FIRST ORDER APPROACH AND INDEX THEOREMS FOR DISCRETE AND METRIC GRAPHS

OLAF POST

ABSTRACT. The aim of the present paper is to introduce the notion of first order (supersymmetric) Dirac operators on discrete and metric ("quantum") graphs. In order to cover all self-adjoint boundary conditions for the associated metric graph Laplacian, we develop systematically a new type of discrete graph operators acting on a decorated graph. The decoration at each vertex of degree \( d \) is given by a subspace of \( \mathbb{C}^d \), generalising the fact that a function on the standard vertex space has only a scalar value.

We develop the notion of exterior derivative, differential forms, Dirac and Laplace operators in the discrete and metric case, using a supersymmetric framework. We calculate the (supersymmetric) index of the discrete Dirac operator generalising the standard index formula involving the Euler characteristic of a graph. Finally, we show that the corresponding index for the metric Dirac operator agrees with the discrete one.

1. INTRODUCTION

In the last years, many attention has been payed in the analysis of metric graph Laplacians, i.e., operators acting as second order differential operators on each edge considered as one-dimensional space, with suitable (vertex) boundary conditions turning the Laplacian into a self-adjoint (unbounded) operator. In most of the works, the second order operator is the starting object for the analysis. For more details on Laplacians on metric graphs, also labelled as “quantum graphs”, we refer to the articles \cite{KS06, Ku04, Ku05} and the references therein.

In this paper whereas, we want to introduce the metric graph Laplacians with general (non-negative) vertex boundary conditions via first order operators, namely via an exterior derivative analogue as in differential geometry. As a by-product, we obtain a new type of discrete graph operators acting on a decorated graph. The decoration at each vertex \( v \) of degree \( \text{deg } v \) is given by a subspace of \( \mathbb{C}^{\text{deg } v} \), generalising the fact that a function \( F \in \ell_2(V) \) on the standard vertex space on \( V \) has only a scalar value \( F(v) \in \mathbb{C} \). In addition, we introduce the notion of a discrete exterior derivative, a discrete Dirac and Laplace operator and show an index theorem generalising the standard index formula involving the Euler characteristic of a graph (cf. Theorem \ref{thm:index}).

In a second part, we define exterior derivatives, Dirac and Laplace operators on a (continuous) metric graph and relate their kernels with the appropriate discrete objects and show that the index agrees with the index of the discrete setting (cf. Theorem \ref{thm:continuous}.

We introduce all Laplacians in a supersymmetric setting, i.e., by appropriate “exterior derivatives” mimicking the corresponding notion for manifolds. The advantage
is the simple structure of these operators; and the use of the abstract supersymmetric setting, e.g., the spectral equality of the Laplacian defined on even and odd “differential forms” (cf. Lemma 1.2).

Index formulas may be used in order to decide whether a metric graph $X_0$ with Laplacian $\Delta_{X_0}$ occurs as limit of a “smooth” space, i.e., a manifold or an open neighbourhood $X_\varepsilon$ of $X_0$ together with a natural Laplacian $\Delta_{X_\varepsilon}$. If $X_\varepsilon$ is homotopy-equivalent to $X_0$ then their Euler characteristics agree, and correspondingly, appropriately defined indices for the operators on $X_\varepsilon$ and $X_0$ must agree if the operators converge. We comment on this observation in Section 6.2.

Spectral graph theory is an active area of research. We do not attempt to give a complete overview here. Results on spectral theory of discrete or combinatorial Laplacians can be found e.g. in [Dod84, MW89, CdV98, Chm97]. For continuous (quantum) graph Laplacians we mention the works [Rot84, Nic87, KS99, Har00, KS03, Ku04, FT04a, Ku05, KS06, Pan06, HP06]. In particular, a heat equation approach for the index formula for certain metric graph Laplacian (with energy-independent scattering matrix) can be found in [KPS07]. In particular, when submitting this work, we learned about a related work on index formulas on quantum graphs proven in a direct way (not using our discrete exterior calculus) by Fulling, Kuchment and Wilson [FKuW07]. Prof. Fulling announced the results in a talk at the Isaac Newton Institute (INI) in Cambridge [F07] where also the first order factorisation of the standard quantum graph Laplacian appears.

When submitting this work, the work [FKuW07] where a similar index formula for quantum graphs is proven in a direct way (not using our discrete exterior calculus).

The paper is organised as follows: In the next subsection, we start with a motivating example of standard boundary conditions in order to illustrate the basic results and ideas. In Section 1.2 we develop the abstract setting of supersymmetry. In Section 2 we define a generalisation for the discrete vertex space $\ell_2(V)$, namely, general vertex spaces. In Section 3 we generalise the notion of the coboundary operator (“exterior derivative”), Dirac and Laplace operators in this context. In Section 4 we calculate the index of the discrete Dirac operator for general vertex spaces and generalise the below discrete Gauß-Bonnet formula (1.6). In Section 5 we develop the theory of “exterior derivatives” on a metric graph and introduce the corresponding notion of Dirac and Laplace operators. In particular, we cover all self-adjoint boundary conditions leading to a non-negative Laplacian. Finally, in Section 6 we show that the discrete and continuous Laplacians agree at the bottom of the spectrum, i.e., the index formula (1.8) for the general case. We conclude with a series of examples showing how an index formula can be used to find “smooth” approximations of metric graph Laplacians.

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1.1. The standard case. In order to motivate our abstract setting, we start with the standard Laplacian in the discrete and continuous setting. Details can be found in the subsequent sections. Let $X = (V, E)$ be an oriented graph with $V$ the set of vertices and $E$ the set of edges $e$, where we denote the initial vertex by $\partial_+ e$ and the terminal vertex by $\partial_- e$. Denote by $\ell_2(V)$ the standard vertex space with weight $\deg v$, the degree of the vertex $v$. We consider a (scalar) function in $\ell_2(V)$ as a
“0-form”. The **coboundary operator** or **(discrete) exterior derivative** is defined as

\[ d : \ell_2(V) \rightarrow \ell_2(E), \quad (dF)_e = F(\partial_+ e) - F(\partial_- e) \]

mapping 0-forms into 1-forms with adjoint operator

\[ d^* : \ell_2(E) \rightarrow \ell_2(V), \quad (d\eta)(v) = \frac{1}{\deg v} \sum_{e \in E_v} \hat{1}_e(v) \eta_e \]

where \( \hat{1}_e(v) = \pm 1 \) if \( v = \partial_+ e \) and \( E_v \) is the set of edges adjacent to \( v \). We call the operator

\[ D(F \oplus \eta) = d^* \eta \oplus dF, \quad \text{i.e.,} \quad D \cong \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix} \]

(1.1)

the associated **Dirac operator** on \( \ell_2(AX) := \ell_2(V) \oplus \ell_2(E) \). The associated Laplacian is defined as \( \Delta_{AX} := D^2 \), and in particular, its component on 0-forms, i.e., on \( \ell_2(V) \) is the standard Laplacian of discrete graph theory, namely

\[ (d^* dF)(v) = (\Delta^0_X F)(v) = \frac{1}{\deg v} \sum_{e \in E_v} (F(v) - F(v_e)) \]

(1.2)

where \( v_e \) denotes the vertex opposite to \( v \) on \( e \in E_v \). For a **finite** graph \( X \), we define the **index** of \( D \) as

\[ \text{ind } D := \dim \ker d - \dim \ker d^*, \quad (1.3) \]

i.e., the index of \( D \) is the Fredholm index of \( d \). It is a classical result from cohomology theory, that the Fredholm-index of the coboundary operator \( d \) equals the **Euler characteristic** \( \chi(X) := |V| - |E| \), namely,

\[ \text{ind } D = \chi(X). \]

(1.4)

If we define the **curvature** at the vertex \( v \in V \) as

\[ \kappa(v) := 1 - \frac{1}{2} \deg v, \]

(1.5)

we can interpret the formula (1.4) as a “discrete Gauß-Bonnet” theorem, namely

\[ \text{ind } D = \sum_{v \in V} \kappa(v) \]

(1.6)

using the classical formula \( 2|E| = \sum_{v \in V} \deg v \). Note that \( \kappa(v) < 0 \) iff \( \deg v \geq 3 \).

Considering \( X \) as a metric graph, our basic Hilbert space is \( L_2(X) \) (cf. (5.1)). On the metric graph, we consider the “exterior” derivative

\[ d : \text{dom } d \rightarrow L_2(X), \quad df = f' = \{f'_e\}_e \]

where \( \text{dom } d = H^1_{\text{max}}(X) \cap C(X) \) is the Sobolev space of functions **continuous** at each vertex. Its \( L_2 \)-adjoint is

\[ d : \text{dom } d^* \rightarrow L_2(X), \quad dg = g' = \{-g'_e\}_e \]

with \( g \in \text{dom } d^* \) iff

\[ \sum_{e \in E_v} \hat{g}_e(v) = 0, \]

(1.7)

where \( \hat{g}_e(v) \) is the **oriented** evaluation at \( v \) (see Eq. (5.2)). As before, we can define a Dirac operator \( D \) on \( L_2(X) \oplus L_2(X) \) and the associated Laplacian \( \Delta_{AX} \) such that its 0-form component is

\[ \Delta^0_X f := d^* df = -f'' = \{-f''_e\}_e \]
with domain
\[ \text{dom } \Delta_X^0 = \{ f \in \text{dom } d \mid f' \in \text{dom } d^* \}, \]
i.e., the standard Laplacian on a metric graph with functions continuous at each vertex and the Kirchoff sum condition for the derivative at each vertex. Although the 0- and 1-forms are formally the same, they differ in their interpretation: We consider 0-forms as \textit{scalar} functions, whereas a 1-form is a \textit{vector-field} with orientation. Then the Kirchoff sum condition Eq. (1.7) is just a “flux” conservation for the flux generated by the “vector field” \( f' \).

Again, we define the index \( \text{ind } D \) of the metric graph Dirac operator \( D \) as the Fredholm-index of \( d \), i.e. in the same way as in Eq. (1.3) and one of our main results in this setting (cf. Theorem 6.1) is
\[ \text{ker } D \cong \text{ker } D \quad \text{and} \quad \text{ind } D = \text{ind } D (= |V| - |E|), \quad (1.8) \]
i.e., an isomorphism between the kernels of the discrete and continuous case.

We want to generalise the above setting to quantum graph Laplacians with \textit{general} self-adjoint operators \( \Delta_X \) (such that \( \Delta_X \geq 0 \)) and derive a similar index formula.

1.2. Supersymmetry. Before defining several operators on a graph, we collect common features shared by several operators. Since in our cases we only define \( p \)-forms for \( p \in \{0, 1\} \), we can identify forms of even and odd degree with the cases \( p = 0 \) and \( p = 1 \), respectively.

\textbf{Definition 1.1.} Let \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \) be a Hilbert space and \( d: \text{dom } d \rightarrow \mathcal{H}_1 \) a closed operator with \( \text{dom } d \subset \mathcal{H}_0 \) (\( d \) may be bounded, in this case we have \( \text{dom } d = \mathcal{H}_0 \)). Then we say that \( d \) has \textit{supersymmetry} or that \( d \) is an \textit{exterior derivative}. A \( p \)-form is an element in \( \mathcal{H}_p \). Furthermore, we define the associated \textit{Dirac operator} as
\[ D(f_0 \oplus f_1) = d^* f_1 \oplus df_0, \quad \text{i.e.,} \quad D \cong \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix} \]
with respect to the decomposition \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \). The associated \textit{Laplacian} is given by \( \Delta := D^2 \). In particular,
\[ \Delta \cong \begin{pmatrix} \Delta_0 & 0 \\ 0 & \Delta_1 \end{pmatrix}, \]
where \( \Delta_0 = d^* d \) and \( \Delta_1 = dd^* \) on their natural domains.

Clearly, \( \Delta \) and \( \Delta^p \) are closed, non-negative operators. Note that \( \text{ker } d = \text{ker } \Delta_0 \) and \( \text{ker } d^* = \text{ker } \Delta_1 \).

We denote the spectral projection of \( \Delta_p \) by \( 1_B(\Delta_p) \). We have the following results on the spectrum away from 0:

\textbf{Lemma 1.2.} Assume that \( d \) has supersymmetry and that \( B \subset [0, \infty) \) is a bounded Borel set. Then
\[ d 1_B(\Delta_0) = 1_B(\Delta_1)d \quad \text{and} \quad d^* 1_B(\Delta_1) = 1_B(\Delta_0)d^*. \]
Furthermore, if 0 is not contained in \( B \), then
\[ d: 1_B(\Delta_0)(\mathcal{H}_0) \rightarrow 1_B(\Delta_1)(\mathcal{H}_1) \quad \text{and} \quad d^*: 1_B(\Delta_1)(\mathcal{H}_1) \rightarrow 1_B(\Delta_0)(\mathcal{H}_0) \]
are isomorphisms. In particular,
\[ \dim 1_B(\Delta_0) = \dim 1_B(\Delta_1) \quad \text{and} \quad \sigma(\Delta_0) \setminus \{0\} = \sigma(\Delta_1) \setminus \{0\}, \]
i.e., the spectra of \( \Delta_0 \) and \( \Delta_1 \) away from 0 agree including multiplicity.
Proof. The first assertion follows from $d\varphi(d^*d) = \varphi(dd^*)d$, first for polynomials $\varphi$, then for functions $\varphi(\lambda) = (\lambda + 1)^{-k}$, $k \geq 1$, and finally by the spectral calculus also for (fast enough decaying) continuous and measurable functions. The second assertion follows since $\ker d = \ker \Delta_0 = 0$ and $\ker d^* = \ker \Delta_1 = 0$. The last statement is a simple consequence of the isomorphisms. □

We have the following result, an abstract version of the Hodge decomposition:

**Lemma 1.3.** Assume that $d$ has supersymmetry and that the associated Dirac operator $D$ has a spectral gap at 0, i.e., $\text{dist}(0, \sigma(D) \setminus \{0\}) > 0$. Then:

$$\mathcal{H} = \ker D \oplus \text{ran } d^* \oplus \text{ran } d,$$

$$\mathcal{H}_0 = \ker d \oplus \text{ran } d^* \quad \text{and} \quad \mathcal{H}_1 = \ker d^* \oplus \text{ran } d.$$

**Proof.** It is a general fact that $\mathcal{H}_0 = \ker d \oplus \text{ran } d^*$ and similarly for $\mathcal{H}_1$. It remains to show that $\text{ran } d$ and $\text{ran } d^*$ are closed. Let $\tilde{D}$ be the restriction of $D$ onto $(\ker D)\perp$. By our assumption, $\tilde{D}$ has a bounded inverse, namely

$$\tilde{D}^{-1} \cong \begin{pmatrix} 0 & \tilde{d}^{-1} \\ (\tilde{d}^*)^{-1} & 0 \end{pmatrix},$$

where $\tilde{d}$ and $\tilde{d}^*$ are the restrictions of $d$ and $d^*$ to $(\ker d)^\perp$ and $(\ker d^*)^\perp$, respectively. In particular, $\tilde{d}^{-1}$ and $(\tilde{d}^*)^{-1}$ are bounded.

Let $g \in \text{ran } d$, then there exists a sequence $\{f_n\} \subset \mathcal{H}_0$ such that $df_n \to g$ in $\mathcal{H}_1$. Without loss of generality, we may assume that $f_n \in (\ker d)^\perp$. Therefore, $\tilde{d}^{-1}df_n = f_n \to \tilde{d}^{-1}g =: f$. Now, $f_n \to f$, $df_n \to g$ and $d$ is closed, so $f \in \text{dom } d$ and in particular, $df = g \in \text{ran } d$. □

**Definition 1.4.** If $\ker d$ and $\ker d^*$ are both finite dimensional (i.e., $0 \notin \sigma_{\text{ess}}(D)$), we define the index of $D$ as

$$\text{ind } D := \dim \ker d - \dim \ker d^*.$$ 

Note that $\text{ind } D$ is the usual Fredholm index of the operator $d$.

We need the following fact in order to calculate the index in concrete examples:

**Lemma 1.5.** Assume that $\{D_t\}_{t \in \mathbb{R}}$ is a family of bounded Dirac operators such that $t \mapsto D_t$ is norm-continuous. Then $\text{ind } D_t$ is constant.

**Proof.** This follows from the fact that the Fredholm index depends continuously on the operator and that a continuous function into $\mathbb{Z}$ is locally constant (see e.g. [Gil95, Lem. 1.4.3]). □

We need the notion of a morphism of this structure.

**Definition 1.6.** Suppose that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ with operator $d$ and $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 \oplus \tilde{\mathcal{H}}_1$ with operator $\tilde{d}$ and associated Dirac operators $D$ and $\tilde{D}$, respectively, have supersymmetry. We say that a linear map $\Phi : \text{dom } D \longrightarrow \text{dom } \tilde{D}$ respects supersymmetry iff $\Phi$ decomposes into $\Phi = \Phi_0 \oplus \Phi_1$ where $\Phi_p$ maps $p$-forms onto $p$-forms.

In some cases we need to enlarge the Hilbert space $\mathcal{H}$ by a space $\mathcal{N}$ on which the exterior derivative acts trivially:

\[\text{ran } d \text{ by } \text{ran } d \text{ and similarly for } d^*, \text{ we can drop this condition.}\]
Definition 1.7. Let $\mathcal{N}$ be a Hilbert space. We set $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{N} \oplus \mathcal{H}_1$. Assume that $d$ is an exterior derivative on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Then we call

$$d_{\gamma_0} = d \oplus 0 : \text{dom} d \oplus \mathcal{N} \longrightarrow \mathcal{H}_1, \quad f \oplus h \mapsto df$$

the exterior derivative trivially $0$-enlarged by $\mathcal{N}$. The associated Dirac operator will be denoted by $D_{\gamma_0}$.

Similarly, we call

$$d_{\gamma_1} = d \oplus 0 : \text{dom} d \longrightarrow \mathcal{H}_1 \oplus \mathcal{N}, \quad f \mapsto df \oplus 0$$

the exterior derivative trivially $1$-enlarged by $\mathcal{N}$. The associated Dirac operator will be denoted by $D_{\gamma_1}$.

Note that $d_{\gamma_0}^* = (d \oplus 0)^* = d^* \oplus 0$ and $d_{\gamma_1}^* = (d \oplus 0)^* = d^* \oplus 0$. Furthermore, $\ker d_{\gamma_0} = \ker d \oplus \mathcal{N}$, $\ker d_{\gamma_0}^* = \ker d^* \text{ and } \ker d_{\gamma_1} = \ker d$, $\ker d_{\gamma_1}^* = \ker d^* \oplus \mathcal{N}$. In particular, we have

$$\text{ind } D_{\gamma_0} = \text{ind } D + \dim \mathcal{N} \quad \text{and} \quad \text{ind } D_{\gamma_1} = \text{ind } D - \dim \mathcal{N}. \quad (1.9)$$

2. Vertex spaces on discrete graphs

2.1. Discrete graphs. Suppose $X$ is a discrete weighted graph given by $(V, E, \partial, \ell)$ where $(V, E, \partial)$ is a usual graph, i.e., $V$ denotes the set of vertices, $E$ denotes the set of edges, $\partial : E \longrightarrow V \times V$ associates to each edge $e$ the pair $(\partial_- e, \partial_+ e)$ of its initial and terminal point (and therefore an orientation). That $X$ is an (edge-)weighted graph means that there is a length or (inverse) edge weight function $\ell : E \longrightarrow (0, \infty)$ associating to each edge $e$ a length $\ell_e$. For simplicity, we consider internal edges only, i.e., edges of finite length $\ell_e < \infty$.

For each vertex $v \in V$ we set

$$E_v^\pm := \{ e \in E \mid \partial_\pm e = v \} \quad \text{and} \quad E_v := E_v^+ \cup E_v^-,$$

i.e., $E_v^\pm$ consists of all edges starting ($-$) resp. ending ($+$) at $v$ and $E_v$ their disjoint union. Note that the disjoint union is necessary in order to allow self-loops, i.e., edges having the same initial and terminal point. The (in/out-)degree of $v \in V$ is defined as

$$\text{deg}^+ v := |E_v^+|, \quad \text{deg}^- v := |E_v^-|, \quad \text{deg} v := |E_v| = \text{deg}^+ v + \text{deg}^- v,$$

respectively. In order to avoid trivial cases, we assume that $\text{deg} v \geq 1$, i.e., no vertex is isolated. On the vertices, we usually consider the canonical (vertex-)weight $\text{deg} v$ (see e.g. the norm definition of $\ell_2(V)$ in (2.5)).

We say that the graph $X$ is $d$-regular, iff $\text{deg} v = d$ for all $v \in V$. Furthermore, $X$ is bipartite, if there is a decomposition $V = V_- \cup V_+$ such that no vertex in $V_-$ is joined with a vertex in $V_+$ by an edge and similar for $V_+$.

We have the following equalities

$$\bigcup_{v \in V} E_v^+ = \bigcup_{v \in V} E_v^- = E \quad \text{and} \quad \bigcup_{v \in V} E_v = E \cup E, \quad (2.1)$$

since each (internal) edge has exactly one terminal vertex and one initial vertex. In addition, a self-loop edge $e$ is counted twice in $E_v$. In particular,

$$\sum_{v \in V} \text{deg} v = 2|E|. \quad (2.2)$$
2.2. **General vertex spaces.** We want to introduce a vertex space allowing us to define Laplace-like operators coming from general vertex boundary conditions for quantum graphs. The usual discrete Laplacian is defined on 0-forms and 1-forms, namely, on sections in the trivial bundles

\[ \Lambda^0 X = V \times \mathbb{C} \quad \text{and} \quad \Lambda^1 X = E \times \mathbb{C}. \]

In order to allow more general vertex boundary conditions in the quantum graph case later on, we need to enlarge the space at each vertex \( v \). We denote \( \mathcal{G}_v^{\max} := \mathbb{C}^E_v \) the maximal vertex space at the vertex \( v \in V \), i.e., a value \( \mathcal{F}(v) \in \mathcal{G}_v^{\max} \) has degree \( v \) components, one for each adjacent edge. A (general) vertex space is a family \( \{ \mathcal{G}_v \}_v \) of subspaces \( \mathcal{G}_v \) of \( \mathcal{G}_v^{\max} \) for each vertex \( v \). We can consider a vertex space as a vector bundle

\[ \Lambda^0 X := \bigcup_{v \in V} \mathcal{G}_v \]

over the discrete base space \( V \) with fibres \( \mathcal{G}_v \) of mixed rank generalising the above setting where \( \mathcal{G}_v \cong \mathbb{C} \) at each vertex. An element of \( \mathcal{G}_v^{\max} \) will generally be denoted by \( \mathcal{F}(v) = \{ F_e(v) \}_{e \in E_v} \). Note that

\[ \mathcal{G}_v^{\max} := \bigoplus_{v \in V} \mathcal{G}_v^{\max} \cong \bigoplus_{e \in E} \mathbb{C}^2 \tag{2.3} \]

since each edge occurs twice in the \( E_v, v \in V \) (cf. Eq. (2.1)).

We denote by

\[ \ell_2(\Lambda^0 X) = \mathcal{G} := \bigoplus_{v \in V} \mathcal{G}_v \quad \text{and} \quad \ell_2(\Lambda^1 X) = \ell_2(E) = \bigoplus_{e \in E} \frac{1}{\ell_e} \mathbb{C} \tag{2.4} \]

the associated Hilbert spaces of 0- and 1-forms with norms defined by

\[ \| F \|_{\mathcal{G}}^2 := \sum_{v \in V} |\mathcal{F}(v)|^2 = \sum_{v \in V} \sum_{e \in E_v} |F_e(v)|^2 \quad \text{and} \quad \| \eta \|_{\ell_2(E)}^2 := \sum_{e \in E} |\eta_e| \frac{1}{\ell_e}. \]

Abusing the notation, we also call the section space \( \mathcal{G} \) a vertex space.

**Definition 2.1.** We say that an operator \( A \) on \( \mathcal{G} \) is local iff \( A \) decomposes with respect to \( \mathcal{G} = \bigoplus_{v \in V} \mathcal{G}_v \), i.e., \( A = \bigoplus_{v \in V} A_v \) where \( A_v \) is an operator on \( \mathcal{G}_v \).

Associated to a vertex space is an orthogonal projection \( P = \bigoplus_{v \in V} P_v \) in \( \mathcal{G}^{\max} \), where \( P_v \) is the orthogonal projection in \( \mathcal{G}_v^{\max} \) onto \( \mathcal{G}_v \). Alternatively, a vertex space is characterised by fixing an orthogonal projection \( P \) in \( \mathcal{G} \) which is local.

**Remark 2.2.** If \( X \) is finite, we can assume without loss of generality that \( P \) is local. If this is not the case, we can pass to a new graph \( \tilde{X} \) by identifying vertices \( v \in V \) for which \( P \) does not decompose with respect to \( \mathcal{G}_v^{\max} \oplus \bigoplus_{v \neq v} \mathcal{G}_v^{\max} \). In the worst case, the new graph \( \tilde{X} \) is a rose, i.e., \( \tilde{X} \) consists of only one vertex with \( |E| \) self-loops attached.

The following notation will be useful:

**Definition 2.3.** The linear operator \( \tau = (\cdot) : \mathcal{G}^{\max} \rightarrow \mathcal{G}^{\max} \) \( F \mapsto \hat{F} \), defined by \( \tau := \bigoplus_{v \in V} \tau_v \) and

\[ \tau_v(F(v)) := \hat{F}(v) = \{ \hat{F}_e(v) \}_{e \in E_v}, \quad \hat{F}_e(v) := \pm F_e(v), \quad \text{if} \ v = \partial_e \]

is called orientation map. We say that \( \tau \) switches from an unoriented evaluation to an oriented evaluation and vice versa.
Clearly, \( \tau \) is a unitary local involution and given by the multiplication with \( \hat{1}(v) \) on \( \mathcal{G}^{\text{max}}_v \) where \( \hat{1}(v) = \pm 1 \) if \( v = \partial_\pm e \).

**Definition 2.4.** Let \( \mathcal{G} = \bigoplus_{v \in V} \mathcal{G}_v \) be a vertex space with associated projection \( P \).

The *dual* vertex space is defined by \( \mathcal{G}^\perp := \mathcal{G}^{\text{max}} \otimes \mathcal{G} \) with projection \( P^\perp = 1 - P \). The oriented version of the vertex space \( \mathcal{G} \) is defined by \( \hat{\mathcal{G}} := \tau \mathcal{G} \) with projection \( \hat{P} = \tau P \).

It can easily be seen that \( \hat{\mathcal{G}} = \mathcal{G} \) iff \( \hat{1}(v) = \pm \mathbb{1}(v) \) for all \( v \in V \), i.e., iff the graph \( X \) is bipartite (with partition \( V = V_- \cup V_+ \)) and the orientation is chosen in such a way that \( \partial_\pm e \in V_\pm \) for all \( e \in E \).

In the following we give several examples of vertex spaces. We will see later on that these spaces are closely related to quantum graph Laplacian where the names come from. We start with two trivial vertex spaces:

**Example 2.5.**

(i) We call the trivial subspace \( \mathcal{G}_v = \mathcal{G}_v^{\text{min}} = 0 \) the minimal or Dirichlet vertex space. The corresponding projection is \( P_v = 0 \).

(ii) We call the maximal subspace \( \mathcal{G}_v = \mathcal{G}_v^{\text{max}} \) the maximal or Neumann vertex space. The corresponding projection is \( P_v = 1 \). Clearly, \( \mathcal{G}_v^{\text{max}} \) is dual to \( \mathcal{G}_v^{\text{min}} \).

These examples are trivial, since every edge decouples from the others:

**Definition 2.6.** Let \( \mathcal{G}_v \) be a vertex space at \( v \) with projection \( P_v \).

(i) We say that \( e_1 \in E_v \) interacts with \( e_2 \in E_v \) in \( \mathcal{G}_v \) iff

\[ p_{e_1,e_2}(v) := \langle \delta_{e_1}(v), P_v \delta_{e_2}(v) \rangle \neq 0 \]

where \( (\delta_{e_1})_v = 1 \) if \( e = e_1 \) and 0 otherwise. If \( p_{e_1,e_2}(v) = 0 \), we say that \( e_1, e_2 \in E_v \) decouple in \( \mathcal{G}_v \).

(ii) We say that \( \mathcal{G}_v \) decouples along \( E_1 \cup E_2 \subseteq E_v \) iff \( e_1 \) and \( e_2 \) decouple in \( \mathcal{G}_v \) for all \( e_1 \in E_1 \) and \( e_2 \in E_2 \).

(iii) We say that \( \mathcal{G}_v \) is completely interacting iff \( e_1 \) and \( e_2 \) are interacting for any \( e_1, e_2 \in E_v \), \( e_1 \neq e_2 \).

**Lemma 2.7.** The edges \( e_1, e_2 \in E_v \) (\( e_1 \neq e_2 \)) are interacting (resp. decouling) in \( \mathcal{G}_v \) iff they are in \( \mathcal{G}_v^\perp \). In particular, \( \mathcal{G}_v \) is completely interacting iff \( \mathcal{G}_v^\perp \) is.

**Proof.** The claim follows immediately from

\[ \langle \delta_{e_1}, P_v \delta_{e_2} \rangle = -\langle \delta_{e_1}, P_v \delta_{e_2} \rangle \]

since \( e_1 \neq e_2 \).

**Remark 2.8.** Let \( \mathcal{G} \) be a vertex space associated to the graph \( X \) such that \( \mathcal{G}_v \) decouples along \( E_1 \cup E_2 = E_v \), then \( \mathcal{G}_v = \mathcal{G}_{1,v} \oplus \mathcal{G}_{2,v} \). Passing to a new graph \( \tilde{X} \) with the same edge set \( E(\tilde{X}) = E(X) \) but replacing \( v \in V(X) \) by two vertices \( v_1, v_2 \) with \( E_{v_1} = E_1 \) and \( E_{v_2} = v_2 \), we obtain a new graph with one more vertex. Repeating this procedure, we can always assume that no vertex space \( \mathcal{G}_v \) decouple along a non-trivial decomposition \( E_v = E_1 \cup E_2 \). It would be interesting to understand the "irreducible" building blocks of this decomposition procedure.

We will define now our main example, since it covers many of classically defined discrete Laplacians on a graph, as we will see later on:
Definition 2.9. We say that a vertex space $\mathcal{G}_v$ is \textit{(weighted) continuous} if \(\dim \mathcal{G}_v = 1\), i.e.,
\[
\mathcal{G}_v = \mathbb{C} \mathbb{P}(v), \quad |\mathbb{P}(v)|^2 = \text{deg } v,
\]
and $\mathcal{G}_v$ is completely interacting, i.e., $p_e(v) \neq 0$ for all $e \in E_v$ where $\mathbb{P}(v) = \{p_e(v)\}_e$.

A vertex space $\mathcal{G}$ is called \textit{(weighted) continuous} if all its components $\mathcal{G}_v$ are (weighted) continuous and if there are uniform constants $p_{\pm} \in (0, \infty)$ such that
\[
p_- \leq |p_e(v)| \leq p_+, \quad e \in E_v, \quad V \in V.
\]
The dual of a continuous vertex space is called an \textit{(unoriented weighted) sum vertex space}.

Applying the procedure of Remark 2.8, any vertex space $\mathcal{G}$ of dimension 1 with generating vector $p(v)$ has a decomposition of $\mathcal{G}_v$ along $E_1 := \{e \in E_v \mid p_e(v) \neq 0\}$ and $E_2 := E_v \setminus E_1$. The corresponding space $\mathcal{G}_{1,v}$ is now a continuous vertex space.

In all of the following examples, we can choose $p_{\pm} = 1$ as uniform bounds.

Example 2.10.

(iii) Choosing $\mathbb{P}(v) = \mathbb{I}(v)$, i.e., $\mathcal{G}_v := \mathcal{G}_v^{\text{std}} := \mathbb{C} \mathbb{I}(v) = \mathbb{C}(1, \ldots, 1)$, we obtain the (uniform) continuous or standard vertex space denoted by $\mathcal{G}_v^{\text{std}}$ where all coefficients $p_e(v) = 1$. The associated projection is
\[
P_v = \frac{1}{\text{deg } v} \mathbb{E}
\]
where $\mathbb{E}$ denotes the square matrix of rank $\text{deg } v$ where all entries equal 1.

(iv) We also have an oriented version of the standard vertex space, namely $\mathcal{G}_v^{\text{std}} = \mathbb{C} \hat{\mathbb{I}}$ where $\hat{\mathbb{I}}$ is defined in Definition 2.3. In particular,
\[
p_e(v) = \pm 1 \quad \text{if} \quad v = \partial_{\pm} e.
\]

(v) We call the dual $\mathcal{G}_v^{\Sigma} := (\mathcal{G}_v^{\text{std}})^\perp = \mathcal{G}_v^{\text{max}} \ominus \mathbb{C}(1, \ldots, 1)$ of the continuous vertex space the (unoriented uniform) sum or $\Sigma$-vertex space. Its associated projection is
\[
P_v = \mathbb{I} - \frac{1}{\text{deg } v} \mathbb{E}.
\]

(vi) The oriented sum vertex space is the dual of the oriented continuous vertex space, i.e., $\mathcal{G}_v^{\Sigma} := (\mathcal{G}_v^{\text{std}})^\perp$.

(vii) A more general case of continuous vertex spaces is given by vectors $p(v)$ such that $|p_e(v)| = 1$, we call such continuous vertex spaces \textit{magnetic}. An example is giving in the following way: Let $\alpha \in \mathbb{R}^E$ be a function associating to each edge $e$ the \textit{magnetic vector potential} $\alpha_e \in \mathbb{R}$ and set
\[
p_e(v) = e^{-i\hat{\alpha}_e(v)/2}
\]
where $\hat{\alpha}_e(v) := \pm \alpha_e$ if $v = \partial_{\pm} e$ as in Definition 2.3. We call the associated vertex space $\mathcal{G}_v^{\text{mag},\alpha}$ magnetic.

Remark 2.11.

(i) Obviously, for the standard vertex space $\mathcal{G}_v^{\text{std}} = \mathcal{G}_v^{\text{mag},0}$. Furthermore, the oriented standard vertex space $\mathcal{G}_v^{\text{std}}$ of (vii) is unitary equivalent to a special case of magnetic vertex spaces in (vi): Choose $\alpha_e = \pi$ for all $e \in E$ then $p_e(\partial_{\pm} e) = \mp 1$, i.e., $p(v) = -i\hat{\mathbb{I}}(v)$ and therefore $\mathcal{G}_v^{\text{std}} = i\mathcal{G}_v^{\text{mag},\pi}$. 


(ii) Note that any magnetic vertex space occurs in the above way: Let \( \mathcal{G} \) be a magnetic vertex space, then \( \hat{p}_e(v) = e^{-i\hat{A}_e(v)} \) for some \( \hat{A} = \{ \hat{A}(v) \} \) with \( \hat{A}(v) \in \mathbb{R}^{E_v} \). Let

\[
\alpha := d\hat{A} : E \longrightarrow \mathbb{R} \quad \text{i.e.,} \quad \alpha_e = \hat{A}_e(\partial_+ e) - \hat{A}_e(\partial_- e),
\]

(we define \( d = d^{\text{max}} \) in the next section). Let \( A_e(v) := \alpha_e(v)/2 \), then \( d\hat{A} = d\hat{A} \), i.e., \( A - \hat{A} \in \ker d \). But the kernel of \( d \) consists of the values \( B \) such that \( B_e(\partial_+ e) = B_e(\partial_- e) =: \beta_e \) for all \( e \in E \) where \( \beta \in \mathbb{R}^E \), in particular, \( A_e(v) = \hat{A}_e(v) + \beta_e \).

Define a unitary map \( F \mapsto \hat{F} \), \( \hat{F}_e(v) := e^{i\beta_e}F_e(v) \) then \( \hat{F} \in \mathcal{G} \) iff \( F \in \mathcal{G} \) where \( \mathcal{G} = \mathcal{G}^{\text{mag},\alpha} \) as defined below. In particular, \( \mathcal{G} \) is unitarily equivalent to \( \mathcal{G}^{\text{mag},\alpha} \) for some vector potential \( \alpha \in \mathbb{R}^E \).

We want to express continuous vertex spaces with respect to the standard space \( \ell_2(V) \), the “classical” space of 0-forms \( \tilde{F} : V \longrightarrow \mathbb{C} \) with norm defined by

\[
\|\tilde{F}\|_{\ell_2(V)}^2 := \sum_{v \in V} |\tilde{F}(v)|^2 \deg v.
\]

In particular, the next lemma shows, that the vertex-weight \( \deg v \) is canonical in the sense of (iii):

**Lemma 2.12.** Let \( \mathcal{G} \) be a continuous vertex space with projection \( P \) and denote by \( [p^{-1}] \) the operator

\[
[p^{-1}] : \mathcal{G}^{\text{max}} \longrightarrow \mathcal{G}^{\text{max}}, \quad F \mapsto \tilde{F} = \{ \tilde{F}(v) \}_v, \quad \tilde{F}_e(v) = \frac{F_e(v)}{p_e(v)}.
\]

(i) The multiplication operators \( [p^{-1}] \) and \( [p] = [p^{-1}]^{-1} \) are bounded on \( \mathcal{G}^{\text{max}} \)

(ii) We have \( [p^{-1}](\mathcal{G}) = \mathcal{G}^{\text{std}} \) and \( [p^{-1}](\mathcal{G}^\perp) = \mathcal{G}^{\Sigma|p|^2} \) where

\[
\mathcal{G}^{\Sigma|p|^2} := \left\{ \tilde{F} \in \mathcal{G}^{\text{max}} \mid \sum_{e \in E_v} |p_e(v)|^2 \tilde{F}_e(v) = 0 \quad \forall v \in V \right\}
\]

for the dual.

(iii) Denote \( \tilde{U} : \mathcal{G}^{\text{std}} \longrightarrow \ell_2(V) \) the local operator mapping \( \tilde{F}(v) = \tilde{F}(v)(1, \ldots, 1) \) onto \( \tilde{F}(v) \in \mathbb{C} \), then \( \tilde{U} \) is unitary. Furthermore,

\[
U : \mathcal{G} \longrightarrow \ell_2(V), \quad U := \tilde{U} \circ [p^{-1}]
\]

is unitary.

(iv) The transformed projection \( \tilde{P} := UP : \mathcal{G}^{\text{max}} \longrightarrow \ell_2(V) \) is given by

\[
(\tilde{P}_vF)(v) = \frac{1}{\deg v} \sum_{e \in E_v} p_e(v)F_e(v) \in \mathbb{C}
\]

and no coefficient \( p_e(v) \) vanishes.

**Proof.** (i) The boundedness follows from the global bounds \( p_{\pm} \) on \( |p_e(v)| \) (cf. Definition 2.9). (ii) \([p^{-1}] \) restricted to \( \mathcal{G}^{\text{max}} \) maps the vector \( \hat{p}(v) \) onto \( (1, \ldots, 1) \), i.e., \( \mathcal{G} \) onto \( \mathcal{G}^{\text{std}} \); a vector \( \hat{F}(v) \in \mathcal{G}^\perp \) satisfies \( \sum_{e \in E_v} \overline{p_e(v)}F_e(v) = 0 \), and therefore \( \tilde{F}(v) \in \mathcal{G}^{\Sigma|p|^2} \). (iii) We have

\[
|\tilde{F}(v)|_{E_v}^2 = |\tilde{F}(v)(1, \ldots, 1)|_{E_v}^2 = |\tilde{F}(v)|^2 \deg v.
\]
and therefore, $\widetilde{U}$ is unitary. Furthermore,
\[
\|\widetilde{F}\|_{\ell_2(V)}^2 = \sum_{v \in V} |\widetilde{F}(v)|^2 \deg v = \sum_{v \in V} \sum_{e \in E_v} |\widetilde{F}(v)p_e(v)|^2 = \sum_{v \in V} \sum_{e \in E_v} |F_e(v)|^2 = \|F\|_{\ell_2}^2
\]
since $|p_e(v)|^2 = \deg v$. The last assertion follows by a straightforward calculation. \qed

Note that the decomposition into $\mathcal{G}^{\text{std}}$ and $\mathcal{G}^{\Sigma}$ is no longer orthogonal if $p^{-1}$ is not unitary (i.e., $|p_e(v)| \neq 1$ for some $e \in E_v$).

The trivial, the uniform continuous and the sum vertex spaces are obviously invariant under permutation of the edges in $E_v$. Indeed, these are the only possibilities for such an invariance:

**Lemma 2.13.** A vertex space $\mathcal{G}_v$ is invariant under permutation of the coordinates $e \in E_v$ iff $\mathcal{G}_v$ is either maximal ($\mathcal{G}_v^{\text{max}} = \mathbb{C}^{E_v}$), minimal ($\mathcal{G}_v^{\text{min}} = 0$), uniform continuous ($\mathcal{G}_v^{\text{std}} = \mathbb{C}(1, \ldots, 1)$) or the sum vertex space ($\mathcal{G}_v^{\Sigma} = \mathbb{C}^{E_v} \oplus \mathbb{C}(1, \ldots, 1)$).

**Proof.** It can be shown, that a square matrix $P$ of dimension $d = \deg v$ is invariant under the symmetric group $S_d$ of order $d$ iff $P$ has the form
\[
P = a \mathbb{I} + b \mathbb{E},
\]
since the only subspaces invariant under $S_d$ are $\mathbb{C}(1, \ldots, 1)$ and its orthogonal complement, and the representation of $S_d$ on the orthogonal complement is irreducible (see e.g. the references in [Ku04]). Using the relations $P = P^*$ and $P^2 = P$ for an orthogonal projection, we obtain that $a$ and $b$ must be real and satisfy the relations $a^2 = a$ and $2ab + (\deg v)b = b$, from which the four cases follow. \qed

### 3. Operators on vertex spaces

In this section, we define a generalised coboundary operator or exterior derivative associated to a vertex space. We use this exterior derivative for the definition of an associated Dirac and Laplace operator in the supersymmetric setting of Section 1.2.

**3.1. Discrete exterior derivatives.** On the maximal vertex space $\mathcal{G}^{\text{max}}$, we define a general coboundary operator or *exterior derivative* as
\[
d = d^{\text{max}} : \mathcal{G}^{\text{max}} \longrightarrow \ell_2(E), \quad (dF)_e := F_e(\partial_+ e) - F_e(\partial_- e),
\]

**Definition 3.1.** Let $\mathcal{G}$ be a vertex space of the graph $X$. The exterior derivative on $\mathcal{G}$ is defined as
\[
d_\mathcal{G} := d^{\text{max}}|_{\mathcal{G}} : \mathcal{G} \longrightarrow \ell_2(E), \quad (dF)_e := F_e(\partial_+ e) - F_e(\partial_- e),
\]
mapping 0-forms onto 1-forms.

We often drop the subscript $\mathcal{G}$ for the vertex space, or use other intuitive notation in order to indicate the vertex space.

We define a multiplication operator $[\ell^{-1}]$ on $\mathcal{G}^{\text{max}}$ and $\ell_2(E)$ by
\[
([\ell^{-1}]F)_e(v) = \frac{1}{\ell_e} F_e(v) \quad \text{and} \quad ([\ell^{-1}]\eta)_e = \frac{1}{\ell_e} \eta_e,
\]
respectively. Clearly, $[\ell^{-1}]$ is bounded on both spaces iff there exists $\ell_0 > 0$ such that
\[
\ell_e \geq \ell_0, \quad e \in E.
\] (3.1)
On a vertex space $\mathcal{G} \leq \mathcal{G}^\text{max}$ with associated projection $P$, we can relax the condition slightly, namely, we assume that $P[\ell^{-1}]$ is bounded, i.e., that

$$\kappa := \sup_{v \in V} |P_v[\ell^{-1}]|_v < \infty \quad (3.2)$$

where $| \cdot |_v$ denotes the operator norm for matrices on $\mathbb{C}^{E_v}$.

**Remark 3.2.**

(i) If (3.1) is fulfilled, then $\kappa \leq 1/\ell_0$. In particular, if $\ell_e = \ell_0$ for all $e \in E$ then $\kappa = 1/\ell_0$.

(ii) For the (uniform) continuous vertex space $\mathcal{G}^\text{std}$, we have

$$|P_v[\ell^{-1}]|_v = \frac{1}{\deg v} \sum_{e \in E_v} \frac{1}{\ell_e}.$$

(iii) If we assume that (3.2) holds for $P$ and $P^\perp$, then (3.1) is also fulfilled. For simplicity, we assume therefore that (3.1) holds (if not stated otherwise).

**Lemma 3.3.** Assume (3.2), then $d$ is norm-bounded by $\sqrt{2\kappa}$. The adjoint

$$d^*: \ell_2(E) \longrightarrow \mathcal{G}$$

fulfills the same norm bound and is given by

$$(d^*\eta)(v) = P_v\left(\left\{\frac{1}{\ell_e} \hat{\eta}_e(v)\right\}\right) \in \mathcal{G}_v,$$

where $\hat{\eta}_e(v) := \pm \eta_e$ if $v = \partial_e^\pm$ denotes the oriented evaluation of $\eta_e$ at the vertex $v$.

**Proof.** We have

$$\|dF\|_{\ell_2(E)}^2 = \sum_{e \in E} \frac{1}{\ell_e} \left|F_e(\partial_+ e) - F_e(\partial_- e)\right|^2$$

$$\leq 2 \sum_{v \in V} \left(\sum_{e \in E_v^+} \frac{1}{\ell_e} \left|F_e(v)\right|^2 + \sum_{e \in E_v^-} \frac{1}{\ell_e} \left|F_e(v)\right|^2\right)$$

$$\leq 2 \sum_{v \in V} \sum_{e \in E_v} \frac{1}{\ell_e} \left|F_e(v)\right|^2$$

$$\leq 2 \sum_{v \in V} \langle [\ell^{-1}]_v F(v), F(v) \rangle$$

$$= 2 \sum_{v \in V} \langle [\ell^{-1}]_v F(v), P_v F(v) \rangle$$

$$\leq 2\kappa \|F\|_{\mathcal{G}}^2$$

using Eq. (2.1) and the fact that $F(v) \in \mathcal{G}_v$. For the second assertion, we calculate

$$\langle dF, \eta \rangle = \sum_{e \in E} \frac{1}{\ell_e} \left(\overline{F_e(\partial_+ e)} - \overline{F_e(\partial_- e)}\right) \eta_e$$

$$= \sum_{v \in V} \left(\sum_{e \in E_v^+} \frac{1}{\ell_e} \overline{F_e(v)} \eta_e - \sum_{e \in E_v^-} \frac{1}{\ell_e} \overline{F_e(v)} \eta_e\right)$$

$$= \sum_{v \in V} \langle P_v F, \left\{\frac{1}{\ell_e} \hat{\eta}_e(v)\right\}_{e \in E_v} \rangle_{\mathcal{G}_v^{\text{max}}} = \langle F, d^*\eta \rangle$$

since $F(v) \in \mathcal{G}_v$, i.e., $P_v F(v) = F(v)$. \qed
Example 3.4.

(i) For the minimal vertex space, we have \( d = 0 \) and \( d^* = 0 \). Obviously, these operators are decoupled, i.e., they do not feel any connection information of the graph.

(ii) For the maximal vertex space, we have (denoting \( d = d^\text{max} \))

\[
(d^* \eta)_e(v) = \frac{1}{\ell_e} \hat{\eta}_e(v).
\]

The operator \( d = d^\text{max} \) decomposes as \( \bigoplus_e d_e \) with respect to the decomposition of \( G^\text{max} \) in Eq. (2.3) and \( \ell_2(E) \) in Eq. (2.4). Here,

\[
(d_e : \mathbb{C}^2 \longrightarrow \mathbb{C}) \cong (1 \quad -1) \quad \text{and} \quad (d^*_e : \mathbb{C} \longrightarrow \mathbb{C}^2) \cong \frac{1}{\ell_e} \left( \begin{array}{c} 1 \\ -1 \end{array} \right)
\]

where \( F_e = (F_e(\partial_+, e), F_e(\partial_-)) \in \mathbb{C}^2 \). Again, the operators are decoupled, since any connection information of the graph is lost.

Remark 3.5. We can always embed the edge space \( \ell_2(E) \) into \( G^\text{max} \) using the operator \( \iota : \ell_2(E) \longrightarrow G^\text{max} \), \( (\iota \eta)_e(v) := \frac{1}{\sqrt{2\ell_e}} \eta_e \).

Indeed, \( \iota \) is an isometry since

\[
\|\iota \eta\|_{G^\text{max}}^2 = \frac{1}{2} \sum_{v \in V} \sum_{e \in E_v} \frac{1}{\ell_e} |\eta_e|^2 = \sum_{e \in E} \frac{1}{\ell_e} |\eta_e|^2 = \|\eta\|_{\ell_2(E)}^2
\]

using Eq. (2.1). Furthermore, the range of \( \iota \) in \( G^\text{max} \) is precisely the kernel of \( d^\text{max} \), i.e.,

\[
\iota(\ell_2(E)) = \ker d^\text{max}
\]

as it can be checked easily. Moreover, we can write the adjoint of the exterior derivative \( d = d_g \) on \( G \) with projection \( P \) as

\[
d^* = P(d^\text{max})^* = \sqrt{2} P\hat{\Pi}_1 \iota^{-1/2}.
\]

We can now calculate the exterior derivative and its adjoint in several general cases. The proofs are straightforward. We start with the relation to the dual vertex space:

Lemma 3.6. Let \( G \) be a vertex space with exterior derivative \( d = d_g \), then

\[
d_g \bigoplus d_{g\perp} = d^\text{max} : G^\text{max} \longrightarrow \ell_2(E), \quad F \bigoplus F^\perp \mapsto d_g F + d_{g\perp} F^\perp
\]

\[
d^*_g \bigoplus d^*_{g\perp} = (d^\text{max})^* : \ell_2(E) \longrightarrow G^\text{max}, \quad \eta \mapsto d^*_g \eta \bigoplus d^*_{g\perp} \eta.
\]

In particular,

\[
((d^*_{g\perp}) \eta)_e(v) = \frac{1}{\ell_e} \hat{\eta}_e(v) - (d^*_g \eta)_e(v).
\]

For a continuous vertex space, it is convenient to use the unitary transformation from \( G \) onto \( \ell_2(V) \) (see Lemma 2.12 (iii)):

Lemma 3.7. For a continuous vertex space, the exterior derivative \( \tilde{d} := d \circ U^{-1} \) transformed back to \( \ell_2(V) \) is given as

\[
(\tilde{d} F)_e = p_e(\partial_+ e) \tilde{F}(\partial_+ e) - p_e(\partial_- e) \tilde{F}(\partial_- e)
\]
and its adjoint $\tilde{d}^* = U \circ d^*$ by

$$ (\tilde{d}^* \eta)(v) = \frac{1}{\deg v} \sum_{e \in E_v} \frac{p_e(v)}{\ell_e} \eta_e(v). $$

Switching the orientation on or off leads to another class of examples:

**Lemma 3.8.** If $\mathcal{G}$ is a vertex space with projection $\hat{P}$ and if we define the “unoriented” exterior derivative $\hat{d}$ via

$$ \hat{d} : \mathcal{G} \rightarrow \ell_2(E), \quad (\hat{d} F)_e := F_e(\partial_+ e) + F_e(\partial_- e), $$

then its adjoint is given by

$$ (\hat{d}^* \eta)(v) = \hat{P}_v \left( \left\{ \frac{1}{\ell_e} \eta_e(v) \right\} \right). $$

In addition, if $\mathcal{G} = \tau \mathcal{G}$ is the vertex space with switched orientation, then $d = \hat{d} \circ \tau$ and $d^* = \tau \circ \hat{d}^*$, i.e., the above “unoriented” exterior derivative $\hat{d}$ occurs as an exterior derivative in the sense of Definition 3.1 for the vertex space $\tau \mathcal{G}$ with switched orientation.

We give now some examples of exterior derivatives on continuous vertex spaces and their duals:

**Example 3.9.**

(iii) For the standard vertex space $\mathcal{G}^{std}$, the exterior derivative and its adjoint are unitarily equivalent to

$$ \tilde{d} : \ell_2(V) \rightarrow \ell_2(E), \quad (\tilde{d} F)_e = F(\partial_+ e) - F(\partial_- e) $$

and

$$ (\tilde{d}^* \eta)(v) = \frac{1}{\deg v} \sum_{e \in E_v} \frac{1}{\ell_e} \eta_e(v), $$

i.e., $\tilde{d}$ is the classical coboundary operator and $\tilde{d}^*$ its adjoint.

(iv) If $\mathcal{G}^{\hat{std}} = \tau \mathcal{G}^{std}$ is the oriented standard vertex space, then the exterior derivative $\tilde{d}$ is unitarily equivalent to

$$ \tilde{d} : \ell_2(V) \rightarrow \ell_2(E), \quad (\tilde{d} F)_e = F(\partial_+ e) + F(\partial_- e) $$

and

$$ (\tilde{d}^* \eta)(v) = \frac{1}{\deg v} \sum_{e \in E_v} \frac{1}{\ell_e} \eta_e(v). $$

(v) For the (unoriented) sum vertex space $\mathcal{G}^{\Sigma} = (\mathcal{G}^{std})^\perp$, we have

$$ (d^* \eta)_e(v) = \frac{1}{\ell_e} \eta_e(v) - \frac{1}{\deg v} \sum_{e' \in E_v} \frac{1}{\ell_{e'}} \eta_{e'}(v) $$

(vi) For the (oriented) sum vertex space $\mathcal{G}^\Sigma = (\mathcal{G}^{\hat{std}})^\perp$, we have

$$ (d^* \eta)_e(v) = \pm \left( \frac{1}{\ell_e} \eta_e(v) - \frac{1}{\deg v} \sum_{e' \in E_v} \frac{1}{\ell_{e'}} \eta_{e'} \right) $$

if $v = \partial_{\pm} e$. 
(vii) For the magnetic vertex space $\mathcal{G}^{\text{mag}, \alpha}$, we have
\[
\tilde{d}: \ell_2(V) \rightarrow \ell_2(E), \quad (\tilde{d}F)_e = e^{-\alpha e/2}F(\partial_+ e) - e^{\alpha e/2}F(\partial_- e)
\]
and
\[
\tilde{d}^*: \ell_2(E) \rightarrow \ell_2(V), \quad (\tilde{d}^* \eta)(v) = \frac{1}{\deg v} \sum_{e \in E_v} \frac{1}{e} e^{\alpha e/2} \tilde{\eta}_e(v).
\]

3.2. Discrete Dirac operators and Laplacians. Let $D = D_\mathcal{G}$ be the Dirac operator associated to the exterior derivative $d = d_\mathcal{G}$ on the vertex space $\mathcal{G}$, i.e.,
\[
D = \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix}
\]
with respect to $\ell_2(AX) := \ell_2(A^0 X) \oplus \ell_2(A^1 X) = \mathcal{G} \oplus \ell_2(E)$ (cf. Definition 1.11).

**Definition 3.10.** We define as in the abstract supersymmetric setting the Laplacians associated to a vertex space $\mathcal{G}$ as
\[
\Delta_{AX} := \Delta_\mathcal{G} := D^2_\mathcal{G}, \quad \Delta_{A^0 X} := \Delta^0_\mathcal{G} := d^* d_\mathcal{G} \quad \text{and} \quad \Delta_{A^1 X} := \Delta^1_\mathcal{G} := d_\mathcal{G} d^*.
\]

In particular, we have
\[
(\Delta^0_\mathcal{G} F)(v) = P_v \left( \left\{ \frac{1}{\ell_e} \left( F_e(v) - F_e(v_e) \right) \right\} \right) \quad (3.3a)
\]
\[
(\Delta^1_\mathcal{G} \eta)_e = \left( P_{\partial_+ e} \left( \left\{ \frac{1}{\ell_e} \tilde{\eta}_e(\partial_+ e) \right\} \right) - P_{\partial_- e} \left( \left\{ \frac{1}{\ell_e} \tilde{\eta}_e(\partial_- e) \right\} \right) \right)_e \quad (3.3b)
\]
where $v_e$ denotes the opposite vertex of $v \in E_v$ on $e$. Here, we see that the orientation plays no role for the 0-form Laplacian.

We have a sort of Hodge decomposition (see Lemma 1.3):

**Lemma 3.11.** Assume that $D$ has a spectral gap at 0, i.e., that $\text{dist}(0, \sigma(D)) \setminus \{0\} > 0$ (e.g., $X$ finite is sufficient). Then
\[
\ell_2(AX) = \ker D \oplus \text{ran} d^* \oplus \text{ran} d, \quad \text{i.e.,}
\]
\[
\ell_2(A^0 X) = \mathcal{G} = \ker d \oplus \text{ran} d^* \quad \text{and} \quad \ell_2(A^1 X) = \ell_2(E) = \ker d^* \oplus \text{ran} d.
\]

Let us start with the Laplacians acting on the trivial vertex spaces:

**Example 3.12.**
(i) For the minimal vertex space, we have $\Delta^p_{\mathcal{G}^0} = 0$ for $p \in \{0, 1\}$.

(ii) For the maximal vertex space, we have $\Delta^p_{\mathcal{G}^\text{max}} := \Delta^p_{\mathcal{G}^\text{max}}$
\[
(\Delta^0_{\mathcal{G}^\text{max}} F)_e(v) = \left\{ \frac{1}{\ell_e} \left( F_e(v) - F_e(v_e) \right) \right\}_{e \in E_v}.
\]
The operator $\Delta^0_{\mathcal{G}^\text{max}}$ decomposes as $\bigoplus_e (\Delta^0_{\mathcal{G}^\text{max}})_e$ with respect to the decomposition of $\mathcal{G}^{\text{max}}$ in Eq. (2.3), where
\[
(\Delta^0_{\mathcal{G}^\text{max}})_e : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \cong \frac{1}{\ell_e} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
\]
Similarly,
\[
(\Delta^1_{\mathcal{G}^\text{max}} \eta)_e = \frac{2}{\ell_e} \eta_e,
\]
i.e., $\Delta^1_{\mathcal{G}^\text{max}} = 2[\ell^{-1}]$ is a multiplication operator on $\ell_2(E)$.

**Lemma 3.13.**
The Laplacian \( \Delta_g^p \) on p-forms associated to the vertex space \( \mathcal{G} \) is a bounded operator with norm bounded by \( 2\kappa \).

(ii) On 1-forms, we have \( \Delta_{\mathcal{G}}^1 = \Delta_g^0 + \Delta_g^1 \) or \( \Delta_{\mathcal{G}}^1 = 2[\ell^{-1}] - \Delta_g^0 \) on \( \ell_2(E) \). In particular, if all length \( \ell_e = 1 \), then
\[
\Delta_{\mathcal{G}}^1 = 2 - \Delta_g^0 \quad \text{and} \quad \sigma(\Delta_{\mathcal{G}}^1) = 2 - \sigma(\Delta_g^0),
\]
i.e., \( \lambda \in \sigma(\Delta_{\mathcal{G}}^1) \) iff \( 2 - \lambda \in \sigma(\Delta_g^0) \).

(iii) Assume that \( \ell_e = 1 \) then we have the spectral relation
\[
\sigma(\Delta_{\mathcal{G}}^0) \setminus \{0, 2\} = 2 - (\sigma(\Delta_g^0) \setminus \{0, 2\})
\]
on 0-forms, i.e., if \( \lambda \neq 0, 2 \), then \( \lambda \in \sigma(\Delta_{\mathcal{G}}^0) \) iff \( 2 - \lambda \in \sigma(\Delta_g^0) \).

Proof. The first assertion follows immediately from Lemma 3.3. The second is a consequence of Lemma 3.6. The last spectral equality follows from the spectral equality for 1-forms and supersymmetry to pass from 1-forms to 0-forms (cf. Lemma 1.2). \( \square \)

In Lemma 4.4 we will prove a relation between the kernels, namely \( \ker \Delta_{\mathcal{G}}^0 \cong \ker \Delta_g^1 \).

Lemma 3.14. Let \( \mathcal{G} \) be a continuous vertex space, \( \widetilde{d} : \ell_2(V) \rightarrow \ell_2(E) \) the unitarily equivalent exterior derivative as defined in Lemma 3.7 and \( \tilde{d}^* \) its adjoint, then \( \widetilde{\Delta}_g^0 := \tilde{d}^* \tilde{d} \) and \( \Delta_g^1 \) are given by
\[
(\widetilde{\Delta}_g^0 F)(v) = \frac{1}{\deg v} \sum_{e \in E_v} \frac{p_e(v)}{\ell_e} (p_e(v) F(v) - p_e(v) F(v))
\]
\[
(\Delta_g^1 \eta)(e) = -\sum_{e' \sim e} (\lambda_{e'}(e)) \tilde{\eta}_{e'}(e) + \left( \frac{1}{\deg \partial_+ e'} \frac{|p_e(\partial_+ e)|^2}{\deg \partial_- e'} + \frac{|p_e(\partial_- e)|^2}{\deg \partial_+ e'} \right) \frac{1}{\ell_e} \eta_e,
\]
where \( e' \sim e \) means that \( e' \neq e \) and \( e' \), \( e \) have the vertex \( e' \cap e \) in common. Furthermore, \( \tilde{\eta}_{e'}(e) = \eta_{e'} \) if the orientation of \( e \), \( e' \) gives an orientation of the path formed by \( e \), \( e' \), and \( \tilde{\eta}_{e'}(e) = -\eta_{e'} \) otherwise.

We have several important special cases of continuous vertex spaces and their duals:

Example 3.15.

(iii) For the standard vertex space \( \mathcal{G}^{\text{std}} \), we have the standard (weighted) Laplacian \( \Delta_{\text{std}}^0 \) transformed to \( \widetilde{\Delta}_{\text{std}}^0 = \Delta_X^0 = \Delta_{(X,\ell^{-1})}^0 \) on \( \ell_2(V) \) and \( \Delta_{\text{std}}^1 \) on \( \ell_2(E) \), where
\[
(\widetilde{\Delta}_{\text{std}}^0 F)(v) = \frac{1}{\deg v} \sum_{e \in E_v} \frac{1}{\ell_e} (F(v) - F(v))
\]
\[
(\Delta_{\text{std}}^1 \eta)(e) = -\sum_{e' \sim e} \frac{1}{\ell_{e'}} \frac{1}{\deg (e' \cap e)} \tilde{\eta}_{e'}(e) + \left( \frac{1}{\deg \partial_+ e'} + \frac{1}{\deg \partial_- e'} \right) \frac{1}{\ell_e} \eta_e.
\]

(iv) For the oriented standard space \( \mathcal{G}^{\text{std}} \), we have
\[
(\widetilde{\Delta}_{\text{std}}^0 F)(v) = \sum_{e \in E_v} \frac{1}{\ell_e} (F(v) + F(v))
\]
\[
(\Delta_{\text{std}}^1 \eta)(e) = \sum_{e' \sim e} \frac{1}{\ell_{e'}} \frac{1}{\deg (e' \cap e)} \eta_{e'} + \left( \frac{1}{\deg \partial_+ e'} + \frac{1}{\deg \partial_- e'} \right) \frac{1}{\ell_e} \eta_e.
Note that
\[ \widetilde{\Delta}^0_{\text{std}} = 2[L^\Sigma] - \Delta^0_{\text{std}}, \]
where \( \Delta^0_{\text{std}} \) is the standard Laplacian of Example 3.13 and \([L^\Sigma]\) is the multiplication operator with
\[ L^\Sigma(v) := \frac{1}{\deg v} \sum_{e \in E_v} \frac{1}{\ell_e}. \]

(v) For the (unoriented) sum vertex space \( G^\Sigma \), the dual of \( G^{\text{std}} \), we have
\[ (\Delta^0_{\Sigma} F)_e(v) = \frac{1}{\ell_e} (F_e(v) - F_e(v_e)) - \frac{1}{\deg v} \sum_{e' \in \tilde{E}_v} \frac{1}{\ell_{e'}} (F_{e'}(v) - F_{e'}(v_{e'})) \]
\[ (\Delta^1_{\Sigma} \eta)_e = \sum_{e' \sim e} \frac{1}{\ell_{e'}} \frac{1}{\deg(e' \cap e)} \widehat{\eta}_{e'}(e) - \left( \frac{1}{\deg \partial_+ e} + \frac{1}{\deg \partial_- e} - 2 \right) \frac{1}{\ell_e} \eta_e. \]

(vi) For the oriented sum vertex space \( \hat{G}^\Sigma \), we have
\[ (\Delta^0_{\Sigma} F)_e(v) = \frac{1}{\ell_e} (F_e(v) - F_e(v_e)) - \frac{1}{\deg v} \sum_{e' \in \tilde{E}_v} \frac{1}{\ell_{e'}} (F_{e'}(v) - F_{e'}(v_{e'})) \]
\[ (\Delta^1_{\Sigma} \eta)_e = -\sum_{e' \sim e} \frac{1}{\ell_{e'}} \frac{1}{\deg(e' \cap e)} \widehat{\eta}_{e'}(e) - \left( \frac{1}{\deg \partial_+ e} + \frac{1}{\deg \partial_- e} - 2 \right) \frac{1}{\ell_e} \eta_e. \]

(vii) For the magnetic vertex space \( G^{\text{mag}, \alpha} \), we have
\[ (\widetilde{\Delta}^0_{\text{mag}, \alpha} F)(v) = \frac{1}{\deg v} \sum_{e \in E_v} \frac{1}{\ell_e} (F(v) - e^{-\hat{\alpha}_e(v)} F(v_e)) \]
\[ (\Delta^1_{\text{mag}, \alpha} \eta)_e = -\sum_{e' \sim e} \frac{e^{\hat{\alpha}_{e',e}}}{\ell_{e'}} \frac{1}{\deg(e' \cap e)} \widehat{\eta}_{e'}(e) + \left( \frac{1}{\deg \partial_+ e} + \frac{1}{\deg \partial_- e} \right) \frac{1}{\ell_e} \eta_e, \]
where \( \hat{\alpha}_{e',e} := (\hat{\alpha}_{e'} - \hat{\alpha}_e)(e \cap e') \) denotes the oriented flux along \( e' \) and \( e \).

Remark 3.16. The 1-form Laplacian of Lemma 3.14 and especially of Example 3.15 (iv) above can be viewed as an operator on the line graph. In order to define the line graph, we assume for simplicity, that \( X \) has no self-loops and multiple edges, and that no edge is isolated (i.e., \( \deg \partial_+ e \) and \( \deg \partial_- e \) are not both equal to 1). Let \( L(X) \) be the line graph associated to the graph \( X \), i.e., \( V(L(X)) = E(X) \) and two “vertices” in the line graph (i.e., edges in the original graph) \( e, e' \) are adjacent iff \( e \neq e' \) and \( e \cap e' \neq \emptyset \), i.e., if they meet in a common vertex. We have
\[ \deg_{L(X)} e = \deg_X \partial_+ e + \deg_X \partial_- e - 2, \]
and in particular, if \( X \) is a \( d \)-regular graph, then \( L(X) \) is \((2d - 2)\)-regular.

The above example of the 1-form Laplacian is a line graph Laplacian (up to a multiplication operator with the complex edge “weight”
\[ \rho_{e,e'} = \frac{(\nabla_{e'} \rho_e)(e \cap e') \hat{\eta}_{e'}(e) \deg e}{\ell_{e'} \deg(e \cap e')}. \]
We will now show how \( \Delta_{L(X)} \) becomes a Laplacian with positive weights.
If $\ell_e = 1$ for all edges, then the 1-form Laplacian is related to the 0-form Laplacian $\Delta^0_{(L(X),\rho)}$ on the line graph with edge weights

$$\rho_{e,e'} = \frac{\deg_{L(X)} e}{\deg_X (e \cap e')} = \frac{\deg_X \partial_+ e + \deg_X \partial_- e - 2}{\deg_X (e \cap e')}$$

via

$$\Delta^1_{\text{std}} = [L] - \Delta^0_{(L(X),\rho)}$$

where $[L]$ is the multiplication operator on $\ell^2(V(L(X)))$ with the function

$$L(e) = \left(\frac{1}{\deg \partial_+ e} + \frac{1}{\deg \partial_- e}\right) + \sum_{e' \sim e} \frac{1}{\deg_X (e \cap e')}.$$ 

In particular, if $X$ is $d$-regular, then $L = 2$. Moreover, $\rho_{e,e'} = (2d - 2)/d$ and

$$\Delta^1_{\text{std}} = 2 - \frac{(2d - 2)}{d} \Delta^0_{L(X)}, \quad (3.6)$$

where now, $\Delta^0_{L(X)} = \Delta^0_{(L(X),1)}$ is the line graph Laplacian with edge weights set to 1.

In addition, we can recover a result of [Shi00, Ogu02] namely a spectral relation for the line graph Laplacian and the Laplacian on the graph itself,

$$\sigma(\Delta^0_{L(X)}) \setminus \{\frac{d}{d-1}\} = \frac{d}{2(d-1)} (\sigma(\Delta^0_X) \setminus \{2\}),$$

using supersymmetry, (3.5) and (3.6). In particular, the spectrum of the line graph is always contained in the interval $[0, d/(d-1)]$ and is therefore not bipartite (if $d \geq 3$).

**Remark 3.17.** There is another interesting example which relates a Dirac operator on $X$ to the (standard) Laplacian on the subdivison graph $S(X)$ defined as follows (cf. [Shi00, Ogu02]). Again, we assume for simplicity, that $X$ has no self-loops and no double edges and that $\ell_e = 1$. As vertices we set $V(S(X)) = V(X) \cup E(X)$, and the edges are given by $\{v,e\}$ if $v \in \partial e$ in the original graph (we do not care about the orientation here). In other words, $S(X)$ is obtained from $X$ by introducing a new vertex on each edge. The subdivision graph $S(X)$ is always bipartite (choose the above decomposition). If $X$ is $d$-regular, then $S(X)$ is $(d,2)$-semiregular, i.e., $\deg v = d$ for vertices $v \in V(X) \subseteq V(S(X))$ and $\deg e = 2$ for vertices $e \in E(X) \subseteq V(S(X))$ with respect to the bipartite decomposition.

The standard Laplacian on $S(X)$ is given as

$$(\Delta^0_{S(X)}) H(e) = -\frac{1}{2} (H(\partial_+ e) + H(\partial_- e)) + H(e)$$

$$(\Delta^0_{S(X)}) H(v) = -\frac{1}{d} \sum_{e \in E_v} H(e) + H(v)$$

for $H \in \ell^2(S(X))$. In particular,

$$\Delta^0_{S(X)} \cong \begin{pmatrix} 1 & -\vec{d}^*_{\text{std}} \\ \vec{d}^*_{\text{std}} & 1 \end{pmatrix} = 1 - \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \vec{D}_{\text{std}}$$

---

Shirai and Ogurisu actually showed more: If $X$ is infinite and $d \geq 3$, then $d/(d-1)$ is contained in the spectrum of the line graph, being an eigenvalue of infinite multiplicity. A corresponding eigenfunction lies in $\ker \Delta^1_{\text{std}} = \ker d^*_{\text{std}}$ by (3.6). For an infinite regular graph, one can see that this space is infinite-dimensional (see also Example 3.5 (iv)).
where \( \tilde{d}_{\text{std}} \) is the (transformed) exterior derivative and \( \tilde{D}_{\text{std}} \) the Dirac operator associated to the oriented standard space (cf. Example 3.9 (iv)). Furthermore,

\[
(\Delta^0_{S(X)} - 1)^2 \approx \frac{1}{2} \begin{pmatrix}
\tilde{\Delta}^0_{g_{\text{std}}} & 0 \\
0 & \Delta^1_{g_{\text{std}}}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
\tilde{\Delta}^0_{g_{\text{std}}} & 0 \\
0 & \Delta^1_{g_{\text{std}}}
\end{pmatrix}
\]

and (for \( \lambda \neq 1 \)) we have \( \lambda \in \sigma(\Delta^0_{S(X)}) \) iff \( 2(\lambda - 1)^2 \in \sigma(\tilde{\Delta}^0_{g_{\text{std}}}) \) by supersymmetry.

But the latter operator equals \( 2 - \Delta^1_X \) by Eq. (3.5) since we assumed \( \ell_e = 1 \).

Therefore

\[
\sigma(\Delta^0_{S(X)}) \setminus \{1\} = \eta^{-1}(\sigma(\Delta^0_X) \setminus \{2\})
\]

where \( \eta(\lambda) = 2 - 2(\lambda - 1)^2 = 2\lambda(2 - \lambda) \). One can show that 1 is also an eigenvalue with infinite multiplicity of the subdivision graph Laplacian. In particular, we recover again a result of [Shi00].

### 4. Indices for Dirac operators on discrete graphs

We start this section with a short excursion into cohomology. Assume that \( d = d_G : \mathcal{G} \rightarrow \ell_2(E) \) is an exterior derivative for the vertex space \( \mathcal{G} \).

**Definition 4.1.** We define the \((\ell_2-)\)cohomology of the graph \( X \) associated to the vertex space \( G \) as

\[
H^0_{\mathcal{G}}(X, \mathbb{C}) := \ker d \quad \text{and} \quad H^1_{\mathcal{G}}(X, \mathbb{C}) := \ker d^* = \ell_2(E) \ominus \text{ran } d.
\]

We call

\[
b_p^\mathcal{G}(X) := \dim_{\mathbb{C}} H^p_{\mathcal{G}}(X, \mathbb{C}) \quad \text{and} \quad \chi_{\mathcal{G}}(X) := b_0^\mathcal{G}(X) - b_1^\mathcal{G}(X)
\]

the \( p \)-th Betti-number and Euler characteristic associated to the vertex space \( \mathcal{G} \), respectively.

From the definition, it follows that

\[
\chi_{\mathcal{G}}(X) = \text{ind } D_{\mathcal{G}}.
\]

In order to derive a sort of “Gauß-Bonnet”-theorem, we need the notion of curvature at a vertex for general vertex spaces:

**Definition 4.2.** We define the curvature of the vertex space \( \mathcal{G}_v \) at the vertex \( v \in V \) as

\[
\kappa_{\mathcal{G}}(v) := \dim \mathcal{G}_v - \frac{1}{2} \deg v.
\]

The reason for the name will become clear in Remark 4.7 (i). Note that there are other notions of curvature, especially for tessellations (see e.g. [BP01]).

In order to calculate the Betti-numbers for a vertex space \( \mathcal{G} \), we need some more notation. For simplicity, we assume that \( X \) is connected. Let \( X' \) be a spanning tree of \( X \), i.e., \( X' \) is simply connected and \( V(X') = V(X) \). For each \( e \in P(X) := E(X) \setminus E(X') \), there exists a unique cycle \( c_e \) (closed path without repetitions) in \( X \) containing \( e \).

**Definition 4.3.** A prime cycle is a cycle \( c_e \) for some \( e \in P(X) \) associated to a spanning tree \( X' \) of \( X \) as above. A cycle \( c \) is said to be even/odd if the number of edges in \( c \) is even/odd.

Before calculating the kernel of the Dirac operator in some examples, we establish a general result on the dual \( \mathcal{G}^\perp \) of a vertex space \( \mathcal{G} \). It shows that actually, \( \mathcal{G}^\perp \) and the oriented version of \( \mathcal{G} \) are related:
Lemma 4.4. Assume that the global length bound

\[ \ell_0 \leq \ell_e \leq \ell_+ \quad \text{for all } e \in E \tag{4.1} \]

holds for some constants \(0 < \ell_0 \leq \ell_+ < \infty\). Then

\[ H^0_{g\perp}(X, \mathbb{C}) = \ker d_{g\perp} \cong \ker d^*_{g\perp} = H^1_{g\perp}(X, \mathbb{C}) \]

are isomorphic. In particular, if \(X\) is finite, then

\[ b^0_{g\perp}(X) = b^1_{g\perp}(X), \quad b^0_{\perp}(X) = b^0_{g\perp}(X) \quad \text{and} \quad \chi_{g\perp}(X) = -\chi_{\perp}(X). \]

If in addition, \(D\) has a spectral gap at 0, then \(\ker d^*_{g\perp} \cong \ker d^\perp_{\perp}\).

Proof. We define \(\psi : \ell_2(E) \rightarrow \mathcal{G}^\perp\) via \((\psi \eta)(v) := P_v\psi\eta\), then

\[ \|\psi\eta\|^2_{\mathcal{G}^\perp} = \sum_{v \in V} \sum_{e \in E_v} \frac{1}{\ell_f^2} |\eta_e|^2 = 2 \sum_{e \in E} \frac{1}{\ell_f^2} |\eta_e|^2 \leq \frac{2}{\ell_0} \|\eta\|^2_{\ell_2(E)}, \]

and therefore \(\psi \eta \in \mathcal{G}^\perp\). Furthermore, \((\psi \ker d^*_{g\perp}) \subset \ker d_{g\perp}\) since \(d^*_{g\perp}\psi\eta = 0\) implies that \(P_v\psi\eta\{\chi\}\) is empty and therefore, \((\psi \eta)_e(v) = \frac{1}{\ell_f} \eta_e\). In particular, the latter expression is independent of \(\partial_e\), so that \(d_{g\perp} (\psi \eta) = 0\). The other inclusion can be shown similarly: Let \(F \in \ker d_{g\perp}\) and set \(\eta := \ell_2 F_v(\psi)\) independent of \(v = \partial_e\). Then \(\eta \in \ell_2(E)\) using the global upper bound \(\ell_e \leq \ell_+\). Furthermore, \((\psi \eta)(v) = P^\perp F(v) = F(v)\) and \(d^*_{g\perp} \psi\eta = \tilde{1} PF = 0\) since \(F \in \mathcal{G}^\perp\). The other assertions follow from the definitions and the fact that \(\perp\) and \(\tilde{\perp}\) are involutions. The isomorphism of the ranges follows from Lemma 3.11. \(\square\)

When writing the index of \(D\), we implicitly assume that the graph is finite, i.e., that \(|E| < \infty\). We calculate the cohomology for the list of our examples. For simplicity, we assume that \(X\) is finite and connected. In particular, the global length bound \((4.1)\), i.e., \(0 < \ell_0 \leq \ell_e \leq \ell_+ < \infty\) is fulfilled.

Example 4.5.

(i) For the minimal vertex space, we have \(\ker d_{g0} = 0\) and \(\ker d^*_{g0} = \ell_2(E)\). In particular, \(\text{ind } D_{g0} = -|E|\).

(ii) For the maximal vertex space, we see from Lemma 4.3 that \(\ker d_{g\max} \cong \ker d^*_{g\max} \cong \ker d_{g0} = 0\). In particular, \(\text{ind } D = |E|\).

(iii) For the standard vertex space \(\mathcal{G}^\text{std}\) we obtain the classical homology groups \(H^p(X, \mathbb{C})\). The 0-th Betti-number counts the number of components, i.e., \(b^0_{\text{std}}(X) = 1\), and the 1-st the number of prime cycles. It is a classical fact that \(b^1_{\text{std}}(X) = |P(E)| = |E| - |V| + 1\) and therefore

\[ \text{ind } D_{g\text{std}} = b^0_{\text{std}}(X) - b^1_{\text{std}}(X) = |V| - |E| = \chi_{\text{std}}(X) \]

(iv) For the oriented standard vertex space \(\mathcal{G}^{\text{std}}\), the 0-th Betti number counts the number of bipartite components of \(X\), i.e., if \(X\) is connected, then \(b^0_{\text{std}} = 1\) if \(X\) is bipartite and 0 otherwise. This can be seen using the characterisation that \(X\) is bipartite iff \(X\) contains no odd cycle. Note that \(\tilde{d} F_e = 0\) for each edge \(e\) in an odd cycle \(c\) implies that \(F\) vanishes on each vertex in \(c\).

The 1-st Betti number counts the number of prime cycles \(|P(E)|\), where one has to subtract 1 if there is an odd prime cycle. But the existence of an odd (prime) cycle is equivalent to the fact that \(X\) is not bipartite. In
particular, $b_{\text{std}}^1 = |E| - |V| + 1$ if $X$ is bipartite and $b_{\text{std}}^1 = |E| - |V|$ otherwise. Again, we have $\text{ind} D_{g_{\text{std}}} = |V| - |E|$.

(v) For the (unoriented) sum vertex space $G$ we can apply Lemma 4.4. In particular,

$$\ker d_{g^\Sigma} = \left\{ F \in G^{\max} \mid F_e(\partial_+ e) = F_e(\partial_- e) : F_e, \sum_{e \in E_v} F_e = 0 \right\}$$

is isomorphic to $\ker d_{g_{\text{std}}}$, i.e., $b_{\Sigma}^0 = b_{\text{std}}^1 = |E| - |V| + 1$ iff $X$ is bipartite and $b_{\Sigma}^1 = |E| - |V|$ otherwise. Furthermore,

$$\ker d_{g^\Sigma}^* = \left\{ \eta \in \ell_2(E) \mid \frac{1}{\ell_e} \hat{n}_e(v) \text{ is independent of } e \in E_v \text{ for } v \in V \right\}$$

is a sort of “oriented” continuity condition. In particular, $b_{\Sigma}^1 = b_{\text{std}}^0 = 1$ iff $X$ is bipartite and $b_{\Sigma}^0 = 0$ otherwise. Finally,

$$\text{ind} D_{g^\Sigma} = -\chi_{\text{std}}(X) = |E| - |V|.$$

(vi) The oriented sum vertex space $G_{\Sigma}$ is dual to the standard vertex space, i.e, $b_{\Sigma}^0 = |E| - |V| + 1$ and $b_{\Sigma}^1 = 1$ by Lemma 4.4. In particular,

$$\text{ind} D_{g^\Sigma} = -\chi_{\text{std}}(X) = |E| - |V|.$$

(vii) Assume that $X$ is finite. For the magnetic vertex space $G_{\text{mag},\alpha}$, we need to define the flux through a circuit $c$: If $c = \sum_{i=1}^n p_i e_i \in H_1(X, \mathbb{Z})$ is a cycle represented in the homology group, then we define the flux of $\alpha$ through $c$ as

$$c \cdot \alpha := \sum_{i=1}^n p_i \hat{\alpha}_e(e_i),$$

where $\hat{\alpha}_e(e_i)$ is defined in Lemma 3.14. Now, $b_{\text{mag},\alpha}^0 = 1$ iff $c \cdot \alpha \in 2\pi \mathbb{Z}$ for all cycles $c \in H_1(X, \mathbb{Z})$ and $b_{\text{mag},\alpha}^0 = 0$ otherwise. In order to calculate $b_{\text{mag},\alpha}^1$, we note that the linear system $(\tilde{d}_{\text{mag},\alpha}^*)(\eta) = 0$ for all $\eta \in V$ consists of $|E|$ variables $\eta_e$ and $|V|$ equations, therefore $b_{\text{mag},\alpha}^1 \geq |E| - |V|$. It remains to show that the rank of the coefficient matrix can increase by 1 iff $c \cdot \alpha \in 2\pi \mathbb{Z}$ for all $c \in H_1(X, \mathbb{Z})$, i.e., $b_{\text{mag},\alpha}^1 = |E| - |V| + 1$ in this case and $b_{\text{mag},\alpha}^1 = |E| - |V|$ otherwise. We do not give a formal proof of this fact here, since the result follows by abstract arguments of the next theorem. Indeed, we have

$$\text{ind} D_{g_{\text{mag},\alpha}} = \chi_{\text{std}}(X) = |V| - |E|.$$

These examples suggest the following theorem:

**Theorem 4.6.** Assume that $X$ is a finite graph, i.e., that $|E| < \infty$, with vertex space $G$. Then the index of the Dirac operator $D$ associated to the exterior derivative $d$ as defined in Definition 2.1 is given by

$$\text{ind} D = \dim G - |E|.$$

**Remark 4.7.**
(i) We can interpret the above theorem as a discrete “Gauß-Bonnet”-theorem for general vertex spaces, namely

\[ \chi_\mathcal{G}(X) = \sum_{v \in V} \kappa_\mathcal{G}(v) \]  \hspace{1cm} (4.2) \]

using Eq. (4.2), where \( \chi_\mathcal{G}(X) \) is defined in Definition 4.1 and \( \kappa_\mathcal{G}(v) \) in Definition 4.2.

(ii) The index \( \text{ind} \, \mathcal{D} \) gives at least some simple information on the vertex space, namely \( \mathcal{G} \) is trivial (i.e., \( \mathcal{G} \) is the maximal or minimal vertex space \( \mathcal{G}_{\text{max}} \) or \( \mathcal{G}^0 = 0 \)) iff \( \text{ind} \, \mathcal{D} = \pm |E| \). This follows from Eq. (2.2) and Example 4.5 (i)–(ii).

(iii) For continuous vertex spaces \( \mathcal{G} \), we obtain the classical case where \( \dim \mathcal{G} = |V| \), i.e., the classical discrete Gauß-Bonnet formula Eq. (1.6).

Before proving our index theorem, we use a deformation argument in order to calculate the index:

**Lemma 4.8.** Let \( \mathcal{G}_0 \) and \( \mathcal{G}_1 \) be two vertex spaces with \( \dim \mathcal{G}_0 = \dim \mathcal{G}_1 = n \), then \( \text{ind} \, \mathcal{D}_{\mathcal{G}_0} = \text{ind} \, \mathcal{D}_{\mathcal{G}_1} \) for the Dirac operators associated to the vertex spaces.

**Proof.** Denote \( P_t \) the associated orthogonal projections, \( t \in \{0, 1\} \). Note that \( P_0 \) and \( P_1 \) can be connected by a (norm-)continuous path \( P_t \) inside the space of orthogonal projections of rank \( n \): This can be seen as follows: Let \( \{ \varphi_{0,k} \}_k \) and \( \{ \varphi_{1,k} \}_k \) be two orthonormal bases such that the first \( n \) vectors span the range of \( P_0 \) and \( P_1 \), respectively. Let \( U_1 \) be the unitary operator mapping \( \varphi_{0,k} \) onto \( \varphi_{1,k} \). Since the space of unitary operators is connected, we can find a (norm-)continuous path \( t \mapsto U_t \) from the identity operator \( U_0 := 1 \) to \( U_1 \) such that all operators \( U_t \) are unitary. Define \( P_t := U_t^* P_0 U_t \), then \( t \mapsto P_t \) is a continuous path from \( P_0 \) to \( P_1 \).

Let \( \varphi_{t,k} := U_t \varphi_{0,k} \), then \( t \mapsto \{ \varphi_{t,k} \}_k \) is a continuous family of orthogonal bases. Let \( D_t \) be the Dirac operator defined with respect to \( \mathcal{G}_t = \text{ran} \, P_t \). Passing to the basis \( \{ \varphi_{t,k} \}_k \) in \( \mathcal{G}_t \), we may assume that the family \( t \mapsto D_t \) is defined on \( \mathbb{C}^n \oplus I_2(E) \). Moreover, the family \( t \mapsto D_t \) is continuous, since \( t \mapsto U_t \) is. The index formula follows from Lemma 1.5. \( \square \)

**Proof of Theorem 4.6.** For each dimension \( n = \dim \mathcal{G} \), we use a simple vertex space \( \mathcal{G} \) of dimension \( n \). In particular, we choose the space \( \mathcal{G}_v := \mathbb{C}^{d_v} \oplus 0 \subset \mathbb{C}^n \) where \( d_v := \dim \mathcal{G}_v \) is the dimension of the original vertex space at \( v \). This vertex space corresponds to \( d_v \) “Neumann” boundary conditions at the first \( d_v \) edges, and \( \deg v - d_v \) “Dirichlet” boundary conditions at the remaining edges in \( E_v \). Due to the stability of the index shown in Lemma 4.8, it suffices to calculate the index of the model vertex space \( \mathcal{G} \).

It is easily seen as in Example 4.5 (i)–(ii) that for a graph consisting of a single edge with two adjacent vertices, we have

\[ \text{ind} \, \mathcal{D}^{DD} = -1, \quad \text{ind} \, \mathcal{D}^{DN} = 0 \quad \text{and} \quad \text{ind} \, \mathcal{D}^{NN} = 1. \]

For example, for the mixed case, \( dF = F(1) \) if the Neumann space is at the vertex 1. In particular, \( d \) and \( d^* \) are both injective, so \( \ker d = 0 \) and \( \ker d^* = 0 \).

Due to the additivity of the index with respect to orthogonal sums, we therefore have

\[ \text{ind} \, \mathcal{D} = \# \{ \text{edges with N-N} \} - \# \{ \text{edges with D-D} \}. \]
for the Dirac operator associated with the vertex space $\tilde{G}$. It remains to show that

$$\#\{\text{edges with N-N}\} - \#\{\text{edges with D-D}\} = \#\{\text{all N}\} - |E|.$$  

In order to show this last equality, we argue by induction over the dimension $n$ of $\tilde{G}$, i.e., the total number of Neumann conditions in $\tilde{G}$. For $n = 0$, the vertex space is the minimal or Dirichlet vertex space, for which the index formula is correct by Example 4.5 (i). For the induction step $n \rightarrow n + 1$ we have to distinguish two cases:

Case A. In an existing edge with two Dirichlet or one Dirichlet and one Neumann boundary space, we replace one Dirichlet space by a Neumann one. This increases the LHS of Eq. (4) as well as the RHS by 1.

Case B. We add an edge with Dirichlet and Neumann vertex space to the graph. The LHS is unchanged, and in the RHS, we increase the number of Neumann conditions by 1, but subtract also one additional edge.

\[ \square \]

5. Exterior derivatives on quantum graphs

In this section, we develop the notion of exterior derivatives on metric graphs. We first start with the definition of a metric graph, and some general results needed later.

5.1. Continuous metric graphs. A (continuous) metric graph $X = (V, E, \partial, \ell)$ is formally given by the same data as a discrete (edge-)weighted graph. The difference is the interpretation of the space $X$: We define $X$ as

$$X := \bigcup_{e \in E} [0, \ell_e] / \sim_{\psi}$$

where we identify $x \sim_{\psi} y$ iff $\psi(x) = \psi(y)$ with

$$\psi: \bigcup_{e \in E} \{0, \ell_e\} \longrightarrow V, \quad 0_e \mapsto \partial_-, \quad \ell_e \mapsto \partial_+.$$  

In the sequel, we often identify the edge $e$ with the interval $(0, \ell_e)$ and use $x = x_e$ as coordinate. In addition, we denote $dx = dx_e$ the Lebesgue measure on $e$ inducing a natural measure on $X$. The space $X$ becomes a metric space by defining the distance of two points to be the length of the shortest path in $X$ joining these points.

We first define several Hilbert spaces associated with $X$. Our basic Hilbert space is

$$L_2(X) := \bigoplus_{e \in E} L_2(e) \quad (5.1)$$

where again $e$ is identified with $(0, \ell_e)$.

Remark 5.1. The interpretation of an edge $e$ as a “continuous” interval is in contrast with the discrete case where $e$ is considered as a single point, e.g. in (2.4). Another point of view is that we use different types of measures; in the discrete case a point measure and in the metric case the Lebesgue measure. This fact of choosing two different types of measures (or even combinations of them) is pointed out in the works of Friedman and Tillich (cf. [FT04a, FT04b]) and also in [BF06, BR07].

More generally, we define the decoupled Sobolev space of order $k$ by

$$H^k_{\text{max}}(X) := \bigoplus_{e \in E} H^k(e).$$
Obviously, for $k = 0$, there is no difference between $L^2_\infty(X)$ and the decoupled space. Namely, evaluation of a function at a point only makes sense if $k \geq 1$ due to the next lemma. We need the following notation: For $f \in H^1_{\max}(X)$, we denote

$$f = \{f(v)\}_{v \in V}, \quad f(v) = \{f_e(v)\}_{e \in E_v}, \quad f_e(v) := \begin{cases} f_e(0), & v = \partial_- e \\ f_e(\ell_e), & v = \partial_+ e \end{cases}$$

the unoriented evaluation at the vertex $v$. Similarly, for $g \in H^1_{\max}(X)$, we denote

$$\hat{g} = \{\hat{g}(v)\}_{v \in V}, \quad \hat{g}(v) = \{\hat{g}_e(v)\}_{e \in E_v}, \quad \hat{g}_e(v) := \begin{cases} -g_e(0), & v = \partial_- e \\ g_e(\ell_e), & v = \partial_+ e \end{cases} \quad (5.2)$$

the oriented evaluation at the vertex $v$.

The following lemma is a simple consequence of a standard estimate for Sobolev spaces:

**Lemma 5.2.** Assume the condition (3.1) on the edge lengths, i.e., there is $\ell_0 \leq 1$ such that $\ell_e \geq \ell_0 > 0$ for all $e \in E$. Then the evaluation maps

$$(\cdot) : H^1_{\max}(X) \longrightarrow \mathcal{G}^\max, \quad f \mapsto \underline{f} \quad \text{and} \quad (\cdot) : H^1_{\max}(X) \longrightarrow \mathcal{G}^\max, \quad g \mapsto \hat{g},$$

are bounded by $2/\sqrt{\ell_0}$.

**Proof.** By density, we can assume that $f$ is smooth on each edge. For $e \in E_v$, let $\chi_{v,e}$ be the affine linear function with value 1 at $v$ and 0 at the other vertex $v_e$. Then

$$f_e(v) = \int_e (f_e \chi_{v,e})'(x) \, dx \hat{1}_e(v) = \int_e (f_e \chi_{v,e})(x) \, dx \hat{1}_e(v) + \frac{1}{\ell_e} \int_e f_e(x) \, dx. \quad (5.3)$$

In order to avoid an upper bound on $\ell_e$, we replace the edge $e$ by the shortened edge $\tilde{e}_v$ of length $\tilde{\ell}_e = \max\{\ell_e, 1\}$ starting at $v$. Then

$$|f_e(v)|^2 \leq 2\tilde{\ell}_e ||f'||^2_{\tilde{e}_v} + \frac{2}{\ell_e} ||f||^2_{\ell_e} \leq 2 \max\{1, \frac{1}{\ell_0}\} ||f||^2_{H^1_e}$$

using Cauchy-Schwarz. Summing the contributions over $e \in E_v$ and $v \in V$ and using (2.1) we are done. The same arguments hold for $\hat{g}$. $\square$

For a general vertex space $\mathcal{G}$, i.e., a closed subspace of $\mathcal{G}^\max := \bigoplus_{v \in V} \mathbb{C}^E_v$, we set

$$H^1_{\mathcal{G}}(X) := \{ f \in H^1_{\max}(X) \mid f \in \mathcal{G} \} = (\cdot)^{-1} \mathcal{G},$$

i.e., the preimage of $\mathcal{G}$ under the (unoriented) evaluation map, and similarly,

$$H^1_{\hat{\mathcal{G}}}(X) := \{ g \in H^1_{\max}(X) \mid \hat{g} \in \mathcal{G} \} = (\cdot)^{-1} \mathcal{G}$$

the preimage of $\mathcal{G}$ under the (oriented) evaluation map. In particular, both spaces are closed in $H^1_{\max}(X)$.

The reason for two different vertex evaluations becomes clear through the following lemma:

**Lemma 5.3.** For $f, g \in H^1_{\max}(X)$, we have

$$\langle f', g \rangle_X = \langle f, -g' \rangle_X + \langle f, \hat{g} \rangle_{\mathcal{G}^\max}.$$
Proof. We have

\[
\langle f', g \rangle_x + \langle f, g' \rangle_x = \sum_{e \in E} (\langle f', g \rangle_e + \langle f, g' \rangle_e)
\]

\[
= \sum_{e \in E} [T_f g]_{\partial_e} = \sum_{v \in V} \sum_{e \in E_v} f_e(v) g_e(v) = \sum_{v \in V} \langle f(v), \hat{g}(v) \rangle_{C E_v} = \langle f, \hat{g} \rangle_{\mathcal{G}^{\max}}
\]

and the latter expression is defined due to Lemma [5.2]. \qed

5.2. Quantum graphs.

Definition 5.4. A Laplacian on a (continuous) metric graph \( X = (V, E, \partial, \ell) \) is an operator \( \Delta_X \) acting as \( (\Delta_X f)_e = -f''_e \) on each edge \( e \in E \).

We have the following characterisation from [Ku04, Thm. 17]:

Theorem 5.5. Assume the condition (3.1) on the edge lengths, namely \( \ell_e \geq \ell_0 > 0 \). Let \( \mathcal{G} \leq \mathcal{G}^{\max} \) be a (closed) vertex space with orthogonal projection \( P \), and let \( L \) be a self-adjoint, bounded operator on \( \mathcal{G} \). Then the Laplacian \( \Delta_{(X,\mathcal{G},L)} \) with domain

\[
H^2(X,\mathcal{G},L) := \{ f \in H^2_{\max}(X) \mid f \in \mathcal{G}, \quad Pf' + Lf = 0 \}
\]

is self-adjoint with associated quadratic form

\[
\tilde{d}_{(X,\mathcal{G},L)}(f) := \|f'\|^2_X + \langle f, Lf \rangle_{\mathcal{G}} = \sum_{e \in E} \|f'_e\|^2_e + \sum_{v \in V} \langle f(v), L(v)f(v) \rangle_{\mathcal{G}_v}
\]

and domain \( \text{dom} \, \tilde{d}_{(X,\mathcal{G},L)} = H^1_{\mathcal{G}}(X) = \{ f \in H^1_{\max}(X) \mid f \in \mathcal{G} \} \).

Remark 5.6.
(i) We have a similar assertion for the “oriented” version, namely, when we replace \( f \) by \( \hat{f} \) and \( f' \) by \( g' \). We will refer to this Laplacian as \( \Delta_{(X,\mathcal{G},L)} \).
(ii) At least for finite graphs, the converse statement is true, i.e., if \( \Delta \) is a self-adjoint Laplacian in the sense of Definition [5.4] then \( \Delta = \Delta_{(X,\mathcal{G},L)} \) for some vertex space \( \mathcal{G} \) and a bounded operator \( L \). For infinite graphs, the operator \( L \) may become unbounded but we do not consider this case here.
(iii) Note that \( \Delta_{(X,\mathcal{G},L)} \geq 0 \) iff \( L \geq 0 \).

We slightly restrict ourselves and consider only those self-adjoint Laplacians on \( X \) that are obtained as in the above theorem:

Definition 5.7. A quantum graph \( X \) is a metric graph together with a self-adjoint Laplacian \( \Delta_{(X,\mathcal{G},L)} \) where \( \mathcal{G} \) (or \( \mathcal{G}^\circ \) for the oriented version) is a vertex space and \( L \) a self-adjoint, bounded operator on \( \mathcal{G} \). The quantum graph is therefore given by \( X = (V, E, \partial, \ell, \mathcal{G}, L) \) or by a metric graph \( X = (V, E, \partial, \ell) \) and the data \( (X, \mathcal{G}, L) \) (resp. \( (X, \mathcal{G}^\circ, L) \)).

Remark 5.8. In [KS99] (see also [KPS07]) there is another way of parametrising all self-adjoint vertex boundary conditions, namely for bounded operators \( A, B \) on \( \mathcal{G}^{\max} \),

\[
\text{dom} \, \Delta_{(A,B)} = \{ f \in H^2_{\max}(X) \mid Af + B\hat{f}' = 0 \}
\]

is the domain of a self-adjoint operator \( \Delta_{(A,B)} \) iff

(i) \( A \oplus B : \mathcal{G}^{\max} \oplus \mathcal{G}^{\max} \longrightarrow \mathcal{G}^{\max} \), \( F \oplus \hat{F} \mapsto AF + B\hat{F} \), is surjective
(ii) \( AB^* \) is self-adjoint, i.e., \( AB^* = BA^* \).
Given a vertex space \( \mathcal{G} \leq \mathcal{G}^{\max} \) and a bounded operator \( L \) on \( \mathcal{G} \), we have \( \Delta_{(A,B)} = \Delta_{(X,\mathcal{G},L)} \) if we choose

\[
A \cong \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = P \cong \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

with respect to the decomposition \( \mathcal{G}^{\max} = \mathcal{G} \oplus \mathcal{G}^\perp \). The associated scattering matrix with spectral parameter \( \mu = \sqrt{\lambda} \) is

\[
S(\mu) = -(A + i\mu B)^{-1}(A - i\mu B) \cong \begin{pmatrix} -(L + i\mu I)^{-1}(L - i\mu I) & 0 \\ 0 & -I \end{pmatrix}.
\]

In particular, \( S(\mu) \) is independent if \( \mu \) iff \( L = 0 \), and in this case, we have \( S(\mu) = 1 + -1 \) for all \( \mu \).

The aim of the subsequent section is to express \( \Delta_{(X,\mathcal{G},L)} \) as \( d^*d \) or \( dd^* \). Of course, to do so, we need \( L \geq 0 \) (since operators \( d^*d \) and \( dd^* \) are always non-negative). Furthermore, for non-trivial \( L \neq 0 \), we need to enlarge the \( L_2 \)-spaces by the vertex space \( \mathcal{G} \).

### 5.3. Differential forms, exterior derivatives and Dirac operators.

**Definition 5.9.** For \( p \in \{0,1\} \), let \( \mathcal{G}^p \) be a vertex space. We call the space

\[
L_2(\Lambda^p X) := L_2(X) \oplus \mathcal{G}^p
\]

the \( L_2 \)-space of \( p \)-forms.

A subspace \( H^1(\Lambda^p X) \) of \( L_2(\Lambda^p X) \) is called an \( H^1 \)-space of \( p \)-forms iff

(i) the space \( H^1(\Lambda^p X) \) is dense in \( L_2(\Lambda^p X) \),

(ii) we have \( \iota^p(H^1_0(X)) \subset H^1(\Lambda^p X) \) and

(iii) we have \( (\iota^p)^*(H^1(\Lambda^p X)) \subset H^1_{\max}(X) \),

where

\[
\iota^p: L_2(X) \hookrightarrow L_2(\Lambda^p X), \quad f \mapsto (f,0), \quad p \in \{0,1\},
\]

denote the natural embedding operators. We set

\[
L_2(\Lambda X) := L_2(\Lambda^0 X) \oplus L_2(\Lambda^1 X) \quad \text{and} \quad H^1(\Lambda X) := H^1(\Lambda^0 X) \oplus H^1(\Lambda^1 X).
\]

Note that \( (\iota^p)^* \) is the projection onto the first factor.

For the definition of an exterior derivative, we need the following decoupled operator \( d_0: H^1_0(X) \longrightarrow L_2(X), \quad d_0 f = f' \). Note that \( d_0 \) is a closed operator with adjoint \( d_1 g = -g' \) and \( \text{dom } d^* = H^1_{\max}(X) \). We now define an exterior derivative associated to spaces of \( p \)-forms. Examples are given below.

**Definition 5.10.** Let \( L_2(\Lambda^p X) \) be an \( L_2 \)-space of \( p \)-forms and \( H^1(\Lambda^0 X) \) be an \( H^1 \)-space of \( 0 \)-forms. We call an operator

\[
d: H^1(\Lambda^0 X) \longrightarrow L_2(\Lambda^1 X)
\]

an exterior derivative on the metric graph \( X \) iff the following conditions are fulfilled:

(i) The operator \( d \) is closed as unbounded operator from the \( 0 \)-form space \( L_2(\Lambda^0 X) \) into the \( 1 \)-form space \( L_2(\Lambda^1 X) \).

(ii) We have \( d d^* = \iota^1 d_0 \), i.e., \( d(\iota^0 f) = \iota^1 f' \) for all \( f \in H^1_0(X) \).

(iii) We have \( \iota^1 d = -d_0^*(\iota^0)^* \), i.e., \( \iota^1* d \tilde{f} = ((\iota^0)^* \tilde{f})' \).

Note that the closeness of \( d \) ensures that we choose the “right” norm on \( H^1(\Lambda^p X) \) (and not an artificially smaller space).
Lemma 5.11. Given the p-form spaces $L_p(\Lambda^pX)$, $p \in \{0, 1\}$, the 0-form space $H^1(\Lambda^0X)$ and an exterior derivative $d: H^1(\Lambda^0X) \to L_2(\Lambda^0X)$, then the adjoint $d^*$ is uniquely defined and closed as operator from $L_2(\Lambda^1X)$ into $L_2(\Lambda^0X)$. Its domain

$$H^1(\Lambda^1X) := \text{dom } d^*$$

is an $H^1$-space of 1-forms (cf. Definition 5.9 (i)–(iii)).

Proof. Since $d$ is densely defined by Definition 5.10 (i), it follows, that $d^*$ is uniquely determined, closed and densely defined, i.e., Definition 5.9 (i) is fulfilled. In order to verify Definition 5.9 (ii), we have to show that for $f \in H^1(X)$, the 1-form $\iota_1 f$ is in $\text{dom } d^*$: Set $h := -\iota_0 d_0 f$. Then $h \in L^2(\Lambda^0X)$ and we have

$$\langle h, g \rangle = \langle -d_0(\iota_0)^* \tilde{f}, f \rangle = \langle (\iota_1)^* d \tilde{f}, f \rangle = \langle d \tilde{f}, \iota_1 f \rangle$$

for all $\tilde{f} \in \text{dom } d$, i.e., $\iota_1 f \in \text{dom } d^*$, where we used Definition 5.10 (iii). Condition (iii) follows similarly from Definition 5.10 (iii). □

Definition 5.12. We call the operator

$$D: H^1(\Lambda X) \to L^2(\Lambda X), \quad D(f, g) := (d^* g, d f)$$

the Dirac-operator associated to the p-form spaces $H^1(\Lambda^pX) \subset L^2(\Lambda^pX)$ and the exterior derivative $d$.

Remark 5.13.

(i) Obviously, $D$ is a closed and self-adjoint operator with the matrix representation

$$D \cong \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix}$$

where we split the space in its 0- and 1-form component.

(ii) In order to define a Dirac operator $D$, it suffices — due to Lemma 5.11 — to determine

$$\mathcal{G}^0, \quad \mathcal{G}^1, \quad H^1(\Lambda^0X), \quad d: H^1(\Lambda^0X) \to L_2(\Lambda^1X)$$

and to ensure that the conditions of Definition 5.9 (i)–(iii), for $p = 0$ and Definition 5.10 (i)–(iii) are fulfilled. In this case, the Dirac operator is uniquely determined.

Again, we have a “baby” version of the Hodge decomposition theorem (see Lemma 1.3):

Lemma 5.14. Assume that $D$ has a spectral gap at 0, i.e., that $\text{dist}(0, \sigma(D) \setminus \{0\}) > 0$ (e.g., $X$ compact is sufficient). Then

$$L_2(\Lambda X) = \ker D \oplus \text{ran } d^* \oplus \text{ran } d, \quad \text{i.e.,} \quad L_2(\Lambda^0X) = \ker d \oplus \text{ran } d^* \quad \text{and} \quad L_2(\Lambda^1X) = \ker d^* \oplus \text{ran } d.$$

We will now give concrete examples of Dirac operators. Since at this stage it is not clear what definition is “natural” we list some reasonable possibilities how to define the $H^1$-spaces:

Lemma 5.15.

(i) The simple case: We set

$$\mathcal{G}^0 := 0, \quad \mathcal{G}^1 := 0, \quad H^1(\Lambda^0X) := H^1_{\mathcal{G}^0}(X), \quad d f := f'.$$

Then

$$H^1(\Lambda^1X) = H^1_{\mathcal{G}^1}(X) \quad \text{and} \quad d^* g = -g'.$$
(ii) **The 0-enlarged space:** Let \( \mathcal{G} \) be a vertex space with bounded operator \( L \geq 0 \) on \( \mathcal{G} \). We set \( \mathcal{G}^0 := \mathcal{G} \), \( \mathcal{G}^1 := 0 \),

\[
H^1(\Lambda^0 X) := \left\{ (f, F) \in H^1_\mathcal{G}(X) \oplus \mathcal{G} \mid f = L^{1/2} F \right\}
\]

and \( d(f, F) = f' \). Then we have

\[
H^1(\Lambda^1 X) := H^1_{\max}(X) \quad \text{and} \quad d^*(g) = (-g', L^{1/2} \overline{P} \overline{g}).
\]

(iii) **The 0-enlarged space with projection:** Let \( \mathcal{G} \) be a vertex space with associated projection \( P \) and bounded operator \( L \geq 0 \) on \( \mathcal{G} \). We set \( \mathcal{G}^0 := \mathcal{G} \), \( \mathcal{G}^1 := 0 \),

\[
H^1(\Lambda^0 X) := \left\{ (f, F) \in H^1_{\max}(X) \oplus \mathcal{G} \mid Pf = L^{1/2} F \right\}
\]

and \( d(f, F) = f' \). Then

\[
H^1(\Lambda^1 X) = H^1_{\mathcal{G}^0}(X) \quad \text{and} \quad d^*(g) = (-g', L^{1/2} \overline{g}).
\]

(iv) **The 1-enlarged space:** Let \( \mathcal{G} \) be a vertex space with bounded operator \( L \geq 0 \). We set

\[
\mathcal{G}^0 = 0, \quad \mathcal{G}^1 := \mathcal{G}, \quad H^1(\Lambda^0 X) := H^1_{\max}(X), \quad df = (f, L^{1/2} Pf).
\]

Then

\[
H^1(\Lambda^1 X) = \left\{ (g, F) \in H^1_{\max}(X) \oplus \mathcal{G} \mid \overline{g} \in \mathcal{G}, \quad \overline{g} + L^{1/2} G = 0 \right\}
\]

and \( d^*(g, F) = -g' \).

(v) **The 1-enlarged space with projection:** Let \( \mathcal{G} \) be a vertex space with associated projection \( P \) and bounded operator \( L \geq 0 \) on \( \mathcal{G} \). We set \( \mathcal{G}^0 := \mathcal{G} \), \( \mathcal{G}^1 := 0 \),

\[
H^1(\Lambda^0 \mathcal{G} X) := H^1_{\mathcal{G}^1}(X), \quad df = (f', L^{1/2} f).
\]

Then

\[
H^1(\Lambda^1 X) = \left\{ (g, F) \in H^1_{\max}(X) \oplus \mathcal{G} \mid P \overline{g} + L^{1/2} G = 0 \right\}
\]

and \( d^*(g, F) = -g' \).

**Proof.** We only check the conditions for (ii) since the other cases are similar. We apply Lemma 5.11 and have to show first that \( H^1(\Lambda^0 X) \) is an \( H^1 \)-space of 0-forms and second, that \( d \) is an exterior derivative. In order to show the first, note that \( H^1(\Lambda^0 X) \) is dense in \( L_2(\Lambda^0 X) \): Let \( (f, F) \in L_2(\Lambda^0 X) \oplus \mathcal{G} \) and \( \varepsilon > 0 \). By density of \( H^1_{\max}(X) \) we can find a function \( f_1 \in H^1_{\max}(X) \) such that \( \|f - f_1\|_{L_2(X)} \leq \varepsilon/2 \). Furthermore, we can change \( f_1 \) to \( f_2 \) near a vertex \( v \) in such a way that \( f_2(v) = F(v) \) and that their norm difference does not exceed \( \varepsilon/2 \). Then \( (f_2, F) \) has distance at most \( \varepsilon \) from \( (f, F) \) in \( L_2(\Lambda^0 X) \).

The second and third condition of Definition 5.9 are obviously fulfilled. In order to show that \( d \) is an exterior derivative we have to check the conditions of Definition 5.10. For the closeness of \( d \) note that the graph norm of \( d \) defined by

\[
\| (f, F) \|^2_d := \| df \|^2_{L_2(\Lambda^1)} + \| (f, F) \|^2_{L_2(\Lambda^0 X)} = \| f' \|^2_{L_2(X)} + \| f \|^2_{L_2(X)} + \| F \|^2_{\mathcal{G}}
\]

is the Sobolev norm. It remains to show that \( H^1(\Lambda^0 X) \) is closed in \( H^1_{\max}(X) \oplus \mathcal{G} \): Note that \( (f, F) \mapsto Pf - L^{1/2} F \) is continuous by Lemma 5.2 and since \( L \) is bounded. Furthermore, \( H^1(\Lambda^0 X) \) is the kernel of this map and therefore closed. The second and third condition of Definition 5.10 follow by an immediate calculation. \( \square \)
Remark 5.16. By splitting the vertex space $\mathcal{G}$ we may assume that $L$ is invertible on a smaller vertex space: For example, in the 0-enlarged case (ii), the condition $\underline{f} = L^{1/2}F$ implies that $\underline{f} \in (\ker L^{1/2})^\perp := \mathcal{G}_1$. Then we have
\[ H^1(\Lambda^0 X) = \{ (f, F) \in H^1_{\mathcal{G}_1}(X) \oplus \mathcal{G}_1 \mid \underline{f} = L^{1/2}F_1 \} \oplus \mathcal{G}_0 \]
where $\mathcal{G}_0 := \ker L$ and $L_1 := L|_{\mathcal{G}_1}$ is invertible.

Remark 5.17. We assume here that $L$ is invertible on $\mathcal{G}$. Then we can pass to the limit $L \to \infty$ in the equation $G = LF$ in the following sense: We consider the limit $L^{-1} \to 0$ in $L^{-1}G = F$, i.e., $F = 0$ and no restriction on $G$. We use this interpretation in order to show how the above different cases are related in the limit case:

(ii) **The 0-enlarged space:** Here, the condition in $H^1(\Lambda^0 X)$ is $\underline{f} = L^{1/2}F$. The limit $L \to \infty$ leads to $F = 0$, i.e., $\mathcal{G}^0 = 0$. Moreover, the second component in $d^*g$ has to vanish, i.e., $P\hat{g} = 0$. In particular, the added space $\mathcal{G}^0 = 0$ vanishes and we arrive at the simple case (i).

(iii) **The 0-enlarged space with projection:** The condition in $H^1(\Lambda^0 X)$ is $P\underline{f} = L^{1/2}F$. The limit $L \to \infty$ leads again to $F = 0$, i.e., $\mathcal{G} = 0$. Furthermore, the second component in $d^*g$ has to vanish, i.e., $\hat{g} = 0$. In particular, we arrive at the simple case (i) with Neumann boundary space $\mathcal{G} = \mathcal{G}^{\max}$.

(iv) **The 1-enlarged space:** The condition in $H^1(\Lambda^1 X)$ is $\hat{g} + L^{1/2}G = 0$. The limit $L \to \infty$ here leads to $G = 0$, i.e., $\mathcal{G}^1 = 0$ and therefore $P\underline{f} = 0$. In particular, we arrive at the simple case (i) with the roles of $\mathcal{G}$ and $\mathcal{G}^1$ interchanged.

(v) **The 1-enlarged space with projection:** Finally, in this case, we arrive at the simple case (i) with Dirichlet vertex space $\mathcal{G} = \mathcal{G}^{\min} = 0$.

As in Section 1.2 we can associate a Laplacian $\Delta_{AX} := D^2$ to the Dirac operator $D$ on the metric graph $X$ with differential form space $H^1(\Lambda X) \subset L_2(\Lambda X)$ with natural domain
\[ H^2(\Lambda X) = \{ \omega \in H^1(\Lambda X) \mid D\omega \in H^1(\Lambda X) \} \]
Furthermore, the operator decomposes into the components
\[ \Delta_{AX} = \Delta_{\Lambda^0 X} + \Delta_{\Lambda^1 X} = d^*d + dd^* \]
with natural domains
\[ H^2(\Lambda^0 X) := \{ f \in H^1(\Lambda^0 X) \mid df \in H^1(\Lambda^1 X) \} \]
\[ H^2(\Lambda^1 X) := \{ g \in H^1(\Lambda^1 X) \mid d^*g \in H^1(\Lambda^0 X) \} \].

**Lemma 5.18.** Assume that $D$ is a Dirac operator on the metric graph $X$ and that $\Delta_{AX} := D^2$ is its associated Laplacian. Then the components $\Delta_{\Lambda^p X}$ act as Laplacians on the metric graph part, i.e.,
\[ (t^0)^*\Delta_{\Lambda^0 X}t^0 = d_0^*d_0 : H^2_D(X) \to L_2(X) \]
\[ (t^1)^*\Delta_{\Lambda^1 X}t^1 = d_0^*d_0 : H^2_D(X) \to L_2(X) , \]
where $d_0^*d_0 = \bigoplus_{e \in E} \Delta_{D_e}^0$ is the sum of the Dirichlet Laplacian on each edge.

**Proof.** The assertion follows directly from Definition 5.10 (ii) \& (iii). \hfill $\square$

We give the concrete domains of the Laplacians in each of the above cases:
Lemma 5.19.

(i) **The simple case:** Here we have
\[
H^2(\Lambda^0 X) := \{ f \in H^2_{\text{max}}(X) \mid f \in \mathcal{G}, \quad \hat{f}' \in \mathcal{G}^\perp \}, \quad \Delta_{A^0 X} f = -f''
\]
\[
H^2(\Lambda^1 X) := \{ g \in H^2_{\text{max}}(X) \mid \hat{g}' \in \mathcal{G}^\perp, \quad \hat{g}' + LP\hat{g} = 0 \},
\]
\[
\Delta_{A^1 X} g = -g''.
\]

In particular, on 0-forms, we have the quantum graph \((X, \mathcal{G}, 0)\), and on 1-forms the quantum graph \((X, \mathcal{G}^\perp, 0)\).

(ii) **The 0-enlarged space:** Here we have
\[
H^2(\Lambda^0 X) := \{ (f, F) \in H^2_{\text{max}}(X) \oplus \mathcal{G} \mid f \in \mathcal{G}, f = L^{1/2} F \},
\]
\[
H^2(\Lambda^1 X) := \{ g \in H^2_{\text{max}}(X) \mid \hat{g}' \in \mathcal{G}^\perp, \quad \hat{g}' + LP\hat{g} = 0 \},
\]
\[
\Delta_{A^0 X} (f, F) = (-f'', L^{1/2} P\hat{f}'), \quad \Delta_{A^1 X} g = -g''.
\]

The 1-form space \(H^2(\Lambda^1 X)\) equals \(H^2(\mathcal{G}^\text{max}, \mathcal{G})\) where \(\mathcal{G}^\text{max} = \mathcal{G} \oplus \mathcal{G}^\perp\). In particular, the 1-form space represents the quantum graph \((X, \mathcal{G}^\text{max}, \mathcal{G})\).

(iii) **The 0-enlarged space with projection:** We have
\[
H^2(\Lambda^0 X) := \{ (f, F) \in H^2_{\text{max}}(X) \oplus \mathcal{G} \mid f \in \mathcal{G}, P^f = L^{1/2} F, \quad \hat{f}' \in \mathcal{G} \}
\]
\[
H^2(\Lambda^1 X) := \{ g \in H^2_{\text{max}}(X) \mid \hat{g} \in \mathcal{G}^\perp, \quad \hat{g}' + L\hat{g} = 0 \},
\]
\[
\Delta_{A^0 X} (f, F) = (-f'', L^{1/2} P\hat{f}'), \quad \Delta_{A^1 X} g = -g''.
\]

(iv) **The 1-enlarged space:** We have
\[
H^2(\Lambda^0 X) := \{ f \in H^2_{\text{max}}(X) \mid \hat{f}' \in \mathcal{G}^\perp, \quad \hat{f}' + LPf = 0 \}
\]
\[
H^2(\Lambda^1 X) := \{ (g, G) \in H^2_{\text{max}}(X) \oplus \mathcal{G} \mid \hat{g} \in \mathcal{G}^\perp, \quad \hat{g} + L^{1/2} G = 0 \},
\]
\[
\Delta_{A^0 X} f = -f'' \quad \text{and} \quad \Delta_{A^1 X} (g, G) = (-g'', -L^{1/2} P\hat{g}').
\]

We have \(H^2(\Lambda^0 X) = H^2(\mathcal{G}^\text{max}, \mathcal{G})\) where \(\mathcal{G}^\text{max} = \mathcal{G} \oplus \mathcal{G}^\perp\). In particular, the 0-form Laplacian defines the quantum graph \((X, \mathcal{G}^\text{max}, \mathcal{G})\).

(v) **The 1-enlarged space with projection:** We have
\[
H^2(\Lambda^0 X) := \{ f \in H^2_{\text{max}}(X) \mid f \in \mathcal{G}, \quad P\hat{f}' + Lf = 0 \}
\]
\[
H^2(\Lambda^1 X) := \{ (g, G) \in H^2_{\text{max}}(X) \oplus \mathcal{G} \mid \hat{g}' \in \mathcal{G}^\perp, \quad \hat{g}' + L^{1/2} G = 0 \},
\]
\[
\Delta_{A^0 X} f = -f'' \quad \text{and} \quad \Delta_{A^1 X} (g, G) = (-g'', -L^{1/2} \hat{g}').
\]

The 0-form Laplacian defines the “classical” quantum graph \((X, \mathcal{G}, \mathcal{G})\).

6. **Index formulas for metric graphs**

6.1. Isomorphism between kernels of discrete and quantum graph Dirac operators. We will now present one of the main results of this article, namely we establish an isomorphism between \(\ker D\) and \(\ker \mathcal{D}\) in the five cases of differential forms mentioned above respecting the supersymmetric space decomposition.

We need some notation: For a bounded operator \(L\) in \(\mathcal{G}\), we set \(\mathcal{G}_0 := \ker L\) and \(\mathcal{G}_1 := \mathcal{G} \ominus \ker L\). Furthermore, \(\mathcal{G}^\perp := \mathcal{G}^\text{max} \ominus \mathcal{G}\). In addition, we denote the projections corresponding to \(\mathcal{G}_i\) and \(\mathcal{G}_i^\perp\) by \(P_i\) and \(P_i^\perp\), respectively. Similarly, we
denote the corresponding exterior derivatives by $d_i : \mathcal{G}_i \rightarrow \ell_2(E)$ and $d_i^\perp : \mathcal{G}_i^\perp \rightarrow \ell_2(E)$.

The discrete Dirac operator needs to be trivially enlarged by the space $\mathcal{N} = \mathcal{G}_0$ in the cases (ii)–(v) (cf. Definition 1.7). Finally, $\eta := \int g$ is defined by $\eta_e := \int_0^\infty g_e(x) \, dx$.

**Theorem 6.1.** Assume (3.1), then

$$\Phi : H^1(\Lambda X) \rightarrow \ell_2(\Lambda X)$$

is a bounded operator with norm bounded by $2/\sqrt{\ell_0}$, and $\Phi(\ker D) = \ker D$ is an isomorphism respecting the supersymmetry (i.e., $\Phi = \Phi_0 \oplus \Phi_1$, cf. Definition 1.6).

In particular, $\Phi_0(\ker d) = \ker d$ and $\Phi_1(\ker d^\perp) = \ker d^\perp$ are isomorphisms and

$$\text{ind } D = \text{ind } D,$$

where $D$ and $D$ are the Dirac operators associated to the exterior derivatives $d$ and $d$, respectively. Furthermore, $d$ and $\Phi$ are given in the following cases:

(i) **The simple case:** Here, $d : \mathcal{G} \rightarrow \ell_2(E)$,

$$\Phi : H^1_\mathcal{G}(X) \oplus H^1_\mathcal{G}(X) \rightarrow \mathcal{G} \oplus \ell_2(E), \quad (f,g) \mapsto (f, \int g),$$

$$\text{ind } D = \text{ind } D = \dim \mathcal{G} - |E|.$$

(ii) **The 0-enlarged space:** Here,

$$d : \mathcal{G}_1 \oplus \mathcal{G}_0 \rightarrow \ell_2(E), \quad F_0 \oplus F_0 \rightarrow d_1 F_1,$$

$$\Phi : H^1(\Lambda X) \rightarrow (\mathcal{G}_1 \oplus \mathcal{G}_0) \oplus \ell_2(E), \quad (f,F,g) \mapsto (P_1 f, F_0 F, \int g),$$

$$\text{ind } D = \text{ind } D = \dim \mathcal{G} - |E|.$$

(iii) **The 0-enlarged space with projection:** We have

$$d : \mathcal{G}_0^\perp \oplus \mathcal{G}_0 \rightarrow \ell_2(E), \quad F_0^\perp \oplus F_0 \rightarrow d_0^\perp F_0^\perp,$$

$$\Phi : H^1(\Lambda X) \rightarrow (\mathcal{G}_0^\perp \oplus \mathcal{G}_0) \oplus \ell_2(E), \quad (f,F,g) \mapsto (P_0^\perp f, P_0 F, \int g),$$

$$\text{ind } D = \text{ind } D = |E|.$$

(iv) **The 1-enlarged space:** We have

$$d : \mathcal{G}_1^\perp \rightarrow \mathcal{G}_0 \oplus \ell_2(E), \quad F_1^\perp \rightarrow 0 \oplus d_1^\perp F_1^\perp,$$

$$\Phi : H^1(\Lambda X) \rightarrow \mathcal{G}_1^\perp \oplus (\mathcal{G}_0 \oplus \ell_2(E)), \quad (f,g,G) \mapsto (P_1^\perp f, P_0 G, \int g),$$

$$\text{ind } D = \text{ind } D = |E| - \dim \mathcal{G}.$$

(v) **The 1-enlarged space with projection:** We have

$$d : \mathcal{G}_0 \rightarrow \mathcal{G}_0 \oplus \ell_2(E), \quad F_0 \rightarrow 0 \oplus d_0 F_0,$$

$$\Phi : H^1(\Lambda X) \rightarrow \mathcal{G}_0 \oplus (\mathcal{G}_0 \oplus \ell_2(E)), \quad (f,g,G) \mapsto (P_0 f, P_0 G, \int g),$$

$$\text{ind } D = \text{ind } D = -|E|.$$

**Remark 6.2.**

(i) Note that in all cases, the index is independent of $L$, i.e., of the decomposition of $\mathcal{G}$ into $\mathcal{G}_0 = \ker L$ and $\mathcal{G}_1 = \mathcal{G} \ominus \ker L$ as one expects since the index should be constant passing to the limit $L \rightarrow 0$. 
(ii) In the first two cases, we obtain the Euler characteristic as index (if $\mathcal{G} = \mathcal{G}_{\text{std}}$). These two cases are the ones we obtain by a limit argument where the metric graph is approached by a manifold (cf. [EP05] and a forthcoming paper) provided the transversal manifold $F$ is simply connected (see also Examples 6.4–6.6).

(iii) If we assume that $L$ is invertible, then the index in each case remains the same when passing to the limit $L \to \infty$ (cf. Remark 5.17).

(iv) We can interprete $\Phi$ in the above theorem as a sort of Hilbert chain morphism (cf. [Liic02, Ch. 1]). For example, in the 0-enlarged case (iii), we have

$$
\begin{align*}
0 & \longrightarrow \mathcal{H}^1(\Lambda^0X) \xrightarrow{d} \mathcal{L}_2(\Lambda^1X) \xrightarrow{\Phi_0} \mathcal{G}_1 \oplus \mathcal{G}_0 \xrightarrow{d_1} \ell_2(E) \xrightarrow{\Phi_1} 0
\end{align*}
$$

where the rows are obviously chain complexes (with bounded maps) and the diagram is commutative. Note that indeed, $\Phi_1 : \mathcal{L}_2(\Lambda^1X) \to \ell_2(E)$, $g \mapsto \int g$, is a bounded map also on the $L_2$-space. The commutativity of the diagram follows from the fact that

$$
((d\Phi_0 - \Phi_1 d)(f,F))_e = (P_1 f)_e(\partial_+ e) - (P_1 f)_e(\partial_- e) - \int_{\mathcal{L}} f'_e \, dx
$$

$$
= -(P_0 f)_e(\partial_+ e) + (P_0 f)_e(\partial_- e)
$$

for $f \in \mathcal{H}^1(\Lambda^0X)$. But note that $f = L^{1/2} F$ implies $f \in (\ker L)^\perp = \mathcal{G}_1$, so that $d\Phi_0 = \Phi_1 d$, i.e., $\Phi$ is a chain morphism. The corresponding homology induces the above isomorphism of the Dirac operator kernels. The other cases can be treated similarly. We will stress this abstract point of view (and also an interpretation of the “enlarged” spaces as twisted chain complexes) in a forthcoming publication.

**Proof.** The boundedness of $\Phi$ in the particular cases follows immediately from Lemma 5.2 and Cauchy-Schwarz. We only prove the second case, since the other ones are similar. First, we note that in Case (iii), we have $(f,F,g) \in \ker D$ iff $f_e, g_e$ are constant, $f \in \mathcal{G}_1, g \in \mathcal{G}_1^\perp$ and $f = L^{1/2} F$. In particular, $df = 0, d^* \int g = P_1 g = 0$ and therefore $\Phi(\ker D) \subset \ker D$.

In order to show that $\Phi$ is injective on $\ker D$, we note that $P_1 f = 0$ implies $f = 0$ (since already $f \in \mathcal{G}_1$). Furthermore, $P_0 F = 0$ and from $0 = f = L^{1/2} F$ we conclude $P_1 F = 0$, i.e., $F = 0$. Finally, $\int g = 0$ and $g_e = \text{const}$ implies that $g = 0$.

It remains to show that $\ker D \subset \Phi(\ker D)$, let $(F_1, F_0, \eta) \in \ker D$. Set $f_e(x) := F_1, e(\partial_x e)$ (both values are the same since $dF_1 = 0$), $F := L^{-1/2} F_1 + F_0$ and $g_e(x) := \eta$. Then $(f,F,g) \in \mathcal{H}^1(\Lambda X)$ since $f \in \mathcal{G}_1 \leq \mathcal{G}$ and $L^{1/2} F = F_1 = f$. Next, $D(f,F,g) = (g^0, L^{1/2} P_0 g, f') = 0$ since $L^{1/2} P_0 g = 0$ and $L^{1/2} P_1 g = L^{1/2} d^* \eta = 0$, and $f_e, g_e$ are constant. Finally, $\Phi(f,F,g) = (P_0 f, P_0 F, \int g) = (F_1, F_0, \eta)$ and the assertion is proven.

The index formulas follow from Theorem 4.6 and Eqs. (1.9) and (2.2). \qed

**Remark 6.3.** We would like to interprete the above index formula together with the discrete index formula of Theorem 4.6 as a “Gauß-Bonnet theorem” on quantum
graphs. To do so, we define the curvature $\kappa := \kappa_{(X,\mathcal{G},L)}$ of the quantum graph $(X,\mathcal{G},L)$ as

$$\kappa_e(x) = 2\left( \frac{\kappa_{\mathcal{G}}(\partial_- e)}{\sum_{e' \in E_{\partial_+ e}} \ell_{e'}} (\ell_e - x) + \frac{\kappa_{\mathcal{G}}(\partial_+ e)}{\sum_{e' \in E_{\partial_- e}} \ell_{e'}} x \right)$$ \hspace{1cm} (6.1)$$

where $\kappa_{\mathcal{G}}(v)$ is the discrete curvature defined in Definition 4.2 but for the vertex space $\mathcal{G}$ given by $\mathcal{G}_v = \mathcal{G}_v$ in the cases $[1]–[11]$, by $\mathcal{G}_v = \mathcal{G}_v^\text{max}$ in the case $[12]$, by $\mathcal{G}_v = \mathcal{G}_v^\text{in}$ in the case $[13]$, and by $\mathcal{G}_v = 0$ in the last case $[14]$ (cf. the index formulas in Theorem 6.1). In particular, we can interpret the index formula $\text{ind } D = \text{ind } D$ as

$$\text{ind } D = \int_X \kappa \, dx$$ \hspace{1cm} (6.2)$$

since

$$\int_X \kappa \, dx = \sum_{v \in V} \kappa_{\mathcal{G}}(v) = \text{ind } D$$ \hspace{1cm} (6.3)$$

by an obvious calculation and Theorem 4.6. Note that the choice of $\kappa_e$ is somehow arbitrary, but it is the unique way to define it if we require that (6.3) holds, that $\kappa_e(v) = c(v)\kappa_{\mathcal{G}}(v)$ for a sequence $c(v) > 0$ and that $\kappa_e'' = 0$.

In particular, we have

$$\kappa(v) = \kappa_{(X,\mathcal{G},L)}(v) = \frac{2\kappa_{\mathcal{G}}(v)}{\sum_{e' \in E_v} \ell_{e'}}.$$ 

If we have a continuous vertex space, i.e., $\dim \mathcal{G}_v = 1$ like the standard vertex space, then $\kappa(v) = 0$ iff $\deg v = 2$. This reflects the fact that a vertex of degree 2 is invisible. Furthermore if $\deg v = 1$ (i.e., a “dead end” with Neumann boundary space), then $\kappa(v) > 0$. Furthermore, if $\deg v \geq 3$, then $\kappa(v) < 0$. Moreover, shorter lengths $\ell_e$ at a vertex $v$ mean a higher absolute value of the curvature.

For example, a dead end $e$ with Dirichlet boundary space at $v \in \partial e$ has negative curvature. In some sense, one could say that high negative curvature forces the function to vanish: If the dead end has length $\ell_e \to 0$ with standard vertex conditions on the other vertex $w \in \partial e$ (of degree $\geq 3$), then $e$ has curvature $\kappa_e \to -\infty$ as $\ell \to 0$, and finally forces the function to vanish also on $w$.

On the other hand a dead end $e$ of length $\ell_e \to 0$ with Neumann boundary space at the endpoint $v$ has curvature tending to $\infty$, but the curvature at the other point $w$ is negative and remains finite. Therefore $\kappa_e(x) = 0$ for a point $x \to w$ as $\ell \to \infty$, and here, the dead end just “disappears” in the limit.

### 6.2. Metric graphs as limits of smooth spaces.

We will give several examples of quantum graph operators on $X_0 = X$ which occur as limits of an appropriate smooth approximation $X_\epsilon$. A simple example is given if $X_0$ is embedded in $\mathbb{R}^2$ and if we choose some open neighbourhood $X_\epsilon$ of $X_0$. Note that

$$\chi(X_\epsilon) = \chi(X_0),$$

since $X_0$ and $X_\epsilon$ are homotopy-equivalent.

In [RS01, KuZ01, KuZ03, EP05, P05, P06], the convergence of the 0-form operators has been established in various situations. We will show in a forthcoming article, that the result extends also to differential forms on $X_\epsilon$ under suitable conditions. Note that in the three first examples below, the “approximating” Laplacian
d^*d_e on Xε (with Neumann boundary conditions on ∂Xε) and its dual d_d_e on 1-forms have index equal to χ(X_0) (more precisely, the Dirac operator associated to the exterior derivative d_e: H^1(Xε) → L^2(A^1Xε) has index equal to χ(X_0)).

We indicate the limit operators acting on a metric graph in several situations:

**Example 6.4** (Standard boundary conditions). If the vertex neighbourhoods do not shrink too slow (e.g., the ε-neighbourhood of the embedded metric graph X_0 ⊂ R^2 is good enough), then the Neumann Laplacian on functions converges to the standard metric graph Laplacian (see the references above), and we will also show that the 1-form Laplacian on Xε converges to the 1-form metric graph Laplacian. In particular, the vertex space of the limit operator is G = G^{std} and the domains are given by

\[ H^2_{\text{std}}(X) := \{ f \in H^2_{\text{max}}(X) \mid f(v) \text{ independent of } e \in E_v, \sum_{e \in E_v} \hat{f}(v) = 0 \}, \]

\[ H^2_{\Sigma}(X) := \{ g \in H^2_{\text{max}}(X) \mid g'(v) \text{ independent of } e \in E_v, \sum_{e \in E_v} \hat{g}(v) = 0 \} \]

as domains for the Laplacian on 0- and 1-forms, respectively. The associated Dirac operator D has index equal to the Euler characteristic (see (1.8)). The same is true for more general Schrödinger operators on Xε like magnetic Laplacians (for the convergence, see e.g. [KuZ01, EP07]). Magnetic Laplacians have been studied throughout in [KS03].

**Example 6.5** (The decoupling case). In [KuZ03] and [EP05, Sec. 6] there is a class of approximations Xε ⊂ R^2 (roughly with slowly decaying vertex neighbourhood volumes of order ε^{-2α} with 0 < α < 1/2). In this case, the limit operator on 0-forms is

\[ \bigoplus_{e \in E} \Delta^D_e \oplus \bigoplus_{v \in V} 0, \]

i.e., the 0-enlarged case (i) with G = G^{std} and operator L = 0. Again, the index of the associated Dirac operator is χ(X) (cf. Theorem 6.1 (ii)). The dual operator is the decoupled Neumann operator.

**Example 6.6** (The borderline case). In [KuZ03] and [EP05, Sec. 7] there is a special class of approximations Xε ⊂ R^2 where the volume of a vertex neighbourhood U_{ε,v} is vol U_{ε,v} = ε vol U_v (i.e., α = 1/2). In this case, the limit operator on 0-forms is of the form Lemma 5.19 (iii) with G = G^{std} and L(v) = (vol U_v)^{-1} (multiplication operator). In particular, the “bizzar” boundary conditions in this case with the enlarged graph space are “natural” in this setting. Again, the index of the associated Dirac operator is χ(X). Note that the dual operator is a “real” quantum graph Laplacian, namely the domain consists of functions g ∈ H^2_{max}(X) such that

\[ g' \text{ is continuous, } (\deg v)(\text{vol } U_v)g'(v) = \sum_{e \in E_v} \hat{g}(v), \]

i.e., a type of δ'-condition with strength given by the local volume (and with oriented evaluation, since we are on 1-forms).

**Example 6.7** (The Dirichlet decoupling case). In [P05] we proved an approximation result for the Laplacian with Dirichlet boundary conditions on a certain set Xε ⊂ R^2 which is “small” around the vertex neighbourhood. The limit operator on functions in this case is the simple decoupled operator \( \bigoplus_e \Delta^D_e \), i.e., the simple case (i) with
$G = 0$. The index formula in this case leads to
\[ \text{ind } D = -|E|. \]

Note that the index of the Dirichlet Laplacian on $X_\varepsilon$ is the relative Euler characteristic
\[ \chi(X_\varepsilon, \partial X_\varepsilon) = \chi(X_\varepsilon) - \chi(\partial X_\varepsilon) = \chi(X_0) = |V| - |E| \]
in this case, which indicates that the 1-form Laplacian on $X_\varepsilon$ in this case does not converge to the 1-form Laplacian $\Delta^1_{g_{\text{min}}} = \bigoplus_e \Delta^N_e$. We will treat this question also in a forthcoming publication.

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