FOURIER UNIQUENESS IN $\mathbb{R}^4$

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ABSTRACT. We show an interrelation between the uniqueness aspect of the recent Fourier interpolation formula of Radchenko and Viazovska and the Heisenberg uniqueness for the Klein-Gordon equation and the lattice-cross of critical density, studied by Hedenmalm and Montes-Rodríguez. This has been known since 2017.

1. INTRODUCTION

1.1. Basic notation in the plane. We write $\mathbb{Z}$ for the integers, $\mathbb{Z}_+$ for the positive integers, $\mathbb{R}$ for the real line, and $\mathbb{C}$ for the complex plane. We write $\mathbb{H}$ for the upper half-plane $\{\tau \in \mathbb{C} : \text{Im} \tau > 0\}$. Moreover, we let $\langle \cdot, \cdot \rangle_d$ denote the Euclidean inner product of $\mathbb{R}^d$.

1.2. The Fourier transform of radial functions. For a function $f \in L^1(\mathbb{R}^d)$, we consider its Fourier transform (with $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$)

$$\hat{f}(y) := \int_{\mathbb{R}^d} e^{-2\pi i \langle x, y \rangle} f(x) \, d\text{vol}_d(x), \quad d\text{vol}_d(x) := dx_1 \cdots dx_d.$$ 

If $f$ is radial, then $\hat{f}$ is radial too. A particular example of a radial function is the Gaussian

$$(1.2.1) \quad G_\tau(x) := e^{i \pi \tau |x|^2},$$

which decays nicely provided that $\text{Im} \tau > 0$, that is, when $\tau \in \mathbb{H}$. The Fourier transform of a Gaussian is another Gaussian, in this case

$$(1.2.2) \quad \hat{G}_\tau(y) := \left(\frac{\tau}{1} \right)^{-d/2} e^{-i \pi |y|^2/\tau} G_{-1/\tau}(y),$$

Here, it is important that $\tau \mapsto -1/\tau$ preserves hyperbolic space $\mathbb{H}$. In the sense of distribution theory, the above relationship extends to boundary points $\tau \in \mathbb{R}$ as well. We now consider the relationship

$$(1.2.3) \quad \Phi(x) := \int_{\mathbb{R}} G_\tau(x) \phi(\tau) \, d\tau = \int_{\mathbb{R}} e^{i \pi \tau |x|^2} \phi(\tau) \, d\tau, \quad x \in \mathbb{R}^d.$$ 

In terms of the Fourier transform, the relationship reads

$$\Phi(x) = \hat{\phi}_1 \left( -\frac{|x|^2}{2} \right),$$

where the subscript signifies that we are dealing with the Fourier transform on $\mathbb{R}^1$. This tells us that $\Phi$ is radial, but pretty arbitrary, if, say, $\phi \in L^1(\mathbb{R})$. In view of the functional identity (1.2.1), the Fourier transform of the radial function $\Phi$ equals

$$(1.2.4) \quad \hat{\Phi}(y) := \int_{\mathbb{R}} \hat{G}_\tau(y) \phi(\tau) \, d\tau = \int_{\mathbb{R}} \left(\frac{\tau}{1} \right)^{-d/2} G_{-1/\tau}(y) \phi(\tau) \, d\tau = \int_{\mathbb{R}} \left(\frac{\tau}{1} \right)^{-d/2} e^{-i \pi |y|^2/\tau} \phi(\tau) \, d\tau.$$
We now rewrite the relationships (1.2.3) and (1.2.4) using integration by parts. If $\phi$ is a tempered test function, integration by parts applied to (1.2.3) gives that

$$\Phi(x) = \frac{i}{\pi|x|^2} \int_{R^d} e^{i\tau|x|^2} \phi'(\tau) d\tau, \quad x \in \mathbb{R}^d \setminus \{0\}. \tag{1.2.5}$$

A similar application of integration by parts to (1.2.4) gives that

$$\hat{\Phi}(y) = \frac{1}{i|y|^2} \int_{R^d} \left(\frac{\tau}{1 + i|\tau|^2}\right) \phi(\tau) d\tau, \quad y \in \mathbb{R}^d \setminus \{0\}. \tag{1.2.6}$$

The setup. We consider $\mathbb{R}^d$ only, and consider for $\psi \in L^1(\mathbb{R})$ the associated function

$$\Psi(x) = -\frac{1}{i|\tau|^2} \int_{R^d} e^{i\tau|x|^2} \psi(\tau) d\tau, \quad x \in \mathbb{R}^d \setminus \{0\}. \tag{2.1.1}$$

This is the same as the relation (1.2.5) only $\psi$ replaces $\phi'$ while $\Psi$ replaces $\Phi$. For real $\tau$, let $H_\tau$ denote the function

$$H_\tau(x) := \frac{e^{i\tau|x|^2}}{i|\tau|^2}, \quad x \in \mathbb{R}^d \setminus \{0\}, \tag{2.1.2}$$

which is locally integrable and decays at infinity. As such, it is a tempered distribution, and its Fourier transform equals

$$\hat{H}_\tau(y) = \frac{1 - e^{-i|y|^2/\tau}}{i|y|^2} = \frac{1}{i|y|^2} - H_{-1/\tau}(y). \tag{2.1.3}$$

This is the integrated version of the Fourier transformation law for Gaussians (1.2.2) in dimension $d = 4$. Indeed, if we differentiate with respect to $\tau$ in (2.1.3), we recover (1.2.2). In other words, differentiation with respect to $\tau$ gives us that $\hat{H}_\tau + H_{-1/\tau}$ is independent of $\tau$. By letting $\tau$ tend to 0, the identification with the Newton kernel as in (2.1.3) follows from the Riemann-Lebesgue lemma. In view of (2.1.3), the Fourier transform of the function $\Psi$ given by (2.1.1) is in the sense of distribution theory

$$\check{\Psi}(y) = -\int_{R^d} H_\tau(y) \psi(\tau) d\tau = -\frac{1}{i|y|^2} \int_{R} \psi(\tau) d\tau + \frac{1}{i|y|^2} \int_{R} e^{-i|\tau|^2/\tau} \psi(\tau) d\tau, \quad y \in \mathbb{R}^d \setminus \{0\}. \tag{2.1.4}$$

This formula extends (1.2.7).
2.2. Fourier uniqueness meets Heisenberg uniqueness and the Klein-Gordon equation. In [3], in the context of the Klein-Gordon equation in 1 + 1 dimensions, Hedenmalm and Montes found discrete uniqueness sets along characteristic directions, based on ideas from dynamical systems and ergodic theory. We apply the approach in [3], [4], [5], and [1] to obtain a uniqueness result for the pair \( \psi, \Psi \) connected by (2.1.1). Let \( H_1^1(\mathbb{R}) \) denote the Hardy space of the upper half-plane. It may be defined as the subspace of functions in \( L^1(\mathbb{R}) \) with Poisson harmonic extension to \( \mathbb{H} \) which is holomorphic.

**Theorem 2.2.1.** Let \( \psi \in L^1(\mathbb{R}) \) and \( \Psi \) be as above. If \( \Psi(x) = \hat{\Psi}(y) = 0 \) holds for all \( x, y \in \mathbb{Z}^4 \setminus \{0\} \), and if \( \Psi(x) = o(|x|^{-2}) \) as \( |x| \to 0 \), then \( \psi \in H_1^1(\mathbb{R}) \) and, as a consequence, \( \Psi(x) \equiv 0 \) on \( \mathbb{R}^4 \setminus \{0\} \).

**Proof.** In view of the assumption that \( \Psi(x) = o(|x|^{-2}) \) as \( |x| \to 0 \), it follows from (2.1.1) that \( \psi \in L^1(\mathbb{R}) \) annihilates the constant function 1. Moreover, by the Lagrange (or Jacobi) four squares theorem, each positive integer may be written as \( |x|^2 \) for some \( x \in \mathbb{Z}^4 \setminus \{0\} \). Consequently, we see from (1.2.5) and (1.2.7) that \( \psi \) also annihilates the subspace of \( L^\infty(\mathbb{R}) \) spanned by the functions \( e^{i\eta m^4} \) and \( e^{-i\eta n^4} \), where \( m, n \in \mathbb{Z}_+ \) and \( \tau \) is the real variable. By Theorem 1.8.2 in [4], which relies on methods developed in [5] and is motivated by [3], we may conclude that \( \psi \in H_1^1(\mathbb{R}) \). Finally, in view of the standard Fourier analysis characterization of \( H_1^1(\mathbb{R}) \), it follows from this and (1.2.7) that \( \Psi = 0 \) on \( \mathbb{R}^4 \setminus \{0\} \).

We return to the initial setup with \( \phi \) and \( \Phi \). We think of \( \phi' = \psi \) and \( \Phi = \Psi \). Let \( C_0(\mathbb{R}) \) denote the space of continuous functions on \( \mathbb{R} \) with limit value 0 at infinity. Then the condition in the origin in Theorem 2.2.1 may be replaced by \( \phi \in C_0(\mathbb{R}) \).

**Corollary 2.2.2.** Let \( \Phi \) be given by (1.2.5), where \( \phi \in C_0(\mathbb{R}) \) with \( \phi' \in L^1(\mathbb{R}) \) and \( d = 4 \). If \( \Phi(x) = \hat{\Phi}(y) = 0 \) for all \( x, y \in \mathbb{Z}^4 \setminus \{0\} \), then \( \phi' \in H_1^1(\mathbb{R}) \) and, as a consequence, \( \Phi(x) \equiv 0 \) on \( \mathbb{R}^4 \setminus \{0\} \).

**Remark 2.2.3.** The above theorem is a four-dimensional analogue of the uniqueness part of the Fourier interpolation formula found by Radchenko and Viazovska [6]. That work in its turn was motivated by Fourier interpolation formulæ associated with optimizing the Cohn-Elkies method for sphere packing [7], [2].

**References**

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