NONLINEAR STOCHASTIC WAVE EQUATION DRIVEN BY
ROUGH NOISE

SHUHUI LIU, YAOZHONG HU, AND XIONG WANG*

Abstract. In this paper, we obtain the existence and uniqueness of the strong solution to one spatial dimension stochastic wave equation
\[ \frac{\partial^2 u(t, x)}{\partial t^2} = \frac{\partial^2 u(t, x)}{\partial x^2} + \sigma(t, x, u(t, x)) \dot{W}(t, x) \]
assuming \( \sigma(t, x, 0) = 0 \), where \( \dot{W} \) is a mean zero Gaussian noise which is white in time and fractional in space with Hurst parameter \( H \in (1/4, 1/2) \).

CONTENTS

1. Introduction 1
2. Preliminaries and Main results 3
3. Uniform moment bounds 7
   3.1. Uniform moment bounds of stochastic convolution 7
   3.2. Uniform moment bounds of the approximate solutions 12
4. Hölder continuity and well-posedness 16
   4.1. Hölder continuity of stochastic convolution 16
   4.2. Hölder continuity of the approximate solutions and well-posedness 28
5. Necessity of \( H > \frac{1}{4} \) 30
Appendix A. Some technical lemmas for wave kernel 34
Appendix B. Lemmas for Proposition 3.3 36
Appendix C. Lemmas for Proposition 4.1 40
References 48

1. INTRODUCTION

In this paper, we consider the following one (spatial) dimensional stochastic nonlinear wave equation (SWE for short) driven by rough spatial Gaussian noise which is white in time and fractional in space:

\[
\begin{cases}
\frac{\partial^2 u(t, x)}{\partial t^2} = \frac{\partial^2 u(t, x)}{\partial x^2} + \sigma(t, x, u(t, x)) \dot{W}(t, x), & t \in [0, T], \ x \in \mathbb{R},
\end{cases}
\]

\[ u(0, x) = u_0(x), \quad \frac{\partial}{\partial t} u(0, x) = v_0(x). \]  

2010 Mathematics Subject Classification. Primary 60H15; secondary 60H05, 60H07, 65M80.

Key words and phrases. Fractional derivative, rough fractional noise, stochastic wave equation, decomposition of wave kernel, sup \( L_p \)-norm, strong solutions, well-posedness, Hölder continuity.

SL was supported by the China Scholarship Council.

YH was supported by the NSERC discovery fund and a startup fund of University of Alberta.

*Corresponding author: xiongwang@ualberta.ca.
Here $W(t, x)$ is a centered Gaussian process with covariance given by
\[
E[W(t, x)W(s, y)] = \frac{1}{2}(s \wedge t)(|x|^{2H} + |y|^{2H} - |x - y|^{2H})
\] (1.2)
and $\hat{W}(t, x) = \frac{\partial^2}{\partial x^2} W(t, x)$. The main feature of this work is our assumption that the Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$. Namely, the noise is rough and fractional in space variable and white in time variable. When the noise is general Gaussian which is white in time and satisfies the so-called Dalang’s condition, there are some results about the well-posedness of the equation and the properties of the solutions (e.g. [4, 5, 10]). If we apply Dalang’s condition to fractional Gaussian noise, then we need to assume the spatial Hurst parameter $H \geq 1/2$. When $H < 1/2$, namely, when the noise is rough in space (in this case the spatial dimension must be one dimensional), there are very limited results. The only result we know, to the best of our knowledge, is the work [1], where the noise coefficient $\sigma(t, x, u) = au + b$ is affine. There has been no work to tackle the case when $\sigma(t, x, u)$ is nonlinear (or not affine) function of $u$. On the other hand, when $\frac{\partial^2}{\partial x^2}$ on the left hand of (1.1) is replaced by $\frac{\partial}{\partial t}$, this is, in the case nonlinear stochastic heat equations (SHE for short) driven by spatial rough noise, the authors of [8] studied the equation in the case $\sigma(t, x, 0) = 0$. They prove the strong existence and uniqueness of solution. This condition $\sigma(t, x, 0) = 0$ is removed in [11], where the authors obtained the existence of weak solution.

Our objective in this paper is to obtain the strong existence and uniqueness of the SHE (1.1) while we still assume $\sigma(t, x, 0) = 0$. The reason we extend the work of [8] under this condition is that one can obtain the existence and uniqueness of strong solution (or mild solution) in a solution space which is much simpler to deal with. It seems too much involved to remove the restriction $\sigma(t, x, 0) = 0$ since in this case we believe that we need to introduce a weighted space for the solution and to study the interaction between the wave Green’s kernel and the weight.

Even in the case $\sigma(t, x, 0) = 0$ there are mainly two difficulties to study (1.1) or its SHE analogue. The first one is that one cannot bound the $L_p$ norm of $\int_0^t \int_R h_t(s, y) W(ds, dy)$ by the $L_p$ norm of $h_t(s, y)$, instead, one has to use the $L_p$ norm of $h_t(s, y)$ itself plus the $L_p$ norm of its fractional derivative, where $h_t(s, y) = G_t(x - y)\sigma(s, y, u(s, y))$ and $G_t(x, y)$ is the heat or wave kernel. This makes things very much sophisticated. In particular, as indicated in [8, 11], due to the existence of our rough noise $W$ we need to bound $|\sigma(u_1) - \sigma(u_2) - \sigma(v_1) + \sigma(v_2)|$ by a multiple of $|u_1 - u_2 - v_1 + v_2|$ (which is possible only in the affine case). To get around this difficulty the authors in [8, 11] use a priori bound of $B_p \times L_\infty$ norm $E\sup_{0 \leq t \leq T} |u(t, x)|_{B_p(L(R))}$ and the similar norm of the fractional derivative of $u(t, x)$ for the solution $u(t, x)$. This immediately poses a new challenge which is our second difficulty since $\int_0^T \int_R h_t(s, y) W(ds, dy)$ is not a martingale in $t$ (nor it is a semi-martingale), it is hard to bound the $L_p$ norm of $\sup_{0 \leq t \leq T} \int_0^t \int_R h_t(s, y) W(ds, dy)$ since we can no longer use the powerful Burkholder-Davis-Gundy inequality. In the case of SHE, this is overcome by a clever exploitation of the semigroup property of the heat kernel. This idea is not reproducible in SWE simply because the wave kernel $G_t(x, y)$ does not have the semigroup property, unfortunately! To surmount this barrier we shall decompose the simple wave kernel $G_t(x - y)$ to four complicated parts so that we can bound the $L_p$ norm of $\sup_{0 \leq t \leq T} \int_0^t \int_R h_t(s, y) W(ds, dy)$ by the $L_p$ norm of $h_t(s, y)$ itself plus the $L_p$ norm of its fractional derivative. Of course,
one also needs to bound $L_p$ norm of the $\sup_{0 \leq t \leq T}$ norm of the fractional derivative of $\int_0^t \int_\mathbb{R} b_1(s, y)W(ds, dy)$. This will be the main effort of this work. After achieving this estimation, the proof of the existence and uniqueness of the mild solution is routine.

In the study of fractional noise, the number $1/4$ seems to be a magic number. It appears in a number of occurrences. Here we are interested in the problem if $H > 1/4$ is necessary for (1.1) to have a classical ($L_2$) solution. We shall provide an affirmative answer. To this end we consider the hyperbolic Anderson model, namely, $\sigma(t, x, u) = u$. In this case the equation (1.1) becomes

\[
\begin{aligned}
\frac{\partial^2 v(t, x)}{\partial t^2} &= \frac{\partial^2 v(t, x)}{\partial x^2} + v(t, x)\dot{W}(t, x), \quad t \in [0, T], \quad x \in \mathbb{R}, \\
v(0, x) &= u_0(x), \quad \frac{\partial}{\partial x}v(0, x) = v_0(x).
\end{aligned}
\]

Under some conditions on the initial data, we shall prove that $v(t, x)$ is square integrable only if $H > 1/4$. After the completion of this work, we discover that the necessity of $H > 1/4$ is implied in [16, Proposition 3.4]. To make the paper more comprehensive, we keep our alternative proof of the necessity of $H > 1/4$. Our method may be useful to study the properties of (1.1) with additive noise ($\sigma \equiv 1$). Let us also mention a recent work [3] that for the parabolic Anderson model when the dimension $d = 1$ and when the noise is white in time and fractional in space with Hurst parameter $H$, then $H > 1/4$ is also the necessary and sufficient condition for the solution to be square integrable.

Here is the organization of this paper. In Section 2 we briefly recall some necessary concept about stochastic integral and wave kernel and so on to fix the notations used in the paper and we also state our main results obtained in this work. Sections 3 and 4 are the core of the paper. In Section 3 we decompose the wave kernel into four parts and then we use this decomposition to obtain the necessary bound of the stochastic integral (stochastic convolution with the wave kernel). There are a lot of computations to obtain the bound for the stochastic convolution. We postpone some of these computations in the Appendix A and B. Section 4 obtains the existence and uniqueness of the strong solution. Some of the computations are moved to Appendix C for the fluency of the proof. Section 5 is about the necessity of $H > 1/4$ for strong solution to exist.

Throughout the paper, $A \lesssim B$ (and $A \gtrsim B$) means that there are universal constants $C_1, C_2 \in (0, \infty)$ such that $A \leq C_1B$ (and $A \geq C_2B$). We also denote throughout the paper

\[
\Delta_x f(t, x) := f(t + \tau, x) - f(t, x),
\]

and

\[
\begin{aligned}
\Box_{h,1} f(t, x) := \mathcal{D}_h \mathcal{D}_h f(t, x) &= \mathcal{D}_h f(t, x + l) - \mathcal{D}_h f(t, x) \\
&= [f(t, x + h + l) - f(t, x + l)] - [f(t, x + h) - f(t, x)].
\end{aligned}
\]

2. Preliminaries and Main results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $W = (W(t, x), t \geq 0, x \in \mathbb{R})$ be a mean zero Gaussian random field whose covariance is given by (1.2). For any $t \geq 0, \mathcal{F}_t = \sigma(W(s, x), s \in [0, t], x \in \mathbb{R})$ be the $\sigma$-algebra generated by the Gaussian
field \( W \). We recall briefly some notations and facts in \([8]\) and refer to that reference for more details.

Denote \( S \) the set of smooth functions on \( \mathbb{R}^+ \times \mathbb{R} \) with compact support. For any \( f, g \in S \), define

\[
\langle f, g \rangle_S = c_H^2 \int_{\mathbb{R}^+ \times \mathbb{R}^2} [f(t, x + y) - f(t, x)][g(t, x + y) - g(t, x)]|y|^{2H - 2}dx dy dt, \tag{2.1}
\]

where

\[
c_H^2 = H \left( \frac{1}{2} - H \right) \left[ \Gamma \left( H + \frac{1}{2} \right) \right]^{-2} \left( \int_0^\infty \left( (1 + t)^{H - \frac{1}{2}} - t^{H - \frac{1}{2}} \right)^2 dt + \frac{1}{2H} \right).
\]

Let \( \mathfrak{H} \) be the Hilbert space obtained by completing \( S \) with respect to the scalar product \( \langle \cdot, \cdot \rangle_S \). Let us start with the stochastic integration of elementary process with respect to \( W \), and then extend it to general process.

**Definition 2.1.** A random field \( f = (f(t, x), (t, x) \in \mathbb{R}^+ \times \mathbb{R}) \) is called adapted to the filtration \( \mathcal{F}_t \) if \( f(t, x) \in \mathcal{F}_t \) for all \((t, x) \in \mathbb{R}^+ \times \mathbb{R}\). An elementary process \( g \) is \( \mathcal{F}_t \)-adapted random field of the following form:

\[
g(t, x) = \sum_{i=1}^n \sum_{j=1}^m X_{i,j} \mathbf{1}_{[a_i, b_i]}(t) \mathbf{1}_{[c_j, d_j]}(x),
\]

where \( n \) and \( m \) are positive integers, \( 0 < a_1 < b_1 < \cdots < a_n < b_n < +\infty \), \( c_j < d_j \) and \( X_{i,j} \) are \( \mathcal{F}_\infty \)-measurable random variables for \( i = 1, \cdots, n, j = 1, \cdots, m \). The stochastic integral of such an elementary process \( g \) with respect to \( W \) is defined as

\[
\int_{\mathbb{R}^+ \times \mathbb{R}} g(t, x) W(dt, dx) = \sum_{i=1}^n \sum_{j=1}^m X_{i,j} W(\mathbf{1}_{[a_i, b_i]} \otimes \mathbf{1}_{[c_j, d_j]})
\]

\[
= \sum_{i=1}^n \sum_{j=1}^m X_{i,j} [W(b_i, d_j) - W(a_i, d_j) - W(b_i, c_j) + W(a_i, c_j)]. \tag{2.2}
\]

In fact, we have the following proposition (e.g. \([8]\)).

**Proposition 2.2.** Let \( \Lambda_H \) be the space of adapted random field \( g \) defined on \( \mathbb{R}^+ \times \mathbb{R} \) such that \( g \in \mathfrak{H} \) a.s. and \( \mathbb{E}[|g|^2] < \infty \). Then we have the following statements.

1. The space of elementary process defined in Definition 2.1 is dense in \( \Lambda_H \);
2. For \( g \in \Lambda_H \), the stochastic integral \( \int_{\mathbb{R}^+ \times \mathbb{R}} g(t, x) W(dt, dx) \) is defined as the \( L^2(\Omega) \)-limit of stochastic integrals of elementary processes approximating \( g(t, x) \) in \( \Lambda_H \), and for this stochastic integral we have the following isometry equality

\[
\mathbb{E} \left[ \left( \int_{\mathbb{R}^+ \times \mathbb{R}} g(t, x) W(dt, dx) \right)^2 \right] = \mathbb{E}[|g|^2].
\]

Now we introduce some norms and spaces used in this paper. Let \( (B, \| \cdot \|_B) \) be a Banach space with the norm \( \| \cdot \|_B \). Let \( \beta \in (0, 1) \) be a fixed number. For any function \( f : \mathbb{R} \to B \) denote

\[
\mathcal{N}_\beta^B f(x) := \left( \int_{\mathbb{R}} \| D_h f(x) \|^2 \| h \|^{-1 - 2\beta} dh \right)^{\frac{1}{2}}, \tag{2.3}
\]
Stochastic wave equation

if the above quantity is finite, where we recall $\mathcal{D}_h f(x) = f(x+h) - f(x)$. When $B = \mathbb{R}$, we abbreviate the notation $\mathcal{N}_h^B f$ as $\mathcal{N}_h f$. With this notation, the norm of the homogeneous Sobolev space $H^p$ can be given by using $\mathcal{N}_h f$: $\|f\|_{\mathcal{N}_h} = \|\mathcal{N}_h f\|_{L^2(\mathbb{R})}$.

As in [8] throughout this paper we are particularly interested in the case $B = L^p(\Omega)$, and in this case we denote $\mathcal{N}_h^B$ by $\mathcal{N}_h$: 

$$
\mathcal{N}_h f(x) := \left( \int_\mathbb{R} \|\mathcal{D}_h f(x)\|_{L^p(\Omega)} |h|^{-1-2\beta} dh \right)^{\frac{1}{2}}.
$$

We shall set $\beta = \frac{1}{2} - H$. The following Burkholder-Davis-Gundy inequality is well-known (see e.g. [8, 11]).

**Proposition 2.3.** Let $W$ be the Gaussian noise defined by the covariance (1.2), and let $f \in \Lambda_H$ be a predictable random field. Then for any $p \geq 2$ we have

$$
\left\| \sup_{0 \leq r \leq t} \int_0^r \int_\mathbb{R} f(s, y) W(ds, dy) \right\|_{L^p(\Omega)} 
\leq C_H \sqrt{p} \left( \int_0^t \int_\mathbb{R} \left[ \mathcal{N}_{\frac{1}{2}-H,p} f(s, y) \right]^2 dy ds \right)^{\frac{1}{2}} \tag{2.5}
\leq C_H \sqrt{p} \left( \int_0^t \int_\mathbb{R} \int_\mathbb{R} \|\mathcal{D}_h f(s, y)\|_{L^p(\Omega)} |h|^{2H-2} dh dy ds \right)^{\frac{1}{2}},
$$

where $C_H$ is a constant depending only on $H$, $\mathcal{N}_{\frac{1}{2}-H,p} f(s, y)$ denotes the application of $\mathcal{N}_{\frac{1}{2}-H,p}$ to the space variable $y$ and $\mathcal{D}_h$ is defined by (1.5).

We introduce the solution space $Z^p(T)$. It consists of all continuous functions $f$ from $[0, T] \times \mathbb{R}$ to $L^p(\Omega)$ such the following norm is finite:

$$
\|f\|_{Z^p(T)} = \|f\|_{Z^p_1(T)} + \|f\|_{Z^p_2(T)} 
:= \sup_{t \in [0, T]} \|f(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})} + \sup_{t \in [0, T]} \mathcal{N}_{\frac{1}{2}-H,p}^* f(t),
$$

where $\|f(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})} = \left( \int_\mathbb{R} \mathbb{E}[|f(t, x)|^p] dx \right)^{1/p}$ and 

$$
\mathcal{N}_{\frac{1}{2}-H,p}^* f(t) := \left( \int_\mathbb{R} \|\mathcal{D}_h f(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2 |h|^{2H-2} dh \right)^{\frac{1}{2}}.
$$

It is proved that $Z^p(T)$ is a Banach space (e.g. [8, Section 4.1]).

After defining the stochastic integral, let us return to the stochastic wave equation. Since we are working in dimension $d = 1$, the Green’s function associated with (1.1) is

$$
G_t(x) = \frac{1}{2} \mathbf{1}_{\{|x| < t\}}, \quad t \in \mathbb{R}_+, x \in \mathbb{R}.
$$

Notice that $G_t(x)$ does not satisfy semigroup property.

Now we give the definitions of strong and weak solutions to (1.1).

**Definition 2.4.** Let $\{u(t, x), t \geq 0, x \in \mathbb{R}\}$ be a real-valued adapted random field such that for all fixed $t \in [0, T]$ and $x \in \mathbb{R}$, the random field

$$
\{G_{t-s}(x-y)\sigma(u(s, y)) \mathbf{1}_{[0, t]}(s), (s, y) \in \mathbb{R}_+ \times \mathbb{R}\}
$$

is integrable with respect to $W$ (namely it is in $\Lambda_H$).
(i) We say that \( u(t,x) \) is a strong (mild, random field) solution to (1.1) if for all \( t \in [0,T] \) and \( x \in \mathbb{R} \) we have almost surely
\[
u(t,x) = \frac{\partial}{\partial t} G_t \ast u_0(x) + G_t \ast v_0(x) + G_t \otimes \sigma(\cdot,\cdot,u)(x)
\]
\[
= I_0(t,x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y) \sigma(s,y,u(s,y)) W(ds,dy),
\]
where
\[
I_0(t,x) := G_t \ast v_0(x) + \frac{\partial}{\partial t} G_t \ast u_0(x)
\]
\[
\frac{1}{2} \int_{x-t}^{x+t} v_0(y) dy + \frac{1}{2} [u_0(x+t) + u_0(x-t)].
\]

(ii) We say (1.1) has a weak solution if there exists a probability space with a filtration \((\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)\), an \( \mathcal{F}_t \)-adapted Gaussian random field \( \tilde{W} \) identical to \( W \) in law, and an \( \mathcal{F}_t \)-adapted random field \( \{u(t,x), (t,x) \in \mathbb{R}_+ \times \mathbb{R}\} \) on this probability space \((\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)\) such that \( u(t,x) \) is a mild solution with respect to \((\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)\) and \( W \).

To obtain the existence and uniqueness of strong (mild) solution to (1.1), we make the following assumptions on \( \sigma \).

**H1** \( \sigma(t,x,u) \) is jointly continuous over \([0,T] \times \mathbb{R}^2 \), \( \sigma(t,x,0) = 0 \), and it is Lipschitz in \( u \) (uniformly in \( t \) and \( x \)). This means \( \forall u,v \in \mathbb{R} \)
\[
\sup_{t \in [0,T], x \in \mathbb{R}} |\sigma(t,x,u) - \sigma(t,x,v)| \leq C|u-v|,
\]
for some constant \( C > 0 \).

One easily observes that the hypothesis (2.10) and the condition \( \sigma(t,x,0) = 0 \) imply that
\[
\sup_{t \in [0,T], x \in \mathbb{R}} |\sigma(t,x,u)| \leq C|u|,
\]
for some constant \( C > 0 \).

**H2** Assume \( \frac{\partial}{\partial u} \sigma(t,x,u) \) and \( \frac{\partial^2}{\partial u \partial \sigma} \sigma(t,x,u) \) exist and are uniformly bounded, i.e. there is some constant \( C' > 0 \) such that
\[
\sup_{t \in [0,T], x \in \mathbb{R}, u \in \mathbb{R}} \left| \frac{\partial}{\partial u} \sigma(t,x,u) \right| \leq C';
\]
\[
\sup_{t \in [0,T], x \in \mathbb{R}, u \in \mathbb{R}} \left| \frac{\partial^2}{\partial u \partial \sigma} \sigma(t,x,u) \right| \leq C.
\]

Moreover, we assume
\[
\sup_{t \in [0,T], x \in \mathbb{R}} \left| \frac{\partial}{\partial u} \sigma(t,x,u_1) - \frac{\partial}{\partial u} \sigma(t,x,u_2) \right| \leq C|u_1 - u_2|.
\]

Notice that (2.10) is a consequence of (2.12). But we keep the former one in the assumption (H1) since we shall use (H1) for the existence of the weak solution and (H2) for the existence and uniqueness of the strong solution.

Now we state the main results of this paper.
Stochastic wave equation

Theorem 2.5. Assume that \( \sigma(t, x, u) \) satisfies the hypothesis (H1) and that \( I_0(t, x) \) is in \( Z^p(T) \) for some \( p > \frac{2}{4H-1} \). Then, there exists a weak solution to (1.1) whose sample paths are in \( C([0, T] \times \mathbb{R}) \) almost surely. Moreover, for any \( \gamma < H - \frac{1}{p} \), the process \( u(t, x) \) is almost surely Hölder continuous of exponent \( \gamma \) with respect to \( t \) and \( x \) on any compact subsets of \( [0, T] \times \mathbb{R} \).

Theorem 2.6. Assume that \( \sigma(t, x, u) \) satisfies the hypothesis (H2) and that \( I_0(t, x) \) is in \( Z^p(T) \) for some \( p > \frac{2}{4H-1} \). Then (1.1) has a unique strong solution whose sample paths are in \( C([0, T] \times \mathbb{R}) \) almost surely. Moreover, the random field \( u(t, x) \) is Hölder continuous a.s. on compact subsets of \( [0, T] \times \mathbb{R} \) with the same exponent as in Theorem 2.5.

Theorem 2.7. If the hyperbolic Anderson model (1.3) has a solution in \( Z^p(T) \) for some \( p \geq 2 \) and for some \( T > 0 \), then the Hurst parameter \( H \) must satisfy \( H > 1/4 \).

3. Uniform moment bounds

In this section, we obtain the uniform moment estimates of the stochastic convolution with the noise \( \dot{W} \) which appears in the definition of the mild solution. These estimates are used later on to prove the existence and uniqueness of solution to SWE (1.1).

3.1. Uniform moment bounds of stochastic convolution. Define

\[
\Phi(t, x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)v(s,y)W(ds,dy), \tag{3.1}
\]

where \( G_t(x) \) is the Green’s function associated with the wave operator (1.1), given by (2.7).

As we mentioned before, the major difficulty here is that the wave Green’s function \( G_t(x) \) does not satisfy the semigroup property so that the stochastic Fubini technique used for stochastic heat equation is no longer applicable (see Remark 4.3 in [11]). To get around this obstacle, we decompose it into sum of convolutions of some ‘nice’ kernels. More precisely, we have the following simple and important lemma which is the key starting point of our approach and which plays the role of semigroup property of the heat kernel when the heat equation is investigated (e.g. [8, 11]).

Lemma 3.1. The wave kernel \( G_t(x) = \frac{1}{2}1_{\{|x|<t\}} \) can be expressed as

\[
\begin{align*}
G_{t-s}(x-y) &= \int_{\mathbb{R}} C_\beta(t-r, x-z)S_{1-\beta}(r-s, z-y)dz \\
&\quad + \int_{\mathbb{R}} S_\alpha(t-r, x-z)C_{1-\alpha}(r-s, z-y)dz \\
&\quad + \int_{\mathbb{R}} S(t-r, x-z)E(r-s, z-y)dz \\
&\quad + \int_{\mathbb{R}} E(t-r, x-z)S(r-s, z-y)dz , \tag{3.2}
\end{align*}
\]

7
where \( \alpha, \beta \in (0, 1) \), \( S(t, x) = S_\alpha(t, x) = G(t, x) = \frac{1}{2} \mathbf{1}_{\{|x| < t\}} \) and

\[
\begin{aligned}
\mathcal{E}(t, x) &:= \frac{1}{\pi t^{1+x}}, \\
S_\alpha(t, x) &:= \frac{\Gamma(1-\alpha)}{2\pi} \cos \left( \frac{2\pi}{} \right) \left[ (t + |x|)^{\alpha-1} + \text{sgn}(t - |x|) |t - |x||^{\alpha-1} \right], \\
C_{1-\alpha}(t, x) &:= \frac{\Gamma(\alpha)}{2\pi} \left[ \cos \left( \frac{2\pi}{} \right) \left[ |t + |x||^{\alpha} + |t - |x||^{-\alpha} \right] \\
&\quad - 2 \cos \left( \alpha \tan^{-1} \left( \frac{|t|}{|x|} \right) \right) |t^2 + x^2|^{-\frac{\beta}{2}} \right].
\end{aligned}
\] (3.3)

**Proof.** We prove (3.2) via Fourier transform

\[ \hat{f}(\xi) = \mathcal{F}[f(x)] = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx, \quad \text{where} \quad i = \sqrt{-1}. \]

The Fourier transform of \( G_{t+s}(x) \) is

\[ \hat{G}_{t+s}(\xi) = \frac{\sin((t+s)|\xi|)}{|\xi|}. \]

We can decompose \( \hat{G}_{t+s}(\xi) \) into the summation of following four items:

\[
\begin{aligned}
\hat{G}_{t+s}(\xi) &= \frac{\sin(t|\xi|) \cos(s|\xi|)}{|\xi|} + \frac{\sin(s|\xi|) \cos(t|\xi|)}{|\xi|} \\
&= \frac{\sin(t|\xi|)}{|\xi|^\alpha} \cdot \frac{\cos(s|\xi|) - e^{-s|\xi|}}{|\xi|^{1-\alpha}} + \frac{\sin(t|\xi|)}{|\xi|} \cdot e^{-s|\xi|} \\
&\quad + \frac{\sin(s|\xi|)}{|\xi|^{\beta}} \cdot \frac{\cos(t|\xi|) - e^{-t|\xi|}}{|\xi|^{1-\beta}} + \frac{\sin(s|\xi|)}{|\xi|} \cdot e^{-t|\xi|}.
\end{aligned}
\]

On the other hand, the Fourier transforms of \( \mathcal{E}(t, x), S_\alpha(t, x) \) and \( C_{1-\alpha}(t, x) \) are given as follows (see Lemma A.1):

\[
\hat{\mathcal{E}}(t, \xi) = e^{-t|\xi|}, \quad \hat{S}_\alpha(t, \xi) = \frac{\sin(t|\xi|)}{|\xi|^\alpha}; \quad \hat{C}_{1-\alpha}(t, \xi) = \frac{\cos(t|\xi|) - e^{-t|\xi|}}{|\xi|^{1-\alpha}}. \] (3.4)

We then conclude the proof of (3.2) by the fact the Fourier transformation transforms the convolution to product. \( \square \)

**Remark 3.2.** Readers may wonder why we don’t use the following simpler decomposition as we originally attempted:

\[
\hat{G}_{t+s}(\xi) = \frac{\sin((t+s)|\xi|)}{|\xi|} \\
= \frac{\sin(t|\xi|)}{|\xi|} \cos(s|\xi|) + \frac{\sin(s|\xi|)}{|\xi|} \cos(t|\xi|) \\
= \frac{\sin(t|\xi|)}{|\xi|^\alpha} \cdot \frac{\cos(s|\xi|)}{|\xi|^{1-\alpha}} + \frac{\cos(t|\xi|)}{|\xi|^\beta} \cdot \frac{\sin(s|\xi|)}{|\xi|^{1-\beta}}.
\]

The reason is that the following quantity

\[ C_\beta(t, x) := \mathcal{F}^{-1} \left[ \frac{\cos(t|\xi|)}{|\xi|^\beta} \right] = c_\beta \left[ (t + |x|)^{\beta-1} + |t - |x||^{\beta-1} \right] \]

is not integrable. \( w.r.t. \ x \in \mathbb{R} \) when \( 0 \leq \beta \leq 1 \).
Analogously to idea used in [8], we shall seek the solution of (1.1) in the space $Z^p(T)$. To this end we need to bound the $\| \cdot \|_{Z^p(T)}$ norm of the stochastic convolution $\Phi(t, x)$ defined by (3.1) and its variant $N_{\frac{p}{2}-H}\Phi(t, x)$ as stated in the following theorem.

**Proposition 3.3.** For the stochastic convolution $\Phi(t, x)$, we have the following estimates:

(i) If $p > \frac{1}{H}$, then
\[
\left\| \sup_{t \in [0, T], x \in \mathbb{R}} |\Phi(t, x)| \right\|_{L^p(\Omega)} \leq C_{T, p, H} \|v\|_{Z^p(T)}.
\] (3.5)

(ii) If $p > \frac{2}{3H-1}$, then
\[
\left\| \sup_{t \in [0, T], x \in \mathbb{R}} \left| N_{\frac{p}{2}-H}\Phi(t, x) \right| \right\|_{L^p(\Omega)} \leq C_{T, p, H} \|v\|_{Z^p(T)}.
\] (3.6)

**Proof.** We shall use Lemma 3.1 to prove this proposition. We divide the proof into two steps.

**Step 1:** In this step, we shall prove part (i) of the proposition. For any $\theta \in (0, 1)$ and $i = 1, 2, 3, 4$, set
\[
J^i_\theta(r, z) := \int_0^r \int_\mathbb{R} (r - s)^{-\theta} \mathcal{K}_i(r - s, z - y)v(s, y)W(dy, ds),
\] (3.7)

where
\[
\mathcal{K}_1 = \mathcal{C}, \quad \mathcal{K}_2 = \mathcal{S}, \quad \mathcal{K}_3 = \mathcal{S}_1, \quad \text{and} \quad \mathcal{K}_4 = \mathcal{E}.
\] (3.8)

And we define $\bar{\mathcal{K}}_i$ to be the complements of $\mathcal{K}_i$ according to (3.2), namely,
\[
\bar{\mathcal{K}}_1 = \mathcal{S}_{1-\alpha}, \quad \bar{\mathcal{K}}_2 = \mathcal{C}_{1-\alpha}, \quad \bar{\mathcal{K}}_3 = \mathcal{E}, \quad \text{and} \quad \bar{\mathcal{K}}_4 = \mathcal{S}.
\] (3.9)

Let us set
\[
\Phi_i(t, x) := \frac{\sin(\theta\pi)}{\pi} \int_0^t \int_\mathbb{R} (t - r)^{\theta - 1} \bar{\mathcal{K}}_i(t - r, x - z)J^i_\theta(r, z)dzdr, \quad i = 1, 2, 3, 4.
\]

Then a stochastic version of Fubini’s theorem and Lemma 3.1 yield
\[
\Phi(t, x) = \int_0^t \int_\mathbb{R} G_{t-s}(x - y)v(s, y)W(ds, dy)
\]
\[
= \frac{\sin(\theta\pi)}{\pi} \int_0^t \int_\mathbb{R} \int_0^s (t - r)^{\theta - 1}(r - s)^{-\theta} dr \times G_{t-s}(x - y)v(s, y)W(dy, ds)
\]
\[
= \sum_{i=1}^4 \frac{\sin(\theta\pi)}{\pi} \int_0^t \int_\mathbb{R} \int_0^s \int_\mathbb{R} (t - r)^{\theta - 1}(r - s)^{-\theta}
\]
\[
\times \bar{\mathcal{K}}_i(t - r, x - z)\mathcal{K}_i(r - s, z - y)dzdr \times v(s, y)W(dy, ds)
\]
\[
= \sum_{i=1}^4 \frac{\sin(\theta\pi)}{\pi} \int_0^t \int_\mathbb{R} (t - r)^{\theta - 1} \bar{\mathcal{K}}_i(t - r, x - z)J^i_\theta(r, z)dzdr
\]
\[
= \sum_{i=1}^4 \Phi_i(t, x),
\] (3.10)

where we have applied the identity
\[
\int_s^t (t - r)^{\theta - 1}(r - s)^{-\theta} dr = \frac{\pi}{\sin(\theta\pi)} \quad \theta \in (0, 1), \quad 0 \leq s \leq t.
\]
This expression is essential for us to derive the desired estimates. In the following, we will use \(\sum_i\) to denote \(\sum_{i=1}^4\) and \(\sup\) to denote \(\sup_{t,x}\).

It is clear by the Hölder inequality with \(1/p + 1/q = 1\) that for \(i = 1, \cdots, 4\)
\[
\sup_{t,x} |\Phi_i(t,x)| \lesssim \sup_{t,x} \int_0^t (t-r)^{\theta-1} \left( \int_{\mathbb{R}} |\tilde{K}_i(t-r,x-z)|^q dz \right)^{\frac{1}{q}} 
\times \|J_{\theta}^{K_i}(r,z)\|_{L_p(\mathbb{R})} dr
\lesssim \left( \sup_{t} \int_0^t \int_{\mathbb{R}} r^{q(\theta-1)} |\tilde{K}_i(r,z)|^q dz dr \right)^{\frac{1}{q}}
\times \left( \int_0^T \|J_{\theta}^{K_i}(r,z)\|_{L_p(\mathbb{R})}^p dr \right)^{\frac{1}{p}}
= (I_1^{(1)})^{1/q} \times (I_1^{(2)})^{1/p},
\] (3.11)
where we change the variables \(r \to t-r\) and \(z \to x-z\) in the second inequality and then it is clear that \(\sup_{t,x}\) becomes \(\sup_t\) thanks to the translation invariance in space variable of the function. This technique will be freely used in the sequel without mention. We shall deal with \(I_1^{(1)}, I_1^{(2)}, i = 1, \cdots, 4\), term by term in the subsequent paragraphs.

First, let us deal with \(I_1^{(1)}\) when \(i = 1, 2\). The cases \(i = 3, 4\) can be treated similarly. When \(i = 1\), \(K_1 = \mathcal{C}_\alpha\) and \(\tilde{K}_1 = \mathcal{S}_1 - \alpha\) defined as (3.3). By the change of variable \(z \to rz\), it is easy to see \(I_1^{(1)}\) can be bounded as
\[
I_1^{(1)} = \sup_{t} \int_0^t \int_{\mathbb{R}} r^{q(\theta-1)} |\mathcal{S}_{1-\alpha}(r,z)|^q dz dr
\lesssim \left[ \sup_{t} \int_0^t \int_{\mathbb{R}} r^{q(\theta-1-\alpha+\frac{1}{q})} dr \right] \times \int_0^\infty \left| (1+|z|)^{-\alpha} + \text{sgn}(1-|z|) (1-|z|)^{-\alpha} \right|^q dz.
\]
In order to make sure the above integrals converge, we need
\[
a q < 1, \quad (\alpha + 1) q > 1 \quad \iff \quad 0 < \alpha < \frac{1}{q} = 1 - \frac{1}{p},
\] (3.12)
and also
\[
q \left( \theta - \alpha - 1 + \frac{1}{q} \right) > -1 \quad \iff \quad \theta > 1 - \frac{2}{q} + \alpha.
\] (3.13)

When \(i = 2\), \(K_2 = \mathcal{S}_\alpha\) and \(\tilde{K}_2 = \mathcal{C}_{1-\alpha}\) which are defined in (3.3), we have
\[
I_2^{(1)} = \sup_{t} \int_0^t \int_{\mathbb{R}} r^{q(\theta-1)} |\mathcal{C}_{1-\alpha}(r,z)|^q dz dr
\lesssim \left[ \sup_{t} \int_0^t \int_{\mathbb{R}} r^{q(\theta-1-\alpha+\frac{1}{q})} dr \right]
\times \int_0^\infty \left[ \cos \left( \frac{\alpha \pi}{2} \right) \left( |1+|z||^{-\alpha} + |1-|z||^{-\alpha} \right) - 2 \cos \left( \alpha \tan^{-1}(z) \right) \left( 1 + z^2 \right)^{-\frac{q}{2}} \right] dz.
\] (3.15)
Stochastic wave equation

By Lemma A.1 in the Appendix A, $C_{1-\alpha}(r,z)$ can be bounded by
\[
|C_{1-\alpha}(r,z)| \lesssim \left\{ \begin{array}{ll}
|r + |z||^{-\alpha} + |r - |z||^{-\alpha} + [r^2 + |z|^2]^{-\frac{\alpha}{2}} & \text{if } |z| \approx r, \\
(r(|z|^2 - r^2))^{-\frac{\alpha}{2} - 1} & \text{if } |z| \approx \infty.
\end{array} \right.
\]

Thus, in order to make sure (3.14) is bounded, we need
\[
aq < 1, \ (\alpha + 2)q > 1 \iff 0 < \alpha < \frac{1}{q},
\]
and
\[
q \left( \theta - \alpha - 1 + \frac{1}{q} \right) > -1 \iff \theta > 1 - \frac{2}{q} + \alpha.
\]

Therefore, to prove part (i) of the proposition we only need to show
\[
\mathbb{E}||J^\alpha_\theta(r,:)||_{L^p(R)} \leq C||v||_{L^p(T)}, \quad i = 1, 2, 3, 4.
\]

This is objective of Lemma B.1, proved in the Appendix B under the following condition:
\[
p > \frac{1}{H} \cdot \frac{1 - \frac{2}{q} + \alpha}{\alpha} < H + \alpha - \frac{1}{2}, \quad 1 - H < \alpha < 1 - \frac{1}{p}.
\]

Therefore, when $p > \frac{1}{H}$, we can choose $\alpha$ such that $1 - H < \alpha < 1 - \frac{1}{p}$, and then
we see (3.12), (3.13), and (3.18) are satisfied since $\frac{1}{H} > \frac{1}{2H - 1}$ if $H < \frac{2}{3}$. Thus we
have proved (i) of the proposition for $\Phi_1(t,x)$, $\Phi_2(t,x)$. The cases for $\Phi_3(t,x)$ and
$\Phi_4(t,x)$ can be proved similarly. Thus, we complete the proof of part (i) of the
proposition.

**Step 2:** Let us now consider part (ii) of the proposition. In order to obtain the
desired decay rate of $J^\lambda_\theta \Phi(t,x)$, we still use the equation (3.10) to express $\Phi(t,x)$ by $J^\alpha_\theta$. Recall our notation $D_h \Phi(t,x) := \Phi(t,x+h) - \Phi(t,x)$ and same notations
for $D_h \bar{K}_i(t-r,z)$, $D_h J^\alpha_\theta(r,z)$. Then
\[
D_h \Phi(t,x) = \frac{\sin(\theta \pi)}{\pi} \sum_i \int_0^t \int_R (t-r)^{\theta-1} D_h \bar{K}_i(t-r,x-z) J^\alpha_\theta(r,z) dz dr
\]
\[
\simeq \sum_i \int_0^t \int_R (t-r)^{\theta-1} \bar{K}_i(t-r,x-z) D_h J^\alpha_\theta(r,z) dz dr,
\]
with the choice of $K$ and $\bar{K}$ defined by (3.8) and (3.9). Invoking Minkowski’s
inequality and then Hölder’s inequality we get
\[
\sup_{t,x} \left( \int_R |D_h \Phi(t,x)|^2 |h|^{2H-2} dh \right)^{\frac{1}{2}}
\]
\[
\lesssim \sup_{t,x} \sum_i \left( \int_R \left| \int_0^t \int_R (t-r)^{\theta-1} \bar{K}_i(t-r,x-z)
\right|^2 |h|^{2H-2} dh \right)^{\frac{1}{2}}
\]
\[
\lesssim \sup_{t,x} \sum_i \int_0^t \int_R (t-r)^{\theta-1} |\bar{K}_i(t-r,x-z)|.
\]
\[
\sum_i \left( \sup_t \int_0^t r^{(\theta-1)} |K_i(t, z)|^q dzdr \right)^rac{1}{q} \times \left( \int_0^T \int_\mathbb{R} \sqrt{2} |h^{2H-2} Eh_{hJ}^{K_i}(r, z)|^\frac{p}{q} dzdr \right)^rac{p}{q} =: (J_1^{(i)})^{\frac{1}{q}} \times (J_2^{(i)})^{\frac{p}{q}}. \tag{3.20}
\]

The first factor \((J_1^{(i)})^{\frac{1}{q}}\) in (3.20) is finite if we require that \(\alpha, \theta, p, q\) satisfy (3.12) and (3.13). Therefore we only need to focus on the second factor \((J_2^{(i)})^{\frac{p}{q}}\) in (3.20). By Lemma B.2, we see

\[
\mathbb{E} \int_\mathbb{R} \left[ \int_\mathbb{R} \sqrt{2} |h^{2H-2} Eh_{hJ}^{K_i}(r, z)|^\frac{p}{q} dz \right] dz \leq C_{T, p, \alpha, \theta} \|v\|_{L^p(\Omega, T)},
\]

under the conditions

\[
p > \frac{1}{H}, \quad 1 - 2/q + \alpha < \theta < 2H + \alpha - 1, \quad \frac{3}{2} - 2H < \alpha < 1 - \frac{1}{2p}. \tag{3.21}
\]

If \(p > \frac{2}{4H-1}\), then we can choose \(\alpha\) such that \(\frac{3}{2} - 2H < \alpha < 1 - \frac{1}{p}\), and then we see (3.12), (3.13) and (3.21) are satisfied since \(\frac{2}{4H-1} > \frac{1}{H}\) when \(H < \frac{1}{2}\). Thus, we complete the proof of part (ii) of the proposition. \(\square\)

3.2. Uniform moment bounds of the approximate solutions. We approximate the noise \(W\) by the following smoothing of the noise with respect to the space variable. That is, for \(\varepsilon > 0\) we define

\[
\frac{\partial}{\partial x} W_\varepsilon(t, x) = \int_\mathbb{R} \rho_\varepsilon(x - y) W(t, dy), \tag{3.22}
\]

where \(\rho_\varepsilon(x) = \frac{1}{\sqrt{2\pi \varepsilon}} \exp(-\frac{x^2}{2\varepsilon})\). The regulated noise \(W_\varepsilon\) induces an approximation of mild solution

\[
u_\varepsilon(t, x) = I_0(t, x) + \int_0^t \int_\mathbb{R} G_{t-s}(x-y) \sigma(s, y, u_\varepsilon(s, y)) W_\varepsilon(ds, dy), \tag{3.23}
\]

where the stochastic integral is understood in the Itô sense. Due to the regularity in space of the noise, the existence and uniqueness of the solution \(u_\varepsilon(t, x)\) to above equation is standard (even the higher dimensional case were known (e.g. [10, 15] and references therein).

The lemma below asserts that the approximate solution \(\{u_\varepsilon(t, x) : \varepsilon > 0\}\) is uniformly bounded in the space \(Z^p(T)\). More precisely, we have

**Lemma 3.4.** Let \(H \in (\frac{1}{4}, \frac{1}{2})\). Assume that \(\sigma(t, x, u)\) satisfies the hypothesis (H1) and assume that \(I_0(t, x)\) is in \(Z^p(T)\). Then the approximate solutions \(u_\varepsilon\) satisfy

\[
\sup_{\varepsilon > 0} \|u_\varepsilon\|_{Z^p(T)} := \sup_{\varepsilon > 0} \|u_\varepsilon(t, \cdot)\|_{Z^p(T)} + \sup_{\varepsilon > 0} \|u_\varepsilon(t, \cdot)\|_{Z^2(T)} < \infty. \tag{3.24}
\]

**Proof.** For notational simplicity we assume \(\sigma(t, x, u) = \sigma(u)\) without loss of generality because of hypothesis (H1). We shall use some thoughts similar to those in [11]. To this end, we define the Picard iteration as follows:

\[
u_\varepsilon^0(t, x) = I_0(t, x),
\]

...
and recursively for \( n = 0, 1, 2, \cdots \),

\[
u^{n+1}_\varepsilon(t, x) = I_0(t, x) + \int_0^t \int_\mathbb{R} G_{t-s}(x-y)\sigma(u^n_\varepsilon(s, y))W(ds, dy) . \tag{3.25}
\]

From [9, Lemma 4.12] it follows that for any fixed \( \varepsilon > 0 \) when \( n \) goes to infinity, the sequence \( u^n_\varepsilon(t, x) \) converges to \( u_\varepsilon(t, x) \) a.s. In the following steps 1 and 2, we shall first bound \( \|u^n_\varepsilon\|_{\mathcal{Z}^p(T)} \) uniformly in \( n \), and \( \varepsilon \). Then, in step 3 we use Fatou’s lemma to show (3.24).

In the following, we will continue to use the notations \( \mathcal{D}_h f(t, x) \) and \( \square_{h, l} f(t, x) \) previously defined in (1.5) and (1.6).

**Step 1.** In this step, we bound the \( L^p(\Omega \times \mathbb{R}) \) norm of \( u^{n+1}_\varepsilon(t, x) \) by the \( \mathcal{Z}^p \) norm of \( u^n_\varepsilon(t, x) \). Rewrite (3.25) as

\[
u^{n+1}_\varepsilon(t, x) = I_0(t, x) + \int_0^t \int_\mathbb{R} \left[ \left(G_{t-s}(x-y)\sigma(u^n_\varepsilon(s, y))\right) + G_{s-t}(y)\right] W(ds, dy) .
\]

Using \( e^{-c|t|^2} \leq 1 \) and the condition (2.11) on \( \sigma \), we have from the Burkholder-Davis-Gundy inequality (2.5)

\[
\mathbb{E}\left[|u^{n+1}_\varepsilon(t, x)|^p \right] \\
\leq C_p |I_0(t, x)|^p + C_p \mathbb{E}\left( \int_0^t \int_\mathbb{R} \mathbb{E}\left[ \left|G_{t-s}(x-y)\sigma(u^n_\varepsilon(s, y))\right|^2 \right] e^{-c|\xi|^2} |\xi|^{1-2H} d\xi ds \right)^{\frac{p}{2}} \\
\leq C_p |I_0(t, x)|^p + C_p \mathbb{E}\left( \int_0^t \int_\mathbb{R} \left|G_{t-s}(x-y-h)\sigma(u^n_\varepsilon(s, y+h)) - G_{t-s}(x-y)\sigma(u^n_\varepsilon(s, y))\right|^2 |h|^{2H-2} d\xi ds \right)^{\frac{p}{2}} \\
\leq C_p \left[ |I_0(t, x)|^p + |\mathcal{D}^{\varepsilon,n}_1(t, x)|^{\frac{p}{2}} + |\mathcal{D}^{\varepsilon,n}_2(t, x)|^{\frac{p}{2}} \right] , \tag{3.26}
\]

where we have used the notations \( \mathcal{D}^{\varepsilon,n}_1(t, x) \) and \( \mathcal{D}^{\varepsilon,n}_2(t, x) \) similar to (B.2) and (B.3), namely,

\[
\mathcal{D}^{\varepsilon,n}_1(t, x) := \int_0^t \int_{\mathbb{R}^2} |\mathcal{D}_h G_{t-s}(y)|^2 \cdot \|u^n_\varepsilon(s, x+y)\|_{L^p(\Omega)}^2 |h|^{2H-2} d\xi ds ,
\]

and

\[
\mathcal{D}^{\varepsilon,n}_2(t, x) := \int_0^t \int_{\mathbb{R}^2} |G_{t-s}(y)||\mathcal{D}_h u^n_\varepsilon(t, x+y)\|_{L^p(\Omega)}^2 |h|^{2H-2} d\xi ds .
\]

This means

\[
\|u^{n+1}_\varepsilon(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2 = \left( \int_\mathbb{R} \mathbb{E}\left[|u^{n+1}_\varepsilon(t, x)|^p \right] dx \right)^{\frac{2}{p}} \\
\leq C_p \left[ |I_0(t, x)|_{L^p(\Omega \times \mathbb{R})}^2 + \mathcal{D}^{\varepsilon,n}_1(t) + \mathcal{D}^{\varepsilon,n}_2(t) \right] , \tag{3.27}
\]

where \( \mathcal{D}^{\varepsilon,n}_1(t) \) and \( \mathcal{D}^{\varepsilon,n}_2(t) \) are defined and can be bounded similar to the argument used in the proof of Lemma B.1:

\[
\mathcal{D}^{\varepsilon,n}_1(t) := \left( \int_\mathbb{R} |\mathcal{D}^{\varepsilon,n}_1(t, x)|^{\frac{2}{p}} dx \right)^{\frac{p}{2}} \leq C_{p, H} \int_0^t (t-s)^{2H} \|u^n_\varepsilon(s, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2 ds , \tag{3.28}
\]
For the term \( J \).

Thus, by Minkowski’s inequality we have

\[
\| u_{n+1}^\varepsilon(t, \cdot) \|^2_{L^p(\Omega \times \mathbb{R})} \leq C_{p,H} \left( \| I_0(t,x) \|^2_{L^p(\Omega \times \mathbb{R})} + \int_0^t (t-s)^{2H} \| u_n^\varepsilon(s, \cdot) \|^2_{L^p(\Omega \times \mathbb{R})} ds 
+ \int_0^t (t-s) \left[ N_{H,p}^* u_n^\varepsilon(s) \right]^2 ds \right). \tag{3.30}
\]

**Step 2.** Next, we bound \( N_{H,p}^* u_n^\varepsilon(t,x) \) by the \( L^p \) norm of \( u_n^\varepsilon(t,x) \). Similar to (3.26) we have

\[
\mathbb{E} \left[ \| \mathcal{D}_h u_{n+1}^\varepsilon(t,x) \|^p \right] \leq C_p | I_0(t,x) - I_0(t,x+h) |^p 
+ C_p \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} \left| \mathcal{D}_h G_{t-s}(x-y-z) \sigma(u_n^\varepsilon(s,y+z)) \right|^2 |y|^{2H-2} dz dy ds \right)^{\frac{p}{2}} 
\leq C_p [ I_0(t,x,h) + I_{1,n}^\varepsilon(t,x,h) + I_{2,n}^\varepsilon(t,x,h) ],
\]

where

\[
I_0(t,x,h) := | I_0(t,x) - I_0(t,x+h) |^p,
\]

\[
I_{1,n}^\varepsilon(t,x,h) := \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} \left| \mathcal{D}_h G_{t-s}(x-y-z) \right|^2 | \mathcal{D}_y \sigma(u_n^\varepsilon(s,z)) |^2 |y|^{2H-2} dz dy ds \right)^{\frac{p}{2}},
\]

and

\[
I_{2,n}^\varepsilon(t,x,h) := \mathbb{E} \left( \int_0^t \int_{\mathbb{R}^2} \left| \Box_{y,h} G_{t-s}(x-y-z) \right|^2 | \sigma(u_n^\varepsilon(s,z)) |^2 |y|^{2H-2} dz dy ds \right)^{\frac{p}{2}}.
\]

Thus, by Minkowski’s inequality we have

\[
\left[ N_{H,p}^* u_{n+1}^\varepsilon(t) \right]^2 = \int_{\mathbb{R}} \| \mathcal{D}_h u_{n+1}^\varepsilon(t,x) \|^2_{L^p(\mathbb{R} \times \Omega)} |h|^{2H-2} dh 
\leq C_p \sum_{j=0}^2 \int_{\mathbb{R}} \left( \int_{\mathbb{R}} I_{j,n}^\varepsilon(t,x,h) dx \right)^{\frac{p}{2}} |h|^{2H-2} dh 
=: J_0 + J_1 + J_2.
\]

For the term \( J_0 \), it is clear that

\[
J_0 = C_p \int_{\mathbb{R}} \left( \int_{\mathbb{R}} | \mathcal{D}_h I_0(t,x) |^p dx \right)^{\frac{p}{2}} |h|^{2H-2} dh = \left[ N_{H,p}^* I_0(t) \right]^2. \tag{3.31}
\]

We can deal with the term \( J_1 \) in the similar manner as that when we deal with (B.14) in the proof of Lemma B.2. An application of Minkowski’s inequality and
then an application of Parseval’s formula yield
\[ J_1 \leq C_{p,H} \int_0^t \int_{\mathbb{R}^2} |\mathcal{D}_h G_{t-s}(z)|^2 |h|^{2H-2}dh \, dz \]
\[ \times \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \mathbb{E} \left[ |\mathcal{D}_y u^n_\varepsilon(t, x)|^p \right] dx \right)^{\frac{2}{p}} |y|^{2H-2}dy \, ds \]  
(3.32)
\[ \leq C_{p,H} \int_0^t (t-s)^{2H} \left[ \mathcal{N}_{\frac{1}{2}-H,p} u^n_\varepsilon(s) \right]^2 ds . \]

Next, we bound \( J_2 \). By the condition \((2.11)\) \(|\sigma(u)| \lesssim |u|\) and by a change of variable \( z \rightarrow x - z \), we obtain
\[ J_2 := \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} \mathcal{I}_2^{z,n}(t, x, h) dx \right)^{\frac{2}{p}} |h|^{2H-2}dh \]
\[ \leq C_{p,H} \int_0^t \int_{\mathbb{R}^3} |\mathcal{D}_y G_{t-s}(z)|^2 |y|^{2H-2}dy|h|^{2H-2}dh \, dz \]
\[ \times \left( \int_{\mathbb{R}} \mathbb{E} |u^n_\varepsilon(s, x - z)|^p dx \right)^{\frac{2}{p}} ds \]
\[ \leq C_{p,H} \int_0^t (t-s)^{4H-1} \| u^n_\varepsilon(s, \cdot) \|^2_{L^p(\Omega \times \mathbb{R})} ds . \]

Thus, we obtain
\[ \left[ \mathcal{N}_{\frac{1}{2}-H,p} u^{n+1}_\varepsilon(t) \right]^2 \leq C_{p,H} \left[ \mathcal{N}_{\frac{1}{2}-H,p} I_0(t) \right]^2 + C_{p,H} \int_0^t (t-s)^{2H} \left[ \mathcal{N}_{\frac{1}{2}-H,p} u^n_\varepsilon(s) \right]^2 ds \]
\[ + C_{p,H} \int_0^t (t-s)^{4H-1} \| u^n_\varepsilon(s, \cdot) \|^2_{L^p(\Omega \times \mathbb{R})} ds . \]
(3.34)

**Step 3.** Set
\[ \Psi^n_\varepsilon(t) := \| u^n_\varepsilon(t, \cdot) \|^2_{L^p(\Omega \times \mathbb{R})} + \left[ \mathcal{N}_{\frac{1}{2}-H,p} u^n_\varepsilon(t) \right]^2 . \]

Then combining the estimates (3.30) and (3.34) yields
\[ \Psi^{n+1}_\varepsilon(t) \leq C_{T,p,H} \left( \| I_0 \|^2_{L^p(T)} + \int_0^t (t-s)^{4H-1} \Psi^n_\varepsilon(s) ds \right) . \]

Now it is relatively easy to see by fractional Gronwall lemma (similar to [2, Lemma A.2])
\[ \sup_{\varepsilon > 0} \sup_{n \geq 1} \sup_{t \in [0,T]} \Psi^n_\varepsilon(t) \leq C_{T,p,H} < \infty . \]

Thus, by the same argument as in the proof of [11, Lemma 4.5], we have that \( \sup_{\varepsilon > 0} \| u_\varepsilon(t, \cdot) \|_{L^p(\Omega \times \mathbb{R})} \) and \( \sup_{\varepsilon > 0} \mathcal{N}_{\frac{1}{2}-H,p} u_\varepsilon(t) \) are finite.

In conclusion, we have proved \( \sup_{\varepsilon > 0} \| u_\varepsilon \|_{L^p(T)} := \sup_{\varepsilon > 0} \sup_{t \in [0,T]} \| u_\varepsilon(t, \cdot) \|_{L^p(\Omega \times \mathbb{R})} + \sup_{\varepsilon > 0} \mathcal{N}_{\frac{1}{2}-H,p} u_\varepsilon(t) \) is finite.
4. Hölder continuity and well-posedness

In this section, we obtain some estimations which imply the Hölder regularity of the stochastic convolution with respect to our noise \( \dot{W} \). Then the similar estimations of the solution to SWE (1.1) follow in a routine way. These estimations are devoted to prove the tightness associated with the solution to (1.1).

4.1. Hölder continuity of stochastic convolution. We have the following regularity results for stochastic convolution \( \Phi(t,x) \) defined by (3.1) and the approximated solution \( u_\varepsilon \) defined by (3.23).

**Proposition 4.1.** Let \( v(\cdot,\cdot) \in Z^p(T) \) and let the stochastic convolution \( \Phi(t,x) \) be defined by (3.1). We have the following Hölder regularity in the space and time variables for \( \Phi(t,x) \):

(i) If \( p > \frac{1}{H} \) and \( 0 < \gamma < H - \frac{1}{p} \), then

\[
\left\| \sup_{t,t+h \in [0,T], x \in \mathbb{R}} |\Phi(t+h,x) - \Phi(t,x)| \right\|_{L^p(\Omega)} \leq C_{T,p,H,\gamma} |h|^{\gamma} \|v\|_{Z^p(T)}.
\]

(ii) If \( p > \frac{1}{H} \) and \( 0 < \gamma < H - \frac{1}{p} \), then

\[
\left\| \sup_{t \in [0,T], x,y \in \mathbb{R}} |\Phi(t,x) - \Phi(t,y)| \right\|_{L^p(\Omega)} \leq C_{T,p,H,\gamma} |x-y|^{\gamma} \|v\|_{Z^p(T)}.
\]

**Proof.** **Step 1:** In this step, we concentrate on the analysis of the following quantity (we denote \( \sup_{t,t+h \in [0,T], x \in \mathbb{R}} \) by \( \sup_{t,x} \))

\[
\sup_{t,x} |\Delta_h \Phi(t,x)| := \sup_{t,x} |\Phi(t+h,x) - \Phi(t,x)|.
\]

Assuming \( h \in (0,1) \) and \( t \in [0,T] \) such that \( t+h \leq T \), then by the representation formula (3.10) and the triangle inequality we have

\[
\Delta_h \Phi(t,x) = \sum_i \frac{\sin(\pi \theta)}{\pi} \left[ \int_0^{t+h} \int_{\mathbb{R}} (t+h-r)^{\theta-1} \mathcal{K}_i(t+h-r,x,z)J^K_\theta(r,z)drdz - \int_0^t \int_{\mathbb{R}} (t-r)^{\theta-1} \mathcal{K}_i(t-r,x,z)J^K_\theta(r,z)drdz \right]
\]

\[
\leq \sum_{j=1}^3 I_j(t,h,x),
\]

where

\[
I_1(t,h,x) := \sum_i I_1^{(i)}(t,h,x)
\]

\[
:= \sum_i \int_0^t \int_{\mathbb{R}} \Delta_h(t-r)^{\theta-1} \mathcal{K}_i(t-r,x,z)J^K_\theta(r,z)drdz
\]

with \( \Delta_h(t-r)^{\theta-1} := (t+h-r)^{\theta-1} - (t-r)^{\theta-1} \);

\[
I_2(t,h,x) := \sum_i I_2^{(i)}(t,h,x)
\]

\[
:= \sum_i \int_0^t \int_{\mathbb{R}} (t+h-r)^{\theta-1} \Delta_h \mathcal{K}_i(t-r,x,z)J^K_\theta(r,z)drdz,
\]

\[
I_3(t,h,x) := \sum_i I_3^{(i)}(t,h,x)
\]

\[
:= \sum_i \int_0^t \int_{\mathbb{R}} (t-r)^{\theta-1} \mathcal{K}_i(t-r,x,z)J^K_\theta(r,z)drdz.
\]

SHUHUI LIU, YAOZHONG HU, AND XIONG WANG*
Stochastic wave equation

\[ \Delta_h \mathcal{K}_r(t-r,x-z) := \mathcal{K}_r(t+h-r,x-z) - \mathcal{K}_r(t-r,x-z); \]

\[ \mathcal{I}_3(t,h,x) := \sum_i \mathcal{I}_3^{(i)}(t,h,x) \]

\[ := \sum_i \int_t^{t+h} \int_\mathbb{R} (t+h-r)^{\theta-1} \mathcal{K}_r(t+h-r,x-z) J^{\mathcal{K}_r}_r(r,z) dr dz. \]

Our goal is to show that

\[ \| \sup_{t,x} \mathcal{I}_j(t,h,x) \|_{L^p(\Omega)} \leq C_{T,p,H,\gamma} |h|^\gamma \| v \|_{Z^p(T)}, \quad j = 1, 2, 3, \quad (4.3) \]

under the conditions

\[ p > \frac{1}{H}, \quad 1 - H < \alpha < 1 - \frac{1}{p}, \quad \gamma < H - \frac{1}{p}. \quad (4.4) \]

We shall first treat \( \mathcal{I}_1(t,h,x) \) and \( \mathcal{I}_3(t,h,x) \). The term \( \mathcal{I}_2(t,h,x) \) is more complicated and shall be handled lastly.

For the term \( \mathcal{I}_1(t,h,x) \), it is easy to see that for any fixed \( \gamma \in (0,1) \),

\[ \Delta_h(t-r)^{\theta-1} = |(t+h-r)^{\theta-1} - (t-r)^{\theta-1}| \lesssim |t-r|^{\theta-1-\gamma} h^\gamma. \quad (4.5) \]

Then by Hölder’s inequality with \( 1/p + 1/q = 1 \) and Lemma B.1, under conditions (3.18) we have for \( i = 1, \cdot \cdot \cdot , 4 \)

\[ \| \sup_{t,x} \mathcal{I}_i^{(1)}(t,h,x) \|_{L^p(\Omega)} \]

\[ \leq \left( \sup_{t,x} \int_0^t \int_\mathbb{R} |\Delta_h(t-r)^{\theta-1}| |\mathcal{K}_r(t-r,x-z)|^q dz dr \right)^{1/q} \times \| v \|_{Z^p(T)} \]

\[ \leq \left( \sup_{t} \int_0^t \int_\mathbb{R} |r|^{(\theta-1-\gamma)q} |\mathcal{K}_r(r,z)|^q dz dr \right)^{1/q} \times \| v \|_{Z^p(T)} \cdot |h|^\gamma, \quad (4.6) \]

where in the last inequality of (4.6) we have used the change of variables \( r \rightarrow t-r \) and \( z \rightarrow z + x \). Now we only need to show

\[ \sup_{t} \int_0^t \int_\mathbb{R} |r|^{(\theta-1-\gamma)q} |\mathcal{K}_r(r,z)|^q dz dr < +\infty. \]

We shall only discuss the situation \( i = 1 \). Other cases \( i = 2, 3, 4 \) can be treated similarly. For \( i = 1 \), we have \( \mathcal{K}_1 = C_\alpha, \mathcal{K}_1 = S_{1-\alpha} \) as defined in (3.3). Hence, by changing variable \( r \rightarrow r z \) we have

\[ \sup_{t} \int_0^t \int_\mathbb{R} |r|^{(\theta-1-\gamma)q} |S_{1-\alpha}(r,z)|^q dz dr \]

\[ \leq \sup_{t} \int_0^t \int_\mathbb{R} |r|^{(\theta-1-\gamma)q+1-\alpha q} dz \cdot \int_0^\infty |z|^{-\alpha} + \text{sgn}(1-|z|) |1-|z||^{-\alpha} |dz|. \]

Then by the same argument as in the proof of part (i) of Proposition 3.3, we have

\[ \| \sup_{t,x} \mathcal{I}_1(t,h,x) \|_{L^p(\Omega)} \leq C_{T,p,H,\gamma} |h|^\gamma \| v \|_{Z^p(T)} \]

under the conditions (3.18) and \( (\theta-1-\gamma)q+1-\alpha q > -1 \), which can be summarized as the following conditions

\[ p > \frac{1}{H}, \quad 1 - H < \alpha < 1 - \frac{1}{p}, \quad 1 + \alpha - \frac{2}{q} + \gamma < \theta < H + \alpha - \frac{1}{2}. \quad (4.7) \]
Since \( p > \frac{1}{\theta} \) implies \( \gamma < H - \frac{1}{p} < H + \frac{2}{q} - \frac{3}{2} \) it is clear that one can choose \( \alpha \) and \( \theta \) satisfying (4.7) under conditions (4.4).

Now let us deal with the term \( \mathcal{I}_3 \). Using Hölder’s inequality, Lemma B.1 and the change of variables \( z \rightarrow z + x \) and \( r \rightarrow r - t - h \), we have

\[
\| \sup_{t,x} \mathcal{I}_3(t, h, x) \|_{L^p(\Omega)} \leq \sum_i \left( \int_0^h \left[ \int_r (\theta - 1) |K_i (t, x - z)|^q dz \right] dr \right)^{1/q} \times \| v \|_{Z^p(T)}.
\]

We want to show that \( \left( \mathcal{I}_3^{(i)}(h) \right)^{1/q} \lesssim h^\gamma \) for \( i = 1, \ldots, 4 \) with \( p, \alpha, \gamma \) satisfying (4.4). As before, we only consider the case \( i = 1 \), i.e. \( K_1 = C_\alpha, \mathcal{K}_1 = S_{1-\alpha} \). The other cases can be handled in similar way. In this case we have

\[
\left( \sup_{t,x} \mathcal{I}_3^{(1)}(h) \right)^{1/q} = \left( \int_0^h \left[ \int_r (\theta - 1) |S_{1-\alpha}(r, z)|^q dz \right] dr \right)^{1/q}
\]

\[
\leq \left( \int_0^h r^{q(\theta - 1) + 2 - q\alpha} dr \right.
\]

\[
\times \left. \int_0^{+\infty} \left[ (1 + |z|)^{-\alpha} + \text{sgn}(1 - |z|) |1 - |z||^{-\alpha} \right]^q dz \right)^{1/q}
\]

\[
\lesssim \left[ \int_0^h r^{q(\theta - 1) + 2 - q\alpha} \right]^{1/q} \leq h^\gamma
\]

if (4.7) is satisfied (and hence so does (4.4)). We have similar bound for \( \mathcal{I}_3^{(i)}(h) \) for \( i = 2, 3, 4 \). Combing these bounds with (4.8) we have

\[
\| \sup_{t,x} \mathcal{I}_3(t, h, x) \|_{L^p(\Omega)} \leq C_{T,p,H,\gamma} \| h \|_{Z^p(T)},
\]

if \( p, \alpha, \gamma \) satisfy (4.4).

Lastly, we are going to deal with \( \mathcal{I}_2 \), which is much more complicated. By Hölder’s inequality,

\[
\mathcal{I}_2(t, h, x) \leq \sum_i \left( \int_0^t \left[ \int_r (t + h - r)^{q(\theta - 1)} |K_i (t - r, x - z)|^q dz \right] dr \right)^{1/q}
\]

\[
\times \left( \int_0^T \| J^K(r, z) \|_{L^p(\mathbb{R})}^p dr \right)^{1/p}.
\]

The second factor inside the summation in (4.10) can be bounded by a multiple of \( \| v \|_{Z^p(T)} \) via Lemma B.1 under the condition (3.18). By the change of variables
Thus, we shall need to show that for $i = 1, 2, 3, 4$

$$\sup_t \mathcal{I}_2(t, h, x) = \sup_t \int_0^t \int_{\mathbb{R}} (r + h)^{q(\theta - 1)} |\Delta_h \mathcal{K}_i(r, z)|^q dz dr \leq C_{T, p, H, \gamma} |h|^\gamma, \quad (4.11)$$

to obtain

$$\left\| \sup_{t,x} \mathcal{I}_2(t, h, x) \right\|_{L^p(\Omega)} \leq C_{T, p, H, \gamma} |h|^\gamma \| v \|_{L^p(\Omega)}. \quad (4.12)$$

Now, we shall deal with $\mathcal{I}_2^{(i)}(t, h)$ for $i = 1, 2, 3, 4$ term by term.

**Case $i=1$.** Recall that $\mathcal{K}_1(r, z) = \mathcal{S}_{1-\alpha}(r, z)$ and $\mathcal{K}_2(r, z) = \mathcal{C}_{\alpha}(r, z)$ are defined by (3.3). We shall show

$$\left\{ \begin{array}{l}
  \sup_t \mathcal{I}_2^{(1)}(t, h) \leq C_{T, p, H, \gamma} |h|^\gamma, \quad \text{where} \\
  \mathcal{I}_2^{(1)}(t, h) = \int_0^t \int_{\mathbb{R}} (r + h)^{q(\theta - 1)} |\Delta_h \mathcal{S}_{1-\alpha}(r, z)|^q dz dr
\end{array} \right. \quad (4.13)$$

for $p, \gamma$ and $\alpha$ satisfying (4.4). Set $A_1 := \|z\| < r$, $A_2 := \|z\| > r + 2h$ and $A_3 := \|r < |z| < r + 2h\). For fixed $\eta \in (0, 1)$, we see

$$\begin{cases}
  \Delta_h [r + |z|]^{-\alpha} = [r + |z| + h]^{-\alpha} - [r + |z|]^{-\alpha} \lesssim |r + |z||^{-\alpha-\eta} h^{\eta}, & \text{on } \mathbb{R} ; \\
  \Delta_h [r - |z|]^{-\alpha} = [r - |z| + h]^{-\alpha} - [r - |z|]^{-\alpha} \lesssim |r - |z||^{-\alpha-\eta} h^{\eta}, & \text{on } A_1 ; \\
  \Delta_h [r - |z|]^{-\alpha} = [r - |z| + h]^{-\alpha} - [r - |z|]^{-\alpha} \lesssim ||z| - r - h||^{-\alpha-\eta} h^{\eta}, & \text{on } A_2 .
\end{cases} \quad (4.14)$$

Then we have

$$|\Delta_h \mathcal{S}_{1-\alpha}(r, z)|^q \leq \left| \Delta_h [r + |z|]^{-\alpha} \right|^q + \left| \Delta_h [r - |z|]^{-\alpha} \right|^q [1_{A_1} + 1_{A_2}]^q + |r + h - |z| |^{-\alpha} + |r - |z| |^{-\alpha}]^q [1_{A_3}]^q \leq \left| [r + |z|]^{-\alpha} h^{\eta} + |r - |z| |^{-\alpha} h^{\eta} \right| [1_{A_1} + 1_{A_2}]^q + |r + h - |z| |^{-\alpha} + |r - |z| |^{-\alpha}]^q [1_{A_3}]^q ,$$
for some $\eta_1, \eta_2, \eta_3 \in (0, 1)$. Therefore,

$$I_2^{(1)}(t, h) \lesssim \int_0^t \int_\mathbb{R} (r + h)^{q(\theta - 1)} |r + |z||(-\alpha - \eta_1)q_k \eta_{2q} dzdr$$

$$+ \int_0^t \int_\mathbb{R} (r + h)^{q(\theta - 1)} |r - |z||(\alpha - \eta_2)q_k \eta_{3q} \cdot 1_{A_2} dzdr$$

$$+ \int_0^t \int_\mathbb{R} (r + h)^{q(\theta - 1)} |z - r - h|(-\alpha - \eta_3)q_k \eta_{4q} \cdot 1_{A_3} dzdr$$

$$+ \int_0^t \int_\mathbb{R} (r + h)^{q(\theta - 1)} |r + h - |z|||\alpha| + |r - |z||(\alpha - \eta)q_k \eta_{5q} \cdot 1_{A_4} dzdr$$

$$=: \sum_{k=1}^4 I_2^{(1)}(t, h). \tag{4.15}$$

The procedures of dealing terms $I_2^{(1)}(t, h), k = 1, 2, 3, 4$ require standard but careful computations which are included in Appendix C. By Lemma C.1, for any $p > \frac{1}{\gamma}$, $\gamma < H - \frac{1}{p}$, $I_2^{(1)}(t, h) \ (k = 1, 2, 3, 4)$ can be bounded by $h^q$ if $\alpha, \theta$ satisfy (II.1) and $\eta_k, k = 1, 2, 3$ satisfy (C.1).

**Case i=2.** In this case, we have $\mathcal{K}_2(r, z) = C_1 - \alpha(r, z)$ defined by (3.3). We want to show when $i = 2$, i.e.

$$\left\{ \begin{array}{l}
\sup_t I_2^{(2)}(t, h) \leq C_{T, p, \theta, \alpha} |h|^q, \\
I_2^{(2)}(t, h) = \int_0^t \int_\mathbb{R} (r + h)^{q(\theta - 1)} |\Delta_h C_1 - \alpha(r, z)|^q dzdr
\end{array} \right. \tag{4.16}$$

with parameters $p, \gamma$ and $\alpha$ satisfying (4.4).

For fixed $\eta \in (0, 1)$, it is not hard to verify

$$|\Delta_h (r^2 + |z|^2)^{-\frac{\alpha}{2}}| \lesssim (r^2 + |z|^2)^{-\frac{\alpha(1 - \eta)}{2}} \frac{(r + \xi h) \cdot h}{|(r + \xi h)^2 + |z|^2|^{\alpha/2 + 1}}$$

$$\lesssim (r^2 + |z|^2)^{-\frac{\alpha(1 - \eta)}{2}} r^2 + |z|^2)^{-\left(\frac{\alpha}{2} + 1\right) \eta} (r + h)^{\eta} h^\eta$$

$$\lesssim \left(r^2 + |z|^2\right)^{-\frac{\alpha}{2} - \eta} (r + h)^\eta h^\eta, \tag{4.17}$$

and

$$|\Delta_h \cos \left(\alpha \tan^{-1} \left(\frac{|z|}{r}\right)\right)| \lesssim \frac{|\theta h|^\eta}{(r^2 + |z|^2)^{\eta}}. \tag{4.18}$$

Then by the above two inequalities (4.17) and (4.18), and the inequalities in (4.14), we have

$$\left|\Delta_h C_1 - \alpha(r, z)\right|^q \lesssim |r + |z||(\alpha - \eta_1)q_k \eta_{2q} + |r - |z||(\alpha - \eta_2)q_k \eta_{3q} \cdot 1_{A_2}$$

$$+ |r + h - |z||(-\alpha - \eta_3)q_k \eta_{4q} \cdot 1_{A_3}$$

$$+ \left| r^2 + |z|^2 \right|(-\frac{\alpha}{2} - \eta)q_k \eta_{5q} \cdot (r + h)^\eta h^\eta q |r + h|^\eta h^\eta + \frac{|z|^\eta h |h|^\eta q |r^2 + |z|^2|^{\eta}}{(r^2 + |z|^2)^{\eta}}$$

$$=: \sum_{k=1}^6 M_2^{(2)}(r, z). \tag{4.19}$$
Stochastic wave equation

Substituting this bound into (4.16), we see that,

$$\sup_t I_2^{(2)}(t, h) \leq \sup_t \sum_{k=1}^{6} I_{2,k}^{(2)}(t, h),$$

where

$$I_{2,k}^{(2)}(t, h) = \int_0^t \int_{\mathbb{R}} (r + h)^{\gamma(\theta-1)} M_k^{(2)}(r, z) dz dr, \quad k = 1, \cdots, 6. \tag{4.20}$$

The first four terms $I_{2,k}^{(2)}(t, h)$, $k = 1, \cdots, 4$ are treated in the same way as Case $i=1$ and require conditions (II.1) and (C.1) to guarantee

$$\sup_t I_{2,k}^{(2)}(t, h) \lesssim |h|^q, \quad k = 1, \cdots, 4.$$ 

We shall deal with the $I_{2,5}^{(2)}(t, h)$ and $I_{2,6}^{(2)}(t, h)$ in Appendix C. By Lemma C.3, $\sup I_{2,k}^{(2)}(t, h) \lesssim |h|^q$ for $k = 5, 6$ under conditions (II.1) and (C.6).

As a result, for any $p > \frac{1}{m}$, $\gamma < H - \frac{1}{p}$, we have $\sup I_{2,k}^{(2)}(t, h) \lesssim |h|^q$ for $k = 1, \cdots, 6$, if $\alpha, \theta$ satisfy (II.1) and $\eta_k (k = 1, \cdots, 6)$ satisfy (C.1) and (C.6).

Case $i=3$. In this case we have $\Delta_3(r, z) = \mathcal{E}(r, z) = \frac{1}{r} \frac{r}{r^2 + z^2}$ and

$$|\Delta h \mathcal{E}(r, z)|^q \approx \left| \frac{r + h}{(r + h)^2 + z^2} - \frac{r}{r^2 + z^2} \right|^q = \left| \frac{h}{(r + h)^2 + z^2} + r \left( \frac{1}{(r + h)^2 + z^2} - \frac{1}{r^2 + z^2} \right) \right|^q \leq \left| \frac{h}{(r + h)^2 + z^2} \right|^q + r^q \cdot \frac{|h|^q \cdot |2r + h|^q}{|r + h|^2 + z^2} \cdot \frac{|r|^q}{|r^2 + z^2|^q}. \tag{4.21}$$

By Hölder’s inequality with $\frac{1}{m} + \frac{1}{n} = 1$ and $|2r + h|^q \leq 2^q |r + h|^q$, we obtain

$$I_2^{(3)}(t, h) = \int_0^t \int_{\mathbb{R}} (r + h)^{\gamma(\theta-1)} |\Delta_h \mathcal{E}(r, z)|^q dz dr \lesssim \int_0^t \int_{\mathbb{R}} (r + h)^{\gamma(\theta-1)} \left| \frac{h}{(r + h)^2 + z^2} \right|^q dz dr + |h|^q \cdot \int_0^t \int_{\mathbb{R}} \left( \frac{|r + h|^q}{(|r + h|^2 + z^2)^q} \cdot \frac{|r|^q}{|r^2 + z^2|^q} \right) dz dr \lesssim \int_0^t \int_{\mathbb{R}} (r + h)^{\gamma(\theta-1)} \left| \frac{h}{(r + h)^2 + z^2} \right|^q dz dr + |h|^q \cdot \left[ \int_0^t \int_{\mathbb{R}} \left( \frac{|r|^q}{(|r + h|^2 + z^2)^q} \right)^m dz dr \right]^{\frac{1}{m}} \cdot \left[ \int_0^t \int_{\mathbb{R}} \left( \frac{|r|^q}{|r^2 + z^2|^q} \right)^n dz dr \right]^{\frac{1}{n}} =: I_{2,1}^{(3)}(t, h) + |h|^q \left[ I_{2,2}^{(3)}(t, h) \right]^{1/m} \left[ I_{2,3}^{(3)}(t, h) \right]^{1/n}. \tag{4.22}$$
By the change of variable $z \to (r+h)z$ in $\mathcal{I}_{2,4}^{(3)}(t,h)$ and $\mathcal{I}_{2,2}^{(3)}(t,h)$, and by the change of variable $z \to rz$ in $\mathcal{I}_{2,3}^{(3)}(t,h)$, we have

$\mathcal{I}_{2}^{(3)} \lesssim |h|^{q} \cdot \int_{0}^{t} \int_{\mathbb{R}} |r + h|^{q(\theta-1)+2} \frac{1}{(1 + z^2)^{q}} dz dr$

\[+ |h|^{q} \cdot \left[ \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{|r + h|^{1+mq\theta-2qm}}{1 + z^2} \right) dz dr \right]^{1/2} \cdot \left[ \int_{0}^{t} \int_{\mathbb{R}} \left( \frac{|r|^{1-nq}}{1 + z^2} \right) dz dr \right]^{1/2} \]

\[\lesssim |h|^{q} \cdot \int_{0}^{t} |r + h|^{q(\theta-1)+2-4q} dr + |h|^{q} \cdot \left[ \int_{0}^{t} |r + h|^{1+mq\theta-2qm} dr \right]^{1/2} \cdot \left[ \int_{0}^{t} |r|^{1-nq} dr \right]^{1/2} \]

\[\lesssim |h|^{q(\theta-1)+2-q} + |h|^{q(\theta-1)+2/m} = |h|^{q(\theta-1)+2-q} + |h|^{q(\theta-1)+2-2n} \lesssim |h|^{q\gamma}, \]

under condition

\[\frac{2}{n} > q, \theta - \gamma > 2 - \frac{2}{q}. \tag{4.23}\]

Then $\mathcal{I}_{2}^{(3)}(t,h) \leq C_{T,p,H,\gamma}|h|^{q\gamma}$ under (4.23).

**Case i=4.** In this case we use $\mathcal{K}_{4}(r,z) = \mathcal{S}(r,z) = \frac{1}{T} \mathbf{1}_{\{|z| < r\}}$. Since $(r+h)^{q(\theta-1)} \leq r^{q(\theta-1)}$, we see

$\mathcal{I}_{2}^{(4)}(t,h) \simeq \int_{0}^{t} \int_{\mathbb{R}} (r + h)^{q(\theta-1)} \mathbf{1}_{\{|z| < r+h\}} dz dr$

\[\lesssim \int_{0}^{t} \int_{-r}^{r} (r + h)^{q(\theta-1)} dz dr + \int_{0}^{t} \int_{-h}^{h} (r + h)^{q(\theta-1)} dz dr \]

\[= \int_{0}^{t} 2h(r+h)^{q(\theta-1)} dr \leq h \int_{0}^{t} 2r^{q(\theta-1)} dr \lesssim |h|^\gamma, \]

where the last inequality requires

\[q(\theta - 1) > -1, \gamma < \frac{1}{q}. \tag{4.24}\]

Then under (4.24), we have

\[\sup_{t,x} \mathcal{I}_{2}^{(4)}(t,h) \lesssim |h|^\gamma. \]

To conclude, with the choice of $1 - H < \alpha < 1 - \frac{1}{p}$, $p > \frac{1}{H}$, $0 < \gamma < H - \frac{1}{p}$, we see that the condition (3.18) to guarantee

\[\left( \int_{0}^{T} \| J_{\theta}^{(i)}(r,z) \|_{L^{p}([\mathbb{R}])}^{p} dr \right)^{1/p} \lesssim \| v \|_{Z^{p}(T)} (i = 1, 2, 3, 4) \]

and the conditions listed in **Case i=1,2,3,4** to guarantee (4.11) are all satisfied, so we have

\[\| \sup_{t,x} \mathcal{I}_{2}(t,h,x) \|_{L^{p}(\Omega)} \leq C_{T,p,H,\gamma}|h|^{\gamma} \| v \|_{Z^{p}(T)}. \]

This finishes the proof of (i).

**Step 2:** In this step, we deal with

\[\sup_{t \in [0,T], x,y \in \mathbb{R}} | \Phi(t,x) - \Phi(t,y) | . \]

\[22\]
By (3.19) in the proof of part (ii) of Proposition 3.3, we have
\[
|\Phi(t, x) - \Phi(t, y)| = \left| \sum_{i=1}^{4} \frac{\sin(\theta \pi)}{\pi} \int_{0}^{t} \int_{\mathbb{R}} (t-r)^{\theta-1} \left[ \Phi(t, x, y) \right]_i (t-r, x, z) \right|
\]
\[
- \sum_{i=1}^{4} \left( \int_{0}^{t} \int_{\mathbb{R}} (t-r)^{\theta-1} \left| \Phi(t, x, y) \right| dz dr \right)^{1/4} \times \left( \int_{0}^{T} \left| \Phi(t, x, y) \right| dz dr \right)^{1/4}
\]
where \( h := |x-y| \) and \( \Phi_i(t, x, y) := \Phi_i(t, x) - \Phi_i(t, y) \). Without loss of generality, we can suppose that \( x > y \) and \( h = |x-y| < 1 \) is sufficiently small. The term \( \left( \int_{0}^{T} \left| \Phi(t, x, y) \right| dz dr \right)^{1/4} \) in (4.25) can be estimated via Lemma B.1 which requires (3.18). Thus, we need to show for \( i = 1, \cdots, 4 \)
\[
\sup_{t, x, y} \mathcal{J}^{(i)}(t, x, y) \leq C_{T, p, H, \gamma} |h|^q ,
\]
where \( \mathcal{J}^{(i)}(t, x, y) \) is defined by (3.3). We shall show that
\[
\sup_{t, x, y} \mathcal{J}^{(i)}(t, x, y) = \sup_{t, x, y} \left( \int_{0}^{t} \int_{\mathbb{R}} (t-r)^{\theta-1} \left| \Phi(t, x, y) \right| dz dr \right)
\]
with \( \alpha, \ p \) and \( \gamma \) satisfying (4.4). We split \( \mathcal{J}^{(i)}(t, x, y) \) into two parts:
\[
\mathcal{J}^{(i)}(t, x, y) = \int_{0}^{t} \int_{\mathbb{R}} (t-r)^{\theta-1} \left| \Phi(t, x, y) \right| dz dr + \int_{0}^{T} \int_{\mathbb{R}} (t-r)^{\theta-1} \left| \Phi(t, x, y) \right| dz dr
\]
Let us treat the term \( \mathcal{J}^{(1)}(t, x, y) \) first. In this case, \( -r+h < r-h \). Set
\[
\begin{align*}
B_1 &= \{ z < -h - r \}, & B_2 &= \{ z > r + h \}, & B_3 &= \{ -r + h < z < r - h \}, \\
B_4 &= \{ -r - h < z < -r + h \}, & B_5 &= \{ r - h < z < r + h \}.
\end{align*}
\]
(4.29)
By the triangle inequality and the inequalities (4.14), we have
\[ |\mathcal{D}_hS_{1-\alpha}(r, z)|^q \simeq \left|(r + |z + h|)^{-\alpha} + \text{sgn}(r - |z + h|) |r - |z + h||^{-\alpha} \right|^q \]
\[ - (r + |z|)^{-\alpha} - \text{sgn}(r - |z||r - |z||^{-\alpha} \right|^q \]
\[ \lesssim |\mathcal{D}_h(r + |z|)^{-\alpha}|^q + |\mathcal{D}_h(r - |z|)^{-\alpha}|^q \cdot (1_{B_1} + 1_{B_2} + 1_{B_3}) \]
\[ + |(r - |z + h|)^{-\alpha} + (r - |z||^{-\alpha}|^q \cdot (1_{B_1} + 1_{B_3}) \]
\[ \lesssim |r + |z||^{-\alpha+\eta_4 q} \cdot (1_{B_1} + 1_{B_2}) + |r - h| - |z||^{-\alpha+\eta_2 q} \cdot 1_{B_3} \]
\[ + |(r - |z + h|)^{-\alpha} + (r - |z||^{-\alpha}|^q \cdot (1_{B_1} + 1_{B_3}) \]
\[ =: \sum_{k=1}^{3} N_{1,k}^{(1)}(t, x, y), \quad (4.30) \]
Then
\[ \mathcal{J}_1^{(1)}(t, x, y) \lesssim \sum_{k=1}^{3} \mathcal{J}_{1,k}^{(1)}(t, x, y) := \sum_{k=1}^{3} \int_{\mathbb{R}}^{t} \int_{\mathbb{R}}^{t} r^{q(\theta-1)} N_{1,k}^{(1)}(t, x, y) dz dr. \quad (4.31) \]
By Lemma C.4, \( \sup_{t,x,y} \mathcal{J}_{1,k}^{(1)}(t, x, y) \lesssim |h|^q \) for \( k = 1, 2, 3 \) if we require (II.1) and (C.9).
Next, we shall deal with \( \mathcal{J}_2^{(1)}(t, x, y) \). In this case, \( -r + h \geq r - h \). Setting
\[ C_1 = [z < -r + h], \quad C_2 = [z > r + h], \quad C_3 = [-r + h < z < r + h], \quad (4.32) \]
then by the inequalities (4.14),
\[ |\mathcal{D}_hS_{1-\alpha}(r, z)|^q \simeq \left|(r + |z + h|)^{-\alpha} + \text{sgn}(r - |z + h|) |r - |z + h||^{-\alpha} \right|^q \]
\[ - (r + |z|)^{-\alpha} - \text{sgn}(r - |z||r - |z||^{-\alpha} \right|^q \]
\[ \lesssim |\mathcal{D}_h(r + |z|)^{-\alpha}|^q + |\mathcal{D}_h(r - |z|)^{-\alpha}|^q \cdot (1_{C_1} + 1_{C_2}) \]
\[ + |(r - |z + h|)^{-\alpha} + (r - |z||^{-\alpha}|^q \cdot (1_{C_1} + 1_{C_2}) \]
\[ \lesssim |r + |z||^{-\alpha+\eta_3 q} + |r - h - |z||^{-\alpha+\eta_4 q} \cdot 1_{C_3} \]
\[ + |(r - |z + h|)^{-\alpha} + (r - |z||^{-\alpha}|^q \cdot (1_{C_1} + 1_{C_2}) \]
\[ =: \sum_{k=1}^{3} N_{2,k}^{(1)}(t, x, y). \quad (4.33) \]
Thus
\[ \mathcal{J}_2^{(1)}(t, x, y) \lesssim \sum_{k=1}^{3} \mathcal{J}_{2,k}^{(1)}(t, x, y) := \sum_{k=1}^{3} \int_{\mathbb{R}}^{t} \int_{\mathbb{R}}^{t} r^{q(\theta-1)} N_{2,k}^{(1)}(t, x, y) dz dr. \quad (4.34) \]
By Lemma C.5, \( \sup_{t,x,y} \mathcal{J}_{2,k}^{(1)}(t, x, y) \lesssim |h|^q \) for \( k = 1, 2, 3 \) under conditions (II.1) and (C.14).
As a result, for any $p > \frac{1}{\eta_1} \gamma < H - \frac{1}{\eta_1}$, we know that (4.28) holds if $\alpha, \theta$ satisfy (I.1) and $\eta_k, k = 1, \ldots, 4$ satisfy (C.9) and (C.14).

**Case i=2.** We consider $\bar{K}_2(r, z) = C_{1-\alpha}(r, z)$ defined by (3.3). We want to obtain

$$\sup_{t, x, y} J^{(2)}(t, x, y) = \sup_{t, x, y} \int_0^t \int_\mathbb{R} r^{q(\theta-1)} |\mathcal{D}_h C_{1-\alpha}(r, z)|^q \, dz \, dr \leq C_{T, p, H, \gamma}|h|^{\gamma q},$$

with parameters $p, \alpha, \gamma$ satisfying (4.4). By the triangle inequality,

$$|\mathcal{D}_h C_{1-\alpha}(r, z)|^q \leq |\mathcal{D}_h| r + |z|^{-\alpha} q + |\mathcal{D}_h| r - |z|^{-\alpha} q + |\mathcal{D}_h(r^2 + z^2)^{-\alpha} q|^q$$

$$+ \left[ 2 \mathcal{D}_h \cos \left( \alpha \tan^{-1} \left( \frac{|z|}{r} \right) \right) \right]^q (r^2 + z^2)^{-\alpha} q$$

$$=: \sum_{k=1}^4 N_k^{(2)}(r, z).$$

Substituting (4.36) into (4.35), we have

$$\sup_{t, x, y} J^{(2)}(t, x, y) \leq \sup_{t, x, y} \sum_{k=1}^4 J_k^{(2)}(t, x, y),$$

where

$$J_k^{(2)}(t, x, y) = \int_0^t \int_\mathbb{R} r^{q(\theta-1)} N_k^{(2)}(r, z) \, dz \, dr, \quad k = 1, \ldots, 4.$$

For the term $J_1^{(2)}(t, x, y)$, since for fixed $\eta_1 \in (0, 1),

$$|\mathcal{D}_h| r + |z|^{-\alpha} q \leq |r + |z|^{-\alpha} q)|h|^{\eta q},$$

similar to the estimation of $I^{(1)}_{2,1}(t, h)$ in (4.15), we have $\sup_{t, x, y} J_1^{(2)}(t, x, y) \leq |h|^\gamma q$ under the condition (C.2).

It is more complicated to deal with the term $J_2^{(2)}(t, x, y)$ since $|\mathcal{D}_h| r - |z|^{-\alpha} q$ has different upper bounds on different domains of $|z|$. Similar to **Case i=1**, we split $J^{(1)}(t, x, y)$ into two parts

$$J_2^{(2)}(t, x, y) = \frac{1}{2} \int_0^t \int_\mathbb{R} r^{q(\theta-1)} (|\mathcal{D}_h| r - |z|^{-\alpha} q) \, dz \, dr$$

$$+ \int_0^t \frac{1}{2} \int_\mathbb{R} r^{q(\theta-1)} (|\mathcal{D}_h| r - |z|^{-\alpha} q) \, dz \, dr$$

$$=: J_{2,1}^{(2)}(t, x, y) + J_{2,2}^{(2)}(t, x, y).$$

We first deal with $J_{2,1}^{(2)}(t, x, y)$ when $r > \frac{h}{2}$, namely $-r < -h$. Let us set

$$\begin{cases} D_1 = [z < -r - h], & D_2 = [-r - h < z < -r], \\ D_3 = [-r < z < -r - h], & D_4 = [r - h < z < r], \\ D_5 = [r < z < r + h], & D_6 = [r > z + h]. \end{cases}$$

The first integral of (4.37) can be bounded by

$$J_{2,1}^{(2)}(t, x, y) \leq \sum_{j=1}^6 \int_0^t \int_{D_j} r^{q(\theta-1)} (|\mathcal{D}_h| r - |z|^{-\alpha} q) \, dz \, dr$$

$$=: \sum_{j=1}^6 J_{2,1,j}^{(2)}(t, x, y).$$
It is not hard to derive that for some $\eta \in (0, 1)$
$$\|D_h |r - |z||^{-\alpha}\| \lesssim \begin{cases} |r - |z||^{-\alpha - \eta \theta_h^0}, & \text{on } D_1 \cup D_5 \cup D_6; \\ |r - |z + \theta_h||^{-\alpha - \eta \theta_h^0}, & \text{on } D_3; \\ |r - |z + \theta_h||^{-\alpha} + |r - |z||^{-\alpha}, & \text{on } D_2 \cup D_4. \end{cases}$$
Substituting this into (4.39) we obtain
$$\sum_{j=1}^{6} J_{2,1,j}(t, x, y) \leq \int_{0}^{t} \int_{D_1 \cup D_5} r^n(t-1) |r - |z||^{-n+\eta \theta_h^0} \frac{dz}{dr} d\theta_h$$
$$+ \int_{0}^{t} \int_{D_6} r^n(t-1) |r - |z||^{-n+\eta \theta_h^0} \frac{dz}{dr} d\theta_h$$
$$+ \int_{0}^{t} \int_{D_3} r^n(t-1) |r - |z + \theta_h||^{-n+\eta \theta_h^0} \frac{dz}{dr} d\theta_h$$
$$+ \int_{0}^{t} \int_{D_2 \cup D_4} r^n(t-1) (|r - |z + \theta_h||^{-\alpha} + |r - |z||^{-\alpha}) \frac{dz}{dr} d\theta_h.$$ 
By Lemma C.6 in Appendix C, we have
$$\sup_{t, x, y} J_{2,1,j}(t, x, y) \lesssim |\theta_h|^{\gamma q}, \ j = 1, \cdots, 6,$$
under conditions (II.1) and (C.17).
In similar way we can obtain the same bound for $J_{2,2}(t, x, y)$ by dividing the domain of $|z|$ into subdomains and estimating each terms. We omit the details here.

Now we turn to the third and last terms $J_{3,2}(t, x, y)$ and $J_{4,2}(t, x, y)$. Analogously to the obtention of (4.17) and (4.18), it is not hard to obtain for fixed $\eta \in (0, 1)$,
$$|D_h (r^2 + z^2)^{-\alpha} | \leq (r^2 + z^2)^{-\alpha} |z + \theta_h|^{\eta} |\theta_h|^{\eta}, \quad (4.40)$$
and
$$|D_h \cos \left( \alpha \tan^{-1} \left( \frac{|z|}{r} \right) \right) | \leq \frac{r^{\eta} |\theta_h|^{\eta}}{(r^2 + z^2)^{\eta}}. \quad (4.41)$$
Then we have
$$J_{3,2}(t, x, y) + J_{4,2}(t, x, y) \leq |\theta_h|^{\eta q} \int_{0}^{t} \int_{R} r^{n(t-1)} (r^2 + z^2)^{-(\alpha + \eta \theta_h^0)} |z + \theta_h|^{\eta} d\theta_h d\theta_h$$
$$+ |\theta_h|^{\eta q} \int_{0}^{t} \int_{R} r^{n(t-1)} \frac{r^{\eta q}}{(r^2 + z^2)^{\eta q}} (r^2 + z^2)^{-\alpha} d\theta_h d\theta_h. \quad (4.42)$$
By Lemma C.7, sup $J_{3,2}(t, x, y)$ and sup $J_{4,2}(t, x, y)$ can be bounded by a multiple of $|\theta_h|^{\gamma q}$ under conditions (II.1) and (C.24).
As a result, for any $p > \frac{n}{n-1}$, $\gamma < H - \frac{1}{p}$, sup $J_{3,2}(t, x, y) \leq |\theta_h|^{\gamma q}$ if $\alpha, \theta$ satisfy (II.1), $\eta_k (k = 1, \cdots, 5)$ satisfy (C.17) and (C.24).
Case i=3. In this case $\tilde{K}_3(r, z) = E(r, z) = \frac{1}{r^2 + z^2}$. Then

$$
\int_0^t \int_\mathbb{R} r^q(|\partial_h \tilde{K}_3(r, z)|^q) \, dz \, dr = \int_0^t \int_\mathbb{R} r^q \left(\frac{1}{r^2 + (z + h)^2} - \frac{1}{r^2 + z^2}\right)^q \, dz \, dr.
$$

(4.43)

The $h = |x - y|$ in (4.43) plays the same role as $h$ in the second term of (4.21). So using the similar method as that in dealing with $|\Delta_h \left(\frac{1}{r^2 + z^2}\right)|^q$ in Case i=3 of Step 1, we have

$$
\int_0^t \int_\mathbb{R} r^q \left| \partial_h \left(\frac{1}{r^2 + z^2}\right) \right|^q \, dz \, dr \lesssim h^\gamma q,
$$

if $\theta - \gamma > 2 - \frac{2}{q}$. Thus, under (4.4) we have

$$
\left( J^{(3)}(t, x, y) \right)_{1/q} \times \left( \int_0^T ||J_{\theta}^{K_3}(r, \cdot)||_{L^p(\mathbb{R})} \, dr \right)_{1/p} \lesssim C_{T, p, H, \gamma} |x - y|^\gamma ||v||_{Z^p(T)}.
$$

Case i=4. In this case $\tilde{K}_4(r, z) = S(r, z) = \frac{1}{2} \mathbb{1}_{\{|z| < r\}}$. Then

$$
\int_0^t \int_\mathbb{R} r^q(|\partial_h \tilde{K}_4(r, z)|^q) \, dz \, dr
$$

$$
= \int_0^t \int_\mathbb{R} r^q \left(\frac{1}{2} \mathbb{1}_{\{|z| + z < r\}} - \frac{1}{2} \mathbb{1}_{\{|z| < r\}}\right)^q \, dz \, dr
$$

$$
\simeq \int_0^t \int_\mathbb{R} r^q (\int_{z-x-r}^r dz + \int_{z-x+r}^r dz) \, dr \simeq h \int_0^t r^q (\theta - 1) \lesssim h^\gamma q,
$$

under the conditions $q(\theta - 1) > -1$ and $\gamma < \frac{1}{q}$. Therefore, under (4.4) we have

$$
\left( J^{(4)}(t, x, y) \right)_{1/q} \times \left( \int_0^T ||J_{\theta}^{K_4}(r, \cdot)||_{L^p(\mathbb{R})} \, dr \right)_{1/p} \lesssim C_{T, p, H, \gamma} |x - y|^\gamma ||v||_{Z^p(T)}.
$$

In conclusion, with the choice of $p > \frac{1}{H}, 1 - H < \alpha < 1 - \frac{1}{p}, 0 < \gamma < H - \frac{1}{p}$, the conditions listed in Case i=1, 2, 3, 4 to ensure

$$
\sup_{t, x, y} J^{(i)}(t, x, y) \lesssim ||h||^{\gamma q},
$$

and the condition (3.18) to ensure

$$
\left( \int_0^T ||J_{\theta}^{K_i}(r, \cdot)||_{L^p(\mathbb{R})} \, dr \right)_{1/p} \lesssim ||v||_{Z^p(T)},
$$

are all satisfied. Thus, we have

$$
\sup_{t \in [0, T], x, y \in \mathbb{R}} |\Phi(t, x) - \Phi(t, y)|_{L^p(\Omega)} \lesssim C_{T, p, H, \gamma} |x - y|^\gamma ||v||_{Z^p(T)}.
$$

This completes the proof of (ii).
4.2. Hölder continuity of the approximate solutions and well-posedness.

Analogous to Proposition 4.1 we have the following regularity results for the approximate solution \( u_\varepsilon \) defined in (3.23). The proof is similar and we omit it.

**Lemma 4.2.** Let \( u_\varepsilon \) be the approximation mild solution defined by (3.23) and assume that \( I_0(t, x) \) belongs to \( \mathcal{Z}^p(T) \).

(i) If \( p > \frac{2}{H-1} \), then
\[
\left\| \sup_{t \in [0, T], x \in \mathbb{R}} |\mathcal{N}_{\frac{1}{2}}u_\varepsilon(t, x)|_{L^p(\Omega)} \right\| \leq C_{T, p, H} \| u_\varepsilon \|_{\mathcal{Z}^p(T)}.
\] (4.44)

(ii) If \( p > \frac{1}{H} \) and \( 0 < \gamma < H - \frac{1}{p} \), then
\[
\left\| \sup_{t, t+h \in [0, T], x \in \mathbb{R}} |u_\varepsilon(t+h, x) - u_\varepsilon(t, x)|_{L^p(\Omega)} \right\| \leq C_{T, p, H, \gamma} |h|^{\gamma} \| u_\varepsilon \|_{\mathcal{Z}^p(T)}.
\] (4.45)

(iii) If \( p > \frac{1}{H} \) and \( 0 < \gamma < H - \frac{1}{p} \), then
\[
\left\| \sup_{t \in [0, T], x, y \in \mathbb{R}} |u_\varepsilon(t, x) - u_\varepsilon(t, y)|_{L^p(\Omega)} \right\| \leq C_{T, p, H, \gamma} |x - y|^{\gamma} \| u_\varepsilon \|_{\mathcal{Z}^p(T)}.
\] (4.46)

Finally, we are in position to prove our main results.

**Proof of Theorem 2.5 and Theorem 2.6.** As we know the uniformly Hölder continuity of the type specified in Lemma 4.2 is the most important ingredient in the proof ([11, Theorem 1.5]) of the existence of weak solution to the nonlinear stochastic heat equation. It is also the most important one to show the existence of weak solution for nonlinear stochastic wave equation (1.1). Hence we omit the details of the proof of Theorem 2.5. Since the pathwise uniqueness implies the existence of strong solution by the Yamada-Watanabe theorem (in the SPDEs setting, e.g. [12, 13]), we only need to focus on the proof of pathwise uniqueness. We follow the same strategy in [8, 11] together with the crucial estimate (3.6) in Proposition 3.3.

Suppose \( u(t, x) \) and \( v(t, x) \) are two solution to (1.1). Define the following stopping times:
\[
\mathcal{T}_k = \inf \left\{ t \in [0, T] : \sup_{0 \leq s \leq t, x \in \mathbb{R}} \mathcal{N}_{\frac{1}{2}}u(s, x) \geq k, \right. \\
\quad \text{or} \quad \sup_{0 \leq s \leq t, x \in \mathbb{R}} \mathcal{N}_{\frac{1}{2}}v(s, x) \geq k \left. \right\}, \quad k = 1, 2, \ldots
\]

Recall that the inequality (3.6) in Proposition 3.3 implies that \( \mathcal{T}_k \uparrow T \) almost surely as \( k \to \infty \). This is a key fact to our method. We need to find appropriate bounds for the following two quantities:
\[
\mathcal{J}_1(t) = \sup_{x \in \mathbb{R}} \mathbb{E}\left[ \mathbf{1}_{\{t < \mathcal{T}_k\}} |u(t, x) - v(t, x)|^2 \right]
\]
and
\[
\mathcal{J}_2(t) = \sup_{x \in \mathbb{R}} \mathbb{E}\left[ \int_{\mathbb{R}} \mathbf{1}_{\{t < \mathcal{T}_k\}} |u(t, x) - v(t, x) - u(t, x + h) + v(t, x + h)|^2 |h|^{2H-2} \, dh \right].
\]

By the elementary properties of Itô’s integral, we have
\[
\mathbf{1}_{\{t < \mathcal{T}_k\}} |u(t, x) - v(t, x)|
\]
\[
= \mathbf{1}_{\{t < \mathcal{T}_k\}} \int_0^t \int_{\mathbb{R}} G_{1-s}(x-y) \mathbf{1}_{\{s < \mathcal{T}_k\}} [\sigma(s, y, u(s, y)) - \sigma(s, y, v(s, y))] W(ds, dy).
\]
Therefore, denoting $\triangle(t, x, y) := \sigma(t, x, u(t, y)) - \sigma(t, x, v(t, y))$ we have
\[
E[\mathbf{1}_{\{t < T_k\}}|u(t, x) - v(t, x)|^2]
\leq E\left( \int_0^t \int_{\mathbb{R}^2} \mathbf{1}_{\{s < T_k\}} |\mathcal{D}_h G_{t-s}(x - y)|^2 |\triangle(s, y, y)|^2 |h|^{2H-2} dh dy ds \right) + E\left( \int_0^t \int_{\mathbb{R}^2} \mathbf{1}_{\{s < T_k\}} G_{t-s}^2(x - y - h) |\triangle(s, y + h, y) - \triangle(s, y, y)|^2 |h|^{2H-2} dh dy ds \right) + E\left( \int_0^t \int_{\mathbb{R}^2} \mathbf{1}_{\{s < T_k\}} G_{t-s}^2(x - y) |\triangle(s, y, y + h) - \triangle(s, y, y)|^2 |h|^{2H-2} dh dy ds \right) =: I_{1,1} + I_{1,2} + I_{1,3}.
\] (4.47)

The assumption (2.10) on $\sigma$ can be used to estimate $I_{1,1}$. This is
\[
I_{1,1} \lesssim E\left( \int_0^t \int_{\mathbb{R}^2} \mathbf{1}_{\{s < T_k\}} |\mathcal{D}_h G_{t-s}(x - y)|^2 |u(s, y) - v(s, y)|^2 |h|^{2H-2} dh dy ds \right) \lesssim \int_0^t (t - s)^{2H} \sup_{y \in \mathbb{R}} E[\mathbf{1}_{\{s < T_k\}}|u(s, y) - v(s, y)|^2] ds \leq \int_0^t (t - s)^{2H} \mathcal{J}_1(s) ds.
\]

Using the property (2.12) of $\sigma$, we have if $|h| > 1$
\[
|\triangle(s, y + h, y) - \triangle(s, y, y)|^2 = \left| \int_{\mathbb{R}^2} \left[ \frac{\partial}{\partial \xi} \sigma(s, y + h, \xi) - \frac{\partial}{\partial \xi} \sigma(s, y, \xi) \right] d\xi \right|^2 \lesssim |u(s, y) - v(s, y)|^2.
\]

If $|h| \leq 1$, with the help of additional property (2.13) we get
\[
|\triangle(s, y + h, y) - \triangle(s, y, y)|^2 = \left| \int_{\mathbb{R}^2} \left[ \frac{\partial}{\partial \xi} \sigma(s, y + h, \xi) - \frac{\partial}{\partial \xi} \sigma(s, y, \xi) \right] d\xi \right|^2 \lesssim \int_0^h \frac{\partial}{\partial \eta} \sigma(s, y + \eta, \xi) d\eta d\xi \lesssim |h|^2 |u(s, y) - v(s, y)|^2.
\]

Thus, the term $I_{1,2}$ in (4.47) is bounded by
\[
I_{1,2} \lesssim \int_0^t \mathcal{J}_1(s) \left( \int_{\mathbb{R}^2} G_{t-s}^2(x - y) dy \right) ds \lesssim \int_0^t (t - s) \mathcal{J}_1(s) ds.
\]

For the last term $I_{1,3}$ in (4.47), by (2.12) and (2.14) we have
\[
|\triangle(s, y, y + h) - \triangle(s, y, y)|^2 = \left| \int_0^1 [u(s, y + h) - v(s, y + h)] \frac{\partial}{\partial \xi} \sigma(s, y, \theta u(s, y + h) + (1 - \theta) v(s, y + h)) d\theta \right|^2 - \left| \int_0^1 [u(s, y) - v(s, y)] \frac{\partial}{\partial \xi} \sigma(s, y, \theta u(s, y) + (1 - \theta) v(s, y)) d\theta \right|^2 \lesssim |u(s, y + h) - v(s, y + h) - u(s, y) + v(s, y)|^2
\]
Thus, we have
\[ u(s, y) = v(s, y) + \frac{\partial}{\partial y} \int_0^y \left( u(t, y) - v(t, y) \right) dt \]

for the solvability of equation (1.1). In this section we shall prove that it is also necessary for some specific stochastic wave equations, namely, the hyperbolic Anderson equation (1.3). It is known that if \( \|v(t, x)\|_{L^2(I)} < \infty \) the solution admits the following unique Wiener chaos expansion (see [7, 14]):

\[
v(t, x) = I_0(t, x) + \sum_{n=1}^{\infty} I_n(g_n(t, x)),
\]

where \( I_n \) denotes the multiple Itô-Wiener integrals and \( g_n(t, x) \) \((n \geq 1)\) are defined by

\[
g_n(\bar{s}, \bar{x}; t, x) = \frac{1}{n!} G_{t-x_{\sigma(n)}}(x - x_{\sigma(n)}) \cdots G_{x_{\sigma(2)} - x_{\sigma(1)}}(x_{\sigma(2)} - x_{\sigma(1)}) I_0(s_{\sigma(1)}, x_{\sigma(1)}) .
\]

where \( \bar{x} = (x_1, \ldots, x_n) \) and \( \bar{s} = (s_1, \ldots, s_n) \) such that \( 0 < s_{\sigma(1)} < s_{\sigma(2)} < \cdots < s_{\sigma(n)} < t \) for a permutation \( \sigma \). Then to verify the existence and uniqueness of the

5. Necessity of \( H > \frac{1}{4} \)

In Theorem 2.5 and Theorem 2.6, we see that \( H > \frac{1}{4} \) is a sufficient condition for the solvability of equation (1.1). In this section we shall prove that it is also necessary for some specific stochastic wave equations, namely, the hyperbolic Anderson equation (1.3). It is known that if \( \|v(t, x)\|_{L^2(I)} < \infty \) the solution admits the following unique Wiener chaos expansion (see [7, 14]):

\[
v(t, x) = I_0(t, x) + \sum_{n=1}^{\infty} I_n(g_n(t, x)),
\]

where \( I_n \) denotes the multiple Itô-Wiener integrals and \( g_n(t, x) \) \((n \geq 1)\) are defined by

\[
g_n(\bar{s}, \bar{x}; t, x) = \frac{1}{n!} G_{t-x_{\sigma(n)}}(x - x_{\sigma(n)}) \cdots G_{x_{\sigma(2)} - x_{\sigma(1)}}(x_{\sigma(2)} - x_{\sigma(1)}) I_0(s_{\sigma(1)}, x_{\sigma(1)}) .
\]

where \( \bar{x} = (x_1, \ldots, x_n) \) and \( \bar{s} = (s_1, \ldots, s_n) \) such that \( 0 < s_{\sigma(1)} < s_{\sigma(2)} < \cdots < s_{\sigma(n)} < t \) for a permutation \( \sigma \). Then to verify the existence and uniqueness of the
Stochastic wave equation

mild solution \( v(t, x) \) is equivalent to show that

\[
\mathbb{E}[|v(t, x)|^2] = \sum_{n=0}^{\infty} n! \|g_n(\cdot; t, x)\|_{\mathcal{H}_0^n}^2 < \infty ,
\]

(5.3)

where \( \mathcal{H} \) is defined by (2.1). In terms of Fourier transformation, we have

\[
\|g_n(\cdot; t, x)\|_{\mathcal{H}_0^n}^2 = \int_{[0,t]^n} \left| \mathcal{F}g_n(\vec{s}; \cdot; t, x)(\vec{\xi}) \right|^2 \mu(d\vec{\xi}) d\vec{s},
\]

with \( \mu(d\vec{\xi}) = \prod_{j=1}^{n} |\xi_j|^{1-2H} d\xi_j \).

For national simplicity, we abbreviate \( I_k(g_k(t, x)) \) as \( I_k(t, x) \) for \( k = 1, 2 \), i.e.

\[
I_1(t, x) = \int_0^t \int_\mathbb{R} G_{t-s}(x-y) I_0(s, y) W(ds, dy),
\]

\[
I_2(t, x) = \int_0^t \int_\mathbb{R} G_{t-s}(x-y) I_1(s, y) W(ds, dy).
\]

Let us select some special initial conditions \( u_0(x) = e^{-x^2} \) and \( v_0(x) \equiv 0 \) to proceed our argument. Then

\[
I_0(t, x) = \frac{1}{2} \int_{x-t}^{x+t} v_0(y) dy + \frac{1}{2} [u_0(x+t) + u_0(x-t)]
\]

\[
= \frac{1}{2} \left[ e^{-(x+t)^2} + e^{-(x-t)^2} \right].
\]

(5.4)

We do not consider the simple case \( u_0(x) = 1 \) and \( v_0(x) = 0 \). Because in this case, \( I_0(t, x) \) is not in the space \( Z^p(T) \) for any \( p \geq 1 \).

**Lemma 5.1.** Suppose \( I_0(t, x) \) are given in (5.4). Then for \( H \in (0, 1/2) \), there exist positive constants \( c_{T, H} \) and \( C_{T, H} \) such that for any \( (t, x) \in [0, T] \times \mathbb{R} \) and \( h \) small enough satisfying \( 0 < h < 1 \wedge \frac{T}{2} \),

\[
c_{t, H} \cdot |h|^{2H} \leq \mathbb{E}[|\mathcal{D}_h I_1(t, x)|^2] \leq C_{T, H} \cdot |h|^{2H}.
\]

(5.5)

**Proof.** At first, from (5.4) we see easily that

\[
|I_0(t, x)| \leq C_T , \quad |\mathcal{D}_t I_0(t, x)| \leq C_T \cdot |t| \wedge 1 .
\]

(5.6)

Moreover, on the set \((t, x) \in [0, T] \times [-T, T] \), we have a lower bound for \( |I_0(t, x)| \):

\[
I_0(t, x) = \frac{1}{2} \left[ e^{-(x+t)^2} + e^{-(x-t)^2} \right] \geq c_T .
\]

(5.7)
Now we are in a position to estimate $\mathbb{E}[|D_h I_1(t, x)|^2]$. Let us consider the lower bound first. Recall an elementary inequality: $(a + b)^2 \geq \frac{3}{4}a^2 - 3b^2$, then

$$
\mathbb{E}[|D_h I_1(t, x)|^2] = \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^2} |D_h G_{t-s}(x-y) \cdot I_0(s, y) W(ds, dy)|^2 \right]
$$

$$
= \int_0^t \int_{\mathbb{R}^2} |D_h G_{t-s}(x-(y+l)) \cdot I_0(s, y+l)
- D_h G_{t-s}(x-y) \cdot I_0(s, y)|^2 \cdot |l|^{2H-2} dl dy ds
$$

$$
\geq \frac{3}{4} \int_0^t \int_{\mathbb{R}^2} \left| \square_{l_h} G_s(y) \cdot I_0(s, y) \right|^2 \cdot |l|^{2H-2} dl dy ds
- 3 \int_0^t \int_{\mathbb{R}^2} |D_h G_s(x-y)|^2 \cdot |D_l I_0(s, y)|^2 \cdot |l|^{2H-2} dl dy ds.
$$

By Hölder’s inequality and (5.6), we see that $\sup_{s \in [0, T], y \in \mathbb{R}} \int_{\mathbb{R}} |D_l I_0(s, y)|^2 \cdot |l|^{2H-2} dl \leq C_{T, H} < \infty$ for $H \in (0, \frac{1}{2})$. Then

$$
\int_0^t \int_{\mathbb{R}^2} \left| \square_{l_h} G_s(y) \cdot I_0(s, y) \right|^2 \cdot |l|^{2H-2} dl dy ds
\lesssim \int_0^t \int_{\mathbb{R}^2} |D_h G_s(y)|^2 dy ds \leq C_T \cdot |h|.
$$

Moreover, we have

$$
\int_0^t \int_{\mathbb{R}^2} \left| \square_{l_h} G_s(y) \right|^2 \cdot |l|^{2H-2} dl dy ds
\geq \int_0^t \int_{y>0} \int_{l \geq h} \left| \square_{l_h} G_s(y) I_0(s, y) \right|^2 \cdot |l|^{2H-2} dl dy ds.
$$

Notice that on the set $\{y > 0\} \times \{l \geq h\}$

$$
|\square_{l_h} G_s(y)|^2 \simeq \left| 1_{\{y+l+1/h\} \times s} - 1_{\{y+l\} \times s} - 1_{\{y+l \times h\} \times s} + 1_{\{y\} \times s} \right|^2
\simeq \left| 1_{\{y+l\} \times s} - 1_{\{y\} \times s} \right|^2
= 1_{\{y+l\} \times s} - 1_{\{y\} \times s}.
$$

Letting $h < 1 \wedge \frac{L}{2}$ be small enough and noticing the lower bound (5.7), we have

$$
\int_0^t \int_{\mathbb{R}^2} \left| \square_{l_h} G_s(y) I_0(s, y) \right|^2 \cdot |l|^{2H-2} dl dy ds
\gtrsim \int_0^t \int_{s-h}^s \int_{l \geq h} |l|^{2H-2} |I_0(s, y)|^2 dl dy ds
\gtrsim \int_0^t \int_{s-h}^s |h|^{2H-1} |I_0(s, y)|^2 dy ds \gtrsim c_{l, H} \cdot h^{2H}.
$$

Thus, we obtain when $H \in (0, 1/2)$ and $|h|$ is relatively small

$$
\mathbb{E}[|D_h I_1(t, x)|^2] \gtrsim c_{l, H} \cdot h^{2H} - C_T \cdot |h| \gtrsim c_{l, H} \cdot h^{2H}.
$$
The upper bound can be derived by the Fourier transformation. By (5.6), we have

\[
\mathbb{E}[|\mathcal{D}_h I_1(t, x)|^2] \leq 2 \int_0^t \int_{\mathbb{R}^2} |\Box_{t,h} G_s(y) I_0(s,y)|^2 \cdot |t|^{2H-2} dl dy ds \\
+ 2 \int_0^t \int_{\mathbb{R}^2} |\mathcal{D}_h G_s(x-y)|^2 \cdot |\mathcal{D}_t I_0(s,y)|^2 \cdot |t|^{2H-2} dl dy ds
\]

The upper bound can be derived by the Fourier transformation. By (5.6), we have

\[
\mathbb{E}[|\mathcal{D}_h I_1(t, x)|^2] \leq 2 \int_0^t \int_{\mathbb{R}^2} |\Box_{t,h} G_s(y) I_0(s,y)|^2 \cdot |t|^{2H-2} dl dy ds \\
+ 2 \int_0^t \int_{\mathbb{R}^2} |\mathcal{D}_h G_s(x-y)|^2 \cdot |\mathcal{D}_t I_0(s,y)|^2 \cdot |t|^{2H-2} dl dy ds
\]

Now we begin to prove Theorem 2.7.

**Proof of Theorem 2.7.** We only need to consider \(\|I_2(t, x)\|_{L^2(\Omega)}^2\) with some special initial data (5.4). Let us denote

\[ F_{t,x}(s,y) := G_{t-s}(x-y)I_1(s,y). \]

Noting that

\[
|\mathcal{D}_h F_{t,x}(s,y)|^2 = |G_{t-s}(x-y)\mathcal{D}_h I_1(s,y) + \mathcal{D}_h G_{t-s}(x-y)I_1(s,y)|^2
\geq \frac{3}{4} |G_{t-s}(x-y)\mathcal{D}_h I_1(s,y)|^2 - 3|\mathcal{D}_h G_{t-s}(x-y)I_1(s,y)|^2,
\]

so we have

\[
\mathbb{E}[|I_2(t, x)|^2] = \mathbb{E} \int_0^t \int_{\mathbb{R}^2} |\mathcal{D}_h F_{t,x}(s,y)|^2 |h|^{2H-2} dh dy ds
\geq \frac{3}{4} \int_0^t \int_{\mathbb{R}^2} |G_{t-s}(x-y)|^2 \mathbb{E}|\mathcal{D}_h I_1(s,y)|^2 |h|^{2H-2} dh dy ds
\tag{5.8}
- 3 \int_0^t \int_{\mathbb{R}^2} |\mathcal{D}_h G_{t-s}(x-y)|^2 \mathbb{E}|I_1(s,y)|^2 |h|^{2H-2} dh dy ds. \tag{5.9}
\]

Without loss of generality, we assume \(t = 2\) and estimate term (5.8) first. By Lemma 5.1 with \(h < 1\) and \(\frac{1}{2} = 1\), it is clear that when \(H \leq \frac{1}{4}\),

\[
\int_0^t \int_{\mathbb{R}^2} |G_{t-s}(x-y)|^2 \mathbb{E}|\mathcal{D}_h I_1(s,y)|^2 |h|^{2H-2} dh dy ds
\geq \int_1^2 \int_{\mathbb{R}^2} |G_{t-s}(x-y)|^2 \mathbb{E}|\mathcal{D}_h I_1(s,y)|^2 |h|^{2H-2} dh dy ds
\geq \int_1^2 \int_{\mathbb{R}^2} |G_{t-s}(x-y)|^2 \cdot |h|^{4H-2} dh dy ds = \infty. \tag{5.10}
\]
For any $H \in (0, 1/2)$ we can get that $\sup_{y \in \mathbb{R}} E |I_1(s, y)|^2 \lesssim s^{2H} + s^2$ in the term (5.9), thus

$$
\int_0^2 \int_{\mathbb{R}^2} |D_h G_{2-s}(x - y)|^2 E |I_1(s, y)|^2 dhdyds \lesssim \int_0^2 (s^{2H} + s^2) \int_{\mathbb{R}} \left( \frac{\sin((2-s)|\xi|)}{|\xi|} \right)^2 |\xi|^{1-2H} d\xi ds = \int_0^2 (s^{2H} + s^2)(2-s)^{2H} ds < \infty.
$$

(5.11)

Plugging (5.10) and (5.11) into (5.8) and (5.9), we obtain that for $t = 2$

$$
E \left[ |I_2(t, x)|^2 \right] = \infty
$$

when $H \leq \frac{1}{4}$. The proof is complete. $\square$

**Appendix A. Some technical lemmas for wave kernel**

In this Appendix, we show some technical lemmas used several times in our work. Let us start by proving the Fourier transform of $\mathcal{E}(t, x)$, $\mathcal{S}_\alpha(t, x)$ and $\mathcal{C}_{1-\alpha}(t, x)$.

**Lemma A.1.** Let $\mathcal{E}(t, x)$, $\mathcal{S}_\alpha(t, x)$ and $\mathcal{C}_{1-\alpha}(t, x)$ be defined by (3.3). Then they are all in $L^1(\mathbb{R})$, and their Fourier transforms are given by (3.4). Consequently, the wave kernel $G_{t-s}(x - y)$ can be expressed as the representation (3.2).

**Proof.** We treat $\mathcal{E}(t, x)$ at first,

$$
\mathcal{E}(t, x) = \mathcal{F}^{-1}[\hat{\mathcal{E}}(t, \cdot)](x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-t|\xi|} e^{tx} \xi d\xi = \frac{1}{\pi} \frac{t}{t^2 + x^2}, \quad (A.1)
$$

which is obviously in $L^1(\mathbb{R})$. Similarly, for $\mathcal{S}_\alpha(t, x)$,

$$
\mathcal{S}_\alpha(t, x) = \mathcal{F}^{-1}[\hat{\mathcal{S}}_\alpha(t, \cdot)](x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\sin(t|\xi|)}{|\xi|^\alpha} e^{tx} \xi d\xi = \frac{1}{\pi} \int_0^\infty \frac{\sin(t\xi)}{\xi^\alpha} \cos(|x|\xi) d\xi
$$

$$
= \frac{1}{2\pi} \int_0^{\infty} \sin \left( (t + |x|)\xi \right) d\xi + \frac{1}{2\pi} \int_0^{\infty} \sin \left( (t - |x|)\xi \right) d\xi
$$

$$
= \frac{\Gamma(1-\alpha)}{2\pi} \cos \left( \frac{\alpha\pi}{2} \right) \left[ (t + |x|)^{\alpha-1} + \text{sgn}(t - |x|)(t - |x|)^{\alpha-1} \right], \quad (A.2)
$$

where the last equality can be found in 17.33(2) in [6]. For fixed $t > 0$, if $|x|$ is close to $t$, $|\mathcal{S}_\alpha(t, x)|$ can be bounded by

$$
(t + |x|)^{\alpha-1} + |t - |x||^{\alpha-1}.
$$

And when $|x|$ is large enough, $|\mathcal{S}_\alpha(t, x)|$ behaves like

$$
(|x| - t)^{\alpha-1} - (|x| + t)^{\alpha-1},
$$

which can be bounded by $t(|x| - t)^{\alpha-2}$. Therefore, $\mathcal{S}_\alpha(t, x)$ is in $L^1(\mathbb{R})$ since $\alpha \in (0, 1)$. 

34
Stochastic wave equation

The last one $C_{1-\alpha}(t, x)$ is more involved because of the term $\mathcal{F}^{-1}\left[\frac{e^{-t|\xi|}}{|\xi|^{2\alpha}}\right]$. But we can apply the formula 17.34(14) in [6] to get

$$C_{1-\alpha}(t, x) = \mathcal{F}^{-1}[\hat{C}_{1-\alpha}(t, \cdot)](x) = \frac{1}{\pi} \int_0^\infty \cos(t\xi) \cos(|x|\xi) d\xi - \frac{1}{\pi} \int_0^\infty \frac{e^{-t|\xi|}}{|\xi|^{1-\alpha}} \cos(|x|\xi) d\xi$$

$$= \frac{\Gamma(\alpha)}{2\pi} \left[ \cos\left(\frac{\alpha\pi}{2}\right) \left[ (t + |x|)^{-\alpha} + |t - |x||^{-\alpha} \right] - 2 \cos\left( \alpha^{-1} \left( \frac{|x|}{t} \right) \right) \left[ t^2 + x^2 \right]^{\frac{\alpha}{2}} \right]. \quad (A.3)$$

Similarly, when $|x|$ is close to $t$, $|C_{1-\alpha}(t, x)|$ can be bounded by

$$|t + |x||^{-\alpha} + |t - |x||^{-\alpha} + \left[ t^2 + x^2 \right]^{-\frac{\alpha}{2}}.$$ 

It is more interesting to know the above asymptotics when $|x|$ is large. Since

$$C_{1-\alpha}(t, x) \simeq \cos\left(\frac{\alpha\pi}{2}\right) \left[ (t + |x|)^{-\alpha} + |t - |x||^{-\alpha} \right] - 2 \cos\left(\frac{\alpha\pi}{2}\right) \left[ t^2 + x^2 \right]^{-\frac{\alpha}{2}}$$

$$+ 2 \left[ \cos\left(\frac{\alpha\pi}{2}\right) - \cos\left( \alpha^{-1} \left( \frac{|x|}{t} \right) \right) \right] \left[ t^2 + x^2 \right]^{-\frac{\alpha}{2}}, \quad (A.4)$$

setting $y_0 = \frac{|x|}{t}$, then for $|x|$ large enough,

$$\cos\left(\frac{\alpha\pi}{2}\right) - \cos\left( \alpha^{-1} \left( \frac{|x|}{t} \right) \right) = \int_{y_0}^{+\infty} \frac{d}{d\omega} \left[ \cos(\alpha^{-1}(\omega)) \right] d\omega$$

$$\leq \alpha \int_{y_0}^{+\infty} \frac{1}{\omega} d\omega \simeq C_\alpha \cdot |t|x|^{-1}. \quad (A.5)$$

Therefore,

$$\left[ 2 \cos\left(\frac{\alpha\pi}{2}\right) - 2 \cos\left( \alpha^{-1} \left( \frac{|x|}{t} \right) \right) \right] \left[ t^2 + x^2 \right]^{-\frac{\alpha}{2}} \simeq C_\alpha \cdot |t|x|^{-1}(t^2 + x^2)^{-\frac{\alpha}{2}}, \quad (A.6)$$

which is integrable with respect to $x$ when $|x|$ is large enough since $\alpha \in (0, 1)$. Moreover, since the following important asymptotic behavior holds, which will be explained in Remark A.2,

$$|t + |x||^{-\alpha} + |t - |x||^{-\alpha} = 2\left[ |x|^2 - t^2 \right]^{-\frac{\alpha}{2}} \cos\left[ \alpha^{-1} \left( \frac{t}{|x|} \right) \right] \sim 2\left( |x|^2 - t^2 \right)^{-\frac{\alpha}{2}}, \quad (A.7)$$

it is clear that

$$t \left( |x|^2 + t^2 \right)^{-\frac{\alpha}{2} - 1} \lesssim \left( |x|^2 - t^2 \right)^{-\frac{\alpha}{2}} - \left( |x|^2 + t^2 \right)^{-\frac{\alpha}{2}} \lesssim t \left( |x|^2 - t^2 \right)^{-\frac{\alpha}{2} - 1}.$$ 

We see that for $\alpha \in (0, 1)$, $C_\alpha(t, x)$ is also integrable with respect to $x$ when $|x|$ sufficiently large. As a result, $C_\alpha(t, x)$ is in $L^1(\mathbb{R})$.

Combining (A.1), (A.2) and (A.3), we can conclude (3.2). \qed

**Remark A.2.** We provide details of the equation (A.7) we used in the above proof of Lemma A.1. Noticing that

$$\arctan(z) = -\frac{t}{2} \ln \left( \frac{t - z}{t + z} \right) = -\frac{1}{2} \ln \left( \frac{1 + tz}{1 - tz} \right),$$
we have
\[
\cos[\alpha \tan^{-1}(z)] = \frac{1}{2} \left\{ \exp \left[ \alpha \tan^{-1}(z) \right] + \exp \left[ -\alpha \tan^{-1}(z) \right] \right\}
\]
\[
= \frac{1}{2} \left\{ \exp \left[ \frac{\alpha}{2} \ln \left( \frac{1 + tz}{1 - tz} \right) \right] + \exp \left[ -\frac{\alpha}{2} \ln \left( \frac{1 + tz}{1 - tz} \right) \right] \right\}
\]
\[
= \frac{1}{2} \left\{ \exp \left[ \frac{\alpha}{2} \ln \left( \frac{1 + tz}{1 - tz} \right) \right] \right\} + \exp \left[ -\frac{\alpha}{2} \ln \left( \frac{1 + tz}{1 - tz} \right) \right]
\]
\[
= \frac{1}{2} \left\{ \left( 1 + \frac{tz}{1 - tz} \right) + \left( 1 - \frac{tz}{1 + tz} \right) \right\}
\]
\[
= \frac{1}{2} \left\{ (1 + z^2)\alpha [(1 - tz)^{-\alpha} + (1 + tz)^{-\alpha}] \right\}.
\]

Letting \( z = \frac{\tan(\theta)}{\cos(\theta)} \), we see the equation (A.7) holds.

**Lemma A.3.** If \( \frac{1}{2} < \alpha < 1 \), then \( \hat{C}_\alpha(t, \xi) := \frac{\cos(t|\xi|) - e^{-t|\xi|}}{|\xi|^\alpha} \) and \( \hat{S}_\alpha(t, \xi) := \frac{\sin(t|\xi|)}{|\xi|^\alpha} \) are in \( L^2(\mathbb{R}) \) for any \( t > 0 \). Hence, \( C_\alpha(t, x) \) and \( S_\alpha(t, x) \) are also in \( L^2(\mathbb{R}) \).

**APPENDIX B. LEMMAS FOR PROPOSITION 3.3**

**Lemma B.1.** If \( p > \frac{1}{H}, \ 1 - H < \alpha < 1 - \frac{1}{H} \) and \( 1 - 2/q + \alpha < \theta < H + \alpha - 1/2 \), then there exists a constant \( C \) independent of \( r \) such that
\[
\mathbb{E}\| J_\alpha^{K_i}(r, \cdot) \|^p_{L^p(\mathbb{R})} \leq C \| v \|^p_{L^p(T)}, \quad i = 1, 2, 3, 4,
\]
where \( J_\alpha^{K_i} \) (depending on \( \alpha, \theta \)) and \( K_i \) (depending on \( \alpha \)) are defined by (3.7) and (3.8) respectively.

**Proof.** We will prove the above bound (B.1) for \( i = 1, 2, 3, 4 \) separately. We deal with the term \( i = 1 \) first. In this case \( K_1 = C_\alpha \) and \( K_1 = S_{1-\alpha} \) as defined by (3.3). From the definition (3.7) of \( J_\alpha^{K_1} \) and from Burkholder-Davis-Gundy’s inequality and the triangle inequality it follows
\[
\int_{\mathbb{R}} \mathbb{E}|J_\alpha^{K_1}(r, z)|^p dz \leq \int_{\mathbb{R}} |D_1(r, z)|^p \| v \|^p_{L^p(T)} dz,
\]
where we have used two notations
\[
D_1(r, z) := \int_0^r \int_{\mathbb{R}^2} (r - s)^{-2\theta} |D_h C_\alpha(r - s, y)|^2 \cdot \| v(s, y + z) \|^2_{L^p(\Omega)} |h|^{2H-2} dh dy ds,
\]
and
\[
D_2(r, z) := \int_0^r \int_{\mathbb{R}^2} (r - s)^{-2\theta} |C_\alpha(r - s, y)|^2 \cdot \| D_h v(s, z + y) \|^2_{L^p(\Omega)} |h|^{2H-2} dh dy ds.
\]
By the definition of $Z^p_1(T)$ in (2.6), we can bound $D_1(r) := \int_{R} |D_1(r,z)|^\frac{p}{2} \, dz$ as follows.

$$D_1(r) \lesssim \left( \int_{0}^{r} \int_{R^2} (r-s)^{-2\theta} |D_h C_\alpha (r-s, y)|^2 \, |h|^{2H-2} \, dhdys \right)^\frac{p}{2} \times \|v\|_{Z^p_1(T)}$$

$$\simeq \left( \int_{0}^{r} \int_{R^2} s^{-2\theta} \left| D_h \hat{C}_\alpha (s, \xi) \right|^2 \, |h|^{2H-2} \, dh\xi ds \right)^\frac{p}{2} \times \|v\|_{Z^p_1(T)}.$$  \hspace{1cm} (B.4)

In the last line of (B.4), we have applied Parseval’s formula which is legitimate since $C_\alpha (s, \cdot)$ is in $L^2(R)$ when

$$\frac{1}{2} < \alpha \leq 1,$$  \hspace{1cm} (B.5)

by Lemma A.3. Through (3.4), we can write (B.4) as

$$D_1(r) \lesssim \left( \int_{0}^{r} \int_{R^2} s^{-2\theta} \left| \frac{\cos(s|\xi|) - e^{-s|\xi|}}{|\xi|^{\alpha}} \right|^2 \, |1 - \cos(h|\xi|)||h|^{2H-2} \, dh\xi ds \right)^\frac{p}{2} \times \|v\|_{Z^p_1(T)}$$

$$\simeq \left( \int_{0}^{r} \int_{R^2} s^{-2\theta} \left| \cos(s|\xi|) - e^{-s|\xi|} \right|^2 \, |\xi|^{1 - 2\alpha - 2H} \, dh\xi ds \right)^\frac{p}{2} \times \|v\|_{Z^p_1(T)}$$

$$\simeq \left( \int_{0}^{r} s^{2(H+\alpha - \theta - 1)} ds \cdot \int_{0}^{\infty} \xi^{1 - 2\alpha - 2H} \left| \cos(\xi) - e^{-\xi} \right|^2 \, d\xi \right)^\frac{p}{2} \times \|v\|_{Z^p_1(T)},$$  \hspace{1cm} (B.6)

which is finite if

$$1 - 2\alpha - 2H < -1, \; 2(H + \alpha - \theta - 1) > -1 \Leftrightarrow \alpha > 1 - H, \; \theta < H + \alpha - \frac{1}{2}.$$  \hspace{1cm} (B.7)

Similarly, by the definition of $Z^p_2(T)$ in (2.6), for $D_2(r) := \int_{R} |D_2(r,z)|^\frac{p}{2} \, dz$, Parseval’s formula implies

$$D_2(r) \lesssim \left( \int_{0}^{r} \int_{R} (r-s)^{-2\theta} |C_\alpha (r-s, y)|^2 \, dyds \right)^\frac{p}{2} \times \|v\|_{Z^p_2(T)}$$

$$\simeq \left( \int_{0}^{r} \int_{R} s^{-2\theta} \left| \hat{C}_\alpha (s, \xi) \right|^2 \, d\xi \, ds \right)^\frac{p}{2} \times \|v\|_{Z^p_2(T)},$$  \hspace{1cm} (B.8)

if $\alpha$ satisfies (B.5). Then plugging (3.4), we have

$$D_2(r) \lesssim \left( \int_{0}^{r} \int_{R} s^{-2\theta} \left| \frac{\cos(s|\xi|) - e^{-s|\xi|}}{|\xi|^{\alpha}} \right|^2 \, d\xi \, ds \right)^\frac{p}{2} \times \|v\|_{Z^p_2(T)}$$

$$\simeq \left( \int_{0}^{r} s^{2(\alpha - \theta) - 1} ds \cdot \int_{0}^{\infty} \xi^{-2\alpha} \cos(s\xi) - e^{-s\xi} \, d\xi \right)^\frac{p}{2} \times \|v\|_{Z^p_2(T)}.$$  \hspace{1cm} (B.9)

which is finite since $\frac{p}{2} < \alpha \leq 1$ and $\alpha > \frac{1}{2} - H + \theta > \theta$ by (B.7).

Thus, with the choice of $\theta < H + \alpha - 1/2$ and $\alpha > 1 - H$, we have finished the proof (B.1) for $i = 1$.

Now let us deal with the case when $i = 2$. Similar to the proof in the case $i = 1$, now we need to show

$$\|J^S_{\theta}(r, z)\|_{L^p(\Omega \times R)} \leq C\|v\|_{Z^p(T)}.$$
From the definition (3.7) of $J_\theta$ and from Burkholder-Davis-Gundy’s inequality it follows

$$
\int_{\mathbb{R}} \mathbb{E}|J_\theta^{S_\alpha}(r,z)|^p dz \lesssim \int_{\mathbb{R}} [\overline{D}_1(r,z)]^{\frac{p}{2}} + [\overline{D}_2(r,z)]^{\frac{p}{2}} dz,
$$

where $\overline{D}_1(r,z)$ and $\overline{D}_2(r,z)$ are defined by (B.2) and (B.3), respectively, with $C_\alpha$ replaced by $S_\alpha$.

By the definition of $Z^p_i(T)$ in (2.6) and Minkowski’s inequality, we have

$$
\overline{D}_1(r) := \int_{\mathbb{R}} |\overline{D}_1(r,z)|^{\frac{p}{2}} dz
$$

$$
\lesssim \left( \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\theta} |\mathcal{D}_h S_\alpha(r-s,y)|^2 |h|^{2H-2} dh dy ds \right)^{\frac{p}{2}} \times \|v\|_{Z^p_i(T)}^{\frac{p}{2}}
$$

$$
\lesssim \left( \int_0^r (r-s)^{-2\theta} |S_\alpha(r-s,y)|^2 dy ds \right)^{\frac{p}{2}} \times \|v\|_{Z^p_i(T)}^{\frac{p}{2}}
$$

which is finite under the condition (B.7). In a similar way we can get

$$
\overline{D}_2(r) := \int_{\mathbb{R}} |\overline{D}_2(r,z)|^{\frac{p}{2}} dz
$$

$$
\lesssim \left( \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\theta} |S_\alpha(r-s,y)|^2 dy ds \right)^{\frac{p}{2}} \times \|v\|_{Z^p_i(T)}^{\frac{p}{2}}
$$

$$
\lesssim \left( \int_0^r (r-s)^{-2\theta} |S_\alpha(r-s,y)|^2 dy ds \right)^{\frac{p}{2}} \times \|v\|_{Z^p_i(T)}^{\frac{p}{2}}
$$

which is clearly bounded by (B.7) since $\frac{1}{2} < \alpha < 1$ and $\alpha > \theta$.

Therefore, with the choice of $\theta \in (1-2/q + \alpha, H + \alpha - 1/2)$, we finish the proof of (B.1) when $i = 2$. The remaining parts of (B.1), i.e. the cases $K_3 = S$ and $K_4 = E$ can be completed in the same spirit and we omit the details since they are actually simpler. □

**Lemma B.2.** If $p > \frac{1}{H}$, $\frac{2}{p} - 2H < \alpha < 1 - \frac{1}{p}$ and $1 - 2/q + \alpha < \theta < 2H + \alpha - 1$, then there exists a constant $C$ independent of $r \in [0,T]$ such that for $i = 1, 2, 3, 4$

$$
\mathbb{E} \left[ \int_{\mathbb{R}} \mathbb{E} \left[ \int_{\mathbb{R}} \left| J^K_{\theta}^i(r, z + h) - J^K_{\theta}^i(r, z) \right|^2 |h|^{2H-2} dh \right]^{\frac{p}{2}} dz \right] \leq C \|v\|_{L^p(\mathbb{R} \times \Omega)}^{p/2}
$$

where $J^K_{\theta}^i$ (depending on $\alpha, \theta$) and $K_i$ (depending on $\alpha$) are defined by (3.7) and (3.8) respectively.

**Proof.** Recall that $\mathcal{D}_h J^K_{\theta}^i(r, z) := J^K_{\theta}^i(r, z + h) - J^K_{\theta}^i(r, z)$. We still first consider the case when $i = 1$, i.e. $K_1 = C_\alpha$ and $K_1 = S_{1-\alpha}$ defined by (3.3). We only need to prove that there exists some constant $C$, independent of $r \in [0,T]$, such that

$$
\int_{\mathbb{R}} \mathbb{E} \left[ \int_{\mathbb{R}} \mathbb{E} \left[ \mathcal{D}_h J^C_{\theta}^1(r, z) \right]^2 |h|^{2H-2} dh \right]^{\frac{p}{2}} dz \leq \left( \int_{\mathbb{R}} \|\mathcal{D}_h J^C_{\theta}^1(r, z)\|_{L^p(\mathbb{R} \times \Omega)}^2 |h|^{2H-2} dh \right)^{\frac{p}{2}} \leq C \|v\|_{L^p(\mathbb{R} \times \Omega)}^{p/2}
$$

where we employed Minkowski’s inequality in the above first inequality.
Thanks to Burkholder-Davis-Gundy’s inequality, the triangle inequality and then a change of variable $y \to z - y$, we have

$$
E[|\mathcal{D}_h J^\infty_\theta (r, z)|^p]
$$

\begin{align*}
&\leq C_p \left( \int_0^r (r - s)^{-2\theta} \int_{\mathbb{R}^2} \left[ \mathbb{E} \left| \mathcal{D}_h c_\alpha (r - s, z - y - l) v(s, y + l) \right|^p \right] \frac{1}{1 - \cdot} \left| l \right|^{2H-2} dl dy ds \right)^{\frac{p}{2}} \\
&\leq C_p \left( \int_0^r (r - s)^{-2\theta} \int_{\mathbb{R}^2} \left| \mathcal{D}_h c_\alpha (r - s, y) \right|^2 \left| l \right|^{2H-2} dl dy ds \right)^{\frac{p}{2}} \\
&+ C_p \left( \int_0^r (r - s)^{-2\theta} \int_{\mathbb{R}^2} \left| \mathcal{D}_h c_\alpha (r - s, y) \right|^2 \left| l \right|^{2H-2} dl dy ds \right)^{\frac{p}{2}} .
\end{align*}

Therefore, by Minkowski’s inequality

\begin{align*}
\int_{\mathbb{R}} \left\| \mathcal{D}_h J^\infty_\theta (r, \cdot) \right\|_{L_p(\mathbb{R} \times \Omega)}^2 \left| h \right|^{2H-2} dh 
&= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \mathbb{E} \left[ \mathcal{D}_h J^\infty_\theta (r, z) \right] |dz| \right)^2 \left| h \right|^{2H-2} dh \\
&\leq \int_0^r \int_{\mathbb{R}^2} (r - s)^{-2\theta} \left| \mathcal{D}_h c_\alpha (r - s, y) \right|^2 \left| l \right|^{2H-2} dl dy ds \times \left\| v \right\|_{L_p(\Omega)}^2 \\
&+ \int_0^r \int_{\mathbb{R}^3} (r - s)^{-2\theta} \left| \mathcal{D}_h c_\alpha (r - s, y) \right|^2 \left| l \right|^{2H-2} dl dy ds \times \left\| v \right\|_{L_p(\Omega)}^2 \\
&= \mathcal{J}_1 (r, z) \times \left\| v \right\|_{L_p(\Omega)}^2 + \mathcal{J}_2 (r, z) \times \left\| v \right\|_{L_p(\Omega)}^2 .
\end{align*}

Applying (3.4) and Parseval’s formula again, one can find

\begin{align*}
\mathcal{J}_1 (r, z) &\approx \int_0^r \int_{\mathbb{R}^3} (r - s)^{-2\theta} \left| \mathcal{D}_h \zeta_\alpha (r - s, \xi) \right|^2 \left| h \right|^{2H-2} dh d\xi ds \\
&\leq \int_0^r s^{2(H+\alpha-\theta-1)} dr \int_0^{\infty} 1^{1-2\alpha-2H} |\cos(\xi) - e^{-\xi}|^2 d\xi , \quad \text{(B.14)}
\end{align*}

which is finite if (B.7) is satisfied. Similarly, we have

\begin{align*}
\mathcal{J}_2 (r, z) &\approx \int_0^r \int_{\mathbb{R}^3} (r - s)^{-2\theta} \left| \xi \right|^{-2\alpha} \left| \cos ((r - s)|\xi|) - e^{-|r-s||\xi|} \right|^2 \\
&\times 1 - \cos(|l||\xi|)[1 - \cos(|h||\xi|)] \times \left| l \right|^{2H-2} \left| h \right|^{2H-2} dl dh d\xi ds \\
&\approx \int_0^r (r - s)^{2(\alpha+2H-\theta)-3} ds \int_0^{\infty} 1^{2(1-\alpha-2H)} |\cos(\xi) - e^{-\xi}|^2 d\xi . \quad \text{(B.15)}
\end{align*}

In order to guarantee the integrals in (B.15) converge, we must have

$$
2(\alpha + 2H - \theta) - 3 > -1, \ 2(1 - \alpha - 2H) < -1
$$

\[\Leftrightarrow \ \theta < \alpha + 2H - 1, \ \alpha > -\frac{3}{2} - 2H . \quad \text{(B.16)}\]

Therefore, with the choice of $\theta \in (1 - 2/q + \alpha, 2H + \alpha - 1)$ and $\alpha \in (\frac{3}{2} - 2H, 1 - \frac{1}{p})$ which implies $p > \frac{1}{H}$, by noting that $\frac{2}{2H-\alpha} > \frac{1}{H}$ when $H < \frac{1}{2}$, then the conditions (B.7) and (B.16) are satisfied. Thus, we complete the proof of (B.13).
Now we show (B.12) for \( i = 2 \), i.e. \( K_2 = \mathcal{S}_a \) and \( \tilde{K}_2 = \mathcal{C}_{1-a} \) only briefly since the idea will be similar as in the above case \( i = 1 \). We only need to show that there exists some constant \( C \) independent of \( r \in [0, T] \), such that

\[
\mathbb{E}\left[ \int_\mathbb{R} |\mathcal{D}_h J^S_n(r, z)|^2 h^{2H-2} dh \right]^{\frac{1}{2}} \leq C \|v\|_{L^p(T)}^p. \tag{B.17}
\]

Using Burkholder-Davis-Gundy’s inequality, Minkowski’s inequality and then the triangle inequality, we have the left hand side of (B.17) is bounded by

\[
\left( \tilde{\mathcal{F}}_1(r, z) \right)^{\frac{1}{2}} \times \|v\|_{L^p(T)}^p + \left( \tilde{\mathcal{F}}_2(r, z) \right)^{\frac{1}{2}} \times \|v\|_{L^p(T)}^p,
\]

Applying (3.4) and Parseval’s formula again, one finds

\[
\tilde{\mathcal{F}}_1(r, z) := \int_0^r \int_\mathbb{R} \int (r-s)^{-2\theta} |\mathcal{D}_h \mathcal{S}_\alpha(r-s, y)|^2 h^{2H-2} dhdyds \leq \int_0^r s^{2(H+\alpha-\theta-1)} ds \cdot \int_0^\infty \xi^{1-2\alpha-2H} |\sin(\xi)|^2 d\xi, \tag{B.18}
\]

which is obviously bounded if (B.7) is satisfied. Similarly, we have

\[
\tilde{\mathcal{F}}_2(r, z) := \int_0^r \int_\mathbb{R} \int (r-s)^{-2\theta} |\mathcal{D}_h \mathcal{S}_\alpha(r-s, y+l) - \mathcal{D}_h \mathcal{S}_\alpha(r-s, y)|^2 \times |l|^{2H-2} h^{2H-2} dl dhdyds \leq \int_0^r (r-s)^{2(\alpha+2H-\theta)-3} ds \cdot \int_0^{\infty} \xi^{2(1-\alpha-2H)} |\sin(\xi)|^2 d\xi, \tag{B.19}
\]

which is finite under (B.16).

Therefore, with the choice \( \theta \in (1 - \frac{2}{p} + \alpha, 2H + \alpha - 1) \) and \( \alpha \in (\frac{3}{2} - 2H, 1 - \frac{1}{p}) \) which implies \( p > \frac{1}{1-H} \), we see the conditions (B.7) and (B.16) are satisfied. So we finish the proof of (B.17). The other cases of (B.12) when \( i = 3 \) and \( i = 4 \) can be done by using the same strategy and we omit them here.

### Appendix C. Lemmas for Proposition 4.1

Our aim is to show for any \( p > \frac{1}{\gamma} \) and \( \gamma < H - \frac{1}{\gamma} \), the temporal-spatial Hölder continuity in Proposition 4.1 hold by selecting appropriate \( \alpha, \theta \) and \( \eta \). Above all, we list some conditions which will be used frequently in our technical lemmas.

\[
\begin{align*}
&\Pi.1 \quad 1 - H < \alpha < \frac{1}{\gamma}, \ \alpha + \gamma < \frac{1}{\gamma}, \ \frac{1}{p} < \theta < H + \alpha - \frac{1}{\gamma}; \\
&\Pi.2 \quad \theta > 1 + \alpha - \frac{2}{\gamma} + 2\eta, \ \eta > \gamma; \\
&\Pi.3 \quad \alpha + \eta > \frac{1}{\gamma}, \ \eta > \gamma; \\
&\Pi.4 \quad \alpha + \eta < \frac{1}{\gamma}, \ \eta > \gamma.
\end{align*}
\]

Throughout Appendix C, we always assume \( p > \frac{1}{\gamma} \) and \( \gamma < H - \frac{1}{\gamma} \).

**Lemma C.1.** Suppose \( \alpha, \theta \) satisfy (\Pi.1) and

\[
\begin{cases}
\eta_1 \text{ satisfies (\Pi.2) and (\Pi.3)}; \\
\eta_2 \text{ satisfies (\Pi.2) and (\Pi.4)}; \\
\eta_3 \text{ satisfies (\Pi.3)}.
\end{cases} \tag{C.1}
\]

Then \( I_{2,k}^{(i)}(t, h), \ k = 1, 2, 3, 4 \) in (4.15) can be bounded by \( |h|^{\gamma \eta} \).
Proof. For $I_{2,1}^{(1)}(t, h)$, since $(r + h)^{q(\theta - 1)} \leq r^{q(\theta - 1)}$ it can be bounded by

$$I_{2,1}^{(1)}(t, h) \lesssim h^{\eta_1 q} \cdot \int_0^t \int_0^r r^{q(\theta - 1)} \cdot r^{1-(\alpha + \eta_1)q} |1 + \tilde{z}|^{-(\alpha - \eta_1)q} dz dr$$

$$\lesssim h^{\eta_1 q} \cdot \int_0^t r^{q(\theta - 1)} \cdot r^{1-(\alpha + \eta_1)q} dr \approx h^{\eta_1 q} \leq h^{\gamma q}$$

where we require $\eta_1$ satisfy

$$\eta_1 > \gamma, \ (\alpha + \eta_1)q > 1, \ q(\theta - 1) + 1 - (\alpha + \eta_1)q > -1,$$

which is

$$\eta_1 > \gamma, \ \alpha + \eta_1 > \frac{1}{q}, \ \theta > 1 + \alpha - \frac{2}{q} + \eta_1. \quad (C.2)$$

Similarly, for $I_{2,2}^{(1)}(t, h)$ we have

$$I_{2,2}^{(1)}(t, h) \approx h^{\eta_2 q} \cdot \int_0^t \int_0^r r^{q(\theta - 1)} (r - z)^{-(\alpha - \eta_2)q} dz dr$$

$$\approx h^{\eta_2 q} \cdot \int_0^t r^{q(\theta - 1)} \cdot r^{1-(\alpha + \eta_2)q} dr \approx h^{\eta_2 q} \leq h^{\gamma q},$$

if we require

$$\eta_2 > \gamma, \ \frac{1}{q} > \alpha + \eta_2, \ \theta > 1 + \alpha - \frac{2}{q} + \eta_2. \quad (C.3)$$

For $I_{2,3}^{(1)}(t, h)$ we have

$$I_{2,3}^{(1)}(t, h) \approx h^{\eta_3 q} \cdot \int_0^t \int_0^\infty r^{q(\theta - 1)} \cdot r^{1-(\alpha + \eta_3)q} dz dr$$

$$\approx h^{\eta_3 q} \cdot \int_0^t r^{q(\theta - 1)} \cdot h^{1-(\alpha + \eta_3)q} dr \approx h^{1-\alpha q} \leq h^{\gamma q},$$

under conditions

$$\frac{1}{q} > \gamma + \alpha, \ \alpha + \eta_3 > \frac{1}{q}, \ \theta > 1 - \frac{1}{q} = \frac{1}{p}. \quad (C.4)$$

For the last term $I_{2,4}^{(1)}(t, h)$ we have

$$I_{2,4}^{(1)}(t, h) \lesssim h \cdot \int_0^t r^{q(\theta - 1)} \cdot \int_0^r [r + h - |z|^{1-\alpha q} + ||z| - r|^{1-\alpha q}] \cdot 1_{A_3} dz dr$$

$$\approx h \cdot \int_0^t r^{q(\theta - 1)} \cdot [2 \int_0^h z^{-\alpha q} dz + \int_0^{2h} z^{-\alpha q} dz] dr$$

$$\leq h^{2-\alpha q} \cdot \int_0^t r^{q(\theta - 1)} dr \approx h^{2-\alpha q} \leq h^{\gamma q}$$

if we set

$$\alpha < \frac{1}{q} = 1 - \frac{1}{p}, \ \theta > 1 - \frac{1}{q} = \frac{1}{p}, \ \alpha + \gamma < \frac{1}{q} < \frac{2}{q}. \quad (C.5)$$

Notice that once $\alpha, \ \theta$ satisfy (II.1) and $\eta_1, \ \eta_2$ and $\eta_3$ satisfy (C.1), then the conditions (C.2)-(C.5) hold automatically. The proof is complete. \(\Box\)

**Remark C.2.** We remark here that the conditions (C.1) for $\alpha, \ \theta, \ \eta$’s are compatible with $p > \frac{1}{H}$ and $\gamma < H - \frac{1}{p}$. Let us summarize all the restrictions in Lemma C.1:
Then the terms (2) and (3) follow.

For any (fixed) \( p > \frac{1}{p} \), we can choose for (small enough) \( \varepsilon_k > 0 \ k = 1, \ldots, 6 \)

\[
\gamma = H - \frac{1}{p} - \varepsilon_k, \quad \alpha = 1 - H - \varepsilon_k, \quad \theta = H - \varepsilon_k,
\]

\[
\eta_1 = H - \frac{1}{p} - \varepsilon_k, \quad \eta_2 = H - \frac{1}{p} - \varepsilon_k, \quad \eta_3 = H - \frac{1}{p} - \varepsilon_k.
\]

For arbitrary (small enough) \( \varepsilon > 0 \) let

\[
\varepsilon_1 = 7\varepsilon, \varepsilon_2 = 4\varepsilon, \varepsilon_3 = \varepsilon, \varepsilon_4 = 3\varepsilon, \varepsilon_5 = 6\varepsilon, \varepsilon_6 = 6\varepsilon.
\]

Then all the restrictions (1)-(5) are satisfied with \( \gamma \) arbitrarily close to \( H - \frac{1}{p} \). The following lemmas can be verified similarly. We omit the details.

**Lemma C.3.** Suppose \( \alpha, \theta \) satisfy (II.1) and \( \eta_4, \eta_5 \) satisfy (II.2) and (II.3).

Then the terms \( I_{2,5}^{(2)}(t, h) \) and \( I_{2,6}^{(2)}(t, h) \) in equation (4.20) can be bounded by \( |h|^{q\eta} \).

**Proof.** For the term \( I_{2,5}^{(2)}(t, h) \), from inequality (4.17) and \( (r + h)^{-\eta q} \leq r^{-\eta q} + h^{-\eta q} \) it follows

\[
I_{2,5}^{(2)}(t, h) \lesssim \int_{0}^{t} \int_{\mathbb{R}} (r + h)^{(\theta - 1)} \left| \Delta_h (r^2 + |z|^2)^{-\frac{\alpha}{2}} \right|^q \, dz \, dr
\]

\[
\lesssim |h|^{\eta q} \int_{0}^{t} \int_{\mathbb{R}} (r + h)^{(\theta - 1)} (r^2 + z^2)^{-\frac{\alpha}{2} + \eta q} (r + h)^{\eta q} \, dz \, dr
\]

\[
\lesssim \eta^{\eta q} \cdot \int_{0}^{t} (r^2 + z^2)^{(\theta - 1) + 1 - (\alpha + \eta q)} \, dr \int_{\mathbb{R}} (1 + z^2)^{-\frac{\alpha}{2} + \eta q} \, dz
\]

\[
\lesssim \eta^{\eta q} \cdot \int_{0}^{t} r^{\theta - 1 + 1 - (\alpha + 2\eta q)} \, dr \int_{\mathbb{R}} (1 + z^2)^{-\frac{\alpha}{2} + \eta q} \, dz \lesssim h^{\eta q}
\]

if \( \eta_4 \) satisfies the following conditions

\[
\eta_4 > \gamma, \quad \theta - 2\eta_4 > 1 + \alpha - \frac{2}{q}, \quad \alpha + 2\eta_4 > \frac{1}{q}.
\]

Now we deal with \( I_{2,6}^{(2)}(t, h) \). For fixed \( \eta \in (0, 1) \) by (4.18) and then by changing of variable \( z \to rz \),

\[
I_{2,6}^{(2)}(t, h) \lesssim \int_{0}^{t} \int_{\mathbb{R}} (r + h)^{(\theta - 1)} \left| \frac{z^{\eta q} |h|^{\eta q}}{(r^2 + z^2)^{\eta q}} (r^2 + z^2)^{-\frac{\alpha}{2} q} \right|^q \, dz \, dr
\]

\[
\lesssim \eta^{\eta q} \int_{0}^{t} \int_{\mathbb{R}} \frac{|z|^{\eta q}}{(r^2 + z^2)^{\eta q}} (r^2 + z^2)^{-\frac{\alpha}{2} q} \, dz \, dr
\]

\[
= \eta^{\eta q} \int_{0}^{t} r^{\theta - 1 - \eta q - \alpha + 1} \, dr \int_{\mathbb{R}} \frac{|z|^{\eta q}}{(1 + z^2)^{\eta q}} (1 + z^2)^{-\frac{\alpha}{2} q} \, dz,
\]
which can be bounded by $h^{\gamma q}$ under conditions (C.2) with $\eta_1$ replaced with $\eta_5$, i.e.

$$
\eta_5 > \gamma, \quad \alpha + \eta_5 > \frac{1}{q}, \quad \theta > 1 + \alpha - \frac{2}{q} + \eta_5. \tag{C.8}
$$

Therefore, under conditions (C.7) and (C.8), we have for $k = 5, 6$,

$$
\sup_t \mathcal{I}_{2,k}^{(2)}(t, h) \leq |h|^{\gamma q}. \tag{C.9}
$$

Notice that once $\alpha, \theta$ satisfy (II.1) and $\eta_4, \eta_5$ satisfy (C.6), then the conditions (C.7)-(C.8) hold automatically. The proof is complete. \hfill \square

**Lemma C.4.** Suppose $\alpha, \theta$ satisfy (II.1) and

\[
\begin{align*}
\eta_1 & \text{ satisfies (II.2) and (II.3); } \\
\eta_2 & \text{ satisfies (II.3); } \\
\eta_3 & \text{ satisfies (II.2) and (II.4). } \tag{C.10}
\end{align*}
\]

Then the term $\mathcal{J}^{(1)}_{1,k}(t, x, y)$ in (4.31) can be bounded as

$$
\sup_{t, x, y} |\mathcal{J}^{(1)}_{1,k}(t, x, y)| \leq C_{T,p,H,\gamma} |h|^{\gamma q} \quad \text{for } k = 1, 2, 3. \tag{C.11}
$$

**Proof.** Similar to the proof of $\mathcal{I}_{2,1}^{(1)}$ in part (i) of Proposition 4.1, $\mathcal{J}^{(1)}_{1,1}(t, x, y)$ can be bounded by $|h|^{\gamma q}$ under the same condition as (C.2) which is implied by conditions on $\eta_1$ in (C.9).

Now we deal with $\mathcal{J}^{(1)}_{1,2}(t, x, y)$. By triangle inequality

\[
\begin{align*}
\mathcal{J}^{(1)}_{1,2}(t, x, y) &= \int_t^t \int_{\mathbb{R}} r^{q(\theta-1)} |\mathcal{D}_h(r - |z|)^{-\alpha}|^q \cdot (1_{B_1} + 1_{B_2} + 1_{B_3}) \, dz \, dr \\
&\leq \int_t^t \int_{z<h} r^{q(\theta-1)} |r - |z||(-\alpha - \eta_2)h^{\eta_2 q} \, dz \, dr \\
&+ \int_t^t \int_{z>r+h} r^{q(\theta-1)} |r - |z||(\alpha - \eta_2)h^{\eta_2 q} \, dz \, dr \\
&+ \int_t^t \int_{z<-r+h} r^{q(\theta-1)} |r - |z||(\alpha - \eta_2)h^{\eta_2 q} \, dz \, dr \\
&=: \sum_{j=1}^3 \mathcal{J}^{(1)}_{1,2,j}(t, x, y). \tag{C.12}
\end{align*}
\]

where $B_1, B_2$ and $B_3$ are defined by (4.29).

For the term $\mathcal{J}^{(1)}_{1,2,1}(t, x, y)$ in (C.12), we have

\[
\begin{align*}
\mathcal{J}^{(1)}_{1,2,1}(t, x, y) &\simeq h^{\eta_2 q} \int_0^t r^{q(\theta-1)} \int_{z<h} z^{-(\alpha + \eta_2)q} \, dz \, dr \\
&\simeq h^{1-(\alpha + \eta_2)q + \eta_2 q} \int_0^t r^{q(\theta-1)} \, dr \simeq h^{1-\alpha q} \lesssim h^{\gamma q},
\end{align*}
\]

under the same conditions as (C.4):

\[
\alpha + \gamma < \frac{1}{q}, \quad \alpha + \eta_2 > \frac{1}{q}, \quad \theta > \frac{1}{p}. \tag{C.13}
\]
Similar to $\mathcal{J}^{(1)}_{1,2,1}(t, x, y)$, if the conditions in (C.11) hold, then we have

$$
\mathcal{J}^{(1)}_{1,2,1}(t, x, y) = \int_{t}^{t + \infty} \int_{h}^{r + h} r^{q(\theta - 1)} |z - r|^{-\alpha - \eta q} h^{\eta q} dz dr \\
= h^{\eta q} \int_{t}^{h} \int_{h}^{r + h} r^{q(\theta - 1)} z^{-(\alpha + \eta q)} dz dr \\
\simeq h^{1 - \alpha q} \int_{r - h}^{r + h} r^{q(\theta - 1)} dz \leq h^{\gamma q}.
$$

To estimate $\mathcal{J}^{(1)}_{1,2,3}(t, x, y)$ in (C.10), letting $\eta_3$ satisfy the conditions (C.3) with $\eta_2$ replaced by $\eta_3$, namely,

$$
\eta_3 > \gamma, \quad \frac{1}{q} > \alpha + \eta_3, \quad \theta > 1 + \alpha - \frac{2}{q} + \eta_3,
$$

we have

$$
\mathcal{J}^{(1)}_{1,2,3}(t, x, y) = \int_{h}^{t} \int_{r - h}^{r + h} r^{q(\theta - 1)} |r - h + z|^{-\alpha - \eta_3 q} h^{\eta_3 q} dz dr \\
+ \int_{h}^{t} \int_{r - h}^{r + h} r^{q(\theta - 1)} |r - h - z|^{-\alpha - \eta_3 q} h^{\eta_3 q} dz dr \\
\simeq h^{\eta_3 q} \int_{h}^{t} \int_{r - h}^{r + h} r^{q(\theta - 1)} z^{-(\alpha - \eta_3 q)} dz dr \\
\leq h^{\eta_3 q} \int_{h}^{t} r^{1 - (\alpha + \eta_3 q)} dr \leq h^{\eta_3 q} \leq h^{\gamma q}.
$$

Now we proceed to deal with $\mathcal{J}^{(1)}_{1,3}(t, x, y)$ in (4.30). By the similar way as dealing with $\mathcal{J}^{(1)}_{1,2}(t, x, y)$, we have with $B_4$ and $B_5$ defined by (4.29).

$$
\int_{h}^{t} \int_{\mathbb{R}} r^{q(\theta - 1)} |(r - |z + h|) - \alpha + (r - |z|) - \alpha|^{q} \cdot (1_{B_4} + 1_{B_5}) dz dr \\
= \int_{h}^{t} \int_{r - h}^{r + h} r^{q(\theta - 1)} (|r - |z + h|| - \alpha q + |r - |z|| - \alpha q) dz dr \\
+ \int_{h}^{t} \int_{r - h}^{r + h} r^{q(\theta - 1)} (|r - |z + h|| - \alpha q + |r - |z|| - \alpha q) dz dr \\
\simeq \int_{h}^{t} \int_{-h}^{h} r^{q(\theta - 1)} |z|^{-\alpha q} dz dr + \int_{h}^{t} \int_{0}^{2h} r^{q(\theta - 1)} |z|^{-\alpha q} dz dr \\
\leq h^{1 - \alpha q} \int_{0}^{t} r^{q(\theta - 1)} dz dr \leq h^{1 - \alpha q} \leq h^{\gamma q},
$$

under the same conditions as (C.5):

$$
\alpha < \frac{1}{q} = 1 - \frac{1}{p}, \quad \theta > \frac{1}{p}, \quad \alpha + \gamma < \frac{1}{q}.
$$

Therefore, if $\alpha$, $\theta$ satisfy (Pi.1) and $\eta_1$, $\eta_2$, $\eta_3$ satisfy (C.9), then we have our desire upper bounds for $\sup_{t, x, y} \mathcal{J}^{(1)}_{1,k}(t, x, y)$ ($k = 1, 2, 3$). \hfill \Box

**Lemma C.5.** Suppose $\alpha$, $\theta$ satisfy (Pi.1), and moreover $\eta_4$ satisfies (Pi.2) and (Pi.3).

$$
\eta_4
$$
Then the terms $J_{2,k}^{(1)}(t, x, y)$ in (4.34) can be bounded as follows

$$\sup_{t, x, y} J_{2,k}^{(1)}(t, x, y) \lesssim C_{T, p, H, \gamma} |h|^{\gamma_q} \text{ for } k = 1, 2, 3.$$ 

**Proof.** Similar to the way when we deal with $J_2^{(1)}$ in the proof of part (i) of Proposition 4.1, $J_{2,1}^{(1)}(t, x, y)$ can be bounded by $h^{\gamma_q}$ under the condition (C.2) which holds under condition (C.14). Let us recall the definitions of $C_1$, $C_2$ and $C_3$ in (4.32), then for $J_{2,2}^{(1)}(t, x, y)$ we have

$$J_{2,2}^{(1)}(t, x, y) = \int_0^h \int_R r^{(q-1)} \left| \mathcal{D}_h(r - |z|)^{-\alpha} \right| \cdot (1 + C_1 + 1 C_2) dz dr$$

$$\leq \int_0^h \int_{z < -r-h} r^{(q-1)} |r - |z|| \langle -\alpha - \eta_4 \rangle h^{\eta_4 q} dz dr$$

$$+ \int_0^h \int_{z > r+h} r^{(q-1)} |r - |z|| \langle -\alpha - \eta_4 \rangle h^{\eta_4 q} dz dr. \quad (C.15)$$

For the first term of the summation in (C.15), we have

$$\int_0^h \int z < -r-h r^{(q-1)} |r - |z|| \langle -\alpha - \eta_4 \rangle h^{\eta_4 q} dz dr$$

$$= h^{\eta_4 q} \int_0^h \int z < -r-h r^{(q-1)} (-z - r)^{-\langle -\alpha - \eta_4 \rangle} dz dr$$

$$= h^{\eta_4 q} \int_0^h \int z > h r^{(q-1)} z^{-\langle -\alpha - \eta_4 \rangle} dz dr$$

$$\approx h^{\eta_4 q} h^{1-(\alpha + \eta_4) q} \int_0^h r^{(q-1)} dr$$

$$\approx h^{\eta_4 q + 1-(\alpha + \eta_4) q + 1+(q-1)} \lesssim h^{\gamma_q},$$

under the same conditions as (C.2) with $\eta_1$ replaced by $\eta_4$.

Similarly, we have for the second term of the sum in (C.15)

$$\int_0^h \int z > r+h r^{(q-1)} |r - |z|| \langle -\alpha - \eta_4 \rangle h^{\eta_4 q} dz dr$$

$$= \int_0^h \int z > r+h r^{(q-1)} (-z - r)^{-\langle -\alpha - \eta_4 \rangle} h^{\eta_4 q} dz dr$$

$$= h^{\eta_4 q} \int_0^h \int z > h r^{(q-1)} z^{-\langle \alpha + \eta_4 \rangle} dz dr$$

$$\approx h^{\eta_4 q} h^{1-(\alpha + \eta_4) q} \int_0^h r^{(q-1)} dr \lesssim h^{\eta_4 q + 1-(\alpha + \eta_4) q + 1+(q-1)},$$

which can be bounded by $h^{\gamma_q}$ if the condition (C.2) with $\eta_1$ replaced by $\eta_4$ holds.

For the last term $J_{2,3}^{(1)}(t, x, y)$, if $\alpha$, $\theta$ satisfy (III.1), then the conditions

$$\alpha < \frac{1}{q} = 1 - \frac{1}{p}, \quad \theta > \frac{1}{q}, \quad \theta > 1 + \alpha - \frac{2}{q} + \gamma, \quad (C.16)$$
are satisfied. So we have
\[ J_{2,3}^{(1)}(t, x, y) = \int_0^h \int_{\mathbb{R}} r^q(q-1) |(r - |z + h|) - \alpha + (r - |z|) - \alpha| q \cdot 1_{c_3} dzdr \]
\[ = \int_0^h \int_{-r-h}^{r-h} r^q(q-1) \left( |r - |z + h|| - \alpha q + |r - |z|| - \alpha q \right) dzdr \]
\[ \simeq \int_0^h \int_0^r r^q(q-1)|z|^{-\alpha q} dzdr + \int_0^h \int_0^h r^q(q-1)|z|^{-\alpha q} dzdr \]
\[ \simeq \int_0^h r^q(q-1)1 - \alpha q dr + h^{1-\alpha q} \int_0^h r^q(q-1) dr \]
\[ \lesssim h^{2-\alpha q + \theta(q-1)} \lesssim h^{\gamma q}. \]

Thus, the proof is complete. \hfill \Box

**Lemma C.6.** Suppose \( \alpha, \theta \) satisfy (II.1) and
\[
\begin{cases}
\eta_2 \text{ satisfies (II.3)}; \\
\eta_3 \text{ satisfies (II.4)}; \\
\eta_4 \text{ satisfies (II.2) and (II.3)}. 
\end{cases}
\]
Then the \( J_{2,1,j}^{(2)}(t, x, y), j = 1, \cdots, 6 \) in (4.39) can be bounded as follows.
\[
\sup_{t,x,y} J_{2,1,j}^{(2)}(t, x, y) \lesssim |h|^{\gamma q}.
\]

**Proof.** Let us recall the definitions of \( D_1, \cdots, D_6 \) in (4.38). Firstly, we deal with \( J_{2,1,1}^{(2)} \) and \( J_{2,1,5}^{(2)} \) on \( D_1 \) and \( D_5 \) successively. We have
\[
J_{2,1,1}^{(2)}(t, x, y) + J_{2,1,5}^{(2)}(t, x, y) \\
= |h|^\eta_2 \int_0^t \int_{z < -r-h} r^{q(q-1)}(z - r)^{-(\alpha + \eta_2)q} dzdr \\
+ |h|^\eta_2 \int_0^t \int_{r+h}^{r} r^{q(q-1)}(z - r)^{-(\alpha + \eta_2)q} dzdr \\
\lesssim |h|^\eta_2 \int_0^t r^{q(q-1)} dr \cdot \int_{z > h} (\bar{z})^{-(\alpha + \eta_2)q} dz \\
+ |h|^\eta_2 \int_0^t r^{q(q-1)} dr \cdot \int_{0}^{\bar{h}} (\bar{z})^{-(\alpha + \eta_2)q} dz, \tag{C.18}
\]
through changing of variables \( \bar{z} = -z - r \) and \( \hat{z} = z - r \). Thus, it can be bounded by \( |h|^{\gamma q} \) if
\[
\alpha + \eta_2 > \frac{1}{q}, \quad \eta_2 > \gamma. \tag{C.19}
\]
In the same way, we can deal with \( J_{2,1,6}^{(2)}(t, x, y) \) by changing of variable \( \hat{z} = z - r \),
\[
|h|^\eta_2 \int_0^t \int_{z > r+h} r^{q(q-1)}(z - r)^{-(\alpha + \eta_3)q} dzdr \\
\lesssim |h|^\eta_2 \int_0^t r^{q(q-1)} dr \cdot \int_{z > h} (\bar{z})^{-(\alpha + \eta_3)q} dz \lesssim |h|^{\gamma q},
\]
46
which requires \( \eta_3 \) satisfying the conditions (C.19) and
\[
\alpha + \eta_3 < \frac{1}{q}, \quad \eta_3 > \gamma. \tag{C.20}
\]

Similarly, by changing of variable \( z \to z + \hbar \) and then \( z \to rz \), we have on \( D_3 \),
\[
\mathcal{J}^{(2)}_{2,1,3}(t, x, y) \lesssim \hbar^{\eta_4} \int_0^t \int_{r-h}^{r-h} r^{q(\theta-1)} |r - |z + \hbar||^{-\alpha+\eta_4} \, dz \, dr \\
\lesssim \hbar^{\eta_4} \int_0^t \int_{r-h}^{r-h} r^{q(\theta-1)} |r - |z||^{-\alpha+\eta_4} \, dz \, dr \\
= \hbar^{\eta_4} \int_0^t r^{q(\theta-1) - \alpha\eta_4 q^2} \frac{1}{1 - \alpha \eta_4} \, dr \\
\lesssim \hbar^{\gamma q}, \tag{C.21}
\]
which requires the same condition as (C.2) with \( \eta_1 \) replaced by \( \eta_4 \) here.

As for \( \mathcal{J}^{(2)}_{2,1,2}(t, x, y) \) and \( \mathcal{J}^{(2)}_{2,1,4}(t, x, y) \), we have
\[
\mathcal{J}^{(2)}_{2,1,2}(t, x, y) + \mathcal{J}^{(2)}_{2,1,4}(t, x, y) \\
= \int_0^t \left( \int_{r-h}^{r-h} + \frac{1}{r-h} \right) r^{q(\theta-1)} |r - |z||^{-\alpha} \, dz \, dr \\
\lesssim \int_0^t \left( \int_{r-h}^{r-h} + \frac{1}{r-h} \right) r^{q(\theta-1)} |r - |z + \hbar||^{-\alpha q} \, dz \, dr \\
+ \int_0^t \left( \int_{r-h}^{r-h} + \frac{1}{r-h} \right) r^{q(\theta-1)} |r - |z||^{-\alpha q} \, dz \, dr \\
\lesssim \int_0^t r^{q(\theta-1) - \alpha q} \, dr \cdot \int_0^h |z|^{-\alpha q} \, dz \lesssim |\hbar|^{1 - \alpha q} \lesssim |\hbar|^{\gamma q}, \tag{C.22}
\]
if we require
\[
\theta > \frac{1}{p}, \quad \alpha < \frac{1}{q}, \quad \alpha + \gamma < \frac{1}{q}. \tag{C.23}
\]
Thus, if \( \alpha, \ \theta \) satisfy (II.1) and if (C.17) holds, then all the restrictions on \( \eta \)'s are satisfied. The proof is then complete. \( \square \)

**Lemma C.7.** Suppose \( \alpha, \ \theta \) satisfy (II.1) and moreover
\[
\eta_4, \eta_5 \text{ satisfy (II.2) and (II.3).} \tag{C.24}
\]
Then the terms \( \sup_{t,x,y} \mathcal{J}^{(2)}_4(t, x, y) \) and \( \sup_{t,x,y} \mathcal{J}^{(2)}_4(t, x, y) \) in (4.42) can be bounded by a constant multiple of \( |\hbar|^{\gamma q} \).

47
Proof. For the term \( \mathcal{J}_3^2(t, x, y) \), by (4.40) and the inequality \(|z + h|^{\eta q} \leq |z|^q + |h|^{\eta q} \), we have

\[
\mathcal{J}_3^2(t, x, y) \lesssim |h|^{\eta q} \int_0^t \int_\mathbb{R} r^{q(\theta - 1)} (r^2 + z^2)^{-\frac{q}{2} \theta + \eta q} |z|^q dz dr
\]

\[
+ |h|^{2\eta q} \int_0^t \int_\mathbb{R} r^{q(\theta - 1)} (r^2 + z^2)^{-\frac{q}{2} \theta + \eta q} dz dr
\]

\[
= |h|^{\eta q} \int_0^t \int_\mathbb{R} r^{q(\theta - 1)} \left( (\alpha + 2\eta q) r^q + q + 1 \right) dr \cdot \int_\mathbb{R} (1 + z^2)^{-\frac{q}{2} \theta + \eta q} |z|^q dz
\]

\[
+ |h|^{2\eta q} \int_0^t \int_\mathbb{R} r^{q(\theta - 1)} (\alpha + 2\eta q) r^q + q + 1 dr \cdot \int_\mathbb{R} (1 + z^2)^{-\frac{q}{2} \theta + \eta q} dz,
\]

which can be bounded by \(|h|^{\gamma q}\) under the following conditions

\[
\eta_4 > \gamma, \quad \theta - 2\eta_4 > 1 + \alpha - \frac{2}{q}, \quad \alpha + \eta_4 > \frac{1}{q}.
\]

As for the term \( \mathcal{J}_4^2(t, x, y) \), by inequality (4.41) and by changing of variable \( z \to rz \),

\[
\mathcal{J}_4^2(t, x, y) \lesssim |h|^{\eta q} \int_0^t \int_\mathbb{R} r^{q(\theta - 1)} \frac{r^{\eta q}}{(r^2 + z^2)^{\eta q}} (r^2 + z^2)^{-\frac{q}{2} \theta} dz dr
\]

\[
\lesssim |h|^{\eta q} \int_0^t \int_\mathbb{R} r^{q(\theta - 1) - \eta q - \alpha q + 1} dr \cdot \int_\mathbb{R} (1 + z^2)^{-\frac{q}{2} \theta - \eta q} dz,
\]

which can be bounded by \(|h|^{\gamma q}\) under conditions (C.2) with \( \eta_4 \) substituted by \( \eta_5 \). So we complete the proof by noticing that (C.26) and (C.2) are implied by (C.24). □

REFERENCES

[1] Raluca M. Balan, Maria Jolis, and Lluís Quer-Sardanyons. SPDEs with affine multiplicative fractional noise in space with index \( \frac{1}{2} < H < \frac{3}{4} \). Electron. J. Probab., 20:no. 54, 36, 2015.

[2] Le Chen, Yaozhong Hu, and David Nualart. Regularity and strict positivity of densities for the nonlinear stochastic heat equation. arXiv preprint arXiv:1611.03909, 2016.

[3] Zhen-Qing Chen and Yaozhong Hu. Solvability of parabolic and fractional evolution equations with fractional Gaussian noise. arXiv preprint arXiv:2101.05997, 2021.

[4] Robert Dalang, Davar Khoshnevisan, Carl Mueller, David Nualart, and Yimin Xiao. A mini course on stochastic partial differential equations, volume 1962 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2009. Held at the University of Utah, Salt Lake City, UT, May 8–19, 2006, Edited by Khoshnevisan and Firas Rassoul-Agha.

[5] Robert C. Dalang and Marta Sanz-Solé. Hölder-Sobolev regularity of the solution to the stochastic wave equation in dimension three. Mem. Amer. Math. Soc., 199(931):vi+70, 2009.

[6] I. S. Gradshteyn and I. M. Ryzhik. Table of integrals, series, and products. Elsevier/Academic Press, Amsterdam, eighth edition, 2015. Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Revised from the seventh edition [MR2360010].

[7] Yaozhong Hu. Analysis on Gaussian spaces. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017.

[8] Yaozhong Hu, Jingyu Huang, Khoa Lê, David Nualart, and Samy Tindel. Stochastic heat equation with rough dependence in space. Ann. Probab., 45(6B):4561–4616, 2017.

[9] Yaozhong Hu, Jingyu Huang, Khoa Lê, David Nualart, and Samy Tindel. Parabolic Anderson model with rough dependence in space. In Computation and combinatorics in dynamics, stochastics and control, volume 13 of Abel Symp., pages 477–498. Springer, Cham, 2018.

[10] Yaozhong Hu, Jingyu Huang, and David Nualart. On Hölder continuity of the solution of stochastic wave equations in dimension three. Stoch. Partial Differ. Equ. Anal. Comput., 2(3):353–407, 2014.
Stochastic wave equation

[11] Yaozhong Hu and Xiong Wang. Stochastic heat equation with general noise. To appear in Ann. Inst. Henri Poincaré Probab. Stat., arXiv preprint arXiv:1912.05624, 2019.

[12] Ioannis Karatzas and Steven E. Shreve. Brownian motion and stochastic calculus, volume 113 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1991.

[13] Thomas G. Kurtz. The Yamada-Watanabe-Engelbert theorem for general stochastic equations and inequalities. Electron. J. Probab., 12:951–965, 2007.

[14] David Nualart. The Malliavin calculus and related topics. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.

[15] Szymon Peszat. The Cauchy problem for a nonlinear stochastic wave equation in any dimension. J. Evol. Equ., 2(3):383–394, 2002.

[16] Jian Song, Xiaoming Song, and Fangjun Xu. Fractional stochastic wave equation driven by a Gaussian noise rough in space. Bernoulli, 26(4):2699–2726, 2020.

School of Mathematics, Shandong University, Jinan, Shandong 250100, China
Email address: shuhuiliusdu@gmail.com

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada
Email address: yaozhong@ualberta.ca

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada
Email address: xiongwang@ualberta.ca