On dually flat \((\alpha, \beta)\)-metrics

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Abstract

The dual flatness for Riemannian metrics in information geometry has been extended to Finsler metrics. The aim of this paper is to study the dual flatness of the so-called \((\alpha, \beta)\)-metrics in Finsler geometry. By doing some special deformations, we will show that the dual flatness of an \((\alpha, \beta)\)-metric always arises from that of some Riemannian metric in dimensional \(n \geq 3\).

1 Introduction

Dual flatness is a basic notion in information geometry. It was first proposed by S.-I. Amari and H. Nagaoka when they study the information geometry on Riemannian spaces\[2\]. Information geometry has emerged from investigating the geometrical structure of a family of probability distributions, and has been applied successfully to various areas including statistical inference, control system theory and multiterminal information theory\[1, 2\].

In 2007, Z. Shen extended the dual flatness in Finsler geometry\[11\]. A Finsler metric \(F\) on a manifold \(M\) is said to be locally dually flat if at any point there is a local coordinate system \((x^i)\) in which \(F = F(x, y)\) satisfies the following PDEs

\[
[F^2]_{x^i y^j y^k} - 2[F^2]_{x^i} = 0.
\]

Such a coordinate system is said to be adapted.

For a Riemannian metric \(\alpha = \sqrt{a_{ij}(x) y^i y^j}\), it is known that \(\alpha\) is locally dually flat if and only if in an adapted coordinate system, the fundamental tensor of \(\alpha\) is the Hessian of some local smooth function \(\psi(x)\)\[1, 2\], i.e.,

\[
a_{ij}(x) = \frac{\partial^2 \psi}{\partial x^i \partial x^j}(x).
\]

The dual flatness of a Riemannian metric can also be described by its spray\[15\]: \(\alpha\) is locally dually flat if and only if its spray coefficients could be expressed in an adapted coordinate system as

\[
G^i_\alpha = 2\theta y^i + \alpha^2 \theta^i
\]

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for some 1-form $\xi := \xi_i(x)y^i$.

The first example of non-Riemannian dually flat Finsler metrics is the co-call *Funk metric*

$$F = \sqrt{\frac{(1 - |x|^2)|y|^2 + \langle x, y \rangle^2}{1 - |x|^2}} + \frac{\langle x, y \rangle}{1 - |x|^2}$$
on the unit ball $\mathbb{B}^n(1)$, which belongs to a special class of Finsler metrics named *Randers metrics*. Randers metrics are expressed as the sum of a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and an 1-form $\beta = b_i(x)y^i$ with the norm $b := \|\beta\|_\alpha < 1$.

Based on the characterization result for locally dually flat Randers metrics given by X. Cheng et al.\cite{6}, the author provide a more direct characterization and prove that the dual flatness of a Randers metric always arises from that of some Riemannian metric\cite{15}: A Randers metric $F = \alpha + \beta$ is locally dually flat if and only if the Riemannian metric $\bar{\alpha} = \sqrt{1 - b^2\sqrt{\alpha^2 - \beta^2}}$ is locally dually flat and the 1-form $\bar{\beta} = -(1 - b^2)\beta$ is dually related with respect to $\bar{\alpha}$. In this case, $F$ can be reexpressed as

$$F = \sqrt{\frac{(1 - b^2)\bar{\alpha}^2 + \bar{\beta}^2}{1 - b^2}} - \frac{\bar{\beta}}{1 - b^2}. \quad (1.2)$$

Recall that an 1-form $\beta$ is said to be *dually related* to a locally dually flat Riemannian metric $\alpha$ if in an adopted coordinate system the spray coefficients of $\alpha$ are in the form (1.1) and the covariant derivation of $\beta$ with respect to $\alpha$ are given by

$$b_{ij} = 2\theta_i b_j + c(x)a_{ij} \quad (1.3)$$

for some scalar function $c(x)$. This concept was first introduced by the author in\cite{15}. In particular, we prove that the Riemannian metrics

$$\bar{\alpha} = \sqrt{\frac{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y \rangle^2}{(1 + \mu|x|^2)^{\frac{1}{2}}}} \quad (1.4)$$

are dually flat on the ball $\mathbb{B}^n(r_\mu)$ and the 1-forms

$$\bar{\beta} = \frac{\lambda\langle x, y \rangle}{(1 + \mu|x|^2)^{\frac{3}{4}}} \quad (1.5)$$

are dually related to $\bar{\alpha}$ for any constant number $\mu$ and $\lambda$, where the the radius $r_\mu$ is given by $r_\mu = \frac{1}{\sqrt{-\mu}}$ if $\mu < 0$ and $r_\mu = +\infty$ if $\mu \geq 0$.

As a result, we construct many non-trivial dually flat Randers metrics as following:

$$F(x, y) = \sqrt{\frac{1 + (\mu + \lambda^2)|x|^2}{1 + \mu|x|^2}} \sqrt{\frac{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y \rangle^2}{(1 + \mu|x|^2)^{\frac{1}{2}}} + \frac{\lambda\langle x, y \rangle}{(1 + \mu|x|^2)^{\frac{1}{4}}} \sqrt{1 + (\mu + \lambda^2)|x|^2}}. \quad (1.6)$$

It is just the Funk metric when $\mu = -1$ and $\lambda = 1$.

(1.2) is just the *navigation expression* for Randers metrics, which play a key role in the research of Randers metrics. For example, D. Bao et al. classified Randers metrics with
constant flag curvature \[5\]: \( F = \alpha + \beta \) is of constant flag curvature if and only if \( \tilde{\alpha} \) in \((1,2)\) is of constant sectional curvature and \( \tilde{\beta} \) is homothetic to \( \tilde{\alpha} \), i.e.,

\[
\frac{1}{2} (\tilde{b}_{ij} + \tilde{b}_{ji}) = c\tilde{a}_{ij}
\]

for some constant \( c \). Similarly, D. Bao et al. gave a characterization for Einstein metric of Randers type\[4\]: \( F = \alpha + \beta \) is Einsteinian if and only if \( \tilde{\alpha} \) is Einsteinian and \( \tilde{\beta} \) is homothetic to \( \tilde{\alpha} \). It seems that most of the properties of Randers metrics become simple and clear if they are described with the navigation form\[8\].

Except for Randers metrics, there is another important class of Finsler metrics defined also by a Riemannian metric and an 1-form and given in the form

\[
F = \alpha \phi(\frac{\beta}{\alpha})
\]

where \( \phi(s) \) is a smooth function. Such kinds of Finsler metrics are called \((\alpha,\beta)\)-metrics. It was proposed by M. Matsumoto in 1972 as a direct generalization of Randers metrics. \((\alpha,\beta)\)-metrics form a special class of Finsler metrics partly because of its computability\[3\]. Recently, many encouraging results about \((\alpha,\beta)\)-metrics, including flag curvature property\[9,12\] and projective property\[10,14\] etc., have been achieved.

2011, Q. Xia give a local characterization of locally dually flat \((\alpha,\beta)\)-metrics on a manifold with dimension \( n \geq 3 \):

**Theorem 1.1.**\[13\] Let \( F = \alpha \phi(\frac{\beta}{\alpha}) \) be a Finsler metric on an open subset \( U \subseteq \mathbb{R}^n \) with \( n \geq 3 \). Suppose \( F \) is not of Riemannian type and \( \phi(0) \neq 0 \). Then \( F \) is dually flat on \( U \) if and only if the following conditions hold:

\[
G^i_\alpha = [2\theta + (3k_1 - 2)\tau \beta]y^i + \alpha^2(\theta^i - \tau b^i) + \frac{3}{2}k_3\tau \beta^2 b^i,
\]

\[
r_{00} = 2\theta \beta + (3\tau + 2\tau b^2 - 2b^k\theta^k)\alpha^2 + (3k_2 - 2 - 3k_3\beta^2)\tau \beta^2,
\]

\[
s_{i0} = \beta \theta_i - \theta b_i,
\]

\[
\tau \left\{ s(k_2 - k_3 s^2)(\phi \phi' - s \phi'^2 - s \phi''') - (\phi'^2 + \phi \phi'') + k_1 \phi (\phi - s \phi') \right\} = 0,
\]

where \( \theta \) is an 1-form, \( \tau \) is a scalar function, and \( k_1, k_2, k_3 \) are constants.

The meaning of some notations here can be found in Section 2.

When \( \tau = 0 \), \((1.6)\) becomes \( G^i_\alpha = 2\theta y^i + \alpha^2 \theta^i \), which implies \( \alpha \) is dually flat. Moreover, \((1.7)\) and \((1.8)\) are equivalent to \( b_{ij} = 2b_i b_j - 2b_i \theta^k a_{ij} \), i.e., \( \beta \) is dually related to \( \alpha \) with \( c(x) + 2b_i \theta^k = 0 \). In fact, this is a trivial case. Because in this case, \( F = \alpha \phi(\frac{\beta}{\alpha}) \) will be always dually flat for any suitable function \( \phi(s) \) by Theorem\[11\]. In this paper, we will focus on the non-trivial case. Thereby, the function \( \phi(s) \) must satisfy a 3-parameters equation

\[
s(k_2 - k_3 s^2)(\phi \phi' - s \phi'^2 - s \phi''') - (\phi'^2 + \phi \phi'') + k_1 \phi (\phi - s \phi') = 0.
\]

It is clear that the geometry meaning of the original data \( \alpha \) and \( \beta \) for the dually flat \((\alpha,\beta)\)-metrics is very obscure. The main aim of this paper is to provide a luminous description for a non-trivial dually flat \((\alpha,\beta)\)-metric. By using a special class of metric deformations called \( \beta \)-deformations, we prove that the dual flatness of an \((\alpha,\beta)\)-metrics always arises from that of some Riemannian metric, just as Randers metrics.
Theorem 1.2. Let $F = \alpha \phi(\frac{\beta}{\alpha})$ be a Finsler metric on an open subset $U \subseteq \mathbb{R}^n$ with $n \geq 3$, where $\phi(s)$ satisfies (1.10). Suppose $F$ is not of Riemannian type and $\phi'(0) \neq 0$. Then $F$ is dually flat if and only if $\alpha$ and $\beta$ can be expressed as

$$
\alpha = \eta(\bar{b}^2) \sqrt{\bar{\alpha}^2 - \frac{(k_2 - k_3 \bar{b})}{1 + k_2 \bar{b}^2 - k_3 \bar{b}^4} \bar{\beta}^2}, \quad \beta = -\frac{\eta(\bar{b}^2)}{(1 + k_2 \bar{b}^2 - k_3 \bar{b}^4)^{1/2}} \bar{\beta},
$$

where $\bar{\alpha}$ is a dually flat Riemannian metric on $U$, $\bar{\beta}$ is dually related to $\bar{\alpha}$, $\bar{b} := \|\bar{\beta}\|_{\bar{\alpha}}$. The deformation factor $\eta(\bar{b}^2)$ is determined by the coefficients $k_1, k_2, k_3$ and given in the following five cases,

1. When $k_3 = 0$, $k_2 = 0$,
   $$\eta(\bar{b}^2) = \exp \left\{ \frac{k_1 \bar{b}^2}{4} \right\};$$

2. When $k_3 = 0$, $k_2 \neq 0$,
   $$\eta(\bar{b}^2) = \left\{ 1 + k_2 \bar{b}^2 \right\}^{\frac{k_1}{k_2}};$$

3. When $k_3 \neq 0$, $\Delta_1 > 0$,
   $$\eta(\bar{b}^2) = \left\{ \frac{\sqrt{\Delta_1 + k_2}}{\sqrt{\Delta_1 - k_2}}, \frac{\sqrt{\Delta_1 - k_2 + 2k_3 \bar{b}^2}}{\sqrt{\Delta_1 + k_2 - 2k_3 \bar{b}^2}} \right\}^{\frac{2k_1 - k_3}{\sqrt{\Delta_1}}};$$

4. When $k_3 \neq 0$, $\Delta_1 = 0$,
   $$\eta(\bar{b}^2) = \sqrt{2} \exp \left\{ \frac{k_3 - 2k_1}{2k_2} \left[ \frac{1}{2 + k_2 \bar{b}^2} - \frac{1}{2} \right] \right\};$$

5. When $k_3 \neq 0$, $\Delta_1 < 0$,
   $$\eta(\bar{b}^2) = \frac{\exp \left\{ \frac{2k_1 - k_2}{4\sqrt{-\Delta_1}} \left( \arctan \frac{k_3 - 2k_3 \bar{b}^2}{\sqrt{-\Delta_1}} - \arctan \frac{k_3}{\sqrt{-\Delta_1}} \right) \right\}}{\sqrt{1 + k_2 \bar{b}^2 - k_3 \bar{b}^4}},$$

where $\Delta_1 := k_2^2 + 4k_3$.

$\beta$-deformations, which play a key role in the proof of Theorem 1.2, are a new method in Riemann-Finsler geometry developed by the author in the research of projectively flat $(\alpha, \beta)$-metrics[14]. Roughly speaking, the method of $\beta$-deformations is aim to make clear the latent light. For an analogy, $\alpha$ and $\beta$ just like two ropes tangles together, and it is possible to unhitch them using $\beta$-deformations. The navigation expression for Randers metrics is a representative example. In fact, it is just a specific kind of $\beta$-deformations. In other words, $\beta$-deformations can be regarded as a natural generalization of the navigation expression for Randers metrics. See also [12] for more applications.

The argument in this paper is similar to that in [15], but we don’t show the original ideas here. One can obtain a more deep analysis in the latter.
In Section 4, we will use a skillful method to solve the basic equation (1.10). As a result, we can construct infinity many non-trivial dually flat \((\alpha, \beta)\)-metrics combining with (1.4) and (1.5). In particular, the following metrics
\[
F = \sqrt{\alpha^2 + 2\varepsilon\alpha\beta + \kappa\beta^2}
\]
is locally dually flat if and only if
\[
\alpha = (1 - \kappa\bar{b}^2)^{-1}\sqrt{(1 - \kappa\bar{b}^2)\bar{\alpha}^2 + \kappa\bar{\beta}^2}, \quad \beta = -(1 - \kappa\bar{b}^2)^{-1}\bar{\beta},
\]
where \(\bar{\alpha}\) is locally dually flat and \(\bar{\beta}\) is dually related to \(\bar{\alpha}\).

Taking \(\kappa = 1\) and \(\varepsilon = 1\), one can see that (1.11) is just the Randers metrics
\[
F = \alpha + \beta.
\]
Taking \(\kappa = 0\) and \(\varepsilon = \frac{1}{2}\), then we can obtain a very simple kind of dually flat \((\alpha, \beta)\)-metrics given in the form
\[
F = \sqrt{\alpha(\alpha + \beta)}.
\]

2 Preliminaries

Let \(M\) be a smooth \(n\)-dimensional manifold. A Finsler metric \(F\) on \(M\) is a continuous function \(F : TM \to [0, +\infty)\) with the following properties:

(i) Regularity: \(F\) is \(C^\infty\) on the entire slit tangent bundle \(TM \setminus \{0\}\);

(ii) Positive homogeneity: \(F(x, \lambda y) = \lambda F(x, y)\) for all \(\lambda > 0\);

(iii) Strong convexity: the fundamental tensor \(g_{ij} := \frac{1}{2}F^2_{x^i y^j} y^k y^l - \frac{1}{2}F^2_{x^k y^l} y^i x^j\) is positive definite for all \((x, y) \in TM \setminus \{0\}\).

Here \(x = (x^i)\) denotes the coordinates of the point in \(M\) and \(y = (y^i)\) denotes the coordinates of the vector in \(T_x M\).

For a Finsler metric, the geodesics are characterized by the geodesic equation
\[
\ddot{c}^i(t) + 2G^i(c(t), \dot{c}(t)) = 0,
\]
where
\[
G^i(x, y) := \frac{1}{4}g^{ij}\left\{[F^2]_{x^k y^j} y^k - [F^2]_{x^j y^i}\right\}
\]
are called the spray coefficients of \(F\). Here \((g^{ij}) := (g_{ij})^{-1}\). For a Riemannian metric \(\alpha\), the spray coefficients are given by
\[
G^i_\alpha(x, y) = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k
\]
in terms of the Christoffel symbols of \(\alpha\).

By definition, an \((\alpha, \beta)\)-metric is a Finsler metric in the form \(F = \alpha\phi(\frac{\beta}{\alpha})\), where \(\alpha = \sqrt{a_{ij}(x)y^i y^j}\) is a Riemannian metric, \(\beta = b_i(x)y^i\) is an 1-form and \(\phi(s)\) is a positive smooth function on some symmetric open interval \((-b_o, b_o)\).
On the other hand, the so-called $\beta$-deformations are a triple of metric deformations in terms of $\alpha$ and $\beta$ listed below:

\[ \tilde{\alpha} = \sqrt{\alpha^2 - \kappa(b^2)\beta^2}, \quad \tilde{\beta} = \beta; \]
\[ \hat{\alpha} = e^{\rho(b^2)}\tilde{\alpha}, \quad \hat{\beta} = \tilde{\beta}; \]
\[ \bar{\alpha} = \hat{\alpha}, \quad \bar{\beta} = \nu(b^2)\hat{\beta}. \]

Some basic formulas for $\beta$-deformations are listed below. Be attention that the notation ‘$\tilde{b}_{ij}$’ always means the covariant derivative of the 1-form ‘$\tilde{\beta}$’ with respect to the corresponding Riemannian metric ‘$\tilde{\alpha}$’, where the symbol ‘$\cdot$’ can be nothing, ‘$\tilde{}$’, ‘$\hat{}$’ or ‘$\bar{}$’ in this paper. Moreover, we need the following abbreviations,

\[ r_{00} := r_{ij}y^iy^j, \quad r_i := r_{ij}y^j, \quad r_0 := r_iy^i, \quad r := r_ib^i, \]
\[ s_{i0} := s_{ij}y^j, \quad s^i_0 := a^ij\bar{s}_{j0}, \quad s_i := s_{ij}y^j, \quad s_0 := s_0b^i, \]
where $r_{ij}$ and $s_{ij}$ are the symmetrization and antisymmetrization of $b_{ij}$ respectively, i.e.,

\[ r_{ij} := \frac{1}{2}(b_{ij} + b_{ji}), \quad s_{ij} := \frac{1}{2}(b_{ij} - b_{ji}). \]

Roughly speaking, indices are raised and lowered by $a_{ij}$, vanished by contracted with $b^i$ and changed to be ‘$0$’ by contracted with $y^i$. Since $b_{ij} - b_{ji} = \frac{\partial b_i}{\partial x} - \frac{\partial b_j}{\partial x}$, $s_{ij} = 0$ implies $\beta$ is closed, and vice versa.

**Lemma 2.1.** [14] Let $\tilde{\alpha} = \sqrt{\alpha^2 - \kappa(b^2)\beta^2}, \tilde{\beta} = \beta$. Then
\[
\tilde{G}^i_{\alpha} = G^i_{\alpha} - \frac{\kappa}{2(1 - \kappa b^2)}\{2(1 - \kappa b^2)\beta s^i_0 + r_{00}b^i + 2\kappa s_0\beta b^i\} \\
\quad + \frac{\kappa'}{2(1 - \kappa b^2)}\{(1 - \kappa b^2)\beta^2(r^i + s^i) + \kappa r\beta^2b^i - 2(r_0 + s_0)\beta b^i\}; \\
\tilde{b}_{ij} = b_{ij} + \frac{\kappa}{1 - \kappa b^2}\{b^2r_{ij} + b_is_j + b_js_i\} \\
\quad - \frac{\kappa'}{1 - \kappa b^2}\{rb_ib_j - b^2b_i(r_j + s_j) - b^2b_j(r_i + s_i)\}. 
\]

**Lemma 2.2.** [14] Let $\hat{\alpha} = e^{\rho(b^2)}\tilde{\alpha}, \hat{\beta} = \tilde{\beta}$. Then
\[
\hat{G}^i_{\alpha} = \tilde{G}^i_{\beta} + \rho'\left\{2(r_0 + s_0)y^i - (\alpha^2 - \kappa\beta^2)\left(r^i + s^i + \frac{\kappa}{1 - \kappa b^2}r\beta b^i\right)\right\}, \\
\tilde{b}_{ij} = \tilde{b}_{ij} - 2\rho'\left\{b_i(r_j + s_j) + b_j(r_i + s_i) - \frac{1}{1 - \kappa b^2}r(a_{ij} - \kappa b_ib_j)\right\}. 
\]

**Lemma 2.3.** [14] Let $\tilde{\alpha} = \hat{\alpha}, \tilde{\beta} = \nu(b^2)\hat{\beta}$. Then
\[
\tilde{G}^i_{\alpha} = \hat{G}^i_{\alpha}, \\
\tilde{b}_{ij} = \nu\tilde{b}_{ij} + 2\nu b_i(r_j + s_j). 
\]
3 Proof of Theorem 1.2

Suppose that \( F = \alpha \phi(\frac{\beta}{\alpha}) \) is a non-trivial dually flat \((\alpha, \beta)\)-metric on \( U \). According to Theorem 1.1 it is easy to obtain the following simple facts:

\[
\begin{align*}
    r_{ij} &= \theta_i b_j + \theta_j b_i + (3\tau + 2\tau b^2 - 2b_k \theta^k)a_{ij} + \tau (3k_2 - 2 - 3k_3 b^2)b_i b_j, \\
    s^i_0 &= \beta \theta^i - \theta b_i, \\
    s_0 &= b_k \theta^k \beta - b^2 \theta, \\
    r_i + s_i &= 3\tau (1 + k_2 b^2 - k_3 b^4) b_i, \\
    b_i s_j + b_j s_i &= 2b_k \theta^k b_i b_j - b^2 (\theta_i b_j + \theta_j b_i), \\
    r &= 3\tau (1 + k_2 b^2 - k_3 b^4) b^2.
\end{align*}
\]

Lemma 3.1. Take \( \kappa(b^2) = -k_2 + k_3 b^2 \), then

\[
\tilde{G}^i_\alpha = [2\theta + \tau \beta (3k_1 - 2)] y^i + \tilde{\alpha}^2 \theta^i + \frac{\tau (3k_2 - 2 - 3k_3 b^2) - 2(k_2 - k_3 b^2) b_k \theta^k}{2(1 + k_2 b^2 - k_3 b^4)} \tilde{\alpha}^2 b^i.
\]

Proof. By (1.0), (3.1)-(3.6) and Lemma 2.1 we have

\[
\begin{align*}
    \tilde{G}^i_\alpha &= [2\theta + (3k_1 - 2) \tau \beta] y^i + \alpha^2 (\theta^i - \tau b^i) + \frac{3}{2} k_3 \tau \beta^2 b^i \\
    &\quad - \frac{\kappa}{2(1 - \kappa b^2)} \left\{ 2(1 - \kappa b^2) \beta (\theta^i - \theta b^i) + 2\theta \beta b^i + (3\tau + 2\tau b^2 - b_k \theta^k) \alpha^2 b^i \\
    &\quad + \tau (3k_2 - 2 - 3k_3 b^2) \beta^2 b^i + 2\kappa (b_k \theta^k \beta - b^2 \theta) \beta b^i \right\} \\
    &\quad + \frac{\kappa'}{2(1 - \kappa b^2)} \left\{ 3\tau (1 - \kappa b^2) (1 + k_2 b^2 - k_3 b^4) \beta^2 b^i \\
    &\quad + 3\tau \kappa (1 + k_2 b^2 - k_3 b^4) b^2 \beta b^i - 6\tau (1 + k_2 b^2 - k_3 b^4) \beta^2 b^i \right\} \\
    &= [2\theta + (3k_1 - 2) \tau \beta] y^i + \tilde{\alpha}^2 \theta^i - \frac{1}{2(1 - k b^2)} \left\{ (3\tau \kappa + 2\tau - 2\kappa b_k \theta^k) \alpha^2 \\
    &\quad + [2\kappa^2 b_k \theta^k - 3\tau k_3 (1 - k b^2) + \tau \kappa (3k_2 - 2 - 3k_3 b^2) + 3\tau \kappa' (1 - k_2 b^2 + k_3 b^4)] \beta^2 \right\} b^i.
\end{align*}
\]

When \( \kappa = -k_2 + k_3 b^2 \), it is easy to verify that

\[
\kappa^2 + k_2 \kappa - k_3 = -\kappa' (1 + k_2 b^2 - k_3 b^4),
\]

and hence \( \tilde{G}^i_\alpha \) is given in the following form,

\[
\tilde{G}^i_\alpha = [2\theta + \tau \beta (3k_1 - 2)] y^i + \tilde{\alpha}^2 \theta^i - \frac{3\tau \kappa + 2\tau - 2\kappa b_k \theta^k}{2(1 - k b^2)} \tilde{\alpha}^2 b^i.
\]

Lemma 3.2. Take \( \rho(b^2) = -\frac{1}{4} \int \frac{k_1 - k_2 + k_3 b^2}{1 + k_2 b^2 - k_3 b^4} \, db^2 \), then

\[
\tilde{G}^i_\alpha = 2\hat{\theta} y^i + \hat{\alpha}^2 \hat{\theta}^i,
\]

where \( \hat{\theta} = \theta - \frac{1}{4} \tau [4 - 3(k_1 + k_2 - k_3 b^2)] \beta. \) In particular, \( \hat{\alpha} \) is dually flat on \( U \).
Proof. by (3.4), (3.6), (3.7) and Lemma 2.2 we have

\[
\hat{G}_\alpha^i = \hat{C}_\alpha^i + \rho' \left\{ 6\tau(1 + k_2b^2 - k_3b^4)\beta y^i - \hat{\alpha}^2 \left( 3\tau(1 + k_2b^2 - k_3b^4)b^i \\
+ \frac{\kappa}{1 - k_2b^2} \cdot 3\tau(1 + k_2b^2 - k_3b^4)b^2b^i \right) \right\}
\]

\[
= \left\{ 2\tau + \tau[3k_1 - 2 + 6\rho'(1 + k_2b^2 - k_3b^4)]\beta \right\} y^i + \hat{\alpha}^2\theta^i
\]

\[- \frac{1}{2(1 - k_2b^2)} \left\{ 3\tau\kappa + 2\tau + 6\tau\rho'(1 + k_2b^2 - k_3b^4) - 2\kappa b_k\theta^k \right\} \hat{\alpha}^2 b^i.
\]

Let

\[
\hat{\theta} := \theta + \frac{1}{2}\tau[3k_1 - 2 + 6\rho'(1 + k_2b^2 - k_3b^4)]\beta.
\]

It is easy to verify that the inverse of \((\hat{a}_{ij})\) is given by

\[
\hat{a}^{ij} = e^{-2\rho} \left( a^{ij} + \frac{\kappa}{1 - k_2b^2}b^i b^j \right),
\]

so \(\hat{\theta}^i := \hat{a}^{ij}\hat{\theta}_j\) are given by

\[
\hat{\theta}^i = e^{-2\rho} \left\{ \theta^i + \frac{1}{2(1 - k_2b^2)} \left[ 2\kappa b_k\theta^k + \tau(3k_1 - 2) + 6\tau\rho'(1 + k_2b^2 - k_3b^4) \right] b^i \right\}.
\]

Hence \(\hat{G}^i_\alpha\) can be reexpressed as

\[
\hat{G}^i_\alpha = 2\hat{\theta} y^i + \hat{\alpha}^2\hat{\theta}^i - \frac{3\tau e^{-2\rho}}{2(1 - k_2b^2)} \left\{ k_1 + \kappa + 4\rho'(1 + k_2b^2 - k_3b^4) \right\} \hat{\alpha}^2 b^i.
\]

Obviously, the deformation factor given in the Lemma satisfies

\[
\rho' = -\frac{k_1 + \kappa}{4(1 + k_2b^2 - k_3b^4)},
\]

thus \(\hat{G}^i_\alpha = 2\hat{\theta} y^i + \hat{\alpha}^2\hat{\theta}^i\).

\[\Box\]

**Lemma 3.3.** Take \(\nu(b^2) = -\sqrt{1 + k_2b^2 - k_3b^4}e^{\rho(b^2)}\), then

\[
\hat{G}_\alpha^i = 2\hat{\theta} y^i + \hat{\alpha}^2\hat{\theta}^i,
\]

\[
\bar{b}_{ij} = 2\hat{\theta}_i \hat{\theta}_j + \bar{c}(x)\hat{a}_{ij},
\]

where \(\bar{c}(x)\) is a scalar function. In particular, \(\bar{\beta}\) is dually related to \(\hat{\alpha}\).

**Proof.** Under the deformations used above, combining with (3.1), (3.4), (3.5) and Lemma 2.2 we can see that

\[
\bar{r}_{ij} = \frac{1}{1 - k_2b^2} \left\{ r_{ij} + 2\kappa b_k\theta^k b_i b_j - k_2b^2(\theta_i b_j + \theta_j b_i) + 3\tau\kappa'(1 + k_2b^2 - k_3b^4)b^2b_i b_j \right\}
\]

\[
= \theta_i b_j + \theta_j b_i + \frac{1}{1 - k_2b^2} \left\{ (3\tau + 2\tau b^2 - 2b_k\theta^k) a_{ij} \\
+ \tau(3k_2 - 2 - 3k_3b^2) + 2\kappa b_k\theta^k + 3\tau\kappa'(1 + k_2b^2 - k_3b^4)b^2b_i b_j \right\}
\]

\[
= \theta_i b_j + \theta_j b_i + \frac{1}{1 - k_2b^2} \left(3\tau + 2\tau b^2 - 2b_k\theta^k\right) \bar{a}_{ij} + \tau(3\kappa + 3k_2 - 2)b_i b_j,
\]

\[
\bar{s}_{ij} = s_{ij} = \theta_i b_j - \theta_j b_i.
\]
Similarly, by (3.4), (3.9) and Lemma 2.2 we get
\[
\hat{r}_{ij} = \tilde{r}_{ij} + \frac{k_1 + \kappa}{2(1 + k_2 b^2 - k_3 b^4)} \left\{ 6\tau(1 + k_2 b^2 - k_3 b^4)b_ib_j - \frac{1}{1 - \kappa b^2} \cdot 3\tau(1 + k_2 b^2 - k_3 b^4)b^2\tilde{a}_{ij} \right\}
\]
\[
= \theta_i b_j + \theta_j b_i + \frac{e^{-2\nu}}{2(1 - \kappa b^2)} \left\{ 6\tau + (4 - 3k_1)\tau b^2 - 3\tau k b^2 - 4b_k \theta^k \right\} \tilde{a}_{ij} + \tau(6\kappa + 3k_1 + 3k_2 - 2)b_i b_j,
\]
\[\hat{s}_{ij} = s_{ij} = \theta_i b_j - \theta_j b_i.\]

If we use \(\hat{\theta}\) instead of \(\theta\) to express \(\hat{r}_{ij}\) and \(\hat{s}_{ij}\), then
\[
\hat{r}_{ij} = \hat{\theta}_i \hat{b}_j + \hat{\theta}_j \hat{b}_i + \frac{e^{-2\nu}}{2(1 - \kappa b^2)} \left\{ 6\tau + \tau b^2 - 3\tau k b^2 - 4b_k \theta^k \right\} \hat{a}_{ij}
\]
\[
+ \frac{3}{2} \tau(5\kappa + k_1 + 2k_2) \hat{b}_i b_j,
\]
\[\hat{s}_{ij} = \hat{\theta}_i \hat{b}_j - \hat{\theta}_j \hat{b}_i,
\]
where \(\hat{b}_i = b_i\) according to the definition of \(\beta\)-deformations.

Finally, by (3.11) and Lemma 2.3 we have
\[
\tilde{r}_{ij} = \nu \tilde{r}_{ij} + 6\tau \nu/(1 + k_2 b^2 - k_3 b^4)b_i b_j,
\]
\[
= \tilde{\theta}_i \tilde{b}_j + \tilde{\theta}_j \tilde{b}_i + \frac{e^{-2\nu}}{2(1 - \kappa b^2)} \left\{ 6\tau + \tau b^2 - 3\tau k b^2 - 4b_k \theta^k \right\} \tilde{a}_{ij}
\]
\[
+ \frac{3}{2} \tau \left\{ (5\kappa + k_1 + 2k_2)\nu + 4(1 + k_2 b^2 - k_3 b^4)\nu' \right\} \tilde{b}_i \tilde{b}_j,
\]
\[\tilde{s}_{ij} = \nu s_{ij} = \nu(\tilde{\theta}_i \tilde{b}_j - \tilde{\theta}_j \tilde{b}_i) = \tilde{\theta}_i \tilde{b}_j - \tilde{\theta}_j \tilde{b}_i,
\]
where \(\tilde{\theta} := \hat{\theta}\). It is easy to verify that the deformation factor in the Lemma satisfies
\[
(5\kappa + k_1 + 2k_2)\nu + 4(1 + k_2 b^2 - k_3 b^4)\nu' = 0,
\]
(3.10)

So
\[
\tilde{r}_{ij} = \tilde{\theta}_i \tilde{b}_j + \tilde{\theta}_j \tilde{b}_i + \tilde{c}(x)\tilde{a}_{ij}
\]
where \(\tilde{c}(x)\) is a scalar function and can be reexpressed as
\[
\tilde{c}(x) = -2\tilde{b}_k \tilde{\theta}^k + \frac{3\tau e^{-2\nu} \nu}{2(1 - \kappa b^2)} \left\{ 2(1 - \kappa b^2) + (k_1 - 1)b^2 \right\}.
\]
(3.11)

Combining with \(\tilde{s}_{ij}\), we have \(\tilde{b}_{ij} = 2\tilde{\theta}_i \tilde{b}_j + \tilde{c}(x)\tilde{a}_{ij}\).

From the equality (3.11) we can see that \(\tilde{c}(x) \neq -2\tilde{b}_k \tilde{\theta}^k\) unless \(\tau = 0\). In other words, when \(\tau \neq 0\), \(\tilde{\beta}\) is non-trivial (see the statements below Theorem 1.1 for the reason).

**Proof of Theorem 1.2**

Due to the above Lemmas, we have show that if \(F = \alpha \phi(\hat{\rho})\) is a non-trivial dually flat Finsler metric with dimension \(n \geq 3\), then the output Riemannian metric \(\hat{\alpha}\) is dually flat and the output 1-form \(\hat{\beta}\) is dually related to \(\hat{\alpha}\).

Conversely, by (3.8) we can see that the norm of \(\tilde{b}\) is related to \(b\) as
\[
\tilde{b}^2 = \nu b_i \nu b_j e^{-2\nu} \left( a^{ij} + \frac{\kappa}{1 - \kappa b^2} b^i b^j \right) = b^2,
\]
which implies that the \( \beta \)-deformations given above are reversible. More specifically, we have

\[
\beta = \nu^{-1}(\bar{\beta}^2) = -\frac{e^{-\rho(\bar{\beta}^2)}}{\sqrt{1 + k_2 b^2 - k_3 b^4}} \bar{\beta}
\]

and

\[
\alpha = \sqrt{e^{-2\rho(\bar{\beta}^2)}\bar{\alpha}^2 + \kappa(\bar{\beta}^2)\bar{\beta}^2} = e^{-\rho(\bar{\beta}^2)} \sqrt{\bar{\alpha}^2 - \frac{(k_2 - k_3 \bar{\beta}^2)}{1 + k_2 b^2 - k_3 b^4}} \bar{\beta}^2.
\]

Denote \( \eta(\bar{\beta}^2) := e^{-\rho(\bar{\beta}^2)} \). By (3.9), \( \eta \) can be chose as

\[
\eta(\bar{\beta}^2) = \exp \left\{ \frac{1}{4} \int_0^{\bar{\beta}^2} \frac{k_1 - k_2 + k_3 t}{1 + k_2 t - k_3 t^2} dt \right\}.
\]

Combining with the discussions in the proofs of Lemma 3.1, Lemma 3.2 and Lemma 3.3, it is not hard to see that if \( \bar{\alpha} \) is dually flat and \( \bar{\beta} \) is dually related to \( \bar{\alpha} \), then the output data \( \alpha \) and \( \beta \) of the reverse \( \beta \)-deformations satisfy (1.10) and hence \( F = \alpha \phi(\frac{\beta}{\alpha}) \) is dually flat. \( \square \)

4 Symmetry and solutions of equation (1.10)

In this section, we will solve the basic equation (1.10) in a nonconventional way. Firstly, let us introduce two special transformations for the function \( \phi \):

\[
g_u(\phi(s)) := \sqrt{1 + us^2} \phi \left( \frac{s}{\sqrt{1 + us^2}} \right), \quad h_v(\phi(s)) := \phi(vs),
\]

where \( u \) and \( v \) are constants with \( v \neq 0 \). The motivation of above transformations can be found in [14], here we just need to know that such transformations satisfy

\[
g_u \circ g_{u_2} = g_{u_1 + u_2}, \quad h_v \circ h_{v_2} = h_{v_1 + v_2}, \quad h_v \circ g_u = g_{v^2} \circ h_v,
\]

and hence generate a transformation group \( G \) under the above generation relationship, which is isomorphism to \([\mathbb{R} \times \mathbb{R} \setminus \{0\}, \cdot]\) under the map \( \pi(g_u \circ h_v) = (u, v) \). For the later, the operation is given by \((u_1, v_1) \cdot (u_2, v_2) = (u_1 + v_1^2 u_2, v_1 v_2)\). In particular,

\[
g_u^{-1} = g_{-u}, \quad h_v^{-1} = h_{v^{-1}}.
\]

The importance of the transformation group \( G \) for our question is that the solution space of the 3-parameters equation (1.10) is invariant under the action of \( G \) as below. The computations are elementary and hence omitted here.

**Lemma 4.1.** If \( \phi(s) \) satisfies (1.10), then the function \( \psi(s) := g_u(\phi) \) satisfies the same kind of equation

\[
s(k_2' - k_3's^2)(\psi\psi' - sv^2 - s\psi\psi''') - (\psi'^2 + \psi'''^2 + k_1'\psi(\psi - sv')) = 0,
\]

where

\[
k_1' = k_1 + u, \quad k_2' = k_2 + 2u, \quad k_3' = k_3 - k_2u - u^2.
\]

Moreover, \( \phi(0) = \psi(0) \) and \( \phi'(0) = \psi'(0) \).
Lemma 4.2. If \( \phi(s) \) satisfies (1.10), then the function \( \varphi(s) := h_v(\phi) \) satisfies the same kind of equation

\[
s(k''_2 - k''_3 s^2)(\varphi \varphi' - s \varphi'^2 - s \varphi\varphi'') - (\varphi'^2 + \varphi''') + k''_1(\varphi - s \varphi') = 0,
\]

where

\[
k''_1 = v^2 k_1, \quad k''_2 = v^2 k_2, \quad k''_3 = v^4 k_3.
\]

Moreover, \( \phi(0) = \varphi(0) \) and \( \phi'(0) = v \varphi'(0) \).

Furthermore, there are some invariants. Denote

\[
\Delta_1 = k_2^2 + 4 k_3, \quad \Delta_2 = k_2 - 2 k_1, \quad \Delta_3 = k_1^2 - k_1 k_2 - k_3.
\]

Then we have

Lemma 4.3. \( \text{Sgn}(\Delta_i) \) \( (i = 1, 2, 3) \) are all invariants under the action of \( G \).

Proof. It’s only need to show that \( \text{Sgn}(\Delta_i) \) are invariant for \( g_\alpha(\phi) \) and \( h_v(\phi) \). It is obvious, because by Lemma 4.1 and Lemma 4.2 we have \( \Delta_1' = \Delta_1, \Delta_2' = \Delta_2, \Delta_3' = \Delta_3 \) and \( \Delta_1'' = v^4 \Delta_1, \Delta_2'' = v^2 \Delta_2, \Delta_3'' = v^4 \Delta_3 \).

Further more, \( \Delta_i \) satisfy \( \Delta_2^2 - 4 \Delta_3 = \Delta_1 \). They will play a basic role for the further research.

Next, we will solve the equation (1.10) with the initial conditions

\[
\phi(0) = 1, \quad \phi'(0) = \varepsilon
\]

combining with the transformation group \( G \). Note that for \((\alpha, \beta)\)-metrics \( F = \alpha \phi(\frac{\beta}{\alpha}) \), the function \( \phi(s) \) must be positive near \( s = 0 \) and hence we can always assume \( \phi(0) = 1 \) after necessary scaling. On the other hand, \( \varepsilon \neq 0 \) by the assumption of Theorem 1.1.

Let \( \psi(s) = g_{-k_1}(\phi) \). According to Lemma 4.1, the function \( \psi(s) \) will satisfies the following equation

\[
\{1 + \Delta_2 s^2 + \Delta_3 s^4\} \psi'' = s \{\Delta_2 + \Delta_3 s^2\} \psi'
\]

with the initial conditions

\[
\psi(0) = 1, \quad \psi'(0) = \varepsilon.
\]

Let \( u(s) = \psi^2(s) \). It is easy to see that (1.1) becomes

\[
\{1 + \Delta_2 s^2 + \Delta_3 s^4\} u'' = s \{\Delta_2 + \Delta_3 s^2\} u'
\]

with the initial conditions

\[
u(0) = 1, \quad u'(0) = 2 \varepsilon.
\]

Hence, \( u'(s) \) is given by

\[
u'(s) = \exp \left\{ \frac{1}{2} \int \frac{\Delta_2 + \Delta_3 s^2}{1 + \Delta_2 s^2 + \Delta_3 s^4} ds^2 \right\} := 2 \varepsilon f(s),
\]

where \( f(s) \) satisfying \( f(0) = 1 \) can be expressed as elementary functions. So we have
Lemma 4.4. The solutions of equation (4.2) with the initial conditions \( u(0) = 1, \ u'(0) = 2\epsilon \) are given by

\[
u(s) = 1 + 2\epsilon \int_0^s f(\sigma) \, d\sigma,
\]

where \( f(s) \) satisfying \( f(0) = 1 \) are given in the following:

1. when \( \Delta_3 = 0, \Delta_1 = 0 \),
   \[f(s) = 1;\]

2. when \( \Delta_3 = 0, \Delta_1 \neq 0 \),
   \[f(s) = \sqrt{1 + \Delta_2 s^2};\]

3. when \( \Delta_3 \neq 0, \Delta_1 > 0 \),
   \[f(s) = \sqrt{1 + \Delta_2 s^2 + \Delta_3 s^4} \left\{ \frac{2 + \Delta_2 + \sqrt{\Delta_1} s^2}{2 + \Delta_2 - \sqrt{\Delta_1} s^2} \right\}^{\frac{\Delta_2}{4\sqrt{\Delta_1}}};\]

4. when \( \Delta_3 \neq 0, \Delta_1 = 0 \),
   \[f(s) = \sqrt{1 + \frac{\Delta_2}{2} s^2} \exp \left\{ \frac{1}{2 + \Delta_2 s^2} - \frac{1}{2} \right\};\]

5. when \( \Delta_3 \neq 0, \Delta_1 < 0 \),
   \[f(s) = \sqrt{1 + \Delta_2 s^2 + \Delta_3 s^4} \exp \left\{ \frac{\Delta_2}{2\sqrt{-\Delta_1}} \left[ \arctan \frac{\Delta_2 + 2\Delta_3 s^2}{\sqrt{-\Delta_1}} - \arctan \frac{\Delta_2}{\sqrt{-\Delta_1}} \right] \right\}.\]

Theorem 4.5. The solutions of equation (1.10) with the initial conditions \( \phi(0) = 1, \phi'(0) = \epsilon \) are given by

\[
\phi(s) = \sqrt{(1 + k_1 s^2) \left\{ 1 + 2\epsilon \int_0^s (1 + k_1 \sigma^2)^{-\frac{3}{2}} f\left( \frac{\sigma}{\sqrt{1 + k_1 \sigma^2}} \right) \, d\sigma \right\}}.
\]

Proof. By assumption,

\[
\psi(s) = \sqrt{u} = \sqrt{1 + 2\epsilon \int_0^s f(\sigma) \, d\sigma},
\]

so

\[
\phi(s) = g_{k_1}(\psi)
= \sqrt{1 + k_1 s^2 \psi\left( \frac{s}{\sqrt{1 + k_1 s^2}} \right)}
= \sqrt{(1 + k_1 s^2) \left( 1 + 2\epsilon \int_0^s (1 + k_1 \sigma^2)^{-\frac{3}{2}} f(\sigma) \, d\sigma \right)},
\]

which can also be expressed as the form given in the Theorem.
Most of the solutions of (1.10) are non-elementary. Some elementary solutions are listed below (except for the last two items). Notice that there is no sum of formula when the sum index \( n = 1 \), and we rule \( m!! = 1 \) when \( m \leq 0 \).

- When \( k_1 = 0, k_2 = 0, k_3 = 0 \),
  \[
  \phi(s) = \sqrt{1 + 2\epsilon s};
  \]

- When \( k_1 = 0, k_2 < 0, k_3 = 0 \),
  \[
  \phi(s) = \sqrt{1 + \epsilon \left( s \sqrt{1 + k_2 s^2} + \frac{1}{\sqrt{-k_2}} \arcsin \sqrt{-k_2 s} \right)};
  \]

- When \( k_1 = 0, k_2 > 0, k_3 = 0 \),
  \[
  \phi(s) = \sqrt{1 + \epsilon \left( s \sqrt{1 + k_2 s^2} + \frac{1}{\sqrt{k_2}} \arcsinh \sqrt{k_2 s} \right)};
  \]

- When \( k_3 = 0, k_1 + k_2 = 0 \),
  \[
  \phi(s) = \sqrt{1 + 2\epsilon s + k_1 s^2};
  \]

- When \( k_1 \neq 0, k_2 = \frac{1}{2n} k_1 \ (n = 1, 2, 3, \ldots), k_3 = 0 \),
  \[
  \phi(s) = \sqrt{1 + k_1 s^2 + \epsilon s \sqrt{1 + k_2 s^2} \left[ \frac{(2n)!!}{(2n - 1)!!} - \sum_{k=1}^{n-1} \frac{2(n-k)(2n-2-k)!!(2k-3)!!}{(2n-1)!!(2k)!!} (1 + k_2 s^2)^{-k} \right]};
  \]

- When \( k_1 > 0, k_2 = \frac{1}{2n+1} k_1 \ (n = 1, 2, 3, \ldots), k_3 = 0 \),
  \[
  \phi(s) = \left\{ (1 + k_1 s^2) \left[ 1 + \frac{(2n-1)!!}{(2n)!!} \frac{\epsilon}{\sqrt{k_2}} \arctan \sqrt{k_2 s} \right] + \sqrt{\frac{2(n+1)!!}{(2n)!!} - \sum_{k=1}^{n-1} \frac{2(n-k)(2n-1-k)!!(2k-2)!!}{(2n)!!(2k+1)!!} (1 + k_2 s^2)^{-k}} \right\}^{\frac{1}{n}};
  \]

- When \( k_1 < 0, k_2 = \frac{1}{2n+1} k_1 \ (n = 1, 2, 3, \ldots), k_3 = 0 \),
  \[
  \phi(s) = \left\{ (1 + k_1 s^2) \left[ 1 + \frac{(2n-1)!!}{(2n)!!} \frac{\epsilon}{\sqrt{-k_2}} \arctanh \sqrt{-k_2 s} \right] + \sqrt{\frac{2(n+1)!!}{(2n)!!} - \sum_{k=1}^{n-1} \frac{2(n-k)(2n-1-k)!!(2k-2)!!}{(2n)!!(2k+1)!!} (1 + k_2 s^2)^{-k}} \right\}^{\frac{1}{n}};
  \]

- When \( k_1 \neq 0, k_2 = -\frac{1}{2n+1} k_1 \ (n = 1, 2, 3, \ldots), k_3 = 0 \),
  \[
  \phi(s) = \sqrt{1 + k_1 s^2 + \epsilon s \left[ \frac{(2n+2)!!}{(2n+1)!!} - \sum_{k=1}^{n} \frac{2(n-k+1)(2n)!!(2k-3)!!}{(2n+1)!!(2k)!!} (1 + k_2 s^2)^{k} \right]};
  \]
• When $k_1 > 0, k_2 = -\frac{1}{2n} k_1$ ($n = 1, 2, 3, \ldots$), $k_3 = 0$, 

\[
\phi(s) = \left\{ (1 + k_1 s^2) \left[ 1 + \frac{(2n - 1)!!}{(2n)!!} \frac{\epsilon}{\sqrt{-k_2}} \arcsin \sqrt{-k_2 s} \right] + \epsilon s \sqrt{1 + k_2 s^2} \left[ \frac{(2n + 1)!!}{(2n)!!} - \sum_{k=1}^{n-1} \frac{2(n-k)(2n-1)!!(2k-2)!!}{(2n)!!(2k+1)!!} (1 + k_2 s^2)^k \right] \right\}^{\frac{1}{2}};
\]

• When $k_1 < 0, k_2 = -\frac{1}{2n} k_1$ ($n = 1, 2, 3, \ldots$), $k_3 = 0$, 

\[
\phi(s) = \left\{ (1 + k_1 s^2) \left[ 1 + \frac{(2n - 1)!!}{(2n)!!} \frac{\epsilon}{\sqrt{k_2}} \arcsinh \sqrt{k_2 s} \right] + \epsilon s \sqrt{1 + k_2 s^2} \left[ \frac{(2n + 1)!!}{(2n)!!} - \sum_{k=1}^{n-1} \frac{2(n-k)(2n-1)!!(2k-2)!!}{(2n)!!(2k+1)!!} (1 + k_2 s^2)^k \right] \right\}^{\frac{1}{2}};
\]

• When $k_1 = 0, k_2 = 0, k_3 \neq 0$, 

\[
\phi(s) = \sqrt{1 + 2\epsilon \int_0^s \sqrt{1 - k_3 \sigma^2} \, d\sigma};
\]

• When $k_1 \neq 0, k_2 = 0, k_3 = 0$, 

\[
\phi(s) = \sqrt{1 + 2\epsilon \int_0^s \frac{\sqrt{\lambda \langle x, y \rangle}}{(1 + \mu |x|^2)^{\frac{1}{2}} (1 + \mu |x|^2)^{\frac{1}{2}}} \, d\sigma}.
\]

### 5 Some explicit examples

We can construct some typical examples below.

**Example 5.1.** Take $k_1 = k_2 = k_3 = 0$ and $\epsilon = 1/2$, then $\phi(s) = \sqrt{1 + s}$ satisfies (1.10). By Theorem 1.2, the Finsler metric 

\[
F = \sqrt{\alpha (\alpha + \beta)}
\]

is locally dually flat if and only if $\alpha$ is locally dually flat and $\beta$ is dually related to $\alpha$. In particular, the following metrics 

\[
F = \sqrt{\frac{(1 + \mu |x|^2)|y|^2 - \mu \langle x, y \rangle^2}{(1 + \mu |x|^2)^{\frac{1}{2}} (1 + \mu |x|^2)^{\frac{1}{2}}} \left( \frac{(1 + \mu |x|^2)|y|^2 - \mu \langle x, y \rangle^2}{(1 + \mu |x|^2)^{\frac{1}{2}} (1 + \mu |x|^2)^{\frac{1}{2}}} + \frac{\lambda \langle x, y \rangle}{(1 + \mu |x|^2)^{\frac{1}{2}} (1 + \mu |x|^2)^{\frac{1}{2}}} \right)}
\]

are dually flat.

**Example 5.2.** Take $k_1 = -k_2 = \kappa$, $k_3 = 0$, then $\phi(s) = \sqrt{1 + 2\epsilon s + \kappa s^2}$ satisfies (1.10). By Theorem 1.2, the Finsler metric 

\[
F = \sqrt{\alpha^2 + 2\epsilon \alpha \beta + \kappa \beta^2}
\]
is locally dually flat if and only if
\[ \alpha = (1 - \kappa b^2)^{-1} \sqrt{(1 - \kappa b^2) \bar{\alpha}^2 + \kappa \bar{\beta}^2}, \quad \beta = -(1 - \kappa b^2)^{-1} \bar{\beta}, \]
where \( \bar{\alpha} \) is locally dually flat and \( \bar{\beta} \) is dually related to \( \bar{\alpha} \).

**Example 5.3.** Take \( k_1 = k_3 = 0, k_2 = -1 \) and \( \varepsilon = 1 \), then \( \phi(s) = \sqrt{1 + s \sqrt{1 - s^2} + \arcsin s} \) satisfies (1.10). By Theorem 1.2, the Finsler metric
\[ F = \sqrt{\alpha^2 + \sqrt{\alpha^2 - \beta^2} \beta + \alpha^2 \arcsin \frac{\beta}{\alpha}} \]
is locally dually flat if and only if
\[ \alpha = (1 - \bar{\beta})^{-\frac{3}{4}} \sqrt{(1 - \bar{\beta}) \bar{\alpha}^2 + \bar{\beta}^2}, \quad \beta = -(1 - \bar{\beta})^{-\frac{3}{4}} \bar{\beta}, \]
where \( \bar{\alpha} \) is locally dually flat and \( \bar{\beta} \) is dually related to \( \bar{\alpha} \).

**Example 5.4.** Take \( k_1 = k_3 = 0, k_2 = 1 \) and \( \varepsilon = 1 \), then \( \phi(s) = \sqrt{1 + s \sqrt{1 + s^2} + \arcsinh s} \) satisfies (1.10). By Theorem 1.2, the Finsler metric
\[ F = \sqrt{\alpha^2 + \sqrt{\alpha^2 + \beta^2} \beta + \alpha^2 \arcsinh \frac{\beta}{\alpha}} \]
is locally dually flat if and only if
\[ \alpha = (1 + \bar{\beta})^{-\frac{3}{4}} \sqrt{(1 + \bar{\beta}) \bar{\alpha}^2 - \bar{\beta}^2}, \quad \beta = -(1 + \bar{\beta})^{-\frac{3}{4}} \bar{\beta}, \]
where \( \bar{\alpha} \) is locally dually flat and \( \bar{\beta} \) is dually related to \( \bar{\alpha} \).

**Example 5.5.** Take \( k_1 = k_2 = 0, k_3 = \pm 1 \) and \( \varepsilon = \frac{1}{2} \), then \( \phi(s) = \sqrt{1 + \int_0^s \sqrt{1 \pm \sigma^2} \, d\sigma} \) satisfies (1.10). By Theorem 1.2, the Finsler metric
\[ F = \sqrt{1 + \int_0^s \sqrt{1 \pm \sigma^4} \, d\sigma} \]
is locally dually flat if and only if
\[ \alpha = (1 \mp \bar{b}^4)^{-\frac{3}{4}} \sqrt{(1 \mp \bar{b}^4) \bar{\alpha}^2 \pm \bar{b}^2 \bar{\beta}^2}, \quad \beta = -(1 \mp \bar{b}^4)^{-\frac{3}{4}} \bar{\beta}, \]
where \( \bar{\alpha} \) is locally dually flat and \( \bar{\beta} \) is dually related to \( \bar{\alpha} \).

**Example 5.6.** Take \( k_2 = k_3 = 0, k_1 = \pm 1 \) and \( \varepsilon = \frac{1}{2} \), then \( \phi(s) = \sqrt{(1 \pm s^2)(1 + \int_0^s \frac{e^{\pm s^2}}{(1 \pm \sigma^2)^2} \, d\sigma)} \) satisfies (1.10). By Theorem 1.2, the Finsler metric
\[ F = \sqrt{(\alpha^2 \pm \beta^2) \left( 1 + \int_0^s \frac{\beta}{\alpha} \frac{e^{\pm s^2}}{(1 \pm \sigma^2)^2} \, d\sigma \right)} \]
is locally dually flat if and only if
\[ \alpha = e^{\pm s^2} \bar{\alpha}, \quad \beta = -e^{\pm s^2} \bar{\beta}, \]
where \( \bar{\alpha} \) is locally dually flat and \( \bar{\beta} \) is dually related to \( \bar{\alpha} \).
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