Presence and Absence of Delocalization-localization Transition in Coherently Perturbed Disordered Lattices

Hiroaki S. Yamada
Yamada Physics Research Laboratory, Aoyama 5-7-14-205, Niigata 950-2002, Japan

Kensuke S. Ikeda
College of Science and Engineering, Ritsumeikan University Noji-higashi 1-1-1, Kusatsu 525-8577, Japan
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A new type of delocalization induced by coherent harmonic perturbations in one-dimensional Anderson-localized disordered systems is investigated. With only a few $M$ frequencies a normal diffusion is realized, but the transition to localized state always occurs as the perturbation strength is weakened below a critical value. The nature of the transition qualitatively follows the Anderson transition (AT) if the number of degrees of freedom $M + 1$ is regarded as the spatial dimension $d$, but the critical dimension is not $d = M + 1 - 2$ of the ordinary AT but $d = 3$.

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Introduction- Since the proposal of Anderson, the localization of electron in disordered lattices has been one of the most fundamental problems associated with the essence of electron conduction process \[^{11}^{13}\]. No matter how high the spatial dimension may be, the Anderson localized state exist prior to the delocalized conducting state, and a transition from localized state to the delocalized state, the so called Anderson transition (AT), occurs as the relative strength of disorder decreases \[^{4}^{8}\]. Theoretical predictions have been obtained by using several theoretical tools such as the one-parameter scaling hypothesis, the self-consistent theory, and so on \[^{9}^{10}\].

On the other hand, in the study of chaotic systems the ergodic transition of quantum maps is equivalent to the AT of disordered lattice \[^{11}^{13}\]. Upon this equivalence, the dynamical AT has been first experimentally observed for the quantum map systems implemented on the optical lattice \[^{14}^{13}\]. In this case the number of dynamical degrees of freedom corresponds to the number of spatial dimension of the disordered lattices, and so the features of AT in high-dimensional lattices can be explored by the quantum maps.

The dynamical interaction among the degrees of freedom thus enables the delocalization transition. Then the following question immediately follows: can the Anderson localization in the disordered lattices be destroyed as it is perturbed by dynamical degrees of freedom such as phonon modes? The perturbation by infinitely many phonon modes can be modeled by a stochastic perturbation, and it is well-known that the stochastic perturbation destroys the localization and induces a normal diffusion \[^{16}^{19}\]. However, the effect of dynamical perturbation composed of finite number of coherent modes has still been unanswered. In the previous papers, we investigated the effect of finite-number harmonic perturbations on one-dimensional disordered lattice (ODDL), and showed that the ODDL exhibits a normal diffusion at least on a finite time scale \[^{20}^{22}\]. On the other hand, numerical and mathematical studies claim that the localization is persistent for finite-number harmonic perturbations Refs. \[^{23}^{24}\], and which of localization and delocalization dominates has still been open to question. It is quite interesting whether or not a coherent dynamical perturbation composed of finite number of harmonic modes can dynamically destroy the localization. In this letter, we present novel results replying the question.

Model- We consider tightly binding ODDL perturbed by coherent periodic perturbations with different incommensurate frequencies. It is given by

$$i\hbar \frac{\partial \Psi_n(t)}{\partial t} = \Psi_{n-1}(t) + \Psi_{n+1}(t) + V_L(n,t)\Psi_n(t), \quad (1)$$

where $V_L(n,t) = V(n)[1 + f(t)]$. The coherently time-dependent part $f(t)$ is given as,

$$f(t) = \frac{\epsilon}{\sqrt{M}} \sum_{i=1}^{M} \cos(\omega_i t), \quad (2)$$

where $M$ and $\epsilon$ are the number of frequency components and the strength of the perturbation, respectively. Note that the long-time average of the total power of the perturbation is normalized to $\langle f(t)^2 \rangle = \epsilon^2/2$. The frequencies $\{\omega_i\}(i = 1,\ldots,M)$ are taken as mutually incommensurate numbers of order $O(1)$. The static on-site disorder potential takes random value $V(n)$ uniformly distributed over the range $[-W,W]$, where $W$ denotes the disorder strength.

It is important to note that the harmonic source can be interpreted as the quantum linear oscillator of the Hamiltonian $\sum_i^M \omega J_i$ interacting with the irregular lattice with the quantum amplitude $\sum_i^M \cos \phi_i$ instead of the classical force $f(t)$, where $(J_i, \phi_i) = (-i\partial/\partial \phi_i, \phi_i)$ are the action-angle operators of the $i$-th oscillator. Each quantum oscillator has the action eigenstates $|n_i> \propto \phi_i$ with the action $I_i = n_i\hbar$ ($n_i$ :integer) and the energy $\hbar \omega_i$. Thus the system \[^{13}\] is regarded as a quantum autonomous system of $(M + 1)$-degrees of freedom spanned by the quantum states $|n > \prod_{i=1}^{M} |n_i >$ \[^{23}\].
We take a lattice-site eigenstate as the initial state $|\Psi(t = 0)\rangle$ and numerically observe the spread of the wavepacket measured by the mean square displacement (MSD), $m_2(t) = \sum (n - n_0)^2 \langle |\Psi(n, t)\rangle^2 \rangle$.

First, we consider the limit $M \to \infty$. In this case $f(t)$ can be identified with the delta-correlated stochastic force $< f(t) f(t') >= \Gamma \delta(t - t')$, where $\Gamma \propto \epsilon^2$ is a noise strength. The localization is surely destroyed and the normal diffusion $m_2(t) = D t$ with the diffusion constant $D$ is recovered for $t \to \infty$ \cite{21, 22}, as was first pointed out by Haken and his coworkers \cite{16, 17}. They predicted analytically for the white Gaussian noise

$$D = \lim_{t \to \infty} \frac{m_2(t)}{t} \propto \frac{\Gamma}{t^2 + W^2/3},$$

(3)

for weak enough $\epsilon$. If $W \gg \Gamma$, $D \propto W^{-2}$ but recently it was shown that $D \propto W^{-4}$ for strong disorder region $W \gg 1$ \cite{18}. The noise-induced diffusion has been extended for a random lattice driven by the colored noise, including the fluctuation of the off-diagonal terms \cite{17–19}.

However, for finite $M$, $f(t)$ can no longer be replaced by the random noise, and it plays as a coherent dynamical perturbation, and the system is a quantum dynamical system with $(M + 1)$-degrees of freedom. The main purpose of this study is to investigate how does the nature of the quantum dynamics of the irregular lattice change as the number $M$ decreases from $\infty$ to 0.

*Delocalized states and normal diffusion-* We show typical examples of time evolution of MSD for $M = 7$ and $M = 3$ in Fig.1(a) and (b), respectively. If $\epsilon$ is large enough, it is evident that $MSD$ follows asymptotically the normal diffusion $m_3 = Dt$, which means that only a finite number of coherent periodic modes plays the same role as the stochastic perturbation in the disordered lattice. The $W$– and $\epsilon$–dependence of the diffusion constant $D$ depicted in Fig.1 follow the main feature of the stochastically induced diffusion constants: as shown in the Fig.1(c) the $W$–dependence changes from $D \propto W^{-2}$ for weak $W$ in Eq.3 to $D \propto W^{-4}$ $W \gg 1$, following the theoretical prediction of stochastic perturbations \cite{18}. Moreover, as depicted in the Fig.1(d), even for $M = 3$ the $\epsilon$–dependence reproduces the characteristic behavior of the stochastically induced $D$, which first increases but finally decreases with $\epsilon$ after reaching to a maximum value. It is a remarkable feature of ODDL that a normal diffusion, which mimics the one induced by a stochastic force composed of infinite number of frequencies, is self-generated by a coherent perturbation composed of only three incommensurate frequencies.

On the other hand, the coherently perturbed ODDL always undergoes a definite phase transition from the diffusing state to a localized state as $\epsilon$ decreases crossing over a critical value $\epsilon_c$. The transition is quite similar to the AT of high-dimensional disordered lattice. As shown in Fig.2(a), at $\epsilon = \epsilon_c$, the MSD exhibits a subdiffusion $m_2(t) \sim t^\alpha$ with a critical diffusion index $\alpha$ ($0 < \alpha < 1$). Close to $\epsilon_c$, typical critical transient phenomena are observed. To show them we define the function $\Lambda(t)$ as the scaled MSD

$$\Lambda(t) = \frac{m_2(t)}{t^\alpha},$$

(4)

divided by the critical MSD. In Fig.2(b) the $\Lambda(t)$ at various $\epsilon$ close to $\epsilon_c$ are displayed for $M = 7$, which form a characteristic fan pattern spreading outward.

As are demonstrated in Fig.3(a), the index of the critical subdiffusion decreases with $M$, following the result of one-parameter scaling hypothesis

$$\alpha = \frac{2}{d} = \frac{2}{M + 1},$$

(5)

for the $d$-dimensional disordered lattice, if we regard $d$ as the total number of degrees of freedom of our system, i.e., $d = M + 1$, which seems to be quite reasonable.

The localization in the side of $\epsilon < \epsilon_c$ is characterized by the localization length $\xi_M$, which diverges at $\epsilon_c$ as
\[ \xi_M(\epsilon) \sim (\epsilon - \epsilon_c)^{-\nu} \] with the critical exponent \( \nu > 0 \).

A remarkable feature of the critical state is that, the fan pattern of \( \Lambda(t) \) are represented by two unified curves depending whether \( \epsilon > \epsilon_c \) or \( \epsilon < \epsilon_c \) by using the scaling variable \( x = \xi_M(\epsilon)t^{\alpha/\nu} \), as demonstrated by Fig. 2(b) for \( M = 3 \). The \( d = M + 1 \) dependence of the critical index \( \nu \) is shown in Fig. 2(d).

Such a remarkable critical sub-diffusion exists at \( \epsilon_c \) for an arbitrary \( M \), but the critical value \( \epsilon_c \) decreases with \( M \) following the diffusion index \( \alpha \):

\[ \epsilon_c \propto \frac{1}{(M - 1)^\delta}, \quad \delta \approx 1, \] (6)

which does not depend upon \( W \) as shown in Fig. 3(c). Thus the ODDL is always localized if \( \epsilon \) is small enough, but no matter how small \( \epsilon \) may be, a normal diffusion mimicking a stochastically induced diffusion is realized if \( M \) is taken large enough.

It is quite interesting that the dependencies of both \( \alpha \) and \( \epsilon_c \) upon \( M \) are the same as those of the AT observed for the quantum maps simulating the high-dimensional disordered lattice \[ [23][27] \]. If we are allowed to extrapolate the above results for the smaller \( M \), \( \epsilon_c \) diverges at \( M = 1 \), at which the critical diffusion index becomes \( \alpha = 1 \). This fact implies that for \( M = 1 \) the critical subdiffusion is realized at \( \epsilon = \epsilon_c = \infty \) as a normal diffusion; namely, that \( M = 1 \) is the critical dimension of the delocalization-localization transition (DLT), which has been established for the quantum maps and high-dimensional disordered lattices. However, our numerical results reject the above conjecture.

**Number of critical modes (\( M = 2 \)).** If the above conjecture is correct, \( M = 2 \) \((d = 3)\) should exhibit the critical phenomenon. In Fig. 3(a) the log-log plot of MSD curves for \( M = 2 \) are shown for various values of \( \epsilon \). Surely, some curve follows the expected critical subdiffusion of the exponent \( \alpha = 2/3 \), which is predicted by Eq. (5), in the initial stage, but it drops from the expected straight line as the time elapses.

To overview the whole feature of the MSD curves, it is instructive to show the time evolution of the diffusion exponent defined as the instantaneous slope of the log-log plot of MSD

\[ \alpha(t) = \frac{d \log m_2(t)}{d \log t}. \] (7)

If DLT happens at a finite \( \epsilon = \epsilon_c \), then the \( \alpha(t) \) should keep a constant value \( \alpha(\epsilon_c) < 1 \). Above \( \epsilon_c \), as time passes, \( \alpha(t) \) increases up to the exponent 1 indicating the normal diffusion, while it decreases to zero indicating the localization below \( \epsilon_c \). Indeed, the expected feature is evident for the \( \alpha(t) \) plot of \( M = 3 \) shown in Fig. 3(b) The same feature is observed also for \( M \geq 4 \).
However, as shown in Fig. 4(c) the $\alpha(t)$-plots of $M = 2$ shows a quite different feature. No curves follow the critical behavior $\alpha(t) = \text{const} < 1$, and all the curves tend to decrease from the initial values, which approaches to $1$ as $\epsilon$ increases. As $\alpha(t)$ comes close to $1$, the time scale beyond which $\alpha(t)$ begins to decrease becomes longer. Certainly it seems as if the normal diffusion $\alpha(t) = 1$, which would be realized in the limit $\epsilon \to \infty$, were the critical diffusion. These facts indicates that the DLT does not exists for $M = 2$ in contradiction with the prediction of the Eqs. (3) and (6), and that $M = 2(d = 3)$ is the critical dimension.

![FIG. 4: (Color online) (a)The double-logarithmic plots of $m_2(t)$ as a function of time for some values of the perturbation strength $\epsilon$ increasing from bottom to top in the trichromatically perturbed 1DDS of $M = 2$. The panels (b) and (c) are the diffusion exponent $\alpha(t)$ as a function of time in the cases $M = 3$ and $M = 2$, respectively. (e)Localization length as a function of $\epsilon$ for $M = 1, 2, 3$ with $W = 2$. Some LL of $M = 2$ are obtained by scaling relation $m_2(t) \sim \xi_M(\epsilon)F(t/\xi_M(\epsilon)^2)$ for $\epsilon \geq 0.5$. Note that the horizontal axis is in logarithmic scale. The dashed lines show $\xi_M \sim e^{5.5\epsilon}$ and $\xi_M \sim e^{13.6\epsilon}$, respectively. The lines $\epsilon = 0.18$ and $\epsilon = 0.6$ are shown as a reference.](image)

Comparison by localization length- Localization, of course, occurs with $M = 1$. Then what is the difference of the localization between the case of $M = 1$ ($d = 2$) and the case of $M = 2$ ($d = 3$). In both cases of $M = d - 1 = 1$ and $M = d - 1 = 2$, the localization length grows exponentially as long as $\epsilon$ is small enough, which coincides with the case of the $d = 2$ ordinary irregular lattice. However with a further increase of $\epsilon$, $\xi_M$ begins to decrease steeply for $M = 1$. Such a behavior is a direct reflection that the inter-site transfer is suppressed by the random potential enhanced with the increasing coupling strength $\epsilon$. Let us remember that as shown in Fig. 1(d) even the recovered diffusion constant of the the system of $d = M + 1 \gg 1$ in general decreases steeply with $\epsilon(\epsilon_c)$. This is the reason why, unlike the ordinary $d$-dimensional irregular lattice, $d = 2$ can not be the critical dimension of our system. The growth of $\xi_M$ with $\epsilon$ takes place only by increasing the dimension from $d = 2$ to $3$.

Indeed, for $d = M + 1 = 3$ the localization still remains, but $\xi$ increases with $\epsilon$ exponentially. The exponential growth rate is further enhanced and a super-exponential growth occurs as $\epsilon$ increases beyond $O(1)$, as is depicted in Fig. 4(d). And it is for $M = 3$ that the divergence of $\xi_M$ is first observed at a finite $\epsilon_c$.

**Summary and discussion**- In the present paper, we investigated the delocalized and the localized motion in 1D irregular lattice coherently perturbed by the harmonic modes. In order to induce a delocalized motion the stochastic perturbation composed of infinite number of harmonic mode is not necessary: the diffusive motion is always induced only by a few number of harmonic modes if the perturbation strength is strong enough. On the other hand, the transition to the localized phase occurs how many the mode number $M$ may be, and the critical perturbation strength as well as the critical diffusion exponent follows those of the Anderson transition in the high-dimensional lattice and many dimensional quantum maps. However, the critical number of the degrees of freedom is not $d = M + 1 = 2$ but $d = 3$ in our system. Thus our system provides with an example demonstrating that the critical dimension of the DLT may be larger than $d = 2$ and depend upon the nature of recovered diffusion, as summarized in the TABLE I.

The threshold number of degrees of freedom, for which the delocalized motion, and more generally the ergodic motion, takes place in quantum systems, is a quite interesting theoretical problem. We hope such approaches may contribute to elucidate the origin of quantum statistical behavior from the side of the study of small quantum systems.

**TABLE I: Dimensionality of the DLT.** For $4 \leq M < \infty$ the result is same as the case of $M = 3$. The lower lines is result of the $d$-dimensional disordered systems. Loc: exponential localization, Diff:Normal diffusion.

| $d = M + 1$ | 1 | 2 | 3 | 4 | 5 | ... | $\infty$ |
|------------|---|---|---|---|---|------|---------|
| this study  | Loc | Loc | Loc | DLT | DLT | ... | DLT     |
| AM[27]     | Loc | Loc | DLT | DLT | DLT | ... | Diff    |
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