The discrepancy of \((n_kx)^\infty_{k=1}\) with respect to certain probability measures

Niclas Technau ∗  
(York)

Agamemnon Zafeiropoulos †  
(TU Graz)

Abstract
Let \((n_k)^\infty_{k=1}\) be a lacunary sequence of integers. We show that if \(\mu\) is a probability measure on \([0,1)\) such that \(|\hat{\mu}(t)| \leq c|t|^{-\eta}\), then for \(\mu\)-almost all \(x\), the discrepancy \(D_N(n_kx)\) satisfies
\[
\frac{1}{4} \leq \limsup_{N \to \infty} \frac{N D_N(n_kx)}{\sqrt{N \log \log N}} \leq C
\]
for some constant \(C > 0\), proving a conjecture of Haynes, Jensen and Kristensen. This allows a slight improvement on their previous result on products of the form \(q\|q\alpha\|q\beta - \gamma\|\).

1 Introduction
Let \((x_n)^\infty_{n=1}\) be a sequence of numbers in the unit interval \([0,1)\). We define the \(N\)-discrepancy of the sequence \((x_n)^\infty_{n=1}\) to be
\[
D_N(x_n) = \sup_{0 \leq \alpha < \beta < 1} \left| \frac{1}{N} Z(N; \alpha, \beta) - (\beta - \alpha) \right|
\]
where \(Z(N; \alpha, \beta) := \#\{1 \leq k \leq N : \alpha \leq x_k \leq \beta\}\). A sequence \((x_n)^\infty_{n=1}\) is by definition uniformly distributed mod 1 if and only if \(D_N(x_n) \to 0\) as \(N \to \infty\). Regarding the order of magnitude of the discrepancy of arbitrary sequences, Schmidt [14] has shown that the discrepancy of any sequence \((x_n)^\infty_{n=1} \subseteq [0,1)\) satisfies
\[
D_N(x_n) \geq c \frac{\log N}{N} \quad \text{for inf. many } N = 1, 2, \ldots
\]
where \(c > 0\) is an absolute constant; thus the discrepancy of an arbitrary sequence cannot tend to 0 arbitrarily fast.

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A case of particular interest is the discrepancy of $(n_k x)_{k=1}^\infty$, where $(n_k)_{k=1}^\infty$ is lacunary and $x \in [0, 1)$. Recall that a sequence $(n_k)_{k=1}^\infty$ of positive integers is called lacunary if there exists some constant $q > 1$ such that

$$\frac{n_{k+1}}{n_k} \geq q, \quad k = 1, 2, \ldots$$  \hspace{1cm} (1)

It is well known that whenever (1) holds, the sequence of functions $(e(n_k x))_{k=1}^\infty$ behaves like a sequence of independent random variables (here and in what follows we use the notation $e(x) = e^{2\pi i x}$); to be more specific, a result of Erdős and Gal [4] states that

$$\limsup_{N \to \infty} \frac{1}{\sqrt{N \log \log N}} \left| \sum_{k=1}^{N} e(n_k x) \right| = 1 \quad \text{for Lebesgue-almost all } x \in [0, 1),$$  \hspace{1cm} (2)

which is an analogue of the Law of the Iterated Logarithm for sequences of independent random variables. The precise order of magnitude of $D_N(n_k x)$ in that case had been an open question for many years, with Philipp [11] giving the final answer.

**Theorem (Philipp).** Let $(n_k)_{k=1}^\infty$ be a lacunary sequence of integers such that (1) is satisfied. Then for Lebesgue-almost all $x \in [0, 1)$ we have

$$\frac{1}{4} \leq \limsup_{N \to \infty} \frac{N D_N(n_k x)}{\sqrt{N \log \log N}} \leq C_q,$$  \hspace{1cm} (3)

where $C_q \leq 166 + 664(q^{1/2} - 1)^{-1}$ is a constant which depends on $q > 1$.

This is in accordance with the Chung-Smirnov Law of the Iterated Logarithm, which states that for any sequence $(X_n)_{n=1}^\infty$ of independent random variables, uniformly distributed on $[0, 1)$, we have

$$\limsup_{N \to \infty} \frac{N D_N(X_1, X_2, \ldots, X_N)}{\sqrt{2N \log \log N}} = \frac{1}{2}$$

with probability 1 (see [15, p.504]), thus further indicating the resemblance of $(n_k x)_{k=1}^\infty$ with a sequence of independent random variables. The exact value of the limsup in (3) for specific choices of the sequence $(n_k)_{k=1}^\infty$ has been calculated by Fukuyama in [5].

In the present article we examine whether Philipp’s metrical result can be generalised for measures which are supported on several fractal subsets of the unit interval. We focus our attention on probability measures $\mu$ such that their Fourier transform defined by

$$\hat{\mu}(t) = \int e^{2\pi i t x} d\mu(x), \quad t \in \mathbb{R}$$

has a prescribed decay rate. In the results to follow, we assume that the Fourier transform of $\mu$ has a polynomial decay rate, that is, an asymptotic relation of the form

$$|\hat{\mu}(t)| \ll |t|^{-n}, \quad |t| \to \infty$$  \hspace{1cm} (4)
holds for some constant $\eta > 0$. The connection of the decay rate of the Fourier transform of $\mu$ with distribution properties is not unexpected, in view of the following theorem of Davenport, Erdős and LeVeque.

**Theorem (Davenport, Erdős & LeVeque).** Let $\mu$ be a probability measure supported on $[0, 1]$ and $(q_n)_{n=1}^{\infty}$ be a sequence of natural numbers. If
\[
\sum_{N=1}^{\infty} \frac{1}{N^3} \sum_{m,n=1}^{N} \hat{\mu}(h(q_m - q_n)) < \infty
\]
for all integers $h \neq 0$, then the sequence $(q_n x)_{n \in \mathbb{N}}$ is uniformly distributed modulo one for $\mu$–almost all $x \in [0, 1)$.

The main result of this paper is the following.

**Theorem 1.** Let $(n_k)_{k=1}^{\infty}$ be a lacunary sequence of integers satisfying (1). Assume $\mu$ is a probability measure on $[0, 1)$ such that (4) holds for some $\eta > 0$. Then the discrepancy $D_N(n_k x)$ satisfies
\[
\frac{1}{4} \leq \limsup_{N \to \infty} \frac{N D_N(n_k x)}{\sqrt{N \log \log N}} \leq C \quad \text{for } \mu\text{-almost all } x \in [0, 1),
\]
where the constant $C > 0$ only depends of the value of $q > 1$ as in (1). Additionally $C \leq 166 + 664(q^{1/2} - 1)^{-1}$.

An application of Theorem 1 is an improvement of a result of Haynes, Jensen and Kristensen in [7] relevant to an inhomogeneous version of Littlewood’s conjecture, which is the statement that for all $\alpha, \beta \in \mathbb{R}$, we have $\liminf q \|q \alpha\| \|q \beta\| = 0$. This is clearly the case when $\alpha$ or $\beta$ is an element of the set $\text{Bad} := \{x \in [0, 1) : \liminf q \|qx\| > 0\}$ of badly approximable numbers. The result proved in [7] is the following:

**Theorem (Haynes, Jensen & Kristensen).** Fix $\varepsilon > 0$ and a sequence $(\alpha_i)_{i=1}^{\infty} \subseteq \text{Bad}$. Then there exists a set $G \subseteq \text{Bad}$ of Hausdorff dimension $\dim G = 1$, such that for all $\beta \in G$ the following holds:
\[
q \|q \alpha_i\| \|q \beta - \gamma\| < 1/(\log q)^{1-\varepsilon} \quad \text{for inf. many } q = 1, 2, \ldots
\]
for all $i \geq 1$ and all $\gamma \in \mathbb{R}$.

The proof of the result in [7] relies on a metric discrepancy estimate with respect to certain probability measures supported on subsets of the set $\text{Bad}$. More precisely, if $F_N := \{x = [a_1, a_2, \ldots] \in [0, 1) : a_n \leq N \text{ for all } n \geq 1\}$ is the set of $x \in [0, 1)$ such that the partial quotients in the continued fraction expansion of $x$ are at most equal to $N$, a theorem of Kaufman [9], later improved by Queffélec and Ramaré [13], states that the sets $F_N$, $N \geq 2$ support probability measures with two key properties:
Theorem (Kaufman & Queffélec-Ramaré). Let $N \geq 2$. If $\varepsilon > 0$ and $\frac{1}{2} < \delta < \dim F_N$, then the set $F_N$ supports a probability measure $\mu = \mu(N, \delta, \varepsilon)$ with the following properties:

(i) $\mu(I) \leq c_1 |I|^{\delta}$ for any interval $I \subseteq [0, 1)$, and

(ii) $|\hat{\mu}(t)| \leq c_2 (1 + |t|)^{-\eta - 8\varepsilon}$ for all $t \in \mathbb{R}$, where $\eta = \frac{\delta(2\delta - 1)}{(2\delta + 1)(4 - \delta)} > 0$.

Here $c_1, c_2 > 0$ are absolute constants.

In the current paper, adapting the method of proof in [7] together with the sharper discrepancy estimate coming from Theorem 1, we are able to obtain a slight improvement to the result of Haynes, Jensen and Kristensen:

Theorem 2. Let $\varepsilon > 0$ be fixed and $(\alpha_j)_{j=1}^\infty \subseteq \text{Bad}$ be a sequence of badly approximable numbers. There exists a subset $G \subseteq \text{Bad}$ of Hausdorff dimension $\dim G = \dim \text{Bad} = 1$ such that for any $\beta \in G$, the following holds:

$$q\|q\alpha_j\|q\beta - \gamma\| < \frac{\left(\log_q q\right)^{1/2 + \varepsilon}}{(\log q)^{1/2}}$$

for inf. many $q = 1, 2, \ldots$ (7)

for all $j = 1, 2, \ldots$ and for all $\gamma \in [0, 1)$.

As another consequence of Theorem 1, we obtain the following discrepancy statement with additional information on the Hausdorff dimension of the set of numbers for which it is satisfied.

Theorem 3. Let $(n_k)_{k=1}^\infty$ be a lacunary sequence of integers satisfying (1). There exists a set $G \subseteq \text{Bad}$ with Hausdorff dimension $\dim G = 1$ such that

$$\limsup_{N \to \infty} \frac{N^2 D_n(n_k x)}{\sqrt{N \log \log N}} \leq C,$$

for all $x \in G$. (8)

2 Proof of Theorem 1

2.1 The upper bound

We employ a classical method of proof for discrepancy estimates, which has been used by Philipp [11], Erdős & Gal [4] and Gal & Gal [6]. For any positive integer $N$, the discrepancy $D_N(n_k x)$ satisfies

$$N^2 D_n(n_k x) = \sup_{0 \leq \alpha < \beta < 1} \left| \sum_{k \leq N} \chi_{[\alpha, \beta)}(n_k x) - N(\beta - \alpha) \right| \leq 2 \sup_{f \in \mathcal{F}} \left| \sum_{k \leq N} f(n_k x) \right|,$$

where $\mathcal{F}$ denotes the set of functions $f : [0, 1) \to \mathbb{R}$ which are 1-periodic with $\int_0^1 f(x)dx = 0$ and have bounded variation. Moreover, since every such function is trivially the sum of an even and an odd function with the same properties, we may restrict our attention to the set $\mathcal{F}^* \subseteq \mathcal{F}$ of functions $f \in \mathcal{F}$ which are additionally even.
2.1.1 Some auxiliary results

Let $f$ be an even function of bounded variation on $[0, 1]$ such that

$$f(x + 1) = f(x) \quad \text{and} \quad \int_0^1 f(x) \, dx = 0 \quad (9)$$

and let

$$f(x) = \sum_{j=1}^{\infty} c_j \cos(jx) = \sum_{|j|=1}^{\infty} c_j e(jx)$$

be its Fourier series expansion. Observe that $c_j = c_{-j}$ for all $j$ and (9) imposes $c_0 = 0$ and $c_j \leq |j|^{-1}$. We set

$$f_n(x) = \sum_{1 \leq |j| \leq n} c_j e(jx).$$

Write $r = q^{1/2}$ (where $q > 1$ is as in (1)) and define

$$\phi_n(x) = f(x) - f_n(x),$$

$$\psi(x; m) = \sum_{r^m \leq |j| < r^{m+1}} c_j e(jx),$$

$$\Phi_N(x; m) = \sum_{k \leq N} \psi(n_k x; m).$$

Also for the arbitrary $f \in L_2(\mu)$ we write $\|f\|_{L_2(\mu)}^2 = \int |f|^2 \, d\mu$. In the following Lemmas we calculate the $\| \cdot \|_{L_2(\mu)}$-norm of sums of the form $\sum_{k \leq N} \phi_T(n_k x)$, first by calculating the norm of the blocks $\Phi_N(x; m)$ in Lemma 1 and then combining these estimates as in Lemma 2.

**Lemma 1.** We have

$$\int |\Phi_N(x; m)|^2 d\mu(x) \ll \frac{N}{r^m \eta}.$$

**Proof.** We calculate

$$\int |\Phi_N(x; m)|^2 d\mu(x) = \sum_{1 \leq k, l \leq N} \sum_{r^m \leq |i|, |j| < r^{m+1}} c_i c_j e((in_k - jn_l)x) \, d\mu(x)$$

$$\ll \sum_{1 \leq k, l \leq N} \sum_{r^m \leq |i|, |j| < r^{m+1}} c_i c_j \delta_{in_k, jn_l} + \sum_{1 \leq k, l \leq N} \sum_{r^m \leq |i|, |j| < r^{m+1}} \frac{c_i c_j}{|in_k - jn_l|^\eta}.$$

Regarding the first of these terms, we can show as in [11, Lemma 1] that it has order of magnitude

$$\ll N \log \left( \frac{r^{m+1}}{r^m} \right) \frac{1}{r^m} \ll \frac{N}{r^m}.$$
The second term is
\[
\sum_{1 \leq k,l \leq N} c_ic_j |m_k - jn_l|^\eta \ll \sum_{r^m \leq |i|,|j| < r^{m+1}} c_ic_j |n_k(j - j)|^{-\eta} + \sum_{l < k \leq N} \sum_{r^m \leq |i|,|j| < r^{m+1}} |in_k - jn_l|^\eta.
\]

The first of the sums in the right hand side of (10) is at most
\[
\sum_{r^m \leq |i|,|j| < r^{m+1}} \sum_{1 \leq l < k \leq N} \sum_{1 \leq i < j < r^{m+1}} c_ic_j |n_k(j - j)|^{-\eta} \ll \sum_{r^m \leq |i|,|j| < r^{m+1}} \sum_{1 \leq i < j < r^{m+1}} (c_i^2 + c_j^2) (j - i)^{-\eta}
\ll \sum_{r^m \leq |i| < r^{m+1}} \frac{1}{i^2} \sum_{j=1}^{r^{m+1}-i} \frac{1}{j^{\eta}}
\ll \sum_{r^m \leq |i| < r^{m+1}} \frac{1}{i^2} (r^{m+1} - i)^{1-\eta}
\ll \sum_{i=r^m}^{\infty} \frac{1}{i^2} (1-\eta)m
\ll \frac{1}{r^{m\eta}}.
\]

Regarding the second sum in the right hand side of (10), under the conditions of summation we get
\[
|in_k - jn_l| = |i|n_k \begin{vmatrix} -\frac{jn_l}{in_k} \end{vmatrix} \geq |i|n_k \begin{vmatrix} 1 - \frac{1}{r} \end{vmatrix} \gg r^m n_k,
\]
whence
\[
\sum_{1 \leq l < k \leq N} \sum_{r^m \leq |i|,|j| < r^{m+1}} c_ic_j |in_k - jn_l|^{-\eta} \ll \sum_{i=r^m}^{r^{m+1}} \sum_{j=r^m}^{N} \frac{1}{j^{\eta}} \sum_{k=l+1}^{r^{m+1}} \frac{1}{r^{m\eta} n_k^{\eta}}
\ll \sum_{i=r^m}^{r^{m+1}} \frac{1}{i^2} r^{m\eta} n_l^{\eta}
\ll \frac{1}{r^{m\eta}}.
\]

\[\square\]
Lemma 2. For any positive integer $N, T$ large enough we have
\[
\int \left( \sum_{k \leq N} \phi_T(n_k x) \right)^2 \, d\mu(x) \ll \frac{N}{T^\eta}.
\]

Proof. Let $m_0$ be the positive integer such that $r^{m_0-1} \leq T < r^{m_0}$. By observing that
\[
\phi_T(n_k x) = r^{m_0-1} \sum_{|j|=T} c_j e(j n_k x) + \sum_{m=m_0}^\infty \psi(n_k x; m),
\]
we conclude that
\[
\left\| \sum_{k \leq N} \phi_T(n_k x) \right\|_{L^2(\mu)} \leq \left\| \sum_{k \leq N} \sum_{|j|=T} c_j e(j n_k x) \right\|_{L^2(\mu)} + \sum_{m=m_0}^\infty \| \Phi_N(x; m) \|_{L^2(\mu)}.
\]
The first of the two terms is, by exploiting the arguments of the proof of Lemma 1, seen to be \( \ll N^{1/2} T^{-\eta/2} \). Furthermore, the second term is, due to Lemma 1, up to a constant at most
\[
\sum_{m=m_0}^\infty \frac{N^{1/2}}{r^{m_0 \eta/2}} \ll \frac{N^{1/2}}{r^{m_0 \eta/2}} \ll N^{1/2} T^{\eta/2}.
\]
Hence the result of the Lemma is shown. \( \Box \)

In what follows $H, P$ and $T$ denote positive integers. We set
\[
g(x) = \sum_{|j|=1}^T c_j e(j x), \quad U_m(x) = \sum_{k=Hm+1}^{H(m+1)} g(n_k x)
\]
Similar to inequality (2.6) of [11] we can write
\[
\|g\|_\infty \leq \text{Var} f + \|f\|_\infty \leq 3.
\]
(11)

Lemma 3. Let $0 < \kappa < 1$ and assume the integers $P, H, T$ are such that
\[
4PT \leq q^{2H} \quad \text{and} \quad 3T^2 H^2 < q^{nH}.
\]
(12)

Then for any $\delta > 0$ we have
\[
\int \exp \left( \kappa \sum_{m=0}^{P-1} U_{2m}(x) \right) \, d\mu(x) \leq \exp \left( \frac{1}{2} (1 + 2\delta) C_0 \kappa^2 \|f\| H(P+1) \right)
\]
and
\[
\int \exp \left( \kappa \sum_{m=1}^{P} U_{2m-1}(x) \right) \, d\mu(x) \leq \exp \left( \frac{1}{2} (1 + 2\delta) C_0 \kappa^2 \|f\| H(P+1) \right),
\]
where $C_0 = \text{is an absolute constant}.$
Proof. We shall employ the inequality

$$|e^z| \leq 1 + z + \frac{1}{2}(1 + \delta)z^2,$$

which is valid for all numbers with $|z| < z_0(\delta)$. Since

$$|\kappa U_{2m}(x)| = \left| \kappa \sum_{k=2Hm+1}^{H(2m+1)} g(n_k x) \right| \leq \kappa H \|g\|_\infty < 1,$$

we can apply (13) to obtain

$$\exp \left( \kappa \sum_{m=0}^{P-1} U_{2m}(x) \right) = \prod_{m=0}^{P-1} \exp(\kappa U_{2m}(x)) \leq \prod_{m=0}^{P-1} \left( 1 + \kappa U_{2m}(x) + \frac{1}{2}(1 + \delta)\kappa^2 U_{2m}^2(x) \right).$$

Observe that

$$U_{2m}(x) = \sum_{k=2mH+1}^{H(2m+1)} g(n_k x) = \sum_{2mH < k < (2m+1)H} \sum_{1 \leq |j| \leq T} c_j e(jn_k x)$$

is a sum of trigonometric terms of frequencies at least $n_{2mH} \geq q^{2mH}$ in absolute value. Write

$$U_{2m}^2(x) = \sum_{2mH < k < (2m+1)H} \sum_{1 \leq |j| \leq T} c_j^2 e(2jn_k x) + \sum_{2mH < k < (2m+1)H} \sum_{1 \leq |j| \leq T} \sum_{1 \leq |l| \leq T} c_j c_j e((j_1 + j_2)n_k x)$$

$$+ 2 \sum_{2mH < k < \ell < (2m+1)H} \sum_{1 \leq |j| \leq T} \sum_{1 \leq |l| \leq T} c_j c_l e((j_1 n_k + j_2 n_\ell)x)$$

$$= W_{2m}(x) + V_{2m}(x),$$

where

$$W_{2m}(x) := \sum_{2mH < k < (2m+1)H} \sum_{1 \leq |j| \leq T} c_j^2 e(2jn_k x) + \sum_{2mH < k < (2m+1)H} \sum_{1 \leq |j_1|, |j_2| \leq T} c_j c_j e((j_1 + j_2)n_k x)$$

$$+ 2 \sum_{2mH < k < \ell < (2m+1)H} \sum_{1 \leq |j_1|, |j_2| \leq T} \sum_{|j_1 n_k + j_2 n_\ell| \geq n_{2mH}} c_j c_l e((j_1 n_k + j_2 n_\ell)x)$$

is the sum of trigonometric terms appearing in $U_{2m}^2(x)$ with frequencies at least $n_{2mH}$, and

$$V_{2m}(x) = U_{2m}^2(x) - W_{2m}(x)$$
is the sum of the remaining terms in $U_{2m}^2(x)$, which have frequencies strictly less than $n_{2mH}$. It is shown in [16] and [11, p.246] that

$$|V_{2m}(x)| \leq C_0 \|f\|H.$$ 

Hence

$$\int \exp \left( \kappa \sum_{m=0}^{P-1} U_{2m}(x) \right) d\mu(x) \leq \int h(x) d\mu(x),$$

where we define the integrand to be

$$h(x) \coloneqq \prod_{m=0}^{P-1} (1 + \frac{1}{2}(1 + \delta)C_0\kappa^2\|f\|H + \kappa U_{2m}(x) + \frac{1}{2}(1 + \delta)\kappa^2 W_{2m}(x))$$

$$= (1 + \frac{1}{2}(1 + \delta)C_0\kappa^2\|f\|H)^P +$$

$$\quad + \left(1 + \frac{1}{2}(1 + \delta)C_0\kappa^2\|f\|H\right)^{P-1} \left( \kappa \sum_{m=0}^{P-1} U_{2m}(x) + \frac{1}{2}(1 + \delta)\kappa^2 \sum_{m=0}^{P-1} W_{2m}(x) \right)$$

$$\quad + \left(1 + \frac{1}{2}(1 + \delta)C_0\kappa^2\|f\|H\right)^{P-2} \sum_{0 \leq m_1 < m_2 < P} \prod_{i=1}^{2} (\kappa U_{2m_i}(x) + \frac{1}{2}(1 + \delta)\kappa^2 W_{2m_i}(x))$$

$$\quad + \ldots$$

$$\quad + \left(1 + \frac{1}{2}(1 + \delta)C_0\kappa^2\|f\|H\right)^{P-s} \sum_{0 \leq m_1 < \ldots < m_s < P} \prod_{i=1}^{s} (\kappa U_{2m_i}(x) + \frac{1}{2}(1 + \delta)\kappa^2 W_{2m_i}(x))$$

$$\quad + \ldots + \prod_{m=0}^{P-1} (\kappa U_{2m}(x) + \frac{1}{2}(1 + \delta)\kappa^2 W_{2m}(x)).$$

If we look at the $s$-th term in the above expansion, every factor

$$\kappa U_{2m_i}(x) + \frac{1}{2}(1 + \delta)\kappa^2 W_{2m_i}(x)$$

is a sum of trigonometric terms, which have frequencies lying between $n_{2m,H}$ and $2Tn_{(2m_i+1)H}$ in absolute value. The number of these terms is at most $3T^2H^2$. Thus any product

$$\prod_{j=1}^{s} (\kappa U_{2m_j}(x) + \frac{1}{2}(1 + \delta)\kappa^2 W_{2m_j}(x))$$

is a sum of at most $3^sT^{2s}H^{2s}$ trigonometric terms, each of them being multiplied by a coefficient at most $\kappa^s$ and having frequency which is at least

$$n_{2m,H} - 2T(n_{2m_1,H} + \ldots + n_{2m_{s-1},H}) = n_{2m,H} \left[ 1 - 2T \left( \frac{n_{2m_1,H}}{n_{2m,H}} + \ldots + \frac{n_{2m_{s-1},H}}{n_{2m,H}} \right) \right]$$

$$\geq q^{2m,H} \left[ 1 - 2T \left( \frac{1}{q^{2(m_{s}-m_1)H}} + \ldots + \frac{1}{q^{2(m_{s}-m_{s-1})H}} \right) \right]$$

$$\geq \frac{1}{2} q^{2m,H},$$
where we used the fact that
\[
\frac{1}{q^{2(m_s-m_1)H}} + \ldots + \frac{1}{q^{2(m_s-m_{s-1})H}} \leq \frac{P}{q^{2H}} \leq \frac{1}{4T}.
\]
We deduce that
\[
\int \prod_{j=1}^{s} \left( \kappa U_{2m_j}(x) + \kappa^2 W_{2m_j}(x) \right) d\mu(x) \leq 2^n \frac{(2\kappa T^2 H^2)^{s}}{q^{2m_s \eta H}} \leq \frac{2^n}{q^{m_s \eta H}}
\]
and
\[
\int \sum_{0 \leq m_1 < \ldots < m_s < P} \prod_{j=1}^{s} \left( \kappa U_{2m_j}(x) + \kappa^2 W_{2m_j}(x) \right) d\mu(x) \leq 2^n \sum_{0 \leq m_1 < \ldots < m_s < P} q^{-m_s \eta H} \leq \frac{2^n q^n}{q^n - 1}.
\]
Thus
\[
\int h(x) d\mu(x) \leq \frac{2^n q^n}{q^n - 1} \sum_{s=0}^{P} \left( 1 + \frac{1}{2} (1 + \delta) C_0 \kappa^2 \|f\| H \right)^{P-s}
\]
\[
\leq \frac{2^n q^n}{q^n - 1} \left( 1 + \frac{1}{2} (1 + \delta) C_0 \kappa^2 \|f\| H \right)^{P+1}
\]
\[
\leq \frac{2^n q^n}{q^n - 1} \exp \left( \frac{1}{2} (1 + \delta) C_0 \kappa^2 \|f\| H (P + 1) \right)
\]
\[
\leq \exp \left( \frac{1}{2} (1 + 2\delta) C_0 \kappa^2 \|f\| H (P + 1) \right).
\]

The second inequality of the Lemma follows in precisely the same way.

**Proposition 4.** Let \(M_0 \geq 0, M \geq 1\) be positive integers and let \(R \geq 1\) be a real number. Assume \(f\) satisfies (9) and \(\|f\| \geq M^{-3/5}\). Consider the set
\[
A = \left\{ x \in [0, 1) : \left| \sum_{k=M_0+1}^{M_0+M} f(n_k x) \right| \geq (1 + 4\delta) C_0 R \|f\|^{1/4} (M \log \log M)^{1/2} \right\},
\]
where \(C_0 > 0\) is the constant from Lemma 3. Then
\[
\mu(A) \ll \exp \left( -(1 + 2\delta) C_0 \|f\|^{-1/2} R \log \log M \right) + \frac{1}{\|f\|^{1/2} R^2 M^4}.
\]

**Proof.** Without loss of generality, we may assume that \(M_0 = 0\). We put \(H = [M^{1/30}], T = M^{4/\eta}\) and set
\[
Q = 3C_0 \|f\|^{1/4} R (M \log \log M)^{1/2}, \quad \kappa = (\|f\|^{-3/2} M^{-1} \log \log M)^{1/2}.
\]
We choose a positive integer \( P \) such that
\[
H(2P + 1) \leq M \leq H(2P + 3).
\] (18)

Observe that \( A \subseteq A_1 \cup A_2 \), where
\[
A_1 = \left\{ x \in [0, 1) : \sum_{k \leq M} g(n_k x) \geq (1 + 2\delta)Q \right\}, A_2 = \left\{ x \in [0, 1) : \sum_{k \leq M} \phi_T(n_k x) \geq 2\delta Q \right\}.
\]

We are going to give estimates for the measure of these sets using the Chebyshev-Markov inequality. In order to do that, we observe that
\[
\kappa \left| \sum_{k \leq M} g(n_k x) - \sum_{m=0}^{2P} U_m(x) \right| \leq \kappa \sum_{k=H(2P+1)}^{M} |g(n_k x)| \leq 6\kappa H = o(1), \quad M \to \infty. \tag{19}
\]

Regarding \( A_1 \), we estimate
\[
\mu(A_1) \leq e^{-(1+2\delta)\kappa Q} \int \exp \left( \kappa \sum_{k \leq M} g(n_k x) \right) d\mu(x)
\]
\[
\ll e^{-(1+2\delta)\kappa Q} \int \exp \left( \kappa \sum_{m=0}^{2P} U_m(x) \right) d\mu(x) \quad \text{(by (19))}
\]
\[
\leq e^{-(1+2\delta)\kappa Q} \exp \left( 2(1 + 2\delta)C_0 \kappa^2 \|f\| H(P + 1) \right)
\]
\[
\leq \exp \left( -(1 + 2\delta)C_0 \|f\|^{-1/2} R \log \log M \right),
\]
while for \( A_2 \), the Chebyshev-Markov inequality again gives
\[
\mu(A_2) \ll \frac{1}{Q^2} \int \left( \sum_{k \leq M} \phi_T(n_k x) \right)^2 d\mu(x)
\]
\[
\ll \frac{1}{Q^2} MT^{-\eta} \quad \text{(by Lemma \[2]}\right)
\]
\[
\ll \frac{1}{\|f\|^{1/2} R^2 M^4}.
\]

2.1.2 Proof of the upper bound

Let \( N \geq 1 \) be a positive integer sufficiently large. We set
\[
H_1 = \left\lfloor \frac{\log N}{\log 4} \right\rfloor + 1. \tag{20}
\]
We define the functions \((\phi_j^{(j)})_{h \leq H_1}^{j \leq 2^h}\) as in [11]. Under this notation, inequality (3.2) in [11] states that for each \(0 \leq \alpha < 1\) there exists some index \(j = j(\alpha) \leq 2^h\) such that

\[
\sum_{h=1}^{H_1-1} \phi_h^{(j)}(x) \leq \chi_{[0,\alpha]}(x) \leq \sum_{h=1}^{H_1} \phi_h^{(j)}(x). \tag{21}
\]

For \(1 \leq h \leq H_1, 1 \leq j \leq 2^h, N \geq 1, M \geq 0\) set

\[
F(M, N, j, h; x) = \left| \sum_{k=M+1}^{M+N} \left( \phi_h^{(j)}(n_k x) - \int \phi_h^{(j)}(t) dt \right) \right|.
\]

The following is a variation of Lemma 4 in [11]. The proof relies on a method of Gal & Gal, see [6, Lemma 3.10].

**Lemma 5.** Let \(n\) be the positive integer such that \(2^n \leq N < 2^{n+1}\). There exist integers \((m_l)_{l=1}^n\) such that \(0 \leq m_l < 2^n - l, 1 \leq l \leq n\) and

\[
F(0, N, j, h; x) \leq F(0, 2^n, j, h; x) + \sum_{l=5n/12}^{n} F(2^n + m_l 2^l, 2^{l-1}, j, h; x) + N^{5/12}.
\]

In what follows we set

\[
\chi(N) = 2(1 + 4\delta)C_0(N \log \log N)^{1/2}.
\]

Define the sets

\[
G(n, j, h) = \{ 0 \leq x < 1 : F(0, 2^n, j, h; x) \geq 2^{-h/8} \chi(2^n) \},
\]

\[
H(n, j, h, l, m) = \{ 0 \leq x < 1 : F(2^n + m_l 2^l, 2^{l-1}, j, h; x) \geq 2^{-h/8} 2^{(l-n-3)/6} \chi(2^n) \},
\]

\[
G_n = \bigcup_{h \leq H, j \leq 2^h} G(n, j, h), \quad H_n = \bigcup_{h \leq H, j \leq 2^h} \bigcup_{l=5n/12}^{m \leq 2^n - l} H(n, j, h, l, m).
\]

**Lemma 6.** Let \(0 < \delta_0 < 1\). There exists \(n_0 = n_0(\delta_0) \in \mathbb{N}\) such that

\[
\mu \left( \bigcup_{n=n_0}^{\infty} (G_n \cup H_n) \right) < \delta_0.
\]

**Proof.** By (20) we have

\[
N^{-1/2} \ll 2^{-(h+1)} \leq \left\| \phi_h^{(j)} - \int \phi_h^{(j)}(t) dt \right\|^2 \leq 2^{-h}, \quad 1 \leq h \leq H_1, 1 \leq j \leq 2^h. \tag{22}
\]

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Applying Proposition 4 with $M = 0$, $N = 2^n$, $R = 1$ and $f = \phi_h^{(j)} - \int \phi_h^{(j)}(t)dt$, we get
\[
\mu \left( G(n, j, h) \right) \ll \exp \left( -(1 + 2\delta)C_0\|f\|^{-1/2} \log \log N \right) + \frac{1}{\|f\|^{1/2}N^4}
\ll \exp \left( -(1 + 2\delta)C_02^{h/4} \log \log 2^n \right) + \frac{2^{h/4}}{2^{4n}},
\]
hence
\[
\mu \left( G_n \right) \ll \sum_{h \leq H_1} \sum_{j \leq 2^h} \mu \left( G(n, j, h) \right) \ll \sum_{h \leq H_1} \sum_{j \leq 2^h} \left( n \log 2 \right)^{-\left(1 + 2\delta\right)C_02^{h/4} + \frac{2^{h/4}}{2^{4n}}} \ll n^{-\left(1 + \delta\right)}.
\]

Applying Proposition 4 with $M = 2^n + m2^l$, $N = 2^{l-1}$, $R = 2^{(n-l)/3}$ and $f = \phi_h^{(j)} - \int \phi_h^{(j)}(t)dt$, we obtain
\[
\mu \left( H(n, j, h, l, m) \right) = \mu \left( \{ x : F(2^n + m2^l, 2^{l-1}, j, h) \geq 2^{-h/82(t-n-3)/6} \chi(2^n) \} \right)
\leq \mu \left( \{ x : F(2^n + m2^l, 2^{l-1}, j, h) \geq \|f\|^{1/4} R \chi(2^l) \} \right)
\ll \exp \left( -(1 + 2\delta)C_02^{h/4} 2^{(n-l)/3} \log \log 2^l \right) + \frac{2^{h/4}}{2^{2n/3}2^{10l/3}}.
\]

We now deduce that
\[
\mu \left( H_n \right) \ll \sum_{h \leq H_1} \sum_{j \leq 2^h} \sum_{l = \frac{5n}{12}}^{2n-1} \sum_{m = 1}^{n} \mu \left( H(n, j, h, l, m) \right)
\ll \sum_{h \leq H_1} \sum_{j \leq 2^h} \sum_{l = \frac{5n}{12}}^{2n-1} \left( \exp \left( -(1 + 2\delta)C_02^{h/4} 2^{(n-l)/3} \log \log l \right) + \frac{2^{h/4}2^{n/3}}{2^{13l/3}} \right)
\ll \sum_{h \leq H_1} \sum_{j \leq 2^h} \sum_{l = \frac{5n}{12}}^{2n-1} \left( n^{-\left(1 + 2\delta\right)C_02^{h/4} 2^{(n-l)/3}} + \frac{2^{h/4}2^{n/3}}{2^{13l/3}} \right)
\ll \sum_{h \leq H_1} \sum_{j \leq 2^h} \left( n^{-\left(1 + 2\delta\right)C_02^{h/4}} + \frac{2^{h/4}}{2^{53n/36}} \right) \ll n^{-\left(1 + \delta\right)}.
\]
The conclusion of the Lemma is now evident. \( \Box \)

We may now proceed to the final part of the proof. Choose an arbitrary $0 \leq \alpha < 1$. By
we obtain
\[
\left| \sum_{k \leq N} \chi_{[0, \alpha)}(n_k x) - N\alpha \right| \leq \sum_{h=1}^{H_1} \sum_{k \leq N} \phi_h^{(j)}(n_k x) - N \int \phi_h^{(j)}(t) dt + 2^{-H_1} N
\]
\[
\leq \sum_{h=H_1} \left( F(0, 2^n, j, h) + \sum_{l=n/2}^n F(2^n + m l^{-1}, 2^{l-1}, j, h) \right) + 2 N^{1/2}
\]
\[
\leq \sum_{h=H_1} 2^{-h/8} \chi(2^n) \left( 1 + \sum_{l=n/2}^n 2^{(l-n-3)/6} \right) + 2 N^{1/2}
\]
\[
\leq (1 + 4\delta)(83 + 332(\sqrt{q} - 1)^{-1})(N \log \log N)^{1/2},
\]
for all $0 \leq x < 1$ lying outside a set of $\mu$-measure at most $\delta_0$. Hence for those $0 \leq x < 1$
we obtain for any $0 \leq \alpha < \beta < 1$
\[
\left| \sum_{k \leq N} \chi_{[\alpha, \beta)}(n_k x) - N(\beta - \alpha) \right| \leq (1 + 4\delta)(166 + 332(\sqrt{q} - 1)^{-1})(N \log \log N)^{1/2}.
\]
Taking the supremum over all $0 \leq \alpha < \beta < 1$, we get
\[
ND_N(n_k x) \leq (1 + 4\delta)(166 + 332(\sqrt{q} - 1)^{-1})(N \log \log N)^{1/2}
\]
for all $x$ in a set of $\mu$ measure at most $\delta_0$. Now letting $\delta \to 0$ and then $\delta_0 \to 0$ we obtain
the requested upper bound in Theorem 1.

2.2 The lower bound

Given a sequence $(n_k)_{k=1}^{\infty}$, Koksma’s Inequality implies that
\[
ND_N(n_k x) \geq \frac{1}{4} \left| \sum_{k=1}^N e(n_k x) \right|, \quad x \in [0, 1),
\]
see [10, p.143] for more details. Thus the lower bound in Theorem 1 will follow immediately
if we prove the following partial generalisation of the result of Erdös and Gaal in [4]:

**Proposition 7.** Let $(n_k)_{k=1}^{\infty}$ be a lacunary sequence of integers such that (1) is satisfied.
If $\mu$ is a probability measure on $[0, 1)$ such that $|\hat{\mu}(t)| \ll |t|^{-\eta}, |t| \to \infty$ then
\[
\limsup_{N \to \infty} \frac{\left| \sum_{k=1}^N e(n_k x) \right|}{\sqrt{N \log \log N}} \geq 1 \quad \text{for } \mu\text{-almost all } x \in [0, 1).
\]
(23)

The proof of Proposition 7 is essentially the same as in [4], with the only modifications
being those relevant to the fact that $\mu$ is a probability measure other than the Lebesgue.
We present here all steps of the proof which are essentially different and refer the reader
to [4] for the remaining parts.
2.2.1 On the number of solutions of certain Diophantine inequalities

In what follows the positive integers \( p, N \) are fixed, the sequence \( (n_k)_{k=1}^{\infty} \) is as in Theorem 1 and
\[
A(x, y) \equiv A(x_1, \ldots, y_p) = (x_1 + \ldots + x_p) - (y_1 + \ldots + y_p)
\]
is a linear form in \( 2p \) variables which are allowed to take values in the set \( \{n_1, \ldots, n_N\} \).

**Lemma 8.** For any \( 0 < \alpha < \beta \),
\[
\#\{1 \leq k \leq N : \alpha \leq n_k \leq \beta\} \leq \frac{\log (\beta \alpha^{-1} q)}{\log q}.
\]

**Proof.** This is Lemma 1 in [4]. \( \square \)

In what follows, given \( s \in \mathbb{R} \) and \( r > 0 \) we write \( B(s, r) := (s - r, s + r) \) for the interval with center \( s \) and length \( 2r \).

**Lemma 9.** For positive integers \( N, K \geq 1 \) and \( s \in \mathbb{Z} \) we write \( \phi(1, N; s, K) \) for the number of pairs \( (n_k, n_l) \) with \( 1 \leq k, l \leq N \) such that \( n_k - n_l \in B(s, 2K - 1) \) and \( n_k \neq n_l \). Then
\[
\phi(1, N; s, K) \leq 2 \frac{\log^2(2Kq^2(q-1)^{-1})}{\log^2 q}.
\]

**Proof.** Since \( \phi(1, N; s, K) = \phi(1, N; -s, K) \), we may assume without loss of generality that \( s \geq 0 \). If \( 0 \leq s \leq 2K^{-1} \) then \( \frac{1}{2}\phi(1, N; s, K) \) is at most equal to the number of pairs \( (n_k, n_l) \) such that \( 1 \leq n_k - n_l \leq 2K \). If \( (n_k, n_l) \) is such a pair, then
\[
2K \geq n_k - n_l \geq n_k \left(1 - \frac{n_l}{n_k}\right) \geq n_k(1 - q^{-1}),
\]
hence \( 1 \leq n_k \leq 2K(1 - q^{-1})^{-1} \) and by Lemma 8 the number of admissible \( n_k \)'s is at most \( \frac{\log(2Kq^2(q-1)^{-1})}{\log q} \). Now we fix an admissible value of \( n_k \) and we count the number of \( n_l \)'s which are acceptable for the specific \( n_k \). Such \( n_l \)'s satisfy \( 1 \leq n_l < n_k \leq 2K(1 - q^{-1})^{-1} \), so by Lemma 8 their number is at most \( \frac{\log(2Kq^2(q-1)^{-1})}{\log q} \). Hence the number of possible pairs \( (n_k, n_l) \) is at most
\[
\frac{\log^2(2Kq^2(q-1)^{-1})}{\log^2 q}.
\]
On the other hand, if \( s > 2K^{-1} \) then \( s + 2K^{-1} \geq n_k - n_l \geq n_k(1 - q^{-1}) \) and \( n_k \geq \max(1, s - 2K^{-1}) = s - 2K^{-1} \), hence
\[
s - 2K^{-1} \leq n_k \leq (s + 2K^{-1})(1 - q^{-1})^{-1}\]
and by Lemma 8 there are at most
\[
\frac{1}{\log q} \log \left( \frac{s + 2^{K-1}}{s - 2^{K-1}} q(1 - q^{-1})^{-1} \right) \leq \frac{\log(2^K q^2 (q - 1)^{-1})}{\log q}
\]
possible values for \( n_k \). Regarding \( n_l \), we get
\[
\frac{1}{\log q} \log \left( \frac{s + 2^{K-1}}{s - 2^{K-1}} q(1 - q^{-1})^{-1} \right) \leq \frac{\log(2^K q^2 (q - 1)^{-1})}{\log q}
\]
for \( n_k \), and the number of pairs \((n_k, n_l)\) is again bounded above by
\[
\frac{\log^2(2^K q^2 (q - 1)^{-1})}{\log^2 q}.
\]

\[ \square \]

**Lemma 10.** For positive integers \( N, K \geq 1 \) and \( s \in \mathbb{Z} \) let \( \phi_p(s, K) \) be the number of pairs \((x_p, y_p)\) such that \( A(x, y) \in B(s, 2^{K-1}) \), subject to the restrictions \( x_1 \leq \ldots \leq x_p \) and \( y_1 \leq \ldots \leq y_p \). Then
\[
\phi_p(s, K) \leq 4pN \frac{\log(2^{K+2}q)}{\log q}.
\]

**Proof.** Since \( \phi_p(s, K) = \phi_p(-s, K) \), we may assume without loss of generality that \( s \geq 0 \). First we count the number of requested pairs \((x_p, y_p)\) for which \( x_p \leq 2s + 2^K \). The assumptions imply that
\[
s - 2^{K-1} \leq (x_1 + \ldots + x_p) - (y_1 + \ldots + y_p) \leq px_p,
\]
hence \( p^{-1}(s - 2^{K-1}) \leq x_p \leq 2^{K+2} \). When \( s > 3 \cdot 2^{K-1} \), we have \( \frac{1}{p}(s - 2^{K-1}) \leq x_p \leq 4(s - 2^{K-1}) \) and by Lemma 8 there are at most
\[
\frac{\log(4pq)}{\log q} \leq p \frac{\log(2^{K+2}q)}{\log q}
\]
possible values for \( x_p \). When \( 0 \leq s \leq 3 \cdot 2^{K-1} \), we have \( 1 \leq x_p \leq 2s + 2^K \leq 2^{K+2} \) and there are at most
\[
\frac{\log(2^{K+2}q)}{\log q} \leq p \frac{\log(2^{K+2}q)}{\log q}
\]
values of \( x_p \). In both cases for \( s \), there are at most \( N \) choices for \( y_p \), so the number of possible pairs \((x_p, y_p)\) with \( x_p \leq 2^{K+2} \) is bounded above by
\[
pN \frac{\log(2^{K+2}q)}{\log q}.
\]
Next we count the number of requested pairs \((x_p, y_p)\) for which \( x_p > 2^{K+2} \). The assumptions now imply that
\[
x_p \leq py_p + s + 2^{K-1} \leq py_p + \frac{1}{2} x_p,
\]
hence
\[ \frac{x_p}{2p} \leq y_p \leq px_p - s + 2^{K-1} \leq px_p + 2^{K-1} < 2px_p \]
and the number of possible values for \( y_p \) is by Lemma 8 at most \( \frac{\log(4p^2 q)}{\log q} \). Since there are at most \( N \) possible choices for \( x_p \), we have the upper bound
\[ N \frac{\log(2p^2 q)}{\log q} \leq 2Np \frac{\log(2^{K+2}q)}{\log q} \]
for the number of pairs \((x_p, y_p)\) with \( x_p > 2s + 2^{K-1} \). Combining the estimates for the two cases we obtain the requested bound \( (25) \).

\[ \forall \]

Lemma 11. For \( 1 \leq p \leq N \) and \( K \geq 1 \) we write \( \phi(p, N) \) for the number of solutions of \( A(x, y) = 0 \) and \( \phi(p, N; s, K) \) for the number of solutions of \( A(x, y) \in B(s, 2^{K-1}) \), both subject to the restrictions \( x_1 \leq \ldots \leq x_p \) and \( y_1 \leq \ldots \leq y_p \). Then
\[ \begin{aligned}
\binom{N}{p} &\leq \phi(p, N) \leq \binom{N}{p} + (cp)^p N^{p-1} \\
0 &\leq \phi(p, N; s, K) \leq 2(4p)^{p-1} N^{p-1} \left( \frac{\log (2^{K+2}q^2(q-1))}{(\log q)^2} \right)^{p+1}.
\end{aligned} \tag{26} \tag{27} \]

Proof. Equation (26) is proved in Lemma 7 of [4]. In order to prove (27), we fix the value of \( K \) and use induction on \( p \geq 1 \). For \( p = 1 \), (27) is implied by (24). Now we assume (27) is true for \( 1, 2, \ldots, p-1 \) and we seek an upper estimate for \( \phi(p, N; s, K) \). To do this, we consider separately two sets of solutions: First, those \( 2p \)-tuples \((x_1, \ldots, y_p)\) with \( x_1 = y_1 \). Then the number of tuples \((x_2, \ldots, x_p, y_2, \ldots, y_p)\) with \( s - 2^{K-1} \leq (x_2 + \ldots + x_p) - (y_2 + \ldots + y_p) \leq s + 2^{K-1} \) is at most \( \phi(p-1, N; s, K) \) and there are \( N \) possible values for \((x_1, y_1)\), hence we have at most
\[ N\phi(p-1, N; s, K) \]
solutions of that kind. Next we consider \( 2p \)-tuples with \( x_1 \neq y_1 \). By (25) the number of \( 2(p-1) \)-tuples \((x_2, \ldots, x_p, y_2, \ldots, y_p)\) is at most
\[ \left( 4pN\frac{2^{K+2}q^2(q-1)}{(\log q)^2} \right)^{p-1}. \]

For each such \( 2(p-1) \)-tuple, the number of acceptable pairs \((x_1, y_1)\) with \( x_1 \neq y_1 \) is given by (24) and is at most
\[ 2\frac{\log^2 (2^{K+2}q^2(q-1))}{(\log q)^2}. \]
Combining the two cases, we obtain
\[
\phi(p, N; s, K) \leq N\phi(p - 1, N; s, K) + 2\frac{\log^2(2K^2q^2(q - 1)^{-1})}{\log^2 q} + 2\frac{\log^2(2K^2q^2(q - 1)^{-1})}{\log^2 q} + 2\log_2 (2K + 2q^2(q - 1)^{-1}) \cdot \log_2 q
\]
\[
\leq 2(4p)^{p-1}N^{p-1} \left( \frac{\log(2K^2q^2(q - 1)^{-1})}{\log^2 q} \right)^{p+1}.
\]
\[
\]

Now we are able to prove the final goal of this subsection, which is giving an estimate for the number of solutions of equations of the form \(A(x, y) \in B(s, 2K^{-1})\). The result follows immediately from Lemma 11 since each solution to the aforementioned equation under the restrictions \(x_1 \leq \ldots x_p, y_1 \leq \ldots \leq y_p\) gives rise to \((p!)^2\) solutions \((x_1, \ldots, y_p)\).

**Lemma 12.** For positive integers \(1 \leq p \leq N\), there exists a constant \(c > 0\) such that for any \(s \in \mathbb{Z}\)
\[
(p!)^2 \binom{N}{p} \leq \sum_{A(x, y) = 0} 1 \leq (p!)^2 \binom{N}{p} + (cp)^{3p}N^{p-1}
\]
and
\[
0 \leq \sum_{A(x, y) \in B(s, 2K^{-1})} 1 \leq 2(4p)^{3p-1}N^{p-1} \left( \frac{\log(2K^2q^2(q - 1)^{-1})}{\log^2 q} \right)^{p+1}.
\]

### 2.2.2 Metrical Estimates on Exponential Sums

The previous Lemmas on the number of solutions of Diophantine equations with linear forms are used to estimate the moments of the function
\[
F(N; x) = \left| \sum_{k=1}^{N} e(n_kx) \right|, \quad x \in [0, 1).
\]

For \(p \geq 1\) and \(0 \leq \alpha < \beta \leq 1\) we need to estimate the integral
\[
I_{2p} = \int_{\alpha}^{\beta} |F(N; x)|^{2p} d\mu(x).
\]

This in turn is used to provide estimates for the function
\[
\phi(t) = \mu \left( \{x \in [\alpha, \beta] : F(N; x) \geq \sqrt{tN\log \log N} \} \right).
\]
The following lemma shows that if a probability measure on $[0, 1)$ has Fourier transform with polynomial decay rate, then the same is true for any restriction of this measure to some subinterval.

**Lemma 13.** Let $\mu$ be a probability measure on $[0, 1)$ and $B = (\alpha, \beta) \subseteq [0, 1)$ be a subinterval with $\mu(B) > 0$. Let $\nu$ be the probability measure defined by $\nu(A) = \frac{1}{\mu(B)} \mu(A \cap B)$ for any subset $A \subseteq [0, 1)$. If (4) holds for some $\eta > 0$, then

$$\hat{\nu}(t) \ll \frac{1}{\mu(B)} |t|^{-\eta}, \quad |t| \to \infty.$$  \tag{31}$$

We include the proof of the Lemma in the Appendix at the end of the paper.

**Proposition 14.** If $\alpha, \beta$ are such that $\mu((\alpha, \beta)) \geq \frac{1}{n_1^2 \sqrt{N}}$ and $1 \leq p \leq 3 \log \log N$, then

$$|I_{2p} - \mu((\alpha, \beta))p!N^p| \leq \mu((\alpha, \beta))N^{p - \frac{1}{2}},$$  \tag{32}$$

for all $N$ large enough.

**Proof.** By definition of $F(N; x)$ we have

$$I_{2p} = \sum_{1 \leq k_1, \ldots, k_p \leq N} \int_{\alpha}^{\beta} e \left( \sum_{j=1}^{p} (n_{k_j} - n_{k_l}) x \right) d\mu(x)$$

$$= \mu((\alpha, \beta)) \sum_{A(x,y)=0} 1 + \sum_{0<|A(x,y)| \leq \frac{1}{2} n_1} \int_{\alpha}^{\beta} e(A(x,y)t) d\mu(t)$$

$$+ O \left( \sum_{K=1}^{\infty} \sum_{2^K \leq A(x,y) < 2^{K+1}} \int_{\alpha}^{\beta} e(A(x,y)t) d\mu(t) \right),$$

where the implicit constant in the $O$-estimate is equal to 1. The first term is estimated by (28), the second term again by (28) and the trivial bound $|e(x)| \leq 1$, while for the third term we use (31) and (29). Thus

$$\left| I_{2p} - \mu((\alpha, \beta))(p!)^2 \left( \frac{N}{p} \right) \right| \leq 2 \mu((\alpha, \beta))(c_1 p)^{3p} + 2 \mu((\alpha, \beta))(4c_1 p)^{3p-1} N^{p-1}$$

$$+ \frac{2(4p)^{p-1} N^{p-1}}{n_1^\eta (\log q)^{2p+2}} \sum_{K=1}^{\infty} \frac{1}{2^n K}$$

$$\leq 2 \mu((\alpha, \beta))(c_1 p)^{3p} N^{p-1} + c_2 \frac{(4p)^{p-1} N^{p-\frac{1}{2}}}{n_1^\eta \sqrt{N}},$$
where \( c_1 = \log(4q^2(q-1)^{-1}) \) and \( c_2 > 0 \) is some constant depending only on \( \eta \) and \( q \). Using the inequality

\[
\left| (p!)^2 \binom{N}{p} - p!N^p \right| \leq \frac{1}{2}N^{p-\frac{1}{2}}
\]

finally yields the requested estimate (32). \( \square \)

Armed with Proposition 14 the analogue of [4, Lemma 8] follows. The proof is omitted, as it involves precisely the same arguments.

**Lemma 15.** Let \( \phi \) be the function defined in (30) and \( 0 < \varepsilon < 1 \). Then

\[
\phi(1 - \varepsilon) > \frac{\mu((\alpha, \beta))}{(\log N)^{1-4\varepsilon^2}}.
\]

The remaining steps for the proof of the lower bound in Theorem 1 go along the lines of [4, p. 77-80], with the appropriate modifications for the measure \( \mu \) instead of the Lebesgue measure.

### 3 Proof of Theorem 2

We utilize the discrepancy estimate coming from Theorem 1 in order to improve the result in [7]. We set

\[
\psi(N) = N^{-1/2} (\log \log N)^{1/2 + \varepsilon}.
\]

For any \( i \geq 1 \) let \( (q_n^{(i)})_{n=1}^\infty \) be the sequence of denominators associated with the continued fraction expansion of \( \alpha_i \), and set

\[
G_i = \bigcap_{\gamma \in \mathbb{R}} \left\{ \beta \in [0, 1) : \|q_n^{(i)} \beta - \gamma\| \leq \frac{(\log_3 q_n^{(i)})^{1/2 + \varepsilon}}{(\log q_n^{(i)})^{1/2}} \text{ for inf. many } n = 1, 2, \ldots \right\}.
\]

The proof of the theorem will be complete as long as we show that the set

\[
G = \bigcap_{i=1}^\infty (G_i \cap \text{Bad}) \subseteq \text{Bad}
\]

has Hausdorff dimension \( \dim G = 1 \). To that end, let \( \mu = \mu(N, \delta, \varepsilon) \) be any of the probability measures as in Theorem of Kaufman and Queffélec-Ramaré. Now for each real \( \gamma \) define the sequence of indices

\[
N_k^\gamma = \min \left\{ N \geq 1 : \# \{ 1 \leq n \leq N : \{ q_n^{(i)} \beta \} \in B(\gamma, \psi(N)) \} = k \right\}, \quad k \geq 1.
\]
Claim: The sequence \( (N_k^\gamma)_{k=1}^\infty \) is well-defined.

Proof of Claim: Since \( \alpha_i \) is badly approximable, the sequence \( (q_n^{(i)})_{n=1}^\infty \) is lacunary and also has a growth rate of the form

\[
\log q_n^{(i)} \asymp n, \quad n \to \infty
\]

(see [12, p.288, 297] for more details). Hence the lacunarity property together with Theorem [1] imply that for finally all \( N \geq 1 \) we have

\[
D_N(q_k^{(i)} \beta) \leq C_i N^{-1/2} (\log \log N)^{1/2} \quad \text{for } \mu\text{-almost all } \beta \in [0, 1).
\]

Here the constant \( C_i > 0 \) depends on \( \alpha_i \in \text{Bad} \) as in Theorem [1]. For these values of \( \beta \) lying in a set of full \( \mu \)-measure, the definition of discrepancy yields

\[
\#\{1 \leq k \leq N : \{q_k^{(i)} \beta\} \in B(\gamma, \psi(N))\} - 2N\psi(N) \leq C_i (N \log \log N)^{1/2}
\]

for all \( \gamma \in \mathbb{R} \). Hence

\[
\#\{1 \leq k \leq N : \{q_k^{(i)} \beta\} \in B(\gamma, \psi(N))\} \geq 2N\psi(N) - C_i (N \log \log N)^{1/2}
\]

\[
\geq N^{1/2} (\log \log N)^{1/2 + \varepsilon},
\]

for all \( N \) sufficiently large. This inequality shows that for all \( k \geq 1 \) there exists \( N_k^\gamma \in \mathbb{N} \) such that

\[
\#\{1 \leq n \leq N_k^\gamma : \{q_n^{(i)} \beta\} \in B(\gamma, \psi(N_k^\gamma))\} = k.
\]

The Claim is proved.

Thus for all \( k = 1, 2, \ldots \) and for all \( \beta \) in a set of full \( \mu \)-measure we have

\[
\|q_{N_k^\gamma}^{(i)} \beta - \gamma\| \leq (N_k^\gamma)^{-1/2} (\log \log N_k^\gamma)^{1/2 + \varepsilon}
\]

\[
\ll \frac{(\log_3 q_{N_k^\gamma}^{(i)})^{1/2 + \varepsilon}}{\left(\log q_{N_k^\gamma}^{(i)}\right)^{1/2}}.
\]

Thus \( \mu(G_i) = 1 \) for all \( i = 1, 2, \ldots \) and hence \( \mu(G) = 1 \). Since \( \mu = \mu(N, \delta, \varepsilon) \) was arbitrarily chosen, by property (ii) of the Theorem of Kaufman and Queffélec–Ramaré together with the mass distribution principle we conclude that

\[
\dim G \geq \delta, \quad \text{for all } \frac{1}{2} < \delta < 1,
\]

so \( \dim G = 1 \) as required.
4 Appendix: The decay rate of the Fourier transform of the restricted measure.

Here we present a proof of Lemma 13 since we have not been able to find a proof in the bibliography. The proof follows the one of an analogous result in [8, p. 252].

Since (4) holds, there exists a constant \( C_1 > 0 \) such that
\[
|\hat{\mu}(t)| \leq C_1(1 + |t|)^{-\eta} \quad \text{for all } t \in \mathbb{R}.
\]

Let \( \phi : \mathbb{R} \to (0, \infty) \) be a \( C^\infty \) function which is equal to 1 on the interval \( B = (\alpha, \beta) \). Since \( \phi \) is \( C^\infty \), we have
\[
\phi(x) = \sum_{k=-\infty}^{+\infty} \hat{\phi}(k)e(kx), \quad x \in \mathbb{R}
\]
where the convergence is uniform for all \( x \in \mathbb{R} \). Furthermore, since \( \phi \) is a \( C^\infty \) function, there exists a constant \( C_\eta > 0 \) such that
\[
|\hat{\phi}(k)| \leq C_\eta(1 + |k|)^{-(1+\eta)}, \quad k \in \mathbb{Z}.
\] (34)

Set \( S = \sum_{k=-\infty}^{+\infty} (1 + |k|)^{-(1+\eta)} < \infty \). For the probability measure \( \nu \) defined as in Lemma 13, we have for \( t > 0 \)
\[
|\hat{\nu}(t)| = \frac{1}{\mu(B)} \left| \int_B e(-tx)d\mu(x) \right| \\
\leq \frac{1}{\mu(B)} \left| \int e(-tx)\phi(x)d\mu(x) \right| \\
= \frac{1}{\mu(B)} \left| \int e(-tx) \sum_{k=-\infty}^{+\infty} \hat{\phi}(k)e(kx)d\mu(x) \right| \\
= \frac{1}{\mu(B)} \left| \sum_{k=-\infty}^{+\infty} \hat{\phi}(k) \int e(-(t-k))d\mu(x) \right| \\
\leq \frac{1}{\mu(B)} \sum_{k=-\infty}^{+\infty} |\hat{\phi}(k)\hat{\mu}(t-k)| \\
= \frac{1}{\mu(B)} \sum_{|k| \leq \frac{1}{2}|t|} |\hat{\phi}(k)\hat{\mu}(t-k)| + \frac{1}{\mu(B)} \sum_{|k| > \frac{1}{2}|t|} |\hat{\phi}(k)\hat{\mu}(t-k)|
\]

We deal with the first of the two terms. The condition of summation implies that \( \frac{1}{2}|t| \leq
\(|t - k| \leq \frac{3}{2}|t|\). Hence employing (4) and (34) we get

\[
\frac{1}{\mu(B)} \sum_{|k| \leq \frac{1}{2}|t|} |\hat{\phi}(k)\hat{\mu}(t - k)| \leq \frac{1}{\mu(B)} \sum_{|k| \leq \frac{1}{2}|t|} C_\eta (1 + |t|)^{-(1+\eta)} \left( 1 + \frac{1}{2}|t| \right)^{-\eta} \\
\leq \frac{1}{\mu(B)} 2^n \eta C \eta S (1 + |t|)^{-\eta}.
\]

Regarding the second term, using the trivial bound \(|\hat{\mu}(t)| \leq 1\) together with (34) we get

\[
\frac{1}{\mu(B)} \sum_{|k| > \frac{1}{2}|t|} |\hat{\phi}(k)\hat{\mu}(t - k)| \leq \frac{1}{\mu(B)} \sum_{|k| > \frac{1}{2}|t|} C_\eta (1 + |k|)^{-(1+\eta)} \leq \frac{1}{\mu(B)} C_\eta 2^n (1 + |t|)^{-\eta}.
\]

Combining the two estimates, we obtain

\[
|\hat{\nu}(t)| \leq \frac{1}{\mu(B)} C (1 + |t|)^{-\eta} \quad \text{for all } t > 0
\]

with \(C = 2^n C_\eta (1 + S) > 0\). The same bound is also true for all real values of \(t\) in view of the relation \(|\hat{\nu}(t)| = |\hat{\nu}(-t)|\), hence the Lemma is proved.

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Niclas Technau: Department of Mathematics, University of York, Heslington, York, YO10 5DD, England. e-mail: niclas.technau@york.ac.uk

Agamemnon Zafeiropoulos: Institute of Analysis and Number Theory, TU Graz, Steyrergasse 30/II, 8010 Graz, Austria. e-mail: zafeiropoulos@math.tugraz.at