RESIDUE FIELDS FOR A CLASS OF RATIONAL E_\infty-RINGS
AND APPLICATIONS

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Abstract. Given a rational E_\infty-ring satisfying a certain noetherian condition on its homotopy groups, we construct commutative “residue fields.” We use them to describe the Galois group and classify the thick subcategories of the module category purely algebraically.

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1. Introduction

The goal of this paper is to describe certain invariants of structured ring spectra in characteristic zero. We start by first reviewing the motivation from stable homotopy theory.

The chromatic picture of stable homotopy theory identifies a class of “residue fields” which play an important role in global phenomena. Consider the following ring spectra:

1. H\mathbb{Q}: rational homology.
2. For each prime p, mod p homology H\mathbb{F}_p.
3. For each prime p and height n, the n-th 2-periodic Morava K-theory K(n).

These all define multiplicative homology theories on the category of spectra satisfying a perfect Künneth isomorphism: they behave like fields. Moreover, as a consequence of the deep nilpotence technology of [DHS88, HS98], they are powerful enough to describe much of the structure of the stable homotopy category. For example, one has the following result:

**Theorem 1.1** (Hopkins-Smith [HS98]). Let R be a ring spectrum and let \alpha \in \pi_*(R). Then \alpha is nilpotent if and only if the Hurewicz image of \alpha in \pi_*(F \otimes R) is nilpotent, as F ranges over all the prime fields above.

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This fundamental result was used in [HS98] to classify the thick subcategories of the category of finite $p$-local spectra for a fixed prime $p$: all thick subcategories are defined by vanishing conditions for the various residue fields. One can attempt to ask such questions not only for spectra but for general symmetric monoidal, stable $\infty$-categories, as Hovey, Palmieri, and Strickland have considered in [HPS97]; whenever one has an analog of Theorem 1.1, it is usually possible to prove results along these lines.

For instance, let $R$ be an $E_\infty$-ring. Then one can try to study such questions in the $\infty$-category $\text{Mod}(R)$ of $R$-modules. If $\pi_*(R)$ is concentrated in even dimensions and is regular noetherian, then it is possible to construct residue fields, prove an analog of Theorem 1.1, and obtain a purely algebraic description of the thick subcategories of perfect $R$-modules. This has been observed independently by a number of authors (e.g., as a piece of forthcoming work of Antieau-Barthel-Gepner). For $E_\infty$-rings (such as the $E_\infty$-ring $\text{TMF}$ of periodic topological modular forms) which are “built up” appropriately from such nice $E_\infty$-rings, it is sometimes possible to construct residue fields as well. We used this to classify thick subcategories for perfect modules over $E_\infty$-rings such as $\text{TMF}$ in [Mat13].

In this paper, we will study such questions over the rational numbers. Let $A$ be a rational $E_\infty$-ring such that the even homotopy groups $\pi_{\text{even}}(A)$ form a noetherian ring and such that the odd homotopy groups $\pi_{\text{odd}}(A)$ form a finitely generated $\pi_{\text{even}}(A)$-module. We will call such rational $E_\infty$-rings noetherian.

For the statement of our first result, we work with $E_\infty$-rings containing a unit in degree two, called 2-periodic. We recall that an $E_\infty$-ring is even periodic if it satisfies the stronger condition that $\pi_1$ should vanish, but we do not need this. In this case, we will produce, for every prime ideal $p \subset \pi_0(A)$, a “residue field” of $A$, which will be an $E_\infty$-$A$-algebra whose homotopy groups form a graded field.

**Theorem 1.2.** Let $A$ be a rational, noetherian, and 2-periodic $E_\infty$-ring. Given a prime ideal $p \subset \pi_0(A)$, there exists an $E_\infty$-$A$-algebra $k(p)$ such that $k(p)$ is even periodic and the map $\pi_0(A) \to \pi_0(k(p))$ induces the reduction $\pi_0(A) \to \pi_0(A)/p\pi_0(A)$. We will prove an analog of Theorem 1.1 in $\text{Mod}(A)$ for these residue fields, and deduce a classification of thick subcategories of the $\infty$-category $\text{Mod}^{\text{even}}(A)$ of perfect $A$-modules. Let $\pi_{\text{even}}(A) = \bigoplus_{i \in \mathbb{Z}} \pi_i(A)$; this is a graded ring, so $\text{Spec} \pi_{\text{even}}(A)$ inherits a $\mathbb{G}_m$-action.

**Theorem 1.3.** Let $A$ be a rational, noetherian $E_\infty$-ring. The thick subcategories of $\text{Mod}^{\text{even}}(A)$ are in natural correspondence with the subsets of $(\text{Spec} \pi_{\text{even}}(A))/\mathbb{G}_m$ closed under specialization.

We will then apply these ideas to the computation of Galois groups, which we introduced in [Mat14] as an extension of Rognes’s work [Rog08]. The use of residue fields in Galois theory goes back to Baker-Richter’s work in [BR08], which studied the Galois groups of Morava $E$-theories at odd primes. We will show that the Galois theory of a noetherian rational $E_\infty$-ring is “almost” entirely algebraic. (The “almost” comes from, e.g., the possibility of adjoining roots of periodicity generators in degrees $2n, n > 1$.) We prove:

**Theorem 1.4.** If $A$ is a noetherian rational $E_\infty$-ring, then the Galois group of $A$ is the étale fundamental group of the graded ring $\pi_{\text{even}}(A)$, or more precisely of the stack $(\text{Spec} \pi_{\text{even}}(A))/\mathbb{G}_m$. 

We will be able to get away with much weaker hypotheses on \( \pi_\ast(A) \) (i.e., nothing close to regularity) because, over characteristic zero, \( E_\infty \)-ring spectra are simpler. They have a more algebraic feel which gives one a wider range of techniques, and they have been studied in detail starting with Quillen’s work on rational homotopy theory [Qui69]. In particular, there are two basic coincidences that will be used in this paper.

(1) The free \( E_\infty \)-ring on a generator in degree zero is equivalent to the suspension spectrum \( \Sigma_+ \mathbb{Z}_{\geq 0} \). In particular, as a result, it is possible to quotient an \( E_\infty \)-ring by an element in degree zero to get a new \( E_\infty \)-ring.

(2) The free \( E_\infty \)-ring on a generator in degree \(-1\) is equivalent to cochains on \( S^1 \).

Both these conditions are specific to the rational numbers. They fail away from characteristic zero, because of the existence of power operations.

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2. Generalities on rational \( E_\infty \)-rings

In this section, we describe the basic characteristic zero techniques needed for this paper. In particular, we discuss the two “coincidences” of \( E_\infty \)-rings \( \Sigma_+ \mathbb{Z}_{\geq 0} \cong \text{Sym}^\ast(\mathbb{Q}) \) and \( C^\ast(S^1; \mathbb{Q}) \cong \text{Sym}^\ast(\mathbb{Q}[−1]) \) in characteristic zero. We discuss the operation of attaching cells in degrees zero and one for rational \( E_\infty \)-ring spectra. The former coincidence enables one to form quotients by degree zero elements and prove a version of the Cohen structure theorem in the complete case. The latter coincidence enables one to use elementary algebraic methods like the Jordan decomposition to classify modules over the \( E_\infty \)-ring \( \text{Sym}^\ast(\mathbb{Q}[−1]) \), by comparing modules with local systems of \( \mathbb{Q} \)-vector spaces on the circle.

Some of the more refined results require the noetherian hypothesis that will be crucial for most of the main results of this paper.

Definition 2.1. We say that a rational \( E_\infty \)-ring \( A \) is noetherian if:

1. The commutative ring \( \pi_{\text{even}}(A) \) is noetherian.
2. The \( \pi_{\text{even}}(A) \)-module \( \pi_{\text{odd}}(A) \) is finitely generated.

2.1. Degree zero elements. Let \( R \) be an \( E_\infty \)-ring and let \( x \in \pi_0(R) \), defining a map of \( R \)-modules \( R \xrightarrow{x} R \). We will write \( R/x \) for the cofiber of this map. One wants to think of this as a homotopy-theoretic “quotient” of \( R \) by the “ideal” generated by \( x \) and, as in algebra, turn this into an \( E_\infty \)-ring under \( R \). There is, in general, no reason to expect this to be possible (or canonical in any way). The sphere \( S^0 \) is the most basic example of an \( E_\infty \)-ring, but it is folklore that the Moore spectrum \( S^0/2 \) cannot even be a ring spectrum up to homotopy. The obstructions to multiplicative structures have been discussed, for example, in [Str99]. Some further obstructions to structured multiplications, via the theory of power operations, are discussed in [MNN14].

To understand this failure, recall how the quotient is constructed in classical commutative algebra. Let \( A \) be a (classical) commutative ring, and fix \( x \in A \). The
(classical) quotient \( A/(x) \) is the pushout of the diagram of commutative rings

\[
\begin{array}{c}
\mathbb{Z}[t] \\
\downarrow \quad \downarrow \\
A \\
\end{array} \quad \begin{array}{c}
t \mapsto 0 \\
\downarrow \\
A/(x) \end{array} \quad \begin{array}{c}
t \mapsto t \\
\downarrow \\
\mathbb{Z} \end{array}
\]

Here \( \mathbb{Z}[t] \) is the free commutative ring on a generator \( t \), and forming the pushout \( A/(x) \) as above amounts to setting \( x = 0 \).

In homotopy theory, one can make a similar construction. There is a free \( E_\infty \)-ring on a single generator, denoted \( S^0 \{t\} \), whose underlying spectrum is given by

\[
S^0 \{t\} \simeq \bigoplus_{n \geq 0} \Sigma^\infty_+ B \Sigma_n,
\]

as \( \{\Sigma_n\}_{n \geq 0} \) ranges over the symmetric groups. Given an \( E_\infty \)-ring \( R \) and an element \( x \in \pi_0 R \), we obtain a map of \( E_\infty \)-rings \( S^0 \{t\} \to R \) sending \( t \mapsto x \), by the universal property: the space of maps of \( E_\infty \)-rings \( S^0 \{t\} \to R \) is precisely \( \Omega^\infty R \).

In particular, we can form a pushout square

\[
\begin{array}{c}
S^0 \{t\} \quad \begin{array}{c}
t \mapsto 0 \\
\downarrow \\
R \\
\end{array} \\
\end{array} \quad \begin{array}{c}
t \mapsto x \\
\downarrow \\
S^0 \{t\} \quad \begin{array}{c}
R \quad \begin{array}{c}
t \mapsto 0 \\
\downarrow \\
R' \end{array} \\
R'' \end{array}
\end{array}
\]

where \( R' \) is, roughly speaking, the free \( R \)-algebra with \( x = 0 \). Given an \( E_\infty \)-\( R \)-algebra \( R'' \), the space \( \text{Hom}_{E_\infty} (R', \ R'') \) is the space of nullhomotopies of \( x \) in \( R'' \) (which is empty unless \( x \) maps to zero in \( \pi_0 R'' \), and in this case is \( \Omega^{\infty+1} R'' \)).

In general, if \( x \in \pi_0 R \) is fixed, the above construction gives

\[
R' \simeq R \otimes_{S^0 \{t\}} S^0,
\]

which is usually very different, as an \( R \)-module, from \( R/x \). For example, the free \( E_\infty \)-ring with \( p^n = 0 \) is not \( S^0/p^n \). (From the “chromatic” point of view, it is actually invisible: its \( E_r \)-localization vanishes for each \( r \), by the main result of [MNN14].)

In fact, the \( S^0 \{t\} \)-module \( S^0 \) is quite complicated, and is not, for example, perfect. As another illustration, if we worked over \( \mathbb{F}_2 \) rather than \( S^0 \), then \( \mathbb{F}_2 \{t\} \) has homotopy groups given by a polynomial ring on the tautological class \( t \) and certain admissible monomials in the Dyer-Lashof algebra applied to \( t \), so that \( \mathbb{F}_2 \) is an infinite quotient of \( \mathbb{F}_2 \{t\} \) by a regular sequence of polynomial generators.

However, there is another \( E_\infty \)-ring which is better behaved in this regard. The \( E_\infty \)-ring \( S^0 \{t\} \) is obtained from the free \( E_\infty \)-space on a single generator by applying \( \Sigma^\infty_+ \). This is the free symmetric monoidal category on one object: the groupoid of finite sets and isomorphisms between them, or topologically \( \bigsqcup_{n \geq 0} B \Sigma_n \). We could also apply \( \Sigma^\infty_+ \) instead to a somewhat less interesting symmetric monoidal groupoid \( \mathbb{Z}_{\geq 0} \), which has objects given by the natural numbers and no nontrivial isomorphisms. The resulting \( E_\infty \)-ring \( \Sigma^\infty_+ \mathbb{Z}_{\geq 0} \), the “monoid algebra” of the natural numbers (as studied in [ABG+08]), will be written \( S^0[t] \) since its homotopy groups actually are given by \( (\pi_* S^0)[t] \). More generally, we will write \( R[t] \) for \( R \otimes \Sigma^\infty_+ \mathbb{Z}_{\geq 0} \), if \( R \) is any \( E_\infty \)-ring.
Now let $R$ be an $E_\infty$-ring, and let $S^0\{t\} \to R$ be a map classifying an element $x \in \pi_0(R)$. Suppose that we have a factorization in the $\infty$-category $\text{CAlg}$

$$
\begin{array}{ccc}
S^0\{t\} & \xrightarrow{x} & R \\
\downarrow & & \downarrow \\
S^0[t] & \xrightarrow{x} & R
\end{array}
$$

In this case, we can form the relative tensor product $R \otimes_{S^0[t]} S^0$ (using the map $S^0[t] \to S^0$ which sends $t \mapsto 0$), as an $E_\infty$-ring. The cofiber sequence of $S^0[t]$-modules

$$
S^0[t] \xrightarrow{x} S^0[t] \to S^0,
$$

shows that this relative tensor product, as an $R$-module spectrum, is actually $R/x$. In other words, by the universal property of the monoid algebra, if there exists a factorization in the diagram of $E_\infty$-spaces

$$
\begin{array}{ccc}
\bigsqcup_{n \geq 0} B\Sigma_n & \xrightarrow{x} & \Omega^\infty R \\
\downarrow & & \downarrow \\
\mathbb{Z}_{\geq 0} & \xrightarrow{x} & \Omega^\infty R
\end{array}
$$

where $\Omega^\infty R$ is given the multiplicative $E_\infty$-structure, then we can place a natural $E_\infty$-structure on $R/x$. (This condition is sometimes called the “strict commutativity” of $x$.)

Unfortunately, in general, describing maps out of $S^0[t]$ is difficult, since $\mathbb{Z}_{\geq 0}$ does not admit a simple presentation as an $E_\infty$-space. In characteristic zero, i.e., over $\mathbb{Q}$, the natural map

$$
\mathbb{Q}\{t\} \to \mathbb{Q}[t],
$$

becomes an equivalence of $E_\infty$-rings, because the $B\Sigma_n$ become indistinguishable from points. In particular, given any rational $E_\infty$-ring $R$, and an element $x \in \pi_0(R)$, we can obtain a map

$$
\mathbb{Q}[t] \to R, \quad t \mapsto x,
$$

and we can form the relative tensor product $R/x \simeq R \otimes_{\mathbb{Q}[t]} \mathbb{Q}$ as an $E_\infty$-ring. This process can be described as attaching a 1-cell to kill the element $x \in \pi_0(R)$.

**Remark 2.2.** Recent work of Hopkins-Lurie has shown that the $E_\infty$-ring spectra $S^0[t]$ are actually tractable if one works under Morava $E$-theory $E_n$, and $K(n)$-locally. For example, if one works $K(1)$-locally and under $p$-adic $K$-theory $\hat{K}U_p$, then $L_{K(1)}(\hat{K}U_p[t])$ has a decomposition as a two-cell complex, obtained by starting with the free $E_\infty$-ring on one generator $t$ and then killing $\theta(t)$, where $\theta$ is the basic power operation for $K(1)$-local $E_\infty$-rings. In particular, they have been able to describe the space of maps $S^0[t^{\pm 1}] \to E_n$.

In many of the cases we are interested in, the element we will be killing is actually nilpotent. It will be useful for us to have the following lemma. We refer to [Mat14, §3-4] for preliminaries on the notion of “admitting descent.”

**Lemma 2.3.** Let $R$ be a rational $E_\infty$-ring and let $x \in \pi_0 R$ be nilpotent. Then the $E_\infty$-$R$-algebra $R/x$ admits descent over $R$. 


Proof. In fact, thanks to the octahedral axiom, the thick subcategory of $\text{Mod}(R)$ generated by $R/x$ contains $R/x^2, R/x^3, \ldots$, and eventually $R/x^N$ where $N$ is so large that $x^N = 0$. But $R/x^N \simeq R \oplus \Sigma R$ for such $N$, and therefore the thick subcategory generated by $R/x$ actually contains $R$.

Finally, we remark that it is similarly possible to quotient by even degree elements of a rational $E_\infty$-ring. (This is even harder to do over the sphere.) It will be convenient to have this for future reference. Let $n$ be an even integer. In this case, $\text{Sym}^*(\mathbb{Q}[n])$, the free $E_\infty$-ring on a degree $n$ class, has homotopy groups given by a polynomial ring on a degree $n$ generator. In particular, if we consider the map of $E_\infty$-rings $\text{Sym}^*(\mathbb{Q}[n]) \to \mathbb{Q}$ classifying the zero class, then there is a cofiber sequence

$$\Sigma^n \text{Sym}^*(\mathbb{Q}[n]) \to \text{Sym}^*(\mathbb{Q}[n]) \to \mathbb{Q},$$

in $\text{Mod}(\text{Sym}^*(\mathbb{Q}[n]))$. Therefore, if $R$ is any rational $E_\infty$-ring and $x \in \pi_n(R)$ is a class, classified by a map $\text{Sym}^*(\mathbb{Q}[n]) \to R$, we can form the pushout

$$R' = R \oplus \text{Sym}^*(\mathbb{Q}[n]) \mathbb{Q},$$

informally the free $E_\infty$-$R$-algebra with $x = 0$, which as an $R$-module is the cofiber of $\Sigma^n R \xrightarrow{x} R$. We will denote this by $R/x$, as before.

To prove properties of quotenting by even degree elements, one can often reduce to the case $n = 0$. To see this, consider the $E_\infty$-ring $\mathbb{Q}[t_{2}^{\pm 1}]$, the free $E_\infty$-ring on an invertible degree two generator, a localization of $\text{Sym}^*(\mathbb{Q}[2])$. Given any $E_\infty$-ring $A$, we have

$$\pi_0(A \otimes \mathbb{Q}[t_{2}^{\pm 1}]) \simeq \pi_{\text{even}}(A).$$

If we want to understand $\pi_{\text{even}}(A/x)$ for $x \in \pi_n A$, this is given by

$$\pi_{\text{even}}(A/x) \simeq \pi_0(A[t_{2}^{\pm 1}]/x) \simeq \pi_0(A[t_{2}^{\pm 1}]/xt_{2}^{-n/2}),$$

where we have used the fact that, in $A[t_{2}^{\pm 1}]$, attaching a cell to force $x = 0$ is equivalent to attaching one to set $xt_{2}^{-n/2} = 0$, and $xt_{2}^{-n/2}$ is concentrated in degree zero. In particular, it follows that questions about $\pi_{\text{even}}$ of the quotient of a rational $E_\infty$-ring by a degree $n$ element can be reduced to questions about $\pi_0$ of the quotient of a rational $E_\infty$-ring by a degree zero element.

2.2. The Cohen structure theorem. Let $(R, m)$ be a complete local noetherian ring with residue field $k$ of characteristic zero. In this case, a basic piece of the Cohen structure theorem (see for instance [Eis95, Ch. 8]) implies that $R$ contains a copy of its residue field:

**Theorem 2.4.** Hypotheses as above, the projection $R \to R/m \to k$ admits a section.

This result comes from the fact that, in characteristic zero, all field extensions are ind-smooth; this argument implies an analogous result in the world of $E_\infty$-ring spectra, and it is the purpose of this subsection to describe that. In particular, we prove:

**Proposition 2.5.** Let $A$ be a noetherian, rational $E_\infty$-ring such that $\pi_0 A$ is a complete local ring with residue field $k$. Then there exists a morphism of $E_\infty$-ring spectra $k \to A$ such that on $\pi_0$, the composite map $k \to \pi_0 A \to k$ is the identity.
Proof. By Theorem 2.4, there is a section \( \phi: k \to \pi_0 A \) of the reduction map. We want to realize this topologically.

To start with, there is a (unique) map \( \mathbb{Q} \to A \) as \( A \) is rational. Let \( \{ t_\alpha \}_{\alpha \in \Gamma} \) be a transcendence basis of \( k/\mathbb{Q} \), so that we have extensions

\[
\mathbb{Q} \subset \mathbb{Q}(\{ t_\alpha \}) \subset k,
\]

where the first extension is purely transcendental and the second extension is algebraic. For each \( \alpha \in \Gamma \), choose \( u_\alpha \in \pi_0 A \) be defined by \( u_\alpha = \phi(t_\alpha) \), so that \( u_\alpha \) projects to \( t_\alpha \) in the residue field. We obtain a map of \( E_\infty \)-ring spectra

\[
\bigotimes_{\Gamma} \mathbb{Q}[t_\alpha] \to A, \quad t_\alpha \mapsto u_\alpha,
\]

where the left-hand-side is a free \( E_\infty \)-ring on \( |\Gamma| \) variables (i.e., a discrete polynomial ring on the \( \{ t_\alpha \} \)). It necessarily factors over the localization \( \mathbb{Q}(\{ t_\alpha \}) \), so we obtain a map \( \mathbb{Q}(\{ t_\alpha \}) \to A \). This realizes on \( \pi_0 \) the restriction \( \phi|_{\mathbb{Q}(\{ t_\alpha \})} \).

Finally, we want to find an extension over \( k \) such that the diagram

\[
\begin{array}{ccc}
\mathbb{Q}(\{ t_\alpha \}) & \to & \pi_0 A \\
\downarrow & & \downarrow \\
A & \to & A
\end{array}
\]

such that the composite \( k \to \pi_0 A \to k \) is the identity. Since \( k \) is a colimit of finite \( \text{étale} \) \( \mathbb{Q}(\{ t_\alpha \}) \)-algebras (i.e., finite separable extensions), it is equivalent to doing this at the level of \( \pi_0 \) (\cite{Lur12, §8.4}), and the map \( \phi: k \to \pi_0(A) \) enables us to do that.

The argument shows that the set of homotopy classes of maps of \( E_\infty \)-rings \( k \to A \) is in bijection with the set of ring-homomorphisms \( k \to \pi_0(A) \).

2.3. Properties of the quotient \( E_\infty \)-ring. In order to make the ideas sketched above work, we will need a few more preliminaries on the operation of coning off a degree zero element and how this behaves on homotopy. If \( A \) is a rational \( E_\infty \)-ring and \( x \in \pi_0 A \), then the homotopy groups of \( A/x \) are determined additively by the short exact sequence

\[
0 \to \pi_j(A)/x\pi_j(A) \to \pi_j(A/x) \to (\ker x)|_{\pi_{j-1}(A)} \to 0,
\]

but this fails to determine the precise multiplicative structure. In this section, we will show that the multiplicative structure is not so different from that of the subring \( \pi_*(A)/x\pi_*(A) \) under noetherian hypotheses. We will not need the full strength of these results in the sequel.

We begin by reviewing the theory of finite radicial extensions (\cite{GD71, §3.5.4}).

Definition 2.6. A map \( R \to R' \) of commutative rings is a finite radicial extension if:

1. \( R' \) is a finitely generated \( R \)-module.
2. For each map \( R \to k \), where \( k \) is a field, the tensor product \( R' \otimes_R k \) is a local artinian \( k \)-algebra.
In particular, a finite radicial extension induces a universal homeomorphism upon applying Spec (a finite radicial extension is the same thing as a finite universal homeomorphism). Examples include finite purely inseparable extensions of fields in characteristic $p > 0$. The composite of two finite radicial extensions is a finite radicial extension. Given a finite map $R \to R'$, in characteristic zero, the condition is equivalent to assuming that, as the maps $R \to k$ ranges over the residue fields of $R$, then each tensor product $R' \otimes_R k$ is local with residue field $k$.

**Proposition 2.7.** Let $A$ be a rational noetherian $E_{\infty}$-ring and let $x \in \pi_0(A)$. The map $\pi_0(A)/(x) \to \pi_0(A/x)$ is a finite radicial extension.

**Proof.** We already know that $\pi_0(A/x)$ is a finitely generated $\pi_0(A)/(x)$-module, by the short exact sequence (1). We will check that the map is a finite radicial extension fiberwise at each prime.

Fix a prime ideal $\mathfrak{p}$ of $\pi_0(A)/(x)$. Let $x_1, \ldots, x_n \in \pi_0(A)$ project to generators of $\mathfrak{p}$. Localizing $A$ at $\mathfrak{p}$, we may assume that $\pi_0(A)$ is local with maximal ideal $\mathfrak{p}$. Completing, we may even assume that $A$ is complete, so that it contains a copy of its residue field $k$ (as $E_{\infty}$-rings) in view of Proposition 2.5.

We need to show that the map of commutative rings

$$\pi_0(A)/(x, x_1, \ldots, x_n) \to \pi_0(A/x)/(x_1, \ldots, x_n)$$

is a finite radicial extension. Since the left-hand-side is the residue field $k$ of $\pi_0(A)$ at the maximal ideal, and the right-hand-side is a finite module over the left-hand-side, it suffices to show that $\pi_0(A/x)/(x_1, \ldots, x_n)$ has no nontrivial idempotents. In fact, this implies that the finite-dimensional $k$-algebra $\pi_0(A/x)/(x_1, \ldots, x_n)$ is local artinian with residue field $k'$ a finite, separable extension of $k$. If this extension was nontrivial, then we could tensor everything up with $k'$ over $k$ and then find ourselves in a similar situation where $\pi_0(A/x)/(x_1, \ldots, x_n)$ had nontrivial idempotents.

As a result, our claim will follow from the next lemma. In fact, it implies that the connected components of $\text{Spec} \pi_0(A)/(x_1, \ldots, x_n)$ are in bijection with those of $\text{Spec} \pi_0(A/(x_1, \ldots, x_n))$, and in turn with those of $\text{Spec} \pi_0(A)/(x, x_1, \ldots, x_n)$, while the latter is just a point. \qed

**Lemma 2.8.** Let $A$ be a rational noetherian $E_{\infty}$-ring and let $x_1, \ldots, x_r \in \pi_0(A)$. Then the map $\pi_0(A)/(x_1, \ldots, x_r) \to \pi_0(A/(x_1, \ldots, x_r))$ induces an isomorphism on connected components of Spec (i.e., of idempotents in the commutative rings).

**Proof.** In fact, we have a map

$$A \to A/(x_1, \ldots, x_r),$$

and if we form the cobar construction on this map, we obtain an augmented cosimplicial object

$$A/(x_1, \ldots, x_r) \xrightarrow{\pi_0} A/(x_1, \ldots, x_r) \otimes_A A/(x_1, \ldots, x_r) \xrightarrow{\pi_0} \ldots,$$

which converges to the $(x_1, \ldots, x_r)$-adic completion of $A$. In particular, the idempotents in the totalization are the same as the idempotents in the $(x_1, \ldots, x_r)$-adic completion of $\pi_0 A$, or equivalently (by the lifting idempotents theorem) in $\pi_0(A)/(x_1, \ldots, x_r)$. 


Since the operation of taking idempotents commutes with homotopy limits, we conclude that the set of idempotents in $\pi_0(A)/(x_1,\ldots,x_r)$ is the reflexive equalizer

$$\text{Idem}(A/(x_1,\ldots,x_r)) \to \text{Idem}(A/(x_1,\ldots,x_r) \otimes_A A/(x_1,\ldots,x_r)).$$

However, we claim that the two maps in the reflexive equalizers are isomorphisms (and thus equal). In fact, $A/(x_1,\ldots,x_r) \otimes_A A/(x_1,\ldots,x_r)$ obtained by attaching 1-cells to kill the classes $x_1,\ldots,x_r$ in $A/(x_1,\ldots,x_r)$ which are already zero; in particular, as an $E_\infty$-ring, we have

$$A/(x_1,\ldots,x_r) \otimes_A A/(x_1,\ldots,x_r) \simeq A/(x_1,\ldots,x_r) \otimes \mathbb{Q}[y_1,\ldots,y_r], \quad |y_i| = 1$$

which has the same idempotents as $A/(x_1,\ldots,x_r)$. \hfill \Box

By induction (and transitivity), one obtains an analogous result for any finite sequence of elements in $\pi_0 A$. Moreover, as explained at the end of Lemma 2.3, we can thus obtain a result for quotients by even degree elements. We find:

**Theorem 2.9.** Let $A$ be a rational noetherian $E_\infty$-ring and let $x_1,\ldots,x_n \in \pi_{\text{even}}(A)$ be a sequence of elements. Then the map

$$\pi_{\text{even}}(A)/(x_1,\ldots,x_n) \to \pi_{\text{even}}(A/(x_1,\ldots,x_n)),$$

is a finite radicial extension.

2.4. **Degree $-1$ elements.** We will also encounter odd degree elements in homotopy, and thus we consider the free $E_\infty$-ring $\text{Sym}^*\mathbb{Q}[-1]$ on a generator in degree $-1$, over $\mathbb{Q}$. More generally, we let $k$ be a field of characteristic zero, and consider $\text{Sym}^*k[-1]$, the free $E_\infty$-ring over $k$ on a degree $-1$ generator.

Recall that this is

$$\text{Sym}^*k[-1] \simeq \bigoplus_{n \geq 0} (k[-1])^\otimes_n.\Sigma_n.$$ Here $k[-1]^\otimes_n \simeq k[-n]$, and the $\Sigma_n$-action is via the sign representation. For $n \geq 2$, this action is nontrivial, and it follows that the homotopy coinvariants are zero. In particular,

$$\pi_i(\text{Sym}^*k[-1]) \simeq \begin{cases} k & \text{if } i = 0 \\ k & \text{if } i = -1 \\ 0 & \text{otherwise} \end{cases},$$

and the multiplication is determined ("square zero" in degree $-1$).

There are two other $E_\infty$-rings which have a similar multiplication law on their homotopy groups:

1. The cochain $E_\infty$-ring on $S^1$, $C^*(S^1;k)$.
2. The square-zero $E_\infty$-ring $k \oplus k[-1]$.

It follows that these three $E_\infty$-rings are in fact all equivalent. In fact, we can produce maps

$$\text{Sym}^*k[-1] \to C^*(S^1;k), \quad \text{Sym}^*k[-1] \to k \oplus k[-1],$$

such that they are isomorphisms on $\pi_{-1}$ (using the universal property of $\text{Sym}^*$), and therefore are equivalences of $E_\infty$-rings. So, all three are equivalent.
Remark 2.10. If one worked over $\mathbb{F}_p$, the symmetric algebra $\text{Sym}^*(\mathbb{F}_p[-1])$ is definitely much too large to be either $C^*(S^1; \mathbb{F}_p)$ or $\mathbb{F}_p \oplus \mathbb{F}_p[-1]$, but $C^*(S^1; \mathbb{F}_p)$ and $\mathbb{F}_p \oplus \mathbb{F}_p[-1]$ have the same square-zero multiplication on homotopy groups. They are not equivalent as $E_\infty$-rings under $\mathbb{F}_p$ because the zeroth reduced power $\mathcal{P}^0$ acts as the identity on $\pi_{-1}$ of the former and zero on the latter.

More generally, if $n$ is any odd integer, then the same reasoning shows that we have an equivalence of $E_\infty$-rings

$$\text{Sym}^*k[-n] \simeq k \oplus k[-n],$$

and, if $n > 0$, then these are equivalent to cochains on the $n$-sphere, $C^*(S^n; k)$.

These ideas show in particular that we have a method of killing elements in a rational $E_\infty$-ring in odd degree. Let $A$ be a rational $E_\infty$-ring and let $y \in \pi_nA$, where $n$ is odd. We can then form the map

$$\text{Sym}^*(\mathbb{Q}[n]) \to A, \quad \mathbb{Q}[n] \xrightarrow{y} A,$$

classifying $y$, and then the relative tensor product

$$A \otimes_{\text{Sym}^*(\mathbb{Q}[n])} \mathbb{Q},$$

where $\text{Sym}^*(\mathbb{Q}[n]) \to \mathbb{Q}$ classifies zero (as it must). This tensor product is an $E_\infty$-algebra under $A$ where $y$ has been “killed.” A priori, it is harder to control this tensor product, because $Q$ is no longer a perfect $\text{Sym}^*(\mathbb{Q}[n])$-module for $n$ odd. However, at least we have:

Proposition 2.11. If $A$ is nonzero and $y \in \pi_nA$ (for $n$ odd), then the tensor product $A' = A \otimes_{\text{Sym}^*(\mathbb{Q}[n])} \mathbb{Q}$ is nonzero: in fact, $A \to A'$ admits descent.

Proof. The map $\text{Sym}^*\mathbb{Q}[n] \to \mathbb{Q}$ admits descent (in the sense of [Mat14, Def. 3.17]), in view of the equivalence $\text{Sym}^*\mathbb{Q}[n] \simeq \mathbb{Q} \oplus \mathbb{Q}[n]$. It follows that the map $A \to A'$ admits descent as well, by base-change, and in particular $A' \neq 0$. □

This would definitely fail for $E_\infty$-rings under $\mathbb{F}_p$. For example, if $p = 2$, then odd degree elements can be invertible (take the Tate spectrum $\mathbb{F}_2^{tZ/2}$). If $p > 2$, odd degree elements can still be “resilient.” In the Tate spectrum $\mathbb{F}_p^{tZ/p}$, we have

$$\pi_*(\mathbb{F}_p^{tZ/p}) \simeq \mathbb{F}_p[t^\pm 1] \otimes E(\alpha_{-1}),$$

where the exterior generator $\alpha_{-1}$ has the property that $\beta \mathcal{P}^0\alpha_{-1} = t^{-1}$ is invertible. Thus, even though $\alpha_{-1}$ squares to zero, a basic power operation goes from it to an invertible element. It follows that in any $E_\infty$-ring under $\mathbb{F}_p^{tZ/p}$, if $\alpha$ goes to zero, the whole $E_\infty$-ring has to be zero. Such phenomena can never happen in characteristic zero.

2.5. Local systems. Moreover, these equivalences enable us to describe the $\infty$-category of modules over the free algebra $\text{Sym}^*k[-1] \simeq C^*(S^1; k)$ for $\text{char} k = 0$. Let $k$ be any field. We will describe modules over the cochain algebra $C^*(S^1; k)$. For now, we do not assume $k$ to be characteristic zero.

As we discussed in [Mat14, §7.2], we have a natural fully faithful imbedding

$$\text{Mod}(C^*(S^1; k)) \subset \text{Loc}_{S^1}(\text{Mod}(k)),$$

from modules over the cochain $E_\infty$-ring into local systems of $k$-modules on the circle $S^1$. The equivalence sends a $C^*(S^1; k)$-module $M$ to the $k$-module $M \otimes_{C^*(S^1; k)} k$,
which lives as a local system over $S^1$ (since there is a $S^1$'s worth of evaluation maps $C^*(S^1; k) \to k$). Moreover, we can describe the essential image. Given a local system $N \in \text{Loc}_{S^1}(\text{Mod}(k))$, the evaluation $N_q$ (for a fixed basepoint $q \in S^1$) is a $k$-vector space, and $N_q$ acquires a monodromy automorphism $\phi$ coming from a choice of generator of $\pi_1(S^1; q)$. As we observed in [Mat14, Remark 7.9], the local system $N$ belongs to the image of $\text{Mod}(C^*(S^1; k))$ if and only if the action of $\phi$ on $\pi_1(N_q)$ is ind-unipotent, which in this context means that the action of $\phi - 1$ on the homotopy groups $\pi_*(N_q)$ (which form a $k$-vector space) is locally nilpotent.

The right adjoint $\text{Loc}_{S^1}(\text{Mod}(k)) \to \text{Mod}(C^*(S^1; k))$ is given, for a local system $N$, by taking its global sections $\varprojlim_{S^1} N$. Explicitly, if $q \in S^1$ and $\phi: N_q \to N_q$ is as above, we have

$$\varprojlim_{S^1} N = \text{fib}(N_q \to N_q),$$

and in particular, we can determine the homotopy groups via a long exact sequence.

**Remark 2.12.** This discussion is special to the case of $S^1$. For any finite complex $X$ and any $E_\infty$-ring $R$, we have an inclusion $\text{Mod}(C^*(X; R)) \subset \text{Loc}_X(\text{Mod}(R))$, and the image always is contained in the subcategory of local systems satisfying an ind-unipotence property, but the precise identification of the image (which is valid for any $R$) relies on the 1-dimensionality of the circle.

Now, to give a local system on $S^1$ in some $\infty$-category $C$ is equivalent to giving an object of that $\infty$-category and an automorphism, via $S^1 \simeq K(\mathbb{Z}, 1)$. Fix two $C$-valued local systems on $S^1$, $(x, \phi_x)$, $(y, \phi_y)$, where $x, y \in C$ and $\phi_x: x \simeq x$, $\phi_y: y \simeq y$ are automorphisms in $C$. Given a map $f: x \to y$ such that the diagram

$$
\begin{array}{ccc}
  x & \xrightarrow{\phi_x} & x \\
  | & f & | \\
  y & \xrightarrow{\phi_y} & y
\end{array}
$$

commutes up to homotopy, then we can produce a map of local systems $(x, \phi_x) \to (y, \phi_y)$. (Specifying such a map amounts in addition to choosing a homotopy to make the diagram commute, and there may be many homotopy classes of such.)

Consider as an example the case of a local system of $k$-modules on $S^1$. Any $k$-module $M$ decomposes as a sum of its homotopy groups, i.e., $M \simeq \bigoplus_{n \in \mathbb{Z}} (\pi_n M)[n]$, and an automorphism $\phi: M \to M$ is determined by what it does on its homotopy groups. It follows that, for each $n$, we can produce squares

$$
\begin{array}{ccc}
  (\pi_n M)[n] & \xrightarrow{\phi_*} & (\pi_n M)[n] \\
  | & | & | \\
  M & \xrightarrow{\phi} & M
\end{array}
$$
which commute up to homotopy. Putting this together, we can produce a square
\[
\begin{array}{ccc}
\bigoplus_{n \in \mathbb{Z}} (\pi_n M)[n] & \xrightarrow{\phi_*} & \bigoplus_{n \in \mathbb{Z}} (\pi_n M)[n] \\
M & \xrightarrow{\phi} & M
\end{array}
\]
which commutes up to homotopy, and where the vertical maps are now equivalences. It follows that the pair \((M, \phi)\) of \(M\) and the self-equivalence is equivalent (non-canonically!) to the direct sum of the pairs \(\left( (\pi_n M)[n], \phi_* \right)\), where each of these is concentrated in a single degree.

We get:

**Proposition 2.13.** Any local system of \(k\)-modules on \(S^1\) is equivalent to a sum of copies of local systems of \(k\)-modules which are shifted discrete.

Using the imbedding \(\text{Mod}(C^*(S^1; k)) \subset \text{Loc}_{S^1}(\text{Mod}(k))\), we find from this:

**Corollary 2.14.** Any \(C^*(S^1; k)\)-module is a direct sum of copies of \(C^*(S^1; k)\)-modules \(M\) such that \(M \otimes_{C^*(S^1; k)} k\) is shifted discrete (and has ind-unipotent action of \(\pi_1(S^1)\)).

In order to give a discrete local system on \(S^1\), it suffices simply to give a \(k\)-vector space (in the classical sense!) with an automorphism. In the case we are interested, i.e., local systems coming from \(C^*(S^1; k)\)-modules, it follows that to give an equivalence class of \(C^*(S^1; k)\)-modules \(M\) equates to giving, for each \(n \in \mathbb{Z}\), a (discrete) \(k[x]\)-module on which \(x\) acts locally nilpotently. The \(n\)th such object corresponds to \(\pi_n (M \otimes_{C^*(S^1; k)} k)\) and \(x\) is the monodromy automorphism minus the identity.

In general, the classification of torsion modules over a PID is nontrivial, but in the finitely generated case, we have a simple complete classification, and we have:

**Corollary 2.15.** There are perfect \(C^*(S^1; k)\)-modules \(N_i, i = 1, 2, \ldots\), such that:

1. \(N_i \otimes_{C^*(S^1; k)} k\) is discrete and \(i\)-dimensional.
2. The monodromy automorphism on \(N_i \otimes_{C^*(S^1; k)} k\) is unipotent with a single Jordan block.

Any perfect \(C^*(S^1; k)\)-module is a direct sum of copies of shifts of \(N_i\) (uniquely determined).

**Proof.** The modules \(N_i\) are determined by the conditions spelled out: they correspond to the local systems on \(S^1\) determined by fixing an \(i\)-dimensional vector space with a unipotent automorphism which has a single Jordan block. The associated discrete \(k[x]\)-module is \(k[x]/x^i\).

Given a perfect \(C^*(S^1; k)\)-module, the associated local system (which is a local system of perfect \(k\)-modules) splits as a direct sum of shifts of local systems of discrete \(k\)-modules, by Proposition 2.13. Each of these is determined by a finite-dimensional \(k\)-vector space with an unipotent automorphism, and these are classified as a direct sum of indecomposable ones determined by their Jordan type. (This corresponds to the decomposition of a finitely generated \(k[x]\)-module as a direct sum of cyclic ones.)
In order to determine the homotopy groups of \( N_1 \), we have to form the associated local system, and take global sections over \( S^1 \). We have

\[
\pi_j(N_1) = \begin{cases} 
  k & \text{if } j = 0 \\
  k & \text{if } j = -1 \\
  0 & \text{otherwise}
\end{cases}
\]

For example, \( N_1 = C^*(S^1; k) \).

We will, however, have to work with certain non-perfect \( C^*(S^1; k) \)-modules in the sequel. For the rest of the subsection, we assume that \( k \) has characteristic zero.

**Example 2.16.** Here is a basic example of a non-perfect \( C^*(S^1; k) \)-module: \( k \) itself, via the map \( C^*(S^1; k) \to k \) of \( E_\infty \)-rings given by evaluation at a point. The base-change \( k \otimes_{C^*(S^1; k)} k \) is the free \( E_\infty \)-ring on a generator in degree zero (since \( \text{char} k = 0 \) and \( C^*(S^1; k) \simeq \text{Sym}^* k[-1] \)), and as a local system of \( k \)-modules on \( S^1 \), it is discrete and is given by a single infinite-dimensional Jordan block. In other words, it corresponds to the \( k[x] \)-module \( k[x^{\pm 1}]/k[x] \) (the “Prüfer” module). This is forced by the following corollary and its proof.

**Corollary 2.17.** Let \( M \) be a \( C^*(S^1; k) \)-module. Suppose that \( \pi_i M = 0 \) if \( i \) is odd. Then \( M \) is a direct sum of copies of even shifts of the \( C^*(S^1; k) \)-module \( k \).

**Proof.** We know, first, that \( M \) is a direct sum of copies of \( C^*(S^1; k) \)-modules whose base-change to \( k \) is shifted discrete, so we may assume this to begin with. In particular, \( M \) comes from a discrete \( k[x] \)-module \( N_0 \), on which \( x \) is nilpotent, (while remembering a shift \( n \)) such that at most one of \( \ker x \), \( \text{coker } x \) is nonzero because of the hypothesis on the homotopy groups.

Since \( x \) is locally nilpotent, \( \ker x \) is always nonzero (if \( N_0 \neq 0 \)), so the hypothesis must be that \( \text{coker } x = 0 \), and we have an even shift of a discrete local system. In other words, \( N_0 \) is a \( x \)-torsion divisible \( k[x] \)-module. Any such is a direct sum of copies of \( k[x^{\pm 1}]/k[x] \) by Theorem 2.18 below. Necessarily, \( k[x^{\pm 1}]/k[x] \) must correspond to the \( C^*(S^1; k) \)-module \( k \) by indecomposability. \( \square \)

In the above proof, we needed to use the following classical algebraic fact with \( R = \mathbb{Q}[x]/(x^r) \).

**Theorem 2.18.** Let \( R \) be a discrete valuation ring with quotient field \( K \). Then any torsion divisible \( R \)-module is a direct sum of copies of \( K/R \).

**Corollary 2.19.** Let \( M \) be a \( C^*(S^1; k) \)-module such that \( \pi_i M = 0 \) if \( i \) is odd. Then the \( \mathbb{Q} \)-module \( M \otimes_{C^*(S^1; k)} k \) has the same property.

**Proof.** This follows from Corollary 2.17, but it could also have been seen directly. \( \square \)

### 3. Residue fields

In this section, we will prove Theorem 1.2, our first main theorem, on the existence of residue fields. It will be convenient to work throughout with an extra assumption of 2-periodicity: this will imply that coning off an odd degree element is equivalent to coning off a degree \(-1\) element.
Let $A$ be a rational, noetherian $E_{\infty}$-ring such that $\pi_2(A)$ contains a unit. We will call such $E_{\infty}$-rings 2-periodic. Note that we are not assuming that $\pi_1(A) = 0$. Fix a prime ideal $p \subset \pi_0(A)$.

**Definition 3.1.** A residue field for $A$ is a map $A \to \kappa(p)$ of $E_{\infty}$-rings such that $\kappa(p)$ is even periodic, with the $\pi_0(A)$-algebra $\pi_0(\kappa(p))$ given by the residue field $\pi_0(A)p/\pi_0(A)p$ of $\pi_0(A)$ at $p$. In particular, $\pi_*k(p)$ is concentrated in even degrees and is a Laurent series ring on a degree two generator over the residue field of $\pi_0A$.

In this section, we will show that residue fields for such $E_{\infty}$-rings exist, and are sufficient to detect nilpotence in $\text{Mod}(A)$. The rest of the paper will use these residue fields to describe the thick subcategories and Galois theory of $\text{Mod}(A)$. We note that the name “residue field” is appropriate because of the perfect Künneth isomorphism

$$k(p)_*(M) \otimes_{k(p)} k(p)_*(N) \simeq k(p)_*(M \otimes_A N), \quad M, N \in \text{Mod}(A);$$

indeed, there is a map from left to right which is an isomorphism for $M = N = A$, and both sides define two-variable homology theories on $\text{Mod}(A)$, so the natural map must be an isomorphism in general. Alternatively, any $k(p)$-module is a sum of shifts of free ones.

We describe the connection with the use of residue fields as in [BR08]. Given an even periodic $E_{\infty}$-ring $R$ (not necessarily over $\mathbb{Q}$) with $\pi_0(R)$ regular noetherian (note that even periodic means that $\pi_1R = 0$), it is possible to form “residue fields” of $R$ as $E_1$-algebras in $\text{Mod}(R)$, by successively quotienting by a regular sequence. These residue fields have analogous properties of detecting nilpotence [Mat13, Cor. 2.6] and are quite useful for describing invariants of $\text{Mod}(R)$. For example, in the “chromatic” setting, the associated residue fields (such as the Morava $K$-theories $K(n)$ for the $E_{\infty}$-ring $E_n$) are almost never $E_{\infty}$.

Over the rational numbers, we are able to produce residue fields without such regularity hypotheses, and as $E_{\infty}$-algebras. However, we will have to work a bit harder: the residue fields of such an $A$ will no longer in general be perfect as $A$-modules (or as $E_{\infty}$-$A$-algebras), and we will have to use a countable limiting procedure, together with the techniques from the previous section.

Let $A$ be as above. In order to construct a residue field for $A$ for the prime ideal $p \in \text{Spec} \pi_0A$, we may first localize at $p$, and assume that $\pi_0A$ is local and that $p$ is the maximal ideal. Then, given generators $x_1, \ldots, x_n \in \pi_0A$ for $p$, we will need to set them equal to zero by attaching cells. That of course will introduce new elements (in both $\pi_0, \pi_1$) and we will have to kill them in turn.

3.1. **Detection of nilpotence.** Given an $E_{\infty}$-ring, we start by reviewing what it means for a collection of $R$-algebras to detect nilpotence, following ideas of [DHS88, HS98].

**Definition 3.2.** Let $R$ be an $E_{\infty}$-ring, and let $R'$ be an $R$-ring spectrum: that is, an associative algebra object in the homotopy category of $\text{Mod}(R)$. We say that $R \to R'$ detects nilpotence if, whenever $T$ is an $R$-ring spectrum, then the map of associative rings

$$\pi_*(T) \to \pi_*(R' \otimes_R T)$$
Example 3.6. Let $\mathcal{S}$ be a collection of $R$-ring spectra $\{R'_\alpha\}_{\alpha \in A}$ is said to detect nilpotence if any $u \in \pi_*(T)$ which maps to nilpotent elements under each map $\pi_*(T) \to \pi_*(R'_\alpha \otimes_R T)$ is itself nilpotent.

For example, the nilpotence theorem (Theorem 1.1) states that the Morava $K$-theories and homology (rational and mod $p$) detect nilpotence for $R = S^0$. The original form (in [DHS88]) states that the $E_\infty$-ring $MU$ of complex bordism detects nilpotence by itself, again over $S^0$.

As in [DHS88, §1], one has the following consequences of detecting nilpotence:

**Proposition 3.3.** Let $\{R'_\alpha\}_{\alpha \in A}$ be a collection of $R$-ring spectra that detect nilpotence.

1. Given a map of perfect $R$-modules $\phi: T \to T'$ such that each $1_{R'_\alpha} \otimes_R \phi: R'_\alpha \otimes_R T \to R'_\alpha \otimes_R T'$ is nullhomotopic as a map of $R$-modules, then $\phi$ is smash nilpotent; $\phi \otimes^N: T \otimes^N \to T' \otimes^N$ is nullhomotopic for $N \gg 0$.

2. Given a self-map of perfect $R$-modules $v: \Sigma^k T \to T$, if each $1_{R'_\alpha} \otimes_R v: \Sigma^k (R'_\alpha \otimes_R T) \to R'_\alpha \otimes_R T$ is nilpotent in $\text{Mod}(R)$, then $v$ itself is nilpotent.

Example 3.4. Suppose $R'$ is an $R$-ring spectrum that detects nilpotence. Then $R'$ cannot annihilate any nonzero perfect $R$-module $M$; in fact, that would force the identity $M \to M$ to be nilpotent.

Moreover, one sees:

**Proposition 3.5.** Let $R$ be an $E_\infty$-ring.

1. Let $R_1 \to R_2 \to R_3 \to \ldots$ be a diagram of $R$-ring spectra, such that the colimit $R_\infty = \lim R_i$ has a compatible structure of an $R$-ring spectrum. If each $R_i$ detects nilpotence over $R$, then $R_\infty$ detects nilpotence over $R$.

2. Let $\{R'_\alpha\}_{\alpha \in A}$ be a collection of $E_\infty$-$R$-algebras that detect nilpotence over $R$. For each $\alpha \in A$, let $\{R''_{\alpha \beta}\}_{\beta \in B_{\alpha}}$ be a collection of $E_\infty$-$R'_\alpha$-algebras that detect nilpotence over $R'_\alpha$. Then the collection $\{R''_{\alpha \beta}\}_{\alpha \in A}$ of $R$-algebras detects nilpotence over $R$.

3. Suppose the $\{R'_\alpha\}_{\alpha \in A}$ are a collection of $E_\infty$-$R$-algebras nilpotence over $R$ such that each $\pi_*(R'_\alpha)$ is a graded field. Then an $R$-ring spectrum $R''$ detects nilpotence over $R$ if and only if $R'' \otimes_R R'_\alpha \neq 0$ for each $\alpha \in A$.

**Proof.** By a graded field, we mean a graded ring which is either a field (concentrated in degree zero) or $k[t^{\pm 1}]$ for $|t| > 0$ and $k$ a field. The third assertion then follows from the second, since any nonzero ring spectrum over each $R'_\alpha$ detects nilpotence over $R'_\alpha$. The proofs of the first and second assertions are straightforward.

Finally, we need an important example of a pair that detects nilpotence.

Example 3.6. Let $R$ be a rational $E_\infty$-ring, and let $x \in \pi_0(R)$. As in the previous section, the cofiber $R/x$ inherits the canonical structure of an $E_\infty$-algebra under $R$. The localization $R[x^{-1}]$ always inherits a natural $E_\infty$-ring structure. The claim is that the pair of $R$-algebras $\{R/x, R[x^{-1}]\}$ detects nilpotence.

To see this, let $T$ be an $R$-ring spectrum, and let $\alpha \in \pi_j(T)$. Suppose $\alpha$ maps to zero in $T[x^{-1}] = T \otimes_R R[x^{-1}]$. This means that $x^N \alpha = 0$ for $N$ chosen large
enough. Suppose also that $\alpha$ maps to zero in $\pi_j(T/x) = \pi_j(T \otimes_R R/x)$. This means that $\alpha = x\beta$ for some $\beta \in \pi_j(T)$. We then have

$$\alpha^{2N} = \alpha^N \alpha^N = (x\beta)^N \alpha^N = \beta^N x^N \alpha^N = 0,$$

since $x^N \alpha = 0$. In other words, $\alpha$ is nilpotent.

This example will be extremely important to us in making induction arguments on the Krull dimension.

**Example 3.7.** Let $R \to R'$ be a map of $\text{E}_\infty$-rings. Suppose that $R \to R'$ admits descent in the sense of [Mat14, Def. 3.17]: in other words, the thick tensor-ideal in $\text{Mod}(R)$ that $R'$ generates contains $R$. Then $R \to R'$ detects nilpotence; see for instance [Mat14, Prop. 3.26].

### 3.2. The main result.

In this subsection, we prove the main technical result of this paper: the existence of residue fields and the detection of nilpotence.

**Theorem 3.8.** Let $A$ be a noetherian, 2-periodic rational $\text{E}_\infty$-ring.

1. For each $p \subset \pi_0(A)$, a residue field $k(p)$ for $A$ at $p$ exists and is unique up to homotopy.
2. The $\text{E}_\infty$-$A$-algebras $k(p)$ detect nilpotence.

The existence and uniqueness up to homotopy should be interpreted in the following sense. Let $k$ be the residue field (in the sense of ordinary commutative algebra) of $\pi_0(A)$ at $p$. By construction (and Proposition 2.5), any residue field $k(p)$, as an $\text{E}_\infty$-ring, must be equivalent to $k[t_2^{\pm 1}]$, the free $\text{E}_\infty$-ring under $k$ on an invertible degree two class. We will show that the space of $\text{E}_\infty$-maps

$$A \to k[t_2^{\pm 1}],$$

which induce the desired map on $\pi_0$ and which carry a specified unit in $\pi_2(A)$ to $t_2$, is connected. (The existence result states that the space of such maps is nonempty.)

**Proof.** We begin with the existence (and uniqueness) of residue fields. By localizing, we may assume that $\pi_0(A)$ is a local ring and that $p \subset \pi_0(A)$ is its maximal ideal; let $k$ be the residue field $\pi_0(A)/p$. Choose ideal generators $x_1, \ldots, x_n \in p$ and use them to construct a map

$$\mathbb{Q}[t_1, \ldots, t_n] \to A, \quad t_i \mapsto x_i.$$

We let $A_0$ be the $\text{E}_\infty$-ring $A \otimes_{\mathbb{Q}[t_1, \ldots, t_n]} \mathbb{Q} \simeq A/(t_1, \ldots, t_n)$. The $\text{E}_\infty$-ring $A_0$ is noetherian, and by Theorem 2.9, and $\pi_0(A_0)$ is actually local artinian with residue field $k$. Thus, $A_0$ admits the structure of an $\text{E}_\infty$-$k$-algebra by Proposition 2.5. Let $k(p)$ be the $\text{E}_\infty$-ring given by $k(p) = k[t_2^{\pm 1}]$.

Our goal is to produce a map of $\text{E}_\infty$-rings from $A_0$ to $k(p)$. For this, we will need to kill the degree zero elements, and the degree $-1$ elements. We will first kill the degree 0 elements by adding cells in dimension one, to reduce to the case where there is nothing (except for $k$) in odd dimensions. Then, we will use a separate argument to kill the odd homotopy.

Given $A_0$, let $m \subset \pi_0(A_0)$ be the maximal ideal, which is a finite-dimensional $k$-vector space. Consider the map

$$\text{Sym}^*(m) \to A_0,$$
of $E_\infty$-$k$-algebras, and form the pushout

$$A_1 \overset{\text{def}}{=} A_0 \otimes_{\text{Sym}^*(m)} k,$$

which has the same property as $A_0$: $\pi_*(A_1)$ satisfies the desired noetherianness assumptions (in fact, all the homotopy groups are finite-dimensional $k$-vector spaces), and $\pi_0(A_1)$ is local artinian with residue field $k$ (by Theorem 2.9). Note that $A_0 \to A_1$ admits descent in view of Lemma 2.3.

Let $m_1 \subset \pi_0(A_1)$ be the maximal ideal, and continue the process with

$$A_2 \overset{\text{def}}{=} A_0 \otimes_{\text{Sym}^*(m_2)} k,$$

and repeating this, we find a sequence of 2-periodic, noetherian $E_\infty$-rings

$$A_0 \to A_1 \to A_2 \to \ldots,$$

where each $A_i$ has the following properties:

1. $\pi_0(A_i)$ is a local artinian ring with residue field $k$ and $\pi_1(A_i)$ is a finite-dimensional $k$-vector space.
2. $A_{i+1}$ is obtained from $A_i$ by attaching a 1-cell for each element in a $k$-basis of the maximal ideal of $\pi_0(A_i)$, and thus $A_i \to A_{i+1}$ admits descent.
3. In particular, the map $\pi_0(A_i) \to \pi_0(A_{i+1})$ annihilates the maximal ideal of the former.

If we take the colimit $A_\infty = \lim_i A_i$, we find an $E_\infty$-$A$-algebra $A_\infty$ such that $\pi_0(A_\infty) = k$. This process of iteratively attaching 1-cells has likely introduced lots of elements in $\pi_{-1}$, and all we know is that $\pi_{-1}(A_\infty)$ is a countably dimensional $k$-vector space. Note that $A_0 \to A_\infty$ detects nilpotence, as it is the sequential colimit of a sequence of $E_\infty$-algebras that admit descent.

**Lemma 3.9.** Let $B$ be a rational $E_\infty$-ring such that:

1. $\pi_0(B)$ is a field $k$.
2. $\pi_2(B)$ contains a unit $u$.
3. $\pi_{-1}(B)$ is a countably dimensional $k$-vector space.

Then there is a natural map of $E_\infty$-rings $B \to k[t_{-1}]_{\pm 1}$, which detects nilpotence.

**Proof.** Note first that, by Proposition 2.5, $B$ naturally admits the structure of an $E_\infty$-$k$-algebra. Let $u_1, u_2, \ldots, \in \pi_{-1}(B)$ be a $k$-basis. We define $B^{(1)} \cong B \otimes_{\text{Sym}^*k[-1]} k$ via the map $\text{Sym}^*k[-1] \to B$ classifying $u_1$. Next, we define the $E_\infty$-$B^{(1)}$-algebra $B^{(2)} \cong B^{(1)} \otimes_{\text{Sym}^*k[-1]} k$ where the map $\text{Sym}^*k[-1] \to B^{(1)}$ classifies $u_2$. Inductively, we obtain a sequence of $E_\infty$-rings

$$B \to B^{(1)} \to B^{(2)} \to \ldots,$$

where $B^{(i+1)} \cong B^{(i)} \otimes_{\text{Sym}^*k[-1]} k$ is obtained by coning off $u_{i+1}$.

Our basic observation is that each $B^{(i)}$ satisfies the conditions of the lemma: that is, no additional homotopy is introduced in even degrees. We need only consider the first case, by induction. Consider the $\text{Sym}^*k[-1]$-module $B$ under the map $\text{Sym}^*k[-1] \to B$ classifying $u_1$. We have a map

$$k[t_{-1}^{\pm 1}] \otimes_k \text{Sym}^*k[-1] \to B,$$

of $\text{Sym}^*k[-1]$-modules. The cofiber $C$, by hypothesis, is a $\text{Sym}^*k[-1]$-module whose homotopy groups are concentrated entirely in odd degrees. In particular, we find
by Corollary 2.17 that $C$ is a direct sum of odd shifts of the $\Sym^* k[-1]$-module $k$, so the cofiber sequence

$$k[t^\pm_2] \to B \otimes \Sym^* k[-1] k \to C \otimes \Sym^* k[-1] k,$$

(which has to induce split exact sequences on the level of homotopy groups) implies that, at the level of homotopy groups, $B^{(1)} = B \otimes \Sym^* k[-1] k$ has the same property as did $B$: the even homotopy groups are given by the Laurent series ring. Moreover, the map $\pi_*(C) \to \pi_*(C \otimes \Sym^* k[-1] k)$ is injective, so the $\{u_2, u_3, \ldots\}$ remain nonzero and linearly independent in $\pi_-(B^{(1)})$. We find inductively that in the sequence (3), all the $E_\infty$-rings have homotopy groups entirely in odd degrees except for the Laurent series over $k$.

Now, given $B$ and the basis $u_1, \ldots$, as above, we let $B_1$ be the colimit $\lim B^{(i)}$ of the sequence (3). By what we have argued, $B_1$ satisfies the hypotheses of the lemma as well. Moreover, the map $\pi_-(B) \to \pi_-(B_1)$ is the zero map. Thus, we can repeat the above sequential construction to produce

$$B_1 \to B_1^{(1)} \to B_1^{(2)} \to \ldots,$$

obtained by coning off the degree $-1$ elements in $\pi_-(B_1)$. Let the colimit be the $E_\infty$-$B_1$-algebra $B_2$. Then $B_2$ satisfies the hypotheses of the lemma, but $\pi_-(B_1) \to \pi_-(B_2)$ is zero. Repeating the process, we get a sequence

$$B \to B_1 \to B_2 \to \ldots,$$

(4)

whose colimit, finally, is the $E_\infty$-ring $k(p) = k[t^\pm_2]$, since each of the maps is zero on $\pi_-$.

Finally, we should argue that $B \to k[t^\pm_2]$ detects nilpotence. For this, it suffices to argue that $B_1 \to B_{i+1}$ in the above sequence detects nilpotence, since detecting nilpotence is preserved in filtered colimits. But $B_i \to B_{i+1}$ is a filtered colimit of maps each of which is obtained by coning off a degree $-1$ element. It now suffices to note that $\Sym^* k[-1] \to k$ admits descent, and so any base-change of it does too, and in particular detects nilpotence. \qed

To recap, let $A$ be a noetherian, 2-periodic $E_\infty$-ring. For every prime ideal $p \in \Spec \pi_0 A$, we have constructed an $E_\infty$-ring $k(p)$ under $A$ with the properties of Definition 3.2: that is, $k(p)$ is even periodic, and $\pi_0 A \to \pi_0 k(p)$ induces the map

$$\pi_0 A \to (\pi_0 A)_p / p(\pi_0 A)_p,$$

of quotienting by the maximal ideal in the localization. The remaining claims are, first, that the residue fields are unique; and second, that the $\{k(p)\}_{p \in \Spec \pi_0(A)}$ detect nilpotence over $A$.

We tackle uniqueness in the same way as we tackled existence. Without loss of generality, $\pi_0(A)$ is local and the prime ideal $p \subset \pi_0(A)$ is the maximal ideal. Let $k$ be the residue field of $\pi_0(A)$. We want to show that the space of maps $A \to k[t^\pm_2]$ inducing the map on $\pi_0$ and which carry a specified unit in $\pi_2(A)$ to $t$ is connected. Let $x_1, \ldots, x_n \in p$ generate the maximal ideal. Then since any map $A \to k[t^\pm_2]$ annihilates $x_1, \ldots, x_n$, it suffices to prove the analogous connectedness assertion for the space of maps $A/(x_1, \ldots, x_n) \to k[t^\pm_2]$. So, we can even assume that $\pi_0(A)$ is local Artinian with residue field $k$. We thus need to prove:
**Lemma 3.10.** Let $A$ be a noetherian, 2-periodic rational $E_\infty$-ring such that $\pi_0A$ is local artinian with residue field $k$. Let $u \in \pi_2(A)$ be a specified unit. Then the space of maps $A \to k[t_2^{\pm 1}]$ carrying $u \to t$ is connected.

**Proof.** As we saw above, given $A$, there is a countable transfinite process that produces from $A$ the $E_\infty$-ring $k[t_2^{\pm 1}]$, where at each step we either cone off a nilpotent degree zero element, cone off a degree $-1$ element, or take $\mathbb{N}$-indexed inductive colimits. Let $\text{CAlg}$ be the $\infty$-category of $E_\infty$-ring spectra. Now, we observe that if $A' \to A''$ is obtained by one of the first two processes, then

$$\pi_0 \text{Hom}_{\text{CAlg}}(A'', k[t_2^{\pm 1}]) \to \pi_0 \text{Hom}_{\text{CAlg}}(A', k[t_2^{\pm 1}])$$

is surjective: that is, any map $A' \to k[t_2^{\pm 1}]$ factors, up to homotopy, over $A''$. Moreover, the inverse limit of a $\mathbb{N}$-indexed family of surjections maps surjectively to each stage. This means that if

$$\cdots \to T_n \to T_{n-1} \to \cdots \to T_0,$$

is a tower of spaces such that each $\pi_0(T_n) \to \pi_0(T_{n-1})$ is surjective, then $\pi_0(\text{holim} T_n) \to \pi_0(T_0)$ is surjective. We thus find that

$$\pi_0 \text{Hom}_{\text{CAlg}}(k[t_2^{\pm 1}], k[t_2^{\pm 1}]) \to \pi_0 \text{Hom}_{\text{CAlg}}(A, k[t_2^{\pm 1}])$$

is surjective. However, the collection of homotopy classes of maps $k[t_2^{\pm 1}] \to k[t_2^{\pm 1}]$ is given by $k^\times$; the only parameter is the action on the unit. \qed

Finally, we need to show that the residue fields detect nilpotence. If $\pi_0A$ is local artinian, then we have already observed this: in fact, we explicitly constructed the residue field by the processes of coning off elements that were either nilpotent degree zero or degree $-1$ elements and forming sequential colimits. As we have seen (Lemma 2.3, Proposition 2.11, and Proposition 3.5), each of these processes preserves the property of detecting nilpotence.

Now, assume the result on detection of nilpotence proved for all noetherian $A$ with the Krull dimension of $A$ at most $n-1$. We will then prove it for dimension $\leq n$. In fact, we may assume that $\pi_0A$ is noetherian local of Krull dimension $\leq n$. Choose $x \in \pi_0A$ such that $\pi_0A/(x)$ has Krull dimension $\leq n-1$. As we saw in Example 3.6, the pair of $E_\infty$-$A$-algebras $A/x, A[x^{-1}]$ detect nilpotence over $A$, and each of these is noetherian with $\pi_0$ having Krull dimension $\leq n-1$. Therefore, the residue fields of the $E_\infty$-rings $A/x, A[x^{-1}]$ are sufficient to detect nilpotence over each of them, and thus detect nilpotence over $A$.

Given any rational noetherian $E_\infty$-ring $A$, in order to prove that the residue fields $\{k(p)\}_{p \in \text{Spec} \pi_0A}$ detect nilpotence over $A$, it suffices to reduce to the case where $\pi_0A$ is local, and thus of finite Krull dimension, so that what we have already done suffices to show that the residue fields detect nilpotence. \qed

**Remark 3.11.** Let $A$ be as above. Given two different prime ideals $p,q \subset \pi_0A$, the tensor product $k(p) \otimes_A k(q)$ vanishes. In fact, if $x \in p \setminus q$ (by symmetry, we may assume this), then $x$ acts as a unit on $k(q)$ but induces the zero map in homotopy on $k(p)$. This forces the tensor product to vanish (e.g., $k(p) \otimes_A k(q) = k(p) \otimes_A k(q)[x^{-1}] = k(p)[x^{-1}] \otimes_A k(q) = 0$).
4. Thick subcategories

The goal of this section is to obtain a classification of thick subcategories of perfect modules over a rational, noetherian $\mathbf{E}_\infty$-ring. In particular, we will be able to prove quickly Theorem 1.3, which we restate here for convenience:

**Theorem 4.1.** Let $A$ be a rational, noetherian $\mathbf{E}_\infty$-ring. The thick subcategories of the $\infty$-category $\text{Mod}^{\omega}(A)$ of perfect $A$-modules are in natural correspondence with the subsets of $(\text{Spec}\pi_{\text{even}}A)/\mathbb{G}_m$ closed under specialization.

Here the $\mathbb{G}_m$-action comes from the grading of $\pi_{\text{even}}(A) = \bigoplus_{i \in \mathbb{Z}} \pi_i(A)$. The points of $(\text{Spec}\pi_{\text{even}}A)/\mathbb{G}_m$ are the homogeneous prime ideals of the commutative, graded ring $\pi_{\text{even}}(A)$. A collection $\mathcal{C}$ of homogeneous prime ideals of $\pi_{\text{even}}(A)$ is closed under specialization if, whenever $p \in \mathcal{C}$ and $q \supset p$ is a larger homogeneous prime ideal, then $q \in \mathcal{C}$ too. Any such is a union of Zariski closed subsets.

Let $M$ be a perfect module over $A$. Then $\pi_* (M)$ is a finitely generated graded $\pi_{\text{even}}(A)$-module; more precisely, it splits into two finitely generated graded $\pi_{\text{even}}(A)$-modules, given by $\pi_{\text{even}}(M), \pi_{\text{odd}}(M)$. In particular, it lives as a $(\mathbb{Z}/2$-graded) coherent sheaf on the stack $(\text{Spec}\pi_{\text{even}}A)/\mathbb{G}_m$, and the support defines a Zariski closed subset of $(\text{Spec}\pi_{\text{even}}A)/\mathbb{G}_m$. In plainer language, we consider the collection of homogeneous prime ideals $p \subset \pi_{\text{even}}(A)$ such that $M(p) \neq 0$, where $M(p)$ denotes the localization of $M$ at the multiplicative subset of $\pi_{\text{even}}(A)$ consisting of the homogeneous elements outside of $p$.\footnote{Not only the degree zero elements in this localization.} This collection is clearly closed under specialization.

**Example 4.2.** Suppose $A$ is 2-periodic. Then the thick subcategories of $\text{Mod}^{\omega}(A)$ are in natural correspondence with the subsets of $\text{Spec}\pi_0A$ closed under specialization.

The deduction of a thick subcategory theorem from a nilpotence result for “residue fields” is well-known, and originates in [HS98] and has been axiomatized in [HPS97]. We will review the conditions for this argument, and then explain how it applies. One of our main tasks is to remove the assumptions of 2-periodicity from the techniques of the previous section.

4.1. **Residue fields without periodicity.** Let $A$ be an $\mathbf{E}_\infty$-ring together with a collection of residue fields $\{k(p)\}$, possibly just multiplicative homology theories on $\text{Mod}(A)$, satisfying perfect Künneth isomorphisms, such that together they detect nilpotence over $A$. In this case, they are sufficient to detect thick subcategories as well. In particular, consider $M, N \in \text{Mod}^{\omega}(A)$. It follows from the Hopkins-Smith (and Hovey-Palmieri-Strickland) argument [HPS97, Th. 5.2.2 and Cor. 5.2.3] that if, whenever $k(p)_* (M) \neq 0$, then $k(p)_* (N) \neq 0$, then the thick subcategory that $N$ generates in $\text{Mod}^{\omega}(A)$ contains $M$.

Every thick subcategory of $\text{Mod}^{\omega}(A)$ is then determined by a subset of the $\{k(p)\}$ (the ones which can be nonzero), and the classification of thick subcategories reduces to the determination of which subsets arise from thick subcategories; or equivalently, which subsets of the $\{k(p)\}$ arise as $\{p : k(p)_* (M) \neq 0\}$ for some $M \in \text{Mod}^{\omega}(A)$.
Now let $A$ be a noetherian rational $E_\infty$-ring and $p \in \pi_{\text{even}}(A)$ a homogeneous prime ideal. We form the $E_\infty$-ring $A' = A[t_{\pm 1}^2]$ (i.e., we enforce 2-periodicity) and we then have

$$\pi_0 A' \simeq \pi_{\text{even}} A.$$ 

In particular, $p$ becomes a prime ideal of $\pi_0 A'$. As a result, in view of Theorem 3.8, we can construct a residue field $k(p)$ of $A'$ at $p$ and we obtain maps $A \to A' \to k(p)$. Rather than considering the $\{k(p)\}$ as $A'$-algebras, we consider them as $E_\infty$-$A$-algebras. They satisfy a perfect Künneth isomorphism as homology theories on $\text{Mod}(A)$, as before.

**Proposition 4.3.** The $\{k(p)\}$, as $p$ ranges over the homogeneous prime ideals in $\text{Spec} \pi_{\text{even}}(A)$, detect nilpotence over $A$.

This is not an immediate consequence of Theorem 3.8 applied to $A'$, because the homogeneous $p$ do not exhaust all the prime ideals of $\pi_0(A')$. In other words, the $k(p)$ in question do not detect nilpotence over $A'$.

**Proof.** Given a commutative, graded and noetherian ring like $\pi_{\text{even}}(A)$, define the *graded Krull dimension* to be the maximum length of an ascending chain of homogeneous prime ideals. We will need certain basic facts about the graded Krull dimension and graded rings.

1. Suppose $R_*$ is graded-local, i.e., it has a unique maximal graded ideal. Then $R_*$ has finite graded Krull dimension, bounded by the ungraded height of the maximal graded ideal.

2. Suppose $R_*$ has graded Krull dimension equal to zero and is graded-local. Then the quotient of $R_*$ by the nilradical is either a field $k$ in degree zero or $k[t_{\pm 1}^2]$ where $|t| > 0$ and where $k$ is a field (in other words, a graded field). In fact, first of all $R_0$ must clearly be local artinian. Quotienting by the nilradical of $R_0$, we may assume $R_0 = k$ is a field. Meanwhile, a positive-degree homogeneous element is either invertible or lies in the unique maximal ideal. If a positive-degree element $u$ is not invertible, then the localization $R_*[u^{-1}]$, if nonzero, has a graded prime ideal, which pulls back to a graded prime ideal of $R_*$ not containing $u$; thus $u$ must be nilpotent. Choosing an invertible element of smallest degree, we find that $R_*$ is as claimed.

3. Let $R_*$ be a graded ring, and let $q \subset R_*$ a prime ideal which is not homogeneous. Then there is a homogeneous prime ideal $p \subset q$ consisting of all $x \in R_*$ all of whose homogeneous components belong to $q$. In particular, $qR_{(p)}$ is an inhomogeneous prime ideal of $R_{(p)}$ and is maximal. This follows as $R_{(p)}/p$ has graded Krull dimension zero, and $q$ defines here a nonzero prime ideal. Therefore, $R_{(p)}/p$ must be $k[t_{\pm 1}]$ for $k$ a field and $|t| > 0$.

We know that the $\{k(q)\}$ for $q \in \text{Spec} \pi_{\text{even}} A = \pi_0 A'$ detect nilpotence over $A'$, and thus over $A$. Thus, in order to prove that the $\{k(p)\}$ for $p$ ranging over the homogeneous prime ideals detect nilpotence over $A$, it suffices to prove:

**Lemma 4.4.** Let $q \subset \pi_{\text{even}}(A)$ be an inhomogeneous prime ideal and let $p \subset \pi_{\text{even}}(A)$ be the homogeneous part. Then $k(q) \otimes_A k(p) \neq 0$.  


Proof. It suffices to prove this after localizing at \( \mathfrak{p} \) (i.e., inverting the homogeneous elements not in \( \mathfrak{p} \)) and after killing a system of generators of \( \mathfrak{p} \) in \( \pi_{\text{even}}(A) \). So, we may assume that \( \pi_{\text{even}}(A) \) is graded-local and has graded Krull dimension zero (in view of Theorem 2.9). We will show that in this case, \( k(\mathfrak{p}) \) detects nilpotence.

By the third item above, there exists a unit in positive degrees, say \( u \in \pi_{2r}(A) \) (we assume \( r \) minimal) and \( \pi_{\text{even}}(A) \) is a nilpotent thickening of \( k[x_{2r}^{\pm 1}] \). This has Krull dimension one, and there are many different residue fields of \( A' \). The homogeneous ideal of \( \pi_0(A') \) in question corresponds to the nilradical. However, we observe that if \( M \) is any module over \( A \), then \( \pi_0(M \otimes_A A') \simeq \pi_{\text{even}}(M) \), so \( \pi_0(M \otimes_A A') \) has support on all of \( k[[u_{2r}^{\pm 1}]] \) if nonzero. In particular, if \( R \) is any \( A \)-ring spectrum, then it follows that \( R \otimes_A A' \) has the property that \( \pi_0(R \otimes_A A') \), as a \( \pi_0(A') \)-module, has nonvanishing stalk at the generic point; thus, \( (R \otimes_A A') \otimes_{A'} k(\mathfrak{p}) = R \otimes_A k(\mathfrak{p}) \neq 0 \) because, for an \( A' \)-algebra, being detected by \( k(\mathfrak{p}) \) is equivalent to being nonzero at the generic point. Thus \( k(\mathfrak{p}) \) detects nilpotence over \( A \).

In particular, it now follows formally that a thick subcategory of \( \text{Mod}^+(A) \) is determined by a subcollection of the \( \{k(\mathfrak{p})\} \) as \( \mathfrak{p} \) ranges over the homogeneous prime ideals of \( \pi_{\text{even}}(A) \). It remains to determine what subsets are allowed to arise. We will show that those subsets are precisely those which are closed under specialization. To do this, we will show in the next subsection that, for a perfect \( A \)-module \( M \), the condition \( k(\mathfrak{p})_*(M) \neq 0 \) is equivalent to an analogous purely algebraic one.

4.2. Completion of the proof. In this subsection, we complete the proof of our main result, Theorem 4.1. We start with the 2-periodic case.

Proposition 4.5. Let \( A \) be a rational, noetherian 2-periodic \( \textbf{E}_\infty \)-ring. Let \( M \) be a perfect \( A \)-module. Then the following are equivalent, for \( \mathfrak{p} \in \text{Spec} \pi_0 A \):

1. \( M_{\mathfrak{p}} \neq 0 \).
2. \( k(\mathfrak{p})_*(M) \neq 0 \) (where \( k(\mathfrak{p}) \) is the residue field of Theorem 3.8).

Proof. Without loss of generality, we can assume that \( \pi_0(A) \) is local and that \( \mathfrak{p} \) is the maximal ideal of \( \pi_0(A) \). Then we need to show that if \( k(\mathfrak{p})_*(M) = 0 \), then \( M \) itself is contractible, a form of Nakayama’s lemma. Without loss of generality, we can assume that \( \pi_0(A) \) is complete local, because the completion is faithfully flat over \( A \).

Let \( x_1, \ldots, x_n \in \pi_0(A) \) be generators for the maximal ideal \( \mathfrak{p} \). Then it suffices to show that \( M/(x_1, \ldots, x_n) \), which is the base-change of \( M \) to \( A/(x_1, \ldots, x_n) \), is contractible, because \( M \) is \( (x_1, \ldots, x_n) \)-adically complete. In particular, we may replace \( A \) with \( A/(x_1, \ldots, x_n) \) and thus assume that \( \pi_0(A) \) is actually local artinian. We thus reduce to this case.

But if \( \pi_0(A) \) is local artinian, we know that the map \( A \to k(\mathfrak{p}) \) actually detects nilpotence: in particular, it cannot annihilate a nonzero perfect \( A \)-module (Example 3.4). So, \( k(\mathfrak{p})_*(M) = 0 \), then \( M \) is contractible.

Now we need the analogous result without 2-periodicity assumptions. Let \( A \) be a noetherian rational \( \textbf{E}_\infty \)-ring. Given a homogeneous prime ideal \( \mathfrak{p} \subset \pi_{\text{even}}(A) \),

we constructed in the previous subsection an even periodic residue field \(k(p)\) as an \(E_{\infty}\)-\(A\)-algebra, and showed that the \(\{k(p)\}\) detect nilpotence.

**Corollary 4.6.** Let \(M\) be a perfect \(A\)-module. Then the following are equivalent:

1. We have \(k(p)_+M \neq 0\).
2. The localization \(M_{(p)} \neq 0\).
3. The homogeneous localization \((\pi_*M)_{(p)} \neq 0\).
4. The inhomogeneous localization \((\pi_*M)_p \neq 0\).

**Proof.** The equivalence of (3) and (4) is pure algebra: (3) states that there exists a homogeneous element \(m \in \pi_*M\) such that \(\text{ann}(m) \subset p\) while (4) states that there exists a possibly inhomogeneous element with this property.

By replacing \(A\) with \(A[t^{\pm 1}]\) and using Proposition 4.5, we are done. \(\square\)

The lemma, and the discussion preceding it, imply that to every perfect module \(M\), when we associate the collection \(\{p: k(p)_+M \neq 0\}\), then this precisely the homogeneous support (in the sense of commutative algebra) of the \(\pi_{\text{even}}(A)\)-module \(\pi_*M\). This implies that if \(M, N\) are perfect \(A\)-modules such that \(\text{Supp}_{\pi_*}(M) \subset \text{Supp}_{\pi_*}(N)\), then the thick subcategory that \(N\) generates contains \(M\). It remains then only to show that every closed subset (associated to a homogeneous ideal \(I \subset \pi_{\text{even}}(A)\)) can be realized as the support of some \(M\), but this follows by forming \(A/(x_1, \ldots, x_n)\) where \(x_1, \ldots, x_n \in \pi_{\text{even}}(A)\) generate \(I\). In particular, this completes the proof of Theorem 4.1.

5. **Galois groups**

Let \(R\) be an \(E_{\infty}\)-ring such that \(\pi_0R\) has no nontrivial idempotents. In [Mat14], we introduced the Galois group \(\pi_1\text{Mod}(R)\) of \(R\), a profinite group defined “up to conjugacy” (canonically as a profinite groupoid), by developing a version of the étale fundamental group formalism. The Galois group \(\pi_1\text{Mod}(R)\) has the property that if \(G\) is a finite group, then giving a continuous group homomorphism \(\pi_1\text{Mod}(R) \rightarrow G\) is equivalent to giving a faithful \(G\)-Galois extension in the sense of Rognes [Rog08].

More generally, we introduced ([Mat14, Def. 6.1]) the notion of a finite cover of an \(E_{\infty}\)-ring \(R\), as a homotopy-theoretic version of the classical notion of a finite étale algebra over a commutative ring. A continuous action of \(\pi_1\text{Mod}(R)\) on a finite set is equivalent to a finite cover of the \(E_{\infty}\)-ring \(R\). The Galois group can be a fairly sensitive invariant of \(E_{\infty}\)-rings; for instance ([Mat14, Ex. 7.21]) two different \(E_{\infty}\)-structures on the same \(E_1\)-ring can yield different Galois groups, and computing it appears to be a subtle problem in general. Here, we will show that the Galois group is much less sensitive over the rational numbers, at least under noetherian hypotheses.

The Galois group comes with a surjection

\[
\pi_1\text{Mod}(R) \twoheadrightarrow \pi_{\text{et}}^0\text{Spec}\pi_0R,
\]

since every algebraic Galois cover of \(\text{Spec}\pi_0R\) can be realized topologically. More generally, to every finite étale \(\pi_0\text{-}R\)-algebra \(R'_0\) one can canonically associate an \(E_{\infty}\)-\(R\)-algebra \(R'\) such that \(\pi_0R' \simeq R'_0\) and such that \(\pi_kR' \simeq R'_0 \otimes_{\pi_0R} \pi_kR\). This yields a full subcategory of the category of finite covers which corresponds to the above surjection. In general, however, it is an insight of Rognes that the above surjection has a nontrivial kernel: that is, there exist finite covers that do not...
arise algebraically in this fashion. A basic example is the complexification map $KO \to KU$.

In \cite{Mat14}, we computed Galois groups in certain instances. Our basic ingredient ([Mat14, Th. 6.30]) was a strengthening of work of Baker–Richter \cite{BR08} to show that the Galois theory is entirely algebraic for even periodic $E_\infty$-rings with regular $\pi_0$, using the theory of residue fields. Over the rational numbers, the methods of the present paper enable one to construct these “residue fields” without regularity assumptions. In particular, we will show in this section that, for noetherian rational $E_\infty$-rings, the computation of the Galois group can be reduced to a problem of pure algebra. (For instance, we will show that (5) is an isomorphism if $R$ is 2-periodic.)

5.1. **Review of invariance properties.** To obtain the results of the present section, we will need some basic tools for working with Galois groups, which will take the form of the “invariance results” of \cite{Mat14}. For example, we will need to know that killing a nilpotent degree zero class does not affect the Galois group.

**Theorem 5.1.** Let $A$ be a rational $E_\infty$-ring and let $x \in \pi_0 A$ be a nilpotent element. Then the map $A \to A/x$ induces an isomorphism on Galois groupoids.

**Proof.** This is \cite[Theorem 8.13]{Mat14}, for the map $\mathbb{Q}[[t]] \to A$ sending $t \mapsto x$. □

**Proposition 5.2.** Let $A$ be a rational $E_\infty$-ring and let $x \in \pi_{-1} A$ be a class. Then the map $A \to A \otimes \text{Sym}^* \mathbb{Q}[{-1}]$, obtained by coning off $x$, induces a surjection on Galois groups.

**Proof.** By \cite[§8.1]{Mat14}, it suffices to show that the map $C^* (S^1; \mathbb{Q}) \simeq \text{Sym}^* \mathbb{Q}[{-1}] \to \mathbb{Q}$ is universally connected: that is, for any $E_\infty$-$C^* (S^1; \mathbb{Q})$-algebra $A$, the natural map $A \to A \otimes C^* (S^1; \mathbb{Q}) \mathbb{Q}$ induces an isomorphism on sets of idempotents.

Since $C^* (S^1; \mathbb{Q}) \to \mathbb{Q}$ admits descent, the set $\text{Idem}(A)$ of idempotents in $A$ is the equalizer of the two maps

$$A \otimes C^* (S^1; \mathbb{Q}) \mathbb{Q} \twoheadrightarrow A \otimes C^* (S^1; \mathbb{Q}) \mathbb{Q} \otimes C^* (S^1; \mathbb{Q}) \mathbb{Q} \simeq (A \otimes C^* (S^1; \mathbb{Q}) \mathbb{Q}) [t].$$

This is a reflexive equalizer, and one of the maps is the natural inclusion

$$A \otimes C^* (S^1; \mathbb{Q}) \mathbb{Q} \to (A \otimes C^* (S^1; \mathbb{Q}) \mathbb{Q}) [t],$$

which induces an isomorphism on idempotents. It follows that all the maps in the reflexive equalizer are isomorphisms and thus the two forward maps are equal, proving that $A \to A \otimes C^* (S^1; \mathbb{Q}) \mathbb{Q}$ induces an isomorphism on idempotents. □

5.2. **The 2-periodic case.** We are now ready to show (Theorem 5.4 below) that the Galois theory of a 2-periodic, noetherian rational $E_\infty$-ring is algebraic.

**Lemma 5.3.** Let $A$ be a rational, noetherian 2-periodic $E_\infty$-ring such that $\pi_0 A$ is local artinian. Then the Galois theory of $A$ is algebraic.
Proof: The strategy is to imitate the proof of Theorem 3.8, while cognizant of the invariance results for Galois groups reviewed in the previous subsection. Namely, we not only showed that $A$ had a residue field, but we constructed it via a specific recipe. Let $k$ be the residue field of $\pi_0 A$.

In proving Theorem 3.8, we first formed a sequence of rational, noetherian, 2-periodic $E_\infty$-rings with artinian $\pi_0$, 

$$A = A_0 \to A_1 \to A_2 \to \cdots \to A_\infty = \varprojlim A_i,$$

where at each stage, in passing from $A_i$ to $A_{i+1}$, we killed a finite number of nilpotent elements. By Theorem 5.1, at no finite stage do we change the Galois group; each map $A \to A_i$ induces an isomorphism on Galois groups. Now, by [Mat14, Th. 6.21], the Galois group is compatible with filtered colimits and therefore $A \to A_\infty$ induces an isomorphism on Galois groups.\footnote{In fact, all we need for the proof of this lemma is that $A \to A_\infty$ induces a surjection on Galois groups. This does not require the obstruction theory used in proving [Mat14, Th. 6.21], and is purely formal.}

Now, the $E_\infty$-ring $A_\infty$ has the properties of Lemma 3.9: it has a unit in degree two, its $\pi_0$ is isomorphic to $k$, and $\pi_1$ is countably dimensional. We showed in the proof of Lemma 3.9 that by killing degree $-1$ cells repeatedly and forming countable colimits, and repeating countably many times, we could start with $A_\infty$ and reach $k[t_2^{\pm1}]$. It follows by Proposition 5.2 (along with the compatibility of Galois groups and filtered colimits, again) that the map 

$$A_\infty \to k[t_2^{\pm1}],$$

induces a surjection on Galois groups. But the Galois group of $k[t_2^{\pm1}]$ is algebraic (i.e., $\text{Gal}(\overline{k}/k)$) in view of the K"unneth isomorphism [Mat14, Prop. 6.28], so the Galois group of $A_\infty$ must be bounded by $\text{Gal}(\overline{k}/k)$, and therefore that of $A$ must be, too.

We can now prove the main result of the present subsection.

**Theorem 5.4.** Let $A$ be a rational, noetherian 2-periodic $E_\infty$-ring. Then the Galois theory of $A$ is algebraic, i.e., $\pi_1 \text{Mod}(A) \simeq \pi_1^\text{et} \text{Spec} \pi_0(A)$.

**Proof:** Fix a finite cover $A \to A'$ of $E_\infty$-rings. We need to show that $A'$ is flat over $A$: that is, the natural map $\pi_0(A) \to \pi_0(A')$ is flat, and the map $\pi_*(A) \otimes_{\pi_0(A)} \pi_0(A') \to \pi_*(A')$ is an isomorphism. This is a local question, so we may assume that $\pi_0 A$ is a local noetherian ring. Moreover, by completing $A$ at the maximal ideal $m \subset \pi_0 A$, we may assume that $\pi_0 A$ is complete; we may do this because the completion of any noetherian local ring is faithfully flat over it.

Let $k$ be the residue field of the (discrete) commutative ring $\pi_0 A$. The étale fundamental group of $\text{Spec} \pi_0 A$ is naturally isomorphic to that of $\text{Spec} k$, via the inclusion $\text{Spec} k \to \text{Spec} \pi_0 A$ as the closed point, since $\pi_0 A$ is a complete local ring. Let $x_1, \ldots, x_n \in \pi_0 A$ be generators for the maximal ideal. Consider the tower of $E_\infty$-$A$-algebras

$$\cdots \to A/(x_1^3, \ldots, x_n^3) \to A/(x_1^2, \ldots, x_n^2) \to A/(x_1, \ldots, x_n),$$

whose inverse limit is given by $A$ itself (by completeness). Observe that the maps at each stage are not uniquely determined. For instance, to give a map $A/(x_1^2, \ldots, x_n^2) \to A/(x_1, \ldots, x_n)$ amounts to giving nullhomotopies of each of
\[x_1^2, \ldots, x_n^2 \text{ in } A/(x_1, \ldots, x_n), \text{ and there are many possible choices of nullhomotopies.} \]

One just has to make choices.

Denote the \(E_{\infty}\)-algebras in this tower by \(\{A_m\}\). As a result, the equivalence \(A \simeq \lim_{\leftarrow} A_m\) leads to a fully faithful imbedding

\[\text{Mod}^\omega(A) \subset \lim_{\leftarrow} \text{Mod}^\omega(A_m),\]

from the \(\infty\)-category \(\text{Mod}^\omega(A)\) of perfect \(A\)-modules into the homotopy limit of the \(\infty\)-categories \(\text{Mod}^\omega(A_m)\) of perfect \(A_n\)-modules. As discussed in [Mat14, \S 7.1], this implies that if we show that the Galois group of each \(A_m\) is equivalent to that of \(k\) (i.e., is algebraic), then the Galois group of \(A\) itself is forced to be algebraic. This, however, is precisely what we proved in Lemma 5.3 above.

\[\square\]

5.3. The general case. In the previous parts of this section, we showed that the Galois theory of a rational 2-periodic, noetherian \(E_{\infty}\)-ring \(R\) was entirely algebraic. In this subsection, we will explain the modifications needed to handle the case where we do not have a unit in \(\pi_2\); in this case, the structure of the entire homotopy ring \(\pi_* R\) (rather than simply \(\pi_0 R\)) intervenes. We will begin with some generalities which, incidentally, shed further light on Galois groups of general \(E_{\infty}\)-rings.

Let \(R_*\) be a commutative, \(\mathbb{Z}\)-graded ring (not graded-commutative!). We start by setting up a Galois formalism for \(R_*\) that takes into account the grading.

**Definition 5.5.** A graded finite étale \(R_*\)-algebra is a commutative, graded \(R_*\)-algebra \(R'_*\) such that, as underlying commutative rings, the map \(R_* \to R'_*\) is finite étale.

**Example 5.6.** Given a finite étale \(R_0\)-algebra \(R'_0\), then one can build from this a graded finite étale \(R_*\)-algebra via \(R'_* \overset{\text{def}}{=} R'_0 \otimes_{R_0} R_*\).

**Example 5.7.** Let \(R_* = \mathbb{Z}[1/n, x_n^{\pm 1}]\) where \(|x_n| = n\). Then the map \(R_* \to R'_* = \mathbb{Z}[1/n, y_n^{\pm 1}], x_n \mapsto y_n^{n}\) is graded finite étale. In other words, one can adjoin \(n\)th roots of invertible generators in degrees divisible by \(n\), over a \(\mathbb{Z}[1/n]\)-algebra.

More generally, given a \(\mathbb{Z}[1/n]\)-algebra \(R\) and an invertible \(R\)-module \(L\), one has a graded algebra \(R_* = \bigoplus_{k \in \mathbb{N}} L^\otimes k\), and the inclusion map

\[\bigoplus_{k \in \mathbb{N}} L^\otimes k \to \bigoplus_{k \in \mathbb{Z}} L^\otimes k,\]

is graded finite étale. When \(L\) is trivial, this reduces to the previous part of this example.

Consider the category \(C_{R_*}\) of graded finite étale \(R_*\)-algebras and graded \(R_*\)-algebra homomorphisms. We start by observing that it is opposite to a Galois category. One can formulate this in the following manner. The grading on \(R_*\) determines an action of the multiplicative group \(\mathbb{G}_m\) on \(\text{Spec} R_*\), in such a manner that to give a quasi-coherent sheaf on the quotient stack \((\text{Spec} R_*)/\mathbb{G}_m\) is equivalent to giving a graded \(R_*\)-module. To give a finite étale cover of the quotient stack \((\text{Spec} R_*)/\mathbb{G}_m\) is equivalent to giving a graded \(R_*\)-algebra which is finite étale over \(R_*\). In other words, graded finite étale \(R_*\)-algebras are equivalent to finite étale covers of the stack \(\text{Spec} R_*/\mathbb{G}_m\).
Definition 5.8. We define the graded étale fundamental group \( \pi_1^\text{ét-gr} \) \( \text{Spec} R_* \) to be the étale fundamental group of the stack \((\text{Spec} R_*)/\mathbb{G}_m\).

Example 5.9. Suppose \( R_* \) contains a unit in degree 1. In this case, \( R_* \cong R_0 \otimes \mathbb{Z}[t^{\pm 1}] \) where \( |t| = 1 \). In particular, the quotient stack \((\text{Spec} R_*)/\mathbb{G}_m\) is simply \( \text{Spec} R_0 \), so the graded étale fundamental group of \( R_* \) is the fundamental group of \( \text{Spec} R_0 \).

Now, let \( A \) be any \( \mathbb{E}_\infty \)-ring, and consider the commutative, graded ring \( \pi_{\text{even}}(A) \) of the evenly graded homotopy groups of \( A \).

Remark 5.10. Here it is important to consider only evenly graded \( \pi_{\text{even}}(A) \)-algebras. That is, the \( \mathbb{G}_m \)-action on \( \pi_{\text{even}}(A) \) should be given by \( u \in \mathbb{G}_m \) acting as \( u^k \) on \( \pi_{2k} \); the odd homotopy groups need to be ignored.

Our first order of business is to show that the graded finite étale \( \pi_{\text{even}}(A) \)-algebras can be used to construct finite covers of \( A \).

Let \( A_{\text{even}} = \pi_{\text{even}}(A) \), and fix a graded, finite étale \( A_{\text{even}} \)-algebra \( A' \). We will construct an \( \mathbb{E}_\infty \)-\( A \)-algebra \( A' \) equipped with an isomorphism \( \pi_{\text{even}} A' \cong A' \) which is a finite cover of \( A \), generalizing the construction that starts with a finite étale \( \pi_0 A \)-algebra and obtains a finite cover of \( A \). As a result, we obtain:

Theorem 5.11. There is a natural fully faithful imbedding from the category of graded, finite étale \( A_{\text{even}} \)-algebras into the category of finite covers of \( A \) in \( \mathbb{E}_\infty \)-ring spectra.

Dually, we obtain surjections of profinite groups

\[ \pi_1 \text{Mod}(A) \twoheadrightarrow \pi_1^\text{ét-gr}(\text{Spec} A_{\text{even}}) \twoheadrightarrow \pi_1^\text{ét}(\text{Spec} \pi_0 A), \]

refining the surjection (5).

Proof. Start with a \( G \)-Galois object \( A'_* \) in the category of graded, finite étale \( A_{\text{even}} \)-algebras (for \( G \) a finite group). Then \( A'_* \) is a projective \( A_{\text{even}} \)-module. Let \( A'_* = A'_* \otimes_{A_{\text{even}}} \pi_*(A) \) be the graded-commutative algebra obtained by tensoring up; it acquires a \( G \)-action. Moreover, it is a finitely generated, projective \( \pi_*(A) \)-module, and the map

\[ \tilde{A}'_* \otimes_{\pi_*(A)} A'_* \twoheadrightarrow \prod_G A'_*, \]

given by all the twisted multiplications \( a_1 \otimes a_2 \mapsto a_1 g(a_2) \) for \( g \in G \), is an isomorphism. The projective \( \pi_*(A) \)-module \( \tilde{A}'_* \) naturally corresponds to a perfect \( A \)-module \( A' \) with \( \pi_*(A') \cong \tilde{A}'_* \).

More precisely, consider the category of graded \( \pi_*(A) \)-modules \( M_* \) whose underlying modules are finitely generated projective. Note that any such is automatically a projective object in the category of graded \( \pi_*(A) \)-modules: the set of all homomorphisms \( \text{Hom}_{\pi_*(A)}(M_*, N_*) \) always has a natural grading for any graded module \( N_* \), and if this is an exact functor in \( N_* \), the degree zero component (i.e., the grading-preserving homomorphisms) form an exact functor in \( N_* \) too. This category is equivalent to the homotopy category of all \( A \)-modules which are retracts of finite direct sums of shifts of \( A \). (We could make similar statements with \( \pi_{\text{even}}(A) \) and retracts of finite direct sums of even shifts of \( A \).)
Moreover, the above multiplication maps and $G$-action induce a $G$-action on $A'$ in the homotopy category of $\text{Mod}(A)$, and a homotopy commutative, associative, and unital multiplication on $A'$ in $\text{Mod}(A)$ such that the corresponding map

$$A' \otimes_A A' \to \prod_G A'$$

is an equivalence. In other words, $A'$ has all the structure of a (faithful) $G$-Galois extension of $A$ that can be seen at the level of the homotopy category of $\text{Mod}(A)$. But by [Mat14, Th. 6.26] (and [HM]), this canonically rigidifies to make $A'$ into an $E_\infty$-$A$-algebra with a $G$-action, such that it is a faithful $G$-Galois extension of $A$.

We can now describe the universal property of $A'$ as an $E_\infty$-$A$-algebra. In fact, we claim that for any $E_\infty$-$A$-algebra $B$, we have a natural homotopy equivalence

$$\text{Hom}_{\text{CAlg},A}(A', B) = \text{Hom}_{A_{\text{even}}}(A'_e, \pi_{\text{even}}(B)),$$

so in particular the left-hand-side is discrete. (This is not surprising; finite covers are always codiscrete objects.) But by Galois descent, this is also the $G$-fixed points of the set of maps

$$\text{Hom}_{\text{CAlg},A}(A' \otimes_A A', B \otimes_A A') \simeq \prod_G \text{Idem}(B \otimes_A A')$$

$$= \text{Hom}_{A'_e}(A'_e \otimes_{\pi_{\text{even}}(A)} A'_e, \pi_{\text{even}}(B) \otimes_{\pi_{\text{even}}(A)} A'_e),$$

since $A'$ has homotopy groups which are flat over $\pi_{\ast}(A)$ and since $A'_e \otimes_{\pi_{\text{even}}(A)} A'_e \simeq \prod_G A'_e$. But using the algebraic form of Galois descent, we get that

$$\text{Hom}_{A'_e}(A'_e \otimes_{\pi_{\text{even}}(A)} A'_e, \pi_{\text{even}}(B) \otimes_{\pi_{\text{even}}(A)} A'_e)^G = \text{Hom}_{A_{\text{even}}}(A'_e, \pi_{\text{even}}(B)),$$

so we get (7) as claimed.

This imbeds the Galois objects in the category of graded, finite étale $\pi_{\text{even}}(A)$-algebras fully faithfully (by (7)) in the category of finite covers of the $E_\infty$-ring. To associate a finite cover to any graded, finite étale $\pi_{\text{even}}(A)$-algebra, one now uses Galois descent: the Galois objects can be used to split any finite étale algebra object. Full faithfulness on these more general covers can now be checked locally, using descent.

With this in mind, we can (re)state and prove our main result.

**Theorem 5.12.** Let $R$ be a noetherian, rational $E_\infty$-ring. Then the natural map $\pi_1 \text{Mod}(R) \to \pi_{1,\text{gr}}^\ast(\text{Spec}\pi_{\text{even}}(R))$ is an isomorphism of profinite group(oid)s.

**Proof.** We have already proved this result if $R$ has a unit in degree two, thanks to Theorem 5.4 (see also Example 5.9). We want to claim that for any $R$ satisfying the conditions of this result, the functor from graded, finite étale $\pi_{\text{even}}(R)$-algebras to finite covers of the $E_\infty$-ring $R$ is an equivalence of categories $C_1 \simeq C_2$. We already know that the functor $C_1 \to C_2$ is fully faithful.

Both categories depend functorially on $R$ and have a good theory of descent via the base-change of $E_\infty$-rings $\mathbb{Q} \to \mathbb{Q}[t_2^{\pm1}]$, where $\mathbb{Q}[t_2^{\pm1}]$ is the free rational $E_\infty$-ring on an invertible degree two class. By descent theory, we can thus reduce to the case of a $\mathbb{Q}[t_2^{\pm1}]$-algebra when, as we already said, we have proved the result in Theorem 5.4. 

$\square$
6. Concluding remarks

6.1. The Picard group. Another natural invariant that one might attempt to study using the theory of residue fields is the Picard group. In fact, these techniques were originally introduced in [HMS94] in the study of the $K(n)$-local Picard group. It is known that if $A$ is an $E_\infty$-ring which is even periodic and whose $\pi_0$ is regular, then the Picard group is algebraic ([BR05]).

We do not if it is possible to describe the Picard group of rational $E_\infty$-rings in a similar global manner. However, the following example suggests caution.

Example 6.1. Consider the $E_\infty$-ring $R = \text{Sym}^* \mathbb{Q}[-1] \otimes \mathbb{Q}[e]/e^2$, which is obtained from the ring of “dual numbers” by freely adding a generator in degree $-1$. We have

$$\text{Mod}(R) \subset \text{Loc}_{S_1}(\text{Mod}(\mathbb{Q}[e]/e^2)),$$

that is, to give an $R$-module is equivalent to giving a $\mathbb{Q}[e]/e^2$-module together with an automorphism whose action on homotopy groups is ind-unipotent.

For example, we might consider the $\mathbb{Q}[e]/e^2$-module which is $\mathbb{Q}[e]/e^2$ and equip it with the automorphism given by $1 + re$, for any $r \in \mathbb{Q}$. For any $r \in \mathbb{Q}$, this defines an $R$-module $M_r$. Since the underlying $\mathbb{Q}[e]/e^2$-module is invertible, it follows that $M_r \in \text{Pic}(\text{Mod}(R))$. Moreover, $M_r \otimes M_{r'} \simeq M_{r+r'}$ (by composing automorphisms). This shows that there is a copy of $\mathbb{Q}$ inside the Picard group of $\text{Mod}(R)$ that one does not see from the homotopy groups of $R$.

6.2. Compact $E_\infty$-rings with big $\pi_*$. Not every compact $E_\infty$-$\mathbb{Q}$-algebra has the noetherianness properties used in this paper. For instance, as pointed out to us by J. Lurie, one can consider the $E_\infty$-ring $A$ of functions on the punctured affine plane $\mathbb{A}^2 \setminus \{(0,0)\}$, which fits into a homotopy pullback

$$\begin{array}{ccc}
A & \rightarrow & \mathbb{Q}[x^{\pm 1}, y] \\
\downarrow & & \downarrow \\
\mathbb{Q}[x, y^{\pm 1}] & \rightarrow & \mathbb{Q}[x^{\pm 1}, y^{\pm 1}] 
\end{array}$$

The homotopy groups $\pi_*(A)$ are given by

$$\pi_i(A) = \begin{cases} 
\mathbb{Q}[x, y] & \text{if } i = 0 \\
\mathbb{Q}[x, y]/(x^\infty, y^\infty) & \text{if } i = -1 \\
0 & \text{otherwise}
\end{cases}$$

where $\mathbb{Q}[x, y]/(x^\infty, y^\infty)$ denotes the cokernel of the map $\mathbb{Q}[x^{\pm 1}, y] \oplus \mathbb{Q}[x, y^{\pm 1}] \rightarrow \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$. In particular, $\pi_{-1}(A)$ is not a finitely generated $\pi_0(A)$-module (though $\pi_0(A)$ is noetherian).

However, the $E_\infty$-ring $A$ is compact. In fact, to give a morphism $A \rightarrow B$, for $B$ a rational $E_\infty$-ring, is equivalent to giving two elements $u, v \in \Omega^\infty B$ which have the property that $B/(x, y)$ is contractible. This follows from the fact that $A$ is the finite localization ([Mil92]) of $\mathbb{Q}[x, y]$ away from the $\mathbb{Q}[x, y]$-module $\mathbb{Q}[x, y]/(x, y)$ (which is supported at the origin). In particular, to give a morphism of $E_\infty$-rings $A \rightarrow B$ is equivalent to giving a map $\mathbb{Q}[x, y] \rightarrow B$ such that $B/(x, y)$ is contractible; note that this condition is detected in a finite stage of a filtered colimit. Forthcoming work of B. Bhatt and D. Halpern-Leinster gives in fact an explicit presentation of
A as an $E_\infty$-ring under $\mathbb{Q}[x,y]$. Consider the $\mathbb{Q}[x,y]$-module $M = \mathbb{Q}[x,y]/(x,y)$ and the natural map $\phi: \mathbb{Q}[x,y] \to M$. The dual gives a map $\psi: \mathbb{D}M \to \mathbb{Q}[x,y]$, where $\mathbb{D}M$ is the Spanier-Whitehead dual of $M$. Then one has:

**Proposition 6.2** (Bhatt, Halpern-Leinster). The $E_\infty$-$\mathbb{Q}[x,y]$-algebra is the pushout

$$
\begin{array}{ccc}
\text{Sym}^*_\mathbb{Q}[x,y](\mathbb{D}M) & \longrightarrow & \mathbb{Q}[x,y] \\
\downarrow & & \downarrow \\
\mathbb{Q}[x,y] & \longrightarrow & A
\end{array}
$$

where:

1. $\text{Sym}^*_\mathbb{Q}[x,y](\mathbb{D}M)$ is the free $E_\infty$-$\mathbb{Q}[x,y]$-algebra on the $\mathbb{Q}[x,y]$-module $\mathbb{D}M$.
2. The two maps $\text{Sym}^*_\mathbb{Q}[x,y](\mathbb{D}M) \to \mathbb{Q}[x,y]$ are adjoint to two maps of $\mathbb{Q}[x,y]$-modules $\mathbb{D}M \to \mathbb{Q}[x,y]$ which are given by $\psi$ and the zero map.

The $\infty$-category of $A$-modules is equivalent to the $\infty$-category of quasi-coherent sheaves on the scheme $A^2(\mathbb{Q})\{\,(0,0)\}$, since this scheme is quasi-affine. In particular, it follows (by a result of Thomason [Tho97]) that the thick subcategories of $\text{Mod}^\omega(A)$ correspond to the subsets of $A^2(\mathbb{Q})\{\,(0,0)\}$ which are closed under specialization. In particular, Theorem 1.3 fails for $A$, as there is no thick subcategory corresponding to the origin in $\text{Spec}_{\mathbb{Q}}A$.

### 6.3. Bousfield decompositions

Given a rational, noetherian $E_\infty$-ring $A$ (say $2$-periodic for simplicity), we constructed a family of residue fields $\{k(p)\}_{p \in \text{Spec}A}$ that detect nilpotence in the $\infty$-category $\text{Mod}(A)$. One could ask if one has in fact a Bousfield decomposition. That is, if an $A$-module $M$ (not necessarily perfect) has the property that $k(p)_*(M) = 0$ for all $p \in \text{Spec}A$, does that force $M$ to be contractible? In the regular and $2$-periodic case, this is known (e.g., [Mat13, Prop. 2.5]). The analog over the sphere fails: there are noncontractible spectra that smash to zero with every residue field, for instance the Brown-Comenentz dual $I$ of the sphere ([HS99, Appendix B]). We do not know what happens in $\text{Mod}(A)$.

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