**CONNECTED $k$-FORCING SETS OF GRAPHS AND SPLITTING GRAPHS**

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**Abstract.** The notion of $k$-forcing number of a graph was introduced by Amos et al. For a given graph $G$ and a given subset $I$ of the vertices of the graph $G$, the vertices in $I$ are known as initially colored black vertices and the vertices in $V(G) - I$ are known as not initially colored black vertices or white vertices. The set $I$ is a $k$-forcing set of a graph $G$ if all vertices in $G$ eventually colored black after applying the following color changing rule: If a black colored vertex is adjacent to at most $k$-white vertices, then the white vertices change to be colored black. The cardinality of a smallest $k$-forcing set is known as the $k$-forcing number $Z_k(G)$ of the graph $G$. If the sub graph induced by the vertices in $I$ are connected, then $I$ is called the connected $k$-forcing set. The minimum cardinality of such a set is called the connected $k$-forcing number of $G$ and is denoted by $Z_{ck}(G)$. This manuscript is intended to study the connected $k$-forcing number of graphs and the splitting graphs.

**Keywords:** zeroforcing number; $k$-forcing number; connected $k$-forcing number.

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1. **INTRODUCTION**

Through out this manuscript, we consider graphs without loops and multiple edges. That is we consider only simple graph $G = (V,E)$ with vertex set $V(G)$ and edge set $E(G)$. The

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splitting graph $S(G)$ of a graph $G$ is the graph derived from a simple graph $G$ by taking a vertex $v'$ corresponds to each vertex $v \in G$ and join $v'$ to all vertices which are adjacent to $v$. The concept of Splitting graph was first defined by E. Sampathkumar et al. in [14]. In [5] and [4] the authors studied about the zero forcing number of the splitting graph of a graph and the $k$-forcing number of graphs and their splitting graphs.

Zero forcing number of graphs were introduced by the AIM Special Work Group (See[11]). The zero forcing number have applications in power network monitoring [10] and quantum physics [3].

In this paper, we introduce the concept of connected $k$-forcing number. This can be regarded as a generalization of connected zero forcing number.

**Definition 1.** $k$-color-changing rule: Let $G$ be a graph in which each vertex is colored either black or white. If a black colored vertex has at most $k$ white neighbors, then change the colors of $k$ white neighbors to black. When the $k$-color changing rule is applied to an arbitrary vertex $v$ to alter the colors of some vertices $w_1, w_2, \ldots, w_k$ to black, then we say the vertex $v$ $k$-forces the vertices $w_1, w_2, \ldots, w_k$ and we denote it as $v \rightarrow w_1, v \rightarrow w_2, \ldots, v \rightarrow w_k$.

The $k$-forcing number of a graph was introduced by D Amos, Y Caro, R Davila and R Pepper in [1].

**Definition 2.** A $k$-forcing set of a graph $G$ is a subset $Z_k$ of vertices such that if at first the vertices in $Z_k$ are colored black and $V(G) - Z_k$ are colored white, the whole graph $G$ may be colored black by continuously applying the $k$-color changing rule. The $k$-forcing number of $G$, denoted by $Z_k(G)$, is the minimum cardinality of a $k$-forcing set in $G$. If the subgraph induced by the vertices in $Z_k$ (that is $(Z_k)$) is connected, then $Z_k$ is known as the $k$-conneted zero forcing set. The minimum size of such a set is called the connected $k$-forcing number of $G$ and is denoted by $Z_{ck}(G)$.

The connected zero forcing set was introduced by M. Khosravi, S. Rashidi and A. Sheikhhosseni (See[13]). When $k = 1$, the definition of connected 1-forcing set is equivalent
to the definition of connected zero forcing set, $Z_c(G)$ (See [11]). In this article, we deal with connected $k$-forcing number of some graphs and their splitting graphs. We use the following definitions for the further development of this article.

- Corona Product: For any two graphs $G$ and $H$, the Corona product $G \circ H$ of the graphs $G$ and $H$ is the graph determined by taking one copy of $G$ and $|V(G)|$ copies of $H$ and by connecting each vertex of the $j^{th}$ copy of $H$ to the $j^{th}$ vertex of $G$, $1 \leq j \leq |V(G)|$.

- Rooted Product: Let $G$ be a connected graph with vertices $v_1, v_2, \ldots, v_n$ and let $H$ be a sequence of $n$-rooted graphs $H_1, H_2, \ldots, H_n$. The rooted product of $G$ and $H$ is defined as the graph obtained by identifying the root of $H_i$, $1 \leq i \leq n$ with the $i^{th}$ vertex of $G$ for all $i$. This graph is denoted by $G(H)$ and is known as the rooted product of $G$ by $H$ (See[8]).

- Square of a Graph: Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Then the square of $G$, denoted by $G^2$, is the graph having the vertex set same as that of $G$ and such that two vertices in $G^2$ are adjacent if the distance between them is at most two in $G$.

- When the $k$-color changing rule is applied to an arbitrary vertex $u$ to change the color of the vertex $v$, we say $u$, $k$-forces (if it is zero forcing, then we say $u$ forces $v$) $v$ and write $u \rightarrow v$.

For more definitions on graphs, we refer to [9]. From the definitions above, we have the following proposition.

**Proposition 3.** Let $P_n$, $n \geq 3$ be a path on $n$ vertices. Then

$$Z_{ck}[S(P_n)] = \begin{cases} 3 & \text{if } k = 1 \\ 1 & \text{if } k \geq 2 \end{cases}$$

**Proof.** Case 1 Assume that $k = 1$. It can be easily observed that if we color any two adjacent vertices as black, it is not possible to obtain a derived coloring. Therefore, $Z_{c1}[S(P_n)] \geq 3$. Now, let $u_1, u_2, \ldots, u_n$ be the vertices of $P_n$ and $u'_1, u'_2, \ldots, u'_n$ be the corresponding vertices in $S(P_n)$. Color the vertices $u_1, u_2$ and $u'_1$ as black. Clearly, the vertex $u_1$ forces $u'_2$ as black, the vertex $u'_2$ forces the vertex $u_3$ as black, the vertex $u_2$ forces the vertex $u'_3$ as black and so on. Therefore,
$Z = \{u_1, u_2, u'_1\}$ forms a connected zero forcing set for the path $P_n$. So, $Z_{c1}(P_n) \leq 3$. Hence the result follows.

**Case 2** Assume that $k \geq 2$. In this case, the vertex $u_1$ forms a connected zero forcing set and hence the result follows. □

It can be observed that any connected $k$-forcing set is a $k$-forcing set. Therefore, we have the following

**Proposition 4.** For any simple graph $G$, and for any fixed $k$, $Z_k(G) \leq Z_{ck}(G)$, where $Z_k$ is the $k$-forcing number and $Z_{ck}(G)$ is the connected $k$-forcing number of $G$.

We consider the next proposition from [5] to prove the result concerning the splitting graph of the cycle $C_n$.

**Proposition 5 ([5]).** If $G$ is the cycle $C_n$ on $n \geq 4$ vertices, then $Z[S(G)] = 4$.

In the succeeding proposition, we consider the splitting graph of the cycle $C_n, n \geq 4$.

**Proposition 6.** Let $S(C_n)$ be the splitting graph of the cycle $C_n$. Then

$$Z_{ck}[S(C_n)] = \begin{cases} 4 & \text{if } k = 1 \\ 3 & \text{if } k = 2 \\ 1 & \text{if } k \geq 3 \end{cases}$$

**Proof. Case 1** Assume that $k = 1$. From Proposition-4 and Proposition-5, we have the following:

$$4 \leq Z_{c1}[S(C_n)]$$

To prove the reverse part, let us consider the vertices of the cycle $C_n$ as $v_1, v_2, \ldots, v_n$ and $v'_1, v'_2, \ldots, v'_n$ be the corresponding vertices of $v_1, v_2, \ldots, v_n$ in $S(C_n)$. Consider the set of vertices $\{v_1, v'_1, v_2, v'_2\}$ as black. Now the vertex $v'_2 \rightarrow v_3$ to black, the vertex $v_2 \rightarrow v'_3$ to black, the
vertex $v_3' \rightarrow v_4$ to black and so on. Therefore, we can obtain a derived coloring with the set of black vertices $\{v_1, v_1', v_2, v_2'\}$. Clearly,

$$4 \geq Z_{c1}[\mathbb{S}(C_n)]$$

Hence the result follows.

**Case 2** Let us assume that $k = 2$ and $Z_{c2}[\mathbb{S}(C_n)] = 2$. Consider a connected 2-forcing set consisting of two vertices. Let $u$ and $v$ be the two adjacent vertices in the connected 2-forcing set of $\mathbb{S}(C_n)$. Then we have two sub cases:

**Subcase 2.1** $\text{deg}(u) = \text{deg}(v) = 4$. Since $u$ and $v$ are adjacent to three white neighbors, color changing rule is not applicable in this case, we get a contradiction to our assumption that $Z_{c2}[\mathbb{S}(C_n)] = 2$.

**Subcase 2.2** $\text{deg}(u) = 2$ and $\text{deg}(v) = 4$. In this case the vertex $u$ can force one more adjacent vertex of degree 4 to black. Therefore, in this case it is not possible to obtain a derived coloring. Hence from subcases 2.1 and 2.2, we have $Z_{c2}[\mathbb{S}(C_n)] \geq 3$.

It can be easily observed that the vertices $\{v_1, v_1', v_2\}$ forms a zero forcing set for $\mathbb{S}(C_n)$ and hence the result follows. For $k = 3$ the result is obvious. \qed

The Friendship graph $F_p$ is the graph obtained by identifying $p$ copies of the cycle graph $C_3$ with a common vertex.

**Proposition 7.** Let $F_p$ denote the friendship graph with $p \geq 2$ triangles. Then $Z_{ck}(F_p) =$

$$\begin{cases} 
 p + 1 - \frac{k}{2} & \text{if } k \text{ is even and } k < \Delta - 2 \\
 \frac{2p-k+3}{2} & \text{if } k \text{ is odd and } k < \Delta - 2 \\
 1 & \text{if } k \geq \Delta - 2 
\end{cases}$$

**Proof.** Case 1 Assume that $k$ is even and $k < \Delta - 2$. Let $v$ be the vertex with maximum degree $\Delta$. It can be noted that $v$ should be a member of any connected zeroforcing set. Otherwise, the zero forcing set will not be connected. Therefore, assume that the vertex
v is there in any connected zero forcing set of $F_p$. If we take one vertex from each of the $p - \frac{k}{2} - 1$ triangles, then it is not possible to obtain a derived coloring since $\text{deg}(v) = 2p$ and by using color changing rule we get $2(p - \frac{k}{2} - 1) = 2p - k - 2$ black vertices which are adjacent to the vertex $v$. Now we have $2p - (2p - k - 2) = k + 2$ white vertices remains. It is not possible to force these $k + 2$ vertices by using the vertex $v$. Therefore, we must take one black vertex from each of the $p - \frac{k}{2}$ triangles since the vertex $v$ is black, $p + 1 - \frac{k}{2} \leq Z_{ck}(F_p)$.

Let us take one vertex from each of the $p - \frac{k}{2}$ triangles as black. Since the vertex $v$ is black, these $p - \frac{k}{2}$ vertices will force the remaining vertices in the $p - \frac{k}{2}$ triangles as black. Now we have $2(p - \frac{k}{2})$ black vertices together with the black vertex $v$ in the connected $k$-forcing set. It can be observed that at this stage we have $2p - (2p - k) = k$ white vertices adjacent to the vertex $v$. Now the vertex $v$ can force these $k$-vertices as black. Therefore we get a derived coloring with $p - \frac{k}{2} + 1$ black vertices. Hence $Z_{ck}(F_p) \leq p - \frac{k}{2} + 1$.

**Case 2** Assume that $k$ is odd and $k < \Delta - 2$. Let $v$ be the vertex with maximum degree $\Delta$. It can be noted that $v$ should be a member of any connected zero-vertex set. Otherwise, the zero forcing set will not be connected. Therefore, assume that the vertex $v$ is there in any connected zero forcing set of $F_p$. Now let us assume that there exist a zero forcing set consisting of $\frac{2p - k + 1}{2}$ vertices. Since the vertex $v$ is black we can distribute the remaining $\frac{2p - k - 1}{2}$ vertices along the triangles. To force the maximum number of vertices as black, we need to distribute one black vertex for each $\frac{2p - k - 1}{2}$-triangles. Now we have $2(\frac{2p - k - 1}{2}) + 1 = 2p - k$ black vertices and $2p + 1 - (2p - k) = k + 1$ white vertices. All these white vertices are adjacent to $v$. Therefore, color changing rule is not applicable since $k + 1$ white vertices are adjacent to the black $v$. Therefore, $\frac{2p - k + 3}{2} \leq Z_{ck}(F_p)$.

Let us take one vertex from each of the $\frac{2p - k + 3}{2} - 1$ triangles as black. Since the vertex $v$ is black, these $\frac{2p - k + 3}{2} - 1$ will force the remaining vertices in the $\frac{2p - k + 3}{2} - 1$ triangles as black. At this stage we have $2(\frac{2p - k + 3}{2} - 1) + 1 = 2p - k + 2$ black vertices remains. Therefore the total number of white vertices remains in this stage is $2p + 1 - (2p - k + 2) = k - 1$. All
these $k - 1$ white vertices are adjacent to $v$. Therefore, $v_k$ forces all these $k - 1$ white vertices as black. Hence $\frac{2n - k + 3}{2} \geq Z_{ck}(F_p)$.

It can be easily observed that if $k \geq \Delta - 2$, then $Z_{ck}(F_p) = 1$. \qed

**Theorem 8.** Let $G$ be a connected graph with $|V(G)| = p_1$ and let $H$ be another connected graph with $Z_{ck}(H) = p_2$. Let $\mathcal{G}$ be the graph obtained by taking the corona product of $G$ and $H$, that is $\mathcal{G} \equiv G \circ H$. Then $Z_{ck}(\mathcal{G}) \leq p_1(1 + p_2)$.

**Proof.** Without loss of generality, assume that $G$ is connected, $|V(G)| = p_1$ and $Z_{ck}(H) = p_2$. Color all vertices of $G$ black. To form the $k$-forcing set for the subgraph induced by $v_1 \cup H_1$, we need a maximum of $1 + p_2$ black vertices. That is, $Z_k(\langle v_1 \cup H_1 \rangle) \leq 1 + p_2$, where $H_1$ is the first copy of $H$ corresponds to the vertex $v_1$ in $\mathcal{G}$. $Z_k(\langle v_2 \cup H_2 \rangle) \leq 1 + p_2$, where $H_2$ is the second copy of $H$ corresponds to the vertex $v_2$ in $\mathcal{G}$. Proceeding like this, we can observe that $Z_k(\langle v_{p_1} \cup H_{p_1} \rangle) \leq 1 + p_2$. Now the graph $\mathcal{G} \equiv \langle v_1 \cup H_1 \rangle \cup \langle v_2 \cup H_2 \rangle \cup \ldots \cup \langle v_{p_1} \cup H_{p_1} \rangle$. Therefore, $Z_k(\mathcal{G}) \leq (1 + p_2) + (1 + p_2) + \ldots + (1 + p_2) - p_1$ times. This follows that $Z_k(\mathcal{G}) \leq p_1(1 + p_2)$. Since each vertex in $G$ is connected to the vertices of all copies of $H$, the $k$-forcing set obtained here forms a connected $k$-forcing set. Therefore, $Z_{ck}(\mathcal{G}) \leq p_1(1 + p_2)$ \qed

**Proposition 9.** Let $G$ be the complete bipartite graph $K_{m,n}$, and $n \geq 2, m \geq 2$. Then the connected zero forcing number of $G$ is $m + n - 2$. That is, $Z_c(G) = m + n - 2$.

**Proof.** Since $G$ is a complete bipartite graph, therefore, the vertex set of $G$ can be partitioned into two sets $X$ and $Y$. Let $u_1, u_2, \ldots, u_m$ be the vertices in $X$ and $v_1, v_2, \ldots, v_n$ be the vertices in $Y$. Note that the vertices in $X$ are non-adjacent. The vertices in $Y$ are also non-adjacent. To start the color changing rule, color any vertex, say $u_1$, in $X$ as black. Since each vertex in $X$ is connected to every vertex in $Y$, we have to color $n - 1$ vertices in $Y$ as black. Let the only white vertex in $Y$ be $v_n$. Now $u_1 \rightarrow v_n$ to black. In $X$ there are $m - 1$ white vertices. Each vertex in $Y$ is joined to $m - 1$ white vertices in $X$. Assign black color to $m - 2$ white vertices in $X$. Then any black vertex in $Y$, say $v_1$ forces the remaining white vertex in $X$ as black. Now the zero forcing set consists of $1 + m - 2 + n - 1$ black vertices, which are connected. Hence the connected zero forcing number of $G$ is $m + n - 2$. That is, $Z_c(G) = m + n - 2$. \qed
We use the following results from [2] and [11] to prove the next result

**Proposition 10.** [2] For any connected graph $G$, $Z(G) \leq Z_c(G)$, where $Z(G)$ is the zero forcing number of $G$.

**Proposition 11.** [11] Let $G$ be the graph obtained by taking the Cartesian product of the cycle $C_n$ with the path $P_m$. Then $Z(G) = \min\{n, 2m\}$

**Proposition 12.** Let $G$ be the graph obtained by taking the Cartesian product of the cycle $C_n$ with the path $P_m$ and let $n \geq 2m$. Then $Z_c(G) = 2m$.

*Proof.* Let $v_1$ and $v_2$ be the two adjacent vertices in the cycle $C_n$. Let $A = \{v_1^1, v_1^2, \ldots, v_1^n\}$ be the vertices corresponding to the vertex $v_1$ in $G$ and let $B = \{v_2^1, v_2^2, \ldots, v_2^m\}$ be the vertices corresponding to the vertex $v_2$ in $G$. Now consider the vertices in the set $A \cup B$ and color these vertices as black in $G$. The vertices in $A \cup B$ forces the remaining vertices in $G$ as black. Clearly these vertices are connected in $G$ and thus forms a connected zero forcing set in $G$. Hence

$$Z_c(G) \leq 2m$$

Also we have from proposition-10 and proposition-11 that

$$Z_c(G) \geq 2m$$

From (1) and (2) the result follows. □

**Proposition 13.** Let $G$ be the star graph $k_{1,n}$ on $n + 1$ vertices and $n > 2$. Then $Z_c(G) = n$. In general, if $n \geq k \geq 2$, then $Z_{ck}(G) = n - k + 1$.

*Proof.* Let $u_1, u_2, \ldots, u_n$ be the vertices of the star graph $k_{1,n}$ with degree 1. Assume that $v$ is the vertex having degree $n$. We generate the connected zero forcing set as follows. Since $\deg(v) = n$, to apply the color changing rule, we have to color $n - 1$ vertices in $G$ adjacent to $v$ as black. Then $v$ forces the only remaining white vertex to black. Therefore, $Z_c(G) = n - 1 + 1 = n$. If $k = 2$, we can easily show that the connected zero forcing number of $G$ is $n - 2 + 1 = n - 1$. Proceeding like this, we obtain $Z_{ck}(G) = n - k + 1$ for any positive integer $n \geq k \geq 2$. □
2. Connected k-Forcing Number of Rooted Product of Graphs

In this section, we deal with the connected k-forcing number of rooted product of cycle with paths, cycle with cycles.

**Proposition 14.** Let $P_1, P_2, \ldots, P_n$ be n-paths (each path is of length $n \geq 3$) rooted at the pendant vertex and $C_n$ be a cycle on $n \geq 3$ vertices. Let $G$ be the graph obtained by taking the rooted product of the cycle $C_n$ with the paths $P_1, P_2, \ldots, P_n$. Then

$$Z_{ck}(G) = \begin{cases} n & \text{if } k = 1 \\ 1 & \text{if } 2 \leq k \leq \Delta, \end{cases}$$

where $\Delta(G) = 3$.

**Proof.** Let $u_1, u_2, \ldots, u_n$ be the vertices of the cycle $C_n, n \geq 3$ and $P_1, P_2, \ldots, P_n$ be the paths rooted at the vertices $u_1, u_2, \ldots, u_n$ respectively. Each path is of length $n, n \geq 3$.

Represent the vertices of $P_1$ by $p^1_1, p^1_2, \ldots, p^1_n$, the vertices of $P_2$ by $p^2_1, p^2_2, \ldots, p^2_n$ and the vertices of $P_n$ by $p^n_1, p^n_2, \ldots, p^n_n$. Let $u_1$ be the vertex identified with the vertex $p^1_1$ in $G$, $u_2$ be the vertex identified with the vertex $p^2_1$ in $G$, $\ldots$, $u_n$ be the vertex identified with the vertex $p^n_1$.

**Case 1.** Assume that $k = 1$. This case is similar to that of the connected zero forcing number of $G$. Color the vertices $u_1, u_2, \ldots, u_n$ in $G$ black. Now one can easily infer that

$$Z_c(G) \leq n$$

It can be worth mentioning that if we start the color changing rule with vertices of $P_i, 1 \leq i \leq n$ other than the vertices identified with the vertices $u_1, u_2, \ldots, u_n$ of $C_n$, we cannot obtain a connected zero forcing set with at least $n$ black vertices. Therefore, we need to consider the vertices in the cycle to force the remaining vertices in $G$.

Now assume that we have a connected zero forcing set consisting of $n - 1$ black vertices. From the above it can be noted that these vertices must be from the cycle $C_n$. Without loss of generality, assume that the vertices are $u_1, u_2, \ldots, u_{n-1}$. Clearly the black vertex $u_2$
can force the vertices of the path \( P_2 \), \( u_3 \) can force the vertices of the path \( P_3 \), ..., the vertex \( u_{n-2} \) can force the vertices of the path \( P_{n-2} \). Since the black vertex \( u_1 \) is adjacent to two white vertices \( u_n \) and \( p^1_2 \), \( u_1 \) cannot force the vertices \( u_n \) and \( p^1_2 \). Similarly the vertex \( u_{n-1} \) is adjacent to two white vertices \( u_n \) and \( p^{n-1}_2 \). Therefore, the vertex \( u_{n-1} \) cannot force \( u_n \) and \( p^{n-1}_2 \), this contradicts our assumption that \( Z_c(G) = n - 1 \). Therefore,

\[(4) \quad Z_c(G) \geq n\]

Now from (3) and (4) the result follows.

**Case 2.** Assume that \( k \geq 2 \). In this case, if we consider any pendant vertex of \( G \) as a black vertex, then it can force the remaining white vertices of \( G \) as black. Hence \( Z_{ck}(G) = 1 \).

\[\square\]

**Proposition 15.** Let \( D_1, D_2, \ldots, D_n \) be the cycles \( C_n \) of order \( n \geq 3 \) rooted at a vertex and \( C_n \) be another cycle of order \( n > 3 \). Let \( G \) be the graph derived from the rooted product of \( C_n \) with the cycles \( D_1, D_2, \ldots, D_n \). Then

\[
Z_{ck}(G) = \begin{cases} 
2n & \text{if } k = 1 \\
n & \text{if } k = 2 \\
1 & \text{if } 3 \leq k \leq \Delta,
\end{cases}
\]

where \( \Delta(G) = 4 \).

**Proof.** Without loss of generality, assume that \( u_1, u_2, \ldots, u_n \) be the vertices of the cycle \( C_n \) in \( G \) and let \( D_1, D_2, \ldots, D_n \) be the cycles rooted at \( u_1, u_2, \ldots, u_n \) respectively. Represent the vertices of the cycle \( D_1 \) in \( G \) by \( d^1_1, d^2_1, \ldots, d^n_1 \). Similarly the vertices of \( D_2 \) in \( G \) by \( d^2_1, d^2_2, \ldots, d^n_2 \) and the vertices of \( D_n \) by \( d^n_1, d^n_2, \ldots, d^n_n \). Assume that the vertex \( d^1_1 \) be rooted at \( u_1 \), the vertex \( d^2_1 \) be rooted at \( u_2 \), ..., the vertex \( d^n_1 \) be rooted at \( u_n \).

**Case 1.** Let us suppose that \( k = 1 \). This case is similar to that of the connected zero forcing number of \( G \). Color the vertices \( u_1, u_2, \ldots, u_n, d^1_2, d^2_2, \ldots, d^n_2 \) as black. Now we can easily see that these black vertices forms a connected zero forcing set. Hence
It can be easily infer that to form a minimum connected zero forcing set for \( G \), we need to color the vertices \( u_1, u_2, \ldots, u_n \) as black and color at least one vertex from each of the cycles \( D_i, 1 \leq i \leq n \) adjacent to each \( u_i, 1 \leq i \leq n \) as black, otherwise we cannot form a connected zero forcing set with at least \( 2n \) black vertices. Clearly,

\[
Z_c(G) \leq 2n
\]

(5)

(5) and (6) concludes the result.

**Case 2.** Let us suppose that \( k = 2 \). Color all vertices of \( C_n \) in \( G \) black. Each vertex \( u_i, 1 \leq i \leq n \), is adjacent to exactly two white vertices of \( D_i \) and \( k = 2 \). Therefore, these vertices forms a 2- forcing set for \( G \). The sub graph induced by these black vertices are connected and hence it forms a connected 2-forcing forcing set for \( G \). Therefore,

\[
Z_{c2}(G) \leq n
\]

(7)

It can be easily infer that to form a minimum connected 2- forcing set for \( G \), we need to color the vertices \( u_1, u_2, \ldots, u_n \) as black, otherwise we cannot form a connected 2-forcing set with at least \( n \) black vertices. Clearly,

\[
Z_{c2}(G) \geq n
\]

(8)

Therefore from (7) and (8), the result follows.

**Case 3.** Let us suppose that \( k \geq 3 \). In this case any arbitrary vertex from the cycle \( D_i, 1 \leq i \leq n \) will \( k \)-forces the remaining vertices as black in \( G \). Therefore, \( Z_{ck}(G) = 1 \). \( \square \)
**Proposition 16.** Let $G$ be the rooted product of $P_n \square P_2$ (the Ladder graph) with $P_t, t \geq 3$ rooted at the pendant vertex. Then

$$Z_{ck}(G) = \begin{cases} 
2n & \text{if } k = 1 \\
n & \text{if } k = 2 \\
1 & \text{if } 3 \leq k \leq \Delta(G) 
\end{cases}$$

where $\Delta(G) = 4$

*Proof.* Represent the vertices of the graph $P_n \square P_2$ by $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$. Let $P_1, P_2, \ldots, P_n$ be the paths rooted at the vertices $u_1, u_2, \ldots, u_n$ respectively. Also let $Q_1, Q_2, \ldots, Q_n$ be the paths rooted at the vertices $v_1, v_2, \ldots, v_n$ respectively. The vertices of the paths $P_1, P_2, \ldots, P_n$ and $Q_1, Q_2, \ldots, Q_n$ in $G$ can be named as follows:

Consider

$$P_1 = \{p_1^1, p_2^1, \ldots, p_t^1\}, \quad Q_1 = \{q_1^1, q_2^1, \ldots, q_t^1\}$$

$$P_2 = \{p_1^2, p_2^2, \ldots, p_t^2\}, \quad Q_2 = \{q_1^2, q_2^2, \ldots, q_t^2\}$$

$$\ldots \ldots \ldots \ldots \ldots$$

$$P_n = \{p_1^n, p_2^n, \ldots, p_t^n\}, \quad Q_n = \{q_1^n, q_2^n, \ldots, q_t^n\}$$

Now color the vertices $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ as black. Clearly these vertices forms a connected zero forcing set for $G$ and hence

$$(9) \quad Z_c(G) \leq 2n$$

Refer Figure 1.
There exists three types of minimum connected zero forcing sets with $Z_c(G) = 2n$. Consider these three sets as follows. We denote them as $A, B$ and $C$

$A = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$

$B = \{u_1, u_2, \ldots, u_n, p_2^1, p_2^2, \ldots, p_2^n\}$

$C = \{v_1, v_2, \ldots, v_n, q_2^1, q_2^2, \ldots, q_2^n\}$

It can be easily observed that if we take $2n$ vertices other than these three sets, then it will not form a minimum connected zero forcing set. Now assume that there exists a connected zero forcing set consisting of $2n - 1$ black vertices.

**Case 1.** Consider the black vertices as depicted in Figure 2. Assume that the black vertices are from the set $A$, except the vertex $u_n$. Consider the vertex $u_n = u_8$ as white. The blue colored vertices represent the vertices which are forced by the black vertices. In this case, if we consider $G$, then there are $3t - 2$ vertices remain as white. Therefore, we cannot obtain a derived coloring, a contradiction to our assumption that there exists a connected zero forcing set consisting of $2n - 1$ vertices. The case is similar if we consider $u_1, v_1$ and $v_n = v_8$ as white vertices.
Case 2. Consider the black vertices as depicted in Figure 1. If we choose any black vertex other than \( u_1, v_1, u_n = u_8, v_n = v_8 \) as white, then one can observe that there are \( 4t - 3 \) white vertices remains in \( G \), a contradiction to our assumption that \( Z_c(G) = 2n - 1 \).
**Case 3.** Consider the black vertices as depicted in Figure-3. Assume that the black vertices are from the set $B$, except the vertex $p^1_2$. Consider the vertex $p^1_2$ as white. The blue colored vertices represent the vertices which are forced by the black vertices. In this case if we consider the graph $G$, then there are $3t-2$ vertices remain as white. Therefore, we cannot obtain a derived coloring with $Z_c(G) = 2n-1$, a contradiction. The case is similar if we consider the vertex $p^2_2$ as white.

**Sub Case 3.1.** Consider the black vertices as depicted in Figure-3. Assume that the black vertices are from the set $B$, except one the vertices $p^i_2, 2 \leq i \leq n-1$. Consider the vertex $p^i_2$ as white. In this case if we consider $G$, then there are $4t-3$ vertices remain as white. Color changing rule is not applicable at this stage, a contradiction to our assumption that $Z_c(G) = 2n-1$.

**Sub Case 3.2.** Assume that the black vertices are from the set $B$, except the vertex $u_i, 1 \leq i \leq n$. In this case we loose the connectivity of the zero forcing set. That is the zero forcing set is not connected, again a contradiction.

**Case 4.** Assume that the black vertices are from the set $C$, except one. This case is similar to that of Case 3. Since the sub graph induced by the connected zero forcing sets $B$ and $C$ are isomorphic. Combining cases 1, 2, 3 and 4,

\begin{equation}
Z_c(G) \geq 2n
\end{equation}

From (9) and (10), $Z_{ck}(G) = 2n$, if $k = 1$.

**Case 5.** Let $k > 1$. If we color any one of the pendant vertices from $G$ as black, then the pendant vertex forms a connected zero forcing set for $G$. Hence $Z_{ck}(G) = 1$ if $1 < k \leq 4$, where $\Delta(G) = 4$. \hfill \square
**Proposition 17.** Let $G$ be the rooted product of $P_n \square P_n$ (The Grid graph) with $P_t, t \geq 3$ rooted at the pendant vertex. Then

$$Z_{ck}(G) = \begin{cases} 
\leq n^2 & \text{if } k = 1 \\
\leq n & \text{if } k = 2 \\
1 & \text{if } 3 \leq k \leq 5.
\end{cases}$$

**Proof.** **Case 1.** Assume that $k = 1$. In this case color all the vertices of the Cartesian product $P_n \square P_n$ in $G$ as black. One can easily observe that these $n^2$- black vertices forms a connected zero forcing set for $G$. Thus $Z_c(G) \leq n^2$.

**Case 2.** Assume that $k = 2$. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path $P_n$ in $P_n \square P_n$ of $G$. Color these vertices as black in $G$. Now one can easily verify that these vertices form a connected zero forcing set for $G$. Thus, $Z_{c2}(G) \leq n$, if $k = 2$.

**Case 3.** Assume that $3 \leq k \leq 5$. Let $P_t$ be the path identified at the vertex $u_1$ in $G$. Now color the pendant vertex of the path $P_t$ in $G$ as black. Let it be the vertex $v$. Clearly the vertex $v$ forces the remaining vertices in $G$ as black. Therefore we can form a derived coloring for $G$. Thus $Z_{ck}(G) = 1$, as desired. □

We strongly believe that the bounds in the above proposition is sharp.

**Proposition 18.** Let $G$ be the rooted product of $P_n \square P_2$ with the cycle $C_n$. Then

$$Z_{ck}(G) = \begin{cases} 
4n & \text{if } k = 1 \\
2n & \text{if } k = 2 \\
1 & \text{if } 3 \leq k \leq 5
\end{cases}$$

**Proof.** **Case 1.** Assume that $k = 1$. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path $P_n$ in $G$ and let $v_1, v_2, \ldots, v_n$ be the vertices corresponding to the copy of the path $P_n$ in $G$. Note that $\deg(u_1) = \deg(v_1) = \deg(u_n) = \deg(v_n) = 4$. The remaining vertices of $P_n \square P_2$ in $G$ have degree 5. It can be noted that any connected zero forcing set of $G$ must contain all the vertices of $P_n \square P_2$. Otherwise the zero forcing set will be disconnected. Without loss of generality,
assume that we have a set consisting of $2n$ connected black vertices from $P_n \Box P_2$ in $G$. To force the white vertices in each cycle, we must select a vertex adjacent to the rooted vertex of each $C_i$, $1 \leq i \leq 2n$. Therefore we need to choose $2n$ black vertices from the cycle $C_n$. Now we have a set of $4n$ black vertices which forces the remaining vertices of $G$, which is connected. Therefore, $Z_{ck}(G) = 4n$.

**Case 2.** Assume that $k = 2$. It can be observed that the connected zero forcing set of $G$ must contain all the vertices of $P_n \Box P_2$. Otherwise, the zero forcing set will be disconnected. If we take the $2n$ black vertices of $P_n \Box P_2$ in $G$, then these black vertices will 2-forces the remaining white vertices as black and hence $Z_{ck}(G) = 2n$.

**Case 3.** Assume that $5 \geq k \geq 3$. Consider the cycle identified with the vertex $u_1$, say $C_1$. Choose a vertex from $C_1$ of degree 2 as black. This vertex will 3-force the remaining vertices in $G$ as black. Hence $Z_{ck}(G) = 1$. □

**Definition 19** ([1]). A connected graph $G$ is defined as a cycle-path graph (CP-graph) if it contains $r$ vertex disjoint cycles that are connected by $r - 1$ edges of the path $P_r$. Thus a CP-graph with $n$ vertices contains $m = n + r - 1$ edges and edge between two cycles is a cut edge.

The zero forcing number of CP-graph was studied in some detail in [5]. Here we study the connected zeroforcing number of the CP-graph considered in [5].

**Proposition 20.** Let $G$ be the CP-graph $C_3P_r, r \geq 3$. Then $Z_c(G) = 2r$. Moreover $Z_{ck}(G) = 1$ if $k = 2, 3$.

**Proof.** Denote the cycles by $C_1, C_2, \ldots, C_r$. Let the vertex sets of the cycles in $C_3P_r$ be

\[
V(C_1) = \{c^1_1, c^2_1, c^3_1\}
\]

\[
V(C_2) = \{c^1_2, c^2_2, c^3_2\}
\]

\[
\ldots
\]

\[
\ldots
\]
\[ V(C_r) = \{ c_1^r, c_2^r, c_3^r \} \]

**Case 1.** Assume that \( k = 1 \). We prove the result by mathematical induction on the number of cycles \( r \) on the \( CP \)-graph. Assume that \( r = 1 \). In this case \( G \) is the cycle \( C_3 \) therefore, \( Z_c(C_3) = 2 \) and the result is true for \( r = 1 \).

Assume that the result is true for all \( C_3P_r \) graphs with \( r - 1 \) cycles \( C_3 \), where \( r \geq 2 \). Let \( C \) be the end cycle connected to the rest of the \( C_3P_r \) graph by an edge \( e = ab \), where \( a \in V(C_3P_r) - V(C) \) and \( b \in V(C) \). Let \( Y = \{ a, b \} \) be the cut set where \( a \in \langle V(C_3P_r) - V(C) \rangle \) and \( b \in V(C) \).

Assume that the result is true for the sub graph induced by \( \langle V(C_3P_r) - V(C) \rangle \). That is \( Z_c(\langle V(C_3P_r) - V(C) \rangle) = 2r - 2 = 2(r - 1) \).

Let \( W \) be the minimum zero forcing set of \( \langle V(C_3P_r) - V(C) \rangle \) with \( |W| = 2r - 2 \). Let \( u_1 \) and \( u_2 \) be two white neighbors of the vertex \( b \) in \( C \). Since the vertex \( a \) is black it forces the vertex \( b \) to black. Since the vertex \( b \) has two white neighbors, further forcing is not possible. In order to make the zero forcing set connected, we have to include the black vertex \( b \) in the connected zero forcing set of \( G \). Therefore, our new connected zero forcing set is \( W \cup \{ b \} \). The set \( W \cup \{ b \} \) cannot force the remaining two white vertices \( (u_1 \text{ and } u_2) \) adjacent to \( b \). Therefore, we need to include either \( u_1 \) or \( u_2 \) in the connected zero forcing set of \( G \). Let it be \( u_1 \). Hence by induction

\[ Z_c(G) = |W \cup \{ b, u_1 \}| \]

\[ = 2r - 2 + 2 = 2r. \]

\( \square \)

**Case 2.** Assume that either \( k = 2 \) or \( k = 3 \). In this case any vertex of degree 2 will form a connected \( k \)-forcing set.
The Cartesian product $C_n \square K_2$ is known as the Prism graph or the circular ladder graph. The length of the shortest cycle in a graph $G$ is called the girth of $G$. We recall the following observation from [6].

**Proposition 21.** [6] Let $G$ be a graph with girth at least 4 and minimum degree $\delta(G) \geq 3$. Then $Z_c(G) \geq \delta(G) + 1$.

**Proposition 22.** Let $G$ be the circular ladder graph of order $n \geq 10$. Then $Z_c(G) = 4$. Also, $Z_{ck}(G) = 2$, if $k = 2$, $Z_{ck}(G) = 1$, if $k = 3$.

**Proof.** Let $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ be the vertices of the circular ladder graph $G$, $u_1, u_2, \ldots, u_n$ being the vertices of the inner circle. By the proposition 21, since $\Delta(G) = \delta(G) = 3$ and the girth is at least 4, we have $Z_c(G) \geq 3 + 1 = 4$.

To establish the reverse inequality, we proceed as follows.

Without loss of generality, choose four vertices $u_1, u_2, u_3$ and $v_1$. Allow these vertices to have black color. Then clearly the black vertex $u_2 \rightarrow v_2$ to black. Now the black vertex $v_2 \rightarrow v_3$ to black. Again, the black vertex $u_3 \rightarrow u_4$ to black, $v_3 \rightarrow v_4$ to black. Apply color changing rule step by step, the black vertex $u_{n-1} \rightarrow u_n$ to black and $v_{n-1} \rightarrow v_n$ to black. Hence $Z = \{u_1, u_2, u_3, v_1\}$ forms a connected zero forcing set for $G$. Here the cardinality of the set $Z$ is 4. So, $Z_c(G) \leq 4$. This concludes the result.

**Case 1** Assume that $k = 2$. In this case, clearly a set consisting of any two adjacent black vertices forms a connected zero forcing set for $G$. Hence, $Z_{c2}(G) = 2$.

**Case 2.** Assume that $k = 3$. It is obvious that any single black vertex gives a derived coloring for $G$. Therefore, the result follows. That is $Z_{c3}(G) = 1$. □

**Proposition 23.** Let $G$ be the rooted product of the circular ladder graph, $C_n \square K_2$ with the path $P_t$, a path of length $t$, $t \geq 4$ rooted at the pendent vertex. Then, $Z_c(G) = 2n$.

**Proof.** Represent the vertices of $C_n \square K_2$ as $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ in $G$ and the paths rooted at the pendent vertex by $P_1, P_2, \ldots, P_n$ of length $t$. Let $v_1 = p_1^1$, $v_2 = p_2^2, \ldots, v_n = p_1^n$, where
$p_1^1, p_1^2, \ldots, p_1^n$ are the pendent vertices of the paths identified at the vertices $v_1, v_2, \ldots, v_n$ respectively, where

$$
P_1 = \{p_1^1, p_2^1, \ldots, p_t^1\}
$$

$$
P_2 = \{p_1^2, p_2^2, \ldots, p_t^2\}
$$

$$
\ldots
$$

$$
\ldots
$$

$$
P_n = \{p_1^n, p_2^n, \ldots, p_t^n\}
$$

We examine the different possibilities of forming a connected zero forcing set as follows.

**Case 1.** Assume that we have a connected zero forcing set consisting of $2n - 1$ black vertices $\{u_1, u_2, \ldots, u_n v_1, v_2, \ldots, v_{n-1}\}$ for $G$. Then, the black vertex $u_n$ has two white neighbors $v_n$ and a vertex of the path rooted at $u_n$. So, the further forcing from the black vertex $u_n$ is not possible, a contradiction.

**Case 2.** Suppose that $Z = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_{n-1}\}$ is a connected zero forcing set for $G$. Then we can easily observe that further forcing from the black vertex $v_n$ is not possible, since it has two white neighbors, a contradiction to our assumption.

**Case 3.** The case of forming a connected zero forcing set by taking the $2n - 1$ black pendent vertices of the paths only is ruled out, since the pendent vertices do not form a connected induced sub graph in $G$.

**Case 4.** Consider a connected zero forcing set of $2n - 1$ black vertices having the following combinations.

**Sub case 4.1.** Combination of the vertices of $u_i, i = 1, 2, \ldots, n$ and the vertices of the path $P_i, i = 1, 2, \ldots, n$, rooted at $u_i$. 

Sub case 4.2. Combination of the vertices of $v_i$ and the vertices of the path $P_i, i = 1, 2, \ldots, n$, rooted at $v_i$.

Sub case 4.3. Combination of the vertices $u_i$ and $v_i$, and the vertices of $P_i, i = 1, 2, \ldots, n$.

Note that the combination of the vertices $u_i$ and the vertices of $P_i$ is not considered, since that combination does not form a connected induced sub graph in $G$. It is easy to verify that none of the above combinations will never form a connected zero forcing set for $G$. Hence from the above cases, we can infer that

\[(11) \quad Z_c(G) \geq 2n.\]

To claim $Z_c(G) \leq 2n$, we proceed as follows. Select $2n$ black vertices $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$. Then the black vertex $v_1 \rightarrow p_2^1$ to black, the black vertex $p_2^1 \rightarrow p_3^1$ to black, $\ldots, p_{t-1}^1 \rightarrow p_t^1$ to black. Similarly, all the vertices of the paths rooted at the black vertices $v_2, v_3, \ldots, v_n$ are colored black. The same argument holds good for the vertices of the paths rooted at the black vertices $u_1, u_2, \ldots, u_n$. Therefore, $Z = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ generates a connected zero forcing set for $G$. Cardinality of $Z$ is $2n$. So,

\[(12) \quad Z_c(G) \leq 2n\]

From (11) and (12), the result follows. \qed

**Proposition 24.** Let $G$ be the rooted product of the circular ladder graph with the cycle $C_k$, $k \geq 4$. Then $Z_c(G) \leq 4n$.

**Proof.** Let $A = \{u_1, u_2, \ldots, u_n\}$ and $B = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of the graph $G$, $A$ being the vertex set of the inner cycle. Suppose that $C_1, C_2, \ldots C_n$ be the cycles rooted at the vertices $v_1, v_2, \ldots, v_n$ and $D_1, D_2, \ldots, D_n$ be the cycles rooted at the vertices $u_1, u_2, \ldots, u_n$. Represent the vertices of cycles $C_1, C_2, \ldots, C_n$ and $D_1, D_2, \ldots, D_n$ as follows.

\[
C_1 = \{c_1^1, c_2^1, \ldots, c_k^1, c_1^1\}
\]

\[
C_2 = \{c_1^2, c_2^2, \ldots, c_k^2, c_1^2\}
\]

\[
\ldots
\]
\[ C_n = \{ c^n_1, c^n_2, \ldots, c^n_k, c^n_1 \} \]

\[ D_1 = \{ d^1_1, d^1_2, \ldots, d^1_k, d^1_1 \} \]

\[ D_2 = \{ d^2_1, d^2_2, \ldots, d^2_k, d^2_1 \} \]

\[ \cdots \]

\[ D_n = \{ d^n_1, d^n_2, \ldots, d^n_k, d^n_1 \} \]

Let \( v_1 = c^1_1, v_2 = c^2_1, \ldots, v_n = c^n_1 \) and \( u_1 = d^1_1, u_2 = d^2_1, \ldots, u_n = d^n_1 \).

We generate a zero forcing set for the graph \( G \) as follows. Consider the set \( \mathcal{X} = \{ v_1, c^1_2, v_2, c^2_2, \ldots, v_n, c^n_2, u_1, u_2, \ldots, u_n, d^1_2, d^2_2, \ldots, d^n_2 \} \). Color the vertices in \( \mathcal{X} \) as black. Now the vertices in \( \mathcal{X} \) can force the remaining white vertices of the cycles \( C_1, C_2, \ldots, C_n \) and \( D_1, D_2, \ldots, D_n \) as black by repeatedly applying the color changing rule. Thus, the set

\[ \mathcal{X} = \{ v_1, c^1_2, v_2, c^2_2, \ldots, v_n, c^n_2, u_1, u_2, \ldots, u_n, d^1_2, d^2_2, \ldots, d^n_2 \} \]

generates a connected zero forcing set for \( G \). The cardinality of the set \( \mathcal{X} \) is \( 4n \). Hence, \( Z_c(G) \leq 4n \). \( \square \)

We strongly believe that the above bound is sharp.

**Proposition 25.** Let \( G \) be the rooted product of the path \( P_n, n \geq 3 \), with \( P_t \), a path of length \( t \), \( t \geq 4 \) rooted at the pendant vertex. Then \( Z_c(G) = n \).

**Proof.** Denote the vertices of the path \( P_n \) by \( u_1, u_2, \ldots, u_n \) in \( G \). Let \( u_1 = P^1_1, u_2 = P^2_1, \ldots, u_n = P^n_1 \), where \( P^1_1, P^2_1, \ldots, P^n_1 \) are the vertices of the path rooted at \( u_1, u_2, \ldots, u_n \).

**Claim:** Any set consisting of \((n - 1)\) black vertices will never form a connected zero
forcing set for the graph $G$. For, consider the following cases.

**Case 1.** Select the pendant vertex of each path rooted at the vertices $u_1, u_2, \ldots, u_{n-1}$.

Clearly they cannot form a connected zero forcing set for $G$.

**Case 2:** Form a set of $n - 1$ black vertices from the vertices of the paths rooted at the vertices $u_1, u_2, \ldots, u_n$. We can easily observe that this set will not form a connected zero forcing set for $G$.

**Case 3:** Assume that $Z = \{u_1, u_2, \ldots, u_{n-1}\}$. Color the vertices in the set $Z$ as black. Then we can see that the vertices of the paths rooted at the vertices $u_1, u_2, \ldots, u_{n-2}$ can be colored as black by applying color changing rule. Note that the forcing from the black vertex $u_{n-1}$ is not possible, since $u_{n-1}$ has two white neighbours. So the set $Z$ cannot generate a zero forcing set for $G$. In view of the above cases, we have $Z_c(G) \geq n$.

To prove the reverse part, let $Z_1 = \{u_1, u_2, \ldots, u_n\}$. Assign black color to the vertices in the set $Z_1$. Then it can be seen that the set $Z_1$ generates a connected zero forcing set for $G$. Therefore, $Z_c(G) \leq n$. Hence the result follows.

Again, when $k = 2$, any black vertex of the graph $G$, other than the vertex having degree 3, gives a derived coloring for $G$. Hence, $Z_c(G) = 1$.

When, $k = 3$, any vertex of $G$ forms a connected zero forcing set, as we wish. \qed

**3. CONNECTED $k$-FORCING NUMBER OF SQUARE OF GRAPHS**

In this section, we deal with the connected $k$-forcing number of square of path graph $P_n, n \geq 4$, the cycle graph $C_n, n \geq 5$.

**Proposition 26.** Let $G$ denotes the square of the path $P_n, n \geq 3$. Then the connected zero forcing number of $G$ is 2.
Proof. Represent the vertices of $G$ by $u_1, u_2, \ldots, u_n$ and let $u_1$ and $u_n$ be the pendant vertices in $G$. The vertices in $G$ and $G^2$ are the same. It is obvious that with one black vertex, we cannot get a derived coloring for $G$. Since $\delta(G) = 2 \leq Z(G) \leq Z_c(G)$. So, $Z_c(G) \geq 2$.

On the other hand, without loss of generality, color the vertices $u_1$ and $u_2$ as black. Then the black vertex $u_1$ forces $u_3$ to black, $u_2$ forces $u_4$ to black, $u_3$ forces $u_5$ to black and so on till all the vertices of $G$ are colored black. So, $Z = \{u_1, u_2\}$ forms a connected zero forcing set for $G$. $|Z| = 2$. Therefore, we have $Z_c(G) \leq 2$. Hence the result follows. □

Proposition 27. The connected zero forcing number of the square of a cycle $C_n$, $n \geq 5$, is 4.

Proof. Let $G$ denotes the square of the cycle $C_n$, $n \geq 5$. It is clear that $G$ is a 4-regular graph. That is, $\Delta(G) = \delta(G) = 4$. This implies that $Z_c(G) \geq 4$.

In order to establish the reverse inequality, choose any four connected vertices of $G$. Let they be $u_1, u_2, u_3$ and $u_n$. Color them as black. In $G$, the white vertices adjacent to the vertex $u_1$ are $u_2, u_3, u_n$ and $u_{n-1}$. So the black vertex $u_1$ forces the vertex $u_{n-1}$ to black. Now consider the black vertex $u_2$. The adjacent vertices of $u_2$ are $u_1, u_n, u_3$, and $u_4$. Of these vertices, $u_1, u_n, u_3$ are already black. So, the vertex $u_2$ forces $u_4$ to black. Again, consider the black vertex $u_3$. At this stage, the vertex $u_3$ has only one white vertex $u_5$. Hence $u_3$ forces $u_5$ to black and so on. Finally, consider the black vertex $u_{n-4}$. The vertex $u_{n-4}$ has 4 neighbours $u_{n-5}, u_{n-6}, u_{n-3}, u_{n-2}$ of which the only one white vertex is $u_{n-2}$. Therefore, the vertex $u_{n-4}$ forces $u_{n-2}$ to black. The vertex $u_{n-3}$ is already colored black by the vertex $u_{n-5}$. Therefore, the set $Z = \{u_1, u_2, u_3, u_4\}$ yields a connected zero forcing set for the graph $G$. Hence, we have $Z_c(G) \leq 4$. This completes the proof. □

4. CONCLUSION AND OPEN PROBLEMS

In this paper we addressed the problem of determining the connected $k$-forcing number of certain graphs. Also we found the exact value of connected zero forcing number of some classes of graphs. In Section 1, we found an upper bound of $Z_{ck}(\mathcal{G})$ for the corona product of two graphs $G$ and $H$. It is an open problem to characterize the connected graphs for which $Z_{ck}(\mathcal{G}) = p_1(1 + p_2)$. In Section 2, we found the exact values of the connected $k$-forcing number
of rooted product of cycles with paths and cycle with cycles. Section 3, deals with the connected k-forcing number of square of graphs such as the paths and cycles.

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

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