Discrete numerical solution for modelling of Phytoplankton growth

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Abstract. Phytoplankton growth model has been observed extensively to track the movement of elements through aquatic food webs and ecological processes. This study is purposed to find numerical solution of the modelling of phytoplankton growth and know the dynamic behavior. The method used to transform the phytoplankton growth model is Finite Difference Euler Method. We focused on the existence and stability of the fixed-points. We break into two cases. The result is that all cases is dynamically consistent with its continous model only for relatively small-step size. We present some numerical simulation to illustrate those cases.

1. Introduction

Ecology can be describe as scientific study of how organism interact with other and their environment. Aquatic ecology, in other hand, includes the study of those relationship in all aquatic environments, including rivers, lakes, and oceans. One focus on studying aquatic ecology is phytoplankton growth, since they supply food to their environment [1–3]. They act as water indicator that show whether the water quality is good or poor [4]. They also have ability to respond to enviromental changes. It means that the quantity of phytoplankton in an aquatic environment plays important thing.

Many research has been done recently to identify the dynamics of phytoplankton, especially on blooming algae phenomenon called HABs (Harmful Algae Blooms). HABs occur when colonies of algae grow out of control and produce toxic effects on nearby environment. Nutrient loadings, pollution, water flow modification and climate change play a role to the HABs phenomenon. Mathematical model can be an effective means to discover the dynamics of plankton using conceptual model [5–7].

Phytoplankton growth was introduced as the following system.

\[
\frac{dN}{dt} = \text{input} - \text{uptake} - \text{loss} \\
\frac{dP}{dt} = \text{uptake} - (\text{death} + \text{sinking})
\]  

(1)
Previous research discuss about the generic mathematical model of phytoplankton blooms [8–10]. With a simple model analysis, it is known that bloom is influenced by nutrient concentration [11–15]. These studies are modeled by using ordinary differential equations (ODE). Dynamic behavior can be seen from each of the models.

Furthermore, our purpose is focused on using the euler method for discretize the continous model to find its numerical solution. In this method, the step-size $h$ plays an important role [16,17]. This method have been applied to various models, such as [18–20]. Furthermore, analyses and numerical simulations can obtain insights into the mechanism of phytoplankton growth, where modifications to the equations via simulations to illustrate our theoretical results. In the following section, we will discuss whether this method will dynamically consistent with its continous model or not.

2. Methods

2.1. A simple phytoplankton growth

The phytoplankton growth model that we used in this paper is written as follow.

\[
\frac{dN}{dt} = I - NP - qN \\
\frac{dP}{dt} = NP - P
\]  

(2)

where $N$ denote nutrient supply in such aquatic environment, $P$ denote phytoplankton biomass, $I$ and $q$ are parameters. We assume that the concentration of the nutrient is given in mg/m$^3$ of water per day. In this paper, we break the parameters into two cases. The first case is $I = q = 0$ which means that there is no nutrient supply and nutrient loss over (2), while the second case is $I > 0$ such that there is nutrient loss $q \geq 0$.

2.2. Euler method

Euler method state that if we have system of differential equation

\[
\frac{dx}{dt} = f(x, y) \\
\frac{dy}{dt} = g(x, y)
\]  

(3)

then the numerical system of (3) can be written as

\[
x_{n+1} = x_n + h f(x_n, y_n) \\
y_{n+1} = y_n + h g(x_n, y_n)
\]

where $h$ is the step-size.

Substituting (2) to (3) for the first equation with step-size $h$, we have

\[
\frac{N_{n+1} - N_n}{h} = I - N_n P_n - qN_n, \\
N_{n+1} - N_n = (I - N_n P_n - qN_n) \\
N_{n+1} = N_n + (I - N_n P_n - qN_n)h.
\]

(4)

Doing the same for the second equation, we have

\[
\frac{P_{n+1} - P_n}{h} = N_n P_n - P_n, \\
P_{n+1} - P_n = (N_n P_n - P_n)h, \\
P_{n+1} = P_n + (N_n P_n - P_n)h.
\]

(5)
Combining (4) and (5), the discrete model of (2) now can be written as

\begin{align*}
N_{n+1} &= N_n + (I - N_n P_n - q N_n) h, \\
P_{n+1} &= P_n + (N_n P_n - P_n) h
\end{align*}  \tag{6}

3. Results and discussion

3.1. First case \((I = q = 0)\)

If we substituting \(I = q = 0\) to (6) then we have

\begin{align*}
N_{n+1} &= N_n - (N_n P_n) h, \\
P_{n+1} &= P_n + (N_n P_n - P_n) h
\end{align*}  \tag{7}

We know that

\begin{align*}
N^* &= N^* + (N^* P^*) h, \\
0 &= (N^* P^*) h. \tag{8}
\end{align*}

And

\begin{align*}
P^* &= P^* + (N^* P^* - P^*) h, \\
0 &= (N^* P^* - P^*) h. \tag{9}
\end{align*}

Since \(h > 0\), then we have

\begin{align*}
(N^* P^*) &= 0 \tag{10}
\end{align*}

or

\begin{align*}
(N P^* - P^*) &= P^* (N^* - 1) = 0 \tag{11}
\end{align*}

With easy manipulation, one can find that \(N^* - 1 = 0\) or \(P^* = 0\). Substituting \(P^* = 0\) to (10) then we have \(N^* = k\), where \(k\) constant derived from initial condition. Equilibrium for the first case is \(E_1 = (k, 0)\).

3.2. Second case \((I > 0, q \geq 0)\)

For the second case, we have

\begin{align*}
N^* &= N^* + (I - N^* P^* - q N^*) h \\
0 &= (I - N^* P^* - q N^*) h
\end{align*}

since \(h > 0\),

\begin{align*}
N^* P^* &= I - q N^* \tag{12}
\end{align*}

and

\begin{align*}
P^* &= P^* + (N^* P^* - P^*) h, \\
P^* &= P^* + (N^* P^* - P^*) h, \\
0 &= (N^* P^* - P^*) h. 
\end{align*}
since $h > 0$, we have

$$N^*P^* - P^* = 0$$

$$(N^* - 1)P^* = 0$$

(13)

From (13), we can conclude that

$$P^* = 0$$

(14)

and

$$N^* = 1.$$ 

Substituting (15) to (12)

$$N^* = \frac{l}{q},$$

(15)

$$E_2 = (N_1^*, P_1^*) = (\frac{l}{q}, 0).$$

(16)

Substituting (15) to (12)

$$P^* = l - q$$

(17)

$$E_3 = (N_2^*, P_2^*) = (1, I - q).$$

(18)

Equilibrium point for second case are $E_2$ and $E_3$.

Next we will discuss the stability of those equilibrium points we found before. Suppose that

$$F(N, P) = N - (NP)h$$

$$G(N, P) = P + (NP - P)h$$

The Jacobian matrix for the model is

$$J = \begin{pmatrix}
\frac{\partial F}{\partial N} & \frac{\partial F}{\partial P} \\
\frac{\partial G}{\partial N} & \frac{\partial G}{\partial P}
\end{pmatrix}
$$

$$= \begin{pmatrix}
1 - Ph & -Nh \\
Ph & 1 + (N - 1)h
\end{pmatrix}$$

(19)

Substituting $E_1 = (k, 0)$ to (19), we have

$$J = \begin{pmatrix}
1 & -kh \\
0 & 1 + (k - 1)h
\end{pmatrix}$$

Lemma 1

If $h < \frac{2}{1-k}$ and $k < 1$ then Equilibrium point $E_1$ saddle, otherwise it is unstable.
Proof:
Notice that the first eigenvalue of $J$, that is $|\lambda_1| = |1| = 1 \geq 1$. It is clear that $1 - k > 0$. Suppose that $h < \frac{2}{1-k}$. Since $h > 0$, we can write as $0 < h < \frac{2}{1-k}$ that leads to $0 < h (1-k) < 2$, similar to $0 < -h (k-1) < 2$ or $-1 < -1 - h (k-1) < 1$. We then rewritten as $-1 < 1 + h (k-1) < 1$, that is $|\lambda_2| =|1 + h (k-1)| < 1$. For the second case, we only consider in analyzing $E_2$.

Lemma 2
If $h < \min\left(\frac{2}{q}, \frac{2}{1-q}\right)$ then equilibrium point $E_2$ is stable.

Proof:
It can be shown that Jacobian matrix of this case is

$$J = \begin{pmatrix}
1 - hq & -\frac{1}{q} h \\
0 & 1 + \left(\frac{1}{q} - 1\right) h
\end{pmatrix}.$$

Suppose that $h < \frac{2}{q}$. Since $h > 0$, we can write $0 < h < \frac{2}{q}$ or $-2 < -hq < 0$ which leads to $-1 < 1 - hq < 1$, that is $|\lambda_1| < 1$. On the other hand, suppose $0 < h < \frac{2}{1-q}$ or $0 < h\left(1 - \frac{1}{q}\right) < 2$. Subtracting 1 from the inequalities then we have $-1 < -1 + \left(1 - \frac{1}{q}\right) h < 1$ or $-1 < -(1 - \left(1 - \frac{1}{q}\right) h) < 1$. We see that $-1 < 1 - \left(1 - \frac{1}{q}\right) h < 1$ or $-1 < 1 + \left(\frac{1}{q} - 1\right) h < 1$, in other word $|\lambda_2| = |1 + \left(\frac{1}{q} - 1\right) h| < 1$.

4. Numerical solution
Having found the equilibrium points and analyze the stability in some points, now we can simulate them to show their behavior. We use step-size $h = 0.05$ for all cases. For the first case in (7), we use $(2.9, 0.0001)$ as initial condition. Figure 2 and 3 shows the result.

Figure 1. The Plot of $N - P$

Figure 2. The Plot of $N(t) - t$ and $P(t) - t$

Figure 2 shows that the numerical result converge to its equilibrium, that is $E_1 = (0.185, 0)$, note that $1 - k = 1 - 0.185 > 0$. Figure 3 shows that the nutrient decreases as time increases and reach the value $k$. We present the simulation for the second case $E_2$. 
5. Conclusion
We have found the equilibrium points of the model, and then analyze them. We found that these equilibrium points need to met certain criteria to be a stable equilibrium. We also found that choosing step-size $h$ is important to reach the correct solution.

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