Qunity
A Unified Language for Quantum and Classical Computing

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We introduce Qunity, a new quantum programming language designed to treat quantum computing as a natural generalization of classical computing. Qunity presents a unified syntax where familiar programming constructs can have both quantum and classical effects. For example, one can use sum types to implement the direct sum of linear operators, exception-handling syntax to implement projective measurements, and aliasing to induce entanglement. Further, Qunity takes advantage of the overlooked $\mathsf{qor}$ subroutine theorem, allowing one to construct reversible subroutines from irreversible quantum algorithms through the uncomputation of “garbage” outputs. Unlike existing languages that enable quantum aspects with separate add-ons (like a classical language with quantum gates bolted on), Qunity provides a unified syntax and a novel denotational semantics that guarantees that programs are quantum mechanically valid. We present Qunity’s syntax, type system, and denotational semantics, showing how it can cleanly express several quantum algorithms. We also detail how Qunity can be compiled into a low-level qubit circuit language like OpenQasm, proving the realizability of our design.

CCS Concepts: • Software and its engineering → Multiparadigm languages; Syntax; Control structures; Data types and structures; Compilers; Functional languages; Patterns; Semantics; • Theory of computation → Denotational semantics; Quantum information theory; Control primitives; Categorical semantics; Quantum query complexity; Assertions.

Additional Key Words and Phrases: algebraic data types, Kraus operators, quantum subroutines, reversible computing

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1 INTRODUCTION
Quantum computing generalizes classical computing. That is, any efficiently-implementable classical algorithm can also be efficiently implemented on a quantum computer. However, quantum programming languages today do not fully leverage this connection. Rather, to varying degrees, they impose a separation between their classical and quantum programming constructs. Such a separation owes in part to the $\mathsf{qram}$ computing model [Knill 1996], and is reflected in slogans

*Work completed before starting at Amazon.

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such as “quantum data, classical control” [Selinger 2004]. While keeping the quantum and classical
separated in the language makes some sense in the near term (the “niso” era [Preskill 2018]),
it artificially limits the long-term potential of quantum algorithm designs. It also fails to take
advantage of a classical programmer’s intuition and limits the reuse of classical code and ideas in
quantum algorithms.

In this paper, we present Qunity (“kymn-tee”), a new programming language whose program-
ing constructs will be familiar to classical programmers but are generalized to include quantum
behavior. Thus, Qunity draws inspiration from prior languages which contain some unified constructs [Altenkirch and
Grattage 2005; Bichsel et al. 2020], but Qunity broadens and deepens that unification.

1.1 Motivating Example: Deutsch’s Algorithm
To give a sense of Qunity’s design, we present Deutsch’s algorithm [Deutsch 1985]. Given black-box
access to a function \( f : \{0, 1\} \to \{0, 1\} \), this algorithm computes whether or not \( f(0) \neq f(1) \) using
only a single query to the function.

\[
\text{deutsch}(f) :=
\begin{align*}
\text{let } x &= \text{Bit} (\text{had } 0) \text{ in } \\
\left( \text{ctrl} (f) \left\{ \begin{array}{ll} 
0 & \mapsto x \\
1 & \mapsto x \cdot \text{gphase}_{\text{Bit}}(\pi) \end{array} \right. \right) \cdot \text{had}
\end{align*}
\]

The algorithm has three steps:

1. Apply a Hadamard operator had to a qubit in the zero state, yielding a qubit in state \( |+\rangle \); the
   \text{let} expression binds this qubit to \( x \).
2. Query an oracle to conditionally flip the phase of the qubit, coherently performing the linear
   map:
   \[
   |x\rangle \mapsto (-1)^{f(x)} |x\rangle
   \]
   This step is implemented by the \	ext{ctrl} expression: if \( fx \) is 0 then \( |x\rangle \) is unchanged, but if \( f x \)
   is 1 then \( x \cdot \text{phase}_{\text{Bit}}(\pi) \) applies a phase of \( e^{i\pi} \) (which is \( -1 \)) to \( |x\rangle \).
3. Finally, apply a Hadamard operator to the qubit output from step 2.

If \( f(0) \neq f(1) \), the output will be \( |1\rangle \) up to global phase; otherwise, it will be \( |0\rangle \).

Qunity’s version of Deutsch’s algorithm reads like a typical functional program and is more
general than typically presented versions. Some presentations [Nielsen and Chuang 2010] require
constructing a two-qubit unitary oracle \( U_f \) from the classical function \( f \) such that \( U_f |x, y\rangle =
|x, y \oplus f(x)\rangle \) for all \( x, y \in \{0, 1\} \). In Qunity, no separate oracle is needed; the “oracle” is \( f \) itself.
Existing programming languages [Amy et al. 2017; Li et al. 2022; Rand et al. 2019] support automatic
construction of oracles \( U_f \) from explicitly-written programs \( f \), but these require that the implemen-
tation of \( f \) is strictly classical. For example, Silq’s [Bichsel et al. 2020] uncomputation construct
requires that the program of interest be “qfree,” with strictly classical behavior, and Quipper [Green
et al. 2013] supports automatic oracle construction only from classical Haskell programs. In Qunity,
\( f \) can be an arbitrary quantum algorithm that takes a qubit as input and produces a qubit as output,
even if it uses measurement or interspersed classical operations.

Qunity takes advantage of the BQP subroutine theorem [Bennett et al. 1997; Watrous 2009], which
allows for the construction of reversible subroutines from arbitrary (not necessarily reversible)
quantum algorithms. If the quantum algorithm has probabilistic behavior, the reversible subroutine
may have a degree of error, but the BQP subroutine theorem places reasonable bounds on this error.
These bounded-error subroutines are commonly used in the design of quantum algorithms [Kothari 2014], but existing programming languages provide no convenient way to construct and compose them, a gap that Qunity fills.

According to Qunity’s type system (Section 3), the following typing judgment is valid:

\[ \vdash f : \text{Bit} \implies \text{Bit} \]

The rule says: “given an arbitrary quantum algorithm for computing a function \( f : \{0, 1\} \rightarrow \{0, 1\} \), our \( \text{deutsch}(f) \) program outputs a bit.” Per Qunity’s formal semantics (Section 4), \( \text{deutsch}(f) \) corresponds to a single-qubit pure state whenever \( f \) corresponds to a single-qubit quantum channel. This is possible because of the unique way that Qunity’s semantics interweaves the usage of pure and mixed quantum states.

1.2 Design Principles
Qunity’s design is motivated by four key principles: generalization of classical constructs, expressiveness, compositionality, and realizability.

Generalization of classical constructs. To make quantum computing easier for programmers, Qunity allows them to draw on intuition from classical computing. Many elements of Qunity’s syntax are simply quantum generalizations of classical constructs: for example, tensor products generalize pairs, projective measurements generalize try-catch, and quantum control generalizes pattern matching. Rather than use linear types (as in QWIRE [Paykin et al. 2017] and the Proto-Quipper languages [Fu et al. 2020; Ross 2015]), Qunity allows variables to be freely duplicated and discarded as in classical languages, but its semantics treats variable duplication as an entangling operation, and variable discarding as a partial trace.

Expressiveness. Qunity allows for writing algorithms at a higher level of abstraction than existing languages. One way that it does this is through algebraic data types: rather than manipulate fixed-length arrays of qubits directly, programmers can work with more complicated types. For example, in our quantum walk algorithm given in Section 5.3, we deal with superpositions of variable-length lists. Qunity’s semantics also allows one to “implement” mathematical objects that are frequently used in algorithm analysis but seldom used in algorithm implementation. For example, the semantics of a Qunity program can be a superoperator [Kaye et al. 2007, p. 57], an isometry [Roman 2008, p. 210], or a projector [Nielsen and Chuang 2010, p. 70], and these can be composed in useful ways.

In detail, the operators that make up Qunity’s semantics are drawn from a broad class called Kraus operators [Kaye et al. 2007, p. 60], which include norm-decreasing operators such as projectors. Projectors are used in quantum algorithms, but more often for analyzing quantum algorithms, and few quantum languages allow projectors to be directly implemented. Motivated by the fact that operators produced by the \( \text{bqp} \) subroutine theorem can be viewed as norm-decreasing rather than norm-preserving, we give Qunity programs norm non-increasing semantics. Expanding the language to include projectors turns out to be quite useful: We can treat the null space of a projector as a sort of “exception space,” allowing us to reason about quantum projectors by a syntactic analogy with classical programming strategies, namely exception handling. This allows us to implement a few more useful program transformations, like a “reflection” in the style of Grover’s diffusion operator. Given the ability to implement a projector \( P \), it is straightforward to implement the unitary reflection \( (2P - I) \), a common feature in quantum algorithms.
**Compositionality.** Qunity programs can be composed in useful ways. For example, our `ctrl` construct uses the `bqp` subroutine theorem [Bennett et al. 1997; Watrous 2009] to reversibly use an irreversible quantum algorithm as a condition for executing another—this happens in step 2 of our version of Deutsch’s algorithm. In general, Qunity’s denotational semantics allows composing programs that are mathematically described in different ways, like defining an expression’s “pure” operator semantics in terms of a subexpression’s “mixed” superoperator semantics. By contrast, prior languages have not managed such a compositional semantics [Grattage 2011, 2006; Ying 2016], as discussed below.

**Realizability.** We have designed Qunity so that it can be compiled into qubit-based unitary circuits written in a lower-level language such as OpenQasm. We have designed such a compilation procedure and proven that the circuits it produces correctly implement Qunity’s denotational semantics. In Section 6, we give an overview of our compilation strategy, and we include the full details and proofs in the supplemental report [Voichick et al. 2022a]. To limit our analysis to finite-dimensional Hilbert spaces, we limit Qunity programs to working with finite types only. Some quantum languages [Ying 2016, Chapter 7] work with infinite-dimensional Hilbert spaces such as Fock spaces, making it possible to coherently manipulate superpositions of unbounded lists, for example. However, since qubit-based circuits work only with finite-dimensional Hilbert spaces, allowing Qunity to work with infinite data types would mean that compilation would have to approximate infinite-dimensional Hilbert spaces with finite ones, which is challenging to do satisfyingly. For this reason, Qunity has no notion of recursive types, though we use classical metaprogramming to define parameterized types.

### 1.3 Related Work

| Feature                  | Qunity | Silq | QML | Spm | Quipper | QuGcl |
|--------------------------|--------|------|-----|-----|---------|-------|
| Decoherence              | ✓      | ✓    | ✓   | ✗   | ✓       | ✓     |
| Denotational Semantics   | ✓      | ✗    | ✓*  | ✗   | ✓       | ✓*    |
| Quantum Sum Types        | ✓      | ✗    | ✓*  | ✗   | ✓       | ✗     |
| Isometries               | ✓      | ✗    | ✓*  | ✗   | ✓       | ✓     |
| Projectors               | ✓      | ✗    | ✗   | ✗   | ✓       |       |
| Classical Uncomputation  | ✓      | ✗    | ✗   | ✓   | ✓       |       |
| Quantum Uncomputation    | ✓      | ✗    | ✗   | ✗   | ✓       | ✗     |

Qunity’s design was inspired by several existing languages, summarized in Table 1: Silq [Bichsel et al. 2020], QML [Altenkirch and Grattage 2005], the symmetric pattern matching language [Sabry et al. 2018] (hereafter Spm), Quipper [Green et al. 2013], and QuGcl [Ying 2016, Chapter 6]. This table shows some of Qunity’s most interesting features and whether these features are present in existing languages. The features listed are:

- Decoherence. All of these languages support some form of measurement or discarding, except for Spm, which is restricted to unitary operators.
- Compositional denotational semantics. Most of these languages define an operational semantics, not a denotational one. QuGcl’s denotational semantics is explicitly non-compositional,
meaning that its equivalence relation does not allow “equivalent” programs to be substituted as subexpressions. Later work on QML [Grattage 2011, 2006] found that the denotational semantics derived from QML’s operational semantics was non-compositional because of entangled “garbage” outputs related to sum types, and QML’s denotational semantics is only partially defined even with sum types removed.

- Quantum sum types. Like QML and SPM, Qunity can coherently manipulate tagged unions, and the “tag” qubit can be in superposition.

- Isometries. Silq is the only other language here with a semantics defined in terms of non-unitary isometries. Quipper and QML allow isometries to be implemented through qubit initialization, but their semantics are defined only in terms of unitaries, meaning there is no notion of equivalence between different unitaries implementing the same isometry.

- Projectors. Like with isometries, Quipper allows projectors to be implemented through assertive termination, but there is no notion of equivalence between different unitaries implementing the same projector.

- Classical uncomputation. Silq and Quipper both have convenient facilities for converting (irreversible) classical programs into reversible quantum subroutines.

- Quantum uncomputation. Silq can only uncompute “qfree” programs that are strictly classical. Quipper does have some facilities for uncomputing values produced by quantum programs, such as the with_computed function. However, this function’s correctness depends on conditions that are nontrivial to verify. In the words of Quipper’s documentation: “This is a very general but relatively unsafe operation. It is the user’s responsibility to ensure that the computation can indeed be undone.” In Qunity, safety is assured.

1.4 Contributions and Roadmap

Our core contribution is a new quantum programming language, Qunity, designed to unify classical and quantum computing through an expressive generalization of classical programming constructs. Qunity’s powerful semantics brings constructions commonly used in algorithm analysis—such as bounded-error quantum subroutines, projectors, and direct sums—into the realm of algorithm implementation. We describe Qunity’s formal syntax (Section 2), an efficiently checkable typing relation (Section 3), a compositional denotational semantics (Section 4), and a strategy for compiling to lower-level unitary quantum circuits (Section 6). We prove that well-typed Qunity programs have a well-defined semantic denotation, and we show that denotation is realizable by proving that our compilation strategy does indeed produce correct circuits. We also show how Qunity can be used to program several interesting examples, including Grover’s algorithm, the quantum Fourier transform, and a quantum walk (Section 5).

2 Syntax

Qunity’s formal syntax is defined in Figures 1–3. Qunity’s types are shown in Figure 1. The algebraic data types $T$ have essentially the same interpretation as in a typical classical programming language, with the caveat that values can be in superposition. The symbols $\oplus$ and $\otimes$ are used because of how these data types will correspond to direct sums and tensor products. The two program types $F$ differ in whether they decohere quantum states: programs of type $T \rightsquigarrow T'$ will have a semantics defined by an (often unitary) linear operator, while programs of type $T \Rightarrow T'$ will have a semantics defined by a superoperator, a completely positive trace-non-increasing map that may involve measurement or discarding.

Qunity’s term language is defined in Figure 3: expressions $e$ are assigned data types $T$ and programs $f$ are assigned program types $F$. Figure 4 shows additional derived forms. In the figure, $x$ ranges over some infinite set $X$ of variables (for example ASCII strings), and $r$ ranges over some representation
of real numbers (one example of which is defined in the supplemental report [Voichick et al. 2022a]). Some syntax elements use type annotations $T$, which are grayed out to reduce visual clutter in the more complex examples. A type inference algorithm could allow these annotations could to be removed, and throughout this paper, we occasionally omit type annotations for brevity.

Qunity makes use of standard classical programming language features generalized to the quantum setting. As examples, pairs generalize to creating tensor products of quantum states, sums generalize to allowing their data to be in superposition, and $\text{ctrl}$ generalizes classical pattern-matching (where in branch $\varepsilon \mapsto \varepsilon'$ the $\varepsilon$ is a pattern which may bind variables in $\varepsilon'$) to quantum control flow using superposition. The type system requires that the left-hand-side patterns are non-overlapping and thus correspond to orthogonal subspaces, and that the right-hand-side expressions appropriately use all of the variables from the condition expression, ensuring that this data is reversibly uncomputed rather than discarded. In general, we allow the left-hand-side patterns to be non-exhaustive, in which case the semantics is a norm-decreasing operator rather than norm-preserving. Operationally, a decrease in norm corresponds to the probability of being in a special “exceptional state.”

Existing languages like Proq [Li et al. 2020] have similarly used norm-decreasing operators such as projectors to describe assertions. Their system is effective for testing and debugging, but Qunity takes these projective assertions a step further by allowing them to be used in the control flow itself. Failed assertions are treated as exceptions, which can be dynamically caught and handled using the $\text{try}-\text{catch}$ construct, generalizing another familiar classical programming construct to the quantum setting. The quantum behavior here is well-defined using the language of projective measurements.
Assuming projector $P$ is implemented by Qunity program $f_P$, and state $|\psi\rangle$ is produced by Qunity program $e_\psi$, the Qunity expression $\text{try } \text{just}_T(f_P e_\psi) \text{ catch nothing}_T$ (using Maybe syntax from Figure 4) produces the mixed state defined by the density operator:

$$P|\psi\rangle\langle\psi| P \oplus (|\psi\rangle \langle \psi| (I - P) |\psi\rangle).$$

This state has $P |\psi\rangle$ in the “just” subspace and the norm of $(I - P) |\psi\rangle$ in the “nothing” subspace. Though we use the language of exception handling, this construct is also useful for non-exceptional conditions, like in the definition of the equals function in Figure 4. This function can be used to measure whether two states are equal; for example, a simple “coin flip” can be implemented by (had 0 = equals_0 (1)), which applies a Hadamard gate to a qubit in the $|0\rangle$ state and then measures whether the result is $|1\rangle$.

Notice that the innermost lambda in the definition of equals uses a non-exhaustive pattern on the left side; the program “$\lambda b \; \text{Bit} \mapsto 1$” implements the projector $|1\rangle \langle 0|$. While a classical operational semantics typically interprets lambdas in terms of substitution of arguments for parameters, Qunity’s semantics is best interpreted as a more general linear mapping. This interpretation means that Qunity’s type system can allow for a much wider range of expressions on the left “parameter” side of the lambda. As an extreme case, consider the definition of “$f^{\triangleright T}$” syntax in Figure 4, which applies a function on the left side. Semantically, this lambda corresponds to the adjoint of $f$, and can be interpreted like this: “given $f(x)$ as input, output $x$.” (However, this interpretation is imprecise when $f$ is norm-decreasing and its adjoint is not its inverse.)

Two of Qunity’s constructs have no classical analog. The $\text{u3}$ construct is a parameterized gate that allows us to implement any single-qubit gate [Cross et al. 2017]; e.g., it is used to implement had, per Figure 4. The rphase construct induces a relative phase of $e^{\lambda r}$, where the value $r$ comes from the first branch for states in the subspace spanned by the first branch’s parameter $e$ and from the second branch for states in the orthogonal subspace. It is used to implement gphase in Figure 4,
used to implement conditional phase flip from Deutsch’s algorithm (Section 1.1). More generally, it can be used to implement reflections, as used in Grover’s search algorithm, the iterator for which is shown below. We see that it includes the same conditional phase flip as Deutsch’s, composed with reflect\(_{\text{Bit}^n}(\text{plus}^n)\) to perform inversion about the mean. If the state \(|\psi\rangle\) is implemented by the expression \(e\), then reflect\(_T\)(\(e\)) implements the reflection \((2|\psi\rangle\langle\psi| - I)\) by coherently applying a phase of \(e^{i\pi} = -1\) to any state orthogonal to \(|\psi\rangle\).

\[
grover_n(f) = \lambda x \xrightarrow{\text{Bit}^n} \text{ctrl } f x \xrightarrow{\text{Bit}^n} \begin{array}{c}
\emptyset & \xrightarrow{x} & \text{Bit}^n
\end{array} \begin{array}{c}
1 & \xrightarrow{x \rightarrow g\text{phase}_{\text{Bit}^n}(\pi)} & \text{Bit}^n
\end{array} \xrightarrow{\text{reflect}_{\text{Bit}^n}(\text{plus}^n)}
\]

Several lambda calculi [Arrighi and Dowek 2017; Selinger and Valiron 2009; van Tonder 2004] have explored the use of higher-order functions in a quantum setting, but Qunity does not. We aim for Qunity’s denotational semantics to closely correspond to existing notations and conventions used in quantum algorithms, and higher-order functions and “superpositions of programs” are uncommon and inconsistently defined. In our experience, quantum mechanical notation (and to some degree quantum computing in general) is ill-suited for higher-order programs. In particular, we interpret programs \(f\) as quantum operations acting on their argument, and expressions \(e\) as quantum operations acting on their free variables, but allowing for higher-order functions means that programs can also have free variables, and one must take some sort of tensor product of a program’s two inputs. To avoid this extra complication, we have first-order functions only, and our typing relation prevents them from containing free variables.

### 3 Typing

This section describes Qunity’s type system. We prove that well-typed programs have a well-defined semantics (given in Section 4), and we have implemented a type checker for Qunity programs [Voichick et al. 2022b].

Qunity’s type system takes the form of three different, interdependent typing judgments: pure expression typing, mixed expression typing, and program typing. The distinction between pure and mixed typing comes from the fact that there are two ways to mathematically represent quantum states depending on whether any classical probability is involved: pure states are described by state vectors and do not involve classical probability, while mixed states are described by density matrices, usually interpreted as a classical probability distribution over pure states. Program types are similarly divided into pure and mixed versions, where pure programs correspond to linear operators from pure states to pure states, and mixed programs correspond to superoperators from mixed states to mixed states. In the supplemental report [Voichick et al. 2022a], we include further discussion on the need for both of these in Qunity.

Judgments are parameterized by typing contexts, which are ordered lists of variable-type pairs, as defined in Fig. 2. We say that a typing context \(x_1 : T_1, \ldots, x_n : T_n\) is well-formed if all variables are distinct; that is, if \(x_j \neq x_k\) whenever \(j \neq k\). Concatenating two typing contexts \(\Gamma_1\) and \(\Gamma_2\) is written \(\Gamma_1 \parallel \Gamma_2\). Within the inference rules, there is an implicit assumption that contexts are well-formed (so concatenation requires \(\text{dom}(\Gamma_1) \cap \text{dom}(\Gamma_2) = \emptyset\)).

#### 3.1 Pure Expression Typing

The pure expression typing judgment is written \(\Gamma \parallel \Delta \vdash e : T,^1\) indicating that expression \(e\) has pure type \(T\) with respect to classical context \(\Gamma\) and quantum context \(\Delta\). Variables in the classical context are in a classical basis state, and are automatically uncomputed, while quantum context

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^1Whether a context is classical or quantum is based on its position in the typing judgment—left of \(\parallel\) is classical and right of it is quantum. We write \(\Gamma\) (\(\Delta\), resp.) for classical (quantum, resp.) variables as a common but not universal convention.
variables are consumed and may be in superposition. Operationally, classical data is still compiled into qubits, but these qubits are only used as control wires for controlled operations, and they are uncomputed when they go out of scope. Whenever $\emptyset \parallel \emptyset \vdash e : T$ holds, the semantics corresponds to a pure state in $H(T)$, the Hilbert space assigned to $T$.

The pure expression typing rules are given in Figure 5. As is typical in quantum computing languages, these rules are substructural [Walker 2004]. In particular, as in QML [Altenkirch and Grattage 2005], quantum variables are relevant, meaning they must be used at least once. This invariant is evident from the T-QVar rule, which requires the quantum context to contain only the variable $x$ of interest, and the T-CVar and T-Unit rules, which require the quantum context to be empty. Indeed, we can prove that $\Gamma \parallel \Delta \vdash e : T$ implies $\text{dom}(\Delta) \subseteq \text{FV}(e)$. On the other hand, the rules do not enforce $\text{dom}(\Gamma) \subseteq \text{FV}(e)$.

By enforcing variable relevance, Qunity’s type system can control what parts of a program are allowed to discard information. Semantically, the non-use of a variable corresponds to a partial trace, a quantum operation described by decoherence. Quantum control flow is ill-defined in the presence of decoherence [Bădescu and Panangaden 2015], and decoherence is inherently irreversible, so we use relevant types wherever computation must be reversible or subject to quantum control.

A quantum lambda calculus [Selinger and Valiron 2009] uses affine types for quantum data, ensuring quantum variables are used at most once, while Proto-Quipper [Ross 2015] and QWire [Paykin et al. 2017] use linear types, requiring them to be used exactly once. This is done as static enforcement of the no-cloning theorem [Wootters and Zurek 1982], which makes sense in the QRAM computational model [Knill 1996]. However, Qunity’s aim (like QML’s [Altenkirch and Grattage 2005]) is to treat quantum computing as a generalization of classical computing. Qunity fixes a particular standard computational basis and treats expression $(x, x)$ as entangling variable $x$ when it is in superposition, essentially as a linear isometry $\alpha \ket{0} + \beta \ket{1} \mapsto \alpha \ket{00} + \beta \ket{11}$. The expression is deemed well typed by T-PurePair—notice that $\Delta$ appears when typing both $e_0$ and $e_1$, allowing duplication (“sharing”) of quantum resources across the pair. In contrast, QRAM-based languages reject $(x, x)$ to avoid confusion with the non-physical cloning function $\alpha \ket{0} + \beta \ket{1} \mapsto (\alpha \ket{0} + \beta \ket{1}) \otimes (\alpha \ket{0} + \beta \ket{1})$.  

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Fig. 5. Pure expression typing rules
The difference between “sharing” and “cloning” affects only non-basis states. Sharing a qubit $\alpha \ket{0} + \beta \ket{1}$ to a second qubit register produces the entangled state $\alpha \ket{00} + \beta \ket{11}$, while cloning would produce the unentangled state $(\alpha \ket{0} + \beta \ket{1})(\alpha \ket{0} + \beta \ket{1}) = \alpha^2 \ket{00} + \alpha \beta \ket{01} + \alpha \beta \ket{10} + \beta^2 \ket{11}$. The former is basis-dependent and implementable in Qunity as “$(x, x)$,” while the latter is basis-independent and physically prohibited by the no-cloning theorem.

The T-PUREAPP rule applies a linear operator $f$ to its argument $e$. The T-PUREPERM rule exists to allow for the usual structural rule of exchange, which is typically implicit. Here and throughout this work, the functions $\pi$ are list permutation functions, arbitrarily permuting the bindings within a context. Making the exchange rule explicit allows compilation to be a direct function of the typing judgment, with SWAP gates introduced at uses of the explicit exchange rule (Section 6.2).

Rule T-CTRL types pattern matching and quantum control. We defer discussing it to Section 3.4, after we have considered the other judgments.

### 3.2 Mixed Expression Typing

We write $\emptyset \vdash e : T$ to indicate that expression $e$ has mixed type $T$ under quantum context $\emptyset$; the rules are in Figure 6. $\emptyset \vdash e : T$ implies $e$‘s semantics corresponds to a mixed state in $\mathcal{H}(T)$.

\begin{align*}
\emptyset \parallel \Delta \vdash e : T & \quad \text{T-Mix} & \quad \Delta \vdash e : T & \quad \text{T-MixedPerm} \\
\Delta \vdash e : T & \quad \text{T-MixedPerm} & \quad \pi(\Delta) \vdash e : T & \\
\Delta, \Delta_0 \vdash e_0 : T_0 & \quad \Delta, \Delta_1 \vdash e_1 : T_1 & \quad \text{T-MixedPair} & \\
\Delta, \Delta_0, \Delta_1 \vdash (e_0, e_1) : T_0 \otimes T_1 & \quad \text{T-MixedPerm} & \\
\Delta \vdash e_0 : T & \quad \Delta \vdash e_1 : T & \quad \text{T-Try} & \\
\Delta_0 \vdash e_1 \text{ catch } e_1 : T & \quad \Delta \vdash f : T \Rightarrow T’ & \quad \text{T-MixedApp} & \\
\end{align*}

Fig. 6. Mixed expression typing rules

Rule T-Mix allows pure expressions to be typed as mixed. (One could equivalently treat pure types as a subtype of mixed types.) Rules T-MixedPerm and T-MixedPair are analogous to the pure-expression versions. Rule T-MixedApp allows applying a quantum channel $f$ on an expression $e$—such an $f$ may perform measurements, as discussed below. Rule T-Try allows exceptions occurring in expression $e_0$ to be caught and replaced by expression $e_1$. Operationally, this is effectively “measuring whether an error occurred” and thus try-catch expressions have no pure type; it also cannot be done without perturbing the input data $\Delta_0$ and thus expression $e_1$ is typed in a separate context.

### 3.3 Program Typing

We write $\vdash f : F$ to indicate that program $f$ has type $F$; the rules are in Figure 7. Whenever $\vdash f : T \Rightarrow T’$ holds, the semantics corresponds to a linear operator mapping $\mathcal{H}(T)$ to $\mathcal{H}(T’)$. Whenever $\vdash f : T \Rightarrow T’$ holds, the semantics corresponds to a superoperator mapping mixed states in $\mathcal{H}(T)$ to mixed states in $\mathcal{H}(T’)$. T-GATE types a single-qubit unitary gate, and T-LEFT and T-RIGHT type sum introduction. These are linear operations. T-PUREABS types a linear abstraction. In abstraction $\lambda e \vdash f : e'$, variables introduced in $e$ are relevant in $e'$—they must be present in quantum context $\Delta$ used to type both $e$ and $e'$. We use the double-lined “$\vdash$” symbol because this typing judgment can be used for measurements, and quantum circuit diagrams conventionally use double-lined wires to carry measurement results.
\[\vdash u_3(r_\theta, r_\phi, r_\psi) : \text{Bit} \rightarrow \text{Bit}\]

\[\vdash \text{left}_{T_0 \oplus T_1} : T_0 \rightarrow T_0 \oplus T_1\]

\[\vdash \text{left}_{T_0 \oplus T_1} : T_0 \rightarrow T_0 \oplus T_1\]

\[\text{T-Gate}\]

\[\vdash \text{right}_{T_0 \oplus T_1} : T_0 \rightarrow T_0 \oplus T_1\]

\[\vdash \text{T-Left}\]

\[\vdash \text{T-Right}\]

\[\vdash \text{T-Channel}\]

\[\vdash \text{T-MixedAbs}\]

\[\vdash \text{T-PureAbs}\]

\[\vdash \text{T-R.phase} \{\frac{e \mapsto r}{e \mapsto r'}\} : T \rightarrow T\]

\[\vdash \text{ctrl}_Q\text{unity allows for quantum control by generalizing pattern matching via }\text{ctrl}_1\text{, which is typed via the T- }\text{ctrl}_\text{rule (Figure 5). The e_j}s in the premises of this rule refer to the indexed expressions in the conclusion. A “prime” symbol should be viewed as part of the variable name. The following is the T-ctrl rule for }n = 2:\]

\[\Gamma, \Delta \vdash e : T \quad \text{ortho}_\Gamma (e_1, e_2) \quad \varnothing \parallel \Gamma_1 \vdash e_1 : T \quad \varnothing \parallel \Gamma_2 \vdash e_2 : T\]

\[\text{eras}_{\Gamma} (x; e_1', e_2') \text{ for all } x \in \text{dom}(\Delta) \quad \Gamma, \Gamma', \Gamma_1 \parallel \Delta \vdash e_1' : T' \quad \Gamma, \Gamma', \Gamma_2 \parallel \Delta \vdash e_2' : T'\]

\[\Gamma, \Gamma' \parallel \Delta \vdash \text{ctrl}_1 e \left\{\begin{array}{l}
e_1 \mapsto e_1' \nonumber \\
e_2 \mapsto e_2'
\end{array}\right\}_{T'}\]

Fig. 7. Program typing rules

and }e'. (It is not hard to prove that }\Delta\text{ can always be uniquely determined.) T-Mixed Abs is more relaxed: variables in }e\text{ can be contained in a context }\Delta_0\text{ which is not used to type }e';\text{ such variables are discarded in }e',\text{ which implies measuring them. So the typing judgment }\vdash \lambda x : \text{Bit} \rightarrow \emptyset : \text{Bit} \rightarrow \text{Bit}\text{ is invalid but the typing judgment }\vdash \lambda x : \text{Bit} \rightarrow \emptyset : \text{Bit} \Rightarrow \text{Bit}\text{ is valid. RPhase uses the expression }e\text{ as a pattern for coherently inducing a phase, either }e_1r\text{ or }e_2r'\text{ depending on whether pattern }e\text{ is matched. T-Channel permits a linear operator to be treated as a superoperator.}

Quinity programs are typed without context to help avoid scenarios where “entangling through variable re-use” might be confusing. For example, the program }\lambda x : \text{Bit} \rightarrow \text{let } (x, x_1) = \text{Bit}@\text{Bit} (x, x)\text{ in }x_0\text{ is valid, with type }\text{Bit} \Rightarrow \text{Bit}.\text{ This program measures a qubit by re-using the variable to share the qubit to a second register and then discarding the entangled qubit, and the valid program makes this sharing explicit. Quinity is designed so that pairing is the only way to perform this kind of entanglement, by using a variable on both sides of the pair.}

3.4 Typing Quantum Control

Quinity allows for quantum control by generalizing pattern matching via }\text{ctrl}_1\text{, which is typed via the T-ctrl rule (Figure 5). The }e_j\text{s in the premises of this rule refer to the indexed expressions in the conclusion. A “prime” symbol should be viewed as part of the variable name. The following is the T-ctrl rule for }n = 2:\]

\[\Gamma, \Delta \vdash e : T \quad \text{ortho}_\Gamma (e_1, e_2) \quad \varnothing \parallel \Gamma_1 \vdash e_1 : T \quad \varnothing \parallel \Gamma_2 \vdash e_2 : T\]

\[\text{eras}_{\Gamma} (x; e_1', e_2') \text{ for all } x \in \text{dom}(\Delta) \quad \Gamma, \Gamma', \Gamma_1 \parallel \Delta \vdash e_1' : T' \quad \Gamma, \Gamma', \Gamma_2 \parallel \Delta \vdash e_2' : T'\]

\[\Gamma, \Gamma' \parallel \Delta \vdash \text{ctrl}_1 e \left\{\begin{array}{l}
e_1 \mapsto e_1' \\
e_2 \mapsto e_2'
\end{array}\right\}_{T'}\]

To ensure realizable circuits, the type rule imposes several restrictions on patterns. First, both pattern and body expressions }e_1\text{ and }e_2'\text{ must be pure expressions; this means they cannot include invocations of superoperators, which might involve measurements. Some recent work has tried to generalize the definition of quantum control to a notion of “quantum alternation” that allows measurements to be controlled. QuGcl [Ying 2016, Chapter 6] attempts this, but the resulting
denotational semantics is non-compositional. Bădescu and Panangaden [Bădescu and Panangaden 2015] shed more light on the problems with quantum alternation, proving that quantum alternation is not monotone with respect to the Löwner order [Ying 2016, p. 17] and concluding that “quantum alternation is a fantasy arising from programming language semantics rather than from physics.”

For ctrl1 semantics to be physically meaningful, T-CTRL depends on two additional judgments applied to the body expressions—“ortho” for the left-hand-side (defined further below), and “erases” for the right-hand-side (defined in Figure 8). The ortho judgment ensures that the left-hand-side patterns $e_j$ are purely classical and non-overlapping. The erases judgment (defined using gphase syntax from Figure 4) ensures that all of the right-hand-side patterns $e_j'$ use the variables from $e$ in a consistent way, and its name comes from the way this judgment is used by the compiler to coherently “erase” these variables. The contexts $\Gamma_j$ are the primary motivation that the typing relation includes classical contexts $\Gamma$ at all. Semantically, the variables they represent are purely classical; operationally, the information on these registers is used without being “consumed,” under the assumption that it will be uncomputed. When typing the expressions $e_j$, these contexts $\Gamma_j$ appear in the “quantum” part of the context so that the type system can enforce variable relevance, but the restrictions placed by the ortho judgment ensure that these variables are still practically classical.

The orthogonality judgment is defined in terms of a “spanning” judgment defined in the supplemental report [Voichick et al. 2022a]. This judgment, written $\text{spanning}_T(e_1, \ldots, e_n)$ and largely inspired by Spm’s [Sabry et al. 2018] “orthogonal decomposition” judgment, denotes that the set of expressions $\{e_1, \ldots, e_n\}$ is a spanning set for type $T$. This judgment enforces that the expressions form an exhaustive set of patterns for the type $T$, which in our quantum setting semantically corresponds to a set of states that span the Hilbert space corresponding to $T$.

We write $\text{ortho}_T(e_1, \ldots, e_n)$ to denote that the set of expressions $\{e_1, \ldots, e_n\}$ is orthogonal. An orthogonal set of expressions is simply a subset of some spanning set. That is, orthogonality judgments can be defined by the following inference rule alone:

$$
\text{spanning}_T(e'_1, \ldots, e'_m) \quad [e_1, \ldots, e_n] \text{ is a subsequence of } [e'_1, \ldots, e'_m] \\
\text{ortho}_T(e_1, \ldots, e_n)
$$
It is not hard to prove that orthogonality holds for a set of expressions regardless of the order they appear in the judgment (e.g., ortho\(\sigma\) \((e_1, e_2)\) iff ortho\(\tau\) \((e_2, e_1)\)).

4 SEMANTICS

In this section, we define Qunity’s denotational semantics in terms of linear operators (for pure expressions) and superoperators (for mixed expressions). These definitions are designed to naturally generalize from a classical semantics defined on a classical sublanguage of Qunity. After presenting the semantics, we present some metatheoretical results, most notably that well-typed Qunity programs are well-defined according to the semantics.

4.1 Classical Sublanguage Semantics

Qunity’s semantics may be more intuitive if we first restrict ourselves to a classical sublanguage.

Definition 4.1 (classical sublanguage). Qunity’s classical sublanguage is defined by removing the u3 and rphase constructs from the language, producing the following:

\[
e := () \mid x \mid (e, e) \mid \text{ctrl } e \left\{ e \mapsto e \right\}_T \mid \text{try } e \text{ catch } e \mid f e
\]

\[
f := \text{left}_{T@T} \mid \text{right}_{T@T} \mid \lambda e^T \mapsto e
\]

We can define a classical denotational semantics for this sublanguage using partial functions over values and valuations.

Definition 4.2 (value and valuation). For any type \(T\), we write \(\mathcal{V}(T)\) to denote the set of expressions that are values of that type.

\[
\mathcal{V}(\text{Void}) = \emptyset
\]

\[
\mathcal{V}(\emptyset) = \{\emptyset\}
\]

\[
\mathcal{V}(T_0 \oplus T_1) = \{\text{left}_{T_0@T_1} v_0 \mid v_0 \in \mathcal{V}(T_0)\} \cup \{\text{right}_{T_0@T_1} v_1 \mid v_1 \in \mathcal{V}(T_1)\}
\]

\[
\mathcal{V}(T_0 \otimes T_1) = \{(v_0, v_1) \mid v_0 \in \mathcal{V}(T_0), v_1 \in \mathcal{V}(T_1)\}
\]

A valuation, written \(\sigma\) or \(\tau\), is a list of variable-value pairs:

\[
\sigma := \emptyset \mid \sigma, x \mapsto v
\]

Like with \(\Gamma\) and \(\Delta\), we will generally use the letter \(\sigma\) for classical data and \(\tau\) for quantum data, but the two are interchangeable. Each typing context has a corresponding set of valuations, defined as follows:

\[
\mathcal{V}(x_1 : T_1, \ldots, x_n : T_n) := \{x_1 \mapsto v_1, \ldots, x_n \mapsto v_n \mid v_1 \in \mathcal{V}(T_1), \ldots, v_n \in \mathcal{V}(T_n)\}
\]

Like with typing contexts, we use a comma to denote the concatenation of valuations, sometimes mixing valuations with explicit variable-value pairs. For example, if \(\tau_0 \in \mathcal{V}(\Delta_0)\) and \(v \in \mathcal{V}(T)\) and \(\tau_1 \in \mathcal{V}(\Delta_1)\), then \(\tau_0, x \mapsto v, \tau_1 \in \mathcal{V}(\Delta_0, x : T, \Delta_1)\).

In the supplemental report [Voichich et al. 2022a], we define a classical denotational semantics for this sublanguage. The semantics of an expression is a partial function from valuations to values, while the semantics of a program is a partial function from values to values. Given a purely-typed expression or program, this partial function will be injective. Stripped of quantum constructs, Qunity thus becomes a (classical) reversible programming language comparable to other programming languages for reversible computing.
Theorem 4.3 (expressiveness of classical sublanguage). Any combinator written in the reversible language \( \Pi \) [James and Sabry 2012] can be translated into a pure program in Qunity’s classical sublanguage, and any arrow computation written in the arrow metalanguage \( \Pi \) [James and Sabry 2012] can be translated into a mixed program in Qunity’s classical sublanguage. These translations preserve types and semantics.

We prove Theorem 4.3 in the supplemental report. We choose the language \( \Pi \) because its typing and semantics are directly comparable with Qunity’s, and because a translation from a more typical \texttt{let}-based language to \( \Pi \) already exists.

Qunity’s quantum semantics, defined in the next section, can be viewed as a generalization of the classical sublanguage semantics. The advantage of this approach is that one does not have to explicitly convert between separate quantum and classical languages, as any program written in the classical sublanguage can be applied to quantum data.

Theorem 4.4 (generalization of classical semantics). The classical semantics of any classical Qunity program coincides with its quantum semantics applied to values treated like basis states.

4.2 Full Semantics

Definition 4.5. We associate Hilbert spaces with types and contexts as follows:

\[
\begin{align*}
H(\text{Void}) &= \{0\} \\
H(\varnothing) &= \mathbb{C} \\
H(T_0 \oplus T_1) &= H(T_0) \oplus H(T_1) \\
H(T_0 \otimes T_1) &= H(T_0) \otimes H(T_1) \\
H(x_1 : T_1, \ldots, x_n : T_n) &= H(T_1) \otimes \cdots \otimes H(T_n)
\end{align*}
\]

On the right-hand side above, we use the symbols \( \oplus \) and \( \otimes \) to refer to the usual direct sum and tensor product of Hilbert spaces [Roman 2008]. For those unfamiliar with the direct sum, we give an overview of its important properties in the supplemental report. Each Hilbert space \( H(T) \) has a canonical orthonormal basis \( \{|v\rangle : v \in \mathbb{V}(T)\} \subset H(T) \) as follows:

\[
\begin{align*}
|\langle\rangle\rangle &= 1 \\
|\text{left}_{T_0 \oplus T_1} v_0\rangle &= |v_0\rangle \oplus 0 \\
|\text{right}_{T_0 \oplus T_1} v_1\rangle &= 0 \oplus |v_1\rangle \\
|(v_0, v_1)\rangle &= |v_0\rangle \otimes |v_1\rangle
\end{align*}
\]

We can do the same for typing contexts \( \Delta \), constructing an orthonormal basis \( \{|\tau\rangle : \tau \in \mathbb{V}(\Delta)\} \subset H(\Delta) \) as follows:

\[
|x_1 \mapsto v_1, \ldots, x_n \mapsto v_n\rangle := |v_1\rangle \otimes \cdots \otimes |v_n\rangle
\]

Using notation from linear algebra [Axler 2015], we write \( L(H_0, H_1) \) to denote the space of linear operators from \( H_0 \) to \( H_1 \), with \( L(H') = L(H', H') \).

Qunity’s semantics is defined by four mutually recursive functions of a valid typing judgment:
\[ [\sigma : \Gamma \parallel \emptyset \triangleright \langle : \emptyset \rangle \triangleright \emptyset ] = \emptyset \]
\[ [\sigma : \Gamma \parallel \emptyset \triangleright x : T \triangleright \emptyset ] = [\sigma(x)] \]
\[ [\sigma : \Gamma \parallel x : T \triangleright x : T] \triangleright [x \mapsto u] = [u] \]
\[ [\sigma : \Gamma \parallel \Delta, \Delta_0, \Delta_1 \triangleright (e_0, e_1 : T_0 \otimes T_1)] = [\sigma : \Gamma \parallel \Delta, \Delta_0 \triangleright e_0 : T_0] \triangleright [r, \tau_0, \tau_1] \]
\[ \otimes [\sigma : \Gamma \parallel \Delta, \Delta_1 \triangleright e_1 : T_1] \triangleright [r, \tau_1] \]
\[ [\sigma, \sigma' : \Gamma, \Gamma' \parallel \Delta, \Delta' + \text{ctrl } e \triangleright \sum_{i=1}^{n} e_i \mapsto e_i'] : T' \triangleright [r, \tau'] \]
\[ = \sum_{i=1}^{n} \sum_{j \in \mathbb{V}(T)} (v) [\sigma_j] [\emptyset : \emptyset \parallel \Gamma_j \triangleright e_j : T] \triangleright [v] \]
\[ [\sigma : \Gamma \parallel \Delta \triangleright f : e : T] \triangleright \sigma : \Gamma \parallel \Delta \triangleright e : T \triangleright \tau \]
\[ [\pi_\alpha(\sigma) : \pi_\alpha(\Gamma) \parallel \pi_\alpha(\Delta) \triangleright e : T] \parallel [\pi_\alpha(\tau)] = [\sigma : \Gamma \parallel \Delta \triangleright e : T] \triangleright \tau \]

Fig. 9. Pure expression semantics

\[ [\Delta, \Delta_0 \triangleright e : T] \triangleright (r, \tau_0, \tau_\|_0) = [\emptyset : \emptyset \parallel \Delta \triangleright e : T] \triangleright (r, \tau_0, \tau_\|_0) \]
\[ [\Delta, \Delta_0 \triangleright e : T] \triangleright (r, \tau_0, \tau_\|_0) = 0 \text{ if } (\emptyset : \emptyset \parallel \Delta \triangleright e : T) \text{ holds and } r \neq r' \]
\[ [\Delta, \Delta_0, \Delta_1 \triangleright (e_0, e_1 : T_0 \otimes T_1)] \triangleright (r, \tau_0, \tau_1, \tau_\|_0, \tau_\|_1) = [\Delta, \Delta_0 \triangleright e_0 : T_0] \triangleright (r, \tau_0, \tau_\|_0, \tau_\|_1) \]
\[ \otimes [\Delta, \Delta_1 \triangleright e_1 : T_1] \triangleright (r, \tau_1, \tau_\|_1) \]
\[ [\Delta_0, \Delta_1 \triangleright \text{try } e_0 \text{ catch } e_1 : T] \triangleright (r_0, \tau_1, \tau_\|_0, \tau_\|_1) = [\Delta_0 \triangleright e_0 : T] \triangleright (r_0, \tau_\|_0) \]
\[ + (1 - \text{tr}([\Delta_0 \triangleright e_0 : T] \triangleright (r_0, \tau_\|_0))) [\Delta_1 \triangleright e_1 : T] \triangleright (r_1, \tau_\|_1) \]
\[ [\Delta \triangleright f : e : T] \triangleright (r, \tau') = [\Delta \triangleright f : T \triangleright T'] \triangleright (r, \tau') \]
\[ [\pi(\Delta) \triangleright e : T] \triangleright (\pi(\tau), \pi(\tau')) = [\Delta \triangleright e : T] \triangleright (\pi(\tau), \pi(\tau')) \]

Fig. 10. Mixed expression semantics

- If \( \Gamma \parallel \Delta \triangleright e : T \) and \( \sigma \in \mathbb{V}(\Gamma) \), then \( [\sigma : \Gamma \parallel \Delta \triangleright e : T] \in \mathcal{L}(\mathcal{H}(\Delta), \mathcal{H}(T)) \) defines the pure semantics of expression \( e \). We give the denotation in Figure 9. The \( \sigma \) is a sort of “classical data,” so the pure expression semantics may be viewed as a two-parameter function \( \mathbb{V}(\Gamma) \times \mathcal{H}(\Delta) \rightarrow \mathcal{H}(T) \), linear in its second argument.
- If \( \Delta \triangleright e : T \), then \( [\Delta \triangleright e : T] \in \mathcal{L}(\mathcal{L}(\mathcal{H}(\Delta)), \mathcal{L}(\mathcal{H}(T))) \) defines the mixed semantics of expression \( e \). We give the denotation in Figure 10.
- If \( \Delta \triangleright f : T \rightsquigarrow T' \), then \( [[\Delta \triangleright f : T \rightsquigarrow T']] \in \mathcal{L}(\mathcal{H}(T), \mathcal{H}(T')) \) defines the pure semantics of program \( f \). We give the denotation in Figure 11.
- If \( \Delta \triangleright f : T \Rightarrow T' \), then \( [[\Delta \triangleright f : T \Rightarrow T']] \in \mathcal{L}(\mathcal{L}(\mathcal{H}(T)), \mathcal{L}(\mathcal{H}(T'))) \) defines the mixed semantics of program \( f \). We give the denotation in Figure 12.
prove the same typing judgment, so we are defining semantics as recursive functions on typing judgments rather than on programs. The operator for each subexpression is applied to the relevant part of the input basis state, and a pure semantics is then used directly as an amplitude semantics. For example, the pure semantics of \( \lambda e \mapsto e' : T \mapsto T' \) is defined as:

\[
[[\lambda e \mapsto e' : T \mapsto T']]([v]([v'])) = e'[[v]([v'])[[f : T \mapsto T']]^
\]

The most complicated definition is the one for `try-catch`, which uses a superoperator to construct an operator. The definition uses \( \langle v | [\Gamma, \Delta \vdash e : T] (|\sigma, \tau\rangle|\sigma, \tau\rangle | v')\), the probability that expression \( e \) outputs \( v \) given input \( (\sigma, \tau) \), as seen by a classical observer. This probability from the mixed semantics is then used directly as an amplitude in the resulting pure semantics. Note that this introduces an unavoidable source of error whenever probabilities between 0 and 1 are involved because there is no square root involved—the resulting semantics will be norm-decreasing even if the original superoperator was trace-preserving.

The definition for `try-catch` uses \( (1 - \text{tr}([\Delta_0 \vdash e_0 : T](|r_0\rangle|r_0\rangle))) \), the probability that \( e_0 \) throws an exception. If \( e_0 \) succeeds, then its results are used, but otherwise, the results from \( e_1 \) are used.

### 4.3 Metatheory

We are defining semantics as recursive functions on typing judgments rather than on programs and expressions directly. Qunity’s typing relation is not syntax-directed, which means that there are multiple ways to type—and give semantics to—the same expression. Thus, we must prove that the semantics is truly a function, i.e., that the different proofs of the same judgment lead to the same semantic denotation.

As an example, note the two deduction trees in Figure 13 prove the same typing judgment, so they should give rise to the same semantics. The semantics takes a tensor product of two state vectors for `T-PurePAIR`, but a tensor product of two density operators for `T-MixedPAIR`. This is not
a problem because a tensor product can equivalently be taken before or after the conversion to density operators: \((|\psi\rangle \otimes |\phi\rangle)(|\psi\rangle \otimes |\phi\rangle) = |\psi\rangle|\psi\rangle \otimes |\phi\rangle|\phi\rangle\) for any states \(|\psi\rangle, |\phi\rangle\). We prove that this kind of equivalence always holds.

**Theorem 4.6 (well-defined semantics).** Qunity has a well-defined semantics:

- Whenever \((\Gamma \parallel \Delta \vdash e : T)\) is valid and \(\sigma \in \forall (\Gamma)\), the denotation \(\models \Gamma \parallel \Delta \vdash e : T\) is uniquely defined.
- Whenever \((\Delta \vdash e : T)\) is valid, the denotation \(\models \Delta \vdash e : T\) is uniquely defined.
- Whenever \((\vdash f : T \rightsquigarrow T')\) is valid, the denotation \(\models \vdash f : T \rightsquigarrow T'\) is uniquely defined.
- Whenever \((\vdash f : T \Rightarrow T')\) is valid, the denotation \(\models \vdash f : T \Rightarrow T'\) is uniquely defined.

Whenever a typing judgment has more than one proof of validity, the derived semantics is independent of the proof.

The proof of Theorem 4.6, given in the supplemental report, was largely inspired by the proof of Newman’s lemma [Huet 1980; Newman 1942], a standard tool for proving global confluence from local confluence when using (operational) reduction semantics. Qunity’s semantics is denotational rather than operational, but one can imagine the operational procedure of *evaluating* the denotational semantics by repeatedly rewriting denotations in terms of the denotations of subexpressions. In this view, we have a terminating sequence of rewrite rules, and global confluence is exactly what is needed to prove the semantics well-defined. We do not use Newman’s lemma directly, but the induction strategy is essentially the same, and this allows us to focus on particular cases of equivalence like the one in Figure 13, which are easy to verify algebraically.

Not all pure Qunity programs have a norm-preserving (isometric) semantics. Rather, programs that may throw an exception are norm-decreasing instead. We can characterize our semantics in terms of *Kraus operators.*

**Definition 4.7 (Kraus operator).** A Kraus operator is a linear operator \(E\) such that \(E^\dagger E \subseteq I\), where \(\subseteq\) denotes the Löwner order [Ying 2016, p. 17].

Note that Kraus operators are typically defined as sets of operators \(E\) whose sum satisfies the property above. For our purposes, a single operator will suffice.

**Theorem 4.8.** \(\models \sigma : \Gamma \parallel \Delta \vdash e : T\) and \(\models \vdash f : T \rightsquigarrow T'\) are Kraus operators whenever well-defined. \(\models \Delta \vdash e : T\) and \(\models \vdash f : T \Rightarrow T'\) are completely positive, trace non-increasing superoperators whenever well-defined.

We do not prove this theorem directly, but it is a direct consequence of the correctness of our compilation procedure discussed in Section 6.

We can also be more precise about the semantics of the orthogonality and spanning judgments. In the absence of variables, the orthogonality judgment describes a set of orthogonal basis states,
and the spanning judgment describes a spanning set of orthogonal basis states. With variables, the picture is a bit more complicated, because we are dealing with projectors onto orthogonal subspaces rather than orthogonal states. An algebraic description of these semantics is given by Lemmas 4.10 and 4.9.

**Lemma 4.9 (spanning semantics).** Suppose $\Theta \parallel \Delta_j \vdash e_j : T$ for all $j \in \{1, \ldots, n\}$ and suppose spanning$_f (e_1, \ldots, e_n)$ is true. Then $\sum_{j=1}^n [\Theta \parallel \Delta_j \vdash e_j : T]\parallel \Theta \parallel \Delta_j \vdash e_j : T]^\dagger = I$.

**Lemma 4.10 (orthogonality semantics).** Suppose $\Theta \parallel \Delta_j \vdash e_j : T$ for all $j \in \{1, \ldots, n\}$ and suppose ortho$_f (e_1, \ldots, e_n)$ is true. Then $\langle v| [\Theta \parallel \Delta_j \vdash e_j : T]|\tau_j \rangle \in \{0, 1\}$ for all $\tau_j \in \forall(\Delta_j), v \in \forall(T)$, and $\sum_{j=1}^n [\Theta \parallel \Delta_j \vdash e_j : T]\parallel \Theta \parallel \Delta_j \vdash e_j : T]^\dagger \subseteq I$.

Finally, Qunity’s semantics uses norm-decreasing operators and trace-decreasing superoperators, which can be interpreted operationally as a sort of “exception.” In the supplemental report, we present an additional judgment “iso” that can be used to statically determine whether a program belongs to a checkable class of exception-free programs.

5 EXAMPLES

We have already shown two examples of programs we can write in Qunity: Deutsch’s algorithm (Section 1.1) and Grover’s algorithm (end of Section 2). This section presents two more—the quantum Fourier transform and the quantum walk—aiming to further illustrate Qunity’s expressiveness. The quantum walk depends on *specialized erasure*, a technique that reverses $\text{ctrl}$ expressions to implement a general form of reversible pattern-matching. We give a third example—the Deutsch-Jozsa algorithm—in the supplemental report.

5.1 Quantum Fourier Transform

In Figure 14, we use a presentation of the quantum Fourier transform that has a symmetric circuit diagram [Griffiths and Niu 1996]. This version uses a two-qubit “coupling” gate that swaps two qubits and coherently induces a particular global phase conditional on both qubits being $|1\rangle$.

5.2 Specialized Erasure

Quantum control—that is, programming with conditionals while maintaining quantum coherence—is a common feature of quantum algorithms, and Qunity’s $\text{ctrl}$ construct can be a powerful tool for implementing this pattern. However, the “erases” requirement of the T-$\text{ctrl}$ typing rule can be a frustrating limitation on practical quantum control. It effectively requires that any part of the input used for quantum control (the $\Delta$ context in the T-$\text{ctrl}$ typing rule) must be present in the output as well. We show here how programmers can get around this limitation with a general approach for “erasing” controlled data in a more customizable way, without adding any new primitives to the Qunity language.

The basic approach is fairly simple, and it is already used in the implementation of existing quantum algorithms [Childs and van Dam 2010, p. 8]. Suppose one has two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ with orthonormal sets $\{|1\rangle, \ldots, |n\rangle\} \subset \mathcal{H}$ and $\{|1\rangle', \ldots, |n\rangle'\} \subset \mathcal{H}'$, and one would like to implement the operator $E := \sum_{j=1}^n |j\rangle'\langle j| \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$. If one can implement operators $E_1 := \sum_{j=1}^n |j\rangle\langle j| \in \mathcal{L}(\mathcal{H}, \mathcal{H} \otimes \mathcal{H}')$ and $E_2 := \sum_{j=1}^n |j\rangle'\langle j'| \in \mathcal{L}(\mathcal{H}', \mathcal{H} \otimes \mathcal{H}')$, then one can implement $E = E_2^\dagger E_1$. This pattern can be useful in Qunity, where the operators $E_1$ and $E_2$ can be easy to implement using a $\text{ctrl}$ expression, but the erases judgment prevents $E$ from being implemented more directly. The $E_2^\dagger$ operator serves as a sort of “specialized erasure,” erasing the input state $|j\rangle$. 

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In particular, the typing rule and semantics in Figure 16 can be inferred.

If compiled without optimizations using the compilation procedure described in Section 5, the semantics of this program are very similar to the symmetric pattern matching language \[ Q_{\text{unity 32:19}} \].

This pattern can be used to implement a “match” program, shown in Figure 15. The typing and semantics of this program are very similar to the symmetric pattern matching language [Sabry et al. 2018]. In particular, the typing rule and semantics in Figure 16 can be inferred.

As another example, Figure 17 shows how one can use this pattern to implement the direct sum of linear operators, defined so that \( [\cdot f_0 \oplus f_1 : T_0 \oplus T_1] \sim [\cdot f_0 : T_0] \oplus [\cdot f_1 : T_1] \oplus T'_0 \oplus T'_1 \).

These examples are admittedly fairly lengthy, and would not lead to an efficient implementation if compiled without optimizations using the compilation procedure described in Section 6. Our
purpose here is to demonstrate the expressiveness of Quunity despite its small number of language primitives, but additional language primitives may make efficient compilation easier.

5.3 Quantum Walk

We use Quunity to implement a quantum walk algorithm for boolean formula evaluation [Ambainis et al. 2010; Childs et al. 2007]. The simplest version of this algorithm treats a NAND formula as a balanced binary tree where each leaf corresponds to a variable in the formula and each vertex corresponds to a NAND application. Given black-box oracle access to a function \( f : \text{Variable} \rightarrow \{0, 1\} \), the task is to evaluate the formula.

This algorithm performs a quantum walk, repeatedly applying a diffusion step and a walk step to two quantum registers: a vertex index and a qutrit “three-sided coin.” The diffusion step uses coherent control to apply a different operator to the coin depending on the vertex index. If the vertex index is a leaf \( a \), then the oracle is used to induce a conditional phase flip of \((-1)^{f(a)}\). Otherwise, a reflection operator \((2\ket{u}\bra{u} - I)\) is applied to the coin register, where \(\ket{u} \) is a coin state whose value depends on whether the vertex index is one of two special root nodes. The walk step then performs a coherent permutation on the two registers, “walking” the vertex index to the direction specified by the coin and setting the coin to be the direction traveled from.

This algorithm presents some interesting challenges for implementation, and Quipper’s [Green et al. 2013] implementation of this algorithm is quite long.\(^3\) One challenge is the representation of the vertex index. The traversed graph is a tree, so a convenient representation is the path taken to get to this vertex from the root of the tree, a list like \([\text{vleft}, \text{vright}, \ldots]\) indicating the sequence of child directions. The problem is that this list has an unknown length, so one cannot simply

\(^3\)See Quipper.Algorithms.BF.BooleanFormula for reference.
use an array of qubits where each qubit corresponds to a direction taken. Quipper has no sum types, and every (quantum) Quipper type is effectively a fixed-length array of qubits, so Quipper’s implementation must directly manipulate an encoding of variable-length lists into bitstrings, which requires developers to work at a lower level of abstraction than they would prefer. Qunity’s sum types make it convenient to coherently manipulate variable-length lists by managing the qubit encoding automatically.

First, we must define the types and values used in this program in terms of existing ones. We use a recursively-defined $\text{Vertex}$ type to represent a vertex in the tree as a variable-length list. This type and others are defined in Figure 18 in terms of Qunity’s base types. Here, the parameter $n$ is the height of the tree, a bound on the depth. A $\text{Vertex}_{n+1}$ is thus either empty or a $\text{Vertex}_n$ with an additional $\text{Child}$ appended and can store a superposition of lists of different lengths. The algorithm augments the tree with two special vertices at the root of the tree, $\text{root}^\prime_n$ and $\text{root}_n$, both of type $\text{Vertex}_{n+2}$, where the “+2” comes from the extra depth incurred by these vertices. The leaves of the traversed tree are those where the path from the root is maximal, so we can use the fixed-length $\text{Leaf}$ type to describe leaves separately from arbitrary vertices.

$\text{Child} = \text{Bit}$
$\text{Vertex}_0 = \text{Void}$
$vleft = \emptyset$
$\text{Vertex}_{n+1} = () \oplus (\text{Vertex}_n \otimes \text{Child})$
$vright = 1$
$\text{root}_n = \text{left}_{\text{Vertex}_{n+1}} ()$
$\text{Coin} = \text{Maybe} (\text{Child})$
$(e \mapsto_n e_0) = \text{right}_{\text{Vertex}_{n+1}} (e, e_0)$
$\text{cdown} = \text{nothing}_{\text{child}}$
$\text{root}^\prime_n = \text{root}_n \mapsto_n vleft$
$\text{cleft} = \text{just}_{\text{Child}} vleft$
$\text{Leaf}_n = \text{Child}^{\oplus n}$
$\text{cright} = \text{just}_{\text{Child}} vright$
$\text{right}_n$

Fig. 18. Types and values used in the quantum walk algorithm (also see Fig. 4)

Figure 19 defines some additional programs used in this algorithm. The $\text{asleaf}$ program is essentially a projector onto the subspace of vertices spanned by leaves; asleaf$_n$ has type $(\text{Vertex}_{n+2} \iff \text{Leaf}_n)$. This program converts the variable-length $\text{Vertex}$ type into the fixed-length $\text{Leaf}$ type, terminating exceptionally if the length is insufficient. The downcast$_n$ program has type $(\text{Vertex}_{n+1} \iff \text{Vertex}_n)$, and is used to convert a vertex to a type with a smaller maximum length, terminating exceptionally if the length is maximal. This program, defined using the specialized erasure technique described in the previous section, is used by the leftchild$_n$ and rightchild$_n$ programs, both of type $(\text{Vertex}_n \iff \text{Vertex}_n)$, which reversibly “step” a vertex to a child in the tree.

We implement the diffusion step of the quantum walk in Figure 20. This implementation resembles the original algorithm description [Childs et al. 2007, p. 4] much more closely than Quipper’s implementation, which involves more complicated qubit encodings and explicit uncomputation. The diffusion program has the following type:

$\vdash f : \text{Leaf}_n \Rightarrow \text{Bit}$

$\vdash \text{diffusion}_n (f) : \text{Coin} \otimes \text{Vertex}_{n+2} \iff \text{Coin} \otimes \text{Vertex}_{n+2}$

We implement the walk step in Figure 21. This program must update both the vertex and the coin while maintaining quantum coherence. To make this work, we use a program $\text{nextcoin}_n$, of type $(\text{Coin} \otimes \text{Vertex}_{n+1} \iff (\text{Coin} \otimes \text{Vertex}_{n+1}) \otimes \text{Coin})$ to both compute the updated coin and
asleaf\(_0\) = \(\lambda\) root\(_0\) \(\xrightarrow{\text{Vertex}_1}\) ()

asleaf\(_{n+1}\) = \(\lambda\) (x \(\xrightarrow{\text{Vertex}_{n+2}}\) x\(_0\)) \(\xrightarrow{\text{Vertex}_{n+3}}\) (x\(_0\), x \(\triangleright\) asleaf\(_n\))

downcast\(_0\) = \(\lambda\) v \(\xrightarrow{\text{ctrl}}\) v \(\xrightarrow{\text{Vertex}_1}\) {} \(\xrightarrow{\text{Vertex}_0}\)

downcast\(_{n+1}\) = \(\lambda\) v \(\xrightarrow{\text{Vertex}_{n+2}}\)

\(\triangleright\) \(\lambda\) \(\xrightarrow{\text{ctrl}}\) v

\(\left\{\begin{array}{l}
\text{root}_{n+1} \mapsto (v, \text{root}_n) \\
\text{vertex}_{n+3} \mapsto (v', x \mapsto (v, \text{downcast}_n v' \mapsto x))
\end{array}\right\}

\(\xrightarrow{\text{Vertex}_{n+2} \xrightarrow{\text{Vertex}_{n+1}}}

\left\{\begin{array}{l}
\text{ctrl}_n \mapsto (\text{root}_{n+1}, v) \\
' \mapsto (\text{downcast}_n (v' \mapsto x), v)
\end{array}\right\}

\(\xrightarrow{\text{Vertex}_{n+2} \xrightarrow{\text{Vertex}_{n+1}}}

\left\{\begin{array}{l}
\text{leftchild}_n \mapsto \lambda\) v \(\xrightarrow{\text{downcast}_n}\) (v \(\xrightarrow{\text{ctrl}}\) \(\triangleright\) vleft)
\text{rightchild}_n \mapsto \lambda\) v \(\xrightarrow{\text{downcast}_n}\) (v \(\xrightarrow{\text{ctrl}}\) \(\triangleright\) vright)
\end{array}\right\}

Fig. 19. Programs and expressions used in the quantum walk implementation

diffusion\(_n\)(f) =

\(\lambda\) (c, v) \(\xrightarrow{\text{Coin} \xrightarrow{\text{Vertex}_{n+2}}\}

\text{ctrl}_n\)

\(\left\{\begin{array}{l}
\text{try just}_n((v' \mapsto x \mapsto x') \mapsto \text{asleaf}_n) \\
\text{catch nothing}_n((v' \mapsto x \mapsto x') \mapsto \text{leaf}_n)
\end{array}\right\}

\(\xrightarrow{\text{try just}_n\text{catch nothing}_n\text{leaf}_n\text{just}_n\text{root}_n\text{ctrl}_n\}

\(\left\{\begin{array}{l}
\text{nothing}_n \mapsto (c \mapsto \text{reflect}_n\text{Coin}(u), v) \\
\text{just}_n \mapsto \text{ctrl}_n(f) \{\emptyset \mapsto (c, v) \\
1 \mapsto (c, v) \mapsto \text{gphase}_n(\pi)
\end{array}\right\}

\(\xrightarrow{\text{try just}_n\text{catch nothing}_n\text{leaf}_n\text{ctrl}_n\}

\text{root}_n \mapsto (c \mapsto \text{reflect}_n\text{Coin}(u'), v)
\text{root}_{n+1} \mapsto (c, v)

Fig. 20. The diffusion step of the boolean formula algorithm. For brevity, we omit some type annotations and the implementation of the expressions u and u'\(_n\). See the Coq implementation [Voichick et al. 2022b] for details.

to uncompute the previous coin. The walk program walk\(_n\) then has type (Coin \(\otimes\) Vertex\(_{n+1}\) \(\mapsto\) Coin \(\otimes\) Vertex\(_{n+1}\)).

6 COMPILATION

We have developed an algorithm for compiling Qunity to a low-level qubit circuit language such as OpenQasm [Cross et al. 2021]. This section provides an overview of the algorithm; the details are given in the supplemental report [Voichick et al. 2022a].
Our compilation algorithm serves two purposes. First, it makes clear that Qunity is realizable on a quantum computer. Realizability is not immediately clear from the definitions of Qunity’s semantics; some quantum languages like Lineal [Arrighi and Dowek 2017] and the \( \text{zx-calculus} \) [van de Wetering 2020] allow for programs to be written with norm-\textit{increasing} semantics and thus have no physical interpretation. Second, it gives an intuition on what is operationally happening step-by-step in the physical computer and answers questions about data representation, automatic uncomputation, and the interplay between Qunity’s pure and mixed modes of computation.

There are some obvious challenges in compiling Qunity to low-level circuits. In particular, the \( \text{ctrl} \) construct can make irreversible programs reversible, and the \( \text{try-catch} \) construct can make trace-decreasing programs trace-preserving. We solve these problems by augmenting the circuit with ancillary “garbage” and “flag” qubits, and then compiling it into a unitary circuit. The garbage qubits are used to store discarded information, taking advantage of the deferred measurement principle [Nielsen and Chuang 2010, p. 186], and the \( \text{ctrl} \) expression performs a reverse computation to “uncompute” this garbage and produce a reversible circuit. The flag qubits are used as a kind of “assertion,” where a nonzero flag qubit corresponds to an error that can be caught by the \( \text{try-catch} \) expression. Our main result, proven in the supplemental report, is summarized by the following theorem:

**Theorem 6.1.** There is a recursive procedure that compiles any well-typed Qunity program to a qubit-based circuit consisting only of single-qubit gates and controlled operators. The unitary semantics of this low-level circuit will correspond closely to the (potentially norm-decreasing) semantics of the Qunity program, as made precise by Definitions 6.2 and 6.3 in Section 6.3 below.

### 6.1 Overview

Compiling Qunity programs to qubit circuits happens in two stages: from Qunity programs to high-level circuits, and from high-level circuits to low-level qubit circuits. More precisely:
The **input** is a valid typing judgment, as defined in Section 3, for either an expression or program, pure or mixed. We assume that the compiler has access to the proof of validity for the typing judgment, and our compiler (like our semantics) is defined as a recursive function of this proof.

An **intermediate representation** uses quantum circuits of the sort found in most quantum computing textbooks, with two notable exceptions:

1. The wires in the circuit correspond to the Hilbert spaces defined in Definition 4.5, rather than single qubits. In our circuit diagrams, we will label wires with the Hilbert space they represent and a “slash” pointing to the labeled wire. This is useful because it is easier to analyze circuit semantics if we can algebraically manipulate vectors in the direct sum of Hilbert spaces rather than the corresponding qubit encodings of these vectors. Using wires to represent the direct sum of Hilbert spaces is unconventional, but not unheard of in graphical quantum languages [Chardonnet et al. 2022].

2. The boxes in the circuit correspond to norm-non-increasing operators and trace-non-increasing superoperators rather than unitary operators and trace-preserving superoperators (cptp maps). Again, this is not unheard of: Fault-tolerant quantum circuits are often drawn and analyzed with norm-decreasing “r-filter” projectors [Gottesman 2010].

Some of our circuits will be “pure,” involving no measurement and described by norm-non-increasing linear operators. Others will be impure, potentially involving measurement and described by trace-non-increasing superoperators. In both cases, a vertical stacking of boxes represents a tensor product, a horizontal stacking of boxes represents function composition, and a bare wire represents the identity. Whenever we use a pure component described by the operator $E$ within an impure circuit, it has superoperator semantics $\rho \mapsto E\rho E^\dagger$.

The **output** is a low-level qubit-based unitary quantum circuit of the sort standard in quantum computing literature. We provide circuit diagrams, and it should be obvious how to implement these circuits in a runnable quantum assembly language such as OpenQasm [Cross et al. 2021]. Here, wire labels indicate the number of qubits in a particular register, and unlabelled wires can be assumed to represent a single qubit. All of these circuits will be “pure” (unitary and measurement-free), and we assume that the controlled gate $|0\rangle\langle0| \otimes 1 + |1\rangle\langle1| \otimes U$ is implementable whenever the unitary $U$ can be, as this sort of quantum control is a primitive “gate modifier” in OpenQasm.

In what follows, we will refer to our intermediate circuit representation as “high-level circuits” and the target qubit language as “low-level circuits.” High-level circuits semantically described by superoperators will be referred to as “decoherence-based,” while those that are measurement-free will be referred to as “pure.” Section 6.2 outlines the compilation from Qunity programs to high-level circuits, and Section 6.3 outlines the compilation of these high-level circuits to low-level circuits.

### 6.2 Qunity Programs to High-Level Circuits

Our compiler, like our denotational semantics, is a recursive function of a valid typing judgment. Sometimes we do not depend on the compiled subcircuit directly, but rather a transformed version of this circuit; we make it clear when we do this and show how to implement these circuit transformers in the supplemental report. For each kind of judgment, we describe below the correspondence between the Qunity semantics and the high-level circuit semantics and give an example of a compiled circuit.

**Pure expressions.** Given a judgment $(\Gamma \parallel \Delta \vdash e : T)$, the compiler produces a pure circuit with input space $\mathcal{H}(\Gamma) \otimes \mathcal{H}(\Delta)$ and output space $\mathcal{H}(\Gamma) \otimes \mathcal{H}(T)$. The semantics of this circuit $C$ corresponds to Qunity expression semantics in the following way:

$$\langle \sigma, \nu \mid C \mid \sigma, \tau \rangle = \langle \nu \mid [\sigma : \Gamma \parallel \Delta \vdash e : T] \mid \tau \rangle$$
Recall that the typing context $\Gamma \parallel \Delta$ is partitioned in two: left of the $\parallel$ is the "classical" context $\Gamma$ and right of it is the "quantum" context $\Delta$. The semantics makes the distinction clear – $\Gamma$ determines the classical parameter to the denotation, while $\Delta$ determines input Hilbert space. In the compilation setting, the story is slightly different. Because even classical computation may need to be done reversibly on quantum data, our compiler uses $\mathcal{H}(\Gamma)$ as an auxiliary Hilbert space holding the variables used in a classical way. Practically, “used in a classical way” means that the wires in this ancilla register are used only as "control" wires, so any data on this register remains unchanged in the standard basis. For this reason, we depict these gates in circuit diagrams with a control on the classical $\Gamma$ register:

$$
\begin{align*}
\mathcal{H}(\Gamma) & \quad \mathcal{H}(\Gamma) \\
\mathcal{H}(\Delta) & \quad \Gamma \parallel \Delta \vdash e : T \\
\mathcal{H}(T) & 
\end{align*}
$$

Seven inference rules define the pure expression typing relation, and in the supplemental report we give a compiled circuit for each of these cases. As an example, Figure 22 shows the circuit for the T-PurePair typing rule, which includes as subcircuits the compiled subexpressions. It uses a “share” gate graphically represented by the "controlled cloud" component that copies quantum data in the standard basis, mapping $|\tau\rangle \mapsto |\tau, \tau\rangle$. We show how to implement this gate later, in Figure 26.

$$
\begin{align*}
\Gamma \parallel \Delta, \Delta_0 \vdash e_0 : T_0 & \quad \Gamma \parallel \Delta, \Delta_1 \vdash e_1 : T_1 \\
\Gamma \parallel \Delta, \Delta_0, \Delta_1 \vdash (e_0, e_1) : T_0 \otimes T_1 & \\
\mathcal{H}(\Gamma) & \quad \mathcal{H}(\Gamma) \\
\mathcal{H}(\Delta) & \quad \Gamma \parallel \Delta \vdash e_0 : T_0 \\
\mathcal{H}(\Delta_0) & \\
\mathcal{H}(\Delta_1) & \\
\mathcal{H}(\Delta) & \\
\mathcal{H}(T_0) & \\
\mathcal{H}(T_1) & \\
\text{T-PurePair} & 
\end{align*}
$$

Fig. 22. T-PurePair compilation

**Pure programs.** Given a judgment $\vdash f : T \rightsquigarrow T'$, the compiler produces a pure circuit with input space $\mathcal{H}(T)$ and output space $\mathcal{H}(T')$. This circuit’s semantics is the same as $\llbracket \vdash f : T \rightsquigarrow T' \rrbracket$. As an example, Figure 23 shows the circuit for the T-PureAbs typing rule. Note that this circuit does not use the compiled circuit for $(\emptyset \parallel \Delta \vdash e : T)$ directly; rather, we use a transformed version: its adjoint.

As described in Section 6.3, norm-decreasing operators in our high-level circuits are implemented by unitary operators in our low-level circuits by treating some of the wires as “prep wires” (initialized to zero) and some of the wires as “flag wires” (asserted to be zero upon termination). Whenever a norm-decreasing operator can be implemented, its adjoint can also be implemented by taking the adjoint of the underlying unitary and swapping the prep and flag wires, reversing the circuit and swapping the processes of qubit initialization and qubit termination. We describe this circuit transformer (and others) in more detail in the supplemental report.
Mixed expressions. Given a judgment \((\Delta \vdash e : T)\), the compiler produces a decoherence-based circuit with input space \(\mathcal{H}(\Delta)\) and output space \(\mathcal{H}(T)\). This circuit’s semantics is the same as \([\Delta \vdash e : T]\). As an example, Figure 24 shows the circuit for the T-MixedPerm typing rule. The \(\pi\) gate here is a series of swap gates that permutes the data in \(\Delta\) according to the permutation function \(\pi\). This example demonstrates the benefit of the explicit exchange rules—exchange of variables corresponds to swap gates in the quantum circuit.

Mixed programs. Given a judgment \((\vdash f : T \Rightarrow T')\), the compiler produces a decoherence-based circuit with input space \(\mathcal{H}(T)\) and output space \(\mathcal{H}(T')\). This circuit’s semantics is the same as \([\vdash f : T \Rightarrow T']\). As an example, Figure 25 shows the circuit for the T-MixedAbs typing rule. This is the same as the T-PureAbs circuit, except that there is an extra context \(\Delta_0\) for unused variables, which are discarded.

6.3 High-Level Circuits to Low-Level Circuits
The high-level circuits constructed in the previous section are convenient for analysis, in particular for our proofs of correctness. For these circuits to be runnable on quantum hardware, however, an additional compilation stage is necessary, transforming these high-level circuits into low-level ones. In particular, our high-level circuits manipulate values in \(\forall(T)\) in Hilbert space \(\mathcal{H}(T)\) with
dim(ℋ(T)) = |𝑉(T)|, but low-level circuits use only qubits, working in the Hilbert space $\mathbb{C}^{2^n}$ where $n$ is the number of qubits on the quantum computer. Our compiler must then translate Qunity programs written for the Hilbert space $ℋ(T)$ into OpenQASM programs that use some $|𝑉(T)|$-dimensional subspace of $\mathbb{C}^{2^n}$, and values in $𝑉(T)$ will be encoded into bitstrings in $\{0,1\}^n$. This number of qubits $n$ is determined from $T$ by a function $\text{size}(\cdot)$, shown below. Values in $𝑉(T)$ will end up encoded into bitstrings whose length is $\text{size}(T)$ using the function $\text{encode}(\cdot)$, where “++” denotes bitstring concatenation.

$$
\begin{align*}
\text{size}(\text{Void}) &= 0 & \text{encode}(()) &= "" \\
\text{size}(() &= 0 & \text{encode}(\text{left}_{T_0 \oplus T_1} v) &= "0" ++ \text{encode}(v) \\
\text{size}(T_0 \oplus T_1) &= 1 + \max\{\text{size}(T_0), \text{size}(T_1)\} & \text{encode}(\text{right}_{T_0 \oplus T_1} v) &= "1" ++ \text{encode}(v) \\
\text{size}(T_0 \otimes T_1) &= \text{size}(T_0) + \text{size}(T_1) & \text{encode}((v_0, v_1)) &= \text{encode}(v_0) ++ \text{encode}(v_1)
\end{align*}
$$

We can now be precise about what it means to implement a norm-non-increasing operator with a low-level circuit.

**Definition 6.2.** We say that it is possible to implement a norm-non-increasing operator $E : ℋ(T) \rightarrow ℋ(T')$ if there is a low-level circuit implementing a unitary operator $U : \mathbb{C}^{2^{\text{size}(T) + n_{\text{pre}}} \otimes n_{\text{flag}}} \rightarrow \mathbb{C}^{2^{\text{size}(T') + n_{\text{flag}}}}$ for some integers $n_{\text{pre}}$ and $n_{\text{flag}}$ such that for all $v \in 𝑉(T), v' \in 𝑉(T')$:

$$
\langle \text{encode}(v'), \theta^{n_{\text{flag}}} | U | \text{encode}(v), \theta^{n_{\text{pre}}} \rangle = \langle v' | E | v \rangle
$$

Here, $n_{\text{pre}}$ is the number of “prep” qubits initialized to $|\theta\rangle$, and $n_{\text{flag}}$ is the number of “flag” qubits asserted to be in the $|\theta\rangle$ state upon termination. This definition requires that $\text{size}(T) + n_{\text{pre}} = \text{size}(T') + n_{\text{flag}}$.

For example, consider the “share” gate graphically represented by the “controlled cloud” in Figure 22. This gate can be defined for any type $T$, where it is mathematically represented by the isometry $\sum_{v \in 𝑉(T)} |v, v⟩⟨v|$, copying a value in the standard basis. By Definition 6.2, one can implement this operator with $n_{\text{pre}} = \text{size}(T)$ prep qubits and $n_{\text{flag}} = 0$ flag qubits, using the series of CNOT gates shown in Figure 26.
Definition 6.2 can be adapted to the setting of decoherence and superoperators by including an additional “garbage” register. This is essentially the Stinespring dilation [Heinosaari and Ziman 2011, p. 186], using the principle of deferred measurement to purify our computation and organize all of our measurements onto one segment of our output.

**Definition 6.3.** We say that it is possible to implement a trace-non-increasing superoperator $\mathcal{E} : \mathcal{L}(\mathcal{H}(T)) \to \mathcal{L}(\mathcal{H}(T'))$ if there is a qubit circuit implementing a unitary operator $U : \mathbb{C}^{\text{size}(T)+n_{\text{PREP}}} \to \mathbb{C}^{\text{size}(T')+n_{\text{FLAG}}+n_{\text{GARB}}}$ for some integers $n_{\text{PREP}}, n_{\text{FLAG}},$ and $n_{\text{GARB}}$ such that for all $\rho \in \mathcal{L}(\mathcal{H}(T)), v'_1, v'_2 \in \forall(T')$:

$$\langle v'_1 | \mathcal{E}(\rho) | v'_2 \rangle = \sum_{b \in \{0,1\}}^{n_{\text{GARB}}} \langle \text{encode}(v'_1), \theta^{\otimes n_{\text{FLAG}}}, b | U (\rho \otimes |0\rangle\langle0|^{\otimes n_{\text{PREP}}}) U^\dagger | \text{encode}(v'_2), \theta^{\otimes n_{\text{FLAG}}}, b \rangle$$

Like before, $n_{\text{PREP}}$ is the number of “prep” qubits initialized to $|0\rangle$, and $n_{\text{FLAG}}$ is the number of “flag” qubits asserted to be in the $|0\rangle$ state upon termination. The new parameter $n_{\text{GARB}}$ is the number of “garbage” qubits discarded after use. This definition requires that $\text{size}(T) + n_{\text{PREP}} = \text{size}(T') + n_{\text{FLAG}} + n_{\text{GARB}}$.

For example, under this definition the “discard” used for the $\mathcal{H}(\Delta_0)$ space in Figure 25 can be implemented by an empty (identity) circuit by setting $n_{\text{GARB}} = \text{size}(T)$. Any implementable pure circuit is also implementable as a decoherence-based circuit by setting $n_{\text{GARB}} = 0$.

### 6.4 Example

As a concrete example of compiling a simple program, consider a “coin flip” expression defined below. This program is defined in terms of a $\text{meas}$ program that measures its argument in the standard basis, by copying in the standard basis and then discarding the copy.

$$\text{meas} : \lambda x \mapsto (x, x) \triangleright \text{fst}_{\text{Bit} \otimes \text{Bit}}$$

$$\text{coin} = \text{meas}_{\text{Bit}}(\text{had } \emptyset)$$

$$= (\cdot \triangleright \text{left}_{\text{Bit}} \triangleright \text{had} \triangleright \lambda x \mapsto (x, x) \triangleright \lambda (x_0, x_1) \mapsto x_0$$

The typing judgment $(\emptyset \vdash \text{coin} : \text{Bit})$ is compiled into the following high-level circuit. Here you can see that the $(x, x)$ expression implements the isometry $(|0\rangle, 0\langle0| + |1\rangle, 1\langle1|)$, and the $\text{fst}$ program implements a partial trace.
This is then compiled into the following low-level circuit, with \( n_{\text{prep}} = 2 \), \( n_{\text{flag}} = 0 \), \( n_{\text{garr}} = 1 \), size(∅) = 0 contextual input wires, and size(Bit) = 1 data output wire.

\[
\begin{align*}
\text{had} & \quad \text{size(Bit) = 1} \\
\text{h} & \quad \text{size(Bit) = 1}
\end{align*}
\]

7 CONCLUSION

Qunity is designed to unify classical and quantum computing through an expressive generalization of classical programming constructs. Its syntax allows programmers to write quantum algorithms using familiar classical programming constructs, like exception handling and pattern matching. Our type system leverages algebraic data types and relevant (substructural) types, differentiating between unitary maps and quantum channels but allowing them to be usefully nested. Qunity’s semantics brings constructions commonly used in algorithm analysis—such as bounded-error quantum subroutines, projectors, and direct sums—into the realm of algorithm implementation.

We have formally defined the Qunity programming language, proven that its semantics is well-defined, and shown how it can be used to implement some complicated quantum algorithms. We have demonstrated a strategy for compiling Qunity programs to low-level qubit-based unitary circuits and proven that our procedure preserves the semantics, demonstrating that this language does indeed have a physical interpretation and could be run on quantum hardware. While classical computers can be modeled by logic gates acting on classical bits, it is far more convenient to use higher-level programming constructs for most tasks. Qunity’s design similarly abstracts away low-level qubit-based gates using techniques from quantum algorithms that are overlooked in existing languages. We hope that Qunity’s features can ease the implementation and analysis of complicated quantum algorithms written at a high level of abstraction.

DATA AVAILABILITY STATEMENT

An extended version of this paper supplements this work with additional background, proofs, and diagrams [Voichick et al. 2022a]. We have implemented and verified a type checker for Qunity programs within the Coq proof assistant [Voichick et al. 2022b].

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