An Explicit Shimura Tower of Function Fields over a Number Field: An Application of Takeuchi’s List

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Abstract

Elkies [E1] proposed a procedure for constructing explicit towers of curves, and gave two towers of Shimura curves as relevant examples. In this paper, we present a new explicit tower of Shimura curves constructed by using this procedure.

Keywords: Function field; Tower of function fields; Belyi map

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1 Introduction

According to a result obtained by Tsfasman, Vladut and Zink, excellent Goppa codes can be obtained from modular towers (more specifically, towers of modular curves) ([TVZ]). However, the construction of such codes requires explicit modular towers. In 1998, Elkies introduced a procedure for defining such towers (see Proposition in [E1]), and thereby constructed several classical (elliptic) modular towers, two Shimura towers and some Drinfeld modular towers ([E1, E3]). In 2012, Hasegawa, Inuzuka and Suzuki presented a number of classical modular towers constructed by the same procedure ([HIS, H]), and Garcia, Stichtenoth, Bassa and Beelen recently constructed several Drinfeld modular towers ([BBGS, BB]). On the other hand, although Elkies suggested new Shimura towers, he did not explicitly construct any (see Section 5 in [E2]). To the author’s knowledge, only two explicit Shimura towers have been constructed thus far.

Let $K$ be a totally real number field, and let $A$ be a quaternion algebra over $K$, namely, a central simple algebra over $K$ of dimension 4. Throughout this paper, we assume that $A$ is ramified at all but one of the infinite places of $K$. Let $D(A)$ denote a discriminant of $A$, that is, the product of all finite places $p$ of $K$ such that $A$ is ramified at $p$. For an ideal $I$ of $K$ coprime to $D(A)$, Shimura curves $X_0(I)$ analogous to classical modular curves $X_0(N)$ are well defined (see Section 2 below).

Note that the prime number $\ell = 3$ is totally ramified in $\mathbb{Q}(\sqrt{3})/\mathbb{Q}$. Our main theorem is as follows.

Main theorem. Let $K$ be the field $\mathbb{Q}(\sqrt{3})$, which is totally real, and let $v_3$ denote the place of $K$ lying above 3. Then, the Shimura curve $X_0(3^2)$ is rational, that is, the genus of $X_0(3^2)$ is equal to 0. For each natural number $n > 1$, the curve $X_0(3^n)$ is defined by $n-1$ coordinates $x_1, x_2, \ldots, x_{n-1}$ which are related by the $n-2$ equations

$$\left(-2\frac{x_{j+1} - 1}{x_{j+1} + 2}\right)^3 = 2\frac{(5 - 3 \sqrt{3})x_j^3 + 4}{x_j^3 - 2(5 - 3 \sqrt{3})} \quad (j = 1, 2, \ldots, n-2),$$

or, equivalently, the curve $X_0(3^n)$ is isomorphic to the locus of $(x_1, x_2, \ldots, x_{n-1})$ in $(\mathbb{P}^1)^{n-1}$ satisfying the above equations (1). In other words, the Shimura tower $\{X_0(3^n)\}_{n>1}$ is defined recursively by the affine equation

$$\left(-2\frac{y - 1}{y + 2}\right)^3 = 2\frac{(5 - 3 \sqrt{3})x^3 + 4}{x^3 - 2(5 - 3 \sqrt{3})}.$$

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Shimura curves $X_0(l)$ always have Atkin-Lehner involutions $\omega_l$. Also, unlike classical modular curves, Shimura curves have no cusps. Thus, although the covering maps between Shimura curves cannot be computed by using $q$-expansions, they can be determined from their ramification behavior. In fact, Elkies illustrated this with two examples (see Third variation in \[E1\]), and also see Section 2. In this paper, we show our main theorem based on the method devised by Elkies (see Section 3).

Beelen, Garcia and Stichtenoth derived a normal form for equations of recursive Kummer towers of function fields ([BCS]). In their notation, the equation of our main theorem is written as

$$Y^3 = \frac{-8(5 - 3 \sqrt{3})(X + 1)^3 + 4(X - 2)^3}{4(X + 1)^3 + (5 - 3 \sqrt{3})(X - 2)^3},$$

where the variables $X, Y$ are defined by $X = (2x - 2)/(x + 2), Y = (2y - 2)/(y + 2)$.

Let $\ell \neq 2, 3$ be a prime number, and let $v_\ell$ denote a place of $K(\sqrt{3})$ lying above $\ell$. Since the coefficients of the above equation are in the ring of integers of $K(\sqrt{3})$, the modulo $v_\ell$ reduction of our tower can be defined, and moreover, the reduction is asymptotically optimal over an extension field of $\mathbb{F}_\ell = \mathbb{Z}/(\ell)$.

Wulfange studied one of the (modified) Shimura towers given by Elkies and proved that this tower is optimal over $\mathbb{F}_{v_\ell}$ (see Section 4 in [W]). Moreover, Brander attempted to construct a Goppa code from the same tower ([B]). In this regard, a Goppa code can be defined by using the tower in our main theorem.

Let $\mathbf{I}$ be an ideal of $K$ coprime to $D(A)$. For an Atkin-Lehner involution $\omega_l$ and a map $j: X_0(l) \to \mathbb{P}^1_C$ (e.g., see Main theorem I (3.2) in [Smr]), the image of a morphism

$$X_0(l) \to \mathbb{P}^1_C \times \mathbb{P}^1_C, \quad z \mapsto (j(z), j(\omega_l(z)))$$

is a closed subvariety of dimension 1, and in the open affine set $(A_C^1 \times A_C^1) \setminus \{\infty\}$, this variety is described by a modular polynomial $\Phi_l(x,y)$. In 1998, Elkies computed a polynomial $\Phi_l(x,y)$ for $N(l) = 3, 8, 9$ ([E1, E3]), where $N$ denotes the norm with respect to $K/Q$, and furthermore, in 2005, Voight determined a polynomial $\Phi_l(x,y)$ for $N(l) = 17$ ([V1]). Polynomials $\Phi_{v_\ell}(x,y)$ and $\Phi_{v_\ell'}(x,y)$ for our curves can be computed by using our theorem.

This paper are organized as follows. In Section 2 two examples of explicit Shimura towers defined by Elkies in \[E1\] are introduced. In Section 3 a new explicit Shimura tower is given.

2 Explicit Shimura towers proposed by Elkies

In this section, we introduce two examples by Elkies (see Third variation in \[E1\]). The reason revisiting the examples of Elkies in this section is as follows: In \[E1\], Elkies left a detailed computational process for classical modular towers (see Example, First and Second variations in \[E1\]), but he omitted the most of a computational process for Shimura towers. Therefore, an important relationship between Shimura towers and arithmetic groups remains unclear. Hence, we shall write a detailed computational process for Shimura towers, and we reveal a relationship between Shimura towers and arithmetic groups (see Remark in this section). Moreover, this section is also a recall of the procedure by Elkies for constructing explicit Shimura towers in the next section.

Takeuchi gave a complete list of arithmetic triangle groups $\Delta$ of signature $(0; e_1, e_2, e_3)$ (see Table (1) in \[T\]). In fact, there exist 85 such groups. In this and the next sections, we will use three of these groups.

We consider the first example given by Elkies. Let $K$ be the number field $Q(\sqrt{3})$, which is totally real. The prime numbers $\ell = 3, 2$ are totally ramified in $K/Q$. Let $v_3$ (resp. $v_2$) denote the place of $K$ lying above 3 (resp. 2), that is, $v_3$ is $(\sqrt{3})$ (resp. $v_2 = (5 - 3 \sqrt{3})$). Also, let $\Delta$ be the arithmetic triangle group of signature $(0; 2, 4, 12)$, and let $A$ be the quaternion algebra associated with $\Delta$, which is ramified at $v_3$ and at exactly one infinite place. Since the discriminant $D(A)$ is equal to $v_3$, the
Shimura curves $X_0(p_2^n)$ are defined. Elkies subsequently constructs the explicit Shimura tower $\{X_0(p_2^n)\}_{n \geq 1}$ as follows:

Let $\mathcal{S}$ be the upper half-plane. The Shimura curve $X(1) = \Delta \setminus \mathcal{S}$ is rational. In fact,

$$2g(\Delta \setminus \mathcal{S}) - 2 = \text{Area}(\Delta \setminus \mathcal{S}) - \sum_{j=1}^{3} \left(1 - \frac{1}{e_j}\right) = \frac{1}{6} - \left(\frac{1}{2} + \frac{3}{4} + \frac{11}{12}\right) = -2,$$

and therefore $g(\Delta \setminus \mathcal{S}) = 0$ (cf. [Smr], and also see p.207 in [IL]). We can choose a coordinate $J$ on $X(1)$ which takes the values 1, 0, $\infty$ at the elliptic points $P_2, P_4, P_{12}$ of order 2, 4, 12, respectively, that is, $J(P_2) = 1, J(P_4) = 0$, and $J(P_{12}) = \infty$.

In general, for each number $n$, the Shimura curve $X_0(p_2^n)$ can be identified with the set

$$\left\{ (P, Q) \in X(1) \times X(1) \setminus \text{a point } P \text{ is } p_2^n\text{-isogenous to a point } Q \right\}$$

(e.g., [IL]), and this curve always has the Atkin-Lehner involution $\omega^{(n)} : (P, Q) \mapsto (Q, P)$. The covering map $\pi : X_0(p_2^n) \to X(1)$ is defined by the projection $\pi((P, Q)) = P$, and its degree equals

$$N(p_2^n) \prod_{p | p_2} \left(1 + \frac{1}{N(p)}\right) = N(p_2)^{n-1}(N(p_2) + 1) = 3 \cdot 2^{n-1},$$

where $N$ denotes the norm with respect to $K/Q$. The map $\pi$ is branched only above the elliptic points $P_2, P_4, P_{12}$ of $X(1)$ and always unramified above the other points. The ramification index at a point $(P, Q)$ of $X_0(p_2^n)$ is equal to the denominator of the irreducible fraction $\text{ord}(Q)/\text{ord}(P)$, where ord represents the order of a point, and the order of a non-elliptic point is equal to 1.

We can determine the ramification of the first map $\pi : X_0(p_2) \to X(1)$ and the involution $\omega^{(1)}$ of the curve $X_0(p_2)$ as follows: The point $P_{12}$ is totally ramified, and the point lying above $P_{12}$ is $(P_{12}, P_4)$. The points lying above $P_4$ (resp. $P_2$) are $(P_4, P_2), (P_4, P_{12})$ (resp. $(P_2, P_4), (P_2, *)$), and thus the ramification indices are equal to 2, 1 (resp. 1, 2), respectively. Here, $*$ denotes a non-elliptic point. Then, the involution $\omega^{(1)}$ must interchange the points $(P_4, P_{12}), (P_{12}, P_4)$ and the points $(P_2, P_4), (P_4, P_{12})$, respectively. Since the Hurwitz genus formula yields

$$2g(X_0(p_2)) - 2 = 3(-2) + (3 - 1) + (2 - 1) + (2 - 1) = -2,$$

the curve $X_0(p_2)$ is again rational, and $J$ is a polynomial of degree 3 with a triple pole, such that $J$ and $J - 1$ have double zeros. Then, we can choose a rational coordinate $t$ on $X_0(p_2)$ such that

$$J = t(4t - 3)^2 \quad \left(\text{so} \quad J - 1 = (t - 1)(4t - 1)^2\right),$$

and thus $\omega^{(1)}$ interchanges the points $t = 0, t = \infty$ and the points $t = 1, t = 3/4$, respectively. Hence,

$$\omega^{(1)}(t) = \frac{3}{4t}. \quad (2)$$

Next, we can determine the ramification of the second map $\pi : X_0(p_2^2) \to X_0(p_2) \to X(1)$ and the involution $\omega^{(2)}$ of the curve $X_0(p_2^2)$ as follows: The point $P_{12}$ is totally ramified, and the point lying above $P_{12}$ is $(P_{12}, P_2)$. The points lying above $P_4$ (resp. $P_2$) are $(P_4, *), (P_4, P_4), (P_4, P_4)$ (resp. $(P_2, P_{12}), (P_2, P_2), (P_2, *), (P_2, *), (P_2, *), (P_2, *))$, and thus the ramification indices are equal to 4, 1, 1 (resp. 1, 1, 2, 2), respectively. Then, the involution $\omega^{(2)}$ interchanges the points $(P_4, P_4), (P_4, P_4)$ and the points $t = 0, t = 1, t = 3/4, t = 1/4$.
(P_2, P_{12}), (P_{12}, P_2)$, respectively, and fixes the point $(P_2, P_2)$. By the Hurwitz genus formula, the curve $X_0(p_2^2)$ is rational. Thus, we can choose a rational coordinate $x$ on $X_0(p_2^2)$ such that

$$t = \frac{x^2 + 3}{4} \quad \text{(and)} \quad J = \frac{x^4(x^2 + 3)}{4},$$

and the involution $\omega^{(2)}$ interchanges the points $x = \sqrt{-3}, x = -\sqrt{-3}$ and the points $x = 1, x = \infty$, respectively, and fixes the point $x = -1$. Hence,

$$\omega^{(2)}(x) = \frac{x + 3}{x - 1}. \quad (4)$$

We now have all the information necessary to determine the curve $X_0(p_2^2)$. It follows from (2) (4) that

$$t \cdot \omega^{(1)}(t) = \frac{3}{4} \iff \frac{x^2 + 3}{4} \cdot \omega^{(1)}\left(\frac{x^2 + 3}{4}\right) = \frac{3}{4}$$

$$\iff (x^2 + 3)(2 + 3) = 12 \iff (x^2 + 3)\left(\frac{y + 3}{y - 1}\right)^2 + 3 = 12,$$

and hence the curve $X_0(p_2^2)$ is defined by $n - 1$ coordinates $x_1, \ldots, x_{n-1}$ satisfying the $n - 2$ relations

$$(x_j^2 + 3)(x_{j+1}^2 + 3) = 12 \quad (j = 1, \ldots, n - 2), \quad z_j := (x_j + 3)/(x_j - 1).$$

Since this tower $\{X_0(p_2^2)\}_{n>1}$ has cyclic steps, and since it becomes unramified after a finite number of steps, this is dominated by a 2-class field tower of the curve $X_0(p_2^2)$ over any finite field of odd characteristic.

Next, we consider the other example given by Elkies. Let $K$ be the number field $\mathbb{Q}(\cos \frac{2\pi}{18})$, which is totally real, and the prime number $\ell = 3$ is totally ramified in $K/\mathbb{Q}$. Also, let $p_3$ denote the place of $K$ lying above 3. Furthermore, let $\Delta$ be the arithmetic triangle group of signature $(0, 2, 3, 9)$, and let $A$ be the quaternion algebra associated with $\Delta$, which is ramified at exactly two infinite places. Since $A$ is not ramified at all finite places of $K$, the discriminant $D(A)$ is equal to 1, and thus the Shimura curves $X_0(p_3^2)$ are defined. Elkies constructs the explicit Shimura tower $\{X_0(p_3^2)\}_{n>1}$ as follows:

The Shimura curve $X(1) = \Delta \backslash \mathbb{H}$ is rational. In fact,

$$2g(\Delta \backslash \mathbb{H}) - 2 = \text{Area}(\Delta \backslash \mathbb{H}) = \sum_{j=1}^{3} \left(1 - \frac{1}{e_j}\right) = \frac{1}{18} - \frac{1}{2} + \frac{2}{3} + \frac{8}{9} = -2, \quad \text{and therefore} \quad g(\Delta \backslash \mathbb{H}) = 0$$

([Smz], and also see p.207 in [T]). We can choose a coordinate $J$ on $X(1)$ which takes the values $1, 0, \infty$ at the elliptic points $P_2, P_3, P_9$ of order 2, 3, 9, respectively. In other words, $J(P_2) = 1, J(P_3) = 0$, and $J(P_9) = \infty$. For each natural number $n$, the Shimura curve $X_0(p_3^n)$ can be defined, and this curve always has the Atkin-Lehner involution $\omega^{(0)}$. The degree of the map $\pi: X_0(p_3^n) \to X(1)$ equals

$$N(p_3^n) \prod_{p|p_3^n} \left(1 + \frac{1}{N(p)}\right) = N(p_3)^{n-1}(N(p_3) + 1) = 4 \cdot 3^{n-1}.$$
We can determine the ramification of the map \( X_0(p_3) \to X(1) \) and the involution \( \omega^{(1)} \) as follows: Note that this map is branched only above \( P_2, P_3, P_9 \) and unramified above the other points. The points lying above \( P_9 \) are \( (P_9, P_9), (P_9, P_3) \), and thus the ramification indices are equal to 1, 3, respectively. Also, the points lying above \( P_3 \) (resp. \( P_2 \)) are \( (P_3, P_9), (P_3, \ast) \) (resp. \( (P_2, \ast), (P_2, \ast) \)), and thus the ramification indices are equal to 1, 3 (resp. 2, 2), respectively. Then, the involution \( \omega^{(1)} \) must interchange the points \( (P_3, P_9), (P_9, P_3) \) and fix the point \( (P_9, P_9) \). Since, from the Hurwitz genus formula

\[
2g(X_0(p_3)) - 2 = 4(-2) + (3 - 1) + (3 - 1) + (2 - 1) + (2 - 1) = -2,
\]

and therefore \( g(X_0(p_3)) = 0 \), the curve \( X_0(p_3) \) is again rational. Then, we can choose a rational coordinate \( t \) on \( X_0(p_3) \) such that

\[
J = \frac{(t - 1)^3(9t - 1)}{64t^3} \quad \text{so} \quad J - 1 = -\frac{(3t^2 + 6t - 1)^2}{64t^3}.
\]

and the involution \( \omega^{(1)} \) interchanges the points \( t = 1, t = 0 \) while fixing the point \( t = \infty \). Hence,

\[
\omega^{(1)}(t) = 1 - t.
\]

Next, the ramification of the map \( X_0(p_2^9) \to X_0(p_3) \to X(1) \) and the involution \( \omega^{(2)} \) can be determined as follows: The points lying above \( P_9 \) are \( (P_9, P_3), (P_9, \ast) \), and the ramification indices are 3, 9, respectively. The points lying above \( P_3 \) (resp. \( P_2 \)) are \( (P_3, P_9), (P_3, P_3), (P_3, \ast), (P_3, \ast) \), \( (P_3, \ast) \) (resp. \( (P_2, \ast), (P_2, \ast), (P_2, \ast) \) ), \( (P_2, \ast) \) (resp. \( (P_2, \ast), (P_2, \ast), (P_2, \ast) \) ), \( (P_2, \ast) \) (resp. \( (P_2, \ast), (P_2, \ast), (P_2, \ast) \) ), and therefore the ramification indices equal 1, 1, 1, 3, 3, 3 (resp. 2, 2, 2, 2, 2), respectively. Then, the involution \( \omega^{(2)} \) interchanges the points \( (P_3, P_9), (P_9, P_3) \) and the points \( (P_3, P_3), (P_3, \ast) \), respectively. By the Hurwitz genus formula, the curve \( X_0(p_2^9) \) is again rational. Thus, we can choose a rational coordinate \( x \) on \( X_0(p_2^9) \) such that

\[
t = x^3 \quad \text{and} \quad J = -\frac{(x^3 - 1)^3(9x^3 - 1)}{64x^9},
\]

and the involution \( \omega^{(2)} \) interchanges the points \( x = 1, x = \infty \) and the points \( x = \zeta_3, x = \zeta_3^2 \), respectively. Here, \( \zeta_3 \) is a cube of unity. Hence, the involution \( \omega^{(2)} \) is given by

\[
\omega^{(2)}(x) = \frac{x + 2}{x - 1}.
\]

We now have all the information necessary to determine the curve \( X_0(p_2^9) \). It follows from (5) and (7) that

\[
t + \omega^{(1)}(t) = 1 \Leftrightarrow x^3 + \omega^{(1)}(x^3) = 1 \Leftrightarrow x^3 + x = x^3 + \omega^{(2)}(y)^3 = 1 \Leftrightarrow x^3 + \left(\frac{y + 2}{y - 1}\right)^3 = 1,
\]

and hence the curve \( X_0(p_2^9) \) is defined by \( n - 1 \) coordinates \( x_1, \ldots, x_{n-1} \) satisfying the \( n - 2 \) relations

\[
x_j^3 + z_{j+1}^3 = 1 \quad \text{for} \quad j = 1, \ldots, n - 2 \quad \text{and} \quad z_j := (x_j + 2)/(x_j - 1).
\]
Since this tower \( \{X_0(p_n^2)\}_{n>1} \) has cyclic steps, and since it become unramified after finitely many steps, this is dominated by a 3-class field tower of the curve \( X_0(p_3^2) \).

**Remark.** It follows from Table (1) in [1] that the Shimura curve \( X_0(p_3) \) corresponds to the triangle group \( \Delta \) of signature \((0; 3, 3, 9)\), that is, \( X_0(p_3) = \Delta \backslash \mathcal{S} \). In fact, \( \Delta \) has exactly three elliptic points \((P_0, P_3), (P_0, P_9), (P_0, P_9)\) of order \(3, 3, 9\), respectively. Voight studied Shimura curves of genus at most 2 [V2]. It follows from Table 4.3 in [V2] that the curve \( X_0(p_3^2) \) corresponds to a group \( \Gamma \) of signature \((0; 3, 3, 3)\), and the elliptic curve \( X_0(p_3^2) \) corresponds to a group of signature \((1; 3, 3, 3)\). In fact, \( \Gamma \) has exactly four elliptic points \((P_3, P_3), (P_3, P_3), (P_3, P_9), (P_9, P_9)\) of order \(3, 3, 3, 3\), respectively.

### 3 A new explicit Shimura tower

In Section 2 we introduced in detail two examples given by Elkies. In this section, using the method proposed by Elkies, we construct a new Shimura tower (see Main theorem in Section 1).

Let \( K \) be the number field \( \mathbb{Q}(\sqrt{3}) \). The prime numbers \( \ell = 3, 2 \) are totally ramified in \( K/\mathbb{Q} \). Let \( p_3 \) (resp. \( p_2 \)) denote the place of \( K \) lying above 3 (resp. 2). Also, let \( \Delta \) be the arithmetic triangle group of signature \((0; 3, 3, 6)\), and let \( A \) be the quaternion algebra associated with \( \Delta \), which is ramified at \( p_2 \) and at exactly one infinite place. Since the discriminant \( D(A) \) equals \( p_2 \), the Shimura curves \( X_0(p_3^4) \) are defined, and then the Shimura tower \( \{X_0(p_3^2)\}_{n>1} \) is constructed as follows:

The Shimura curve \( X(1) = \Delta \backslash \mathcal{S} \) is rational. In fact,

\[
2g(\Delta \backslash \mathcal{S}) - 2 = \text{Area}(\Delta \backslash \mathcal{S}) - \sum_{n=1}^{3} \left( 1 - \frac{1}{v_n} \right) = \frac{1}{6} - \left( \frac{2}{3} + \frac{2}{3} + \frac{5}{6} \right) = -2, \quad \text{and therefore} \quad g(\Delta \backslash \mathcal{S}) = 0
\]

(see [Smz], and also see p.207 in [1]). We can choose a coordinate \( j \) on \( X(1) \) which takes the values \( 1, 0, \infty \) at the elliptic points \( P_3, P_3, P_6 \) of order \( 3, 3, 6 \), respectively. Specifically, \( J(P_3) = 1, J(P_3') = 0, \) and \( J(P_6) = \infty \). For each natural number \( n \), the Shimura curve \( X_0(p_3^n) \) can be defined, and this curve has the Atkin-Lehner involution \( \omega^{(0)} \). The degree of the map \( \pi : X_0(p_3^n) \to X(1) \) is equal to

\[
N(p_3^n) \prod_{p \mid p_3} \left( 1 + \frac{1}{N(p)} \right) = N(p_3)^{n-1}(N(p_3) + 1) = 4 \cdot 3^{n-1}.
\]

We can determine the ramification of \( X_0(p_3) \to X(1) \) and the involution \( \omega^{(1)} \) as follows: Note that this map is branched only above \( P_3, P_3', P_6 \) and unramified above the other points. The points lying above \( P_6 \) are \((P_6, P_3), (P_6, P_3'), (P_6, P_3)\), and thus the ramification indices are equal to 2, 2, respectively. The points lying above \( P_3' \) (resp. \( P_3 \)) are \((P_3', P_6), (P_3, P_6)\) (resp. \((P_3, P_6), (P_3, P_3)\)), and therefore the ramification indices are equal to 1, 3 (resp. 1, 3), respectively. Then, the involution \( \omega^{(1)} \) interchanges the points \((P_3', P_6), (P_6, P_3')\) and the points \((P_3, P_6), (P_6, P_3)\), respectively. Since by the Hurwitz genus formula

\[
2g(X_0(p_3)) - 2 = 4(-2) + (2 - 1) + (2 - 1) + (3 - 1) + (3 - 1) = -2, \quad \text{and therefore} \quad g(X_0(p_3)) = 0,
\]

the curve \( X_0(p_3) \) is also rational. Then, we can choose a rational coordinate \( t \) on \( X_0(p_3) \) such that

\[
j = \frac{4(2t + 1)^3}{(t^2 + 10t - 2)^2}, \quad \text{so} \quad j - 1 = -\frac{t(t - 4)^3}{(t^2 + 10t - 2)^2},
\]

and the involution \( \omega^{(1)} \) interchanges the points \( t = \infty, t = -5 + 3 \sqrt{3} \) and \( t = 0, t = -5 - 3 \sqrt{3} \), from which

\[
\omega^{(1)}(t) = -\frac{(5 - 3 \sqrt{3})t - 2}{t + (5 - 3 \sqrt{3})}.
\]
At this stage, the ramification of $X_0(p)^2 \to X_0(p_3) \to X(1)$ and the involution $\omega^{(2)}$ can be determined as follows: The points lying above $P_0$ are $(P_0,\ast), (P_0, \ast)$, and the corresponding indices are equal to 6,6, respectively. The points lying above $P_3$ (resp. $P_3$) are $(P_3, P_3), (P_3, P_3)$, $(P_3, P_3), (P_3, P_3)$, $(P_3, P_3), (P_3, P_3)$, $(P_3, P_3), (P_3, P_3)$, $(P_3, P_3)$, and thus the corresponding indices are equal to 1,1,1,3,3,3 (resp. 1,1,1,3,3,3), respectively. Then, the involution $\omega^{(2)}$ interchanges the points $(P_3, P_3), (P_3, P_3)$, the points $(P_3, P_3), (P_3, P_3)$ and the points $(P_3, P_3), (P_3, P_3)$, respectively. By the Hurwitz genus formula, the curve $X_0(p^2)$ is rational. Thus, we can choose a rational coordinate $x$ on $X_0(p^2)$ such that

$$t = -\frac{(5 + 3\sqrt{3})x^3 + 4}{x^3 - 2(5 + 3\sqrt{3})} \tag{9}$$

whereby the involution $\omega^{(2)}$ interchanges the points $x = 1 - \sqrt{3}, x = 1 + \sqrt{3}$, the points $x = (1 - \sqrt{3})z_3, x = (1 - \sqrt{3})z_3^2$ and the points $x = (1 + \sqrt{3})z_3, x = (1 + \sqrt{3})z_3^2$, respectively. Here, $z_3$ is a cube of unity. Hence, the involution $\omega^{(2)}$ is given by

$$\omega^{(2)}(x) = -2 \frac{x - 1}{x + 2} \tag{10}$$

We now have all the information necessary to determine the curve $X_0(p_3^2)$. It follows by (8) (9) (10) that

$$\omega^{(1)}(t) = -\frac{(5 - 3\sqrt{3})t - 2}{t + (5 - 3\sqrt{3})} \iff \omega^{(1)}(x) = \frac{4}{x^3} \iff \omega^{(2)}(y)^3 + 4 = \frac{4}{x^3} \iff \left(\frac{-2y - 1}{y + 2}\right)^3 = 2\frac{(5 - 3\sqrt{3})x^3 + 4}{x^3 - 2(5 - 3\sqrt{3})},$$

and hence the curve $X_0(p_3^2)$ is defined by $n - 1$ coordinates $x_1, \ldots, x_{n-1}$ satisfying the $n - 2$ relations

$$\sum_{j=1}^{n-1} = \frac{2(5 - 3\sqrt{3})x_j^3 + 4}{x_j^3 - 2(5 - 3\sqrt{3})} \quad (j = 1, \ldots, n - 2), \quad z_j := -\frac{x_j - 1}{x_j + 2}.$$

**Remark.** Voight gave a list of Shimura curves of genus at most two ([V2]). It follows from Table 4.2 in [V2] that our Shimura curve $X_0(p_3)$ corresponds to a group $\Gamma_1$ of signature $(0;3,3,3,3)$, and our Shimura curve $X_0(p_3^2)$ corresponds to a group $\Gamma_2$ of signature $(0;3,3,3,3,3,3)$. In fact, $\Gamma_1$ has exactly four elliptic points $(P_3, P_3), (P_3, P_3), (P_3, P_3), (P_3, P_3)$ of order $3,3,3,3$, respectively, and $\Gamma_2$ has exactly six elliptic points $(P_3, P_3), (P_3, P_3), (P_3, P_3), (P_3, P_3), (P_3, P_3), (P_3, P_3)$ of order $3,3,3,3,3,3$, respectively.

**Remark.** While the curves given by Elkies are defined over $\mathbb{Q}$, ours is defined over $\mathbb{Q}(\sqrt{3})$. In general, ours also has a model over $\mathbb{Q}$ (see Proposition 5.1.2 in [V1]). However, such a model does not perhaps have a simple equation as in our theorem.
Remark. Takeuchi gave a complete list of arithmetic triangle groups of signature $(0; e_1, e_2, e_3)$: There exist 85 such groups, of which 76 groups have no cusps (see Cases II, . . . , XIX of Table (1) in [T]). Note that Shimura curves have no cusps. Elkies constructed two Shimura towers from two groups $(0; 2, 4, 12)$ and $(0; 2, 3, 9)$, respectively ([E1], and also see Section 2). This time, we constructed a Shimura tower from the group $(0; 3, 3, 6)$ (see Section 3). I think that by using the same procedure as Elkies, explicit Shimura towers can not be constructed from the other 73 groups any longer.

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