STABILITY OF CRAMER’S CHARACTERIZATION OF NORMAL LAWS IN INFORMATION DISTANCES

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Abstract. Optimal stability estimates in the class of regularized distributions are derived for the characterization of normal laws in Cramer’s theorem with respect to relative entropy and Fisher information distance.

1. Introduction

If the sum of two independent random variables has a nearly normal distribution, then both summands have to be nearly normal. This property is called stability, and it depends on distances used to measure “nearness”. Quantitative forms of this important theorem by P. Lévy are intensively studied in the literature, and we refer to [B-C-G3] for historical discussions and references. Most of the results in this direction describe stability of Cramer’s characterization of the normal laws for distances which are closely connected to weak convergence. On the other hand, there is no stability for strong distances including the total variation and the relative entropy, even in the case where the summands are equally distributed. (Thus, the answer to a conjecture from the 1960’s by McKean [MC] is negative, cf. [B-C-G1-2].) Nevertheless, the stability with respect to the relative entropy can be established for regularized distributions in the model, where a small independent Gaussian noise is added to the summands. Partial results of this kind have been obtained in [B-C-G3], and in this note we introduce and develop new technical tools in order to reach optimal lower bounds for closeness to the class of the normal laws in the sense of relative entropy. Similar bounds are also obtained for the Fisher information distance.

First let us recall basic definitions and notations. If a random variable $X$ with finite second moment has a density $p$, the entropic distance from the distribution $F$ of $X$ to the normal is defined to be

$$D(X) = h(Z) - h(X) = \int_{-\infty}^{\infty} p(x) \log \frac{p(x)}{\varphi_{a,b}(x)} \, dx,$$

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where
\[ \varphi_{a,b}(x) = \frac{1}{b\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2b^2}}, \quad x \in \mathbb{R}, \]
denotes the density of a Gaussian random variable \( Z \sim N(a, b^2) \) with the same mean \( a = \mathbb{E}X = \mathbb{E}Z \) and variance \( b^2 = \text{Var}(X) = \text{Var}(Z) \) as for \( X \) (\( a \in \mathbb{R}, b > 0 \)). Here
\[ h(X) = -\int_{-\infty}^{\infty} p(x) \log p(x) \, dx \]
is the classical Shannon entropy, which is well-defined and is bounded from above by the entropy of \( Z \), so that \( D(X) \geq 0 \). The quantity \( D(X) \) represents the Kullback-Leibler distance from \( F \) to the family of all normal laws on the line; it is affine invariant, and so it does not depend on the mean and variance of \( X \).

One of the fundamental properties of the functional \( h \) is the entropy power inequality
\[ N(X + Y) \geq N(X) + N(Y), \]
which holds for independent random variables \( X \) and \( Y \), where \( N(X) = e^{2h(X)} \) denotes the entropy power (cf. e.g. [D-C-T], [J]). In particular, if \( \text{Var}(X + Y) = 1 \), it yields an upper bound
\[ D(X + Y) \leq \text{Var}(X)D(X) + \text{Var}(Y)D(Y), \quad (1.1) \]
which thus quantifies the closeness to the normal distribution for the sum in terms of closeness to the normal distribution of the summands. The generalized Kac problem addresses (1.1) in the opposite direction: How can one bound the entropic distance \( D(X + Y) \) from below in terms of \( D(X) \) and \( D(Y) \) for sufficiently smooth distributions? To this aim, for a small parameter \( \sigma > 0 \), we consider regularized random variables
\[ X_\sigma = X + \sigma Z, \quad Y_\sigma = Y + \sigma Z', \]
where \( Z, Z' \) are independent standard normal random variables, independent of \( X, Y \). The distributions of \( X_\sigma \) and \( Y_\sigma \) will be called \emph{regularized} as well. Note that the additive white Gaussian noise is a basic statistical model used in information theory to mimic the effect of random processes that occur in nature. In particular, the class of regularized distributions contains a wide class of probability measures on the line which have important applications in statistical theory.

As a main goal, we prove the following reverse of the upper bound (1.1).

**Theorem 1.1.** Let \( X, Y \) be independent random variables with \( \text{Var}(X + Y) = 1 \). Given \( 0 < \sigma \leq 1 \), the regularized random variables \( X_\sigma \) and \( Y_\sigma \) satisfy
\[ D(X_\sigma + Y_\sigma) \geq c_1(\sigma) \left( e^{-c_2(\sigma)/D(X_\sigma)} + e^{-c_2(\sigma)/D(Y_\sigma)} \right), \quad (1.2) \]
where \( c_1(\sigma) = \exp\{c\sigma^{-6} \log \sigma\} \), \( c_2(\sigma) = c\sigma^{-6} \) with an absolute constant \( c > 0 \).

Thus, when \( D(X_\sigma + Y_\sigma) \) is small, the entropic distances \( D(X_\sigma) \) and \( D(Y_\sigma) \) have to be small, as well. In particular, if \( X + Y \) is normal, then both \( X \) and \( Y \) are normal, so we recover Cramer’s theorem. Moreover, the dependence with respect to the couple \((D(X_\sigma), D(Y_\sigma))\) on the right-hand side of (1.2) can be shown to be essentially optimal, as stated in Theorem 1.3 below.
Theorem 1.1 remains valid even in extremal cases where \( D(X) = D(Y) = \infty \) (for example, when both \( X \) and \( Y \) have discrete distributions). However, the value of \( D(X_\sigma) \) for the regularized variables \( X_\sigma \) cannot be arbitrary. However, for the regularized distributions the value of \( D(X_\sigma) \) cannot be arbitrary. Indeed, \( X_\sigma \) has always a bounded density \( p_\sigma(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-x^2/2\sigma^2} \leq \frac{1}{\sigma \sqrt{2\pi}} \), so that \( h(X_\sigma) \geq -\log \frac{1}{\sigma \sqrt{2\pi}} \). This implies an upper bound

\[
D(X_\sigma) \leq \frac{1}{2} \log \frac{e \text{Var}(X_\sigma)}{\sigma^2} \leq \frac{1}{2} \log \frac{2e}{\sigma^2},
\]

describing a general possible degradation of the relative entropy for decreasing \( \sigma \). If \( D_\sigma \equiv D(X_\sigma + Y_\sigma) \) is known to be sufficiently small, say, when \( D_\sigma \leq c_1^2(\sigma) \), the inequality (1.2) provides an additional constraint in terms of \( D_\sigma \), namely,

\[
D(X_\sigma) \leq \frac{c}{\sigma^6 \log(1/D_\sigma)}.
\]

Let us also note that one may reformulate (1.2) as an upper bound for the entropy power \( N(X_\sigma + Y_\sigma) \) in terms of \( N(X_\sigma) \) and \( N(Y_\sigma) \). Such relations, especially those of the linear form

\[
N(X + Y) \leq C(N(X) + N(Y)),
\]

are intensively studied in the literature for various classes of probability distributions under the name “reverse entropy power inequalities”, e.g. [C-Z], [B-M1], [B-M2], [B-N-T]. However, (1.3) cannot be used as a quantitative version of Cramér’s theorem, since it loses information about \( D(X_\sigma + Y_\sigma) \), when \( D(X_\sigma) \) and \( D(Y_\sigma) \) approach zero.

A result similar to Theorem 1.1 also holds for the Fisher information distance, which may be more naturally written in the standardized form

\[
J_{std}(X) = b^2(I(X) - I(Z)) = b^2 \int_{-\infty}^{\infty} \left( \frac{p'(x)}{p(x)} - \frac{\varphi'_a b(x)}{\varphi_a b(x)} \right)^2 p(x) \, dx
\]

with parameters \( a \) and \( b \) as before. Here

\[
I(X) = \int_{-\infty}^{\infty} \frac{p'(x)^2}{p(x)} \, dx,
\]

denotes the Fisher information of \( X \), assuming that the density \( p \) of \( X \) is (locally) absolutely continuous and has a derivative \( p' \) in the sense of Radon-Nikodym. Similarly to \( D \), the standardized Fisher information distance is an affine invariant functional, so that \( J_{std}(\alpha + \beta X) = J_{std}(X) \) for all \( \alpha, \beta \in \mathbb{R}, \beta \neq 0 \). In many applications it is used as a strong measure of \( X \) being non Gaussian. For example, \( J_{std}(X) \) dominates the relative entropy; more precisely, we have

\[
\frac{1}{2} J_{std}(X) \geq D(X).
\]

This relation may be regarded as an information theoretic variant of the logarithmic Sobolev inequality for the Gaussian measure due to Gross. Indeed it may be derived from an isoperimetric inequality for entropies due to Stam (cf. [S], [C], [B-G-R-S]). Moreover,
in [S] Stam established an analog for the entropy power inequality, \( \frac{1}{\mathcal{I}(X+Y)} \geq \frac{1}{\mathcal{I}(X)} + \frac{1}{\mathcal{I}(Y)} \), which implies the following counterpart of the inequality (1.1)

\[
J_{st}(X + Y) \leq \text{Var}(X)J_{st}(X) + \text{Var}(Y)J_{st}(Y),
\]

for any independent random variables \( X \) and \( Y \) with \( \text{Var}(X + Y) = 1 \). We will show that this upper bound can be reversed in a full analogy with (1.2).

**Theorem 1.2.** Under the assumptions of Theorem 1.1,

\[
J_{st}(X_\sigma + Y_\sigma) \geq c_3(\sigma) \left( e^{-c_4(\sigma)/J_{st}(X_\sigma)} + e^{-c_4(\sigma)/J_{st}(Y_\sigma)} \right),
\]

where \( c_3(\sigma) = \exp\{c\sigma^{-6}(\log \sigma)^3\} \), \( c_4(\sigma) = c\sigma^{-6} \) with an absolute constant \( c > 0 \).

Let us also describe in which sense the lower bounds (1.2) and (1.7) may be viewed as optimal.

**Theorem 1.3.** For every \( T \geq 1 \), there exist independent identically distributed random variables \( X = X_T \) and \( Y = Y_T \) with mean zero and variance one, such that \( J_{st}(X_\sigma) \to 0 \) as \( T \to \infty \) for \( 0 < \sigma \leq 1 \) and

\[
D(X_\sigma - Y_\sigma) \leq e^{-c(\sigma)/D(X_\sigma)} + e^{-c(\sigma)/D(Y_\sigma)},
\]

\[
J_{st}(X_\sigma - Y_\sigma) \leq e^{-c(\sigma)/J_{st}(X_\sigma)} + e^{-c(\sigma)/J_{st}(Y_\sigma)}
\]

with some \( c(\sigma) > 0 \) depending on \( \sigma \) only.

The paper is organized as follows. In Section 2 we describe preliminary steps by introducing truncated random variables \( X^* \) and \( Y^* \). Since their characteristic functions represent entire functions, this reduction of Theorems 1.1-1.2 to the case of truncated random variables allows to invoke powerful methods of complex analysis. In Section 3, \( D(X_\sigma) \) is estimated in terms of the entropic distance to the normal distribution for the regularized random variables \( X^*_\sigma \), while an analogous result for the Fisher information distance is obtained in Section 4. In Section 5, the product of the characteristic functions of \( X^* \) and \( Y^* \) is shown to be close to the normal characteristic function in a disk of large radius depending on \( 1/D(X_\sigma + Y_\sigma) \) in the proof of Theorem 1.1 and on \( 1/J_{st}(X_\sigma + Y_\sigma) \) in the proof of Theorem 1.2. In Section 6, we deduce by means of saddle-point methods a special representation for the derivatives of the density of the random variables \( X_\sigma \), which is needed in Sections 7-8. Based on the resulting bounds for the density of \( X^*_\sigma \), we establish the desired upper bounds for \( D(X^*_\sigma) \) and \( J_{st}(X^*_\sigma) \) in Sections 9 and 10, respectively. In Section 11 we construct an example, showing the sharpness of the estimates of Theorems 1.1-1.2.

### 2. Truncated random variables

Turning to the proof of Theorem 1.1, let us fix several standard notations. By

\[
(F * G)(x) = \int_{-\infty}^{\infty} F(x - y) dG(y), \quad x \in \mathbb{R},
\]

we denote the convolution of given distribution functions \( F \) and \( G \). This operation will only be used when \( G = \Phi_b \) is the normal distribution function with mean zero and a
standard deviation $b > 0$. We omit the index in case $b = 1$, so that $\Phi_b(x) = \Phi(x/b)$ and $\varphi_b(x) = \frac{1}{b} \varphi(x/b)$.

The Kolmogorov (uniform) distance between $F$ and $G$ is denoted by

$$\|F - G\| = \sup_{x \in \mathbb{R}} |F(x) - G(x)|,$$

and $\|F - G\|_{TV}$ denotes the total variation distance. In general, $\|F - G\| \leq \frac{1}{2} \|F - G\|_{TV}$, while the well-known Pinsker inequality provides an upper bound for the total variation in terms of the relative entropy. Namely,

$$\|F - G\|_{TV}^2 \leq 2 \int_{-\infty}^{\infty} p(x) \log \frac{p(x)}{q(x)} \, dx,$$

where $F$ and $G$ are assumed to have densities $p$ and $q$, respectively.

In the required inequality (1.2) of Theorem 1.1, we may assume that $X$ and $Y$ have mean zero, and that $D(X_\sigma + Y_\sigma)$ is small. Thus, from now on our basic hypothesis may be stated as

$$D(X_\sigma + Y_\sigma) \leq 2\varepsilon \quad (0 < \varepsilon \leq \varepsilon_0),$$

(2.1)

where $\varepsilon_0$ is a sufficiently small absolute constant. By Pinsker’s inequality, this yields bounds for the total variation and Kolmogorov distances

$$\|F_\sigma \ast G_\sigma - \Phi_{\sqrt{1 + 2\sigma^2}}\| \leq \frac{1}{2} \|F_\sigma \ast G_\sigma - \Phi_{\sqrt{1 + 2\sigma^2}}\|_{TV} \leq \sqrt{\varepsilon} < 1,$$

(2.2)

where $F_\sigma$ and $G_\sigma$ are the distribution functions of $X_\sigma$ and $Y_\sigma$, respectively. Moreover, without loss of generality, one may assume that

$$\sigma^2 \geq \hat{c}(\log \log(1/\varepsilon)/\log(1/\varepsilon))^{1/3}$$

(2.3)

with a sufficiently large absolute constant $\hat{c} > 0$. Indeed if (2.3) does not hold, the statement of the theorem obviously holds.

In the inequality (1.7) of Theorem 1.2, we assume that $X$ and $Y$ have mean zero, and that $J_{st}(X_\sigma + Y_\sigma)$ is small, i.e.,

$$J_{st}(X_\sigma + Y_\sigma) \leq 2\varepsilon \quad (0 < \varepsilon \leq \varepsilon_0),$$

(2.4)

where $\varepsilon_0$ is a sufficiently small absolute constant. Then, according to the Stam inequality (1.5), the relative entropy has to be small as well, namely $D(X_\sigma + Y_\sigma) \leq \varepsilon$.

Moreover, without loss of generality, we assume that

$$\sigma^2 \geq \hat{c} \log \log(1/\varepsilon)/(\log(1/\varepsilon))^{1/3}$$

(2.5)

with a sufficiently large absolute constant $\hat{c} > 0$.

We shall need some auxiliary assertions about truncated random variables. Let $F$ and $G$ be the distribution functions of independent, mean zero random variables $X$ and $Y$ with second moments $\mathbb{E}X^2 = v_1^2$, $\mathbb{E}Y^2 = v_2^2$, such that $\text{Var}(X + Y) = 1$. Put

$$N = N(\varepsilon) = \sqrt{1 + 2\sigma^2} \left(1 + \sqrt{2 \log(1/\varepsilon)}\right)$$

with a fixed parameter $0 < \sigma \leq 1$. 
Introduce truncated random variables at level $N$. Put $X^* = X$ in case $|X| \leq N$, $X^* = 0$ in case $|X| > N$, and similarly $Y^*$ for $Y$. Note that

$$\mathbb{E}X^* \equiv a_1 = \int_{-N}^N x \, dF(x), \quad \text{Var}(X^*) \equiv \sigma_1^2 = \int_{-N}^N x^2 \, dF(x) - a_1^2,$$

$$\mathbb{E}Y^* \equiv a_2 = \int_{-N}^N x \, dG(x), \quad \text{Var}(Y^*) \equiv \sigma_2^2 = \int_{-N}^N x^2 \, dG(x) - a_2^2.$$

By definition, $\sigma_1 \leq v_1$ and $\sigma_2 \leq v_2$. In particular,

$$\sigma_1^2 + \sigma_2^2 \leq v_1^2 + v_2^2 = 1.$$

Denote by $F^*, G^*$ the distribution functions of the truncated random variables $X^*, Y^*$, and respectively by $F^*_\sigma, G^*_\sigma$ the distribution functions of the regularized random variables $X^*_\sigma = X^* + \sigma Z$ and $Y^*_\sigma = Y^* + \sigma Z'$, where $Z, Z'$ are independent standard normal random variables that are independent of $(X, Y)$.

**Lemma 2.1.** With some absolute constant $C$ we have

$$0 \leq 1 - (\sigma_1^2 + \sigma_2^2) \leq CN^2 \sqrt{\epsilon}.$$

For the proof of Lemma 2.1 we use arguments from [B-C-G3]. It will be convenient to divide the proof into several steps.

**Lemma 2.2.** For any $M > 0$,

$$1 - F(M) + F(-M) \leq 2 \left( 1 - F_\sigma(M) + F_\sigma(-M) \right) \leq 4\Phi_{\sqrt{1+2\sigma^2}}(-M-2) + 4\sqrt{\epsilon}.$$

The same inequalities hold true for $G$.

**Proof.** Since both $F_\sigma$ and $G_\sigma$ are continuous functions, one may assume that $\pm M$ are points of continuity for $F$ and $G$. First note that

$$\frac{1}{2} F(-M) = \frac{1}{2} \int_{-\infty}^{-M} dF(u) = \int_{-\infty}^{-M} dF(u) \int_{-\infty}^{0} \varphi_\sigma(s) \, ds \leq \int_{-\infty}^{\infty} dF(u) \int_{-\infty}^{-M} \varphi_\sigma(s-u) \, ds = F_\sigma(-M).$$

In the same way, $\frac{1}{2} (1 - F(M)) \leq 1 - F_\sigma(M)$, thus proving the first inequality of the lemma. For the second one, first note that $\mathbb{E}Y^*_\sigma = v_2^2 + \sigma^2 \leq 2$, so that $\mathbb{P}(|Y_\sigma| \geq 2) \leq \frac{1}{2}$, by Chebyshev’s inequality. Hence,

$$\frac{1}{2} F_\sigma(-M) = \frac{1}{2} \mathbb{P}(X_\sigma \leq -M) \leq \mathbb{P}(X_\sigma \leq -M, |Y_\sigma| \leq 2) \leq \mathbb{P}(X_\sigma + Y_\sigma \leq -M + 2) = (F_\sigma * G_\sigma)(-M + 2) \leq \Phi_{\sqrt{1+2\sigma^2}}(-M + 2) + \sqrt{\epsilon},$$

where we used (2.2) on the last step. By a similar argument,

$$\frac{1}{2} (1 - F_\sigma(M)) \leq \Phi_{\sqrt{1+2\sigma^2}}(-M + 2) + \sqrt{\epsilon},$$
Lemma 2.3. With some positive absolute constant \( C \) we have
\[
\|F^* - F\|_{TV} \leq C\sqrt{\varepsilon}, \quad \|G^* - G\|_{TV} \leq C\sqrt{\varepsilon},
\]
\[
\|F^*_\sigma * G^*_\sigma - \Phi_{1+2\sigma^2}\|_{TV} \leq C\sqrt{\varepsilon}.
\]

Proof. The distribution \( F^* \) of \( X^* \) is supported on the interval \([-N, N]\), where it coincides with \( F \) as a measure up to an atom at zero of size \( P(|X| > N) \). Applying Lemma 2.2 with \( M = N \), we therefore obtain that
\[
\|F^* - F\|_{TV} \leq 2P(|X| > N) \leq 2(F(-N) + (1 - F(N))) \leq 8\Phi_{1+2\sigma^2}(-N + 2) + 8\sqrt{\varepsilon}.
\]
By the choice of the function \( N \), the latter expression does not exceed \( c\sqrt{\varepsilon} \) up to some absolute constant \( c > 0 \), so \( \|F^* - F\|_{TV} \leq c\sqrt{\varepsilon} \). Similarly, we have \( \|G^* - G\|_{TV} \leq c\sqrt{\varepsilon} \).
From this, using the triangle inequality, we conclude that
\[
\|F^*_\sigma * G^*_\sigma - F^*_\sigma * G^*_\sigma\|_{TV} \leq 2c\sqrt{\varepsilon}
\]
and then, by (2.2), \( \|F^*_\sigma * G^*_\sigma - \Phi_{1+2\sigma^2}\|_{TV} \leq (2c + 2)\sqrt{\varepsilon} \). □

Proof of Lemma 2.1. Since \( E(X^*_\sigma + Y^*_\sigma) = E(X^* + Y^*) = a_1 + a_2 \), we have, integrating by parts,
\[
a_1 + a_2 = \int_{-\infty}^{\infty} x d((F^*_\sigma * G^*_\sigma)(x) - \Phi_{1+2\sigma^2}(x))
\]
\[
= -\int_{-\infty}^{\infty} ((F^*_\sigma * G^*_\sigma)(x) - \Phi_{1+2\sigma^2}(x)) \, dx.
\]
The modulus of the last integral does not exceed
\[
\int_{-4N}^{4N} |(F^*_\sigma * G^*_\sigma)(x) - \Phi_{1+2\sigma^2}(x)| \, dx + \int_{|x|>4N} |(F^*_\sigma * G^*_\sigma)(x) - \Phi_{1+2\sigma^2}(x)| \, dx,
\]
where, by Lemma 2.3, the first integral may be bounded by \( 8NC\sqrt{\varepsilon} \). Using the property that \( (F^*_\sigma * G^*_\sigma)(s) = 0 \) for \( s < -2N \) and \( (F^*_\sigma * G^*_\sigma)(s) = 1 \) for \( s > 2N \), the second integral may be bounded by
\[
\int_{|x|>4N} dx \left( \int_{-2N}^{2N} \varphi_{1+2\sigma^2}(x-s) |(F^*_\sigma * G^*_\sigma)(s) - \Phi(s)| \, ds \right)
\]
\[
+ 2 \int_{|s|>2N} \varphi_{1+2\sigma^2}(x-s)(1 - \Phi(|s|)) \, ds \leq \int_{|x|>4N} dx \int_{-2N}^{2N} \varphi_{1+2\sigma^2}(x-s) \, ds
\]
\[
+ 2 \int_{|s|>2N} (1 - \Phi(|s|)) \, ds \int_{|x|>4N} \varphi_{1+2\sigma^2}(x-s) \, dx \leq C\sqrt{\varepsilon}
\]
with some positive absolute constant \( C \). Hence
\[
|a_1 + a_2| \leq C_1 N\sqrt{\varepsilon} \tag{2.6}
\]
with an absolute constant \( C_1 \).
An estimation of the second moment $E((X_\sigma^* + Y_\sigma^*)^2)$ is based on the identity
\[ \int_{-\infty}^{\infty} x^2 d((F_\sigma^* \ast G_\sigma^*)(x) - \Phi_{\sqrt{1+2\sigma^2}}(x)) = -2 \int_{-\infty}^{\infty} x ((F_\sigma^* \ast G_\sigma^*)(x) - \Phi_{\sqrt{1+2\sigma^2}}(x)) dx. \]

Using the previous arguments, we obtain that
\[ \left| \int_{-4N}^{4N} x ((F_\sigma^* \ast G_\sigma^*)(x) - \Phi_{\sqrt{1+2\sigma^2}}(x)) dx \right| \leq 32N^2C\sqrt{\epsilon}, \]

while
\[ \left| \int_{|x|>4N} x ((F_\sigma^* \ast G_\sigma^*)(x) - \Phi_{\sqrt{1+2\sigma^2}}(x)) dx \right| \]
\[ \leq \int_{|x|>4N} |x| dx \left( \int_{-2N}^{2N} \varphi_{\sqrt{2\sigma}}(x-s)|(F_\sigma^* \ast G_\sigma^*)(s) - \Phi(s)| ds \right. \]
\[ + 2 \int_{|s|>2N} \varphi_{\sqrt{2\sigma}}(x-s)(1 - \Phi(|s|)) ds \left. \right) \leq \int_{|x|>4N} |x| dx \int_{-2N}^{2N} \varphi_{\sqrt{2\sigma}}(x-s) ds \]
\[ + 2 \int_{|s|>2N} (1 - \Phi(|s|)) ds \int_{|x|>4N} |x| \varphi_{\sqrt{2\sigma}}(x-s) dx \leq C_1 \sqrt{\epsilon}. \]

Hence, we finally get
\[ |E((X^* + Y^*)^2 - 1)| = |E((X_\sigma^* + Y_\sigma^*)^2 - (1 + 2\sigma^2))| \leq C_2 N^2 \sqrt{\epsilon} \quad (2.7) \]

with some absolute constant $C_2$. The assertion of the lemma follows immediately from (2.6) and (2.7).

**Corollary 2.4.** With some absolute constant $C$, we have
\[ \int_{|x|>N} x^2 dF(x) \leq CN^2\sqrt{\epsilon}, \quad \int_{|x|>2N} x^2 d(F_\sigma(x) + F_\sigma^*(x)) \leq CN^2\sqrt{\epsilon}, \]

and similarly for $G$ replacing $F$.

**Proof.** By the definition of truncated random variables,
\[ v_1^2 = \sigma_1^2 + a_1^2 + \int_{|x|>N} x^2 dF(x), \quad v_2^2 = \sigma_2^2 + a_2^2 + \int_{|x|>N} x^2 dG(x), \]

so that, by Lemma 2.1,
\[ \int_{|x|>N} x^2 d(F(x) + G(x)) \leq 1 - (\sigma_1^2 + \sigma_2^2) \leq CN^2\sqrt{\epsilon}. \]
As for the second integral of the corollary, we have
\[
\int_{|x|>2N} x^2 dF_\sigma(x) = \int_{|x|>2N} x^2 \left[ \int_{-\infty}^\infty \varphi_\sigma(x-s) dF(s) \right] dx \\
= \int_{-\infty}^\infty dF(s) \int_{|x|>2N} x^2 \varphi_\sigma(x-s) dx \\
\leq 2 \int_{-N}^N s^2 dF(s) \int_{|u|>N} \varphi_\sigma(u) du + 2 \int_{|s|>N} s^2 dF(s) \int_{-\infty}^\infty \varphi_\sigma(u) du \\
+ 2 \int_{-N}^N dF(s) \int_{|u|>N} u^2 \varphi_\sigma(u) du + 2 \int_{|s|>N} dF(s) \int_{-\infty}^\infty u^2 \varphi_\sigma(u) du.
\]
It remains to apply the previous step together with the estimate \( \int_{N}^\infty u^2 \varphi_\sigma(u) du \leq c \sigma N e^{-N^2/(2\sigma^2)} \). The same estimate holds for \( \int_{|x|>2N} x^2 dF_\sigma(x) \).

3. Entropic distance to normal laws for regularized random variables

We keep the same notations as in the previous section and use the relations (2.1) and (2.4) when needed. In this section we obtain some results about the regularized random variables \( X_\sigma \) and \( X^*_\sigma \), which also hold for \( Y_\sigma \) and \( Y^*_\sigma \). Denote by \( p_{X_\sigma} \) and \( p_{X^*_\sigma} \) the (smooth positive) densities of \( X_\sigma \) and \( X^*_\sigma \), respectively.

**Lemma 3.1.** With some absolute constant \( C \) we have, for all \( x \in \mathbb{R} \),
\[
|p_{X_\sigma}^{(k)} (x) - p_{X^*_\sigma}^{(k)} (x)| \leq C \sigma^{-2k} \sqrt{\varepsilon}, \quad k = 0, 1, 2.
\]

**Proof.** Write
\[
p_{X_\sigma}(x) = \int_{-\infty}^\infty \varphi_\sigma(x-s) dF(s) = \int_{-N}^N \varphi_\sigma(x-s) dF(s) + \int_{|s|>N} \varphi_\sigma(x-s) dF(s),
\]
\[
p_{X^*_\sigma}(x) = \int_{-\infty}^\infty \varphi_\sigma(x-s) dF^*(s) = \int_{-N}^N \varphi_\sigma(x-s) dF(s) \\
\quad + (1 - F(N) + F((-N)-) \varphi_\sigma(x).
\]
Hence
\[
|p_{X_\sigma}(x) - p_{X^*_\sigma}(x)| \leq \frac{1}{\sqrt{2\pi\sigma}} \left( 1 - F(N) + F(-N) \right).
\]
But, by Lemma 2.2 and recalling the definition of \( N = N(\varepsilon) \), we have
\[
1 - F(N) + F(-N) \leq 2(1 - F_\sigma(N) + F_\sigma(-N)) \leq C \sqrt{\varepsilon}
\]
with some absolute constant \( C \). Therefore, \( |p_{X_\sigma}(x) - p_{X^*_\sigma}| \leq C \sigma^{-1} \sqrt{\varepsilon} \), which is the assertion (3.1) of the lemma in case \( k = 0 \). We obtain (5.1) for \( k = 1, 2 \) in the same way. The lemma is proved.

**Lemma 3.2.** With some absolute constant \( C > 0 \) we have
\[
D(X_\sigma) \leq D(X^*_\sigma) + C \sigma^{-3} N^3 \sqrt{\varepsilon}.
\]
\( (3.2) \)
Proof. In general, if a random variable $U$ has density $u$ with finite variance $b^2$, then, by the very definition,

$$D(U) = \int_{-\infty}^{\infty} u(x) \log u(x) \, dx + \frac{1}{2} \log(2\pi e b^2).$$

Hence,

$$D(X_\sigma) - D(X_\sigma^*) = \int_{-\infty}^{\infty} p_{X_\sigma}(x) \log p_{X_\sigma}(x) \, dx - \int_{-\infty}^{\infty} p_{X_\sigma^*}(x) \log p_{X_\sigma^*}(x) \, dx$$

$$+ \frac{1}{2} \log \frac{\sigma^2}{\sigma_1^2 + \sigma^2} = \int_{-\infty}^{\infty} (p_{X_\sigma}(x) - p_{X_\sigma^*}(x)) \log p_{X_\sigma}(x) \, dx$$

$$+ \int_{-\infty}^{\infty} p_{X_\sigma}(x) \log \frac{p_{X_\sigma}(x)}{p_{X_\sigma^*}(x)} \, dx + \frac{1}{2} \log \frac{\sigma^2}{\sigma_1^2 + \sigma^2}. \quad (3.3)$$

Since $EX^2 \leq 1$, necessarily $F(-2) + 1 - F(2) \leq \frac{1}{2}$, so

$$\frac{1}{2} e^{-\left|\frac{x}{2\sigma}\right|^2/(\frac{3}{2}\sigma^2)} \leq p_{X_\sigma}(x) \leq \frac{1}{\sigma \sqrt{2\pi}}, \quad (3.4)$$

and therefore

$$|\log p_{X_\sigma^*}(x)| \leq C\sigma^{-2}(x^2 + 4), \quad x \in \mathbb{R}, \quad (3.5)$$

with some absolute constant $C$. The same estimate holds for $|\log p_{X_\sigma}(x)|$.

Splitting the integration in

$$I_1 = \int_{-\infty}^{\infty} (p_{X_\sigma}(x) - p_{X_\sigma^*}(x)) \log p_{X_\sigma}(x) \, dx = I_{1,1} + I_{1,2}$$

$$= \left( \int_{|x| \leq 2N} + \int_{|x| > 2N} \right) (p_{X_\sigma}(x) - p_{X_\sigma^*}(x)) \log p_{X_\sigma}(x) \, dx,$$

we now estimate the integrals $I_{1,1}$ and $I_{1,2}$. By Lemma 3.1 and (3.5), we get

$$|I_{1,1}| \leq C' \sigma^{-3} N^3 \sqrt{\varepsilon}$$

with some absolute constant $C'$. Applying (3.5) together with Corollary 2.4, we also have

$$|I_{1,2}| \leq 4C \sigma^{-2} \left( 1 - F_\sigma(2N) + F_\sigma(-2N) + 1 - F_\sigma^*(2N) + F_\sigma^*(-2N) \right)$$

$$+ C \sigma^{-2} \left( \int_{|x| > 2N} x^2 dF_\sigma(x) + \int_{|x| > 2N} x^2 dF_\sigma^*(x) \right) \leq C'' \sigma^{-2} N^2 \sqrt{\varepsilon}.$$

The two bounds yield

$$|I_1| \leq C''' \sigma^{-3} N^3 \sqrt{\varepsilon} \quad (3.6)$$

with some absolute constant $C'''$.

Now consider the integral

$$I_2 = \int_{-\infty}^{\infty} p_{X_\sigma^*}(x) \log \frac{p_{X_\sigma}(x)}{p_{X_\sigma^*}(x)} \, dx = I_{2,1} + I_{2,2}$$

$$= \left( \int_{|x| \leq 2N} + \int_{|x| > 2N} \right) p_{X_\sigma^*}(x) \log \frac{p_{X_\sigma}(x)}{p_{X_\sigma^*}(x)} \, dx,$$
which is non-negative, by Jensen’s inequality. Using \( \log(1 + t) \leq t \) for \( t \geq -1 \), and Lemma 3.1, we obtain

\[
I_{2,1} = \int_{|x| \leq 2N} p_{X^*_t}(x) \log \left( 1 + \frac{p_{X^*_t}(x) - p_{X^*_t}(x)}{p_{X^*_t}(x)} \right) dx \\
\leq \int_{|x| \leq 2N} |p_{X^*_t}(x) - p_{X^*_t}(x)| dx \leq 4C^{-1}N\sqrt{\varepsilon}.
\]

It remains to estimate \( I_{2,2} \). We have as before, using (3.3) and Corollary 2.4,

\[
|I_{2,2}| \leq C \int_{|x| > 2N} p_{X^*_t}(x) \frac{x^2 + 4}{\sigma^2} dx \leq C' \sigma^{-2} N^2 \sqrt{\varepsilon}
\]

with some absolute constant \( C' \). These bounds yield

\[
I_2 \leq C'' \sigma^{-2} N^2 \sqrt{\varepsilon}.
\]

In addition, by Lemma 2.1

\[
\log \frac{\nu^2 + \sigma^2}{\sigma^2} \leq \frac{\nu^2 - \sigma^2}{\sigma^2} \leq C \sigma^{-2} N^2 \sqrt{\varepsilon}.
\]

It remains to combine this bound with (3.6)-(3.7) and apply them in (3.3). \( \square \)

4. Fisher information for regularized random variables

To compare the standardized Fisher information of \( X_\sigma \) and \( X^*_\sigma \), we need the following simple lemmas. These lemmas hold for both random variables \( Y_\sigma \) and \( Y^*_\sigma \).

**Lemma 4.1.** For \( j = 0, 1, 2 \),

\[
p_{X_t}^{(j)}(x)(x^2 + 1) \to 0, \quad p_{X^*_t}^{(j)}(x)(x^2 + 1) \to 0, \quad \text{as} \quad x \to \pm \infty;
\]

\[
\int_{-\infty}^{\infty} |p_{X_t}^{(j)}(x)| (x^2 + 1) dx < \infty, \quad \int_{-\infty}^{\infty} |p_{X^*_t}^{(j)}(x)| (x^2 + 1) dx < \infty.
\]

**Proof.** In view of the formula \( p_{X_t}^{(j)}(x) = \int_{-\infty}^{\infty} \varphi^{(j)}(x-s) dF(s) \), \( x \in \mathbb{R} \), we have the simple upper bound, for all \( x \in \mathbb{R} \),

\[
|p_{X_t}^{(j)}(x)| \leq \sigma^{-2} \left( \int_{|x-s| < |x|/2} + \int_{|x-s| \geq |x|/2} \right) \left( 1 + \frac{(x-s)^2}{\sigma^2} \right) \varphi_\sigma(x-s) dF(s)
\]

\[
\leq C \sigma^{-2} \left( 1 - F(|x|/2) + F(-|x|/2) + \left( 1 + \frac{|x|^2}{2\sigma^2} \right) \varphi_\sigma(|x|/2) \right), \quad j = 0, 1, 2,
\]

with some absolute constant \( C > 0 \). Since \( \int_{-\infty}^{\infty} s^2 dF(s) < \infty \), the first relation of (4.1) follows immediately from this bound. We prove the second relation of (4.1) in the same way.

Let us prove the first inequality of (4.2). We have

\[
\int_{-\infty}^{\infty} |p_{X_t}^{(j)}(x)| (x^2 + 1) dx \leq \int_{-\infty}^{\infty} (x^2 + 1) \left[ \int_{-\infty}^{\infty} |\varphi^{(j)}_\sigma(x-s)| dF(s) \right] dx
\]

\[
\leq 2 \int_{-\infty}^{\infty} s^2 dF(s) \int_{-\infty}^{\infty} |\varphi^{(j)}_\sigma(u)| du + 2 \int_{-\infty}^{\infty} dF(s) \int_{-\infty}^{\infty} (u^2 + 1) |\varphi^{(j)}_\sigma(u)| du < \infty.
\]
The second relation in (4.2) follows similarly. □

**Lemma 4.2.** With some absolute constant $C$

\[
\int_{|x| > 2N} |p''_{X_\sigma}(x)|(x^2 + 1) \, dx \leq C\sigma^{-2}N^2\sqrt{\varepsilon}, \quad \int_{|x| > 2N} |p''_{X^*_\sigma}(x)|(x^2 + 1) \, dx \leq C\sigma^{-2}N^2\sqrt{\varepsilon}.
\]

**Proof.** Note that

\[
\int_{|x| > 2N} |p''_{X_\sigma}(x)|(x^2 + 1) \, dx \leq \int_{-\infty}^{\infty} dF(s) \int_{|x| > 2N} (x^2 + 1)|\varphi^{(2)}(x - s)| \, dx
\]

\[
\leq 2 \int_{-N}^{N} s^2 dF(s) \int_{|u| > N} |\varphi^{(2)}(u)| \, du + 2 \int_{|u| > N} s^2 dF(s) \int_{-\infty}^{\infty} |\varphi^{(2)}(u)| \, du
\]

\[
+ 2 \int_{-N}^{N} dF(s) \int_{|u| > N} (u^2 + 1)|\varphi^{(2)}(u)| \, du + 2 \int_{|u| > N} dF(s) \int_{-\infty}^{\infty} (u^2 + 1)|\varphi^{(2)}(u)| \, du.
\]

(4.3)

Since

\[|\varphi^{(2)}(u)| \leq \frac{1}{\sigma^2} \left(1 + \frac{u^2}{\sigma^2}\right) \varphi(u), \quad u \in \mathbb{R},\]

we get the following estimates

\[
\int_{|u| > N} |\varphi^{(2)}(u)| \, du \leq \frac{1}{\sigma^2} \int_{|u| > N} \left(1 + \frac{u^2}{\sigma^2}\right) \varphi(u) \, du \leq C\frac{N}{\sigma^3} e^{-N^2/(2\sigma^2)} \leq \frac{C}{\sigma^2}\varepsilon,
\]

\[
\int_{|u| > N} (u^2 + 1)|\varphi^{(2)}(u)| \, du \leq \frac{1}{\sigma^2} \int_{|u| > N} \left(1 + \frac{u^2}{\sigma^2}\right)^2 \varphi(u) \, du \leq C\frac{N^3}{\sigma^5} e^{-N^2/(2\sigma^2)} \leq \frac{C}{\sigma^2}N^2\varepsilon.
\]

Applying these upper bounds and Collorary 2.4 to (4.3) we easily obtain the first assertion of the lemma. We may prove the second assertion of the lemma in the same way. □

The next representations are well-known; they are obtained, using Lemma 4.1 and the bound (3.5), via integration by parts in the integral which defines the Fisher information.

**Lemma 4.3.** The following formulas hold

\[
I(X_\sigma) = -\int_{-\infty}^{\infty} p''_{X_\sigma}(x) \log p_{X_\sigma}(x) \, dx, \quad I(X^*_\sigma) = -\int_{-\infty}^{\infty} p''_{X^*_\sigma}(x) \log p_{X^*_\sigma}(x) \, dx.
\]

(4.4)

We are now prepared to prove the following bound.

**Lemma 4.4.** With some absolute constant $C > 0$ we have

\[(v_1^2 + \sigma^2)^{-1}J_{st}(X_\sigma) \leq (\sigma_1^2 + \sigma^2)^{-1}J_{st}(X^*_\sigma) + C\sigma^{-7}N^3\sqrt{\varepsilon}.
\]
Proof. Write
\[ \frac{J_{st}(X_0)}{v_1^2 + \sigma^2} - \frac{J_{st}(X_0^*)}{v_1^2 + \sigma^2} = - \int_{-\infty}^{\infty} p''_{X_0}(x) \log p_{X_0}(x) \, dx + \int_{-\infty}^{\infty} p''_{X_0^*}(x) \log p_{X_0^*}(x) \, dx \]
\[+ \frac{v_1^2 - \sigma_1^2}{(v_1^2 + \sigma^2)(\sigma_1^2 + \sigma^2)} = - \int_{-\infty}^{\infty} (p''_{X_0}(x) - p''_{X_0^*}(x)) \log p_{X_0}(x) \, dx \]
\[+ \int_{-\infty}^{\infty} \frac{p''_{X_0^*}(x)}{p_{X_0^*}(x)} \log \frac{p_{X_0}(x)}{p_{X_0^*}(x)} \, dx + \frac{v_1^2 - \sigma_1^2}{(v_1^2 + \sigma^2)(\sigma_1^2 + \sigma^2)}. \] (4.5)

Splitting integration in the first integral on the right-hand side of (4.5),
\[ J_1 = \int_{-\infty}^{\infty} (p''_{X_0}(x) - p''_{X_0^*}(x)) \log p_{X_0}(x) \, dx = J_{1,1} + J_{1,2} \]
\[= \left( \int_{|x|\leq 2N} + \int_{|x|>2N} \right) (p''_{X_0}(x) - p''_{X_0^*}(x)) \log p_{X_0}(x) \, dx, \]
we now estimate the integrals \( J_{1,1} \) and \( J_{1,2} \).

By Lemma 3.1 and (3.5),
\[ |J_{1,1}| \leq C'\sigma^{-7}N^2\sqrt{\varepsilon} \] (4.6)
with some absolute constant \( C' \), while, by (3.5) and Lemma 4.2,
\[ |J_{1,2}| \leq C'\sigma^{-4}N^2\sqrt{\varepsilon}. \] (4.7)

Now consider the integral
\[ J_2 = \int_{-\infty}^{\infty} p''_{X_0^*}(x) \log \frac{p_{X_0}(x)}{p_{X_0^*}(x)} \, dx = J_{2,1} + J_{2,2} \]
\[= \left( \int_{|x|\leq 2N} + \int_{|x|>2N} \right) p''_{X_0^*}(x) \log \frac{p_{X_0}(x)}{p_{X_0^*}(x)} \, dx. \]

It is easy to verify that
\[ p''_{X_0^*}(x) = -\sigma^{-2}p_{X_0^*}(x) + \sigma^{-4}\int_{-\infty}^{\infty} (x-s)^2 \varphi(x-s) \, dF^*(s). \]

Therefore, for \( x \in [-2N, 2N] \),
\[ |p''_{X_0^*}(x)| \leq \sigma^{-2}(1 + 9\sigma^{-2}N^2) p_{X_0^*}(x), \]
which leads to the upper bound
\[ |J_{2,1}| \leq \sigma^{-2}(1 + 9\sigma^{-2}N^2) \left( \int_{E_+} p_{X_0^*}(x) \log \frac{p_{X_0^*}(x)}{p_{X_0}(x)} \, dx + \int_{E_-} p_{X_0^*}(x) \log \frac{p_{X_0}(x)}{p_{X_0^*}(x)} \, dx \right), \] (4.8)
where \( E_+ = \{ x \in [-2N, 2N] : p_{X_0}(x) > p_{X_0^*}(x) \} \) and \( E_- = [-2N, 2N] \setminus E_+ \). As in the proof of the estimate on \( I_{2,1} \) in the previous section, we obtain
\[ \int_{E_+} p_{X_0^*}(x) \log \frac{p_{X_0}(x)}{p_{X_0^*}(x)} \, dx \leq 4C\sigma^{-1}N\sqrt{\varepsilon}. \] (4.9)
From (3.4) and from the definition of $p_{X^*}(x)$ and $p_{Y^*}(x)$, we see that, for $x \in E_-$,
\[
\frac{1}{10} p_{X^*}(x) \geq (1 - F(N) + F((-N)-)) \varphi_\sigma(x) \geq \int_{|s| \geq N} \varphi_\sigma(x - s) \, dF(s)
\]
and we have $p_{X^*}(x)/p_{X^*}(x) \leq 2$ for $x \in E_-$. Therefore, as above, we get
\[
\int_{E_-} p_{X^*}(x) \log \frac{p_{X^*}(x)}{p_{X^*}(x)} \, dx \leq 2 \int_{E_-} p_{X^*}(x) \log \frac{p_{X^*}(x)}{p_{X^*}(x)} \, dx \leq 8C\sigma^{-1} N \sqrt{\varepsilon}. \tag{4.10}
\]
Applying (4.9) and (4.10) to (4.8), we finally obtain
\[
|J_{2,1}| \leq C'\sigma^{-5} N^3 \sqrt{\varepsilon} \tag{4.11}
\]
with some absolute constant $C'$.

It remains to estimate $J_{2,2}$. We have, using (3.5) and Lemma 4.2
\[
|J_{2,2}| \leq C \int_{|x| > 2N} p_{X^*}''(x)(x^2 + e^2) \sigma^{-2} \, dx \leq C'' \sigma^{-4} N^2 \sqrt{\varepsilon} \tag{4.12}
\]
with some absolute constant $C''$. Applying Lemma 2.1 (4.6)–(4.7) and (4.11)–(4.12) in the representation (4.3), we arrive at the assertion of the lemma. \hfill \square

5. Characteristic functions of truncated random variables

Denote by $f_{X^*}(t)$ and $f_{Y^*}(t)$ the characteristic functions of the random variables $X^*$ and $Y^*$, respectively. As integrals over finite intervals they admit analytic continuations as entire functions to the whole complex plane $\mathbb{C}$. These continuations will be denoted by $f_{X^*}(t)$ and $f_{Y^*}(t)$, $(t \in \mathbb{C})$.

Put $T = \frac{1}{64} N = \frac{1}{64} \sqrt{1 + 2\sigma^2 \left(1 + \sqrt{2 \log \frac{1}{\varepsilon}}\right)}$. We assume without loss of generality that $0 < \varepsilon \leq \varepsilon_0$, where $\varepsilon_0$ is a sufficiently small absolute constant.

**Lemma 5.1.** For all $t \in \mathbb{C}$, $|t| \leq T$,
\[
\frac{1}{2} |e^{-t^2/2}| \leq |f_{X^*}(t)| |f_{Y^*}(t)| \leq \frac{3}{2} |e^{-t^2/2}|. \tag{5.1}
\]

**Proof.** For all complex $t$,
\[
\left| \int_{-\infty}^{\infty} e^{itx} \, d(F^*_\sigma * G^*_\sigma)(x) - \int_{-\infty}^{\infty} e^{itx} \, d(\Phi_{\sqrt{1 + 2\sigma^2}})(x) \right| \leq \left| \int_{-4N}^{4N} e^{itx} \, d(F^*_\sigma * G^*_\sigma - \Phi_{\sqrt{1 + 2\sigma^2}})(x) \right|
\]
\[\quad + \int_{|x| \geq 4N} e^{-x \Im(t)} \, d(F^*_\sigma * G^*_\sigma)(x) + \int_{|x| \geq 4N} e^{-x \Im(t)} \varphi_{\sqrt{1 + 2\sigma^2}}(x) \, dx. \tag{5.2}\]

Integrating by parts, we have
\[
\int_{-4N}^{4N} e^{itx} \, d(F^*_\sigma * G^*_\sigma - \Phi_{\sqrt{1 + 2\sigma^2}})(x) = e^{4itN} (F^*_\sigma * G^*_\sigma - \Phi_{\sqrt{1 + 2\sigma^2}})(4N)
\]
\[\quad - e^{-4itN} (F^*_\sigma * G^*_\sigma - \Phi_{\sqrt{1 + 2\sigma^2}})(-4N) - it \int_{-4N}^{4N} (F^*_\sigma * G^*_\sigma - \Phi_{\sqrt{1 + 2\sigma^2}})(x) \, e^{itx} \, dx.
\]
In view of the choice of $T$ and $N$, we easily obtain, using Lemma 2.3 for all $|t| \leq T$, 
\[
\left| \int_{-4N}^{4N} e^{itx} d(F_{\sigma}^*G_{\sigma}^* - \Phi_{\sqrt{1+2\sigma^2}})(x) \right| \leq 2C \sqrt{\varepsilon} e^{4N|\text{Im}(t)|} + 8C |t| \sqrt{\varepsilon} e^{4N|\text{Im}(t)|} \leq \frac{1}{6} e^{-(1/2+\sigma^2)T^2}. \tag{5.3}
\]

The second integral on the right-hand side of (5.2) does not exceed, for $|t| \leq T$,
\[
\left(5.4\right)
\]
\[
\int_{-2N}^{2N} d(F^* G^*)(s) \int_{|x|\geq 4N} e^{-x \text{Im}(t)} \varphi_{\sqrt{2\sigma}}(x-s) \, dx \leq \int_{-2N}^{2N} e^{-u \text{Im}(t)} d(F^* G^*)(s) \cdot \int_{|u|\geq 2N} e^{-u \text{Im}(t)} \varphi_{\sqrt{2\sigma}}(u) \, du \leq e^{2NT} \cdot \frac{1}{\sqrt{\pi}} \int_{2N/\sigma}^{\infty} e^{\sigma^2 T u - u^2/4} \, du \leq \frac{1}{6} e^{-(1/2+\sigma^2)T^2}. \tag{5.4}
\]

The third integral on the right-hand side of (5.2) does not exceed, for $|t| \leq T$,
\[
\left(5.5\right)
\]
\[
\sqrt{\frac{2}{\pi}} \int_{4N}^{\infty} e^{T u - u^2/6} \, du \leq \frac{1}{6} e^{-(1/2+\sigma^2)T^2}. \tag{5.5}
\]

Applying (5.3)-(5.5) in (5.2), we arrive at the upper bound
\[
\left| e^{-\sigma^2 t^2/2} f_X^*(t) e^{-\sigma^2 t^2/2} f_Y^*(t) - e^{-(1/2+\sigma^2)t^2} \right| \leq \frac{1}{2} e^{-(1/2+\sigma^2)T^2} \leq \frac{1}{2} \left| e^{-(1/2+\sigma^2)t^2} \right| \tag{5.6}
\]
from which (5.1) follows. \hfill \Box

The bounds in (5.1) show that the characteristic function $f_X^*(t)$ does not vanish in the circle $|t| \leq T$. Hence, using results from ([L-O], pp. 260–266), we conclude that $f_X^*(t)$ has a representation
\[
\left(5.7\right)
\]
\[
f_X^*(t) = \exp\{g_X^*(t)\}, \quad g_X^*(0) = 0,
\]
where $g_X^*(t)$ is analytic on the circle $|t| \leq T$ and admits the representation
\[
\left(5.8\right)
\]
\[
g_X^*(t) = i a_1 t - \frac{1}{2} \sigma_1^2 t^2 - \frac{1}{2} t^2 \psi_X^*(t),
\]
where
\[
\psi_X^*(t) = \sum_{k=3}^{\infty} t^k c_k \left( \frac{t}{T} \right)^{k-2}
\]
with real-valued coefficients $c_k$ such that $|c_k| \leq C$ for some absolute constant $C$. In the sequel without loss of generality we assume that $a_1 = 0$. An analogous representation holds for the function $f_Y^*(t)$.

6. Derivatives of the density of the random variable $X_\sigma^*$

We shall use the following inversion formula
\[
\left(5.9\right)
\]
\[
P_{X_\sigma^*}^{(k)}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-it)^k e^{-itx} e^{-\sigma^2 t^2/2} f_X^*(t) \, dt, \quad k = 0, 1, \ldots, \quad x \in \mathbb{R},
\]
for the derivatives of the density $p_{X^2}(x)$. By Cauchy’s theorem, one may change the path of integration in this integral from the real line to any line $z = t + iy$, $t \in \mathbb{R}$, with parameter $y \in \mathbb{R}$. This results in the following representation

$$p^{(k)}_{X^2}(x) = e^{y \cdot x} e^{\sigma^2 y^2 / 2} f_{X^*}(iy) \cdot I_k(x, y), \quad x \in \mathbb{R}. \tag{6.1}$$

Here

$$I_k(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-i(t + iy))^k R(t, x, y) \, dt, \tag{6.2}$$

where

$$R(t, x, y) = f_{X^*}(t + iy) e^{-it(x + \sigma^2 y) - \sigma^2 t^2 / 2} / f_{X^*}(iy). \tag{6.3}$$

Let us now describe the choice of the parameter $y \in \mathbb{R}$ in (6.1). It is well-known, that the function $\log f_{X^*}(iy), y \in \mathbb{R}$, is convex. Therefore, the function $\frac{d}{dy} \log f_{X^*}(iy) + \sigma^2 y$ is strictly monotone and tends to $-\infty$ as $y \to -\infty$ and tends to $\infty$ as $y \to \infty$. By (5.7) and (5.8), this function is vanishing at zero. Hence, the equation

$$\frac{d}{dy} \log f_{X^*}(iy) + \sigma^2 y = -x \tag{6.4}$$

has a unique continuous solution $y = y(x)$ such that $y(x) < 0$ for $x > 0$ and $y(x) > 0$ for $x < 0$. Here and in the sequel we use the principal branch of log $z$.

We shall need one representation of the solution $y(x)$ in the interval $x \in [-\sigma_1^2 + \sigma^2)T_1, (\sigma_1^2 + \sigma^2)T_1]$, where $T_1 = c'(\sigma_1^2 + \sigma^2)T$ with a sufficiently small absolute constant $c' > 0$. We see that

$$q_{X^*}(t) \equiv \frac{d}{dt} \log f_{X^*}(t) - \sigma^2 t = -(\sigma_1^2 + \sigma^2)t - r_1(t) - r_2(t)$$

$$= -(\sigma_1^2 + \sigma^2)t - t\psi_{X^*}(t) - \frac{1}{2} t^2 \psi'_{X^*}(t). \tag{6.5}$$

The functions $r_1(t)$ and $r_2(t)$ are analytic in the circle $\{ |t| \leq T/2 \}$ and there, by (5.8), they may be bounded as follows

$$|r_1(t)| + |r_2(t)| \leq C|t|^2/T \tag{6.6}$$

with some absolute constant $C$. Using (6.5), (6.6) and Rouché’s theorem, we conclude that the function $q_{X^*}(t)$ is univalent in the circle $D = \{ |t| \leq T_1 \}$, and $q_{X^*}(D) \supset \frac{1}{2} (\sigma_1^2 + \sigma^2)D$. By the well-known inverse function theorem (see [S-G], pp. 159-160), we have

$$q_{X^*}^{-1}(w) = b_1 w + ib_2 w^2 - b_3 w^3 + \ldots, \quad w \in \frac{1}{2} (\sigma_1^2 + \sigma^2)D, \tag{6.7}$$

where

$$i^{n-1} b_n = \frac{1}{2\pi i} \int_{|\zeta| = \frac{1}{2} T_1} \frac{\zeta \cdot q'_{X^*} (\zeta)}{q_{X^*} (\zeta)^{n+1}} \, d\zeta, \quad n = 1, 2, \ldots. \tag{6.8}$$

Using this formula and (6.5)–(6.6), we note that

$$b_1 = -\frac{1}{\sigma_1^2 + \sigma^2} \tag{6.9}$$
and that all remaining coefficients $b_2, b_3, \ldots$ are real-valued. In addition, by (6.5)–(6.6),

$$-rac{q_X(t)}{(\sigma_1^2 + \sigma^2)t} = 1 + q_1(t) \quad \text{and} \quad -\frac{q'_X(t)}{\sigma_1^2 + \sigma^2} = 1 + q_2(t),$$

where $q_1(t)$ and $q_2(t)$ are analytic functions in $D$ satisfying there $|q_1(t)| + |q_2(t)| \leq \frac{1}{2}$. Therefore, for $\zeta \in D$,

$$\frac{q'_X(\zeta)}{q_X(\zeta)^{n+1}} = (-1)^n \frac{q_3(\zeta)}{(\sigma_1^2 + \sigma^2)^n \zeta^{n+1}} = (-1)^n \frac{1 + q_2(\zeta)}{(\sigma_1^2 + \sigma^2)^n (1 + q_1(\zeta))^{n+1} \zeta^{n+1}},$$

where $q_3(\zeta)$ is an analytic function in $D$ such that $|q_3(\zeta)| \leq 3 \cdot 2^n$. Hence, $q_3(\zeta)$ admits the representation

$$q_3(\zeta) = 1 + \sum_{k=1}^{\infty} d_k \frac{\zeta^k}{T_k^k},$$

with coefficients $d_k$ such that $|d_k| \leq 3 \cdot 2^n$. Using this equality, we obtain from (6.8) that

$$b_n = \frac{d_{n-1}}{(\sigma_1^2 + \sigma^2)^n T_1^{n-1}} \quad \text{and} \quad |b_n| \leq \frac{3 \cdot 2^n}{(\sigma_1^2 + \sigma^2)^n T_1^{n-1}}, \quad n = 2, \ldots.$$  \hfill (6.10)

Now we can conclude from (6.7) and (6.10) that, for $|x| \leq T_1/(4|b_1|),$

$$y(x) = -i q_X^{-1}(ix) = b_1 x - b_2 x^2 + R(x), \quad \text{where} \quad |R(x)| \leq 48 |b_1|^3 |x|^3 / T_1^2. \hfill (6.11)$$

In the sequel we denote by $\theta$ a real-valued quantity such that $|\theta| \leq 1$. Using (6.11), let us prove:

**Lemma 6.1.** In the interval $|x| \leq c'' T_1/|b_1|$ with a sufficiently small positive absolute constant $c''$,

$$y(x)x + \frac{1}{2} \sigma^2 y(x)^2 + \log f_X(iy(x)) = \frac{1}{2} b_1 x^2 + \frac{c_3 b_1^3}{2T} x^3 + \frac{c \theta b_1^5}{T^2} x^4,$$  \hfill (6.12)

where $c$ is an absolute constant.

**Proof.** From (6.10) and (6.11), it follows that

$$\frac{1}{2} |b_1 x| \leq |y(x)| \leq \frac{3}{2} |b_1 x|. \hfill (6.13)$$

Therefore,

$$\frac{1}{2} y(x)^2 \sum_{k=1}^{\infty} |c_k| \left(\frac{|y(x)|}{T}\right)^{k-2} \leq C \left(\frac{3}{2}\right)^4 b_1^4 x^4 / T^2.$$

On the other hand, with the help of (6.10) and (6.11) one can easily deduce the relation

$$y(x)x + \frac{1}{2} (\sigma^2 + \sigma^2) y(x)^2 + \frac{1}{2} c_3 y(x)^3 / T = \frac{1}{2} b_1 x^2 + \frac{1}{2} c_3 b_1^3 x^3 / T + \frac{c \theta b_1^5}{T^2} x^4,$$

with some absolute constant $c$. The assertion of the lemma follows immediately from the two last relations.  \hfill $\square$
Now, applying Lemma 6.1 to (6.1), we may conclude that in the interval $|x| \leq c''T_1/|b_1|$, the derivative $p_{X_2}^{(k)}(x)\right.$ admits the representation

$$p_{X_2}^{(k)}(x) = \exp \left\{ \frac{1}{2} b_1 x^2 + \frac{1}{2} c_3 b_1^3 \frac{x^3}{T} + \frac{c \theta b_1^5}{T^2} x^4 \right\} \cdot I_k(x, y(x)) \tag{6.14}$$

with some absolute constant $c$.

As for the values $|x| > c''T_1/|b_1|$, in (6.1) we choose $y = y(x) = y(c''T_1/|b_1|)$ for $x > 0$ and $y = y(x) = y(-c''T_1/|b_1|)$ for $x < 0$. In this case, by (6.13), we note that $|y| \leq 3c''T_1/2$, and we have

$$\frac{1}{2} \sigma^2 y^2 + \log f_{X\ast}(iy) \leq \frac{y^2}{2|b_1|} + \frac{C}{2} \frac{|y|^3}{T} \sum_{k=3}^{\infty} \left( \frac{|y|}{T} \right)^{k-3} \leq \frac{1}{2} |y| \left( \frac{3}{2} |x| + \frac{1}{4} |x| \right) \leq \frac{7}{8} |yx|.$$  

As a result, we obtain from (6.1) an upper bound $|p_{X_2}^{(k)}(x)| \leq e^{-\frac{1}{2}|b(x)|} |I_k(x, y(x))|$ for $|x| > c''T_1/|b_1|$, which with the help of left-hand side of (6.13) yields the estimate

$$|p_{X_2}^{(k)}(x)| \leq e^{-c'T|x|/|b_1|} |I_k(x, y(x))|, \quad |x| > c''T_1/|b_1|, \tag{6.15}$$

with some absolute constant $c > 0$.

7. The estimate of the integral $I_0(x, y)$

In order to study the behavior of the integrals $I_k(x, y)$, we need some auxiliary results. We use the letter $c$ to denote absolute constants which may vary from place to place.

**Lemma 7.1.** For $t, y \in [-T/4, T/4]$ and $x \in \mathbb{R}$, we have the relation

$$\log |R(t, x, y)| = -\gamma(y)t^2/2 + r_1(t, y), \tag{7.1}$$

where

$$\gamma(y) = |b_1|^{-1} + \psi_{X\ast}(iy) + 2iy \psi_{X\ast}'(iy) \tag{7.2}$$

and

$$|r_1(t, y)| \leq c t^2 (t^2 + y^2)^{T^2} \quad \text{with some absolute constant } c. \tag{7.3}$$

**Proof.** From the definition of the function $R(t, x, y)$ it follows that

$$\log |R(t, x, y)| = \frac{1}{2} \left( b_1 - \psi_{X\ast}(iy) - 2iy \psi_{X\ast}'(iy) \right) t^2 - \frac{1}{2} (\text{Im} \psi_{X\ast}(t + iy) - \psi_{X\ast}(iy)) (t^2 - y^2)$$

$$+ (\text{Im} \psi_{X\ast}(t + iy) + it \psi_{X\ast}'(iy)) ty. \tag{7.4}$$

Since, for $t, y \in [-T/4, T/4]$ and $k = 4, \ldots,$

$$|\Re(\psi^k(t + iy)^{k-2} - \psi^k(iy)^{k-2})| = \left| \sum_{l=0}^{(k-2)/2} (-1)^{k+1+l} \binom{k-2}{2l} t^{2l} y^{k-2-2l} - (-1)^{k+1} y^{k-2} \right|$$

$$\leq t^2 (T/4)^{k-4} \sum_{l=1}^{(k-2)/2} \binom{k-2}{2l} \leq 4t^2 (T/2)^{k-4},$$

...
we obtain an upper bound, for the same \( t \) and \( y \), namely

\[
|\Re \psi_X(t + iy) - \psi_X(iy)| \leq \sum_{k=4}^{\infty} \frac{|c_k|}{T^{k-2}} |\Re(i^k(t + iy)^{k-2} - i^k(iy)^{k-2})| \leq \frac{23Ct^2}{T^2}.
\] (7.5)

Since, for \( t, y \in [-T/4, T/4] \) and \( k = 5, \ldots, \)

\[
|\Im(i^k(t + iy)^{k-2} - i^k(k-2)t(iy)^{k-3})| = \left| \sum_{l=1}^{(k-3)/2} \left( \frac{k-2}{2l+1} \right) (-1)^{k+l} t^{2l+1} y^{k-3-2l} \right|
\leq |t|^3 (T/4)^{k-5} \sum_{l=1}^{(k-3)/2} \left( \frac{k-2}{2l+1} \right) \leq 8|t|^3 (T/2)^{k-5},
\]

we have

\[
|\Im \psi_X(t + iy) + it\psi_X'(iy)| \leq \sum_{k=5}^{\infty} \frac{|c_k|}{T^{k-2}} |\Im(i^k(t + iy)^{k-2} - ti^k(k-2)(iy)^{k-3})| \leq \frac{24C|t|^3}{T^3}
\] (7.6)

for the same \( t \) and \( y \). Applying (7.5) and (7.6) in (7.4), we obtain the assertion of the lemma.

**Lemma 7.2.** For \( |t| \leq c''T/\sqrt{|b_1|} \) and \( |y| \leq c''T/|b_1| \), we have the estimates

\[
\frac{3}{4|b_1|} \leq \gamma(y) \leq \frac{5}{4|b_1|}
\] (7.7)

and

\[
|r_1(t, y)| \leq l^2/(8|b_1|).
\] (7.8)

**Proof.** Recall that the positive absolute constant \( c'' \) is chosen to be sufficiently small. Using the following simple bounds

\[
|\psi_X(iy)| \leq \sum_{k=3}^{\infty} |c_k| \left( \frac{|y|}{T} \right)^{k-2} \leq C \left( \frac{|y|}{T} \right) \sum_{k=3}^{\infty} \left( \frac{c''}{|b_1|} \right)^{k-3} \leq \frac{1}{8|b_1|},
\] (7.9)

\[
2|y\psi_X'(iy)| \leq \frac{2|y|}{T} \sum_{k=3}^{\infty} |c_k|(k-2) \left( \frac{|y|}{T} \right)^{k-3} \leq C \frac{2|y|}{T} \sum_{k=3}^{\infty} (k-2) \left( \frac{c''}{|b_1|} \right)^{k-3} \leq \frac{1}{8|b_1|}.
\] (7.10)

we easily obtain

\[
\frac{3}{4|b_1|} \leq \frac{1}{|b_1|} - |\psi_X(iy)| - 2|y\psi_X'(iy)| \leq \gamma(y)
\leq \frac{1}{|b_1|} + |\psi_X(iy)| + 2|y\psi_X'(iy)| \leq \frac{5}{4|b_1|},
\]

and thus (7.7) is proved. The bound (7.8) follows immediately from (7.3).

**Lemma 7.3.** For \( t \in [-T/4, T/4] \) and \( x \in [-c''T_1/|b_1|, c''T_1/|b_1|] \), we have

\[
\Im \log R(t, x, y(x)) = \frac{i}{2} t^3 \psi_X'(iy(x)) + r_2(t, x),
\] (7.11)
where
\[ |r_2(t, x)| \leq c(|t| + |y(x)|)|t|^3T^{-2} \quad \text{with some absolute constant } c. \quad (7.12) \]

Proof. Write, for \( t, y \in [-T/4, T/4] \) and \( x \in \mathbb{R} \),
\[ \text{Im} \log R(t, y, x) = -tx + \frac{1}{b_1}ty - ty \Re \psi_X \cdot (t + iy) - \frac{1}{2}(t^2 - y^2) \text{Im} \psi_X \cdot (t + iy). \quad (7.13) \]

Now we choose in this formula \( y = y(x) \), where \( y(x) \) is a solution of the equation of (6.4) for \( x \in [-c''T_1/|b_1|, c''T_1/|b_1|] \). For such \( x \), in view of (6.13), we know that \( |y(x)| \leq T/4 \).

Let us rewrite (6.4) (see as well (6.5)) in the form
\[ -\frac{1}{b_1}y(x) + y(x) \psi_X \cdot (iy(x)) + \frac{i}{2}y^2 \psi_X' \cdot (iy(x)) = -x. \]

Applying this relation in (7.13), we obtain the formula
\[ \text{Im} \log R(t, x, y(x)) = -ty(x)(\Re \psi_X \cdot (t + iy(x)) - \psi_X \cdot (iy(x))) + \frac{i}{2}t^3 \psi_X' \cdot (iy(x)) \]
\[ -\frac{1}{2}(t^2 - y(x)^2)(\text{Im} \psi_X \cdot (t + iy(x)) + it \psi_X' \cdot (iy(x))). \]

In view of (7.5) and (7.6), we can conclude that
\[ \text{Im} \log R(t, x, y(x)) = \frac{i}{2}t^3 \psi_X' \cdot (iy(x)) + r_2(t, x), \]
where
\[ |r_2(t, x)| \leq 8C|t^3|y(x)|T^{-2} + 8C|t^3(t^2 + y(x)^2)T^{-3} \leq 16C(|t| + |y(x)|)|t|^3T^{-2} \]
for \( |t| \leq T/4 \) and \( |y(x)| \leq T/4 \). Thus, the lemma is proved. \( \square \)

Our next step is to estimate the integrals
\[ I_p(x) \equiv \Re \int_{\mathbb{R}} (it)^p R(t, x, y(x)) \, dt, \quad p = 0, 1, 2. \]
To this aim, we need the following lemmas.

Lemma 7.4. For \( p = 0, 2, \)
\[ (-1)^{p/2}I_p(x) = \frac{p!}{2^{p/2}(p/2)!} \frac{\sqrt{2\pi}}{\gamma(y(x))(p+1)/2} + r_p(x), \quad |x| \leq c''T_1/|b_1|, \]
where
\[ |r_p(x)| \leq c(|b_1|^{7/2} + |b_1|^{(p+3)/2}y(x)^2)T^{-2} \quad (7.14) \]
with some absolute constants \( c \).

Proof. For short we write \( y \) in place of \( y(x) \). Put \( T_2 = c''T/\sqrt{|b_1|} \) and write
\[ \int_{-\infty}^{\infty} t^p \Re R(t, x, y) \, dt = I_{p1} + I_{p2} = \left( \int_{-T_2}^{T_2} + \int_{|t| \geq T_2} \right) t^p \Re R(t, x, y) \, dt. \]
First consider the integral $I_{p_1}$. We have

$$I_{p_1} = I_{p_1,1} - I_{p_1,2} = \int_{-T_2}^{T_2} t^p |R(t, x, y)| \, dt - 2 \int_{-T_2}^{T_2} t^p |R(t, x, y)| \sin^2 \left( \frac{1}{2} \text{Im} \log R(t, x, y) \right) \, dt.$$

By (7.1), we see that

$$I_{p_1,1} = \int_{-T_2}^{T_2} t^p e^{-\frac{\gamma(y)}{2} t^2} \, dt + \int_{-T_2}^{T_2} t^p e^{-\frac{\gamma(y)}{2} t^2} (e^{r_1(t, y)} - 1) \, dt.$$

Using the inequality $|e^z - 1| \leq |z| |e^z|$, $z \in \mathbb{C}$, and applying Lemma 7.1 together with (7.3), (7.7), (7.8), we have

$$\left| \int_{-T_2}^{T_2} t^p e^{-\frac{\gamma(y)}{2} t^2} (e^{r_1(t, y)} - 1) \, dt \right| \leq \int_{-T_2}^{T_2} t^p e^{-\frac{\gamma(y)}{2} t^2} |r_1(t, y)| |e^{r_1(t, y)}| \, dt$$

$$\leq \int_{-T_2}^{T_2} t^p e^{-\frac{1}{\pi y_1} t^2} |r_1(t, y)| \, dt \leq c \int_{-T_2}^{T_2} t^{p+2} e^{-\frac{1}{\pi y_1} t^2 + y^2 / T^2} \, dt$$

$$\leq c |b_1|^{(p+3)/2} (|b_1| + y^2) T^{-2}. \quad (7.15)$$

On the other hand

$$\int_{-T_2}^{T_2} t^p e^{-\frac{\gamma(y)}{2} t^2} \, dt = \frac{\sqrt{2\pi} p!}{2^{p/2} (p/2)! \gamma(y)^{(p+1)/2}} - \int_{|t| \geq T_2} t^p e^{-\frac{\gamma(y)}{2} t^2} \, dt, \quad (7.16)$$

where, by (7.7) and by the assumption (2.3) in the proof of Theorem 2.1 and by the assumption (2.5) in the proof of Theorem 2.2,

$$\int_{|t| \geq T_2} t^p e^{-\frac{\gamma(y)}{2} t^2} \, dt \leq \frac{c T_2^{p-1}}{\gamma(y)^{(p+1)/2}} e^{-\frac{1}{4} (T_2 \sqrt{\gamma(y)})^2} \leq c |b_1|^{(3-p)/2} T^{p-1} e^{-\frac{1}{4} \gamma(y) T^2} \leq c T^{-4}. \quad (7.17)$$

Therefore in view of (7.15)–(7.17), we deduce

$$I_{p_1,1} = \frac{T_2^{p-1}}{2^{p/2} (p/2)! \gamma(y)^{(p+1)/2}} + c |b_1|^{(p+3)/2} (|b_1| + y^2) T^2. \quad (7.18)$$

Now let us turn to the integral $I_{p_1,2}$. By (7.11), we have

$$|I_{p_1,2}| \leq \frac{1}{2} \int_{-T_2}^{T_2} |R(t, x, y)| (\text{Im} \log R(t, x, y))^2 \, dt$$

$$\leq 2 \int_{-T_2}^{T_2} |R(t, x, y)| (t^6 |y|^2 |X^* \cdot (iy)|^2 + |r_2(t, x)^2|) \, dt.$$

By Lemmas (7.1) (7.3) and by the assumptions (2.3) and (2.5), and by (7.10), we arrive at the upper bound

$$|I_{p_1,2}| \leq c \int_{-\infty}^{\infty} t^6 \left( \frac{t^2 + y^2}{T^2} + 1 \right) e^{-\frac{1}{\pi y_1} t^2} \, dt \leq \frac{c}{T^2} |b_1|^{7/2} \left( \frac{|b_1| + y^2}{T^2} + 1 \right) \leq \frac{c |b_1|^{7/2}}{T^2}. \quad (7.19)$$
It remains to estimate the integral $I_{p2}$. By (2.3) in the proof of Theorem 2.1 and by (2.5) in the proof of Theorem 2.2,

$$|I_{p2}| \leq 2 \int_{T_2}^{\infty} t^p |R(t, x, y)| dt \leq 2 \int_{T_2}^{\infty} t^p e^{-\frac{t^2}{2}} dt$$

$$\leq 2 \int_{e^{-\sigma T}}^{\infty} t^p e^{-\frac{t^2}{2}} dt \leq c e^{p-3T} p^{-1} e^{-(c')^2 \sigma^4 T^2} \leq cT^{-4}. \quad (7.20)$$

The assertion of the lemma follows from (7.18)–(7.20).

Let us return to the definition of the integrals $I_k(x, y(x))$, $k = 0, 1, 2$, see (6.2). We note that $I_0(x, y(x)) = \frac{1}{2\pi} I_0(x)$ and, by Lemma 7.4 for $|x| \leq c''T_1/|b_1|$,

$$I_0(x, y(x)) = \frac{1}{\sqrt{2\pi \gamma(y(x))}} + \frac{1}{2\pi} r_0(x), \quad (7.21)$$

where $r_0(x)$ satisfies the inequality (7.14).

Since for $|x| > c''T_1/|b_1|$ we choose $y(x) = y(\pm c''T_1/|b_1|)$ and since $|y(x)| \leq c''T/|b_1|$ for such $x$, we obtain, using Lemmas 7.1 and 7.2 and the assumptions (2.3) or (2.5), that

$$|I_k(x, y(x))| \leq \int_{|t| \leq T_2} |t|^k |R(t, x, y(x))| dt + \int_{|t| > T_2} |t|^k |R(t, x, y(x))| dt$$

$$\leq \int_{-\infty}^{\infty} |t|^k e^{-\frac{t^2}{4|b_1|}} dt + \int_{|t| > T_2} |t|^k e^{-\frac{t^2x^2}{2}} dt \leq c \left( |b_1|^{\frac{k+1}{2}} + T_2^{k-1} \sigma^{-2} e^{-\frac{x^2y^2}{2}} \right) \leq c |b_1|^{\frac{k+1}{2}} \quad (7.22)$$

with some absolute constant $c$. The bound (7.22) holds for $|x| \leq c''T_1/|b_1|$ as well. Thus (7.22) is valid for all real $x$.

The relations (7.21) and (7.22) allow us to control the behaviour of the integral $I_0(x, y(x))$.

8. Estimation of the Integrals $I_1(x, y(x))$ and $I_2(x, y(x))$

In Section 8 we assume that (2.5) holds. As before we use the letter $c$ to denote absolute constants which may vary from place to place.

In order to get estimates on $I_1(x, y(x))$ and $I_2(x, y(x))$, which would be similar to (7.21) and (7.22), we need to bound the integral $I_1(x) = -\int_\mathbb{R} t \text{Im} R(t, x, y(x)) dt$. Let us prove the following lemma.

Lemma 8.1. For $|x| \leq c''T_1/|b_1|$,

$$- I_1(x) = 3\sqrt{\pi} \gamma(y(x))^{-5/2} i\psi_1''(ix(y)) + r_1(x),$$

where

$$|r_1(x)| \leq c |b_1|^{7/2} (|b_1|^{1/2} + |y(x)|) T^{-2} \quad (8.1)$$

with some absolute constant $c$. 

Proof. Put $T_3 = c'' T^{1/3}$ and rewrite $I_1(x)$ in the form

$$-I_1(x) = I_{11} + I_{12} \equiv \left( \int_{-T_3}^{T_3} + \int_{|t|>T_3} \right) t \Im R(t, x, y(x)) \, dt. \tag{8.2}$$

Below in the proof of this lemma we simply write $y$ instead of $y(x)$. For $|t| \leq T_3$, by Lemma 7.3,

$$\sin(\Im \log R(t, x, y)) = \sin \left( \frac{i}{2} t^3 \psi_X^\prime(iy) \right) - 2 \sin \left( \frac{i}{2} t^3 \psi_X^\prime(iy) \right) \sin^2 \left( \frac{1}{2} r_2(t, x) \right)$$

$$+ \cos \left( \frac{i}{2} t^3 \psi_X^\prime(iy) \right) \sin(r_2(t, x)).$$

Using $|\sin x| \leq |x|$ and $|\sin x - x| \leq \frac{1}{6}|x|^3$ for $x \in \mathbb{R}$, we have

$$\sin(\Im \log R(t, x, y)) = \frac{i}{2} t^3 \psi_X^\prime(iy) + r_{1,1}(t, x),$$

where

$$|r_{1,1}(t, x)| \leq \frac{1}{48} |t|^9 |\psi_X^\prime(iy)|^3 + |r_2(t, x)| \left( \frac{1}{4} |r_2(t, x)||t^3 \psi_X^\prime(iy)| + 1 \right). \tag{8.3}$$

Therefore, one can rewrite the integral $I_{11}$ in the form

$$I_{11} = \frac{i}{2} \psi_X^\prime(iy) \int_{-T_3}^{T_3} t^4 |R(t, x, y)| \, dt + \int_{-T_3}^{T_3} t |R(t, x, y)| r_{1,1}(t, x) \, dt, \tag{8.4}$$

where the second integral on the right-hand side of (8.4) does not exceed, by (8.3) and by (7.10) and (7.12), the quantity

$$c \int_{-T_3}^{T_3} \left[ \frac{10}{T^3} + \frac{(|t|+|y|)t^4}{T^2} \left( \frac{(|t|+|y|)|t|^3}{T^3} + 1 \right) \right] |R(t, x, y)| \, dt$$

with some absolute constant $c$. We see, by Lemmas 7.1 and 7.2 and by (2.5), that

$$|R(t, x, y)| \leq e^{-\frac{3}{8} |t|^2} = e^{-\frac{3}{8} |y|^2} \quad \text{for} \quad |t| \leq T_3 \leq c'' T / \sqrt{|b_1|}, \quad |y| \leq c'' T / |b_1|, \tag{8.5}$$

so, using again (2.5), the above integral does not exceed

$$\frac{c}{T^2} \left( |b_1|^{5/2} (|b_1|^{1/2} + |y|) + \frac{|b_1|^{11/2}}{T} + \frac{|b_1|^{1}(|b_1| + y^2)}{T^3} \right) \leq \frac{c}{T^2} |b_1|^{7/2} (|b_1|^{1/2} + |y|). \tag{8.6}$$

Repeating the arguments which we used in the proof of (7.18), we easily obtain that the first summand on the right-hand side of (8.4) is equal to

$$3 \sqrt{\frac{\pi}{2}} \gamma(y)^{-5/2} i \psi_X^\prime(iy) + c \theta |\psi_X^\prime(iy)| \frac{|b_1|^{7/2} (|b_1| + y^2)}{T^2}$$

$$= 3 \sqrt{\frac{\pi}{2}} \gamma(y)^{-5/2} i \psi_X^\prime(iy) + c \theta \frac{|b_1|^{7/2} (|b_1| + y^2)}{T^3}. \tag{8.7}$$
It remains to estimate the integral $I_{12}$ similarly as in the proof of (7.20). Namely, using the assumption (2.5), we obtain that

$$|I_{12}| \leq 2 \int_{T_3}^{\infty} t|R(t, x, y)| dt \leq 2 \int_{T_3}^{\infty} t e^{-t^2/2} dt = \frac{2}{\sigma^2} e^{-t^2/2} = \frac{2}{\sigma^2} e^{-t^2/(\sigma c)^2} \leq \frac{1}{T^4}.$$  

(8.8)

Taking into account (2.5), we see that the assertion of the lemma follows immediately from (8.6)–(8.8).

Recalling the definition of the integrals $I_1(x, y(x))$ and $I_2(x, y(x))$, we see that

$$I_1(x, y(x)) = \frac{1}{2\pi} (y(x)I_0(x) - I_1(x)),$$

$$I_2(x, y(x)) = \frac{1}{2\pi} (y(x)^2 I_0(x) - 2y(x)I_1(x) + I_2(x)).$$  

(8.9)

Using Lemma 7.4 and Lemma 8.1, we conclude from that, for $|x| \leq c''T_1/|b_1|$, $I_2(x, y(x)) = \frac{1}{\sqrt{2\pi \gamma(y(x))}} (y(x)^2 + \frac{3y(x)}{\gamma(y(x))^2} iy'_{X^*}(iy(x))) - \frac{1}{\gamma(y(x))}$$

$$+ \frac{1}{2\pi} (r_0(x)y^2(x) + 2r_1(x)y(x) - r_2(x)),$$

(8.10)

where $r_0(x), r_2(x)$ and $r_1(x)$ admit the upper bounds (7.14), (8.1), respectively.

9. END OF THE PROOF OF THEOREM 1.1

Starting from the hypothesis (2.1), we need to derive a good upper bound for $D(X_\sigma)$, which is equivalent to bounding the relative entropy $D(X^*_\sigma)$, according to Lemma 3.2.

This will be done with the help of the relations (6.14), (6.15) and (7.21), (7.22) for the density $p_{X^*_\sigma}(x)$ of the random variable $X^*_\sigma$. First, let us prove the following lemma.

**Lemma 9.1.** For $|x| \leq c''T_1/|b_1|$,  

$$\log \frac{p_{X^*_\sigma}(x)}{\psi_1(x)} = \frac{c_3}{2T} \left((b_1x)^3 + 3b_1y(x)\right) + \tilde{r}(x),$$

where with some absolute constant $c$

$$|\tilde{r}(x)| \leq \frac{c}{T^2} (b_1^2y(x)^2 + |b_1|^3 + |b_1|^5x^4).$$  

(9.1)

**Proof.** By (6.14) and (7.21), we have, for $|x| \leq c''T_1/|b_1|$,  

$$\log \frac{p_{X^*_\sigma}(x)}{\psi_1(x)} = \frac{1}{2} c_3b_1^3 \frac{x^3}{T} + \frac{cbb_1^2}{T^2} x^4 - \frac{1}{2} \log(|b_1|\gamma(y(x)) + \log \left(1 + \frac{\gamma(y(x))}{2\pi} r_0(x)\right).$$

(9.2)

Recalling (7.2) and (5.8), we see that  

$$|b_1|\gamma(y(x)) = 1 + |b_1|(|\psi_{X^*}(iy(x)) + 2iy(x)\psi'_{X^*}(iy(x))) = 1 + 3c_3|b_1|y(x)T^{-1} + \rho_1(x),$$

(9.3)

where

$$\rho_1(x) \equiv |b_1| \sum_{k=4}^{\infty} i^k(2k - 3) c_k \left(\frac{iy(x)}{T}\right)^{k-2}.$$
It is easy to see that
\[ |\rho_1(x)| \leq 8C|b_1| \left( \frac{y(x)}{T} \right)^2 \leq \frac{1}{4}, \quad (9.4) \]

Since \[ \frac{3c_3|b_1y(x)|}{T} \leq \frac{1}{4}, \] and using \[ |\log(1 + u) - u| \leq u^2 \ (|u| \leq 1/2), \] we get from \[ (9.3) \] that
\[ \log(|b_1| \gamma(y(x))) = \frac{3c_3|b_1y(x)|}{T} + c\theta \left( \frac{b_1y(x)}{T} \right)^2 \quad (9.5) \]
with some absolute constant \( c \). Now we conclude from \[ (7.7) \] and \[ (7.14) \] that
\[ \sqrt{\frac{\gamma(y(x))}{2\pi}} |r_0(x)| \leq c |b_1| \frac{b_1^2 + y(x)^2}{T^2} \leq \frac{1}{4} \]
and arrive as before at the upper bound
\[ |\log \left( 1 + \sqrt{\frac{\gamma(y(x))}{2\pi}} r_0(x) \right)| \leq c |b_1| \frac{b_1^2 + y(x)^2}{T^2}. \quad (9.6) \]

Applying \[ (9.5) \] and \[ (9.6) \] to \[ (9.2) \], we obtain the assertion of the lemma.  \( \square \)

To estimate the quantity \( D(X_{\sigma}^*) \), we represent it as

\[ D(X_{\sigma}^*) = J_1 + J_2 = \left( \int_{-c''T_1/|b_1|}^{c''T_1/|b_1|} + \int_{|x| > c''T_1/|b_1|} \right) p_{X_{\sigma}^*}(x) \log \frac{p_{X_{\sigma}^*}(x)}{\varphi \sqrt{1/|b_1|}}(x) \ dx. \quad (9.7) \]

First let us estimate \( J_1 \). Using the letters \( c, C' \) to denote absolute positive constants which may vary from place to place. By Lemma \[ 9.1 \]
\[ J_1 = \frac{c_3}{T} J_{1,1} + \frac{c_3}{2T} \int_{-c''T_1/|b_1|}^{c''T_1/|b_1|} p_{X_{\sigma}^*}(x) \left( (b_1 x)^3 + 3b_1 y(x) \right) \ dx + \int_{-c''T_1/|b_1|}^{c''T_1/|b_1|} p_{X_{\sigma}^*}(x) \tilde{r}(x) \ dx. \quad (9.8) \]

Using \[ (6.14) \] and \[ (7.21) \], we note that
\[ \int_{-c''T_1/|b_1|}^{c''T_1/|b_1|} x^3 p_{X_{\sigma}^*}(x) \ dx = \int_{-c''T_1/|b_1|}^{c''T_1/|b_1|} x^3 \left( p_{X_{\sigma}^*}(x) - \varphi \sqrt{1/|b_1|}(x) \right) \ dx = J_{1,1,1} + J_{1,1,2} + J_{1,1,3} \]
\[ = \int_{-c''T_1/|b_1|}^{c''T_1/|b_1|} x^3 \varphi \sqrt{1/|b_1|}(x) \left( \frac{1}{\sqrt{\gamma(y(x))}|b_1|} - 1 \right) e^{c_3b_1^3x^3/(2T) + c\theta b_1^4x^4/T^2} \ dx \]
\[ + \int_{-c''T_1/|b_1|}^{c''T_1/|b_1|} x^3 \varphi \sqrt{1/|b_1|}(x) \left( e^{c_3\theta b_1^3x^3/(2T) + c\theta b_1^4x^4/T^2} - 1 \right) \ dx \]
\[ + \frac{1}{\sqrt{2\pi |b_1|}} \int_{-c''T_1/|b_1|}^{c''T_1/|b_1|} x^3 \varphi \sqrt{1/|b_1|}(x) e^{c_3b_1^3x^3/(2T) + c\theta b_1^4x^4/T^2} r_0(x) \ dx. \]

It is easy to see that
\[ \frac{|c_3| |b_1|^3 |x|^3}{2T} + \frac{c|b_1|^5 x^4}{T^2} \leq \frac{|b_1|x^2}{4} \quad \text{for} \quad |x| \leq \frac{c''T_1}{|b_1|}. \quad (9.9) \]
With the help of (9.3)–(9.4) and using the bound \(|(1 + u)^{-1/2} - 1| \leq |u|, |u| \leq \frac{1}{2}\), we get
\[ |(\gamma(x)|b_1|)^{-1/2} - 1| \leq c \frac{|b_1| |y(x)|}{T}. \]

The last estimates and (6.13) lead to
\[
|J_{1,1,1}| \leq \frac{c|b_1|}{T} \int_{-c''T_1/b_1}^{c''T_1/b_1} x^3 |y(x)| \sqrt{|b_1|} e^{-|b_1|x^2/4} \, dx \leq \frac{c b_1^{5/2}}{T} \int_{-\infty}^{\infty} x^4 e^{-|b_1|x^2/4} \, dx \leq \frac{c}{T}. \tag{9.10}
\]

Applying \(|e^u - 1| \leq |u|e^{|u|}\), we have, for \(|x| \leq c'T_1/|b_1|\),
\[
|e^{cT_1^3/(2T^2) + c\theta b_1^2 x^2/2T^2} - 1| \leq c|b_1|^3 |x|^3 \left( \frac{1}{2T} + \frac{b_1^2 |x|}{T^2} \right) e^{b_1|x|^2/4}.
\]

Therefore, we deduce the estimate
\[
|J_{1,1,2}| \leq c|b_1|^{7/2} \int_{-\infty}^{\infty} x^6 \left( \frac{1}{T} + \frac{b_1^2 |x|}{T^2} \right) e^{-b_1|x|^2/4} \, dx \leq c \left( \frac{1}{T} + \frac{|b_1|^{3/2}}{T^2} \right). \tag{9.11}
\]

By (6.13) and (7.14), we immediately get
\[
|J_{1,1,3}| \leq cT^{-2} \int_{-c''T_1/b_1}^{c''T_1/b_1} |x|^3 \left(|b_1|^{7/2} + |b_1|^{3/2} y(x)^2\right) e^{-b_1|x|^2/4} \, dx \leq c|b_1|^{3/2} T^{-2}. \tag{9.12}
\]

Hence, by (9.10)–(9.12) and (2.3),
\[
\left| \int_{-c''T_1/b_1}^{c''T_1/b_1} x^3 p_{X^*_2}(x) \, dx \right| \leq c \left( \frac{1}{T} + \frac{|b_1|^{3/2}}{T^2} \right) \leq \frac{c}{T}. \tag{9.13}
\]

In the same way,
\[
\left| \int_{-c''T_1/b_1}^{c''T_1/b_1} x p_{X^*_2}(x) \, dx \right| \leq c \left( \frac{|b_1|}{T} + \frac{|b_1|^{5/2}}{T^2} \right) \leq c \frac{|b_1|}{T}. \tag{9.14}
\]

Recalling (6.11), we see that \(y(x) = b_1 x + c\theta b_1^2 x^2 / T_1\). As a result, using (9.13)–(9.14) and the property \(\text{Var}(X) \leq 1\), we come to the upper bound
\[
|J_{1,1,1}| \leq c |b_1|^3 T^{-1}. \tag{9.15}
\]

In order to estimate \(J_{1,2}\), we employ the inequality (9.4). Recalling (6.14), (7.22) and (9.15), we then have
\[
|J_{1,2}| \leq \frac{c}{T^2} \int_{-c''T_1/b_1}^{c''T_1/b_1} \left( b_1^2 y(x)^2 + |b_1|^3 + |b_1|^{5/2} x^4 \right) \sqrt{|b_1|} e^{-|b_1|x^2/4} \, dx \leq c|b_1|^3 T^{-2}. \tag{9.16}
\]

Combining (9.13) and (9.16), we arrive at
\[
|J_1| \leq c|b_1|^3 T^{-2}. \tag{9.17}
\]

Let us estimate \(J_2\). From (6.15), (7.22), we have, for all \(|x| > c''T_1/|b_1|\),
\[
px_2(x) \leq C' \sqrt{|b_1|} e^{-cT|x|/|b_1|} \leq C' \sqrt{|b_1|} e^{-cc''T^2/|b_1|^3} < 1. \tag{9.18}
\]
Here we also used (2.3) and the assumption that $0 < \varepsilon \leq \varepsilon_0$, where $\varepsilon_0$ is a sufficiently small absolute constant. Using (9.18) and (2.3), we easily obtain

$$J_2 \leq - \int_{|x| > c' T_1 / |b_1|} p_{X^*_2}(x) \log \varphi \sqrt{1 / b_1}(x) \, dx = \frac{1}{2} \log \frac{2\pi}{|b_1|} \int_{|x| > c' T_1 / |b_1|} p_{X^*_2}(x) \, dx$$

$$+ \frac{|b_1|}{2} \int_{|x| > c' T_1 / |b_1|} x^2 p_{X^*_2}(x) \, dx \leq C' \sqrt{|b_1|} \int_{|x| > c' T_1 / |b_1|} \frac{1}{2} (\log(4\pi) + |b_1| x^2) e^{-c' |x| / |b_1|} \, dx$$

$$\leq C' (|b_1|^{3/2} T^{-1} + |b_1|^{3/2} T) e^{-c' c'' T^2 / |b_1|^3} \leq C' T^{-2}. \quad (9.19)$$

Thus, we derive from (9.17) and (9.19) the inequality $D(X^*_\sigma) \leq c |b_1|^3 T^{-2}$. Recalling (3.2) and Lemma 2.1, we finally conclude, using (2.3), that

$$D(X_\sigma) \leq c |b_1|^3 T^{-2} + c \left( \frac{N}{\sigma} \right)^3 \sqrt{\varepsilon} \leq \frac{c}{(v^2 + \sigma^2)^3 T^2} + c \left( \frac{N}{\sigma} \right)^3 \sqrt{\varepsilon} \leq \frac{c}{(v^2 + \sigma^2)^3 T^2}. \quad (9.20)$$

An analogous inequality also holds for the random variable $Y_\sigma$, and so Theorem 1.1 follows from these estimates.

**Remark 9.2.** Under the assumptions of Theorem 1.1, a stronger inequality than (1.2) follows from (9.20), namely

$$D(X_\sigma + Y_\sigma) \geq e^{c \sigma^{-6} \log \sigma} \left[ \exp \left\{ - \frac{c}{(\text{Var}(X_\sigma))^3 D(X_\sigma)} \right\} + \exp \left\{ - \frac{c}{(\text{Var}(Y_\sigma))^3 D(Y_\sigma)} \right\} \right]$$

with some positive absolute constant $c$.

10. **Proof of Theorem 1.2**

The proof of Theorem 1.2 is very similar to the proof of Theorem 1.1. Therefore, keeping the same notations, we omit some routine calculations.

Let $J_{st}(X_\sigma + Y_\sigma) \leq \varepsilon < 1$. Then, by (10.1), $D(X_\sigma + Y_\sigma) \leq \frac{1}{2} \varepsilon$, so that one can use the previous arguments of the proof of Theorem 1.1. Without loss of generality we assume that (2.25) holds. As before, we use the letter $c$ to denote absolute constants which may vary from place to place.

To derive an upper bound for the standardized Fisher information $J_{st}(X^*_\sigma)$, we represent it as

$$\frac{J_{st}(X^*_\sigma)}{(\sigma^2 + \sigma^2)} = J_1 + J_2 = \left( - \int_{c' T_1 / |b_1|}^{c T_1 / |b_1|} - \int_{|x| > c' T_1 / |b_1|} \right) p_{X^*_2}(x) \log \frac{p_{X^*_2}(x)}{\varphi \sqrt{1 / |b_1|}(x)} \, dx, \quad (10.1)$$

which is analogous to (9.7).

Now, using (7.10), (7.14) and (8.1), rewrite (8.10) in the form, for $|x| \leq c' T_1 / |b_1|,$

$$I_2(x, y(x)) = Q_1(x) + \frac{c\theta}{T} Q_2(x) + \frac{c\theta}{T^2} Q_3(x) \equiv \frac{1}{\sqrt{2\pi \gamma(y(x))}} (y(x)^2 - \gamma(y(x))^{-1})$$

$$+ \frac{c\theta}{T} \gamma(y(x))^{5/2} + \frac{c\theta}{T^2} \left[ |b_1|^{7/2}(y(x)^2 + 1 + |b_1|^{1/2} |y(x)|) + |b_1|^{3/2} |y(x)|^4 \right]. \quad (10.2)$$
Recalling (6.11), we see that $y(x) = b_1 x + \frac{eb_1^3 x^2}{T}$ and $y(x)^2 = b_1^2 x^2 + \frac{eb_1^5 x^3}{T}$. Therefore from these relations and from (9.3)–(9.4), we deduce the representation

$$Q_1(x) = \sqrt{\frac{|b_1|}{2\pi}} |b_1|(|b_1| x^2 - 1) + \frac{c\theta}{T} |b_1| |x|(|b_1| x^2 + 1).$$  

(10.3)

By Lemma 9.1, we obtain an analog of (9.8),

$$J_1 = -\frac{c_3}{T} J_{1,1} - J_{1,2}$$

$$= \frac{c_3}{2T} \int_{-\epsilon_{T_1}/|b_1|}^{\epsilon_{T_1}/|b_1|} p_{X^\sigma}(x) \left( (b_1 x)^3 + 3b_1 x(y(x)) \right) dx + \int_{-\epsilon_{T_1}/|b_1|}^{\epsilon_{T_1}/|b_1|} p_{X^\sigma}(x) \tilde{r}(x) dx.

Put

$$J_{1,1} = J_{1,1,1} + J_{1,1,2}$$

$$= -\frac{1}{2} \int_{-\epsilon_{T_1}/|b_1|}^{\epsilon_{T_1}/|b_1|} p_{X^2}(x) \left( (b_1 x)^3 + 3b_1^2 x \right) dx - \frac{cb_1^4}{T} \int_{-\epsilon_{T_1}/|b_1|}^{\epsilon_{T_1}/|b_1|} p_{X^2}(x) \theta x^2 dx.$$

Taking into account (6.14), (7.22) and (9.9), we obtain

$$|J_{1,1,2}| \leq c \frac{|b_1|^{11/2}}{T} \int_{-\epsilon_{T_1}/|b_1|}^{\epsilon_{T_1}/|b_1|} x^2 e^{-|b_1| x^2/4} dx \leq c b_1^4.$$  

(10.4)

Consider the integral

$$\tilde{J}_1 = \frac{1}{2} \int_{-\epsilon_{T_1}/|b_1|}^{\epsilon_{T_1}/|b_1|} |b_1|(|b_1| x^2 - 1)((b_1 x)^3 + 3b_1^2 x) \varphi \frac{1}{\sqrt{|b_1|}}(x) e^{c b_1^3 x^3 + e b_1^5 x^4} dx.$$  

(10.5)

Using (2.5), we obtain, for $k = 0, 1, \ldots$,

$$\left| \int_{-\epsilon_{T_1}/|b_1|}^{\epsilon_{T_1}/|b_1|} x^{2k+1} \varphi \frac{1}{\sqrt{|b_1|}}(x) e^{c b_1^3 x^3 + e b_1^5 x^4} dx \right|$$

$$= \left| \int_{-\epsilon_{T_1}/|b_1|}^{\epsilon_{T_1}/|b_1|} x^{2k+1} \varphi \frac{1}{\sqrt{|b_1|}}(x) \left( e^{c b_1^3 x^3 + e b_1^5 x^4} - 1 \right) dx \right|$$

$$\leq c |b_1|^{1/2} \int_{-\epsilon_{T_1}/|b_1|}^{\epsilon_{T_1}/|b_1|} |x|^{2k+1} (|b_1| x)^3 e^{-|b_1| x^2/4} dx$$

$$\leq c(k) |b_1|^{1-k} (T^{-1} + |b_1|^{3/2} T^{n-2}) \leq c(k) |b_1|^{1-k} T^{-1}$$

with some constants $c(k)$ depending on $k$ only. We easily conclude from this estimate that

$$|\tilde{J}_1| \leq c b_1^4 T^{-1}.$$  

(10.6)

For the integral

$$\tilde{J}_2 = \int_{-\epsilon_{T_1}/|b_1|}^{\epsilon_{T_1}/|b_1|} \theta \frac{T}{\theta} |b_1|^3 x |(|b_1| x^2 + 1)((b_1 x)^3 + 3b_1^2 x) \varphi \frac{1}{\sqrt{|b_1|}}(x) e^{c b_1^3 x^3 + e b_1^5 x^4} dx$$
we have a straightforward upper bound

\[ |\bar{J}_2| \leq \frac{c b_1^{11/2}}{T} \int_{-\infty}^{\infty} x^2(|b_1| x^2 + 1)^2 e^{-|b_1| x^2/4} \, dx \leq \frac{c b_1^4}{T}. \]  

(10.7)

From (10.6), (10.7) we deduce the estimate

\[ \frac{1}{|b_1|} \left| \int_{cTr_1/|b_1|}^{\infty} Q_1(x) ((b_1 x)^3 + 3b_1^2x) \phi(x) e^{c b_1^4 x^3 + c b_1^2 x^4} \, dx \right| \leq \frac{c b_1^4}{T}. \]  

(10.8)

In the same way,

\[ \frac{1}{|b_1|} \left| \int_{-cTr_1/|b_1|}^{-\infty} Q_2(x)(((b_1 x)^3 + 3b_1^2x) \phi(x) e^{c b_1^4 x^3 + c b_1^2 x^4} \, dx \right| \leq \frac{c b_1^{11/2}}{T^2} \leq \frac{c b_1^4}{T}. \]  

(10.9)

and, taking into account (2.5),

\[ \frac{1}{|b_1|} \left| \int_{cTr_1/|b_1|}^{\infty} Q_3(x)((b_1 x)^3 + 3b_1^2x) \phi(x) e^{c b_1^4 x^3 + c b_1^2 x^4} \, dx \right| \leq \frac{c b_1^{11/2}}{T^2} \leq \frac{c b_1^4}{T}. \]  

(10.10)

The bounds (10.8)–(10.10) yield \(|J_{1,1,1}| \leq c b_1^4 T^{-1}\), and taking into account (10.4) we have

\[ |J_{1,1}| \leq c b_1^4 T^{-1}. \]  

(10.11)

In view of (6.14), (7.22), (9.1) and (9.9), we get

\[ |J_{1,2}| \leq \frac{c |b_1|^{9/2}}{T^2} \int_{cTr_1/|b_1|}^{\infty} (|b_1| x^2 + 1 + b_1^2 x^4) e^{-|b_1| x^2/4} \, dx \leq c b_1^4 T^{-2}. \]  

(10.12)

To estimate the integral \(J_2\), we use (3.5), (6.15) and (7.22). We have, taking into account (2.5),

\[ |J_2| \leq c |b_1|^{3/2} \int_{|x|>cTr_1/|b_1|} \left( \frac{x^2 + 1}{\sigma^2} + |b_1| x^2 + |\log |b_1|| \right) e^{-cT|x|/|b_1|} \, dx \]

\[ \leq c |b_1|^{3/2} \left( \frac{T}{|b_1|^2} + \frac{1}{b_1^2} + \frac{|b_1| |\log |b_1||}{T} \right) e^{-c c' T^2/|b_1|^3} \leq c T^{-2}. \]  

(10.13)

Applying (10.11)–(10.13) to (10.4), we get

\[ J_{st}(X^*_\sigma) \leq c |b_1|^{3} T^{-2}. \]

Now, by Lemmas 2.1, 4.4 and by (2.5), the above inequality gives

\[ J_{st}(X_\sigma) \leq \left( 1 + \frac{v_1^2 - \sigma_1^2}{\sigma_1^2 + \sigma^2} \right) J_{st}(X^*_\sigma) + c \sigma^{-7} N^3 \sqrt{\varepsilon} \leq \frac{C'}{(v_1^2 + \sigma^2)^3 T^2} \]

with some positive absolute constants \(c, C'\). An analogous inequality also holds for the random variable \(Y_\sigma\), and thus Theorem 1.2 is proved.
Remark 10.1. Under the assumptions of Theorem 1.2, a stronger inequality than \([11.7]\) follows from \([10.14]\),
\[
J_{st}(X_\sigma + Y_\sigma) \geq e^{c \alpha^{-6}(\log \sigma)^3} \left[ \exp \left\{ - \frac{c}{(\text{Var}(X_\sigma))^3 J_{st}(X_\sigma)} \right\} + \exp \left\{ - \frac{c}{(\text{Var}(Y_\sigma))^3 J_{st}(Y_\sigma)} \right\} \right]
\]
with some positive absolute constant \(c\).

11. Proof of Theorem 1.3

In order to construct random variables \(X\) and \(Y\) with the desired properties, we need some auxiliary results. We use the letters \(c, c', \tilde{c}\) (with indices or without) to denote absolute positive constants which may vary from place to place, and \(\theta\) may be any number such that \(|\theta| \leq 1\). First we analyze the function \(v_\sigma\) with Fourier transform
\[
f_\sigma(t) = \exp\{- (1 + \sigma^2) t^2 / 2 + it^3 / T\}, \quad t \in \mathbb{R}
\]

Lemma 11.1. If the parameter \(T > 1\) is sufficiently large and \(0 \leq \sigma \leq 2\), the function \(f_\sigma\) admits the representation
\[
f_\sigma(t) = \int_{-\infty}^{\infty} e^{itx} v_\sigma(x) \, dx \quad (11.1)
\]
with a real-valued infinitely differentiable function \(v_\sigma(x)\) which together with its all derivatives are integrable and satisfies
\[
v_\sigma(x) > 0, \quad \text{for} \quad x \leq (1 + \sigma^2)^2 T / 16; \quad (11.2)
\]
\[
|v_\sigma(x)| \leq e^{-(1+\sigma^2)Tx/32}, \quad \text{for} \quad x \geq (1 + \sigma^2)^2 T / 16. \quad (11.3)
\]
In addition, for \(|x| \leq (1 + \sigma^2)^2 T / 16,
\[
C_1 e^{-2(5 - \sqrt{7}) |xy(x)| / 4} \leq |v_\sigma(x)| \leq C_2 e^{-4|x/9|}, \quad (11.4)
\]
where
\[
y(x) = \frac{1}{6} T \left( - (1 + \sigma^2) + \sqrt{(1 + \sigma^2)^2 - 12x / T} \right). \quad (11.5)
\]
The right-hand side of the inequality \([11.4]\) continues to hold for all \(x \leq (1 + \sigma^2)^2 T / 16\).

Proof. Since \(f_\sigma(t)\) decays very fast at infinity, the function \(v_\sigma\) is given according to the inversion formula by
\[
v_\sigma(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} f_\sigma(t) \, dt, \quad x \in \mathbb{R}. \quad (11.6)
\]
Clearly, it is infinitely many times differentiable and all its derivatives are integrable. It remains to prove \([11.2]\)–\([11.4]\). By the Cauchy theorem, one may also write
\[
v_\sigma(x) = e^{yx} f_\sigma(iy) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} R_\sigma(t, y) \, dt, \quad \text{where} \quad R_\sigma(t, y) = \frac{f_\sigma(t + iy)}{f_\sigma(iy)}, \quad (11.7)
\]
for every fixed real \(y\). Here we choose \(y = y(x)\) according to the equality in \([11.5]\) for \(x \leq (1 + \sigma^2)^2 T / 16\). In this case, it is easy to see,
\[
e^{-ixt} R_\sigma(t, y(x)) = \exp \left\{ - \frac{(1 + \sigma^2)t^2}{2} \left( 1 + \frac{6y(x)}{(1 + \sigma^2)T} \right) + \frac{it^3}{T} \right\} \equiv \exp \left\{ - \frac{\alpha(x)}{2} t^2 + \frac{t^3}{T} \right\}.
\]
Note that $\alpha(x) \geq (1 + \sigma^2)/2$ for $x$ as above.

For a better understanding of the behaviour of the integral in the right-hand side of (11.7), put

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i xt} R_{\sigma}(t, y) \, dt$$

and rewrite it in the form

$$I = \bar{I}_1 + \bar{I}_2 = \frac{1}{2\pi} \left( \int_{|t| \leq T^{1/3}} + \int_{|t| > T^{1/3}} \right) e^{-i xt} R_{\sigma}(t, y(x)) \, dt.$$

Using $|\cos u - 1 + u^2/2| \leq u^4/4!$ ($u \in \mathbb{R}$), we easily obtain the representation

$$\bar{I}_1 = \frac{1}{2\pi} \int_{|t| \leq T^{1/3}} \left( 1 - \frac{t^6}{2T^2} \right) e^{-\alpha(x) t^2/2} \, dt + \frac{\theta}{4! T^4} \frac{1}{2\pi} \int_{|t| \leq T^{1/3}} t^{12} e^{-\alpha(x) t^2/2} \, dt$$

$$= \frac{1}{\sqrt{2\pi \alpha(x)}} \left( 1 - \frac{15}{2\alpha(x)^3 T^2} + \frac{c \theta}{\alpha(x)^6 T^4} \right) - \frac{1}{2\pi} \int_{|t| > T^{1/3}} \left( 1 - \frac{t^6}{2T^2} \right) e^{-\alpha(x) t^2/2} \, dt.$$  

(11.8)

The absolute value of last integral does not exceed $c(T^{1/3} \alpha(x))^{-1} e^{-\alpha(x) T^{2/3}/2}$. The integral $\bar{I}_2$ admits the same estimate. Therefore, we obtain from (11.8) the relation

$$\bar{I} = \frac{1}{\sqrt{2\pi \alpha(x)}} \left( 1 - \frac{15}{2\alpha(x)^3 T^2} + \frac{c \theta}{\alpha(x)^6 T^4} \right).$$

(11.10)

Applying (11.10) in (11.7), we deduce for the half-axis $x \leq (1 + \sigma^2)^2 T/16$, the formula

$$v_{\sigma}(x) = \frac{1}{\sqrt{2\pi \alpha(x)}} \left( 1 - \frac{15}{2\alpha(x)^3 T^2} + \frac{c \theta}{\alpha(x)^6 T^4} \right) e^{\frac{y(x)}{x}} f_{\sigma}(iy(x)).$$

(11.11)

We conclude immediately from (11.11) that (11.2) holds. To prove (11.3), we use (11.7) with $y = y_0 = -(1 + \sigma^2)T/16$ and, noting that

$$x + \frac{1 + \sigma^2}{2} y_0 \geq \frac{x}{2} \quad \text{for} \quad x \geq \frac{(1 + \sigma^2)^2}{16} T,$$

we easily deduce the desired estimate

$$|v_{\sigma}(x)| \leq e^{-(1+\sigma^2)T x/32} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-5(1+\sigma^2) t^2/16} \, dt \leq e^{-(1+\sigma^2)T x/32}.$$

Finally, to prove (11.4), we apply the formula (11.11). Using the explicit form of $y(x)$, write

$$e^{\frac{y(x)}{x}} f_{\sigma}(iy(x)) = \exp \left\{ y(x) x + \frac{1 + \sigma^2}{2} y^2(x) + \frac{y(x)^3}{T} \right\} = \exp \left\{ \frac{y(x)}{3} (2x + \frac{1 + \sigma^2}{2} y(x)) \right\}$$

(11.12)

for $x \leq \frac{(1 + \sigma^2)^2}{16} T$. Note that the function $y(x)/x$ is monotonically decreasing from zero to $-\frac{4}{3} (1 + \sigma^2)^{-1}$ and is equal to $-\frac{8}{3} \left( 1 + \sqrt{\frac{1}{4}} \right) (1 + \sigma^2)^{-1}$ at the point $x = -\frac{(1 + \sigma^2)^2}{16} T$.

Using these properties in (11.12), we conclude that in the interval $|x| \leq \frac{(1 + \sigma^2)^2}{16} T$,

$$e^{-2(5-\sqrt{7})|y(x)|/9} \leq e^{\frac{y(x)}{x}} f_{\sigma}(iy(x)) \leq e^{-4|y(x)|/9},$$

(11.13)
where the right-hand side continues to hold for all \( x \leq \frac{(1+\sigma^2)^2}{16} \). The inequalities in (11.4) follow immediately from (11.11) and (11.13).

Now, introduce independent identically distributed random variables \( U \) and \( V \) with density

\[
p(x) = d_0 v_0(x) I_{(-\infty,x/16]}(x), \quad d_0 = 1/ \int_{-\infty}^{T/16} v_0(u) \, du,
\]

where \( I_A \) denotes the indicator function of a set \( A \). The density \( p \) depends on \( T \), but for simplicity we omit this parameter. Note that, by Lemma 11.1, \( |1 - d_0| \leq e^{-cT^2} \).

Consider the regularized random variable \( U_\sigma \) with density \( p_\sigma = p * \varphi_\sigma \), which we represent in the form

\[
p_\sigma(x) = d_0 v_\sigma(x) - w_\sigma(x), \quad \text{where} \quad w_\sigma(x) = d_0 \left( (v_0 I_{(T/16,\infty)}) * \varphi_\sigma \right)(x).
\]

The next lemma is elementary, and we omit its proof.

**Lemma 11.2.** We have

\[
|w_\sigma(x)| \leq \varphi_\sigma(x + T/16) e^{-cT^2}, \quad x \leq 0,
\]

\[
|w_\sigma(x)| \leq e^{-cT^2}, \quad 0 < x \leq T/16,
\]

\[
|w_\sigma(x)| \leq e^{-cT^2}, \quad x > T/16.
\]

**Lemma 11.3.** For all sufficiently large \( T > 1 \) and \( 0 < \sigma \leq 2 \),

\[
D(U_\sigma) = \frac{3}{(1 + \sigma^2)^3 T^2} + \frac{c\theta}{T^3}.
\]

**Proof.** Put \( EU_\sigma = a_\sigma \) and \( \text{Var}(U_\sigma) = b_\sigma^2 \). By Lemma 11.2, \( |a_\sigma| + |b_\sigma^2 - 1 - \sigma^2| \leq e^{-cT^2} \).

Write

\[
D(U_\sigma) = J_1 + J_2 + J_3 = d_0 \int_{|x| \leq e^{cT}} v_\sigma(x) \log \frac{p_\sigma(x)}{\varphi_{a_\sigma,b_\sigma}(x)} \, dx
\]

\[
- \int_{|x| \leq e^{cT}} w_\sigma(x) \log \frac{p_\sigma(x)}{\varphi_{a_\sigma,b_\sigma}(x)} \, dx + \int_{|x| > e^{cT}} p_\sigma(x) \log \frac{p_\sigma(x)}{\varphi_{a_\sigma,b_\sigma}(x)} \, dx,
\]

where \( c' > 0 \) is a sufficiently small absolute constant. First we find lower and upper bounds of \( J_1 \), which are based on some additional information about \( v_\sigma \).

Using a Taylor expansion for the function \( \sqrt{T - u} \) about zero in the interval \(-\frac{3}{4} \leq u \leq \frac{3}{4}\), we easily obtain, for \( |x| \leq (1 + \sigma^2)^2 T/16 \),

\[
\frac{6y(x)}{(1 + \sigma^2)T} = -1 + \sqrt{1 - \frac{12x}{(1 + \sigma^2)^2 T}}
\]

\[
= -1 + \frac{6x}{(1 + \sigma^2)^2 T} - \frac{18x^2}{(1 + \sigma^2)^4 T^2} - \frac{108x^3}{(1 + \sigma^2)^6 T^3} + \frac{c\theta x^4}{T^4},
\]

which leads to the relation

\[
y(x) + \frac{1 + \sigma^2}{2} y(x)^2 + \frac{y(x)^3}{T} = -\frac{x^2}{2(1 + \sigma^2)} - \frac{x^3}{(1 + \sigma^2)^3 T} - \frac{9x^4}{2(1 + \sigma^2)^5 T^2} + \frac{c\theta x^5}{T^3}. \tag{11.16}
\]
In addition, it is easy to verify that

\[
\alpha(x) = (1 + \sigma^2) \left(1 - \frac{6x}{(1 + \sigma^2)^2 T} - \frac{18x^2}{(1 + \sigma^2)^4 T^2} + \frac{c\theta x^5}{T^3}\right).
\] (11.17)

Finally, using (11.16) and (11.17), we conclude from (11.11) that \(v_\sigma\) is representable as

\[
v_\sigma(x) = g(x) \varphi_{\sqrt{1+\sigma^2}}(x) e^{h(x)}
\]

\[=
\left(1 + \frac{3x}{(1 + \sigma^2)^2 T} + \frac{15}{2} \frac{3x^2 - (1 + \sigma^2) + c\theta|x|(1 + x^2)}{(1 + \sigma^2)^4 T^2} \frac{x^3}{(1 + \sigma^2)^3 T} - \frac{9x^4}{2(1 + \sigma^2)^5 T^2} + \frac{c\theta x^5}{T^3}\right)
\] (11.18)

for \(|x| \leq (1 + \sigma^2)^2 T / 16\).

Now, from (11.18) and Lemma 11.2, we obtain a simple bound

\[
|w_\sigma(x)/v_\sigma(x)| \leq 1/2 \quad \text{for} \quad |x| \leq c'T.
\] (11.19)

Therefore we have this relation, using again Lemma 11.1 and Lemma 11.2

\[
\tilde{J}_1 = \int_{|x| \leq c'T} v_\sigma(x) \log \frac{v_\sigma(x)}{\varphi_{\alpha, \beta_\sigma}(x)} dx + 2\theta \int_{|x| \leq c'T} |w_\sigma(x)| dx
\]

\[
= \int_{|x| \leq c'T} v_\sigma(x) \log \frac{v_\sigma(x)}{\varphi_{\sqrt{1+\sigma^2}}(x)} dx + \theta e^{-c'T}.
\] (11.20)

Let us denote the integral on the right-hand side of (11.20) by \(\tilde{J}_{1,1}\). With the help of (11.18) it is not difficult to derive the representation

\[
\tilde{J}_{1,1} = \int_{|x| \leq c'T} \varphi_{\sqrt{1+\sigma^2}}(x) e^{h(x)} \left(-\frac{x^3}{(1 + \sigma^2)^3 T} - \frac{15x^4}{2(1 + \sigma^2)^5 T^2} + \frac{3x}{(1 + \sigma^2)^2 T} + \frac{54x^2 - 15(1 + \sigma^2)}{2(1 + \sigma^2)^4 T^2} + \frac{c\theta|x|(1 + x^4)}{T^3}\right) dx.
\] (11.21)

Since

\[
|h(x) - 1 - h(x)| \leq \frac{1}{2} h(x)^2 e^{|h(x)|},
\]

and \(\varphi_{\sqrt{1+\sigma^2}}(x) e^{h(x)} \leq \sqrt{\varphi_{\sqrt{1+\sigma^2}}(x)}\) for \(|x| \leq c'T\), we easily deduce from (11.21) that

\[
\tilde{J}_{1,1} = \int_{|x| \leq c'T} \varphi_{\sqrt{1+\sigma^2}}(x) \left(\frac{3(1 + \sigma^2)x - x^3}{(1 + \sigma^2)^3 T} + \frac{54x^2 - 15(1 + \sigma^2)}{2(1 + \sigma^2)^4 T^2} - \frac{21(1 + \sigma^2)x^4 - 2x^6}{2(1 + \sigma^2)^6 T^2}\right) dx + \frac{c\theta}{T^3} = \frac{3}{(1 + \sigma^2)^3 T^2} + \frac{c\theta}{T^3}.
\] (11.22)

It remains to estimate the integrals \(\tilde{J}_2\) and \(\tilde{J}_3\). By (11.19) and Lemma 11.2

\[
|\tilde{J}_2| \leq \int_{|x| \leq c'T} |w_\sigma(x)|(-\log\varphi_{\alpha, \beta_\sigma}(x) + \log \frac{3}{2} + |\log v_\sigma(x)|) dx \leq cT^3 e^{-cT^2} \leq e^{-cT^2},
\] (11.23)
while by Lemma 11.1 and Lemma 11.2
\[
|\tilde{J}_3| \leq \int_{|x| > cT} (|v_\sigma(x)| + |w_\sigma(x)|)(\sqrt{2\pi b_\sigma} + \frac{x^2}{2b_\sigma^2} + |\log(|v_\sigma(x)| + |w_\sigma(x)|)) \, dx
\]
\[
\leq c \int_{|x| > cT} (1 + x^2)e^{-cT|x|} \, dx + \int_{|x| > cT} (|v_\sigma(x)| + |w_\sigma(x)|)^{1/2} \, dx \leq e^{-cT^2}. \quad (11.24)
\]
The assertion of the lemma follows from (11.22)–(11.24).

To complete the proof of Theorem 1.3, we need yet another lemma.

**Lemma 11.4.** For all sufficiently large \( T > 1 \) and \( 0 < \sigma \leq 2 \), we have
\[
D(U_\sigma - V_\sigma) \leq e^{-cT^2}.
\]

**Proof.** Putting \( \bar{p}_\sigma(x) = p_\sigma(-x) \), we have
\[
D(U_\sigma - V_\sigma) = \int_{-\infty}^{\infty} (p_\sigma * \bar{p}_\sigma)(x) \log \frac{(p_\sigma * \bar{p}_\sigma)(x)}{\varphi(\sqrt{2(1+\sigma^2)}(x))} \, dx
\]
\[
+ \int_{-\infty}^{\infty} (p_\sigma * \bar{p}_\sigma)(x) \log \frac{\varphi(\sqrt{2(1+\sigma^2)}(x))}{\varphi(\sqrt{\text{Var}(X_\sigma - Y_\sigma)}(x))} \, dx. \quad (11.25)
\]
Note that \( \tilde{p}_\sigma(x) = d_0 \tilde{v}_\sigma(x) - \tilde{w}_\sigma(x) \) with \( \tilde{v}_\sigma(x) = v_\sigma(-x), \tilde{w}_\sigma(x) = w_\sigma(-x) \), and
\[
p_\sigma * \bar{p}_\sigma = d_0^2(v_\sigma * \tilde{v}_\sigma)(x) - d_0(v_\sigma * \tilde{w}_\sigma)(x) - d_0(\tilde{v}_\sigma * w_\sigma)(x) + (w_\sigma * \tilde{w}_\sigma)(x). \quad (11.26)
\]
By the very definition of \( v_\sigma, v_\sigma * \tilde{v}_\sigma = \varphi(\sqrt{2(1+\sigma^2)} \). Since \( |\text{Var}(U_\sigma - V_\sigma) - 2(1+\sigma^2)| \leq e^{-cT^2} \),
using Lemma 11.1 we note that the second integral on the right-hand side of (11.25) does not exceed \( e^{-cT^2} \). Using Lemma 11.1 and Lemma 11.2 we get
\[
|(v_\sigma * \tilde{w}_\sigma)(x)| + |(\tilde{v}_\sigma * w_\sigma)(x)| + |w_\sigma * \tilde{w}_\sigma(x)| \leq e^{-cT^2}, \quad |x| \leq \tilde{c}T, \quad (11.27)
\]
\[
|(v_\sigma * \tilde{w}_\sigma)(x)| + |(\tilde{v}_\sigma * w_\sigma)(x)| + |(w_\sigma * \tilde{w}_\sigma)(x)| \leq e^{-cT^2(x)}, \quad |x| > \tilde{c}T. \quad (11.28)
\]
It follows from these estimates that
\[
\frac{(p_\sigma * \bar{p}_\sigma)(x)}{\varphi(\sqrt{2(1+\sigma^2)}(x))} = 1 + c\theta e^{-cT^2} \quad (11.29)
\]
for \( |x| \leq cT \). Hence, with the help of the Lemma 11.1 and Lemma 11.2 we may conclude that
\[
\left| \int_{|x| \leq cT} (p_\sigma * \bar{p}_\sigma)(x) \log \frac{(p_\sigma * \bar{p}_\sigma)(x)}{\varphi(\sqrt{2(1+\sigma^2)}(x))} \, dx \right| \leq e^{-cT^2}. \quad (11.30)
\]
A similar integral over the set \( |x| > cT \) can be estimated with the help of (11.27) and (11.28), and here we arrive at the same bound as well. Therefore, the assertion of the lemma follows from (11.25). \( \square \)

Introduce the random variables \( X = (U - a_0)/b_0 \) and \( Y = (V - a_0)/b_0 \). Since \( D(X_\sigma) = D(U_{b_0\sigma}) \) and \( D(X_\sigma - Y_\sigma) = D(U_{b_0\sigma} - V_{b_0\sigma}) \), the statement of Theorem 1.3 for the entropic distance \( D \) immediately follows from Lemma 11.3 and Lemma 11.4.
As for the distance $J_{st}$, we need to prove corresponding analogs of Lemma 11.3 and Lemma 11.4 for $J_{st}(U_\sigma)$ and $J_{st}(U_\sigma - V_\sigma)$, respectively. By the Stam inequality (1.5) and Lemma 11.3, we see that

$$J_{st}(U_\sigma) \geq c(\sigma) T^{-2}$$

for sufficiently large $T > 1$, (11.31)

where $c(\sigma)$ denote positive constants depending on $\sigma$ only. We estimate the quantity $J_{st}(U_\sigma - V_\sigma)$, by using the formula

$$J_{st}(U_\sigma - V_\sigma) = -\int_{-\infty}^{\infty} (p_\sigma \ast \tilde{p}_\sigma)'(x) \log \frac{(p_\sigma \ast \tilde{p}_\sigma)(x)}{\varphi_{\sqrt{\text{Var}(U_\sigma - V_\sigma)}}(x)} dx.$$ (11.32)

It is not difficult to conclude from (11.26), using our previous arguments, that

$$(p_\sigma \ast \tilde{p}_\sigma)'(x) = d_0^2 \varphi''(2(1+\sigma^2)(x) + R_\sigma(x),$$ (11.33)

where $|R_\sigma(x)| \leq c(\sigma)e^{-cT^2}$ for $|x| \leq \tilde{c}T$ and $|R_\sigma(x)| \leq c(\sigma)e^{-cT|x|}$ for $|x| > \tilde{c}T$. Applying (11.33) in the formula (11.32) and repeating the argument that we used in the proof of Lemma 11.3 we obtain the desired result, namely

$$J_{st}(U_\sigma - V_\sigma) \leq c(\sigma) \text{Var}(X_\sigma)e^{-cT^2}$$

for sufficiently large $T > 1$. (11.34)

By Theorem 1.2, $J_{st}(U_\sigma) \leq -c(\sigma)/(\log J_{st}(U_\sigma - V_\sigma))$, which implies $J_{st}(U_\sigma) \to 0$ as $T \to \infty$. Since $J_{st}(X_\sigma) = J_{st}(U_{b_0}\sigma)$ and $J_{st}(X_\sigma - Y_\sigma) = J_{st}(U_{b_0}\sigma - V_{b_0}\sigma)$, the statement of Theorem 1.3 for $J_{st}$ follows from (11.31) and (11.34).

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