Leptonic contribution to the bulk viscosity of nuclear matter

Mark G. Alford and Gerald Good

Physics Department, Washington University, St. Louis, MO 63130, USA

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I. INTRODUCTION

The bulk viscosity of nuclear matter plays an important role in the damping of oscillations in neutron stars. One well-known example is \( r \)-modes, which, if the interior of the star is a perfect (dissipationless) fluid, become unstable with respect to the emission of gravitational waves \[1\]–\[3\]. This emission acts as a brake on the rotation of the star. However, \( r \)-mode spindown will not occur if the \( r \)-mode is sufficiently strongly damped, for example by shear or bulk viscosity of the matter in the interior of the star. It is therefore important to calculate the bulk viscosity of the various candidate phases in a neutron star. Several calculations exist in the literature, for nuclear \[4\]–\[9\] and hyperonic \[10\]–\[12\] as well as for unpaired quark matter \[13\]–\[15\] and various color-superconducting phases \[16\]–\[21\].

In this paper we will study \( \beta \)-equilibrated nuclear matter. We define the chemical potential for charged leptons to be \( \mu_l = -\phi/e \) where \( \phi \) is the electrostatic potential and \( e \) is the positron charge. We will assume that the density is high enough that \( \mu_l \) is greater than the mass of the muon, so the matter consists of neutrons, protons, electrons and muons. Such matter is expected to exist in the core of the star. In previous calculations of bulk viscosity of \( npe\mu \) nuclear matter the focus has been on the contribution from interconversion of neutrons and protons via weak interactions. But nuclear matter at neutron-star densities is expected to show Cooper pairing of protons (superconductivity) or neutrons (superfluidity) \[22\]–\[24\] either of which will suppress interconversion by a factor of order \( \exp(-\Delta/T) \), where \( \Delta \) is the energy gap at the Fermi surface and \( T \) is the temperature. This opens up the possibility that, in superfluid or superconducting phases, the dominant contribution to the bulk viscosity might come from purely leptonic processes. The relevant process is conversion of electrons to muons (and vice versa) via either the direct Urca process or the modified Urca process. The direct Urca leptonic conversion process is forbidden by energy and momentum conservation: in converting an electron near its Fermi surface to a muon near its Fermi surface, the change in free energy is very small (of order \( T \)), so the emitted neutrinos carry momentum and energy of this order; but the change of momentum of the charged lepton is large, at least \( q_{\text{min}} = \mu_l - \sqrt{\mu_l^2 - m_{\mu}^2} \), and the low-energy neutrino cannot carry this much momentum. However, the modified Urca process can occur; for example, two electrons with energy slightly above the Fermi energy can scatter to an electron and a muon with energies near the Fermi energy, or an electron and muon can scatter to two muons. The strongest interaction between leptons is electromagnetism, so this process proceeds via exchange of a photon, whose propagator should include the effects of screening by the nuclear medium. As the temperature decreases, the process will become suppressed as the Fermi distributions assume their zero-temperature step function profiles, but at finite temperature the modified Urca process will result in a non-zero contribution to the bulk viscosity.

We calculate the leptonic bulk viscosity arising from the processes \( e + \ell \leftrightarrow \mu + \ell + \nu + \bar{\nu} \), where \( \ell = e \) or \( \mu \). All our calculations are in the “subthermal” regime where the density oscillation has a small amplitude, and the bulk viscosity is independent of that amplitude. We conclude that, if the protons and neutrons are both ungapped, i.e if there is neither superfluidity nor superconductivity, then the bulk viscosity from these purely leptonic processes is several orders of magnitude smaller than that from the nucleonic processes. However, once the temperature drops below the critical value for Cooper pairing of the protons or neutrons, the nucleonic bulk viscosity at frequencies \( \gtrsim 10 \text{ Hz} \) is strongly suppressed, and leptonic processes become the dominant source of bulk viscosity at those frequencies.

In section [II] we lay out the process for calculating the bulk viscosity of a two-component leptonic system under application of a periodic volume and pressure perturbation. A crucial component of this calculation is the conversion rate between electrons and muons, which is discussed in [III]. In section [IV] we show the numerical results of our calculations and how they compare to the bulk viscosity resulting from modified Urca equilibration of the nucleon population.
II. BULK VISCOSITY OF LEPTONS

First we write down a general expression for bulk viscosity in a two-species system, arising from interconversion of the two species. Then we specialize to the case of electrons and muons in nuclear matter.

A. Bulk viscosity of a two-species system

We assume that the system experiences a small-amplitude driving oscillation

\[ V(t) = \bar{V} + \text{Re}(\delta V e^{i\omega t}) \]
\[ p(t) = \bar{p} + \text{Re}(\delta p e^{i\omega t}) \]  

(1)

where the volume amplitude \( \delta V \ll \bar{V} \) is real by convention, and the resultant pressure oscillation \( p(t) \) is complex. The average power dissipated per unit volume is

\[ \frac{dE}{dt} = -\frac{1}{\tau} \int_0^\tau p(t) \frac{dV}{dt} dt = -\frac{1}{2} \omega \text{Im}(\delta p) \frac{\delta V}{V} , \]

(2)

where \( \tau = 2\pi/\omega \), so the bulk viscosity is \([14]\)

\[ \zeta = \frac{2\bar{V}^2}{\omega^2(\delta V)^2} \frac{dE}{dt} = -\frac{\text{Im}(\delta p)}{\delta V} \frac{\bar{V}}{\omega} . \]

(3)

We will determine \( \text{Im}(\delta p) \), which will be negative. We will assume that heat arising from dissipation is conducted away quickly, so the whole calculation is performed at constant temperature \( T \). We assume that our system contains two particle species \( e \) and \( \mu \), and the state of the system is determined by the corresponding chemical potentials \( \mu_e \) and \( \mu_\mu \). The total number of electrons and muons is conserved, and equilibrium is established via the conversion process \( e \leftrightarrow \mu \). For simplicity of presentation and of the final expressions, it is better to work in terms of charged lepton number \( l \) and electron-muon asymmetry \( a \), so pressure is a function of \( \mu_l \) and \( \mu_a \), where

\[ \mu_l = \frac{1}{2}(\mu_e + \mu_\mu) \quad n_l = n_e + n_\mu = \frac{\partial p}{\partial \mu_l} |_{\mu_a} \]
\[ \mu_a = \frac{1}{2}(\mu_e - \mu_\mu) \quad n_a = n_e - n_\mu = \frac{\partial p}{\partial \mu_a} |_{\mu_l} \]

(4)

From now on all partial derivatives with respect to \( \mu_l \) will be assumed to be at constant \( \mu_a \), and vice versa. In beta-equilibrium, \( \mu_a \) is zero. The variations in the chemical potentials are expressed in terms of complex amplitudes \( \delta \mu_l \), and \( \delta \mu_a \),

\[ \mu_l(t) = \bar{\mu}_l + \text{Re}(\delta \mu_l e^{i\omega t}) , \]
\[ \mu_a(t) = \text{Re}(\delta \mu_a e^{i\omega t}) . \]

(5)

The pressure amplitude is then

\[ \delta p = \frac{\partial p}{\partial \mu_l} |_{\mu_a} \delta \mu_l + \frac{\partial p}{\partial \mu_a} |_{\mu_l} \delta \mu_a = n_l \delta \mu_l + n_a \delta \mu_a , \]

(6)

From (6) and (3) we find

\[ \zeta = -\frac{1}{\omega} \frac{\bar{V}}{\delta V} \left( \bar{n}_l \text{Im}(\delta \mu_l) + \bar{n}_a \text{Im}(\delta \mu_a) \right) . \]

(7)

To obtain the imaginary parts of the chemical potential amplitudes, we write down the rate of change of the corresponding conserved quantities,

\[ \frac{dn_l}{dt} = \frac{\partial n_l}{\partial \mu_l} \frac{d\mu_l}{dt} + \frac{\partial n_l}{\partial \mu_a} \frac{d\mu_a}{dt} = -\frac{n_l}{\bar{V}} \frac{dV}{dt} , \]
\[ \frac{dn_a}{dt} = \frac{\partial n_a}{\partial \mu_l} \frac{d\mu_l}{dt} + \frac{\partial n_a}{\partial \mu_a} \frac{d\mu_a}{dt} = -\frac{n_a}{\bar{V}} \frac{dV}{dt} - \Gamma_{\mu \rightarrow e} . \]

(8)
All the partial derivatives are evaluated at equilibrium, $\mu_l = \bar{\mu}_l$ and $\mu_a = 0$. The right hand term on the first line expresses the fact that charge is conserved, so when a volume is compressed, the density of charged leptons rises. On the second line, there is such a term from the compression of the existing population of particles, but there is also a rate of conversion $\Gamma_{e \rightarrow \mu}^{\text{total}}$ of electrons to muons, which reflects the fact that weak interactions will push the lepton densities towards their equilibrium value. For small deviations from equilibrium we expect $\Gamma_{e \rightarrow \mu}^{\text{total}}$ to be linear in $\mu_a$, so it is convenient to write the rate in terms of an average width $\gamma_a$, which is defined in terms of the total rate by writing

$$\Gamma_{e \rightarrow \mu}^{\text{total}} = \gamma_a \frac{\partial n_a}{\partial \mu_a} \mu_a .$$

We now substitute the assumed oscillations (1) and (5) in to (8), and solve to obtain the amplitudes $\delta \mu_l$ and $\delta \mu_a$ in terms of the amplitude $\delta V$ and frequency $\omega$ of the driving oscillation. Inserting their imaginary parts in (7) we obtain the bulk viscosity, which is conveniently expressed in terms of the susceptibilities

$$\chi_{ll} = \frac{\partial n_l}{\partial \mu_l} ,$$

$$\chi_{la} = \frac{\partial n_l}{\partial \mu_a} = \frac{\partial n_a}{\partial \mu_l} ,$$

$$\chi_{aa} = \frac{\partial n_a}{\partial \mu_a} ,$$

all evaluated at equilibrium, $\mu_l = \bar{\mu}_l$, $\mu_a = 0$. Note that $\chi_{al}$ is the same as $\chi_{la}$ from (4). Defining

$$\gamma_{\text{eff}} = \frac{\chi_{ll} \chi_{aa} - \chi_{la}^2}{\chi_{ll} \chi_{aa} - \chi_{la}^2} \gamma_a = \frac{\chi_{ll}}{\chi_{ll} \chi_{aa} - \chi_{la}^2} \frac{\partial \Gamma_{e \rightarrow \mu}^{\text{total}}}{\partial \mu_a} \bigg|_{\mu_a = 0} ,$$

$$C = \frac{(\chi_{ll} n_a - \chi_{la} n_l)^2}{\chi_{ll} (\chi_{ll} \chi_{aa} - \chi_{la}^2)} ,$$

we obtain the final result for the bulk viscosity in a two-species system,

$$\zeta = C \frac{\gamma_{\text{eff}}}{\omega^2 + \gamma_{\text{eff}}^2} .$$

From (12) we can already see how the bulk viscosity of a two-species system depends on the frequency $\omega$ of the oscillation and the effective equilibration rate $\gamma_{\text{eff}}$.

At fixed equilibration rate, the bulk viscosity decreases monotonically as the oscillation frequency rises; it is roughly constant for $\omega \lesssim \gamma_{\text{eff}}$, and then drops off quickly as $1/\omega^2$ for $\omega \gg \gamma_{\text{eff}}$.

At fixed oscillation frequency $\omega$, the bulk viscosity is a non-monotonic function of the rate $\gamma_{\text{eff}}$. It is peaked at $\gamma_{\text{eff}} = \omega$, with a value

$$\zeta_{\text{max}} = \frac{1}{2} C / \omega .$$

For $\gamma_{\text{eff}} \ll \omega$ or $\gamma_{\text{eff}} \gg \omega$ the bulk viscosity tends to zero. Thus very fast and very slow processes are not an important source of bulk viscosity. As we will see below, for leptons in nuclear matter the equilibration rate is sensitive to temperature but the coefficient $C$ is not, so we expect $\zeta(T)$ to be peaked at $\gamma_{\text{eff}}(T) = \omega$, where the oscillation frequency $\omega$ is of order kHz for typical oscillation modes of neutron stars.

### B. Leptons in nuclear matter

In nuclear matter the leptonic chemical potential $\mu_l = \mu_e = \mu_\mu$ is much greater than the temperature and the electron mass, so we can evaluate the susceptibilities (10) at $m_e = T = 0$. Temperature dependence will come in only via the equilibration rate $\gamma_a$. Treating the electrons and muons as free fermions, we find

$$\gamma_{\text{eff}} = \gamma_a \frac{(\mu_l + k_F)^2}{4\mu_l k_F} ,$$

$$C = \frac{1}{9\pi^2 m_e^2 k_F (\mu_l - k_F)} .$$
where the muon Fermi momentum is given by $k_f^2 = \mu_f^2 - m_\mu^2$. Note that the bulk viscosity goes to zero as $m_\mu \to 0$ ($m_\mu \to m_e$, really). This is because if the muons and electrons have equal mass then under compression their relative densities do not change, and there is no need for any equilibrating process, so the pressure is always in phase with the volume and no dissipation occurs.

Even without calculating the rate of lepton number equilibration, we can now estimate the amount of bulk viscosity that could possibly arise from leptons. If the equilibrating weak interaction at some temperature happens to be absent, and no dissipation occurs.

The partial rates are

$$\Gamma_{ee \to e\mu} = \Gamma_{e\mu \to e\mu} + \Gamma_{e\mu \to \mu\mu}$$

(15)

The partial rates are

$$\Gamma_{ab \to cd} = \int \frac{d^3p_1 d^3p_2 d^3p_3 d^3p_4 d^3k_1 d^3k_2}{64(2\pi)^4} \frac{\delta^4(p_1 + p_2 - p_3 - p_4 - k_1 - k_2)}{\omega_1 \omega_3 \omega_4} \frac{\delta^4(p_1 p_2 p_3 p_4)}{W_{ab \to cd}(p_1 p_2 \to p_3 p_4 k_1 k_2)}$$

(16)

where $a, b, c, d$ are either $e$ or $\mu$, $W_{ab \to cd}$ is the spin-summed and averaged matrix element. The charged lepton of flavor $j$ has energy $\omega_j = \sqrt{p_j^2 + m_j^2}$, the neutrino of flavor $j$ has energy $\Omega_j = |k_j|$, and $f_b(\omega_j)$ is the Fermi distribution function

$$f_b(\omega_j) = \left[1 + \exp\left(\frac{\omega_j - \mu_b}{T}\right)\right]^{-1}$$

(17)

Using the previous definitions for $\mu_t$ and $\mu_a$, we have

$$\mu_e = \mu_t + \mu_a, \quad \mu_\mu = \mu_t - \mu_a$$

(18)

and since $\mu_a$ is small, to first order in $\mu_a$ we have

$$f_e(\omega_1) f_e(\omega_2) (1 - f_e(\omega_3)) (1 - f_\mu(\omega_4)) - f_e(\omega_1) f_\mu(\omega_2) (1 - f_e(\omega_3)) (1 - f_\mu(\omega_4)) = F(\omega_1, \omega_2, \omega_3, \omega_4, \mu_a/T)$$

(19)

and

$$f_\mu(\omega_1) f_e(\omega_2) (1 - f_\mu(\omega_3)) (1 - f_\mu(\omega_4)) - f_\mu(\omega_1) f_\mu(\omega_2) (1 - f_\mu(\omega_3)) (1 - f_\mu(\omega_4)) = F(\omega_1, \omega_2, \omega_3, \omega_4, \mu_a/T)$$

(20)

$$F(\omega_1, \omega_2, \omega_3, \omega_4) = \frac{2 \exp\left[\omega_3 + \omega_4 - 2\mu_1/T\right]}{(1 + \exp[\omega_1 - \mu_1/T]) (1 + \exp[\omega_2 - \mu_1/T])} \frac{\exp\left[\omega_3 + \omega_4 - 2\mu_1/T\right]}{(1 + \exp[\omega_2 - \mu_1/T])} \frac{\exp\left[\omega_3 + \omega_4 - 2\mu_1/T\right]}{(1 + \exp[\omega_3 - \mu_1/T])} \frac{\exp\left[\omega_3 + \omega_4 - 2\mu_1/T\right]}{(1 + \exp[\omega_4 - \mu_1/T])}$$

(21)

To determine the content of the matrix elements, we draw the Feynman diagrams for each possible way the reaction can occur. We can draw two different diagrams for each process, depending on the whether the weak conversion of the electron to muon occurs before the electromagnetic scattering, or in the reverse order (Fig. 1 and Fig. 2). However, because there are identical particles involved, and we are integrating over all initial and final momenta, we need to add two additional diagrams for each process. For the process $e + e \leftrightarrow e + e + \nu + \bar{\nu}$, we must add two diagrams where the labels on the initial state electron momenta are reversed; and for the process $e + \mu \leftrightarrow e + \mu + \nu + \bar{\nu}$, we must add two diagrams where the labels on the final state muon momenta are reversed. These diagrams get an additional negative sign for the interchange of fermions [25]. For similar calculations, see [26, 27].

### III. MUON-ELECTRON CONVERSION RATE

The muon-electron conversion rate $\Gamma_{e\mu}$ consists of two partial rates,

$$\Gamma_{e\mu} = \Gamma_{ee \to e\mu} + \Gamma_{e\mu \to \mu\mu}$$

(15)
Since we have four diagrams for each process, the spin summed-and-averaged matrix elements are

\[
W_{ee \rightarrow e\mu} = \frac{1}{8} \sum_{\text{spins}} |E_1 + E_2 - E_3 - E_4|^2
\]

\[
W_{e\mu \rightarrow \mu\mu} = \frac{1}{8} \sum_{\text{spins}} |M_1 + M_2 - M_3 - M_4|^2
\]

(22)

Here \(E_1, E_2, E_3, E_4\) are the amplitudes corresponding to the diagrams of Fig. 1 and \(M_1, M_2, M_3, M_4\) are the amplitudes corresponding to the diagrams of Fig. 2:

\[
E_1 = \frac{e^2 G_F}{\sqrt{2}(q^2 - q_s^2)} \bar{e}(p_3)\gamma^\mu e(p_1)\bar{\nu}_e(k_1)\gamma^\lambda \left(1 - \gamma^5\right) \frac{\not{p}_2 + \not{q} + m_e}{(p_2 + q)^2 - m_e^2} \gamma_\mu e(p_2)\mu(p_4)\gamma_\lambda \left(1 - \gamma^5\right) \nu_\mu(k_2)
\]

\[
E_2 = \frac{e^2 G_F}{\sqrt{2}(q^2 - q_s^2)} \bar{e}(p_3)\gamma^\mu e(p_1)\bar{\nu}_e(k_1)\gamma^\lambda \left(1 - \gamma^5\right) e(p_2)\mu(p_4)\gamma_\mu \frac{\not{p}_4 - \not{q} + m_\mu}{(p_4 - q)^2 - m_\mu^2} \gamma_\lambda \left(1 - \gamma^5\right) \nu_\mu(k_2)
\]

\[
E_3 = \frac{e^2 G_F}{\sqrt{2}(w^2 - q_s^2)} \bar{e}(p_3)\gamma^\mu e(p_2)\bar{\nu}_e(k_1)\gamma^\lambda \left(1 - \gamma^5\right) \frac{\not{p}_1 + \not{w} + m_e}{(p_1 + w)^2 - m_e^2} \gamma_\mu e(p_1)\mu(p_4)\gamma_\lambda \left(1 - \gamma^5\right) \nu_\mu(k_2)
\]

\[
E_4 = \frac{e^2 G_F}{\sqrt{2}(w^2 - q_s^2)} \bar{e}(p_3)\gamma^\mu e(p_2)\bar{\nu}_e(k_1)\gamma^\lambda \left(1 - \gamma^5\right) e(p_1)\mu(p_4)\gamma_\mu \frac{\not{p}_4 - \not{w} + m_\mu}{(p_4 - w)^2 - m_\mu^2} \gamma_\lambda \left(1 - \gamma^5\right) \nu_\mu(k_2)
\]

(23)
where \( w = p_2 - p_3 \), and \( s = p_1 - p_4 \).

The only parameter in our calculation that depends on details of the baryonic matter in the neutron star is the plasma screening momentum \( q_s \). In a full treatment one would have to use the appropriate in-medium propagator which is a complicated function of the photon momentum.

In this paper we greatly simplify the calculation by assuming that the longitudinal and transverse photons have a common screening mass

\[
q_s^2 = 5\alpha \mu^2.
\]  

We argue in Appendix [A] that this leads to an estimate of the bulk viscosity that is correct to within an order of magnitude at reasonable densities for nuclear matter. As a further test we also performed calculations with no screening at all \((q_s^2 = 0)\) and found that the bulk viscosity shifted by no more than one order of magnitude.

To obtain the equilibration rates, we first multiply out the right hand sides of (22) and define partial matrix elements by

\[
W_{ee \rightarrow e\mu} = \sum_{i,j \leq 4} W_{ee \rightarrow e\mu}^{ij}, \quad W_{e\mu \rightarrow e\mu} = \sum_{i,j \leq 4} W_{e\mu \rightarrow e\mu}^{ij},
\]

\[
W_{ee \rightarrow e\mu}^{11} = -\frac{1}{8} \sum_{spins} |E_1|^2, \quad W_{ee \rightarrow e\mu}^{12} = \frac{1}{8} \sum_{spins} (E_1^1 E_2^2 + E_1^2 E_2^1), \quad W_{ee \rightarrow e\mu}^{13} = -\frac{1}{8} \sum_{spins} (E_1^1 E_3^2 + E_1^2 E_3^1), \text{ etc.}
\]  

The traces resulting from the spin sums are easily evaluated with a computer algebra package; we used the FeynCalc package for Mathematica [29]. In the next few paragraphs, we will describe the steps used to analytically integrate 10 of the 18 integrals, and list the expressions that we subsequently integrated numerically in Appendix [B].

We make use of the fact that the neutrino energies are \( \sim T \ll \mu_e, \mu_\mu \) by approximating the momentum and energy conserving delta functions as

\[
delta^3(p_1 + p_2 - p_3 - p_4 - k_1 - k_2) \approx \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4 - \Omega_1 - \Omega_2)\delta^3(p_1 + p_2 - p_3 - p_4). 
\]  

We then note that \( k_1 \) and \( k_2 \) occur exactly once in each term, dotted into one of the other 4-momenta \( p_i \). Writing

\[
k_j = \Omega_j (1, \sin \xi_j \cos \eta_j, \sin \xi_j \sin \eta_j, \cos \xi_j)
\]

we can see that any dot product with another 4-momentum \( p_i \) is

\[
p_i \cdot k_j = \Omega_j (\omega_i - (p_i)_z \sin \xi_j \cos \eta_j - (p_i)_y \sin \xi_j \sin \eta_j - (p_i)_z \cos \xi_j).
\]

The integrals over the \( k_1 \) and \( k_2 \) angular variables then become trivial:

\[
\int \frac{d^3k_j}{\Omega_j} p_i \cdot k_j = \int k_j^2 dk_j d(\cos \xi_j) d\eta_j p_i \cdot \hat{k}_j = 4\pi \omega_i \int_0^\infty \Omega_j^2 d\Omega_j
\]

because all of the integrations over one of the angles \( \xi_j \) or \( \eta_j \) are zero.

The energy-momentum conserving delta function allows us to use relations like \( p_1 - p_3 = p_4 - p_2 \) to rewrite some of the denominators of the matrix elements. For example, in \( W_{ee \rightarrow e\mu} \) we can substitute variables so that \( p_3 \) does not appear in the denominators of any of the terms; then we can integrate out the \( p_3 \) 3-momentum variables easily. Similarly, in \( W_{e\mu \rightarrow e\mu} \) we can substitute variables so that \( p_2 \) does not appear in the denominators and integrate out
the $p_2$ 3-momentum variables. However, our matrix elements have many terms containing the four-momentum $p_3$ ($p_2$), so it would be easier if we could integrate over $d^4p_3$ ($d^4p_2$). This is accomplished by replacing

$$\int \frac{d^4p_3}{\omega_3} = \int \frac{d^4p_3}{(p_3)_0} \delta ((p_3)_0 - \sqrt{p_3^2 + m_\mu^2}) \approx \int \frac{d^4p_3}{(p_3)_0} \delta ((p_3)_0 - \mu_t)$$

(31)

in $W_{ee \rightarrow e\mu}$ and similarly for $p_2$ in $W_{ee \rightarrow \mu\mu}$. In the last approximation we are using the fact that the Fermi distribution function is sharply peaked at low temperatures. Then we integrate over $d^4p_3$ ($d^4p_2$) using four of the delta functions.

We can further approximate that the medium is isotropic, by taking one of the remaining momentum variables to be in a fixed direction (the $z$-axis for convenience). The electrons are relativistic, so $\omega_i = |p_i|$ and $d^3p_i = \omega_i^2 d\omega_i d\cos \theta_i d\phi_i$ when particle $i$ is an electron. The muons may not be relativistic, so $\omega_i = \sqrt{p_i^2 + m_\mu^2}$ and $d^3p_i = \omega_i \sqrt{\omega_i^2 - m_\mu^2} d\omega_i d\cos \theta_i d\phi_i$ when particle $i$ is a muon. We then use the remaining delta function to integrate over the magnitude of this isotropic momentum variable.

The remainder of the integrations are performed numerically. The only further approximation made was to again take advantage of the sharply peaked Fermi distribution function, and set $\omega_i = \mu_t$ everywhere inside the integral, except for inside the Fermi function itself. This allows a separation of the eight-dimensional integral into a four-dimensional energy integral and a four-dimensional integral over the angular variables. The integration variables are also changed to dimensionless variables by scaling them with respect to $\mu_t$.

The final expression for each term in the rate has the form

$$\Gamma^{ij}_{\ell \rightarrow \mu \ell} = \frac{e^4 G_F^2 \mu_t^4}{128 \pi^3 m_\mu^4} \left( \frac{\mu_t}{T} \right) \times I^\ell_\omega \times I^{ij}_{\delta \delta}$$

(32)

where $\ell$ is the species of the spectator lepton, and $I^\ell_\omega$ and $I^{ij}_{\delta \delta}$ are dimensionless energy and angular integrals, respectively. These integrals are listed in Appendix [B].

IV. NUMERICAL RESULTS AND CONCLUSIONS

The remaining part of the rate calculation is performed numerically. The dimensionless energy integrals are nearly the same; a power-law fit of the results yields

$$I^\ell_\omega \approx 78.86 \left( \frac{T}{\mu_t} \right)^8, \quad I^{ij}_{\delta \delta} \approx 78.62 \left( \frac{T}{\mu_t} \right)^8$$

(33)

In our approximation, the angular integrals only have dependence on $\mu_t$. We determined an analytical fit for the $\mu_t$-dependence of $I^{ij}_{\delta \delta}$ and $I^{ij}_{\delta \delta}$ (accurate within 5%) over the range $120 \text{ MeV} < \mu_t < 300 \text{ MeV}$ by curve-fitting the numerical data with sixth-order polynomials:

$$\sum_{ij} I^{ij}_{\delta \delta} \approx \left( 1 - \frac{m_\mu^2}{\mu_t^2} \right)^{1/2} \sum_{i=0}^6 c_i \left( \frac{\mu_t}{m_\mu} \right)^i,$$

$$c_0 = -1.7363 \times 10^4, \quad c_1 = 5.0189 \times 10^3, \quad c_2 = -4.7644 \times 10^4, \quad c_3 = 1.3224 \times 10^4,$$

$$c_4 = 4.4203 \times 10^3, \quad c_5 = -2.7199 \times 10^3, \quad c_6 = 3.5119 \times 10^2$$

(34)

$$\sum_{ij} I^{ij}_{\delta \delta} \approx \left( 1 - \frac{m_\mu^2}{\mu_t^2} \right)^{3/2} \sum_{i=0}^6 c_i \left( \frac{\mu_t}{m_\mu} \right)^i,$$

$$c_0 = 1.2433 \times 10^6, \quad c_1 = -3.6329 \times 10^6, \quad c_2 = 4.4365 \times 10^6, \quad c_3 = -2.8702 \times 10^6,$$

$$c_4 = 1.0354 \times 10^6, \quad c_5 = -1.9728 \times 10^6, \quad c_6 = 1.5507 \times 10^4$$

(35)

Fig. 3 shows the $\mu_t$ dependence of the effective rate $\gamma_{\ell \mu}$ defined in (11). As $\mu_t$ approaches $m_\mu$, the rate quickly drops to zero as the muon population disappears. The overall $T^7$ dependence is also illustrated in the sizable difference in order of magnitude of the rate for the three different temperatures.

Fig. 4 shows the temperature dependence of the leptonic bulk viscosity $\zeta$ as defined in (12), for an oscillation frequency $\omega = 2\pi \times 1\text{kHz}$. The three approximately straight lines on the log-log plot illustrate the power-law
FIG. 3: Dependence of the effective rate of electron/muon conversion $\gamma_{\text{eff}}$ (see (11)) on the charged-lepton chemical potential $\mu_l$ at three different temperatures. As $\mu_l$ drops towards $m_\mu$, the muon population decreases and the conversion rate drops to zero. The temperature dependence is $T^7$, hence $\gamma_{\text{eff}}$ is much larger at higher temperatures.

Dependence on $T$ for three different values of $\mu_l$. Also plotted are dotted curves showing the nucleonic bulk viscosity for two different values of the critical temperature $T_c$. These are obtained from Ref. [5] in a model where the neutrons are superfluid, pairing in the spin triplet state, the protons are superconducting, pairing in the spin singlet state, and they have a common critical temperature $T_c = T_{cp} = T_{cn}$. Also, it is assumed that only modified Urca processes are available for damping of pulsations (although direct Urca processes would become possible at higher densities).

Above the critical temperature for superfluidity/superfluidity, the bulk viscosity for 1 kHz oscillations due to leptons is several orders of magnitude less than the bulk viscosity due to nucleons. Below the critical temperature, the nucleonic bulk viscosity quickly decreases and at a low enough temperature, the leptonic contribution becomes dominant. Based on our calculations, this crossover temperature appears to be of order 0.01 to 0.1 MeV ($10^8$ to $10^9$ K) for an oscillation frequency in the kHz range. Such a suppression of the nucleonic contribution can arise either from superfluidity of neutrons or from superconductivity of protons. It is therefore quite possible that for many cold neutron stars, the bulk viscosity of the superconducting or superfluid region comes mainly from leptonic processes. In regions that are neither superconducting nor superfluid (more strictly, where $T \gtrsim T_{cp}$ and $T \gtrsim T_{cn}$) the nucleonic bulk viscosity will likely dominate.

The viscosity curves in Fig. 4 all slope upwards because the equilibration rate $\gamma_{\text{eff}}(T)$ is well below the oscillation frequency $\omega$, so we are in the slow-equilibration (high frequency) regime of (12), where

$$\zeta \approx C \frac{\gamma_{\text{eff}}(T)}{\omega^2}.$$  \hspace{1cm} (36)

This is true for both leptonic and nuclear viscosities. In this regime one can simply add the two bulk viscosities to get the total bulk viscosity (see, for example, appendix A of Ref. [17]). As the temperature rises, the equilibration rate and hence the bulk viscosity rise. When $\gamma_{\text{eff}}(T)$ comes close to $\omega$, (36) becomes a poor approximation to (12): $\zeta$ reaches a maximum when $\gamma_{\text{eff}}(T) = \omega$. Those maxima, for both leptonic and nuclear bulk viscosities, are beyond the right hand limit of Fig. 4 for $\mu_l = 200$ MeV, the peak occurs at $T \approx 40$ MeV.

We can now see how our results depend on the frequency of the oscillations. Decreasing $\omega$ moves each $\zeta(T)$ curve to the left, shifting the viscosity curves in Fig. 4 upwards. The largest value we find for the leptonic effective rate (at $T = 10$ MeV, for $\mu_l = 300$ MeV) is $\gamma_{\text{eff}} \sim 2$ rad/s, so for the leptonic bulk viscosity (36) is valid for oscillation
FIG. 4: (Color online) Dependence of the leptonic bulk viscosity $\zeta$ on temperature for three different values of the lepton chemical potential, and an oscillation frequency of 1 kHz; for frequency dependence, see the discussion after (36). We also show the nucleonic bulk viscosity [5] due to modified-Urca processes, for two values of the critical temperature.

It will be interesting to see whether the leptonic contribution that we have calculated here has any impact on oscillations of neutron stars. In the case of r-modes, shear viscosity becomes the dominant source of damping in the low temperature regime, so the leptonic contributions to the bulk viscosity at low temperature are not likely to be an important source of r-mode damping. Also the shear viscosity $\eta$ of superfluid nuclear matter is much larger than the leptonic bulk viscosity we have calculated: $\eta \sim 10^{16} \text{g cm}^{-1}\text{s}^{-1}$ at $T \sim 0.1$ MeV, rising to $\eta \sim 10^{22} \text{g cm}^{-1}\text{s}^{-1}$ at $T \sim 0.001$ MeV (see Fig. 5 of Ref. [30]) so bulk viscosity would only dominate the damping of modes with very little shear flow. Radial pulsations [31, 32] would be an interesting example to investigate. We used a rough approximation (25) to treat the photon screening; we argued (Appendix A) that this is valid to within about an order of magnitude, but if a more precise estimate of the bulk viscosity were required, one could improve on our treatment by replacing the approximation (25) with separate propagators for the transverse and longitudinal photons, incorporating their separate screening mechanisms [33]. It should be noted that our calculation is limited to the small-amplitude regime ($\mu_a \ll T$). If the leptonic bulk viscosity is insufficient to damp an unstable oscillation such as an r-mode then the amplitude will rise and it will be necessary to repeat our calculation in the large-amplitude (“supra-thermal”) regime [34] to see whether leptonic bulk viscosity can stop the growth of the mode once it reaches a large enough amplitude.

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Appendix A: Photon screening

In this appendix we discuss the adequacy of our approximation (25) for the internal photon propagator in the modified Urca process for leptons. The energy $\omega$ of the photon is $\sim T$ because all the initial and final state particles have energies within $T$ of their Fermi energies; however, the photon 3-momentum $q$ must be large enough to move a lepton between the muon and electron Fermi surfaces, so $q \geq q_{\text{min}}$ where $q_{\text{min}} = p_{F,e} - p_{F,\mu}$. Thus $\omega \ll q$ and we can write the photon self-energy in the static limit where it only depends on $q$. There are contributions to the longitudinal and transverse self-energies from protons, electrons, and muons. If the protons are superconducting, as they are at the temperatures of interest in this paper, then they provide an additional contribution to the transverse photon self-energy. The complete expressions are

$$
\Pi_L(q) = M^2_{D,p} \xi_L \left( \frac{q}{k_{F,p}} \right) + M^2_{D,e} \xi_L \left( \frac{q}{k_{F,e}} \right) + M^2_{D,\mu} \xi_L \left( \frac{q}{k_{F,\mu}} \right),
$$

$$
\Pi_T(q) = M^2_{D,p} \xi_T \left( \frac{q}{k_{F,p}} \right) + M^2_{D,e} \xi_T \left( \frac{q}{k_{F,e}} \right) + M^2_{D,\mu} \xi_T \left( \frac{q}{k_{F,\mu}} \right) + \Pi^{(sc)}_p(q).
$$

(A1)

The Debye mass for a given species is (see, for example, \[35\])

$$
M^2_D = 4\pi\alpha\mu k_F
$$

(A2)

where $\mu$ is the Fermi energy (defined relativistically, so $\mu^2 = k^2_F + m^2$) and $k_F$ is the Fermi momentum. The screening functions $\xi_L$ and $\xi_T$ in the static limit are real, and are given by

$$
\xi_L(q) = \frac{1}{2} + \frac{1}{2q} \left( 1 - \frac{q^2}{4} \right) \log \left| \frac{\tilde{q} + 2}{\tilde{q} - 2} \right|,
$$

$$
\xi_T(q) = \frac{1}{8} \left( 1 + \frac{q^2}{4} - \frac{1}{q} \left( 1 - \frac{q^2}{4} \right)^2 \log \left| \frac{\tilde{q} + 2}{\tilde{q} - 2} \right| \right).
$$

(A3)

The full expressions for photon screening by a degenerate gas of charged fermions were first obtained by Lindhard \[33\]. Eq. (A3) was obtained from the version of Lindhard’s expressions for the dielectric permittivities $\varepsilon$ given in Ref. \[37\], using the fact that $\Pi_L(\omega,q) = (\omega^2 - q^2)(1 - \varepsilon_L)$ and $\Pi_T(\omega,q) = \omega^2(1 - \varepsilon_T)$ (see Sec. (6.4) of Ref. \[33\]). Note that $\xi_L$ and $\xi_T$ above are defined in the static limit, where $\omega/q \rightarrow 0$ at fixed $q$. They are therefore different from the quantities $\chi'(x)$ and $\chi(x)$ which are commonly given in the literature \[33\ \[39\], and are calculated at $\omega = xq$ in the limit $q \rightarrow 0$.

In our calculations of the leptonic flavor equilibration rate we use the rough approximation $\Pi_L = \Pi_T = q^2$ (25) instead of the correct screening expressions given above. We now explain why this is a reasonable approximation.

First we discuss the longitudinal photons. Their momentum varies from $q_{\text{min}}$ up to $k_{F,e} + k_{F,\mu}$, but the momentum dependence of $\Pi_T$ is very moderate: from (A3) we see that as $\tilde{q}$ varies from 0 to 2, $\xi_L$ varies from 1 to $\frac{1}{2}$. In order to judge whether, for the denominator of the longitudinal photon propagator, $q^2 + \Pi_L(q)$ is a good approximation to $q^2 + \Pi_L(q)$, we use a naive free particle model for nuclear matter. In Table 1 we show the results. At each value of the baryon chemical potential $\mu_B$, the negative-charge chemical potential $\mu_\ell$ is determined by requiring overall electrical neutrality. This then fixes the Fermi momenta of the protons, electrons, and muons. In Table 1 we see that when $q = q_{\text{min}}$ (which is where there is greatest sensitivity to the exact form of the screening), the difference between $q^2 + \Pi_L(q)$ and $q^2 + \Pi_L(q_{\text{min}})$ is a few percent at low density, and still less than a factor of 2 at very high densities.

For the transverse photons, $\xi_T$ varies from 0 at $q = 0$ to $\frac{1}{3}$ at $q = \infty$, so the normal-fermion contribution to the transverse screening is more important at higher momenta. The other contribution to $\Pi_T$ comes from the superconducting protons, and it is more important at low momentum. At zero momentum we have Meissner screening, but as the momentum rises the effective screening mass drops slowly: this is seen in the calculation of Ref. \[33\] which finds that, for $q \gg \xi^{-1}$ (where the correlation length $\xi = p_{F,p}/(m_p T_{c,p})$), and assuming the static limit,

$$
\Pi^{(sc)}_T(q) \approx \frac{\pi \alpha p_{F,p} T_{c,p}}{q}.
$$

(A4)

(This result follows from Ref. \[33\] eqn (49), taking $\omega \rightarrow 0$ and using $Q = \pi^2$ as specified in the preceding paragraph.) In Table 1 we show numerical results for the naive free-nucleon model of nuclear matter. We assumed $T_{c,p} = 1$ MeV.
(see Ref. [23], fig. 10, and Ref. [40], fig. 2). At the lowest allowed photon momentum \(q = q_{\min}\), which is where there is greatest sensitivity to the exact form of the screening, the difference between \(q^2 + q_{\min}^2\) and \(q^2 + \Pi^2\) is a few percent at low density, but rises to a factor of 3 at density \(n/n_{\text{sat}} = 12\), and a factor of 10 at \(n/n_{\text{sat}} = 27\).

We conclude that our rough approximation of using a photon self-energy \(q^2_{\min} = 5\alpha \mu^2\) [23] gives a reasonable estimate of the in-medium photon propagator. At low densities it is accurate to within 10%. At higher densities, up to 10 times nuclear saturation density in the simple model of Table I, our approximation underestimates the screening of longitudinal photons by a factor of about 2 and overestimates the screening of transverse photons by a factor of about 3. (At even higher densities, where a description in terms of nucleons is probably no longer appropriate, our approximation for transverse screening deviates further from the free nucleon model.) Since the rate involves the square of the photon propagator, we conclude that our approximate treatment of the photon propagator affects the rate by less than an order of magnitude at reasonable densities for nuclear matter.

| \(\mu B\) | \(n/n_{\text{sat}}\) | \(\mu\) | \(q^2_{\min}\) | \(\Pi_L(q_{\min})\) | \(\Pi_T(q_{\min})\) | \(q^2_{\min} + \Pi_L(q_{\min})\) | \(q^2_{\min} + \Pi_T(q_{\min})\) | \(q^2_{\min} + q^2_{\min}\) |
|---|---|---|---|---|---|---|---|---|
| 1056 | 3.164 | 111.1 | 5908 | 1067 | 55.45 | 450.1 | 6974 | 5963 | 6358 |
| 1125 | 6.76 | 167.6 | 1406 | 2150 | 30.13 | 1025 | 3557 | 1436 | 2537 |
| 1200 | 12.03 | 224.4 | 698.3 | 3300 | 65.06 | 1838 | 3999 | 763.4 | 2537 |
| 1350 | 26.93 | 328.7 | 304.3 | 5783 | 215 | 3943 | 6087 | 519.3 | 4247 |

**TABLE I:** Screening parameters in MeV or MeV^2 for a free-nucleon model of \(npe\mu\) nuclear matter; \(\mu_B\) is the baryon number chemical potential, \(n/n_{\text{sat}}\) is the baryon density relative to nuclear saturation density; \(\mu\) is the Fermi energy of the electrons and muons; \(q_{\min}\) is the lowest photon momentum that contributes to the modified Urca process; \(\Pi_L\) and \(\Pi_T\) are defined in (A1). The last three columns compare our approximate photon propagator at \(q = q_{\min}\) (final column) with the photon propagator using the full screening expressions given in Appendix A.

**Appendix B: Partial Rate Integrals**

The following abbreviations are used throughout this appendix:

\[
x_m = \frac{m_{\mu}}{\mu}, \quad x_s = \frac{q_s}{\mu}, \quad t = \frac{T}{\mu}
\]

\[
C_{12} = 1 - \cos \theta_2, \quad C_{14} = 1 - \sqrt{1 - x_m^2} \cos \theta_4
\]

\[
C_{24} = 1 - \sqrt{1 - x_m^2} (\sin \theta_2 \sin \theta_4 (\sin \phi_2 \sin \phi_4 + \cos \phi_2 \cos \phi_4) + \cos \theta_2 \cos \theta_4)
\]

\[
C_{13} = 1 - (1 - x_m^2) (\sin \theta_1 \sin \theta_3 (\sin \phi_1 \sin \phi_3 + \cos \phi_1 \cos \phi_3) + \cos \theta_1 \cos \theta_3)
\]

\[
C_{14} = 1 - (1 - x_m^2) \cos \theta_1, \quad C_{34} = 1 - (1 - x_m^2) \cos \theta_3
\]

\[
F(x_a, x_b, x_c, x_d) = \frac{2 \exp [(x_a + x_d - 2)/t] [1 + 2 \exp [(x_b + x_c - 2)/t] + \exp [(x_b + x_d - 2)/t]]}{(1 + \exp [(x_a - 1)/t]) (1 + \exp [(x_b + x_c - 2)/t]) (1 + \exp [(x_c - 1)/t]) (1 + \exp [(x_d - 1)/t])^2}
\]

\[
I_{\alpha}^m = \int dx_2 dx_3 dy_1 dy_2 \, y_1^2 y_2^2 F(x_4 + y_1 + y_2 - x_2 + 1, x_2, 1, x_4)
\]

\[
I_{\alpha}^s = \int dx_1 dx_3 dy_1 dy_2 \, y_1^2 y_2^2 F(x_1, x_2, x_3 - x_4 - 1 + y_1 - y_2 + 1)
\]

\[
I_{\alpha 1}^{11} = \sqrt{1 - x_m^2} \int d\Omega_2 d\Omega_4 \frac{4C_{12}C_{14} + 2C_{12}C_{24} - 2C_{14}C_{24} - 4x_m^2 C_{12} + x_m^2 C_{24}}{(x_m^2 - 2C_{24} - x_s^2)^2}
\]

\[
I_{\alpha 1}^{12} = -\sqrt{1 - x_m^2} \int d\Omega_2 d\Omega_4 \left[\frac{-2C_{12}C_{24} + 2C_{14}C_{24} + 8C_{12}C_{14} + 4C_{12}C_{24} - 4C_{12}C_{14}C_{24} - 4C_{14}C_{24}}{(x_m^2 - 2C_{24} - x_s^2)^2}ight.
\]

\[
+ \left.\frac{x_m^2 (4C_{12}^2 - 8C_{12}C_{14} + 4C_{12}C_{24} - 4C_{14}C_{24}) - x_m^4}{(x_m^2 - 2C_{24} - x_s^2)^2}\right]
\]
\[ I_{\alpha_1}^{13} = -\sqrt{1 - x_m^2} \int d\Omega_2 d\Omega_4 \frac{-4C_{12}C_{14} - 4C_{12}C_{24} + 6x_m^2 C_{12}}{(x_m^2 - 2C_{24} - x_s^2)(x_m^2 - 2C_{14} - x_s^2)} \]  

(B5)

\[ I_{\alpha_1}^{14} = \sqrt{1 - x_m^2} \int d\Omega_2 d\Omega_4 \left[ \frac{2C_{12}C_{14}^2 - 4C_{12}C_{14} - 4C_{12}C_{24} + 2C_{12}C_{14}C_{24}}{(x_m^2 - 2C_{24} - x_s^2)(x_m^2 - 2C_{14} - x_s^2)} \right. 
+ \left. \frac{x_m^2 (-C_{12}^2 + 6C_{12} - 2C_{12}C_{24} - 2C_{14} + 2C_{24} + x_m^4/2)}{(x_m^2 - 2C_{24} - x_s^2)(x_m^2 - 2C_{14} - x_s^2)} \right] \]  

(B6)

\[ I_{\alpha_1}^{22} = \sqrt{1 - x_m^2} \int d\Omega_2 d\Omega_4 \frac{4C_{12}C_{14} + 2C_{12}C_{24} - 2C_{14}C_{24} - 4x_m^2 C_{12} + 4x_m^2 C_{14} + x_m^4 C_{24} - x_m^4}{(x_m^2 - 2C_{24} - x_s^2)^2} \]  

(B7)

\[ I_{\alpha_1}^{23} = \sqrt{1 - x_m^2} \int d\Omega_2 d\Omega_4 \left[ \frac{2C_{12}C_{14}^2 - 4C_{12}C_{14} - 4C_{12}C_{24} + 2C_{12}C_{14}C_{24}}{(x_m^2 - 2C_{24} - x_s^2)(x_m^2 - 2C_{14} - x_s^2)} \right. 
+ \left. \frac{x_m^2 (-C_{12}^2 + 6C_{12} - 2C_{12}C_{24} - 2C_{14} + 2C_{24} + x_m^4/2)}{(x_m^2 - 2C_{24} - x_s^2)(x_m^2 - 2C_{14} - x_s^2)} \right] \]  

(B8)

\[ I_{\alpha_1}^{24} = -\sqrt{1 - x_m^2} \int d\Omega_2 d\Omega_4 \left[ \frac{2C_{12}C_{14}^2 - 4C_{12}C_{14} - 4C_{12}C_{24} + 2C_{12}C_{14}C_{24}}{(x_m^2 - 2C_{24} - x_s^2)(x_m^2 - 2C_{14} - x_s^2)} \right. 
+ \left. \frac{x_m^2 (-C_{12}^2 + 5C_{12} + 12C_{12}C_{24} + 2C_{14} + C_{24})}{(x_m^2 - 2C_{24} - x_s^2)(x_m^2 - 2C_{14} - x_s^2)} \right] \]  

(B9)

\[ I_{\alpha_1}^{33} = \sqrt{1 - x_m^2} \int d\Omega_2 d\Omega_4 \frac{2C_{12}C_{14} + 4C_{12}C_{24} - 2C_{14}C_{24} + x_m^2 (-4C_{12} + C_{14})}{(x_m^2 - 2C_{14} - x_s^2)^2} \]  

(B10)

\[ I_{\alpha_1}^{34} = -\sqrt{1 - x_m^2} \int d\Omega_2 d\Omega_4 \left[ \frac{-2C_{12}C_{14}^2 + 4C_{12}C_{14} + 2C_{14}^2 C_{24} + 8C_{12}C_{24} - 4C_{12}C_{14}C_{24} - 4C_{14}C_{24}}{(x_m^2 - 2C_{14} - x_s^2)^2} \right. 
+ \left. \frac{x_m^2 (2C_{14}^2 - 8C_{12} + 12C_{12}C_{14} - 14C_{24} + 4C_{24}) - x_m^4}{(x_m^2 - 2C_{24} - x_s^2)^2} \right] \]  

(B11)

\[ I_{\alpha_1}^{44} = \sqrt{1 - x_m^2} \int d\Omega_2 d\Omega_4 \frac{2C_{12}C_{14} + 4C_{12}C_{24} - 2C_{14}C_{24} + x_m^2 (-4C_{12} + C_{14} + 4C_{24}) - x_m^4}{(x_m^2 - 2C_{14} - x_s^2)^2} \]  

(B12)

\[ I_{\alpha_1}^{11} = (1 - x_m^2)^{3/2} \int d\Omega_1 d\Omega_3 \frac{-2C_{13}C_{14} + 2C_{13}C_{34} + 4C_{14}C_{34} + x_m^2 (3C_{14} - 5C_{34}) - x_m^4}{(2x_m^2 - 2C_{13} - x_s^2)^2} \]  

(B13)

\[ I_{\alpha_1}^{12} = -(1 - x_m^2)^{3/2} \int d\Omega_1 d\Omega_3 \left[ \frac{-2C_{13}C_{14}^2 + 4C_{13}C_{34} - 2C_{13}C_{34} - 4C_{14}^2 + 8C_{13}C_{34} + 8C_{14}C_{34}}{(2x_m^2 - 2C_{13} - x_s^2)^2} \right. 
+ \left. \frac{x_m^2 (4C_{14}^2 + 4C_{34}^2 - 8C_{13} + 2C_{13}C_{14} - 8C_{14} - 2C_{13}C_{34} - 4C_{14}C_{34} - 8C_{34})}{(2x_m^2 - 2C_{13} - x_s^2)^2} \right. 
+ \left. \frac{x_m^4 (3C_{14}^2 + 2C_{14} + 2C_{34} + 4C_{34} - 3x_m^4)}{(2x_m^2 - 2C_{13} - x_s^2)^2} \right] \]  

(B14)
\[ I_{d\Omega}^{\mu 13} = -(1 - x_m^2)^{3/2} \int d\Omega_1 d\Omega_3 \left[ \frac{2 \tilde{C}_{13} \tilde{C}_{34}^2 + 2 \tilde{C}_{14} \tilde{C}_{34}^2 - 4 \tilde{C}_{13} \tilde{C}_{34} - 4 \tilde{C}_{14} \tilde{C}_{34}}{(2 x_m^2 - 2 C_{13} - x_s^2)(2 x_m^2 - 2 C_{14} - x_s^2)} \right. \\
+ \left. \frac{x_m^2 (-4 \tilde{C}_{34}^2 + 5 \tilde{C}_{13} + 5 \tilde{C}_{14} + \tilde{C}_{13} \tilde{C}_{34} + \tilde{C}_{14} \tilde{C}_{34} + 5 \tilde{C}_{34})}{(2 x_m^2 - 2 C_{13} - x_s^2)(2 x_m^2 - 2 C_{14} - x_s^2)} \right. \\
+ \left. \frac{x_m^4 (-3 \tilde{C}_{13} / 2 - 3 \tilde{C}_{14} / 2 - \tilde{C}_{34} - 6) + 2 x_m^6}{(2 x_m^2 - 2 C_{13} - x_s^2)(2 x_m^2 - 2 C_{14} - x_s^2)} \right] \tag{B15} \]

\[ I_{d\Omega}^{\mu 14} = (1 - x_m^2)^{3/2} \int d\Omega_1 d\Omega_3 \left[ \frac{2 \tilde{C}_{13} \tilde{C}_{34}^2 + 4 \tilde{C}_{34}^2 - 8 \tilde{C}_{13} \tilde{C}_{34} - 8 \tilde{C}_{14} \tilde{C}_{34}}{(2 x_m^2 - 2 C_{13} - x_s^2)(2 x_m^2 - 2 C_{14} - x_s^2)} \right. \\
+ \left. \frac{x_m^2 (-\tilde{C}_{14}^2 - 3 \tilde{C}_{34}^2 + 6 \tilde{C}_{13} + 8 \tilde{C}_{14} + 2 \tilde{C}_{13} \tilde{C}_{34} + 2 \tilde{C}_{14} \tilde{C}_{34} + 8 \tilde{C}_{34})}{(2 x_m^2 - 2 C_{13} - x_s^2)(2 x_m^2 - 2 C_{14} - x_s^2)} \right. \\
+ \left. \frac{x_m^4 (-3 \tilde{C}_{13} / 2 - 2 \tilde{C}_{34} - 9) + 3 x_m^6 / 2}{(2 x_m^2 - 2 C_{13} - x_s^2)(2 x_m^2 - 2 C_{14} - x_s^2)} \right] \tag{B16} \]

\[ I_{d\Omega}^{\mu 22} = (1 - x_m^2)^{3/2} \int d\Omega_1 d\Omega_3 \left[ \frac{2 \tilde{C}_{14} \tilde{C}_{34}^2 - 2 \tilde{C}_{34}^2 + 4 \tilde{C}_{13} \tilde{C}_{34} + 4 \tilde{C}_{14} \tilde{C}_{34} + x_m^4 (-4 \tilde{C}_{13} - 4 \tilde{C}_{34}) + 4 x_m^4}{(2 x_m^2 - 2 C_{13} - x_s^2)^2} \right] \tag{B17} \]

\[ I_{d\Omega}^{\mu 23} = (1 - x_m^2)^{3/2} \int d\Omega_1 d\Omega_3 \left[ \frac{2 \tilde{C}_{14} \tilde{C}_{34}^2 + 4 \tilde{C}_{34}^2 - 8 \tilde{C}_{13} \tilde{C}_{34} - 8 \tilde{C}_{14} \tilde{C}_{34}}{(2 x_m^2 - 2 C_{13} - x_s^2)(2 x_m^2 - 2 C_{14} - x_s^2)} \right. \\
+ \left. \frac{x_m^2 (-\tilde{C}_{13}^2 - 3 \tilde{C}_{34}^2 + 10 \tilde{C}_{13} + 13 \tilde{C}_{14} + 6 \tilde{C}_{13} \tilde{C}_{34} + 2 \tilde{C}_{13} \tilde{C}_{34} + 2 \tilde{C}_{14} \tilde{C}_{34} + 8 \tilde{C}_{34})}{(2 x_m^2 - 2 C_{13} - x_s^2)(2 x_m^2 - 2 C_{14} - x_s^2)} \right. \\
+ \left. \frac{x_m^4 (-\tilde{C}_{13} - 2 \tilde{C}_{14} - 2 \tilde{C}_{34} - 10) + 2 x_m^6}{(2 x_m^2 - 2 C_{13} - x_s^2)(2 x_m^2 - 2 C_{14} - x_s^2)} \right] \tag{B18} \]

\[ I_{d\Omega}^{\mu 24} = -(1 - x_m^2)^{3/2} \int d\Omega_1 d\Omega_3 \left[ \frac{4 \tilde{C}_{34}^2 - 8 \tilde{C}_{13} \tilde{C}_{34} - 8 \tilde{C}_{14} \tilde{C}_{34} + x_m^2 (8 \tilde{C}_{13} + 8 \tilde{C}_{14} + 8 \tilde{C}_{34}) - 8 x_m^4}{(2 x_m^2 - 2 C_{13} - x_s^2)(2 x_m^2 - 2 C_{14} - x_s^2)} \right] \tag{B19} \]

\[ I_{d\Omega}^{\mu 33} = (1 - x_m^2)^{3/2} \int d\Omega_1 d\Omega_3 \left[ -\frac{2 \tilde{C}_{14} \tilde{C}_{34}^2 + x_m^2 (\tilde{C}_{13} + 6 \tilde{C}_{14} - 3 \tilde{C}_{34}) - 3 x_m^4}{(2 x_m^2 - 2 C_{14} - x_s^2)^2} \right] \tag{B20} \]

\[ I_{d\Omega}^{\mu 34} = -(1 - x_m^2)^{3/2} \int d\Omega_1 d\Omega_3 \left[ \frac{2 \tilde{C}_{14}^2 - 2 \tilde{C}_{13} \tilde{C}_{14} + 2 \tilde{C}_{13} \tilde{C}_{14} - 2 \tilde{C}_{13} \tilde{C}_{34} + 2 \tilde{C}_{13} \tilde{C}_{14} \tilde{C}_{34}}{(2 x_m^2 - 2 C_{14} - x_s^2)^2} \right. \\
+ \left. \frac{x_m^2 (-\tilde{C}_{13}^2 - \tilde{C}_{14}^2 + 4 \tilde{C}_{13} \tilde{C}_{14} + 2 \tilde{C}_{14})}{(2 x_m^2 - 2 C_{14} - x_s^2)^2} \right. \\
+ \left. \frac{x_m^4 (-2 \tilde{C}_{13} - 2 \tilde{C}_{14} - 2) + 2 x_m^6}{(2 x_m^2 - 2 C_{14} - x_s^2)^2} \right] \tag{B21} \]

\[ I_{d\Omega}^{\mu 44} = (1 - x_m^2)^{3/2} \int d\Omega_1 d\Omega_3 \left[ \frac{2 \tilde{C}_{13} \tilde{C}_{34}^2 + 4 \tilde{C}_{13} \tilde{C}_{34} + 4 \tilde{C}_{14} \tilde{C}_{34} + x_m^2 (-4 \tilde{C}_{14} - 4 \tilde{C}_{34}) + 4 x_m^4}{(2 x_m^2 - 2 C_{14} - x_s^2)^2} \right] \tag{B22} \]

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