Multiparameter quantum Pfaffians

Naihuan Jing and Jian Zhang

Abstract. The multiparameter quantum Pfaffian of the \((p, \lambda)\)-quantum group is introduced and studied together with the quantum determinant, and an identity relating the two invariants is given. Generalization to the multiparameter hyper-Pfaffian and relationship with the quantum minors are also considered.

1. Introduction

In the early study of quantum groups, multiparameter quantum groups and quantum enveloping algebras were considered along the line of one-parameter quantum groups \([13, 12, 10]\). It was clear that many of their properties are similar to their one-parameter analog, for example, Artin-Schelter-Tate \([1]\) showed that the multiparameter general linear quantum group has the same Hilbert function of the polynomial functions in \(n^2\) variables under the so-called \((p, \lambda)\)-condition (see \((2.19)\)). Further results have been established for two- and multi-parameter quantum groups \([11, 2, 4, 8]\) \([5]\) such as the existence of the quantum determinant, which helps to transform quantum semigroups into quantum groups. Recently it is known that the quantum Pfaffians can be extended to two-parameter quantum groups as well \([7]\).

In this paper, we generalize our recent study of quantum determinants and Pfaffians from two-parameter quantum groups to multiparameter cases. We will adopt the same approach of quadratic algebras \([9]\) to study quantum determinants and quantum Pfaffians, and express them as the scaling constants of quantum differential forms (cf. \([6]\)). In particular, we will prove that the multiparameter quantum Pfaffian can be defined for a more general class of multiparameter quantum matrices and prove the identity between the quantum determinant and quantum Pfaffian, and also establish their integrality property for quantum groups under the \((p, \lambda)\)-conditions.

MSC (2010): Primary: 17B37; Secondary: 58A17, 15A75, 15B33, 15A15.

Keywords: multiparameter quantum groups, \(q\)-determinants, \(q\)-Pfaffians, \(q\)-hyper-Pfaffians

Supported by NSFC (11531004), Fapesp (2015/05927-0) and Humboldt Foundation.
We also formulate the multiparameter quantum determinants in terms of the quasideterminant of the generating matrix. Generalizing the one-form and two-form, we obtain higher degree analogs of the multiparameter Pfaffians and establish the transformation rule of the multiparameter quantum hyper-Pfaffian under the quantum determinant, which extends some of the transformation rules of Pfaffians in linear algebra.

2. Quantum determinants

2.1. Quantum semigroup A. Let \( p = (p_{ij}), q = (q_{ij}) \) be two sets of \( n^2 \) parameters in the complex field \( \mathbb{C} \) arranged in matrix forms satisfying the following relations:

\[
p_{ij}p_{ji} = 1, \quad p_{ii} = 1; \quad q_{ij}q_{ji} = 1, \quad q_{ii} = 1.
\]

For a scalar \( v \), the \( v \)-commutator \([x, y]_v\) is defined by

\[
[x, y]_v = xy - vyx.
\]

Therefore two elements \( x \) and \( y \) are \( q \)-commutative if \([x, y]_{q_{ij}} = 0\) for a parameter \( q_{ij} \).

We define the unital algebra \( A \) as an associative complex algebra generated by \( a_{ij} \), \( 1 \leq i, j \leq n \) subject to the following relations:

\[
a_{ik}a_{jl} - q_{kl}q_{ij}a_{il}a_{jk} = q_{kl}a_{il}a_{jk} - q_{ij}a_{jk}a_{il},
\]

where \( i < j \) and \( k < l \). These can be paraphrased as that the quantum matrix \( A = (a_{ij}) \) is row \( q \)-commutative, column \( p \)-commutative, and satisfies the equality between the \( q \)-commutator (resp. \( p \)-commutator) of the diagonal entries and the \( q \)-commutator (resp. \( p \)-commutator) of the skew diagonal entries up to a \( q \)-factor (resp. \( p \)-factor).

The algebra \( A \) has a bialgebra structure under the comultiplication \( \Delta : A \rightarrow A \otimes A \) given by

\[
\Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj},
\]

and the counit given by \( \varepsilon(a_{ij}) = \delta_{ij} \), the Kronecker symbol.

For any permutation \( \sigma \) in \( S_n \), the \( q \)-inversion associated to the parameters \( q_{ij} \) is defined as

\[
(-q)_\sigma = (-1)^{l(\sigma)} \prod_{i<j, \sigma_i > \sigma_j} q_{\sigma_j \sigma_i},
\]

where \( l(\sigma) = |\{(i, j) | i < j, \sigma_i > \sigma_j\}| \) is the classical inversion number of \( \sigma \).
The \( v \)-based quantum number is defined to be:

\[
[n]_v = 1 + v + \cdots + v^{n-1},
\]

and the quantum factorial \([n]_v! = [1]_v[2]_v \cdots [n]_v\).

Let \( A = (a_{ij}) \) be the matrix with entries satisfying (2.2)-(2.5). We define the quantum row-determinant and column-determinant of \( A \) as follows.

\[
\text{rdet}(A) = \sum_{\sigma \in S_n} (-q)_{\sigma} a_{1,\sigma_1} \cdots a_{n,\sigma_n},
\]

\[
\text{cdet}(A) = \sum_{\sigma \in S_n} (-p)_{\sigma} a_{\sigma_1,1} \cdots a_{\sigma_n,n}.
\]

The first property we show is that both are group-like elements:

\[
\Delta(\text{rdet}(A)) = \text{rdet}(A) \otimes \text{rdet}(A),
\]

\[
\Delta(\text{cdet}(A)) = \text{cdet}(A) \otimes \text{cdet}(A).
\]

To do this we introduce two copies of commuting quantum exterior algebras associated to the parameters \( p_{ij} \) and \( q_{ij} \) respectively. The first one is

\[
\Lambda_q(x) = \mathbb{C}\langle x_1, \ldots, x_n \rangle / I
\]

where \( I \) is the ideal \((x_i^2, q_{ij}x_i x_j + x_j x_i | 1 \leq i < j \leq n)\) and one writes \( x \wedge x' = x \otimes x' \mod I \). Then we have that

\[
x_j \wedge x_i = -q_{ij} x_i \wedge x_j,
\]

\[
x_i \wedge x_i = 0,
\]

where \( i < j \). Then for \( \sigma \in S_n \),

\[
x_{\sigma_1} \wedge \cdots \wedge x_{\sigma_n} = (-q)_{\sigma} x_1 \wedge \cdots \wedge x_n.
\]

Clearly \( \Lambda_q \) is a left \( \mathcal{A} \)-comodule with the coaction \( \mu_q : \Lambda_q \to \mathcal{A} \otimes \Lambda_q \) given by

\[
\mu_q(x_i) = \sum a_{ij} \otimes x_j.
\]

The row determinant can be computed via the coaction:

\[
\mu_q(x_1 \wedge \cdots \wedge x_n) = \text{rdet}(A) \otimes x_1 \wedge \cdots \wedge x_n.
\]

Subsequently the comodule structure map \((\text{id} \otimes \mu_q)\mu_q = (\Delta \otimes \text{id})\mu_q \) implies that \( \Delta(\text{rdet}(A)) = \text{rdet}(A) \otimes \text{rdet}(A) \).

Let \( \Lambda_p = \Lambda_p(y) \) be the unital associative algebra \( \mathbb{C}\langle y_1, \ldots, y_n \rangle / J \), where \( J \) is the ideal \((y_i^2, p_{ij}y_i y_j + y_j y_i | 1 \leq i < j \leq n)\). Using similar convention for \( x_i \)'s, the relations are

\[
y_j \wedge y_i = -p_{ij} y_i \wedge y_j,
\]

\[
y_i \wedge y_i = 0,
\]
where $1 \leq i < j \leq n$. The space $\Lambda_p$ is a right $\mathcal{A}$-comodule with coaction $\mu'_p : \Lambda_p \to \Delta_p \otimes \mathcal{A}$ given by

$$\mu'_p(y_i) = \sum_{j=1}^{n} y_j \otimes a_{ji}. \quad (2.18)$$

Similar to rdet one has that $\Delta(\text{cdet}(A)) = \text{cdet}(A) \otimes \text{cdet}(A)$. In general, $\text{rdet} \neq \text{cdet}$. However we will consider a special case while the two determinants are equal.

From now on we assume that $(p_{ij}, q_{ij})$ live on the parabola

$$p_{ij}q_{ij} = \lambda, \quad \lambda \neq -1, \quad i < j. \quad (2.19)$$

These relations (2.19) are referred as the $(p, \lambda)$ conditions [1]. In the following whenever we talk about the multiparameter quantum groups we always consider those satisfying the $(p, \lambda)$ relations.

**Theorem 2.1.** In the bialgebra $\mathcal{A}$ with the $(p, \lambda)$ relations, one has that $\text{rdet}(A) = \text{cdet}(A)$.

**Proof.** Consider the following special linear element $\Phi$ in $\mathcal{A} \otimes \Lambda_q \otimes \Lambda_p$:

$$\Phi = \sum_{i,j=1}^{n} a_{ij} \otimes y_i \otimes x_j = y^T A x, \quad (2.20)$$

where we have set $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$.

Put further $\delta = (\delta_1, \ldots, \delta_n)^T$, $\partial = (\partial_1, \ldots, \partial_n)^T$, and consider the following elements:

$$\delta = A x, \quad (2.21)$$

$$\partial = A^T y. \quad (2.22)$$

Let $\omega_i = y_i \delta_i = \sum_{j=1}^{n} y_i a_{ij} x_j$. It follows from the relations (2.2)-(2.5) and the commutation relations of $\Lambda_q$ and $\Lambda_p$ that

$$\omega_i \wedge \omega_i = 0, \quad 1 \leq i \leq n, \quad (2.23)$$

$$\omega_j \wedge \omega_i = \lambda \omega_i \wedge \omega_j, \quad 1 \leq i < j \leq n. \quad (2.24)$$

It follows from (2.23)-(2.24) that

$$\wedge^n \Phi = \left( \sum_{\sigma \in S_n} \lambda^{l(\sigma)} \right) \omega_1 \wedge \cdots \wedge \omega_n$$

$$= [n]_\lambda ! (x_1 \wedge \cdots \wedge x_n) (\partial_1 \wedge \cdots \wedge \partial_n)$$

$$= [n]_\lambda ! \text{rdet}(A)(y_1 \wedge \cdots \wedge y_n)(x_1 \wedge \cdots \wedge x_n).$$

Note that one can also rewrite $\Phi = \sum_{i=1}^{n} \omega'_i$ with $\omega'_i = \partial_i x_i = \sum_{j=1}^{n} a_{ji} y_j x_i$. Then it follows that

$$\wedge^n \Phi = [n]_\lambda ! \text{cdet}(A)(y_1 \wedge \cdots \wedge y_n)(x_1 \wedge \cdots \wedge x_n),$$
which implies that
\[ \text{rdet}(A) = \text{cdet}(A). \]
\[ \square \]

Due to this identity, from now on, we will define the multiparameter quantum determinant for the \((p, \lambda)\)-quantum group as

\begin{equation}
\det_q(A) = \sum_{\sigma \in S_n} (-q)_{\sigma}a_{1,\sigma_1} \cdots a_{n,\sigma_n}
\end{equation}

(2.25)

\[ = \sum_{\sigma \in S_n} (-p)_{\sigma}a_{1,\sigma_1} \cdots a_{n,\sigma_n}. \]

For a pair of \(t\) indices \(i_1, \ldots, i_t\) and \(j_1, \ldots, j_t\), we define the quantum row-minor \(\det_q(A_{i_1 \ldots i_t})\) as in (2.25). Like the determinant, the quantum row minor also equals to the quantum column minor for any pairs of ordered indices \(1 \leq i_1 < \cdots < i_t \leq n\) and \(1 \leq j_1 < \cdots < j_t \leq n\), which justifies the notation.

For any \(t\) indices \(i_1, \ldots, i_t\)

\begin{equation}
\delta_{i_1} \wedge \cdots \wedge \delta_{i_t} = \sum_{j_1 < \cdots < j_t} \det_q(A_{j_1 \ldots j_t}) x_{j_1} \wedge \cdots \wedge x_{j_t},
\end{equation}

(2.26)

where the sum runs through all indices \(1 \leq j_1 < \cdots < j_t \leq n\). This implies that \(\det_q(A_{j_1 \ldots j_t}) = 0\) whenever there are two identical rows.

As \(\delta_i\)'s obey the wedge relations (2.11)-(2.12), for any \(t\)-shuffle \(\sigma \in S_n\) : \(1 \leq \sigma_1 < \cdots < \sigma_t, \sigma_{t+1} < \cdots < \sigma_n \leq n\), one has that

\[ \delta_{\sigma_1} \wedge \cdots \wedge \delta_{\sigma_t} \wedge \delta_{\sigma_{t+1}} \wedge \cdots \wedge \delta_{\sigma_n} = (-q)_{\sigma} \delta_1 \wedge \cdots \wedge \delta_n. \]

Note that \(x_j\)'s also satisfy the same wedge relations. This then implies the following Laplace expansion by invoking (2.26): for each fixed \(t\)-shuffle \(\sigma_1 < \cdots < \sigma_t, \sigma_{t+1} < \cdots < \sigma_n\), one has that

\begin{equation}
\det_q(A) = \sum_{\alpha} \frac{(-q)_{\alpha}}{(-q)_{\sigma}} \det_q(A_{\alpha_1 \ldots \alpha_t}) \det_q(A_{\sigma_{t+1} \ldots \sigma_n}),
\end{equation}

(2.27)

where the sum runs through all \(t\)-shuffles \(\alpha \in S_n\) such that \(\alpha_1 < \cdots < \alpha_t, \alpha_{t+1} < \cdots < \alpha_n\).

In particular, for fixed \(i, k\)

\[ \delta_{ik} \det_q(A) = \sum_{j=1}^{n} \frac{\prod_{l<i} (-q_{jl})}{\prod_{l<i} (-q_{li})} a_{ij} \det_q(A^\hat{k}_j)
\]

(2.28)

\[ = \sum_{j=1}^{n} \frac{\prod_{l>j} (-q_{jl})}{\prod_{l>j} (-q_{li})} \det_q(A^\hat{k}_j) a_{ij}, \]

where \(\hat{i}\) means the indices \(1, \ldots, i - 1, i + 1, \ldots, n\) for brevity.
As for the quantum (column) determinant or column-minor, the corresponding Laplace expansion for a fixed $r$-shuffle $(\tau_1 \ldots \tau_n)$ of $n$ such that $\tau_1 < \cdots < \tau_r, \tau_{r+1} < \cdots < \tau_n$

\begin{equation}
\det_q(A) = \sum_{\beta} \frac{(-p)^\beta}{(-p)^r} \det_q(A^{\beta_1 \ldots \beta_r}) \det_q(A^{\beta_{r+1} \ldots \beta_n}),
\end{equation}

where the sum runs through all $r$-shuffles $\beta \in S_n$ such that $\beta_1 < \cdots < \beta_r, \beta_{r+1} < \cdots < \beta_n$.

In particular, we have that for fixed $i, k$

\begin{equation}
\delta_{ik} \det_q(A) = \sum_{j=1}^n \prod_{l<j} (-q_{lj}) \prod_{l<i} (-q_{li}) a_{ij} \det_q(A^{\hat{j} \hat{k}}),
\end{equation}

where the sum runs through all $r$-shuffles $\beta \in S_n$ such that $\beta_1 < \cdots < \beta_r, \beta_{r+1} < \cdots < \beta_n$.

**Theorem 2.2.** In the bialgebra $A$ one has that

\begin{equation}
a_{ij} \det_q(A) = \lambda^{j-i} \prod_{l=1}^n q_{li} \prod_{l=1}^n q_{lj} \det_q(A) a_{ij}.
\end{equation}

**Proof.** Let $X = (x_{ij}), A' = (a'_{ij}), A'' = (a''_{ij})$ be the matrices with entries in $A$ defined by

\begin{align}
x_{ij} &= \det_q(A^{ij}), \\
a'_{ij} &= \prod_{l<i} (-q_{li}) a_{ij}, \\
a''_{ij} &= \prod_{l>j} (-p_{jl}) a_{ij},
\end{align}

It follows from the Laplace expansion that $A' \det_q = A' X A'' = \det_q A''$. Therefore

\begin{equation}
\prod_{l<i} (-a_{ij}) a_{ij} \det_q(A) = \prod_{l>j} (-p_{jl}) \det_q(A) a_{ij}.
\end{equation}

This is exactly

\begin{equation}
\prod_{l=1}^n q_{li} \det_q(A) a_{ij} = \lambda^{j-i} \prod_{l=1}^n q_{li} \det_q(A) a_{ij}.
\end{equation}

**Remark 2.3.** $\det_q(A)$ is central if and only if $\lambda^{j-i} \prod_{l=1}^n q_{li} = \prod_{l=1}^n q_{lj}$ for any $i, j$.

Theorem 2.2 implies that $\det_q(A)$ is a regular element in the bialgebra $A$, therefore we can define the localization $A[\det_q^{-1}]$, which will be denoted as $GL_{p,\lambda}(n)$. In fact, Theorem 2.2 gives the following identity:
(2.35) \[ \det_q(A)^{-1} a_{ij} = \chi^i j \prod_{l=1}^n q_{il} a_{lj} \det_q(A)^{-1}. \]

By defining the antipode
\[ S(a_{ij}) = \frac{1}{\prod_{l<j} -q_{li}} \det_q(A^\hat{j}_i - 1) \]
(2.36)
\[ = \prod_{l>i} -p_{jl} \det_q(A) \det_q(A^\hat{j}_i - 1) \]
the bialgebra \( A[\det_q^{-1}] \) becomes a Hopf algebra, thus a quantum group in the sense of Drinfeld.

In fact, the second equation follows from (2.35). Therefore,
\[ AS(A) = S(A)A = I \]
by the Laplace expansions. Subsequently
\[ (id \otimes S) \Delta = (S \otimes id) \Delta = \varepsilon. \]

3. Quasideterminants

In this section we will work with the ring of fractions of noncommutative elements. First of all let us recall some basic facts about quasideterminants. Let \( X \) be the set of \( n^2 \) elements \( x_{ij}, 1 \leq i, j \leq n \). For convenience, we also use \( X \) to denote the matrix \( (x_{ij}) \) over the ring generated by \( x_{ij} \).

Denote by \( F(X) \) the free division ring generated by \( 0, 1, x_{ij}, 1 \leq i, j \leq n \). It is well-known that the matrix \( X = (x_{ij}) \) is an invertible element over \( F(X) \).

Let \( I, J \) be two finite subsets of cardinality \( k \leq n \) inside \( \{1, \ldots, n\} \). Following [3], we introduce the notion of quasideterminant.

**Definition 3.1.** For \( i \in I, j \in J \), the \((i, j)\)-th quasideterminant \( |X|_{ij} \) is the following element of \( F(X) \):
\[ |X|_{ij} = y_{ji}^{-1}, \]
where \( Y = X^{-1} = (y_{ij}) \).

If \( n = 1 \), \( I = i, J = j \). Then \( |X|_{ij} = x_{ij} \).

When \( n \geq 2 \), and let \( X^{ij} \) be the \((n-1) \times (n-1)\)-matrix obtained from \( X \) by deleting the \( i \)th row and \( j \)th column. In general \( X^{i_1 \ldots i_r, j_1 \ldots j_r} \) denotes the submatrix obtained from \( X \) by deleting the \( i_1, \ldots, i_r \)-th rows, and \( i_1, \ldots, i_r \)-th columns. Then
\[ |X|_{ij} = x_{ij} - \sum_{i' j'} x_{ii'} (|X|^{ij}_{j'} x_{j'j}), \]
where the sum runs over \( i' \notin I \setminus \{i\}, j' \notin J \setminus \{j\} \).

**Theorem 3.2.** Let \( A \) be the matrix of generators of \( GL_{p, \lambda}(n) \). In the ring of fractions of elements of \( GL_{p, \lambda}(n) \), one has that
\[ \det_q(A) = |A|_{11} A^{11}_{12} A^{12}_{13} \cdots a_{nn} \]
and the quasi-minors in the right-hand side commute with each other. More generally, for two permutations \( \sigma \) and \( \tau \) of \( S_n \), one has that

\[
\det_q(A) = (-q)_{\sigma_1 \tau_1} A_{\sigma_2 \tau_2} \cdots A_{\sigma_n \tau_n} 
\]

**Proof.** By definition the quasi-determinants of \( A \) are inverses of the entries of the antipode \( S(A) \),

\[
|A|_{ij} = S(a_{ji})^{-1} = \prod_{l<i} (-q_{li}) \det_q(A) \det_q(A_{ij})^{-1} 
= \prod_{l>i} (-p_{il}) \det_q(A_{1i})^{-1} \det_q(A), 
\]

then

\[
\prod_{l<i} (-q_{li}) |A|_{ij} \det_q(A_{ij}) = \det_q(A) 
\]

\[
\prod_{l<i} (-p_{il}) \det_q(A_{ij}) |A|_{ij} = \det_q(A), 
\]

By induction on the size of the matrix \( A \), one sees that (3.1) and (3.2) hold.

It follows from (2.34) that \( \det_q(A_{11} \ldots 1, 21 \ldots 2) \) and \( \det_q(A_{11} \ldots s, 21 \ldots s) \) commute for \( 1 \leq s, t \leq n-1 \). Any factor on the right hand side of (3.1) can be expressed as \( \det_q(A_{11} \ldots 1, 21 \ldots 2) \) multiplied by a scalar, therefore they commute with each other.

**4. Multiparameter quantum Pfaffians**

**Definition 4.1.** Let \( B = (b_{ij}) \) be an \( 2n \times 2n \) square \( p \)-antisymmetric matrix with noncommutative entries such that \( b_{ji} = -p_{ij} b_{ij}, i < j \). The multiparameter quantum \( q \)-Pfaffian is defined by

\[
Pf_q(B) = \sum_{\sigma \in \Pi} (-q)_{\sigma(1)\sigma(2)} b_{\sigma(3)\sigma(4)} \cdots b_{\sigma(2n-1)\sigma(2n)},
\]

where \( p = (p_{ij}), q = (q_{ij}), i < j \), and the sum runs through the set \( \Pi \) of permutations \( \sigma \) of \( 2n \) such that \( \sigma(2i-1) < \sigma(2i), i = 1, \ldots, n \).

Note that the parameters \( q_{ij} \) and \( p_{ij} \) satisfy the \((p, \lambda)\) condition: \( p_{ij} q_{ij} = \lambda \).

**Proposition 4.2.** For any \( 0 \leq t \leq n \),

\[
Pf_q(B) = \sum_I \text{inv}(I, I^c) \Pf_q(B_I) \Pf_q(B_{I^c}),
\]

where \( \text{inv}(I, I^c) \) is the number of inversions in \( I \) with respect to \( I^c \).
where the sum is taken over all subsets $I = \{i_1 \cdots i_2t | i_1 < \cdots < i_{2t} \}$ of $[1,2n]$, and

\begin{equation}
\text{inv}(I, J) = \prod_{i \in I, j \in J, i > j} (-q_{ji}).
\end{equation}

**Proof.** Let $\Omega = \sum_{i<j} b_{ij}x_ix_j$, where $x_i \in \Lambda_q(x)$. Then

\begin{equation}
\bigwedge^n \Omega = \text{Pf}_q(B)x_1 \wedge \cdots \wedge x_{2n}.
\end{equation}

On the other hand,

\begin{equation}
\bigwedge^n \Omega = \Omega^t \bigwedge^{n-t}
\end{equation}

\begin{equation}
= \sum_{I,J} \text{Pf}(B_I)x_J \text{Pf}(B_{I^c})x_J
\end{equation}

\begin{equation}
= \sum_{I,J} \text{Pf}(B_I)\text{Pf}(B_{I^c})x_Ix_J
\end{equation}

It is easy to see that $x_Ix_J$ vanishes unless $J = I^c$. Therefore

\begin{equation}
\bigwedge^n \Omega = \sum_{I} \text{Pf}(B_I)\text{Pf}(B_{I^c})x_Ix_{I^c}
\end{equation}

\begin{equation}
= \sum_{I} \text{inv}(I, I^c)\text{Pf}(B_I)\text{Pf}(B_{I^c})x_1 \wedge \cdots \wedge x_{2n}.
\end{equation}

Thus we conclude that

\begin{equation}
\text{Pf}(B) = \sum_{I} \text{inv}(I, I^c)\text{Pf}(B_I)\text{Pf}(B_{I^c}).
\end{equation}

\begin{flushright}
\square
\end{flushright}

**Theorem 4.3.** Let $B = (b_{ij})_{1 \leq i,j \leq 2n}$ be the $p$-antisymmetric matrix such that $b_{ji} = -p_{ij}b_{ij}, i < j$, and assume that the entries of $B$ commute with those of a $(p,\lambda)$-matrix $A = (a_{ij})_{1 \leq i,j \leq 2n}$. Let $C = A^TBA$. Then

\begin{equation}
c_{ij} = -p_{ij}c_{ij}, \quad i < j
\end{equation}

and

\begin{equation}
\text{Pf}_q(C) = \det_q(A)\text{Pf}_q(B).
\end{equation}
Proof. We first check that $c_{ij}$ also form anti-symmetric matrix. We compute that

$$c_{ii} = \sum_{k,l} a_{ki}b_{kl}a_{li} = \sum_{k<l} a_{ki}b_{kl}a_{li} + a_{li}b_{lk}a_{ki}$$

$$= \sum_{k<l} (a_{ki}a_{li} - p_{kl}a_{li}a_{ki})b_{kl} = 0.$$ 

For $i < j$,

$$c_{ij} = \sum_{k,l} a_{ki}b_{kl}a_{lj} = \sum_{k<l} (a_{ki}b_{kl}a_{lj} + a_{li}b_{lk}a_{kj})$$

$$= \sum_{k<l} (a_{ki}a_{lj} - p_{kl}a_{li}a_{kj})b_{kl} = \sum_{k<l} \det_q(A_{ij}^{kl})b_{kl},$$

$$c_{ji} = \sum_{k,l} a_{kj}b_{kl}a_{li} = \sum_{k<l} (a_{kj}a_{li} - p_{kl}a_{cj}a_{ki})b_{kl}$$

$$= \sum_{k<l} -p_{ij}(a_{ki}a_{lj} - p_{kl}a_{cj}a_{ki})b_{kl} = -p_{ij} \sum_{k<l} \det_q(A_{ij}^{kl})b_{kl}$$

$$= -p_{ij}c_{ij}.$$ 

Consider the element

$$\Omega = x^t C x,$$

where we recall that $x = (x_1, \ldots, x_n)^t$ and $x_i \in \Lambda_q(x)$. Explicitly we have that

$$\Omega = \sum_{1 \leq i,j \leq n} c_{ij}x_i x_j = \sum_{i<j}(1 + \lambda)c_{ij}x_i x_j,$$

therefore

$$\bigwedge^n \Omega = (1 + \lambda)^n \text{Pf}_q(C)x_1 \wedge \cdots \wedge x_{2n}.$$ 

On the other hand, let $\omega_i = \sum_{j=1}^n a_{ij}x_j$. Then

$$\omega_j \omega_i = -q_{ij}\omega_i \omega_j, \quad i < j,$$

$$\omega_i \omega_i = 0.$$ 

As $Ax = (\omega_1, \ldots, \omega_{2n})^t$, one has that

$$\Omega = x^t A^t B A x = (Ax)^t B(Ax)$$

$$= \sum_{1 \leq i,j \leq n} b_{ij}\omega_i \omega_j$$

$$= \sum_{i<j} (1 + \lambda)b_{ij}\omega_i \omega_j.$$ 

Therefore
\begin{equation}
\bigwedge^n \Omega = (1 + \lambda)^n \Pf_q(B) \omega_1 \wedge \cdots \wedge \omega_{2n} \\
= (1 + \lambda)^n \Pf_q(B) \det_q(A) x_1 \wedge \cdots \wedge x_{2n}
\end{equation}

Subsequently we have proved that
\[ \Pf_q(C) = \det_q(A) \Pf_q(B). \]

The following column analog is clear.

**Remark 4.4.** Let \( B \) be any matrix with entries \( b_{ij}, 1 \leq i, j \leq 2n \) commuting with \( a_{ij} \) and \( b_{ji} = -q_{ij} b_{ij}, i < j \). Let \( C = ABA^t \). Then \( c_{ji} = -q_{ij} c_{ij}, i < j \) and \( \Pf_p(C) = \det_q(A) \Pf_p(B) \).

### 5. Multiparameter quantum hyper-Pfaffians

We now generalize the notion of the quantum multiparameter Pfaffian to the quantum hyper-Pfaffian. A hypermatrix \( A = (A_{i_1 \cdots i_n}) \) is an array of entries indexed by several indices, while a matrix is indexed by two indices.

**Definition 5.1.** Let \( B \) be a hypermatrix with noncommutative entries \( b_{i_1 \cdots i_m}, 1 \leq i_k \leq mn, k = 1, \ldots, m \). Multiparameter quantum hyper-Pfaffian is defined by

\[ \Pf_q(B) = \sum_{\sigma \in \Pi} (-q)_{\sigma} b_{\sigma(1) \cdots \sigma(m)} \cdots b_{\sigma(m(n-1)+1) \cdots \sigma(mn)}, \]

Here \( \Pi \) is the set of permutations \( \sigma \) of \( mn \) such that \( \sigma((k-1)m + 1) < \sigma((k-1)m + 2) < \sigma(km), k = 1, \ldots, n \).

Note that the multiparameter Pfaffian uses only the entries \( b_{i_1 \cdots i_m} \), where \( i_1 < \cdots < i_m \).

Similar to Proposition 4.2, one has the following result.

**Proposition 5.2.** For any \( 0 \leq t \leq n \),
\begin{equation}
Pf(B) = \sum_I \text{inv}(I, I^c) \Pf(B_I) \Pf(B_{I^c}),
\end{equation}

where \( I \) runs through subsets of \([1, mn]\) such that \( |I| = mt \).

**Proof.** Let \( \Omega = \sum_{i_1 < \cdots < i_m} b_{i_1 \cdots i_m} x_{i_1} \wedge \cdots \wedge x_{i_m} \), then one has that
\begin{equation}
\bigwedge^n \Omega = \Pf_q(B) x_1 \wedge \cdots \wedge x_{2n}.
\end{equation}

and
\begin{equation}
\bigwedge^n \Omega = \Omega^t \bigwedge^{n-t} = \sum_{I, J} \Pf(B_I) \Pf(B_{I^c}) x_I x_J.
\end{equation}
where as usual we have put \( x_I = x_{i_1} \land \cdots \land x_{i_m} \). Comparing (5.2) and (5.3), one has the statement.

**Theorem 5.3.** Let \( B = (b_{i_1 \cdots i_m})_{1 \leq i,j \leq n} \) be any hypermatrix with noncommutative entries commuting with those of the matrix \( A = (a_{ij}) \). Let

\[
c_I = \sum_J \det_q(A_J^I) b_J,
\]

then \( \text{Pf}_q(C) = \det_q(A) \text{Pf}_q(B) \).

**Proof.** Let \( \delta_i = \sum_{j=1}^{mn} a_{ij} x_j \), and consider the element \( \Omega = \sum c_I x_I \). It is clear that

\[
\Omega^n = \text{Pf}_q(C) x_1 \land \cdots \land x_{2n}.
\]

On the other hand, \( \Omega = \sum \delta_J \delta_J \). Then

\[
\Omega^n = \text{Pf}(B) \delta_1 \land \cdots \land \delta_{2n} = \text{Pf}(B) \det_q(A) x_1 \land \cdots \land x_{2n}.
\]

Comparing (5.4) and (5.5) we conclude that

\[
\text{Pf}_q(C) = \det_q(A) \text{Pf}_q(B).
\]

**Remark 5.4.** The column-analog is also true. In fact, one has the following result. Let \( B = (b_{i_1 \cdots i_m})_{1 \leq i,j \leq n} \) be any hypermatrix with noncommutative entries commuting with those of the matrix \( A = (a_{ij}) \). Let

\[
c_I = \sum_J \det_q(A_J^I) b_J,
\]

then \( \text{Pf}_p(C) = \det_q(A) \text{Pf}_p(B) \).

**Acknowledgments**

The work is supported by National Natural Science Foundation of China (11531004), Fapesp (2015/05927-0) and Humboldt foundation. Jing acknowledges the support of Max-Planck Institute for Mathematics in the Sciences, Leipzig. Both authors also thank South China University of Technology for support during the work.

**References**

[1] Michael Artin, William Schelter, John Tate, *Quantum deformations of \( GL_n \)*, Comm. Pure Appl. Math. 44 (1991), 879-895.
[2] Ken A. Brown, Ken R. Goodearl, *Lectures on algebraic quantum groups*, Birkhäuser, 2012.
[3] Israel Gelfand, Sergei Gelfand, Vladimir Retakh, Robert Lee Wilson, *Quasideterminants*, Adv. Math. 193 (2005), 56-141.
[4] Timothy J. Hodges, Thierry Levasseur, Margarita Toro, *Algebraic structure of multiparameter quantum groups*, Adv. Math. 126 (1997), 52-92.
[5] Naihuan Jing, Ming Liu, *R-matrix realization of two-parameter quantum group \( U_{r,s}(\mathfrak{g}_m) \)*, Commun. Math. Stat. 2 (2014), 211-230.
[6] Naihuan Jing, Jian Zhang, *Quantum Pfaffians and hyper-Pfaffians*, Adv. Math. 265 (2014), 336-361.

[7] Naihuan Jing, Jian Zhang, *Quantum permanents and Hafnians via Pfaffians*, Lett. Math. Phys. 106 (2016), 1451-1464.

[8] T. Levasseur, J. T. Stafford, *The quantum coordinate ring of the special linear group*, J. Pure Appl. Algebra 86 (1993), 181-186.

[9] Yu. I. Manin, *Notes on quantum groups and quantum de Rham complexes*, Teoret. Mat. Fiz. 92 (1992), 425–450; English transl. in: Theoret. Math. Phys. 92 (1992), 997-1023.

[10] N. Reshetikhin, *Multiparameter quantum groups and twisted quasitriangular Hopf algebras*, Lett. Math. Phys. 20 (1990), 331-335.

[11] Arne Schirrmacher, *The multiparametric deformation of GL(n) and the covariant differential calculus on the quantum vector space*, Z. Phys. C 50 (1991), 321-327.

[12] A. Sudbery, *Consistent multiparameter quantization of GL(n)*, J. Phys. A 23 (1990), L697-L704.

[13] Mitsuhiro Takeuchi, *Two parameter quantization of GL (summary)*, Proc. Japan Acad. 66, Ser. A, (1990), 112-114.

NJ: Department of Mathematics, Shanghai University, Shanghai 200444, China and Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA
E-mail address: jing@math.ncsu.edu

JZ: Institute of Mathematics and Statistics, University of Sao Paulo, Sao Paulo, Brazil 05315-970
E-mail address: zhang@ime.usp.br