DOUBLE COSETS \( NgN \) OF NORMALIZERS OF MAXIMAL TORI OF SIMPLE ALGEBRAIC GROUPS AND ORBITS OF PARTIAL ACTIONS OF CREMONA SUBGROUPS

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Abstract. Let \( G \) be a simple algebraic group over an algebraically closed field \( K \) and let \( N = N_G(T) \) be the normalizer of a fixed maximal torus \( T \leq G \). Further, let \( U \) be the unipotent radical of a fixed Borel subgroup \( B \) that contains \( T \) and let \( U^- \) be the unipotent radical of the opposite Borel subgroup \( B^- \). The Bruhat decomposition implies the decomposition \( G = NU^-UN \). The Zariski closed subset \( U^-U \subset G \) is isomorphic to the affine space \( A_m K \) where \( m = \dim G - \dim T \) is the number of roots in the corresponding root system. Here we construct a subgroup \( N \leq C_m(K) \) that “acts partially” on \( A_m K \approx U \) and we show that there is one-to-one correspondence between the orbits of such a partial action and the set of double cosets \( \{NgN\} \). Here we also calculate the set \( \{g_\alpha\}_\alpha \subset U \) in the simplest case \( G = SL_2(C) \).

Introduction

Double cosets of transformation groups. Let \( G \) be a group which acts on a set \( X \). Further, let \( x_1, x_2 \in X \), \( H_1 = \text{St}_{x_1}, H_2 = \text{St}_{x_2} \) be stabilizers of \( x_1, x_2 \) and let \( O_{x_1}, O_{x_2} \) be orbits of \( x_1, x_2 \). Then there is one-to-one correspondence between the set of \( G \)-orbits of the natural action on \( O_{x_2} \times O_{x_1} \) and the set of double cosets \( \{H_1g_\alpha H_2\}_\alpha \) (it is a simple and well-known fact). Namely,

\[ \{(g_\alpha(x_2), x_1)\}_\alpha \] is the minimal set of representations of \( G \)-orbits of \( O_{x_2} \times O_{x_1} \). (*)

The pairs of maximal tori of simple algebraic groups. In this paper we consider the case when \( G \) is a simple algebraic group over an algebraically closed field \( K \). The decomposition of a group \( G \) into the union of double cosets \( G = \bigcup g_iH_2g_iH_1 \) is a very important construction in the theory of algebraic groups, especially in the case when \( H_1, H_2 \) are parabolic subgroups. For these cases the decomposition is finite. Here we consider the case when \( H_1 = H_2 = N = N_G(T) \) is the normalizer of a fixed maximal torus \( T \). Now let \( X \) be the set of all maximal tori of \( G \). The group \( G \) acts on \( X \) by conjugation. Then \( X \) is just one \( G \)-orbit of \( T \in X \) and \( N := N_G(T) = \text{St}_T \). Thus, we have one-to-one correspondence between the set of \( G \)-orbits of the set \( X \times X \) and the set of double cosets \( \{Ng_{\alpha}N\}_\alpha \). Further, we have the decomposition

\[ G = NU^-UN, \quad \text{where} \quad U = R_a(B), \quad U^- = R_a(B^-) \]
(here $B$ is a fixed Borel subgroup that contains $T$, $B^- = w_0Bw_0$ is the opposite Borel subgroup, $R_u(B)$ is the unipotent radical of $B$). Note, 

$$U := U^- U \approx A^m_K$$

where $m$ is the number of roots in the corresponding root system and $A^m_K$ is the $m$-dimensional affine space over $K$. Thus we have one-to-one correspondence between the set of $G$-orbits of $X \times X$ and the set of double cosets $\{NuN\}_{u \in U}$.

The group $N \leq \text{Cr}_m(K)$. To emphasize the minimal set of representatives $\{u_\alpha \in U\}$ of double cosets $\{NuN\}_{u \in U}$ we use the following construction. Using the multiplication of $G$ by the elements of the group $N$ on left-right sides we construct a subgroup $N \leq \text{Cr}_m(K)$ of the Cremona group $\text{Cr}_m(K)$ which acts partially (see, section 1) on $U \approx A^m_K$ and we get the following

**Theorem 1.** Elements $u_1, u_2 \in U$ belong to the same double coset $NuN$ if and only if they are in the same $N$-orbit.

Hence we have

**Corollary 1.** There is one-to-one correspondence between the set of $G$-orbits of the pairs of maximal tori of $G$ and the $N$-orbits of the subgroup $N \leq \text{Cr}_m(N)$ with respect to the partial action on $A^m_K$.

The definition of the group $N$ depends on the group $W \times W$ but it is not unique. In particular, it depends on the choices of the preimages $\dot{w}$ of elements of Weyl group $W = N/T$. 

**The problem of pair matrices.** There is a reduction of the well-known “wild” problem of the classification of the pairs of matrices $(A, B)$ up to conjugation by a single element of $\text{GL}_n(C)$, where $A, B \in M_n(C)$. Namely, we may change the description of all $\text{GL}_n(K)$-orbits of the action $(A, B) \to (gAg^{-1}, gBg^{-1})$ to the description of fibers

$$\pi : M_n(C) \times M_n(C) \to (M_n(C) \times M_n(C))/G$$

where $\pi$ is the quotient map and $(M_n(C) \times M_n(C))/G$ is the corresponding algebraic factor (see, [VP], 9.5).

We also may formulate the following “subproblem”: to classify $\text{GL}_n(C)$-orbits on the set $C_A \times C_B$ where $C_A, C_B \subset M_n(C)$ are the $\text{GL}_n(K)$-orbits of $A, B$ for given $A, B$. We also may give such a classification “up to an isomorphism of centralizers”. Say, if we take $A, B$ to be regular semisimple unimodular matrices the classification of such pairs “up to an isomorphism of centralizers” is exactly the classification of the pairs of maximal tori of $\text{SL}_n(C)$.

**Case $G = \text{SL}_2(C)$.** Here we calculate the system of representatives of maximal tori for the simplest case $G = \text{SL}_2(C)$ (see, Corollary 4.7). Namely, let

$$g_\alpha := \begin{pmatrix} 1 & \alpha \\ 1 & 1 + \alpha \end{pmatrix} \in U \subset \text{SL}_2(C)$$

and let
\[
\mathcal{K} := \{ \alpha = a + bi \in \mathbb{C} \mid a \geq -\frac{1}{2} \} \setminus \{ \alpha = -\frac{1}{2} + bi \in \mathbb{C} \mid b < 0 \}.
\]

**Theorem 2.** The set of pairs
\[
(T, T) \cup (g_\alpha T g^{-1}_\alpha, T)_{\alpha \in \mathcal{K}}
\]
is a minimal set of representatives of the orbits of the pairs of tori of \( G \times G \) under conjugation by elements of \( G \).

Note, that \( \mathcal{K} \) is a subset of \( \mathbb{C} \) which is homeomorphic to \( \mathbb{C} \) in the standard topology (indeed, the map \( \varepsilon : \mathcal{K} \to \mathbb{C} \) which is given by the formula \( \varepsilon(z) = (z + \frac{1}{2})^2 \) defines the corresponding homeomorphism).

At the end of the paper we also consider the description of \( \text{SL}_2(\mathbb{C}) \)-orbits of pairs \((s, t)\) of non-central semisimple elements of \( \text{SL}_2(K) \).

**Notation and terminology.**
Here
- \( K \) is an algebraically closed field;
- \( \mathbb{C} \) is a complex number field;
- \( G \) is a simple algebraic group over \( K \) (here we identify \( G \) with the group of \( K \) points \( G(K) \)) that corresponds to the root system \( R \) of the rank \( r \), \( R = R^+ \cup R^- \) is the fixed decomposition into the union of positive and negative roots;
- we consider only Zariski topology on \( G \);
- \( T \) is a fixed maximal torus of \( G \), \( B \leq G \) is a fixed Borel subgroup that contains \( T \), \( B^- = w_0 B w_0 \) is the opposite Borel subgroup (here \( w_0 \) is the longest element of the Weyl group);
- \( U = R_u(B) \), \( U^- = R_u(B^-) \) are the unipotent radicals of \( B, B^- \);
- \( N = N_G(T) \) is the normalizer of \( T \);
- \( N/T = W \) is the Weyl group of \( G \);
- for \( w \in W \) by \( \dot{w} \in N \) we denote any preimage of \( w \).

1. Partial action of the Cremona subgroups

The partial action of the group is used in different cases (see, for instance, [A], [E]). Here we give the definition of the partial action in a little bit different form.

**Definition 1.1.** Let \( \Gamma \) be a group and let \( X \) be a set. We say that a partial action of \( \Gamma \) on \( X \) is defined if for every \( x \in X \) a subset \( \Gamma(x) \subset \Gamma \) is fixed and the following conditions hold:
- i. for every \( \sigma \in \Gamma(x) \) an element \( \sigma(x) \in X \) is defined;
- ii. the identity \( e \in \Gamma \) belongs to every \( \Gamma(x) \) and \( \epsilon(x) = x \);
- iii. if \( \tau \in \Gamma(\sigma(x)) \) then \( \tau \sigma \in \Gamma(x) \) and \( \tau(\sigma(x)) = \tau \sigma(x) \);
- iv. \( \sigma^{-1} \in \Gamma(\sigma(x)) \).

**Orbits of partial action.** For partial actions we may define orbits of such actions. Namely, we say that \( x, y \in X \) belong to the same \( \Gamma \)-orbit if and only if one can find an element
Let $\sigma \in \Gamma(x)$ such that $\sigma(x) = y$. Obviously, the conditions $i.$ – $iv.$ of the Definition can guarantee that the $\Gamma$-orbits are classes of equivalence under the equivalence
\[ x \sim_{\Gamma} y \iff \sigma(x) = y \text{ for some } \sigma \in \Gamma(x). \]

The Cremona group action on the affine space. Here we consider the Cremona group $\text{Cr}_n(K)$ as the group of automorphisms of the field $K(x_1, \ldots, x_n)$ (see, for instance, [S]). Let
\[ A^n_K = \{ a = (\alpha_1, \ldots, \alpha_n) \mid \alpha_i \in K \} \]
be the $n$-dimensional affine space. Every element $\sigma \in \text{Cr}_n(K)$ is presented by the sequence of rational functions
\[ \sigma = \left( \frac{\varphi_1}{\psi_1}, \ldots, \frac{\varphi_n}{\psi_n} \right), \]
where $\varphi_i, \psi_i \in K[x_1, \ldots, x_n]$, $\psi_i \neq 0$, $(\varphi_i, \psi_i) = 1$, and we may define
\[ \sigma(a) := \left( \frac{\varphi_1(a)}{\psi_1(a)}, \ldots, \frac{\varphi_n(a)}{\psi_n(a)} \right) \]
for every point $a \in A^n_K$ such that
\[ a \in U_\sigma := \{ a' \in A^n_K \mid \psi_i(a') \neq 0 \text{ for every } i \}. \]
The set $U_\sigma$ is an open subset of $A^n_K$ where the rational map $\sigma$ is regular.

Let
\[ \tau = \left( \frac{\mu_1}{\nu_1}, \ldots, \frac{\mu_n}{\nu_n} \right) \in \text{Cr}_n(K). \]
If $\sigma(a) \in U_\tau := \{ a' \in A^n_K \mid \nu_i(a') \neq 0 \text{ for every } i \}$ then the image $\tau(\sigma(a))$ is defined and equal to $(\tau\sigma)(a)$ where
\[ \tau\sigma = \left( \frac{\mu_1(\varphi_1, \varphi_2, \ldots, \varphi_n)}{\nu_1(\psi_1, \psi_2, \ldots, \psi_n)}, \ldots, \frac{\mu_n(\varphi_1, \varphi_2, \ldots, \varphi_n)}{\nu_n(\psi_1, \psi_2, \ldots, \psi_n)} \right). \]

Now let $\Gamma \leq \text{Cr}_n(K)$. For every $x \in A^n_K$ we put $\Gamma(x) := \{ \sigma \in \Gamma \mid x \in U_\sigma \}$. Obviously, conditions $i.$–iii. hold for $\Gamma$. However, the condition iv. does not necessarily hold. For instance, let $n = 2$ and $\Gamma = \langle \sigma = (x_1, \frac{x_1}{x_2}) \rangle$. Then $\sigma^{-1} = x$ and $\sigma^{-1} \notin \Gamma(\sigma((1, 1)))$.

Below we will consider some subgroups of Cremona group $\text{Cr}_n(K)$ which act partially on the affine space $A^n_K$.

2. The decomposition $G = NU^-UN$

We have the decomposition
\[ G = NU^-UN. \tag{2.1} \]
Indeed, for every Bruhat cell $BwB$ we have $BwB = U_w\dot{w}TU$ where $U_w$ is the product (in any fixed order) of the root subgroups $X_\alpha$ such that $\alpha \in R^+$ and $w^{-1}(\alpha) \in R^-$. Hence $\dot{w}^{-1}U_w\dot{w} \leq U^-$ and we have
\[ BwB \leq \dot{w}\dot{w}^{-1}U_w\dot{w}TU = w(U \dot{w}^{-1}U_w\dot{w})UT \leq \dot{w}U^-UT \leq \dot{w}TU^-UN \leq NU^-UN. \]

Put
\[ \mathcal{U} = U^-U, \quad \mathcal{U}_G = U^-TU. \tag{2.2} \]
Then $\mathcal{U}_G$ is the Big Gauss cell of $G$ which corresponds to the Borel subgroup $B$ and $\mathcal{U} \approx A^m_K$ where $m = \dim G - \dim T = |R|$

2.1. The equations which define $\mathcal{U} = U^r - U$. 

For a dominant weight $\lambda : T \rightarrow K^*$ there is a regular function $\delta_\lambda$ on $G$ such that the restriction of $\delta_\lambda$ on $T$ coincides with $\lambda$ (see, for instance, [EG]). We recall the construction of $\delta_\lambda$. Let $V_\lambda$ be the irreducible $G$-module with the highest weight $\lambda$. Further, let $\mathfrak{B} = \{e_1, \ldots, e_d\}$ be the basis which consists of weight vectors of $T$ where $e_1$ corresponds to $\lambda$, and let $\rho_\lambda : G \rightarrow \text{GL}_n(K)$ be the matrix representation which corresponds to the basis $\mathfrak{B}$. The regular function $\delta_\lambda : G \rightarrow K$ which is defined by the formula $\delta_\lambda(g) := g_{11}$, where $g_{11}$ is the $(1, 1)$-entry of the matrix $\rho_\lambda(g)$, satisfies the following condition:

$$\delta_\lambda(vtu) = \lambda(t) \quad \text{for every } t \in T \quad \text{and for every } v \in U^r, u \in U.$$ 

Further, let $\delta_{\lambda_1}, \ldots, \delta_{\lambda_r}$ be regular functions on $G$ that correspond to the fundamental weights $\lambda_1, \ldots, \lambda_r$. Then the Big Gauss cell $\mathcal{U}_G$ is defined by the inequalities

$$g \in \mathcal{U}_G \Leftrightarrow \delta_{\lambda_i}(g) \neq 0 \quad \text{for every } i = 1, \ldots, r. \quad (2.3)$$

The closed subset $\mathcal{U} \subset G$ is defined by equations

$$g \in \mathcal{U} \Leftrightarrow \delta_{\lambda_i}(g) = 1 \quad \text{for every } i = 1, \ldots, r. \quad (2.4)$$

Remark 2.1. Note, that in the case $G = \text{SL}_{r+1}(K)$ the value $\delta_{\lambda_i}(g)$ is the principal $i^{th}$-minor of the matrix $g \in \text{SL}_{r+1}(K)$.

2.2. The rational map $\delta^* : G \rightarrow T$.

Consider the regular map $\delta : G \rightarrow A^r_K$

which is defined by the formula

$$\delta(g) = (\delta_{\lambda_1}(g), \ldots, \delta_{\lambda_r}(g)).$$

The definition of $\delta$ implies that

$$\delta(vtu) = \delta(t) \quad \text{for every } v \in U^r, t \in T, u \in U \quad (2.5)$$

The restriction of the map $\delta$ on $T$ gives us an isomorphism

$$T \overset{\delta}{\approx} (A^r_K)^* := \{(a_1, \ldots, a_r) \mid a_i \neq 0 \quad \text{for every } i\} \approx K^*.$$ 

We will identify the group $G$ with the closed subgroup of $\text{SL}_n(K)$ for some $n$ where the fixed torus $T$ is a subgroup of the group of diagonal matrices of $\text{SL}_n(K)$. Thus, if $t \in T$ then

$$t = \text{diag}(t_1, t_2, \ldots, t_n) \quad \text{for some } t_i \in K.$$ 

Let

$$\kappa : (A^r_K)^* \rightarrow T$$

be the regular map such that

$$\kappa \circ \delta(t) = t \quad \text{for every } t \in T.$$
Since the restriction $\delta|_T : T \rightarrow (A^r)^*$ is a rational homomorphism of the torus $T$ (recall, that $\delta(t) = (\delta_{\lambda_1}(t), \ldots, \delta_{\lambda_r}(t)) = (\lambda_1(t), \ldots, \lambda_r(t))$ for every $t \in T$), then $\kappa$ is also a rational homomorphism of the torus $(A^r)^*$. Hence
\[
\kappa((a_1, \ldots, a_r)) = \left( \prod_{i=1}^{r} a_i^{z_{1i}}, \ldots, \prod_{i=1}^{r} a_i^{z_{ni}} \right)
\]  
(2.6)
for some $z_{ij} \in \mathbb{Z}$. Since $\det \text{diag}(t_1, \ldots, t_n) = 1$ we have
\[
\sum_{j=1}^{n} z_{ji} = 0 \text{ for every } i = 1, \ldots, r.
\]
(2.7)
Thus we may consider the map $\kappa$ as a rational map
\[
\kappa : A^r_K \rightarrow A^n_K
\]
which is defined by formula (2.6). Then the map $\kappa$ is regular at the point $(a_1, \ldots, a_r)$ if and only if $(a_1, \ldots, a_r) \in (A^r_K)^*$ (see, 2.7).

**Example.** Let $G = \text{SL}_{r+1}(K)$. Let $t = \text{diag}(t_1, \ldots, t_r, \frac{1}{t_1t_2 \cdots t_r}) \in T$. Then $\delta(t) = (t_1, t_1t_2, \ldots, t_1t_2 \cdots t_r)$ and therefore $\kappa((a_1, \ldots, a_r)) = (a_1, a_2, \ldots, a_r, 1, \frac{1}{a_r})$.

Now we define the rational map
\[
\delta^* = \kappa \circ \delta : G \rightarrow A^r_K.
\]
This map is regular only at points of the Big Gauss cell $U_G$ and the image $\text{Im} \delta^*$ is isomorphic to $T$. We will identify $\text{Im} \delta^*$ with the torus $T$. It follows directly from the definition
\[
\delta^*(t) = t \text{ for every } t \in T.
\]
(2.8)
From (2.5) we get
\[
\delta^*(t_1vtut_2) = t_1tt_2 \text{ for every } t_1, t_2, t \in T, v \in U^-, u \in U.
\]
(2.9)

### 2.3. The rational map $w_{\tilde{w}_1, \tilde{w}_2} : \mathcal{U} \rightarrow \mathcal{U}$.

Let $\omega : G \rightarrow G$ be an isomorphism of the affine variety $G$. Recall, that the closed subset $\mathcal{U} = U^- U$ we consider also as the affine variety $A^m_K \approx A^m_L \times A^m_K$.

Consider the rational map $w_\omega : G \rightarrow G$ which is given by the formula
\[
w_\omega(g) = \left( \delta^*(\omega(g)) \right)^{-1} \omega(g).
\]
Let map $w_\omega$ be regular at a point $g \in \mathcal{U}$. Then the element $\omega(g)$ belongs to the Big Gauss cell $U_G \subset G$ and therefore $\omega(g) = vtu$ for some $v \in U^-, t \in T, u \in U$. Hence $\delta^*(\omega(g)) = t$ (see, 2.9) and therefore
\[
w_\omega(g) = t^{-1}vtu = (t^{-1}vt)t^{-1}tu = (t^{-1}vt)u \in \mathcal{U},
\]
Let
\[
\mathcal{U}_\omega := \{ u \in \mathcal{U} \mid w_\omega \text{ is regular at the point } u \}.
\]
Consider the restriction of $w_\omega$ on the closed subset $U$. If $U_\omega \neq \emptyset$ then we may and we will consider $w_\omega$ as a rational map $w_\omega : U \to U$. Since $\omega$ is an isomorphism of varieties we have from the definition of $w_\omega$ the equivalence
\[ g \in U_\omega \Leftrightarrow \omega(g) \in U_G. \]
Hence
\[ U_\omega = U \cap \omega^{-1}(U_G). \]
(2.10)

Now we consider the map $\omega : U \to U$ which is defined by left-right multiplication of elements from subgroup $N$. Namely, let $w_1, w_2 \in W$ and let $\dot{w}_1, \dot{w}_2 \in N$ be fixed preimages. Further, let $\omega : G \to G$ be the isomorphism (as a variety) which is given by the formula $\omega(g) = \dot{w}_1 g \dot{w}_2$. Then we put
\[ w_{\dot{w}_1, \dot{w}_2} := w_\omega, \quad U_{w_1, w_2} := U_\omega. \]

The set $U_{w_1, w_2} = U \cap \dot{w}_1 U_G \dot{w}_2^{-1}$ is an intersection of a closed and an open subsets of $G$ and therefore it is an open subset in $U$. Since $U_G = U^{-1} U$ the set $\dot{w}_1 U_G \dot{w}_2^{-1}$ does not depend on the choice of preimages $\dot{w}_1, \dot{w}_2$ of $w_1, w_2$. Hence the set $U_{w_1, w_2}$ does not also depend on the choice of preimages $w_1, \dot{w}_2$.

**Lemma 2.2.**
\[ U_{w_1, w_2} \neq \emptyset. \]

**Proof.** Since $U_G \neq \emptyset$ is an open subset in $G$ we have $\dot{w}_1 U_G \dot{w}_2 \cap U_G \neq \emptyset$. Let
\[ \dot{w}_1 v t u \dot{w}_2 = v' t' u' \in \dot{w}_1 U_G \dot{w}_2 \cap U_G \]
where $v, v' \in U^-, t, t' \in T, u, u' \in U$. Let $s = \dot{w}_1 t^{-1} \dot{w}_1^{-1}$. Then $s \in T$ and
\[ s \dot{w}_1 v t u \dot{w}_2 = (\dot{w}_1 t^{-1} \dot{w}_1^{-1}) \dot{w}_1 v t u \dot{w}_2 = \dot{w}_1 \left( \underbrace{t^{-1} v t}_{\in U^-} \right) u \dot{w}_2 = \]
\[ = s v' t' u' = \underbrace{(s v' s^{-1})}_{\in U^-} \underbrace{(s t')}_{\in T} u' \in U_G \Rightarrow U \cap \dot{w}_1 U_G \dot{w}_2^{-1} \neq \emptyset. \]
\[ \square \]

Thus we have the rational map
\[ w_{\dot{w}_1, \dot{w}_2} : U \to U \]
such that the set of points $u \in U$ where $w_{\dot{w}_1, \dot{w}_2}$ is regular coincides with a non-empty open subset $U_{w_1, w_2} = U \cap \dot{w}_1 U_G \dot{w}_2^{-1}$.

**Proposition 2.3.** The map
\[ w_{\dot{w}_1, \dot{w}_2} : U_{w_1, w_2} \to w_{\dot{w}_1, \dot{w}_2}(U_{w_1, w_2}) \]
is an isomorphism of open subsets of $U$ and $w_{\dot{w}_1, \dot{w}_2}(U_{w_1, w_2}) = U_{w_1, w_2}^{-1}$. 

\[ \square \]
Proof. Let \( u \in \mathcal{U}_{w_1, w_2} \). Further, let

\[
\dot{w}_1 u \dot{w}_2 = vtu
\]

for some \( v \in U^{-}, u \in U, t \in T \). Then

\[
u' = w_{\dot{w}_1, \dot{w}_2}(u) = t^{-1} vtu.
\]

Further,

\[
w_{\dot{w}_1^{-1}, \dot{w}_2^{-1}}(u') = \left(\delta^* (\dot{w}_1^{-1} u' \dot{w}_2^{-1})\right)^{-1} \dot{w}_1^{-1} u' \dot{w}_2^{-1} = \left(\delta^* (\dot{w}_1^{-1} u' \dot{w}_2^{-1})\right)^{-1} \underbrace{(\dot{w}_1^{-1} t^{-1} \dot{w}_1)}_{:= s \in T} \underbrace{(\dot{w}_1^{-1} vtu \dot{w}_2^{-1})}_{:= u; \text{ see } (2.11)} = \left(\delta^* (su)\right)^{-1} su = u.
\]

Hence \( w_{\dot{w}_1^{-1}, \dot{w}_2^{-1}}(u') = u \) and therefore

the map \( w_{\dot{w}_1^{-1}, \dot{w}_2^{-1}} \circ w_{\dot{w}_1, \dot{w}_2} : \mathcal{U}_{w_1, w_2} \to \mathcal{U}_{w_1, w_2} \) is the identity. \hspace{1cm} (2.12)

Since (2.12) holds for every pair of \( \dot{w}_1, \dot{w}_2 \) we have

\[
w_{\dot{w}_1, \dot{w}_2} \circ w_{\dot{w}_1^{-1}, \dot{w}_2^{-1}} : \mathcal{U}_{w_1^{-1}, w_2^{-1}} \to \mathcal{U}_{w_1^{-1}, w_2^{-1}} \text{ is the identity.} \hspace{1cm} (2.13)
\]

Since the map \( w_{\dot{w}_1^{-1}, \dot{w}_2^{-1}} \) is regular at every point \( u' = w_{\dot{w}_1, \dot{w}_2}(u) \) we have the inclusion

\[
w_{\dot{w}_1, \dot{w}_2}(\mathcal{U}_{w_1, w_2}) \subset \mathcal{U}_{w_1^{-1}, w_2^{-1}}. \hspace{1cm} (2.14)
\]

Then the inclusion (2.14) holds if we change \( w_1 \) for \( w_1^{-1} \)

\[
w_{\dot{w}_1^{-1}, \dot{w}_2^{-1}}(\mathcal{U}_{w_1^{-1}, w_2^{-1}}) \subset \mathcal{U}_{w_1, w_2}. \hspace{1cm} (2.15)
\]

We may consider maps

\[
\mathcal{U}_{w_1, w_2} \xrightarrow{w_{\dot{w}_1, \dot{w}_2}} \mathcal{U}_{w_1^{-1}, w_2^{-1}} \xrightarrow{w_{\dot{w}_1^{-1}, \dot{w}_2^{-1}}} \mathcal{U}_{w_1, w_2} \xrightarrow{w_{\dot{w}_1, \dot{w}_2}} \mathcal{U}_{w_1^{-1}, w_2^{-1}}.
\]

(see, (2.14) and (2.15). Thus, (2.12) and (2.13) imply that

\[
w_{\dot{w}_1, \dot{w}_2} : \mathcal{U}_{w_1, w_2} \to w_{\dot{w}_1, \dot{w}_2}(\mathcal{U}_{w_1, w_2})
\]

is an isomorphism of open subsets of \( \mathcal{U} \) and \( w_{\dot{w}_1, \dot{w}_2}(\mathcal{U}_{w_1, w_2}) = \mathcal{U}_{w_1^{-1}, w_2^{-1}}. \]

\[\square\]

Now we get

**Corollary 2.4**. \( w_{\dot{w}_1, \dot{w}_2} \in \text{Cr}_m(K) \) and \( w_{\dot{w}_1^{-1}, \dot{w}_2^{-1}}^{-1} = w_{\dot{w}_1^{-1}, \dot{w}_2^{-1}} \).

**Remark 2.5.** Note, that \( w_{\dot{w}_1^{-1}, \dot{w}_2^{-1}} \) is the map that corresponds to \( \dot{w}_1^{-1}, \dot{w}_2^{-1} \), that is, we take here fixed preimages \( \dot{w}_1, \dot{w}_2 \) of \( w_1, w_2 \) and then take the inverse elements of these preimages. If we take other preimages \((w_1^{-1}), (w_2^{-1})\) we get the corresponding map \( w_{(w_1^{-1}), (w_2^{-1})} \) which differs from \( w_{\dot{w}_1^{-1}, \dot{w}_2^{-1}} \) on a multiplier from \( T \).
3. Definition of the group $\mathcal{N}$

Now we define some group $\mathcal{N} \leq Cr_m(K)$ which partially acts on $\mathcal{U} \approx A^m$.

**Lemma 3.1.** Let $u \in \mathcal{U}$ and let $t, s \in T$. Then
\[
\text{tus} \in \mathcal{U} \iff s = t^{-1}.
\]

*Proof.* Let $u = vu$ where $v \in U^-$, $u \in U$. Then
\[
\text{tus} = (tvt^{-1})s^{-1}us \in \mathcal{U} \iff ts = 1
\]
(recall, that the Gauss decomposition $g = vtu$ of every element $g \in \mathcal{U}_G$ is unique). \hfill \Box

For every $s \in T$ we define the transformation of
\[
t_s : \mathcal{U} \to \mathcal{U}
\]
that is defined by the formula
\[
t_s(u) = sus^{-1}.
\]

Let
\[
\mathcal{T} := \langle t_s \mid s \in T \rangle.
\]

Then we may consider $\mathcal{T}$ as a subgroup of $Cr_m(K)$. Further, we assume that we have the set of fixed preimages $\{\tilde{w}\}_{w \in \mathcal{W}}$. Put
\[
\mathcal{N} = \langle \mathcal{T}, w_{\tilde{w}_1, \tilde{w}_2} \mid (w_1, w_2) \in W \times W \rangle \leq Cr_m(K).
\]

Note, that $\mathcal{N} \leq Cr_m(K)$ is a group that depends on the choice of preimages $\{\tilde{w}\}_{w \in \mathcal{W}}$. Moreover, the inverse elements $w_{\tilde{w}_1, \tilde{w}_2}^{-1}$ do not necessary coincide with $w_{(w_1^{-1}, (w_2^{-1})}$ (see, Remark 2.5). More precisely

**Lemma 3.2.** Let $\{\tilde{w}\}_{w \in \mathcal{W}}$ be a fixed set of preimages in $N$ elements of the Weyl group. Further, let
\[
\omega_1 = w'_1w_1, \omega_2 = w_2w'_2
\]

Then there exists an element $t_s \in \mathcal{T}$ such that
\[
w_{\tilde{w}_1, \tilde{w}_2}w_{\tilde{w}_1, \tilde{w}_2} = t_s\tilde{w}_1\tilde{w}_2.
\]

*Proof.* We have
\[
tag_1\tilde{w}_1 = \tilde{w}_1', \tag_2\tilde{w}_2 = \tilde{w}_2'\text{ for some } t_1, t_2 \in T. \tag{3.1}
\]

Let
\[
u \in \mathcal{U}_{w_1, w_2} \cap w_1^{-1}_{\tilde{w}_1, \tilde{w}_2} \cap \mathcal{U}_{w_2^{-1}, w_2^{-1}}
\]
(note, that the intresections of non-empty open subsets of $\mathcal{U}$ are non-empty open subsets).

Hence the map $w_{\tilde{w}_1, \tilde{w}_2}$ is regular at the point $w_{\tilde{w}_1, \tilde{w}_2}(u)$ and we have
\[
w_{\tilde{w}_1', \tilde{w}_2'}w_{\tilde{w}_1, \tilde{w}_2}(u) = w_{\tilde{w}_1', \tilde{w}_2'}(\left((\delta^*(\tilde{w}_1'\tilde{w}_2))^{-1}\tilde{w}_1\tilde{w}_2\right)_{t \in \mathcal{T}} =
\]
\[
= (\delta^*(\tilde{w}_1't\tilde{w}_1'^{-1})\tilde{w}_1'(\tilde{w}_1'\tilde{w}_2)\tilde{w}_2'))^{-1}(\tilde{w}_1't\tilde{w}_1'^{-1})\tilde{w}_1'(\tilde{w}_1'\tilde{w}_2)\tilde{w}_2' \tag{2.9}
\]
\[
= (\delta^*(\tilde{w}_1'(\tilde{w}_1'\tilde{w}_2)\tilde{w}_2'))^{-1}\tilde{w}_1'(\tilde{w}_1'\tilde{w}_2)\tilde{w}_2' \tag{3.1}
\]
\[
= (\delta^*(t_1\tilde{w}_1\tilde{w}_2)\tilde{w}_2)\tilde{w}_2' = (\delta^*(t_1\tilde{w}_1\tilde{w}_2)\tilde{w}_2)\tilde{w}_2' =
\]
\[
= (\delta^*(t_1\tilde{w}_1\tilde{w}_2)\tilde{w}_2)\tilde{w}_2' =
\]
2.9 \[ t_2^{-1} (\delta^*(\hat{w}_1u\hat{w}_2))^{-1} \hat{w}_1u\hat{w}_2 t_2 = t_s w_{\hat{w}_1,\hat{w}_2} \] where \( s = t_2^{-1} \).
\\

Lemma 3.3. The subgroup \( T \) is normal in \( N \).

Proof. Let \( t_s \in T \), \( w_{\hat{w}_1,\hat{w}_2} \in N \) and let \( u \in U_{\hat{w}_1,\hat{w}_2} \). Then

\[
t_s w_{\hat{w}_1,\hat{w}_2}(u) = s \left( (\delta^*(\hat{w}_1 u \hat{w}_2))^{-1}(\hat{w}_1 u \hat{w}_2) \right) s^{-1} = t(\hat{w}_1 u \hat{w}_2) s^{-1}. \]
\\

Then

\[
w_{\hat{w}_1,\hat{w}_2}^{-1} t_s w_{\hat{w}_1,\hat{w}_2}(u) = \left( (\delta^*(\hat{w}_1^{-1} t(\hat{w}_1 u \hat{w}_2) s^{-1} \hat{w}_2^{-1}))^{-1}(\hat{w}_1^{-1} t(\hat{w}_1 u \hat{w}_2) s^{-1} \hat{w}_2^{-1}) \right) t_1 u t_2 \text{ Lem.} 3.1
\\

\[ = t_2^{-1} u t_2 = t_s(u) \text{ where } s' = t_2^{-1}. \]
\\

Now we show that the group \( N \) acts partially on the affine space \( U \). For every \( u \) we define \( N(u) \) as the set of elements of \( N \) which are regular at the point \( u \). Then conditions i. - iii. of the Definition 1.1 obviously hold. From Proposition 2.3 and Corollary 2.4 we have condition iv:

\[ n^{-1} \in N(n(u)) \text{ for every } n \in N(u). \]
\\

Now we may prove Theorem 1.

Proof. The implication \( \Leftarrow \) follows directly from the definition of double cosets.

Let \( u_1, u_2 \in U \). Then

\[ u_1, u_2 \in NuN \overset{2.1}{\Leftrightarrow} n_1 u_1 n_2 = u_2 \text{ for some } n_1, n_2 \in N. \]
\\

Let \( n_1 u_1 n_2 = u_2 \). Then \( n_1 = t_1 \hat{w}_1, n_2 = \hat{w}_2 t_2 \) for some \( t_1, t_2 \in T \). Since \( u_1, u_2 \in U \) then \( u_1 \in U_{\hat{w}_1,\hat{w}_2} \). We have

\[
n_1 u_1 n_2 = t_1 \hat{w}_1 u_1 \hat{w}_2 t_2 = t_1 (\delta^*(\hat{w}_1 u \hat{w}_2)) w_{\hat{w}_1,\hat{w}_2}(u) t_2 = u_2 \in U \text{ Lem.} 3.1 \Rightarrow t_1' t_2 = 1 \Rightarrow
\\

\[
\Rightarrow u_2 = t_s w_{\hat{w}_1,\hat{w}_2}(u_1) \text{ where } s = t_2^{-1}. \]
\\

□
4. Example. Case $G = \text{SL}_2(\mathbb{C})$

4.1. Group $\mathcal{N}$.

Let $G = \text{SL}_2(K)$. Then

$$
\mathcal{U} = \left\{ \begin{pmatrix} 1 & \alpha \\ \beta & 1 + \alpha \beta \end{pmatrix} \mid \alpha, \beta \in K \right\}.
$$

Here $W = \{e, w\}$ is the group consisting of two elements – the identity $e$ and the involution $w$. We take

$$
\dot{e} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \dot{w} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
$$

Hence

$$
\begin{aligned}
\dot{w} \dot{e} \dot{w}^{-1} & = \begin{pmatrix} 1 & \alpha \\ \beta & 1 + \alpha \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ \beta & 1 + \alpha \beta \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\beta^{-1}(1 + \alpha \beta) \\ -\beta & 1 + \alpha \beta \end{pmatrix}, \\
\dot{w} \dot{e} \dot{w}^{-1} & = \begin{pmatrix} 1 & -\alpha^{-1} \\ 0 & 1 + \alpha \beta \end{pmatrix} \begin{pmatrix} -\alpha^{-1} & 0 \\ 0 & 1 + \alpha \beta \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\alpha^{-1} \\ 0 & 1 + \alpha \beta \end{pmatrix}, \\
\dot{w} \dot{w}^{-1} & = \begin{pmatrix} 1 & -\alpha \beta^{-1} \alpha^{-1} \\ 0 & 1 + \alpha \beta \end{pmatrix} = \begin{pmatrix} 1 & -\beta(1 + \alpha \beta)^{-1} \\ 0 & 1 + \alpha \beta \end{pmatrix}.
\end{aligned}
$$

It is easy to check

$$
\dot{w} \dot{e} \dot{w} = \dot{w} \dot{e} \dot{w} \dot{e} = \dot{e} \dot{w} \dot{e} \dot{w} = \dot{w} \dot{w} = \dot{w} \dot{e} \dot{w} = \dot{e} \dot{w} = \dot{e} = e
$$

(4.1)

(here $e$ is the identity of $\text{Cr}_2(K)$). Put

$$
\mathbf{w}_t := \dot{w} \dot{e} \dot{w}, \quad \mathbf{w}_r := \dot{w} \dot{e} \dot{w}, \quad \mathbf{w}_d := \dot{w} \dot{w}.
$$

Thus, $\mathbf{w}_t, \mathbf{w}_r, \mathbf{w}_d \in \text{Cr}_2(K)$ are birational transformations of the affine plane

$$
A^2_K = \{ (\alpha, \beta) \mid \alpha, \beta \in K \}.
$$

Namely,

$$
\mathbf{w}_t((\alpha, \beta)) = (\beta^{-1}(1 + \alpha \beta), -\beta), \quad \mathbf{w}_r((\alpha, \beta)) = (-\alpha^{-1}, \alpha(1 + \alpha \beta)),
$$

$$
\mathbf{w}_d((\alpha, \beta)) = (-\beta(1 + \alpha \beta)^{-1}, -\alpha(1 + \alpha \beta)).
$$

The element $t_s \in \mathcal{T} \leq \text{Cr}_2(K)$ acts on $A^2_K$ according to the following formula

$$
\mathbf{t}_s((\alpha, \beta)) = (s^2 \alpha, s^{-2} \beta).
$$

Also

$$
\mathbf{w}_t((\alpha, \beta)) \mathbf{t}_s \mathbf{w}_t((\alpha, \beta)) = \mathbf{w}_t(s^2 \alpha^{-1}, -s^{-2} \beta) = (s^2 \alpha, s^{-2} \beta) = \mathbf{t}_s((\alpha, \beta)),
$$

$$
\mathbf{w}_r((\alpha, \beta)) \mathbf{t}_s \mathbf{w}_r((\alpha, \beta)) = \mathbf{w}_r((-s^2 \alpha^{-1}, s^{-2} \alpha(1 + \alpha \beta))) = (s^2 \alpha, s^{-2} \beta) = \mathbf{t}_s^{-1}((\alpha, \beta)).
$$

Hence

$$
\mathcal{N} = \langle \mathcal{T}, \mathbf{w}_r \rangle \times \langle \mathbf{w}_t \rangle \approx D_\infty \times \mathbb{Z}_2
$$

(here $D_\infty$ is the infinite dihedral group and $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ is the group of order 2).
Let
\[ \mathcal{M} = \{ (\alpha, \beta) \mid \alpha \neq 0, \beta \neq 0, \alpha \beta \neq -1 \} \]

**Lemma 4.1.** Every element \( g \in \mathcal{N} \) stabilizes the open set \( \mathcal{M} \).

*Proof.* We have to check: if \((\alpha, \beta) \in \mathcal{M}\) then \( g((\alpha, \beta)) \in \mathcal{M} \) if \( g = t_s, w_l, w_r \). It is obvious for \( t_s \). Further,
\[ w_l((\alpha, \beta)) = (\beta^{-1}(1 + \alpha \beta), -\beta), \quad w_r((\alpha, \beta)) = (-\alpha^{-1}, \alpha(1 + \alpha \beta)). \]

In both cases \( \alpha' \beta' = -(1 + \alpha \beta) \). Thus, \( 1 + \alpha' \beta' = -\alpha \beta \neq 0 \) and therefore
\[ w_l((\alpha, \beta)), w_r((\alpha, \beta)) \in \mathcal{M}. \]

\[ \Box \]

Let
\[ \mathcal{M}_{0,1} = \{ (0, \beta) \mid \beta \neq 0 \}, \mathcal{M}_{1,0} = \{ (0, \alpha) \mid \alpha \neq 0 \}, \mathcal{M}_{-1} = \{ (\alpha, \beta) \mid \alpha \beta = -1 \}. \]

From the definition of \( w_l, w_r, w_d \) we have the following formulas
\[ w_l(M_{0,1}) = \{ (\beta^{-1}, -\beta) \mid \beta \neq 0 \} = \mathcal{M}_{-1}, \quad \tag{4.2} \]
\[ w_l(M_{-1}) = \{ (\beta^{-1}(1 + \alpha \beta), -\beta) = (0, -\beta) \mid \beta \neq 0 \} = \mathcal{M}_{0,1}, \]
\[ w_r(M_{1,0}) = \{ (-\alpha^{-1}, \alpha) \mid \alpha \neq 0 \} = \mathcal{M}_{-1}, \quad w_r(M_{-1}) = \{ (-\alpha^{-1}, 0) \mid \alpha \neq 0 \} = \mathcal{M}_{1,0}, \]
\[ w_d(M_{1,0}) = \mathcal{M}_{0,1}, \quad w_d(M_{0,1}) = \mathcal{M}_{1,0}. \]

**Lemma 4.2.** The set
\[ \tilde{\mathcal{M}} = \mathcal{M}_{0,1} \cup \mathcal{M}_{1,0} \cup \mathcal{M}_{-1} \]

is just one \( \mathcal{N} \)-orbit. The point \((0, 1)\) can be taken as a representative of this orbit.

*Proof.* Since \( K \) is an algebraically closed field the sets \( \mathcal{M}_{0,1}, \mathcal{M}_{0,1}, \mathcal{M}_{-1} \) are three \( \mathcal{T} \)-orbits of points \((0, 1), (1, 0), (1, -1)\) respectively. But these points are in the same \( \mathcal{N} \)-orbit (that follows from 4.2). \[ \Box \]

**4.3. The representatives of \( \mathcal{N} \)-orbits on \( A_K^2 = \mathcal{M} \cup \tilde{\mathcal{M}} \cup \{ (0, 0) \} \).**

Put
\[ \mathcal{M}^1 := \{ (\alpha, 1) \mid \alpha \neq 0, -1 \} \subset \mathcal{M}. \]

**Lemma 4.3.** If \((\alpha', \beta') \in \mathcal{M} \) then there is an element \((\alpha, 1) \in \mathcal{M}^1 \) which belongs to the same orbit as \((\alpha', \beta') \).

*Proof.* We have \( t_s((\alpha', \beta')) = (s^2 \alpha', s^{-2} \beta') \). Since \( K \) is an algebraically closed field we can find \( s \in K \) such that \( s^{-2} \beta' = 1 \). Hence in every \( \mathcal{N} \)-orbit of the set \( \mathcal{M} \) there is an element of the form \((\alpha', 1)\). \[ \Box \]

**Lemma 4.4.** The elements \((\alpha, 1) \neq (\alpha', 1) \in \mathcal{M}^1 \) are in the same \( \mathcal{N} \)-orbit if and only if \( \alpha' = -1 - \alpha \).
Proof. The elements \((\alpha, 1), (\alpha', 1) \in M^1\) are in the same \(\mathcal{N}\)-orbit if and only if
\[
\mathbf{t}_s \mathbf{w}((\alpha, 1)) = (\alpha', 1) \quad \text{for some } s = s(\alpha, \mathbf{w}) \in K \quad \text{and where }
\]
\[
\mathbf{w} \quad \text{is one of the following elements: } \mathbf{e}, \mathbf{w}_l, \mathbf{w}_r, \mathbf{w}_d
\]
(it follows from Lemmas 3.2, 3.3, recall, \(e \in C_{r2}(K)\) is the identity). Let \(\mathbf{w} = \mathbf{e}\). Then
\[
\mathbf{t}_s \mathbf{e}((\alpha, 1)) = \mathbf{t}_s(((\alpha, 1)) \in M^1 \iff \mathbf{t}_s = \mathbf{e} \iff \alpha' = \alpha.
\]
We have
\[
w_l((\alpha, 1)) = ((1 + \alpha), -1), \quad w_r((\alpha, 1)) = (-\alpha^{-1}, \alpha(1 + \alpha)),
\]
\[
w_d((\alpha, 1)) = (-1 + \alpha)^{-1}, -\alpha(1 + \alpha)).
\]
For \(\mathbf{w} = \mathbf{w}_l\), or \(\mathbf{w} = \mathbf{w}_r\), or \(\mathbf{w} = \mathbf{w}_d\) and for the fixed \(\alpha\) there is only one element \(\mathbf{t}_s\) such that \(\mathbf{t}_s \mathbf{w}((\alpha, 1)) \in M^1\). From [4.3]
\[
s = \begin{cases} \sqrt{-1} \text{ for the case } \mathbf{w} = \mathbf{w}_l, \\ \sqrt{\alpha(1 + \alpha))} \text{ for the case } \mathbf{w} = \mathbf{w}_r, \\ \sqrt{-\alpha(1 + \alpha)} \text{ for the case } \mathbf{w} = \mathbf{w}_d 
\end{cases}
\]
(recall, that \(\mathbf{t}_s((\alpha, \beta)) = (s^2\alpha, s^{-2}\beta)\)) and therefore \(t_{s_1} = t_{s_2}\) if and only if \(s_1 = \pm s_2\).
From [4.4, 4.5] for corresponding \(s\) we get
\[
\begin{cases} 
\mathbf{t}_s \mathbf{w}_l((\alpha, 1)) = (-1 + \alpha), 1 \\
\mathbf{t}_s \mathbf{w}_r((\alpha, 1)) = (-1 + \alpha), 1) \\
\mathbf{t}_s \mathbf{w}_d((\alpha, 1)) = (\alpha, 1).
\end{cases}
\]
Now the statement of the Lemma follows from [4,3 and 4,6] \(\square\)

Let \(b : K \to K\) be the map which is given by the formula
\[
b(\alpha) = -1 - \alpha
\]
for every \(\alpha \in K\). Then \(b^2\) is the identity on \(K\) and there is only one element \(\alpha\) such that \(b(\alpha) = \alpha\), namely \(\alpha = -\frac{1}{2}\). Then we may decompose \(K = K^+ \cup K^-\) into the union of two disjoint subsets \(K^+, K^-\) where \(-\frac{1}{2}, 0 \in K^-\) and if \(-\frac{1}{2} \neq \alpha \in K^-\) then \(b(\alpha) \in K^-.\) Now let us fix such a decomposition \(K = K^- \cup K^+\). Put
\[
\Omega_K := \{(\alpha, 1)\}_{\alpha \in K^+} \cup \{(0, 0)\}.
\]

**Theorem 4.5.** The set \(\Omega_K\) is a smallest set of the representatives of the \(\mathcal{N}\)-orbits on \(A_K^2\), that is, for every \(\mathcal{N}\)-orbit there is only one its representative which is contained in \(\Omega_K\).

**Proof.** Let \((\alpha, \beta) \in A_K^2\) and let \(O_{\alpha, \beta}\) be the \(\mathcal{N}\)-orbit of \((\alpha, \beta)\). Suppose \((\alpha, \beta) \in M\). Then \(O_{\alpha, \beta} = O_{\alpha', 1}\) where \(\alpha' \in K^+_\sim\) (see, Lemmas [4.3, 4.4]). Moreover, the definition of \(K^\sim\) and Lemma [4.4] imply that such element \(\alpha' \in K^\sim\) is unique. Suppose \((\alpha, \beta) \in \tilde{M}\) then \(O_{\alpha, \beta} = \tilde{M}\) and we may take a representative \((0, 1) \in K^\sim\) of this orbit (Lemma [4.2]). Note, that only elements of the form \(\mathbf{t}_s\) and \(\mathbf{w}_d\) are regular at the point \((0, 0)\) and in these cases the point \((0, 0)\) is invariant. Hence the set \(\{(0, 0)\}\) is just one \(\mathcal{N}\)-orbit. \(\square\)
From Theorems 1 and 4.5 we get

**Corollary 4.6.**

\[
\text{SL}_2(K) = N \cup \left( \bigcup_{\alpha \in K} N \begin{pmatrix} 1 & \alpha \\ 1 & 1 + \alpha \end{pmatrix} N \right).
\]

Put

\[
g_\alpha := \begin{pmatrix} 1 & \alpha \\ 1 & 1 + \alpha \end{pmatrix}.
\]

From Corollary 4.6 and (*) (see, Introduction) we get

**Corollary 4.7.** The set of pairs

\[
(T, T) \cup (g_\alpha T g_\alpha^{-1}, T)_{\alpha \in K}
\]

is a minimal set of representatives of the orbits of pairs of tori of \(G \times G\) under conjugation by elements of \(G\).

### 4.4. Case \(K = \mathbb{C}\).

**Lemma 4.8.** Let

\[
\mathcal{K} = \{ z = a + bi \in \mathbb{C} \mid a \geq -\frac{1}{2} \} \setminus \{ z = -\frac{1}{2} + bi \in \mathbb{C} \mid b < 0 \}.
\]

Then we may take \(\mathcal{K}\) as the set \(\mathbb{C}_b\).

**Proof.** We have \(-\frac{1}{2}, 0 \in \mathcal{K}\). Let \(z = a + bi \in \mathcal{K}\). Suppose \(a \neq -\frac{1}{2}\). Then

\[
b(z) = (-1 - a) - bi \in \mathbb{C} \setminus \mathcal{K}.
\]

If \(a = -\frac{1}{2}\) then \(b \geq 0\) and therefore

\[
b(z) = -\frac{1}{2} - b < 0 \quad i \in \mathbb{C} \setminus \mathcal{K}.
\]

\[\quad \Box\]

### 4.5. Orbits of pairs of semisimple matrices.

Put

\[
U^w := \dot{w}U = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \mid \beta \in K \right\} = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & -\beta \end{pmatrix} \mid \beta \in K \right\}.
\]

We have

\[
G = \dot{w}G = \dot{w}B \cup \dot{w}B \dot{w}^{-1}B = T \dot{w}U \cup (\dot{w}U \dot{w}^{-1})UT = T(U^w \cup U)T.
\]

Obviously, for every \(v \in U^w\) and \(t_1, t_2 \in T\) the equality \(t_1 vt_2 \in U^w\) implies \(t_1 = t_2\). Thus only two \(T \times T\)-orbits intersect \(U^w\) – the orbit of matrix \(\dot{w} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\) and the orbit of \(\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\).
By Lemma 3.1 for \( u \in \mathcal{U} \) we have the inclusion \( t_1ut_2 \in \mathcal{U} \) if and only if \( t_2 = t_1^{-1} \). Thus, the minimal set of the representatives of double cosets of \( TyT \) in the group \( G \) is

\[
\{g_0\} \in \mathbb{K} \cup \{(0 1 \mid -1 1), (0 1 \mid -1 0), (1 1 \mid 0 1), (1 0 \mid 0 1)\}. \tag{4.7}
\]

Let \( t = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \). Then

\[
g_\alpha t g_\alpha^{-1} = \begin{pmatrix} s + \alpha(s - s^{-1}) & \alpha(s^{-1} - s) \\ (1 + \alpha)(s - s^{-1}) & s^{-1} + \alpha(s^{-1} - s) \end{pmatrix} = \begin{pmatrix} s + \alpha \Delta_t & -\alpha \Delta_t \\ (1 + \alpha) \Delta_t & s^{-1} - \alpha \Delta_t \end{pmatrix} \tag{4.8}
\]

where \( \Delta_t = s - s^{-1} \). Note, that the given \( \Delta \in K \) corresponds only to matrices \( t \) and \( \bar{t} = \begin{pmatrix} -s^{-1} & 0 \\ 0 & -s \end{pmatrix} \).

Now let \( t' = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \). Suppose that \( s, r \neq \pm 1 \) then the centralizer of the matrix \( t \) and also the matrix \( t' \) is \( T \). From 4.7 a smallest set of representatives of \( G \)-orbits (under conjugation) of \( C_t \times C_{t'} \), where \( C_t, C_{t'} \) are conjugacy classes of \( t, t' \) respectively, consists of the following pairs:

\( (g_\alpha t g_\alpha^{-1}, t')_{\alpha \in \mathbb{K}}, (\bar{w}(1)t \bar{w}(1)^{-1}, t'), (\bar{w}t \bar{w}^{-1}, t'), (u(1)tu(1)^{-1}, t'), (t, t') \).

Thus we have the following

**Proposition 4.9.** Let \( \pm \bar{e} \neq t, t' \in T \) and let \( C_t, C_{t'} \) be corresponding conjugacy classes. Then there are only the following \( G \)-orbits on \( C_t \times C_{t'} \):

\[
\mathcal{O}_\alpha := \{ A \left( \begin{pmatrix} s & \alpha \Delta_t \\ (1 + \alpha) \Delta_t & s^{-1} - \alpha \Delta_t \end{pmatrix}, \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \right) A^{-1} \mid \alpha \in \mathbb{K}, A \in \text{SL}_2(\mathbb{K}) \}
\]

(\text{the orbits of } (g_\alpha t g_\alpha^{-1}, t')_{\alpha \in \mathbb{K}});

\[
\mathcal{O}_U^+ := \{ A \left( \begin{pmatrix} s & 0 \\ 0 & \Delta_t \end{pmatrix}, \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \right) A^{-1} \mid A \in \text{SL}_2(\mathbb{K}) \}
\]

(\text{the orbit } (u(1)tu(1)^{-1}, t'));

\[
\mathcal{O}_V^- := \{ A \left( \begin{pmatrix} s^{-1} & 0 \\ \Delta_t & s \end{pmatrix}, \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \right) A^{-1} \mid A \in \text{SL}_2(\mathbb{K}) \}
\]

(\text{the orbit of } (\bar{w}(1)t \bar{w}(1)^{-1}, t'));

\[
\mathcal{O}_T^+ := \{ A \left( \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}, \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \right) A^{-1} \mid A \in \text{SL}_2(\mathbb{K}) \}
\]

(\text{the orbit of } (t, t'));

\[
\mathcal{O}_T^- := \{ A \left( \begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix}, \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \right) A^{-1} \mid A \in \text{SL}_2(\mathbb{K}) \}
\]
(the orbit of \((\hat{w}t\hat{w}^{-1}, t')\)).

**Adherence of orbits.** Consider the adherence of \(G\)-orbits on \(C_t \times C_{t'}\). Note, that the invariant algebra of \(M_2(\mathbb{C}) \times M_2(\mathbb{C})\), which corresponds to the action by conjugation of \(SL_2(\mathbb{C})\), is generated by \(\text{tr}X, \text{tr}Y, \text{tr}XY\) where \((X, Y) \in M_2(\mathbb{C}) \times M_2(\mathbb{C})\) (see, for instance, [VP], 9.5). The algebraic factor \(\left(\frac{M_2(\mathbb{C}) \times M_2(\mathbb{C})}{SL_2(\mathbb{C})}\right)\) is isomorphic to \(A^3_\mathbb{C}\). Thus we have the quotient map \(\pi : M_2(\mathbb{C}) \times M_2(\mathbb{C}) \to A^3_\mathbb{C}\) where \(\pi : (X, Y) = (\text{tr}X, \text{tr}XY, \text{tr}Y)\).

In every fiber of this map there is only one closed orbit.

Now we consider the restriction of \(\pi_{t,t'}\) on the closed subset \(C_t \times C_{t'} \subset M_2(\mathbb{C}) \times M_2(\mathbb{C})\) (recall, that the conjugacy class of a semisimple element is a closed subset of \(M_2(\mathbb{C})\)). Since \(\text{tr}X, \text{tr}Y\) are constants on \(C_t \times C_{t'}\) we will consider the map \(\pi_{t,t'}\) as

\[
\pi_{t,t'} : C_t \times C_{t'} \to A^1_\mathbb{C} = \mathbb{C} \quad \text{where} \quad \pi_{t,t'}((X, Y)) = \text{tr}(XY).
\]

Every fiber of \(\pi_{t,t'}\) contains only one closed \(G\)-orbit in \(C_t \times C_{t'}\). Also, every fiber of \(\pi_{t,t'}\) has the dimension \(\geq 3\) (indeed, \(\dim C_t \times C_{t'} - \dim \text{Im} \pi_{t,t'} = 3\)).

Consider an orbit of the type \(O_\alpha\).

Let \((X_\alpha, Y) \in O_\alpha\) be the representative which is pointed out in Proposition 4.9.

Here

\[
\text{tr}(X_\alpha Y) = \text{tr}\left(\begin{pmatrix} s + \alpha \Delta_t & -\alpha \Delta_t \\ (1 + \alpha) \Delta_t & s^{-1} - \alpha \Delta_t \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \right) = sr + s^{-1}r^{-1} + \alpha \Delta_t \Delta_{t'}.
\]

(4.9)

Further, let \(Y = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}\). Then for every \(a \in K\)

\[
\text{tr}(XY) = \begin{cases} 
sr + s^{-1}r^{-1} & \text{if } X = \begin{pmatrix} s & a \\ 0 & s^{-1} \end{pmatrix}, \begin{pmatrix} s & 0 \\ a & s \end{pmatrix} \\
 s^{-1}r + sr^{-1} & \text{if } X = \begin{pmatrix} s^{-1} & a \\ 0 & s \end{pmatrix}, \begin{pmatrix} s^{-1} & 0 \\ a & s \end{pmatrix}.
\end{cases}
\]

(4.10)

Note,

\[
(sr + s^{-1}r^{-1}) - (s^{-1}r + sr^{-1}) = (s - s^{-1})(r - r^{-1}) = \Delta_t \Delta_{t'} \neq 0 \Rightarrow
\]

\[
sr + s^{-1}r^{-1} + (-1) \Delta_t \Delta_{t'} = (s^{-1}r + sr^{-1})
\]

(4.11)

Let \(\alpha \neq 0, -1\). From (4.9) (4.11) we get

\[
\text{tr}(X_\alpha Y) \neq sr + s^{-1}r^{-1}, \ s^{-1}r + sr^{-1}.
\]

(4.12)

Now (4.10) (4.12) imply that \(G\)-orbits \(O^+_t, O^-_t, O^+_T, O^-_T\) cannot be in the same fiber of \(\pi_{t,t'}\) that contains \((X_\alpha, Y)\). Since the stabilizer of \((X_\alpha, Y)\) is equal to \(\{\pm \hat{e}\}\), the dimension of \(G\)-orbit is equal to 3. Hence the orbit of \((X_\alpha, Y)\) is closed and coincides with the fiber of \(\pi_{t,t'}\) which contains \((X_\alpha, Y)\).
Consider the orbits $\mathcal{O}_0, \mathcal{O}_U^+$. Both orbits have dimension 3 and belong to the fiber $\pi^{-1}(sr + s^{-1}r^{-1})$ (see, [4,10]). Moreover, in the same fiber we have 2-dimensional closed orbit $\mathcal{O}_T^+$ and

$$\overline{\mathcal{O}_0} \setminus \mathcal{O}_0 = \overline{\mathcal{O}_U^+} \setminus \mathcal{O}_U^+ = \mathcal{O}_T^+.$$ 

Indeed, for any fixed $0 \neq a, b \in K$ we have

$$\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid 0 \neq c \in K \} \setminus \{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid 0 \neq c \in K \} = \{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \}$$

(here $\overline{X}$ is the closure in Zariski topology). The same equality hold for lower triangular matrices.

The analogical result we can get for $\mathcal{O}_{-1}, \mathcal{O}_V^+, \mathcal{O}_T^-$.

Now we summarize the facts on adherence of orbits

**Proposition 4.10.**

i. The orbits $\mathcal{O}_\alpha$, where $\alpha \neq 0, -1$, are closed 3-dimensional $G$-orbits which coincide with fibers $\pi^{-1}_t(l_\alpha)$ for $l_\alpha = \text{tr}(X_\alpha Y) \neq sr + s^{-1}r^{-1}, s^{-1}r + sr^{-1}$.

ii. The fiber $\pi^{-1}_t(l_0)$ where $l_0 = \text{tr}(X_0 Y) = sr + s^{-1}r^{-1}$ consists of two 3-dimensional orbits

$$\mathcal{O}_0 = \{ A( \begin{pmatrix} s & 0 \\ \Delta_t & s^{-1} \end{pmatrix}, \begin{pmatrix} r & 0 \\ 0 & r_1^{-1} \end{pmatrix} ) A^{-1} \mid A \in \text{SL}_2(K) \},$$

$$\mathcal{O}_U^+ = \{ A( \begin{pmatrix} s & -\Delta_t \\ 0 & s^{-1} \end{pmatrix}, \begin{pmatrix} r & 0 \\ 0 & r_1^{-1} \end{pmatrix} ) A^{-1} \mid A \in \text{SL}_2(K) \}$$

and the closed 2-dimensional orbit

$$\mathcal{O}_T^+ = \{ A( \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}, \begin{pmatrix} r & 0 \\ 0 & r_1^{-1} \end{pmatrix} ) A^{-1} \mid A \in \text{SL}_2(K) \}$$

which coincides with $\overline{\mathcal{O}_0} \cap \overline{\mathcal{O}_U^+}$.

iii. The fiber $\pi^{-1}_{t,t'}(l_{-1})$ where $l_{-1} = \text{tr}(X_{-1} Y) = s^{-1}r + sr^{-1}$ consists of two 3-dimensional orbits

$$\mathcal{O}_{-1} = \{ A( \begin{pmatrix} s^{-1} & \Delta_t \\ 0 & s \end{pmatrix}, \begin{pmatrix} r & 0 \\ 0 & r_1^{-1} \end{pmatrix} ) A^{-1} \mid A \in \text{SL}_2(K) \},$$

$$\mathcal{O}_V^+ = \{ A( \begin{pmatrix} s^{-1} & 0 \\ \Delta_t & s \end{pmatrix}, \begin{pmatrix} r & 0 \\ 0 & r_1^{-1} \end{pmatrix} ) A^{-1} \mid A \in \text{SL}_2(K) \}$$

and the closed 2-dimensional orbit

$$\mathcal{O}_T^- = \{ A( \begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix}, \begin{pmatrix} r & 0 \\ 0 & r_1^{-1} \end{pmatrix} ) A^{-1} \mid A \in \text{SL}_2(K) \}$$

which coincides with $\overline{\mathcal{O}_{-1}} \cap \overline{\mathcal{O}_V^+}$. 
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