On complete embedded translating solitons of the mean curvature flow that are of finite genus.

16th January 2015

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Abstract: We desingularise the union of 3 Grim paraboloids along Costa-Hoffman-Meeks surfaces in order to obtain what we believe to be the first examples in $\mathbb{R}^3$ of complete embedded translating solitons of the mean curvature flow of arbitrary non-trivial genus. This solves a problem posed by Martín, Savas-Halilaj and Smoczyk.

Key Words: Mean curvature flow, soliton, singular perturbation, Costa-Hoffman-Meeks surface, Grim paraboloid.

AMS Subject Classification: 53C44
1 - Introduction.

1.1 - Background and Main Result. Mean curvature flow (MCF) solitons are complete surfaces in $\mathbb{R}^{n+1}$ whose evolution under the mean curvature flow is given by translation. As such, they are solutions of the quasi-linear second-order elliptic PDE,

$$ H + \langle N, e_z \rangle = 0, $$

where $H$ here denotes the mean curvature of the surface, and $N$ denotes its unit normal vector field. MCF solitons are of particular interest, partly because they provide relatively non-trivial examples of complete and eternal mean curvature flows in Euclidean space, but mainly because they also arise as the limits of blow-ups of more general mean curvature flows, thereby providing an indication of the types of singularities that may exist.

The classification of MCF solitons therefore presents an interesting problem which is analogous to that of minimal surfaces, but which, despite being a small perturbation of the latter, exhibits substantially different phenomena. Naturally, when the ambient space is 2-dimensional, the picture is completely understood, and the only MCF solitons are the vertical lines, and the Grim reaper curves, which are translates of graphs of the function $g(x) = \log(\sec(x))$ over the interval $]-\pi/2, \pi/2[$. However, already when the ambient space is 3-dimensional, MCF solitons exhibit an immense variety of phenomena. A relatively complete review of the state of the art is provided by Martín, Savas-Halilaj and Smoczyk in [14], which, for convenience, we briefly recall here.

(1) Trivial examples are given by the Cartesian products of horizontal lines with MCF solitons in 2-dimensional space. The surfaces thereby constructed comprise the vertical planes and those surfaces which we henceforth refer to as Grim planes.

(2) Various graphical examples have been constructed, for example, by Altschuler and Wu in [1] and Wang in [18].

(3) Radially symmetric examples are obtained by solving an appropriate ODE, and are studied by Altschuler and Wu in [1] and Clutterbuck, Schnürer and Schulze in [2]. We recall that radially symmetric surfaces are either simply or doubly connected. The simply connected examples, which we henceforth refer to as Grim paraboloids are convex graphs over $\mathbb{R}^2$, whilst the doubly connected examples, which we henceforth refer to as Grim catenoids are unions of two graphs over an unbounded annulus. Each of these surfaces is asymptotic at infinity to a vertical translate of the graph of $\frac{1}{2}r^2 - \log(r)$, where $r$ here denotes the radial distance in $\mathbb{R}^2$ to the origin.

(4) Examples of helicoidal type are constructed by Halldorsson in [8].

(5) Examples of infinite genus invariant under a discrete group of translations are constructed by Nguyen in [16] and [17]. In these beautiful papers, following the ideas of Kapouleas (see below), unions of families of vertical planes and Grim planes are desingularised along Sherk surfaces. As an example, the simplest case is the desingularisation of the union of 1 vertical plane with 1 Grim plane. This heuristically takes the form of a pitchfork and is referred to as a translating trident.
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Nevertheless, as highlighted by Martín, Savas-Halilaj and Smoczyk, despite this considerable variety of examples, there exists to date no construction of MCF solitons of finite non-trivial genus. It is our aim in this paper to fill this gap. We first recall that, for any positive integer $g$, the Costa-Hoffman-Meeks surface of genus $g$, which we henceforth denote by $C_g$, is an embedded minimal surface in $\mathbb{R}^3$ with finite genus equal to $g$ and with 3 ends, each of which is a graph over $\mathbb{R}^2$ (c.f [10]). Importantly, for all $0 \leq k \leq g$, $C_g$ is invariant under reflection in the plane spanned by the $z$-axis and the linear subspace of the $x-y$-plane which makes an angle of $k\pi/(g+1)$ with the $x$-axis. The group of symmetries of $\mathbb{R}^3$ generated by these reflections will play an important rˆle in the later stages of our construction, and will henceforth be referred to as the group of horizontal symmetries of the Costa-Hoffman-Meeks surface $C_g$. By desingularising the union of a Grim paraboloid and a Grim catenoid about Costa-Hoffman-Meeks surfaces, we obtain,

**Theorem A**

Fix $g \in \mathbb{N}$ and $\eta < 1$. For all sufficiently large $\Delta$, and for all $\epsilon$, and for all $R$ satisfying

$$\left(\epsilon R^{4+\eta} + \frac{1}{R^{1-\eta}}\right) \leq \frac{1}{\Delta}, \quad \epsilon R^{5-\eta} \geq \Delta,$$

there exists a complete embedded MCF soliton $\Sigma$ of genus $g$ with 3 ends. Furthermore,

(1) $\Sigma$ is preserved by the horizontal symmetries of the Costa-Hoffman-Meeks surface $C_g$,

(2) $\Sigma \setminus B(\epsilon R)$ consists of three disjoint Grim ends each of which converges towards the Grim paraboloid as $\Delta$ tends to infinity; and

(3) upon rescaling by $1/\epsilon$, $\Sigma \cap B(2\epsilon R)$ converges towards the Costa-Hoffman-Meeks surface $C_g$ as $\Delta$ tends to infinity.

Theorem A follows immediately from the results of Section 7.1 from which the precise modes of convergence employed in (2) and (3) should also become clear.

**1.2 - Techniques.** Our approach follows the seminal work [11], [12] and [13] of Kapouleas, which we believe to be best explained in 3 steps with the help of a simple example. Consider the desingularisation of two Grim paraboloids about a minimal catenoid. In the first step, for large $R$, we consider the cylinder $B(2R)$ of radius $2R$ about the $z$-axis and the annular prism $A(R, \infty)$ of inner radius $R$ about the $z$-axis. For small $\epsilon$, an approximate soliton - moving at speed $1/\epsilon$ - is constructed by simply replacing each of the ends of the catenoid on the outside of this cylinder by slightly perturbed copies of the outside part of the Grim paraboloid, translated in the vertical direction and rescaled by $1/\epsilon$. In Kapouleas’ approach, this is achieved by simply using cut-off functions to interpolate between the two surfaces over the transition region $B(2R) \cap A(R, \infty)$ (though other approaches are possible, see below). In particular, by minimality of the catenoid, the value of the MCFS functional of the joined surface should also be of order $\epsilon$. The second step, which constitutes the main mathematical challenge of the technique, involves the construction of the Green’s operator of this approximate soliton along with estimates of its norm. Having achieved this, existence is then proven in the third step via a relatively straightforward application of the Schauder fixed point theorem (c.f. [7]).
Later work has shown how, in many important cases, this highly technical construction of the Green’s operator can be substituted for by simpler approaches. For example, in the work [16] and [17] of Nguyen, the Jacobi operator of the approximate soliton is shown to become arbitrarily close to that of the classical Sherk surface as the parameter tends to infinity, and thus, since the properties of the latter are well known, the required norm estimates for the Green’s operator immediately follow. On the other hand, in [9], Hauswirth and Pacard follow a more subtle and elegant approach. First, in order to construct the approximate soliton, instead of using cut-off functions as Kapouleas does, the authors perturb and then line up the boundary curves of the different components in order to obtain a continuous approximate soliton, which exactly solves the soliton equation wherever it is smooth, but which has tangent planes that are not necessary continuous along the common boundary curve. The problem then becomes one of perturbing this surface to one that is smooth throughout. This in turn is achieved in the usual manner via a fixed-point technique using uniform bounds for the norm of the inverse of the first order Cauchy jump operator along this curve. The key observation is that this Cauchy jump operator is simply the difference between the scattering (that is, Dirichlet to Neumann) operators of the two components on each side. In particular, if the scattering operators of the new ends become arbitrarily close to those of the original ends as the parameters tend to infinity, then the problem reduces in the limit to one of inverting the Cauchy jump operator of a known closed curve in a classical minimal surface, and it is by this means that the authors construct new embedded Riemann-type surfaces of arbitrary non-trivial genus in $\mathbb{R}^3$.

In the present case, however, neither of these simplifications are feasible. Firstly, the Jacobi operator of the approximating surface is qualitatively different to that of the original Costa-Hoffman-Meeks surface. Indeed, it can be shown to be Fredholm of index 0 over suitably weighted spaces, which is known not to be the case for Costa-Hoffman-Meeks surfaces. Nguyen’s approach therefore does not apply. Likewise, there is no a-priori reason for the scattering operator of the unbounded part of a Grim paraboloid to bear any relation to that of a catenoidal or planar end of a minimal surface. Furthermore, since the proof of such a result would involve a global argument that would essentially amount to constructing the Green’s operator of the approximating soliton anyway, it would be unlikely to constitute a genuine simplification, and so the technique employed by Hauswirth and Pacard is most likely of no assistance in this case.

We are therefore obliged to directly construct the Green’s operator of the approximating soliton using Kapouleas’ original argument. In the present case, this requires what we consider to be two mathematical novelties. First, to our knowledge, this technique has never been applied directly to Costa-Hoffman-Meeks surfaces. In particular, since symmetries play an important rôle in providing the necessary decay estimates, and since the symmetry groups of low-genus Costa-Hoffman-Meeks surfaces are relatively small, they would appear to lie close to the limit of the range of applicability of Kapouleas’ approach, and it appears that considerable extra care and more subtle estimates are required at many steps in the argument. Second, the construction of the Green’s operator of the approximating soliton requires already an in-depth understanding of the Green’s operators of Grim ends themselves, which are currently completely unstudied in the literature from this point of view. Consequently, the main body of this paper (Sections 2, 3 and 4) is actu-
ally devoted to the study of these operators. At this point, two more fundamental insights guide our work. First, upon conjugating by the operator of multiplication by a suitably chosen positive function (as in [3], [4], [5] and [6]), we discover that the Jacobi operator of the Grim paraboloid transforms into a self-adjoint operator which, in particular and above all, models a potential well. Once having made this key observation proving invertibility becomes a straightforward using classical PDE techniques. Second, upon conjugating by the operator of multiplication by a different positive function, we ensure that the Jacobi operators of suitably controlled Grim ends converge in the operator norms towards the Jacobi operator of the Grim paraboloid and the uniform invertibility, which plays a central rôle in our result, follows immediately.

1.3 - Overview and Acknowledgements. The paper is structured as follows.

(1) In Section 2, we employ classical ODE techniques to determine asymptotic expansions for the profiles of Grim ends both near infinity, and near their finite boundaries. For completeness, we determine these asymptotic expansions up to arbitrary order, although this is far more than what we actually require.

(2) In Section 3, we prove the invertibility of the Jacobi operator of the Grim paraboloid over suitably weighted Hölder and Sobolev spaces, both of which will be required in the sequel. Our fundamental insight is that, upon conjugating with the operator of multiplication by the conformal scaling factor of the soliton metric over $\mathbb{R}^3$, the Jacobi operator transforms into a self-adjoint operator modelling a potential well and invertibility is then readily obtained by standard PDE techniques.

(3) In Section 4, we prove invertibility of the Jacobi operators of suitably controlled Grim ends over correspondingly weighted Hölder and Sobolev spaces. We achieve this by showing that their Jacobi operators converge in the operator norms towards the Jacobi operator of the Grim paraboloid as the parameter tends to infinity. However, the existence of a divergent term obstructs the direct application of this technique. Our fundamental insight is that, upon conjugating with the operator of multiplication by the Jacobi field of vertical translation, the rate of divergence of this term is reduced, allowing it to then be eliminated via an averaging argument. It is in proving convergence of these operators in the Hölder norms that the horizontal symmetries of Costa-Hoffman-Meeks surfaces are first required.

(4) In Section 5 we review in detail the now classical technique by which the Costa-Hoffman-Meeks surface is joined to the Grim ends, and we introduce the family of smooth perturbations used to prove existence in the Schauder fixed-point theorem. In addition, we study the basic properties of various operators that are used in this construction.

(5) In Section 6, we apply Kapouleas’ “ping-pong” argument to construct the Green’s operator of the approximate soliton. More care than normal is required due to the low order of symmetry of low-genus Costa-Hoffman-Meeks surfaces. A key result is Lemma 6.1.1, which determines strong first order estimates of functions via a marriage of the Sobolev embedding theorem and Hölder interpolation estimates.

(6) Theorem A is proven in Section 7 using the Schauder fixed-point theorem. This final stage is still not yet completely trivial, and a further novelty is required in the form...
of Lemma 7.1.2, which, using once again the Sobolev inequality and Hölder interpolation estimates, along with the quasi-linearity of the MCFS functional itself, allows us to improve our estimate of the quadratic error of the MCFS functional over the Grim end by a sufficient factor for existence to be proven.

(7) Finally, in Appendix A, we provide a complete description of the notation, conventions and terminology used throughout the paper. In addition, for the reader’s convenience, various well-known geometric and analytic results are collected here.

The author is grateful to Knut Smoczyk for drawing this interesting problem to his attention. He is likewise grateful to Detang Zhou and Andrew Clarke for helpful conversations and their invaluable insights.

2 - Grim Surfaces.

2.1 - The Large Scale. We define a Grim surface to be any unit speed MCF soliton which is a graph over some annulus $A(a,b)$. We define a Grim end to be a Grim surface which is defined over some unbounded annulus $A(a,\infty)$, and which is invariant by rotation about the $z$-axis. We first recall the general formula for such surfaces. Let $u$ be a $C^2$ function defined over some closed interval $[a,b]$ and let $G$ be the surface of revolution generated by rotating its graph about the $z$-axis. The principle curvatures of $G$ in the radial and angular directions are respectively

\[
c_r = \frac{-\ddot{u}}{\sqrt{1 + \dot{u}^2}}, \quad c_\theta = \frac{-\ddot{u}}{r\sqrt{1 + \dot{u}^2}},
\]

(2.1)

where the dot here denotes differentiation with respect to the radial parameter, which we henceforth denote by $r$. The vertical component of the upward-pointing unit normal vector over $G$ is

\[
\langle N_G, e_z \rangle = \frac{1}{\sqrt{1 + \dot{u}^2}},
\]

(2.2)

so that $G$ is a Grim surface whenever

\[
\dot{r}\dddot{u} + (\dddot{u} - r)(1 + \dot{u}^2) = 0.
\]

(2.3)

In this section, we derive an asymptotic formula for solutions to (2.3) over the large scale. Since (2.3) is actually a first-order equation in $\dot{u}$, we define the non-linear operator, which we also denote by $G$, by

\[
Gv := \dot{r}\dddot{v} + (\dddot{v} - r)(1 + v^2).
\]

(2.4)

We begin by deriving algebraic approximations to solutions of (2.4) from which the asymptotic formula will follow. To this end, we define a Laurent polynomial in $r$ to be a formal sum of the form

\[
V := \sum_{m=-\infty}^{k} V_m r^m,
\]

(2.5)

for some finite $k$, where, for all $m$, $V_m$ is a real number, and we henceforth refer to $k$ as the order of $V$. Observe that, using formal multiplication and formal differentiation of infinite series, the operator $G$ extends in a well-defined manner over the space of Laurent polynomials. We now construct a formal solution to (2.4).
Lemma 2.1.1

There exists a unique Laurent polynomial $V$ such that $GV = 0$. Furthermore,

(1) $V$ has order 1;
(2) $V_1 = 1$, $V_{-1} = -1$;
(3) if $m$ is even, then $V_m = 0$;
(4) if $v_n := \sum_{m=1-2n}^1 V_m r^m$ is the $n$'th partial sum of $V$, then $Gv_n$ is a finite Laurent polynomial of order $1 - 2n$.

Proof: Consider the ansatz (2.5). If $k \leq -1$, then the highest order term in $GV$ is equal to $-r$, if $k = 0$, then it is equal to $-r(1 + V_0^2)$, and if $k \geq 2$, then it is equal to $V_k r^{3k}$. Since none of these vanish, it follows that $V$ is of order 1. In this case, the highest order term in $GV$ is $V_1 r^3$, and so $V_1 = 1$. We now have

$$ rV + (V - r)(1 + V^2) = r + \sum_{m=0}^0 (m + 1)V_m r^m + \sum_{m=-\infty}^2 \left( \sum_{p+q+r=m, \atop p\leq0, q,r\leq1} V_p V_q V_r \right) r^m. $$

Setting the coefficients of $r^2$ and $r$ respectively equal to 0 yields

$$ V_0 = 0, \ V_{-1} = -1. $$

For all $m \leq -2$, setting the coefficients of $r^{m+2}$ equal to 0 yields

$$ V_m + \left( \sum_{p+q+r=m+2, \atop m+1\leq p\leq-1, \atop m+2\leq q,r\leq1} V_p V_q V_r \right) + (m + 3)V_{m+2} = 0. $$

The existence and uniqueness of $V$ now follow from this recurrence relation. Furthermore, if $p + q + r = m + 2$, and if $m$ is even, then so too is at least one of $p$, $q$ and $r$. Since $V_0 = 0$, it follows by induction that $V_m = 0$ for all even $m$. In particular, for all $n$, $V_{-2n} = 0$, so that the partial sum $v_n$ is defined by setting the coefficients of $r^{m+2}$ equal to 0 for all $m \geq -2n$. The function $Gv_n$ is therefore a finite Laurent polynomial of order $1 - 2n$, and this completes the proof. \(\square\)

Lemma 2.1.2

If $v : [R, \infty] \to \mathbb{R}$ solves (2.4), then

$$ |v_0 - v| \lesssim \frac{1}{r}, \quad (2.6) $$
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**Proof:** Consider the family of polynomials $p_t(y) = (y - 1)(t^2 + y^2)$. For all $t > 0$, $y = 1$ is the unique real root of $p_t$. Since $y = 0$ is the unique local maximum of $p_0$, for sufficiently small $t$, the unique local maximum of $p_t$ is near 0, and the value of $p_t$ at this point is less than $-t^2/2$. Since $p_0$ is convex over the interval $[1/3, \infty]$, for $1/3 < y < 1$, $p_0(y) \leq (3/2)(1 - y)p_0(1/3) = (y - 1)/9$. Thus, for sufficiently small $t$, over the smaller interval $[1/2, 1]$, $p_t(y) \leq (y - 1)/18$.

Now let $v$ be a solution of $Gv = 0$. In particular, $\dot{v} = -r^2p_{1/r}(v/r)$. Suppose furthermore that $r \gg 1$ so that the estimates of the preceding paragraph hold for $p_{1/r}$. When $v > r$, $\dot{v} < 0$, and so, for sufficiently large $r$, $v(r) \leq r$. If $v \leq r/2$, then $\dot{v} \geq r/2$, and so, for sufficiently large $r$, $v(r) \geq r/3$. Finally, if $r/3 \leq v \leq r$, then $\dot{v} \geq r(r - v)/18$. It follows that if $w = r(v_0 - v) = r(r - v)$, then $w > 0$ and $\dot{w} = 2r - v - r\dot{v} \leq r + w/r - rw/18$. Since this is negative when $w \geq 36$ and $r > 6$, the function $w$ is bounded, and the result follows. □

**Lemma 2.1.3**

If $v : [R, \infty) \to \mathbb{R}$ solves (2.4), then, for all $n$,

$$|v_n - v| \lesssim r^{-(2n+1)}. \quad (2.7)$$

**Proof:** For all $n$, let $w_n := r^{2n-1}(v_n - v)$ be the rescaled error. We prove by induction that $|w_n| \lesssim r^{-2}$ for all $n$. Indeed, the case $n = 0$ follows from (2.6). We suppose therefore that $n \geq 1$. Since $w_n = r^2w_{n-1} + v_{2n-1}$, it follows by the inductive hypothesis that $w_n$ is bounded. Let $P_3(r^{-1}, w_n)$ denote any term which is a polynomial in $r^{-1}$ with no constant term and with coefficients that are polynomials of order at most 3 in $w_n$. Since $Gv_{2n-1}$ is a finite Laurent polynomial of order $1 - 2n$,

$$\dot{w}_n = \frac{2n-1}{r}w_n + r^{2n-2}(r\dot{v}_n - r\dot{v}),$$

$$= P_3(r^{-1}, w_n) - r^{2n-2}((v_n - r)(1 + v^2_n) - (v - r)(1 + v^2)),$n

$$= P_3(r^{-1}, w_n) - \frac{1}{r}w_n (1 - r(v_n + v)(v^2_n + v_nv + v^2)).$$

Since $v = v_n - r^{-(2n-1)}w_n$, and since $v_n - r$ is also a polynomial in $r^{-1}$ with no constant term, this yields

$$\dot{w}_n = P_3(r^{-1}, w_n) - rw_n.$$

Since $w_n$ is bounded, there therefore exists a constant $B > 0$ such that for all $r \geq 1$,

$$|\dot{w}_n + rw_n| \leq Br^{-1}.$$

It follows that for $r \geq 2$ and $r^2w_n \geq 2B$,

$$\frac{d}{dr}r^2w_n = r^2(\dot{w} + rw_n) + (2r - r^2)w_n \leq Br - \frac{1}{2}r^3w_n \leq 0,$$

and it follows that $r^2w_n$ is bounded from above. In like manner we show that $r^2w_n$ is bounded from below, and this completes the proof. □
Lemma 2.1.4

If \( v : [R, \infty[ \to \mathbb{R} \) solves (2.4), then, for all \( n \),

\[
v_n - v = O(r^{-(2n+1+k)}).
\] (2.8)

Proof: For all \( n \), we denote \( w_n = v_n - v \). By (2.4),

\[
\frac{d}{dr} w_n = w_n P_1(r^{-1}, w_n) + G v_n,
\]

where \( P_1(r^{-1}, w_n) \) is a polynomial in \( r^{-1} \) and \( w_n \). Observe that \( G v_n \) is a finite Laurent polynomial of order \( 1 - 2n \). By induction, it follows that for all \( l \geq 1 \),

\[
\frac{d^l}{dr^l} w_n = w_n P_l(r^{-1}, w_n) + Q_l(r^{-1}),
\]

where \( P_l(r^{-1}, w_n) \) is a polynomial in \( r^{-1} \) and \( w_n \), and \( Q_l(r^{-1}) \) is a finite Laurent polynomial of order \( l - 2n \). Thus, for all \( l \),

\[
\left| \frac{d^l}{dr^l}(v_{n+l+1} - v) \right| \lesssim r^{-(2n+1+2l)}.
\]

However, since \( v_{n+l+1} - v \) is a finite Laurent polynomial of order \( -(2n+1) \), it follows that

\[
\left| \frac{d^l}{dr^l}(v_n - v) \right| \lesssim r^{-(2n+1+l)},
\]

and so \( v_n - v = O(r^{-(2n+1+l)}) \), as desired. \( \square \)

2.2 - The Small Scale - Formal Solutions. We now derive an asymptotic formula for solutions to (2.4) over the small scale. This will be more involved than the previous case. Indeed, we first require a precise notion of what “small-scale” means. Fix positive constants \( C \gg 1 \) and \( \eta \ll 1 \) which we henceforth consider to be universal. Let \( \Delta \) be a large, positive real number, and let \( \epsilon, A > 0 \) and \( c \in \mathbb{R} \) be such that

\[
(\epsilon A^{4+\eta} + \frac{1}{A^{1-\eta}}) \leq \frac{1}{\Delta}, \ \epsilon A^{5-\eta} \geq \Delta, \ |c| \leq C.
\] (2.9)

Observe in particular that these conditions imply that \( A \) becomes large as \( \Delta \) tends to infinity. We study solutions to (2.4) over the small scale \([\epsilon A, \epsilon A^4]\) with initial value determined by \( c \). We call the constant \( c \) the logarithmic coefficient for reasons that will become clear presently. It will be convenient for this initial value to be equal to a certain non-trivial function of \( c \) rather than to \( c \) itself. This function, which is a small perturbation of the identity, will depend on the desired order of asymptotic approximation and is defined explicitly in Section 2.4, below.
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By applying the change of variables \( r = \epsilon Ae^x \) we obtain
\[
Gv = D - \epsilon Ae^x + (v - \epsilon Ae^x)v^2,
\]
(2.10)
where
\[
Dv = v' + v,
\]
(2.11)
and a prime here denotes differentiation with respect to \( x \). We are now concerned with functions defined over the interval \([0, 3\log(A)]\). As in Section 2.1, we first derive algebraic approximations for solutions to (2.10) from which the asymptotic formula will follow. To this end, we introduce the following framework. Let \( \mathbb{R}[X, M, N] \) be the ring of polynomials with real coefficients in the variables \( X, M \) and \( N \). We consider a general element \( V \) of \( \mathbb{R}[X, M, N] \) as a sum of the form
\[
V = \sum_{p, q \leq k} V_{p, q}(X)M^pN^q,
\]
(2.12)
for some finite \( k \), where for all \( p, q \), \( V_{p, q} \) is a polynomial in the variable \( X \). We henceforth refer to \( k \) as the order of \( V \). There is a natural correspondence which sends \( \mathbb{R}[X, M, N] \) into the space of continuous functions over \([0, 3\log(A)]\) and is given by setting
\[
V \mapsto v(x) := \sum_{p+q=k} V_{p, q}(x) (\epsilon Ae^x) \left( \frac{c}{A} e^{-x} \right).
\]
(2.13)
Observe that this correspondence is simply the unique \( \mathbb{R}[X] \)-ring homomorphism sending \( M \) to \( \epsilon Ae^x \) and \( N \) to \( \frac{c}{A} e^{-x} \) and, although it is not injective, it keeps track of the parameters \( \epsilon, A \) and \( c \), for which reason it serves our purposes. There also exist natural operators over \( \mathbb{R}[X, M, N] \) which are mapped through this correspondence into \( D_1 \) and \( G \). These operators, which we also denote by \( D_1 \) and \( G \) respectively, are given by
\[
(D_1V)_{p, q} = \left( \frac{d}{dx} + 1 + (p - q) \right) V_{p, q},
\]
\[
GV = D_1V - M + (V - M)V^2.
\]
(2.14)
Observe that \( D_1 \) defines a surjective linear map from \( \mathbb{R}[X, M, N] \) to itself, and that its kernel consists of finite sums of the form
\[
V = \sum_{p \leq k} a_p M^p N^{p+1},
\]
where \( a_0, ..., a_k \) are real constants. Since this is mapped by the natural correspondence to the space of functions of the form
\[
v = \sum_{p \leq k} a_p (\epsilon A)^p \frac{c}{A} e^{-x},
\]
and since we are only interested in small values of the parameter \( \epsilon \), this justifies to some extent the restriction that we are about to make on possible formal solutions in Lemma 2.2.1, below. Let \( \mathbb{R}[X][[M, N]] \) be the ring of all formal power series over the variables \( M \) and \( N \) with coefficients that are polynomials in the variable \( X \). Observe that the operators \( D_1 \) and \( G \) naturally extend again to well defined operators over this space.
Lemma 2.2.1

There exists a unique formal power series $V$ in $\mathbb{R}[X][[M,N]]$ such that

1. $V_{0,1} = 1$;
2. $V_{p,p+1}(0) = 0$ for all $p \geq 1$; and
3. $GV = 0$.

Furthermore,

4. $V_{1,0} = \frac{1}{2}$;
5. if $p + q$ is even, then $V_{p,q} = 0$; and
6. if $p + q = 2k + 1$ is odd, then $V_{p,q}$ has order at most $k$ in $X$.

Proof: Let $V = \sum_{p,q} V_{p,q}(X)M^pN^q$ be an element of $\mathbb{R}[X][[M,N]]$ which solves $GV = 0$. For all $p$ and for all $q$,

$$
\left(\frac{d}{dx} + (1 + (p - q))\right)V_{p,q} = \delta_{p1}\delta_{q0} - \sum_{p_1+p_2+p_3=p, q_1+q_2+q_3=q} V_{p_1,q_1}V_{p_2,q_2}V_{p_3,q_3}
+ \sum_{p_1+p_2=p-1, q_1+q_2=q} V_{p_1,q_1}V_{p_2,q_2}.
$$

(2.15)

In particular,

$$
V'_{0,0} + V_{0,0}(1 + V_{0,0}^2) = 0,
$$

and since there exists no non-trivial polynomial solution to this equation, we infer that $V_{0,0} = 0$. It now follows that the two summations in (2.15) only involve terms of order at most $p + q - 2$ in $M$ and $N$. In particular, $V_{0,1}$ satisfies

$$
V'_{0,1} = 0,
$$

so that $V_{0,1}$ may assume any constant value. It now follows by recurrence that there exists a unique sequence $(V_{p,q})$ satisfying (2.15) such that $V_{0,1} = 1$ and $V_{p,p+1}(0) = 0$ for all $p \geq 1$. The corresponding formal power series $V$ is the desired solution and existence follows. Observe now that $V_{1,0}$ satisfies

$$
V'_{1,0} + 2V_{1,0} = 1,
$$

from which it follows that $V_{1,0} = \frac{1}{2}$. Next, if $p + q$ is even, then every summand on the right hand side involves at least one term of the form $V_{p',q'}$, where $p' + q'$ is an even number no greater than $p + q - 2$. Since $V_{0,0} = 0$, it follows by induction that $V_{p,q} = 0$ whenever $p + q$ is even. Finally, suppose that for all $l < k$, and for $p + q = 2l + 1$, the polynomial $V_{p,q}$ has order at most $l$ in $X$. By (2.15), for all $p + q = 2k + 1$, the polynomial $V_{p,q}$ is obtained by integrating terms of order at most $k - 1$ in $X$. It therefore follows by induction that $V_{p,q}$ has order at most $k$ in $X$, and this completes the proof. $\square$
2.3 - The Small Scale - Exact Solutions I. Let $V$ be the formal power series constructed in Lemma 2.2.1. We now show that solutions to (2.10) are asymptotic to $V$. We first use an analytic argument to obtain rough global estimates. Fix a non-negative integer $k$, and let $V_k$ be the $k$'th partial sum of $V$. That is,

$$V_k := \sum_{p+q \leq 2k+1} V_{p,q}(X) M^p N^q. \quad (2.16)$$

For a general element $W$ of $\mathbb{R}[X, M, N]$, we write $W = O((M + N)^l)$ to mean that all terms of $W$ of order less than $l$ vanish. By Lemma 2.2.1,

$$GV_k = O((M + N)^{2k+3}). \quad (2.17)$$

Let $v_k$ be the function corresponding to $V_k$. That is

$$v_k(x) := \sum_{p+q \leq 2k+1} V_{p,q}(x) (\epsilon A e^x) \left( \frac{c}{A} e^{-x} \right). \quad (2.18)$$

Observe that for all $k < l$,

$$v_l - v_k = O \left( x^{k+1} \left( \epsilon A e^x + \frac{1}{A} e^{-x} \right)^{2k+3} \right). \quad (2.19)$$

Furthermore, it follows from (2.17) that, for all $k$,

$$Gv_k = O \left( x^{k+1} \left( \epsilon A e^x + \frac{1}{A} e^{-x} \right)^{2k+3} \right). \quad (2.20)$$

Observe that $v_k(0)$ is a polynomial in $c$ with coefficients that depend on $\epsilon$ and $A$. Furthermore, this polynomial converges locally uniformly to the identity polynomial as $\Delta$ tends to infinity. In particular, for sufficiently large $\Delta$, it defines an invertible function over the interval $[-C, C]$. We define $\hat{v}_k$ to be the unique solution to $G\hat{v}_k = 0$ with initial condition

$$\hat{v}_k(0) = v_k(0), \quad (2.21)$$

and we refer to $\hat{v}_k$ as the exact solution of order $k$ with logarithmic coefficient $c$.

We use the contraction mapping theorem to obtain global estimates for the difference between $v_k$ and $\hat{v}_k$. We first recall the requisite functional analytic framework. For $T \in [0, 3\log(A)]$, let $C^0([0, T])$ be the Banach space of continuous functions over the interval $[0, T]$ furnished with the uniform norm and let $C^1_0([0, T])$ be the Banach space of $C^1$ functions over the same interval with initial value 0, furnished with the norm

$$\|w\|_1 := \|w'\|_{L^\infty}. \quad (2.22)$$

Observe, in particular, that for all $w \in C^1_0([0, T])$,

$$\|w\|_{L^\infty} \leq T \|w\|_1. \quad (2.23)$$
Lemma 2.3.1

The operator \( D_1 \) defines a linear isomorphism from \( C^1_0([0,T]) \) into \( C^0([0,T]) \). Furthermore, the operator norms of \( D_1 \) and its inverse satisfy

\[
\|D_1\| \leq 1 + T, \quad \|D_1^{-1}\| \leq 2. \tag{2.24}
\]

Proof: First, by (2.23),

\[
\|D_1 w\|_{L^\infty} \leq \|w'\|_{L^\infty} + \|w\|_{L^\infty} \leq (1 + T)\|w\|_1,
\]

and so \( \|D_1\| \leq 1 + T \). By inspection, for all \( w \),

\[
(D_1^{-1}w)(x) = e^{-x} \int_0^x e^y w(y)dy.
\]

In particular,

\[
\|D_1^{-1}w\|_{L^\infty} \leq \|w\|_{L^\infty}.
\]

Thus,

\[
\|D_1^{-1}w\|_1 = \|(D_1^{-1}w)'\|_{L^\infty} \leq \|D_1 D_1^{-1}w\|_{L^\infty} + \|D_1^{-1}w\|_{L^\infty} \leq 2\|w\|_{L^\infty},
\]

and so \( \|D_1^{-1}\| \leq 2 \). This completes the proof. \( \square \)

We define the functional \( H : C^1_0([0,T]) \to C^0([0,T]) \) by

\[
Hw = G(v_k + w). \tag{2.25}
\]

the Frechet derivative of \( H \) at \( w \) is

\[
D_w H f = D_1 f + E(w)f, \tag{2.26}
\]

where

\[
E(w)f = 3(v_k + w)^2 f - 2\varepsilon Ae^x(v_k + w)f. \tag{2.27}
\]

Lemma 2.3.2

The operator norm of \( E(w) \), considered as a map from \( C^1_0([0,T]) \) into \( C^0([0,T]) \), satisfies

\[
\|E(w)\| \lesssim T \left( (\varepsilon Ae^T)^2 + \frac{1}{A^2} + T^2\|w\|_1^2 \right). \tag{2.28}
\]

Proof: Indeed, over \([0,T]\),

\[
\|\varepsilon Ae^x\|_{L^\infty} \leq \varepsilon Ae^T, \quad \left\| \frac{c}{A}e^{-x} \right\|_{L^\infty} \leq \frac{c}{A}.
\]
On complete embedded translating solitons...

Thus, by Lemma 2.2.1, and (2.9),
\[ \|v_k\|_{L^\infty} \lesssim \sum_{i=0}^{k} T^k \left( \epsilon A e^T + \frac{1}{A} \right)^{2k+1} \lesssim \epsilon A e^T + \frac{1}{A}. \]

Thus, by (2.23) and (2.27),
\[ \|E(w)f\|_{L^\infty} \lesssim \left( (\epsilon A e^T)^2 + \frac{1}{A^2} + \|w\|_{L^\infty}^2 \right) \|f\|_{L^\infty} \lesssim T \left( (\epsilon A e^T)^2 + \frac{1}{A^2} + T^2 \|w\|_1^2 \right) \|f\|_1. \]

The result follows. \( \square \)

We define the mapping \( \Phi : C_0^1([0,T]) \to C_0^1([0,T]) \) by
\[ \Phi(w) = w - D_1^{-1} H(w). \] (2.29)

**Lemma 2.3.3**

For \( w, \overline{w} \in C_0^1([0,T]), \)
\[ \|\Phi(w) - \Phi(\overline{w})\|_1 \lesssim T \left( (\epsilon A e^T)^2 + \frac{1}{A^2} + T^2 \|w\|_1^2 + T^2 \|\overline{w}\|_1^2 \right) \|w - \overline{w}\|_1. \] (2.30)

**Proof:** Indeed, for \( w, \overline{w} \in C_0^1([0,T]), \) using (2.26),
\[ \Phi(w) - \Phi(\overline{w}) = w - \overline{w} - D_1^{-1} (H(w) - H(\overline{w})) \]
\[ = D_1^{-1} (H(w) - H(\overline{w}) - D_1 (w - \overline{w})) \]
\[ = D_1^{-1} \left( \int_0^1 E(tw + (1-t)\overline{w}) dt \right) (\overline{w} - w). \]

Thus, by (2.23), (2.24) and (2.28),
\[ \|\Phi(w) - \Phi(\overline{w})\|_1 \lesssim T \left( (\epsilon A e^T)^2 + \frac{1}{A^2} + T^2 \|w\|_1^2 + T^2 \|\overline{w}\|_1^2 \right) \|w - \overline{w}\|_1, \]
as desired. \( \square \)

Applying the contraction-mapping theorem now yields
Lemma 2.3.4

For sufficiently large $\Delta$, there exists a unique function $\hat{v}_k$ in $C^1_0([0, T])$ such that $\hat{v}_k(0) = v_k(0)$ and $G\hat{v}_k = 0$. Furthermore,

$$
\|\hat{v}_k - v_k\|_1 \lesssim T^{k+1} \left( \epsilon A e^T + \frac{1}{A} \right)^{2k+3}.
$$

(2.31)

**Proof:** Uniqueness follows by the uniqueness of solutions to ODEs with prescribed initial values. By (2.20) and (2.24), there exists $C > 0$, which we may consider to be universal, such that

$$
\|\Phi(0)\|_1 = \|D^{-1}Gv_k\|_1 \leq CT^{k+1} \left( \epsilon A e^T + \frac{1}{A} \right)^{2k+3}.
$$

Let $X$ be the closed ball of radius $2CT^{k+1} \left( \epsilon A e^T + A^{-1} \right)^{2k+3}$ about 0 in $C^1_0([0, T])$. By (2.9), if $w, \overline{w} \in X$, then, in particular,

$$
T\|w\|_1, T\|\overline{w}\|_1 \lesssim \left( \epsilon A e^T + \frac{1}{A} \right),
$$

and so, by (2.30),

$$
\|\Phi(w) - \Phi(\overline{w})\|_1 \lesssim \frac{1}{\Delta} \|w - \overline{w}\|_1.
$$

In particular, for sufficiently large $\Delta$,

$$
\|\Phi(w) - \Phi(\overline{w})\|_1 \leq \frac{1}{2} \|w - \overline{w}\|_1.
$$

The mapping $\Phi$ is therefore a contraction mapping from $X$ to itself, and there therefore exists $w \in X$ such that $\Phi(w) = w$. In particular $Hw = 0$, and,

$$
\|w\|_1 \lesssim T^{k+1} \left( \epsilon A e^T + \frac{1}{A} \right)^{2k+3}.
$$

Existence follows with $\hat{v}_k = v_k + w$. This completes the proof. □

2.4 - The Small Scale - Exact Solutions II. We now refine this estimate via an algebraic argument.

**Lemma 2.4.1**

The function $v_k$ satisfies

$$
\hat{v}_k - v_k = O \left( T^{k+2} \left( \epsilon A e^T + \frac{1}{A} \right)^{2k+3} \right).
$$

(2.32)
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**Proof:** As in the proof of Lemma 2.3.4, we denote \( w = \hat{v}_k - v_k \). Recall that \( v_k \) is an \( \mathbb{R}[X] \)-polynomial in \( \epsilon Ae^x \) and \( \frac{c}{A} e^{-x} \). Thus, since \( Hw = 0 \),

\[
\frac{d}{dx} w = wP_1 \left( w, \epsilon Ae^x, \frac{c}{A} e^{-x} \right) + Gv_k,
\]

for some \( \mathbb{R}[X] \)-polynomial \( P_1 \). Since \( Gv_k \) is also an \( \mathbb{R}[X] \)-polynomial in \( \epsilon Ae^x \) and \( \frac{c}{A} e^{-x} \), it follows by induction that for all \( l \),

\[
\frac{d^l}{dx^l} w = wP_l \left( w, \epsilon Ae^x, \frac{c}{A} e^{-x} \right) + \sum_{p=0}^{l-1} Q_{p,l} \left( \epsilon Ae^x, \frac{c}{A} e^{-x} \right) \frac{d^p}{dx^p} Gv_k,
\]

(2.33)

for suitable \( \mathbb{R}[X] \)-polynomials \( P_l \) and \( Q_{p,l} \). By (2.9) and (2.31), for all \( l \) and for all \( p \),

\[
\| P_l \left( w, \epsilon Ae^x, \frac{c}{A} e^{-x} \right) \|_{L^\infty}, \| Q_{p,l} \left( \epsilon Ae^x, \frac{c}{A} e^{-x} \right) \| \lesssim 1.
\]

Furthermore, by (2.23) and (2.31),

\[
\| w \|_{L^\infty} \lesssim T^{k+2} \left( \epsilon Ae^T + \frac{1}{A} \right)^{2k+3}.
\]

Finally, by (2.20), for all \( l \),

\[
\left\| \frac{d^{l-1}}{dx^{l-1}} Gv_k \right\|_{L^\infty} \lesssim T^{k+2} \left( \epsilon Ae^T + \frac{1}{A} \right)^{2k+3}.
\]

The result follows upon combining these relations with (2.33). □

**Lemma 2.4.2**

The function \( v_k \) satisfies

\[
\hat{v}_{4k+9} - v_k = O \left( x^{k+1} \left( \epsilon Ae^x + \frac{1}{A} e^{-x} \right)^{2k+3} \right), \tag{2.34}
\]

**Proof:** Since (2.32) holds for all \( T \in [0, 3\log(A)] \), it follows by uniqueness that,

\[
\hat{v}_k - v_k = O \left( x^{k+2} \left( \epsilon Ae^x + \frac{1}{A} \right)^{2k+3} \right).
\]

In particular, over \([0, 3\log(A)]\), by (2.9)

\[
\hat{v}_{4k+9} - v_{4k+9} = O \left( x^{4k+11} \left( \epsilon Ae^x + \frac{1}{A} e^{-x} \right)^{2k+5} \right).
\]

Thus, by (2.9) and (2.19),

\[
\hat{v}_{4k+9} - v_k = (\hat{v}_{4k+9} - v_{4k+9}) + (v_{4k+9} - v_k) = O \left( x^{k+1} \left( \epsilon Ae^x + \frac{1}{A} e^{-x} \right)^{2k+3} \right),
\]

as desired. □

Substituting \( r = \epsilon Ae^x \) and recalling that, by the chain rule, \( \frac{d}{dr} = \frac{1}{r} \frac{d}{dx} \), we obtain,
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**Theorem 2.4.3**

The function $v_k$ satisfies

$$\hat{v}_{4k+9} - v_k = O \left( \log \left( \frac{r}{\epsilon A} \right)^{k+1} \frac{1}{r^1} \left( r + \frac{\epsilon}{r} \right)^{2k+3} \right). \quad (2.35)$$

In this paper, we only require the asymptotic expansion up to order 1 in $M + N$. We therefore set $\hat{v} = \hat{v}_9$, and we refer to this solution as the exact solution to (2.4) with logarithmic parameter $c$. Combining the above results yields

$$\hat{v} = \frac{1}{2} r + c \frac{\epsilon}{r} + O \left( \log \left( \frac{r}{\epsilon A} \right) \frac{1}{r^k} \left( r + \frac{\epsilon}{r} \right)^3 \right) \quad (2.36)$$

**2.5 - The Small Scale - Jacobi Fields.** There are two types of Jacobi fields over Grim surfaces: those arising from vertical translations, and those arising by variation of the logarithmic parameter. The former are essentially trivial, whilst asymptotic approximations of the latter over the small scale are readily derived from the results of the preceding sections, as we will now show. Let $\mathbb{R}[X][[M,N]]$ be as in Section 2.2 and define the operator $\partial_N$ over this space by

$$(\partial_N V)_{p,q} := (q + 1)V_{p,q+1}, \quad (2.37)$$

so that $\partial_N$ is simply the operator of formal differentiation with respect to $N$. By explicit calculation, we show that $N\partial_N$ commutes with $D$. Now let $V$ be the formal power series constructed in Lemma 2.2.1 and define

$$W := N\partial_N V. \quad (2.38)$$

Applying $N\partial_N$ to the relation $GV = 0$ yields

$$D_1 W + 3V^2 W - 2MVW = 0. \quad (2.39)$$

Fix a non-negative integer $k$ and let $V_k$ and $W_k$ be the $k$’th partial sums of $V$ and $W$ respectively, so that $V_k$ is given by (2.16), and

$$W_k := \sum_{p+q \leq 2k+1} W_{p,q}(X)M^p N^q. \quad (2.40)$$

It follows from (2.39) that

$$D_1 W_k + 3V_k^2 W_k - 2MV_k W_k = O((M + N)^{2k+3}). \quad (2.41)$$

Let $v_k$ and $w_k$ be the functions corresponding to $V_k$ and $W_k$ respectively. For all $k < l$,

$$w_l - w_k = O \left( x^{k+l} \left( \epsilon A e^x + \frac{1}{A} e^{-x} \right)^{2k+3} \right), \quad (2.42)$$

and it follows from (2.41) that, for all $k$,

$$D_1 w_k + 3v_k^2 w_k - 2(\epsilon A e^x)v_k w_k = O \left( x^{k+1} \left( \epsilon A e^x + \frac{1}{A} e^{-x} \right)^{2k+3} \right). \quad (2.43)$$

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Lemma 2.5.1

For sufficiently large $\Delta$ and for all $T \in [0, 3\log(A)]$, there exists a unique function $\hat{w}_k$ in $C^1([0, T])$ such that $\hat{w}_k(0) = w_k(0)$, and
\[
D_1 \hat{w}_k + 3\hat{v}_k^2 \hat{w}_k - 2\epsilon A e^x \hat{v}_k \hat{w}_k = 0. \tag{2.44}
\]
Furthermore,
\[
\|\hat{w}_k - w_k\|_1 \lesssim T^{k+1} \left( \epsilon A e^T + \frac{1}{A} \right)^{2k+3}. \tag{2.45}
\]

**Proof:** Existence and uniqueness follow by the existence and uniqueness of solutions to linear ODEs with prescribed initial values. We now estimate the norm of $\hat{w}_k - w_k$. By (2.43),
\[
\|D_1 w_k + 3v_k^2 w_k - 2(\epsilon A e^x) v_k w_k\|_0 \lesssim T^{k+1} \left( \epsilon A e^T + \frac{1}{A} \right)^{2k+3}.
\]
Observe that $\|v_k\|_0, \|\hat{v}_k\|_0, \|w_k\|_0 \lesssim 1$. Thus by (2.23) and (2.31),
\[
\|3\hat{v}_k^2 w_k - 3v_k w_k\|_0 = 3\| (\hat{v}_k - v_k) (\hat{v}_k + v_k) w_k\|_0 \lesssim T^{k+1} \left( \epsilon A e^T + \frac{1}{A} \right)^{2k+3},
\]
and
\[
\|2(\epsilon A e^x) \hat{v}_k w_k - 2(\epsilon A e^x) v_k w_k\|_0 \lesssim T^{k+1} \left( \epsilon A e^T + \frac{1}{A} \right)^{2k+3}.
\]
Thus
\[
\|D_1 (\hat{w}_k - w_k) + 3\hat{v}_k (\hat{w}_k - w_k) - 2 (\epsilon A e^x) \hat{v}_k (\hat{w}_k - w_k)\|_0
\]
\[
= \|D_1 w_k + 3\hat{v}_k w_k - 2(\epsilon A e^x) \hat{v}_k w_k\|_0
\]
\[
\lesssim T^{k+1} \left( \epsilon A e^T + \frac{1}{A} \right)^{2k+3}.
\]
Observe that $3\hat{v}_k w_k - 2(\epsilon A e^x) \hat{v}_k w_k = E(\hat{v}_k - v_k) w_k$, where $E$ is defined as in Section 2.3. In particular, by (2.9), (2.28) and (2.31), the operator norm of $E(\hat{v}_k - v_k)$, considered as a mapping from $C^1_0([0, T])$ into $C^0([0, T])$ satisfies
\[
\|E(\hat{v}_k - v_k)\| \lesssim T \left( (\epsilon A e^T)^2 + \frac{1}{A^2} \right).
\]
Thus, by (2.24), for sufficiently large $\Delta$, the operator $D_1 + E(\hat{v}_k - v_k)$ defines an invertible mapping from $C^1_0([0, T])$ into $C^0([0, T])$ and furthermore,
\[
\| (D_1 + E(\hat{v}_k - v_k))^{-1} \| \leq 4.
\]
Thus,
\[
\|\hat{w}_k - w_k\|_1 \lesssim T^{k+1} \left( \epsilon A e^T + \frac{1}{A} \right)^{2k+3},
\]
as desired. □

Proceeding exactly as in Section 2.4, we obtain

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Theorem 2.5.2
The function $w_k$ satisfies
\[
\hat{w}_{4k+9} - w_k = O \left( \log \left( \frac{r}{\epsilon A} \right)^{k+1} \frac{1}{r^k} \left( r + \frac{\epsilon}{r} \right)^{2k+3} \right).
\]

As in Section 2.4, we set $\hat{w} = \hat{w}_9$. Observe that, by uniqueness, $\hat{w}$ coincides with the derivative of $\hat{v}$ with respect to the logarithmic parameter. We readily obtain
\[
\hat{w} = \frac{\epsilon}{r} + O \left( \log \left( \frac{r}{\epsilon A} \right)^3 \left( r + \frac{\epsilon}{r} \right)^{3} \right).
\]

3 - The Grim Paraboloid.

3.1 - The MCFS Jacobi Operator. There exists a unique solution $v$ to the ODE problem $Gv = 0$ which is defined over the whole interval $[0, \infty]$. Furthermore $v(x)$ tends to 0 as $x$ tends to 0. Let $u$ be its integral with initial value 0, and let $G$ denote the surface of revolution of its graph. This surface is known as the Grim paraboloid. Let $J$ be its MCFS Jacobi operator as defined in Appendix A.2. In this section, we prove that $J$ defines an invertible operator over suitably weighted Sobolev and Hölder spaces over $G$, both of which will play an important role in the sequel. Indeed, for all $m$ and for all $\alpha$, we denote by $\| \cdot \|_{C^{m, \alpha}(G)}$ and $\| \cdot \|_{H^m(G)}$ the $m$'th order Hölder and Sobolev norms of functions over $G$ respectively (c.f. Appendix A.5). We denote by $C^{m, \alpha}(G)$ the completion of $C^\infty(G)$ with respect to the $m$'th order Hölder norm, and we denote by $H^m(G)$ the completion of $C^\infty_0(G)$ with respect to the $m$'th order Sobolev norm. For all real $\gamma$, we define following weight.
\[
\phi_\gamma := e^{(1+\gamma)u(x)}.
\]
Observe that $\phi_\gamma$ is strictly positive. We recall that, for all $m$, the $\phi_\gamma$-weighted Hölder and Sobolev spaces over $G$ are defined by
\[
C^{m, \alpha}_{G, \gamma} := \{ f \mid \phi_\gamma f \in C^{m, \alpha}(G_0) \},
H^m_{G, \gamma} := \{ f \mid \phi_\gamma f \in H^m(G_0) \},
\]
and that these spaces are furnished with the obvious norms. We now define,
\[
J_\gamma := M_\gamma J M_\gamma^{-1},
\]
where $M_\gamma$ is the operator of multiplication by $\phi_\gamma$. We observe that $J$ defines an invertible linear map over $C^{2, \alpha}_{G, \gamma}$ and $H^2_{G, \gamma}$ if and only if the conjugate operator $\hat{J}_\gamma$ defines an invertible linear map over $C^{2, \alpha}(G)$ and $H^2(G)$ respectively.* It is this perspective that we will adopt, and we will henceforth work with the conjugate operator $J_\gamma$. In fact, this operator is none other than the $\phi_\gamma$-Jacobi operator of the Grim paraboloid, which has been studied in detail in [3], [4], [5] and [6].

* in analogy to the relationship between the Heisenberg and Schrödinger pictures in quantum mechanics.
On complete embedded translating solitons...

**Lemma 3.1.1**

For all \( f \),

\[
J_\gamma f = \Delta^G f - \gamma \langle e_z, \nabla^G f \rangle + \frac{(\gamma^2 - 1)}{4} f - \frac{(1 + \gamma)^2}{4} \langle e_z, N_G \rangle^2 f + \text{Tr}(A_G^2) f. \tag{3.3}
\]

**Proof:** We think of \( \phi_\gamma \) as the restriction to \( G \) of the function \( e^{(1+\gamma)z/2} \). By (A.3),

\[
\nabla^G \phi_\gamma^{-1} = -\frac{(1 + \gamma)}{2\phi_\gamma} \pi^G(e_z),
\]

\[
\text{Hess}^G \phi_\gamma^{-1} = \frac{(1 + \gamma)^2}{4\phi_\gamma} dz \otimes dz + \frac{(1 + \gamma)}{2\phi_\gamma} \langle e_z, N_G \rangle \Pi^G.
\]

However, since \( G \) is an MCF soliton, \( H_G = -\langle e_z, N_G \rangle \), and taking the trace therefore yields

\[
\Delta^G \phi_\gamma^{-1} = \frac{(1 + \gamma)^2}{4\phi_\gamma} - \frac{(1 + \gamma)(3 + \gamma)}{4\phi_\gamma} \langle e_z, N_G \rangle^2.
\]

Thus, by (A.2),

\[
\phi_\gamma J_0 \phi_\gamma^{-1} = \frac{(\gamma^2 - 1)}{4} - \frac{(1 + \gamma)^2}{4} \langle e_z, N_G \rangle^2 + \text{Tr}(A_G)^2.
\]

The result now follows by (A.4). \( \square \)

Observe, in particular, that \( J_0 \) is self-adjoint. We will henceforth only concern ourselves with this case. By (A.6) and (2.8), for large values of \( r \),

\[
\langle e_z, N_G \rangle^2 = O(r^{-(2+k)}), \quad \text{Tr}(A_G^2) = O(r^{-(2+k)}). \tag{3.4}
\]

Let \( c : [0, \infty) \to \mathbb{R} \) be the geodesic curvature of the circle \( C(r) \) with respect to the induced metric of \( G \).

**Lemma 3.1.2**

The function \( c \) is given by

\[
c = \frac{1}{r} \langle e_z, N_G \rangle. \tag{3.5}
\]

In particular, for large values of \( r \),

\[
c = O(r^{-(2+k)}). \tag{3.6}
\]

**Proof:** Let \( D \) denote the Levi-Civita covariant derivative of the Euclidean metric over \( \mathbb{R}^3 \). Think of \( C(r) \) as a horizontal circle in \( \mathbb{R}^3 \) at height \( u(r) \). In particular, \( D_{e_\theta} e_\theta = \frac{1}{r} e_r \). Since
the geodesic curvature of $C(r)$ with respect to the induced metric over $G$ is equal to the tangential component of this vector, the function $c$ is given by

$$c = \frac{1}{r} \sqrt{1 - (e_r, N_G)^2} = \frac{1}{r} (e_z, N_G),$$

as desired. In particular, (3.6) follows from (3.4), and this completes the proof. □

Let $\rho : [0, \infty) \to \mathbb{R}$ be the intrinsic distance along $G$ of any point on the circle $C(r)$ from the origin. Since $\rho$ is obtained by integrating $\sqrt{1 + \|Du\|^2}$, by the asymptotic formula (2.8) for $r$ again, for large values of $r$,

$$\rho = r^2 + O(r^{-k}). \quad (3.7)$$

**Lemma 3.1.3**

Away from the $z$-axis,

$$J_0 f = f_{\rho\rho} + f_{\theta\theta} + cf_{\rho} - \frac{1}{4} f + \psi f, \quad (3.8)$$

where the subscripts $\rho$ and $\theta$ denote differentiation along unit radial and angular directions in $G$, and, for large $r$,

$$\psi \lesssim r^{-2}. \quad (3.9)$$

**Proof:** Indeed, away from the $z$-axis,

$$\Delta^G f = f_{\rho\rho} + f_{\theta\theta} + cf_{\rho}.$$

The result now follows from (3.3), (3.4) and (3.7). □

**3.2 - Invertibility over Sobolev Spaces.** We first show that $\hat{J}_0$ defines a Freholm map of index 0 from $H^2(G_0)$ into $L^2(G_0)$. The main technical difficulty here arises from the non-compactness of the ambient space. This is made up for by the following integral formula.

**Lemma 3.2.1**

There exist $B, R > 0$ such that for all $f$ in $H^2(G)$,

$$\|f\|_{A(R, \infty)} \leq B \left( \|f\|_{A(R-1, R+1)} \|L^2 + \|J_0 f\|_{A(R-1, \infty)} \|L^2 \right). \quad (3.10)$$

**Proof:** Since $C_0^\infty(G_0)$ is dense in $H^2(G_0)$, it is sufficient to prove the result when $f$ is smooth and has compact support. We denote $g = J_0 f$ and we define $\alpha, \beta : [0, \infty) \to \mathbb{R}$ by

$$\alpha(\rho) = \int_{C(\rho)} f^2 \, dl,$$

$$\beta(\rho) = \int_{C(\rho)} g^2 \, dl,$$
where $C(\rho)$ here denotes the circle of points lying at intrinsic distance $\rho$ along $G$ from the origin. Twice differentiating $\alpha$ yields

$$\alpha' = \int_{C(\rho)} 2ff_\rho + f^2 cdl,$$

$$\alpha'' = \int_{C(\rho)} 2f^2_\rho + 2ff_\rho + 4ff_\rho c + f^2 c_\rho + f^2 c^2 cdl.$$

By (3.8), and by definition of $g$,

$$\alpha'' = \int_{C(\rho)} 2f^2_\rho - 2f f_\theta \theta + \frac{1}{2} f^2 - 2\psi f^2 + 2fg + 2ff_\rho c + f^2 c_\rho + f^2 c^2 cdl.$$

Integrating by parts the term $2ff_\theta \theta$ and applying the algebraic-geometric mean inequality, we obtain

$$\alpha'' \geq \int_{C(\rho)} \left( \frac{1}{4} - 2\psi + c_\rho - c^2 \right) f^2 - 4g^2 cdl.$$

However, by (3.6), (3.9) and (3.7), $\psi$, $c$ and $c_\rho$ all tend to 0 as $\rho$ tends to $\infty$. Thus, for sufficiently large $\rho$

$$\alpha'' \geq \frac{1}{8} \alpha - 4\beta.$$

Since $f$ has compact support, upon integrating this relation, we obtain

$$\|f\|_{L^2(G)} \leq \int_R^\infty \alpha d\rho \leq 32 \int_R^\infty \beta d\rho - 8\alpha'(R) = 32\|\hat{J}_0 f\|_{L^2(G)}^2 - 8\alpha'(R).$$

However, by the Sobolev trace formula, and classical elliptic estimates,

$$\alpha'(R) \leq B \|f\|_{H^2(G)}^2,$$

for a suitable constant $B$. The result now follows upon combining the last two relations. □

**Lemma 3.2.2**

There exist $B, R > 0$ such that for all $f$ in $H^2(G)$,

$$\|f\|_{H^2(G)} \leq B \left( \|f\|_{L^2(G)} + \|\hat{J}_0 f\|_{L^2(G)} \right).$$

**Proof:** By classical elliptic theory, there exists $B > 0$ such that

$$\|f\|_{H^2(G)} \leq B \left( \|f\|_{L^2(G)} + \|\hat{J}_0 f\|_{L^2(G)} \right).$$

The result now follows upon combining this relation with (3.10). □
Lemma 3.2.3

\( J_0 \) defines a Fredholm map from \( H^2(G) \) into \( L^2(G) \) of Fredholm index equal to 0.

**Proof:** By Rellich’s compactness theorem, the canonical embedding of \( H^2(G) \) into the space \( L^2(B(0,R)) \) is compact. Thus, by (3.11), \( J_0 \) satisfies an elliptic estimate. It follows by Theorem A.6.1 that \( J_0 \) has finite-dimensional kernel and closed image. Observe now that \( J_0 \) is self adjoint, and so Ker(\( J_0 \)) is contained within the annihilator of Im(\( J_0 \)). Conversely, suppose that \( u \) is an element of \( L^2(G) \) which annihilates Im(\( J_0 \)). In particular, \( J_0u = 0 \) in the distributional sense. It follows by classical elliptic theory that \( u \) is an element of \( H^2(G) \), and, in particular, is an element of Ker(\( J_0 \)). We conclude that the cokernel of \( J_0 \) has finite-dimension equal to that of Ker(\( J_0 \)), and so \( J_0 \) is a Freholm map of Fredholm index equal to 0, as desired. □

The Fredholm alternative therefore holds for \( J_0 \) and in order to prove invertibility it thus suffices to prove that \( J_0 \) has trivial kernel in \( H^2(G) \).

Lemma 3.2.4

There exists no non-trivial bounded function \( f : \mathbb{R}^2 \to \mathbb{R} \) such that \( J_0f = 0 \).

**Proof:** Indeed, suppose that there exists a non-trivial bounded function \( f : \mathbb{R}^2 \to \mathbb{R} \) such that \( J_0f = 0 \). Upon multiplying by \(-1\), we may suppose that \( f \) is positive at some point. Now, since all vertical translations of \( G \) are also MCF solitons, the function \( \mu = \langle e_z, N_G \rangle \) is a Jacobi field over \( G \). That is,

\[ J_0 \phi_0 \mu = \phi_0 J \mu = 0. \]

Since \( G \) is a graph, the function \( \mu \) is everywhere strictly positive, and therefore so too is \( \phi_0 \mu \). Since \( \phi \gtrsim e^{-r^2/4} \) and \( \mu = O(r^{-1}) \), the function \( \phi_0 \mu \) also tends to infinity as \( r \) tends to infinity. Thus, since \( f \) is bounded, the quotient \( f/\phi_0 \mu \) attains its maximum value at some point \( x \), say, of \( G \). That is, upon rescaling, we may suppose that \( f/\phi_0 \mu \leq 1 \) and \( f(x)/\phi_0(x)\mu(x) = 1 \).

Since \( \mu \) is positive, we define the operator \( J_\mu := M_\mu^{-1}JM_\mu \), where \( M_\mu \) is the operator of multiplication by \( \mu \). Since \( J_\mu = 0 \), by (A.4), this operator has no zeroeth order term. Thus, since \( J_\mu (f/\mu \phi_0) = (1/\mu \phi_0)J_0f = 0 \), it follows by the strong maximum principle that \( f/\phi_0 \mu \) is constant and equal to 1. However, since \( \phi_0 \mu \) is unbounded, this is absurd, and the result follows. □

Lemma 3.2.5

\( J_0 \) has trivial kernel in \( H^2(G) \).

**Proof:** Indeed, by the Sobolev embedding theorem, every element of \( H^2(G) \) is bounded, and the result now follows by Lemma 3.2.4. □

**Theorem 3.2.6, Invertibility over Sobolev Spaces**

For sufficiently small \( \gamma \), \( J \) defines an invertible linear map from \( H^2_{G,\gamma} \) into \( L^2_{G,\gamma'} \).

**Proof:** Indeed, by Lemmas 3.2.3 and 3.2.5, \( J \) defines an invertible linear map from \( H^2_{G,0} \) into \( L^2_{G,0} \). However, by (3.3) and (3.4), as \( \gamma \) tends to 0, \( J_\gamma \) converges to \( J_0 \) in the operator norm, and the result follows. □
3.3 - Invertibility over Hölder Spaces. We prove the invertibility of $J_0$ over $C^{2, \alpha}(G)$ in essentially the same manner. We first require the following preliminary result.

**Lemma 3.3.1**

Let $\alpha$ and $\beta$ be positive constants. If $\phi : [0, \infty[ \rightarrow ]0, \infty[ \text{ is a bounded, positive function such that } \phi'' \geq \alpha^2 \phi - \beta \text{ in the viscosity sense, then for all } t$,

$$\phi(t) \leq \text{Max}(\phi(0) - \beta/\alpha^2, 0)e^{-\alpha t} + \beta/\alpha^2.$$  \hspace{1cm} (3.12)

**Proof:** Let $A = \text{Max}(\phi(0) - \beta/\alpha^2, 0)$ and let $B = \text{Sup}_{t \in [0, \infty]} \phi(t)$. Fix $T > 0$ and define

$$f = \frac{Be^{\alpha T} - A}{e^{2\alpha T} - 1} e^{\alpha t} + \frac{A - Be^{-\alpha T}}{1 - e^{-2\alpha T}} e^{-\alpha t} + \beta/\alpha^2.$$  

In other words, $f$ is the unique solution to the ODE problem, $f'' = \alpha^2 f - \beta$, with boundary values $f(0) = A + \beta/\alpha^2 \geq \phi(0)$ and $f(T) = B + \beta/\alpha^2 \geq \phi(T)$. Let $C$ be the minimum of $f - \phi$ over $[0, T]$ and let $t \in [0, T]$ be the point at which this minimum is attained. If $t$ is a boundary point of this interval, then $C \geq 0$. Otherwise, $f - C \geq \phi$ and $f(t) - C = \phi(t)$. Thus, since $\phi$ is a viscosity solution of $\phi'' \geq \alpha^2 \phi - \beta$, the function $f - C$ satisfies $(f - C)'' \geq \alpha^2 (f - C) - \beta$ at this point, and so $C \geq 0$. In each case, we obtain

$$\phi \leq f = \frac{Be^{\alpha T} - A}{e^{2\alpha T} - 1} e^{\alpha t} + \frac{A - Be^{-\alpha T}}{1 - e^{-2\alpha T}} e^{-\alpha t} + \beta/\alpha^2.$$  

The result now follows upon taking the limit as $T$ tends to infinity. □

As in the Sobolev case, the non-compactness of the ambient space is now made up for by the following estimate.

**Lemma 3.3.2**

There exist $B, R > 0$ such that for all $f$ in $C^{2, \alpha}(G)$,

$$\|f\|_{A(\infty, \infty)} \|C^0 \leq B \left( \|f\|_{C(R)} \|C^0 + \|J_0 f\|_{A(R-1, \infty)} \|C^0 \right).$$  \hspace{1cm} (3.13)

**Proof:** We define $\alpha : ]0, \infty[ \rightarrow \mathbb{R}$ by

$$\alpha(\rho) = \text{Sup}_{x \in C(\rho)} f(x)^2.$$  

We denote $g := J_0 f$, and we define $B \geq 0$ by

$$B = \|g^2\|_{A(\infty, \infty)} \|C^0.$$  

Choose $x \in C(\rho)$ maximising $f^2$. Observe that $\rho f_{\rho \theta}$ is non-positive at this point. Thus, bearing in mind (3.8),

$$(f^2)_{\rho \rho} = 2f_{\rho}^2 + 2ff_{\rho \rho},$$

$$\geq 2f_{\rho}^2 + 2fg - 2cf_{\rho} + \frac{1}{2} f^2 - 2\psi f^2,$$

$$\geq \left( \frac{1}{4} - \frac{1}{2} c^2 - 2\psi \right) f^2 - 4g^2.$$  

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By (3.6) and (3.9), for sufficiently large $\rho$

$$(f^2)_{\rho\rho} \geq \frac{1}{8} f^2 - 4g^2.$$ 

Since $\alpha$ is the envelope of the restriction of $f(x)^2$ to each radial line, it follows that over $[R, \infty[$,

$$\alpha'' \geq \frac{1}{8} \alpha - 4B,$$

in the viscosity sense. Thus, by Lemma 3.3.1,

$$\sup_{x \in A(R, \infty)} f^2(x) = \sup_{\rho \geq R} \alpha(\rho) \leq \max(\|f^2\|_{C(R)}\|C^0 - 32B, 0\| + 32B).$$

The result follows. □

**Lemma 3.3.3**

There exist $B, R > 0$ such that for all $f$ in $C^{2,\alpha}(G)$,

$$\|f\|_{C^{2,\alpha}(G)} \leq B (\|f\|_{B(0, R)}\|C^0(G)\| + \|J_0 f\|_{C^{0,\alpha}(G)}). \tag{3.14}$$

**Proof:** By classical elliptic theory, there exists $B > 0$ such that

$$\|f\|_{C^{2,\alpha}} \leq B (\|f\|_{C^0(G)} + \|J_0 f\|_{C^{0,\alpha}(G)}),$$

The result now follows upon combining this relation with (3.13). □

**Theorem 3.3.4**

For all sufficiently small $\gamma$, $J$ defines an invertible mapping from $C^{2,\alpha}_{G,\gamma}$ into $C^{0,\alpha}_{G,\gamma}$.

**Proof:** By the Arzela-Ascoli theorem, the canonical embedding of $C^{2,\alpha}(G)$ into the space $C^0(B(0, R))$ is compact. Thus, by (3.14), $J_0$ satisfies an elliptic estimate. It follows from Theorem A.6.1 that the image of $J_0$ is closed, and, in particular, is a Banach subspace of $C^{0,\alpha}(G)$. However, by Lemma 3.2.4, the kernel of $J_0$ in $C^{2,\alpha}(G)$ is trivial. It follows from the closed graph theorem that $J_0$ defines a linear isomorphism from $C^{2,\alpha}(G)$ into its image. In particular, there exists a constant $B > 0$ such that for all $u \in C^{2,\alpha}(G)$,

$$\|u\|_{2,\alpha} \leq B \|J_0 u\|_{0,\alpha}. \tag{3.15}$$

We now prove surjectivity. Fix $v \in C^{0,\alpha}(G)$ and let $(v_m)$ be a sequence of smooth functions of compact support in $G$ which is bounded in $C^{0,\alpha}$ and which converges to $v$ in the $C^{0,\alpha}$ sense. For all $m$, since it is smooth with compact support, $v_m$ is an $L^2$ function over $G$, and so, by Theorem 3.2.6, there exists a unique $H^2$ function, $u_m$ such that $J_0 u_m = v_m$. Furthermore, by elliptic regularity $u_m$ is an element of $C^{2,\alpha}(G)$ and so, by (3.15), for all $m$,

$$\|u_m\|_{C^{2,\alpha}(G)} \leq B \|v_m\|_{C^{0,\alpha}(G)}.$$ 

In particular, the sequence $(u_m)$ is uniformly bounded in $C^{2,\alpha}(G)$, and it follows by the Arzela-Ascoli theorem that there exists $u \in C^{2,\alpha}(G)$ towards which $(u_m)$ subconverges in the $C^{2,\beta}_{loc}$-topology for all $\beta < \alpha$. In particular, by continuity, $J_0 u = v$ and since $v$ is arbitrary, it follows that $J_0$ is surjective as desired. Finally, by (3.3) and (3.4), $J_\gamma$ converges to $J_0$ in the operator norm as $\gamma$ tends to 0, and this completes the proof. □
4 - Grim Ends.

4.1 - The Modified MCFS Jacobi Operator. We now return to the case of general Grim surfaces. Let $\Delta$ be a large, positive real number, and let $\epsilon, A > 0$ and $c \in \mathbb{R}$ satisfy (2.9). Let $v : [\epsilon A, \infty] \rightarrow \mathbb{R}$ be the solution to (2.4) with logarithmic coefficient $c$ (as defined in Section 2.4). We recall from (2.36) that, over the interval $[\epsilon A, \epsilon A^4]$,

$$v = \frac{1}{2} r + \frac{c\epsilon}{r} + O \left( \log \left( \frac{r}{\epsilon A} \right) \frac{1}{r^k} \left( r + \frac{\epsilon}{r} \right)^3 \right). \quad (4.1)$$

Let $u : [\epsilon A, \infty] \rightarrow \mathbb{R}$ be a primitive of $v$ and let $G$ be the surface of revolution generated by rotating the graph of $u$ about the $z$-axis so that $G$ is a Grim end.

Let $J$ be the MCFS Jacobi operator of $G$, as defined in Appendix A.2. Ideally, we would like a uniform bound for the norm of the Green’s operator of $J$ that is independent of $\Delta$. However, the zero’th order coefficients of $J$ diverge rapidly over the annulus $A(\epsilon A, \epsilon A^4)$ as $\Delta$ tends to infinity. We therefore first reduce the order of divergence by conjugating $J$ with the operator of multiplication by a suitably chosen positive function before eliminating the result lower-order terms via an averaging argument. Thus, let $\chi_1$ be the cut-off function of the transition region $A(1, 2)$ and define $\psi : A(\epsilon A, \infty) \rightarrow \mathbb{R}$ by

$$\psi(r) = \chi_1 \langle e_z, N_G \rangle + (1 - \chi_1). \quad (4.2)$$

In particular, since every vertical translate of $G$ is also an MCF soliton, $\langle e_z, N_G \rangle$ is a Jacobi field and so, over $A(\epsilon A, 1)$,

$$J\psi = 0. \quad (4.3)$$

We now define the modified MCFS Jacobi operator of $G$ by,

$$\hat{J} := M_{\psi}^{-1} JM_{\psi}, \quad (4.4)$$

where $M_{\psi}$ denotes the operator of multiplication by $\psi$. Observe that, over $A(\epsilon A, 1)$, $\hat{J}$ is in fact simply the linearisation of the MCFS operator for graphs of functions. We aim to prove the obtain uniform norm bounds for the Green’s operator of $\hat{J}$ over $H^2_{G, \gamma}$ and $C^{2, \alpha}_{G, \gamma}$ for all sufficiently small $\gamma$. In order to apply the results of the previous section, we set $\chi_2$ to be the cut-off function of the transition region $A(2, 4)$, and we define,

$$\hat{J}_\gamma = M_{\gamma}^{-1} \hat{J} M_{\gamma},$$

where $M_{\gamma}$ is the operator of multiplication by $\chi_2 + (1 - \chi_2)\phi_\gamma$ and $\phi_\gamma$ is given by (3.1). Observe that $\hat{J}_\gamma$ coincides with $\hat{J}$ over $B(1)$. Furthermore, since $\phi_\gamma$ and $\psi$ are functions of $u$ and $u'$, it follows by (A.2) that the coefficients of $\hat{J}_\gamma$ are functions of $u$ and its derivatives up to order 3.
Lemma 4.1.1

Over $A(\epsilon A, 1)$, the modified MCFS Jacobi operator of $G$ is given by

$$\hat{J} f = g^{ij} f_{ij} - 2\mu g^{ip} g^{jq} u_{pq} u_{ij} f_i.$$  \hspace{1cm} (4.5)

**Proof:** First observe that for every tangent vector, $X$, over $G$,

$$\langle \nabla^G \psi, X \rangle = X \psi = X \langle N_G, e_z \rangle = \langle D_X N_G, e_z \rangle = \langle A_G X, e_z \rangle = \langle X, A_G \pi^G(e_z) \rangle,$$

and so,

$$\nabla^G \psi = A_G \pi^G(e_z).$$

Thus, by (4.3) and (A.4),

$$\hat{J} f = \Delta^G f + \langle e_z, \nabla^G f \rangle + 2\psi^{-1} \langle A_G \nabla^G f, e_z \rangle.$$

By (A.3),

$$\text{Hess}^G f = \text{Hess}(f) \circ \pi - \langle D(f \circ \pi), N \rangle \Pi_G.$$

Furthermore, since $D(f \circ \pi)$ is horizontal

$$\langle D(f \circ \pi), N_G \rangle = -\frac{1}{\langle N_G, e_z \rangle} \langle D(f \circ \pi), e_z - \langle N_G, e_z \rangle N_G \rangle = -\frac{1}{\langle N_G, e_z \rangle} \langle \nabla^G f, e_z \rangle.$$

Taking the trace therefore yields

$$\Delta^G f = g^{ij} f_{ij} + \frac{1}{\langle N_G, e_z \rangle} \langle \nabla^G f, e_z \rangle H_G.$$

However, since $G$ is an MCF soliton, $H_G = -\langle N, e_z \rangle$, and so

$$\Delta^G f = g^{ij} f_{ij} - \langle \nabla^G f, e_z \rangle.$$

We conclude that

$$\hat{J} f = g^{ij} f_{ij} + 2\psi^{-1} \langle A_G \nabla^G f, e_z \rangle,$$

and the result now follows by (A.6). □
Lemma 4.1.2
Over $A(\epsilon A, 2\epsilon A^4)$, the modified MCFS Jacobi Operator of $G$ satisfies

$$
\hat{J} f = \Delta f - \left( \frac{1}{2} + \frac{\epsilon}{r^2} \right)^2 x^i x^j f_{ij} - \left( \frac{1}{2} - \frac{2\epsilon^2}{r^4} \right) x^i f_i + \mathcal{E}_G f,
$$

(4.6)

where $\mathcal{E}_G f = a^{ij} f_{ij} + b^i f_i$ and $a$ and $b$ satisfy

$$
a = O \left( \log \left( \frac{r}{\epsilon A} \right) \frac{1}{r^k} \left( r + \frac{\epsilon}{r} \right)^4 \right),
$$

$$
b = O \left( \log \left( \frac{r}{\epsilon A} \right) \frac{1}{r^{k+1}} \left( r + \frac{\epsilon}{r} \right)^4 \right).
$$

(4.7)

**Proof:** Indeed, by (4.1),

$$
\frac{1}{2} x_i + \frac{\epsilon}{r^2} x_i + O \left( \log \left( \frac{r}{\epsilon A} \right) \frac{1}{r^k} \left( r + \frac{\epsilon}{r} \right)^3 \right).
$$

Thus, by (A.6),

$$
\mu^2 = 1 - \left( \frac{r}{2} + \frac{\epsilon}{r} \right)^2 + O \left( \log \left( \frac{r}{\epsilon A} \right) \frac{1}{r^k} \left( r + \frac{\epsilon}{r} \right)^4 \right),
$$

$$
g^{ij} = \delta_{ij} - \left( \frac{1}{2} + \frac{\epsilon}{r^2} \right)^2 x^i x^j + O \left( \log \left( \frac{r}{\epsilon A} \right) \frac{1}{r^{k+1}} \left( r + \frac{\epsilon}{r} \right)^4 \right).
$$

It follows that

$$
g^{ij} f_{ij} = \Delta f - \left( \frac{1}{2} + \frac{\epsilon}{r^2} \right)^2 x^i x^j f_{ij} + \alpha^{ij} f_{ij},
$$

where $a = O \left( \log(r/\epsilon A) r^{-k}(r + \epsilon/r)^4 \right)$, and since $r^{-1}(r + \epsilon/r)^4$ bounds $(r + \epsilon/r)^3$,

$$
-2 \mu g^{ip} g^{jq} u_{pq} u_i f_j = - \left( \frac{1}{2} - \frac{2\epsilon^2}{r^4} \right) x^i f_i + \beta^i f_i,
$$

where $b = O \left( \log(r/\epsilon A) r^{-(k+1)}(r + \epsilon/r)^4 \right)$. The result follows. □

4.2 - Harmonic Extensions. Let $v_p : [0, \infty[ \to \mathbb{R}$ be the unique solution to (2.4) which is defined over the whole positive half-line, as introduced in Section 3.1. Let $u_p$ be any primitive of $v_p$. Let $G_p$ be the surface of revolution obtained by rotating the graph of $u_p$ about the $z$-axis so that $G_p$ is the Grim paraboloid. Let $\hat{J}_p$ be its modified MCFS Jacobi operator, as constructed in Section 4.1. Observe that

$$
v_p(r) = \frac{1}{2} r + O(r^{3-k}).
$$

(4.8)
On complete embedded translating solitons...

Thus, as in Lemma (4.6), over $B(0, 2\epsilon A^4)$,

$$\hat{J}_p f = \Delta f - \frac{1}{2} x^i x^j f_{ij} - \frac{1}{2} x^i f_i + \mathcal{E}_p f,$$

where $\mathcal{E}_p f = a^{ij} f_{ij} + b^i f_i$ and $a$ and $b$ satisfy

$$a = O(r^{4-k}),$$
$$b = O(r^{3-k}).$$

We aim to estimate the difference between the modified MCFS Jacobi operators of $G$ and $G_p$. However, since the operator $\hat{J}_0$ is only defined over the annulus $A(\epsilon A, \infty)$, it is necessary to construct a natural extension to the whole of $\mathbb{R}^2$. We proceed as follows. Given an operator $L = a^{ij} \partial_i \partial_j + b^i \partial_i + c$ defined over $A(\epsilon A, \infty)$, we extend the coefficients $a$, $b$ and $c$ to functions defined over the whole of $\mathbb{R}^2$ by setting them to be harmonic over $B(\epsilon A)$. We call the resulting operator the harmonic extension of $L$. Observe that if $L$ is elliptic, then so too is its harmonic extension. Likewise, if $L$ has any rotational symmetry, or if $L$ has locally Lipschitz coefficients, then so too does its harmonic extension. Now let $\chi$ be the cut-off function of the transition region $A(\epsilon A^4, 2\epsilon A^4)$. We define the operators $D$ and $E$ over $A(\epsilon A, \infty)$ by

$$Df := (\hat{J}_0 - E) f - \hat{J}_{p,0} f,$$
$$Ef := \chi \frac{2c^2 \epsilon^2}{r^4} x^i f_i,$$

and we extend these operators harmonically to the whole of $\mathbb{R}^2$. The extension of $\hat{J}_0$ is then naturally given by

$$\hat{J}_0 := \hat{J}_{p,0} + D + E.$$

We first show that the coefficients of $D$ tend to zero in all norms that concern us as $\Delta$ tends to infinity. Indeed, by (4.6), (4.7), (4.9) and (4.10),

$$Df = a^{ij} f_{ij} + b^i f_i,$$

where, over $A(\epsilon A, 2\epsilon A^4)$, $a$ and $b$ satisfy

$$a^{ij} = -\frac{c\epsilon}{r^2} x^i x^j - \frac{c^2 \epsilon^2}{r^4} x^i x^j + O \left( \log \left( \frac{r}{\epsilon A} \right) \frac{1}{r^k} \left( r + \frac{\epsilon}{r} \right)^4 \right),$$
$$b^i = (1 - \chi) \frac{2c^2 \epsilon^2}{r^4} x^i + O \left( \log \left( \frac{r}{\epsilon A} \right) \frac{1}{r^k+1} \left( r + \frac{\epsilon}{r} \right)^4 \right).$$
Lemma 4.2.1
For sufficiently small $\alpha$,
\[ \|a|_{B(\varepsilon A)}\|_{C^{0,\alpha}}, \|b|_{B(\varepsilon A)}\|_{C^{0,\alpha}} \to 0, \quad (4.14) \]
as $\Delta$ tends to infinity.

Proof: Indeed, by (4.13), since $\chi$ equals 1 near $C(\varepsilon A)$, over this circle,
\[
a = O\left( \frac{1}{(\varepsilon A)^k} \left( \varepsilon + \frac{1}{A^2} + (\varepsilon A)^4 + \frac{1}{A^4} \right) \right),
\[
b = O\left( \frac{1}{(\varepsilon A)^{k+1}} \left( (\varepsilon A)^4 + \frac{1}{A^4} \right) \right).
\]

By the maximum principle, these relations continue to hold over the whole of $B(\varepsilon A)$. Thus, by (A.12), for all $\alpha \in [0, 1]$,
\[
\|a|_{B(\varepsilon A)}\|_{C^{0,\alpha}} \lesssim \frac{\varepsilon^{1-\alpha}}{A^\alpha} + \frac{1}{\varepsilon^\alpha A^{2+\alpha}} + (\varepsilon A)^{4-\alpha} + \frac{1}{\varepsilon^\alpha A^{4+\alpha}},
\]
\[
\|b|_{B(\varepsilon A)}\|_{C^{0,\alpha}} \lesssim (\varepsilon A)^{3-\alpha} + \frac{1}{\varepsilon^{1+\alpha} A^{5+\alpha}}.
\]

By (2.9), for sufficiently small $\alpha$, these both tend to 0 as $\Delta$ tends to infinity, as desired. $\square$

Lemma 4.2.2
For sufficiently small $\alpha$,
\[ \|a|_{A(\varepsilon A, 2\varepsilon A^4)}\|_{C^{0,\alpha}}, \|b|_{A(\varepsilon A, 2\varepsilon A^4)}\|_{C^{0,\alpha}} \to 0, \quad (4.15) \]
as $\Delta$ tends to infinity.

Proof: Indeed, by (4.13), over $A(\varepsilon A, 2\varepsilon A^4)$,
\[
a = O\left( \frac{1}{r^k} \left( \varepsilon + \frac{\varepsilon^2}{r^2} \right) \right) + O\left( \log\left( \frac{r}{\varepsilon A} \right) \frac{1}{r^k} \left( \varepsilon + \frac{\varepsilon}{r} \right)^4 \right),
\]
and $b = b_1 + (1 - \chi)c$, where
\[
b_1 = O\left( \log\left( \frac{r}{\varepsilon A} \right) \frac{1}{r^{k+1}} \left( r^4 + \frac{\varepsilon^4}{r^4} \right) \right),
\]
\[c = \frac{2\varepsilon^2 \varepsilon^2}{r^4} x^i.
\]

Thus, by (A.12) and (A.22), for all $\alpha \in [0, 1]$,
\[
\|a|_{A(\varepsilon A, 2\varepsilon A^4)}\|_{C^{0,\alpha}} \lesssim \frac{\varepsilon^{1-\alpha}}{A^\alpha} + \frac{1}{\varepsilon^\alpha A^{2+\alpha}} + \log(A)(\varepsilon A^4)^{4-\alpha} + \frac{1}{\varepsilon^\alpha A^{4+\alpha}},
\]
\[
\|b_1|_{A(\varepsilon A, 2\varepsilon A^4)}\|_{C^{0,\alpha}} \lesssim \log(A)(\varepsilon A^4)^{3-\alpha} + \frac{1}{\varepsilon^{1+\alpha} A^{5+\alpha}}.
\]
By (2.9), for sufficiently small $\alpha$, these both tend to 0 as $\Delta$ tends to infinity. Finally, over $A(\epsilon A^4, 2\epsilon A^4)$,
\[
c = O(\epsilon^2 r^{-(k+3)}),
\]
\[
(1 - \chi) = O((\epsilon A^4)^{-k}).
\]
Thus, by (A.12),
\[
\|c\|_{A(\epsilon A^4, 2\epsilon A^4)} C^{0, \alpha} \lesssim \frac{1}{\epsilon^{1+\alpha} A^{12+4\alpha}},
\]
\[
\|(1 - \chi)\|_{A(\epsilon A^4, 2\epsilon A^4)} C^{0, \alpha} \lesssim \frac{1}{\epsilon^{\alpha} A^{4\alpha}}.
\]
By (A.14), combining these relations yields
\[
\|(1 - \chi)\gamma\|_{A(\epsilon A^4, 2\epsilon A^4)} C^{0, \alpha} \lesssim \frac{1}{\epsilon^{1+\alpha} A^{12+4\alpha}}.
\]
By (2.9), for sufficiently small $\alpha$, this tends to 0 as $\Delta$ tends to infinity, and the result follows. □

**Lemma 4.2.3**

*If $\epsilon A < s < t < \sqrt{2}$, then*
\[
|v(t) - v_p(t)| \leq |v(s) - v_p(s)|. \tag{4.16}
\]

**Proof:** Indeed, by (2.4),
\[
r(\dot{v} - \dot{v}_p) = -(v - v_p)(1 - r(v + v_p) + (v^2 + vv_p + v_p^2)).
\]
However
\[
1 - r(v + v_p) + (v^2 + vv_p + v_p^2) \geq 1 - \frac{r^2}{2}.
\]
Thus, for $r \leq \sqrt{2}$, $|v - v_p|$ is decreasing, and the result follows. □

**Lemma 4.2.4**

\[
\|a\|_{A(\epsilon A^4, 1)} C^{1}, \|b\|_{A(\epsilon A^4, 1)} C^{1} \to 0, \tag{4.17}
\]
as $\Delta$ tends to infinity.

**Proof:** By (4.1) and (4.8), over $C(2\epsilon A^4)$,
\[
v - v_p \lesssim \frac{1}{A^4} + \log(A)(\epsilon A^4)^3 + \log(A)\frac{1}{A^{12}}.
\]
By Lemma 4.2.3, this inequality continues to hold over the whole of $A(2\epsilon A^4, 1)$. Since $v$ and $v_0$ both solve (2.4), it follows that, over this annulus,
\[
v - v_p = O\left(\frac{1}{(\epsilon A^4)^k} \left(\frac{1}{A^4} + \log(A)(\epsilon A^4)^3 + \log(A)\frac{1}{A^{12}}\right)\right),
\]

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Thus,
\[ \|(v - v_p)\|_{[2\epsilon A^4, 1]} \lesssim \frac{1}{\epsilon^2 A^{12}} + \log(A)\epsilon A^4 + \log(A) \frac{1}{\epsilon^2 A^{20}}. \]

It follows by (2.9) that
\[ \|(v - v_p)\|_{[2\epsilon A^4, 1]} \to 0 \]
as \( \Delta \) tends to infinity. However, by (4.5), over \( A(\epsilon A^4, 1) \), the coefficients \( a \) and \( b \) only depend on the first derivatives of \( v \) and \( v_p \), and so
\[ \|a|_{A(\epsilon A^4, 1)}\|_{C^1}, \|b|_{A(\epsilon A^4, 1)}\|_{C^1} \to 0, \]
as \( \Delta \) tends to infinity, as desired. ☐

**Lemma 4.2.5**

*For all \( \epsilon > 0 \), there exists \( R > 0 \) such that if \( |v(1) - v_p(1)| \leq 1 \), then*
\[ \|a|_{A(R, \infty)}\|_{C^1(G)}, \|b|_{A(R, \infty)}\|_{C^1(G)} \leq \epsilon. \]  

**(4.18)**

**Proof:** Indeed, over \( A(4, \infty) \), both \( \hat{J}_0 \) and \( \hat{J}_{p,0} \) are given by (3.8). The result now follows by local uniform dependence of the estimates in (3.9) on the initial value. ☐

**Lemma 4.2.6**

*For all \( R > 1 \),*
\[ \|a|_{A(1, R)}\|_{C^1}, \|b|_{A(1, R)}\|_{C^1} \to 0, \]  

**(4.19)**
as \( \Delta \) tends to infinity.

**Proof:** By (4.1), (4.8) and (4.16), over \( C(1) \),
\[ |v - v_p| \lesssim \frac{1}{A^4} + \log(A) (\epsilon A^4)^3 + \log(A) \frac{1}{A^{12}}. \]
Since solutions of first order ODEs vary smoothly with their parameters,
\[ \|(v - v_p)\|_{[1, R]} \lesssim 0, \]
as \( \Delta \) tends to \( \infty \). However, over \( A(1, R) \), \( a \) and \( b \) only depend on \( v \) and \( v_0 \) and their derivatives up to order 2, and the result follows. ☐

Combining these results yields,

**Lemma 4.2.7**

*(1) The operator norm of \( D \), considered as a mapping from \( H^2(G) \) into \( L^2(G) \) converges to zero as \( \Delta \) tends to infinity; and*

*(2) For sufficiently small \( \alpha \), the operator norm of \( D \), considered as a mapping from \( C^{2,\alpha}(G) \) into \( C^{0,\alpha}(G) \) converges to zero as \( \Delta \) tends to infinity.*

**Proof:** Indeed, by (4.14), (4.15), (4.17), (4.18) and (4.19), \( \|a\|_{C^{0,\alpha}(G)} \) and \( \|b\|_{C^{0,\alpha}(G)} \) both converge to 0 as \( \Delta \) tends to infinity. In particular, \( \|a\|_{L^\infty} \) and \( \|b\|_{L^\infty} \) tend to 0 as \( \Delta \) tends to infinity, and (1) follows. (2) follows by (A.12), and this completes the proof. ☐
4.3 - The Singular Part. We now write

$$Ef = a^i f_i,$$  \tag{4.20}

so that, by harmonicity, over $$B(\epsilon A),$$

$$a^i = \frac{2c^2}{\epsilon^2 A^4} x^i.$$ \tag{4.21}

Lemma 4.3.1

For all $$1 \leq p < \infty$$

$$\|\alpha\|_{L^p} \lesssim \epsilon^{\frac{2}{p} - 1} A^{\frac{2}{p} - 3}. \tag{4.22}$$

Proof: By (4.21),

$$\int_{B(\epsilon A)} |a|^p dVol = \frac{2\pi}{p + 2} (2c^2)^p \epsilon^{2-p} A^{2-3p}.\tag{4.21}$$

By (4.11), since $$p > 1,$$

$$\int_{A(\epsilon A, 2\epsilon A^4)} |a|^p dVol \leq \frac{2\pi}{3p - 2} (2c^2)^p \epsilon^{2-p} A^{2-3p}.$$

The result follows. \Box

Lemma 4.3.2

For all $$p > 1,$$ the operator norm of $$E$$ as a mapping from $$H^2(G)$$ into $$L^2(G)$$ satisfies

$$\|E\| \lesssim \epsilon^{\frac{1}{p} - 1} A^{\frac{1}{p} - 3}. \tag{4.23}$$

In particular, for $$p$$ sufficiently close to 1, this tends to 0 as $$\Delta$$ tends to infinity.

Proof: Choose $$p > 1$$ and let $$q$$ be such that $$(2p)^{-1} + (2q)^{-1} = 1.$$ By the Sobolev embedding theorem, for all $$f \in H^2(G),$$

$$\|Df\|_{L^q(B(1))} \lesssim C\|f\|_{H^2(G)}.$$\tag*{R.11}

Thus, by Hölder’s inequality,

$$\|Ef\|_{L^2}^2 = \int_{B(1)} (a^i f_i)^2 dVol \leq \|a\|_{L^{2p}(B(1))}^2 \|Df\|_{L^{2q}(B(1))}^2$$

$$\lesssim \epsilon^{\frac{1}{2p} - 2} A^{\frac{1}{2p} - 6}\|f\|_{H^2(G)}^2.$$

The result follows. \Box

In order to average out the contribution of $$E$$ over Hölder spaces, it is necessary to impose symmetries. Thus, for all $$\theta,$$ we denote by $$C^{2, \alpha}_\theta(G)$$ the subspace of $$C^{2, \alpha}(G)$$ consisting of those functions which are symmetric by rotation by an angle of $$\theta$$ about the origin.
Lemma 4.3.3

If \( \theta \) is not an integer multiple of \( 2\pi \), and if \( f \in C^{2,\alpha}_\theta(G) \), then there exists a matrix valued function \( M(x) \) such that

\[
\langle Df(x), x \rangle = \langle M(x)x, x \rangle,
\]

and

\[
\|M\|_{B(0,1)} \lesssim \|f\|_{C^{2,\alpha}(G)}.
\]

**Proof:** Since \( \theta \) is not an integer multiple of \( 2\pi \), \( Df(0) = 0 \). It follows that \( Df(x) = M(x)x \), where

\[
M(x) := \int_0^1 D^2 f(tx) dt.
\]

The result follows. \( \square \)

Lemma 4.3.4

If \( \theta \) is not an integer multiple of \( 2\pi \), then for sufficiently small \( \alpha \), the operator norm of \( E \) considered as a mapping from \( C^{2,\alpha}_\theta(G) \) to \( C^{0,\alpha}_\theta(G) \) tends to zero as \( \Delta \) tends to infinity.

**Proof:** Fix \( f \in C^{2,\alpha}_\theta(G) \) and let \( M \) be as in Lemma 4.3.3. By (4.21), over \( B(\epsilon A) \),

\[
Ef = \frac{2c^2}{\epsilon^2 A^4} \langle M(x)x, x \rangle.
\]

However, bearing in mind (A.12),

\[
\|x \otimes x\|_{C^0} \leq \epsilon^2 A^2,
\]

\[
[x \otimes x]_{C^0} \leq 2^{1-\alpha} \|x \otimes x\|_{B(\epsilon A)} \|x \otimes x\|_{B(\epsilon A)}^{1-\alpha} \lesssim 2^{2-\alpha} A^{2-\alpha}.
\]

Thus, by (A.14) and (4.25),

\[
\|Ef\|_{B(\epsilon A)} \lesssim \frac{1}{\epsilon^2 A^{2+\alpha}} \|M\|_{B(1)} \|C^0,\alpha \| \lesssim \frac{1}{\epsilon^2 A^{2+\alpha}} \|f\|_{C^{2,\alpha}(G)}.
\]

Likewise, by (4.11), over \( A(\epsilon A, 2\epsilon A^4) \),

\[
Ef = \chi \frac{2c^2 \epsilon^2}{r^4} \langle M(x)x, x \rangle.
\]

However, bearing in mind (A.12) again,

\[
\|x \otimes x\|_{A(\epsilon A, 2\epsilon A^4)} \leq \frac{1}{\epsilon^2 A^2},
\]

\[
[x \otimes x]_{A(\epsilon A, 2\epsilon A^4)} \leq 2^{1-\alpha} \|x \otimes x\|_{A(\epsilon A, 2\epsilon A^4)} \|x \otimes x\|_{A(\epsilon A, 2\epsilon A^4)}^{1-\alpha} \lesssim \frac{1}{\epsilon^{2+\alpha} A^{2+\alpha}}.
\]
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Thus, as before, by (A.2) and (4.25),

\[ \| Ef \|_{A(\epsilon A, 2\epsilon A^4)} \lesssim \frac{1}{\epsilon^\alpha A^{2+\alpha}} \| f \|_{C^{2,\alpha}(G)}. \]

Upon combining these relations, we see that the operator norm of \( E \), considered as a mapping from \( C^{2,\alpha}_0(G) \) into \( C^{0,\alpha}_0(G) \) satisfies

\[ \| E \| \lesssim \frac{1}{\epsilon^\alpha A^{2+\alpha}}. \]

By (2.9), for sufficiently small \( \alpha \), this tends to zero as \( \Delta \) tends to infinity, as desired. □

Combining these results yields,

**Lemma 4.3.5**

(1) For sufficiently small \( \gamma \), the operator norm of \( \hat{J}_\gamma - \hat{J}_{p,\gamma} \), considered as a mapping from \( H^2(G) \) into \( L^2(G) \) tends to zero as \( \Delta \) tends to infinity; and

(2) if \( \theta \) is a non-integer multiple of \( 2\pi \), then for sufficiently small \( \alpha \) and sufficiently small \( \gamma \), the operator norm of \( \hat{J}_\gamma - \hat{J}_{p,\gamma} \), considered as a mapping from \( C^{2,\alpha}_\theta(G) \) into \( C^{0,\alpha}_\theta(G) \) tends to 0 as \( \Delta \) tends to infinity.

**Proof:** By definition, \( \hat{J}_0 - \hat{J}_{p,0} = D + E \), and the result now follows by Lemmas 4.2.7, 4.3.2 and 4.3.4. □

In summary, we have,

**Theorem 4.3.6**

For \( \theta \) a non-integer multiple of \( \pi \), for sufficiently small \( \alpha \) and \( \gamma \) and for sufficiently large \( \Delta \), \( \hat{J} \) defines invertible mappings from \( H^2_{G,\gamma} \) and \( C^{2,\alpha}_{G,\gamma,\theta} \) into \( L^2_{G,\gamma} \) and \( C^{2,\alpha}_{G,\gamma,\theta} \) respectively. Furthermore, the operator norms of these mappings as well as their inverses are uniformly bounded for large values of \( \Delta \).

5 - Surgery and the Perturbation Family.

5.1 - The Basic Surgery Operation. We join complete minimal surfaces to Grim ends as follows. Let \( \Delta \) be a large positive number, let \( R_0 > 0 \) be relatively small, and choose \( \epsilon, R > 0 \) and \( c \in \mathbb{R} \) satisfying (2.9). Let \( F : A(R_0, \infty) \rightarrow \mathbb{R} \) be the profile of a catenoidal end of a minimal surface with logarithmic parameter \( c \) and constant term \( A \). We recall that,

\[ F = A + c \log(r) + O \left( r^{-(2+k)} \right). \quad (5.1) \]

We denote the graph of \( F \) by \( C \). Let \( G : A(R/4, \infty) \rightarrow \mathbb{R} \) be the profile of a Grim end with logarithmic parameter \( c \), constant term \( A \) and speed \( \epsilon \). Upon rescaling (2.36), and then integrating, we obtain, over \( A(R/4, 2R^4) \),

\[ G = A + c \log(r) + \frac{1}{4} \epsilon r^2 + O \left( \log \left( \frac{r}{R} \right) r^{1-k} \left( \epsilon r + \frac{1}{r} \right)^3 \right). \quad (5.2) \]
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We denote the graph of $G$ also by $G$. Let $\chi_c$ be the cut-off function of the central transition region $A(R, 2R)$. We define the function $H$ over $A(R_0, \infty)$ by

$$H = \chi_c F + (1 - \chi_c)G. \quad (5.3)$$

Observe that, over $A(R_0, R)$ and $A(2R, \infty)$, $H$ simply coincides with $F$ and $G$ respectively. Furthermore, since $\chi_c = O(R^{-k})$, and bearing in (2.9), over $A(R, 2R)$,

$$H = A + c \log(R) + \frac{1}{4} \epsilon(1 - \chi_c) r^2 + O\left( R^{-(k+2)} \right). \quad (5.4)$$

We call the graph of $H$ the joined end and we denote it by $S$. Observe that $S$ is entirely determined by $F$ and the parameters $\epsilon$ and $R$.

5.2 - The Perturbation Family and the Jacobi Operator. We now describe the family of smooth perturbations of the joined surface that will be used in the sequel. Since every way of perturbing a fixed compact region is essentially the same, it is sufficient to describe a family of perturbations of each ends. Thus, let $F_c : A(R_0, \infty) \to \mathbb{R}$ be a family of profiles of catenoidal ends parameterised smoothly by their logarithmic parameters. For all $c$, let $G_c : A(R/4, \infty) \to \mathbb{R}$ be the profile of a Grim end with logarithmic parameter $c$, speed $\epsilon$ and constant term chosen to match that of $F_c$. For all $c$, let $H_c = \chi_c F_c + (1 - \chi_c)G_c$ be the profile of the joined surface which we denote by $S_c$.

Over each surface in the above constructed family, one would usually consider perturbations in the normal direction. However, in keeping with the results of Section 4, we prefer to study perturbations in the direction of a slightly different vector field. In general, let $\Phi : A(R_0, \infty) \to \mathbb{R}$ be any smooth function and let $\Sigma$ be the graph of $\Phi$. We define the modified normal vector field over $\Sigma$ by

$$\hat{N}_\Sigma := \frac{\sin(\phi)}{\sin(\theta)}e_z + \frac{\sin(\theta - \phi)}{\sin(\theta)}N_\Sigma, \quad (5.5)$$

where

$$\cos(\theta) = \langle N_\Sigma, e_z \rangle, \quad \cos(\phi) = (1 - \chi_\epsilon) + \chi_\epsilon \cos(\theta), \quad (5.6)$$

and $\chi_\epsilon$ is the cut-off function of the transition region $A(1/\epsilon, 2/\epsilon)$. Observe that $\hat{N}_\Sigma$ has been constructed so that

$$\langle \hat{N}_\Sigma, N_\Sigma \rangle = \chi_\epsilon \langle N_\Sigma, e_z \rangle + (1 - \chi_\epsilon). \quad (5.7)$$

For all $c$, we define $\hat{N}_c$ to be the modified normal vector field of the joined surface $S_c$. We now define the smooth family $\hat{H} : c_0 - \eta, c_0 + \eta[|x| - \eta, \eta \times C^\infty_0(A(R_0, \infty))] \to C^\infty(A(R_0, \infty), \mathbb{R}^3)$ by

$$\hat{H}(l, v, u)(x) = (x, H_{c_0 + l}(x)) + ve_z + u(x)\hat{N}_{c_0 + l}(x). \quad (5.8)$$

A perturbations in the direction of $l$ correspond to a variation of the logarithmic parameter whilst a perturbation in the direction of $v$ corresponds to a vertical translation. We call
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these macroscopic perturbations, as they become large at infinity. Perturbations in the direction of \( u \) correspond to general local perturbations of the surface, and we call these microscopic perturbations.

We define \( M : \mathbb{C}_0 - \eta, c_0 + \eta \times C_0^\infty(A(R_0, \infty)) \to C_0^\infty(A(R_0, \infty), \mathbb{R}) \) such that for all \((l, v, u)\), and for all \( x \), \( M(l, v, u)(x) \) is the value of the MCFS functional of the image of \( H \) at the point \( x \). We define the operators \( X : \mathbb{R} \to C_0^\infty(A(R_0, \infty)), \ Y : \mathbb{R} \to C_0^\infty(A(R_0, \infty)) \) and \( \hat{J} : C_0^\infty(A(R_0, \infty)) \to C_0^\infty(A(R_0, \infty)) \) by

\[
(Xl)(x) = \frac{d}{dt}M(c_0 + tl, 0, 0)(x)|_{t=0},
\]

\[
(Yv)(x) = \frac{d}{dt}M(c_0, tv, 0)(x)|_{t=0},
\]

\[
(\hat{J}f)(x) = \frac{1}{\langle N_S, N_S \rangle} \frac{d}{dt}M(c_0, 0, tf)(x)|_{t=0}.
\]

For all \( l \), \( M(c_0 + l, 0, 0) \) vanishes outside \( B(2R) \), and so \( Xl \) is also supported in this ball. Likewise, \( Yv \) vanishes over the whole annulus \( A(R_0, \infty) \), although, since this family of perturbations will be extended in some manner across the compact region, it will actually be supported in \( B(R_0) \) in a manner that is uniformly bounded independent of the parameters. Finally, over \( A(R/4, 1/\epsilon) \), \( \hat{J} \) is merely \( \langle N, e_z \rangle^{-1} \) times the linearisation of the MCFS functional for graphs. Thus, upon differentiating (A.7), we obtain, over \( A(R_0, 1/\epsilon) \),

\[
\hat{J}f = g^{ij}f_{ij} - \mu^2 g^{ij}F_{ij}F_kF_k + 2\mu^4 F_iF_jF_kF_{ij}F_k - 2\mu^2 F_{ij}F_{ij}f_j + 6\mu^2 F_{ij}f_{ij} + \epsilon \mu^2 F_{ij}f_{ij}.
\]

(5.10)

5.3 - Modified Jacobi Operators. We now derive asymptotic formulae for the modified MCFS Jacobi operators of the surfaces of interest to us, as well as the relationships between them. Continuing to use the notation of the previous sections, we have

**Lemma 5.3.1**

Over \( A(R/4, 2R^4) \), the modified MCFS Jacobi operator of \( C \) satisfies

\[
\hat{J}_C f = \Delta f - \frac{c^2}{r^4} x^i x^j f_{ij} + \frac{\epsilon c}{r^2} x^i f_i + \frac{2c^2}{r^4} x^i f_i + \mathcal{E}_C f,
\]

(5.11)

where \( \mathcal{E}_C f = a^{ij}f_{ij} + b^i f_i \) and \( a \) and \( b \) satisfy

\[
a = O\left( r^{-(k+4)} \right), \\
b = O\left( r^{-(k+4)} \left( \epsilon r + \frac{1}{r} \right) \right).
\]

(5.12)

**Proof:** By (5.1),

\[
F_i = \frac{c}{r^2} x^i + O\left( r^{-(k+3)} \right).
\]
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Thus, by (A.6),
\[ \mu^2 = 1 - \frac{c^2}{r^2} + O \left( r^{-(k+4)} \right), \]
\[ g^{ij} = \delta_{ij} - \frac{c^2}{r^4} x^i x^j + O \left( r^{-(k+4)} \right). \]

Thus,
\[ g^{ij} f_{ij} = \Delta f - \frac{c^2}{r^4} x^i x^j f_{ij} + a^{ij} f_{ij}, \]
where \( a = O \left( r^{-(k+4)} \right) \). Likewise,
\[ \mu^2 g^{ij} F_{ij} F_k f_k = b^i_1 f_i, \]
\[ 2\mu^4 F_i F_j F_k f_{ij} f_k = b^i_2 f_i, \]
\[ -2\mu^2 F_{ij} F_i f_j = \frac{2c^2}{r^4} x^i f_i + b^i_3 f_i, \]
where \( b^i_1, b^i_2, b^i_3 = O \left( r^{-(k+5)} \right) \). Finally,
\[ \epsilon \mu^2 F_i f_i = \frac{\epsilon c}{r^2} x^i f_i + b^i_4, \]
where \( b^i_4 = O \left( \epsilon r^{-(k+3)} \right) \). The result follows. \( \square \)

Up to rescaling, the modified MCFS Jacobi operator of \( G_\epsilon \) has already been determined in Section 4.1, and over \( A(R/4,2R^4) \), we have
\[ \hat{J}_G F = \Delta f - \left( \frac{\epsilon}{2} + \frac{c}{r^2} \right)^2 x^i x^j f_{ij} - \left( \frac{c^2}{2} - \frac{2c^2}{r^4} \right) x^i u f_i + \mathcal{E}_G f. \] (5.13)

where \( \mathcal{E}_G f = a^{ij} f_{ij} + b^i f_i \), and \( a \) and \( b \) satisfy,
\[ a = O \left( \log \left( \frac{r}{R} \right) \frac{1}{r^k} \left( \epsilon r + \frac{1}{r} \right)^4 \right), \]
\[ b = O \left( \log \left( \frac{r}{R} \right) \frac{1}{r^{k+1}} \left( \epsilon r + \frac{1}{r} \right)^4 \right). \] (5.14)

**Lemma 5.3.2**

Over \( A(R,2R) \),
\[ \left( \hat{J}_S - \hat{J}_C \right) f = a^{ij}_1 f_{ij} + b^i_1 f_i, \]
\[ \left( \hat{J}_S - \hat{J}_C \right) f = a^{ij}_2 f_{ij} + b^i_2 f_i. \]
where \( a_1, a_2, b_1 \) and \( b_2 \) satisfy,

\[
\begin{align*}
    a_1, a_2 &= O \left( R^{- (4 + k)} \right), \\
    b_1, b_2 &= O \left( R^{- (5 + k)} \right).
\end{align*}
\] (5.15)

**Proof:** By (5.1), (5.4) and (2.9), over \( A(R, 2R) \),

\[
\begin{align*}
    H_i - F_i &= O \left( R^{-k} \left( \epsilon R + \frac{1}{R^3} \right) \right), \\
    F_i, H_i &= O \left( R^{-(k+1)} \right).
\end{align*}
\]

Thus, by (A.6),

\[
\begin{align*}
    \mu_H - \mu_H &= O \left( R^{-k} \left( \epsilon + \frac{1}{R^4} \right) \right), \\
    g_{ij}^H - g_{ij}^F &= O \left( R^{-k} \left( \epsilon + \frac{1}{R^4} \right) \right).
\end{align*}
\]

The result follows for \( \hat{J}_S - \hat{J}_C \) by (2.9). The result for \( \hat{J}_S - \hat{J}_G \) follows in a similar manner, and this completes the proof. □

Finally, we denote by \([J_C, \chi_l]\) the commutator of \( J_C \) with the operator of multiplication by the cut-off function \( \chi_l \) of the lower transition region \( A(R/4, R/2) \). Since \( \chi_l = O(R^{-k}) \), it follows by (5.11) and (5.12) that

\[
[J_C, \chi_l]f = a^i f_i + bf,
\]

where

\[
\begin{align*}
    a &= O(R^{-(k+1)}), \\
    b &= O(R^{-(k+2)}).
\end{align*}
\] (5.16)

We likewise denote by \([J_G, \chi_u]\) the commutator of \( J_G \) with the operator of multiplication by the cut-off function \( \chi_u \) of the upper transition region \( A(R^4, 2R^4) \). Since \( \chi_u = O(R^{-4k}) \), it follows by (5.13) and (5.14) that

\[
[J_G, \chi_u]f = a^i f_i + bf,
\]

where

\[
\begin{align*}
    a &= O(R^{-4(k+1)}), \\
    b &= O(R^{-4(k+2)}).
\end{align*}
\] (5.17)
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5.4 - Controlling Macroscopic Perturbations. We now estimate the norm of \( X \).

Observe that, over \( A(R_0, 2R) \),

\[
X = \hat{J}_S W,
\]

where

\[
W := \frac{d}{dt} H(c + t, 0, 0)|_{t=0}.
\]

We define \( W_0 \) by

\[
W_0 := \frac{d}{dt} F_{c+t}|_{t=0}.
\]

Lemma 5.4.1

\[
\|\hat{J}_C W_0|_{A(R_0, 2R)}\|_{C^{0, \alpha}_{2+\delta, \text{Cyl}}} \lesssim \epsilon R^\delta.
\]  (5.18)

Proof: Since the graph of \( F_c \) is a minimal surface for all \( c \), \( W_0 \) is a Jacobi field. Thus, by (A.2),

\[
\hat{J}_C W_0 = \epsilon \mu^2 F_{c,i} W_i.
\]

Moreover, by (5.1) and (5.4),

\[
F_{c,i} = O(r^{-(k+1)}),
\]

\[
W_i = O(r^{-(k+1)}).
\]

Thus, by (A.6),

\[
\mu = 1 + O(r^{-(k+2)}).
\]

Thus, over \( A(R_0, 2R) \),

\[
\left| r^{2+\delta} \hat{J}_C W_0 \right| \lesssim \epsilon r^\delta.
\]

Likewise, by the chain rule, over this annulus,

\[
\left| r^{2+\delta} D_{\text{Cyl}} \hat{J}_C W_0 \right| \lesssim \epsilon r^\delta.
\]

Thus, by (A.12), over this annulus,

\[
\left| r^{2+\delta} \delta_{\text{Cyl}} \hat{J}_C W_0|_{A(R_0, 2R)} \right| \lesssim \epsilon r^\delta.
\]

The result follows. □
Lemma 5.4.2

\[ \|X\|_{C^{0,\alpha}_{2+\delta,\text{Cyl}}} \lesssim \frac{1}{R^{2-\delta}}. \] (5.19)

**Proof:** Over \( A(R_0, R) \), \( X \) coincides with \( \hat{J}_C W_0 \). Thus, by (5.18) and (2.9),

\[ \|X|_{A(R_0, 2R)}\|_{C^{0,\alpha}_{2+\delta,\text{Cyl}}} \lesssim \epsilon R^{\delta} \lesssim \frac{1}{R^{4-\delta}}. \]

By (5.4), over \( A(R, 2R) \),

\[ W_i - W_{0,i} = O \left( R^{-(3+k)} \right), \]
\[ W_i, W_{0,i} = O \left( R^{-1} \right). \]

Thus, by (5.11) and (5.12), over this annulus,

\[ \left| \hat{J}_C(W - W_0) \right| \lesssim \frac{1}{R^4}. \]

Likewise, by the chain rule, over this annulus,

\[ \left| D\hat{J}_C(W - W_0) \right| \lesssim \frac{1}{R^5}. \]

Thus, by (A.12),

\[ \|\hat{J}_C(W - W_0)|_{A(R, 2R)}\|_{C^{0,\alpha}_{2+\delta,\text{Cyl}}} \lesssim \frac{1}{R^{2-\delta}}. \]

In like manner, using (5.15), we obtain,

\[ \| \left( \hat{J}_S - \hat{J}_C \right) W|_{A(R, 2R)}\|_{C^{0,\alpha}_{2+\delta}} \lesssim \frac{1}{R^{4+\delta}}. \]

Since \( X \) vanishes over \( A(2R, \infty) \), the result follows upon combining these relations. □

6 - Constructing the Green’s Operator.

6.1 - The Cylindrical and Grim Norms. We recall that \( D \) denotes the standard operator of differentiation over \( \mathbb{R}^3 \), as well as its restriction to any surface \( \Sigma \) contained in \( \mathbb{R}^3 \). We define the operators \( D_{\text{Cyl}} \) and \( D_G \) by

\[ D_{\text{Cyl}} := rD, \quad D_G := \frac{1}{\epsilon} D. \] (6.1)

For all functions, \( f \), we likewise define the functions

\[ \delta_{\text{Cyl}}^\alpha f(r) = r^\alpha [f|_{A(r/2, 2r)}], \]
\[ \delta_G^\alpha f(x) = \frac{1}{\epsilon^\alpha}[f|_{B(x, 1/\epsilon)}]. \] (6.2)
We define the weighted Hölder norms

\[ \|f\|_{C^{m,\alpha}_{\delta,\text{Cyl}}} := \sum_{n=0}^{n} \|r^{\delta} \partial_{\text{Cyl}} f\|_{C^{m}} + \|r^{\delta} \partial_{\text{Cyl}}^{m} f\|_{C^{m}}. \]  

(6.3)

Now fix a positive integer, \( g \), and let \( C \) be a Costa-Hoffman-Meeks surface of genus \( g \). Denote by \( C^{m,\alpha}_{\delta,\text{Cyl},g}(C) \) the completion with respect to \( \| \cdot \|_{C^{m,\alpha}_{\delta,\text{Cyl}}} \) of the space of smooth functions \( f \) over \( c \) such that for every horizontal symmetry \( \sigma \) of \( C \), \( f \circ \sigma = f \). We recall that, with these symmetries imposed, for all \( \delta \in [0,2] \), the operator \( \hat{J}_{C} \) defines an injective Fredholm mapping of Fredholm index \(-3\) from \( C^{2,\alpha}_{\delta,\text{Cyl}}(C) \) into \( C^{0,\alpha}_{2,\delta,\text{Cyl}}(C) \).* Furthermore, the complementary space to the image of \( \hat{J}_{C} \) is spanned by varying independently the heights and the logarithmic parameters of each of the ends. That is, if we define \( X_{0}, Y_{0} : \mathbb{R}^{3} \rightarrow C^{\infty}_{0}(C) \) such that \( X_{0}(l_{1}, l_{2}, l_{3}) \) is the infinitesimal variation of the mean curvature of \( C \) resulting from a perturbation of the logarithmic parameters of the three separate ends by \( c_{1}, c_{2} \) and \( c_{3} \) respectively, and \( Y_{0}(v_{1}, v_{2}, v_{3}) \) is the infinitesimal variation of the mean curvature of \( C \) resulting from infinitesimal vertical translations of each of the catenoidal ends by \( v_{1}, v_{2} \) and \( v_{3} \) respectively (all extended in some manner across the compact region), then \( X_{0} \oplus Y_{0} \oplus \hat{J}_{C} \) defines a surjective Fredholm map of Fredholm index \( 3 \) from \( \mathbb{R}^{3} \oplus \mathbb{R}^{3} \oplus C^{2,\alpha}_{\delta,\text{Cyl}}(C) \) into \( C^{0,\alpha}_{2,\delta,\text{Cyl}}(C) \). Now let \( X, Y : \mathbb{R}^{3} \rightarrow C^{\infty}_{0}(C) \) be the operators defined in Section 5.2. By (5.19), \( X \) converges to \( X_{0} \) in the operator norm as \( \Delta \) tends to infinity, and since \( Y \) simply coincides with \( Y_{0} \) over the ends, it follows that for sufficiently large \( \Delta \), \( X \oplus Y \oplus \hat{J}_{C} \) also defines a surjective Fredholm map of Fredholm index \( 3 \) from \( \mathbb{R}^{3} \oplus \mathbb{R}^{3} \oplus C^{2,\alpha}_{\delta,\text{Cyl}}(C) \) into \( C^{0,\alpha}_{2,\delta,\text{Cyl}}(C) \). Furthermore, the right inverse of this mapping can be chosen in such a manner that its operator norm is uniformly bounded independent of large values of \( \Delta \).

We recall that \( \text{dVol} \) denotes the canonical volume form over \( \mathbb{R}^{3} \), as well as its restriction to any surface \( \Sigma \) contained in \( \mathbb{R}^{3} \). As above, we define the volume forms

\[ \text{dVol}_{\text{Cyl}} := \frac{1}{r^{2}} \text{dVol}, \quad \text{dVol}_{G} := \varepsilon^{2} \text{dVol}. \]  

(6.4)

For any function, \( f \), we define the weighted Hölder and Sobolev norms

\[ \|f\|_{C^{m,\alpha}_{G,\gamma}} := \|f(\cdot/\varepsilon)\|_{C^{m,\alpha}_{G,\gamma}(G)}, \]  

\[ \|f\|_{H^{m,\alpha}_{G,\gamma}} := \|f(\cdot/\varepsilon)\|_{H^{m,\alpha}_{G,\gamma}(G)}. \]  

(6.5)

As above, we denote by \( C^{m,\alpha}_{G,\gamma,g} \) the completion with respect to \( \| \cdot \|_{C^{m,\alpha}_{G,\gamma}} \) of the space of smooth functions over \( G \) that are invariant under every horizontal symmetry of \( C \). Likewise, we denote by \( H^{m,\alpha}_{G,\gamma,g} \) the completion with respect to \( \| \cdot \|_{H^{m,\alpha}_{G,\gamma}} \) of the space of compactly supported smooth functions over \( G \) that are also invariant under these symmetries. By

* We aim to include an overview of the perturbation theory of these surfaces in forthcoming work, as we are not aware of any satisfactory and straightforward account extant in the literature.

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Theorem 4.3.6, upon rescaling, we find that for all sufficiently small $\alpha$ and $\gamma$, $\epsilon^2 \hat{J}_G$ defines an invertible linear map from $C^{2,\alpha}_{G,\gamma,g}$ and $H^2_{G,\gamma,g}$ into $C^{0,\alpha}_{G,\gamma,g}$ and $L^2_{G,\gamma,g}$ respectively, and that the operator norms of these mappings and their inverses are uniformly bounded independent of large values of $\Delta$.

The following straightforward result will play an important role throughout this section.

**Lemma 6.1.1**

If $f$ is such that $\|f\|_{C^{2,\alpha}_{G,\gamma}} \leq A$ and $\|f\|_{H^2_{G,\gamma}} \leq BA$, where $B \leq 1$, then

$$|D_G f|_{C^{0}_{G,\gamma}} \lesssim B^{1-\alpha} A. \quad (6.6)$$

Furthermore, over $B(2R)$

$$|f|_{B(R)} \lesssim (\epsilon R) B^{1-\alpha} A. \quad (6.7)$$

**Proof:** Indeed, by the Sobolev embedding theorem, for all small $\beta$,

$$\|D_G f\|_{C^{0}_{G,\gamma}} \lesssim \alpha A.$$  

Thus, by (A.12) and (A.13),

$$|D_G f| \lesssim \alpha^{\frac{1+\alpha}{1+\alpha+\beta}} A,$$

The first relation now follows upon choosing $\beta$ sufficiently small, and the second then follows by integration. □

### 6.2 - Kapouleas’ Ping-Pong Argument - Part I.

We use the following seminorms over $C^m(S)$.

$$\|f\|_{m,C} := \|f\|_{B(4R)} \|f\|_{C^{m,\alpha}_{G,\gamma}},$$

$$\|f\|_{m,G,H} := \|f\|_{A(R,\infty)} \|f\|_{C^{m,\alpha}_{G,\gamma}},$$

$$\|f\|_{m,G,S} := \|f\|_{A(R,\infty)} \|f\|_{H^{m}_{G,\gamma}},$$

$$\|f\|_{m,G} := \|f\|_{m,G,H} + \frac{1}{\epsilon R} \|f\|_{m,G,S}. \quad (6.8)$$

As discussed in Section 6.1, there exists a bounded linear map $L \oplus V \oplus \Phi : C^{2,\alpha}_{2+\delta,\gamma}(C) \to R^3 \oplus R^3 \oplus C^{2,\alpha}_{\delta,\gamma}(C)$ such that for all $e$,

$$e = XLe + YVe + \hat{J}_C \Phi e. \quad (6.9)$$

Let $E_\gamma$ be the closure with respect to $\|\cdot\|_{0,C}$ of the space of functions supported in the interior of $B(3R)$ that are invariant under every horizontal symmetry of the Costa-Hoffman-Meeks surface $C_g$. Likewise, let $F_\gamma$ be the closure with respect to $\|\cdot\|_{0,G}$ of the space of functions supported in the interior of $A(R,\infty)$ that are also invariant under these symmetries. We define the operator $A : E_\gamma \to F_\gamma$ by

$$Ae := \hat{J}_S \Phi e + XLe + YVe - e, \quad (6.10)$$

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where \( \chi_u \) is the cut-off function of the upper transition region \( A(R^4, 2R^4) \). Observe that since \( \hat{J}_S \) coincides with \( \hat{J}_C \) over \( B(0, R) \), \( Ae \) is indeed supported in the interior of \( A(R, \infty) \) and is therefore an element of \( F_g \). Furthermore, by definition of \( \hat{J}_S \), and bearing in mind that \( X \) and \( Y \) are both supported in \( B(2R) \),

\[
Ae = [\hat{J}_G, \chi_u] \Phi e + (1 - \chi_c) \chi_u (\hat{J}_S - \hat{J}_C) \Phi e,
\]

(6.11)

where \( \chi_c \) is the cut-off function of the central transition region \( A(R, 2R) \). For convenience, throughout the rest of this section, we will denote \( f := Ae \) and \( \phi := \Phi e \).

**Lemma 6.2.1**

\[
\| \chi_u \phi \|_{2, G} \lesssim \frac{1}{(\epsilon R)^{\alpha}} \frac{1}{\epsilon^2 R^{2+\delta}} \| e \|_{0, C}.
\]

(6.12)

**Proof:** Indeed, since \( \chi_u = O(R^{-4k}) \) over \( A(R, 2R^4) \), \( \| \chi_u \|_{C^2, \alpha, \infty} \lesssim 1 \). Thus

\[
\| \chi_u \phi \|_{C^2, \alpha, \infty} \lesssim \| \phi \|_{C^2, \alpha, \infty} \lesssim \| e \|_{0, C}.
\]

Thus, by (6.1), for \( k \in \{0, 1, 2\} \), over \( A(R, 2R^4) \),

\[
|D^k_G \chi_u \phi| \lesssim \frac{1}{(\epsilon r)^k r^\delta} \| e \|_{0, C}.
\]

Likewise, by (6.2), for all \( r \in [2R, R^4] \),

\[
|\delta^\alpha_G (D^2_G \chi_u \phi|_{A(r/2, 2r)})| \lesssim \frac{1}{(\epsilon r)^{2+\alpha}} \frac{1}{r^\delta} \| e \|_{0, C}.
\]

Thus, by (A.12),

\[
|\delta^\alpha_G (D^2_G \chi_u \phi|_{A(R, 2R^4)})| \lesssim \frac{1}{(\epsilon R)^{2+\alpha}} \frac{1}{R^{2+\delta}} \| e \|_{0, C}.
\]

It follows that

\[
\| \chi_u \phi \|_{2, G, H} \lesssim \frac{1}{(\epsilon R)^{\alpha}} \frac{1}{\epsilon^2 R^{2+\delta}} \| e \|_{0, C}.
\]

Likewise, for all \( k \), over \( A(R, 2R^4) \),

\[
|D^k_G \chi_u \phi|^2 \text{dVol}_G \lesssim \frac{1}{(\epsilon r)^{2k}} \frac{1}{r^{2\delta}} \| e \|_{0, C} \| e \|_{0, C} \text{dVol}_C yl.
\]

Since \( \delta > 1 \), it follows by (A.23) that

\[
\| \chi_u \phi \|_{2, G, S} \lesssim \frac{1}{\epsilon R^{1+\delta}} \| e \|_{0, C}.
\]

The result follows by definition of \( \| \cdot \|_{2, G} \). \( \square \)
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Lemma 6.2.2

\[ \| (\hat{J}_S - \hat{J}_G) \Phi e|_{A(R,2R^4)} \|_{C^0_{\alpha,\gamma}} \lesssim \frac{1}{(\epsilon R)^\alpha} \frac{1}{R^{6+\delta}} \|e\|_{0,C}. \]  (6.13)

Proof: Indeed, by (6.1), for \( k \in \{0,1,2\} \), over \( A(R,2R^4) \),

\[ |D^k \phi| \lesssim \frac{1}{r^{k+\delta}} \|\phi\|_{C^{2,\alpha}_{\delta,Cyl}} \lesssim \frac{1}{r^{k+\delta}} \|e\|_{0,C}. \]

Likewise, by (6.2), for all \( r \in [2R,R^4] \),

\[ |\delta^\alpha (D^2 \phi|_{A(r/2,2r)})| \lesssim \frac{1}{r^{\alpha}} \frac{1}{r^{k+\delta}} \|e\|_{0,C}. \]

Thus, by (5.11), (5.12), (5.13), (5.14) and (5.15) over \( A(R,2R^4) \),

\[ \left| (\hat{J}_S - \hat{J}_G) \phi \right| \lesssim \left( \frac{\epsilon}{r^{2+\delta}} + \frac{\epsilon^2}{r^6} + \log \left( \frac{r}{R} \right) \epsilon^4 r^{2-\delta} + \log \left( \frac{r}{R} \right) \frac{1}{r^{6+\delta}} \right) \|e\|_{0,C}. \]  (6.14)

Likewise, by (A.12), for \( r \in [2R,R^4] \),

\[ |\delta^\alpha \left( (\hat{J}_S - \hat{J}_G) \phi|_{A(r/2,2r)} \right)| \lesssim \frac{1}{r^{\alpha}} \left( \frac{\epsilon}{r^{2+\delta}} + \frac{\epsilon^2}{r^6} + \log \left( \frac{r}{R} \right) \epsilon^4 r^{2-\delta} + \log \left( \frac{r}{R} \right) \frac{1}{r^{6+\delta}} \right) \|e\|_{0,C}. \]

By (6.2), for \( r \in [2R,R^4] \),

\[ |\delta^\alpha_G \left( (\hat{J}_S - \hat{J}_G) \phi|_{A(r/2,2r)} \right)| \lesssim \frac{1}{(\epsilon r)^\alpha} \left( \frac{\epsilon}{r^{2+\delta}} + \frac{\epsilon^2}{r^6} + \log \left( \frac{r}{R} \right) \epsilon^4 r^{2-\delta} + \log \left( \frac{r}{R} \right) \frac{1}{r^{6+\delta}} \right) \|e\|_{0,C}. \]

It follows by (A.22) and (2.9) that

\[ \left| (\hat{J}_S - \hat{J}_G) \phi|_{A(R,2R^4)} \right| \lesssim \frac{1}{R^{6+\delta}} \|e\|_{0,C}; \]

\[ |\delta^\alpha_G \left( (\hat{J}_S - \hat{J}_G) \phi|_{A(R,2R^4)} \right)| \lesssim \frac{1}{(\epsilon R)^\alpha} \frac{1}{R^{6+\delta}} \|e\|_{0,C}. \]

The result follows. \( \square \)
Lemma 6.2.3

\[ \| (\hat{J}_S - \hat{J}_G) \Phi e |_{A(R,2R^4)} \|_{L^2_{G,\gamma}} \lesssim \frac{(\epsilon R)}{R^{6+\delta}} \| e \|_{0,C}. \]  \tag{6.15}

**Proof:** By (6.4) and (6.14), over \( A(R, 2R^4) \),

\[ \left| (\hat{J}_S - \hat{J}_G) \phi \right|^2 d\text{Vol}_G \lesssim \left( \frac{\epsilon^4}{r^{2+2\delta}} + \epsilon^6 r^{2-2\delta} + \log \left( \frac{r}{R} \right)^2 \epsilon^{10} r^{6-2\delta} \right. \\
+ \log \left( \frac{r}{R} \right)^2 \epsilon^2 \left. \right\| e \|^2_{0,C} d\text{Vol}_{Cyl}. \]

Thus, by (A.23) and (2.9),

\[ \int_{A(R, 2R^4)} \left| (\hat{J}_S - \hat{J}_G) \phi \right|^2 d\text{Vol}_G \lesssim \frac{(\epsilon R)^2}{R^{12+2\delta}} \| e \|^2_{0,C}. \]

The result follows. \( \square \)

Lemma 6.2.4

\[ \| [\hat{J}_G, \chi_u] \Phi e \|_{C^{0,\alpha}_{G,\gamma}} \lesssim \frac{1}{(\epsilon R)^\alpha} \frac{1}{R^{8+4\delta}} \| e \|_{0,C}. \]  \tag{6.16}

**Proof:** By (6.1) for \( k \in \{0, 1, 2\} \), over \( A(R^4, 2R^4) \),

\[ |D^k \phi| \lesssim \frac{1}{R^{4k+4\delta}} \| \phi \|_{C_{G,\gamma}^{0,\alpha}} \lesssim \frac{1}{R^{4k+4\delta}} \| e \|_{0,C}. \]

Since \( \delta > 1 \), by (5.17), for \( k \in \{0, 1\} \), over this annulus,

\[ |D^k [\hat{J}_G, \chi_u] \phi| \lesssim \frac{1}{R^{8+4k+4\delta}} \| e \|_{0,C}. \]  \tag{6.17}

Thus, by (6.1), for \( k \in \{0, 1\} \), over this annulus,

\[ |D^k_G [\hat{J}_G, \chi_u] \phi| \lesssim \frac{1}{(\epsilon R^4)^\alpha} \frac{1}{R^{8+4\delta}} \| e \|_{0,C}. \]

The result follows by (A.12). \( \square \)

Lemma 6.2.5

\[ \| [\hat{J}_G, \chi_u] \Phi e \|_{L^2_{G,\gamma}} \lesssim \frac{(\epsilon R)}{R^{5+4\delta}} \| e \|_{0,C}. \]  \tag{6.18}

**Proof:** By (6.17) and (6.4), over \( A(R^4, 2R^4) \),

\[ \left| [\hat{J}_G, \chi_u] \phi \right|^2 d\text{Vol}_G \lesssim \frac{\epsilon^2}{R^{8+8\delta}} \| e \|^2_{0,C} d\text{Vol}_{Cyl}. \]

Thus,

\[ \int_{A(R^4, 2R^4)} \left| [\hat{J}_G, \chi_u] \phi \right|^2 d\text{Vol}_G \lesssim \frac{\epsilon^2}{R^{8+8\delta}} \| e \|_{0,C}. \]

The result follows. \( \square \)
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**Theorem 6.2.6**

\[ \| A e \|_{0,G} \lesssim \frac{1}{(\epsilon R)^{3\alpha}} \frac{1}{R^{6+\delta}} \| e \|_{0,C}. \]  

(6.19)

**Proof:** Observe that, by (A.12), \(|\chi u|_{C^{0,\alpha}_G} \lesssim \frac{1}{(\epsilon R)^{3\alpha}} \lesssim \frac{1}{(\epsilon R)^{2\alpha}}\). The result now follows by (6.13), (6.15), (6.16), (6.18), and (A.14) and the definition of \( \| \cdot \|_G \). □

6.3 - Kapouleas' Ping-Pong Argument - Part II. As discussed in Section 6.1, there exists a linear map \( \Psi : C^{0,\alpha}_{G,\gamma,g} \cap L^2_{G,\gamma,g} \to C^{2,\alpha,g}_{G,\gamma} \cap H^2_{G,\gamma,g} \) such that for all \( f \),

\[ f = \hat{J}_G \Psi f, \]

and,

\[ \| \Psi f \|_{C^{2,\alpha}_{G,\gamma}} \lesssim \epsilon^{-2} \| f \|_{0,G,H}, \]

\[ \| \Psi f \|_{H^2_{G,\gamma}} \lesssim \epsilon^{-2} \| f \|_{0,G,S}. \]  

(6.20)

We now define the operators \( B : F_g \to E_g \) and \( V : F_g \to \mathbb{R}^3 \) by,

\[ Bf := \hat{J}_S \chi_l(\Psi f - (\Psi f)(0)) + YV f - f, \]

\[ Vf := (\Psi f)(0). \]  

(6.21)

where \( \chi_l \) is the cut-off function of the lower transition region \( A(R/4, R/2) \). Observe that, over \( A(R/4,2R) \), the constant function is the Jacobi field of vertical translations, and, in particular, \( \hat{J}_S(\Psi f)(0) \) vanishes throughout. Since, furthermore, \( \hat{J}_S \) coincides with \( \hat{J}_G \) over \( A(2R, \infty) \), we infer that \( Bf \) is supported in the interior of \( B(R) \) making it indeed an element of \( E_g \). Finally, by definition of \( \hat{J}_S \),

\[ Bf = [\hat{J}_G, \chi_l](\Psi f - (\Psi f)(0)) - \chi_c(1 - \chi_l)(\hat{J}_S - \hat{J}_G) \Psi f + YV f. \]  

(6.22)

Observe that, by the Sobolev embedding theorem,

\[ \| Vf \| \lesssim \| \Psi f \|_{H^2_{G,\gamma}} \lesssim \frac{R^2}{(\epsilon R)} \| f \|_{0,G,S}. \]  

(6.23)

For convenience, throughout the rest of this section, we will denote \( \psi := \Psi f \).

**Lemma 6.3.1**

\[ \| \Psi f - (\Psi f)(0) \|_{A(R/4,2R)} \|_{2,G} \lesssim \frac{1}{(\epsilon R)^{\alpha} R^{2+\delta}} \| f \|_{0,G}. \]  

(6.24)

**Proof:** By (6.7), over \( A(R/4,2R) \),

\[ [\psi]_0 \lesssim \frac{1}{(\epsilon R)^{\alpha}(\epsilon R)^2} \left( \| \phi \|_{C^{2,\alpha}_{G,\gamma}} + \frac{1}{(\epsilon R)} \| \phi \|_{H^2_{G,\gamma}} \right) \lesssim \frac{1}{(\epsilon R)^{\alpha}} R^2 \| f \|_{0,G}. \]
Likewise, by (6.6), over this annulus,
\[ |D_G \psi| \lesssim \frac{R}{(\epsilon R)^\alpha} \epsilon \|f\|_{0,G}. \]

Finally, over this annulus,
\[ |D_G^2 \psi| \lesssim \|\phi\|_{C^{2,\alpha}_{G,\gamma}} \lesssim \frac{1}{\epsilon^2} \|f\|_{0,G}, \]
and
\[ |\delta_G^\alpha (D_G^2 \psi|_{A(R/4,2R)})| \lesssim \frac{1}{\epsilon^2} \|f\|_{0,G}. \]

The result now follows by (6.1) and (6.2). □

**Lemma 6.3.2**
\[ \|\hat{J}_S - \hat{J}_D\|_{A(R/4,2R)} \|_{C^{0,\alpha}_{2+\delta,\text{Cyl}}} \lesssim \frac{1}{(\epsilon R)^\alpha} \frac{1}{R^{2-\delta}} \|f\|_{0,G}. \] (6.25)

**Proof:** Indeed, by (6.1), for \( k \in \{0, 1, 2\} \), over \( A(R/4,2R) \),
\[ |D^k \psi| \lesssim \epsilon^k \|\psi\|_{C^{2,\alpha}_{G,\gamma}} \lesssim \frac{1}{\epsilon^{2-k}} \|f\|_{0,G}. \]

Likewise,
\[ |\delta^\alpha (D^2 \psi|_{A(R/4,2R)})| \lesssim \epsilon^\alpha \|f\|_{0,G}. \]

Thus, by (5.11), (5.12), (5.13), (5.14), (5.15) and (2.9), over \( A(R/4,2R) \),
\[ |(\hat{J}_S - \hat{J}_C) \psi| \lesssim \frac{1}{(\epsilon R)^\alpha} \left( \epsilon^2 R^2 + \epsilon + \epsilon^4 R^4 + \frac{1}{R^4} \right) \|f\|_{0,G} \lesssim \frac{1}{(\epsilon R)^\alpha} \frac{1}{R^4} \|f\|_{0,G}. \]

Likewise, using also (A.12) and (A.14),
\[ |\delta^\alpha ((\hat{J}_S - \hat{J}_C) \psi|_{A(R/4,2R)})| \lesssim \frac{1}{(\epsilon R)^\alpha} \frac{1}{R^{4+\alpha}} \|f\|_{0,G}. \]

Thus, by (6.2),
\[ |\delta^\alpha_{\text{Cyl}} ((\hat{J}_S - \hat{J}_C) \psi|_{A(R/4,2R)})| \lesssim \frac{1}{(\epsilon R)^\alpha} \frac{1}{R^3} \|f\|_{0,G}. \]

The result follows. □

**Lemma 6.3.3**
\[ ||\hat{J}_C, \chi_l||(\Psi f - (\Psi f)(0))||_{C^{0,\alpha}_{2+\delta,\text{Cyl}}} \lesssim \frac{1}{(\epsilon R)^\alpha} R^{2+\delta} \|f\|_{0,G}. \] (6.26)

**Proof:** This follows from (6.24) and (5.16). □
Theorem 6.3.4
\[ \|Bf\|_{0,C} \lesssim \frac{R^2}{\epsilon R} \|f\|_{0,G}. \]  

**Proof:** Observe that \( \|\chi_c(1 - \chi_l)\|_{C^{0,\alpha}_{0,\text{cyl}}} \lesssim 1 \). The result now follows by (6.23), (6.25), (6.26), and (A.14). \( \square \)

6.4 - Kapouleas’ Ping-Pong Argument - Part III. By (6.19) and (6.27), and bearing in mind (2.9), the operator norms of the products \( AB \) and \( BA \) satisfy
\[ \|AB\|, \|BA\| \lesssim \frac{1}{(\epsilon R)^{3\alpha}} \frac{1}{\epsilon R^{5+\delta}} \lesssim \frac{1}{\Delta}. \]
We therefore define \( Q_E : \mathcal{E}_g \to \mathcal{E}_g \) and \( Q_F : \mathcal{F}_g \to \mathcal{F}_g \) by
\[ Q_E := \sum_{m=0}^{\infty} (BA)^m, \quad Q_F := \sum_{m=0}^{\infty} (AB)^m, \]  
and it follows that the operator norms of both \( Q_E \) and \( Q_F \) are uniformly bounded for large values of \( \Delta \). We now define \( L_C, L_G, V_C, V_G, P_C \) and \( P_G \) by
\[ L_C e = LQ_E e, \]
\[ L_G f = -LBQ_F f, \]
\[ V_C e = VQ_E e + (\Psi A Q_E e)(0), \]
\[ V_G f = -VBQ_F f - (\Psi Q_F f)(0), \]
\[ P_C e = \chi_u \Phi Q_E e - (1 - \chi_l)((\Psi A Q_E e) - (\Psi A Q_E e)(0)), \]
\[ P_G f = -\chi_u \Phi B Q_F f + (1 - \chi_l)((\Psi Q_F f) - (\Psi Q_F f)(0)). \]

**Lemma 6.4.1**
\[ \hat{J}_S P_C e + XL_C e + YV_C e = e, \]
\[ \hat{J}_S P_G f + XL_G f + YV_G f = f. \]  

**Proof:** Indeed, bearing in mind (6.10) and (6.21),
\[ \hat{J}_S P_C e + XL_C e + YV_C e = \hat{J}_S \chi_u \Phi Q_E e + XLQ_E e + YVQ_E e \]
\[ = A Q_E e + Q_E e - B A Q_E e - A Q_E e \]
\[ = e. \]

The second relation follows in a similar manner, and this completes the proof. \( \square \)

Now let \( \chi \) be the cut-off function of the transition region \( A(2R,4R) \). Observe that \( \chi = O(R^{-k}) \), and so, for all \( f \),
\[ \|\chi f\|_{0,C} \lesssim \|f\|_{0,C}, \quad \|\chi f\|_{0,G} \lesssim \frac{1}{(\epsilon R)^{\alpha}} \|f\|_{0,G}. \]  

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We define

\[ Lf := L_C \chi f + L_G (1 - \chi) f, \]
\[ Vf := V_C \chi f + V_G (1 - \chi) f, \]
\[ Pf := P_C \chi f + P_G (1 - \chi) f. \]  \hspace{1cm} (6.32)

In particular, by (6.30),

\[ \hat{J}_S Pf + XLf + YVf = f. \]  \hspace{1cm} (6.33)

Since the operator norm of \( L \) is uniformly bounded, by (6.27) and (6.31),

\[ \| Lf \| \lesssim \| f\|_{0,C} + \frac{1}{(\epsilon R)^{\alpha}} \frac{R^2}{(\epsilon R)^{\alpha}} \| f\|_{0,G}. \]  \hspace{1cm} (6.34)

Likewise, by (6.19) and (6.23), for sufficiently small \( \alpha \),

\[ \| Vf \| \lesssim \| f\|_{0,C} + \frac{1}{(\epsilon R)^{\alpha}} \frac{R^2}{(\epsilon R)^{\alpha}} \| f\|_{0,G}. \]  \hspace{1cm} (6.35)

**Theorem 6.4.2**

For sufficiently small \( \alpha \),

\[ \| Pf \|_{2,C} \lesssim \| f\|_{0,C} + \frac{1}{(\epsilon R)^{\alpha}} \frac{R^2}{(\epsilon R)^{\alpha}} \| f\|_{0,G}. \]  \hspace{1cm} (6.36)

**Proof:** Indeed, by (6.19) and (6.24),

\[ \| P_C \chi f \|_{2,C} \lesssim \| f\|_{0,C} + \frac{1}{(\epsilon R)^{2\alpha}} \frac{1}{R^6} \| f\|_{0,G}. \]

Likewise, by (6.24), (6.27) and (6.31),

\[ \| P_G (1 - \chi) f \|_{2,C} \lesssim \frac{1}{(\epsilon R)^{\alpha}} \frac{R^2}{(\epsilon R)^{\alpha}} \| f\|_{0,G} + \frac{1}{(\epsilon R)^{2\alpha}} \frac{R^{2+\delta}}{(\epsilon R)^{2\alpha}} \| f\|_{0,G}. \]

The result now follows by (2.9). \( \square \)

**Theorem 6.4.3** For sufficiently small \( \alpha \),

\[ \| Pf \|_{2,G} \lesssim \frac{1}{(\epsilon R)^{\alpha}} \frac{1}{(\epsilon R)^{2^\delta}} \| f\|_{0,C} + \frac{1}{(\epsilon R)^{2\alpha}} \frac{1}{\epsilon^2 (\epsilon R)^{R^6}} \| f\|_{2,G}. \]  \hspace{1cm} (6.37)

**Proof:** By (6.19) and (6.12),

\[ \| P_C \chi f \|_{2,G} \lesssim \frac{1}{(\epsilon R)^{\alpha}} \frac{1}{(\epsilon R)^{2^\delta}} \| f\|_{0,C} + \frac{1}{(\epsilon R)^{2\alpha}} \frac{1}{\epsilon^2 (\epsilon R)^{R^6}} \| f\|_{0,C}. \]

Likewise, by (6.12), (6.27) and (6.31),

\[ \| P_G (1 - \chi) f \|_{2,G} \lesssim \frac{1}{(\epsilon R)^{2\alpha}} \frac{1}{\epsilon^2 (\epsilon R)^{R^6}} \| f\|_{2,G} + \frac{1}{(\epsilon R)^{\alpha}} \frac{1}{\epsilon^2} \| f\|_{2,G}. \]

The result now follows by (2.9). \( \square \)
7 - Existence and Embeddedness.

7.1 - The Schauder Fixed-Point Theorem. At this point, it is convenient to modify slightly the norms introduced in (6.8). We define,
\[
\begin{align*}
\|f\|'_{m,G,H} &:= \|f|_{A(2R,\infty)}\|_{C_{G,G}^{m}\alpha}, \\
\|f\|'_{m,G,S} &:= \|f|_{A(2R,\infty)}\|_{H_{G,G}^{m}\gamma}, \\
\|f\|'_{m,G} &:= \|f\|'_{m,G,H} + \frac{1}{(\epsilon R)} \|f\|'_{m,G,S}.
\end{align*}
\]
(7.1)

Observe that, by the definition (6.32) of \( L, V \) and \( P \), this does not affect the estimates of Section 6.4.

For all \( m, \alpha \) and \( \gamma \), we define \( X_{m,\alpha,\gamma} \) to be the completion of \( C_{\infty}(S) \) with respect to the seminorms \( \| \cdot \|_{m,C} \) and \( \| \cdot \|_{m,G} \). Observe that this makes \( X_{m,\alpha,\gamma} \) into a Frechet space. As in Section 5.2, let \( M: \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus X_{2,\alpha,\gamma} \to X_{0,\alpha,\gamma} \) denote the MCFS functional.

It is a straightforward exercise to show that for \( \|l\|, \|v\| \) and \( \|f\|_{2,C} \) sufficiently small, independent of \( \Delta \),
\[
\|M(l,v,f) - M(0,0,0) - \hat{J}S f - Xl - Yv\|_{0,C} \lesssim \|f\|_{2,C}^2 + \|l\|^2 + \|v\|^2.
\]
(7.2)

The corresponding estimate over the Grim end is more subtle. We first require the following preliminary estimate, which will also serve in the proof of embeddedness.

Lemma 7.1.1

Indeed, for all \( f \),
\[
\|f\|'_{1,G,H} \lesssim (\epsilon R)^{1-2\alpha} \left( \frac{1}{\epsilon^2 R^{2+\delta}} \|f\|_{2,C} + \|f\|'_{2,G} \right).
\]
(7.3)

Proof: For all \( f \),
\[
\|f|_{A(R,\infty)}\|_{H_{G,G}^{2}\gamma} \lesssim (\epsilon R) \left( \frac{1}{\epsilon^2 R^{2+\delta}} \|f\|_{2,C} + \|f\|_{2,G}' \right).
\]

We apply the Sobolev embedding theorem over \( A(R,\infty) \) in \( \mathbb{R}^2 \) by first applying the Sobolev trace formula over half planes and then applying the Sobolev embedding theorem over complete straight lines. In this manner, we obtain,
\[
\|f|_{A(2R,\infty)}\|_{C_{G,G}^{0,1-\alpha}} \lesssim (\epsilon R) \left( \frac{1}{\epsilon^2 R^{2+\delta}} \|f\|_{2,C} + \|f\|_{2,G}' \right).
\]

Thus, by (A.12) and (A.13),
\[
\|f\|'_{1,G,H} \lesssim (\epsilon R)^{1-2\alpha} \left( \frac{1}{\epsilon^2 R^{2+\delta}} \|f\|_{2,C} + \|f\|_{2,G}' \right)^{\frac{1}{1+2\alpha}} \left( \|f\|_{2,G}' \right)^{\frac{2\alpha}{1+2\alpha}}
\]
\[
\lesssim (\epsilon R)^{1-2\alpha} \left( \frac{1}{\epsilon^2 R^{2+\delta}} \|f\|_{2,C} + \|f\|_{2,G}' \right),
\]

as desired. □
Lemma 7.1.2

There exists $\eta > 0$ such that, for sufficiently large $\Delta$, if $\epsilon(\epsilon R)^{1-2\alpha}\|f\|_{2,G} < \eta$ and if $\|f\|_{0,C}/(\epsilon R)^{2\alpha} R^{1+\delta} < \eta$, then

$$\|M(l, v, f) - M(0, 0, 0) - \hat{J}_S f\|_{0,G} \lesssim \epsilon^3 (\epsilon R)^{1-2\alpha} \left( \frac{1}{\epsilon^2 R^{2+\delta}} \|f\|_{2,G} + \|f\|_{2,G}' \right) \|f\|_{2,G}. \quad (7.4)$$

Proof: Indeed, since $M$ is a second-order quasi-linear functional, if $\|\epsilon f\|_{1,G,H}'$ is sufficiently small, then, by rescaling, we obtain

$$\|M(l, v, f) - M(0, 0, 0) - \hat{J}_S f\|_{0,G} \lesssim \epsilon^3 \|f\|_{1,G,H}' \|f\|_{2,G},$$

and the result now follows by (7.3). □

Finally, before applying the fixed-point theorem, we obtain an estimate of the error in the MCFS functional of the joined surface.

Lemma 7.1.3

$$\begin{align*}
\|M(0, 0, 0)\|_{0,C} &\lesssim \epsilon R^{2+\delta}, \\
\|M(0, 0, 0)\|_{0,G} &= 0. \quad (7.5)
\end{align*}$$

Proof: We denote $\psi = M(0, 0, 0)$. Since $C$ is minimal, over $B(R)$,

$$\psi = \epsilon \mu.$$

Thus, by (5.1) and (A.6),

$$\|\psi\|_{B(R)} \|\psi\|_{C^{0,\alpha}_{2+\delta,cyl}} \lesssim \epsilon R^{2+\delta}.$$

By (5.4) and (2.9), over $A(R, 2R)$,

$$H_i = O(R^{-(k+1)}).$$

Thus, by (A.6),

$$\mu = 1 + O(R^{-2}), \quad g^{ij} = \delta_{ij} + O(R^{-2}).$$

Thus, by (6.1) and (A.7),

$$\|\psi\|_{A(R,2R)} \|\psi\|_{C^{0,\alpha}_{2+\delta,cyl}} \lesssim \epsilon R^{2+\delta}.$$

Upon combining these relations, the first estimate follows. However, by construction, $f$ vanishes over $A(2R, \infty)$, and so $\|\psi\|_{0,G} = 0$. This completes the proof. □
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**Theorem 7.1.4**

For \( \alpha \) and \( \gamma \) sufficiently small, and for \( \Delta \) sufficiently large, there exist \( l, v \) and \( f \) such that \( M(l, v, f) = 0 \) and,

\[
\|l\|, \|v\|, \|f\|_{2,C} \lesssim \epsilon R^{2+\delta}, \quad \|f\|_{2,G} \lesssim \frac{1}{\epsilon}.
\]  

(7.6)

**Proof:** Fix \( \alpha, \gamma \ll 1 \). Set \( \psi_0 := M(0, 0, 0) \) and denote \((l_0, v_0, f_0) := \phi_0 := (L\psi_0, V\psi_0, P\psi_0)\).

By (7.5) and the norm estimates of Section 6.4, there exists a constant \( C > 0 \), such that, for all large \( \Delta \),

\[
\|l_0\|, \|v_0\|, \|f_0\|_{2,C} \leq C\epsilon R^{2+\delta}, \quad \|f_0\|_{2,G} \leq \frac{C}{\epsilon}.
\]

We define \( \Omega \subseteq \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus X_{2,\alpha,\gamma} \) to be the set of all triplets \((l, v, f)\) such that,

\[
\|l_0\|, \|v_0\|, \|f_0\|_{2,C} \leq 2C\epsilon R^{2+\delta}, \quad \|f_0\|_{2,G} \leq \frac{2C}{\epsilon}.
\]

Observe that \( \Omega \) is convex. Furthermore, by the Arzela-Ascoli Theorem, for all \( \alpha' < \alpha \) and \( \gamma' < \gamma \), \( \Omega \) is a compact subset of \( X_{m,\alpha',\gamma'} \). We now define a mapping \( \Phi : \Omega \to \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus X_{2,\alpha,\gamma} \) as follows. Fix \( \phi := (l, v, f) \) in \( \Omega \) and denote \( \psi := M(l, v, f) - M(0, 0, 0) - \hat{J}_S f - Xl - Xv \).

By (7.2) and (7.4),

\[
\|\psi\|_{0,C} \lesssim \left(\epsilon R^{2+\delta}\right)^2, \quad \|\psi\|_{0,G} \lesssim \epsilon R^{1-2\alpha}.
\]

We now set \( \Phi(\phi) := \phi_0 + (L\psi, V\psi, P\psi) \). By the norm estimates of Section 6.4 again and by (2.9), for sufficiently large \( \Delta \), \( \Phi \) maps \( \Omega \) to itself. Furthermore, for all \( \alpha' < \alpha \) and \( \gamma' < \gamma \), \( \Phi \) is continuous with respect to the topology of \( X_{2,\alpha',\gamma'} \). It follows by the Schauder fixed point-theorem (c.f. [7]) that there exists a fixed point \( \phi \) of \( \Phi \) in \( \Omega \). We readily verify that \( M(\phi) = 0 \), and this completes the proof. \( \square \)

**Theorem 7.1.5**

Let \((l, v, f)\) be as in Theorem 7.1.4. For sufficiently large \( \Delta \), the surface \( H(l, v, f) \) is embedded.

**Proof:** We denote the joined surface by \( S \), we denote the image of \( H(l, v, f) \) by \( S' \), and we rescale both \( S \) and \( S' \) by \( \epsilon \). Observe that the intersection of \( S \) with \( A(\epsilon R) \) consists of 3 distinct Grim ends which we denote by \( G_+, G_0 \) and \( G_- \) respectively. Let \( u_+, u_0 \) and \( u_- \) be the respective profiles of these ends, and let \( v_+, v_0 \) and \( v_- \) be their derivatives in the radial direction. Observe that

\[
u_+(\epsilon R) > u_0(\epsilon R) > u_-(\epsilon R),
\]

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and

$$v_+(\epsilon R) > v_0(\epsilon R) > v_-(\epsilon R).$$

Since $$v_+, v_0$$ and $$v_-$$ are all solutions of the same first order ODE, it follows that $$v_+(r) > v_0(r) > v_- (r)$$ for all $$r$$. In particular, the ends $$G_+, G_0$$ and $$G_-$$ are separated vertically by a distance of no less that $$\eta$$, for some $$\eta > 0$$. In fact $$\eta \sim \epsilon \log(\epsilon R)$$.

Now let $$G'_+, G'_0$$ and $$G'_-$$ be the three ends of $$S'$$. Since $$\| \epsilon f \|_{0, C} \lesssim \epsilon^2 R^{2 + \delta}$$, over the circle $$C(2R)$$, $$G'_+$$ lies strictly above $$G'_0$$, and $$G'_0$$ lies strictly above $$G'_-$$.

However, by (7.3),

$$\| \epsilon f \|_{1, G, H} \lesssim (\epsilon R)^{1 - 2\alpha}.$$  

Bearing in mind the definition of the norm $$\| \cdot \|_{1, G, H}$$, it follows that for sufficiently large $$\Delta$$, $$G'_+, G'_0$$ and $$G'_-$$ are all graphs over $$A(\epsilon R, \infty)$$. Furthermore, for some large $$R'$$, the restrictions of $$G'_+, G'_0$$ and $$G'_-$$ to $$A(R', \infty)$$ lie within $$\eta/2$$-neighbourhoods of $$G_+, G_0$$ and $$G_-$$ respectively. In particular, outside $$B(R')$$, $$G'_+$$ lies strictly above $$G'_0$$ and $$G'_0$$ lies strictly above $$G'_-$$.

Since vertical translates of MCF solitons are also MCF solitons, it now follows by the strong maximum principle that, over the whole of $$A(\epsilon R, \infty)$$, $$G'_+$$ lies strictly above $$G'_0$$ and $$G'_0$$ lies strictly above $$G_-$$.

This completes the proof. □

A - Terminology, Conventions and Standard Results.

A.1 - General Definitions. Let $$\mathbb{R}^2$$ and $$\mathbb{R}^3$$ denote respectively 2 and 3 dimensional Euclidean space and let $$\Sigma$$ be an embedded surface in $$\mathbb{R}^3$$. We consider $$\mathbb{R}^2$$ as the $$x - y$$ plane in $$\mathbb{R}^3$$.  

(1) $$\pi : \mathbb{R}^3 \to \mathbb{R}^2$$ denotes the canonical projection.

(2) $$r$$ denotes a smooth positive function over $$\mathbb{R}^2$$ which is equal to the distance to the origin outside some (suitably large) compact set. We denote the composition of $$r$$ with $$\pi$$ also by $$r$$.

(3) $$e_x$$, $$e_y$$ and $$e_z$$ denote the vectors of the canonical basis of $$\mathbb{R}^3$$.

(4) $$e_r$$, $$e_\theta$$ denote respectively the unit radial and unit angular vector fields over $$\mathbb{R}^2$$. We denote the pull-backs of these vector fields through $$\pi$$ also by $$e_r$$ and $$e_\theta$$ respectively.

(5) $$D$$ denotes the canonical differentiation operator over $$\mathbb{R}^2$$ and $$\mathbb{R}^3$$.

(6) $$\Delta$$ denotes the canonical Laplacian over $$\mathbb{R}^2$$ (not to be confused with $$\Delta^2$$, defined below).

(7) $$C(a)$$ denotes the circle of radius $$a$$ about the origin in $$\mathbb{R}^2$$. It also denotes the cylinder of radius $$a$$ about the $$z$$-axis in $$\mathbb{R}^3$$ as well as the intersection of this cylinder with $$\Sigma$$.

(8) $$B(a)$$ denotes the closed disk of radius $$a$$ about the origin in $$\mathbb{R}^2$$. It also denotes the closed solid cylinder of radius $$a$$ about the $$z$$-axis in $$\mathbb{R}^3$$ as well as the intersection of this cylinder with $$\Sigma$$.

(9) $$A(a, b)$$ denotes the closed annulus of inner radius $$a$$ and outer radius $$b$$ about the origin in $$\mathbb{R}^2$$. It also denotes the closed annular prism of inner radius $$a$$ and outer radius $$b$$ about the $$z$$-axis in $$\mathbb{R}^3$$ as well as the intersection of this prism with $$\Sigma$$.  

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Let $\chi : [0, \infty] \to \mathbb{R}$ be a non-negative, non-increasing function such that $\chi = 1$ over $[0, 1]$ and $\chi = 0$ over $[2, \infty]$. For all $a$, define $\chi_a : \mathbb{R}^2 \to \mathbb{R}$ by $\chi_a(x) = \chi(\|x\|/a)$. We call $\chi_a$ the cut-off function of the transition region $A(a, 2a)$. Composing with $\pi$, we likewise consider $\chi_a$ as a function over $\mathbb{R}^3$ and $\Sigma$.

**A.2 - Surface Geometry.**

1. $N_\Sigma$ denotes the unit normal vector field over $\Sigma$.
2. $\pi_\Sigma$ denotes the orthogonal projection onto the tangent space of $\Sigma$.
3. $\nabla_\Sigma$ denotes the gradient operator as well as the Levi-Civita covariant derivative of $\Sigma$.
4. $\text{Hess}_\Sigma$ denotes the intrinsic Hessian operator of $\Sigma$.
5. $\Delta_\Sigma$ denotes the intrinsic Laplacian of $\Sigma$.
6. $\Pi_\Sigma$ denotes the second fundamental form of $\Sigma$.
7. $A_\Sigma$ denotes the shape operator of $\Sigma$.
8. $H_\Sigma$ denotes the mean curvature of $\Sigma$ (taken to be the sum of the principle curvatures, or the trace of the shape operator).
9. $M_\Sigma$ denotes the MCFS operator of $\Sigma$ (with speed $\epsilon$). It is given by

$$M_\Sigma := H_\Sigma + \epsilon \langle N_\Sigma, e_z \rangle.$$  \hspace{1cm} (A.1)

10. $J_\Sigma$ denotes the MCFS Jacobi operator (with speed $\epsilon$) of $\Sigma$. That is, $J_\Sigma$ is the linearisation of the MCFS operator of $\Sigma$. It is given by

$$J_\Sigma f = \Delta_\Sigma f + \text{Tr}(A_\Sigma^2) f + \epsilon \langle \nabla_\Sigma f, e_z \rangle.$$  \hspace{1cm} (A.2)

We recall in addition the following elementary relations. For any function $f$ defined over a neighbourhood of $\Sigma$,

$$\nabla_\Sigma f = Df - \langle Df, N_\Sigma \rangle N_\Sigma,$$
$$\text{Hess}_\Sigma(f) = \text{Hess}(f) - \langle Df, N_\Sigma \rangle \Pi_\Sigma.$$  \hspace{1cm} (A.3)

Given any positive function $\phi$ defined over $\Sigma$, if $\hat{J}_\Sigma := M_\phi^{-1}J_\Sigma M_\phi$ denotes the conjugate of $J_\Sigma$ with the operator of multiplication by $\phi$, then

$$\hat{J}_\Sigma f = \Delta_\Sigma f + 2\phi^{-1} \langle \nabla_\Sigma \phi, \nabla_\Sigma f \rangle + \epsilon \langle \nabla_\Sigma f, e_z \rangle + \phi^{-1}J_\Sigma \phi f.$$  \hspace{1cm} (A.4)
A.3 - Surface Geometry of Graphs. (1) If $\Sigma$ is the graph of a function $u$ over a subset of $\mathbb{R}^2$, then we call $u$ the profile of $\Sigma$. In this case, $\pi$ defines a coordinate chart of $\Sigma$ in $\mathbb{R}^2$. It will be more convenient to work, sometimes over $\Sigma$, and sometimes over $\mathbb{R}^2$, and we will move freely between these two perspectives.

(2) $g_{ij}$ denotes the intrinsic metric of $\Sigma$. Its inverse is denoted by $g^{ij}$.

(3) $\Gamma^k_{ij}$ denotes the Christoffel symbols of the Levi-Civita covariant derivative of $g_{ij}$.

Denoting

$$\mu := \langle e_z, N_\Sigma \rangle,$$

we readily verify the following relations.

$$\mu = \frac{1}{\sqrt{1 + \|Du\|^2}},$$

$$g_{ij} = \delta_{ij} + u_iu_j,$$

$$g^{ij} = \delta_{ij} - \mu^2 u^i u^j,$$

$$\Gamma^k_{ij} = g^{kp}u_{ij}u_p,$$

$$\text{Hess}^\Sigma (f)_{ij} = f_{ij} - g^{kp}u_{ij}u_p f_k,$$

$$\Delta^\Sigma (f) = g^{ij} f_{ij} - g^{ij} g^{kp}u_{ij}u_p f_k,$$

$$\Pi^\Sigma_{ij} = -\mu u_{ij},$$

$$(A^\Sigma)^i_j = -\mu g^{jp} u_{pj},$$

$$H^\Sigma = -\mu g^{ij} u_{ij},$$

$$\pi^T (e_z)_i = \mu^2 u_i.$$

(4) When $\Sigma$ is a graph, the MCFS functional is given by,

$$M_\Sigma = \mu g^{ij} u_{ij} + \epsilon \mu.$$

A.4 - Analysis. (1) Given two variable quantities $a$ and $b$, we write

$$a \lesssim b$$

whenever there exists a constant $C$, which for the purposes of this paper we consider to be universal, such that

$$a \leq C b.$$

(2) Given a function $f$ and a sequence of functions $(g_m)$, we write

$$f = O(g_m)$$

whenever there exists a sequence $(C_m)$ of constants, which for the purposes of this paper we consider to be universal, such that the relation

$$|D^m f| \leq C_m g_m$$

holds pointwise for all $m$. The indexing variable of the sequence $(g_m)$ should be clear from the context. In some cases, as in, for example (2.32), every element of the sequence $(g_m)$ is the same.
A.5 - Function Spaces. Let $X$ be a metric space.

(1) For all $\alpha \in [0, 1]$, we define the H"older seminorm of order $\alpha$ over $X$ by

$$[f]_\alpha := \sup_{x \neq y \in X} \frac{|f(x) - f(y)|}{d(x, y)\alpha}.$$  \hfill (A.10)

Observe that $[f]_0$ measures the total variation of $f$. In particular,

$$[f]_0 \leq 2\|f\|_{C^0}.$$  \hfill (A.11)

We readily prove that for all $\alpha \in [0, 1]$,

$$[f]_\alpha \leq [f]_0^{1-\alpha}[f]_1^{\alpha} \leq 2^{1-\alpha}\|f\|_{C^0}^{1-\alpha}[f]_1^{\alpha}. \hfill (A.12)$$

If $X$ is a complete manifold, and if $f$ is differentiable over $X$, then for all $\alpha \in [0, 1]$ and for all $\beta \in ]0, 1[$,

$$\|Df\|_{C^0} \leq 2[f]_\alpha^{1+\beta-\alpha}[Df]_\beta^{1-(\beta-\alpha)}, \hfill (A.13)$$

which merely expresses the (almost) log-convexity of the H"older seminorms. For all $\alpha$, we have the following variant of the product rule.

$$[fg]_\alpha \leq \|f\|_{C^0} [g]_\alpha + [f]_\alpha \|g\|_{C^0}. \hfill (A.14)$$

Finally, if $X = X_1 \cup \ldots \cup X_m$, then, for all $\alpha$,

$$[f]_\alpha \leq m^{1-\alpha} \sup_{1 \leq k \leq m} [f|_{X_k}]_\alpha. \hfill (A.15)$$

If, in particular, $X = [0, m+1] \times S^1$ is a cylinder and $X_i = [i, i+1] \times S^1$ for all $i$, then (A.15) refines to

$$[f]_\alpha \leq \sum_{i=1}^m [f|_{X_i}]_\alpha. \hfill (A.16)$$

(2) For a continuous function $f$ over $X$, for all $\alpha$, we define

$$\delta^\alpha f(x) := [f|_{B_1(x)}]_\alpha. \hfill (A.17)$$

Now suppose that $X$ is a smooth Riemannian manifold.

(3) For all $k, \alpha$, we define the $C^{k,\alpha}$-H"older norm over $C^\infty(M)$ by

$$\|f\|_{C^{k,\alpha}} := \sum_{i=0}^k \|D^i f\|_{C^0} + \|\delta^\alpha D^k f\|_{C^0}. \hfill (A.18)$$

We define the space $C^{k,\alpha}(X)$ to be the closure of $C^\infty(X)$ with respect to this norm.
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(4) For all $p$, we define the $L^p$-norm over $C^\infty_0(M)$ by

$$\|f\|_{L^p}^p := \int_X |f|^p \, d\text{Vol}. \quad (A.19)$$

We define the space $L^p(X)$ to be the closure of $C^\infty_0(X)$ with respect to this norm.

(5) For all $k$, we define the $H^k$-Sobolev norm over $C^\infty_0(M)$ by

$$\|f\|_{H^k} := \sum_{i=0}^{k} \|D^i f\|_{L^2}. \quad (A.20)$$

The reader may verify that all surfaces studied in this paper are sufficiently regular at infinity for the Sobolev embedding theorem to hold. That is for all $l$, and for all $k+\alpha < l-1$,

$$\|f\|_{C^{k,\alpha}} \lesssim \|f\|_{H^l}. \quad (A.21)$$

(6) We will make use of the following readily verified formulae.

$$\sup_{t \in [1,T]} \log(t) t^\alpha \lesssim \begin{cases} \log(T) T^\alpha & \text{if } \alpha > 0, \\ 1 & \text{if } \alpha < 0. \end{cases} \quad (A.22)$$

Likewise

$$\int_{A(1,T)} \log(r)^m r^\alpha \, d\text{Vol}_{\text{Cyl}} \lesssim \begin{cases} \log(T)^m T^\alpha & \text{if } \alpha > 0, \\ 1 & \text{if } \alpha < 0. \end{cases} \quad (A.23)$$

A.6 - Elliptic Estimates. Let $E$ and $F$ be Banach spaces and let $A : E \rightarrow F$ be a bounded linear map. We say that $A$ satisfies an elliptic estimate whenever there exists a normed vector space $G$, a compact map $K : E \rightarrow G$, and a constant $C$ such that for all $e$ in $E$,

$$\|e\| \leq C (\|Ke\| + \|Ae\|). \quad (A.24)$$

The following straightforward result plays an important rôle in Fredholm theory.

**Theorem A.6.1**

If $A$ satisfies an elliptic estimate, then the kernel of $A$ is finite-dimensional and its image is a closed subset of $F$.

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