Mixed means of commutators
of central integral means and $CMO$

Shunchao Long and Jian Wang

In this paper we obtain some mixed means and weighted $L^p$ estimates for the commutators generating $r$ order central integral means operators with $CMO$ functions.

1. Introduction

Let $T$ be a sublinear operator and $b(x)$ a function, the commutator $[T, b]$ is defined by

$$[T, b]f(x) = T(fb)(x) - b(x)Tf(x).$$

A famous result of Coifman, Rochberg and Weiss stated that the commutators of Calderon-Zygmund singular integral operators and $BMO$ functions are bounded on $L^p$ for $1 < p < \infty$ (see [5]). Since then, the $L^p$ estimates and applications of these type commutators were studied by many authors (see [2,21,19,4] and [3,15,20,23]).

Recently the authors of this paper proved the $L^p$-boundedness of the commutators of Hardy operators and one-side $CMO$ functions for $1 < p < \infty$ in [18]. And this result was extended in [16,11]. In this paper we establish some mixed means estimates and weighted $L^p$-estimates of the commutators of central integral means and $CMO$ functions, and this extends the boundedness results in [18,11].

Let $1 \leq p < \infty$. Denote

$$\sup_{B \subset \mathbb{R}^n} \left( \frac{1}{|B|} \int_B |b(x) - b_B|^p dx \right)^{1/p} = \left\{ \begin{array}{ll} \|b\|_{BMO^p}, & \text{if } B \text{ are arbitrary balls,} \\ \|b\|_{CMO^p}, & \text{if } B \text{ are balls centered at origin,} \end{array} \right.$$ 

where $b_B = \frac{1}{|B|} \int_B b(x) dx$. If $\|b\|_{BMO^p} < \infty$, we say $b \in BMO^p$, the Bounded Mean Oscillation. It is well-known that $BMO^p = BMO^1 = BMO$ for all $1 \leq p < \infty$. If $\|b\|_{CMO^p} < \infty$, we say $b \in CMO^p$, the Central Mean Oscillation. Obviously, if $1 \leq p < q < \infty$, then $CMO^q \subset CMO^p$, and $\|b\|_{CMO^p} \leq \|b\|_{CMO^q}$.

We say $b \in CMO$, if $\|b\|_{CMO} = \sup_{1 \leq p < \infty} \|b\|_{CMO^p} < \infty$.

The spaces $CMO^p$ bear a simple relationship with $BMO$: $g \in BMO$ precisely when $g$ and all of its translates belong to $CMO^p$ uniformly a.e., so the classical $BMO$ space $\subset CMO^p$ for all $1 \leq p < \infty$. 
Many precise analogies exist between $CMO^p$ and $BMO$ from the point of view of real Hardy spaces, for example, the duality: $CMO^{p'}, p' = p/(p - 1)$, is the dual of the Beurling-Herz-Hardy spaces $HA^p, 1 \leq p < \infty$, that is then analogous to the $H^1 \leftrightarrow BMO$, (see [11-12]); the constructive decomposition (see [17]); and so on.

If $R > 0$, denote $B(R) = B(0, R)$ be the ball in $\mathbb{R}^n$ centered at origin and of radius $R$. Let $r, \alpha \in \mathbb{R}, f \in L_{\text{loc}}^r(\mathbb{R}^n, |x|^{\alpha(-1)})$, the central integral mean of order $r$, with the power weight, of $f$, be defined by

$$M_r(f, \alpha)(|y|) = \left[ \frac{1}{|B(|y|)|^\alpha} \int_{B(|y|)} |B(|x|)|^{\alpha - 1} |f(x)|^r dx \right]^{1/r},$$

and the companion mean of order $r$, with the power weight, of $f$, by

$$M_r^*(f, \alpha)(|y|) = \left[ \frac{1}{|B(|y|)|^\alpha} \int_{\mathbb{R}^n \setminus B(|y|)} |B(|x|)|^{\alpha - 1} |f(x)|^r dx \right]^{1/r}.$$

The properties and applications of these types of integral means can be found in many literatures. Firstly, the limits of $(M_2(f, 1)(y))^2 = (1/2y) \int_0^y |f|^2(y > 0$, the one-dimensional case), were used to study the almost periodic functions, and the spectrum and ergodicity of sample paths of certain stochastic processes by Wiener in [22]. Secondly, the functions spaces of bounded integral mean of $r$ order, introduced firstly by Beurling,

$$B^r = B^{r, \infty} = \{ f : M_r(f, 1)(|y|) \in L^\infty \},$$

(both homogeneous spaces ($|y| > 0$) and non-homogeneous spaces ($|y| > 1$)), together with their corresponding Beurling algebra $A^r$ and the Hardy space $HA^r$ [cf, 1, 12, 6] had rich contents; Thirdly, the Hardy type inequalities associating with $M_1(f, 1)(|y|)$ and $M_1^*(f, 1)(|y|)$ generalized the classical ones to $n$-dimension ball [9-10, 7]. And the mixed means inequalities were used to derive the generalizing Hardy and Levin-Cochran-Lee type inequalities in [7] (see also [8]).

We state the mixed means inequalities as following:

**Theorem 1** ([7: Theorem 5]). Let $r, s, R, \alpha, \gamma \in \mathbb{R}$, and let $r < s, r, s \neq 0, R > 0(f \neq 0$ in the case of $r < 0)$). Then,

$$M_s((M_r(f, \alpha), \gamma)(R) \leq M_r((M_s(f, \gamma), \alpha)(R), \quad (1)$$

$$M^*_s((M^*_r(f, \alpha), \gamma)(R) \leq M^*_r((M^*_s(f, \gamma), \alpha)(R). \quad (2)$$

From Theorem 1, we can obtain the $L^p$-boundedness estimates of these types of integral means above as following.

**Theorem 2.** Let $r, s, \alpha, \gamma \in \mathbb{R}$, and let $r < s, r, s \neq 0, s > 0(f \neq 0$ in the case of $r < 0$). Then,

$$\int_{\mathbb{R}^n} |B(|y|)|^{-1} (M_r(f, \alpha)(|y|))^s dy \leq \frac{1}{(\alpha - \gamma r/s)^{s/r}} \int_{\mathbb{R}^n} |B(|y|)|^{-1} |f(y)|^s dy \quad (3)$$
if $\alpha - \gamma r/s > 0$, and

$$\int_{\mathbb{R}^n} |B(|y|)|^{-1}(M_r^*(f, \alpha)(|y|))^sdy \leq \frac{1}{(\gamma r/s - \alpha)^s/r} \int_{\mathbb{R}^n} |B(|y|)|^{-1}f(y)|^sdy \quad (4)$$

if $\alpha - \gamma r/s < 0$.

**Proof.** For $y \in B(R)$, $\int_{B(|y|)} |B(|x|)|^{-1}f(x)|^sdx \leq \int_{B(|y|)} |B(|x|)|^{-1}f(x)|^sdx$ and $\int_{B(|y|)} |B(|y|)|^{\alpha - \gamma r/s}dy = \frac{1}{\alpha - \gamma r/s} |B(R)|^{\alpha - \gamma r/s}$ when $\alpha - \gamma r/s > 0$, and using (1), we have

$$\int_{B(R)} |B(|y|)|^{-1}(M_r(f, \alpha)(|y|))^sdy$$

$$\leq |B(R)|^{-\gamma s s/r} \left[ \int_{B(R)} |B(|y|)|^{-1} \left( \frac{1}{|B(|y|)|^s} \int_{B(|y|)} |B(|x|)|^{-1}|f(x)|^sdx \right)^{r/s} dy \right]^{s/r}$$

$$\leq \frac{1}{(\alpha - \gamma r/s)^{s/r}} \int_{B(R)} |B(|x|)|^{-1}|f(x)|^sdx,$$

(3) follows by taking the $\lim_{R \to \infty}$. And (4) is the consequence of (2), derived by the same technique as (3) from (1), except for taking the $\lim_{R \to 0}$.

**Remark** It is easy to see that $L^\infty \subset B^{r, \infty}$ for $0 < r < \infty$ (see also [6,12-13]). Let

$$B^{r,s}(\alpha, \gamma) = \{ f : M_r(f, \alpha)(|y|) \in L^s_{|x|^\gamma} \}.$$  

Then $r, s, \alpha, \gamma$ are under the conditions of Theorem 2 and $\alpha - \gamma r/s > 0$. Thus, we have

$$L^s_{|x|^\gamma} \subset B^{r,s}(\alpha, \gamma).$$

Further, we can obtain the boundedness estimates of the commutators generating $r$ order central integral means operators and $CMO$ functions.

From the point of view on Hardy spaces, the central integral means bear some relationships with $CMO^p$. In fact, $(A^p)^* = B^p$ and $(HA^p)^* = CMO^p$.

**2. COMMUTATOR THEOREMS**

Let $r > 0, \alpha \in \mathbb{R}, f \in L^r_{\text{loc}}(\mathbb{R}^n, |x|^\alpha)$, and $b$ be local integral functions on $\mathbb{R}^n$. We define the integral mean commutators of order $r$, with the power weight, by

$$M_{r,b}(f, \alpha)(|y|) = \left[ \frac{1}{|B(|y|)|^\alpha} \int_{B(|y|)} |B(|x|)|^{-1}(|b(x) - b(y)||f(x)||^r dx \right]^{1/r},$$

and the companion mean commutators of order $r$, with the power weight, by

$$M_{r,b}^*(f, \alpha)(|y|) = \left[ \frac{1}{|B(|y|)|^\alpha} \int_{\mathbb{R}^n \setminus B(|y|)} |B(|x|)|^{-1}(|b(x) - b(y)||f(x)||^r dx \right]^{1/r}.$$  

When $r \geq 1$, by the Minkowski inequality, it is easy to calculate that,

$$||M_r(\bullet, \alpha), b||f(|y|) || \overset{def}{=} |M_r(bf, \alpha)(|y|) - bf(|y|)M_r(f, \alpha)(|y|)| \leq M_{r,b}(f, \alpha)(|y|),$$

where
if $\alpha > 1$ and $\gamma < s/r$, where $c_2 = \frac{1}{|1-\gamma/s|^{s/r}}[2^n|\alpha/2|\alpha-1|3^r \times 4 \times 2^{nr} \sum_{k=0}^{\infty} 2^{-kn|\alpha-1|k}]^{s/r}.$

**Proof of Theorem 3** Let us prove (5). Let $2^{N-1} < R < 2^N$, denote

$$B_i = \begin{cases} B(2^i), & \text{if } i \leq N-1, \\ B(R), & \text{if } i = N, \end{cases}, C_i = B_i \setminus B_{i-1}, i = -\infty, \ldots, N;$$

if $x \in C_i$, denote

$$\overline{B_j} = \begin{cases} B_j, & \text{if } j \leq i-1, \\ B(|x|), & \text{if } j = i, \end{cases}, \overline{C_j} = \overline{B_j} \setminus \overline{B_{j-1}}, j = -\infty, \ldots, i.$$ (10)

Then, we have

$$h(x) = [(M_{r,b}(f,\alpha)(|x|))^s] = \frac{1}{|B(|x|)|^{\alpha}} \sum_{j=-\infty}^{i} \int_{C_j} |B(|y|)|^{\alpha-1}(|b(x) - b(y)||f(y)||^{\gamma} dy,$$

and

$$[M_s((M_{r,b}(f,\alpha),\gamma)(R))^s = \frac{1}{|B(R)|^{\gamma}} \sum_{i=-\infty}^{N} \int_{C_i} |B(|x|)|^{\gamma-1} [h(x)]^{s/r} dx.$$ (11)

If $x \in C_i$ or $\overline{C_i}$, using $|B(R)| = R^n|B(1)|$, we have

$$|B(|x|)|^{\alpha} \leq \begin{cases} |B(1)|^{\alpha 2^{i_0}a}, & \text{if } \alpha \geq 0, \\ |B(1)|^{\alpha 2^{i_0}a}, & \text{if } \alpha < 0, \end{cases} \leq |B(1)|^{\alpha 2^{i_0}a},$$ (12)
and for $r > 0$ ,

$$|a + b + c|^r \leq \begin{cases} 
|a|^r + |b|^r + |c|^r, & \text{if } 0 < r \leq 1, \\
3^{r-1}(|a|^r + |b|^r + |c|^r), & \text{if } 1 < r,
\end{cases} \leq 3^r(|a|^r + |b|^r + |c|^r). \quad (13)$$

The first inequality is obvious when $0 < r \leq 1$, from the property of convex function [14] when $r > 1$ since $g(x) = x^r$ is convex function. Noticing that $|b(x) - b(y)| \leq |b(x) - b_{B_i}| + |b(y) - b| |_{B_j}$, using (13) and (12), we have

$$h(x) \leq 2^{n|\alpha|2n|\alpha-1|3^r} \frac{1}{2^{\alpha n}|B(1)|} \sum_{j=-\infty}^{i} 2^{jn(\alpha-1)} \int_{C_j} |b(x) - b_{B_i}|^r |f(y)|^r dy$$

$$+ 2^{n|\alpha|2n|\alpha-1|3^r} \frac{1}{2^{\alpha n}|B(1)|} \sum_{j=-\infty}^{i} 2^{jn(\alpha-1)} \int_{C_j} |b(y) - b_{B_j}|^r |f(y)|^r dy$$

$$+ 2^{n|\alpha|2n|\alpha-1|3^r} \frac{1}{2^{\alpha n}|B(1)|} \sum_{j=-\infty}^{i} 2^{jn(\alpha-1)} \int_{C_j} |b_{B_i} - b_{B_j}|^r |f(y)|^r dy$$

$$= 2^{n|\alpha|2n|\alpha-1|3^r}(I_1 + I_2 + I_3).$$

By (10), we see that

$$C_j \subset B_i = B(|x|) \text{ for } j \leq i \text{ and } x \in C_i; \quad \text{and } \frac{1}{2^\alpha |B(1)|} \leq \frac{1}{|B_i|}.$$

Let $x \in C_i$, then we have

$$I_1 \leq |b(x) - b_{B_i}|^r \frac{1}{2^{\alpha n}|B(1)|} \int_{B_i} |f(y)|^r dy \sum_{j=-\infty}^{i} 2^{-(i-j)n(\alpha-1)}$$

$$\leq \frac{1}{1 - 2^{-n(\alpha-1)}} |b(x) - b_{B_i}|^r \frac{1}{|B_i|} \int_{B_i} |f(y)|^r dy,$$

noticing that $\alpha - 1 > 0$. For $I_2$, since $f$ is local bounded, by Holder’s inequality and Lebesque’s control convergence theorem, we have

$$\int_{C_j} |b(y) - b_{B_j}|^r |f(y)|^r dy \leq \left( \int_{C_j} |b(y) - b_{B_j}|^{r'} |y|^{r''} dy \right)^{1/r'} \left( \int_{C_j} |f(y)|^{r''} dy \right)^{1/r''}$$

$$\leq |B_j|^{1/r'} \|b\|_{C_{MO}}^{r'} \left( \int_{C_j} |f(y)|^{r''} dy \right)^{1/r''}$$

$$\rightarrow \|b\|_{C_{MO}}^{r'} \int_{C_j} |f(y)|^{r''} dy, \quad \text{(when } l \rightarrow 1). \quad (14)$$

Thus, as $I_1$,

$$I_2 \leq \|b\|_{C_{MO}}^{r'} \frac{1}{1 - 2^{-n(\alpha-1)}} \frac{1}{|B_i|} \int_{B_i} |f(y)|^r dy.$$
For $I_3$, by (10),

\[
|b_{B_i} - b_{B_j}| \leq \frac{1}{|B_j|} \int_{B_j} |b(x) - b_{B_i}| dx
\]

\[
\leq \left\{ \begin{array}{ll}
\frac{1}{|B_j|} \int_{B_j} |b(x) - b_{B_i}| dx, & \text{if } j \leq i - 1, \\
\frac{1}{|B_{j-1}|} \int_{B_j} |b(x) - b_{B_i}| dx, & \text{if } j = i,
\end{array} \right.
\]

\[
\leq \frac{2^n}{|B_j|} \int_{B_j} |b(x) - b_{B_i}| dx
\]

\[
\leq \frac{2^n}{|B_j|} \int_{B_j} |b(x) - b_{B_i}| dx + 2^n \sum_{h=0}^{i-1} |b_{B_h} - b_{B_{h+1}}| + 2^n \|b\|_{CMO(i - j)},
\]

thus

\[
I_3 \leq \|b\|_{CMO} \frac{1}{2^n |B(1)|} \int_{B_1} |f(y)|^r dy \sum_{j=-\infty}^{i} 2^{-(i-j)n(a-1)} (i - j)^r
\]

\[
\leq c \|b\|_{CMO} \frac{1}{|B_i|} \int_{B_i} |f(y)|^r dy,
\]

where $c = 2^{2nr} \sum_{j=-\infty}^{i} 2^{-(i-j)n(a-1)} (i - j)^r$. When $x \in C_i$, let

\[
g(x) = \left( \frac{1}{|B_i|} \int_{B_i} |f(y)|^r dy \right)^{1/r} = \left( \frac{1}{|B(|x|)|} \int_{B(|x|)} |f(y)|^r dy \right)^{1/r},
\]

combining to the estimates of $I_1, I_2, I_3$ above, and noticing that $(|b(x) - b_{B_i}|^r + \|b\|_{CMO}^s)^{s/r} \leq 2^{s/r} (|b(x) - b_{B_i}|^s + \|b\|_{CMO}^s)$ for $s > 0, r > 0$, we see that

\[
\int_{C_i} |B(|x|)|^{\gamma - 1} |h(x)|^{s/r} dx \leq c_0 \int_{C_i} |B(|x|)|^{\gamma - 1} |b(x) - b_{B_i}|^s + \|b\|_{CMO}^s \ g^s(x) dx, \tag{15}
\]

where $c_0 = [2^{n(a-1)} 3^n \times 4 \times 2^{2nr} \sum_{j=-\infty}^{i} 2^{-(i-j)n(a-1)} (i - j)^r]^{s/r}$. While, since $f$ local bound implies $g$ local bound, as (14),

\[
\int_{C_i} |B(|x|)|^{\gamma - 1} |b(x) - b_{B_i}|^s g^s(x) dx \leq \|b\|_{CMO}^s \int_{C_i} |B(|x|)|^{\gamma - 1} g^s(x) dx. \tag{16}
\]

Thus, combining to (11), (15) and (16), we obtain

\[
M_s((M_{r,b}(f, \alpha), \gamma))(R) \leq c_0^{1/s} \|b\|_{CMO} \left( \frac{1}{|B(R)|^{\gamma}} \sum_{i=-\infty}^{N} \int_{C_i} |B(|x|)|^{\gamma - 1} g^s(x) dx \right)^{1/s}
\]

\[
= c_0^{1/s} \|b\|_{CMO} M_s((M_{r,f,1}, \gamma))(R).
\]

Using (1), we obtain (5).
The proof of (6). Replace (9) and (10) by
\[ B_i = \begin{cases} B(2^i), & \text{if } i \geq N, \\ B(R), & \text{if } i = N - 1, \end{cases} \quad C_i = B_i \setminus B_{i-1}, i = N, N + 1, \ldots, \infty; \]
and if \( x \in C_i \),
\[ C_j = B_j \setminus B_{j-1}, j = i, i + 1, \ldots, \infty. \]

Then,
\[ h^*(x) = \left[ (M_{r,b}^* (f, \alpha))(|x|) \right]^r = \frac{1}{|B(|x|)|^\alpha} \sum_{j=i}^{\infty} \int_{C_j} |B(|y|)|^{\alpha-1} (|b(x) - b(y)||f(y)|)^r dy, \]
and
\[ \left[ M_{s, \gamma}^* ((M_{r,b}^* (f, \alpha), \gamma)(R)) \right]^s = \frac{1}{|B(|x|)|^\gamma} \sum_{i=N}^{\infty} \int_{C_i} |B(|x|)|^{\gamma-1} [h^*(x)]^{s/r} dx. \]

The rest of the proof of (6) is exactly similar to that of (5).

Thus, we finish the proof of Theorem 3.

The proofs of Theorem 4 by using (5)(6) are exactly similar to that of Theorem 2 by using (1)(2).

References

[1] A. Beurling, Construction and analysis of some convolution algebras, *Ann. Inst. Fourier (Grenoble)* **14**, (1964), 1-32.

[2] S. Bloom, A commutator theorem and weighted BMO, *Trans. Amer. Math. Soc.* **292**(1985), 103-122

[3] A.P. Calderon, Commutators of singular integral operators, *Proc. Natl. Acad. Sci. USA* 53 (1965) 1092-1099.

[4] R.Coiñman, P. L. Lions, Y. Meyer and S. Semmes, Compensated compactness and Hardy spaces, *J. Math. Pures. Appl.* **72**, (1993),247-286.

[5] R. R. Coifman, R. Rochberg and G. Weiss, Factorization theorems for Hardy spaces in several variables variables, *Ann.ofMath.* **103**(1976), 611-635.

[6] Y. Z. Chen and K. S. Lau, Some new classes of Hardy spaces, *J.Functional Anal.* **84**(1989), 255-278.

[7] A. Cizmesija, J. Pecaric and I. Peric, Mixed means and inequalities of Hardy and Levin-Cochran-Lee type for multidimensional balls, *Proc. Amer. Math. Soc.* **128**, No.9 (2000), 2543-2552.
[8] A. Cizmesija and J. Pecaric, Mixed means and Hardy’s inequality, *Math. Inequal. Appl.* 1, No.4 (1998), 497-506.

[9] M. Christ, and L. Grafakos, Best constants for two nonconvolution inequalities, *Proc. Amer. Math. Soc.* 123, No.6 (1995), 1687-1693.

[10] P. Drabek, H. P. Heinig and A. Kufner, Higher dimensional Hardy’s inequality, *Int. Ser. Num. Math.* 123(1997), 3-16.

[11] Z.Fu, Z.Liu, S.Lu, H.Wang, Characterization of the commutators of n-dim fractional order Hardy operator; Science in China Series A37, No.6, (2007), 651-659.

[12] J. Garcia-Cuerva, Hardy spaces and Beurling algebras, *J.LondonMath.Soc.* 39(1989), 499-513.

[13] J. Garcia-Cuerva and M. J. L. Herrero, A theory of Hardy spaces associated to the Herz spaces, *Proc.LondonMath.Soc.* 69, No.3(1994), 605-628.

[14] G. H. Hardy, J. E. Littlewood, and G. Polya , Inequalities, Cambridge Univ.Press, Cambridge, UK, 1959.

[15] S. Janson, Mean oscillation and commutators of singular integral operators, *Ark. Math.* 16, (1978), 263-270.

[16] Y. Komori, Notes on commutators of Hardy operators [M]; Intern J Pure Appl Math, 1, (2003), 329-334.

[17] J. D. Lakey, Constructive decomposition of functions of finite central mean oscillation, *Proc. Amer. Math. Soc.* 127, No.8 (1999), 2375-2384.

[18] Long Shunchao and Wang Jian, Commutators of Hardy operators, *J. Math. Anal. Appl.* 274, No.2 (2002), 626-644.

[19] C. Perez, Endpoint estimates for commutators of singular integral operators, *J.Functional Anal.* 128(1995), 163-195.

[20] M. Paluszynski, Characterization of the Besov spaces via commutators operator of Coifman, Rochberg and Weiss, *Indiana Univ. Math. J.* 44, (1995), 1-17.

[21] Carlos Segovia and Jose L. Torrea, Higher order commutators for vector-valued Calderon-Zygmund operators, *Trans. Amer. Math. Soc.* 336(1993),537-556.

[22] N. Wiener, Generalized harmonic analysis, *Acta Math.* 55, (1930), 117-258.

[23] A. Youssfi, Regularity properties of commutators and BMO-Triebel-Lizorkin spaces, *Ann. Inst. Fourier, Grenoble.* , No.3 (1995), 795-807.
Long Shunchao, Jian Wang,
Mathematics Department,
Xiangtan University,
Xiangtan, 411105, China,
E-mail address: sclong@xtu.edu.cn