The effect of the three-spin interaction and the next nearest neighbor interaction on the quenching dynamics of a transverse Ising model

Uma Divakaran and Amit Dutta

Department of Physics, Indian Institute of Technology Kanpur, Kanpur 208016, India
E-mail: udiva@iitk.ac.in and dutta@iitk.ac.in

Received 14 September 2007
Accepted 10 October 2007
Published 9 November 2007

Abstract. We study the zero-temperature quenching dynamics of various extensions of the transverse Ising model (TIM) for when the transverse field is linearly quenched from \(-\infty\) to \(+\infty\) (or zero) at a finite and uniform rate. The rate of quenching is dictated by a characteristic scale given by \(\tau\). The density of kinks produced in these extended models while crossing the quantum critical points during the quenching process is calculated using a many-body generalization of the Landau–Zener transition theory. The density of kinks in the final state is found to decay as \(\tau^{-1/2}\). In the first model considered here, the transverse Ising Hamiltonian includes an additional ferromagnetic three-spin interaction term of strength \(J_3\). For \(J_3 < 0.5\), the kink density is found to increase monotonically with \(J_3\) whereas it decreases with \(J_3 > 0.5\). The point with \(J_3 = 0.5\) and the transverse field \(h = -0.5\) is multicritical, where the density shows a slower decay given by \(\tau^{-1/6}\). We also study the effect of ferromagnetic or antiferromagnetic next nearest neighbor (NNN) interactions on the dynamics of the TIM under the same quenching scheme. In a mean field approximation, the transverse Ising Hamiltonians with NNN interactions are identical to the three-spin Hamiltonian. The NNN interactions non-trivially modify the dynamical behavior; for example an antiferromagnetic NNN interaction results in a larger number of kinks in the final state in comparison to the case when the NNN interaction is ferromagnetic.

Keywords: integrable spin chains (vertex models), spin chains, ladders and planes (theory), quantum phase transitions (theory)
1. Introduction

The critical dynamics of classical systems have been studied extensively in the last three decades while the dynamics of a quantum system when swept across a quantum critical point (QCP) is a fairly recent object of study and not yet fully understood. The vanishing of the energy gap between the ground state and the first excited state of the quantum Hamiltonian signals the existence of a QCP [1, 2]. At a QCP, the correlation length as well as the relaxation time diverge, a phenomenon known as critical slowing down. This diverging timescale makes it impossible for any system to cross the quantum critical point without excitations from the ground state. The dynamics therefore is non-adiabatic in contrast to an adiabatic evolution where the system sticks to the instantaneous ground state throughout the quenching process. In recent years, there has been an upsurge in the study of dynamics close to a quantum critical point clearly indicating a growing interest in the field [3]–[13].

One such attempt was to extend Kibble’s theory of defect production introduced to explain early universe behavior [14] to the second-order quantum phase transitions. This method of calculating the density of defects is known as the Kibble–Zurek mechanism (KZM) [15]. The theory of the KZM for a classical second-order phase transition is based on the universality of the critical slowing down and leads to the prediction that the linear dimension of the ordered domains scales with the transition time \( \tau \) as \( \tau^w \) where \( w \) is some combination of critical exponents. The KZM has been confirmed by numerical simulations of the time dependent Ginzburg–Landau model [16] and also for various experimental systems [17]. On the other hand, for the He-4 superfluid transition, the KZM could not be verified experimentally [18]. Hence more experiments are clearly needed to put KZM theory on a stronger footing. The same idea has been applied to study the dynamics across a zero-temperature QCP by different groups [3]–[10].

The extended KZM for the zero-temperature quantum transitions relies on the fact that during the evolution when the system is close to the static critical point, the relaxation time diverges in a power-law fashion. The non-adiabatic effects become prominent when the time scale associated with the change of the Hamiltonian is of the order of the
The quenching dynamics of a transverse Ising model relaxation time. The loss of adiabaticity while crossing a quantum critical point can be quantified by estimating either the density of defects (e.g., the density of oppositely oriented spins in Ising models) in the final state [3]–[7] or the fidelity of the final state with respect to the ground state [3] or the residual energy [19]–[24]. The argument given above immediately leads to a \((1/\sqrt{\tau})\) dependence of the density of defects on the characteristic timescale \(\tau\) of the quenching. The residual energy is defined as the difference between the energy of the evolved ground state and the true ground state. This residual energy for the integrable disorder free systems is trivially proportional to the density of kinks with the proportionality constant being equal to the strength of interaction. In an optimization approach popularly known as the ‘quantum annealing’ [19]–[24], the strength of the quantum fluctuations is slowly reduced to zero so that a disordered and frustrated system of finite size is expected to reach adiabatically its true classical ground state. In the present literature, the expressions ‘annealing’ and ‘quenching’ are used synonymously. The residual energy turns out to be a more appropriate measure of non-adiabaticity for the annealing approach. In a recent work by Caneva et al [25], it has been shown that for a disordered quantum Ising spin chain, the residual energy and the density of kinks show different scaling behaviors with \(\tau\). Recently a general analysis has been carried out of the breakdown of the adiabatic limit in low-dimensional gapless systems [26].

In this paper, we will concentrate on the estimation of the density of defects produced during the dynamics of three different types of model Hamiltonian, all of them being exactly solved, at least at a mean field level, via the Jordan–Wigner transformation. These three Hamiltonians are extensions of the TIM with an additional interaction term in each and our aim is to study the effect of such interactions on the density of defects produced during the quenching. The additional terms are (i) a ferromagnetic three-spin interaction [27], (ii) an antiferromagnetic next nearest neighbor interaction and (iii) a ferromagnetic next nearest neighbor interaction. We consider the unitary evolution of the system prepared in the ground state of the initial Hamiltonian which crosses its equilibrium critical line as the system evolves. As described later, in all the cases, the fermionization of the Hamiltonian reduces it to a quadratic form and hence one can reduce the dynamics of a many-body Hamiltonian effectively to a 2 \times 2 Landau–Zener problem [28] in the Fourier representation.

The paper is organized as follows. Section 2 includes a detailed discussion on the analytical diagonalization of the transverse Ising Hamiltonian with a three-spin interaction term. In section 3, we have described the transverse quenching scheme along with the results for the above model. We have presented a comparison between the three-spin Hamiltonian and the Hamiltonians with next nearest neighbor interactions when treated at a mean field level in section 4. A brief summary of the work is presented in the concluding section with a brief discussion based on the recent developments in this field.

2. Model and the phase diagram

The Hamiltonian of a one-dimensional three-spin interacting transverse Ising system is given by [27]

\[
H = -\frac{1}{2} \left\{ \sum_i \sigma_i^z [h + J_3 \sigma_{i-1}^x \sigma_{i+1}^x] + J_x \sum_i \sigma_i^x \sigma_{i+1}^x \right\},
\]

(1)

\[\text{doi:10.1088/1742-5468/2007/11/P11001}\]
The quenching dynamics of a transverse Ising model

where $\sigma^z$ and $\sigma^x$ are non-commuting Pauli spin matrices, $J_x$ is the strength of the nearest neighbor ferromagnetic interaction while $J_3$ denotes the strength of the three-spin interaction. In the limit $J_3 \to 0$, the model reduces to the celebrated transverse Ising model studied extensively in recent years [1, 2]. By a duality transformation [31], the above Hamiltonian can be mapped to a transverse $XY$ model with competing (ferro–antiferromagnetic) interactions in the $x$ and $y$ components of the spin [27]. Interestingly, even in the presence of the three-spin interaction term, the Hamiltonian given by equation (1) is exactly solved via the Jordan–Wigner (JW) transformation [29, 31] which maps this interacting spin system to a system of non-interacting spinless fermions. Moreover, this three-spin term is found to be irrelevant in determining the quantum critical behavior of the system. The critical exponents are the same as that of the Ising model in a transverse field except for the case $J_x = 0$, and $J_3 = h$. For the sake of completeness, let us now provide a brief discussion on the diagonalization of the Hamiltonian.

In the JW transformation, the Pauli matrices are transformed to a set of fermionic operators ($c_i$) defined as

$$c_i = \sigma^{-}_i \exp \left( -i \pi \sum_{j=1}^{i-1} \sigma^+_j \sigma^-_j \right)$$

$$\sigma^z_i = 2c_i^\dagger c_i - 1$$

with $\sigma^\dagger = (\sigma^x + i \sigma^y)/2$ and $\sigma^- = (\sigma^x - i \sigma^y)/2$, and satisfy the standard anticommutation relations

$$\{c_i, c_j\} = \delta_{ij}, \quad \{c_i^\dagger, c_j^\dagger\} = \{c_i, c_j\} = 0.$$

We shall work in the basis in which $\sigma^z$ is diagonal so that the presence of a fermion at a particular site $i$ corresponds to an up spin (i.e., eigenvalue +1 of the operator $\sigma^-_i$) at that site. Using a periodic boundary condition, the Fourier transform of the Hamiltonian can be cast in the form

$$H = - \sum_{k>0} \left( h + \cos k - J_3 \cos 2k \right) (c_k^\dagger c_k + c_{-k}^\dagger c_{-k}) + i \left( \sin k - J_3 \sin 2k \right) (c_k^\dagger c_{-k}^\dagger + c_k c_{-k}).$$

Clearly, in the momentum representation of c-fermions, the Hamiltonian is quadratic and is translationally invariant. Using the Bogoliubov transformation, the Hamiltonian can be diagonalized to the form $- \sum_k \epsilon_k \eta_k^\dagger \eta_k$ where $\eta_k$ are the Bogoliubov quasiparticles and $\epsilon_k$ is the excitation energy or gap given by [27, 29]

$$\epsilon_k = (h^2 + 1 + J_3^2 + 2h \cos k - 2hJ_3 \cos 2k - 2J_3 \cos k)^{1/2}$$

with $J_x$ set equal to unity.

It can be easily shown that the gap of the spectrum vanishes at $h = J_3 + 1$ and also at $h = J_3 - 1$ with ordering (or mode softening) wavevectors $\pi$ and 0 respectively. These two lines correspond to quantum phase transitions from a ferromagnetically ordered phase to a quantum paramagnetic phase with the associated exponents being the same as in the transverse Ising model [30]. The wavevector at which the minimum of $\epsilon_k$ (equation (4)) occurs gets shifted from $k = 0$ to $\pi$ when one crosses the line $h = J_3$. Moreover, there is an additional phase transition at $h = -J_3$. This transition belongs to the universality class of...
The quenching dynamics of a transverse Ising model

J 3\hbar=1+

J 3 –1

Ferro

Para

Incommensurate

Para

Incommensurate

Commensurate

3-spin dominated

Figure 1. Equilibrium phase diagram of the three-spin interacting Ising model. Solid lines show phase boundaries and the dotted line marks the boundary between the incommensurate and the commensurate phase.

The anisotropic transition observed in the transverse XY model dual to the Hamiltonian (1) [32] and the phase boundary is flanked by the incommensurate phases on either side with ordering wavevector given by

\[
\cos k = \frac{h - J_3}{4hJ_3}. \tag{5}
\]

This incommensurate wavevector picks up a value \(k_0\) such that \(\cos k_0 = 1/2J_3\) at the phase boundary. Obviously, for \(J_3 < 0.5\), the anisotropic phase transition cannot occur. The equilibrium phase diagram of the model is shown in figure 1.

3. Transverse quenching and results

The dynamics of the three-spin interacting TIM is found to be very interesting due to the fact that the system crosses various quantum critical lines during the process of the dynamics. As mentioned already, the system deviates from the adiabatic evolution in the neighborhood of a quantum critical point where non-adiabaticity dominates due to the divergence of the relaxation time. We shall introduce the time dependence in the Hamiltonian through the transverse field which is linearly quenched from \(-\infty\) to \(+\infty\) at a steady finite rate given by \(h(t) \sim t/\tau\), where the quenching time \(\tau\) determines the rate of quenching [3–5]. At time \(t = -\infty\) the transverse field \(h = -\infty\) and hence all the spins are pointing in the negative z-direction. By virtue of the duality transformation, the transverse quenching of the three-spin Hamiltonian corresponds to the anisotropic
The quenching dynamics of a transverse Ising model

quenching of the transverse XY model where the interaction term of the later Hamiltonian is adiabatically changed from $-\infty$ to $\infty$ [10].

Let us recall the Hamiltonian given in equation (3) with a time dependent transverse field $h(t)$. This Hamiltonian can be split into a sum of independent terms, $H(t) = \sum_{k>0} H_k(t)$, where each $H_k(t)$ operates on a four-dimensional Hilbert space spanned by the basis vectors $|0\rangle, |k, -k\rangle, |k\rangle$ and $|-k\rangle$. The vacuum state where no $c$-particle is present is denoted by $|0\rangle$ which corresponds to a spin configuration with all spins pointing in the $-z$-direction. The form of the Hamiltonian readily suggests that the parity (even or odd) of total number of fermions (given by $n_k = c_k^\dagger c_k + c_{-k}^\dagger c_{-k}$) for each mode is conserved. Therefore, to study the quenching dynamics, it is convenient to project the Hamiltonian $H_k(t)$ in the subspace spanned by $|0\rangle$ and $|k, -k\rangle$. The projected Hamiltonian has a form

$$\begin{bmatrix}
h(t) + \cos k - J_3 \cos 2k & i(\sin k - J_3 \sin 2k) \\
-i(\sin k - J_3 \sin 2k) & -(h(t) + \cos k - J_3 \cos 2k)
\end{bmatrix}.
$$

In the reduced Hilbert space, any general state can be represented as a superposition of $|0\rangle$ and $|k, -k\rangle$ with time dependent amplitudes $u_k(t)$ and $v_k(t)$ such that $\psi_k(t) = u_k(t)|0\rangle + v_k(t)|k, -k\rangle$. The time evolution of the state is given by the Schrödinger equation

$$i\partial_t \psi_k(t) = H_k(t)\psi_k(t),
$$

We shall here use the initial conditions $u_k(-\infty) = 1$ and $v_k(-\infty) = 0$ which in the spin language corresponds to the state with all spins down. The off-diagonal term $\Delta = \sin k - J_3 \sin 2k$ represents the interaction between the two time dependent levels with energies $E_{1,2} = \pm [h(t) + \cos(k) - J_3 \cos 2k]$. The zeros of the off-diagonal term yield the mode softening wavevectors $k = 0, \pi$ and $\cos^{-1} 1/(2J_3)$ (provided $J_3 > 0.5$) at which the system becomes quantum critical for appropriate parameter values. At these parameter values and wavevectors, the system undergoes a non-adiabatic transition from its instantaneous ground state. A measure of non-adiabaticity can be obtained by comparing the two-level problem to the corresponding Landau-Zener transition equations [5, 7]. For a completely adiabatic transition, we expect the final state to be described by the probability amplitudes $u_k(+\infty) = 0$ and $v_k(+\infty) = 1$, i.e., the complete spin flip from down to up occurs. The non-adiabatic transition probability $p_k$ is directly given by $|u_k(+\infty)|^2$ where the probability amplitudes $u_k(t)$ and $v_k(t)$ are normalized at each instant of time. Equivalently, $p_k$ also measures the probability that the system remains in its initial state $|0\rangle$ even at the final time. Using the results of Landau-Zener transitions [21, 28], $p_k$ is found to be

$$p_k = |u_k(+\infty)|^2 = \exp(-2\pi\gamma) \quad \text{where} \quad \gamma = \frac{\Delta^2}{(d/dt)(E_1 - E_2)}.
$$

Therefore, in this model

$$p_k = \exp[-\pi\tau(\sin k - J_3 \sin 2k)^2].
$$

The variation of $p_k$ as a function of $k$ for different values of quenching time $\tau$ is shown in figure 2. It is to be noted that for $J_3 < 0.5$, there are peaks at $-\pi$, 0 and $\pi$ in the whole range of wavevectors from $-\pi$ to $\pi$ whereas for $J_3 > 0.5$ there are additional peaks at the incommensurate values $\pm \cos^{-1}(1/2J_3)$.

doi:10.1088/1742-5468/2007/11/P11001
The quenching dynamics of a transverse Ising model

As mentioned already, the degree of non-adiabaticity can be quantified through the density of kinks $n$ generated at $t = +\infty$ which is obtained by integrating the probability $p_k$ over the entire range of wavevector

$$n = \sum_k p_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk p_k. \quad (9)$$

A close inspection of equation (8) (see also figure 2) shows that for sufficiently slow quenching (i.e., large $\tau$), only modes close to the critical modes are excited. One can therefore, to the lowest order in $k$, replace $\sin k$ by $k$ in the exponential of equation (8) and arrive at an approximate analytical expression for density of kinks in the large $\tau$ limit, given by

$$n = \frac{1}{2\pi(1 - 2J_3)\sqrt{\tau}} + \frac{1}{2\pi(1 + 2J_3)\sqrt{\tau}} \quad (for \ J_3 < 0.5) \quad (10a)$$

$$n = \frac{2}{2\pi(2J_3 - 1)\sqrt{\tau}} + \frac{2}{2\pi(1 + 2J_3)\sqrt{\tau}} \quad (for \ J_3 > 0.5). \quad (10b)$$

In equation (10a), the first term corresponds to the contribution from modes close to $k = 0$ whereas the second term is due to the peaks at $k = \pi, -\pi$. For the case $J_3 > 0.5$, the contribution from peaks at $k = 0, \pi$ and $-\pi$ happens to be the same as equation (10a) whereas the contribution $n_1$ from the modes close to $k = k_0$ is also equal to (10a) in the

Figure 2. Non-adiabatic transition probability $p_k$ for the three-spin interacting Hamiltonian with $J_3 = 0.1$ in (a) and $J_3 = 1$ in (b) for various $\tau$. It should be noted that for $J_3 = 1$, the system undergoes a non-adiabatic transition at an incommensurate wavevector $k = \pi/3$ and therefore, there is an additional peak at this wavevector. For large $\tau$, $p_k$ is non-zero only for wavevectors very close to the critical modes. On the other hand, for small values of $\tau$, levels cross quickly resulting to a non-zero value of $p_k$ for all values of $k$.  

As mentioned already, the degree of non-adiabaticity can be quantified through the density of kinks $n$ generated at $t = +\infty$ which is obtained by integrating the probability $p_k$ over the entire range of wavevector

$$n = \sum_k p_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk p_k. \quad (9)$$

A close inspection of equation (8) (see also figure 2) shows that for sufficiently slow quenching (i.e., large $\tau$), only modes close to the critical modes are excited. One can therefore, to the lowest order in $k$, replace $\sin k$ by $k$ in the exponential of equation (8) and arrive at an approximate analytical expression for density of kinks in the large $\tau$ limit, given by

$$n = \frac{1}{2\pi(1 - 2J_3)\sqrt{\tau}} + \frac{1}{2\pi(1 + 2J_3)\sqrt{\tau}} \quad (for \ J_3 < 0.5) \quad (10a)$$

$$n = \frac{2}{2\pi(2J_3 - 1)\sqrt{\tau}} + \frac{2}{2\pi(1 + 2J_3)\sqrt{\tau}} \quad (for \ J_3 > 0.5). \quad (10b)$$

In equation (10a), the first term corresponds to the contribution from modes close to $k = 0$ whereas the second term is due to the peaks at $k = \pi, -\pi$. For the case $J_3 > 0.5$, the contribution from peaks at $k = 0, \pi$ and $-\pi$ happens to be the same as equation (10a) whereas the contribution $n_1$ from the modes close to $k = k_0$ is also equal to (10a) in the
The quenching dynamics of a transverse Ising model

Figure 3. The variation of the kink density with $\tau$ for different $J_3$ ($<0.5$) is shown in (a). The kink density increases with $J_3$. On the other hand for $J_3 > 0.5$, this variation decreases with $J_3$ as shown in (b). The thick line has slope $-0.5$ and the slope of the dotted line is $-1/6$, a behavior observed at $J_3 = 0.5$.

following way:

$$ n_1 = \frac{1}{\pi} \int_{0}^{\pi} \exp\{-\pi \tau \{ (\cos k_0 - 2J_3 \cos 2k_0)(k - k_0) \}^2 \} $$

$$ = \frac{1}{2\pi \sqrt{\tau}} \left[ \frac{1}{2J_3 + 1} + \frac{1}{2J_3 - 1} \right]. $$

(11)

The density of kinks monotonically increases with increasing $J_3$ provided $J_3 < 0.5$ because of the decrease in the off-diagonal term making the probability of excitations higher. On the other hand, for $J_3 > 0.5$, the off-diagonal term monotonically increases with increasing $J_3$ resulting in an overall decrease in the density of kinks, see figure 3. These results can also be seen from the approximate analytical expression of the kink density given in equations (10a) and (10b) for both cases.

We shall now focus our attention on the case $J_3 = 0.5$. In the process of the transverse quenching, the system crosses the multicritical point at $h = -0.5, J_3 = 0.5$ as shown in figure 1 and a special power-law behavior of the kink density is observed at these parameter values. The transition probability $p_k$ maximizes at $k = 0$ as shown above. The argument of the exponential in $p_k$ is expanded about $k = 0$ at $J_3 = 0.5$, leading to a form $p_k = \exp[-\pi \tau k^6/4]$. The contribution to the kink density scales as $1/\tau^{1/6}$ which can be obtained by simply integrating this $p_k$, see figure 3. This relatively slow decay of density is a special characteristic of quenching through a multicritical point. A similar behavior is also seen in the anisotropic quenching of the transverse $XY$ model [10].

One can also study the effect of the anisotropic quenching which involves quenching of the nearest neighbor Ising interaction term $J^x(t)(\sim t/\tau)$ from $-\infty$ to $+\infty$ instead of the transverse field with the three-spin interaction term set to unity. At $t \rightarrow -\infty$, the ground state of the system is antiferromagnetic along $x$. The probability of the non-adiabatic
transition is similarly given by

\[ p_k = \exp \left[ -\pi \tau \{(h + 1) \sin k\}^2 \right]. \] (12)

It is interesting to note that for \( h = -1 \), \( p_k \) is unity for all values of \( k \). The density of kinks for the anisotropic quenching is given as

\[ n = \frac{1}{2\pi} \int dk p_k = \exp\left( (-\pi \tau (h + 1) \sin k)^2 \right) \] (13)

with an approximate analytical form given as

\[ n = \frac{1}{\pi (h + 1) \sqrt{\tau}} \] (14)

which shows that the density of kinks decreases monotonically with \( h \). This can be attributed to an increase in the off-diagonal term of the Hamiltonian.

4. Connection to the transverse quenching of the models with next nearest neighbor interactions

We shall now use the results of the previous section to study the transverse quenching of a quantum Ising model with uniform ferromagnetic nearest neighbor interaction and also an additional NNN interaction which is either antiferromagnetic or ferromagnetic. The model with NNN antiferromagnetic interaction has regular frustrations and is popularly known as the axial next nearest neighbor Ising (ANNNI) model [33] in a transverse field. We shall show below that within a mean field approximation, the three-spin model has a close resemblance to the one-dimensional NNN interacting TIMs.

The Hamiltonian of the transverse ANNNI model is given by

\[ H = -\frac{1}{2} \sum_{i=1}^{N} \left[ h \sigma_i^z + J_1 \sigma_i^x \sigma_{i+1}^x + J_2 \sigma_i^x \sigma_{i+2}^x \right] \] (15)

where \( J_1, J_2 > 0 \). Henceforth, without loss of any generality, we shall set \( J_1 = 1 \). At \( h = 0 \), the ground state is ferromagnetically ordered for \( J_2 < 0.5 \), whereas the system shows an ‘antiphase’ ordering (where two up spins are followed by two down spins) for \( J_2 > 0.5 \). The two phases meet at an infinitely degenerate multicritical point \( J_2 = 0.5 \) and \( h = 0 \). The quantum fluctuations introduced by the transverse field \( h \) compete with the ferromagnetic (or the antiphase) order and eventually the system undergoes a quantum phase transition to a paramagnetic phase at a critical value of the transverse field given by \( h_c \) which is a function of the NNN interaction \( J_2 \). The one-dimensional quantum ANNNI model shows a rich phase diagram which is not fully understood to date [2,33,34].

When mapped to the corresponding fermionic Hamiltonian via a JW transformation, the NNN interaction leads to a four-fermion term in the fermionic version of the Hamiltonian. In the limit \( J_2 \to 0 \), this term vanishes so that the model is exactly solvable in terms of non-interacting fermions. For non-zero \( J_2 \), the fermionic Hamiltonian is
The quenching dynamics of a transverse Ising model

written as

\[ H = \sum_i h(2c_i^\dagger c_i - 1) + (c_i^\dagger - c_i)(c_{i+1}^\dagger + c_{i+1}) \]

\[ - J_2(c_i^\dagger - c_i)(1 - 2c_{i+1}^\dagger c_{i+1})(c_{i+2}^\dagger + c_{i+2}) \]. \hspace{1cm} (16)\]

The occurrence of the four-fermion term renders the model analytically intractable though an approximate analytical solution is possible at least in the limit of small \( J_2 \). Deep in the paramagnetic phase all the spins are oriented in the direction of the transverse field \( \langle \sigma_z \rangle = 1 \) or in the fermionic language \( 1 - 2c_i^\dagger c_{i+1} = -1 \). We shall approximate \( \langle \sigma_z \rangle = 1 \) for all positive values of \( h \), including \( h \sim 0 \). This approximation, though crude, transforms the four-fermion term \((c_i^\dagger - c_i)(1 - 2c_{i+1}^\dagger c_{i+1})(c_{i+2}^\dagger + c_{i+2})\) to a quadratic form.

The Hamiltonian becomes exactly solvable but the rich phase diagram of the model is not captured in this approximation \[35,36\]. Within this approximation, we shall explore the role of small NNN antiferromagnetic interaction in the density of kinks produced during the transverse quenching. As described below, this approximation at least shows a decrease of critical field \( h_c \) with \( J_2 \) for \( J_2 < 0.5 \).

The mean field Hamiltonian in the momentum space is

\[ H = -\sum_{k>0} (h + \cos k + J_2 \cos 2k)(c_k^\dagger c_k + c_{-k}^\dagger c_{-k}) + i(\sin k + J_2 \sin 2k)(c_k^\dagger c_{-k}^\dagger + c_{-k} c_k) \]. \hspace{1cm} (17)\]

Comparing equation (17) with (3), one finds that the transverse ANNNI chain Hamiltonian in the mean field approximation is identical to the three-spin Hamiltonian if the antiferromagnetic interaction \( J_2 \) of the former is replaced by the negative of the three-spin interaction term \( (J_3) \) in the latter. Using the results of the previous section, the phase diagram of the mean field ANNNI model can be found for \( h > 0 \) (see figure 4). The phase boundary between the ferromagnetic phase and the paramagnetic phase is given by \( h = 1 - J_2 \) with an ordering wavevector \( \pi \) (this corresponds to the Ising transition at \( h = J_3 + 1 \) of figure 1). For \( J_3 > 0.5 \), i.e., the transition between the antiphase and the paramagnetic phase, we have the corresponding anisotropic transition of the three-spin model with the phase boundary given by the equation \( h = J_2 \) and the ordering wavevector has an incommensurate value as given in equation (5).

The approximation \( 1 - 2c_i^\dagger c_i = -1 \) is valid for positive \( h \) only; we choose a quenching scheme where the transverse field has a functional dependence \( h(t) \sim -t/\tau \) with \( t \) going from \(-\infty\) to 0 so that \( h(t) \) remains positive for the entire quenching period and vanishes at the end of the quenching. Therefore the system does not cross the Ising critical line \(-h = J_2 + 1\).

In the final state at \( t = 0 \), all the spins are expected to orient in the \( x \)-direction with a ferromagnetic order. The density of oppositely oriented spins at \( t \to 0 \) is related to \( J_2 \) as

\[ n = \frac{1}{2\pi(1 - 2J_2)^2} \]

which shows that the density of kinks increases monotonically with \( J_2 \).

doi:10.1088/1742-5468/2007/11/P11001
Figure 4. Mean field phase diagram of the ANNNI model in the $h$–$J_2$ plane. We study the quenching dynamics across the phase boundary close to $J_2 \to 0$.

It should be noted that if we follow a quenching scheme in which the transverse field is changed from $-\infty$ to zero, we must approximate the term $1 - 2c_{i+1}^\dagger c_{i+1}$ with +1 rather than −1 for the above calculations to be viable. In the process of dynamics, the system crosses the quantum critical line $-h = J_2 + 1$ with the modes close to $k = 0$ getting excited. This approach also leads to an identical result for the kink density (as given in equation (18)). Therefore, the presence of a small antiferromagnetic NNN interaction adds to the kink production in comparison to the ferromagnetic transverse Ising model ($J_2 = 0$) with the same quenching scheme.

One can also study, in a similar spirit, a model with a small ferromagnetic NNN interaction $J_{FM}$. We use the same mean field approximation for $h \geq 0$ so that this model is identical to the three-spin model with $J_3 = J_{FM}$. A similar calculation leads once again to a $1/\sqrt{\tau}$ fall of the density of kinks given by

$$n = \frac{1}{2\pi(1 + 2J_{FM})\sqrt{\tau}}. \tag{19}$$

This is expected because the NNN ferromagnetic interaction enhances the strength of the ferromagnetic ordering and hence the probability of excitations or density of kinks is lowered.

5. Conclusions

In this paper, we have studied the effect of various additional interactions on the dynamics of the transverse Ising model when swept across the quantum critical lines. The defect density scales with the time scale $\tau$ as $\tau^{-1/2}$, like in the transverse Ising case, with a prefactor which varies from model to model. The first of the variants includes a three-spin interaction with strength $J_3$. Here, the phase diagram indicates the existence of an
anisotropic phase transition at an incommensurate value of the wavevector in addition to the normal Ising transition for $J_3 > 0.5$. Interestingly, we observe that the density of kinks increases monotonically with $J_3$ for $J_3 < 0.5$ whereas it decreases for $J_3 > 0.5$. On the other hand, at $J_3 = 0.5$, the contribution to the kink density scales as $\tau^{-1/6}$ due to the existence of a multicritical point at $J_3 = 0.5$. The other set of Hamiltonians includes a ferromagnetic or an antiferromagnetic next nearest neighbor interaction. The presence of the four-fermion term makes such a Hamiltonian analytically intractable. We have used a mean field approximation to reduce the four-fermion term in the fermionized representation to a quadratic term. The quenching scheme is chosen carefully so that the regions where the approximation is not valid are avoided in the process of dynamics. Using the similarity between the fermionized next nearest neighbor interacting Hamiltonians under the mean field approximation, and the three-spin interacting model, the density of kink in the final state is estimated. It is observed that the ferromagnetic next nearest neighbor interaction reduces the density of kinks produced as opposed to the case of antiferromagnetic next nearest neighbor interaction because such a ferromagnetic interaction enhances the ferro-ordering discouraging the production of kinks. On the other hand, frustration leads to enhanced non-adiabatic transitions. We should mention in conclusion that it is in principle possible to construct a better mean field theory for the ANNNI model [36]; however, no qualitative change in the dynamical behavior in the region $J_2 \to 0$ is expected.

We conclude with the comment that the models studied in the present work are integrable (at least in the mean field limit) which leads to a $1/\sqrt{\tau}$ scaling behavior of the defect density. However, in a random or a non-integrable system such a behavior need not be expected [25]. The quenching and annealing dynamics of several non-integrable systems along with the dependence of the defect density on the integrability of the model are yet to be completely understood. We have also observed a much slower decay of the form $1/\tau^{1/6}$ when quenched through the multicritical point of the three-spin model as in the anisotropic quenching of the transverse $XY$ chain [10].

**Acknowledgments**

We acknowledge Victor Mukherjee and Diptiman Sen for collaboration in related works.

**References**

[1] Sachdev S, 1999 *Quantum Phase Transitions* (Cambridge: Cambridge University Press)

[2] For a review on phase transitions in TIMs see: Chakrabarti B K, Dutta A and Sen P, 1996 *Quantum Ising Phases and Transitions in Transverse Ising Models* vol m41 (Heidelberg: Springer)

[3] Zurek W H, Dorner U and Zoller P, 2005 *Phys. Rev. Lett.* 95 105701

[4] Dziarmaga J, 2005 *Phys. Rev. Lett.* 95 245701

[5] Damski B, 2005 *Phys. Rev. Lett.* 95 035701

[6] Damski B and Zurek W H, 2006 *Phys. Rev. A* 73 063405

[7] Polkovnikov A, 2005 *Phys. Rev. B* 72 161201

[8] Cherng R W and Levitov L S, 2006 *Phys. Rev. A* 73 043614

[9] Cincio L, Dziarmaga J, Rams Marek M and Zurek W H, 2007 *Phys. Rev. A* 75 052321

[10] Mukherjee V, Divakaran U, Dutta A and Sen D, 2007 *Preprint cond-mat/0703314*

[11] Cramer M, Dawson C M, Eisert J and Osborne T J, 2007 *Preprint cond-mat/0703314*

[12] Das A, Sengupta K, Sen D and Chakrabarti B K, 2006 *Phys. Rev. B* 74 144423

[13] Kibble T B W, 1976 *J. Phys. A: Math. Gen.* 9 1387

doi:10.1088/1742-5468/2007/11/P11001
The quenching dynamics of a transverse Ising model

[15] Zurek A H, 1985 Nature 317 505
Zurek A H, 1996 Phys. Rep. 276 177
[16] Laguna P and Zurek W H, 1997 Phys. Rev. Lett. 78 2519
Laguna P and Zurek W H, 1998 Phys. Rev. D 58 5021
Yates A and Zurek W H, 1998 Phys. Rev. Lett. 80 5477
Stephens G J et al, 1999 Phys. Rev. D 59 045009
Antunes N D et al, 1999 Phys. Rev. Lett. 82 2824
Dziarmaga J, Laguna P and Zurek W H, 1999 Phys. Rev. Lett. 82 4749
Hindmarsh M B and Rajantie A, 2000 Phys. Rev. Lett. 85 4660
Stephens G J, Bettencourt L M A and Zurek W H, 2002 Phys. Rev. Lett. 88 137004
[17] Ruutu V M H et al, 1996 Nature 382 334
Bairle C et al, 1996 Nature 382 332
[18] Hendry P C et al, 1994 Nature 368 315
Dodd M E et al, 1998 Phys. Rev. Lett. 81 3703
[19] Kadawaki T and Nishimori H, 1998 Phys. Rev. E 58 5355
[20] Das A and B K Chakrabarti (ed), 2005 Quantum Annealing and Related Optimization Methods
(Berlin: Springer)
[21] Suzuki S and Okada M, 2005 Quantum Annealing and Related Optimization Methods ed A Das and
B K Chakrabarti (Berlin: Springer)
[22] Brohee J, Bitko D, Rosenbaum T F and Aeppli G, 1999 Science 284 779
[23] Santoro G E, Martonak R, Tosatti E and Car R, 2002 Science 295 2427
[24] Santoro G E and Tosatti E, 2006 J. Phys. A: Math. Gen. 39 R393
[25] Caneva T, Fazio R and Santoro G E, 2007 Preprint 0706.1832
[26] Polkovnikov A and Gritsev V, 2007 Preprint cond-mat/0706.0212
[27] Kopp A and Chakravarty S, 2005 Nat. Phys. 1 53
[28] Zener C, 1932 Proc. R. Soc. A 137 696
Landau L D and Lifshitz E M, 1965 Quantum Mechanics: Non Relativistic Theory 2nd edn (Oxford: Pergamon)
[29] Lieb E, Schultz E and Mattis D, 1961 Ann. Phys., NY 61 407
[30] Pfleuty P, 1970 Ann. Phys., NY 57 79
[31] Kogut J B, 1979 Rev. Mod. Phys. 51 659
[32] Bunder J E and McKenzie R H, 1999 Phys. Rev. B 60 344
[33] Selke W, 1988 Phys. Rep. 170 213
[34] Villain J and Bak P, 1981 J. Physique 42 657
Selke W, 1992 Phase Transitions and Critical Phenomena vol 12, ed C Domb and J L Lebowitz (New York: Academic)
Selke W, 1992 Phase Transitions and Critical Phenomena vol 15, ed C Domb and J L Lebowitz (New York: Academic)
Yeomans J i, 1987 Solid State Physics vol 41, ed H Ehrenreich and J L Turnbull (New York: Academic)
[35] Dutta A and Sen D, 2003 Phys. Rev. B 67 094435
Shirahata T and Nakamura T, 2001 Phys. Rev. B 65 024402
[36] Chandra A K and Das Gupta S, 2007 J. Phys. A: Math. Theor. 40 6251
Beccaria M, Camostri M and Feo A, 2007 Preprint cond-mat/0702676
[39] Sen P and Chakrabarti B K, 1989 Phys. Rev. B 40 760
[40] Sen P and Chakrabarti B K, 1991 Phys. Rev. B 43 13559