Intrinsic invariants of cross caps

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Published online: 30 July 2013
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Abstract It is classically known that generic smooth maps of $\mathbb{R}^2$ into $\mathbb{R}^3$ admit only isolated cross cap singularities. This suggests that the class of cross caps might be an important object in differential geometry. We show that the standard cross cap $f_{\text{std}}(u, v) = (u, uv, v^2)$ has non-trivial isometric deformations with infinite-dimensional freedom. Since there are several geometric invariants for cross caps, the existence of isometric deformations suggests that one can ask which invariants...

The second and third authors were partly supported by the Grant-in-Aid for JSPS Fellows. The fourth and fifth authors were partially supported by Grant-in-Aid for Scientific Research (A) No. 22244006, and Scientific Research (B) No. 21340016, respectively, from the Japan Society for the Promotion of Science.

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of cross caps are intrinsic. In this paper, we show that there are three fundamental intrinsic invariants for cross caps. The existence of extrinsic invariants is also shown.

**Keywords**  Cross cap · Curvature · Isometric deformation

**Mathematics Subject Classification (2010)**  Primary 57R45; Secondary 53A05

## 1 Introduction

Let $U$ be a domain in $\mathbb{R}^2$ and $f : U \to \mathbb{R}^3$ a $C^\infty$-map. A point $p(\in U)$ is called a *singular point* if the rank of the Jacobi matrix of $f$ at $p$ is less than 2. Consider such a map given by

$$f_{\text{std}}(u, v) = (u, uv, v^2),$$

which has an isolated singular point at the origin $(0, 0)$ and is called the *standard cross cap* (see Fig. 1, left). A singular point $p$ of a map $f : U \to \mathbb{R}^3$ is called a *cross cap* or a *Whitney umbrella* if there exist a local diffeomorphism $\phi$ on $\mathbb{R}^2$ and a local diffeomorphism $\Phi$ on $\mathbb{R}^3$ such that $\Phi \circ f = f_{\text{std}} \circ \phi$. Whitney proved that a $C^\infty$-map $f : U \to \mathbb{R}^3$ has a cross cap singularity at $p \in U$ if there exists a local coordinate system $(u, v)$ centered at $p$ such that

$$f_v(0, 0) := \frac{\partial f}{\partial v}(0, 0) = 0$$

and the three vectors

$$f_u(0, 0) := \frac{\partial f}{\partial u}(0, 0), \quad f_{uv}(0, 0) := \frac{\partial^2 f}{\partial u \partial v}(0, 0), \quad f_{vv}(0, 0) := \frac{\partial^2 f}{\partial v^2}(0, 0)$$

are linearly independent. By a rotation, a translation in $\mathbb{R}^3$ and a suitable orientation preserving coordinate change of the domain $U \subset \mathbb{R}^2$, we have the following Maclaurin expansion of $f$ at a cross cap singularity $(0, 0)$ (cf. [5] or [13])

![Fig. 1](image)  An isometric deformation of the standard cross cap
\[ f(u, v) = \left( u, uv + \sum_{i=3}^{n} \frac{b_i}{i!} v^i, \sum_{r=2}^{n} \sum_{j=0}^{r} \frac{a_{r-j}}{j!(r-j)!} u^j v^{r-j} \right) + O(u, v)^{n+1}, \quad (2) \]

where \( a_{02} \neq 0 \). By orientation preserving coordinate changes \((u, v) \mapsto (-u, -v)\) and \((x, y, z) \mapsto (-x, y, -z)\), we may assume that

\[ a_{02} > 0, \quad (3) \]

where \((x, y, z)\) is the usual Cartesian coordinate system of \( \mathbb{R}^3 \). After this normalization (3), one can easily verify that all of the coefficients \( a_{jk} \) and \( b_i \) are uniquely determined. An oriented local coordinate system \((u, v)\) giving such a normal form is called the \textit{canonical coordinate system} of \( f \) at the cross cap singularity. This unique expansion of a cross cap implies that the coefficients \( a_{jk} \) and \( b_i \) can be considered as geometric invariants of the cross cap \( f \). A cross cap is called \textit{non-degenerate} (resp. \textit{degenerate}) if \( a_{20} \neq 0 \) (resp. \( a_{20} = 0 \)). On the other hand, a real analytic cross cap is called \textit{quadratic} if \( a_{jk} = 0 \) for \( j + k \geq 3 \) and \( b_i = 0 \) for \( i \geq 3 \). The standard cross cap is a typical example of a degenerate quadratic cross cap.

Let \( f \) be a degenerate quadratic cross cap. Using the classically known isometric deformations of ruled surfaces, we show that each degenerate quadratic cross cap induces a non-trivial family of isometric deformations with infinite-dimensional freedom. Moreover, using such a deformation, we show that the invariants \( a_{03}, a_{12} \) and \( b_3 \) in (2) are extrinsic, namely, these invariants change according to the isometric deformation. It should be remarked that, using the same method, the existence of non-trivial isometric deformations is shown for other typical singularities on surfaces, that is, cuspidal edges, swallowtails and cuspidal cross caps (cf. Remark 3).

The differential geometry of cross caps in \( \mathbb{R}^3 \) has been discussed by several authors (cf. [1,3–7,9,10,12] and [13]). However, the distinction between intrinsic and extrinsic invariants has not been clearly discussed before. When \( f : U \rightarrow \mathbb{R}^3 \) is an immersion, it induces a Riemannian metric on \( U \) (called the first fundamental form) and we know that ‘intrinsic’ means that a given invariant is described in terms of this Riemannian structure on \( U \). Similarly, each cross cap induces a positive semidefinite symmetric tensor \( ds^2 \) as a pullback of the ambient metric. Then, a given invariant of a cross cap is called \textit{intrinsic} if it can be described in terms of this positive semidefinite metric \( ds^2 \).

In the case of cuspidal edges in \( \mathbb{R}^3 \), such an intrinsic invariant is defined as a ‘singular curvature’ along these singular points (cf. [11]).

We then show that \( a_{02} \), \( a_{20} \) and \( a_{11} \) are intrinsic invariants. In fact, if \( a_{20} \) is negative, then the Gaussian curvature of a given cross cap is negative and having no lower bound. On the other hand, if \( a_{20} \) is positive, then the Gaussian curvature is not bounded neither from below nor from above. Fukui and the first author [5] found an important concept of the ‘focal conic’ of a cross cap, as a section of its caustic by the normal plane (see the explanation after (16) and also [3]). They also showed that focal conics have the expression

\[ y^2 + 2a_{11}yz - (a_{20}a_{02} - a_{11}^2)z^2 + a_{02}z = 0. \]
The focal conic is a hyperbola (resp. an ellipse) if and only if $a_{20}$ is positive (resp. negative). Since we have seen that $a_{02}$, $a_{20}$ and $a_{11}$ are intrinsic, we can say that focal conics live in the intrinsic geometry of cross caps, although caustics themselves are extrinsic objects. It should be remarked that the Gauss–Bonnet type formula for closed surfaces which admit only cross cap singularities in $\mathbb{R}^3$ has no defect at each singular point (cf. Kuiper [8, p. 92]).

2 Isometric deformations of degenerate quadratic cross caps

As mentioned in the introduction, a quadratic cross cap can be expressed as

$$\left(u, uv, \frac{1}{2} (a_{20} u^2 + 2a_{11} uv + a_{02} v^2)\right) \quad (a_{02} > 0),$$

where $(u, v)$ is the canonical coordinate system. It should be remarked that the set of self-intersections lies in a straight line (see Theorem 11).

As defined in the introduction, the cross cap $(0, 0)$ is degenerate if $a_{20} = 0$. A degenerate quadratic cross cap has the following expression

$$f_0(u, v) := \frac{1}{2} \left(0, 0, a_{02} v^2\right) + u (1, v, a_{11} v) \quad (a_{02} > 0). \quad (4)$$

In particular, it is a ruled surface. The first fundamental form

$$ds^2 := E_0 du^2 + 2F_0 du dv + G_0 dv^2$$

of $f_0$ is given by

$$E_0 := (f_0)_u \cdot (f_0)_u = 1 + (1 + a_{11}^2)v^2,$$

$$F_0 := (f_0)_u \cdot (f_0)_v = (1 + a_{11}^2)uv + a_{02}a_{11}v^2,$$

$$G_0 := (f_0)_v \cdot (f_0)_v = (1 + a_{11}^2)u^2 + 2a_{02}a_{11}uv + a_{02}^2 v^2,$$

where the dot indicates the canonical inner product of $\mathbb{R}^3$.

Definition 1 Let $U$ be a domain in $(\mathbb{R}^2; u, v)$ containing the origin, and let $f_i : U \rightarrow \mathbb{R}^3 \ (i = 0, 1)$ be two $C^\infty$-maps having a cross cap singularity at $(0, 0)$. If $f_0$ and $f_1$ satisfy

$$(f_0)_u \cdot (f_0)_u = (f_1)_u \cdot (f_1)_u, \quad (f_0)_v \cdot (f_0)_v = (f_1)_v \cdot (f_1)_v,$$

then we say that $f_0$ is isometric to $f_1$. On the other hand, let $f_i : U \rightarrow \mathbb{R}^3 \ (|t| < \epsilon)$ be a smooth 1-parameter family of $C^\infty$-maps having a cross cap singularity at $(0, 0)$, where $\epsilon$ is a positive constant. Then, $\{f_i\}_{|t| < \epsilon}$ is called an isometric deformation of $f_0$ if each $f_i$ is isometric to $f_0$. An isometric deformation $\{f_i\}_{|t| < \epsilon}$ of $f_0$ is non-trivial if each $f_t$ is not congruent to $f_0$. 
It is classically known that ruled surfaces admit non-trivial isometric deformations in general. As pointed out in Remark 3, several singularities (i.e. cross caps, cuspidal edges, swallowtails and cuspidal cross caps) may admit isometric deformations as ruled surfaces. The following assertion gives a characterization of the degenerate quadratic cross caps:

**Theorem 2** Let $c(s) (|s| < \pi/2)$ be a regular curve in the unit sphere $S^2 (\subset \mathbb{R}^3)$ with arc length parameter. We set

$$
\xi(v) := \sqrt{1 + (1 + a_{11}^2)v^2} \hat{c}(v), \quad \hat{c}(v) := c\left(\arctan(v\sqrt{1 + a_{11}^2})\right),
$$

(5)

for each $v \in \mathbb{R}$, and

$$
\gamma(v) := \frac{a_{02}}{1 + a_{11}^2} \int_0^v t B(t) \, dt, \quad B(v) := a_{11} \xi'(v) + \xi(v) \times \xi'(v),
$$

where the prime means the derivative with respect to $v$ and the cross ‘×’ denotes the vector product in $\mathbb{R}^3$. Then, a ruled surface $f_c : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$
f_c(u, v) := \gamma(v) + u\xi(v)
$$

has a cross cap singularity at the origin such that $f_c$ is isometric to a degenerate quadratic cross cap $f_0$. Moreover, let $c_i(s) (|s| < \pi/2; \ i = 1, 2)$ be two regular curves in $S^2$ with arc length parameter. Then, $f_{c_1}$ is congruent to $f_{c_2}$ if and only if $c_1$ is congruent to $c_2$ in $S^2$. In this correspondence $c \mapsto f_c$ between spherical curves and cross caps, the initial degenerate quadratic cross cap corresponds to the geodesic in $S^2$. More precisely, $f_c$ is congruent to $f_0$ as in (4) if and only if $c(s)$ is a geodesic in $S^2$.

**Proof** By (5), we have that

$$
\xi(v) \cdot \xi(v) = 1 + (1 + a_{11}^2)v^2, \quad (6)
$$

$$
\xi'(v) \cdot \xi'(v) = 1 + a_{11}^2, \quad (7)
$$

$$
\xi(v) \cdot \xi'(v) = (1 + a_{11}^2)v, \quad (8)
$$

where we used the fact that $\hat{c}'(v)$ is orthogonal to $\hat{c}(v)$. Since $\xi(v) \times \xi'(v)$ is orthogonal to $\xi(v)$ and $\xi'(v)$, the Eqs. (7) and (8) yield that

$$
\xi(v) \cdot B(v) = a_{11}(1 + a_{11}^2)v, \quad \xi'(v) \cdot B(v) = a_{11}(1 + a_{11}^2).
$$

Finally, we have

$$
B \cdot B = |\xi \times \xi'|^2 + a_{11}^2 |\xi'|^2 = |\xi|^2 |\xi'|^2 - (\xi \cdot \xi')^2 + a_{11}^2 |\xi'|^2 = (1 + a_{11}^2)^2.
$$
From now on, we denote $f := f_c$ for the sake of simplicity. One can prove that $f_u \cdot f_u, f_u \cdot f_v$ and $f_v \cdot f_v$ coincide with $E_0, F_0$ and $G_0$, respectively, using the above relations.

The unit normal vector field $v(u, v)$ is given by

$$v(u, v) = \frac{1}{\delta} \left( - \left( a_{02}v \sqrt{1 + (1 + a_{11}^2)v^2} \right) e(v) + \left( (1 + a_{11}^2)u + a_{02}a_{11}v \right) n(v) \right),$$

where

$$\delta := \sqrt{1 + a_{11}^2(1 + a_{11}^2)v^2 + 2a_{02}a_{11}uv + a_{02}^2v^2(1 + v^2)},$$

$e = dc/ds$ and $n = c \times e$. By a straightforward calculation, the second fundamental form $L du^2 + 2M du dv + N dv^2$ of $f$ is given by

$$L := 0, \quad M := -\frac{a_{02} \sqrt{1 + a_{11}^2}}{\delta},$$

$$N := \frac{a_{02} \sqrt{1 + a_{11}^2}}{\delta} u + \frac{\delta \kappa(v)}{(1 + (1 + a_{11}^2)v^2)^{3/2}},$$

where $\kappa(s)$ is the geodesic curvature function of $c(s)$. Let $c_i(s)$ ($|s| < \pi/2; \ i = 1, 2$) be two regular curves in $S^2$ with arc length parameter and $\kappa_i(s)$ the geodesic curvature function of $c_i(s)$. Then, $f_{c_1}$ is congruent to $f_{c_2}$ if $c_1$ is congruent to $c_2$ in $S^2$, since (9) implies that the second fundamental form of $f_{c_1}$ coincides with that of $f_{c_2}$ if and only if $\kappa_1$ coincides with $\kappa_2$. Finally, as seen in the following example, degenerate quadratic cross caps correspond to the great circles, so we get the assertion. □

**Example 1** Take a constant $\kappa$ and set

$$c_\kappa(s) := \frac{1}{\mu^2} \left( \kappa^2 + \cos(\mu s), \mu \sin(\mu s), \kappa \left( 1 - \cos(\mu s) \right) \right) \quad (|s| < \frac{\pi}{2}, \ \mu := \sqrt{1 + \kappa^2}),$$

which gives a circle in $S^2$ with arc length parameter and of constant geodesic curvature $\kappa$. Then, it produces a deformation of the standard cross cap, where $c_0$ corresponds to $f_{\text{std}}$ as in (1). Fig. 1 indicates the cross caps corresponding to $\kappa = 0, 1$ and 3, respectively.

**Remark 3** (Isometric deformations of ruled surfaces with singularities) Let $\gamma(v)$ be a curve in $\mathbb{R}^3$ defined near $v = 0$, and $\xi(v)$ a vector field along the curve $\gamma$ such that $\xi$ does not vanish and $\xi'(0) \neq 0$. Then, the ruled surface $f(u, v) := \gamma(v) + u \xi(v)$ has non-trivial isometric deformations as follows: By the coordinate change $(u, v) \mapsto (u/|\xi(v)|, v)$, we may assume that $|\xi(v)| = 1$. Since $\xi'(0) \neq 0$, we may also assume that $|\xi'(v)| = 1$. Then, $(\xi(v), \xi'(v), \xi(v) \times \xi'(v))$ forms an orthonormal frame field, and the derivative of $\gamma(v)$ has the following expression

$$\gamma'(v) = a(v)\xi(v) + b(v)\xi'(v) + c(v)(\xi(v) \times \xi'(v)).$$
Let $\tilde{\xi}(v)$ be an arbitrarily given spherical curve with arc length parameter, and let $\tilde{\gamma}$ be the curve whose derivative is given as

$$\tilde{\gamma}'(v) = a(v)\tilde{\xi}(v) + b(v)\tilde{\xi}'(v) + c(v)(\tilde{\xi}(v) \times \tilde{\xi}'(v)).$$

Then, $\tilde{f}(u, v) := \tilde{\gamma}(v) + u\tilde{\xi}(v)$ has the same first fundamental form as $f(u, v)$. By computing the second fundamental form, one can easily verify that $f$ and $\tilde{f}$ are congruent if $\xi$ and $\tilde{\xi}$ are congruent as spherical curves. This implies that several well-known singularities on surfaces admit isometric deformations like as in the case of cross caps in Theorem 2. For example,

(i) $(0, 0)$ is a cross cap if and only if $\gamma'(0) = 0$ and $\det(\gamma''(0), \xi(0), \xi'(0)) \neq 0$.

(ii) $f$ is a developable map (i.e. the Gaussian curvature of $f$ vanishes identically at each regular point of $f$) having a cuspidal edge (cf. Fig. 2, left) at $(0, 0)$ if $b(v)$ and $c(v)$ vanish identically and $a(0) \neq 0$ holds (cf. [2, Fact (1)]).

(iii) $f$ is a developable map having a swallowtail (cf. Fig. 2, center) at $(0, 0)$ if $b(v)$ and $c(v)$ vanish identically, and $a(0) = 0$, $a'(0) \neq 0$ hold (cf. [2, Fact (2)]).

(iv) We set $v = \xi \times \xi'$. Then, $f$ is a developable map having a cuspidal cross cap (cf. Fig. 2, right) at $(0, 0)$ if $b(v)$ and $c(v)$ vanish identically, and

$$\det(\xi(0), v(0), v'(0)) = 0, \quad a(0) \neq 0, \quad \det(\xi(0), v(0), v''(0)) \neq 0$$

hold. This criterion can be proved by applying [2, Corollary 1.5], using the fact that $v$ gives the unit normal vector field when $f$ is a developable map.

One can choose $\gamma$ and $\xi$ so that they satisfy each of the above criteria. This implies that cross caps, cuspidal edges, swallowtails and cuspidal cross caps actually admit non-trivial isometric deformations in a certain class of ruled surfaces.

Using the existence of non-trivial isometric deformations of degenerate quadratic cross caps, we can prove the following assertion.

**Theorem 4** There exists an isometric deformation of a cross cap singularity which changes the three invariants $a_{12}$, $a_{03}$ and $b_3$, that is, these invariants are extrinsic.
Proof Let \( c(s) \) be a spherical curve with arc length parameter \( s \) so that
\[
c(0) = (1, 0, 0), \quad \frac{dc}{ds}(0) = \frac{1}{\sqrt{1 + a_{11}^2}}(0, 1, a_{11}),
\]
and set \( e := dc/ds, \ n := c \times e. \) Then,
\[
\frac{de}{ds}(s) = \kappa(s)n(s) - c(s), \quad \frac{dn}{ds}(s) = -\kappa(s)e(s),
\]
hold, where \( \kappa(s) \) is the geodesic curvature function of \( c. \) Using these, one can see that the cross cap \( f = f_c \) given in Theorem 2 has the following expansion:
\[
f(u, v) = \left(u, uv, a_{11}uv + \frac{1}{2}a_{02}v^2\right)
+ \frac{\kappa(0)\sqrt{1 + a_{11}^2}}{6} \left(0, -3a_{11}uv^2 - 2a_{02}v^3, 3uv^2\right) + O(u, v^4). \tag{10}
\]
By a parameter change \( v = w + \frac{1}{2}a_{11}\kappa(0)\sqrt{1 + a_{11}^2}w^2, \) (10) is rewritten as
\[
f(u, w) = \left(u, uw, a_{11}uw + \frac{1}{2}a_{02}w^2\right)
+ \frac{\kappa(0)\sqrt{1 + a_{11}^2}}{6} \left(0, -2a_{02}w^3, 3(1 + a_{11}^2)uw^2 + 3a_{11}a_{02}w^3\right) + O(u, w^4).
\]
Thus, \( (u, w) \) forms the canonical coordinate system up to the third-order terms, and the invariants \( a_{12}, a_{03} \) and \( b_3 \) are expressed as
\[
a_{12} = \kappa(0)(1 + a_{11}^2)^{3/2}, \quad a_{03} = 3a_{02}a_{11}\kappa(0)\sqrt{1 + a_{11}^2},
b_3 = -2a_{02}\kappa(0)\sqrt{1 + a_{11}^2}, \tag{11}
\]
which depend on the initial value \( \kappa(0) \) of the geodesic curvature function, and thus, they are extrinsic.
\[\square\]

Remark 5 By a straightforward calculation, one can also check that \( a_{0j}, a_{1j} \) and \( b_j \) for \( j = 3, 4, 5, \ldots \) all change values by the same isometric deformation as in the proof of Theorem 4.

3 Differential geometry of cross caps

Let \( f : U \to \mathbb{R}^3 \) be a \( C^\infty \)-map and \( p \in U \) a cross cap singularity. A local coordinate system \( (u, v) \) centered at \( p \) is said to be admissible if it satisfies \( f_v(0, 0) = 0. \) Canonical
coordinate systems of cross caps are admissible. The concept of admissible coordinate systems is intrinsic, since $\partial/\partial v$ at $(0, 0)$ points the degenerate directions of the induced metrics. In contrast to Theorem 4, the following assertion holds as follows:

**Theorem 6** The coefficients $a_{02}$, $a_{20}$ and $a_{11}$ are intrinsic invariants of cross caps.

**Proof** Let $(0, 0)$ be a cross cap singularity of a $C^\infty$-map $f : (U; u, v) \rightarrow \mathbb{R}^3$, and $(u, v)$ be an admissible coordinate system. Without loss of generality, we may assume that

$$[fu, fuv, fvv] > 0$$

by applying the coordinate change $(u, v) \mapsto (-u, -v)$ if necessary, where

$$[a, b, c] := \det(a, b, c) = (a \times b) \cdot c \quad (a, b, c \in \mathbb{R}^3).$$

Then, we have that

$$a_{02} = \frac{|fu||fu \times fvv|^3}{(fu, fuv, fvv)^2}$$

$$a_{20} = \frac{|fu \times fvv|}{4|fu|^3(fu, fuv, fvv)^2}\left([fu, fuv, fvv]^2 + 4[fu, fuv, fvv][fu, fuv, fuu]\right),$$

$$a_{11} = \frac{1}{2|fu|^2(fu, fuv, fvv)^2}\left(2[fu, fuv, fvv] \det\left(\begin{array}{ccc} fu \cdot fu & fu \cdot fu & fu \\ fuv \cdot fu & fuv \cdot fuv & fuv \\ fvv \cdot fuv & fvv \cdot fuv & fvv \end{array}\right) \right)$$

(14)

hold at $(u, v) = (0, 0)$. One can prove these identities immediately: In fact, the right-hand sides of these identities are independent of the choice of admissible coordinate systems, and these identities themselves can be directly verified for the canonical coordinate system of $f$.

We now set

$$E := fu \cdot fu, \quad F := fu \cdot fv, \quad G := fv \cdot fv,$$

which are the coefficients of the induced metric of the cross cap. It is sufficient to show that the right-hand sides of (12), (13) and (14) are written in terms of derivatives of $E$, $F$ and $G$ at $(0, 0)$.

We first show that $a_{02}$ is intrinsic: Since $fv(0, 0) = 0$, it holds that
\[ [f_u, f_{uv}, f_{vv}]^2 = \det \begin{pmatrix} f_u & f_{uv} & f_{vv} \\ f_{uv} & f_{uu} & f_{uv} \\ f_{vv} & f_{vv} & f_{vv} \end{pmatrix} = \det \begin{pmatrix} E & F_u & F_v \\ F_u & G_{uu}/2 & G_{uv}/2 \\ F_v & G_{uv}/2 & G_{vv}/2 \end{pmatrix} \] (15)

at \((u, v) = (0, 0)\), where we used the identities

\[ f_u \cdot f_u = E, \quad f_u \cdot f_{uv} = F_u, \quad f_u \cdot f_{vv} = F_v, \]
\[ f_{uv} \cdot f_{uv} = \frac{G_{uu}}{2}, \quad f_{uv} \cdot f_{vv} = \frac{G_{uv}}{2}, \quad f_{vv} \cdot f_{vv} = \frac{G_{vv}}{2} \]

at \((u, v) = (0, 0)\). Since

\[ |f_u|^2 = E, \quad |f_u \times f_{vv}|^2 = (f_u \cdot f_u)(f_{vv} \cdot f_{vv}) - (f_u \cdot f_{vv})^2 = \frac{EG_{vv}}{2} - (F_v)^2 \]

at \((u, v) = (0, 0)\), we can conclude that \(a_{02}\) is an intrinsic invariant.

Similarly, to prove \(a_{20}\) and \(a_{11}\) are intrinsic, it is sufficient to show that \([f_u, f_{uu}, f_{vv}]\) and \([f_u, f_{uv}, f_{uu}]\) are both written in terms of derivatives of \(E, F\) and \(G\) at \((0, 0)\). In fact, (15) implies that \([f_u, f_{uu}, f_{vv}]\) is intrinsic, because

\[ [f_u, f_{uu}, f_{vv}] = \frac{1}{[f_u, f_{uv}, f_{vv}]} \det \begin{pmatrix} f_u & f_{uu} & f_{vv} \\ f_{uv} & f_{uv} & f_{uv} \\ f_{vv} & f_{uu} & f_{vv} \end{pmatrix} = \frac{1}{[f_u, f_{uv}, f_{vv}]} \det \begin{pmatrix} f_u \cdot f_u & f_u \cdot f_{uu} & f_u \cdot f_{vv} \\ f_{uv} \cdot f_u & f_{uv} \cdot f_{uu} & f_{uv} \cdot f_{vv} \\ f_{vv} \cdot f_u & f_{vv} \cdot f_{uu} & f_{vv} \cdot f_{vv} \end{pmatrix} = \frac{1}{[f_u, f_{uv}, f_{vv}]} \det \begin{pmatrix} E & f_u \cdot f_{uu} & F_v \\ F_u & f_{uv} \cdot f_{uu} & G_{uv}/2 \\ F_v & f_{vv} \cdot f_{uu} & G_{vv}/2 \end{pmatrix} \]

holds at \((u, v) = (0, 0)\), and

\[ f_u(0, 0) \cdot f_{uu}(0, 0) = \frac{E_u(0, 0)}{2}, \]
\[ f_{uv}(0, 0) \cdot f_{uu}(0, 0) = \frac{E_{uv}(0, 0)}{2}, \]
\[ f_{vv}(0, 0) \cdot f_{uu}(0, 0) = \frac{E_{vv}(0, 0)}{2}. \]

Similarly, \([f_u, f_{uv}, f_{uu}]\) is intrinsic, because of the identity

\[ [f_u, f_{uv}, f_{uu}] = \frac{1}{[f_u, f_{uv}, f_{vv}]} \det \begin{pmatrix} f_u & f_{uv} & f_{uu} \\ f_{uv} & f_{uv} & f_{uv} \\ f_{vv} & f_{vv} & f_{vv} \end{pmatrix} \].
Remark 7 The value $\Delta := [f_u(0, 0), f_{uv}(0, 0), f_{vv}(0, 0)]$ is a criterion of cross cap singularities, that is, $(0, 0)$ is a cross cap singularity of $f$ if and only if $f_v(0, 0) = 0$ and $\Delta \neq 0$. In the above proof (cf. (15)), we showed the identity

$$\Delta^2 = \det \begin{pmatrix} E & F_u & F_v \\ F_u & G_{uu}/2 & G_{uv}/2 \\ F_v & G_{uv}/2 & G_{vv}/2 \end{pmatrix}$$

at $(u, v) = (0, 0)$, which implies that $\Delta$ is intrinsic. Moreover, we set

$$h(u, v) := E(u, v)G(u, v) - (F(u, v))^2.$$ 

Using the fact that $(u, v)$ is admissible, one can easily prove

$$\Delta^2 = \frac{1}{4E(0, 0)} \left( h_{uu}(0, 0)h_{vv}(0, 0) - (h_{uv}(0, 0))^2 \right),$$

that is, $\Delta$ is closely related to the Hessian of $h$. Since $h(u, v)$ is non-negative and $h(0, 0) = 0$, it holds that $h_{vv}(0, 0) \geq 0$. Moreover, the identity

$$a_{02} = \frac{\sqrt{E(0, 0)(h_{vv}(0, 0))^{3/2}}}{2\Delta^2}$$

holds.

In [12,13] and [5], the ellipticity, hyperbolicity and parabolicity of cross caps are defined. The following assertion holds as follows:

**Corollary 8** The ellipticity, hyperbolicity and parabolicity of cross caps in $\mathbb{R}^3$ are all intrinsic properties.

**Proof** A cross cap is elliptic (resp. hyperbolic) if $a_{20} > 0$ (resp. $a_{20} < 0$) (cf. [5]). Since we have already seen that $a_{20}$ is intrinsic, the ellipticity and hyperbolicity are as well. In [5], it was shown that a cross cap is parabolic if and only if $a_{20} = 0$ and the zero set $Z_K$ of the Gaussian curvature gives a regular curve in the $r\theta$-plane which is tangent to the line $r = 0$, where

$$u = r \cos \theta, \quad v = r \sin \theta$$

and $(u, v)$ is a canonical coordinate system. Since the set $Z_K$ is intrinsic and this tangency property does not depend on the choice of an admissible coordinate system, we get the assertion. \( \square \)

We fix a cross cap $f : (U; u, v) \to \mathbb{R}^3$, where $(u, v)$ is an admissible coordinate system, that is, $(u, v) = (0, 0)$ is a cross cap singularity and $f_v(0, 0) = 0$. We call the line

$$\{ f(0, 0) + tf_u(0, 0); \ t \in \mathbb{R} \}$$
the tangential line at the cross cap and a nonzero vector at $T_{f(0,0)}R^3$ proportional to $f_u(0,0)$ is called the tangential direction. The plane passing through $f(0,0)$ spanned by $f_u(0,0)$ and $f_{uv}(0,0)$ is called the principal plane. On the other hand, the plane passing through $f(0,0)$ perpendicular to the proper tangential direction is called the normal plane. The unit normal vector $v(u,v)$ near the cross cap at $(u,v) = (0,0)$ can be extended as a $C^\infty$-function of $r, \theta$ by setting $u = r \cos \theta$ and $v = r \sin \theta$, and the limiting normal vector

$$v(\theta) := \lim_{r \to 0} v(r \cos \theta, r \sin \theta) \in T_{f(0,0)}R^3$$

lies in the normal plane. Then, one can consider the parallel family of cross caps

$$f_t(r,\theta) := f(r,\theta) + tv(r,\theta) \quad (t \in \mathbb{R})$$

which are $C^\infty$-maps with respect to $(r,\theta)$, even at $r = 0$. The focal surface of this parallel family meets the normal plane at the focal conic as mentioned in the introduction. On the other hand, the principal plane has the following property:

**Proposition 9** The initial velocity vector of the space curve emanating from the cross cap singularity which parametrizes the self-intersection is contained in the principal plane.

**Proof** Since the principal plane is invariant under diffeomorphisms of $R^3$, this assertion can be verified by the standard cross cap.

To get much precise information to the set of cross caps, we give the following definition:

**Definition 10** A germ of cross cap $f : U \to R^3$ is called normal if the set of self-intersections is contained in the intersection of the principal plane and the normal plane.

The quadratic cross caps defined in Sect. 2 are all normal. We get the following criterion of normal cross caps:

**Theorem 11** The germ of a real analytic cross cap is normal if and only if all of the invariants $b_j (j = 3, 4, 5, \ldots)$ associated with its normal form (2) vanish simultaneously.

**Proof** Without loss of generality, we may assume that the given cross cap $f$ has an expression as in (2). We set

$$\beta(v) := \sum_{i=3}^{\infty} \frac{b_i v^i}{i!}.$$ 

Since $f$ is real analytic, $\beta$ is a real analytic function. If $\beta$ vanishes identically, then

$$f(0,v) = \left(0,0,\sum_{n=2}^{\infty} \frac{a_0 n v^n}{n!}\right).$$
Since $a_{02} > 0$,

$$w := \sqrt{\sum_{n=2}^{\infty} a_{0n} v^n n!}$$

is well-defined and gives a real analytic function. Replacing the coordinate system $(u, v)$ by $(u, w)$, we have that

$$f(0, w) = (0, 0, w^2) = f(0, -w),$$

which implies that the set of self-intersection of $f$ lies in the third axis, namely, the set of self-intersections is contained in the intersection of the principal plane and the normal plane. Conversely, we assume that the set $S$ of self-intersection lies in the third axis. Then, the first and the second components of (2) yield that

$$u = 0, \quad uv + \beta(v) = 0$$

hold along $S$. Thus, the $v$-axis parametrizes the set $S$ and $\beta(v)$ vanishes identically, which proves the assertion.

As pointed out in the introduction, $a_{20}$ is an important intrinsic invariant of cross caps related to the sign of the Gaussian curvature. The following assertion can be proved easily, which gives a geometric meaning for $a_{20}$:

**Proposition 12** The section of a cross cap by its principal plane contains a regular curve $\gamma$ whose velocity vector is $f_u(0, 0)$, and then, the curvature of $\gamma$ as a plane curve at the cross cap is equal to $a_{20}$, where we give the orientation to the principal plane so that $\{f_u(0, 0), f_{vv}(0, 0)\}$ is a positive frame.

Intersections of a cross cap with planes are discussed in [3].

At the end of this section, we discuss cross caps in an arbitrary Riemannian 3-manifold $(N^3, g)$. Let $f : M^2 \to (N^3, g)$ be a $C^\infty$-map having a cross cap singularity at $p \in M^2$, where $M^2$ is a 2-manifold.

Since the $\text{SO}(3)$-rotation of a normal coordinate system gives a new normal coordinate system, there exists a local coordinate system $(u, v)$ of $M^2$ centered at $p$ and a normal coordinate system $(x^1, x^2, x^3)$ of $(N^3, g)$ centered at $f(p)$ such that (cf. (2))

$$f(u, v) = \left( u, uv, \frac{a_{20}}{2} u^2 + a_{11} uv + \frac{a_{02}}{2} v^2 \right) + O(u, v)^3.$$  

Like as in the case of the Euclidean 3-space, one can easily verify that $a_{20}$, $a_{02}$ and $a_{11}$ are all intrinsic invariants: In fact, we set

$$[a, b, c] := \Omega(a, b, c) = g(a \times b, c),$$

where $\Omega$ is the Riemannian volume form of $(N^3, g)$ and $a, b, c$ are vector fields of $N^3$ along the $C^\infty$-map $f$. We denote by $D$ the Levi-Civita connection of $g$. By replacing

$$f_{uu} \mapsto D_u f_u, \quad f_{uv} \mapsto D_u f_v, \quad f_{vv} \mapsto D_v f_v,$$
the right-hand sides of (12), (13) and (14) do not depend on a choice of admissible coordinate systems (i.e. a local coordinate system \((u, v)\) such that \(f_v(0, 0) = 0\)). In particular, it can be directly checked that (12), (13) and (14) hold at the cross cap singularity of \(N^3\) with respect to an arbitrarily given admissible coordinate system. We set \(g = \sum_{i,j=1}^{3} g_{ij} dx^i dx^j\). Since \((x^1, x^2, x^3)\) is a normal coordinate system, it holds that

\[
g_{ij} = \delta_{ij} + O(x^1, x^2, x^3)^2 \quad (i,j = 1, 2, 3),
\]

where \(\delta_{ij}\) is the Kronecker’s delta. Then, the unit normal vector satisfies

\[
v = \frac{1}{A_{\theta}} (0, -a_{11} \cos \theta - a_{02} \sin \theta, \cos \theta) + O(r),
\]

where \(u = r \cos \theta, v = r \sin \theta\) and

\[
A_{\theta} := \sqrt{\cos^2 \theta + (a_{11} \cos \theta + a_{02} \sin \theta)^2}.
\]

By a straightforward calculation, the first and the second fundamental forms

\[
E du^2 + 2F du dv + G dv^2, \quad L du^2 + 2M du dv + N dv^2
\]

have the following expressions

\[
E = 1 + O(r^2), \quad F = O(r^2), \quad G = r^2 (A_{\theta}^2 + O(r)),
\]

\[
L = \frac{a_{20} \cos \theta}{A_{\theta}} + O(r), \quad M = - \frac{a_{02} \sin \theta}{A_{\theta}} + O(r), \quad N = \frac{a_{02} \cos \theta}{A_{\theta}} + O(r).
\]

We denote by \(K_{\text{ext}}\) the determinant of the shape operator of \(f\), which is called the extrinsic curvature function.

Since \(EG - F^2 = r^2 (A_{\theta}^2 + O(r))\), the mean curvature function \(H\), the extrinsic curvature function \(K_{\text{ext}}\) and the Gaussian curvature function \(K\) are given by

\[
H = \frac{1}{r^2} \left( \frac{a_{02} \cos \theta}{2 A_{\theta}} + O(r) \right),
\]

\[
K_{\text{ext}} = \frac{a_{02}}{r^2 A_{\theta}^3} \left( a_{20} \cos^2 \theta - a_{02} \sin^2 \theta + O(r) \right),
\]

\[
K = K_{\text{ext}} + c_g(r, \theta) = \frac{a_{02}}{r^2 A_{\theta}^3} \left( a_{20} \cos^2 \theta - a_{02} \sin^2 \theta + O(r) \right),
\]

where \(c_g(r, \theta)\) is a suitable \(C^\infty\)-function at \(p\) with respect to the sectional curvature of the Riemannian metric \(g\) appeared in the Gauss equation. When \((N^3, g)\) is the Euclidean space, the formulas (18) and (19) (and also the description of principal curvatures) have been given in [5]. It should be remarked that the top terms of the curvature functions \(K\) and \(H\) are determined by the three invariants \(a_{20}, a_{02}\) and \(a_{11}\).
Moreover, (19) and (20) imply that the top term of $K$ is equal to that of $K_{\text{ext}}$ at cross cap singularities.

The following is a generalization of the assertion proved in Fukui-Ballesteros [4] and Tari [12] when $(N^3, g)$ is the Euclidean 3-space:

**Proposition 13** Umbilical points do not accumulate to a cross cap singularity in $(N^3, g)$.

**Proof** By (19), we know that $K_{\text{ext}} < 0$ if $\theta = \pm \pi/2$. Thus, it is sufficient to show that $H^2 - K_{\text{ext}}$ does not vanish under the assumption $\cos \theta \neq 0$. In fact,

$$H^2 - K_{\text{ext}} = \frac{1}{r^4} \left( \frac{(a_{02} \cos \theta)^2}{4A_\theta^6} + O(r) \right)$$

diverges if $\cos \theta \neq 0$ as $r$ tends to zero. \hfill \Box

4 Invariants of cross caps under isometric deformations

It was classically known that regular surfaces (not only ruled surfaces) admit non-trivial isometric deformations in general, and such deformations can be expected even at cross cap singularities. In this section, we shall give further invariants under isometric deformation of cross caps. The following assertion holds as follows:

**Proposition 14** The following four quantities written in terms of coefficients of the normal form as in (2)

$$a_{03} + \frac{3a_{11}b_3}{2}, \quad a_{12} + \frac{(1 + a_{11}^2)b_3}{2a_{02}}, \quad a_{21} - \frac{a_{11}a_{20}b_3}{6a_{02}}, \quad a_{30} - \frac{(1 + a_{11}^2)a_{20}b_3}{2a_{02}^2}$$

are common for two cross caps having the same first fundamental form. In particular, they do not change under isometric deformations of a cross cap.

**Proof** Let $f_0$ and $f_1$ be two cross caps having the following normal forms, respectively;

$$f_0(u, v) = \left( u, \ uv + \frac{b_3}{3!} v^3, \sum_{r=2}^{3} \sum_{j=0}^{r} \frac{a_{jr-j}}{j!(r-j)!} u^j v^{r-j} \right) + O(u, v)^4, \quad (23)$$

$$f_1(x, y) = \left( x, \ xy + \frac{B_3}{3!} y^3, \sum_{r=2}^{3} \sum_{j=0}^{r} \frac{A_{jr-j}}{j!(r-j)!} x^j y^{r-j} \right) + O(x, y)^4. \quad (24)$$

Since $(x, y)$ and $(u, v)$ are both local coordinate systems of $R^2$, the mapping $(u, v) \mapsto (x(u, v), y(u, v))$ is a diffeomorphism. Since two coordinates give a normal form of $f_0$ and $f_1$, we may set
Suppose that $f_0$ and $f_1$ share the same first fundamental form. Since we have seen that $a_{02}, a_{20}$ and $a_{11}$ are intrinsic (by Theorem 6), we have that

$$a_{20} = A_{20}, \quad a_{11} = A_{11}, \quad a_{02} = A_{02}.$$ 

Moreover, it holds that

$$E_0 = E_1(x_u)^2 + 2F_1x_u y_u + G_1(y_u)^2, \quad \quad \quad (25)$$

$$F_0 = E_1x_u x_v + F_1(x_u y_v + x_v y_u) + G_1 y_u y_v, \quad \quad \quad (26)$$

$$G_0 = E_1(x_v)^2 + 2F_1x_v y_v + G_1(y_v)^2, \quad \quad \quad (27)$$

where

$$E_i := (f_i)_u \cdot (f_i)_u, \quad F_i := (f_i)_u \cdot (f_i)_v, \quad G_i := (f_i)_v \cdot (f_i)_v \quad (i = 0, 1). \quad (28)$$

Computing the first- and second-order terms of the Taylor expansions of the left and right-hand sides of (25), (26) and (27), we get the following relations:

$$x_{uu}(0, 0) = x_{uv}(0, 0) = x_{vv}(0, 0) = 0,$$

$$x_{uuu}(0, 0) = x_{uuv}(0, 0) = x_{uvv}(0, 0) = x_{vvv}(0, 0) = 0.$$

Similarly, computing the Taylor expansions of the third-order derivatives

$$\frac{\partial^3}{\partial u^3}, \quad \frac{\partial^3}{\partial u^2 \partial v}, \quad \frac{\partial^3}{\partial v \partial u^2}, \quad \frac{\partial^3}{\partial v^3},$$

one gets explicit expressions for $y_{uu}(0, 0)$, $y_{uv}(0, 0)$, $y_{vv}(0, 0)$, and

$$a_{03} - A_{03}, \quad a_{12} - A_{12}, \quad a_{21} - A_{21}, \quad a_{30} - A_{30}$$

in terms of $a_{02}, a_{11}, a_{20}$ and $b_3 - B_3$, which proves the assertion. \qed

**Remark 15** The above conclusion implies that $a_{30}$ and $a_{21}$ do not change under isometric deformations when $a_{20} = 0$. Moreover, (21) and (22) imply that

$$2a_{03} + 3a_{11} b_3 = 0, \quad 2a_{02} a_{12} + (1 + a_{11}^2) b_3 = 0 \quad (29)$$

hold for non-trivial isometric deformations of quadratic cross caps even when $a_{20} \neq 0$. The relations (29) also follow from (11) in the case $a_{20} = 0$.

One can continue the same calculation for the fourth-order terms. The authors checked using Mathematica that

$$a_{04} - A_{04}, \quad a_{13} - A_{13}, \quad a_{22} - A_{22}, \quad a_{31} - A_{31}, \quad a_{40} - A_{40}$$

can be written in terms of $a_{02}, a_{11}, a_{20}, b_3, b_3 - B_3$ and $b_4 - B_4$. Using this, it can be observed that $a_{ij} = A_{ij} (2 \leq i + j \leq 4)$ hold if $b_3 = B_3$ and $b_4 = B_4$. 
Acknowledgments  The authors thank Shyuichi Izumiya, Toshizumi Fukui and Wayne Rossman for valuable comments. The fourth author thanks Huili Liu for fruitful discussions on this subject at 8th Geometry Conference for Friendship of China and Japan at Chengdu.

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