A Conformal Mapping and Isothermal Perfect Fluid Model

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Abstract
Instead of conformal to flat spacetime, we take the metric conformal to a spacetime which can be thought of as “minimally” curved in the sense that free particles experience no gravitational force yet it has non-zero curvature. The base spacetime can be written in the Kerr-Schild form in spherical polar coordinates. The conformal metric then admits the unique three parameter family of perfect fluid solution which is static and inhomogeneous. The density and pressure fall off in the curvature radial coordinates as $R^{-2}$, for unbounded cosmological model with a barotropic equation of state. This is the characteristic of isothermal fluid. We thus have an ansatz for isothermal perfect fluid model. The solution can also represent bounded fluid spheres.

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1. Introduction

It is well-known [1] that the Kerr-Schild form of the metric plays an important role in finding vacuum, pure radiation and Einstein-Maxwell solutions in general relativity (GR). It turns out that unless the Kerr-Schild ansatz is generalised the perfect fluid solutions cannot be found [2]. Senovilla and coworkers [3-5] have obtained perfect fluid solutions by taking conformally flat metric as the seed metric in place of the original flat metric and thus generalising the KS ansatz. It reads as

\[ \mathcal{g}_{ij} = g_{ij} + 2Hl_il_j \] (1.1)

where \( g_{ij} \) is the seed metric, \( H \) is a scalar field and \( l^i \) is a null vector relative to both \( g_{ij} \) and \( g_{ij} \). When \( g_{ij} = \eta_{ij} \), we have the original KS form. No perfect fluid solution can be admitted so long as the seed metric is vacuum or its energy momentum tensor has \( l^i \) as an eigenvector [2].

For all the perfect fluid solutions for the form (1.1), the seed metric is taken as conformally flat and \( l^i \) as geodetic. That means the solution can be written as conformal to a metric in the original KS form, which may or may not be a solution of the Einstein equation. We shall here consider a metric which is conformal to a KS metric. The original metric need not be a solution of the Einstein equation but has to be chosen on some physical and geometric considerations. All conformally flat spacetimes will have \( g_{ij} = e^{2U} \eta_{ij}, \quad U = U(x^i) \) and \( H = 0 \) in (1.1). There are well-known conformally flat perfect fluid solutions; e.g. Friedman- Robertson-Walker (FRW) and the Schwarzschild interior spacetimes.

The question is what should be the metric in place of flat to conform to? What we wish to consider is,

\[ \mathcal{g}_{ij} = e^{2U}(\eta_{ij} + 2Hl_il_j), \quad U = U(x^i) \] (1.2)

in which \( H = 0 \) gives conformally flat. The various forms of \( H \) will, for the base spacetime, give vacuum, Einstein-Maxwell and pure radiation solutions. What happens when \( H \) is a constant \( \neq 0 \)? Does the original metric now become flat?
No, it doesn’t. In fact it represents an interesting situation which is free of the Newtonian gravity (free particles experience no acceleration), yet spacetime is curved and hence it can be thought of as “minimally” curved [6,7]. For vacuum solution $H$ satisfies the Laplace equation corresponding to $R^0_0 = 0$ (note that $R^0_0 = 0$ is supposed to be the analogue of the Newtonian equation $\nabla^2 \phi = 0$) and hence $H = \text{const.} \neq 0$ will present so to say a spacetime arising out of a constant “gravitational potential”. This is why it is free of the usual gravitational force and hence very close (“minimally” curved) to flat spacetime. How about considering a metric conformal to it and enquire whether it is compatible with the perfect fluid conditions? This is exactly what we wish to do in this paper.

For the metric (1.2) with $H = \text{const.} \neq 0$, we shall find the most general spherically symmetric perfect fluid solution which will turn out to be static. It may be noted that when $H = 0$, non-static perfect fluid solution like FRW is admitted but for $H$ different from zero, even when constant, only static solution is admitted. That means $H \neq 0$ implies severe constraints on the fluid solutions. The solution is uniquely given by a three parameter family which can represent both bounded as well as unbounded distributions. In the case of unbounded cosmological model density and pressure satisfy a barotropic equation of state and fall off as inverse square of the (curvature) radius. This is the characteristic behaviour for an isothermal fluid model.

In section 2, we derive the metric and set up the field equations in section 3 and find the general perfect fluid solution. We give some examples of fluid models in section 4. We conclude with a discussion.

2. The metric

Consider the metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 + 2H(dt + dr)^2, \quad d\Omega^2 = d\theta^2 + \sin^2 \theta \; d\varphi^2$$  \hspace{1cm} (2.1)

which is in the KS form and we take $H = \text{const} \neq 0$. This metric can be transformed to the form
\[ ds^2 = -dt^2 + K^2dr^2 + r^2d\Omega^2, \quad K^2 = (1 + 2H)^2/(1 - 2H). \] (2.2)

The coordinate \( t \) has been redefined to absorb the constant coefficient. Recall that for the vacuum solution \( H \) satisfies the Laplace equation for \( R^0_0 = 0 \) and hence behaves like a gravitational potential. It is not that \( H \) is a solution of any Laplace equation but is the solution of \( R^0_0 = 0 \) that can be written as the Laplace equation. It is this equation which is the analogue of the Newtonian equation defining potential. Hence \( H \) may be considered as gravitational potential. We are taking \( H \) constant which should be equivalent to constant potential and hence spacetime should be free of the Newtonian gravity as can be verified by considering the geodesics of the above metric. However the curvature is non-zero (it has only one curvature, \( R^2_{23} = -(K^2 - 1)/K^2r^2 \), which is an invariant of spherical symmetry[7]) which will make its presence felt only in geodesic deviation (tidal force) alone. Note that we have just taken the constant potential solution of the Laplace equation \( R^0_0 = 0 \) and that has generated a curved spacetime and hence we would like to term it as “minimally” curved[6,7]. Physically this can be thought of as the closest curved spacetime to flat spacetime.

We shall consider the metric conformal to (2.2) and write it as

\[ ds^2 = e^{2U}(-dt^2 + K^2dr^2 + r^2d\Omega^2), \quad U = U(r, t). \] (2.3)

Note that both the Ricci and the curvature tensors for the metric (2.2) have the only one non-zero component and they are given by [7],

\[ R^2_2 = \frac{K^2 - 1}{K^2r^2} = -R^2_{23}. \] (2.4)

Let us write from [1],

\[ \overline{R}_{ab} = R_{ab} - 2Y_{ab} - g_{ab}Y^c_c \] (2.5)
\[ Y_{ab} = U_{;ab} - U_{;a}U_{;b} + \frac{1}{2}g_{ab}U_{;c}U^{;c} \] (2.6)
where $R_{ab}$ and $\overline{R}_{ab}$ refer to $g_{ij}$ and $\overline{g}_{ij} = e^{2U} g_{ij}$. Then for the metric (2.3) we have

$$e^{2U} \overline{R}_0^0 = -\frac{1}{K^2} \left( U'' + 2U' + \frac{2U'}{r} \right) + 3\ddot{U}$$  \hspace{1cm} (2.7)$$

$$e^{2U} \overline{R}_1^1 = -\frac{1}{K^2} \left( 3U'' + \frac{2U'}{r} \right) + \ddot{U} + 2\dot{U}^2$$  \hspace{1cm} (2.8)$$

$$e^{2U} \overline{R}_2^2 = -\frac{1}{K^2} \left( U'' + 2U' + \frac{4U'}{r} - \frac{K^2 - 1}{r^2} \right) + \ddot{U} + 2\dot{U}^2$$  \hspace{1cm} (2.9)$$

$$\overline{R}_{01} = -2(\dot{U}' - \dot{U}''')$$  \hspace{1cm} (2.10)$$

where $\dot{U} = \partial U/\partial t$ and $U' = \partial U/\partial r$.

3. The general perfect fluid solution

The field equation for a perfect fluid distribution reads as

$$R_{ik} = 8\pi \left[ (\rho + p)u_i u_k + \frac{1}{2}(\rho - p)g_{ik} \right]$$  \hspace{1cm} (3.1)$$

which for a comoving velocity field $u = e^U dt$ implies $R_{01} = 0$ and $R_1^1 = R_2^2$ (dropping overhead bar in (2.7) - (2.10)). From (2.10) we get

$$\dot{U}' - \dot{U}'' = 0$$

which integrates to give

$$e^{-U} = f(t) + g(r).$$  \hspace{1cm} (3.2)$$

Though we began with the KS form but we are ultimately considering the metric (2.3) which is orthogonal and spherically symmetric. Hence we are justified in employing the comoving coordinates. Substituting (3.2) in $R_1^1 = R_2^2$, following from (2.8) and (2.9), leads to

$$\left( -g'' + \frac{g'}{r} + \frac{K^2 - 1}{K^2 r^2} \right) \frac{K^2 r^2}{K^2 - 1} = -f(t)$$  \hspace{1cm} (3.3)$$
which clearly implies \( f(t) = \text{const.} \). This means the conformal function \( U \) must only be a function of \( r \) alone and the spacetime then becomes static. Note that this conclusion follows from (3.3) only when \( K \neq 1 \), and when \( K = 1 \), (3.3) does admit non-static solutions as for FRW. This precisely demonstrates the role played by the non-flat character (which is introduced through \( K \neq 1 \)) of the original metric.

With \( U = U(r) \), the only differential equation to be solved is

\[
U'' - U' r^2 - \frac{U'}{r} + \frac{K^2 - 1}{2r^2} = 0 \tag{3.4}
\]

which admits the general solution

\[
e^{-U} = c_1 r^n + c_2 r^{2-n}, \quad K^2 = 1 + 2n(n-2) \tag{3.5}
\]

\[
8\pi\rho K^2 e^{2U} r^2 = 2\frac{(2n - 1)nc_1 + (2 - n)(3 - 2n)c_2\mu}{c_1 + c_2\mu} - 3\left(\frac{nc_1 + (2 - n)c_2\mu}{c_1 + c_2\mu}\right)^2 \tag{3.6}
\]

\[
8\pi\rho K^2 e^{2U} r^2 = -2\frac{n^2c_1 + (2 - n)^2c_2\mu}{c_1 + c_2\mu} + 3\left(\frac{nc_1 + (2 - n)c_2\mu}{c_1 + c_2\mu}\right)^2 \tag{3.7}
\]

where \( \mu = r^{2(1-n)} \).

From (3.5), the window \( 1 - 1/\sqrt{2} < n < 1 + 1/\sqrt{2} \) is prohibited for \( n \). \( K = 1 \) for \( n = 0, 2 \) and then the metric turns conformally flat. \( \rho \) will always remain positive for \( c_1/c_2 > 0 \) and \( n < 0 \) or \( n > 2 \). It may be noted from (3.5) that we need to consider either \( n < 0 \) or \( n > 2 \) because one can be transformed into the other simply by \( c_1 \leftrightarrow c_2 \). From (3.6) and (3.7) it can be seen that the fluid cannot admit a barotropic equation of state, \( \rho = kp \) unless either \( c_1 \) or \( c_2 \) vanish. In that case the solution is cosmological for \( p \) cannot be made zero for a finite \( r \).

Let us examine the condition for a bounded sphere, i.e. when \( p = 0 \) which implies

\[
(2 - n)^2\mu^2 + 2\frac{c_1}{c_2} \left(2(2 - n)(2n - 1) - n^2\right)\mu + n^2\frac{c_1^2}{c_2^2} = 0. \tag{3.8}
\]
It will admit a real solution only if $3n^2 - 6n + 2 \geq 0$ which rules out the window $1 - 1/\sqrt{3} \leq n \leq 1 + 1/\sqrt{3}$ for $n$. This range is entirely covered by the earlier forbidden range $1 - 1/\sqrt{2} < n < 1 + 1/\sqrt{2}$. Hence so long as $c_1$ and $c_2$ are non-zero $p$ will always vanish at some finite $r$ defining the boundary of fluid sphere. It is obvious that when one of $c_1$ and $c_2$ is zero, $p$ cannot vanish at any $r$ and hence the fluid will have no boundary. Thus we have a bounded fluid sphere for $c_1$ and $c_2 \neq 0$ and unbounded for one of them being zero.

4. Fluid models

4.1 Unbounded and cosmological:

In here we have to take one of $c_1$ and $c_2$ zero and then (3.5) will represent the only one solution with $n$ or $(2 - n)$ as the free parameter.

Without any loss of generality let us set $c_1 = 1$ and $c_2 = 0$ and have

$$ds^2 = r^{-2n}(-dt^2 + K^2dr^2 + r^2d\Omega^2) \quad (4.1)$$

and

$$8\pi\rho = \frac{n^2 - 2n}{K^2r^2(1-n)}, \quad 8\pi p = \frac{n^2}{K^2r^2(1-n)}, \quad K^2 = 1 + 2n(n-2). \quad (4.2)$$

Note that physically relevant radial coordinate is the one that occurs as $R^2$ coefficient of $d\Omega^2$ in the metric; i.e. $R = r^{1-n}$. Thus in terms of the curvature radial coordinate density and pressure fall off as $R^{-2}$ (This $R$ should not be confused with the scalar curvature $R$). The inverse square fall off is characteristic of isothermal equilibrium [8] of fluid with a barotropic equation of static.

The positivity of $\rho$ is ensured by $n > 2$ or $n < 0$ which is permissible for it lies outsided the prohibited window. The equation of state for fluid is given by

$$\rho = \frac{n-2}{n} p \quad (4.3)$$

Now $\rho \geq 3p$ requires $n \leq -1$ which means $k^2 \geq 7$. The spacetime becomes flat for $n = 0$. It may be noted that $\rho \geq 3p$ automatically ensures $\rho \geq 0$ (i.e. $n \leq -1$).
Though $\rho$ diverges as $R \to 0$ but total mass contained inside a radius $R$ remains always finite and goes to zero with $R$. Also note that $dp/d\rho$ is always less than 1, indicating velocity of light is always greater than velocity of sound.

4.2 Fluid spheres

For the bounded fluid distributions, $c_1$ and $c_2$ must be non-zero. For positivity of density $c_1/c_2 > 0$ and $n \leq 0$ or $n \geq 2$, where the equality implies $K = 1$ and will make the spacetime conformally flat. For both $n = 0$ and $n = 2$, we get the same solution with the role of $c_1$ and $c_2$ interchanged.

When $n = 0$, $K = 1$, we have

\begin{equation}
    e^{-U} = c_1 + c_2 r^2, \tag{4.4}
\end{equation}

\begin{equation}
    2\pi \rho = 3c_1c_2 \tag{4.5}
\end{equation}

\begin{equation}
    2\pi p = c_2(-2c_1 + c_2 r^2) \tag{4.6}
\end{equation}

which means $c_2 r^2 \geq 2c_1$. This will imply an upper (lower) bound on $r$ depending upon both $c_1, c_2 < 0 (> 0)$. The boundary of the sphere is defined by $r_0^2 = 2c_1/c_2$. The solution is conformally flat because $K = 1$. It is a uniform density conformally flat fluid sphere. It is therefore a transform of the Schwarzschild interior solution.

Let us consider an example of conformally non-flat fluid sphere. Take $n = -1$, say, then $K^2 = 7$, and

\begin{equation}
    e^{-U} = c_1 r^{-1} + c_2 r^3 \tag{4.7}
\end{equation}

\begin{equation}
    8\pi \rho = \frac{3}{7}(c_2^2 r^4 + 18c_1c_2 + c_1^2 r^{-4}) \tag{4.8}
\end{equation}

\begin{equation}
    8\pi p = \frac{1}{7}(9c_2^2 r^4 - 38c_1c_2 + c_1^2 r^{-4}) \tag{4.9}
\end{equation}
The fluid boundary is defined by \( r_0^4 = (19 \pm \sqrt{10})c_1/9c_2 \). This means either \( r^4 \leq \left( \frac{19-\sqrt{10}}{9} \right) c_1 \) or \( r^4 \geq \left( \frac{19+\sqrt{10}}{9} \right) c_1 \). But \( \rho \geq p \) requires \( r^4 \leq \left( \frac{46}{3} \right) c_1 \) which rules out the latter case. There are two free parameters \( c_1, c_2 \) in the solution and hence it could easily be matched to the Schwarzschild exterior solution at \( r_0^4 = \left( \frac{19-\sqrt{10}}{9} \right) c_1 \). Thus this solution can describe the interior of a star in hydrostatic equilibrium. As in the cosmological case \( \rho \) and \( p \) diverge as \( r \to 0 \) but the mass contained in a radius \( r \) remains finite and goes to zero with \( r \).

5. Discussion

The solution obtained above is of the Tolman type [9] but the important point is that it follows as the unique solution of the metric ansatz (2.3); i.e. the metric conformal to a KS spherically symmetric metric free of the Newtonian gravity. There are well-known conformally flat perfect fluid solutions. Here we retain the conformal character but let go flatness of the original metric and yet essentially keeping it free of usual gravity. It exhibits in the context of perfect fluid spacetimes what happens when flatness of the base metric is relaxed yet retaining the essential physical character of spacetime. It is interesting that it leads to the unique solution representing a three parameter \((n, c_1 \text{ and } c_2)\) family.

The perfect fluid is in hydrostatic equilibrium resulting from pressure gradient balancing radial acceleration. The general solution can provide both the bounded fluid spheres as well as unbounded cosmological distribution depending upon the choice of free parameters.

The remarkable feature of this family is that in the cosmological case it gives \( \rho \) and \( p \) satisfying a barotropic equation of state and falling off inverse square of the curvature radial coordinate which is the physically relevant radius. This behaviour is characteristic of an isothermal fluid sphere [8]. Recently it has been argued [10] that the ultimate end state of the Einstein-deSitter model would approximate to an isothermal fluid sphere. As mentioned earlier, in such models though density may diverge, but total mass contained inside a finite radius will however be finite and will go to zero as radius goes to zero.
Finally we would like to say that the ansatz (2.3) is a remarkably simple way of obtaining a static inhomogeneous and isothermal perfect fluid model with a barotropic equation of state and density falling off as $R^{-2}$. It is remarkable that (2.3) admits the unique solution with these specific properties. Our ansatz may be taken as characterising such a behaviour.

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