Takano’s Theory of Quantum Painlevé Equations

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Abstract. Recently, a quantum version of Painlevé equations from the point of view of their symmetries was proposed by H. Nagoya. These quantum Painlevé equations can be written as Hamiltonian systems with a (non-commutative) polynomial Hamiltonian $H_J$. We give a characterization of the quantum Painlevé equations by certain holomorphic properties. Namely, we introduce canonical transformations such that the Painlevé Hamiltonian system is again transformed into a polynomial Hamiltonian system, and we show that the Hamiltonian can be uniquely characterized through this holomorphic property.

1 Introduction

The Painlevé equations $P_J$ ($J = I, \cdots , VI$) are second-order nonlinear ordinary equations without movable singular points. K. Okamoto revealed the Hamiltonian structures of the Painlevé equations and showed that there are affine Weyl group symmetries which function as a group of Bäcklund transformations. In recent papers [4, 5], H. Nagoya showed that there are quantum versions of Painlevé equations $P_{II}, P_{III}, P_V, P_{VI}$ which have the affine Weyl group symmetries. The relation with the KZ equation with irregular singularities is discussed by M. Jimbo, H. Nagoya and J. Sun[1], where $P_I$ is also considered.

In this paper, we show another construction and characterization of quantum Painlevé equations by a certain kind of the holomorphic properties. This result can be viewed as a quantum version of (the simpler part of) Takano’s theory [9, 8, 2, 3]. Our quantum Painlevé equations are given by the quantum Hamiltonian systems:

$$\frac{df}{dt} = \frac{1}{\hbar} [f, H_J] + \frac{\partial f}{\partial t} \quad (J=II, IV),$$

$$\frac{df}{dt} = \frac{1}{\hbar} [f, H_J] + t \frac{\partial f}{\partial t} \quad (J=III, V),$$

$$\frac{df}{dt} = \frac{1}{\hbar} [f, H_J] + t(t - 1) \frac{\partial f}{\partial t} \quad (J=VI),$$

where $[\cdot, \cdot]$ is the commutator defined by $\{q, p\} := qp - pq = \hbar (\hbar \in \mathbb{C})$. The
Hamiltonians $H_j$ are as follows:

\[ H_{II}(q, p, t) = \frac{1}{2} p^2 - (q^2 + \frac{t}{2}) - bq \]
\[ (a + b + 2h = 1), \]

\[ H_{III}(q, p, t) = q^2 p^2 - q^2 p + (a + b)qp - bq + tp \]
\[ (a + b + c + 2h = 1), \]

\[ H_{IV}(q, p, t) = tqp - qp^2 - q^2 p + ap - bq \]
\[ (a + b + c + h = 1), \]

\[ H_{V}(q, p, t) = tqp - qp^2 - q^2 p + ap - bq \]
\[ (a + b + c + d = 1), \]

\[ H_{VI}(q, p, t) = q^3 p^2 - (1 + t)q^2 p^2 - (a + b + c)q^2 p + tqp^2 \]
\[ + (b + c + (a + b)t)qp - d(a + b + c + d - h)q - btp \]
\[ (e = -a - b - c - 2d + 2h), \]

where $a, b, c, d, e$ are parameters with the above relations. We note that our resulting systems are consistent with Nagoya’s Hamiltonian systems.

The contents of this paper are the following. In section 2 we give quantum versions of Takano’s coordinates for the system given by equation (2), which are birational canonical transformations preserving the holomorphic of the system. The explicit forms of transformed Hamiltonians are given in section 3. In section 4 we show that the system (2) is uniquely characterized by the condition on the holomorphic property in section 2 and 3. This is the main result of this paper. In section 5 we show the consistency of our result with that of H. Nagoya, which shows that the quantum Painlevé equations determined by the holomorphic have affine Weyl group symmetry.

2 Canonical transformations

In this section, we will give canonical transformations such that the holomorphic property of the system (2) is preserved. They are explicitly given as follows.

The case of $P_{II}$.

\[ q = \frac{1}{x_0}, \quad p = -bx_0 - x_0^2 y_0, \quad x_0 = \frac{1}{q}, \quad y_0 = -bq - q^2 p, \]
\[ q = \frac{1}{x_1}, \quad p = t + \frac{2}{x_1^2} - ax_1 - x_1^2 y_1, \quad x_1 = \frac{1}{q}, \quad y_1 = 2q^4 - q^2 p + t q^2 - aq. \]

(3)

The case of $P_{III}$.

\[ q = \frac{1}{x_0}, \quad p = -bx_0 - x_0^2 y_0, \quad x_0 = \frac{1}{q}, \quad y_0 = -bq - q^2 p, \]
\[ q = x_1, \quad p = y_1 + \frac{c}{x_1} - \frac{t}{x_1^2}, \quad x_1 = q, \quad y_1 = p - \frac{c}{q} + \frac{t}{q^2}, \]
\[ q = \frac{1}{x_2}, \quad p = 1 - ax_2 - x_2^2 y_2, \quad x_2 = \frac{1}{q}, \quad y_2 = q^2 - aq - q^2 p. \]
The case of \( P_{IV} \).

\[
q = \frac{1}{x_0}, \quad p = -\frac{1}{x_0} + t - cx_0 - x_0^3y_0, \quad x_0 = \frac{1}{q}, \quad y_0 = -q^3 + tq^2 - q^2p - cq,
\]
\[
q = ay_1 - x_1y_1^2, \quad p = \frac{1}{y_1}, \quad x_1 = ap - qp^2, \quad y_1 = \frac{1}{p},
\]
\[
q = \frac{1}{x_2}, \quad p = -bx_2 - x_2^2y_2, \quad x_2 = \frac{1}{q}, \quad y_2 = -bq - q^2p.
\]

The case of \( P_{V} \).

\[
q = \frac{1}{x_0}, \quad p = -t - dx_0 - x_0^2y_0, \quad x_0 = \frac{1}{q}, \quad y_0 = -tq^2 - dq - q^2p,
\]
\[
q = ay_1 - x_1y_1^2, \quad p = \frac{1}{y_1}, \quad x_1 = ap - qp^2, \quad y_1 = \frac{1}{p},
\]
\[
q = \frac{1}{x_2}, \quad p = -bx_2 - x_2^2y_2, \quad x_2 = \frac{1}{q}, \quad y_2 = -bq - q^2p,
\]
\[
q = cy_3 - x_3y_3 + 1, \quad p = \frac{1}{y_3}, \quad x_3 = cp - qp^2 + p^2, \quad y_3 = \frac{1}{p}.
\]

The case of \( P_{VI} \).

\[
q = \frac{1}{x_0}, \quad p = -dx_0 - x_0^2y_0, \quad x_0 = \frac{1}{q}, \quad y_0 = -dq - q^2p,
\]
\[
q = ay_1 - x_1y_1^2, \quad p = \frac{1}{y_1}, \quad x_1 = ap - qp^2 + p^2, \quad y_1 = \frac{1}{p},
\]
\[
q = by_2 - x_2y_2^2, \quad p = \frac{1}{y_2}, \quad x_2 = bp - qp^2, \quad y_2 = \frac{1}{p},
\]
\[
q = t + cy_3 - x_3y_3, \quad p = \frac{1}{y_3}, \quad x_3 = cp - qp^2 + bp^2, \quad y_3 = \frac{1}{p},
\]
\[
y_0 = \frac{1}{y_4}, \quad x_0 = ex_4 - x_4^2y_4.
\]

**Proposition 2.1** The system (2) is transformed into a polynomial Hamiltonian system under the transformations (3)-(7).

The proof of this proposition is given in the next section, where we will give the transformed Hamiltonian in each chart explicitly.

## 3 Hamiltonians on the charts

In this section, we will prove the holomorphic property (Proposition 2.1). The proof is given by explicit computations. Since the method is similar in all cases, we will give the case of \( P_{II} \) as an example, where \( x, y \) are used instead of \( x_i, y_i \):

Our \( P_{II} \) system can be written as

\[
\begin{align*}
\frac{dq}{dt} &= p - q^2 - \frac{t}{2}, \\
\frac{dp}{dt} &= 2qp + b.
\end{align*}
\]
We will transform this in terms of new coordinates given by the first equation in (3). Since \( q = \frac{1}{x} \), we have
\[
\frac{dq}{dt} = -\frac{1}{x} \frac{dx}{dt}.
\]
(9)

From (8) and (9), we get
\[
\frac{dx}{dt} = x^4 y + (b - h)x^3 + \frac{t}{2} x^2 + 1.
\]
(10)

Similarly, since \( p = -bx - x^2 y \), we have
\[
\frac{dp}{dt} = -b \frac{dx}{dt} - (x \frac{dx}{dt}) y - x^2 \frac{dy}{dt}.
\]
(11)

Together with (8), we obtain
\[
\frac{dy}{dt} = 3(h - b)x^2 y - b(b - h)x - \frac{t}{2} b - 2x^3 y^2 - txy.
\]
(12)

Namely, we proved that the transformed system in the \((x, y)\) variables can be written again as a Hamiltonian system with the following polynomial Hamiltonian:
\[
H = \frac{1}{2} x^4 y^2 - (h - b)x^3 y + \frac{1}{2} b(b - h)x^2 + \frac{t}{2} x^2 y + \frac{t}{2} bx + y.
\]
(13)

In the same way as above, we can get Hamiltonians \( H_i = H_{J_i}, (x_i, y_i, t, \alpha) \) on all the other charts (3)-(7). The results are as follows, where \( x, y \) are used instead of \( x_i, y_i \):

The case of \( P_{II} \).
\[
H_0 = \frac{1}{2} x^4 y^2 + (b - h)x^3 y + \frac{1}{2} b(b - h)x^2 + \frac{1}{2} tx^2 y + \frac{t}{2} bx + y,
\]
\[
H_1 = \frac{1}{2} x^4 y^2 + (a - h)x^3 y + \frac{1}{2} a(a - h)x^2 - \frac{1}{2} tx^2 y - \frac{1}{2} ax - y.
\]
(14)

The case of \( P_{III} \).
\[
H_0 = x^2 y^2 - tx^2 y + (-a + b - 2h)xy - bt x + y,
\]
\[
H_1 = x^2 y^2 - x^2 y + (a + b + 2c)xy + (b - c)x - ty,
\]
\[
H_2 = x^2 y^2 - tx^2 y + (a - b - 2h)xy - at x - y.
\]
(15)

The case of \( P_{IV} \).
\[
H_0 = -x^2 y^2 + (a - 2c + 2h)x^2 y + txy - c(a + c - h)x - y,
\]
\[
H_1 = -x^2 y^2 + (2a + b + 2h)xy^2 - txy + x - a(a + b + h)y,
\]
\[
H_2 = -x^2 y^2 + (a - 2b + 2h)x^2 y - txy - b(a + b - h)x + y.
\]
(16)
The case of $P_V$.

$$H_0 = -x^3y^2 + x^2y^2 + (-a - 2d + 2h)x^2y + (a + c + 2d - 2h - t)xy - d(a + d - h)x + ty,$$

$$H_1 = tx^2y^3 + x^2y^2 - (a + b + 2h)txy^2 + (c - a - 2h + t)xy + x + a(a + b + h)ty,$$

$$H_2 = -x^3y^2 + x^2y^2 + (-a - 2b + 2h)x^2y + (a + 2b + c - 2h + t)xy - b(a + b + h)x - ty,$$

$$H_3 = tx^2y^3 + x^2y^2 - (b + 2(c + h))txy^2 + (a - c - 2h - t)xy - c(b + c + h)ty. \quad (17)$$

The case of $P_{VI}$.

$$H_0 = tx^3y^2 - (1 + t)x^2y^2 + (b + 2d - 2h)tx^2y + xy^2$$

$$+ (-b - c - 2d + 2h - (a + b + 2d - 2h)t)xy$$

$$+ (b(a + b + c + 2d - 2h)y,$$

$$H_1 = -x^3y^4 + (2a - b - c + 6h)x^2y^3 + (2 - t)x^2y^2 + [-a^2 + a(2b + 2c + d - 7h)$$

$$+ (b + c + d - 3h)(d + 2h)]xy^2 + (-2a + b + c - 4h + (a - b + 2h)t)xy + (t - 1)x$$

$$- a(b + c + d - 2h)(a + d + h)y,$$

$$H_2 = -x^3y^4 + (-a - b + 2c - 6h)x^2y^3 - (1 + t)x^2y^2$$

$$+ [-b^2 + b(2c + d - 7h) + (c + d - 3h)(d + 2h) + a(2b + d + 2h)]xy^2$$

$$+ (b - c + 2h - (a - b - 2h)t)xy - tx - b(a + c + d - 2h)(b + d + h)y,$$

$$H_3 = -x^3y^4 + (-a - b + 2c + 6h)x^2y^3 + (2t - 1)x^2y^2$$

$$+ [c^2 + cd + d^2 - 7ch - 6h^2 - dh + a(2c + d + 2h) + b(2c + d + 2h)]xy^2$$

$$+ [-b + c + 2h + (a + b - 2c - 4h)t]xy + t(1 - t)x - c(a + b + d - 2h)(c + d + h)y,$$

$$H_4 = -tx^3y^4 - (3a + 2b + 3c + 4d - 10h)tx^2y^3 - (t + 1)x^2y^2$$

$$+ (3a^2 + b^2 + 3c^2 + 5d^2 + 24h^2 + 4ab + 4bc + 6ca + 8ad + 5bd$$

$$+ 8cd - 17ah - 11bh - 17ch - 23dh)txy^2$$

$$+ (-2a - b - c - 2d - 4h - (a + b + 2c + 2d - 4h)t)xy - x$$

$$- (a + b + c + d - 2h)(a + c + d - 2h)(a + b + c + 2d - 2h)ty. \quad (18)$$

4 Characterization of $H_J$ by Takano’s theory

In this section, we characterize $H_J$ by the holomorphic property (Takano’s theory \[9 \( \|9 \) \|2 \( \|3 \)).

**Theorem 4.1** In a polynomial Hamiltonian system for each variables $q, p$, the Hamiltonian $H_J$ can be uniquely characterized through the holomorphic property under the transformations given in \(6 \( \|3 \)).

We show only the case of $J = II$, since the other cases are similar. For example, we first consider the case of polynomials of order 4. We parametrize such a general polynomial as
\[ H = k_1 q^4 p^4 + k_2 q^4 p^3 + k_3 q^4 p^2 + k_4 q^4 p + k_5 q^4 + k_6 q^3 p^4 + k_7 q^3 p^3 + k_8 q^3 p^2 + k_9 q^3 p + k_{10} q^3 + k_{11} q^2 p^4 + k_{12} q^2 p^3 + k_{13} q^2 p^2 + k_{14} q^2 p + k_{15} q^2 + k_{16} q p^4 + k_{17} q p^3 + k_{18} q p^2 + k_{19} q p + k_{20} q + k_{21} p^4 + k_{22} p^3 + k_{23} p^2 + k_{24} p \]  

The transformations in the first equation of (3) are computed in a similar way as in section 3. Then, we find poles up to order \( x^{-5} \). Similarly, for the second equation (3), we have poles up to order \( x^{-13} \).

Solving the vanishing conditions of these residues, we have the following results for unknown coefficients \( k_1, \cdots, k_{24} \):

\[
\begin{aligned}
  k_{14} &= \frac{1}{2(a + b + 2h)}, \\
  k_{20} &= -\frac{b}{a + b + 2h}, \\
  k_{23} &= \frac{2(a + b + 2h)}{1}, \\
  k_{24} &= -\frac{1}{2(a + b + 2h)}, \\
  k_i &= 0 \text{ (otherwise)}.
\end{aligned}
\]  

This shows that the Hamiltonian systems with the desired holomorphic property are uniquely determined as follows:

\[ H_{11} = -\frac{tp + 2bq - p^2 + 2q^2 p}{2(a + b + 2h)}. \]  

By normalizing the parameters as \( a + b + 2h = 1 \), we obtain equation (2).

The proof of Theorem 4.1 in the case of general degree is as follows. The equation for undetermined coefficients \( \vec{k} \) is a linear inhomogeneous equation

\[ A(h)\vec{k} = \vec{c}, \]  

where the coefficients \( A \) are polynomials in \( h \) and the inhomogeneous term \( \vec{c} \) coming from the second term in equation (1) is independent of \( h \). We note that the solution of this equation reduces to that of the analogous problem in the classical version of Takano’s theory, in the limit as \( h \to 0 \). To prove the uniqueness of the solution (22) for the generic parameter \( h \), we need to show that \( \det(A(h)) \) is not identically zero. The last condition follows from the classical result, where \( \det(A(0)) \neq 0 \).

5 Affine Weyl group symmetry

In this section, we compare our Hamiltonian systems with the quantum Painlevé equations proposed by H. Nagoya. As a result, we find that our system is consistent with that of H. Nagoya, up to redefinition of parameters (and rescaling
of canonical variables). This means that our system has the affine Weyl group symmetry of type $A_1^{(1)}$, $C_2^{(1)}$, $A_2^{(1)}$, $A_3^{(1)}$ and $D_4^{(1)}$ for $P_{II}, P_{III}, P_{IV}, P_V$ and $P_{VI}$, respectively.

Let us recall the Hamiltonians $\hat{H}_J$ ($J = II, \cdots , VI$) given by H. Nagoya \cite{4, 5} (see also \cite{1}).

The case of $P_{II}$.

$$\dot{H}_\text{II} = -qpq + \frac{1}{2}p^2 - \frac{t}{2}p - 2\alpha_1 q, \quad (23)$$

where $\alpha_0 + \alpha_1 = 1$.

The case of $P_{III}$.

$$\dot{H}_\text{III} = \frac{1}{4}[pq(p-1)q + (p-1)qpq + qpq(p-1) + q(p-1)qp] + \frac{1}{2}(\alpha_0 + \alpha_2)(qp + pq) - \alpha_0 q + tp, \quad (24)$$

where $\alpha_0 = 1 - 2\alpha_1 - \alpha_2$.

The case of $P_{IV}$.

$$\dot{H}_\text{IV} = -qpq - pqp + 2tpq - \frac{1}{2}(\alpha_0 + \alpha_1 - 4)p - \frac{\alpha_1}{2}q + \frac{1}{3}(\alpha_0 + \alpha_1 - 4)t, \quad (25)$$

where $\alpha_0 + \alpha_1 + \alpha_2 = 1$.

The case of $P_{V}$.

$$\dot{H}_V = \frac{1}{2}[pqqp + pqpq] - pqp + tpq - \frac{t}{2}(qp + pq) + \alpha_1 p + \alpha_2 t q - \frac{1}{2}(\alpha_1 + \alpha_3)(qp + pq), \quad (26)$$

where $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1$.

The case of $P_{VI}$ (see \cite{5}).

$$t(t-1)\dot{H}_\text{VI} = \frac{1}{6}[qp(q-1)p(q-t) + (q-1)p(q-t)pq + (q-t)qpq(q-1) + (q-t)p(q-1)pq + (q-1)pqp(q-t) + qp(q-t)p(q-1)]$$

$$+ \frac{1}{2}[(\alpha_0 - 1)(qp(q-1) + (q-1)pq) + \alpha_3(qp(q-t) + (q-t)pq) + \alpha_4((q-1)p(q-t) + (q-t)p(q-1))] + \alpha_2(\alpha_1 + \alpha_2)(q-t), \quad (27)$$

where $\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$.

**Proposition 5.1** The Hamiltonians \cite{2} and Nagoya’s Hamiltonians \cite{23, 27} coincide, up to redefinitions of parameters by additive constants.

We will list the relations of parameters.

The case of $P_{II}$.

$$\alpha_1 = \frac{b-h}{2}. \quad (28)$$

The case of $P_{III}$.

$$\alpha_0 = b + h, \quad \alpha_1 = \frac{1}{2}(1-a-b-2h), \quad \alpha_2 = a + h. \quad (29)$$

\footnote{The variables $p, q$ correspond to $\hat{p}, \hat{q}$ in \cite{2}, except for the case of $P_{III}$, where $p = -\hat{p}, q = \hat{q}$.}
The case of $P_{IV}$.

$$\alpha_0 = -2(-2 + a + b), \quad \alpha_1 = 2(b + h).\quad (30)$$

The case of $P_{V}$.

$$\alpha_1 = a - h, \quad \alpha_2 = b + h, \quad \alpha_3 = c - h.\quad (31)$$

The case of $P_{VI}$.

$$\begin{cases}
\alpha_0 = 1 - c + h, \alpha_1 = a + b + c + 2d - h, \alpha_2 = -d - h, \alpha_3 = -a + h, \alpha_4 = -b + h, \\
\alpha_0 = 1 - c + h, \alpha_1 = -a - b - c - 2d + h, \alpha_2 = a + b + c - 2h, \alpha_3 = -a + h, \alpha_4 = -b + h.
\end{cases}\quad (32)$$

From this Proposition 5.1, we find that our system defined from the holomorphic property has affine Weyl group symmetry. We write down the symmetry transformations in the notation of Nagoya’s Hamiltonian for convenience.

The case of $P_{II}$.

From this Proposition 5.1, we find that our system defined from the holomorphic property has affine Weyl group symmetry. We write down the symmetry transformations in the notation of Nagoya’s Hamiltonian for convenience.

The case of $P_{III}$.

$$\begin{array}{c|c|c|c|c}
\alpha_0 & \alpha_1 & q & p \\
\hline
s_0 & -\alpha_0 & \alpha_1 + 2\alpha_0 & q + \frac{\alpha_0}{p - q^2 - \frac{1}{p}} & p - q - \alpha_0 q - \frac{\alpha_0 q}{p - q^2 - \frac{1}{p}} - \frac{\alpha_0 q}{p - q^2 - \frac{1}{p}} \\
s_1 & \alpha_0 + 2\alpha_1 & -\alpha_1 & q - \frac{\alpha_1}{p} \\
\end{array}$$

The case of $P_{IV}$.

$$\begin{array}{c|c|c|c|c|c}
\alpha_0 & \alpha_1 & \alpha_2 & q & p \\
\hline
s_0 & -\alpha_0 & \alpha_1 + \alpha_0 & \alpha_2 + \alpha_0 & q + \frac{\alpha_0}{p - q} & p - \frac{\alpha_0}{p - q} \\
s_1 & \alpha_0 + \alpha_1 & -\alpha_1 & \alpha_2 + \alpha_1 & q & p + \frac{\alpha_1}{q} \\
s_2 & \alpha_0 + \alpha_2 & \alpha_1 + \alpha_2 & -\alpha_2 & q - \frac{\alpha_2}{p} & p \\
\end{array}$$

The case of $P_{V}$.

$$\begin{array}{c|c|c|c|c|c}
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & q & p \\
\hline
s_0 & -\alpha_0 & \alpha_1 + \alpha_0 & \alpha_2 & \alpha_3 + \alpha_0 & q - \frac{\alpha_0}{p - q} \\
s_1 & \alpha_0 + \alpha_1 & -\alpha_1 & \alpha_2 + \alpha_1 & \alpha_3 & q + \frac{\alpha_1}{p} & p \\
s_2 & \alpha_0 & \alpha_1 + \alpha_2 & -\alpha_2 & \alpha_3 + \alpha_2 & q & p - \frac{\alpha_2}{q} \\
s_3 & \alpha_0 + \alpha_3 & \alpha_1 & \alpha_2 + \alpha_3 & -\alpha_3 & q - \frac{\alpha_3}{1 - p} & p \\
\end{array}$$

The case of $P_{VI}$.

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\[
\begin{array}{cccccc}
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & q \\
\hline
s_0 & -\alpha_0 & \alpha_1 & \alpha_2 + \alpha_0 & \alpha_3 & \alpha_4 & q \\
s_1 & \alpha_0 & -\alpha_1 & \alpha_2 + \alpha_1 & \alpha_3 & \alpha_4 & p \\
s_2 & \alpha_0 + \alpha_2 & \alpha_1 + \alpha_2 & -\alpha_2 & \alpha_3 + \alpha_2 & \alpha_4 + \alpha_2 & q + \frac{\alpha_0}{p} \\
s_3 & \alpha_0 & \alpha_1 & \alpha_2 + \alpha_3 & -\alpha_3 & \alpha_4 & q - \frac{\alpha_0}{q-1} \\
s_4 & \alpha_0 & \alpha_1 & \alpha_2 + \alpha_4 & \alpha_3 & -\alpha_4 & q - \frac{\alpha_0}{q-1}
\end{array}
\]

6 Conclusions

In this paper, we gave a construction and characterization of quantum Painlevé equations by the holomorphic properties. This may be considered as a first step toward the study of a "quantum Painlevé property". Recently, Y. Sasano extended Takano’s theory and discovered new equations (Sasano systems) as Hamiltonian systems with holomorphic [7]. In particular, the series of equations having the symmetry of type of \( D^{(1)}_n \) can be regarded as extensions of the Painlevé V,VI equations. It is natural to expect a quantum analog of Sasano’s results as well.

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