Fundamental Performance of Integrated Sensing and Communications (ISAC) Systems

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Abstract

This letter analyzes the fundamental performance of integrated sensing and communications (ISAC) systems. For downlink and uplink ISAC, the diversity orders are analyzed to evaluate the communication rate (CR) and the high signal-to-noise ratio (SNR) slopes are unveiled for the CR as well as the sensing rate (SR). Furthermore, the achievable downlink and uplink CR-SR regions are characterized. It is shown that ISAC can provide more degrees of freedom for both the CR and the SR than conventional frequency-division sensing and communications systems where isolated frequency bands are used for sensing and communications, respectively.

Index Terms

Fundamental performance, integrated sensing and communications (ISAC), rate region.

I. INTRODUCTION

Enabling share of spectrum and hardware resources, integrated sensing and communications (ISAC) systems can perform dual-function sensing-communications within the same time-frequency resource block, which is expected to play a key role in the future wireless network market [1]–[3]. Recently, ISAC has received considerable research attention due to its superior hardware- and spectral-efficiency compared to conventional frequency-division sensing and communications (FDSAC) systems where isolated frequency bands are used for sensing and communications, respectively [4], [5].

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Several works discussed the main features of ISAC and analyzed its performance from an information-theoretic perspective [5]–[7]. Typical information-theoretic performance metrics of ISAC include the estimation or sensing rate (SR) for radar sensing and the communication rate (CR) for communications. For more details about the performance of ISAC, please refer to the recent overview paper [5] and references therein. Yet, it is worthy of mentioning that most previous ISAC researchers did not take account of the influence of channel fading when analyzing the performance of ISAC [5]–[7]. In addition, a rigorous discussion on the in-depth system insights of ISAC, including the diversity order and the high signal-to-noise ratio (SNR) slope, is still missing.

The aim of this letter is to present fundamental performance of downlink and uplink ISAC from an information-theoretic perspective. To this aim, we discuss in detail the CR, SR, and achievable CR-SR region of ISAC by taking into account the capacity-achieving coding/decoding and the SR-optimal radar waveforming as well as the influence of channel fading. We further analyze the high-SNR CR and SR in order to unveil the diversity order and high-SNR slope. Theoretical analyses and numerical results indicate that ISAC is capable of providing more degrees of freedom for both the CR and the SR than conventional FDSAC.

II. SYSTEM MODEL

In an ISAC system shown in Fig. 1(a), one radar-communications (RadCom) base station (BS) serves $K$ single-antenna communication users (CUs) while simultaneously sensing the radar targets (RTs) in the near environment. The BS is equipped with two spatially widely
separated antenna arrays, i.e., \( M (M \geq K) \) transmit antennas and \( N (N \geq K) \) receive antennas, whose structure is illustrated in Fig. [1(b)]. In this letter, a structure of statistical multiple-input multiple-output (MIMO) radar is considered, where the receive antennas are widely separated [8]. In this case, we can ignore the spatial correlation between receive antennas [3].

A. Downlink ISAC

The downlink ISAC (D-ISAC) comprises two stages. In the first stage, the BS broadcasts the communication signal plus radar waveform to the CUs and RTs. Accordingly, the received signal at CU \( k \) is given by

\[
\mathbf{y}_k^H = \mathbf{h}_k^H (\mathbf{X} + \mathbf{S}) + \mathbf{n}_k^H,
\]

where \( \mathbf{h}_k \in \mathbb{C}^{M \times 1} \) denotes the channel vector from CU \( i \) to the transmit array at the BS; \( \mathbf{n}_k \sim \mathcal{C}\mathcal{N} (0, \mathbf{I}_M) \) denotes the additive white Gaussian noise (AWGN); \( \mathbf{S} = [\mathbf{s}_1 \cdots \mathbf{s}_L] \in \mathbb{C}^{M \times L} \) denotes the radar waveform with \( \mathbf{s}_l \in \mathbb{C}^{M \times 1} \) representing the waveform at the \( l \)th time instant; \( \mathbf{X} = [\mathbf{x}_1 \cdots \mathbf{x}_L] \in \mathbb{C}^{M \times L} \) denotes the communication signal matrix with \( \mathbf{x}_l \in \mathbb{C}^{M \times 1} \) representing the downlink communication signal at the \( l \)th time instant. Moreover, the communication signal and the radar waveform are subject to the power budget

\[
\mathbb{E} \{ \mathbf{x}_l^H \mathbf{x}_l \} \leq p_c \quad \text{and} \quad \text{tr} (\mathbf{S} \mathbf{S}^H) \leq p_s,
\]

respectively, where \( p_c \) and \( p_s \) denote the communication SNR and sensing SNR, respectively. In the second stage of the downlink ISAC, the BS uses the reflected signals from the RTs to sense the nearby environment. It is worth noting that the signals reflected by the CUs are assumed to be perfectly removed at the BS for simplicity. Therefore, the BS observes the following superposed signal matrix:

\[
\mathbf{Y} = \mathbf{G}^H (\mathbf{S} + \mathbf{X}) + \mathbf{N}^H,
\]

where \( \mathbf{G} = [\mathbf{g}_{i,j}] = [\mathbf{g}_1 \cdots \mathbf{g}_N] \in \mathbb{C}^{M \times N} \) denotes the channel matrix to be sensed with \( \mathbf{g}_{i,j} \) representing the target response from the \( i \)th transmit antenna to the \( j \)th receive antenna; \( \mathbf{N} = [\mathbf{n}_1 \cdots \mathbf{n}_N] \in \mathbb{C}^{L \times N} \) denotes the AWGN. Particularly, we assume that \( \mathbf{h}_i \sim \mathcal{C}\mathcal{N} (0, \mathbf{R}_i) \) with \( \mathbf{R}_i \succ 0 \) denoting the transmit correlation matrix and \( \mathbb{E} \{ \mathbf{h}_i \mathbf{h}_j^H \} = 0 \) (\( \forall i \neq j \)). Furthermore, we assume \( \mathbf{n}_n \sim \mathcal{C}\mathcal{N} (0, \mathbf{I}_L) \), \( \mathbb{E} \{ \mathbf{n}_n \mathbf{n}_m^H \} = 0 \) (\( \forall n \neq m \)), \( \mathbf{g}_n \sim \mathcal{C}\mathcal{N} (0, \mathbf{R}_T) \) (\( \mathbf{R}_T \succ 0 \)), and \( \mathbb{E} \{ \mathbf{g}_n \mathbf{g}_m^H \} = 0 \) (\( \forall n \neq m \)). Moreover, full channel state information (CSI) of CU \( i \) (\( \forall i \)) is assumed to be available at the BS and CU \( i \). By contrast, since \( \mathbf{G} \) needs to be sensed, the BS is assumed to know only the spatial correlation matrix, \( \mathbf{R}_T \).
B. Uplink ISAC

The uplink ISAC (U-ISAC) also includes two stages. Firstly, the BS broadcasts the radar waveform \( S \) for sensing the nearby environment. Secondly, the BS receives the radar waveform reflected by the RTs and the communication messages sent by the CUs simultaneously. Like the downlink case, we assume the BS can remove the signals reflected by the CUs. As a result, the BS observes the following superposed signal:

\[
Y = \sum_{k=1}^{K} \bar{h}_k \bar{x}_k^H + G^H S + N^H, \tag{2}
\]

where \( \bar{h}_i \in \mathbb{C}^{N \times 1} \) denotes the channel vector from CU \( i \) to the receive antenna array of the BS; \( \bar{x}_i = [x_{i_1}, \cdots, x_{i_L}]^H \) denotes the message sent by CU \( i \) subject to the power budget \( \mathbb{E} \{|x_{i,j}|^2\} \leq p_c \). As explained earlier, the correlation between receive antennas can be omitted, and thus we can assume \( \bar{h}_i \sim \mathcal{CN}(0, I_N) \) and \( \mathbb{E} \{\bar{h}_i \bar{h}_j^H\} = 0 \) (\( \forall i \neq j \)). Besides, we assume the BS knows the full information of \( \bar{h}_i \) (\( \forall i \)) and \( \mathbf{R}_T \). After the BS receives \( Y \) presented in (2), it can leverage a successive interference cancellation (SIC)-based framework to decode the communication signal, \( \bar{x}_i \), as well as sensing the channel matrix, \( G \) [6]. Specifically, the BS first decodes \( \bar{x}_i \) by treating the radar waveform as interference. Then, \( \bar{x}_i \) can be subtracted from \( Y \) and the rest part will be used for sensing.

III. Downlink Performance

A. Performance of Communications

In this letter, we assume only the statistical information of the communication signal is used during the design of the radar waveform. Thus, CU \( k \) can know the designed waveform in advance and remove the term \( h_k^H S \) from \( y_k^H \) before decoding the information bits. Besides, to explore the fundamental communication performance, we exploit dirty paper coding (DPC) to generate \( \mathbf{X} \), which can achieve the sum CR capacity of broadcast channels. Under the uplink-downlink duality, the maximal downlink sum CR satisfies \( R_d = \max \sum_{k=1}^{K} p_k \log_2 \det \left( \mathbf{I}_M + \sum_{k=1}^{K} p_k \mathbf{h}_k \mathbf{h}_k^H \right) \) [9].

1) Outage Probability: The outage probability (OP) of the sum CR is given by

\[
P_d = \Pr (R_d < \mathcal{R}),
\]

where \( \mathcal{R} \) denotes the target sum rate. Yet, \( R_d \) lacks any closed-form solutions, which together with the fact that \( \{\mathbf{h}_k\}_{k=1}^{K} \) are independent but not identically distributed random vectors, makes
the quantitative analysis of $P_d$ an intractable problem. As a compromise, we assume all the CUs share the same correlation matrix to glean further insights. In this case, the following theorem describes the high-SNR behaviour of the OP.

**Theorem 1.** When $R_k = R$ ($\forall k$), the outage probability satisfies $\lim_{p_c \to \infty} P_d = O \left( p_c^{-MK} \right)$.

*Proof:* Please refer to Appendix [A] for more details.

**Remark 1.** When $R_k = R$ ($\forall k$), a diversity order of $KM$ is achievable for the sum communication rate of the CUs, which increases with the number of transmit antennas of the BS.

2) Ergodic Rate: The ergodic CR (ECR) of the CUs is given as $\bar{R}_c = \mathbb{E}\{R_d\}$. Let $\tilde{H} = [\mathbf{h}_1 \cdots \mathbf{h}_K] \in \mathbb{C}^{M\times K}$ denote the concatenation of the channels. Then, the following theorem describes the high-SNR behaviour of the ECR.

**Theorem 2.** The downlink ECR satisfies $\lim_{p_c \to \infty} \bar{R}_c = K \log_2 \frac{p_c}{K} + \mathbb{E}\{\log_2 \det (\tilde{H}^H \tilde{H})\}$.

*Proof:* Please refer to Appendix [B] for more details.

Note that $\mathbb{E}\{\log_2 \det (\tilde{H}^H \tilde{H})\}$ is a constant independent of $p_c$, which lacks any closed-form expressions. Yet, when $R_k = I_M$ ($\forall k$), the following corollary can be found.

**Corollary 1.** When $R_k = I_M$ ($\forall k$), $\mathbb{E}\{\log_2 \det (\tilde{H}^H \tilde{H})\} = \frac{1}{m^2} \sum_{l=0}^{K-1} \left( \sum_{i=1}^{M-l-1} \frac{1}{i} - C \right)$, where $C$ is the Euler constant.

*Proof:* Please refer to Appendix [B] for more details.

**Remark 2.** A high-SNR slope of $K$ is achievable for the $K$ CUs. When $K = M$, the high-SNR slope can be improved by increasing the number of transmit antennas of the BS.

**B. Performance of Sensing**

Having analyzed the performance of the downlink communication signal, we now turn our attention to the sensing signal. Particularly, the BS can use the received signal presented in (1) to sense the channel $\mathbf{G}$. The SR is evaluated by the sensing mutual information (MI) per unit time [4], [8]. In this letter, we assume that each waveform symbol lasts 1 unit time. Accordingly, the downlink SR can be mathematically characterized as $\bar{I}_L / L$, where $\bar{I}_L$ denotes the sensing MI over the duration of $L$ symbols. By definition, we can obtain $\bar{I}_L = I(\mathbf{Y};\mathbf{G}|\mathbf{S})$ [4], [8], [10], where $I(X;Y|Z)$ denotes the MI between $X$ and $Y$ conditioned on $Z$. To simplify the expression of $\bar{I}_L$ as well as the corresponding analyses, we treat $\mathbf{G}^H \mathbf{X}$ as interference, which thus
yields a sensing performance lower bound. Moreover, from a worst-case design perspective [11],
the aggregate interference-plus-noise \( \mathbf{G}^H \mathbf{X} + \mathbf{N} \) is treated as the Gaussian noise. Since \( \mathbf{X} \) is dirty paper coded, we have \( \mathbb{E} \{ \mathbf{X} \mathbf{X}^H \} = \mathbf{0} (\forall i \neq j) \) and \( \mathbf{x}_i \sim \mathcal{CN} (0_M, \mathbf{J}_M) (\forall \mathbf{l}) \) with \( \text{tr} (\Sigma_{\mathbf{H}}) \leq p_c \), where \( \Sigma_{\mathbf{H}} \) is obtained by the iterative water-filling method as well as the uplink-downlink duality [9]. The following lemma describes the sensing MI.

**Lemma 1.** The sensing MI can be written as \( \mathcal{I}_L = N \log_2 \det \left( \mathbf{I}_L + \frac{1}{\sigma^2} \mathbf{S}^H \mathbf{R}_x \mathbf{S} \right) \) with \( \sigma^2 = 1 + \text{tr} (\mathbf{R}_x \Sigma) \) and \( \Sigma = \mathbb{E}_{\mathbf{H}} (\Sigma_{\mathbf{H}}) \leq p_c \).

**Proof:** Please refer to Appendix C for more details. \( \blacksquare \)

We comment that \( \Sigma \) lacks any closed-form expressions, which can be evaluated numerically. Based on Lemma 1, the maximal downlink SR can be expressed as

\[
\mathcal{R}_d = \frac{N}{L} \max_{\text{tr}(\mathbf{SS}^H) \leq p_s} \log_2 \det \left( \mathbf{I}_L + \frac{1}{\sigma^2} \mathbf{S}^H \mathbf{R}_x \mathbf{S} \right).
\]

The following theorem provides an exact expression for \( \mathcal{R}_d \) as well as its high-SNR approximation.

**Theorem 3.** The maximal downlink SR of the considered ISAC system is given by \( \mathcal{R}_d = \frac{N}{L} \sum_{m=1}^M \log_2 \left( 1 + \frac{\lambda_m s_m^2}{\sigma^2} \right) \), where \( \{\lambda_m\}_{m=1}^M \) denote the eigenvalues of \( \mathbf{R}_x \) and \( s_m^* = \max \left\{ 0, \frac{1}{\nu} - \frac{\sigma^2}{\lambda_i} \right\} \) with \( \sum_{m=1}^M \max \left\{ 0, \frac{1}{\nu} - \frac{\sigma^2}{\lambda_i} \right\} = p_s \). The maximal SR is achieved when the eigendecomposition (ED) of \( \mathbf{SS}^H \) satisfies \( \mathbf{SS}^H = \mathbf{U}_x \Delta^* \mathbf{U}_x \), where \( \mathbf{U}_x \Delta \{\lambda_1, \cdots, \lambda_M\} \) \( \mathbf{U}_x \) denotes the ED of \( \mathbf{R}_x \) with \( \lambda_1 \geq \cdots \geq \lambda_M > 0 \) and \( \Delta^* = \text{diag} \{s_1^*, \cdots, s_M^*\} \). If the sensing SNR \( p_s \) approaches infinity, we can obtain \( \mathcal{R}_d \approx \frac{NM}{L} \left( \log_2 p_s + \frac{1}{M} \sum_{m=1}^M \log_2 \left( \frac{\lambda_m}{\lambda_m \sigma^2} \right) \right) \).

**Proof:** Please refer to Appendix D for more details. \( \blacksquare \)

**Remark 3.** A high-SNR slope of \( \frac{NM}{L} \) is achievable for the maximal downlink SR, which can be improved by increasing the number of BS antennas.

In the following, we set \( \mathbf{R}_x = \mathbf{I}_M \) and \( L = M = N \) to unveil more system insights.

**Corollary 2.** When \( \mathbf{R}_x = \mathbf{I}_M \) and \( L = M = N \), the maximal SR is \( \mathcal{R}_d = N \log_2 \left( 1 + \frac{p_s}{\text{tr} (\Sigma)} \right) \triangleq \mathcal{R}(N) \).

By checking the first-order derivative of \( \mathcal{R}(x) \) with respect to \( x \geq 1 \), we find that \( \frac{d}{dx} \mathcal{R}(x) \geq 0 \) \( (x \geq 1) \). Moreover, it can be found that \( \lim_{N \to \infty} \mathcal{R}(N) = \frac{p_s}{\text{tr}(\Sigma)} \log_2 e \geq \frac{p_s}{1+p_c} \log_2 e \).

**Remark 4.** The fact of \( \frac{d}{dx} \mathcal{R}(x) \geq 0 \) \( (x \geq 1) \) suggests that the maximal achievable SR, \( \mathcal{R}(N) \),
increases with the RadCom BS antenna number, $N$, monotonically.

**Remark 5.** The facts of \( \lim_{N \to \infty} \mathcal{R}(N) = \frac{p_c}{1 + \text{tr}(\Sigma)} \log_2 e \) and \( \frac{d}{dx} \mathcal{R}(x) \geq 0 \ (x \geq 1) \) indicate that the downlink SR is limited by the power used for both radar sensing and communications. The reason lies in that the communication signal serves as interference during the sensing procedure.

### C. Performance of FDSAC

Turn now to the performance of downlink FDSAC (D-FDSAC) systems, where the total bandwidth is separated into two sub-bands, one for sensing only and the other for communications. It is assumed that $\alpha \in [0, 1]$ fraction of the total bandwidth is used for communications. In this case, the ECR is given by $\mathcal{R}_\alpha^c = \mathbb{E} \{ \mathcal{R}_{\alpha,c,d} \}$ with $\mathcal{R}_{\alpha,c,d} = \alpha \max_{\sum_{i=1}^K p_i \leq p_c} \log_2 \det \left( \mathbf{I}_M + \sum_{k=1}^K \frac{p_c}{\alpha} \mathbf{h}_k \mathbf{h}_k^H \right)$. As for radar sensing, the maximal downlink SR is

$$\mathcal{R}_\alpha^s = \frac{N(1 - \alpha)}{L} \max_{\text{tr}(\mathbf{SS}^H) \leq p_s} \log_2 \det \left( \mathbf{I}_L + \frac{1}{1 - \alpha} \mathbf{S}^H \mathbf{R}_T \mathbf{S} \right).$$

It is worth noting that $(\mathcal{R}_\alpha^c, \mathcal{R}_\alpha^s)$ can be analyzed in a similar way we analyze $(\mathcal{R}_c, \mathcal{R}_s)$. We find that $\mathcal{R}_\alpha^c$ (or $\mathcal{R}_\alpha^s$) achieves a smaller high-SNR slope than $\mathcal{R}_c$ (or $\mathcal{R}_s$), whereas $\mathcal{R}_{\alpha,c,d}$ yields the same diversity order as $\mathcal{R}_d$.

### IV. Uplink Performance

#### A. Performance of Communications

At the $i$th time instant of uplink ISAC, the BS receives $\mathbf{y}_i = \sum_{k=1}^K \mathbf{h}_k x_{k,i} + \mathbf{G}^H \mathbf{s}_i + \mathbf{n}_i$, where $\mathbf{n}_i \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_N)$ denotes the $i$th column of $\mathbf{N}^H$. To show the fundamental communication performance, we use the minimum mean-square error (MMSE)-SIC decoder to detect the information bits, which is capacity-achieving [9]. Moreover, from a worst-case design perspective [11], the aggregate interference-plus-noise $\mathbf{G}^H \mathbf{s}_i + \mathbf{n}_i$ is treated as the Gaussian noise. Accordingly, the uplink sum CR at the $i$th time instant is given by $\mathcal{R}_{u,i} = \log_2 \det \left( \mathbf{I}_N + p_c \mathbf{H} \mathbf{H}^H \mathbf{W}_i^{-1} \right)$, where $\mathbf{H} = [\mathbf{h}_1 \cdots \mathbf{h}_K] \in \mathbb{C}^{N \times K}$ and $\mathbf{W}_i = \mathbb{E} \left\{ (\mathbf{G}_i \mathbf{s}_i + \mathbf{n}_i) (\mathbf{s}_i^H \mathbf{G} + \mathbf{n}_i^H) \right\} = \mathbf{G}_i^2 \mathbf{I}_N$ with $\mathbf{G}_i^2 = 1 + |\mathbf{s}_i^H \mathbf{R}_T \mathbf{s}_i|$. As a result, the uplink sum CR can be simplified to $\mathcal{R}_{u,i} = \log_2 \det \left( \mathbf{I}_N + \frac{\mathbf{G}_i^2}{\mathbf{G}_i^2} \mathbf{H} \mathbf{H}^H \right)$. It is worth mentioning that the uplink sum CR varies with the index of time instant, $i$. For brevity, we leverage the expectation of $\mathcal{R}_{u,i}$ with respect to $i$ to evaluate the uplink performance of communication signals, namely $\mathcal{R}_u = \frac{1}{L} \sum_{i=1}^L \mathcal{R}_{u,i}$. 
1) Outage Probability: The OP of the uplink sum CR is given as

\[ P_u = \Pr (R_u < \overline{R}) , \]

where \( \overline{R} \) denotes the target rate. Using similar steps as those outlined in Appendix A we characterize the high-SNR behaviour of the OP as follows.

**Theorem 4.** In the high-SNR regime, the outage probability satisfies 
\[ \lim_{p_c \to \infty} P_u = O (p_c^{-NK}) . \]

**Remark 6.** A diversity order of \( KN \) is achievable for the uplink sum communication rate, which can be improved by increasing the number of receive antennas at the BS.

2) Ergodic Rate: The uplink ECR of the CUs is given as 
\[ R_c = \mathbb{E} \{ R_u \} . \]

Bearing the same idea built in Appendix B in mind, we obtain Theorem 5.

**Theorem 5.** The ECR satisfies 
\[ \lim_{p_c \to \infty} R_c = K \log_2 p_c + \sum_{i=1}^{K-1} \sum_{C=0}^{M-1} \log_2 \left( \frac{1}{i+\frac{1}{C}} \right) - \frac{L}{K} \sum_{i=1}^{K} \log_2 \vartheta_i^2 . \]

**Remark 7.** A high-SNR slope of \( K \) is achievable for the uplink sum CR, which is the same as that of the downlink sum CR. When \( K = N \), the high-SNR slope can be improved by increasing the number of receive antennas of the BS.

B. Performance of Sensing

After decoding all the information bits sent by the CUs, the BS can remove \( \sum_{k=1}^{K} h_k x_k H_k \) from \( Y \) in (2). The rest part can be used for radar sensing [6], which is expressed as \( Y_s = G H S + N H \). The corresponding SR can be expressed as \( \overline{I}_L / L \), where \( \overline{I}_L \) denotes the sensing MI over the duration of \( L \) symbols. Following similar steps as those outlined in Appendix C we can get 
\[ \overline{I}_L = I (Y_s; G|S) = N \log_2 \det (I + S^H R_T S) . \]

Therefore, the maximal uplink SR satisfies 
\[ \overline{R}_s = \frac{N}{L} \max_{\text{tr}(SS^H) \leq p_s} \log_2 \det (I_L + S^H R_T S) . \]

By the method we derive Theorem 3 we obtain Theorem 6.

**Theorem 6.** The maximal uplink SR is given as 
\[ \overline{R}_s = \frac{N M}{L} \sum_{m=1}^{M} \log_2 \left( 1 + \lambda_m a_m^* \right) , \]

where \( a_m^* = \max \{ 0, 1/\nu - 1/\lambda_i \} \) with \( \sum_{m=1}^{M} \max \{ 0, 1/\nu - 1/\lambda_i \} = p_s \). The SR is maximized when \( SS^H = U_T^H \Theta^* U_T \) with \( \Theta^* = \text{diag} \{ a_1^*, \ldots, a_M^* \} \). When \( p_s \to \infty \), we can obtain 
\[ \overline{R}_s \approx \frac{NM}{L} \left( \log_2 p_s + \frac{1}{M} \sum_{m=1}^{M} \log_2 \left( \frac{\lambda_m}{M} \right) \right) . \]

**Remark 8.** The uplink SR achieves the same high-SNR slope, namely \( \frac{NM}{L} \), as the downlink SR.
Corollary 3. When $R_T = I_M$ and $L = M = N$, the uplink SR is given by $R_s = N \log_2 \left( 1 + \frac{p_s}{N} \right) \triangleq \mathcal{R}(N)$.

Noticeable, $\mathcal{R}(N)$ presents a similar form as $\mathcal{R}(N)$.

Remark 9. The facts of $\lim_{N \to \infty} \mathcal{R}(N) = p_s \log_2 e$ and $\frac{d}{dx} \mathcal{R}(x) \geq 0 \ (x \geq 1)$ indicate that the uplink SR is mainly limited by the power used for radar sensing, which is different from the downlink SR. The reason lies in that the communication signal can be removed by the SIC technique [6] before processing the uplink sensing signal.

C. Performance of FDSAC

Turn now to the uplink FDSAC (U-FDSAC) system where the SR and the ECR are given by $R_s = N \log_2 \left( 1 - \frac{N}{1-\alpha} L \max_{tr[SS^H]} \left( I_N + \frac{p_c}{\alpha} \mathbb{H} \mathbb{H}^H \right) \right) \triangleq R_s^\alpha$ and $R_c = \mathbb{E} \{ R_{c,u}^\alpha \}$ with $R_{c,u}^\alpha = \alpha \log_2 \left( I_N + \frac{p_s}{\alpha} \mathbb{H} \mathbb{H}^H \right)$, respectively. Note that $(\mathcal{R}^\alpha_c, \mathcal{R}^\alpha_s)$ can be discussed in the way we discuss $(\mathcal{R}_c, \mathcal{R}_s)$. In particular, we find the high-SNR slope of $\mathcal{R}^\alpha_c$ (or $\mathcal{R}^\alpha_s$) is no larger than that of $\mathcal{R}_c$ (or $\mathcal{R}_s$). Moreover, we note that $\mathcal{R}_u$ yields the same diversity order as $\mathcal{R}_{c,u}^\alpha$.

Remark 10. The results in Section III-C and Section IV-C demonstrate that the ISAC system achieves a larger high-SNR slope than the FDSAC system in terms of both the CR and the SR. In other words, ISAC can provide more degrees of freedom [12] for both the CR and the SR than FDSAC.

V. Rate Region Characterization

We now characterize the communication-sensing rate region of the considered ISAC and FDSAC systems when $p_c \in [0, \hat{p}_c]$ and $p_s \in [0, \hat{p}_s]$, where $\hat{p}_c$ and $\hat{p}_s$ denote the maximal communication SNR and the maximal sensing SNR, respectively. Let $\mathcal{R}_c^e$ and $\mathcal{R}_s^e$ denote the achievable ECR and SR, respectively. According to our previous discussions, for ISAC systems, the achievable downlink rate region satisfies

\[
\{(\mathcal{R}_c^e, \mathcal{R}_s^e) | \mathcal{R}_c^e \in [0, \mathcal{R}_c], \mathcal{R}_s^e \in [0, \mathcal{R}_s], p_c \in [0, \hat{p}_c], p_s = \hat{p}_s \},
\]

whereas the achievable uplink rate region satisfies

\[
\{(\mathcal{R}_c^e, \mathcal{R}_s^e) | \mathcal{R}_c^e \in [0, \mathcal{R}_c], \mathcal{R}_s^e \in [0, \mathcal{R}_s], p_c = \hat{p}_c, p_s \in [0, \hat{p}_s] \}.
\]
Fig. 2: Performance of communications. $\bar{R} = \bar{R}_c = 5$ bps/Hz, $\alpha = 0.5$, and $p_c = 10$ dB.

As for FDSAC systems, the downlink rate region satisfies

$$\left\{ \left( R^c, R^s \right) \left| R^c \in [0, \bar{R}_c^\alpha], R^s \in [0, \bar{R}_s^\alpha] \right. \right\}$$

whereas the uplink rate region satisfies

$$\left\{ \left( R^c, R^s \right) \left| R^c \in [0, \bar{R}_c^\alpha], R^s \in [0, \bar{R}_s^\alpha] \right. \right\}$$

VI. NUMERICAL RESULTS

Numerical analysis is presented to evaluate the performance of ISAC systems. The parameters used for simulation are listed as follows: $N = 2$, $M = 2$, $L = 4$, and $K = 2$. The $(i, j)$th element of $R_1$ is set as $0.7|i - j|$, whereas the $(i, j)$th element of $R_k = R$ ($\forall k$) is set as $0.8|i - j|$.

Fig. 2(a) and Fig. 2(b) plot the OP and the ECR versus the communication SNR, $p_c$, respectively. As shown in Fig. 2(a), ISAC achieves a lower OP than FDSAC for both downlink and uplink transmissions. Besides, in the high-SNR regime, the OP curves for ISAC and FDSAC are mutually parallel, which suggests that these two systems achieve the same diversity order. This observation agrees with the conclusions drawn in Section III-C and Section IV-C. Note that the curve for $p_c^{-4}$ is also provided to demonstrate the achievable diversity order. As shown, in the high-SNR regime, the curves for OP are parallel to the one for $p_c^{-4}$, suggesting the achievable diversity order obtained in the previous section is tight. Turn now to Fig. 2(b). It can be observed that in the low-SNR regime, ISAC achieves virtually the same ECR as FDSAC, whereas in the
high-SNR regime, ISAC achieves a higher ECR than FDSAC. The reason lies in that the ECR of ISAC systems yields a larger high-SNR slope than the ECR of FDSAC systems.

Fig. 3(a) and Fig. 3(b) plot the downlink and the uplink SRs versus the sensing SNR, $p_s$, respectively. As shown, the SR curve of ISAC yields a larger high-SNR slope than the SR curve of FDSAC. Besides, as Fig. 3(a) shows, in low and moderate SNR regions, D-FDSAC achieves a higher SR than D-ISAC. This is because the communication signal interferes in D-ISAC’s sensing procedure, thus reducing the SR. Yet, involving a larger high-SNR slope, the SR of D-ISAC will exceed that of D-FDSAC as $p_s$ increases, which agrees with the observation from Fig. 3(a). By contrast, as shown in Fig. 3(b), the SR of U-ISAC is higher than that of U-FDSAC in regions of all SNR. This is because the communication signal has no influence on U-ISAC’s sensing procedure under the SIC-based framework [6].

Notably, it is challenging to provide a rigorous comparison of the CR-SR regions achieved by
ISAC and FDSAC. As a compromise, we provide some numerical results in Fig. 4 for a heuristic exploration. In particular, Fig. 4(a) compares the downlink rate regions of ISAC (presented in (3)) and FDSAC (presented in (5)). As shown in this graph, the achievable rate region of D-FDSAC is entirely included in the achievable rate region of D-ISAC, which highlights the superiority of D-ISAC over conventional D-FDSAC. Fig. 4(b) compares the uplink rate regions of ISAC (presented in (4)) and FDSAC (presented in (6)). It can be observed that the rate region of U-FDSAC is mostly covered by that of U-ISAC. However, as Fig. 4(b) shows, in the high-ECR region, U-FDSAC yields a slightly higher SR than U-ISAC.

VII. CONCLUSION

Fundamental performance have been analyzed for downlink and uplink ISAC. Theoretical analyses have shown that ISAC can provide more degrees of freedom for both the communication rate and the sensing rate than conventional FDSAC.

APPENDIX A
PROOF OF THEOREM 1

Proof: The sum rate by DPC satisfies \( \lim_{p_c \to \infty} R_d = \log_2 \det \left( I_K + \frac{p_c}{K} H^H H \right) \) \[13\], which yields \( \lim_{p_c \to \infty} P_d = \lim_{p_c \to \infty} \Pr \left( \det \left( I_K + \frac{p_c}{K} H^H H \right) < 2^R \right) \). When \( R_k = R \ (\forall k) \), it follows that \( \lim_{p_c \to \infty} P_d = O \left( p_c^{-MK} \right) \) \[14\].

APPENDIX B
PROOF OF THEOREM 2

Proof: Since \( \lim_{p_c \to \infty} R_d = \log_2 \det \left( I_K + \frac{p_c}{K} H^H H \right) \), we have

\[
\lim_{p_c \to \infty} \mathcal{R}_c = \lim_{p_c \to \infty} \mathbb{E} \left\{ \det \left( \frac{p_c}{K} H^H H \right) \right\}.
\]

It follows that \( \lim_{p_c \to \infty} \mathcal{R}_c = K \log_2 \frac{p_c}{K} + \mathbb{E} \left\{ \log_2 \det \left( H^H H \right) \right\} \). When \( R_k = I_M \ (\forall k) \), we can get

\[
\mathbb{E} \left\{ \log_2 \det \left( H^H H \right) \right\} = \frac{1}{\ln 2} \sum_{l=0}^{K-1} \left( \sum_{i=1}^{M-l-1} \frac{1}{i} - C \right) \[15\].
\]

APPENDIX C
PROOF OF LEMMA 1

Proof: Denote \( Z = G^H X + N = [z_1 \cdots z_N]^H \in \mathbb{C}^{N \times L} \), where \( z_n^H = g_n^H x_n + n_n^H \). It is worth noting that the row vectors of \( Z \) are mutually independent, which satisfy \( \mathbb{E} \{ z_n z_n^H \} = \mathbb{E} \{ X^H R_T X \} + \)}
As stated before, $\mathbf{X} = [\mathbf{x}_1 \cdots \mathbf{x}_L]$ is generated by DPC, where $\mathbb{E}\{\mathbf{x}_i \mathbf{x}_j^H\} = 0$ ($\forall i \neq j$) and $\mathbb{E}\{\mathbf{x}_i \mathbf{x}_j^H\} = \mathbf{H} (\forall l)$. It follows that $\mathbb{E}\{\mathbf{Y}^H \mathbf{Y}\} = \mathbb{E}\{\text{tr} (\mathbf{R}_T \mathbf{H}) \mathbf{I}_L\} = \text{tr} (\mathbf{R}_T \mathbf{H}) \mathbf{I}_L \preceq \mathbf{0}$ and thus $\mathbb{E}\{\mathbf{z}_n \mathbf{z}_n^H\} = \sigma^2 \mathbf{I}_L$. Denote $\mathbf{Y}^H = [\mathbf{y}_1 \cdots \mathbf{y}_N] \in \mathbb{C}^{L \times N}$, where $\mathbf{y}_n = g_n^H \mathbf{s} + \mathbf{z}_n^H$. When $\mathbf{z}_n$ is treated as Gaussian noise following $\mathcal{C}\mathcal{N}(0, \sigma^2 \mathbf{I}_L)$, we have $\mathbf{y}_n \sim \mathcal{C}\mathcal{N}(0, \mathbf{S}^H \mathbf{R}_T \mathbf{S} + \sigma^2 \mathbf{I}_L)$. By the definition of $I(\mathbf{Y}; \mathbf{G}|\mathbf{S})$, we can get Lemma 1 \cite{10}.

**APPENDIX D**

**PROOF OF THEOREM 3**

*Proof:* Note that $\mathcal{I}_T \triangleq \log_2 \det (\mathbf{I}_L + \frac{1}{\sigma^2} \mathbf{S}^H \mathbf{R}_T \mathbf{S})$ equals the MI of a virtual MIMO channel $\hat{\mathbf{y}} = \mathbf{R}_T^{1/2} \mathbf{x} + \hat{\mathbf{n}}$ with $\mathbb{E}\{\hat{\mathbf{x}} \hat{\mathbf{x}}^H\} = \mathbf{S}^H \mathbf{S}$ and $\mathbf{\hat{n}} \sim \mathcal{C}\mathcal{N}(0, \sigma^2 \mathbf{I}_M)$. Thus, when $\mathcal{I}_T$ is maximized, the eigenvectors of $\mathbf{S}^H \mathbf{S}$ should equal the left eigenvectors of $\mathbf{R}_T^{1/2}$, with the eigenvalues chosen by the water-filling procedure \cite{9}, which yields $\mathcal{R}_s = \frac{N}{L} \sum_{m=1}^{M} \log_2 \left(1 + \frac{1}{\sigma^2} \lambda_m s_m^*\right)$. Here, $\{\lambda_i\}_{i=1}^M$ denote eigenvalues of $\mathbf{R}_T$ and $s_m^* = \max \left\{0, \frac{1}{\nu} - \frac{\sigma^2}{\lambda_m} \right\}$ with $\sum_{m=1}^{M} \max \left\{0, \frac{1}{\nu} - \frac{\sigma^2}{\lambda_m} \right\} = p_s$. When $p_s \to \infty$, we have $\nu \to 0$ and thus $\sum_{m=1}^{M} \max \left\{0, \frac{1}{\nu} - \frac{\sigma^2}{\lambda_m} \right\} = \frac{M}{\nu} - \sum_{m=1}^{M} \frac{\sigma^2}{\lambda_m} = p_s$. Hence, $\lim_{p_s \to \infty} \mathcal{R}_s = \frac{N}{L} \sum_{m=1}^{M} \log_2 \left(\frac{\lambda_m}{M \sigma^2}\right) + \frac{NM}{L} \log_2 \left(p_s + \sum_{m=1}^{M} \frac{\sigma^2}{\lambda_m}\right)$. The final result follows immediately.

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