RIEMANN GEOMETRY IN THEORY OF THE FIRST ORDER SYSTEMS OF EQUATIONS

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Abstract

Theory of Riemann extensions of the spaces with constant affine connection is proposed to study of the properties of nonlinear the first order systems of differential equations.

As example quadratic system of differential equations

\[ \frac{dx}{ds} = P(x, y), \quad \frac{dy}{ds} = Q(x, y), \quad (1) \]

where

\[ P(x, y) = a_0 + a_1 x + a_2 y + a_{11} x^2 + a_{12} xy + a_{22} y^2, \]
\[ Q(x, y) = b_0 + b_1 x + b_2 y + b_{11} x^2 + b_{12} xy + b_{22} y^2, \]

and \( a, b \) are the parameters, is presented in homogeneous form and is considered as geodesic of three-dimensional space with constant affine connection depending on the parameters \( a, b \). After the Riemann extension one get a six-dimensional space and its properties in relation to the parameters are investigated.

The Lorenz system of equations

\[ \frac{dx}{ds} = \sigma (y - x), \quad \frac{dy}{ds} = rx - y - xz, \quad \frac{dz}{ds} = xy - bz \]

after presentation in homogeneous form is considered as geodesic equations of four dimensional space with constant affine connection. Based on the eight-dimensional Riemann extension of a given type space the properties of the Lorenz system are studied.

1 Introduction

The subject of consideration is the first order polynomial systems of differential equations

\[ \frac{dx}{ds} = c^i + a^i_j x^j + b^i_{jk} x^j x^k \quad (2) \]

depending on the parameters \( a, b, c \).

They play an important role in various branches of modern mathematics and its applications.

In particular case of the system of two equations

\[ \frac{dx}{ds} = a_0 + a_1 x + a_2 y + a_{11} x^2 + a_{12} xy + a_{22} y^2, \]
\[ \frac{dy}{ds} = b_0 + b_1 x + b_2 y + b_{11} x^2 + b_{12} xy + b_{22} y^2 \]
there are many unsolved problems.

The spatial first order system of differential equations

\[
\begin{align*}
\frac{dx}{ds} &= P(x, y, z), \\
\frac{dy}{ds} &= Q(x, y, z), \\
\frac{dz}{ds} &= R(x, y, z)
\end{align*}
\]

(3)

with the functions \( P, Q, R \) polynomial on variables \( x, y, z \) are still more complicated object for the
studying of their properties.

As example the studying of the Lorenz system of equations

\[
\begin{align*}
\frac{dx}{ds} &= \sigma(y - x), \\
\frac{dy}{ds} &= rx - y - xz, \\
\frac{dz}{ds} &= xy - bz
\end{align*}
\]

(4)

and the Rössler system

\[
\begin{align*}
\frac{dx}{ds} &= -y - z, \\
\frac{dy}{ds} &= x + ay, \\
\frac{dz}{ds} &= bx - cz + xz
\end{align*}
\]

(5)

which are the simplest examples of the spatial systems have chaotic behaviour at some values of pa-
rameters represent the difficult task.

2 From the first order system of equations to the second order systems of ODE

The systems of the first order differential equations are not suitable object of consideration from the
usually point of Riemann geometry.

The systems of the second order differential equations in form

\[
\frac{d^2x^i}{ds^2} + \Pi^i_{kj}(x)\frac{dx^k}{ds}\frac{dx^j}{ds} = 0
\]

(6)

are best suited to do that.

They can be considered as geodesics of the affinely connected space \( M^k \) in local coordinates \( x^k \). The
values \( \Pi^i_{jk} = \Pi^i_{kj} \) are the coefficients of affine connections on \( M^k \).

With the help of such coefficients can be constructed curvature tensor and others geometrical objects
defined on variety \( M^k \).

There are many possibilities to present a given system of the first order of equations in the form of
(6).

One of them is a following naive presentation.

For the system

\[
\begin{align*}
\frac{dx}{ds} &= P(x, y), \\
\frac{dy}{ds} &= Q(x, y)
\end{align*}
\]

(7)

after differentiation with respect the parameter \( (s) \) we get the second order system of differential equa-
tions of the form (6)

\[
\begin{align*}
\frac{d^2x}{ds^2} &= \frac{1}{P}(P_x\frac{dx}{ds} + P_y\frac{dy}{ds}) \\
\frac{d^2y}{ds^2} &= \frac{1}{Q}(Q_x\frac{dx}{ds} + Q_y\frac{dy}{ds}).
\end{align*}
\]

Such type of the system contains the integral curves of the system (7) as part of its solutions and can be considered as the equations of geodesics of two dimensional space \( M^2(x, y) \) equipment by affine
connections with coefficients

\[
\begin{align*}
\Pi^1_{11} &= -\frac{P_x}{P}, & \Pi^1_{12} &= -\frac{P_y}{2P}, & \Pi^1_{22} &= -\frac{Q_x}{2Q} \\
\Pi^2_{11} &= -\frac{Q_y}{Q}.
\end{align*}
\]

(8)
It is apparent that the properties of the system (7) have an influence on geometry of the variety $M^2(x, y)$.

Remark that a given system is equivalent the second order differential equation

$$\frac{d^2y}{dx^2} = \ln(Q/P)\left(\frac{dy}{dx}\right)^2 + \ln(Q/P)\frac{dy}{dx}$$

which has the solution in form

$$\frac{dy}{dx} = \frac{Q}{P}.$$

By analogy the spatial system of the first order differential equations can be written. As example the Lorenz system of equations is equivalent the second order ODE

$$\frac{d^2y}{dx^2} - \frac{3}{y}\left(\frac{dy}{dx}\right)^2 + \left(\alpha y - \frac{1}{x}\right)\frac{dy}{dx} + cxy^4 - \gamma y^3 - \beta x^3y^4 - \beta x^2y^3 + \delta \frac{y^2}{x} = 0,$$

where

$$\alpha = \frac{b + \sigma + 1}{\sigma}, \quad \beta = \frac{1}{\sigma^2}, \quad \gamma = \frac{b(\sigma + 1)}{\sigma^2}, \quad \delta = \frac{\sigma + 1}{\sigma}, \quad \epsilon = \frac{b(r - 1)}{\sigma^2},$$

which can be obtained by the elimination of variable $z$ from the system

$$\frac{dy}{dx} = \frac{rx - y - xz}{\sigma(y - x)}, \quad \frac{dz}{dx} = \frac{xy - bz}{\sigma(y - x)}.$$

The second order ODE’s

$$\frac{d^2y}{dx^2} + a_1(x, y)\left(\frac{dy}{dx}\right)^3 + 3a_2(x, y)\left(\frac{dy}{dx}\right)^2 + 3a_3(x, y)\frac{dy}{dx} + a_4(x, y) = 0,$$

with arbitrary coefficients $a_i(x, y)$ are form-invariant under the change of the coordinates

$$x = f(u, v), \quad y = h(u, v)$$

and are equivalent to the system

$$\frac{d^2x}{ds^2} - a_3(x, y)\left(\frac{dx}{ds}\right)^2 - 2a_2(x, y)\left(\frac{dx}{ds}\right)\left(\frac{dy}{ds}\right) - a_1(x, y)\left(\frac{dy}{ds}\right)^2 = 0,$$

$$\frac{d^2y}{ds^2} + a_4(x, y)\left(\frac{dx}{ds}\right)^2 + 2a_3(x, y)\left(\frac{dx}{ds}\right)\left(\frac{dy}{ds}\right) + a_2(x, y)\left(\frac{dy}{ds}\right)^2 = 0,$$

having the form of geodesics of two dimensional affinely connected space (with this aim the formula

$$\frac{d^2y}{dx^2} = \frac{\dot{y}\ddot{x} - \ddot{y}\dot{x}}{(\dot{x})^3}$$

was used.

### 3 From the affinely connected space to the Riemann space

Now we shall construct the Riemann space starting from a given affinely connected space defined by the second order ODE’s.

With this aim we use the notion of the Riemann extension of nonriemannian space which was used earlier in the articles of author.

Remind the basic properties of this construction.
With help of the coefficients of affine connection of a given n-dimensional space can be introduced 2n-dimensional Riemann space $D^{2n}$ in local coordinates $(x^i, \Psi_k)$ having the metric of form

$$2^n ds^2 = -2\Pi^k_l(x^i)\Psi_k dx^i dx^j + 2d\Psi_k dx^k$$

(9)

where $\Psi_k$ are additional coordinates.

The important property of such type metric is that the geodesic equations of metric (9) decomposes into two parts

$$\ddot{x}^k + \Pi^k_{ij}\dot{x}^i\dot{x}^j = 0,$$

(10)

and

$$\frac{\delta^2\Psi_k}{ds^2} + R_{kji}^l\dot{x}^i\dot{x}^j\Psi_l = 0,$$

(11)

where

$$\frac{\delta\Psi_k}{ds} = \frac{d\Psi_k}{ds} - \Pi^l_{jk}\frac{dx^l}{ds}\Psi_k$$

and $R_{kji}^l$ are the curvature tensor of n-dimensional space with a given affine connection.

The first part (10) of the full system is the system of equations for geodesic of basic space with local coordinates $x^i$ and it do not contains the supplementary coordinates $\Psi_k$.

The second part (11) of the system has the form of linear $N \times N$ matrix system of second order ODE’s for supplementary coordinates $\Psi_k$

$$\frac{d^2\Psi}{ds^2} + A(s)\frac{d\Psi}{ds} + B(s)\Psi = 0.$$

(12)

Remark that the full system of geodesics has the first integral

$$-2\Pi^k_l(x^i)\Psi_k\frac{dx^i}{ds}\frac{dx^j}{ds} + 2\frac{d\Psi_k}{ds}\frac{dx^k}{ds} = \nu$$

(13)

which is equivalent to the relation

$$2\Psi_k\frac{dx^k}{ds} = \nu s + \mu$$

(14)

where $\mu, \nu$ are parameters.

The geometry of extended space connects with geometry of basic space. For example the property of the space to be Ricci-flat $R_{ij} = 0$ or symmetrical $R_{ijkl} = 0$ keeps also for the extended space.

It is important to note that for extended space having the metric (9) all scalar curvature invariants are vanished.

As consequence the properties of linear system of equation (11-12) depending from the the invariants of $N \times N$ matrix-function

$$E = B - \frac{1}{2}\frac{dA}{ds} - \frac{1}{4}A^2$$

under change of the coordinates $\Psi_k$ can be of used for that.

First applications the notion of extended spaces for the studying of nonlinear second order ODE’s connected with nonlinear dynamical systems have been considered by author (V.Dryuma 2000-2008).

4 Rigorous approach to geometry of planar systems

The system of paths of two-dimensional space $S_2$ in general form looks as

$$\ddot{x} + \Pi^1_{11}(\dot{x})^2 + 2\Pi^1_{12}\dot{x}\dot{y} + \Pi^1_{22}(\dot{y})^2 = 0,$$

$$\ddot{y} + \Pi^2_{11}(\dot{x})^2 + 2\Pi^2_{12}\dot{x}\dot{y} + \Pi^2_{22}(\dot{y})^2 = 0,$$

where the coefficients $\Pi^k_{ij} = \Pi^k_{ji}$. 
The Riemann extension of the space $S^2$ is determined by the metric

$$4ds^2 = -2z\Pi^1_{11}dx^2 - 2t\Pi^1_{12}dx^2 - 4z\Pi^1_{12}dxdy - 4t\Pi^2_{12}dxdy -$$

$$-2z\Pi^2_{22}dy^2 - 2\Pi^2_{22}dxdy + 2dxdz + 2dydt.$$ 

We shall apply such system of equations for the studying of the first order planar system of ODE’s. With this aim we use the facts about the geometries of paths for which the equations of the paths admits independent linear first integrals.

L.P.Eisenhart, 1925

A necessary condition to geodesic admit the linear first integral

$$a_i(x, y) \frac{dx^i}{ds} = \text{const}$$

is

$$a_{ij} + a_{ji} = 0,$$

where

$$a_{ij} = \frac{\partial a_i}{\partial x^j} - a_k \Gamma_{ij}^k,$$

and $\Gamma_{ij}^k$ are the Christoffel symbols of the metric.

We apply this conditions and their consequence

$$a_{ij;k} + R_{ijkl}^m a_m = 0$$

where $R_{ijkl}$ is the curvature tensor of the space to determination of the coefficients of equations $\Pi_{ij}^k$ using the vector $a_i$ in form

$$a_i = [Q(x, y), -P(x, y), 0, 0].$$

By this means that the first order of equation

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}$$

or

$$Q(x, y)dx - P(x, y)dy = 0$$

is an integral of the paths equations.

With the help of these conditions it is possible to state only three coefficients of affine connections $\Pi_{ij}^k$.

For determination of others coefficients we use yet another the first order equation

$$\frac{dy}{dx} = -\frac{y(y-1)}{x(x-1)}$$

with a first integral

$$y(x) = \frac{C(x-1)}{x-C}.$$ 

In this case a second vector $b_i$ is in form

$$b_i = [y(y-1), x(x-1), 0, 0].$$

The equation plays an important role in theory of of planar the first order system of equations and was used in a famous article of Petrovsky-Landis (1956) ([?]).
5 THE SECOND ORDER ODE’S CUBIC ON THE FIRST DERIVATIVE

So, from the conditions on the metrics to admits two linear first integrals the coefficients $\Pi_{ij}^{1}$ of the paths equation are uniquely determined and have the form

$$\Pi_{11}^{1} = \frac{(\frac{\partial}{\partial y} Q(x, y)) x (x - 1)}{y^2 P(x, y) - y P(x, y) + Q(x, y)x^2 - Q(x, y)x}$$

$$\Pi_{22}^{1} = -\frac{(\frac{\partial}{\partial y} P(x, y)) x (x - 1)}{y^2 P(x, y) - y P(x, y) + Q(x, y)x^2 - Q(x, y)x}$$

and corresponding expressions for the coefficients $\Pi_{11}^{2}, \Pi_{12}^{2}, \Pi_{22}^{2}$

Remark that last two equations are reduced at the independent equations

$$\frac{d^2 z}{ds^2} + M(s) \frac{dz}{ds} + N(s)z(s) + F(s) = 0$$

and

$$\frac{d^2 t}{ds^2} + U(s) \frac{dt}{ds} + V(s)t(s) + H(s) = 0$$

with the help of the first integral of geodesics

$$z(s) \frac{dx}{ds} + t(s) \frac{dy}{ds} - \frac{\alpha}{2} s - \beta = 0$$

of the metric (18).

It is interested to note that such type of non homogeneous linear second order ODE’s are connected with theory of first order systems of ODE’s as the equations on the periods of corresponding Abel integrals.

5 The second order ODE’s cubic on the first derivative

The first two equations of geodesic of the metric are equivalent to the one second order differential equation

$$\frac{d^2}{dx^2} y(x) + \left(\left(\frac{\partial}{\partial y} P(x, y)\right) x^2 - \left(\frac{\partial}{\partial y} P(x, y)\right) x \right) \left(\frac{dy}{dx} y(x)\right)^3 +$$

$$\left(\frac{\partial}{\partial y} P(x, y)\right) y^2 P(x, y) - y P(x, y) + Q(x, y)x^2 - Q(x, y)x + (P_x - Q_y) x^2 + (Q_y - P_x - 2 P) x + (P_y) y^2 + (-2 P - P_y) y + 2 P \right) \left(\frac{dy}{dx}\right)^2 +$$

$$\left((-Q_x) x^2 + (Q_x + 2 Q) x + (P_x - Q_y) y^2 + (2 Q - P + Q_y) y - 2 Q\right) \frac{d}{dx} y +$$

$$\left(\frac{\partial}{\partial y} Q(x, y)\right) y^2 + \left(\frac{\partial}{\partial y} Q(x, y)\right) y \right) \frac{dy}{dx} + \left(\frac{\partial}{\partial y} P(x, y)\right) y^2 P(x, y) - y P(x, y) + Q(x, y)x^2 - Q(x, y)x = 0. \quad (14)$$

The equation (14) is of the form

$$\frac{d^2 y}{dx^2} + a_1(x, y) \left(\frac{dy}{dx}\right)^3 + 3a_2(x, y) \left(\frac{dy}{dx}\right)^2 + 3a_3(x, y) \frac{dy}{dx} + a_4(x, y) = 0,$$

and it has the invariants depending from the coefficients $a_i(x, y)$ (R.Liouville, 1880, T.Tresse, 1886) under transformations of variables $(x, y)$.

As it was shown by (E.Cartan, 1924) these invariants in general case are same with the invariants of two-dimensional surface $V^2(x, y)$ in a four-dimensional projective space $RP^4(\xi^i)$ (under the reparametrization and the change of coordinates $\xi^i$).
For example the invariants of equations with condition on coefficients \( \nu_5 = 0 \) same with the invariants of developing surfaces in a three-dimensional projective space \( RP^3 \).

The invariants of R.Liouville have been used successful in theory of ODE’s and their applications in works author (V.Dryuma, 1984-). In particular the second order ODE with a Painleve property have the condition \( \nu_5 = 0 \).

In our case the equation has the particular integral

\[
\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}
\]

and the function

\[ y(x) = \frac{C(x-1)}{x-C} \]

as the first integral.

From geometrical point of view the equation (14) corresponds two-dimensional surface in the \( RP^4 \)-space.

In this context it is interested to note the relation with the Petrovsky-Landis theory of limit cycles of the equation

\[
\frac{dy}{dx} = \frac{a_0 + a_1 x + a_2 y + a_{11} x^2 + a_{12} xy + a_{22} y^2}{b_0 + b_1 x + b_2 y + b_{11} x^2 + b_{12} xy + b_{22} y^2}.
\]

In the famous article of \( P - L \) was developed approach to the studying of the problem of the limit cycles of the first order quadratic equation.

Let us recall the basic facts of the Petrovsky-Landis theory.

For studying of the closed curves of quadratic the first order equation is considered the equation

\[
\frac{dy}{dx} = \frac{y(y-1)}{x(x-1)}
\]

with solution

\[ y(x) = \frac{C(x-1)}{(x-C)} \]

As it was showed for the closed curves the parameter \( C \) satisfies the algebraic equations

\[
\sum a_n(\mu_i)C^n = 0
\]

where the coefficients \( a_n(\mu_i) \) are dependent from the parameters of the quadratic equation.

Such type of equation arises from the condition

\[
\int_c \frac{(x-C)^2[x(x-1)P(x,y) + y(y-1)Q(x,y)]}{x^3(x-1)^3} dx = 0,
\]

which lead to determination of the closed integral curves of the first order system of equations

\[
\frac{dy}{dx} = \frac{a_0 + a_1 x + a_2 y + a_{11} x^2 + a_{12} xy + a_{22} y^2}{b_0 + b_1 x + b_2 y + b_{11} x^2 + b_{12} xy + b_{22} y^2}.
\]

As it was shown by Petrovsky-Landis in this case the parameter \( C \) is determined from calculations of the residues of integral after substitution of the expression

\[ y = \frac{C(x-1)}{(x-C)} \]

in it.

The simplest of them are
The function defined by the equation is equal three.

According to the results of \((P - L)\) general quantity of the values \(C\) defined by a such type of equations is equal 14 and this number coincides with the quantity of closed solutions determined by the quadratic equation.

On other side 11 curves from 14 can be transformed into the small neighborhood of the essential singular points of the equation

\[
\frac{dy}{dx} = -\frac{y(y - 1)}{x(x - 1)}.
\]

They are: \((0, 1), (1, 0), (0, 0), (1, 1)\) and so on...

As result only three closed curves do not be transformed into the neighborhood of the singular points and so the quantity of the limit cycles defined by the equation is equal three.

It is interested to note that some conditions on the parameter \(C\) are appeared in context of the second order ODE.

In fact the result of joint consideration of the first order equations

\[
y' = \frac{Q(x, y)}{P(x, y)}, \quad y' = \frac{y(y - 1)}{x(x - 1)}
\]

the function

\[
y(x) = \frac{C(x - 1)}{(x - C)}
\]

and the second order ODE (13) we get the conditions

\[
\alpha(x, y)C^5 + \beta(x, y)C^4 + \gamma(x, y)C^3 + \delta(x, y)C^2 + \epsilon(x, y)C + \mu(x, y) = 0,
\]

where

\[
\alpha(x, y) = (a_{12} + b_{12})y^2 + \\
+ (b_1 + b_2 + (2a_{22} + 2a_{11} + 2b_{22} + 2b_{11})x + a_2 + a_1)y + \\
+ (a_{12} + b_{12})x^3 + (a_1 + b_2 + b_1 + a_2)x + 2a_0 + 2b_0,
\]

\[
\beta(x, y) = 2b_{22}y^3 + ((-5a_{12} - b_{12} - 2b_{22})x + 2b_2 - b_{12})y^2 + \\
+ ((-4a_{22} - 3b_2 - 5a_1 + 3a_2 - 2b_{11} - 6b_{22} - b_1)x - a_2) + \\
+ ((-4b_{11} + a_{12} - 10a_{11} - b_{12} - 4a_{22})x^2 + 2b_0 - 2b_2 - b_1)y + \\
+ (-2a_{12} + 2a_{11})x^3 + (-2a_{12} - 2b_1 - 2a_1 - 3b_{12} - 2a_2)x^2 - 4b_0 + \\
+ (-6a_0 - 2b_1 - 4b_0 - 3b_2 - a_1 - 2a_2)x - 2a_0,
\]

\[
\mu(x, y) = -y^2x^5a_{12} + (-2x^6a_{11} + (a_{12} - a_1)x^5)y + 2x^6a_{11} + x^5a_1.
\]
From these relation we get the relations between the parameter $C$ and coefficients $a_{ij}$, $b_{ij}$, $a_i$, $b_i$
As example at the values $x = 0$ and $x = 1$ we get conditions on the value $C$

\[
\begin{align*}
(a2 + a12 + b1 + b12 + b2 + 2a0 + a1 + 2b0)C^2 + \\
+ (-2b0 - 2a0 - b1 - a2 - b12 + 2b22)C - b2 - 2b22 = 0, \\
(a2 + a12 + b1 + b12 + b2 + 2a0 + a1 + 2b0)C^2 + \\
+ (2a11 - 2b0 - b1 - 2a0 - a12 - a2)C - 2a11 - a1 = 0.
\end{align*}
\]

The substitution $(x = C, y = 1 - C)$ lead to the conditions on the value $C$

\[
(b12 - 2b22)C + 2b22 + b2 = 0.
\]

After substitution $(y = 1 - x, x = 1, C = 1/C1)$ we get

\[
(\begin{align*}
(a1 + 2a11)C1^2 + (a12 - 2a11 + b1 + a2 + 2a0 + 2b0)C1 - \\
- a1 - a2 - a12 - 2b0 - b12 - b1 - b2 - 2a0 = 0,
\end{align*}
\]

and the substitution $(y = 1 - x, x = 0, C = 1/C1)$ lead to the condition

\[
(-2b22 - b2)C1^2 + (-2b0 - b12 - b1 - a2 - 2a0 + 2b22)C1 + \\
+ a1 + a2 + a12 + 2b0 + b12 + b1 + b2 + 2a0 = 0.
\]

In result of a such type consideration we have got the conditions of Petrovskii-Landis article on parameter $C$.

6 Three dimensional homogeneous system

As it was shown in article (V.Dryuma,2006) between the planar system

\[
\begin{align*}
\frac{dx}{dt} &= a_0 + a_1x + a_2y + a_{11}x^2 + a_{12}xy + a_{22}y^2, \\
\frac{dy}{dt} &= b_0 + b_1x + b_2y + b_{11}x^2 + b_{12}xy + b_{22}y^2,
\end{align*}
\]

and a spatial homogeneous quadratic system of equations

\[
\begin{align*}
\frac{dx}{dt} &= P(x, y, z), \\
\frac{dy}{dt} &= Q(x, y, z), \\
\frac{dz}{dt} &= Q(x, y, z)
\end{align*}
\]

of the form

\[
\begin{align*}
\frac{dx}{dt} &= 4a_0z^2 + 4a_2yz + (3a_1 - b_2)xz + 4a_{22}y^2 + (3a_{12} - 2b_{22})xy + (2a_{11} - b_{12})x^2, \\
\frac{dy}{dt} &= 4b_0z^2 + 4b_1xz + (3b_2 - a_1)yz + 4b_{11}x^2 + (3b_{12} - 2a_{11})xy + (2b_{22} - a_{12})y^2, \\
\frac{dz}{dt} &= -(a_1 + b_2)z^2 - (2b_{22} + a_{12})yz - (2a_{11} + b_{12})xz
\end{align*}
\]

exists some connections.

For a such system the condition

\[
\frac{\partial P(x, y, z)}{\partial x} + \frac{\partial Q(x, y, z)}{\partial y} + \frac{\partial R(x, y, z)}{\partial z} = 0
\]
is fulfilled and in variables
\[ \xi(t) = \frac{x(t)}{z(t)}, \quad \eta(t) = \frac{y(t)}{z(t)} \]
it takes the form of the equation
\[ \frac{d\xi}{d\eta} = \frac{a_0 + a_1\xi + a_2\eta + a_{11}\xi^2 + a_{12}\xi\eta + a_{22}\eta^2}{b_0 + b_1\xi + b_2\eta + b_{11}\xi^2 + b_{12}\xi\eta + b_{22}\eta^2} \]
equivalent the planar system.

It is significant that the spatial system in the variables
\[ 7 \text{ Six-dimensional Riemann space} \]
For the
\[ 7 \text{ Six-dimensional Riemann space} \]

\[ \chi = (x, y, z) \]

in local coordinates
\[ \chi = (x, y, z) \]

takes the form of geodesic equations of a three dimensional space
\[ \frac{d^2X^i}{dt^2} + \Gamma^i_{jk} \frac{dX^j}{dt} \frac{dX^k}{dt} = 0 \quad (15) \]
in local coordinates \( X^i = (X(t), Y(t), Z(t)) \).

In doing so the coefficient \( \Gamma^i_{jk} \) are constant and depend on the parameters \( a_i, a_{ij}, b_i, b_{ij} \).
The set of such coefficients can be considered as the coefficients of an affine connection on a three dimensional space \( H_3 \) in local coordinates \( X^i \).

From geometrical point of view through the equations (5) on the space \( H_3 \) the structure of an affinely connected space with constant coefficients of connection \( \Gamma^i_{jk} \) is determined.

### 7 Six-dimensional Riemann space

For the \( n \)-dimensional space equipped with an affine connection \( \Gamma^i_{jk} \) the metrics of the Riemann extension \( V_{2n} \) has the form
\[ 2^n ds^2 = -2\Gamma^i_{jk} dX^j dX^k d\xi^i - 2d\xi_i dX^i \quad (16) \]
where \( \chi_i \) are additional coordinates.

Non zero coefficients of affine connections of the space \( H_3 \) are
\[ \Gamma^1_{33} = -4a_0, \quad \Gamma^1_{23} = -2a_2, \quad \Gamma^1_{13} = \frac{1}{2}(b_2 - 3a_1), \quad \Gamma^1_{22} = -4a_22, \]
\[ \Gamma^1_{12} = \frac{1}{2}(2b_2 - 3a_{12}), \quad \Gamma^1_{11} = b_{12} - 2a_{11}, \]
\[ \Gamma^2_{33} = -4b_0, \quad \Gamma^2_{23} = -2b_2, \quad \Gamma^2_{22} = a_{11} - 3b_2, \quad \Gamma^2_{11} = -4b_{11}, \]
\[ \Gamma^2_{12} = \frac{1}{2}(2a_{11} - 3b_{12}), \quad \Gamma^2_{22} = a_{12} - 2b_{22}, \]
\[ \Gamma^2_{13} = a_1 + b_2, \quad \Gamma^2_{23} = \frac{1}{2}(a_{12} + 2b_{22}), \quad \Gamma^3_{13} = \frac{1}{2}(2a_{11} + b_{12}). \]

According with the (16) the Riemann metric of six dimensional extended space \( V_6 \) has the form
\[ 6 ds^2 = -2Ub_{12} + 4u_{a11} + 8b_{11} V) dx^2 + (8a_22 U - 2 Va_{12} + 4Vb_{22}) dy^2 + \]
\[ + (8b0 V - 2Wb2 + 8a0 U - 2 Wa1) dz^2 + \]
\[ + u_{a22} - 4Va_{12} + 6Vb_{12} + 6u_{a22} dz dy + \]
\[ + (6ua1 - 2Ub2 - 4Wa11 - 2Wb12 + 8b1 V) dx dz + \]
\[ + (2Wa12 + 8a2U + 6Vb2 - 2Va1 - 4Wb22) dy dz + 2 dx dU + 2 dy dV + 2 dz dW. \]
Remark that we use the denotes \(x, y, z\) for the coordinates \((X, Y, Z)\).

Geodesic of the metric consist from two parts. Nonlinear system of coupled equations on coordinates \((x, y, z)\)

\[
\frac{d^2}{ds^2} x(s) + ( -2a11 + b12 ) \left( \frac{d}{ds} x(s) \right)^2 + 2 \left( -3/2a12 + b22 \right) \left( \frac{d}{ds} x(s) \right) \frac{d}{ds} y(s) + \\
+ 2 \left( -3/2a1 + 1/2b2 \right) \left( \frac{d}{ds} x(s) \right) \frac{d^2}{ds^2} y(s) - 4a22 \left( \frac{d}{ds} y(s) \right)^2 - 4a2 \left( \frac{d}{ds} y(s) \right) \frac{d}{ds} z(s) - 4a0 \left( \frac{d}{ds} z(s) \right)^2 = 0,
\]

\[
\frac{d^2}{ds^2} y(s) - 4b11 \left( \frac{d}{ds} x(s) \right)^2 + 2 \left( -3/2b12 + a11 \right) \left( \frac{d}{ds} x(s) \right) \frac{d}{ds} y(s) - 4b1 \left( \frac{d}{ds} y(s) \right) \frac{d}{ds} z(s) + \\
+ (-2b22 + a12) \left( \frac{d}{ds} y(s) \right)^2 + 2 \left( -3/2b2 + 1/2a1 \right) \left( \frac{d}{ds} y(s) \right) \frac{d}{ds} z(s) - 4b0 \left( \frac{d}{ds} z(s) \right)^2 = 0,
\]

\[
\frac{d^2}{ds^2} z(s) + 2 \left( a11 + 1/2b12 \right) \left( \frac{d}{ds} x(s) \right) \frac{d^2}{ds^2} z(s) + 2 \left( 1/2a12 + b22 \right) \left( \frac{d}{ds} y(s) \right) \frac{d}{ds} z(s) + \left( a1 + b2 \right) \left( \frac{d}{ds} z(s) \right)^2 = 0.
\]

And the linear system of equations on coordinates \((U, V, W)\)

\[
\frac{d^2}{ds^2} U(s) + A1 \left( \frac{d}{ds} U(s) \right) + B1 \left( \frac{d}{ds} V(s) \right) + C1 \left( \frac{d}{ds} W(s) \right) + E1 U(s) + F1 V(s) + H1 W(s) = 0
\]

\[
\frac{d^2}{ds^2} V(s) + A2 \left( \frac{d}{ds} U(s) \right) + B2 \left( \frac{d}{ds} V(s) \right) + C2 \left( \frac{d}{ds} W(s) \right) + E2 U(s) + F2 V(s) + H2 W(s) = 0
\]

\[
\frac{d^2}{ds^2} W(s) + A3 \left( \frac{d}{ds} U(s) \right) + B3 \left( \frac{d}{ds} V(s) \right) + C3 \left( \frac{d}{ds} W(s) \right) + E3 U(s) + F3 V(s) + H3 W(s) = 0,
\]

where the coefficients \((A_i, B_i, C_i)\) are depended from the parameters \((a, b)\) and derivatives \((\dot{x}, \dot{y}, \dot{z})\) with respect to the parameter \(s\).

Taking in consideration \(x, y, z\)-equations the \(U, V, W\)-equations can be one time integrated and take the form

\[
\frac{d}{ds} U(s) = (2a11 + b12) z(s) W(s) + \\
+ \left( (-3b12 + 2a11) y(s) - 8b11 x(s) - 4b1 z(s) \right) V(s) + \\
+ \left( (2b12 - 4a11) x(s) + (-3a12 + 2b22) y(s) + (b2 - 3a1) z(s) \right) U(s),
\]

\[
\frac{d}{ds} V(s) = (a12 + 2b22) z(s) W(s) + \\
+ \left( (-3a12 + 2b22) x(s) - 8a22 y(s) - 4a2 z(s) \right) U(s) + \\
+ \left( (-3b12 + 2a11) x(s) + (2a12 - 4b22) y(s) + (-3b2 + a1) z(s) \right) V(s),
\]

\[
\frac{d}{ds} W(s) = \left( (b2 - 3a1) x(s) - 8a0 z(s) - 4a2 y(s) \right) U(s) + \\
+ \left( (-3b2 + a1) y(s) - 8b0 z(s) - 4b1 x(s) \right) V(s) + \\
+ \left( (2a11 + b12) x(s) + (a12 + 2b22) y(s) + 2 (a1 + b2) z(s) \right) W(s).
\]

In result we have got a six-dimensional Riemann space associated with a quadratic the first order system of equations.

An investigation the properties of the metric at the change of parameters \(a, b\) may be useful for the theory of a such type of the systems.

In particular a study of the Killing properties of the metric allow us to get information on particular integrals of geodesic equations.
8 On the surfaces defined by spatial system of equations

The equation of surfaces \( z = z(x, y) \) defined by the spatial system of equations \((4)\) has the form of the first order p.d.e:

\[
z_x \left( 4a \theta \ (z)^2 + (4a_2 y + (3a_1 - b_2) x) z + 4a_22 y^2 \right) + z_y \left( (3a_12 - 2b_22) xy + (2a_11 - b_12) x^2 \right) + \\
+ z_y \left( 4b_0 z^2 + (3b_2 - al) yz + 4b_11 x^2 + 4b_1 xz \right) + z_y \left( (2b_22 - al2) y^2 + (-2a_11 + 3b_12) xy \right) + \\
+ (b_2 + al) \ (z)^2 + ((2b_22 + al2) y + (2a_11 + b_12) x) z. \tag{17}
\]

An examples of solutions of this equation were obtained by the method of \((u,v)\)-transformation developed earlier by author.

To integrate the partial nonlinear first order differential equation

\[
F(x, y, z(x, y), z_x, z_y) = 0 \tag{18}
\]

we use a following change of the functions and variables

\[
z(x, y) \rightarrow u(x, t), \quad y \rightarrow v(x, t), \quad z_x \rightarrow u_x - \frac{v_x}{v_t} u_t, \quad v_y \rightarrow \frac{u_t}{v_t}. \tag{19}
\]

In result instead of the equation \((18)\) one get the relation between the new variables \(u(x, t)\) and \(v(x, t)\) and their partial derivatives

\[
\Phi(u, v, u_x, u_t, v_x, v_t) = 0. \tag{20}
\]

In many cases the solution of last equation is a more simple problem than solution of the equation \((18)\).

In result of application of the \((u,v)\)-transformation the equation for \(z = z(x, y)\) takes the form of \((20)\).

Using the substitution

\[
u(x, t) = t \frac{\partial}{\partial t} \omega(x, t) - \omega(x, t), \quad v(x, t) = \frac{\partial}{\partial t} \omega(x, t),
\]

where \(\omega(x, t) = xA(t)\), we find from a given relation the equation for the function \(A(t)\)

\[
\left( -t^2b_2 - tb_22 + a_22 A(t) - t^3b_0 + a_0 A(t)t^2 + a_2 t A(t) \right) \left( \frac{d}{dt} A(t) \right)^2 + \\
+ \left( -2a_0 \ (A(t))^2 t - tb_12 - a_2 \ (A(t))^2 - t^2b_1 + ta_1 A(t) \right) \frac{d}{dt} A(t) + \\
+ \left( 2t^2b_0 A(t) + tb_2 A(t) + a_12 A(t) \right) \frac{d}{dt} A(t) - \\
- tb_11 - tb_0 \ (A(t))^2 - a_1 \ (A(t))^2 + tb_1 A(t) + a_11 A(t) + a_0 \ (A(t))^3 = 0.
\]

Solutions of this equation depend from the parameters \(a, \ b\) and can play a key value in theory of the quadratic systems.

Another type of substitution

\[
v(x, t) = t \frac{\partial}{\partial t} \omega(x, t) - \omega(x, t), \quad u(x, t) = \frac{\partial}{\partial t} \omega(x, t),
\]

where \(\omega(x, t) = xB(t)\), lead to the equation on the function \(B(t)\)

\[
\left( a_22 B(t)t^2 + a_2 B(t)t + a_0 B(t) + b_22 t^2 + tb_2 + b_0 \right) \left( \frac{d}{dt} B(t) \right)^2 +
\]

\[
- tb_11 - tb_0 \ (B(t))^2 - a_1 \ (B(t))^2 + tb_1 B(t) + a_11 B(t) + a_0 \ (B(t))^3 = 0.
\]
which is a more simple than previous and also can be useful in theory of quadratic systems.

Let us consider some examples.

In the variables

\[ B(t) = S, \quad \frac{dB(t)}{dt} = T \]

the equation (21) determines algebraic curve

\[ F(S, T, t, a, b) = 0 \]

having genus \( g = 1 \) or \( g = 0 \) in depending on the parameters \( a, b \).

As example at the conditions

\[ b_0 = 0, \quad a_0 = 0, \quad b_2 = 0, \quad a_1 = 0, \quad a_2 = 0, \quad a_{22} = 0, \quad b_{11} = 0, \quad a_{12} = 0, \quad a_{11} = b_{12}, \quad b_1 = 0 \]

the equation (21) takes the form

\[ b_{22} t^2 \left( \frac{d}{dt} A(t) \right)^2 + \left( -2 b_{22} t A(t) + tb_{12} \right) \frac{d}{dt} A(t) + b_{22} (A(t))^2 = 0 \]

and defines algebraic curve of the genus \( g = 0 \).

Integral of this equation is defined by the relation

\[
\ln(t) + \ln\left( -4 b_{22} B(t) b_{12} + b_{12}^2 - b_{12} \right) - \frac{b_{12}}{-4 b_{22} B(t) b_{12} + b_{12}^2 - b_{12}} - \\
\ln\left( -4 b_{22} B(t) b_{12} + b_{12}^2 + b_{12} \right) - \frac{b_{12}}{-4 b_{22} B(t) b_{12} + b_{12}^2 + b_{12}} - \frac{1}{2} \frac{b_{12}}{b_{22} B(t)} - \ln(B(t)) - C_1 = 0.
\]

After the inverse \((u, v)\)-transformation with the help of the function \( B(t) \) can be find the solution of the equation (17) at indicated above the values of parameters.

9 Eight-dimensional Riemann space for the Lorenz system of equations

To investigation of the properties of classical Lorenz equations

\[
\frac{dx}{ds} = \sigma(y - x), \quad \frac{dy}{ds} = rx - y - xz, \quad \frac{dz}{ds} = -bz + xy \tag{22}
\]

we use its presentation in the form

\[
\frac{d}{ds} \xi(s) = 1/5 (b - 4 \sigma + 1) \xi \rho + \sigma \eta \rho, \\
\frac{d}{ds} \eta(s) = -\xi \theta + r \xi \rho + 1/5 (b + \sigma - 4) \eta \rho, \\
\frac{d}{ds} \theta(s) = \xi \eta + 1/5 (\sigma - 4b + 1) \rho \theta,
\]
\[
\frac{d}{ds} \rho(s) = 1/5 \left(\sigma + 1 + b\right) \rho^2.
\]

The relation between both systems is defined by the conditions
\[
x(s) = \frac{\xi}{\rho}, \quad y(s) = \frac{\eta}{\rho}, \quad z(s) = \frac{\theta}{\rho}.
\]

Four dimensional system can be presented in the form
\[
\frac{d^2X^i}{ds^2} + \Gamma_{jk}^i \frac{dX^j}{ds} \frac{dX^k}{ds} = 0,
\]
which allow us to consider it as geodesic equations of the space with constant affine connection.

Nonzero components of connection are
\[
\Gamma_{14}^1 = 4\sigma - b - 2, \quad \Gamma_{24}^1 = -\frac{\sigma}{2}, \quad \Gamma_{13}^2 = 1/2,
\]
\[
\Gamma_{14}^2 = -\frac{r}{2}, \quad \Gamma_{34}^2 = -\frac{4 - \sigma - b}{10}, \quad \Gamma_{34}^3 = \frac{4b - \sigma - 1}{10},
\]
\[
\Gamma_{12}^3 = -\frac{1}{2}, \quad \Gamma_{44}^4 = -\frac{\sigma + b + 1}{5}.
\]

The metric of associated space is
\[
8 ds^2 = 2/5 \left(b + \sigma + 1\right) du^2 V +
+ 2 dx du + 2 dx dy - 8/5 dx du b + 2/5 dz du \sigma) U +
+ 2 r dx du + 2/5 dz du \sigma - 2 dx dz - 8/5 dz du + 2/5 dz du b) Q +
+ (-8/5 dx du \sigma + 2 \sigma dy du + 2/5 dx du b + 2/5 dx du) P +
+ 2 dy dQ + 2 dz dU + 2 du dV + 2 dx dP.
\]

After integration geodesic of additional coordinates take the form
\[
\frac{d}{dt} P(t) - (z(t) - ru(t)) Q(t) - \frac{1}{5} (-b - 1 + 4\sigma) u(t) P(t) + y(t) U(t) = 0,
\]
\[
\frac{d}{dt} Q(t) + x(t) U(t) + \sigma u(t) P(t) = 0,
\]
\[
\frac{d}{dt} U(t) + \frac{1}{5} ((\sigma + b - 4) u(t) - x(t)) Q(t) + \frac{1}{5} (-4b + 1 + \sigma) u(t) U(t) = 0,
\]
\[
\frac{d}{dt} V(t) + 1/5 \left(b + \sigma + 1\right) u(t) V(t) - \frac{x(t) u(t) U(t)}{u(t)} + \frac{x(t) Q(t) z(t)}{u(t)} = 0.
\]

Properties of this system from the parameters are dependent and can be investigated with the help of the Wilczynski invariants.
10 Laplace operator

In theory of Riemann spaces the equation

\[ L\psi = g^{ij}(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial}{\partial x^k})\psi(x) = 0 \]  

(24)

can be used to the study of the properties of the space.

For the eight-dimensional space with metric (23) corresponded the Lorenz system we get the equation on the function \( \psi(x, y, z, u, P, Q, U, V) \)

\[ -2U \frac{\partial^2}{\partial P\partial Q} \psi + 1/5 \left( \frac{\partial}{\partial V} \psi \right) \sigma + 2Q \frac{\partial^2}{\partial P\partial U} \psi - 2/5 \left( \frac{\partial}{\partial W} \psi \right) P\sigma - 2/5 \left( \frac{\partial}{\partial W} \psi \right) U\sigma + 2/5 \left( \frac{\partial}{\partial W} \psi \right) Q\sigma = 0. \]

This equation has varies type of particular solutions.

As example

\[ \psi(P, Q, U, V) = H(P, Q, U) + VP \]

where the function \( \psi(P, Q, U, V) \) satisfies the equation

\[ 2Q \frac{\partial^2}{\partial P\partial U} H(P, Q, U) + 9/5 P \sigma - 2U \frac{\partial^2}{\partial P\partial Q} H(P, Q, U) - 1/5 P b - 2 r Q - 6/5 P = 0. \]

Its solution is in form

\[ H(P, Q, U) = \left(-3/10 - 1/20 b + \frac{9}{20} \sigma\right) \arctan(\frac{Q}{U}) P^2 + P r U + F1(Q, U) + F2(P, Q^2 + U^2) \]

where \( F_1 \) are arbitrary functions.

At the condition

\[ b = 9 \sigma - 6 \]

this solution takes a more simple form.

In the case

\[ \psi(P, Q, U, V) = H(P, Q) + V \]

we get the solution

\[ H(P, Q) = F2(P) + F1(Q) + \frac{(1/10 b - 2/5 + 1/10 \sigma) PQ}{U} \]

for which the relation

\[ b = 4 - \sigma \]

is a special.
11 EIKONAL EQUATION

11 Eikonal equation

Solution of the eikonal equation

$$g^{ij} \frac{\partial F}{\partial x^i} \frac{\partial F}{\partial x^j} = 0$$

also give us useful information about the properties of Riemann space.

In the case of the space with the metric (23) we get the equation

$$2 \left( \frac{\partial}{\partial x} \right) \frac{\partial}{\partial P} \phi + 2 \left( \frac{\partial}{\partial y} \right) \frac{\partial}{\partial Q} \phi + 2 \left( \frac{\partial}{\partial z} \right) \frac{\partial}{\partial U} \phi + 2 \left( \frac{\partial}{\partial u} \right) \frac{\partial}{\partial V} \phi - 2 U \left( \frac{\partial}{\partial P} \phi \right) \frac{\partial}{\partial Q} \phi +$$

$$+ 2 Q \left( \frac{\partial}{\partial P} \phi \right) \frac{\partial}{\partial U} \phi + 8/5 \left( \frac{\partial}{\partial P} \phi \right) P \sigma - 2/5 \left( \frac{\partial}{\partial P} \phi \right) P b - 2/5 \left( \frac{\partial}{\partial P} \phi \right) P -$$

$$- 2 \left( \frac{\partial}{\partial P} \phi \right) \frac{\partial}{\partial U} \phi + 8/5 \left( \frac{\partial}{\partial U} \phi \right) P \sigma - 2/5 \left( \frac{\partial}{\partial U} \phi \right) U \sigma -$$

$$- 2/5 V \left( \frac{\partial}{\partial V} \phi \right)^2 b - 2/5 V \left( \frac{\partial}{\partial V} \phi \right)^2 \sigma - 2/5 V \left( \frac{\partial}{\partial V} \phi \right)^2 = 0.$$

In particular case

$$\phi(x, y, z, u, P, Q, U, V) = A(Q, U, V) + PU$$

one get the equations to determination of the function $A(Q, U, V)$

$$- 2 U^2 \frac{\partial}{\partial Q} A(Q, U, V) + 2 QU \frac{\partial}{\partial U} A(Q, U, V) + 2 QU P + 6/5 U \left( \frac{\partial}{\partial V} A(Q, U, V) \right) P \sigma + 6/5 U \left( \frac{\partial}{\partial V} A(Q, U, V) \right) P b -$$

$$- 4/5 U \left( \frac{\partial}{\partial V} A(Q, U, V) \right) P - 2 U \left( \frac{\partial}{\partial V} A(Q, U, V) \right) r Q - 2 \sigma P \left( \frac{\partial}{\partial Q} A(Q, U, V) \right) \frac{\partial}{\partial V} A(Q, U, V) +$$

$$+ 8/5 \left( \frac{\partial}{\partial U} A(Q, U, V) \right) \frac{\partial}{\partial V} A(Q, U, V) Q + 8/5 \left( \frac{\partial}{\partial U} A(Q, U, V) \right) Q P -$$

$$- 2/5 \left( \frac{\partial}{\partial U} A(Q, U, V) \right) \frac{\partial}{\partial V} A(Q, U, V) Q \sigma - 2/5 \left( \frac{\partial}{\partial U} A(Q, U, V) \right) Q \sigma P -$$

$$- 2/5 \left( \frac{\partial}{\partial U} A(Q, U, V) \right) \frac{\partial}{\partial V} A(Q, U, V) Q b - 2/5 \left( \frac{\partial}{\partial U} A(Q, U, V) \right) Q b P +$$

$$- 8/5 \left( \frac{\partial}{\partial U} A(Q, U, V) \right) \frac{\partial}{\partial V} A(Q, U, V) U b - 2/5 \left( \frac{\partial}{\partial U} A(Q, U, V) \right) \frac{\partial}{\partial V} A(Q, U, V) U \sigma -$$

$$- 2/5 \left( \frac{\partial}{\partial U} A(Q, U, V) \right) \frac{\partial}{\partial V} A(Q, U, V) U - 2/5 V \left( \frac{\partial}{\partial V} A(Q, U, V) \right)^2 b - 2/5 V \left( \frac{\partial}{\partial V} A(Q, U, V) \right)^2 \sigma -$$

$$- 2/5 V \left( \frac{\partial}{\partial V} A(Q, U, V) \right)^2 = 0.$$

The solution of this equation is

$$A(Q, U, V) =$$

$$= \mathcal{F}00(U) \mathcal{F}3 V + 1/2 \frac{Q^2 U}{\Sigma_T 00(U) \sigma} + 3/5 QU + 3/5 \frac{b U Q}{\sigma} - 2/5 \frac{QU}{\sigma} + 2/5 \frac{QU}{\sigma} - 1/10 Q^2 - 1/10 \frac{Q^2 b}{\sigma} + \mathcal{F}5(U)$$
where
\[
\mathcal{F}_{00}(U) = \frac{U}{-4c_3 + \sigma c_3 + \sigma c_3 b},
\]
\[
\mathcal{F}_5(U) = \frac{1}{2} \frac{U^2 (-2 + \sigma)}{1 + \sigma} + C1
\]
and
\[
b = 2/3 + 2/3 \sigma, \quad r = 1/3 + 1/3 \sigma.
\]

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