix $S_\eta$ arises after
introduction of the adiabatic switching function $f(\eta) = e^{-i\eta}$
in the QED interaction Hamiltonian. Assuming that no
dynamic excitation of the system takes place during switching
off and on the interaction, the adiabatic $S$-matrix remains
unitary \[14,47\]. Moreover, all observable quantities calculated
on the basis of adiabatic approach should not depend on the
specific form used for the adiabatic factor after the limiting
process $\eta \to 0$ has been performed. Therefore, we will apply
the ‘optical theorem’ relations, see details in \[19\]:

$$-2\text{Re}\langle A|S^{(2)}_\eta|A\rangle = \sum_{F \neq A} \langle F|S^{(1)}_\eta|A\rangle^2, \quad \text{(A9)}$$

$$-2\text{Re}\langle A|S^{(4)}_\eta|A\rangle = \left| \langle A|S^{(2)}_\eta|A\rangle \right|^2 + \sum_{F \neq A} \langle F|S^{(2)}_\eta|A\rangle^2 + 2\sum_{F \neq A} \text{Re}\langle A|S^{(1)}_\eta|F\rangle\langle F|S^{(1)}_\eta|A\rangle. \quad \text{(A10)}$$

to the adiabatic formulas (A5), (A6) and (A8).

Then for the excited state $A$ without photons for the pure
one-photon width one can find

$$\Gamma^{(1)}_A = \lim_{\eta \to 0} \eta \sum_{F \neq A} \langle F|S^{(1)}_\eta|A\rangle^2 \quad \text{(A11)}$$

and for the two-photon width

$$\Gamma^{(2)}_A = \lim_{\eta \to 0} \eta \left[ 2\sum_{F \neq A} \langle F|S^{(2)}_\eta|A\rangle^2 + 4\left( \text{Re}\langle A|S^{(2)}_\eta|A\rangle \right)^2 \right] \quad \text{(A12)}$$
or, employing equation (A9),

$$\Gamma^{(2)}_A = \lim_{\eta \to 0} \eta \left[ 2\sum_{F \neq A} \langle F|S^{(2)}_\eta|A\rangle^2 + 2\sum_{A, 2} \langle A, 2|S^{(2)}_\eta|A\rangle^2 + \left( \sum_{F \neq A} \langle F'|S^{(1)}_\eta|A\rangle \right)^2 \right]. \quad \text{(A13)}$$

\section*{Appendix B. Derivation of the thermal correction for one-photon radiation of the vertex type}

Substitution of equation (16) into (13) allows the integration
over $x_3$ that leads to the Fourier transform of the external
current $\hat{J}(x_3)$. Then

$$S^{(3)}_{3\omega} = -iZe^3 \int d^4x_1 d^4x_2 \psi_1(x_1)\gamma^\mu A_\mu(x_1)e^{-i\eta_1}$$

$$\times S(x_1, x_2)e^{-i\eta_2}(-4\pi i) \int \frac{d^4q}{C_\gamma}$$

$$\times \frac{e^{i\eta_1}}{q^2}n_\gamma(|\vec{q}|)$$

$$\times S(x_1, x_2)\psi_1(x_2). \quad \text{(B1)}$$

Employing the static limit to the nuclear current
$\hat{J}(q) = 2\pi\delta(q_0)\delta(\vec{q}) \approx 2\pi\delta(q_0)$ (in approximation of point-like nucleus), where $\delta(q_0)$ is the Dirac $\delta$-function, one can find

$$S^{(3)}_{3\omega} = -4\pi Ze^3 \int d^4x_1 d^4x_2 \psi_1(x_1)\gamma^\mu A_\mu(x_1)e^{-i\eta_1}$$

$$\times S(x_1, x_2)e^{-i\eta_2} \int \frac{d^4q}{C_\gamma}$$

$$\times \frac{e^{i\eta_1}}{q^2}n_\gamma(|\vec{q}|)\psi_1(x_2). \quad \text{(B2)}$$

It should be noted here that the same expression could imme-
diately be written in the thermal Coulomb gauge, see \[10, 13\].

The following evaluation corresponds to the integration
over the time variables $t_1$ and $t_2$. Substituting equation (15)
into (B2) with the use of equations (4) and (5), we arrive at

$$S^{(3)}_{3\omega} = -2iZe^3 \int d^3r_1 d^3r_2 \psi_1(r_1)\frac{2\pi}{\omega}$$

$$\times \sum_n \psi_n(r_1)\psi_n(r_2) \int \frac{d^3q}{(2\pi)^3}$$

$$\times \frac{e^{i\eta_1}}{q^2}$$

$$\times \frac{4\pi \eta_1}{(E_i - E_f - \omega^2)^2 + 4\eta_2^2}$$

$$\times \frac{E_i + E_f - 2E_\omega + \omega + 4i\eta_1}{(E_i - E_\omega + i\eta)(E_f - E_\omega + \omega + i\eta)}. \quad \text{(B3)}$$
where the relation \( \psi_n(x) = \psi_n(\vec{r})e^{-iE_nt} \) and integration over the frequency \( \Omega \), see equation (15), were employed. Repeating all the calculations above, for the graph (b) in figure 2 one can find

\[
S_{n,b}^{(3)} = -2iZe^3 \int d^3r_1 d^3r_2 \frac{\omega \psi_1(\vec{r}_1) \psi_2(\vec{r}_2)}{d^3q} \int \frac{d^3q}{(2\pi)^3} \eta_q(\vec{k})
\]

\[
\times \sum_n \psi_n(\vec{r}_1) \psi_n(\vec{r}_2) \sqrt{\frac{2\pi}{\omega}} \sum_n \psi_n(\vec{r}_1) \psi_n(\vec{r}_2) \int \frac{4\pi\eta}{(E_i - E_f - \omega)^2 + 4\eta^2}
\]

\[
\times \frac{1}{(E_i - E_f - \omega - \omega + i\eta)}
\]

Then, according to equation (12), the summary contribution is

\[
\Delta W_{rad}^{(1)} = \frac{2\pi^2 Ze^3}{\omega} \frac{d^3k}{(2\pi)^3} \Re \langle i|\vec{A}|f \rangle
\]

\[
\times \lim_{\eta \to 0} \left( \frac{(E_i - E_f - \omega)^2 + \eta^2}{(E_i - E_f - \omega)^2 + 4\eta^2} \right)
\]

\[
\times \sum_n \left( \frac{\langle f|\vec{A}^\dagger|n \rangle \langle n|V^j(\vec{r})|i \rangle (E_f + E_i - 2E_n - \omega + 4i\eta)}{(E_i - E_n + i\eta)(E_f - E_n + \omega + i\eta)} \right)
\]

\[
\times \left( \frac{\langle f|V^j(\vec{r})|n \rangle \langle n|\vec{A}\rangle|i \rangle (E_f + E_i - 2E_n - \omega + 4i\eta)}{(E_f - E_n + i\eta)(E_i - E_n - \omega + i\eta)} \right)
\]

\[\text{B4}\]

To determine the radiative correction to the transition probability, it is also necessary to calculate \( S_{n}^{(1)} \):

\[
S_{n}^{(1)} = \frac{-ie2\sqrt{2}}{(E_i - E_f - \omega)^2 + \eta^2} \int d^3r \psi_j(\vec{r}) (\vec{\alpha}\vec{A}(\vec{k}, \vec{r})) \psi_i(\vec{r})
\]

\[\text{B5}\]

The contributions to the thermal radiative correction, equation (B8), can be considered as reducible and irreducible parts. The first one corresponds to the case of \( n = i \) and \( n = f \) in the first and second terms, and the second, irreducible contribution, is represented by \( n \neq i \) and \( n \neq f \), respectively. The calculations are not difficult, but in the presence of poles in the energy denominators for the reducible part, one should at first to use the series expansion over the vanishing energy difference. Taking into account the limit \( \eta \to 0 \) and that the product of operators in numerators are purely real, the result reforms to

\[
\Delta W_{rad}^{(1)} = \frac{8Ze^4}{3} \lim_{\eta \to 0} \Re \langle i|\vec{A}|f \rangle
\]

\[
\times \sum_n \left( \frac{\langle f|\vec{A}^\dagger|n \rangle \langle n|V^j(\vec{r})|i \rangle (E_f + E_i - 2E_n - \omega + 4i\eta)}{(E_i - E_n + i\eta)(E_f - E_n + \omega + i\eta)} \right)
\]

\[
\times \left( \frac{\langle f|V^j(\vec{r})|n \rangle \langle n|\vec{A}\rangle|i \rangle (E_f + E_i - 2E_n - \omega + 4i\eta)}{(E_f - E_n + i\eta)(E_i - E_n - \omega + i\eta)} \right) d\vec{\nu}.
\]

\[\text{B8}\]

where \( \omega_{if} \equiv E_f - E_i \), the prime of the sum sign means the absence of corresponding state in the summation over the entire spectrum.

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