Covariance Estimation for Multivariate Conditionally Gaussian Dynamic Linear Models

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Abstract

In multivariate time series, the estimation of the covariance matrix of the observation innovations plays an important role in forecasting as it enables the computation of the standardized forecast error vectors as well as it enables the computation of confidence bounds of the forecasts. We develop an on-line, non-iterative Bayesian algorithm for estimation and forecasting. It is empirically found that, for a range of simulated time series, the proposed covariance estimator has good performance converging to the true values of the unknown observation covariance matrix. Over a simulated time series, the new method approximates the correct estimates, produced by a non-sequential Monte Carlo simulation procedure, which is used here as the gold standard. The special, but important, vector autoregressive (VAR) and time-varying VAR models are illustrated by considering London metal exchange data consisting of spot prices of aluminium, copper, lead and zinc.

Some key words: Multivariate time series, dynamic linear model, Kalman filter, vector autoregressive model, London metal exchange.

Introduction

Multivariate time series receive considerable attention because a great deal of time series data arrive in vector form. Whittle (1984) and Lütkepohl (1993) discuss VARMA models for vector responses, whilst Harvey (1989, Chapter 8), West and Harrison (1997, Chapter 16) and Durbin and Koopman (2001, Chapter 3) extend this work to state space models for observation vectors. In econometrics most studies of state space models focus on trend estimation, signal extraction and volatility. A review of recent developments of state space models in econometrics can be found in Pollock (2003). Barassi et al. (2005) and Gravelle and Morley (2005) give applications of the Kalman filter to interest rates data and Harvey et al. (1994) use Kalman filter techniques to estimate the volatility of foreign exchange rates using multivariate stochastic volatility (MSV) models. With the exception of multivariate GARCH and MSV models, which focus on the prediction of the volatility, it is usually desirable to use a structural state space model to forecast time series vectors (e.g. foreign exchange rates, monthly sales, interest rates, etc) and to estimate the observation innovation covariance matrix of the underlying time series. For such applications and for short term forecasting the

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above covariance matrix can be assumed time-invariant, but unknown, and its estimation is the main aim of this paper.

The estimation of the observation covariance matrix plays an important role in forecasting. Firstly we note that, under the general multivariate dynamic linear model (see equation (1) below), the multi-step forecast mean of the response time series vector is a non-linear function of the observation covariance matrix (West and Harrison, 1997, Chapter 16). Secondly, the computation of the standardized forecast error vectors requires a precise estimation of the observation covariance matrix and thus a miss-specification of the observation covariance matrix can lead to false results regarding the evaluation and judgement of the model. Thirdly, the multi-step forecast covariance matrix is a linear function of the observation covariance matrix and the former is of particular interest; the forecast covariance matrix can explain the variability of the forecasts and hence it can enable the computation of confidence bounds for the forecasts. Finally, the precise estimation of the observation covariance matrix gives an accurate estimation of the cross-correlation structure of the several component time series, which is particularly useful, especially for financial time series. For all the above reasons the study of the estimation of the observation covariance matrix is worthwhile and its contribution to forecasting for multivariate time series is paramount.

The problem of the estimation of the observation innovation variance for univariate state space models has been well reported (West and Harrison, 1997, §4.5; Durbin and Koopman, 2001, §2.10), however, for vector time series this problem becomes considerably more complex and the available methodology consists of special cases, approximations and iterative procedures.

Let \( y_t \) be a \( p \)-dimensional observation vector following the Gaussian dynamic linear model (DLM):

\[
y_t = F^t \theta_t + \epsilon_t \quad \text{and} \quad \theta_t = G \theta_{t-1} + \omega_t,
\]

where \( \theta_t \) is a \( d \)-dimensional Markovian state vector, \( F \) is a known \( d \times p \) design matrix and \( G \) a known \( d \times d \) transition matrix. The notation \( F^t \) is used for the transpose matrix of \( F \). The distributions usually adopted for \( \{ \epsilon_t \} \), \( \{ \omega_t \} \) and \( \theta_0 \) are the multivariate Gaussian, i.e. \( \epsilon_t \sim \mathcal{N}_p(0, \Sigma) \), \( \omega_t \sim \mathcal{N}_d(0, \Omega) \) and \( \theta_0 \sim \mathcal{N}_d(m_0, P_0) \), for some known priors \( m_0 \) and \( P_0 \). The innovation vectors \( \{ \epsilon_t \} \) and \( \{ \omega_t \} \) are assumed individually and mutually uncorrelated and they are also assumed uncorrelated with the initial state vector \( \theta_0 \), i.e. for all \( t \neq s \): \( \mathbb{E}(\epsilon_t \epsilon_s^t) = 0 \), \( \mathbb{E}(\omega_t \omega_s^t) = 0 \), and for all \( t, s > 0 \): \( \mathbb{E}(\epsilon_t \omega_s^t) = 0 \), \( \mathbb{E}(\epsilon_t \theta_0^t) = 0 \) and \( \mathbb{E}(\omega_t \theta_0^t) = 0 \), where \( \mathbb{E}(\cdot) \) denotes expectation. The covariance matrices \( \Sigma \) and \( \Omega \) are typically unknown and their estimation or specification is a well known problem. The interest is centered on the estimation of \( \Sigma \), while \( \Omega \) can be specified \textit{a priori} (West and Harrison, 1997, Chapter 6; Durbin and Koopman, 2001, §3.2.2).

Several methods have been proposed, for the estimation of \( \Sigma \). Harvey (1986) and Quintana and West (1987) independently introduce matrix-variate DLMs, which are matrix-variate linear state space models allowing for covariance estimation. Harvey (1986) proposes a likelihood estimator, while Quintana and West (1987) propose a Bayesian estimation modelling \( \Sigma \) with an inverted Wishart distribution. Harvey (1986)’s model is reported and further developed in Harvey (1989), Fernández and Harvey (1990), Harvey and Koopman (1997) and Moauro and Savio (2005), while Quintana and West (1987)’s model is reported and further developed in Quintana and West (1988), Queen and Smith (1992), West and Harrison (1997), Salvador et al. (2003), Salvador and Gargalo (2004) and Salvador et al. (2004). However, both suggestions (Harvey (1989)’s and Quintana and West (1987)’s) are criticized in Barbosa...
and Harrison (1992) where it is shown that the above models are restrictive in the sense that one can decompose the response vector $y_t$ into several scalar time series and model each of these time series individually, using univariate DLMs. Barbosa and Harrison (1992) propose an approximate algorithm for the general DLM (1), but their main assumption seems rather unjustified, since it suggests that for any $p \times p$ matrix $C$ it is $\Sigma^{1/2} \Sigma^{-1/2} = \tilde{\Sigma}^{1/2} C \tilde{\Sigma}^{-1/2}$, where $\tilde{\Sigma}$ is a point estimate of $\Sigma$ and the notation $\Sigma^{1/2}$ stands for the symmetric square root of $\Sigma$ (Gupta and Nagar, 1999, p. 7). This assumption holds clearly when $\tilde{\Sigma}^{1/2}, C$ commute and when $\Sigma^{1/2} = \Sigma^*, C$ commute, where $(\Sigma^*)^2$ is any particular realization of $\Sigma$. However, in general the above assumption is difficult to check since $\Sigma$ is the unknown covariance matrix subject to estimation. In addition, that assumption seems to be probabilistically quite inappropriate, since it translates that the non-stochastic quantity $\tilde{\Sigma}^{1/2} C \tilde{\Sigma}^{-1/2}$ equals the stochastic quantity $\Sigma^{1/2} C \Sigma^{-1/2}$ with probability 1. A possible analysis can be obtained in special cases where $\Sigma$ is diagonal or when the off-diagonal elements of $\Sigma$ are all common. Triantafyllopoulos and Pikoulas (2002) and Triantafyllopoulos (2006) adopt the model of Harvey (1986) and they provide an improved on-line estimator for $\Sigma$ based on a standard maximum likelihood technique. The problem is again that the models discussed lack the general formulation of the state space model (1); e.g. one can easily show that all above models are special cases of model (1). Iterative procedures via maximum likelihood and Markov chain Monte Carlo (MCMC) techniques are available, but they tend to be slow, especially as the dimension of the observation vector $p$ increases. Kitagawa and Gersh (1996), Shumway and Stoffer (2000, Chapter 4), Durbin and Koopman (2001, Chapter 7) and Doucet et al. (2001) discuss univariate modelling with iterative methods, but their efficiency in multivariate time series is not yet explored. Barbosa and Harrison (1992) and West and Harrison (1997, §16.2.3) discuss the problem of inefficiency of iterative methods and they point out that the number of parameters to be estimated in $\Sigma$ is $p(p + 1)/2$, which rapidly increases with the dimension $p$ of the response vector, e.g. for $p = 10$ there are 55 distinct parameters in $\Sigma$ to be estimated.

In this paper we propose a new non-iterative Bayesian procedure for estimating $\Sigma$ and forecasting $y_t$. This procedure offers a novel estimator of $\Sigma$ for the general DLM (1). The proposed estimator is empirically found to converge to the true value of $\Sigma$ and this estimator approximates well the respective estimators in the special cases of the conjugate univariate and matrix-variate DLMs. A comparison with a non-sequential Monte Carlo simulation shows that the new method produces estimates close to the MCMC. The focus and the benefit employing the new method is on on-line estimation and therefore no attempt has been made to compare the proposed algorithms with sequential iterative procedures. The reason for this is justified by the above discussion and the interested reader should refer to Dickey et al. (1986) and West and Harrison (1997, §16.2.3). The proposed forecasting procedure for model (1) is applied to the important model subclasses of vector autoregressive (VAR) and VAR with time-dependent parameters. These models are illustrated by considering London metal exchange data, consisting of spot prices of aluminium, copper, lead and zinc (Watkins and McAleer, 2004).

We begin by developing the main idea of the paper and giving the proposed algorithm. The performance of this algorithm is illustrated in the following section by considering simulated time series data; a comparison with a Monte Carlo simulation is performed. The proceeding section gives an application to vector autoregressive modelling, which is used to analyze London metal exchange data, in the following section. The appendix details a proof of a
Main Results

Denote with $y^t = (y_1, y_2, \ldots, y_t)$ the information set comprising data up to time $t$, for some positive integer $t > 0$. Let $m_t$ and $P_t$ be the posterior mean and covariance matrix of $\theta_t | y^t$ and $S_t$ be the posterior expectation of $\Sigma$, i.e. $E(\Sigma | y^t) = S_t$. Let $y_t(1) = E(y_{t+1} | y^t) = F^Gm_t$ be the one-step forecast mean at time $t$ and $Q_{t+1} = \text{Var}(y_{t+1} | y^t) = F^GR_{t+1}F + S_t$ be the one-step forecast covariance matrix at $t$, where $R_{t+1} = GP_tG' + \Omega$. Upon observing $y_{t+1}$, we define the one-step forecast error vector as $e_{t+1} = y_{t+1} - y_t(1)$. The next result (proved in the appendix) gives an approximate property of $S_t$.

**Theorem 1.** Consider the dynamic linear model [1]. Let $\Sigma$ be the covariance matrix of the observation innovation $e_t$ and assume that $\lim_{m \to \infty} S_t = \Sigma$, where $E(\Sigma | y^t) = S_t$ is the true posterior mean of $\Sigma$ given $y^t$. Let $n_0$ be a positive scalar and $S_0 = E(\Sigma)$ be the prior expectation of $\Sigma$. If $\Sigma$ is bounded, then for large $t$ the following holds approximately

$$S_t = \frac{1}{n_0 + t} \left( n_0 S_0 + \sum_{i=1}^{t} S_{i-1}^{1/2} Q_{i-1}^{-1/2} e_i e_i' Q_{i-1}^{-1/2} S_{i-1}^{1/2} \right),$$

where $e_i, Q_i$ are defined above and $S_{i-1}^{1/2}, Q_i^{-1/2}$ denote respectively the symmetric square roots of the matrices $S_{i-1}, Q_i^{-1}$ based on the spectral decomposition factorization of symmetric positive definite matrices ($i = 1, 2, \ldots, t$).

Conditionally now on $\Sigma = S$, for a particular value $S$, we can apply the Kalman filter to the DLM [1] and obtain the posterior and predictive distributions of $\theta_t | \Sigma = S, y^t$ and $y_{t+h} | \Sigma = S, y^t$, for a positive integer $h > 0$, known as the forecast horizon. Theorem 1 motivates approximating the true posterior mean $S_t$ by $S = \tilde{S}_t$, which is produced from application of equation (2), given a particular data set $y^t = (y_1, y_2, \ldots, y_t)$. Thus we obtain the following algorithm:

**Algorithm 1.** (a) Prior distribution at time $t = 0$: $\theta_0 | \Sigma = \tilde{S}_0 \sim N_d(\tilde{m}_0, \tilde{P}_0)$, for some $\tilde{m}_0$, $\tilde{P}_0$ and $\tilde{S}_0$.

(b) Posterior distribution at time $t$: $\theta_t | \Sigma = \tilde{S}_t, y^t \sim N_d(\tilde{m}_t, \tilde{P}_t)$, where $\tilde{e}_t = y_t - \tilde{y}_{t-1}(1)$ and

$$\tilde{m}_t = G\tilde{m}_{t-1} + A_t \tilde{e}_t, \quad \tilde{P}_t = G\tilde{P}_{t-1}G' + \Omega - A_t \tilde{Q}_t A_t', \quad A_t = (G\tilde{P}_{t-1}G' + \Omega)F\tilde{Q}_t^{-1},$$

$$\tilde{S}_t = \frac{1}{n_0 + t} \left( n_0 \tilde{S}_0 + \sum_{i=1}^{t} \tilde{S}_{i-1}^{1/2} \tilde{Q}_i^{-1/2} \tilde{e}_i \tilde{e}_i' \tilde{Q}_i^{-1/2} \tilde{S}_{i-1}^{1/2} \right).$$

(c) $h$-step forecast distribution at $t$: $y_{t+h} | \Sigma = \tilde{S}_t, y^t \sim N_p(\tilde{y}_t(h), \tilde{Q}_t(h))$, where $\tilde{y}_t(h) = F^Gh \tilde{m}_t$ and

$$\tilde{Q}_t(h) = F^Gh \tilde{P}_t(G^h)'F + \sum_{i=0}^{h-1} F^G i Q_i G^i F + \tilde{S}_t.$$
In the special case of matrix-variate DLMs (Harvey, 1986; West and Harrison, 1997, §16.4) the estimator $S_t$ approximates the true posterior mean of $\Sigma$ produced by an application of Bayes’ theorem, assuming a prior inverted Wishart distribution for $\Sigma$. To see this, note that in the matrix-variate DLM (this model is briefly in page 15, see equation (8)), $F$ is a $d$-dimensional design vector and $Q_t = U_t S_{t-1}$ with $U_t = F'R_tF + 1$ and so equation (2) can be written recursively as

$$S_t = n_t^{-1}(n_{t-1}S_{t-1} + e_t e_t'/U_t) \quad \text{and} \quad n_t = n_{t-1} + 1 = n_0 + t. \quad (3)$$

It is easy to verify that the assumption $\lim_{t \to \infty} S_t = \Sigma$ is satisfied, since $\lim_{t \to \infty} S_t = \lim_{t \to \infty} \mathbb{E}(\Sigma|y^t)$ and $\lim_{t \to \infty} \text{Var}\{\text{vech}(\Sigma)|y^t\} = 0$, where $\text{vech}(\cdot)$ denotes the column stacking operator of a lower portion of a symmetric matrix. For $p = 1$ the matrix-variate DLM is reduced to the conjugate Gaussian/gamma DLM (West and Harrison, 1997, §4.5). It turns out that the estimator $S_t$ of equation (2) approximates the analogous estimators of all existing conjugate Gaussian dynamic linear models.

It is worth noting that Theorem 1 and Algorithm 1 have been presented for the state space model (1) having time-invariant components $F$, $G$ and $\Omega$. However, these results apply if some or all of the above components change with time. In addition, if the evolution covariance matrix $\Omega_t$ is time-dependent, it can be specified via discount factors (West and Harrison, 1997, Chapter 6). This is a useful consideration, because in practice the signal $\theta_t$ is unlikely to have the same variability over time.

For the application of Algorithm 1 the initial values $\tilde{m}_0$, $\tilde{P}_0$, $n_0$ and $\tilde{S}_0$ must be specified. $\tilde{m}_0$ can be specified from historical information from the underlying experiment and $\tilde{P}_0$ can be set as a typically large diagonal matrix, e.g. $\tilde{P}_0 = 1000I_p$, reflecting a low precision (or high uncertainty) on the specification of the moments of $\theta_0$. The scalar $n_0$ can be set to $n_0 = 1$ (in the special case of matrix-variate DLMs, $n_0$ is the prior degrees of freedom). $\tilde{S}_0$ is a prior estimate of $\Sigma$ and requires at least a rough specification. As information is deflated in time series, a miss-specification of $\tilde{S}_0$ may not affect much the posterior estimate $\tilde{S}_t$, especially in the presence of large data sets. However, in many cases and especially in financial time series, a miss-specification of $\tilde{S}_0$ can lead to poor estimates of $\Sigma$. Here we suggest that a diagonal covariance matrix can be used, where the diagonal elements of $\tilde{S}_0$ reflect the empirical expectation of the diagonal elements of $\Sigma$. This expectation can be obtained by studying historical data and other qualitative pieces of information, which are usually available to practicing experts of the experiment or of the application of interest.

Simulation Studies

Empirical Convergence of $\tilde{S}_t$

We have generated 1000 bivariate time series $\{y_{it}\}_{t=1,2,...,500}$ from several state space models and then we have averaged the 1000 estimates $\tilde{S}_{i,t}$ (produced by each of the 1000 time series) and compared the average $\tilde{S}_t = 1000^{-1} \sum_{i=1}^{1000} \tilde{S}_{i,t}$ with the true value of $\Sigma$.

Since in practice complicated models are decomposed into simple models comprising local level, polynomial trend and seasonal components (Godolphin and Triantafyllopoulos, 2006), we consider estimation separately in such different component models. We have three modelling situations of interest: situation 1 (bivariate local level models); situation 2 (bivariate linear trend models); and situation 3 (bivariate seasonal models). For each of the above
three situations we have generated 1000 bivariate time series, each of length 500, using three different covariance matrices \( \Sigma_i \), i.e.

\[
\Sigma_1 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} 100 & 85 \\ 85 & 80 \end{bmatrix} \quad \text{and} \quad \Sigma_3 = \begin{bmatrix} 1 & 7 \\ 7 & 50 \end{bmatrix}.
\]

Throughout the simulations we have chosen high correlations for each \( \Sigma_i \) \((i = 1, 2, 3)\), since uncorrelated or approximately uncorrelated state space models can be handled easily by employing several univariate state space models. The priors of \( \Sigma_i \) are chosen as \( \tilde{S}_{0,1} = I_2 \), \( \tilde{S}_{0,2} = 150I_2 \) and \( \tilde{S}_{0,3} = \text{diag}\{3, 40\} \) \((i = 1, 2, \ldots, 1000)\). The diagonal choice for the priors \( \tilde{S}_{0,j} \) has been done for: (a) operational simplicity (the user is likely to expect rough values for the diagonal elements of \( \Sigma_i \), rather than for the associated correlations) and (b) judging how the estimation of \( \Sigma_i \) is affected by improper priors in the sense of setting the off-diagonal elements of \( \tilde{S}_{0,j} \) to zero, while the true values of \( \Sigma_i \) posses high correlations. Throughout the models the remaining settings are \( n_0 = 1, \Omega = I_2, m_0 = [0 \ 0]' \) and \( P_0 = 1000I_2 \), for all models. Table 1 shows the results. There are three blocks of columns, each showing results of the state space model considered, namely local level model \( \text{LL} \) or block 1), linear trend model \( \text{LT} \) or block 2) and seasonal model \( \text{SE} \) or block 3). In each block the first column shows the mean of the average \( \overline{S} = 500^{-1} \sum_{t=1}^{500} \overline{S}_t \) of all \( S_t \). The second column shows the average \( \overline{S}_{100} \) at time point \( t = 100 \). Likewise the third column shows the respective \( \overline{S}_{500} \) averaged over all 1000 series. The rows in Table 1 show the picture of \( \overline{S}_t \) over the three different values of \( \Sigma \), e.g. \( \Sigma_1 \), \( \Sigma_2 \) and \( \Sigma_3 \). The average estimate of the correlations is also shown and it is marked in the table by \( \rho \). The results suggest that, generally, the LL model has the best performance as opposed to the LT and the SE model, although we note that \( \sigma_{12} = 3 \) (covariance in \( \Sigma_1 \)) is estimated better from the LT model. It appears that the estimator \( \overline{S}_t \) for all models converges to the true values of \( \Sigma \), but the rate of convergence depends on the underlying state space model (here LL performs faster convergence) and on the prior \( \tilde{S}_0 \).

Table 2 shows the averaged (over all 1000 simulated time series) mean vector of squared standardized one-step forecast errors \( \text{MSSE}^{(1)} \), for each of the three models \( \text{LL, LT, SE} \) and for each of \( \Sigma \) \( (\Sigma_1, \Sigma_2, \Sigma_3) \). For comparison purposes, Table 2 also shows the respective values of the \( \text{MSSE}^{(2)} \) when \( \Sigma_i \) is the true value. The target value of the \( \text{MSSE}^{(i)} \) is [1 1]. We see that the \( \text{MSSE}^{(1)} \) approaches the respective \( \text{MSSE}^{(2)} \) and this demonstrates the accuracy of the estimator \( \overline{S}_t \). We observe that under \( \Sigma_3 \), the \( \text{MSSE}^{(1)} \) has values significantly smaller than 1 as compared to the \( \text{MSSE}^{(2)} \) using the true value of \( \Sigma_3 \).

**Comparison of the Local Level Model with MCMC**

We have simulated a single local level model under the observation covariance matrix \( \Sigma = \Sigma_1 \) and the relevant model components of the local level model of the previous sub-section. We apply Algorithm 1 and we compare it with a state of the art MCMC estimation procedure based on a blocked Gibbs sampler suitable for state space models (Gamerman, 1997, p. 149); the MCMC procedure we use is described in the appendix. The MCMC estimation procedure is an iterative non-sequential MCMC procedure and its role in this section is to provide a means of comparison with the non-iterative procedure of Algorithm 1. MCMC is the gold standard, since it produces (given enough computation) exact computation of \( S_t \). But MCMC is impractical; the new proposed method is a quick, practical and easily implemented
Table 1: Performance of the estimator $\tilde{S}_t$ (of Algorithm 1) for 1000 simulated bivariate dynamic models generated from a local level model (LL), a linear trend model (LT) and a seasonal model (SE) under three observation covariance matrices $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$.

| Model | LL $\Sigma = \Sigma_1$ | LT $\Sigma = \Sigma_1$ | SE $\Sigma = \Sigma_1$ |
|-------|------------------------|------------------------|------------------------|
| $\sigma_{11} = 2$ | $\bar{S}$ 1.945 $\bar{S}_{100}$ 1.997 $\bar{S}_{500}$ 2.572 | $\bar{S}$ 2.798 $\bar{S}_{100}$ 2.920 | $\bar{S}$ 4.722 $\bar{S}_{100}$ 4.899 |
| $\sigma_{12} = 3$ | $\bar{S}$ 2.798 $\bar{S}_{100}$ 2.770 | $\bar{S}$ 2.988 $\bar{S}_{100}$ 3.029 | $\bar{S}$ 4.547 $\bar{S}_{100}$ 4.777 |
| $\sigma_{22} = 5$ | $\bar{S}$ 4.722 $\bar{S}_{100}$ 4.685 | $\bar{S}$ 4.547 $\bar{S}_{100}$ 4.899 | $\bar{S}$ 4.399 $\bar{S}_{100}$ 4.283 |
| $\rho = 0.948$ | $\bar{S}$ 0.923 $\bar{S}_{100}$ 0.919 | $\bar{S}$ 0.874 $\bar{S}_{100}$ 0.867 | $\bar{S}$ 0.748 $\bar{S}_{100}$ 0.711 |
| $\sigma_{11} = 100$ | $\bar{S}$ 100.039 $\bar{S}_{100}$ 99.931 $\bar{S}_{500}$ 98.271 | $\bar{S}$ 83.133 $\bar{S}_{100}$ 83.028 $\bar{S}_{500}$ 79.471 | $\bar{S}$ 80.430 $\bar{S}_{100}$ 80.277 $\bar{S}_{500}$ 78.917 |
| $\sigma_{12} = 85$ | $\bar{S}$ 80.430 $\bar{S}_{100}$ 80.277 | $\bar{S}$ 80.917 $\bar{S}_{100}$ 80.353 | $\bar{S}$ 80.917 $\bar{S}_{100}$ 80.353 |
| $\sigma_{22} = 80$ | $\bar{S}$ 0.927 $\bar{S}_{100}$ 0.927 | $\bar{S}$ 0.902 $\bar{S}_{100}$ 0.927 | $\bar{S}$ 0.902 $\bar{S}_{100}$ 0.927 |
| $\rho = 0.950$ | $\bar{S}$ 0.927 $\bar{S}_{100}$ 0.927 | $\bar{S}$ 0.902 $\bar{S}_{100}$ 0.927 | $\bar{S}$ 0.902 $\bar{S}_{100}$ 0.927 |
| $\sigma_{11} = 1$ | $\bar{S}$ 1.124 $\bar{S}_{100}$ 1.135 | $\bar{S}$ 1.200 $\bar{S}_{100}$ 1.234 | $\bar{S}$ 1.002 $\bar{S}_{100}$ 1.034 |
| $\sigma_{12} = 7$ | $\bar{S}$ 6.506 $\bar{S}_{100}$ 6.457 | $\bar{S}$ 5.388 $\bar{S}_{100}$ 5.177 | $\bar{S}$ 5.764 $\bar{S}_{100}$ 5.623 |
| $\sigma_{22} = 50$ | $\bar{S}$ 49.305 $\bar{S}_{100}$ 49.375 | $\bar{S}$ 48.518 $\bar{S}_{100}$ 49.375 | $\bar{S}$ 48.518 $\bar{S}_{100}$ 49.375 |
| $\rho = 0.989$ | $\bar{S}$ 0.784 $\bar{S}_{100}$ 0.862 | $\bar{S}$ 0.706 $\bar{S}_{100}$ 0.668 | $\bar{S}$ 0.758 $\bar{S}_{100}$ 0.732 |

Table 2: Mean vector of squared standardized one-step forecast errors (MSSE$^{(i)}$) of the multivariate dynamic model of Algorithm 1. The index $i = 1, 2$ refers to when $\Sigma$ is estimated by the data ($i = 1$) and when $\Sigma$ is assumed known (according to the simulations) for comparison purposes ($i = 2$). The notation LL, LT, SE and $\Sigma_1$, $\Sigma_2$, $\Sigma_3$ is the same as in Table 1.

| Model | MSSE$^{(1)}$ | MSSE$^{(2)}$ |
|-------|-------------|-------------|
| LL $\Sigma_1$ | 0.994 1.071 | 0.999 0.999 |
| LL $\Sigma_2$ | 0.939 0.914 | 0.999 0.999 |
| LL $\Sigma_3$ | 0.773 1.026 | 0.998 0.997 |
| LT $\Sigma_1$ | 0.875 1.141 | 0.992 0.996 |
| LT $\Sigma_2$ | 0.900 0.895 | 1.002 0.997 |
| LT $\Sigma_3$ | 0.774 1.031 | 1.000 0.996 |
| SE $\Sigma_1$ | 0.930 1.092 | 0.997 0.999 |
| SE $\Sigma_2$ | 0.903 0.864 | 0.998 0.996 |
| SE $\Sigma_3$ | 0.805 1.026 | 0.998 0.996 |
approximation. In this section we compare the new method with the gold standard in order to show how good is the approximation. Tables 3 and 4 give the results; the former shows the estimates of $\Sigma$ with both methods (MCMC and Algorithm 1) and the latter shows the performance of the one-step forecast errors for both methods. In Table 4 the one-step forecast error vector $e_t = [e_{1t} e_{2t}]'$ and the mean vector of squared one-step forecast errors are shown for several values of $t$ under both estimation methods. We observe that the new method (of Algorithm 1) approximates well the MCMC estimates, especially for large values of time $t = N$.

We note that MCMC should not be considered as a better method as compared to the proposal of Algorithm 1 since MCMC is an iterative and in particular in this paper it is a non-sequential estimation procedure. The application of sequential MCMC estimation (Doucet et al., 2001) often experience several challenges as for example time-constraints, availability for general purpose algorithms, prior-specification, prior-sensitivity, fast monitoring and expert intervention features. The proposal of this paper provides a strong modelling approach allowing for variance estimation in a wide class of conditionally Gaussian dynamic linear models and this section shows that for large time periods its performance is close to Monte Carlo estimation.

### Application to VAR and TVVAR Time Series Models

The dynamic model 1 is very general and an important subclass of 1 is the popular vector ARMA model. In recent years vector autoregressive (VAR) models have been extensively developed and used, especially for economic time series, as in Doan et al. (1984), Litterman (1986), Kadiyala and Karlsson (1993, 1997), Ooms (1994), Johansen (1995), Uhlig (1997), Ni and Sun (2003), Sun and Ni (2004) and Huerta and Prado (2006).

Our discussion in this section includes two important subclasses of model 1, which can be used for a wide-class of stationary and non-stationary time series forecasting. The first is
Table 4: Bivariate simulated local level dynamic linear model. Showed are: one-step forecast errors at time \( t = N \) and the squared sums of the forecasting errors up to time \( N \).

|   | 100 | 150 | 200 | 250 | 300 | 350 | 400 | 450 | 500 |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|   | MCMC | New | MCMC | New | MCMC | New | MCMC | New | MCMC | New |
| \( e_{1N} \) | | | | | | | | | |
|       | -2.14 | -0.14 | -1.02 | -0.24 | -1.72 | -0.42 | -3.91 | 1.48 | |
|       | -2.22 | -0.28 | -1.09 | -0.20 | -1.79 | -0.46 | -4.02 | 1.68 | |
| \( e_{2N} \) | | | | | | | | | |
|       | -2.46 | -3.88 | -0.12 | -1.45 | -0.61 | -1.91 | -2.26 | -0.93 | |
|       | -2.49 | -3.92 | 0.05 | -1.47 | -0.71 | -1.91 | -2.29 | -0.98 | |
| \( N^{-1} \sum_{t=1}^{N} e_{1t}^2 \) | | | | | | | | | |
|       | 4.85 | 4.83 | 5.02 | 5.25 | 5.01 | 5.12 | 5.21 | 5.24 | |
|       | 4.84 | 4.86 | 5.04 | 5.30 | 5.12 | 5.12 | 5.22 | 5.24 | |
| \( N^{-1} \sum_{t=1}^{N} e_{2t}^2 \) | | | | | | | | | |
|       | 8.54 | 8.21 | 8.10 | 8.72 | 8.91 | 9.01 | 9.09 | 9.32 | |
|       | 8.59 | 8.26 | 8.13 | 8.76 | 9.04 | 9.04 | 9.12 | 9.33 | |

the VAR model of known order \( \ell \geq 1 \), defined by

\[
y_t = \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \cdots + \Phi_\ell y_{t-\ell} + \epsilon_t, \quad \epsilon_t \sim N_p(0, \Sigma),
\]

where \( \Phi_1, \Phi_2, \ldots, \Phi_\ell \) are \( p \times p \) matrices of parameters. In the usual estimation of VAR, stationarity has to be assumed and so the roots of the polynomial (in \( z \))

\[
|I_p - \Phi_1 z - \Phi_2 z^2 - \cdots - \Phi_\ell z^\ell| = 0
\]

should lie outside the unit circle. In standard theory (4) may not assume a Gaussian distribution for \( \epsilon_t \), although in practice this is used for operational simplicity. It is also known that for a high order \( \ell \) model (4) approximates multivariate moving average models, which are typically difficult to estimate and this makes the VAR even more attractive in applications. It is also known that for general on-line estimation and forecasting, the covariance matrix \( \Sigma \) either has to be assumed known or it has to be diagonal. This is a major limitation, because it means that either the modeller knows a priori the cross-correlation between the series \( \{y_{1t}\}, \{y_{2t}\}, \ldots, \{y_{pt}\} \), where \( y_t = [y_{1t} \ y_{2t} \ \cdots \ y_{pt}]' \), or that the \( p \) scalar time series are all stochastically uncorrelated, in which case it is more sensible to use several univariate AR models instead. Recently, the need for estimation of \( \Sigma \) as a full covariance matrix (e.g. where \( \Sigma \) has \( p(p+1)/2 \) elements to be estimated) is considered, but the existing estimation procedures include necessarily iterative estimation via importance sampling (Kadiyala and Karlsson, 1997). Ni and Sun (2003) point out that from a frequentist standpoint ordinary least squares and maximum likelihood estimators of (4) are unavailable. These authors state that asymptotic theory estimators may not be applicable for VAR (especially when \( \{y_t\} \) is a short-length time series). Ni and Sun (2003), Sun and Ni (2004) and Huerta and Prado (2006) propose Bayesian estimation of the autoregressive parameters \( \Phi_1 \) and \( \Sigma \), based on MCMC. It follows that for model (4) when \( \Sigma \) is unknown, only iterative estimation procedures can be applied. Our proposal for on-line estimation of \( \Sigma \) gives a step forward to the estimation and forecasting of VAR models and it is outlined below.

We propose a generalization of the univariate state space representation considered in West and Harrison (1997, §9.4.6). Other state space representations of the VAR are considered in
Huerta and Prado (2006), but these representations, usually referred to as canonical representations of the VAR model (Shumway and Stoffer, 2000) are not convenient for the estimation of Σ, because Σ is embedded into the evolution equation of the states θ. First note that we can rewrite (4) as

\[ y_t = F_t' \theta + \epsilon_t = \left( X_t' \otimes I_p \right) \text{vec}(\Phi) + \epsilon_t, \]

where \text{vec}(\cdot) denotes the column stacking operator of a portion of a matrix and \( \otimes \) denotes the Kronecker or tensor product of two matrices. Model (3) can be seen as a regression-type time series model and it can be handled by the general Algorithm 1 for model (1) if we set \( G = I_p, \, \Omega = 0 \) and if we replace \( F \) by the time-varying \( F_t = X_t \otimes I_p \). Thus we can readily apply Algorithm 1 to estimate \( \Sigma \) and \( \theta \) or \( \Phi_1, \Phi_2, \ldots, \Phi_\ell \).

Moving to the time-varying vector autoregressive (TVVAR) time series, in recent years there has been a growing literature for TVVAR time series. Kitagawa and Gersch (1996), Dahlhaus (1997), Francq and Gautier (2004) and Anderson and Meerschaert (2005) study parameter estimation based on the asymptotic behaviour of TVVAR and time-varying ARMA models. From a state space standpoint West et al. (1999) propose a state space formulation for a univariate time-varying AR model applied to electroencephalographic data. In this section we extend this state space formulation to a vector of observations and hence we can propose the application of Algorithm 1 in order to estimate the covariance matrix of the error drifts of the TVVAR model.

Consider that the \( p \)-vector time series \( \{y_t\} \) follows the TVVAR model of known order \( \ell \) defined by

\[ y_t = \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \cdots + \Phi_\ell y_{t-\ell} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}_p(0, \Sigma), \]  

where \( \Phi_1, \Phi_2, \ldots, \Phi_\ell \) are the time-varying autoregressive parameter matrices. The model can be stationary, locally-stationary or non-stationary depending on the roots of the \( \ell \) polynomials (in \( z \))

\[ |I_p - \Phi_1 z - \Phi_2 z^2 - \cdots - \Phi_\ell z^\ell| = 0. \]

Typical considerations include the local stationarity where there are several regimes for which, locally, \( \{y_t\} \) is stationary, but globally \( \{y_t\} \) is non-stationary. Also the time-dependent parameter matrices \( \Phi_\ell \) can allow for an improved dynamic fit as opposed to the static parameters of the VAR.

In our development we adopt a random walk for the evolution of the parameters \( \Phi_\ell \) \((i = 1, 2, \ldots, \ell)\), although the modeller might suggest other Markovian stochastic evolution formulae for \( \Phi_\ell \). The random walk evolution is the natural consideration when \( \{y_t\} \) is assumed locally stationary. Hence we can rewrite model (4) in state-space form as

\[ y_t = \Phi_t X_t + \epsilon_t = F_t' \theta_t + \epsilon_t \quad \text{and} \quad \theta_t = \theta_{t-1} + \omega_t, \]

where \( X_t = \left[ y_{t-1}' \, y_{t-2}' \, \cdots \, y_{t-\ell}' \right]' \), \( F_t = X_t \otimes I_p, \, \Phi_t = [\Phi_1 \, \Phi_2 \, \cdots \, \Phi_\ell], \, \theta_t = \text{vec}(\Phi_t) \) and \( \omega_t \sim \mathcal{N}_{p \cdot \ell^2}(0, \Omega) \), for some transition covariance matrix \( \Omega \). Model (7) is reduced to (5) when \( \Omega = 0 \), in which case \( \theta_t = \theta_{t-1} = \theta \). After specifying \( \Omega \), we can directly apply Algorithm 1 to the state space model (7) and thus we can obtain an algorithm for the estimation of \( \Sigma \), for the estimation of \( \theta_t \) or \( \Phi_1, \Phi_2, \ldots, \Phi_\ell \) and for forecasting the series \( \{y_t\} \).
Figure 1: LME data, consisting of aluminium, copper, lead and zinc spot prices (in US dollars per tonne of each metal).
London Metal Exchange Data

In this section we analyze London metal exchange (LME) data consisting of official spot prices (US dollars per tonne of metal). LME is the world’s leading non-ferrous metals’ market, trading currently highly liquid contracts for metals, such as aluminium, aluminium alloy, copper, lead, nickel, tin and zinc. According to the LME website \( \text{http://www.lme.co.uk/} \) “LME is highly successful with a turnover in excess of US$3,000 billion per annum. It also contributes to the UKs invisible earnings to the sum of more than £250 million in overseas earnings each year.” More information about the functions of the LME can be found via its website (see above); the recently growing literature on the econometrics modelling of the LME can be found in the review of Watkins and McAleer (2004).

We consider forecasting for four metals exchanged in the LME, namely aluminium, copper, lead and zinc. The data are provided from the LME website for the period of 4 January 2005 to 31 October 2005. After excluding weekends and bank holidays there are \( N = 210 \) trading days. We store the data into the \( 4 \times 1 \) vector time series \( \{ y_t \}_{t=1}^{210} \) and

\[
\begin{bmatrix}
 y_{1t} \\
 y_{2t} \\
 y_{3t} \\
 y_{4t}
\end{bmatrix}
\]

where \( y_{1t} \) denotes the spot price at time \( t \) of aluminium, \( y_{2t} \) denotes the spot price at time \( t \) of copper, \( y_{3t} \) denotes the spot price at time \( t \) of lead and \( y_{4t} \) denotes the spot price at time \( t \) of zinc. The data are plotted in Figure 1.

We propose the VAR and TVVAR models of the previous section; the motivation of this being that from Figure 1 the evolution of the data seems to follow roughly an autoregressive type model. Indeed there is an apparent trend with no seasonality, which can be modelled with a trend model or with a VAR or TVVAR model of the previous section. Here we illustrate the proposal of VAR and TVVAR models, which, according to the previous section, can estimate the covariance matrix of \( y_t \), given the state parameters, and thus the correlation structure of \( \{ y_t \} \) can be studied. Other models for this kind of data have been applied in Triantafyllopoulos (2006) and we can envisage that the models of West and Quintana (1987) can also be applied to the LME data.

First we apply the algorithms of the previous section to several VAR and TVVAR models of different orders in order to find out which model gives the best performance. Performance here is measured via the mean vector of squared standardized one-step forecast errors (MSSE) and the mean vector of absolute percentage one-step forecast errors (MAPE). The first is chosen as a general performance measure taking into account the estimation of the covariance matrix \( \Sigma \) and the second is chosen as a generally reliable percent performance measure. Table 5 shows the results of 10 VAR(\( i \)) and TVVAR(\( i \)) models (first column) of order \( i = 1, 2, \ldots, 10 \). The discount factor \( \delta \) refers to the discounting of the evolution covariance matrix of the state parameters \( \theta_t \); \( \delta = 1 \) refers to a static \( \theta_t = \theta \) (VAR model), while \( \delta < 1 \) refers to a dynamic local level evolution of \( \theta_t = \theta_{t-1} + \omega_t \) (TVVAR model). Table 5 shows that the performance of the TVVAR is remarkable compared with the performance of VAR, which produces very high MSSE throughout the range of \( i \). Out of the VAR models, the best is the VAR(1), which still produces very large MSSE. This indicates that a moving average (MA) model is unlikely to produce good results at all, as the MSSE of the VAR increases with the order \( i \). Also the approximation of a MA model with a high order VAR model will include a large number of state parameters to be estimated and this will introduce computational problems.

Therefore, our attention is focused on the TVVAR models. From a computational standpoint we note that as the order increases \( \delta \) can not be too low, because then there are computational difficulties in the calculation of the symmetric square root of \( \tilde{Q}_t \), used for the estimation of \( \tilde{S}_t \) (the estimate of \( \Sigma \)). Lower values of \( \delta \) work better (Triantafyllopoulos, 2006)
Table 5: Mean vector of squared standardized one-step forecast errors (MSSE) and mean vector of absolute percentage one-step forecast errors (MAPE) of the multivariate LME time series \( \{y_t\} \). The first column indicates several VAR and TVVAR models.

|          | MSSE      | MAPE      |
|----------|-----------|-----------|
| VAR(1)   | 6.614     | 0.033     |
|          | 16.782    | 0.071     |
|          | 7.655     | 0.143     |
|          | 18.370    | 0.057     |
| TVVAR(1): \( \delta = 0.1 \) | 2.430 | 0.059 |
|          | 1.764     | 0.053     |
|          | 0.622     | 0.084     |
|          | 1.852     | 0.076     |
| VAR(2)   | 19.610    | 0.081     |
|          | 15.966    | 0.226     |
|          | 11.934    | 0.201     |
|          | 10.271    | 0.101     |
| TVVAR(2): \( \delta = 0.35 \) | 1.296 | 0.065 |
|          | 1.743     | 0.057     |
|          | 1.228     | 0.116     |
|          | 1.822     | 0.095     |
| VAR(3)   | 11.777    | 0.585     |
|          | 23.715    | 0.345     |
|          | 9.906     | 0.480     |
|          | 9.058     | 0.246     |
| TVVAR(3): \( \delta = 0.65 \) | 2.254 | 0.074 |
|          | 3.149     | 0.053     |
|          | 2.222     | 0.132     |
|          | 2.180     | 0.108     |
| VAR(4)   | 39.979    | 0.159     |
|          | 54.892    | 0.103     |
|          | 28.407    | 0.235     |
|          | 19.169    | 0.161     |
| TVVAR(4): \( \delta = 0.6 \) | 1.389 | 0.101 |
|          | 1.802     | 0.072     |
|          | 1.210     | 0.179     |
|          | 1.329     | 0.147     |
| VAR(5)   | 18.592    | 0.203     |
|          | 16.605    | 0.076     |
|          | 15.474    | 0.392     |
|          | 12.570    | 0.248     |
| TVVAR(5): \( \delta = 0.7 \) | 1.429 | 0.114 |
|          | 2.269     | 0.079     |
|          | 1.651     | 0.208     |
|          | 1.677     | 0.171     |
| VAR(6)   | 24.910    | 0.206     |
|          | 19.085    | 0.134     |
|          | 14.584    | 0.320     |
|          | 17.784    | 0.197     |
| TVVAR(6): \( \delta = 0.75 \) | 1.828 | 0.132 |
|          | 2.705     | 0.089     |
|          | 1.683     | 0.243     |
|          | 1.757     | 0.197     |
| VAR(7)   | 21.722    | 0.330     |
|          | 38.054    | 0.092     |
|          | 14.597    | 0.422     |
|          | 14.180    | 0.490     |
| TVVAR(7): \( \delta = 0.75 \) | 1.366 | 0.148 |
|          | 2.044     | 0.101     |
|          | 1.191     | 0.280     |
|          | 1.531     | 0.223     |
| VAR(8)   | 28.985    | 0.515     |
|          | 35.867    | 0.325     |
|          | 11.291    | 0.812     |
|          | 16.370    | 0.563     |
| TVVAR(8): \( \delta = 0.8 \) | 2.130 | 0.168 |
|          | 2.900     | 0.111     |
|          | 1.637     | 0.326     |
|          | 1.910     | 0.249     |
| VAR(9)   | 40.229    | 0.393     |
|          | 53.798    | 0.184     |
|          | 12.249    | 0.416     |
|          | 19.691    | 0.411     |
| TVVAR(9): \( \delta = 0.95 \) | 14.042 | 0.207 |
|          | 21.011    | 0.124     |
|          | 6.724     | 0.352     |
|          | 8.708     | 0.284     |
| VAR(10)  | 46.791    | 0.611     |
|          | 49.869    | 0.306     |
|          | 16.240    | 0.751     |
|          | 23.974    | 0.694     |
| TVVAR(10): \( \delta = 0.9 \) | 4.273 | 0.205 |
|          | 7.541     | 0.124     |
|          | 3.637     | 0.391     |
|          | 5.629     | 0.296     |
and here we have chosen the lowest values of $\delta$, which are allowed. Our decision on the best TVVAR model is based on the following four criteria.

1. low order models are preferable as they have fewer state parameters;
2. $\delta$ should not be too low, because then the covariance matrix of $\theta_t$ will be too large;
3. the MSSE vector should be close to $[1 1 1 1]'$;
4. the MAPE vector should be as low as possible.

Considering the above criteria we favor the TVVAR(2). Figure 2 shows the estimate of the observation covariance matrix $\Sigma$. From the right graph we observe that the estimate of the correlations of $y_{1t}$ and $y_{jt}$, given $\theta_t$ are very high (close to 1) and this means that in forecasting; this provides useful information about the cross-dependence of the four metal prices over time.

As mentioned before two competitive models to our TVVAR modelling for the LME data are the matrix-variate DLMs (MV-DLMs) of Quintana and West (1987) and the discount weighted regression (DWR) of Triantafyllopoulos (2006). Next we compare the TVVAR(2) model discussed above with these two modelling approaches. We start by briefly describing the MV-DLM and the DWR.
Table 6: Mean vector of squared standardized one-step forecast errors (MSSE) and mean vector of absolute percentage one-step forecast errors (MAPE) for the LME data and for three multivariate models: TVVAR, MV-DLM and DWR.

| Model  | MSSE          | MAPE          |
|--------|---------------|---------------|
| TVVAR(2) | 1.296 1.743 1.228 1.822 | 0.065 0.057 0.116 0.095 |
| MV-DLM   | 1.306 2.436 0.984 1.887 | 0.019 0.022 0.026 0.025 |
| DWR      | 2.202 1.610 1.590 1.868 | 0.013 0.015 0.017 0.017 |

The MV-DLM is defined by

\[ y_t' = F'y_t + \epsilon_t \quad \text{and} \quad \Theta_t = G\Theta_{t-1} + \omega_t, \quad (8) \]

where \( F \) is a \( d \times 1 \) design vector, \( \Theta_t \) is a \( d \times p \) state matrix, \( G \) is a \( d \times d \) transition matrix, \( \epsilon_t \sim N_p(0, \Sigma) \) and \( \text{vec}(\omega_t) \sim N(dp, \Sigma \otimes \Omega) \), where \( \text{vec}(\cdot) \) denotes the column stacking operator of a lower portion of a matrix and \( \otimes \) denotes the Kronecker product of two matrices. A prior inverted Wishart distribution is assumed for \( \Sigma \) and the resulting posterior distributions as well as further details on the model can be found in Quintana and West (1987) and West and Harrison (1997, Chapter 16) (for more references on this model, see also the Introduction).

In the application of MV-DLMs it is necessary to specify \( F \) and \( G \). Following Quintana and West (1987), who consider international exchange rates data, and by consulting the plots of Figure 4, we propose a linear trend model for the LME data. Thus we can set

\[ F = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \]

The DWR is defined by

\[ y_t = y_{t-1} + \psi_t + \epsilon_t \quad \text{and} \quad \psi_t = \psi_{t-1} + \zeta_t, \]

with \( \epsilon_t \sim N_p(0, \Sigma) \) and \( \zeta_t \sim N_p(0, \Omega_t) \). This model can be put into state space form as in

\[ y_t = \left[ y_{t-1} I_p \right] \begin{bmatrix} 1 \\ \psi_t \end{bmatrix} + \epsilon_t = F_t' \theta_t + \epsilon_t, \quad \theta_t = \begin{bmatrix} 1 \\ \psi_t \end{bmatrix} = \begin{bmatrix} 1 \\ \psi_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \zeta_t \end{bmatrix} = \theta_{t-1} + \omega_t. \]

The covariance matrix \( \Omega_t \) is modelled with a discount factor \( \delta \) and \( \Sigma \) is estimated following Triantafyllopoulos and Pikoulas (2002) and Triantafyllopoulos (2006).

Table 6 shows the MSSE and the MAPE of the three models. We see that all models produce reasonable results. For the MSSE the best model is the TVVAR(2) (with the exception of the lead variable where the MV-DLM produces MSSE closer to 1). For the MAPE the best model is the DWR with the TVVAR(2) producing the highest MAPE. Out of the three models, the MV-DLM is limited by its mathematical form, which is constructed to give conjugate analysis (see also the Introduction). The DWR suffers from similar limitations as the MV-DLM, but it provides good results, for linear trend time series without seasonality. The TVVAR model provides a good modelling alternative and considering the numerous applications of VAR time series models in econometrics, it is believed that the TVVAR has a great potential.
In conclusion, the TVVAR model can produce forecasts with good forecast accuracy, while the correlation of the series can be estimated on-line with a fast linear algorithm. A criticism of the model is that its efficiency depends on its order and if high order TVVAR models are required (e.g. as in approximating moving average processes with time-dependent parameters) its efficiency will be similar of that of a vector MA, since the discount factor will have to be close to 1. It will be interesting to know how the order of the TVVAR model is related to the boundness of the eigenvalues of the covariance estimator $\tilde{S}_t$.

Concluding Comments

This paper develops an algorithm for covariance estimation in multivariate conditionally Gaussian dynamic linear models, assuming that the observation covariance matrix is fixed, but unknown. This is a general estimation procedure, which can be applied to any Gaussian linear state space model. The algorithm is empirically found to have good performance providing a covariance estimator which converges to the true value of the observation covariance matrix. The proposed methodology compares well with a non-sequential state of the art MCMC estimation procedure and it is found that the proposed estimates are close to the estimates of the MCMC. The new algorithm is applied (but not limited to) model subclasses of VAR and VAR with time-dependent parameters (TVVAR), which have great application in financial time series. Considering the London metal exchange data, it is found that the TVVAR model has outstanding performance as opposed to the VAR model. It is believed that the development of the TVVAR model is a worthwhile project and the proposed fast, on-line algorithm for the estimation of the observation covariance matrix, is a step forward opening several paths for practical forecasting.

The focus in this paper is on facilitating and advancing non-iterative covariance estimation procedures for vector time series. Such procedures are particularly appealing, because of their simplicity and ease in use. For such wide class of models such us the conditionally Gaussian dynamic linear models, the proposed on-line algorithm enables the computation of the mean vector of standardized errors as well as it enables the computation of the multi-step forecast covariance matrix. Both these computations are valuable considerations in forecasting and they attract interest by academics and practitioners alike.

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Appendix

Proof of Theorem 1

Let $vech(\cdot)$ denote the column stacking operator of a lower portion of a symmetric square matrix and let $\otimes$ denote the Kronecker product of two matrices. First we prove that for large $t$, it is approximately

$$E(\Sigma - A_t e_t e'_t A'_t | y^t) = E(\Sigma - A_t e_t e'_t A'_t | y^{t-1}),$$

(A-1)
where \( A_t = n_t^{-1/2} S_{t-1}^{1/2} \). Conditional on \( \Sigma \), we have from an application of the Kalman filter that \( \text{Cov}(e_{it} e_{jt}, e_{kt} e_{lt}) | \Sigma, y^{t-1} \) is bounded, where \( e_t = [e_{1t}, e_{2t}, \ldots, e_{pt}]' \). Since \( \Sigma \) is bounded, \( S_t \) is also bounded (\( \lim_{t \to \infty} S_t = \Sigma \)), and so all the covariances of \( e_{it} e_{jt} \) and \( e_{kt} e_{lt} \) unconditional on \( \Sigma \) are also bounded. This means that all the elements of \( \text{Var}\{\text{vech}(e_{it} e_{jt})\} \) are bounded and so \( \text{Var}\{\text{vech}(e_{it} e_{jt})\} \) is bounded. Now let

\[
X_1 = \mathbb{E}(\Sigma - A_t e_t e_t' A_t' | y, y^{t-1}) = S_t - A_t e_t e_t' A_t
\]

and

\[
X_2 = \mathbb{E}(\Sigma - A_t e_t e_t' A_t' | y^{t-1}) = S_{t-1} - A_t Q_t A_t'.
\]

Then, since \( \lim_{t \to \infty} S_t = \Sigma \), there exists appropriately a large integer \( t(L) > 0 \) such that for every \( t > t(L) \) it is \( \mathbb{E}(X_1 - X_2 | y^{t-1}) \approx 0 \). Also

\[
\text{Var}\{\text{vech}(X_1 - X_2)|y^{t-1}\} = \text{Var}\{(A_t \otimes A_t) D_p \text{vech}(e_t e_t')|y^{t-1}\} = \frac{1}{n_t} E_t \to 0,
\]

with

\[
E_t = \left[ S_{t-1}^{1/2} Q_t^{-1/2} \otimes S_{t-1}^{1/2} Q_t^{-1/2} \right] D_p \text{vech}(e_t e_t') D_p^\top \left[ Q_t^{-1/2} S_{t-1}^{1/2} \otimes Q_t^{-1/2} S_{t-1}^{1/2} \right],
\]

where \( D_p \) is the duplication matrix and from the first part of the proof we have that \( E_t \) is bounded. It follows that for any \( t > t(L) \) it is \( X_1 \approx X_2 \) with probability 1 and so we have proved equation (A-1). Using \( \mathbb{E}(\Sigma|y^t) = S_t \), from equation (A-1) we have

\[
\mathbb{E}(\Sigma|y^t) - A_t e_t e_t' A_t' = \mathbb{E}(\Sigma|y^{t-1}) - A_t \mathbb{E}(e_t e_t'|y^{t-1}) A_t'
\]

\[
\Rightarrow S_t = S_{t-1} + \frac{1}{n_t} S_{t-1}^{1/2} Q_t^{-1/2} (e_t e_t' - Q_t) Q_t^{-1/2} S_{t-1}^{1/2}
\]

\[
\Rightarrow S_t = S_{t-1} - \frac{1}{n_t} S_{t-1} + \frac{1}{n_t} S_{t-1}^{1/2} Q_t^{-1/2} e_t e_t' Q_t^{-1/2} S_{t-1}^{1/2}
\]

\[
\Rightarrow n_t S_t = n_{t-1} S_{t-1} + \frac{1}{n_t} S_{t-1}^{1/2} e_t e_t' Q_t^{-1/2} S_{t-1}^{1/2} = n_0 S_0 + \sum_{i=1}^t S_{i-1}^{1/2} Q_i^{-1/2} e_t e_t' Q_i^{-1/2} S_i^{1/2}
\]

and by dividing by \( n_t = n_0 + t = n_{t-1} + 1 \) we obtain equation (2) as required.

### The Gibbs Sampler for Multivariate Conditionally Gaussian DLMs

The following procedure applies to any conditionally Gaussian dynamic linear model in the form of equation (1). For the simulation studies considered in this paper, given data \( y^N = (y_1, y_2, \ldots, y_N) \), we are interested in sampling a set of state vectors, \( \theta_1, \theta_2, \ldots, \theta_N \) and the observation covariance matrix \( \Sigma \) from the full, multivariate posterior distribution of \( \theta_1, \theta_2, \ldots, \theta_N, \Sigma|y^N \).

Gibbs sampling involves iterative sampling from the full conditional posterior of each \( \theta_t, | \theta_{-t}, \Sigma, y^N \), for all \( t = 1, 2, \ldots, N \), and \( \Sigma|\theta_1, \theta_2, \ldots, \theta_N, y^N \); in our notation, \( \theta_{-t} \) means that we are conditioning upon all the components \( \theta_1, \theta_2, \ldots, \theta_N \) but \( \theta_t \). Given the conditionally normal and linear structure of the system, such full conditional distributions are standard, and therefore easily sampled. However, such an implementation of the Gibbs sampler, where each component is updated once at a time, could be very inefficient when applied to the multivariate DLMs discussed in this paper; in fact, the high-correlation of the dynamic system
will most likely bring convergence problems. In order to overcome such difficulties, following the early suggestions of Carter and Kohn (1994) and Frühwirth-Schnatter (1994), we have chosen to implement a blocked Gibbs sampler Gamerman (1997, p. 149); within this context, this sampling scheme is better known as the forward filtering, backward sampling algorithm. Following is a concise description of the algorithm used in our studies; for more details, the reader should consult the references above, as well as West and Harrison (1997, Chapter 15).

The first step of the Gibbs sampler involves sampling from the updating distribution of \( \theta_N | \Sigma, y^N \), which is given by the multivariate normal \( N_{d}(m^M_N, P^M_N) \). This is done in the forward filtering phase of the sampler, as follows. Starting at time \( t = 0 \) with some given initial values \( m^M_0, P^M_0 \) and \( \Sigma \) we compute the following quantities at each time \( t \), for \( t = 1, 2, \ldots, N \):

(a) the prior mean vector and covariance matrix of \( \theta_t | \Sigma, y^{t-1} \),
\[
a_t = G m_{t-1}^M \quad \text{and} \quad P_t^M = G P_{t-1}^M G' + \Omega.
\]

(b) the mean vector and covariance matrix of the one-step ahead forecast of \( y_t | y^{t-1} \),
\[
y_{t-1}^M(1) = F' a_t \quad \text{and} \quad Q_t^M = F' R_t^M F + \Sigma.
\]

(c) the posterior mean vector and covariance matrix of \( \theta_t | y^t \),
\[
m_t^M = a_t + A_t^M \epsilon_t^M \quad \text{and} \quad P_t^M = R_t^M - A_t^M Q_t^M (A_t^M)' ,
\]

where \( A_t^M = R_t^M F (Q_t^M)^{-1} \) is the Kalman gain and \( \epsilon_t^M = y_t - y_{t-1}^M(1) \) is the one-step ahead forecast error vector.

An updated vector \( \theta_N \) is thus obtained, and the filtering part of the algorithm is completed. The backwards sampling phase involves sampling from the distribution of \( \theta_t | \theta_{t+1}, \Sigma, y^t \) at all times \( t = N - 1, \ldots, 1, 0 \). Each of such vectors is drawn from a multivariate normal \( N_{d}(h_t, H_t) \), where
\[
h_t = m_t^M + P_t^M G'(R_{t+1}^M)^{-1}(\theta_{t+1} - a_{t+1}) \quad \text{and} \quad H_t = P_t^M (I_d - G(R_{t+1}^M)^{-1} G P_t^M ) ,
\]

with \( I_d \) being the \( d \times d \) identity matrix. At each time \( t \), we also compute \( \epsilon_t^* = y_t - F' \theta_t \). Once the backwards sampling phase is completed, we set
\[
\hat{\Sigma} = N^{-1} \sum_{t=1}^{N} \epsilon_t^* (\epsilon_t^*)'.
\]

Finally, with \( n_0^M \) being the prior degrees of freedom and \( S_0^M \) being the prior estimate of \( \Sigma \), we sample from the full conditional density of \( \Sigma | \Theta, y^N \), which is an inverted Wishart distribution \( IW_p(n_0^M + N + 2p, N \hat{\Sigma} + n_0^M S_0^M) \), whose simulation is also standard. This concludes an iteration of the Gibbs sampler.

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