Meromorphic Solutions of Linear Differential Systems, Painleve type functions

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Abstract

We consider the \( n \times n \) matrix linear differential systems in the complex plane. We find necessary and sufficient conditions under which these systems have meromorphic fundamental solutions. Using the operator identity method we construct a set of systems which have meromorphic solutions. We prove that the well known operator with the sine kernel generates a class of meromorphic Painleve type functions. The fifth Painleve function belongs to this class. Hence we obtain a new and simple proof that the fifth Painleve function is meromorphic.

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1 Introduction

Let us consider the \( n \times n \) matrix system of the form

\[
\frac{dW}{dx} = A(x)W,
\]

where \( A(x) \) is the \( n \times n \) matrix function. Further we assume that the matrix function \( A(x) \) is holomorphic and single valued in a punctured neighborhood of a point \( x_0 \).

Every fundamental solution \( W(x) \) of system (1.1) has the form (see [1],[20])

\[
W(x) = S(x)(x - x_0)^\Phi,
\]

where the matrix \( S(x) \) is holomorphic and single valued in the domain \( 0 < |x - x_0| < \rho \) and \( \Phi \) is a constant matrix.
Definition 1.1. (see [1],[20]) The point \( x_0 \) is called a regular point of system (1.1) if the corresponding matrix \( S(x) \) is either holomorphic in a neighborhood of \( x_0 \) or has a pole in \( x_0 \).

Definition 1.2. We shall name the regular point \( x_0 \) of system (1.1) a strong regular if \( \Phi = 0 \) in formula (1.2).

We use the following transformation

\[
W(x) = F(x)Y(x),
\]

where

\[
F(x) = \sum_{k=\ell}^{m} f_k(x - x_0)^k,
\]

\( f_k \) are constant \( n \times n \) matrices, \( \det F(x) \neq 0, \quad x \neq x_0 \). Then system (1.1) takes the form

\[
\frac{dY}{dx} = B(x)Y,
\]

where

\[
B(x) = F^{-1}(x)A(x)F(x) - F^{-1}(x)\frac{dF}{dx}.
\]

The following theorem gives the condition of regularity.

**Theorem 1.1.** (Horn’s theorem (see [1])) The point \( x_0 \) is regular for system (1.1) if and only if there exists transformation (1.3) such that the corresponding matrix \( B(x) \) has the form

\[
B(x) = \frac{B_1(x)}{x - x_0},
\]

where \( B_1(x) \) is holomorphic in the domain \( 0 \leq |x - x_0| < \rho \).

The conditions of Horn’s theorem are necessary conditions of the strong regularity. In the present paper we give necessary and sufficient conditions of strong regularity.

Separately we consider the case when the entries of \( A(x) \) are meromorphic functions.

Definition 1.3. We name the fundamental solution \( W(x) \) of system (1.1) global strong regular if this solution is strong regular for all singular points of \( A(x) \).

It is easy to see that the global strong solution is meromorphic. We apply the
obtained results to the canonical differential systems [15] with the spectral parameter $\rho$:

$$\frac{dW(x, \rho)}{dx} = [P(x) + \rho Q(x)]W(x, \rho).$$  \hspace{1cm} (1.8)

We investigate in detail the special case when $n = 2$, $P(x) = 0$, and

$$Q(x) = \begin{bmatrix} 0 & r^{-2}(x) \\ r^2(x) & 0 \end{bmatrix}.$$  \hspace{1cm} (1.9)

Now we shall explain the connection of system (1.8), (1.9) with the classical second order equations. The solution of this system

$$U(x, \rho) = \text{col}[u_1(x, \rho), u_2(x, \rho)]$$  \hspace{1cm} (1.10)

satisfies the relations

$$\frac{du_1}{dx} = i\rho r^{-2}(x)u_2(x, \rho), \quad \frac{du_2}{dx} = i\rho r^2(x)u_1(x, \rho).$$  \hspace{1cm} (1.11)

System (1.8), (1.9) reduces to two equations of the second order.

$$-\frac{d}{dx}r^2(x)\frac{du_1}{dx} = \rho^2 r^2(x)u_1(x, \rho),$$  \hspace{1cm} (1.12)

$$-\frac{d}{dx}r^{-2}(x)\frac{du_2}{dx} = \rho^2 r^{-2}(x)u_2(x, \rho).$$  \hspace{1cm} (1.13)

Let us note that equations (1.12) and (1.13) are mutually dual [5],[8],[16] and play an important role in a number of theoretical and applied problems (prediction theory [11], vibration of a thin straight rod [3],generalized string equation [15]).

Using the operator identity method [15] we construct classes $r(x)$ such that the corresponding equations (1.12) and (1.13) have meromorphic solutions in respect to $x$ for all $\rho$. In particular we construct a class of the rational functions $r(x)$ with this property.

The operator identity method allows to construct an analytic continuation of $r^2(x)$ from half-axis $(0, \infty)$ onto the complex plane. We have applied this approach to the third and the fifth Painleve functions. In particular we have obtained a new and simple proof that the fifth Painleve function is meromorphic (see [7]).

**Remark 1.1.** The global Fuchsian theory (see [1],[20]) requires that the
regularity condition be met at infinity as well. In our approach this condition can fail for \( x = \infty \). Thus our theory can be applied to the important examples (see sections 8-10) in which classical Fuchsian theory does not work.

**Remark 1.2.** The meromorphic solutions of the differential systems play an important role in the spectral theory in the space with indefinite metric [13].

## 2 Conditions of strong regularity

Taking into account Horn’s theorem we begin with the matrix function \( A(x) \) which can be represented in the form

\[
A(x) = \frac{a_{-1}}{x - x_0} + a_0 + a_1(x - x_0) + ..., \tag{2.1}
\]

where \( a_k \) are \( n \times n \) matrices. We investigate the case when \( x_0 \) is either a regular point of \( W(x) \) or a pole. Hence the following relation

\[
W(x) = \sum_{k \geq m} b_k (x - x_0)^k, \quad b_m \neq 0 \tag{2.2}
\]

is true. Here \( b_k \) are \( n \times n \) matrices. We note that \( m \) can be negative. From formulas (1.1), (2.1) and (2.2) we deduce that

\[
(k + 1)b_{k+1} = \sum_{j + \ell = k} a_j b_\ell, \tag{2.3}
\]

where \( j \geq -1, \quad \ell \geq m \). Relation (2.3) can be rewritten in the recurrent form

\[
[(k + 1)I_n - a_{-1}]b_{k+1} = \sum_{j + \ell = k} a_j b_\ell, \quad k \geq m, \tag{2.4}
\]

where \( j \geq 0, \quad \ell \geq m \). When \( k = m - 1 \) we have

\[
(mI_n - a_{-1})b_m = 0. \tag{2.5}
\]

From relation (2.5) we deduce the following assertion.

**Proposition 2.1.** (necessary condition) If the solution of system (1.1) has form (2.2) then \( m \) is an eigenvalue of \( a_{-1} \).

We denote by \( M \) the greatest integer eigenvalue of the matrix \( a_{-1} \). Using
relations (2.5) we obtain the assertion.

**Proposition 2.2.** (sufficient condition) If the matrix system

\[
[(k + 1)I_n - a_{-1}]b_{k+1} = \sum_{j+\ell=k} a_j b_{\ell},
\]  

(2.6)

where \( m \leq k + 1 \leq M \) has a solution \( b_m, b_{m+1}, ..., b_M \) and \( b_m \neq 0 \) then system (1.1) has a solution of form (2.2).

We consider the system of equations

\[
\frac{dY}{dx} = -YA(x),
\]  

(2.7)

where \( Y(x) \) has the form

\[
Y(x) = \sum_{k \geq p} c_k (x - x_0)^k, \quad c_p \neq 0.
\]  

(2.8)

Formulas (2.1), (2.7) and (2.8) imply that

\[
(k + 1)c_{k+1} = -\sum_{j+\ell=k} c_{\ell}a_j,
\]  

(2.9)

where \( j \geq -1, \quad \ell \geq p \). We rewrite relation (2.9) in the form

\[
c_{k+1}[(k + 1)I_n + a_{-1}] = -\sum_{j+\ell=k} c_{\ell}a_j,
\]  

(2.10)

where \( j \geq 0, \quad \ell \geq p \). In the same way as Propositions 2.1 and 2.2 we obtain the following results.

**Proposition 2.3.** If the solution of system (1.1) has form (2.2) then \((-p)\) is an eigenvalue of \( a_{-1} \).

**Proposition 2.4.** Let \((-P)\) be the smallest integer eigenvalue of the matrix \( a_{-1} \). If the matrix system

\[
c_{k+1}[(k + 1)I_n + a_{-1}] = -\sum_{j+\ell=k} c_{\ell}a_j,
\]  

(2.11)

where \( p \leq k + 1 \leq P \) has a solution \( c_p \neq 0, c_{p+1}, ..., c_P \), then system (1.1) has a solution of form (2.2).

**Remark 2.1.** If \( W(x) \) is a fundamental solution of system (1.1), then \( Y(x) = \)
$W^{-1}(x)$ is a fundamental solution of system (2.7).

**Proposition 2.5.** If $W(x)$ and $W^{-1}(x)$ satisfy relations (1.1), (2.1) and (2.7), (2.8) respectively, then $m$ and $-p$ are eigenvalues of the matrix $a_{-1}$. The corresponding matrix $a_{-1}$ is either scalar or has at least two different integer eigenvalues.

**Proof.** Let the matrix $a_{-1}$ not be a scalar one. Then it follows from relation (2.5) that

$$\det b_m = 0.$$  \hspace{1cm} (2.12)

Let us suppose that $a_{-1}$ doesn’t have integer eigenvalues different from $m$. In view of Propositions 2.1 and 2.3 the equality $p = -m$ is true. From relations (2.2), (2.8) and the equality $W(x)W^{-1}(x) = I_n$ we have $b_me_m = I_n$ which contradicts relation (2.12). This proves the proposition.

## 3 Integer eigenvalues

We consider again differential system (1.1), where $A(x)$ has form (2.1). Let $T$ be a constant matrix such that

$$T^{-1}a_{-1}T = b_{-1},$$  \hspace{1cm} (3.1)

where $b_{-1}$ is Jordan matrix, i.e. $b_{-1}$ has the following structure $b_{-1} = \text{diag}(J_1, J_2, ..., J_s)$, $s \leq n$. Here $J_k = \lambda_k I_k + H_k$, $1 \leq k \leq s$ and

$$H_k = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & 1 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}.$$

We reduce system (1.1) to the form

$$\frac{dV}{dx} = B(x)V,$$  \hspace{1cm} (3.2)

where $W(x) = TV$, $B(x) = T^{-1}A(x)T$. Now we describe the "shearing" transformation (see [20],Ch.5) which lowers the eigenvalue $\lambda_s$ of the matrix $b_{-1}$ by one, while leaving the others unchanged.

We denote by $q$ the order of Jordan matrix $J_s$ and represent $B(x)$ in the form

$$B(x) = \frac{1}{x - x_0} \begin{bmatrix} \tilde{b}_{-1} & 0 \\ 0 & J_s \end{bmatrix} + \tilde{B}(x),$$  \hspace{1cm} (3.3)
where $\tilde{B}(x)$ is holomorphic at $x_0$ and
\[
\tilde{b}_{-1} = \text{diag}(J_1, J_2, \ldots, J_{s-1}).
\] (3.4)

The "shearing" transformation is defined by the relation (see [1],[20])
\[
V = S(x)U.
\] (3.5)

Here
\[
S(x) = \begin{bmatrix}
I_{n-q} & 0 \\
0 & (x-x_0)I_q
\end{bmatrix}.
\] (3.6)

Using (3.5) we deduce that
\[
\frac{dU}{dx} = C(x)U,
\] (3.7)

where
\[
C(x) = S^{-1}(x)B(x)S(x) - S^{-1}(x)\frac{d}{dx}S(x).
\] (3.8)

It follows from (3.3) and (3.6) that
\[
c_{-1} = \begin{bmatrix}
\tilde{b}_{-1} & 0 \\
\Gamma & J_s - I_q
\end{bmatrix}.
\] (3.9)

It is easy to see that the matrix $c_{-1}$ has the same eigenvalues as $b_{-1}$ except that the eigenvalue $\lambda_s$ has been decreased by unity.

**Theorem 3.1.** If the fundamental solution of system (1.1) is strong regular then all the eigenvalues of the corresponding matrix $a_{-1}$ are integer.

**Proof.** Using a finite number of pairs of constant and "shearing" transformations we can reduce system (1.1) to the system
\[
\frac{d}{dx}\tilde{W}(x) = \tilde{A}(x)\tilde{W}(x),
\] (3.10)

where all the integer eigenvalues of $\tilde{a}_{-1}$ coincide with the smallest integer eigenvalue of $a_{-1}$, the non integer eigenvalues of $a_{-1}$ and $\tilde{a}_{-1}$ coincide. If the fundamental solution $W(x)$ of system (1.1) is strong regular then the fundamental solution $\tilde{W}(x)$ of system (3.10) is strong regular as well. If $\tilde{a}_{-1}$ has non integer eigenvalues then according to Proposition 2.5 the matrix $\tilde{a}_{-1}$ has at least two different integer eigenvalues. The theorem is proved.
4 Examples

Example 4.1. V.Katsnelson and D.Volok [10] investigated the case when the point \( x_0 \) is a simple pole of \( W(x) \) and a holomorphic point of the inverse matrix function \( W^{-1}(x) \). They proved that in this case

\[
a_{-1}^2 = -a_{-1}, \quad a_{-1}a_0a_{-1} = -a_0a_{-1}.
\] (4.1)

It follows from the first of the relations (4.1) that the eigenvalues of \( a_{-1} \) are equal \(-1\) or 0. From Proposition 2.2 we deduce the assertion.

Proposition 4.1. Let conditions (4.1) be fulfilled. Then system (1.1) has a strong regular solution, where \( m = -1 \).

Proof. In the case under consideration we have \( m = -1, \quad M = 0 \). Hence system (2.6) takes the form

\[
(I_n + a_{-1})b_{-1} = 0, \quad -a_{-1}b_0 = a_0b_{-1}.
\] (4.2)

Comparing the first relations of (4.1) and (4.2) we obtain the equality

\[
b_{-1} = a_{-1}c,
\] (4.3)

where \( c \) is an arbitrary invertible matrix. It follows from the second relation of (4.1) that

\[
b_0 = a_0a_{-1}c
\] (4.4)

satisfies the second equality of (4.2). The proposition is proved.

Example 4.2. Let us consider the case when \( A(x) \) has a pole of the second order. We suppose that the matrix \( A(x) \) has the form

\[
A(x) = \begin{bmatrix}
a_{11}(x) & a_{12}(x) \\
a_{21}(x) & a_{22}(x)
\end{bmatrix},
\] (4.5)

where

\[
a_{11}(x) = a_0 + a_1(x - x_0) + ..., \quad a_{22}(x) = \beta_0 + \beta_1(x - x_0) + ..., \] (4.6, 4.7)

\[
a_{12}(x) = \gamma_{-2}(x - x_0)^{-2} + \gamma_{-1}(x - x_0)^{-1} + ..., \] (4.8)

\[
a_{21}(x) = \mu_2(x - x_0)^2 + \mu_3(x - x_0)^3 + .... \] (4.9)
We introduce the matrix function
\[
\tilde{A}(x) = F^{-1}(x)A(x)F(x) - F^{-1}(x)\frac{d}{dx}F(x),
\] (4.10)
where
\[
F(x) = \begin{bmatrix}
\frac{1}{x-x_0} & 0 \\
0 & 1
\end{bmatrix}
\] (4.11)
It is easy to see that the matrix function \(V(x) = F^{-1}(x)W(x)\) satisfies the equation
\[
\frac{d}{dx}V = \tilde{A}(x)V.
\] (4.12)
It is important that the matrix function \(\tilde{A}(x)\) has a pole of the first order. Indeed it follows from formulas (4.5)-(4.10) that
\[
\tilde{A}(x) = \frac{\tilde{a}_{-1}}{x-x_0} + \tilde{a}_0 + \ldots,
\] (4.13)
where
\[
\tilde{a}_{-1} = \begin{bmatrix}
1 & \gamma - 2 \\
0 & 0
\end{bmatrix}, \quad \tilde{a}_0 = \begin{bmatrix}
\alpha_0 & \gamma - 1 \\
0 & \beta_0
\end{bmatrix}.
\]
We shall consider the case when
\[
V(x) = \sum_{k \geq m} \tilde{b}_k(x-x_0)^k, \quad \tilde{b}_m \neq 0
\] (4.14)
is true. Here \(\tilde{b}_k\) are \(n \times n\) matrices.

**Proposition 4.2.** Let the matrix \(A(x)\) have the form defined by relations (4.6) – (4.9). Then system (1.1) has a strong regular solution if and only if
\[
\gamma_2(\alpha_0 - \beta_0) = \gamma_1.
\] (4.15)

**Proof.** In this case we have \(m = 0, \quad M = 1\). From equality \(\tilde{a}_{-1}\tilde{b}_0 = 0\) we deduce that \(\tilde{b}_0\) has the following form
\[
\tilde{b}_0 = \begin{bmatrix}
-s\gamma_2 & -t\gamma_2 \\
s & t
\end{bmatrix}.
\] (4.16)
In view of (2.4) we have
\[
(I_m - \tilde{a}_{-1})\tilde{b}_1 = \tilde{a}_0\tilde{b}_0
\] (4.17)
Equation (4.17) has a solution \(\tilde{b}_1\) if and only if relation (4.15) is fulfilled. From Proposition 2.2 we deduce the desired assertion.

**Corollary 4.1.** In addition to the conditions of Proposition 4.2 we suppose that \(\alpha_0 = \beta_0\). System (1.1) has a strong regular solution if and only if \(\gamma_1 = 0\).
5 \hspace{1cm} \textbf{Differential Systems with spectral parameter}

We consider the differential system with the parameter $\rho$:

$$
\frac{dW(x, \rho)}{dx} = [P(x) + \rho Q(x)]W(x, \rho),
$$

(5.1)

where the $n \times n$ matrix functions $P(x)$ and $Q(x)$ can be represented in the forms

$$
P(x) = \frac{p_{-1}}{x - x_0} + p_0 + ... ,
$$

(5.2)

$$
Q(x) = \frac{q_{-1}}{x - x_0} + q_0 + ... .
$$

(5.3)

Systems (5.1) play an important role in the spectral theory of the canonical differential systems with the spectral parameter $\rho$ (see [15]). Due to Theorem 3.1 the following assertion is true.

**Proposition 5.1.** (necessary condition) If system (5.1) – (5.3) has a strong regular fundamental solution $W(x, \rho)$ for all $\rho$ then all the eigenvalues of the matrix $p_{-1} + \rho q_{-1}$ are integer and do not depend on $\rho$.

**Example 5.1.** Let $n = 2$ and $p_{-1} = \begin{bmatrix} \lambda_1 & \Gamma_1 \\
0 & \lambda_2 \end{bmatrix}$, $q_{-1} = \begin{bmatrix} 0 & \Gamma_2 \\
0 & 0 \end{bmatrix}$.
We assume that $\lambda_1$ and $\lambda_2$ are integer numbers. The eigenvalues of the matrix $p_{-1} + \rho q_{-1}$ are equal to $\lambda_1$ and $\lambda_2$, i.e. these eigenvalues are integer and do not depend on $\rho$.

**Example 5.2.** We consider the system

$$
\frac{d}{dx}W(x, \rho) = \rho A(x)W(x, \rho),
$$

(5.4)

where the matrix function $A(x)$ is defined by relations (4.6)-(4.9). We introduce the matrix

$$
\tilde{A}(x, \rho) = \rho F^{-1}(x)A(x)F(x) - F^{-1}(x)\frac{d}{dx}F(x),
$$

(5.5)

where

$$
F(x) = \begin{bmatrix} \frac{1}{x - x_0} & 0 \\
0 & 1 \end{bmatrix}
$$

(5.6)
The matrix function $V(x, \rho) = F^{-1}(x)W(x, \rho)$ satisfies the equation
\[
\frac{d}{dx}V = [P(x) + \rho Q(x)]V, \quad (5.7)
\]
where
\[
P(x) = -F^{-1}(x)\frac{d}{dx}F(x) = \begin{bmatrix} \frac{1}{x-x_0} & 0 \\ 0 & 1 \end{bmatrix}, \quad (5.8)
\]
\[
Q(x) = F^{-1}(x)A(x)F(x). \quad (5.9)
\]
It follows from (5.8) and (5.9) that
\[
p_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad p_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad q_{-1} = \rho \begin{bmatrix} 0 & \gamma_{-2} \\ 0 & 0 \end{bmatrix}, \quad q_0 = \rho \begin{bmatrix} \alpha_0 & \gamma_{-1} \\ 0 & \beta_0 \end{bmatrix}.
\]
Condition (4.15) takes the form
\[
\rho \gamma_{-2}(\alpha_0 - \beta_0) = \gamma_{-1}. \quad (5.10)
\]
Using Propositions 4.2 and 5.1 we obtain the following assertion.
\begin{proposition}
Let the matrix function $A(x)$ be defined by relations (4.6)–(4.9). System (5.1) has a strong regular fundamental solution for all $\rho$ if and only if
\[
\alpha_0 = \beta_0, \quad \gamma_{-1} = 0.
\]
\end{proposition}

6 Global strong regular solutions

In sections 1-5 we investigated the strong regular solutions in a punctured neighborhood of the singular point $x_0$. Now we deduce the conditions under which the solution of system (5.1) is strong regular for all complex $x \neq \infty$ and all complex $\rho \neq \infty$ (global strong regular solution). It is obvious that the global strong solution is meromorphic in $x$ and entire in $\rho$.

Let us consider the differential system
\[
\frac{dW}{dx} = \rho A(x)W, \quad (6.1)
\]
where the $2 \times 2$ matrix function $A(x)$ has the form
\[
A(x) = \begin{bmatrix} 0 & r(x)^{-2} \\ r(x)^2 & 0 \end{bmatrix}. \quad (6.2)
\]
Here \( r(x) \) is a meromorphic function in the complex plane. We denote by \( x_k, \quad 1 \leq k \leq n \leq \infty \) and by \( y_\ell, \quad 1 \leq \ell \leq m \leq \infty \) the different roots of \( r(x) \) and of \( r^{-1}(x) \) respectively. Proposition 5.2 implies the following result.

**Theorem 6.1.** Let all the roots \( x_k \) and \( y_\ell \) of \( r(x) \) and of \( r^{-1}(x) \) respectively be simple. The fundamental solution \( W(x, \rho) \) of system (6.1), (6.2) is strong regular for all \( x \) and \( \rho \) if and only if

\[
\begin{align*}
    r(x_k) &= 0, \quad r'(x_k) \neq 0, \quad r''(x_k) = 0, \quad (1 \leq k \leq n), \\
    q(y_\ell) &= 0, \quad q'(y_\ell) \neq 0, \quad q''(y_\ell) = 0, \quad (1 \leq \ell \leq m), \quad q(x) = r^{-1}(x)
\end{align*}
\]

The following assertions can be proved by the direct calculation.

**Proposition 6.1.** The functions

\[
\begin{align*}
    I) \quad r_1(x) &= x, & II) \quad r_2(x) &= \tan x
\end{align*}
\]

satisfy all the conditions of Theorem 6.1. The corresponding \( x_k \) and \( y_\ell \) are defined by the relations:

\[
\begin{align*}
    I) n = 1, m = 0, x_1 = 0, & \quad II) x_k = k\pi, \quad y_\ell = \ell\pi + \frac{\pi}{2}, \quad -\infty < k, \ell < \infty.
\end{align*}
\]

**Proposition 6.2.** If \( r(x) \) is a polynomial and \( \deg r(x) \geq 2 \) then \( r(x) \) does not satisfy conditions (6.3).

**Proposition 6.3.** The functions

\[
\begin{align*}
    r_3(x) &= \frac{x - \lambda_1}{x - \lambda_2}, & \lambda_1 \neq \lambda_2, \\
    r_4(x) &= \frac{(x - \lambda_1)(x - \lambda_2)}{x - \mu_1}, & \lambda_1 \neq \lambda_2, \quad \lambda_1, \lambda_2 \neq \mu_1
\end{align*}
\]

do not satisfy conditions (6.3), (6.4).

Theorem 6.1 and Propositions 6.1-6.3 lead to the following problems.

**Problem 6.1.** To construct meromorphic functions \( r(x) \) which satisfy conditions (6.3) and (6.4).

**Problem 6.2.** To construct rational functions \( r(x) \) which satisfy conditions (6.3) and (6.4).

These problems will be investigated in the next sections.

The connection of system (6.1), (6.2) with the classical second order equations is explained in the introduction.
7 Operator Identity

To solve Problems 6.1 and 6.2 we use the operator identity method (see [15]).

We introduce the operators

\[ Af = i \int_0^x f(t) dt, \quad f(x) \in L^2(0, a) \]  
(7.1)

and

\[ Sf = f(x) + \int_0^a f(t) k(x - t) dt, \]  
(7.2)

where the function \( k(x), \quad (-a \leq x \leq a) \) is continuous and

\[ k(x) = k(-x) = k(x). \]  
(7.3)

We use the following operator identity

\[ AS - SA^* = i(\Phi_1 \Phi_2^* + \Phi_2 \Phi_1^*). \]  
(7.4)

Here the operators \( \Phi_1 \) and \( \Phi_2 \) are defined by the relations

\[ \Phi_1 g = M(x) g, \quad \Phi_2 g = g, \]  
(7.5)

where

\[ M(x) = \int_0^x k(u) du + \frac{1}{2}, \quad 0 \leq x \leq a. \]  
(7.6)

Thus the operators \( \Phi_1 \) and \( \Phi_2 \) map the one-dimensional space of constant numbers \( g \) into \( L^2(0, a) \). Let us consider the operator

\[ S_\xi f = f(x) + \int_0^\xi f(t) k(x - t) dt, \quad f(x) \in L^2(0, \xi) \quad 0 \leq \xi \leq a. \]  
(7.7)

Let us formulate the following results (see [13]).

**Theorem 7.1.** We assume:
There are points \( 0 < x_1 < x_2 < ... \) having no limit points in \([0, a]\) and such that the operator \( S_\xi \) is invertible on \( L^2(0, \xi) \) for each \( \xi \in [0, a)/\{x_1, x_2, ...\} \).

Then the matrix function

\[ B(\xi) = \Pi S_\xi^{-1} P_\xi \Pi, \quad \Pi = [\Phi_1, \Phi_2] \]  
(7.8)
is continuous and nondecreasing in each of the intervals \((x_k, x_{k+1})\). The matrix function
\[ W(\xi, \rho) = I_2 + i\rho J^* S_\xi^{-1} P_\xi (I - \rho A)^{-1} \Pi \] (7.9)
is a fundamental solution for the system
\[ W(\xi, \rho) = I_2 + i\rho J \int_0^\xi [dB(t)W(t, \rho)], \] (7.10)
where
\[ J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \] (7.11)

**Theorem 7.2.** Let \(B(x)\) be constructed by (7.8). Then \(B(x)\) is continuously differentiable in the intervals between the singularities, and in these intervals
\[ H(\xi) = B'(\xi) = [h^*_1(\xi)h_j(\xi)]^2, \] (7.12)

where
\[ h_1(\xi) = M(\xi) + \int_0^\xi \Gamma_\xi(\xi, t) M(t) dt. \] (7.13)
\[ h_2(\xi) = 1 + \int_0^\xi \Gamma_\xi(\xi, t) dt, \] (7.14)

We use here the formula
\[ S_\xi^{-1} f = f(x) + \int_0^\xi \Gamma_\xi(x, t) f(t) dt. \] (7.15)

We remark that \(H(\xi)\) has the special form [16]
\[ H(\xi) = \frac{1}{2} \begin{bmatrix} Q(\xi) & 1 \\ 1 & Q^{-1}(\xi) \end{bmatrix}. \] (7.16)

It follows from relations (7.10) and (7.11) that
\[ \frac{dW(x, \rho)}{dx} = i\rho JH(x)W(x, \rho). \] (7.17)

Introducing \(U(x, \rho) = W(2x, \rho)e^{-i\rho x}\) we reduce system (7.17) to the form
\[ \frac{dU(x, \rho)}{dx} = i\rho JH_1(x)U(x, \rho), \] (7.18)
where
\[ H_1(x) = \begin{bmatrix} r^2(x) & 0 \\ 0 & r^{-2}(x) \end{bmatrix}, \] (7.19)
\[ R(x) = Q(2x). \] (7.20)

Let us note that obtained system (7.18), (7.19) coincides with system (6.1), (6.2).

8  **Rational \( r(x) \)**

Let us consider the operator \( S_\xi \) (see (7.7)), where \( k(x) \) satisfies conditions (7.3) and is a polynomial of degree \( 2m \). The kernel \( k(x-t) \) can be represented in the form
\[ k(x-t) = \sum_{s=0}^{2m} x^s p_s(t), \] (8.1)
where \( p_s(t) \) are the polynomials \( \deg p_s(t) \leq (2m - s) \). We introduce the matrix
\[ A_\xi = [\delta_{j,s} + (x^s,p_j(x))_\xi]_{0}^{2m} \] (8.2)
and the determinant
\[ \Delta_\xi = \det A_\xi. \] (8.3)

In formula (8.2) we used the notation \( (f,g)_\xi = \int_0^\xi f(t)g(t)dt \). The solution \( g(x,\xi) \) of the equation
\[ S_\xi g = f(x) \] (8.4)
has the form
\[ g(x,\xi) = f(x) - \sum_{s=0}^{2m} c_s(\xi)x^s, \] (8.5)
where \( c_s(\xi) = (g,p_s)_\xi \). It follows from (8.4) and Cramer’s rule that
\[ c_s(\xi) = \frac{d_s(\xi)}{\Delta_\xi}, \] (8.6)
where the determinant \( d_s(\xi) \) is formed by replacing the column under number \( s \) in \( \Delta_\xi \) by the column
col[(f, p_0) \xi, (f, p_1) \xi, \ldots, (f, p_{2m}) \xi].

Using (8.5) and (8.6) we have

\[ g(x) = f(x) + \int_0^\xi \Gamma_\xi(x, t) f(t) dt, \quad (8.7) \]

where

\[ \Gamma_\xi(x, t) = \frac{1}{\Delta_\xi} \sum_{s=0}^{2m} D_s(\xi, t) x^s. \quad (8.8) \]

Here the determinant \( D_s(\xi, t) \) is formed by replacing the column under number \( s \) in \( \Delta_\xi \) by the column

\[ \text{col}[p_0(t), p_1(t), \ldots, p_{2m}(t)]. \]

The expression

\[ \Gamma_\xi(0, \xi) = \frac{D_0(\xi, \xi)}{\Delta_\xi} \quad (8.9) \]

plays an important role in our theory.

From (8.7) and (8.8) we deduce the following assertion.

**Theorem 8.1.** If \( k(x) \) satisfies conditions (7.3) and is a polynomial then the corresponding function \( Q(\xi) \) (see (7.16)) is rational.

**Proof.** Let \( f(x) = 1 \). In this case formula (8.5) gives

\[ g(x, \xi) = 1 + \sum_{s=0}^{2m} x^s R_s(\xi), \quad (8.10) \]

where the functions \( R_s(\xi) \) are rational. Hence the function \( g(\xi, \xi) = h_2(\xi) \) is rational too. The assertion of the theorem follows directly from the equality

\[ Q^{-1}(x) = 2h_2^2(x), \quad (8.11) \]

which can be obtained from (7.12) and (7.16).

We denote by

\[ \xi_1, \xi_2, \ldots, \xi_n \]

the roots of the polynomial \( \Delta_\xi \).

**Theorem 8.2.** If \( k(x) \) satisfies conditions (7.3) and is a polynomial then the corresponding matrix function \( W(\xi, \rho) \) defined by relation (7.9) is entire in respect to \( \rho \) and meromorphic in respect to \( \xi \) with the poles in the points \( \xi_1, \xi_2, \ldots, \xi_n \).

**Proof.** According to (7.6) the function \( M(x) \) is a polynomial. Hence the function \( (I - A\rho)^{-1}M(x) \) is an entire function of \( \rho \) and \( x \). Using (7.9) and
(8.8) we deduce the assertion of the theorem.

**Remark 8.2.** We consider $Q(\xi)$ and $W(\xi, \rho)$ for all the complex $\xi \neq \xi_k$, $(1 \leq k \leq n)$ and for all the complex $\rho$.

Due to analytic continuation the equality

$$\frac{dW}{d\xi} = i\rho JH(\xi)W(\xi, \rho)$$  \hspace{1cm} (8.12)

is true for all complex $\xi \neq \xi_k$, $(1 \leq k \leq n)$ and for all the complex $\rho$. Here the matrix function $H(\xi)$ is defined by formula (7.16). Theorems 7.2 and 8.2 imply the following assertion.

**Corollary 8.2** The function $r(x) = 1/\sqrt{2h_2(2x)}$ is a rational function. The union of the sets of the roots and the poles of $r(x)$ coincides with the set $\frac{1}{2}\xi_1, \frac{1}{2}\xi_2, ..., \frac{1}{2}\xi_n$.

We shall use the relation (see [7], Ch. 4.)

$$\frac{dg(\xi, \xi)}{d\xi} = \Gamma_\xi(0, \xi)g(\xi, \xi), \quad g(\xi, \xi) = h_2(\xi).$$  \hspace{1cm} (8.13)

From relation (8.13) and Corollary 8.2 we deduce the assertion.

**Corollary 8.3** If all roots of $\Delta_\xi$ are simple then the corresponding function $r(x)$ satisfies the conditions (6.3), (6.4).

**Example 8.1.** Let us consider the case when $k(x) = x^2$.  \hspace{1cm} (8.14)

In this case we have $p_0(t) = t^2, p_1(t) = -2t, p_2(t) = 1$. Hence the determinants $\Delta_\xi$ and $d_0(\xi)$ are defined by the relations

$$\Delta_\xi = \begin{vmatrix} 1 + \xi^3/3 & \xi^4/4 & \xi^5/5 \\ -\xi^2 & 1 - 2\xi^3/3 & -\xi^4/4 \\ \xi & \xi^2 & 1 + \xi^3/3 \end{vmatrix},$$  \hspace{1cm} (8.15)

$$d_0(\xi) = \begin{vmatrix} -\xi^2 & \xi^4/4 & \xi^5/5 \\ 2xi & 1 - 2\xi^3/3 & -\xi^4/4 \\ -1 & \xi^2 & 1 + \xi^3/3 \end{vmatrix}.$$  \hspace{1cm} (8.16)

It follows from (8.15) and (8.16) that

$$\Delta_\xi = \frac{1}{1080}\xi^9 - \frac{1}{30}\xi^6 + 1,$$  \hspace{1cm} (8.17)
\[ d_0(\xi) = -\xi^2[(1 - \xi^3/6)^2 + \xi^3(1 - \xi^3/15)/2 - \xi^3(1 - \xi^3/24)/5]. \quad (8.18) \]

The polynomial \( \Delta_\xi \) has nine different roots
\[ x_k^3 = 6, \quad (1 \leq k \leq 3), \quad x_k^3 = 15 + 9\sqrt{5}, \quad (4 \leq k \leq 6), \quad x_k^3 = 15 - 9\sqrt{5}, \quad (7 \leq k \leq 9). \]

By the direct calculation we prove the following assertion.

**Proposition 8.1.** The poles of \( \Gamma_\xi(0, \xi) \) coincide with \( x_k, \quad (1 \leq k \leq 9) \). These poles are simple and the residues in the points \( x_k, \quad (1 \leq k \leq 3) \) are equal to \( 1 \) and in the points \( x_k, \quad (4 \leq k \leq 9) \) are equal to \( -1 \).

From relation (8.17) and Proposition 8.1 we deduce that
\[ h_2(x) = \frac{\frac{1}{6}x^3 - 1}{\frac{1}{180}x^6 - \frac{1}{6}x^3 - 1}. \quad (8.19) \]

Hence the corresponding function \( r(x) = 1/[\sqrt{2}h_2(2x)] \) is rational and satisfies conditions (6.3) and (6.4) (see Problem 6.2).

### 9 Exponential \( r(x) \)

The following example was considered in the paper [13].

**Example 9.1.** Let the operator \( S_\xi \) have the form
\[ S_\xi f = f(x) + \beta \int_0^\xi \left[e^{i\lambda(x-t)} + e^{-i\lambda(x-t)}\right] f(t)dt, \quad (9.1) \]
where \( \beta = \overline{\beta} \neq 0, \quad \lambda > 0. \) We find
\[ S_\xi^{-1} f = f(x) - K(x)T^{-1}(\xi) \int_0^\xi K^*(t)f(t)dt, \quad (9.2) \]
where \( K(x) = [e^{i\lambda x}, e^{-i\lambda x}] \) and
\[ T(\xi) = \begin{bmatrix} \xi + \beta^{-1} & \lambda^{-1}e^{-i\lambda\xi}\sin\lambda\xi \\ \lambda^{-1}e^{i\lambda\xi}\sin\lambda\xi & \xi + \beta^{-1} \end{bmatrix}. \quad (9.3) \]
By direct calculation we have \( h_1(x) = \frac{1}{2h_2(x)} \) and
\[ h_2(x) = \frac{u(x)}{v(x)}, \quad (9.4) \]
where
\[ u(x) = x + \beta^{-1} - \lambda^{-1} \sin \lambda x, \quad v(x) = x + \beta^{-1} + \lambda^{-1} \sin \lambda x. \] (9.5)

It is easy to see that all the roots and the poles of \( h_2(x) \) are simple. In the same way as Corollary 8.3, we deduce the following assertion.

**Proposition 9.1.** The corresponding function
\[ r(x) = \frac{2x + \beta^{-1} + \lambda^{-1} \sin 2 \lambda x}{\sqrt{2(2x + \beta^{-1} + \lambda^{-1} \sin 2 \lambda x)}}. \] (9.6)

is rational and satisfies conditions (6.3) and (6.4)

## 10 Analytic continuation, Painleve transcendents

1. Let us consider the operator
\[ (S_\xi f)(x) = f(x) + \int_0^\xi k(x, t) f(t) \, dt \] (10.1)
on \( L^2(0, \xi) \).

**Theorem 10.1.** Let the kernel \( k(x, t), \quad 0 < x, t < \infty \) have an extension to a function \( k(z, w) \) which is analytic as function of \( z \) and \( w \) in a region \( G \) such that \( G \) contains the set \((0, \infty)\) and \( zt \in G \) whenever \( z \in G, \quad 0 < t < 1 \). Then the function
\[ \sigma(\xi, f, g) = (S_\xi^{-1} f, g) \xi \] (10.2)
where \( f(x) \) and \( g(x) \) are entire functions of \( x \), has an extension to a function \( \sigma(z, f, g) \) which is analytic in \( G \) except at isolated points. All finite singular points of \( \sigma(z, f, g) \) are poles.

**Proof.** For small \( \xi \), the operator \( S_\xi \) differs from the identity operator by an operator of norm less than one. Therefore \( S_\xi \) is invertible for \( 0 \leq \xi < \varepsilon \) for some \( \varepsilon > 0 \). For each \( \xi \) in \((0, \infty)\), define \( U_\xi \) from \( L^2(0, 1) \) to \( L^2(0, \xi) \) by
\[ (U_\xi f)(t) = \sqrt{\frac{1}{\xi}} f \left( \frac{x}{\xi} \right), \quad 0 < x < \xi. \]

Then \( U_\xi \) maps \( L^2(0, 1) \) isometrically onto \( L^2(0, \xi) \), and
\[ (U_\xi^{-1} g)(x) = \sqrt{\xi} g(t\xi) \quad 0 < t < 1. \]
Hence $\tilde{S}_\xi = U_\xi^{-1}S_\xi U_\xi$ is a bounded operator on $L^2(0,1)$ given by

$$\tilde{S}_\xi f(x) = f(x) + \xi \int_0^1 k(\xi x, \xi t) f(t) \, dt. \quad (10.3)$$

Clearly $S_\xi$ is invertible if and only if $\tilde{S}_\xi$ is invertible. Write

$$\tilde{S}_\xi = I + T_\xi. \quad (10.4)$$

The assumptions of the theorem allow us to define an operator $T(z)$ on $L^2(0,1)$ by

$$T(z)f = z \int_0^1 k(zx, zt) f(t) \, dt. \quad (10.5)$$

The operator $T(z)$ is compact and depends holomorphically on $z$, and $T(z)$ agrees with the operator $T(\xi)$ defined by (10.4) when $z = \xi$ is a point of $(0, \infty)$. Since $I + T(\xi)$ is invertible for small positive $\xi$, $I + T(z)$ is invertible except at isolated points of $G$ (see Kato [9] Theorem 1.9 on p. 370) in which $[I + T(z)]^{-1}$ has the poles. Hence the function $([I + T(z)]^{-1}x^m, x^n)_1$ is meromorphic if $m$ and $n$ are non-negative integers. The assertion of the theorem follows from the relation

$$(S_\xi^{-1}x^m, x^n)_\xi = \xi^{m+n+1}(\tilde{S}_\xi^{-1}x^m, x^n)_1. \quad (10.6)$$

**Remark 10.1.** The arguments close to Theorem 10.1 are contained in the article [13]
The following kernels satisfy the conditions of Theorem 10.1:

$$k_1(x, t) = \gamma \frac{\sin \pi(x - t)}{\pi(x - t)}, \quad \gamma = \overline{\gamma}, \quad (10.7)$$

$$k_2(x, t) = \gamma \frac{Ai(x)Ai'(t) - Ai(t)Ai'(t)}{x - t}, \quad \gamma = \overline{\gamma}, \quad (10.8)$$

where $Ai(x)$ is the Airy function.

Let us introduce the functions $\phi(x) = J_\alpha(\sqrt{x})$, $\psi(x) = x\phi'$, $x \geq 0$ and the kernel

$$k_3(x, t) = \gamma \frac{\phi(x)\psi(t) - \phi(t)\psi(x)}{x - t}, \quad \gamma = \overline{\gamma}, \quad (10.9)$$

where $J_\alpha(x)$ is the Bessel function of order $\alpha$ ($\alpha > -1$).

**Remark 10.2.** The sine-kernel $k_1(x, t)$, the Airy-kernel $k_2(x, t)$ and the
Bessel-kernel \(k_3(x,t)\) play an important role in the random matrix theory (see [4], [12], [17-19].)

**Remark 10.3.** The region \(G\) in the cases \(k_1(x,t)\) and \(k_2(x,t)\) is the complex plane. The region \(G\) in the case \(k_3(x,t)\) is the complex plane cut by the half-axis \([0, \infty)\).

2. **Example 10.1 (fifth Painleve transcendent)**

Let us consider the operator

\[
S_t f = f(x) + \gamma \int_{-\tau}^{t} k(x - u)f(u)du, \quad f(u) \in L^2(-a, a),
\]

where \(|\tau| \leq a\), \(\gamma = \gamma_0\) and

\[
k(x) = \frac{\sin x\pi}{x\pi}.
\]

The operator \(S_t\) is invertible (see [4,p.167], when \(|\gamma| \leq 1\). Hence we have

\[
S_t^{-1} f = f(x) + \int_{-\tau}^{t} \Gamma_t(x, u, \gamma)f(u)du, \quad f(u) \in L^2(-t, t),
\]

where the kernel \(\Gamma_t(x, u, \gamma)\) is jointly continuous to the variables \(x, t, u, \gamma\).

Together with the operator \(S_t\) we shall consider the operator

\[
\tilde{S}_{2t} f = f(x) + \gamma \int_{0}^{2t} k(x - v)f(v)dv, \quad f(u) \in L^2(0, 2t).
\]

The operator

\[
U_t f(x) = f(u + t)
\]

maps unitarily the space \(L^2(0, 2t)\) onto \(L^2(-t, t)\). It is easy to see that

\[
U_t^{-1} S_t U_t f = \tilde{S}_{2t} f.
\]

In view of (10.12) and (10.15) we have

\[
\tilde{S}_{2t}^{-1} f = f(x) + \int_{0}^{2t} \tilde{\Gamma}_{2t}(x, u, \gamma)f(u)du, \quad f(u) \in L^2(0, 2t),
\]

where

\[
\tilde{\Gamma}_{2t}(x, y, \gamma) = \Gamma_t(x - t, y - t, \gamma).
\]

It follows from (10.17) that

\[
\tilde{\Gamma}_{2t}(2t, 2t, \gamma) = \Gamma_t(t, t, \gamma), \quad \tilde{\Gamma}_{2t}(2t, 0, \gamma) = \Gamma_t(t, -t, \gamma).
\]
Now we consider the case when $\gamma = -1$. For brevity we omit the parameter $\gamma = -1$ in the notation $\Gamma_t(x, u, -1)$. Following C.Trace and H.Widom [17] we introduce the functions

$$r(t) = e^{it\pi} + \int_{-t}^{t} \Gamma_t(t, u)e^{iu\pi} du$$  \hspace{1cm} (10.19)$$

and

$$q(t) = e^{it\pi} + \int_{0}^{t} \tilde{\Gamma}_t(t, u)e^{iu\pi} du$$  \hspace{1cm} (10.20)$$

Relations (10.17) and (10.19), (10.20) imply that

$$q(2t) = r(t)e^{it\pi}.$$  \hspace{1cm} (10.21)$$

We use the following relation (see [17])

$$\frac{d}{dt}[tR(t, t)] = |r(t)|^2,$$  \hspace{1cm} (10.22)$$

where $R(t, t) = \Gamma_t(t, t)$. From (10.21) and (10.22) we have

$$tR(t, t) = \frac{1}{2} \int_{0}^{2t} |q(v)|^2 dv.$$  \hspace{1cm} (10.23)$$

To prove the relation

$$tR(t, t) = \frac{1}{2}(S_{t}^{-1}e^{u\pi}, e^{u\pi})_{2t}$$  \hspace{1cm} (10.24)$$

we use the notion of the triangular factorization (see [7], Ch.4; [14]).

**Definition 10.1.** The positive operator $S$ acting in $L^2(0, a)$ admits the *triangular factorization* if it can be represented in the form

$$S = S_- S^*_-. \hspace{1cm} (10.25)$$

Here $Q_\xi S_{\pm}^{\pm} = Q_\xi S_{\pm}^{\pm} Q_\xi$, where $Q_\xi = I - P_\xi$, $P_\xi f = f(x)$, $0 \leq x < \xi$ and $P_\xi f = 0$, $\xi \leq x \leq a$, $f(x) \in L^2_k(0, a)$. Using M.G.Krein result (see [7], Ch.4) on the triangular factorization of the operator $S$ with continuous kernel we obtain the assertion.

**Theorem 10.2.** The operator

$$Sf = f(x) - \frac{1}{\pi} \int_{0}^{\pi} \frac{\sin \pi(x - t)}{x - t} f(t) dt$$  \hspace{1cm} (10.26)$$
admits triangular factorization (10.25) and

\[ S^{-1}f = f(v) + \int_{0}^{v} \tilde{\Gamma}_v(v,u)f(u)du. \]  
\(10.27\)

Hence formula (10.20) can be written in the form

\[ q(x) = S^{-1}e^{iu\pi}. \]  
\(10.28\)

**Remark 10.4.** Representation (10.28) of \(q(x)\) which contains the factorizing operator \(S_-\) plays an essential role in our approach.

We use the notations

\[ D(\xi) = \det \tilde{S}_\xi, \]  
\(10.29\)

\[ \sigma(x) = \frac{x}{\pi}D'(\frac{x}{\pi})/D(\frac{x}{\pi}). \]  
\(10.30\)

It is known (see[17]) that

\[ \sigma(x) = -2tR(t,t), \quad x = 2\pi t. \]  
\(10.31\)

Relations (10.24) and (10.31) imply that

\[ \sigma(x) = -\left(\tilde{S}_2^{-1}e^{iu\pi}, e^{iu\pi}\right)_{2t}, \quad x = 2\pi t. \]  
\(10.32\)

We note that the function \(\sigma(x)\) is the fifth Painleve transcendent (see[17]). Using Proposition 10.1 and relation (10.31) we have obtained the new proof of the following well-known fact (see[6]).

**Corollary 10.1.** The fifth Painleve transcendent \(\sigma(\xi)\) can be extended to the meromorphic function \(\sigma(z)\).

The function \(\sigma(\xi)\) is a solution of the Painleve equation \((P_5\text{ in the sigma form, see [17]})

\[ (\xi\sigma'')^2 + 4(\xi\sigma' - \sigma)(\xi\sigma' - \sigma + \sigma'^2) = 0. \]  
\(10.33\)

**Proposition 10.1.** All the poles \(z_k\) of \(\sigma(z)\) are simple with residues \(z_k\).

**Proof.** Looking at the Laurent expansion of \(\sigma(z)\) at the poles \(z_k\) we observe by (10.33) that the principal term of \(\sigma(z)\) has to be \(z_k/(z - z_k)\). The proposition is proved.

According to (10.4) and (10.5) the function \(D(\xi)\) can be extended to the entire function \(D(z)\). From Proposition 10.1 and relation (10.29) we obtain
the assertions.

**Corollary 10.2.** All the zeroes of $D(z)$ are simple.

**Corollary 10.3.** All the eigenvalues of $T(z)$ are simple.

3. **Example 10.2.** (Painleve type functions)

Let us consider the operator $S_{\xi}$ of form (10.1), where $k(x, t) = k_1(x, t)$. We introduce the functions

$$\sigma_1(\xi, \gamma, \lambda) = (S_{\xi}^{-1} f, g)_{\xi}, \quad (10.34)$$

where $f(x) = g(x) = e^{ix\lambda}$, $\lambda = \bar{\lambda}$. Using Theorem 10.1 and Corollary 10.3 we obtain the following assertion.

**Proposition 10.2.** The function $\sigma_1(\xi, \gamma, \lambda)$ can be extended to the meromorphic function $\sigma_1(z, \gamma, \lambda)$, all the poles of $\sigma_1(z, \gamma, \lambda)$ are simple.

**Definition 10.2.** We call the functions $\sigma_1(z, \gamma, \lambda)$ the Painleve type functions.

We note that the fifth Painleve transcendent $\sigma(z)$ is connected with the functions of form (10.34) by the relation

$$\sigma(z) = -\sigma_1(z/\pi, -1, \pi). \quad (10.35)$$

4. We separately consider the function

$$\sigma_2(z, \gamma) = \sigma_1(z, \gamma, 0). \quad (10.36)$$

It follows from (10.34) and (10.36) that

$$\sigma_2(z) = (S_{\xi}^{-1}, 1)_{\xi}, \quad \xi > 0, \quad (10.37)$$

where the operator $S_{\xi}$ and the kernel $k(x, y)$ are defined by relations (10.1) and (10.7) respectively. We introduce the operators of form (10.10) with the kernels $k(x, y)$ and

$$k_{\pm}(x, y) = \frac{1}{2}[k(x, y) \pm k(\bar{x}, \bar{y})]. \quad (10.38)$$

Let us denote the Fredholm determinants corresponding to $k(x, y)$, $k_+(x, y)$ and $k_-(x, y)$ by $D(\gamma, t)$, $D_+(\gamma, t)$ and $D_-(\gamma, t)$ respectively. We use the following relations (see [12], Ch.21.)

$$D(\gamma, t) = D_+(\gamma, t)D_-(\gamma, t), \quad (10.39)$$
\[
\frac{D_-(\gamma, t)}{D_+(\gamma, t)} = 1 + \int_{-t}^{t} \Gamma_t(t, y, \gamma)dy. \tag{10.40}
\]

It follows from (10.17) and (10.40) that
\[
\frac{D_-(\gamma, t)}{D_+(\gamma, t)} = 1 + \int_{0}^{2t} \bar{\Gamma}_2(t, y, \gamma)dy. \tag{10.41}
\]

In view of (7.14) and (10.40) the relation
\[
\frac{D_-(\gamma, t)}{D_+(\gamma, t)} = h_2(2t) \tag{10.42}
\]
is true. Using formulas (10.3)-(10.5) we deduce the assertion

**Proposition 10.3.** The functions \(D_-(x, t)\) and \(D_-(x, t)\) can be extended to the entire functions \(D_-(z, t)\) and \(D_-(z, t)\) respectively. Hence we have

**Corollary 10.4.** The function \(h_2(2t)\) can be extended to the meromorphic function
\[
h_2(2z) = \frac{D_-(\gamma, z)}{D_+(\gamma, z)}. \tag{10.43}
\]

According to representation (7.9) and Theorem 10.1 equations (1.12) and (1.13) have the strong regular solutions \(u_1(x, \rho)\) and \(u_2(x, \rho)\) respectively when
\[
r^{-2}(x) = 2h_2^2(2x). \tag{10.44}
\]

From Theorem 6.1, Corollary 10.4 and relation (10.44) we obtain

**Corollary 10.5.** The function
\[
r(z) = \frac{D_-(\gamma, z)}{D_+(\gamma, z)} \tag{10.45}
\]
satisfies conditions (6.3), (6.4).

4. **Example 10.3. (third Painleve function).**

Let us consider the operator (10.1), where \(k(x, t)\) is the Bessel kernel.

**Proposition 10.4.** The operator \(S_\xi\) defined by relations (10.1) and (10.9) is invertible on \(L^2(0, \xi)\), when \(|\gamma| \leq 1\).

**Proof.** The kernel \(k(x, t)\) has the form \(k(x, t) = \gamma K(x, t)\), where
\[
K(x, t) = \frac{1}{4} \int_{0}^{1} \phi(xs)\phi(ts)ds, \quad \phi(x) = J_\alpha(\sqrt{x}). \tag{10.46}
\]
The operator $T_\xi = S_\xi - I$ has the kernel $k(x, t)$ and is self-adjoint. It follows from (10.1) and (10.46) that

$$(T_\xi f, f) = \frac{\gamma}{4} \int_0^1 |J_\alpha(\sqrt{xs})f(x)|^2 ds.$$  

(10.47)

The last relation can be written in the form

$$(T_\xi f, f) = \int_0^1 |F(s)|^2 ds,$$  

(10.48)

where

$$F(s) = \int_0^\sqrt{\xi} \sqrt{xs} J_\alpha(sx) f(x^2) \sqrt{2x} dx.$$  

(10.49)

The Hankel transformation (10.49) is unitary. So we have

$$|(T_\xi f, f)| \leq |\gamma| \int_0^\infty |F(s)|^2 ds = |\gamma| \int_0^\xi |f(x)|^2 dx.$$  

(10.50)

Hence $||T_\xi|| \leq |\gamma|$. If $|\gamma| < 1$ then the operator $S_\xi$ is invertible. We shall consider separately the case when $\gamma = \pm 1$. Let us assume that $||T_\xi|| = 1$. In this case we have for some $f$ the equality

$$T_\xi f = \pm f, \quad ||f|| \neq 0.$$  

(10.51)

From relations (10.48), (10.50) and (10.51) we deduce that

$$F(s) = 0, \quad s > 1.$$  

(10.52)

But the function $F(s)$ is analytic when $\text{Res} > 0$. Hence the equality $F(s) = 0, (s > 0)$ is true. It means that $||f(x)|| = 0$. We have obtained a contradiction, i.e. $||T_\xi|| < 1$. The proposition is proved.

The operator $S_\xi^{-1}$ has the form

$$S_\xi^{-1} f = f(x) + \int_0^\xi \Gamma_\xi(x, t)f(t) dt.$$  

(10.53)

We consider the functions

$$q(\xi) = \phi(\xi) + \int_0^\xi \Gamma_\xi(\xi, t)\phi(t) dt,$$  

(10.54)
We shall use the following relations (see [18])

\[ [sR(s)]' = \frac{1}{4}q^2(s) \]  

(10.55)

\[ R(t) = -\frac{d}{dt}\log\det S_t. \]  

(10.56)

Using M.G.Krein result (see [7], Ch.4) on the triangular factorization of the operator \( S \) with continuous kernel we obtain the assertion.

**Theorem 10.3.** The operator \( S_\alpha \) defined by (10.1), (10.9) when \( \alpha \geq 0 \) admits triangular factorization (10.25) and

\[ S^{-1}_\alpha f = f(v) + \int_0^v \Gamma_v(v, u) f(u) du. \]  

(10.57)

Formula (10.54) can be written in the form

\[ q(x) = S^{-1}_\alpha \phi. \]  

(10.58)

We introduce the notation

\[ \sigma(s) = sR(s, s). \]  

(10.59)

Relations (10.55) and (10.58) imply that

\[ \sigma(x) = \frac{1}{4}(S^{-1}_\xi \phi, \phi)_\xi. \]  

(10.60)

Further we consider only the important case when \( \gamma = -1, \alpha \geq 0 \).

We note that in this case the function \( \sigma(x) \) is the third Painleve transcendent (see [17]). Using Proposition 10.1 and relation (10.60) we obtain the following fact (see[7]).

**Corollary 10.6.** The third Painleve transcendent \( \sigma(\xi) \) can be extended to the function \( \sigma(z) \) which is analytic in \( G \) except at isolated points. All finite singular points in \( G \) are poles. (The domain \( G \) is defined in Remark 10.3.)

The function \( \sigma(\xi) \) is a solution of the Painleve equation \((P_3)\)\n
\[(\xi \sigma')^2 + \sigma'(\sigma - \xi \sigma')(4\sigma' - 1) - \alpha^2 \sigma'^2 = 0. \]  

(10.61)

**Proposition 10.5.** All the poles \( z_k \) of \( \sigma(z) \) are simple with residues \( z_k \).

**Proof.** Looking at the Laurent expansion of \( \sigma(z) \) at the poles \( z_k \) we observe by (10.61) that the principal term of \( \sigma(z) \) has to be \( z_k/(z - z_k) \). The proposition is proved.
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