Geometrical Proof of Generalized Mirror Transformation of Projective Hypersurfaces

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Abstract
In this paper, we propose a geometrical proof of the generalized mirror transformation of genus 0 Gromov-Witten invariants of degree \( k \) hypersurface in \( CP^{N-1} \).

1 Introduction

1.1 Notation and Main Theorem
Let \( N, k \) be positive integers and \( M^k_N \) be a degree \( k \) hypersurface in \( CP^{N-1} \). We denote by \( \overline{M}_{0,n}(CP^{N-1}, d) \) the moduli space of stable maps of degree \( d \) from genus 0 semi-stable curve with \( n \) marked points to \( CP^{N-1} \) [16]. The genus 0 \( n \)-pointed Gromov-Witten invariant of \( M^k_N \) used in this paper is defined as follows [16, 10],

\[
\langle \prod_{j=1}^n O_{h^{a_j}} \rangle_{0,d} = \int_{\overline{M}_{0,n}(CP^{N-1}, d)} c_{top}(\overline{E}_d^k) \wedge \left( \bigwedge_{j=1}^n ev_j^*(h^{a_j}) \right),
\]

(1.1)

where \( h \) is the hyperplane class in \( H^*(CP^{N-1}, C) \) and \( ev_j : \overline{M}_{0,n}(CP^{N-1}, d) \to CP^{N-1} \) \( (j = 1, 2, \cdots, n) \) is the evaluation map at the \( j \)-th marked point. \( \overline{E}_d^k \) is the vector bundle on \( \overline{M}_{0,2}(CP^{N-1}, d) \) that impose the condition that image of the stable map is contained in the hypersurface.\(^1\) It is non-vanishing only if the following condition is satisfied,

\[
\sum_{j=1}^n a_j = N - 5 + (N - k)d + n.
\]

(1.2)

On the other hand, we also introduce the compactified moduli space of quasimaps from \( CP^1 \) with two marked points (0 and \( \infty \)) to \( CP^{N-1} \) of degree \( d \), which we denote by \( \overline{M}_{0,2}(CP^{N-1}, d) \). See [9] and [22] for details of construction. We then define intersection number \( w(O_{h^a}O_{h^b})_{0,d} \) of \( \overline{M}_{0,2}(CP^{N-1}, d) \), which is an analogue of \( \langle O_{h^a}O_{h^b} \rangle_{0,d} \) of \( \overline{M}_{0,2}(CP^{N-1}, d) \), as follows.

\[
w(O_{h^a}O_{h^b})_{0,d} = \int_{\overline{M}_{0,2}(CP^{N-1}, d)} c_{top}(\overline{E}_d) \wedge ev_0^*(h^a) \wedge ev_{\infty}^*(h^b).
\]

(1.3)

\(^1\)This bundle is rigorously described by direct image sheaf of pull-back of \( O_{CP^{N-1}}(k) \) by evaluation map [16, 10], but we omit here this lengthy notation.
Here, $ev_0$ and $ev_{\infty}$ is the evaluation maps at 0 and $\infty$ respectively and $E^k_d$ is the vector bundle on $\widetilde{M}_{0,d}(N, d)$ which has the same geometrical meaning as $\tilde{E}_d$. It is non-vanishing only if the following condition is satisfied.

$$a + b = N - 3 + (N - k)d. \quad (1.4)$$

In [22], Saito constructed explicit toric data of $\widetilde{M}_{0,d}(N, d)$ and showed that it is a compact toric orbifold. Moreover, he showed that its Chow ring is generated by $\tilde{E}_i$. Therefore, we can compute $w(\mathcal{O}_{h^* \mathcal{O}_{h^*}})_{0,d}$ explicitly.

Let $P_g$ be set of partitions of positive integer $g$:

$$P_g = \{ \sigma_g = (g_1, \cdots, g_l(\sigma_g)) \mid 1 \leq g_1 \leq g_2 \leq \cdots \leq g_l(\sigma_g), \sum_{j=1}^{l(\sigma_g)} g_j = g \}. \quad (1.7)$$

For a partition $\sigma_g \in P_g$, we define multiplicity $mul(i, \sigma_g)$ of $\sigma_g$ as follows.

$$mul(i, \sigma_g) = \text{(number of subscript } j \text{ that satisfies } g_j = i). \quad (1.8)$$

We define combinatorial factor $S(\sigma_g)$ as follows,

$$S(\sigma_g) = \prod_{i=1}^{g} \frac{1}{(mul(i, \sigma_g))^!}. \quad (1.9)$$

In this paper, we prove the following theorem that describes relation between intersection numbers $w(\mathcal{O}_{h^* \mathcal{O}_{h^*}})_{0,d}$ and Gromov-Witten invariants $\prod_{i=1}^{d} \mathcal{O}_{h^*} \cdot \mathcal{O}_{h^*} \cdot \mathcal{O}_{h^*}$.

**Theorem 1.1.**

$$w(\mathcal{O}_{h^* \mathcal{O}_{h^*}})_{0,d} - w(\mathcal{O}_{h^{N-3-(k-N)d}})_{0,d}
= (\mathcal{O}_{h^* \mathcal{O}_{h^*}})_{0,d} + \sum_{g=1}^{d-1} \sum_{\sigma_g \in P_g} S(\sigma_g) \prod_{i=1}^{l(\sigma_g)} \mathcal{O}_{h_i+(k-N)g_i} \cdot \prod_{i=1}^{l(\sigma_g)} \frac{w(\mathcal{O}_{h^{N-3+(k-N)g_i}} \mathcal{O}_1)}{k},
(a + b = N - 3 + (N - k)d). \quad (1.10)$$

This theorem corresponds to the most concise version of “generalized mirror transformation” for Kähler sub-ring of the small quantum cohomology ring of $M_k^h$.

Generalized mirror transformation was first observed in [10] for low degrees and was generalized to arbitrary degree in [11] in the context of virtual structure constants [12]. Rigorous proof of generalized mirror transformation was given by Iritani [8] by using Birkhoff factorization technique invented by Coates.
and Givental [1] and also by Guest [6]. But these results are based on three pointed Gromov-Witten invariants or $J$-function and process of explicit computation was quite complicated.

Later, we found that fundamental invariants to describe generalized mirror transformation are two pointed Gromov-Witten invariants. Therefore, we present here the version given in (1.10) that is written in terms of $(\mathcal{O}_h^* \mathcal{O}_{h^*})_{0,d}$ and $w(\mathcal{O}_h^* \mathcal{O}_{h^*})_{0,d}$. As was mentioned before, we think that it is the most compact form of the generalized mirror transformation for small quantum cohomology ring. As for big quantum cohomology ring, generalized mirror transformation of projective hypersurfaces was conjectured in [15] and this version is much simpler for explicit computation of Gromov-Witten invariants.

In this paper, we give a geometrical proof of the generalized mirror transformation for small quantum cohomology ring. The word "geometrical" means that we do not use localization technique. Instead, we go back to our original motivation given in [14]. That is to say, "generalized mirror transformation is nothing but the process of removing contributions of quasi maps that are not actual maps to $\mathbb{C}P^{1}$ to $\mathbb{C}P^{N-1\ast}$. Our proof given in this paper clarifies geometrical meaning of the generalized mirror transformation.

### 1.2 Usage of Theorem 1.1 and Historical Background of Quasimap

We explain briefly usage of Theorem 1.1. Let us first discuss the case of Fano hypersurface with $N-k \geq 2$. Since $N-3 - (k-N) d > N-2$ ($d \geq 1$), (1.10) reduces to the following equality,

$$w(\mathcal{O}_h^* \mathcal{O}_{h^*})_{0,d} = (\mathcal{O}_h^* \mathcal{O}_{h^*})_{0,d}, \quad (a + b = N - 3 + (N-k)d).$$

(1.11)

This says that Gromov-Witten invariant is correctly evaluated by using the moduli space of quasimaps $\bar{M}_{0,d}(N,d)$. This fact was implied in [13] and follows from Theorem 9.1 in [4] proved by Givental. Explicit statement of the above equality was given in [12] in terms of the virtual structure constant:

$$\tilde{L}^N_{n,k,d} = \frac{d}{k} w(\mathcal{O}_h^{N-2-n} \mathcal{O}_{h^{N-1*(N-k)d})}_{0,d}. \quad (1.12)$$

(1.12) was proved for arbitrary $N$ and $k$ in [9].

If $N-k = 1$, the above equality is slightly modified only in the $d = 1$ case.

$$w(\mathcal{O}_h^* \mathcal{O}_{h^*}) - w(\mathcal{O}_h^{N-2} \mathcal{O}_1)_{0,1} = w(\mathcal{O}_h^* \mathcal{O}_{h^*}) - k \cdot k! = (\mathcal{O}_h^* \mathcal{O}_{h^*})_{0,1}. \quad (a + b = N - 2). \quad (1.13)$$

It was also fundamentally proved in [4] and explicitly stated in [2].

In the case of $N = k$ where the hypersurface is a Calabi-Yau hypersurface, we introduce the following generating function:

$$w(\mathcal{O}_h^* \mathcal{O}_{h^*})_{0,d}(x) := kx + \sum_{d=1}^{\infty} w(\mathcal{O}_h^* \mathcal{O}_{h^*})_{0,d} e^{dx} \quad (a + b = N - 3). \quad (1.14)$$

In [9], we proved the following equality:

$$w(\mathcal{O}_h^{N-2} \mathcal{O}_1)_{0,d}(x) = kt(x), \quad (1.15)$$

where

$$t(x) := \left(x + \frac{w_1(x)}{w_0(x)}\right), \quad (1.16)$$

$$w_0(x) = \sum_{d=0}^{\infty} \frac{(kd)!}{(d)!^k} e^{dx}, \quad w_1(x) = \sum_{d=1}^{\infty} \frac{(kd)!}{d!^k} \left(\sum_{i=1}^{d} \sum_{l=1}^{k-1} \frac{l}{i(ki-l)}\right) e^{dx}. \quad (1.17)$$

This $t(x)$ is nothing but the mirror map (in physics terminology, redefinition of the coupling constant of the Gauged Linear Sigma Model) used in the mirror computation of genus 0 Gromov-Witten invariants of the Calabi-Yau hypersurface. Then Theorem 1.1 is equivalent to the following equality:

$$w(\mathcal{O}_h^* \mathcal{O}_{h^*})_{0,d}(x) = (\mathcal{O}_h^* \mathcal{O}_{h^*})_{0,d}(t(x)), \quad (1.18)$$

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Therefore by combining the results given in [9] and [12], Theorem 1 gives a proof of the mirror theorem of genus 0 Gromov-Witten invariants of the Calabi-Yau hypersurface. Of course, the mirror theorem in this case was already proved in [4] and [18], but their treatment of the mirror transformation (1.18) is analytic or complicated. Therefore, the derivation of the mirror transformation from the point of view of quasimap is not clear in these works. Our proof of Theorem 1 provides a short and geometrically clear proof of the mirror theorem of the Calabi-Yau hypersurface.

In the $N - k < 0$ case where the hypersurface is general type, Theorem 1.1 enables us to write down $\langle O_{h^s}O_{h^b}\rangle_{0,d} (a + b = N - 3 + (N - k)d)$ in terms of the virtual structure constant $\tilde{L}^{N,k,d'}_n (d' \leq d)$. The process is briefly given as follows. First, note that the equality:

$$d(\langle O_{h^s}O_{h^b}\rangle_{0,d}) = \langle O_{h^s}O_{h^b}O_{h^t}\rangle_{0,d}.$$  

(1.20)

According to the reconstruction theorem of Kontsevich-Manin [17], we can compute all the multi-point genus 0 Gromov-Witten invariants $(\prod_{i=1}^n \langle O_{h^s_{i}}\rangle_{0,d'}) (d' \leq d)$ from the initial data $\langle O_{h^s}O_{h^b}O_{h^t}\rangle_{0,d'} (d' \leq d)$. Moreover, in the $d = 1$ case, Theorem 1.1 gives us,

$$\langle O_{h^{N-2-n}}O_{h^{N-1+(N-k)}}\rangle_{0,1} = w(\langle O_{h^{N-2-n}}O_{h^{N-1+(N-k)}}\rangle_{0,d}) - w(\langle O_{h^{N-3-(k-N)}O_{1}}\rangle_{0,d})$$

$$= k(\tilde{L}^{N,k,1}_n - \tilde{L}^{N,k,1}_{1+k-N}).$$  

(1.21)

Hence by induction of $d$, we can express $\langle O_{h^s}O_{h^b}\rangle_{0,d} (a + b = N - 3 + (N - k)d)$ in terms of the virtual structure constant $\tilde{L}^{N,k,d'}_n (d' \leq d)$ with the aid of the equality (1.12). This procedure derives all the conjectures given in [10] and [11], and completes the proof of the genus 0 mirror theorem for general type hypersurfaces in our formulation. The intersection number $w(\langle O_{h^{N-3-(k-N)}O_{1}}\rangle_{0,d})$ also appears as the expansion coefficient of the mirror map of big quantum cohomology ring in the context of Iritani [8], but his proof is also analytic. Therefore, our proof clarifies geometrically the meaning of the generalized mirror transformation also in this case.

Lastly, we briefly review historical background of the theory of quasimap. The idea of quasimap was first introduced by Witten [23] and was also used by ourselves [14] in order to derive the leading term of the $(N - 2)$-point genus 0 correlation function of Calabi-Yau hypersurface in $CP^{N-1}$. Then this idea was realized as Gauged Linear Sigma Model [23, 21] in physics terminology and studied in the context of quantum field theory. This line of study was extended to the celebrated work of Morrison and his collaborators [7]. It seems that they succeeded in deriving hypergeometric series used in the mirror computation from the context of Gauged Linear Sigma Model, but derivation of redefinition coupling constant was not done. Relation between the mirror transformation and the procedure of removing contributions from quasimaps that are not actual maps (in the context of [19, 20], these are called freckled instantons) was qualitatively suggested in [23, 4, 19, 20], but these works lacks quantitative derivation of the mirror transformation. We think that key of quantitative derivation of the mirror transformation is the intersection number:

$$w(\langle O_{h^{N-3-(k-N)}O_{1}}\rangle_{0,d}),$$  

(1.22)

which does not vanish even with the trivial operator insertion $O_1$. The reason why it does not vanish only comes from our construction of the moduli space $\bar{M}_{0,d}(N, d)$. This fact enables us to write down the following short proof of the generalized mirror transformation. Of course, $\bar{M}_{0,d}(N, d)$ is also independently constructed by Ciocan-Fontanine and Kim [3]. But their idea of construction is based on stability condition and quite different from our construction. Hence their derivation of the mirror transformation is quite different from ours. We hope that relation between the two different approaches of construction of the moduli space of quasi maps will be clarified in the future.
As for mirror symmetry of general type hypersurface, Landau-Ginzburg model is considered as its mirror counterpart [24]. In [5], coincidence of hodge numbers of general hypersurface and corresponding Landau-Ginzbrug model is shown. Relation between intersection number of moduli space of quasi maps and Landau-Ginzburg model seems to be also important.

This paper is organized as follows, In Section 2, we introduce the moduli space $\tilde{M}_{0,2}(N, d)$, which plays the central role in the proof of Theorem 1.1. In Section 3, we prove Theorem 1.1. In Appendix A, we explain the reason why the “perturbation space” introduced in Section 3 can be used to evaluate the contributions from the excess intersections to $w(\mathcal{O}_h \cdot \mathcal{O}_{h^*})_{0,d}$.

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## 2 The Moduli Space $\tilde{M}_{0,2}(N, d)$

Let $a_j$, $(j = 0, 1, \ldots, d)$ be vectors in $\mathbb{C}^N$ and let $\pi_N: \mathbb{C}^N \setminus \{0\} \to CP^{N-1}$ be a projection map. In this section, we define a degree $d$ quasimap $p$ from $\mathbb{C}^2$ to $\mathbb{C}^N$ as a map that consists of $\mathbb{C}^N$-vector-valued degree $d$ homogeneous polynomials in two coordinates $s, t$ of $\mathbb{C}^2$:

$$p(s, t) = a_0 s^d + a_1 s^{d-1} t + a_2 s^{d-2} t^2 + \cdots + a_d t^d.$$  \tag{2.23}

The parameter space of quasi maps is given by $\mathbb{C}^{N(d+1)} = \{(a_0, a_1, \ldots, a_d)\}$. We denote by $M_{0,2}(N, d)$ the space obtained from dividing $\{(a_0, \ldots, a_d) \in \mathbb{C}^{N(d+1)} | a_0 \neq 0, a_d \neq 0\}$ by two $\mathbb{C}^\times$ actions induced from the following two $\mathbb{C}^\times$ actions on $\mathbb{C}^2$ via the map $p$ in (2.23).

$$\begin{align*}
(s, t) &\to (\mu s, \mu t), \\
(s, t) &\to (s, \nu t).
\end{align*}$$  \tag{2.24}

With the above two torus actions, $M_{0,2}(N, d)$ can be regarded as the parameter space of degree $d$ quasi maps from $CP^1$ to $CP^{N-1}$ with two marked points in $CP^1$: $0 (= (1 : 0))$ and $\infty (= (0 : 1))$. Set theoretically, it is given as follows:

$$M_{0,2}(N, d) = \{(a_0, a_1, \ldots, a_d) \in \mathbb{C}^{N(d+1)} | a_0, a_d \neq 0\}/(\mathbb{C}^\times)^2,$$  \tag{2.25}

where the two $\mathbb{C}^\times$ actions are given by:

$$\begin{align*}
(a_0, a_1, \ldots, a_d) &\to (\mu a_0, \mu a_1, \ldots, \mu a_{d-1}, \mu a_d) \\
(a_0, a_1, \ldots, a_d) &\to (a_0, \nu a_1, \ldots, \nu^{d-1} a_{d-1}, \nu^d a_d)
\end{align*}$$  \tag{2.26}

The condition $a_0, a_d \neq 0$ assures that the images of 0 and $\infty$ are well-defined in $CP^{N-1}$.

At this stage, we have to note the difference between the moduli space of holomorphic maps from $CP^1$ to $CP^{N-1}$ and the moduli space of quasi maps from $CP^1$ to $CP^{N-1}$. In short, the latter includes the points that are not actual maps from $CP^1$ to $CP^{N-1}$ but rational maps from $CP^1$ to $CP^{N-1}$. Let us consider a quasi map $\sum_{j=0}^d a_j s^j t^{d-j}$ which can be factorized as

$$\sum_{j=0}^d a_j s^j t^{d-j} = p_{d-d_1}(s, t) \cdot \left(\sum_{j=0}^{d_1} c_j s^j t^{d_1-j}\right),$$  \tag{2.27}

where $p_{d-d_1}(s, t)$ is a homogeneous polynomial of degree $d - d_1 (> 0)$ and $\sum_{j=0}^{d_1} c_j s^j t^{d_1-j}$ represents an actual holomorphic map of degree $d_1$ from $CP^1$ to $CP^{N-1}$. If we consider $\sum_{j=0}^d a_j s^j t^{d-j}$ as a map from
$CP^1$ to $CP^{N-1}$, it should be regarded as a rational map whose images of the zero points of $p_{d_i}$ is undefined. Moreover, the closure of the image of this map is a rational curve of degree $d_1(< d)$ in $CP^{N-1}$.

The reason why we include this kind of quasimap is that we can obtain simpler compactification of the moduli space than the moduli space of the stable maps $\bar{M}_{0,2}(CP^{N-1}, d)$, the standard moduli space used to define the two-point Gromov-Witten invariants.

Now, let us turn into the problem of compactification of $M_{p_{0,2}}(N, d)$. If $d = 1$, $M_{p_{0,2}}(N, 1)$ is given by,

$$M_{p_{0,2}}(N, 1) = \{(a_0, a_1) \in C^{2N} | a_0, a_1 \neq 0\} / (C^\times)^2,$$

where $(C^\times)^2$ action is given as follows.

$$(a_0, a_1) \rightarrow (\mu a_0, \mu a_1)$$

$$\mathbf{(a_0, a_1) \rightarrow (a_0, \nu a_1).}$$

Therefore, $M_{p_{0,2}}(N, 1)$ is nothing but $CP^{N-1} \times CP^{N-1}$ and is already compact. If $d \geq 2$, we have to use the two $C^\times$ actions in (2.26) to turn $a_0$ and $a_d$ into the points in $CP^{N-1}$, $[a_0]$ and $[a_d]$. Therefore, we can easily see,

$$M_{p_{0,2}}(N, d) = \{(a_0, a_1, \ldots, a_{d-1}, [a_d]) \in CP^{N-1} \times C^{N(d-1)} \times CP^{N-1} \} / Z_d.$$  \hspace{1cm} (2.30)

In (2.30), the $Z_d$ acts on $C^{N(d-1)}$ as follows.

$$(a_1, a_2 \cdots, a_{d-1}) \rightarrow ((\zeta_d)^j a_1, (\zeta_d)^j a_2 \cdots, (\zeta_d)^{(d-1)j} a_{d-1}),$$

where $\zeta_d$ is the $d$-th primitive root of unity. In this way, we can see that $M_{p_{0,2}}(N, d)$ is not compact if $d \geq 2$. In order to compactify $M_{p_{0,2}}(N, d)$, we imitate the stable map compactification and add the following chains of quasi maps

\[\cup_{j=1}^{l(\sigma_d)} \sum_{m_j=0}^{d_j-d_j-1} a_{d_j-1+m_j} (s_j)^{m_j} (t_j)^{d_j-d_j-1-m_j}, \quad (a_{d_j} \neq 0, \ j = 0, 1, \cdots, l(\sigma_d)) \] \hspace{1cm} (2.32)

at the infinity locus of $M_{p_{0,2}}(N, d)$. In (2.32), $d_j$’s are integers that satisfy,

$$1 \leq d_1 < d_2 < \cdots < d_{l(\sigma_d)} \leq d - 1.$$ \hspace{1cm} (2.33)

We denote by $\bar{M}_{p_{0,2}}(N, d)$ the space obtained after this compactification. This $\bar{M}_{p_{0,2}}(N, d)$ is the moduli space we use in this paper. It is explicitly constructed as a toric orbifold by introducing boundary divisor coordinates $u_1, u_2, \cdots u_{d-1}$ as follows.

$$\bar{M}_{p_{0,2}}(N, d) = \{(a_0, a_1, \ldots, a_d, u_1, u_2, \ldots, u_{d-1}) \in C^{N(d+1)+d-1} | a_0, (a_1, u_1), \cdots, (a_{d-1}, u_{d-1}), a_d \neq 0\} / (C^\times)^{d+1},$$ \hspace{1cm} (2.34)

where the $(d+1)$ $C^\times$ actions are given by,

$$(a_0, a_1, \cdots, a_d, u_1, \cdots, u_{d-1}) \rightarrow (\mu a_0, \cdots, \mu^{-1} u_1, \cdots),$$

$$(a_0, a_1, \cdots, a_d, u_1, \cdots, u_{d-1}) \rightarrow (\cdots, \mu a_1, \cdots, \mu^{-2} u_1, \mu^{-1} u_2, \cdots),$$

$$(a_0, a_1, \cdots, a_d, u_1, \cdots, u_{d-1}) \rightarrow (\cdots, \mu_i a_i, \cdots, \mu_{i-1} u_i, \mu_{i+1} u_{i+1}, \cdots), \quad (i = 2, \cdots, d - 1),$$

$$(a_0, a_1, \cdots, a_d, u_1, \cdots, u_{d-1}) \rightarrow (\cdots, \mu_{d-1} a_{d-1}, \cdots, \mu_{d-1} u_{d-2}, \mu_{d-2} u_{d-1}),$$

$$(a_0, a_1, \cdots, a_d, u_1, \cdots, u_{d-1}) \rightarrow (\cdots, \mu a_d, \cdots, \mu_d u_{d-1}).$$ \hspace{1cm} (2.35)
In (2.35), "\(\cdots\)" in the r.h.s indicates that the \(\mathbf{C}^\times\) actions are trivial. These torus actions are represented by a \((d + 1) \times 2d\) weight matrix \(W_d:\)

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 2 & -1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & -1 & 2 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & -1 \\
\end{pmatrix}
\]

(2.36)

Notice that the \(A_{d-1}\) Cartan matrix appears in \(W_d\). If \(u_1, u_2, \cdots, u_{d-1} \neq 0\), we can set all the \(u_i\)'s to 1 by using the \((d + 1)\) torus actions. The remaining two torus actions that leave them invariant are nothing but the ones given in (2.26). Therefore, the subspace given by the condition \(u_1, u_2, \cdots, u_{d-1} \neq 0\) corresponds to \(M_{p_0,2}(N, d)\). If \(u_d = 0, u_j \neq 0\) \((j \neq d)\), we have to delete the \(u_d\) column of matrix \(W_d\). This operation turns the \(A_{d-1}\) Cartan matrix into the \(A_{d-1} \times A_{d-1} \times A_{d-1}\) Cartan matrix and results in chains of two quasi maps:

\[
\left\{ \sum_{j=0}^{d_1} a_j s_1^{(d_1 - j)} \right\} \cup \left\{ \sum_{j=0}^{d-d_1} a_j s_2^{(d-d_1 - j)} \right\}, \quad (a_0, a_{d_1}, a_d \neq 0).
\]

(2.37)

Therefore, the corresponding boundary locus is given by \(M_{p_0,2}(N, d_1) \times_{CP^{N-1}} M_{p_0,2}(N, \delta_1 - d_1)\), where \(\times_{CP^{N-1}}\) is the fiber product with respect to the following projection maps:

\[
e_{v_\infty} : M_{p_0,2}(N, d_1) \to CP^{N-1}, \quad e_{v_\infty}(a_0, \cdots, a_{d_1}) = [a_{d_1}]
\]

(2.38)

In general, the subspace given by the condition

\[
u_{d_i} = 0, \quad (1 \leq d_1 < d_2 < \cdots < d_{l(\sigma_d)} - 1 \leq d - 1), \quad u_j \neq 0, \quad (j \notin \{d_1, d_2, \cdots, d_{l(\sigma_d)} - 1\}),
\]

(2.39)

corresponds to chains of quasi maps labeled by ordered partition \(\sigma_d = (d_1 - d_0, d_2 - d_1, d_3 - d_2, \cdots, d_{l(\sigma_d)} - d_{l(\sigma_d) - 1})\):

\[
\left\{ \sum_{m_j=0}^{d_j - d_{j-1}} a_{d_{j-1} + m_j} s_j^{m_j} (t_j)^{d_j - d_{j-1} - m_j} \right\}, \quad (a_{d_j} \neq 0, \quad j = 0, 1, \cdots, l(\sigma_d))
\]

(2.40)

where we set \(d_0 = 0, d_{l(\sigma_d)} = d\). In this case, the corresponding boundary locus is,

\[
M_{p_0,2}(N, d_1 - d_0) \times_{CP^{N-1}} M_{p_0,2}(N, d_2 - d_1) \times_{CP^{N-1}} \cdots \times_{CP^{N-1}} M_{p_0,2}(N, \delta_{l(\sigma_d)} - d_{l(\sigma_d) - 1}).
\]

(2.41)

Since the lowest dimensional boundary:

\[
M_{p_0,2}(N, 1) \times_{CP^{N-1}} M_{p_0,2}(N, 1) \times_{CP^{N-1}} \cdots \times_{CP^{N-1}} M_{p_0,2}(N, 1),
\]

(2.42)

is identified with the compact space \((CP^{N-1})^{d+1}\), we can expect that \(\tilde{M}_{p_0,2}(N, d)\) is compact. As was mentioned before, the fact that \(\tilde{M}_{p_0,2}(N, d)\) is compact was proved by Saito [22].
3 Proof of Theorem 1.1

As was shown in the previous section, \( \overline{M}_{0,2}(N, d) \) has the following stratification.

\[
\overline{M}_{0,2}(N, d) = \prod_{l=1}^{d} \prod_{d_{0} < d_{1} < \cdots < d_{l-1} < d_{l}=d} \left( \overline{M}_{0,2}(N, d_{l} - d_{0}) \times \cdots \times \overline{M}_{0,2}(N, d_{1} - d_{0}) \right).
\]

(3.43)

Let us consider the stratum of highest dimension,

\[
\overline{M}_{0,2}(N, d) = \{(a_{0}, a_{1}, \cdots, a_{d}) \mid a_{i} \in \mathbb{C}^{N}, a_{0} \neq 0, a_{d} \neq 0 \}/(\mathbb{C}^{\times})^{2},
\]

(3.44)

where the two \( \mathbb{C}^{\times} \) actions are given by,

\[
(a_{0}, a_{1}, \cdots, a_{d-1}, a_{d}) \rightarrow (\lambda^{d}a_{0}, \lambda^{d-1}a_{1}, \cdots, \lambda a_{d-1}, a_{d}),
\]

\[
(a_{0}, a_{1}, \cdots, a_{d-1}, a_{d}) \rightarrow (a_{0}, \nu a_{1}, \cdots, \nu^{d-1}a_{d-1}, \nu^{d}a_{d}).
\]

(3.45)

\([a_{0}, a_{1}, \cdots, a_{d-1}, a_{d}]\) represents a rational map \( \varphi(s : t) = \pi_{N}(\sum_{j=0}^{d} a_{j} s^{j} t^{d-j}) \) from \( CP^{1} \) to \( CP^{N-1} \) modulo \( \mathbb{C}^{\times} \) action on \( CP^{1} \) that fixes \( 0 = (0 : 1), \infty = (1 : 0) \in CP^{1} \):

\[
(s : t) \rightarrow (s : \lambda t).
\]

(3.46)

Here, \( \pi_{N} : \mathbb{C}^{N} \backslash \{0\} \rightarrow CP^{N-1} \) is the projective equivalence. Note that \( \sum_{j=0}^{d} a_{j} s^{j} t^{d-j} \) is factorized into the following form up to \( \mathbb{C}^{\times} \) multiplication.

\[
\sum_{j=0}^{d} a_{j} s^{j} t^{d-j} = \left( \prod_{j=1}^{l(\sigma_{g})} (\beta_{j}s - \alpha_{j}t)^{g_{j}} \right) \cdot \left( \sum_{j=0}^{d-g} c_{j} s^{j} t^{d-g-j} \right),
\]

(3.47)

where \( \pi_{N}(\sum_{j=0}^{d-g} c_{j} s^{j} t^{d-g-j}) \) defines a well-defined map from \( CP^{1} \) to \( CP^{N-1} \).

Since \( a_{0}, a_{d} \neq 0 \), it follows that \( z_{j} := (\alpha_{j} : \beta_{j}) \) never coincides with 0 and \( \infty \) in \( CP^{1} \). These distinct points \( (z_{j} \mid j = 1, \cdots, l(\sigma_{g})) \) also represent points where \( \varphi(s : t) \) is ill-defined. Since \([a_{0}, a_{1}, \cdots, a_{d-1}, a_{d}]\) represents \( \varphi(s : t) \) modulo the \( \mathbb{C}^{\times} \) action on \( CP^{1} \), configuration of \( z_{j} \)'s should be considered modulo the \( \mathbb{C}^{\times} \) action given in (3.46).

We can easily see that the factorization in (3.47) is invariant under permutation of \( z_{j} \)'s with subscript \( j \) that has the same value of \( g_{j} \). With these considerations, we obtain the following decomposition of \( \overline{M}_{0,2}(N, d) \).

\[
\overline{M}_{0,2}(N, d) = \prod_{g=0}^{d} \prod_{\sigma_{g} \in \mathcal{P}_{g}} M_{0,2+l(\sigma_{g})}(CP^{N-1}, d - g, \sigma_{g}).
\]

(3.48)

In the above decomposition, \( M_{0,2+l(\sigma_{g})}(CP^{N-1}, d - g, \sigma_{g}) \) is uncompactified moduli space of degree \( d - g \) holomorphic map from \( CP^{1} \) to \( CP^{N-1} \) with \( 2 + l(\sigma_{g}) \) distinct marked points divided by \( \prod_{i=1}^{g} \text{Sym} (\text{mul}(i, \sigma_{g})) \) action.

\[
M_{0,2+l(\sigma_{g})}(CP^{N-1}, d - g, \sigma_{g}) := \{[(\pi_{N}(\sum_{j=0}^{d-g} c_{j} s^{j} t^{d-g-j}), (0, \infty, z_{1}, z_{2}, \cdots, z_{l(\sigma_{g})}))]/\left(\prod_{i=1}^{g} \text{Sym} (\text{mul}(i, \sigma_{g}))\right)\}.
\]

(3.49)

In the above definition, the tuple \([(\pi_{N}(\sum_{j=0}^{d-g} c_{j} s^{j} t^{d-g-j}), (0, \infty, z_{1}, z_{2}, \cdots, z_{l(\sigma_{g})}))]/\left(\prod_{i=1}^{g} \text{Sym} (\text{mul}(i, \sigma_{g}))\right)\) is considered modulo the \( \mathbb{C}^{\times} \) action on \( CP^{1} \) and the symmetric group \( \text{Sym} (\text{mul}(i, \sigma_{g})) \) permutes \( z_{j} \)'s that satisfy \( g_{j} = i \).
Let $\mathcal{E}_d^k$ be a vector bundle on $M_{0,2}(N, d)$ that impose on $\varphi \in M_{0,2}(N, d)$ the condition that $\varphi(CP^1)$ is contained in a generic degree $k$ hypersurface in $CP^{N-1}$. $\mathcal{E}_d^k$ is extended to a rank $kd + 1$ vector bundle $\tilde{\mathcal{E}}_d^k$ on $\tilde{M}_{0,2}(N, d)$. Then we introduce the following intersection number of $\tilde{M}_{0,2}(N, d)$.

$$w(O_{h^a} \mathcal{O}_{h^b})_{0, d} = \int_{\tilde{M}_{0,2}(N, d)} c_{\text{top}}(\tilde{\mathcal{E}}_d^k) \wedge ev_0^*(h^a) \wedge ev_\infty^*(h^b).$$

$$a + b = N - 3 + (N - k)d$$

In the above definition, $ev_0$ and $ev_\infty$ are the evaluation maps at $(0: 1) = 0$ and $(1: 0) = \infty$ respectively. On the locus $M_{0,2 + l(\sigma_g)}(CP^{N-1}, d - g, \sigma_g)$, the condition that the section of $\tilde{\mathcal{E}}_d^k$ induced from defining equation of the hypersurface vanishes, lowers the dimension only by $k(d - g) + 1$. Hence if $k(d - g) + 1 + N - 3 + (N - k)d = N - 2 + N d - k g \leq N - 1 + N(d - g) + l(\sigma_0) - 1 \iff (k - N) g + l(\sigma_0) \geq 0$, this locus contributes to the intersection number. Moreover, if $(k - N) g + l(\sigma_0) > 0$, we have excess intersection on the locus. In order to evaluate contribution from this excess intersection, we introduce “perturbation space” defined as follows.

$$\tilde{M}_{0,2}^{\text{pert}}(N, d, \sigma_g)$$

$$:= \left( M_{0,2 + l(\sigma_g)}(CP^{N-1}, d - g) \left( \prod_{i=1}^{l(\sigma_g)} \times CP_{N-1} \tilde{M}_{0,2}(N, g_i) \right) \right) / \left( \prod_{i=1}^{g} \text{Sym} \left( \text{mul} (i, \sigma_g) \right) \right).$$

where $M_{0,2 + l(\sigma_g)}(CP^{N-1}, d - g)$ is the uncompactified moduli space of holomorphic maps from $CP^1$ with $2 + l(\sigma_g)$ marked points to $CP^{N-1}$.

$$M_{0,2 + l(\sigma_g)}(CP^{N-1}, d - g) := \{ \langle \pi_N \left( \sum_{j=0}^{d-g} c_j s^j t^{d-g-j} \right), (0, \infty, z_1, z_2, \cdots, z_{l(\sigma_g)}) \rangle \},$$

and the $i$-th fiber product is defined via the following diagrams,

$$M_{0,2 + l(\sigma_g)}(CP^{N-1}, d - g) \overset{(ev_i)_!}{\rightarrow} (CP^{N-1})^{l(\sigma_g)},$$

and,

$$\prod_i \tilde{M}_{0,2}(N, g_i) \overset{\prod_i \text{ev}_0}{\rightarrow} (CP^{N-1})^{l(\sigma_g)}.$$

In the diagram (3.53), $ev_i$ is the evaluation map at $z_i$. $\prod_{i=1}^{g} \text{Sym} \left( \text{mul} (i, \sigma_g) \right)$ permutes $z_j$ together with $\times CP_{N-1} \tilde{M}_{0,2}(N, g_j)$ in the same way as the definition of $M_{0,2 + l(\sigma_g)}(CP^{N-1}, d - g, \sigma_g)$.

Let us explain geometrical meaning of $\tilde{M}_{0,2}^{\text{pert}}(N, d, \sigma_g)$. Roughly speaking, it generates “infinitesimal deformation of quasimap in $M_{0,2 + l(\sigma_g)}(CP^{N-1}, d - g, \sigma_g)$” that evaluates contribution of the excess intersection coming from $M_{0,2 + l(\sigma_g)}(CP^{N-1}, d - g, \sigma_g)$, to $w(O_{h^a} \mathcal{O}_{h^b})_{0, d}$. In order to illustrate the idea, we introduce new homogeneous coordinates ($\tilde{s} : \tilde{t} := (\beta s - \alpha t : t)$ of $CP^1$. When $\sum_{j=0}^{d} \tilde{a}_j \beta^j \beta^d - j$ is factorized in the form given in (3.47), it is written in terms of the new homogeneous coordinates as follows:

$$\sum_{j=g_i}^{d} \tilde{a}_j \tilde{\beta}^{j-d} = \tilde{g}_n \left( \sum_{j=g_i}^{d} \tilde{a}_j \tilde{\beta}^{j-g_i} \tilde{\beta}^{d-j} \right), \quad (\tilde{a}_{g_i}, \tilde{a}_d \neq 0).$$

2The reason why this space can be used to evaluate contribution of the excess intersection will be explained in the proof of Lemma A.1.
Therefore, deformation of this quasimap is given by,
\[
\sum_{j=0}^{g_i} \tilde{a}^j \tilde{t}^{d-j} = \tilde{t}^{d-g_i} \left( \sum_{j=0}^{g_i} \tilde{a}^j \tilde{t}^{g_i-j} \right),
\]
which can be considered as a point in $\overline{MP}_{0,2}(N, g)$. Since $\tilde{a}_g$ appears both in (3.35) and (3.36), we use fiber product $\times_{CP^{N-1}}$. Note here that the quasimap $\sum_{j=0}^{g_i} \tilde{a}^j \tilde{t}^{g_i-j}$ should be regarded as a quasimap from different $CP^1$ component attached to the original $CP^1$ at $z_i$, in order to describe “infinitesimalness” of the deformation. Remaining construction will be given in Appendix A.

Dimension of $\overline{MP}_{0,2}(N, d, \sigma_g)$ is given by,
\[
N - 1 + N(d - g) - 1 + l(\sigma_g) + \sum_{i=1}^{l(\sigma_g)} (N - 2 + Ng_i) - l(\sigma_g)(N - 1)
= N - 2 + N d = \dim_{\mathbb{C}}(\overline{MP}_{0,2}(N, d)).
\]

Hence this space can give us non-vanishing result. We can evaluate the contribution of the excess intersection coming from $M_{0,2+1(\sigma_g)}(CP^{N-1}, d - g, \sigma_g)$ to the intersection number $w(O_{h\ast}O_h)_{0,d}$ by extending the vector bundle $\tilde{E}_d$ to $\overline{MP}_{0,2}(N, d, \sigma_g)$. Then the contribution is given as follows,
\[
\int_{\overline{MP}_{0,2}(N, d, \sigma_g)} c_{top}(\tilde{E}_d) \wedge ev^*_0(h^a) \wedge ev^*_\infty(h^b)
= \langle O_{h\ast}O_h \prod_{i=1}^{l(\sigma_g)} O_{h^{1+(k-N)g_i}} \rangle_{0,d-g} \left( \prod_{i=1}^{l(\sigma_g)} w(O_{h^{N+3+(N-k)g_i}}O_1)_{0,g_i} \right) \left( \prod_{i=1}^{g} \frac{1}{\text{mul}(i, \sigma_g)} \right),
\]

since $w(O_{h^{N+3+(N-k)g_i}}O_1)_{0,g_i}$ (that contains insertion of “1”, or insertion of nothing) is non-vanishing. In deriving the above formula, we also used splitting axiom [17] of the fiber product:
\[
\times_{CP^{N-1}} \underbrace{O_{h\ast}O_h \prod_{i=1}^{l(\sigma_g)} O_{h^{1+(k-N)g_i}}}_{M_{0,2+1(\sigma_g)}(CP^{N-1}, d - g)} \xrightarrow{ev_0} CP^{N-1} \xrightarrow{ev_0} \overline{MP}_{0,2}(N, g_i)
= \frac{1}{k} \sum_{c_i=0}^{N-2} ev^*_i(h^{c_i}) \wedge ev^*_0(h^{N-2-c_i}),
\]

which follows from description of Poincaré dual of diagonal $\Delta$ in $CP^{N-1} \times CP^{N-1}$ (precisely speaking $M_{\overline{N}} \times M_{\overline{N}}$) by cohomology elements. Then we are naturally led to consider,
\[
\sum_{c_1=0}^{N-2} \cdots \sum_{c_{(\sigma_g)}=0}^{N-2} \langle O_{h\ast}O_h \prod_{i=1}^{l(\sigma_g)} O_{h^{1+(k-N)g_i}} \rangle_{0,d-g} \left( \prod_{i=1}^{l(\sigma_g)} w(O_{h^{N+3+(N-k)g_i}}O_1)_{0,g_i} \right),
\]

but we are forced to set $c_i = 1 + (k - N)g_i$ because we have the condition (1.4).

$\langle O_{h\ast}O_h \prod_{i=1}^{l(\sigma_g)} O_{h^{1+(k-N)g_i}} \rangle_{0,d-g}$ in (3.38) is the Gromov-Witten invariant of the hypersurface which is defined by,
\[
\langle O_{h\ast}O_h \prod_{i=1}^{l(\sigma_g)} O_{h^{1+(k-N)g_i}} \rangle_{0,d-g} := \int_{M_{0,2+1(\sigma_g)}(CP^{N-1}, d - g)} c_{top}(\tilde{E}_d) \wedge ev^*_0(h^a) \wedge ev^*_\infty(h^b) \wedge \left( \prod_{i=1}^{l(\sigma_g)} ev^*_i(h^{1+(k-N)g_i}) \right).
\]
In the above formula, the moduli space $M_{0,2+\ell}(\sigma_g)(CP^{N-1}, d-g)$ is not compactified. But note that boundary components added in compactification by stable maps do not contribute to intersection numbers because they have positive codimensions under insertion of $c_{top}(E_{d-g})$. The last factor of the r.h.s. of (3.58) appears as the effect of dividing by the group $\prod_{i=1}^\ell \text{Sym}(\text{mult}(i, \sigma_g))$.

In the $d=g$ case, $\langle \mathcal{O}_{h^*}^\bullet \mathcal{O}_h \prod_{i=1}^{l(\sigma_g)} \mathcal{O}_{h_1+(k-N)g_i} \rangle_{0,0} = 0$ vanishes if $l(\sigma_g)$ is greater than 1. If $l(\sigma_g) = 1$, we have,

$$\langle \mathcal{O}_{h^*}^\bullet \mathcal{O}_h \prod_{i=1}^{l(\sigma_g)-1} \mathcal{O}_{h_1+(k-N)g_i} \rangle_{0,0} = k. \quad (3.62)$$

Of course, we can introduce perturbation spaces for the lower dimensional strata of $\widetilde{M}_{p_0,2}(N, d)$, but they are irrelevant to the intersection number because they have positive codimensions.

In this way, we obtain the following equality:

$$w(\mathcal{O}_{h^*}^\bullet \mathcal{O}_h)_0 = \langle \mathcal{O}_{h^*}^\bullet \mathcal{O}_h \prod_{i=1}^{l(\sigma_g)} \mathcal{O}_{h_1+(k-N)g_i} \rangle_{0,0} = 0.$$

$$+ w(\mathcal{O}_{h^*}^\bullet \mathcal{O}_h \prod_{i=1}^{l(\sigma_g)-1} \mathcal{O}_{h_1+(k-N)g_i})_0 \prod_{i=1}^g \frac{1}{\text{mult}(i, \sigma_g)} = 0.$$  \quad (3.63)

This is nothing but the generalized mirror transformation given in (1.10)!

## A Some Comments on the Perturbation Space

In this part, we add some comments on background idea of construction of the perturbation space:

$$\widetilde{M}_{p_0,2}^{\text{pert}}(N, d, \sigma_g) = \left( \left( \bigcup_{i=1}^\ell \mathcal{M}_{p_0,2}(N_i, g_i) \right) \left( \bigcup_{i=1}^\ell \mathcal{M}_{p_0,2}(N_i, d_i) \right) \big/ \prod_{i=1}^\ell \text{Sym}(\text{mult}(i, \sigma_g)) \right). \quad (A.64)$$

As was suggested in (3.55), the vector valued polynomial $\sum_{j=g_i}^d \tilde{a}_j s_j^{g_i} t^{d-g_i}$ defines a quasi map in $\mathcal{M}_{p_0,2}(N, d-g_i)$. If we regard $(\tilde{s} : \tilde{t})$ as the original homogeneous coordinates $(s : t)$ used in the construction of $\widetilde{M}_{p_0,2}(N, d)$, the quasi map $s^{g_i} \left( \sum_{j=g_i}^d \tilde{a}_j s_j^{g_i} t^{d-g_i} \right)$ corresponds to the following boundary components of $\widetilde{M}_{p_0,2}(N, d)$,

$$\prod_{i=1}^\ell \prod_{d_0<d_1<\cdots<d_{d_i}=g_i} \left( \bigcup_{C_{P^{N-1}}} \left. \mathcal{M}_{p_0,2}(N, d_i - d_0) \times \cdots \times \bigcup_{C_{P^{N-1}}} \mathcal{M}_{p_0,2}(N, d_i - d_{i-1}) \times \bigcup_{C_{P^{N-1}}} \mathcal{M}_{p_0,2}(N, d - g_i) \right) \big/ \prod_{i=1}^\ell \text{Sym}(\text{mult}(i, \sigma_g)) \right). \quad (A.65)$$

Of course, the part $\widetilde{M}_{p_0,2}(N, g_i) \times C_{P^{N-1}}$ is not contained in the original $\widetilde{M}_{p_0,2}(N, d)$. But by applying $\widetilde{M}_{p_0,2}(N, g_i) \times C_{P^{N-1}}$, we can create the perturbation without changing $\sum_{j=g_i}^d \tilde{a}_j s_j^{g_i} t^{d-g_i}$ that produces
the corresponding boundary components of $\overline{M}_{0,2}(N, d)$ in case that we regard $z_i = (\alpha_i : \beta_i)$ ($\leftrightarrow (\hat{s} : \hat{t}) = (0 : 1)$) as the 0 = (0 : 1) in the original construction of $\overline{M}_{0,2}(N, d)$. Successive operation of $\overline{M}_{0,2}(N, g_i) \times_{CP^{N-1}} (i = 1, 2, \cdots, l(\sigma_y))$ is the idea behind construction of the perturbation space.

Next, we explain why the moduli space $\overline{M}_{0,2}(N, d)$ can evaluate the contribution from excess intersection. Let us take for example a subset of $M_{0,2}(N, d)$:

$$U = \{ [\bar{a}_0 (s-t)^d] (= [\bar{a}_0 (\lambda s-t)^d] (\lambda \in C^\times)) | [\bar{a}_0] \in CP^{N-1} \} \quad (A.66)$$

which corresponds to maximally degenerated locus. If we introduce new homogeneous coordinates $(\hat{s} : \hat{t})$ with $\hat{s} = s - t$, the above quasimap is rewritten as $[\bar{a}_0 \hat{s}^d]$. Then the reason comes from the following lemma.

**Lemma A.1.** The contribution from excess intersection coming from the subset $U$ in (A.66) to $w(\mathcal{O}_{h^a} \mathcal{O}_{h^b})_{0,d}$ $(a + b = N - 3 + (N - k)d)$ equals to $w(\mathcal{O}_{h^{N-3+(N-k)d}} \mathcal{O}_1)_{0,d}$.

*Proof.* Since the image of the quasimap in $U$ is a point $[\bar{a}_0] \in CP^{N-1}$, the excess intersection of $w(\mathcal{O}_{h^a} \mathcal{O}_{h^b})_{0,d}$ caused by $U$ is given by intersection of a codimension $a + b = N - 3 + (N - k)d$ hypersurface in $CP^{N-1} = \{ [\bar{a}_0] \}$ and a degree $k$ hypersurface in the same $CP^{N-1}$.

On the other hand, let us consider $w(\mathcal{O}_{h^{N-3+(N-k)d}} \mathcal{O}_1)_{0,d}$. It is defined by,

$$w(\mathcal{O}_{h^{N-3+(N-k)d}} \mathcal{O}_1)_{0,d} = \int_{\overline{M}_{0,2}(N, d)} c_{top}(\overline{\mathcal{E}}^h_d) \wedge ev_0^*(h^{N-3+(N-k)d}). \quad (A.67)$$

A point in $M_{0,2}(N, d)$ is represented by a quasimap:

$$a_0 s^d + a_1 s^{d-1} t + a_2 s^{d-2} t^2 + \cdots + a_d t^d \quad (a_0, a_1 \neq 0), \quad (A.68)$$

where we can assume $||a_0|| = ||a_d|| = 1$ by using the two $C^\times$ action in (3.45). But we have equivalence relation under the two $C^\times$ action, and we can use another representative:

$$a_0 s^d + (\lambda) a_1 s^{d-1} t + (\lambda)^2 a_2 s^{d-2} t^2 + \cdots + (\lambda)^d a_d t^d \quad (||a_0|| = ||a_1|| = 1), \quad (A.69)$$

with fixed $\lambda \in C^\times$. Then let us take the limit $\lambda \to 0$. This is possible because we have no operator insertion at $(s : t) = (0 : 1)$. Then the moduli space degenerates to $\{ [a_0 s^d] | [a_0] \in CP^{N-1} \}$ that equals to $U$ and the intersection represented by $w(\mathcal{O}_{h^{N-3+(N-k)d}} \mathcal{O}_1)_{0,d}$ reduces to intersection of a codimension $N - 3 + (N - k)d$ hypersurface in $CP^{N-1} = \{ [a_0] \}$ and a degree $k$ hypersurface in the same $CP^{N-1}$. Hence we obtain the assertion of the lemma. \(\square\)
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