Algebraic Cobordism as a module over the Lazard ring

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Abstract

In this paper we study the structure of the Algebraic Cobordism ring of a variety as a module over the Lazard ring, and show that it has relations in positive codimensions. We actually prove the stronger graded version. This extends the result of M.Levine-F.Morel [7] claiming that this module has generators in non-negative codimensions. As an application we compute the Algebraic Cobordism ring of a curve. The main tool is Symmetric Operations in Algebraic Cobordism - [12].

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1 Introduction

The Algebraic Cobordism theory of Levine-Morel Ω∗ - [7] provides the universal oriented cohomology theory in the algebro-geometric context. It is much larger than CH∗ and K0, and contains these classical theories as little faces. Since rationally any formal group law, including the universal one, is isomorphic to an additive FGL, one obtains that Ω∗ ⊗Z Q splits into the direct sum of copies of CH∗ ⊗Z Q, and for the respective graded (by codimension of support) theory we have a natural identification:

\[ GrΩ∗ ⊗Z Q = CH∗ ⊗Z L ⊗Z Q. \]

So, all the relations in GrΩ∗ (modulo the relations in Chow groups) are torsion. But integrally, the structure of Algebraic Cobordism is much more complex, and little is known about it. The number of types of non-cellular varieties for which Ω∗(X) was computed is close to zero. In this article we study the general structure of Ω∗(X) as a module over the Lazard ring. Namely, the relations of this module. It appears that these relations are all concentrated in

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positive codimensions - Theorem 4.3. This fits nicely with the result of M.Levine-F.Morel claiming that the generators of this module are in non-negative codimensions. As an application we compute the Algebraic Cobordism ring of a curve - Theorem 4.4. We also make a guess how the syzygies of our L-module should behave - see Conjecture 4.5 and the discussion after it.

Our main result follows from the stronger graded version. The main tool here is the use of the Symmetric operations - see [12]. These unstable operations are more subtle than the Landweber-Novikov operations and permit to work effectively with the torsion effects, which the current article should demonstrate.

2 BP-theory

Algebraic Cobordism theory combines effects related to all prime numbers. It is convenient to separate the information concerning particular prime $p$ in order to simplify the situation. In Topology this is achieved with the help of the Brown-Peterson theory $BP^*$ constructed in [2]. It is obtained from the localized at $p$ complex-oriented cobordism $MU^*_\mathbb{Z}(p)$ by an explicit multiplicative projector produced out of Adams operations (see, for example, [1] and [16, I.3]). Exactly the same projector can be applied in the algebro-geometric context to $\Omega^*_\mathbb{Z}(p)$ producing the algebraic variant of $BP^*$ which we still call by the same name. Then, as in Topology - see [8, Theorem 4], $\Omega^*_\mathbb{Z}(p)$ additively splits into a direct sum of copies of $BP^*$. It follows from [11, Proposition 4.11] that $BP^*$ can be obtained from Algebraic Cobordism $\Omega^*$ by change of coefficients: $BP^*(X) = \Omega^*(X) \otimes L BP$. The coefficient ring $BP = BP^*(\text{Spec}(k))$ will be the same as for the topological counterpart (notice, that for odd $p$, it will have only even-dimensional elements).

We have $BP = \mathbb{Z}(p)[P_1, P_2, \ldots]$, where $P_1$ is a $\nu_1$-element. In particular, $\text{dim}(P_1) = p^1 - 1$. We can also set $P_0 = p$. These generators can be chosen as the coefficients of the formal $p$ (see [16, Lemma 3.17]):

$$[p] = \frac{p \cdot BP^*}{t} \equiv \sum_{l \geq 0} P_l t^{p^l - 1} \pmod{I(p)^2}.$$

It is convenient to introduce the notation: $[p]_{\leq i} := \sum_{l=0}^i P_l t^{p^l - 1} \in BP[t]$.

3 Symmetric and Steenrod Operations

Our principal method is based on the use of Symmetric operations in algebraic cobordism. These operations are related to Steenrod operations of Quillen’s type there. Let me start with the latter (where we need a $\mathbb{Z}(p)$-local version only).

Let $p$ be a prime. Choose representatives $\overline{i} = \{i_1, \ldots, i_{p-1}\}$ of all non-zero cosets ($mod$ $p$), and denote as $i$ their product. Then we have the multiplicative Total Steenrod operation:

$$\Omega_{\mathbb{Z}(p)} \xrightarrow{St(\overline{i})} \Omega_{\mathbb{Z}(p)} [[t]][t^{-1}],$$
with the inverse Todd genus \( \gamma_{St}(x) = x \prod_{i=1}^{p-1} (x + \Omega (i) \Omega t) \) - see [12, 6.4]. This is just some specialization of the Total Landweber-Novikov operation - [7, Example 4.1.25]. Below the choice of the coset representatives will not be important, so I will omit it from the notations.

Denote as \( \square^p \) the \( p \)-th power operation. Then it can be shown - see [12] that the non-positive degree (in \( t \)) part of the operation \( (\square^p - St) \) is divisible by formal \( [p] \). Moreover, it was shown in [12, Theorem 7.1] that one can divide canonically and obtain the Total Symmetric operation for a given prime:

\[
\Omega_{\mathbb{Z}(p)} \xrightarrow{\Phi} \Omega_{\mathbb{Z}(p)}[t^{-1}].
\]

Such an operation controls all \( p \)-primary divisible numbers of characteristic numbers, and this is exactly what we are exploiting. Operation \( \Phi \) is not multiplicative, but is almost additive - it becomes additive if we cut off the component of degree zero.

The above operations can be extended from \( \Omega^* \) to any theory obtained from it by a multiplicative projector. In particular, to the theory \( BP^* \). Considering the composition

\[
BP^* \hookrightarrow \Omega^*_{\mathbb{Z}(p)} \xrightarrow{St} \Omega^*_{\mathbb{Z}(p)}[t][t^{-1}] \rightarrow BP^*[t][t^{-1}]
\]

we get the multiplicative operation on \( BP^* \)-theory which we still denote as \( St \). Similarly, we get a Total Symmetric Operation \( BP^* \xrightarrow{\Phi} BP^*[t^{-1}] \).

**Proposition 3.1** Let \( \prod_{i=1}^{m} P_{ki} \) be some monomial of dimension \( d = \sum_{i=1}^{m} (p^{ki} - 1) \). Then

\[
St(\prod_{i=1}^{m} P_{ki}) \equiv t^{-pd} \cdot \prod_{i=1}^{m} [p]_{\leq ki} \pmod{I(p)^{m+1}}.
\]

**Proof:** Since \( St \) is multiplicative, it is sufficient to prove the case \( m = 1 \). Recall that \( \gamma_{St}(x) = x \cdot \prod_{j=1}^{p-1} (x + BP i_j \cdot BP t) \). Then \( St(P_k) = \pi_{s}(\gamma_{St}(x)(-T_{P_k} - 1 P_k)) \). Since we are working ( \( mod I(p)^2 \)), the choice of the coset representatives \( i_j \) will not be important for us. Moreover, since \( St \) can be expressed in terms of the Landweber-Novikov operations where coefficients depend on the power series \( \gamma_{St} \), \( P_k \in I(p) \), and the latter ideal is stable under the Landweber-Novikov operations, any element from \( I(p) \) (in particular, any element of positive dimension) from the above formal sums and products will give terms from \( I(p)^2 \).

So, we can assume that FGL we use in \( \gamma_{St} \) is just additive ( \( mod p \)). In other words, that \( \gamma(x) = x^p - tx^{p-1} \). Then from the equation

\[
St(p \cdot BP \gamma(x)) = \gamma(p \cdot BP x)
\]

(which reflects the fact that \( St \) is multiplicative, and so, defines a morphism of FGL’s), we obtain:

\[
St\left( \sum_{l \geq 0} P_l (x^p - tx^{p-1})^l \right) \equiv \left( \sum_{l \geq 0} P_l x^l \right)^p - \left( \sum_{l \geq 0} P_l x^l \right) t^{p-1} \pmod{I(p)^2}.
\]

This immediately gives:

\[
St(P_k) \equiv P_k t^{(p^{ki} - 1) - p^{(ki - 1)}} + P_{k-1} t^{(p^{ki - 1} - 1) - p(p^{ki - 1})} + \ldots + P_0 t^{p(p^{ki - 1})} \equiv t^{-p^{(ki - 1)}} \cdot [p]_{\leq k} \pmod{I(p)^2}.
\]
In particular, modulo $I(p)^{m+1}$, $St(I(p)^m)$ is concentrated in non-positive degrees of $t$.

**Corollary 3.2** The $t^{-d(p-1)}$-component of $St$ is the identity map on $I(p)^m/I(p)^{m+1}$.

**Proposition 3.3**

$$\Phi(I(p)^{m+1}) \subset I(p)^m[t^{-1}].$$

**Proof:** Since $\Phi(x + y) = \Phi(x) + \Phi(y) + f_p(x, y)$, where $f_p(x, y) = \sum_{i=1}^{p-1} \frac{(\binom{p}{i})}{p^i} x^i y^{p-i}$ is a polynomial in $x$ and $y$ of degree $p$ - see [12, Proposition 7.12], it is enough to consider the case of additive generators of $I(p)^{m+1}$ - the monomials $\prod_{i=1}^{m+1} P_{k_i}$.

Since $[p] \cdot \Phi(x) = (x^p - St(x))$, from Proposition 3.1 we get that $[p] \cdot \Phi(\prod_{i=1}^{m+1} P_{k_i}) \in I(p)^{m+1}[[t]]t^{-1}$. Since $P_i$ are algebraically independent variables in $BP$, we obtain by an increasing induction on the power of $t$ that all coefficients of $\Phi(\prod_{i=1}^{m+1} P_{k_i})$ belong to $I(p)^m$. □

4 Graded Algebraic Cobordism

On $\Omega^*$, as well as on any other theory, we have the filtration by codimension of support, which is respected by all cohomological operations. Thus, such operations also act on the respective graded theory $Gr\Omega^*$ (for non-additive operation need to consider a ”discrete derivative” here - see [13]). Moreover, this action is substantially simpler, which is exploited below.

**Theorem 4.1** Let $X$ be a smooth quasi-projective variety. Then $Gr\Omega^*(X)$ as a module over $\mathbb{L}$ has relations in positive codimensions.

**Proof:**

It is sufficient to prove this statement $\otimes \mathbb{Z}(p)$, for all primes $p$. Then $\Omega^*_{\mathbb{Z}(p)}$ and $Gr\Omega^*_{\mathbb{Z}(p)}$ split into direct sum of copies of $BP^*$ and $GrBP^*$. So, it is sufficient to prove the statement for $GrBP^*$.

The problem splits into components corresponding to various codimensions, where the component of codimension $r$ can be covered as: $CH^r(X) \otimes \mathbb{Z} BP \xrightarrow{i^r} BP^r(X)$ (here we already imposed some relations of codimension $r$, as $CH^r(X)$ is not free abelian, in general). Can clearly assume that $r$ is positive (since $BP^*$ is a ”constant” theory, and so $BP \cdot 1_X$ is a direct summand).

We have an action of Steenrod and Symmetric operations on $BP^r(X)$ - see [12, Proposition 7.14], given by: for $z \in CH^r(X)$, and $u \in BP_d$,

$$St(z \cdot u) = z \cdot i^r \cdot t^{r(p-1)} \cdot St(u); \quad \Phi(z \cdot u) = z \cdot i^r \cdot t^{r(p-1)} \cdot \Phi(u)_{\leq -r(p-1)},$$

where $\Phi(u)_{\leq -r(p-1)}$ is the $t^{\leq -r(p-1)}$-part of $\Phi(u)$.

Let $\alpha = \sum_{j \in J} z_j \otimes u_j$ be an element of non-positive codimension $c = (r - d)$ such that $\sum_{j \in J} z_j \cdot u_j = 0 \in BP^r(X)$. Let us say that $\alpha$ is supported in $\{z_j, j \in J\}$. We want to show
that our element belongs to the $BP$-submodule $M_{>c}$ generated by similar relations of larger codimension.

We will prove by the increasing induction on $m$ that this is true modulo $I(p)^m$. The case $m = 0$ is evident (and so is $m = 1$, by dimensional considerations). Suppose that all $u_j$’s belong to $I(p)^m$. Denote as $\Phi(\alpha)$ the expression

$$
\sum_{j \in J} z_j \otimes t^r \cdot t^{r(p-1)} \cdot \Phi(u_j)_{< -r(p-1)} \in \text{CH}^r(X) \otimes \mathbb{Z} BP[t^{-1}].
$$

It is a "relation" (becomes zero if we substitute $\otimes$ by $\cdot$, because $\rho(\Phi(\alpha)) = \Phi(\rho(\alpha)) = 0$). Moreover, it belongs to $\text{CH}^r(X) \otimes I(p)^m[t^{-1}]$ by the Proposition 3.3. We have:

$$(\square^p - St - [p] \cdot \Phi) : BP^* \rightarrow BP^*[t][t].$$

Since the $p$-th power $\square^p$ is zero on $BP^*_{(r)}$, for $r > 0$, we obtain that the $t$-non-positive components of $St$ and $-[p] \cdot \Phi$ coincide. But by Corollary 3.2, the $t^{(r-d)(p-1)}$-component of $St(\alpha)$ is congruent to $\pm \alpha$ modulo $I(p)^{m+1}$ (notice that $i \equiv -1 \pmod{p}$). Since $(r-d)(p-1) \leq 0$, this component can be written as a sum of various components of $-\Phi(\alpha)$ (which are all "relations") multiplied by various $I_l$, $l \geq 0$. Notice, that all our elements are still supported in $\{z_j, j \in J\}$. Separating $P_0$ from $P_l$, $l \geq 0$, we get: $\alpha \equiv p\alpha_1 + \beta_1$ $(\mod I(p)^{m+1})$, where $\alpha_1$ and $\beta_1$ are relations supported in $\{z_j, j \in J\}$, where $\beta_1 \in M_{>c}$ and the coefficients of $\beta_1$ are in $I(p)^m$. Repeating this arguments for $\alpha_1, \alpha_2, \ldots$, we get that $\alpha \equiv \beta$ $(\mod I(p)^{m+1})$, where $\beta \in M_{>c}$ has coefficients in $I(p)^m$ (and the same support). Applying this inductively, and using the fact that $\dim(P_i) > 0$, for $i > 0$, we obtain that for arbitrary codimension $c \leq 0$ relation $\alpha$ with support $\{z_j; j \in J\}$, and for arbitrary $n$, there exists $\beta \in M_{>c}$ with the same support, such that $(\alpha - \beta) = p^n \cdot \gamma$ for some relation $\gamma$ with mentioned support.

Consider $N = \bigoplus_{j \in J} z_j \cdot BP_d$ - the free $\mathbb{Z}(p)$-module of finite rank. Let $K$ be the submodule of relations, and $L$ be the submodule consisting of elements from $M_{>c}$. Then $K \supset L$, and as we saw above, $K/L$ is infinitely divisible by $p$. Hence, $K = L$, and so, any relation of non-positive codimension can be expressed through relations of larger codimension. Theorem is proven.

\[\square\]

**Remark 4.2** Notice, that without the use of the non-additive 0-th Symmetric operation $\Phi^{(0)}$ we would be able only to prove that relations are concentrated in non-negative codimensions.

**Theorem 4.3** Let $X$ be a smooth quasi-projective variety. Then $\Omega^* (X)$ as a module over $\mathbb{L}$ has relations in positive codimensions.

**Proof:** The statement follows from the graded case by induction on the dimension of support, using the fact that any relation in the graded $BP_{(r)}$ can be lifted to one in $BP$ (proven, again, by a similar induction).

This extends the result of M.Levine and F.Morel - [7] Theorem 1.2.19] claiming that the generators of $\Omega^* (X)$ are in non-negative codimensions.

Applying the above Theorem to the case of a curve, we obtain:
Theorem 4.4 Let $C$ be a smooth curve over $k$. Then:

$$\Omega^*(C) = \mathbb{L} \cdot 1 \oplus \text{CH}_0(C) \otimes \mathbb{Z} \mathbb{L}$$

Notice, that this result can be obtained also with the help of the algebro-geometric version of Atiyah-Hirzerbruch spectral sequence (introduced in [3] and [4]) using arguments similar to that of [6]. At the same time, the author does not see how to prove Theorem 4.3 using this method.

On the base of the small knowledge we possess about $\Omega^*$, I would make the following guess about syzygies of our module:

Conjecture 4.5 Let $X$ be a smooth variety of dimension $d$. Then $\Omega^*(X)$ as a module over $\mathbb{L}$ has a free resolution whose $j$-th term has generators concentrated in codimensions between $j$ and $d$. In particular, the cohomological dimension of the $\mathbb{L}$-module $\Omega^*(X)$ is $\leq d$.

Aside from the case of a curve, one of the few cases where the ring $\Omega^*(X)$ was computed is that of a Pfister quadric $Q_\alpha$, where $\alpha = \{a_1, \ldots, a_n\}$ is a pure symbol in $K_n^M(k)/2$. The cobordism motive $M^{\Omega}(Q_\alpha)$ splits as a direct sum of (cobordism) Rost-motives - see [14]:

$$M^{\Omega}(Q_\alpha) = \bigoplus_{i=0}^{2n-1} M^{\Omega}_\alpha(i)[2i].$$

The Rost-motive $M^{\Omega}_\alpha$ can be realized as a motive of affine $(2n-1)$-dimensional quadric $A = P \setminus Q_\beta$, where $q_\beta = \langle a_1, \ldots, a_{n-1} \rangle$ is the $(n-1)$-fold Pfister form, and $p = q_\beta \perp \langle -a_n \rangle$ is the "small" Pfister form. Over algebraic closure $\overline{k}$ the Rost-motive splits into the sum of two Tate-motives: $M^{\Omega}_\alpha|_{\overline{k}} = \mathbb{L} \oplus \mathbb{L}(2n-1)[2n-2]$.

It was shown in [14] Theorem 3.5 that the restriction to the algebraic closure homomorphism is injective on $\Omega^*(M^{\Omega}_\alpha)$ and

$$\Omega^*(M^{\Omega}_\alpha) = \mathbb{L} \cdot e^0 \oplus I(2, n-2) \cdot e_0,$$

where $e^0$ is the generic cycle and $e_0$ is the class of a rational point on $A|_{\overline{k}}$, and $I(2, n-2) \subset \mathbb{L}$ is the ideal generated by the classes of the varieties of dimension $\leq 2n-2-1$ whose all characteristic numbers are even.

The ideal $I(2, n-2)$ is generated by $2 = v_0, v_1, \ldots, v_{n-2}$, where $\dim(v_i) = 2^i - 1$. These $v_i$’s correspond to the classes $P_i$ in the $BP$-theory. Since $v_i$ form a regular sequence in $\mathbb{L}$, the Koszul complex $R_s = \Lambda_{\mathbb{L}}^{-1} (v_0, \ldots, v_{n-2})$ (where $\Lambda^s$ is the positive degree part of the external algebra $\Lambda^s$) with the natural differential is the free $\mathbb{L}$-resolution of $I(2, n-2)$.

For $I \subset \{0, \ldots, n-2\}$, the codimension of the generator $(\Lambda_{i \in I} v_i) \cdot e_0$ of our resolution will be $(2n-1) - \sum_{i \in I} (2^i - 1)$. Notice, that this number belongs to the range between $|I|$ and $(2n-1)$. Moreover, for the highest generator of the external algebra $(\Lambda_{i \in \{0, \ldots, n-2\}} v_i) \cdot e_0$ the codimension will be exactly $(n-2)$. And the respective group $\text{Tor}^{\mathbb{L}}_{n-2}(\Omega^*(M^{\Omega}_\alpha), \mathbb{Z}/2)$ is really non-trivial. This shows that the estimate in the Conjecture above can not be improved.

Since the (cobordism) motives of the Pfister qudrics, and more generally, excellent qudrics (see [5], or [10] Introduction] for the definition) are direct sums of Rost-motives (see [9] Proposition 2.4] and [14] Corollary 2.8)) we easily get:
Proposition 4.6  The Conjecture 4.5 is true for excellent quadrics. In particular, for Pfister quadrics.

The same can be seen for the generalized Rost motive for odd primes, at least $p$-adically, from the computations of N.Yagita - see [15].

This provides a tiny bit of evidence in support of the Conjecture.

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