The Analysis of Initial Conditions for the LTB Model

by

Alexander Gromov

St. Petersburg State Technical University
Faculty of Technical Cybernetics, Dept. of Computer Science
29, Polytechnicheskaya str. St.-Petersburg, 195251, Russia

and

Istituto per la Ricerca di Base
Castello Principe Pignatelli del Comune di Monteroduni
I-86075 Monteroduni(IS), Molise, Italia

e-mail: gromov@natus.stud.pu.ru

Abstract

The LTB model is studied as the Caushy problem for the equations defined two metrical functions $\lambda(\mu, \tau)$ and $\omega(\mu, \tau)$. The initial conditions throught metrical functions are presented. The rules of calculating three undetermined functions $f(\mu)$, $F(\mu)$ and $F(\mu)$ are obtained. The general expressions for the density and Hubble function in the LTB model are written by the metrical functions.

PACS number(s):98.80
1 The Introduction

The LTB model is one of the most known spherical symmetry model in general relativity. It was created by Lemaitre [1], Tolman [2], and Bondy [3] during the period of time from 1933 to 1947. The exact solution have been obtained by Bonnor [4] in 1972 and [5] in 1974. The LTB model represented one of the simplest nonhomogeneous nonstationary cosmological models and due to this fact is used to study some new ideas in the cosmology.

The present interest to LTB was risen by the observations shown the fractal structure of the Universe in the large scale [6], [7], and discuss in the set of article (the modern review is presented in §8). In a set of papers the LTB model is used to study the observational datas and redshift as a main cosmological test [9] - [14].

The central problem of using the LTB model is in calculation of three undetermined functions which defined the solution. There is a set of ways how to solve this problem from the physical point of view in the mentioned papers. This article is devoted to the mathematical point of view on this matter §13.

2 The LTB Model

This section is devoted to presentation the Lemaitre-Tolman-Bondy model and section 2 of the paper [2] is cited. The co-moving system of coordinate is used in this model where the interval has the form

\[ ds^2(r, t) = -e^{\lambda(r, t)} dt^2 - e^{\omega(r, t)} \left( d\theta^2 + \sin^2 \theta d\phi \right) + dr^2. \]  

(1)

\( \lambda(r, t) \) and \( \omega(r, t) \) are metrical functions [2] defining the solution. Following the [2] we will omit the arguments in the functions \( \omega \) and \( \lambda \) in this section. In the co-moving system of coordinate with the line element (1) the energy-momentum tensor

\[ T^\alpha_\beta = \rho \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} \]

has only one non zero component

\[ T^4_4 = \rho \quad T^\beta_4 = 0, \quad \alpha \text{ or } \beta = 4. \]

Using them together with Dingle results [14] we obtain the system of equations of the LTB model:

\[ 8\pi T^1_1 = e^{-\omega} - e^{-\lambda} \frac{\omega'^2}{4} + \frac{3}{4} \dot{\omega}^2 - \Lambda = 0 \]

(2)

\[ 8\pi T^2_2 = 8\pi T^3_3 = e^{-\lambda} \left( \frac{\omega'^2}{2} + \frac{\lambda' \omega'}{4} \right) + \frac{\lambda}{4} + \frac{\dot{\omega}}{2} + \frac{\dot{\lambda} \omega}{4} - \Lambda = 0 \]

(3)

\[ 8\pi T^4_4 = e^{-\omega} - e^{-\lambda} \left( \frac{\omega'^2}{4} - \frac{\lambda' \omega'}{2} \right) + \frac{\dot{\omega}^2}{2} + \frac{\dot{\lambda} \omega}{2} - \Lambda = 8\pi \rho \]

(4)

\[ 8\pi e^{\lambda} T^1_4 = -8\pi T^4_1 = \frac{\omega' \dot{\omega}}{2} - \frac{\dot{\lambda} \omega'}{2} + \omega' = 0, \]

(5)

where

\[ ' = \frac{\partial}{\partial r} \quad = \frac{\partial}{\partial t} \]

The equation (3) has the solution

\[ e^\lambda = e^{\omega} \frac{\omega'^2}{4f'^2(r)}, \]

(6)
where \( f(r) \) is undetermined function. Substituting (3) into (2) we obtain

\[
e^\omega \left( \dot{\omega} + \frac{3}{4} \dot{\omega}^2 - \Lambda \right) + \left[ 1 - f^2(r) \right] = 0.
\]

This equation is integrated twice. First integral gives the equation

\[
e^{3\omega/2} \left( \frac{\dot{\omega}^2}{2} - \frac{2}{3} \Lambda \right) + 2e^{\omega/2} \left[ 1 - f^2(r) \right] = F(r),
\]

and the second one gives the equation

\[
\int \frac{de^{\omega/2}}{\sqrt{l^2(r) - 1 + \frac{1}{3} F(r)e^{-\omega/2} + \frac{4}{3} e^\omega}} = t + F(r)
\]

The equations (9) and (10) hold undetermined functions \( F(r) \). The substitution of (3) into (4) together with (8) gives the equation for density

\[
8\pi \rho = \frac{1}{\omega e^{3\omega/2}} \frac{\partial F(r)}{\partial r}
\]

3 The Cauchy Problem for the LTB Model

Before we study the LTB model let us introduce the following characteristic values: a velocity of light \( c \), an observational meaning of the Hubble constant \( \tilde{H} \), a characteristic time \( 1/\tilde{H} \) and characteristic length \( c/\tilde{H} \). We use the co-moving system of coordinates in the LTB model, so the radial coordinate \( r \) has the sense of Lagrangian mass coordinate [18], [17]. Two dimensionless variables \( \mu \) and \( \tau \) are defined by the rules

\[
\mu = \frac{r}{m}, \quad \tau = \tilde{H}t,
\]

where \( m \) is a full mass of the ”gas”. The dimensionless Hubble function \( h(\mu, \tau) \) and density \( \delta(\mu, \tau) \) will be also used:

\[
h(\mu, \tau) = \frac{H(\mu, \tau)}{H}, \quad \delta(\mu, \tau) = \frac{\rho(r, t)}{\rho(0, 0)},
\]

where \( H(0, 0) = \tilde{H} \).

Let us write the interval (1) as

\[
ds^2(r, t) = -\Lambda e^{\lambda(r, t)} dr^2 - B e^{\omega(r, t)} (d\theta^2 + \sin^2 \theta d\phi) + c^2 dt^2,
\]

where two constants \( A \) and \( B \) are introduced to take into account the fact that (11) is dimension equation.

The dimension of \([ds^2]\) is \( L^2 \), dimension of \([A]\) is \( L^2 M^{-2} \) and dimension of \([B]\) is \( L^2 \), so

\[
A = \left( \frac{c}{Hm} \right)^2, \quad B = \left( \frac{c}{H} \right)^2.
\]

The interval (11) has now the form

\[
\left( \frac{\tilde{H}}{c} \right)^2 ds^2(r, t) = -e^{\lambda(r, t)} dr^2 - e^{\omega(r, t)} (d\theta^2 + \sin^2 \theta d\phi) + d\tau^2,
\]

3
Together with metrical functions $\omega(\mu, \tau)$ and $\lambda(\mu, \tau)$, introduced by Tolman, it is conveniently to use the Bonnor’s function

$$R(\mu, \tau) = e^{\omega(\mu, \tau)/2}. \quad (13)$$

In the Bonnor’s notation the interval $(12)$ takes the form

$$\left( \frac{H}{c} \right)^2 ds^2(\mu, \tau) = -\frac{[R'(\mu, \tau)]^2}{f^2}d\mu^2 - R^2(\mu, \tau) \left( d\theta^2 + \sin^2 \theta d\phi \right) + d\tau^2$$

As it is shown in [17] and [18], the Bonnor’s coordinate $R(\mu, \tau)$ has a sense of Euler coordinate, so the equation (13) correlates geometrical radius of the sphere $R(\mu, \tau)$ where the particle is located, and the Lagrangian coordinate $\mu$ of this sphere.

To describe the radial motion we will use the Habble function connected with variation of the radial length $dl$:

$$h = \frac{dl}{d\mu}. \quad (14)$$

where, according to the (12), for $dl^2$ we read:

$$dl^2 = e^{\lambda(\mu, \tau)} d\mu^2 = \left( \frac{R'(\mu, \tau)}{f(\mu)} \right)^2. \quad (15)$$

By the substitution (15) into the definition of the Habble function (14) we obtain

$$h(\mu, \tau) = \frac{\dot{\lambda}(\mu, \tau)}{2} = \frac{\dot{R}'(\mu, \tau)}{R'(\mu, \tau)} = \frac{\partial \ln R'(\mu, \tau)}{\partial \tau}. \quad (16)$$

By the integration of the equation (16) we obtain the formula for metrical function $\lambda(\mu, \tau)$:

$$\lambda(\mu, \tau) = 2 \int_0^\tau h(\mu, \tau) d\tau + \lambda(\mu, 0).$$

A solution in the LTB model is defined by the functions $f(\tau)$, $F(\tau)$ and $\mathbf{F}(\tau)$. These functions are obtained in the process of solution of the system of PDE, so to define them the initial/boundary conditions should definitely be used. The metrical function $\omega(\mu, \tau)$ is the solution of the equation (3). The equations of the LTB model are obtained in [3] and solved in the parametric form for the three cases $f^2(\mu) < 1$, $f^2(\mu) = 1$ and $f^2(\mu) > 1$ in [4] and [5].

The equations (6) - (9) are valid for every $\tau$, and due to this fact in the Cauchy problem they define the functions $f^2(\mu)$, $F(\mu)$ and $\mathbf{F}(\mu)$ at the moment of time $\tau = 0$:

$$f^2(\mu) - 1 = e^{\omega(\mu)} \left( \frac{\dot{\omega}_0(\mu)}{2} + \frac{3}{4} \ddot{\omega}_0(\mu) - \Lambda \right) \quad (17)$$

$$F(\mu) = e^{3\omega(\mu)/2} \left( \frac{\omega^2_0(\mu)}{2} - \frac{2}{3} \Lambda \right) + 2e^{\omega(\mu)/2} \left[ 1 - f^2(\mu) \right] \quad (18)$$

$$\mathbf{F}(\mu) = \int_{e^{\omega(\mu)/2}}^{dx} \sqrt{\frac{f^2(\mu) - 1 + \frac{F(\mu)}{2x} + \frac{4}{3} x^2} {\left( \frac{\dot{\omega}_0(\mu)}{2} + \frac{3}{4} \ddot{\omega}_0(\mu) - \Lambda \right)}} \quad (19)$$

At the time $\tau = 0$ the equation (13) defines the function $\lambda_0(\mu)$:

$$e^{\lambda_0(\mu)} = e^{\omega(\mu)} \left( \frac{\dot{\omega}_0(\mu)}{2} + \frac{3}{4} \ddot{\omega}_0(\mu) - \frac{4}{3} \Lambda \right). \quad (20)$$

Substituting (17) into (18), we obtain

$$F(\mu) = e^{3\omega(\mu)/2} \left( -2\dot{\omega}_0(\mu) - \ddot{\omega}_0(\mu) + \frac{4}{3} \Lambda \right). \quad (21)$$
Comparing (8) - (11) with (17) - (18), we find out that

\[ f^2(\mu) - 1 = e^{\omega_0(\mu)} \left( \ddot{\omega}_0(\mu) + \frac{3}{4} \omega_0^2(\mu) - \Lambda \right) = e^{\omega(\mu, \tau)} \left( \ddot{\omega}(\mu, \tau) + \frac{3}{4} \omega^2(\mu, \tau) - \Lambda \right) \]  

(22)

and

\[ F(\mu) = e^{3\omega_0(\mu)/2} \left( -2\ddot{\omega}_0(\mu) - \dot{\omega}_0^2(\mu) + \frac{4}{3} \Lambda \right) = e^{3\omega(\mu, \tau)/2} \left( -2\ddot{\omega}(\mu, \tau) - \dot{\omega}(\mu, \tau)^2 + \frac{4}{3} \Lambda \right) \]  

(23)

are not dependent on time. Let's use the previous results to calculate the functions \( F(\mu) \) and integral in the equation (13). Substituting the definitions (17) and (21) into (13) we obtain:

\[ F(\mu) = \frac{\omega_0(\mu)}{\omega_0(0)} \int \frac{d\dot{\omega}}{\dot{\omega}} \]  

(24)

The function \( F(\mu) \) is equal to zero at the moment of time \( \tau \equiv 0 \) according the definition. Substituting the right part of the equations (22) and (23) into the (13), we obtain the equation

\[ \pm \int_{\omega(\mu, 0)}^{\omega(\mu, \tau)} \frac{d\dot{\omega}}{\dot{\omega}} = \pm \int_{\omega_0(0)}^{\omega(\mu)} \frac{d\dot{\omega}}{\dot{\omega}} + \tau. \]  

(25)

This analysis of the LTB model shows that the functions

\[ \omega(\mu, \tau) \big|_{\tau=0} = \omega_0(\mu) \quad \dot{\omega}(\mu, \tau) \big|_{\tau=0} = \dot{\omega}_0(\mu) \]  

(26)

and constants

\[ \omega(\mu, 0) \big|_{\mu=0} = \omega_0(0) \quad \dot{\omega}(\mu, 0) \big|_{\mu=0} = \dot{\omega}_0(0) \]  

\[ \ddot{\omega}(\mu, 0) \big|_{\mu=0} = \dot{\omega}_0(0), \quad \Lambda, \]  

(27)

are included into the definitions (17) - (19) and they form the initial conditions of the Cauchy problem for the equations (2) - (3). In accordance with (21) the function \( \lambda_0(\mu) \) is not include in the set of initial conditions.

Substituting (21) into (10), we obtain the general expression for the density of "gas" in the LTB model:

\[ 8\pi\delta(\mu, \tau) = e^{\frac{2}{3} [\omega_0(\mu) - \omega(\mu, \tau)]} \times \]  

\[ \left\{ 3 [\omega_0(\mu)]' \left[ -\ddot{\omega}_0(\mu) - \frac{1}{2} \dot{\omega}_0^2(\mu) + \frac{\Lambda}{6} \right] - 2 [\ddot{\omega}_0(\mu)]' - 2\dot{\omega}_0(\mu) [\dot{\omega}_0(\mu)]' \right\} \]  

(28)

We obtain the formul for Hubble’s function using (3), (13) and (16):

\[ h(\mu, \tau) = \frac{\frac{\partial K(\mu, \tau)}{\partial \mu}}{\frac{\partial}{\partial \mu} \int_0^\tau K(\mu, \tau) d\tau + R'(\mu, 0)} \]  

(29)
where
\[ K(\mu, \tau) = \sqrt{f^2(\mu) - 1 + \frac{F(\mu)}{2R(\mu, \tau)} + \frac{\Lambda}{3} R^3(\mu, \tau)}. \]

The dependence of cosmological parameter on the initial conditions of the LTB model is represented by the follow formula:
\[ \Omega(\mu, \tau) = \frac{\delta(\mu, \tau)}{\delta_c(\mu, \tau)}, \]
where the critical density defined by formulm
\[ \delta_c(\mu, \tau) = \frac{3H^2}{8\pi G \rho_0} h^2(\mu, \tau). \]

The function \( \omega(\mu, \tau) \) from the equation (28) is the solution of the equation (4).

4 Results

The LTB model is the Cauchy problem for the PDE (2) - (5). Traditionally, three functions \( f(\mu) \), \( F(\mu) \) and \( F'(\mu) \) are used to chose some physical propriety of the problem solving in the LTB model. These functions play a role of the initial conditions in the LTB model. But the input equations (2) - (5) are written through the metrical functions \( \lambda(\mu, \tau) \) and \( \omega(\mu, \tau) \). So, should definitly be studied the dependence functions \( f(\mu) \), \( F(\mu) \) and \( F'(\mu) \) on metrical functions. The present article learns this problem and three initial conditions (26) - (27) defined three function by the rules (17) - (19). The general expressions for the density (28) and Habble fuction (29) show the dependence on the initial conditions (26).

5 Acknowledgements

I’m grateful to Prof. Arthur D. Chernin for encouragement and discussion. Dr. Yurij Barishev has initiated my interest to the modern Cosmology as fractal structure of the Universe and interpretation of the observations by calculation redshift in the LTB model. This paper was financially supported by "COSMION" Ltd., Moscow.

References

[1] Lemaître, Ann. de la Soc.Scient. de Bruxelles A53, 51 (1933).
[2] R.C.Tolman, Proc.Nat.Acad.Sci (Wash), 20,(1934).
[3] H.Bondy, MNRAS 107, p.p. 410 - 425 (1947).
[4] W.B.Bonnor, MNRAS 159, 261 - 268 (1972)
[5] W.B.Bonnor, MNRAS 167, 55 (1974).
[6] P.H.Coleman and L.Pietronero, Physics Reports, 213, 6, (1992).
[7] L.Pietronero, Physica, 144A, 257 (1987).
[8] Yu.V.Baryshev, F.Sylos Labini, M.Montuori, and L.Pietronero, Vistas in Astronomy, 38, 4 (1994).

[9] M.B.Ribeiro, ApJ.388,1 (1992).

[10] M.B.Ribeiro, ApJ.395,29 (1992).

[11] M.B.Ribeiro, ApJ.415 (1993).

[12] J.W.Moffat and D.S.Tatarski, Phys.Rev.D 45,10 (1992)

[13] J.W.Moffat and D.S.Tatarski, Proceeding of the XXIXth RENCONTRE DE MORIOND, Series:Moriond Workshops, Villars sur Ollon, Switzerland, January 22-29, 1994. Editions Frontieres.

[14] J.W.Moffat and D.S.Tatarski, preprint UTPT-94-19 (1994).

[15] A.Gromov, astro-ph/9605201 in astro-ph@xxx.lanl.gov

[16] Dingle, Proc.Nat.Acad. 19, 559, (1933).

[17] L.D.Landau, E.M.Lifshits, ”The Field Theory”, Moscow, ”Nauka”, (1973).

[18] L.E.Gurevich and A.D.Chernin, ”The Introduction into Cosmology”, Moscow,”Mir”, (1978).

[19] H.B. Dwight, ”Tables of Integrals”, NY, (1961).

[20] G.A.Corn, T.M.Corn, ”Mathematical Handbook”, McGraw-Hill Company, NY, (1968).