Efficiency fluctuations of a quantum Otto engine

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We derive the probability distribution of the efficiency of a quantum Otto engine. We explicitly compute the quantum efficiency statistics for an analytically solvable two-level model. We analyze the occurrence of values of the stochastic efficiency above unity, in particular at infinity, in the nonadiabatic regime and further determine mean and variance in the case of adiabatic driving. We finally investigate the classical-to-quantum transition as the temperature is lowered.

Efficiency is a key performance measure of thermal machines. For heat engines that cyclically convert heat into useful work, it is defined as the ratio of work output and heat input \[ \eta \]. For macroscopic systems consisting of a huge number of degrees of freedom, heat, work and, consequently, efficiency are deterministic quantities. The second law of thermodynamics imposes an upper bound to the efficiency of any heat engine given by the familiar Carnot expression, \[ \eta_{\text{Carnot}} = 1 - T_1/T_2, \] where \( T_1, T_2 \) are the respective temperatures of the cold and hot heat reservoirs [1]. By contrast, these variables become random at the microscopic scale owing to the presence of nonnegligible thermal fluctuations [2-4] and additional quantum fluctuations at low enough temperatures [4-5]. A central question is then to determine their probability distributions in order to assess their stochastic properties.

For classical heat engines, efficiency fluctuations have been theoretically investigated in a number of recent studies [6-15]. In particular, the stochastic efficiency of a Carnot engine has been shown to admit values larger than the Carnot bound and, remarkably, to be the least likely in the long-time limit [6]. These predictions have been experimentally verified in a stochastic Carnot engine based on an optically trapped colloidal particle [16]. In the quantum regime, work distributions of driven oscillators have been analyzed both theoretically [17-19] and experimentally using a trapped ion [20]. The quantum work statistics of driven two-level systems have been similarly computed analytically [21-22] and determined experimentally in NMR [23] and cold-atom [24] setups. On the other hand, the quantum heat statistics has been calculated theoretically for harmonic oscillators [25] as well as for two-level models [20, 27], and reconstructed experimentally in the latter case [29]. However, to our knowledge, the effects of quantum fluctuations on the efficiency of a heat engine have not been examined so far.

In this paper, we analytically evaluate the efficiency statistics of a quantum Otto engine, a paradigmatic model of quantum thermal machines [28]. The quantum Otto cycle, a generalization of the ordinary four-stroke motor, has been extensively studied in the past thirty years [29-35]. The experimental realization of a quantum Otto spin engine in a NMR system has been reported lately [35]. In the following, we concretely determine the respective work and heat probability densities of the different branches of the engine cycle within the usual two-projective measurement scheme [36]. We use these distributions to derive a general formula for the quantum efficiency statistics that explicitly depends on the time evolution operators of expansion and compression steps. We treat in detail an analytically solvable model of a two-level engine and obtain an explicit expression for the quantum efficiency distribution. We discuss the appearance of values of the stochastic efficiency above unity in the nonadiabatic regime and, in particular, at infinity when no heat is absorbed although non-zero work is produced. This peculiar behavior stems from the discrete quantum nature of the engine. We further determine mean and variance of the efficiency in the case of adiabatic driving. We concretely investigate their evolution from a regime dominated by thermal fluctuations at high temperatures to a domain characterized by quantum fluctuations at low temperatures. We finally demonstrate that the average quantum efficiency is always smaller than the conventional thermodynamic efficiency owing to the presence of positive correlations between stochastic efficiency and absorbed heat.

Quantum Otto engine. We consider a generic quantum system with a time-dependent Hamiltonian \( H_t \) as the working fluid of a quantum Otto engine. The system is initially thermalized at \( t = 0 \) by weakly coupling it to a cold heat reservoir at inverse temperature \( \beta_1 \). The Otto cycle consists of the following four steps (Fig. 1): (1) Unitary expansion (AB) during which the Hamiltonian is changed from \( H_0 \) to \( H_{\tau_1} \) in a time \( \tau_1 \), consuming an amount of work \( W_1 \), (2) Hot isochore (BC) during which the system is put into contact with a heat bath at inverse temperature \( \beta_2 \) to absorb heat \( Q_2 \) in a time \( \tau_2 \), (3) Unitary compression (CD) that drives the isolated system from \( H_{\tau_2} \) back to \( H_0 \) in a time \( \tau_3 \), producing an amount of work \( W_3 \), and (4) Cold isochore (DA), which closes the cycle by weakly coupling the system to the cold bath at inverse temperature \( \beta_1 \), thus releasing heat \( Q_4 \) in a time \( \tau_4 \). We further assume that heating and cooling times, \( \tau_{2,4} \), are longer than the relaxation time of the system, so that thermalization is achieved after each isochore.

Quantum work and heat distributions are commonly determined with the help of the so-called two-projective-measurement method [30] or variants thereof [17, 27]. In this approach, energy changes of a quantum system during single realizations of a process are associated with the difference of eigenvalues obtained though projective energy measurements at the beginning and at the end of the process. In the quantum Otto cycle, work is
performed during the unitary expansion and compression steps, while heat is exchanged during the nonunitary heating and cooling stages. We obtain the corresponding distributions by applying the two-projective-measurement scheme to the respective expansion, hot isochore and compression branches. The probability distribution of the expansion work $W_1$ is accordingly \[ \text{Expression (4)} \]

\[ P(W_1) = \sum_{n,m} \delta [W_1 - (E_n^r - E_n^0)] P^{\tau}_{n \rightarrow m} P^0_n(\beta_1), \] (1)

where $E_n^0$ and $E_n^r$ are the respective energy eigenvalues at the beginning and at the end of the expansion step, $P^0_n(\beta_1) = \exp(-\beta_1 E_n^0)/Z^0$ is the initial thermal occupation probability with partition function $Z^0$ and $P^{\tau}_{n \rightarrow m} = \langle |n \rangle U_{\text{exp}} |m \rangle |^2$ the transition probability from eigenstate $|n \rangle$ to $|m \rangle$. The corresponding unitary time evolution operator is denoted by $U_{\text{exp}}$. The occupation probability $P^0_n(\beta_1)$ embodies the influence of thermal fluctuations, whereas the transition probability $P^{\tau}_{n \rightarrow m}$ accounts for the effects of quantum fluctuations.

Similarly, the probability density of the heat $Q_2$ during the ensuing hot isochore, given the expansion work $W_1$, is equal to the conditional distribution \[ \text{Expression (5)} \]

\[ P(Q_2|W_1) = \sum_{k,l} \delta [Q_2 - (E_k^l - E_k^0)] P^{\tau}_{k \rightarrow l} P^0_k, \] (2)

where the occupation probability at time $\tau$ is $P^0_k = \delta_{km}$ when the system is in eigenstate $|m \rangle$ after the second projective energy measurement. Noting that the state of the system is thermal with inverse temperature $\beta_2$ at the end of the isochore, we further have $P^{\tau}_{k \rightarrow l} = P^{\tau}_{k \rightarrow l}(\beta_2) = \exp(-\beta_2 E_k^l)/Z^\tau$, with the partition function $Z^\tau$.

The quantum work distribution for compression, given the expansion work $W_1$ and the heat $Q_2$, is furthermore, \[ \text{Expression (6)} \]

\[ P(W_3|W_1, Q_2) = \sum_{i,j} \delta [W_3 - (E_j^0 - E_i^0)] P^{\tau}_{i \rightarrow j} P^{\tau + \tau_2}_{l \rightarrow i}, \] (3)

with the occupation probability $P^{\tau + \tau_2}_{l \rightarrow i} = \delta_{il}$ when the system is in eigenstate $|l \rangle$ after the third projective energy measurement. The transition probability $P^{\tau}_{i \rightarrow j} = \langle |i \rangle U_{\text{com}} |j \rangle |^2$ is fully specified by the unitary time evolution operator for compression $U_{\text{com}}$.

The joint probability of having certain values of $W_3$, $Q_2$ and $W_1$ during a cycle of the Otto engine readily follows from the chain rule for conditional probabilities \[ \text{Expression (7)} \]

\[ P(W_1, Q_2, W_3) = \sum_{n,m,k,l} \delta [W_1 - (E_n^r - E_n^0)] \times \delta [Q_2 - (E_k^l - E_k^0)] \times \delta [W_3 - (E_m^0 - E_m^0)] \times \frac{\exp(-\beta_1 E_n^0) - \exp(-\beta_2 E_k^l)}{Z^0 Z^\tau} |\langle m \rangle U_{\text{exp}} |n \rangle |^2 |\langle k \rangle U_{\text{com}} |l \rangle |^2. \] (4)

Equation (4) is the essential quantity needed to determine the quantum efficiency statistics.

Quantum efficiency distribution. The stochastic efficiency of the quantum Otto engine is defined as, \[ \text{Expression (8)} \]

\[ \eta = -\frac{W_1 + W_3}{Q_2}. \] (5)

It should not be confused with the thermodynamic efficiency which is defined in terms of the averaged values of work and heat, $\eta_{\text{th}} = -(\langle W_1 \rangle + \langle W_3 \rangle)/\langle Q_2 \rangle$. The probability distribution $P(\eta)$ is obtained by integrating over all possible values of $W_3$, $Q_2$ and $W_1$ via,

\[ \text{Expression (9)} \]

\[ P(\eta) = \int dW_3 dQ_2 dW_1 P(W_1, Q_2, W_3) \delta \left( \eta + \frac{W_1 + W_3}{Q_2} \right) \] (6)

Using Eq. (4) and the properties of the delta, we find,

\[ \text{Expression (10)} \]

\[ P(\eta) = \sum_{n,m,k,l} \delta \left( \eta + \frac{E_n^r - E_n^0 + E_k^l - E_k^0}{E_k^l - E_m^0} \right) \times \frac{\exp(-\beta_1 E_n^0) - \exp(-\beta_2 E_k^l)}{Z^0 Z^\tau} |\langle m \rangle U_{\text{exp}} |n \rangle |^2 |\langle k \rangle U_{\text{com}} |l \rangle |^2. \] (7)

Expression (10) is our main result. It shows that the efficiency statistics of the quantum Otto engine is fully determined by the unitary time evolution operators for expansion and compression, $U_{\text{exp}}$ and $U_{\text{com}}$, and by the inverse temperatures, $\beta_1$ and $\beta_2$, of the two reservoirs, when complete thermalization is assumed.

Example of a spin heat engine. Formula (10) is valid for any working fluid. As an illustration, we now investigate the fluctuating properties of the stochastic efficiency, $\eta$, for an analytical solvable two-level heat engine. Compression and expansion are implemented by driving...
a spin-1/2 with a constant magnetic field with strength \( \omega/2 \) along the \( z \)-axis and a rotating magnetic field with varying strength \( \gamma(t) \) in the \((x, y)\)-plane. This driving changes both the eigenenergies and the occupation probabilities of the system and could be realized in a NMR setup [39]. The expansion Hamiltonian reads,

\[
H_{\text{exp}}(t) = \gamma(t) (\cos \omega t \sigma_x + \sin \omega t \sigma_y) + \frac{\omega}{2} \sigma_z,
\]

(8)

where \( \sigma_i, i = (x, y, z) \), are the usual Pauli operators. The rotation frequency is chosen to be \( \omega = \pi/2\tau \) to ensure a complete rotation from the \( x \)-axis to the \( y \)-axis during the expansion step of duration \( \tau \). The amplitude of the rotating field, \( \gamma(t) = \gamma_1 (1 - t/\tau) + \gamma_2 (t/\tau) \), is increased from \( \gamma_1 \) at time zero to \( \gamma_2 \) at time \( \tau \). This driving leads to a widening of the energy spacing of the two-level system from \( \nu^0 = \sqrt{4\gamma(0)^2 + \omega^2}/2 \) to \( \nu^\tau = \sqrt{4\gamma(\tau)^2 + \omega^2}/2 \). For simplicity, we will expand and compression times to be equal, \( \tau_1 = \tau_2 = \tau \). The compression stroke is then simply obtained from the time reversed process, \( H_{\text{com}}(t) = -H_{\text{exp}}(\tau - t) \). The corresponding expansion time evolution operator reads [43],

\[
U_{\text{exp}} = \begin{pmatrix}
e^{-i\omega t/2} \cos I & i e^{-i\omega t/2} \sin I \\
i e^{i\omega t/2} \sin I & e^{i\omega t/2} \cos I
\end{pmatrix},
\]

(9)

where \( I = -\int_0^\tau dt' \gamma(t') \) is the integral over the increasing strength of the rotating magnetic field. The operator \( U_{\text{com}} \) follows from \( U_{\text{exp}} \) by replacing \( t \) with \( \tau - t \).

The probability of no level transition during expansion or compression steps may be inferred from [39] as,

\[
u = u_{\text{exp}} = u_{\text{com}} = \cos^2 I.
\]

(10)

The two are identical since \( I \) is the same for both cases apart from a minus sign. The probability of a level transition during either driving phases is accordingly \( v = 1 - u \).

In order to operate as an engine, the mean heat absorbed during (BC) should be positive, \( \langle Q_2 \rangle > 0 \), as well as the total mean work output during the cycle, \(-((W_1) + (W_3)) > 0 \). Specifically, we have [43],

\[
\langle W_1 \rangle = (\nu^t A^* + \nu^0) \tanh(\beta_1 \nu^0),
\]

(11)

\[
\langle W_3 \rangle = (\nu^t A^* + \nu^0) \tanh(\beta_2 \nu^0),
\]

(12)

\[
\langle Q_2 \rangle = -\nu^t \left[ \tanh(\beta_2 \nu^0) + \tanh(\beta_1 \nu^0) A^* \right],
\]

(13)

where we have introduced the adiabaticity parameter \( A^* = 1 - 2u \in [-1, 1] \). For adiabatic driving, when the system remains in the same state \( (u = 1), A^* = -1 \), while \( A^* = 1 \) when a transition occurs with certainty \( (v = 1) \). The value of \( A^* \) depends on the driving protocol \( \gamma(t) \) as well as on the driving time \( \tau \). The two heat engine conditions then lead to the inequalities,

\[
A^* \leq -\frac{\tanh(\beta_2 \nu^0)}{\tanh(\beta_1 \nu^0)};
\]

(14)

\[
A^* \leq -\frac{\nu^0 \tanh(\beta_1 \nu^0) + \nu^t \tanh(\beta_2 \nu^0)}{\nu^t \tanh(\beta_1 \nu^0) + \nu^0 \tanh(\beta_2 \nu^0)}.
\]

(15)

Equations (14) and (15) impose constraints on the allowed values of the time \( \tau \) for a given protocol \( \gamma(t) \). The thermodynamic efficiency \( \eta_{th} \) is further given by,

\[
\eta_{th} = 1 + \frac{\nu^0 \tanh(\beta_2 \nu^0) A^* + \tanh(\beta_1 \nu^0)}{\nu^t \tanh(\beta_2 \nu^0) + \tanh(\beta_1 \nu^0) A^*}.
\]

(16)

It reduces to the known adiabatic Otto efficiency, \( \eta_{ad} = 1 - \nu^0/\nu^t \), in the limit \( A^* = -1 \), as expected [28–38].

Using the above results, the quantum efficiency distribution [7] may be analytically evaluated as,

\[
P(\eta) = \frac{2}{Z^0 Z^\tau} \int \left\{ \left[ u^2 \cos(\beta_1 \nu^0 + \beta_2 \nu^t) + u^0 \cosh(\beta_1 \nu^0 - \beta_2 \nu^t) \right] \delta(\eta) \right. \]

\[
+ u^2 \cosh(\beta_1 \nu^0 - \beta_2 \nu^t) \left[ \eta - \left( 1 + \frac{\nu^0}{\nu^t} \right) \right] \]

\[
+ \left. u^0 \cosh(\beta_1 \nu^0 - \beta_2 \nu^t) \left[ \eta - \left( 1 - \frac{\nu^0}{\nu^t} \right) \right] \right\} \left[ \nu \left[ \delta(\eta - 1) \right] \right].
\]

(17)

This distribution is normalized to one, as it should. We observe that the stochastic efficiency \( \eta \) can take six different discrete values as seen in Fig. [2]. Four are particularly notable: (i) the value at zero is obtained when the produced work \(-W_1 + W_3 + Q_2 \) vanishes [43], (ii) on the other hand, the value at one corresponds to the case where the eigenvalue at point \( A \) is the same as the one at point \( D \), implying \( W_1 + W_3 + Q_2 = 0 \), (iii) finally the values at infinity occur when the heat \( Q_2 \) is zero. All four follow
from the discrete quantum nature of the energy spectrum. The values at infinity are particularly intriguing, since the efficiency is not defined at these points.

The efficiency statistics (17) depends on the driving time \( \tau \) (see Fig. 2). For adiabatic driving, \( u = 1 \) (\( v = 0 \)) (blue squares), all values of the stochastic efficiency are smaller or equal than the adiabatic Otto efficiency \( \eta_{th}^{ad} \), with the largest peak at zero and the second largest at \( \eta_{th}^{ad} \). The value at infinity does not appear in this case. By contrast, for nonadiabatic driving, \( v > 0 \) \((u < 1)\) (red dots), three peaks at and above unit efficiency are visible, including the one at infinity. As a result, an average efficiency is not defined. Specifically, Eq. (17) reveals that the values at one and infinity only disappear when \( uv = 0 \), that is, for certain events. They may thus be regarded as following from quantum indeterminacy.

In the adiabatic case, \( u = 1 \), the mean efficiency reads,

\[
\langle \eta \rangle = \frac{2}{Z_{\omega}^2} \cosh(\beta_1 \nu^0 - \beta_2 \nu^\tau) \left(1 - \frac{\nu^0}{\nu^\tau}\right) < \eta_{th}.
\]  

It is always smaller than the thermodynamic efficiency \( \eta_{th} \) (see Fig. 3). This can be understood by noting that generally \( \langle W/Q_2 \rangle = \langle W \rangle / \langle Q_2 \rangle = \text{cov}(W/Q_2, Q_2) / \langle Q_2 \rangle \), where \( W = -(W_1 + W_2) \) and \( \text{cov}(W/Q_2, Q_2) \) denotes the covariance between the ratio \( W/Q_2 \) and \( Q_2 \). The inequality \( \langle \eta \rangle < \eta_{th} \) is then obeyed when stochastic efficiency and absorbed heat are positively correlated \( \left[47\right] \). The high-temperature \((\beta_i \nu^i \ll 1)\) and low-temperature \((\beta_i \nu^i \gg 1)\) limits, \( i = (1, 2) \) and \( j = (0, \tau) \), of the mean efficiency \( \langle \eta \rangle \) are readily evaluated. We obtain,

\[
\langle \eta \rangle_{\text{high}} = \frac{1}{2} \left(1 - \frac{\nu^0}{\nu^\tau}\right) = \frac{\eta_{th}}{2},
\]

\[
\langle \eta \rangle_{\text{low}} = \eta_{th} \left(e^{-2 \beta_2 \nu^\tau} + e^{-2 \beta_1 \nu^0}\right). \tag{20}
\]

The variance, \( \sigma_{\eta}^2 = \langle \eta^2 \rangle - \langle \eta \rangle^2 \), of the stochastic efficiency may be evaluated in a similar manner, yielding,

\[
\sigma_{\eta}^2 = \frac{1}{4} \left(1 - \frac{\nu^0}{\nu^\tau}\right)^2 \left[1 - \tanh^2(\nu^0 \beta_1) \tanh^2(\nu^\tau \beta_2)\right]. \tag{21}
\]

Its respective high and low temperatures limits are,

\[
\sigma_{\eta_{\text{high}}}^2 = \frac{1}{4} \left(1 - \frac{\nu^0}{\nu^\tau}\right)^2 = \frac{\eta_{th}^2}{4}, \tag{22}
\]

\[
\sigma_{\eta_{\text{low}}}^2 = \eta_{th}^2 \left(\frac{e^{2 \beta_2 \nu^\tau} + e^{2 \beta_1 \nu^0}}{e^{2 \beta_1 \nu^0} + 2 e^{2 \beta_2 \nu^\tau} + 2 e^{2 \beta_1 \nu^0} + 2 e^{2 \beta_2 \nu^\tau}}\right). \tag{23}
\]

The behavior of both the average \( \langle \eta \rangle \) (inset) and the variance \( \sigma_{\eta}^2 \) of the stochastic efficiency are represented as a function of inverse temperature in Fig. 4. The transition from a regime dominated by thermal fluctuations at high temperatures to a domain characterized by quantum fluctuations at low temperatures is clearly visible. In particular, the mean efficiency sharply drops as the ratio \( \text{cov}(W/Q_2, Q_2) / \langle Q_2 \rangle \) increases \( \left[43\right] \), while the variance gets reduced when thermal fluctuations are replaced by smaller quantum fluctuations.

**Conclusions.** We have developed a general framework allowing to calculate the distribution of the efficiency of a quantum Otto engine. We have shown that it is fully determined by the time evolution operators of the two
isothermal compression and expansion steps, and by the
two bath temperatures when complete thermalization is
considered. The fluctuation statistics will additionally
depend on the nonunitary relaxation dynamics in the
case of incomplete thermalization. We have applied our
results to an analytically solvable two-level engine and
evaluated the discrete efficiency distribution in closed
form. We have established the existence of peaks at infinity
which follow from the quantum nature of the engine in
the nonadiabatic regime. An average efficiency is thus
not defined for nonadiabatic driving. We have additionally
computed the first two cumulants of the stochastic
efficiency in the adiabatic limit and found that the
mean is always smaller than the corresponding thermo-
dynamic efficiency since efficiency and heat are positively
correlated. We have finally observed the crossover of the
variance from the classical to the quantum domain.

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Supplemental Material

Solution of the two-level model. We here present the
detailed solution of the expansion and compression dy-
namics of the spin-1/2 engine. The Hamiltonian of a
two-state system can be written in the general form,

\[ H = b_x(t)\sigma_x + b_y(t)\sigma_y + b_z(t)\sigma_z. \]  

(24)

In order to describe a rotating magnetic field with
frequency \( \omega \) and time-dependent strength \( \gamma(t) \), we set,

\[ b_x(t) = \gamma(t)\cos(\omega t), \quad b_y(t) = \gamma(t)\sin(\omega t). \]  

(25)

We leave the function \( b_z(t) \) unspecified for the time being.
A method to evaluate the corresponding time-evolution
operator, given some constraints, has been proposed in
Refs. [45 46]. The first step is to write the time-evolution
operator in the form,

\[ U = \begin{pmatrix} u_{11} & -u_{21} \\ u_{21} & u_{11} \end{pmatrix}, \]  

(26)

where \( u_{11} = \cos \chi \cdot e^{i\xi_2 - i\phi/2} \) and \( u_{21} = i\eta \sin \chi \cdot e^{i\xi_2 + \phi/2} \).
Here \( \eta = 1 \) and the parameters \( \xi_\pm \) are given by,

\[ \xi_\pm = \int_0^t dt' \sqrt{1 - \frac{\xi^2}{\beta^2}} \csc(2\chi) \pm \frac{1}{2} \sin^{-1} \left( \frac{\chi}{\beta} \right) \pm \eta \frac{\pi}{4}. \]  

(27)

The given constraints allow the variables \( \chi, \beta, \) and \( \phi \)
to be chosen arbitrarily. Using Eqs. (25) and (27) the
prefactors in the Hamilton operator (24) are,

\[ b_x(t) = \beta \cos \phi, \quad b_y(t) = \beta \sin \phi, \]  

(28)

\[ b_z(t) = \frac{\ddot{\chi} - \dot{\chi}\dot{\beta}/\beta}{2\beta\sqrt{1 - \chi^2/\beta^2}} - \beta \sqrt{1 - \chi^2/\beta^2} \cot(2\chi) + \frac{\dot{\phi}}{2}. \]

The choice (25) is reproduced by taking \( \beta = \gamma(t) \) and
\( \phi = \omega t \). By further setting \( \chi = -\eta \int_0^t dt' \beta(t') \), we obtain
\( b_z(t) = \omega/2 \). The evolution operator (26) then follows as,

\[ U = \begin{pmatrix} e^{-i\omega t/2} \cos I & ie^{-i\omega t/2} \sin I \\ ie^{i\omega t/2} \sin I & e^{i\omega t/2} \cos I \end{pmatrix}, \]  

(29)

with the quantity \( I = -\int_0^t dt' \gamma(t') \).

Zero-over-zero efficiency. The computation of the effi-
ciency distribution for the two-level engine, Eq. (17)
of the main text, involves expressions of the form \( 0/0 \),
which occur with finite probability. Since these are ill
defined mathematically, we determine their values physically
by concretely considering the case of adiabatic driving,
\( u = 1 \). The corresponding efficiency distribution is,

\[ P(\eta) = P_0^0 P_r^r \delta \left( \eta - \frac{0}{0} \right) + P_1^0 P_1^r \delta \left( \eta - \frac{0}{0} \right) \]  

\[ + \left( P_0^0 P_1^r + P_1^0 P_0^r \right) \delta \left[ \eta - \left( 1 - \frac{\nu^0}{\nu^r} \right) \right], \]  

(30)

since \( \langle n| U_{\text{exp}} | m \rangle^2 = \delta_{nm} \) and \( \langle k| U_{\text{com}} | l \rangle^2 = \delta_{kl} \).
The first term, \( P_0^0 P_r^r = 1 \), corresponds to two baths with
equal and vanishingly small temperatures, \( \beta_1 = \beta_2 \to \infty \)
(and the engine always remains in the ground state).
Similarly, the second term, \( P_0^1 P_1^r = 1 \), corresponds to two
baths with equal and extremely large temperatures, \( \beta_1 = \beta_2 \to 0 \)
(and the engine always remains in the excited state).
In both cases, the Carnot formula implies that the
efficiency vanishes and we therefore set \( 0/0 = 0 \).

Calculation of the mean values. The average values
of work and heat along the different branches of the heat
engine cycle are obtained by direct integration of the cor-
responding probability distributions. We find,

$$\langle W_1 \rangle = \int_{-\infty}^{+\infty} dW_1 W_1 P(W_1)$$

$$= \sum_{n,m} (\nu^m_n - \nu^0_n) \frac{e^{-\beta_1 \nu^0_n}}{Z^0} P^\tau_{n\rightarrow m}$$

$$= [\nu^0 + \nu^\tau (1 - 2u)] \tanh(\beta_1 \nu^0),$$

and in an analogous manner,

$$\langle W_3 \rangle = [\nu^\tau + \nu^0 (1 - 2u)] \tanh(\beta_2 \nu^\tau).$$

At the same time, the absorbed heat reads,

$$\langle Q_2 \rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dW_1 dQ_2 P(Q_2|W_1)p(W_1)$$

$$= \sum_{k,m,n} (\nu^k_n - \nu^m_m) \frac{e^{-\beta_2 \nu^k_n - \beta_1 \nu^m_m}}{Z^\tau Z^0} P^\tau_{n\rightarrow m}$$

$$= -\nu^\tau \left[ \tanh(\beta_2 \nu^\tau) + \tanh(\beta_1 \nu^0)(1 - 2u) \right].$$

We mention that averaged work and heat can also be calculated by considering the energy changes along individual branches of the Otto cycle, that is, by only performing projective energy measurements at the beginning and at the end of one given step, instead of the first three consecutive steps as done above. The two methods give the same results. This may be understood by noting that work and heat only depend on (diagonal) energy differences which do not depend on the (nondiagonal) coherences of the two-level system. The presence or absence of intermediate projective energy measurements hence do not affect the value of the averaged work and heat.

**Covariance.** The covariance between absorbed heat $Q_2$ and stochastic efficiency $\eta$ reads

$$\text{Cov}(Q_2, \eta) = -\langle (W_1 + W_3) \rangle - \langle Q_2 \rangle \langle \eta \rangle.$$ (38)

We have, as a result,

$$\langle \eta \rangle = -\frac{\langle (W_1 + W_3) \rangle - \text{Cov}(Q_2, \eta)}{\langle Q_2 \rangle}$$ (39)

The covariance is shown in Fig. 5 as a function of the inverse temperature. We notice that it first increases and then decreases as the temperature is lowered. It is moreover always positive, indicating that heat and efficiency are always positively correlated. On the other hand, the ratio $\text{cov}(W/Q_2, Q_2)/\langle Q_2 \rangle$ increases. Consequently, the mean efficiency is smaller than the corresponding thermodynamic efficiency, $\langle \eta \rangle \leq \eta_{th}$. 

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