The Sufficient Conditions of The $C^*$-Module $C^*m$ to become a $C$-Subcomodule

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Abstract. Let $R$ be a commutative ring with identity and $M$ be a comodule over $R$-coalgebra $C$. It was already well-known that any $C$-comodule $M$ is a module over dual algebra $C^*$ where $C^*$ is the set of all $R$-module homomorphisms from $C$ to $R$. Furthermore, the category of comodule is a subcategory of the category of $C^*$-module. Hence, any $C$-subcomodule of $M$ is a $C^*$-submodule of $M$, and the conversely is not true. For any non zero element $m$ in $M$, $C^*m$ is a $C^*$-submodule of $M$. In general, $C^*m$ is not to become a $C$-subcomodule of $M$. By using the theory of exact sequences in modules and the theory of categories, we give a condition such that $C^*m$ to be a $C$-subcomodule of $M$.

Keyword: submodule, subcomodule, dual algebra, coalgebra, comodule.

1. Introduction

Let $R$ be a commutative ring with the multiplicative identity, $(C, \Delta, \epsilon)$ be a coassociative and counital $R$-coalgebra. Coalgebras and Comodules over a commutative ring were introduced by [1]. An $R$-module $M$ is called a (right) comodule over $R$-coalgebra $C$ if there exists a (right) coaction $\rho^M: M \to M \otimes_R C$. Using the Sweedler notation [2], for any $m \in M$, we define $\rho^M(m)=\sum m_i \otimes m_i$. Moreover, an $R$-module homomorphism $f: M \to N$ is said to be a $C$-comodule morphism provided satisfies the condition that $\rho^N \circ f = (f \otimes 1_C) \circ \rho^M$ [1]. The all of right $C$-comodule morphism from $M$ to $N$ denoted by $Hom_C(M,N)$ and $Hom_R(M,N)$ is the notation of $R$-module homomorphism from $M$ to $N$. The category of $C$-comodules denoted by $M^C$ and the category of $R$-module is $M_R$. Throughout, in this paper $M \in M^C$.

For any $R$-coalgebra $C$, we can construct all of the $R$-module homomorphisms from $M$ to $R$, i.e., $C^*=Hom_R(M,R)$. In [1], we can construct a ring from $C^*$. For any $f,g \in C^*$, define a convolution product between $f$ and $g$, i.e., $f \ast g = (f \otimes g) o \Delta$. Therefore, $(C^*,+,o)$ is an $R$-algebra. In [1], there are many relations between $C$ and $C^*$ as an example $(C^*,+,o)$ is associative and with an identity element if and only if $(C, \Delta, \epsilon)$ is coassociative and counital. We are called the $R$-algebra $(C^*,+,o)$ as an $R$-algebra dual of $C$. 
In this paper, we will discuss some topics related to a comodule over \( C \) and module over dual \( R \)-algebra \( C^* \). It was interesting that any \( M \in \mathbf{M}^C \) can be considered as a module over \( C^* \) by a left action, i.e.,

\[
f \cdot m = (I_M \otimes f) \circ \rho^M(m) = \sum m_2 \otimes m_1 = \sum f(m_1) \in M \text{ (since } f \text{ in } C^* = \text{Hom}_R(C,R))
\]

Furthermore, any element of \( \text{Hom}^C(M,N) \) is a \( C^* \)-module homomorphism. It implies the category of \( \mathbf{M}^C \) is a subcategory of the category of left \( C^* \)-module \( \mathbf{C} \). One of the consequences of the relationship between category \( \mathbf{M}^C \) and \( \mathbf{C} \) i.e. any \( C \)-subcomodule of \( M \) is a submodule of \( M \) as a \( C^* \)-module.

In [1], the sufficient condition that will make the category \( \mathbf{M}^C \) become a full subcategory of \( \mathbf{C} \)-module \( \mathbf{M} \) is the \( \alpha \)-condition of coalgebra \( C \). The \( \alpha \)-condition arises based on a question, when is the \( \mathbf{M}^C \) be a full subcategory of \( \mathbf{C} \)-module or \( \text{Hom}^C(M,N) = \mathbf{C} \text{-Hom}(M,N) \). The definition of \( \alpha \)-condition give as below:

**Definition 1.1** [1] An \( R \)-coalgebra \( C \) is said to satisfy the \( \alpha \)-condition if the map \( \alpha : N \otimes_R C \rightarrow \text{Hom}_C(C^*,N), \alpha_N(n \otimes c) = [f \mapsto f(c) n], \)

is injective, for any \( N \in \mathbf{M}_R \).

Moreover, in [1] we also see the properties related to the \( \alpha \)-condition as the following statements.

**Theorem 1.2** [1] The following conditions are equivalent:
1. \( C \) satisfies the \( \alpha \)-condition;
2. for any \( N \in \mathbf{M}_R \) and \( u \in N \otimes_R C \) if \( (I_N \otimes f)(u) = 0 \) for all \( f \in C^* \), implies \( u = 0 \);
3. \( C \) is locally projective as an \( R \)-module.

The implication of the \( \alpha \)-condition of \( C \) is not only making the category \( \mathbf{M}^C \) be a full subcategory of \( \mathbf{C} \)-module. The readers can find the important theorem related to the condition \( \alpha \) of coalgebra \( C \) as the following theorem.

**Theorem 1.3** [1] The following are equivalent:
(a) \( \mathbf{M}^C = \sigma[C^*C] \);
(b) \( \mathbf{M}^C \) is a full subcategory of \( \mathbf{C} \)-module;
(c) for all \( M,N \in \mathbf{M}^C \), \( \text{Hom}^C(M,N) = \mathbf{C} \text{-Hom}(M,N) \);
(d) \( \sigma C \) is locally projective;
(e) every left \( C^* \)-submodule of \( C \), \( n \in N \), is a submodule of \( C^* \).

In general, any \( C^* \)-module \( M \) is not always to become a \( C \)-comodule. Let \( m \) be a non zero element of \( M \in \mathbf{M}^C \). Consider \( M \) as a \( C^* \)-module, the \( C^* \)-submodule generated by the element \( m \) denoted \( C^*m \). For example, suppose that \( R = \mathbb{Z} \) (the set of integer numbers) and \( M = \mathbb{R} \) (The set or real numbers). Let \( R \) be a \( \mathbb{Z} \)-comodule. If we put \( m = 1 \), we cannot guarantee that \( \rho^R(\mathbb{Z}^*) \) is well defined. Therefore, we need to find a \( C \)-coaction for \( \mathbb{Z}^* \).

In this paper, we want to show the sufficient condition that will make the \( C^* \)-submodule \( C^*m \) be a \( C \)-subcomodule of \( M \) for any non zero element \( m \in M \). On the proof of Theorem 1.4 [1], we have already seen that if \( C \) satisfies the \( \alpha \)-condition, then it implies that any \( C^* \)-submodule of \( M \) is also a \( C \)-subcomodule. As the fact, the \( R \)-coalgebra \( C \) satisfy \( \alpha \)-condition if and only if every \( C^* \)-module of \( M \) is a \( C \)-subcomodule of \( M \). In this paper, we going to find the condition which is smoother then the \( \alpha \)-condition to solve our problem.

Our goal is to find the \( C \)-coaction of \( C^*m \). However, we need to understand the kernel concept in categories theory and we refer to [3] and [4]. For exact sequences in module theory, we use some sources, i.e. [5], [6], and [7]. The result of this paper use to find the necessary and sufficient condition that will make \( C \)-comodule \( M \) be a clean comodule.
2. Main Results

Let \( M \in \mathbf{M} \) and \( m \) be a non zero element of \( M \). As the fact, \( M \in \mathbf{M} \) is a (left) \( C^* \)-module where \( C^* \) is an \( R \)-algebra. Given an \( R \)-module \( C^m = \{ f \mapsto m | f \in C^* \} \). For any \( r \in R \) and \( f \mapsto m \in C^* m \), \( rf \mapsto m \) such that \( C^* m \) is an \( R \)-module. In particular, it is an \( R \)-submodule of \( M \). In [1], an \( R \)-submodule \( K \) of \( M \) is said to be a \( C \)-submodule of \( M \) provided \( K \) has a (right) \( C \)-comodule structure such that the inclusion is a \( C \)-comodule morphism. In this section, we will observe the properties of \( C^* m \) as a submodule of \( M \). For starting our investigation we prove that \( C^* m \) is a submodule of \( M \), on the following lemma we are going to prove that \( C^* m \) is a submodule of \( M \) as a \( C^* \)-module.

Theorem 2.1 Let \((M, \rho^M)\) be a \( C \)-comodule and \( C^* \) be a dual \( R \)-algebra of \( C \). For any non zero element \( m \in M \), the \( R \)-module \( C^* m \) is a \( C^* \)-submodule of \( M \).

Proof.

Given a non zero element \( m \in M \) and \( C^* m = \{ f \mapsto m | f \in C^* \} \). We have some facts below:

1. By definition of scalar multiplication in \( C^* \)-module \( M \), it is obviously that \( C^* m \) is a subset of \( M \).
2. For any \( r \in \mathbb{R} \) and \( f \mapsto m \in C^* m \), \( rf \mapsto m \in M \). By using scalar operation in \( R \)-module \( M \), it is trivial that \( C^* m \) is a submodule of \( R \)-module \( M \).
3. We want to prove that \( C^* m \) is a \( C^* \)-module of \( M \). Given any \( f \mapsto m \), \( h \mapsto m \in C^* m \). We have,
\[
(f \mapsto m) - (h \mapsto m) = (f \circ h) o \rho^M (m) = \sum m_0 g (m_1) - \sum m_0 h (m_1) = (f - h) \mapsto m,
\]

Since \( f \circ h \in C^* \), \((f \mapsto m) - (h \mapsto m) = (f \circ h) \mapsto m \) in \( C^* m \).

Therefore, \( C^* m \) is a \( C^* \)-submodule of \( M \).

As a right \( C \)-comodule, \( M \) is a module over \( R \). Let \( C^* m \) be an \( R \)-submodule of \( M \). It is clear that any \( C \)-subcomodule of \( M \) is a \( C^* \)-submodule of \( M \). However, the conversely is not directly valid. We were already known that \( C^* m \) is a \( C^* \)-submodule of \( M \). In this paper, we give the condition that will be making \( C^* m \) is not only a \( C^* \)-submodule of \( M \) but also can be a \( C \)-subcomodule of \( M \).

Theorem 2.2 Let \((M, \rho^M)\) be a \( C \)-comodule and \( m \in M \) be a non zero element. If
\[
\alpha_{M/C^* m} : M/C^* m \otimes_R C \to \text{Hom}_R(C^*, M/C^* m), \alpha_{M/C^* m} (x \otimes c) = [f \mapsto f(c)x]
\]
is an injective \( R \)-module homomorphism and \( C \) is a flat \( R \)-module, then the set \( C^* m \) is a \( C \)-subcomodule of \( M \).

Proof.

Suppose that \( M \) is a (right) \( C \)-comodule, then \( M \) is a left \( C^* \)-module with scalar multiplication define as
\[
f \mapsto m = (I_M \otimes f) o \rho^M (m) \in M, \text{ for any } f \in C^* \text{ and } m \in M.
\]
Our goal is to prove that $C^*-\text{submodule } C^*m$ is a $C$-subcomodule of $M$ as below: 
Let $N=C^*m$. The inclusion map $i : N \to M$ yields the diagram with exact rows follows as below.

![Diagram](image)

**Figure 1.** Exact Rows and Commutative Diagram between $M$ and $C^*M$

Hence, $p$ is a natural morphism from $M$ to the factor module $M/N$ and $I_C$ is an identity map of $R$-module $C$. In Figure 1 we have some information as below i.e., The first row in Figure 1 is exact [5] and the exactness of the first row implies that $p \otimes I_C$ is surjective, since $p$ is a natural homomorphism and $I_C$ is an identity map.

Define a map

$$\rho^{C^*m} : C^*m \to Hom_R(C^*, C^*m), f \mapsto [g \mapsto g \mapsto (f \mapsto m)],$$

then we have another diagram as follow

![Diagram](image)

**Figure 2.** Diagram $\rho^{C^*m}$

With with $\rho^N = \rho^{C^*m}$, $N=C^*m$ and

$$\alpha_M = M \otimes C \to Hom_R(C^*, M), m \otimes c \mapsto [f \mapsto mf(c)].$$

Therefore, we have some facts as below:

1. Figure 2 is a commutative diagram. For any $f \mapsto m$ in $N = C^*m$,

   $Hom_R(C^*, i) \circ \rho^{C^*m} (f \mapsto m) = Hom_R(C^*, i)(g), \text{ (where } g \mapsto g \mapsto (f \mapsto m))$

   $= i \circ g, \text{ (where } g \mapsto g \mapsto \sum m_2 f(m_1) )$

   $= i \circ g \in Hom_R(C^*, M) \text{ (where } g \mapsto \sum \sum m_{00} g(m_{01}) f(m_1) )$

   and

   $\alpha_M \circ \rho^{C^*m}(f \mapsto m) = \alpha_M \circ \rho^M(f \mapsto m)$

   $= \alpha_M (\sum \rho^M(m_1) f(m_1))$

   $= \alpha_M (\sum (\sum m_{00} \otimes m_{01}) f(m_1) )$

   $= g \in Hom_R(C^*, M), \text{ (where } g \mapsto \sum (\sum m_{00} g(m_{01}) f(m_1))$.

Since $i$ is an inclusion map, by considering the element $C^*m$ as the element of $M$, then the image of $g$ can be equal to the image of $iog$. Hence, we have that
\[ \alpha_{M/C^m} = \text{Hom}(C^*, i) \circ \rho^C = \text{Hom}(C^*, i) \circ \rho_N. \]

In particularly, Figure 2 is a commutative diagram.

2. From Figure 1 and commutativity of Figure 2, we have
\[ \alpha_{M/C^m} o (p \otimes I_c) o \rho^M = \text{Hom}_R(C^*, p) o \text{Hom}_R(C^*, i) o \rho^C. \]

For any \( f \rightarrow m \) in \( C^*m \),
\[ \text{Hom}_R(C^*, p) o \text{Hom}_R(C^*, i) o \rho^C \]
\[ (f \rightarrow m) = \text{Hom}_R(C^*, p) o \text{Hom}_R(C^*, i)(g) \]
\[ = (g o i o p) \]
\[ = go(i o p) \]
\[ = 0. \]

Therefore, \( \alpha_{M/C^m} o (p \otimes I_c) o \rho^M = 0 \). Moreover, the injectivity of \( \alpha_{M/C^m} \) implies that
\( (p \otimes I_c) o \rho^M = 0 \).

3. In Figure 1, since \( C \) is a flat \( R \)-module and the first row is exact, the second row is exact (by exactness of the first row) such that
\[ \text{Ker}(p \otimes I_c) = \text{Im}(I \otimes I_c) = C^*m \otimes_R M \]
or
\[ I \otimes I_c : N \otimes_R C \rightarrow M \otimes_R C \]
is kernel of
\[ p \otimes I_c : M \otimes_R C \rightarrow M/N \otimes_R C. \]

It implies \( (p \otimes I_c) o (I \otimes I_c) = 0 \). By using the definition of kernel on the category theory [4], for a morphism
\[ \rho^M : C^*m \rightarrow M \rightarrow M \otimes_R C \]
with \( (p \otimes I_c) o \rho^M = 0 \),

there exist a unique a map (as a \( C \)-comodule morphism), i.e.,
\[ \rho^C : C^*m \rightarrow C^*m \otimes_R C. \]

The \( C \)-comodule morphism \( \rho^C \) will be the \( C \)-coaction of \( C^*m \). In particularly, \( C^*m \) is the right \( C \)-comodule of \( M \).

4. For the end, we need to prove that the inclusion map \( i \) is a \( C \)-comodule morphism from \( C^*m \) to \( M \), i.e., \( \rho^M o i = (f \otimes I_c) o \rho^C \). Let see figure 1. By using commutativity of Figure 1 (see the first and second rows), it is obviously that the inclusion map \( i \) is a comodule morphism. It means \( C^*m \) is a \( C \)-subcomodule of \( M \).

The Theorem 2.2 explained that by adding flatness property of \( C \) as an \( R \)-module, \( C^*m \) is going to be \( C \)-subcomodule of \( M \) provided \( \alpha_{M/C^m} \) is a monomorphism. Theorem 2.1 is really important for the research of clean comodules. We use this property for finding necessary and sufficient conditions of clean comodules.

3. Conclusions

Let \( M \) be a comodule over \( R \)-coalgebra \( C \). For any non zero element \( m \) in \( M \), based on the main result, we have the important conclusion to make the \( C^*m \)-submodule \( C^*m \) become a subcomodule of \( M \) over \( C \). The sufficient condition that will make \( C^*m \) be a subcomodule of the \( C \)-comodule \( M \) is \( C \) need to be a flat \( R \)-module and the map \( \alpha_{M/C^m} : M/C^*m \otimes_R C \rightarrow \text{Hom}_R(C^*, M/C^*m) \) is injective. On the next project, we will use the result on this paper for finding the necessary and sufficient conditions of clean comodules.

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