Space-time evolution of Gaussian wave packets through superlattices containing left-handed layers

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Abstract. We study the space-time evolution of Gaussian electromagnetic wave packets moving through $(L/R)^n$ superlattices, containing alternating layers of left and right-handed materials. We show that the time spent by the wave packet moving through arbitrary $(L/R)^n$ superlattices are well described by the phase time. We show that in the particular case where the thicknesses $d_{L,R}$ and indices $n_{L,R}$ of the layers satisfy the condition $d_L n_L = d_R n_R$, the usual band structure becomes a sequence of isolated and equidistant peaks with negative phase times.

1. Introduction

The possibility of having negative refraction index (left-handed) materials (with electric permittivity $\epsilon$ and magnetic permeability $\mu$ simultaneously negative), as a simple extension of the well established electromagnetic systems, has become lately an active and controversial research field, where striking and new effects are expected. Veselago[1] already predicted in 1968 some of the unique properties of EM wave propagation in left-handed media (LHM): a) that the waves appear to propagate towards the source and not away from it, b) that their group velocity is negative and, c) because waves incident on right-handed/left-handed (RH/LH) interfaces are refracted to the same side of the normal, converging and diverging lenses exchange their roles. Furthermore, Veselago proposed the constraints:

$$\frac{\partial \epsilon \omega}{\partial \omega} > 0 \quad \frac{\partial \mu \omega}{\partial \omega} > 0$$

with $\epsilon$ and $\mu$ depending on the frequency, for the energy transferred from the source to the load to be positive and to avoid causality violations. Smith and Kroll[2] on the other hand maintain that while a reversed $k$ resembles a time-reversed propagation towards the source, the work done is nevertheless positive. Incidentally, by analyzing the implications of Eqs. (1) these authors reach the conclusion that the group velocity ought to be positive for both types of materials. A simplified approach can easily lead to the conclusion that the sign in $k$ causes opposing group velocities.
Besides the blazing presumption of perfect lenses[3], and problems like the sign selection and directions of motion of both the energy and the electromagnetic field, it has been previously shown [4] that the transmission amplitude

\[ t_L = \left[ \cos (k_1 d_1 \cos \theta_1) + \frac{i}{2n_1n_2} \left( n_2^2 \cos \theta_1 \cos \theta_2 + n_2^2 \frac{\cos \theta_2}{\cos \theta_1} \right) \sin (|k_1|d_1 \cos \theta_1) \right]^{-1} = |t_L| e^{-i\theta_L} \]  

(2)

of a single left-handed slab is just the complex conjugate of the transmission amplitude

\[ t_R = \left[ \cos (k_1 d_1 \cos \theta_1) - \frac{i}{2n_1n_2} \left( n_2^2 \cos \theta_1 \cos \theta_2 + n_2^2 \frac{\cos \theta_2}{\cos \theta_1} \right) \sin (|k_1|d_1 \cos \theta_1) \right]^{-1} = |t_L| e^{i\theta_L} \]  

(3)

of a similar but right-handed slab[4]. An important consequence of this property is that the phase time \( \tau \) of a single slab, defined as the frequency derivative of the transmission-amplitude’s phase \( \theta_L \), becomes the negative of the corresponding phase time of a right-handed slab, which arises concerns of a possible causality violation. Gupta et al. studied also the transmission of electromagnetic pulses across a parallel slab of a medium where \( \epsilon \) and \( \mu \) are functions of the frequency \( \omega \). For the frequency range where both quantities are negative, they found a positive phase time[5]. We are interested here on this problem. As will be seen in the next section, the analysis in multilayered structures (MLS) might be more involved. It is no longer true that, except for specific cases, the transmission amplitude \( t_L \) of a MLS containing left-handed layers (LHL’s) will be the complex conjugate of \( t_R \). Therefore the phase times are not necessarily negative. But what happens when the phase time is negative?.

In this paper we will discuss the transport and transmission time of Gaussian electromagnetic wave packets through (L/R)^n superlattices (LRSL’s), where left-hand layers (LHL’s) of refractive index \( n_1 \equiv n_L = -|n_L| \) and width \( d_L \), alternate with right-handed layers (RHL’s) having refractive index \( n_2 \equiv n_R \) and width \( d_R \). We will show that in the particular case where \( d_L n_L = d_R n_R \), the usual transmission coefficient band structure becomes a sequence of isolated and equidistant resonances (IER).

2. Transport of EM waves in an air(L/R)^n air system

In a (L/R)^n superlattice bounded by semi-infinite air layers, an electromagnetic wave moving across it has a total transmission amplitude \( t_T \) given by

\[ t_T = \frac{1}{\alpha_T} \]  

(4)

with (assuming \( \mu_L = \mu_R \approx \mu_0 \)):

\[ \alpha_T = \alpha_{nr} + i \left( \frac{1}{2} n_R^2 \alpha_{ni} + \frac{n_R^2 - 1}{2n_R} \beta_{ni} \right) \]  

(5)

and[6]

\[ \alpha_{nr} = U_n - \alpha_r U_{n-1} \quad \alpha_{ni} = -\alpha_i U_{n-1} \]  

(6)

\[ \beta_{nr} = \beta_r U_{n-1} \quad \beta_{ni} = \beta_i U_{n-1} \]  

(7)

Here \( \alpha_r \) and \( \alpha_i \), \( \beta_r \) and \( \beta_i \) are real and imaginary parts of

\[ \alpha = e^{ik_R d_R \cos \theta_R} \left( \cos (k_L d_L \cos \theta_L) + \frac{i}{2n_L n_R} \left( n_R^2 \cos \theta_{LR} + n_R^2 \cos \theta_{LR}^* \right) \sin (k_L d_L \cos \theta_L) \right), \]  

(8)
\[ \beta = \frac{i e^{i k_R d_R \cos \theta_R}}{2 n_L n_R} \left( n_L^2 \cos \theta_R \cos \theta_L - n_R^2 \cos \theta_R \right) \sin (k_L d_L \cos \theta_L), \]  
\tag{9} 

and \( U_n \) is the Chebyshev polynomial of the second kind and order \( n \), evaluated at \( \alpha_r \).

It is apparent from these results that, in a system like the one considered here, it is not as simple to conclude in general, as it was for a single slab, whether the phase time sign becomes negative or not. It depends on many factors. In the next section we will present the space-time evolution of specific Gaussian wave packets whose centroid will be defined at qualitatively different points. We will consider two examples. In the first one we will have a LRSL with \( d_L = d_R, |n_L| \neq n_R \), and the Gaussian centroid in the photonic gap. In the second example we will have a LRSL with layer thickness and refraction parameters such that \( d_L |n_L| = d_R n_R \). In this case we choose the Gaussian-wave-packet centroid coinciding with one of the isolated resonances.

3. Space-time evolution of Gaussian WP

![Figure 1](image1.png)  
**Figure 1.** Transmission coefficient as function of the wave’s angular frequency. The fields are assumed to move through the superlattice \((L/R)^9\) with \( n_L = -2.22, n_R = 1.41, d_L = d_R = 140\text{nm} \).

![Figure 2](image2.png)  
**Figure 2.** Phase time for the same superlattice of Figure 1, containing left-handed layers. The phase time predictions are positive.

It is well known that the superlattice phase time, generally follows the characteristic resonant band structure of the superlattice transmission coefficients\cite{7}. However, based on the examples presented here, we can state that positive and negative phase times are likely possible. In the first example (see Figures 1 and 2), the phase time is positive for all frequencies. It will be shown that a WP defined inside the photonic gap will behave very much like in a right handed superlattice\cite{7}. In the second example, we will have an IER LRSL, which for frequencies around

![Figure 3](image3.png)  
**Figure 3.** Transmission coefficient as function of frequency. The fields are assumed to move through the superlattice \((L/R)^9\) with \( n_L = -2.22, n_2 = 1.41, d_L = d_R n_R/|n_L| = 316.5\text{nm} \).

![Figure 4](image4.png)  
**Figure 4.** Phase time for the superlattice in Figure 3. The phase times obtained near the resonant frequencies are negative.
the isolated resonances, has negative phase times (see Figures 3 and 4). We will study the transmission of an electromagnetic WP through this system. At \( t = 0 \), we will locate the wave-packet centroid at a distance \( z_0 \) from the left edge of the LRSL. Moving with the light velocity, the WP centroid reaches the left edge at \( t_a = z_0/c \), therefore a snapshot of the wave packet at \( t_b = 2t_a + \tau \), with \( \tau \) the transmission time, should show the transmitted WP on the other side, at a distance \( z_0 \) from the right edge of the IER LRSL. We will show that even that \( \tau \) is negative, the electromagnetic WP at \( t_b \) will certainly appear at \( z = z_0 + L \), with \( L \) the IER LRSL length. We shall now discuss in detail these specific examples.

3.1. The Gaussian WP in arbitrary \((L/R)^n\) superlattices

To simplify the discussion, let us consider a WP moving to the right in the parallel polarization and with normal incidence. At \( t = 0 \) the centroid of the Gaussian packet is at \(-z_0\). If we choose \( d_L = d_R = 140\text{nm} \) with \( n_L = -2.22 \) and \( n_R = 1.41 \), the transmission coefficient has the resonant bandstructure shown in figure 1, as function of the frequency \( w \), and the phase time shown in figure 2.

As shown in Fig. 1, there is a wide photonic gap between \( w = 1.610^{16}\text{s}^{-1} \) and \( w = 1.7210^{16}\text{s}^{-1} \). The Gaussian WP is placed at the center of this gap. In figure 5 we plot the WP at different instants of time. The phase time predicts a positive transmission time \( \tau \) of 0.9223 fs for the centroid, which is much less than the time that would require to cross the LRSL were it to move at the speed of light (5.6 ps for a length of 1680 nm). Since the WP is defined almost completely inside the gap, there will be no distortion after it passes or gets reflected from the metamaterial superlattice.

In figure 5 we show snapshots of the WP at different times. In the upper frame (at \( t = 0 \)) the WP is moving with group velocity \( \nu_g = c \) towards the metamaterial superlattice. At \( t = z_0/\nu_g \) the wave-packet centroid reaches the left edge of the LRSL. The WP spends some time (the transmission time \( \tau \)) inside the superlattice and, at \( t = z_0/\nu_g + \tau \) it begins to leave the LRSL. In the bottom frame the wave packet is plotted when presumably it is back to \(-z_0\) or transmitted at \( z = L + z_0 \).

3.2. The Gaussian WP in the IER \((L/R)^n\) superlattice

A different behavior in the transport properties of metamaterial superlattices \((L/R)^n\) occurs when the layer-widths fulfill the relation \( d_L = d_R n_R/|n_L| \). We choose \( d_R n_R = \lambda_0/4 \) with \( \lambda_0 \) the wave length at the center of the photonic gap. In figure 3 we plot the transmission coefficient, it is clear that the band structure collapses into a periodic sequence of single resonances, independent of the number of cells in the LRSL. In this case the phase time is negative for frequencies around the resonances (see figure 4). To visualize this property and to see whether we have some causality problem, we shall study the evolution of a Gaussian wave packet defined precisely at a resonance, as shown in figure 3. In figure 6 we plot the packet snapshots as the time \( t \) increases. Since the phase time prediction is negative (\( \tau = -1.1129\text{fs} \)), we have \( z_0/\nu_g + \tau < z_0/\nu_g + \tau/2 < z_0/\nu_g \). As mentioned before, one would expect that at \( t_a = z_0/\nu_g \), the WP will be arriving at the left of the LRSL, and leaving it at \( t_l = z_0/\nu_g + \tau \). Suppose that the WP is leaving the SL at \( t_l \), at latter times like \( z_0/\nu_g + \tau/2 = z_0/\nu_g + \tau + |\tau|/2 \) and \( z_0/\nu_g = z_0/\nu_g + \tau + |\tau| \) the WP should be farther away. Although it is difficult to distinguish the differences between the three graphs in the middle, one can perceive that, for \( t = z_0/\nu_g \), part of the reflected wave packet is growing at the left hand side of the superlattice as part of the reflected WP, which is much clearly seen later at \( t = 2z_0/\nu_g + \tau \). In fact, if the phase time prediction is correct, the WP at \( t = 2z_0/\nu_g + \tau \) should be back in the position where it was at \( t = 0 \) (when it is reflected), or at \( z = L + z_0 \) defined by the peak of the Gaussian pink curve at the right (when it is transmitted). As can be seen in the graph at the bottom, this is precisely what happens. The WP seems to travel from the far edge of the LM layers towards the source.
Figure 5. Time series of a WP defined in the gap with centroid at $\omega = 1.66 \times 10^{16} \text{s}^{-1}$. At $t = 0$, the WP moves to the right with group velocity $v_g = c$. The phase time predicts a transmission time $\tau = 0.9223 \text{fs}$ through a superlattice (L/R)$^6$. The WP is shown also when the centroid is presumably touching, inside and when it is presumably leaving the LRSL. At $t = 2z_0/v_g + \tau$ the reflected and the transmitted WP’s should be in the position of the pink Gaussians. The black Gaussian envelope on the RHS of the bottom frame indicates the position where the WP would be were it to move with $v_g = c$. The rectangular inset shows the spatial extent of the LRSL.

before it reaches the LRSL, just as predicted by Veselago. This is the main result that we report in this paper.

4. Conclusion
We have studied the space-time evolution of Gaussian wave packets through left- and right-handed (L/R)$^n$ superlattices. We have shown that the phase time describes the transmission time on these systems. We have shown also that, even though the phase time of a single LHM slab is the negative of the RHM slab phase time, the (L/R)$^n$ superlattice phase times are generally positive. We have found that in the particular (L/R)$^n$ superlattice with $d_L|n_L| = d_Rn_R$, the usual band structure of the transmission coefficient becomes a sequence of isolated and equidistant peaks with negative phase times. Our results show that in this case no apparent
Figure 6. Time series of a WP defined in a resonance with centroid at $\omega = 5.38 \times 10^{16} \text{s}^{-1}$. At $t = 0$, the WP moves to the right with group velocity $v_g = c$. The phase time predicts a transmission time $\tau = -1.1129 \text{fs}$ through a superlattice (L/R). The WP is shown also at $t = z_0/v_g + \tau$, at $t = z_0/v_g + \tau/2$ and at $t = z_0/v_g$. Because of the negative sign of the phase time the sequence is inverted and it is interesting to see that at $t = 2z_0/v_g + \tau$ the transmitted WP coincides with the pink Gaussian peak at $z_0 + L$. Causality violation occurs. The negative phase time of left-handed slabs deserves further research.

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