The Size of $p$-Branes

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Abstract

We obtain form factors for scattering of gravitons and anti-symmetric tensor particles off Dirichlet $p$-branes in Type II superstring theory. As expected, the form factor of the $-1$-brane (the D-instanton) exhibits point-like behavior: in fact it is saturated by the dilaton tadpole graph. In contrast, $p$-branes with $p > -1$ acquire size of order the string scale due to quantum effects and exhibit Regge behavior. We find their leading form factors in closed form and show that they contain an infinite sequence of poles associated with $p$-brane excitations. Finally, we argue that the $p$-brane form factors for scattering of R-R bosons will have the same stringy features found with NS-NS states.
String theory appears to contain more than just strings. One lesson of the exciting recent developments is that higher dimensional structures, so called \( p \)-branes, play a prominent role in the strongly coupled dynamics [1, 2, 3, 4]. The \( p \)-branes arise as non-perturbative solutions of the low-energy effective field theory of closed string theories [5, 6] and are a crucial ingredient in various string-string dualities [7, 5, 6]. In a seminal paper, Polchinski has proposed a realization of \( p \)-branes in terms of mixed Dirichlet and Neumann boundary conditions in Type-II superstring theory [10]. Such a worldsheet approach allows one to study these extended objects using well understood tools of two-dimensional quantum field theory. This insight has already produced rapid development in the field [11, 12].

The issue we wish to address here is that of the physical size of \( p \)-branes. This question is of special interest in light of Shenker’s recent suggestion [13], that certain non-perturbative states carrying RR charge may introduce a new dynamical length scale into string theory, which is much shorter than the string scale if the theory is weakly coupled. Our main conclusion is that, at least as far as their gravitational properties are concerned, the Dirichlet \( p \)-branes with \( p > -1 \) acquire a size of the order of the string scale through quantum effects (the \(-1\)-brane is a special case, as we shall see below).

We study the size of the Dirichlet \( p \)-branes by scattering off them massless string states, whose vertex operators are

\[
V(z, \bar{z}) = \varepsilon_{ij} \left( i X^i + k \cdot \psi^i \right) \left( i \bar{X}^j + k \cdot \bar{\psi}^j \right) e^{ik \cdot X},
\]

with \( \varepsilon_{ij} \) symmetric for gravitons and dilatons, and anti-symmetric for anti-symmetric tensor particles. The resulting form factor provides a measure of the effective thickness of the \( p \)-brane as seen by stringy probes. In this paper we restrict our attention to amplitudes involving NS-NS sector string states, whereas one would employ R-R sector photons to measure the R-R charge radius of a Dirichlet \( p \)-brane. There are technical reasons to believe our result, that \( p > -1 \) Dirichlet-branes are typically of string scale thickness, will extend to their R-R charge as well. If this is the case, one may have to look elsewhere for objects much smaller than strings. We will comment on this issue below and hope to present a calculation of the scattering of R-R states off \( p \)-branes at a later date.

The leading contribution to the \( p \)-brane form factor may be read off from the two-point function of closed-string vertex operators on a disk with the appropriate mixed Dirichlet and Neumann boundary conditions. This process may be thought of as follows: an open string with both end-points glued to the \( p \)-brane is created from vacuum, it absorbs and emits a graviton (or anti-symmetric tensor particle) and is subsequently annihilated. Since the open strings with end-points glued to the \( p \)-brane describe excitations of the \( p \)-brane itself [14] there is another physical picture of the same process: a particle collides with a \( p \)-brane creating an excitation propagating along its world volume and the \( p \)-brane subsequently returns to its ground state by emitting another particle.

Let us first consider this calculation for the simplest Dirichlet brane with \( p = -1 \), also known as the D-instanton [15, 16]. Since this object has no world volume one might expect
the scattering process described above to degenerate and we shall see that this is indeed the case. In discussing the $-1$-brane one imposes Dirichlet boundary conditions on all ten superstring coordinates. The required Green function on a disk of radius equal to 1 is given by

$$\langle X^\mu(z, \bar{z}) X^\nu(w, \bar{w}) \rangle = \eta^{\mu\nu} (-2 \ln |z - w| + 2 \ln |1 - z \bar{w}|).$$  \hspace{1cm} (1)

For the fermions we correspondingly take

$$\langle \psi^\mu(z) \psi^\nu(w) \rangle = \frac{\eta^{\mu\nu}}{z - w}, \hspace{1cm} \langle \psi^\mu(z) \bar{\psi}^\nu(\bar{w}) \rangle = \frac{i \eta^{\mu\nu}}{1 - z \bar{w}}. \hspace{1cm} (2)$$

One needs to evaluate the correlation function of two vertex operators, one with polarization $\varepsilon^1$ and momentum $k$ and the other with polarization $\varepsilon^2$ and momentum $p$. The usual kinematics of massless closed-string states applies: $\varepsilon^1_{\mu\nu} k^\nu = 0 = \varepsilon^2_{\mu\nu} p^\nu$, $\varepsilon^1_\mu = 0 = \varepsilon^2_\mu$. We gauge fix the residual symmetry of the disk amplitude by integrating only over the location of one of the vertex operators, placing the other at the origin. This, in fact, overcounts by a factor of $2\pi$ (from the angular integration) which is easily divided out. The result may be expressed as

$$\int d^{10}q e^{iq \cdot Y} F(q)$$

where $q = p + k$, and $Y$ is the position of the D-instanton.

In bosonic string theory the form factor $F(q)$ was first calculated by Green and Wai [17] using string field theory techniques. They found a simple answer with poles only at $q^2 = -2, 0, 2$. This expression exhibited a power law behavior at large $q^2$ indicative of a point-like nature of the D-instanton. It thus seems that string theory contains objects which, at least in perturbation theory, are point-like events.

We have calculated the D-instanton form factor in the Type-II superstring theory and found it to be even simpler than in the bosonic theory. As one might have expected, all poles at $q^2 = \pm 2$ cancel out. In fact, for gravitons the entire two-point function vanishes! For two anti-symmetric tensor particles the form factor turns out to be

$$F_B(q) = \varepsilon^1_{\mu\nu} \varepsilon^2_{\nu\mu} + 4 \varepsilon^1_{\mu\lambda} \varepsilon^2_{\nu\lambda} \frac{q^\mu q^\nu}{q^2}. \hspace{1cm} (4)$$

This expression has a purely field theoretic interpretation: it is due to a dilaton emitted by the $H^2 e^{-2\phi}$ vertex of the low-energy effective field theory (where $H_{ijk}$ is the anti-symmetric tensor field strength) and absorbed by a dilaton tadpole created by the D-instanton. This dilaton tadpole does not affect the graviton two-point function because in the physical “Einstein frame” there is no tree level vertex coupling two on-shell gravitons to a dilaton.

We conclude that in the Type-II superstring theory a D-instanton is indeed point-like, its form factors being miraculously saturated by the field theoretic dilaton tadpole graph. While a fixed D-instanton produces such a tadpole, in the complete theory one has to integrate over the collective coordinates. In addition to the space-time position $X^\mu$ one finds its
superpartner \([\Psi_1, \Psi_2] \), a Majorana-Weyl coordinate \(\theta^a\), \(a = 1, 2, \ldots, 16\). Integration over \(\theta^a\) sets the dilaton tadpole and the associated cosmological constant to zero.

We now move on to consider Dirichlet \(p\)-branes with \(p > -1\). These objects are more physically familiar than the D-instanton: the 0-brane is a stringy description of a solitonic particle, the 1-brane describes a solitonic string, etc. The gravitational form factors of these objects turn out to have exponential rather than power law decay at high energy, and do not exhibit any obvious point-like structure. This behavior is very different from the D-instanton case.

To see what makes \(p > -1\) so different from \(p = -1\) let us consider the 0-brane. Here the spatial coordinates \(X^i\), with \(i = 1, 2, \ldots, 9\), and the corresponding fermionic fields, satisfy Dirichlet boundary conditions on the disk with Green functions given in (4)-(2). The time coordinate, however, has Neumann boundary conditions, so this time the Green functions for \(X^0\) and \(\psi^0\) are

\[
\langle X^0(z, \bar{z})X^0(w, \bar{w}) \rangle = 2 \ln |z - w| + 2 \ln \left|1 - \frac{1}{z\bar{w}}\right| \tag{5}
\]

\[
\langle \psi^0(z)\psi^0(w) \rangle = \frac{1}{z - w}, \quad \langle \psi^0(z)\bar{\psi}^0(\bar{w}) \rangle = \frac{i}{1 - z\bar{w}}. \tag{6}
\]

The change in sign of \(\langle \psi^0(z)\bar{\psi}^0(\bar{w}) \rangle\) compared to (2) is required by worldsheet supersymmetry. The asymmetry between the time and the spatial directions gives rise to some interesting effects. For example, the normal ordering of \(e^{ik\cdot X}\) introduces a factor \(|1 - z\bar{z}|^{-2k_0^2}\) into the correlation function. This factor, which was absent for the D-instanton, leads to singular behavior near the edge of the disk. The interplay between this singularity and the one arising from the collision of the two vertex operators makes the amplitude stringy rather than field theoretic.

Let us break up the momenta into spatial and time components,

\[
p = (p_0, \vec{p}), \quad k = (k_0, \vec{k}). \tag{7}
\]

For simplicity, we choose \(\varepsilon_1^{ij}\) and \(\varepsilon_2^{ij}\) to have non-zero components only in the spatial directions \((1, 2, \ldots, 9)\). Calculations with general polarizations are somewhat more involved and this restricted kinematics is sufficient to get at the physical result we are interested in. Since the energy is conserved, \(p_0 = -k_0\), the form factor is a function of two quantities, \(k_0^2\) and \(\vec{k} \cdot \vec{p}\). The on-shell conditions are

\[
p^2 = \vec{k}^2 = k_0^2, \quad \varepsilon_1^{ij} = \varepsilon_2^{ij} = 0, \quad k^i \varepsilon_1^{ij} = p^i \varepsilon_2^{ij} = 0. \tag{8}
\]

As a warm up, let us perform the calculation in bosonic string theory. In calculating the term \(\sim \varepsilon_1^{ij}\varepsilon_2^{ij}\) of the two-graviton amplitude we fix one of the graviton vertex operators at the center of the disk and find the following integral in the variable \(x = r^2\),

\[
\frac{1}{2} \varepsilon_1^{ij}\varepsilon_2^{ij} \int_0^1 dx \left[\frac{1}{x^2} + 1\right] (1 - x)^{-2k_0^2} x^{\vec{k} \cdot \vec{p} + k_0^2}. \tag{9}
\]

\[^1\text{We thank E. Witten for pointing this out.}\]
Each of the necessary integrals yields a beta function as is common in open string calculations. This is not too surprising since in the s channel this process is mediated by an open string sliding along the p-brane world volume. The complete answer for the gravitational form factor turns out to be

\[ F^\text{bosonic}_g = \frac{\Gamma(2 - 2k_0^2)\Gamma(\vec{k} \cdot \vec{p} + k_0^2 - 1)}{\Gamma(\vec{k} \cdot \vec{p} - k_0^2 + 2)} \left[ (\vec{k} \cdot \vec{p} + k_0^2 - 1) - 2(1 - 2k_0^2)(\vec{k} \cdot \vec{p})^2 + p^i \varepsilon_{ij}^2 \varepsilon_{il}^2 k^l (1 - k_0^2)(1 - 2k_0^2) \right] \]  

(9)

The anti-symmetric tensor form factor is

\[ F^\text{bosonic}_B = \frac{\Gamma(2 - 2k_0^2)\Gamma(\vec{k} \cdot \vec{p} - k_0^2 - 1)}{\Gamma(\vec{k} \cdot \vec{p} - k_0^2 + 2)} \left[ \varepsilon_{ij}^1 \varepsilon_{ij}^2 \varepsilon_{ij}^3 \varepsilon_{ij}^4 \varepsilon_{ij}^5 k^j (1 - k_0^2)(1 - 2k_0^2) \right] \]  

(10)

This expression can be cast in a manifestly gauge invariant form,

\[ F^\text{bosonic}_B = \frac{\Gamma(2 - 2k_0^2)\Gamma(\vec{k} \cdot \vec{p} - k_0^2 - 1)}{\Gamma(\vec{k} \cdot \vec{p} - k_0^2 + 2)} H_{ijl}^1(h) H_{ijl}^2(p), \]  

(11)

where

\[ H_{ijl}^1(k) = i(k_\alpha \varepsilon_{ij}^1 \varepsilon_{\gamma l}^1 + k_\beta \varepsilon_{ij}^1 \varepsilon_{\gamma l}^1 + k_\gamma \varepsilon_{ij}^1 \varepsilon_{\alpha l}^1). \]  

(12)

In proceeding to the type-II string one finds, as usual, that the calculations are longer, but the answers are shorter. We find the following expression for the gravitational form factor,

\[ F_g = \frac{1}{2} \varepsilon_{ij}^1 \varepsilon_{ij}^2 \int_0^1 dx \left[ \frac{(1 - k_0^2 - \vec{k} \cdot \vec{p})^2}{x^2} + 2 \frac{k_0^2 - (\vec{k} \cdot \vec{p})^2}{x} + (1 - k_0^2, \vec{k} \cdot \vec{p})^2 \right] x^{-\vec{k} \cdot \vec{p} + k_0^2}(1 - x)^{-2k_0^2}. \]  

(13)

Each of the necessary integrals is again a beta function and can be explicitly evaluated to give the complete form factor,

\[ F_g = \varepsilon_{ij}^1 \varepsilon_{ij}^2 k_0^2 \frac{\Gamma(1 - 2k_0^2)\Gamma(\vec{k} \cdot \vec{p} + k_0^2)}{\Gamma(\vec{k} \cdot \vec{p} - k_0^2 + 1)}. \]  

(14)

We note that the terms involving contractions between the polarization tensors and the transverse momenta have cancelled in this form factor. The corresponding calculation for two anti-symmetric tensor particles also yields a simple result,

\[ F_B = \left[ 2 \varepsilon_{ij}^1 \varepsilon_{ij}^2 \varepsilon_{ij}^3 \varepsilon_{ij}^4 \varepsilon_{ij}^5 k^j \right] \frac{\Gamma(1 - 2k_0^2)\Gamma(\vec{k} \cdot \vec{p} + k_0^2)}{\Gamma(\vec{k} \cdot \vec{p} - k_0^2 + 1)}, \]  

(15)

whose manifestly gauge invariant form is

\[ F_B = \frac{1}{3} H_{ijl}^1(h) H_{ijl}^2(p) \frac{\Gamma(1 - 2k_0^2)\Gamma(\vec{k} \cdot \vec{p} + k_0^2)}{\Gamma(\vec{k} \cdot \vec{p} - k_0^2 + 1)}. \]  

(16)
Equations (14)-(16) have structure typically found in string theoretic four-point functions. In the physical region there is an infinite sequence of poles at \(2k_0^2 = 1, 2, 3, \ldots\). The positions of these poles coincide with the excitation energies of an open string whose ends are attached to the particle (0-brane). From this point of view the infinite energy spectrum is perfectly natural. We should emphasize, however, that it is somewhat unusual to find a “particle” with an infinite spectrum of excitations. Thus, the string solitons which are described by the 0-branes do not behave like typical field theoretic particles. It appears that in string theory all objects, even those that at low energies are best described as particles, acquire stringy properties.

The difference between the D-instanton and the other \(p\)-branes could be predicted on purely kinematical grounds. For the D-instanton, the \(\varepsilon_{ij}\varepsilon_{ij}^2\) term in the form factor is \textit{a priori} a function of \(t = -(p + k)^2\) only. For the 0-brane it is a function of two variables, \(t = -(p + k)^2\) and the conserved quantity \(s = k_0^2\). This is what makes the Veneziano-type formulae with the interplay between the \(s\) and \(t\) channels, possible. For instance, the graviton form factor (14) may be rewritten as

\[
F_g = s \varepsilon_{ij} \varepsilon_{ij}^2 \frac{\Gamma(1 - 2s)\Gamma(-t/2)}{\Gamma(1 - 2s - t/2)} \tag{17}
\]

Here we find a somewhat unusual situation: the poles in the \(s\)-channel correspond to an open string glued to the \(p\)-brane, while the poles in the \(t\)-channel correspond to closed strings created in a collision of two massless particles and absorbed by the \(p\)-brane. In the \(u\)-channel the process appears essentially the same as in the \(s\)-channel and is also mediated by the \(p\)-brane excitations.

As we proceed to \(p\)-branes with \(p > 0\), the number of variables in the form factor remains equal to two. For the 1-brane, for example, in addition to \(t = -(p + k)^2\) we have the invariant conserved quantity \(s = k_0^2 - k_1^2\). For gravitons polarized transversely to the 1-brane the form factor is still given by (17). It is straightforward to generalize the calculation of form factors to 1-branes, 2-branes, \textit{etc}. For gravitons and anti-symmetric tensor particles polarized transversely to the \(p\)-brane world volume, the results are given by (14)-(15) with \(k_0^2\) replaced by an appropriate conserved quantity: \(k_0^2 \rightarrow k_0^2 - k_1^2\) for 1-branes; \(k_0^2 \rightarrow k_0^2 - k_1^2 - k_2^2\) for 2-branes; \textit{etc}. This demonstrates a remarkable universality in the properties of the Dirichlet \(p\)-branes with \(p > -1\): to gravitons or antisymmetric tensor particles polarized transversely to the \(p\)-brane world volume all \(p\)-branes appear essentially the same. In particular, for gravitons polarized transversely to the 1-brane (string) the form factor is still given by (17). This remarkably simple expression for scattering off a long string may serve as a guide to similar calculations where the long string is described by a soliton. Our conclusion from all of the above is that, as far as their gravitational properties are concerned, all \(p\)-branes with \(p > -1\) behave in a stringy, rather than point-like manner.

In this paper we presented detailed results for scattering of NS-NS closed string states off the Dirichlet \(p\)-branes of type-II superstring theory. An equally interesting exercise is to scatter R-R closed string states off \(p\)-branes. While this is technically more difficult, we
may anticipate the structure of the result simply by noting that R-R vertex operators also contain a factor $e^{ik \cdot X}$. The normal ordering of this factor introduces $|1 - r^2|^{-2s}$ into the integrand, which is singular near the edge of the disk ($s = k_0^2 - \sum_{i=1}^{p} k_i^2$ is one of the two kinematical invariants for the scattering process). Thus, we anticipate that the result will have a stringy structure similar to (14)-(17), with an interplay between the $s$ and $t$ channels. Another argument in favor of this is the fact that the R-R scattering amplitudes are related to the NS-NS scattering amplitudes by space-time supersymmetry. Our tentative conclusion is therefore that quantum fluctuations smear the R-R charge of the $p$-branes over the string scale. The R-R charge radius, like the gravitational radius, will then exhibit Regge behavior and grow with an increasing energy of the probe.

Another important issue that needs to be addressed is that of $p$-brane recoil. In the leading order calculations we have performed, $p$-branes act as fixed objects which can absorb any amount of transverse momentum. In the full quantum theory, however, we expect a phenomenon similar to quantization of field theoretic collective coordinates. As a result, the 0-branes or the fully compactified $p$-branes should recoil with velocity which depends on the soliton mass. Some insight into this issue has recently been obtained [18, 19], but the Dirichlet $p$-brane formalism should be helpful in performing detailed calculations.

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