Abstract

We prove that the negative generator \( L \) of a semigroup of positive contractions on \( L^\infty \) has bounded \( H^\infty(S_\eta) \)-calculus on BMO(\( \sqrt{L} \)) for any angle \( \eta > \pi/2 \), provided \( L \) satisfies Bakry-Émery’s \( \Gamma^2 \geq 0 \) criterion. Our arguments only rely on the properties of the underlying semigroup and works well in the noncommutative setting. A key ingredient of our argument is a quasi monotone property for the subordinated semigroup \( T_{t,\alpha} = e^{-t L^{\alpha}} \), \( 0 < \alpha < 1 \), that is proved in the first part of this article.

Introduction

Let \( \Delta = -\partial_x^2 \) be the negative Laplacian operator on \( \mathbb{R}^n \). The associated Poisson semigroup of operators \( P_t = e^{-t\sqrt{\Delta}} , t \geq 0 \) has many nice properties that make it a very useful tool in the classical analysis. In particular, the Poisson semigroup has a quasi monotone property that there exist constants \( c_{r,j} \) such that, for any nonnegative function \( f \in L^1(\mathbb{R}^n, \frac{1}{1+|x|^2} \, dx) \),

\[
|t^j \partial_x^j P_t f| \leq c_{r,j} P_{rt} f,
\]

for any \( 0 < r < 1, j = 0,1,2, \ldots \). As a first result of this article, we show that the quasi monotone property \( (1) \) extends to all subordinated semigroups \( T_{t,\alpha} = e^{-t L^{\alpha}} \) for all \( 0 < \alpha < 1 \) if \( L \) generates a semigroup of positive preserving operators on a Banach lattice \( X \). The case of \( 0 < \alpha \leq \frac{1}{2} \) is easy and is previously known because of a precise subordination formula (see e.g. [21, 20]).

Functional calculus is a theory of studying functions of operators. The so-called \( H^\infty \)-calculus is a generalization of the Riesz-Dunford analytic functional calculus and defines \( \Phi(L) \) via a Cauchy-type integral for an (unbounded) sectorial operator \( L \) and a function \( \Phi \) that is bounded and holomorphic in a sector \( S_\eta \) of the complex plane. \( L \) is said to have the bounded \( H^\infty \)-calculus property if the so-defined \( \Phi(L) \) extends to bounded operators on \( X \) and \( \| \Phi(L) \| \leq c \| \Phi \|_\infty \) for all such \( \Phi \)'s. The theory of bounded \( H^\infty \)-calculus has developed rapidly in the last thirty years with many applications and interactions with harmonic

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analysis, Banach space theory, and the theory of evolution equations, starting with A. McIntosh’s seminal work in 1986 ([23], [14], [22], [31]). It is a major task in the study in the bounded $H^\infty$-calculus theory to determine which operators have such a strong property. Cowling, Duong, and Hieber & Prüß ([7, 11, 16]) prove that the infinitesimal generator of a semigroup of positive contractions on $L^p$, $1 < p < \infty$ always has the bounded $H^\infty(S_\eta)$-calculus on $L^p$ for any $\eta > \frac{\pi}{2}$. When the semigroup is symmetric, the angle can be reduced to $\eta > \omega_p = |\frac{\pi}{2} - \frac{\pi}{p}|$ by interpolation. It is not surprising that this result fails for $L^\infty$ in general. One may want to seek a BMO-type space that could be an appropriate alternative for $p = \infty$ case. The main theorem of this article states that the negative generator $L$ of a semigroup of positive contractions on $L^\infty$ always has bounded $H^\infty$-calculus on the space $BMO(\sqrt{L})$, provided $L$ satisfies Bakry-Émery’s $\Gamma^2$ criterion.

The classical BMO norm of a function $f \in L^1(\mathbb{R}^n, \frac{1}{1+|x|^2} dx)$ can be defined as
\[
\|f\|_{BMO(\sqrt{\Delta})} = \sup_{t > 0} \left\| e^{-t\sqrt{\Delta}} |f - e^{-t\sqrt{\Delta}} f|^{\frac{1}{2}} \right\|_{L^\infty}.
\] (2)

BMO spaces associated with semigroup generators have been intensively studied recently (e.g. [12] and the after-works). When a cubic-BMO is available, one can often compare it with the semigroup BMO and they are equivalent in many cases. In this article, we follow the approach from [20, 24] and consider the $BMO(\sqrt{L})$-(semi)norm defined similarly to (2), merely replacing $\Delta$ with the studied semigroup generator $L$. The corresponding space $BMO(\sqrt{L})$ interpolates well with $L^p$-spaces when the semigroup is symmetric Markovian (see Lemma [14]).

Under the assumptions of our main theorem, we also study semigroup-BMO spaces $BMO(L^\alpha)$, $0 < \alpha < 1$ and prove that they are all equivalent. We further prove that the imaginary power $L^{is}$ is bounded on the associated semigroup-BMO space $BMO(L^\alpha)$ with a bound $\lesssim (1 + |s|)^{\frac{1}{2}} \exp(|s|^{\frac{3}{2}})$ (see [71], [72]). This complements Cowling’s $L^p$-estimate (see [7] Corollary 1) and fixes a mistake in [20] (see the Remark in Section 3).

The related topics and estimates on semigroup generators have been studied with geometric/metric assumptions on the underlying measure space. This article is from a functional analysis point of view and tries to obtain a general result by abstract arguments. Cowling and Hieber/Prüß’s method for their $H^\infty$-calculus results on $L^p$ is based on the transference techniques of Coifman and Weiss, which does not work for non-UMD Banach spaces, such as BMO. Our method is to consider the fractional power of the generator to take the advantages of the quasi-monotone property [1]. Our argument works well for the noncommutative case, that is, for $L$ that generates a semigroup of completely positive contractions on a semifinite von Neumann algebra.

We analyze a few examples to illustrate our results at the end of the article. We use $c$ for an absolute constant which may differs from line to line.
1 The complete monotonicity of a difference of exponential power functions

A nonnegative $C^\infty$-function $f(t)$ on $(0, \infty)$ is completely monotone if

$(-1)^k \partial_t^k f(t) \geq 0$

for all $t$. Easy examples are $f(t) = e^{-\lambda t}$ for any $\lambda > 0$. It is well-known that completely monontonicity is preserved by addition, multiplication, and taking pointwise limits. So the Laplace transform of a positive Borel measures on $[0, \infty)$, which is an average of $e^{-\lambda t}$ in $\lambda$, is completely monotone. The Hausdorff-Bernstein-Widder Theorem says that the reverse is also true; namely that a function is completely monotone if and only if it is the Laplace transform of a positive Borel measures on $[0, \infty)$. In particular, $g_s(t) = e^{-st^\alpha}$ is completely monotone and is the Laplace transform of a positive integrable $C^\infty$ function $\phi_{s,\alpha}$ on $(0, \infty)$ for all $s > 0, 0 < \alpha < 1$.

$e^{-st^\alpha} = \int_0^\infty e^{-\lambda t} \phi_{s,\alpha}(\lambda) d\lambda = \int_0^\infty e^{-s^{\frac{1}{\alpha}} \lambda t} \phi_{1,\alpha}(\lambda) d\lambda. \quad (3)$

The function $\phi_{s,\alpha}$ is uniquely determined by the inverse Laplace transform

$\phi_{s,\alpha}(\lambda) = s^{-\frac{\alpha}{\alpha}} \phi_{1,\alpha}(s^{-\frac{1}{\alpha}} \lambda) = \mathcal{L}^{-1}(e^{-sz^\alpha})(\lambda) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{sz} e^{-sz^\alpha} dz, \quad (4)$

for $\sigma > 0, \lambda > 0$. The derivative $\partial_s \phi_{s,\alpha}$ is again an integrable function (see e.g. [32] page 263), and

$-t^\alpha e^{-st^\alpha} = \int_0^\infty e^{-\lambda t} \partial_s \phi_{s,\alpha}(\lambda) d\lambda. \quad (5)$

The properties of $\phi_{s,\alpha}$ are important in the study of the fractional powers of semigroup generators.

The goal of this section is to prove a few pointwise inequalities for $\phi_{s,\alpha}$, which will be used in the next section. For that purpose, we first prove the complete monotonicity of several variants of $e^{-st^\alpha}$.

For $k, n \in \mathbb{N}, 1 \leq k \leq n$, let $a_k^{(n)}$ be the real coefficients in the expansion

$\frac{d^n}{dt^n} e^{-t^\alpha} = (-1)^n \sum_{k=1}^n a_k^{(n)} t^{-n+k\alpha} e^{-t^\alpha}.$

It is easy to see that

$\frac{d^n}{dt^n} e^{-ct^\alpha} = (-1)^n \sum_{k=1}^n c^k a_k^{(n)} t^{-n+k\alpha} e^{-ct^\alpha}.$

We define $a_k^{(n)} = 0$ if $k > n$ or $k \leq 0$. The proof of the following lemma is simple and elementary. We leave it for the reader to verify.
Lemma 1. The $a_k^{(n)}$'s satisfy the relation
\[ a_k^{(n+1)} = (n - k\alpha)a_k^{(n)} + \alpha a_{k-1}^{(n)} \]  \hspace{1cm} (6)
for all $k \in \mathbb{Z}, n \in \mathbb{N}$.

Lemma 2. Let $K_i, i = 1, 2$ be the first integer such that $K_i > \alpha$. Then, for all $j \in \mathbb{Z}, n \in \mathbb{N}$, we have
\[ a_k^{(n)} = (j + 1)a_k^{(n)} \geq 0 \quad \text{if} \quad k \geq K_1 \]  \hspace{1cm} (7)
and
\[ (j + 1)(a_{k+1}^{(n)} - (j + 2)a_{k+2}^{(n)}) \leq a_k^{(n)} - (j + 1)a_{k+1}^{(n)} \quad \text{if} \quad k \geq K_2 \]  \hspace{1cm} (8)

Proof. We only need to prove the case $j \geq 0$. Let $D$ be the right derivative for discrete functions: $Df = f(j + 1) - f(j)$. It is easy to see that the product rule holds $D(jf)(j) = jD_j f(j) + f(j + 1)$. Fix $k \in \mathbb{Z}$. Let
\[ f_n(j) = a_k^{(n)}j! \]  \hspace{1cm} (9)
for $j \geq 0$, where we use the convention that $0! = 1$. By (9), we have
\[ f_{n+1}(j) = (n - (k + j)\alpha)f_n(j) + \alpha j f_n(j - 1), \]
for all $j \geq 1$ and $f_{n+1}(0) = (n - k\alpha)f_n(0) + \alpha a_k^{(n)}$. Taking the discrete derivative on both sides, we get
\[ Df_{n+1}(j) = (n - (k + j)\alpha)Df_n(j) - \alpha f_n(j + 1) + \alpha j Df_n(j + 1) - \alpha f_n(j), \]
for $j \geq 1$ and $Df_{n+1}(0) = (n - (k + 1)\alpha)Df_n(0) - \alpha a_k^{(n)}$. By induction, we get
\[ D^i f_{n+1}(j) = (n - (k + j + i)\alpha)D^i f_n(j) + \alpha_j D^i f_n(j - 1). \]  \hspace{1cm} (11)
for all $i \geq 1, j \geq 1$ and $D^i f_{n+1}(0) = (n - (k + i)\alpha)D^i f_n(0) + (-1)^i \alpha a_k^{(n)}$.

Let $k = K_1$ in (9). Note that the condition $Df_n(j) \leq 0$ trivially holds for $n \leq K_1 + j$ because $a_i^{(j)} = 0$ for $i > j$. In particular, $Df_n(j) \leq 0$ for all $j \geq 0, n = K_1$. We apply induction on $n$. Assume $Df_n(j) \leq 0$ holds for all $j \geq 0$. The equality (10) implies that $Df_{n+1}(j) \leq 0$ for all $j \geq 0$ satisfying $n \geq (K_1 + j + 1)\alpha$, which holds if $n + 1 \geq K_1 + j + 1$ since $\frac{n}{\alpha} \geq \alpha$. On the other hand, if $n + 1 \leq K_1 + j$ we have $Df_{n+1}(j) \leq 0$ trivially. So $Df_{n+1}(j) \leq 0$ for all $j \geq 0$. Therefore, $Df_n(j) \leq 0$ and equivalently (7) holds for all $n \in \mathbb{N}, j \geq 0$.

The argument for (8) is similar. Let $k = K_2$ in (9). Note that $D^2 f_n(j) \geq 0$ is equivalent to (5) for $j \geq 0$, which trivially holds for $n \leq K_2 + j$ since $K_2 \geq K_1$ and $a_k^{(n)}(j + 1)a_{k+1}^{(n)} \geq 0$. In particular, (5) holds for $n = K_2, j \geq 0$. Assume that (5) holds for $n = m, j \geq 0$. We consider the case $n = m + 1$. If $n = m + 1 \leq K_2 + j$, (8) holds trivially. Otherwise, $m + 1 \geq K_2 + j + 1$ and by applying (11) we see that $D^2 f_{n+1} \geq 0$. By induction, (5) holds for all $n \in \mathbb{N}, j \geq 0$. \hfill $\square$
Remark. The argument of the previous lemma shows that $(-1)^j D^j f_n(j) \geq 0$ for all $n \in \mathbb{N}, j \geq 0$ if we choose $k$ so that $\frac{k}{k+1} \leq \alpha$.

For a fixed $K \geq K_1$, let

$$F_n(x) = x^{-K} \sum_{j=0}^{n} a_j^{(n)} x^j = \sum_{j=-\infty}^{\infty} a_{K+j}^{(n)} x^j.$$  \hspace{1cm} (12)

and for a fixed $K \geq K_2$, let

$$G_n(x) = x^{-K} \sum_{j=0}^{n} (a_j^{(n)} - ja_{j+1}^{(n)}) x^j = \sum_{j=-\infty}^{\infty} (a_{K+j}^{(n)} - ja_{K+j+1}^{(n)}) x^j.$$ \hspace{1cm} (13)

Lemma 3. Let $f(x) = F_n(x)$, or $G_n(x)$ for the given suitable $K$. We have $(f(x)e^{-x})' \leq 0$ and $f(x+rx) \leq e^{rx} f(x)$ for all $r, x > 0$.

Proof. It is easy to see that $f(x) - f'(x) \geq 0$ for $x > 0$ by Lemma \[2\] So $(f(x)e^{-x})' = (f' - f)e^{-x} \leq 0$ and hence $f(x+rx) \leq e^{rx} f(x)$ for $r > 0$. \[\square\]

We now come to the main result of this section.

Theorem 1. Let $0 < \alpha, c < 1$, and $s \geq 0$ be fixed. Then

(i) $e^{-ct^n} - e^{K_1 e^{-st^n}}$ is completely monotone in $t$.

(ii) $K_1 e^{-st^n} + st^n e^{-st^n}$ is completely monotone in $t$.

(iii) $\frac{1}{c^n (1-c)} e^{-ct^n} - st^a e^{-st^n}$ is completely monotone in $t$.

(iv) $(\max\{\frac{K_1}{c^n}, \frac{1}{c^n (1-c)}\})^j e^{-ct^n} \pm st^n e^{-st^n}$ are completely monotone in $t$ for any $j \in \mathbb{N}$.

Proof. By dilation, we may assume $s = 1$. We prove (i) first. Let $x = t^n$ and $F_n$ be as in \[12\]

$$\frac{d^n}{dt^n} e^{-t^n} = (-1)^n t^{-n} \sum_{k=1}^{n} a_k^{(n)} x^{k} e^{-x} = (-1)^n t^{-n+K \alpha} e^{-x} F_n(x)$$

and

$$\frac{d^n}{dt^n} c e^{-ct^n} = (-1)^n t^{-n} \sum_{k=1}^{n} c e^{k} a_k^{(n)} x^{k} e^{-x} e^{-rx} = (-1)^n t^{-n+K \alpha} e^{-cK} e^{-cx} F_n(cx).$$ \hspace{1cm} (14)

Applying Lemma \[2\] and Lemma \[3\] to $F_n$ gives us

$$\frac{d^n}{dt^n} e^{-ct^n} \geq cK,$$

for any $K \geq K_1$. This implies (i) since $e^{-ct^n}$ is completely monotone for any $0 < \alpha \leq 1$.\[5\]
We now prove (ii). Let \( g(s, t) = e^{-st^\alpha} s^{-K_1} \). Then \(-\partial_s g(s, t)\), is the limit of the family of functions

\[
\frac{1}{s^{K_1+1}(c-1)} (e^{-st^\alpha} - e^{-K_1 e^{-ct^\alpha}})
\]
as \( c \to 1 \), which are completely monotone in \( t \) by (i). So

\[
K_1 e^{-st^\alpha} + st^\alpha e^{-st^\alpha} = -s^{K_1+1} \partial_s g(s, t)
\]
is completely monotone in \( t \).

For (iii), we denote by \( f^{(n)}(t) = \partial_t^{\alpha} f(t) \) and, for \( K \ge K_2 \ge K_1 \), write

\[
(t^{\alpha} e^{-t^\alpha})^{(n)} + K (e^{-t^\alpha})^{(n)} = \frac{1}{\alpha} [t(e^{-t^\alpha})']^{(n)} + K (e^{-t^\alpha})^{(n)}
\]

\[
= \frac{1}{\alpha} [t(e^{-t^\alpha})^{(n+1)} + n(e^{-t^\alpha})^{(n)}] + K (e^{-t^\alpha})^{(n)}
\]

\[
= (-1)^n t^{-n} \sum_{k=1}^{\infty} (a_k^{(n+1)} - (n - K\alpha)a_k^{(n)}) t^{k\alpha} e^{-t^\alpha}
\]

\[
= (-1)^n t^{-n+K\alpha} \sum_{k=-\infty}^{\infty} (a_{k+1}^{(n)} - ka_k^{(n)}) t^{k\alpha} e^{-t^\alpha}
\]

\[
= (-1)^n t^{-n+K\alpha} e^{-x} G_n(x)
\]

with \( x = t^\alpha \) and \( G_n(x) \) defined as in (13) which depends on \( K \). Lemma 3 says that \( G_n(x)e^{-x} \) decreases in \( x \) if \( K \ge K_2 \) and note that \( G_n(x)e^{-x} = -(F_n(x)e^{-x})' \ge 0 \).

We have

\[
x G_n(x)e^{-x} \le \frac{1}{1-c} \int_{cx}^{x} G_n(s)e^{-s} ds
\]

\[
= \frac{1}{1-c} \int_{cx}^{x} -(F_n(s)e^{-s})' ds
\]

\[
\le \frac{1}{1-c} F_n(cx)e^{-cx},
\]

for \( 0 < c < 1 \). Combing this inequality with (13) and (15) we get

\[
\frac{(-1)^n \frac{d^n}{dt^n}(t^{\alpha} e^{-t^\alpha} + K_2 e^{-t^\alpha})}{(-1)^n \frac{d^n}{dt^n} e^{-t^\alpha}} \le \frac{1}{e^{K_2(1-c)}}.
\]

This proves (iii) since \( e^{-ct^\alpha} \) and \( e^{-t^\alpha} \) are completely monotone.

For (iv), let \( f(t) = \max\{\frac{K_1}{e^{ct^\alpha}}, \frac{1}{e^{ct^\alpha}}\} e^{-ct^\alpha}, g(t) = st^\alpha e^{-st^\alpha} \). By (i), (ii) and (iii) we have that both \( f + g, f - g \) are completely monotone in \( t \). Recall
that complete monotonicity is preserved by multiplication. Note that
\[ f^{j+1} + g^{j+1} = \frac{1}{2} [(f^j - g^j)(f - g) + (f^j + g^j)(f + g)] \]
\[ f^{j+1} - g^{j+1} = \frac{1}{2} [(f^j - g^j)(f + g) + (f^j + g^j)(f - g)]. \]

We get, by induction, that \((\max\{\frac{K_i}{c_{K_i}}, \frac{1}{c_{K_i}(1-c)|\alpha|}\})^j e^{-j\omega t} - s^j \Phi^{j\alpha} e^{-jst\alpha}\) is completely monotone for any \(s > 0\), which implies (iv).

We will apply Theorem 1 to pointwise estimates of \(\phi_s,\alpha(\lambda)\). Let us first list a few basic properties of \(\phi_s,\alpha\).

**Lemma 4.** For any \(s > 0, 0 < \alpha, \beta < 1\), we have
\[ \phi_s^{1/2}(\lambda) = \frac{1}{2\sqrt{\pi}} s e^{-\frac{s^2}{4\lambda}} \lambda^{-\frac{3}{2}}. \] (16)
\[ \phi_{1,\alpha\beta}(\lambda) = \int_0^\infty \phi_{s,\alpha}(\lambda) \phi_{1,\beta}(s) ds. \] (17)
\[ \phi_{s,\alpha}(\lambda) = s^{-\frac{1}{2}} \phi_{1,\alpha}(s^{-\frac{1}{2}} \lambda), \] (18)
\[ -\alpha s \partial_s \phi_{s,\alpha}(\lambda) = \phi_{s,\alpha}(\lambda) + \lambda \partial \lambda \phi_{s,\alpha}(\lambda). \] (19)

**Proof.** (16) is well-known (see e.g. [32], page 268). (17), (18) can be easily seen from (3) and (4). (18) implies (19).

**Corollary 1.** For all \(\lambda, s > 0, 0 < c < 1, j \in \mathbb{N}\), we have
\[ c^{K_i} \phi_{s,\alpha}(\lambda) \leq \phi_{cs,\alpha}(\lambda) \] (20)
\[ 0 \leq \partial \lambda (\lambda^{1+s K_i} \phi_{s,\alpha}(\lambda)), \] (21)
\[ |s^j \partial_s^j \phi_{s,\alpha}(\lambda)| \leq \left( \max \left\{\frac{j K_i}{c K_i}, \frac{1}{c K_i(1-c)|\alpha|}\right\} \right)^j \phi_{cs,\alpha}, \] (22)
\[ |s \partial_s \phi_{s,\alpha}(\lambda)| \leq \left( \frac{10}{1-\alpha} \right)^{\frac{j}{1-\alpha}} \phi_{s,\alpha}(\lambda), \] (23)
\[ |s^j \partial_s^j \phi_{s,\alpha}(\lambda)| \leq \left( \frac{10j}{1-\alpha} \right)^{\frac{j}{1-\alpha}} \phi_{s,\alpha}(\lambda). \] (24)

**Proof.** These are direct consequences of Theorem 1, the identity (3), and the Hausdorff-Bernstein-Widder Theorem because \(K_i \leq \frac{1}{(1-\alpha)^\alpha}\), except that (21) requires a little more calculation. To prove (21), note that (3) and Theorem 1 (ii) imply that
\[ \partial_s \frac{\phi_{s,\alpha}(\lambda)}{s^{K_i}} = -s^{1-K_i}(K_i \phi_{s,\alpha}(\lambda) - s \partial_s \phi_{s,\alpha}(\lambda)) \leq 0. \]
Since \(\phi_{s,\alpha}(\lambda) = s^{-\frac{1}{2}} \phi_{1,\alpha}(s^{-\frac{1}{2}} \lambda)\), we get
\[ -\left( \frac{1}{\alpha} + K_i \right) s^{-\frac{1}{2} - K_i} \phi_{1,\alpha}(s^{-\frac{1}{2}} \lambda) - \frac{1}{\alpha} s^{-\frac{1}{2} - K_i} \phi_{1,\alpha}(s^{-\frac{1}{2}} \lambda) \leq 0. \]
That is
\[(1 + K_1 \alpha) \phi_{1,\alpha}(s^{-\frac{1}{\alpha}} \lambda) + \lambda s^{-\frac{1}{\alpha}} (\partial_\lambda \phi_{1,\alpha})(s^{-\frac{1}{\alpha}} \lambda) \geq 0.\]
Therefore
\[(1 + K_1 \alpha) \phi_{s,\alpha}(\lambda) + \lambda \partial_\lambda \phi_{s,\alpha}(\lambda) \geq 0,
\]since \(\partial_\lambda \phi_{s,\alpha}(\lambda) = s^{-\frac{1}{\alpha}} \partial_\lambda \phi_{s,\alpha}(s^{-\frac{1}{\alpha}} \lambda)\). This is (21).

**Lemma 5.** For any \(s > 0, 0 < \beta < \alpha < 1\), we have that
\[
\int_0^\infty \int_0^\infty \left| \ln \left( s^{-\frac{1}{\alpha}} u \right) \right| \phi_{s,\alpha}(u) du < c \beta.
\] (25)
\[
\int_0^\infty \int_0^\infty \left| \ln \left( \frac{u}{v} \right) \right| \phi_{s,\alpha}(u) \phi_{s,\alpha}(v) dudv < \frac{c}{\beta^2}.
\] (26)

**Proof.** Since \(\phi_{s,\alpha}(u) = s^{-\frac{1}{\alpha}} \phi_{1,\alpha}(s^{-\frac{1}{\alpha}} u)\), the left hand side of (25) is independent of \(s\). We only need to prove the case \(s = 1\). For \(\alpha = \frac{1}{2}\), we can verify directly from (16) that (25) holds. Denote by \(u(\alpha)\) the left hand side of (25). We then get \(u(\frac{1}{2}) < \infty\). Using (17), we get \(u(\frac{1}{2^n}) < \infty\). Now, for \(\alpha > \frac{1}{2^n}\), we use (17) again and get
\[
\phi_{1,\frac{1}{2^n}}(\lambda) = \int_0^\infty \phi_{s,\alpha}(\lambda) \phi_{1,\frac{1}{2^n}}(s) ds
\geq \int_0^1 \phi_{s,\alpha}(\lambda) \phi_{1,\frac{1}{2^n}}(s) ds
\geq \phi_{1,\alpha}(\lambda) \int_0^1 s^{K_1(\alpha)} \phi_{1,\frac{1}{2^n}}(s) ds
\geq c_\alpha \phi_{1,\alpha}(\lambda).
\]

We conclude that \(u(\alpha) < \infty\) for all \(0 < \alpha < 1\). Since \(\phi_{1,\alpha}(\lambda)\) is continuous as a function in \(\alpha\) and this continuity is uniform for \(\lambda \in [\delta, N]\) for any \(0 < \delta < N < \infty\), one can easily see that \(u(\alpha)\) is continuous in \(\alpha\) for \(\alpha \in (0, 1)\). We conclude that \(u(\alpha)\) is bounded on \([\frac{1}{2^n}, \frac{1}{2^{n+1}}]\) for any \(n \in \mathbb{N}\). Note that (17) also implies that
\[
\int_0^\infty \phi_{1,\alpha}(\lambda) |\ln \lambda| d\lambda
= \int_0^1 \int_0^\infty \phi_{s,\alpha}(\lambda) |\ln \lambda| d\lambda \phi_{1,\alpha}(s) ds
= \int_0^\infty \int_0^\infty \phi_{1,\alpha}(v) |\ln (s^{\frac{1}{\alpha}} v)| dv \phi_{1,\alpha}(s) ds
\geq \pm \int_0^\infty \int_0^\infty \phi_{1,\alpha}(v) \left( \frac{1}{\alpha} |\ln s| - |\ln v| \right) dv \phi_{1,\alpha}(s) ds (27)
\leq \int_0^\infty \int_0^\infty \phi_{1,\alpha}(v) \left( \frac{1}{\alpha} |\ln s| + |\ln v| \right) dv \phi_{1,\alpha}(s) ds (28)
\]
Our change in the order of integration is justified because all the terms are positive. Note \( \int_0^\infty \phi_{t,\alpha}(s)\,ds = 1 \) for any \( t, \alpha \).

(27) and (28) imply that

\[
|u(\alpha) - \frac{1}{\alpha}u(\beta)| \leq u(\alpha\beta) \leq u(\alpha) + \frac{1}{\alpha}u(\beta)
\]  

(29)

We then obtain (25). (26) follows from (25).

**Remark (Bell Polynomials).** We define the complete Bell polynomial \( B_n(x_1, \ldots, x_n) \) by its generating function

\[
\exp \left( \sum_{j=1}^\infty \frac{x^j}{j!} u^j \right) = \sum_{n=0}^\infty \frac{B_n(x_1, \ldots, x_n) u^n}{n!}
\]

From this, we get the formula

\[
B_n(x_1, \ldots, x_n) = \frac{d^n}{du^n} \exp \left( \sum_{j=1}^\infty \frac{x^j}{j!} u^j \right) \bigg|_{u=0}
\]

Now, for \( s > 0 \), let

\[
x_j = -s \frac{d^j}{dt^j} t^\alpha = -s(\alpha)_j t^{\alpha-j},
\]

(30)

where \((\alpha)_j\) denotes the falling factorial. Then

\[
\sum_{j=1}^\infty \frac{x^j}{j!} = -st^\alpha \sum_{j=1}^\infty \frac{(\alpha)_j}{j!} \left( \frac{u}{t} \right)^j = st^\alpha - st^\alpha \left( 1 + \frac{u}{t} \right)^\alpha = st^\alpha - s(t + u)^\alpha
\]

Applying Theorem 1 part (i), we see that for all \( n \in \mathbb{N} \), \( c \in (0, 1) \), and \( t > 0 \) it holds that

\[
\frac{d^n}{du^n} e^{-sc(t+u)^\alpha} \bigg|_{u=0} \geq c^{K_1},
\]

where \( K_1 \) is as in Lemma 2. We can rewrite this inequality as

\[
e^{(1-c)st^\alpha} \frac{d^n}{du^n} e^{sc(t+u)^\alpha} \bigg|_{u=0} \geq c^{K_1},
\]

We conclude that if we define \( x_j \) by (30), then

\[
e^{(1-c)st^\alpha} B_n(cx_1, \ldots, cx_n) \geq c^{K_1}
\]

(31)

for all \( n \in \mathbb{N} \), \( c \in (0, 1) \), and \( t > 0 \). All of these calculations are easily reversible, and we conclude that (31) is actually equivalent to part (i) of Theorem 1.
2 Positive semigroups and BMO

Let \((M, \sigma, \mu)\) be a sigma-finite measure space. Let \(L^1(M)\) be the space of all complex valued integrable functions and \(L^\infty(M)\) be the space of all complex valued measurable and essentially bounded functions on \(M\). Denote by \(f^*\) the pointwise complex conjugate of a function \(f\) on \(M\) and by \(\langle f, g \rangle\) the duality bracket \(\int fg^*\).

**Definition 1.** A map \(T\) from \(L^\infty(M)\) to \(L^\infty(M)\) is called *positive* if \(Tf \geq 0\) for \(f \geq 0\). If \(T\) is positive on \(L^\infty(M)\), then \(T \otimes \text{id}\) is positive on matrix valued function spaces \(L^\infty(M) \otimes M_n\) for all \(n \in \mathbb{N}\), i.e. \(T\) is *completely positive*.

A positive map \(T\) commutes with complex conjugation, i.e. \(T(f^*) = T(f)^*\). For two positive maps \(S, T\), we will write \(S \geq T\) if \(S - T\) is positive.

We will need the following Kadison-Schwarz inequality for completely positive maps \(T\),

\[
|T(f)|^2 \leq \|T(1)\|_{L^\infty} T(|f|^2), \quad f \in L^\infty(M). \tag{32}
\]

**2.1 Positive semigroups**

We will consider a semigroup \((T_t)_{t \geq 0}\) of positive, weak*-continuous contractions on \(L^\infty\) with the weak* continuity at \(t = 0^+\). That is a family of positive, weak*-continuous contractions \(T_t, t \geq 0\) on \(L^\infty\) such that \(T_s T_t = T_{s+t}\), \(T_0 = \text{id}\) and \(\langle T_t(f), g \rangle \to \langle f, g \rangle\) as \(t \to 0^+\) for any \(f \in L^\infty\), \(g \in L^1\).

Such a semigroup \((T_y)\) always admits an infinitesimal negative generator \(L = \lim_{y \to 0} \frac{1}{y} (T_y - 1)\) which has a weak*-dense domain \(D(L) \subset L^\infty\). We will write \(T_y = e^{-yL}\). These definitions and facts extend to the noncommutative setting. Namely, given a semifinite von Neumann algebra \(M\) and a normal semifinite faithful trace \(\tau\), we let \(L^\infty(M) = M\) and \(L^1(M)\) be the completion of \(\{f \in M : \|f\|_1 = \tau(f)^{\frac{1}{2}} < \infty\}\) Here \(|g| = (g^*g)^{\frac{1}{2}}\) and \(g^*\) denotes the adjoint operators of \(g\) and we set \(\langle f, g \rangle = \tau(fg^*)\). We say a map \(T\) on \(M\) is completely positive if \((T \otimes \text{id})(f) \geq 0\) for any \(f \geq 0, f \in M \otimes M\). We say \(f_\lambda\) weak* converges to \(f\) if \(\lim_{\lambda} \langle f_\lambda, g \rangle = \langle f, g \rangle\) for all \(g \in L^1(M)\) (see [21] for details).

The so-called subordinated semigroups \(T_{y,\alpha} = e^{-yL^\alpha}, 0 < \alpha < 1\) are defined as

\[
T_{t,\alpha} f = \int_0^\infty T_u f \phi_{t,\alpha}(u) du = \int_0^\infty T_{t+u} f \phi_{1,\alpha}(u) du, \tag{33}
\]

with \(\phi_{t,\alpha}\) given in Section 1. The generator \(L^\alpha\) is given by

\[
L^\alpha(f) = \Gamma(-\alpha)^{-1} \int_0^\infty (T_t - \text{id})(f) t^{-1-\alpha} dt, \tag{34}
\]

for \(f \in D(L)\). There are other (equivalent) formulations for \(L^\alpha\). The formula (34) is due to Balakrishnan (see [5] and [32, page 260]). For \(T_t = e^{-tz}id\) with \(Re(z) \geq 0\), \(L^\alpha = z^\alpha\) with a chosen principal value so that \(Re(z^\alpha) \geq 0\).
$(T_{y,\alpha})$ is again a semigroup of positive weak*-continuous contractions. The semigroup has an analytic extension and has the well-known norm estimate that

$$\sup_{y>0} \| y^k \partial_y^k T_{y,\alpha} \| < c_k.$$  

What we wish is a pointwise estimate.

Note that (33) implies

$$\frac{T_{y,\frac{1}{t}}}{y} (f) \leq \frac{T_{t,\frac{1}{t}}}{t} (f) \quad \text{and} \quad |y^k \partial_y^k T_{y,\frac{1}{t}} f| \leq c_{k,t} T_{t,\frac{1}{t}} f,$$

for any $0 \leq t \leq y$ and $f \geq 0$ because of the positivity of $T_u$ and the precise formulation of $\phi_{y,\frac{1}{t}}$.

Corollary 1 and the identity (33) actually imply the following corollary.

**Corollary 2.** For all $f \geq 0, s > 0, 0 < c, \alpha < 1$, and $j \in \mathbb{N}$, we have

$$c^{K_1} T_{s,\alpha} f \leq T_{c,s,\alpha} f \quad \text{(37)}$$

$$|s^j \partial^j y T_{s,\alpha} (f)| \leq (\frac{10j}{1-\alpha})^{j} T_{a,s,\alpha} (f). \quad \text{(38)}$$

**Remark.** When $\alpha = 1$, a similar estimate to Corollary 2 may hold for some special semigroups. For example, the heat semigroups generated by the Laplacian operator on $\mathbb{R}^n$ has a similar estimate with $c > 1$. But one can not hope this in general since (38) is stronger then the analyticity on $L^\infty$.

### 2.2 $\Gamma^2$ criterion

P. A Meyer’s gradient form $\Gamma$ (also called “Carré du Champ”) associated with $T_t$ is defined as

$$2\Gamma_L (f, g) = -L(f^*g) + (L(f^*)g) + f^*(L(g)),$$

for $f, g$ with $f^*, g, f^*g \in D(L)$. It is easy to verify that for $L = -\Delta = -\frac{d^2}{dx^2}$,

$$\Gamma_L (f, g) = \nabla f^* \cdot \nabla g.$$

**Convention.** We will write $\Gamma(f)$ for $\Gamma_L (f, f)$.

It is well known that the completely positivity of the operators $T_t$ implies that $\Gamma(f, g)$ is a completely positive bilinear form. We then have the Cauchy-Schwartz inequality

$$\Gamma \left( \int_0^\infty a_s d\mu(s), \int_0^\infty a_s d\mu(s) \right) \leq \int_0^\infty d|\mu|(s) \int_0^\infty \Gamma(a_s, a_s) d|\mu|(s) \quad \text{(40)}$$

Bakry-Émery’s $\Gamma^2$ criterion plays an important role in this article. We use an equivalent definition.
Definition 2. A semigroup of positive operator \((T_t)\) satisfies the \(\Gamma^2 \geq 0\) criterion if \(\Phi(s) = T_{s-u}|T_u f|^2, s > u\) is (midpoint) convex in \(u\), i.e.

\[
T_t|T_u f|^2 - |T_t T_u f|^2 \leq T_u(T_t|f|^2 - |T_t f|^2)
\]

for all \(t, u > 0\) and \(f \in L^\infty\).

For \(L\) equal to the Laplace-Beltrami operator on a complete manifold, the \(\Gamma^2 \geq 0\) criterion holds if the manifold has nonnegative Ricci curvature everywhere. The “\(\Gamma^2\)” criterion is satisfied by a large class of semigroups including the heat, Ornstein-Uhlenbeck, Laguerre, and Jacobi semigroups (see [2]), and also by the semigroups of completely positive contractions on group von Neumann algebras. We refer the reader to [3] and references therein for the so-called curvature-dimension criterion which is more general than the “\(\Gamma^2\)” criterion.

D. Bakry usually assumes that there exists a \(*\)-algebra \(A\) which is weak\(^*\) dense in \(L^\infty(M)\) such that \(T_s(A) \subset A \subset D(L)\). This is not needed in this article because we will only use the form \(T_{t,\alpha} \Gamma_{L^\alpha}(T_{s,\alpha} f, T_{s,\alpha} g), 0 < \alpha < 1, \alpha \leq \beta \leq 1\) which is well defined as

\[
-L^\beta T_{t,\alpha}((T_{s,\alpha} f^*)(T_{s,\alpha} g)) + T_{t,\alpha}((T_{s,\alpha} f^*)((L^\beta T_{s,\alpha} g)) + T_{t,\alpha}[(L^\beta T_{s,\alpha} f^*)(T_{s,\alpha} g)]
\]

for all \(f, g \in L^\infty\) since \(T_{s,\alpha}(L^\infty) \subset D(L) \subset D(L^\alpha)\) because of (33).

We will need the following Lemma due to P.A. Meyer. We add a short proof for the convenience of the reader.

Lemma 6. For any \(f \in L^\infty\) such that \(T_s f, T_s f^*, T_s|f|^2 \in D(L)\) for all \(s > 0\), we have

\[
T_s|f|^2 - |T_s f|^2 = 2 \int_0^s T_{s-t} \Gamma(T_t f) dt.
\]

In particular, for \(0 < \alpha < 1\),

\[
T_{s,\alpha}|f|^2 - |T_{s,\alpha} f|^2 = 2 \int_0^s T_{s-t,\alpha} \Gamma_{L^\alpha}(T_t f) dt
\]

for any \(f \in L^\infty\).

Proof. For \(s\) fixed, let

\[
F_t = T_{s-t}(|T_t f|^2).
\]

Then

\[
\frac{\partial F_t}{\partial t} = \frac{\partial T_{s-t}}{\partial t}(|T_t f|^2) + T_{s-t}[(\frac{\partial T_t}{\partial t} f^*) f] + T_{s-t} [f^* (\frac{\partial T_t}{\partial t} f)]
\]

\[
= -2T_{s-t} \Gamma(T_t f).
\]

Therefore

\[
T_s|f|^2 - |T_s f|^2 = -F_s + F_0 = 2 \int_0^s T_{s-t} \Gamma(T_t f) dt.
\]

Since \(T_{s,\alpha}(L^\infty) \subset D(L^\alpha)\) we get (33) for all \(f \in L^\infty\). □
Applying the subordination formula that $T_v \Gamma(T_{v+t}f) \leq T_{v+s}(\Gamma(T_t f))$ for all $v, s, t > 0$ and $f \in L^\infty$ such that $T_s f, T_s f^*, T_s |f|^2 \in D(L)$ for all $s > 0$.

The following lemma says that the $\Gamma^2 \geq 0$ criterion passes to fractional powers, which could be known to some experts. We add a proof as we do not find a reference.

**Lemma 7.** If $T_t = e^{-t L}$ satisfies the $\Gamma^2 \geq 0$ criterion (41), then $T_{t,\alpha} = e^{-t L^\alpha}$ satisfies (44) and (45) for all $f \in L^\infty$ and $0 < \alpha < 1$. Moreover,

$$\Gamma_{L^\alpha}(s^\alpha \partial_s^j T_s f, f) \leq \left(\frac{10}{1-\alpha}\right)^j T_{s,\alpha} \Gamma_{L^\alpha}(f)$$

**Proof.** Applying (41), we have that, with $c_\alpha = - (\Gamma(-\alpha))^{-1} > 0$,

$$\Gamma_{L^\alpha}(f, f) = c_\alpha \int_0^\infty (T_t |f|^2 - (T_t f^*) f - f^*(T_t f) + |f|^2) t^{-1-\alpha} dt $$

$$= c_\alpha \int_0^\infty (T_t |f|^2 - |T_t f|^2 + |T_t f - f|^2) t^{-1-\alpha} dt. \quad (47)$$

if $f, f^*, |f|^2 \in D(L)$. The integration converges because

$$|T_t |f|^2 - |T_t f|^2| \leq c \min\{t, 1\}, \quad (48)$$

for $f \in D(L)$. In fact, by the $\Gamma^2 \geq 0$ criterion (41), we see that

$$T_t |T_t f|^2 - |T_{2t} f|^2 \leq \frac{1}{2} (T_{2t} |f|^2 - |T_{2t} f|^2).$$

So

$$\|T_t |f|^2 - |T_t f|^2\|^{\frac{1}{2}} \leq \|T_t |f - T_t f|^2 - |T_t(f - T_t f)|^2\|^{\frac{1}{2}} + \|T_t |T_t f|^2 - |T_{2t} f|^2\|^{\frac{1}{2}}$$

$$\leq ct + 2 t^{\frac{1}{2}} \|T_{2t} |f|^2 - |T_{2t} f|^2\|^{\frac{1}{2}}.$$  

Let $u(t) = t^{-\frac{1}{2}} ||T_t |f|^2 - |T_t f|^2||^{\frac{1}{2}}$. We get

$$u(t) \leq ct^{\frac{1}{2}} + u(2t).$$

Since $u(t)$ is uniformly bounded on $[1, \infty)$, we get $u(t)$ is uniformly bounded on $[0, \infty)$ by iteration. This proves (48). Therefore, by the Cauchy-Schwartz inequality (40) and the $\Gamma^2 \geq 0$ criterion for $T_t$ we get

$$\Gamma_{L^\alpha}(T_u f, T_u f) \leq T_u \Gamma_{L^\alpha}(f, f). \quad (49)$$

Applying the subordination formula that $T_{t,\alpha} = \int_0^\infty T_u \phi_t,\alpha(u) du$ and the Cauchy-Schwartz inequality (40), we obtain

$$\Gamma_{L^\alpha}(T_{t,\alpha} f, T_{t,\alpha} f) \leq T_{t,\alpha} \Gamma_{L^\alpha}(f, f). \quad (50)$$
One can easily adapt the proof to get
\[ T_{u,\alpha} \Gamma L^\alpha(T_{v,\alpha} g, T_{u,\alpha} g) \leq T_{u,\alpha} T_{t,\alpha} \Gamma L^\alpha(T_{v,\alpha} g, T_{v,\alpha} g). \] (51)
for all \( g \in L^\infty \) since \( T_{v,\alpha} g, T_{u,\alpha} |T_{v,\alpha} g|^2 \in D(L) \). Applying (43), we get (45) for
\( T_{t,\alpha} \).
Now, apply (40) to \( \Gamma L^\alpha \) and \( a(s) = T_s f, d\mu(s) = s^j \partial_j \phi_{t,\alpha}(s) ds \); we get (46) from (33), (24), and (51).

2.3 BMO spaces associated with semigroups of operators
BMO spaces associated with semigroup generators have been intensively studied recently (see [12]). In this article, we follow the ones studied in [20] and [24] because they are defined in a pure semigroup language. Set
\[ \|f\|_{\text{bmo}(L^\alpha)} = \sup_{0 < t < \infty} \|T_{t,\alpha} |f|^2 - |T_{t,\alpha} f|^2\|_{L^\infty}, \] (52)
\[ \|f\|_{\text{BMO}(L^\alpha)} = \sup_{0 < t < \infty} \|T_{t,\alpha} |f - T_{t,\alpha} f|^2\|_{L^\infty}, \] (53)
for \( f \in L^\infty, 0 < \alpha \leq 1 \).

We wish to define the space \( \text{BMO}(L^\alpha) \), \( 0 < \alpha \leq 1 \) so that it is a dual space and \( L^\infty_0(M) = \text{ker} L^\alpha \) is weak* dense in it, to be consistent with the classical ones (where \( L^\infty_0(M) = L^\infty(M)/\text{ker} L^\alpha \)). In [20] and [24], this is done by using a SOT-topology in the corresponding Hilbert C* modulars. In this article, we prefer to use the following detour to avoid introducing the theory of Hilbert C* modulars. Define, for \( g \in L^1 \),
\[ \|g\|_{H^1(L^\alpha)} = \sup \{|\langle f, g \rangle| : f \in L^\infty, \|f\|_{\text{BMO}(L^\alpha)}, \|f^*\|_{\text{BMO}(L^\alpha)} \leq 1\}. \] (54)
Let \( H^1(L^\alpha) = \{g \in L^1; \|g\|_{H^1} < \infty\} \). For a net \( f_\lambda \in L^\infty_0(M) \), we say \( f_\lambda \) converges in the weak* topology if \( \langle f_\lambda, g \rangle \) converges for any \( g \in H^1(L^\alpha) \). Let \( \text{BMO}(L^\alpha) \) be the abstract closure of \( L^\infty_0(M) \) with respect to this weak* topology, that is the linear space of all weak* convergent nets \( f_\lambda \in L^\infty_0(M) \). For a weak* convergent \( f_\lambda \), let
\[ \|\lim_\lambda f_\lambda\|_{\text{BMO}(L^\alpha)} = \sup \lim_\lambda |\langle f_\lambda, g \rangle|. \]
It is easy to see that this coincides with (53) if \( \lim_\lambda f_\lambda \in L^\infty \).

As an application of Corollary 2, we show that these BMO and bmo norms with different \( 0 < \alpha < 1 \) are all equivalent if we assume the \( \Gamma^2 \geq 0 \) criterion.

Lemma 8. Suppose \( L \) generates a weak* continuous semigroup of positive contractions, we have
\[ \|f\|_{\text{BMO}(L^\beta)} \leq \frac{c\alpha}{\beta} \|f\|_{\text{BMO}(L^\alpha)}, \] (55)
\[ \|f\|_{\text{BMO}(L^\beta)} \leq \frac{4}{1-\beta} \|f\|_{\text{bmo}(L^\beta)}, \] (56)
for any $0 < \beta < \alpha \leq 1$. Assuming in addition that the semigroup $T_t = e^{-tL}$ satisfies the $\Gamma^2 \geq 0$ criterion [19], we have that

$$\|f\|_{BMO(L^\alpha)} \simeq \|f\|_{bmo(L^\alpha)} \simeq \|f\|_{bmo(L^\beta)},$$

(57)

for all $0 < \beta, \alpha < 1$. In particular,

$$c(1-\alpha)^2\|f\|_{BMO(L^\alpha)} \leq \|f\|_{BMO(\sqrt{T})} \leq c\|f\|_{BMO(L^\alpha)},$$

(58)

for all $\frac{1}{2} < \alpha < 1$.

Proof. The argument for (55) is the same as that for the second inequality of [20, Theorem 2.6]. We sketch it here. By the Cauchy-Schwartz inequality,

$$T_{t,\beta}|f - T_{t,\beta}f|^2 = \int_0^\infty \phi_{t,\frac{u}{\beta}}(u)T_u\alpha|\int_0^\infty \phi_{t,\frac{u}{\beta}}(v)(f - T_{v,\alpha}f)dv|^2 du$$

$$\leq \int_0^\infty \int_0^\infty \phi_{t,\frac{u}{\beta}}(u)\phi_{t,\frac{v}{\beta}}(v)T_u\alpha|f - T_{v,\alpha}f|^2 du dv.$$

It is easy to see that $\|T_{u,\alpha}|f - T_{v,\alpha}f|^2\| \leq (1 + |\ln \frac{u}{v}|)\|f\|_{BMO(L^\alpha)}^2$, so we get (55) from (56).

For the rest of this proof, we use $\Gamma$ for $\Gamma_{L^\alpha}$, $T_t$ for $T_{t,\beta}$ and $P_t$ for $T_{t,\frac{u}{\beta}}$ to simplify the notation. Since $T_1$ has the quasi monotone property (37), we have

$$P_1 = \int_0^\infty T_u\phi_{t,\frac{u}{\beta}}(u)du \geq \int_0^{t^2} \left(\frac{u}{12}\right)^K\phi_{t,\frac{u}{\beta}}(u)du \geq \frac{1}{100K^2}T_{12}.$$  

(59)

We now prove (56). Note

$$\|T_t|f - T_lf|^2\| = \|T_t|f - T_lf|^2 - |T_t|f - T_lf|^2 + |T_l|f - T_lT_t|^2\|$$

$$\leq \|T_t|f - T_lf|^2\|_{bmo(L^\beta)} + \|T_l|f - T_{2l}|f|^2\|.$$  

Let $\gamma = 2\frac{1}{\alpha^2}$ and $S = 2T_t - T_{\gamma t}$. Then $S$ is a unital completely positive map because of (37). We have

$$|T_tf - T_{\gamma t}f|^2 + |Sf - T_lf|^2 = -2|T_lT_tf|^2 + |T_{\gamma t}f|^2 + |Sf|^2$$

$$\leq -2|T_lT_tf|^2 + |T_{\gamma t}f|^2 + |Sf|^2$$

$$\leq -2|T_lT_tf|^2 + 2|T_{l}|f|^2$$

$$\leq 2\|f\|^2_{bmo(L^\beta)}.$$  

We get by the triangle inequality that

$$\|T_t|f - T_{2l}|f\| \leq K_1\sup_s\|T_s|f - T_{\gamma s}|f\| \leq \sqrt{2}K_1\|f\|_{bmo(L^\beta)}.$$  

Therefore,

$$\|f\|_{BMO(L^\beta)} \leq \sqrt{4 + 2K_1^2\|f\|^2_{bmo(L^\beta)}}.$$  

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To prove (57), we note that the $\Gamma^2 \geq 0$ assumption for $L$ passes to $L^\alpha$ by Lemma 7. The inequality $\|f\|_{bmo} \leq (2 + \sqrt{2})\|f\|_{BMO}$ is proved in [20, Proposition 2.4] assuming the $\Gamma^2 \geq 0$ criterion. Together with (56), we get $\|f\|_{BMO(L^\alpha)} \simeq \|f\|_{bmo(L^\alpha)}$. We now show the second equivalence in (57). Note,

$$
\int_0^t T_{t-s} \Gamma(T_s \sqrt{t} f) ds = \int_0^t T_{t-s} \Gamma \left( \int_s^\infty \phi_{\sqrt{t}/2}(v) T_v T_s f dv \right) ds
\leq \int_0^\infty \phi_{\sqrt{t}/2}(v) \int_0^t T_{t-s} \Gamma(T_v T_s f) ds dv
\leq \int_0^\infty \phi_{\sqrt{t}/2}(v) \int_0^t T_{t+v-t-s} \Gamma(T_{t+s} f) ds dv
\quad (u = t + v/t - s)
\leq \int_0^\infty \phi_{\sqrt{t}/2}(v) \frac{t}{t+v} \int_0^t T_{t+v-t} \Gamma(T_u f) ds dv
\quad (43)
\leq \int_0^\infty \phi_{\sqrt{t}/2}(v) \frac{t}{t+v} \|f\|_{bmo(L^\alpha)}^2 dv < \frac{5}{6} \|f\|_{bmo(L^\alpha)}^2.
$$

We then have

$$
(T_i |f|^2 - |T_i f|^2)^{\frac{1}{2}}
\leq (T_i |f| - P_{\sqrt{t}} f|^2 - |T_i f - T_i P_{\sqrt{t}} f|^2)^{\frac{1}{2}} + (T_i |P_{\sqrt{t}} f|^2 - |T_i P_{\sqrt{t}} f|^2)^{\frac{1}{2}}
\leq 100 K_1 (P_{\sqrt{t}} f - P_{\sqrt{t}} f)|^2 + \sqrt{\frac{5}{6}} \|f\|_{bmo(L^\alpha)}^2
\leq 100 K_1 \|f\|_{bmo(L^\alpha)} + \sqrt{\frac{5}{6}} \|f\|_{bmo(L^\alpha)}^2.
$$

so

$$
\|f\|_{bmo(L^\alpha)} \leq 1200 K_1 \|f\|_{bmo(L^\alpha)}^2.
$$

Therefore,

$$
\|f\|_{BMO(L^\alpha)} \leq 10000 K_1^2 \|f\|_{BMO(L^\alpha)}^2.
$$

Applying (55), we have $\|f\|_{BMO(L^\alpha)} \simeq \|f\|_{BMO(L^\alpha)}$ for all $0 < \beta, \alpha < 1$.

**Remark.** The equivalence (57) fails for $\alpha = 1$ in general. See Section 4, Example 2.

### 3 Imaginary powers and $H^\infty$-calculus

#### 3.1 $H^\infty$-calculus.

Let us review some definitions and basic facts about $H^\infty$-calculus. We refer the readers to [8, 21, 14] for details. For $0 < \theta < \pi$, let $S_\theta$ be the following open
sector of the complex plane:

\[ S_\theta = \{ z \in \mathbb{C}, |\arg z| < \theta \}. \]

Recall that we say a closed operator \( A \) on a Banach space \( X \) is a sectorial operator of type \( \omega < \pi \) if the spectrum of \( A \) is contained in \( S_\omega \), the closure of \( S_\omega \), and for any \( \theta, \omega < \theta < \pi, z \notin S_\theta \), there exists \( c_\theta \) such that

\[ \|z(z - A)^{-1}\| \leq c_\theta. \]

We will assume that the domain of \( A \) is dense in \( X \) (or weak* dense in \( X \) when \( X \) is a dual space). We may also assume that \( A \) has dense range and is one to one by considering \( A + \varepsilon \) (see [21, Lemma 3.2, 3.5]).

Let \( H^\infty(S_\eta) \) be the space of all bounded analytic functions on \( S_\eta \) and \( H^\infty_0(S_\eta) \) be the subspace of the functions \( \Phi \in H^\infty(S_\eta) \) with an extra decay property that

\[ |\Phi(z)| \leq \frac{c|z|^r}{(1 + |z|)^{2r}}, \]

for some \( c, r > 0 \). Then for any \( \Phi \in H^\infty_0(S_\eta) \), and \( \theta > \eta \),

\[ \Phi(A) = \frac{1}{2\pi i} \int_{\gamma_\theta} \Phi(z)(z - A)^{-1} \, dz \tag{60} \]

is a well defined bounded operator on \( D(A) \) and its (weak*) extension is bounded on \( X \). Here \( \gamma_\theta \) is the boundary of \( S_\theta \) oriented counterclockwise. For general \( \Phi \in H^\infty(S_\eta) \), set

\[ \Phi(A) = \psi(A)^{-1}(\Phi \psi)(A), \tag{61} \]

with \( \psi(z) = \frac{z}{(1 + |z|)^2} \). It turns out that the so defined \( \Phi(A) \) is a closed (weak*) densely defined operator, which may not be bounded, and it coincides with \( \Phi(A) \) defined as in (60) for \( \Phi \in H^\infty_0(S_\eta) \). Moreover, these definitions are consistent with the definitions in the “older” functional calculus.

**Definition 3.** We say a (weak*) densely defined sectorial operator \( A \) of type \( \omega \) has bounded \( H^\infty(S_\eta) \)-calculus, \( \omega < \eta < \pi \), if the map \( \Phi(A) \) extends to a bounded operator on \( X \) and there is a constant \( C \) such that

\[ \|\Phi(A)\| \leq C\|\Phi\|_{H^\infty(S_\eta)} \tag{62} \]

for any bounded analytic function \( \Phi \in H^\infty(S_\eta) \).

**Remark.** Suppose a densely defined sectorial \( A \) has bounded \( H^\infty(S_\eta) \)-calculus on \( Y \) and suppose \( Y \) is a weak* dense subspace of a dual Banach space \( X \). Then the weak* extension of \( \Phi(A) \) onto \( X \), still denoted by \( \Phi(A) \), is bounded and satisfies (62) with the same constant. So a weak* dense sectorial operator \( A \) has \( H^\infty \)-calculus on \( X \) if and only if it has \( H^\infty \)-calculus on the norm closure of \( D(A) \).
The negative infinitesimal generator $L$ of any uniformly bounded (weak*) strong continuous semigroup on a dual Banach space $X$ is actually a (weak*) densely defined sectorial operator of type $\frac{\pi}{2}$ and $L^\alpha$ is of type $\frac{\alpha \pi}{2}$ on $X$. Cowling, Duong, and Hieber & Prüss ([7, 11, 16]) prove that the negative infinitesimal generator of a semigroup of positive contractions on $L^p$, $1 < p < \infty$ always has the bounded $H^\infty(S_\eta)$-calculus for any $\eta > \frac{\pi}{2}$. One cannot hope to extend this to $p = \infty$. We will prove that the associated BMO($\sqrt{L}$) space is a good alternative, as desired.

**Lemma 9.** Suppose $A$ is a densely defined sectorial operator of type $\omega < \frac{\pi}{2}$ on a Banach space $X$. Assume $\int_0^\infty A e^{-tA}a(t)dt$ is bounded on $X$ with norm smaller than $C$ for any function $a(t)$ with values in $\pm 1$. Then $A$ has a bound $H^\infty(S_\eta)$ calculus for any $\eta > \frac{\pi}{2}$.

**Proof.** This is a consequence of [8, Example 4.8] by setting $a(t)$ to be the sign of $\langle Te^{-tA}u, v \rangle$ for any pair $(u, v)$ in a dual pair $(X, Y)$. □

We are going to prove that the negative generator $L$ of a semigroup of positive contractions satisfies the assumptions of Lemma 9. We follow an idea of E. Stein and consider scalar valued functions $a(t)$ such that

$$s \int_s^\infty \frac{|a(v-s)|^2}{v^2}dv \leq c_s^2,$$  \hspace{1cm} (63)

for all $s > 0$ and some constant $c_s$. Define $M_a$ by

$$M_a(f) = \int_0^\infty a(t)\frac{\partial T_t f}{\partial t}dt = \int_0^\infty a(t)L^\alpha T_t f dt,$$  \hspace{1cm} (64)

for $f \in L^\infty$, $0 < \alpha < 1$. For now, we assume $a$ is supported on a compact subset of $(0, \infty)$ so we do not worry about the convergence of the integration.

**Lemma 10.** Assume that $L$ generates a weak* continuous semigroup of positive contractions on $L^\infty$ satisfying the $\Gamma^2 \geq 0$ criterion ([7]). We have

$$\|M_a(f)\|_{bmo(L^\alpha)} \leq \frac{cc_s}{(1-\alpha)^2}\|f\|_{bmo(L^\alpha)},$$  \hspace{1cm} (65)

$$\|M_a(f)\|_{BMO(L^\alpha)} \leq \frac{cc_s}{(1-\alpha)^2}\|f\|_{L^\infty},$$  \hspace{1cm} (66)

for any $f \in L^\infty$, $0 < \alpha < 1$.

**Proof.** We consider the case $\alpha \geq \frac{3}{4}$ only. The case $\alpha < \frac{3}{4}$ is easier and follows from this case by subordination. Recall that the $\Gamma^2 \geq 0$ assumption for $L$ passes to $L^\alpha$ by Lemma [7] and $T_{t,\alpha}(L^\infty) \subset D(L^{2\alpha}), L^{2\alpha}T_{t,\alpha} = \partial^2_t T_{t,\alpha}$. In this proof, we use $\Gamma$ for $\Gamma_{L^\alpha}$ the gradient form associated with $L^\alpha$, $T_t$ for $T_{t,\alpha}$ and $P_t$ for
$T_{t, \frac{r}{2}}$ to simplify the notation. Let $r = \frac{1}{1 + \alpha} > 4$. We have that

$$
\int_0^t T_{t-s} \Gamma(T_s f) ds = \int_0^t T_{t-s} \Gamma \left( \int_s^\infty L^\alpha T_v f dv \right) ds
$$

$$
\leq \int_0^t T_{t-s} \int_s^\infty \Gamma(L^\alpha T_v f) v^{\frac{\alpha}{2}} dv \int_s^\infty v^{-\frac{\alpha}{2}} dv ds
$$

$$
= \int_0^t T_{t-s} \int_s^\infty \Gamma(L^\alpha T_v f) v^{\frac{\alpha}{2}} dv 2s^{-\frac{\alpha}{2}} ds
$$

$$
= \int_0^\infty \int_0^t 2s^{-\frac{\alpha}{2}} T_{t-s} ds \Gamma(L^\alpha T_v f) v^{\frac{\alpha}{2}} dv
$$

Let $S_v = \int_0^t 2s^{-\frac{\alpha}{2}} T_{t-s} ds$. So by Lemma 15 and the $\Gamma^2 \geq 0$ criterion,

$$
\|f\|^2_{bmo} = \sup_t \left\| \int_0^t T_{t-s} \Gamma(T_s f) ds \right\|
$$

$$
\leq \sup_t \left\| \int_0^t T_{t-s} \Gamma(T_s f) ds \right\|
$$

$$
= \sup_t \left\| \int_0^t T_{t-s} \Gamma(T_s f) ds \right\|
$$

$$
\leq \sup_r \left\| \int_0^t S_v \Gamma(L^\alpha T_v) v^{\frac{\alpha}{2}} dv \right\|.
$$

So,

$$
\frac{1}{r} \|Ma f\|^2_{bmo} \leq \left\| \int_0^\infty S_v \Gamma(L^\alpha T_v) a v^{\frac{\alpha}{2}} dv \right\|
$$

$$
= \left\| \int_0^\infty S_v \Gamma(T_v) a u L^\alpha T_u f du v^{\frac{\alpha}{2}} dv \right\|
$$

$$
= \left\| \int_0^\infty S_v \Gamma(\int_0^\infty a(u) L^\alpha T_u f du) v^{\frac{\alpha}{2}} dv \right\|
$$

$$
= \left\| \int_0^\infty S_v \Gamma(\int_0^\infty a(u) L^\alpha T_u f du) v^{\frac{\alpha}{2}} dv \right\|
$$

(Inequality 40)

$$
\leq \left\| \int_0^\infty v^{\frac{\alpha}{2}} S_v \left( \int_0^\infty \frac{|a|^2}{u^2} du \int_0^\infty \Gamma(u L^\alpha T_u f) du \right) dv \right\|
$$

$$
\leq C^2 \left\| \int_0^\infty S_v \left( \int_0^\infty \Gamma(u L^\alpha T_u f) du \right) v^{\frac{\alpha}{2}} dv \right\|
$$

$$
= C^2 \left\| \int_0^\infty S_v dv \Gamma(u L^\alpha T_u f) du \right\|.
$$

Note $K_1 \leq r$ and $\sup_{r>4} \left( \frac{2}{1 + \alpha} \cdot \frac{r}{r-1} \right) \leq c$. By (47), we have, for $u \leq t$,

$$
\int_0^{t-u} v^{\frac{\alpha}{2}} S_v dv \leq \int_0^{t-u} v^{\frac{\alpha}{2}} \int_0^{t-u} v^{s} T_{s-u} \left( \frac{2}{1 + \alpha} \cdot \frac{r}{r-1} \right) dv ds
$$

$$
\leq C T_{\frac{1}{\alpha} (rt-u)} t^2 \wedge u^2.
$$

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Applying (46), we get
\[
\left\| \int_0^t v^2 S_v d\Gamma(uL^{2\alpha} T_{\psi} T_{\psi} f) \right\| \leq cr^2 \left\| \int_0^t \frac{T_{\psi}}{u^2} \int_0^{u\wedge t} v^2 S_v d\Gamma(T_{\psi} f) du \right\| \\
\leq cr^2 \left\| \int_0^t T_{(1+\alpha)t-\frac{r}{2}} \Gamma(T_{\psi} f) du \right\| \\
\leq cr^2 \left\| \int_0^t T_{\psi}^{-\alpha} \Gamma(T_s f) ds \right\| \leq cr^2 \| f \|_{bmo}.
\]

For \( \alpha^{-n}t < u \leq \alpha^{-n-1}t, n \geq 0 \), we use
\[
\int_0^{u\wedge t} v^2 S_v dv \leq \int_0^{u\wedge t} v^2 \int_0^{u\wedge t} 2s^{-\frac{1}{2}} T_{rt-t} \left( \frac{r}{r-1} \right)^r ds dv \leq cT_{rt-t}^2 \wedge u^2.
\]

Similar to (46), we get \( \Gamma(uL^{2\alpha} T_{\alpha-n} T_{\alpha-n} f) \leq cr^2 T_{2\alpha-n-t-u} \Gamma(f) \) because \( \frac{r-1}{r-2} = \frac{\alpha^{-n}}{2-\alpha^{-n}} \leq 2 \alpha^{-n} \leq 1 \). So
\[
\Gamma(uL^{2\alpha} T_{\alpha-n} T_{u-\alpha-n} T_{\alpha-n} f) \leq cr^2 T_{2\alpha-n-t-u} \Gamma(T_{u-\alpha-n} T_{\alpha-n} f).
\]

Therefore,
\[
\left\| \int_0^{\alpha^{-n-1}t} \int_0^{u\wedge t} v^2 S_v d\Gamma(uL^{2\alpha} T_{\psi} T_{\psi} f) du \right\| \\
\leq cr^2 \left\| \int_0^{\alpha^{-n-1}t} \frac{T_{\alpha-n} T_{u} \Gamma(T_{\alpha-n} T_{\alpha-n} f) du} {u^2} \right\| \\
= cr^2 \alpha^{2n} \left\| \int_0^{\alpha^{-n-1}t} \frac{T_{\alpha-n} T_{\alpha-n} f} {T_{\alpha-n} T_{\alpha-n} f} \right\| ds \\
\leq cr^2 \alpha^{2n} \| f \|_{bmo}^2.
\]

Summing up for \( n \geq 0 \), we get
\[
\left\| \int_t^{u\wedge t} v^2 S_v d\Gamma(uL^{2\alpha} T_{\psi} T_{\psi} f) du \right\| \leq cr^3 \| f \|_{bmo}.
\]

Combining the estimates above, we conclude that
\[
\| M_a(f) \|_{bmo(L^\alpha)} \leq cc_a r^2 \| f \|_{bmo(L^\alpha)}.
\]

Applying (57), we actually get
\[
\| M_a(f) \|_{BMO(L^\alpha)} \leq cc_a r^3 \| f \|_{BMO(L^\alpha)} \leq cc_a r^3 \| f \|_{L^\infty}.
\]
But we wish to get a better estimate. Note

\[
(T_t - T_{2t}) M_a(f) = \int_0^\infty a(s) \partial_s (T_{t+s} - T_{2t+s}) f ds \\
= \int_0^\infty a(s-t) \partial_s (T_s - T_{t+s}) f ds \\
\leq \left( \int_0^\infty \frac{|a(t)|^2}{s^2} ds \right)^{\frac{1}{2}} \left( \int_0^\infty s^2 \left| \int_0^t \partial^2_s T_{v+s} f dv \right|^2 ds \right)^{\frac{1}{2}} \\
\leq c_a \left( \int_0^\infty s^2 \int_0^t \left| \partial^2_s T_{v+s} f \right|^2 dv ds \right)^{\frac{1}{2}} \\
(by (38)) \leq \frac{25c_a}{(1-\alpha)^2} \left( \int_0^\infty s^{-2} \int_0^t T_{\alpha(v+s)} |f|^2 dv ds \right)^{\frac{1}{2}}.
\]

Therefore

\[\|(T_t - T_{2t}) M_a(f)\|_{L^\infty} \leq \frac{25c_a}{(1-\alpha)^2} \|f\|_{L^\infty},\]

and hence

\[\|M_a(f)\|_{BMO(L^\alpha)} \leq \|M_af\|_{bmo(L^\alpha)} + \sup_t \|(T_t - T_{2t}) M_a(f)\|_{L^\infty} \leq c r^2 c_a \|f\|_{L^\infty}.\]

Given \(f \in L^\infty, g \in H^1(L^\alpha)\), let \(\hat{a}(t) = \text{sign}(\langle L^\alpha T_{t,\alpha} f, g \rangle) a(t)\). Then \(\hat{a}\) satisfies (63) if \(a\) does. We have from Lemma 10 that

\[
\int_0^\infty |\langle a(t) L^\alpha T_{t,\alpha} f, g \rangle| dt = \lim_{N,M \to \infty} \int_0^N |\langle a(t) L^\alpha T_{t,\alpha} f, g \rangle| dt \\
= \lim_{N,M \to \infty} \left\langle \int_0^N \hat{a}(t) L^\alpha T_{t,\alpha} f dt, g \right\rangle \\
\leq cc_a \|M_a f\|_{BMO(L^\alpha)} \|g\|_{H^1} \\
\leq \frac{cc_a}{(1-\alpha)^2} \|f\|_{L^\infty} \|g\|_{H^1}.
\]

This shows that \(\lim_{N,M \to \infty} \int_0^N \langle a(t) L^\alpha T_{t,\alpha} f, g \rangle dt\) exists and \(\int_0^N \langle a(t) L^\alpha T_{t,\alpha} f, g \rangle dt\) weak* converges in \(BMO(L^\alpha)\) as \(N,M \to \infty\). So the integration in (64) weak* converges and \(M_a\) is well defined for all \(f \in L^\infty\) and \(a(t)\) satisfying (63). The weak* extension of \(M_a\) is then a bounded map from \(BMO(L^\alpha)\) to \(BMO(L^\alpha)\).

**Theorem 2.** Suppose \(a(t)\) satisfies (63). \(M_a\) extends to a bounded operator from \(BMO(L^\alpha)\) to \(BMO(L^\alpha)\) for \(0 < \alpha < 1\). The estimates are as in Lemma 10.
**Theorem 3.** Suppose \( T_t = e^{-tL} \) is a weak* continuous semigroup of positive contractions on \( L^\infty \) satisfying the \( \Gamma^2 \geq 0 \) criterion. Then \( L \) has a complete bounded \( H^\infty(S_\eta) \) calculus on \( BMO(\sqrt{L}) \) for any \( \eta > \frac{\pi}{2} \).

**Proof.** Given \( \alpha \in (\frac{1}{2}, 1) \), let \( Y^\alpha \) be the norm closure of \( D(L) \) in \( BMO(L^\alpha) \). It is easy to check that \( T_{t, \alpha} = e^{-tL^\alpha} \) are contractions on \( Y^\alpha \). Then \( L^\alpha \) is a densely defined sectorial operator of type \( \frac{\pi}{2} \) in \( Y^\alpha \). Lemma 9 and Lemma 10 imply that \( L^\alpha \) has a bounded \( H^\infty(S_\eta) \) calculus on \( Y^\alpha \) for any \( \eta > \frac{\pi}{2} \alpha \). Note \( \Phi(z) = \Psi(z^{\frac{1}{\alpha}}) \in S_\eta \) if \( \Psi \in S_{\frac{\pi}{\alpha}} \) and \( \Phi(L^\alpha) = \Psi(L) \). We conclude that \( L \) has a bounded \( H^\infty(S_\eta) \) calculus on \( Y^\alpha \) for any \( \eta > \frac{\pi}{2} \alpha \). Given \( \theta > \frac{\pi}{2} \), choose \( \frac{1}{2} < \alpha < 1 \) so that \( \alpha \theta > \frac{\pi}{2} \). Then \( L \) has a bounded \( H^\infty(S_\theta) \) calculus on \( Y^\alpha \). Lemma 8 then implies that \( L \) has a bounded \( H^\infty(S_\theta) \) calculus on \( Y^\alpha \) and on \( BMO(\sqrt{L}) \) for any \( \theta > \frac{\pi}{2} \), since \( Y^\frac{1}{2} \alpha \) is weak* dense in \( BMO(\sqrt{L}) \) and \( \Phi(L) \) is the weak* extension of its restriction on \( Y^\frac{1}{2} \alpha \) by definition. The same argument applies to \( \text{id} \otimes L \). We then obtain the completely bounded \( H^\infty(S_\eta) \) calculus as well.

### 3.2 Imaginary Power and Interpolation.

Given \( 0 < \alpha < 1 \), choose \( \frac{\pi}{2} < \theta < \frac{\pi}{2} \alpha \). By (61), we have the identities

\[
L^\alpha e^{-tL^\alpha} = \frac{1}{2\pi i} \psi(L)^{-1} \int_{\gamma_\theta} z^\alpha \psi(z) e^{-tz^\alpha} (z - L)^{-1} dz, \tag{67}
\]

\[
L^{i\alpha s} = \frac{1}{2\pi i} \psi(L)^{-1} \int_{\gamma_\theta} z^{i\alpha s} \psi(z) (z - L)^{-1} dz. \tag{68}
\]

Note

\[
z^{i\alpha s} = \Gamma(1 - is)^{-1} \int_0^\infty t^{-is} z^\alpha e^{-tz^\alpha} dt. \tag{69}
\]

Since these integrals converge absolutely, we can exchange the order of the integrations and get

\[
L^{i\alpha s} = \Gamma(1 - is)^{-1} \int_0^\infty t^{-is} L^\alpha e^{-tL^\alpha} dt. \tag{70}
\]

Lemma 10 implies that

\[
\|L^{i\alpha s} f\|_{BMO(L^\alpha)} \leq \frac{c}{(1 - \alpha)^2} \Gamma(1 - is)^{-1} \|f\|_{L^\infty}.
\]

Thus for \( \alpha > \frac{1}{2} \),

\[
\|L^{is} f\|_{BMO(L^\alpha)} \leq c \Gamma \left( \frac{1 - is}{\alpha} \right)^{-1} \|f\|_{L^\infty},
\]

\[
\leq \frac{c}{(1 - \alpha)^2 (1 + |s|)^\frac{\pi}{2\alpha}} \exp \left( \frac{\pi |s|}{2\alpha} \right) \|f\|_{L^\infty}.
\]
Choosing $\alpha = \frac{|s|}{|s| + 1}$ for $s$ large, we get

$$\|L^{is}f\|_{BMO(L^\alpha)} \leq c(1 + |s|)\frac{2}{\pi} \exp\left(\frac{\pi|s|}{2}\right) \|f\|_{L^\infty}. \quad (71)$$

The same estimate holds with $\text{bmo}(L^\alpha)$-norms putting on both sides of (71) because of (65). One can improve such estimates for concrete example of semigroups, see [?] for example.

**Definition 4.** We say a weak* continuous semigroup of positive contractions is a symmetric Markov semigroup if $\langle T_t f, g \rangle = \langle f, T_t g \rangle$ for $f \in L^\infty, g \in L^1$ and it admits a standard Markov dilation in the sense of [20, page 717].

**Remark.** The Markov dilation assumption in the above definition holds automatically in many cases. In the commutative case (i.e the underlying von Neumann algebra $\mathcal{M} = L^\infty(M)$), this is due to Rota (see [30, page 106, Theorem 9]). Therefore every weak* continuous semigroup of unital symmetric positive contractions is automatically a symmetric Markov semigroup. In [29] it is proven that this is the case for convolution semigroups on group von Neumann algebras. In [9, 19] it is proven that this holds for the finite von Neumann algebras case. The case of a general semifinite von Neumann algebra is conjectured but there has not been a written proof.

**Lemma 11.** ([JM12]) Assume that $T_t = e^{-tA}$ (e.g. $A = L^\alpha$) is a symmetric Markov semigroup on a semifinite von Neumann algebra $\mathcal{M}$. Then, the following interpolation result holds

$$[BMO(A), L^0_\eta(\mathcal{M})]_\frac{1}{p} = L^0_p(\mathcal{M})$$

for $1 < p < \infty$. Here $L^0_p(\mathcal{M}) = L^p(\mathcal{M})/\ker A$.

Since $\|L^{is}\|_{L^2 \to L^2} = 1$ if $L$ generates a symmetric Markov semigroup, by interpolation, we get from (71) the following result.

**Corollary 3.** Suppose $T_t = e^{-tL}$ is a symmetric Markov semigroup of operators on a semifinite von Neumann algebra $\mathcal{M}$ and satisfies the $\Gamma^2 \geq 0$ criterion. Then, $L$ has the completely bounded $H^\infty(S_\eta)$-calculus on $L^p$ for any $\eta > \omega_p = |\frac{\pi}{2} - \frac{\pi}{p}|$, $1 < p < \infty$ and

$$\|L^{is}\|_{L^p \to L^p} \leq c(1 + |s|)\frac{2}{\pi} \exp\left(\frac{\pi|s|}{2}\right) \left|\frac{\pi}{2} - \frac{\pi}{p}\right|, \quad (72)$$

for all $1 < p < \infty$.

**Remark.** Let us point out that the left-hand side of the inequality on [20, line 4, page 728] misses a “$\frac{1}{2}$”. It should be $\|L^{\frac{1}{2}}f\|$ instead of $\|L^{is}f\|$, because Theorem 3.3 of [20] is for the semigroup generated by $\sqrt{L}$. So the estimate of the constants $c_{\eta,p}$ given in [20, Corollary 5.4] is not correct. Also [18] contains a
similar estimate to (72) without assuming the $\Gamma^2 \geq 0$ criterion. Their method is the transference principle and works for $L^p$ only.

Junge, Le Merdy, and Xu (21) studied the $H^\infty$-calculus in the noncommutative setting. In particular, they prove a $H^\infty(S_\eta)$-calculus property of $L : \lambda_y \mapsto |g|_\lambda y$ on $L^p(\hat{F}_n)$ for all $1 < p < \infty, \eta > \frac{|\pi_2 - \pi_\eta|}{p}$.

### 4 Examples

The $\Gamma^2 \geq 0$ criterion is known to be satisfied by a large class of semigroups including the heat, Ornstein-Uhlenbeck, and Jacobi semigroups (see [2]). The results proved in this article apply to all of them. The main example in the noncommutative setting, is the semigroup of operators on a group von Neumann algebra, generated from a conditionally negative function on the underlying group (see Example 4). We will analyze a few of them in the following.

**Example 1.** Let $-L = \Delta$ be the Laplace-Beltrami operator on a complete Riemannian manifold with nonnegative Ricci curvatures. Then the associated heat semigroup $T_t = e^{-tL}$ is symmetric Markovian and satisfies the $\Gamma^2 \geq 0$ criterion. All the theorems of this article hold for $L$, and it has bounded $H^\infty(S_\eta)$ calculus on $BMO(\sqrt{L})$ for any $\eta > \frac{\pi_2}{2}$.

In the special case that $L = -\partial_x^2$ the Laplacian on Euclidean space $\mathbb{R}^n$, the $BMO(L), bmo(L)$, and $BMO(\sqrt{L})$ spaces are all equivalent to the classical $BMO(\mathbb{R}^n)$ space. Indeed, by the subordination formula, we get the following integral representation for $T_{t, \frac{1}{2}} = e^{-t\sqrt{L}}$:

$$T_{t, \frac{1}{2}} f(x) = \frac{1}{2\sqrt{\pi}} \int_0^\infty f(x - s)te^{-\frac{s^2}{2}}s^{-\frac{3}{2}}ds.$$  

From this, it is easy to check that, for $I_{x,k} = [\frac{x - 2^k L^2}{L}, x - \frac{2^k L^2}{L}], k \in \mathbb{N},$

$$c^{-1}E_{I_{x,1}} f \leq T_{t, \frac{1}{2}} f(x) \leq c \sum_k 2^{-\frac{3}{2}}E_{I_{x,k}} |f|.$$  

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After an elementary calculation and using the fact that \( |E_B f - E_{kB} f| \lesssim k \|f\|_{BMO(\mathbb{R}^n)} \), one can see that \( \| \cdot \|_{BMO(\sqrt{T})} \simeq \| \cdot \|_{BMO(\mathbb{R}^n)} \), thus \( \| \cdot \|_{BMO(L^\alpha)} \simeq \| \cdot \|_{BMO(\mathbb{R}^n)} \) for all \( 0 < \alpha < 1 \) by Lemma 8.

By Theorem 3, \( L \) has \( H^\infty(S_\theta) \)-calculus on \( BMO(\sqrt{T}) \simeq BMO(\mathbb{R}^n) \) for any \( \eta > \frac{n}{2} \). It is easy to see that

\[
L^{is} = P_x e^{-\frac{s^2}{2} \Delta} + P_x e^{\frac{s^2}{2} \Delta}.
\]

So \( L \) does not have \( H^\infty(S_\theta) \)-calculus on \( BMO(L) \simeq L^\infty(\mathbb{R})/\mathbb{C} \) for any positive \( \theta \) and

\[
\|L^{is}\|_{BMO \to BMO} \simeq e^{\frac{|s|^2}{4 \eta}} \|\Delta\|_{BMO \to BMO}
\]

for \( |s| \) large. This also shows that \( L \) cannot have bounded \( H^\infty(S_\theta) \)-calculus on \( BMO(\mathbb{R}) \) for any \( \eta \leq \frac{n}{2} \).

**Example 3.** Let \( -L = \frac{\partial^2}{\partial x} - x \cdot \partial_x \) be the Ornstein-Uhlenbeck operator on \( (\mathbb{R}^n, e^{-|x|^2} \, dx) \). Let \( O_t f = O_{t,1} \) be \( e^{-tL} \). \( O_t \) is a symmetric Markov semigroup with respect to the Gaussian measure \( d\mu = e^{-|x|^2} \, dx \) and satisfies the \( \Gamma^2 \geq 0 \) criterion. Theorem 3 says that \( L = -\frac{\partial^2}{\partial x} + x \cdot \partial_x \) has bounded \( H^\infty(S_\theta) \)-calculus on \( BMO(\sqrt{T}) \) for any \( \eta > \frac{n}{2} \).

Mauceri and Meda (see [27]) introduced the following BMO space for the Ornstein-Uhlenbeck semigroup

\[
\|f\|_{BMO(\mu M M)} = \sup_{r_B \leq \min\{1, \frac{\sqrt{t}}{e^{r_B}}\}} (E_B^\mu |f - E_B^\mu f|^2)^{\frac{1}{2}}, \tag{73}
\]

with \( r_B, c_B \) the radius and the center of \( B \), and \( E_B^\mu = \frac{1}{\mu(B)} \int B \, d\mu \) the mean value operator with respect to the Gaussian measure \( d\mu \). Note, for the balls \( B \) satisfying \( r_B \leq \min\{1, \frac{\sqrt{t}}{e^{r_B}}\} \), we have the equivalence \( E_B^\mu |f| \simeq E_B |f| \). One may replace \( E_B^\mu \) by \( E_B \), the mean value operator with respect to the Lebesque measure \( dx \) in (73). The resulted BMO norms are equivalent to each other. From the integral presentation

\[
O_t f = \frac{1}{(\pi - \pi e^{-2t})^{\frac{n}{2}}} \int_{\mathbb{R}^n} \exp\left(-\frac{|e^{-t}x - y|^2}{1 - e^{-2t}}\right) f(y) \, dy, \tag{74}
\]

one easily see that, for \( t \leq 4 \) and \( \sqrt{t}|x| \leq 1 \),

\[
O_t |f|(x) \geq \frac{1}{(\pi - \pi e^{-2t})^{\frac{n}{2}}} \int_{B(x, \sqrt{t})} \exp\left(-\frac{2|x - y|^2}{1 - e^{-2t}}\right) f(y) \, dy
\]

\[
\geq c_n E_{B(x, \sqrt{t})} |f|(x). \tag{75}
\]

Note \( E_{B(x, \sqrt{t})} |f| \leq c_n E_{B(x, \sqrt{t})} |f| \) for all \( s < 2t \). We then have from (75) that, for \( O_t \frac{t}{4} = e^{-tL} \), \( t \leq 1, tx \leq 1 \),

\[
O_t \frac{t}{4} |f|(x) = \int_0^\infty O_t |f|(x) \phi_{t, \frac{t}{4}}(s) \, ds \geq \frac{c}{\sqrt{t}} \int_0^{4t} O_s |f|(x) \, ds \geq c_n E_{B(x,t)} |f|(x).
\]
We then easily get
\[ 4O_{t,a}|f - O_{t,a} f|^2(x) \geq c_a E_{B(x,t^{4/5})} |f - E_{B(x,t^{4/5})} f(x)|^2(x), \]
by the convexity of $|\cdot|^2$, for $\alpha = \frac{1}{2}, 1$. Therefore,
\[ \|\cdot\|_{\text{BMO}(MM)} \lesssim \|\cdot\|_{\text{BMO}(L^\alpha)} \cdot \|bmo(L^\alpha)\| \cdot \|\cdot\|_{\text{BMO}(L^\alpha)} \]
and by Lemma \[ \|\cdot\|_{\text{BMO}(MM)} \lesssim \|\cdot\|_{\text{BMO}(L^\alpha)} \]
for all $0 < \alpha \leq 1$. By Theorem \[ the Ornstein-Uhlenbeck operator $L = -\frac{d^2}{dt^2} + x \cdot \partial_x$ has bounded $H^\infty(S_\eta)$ calculus from $L^\infty(\mathbb{R}^n)$ to Mauceri-Meda’s BMO$(MM)$ for any $\eta > \frac{n}{2}$.

Let $f(y) = \frac{1}{\sqrt{4\pi s}} \exp(-\frac{|y|^2}{4s})$, with $s > 100$. We have
\[
\begin{align*}
(O_t|f|^2 - |O_t f|^2)(x) &= \frac{1}{4\pi \sqrt{(s+2v)s}} \exp\left( -\frac{|x-t|^2}{2s+4v} \right) - \frac{1}{4\pi (s+v)} \exp\left( -\frac{|x-t|^2}{2s+2v} \right) \\
&= \left( \frac{1}{4\pi \sqrt{(s+2v)s}} - \frac{1}{4\pi (s+v)} \right) \exp\left( -\frac{|x-t|^2}{2s+4v} \right) \\
&\quad + \frac{1}{4\pi (s+v)} \left( \exp\left( -\frac{|x-t|^2}{2s+2v} \right) - \exp\left( -\frac{|x-t|^2}{2s+4v} \right) \right) \\
&\lesssim \frac{1}{s^3} + \frac{1}{s^2} \lesssim \frac{1}{s^2}.
\end{align*}
\]

On the other hand, for $v = \frac{1-e^{-2t}}{4}, v' = \frac{1-e^{-4t}}{4}$,
\[
(O_t f - O_{2t} f)(x) = \frac{1}{\sqrt{4\pi (s+v)}} e^{-\frac{|x-t|^2}{4s+4v}} - \frac{1}{\sqrt{4\pi (s+v')}} e^{-\frac{|x-t|^2}{4s+4v'}}.
\]

For $x^2 = e^{2t}(4s+4v)$, $t = 10$, we get
\[
|\langle O_t f - O_{2t} f \rangle(x)| \geq \left| \frac{1}{\sqrt{4\pi (s+v)}} e^{-\frac{1}{4s+4v}} - \frac{1}{\sqrt{4\pi (s+v')}} e^{-\frac{1}{4s+4v'}} \right| \\
\geq \frac{1}{2\sqrt{4\pi (s+v')}} \geq \frac{1}{10\sqrt{s}}.
\]

So,
\[
\|f\|_{\text{BMO}(L^\alpha)} \geq \sup_{t>0} \|O_t f - O_{2t} f\|_{L^\infty} \geq \frac{\sqrt{s}}{5} \|f\|_{bmo(L^\alpha)}.
\]

Therefore, the BMO$(L^\alpha)$ and bmo$(L^\alpha)$-norms are not equivalent for the Ornstein-Uhlenbeck semigroup, by letting $s \to \infty$. This shows that one cannot extend Lemma \[ to the case of $\alpha = 1$.
Example 4. Let \((G, \mu)\) be a locally compact unimodular group with its Haar measure. Let \(\lambda_g, g \in G\) be the translation-operator on \(L^2(G)\) defined as
\[
\lambda_g(f)(h) = f(g^{-1}h).
\]
The so-called group von Neumann algebra \(L^\infty(\hat{G})\) is the weak* closure in \(B(L_2(G))\) of the operators \(f = \int_G f(g)\lambda_g d\mu(g)\) with \(f \in C_c(G)\). The canonical trace \(\tau\) on \(L^\infty(\hat{G})\) is defined as \(\tau f = \hat{f}(e)\). If \(G\) is abelian, then \(L^\infty(\hat{G})\) is the canonical \(L^\infty\) space of functions on the dual group \(\hat{G}\). In particular, if \(G = \mathbb{Z}\), the integer group, then \(\lambda_k = e^{ikt}, k \in \mathbb{Z}\) and \(L^p(\hat{\mathbb{Z}}) = L^p(\mathbb{T})\), the function space on the unit circle. Please refer to [28] for details on noncommutative \(L^p\) spaces.

Let \(\varphi\) be a scalar valued function on \(G\). We say \(\varphi\) is conditionally negative if \(\varphi(g^{-1}) = \varphi(g)^*\) and
\[
\sum_{g,h} a_{g,h} \varphi(g^{-1}h) \leq 0
\]
for any finite collection of coefficients \(a_{g,h} \in \mathbb{C}\) with \(\sum_g a_g = 0\). Schöenberg’s theorem says that
\[
T_t : \lambda_g = e^{-t\varphi(g)}\lambda_g
\]
extends to a Markov semigroups of operators on the group von Neumann algebra \(L^\infty(\hat{G})\) if and only if \(\varphi\) is a conditionally negative function with \(\varphi(e) = 0\). The negative generator of the semigroup is the unbounded map \(L : \lambda_g \mapsto \varphi(g)\lambda_g\) which is weak* densely defined on \(L^\infty(\hat{G})\).

Let \(K_{\varphi}(g,h) = \frac{1}{2}(\varphi(g) + \varphi(h) - \varphi(g^{-1}h))\), the Gromov form associated with \(\varphi\). Then one can directly verify from (77) that \(K_{\varphi}\) is a positive definite function on \(G \times G\). Thus \(K_{\varphi}^2\) is a positive definite function too. This is equivalent to the \(\Gamma^2 \geq 0\) criterion for \(T_t\), and therefore Theorem 3 applies to all such \((T_t)'s.\) If in addition, \(\varphi\) is real valued, then \((T_t)\) is a symmetric Markov semigroup. We then obtain the following corollary.

Corollary 4. Let \(G\) be a locally compact unimodular group. Suppose \(\varphi\) is a conditionally negative function on \(G\) with \(\varphi(e) = 0\). Let \(L\) be the weak* densely defined linear map on \(L^\infty(\hat{G})\) such that \(L(\lambda_g) = \varphi(g)\lambda_g\). Then,

(i) For any \(\eta > \frac{n}{2}\) and any bounded analytic \(\Phi\) on \(S_\eta\), the map \(\Phi(L) : \lambda_g \mapsto \Phi(\varphi(g))\lambda_g\) extends to a completely bounded operator on \(BMO(\sqrt{L})\) and
\[
\|\Phi(L)\| \leq C_\eta \|\Phi\|_\infty.
\]

(ii) Suppose in addition that \(\varphi\) is real valued. If \(\Phi\) is a bounded analytic function on \(S_{\eta}\) with \(\eta > |\frac{n}{2} - \frac{1}{p}|\), then the map \(\Phi(L)\) extends to a completely bounded operator on \(L^p(\hat{G})\) for \(1 < p < \infty\).

Remark. Corollary 3 (i) was proved in [26] for \(L : \lambda_g \mapsto \sqrt{\varphi(g)}\lambda_g\) with \(\varphi\) a symmetric conditionally negative function on \(G\).

Example 5. Let \(G = \mathbb{F}_\infty\) be the nonabelian free group with a countably infinite number of generators. Let \(|g|\) be the reduced word length of \(g \in G\).
Then \( \varphi : g \to |g| \) is a conditionally negative function (see [15]) and \( L : \lambda g \mapsto |g| \lambda g \) generates a symmetric Markov semigroup on the free group von Neumann algebra. Fix \( \theta \in (\frac{\pi}{2}, \pi) \), let \( \Phi(z) = (\ln(z+2))^{-1} \) for \( z \in S_\theta \). Then \( \Phi \in H^\infty(S_\theta) \). Corollary 4 then implies that the Fourier multiplier

\[ \lambda g \mapsto \frac{1}{\ln(|g| + 2)} \lambda g \]

extends to a bounded operator on \( \text{BMO}(\sqrt{L}) \). By the interpolation result Lemma [11], we conclude that this multiplier is bounded on \( L^p(\hat{F}_\infty) \) with constant \( \lesssim p^{2p-1} \).

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