ON I-NULL LIE ALGEBRAS

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Abstract. We consider the class of complex Lie algebras for which the Koszul 3-form is zero, and prove that it contains all quotients of Borel subalgebras, or of their nilradicals, of finite dimensional complex semisimple Lie algebras. A list of Kac-Moody types for indecomposable nilpotent complex Lie algebras of dimension ≤ 7 is given.

1. Introduction

Leibniz algebras are non-antisymmetric versions $\mathfrak{g}$ of Lie algebras: the commutator is not required to be antisymmetric, and the right adjoint operations $[.,Z]$ are required to be derivations for any $Z \in \mathfrak{g}$ (10). In the presence of antisymmetry, that is equivalent to the Jacobi identity. Leibniz algebras have a cohomology of their own, the Leibniz cohomology $HL^* (\mathfrak{g}, \mathfrak{g})$, associated to the complex $CL^* (\mathfrak{g}, \mathfrak{g}) = \text{Hom} (\mathfrak{g} \otimes^* \mathfrak{g}, \mathfrak{g}) = \mathfrak{g} \otimes (\mathfrak{g}^*)^\otimes$ and the Leibniz coboundary $\delta$ defined for $\psi \in CL^n (\mathfrak{g}, \mathfrak{g})$ by

$$(\delta \psi)(X_1, X_2, \cdots, X_{n+1}) =$$
$$[X_1, \psi(X_2, \cdots, X_{n+1})] + \sum_{i=2}^{n+1} (-1)^i [\psi(X_1, \cdots, \hat{X}_i, \cdots, X_{n+1}), X_i]$$
$$+ \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \psi(X_1, \cdots, X_{i-1}, [X_i, X_j], X_{i+1}, \cdots, \hat{X}_j, \cdots, X_{n+1})$$

(If $\mathfrak{g}$ is a Lie algebra, $\delta$ coincides with the usual coboundary $d$ on $C^* (\mathfrak{g}, \mathfrak{g}) = \mathfrak{g} \otimes \wedge^* \mathfrak{g}^*$.) Since Lie algebras are Leibniz algebras, a natural question is, given some fixed Lie algebra, whether or not it has more infinitesimal Leibniz deformations (i.e. deformations as a Leibniz algebra) than infinitesimal deformations as a Lie algebra. That amounts to the comparison of the adjoint Leibniz 2-cohomology group $HL^2 (\mathfrak{g}, \mathfrak{g})$ and the ordinary one $H^2 (\mathfrak{g}, \mathfrak{g})$, and was addressed by elementary methods in [5]. There we proved that

$$HL^2 (\mathfrak{g}, \mathfrak{g}) = H^2 (\mathfrak{g}, \mathfrak{g}) \oplus ZL^2_0 (\mathfrak{g}, \mathfrak{g}) \oplus C,$$

where $ZL^2_0 (\mathfrak{g}, \mathfrak{g})$ is the space of symmetric Leibniz 2-cocycles and $C$ is a space consisting of coupled Leibniz 2-cocycles, i.e. the nonzero elements have the property that their symmetric and antisymmetric parts are not cocycles. The Lie algebra $\mathfrak{g}$ is said to be (adjoint) $ZL^2$-uncoupling if $C = \{0\}$. That is best understood in terms of the Koszul map $\mathcal{I}$ which associates to any invariant bilinear form $B$ on the Lie algebra $\mathfrak{g}$ the Koszul form $(X, Y, Z) \mapsto I_B (X, Y, Z) = B ([X, Y], Z)$ $(X, Y, Z \in \mathfrak{g})$. Then $ZL^2_0 (\mathfrak{g}, \mathfrak{g}) = \mathfrak{c} \otimes \ker \mathcal{I}$ ($\mathfrak{c}$ the center of $\mathfrak{g}$) and $C \cong (\mathfrak{c} \otimes \text{Im} \mathcal{I}) \cap B^3 (\mathfrak{g}, \mathfrak{g})$. Hence $\mathfrak{g}$ is $ZL^2$-uncoupling if and only if $(\mathfrak{c} \otimes \text{Im} \mathcal{I}) \cap B^3 (\mathfrak{g}, \mathfrak{g}) = \{0\}$. The class of (adjoint)
$ZL^2$-uncoupling Lie algebras is rather extensive since it contains, beside the class of zero center Lie algebras, the class of Lie algebras having zero Koszul form, which we call $\mathcal{I}$-null Lie algebras.

In the present paper, we examine some properties of the class of $\mathcal{I}$-null Lie algebras. First, after proving basic properties of $\mathcal{I}$-null Lie algebras, we state in Proposition 2.6 a result for Lie algebras having a codimension 1 ideal, connecting $\mathcal{I}$-nullity of the ideal and $\mathcal{I}$-nullity or $\mathcal{I}$-exactness (i.e. the Koszul form is a coboundary) of the Lie algebra itself. Several corollaries are given, and fundamental examples are treated in detail. We also give a table (Table 1) for all non $\mathcal{I}$-null complex Lie algebras of dimension $\leq 7$. This table is a new result. Then comes the main result of the paper, Theorem 3.1, which states that any nilradical of a Borel subalgebra of a finite-dimensional semi-simple Lie algebra is $\mathcal{I}$-null.

We also give a list of Kac-Moody types for indecomposable nilpotent Lie algebras of dimension $\leq 7$ (Table 2). Again, that result is new.

Throughout the paper, the base field is $\mathbb{C}$.

2. The Koszul map and $\mathcal{I}$-null Lie algebras

Let $g$ be any finite dimensional complex Lie algebra. Recall that a symmetric bilinear form $B \in S^2 g^*$ is said to be invariant (see [9]), i.e. $B \in (S^2 g^*)^g$ if and only if $B([Z,X],Y) = -B(X,[Z,Y]) \forall X,Y,Z \in g$. The Koszul map $\mathcal{I} : (S^2 g^*)^g \to \bigotimes^3 g^g \subset Z^3(g,\mathbb{C})$ is defined by $\mathcal{I}(B) = I_B$, with $I_B(X,Y,Z) = B([X,Y]_I,Z) \forall X,Y,Z \in g$.

**Lemma 2.1.** Denote $C^2 g = [g,g]$. The projection $\pi : g \to g/C^2 g$ induces an isomorphism $$\varpi : \ker \mathcal{I} \to S^2 (g/C^2 g)^g.$$ **Proof.** For $B \in \ker \mathcal{I}$, define $\varpi(B) \in S^2 (g/C^2 g)^g$ by $$\varpi(B)(\pi(X),\pi(Y)) = B(X,Y), \forall X,Y \in g.$$ $\varpi(B)$ is well-defined since for $X,Y,U,V \in g$ $$B(X+[U,V],Y) = B(X,Y) + B([U,V],Y) = B(X,Y) + I_B(U,V,Y) = B(X,Y) \text{ (as } I_B = 0).$$

The map $\varpi$ is injective since $\varpi(B) = 0$ implies $B(X,Y) = 0 \forall X,Y \in g$. To prove that it is onto, let $\bar{B} \in S^2 (g/C^2 g)^g$, and let $B_\pi \in S^2 g^g$ defined by $B_\pi(X,Y) = B(\pi(X),\pi(Y))$. Then $B_\pi([X,Y],Z) = B(\pi([X,Y]),\pi(Z)) = B(0,\pi(Z)) = 0 \forall X,Y,Z \in g$, hence $B_\pi \in (S^2 g^g)^g$ and $B_\pi \in \ker \mathcal{I}$. Now, $\varpi(B_\pi) = B$. $\square$

From Lemma 2.1, $\dim (S^2 g^g)^g = \frac{\ell(\ell+1)}{2} + \dim \text{Im } \mathcal{I}$, where $\ell = \dim H^1(g,\mathbb{C}) = \dim (g/C^2 g)$. For reductive $g$, $\dim (S^2 g^g)^g = \dim H^3(g,\mathbb{C})$ ([9]).

**Definition 2.2.** $g$ is said to be $\mathcal{I}$-null (resp. $\mathcal{I}$-exact) if $\mathcal{I} = 0$ (resp. $\text{Im } \mathcal{I} \subset B^3(g,\mathbb{C})$).

$g$ is $\mathcal{I}$-null if and only $C^2 g \subset \ker B \forall B \in (S^2 g^g)^g$. It is standard that for any $B \in (S^2 g^g)^g$, there exists $B_1 \in (S^2 g^g)^g$ such that $\ker (B+B_1) \subset C^2 g$. Hence $\bigcap_{B \in (S^2 g^g)^g} \ker B \subset C^2 g$, and $g$ is $\mathcal{I}$-null if and only $\bigcap_{B \in (S^2 g^g)^g} \ker B = C^2 g$. 

$\square$
Lemma 2.3. (i) Any quotient of a (not necessarily finite dimensional) I-null Lie algebra is I-null.
(ii) Any finite direct product of I-null Lie algebras is I-null.

Proof. (i) Let \( g \) be any I-null Lie algebra, \( h \) an ideal of \( g \), \( \bar{g} = g/h \), \( \pi : g \to \bar{g} \) the projection, and \( \bar{B} \in (S^2\bar{g}^*)^\theta \). Define \( B_\pi \in S^2g^* \) by \( B_\pi(X,Y) = \bar{B}(\pi(X),\pi(Y)) \), \( X,Y \in g \). Then \( B_\pi([X,Y],Z) = B_\pi(\pi([X,Y]),\pi(Z)) = B_\pi((\pi(X),\pi(Y)),\pi(Z)) = B_\pi(\pi(X),\pi([Y,Z])) = B_\pi(X,Y,Z) \in g \), hence \( B_\pi \in (S^2g^*)^\theta \) and \( I_B \circ (\pi \times \pi \times \pi) = I_{B_\pi} = 0 \) since \( g \) is I-null. Hence \( I_B = 0 \).

(ii) Let \( g = g_1 \times g_2 \) (I-null) and \( B \in (S^2g^*)^\theta \). As \( B(X_1,[Y_2,Z_2]) = B([X_1,Y_2],Z_2) = 0 \) \( \forall X_1 \in g_1, Y_2, Z_2 \in g_2 \), \( B \) vanishes on \( g_1 \times C^2g_2 \) and on \( C^2g_1 \times g_2 \) as well, hence \( I_B = 0 \). \( \square \)

Lemma 2.4. Let \( g \) be a finite dimensional semi-simple Lie algebra, with Cartan subalgebra \( h \), simple root system \( S \), positive roots \( \Delta_+ \), and root subspaces \( g^\alpha \). Let \( \mathfrak{t} \not= \{0\} \) be any subspace of \( h \), and \( \Gamma \subset \Delta_+ \) such that \( \alpha + \beta \in \Gamma \) for \( \alpha, \beta \in \Gamma \), \( \alpha + \beta \in \Delta_+ \). Consider \( u = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Gamma} g^\alpha \).

(i) Suppose that \( \alpha|_{\mathfrak{t}} \not= 0 \) \( \forall \alpha \in \Gamma \). Then \( u \) is I-null.

(ii) Suppose that \( \alpha|_{\mathfrak{t}} = 0 \) \( \forall \alpha \in \Gamma \cap S \), and \( \alpha|_{\mathfrak{t}} \not= 0 \) \( \forall \alpha \in \Gamma \setminus S \). Then \( u \) is I-null.

Proof. (i) Let \( u_+ = \bigoplus_{\alpha \in \Gamma} g^\alpha \), and \( X_\alpha \) a root vector in \( g^\alpha : g^\alpha = \mathbb{C}X_\alpha \) \( \forall \alpha \in \Gamma \). Let \( B \in (S^2u^*)^u \). First, \( B(H,X) = 0 \forall H \in \mathfrak{t}, X \in u_+ \). In fact, for any \( \alpha \in \Gamma \), since there exists \( H_\alpha \in \mathfrak{t} \) such that \( \alpha(H_\alpha) \not= 0 \), \( B(H,X_\alpha) = \frac{1}{\alpha(H_\alpha)} B([H,X_\alpha],X_\alpha) = \frac{1}{\alpha(H_\alpha)} B(0,X_\alpha) = 0 \). Second, that entails that the restriction of \( B \) to \( u_+ \times u_+ \) is zero, since for any \( \alpha, \beta \in \Gamma \),
\[
B(X_\alpha,X_\beta) = \frac{1}{\alpha(H_\alpha)} B([H_\alpha,X_\alpha],X_\beta) = \frac{1}{\alpha(H_\alpha)} B(H_\alpha,[X_\alpha,X_\beta]) = 0
\]
as \( [X_\alpha,X_\beta] \in u_+ \). Then \( u \) is I-null.

(ii) In that case, \( X_\alpha \not\in \mathbb{C}u \) \( \forall \alpha \in \Gamma \cap S \), and \( \dim (u/C^2u) = \dim \mathfrak{t} + \#(\Gamma \cap S) \). For \( u \) to be I-null, one has to prove that, for any \( B \in (S^2u^*)^u \):

(2.1) \( B(H,X_\beta) = 0 \forall H \in \mathfrak{t}, \beta \in \Gamma \setminus S \);
(2.2) \( B(X_\alpha,X_\beta) = 0 \forall \alpha \in \Gamma \cap S, \beta \in \Gamma \setminus S \);
(2.3) \( B(X_\beta,X_\gamma) = 0 \forall \beta, \gamma \in \Gamma \setminus S \).

(2.1) is proved as in case (i). To prove (2.2), let \( H_\beta \in \mathfrak{t} \) such that \( \beta(H_\beta) \not= 0 \). Then
\[
B(X_\alpha,X_\beta) = \frac{1}{\beta(H_\beta)} B(X_\alpha,[H_\beta,X_\beta]) = \frac{1}{\beta(H_\beta)} B([X_\alpha,H_\beta],X_\beta) = -\frac{1}{\beta(H_\beta)} B(\alpha(H_\beta)X_\alpha,X_\beta) = -\frac{1}{\beta(H_\beta)} B(0,X_\beta) = 0.
\]

As to (2.3),
\[
B(X_\beta,X_\gamma) = \frac{1}{\beta(H_\beta)} B([H_\beta,X_\beta],X_\gamma) = \frac{1}{\beta(H_\beta)} B(H_\beta,[X_\beta,X_\gamma]) = 0 \text{ from (2.1)}.
\]

\( \square \)

Example 2.5. Any Borel subalgebra is I-null.
Proposition 2.6. Let \( \mathfrak{g}_2 \) be a codimension 1 ideal of the Lie algebra \( \mathfrak{g} \), \((x_1, \cdots, x_N)\) a basis of \( \mathfrak{g} \) with \( x_1 \notin \mathfrak{g}_2, x_2, \cdots, x_N \in \mathfrak{g}_2 \), \( \pi_2 \) the corresponding projection onto \( \mathfrak{g}_2 \), and \((\omega^1, \cdots, \omega^N)\) denote the dual basis for \( \mathfrak{g}^* \). Let \( B \in (S^2 \mathfrak{g}_2)^* \), and denote \( B_2 \in (S^2 \mathfrak{g}_2^*)^* \) the restriction of \( B \) to \( \mathfrak{g}_2 \times \mathfrak{g}_2 \). Then:

(i) \( (2.4) \)

\[
I_B = d(\omega^1 \wedge f) + I_{B_2} \circ (\pi_2 \times \pi_2 \times \pi_2).
\]

where \( f = B(\cdot, x_1) \in \mathfrak{g}^* \);

(ii) Let \( \gamma \in \Lambda^2 \mathfrak{g}_2^* \subset \Lambda^2 \mathfrak{g}^* \), and denote \( d_{\mathfrak{g}_2} \) the coboundary operator of \( \mathfrak{g}_2 \). Then

\[
(2.5) \quad d\gamma = \omega^1 \wedge \theta_{x_1}(\gamma) + d_{\mathfrak{g}_2} \gamma \circ (\pi_2 \times \pi_2 \times \pi_2)
\]

where \( \theta_{x_1} \) stands for the coadjoint action of \( x_1 \) on the cohomology of \( \mathfrak{g} \);

(iii) Suppose \( I_{B_2} \in B^3(\mathfrak{g}_2, \mathbb{C}) \), and let \( \gamma \in \Lambda^2 \mathfrak{g}_2^* \subset \Lambda^2 \mathfrak{g}^* \) such that \( I_{B_2} = d_{\mathfrak{g}_2} \gamma \). Then \( I_B \in B^3(\mathfrak{g}, \mathbb{C}) \) if and only if \( \omega^1 \wedge \theta_{x_1}(\gamma) \in B^3(\mathfrak{g}, \mathbb{C}) \). In particular, the condition

\[
(2.6) \quad \theta_{x_1}(\gamma) = df
\]

implies \( I_B = d\gamma \).

Proof. (i) For \( X, Y, Z \in \mathfrak{g} \) one has

\[
(2.7) \quad B([X,Y],Z) = B(\omega^1(X)x_1 + \pi_2(X), \omega^1(Y)x_1 + \pi_2(Y), \omega^1(Z)x_1 + \pi_2(Z))
\]

\[
= B(\omega^1(X)[x_1, \pi_2(Y)] - \omega^1(Y)[x_1, \pi_2(X)] + [\pi_2(X), \pi_2(Y)], \omega^1(Z)x_1 + \pi_2(Z))
\]

\[
= \omega^1(X)\omega^1(Z)B([x_1, \pi_2(Y)], x_1) - \omega^1(Y)\omega^1(Z)B([x_1, \pi_2(X)], x_1)
\]

\[
\quad + \beta(X,Y,Z) + B([\pi_2(X), \pi_2(Y)], \pi_2(Z))
\]

\[
= \beta(X,Y,Z) + B([\pi_2(X), \pi_2(Y)], \pi_2(Z))
\]

where

\[
\beta(X,Y,Z) = \omega^1(Z)B([\pi_2(X), \pi_2(Y)], x_1) + \omega^1(X)B([x_1, \pi_2(Y)], \pi_2(Z))
\]

\[
- \omega^1(Y)B([x_1, \pi_2(X)], \pi_2(Z))
\]

\[
= \omega^1(Z)B([\pi_2(X), \pi_2(Y)], x_1) + \omega^1(X)B(x_1, [\pi_2(Y), \pi_2(Z)])
\]

\[
= \omega^1(Y)B(x_1, [\pi_2(X), \pi_2(Z)]).
\]

Now

\[
df(X,Y) = -B([X,Y], x_1)
\]

\[
= -B(\omega^1(X)x_1 + \pi_2(X), \omega^1(Y)x_1 + \pi_2(Y), x_1)
\]

\[
= -B(\omega^1(X)[x_1, \pi_2(Y)] - \omega^1(Y)[x_1, \pi_2(X)] + [\pi_2(X), \pi_2(Y)], x_1)
\]

\[
= -B([\pi_2(X), \pi_2(Y)], x_1),
\]

hence

\[
\beta(X,Y,Z) = -(\omega^1(Z)df(X,Y) + \omega^1(X)df(Y,Z) - \omega^1(Y)df(X,Z))
\]

\[
= -(\omega^1 \wedge df)(X,Y,Z).
\]

Since \( d\omega^1 = 0 \), \( (2.7) \) then reads

\[
(2.8) \quad I_B = d(\omega^1 \wedge f) + I_{B_2} \circ (\pi_2 \times \pi_2 \times \pi_2).
\]
(ii) One has for any $X, Y, Z \in \mathfrak{g}$

$$d\gamma(X, Y, Z) = d\gamma(\pi_2(X), \pi_2(Y), \pi_2(Z)) + \omega^1(X)d\gamma(x_1, \pi_2(Y), \pi_2(Z))$$

$$+ \omega^1(Y)d\gamma(\pi_2(X), x_1, \pi_2(Z)) + \omega^1(Z)d\gamma(\pi_2(X), \pi_2(Y), x_1).$$

Now, since $\gamma$ vanishes if one of its arguments is $x_1$,

$$d\gamma(x_1, \pi_2(Y), \pi_2(Z)) = -\gamma([x_1, \pi_2(Y)], \pi_2(Z)) + \gamma([x_1, \pi_2(Z)], \pi_2(Y))$$

$$d\gamma(\pi_2(X), x_1, \pi_2(Z)) = -\gamma([\pi_2(X), x_1], \pi_2(Z)) - \gamma([x_1, \pi_2(Z)], \pi_2(X))$$

$$d\gamma(\pi_2(X), \pi_2(Y), x_1) = \gamma([\pi_2(X), x_1], \pi_2(Y)) - \gamma([\pi_2(Y), x_1], \pi_2(X)),$$

hence

$$d\gamma(X, Y, Z) = d\gamma(\pi_2(X), \pi_2(Y), \pi_2(Z)) + \omega^1(X)\theta_{x_1}\gamma(\pi_2(Y), \pi_2(Z))$$

$$- \omega^1(Y)\theta_{x_1}\gamma(\pi_2(X), \pi_2(Z)) + \omega^1(Z)\theta_{x_1}\gamma(\pi_2(X), \pi_2(Y))$$

$$= d\gamma(\pi_2(X), \pi_2(Y), \pi_2(Z)) + (\omega^1 \otimes \theta_{x_1}) \gamma(X, Y, Z)$$

since $\theta_{x_1}\gamma(\pi_2(U), \pi_2(V)) = \theta_{x_2}\gamma(U, V)$ for all $U, V \in \mathfrak{g}$.

(iii) Results immediately from (i) and (ii). \hfill \square

**Corollary 2.7.** Under the hypotheses of Proposition 2.6, suppose that $x_1$ commutes with every $x_i$ $(2 \leq i \leq N)$ except for $x_{i_1}, \ldots, x_{i_r}$ and that $x_{i_1}, \ldots, x_{i_r}$ commute to one another. Then, if $\mathfrak{g}_2$ is $\mathcal{I}$-null, $\mathfrak{g}$ is $\mathcal{I}$-null.

**Proof.** From Equation (2.4), one has to prove that for any invariant bilinear symmetric form $B$ on $\mathfrak{g}$, $f = B(\cdot, x_1) \in \mathfrak{g}^*$ verifies $df = 0$, i.e., for any $2 \leq i, j \leq N$, $B(x_i, [x_1, x_j]) = 0$. For $i \neq i_1, \ldots, i_r$, and any $j \geq 2$, $B(x_i, [x_1, x_j]) = B(x_1, x_i)] = 0$. For $i, j \in \{i_1, \ldots, i_r\}$, $B(x_i, [x_i, x_j]) = B(x_1, 0) = 0$. \hfill \square

**Definition 2.8.** The $n$-dimensional standard filiform Lie algebra is the Lie algebra with basis $\{x_1, \ldots, x_n\}$ and commutation relations $[x_1, x_i] = x_{i+1}$ ($1 \leq i < n$).

**Corollary 2.9.** Any standard filiform Lie algebra or any Heisenberg Lie algebra is $\mathcal{I}$-null.

**Corollary 2.10.** Any Lie algebra containing some $\mathcal{I}$-null codimension 1 ideal is $\mathcal{I}$-exact.

**Corollary 2.11.** Suppose that the Lie algebra $\mathfrak{g}$ is such that $\dim \text{Im}\mathcal{I} = 0$ or 1. Let $\tau \in \text{Der} \mathfrak{g}$ such that $\tau x_k \in C^2 \mathfrak{g}$ $\forall k \geq 2$ where $(x_1, \ldots, x_N)$ is some basis of $\mathfrak{g}$. Denote $\tilde{\mathfrak{g}} = \mathcal{C} \tau \oplus \mathfrak{g}$ the Lie algebra obtained by adjoining the derivation $\tau$ to $\mathfrak{g}$, and by $\tilde{\omega}$ the Koszul map of $\tilde{\mathfrak{g}}$. Then $\dim \text{Im}\tilde{\mathcal{I}} = 0$ if $\dim \text{Im}\mathcal{I} = 0$, and $\dim \text{Im}\tilde{\mathcal{I}} = 0$ or 1 if $\dim \text{Im}\mathcal{I} = 1$.

**Proof.** Let $B \in (S^2\tilde{\mathfrak{g}}^*)^{\tilde{\mathfrak{g}}}$. One has

$$I_B = \omega^1 \otimes df + I_{B_2} \circ (\pi_2 \times \pi_2 \times \pi_2)$$

where $(\tau, x_1, \ldots, x_N)$ is the basis of $\tilde{\mathfrak{g}}$, $(\omega^1, \omega^1, \ldots, \omega^1)$ the dual basis, $B_2$ the restriction of $B$ to $\mathfrak{g}$, $f_\tau = B(\cdot, \cdot)$ and $\pi_2$ the projection on $\mathfrak{g}$. We will also use the projection $\pi_3$ on $\text{vect}(x_2, \ldots, x_N)$. For $X, Y \in \tilde{\mathfrak{g}}$, $X = \omega^1(\tilde{X})\tau + X, Y = \omega^1(\tilde{Y})\tau + Y$, $X = \pi_2(\tilde{X}), Y = \pi_2(\tilde{Y})$, so that $df_\tau(X, Y) = -B(\tau, [\tilde{X}, \tilde{Y}]) = -B(\tau, [X, Y]) = \omega^1(X)df_\tau(Y) + \omega^1(Y)df_\tau(X) - df_\tau(\tilde{X}, \tilde{Y})$.\hfill \square
$-B(\tau, [\omega^1(X)x_1 + \pi_3(X), \omega^1(Y)x_1 + \pi_3(Y)]) = -\omega^1(X)B(\tau, [x_1, \pi_3(Y)])$

$$+\omega^1(Y)B(\tau, [x_1, \pi_3(X)]) - B(\tau, [\pi_3(X), \pi_3(Y)]),$$

hence

(2.10)

$$df_x(X, Y) = \omega^1(X)B_2(\tau\pi_3(Y), x_1) - \omega^1(Y)B_2(\tau\pi_3(X), x_1) - B_2(\tau\pi_3(X), \pi_3(Y)).$$

Note that $\tau\pi_3(X), \tau\pi_3(Y) \in \mathcal{C}^2 g$ by the hypotheses. Suppose first that $g$ is $I$-null. Then $B_2(\tau\pi_3(Y), x_1), B_2(\tau\pi_3(X), x_1), B_2(\tau\pi_3(X), \pi_3(Y))$ all vanish. From Equations (2.12), (2.11), $g$ is $I$-null. Suppose now that $g$ verifies $\dim \mathcal{I} = 1$ and let $C \in (S^2g)^0$ with $I_C \neq 0$. If $g_\tau$ is not $I$-null we may suppose that $I_B \neq 0$. There exists $\lambda \in \mathbb{C}$ such that $I_{B_2} = \lambda I_C$. Then $B_2(\tau\pi_3(Y), x_1) = \lambda C(\tau\pi_3(Y), x_1)$.

It follows from Equations (2.12), (2.11), that $\dim \mathcal{I} = 1$.

**Definition 2.12.** A Lie algebra $g$ is said to be quadratic if there exists a nondegenerate invariant bilinear form on $g$.

Clearly, quadratic nonabelian Lie algebras are not $I$-null.

**Example 2.13.** This example is an illustration to Corollary 2.11. The nilpotent Lie algebra $g_{7,2.4}$ has commutation relations $[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5, [x_1, x_5] = x_6, [x_2, x_3] = -x_7, [x_3, x_4] = x_7$. $g_{7,2.4}$ is quadratic and $\dim \mathcal{I} = 1$.

The elements of $\text{Der} g_{7,2.4}$ are

$$\tau = \begin{pmatrix}
\xi_1^1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\xi_1^2 & \xi_2^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \xi_1^3 + \xi_2^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2\xi_1^4 + \xi_2^4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \xi_1^5 + \xi_2^5 & 0 & 0 \\
\xi_1^6 & \xi_2^6 & \xi_1^7 & 0 & 0 & 0 & 0 \\
\xi_1^7 & 0 & 0 & 0 & 0 & \xi_1^5 + 2\xi_2^5 & 0 \\
\xi_1^8 & 0 & 0 & 0 & 0 & 0 & \xi_1^5 + 2\xi_2^5 \\
\end{pmatrix}$$

$\tau$ is nilpotent if $\xi_1^1 = \xi_2^2 = 0$. Denote the nilpotent $\tau$ by $(\xi_1^2; \xi_1^3, \xi_1^4, \xi_1^5; \xi_1^6, \xi_1^7)$. Now, projectively equivalent derivations $\tau, \tau'$ (see [12]) give isomorphic $g_\tau, g_{\tau'}$. By reduction using projective equivalence, we are reduced to the following cases: Case 1. $\xi_1^1 \neq 0 : (1; \varepsilon; 0, \eta, 0)$; Case 2. $\xi_2^2 = 0 : (0; \varepsilon; 0, \eta; \lambda)$; where $\varepsilon, \eta, \lambda = 0, 1$. In both cases $g_\tau$ is $I$-null, except when $\tau = 0$ in case 2 where $g_\tau$ is the direct product $\mathbb{C} \times g_{7,2.4}$ which is quadratic. Hence any indecomposable 8-dimensional nilpotent Lie algebra containing a subalgebra isomorphic to $g_{7,2.4}$ is $I$-null, though $g_{7,2.4}$ is quadratic. That is in line with the fact that, from the double extension method of [16], [15], any indecomposable quadratic solvable Lie algebra is a double extension of a quadratic solvable Lie algebra by $\mathbb{C}$.

**Example 2.14.** Among the 170 (non isomorphic) nilpotent complex Lie algebras of dimension $\leq 7$, only a few are not $I$-null. Those are listed in Table 1 in the classification of [11], [13] (they are all $I$-exact). Table 1 gives for each of them $\dim (S^2 g)^0$, a basis for $(S^2 g)^0 / \ker \mathcal{I}$ (which in those cases is one-dimensional), and the corresponding $I_{B_S}$. The results in Table 1 are new and have been obtained, first by explicit computation of all invariant bilinear forms on each one of the 170 Lie algebras with the computer algebra system Reduce and a program similar to those in [12], [13], and second by hand calculation of $I_B$ for non-$I$-null Lie algebras. $\otimes$ denotes quadratic Lie algebras; for $\omega, \pi \in g^*$, $\otimes$ stands for the symmetric product $\omega \otimes \pi = \omega \otimes \pi + \pi \otimes \omega$; $\omega^{i,j,k}$ stands for $\omega^i \wedge \omega^j \wedge \omega^k$. 

\[ \omega \otimes \pi = \omega \otimes \pi + \pi \otimes \omega. \]
Example 2.16. The quadratic 5-dimensional nilpotent Lie algebra $\mathfrak{g}$ consists in an inductive use of Corollary 2.7. In the exceptional case $s$, we either see $[5]$. The 4-dimensional solvable "diamond" Lie algebra $\mathfrak{g}$ cannot be obtained as in Lemma 2.4. Here $\dim \mathfrak{g} = 4$, and we verify that all other solvable 4-dimensional Lie algebras are $\mathcal{I}$-exact. In fact, one verifies that all other solvable 4-dimensional Lie algebras are $\mathcal{I}$-exact. For example, in the case of the 10 dimensional Lie algebra $\mathfrak{g}$ with basis $\{x_1, x_2, x_3, x_4\}$ and commutation relations $[x_1, x_2] = x_5, [x_1, x_3] = x_4, [x_2, x_3] = x_5$, consider the 10-dimensional direct product $\mathfrak{g} \times \mathfrak{g}$, with the commutation relations: $[x_1, x_2] = x_5, [x_1, x_5] = x_6, [x_2, x_5] = x_7, [x_3, x_4] = x_8, [x_3, x_5] = x_9, [x_4, x_5] = x_{10}$. The only 11-dimensional nilpotent Lie algebra with an invariant bilinear form which reduces to $B_1 = \omega^1 \circ \omega^2 - \omega^2 \circ \omega^3 + \omega^3 \circ \omega^1$, $B_2 = \omega^3 \circ \omega^4 - \omega^4 \circ \omega^5 + \omega^5 \circ \omega^3$, the first and second factor is the direct product $\mathbb{C} \times \mathfrak{g} \times \mathfrak{g}$.
utilize directly the commutation relations \((G_2, F_1)\), or make use of a certain property of the pattern of positive roots, which we call property \((P)\) \((E_6, E_7, E_8)\).

**Theorem 3.1.** Any nilradical \(\mathfrak{g}\) of a Borel subalgebra of a finite-dimensional semisimple Lie algebra is \(I\)-null.

**Proof.** It is enough to consider the case of a simple Lie algebra, hence one of the 4 classical types plus the 5 exceptional ones.

**Case \(A_n\).** Denote \(E_{i,j}, 1 \leq i, j \leq n + 1\) the canonical basis of \(\mathfrak{gl}(n + 1, \mathbb{C})\). One may suppose that the Borel subalgebra of \(A_n = \mathfrak{sl}(n + 1)\) is comprised of the upper triangular matrices with zero trace, and the Cartan subalgebra \(\mathfrak{h}\) is \(\bigoplus_{i=1}^{n} \mathbb{C}H_i\) with \(H_i = E_{i,i} - E_{i+1,i+1}\). The nilradical is \(\mathfrak{g} = A_n^+ = \bigoplus_{1 \leq i < j \leq n+1} \mathbb{C}E_{i,j}\). For \(n = 1\), \(\mathfrak{g} = \mathbb{C}\) is \(I\)-null. Suppose the result holds for the nilradical of the Borel subalgebra of \(A_{n-1} = \mathfrak{sl}(n)\). One has \(\mathfrak{g} = CE_{1,2} \oplus \cdots \oplus CE_{1,n+1} \oplus g'_2 = \bigoplus_{2 \leq i < j \leq n+1} CE_{i,j}\) being the nilradical of the Borel subalgebra of \(A_{n-1}\), hence \(I\)-null. \(E_{1,n+1}\) commutes with \(g'_2\), hence \(g'_2\) is a codimension 1 ideal of \(CE_{1,n+1} \oplus g'_2\), and, from Corollary \(2.7\), \(CE_{1,n+1} \oplus g'_2\) is \(I\)-null. Now \(E_{1,n}\) commutes with all members of the basis of \(CE_{1,n+1} \oplus g'_2\), except for \(E_{n,n+1}\), and \(\{E_{1,n}, E_{n,n+1}\} = E_{1,n+1}\). Then \(CE_{1,n+1} \oplus g'_2\) is a codimension 1 ideal of \(CE_{1,n} \oplus (CE_{1,n+1} \oplus g'_2)\), and from Corollary \(2.7\) \(CE_{1,n} \oplus (CE_{1,n+1} \oplus g'_2)\) is \(I\)-null. Consider \(CE_{1,n-1} \oplus (CE_{1,n} \oplus CE_{1,n+1} \oplus g'_2)\). \(CE_{1,n-1}\) commutes with all members of the basis of \(CE_{1,n} \oplus CE_{1,n+1} \oplus g'_2\), except for \(E_{n-1,n}, E_{n-1,n+1}\), and yields respectively \(E_{1,n}, E_{n+1,n+1}\). Then \(CE_{1,n} \oplus CE_{1,n+1} \oplus g'_2\) is a codimension 1 ideal of \(CE_{1,n-1} \oplus CE_{1,n} \oplus CE_{1,n+1} \oplus g'_2\), and since \(E_{n-1,n}, E_{n-1,n+1}\) commute, we get from Corollary \(2.7\) that \(CE_{1,n-1} \oplus CE_{1,n} \oplus CE_{1,n+1} \oplus g'_2\) is \(I\)-null. The result then follows by induction.

**Case \(D_n\).** We may take \(D_n\) as the Lie algebra of matrices

\[
(3.1) \quad \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -^tZ_1 \end{pmatrix}
\]

with \(Z_i \in \mathfrak{gl}(n, \mathbb{C}), Z_2, Z_3\) skew symmetric (see [7], p. 193). Denote \(\hat{E}_{i,j} = \begin{pmatrix} E_{i,j} & 0 \\ 0 & -E_{i,j} \end{pmatrix}, \hat{F}_{i,j} = \begin{pmatrix} 0 & -E_{i,j} \\ E_{i,j} & 0 \end{pmatrix}\) \((E_{i,j}, 1 \leq i, j \leq n\) the canonical basis of \(\mathfrak{gl}(n, \mathbb{C})\)). The Cartan subalgebra \(\mathfrak{h}\) is \(\bigoplus_{i=1}^{n} \mathbb{C}H_i\) with \(H_i = \hat{E}_{i,i}\) and the nilradical of the Borel subalgebra is

\[
(3.2) \quad D_n^+ = \bigoplus_{1 \leq i < j \leq n} \mathbb{C}\hat{E}_{i,j} \oplus \bigoplus_{1 \leq i < j \leq n} \mathbb{C}\hat{F}_{i,j}.
\]

All \(\hat{F}_{i,j}\)'s commute to one another, and one has:

\[
(3.3) \quad [\hat{E}_{i,j}, \hat{F}_{k,l}] = \delta_{j,k}\hat{F}_{i,l} - \delta_{j,l}\hat{F}_{i,k}.
\]

We identify \(D_{n-1}\) to a subalgebra of \(D_n\) by simply taking the first row and first column of each block to be zero in \((3.1)\). For \(n = 2\), \(D_2^+ = \mathbb{C}^2\) is \(I\)-null. Suppose the result holds true for \(D_{n-1}^+\). One has

\[
D_n^+ = CE_{1,2} \oplus CE_{1,3} \oplus \cdots \oplus CE_{1,n} \oplus CE_{1,n} \oplus \cdots \oplus CE_{1,n} \oplus D_{n-1}^+.
\]

Start with \(CE_{1,2} \oplus D_{n-1}^+\). From \((3.3)\), \(\hat{F}_{1,2}\) commutes with all \(\hat{E}_{i,j}\) \((2 \leq i < j \leq n)\) hence with \(D_{n-1}^+\). Then \(D_{n-1}^+\) is a codimension 1 ideal of \(CE_{1,2} \oplus D_{n-1}^+\) and \(CE_{1,2} \oplus D_{n-1}^+\) is \(I\)-null from Corollary \(2.7\). Consider now \(CE_{1,3} \oplus (CE_{1,2} \oplus D_{n-1}^+)\). Again from \((3.3)\), \(\hat{F}_{1,3}\) commutes with all elements of the basis of \(D_{n-1}^+\) except \(\hat{E}_{2,3}\) and
Then $\tilde{E}_{2,3}, \tilde{F}_{1,3}$ is $\tilde{F}_{1,2} \oplus D_{n-1}^+$ is a codimension 1 ideal of $\mathbb{C}\tilde{F}_{1,3} \oplus (\mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+)$, and the latter is $T$-null. Suppose that $\mathbb{C}\tilde{F}_{1,s-1} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+$ is a codimension 1 ideal of $\mathbb{C}\tilde{F}_{1,s} \oplus (\mathbb{C}\tilde{F}_{1,s-1} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+)$, and that the latter is $T$-null. Consider $\mathbb{C}\tilde{F}_{1,s+1} \oplus (\mathbb{C}\tilde{F}_{1,s} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+)$, and that the latter is $T$-null. Consider now $\mathbb{C}\tilde{E}_{1,n} \oplus (\mathbb{C}\tilde{F}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+)$. One has for $2 \leq i < j \leq n$, $[\tilde{E}_{1,n}, \tilde{F}_{1,i}] = 0$. Hence $\mathbb{C}\tilde{F}_{1,n} \oplus (\mathbb{C}\tilde{F}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+)$ is an ideal of $\mathbb{C}\tilde{E}_{1,n} \oplus (\mathbb{C}\tilde{F}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+)$ and the latter is $T$-null. For $2 \leq i < j \leq n$, $1 \leq k \leq n-2,$

$$
\begin{align*}
[\tilde{E}_{1,n-k}, \tilde{E}_{1,j}] &= \delta_{n-k,i} \tilde{E}_{1,j}, \\
[\tilde{E}_{1,n-k}, \tilde{F}_{1,j}] &= \delta_{n-k,i} \tilde{F}_{1,j} - \delta_{n-k,j} \tilde{F}_{1,i}, \\
[\tilde{E}_{1,n-k}, \tilde{E}_{1,n}] &= 0, \\
[\tilde{E}_{1,n-k}, \tilde{F}_{1,j}] &= \delta_{n-k,j} \tilde{F}_{1,j}.
\end{align*}
$$

$[\tilde{E}_{1,n-1}, \tilde{E}_{1,j}]$ is nonzero only for $(i = n-1, j = n)$ and then yields $\tilde{E}_{1,n}; [\tilde{E}_{1,n-1}, \tilde{F}_{1,j}]$ is nonzero only for $(i = n-1, j = n)$ or $(i < j = n - 1)$ and yields respectively $\tilde{F}_{1,n}$ or $-\tilde{F}_{1,i}$, $[\tilde{E}_{1,n-1}, \tilde{E}_{1,n}]$ and $[\tilde{E}_{1,n-1}, \tilde{F}_{1,j}]$ are zero for $n \geq 3$. Hence, first $\mathbb{C}\tilde{F}_{1,n} \oplus (\mathbb{C}\tilde{F}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+)$ is a codimension 1 ideal of $\mathbb{C}\tilde{E}_{1,n-1} \oplus (\mathbb{C}\tilde{F}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+)$, and second the latter is $T$-null, since $\tilde{E}_{n-1,n}$ commutes with $\tilde{F}_{n-1,n}$. Suppose that $\mathbb{C}\tilde{E}_{1,n-k+1} \oplus (\mathbb{C}\tilde{E}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+)$ is a codimension 1 ideal of $\mathbb{C}\tilde{E}_{1,n-k} \oplus (\mathbb{C}\tilde{E}_{1,n-k+1} \oplus \cdots \oplus \mathbb{C}\tilde{E}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+)$ and that the latter is $T$-null. Consider $\mathbb{C}\tilde{E}_{1,n-k-1} \oplus (\mathbb{C}\tilde{E}_{1,n-k} \oplus \cdots \oplus \mathbb{C}\tilde{E}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+)$.

**Case $B_n$.** We may take $B_n (n \geq 2)$ as the Lie algebra of matrices

\begin{equation}
\begin{pmatrix}
0 & u & v \\
-\tau v & Z_1 & Z_2 \\
-\tau u & Z_3 & -\tau Z_1
\end{pmatrix}
\end{equation}

with $u, v$ complex $(1 \times n)$-matrices, $Z_i \in \mathfrak{gl}(n, \mathbb{C})$, $Z_2, Z_3$ skew symmetric, i.e.

\begin{equation}
\begin{pmatrix}
0 & u & v \\
-\tau v & A & \\
-\tau u & & 
\end{pmatrix}
\end{equation}
with $A \in D_n$. We identify $A \in D_n$ to the matrix
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & A & 0 \\
0 & 0 & 0
\end{pmatrix} \in B_n.
\]

The Cartan subalgebra of $B_n$ is then simply that of $D_n$. $B_n^+$ consists of the matrices
\[
\begin{pmatrix}
0 & 0 & 0 \\
-tv & A & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
with $v$ complex $(1 \times n)$-matrix and $A \in D_n^+$. For $1 \leq q \leq n$, let $v_q$ the $(1 \times n)$-matrix $(0, \cdots, 1, \cdots, 0)$ (1 in $q^{th}$ position), and
\[
\tilde{v}_q = \begin{pmatrix}
0 & 0 \\
-tv_q & 0 \\
0 & 0
\end{pmatrix}
\]
Hence $B_n^+ = (\bigoplus_{q=1}^n C\tilde{v}_q) \oplus D_n^+$. One has for $1 \leq q \leq n$, $1 \leq i < j \leq n$
\[
[\tilde{v}_q, \tilde{E}_{i,j}] = -\delta_{q,j} \tilde{v}_i \\
[\tilde{v}_q, \tilde{F}_{i,j}] = 0
\]
and for $1 \leq s < q \leq n$
\[
[\tilde{v}_q, \tilde{v}_s] = \tilde{F}_{s,q}.
\]

Consider $C\tilde{v}_1 \oplus D_n^+$. As $\tilde{v}_1$ commutes with $\tilde{E}_{i,j}$ and $\tilde{F}_{i,j}$, $D_n^+$ is an ideal of $C\tilde{v}_1 \oplus D_n^+$ and the latter is $I$-null. Suppose that $C\tilde{v}_{s-1} \oplus \cdots \oplus C\tilde{v}_1 \oplus D_n^+$ is an ideal of $C\tilde{v}_s \oplus (C\tilde{v}_{s-1} \oplus \cdots \oplus C\tilde{v}_1 \oplus D_n^+)$ and the latter is $I$-null. Consider $C\tilde{v}_{s+1} \oplus (C\tilde{v}_{s} \oplus C\tilde{v}_{s-1} \oplus \cdots \oplus C\tilde{v}_1 \oplus D_n^+)$. $[\tilde{v}_{s+1}, \tilde{E}_{i,j}] = -\delta_{s+1,j} \tilde{v}_i$ hence $\tilde{v}_{s+1}$ commutes to all $\tilde{E}_{i,j}$’s except for $\tilde{E}_{i,s+1}$ ($i \leq s$) and then yields $-\tilde{v}_i$. For $t \leq s$, $[\tilde{v}_{s+1}, \tilde{v}_t] = \tilde{F}_{t,s+1}$. Hence $C\tilde{v}_s \oplus C\tilde{v}_{s-1} \oplus \cdots C\tilde{v}_1 \oplus D_n^+$ is an ideal of $C\tilde{v}_{s+1} \oplus (C\tilde{v}_s \oplus C\tilde{v}_{s-1} \oplus \cdots C\tilde{v}_1 \oplus D_n^+)$. Now we cannot apply directly Corollary 2.7 to conclude that the latter is $I$-null as the family $\mathcal{F} = \{\tilde{E}_{i,s+1}, \tilde{v}_t; 1 \leq i \leq s, 1 \leq t \leq s\}$ is not commutative. The $\tilde{E}_{i,s+1}$’s ($i \leq s$) commute to one another and to the $\tilde{v}_t$’s, but the $\tilde{v}_t$’s do not commute to one another. However, recall from the proof of Corollary 2.7 that one has to check that, for any invariant bilinear form $B$ on $C\tilde{v}_{s+1} \oplus (C\tilde{v}_s \oplus C\tilde{v}_{s-1} \oplus \cdots C\tilde{v}_1 \oplus D_n^+)$, $B(\tilde{v}_{s+1}, [X,Y]) = 0$ for all $X, Y \in \mathcal{F}$. That reduces to $B(\tilde{v}_{s+1}, [\tilde{v}_s, \tilde{v}_t]) = 0 \forall t, t', 1 \leq t < t' \leq s$. Now, $B(\tilde{v}_{s+1}, [\tilde{v}_s, \tilde{v}_t]) = B([\tilde{v}_{s+1}, \tilde{v}_s], \tilde{v}_t) = B(\tilde{F}_{s,s+1}, \tilde{v}_t) = B(\tilde{E}_{s,s+1}, \tilde{v}_t) = 0$ since $\tilde{E}_{s,s+1}, \tilde{F}_{s,s+1}, \tilde{v}_t \in C\tilde{v}_s \oplus C\tilde{v}_{s-1} \oplus \cdots C\tilde{v}_1 \oplus D_n^+$ which is $I$-null. We conclude that $C\tilde{v}_{s+1} \oplus (C\tilde{v}_s \oplus C\tilde{v}_{s-1} \oplus \cdots C\tilde{v}_1 \oplus D_n^+)$ is $I$-null. By induction the property holds for $s = n$ and $B_n^+$ is $I$-null. 

Case $C_n$. This case is pretty similar to the case $D_n$. We may take $C_n$ as the Lie algebra of matrices
\[
\begin{pmatrix}
Z_1 & Z_2 \\
Z_3 & -tZ_1
\end{pmatrix}
\]
with $Z_i \in \mathfrak{g}(n, C)$, $Z_2, Z_3$ symmetric. \( \hat{E}_{i,j} \) and the Cartan subalgebra are identical to those of $D_n$. We denote for $1 \leq i, j \leq n : \hat{F}_{i,j} = (0 \ E_{i,j} + E_{j,i})$. Then

\[
(3.9) \quad C^+_n = \bigoplus_{1 \leq i < j \leq n} \mathbb{C}\hat{E}_{i,j} \oplus \bigoplus_{1 \leq i \leq j \leq n} \mathbb{C}\hat{F}_{i,j}.
\]

All \( \hat{F}_{i,j} \)'s commute to one another, and one has:

\[
(3.10) \quad [\hat{E}_{i,j}, \hat{F}_{k,l}] = \delta_{j,k}\hat{F}_{i,l} + \delta_{j,k}\hat{F}_{i,k}.
\]

The case is step by step analogous to the case of $D_n$ with (3.10) instead of (3.9) and (3.9) instead of (3.3).

Case $G_2$. The commutation relations for $G_2$ appear in [5], p. 346. $G_2$ is 6-dimensional with commutation relations $[x_1, x_2] = x_3; [x_1, x_3] = 2x_4; [x_1, x_4] = -3x_5; [x_2, x_5] = -x_6; [x_3, x_4] = -3x_6$. $G_2$ has the same adjoint cohomology $(1, 4, 7, 8, 7, 5, 2)$ as, and is isomorphic to, \( \mathfrak{g}_{6,18} \), which is I-null.

Case $F_4$. $F_4^+$ has 24 positive roots, and root vectors $x_i (1 \leq i \leq 24)$. From the root pattern, one gets with some calculations the commutation relations of $F_4^+$ :

\[
[\hat{E}_{x_1, x_2}] = x_5; [\hat{E}_{x_1, x_3}] = x_4; [\hat{E}_{x_1, x_4}] = -x_6; [\hat{E}_{x_1, x_5}] = -x_7; [\hat{E}_{x_1, x_6}] = -x_9; [\hat{E}_{x_1, x_7}] = -x_{23}; [\hat{E}_{x_1, x_8}] = x_9; [\hat{E}_{x_1, x_9}] = x_7; [\hat{E}_{x_1, x_{10}}] = x_9; [\hat{E}_{x_1, x_{11}}] = x_7; [\hat{E}_{x_1, x_{12}}] = -x_9; [\hat{E}_{x_1, x_{13}}] = x_7; [\hat{E}_{x_1, x_{14}}] = -x_9; [\hat{E}_{x_1, x_{15}}] = x_7; [\hat{E}_{x_1, x_{16}}] = -x_9; [\hat{E}_{x_1, x_{17}}] = x_7; [\hat{E}_{x_1, x_{18}}] = -x_9; [\hat{E}_{x_1, x_{19}}] = x_7; [\hat{E}_{x_1, x_{20}}] = -x_9; [\hat{E}_{x_1, x_{21}}] = x_7; [\hat{E}_{x_1, x_{22}}] = -x_9; [\hat{E}_{x_1, x_{23}}] = x_7; [\hat{E}_{x_1, x_{24}}] = -x_9.
\]

Then the computation of all invariant bilinear forms on $F_4^+$ with the computer algebra system Reduce yields the conclusion that $F_4^+$ is I-null.

Case $E_6$. In the case of $E_6^+$ the set $\Delta_+$ of positive roots (associated to the set $S$ of simple roots) has cardinality 36 ([2], p. 333):

\[
\Delta_+ = \{ \epsilon_i + \epsilon_j : 1 \leq i < j \leq 5 \} \cup \{ \epsilon_i - \epsilon_j : 1 \leq j < i \leq 5 \} \cup \{ \frac{1}{2}(\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4 \pm \epsilon_5 \pm \sqrt{3}\epsilon_6) : \# \text{ minus signs even} \}
\]

\[
\Delta_+ \cup \{ \frac{1}{2}(\pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4 \pm \epsilon_5 \pm \sqrt{3}\epsilon_6) : \# \text{ minus signs even} \}
\]

\[
\text{the (}\epsilon_j)\text{'s an orthogonal basis of the Euclidean space). Instead of computing the commutation relations, we will use the following property (P) of $\Delta_+$.}
\]

\[
(P) : \text{ for } \alpha, \beta, \gamma \in \Delta_+, \text{ if } \alpha + \beta \in \Delta_+ \text{ and } \alpha + \gamma \in \Delta_+, \text{ then } \beta + \gamma \notin \Delta_+.
\]

Introduce some Chevalley basis ([4], p. 19 ex. 7) of $E_6^+ : (X_\alpha)_{\alpha \in \Delta_+}$. One has

\[
[X_\alpha, X_\beta] = N_{\alpha, \beta}X_{\alpha + \beta} \forall \alpha, \beta \in \Delta_+
\]

\[
N_{\alpha, \beta} = 0 \text{ if } \alpha + \beta \notin \Delta_+, N_{\alpha, \beta} \in \mathbb{Z} \setminus \{0\} \text{ if } \alpha + \beta \in \Delta_+.
\]

Define inductively a sequence $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_{36} = E_6^+$ of I-null subalgebras, each of which a codimension 1 ideal of the following, as follows. Start with $\mathfrak{g}_1 = \mathbb{C}X_\delta,$
\[ \delta_1 \in \Delta_+ \] of maximum height. Suppose \( \mathfrak{g}_i \) defined. Then take \( \mathfrak{g}_{i+1} = \mathbb{C}X_{\delta_{i+1}} \oplus \mathfrak{g}_i \) with \( \delta_{i+1} \in \Delta_+ \setminus \{ \delta_1, \ldots, \delta_i \} \) of maximum height. Clearly, \( \mathfrak{g}_i \) is a codimension 1 ideal of \( \mathfrak{g}_{i+1} \). To prove that it is \( T \)-null we only have to check that, for \( 1 \leq s, t \leq i \), if \( \delta_{i+1} + \delta_s \in \Delta_+ \) and \( \delta_{i+1} + \delta_t \in \Delta_+ \) then \( \delta_s + \delta_t \not\in \Delta_+ \). That holds true because of property \((P)\).

**Case \( E_6^+ \).** In the case of \( E_6^+ \) the set \( \Delta_+ \) of positive roots has cardinality 63 \([6], \ p. \ 333\):

\[
\Delta_+ = \{ \varepsilon_i + \varepsilon_j; 1 \leq i < j \leq 6 \} \cup \{ \varepsilon_i - \varepsilon_j; 1 \leq j < i \leq 6 \} \cup \{ \pm \sqrt{2} \varepsilon_7 \} \\
\cup \{ \pm (\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm 2 \varepsilon_4 \pm \varepsilon_5 \pm \varepsilon_6 \pm \sqrt{2} \varepsilon_7); \# \text{ minus signs odd} \}.
\]

Property \((P)\) holds true for \( E_6^+ \) \((see \ [14])\). Hence the conclusion follows as in the case of \( E_6^+ \).

**Case \( E_8^+ \).** In the case of \( E_8^+ \) the set \( \Delta_+ \) of positive roots has cardinality 120 \([6], \ p. \ 333\):

\[
\Delta_+ = \{ \varepsilon_i + \varepsilon_j; 1 \leq i < j \leq 8 \} \cup \{ \varepsilon_i - \varepsilon_j; 1 \leq j < i \leq 8 \} \\
\cup \{ \pm (\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 \pm \varepsilon_5 \pm \varepsilon_6 \pm \varepsilon_7 \); \# \text{ minus signs even} \}.
\]

Property \((P)\) holds true for \( E_8^+ \) \((see \ [14])\). Hence the conclusion follows as in the case of \( E_6^+ \).

**Remark 3.2.** Property \((P)\) holds for \( A_n^+ \), hence we could have used it. However, it does not hold for \( F_4^+ \). One has for example in the above commutation relations of \( F_4^+ \) (with root vectors) \( [x_3, x_4] \neq 0, [x_3, x_9] \neq 0, \) yet \( [x_4, x_9] \neq 0 \).

**Remark 3.3.** In the transversal to dimension approach to the classification problem of nilpotent Lie algebras initiated in \([18]\), one first associates a generalized Cartan matrix (abbr. GCM) \( A \) to any nilpotent finite dimensional complex Lie algebra \( \mathfrak{g} \), and then looks at \( \mathfrak{g} \) as the quotient \( \mathfrak{g}(A)_+/\mathfrak{I} \) of the nilradical of the Borel subalgebra of the Kac-Moody Lie algebra \( \hat{\mathfrak{g}}(A) \) associated to \( A \) by some ideal \( \mathfrak{I} \). Then one gets for any GCM \( A \) the subproblem of classifying (up to the action of a certain group) all ideals of \( \mathfrak{g}(A)_+ \), thus getting all nilpotent Lie algebras of type \( A \) \((see \ [2, \ 3, \ 4, \ 19], \ and \ the \ references \ therein)\). Any indecomposable GCM is of exactly one of the 3 types finite, affine, indefinite \((among \ that \ last \ the \ hyperbolic \ GCMs, \ with \ the \ property \ that \ any \ connected \ proper \ subdiagram \ of \ the \ Dynkin \ diagram \ is \ of \ finite \ or \ affine \ type) \((1, \ 8, \ 20)\). From Theorem 3.1 the nilpotent Lie algebras that are not \( T \)-null all come from affine or indefinite types. Unfortunately, that is the case of many nilpotent Lie algebras, see Table 2. Finally, let us add some indications on how Table 2 was computed. The commutation relations for the nilpotent Lie algebras \( \mathfrak{g} \) in Table 2 are given in \([12, \ 13]\) in terms of a basis \( (x_j)_{1 \leq j \leq n} \) \((n = \dim \mathfrak{g})\) which diagonalizes a maximal torus \( T \). We may suppose here that \( (x_j)_{1 \leq j \leq \ell} \), \( \ell = \dim (\mathfrak{g}/C^2\mathfrak{g}) \), is a basis for \( \mathfrak{g} \) modulo \( C^2\mathfrak{g} \). The associated weight pattern \( \mathcal{R}(T) \) and weight spaces decomposition \( \mathfrak{g} = \oplus_{\beta \in \mathcal{R}(T)} \mathfrak{g}\beta \) appear in \([13]\). As in \([18]\), one first introduces \( R_1(T) = \{ \beta \in \mathcal{R}(T); \mathfrak{g}\beta \not\subset C^2\mathfrak{g} \} = \{ \beta_1, \ldots, \beta_s \} \), \( \ell_a = \dim (\mathfrak{g}\beta_a / (\mathfrak{g}\beta_a \cap C^2\mathfrak{g})) \), \( d_a = \dim \mathfrak{g}\beta_a \) \((1 \leq a \leq s)\). By definition the GCM associated to \( \mathfrak{g} \) is \( A = (a_{ij})_{1 \leq i, j \leq \ell} \) with \( a_{ii} = 2 \) and, for \( i \neq j, \ -a_{ij} \) defined as follows. In the simplest case where \( d_a = 1 \forall a \) \((1 \leq a \leq s)\), then, for \( i \neq j, \ -a_{ij} \)
is the lowest \( k \in \mathbb{N} \) such that \( ad(x_i)^{k+1}(x_j) = 0 \). If \( d_a > 1 \) for some \( 1 \leq a \leq s \) (Lie algebras having that property are signalled by a ↑ in Table 2), one has (if \( l_a > 1 \) as well) to reorder \( x_1, \ldots, x_l \) according to weights as \( y_1, \ldots, y_l \) with \( y_i \) of weight \( \beta_j(x_i) \cdot f : \{1,\ldots,s\} \rightarrow \{1,\ldots,s\} \) some step function. Then, for \( i \neq j \),
\[ -a_j^i = \inf \{ k \in \mathbb{N} ; ad(v)^{k+1}(w) = 0 \forall v \in g^{\beta_j(x_i)} \forall w \in g^{\beta_j(x_j)} \}. \]
The GCM \( A \) is an invariant of \( g \), up to permutations of \( \{\beta_1, \ldots, \beta_n\} \) that leave the \( d_j \)'s invariant. The type of the GCM was identified either directly or through the associated Dynkin diagram. As an example to Table 2, there are (up to isomorphism) three 7-dimensional nilpotent Lie algebras that can be constructed from the GCM \( D_4^{(3)} \): \( g_{7,2,1}^{(ii)} \), \( g_{7,2,10}^{(0)} \), \( g_{7,3,2} \). The 7-dimensional nilpotent Lie algebra \( D_{4,42}^{(3)} \) constructed from the GCM \( D_4^{(3)} \) in \( \mathfrak{X} \) is isomorphic to \( g_{7,3,2} \).

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Table 2. Kac-Moody types for indecomposable nilpotent Lie algebras of dimension \( \leq 7 \). Notations for indefinite hyperbolic are those of [20], supplemented in parentheses for rank 3, 4 by the notations of [1] (as there are misprints and omissions in [20]).

| Algebra | GCM | Finite | Affine | Indefinite Hyperbolic | Indefinite Not Hyperbolic |
|---------|-----|--------|--------|-----------------------|----------------------------|
| \( \mathfrak{g}_3 \) | \( \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \) | | | \( A_2 \) | \( \mathfrak{g}_3 \) |
| \( \mathfrak{g}_4 \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \) | \( C_2 \) | | | |
| \( \mathfrak{g}_{5.1} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \) | \( A_2 \times A_2 \) | | | |
| \( \mathfrak{g}_{5.2} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | \( A_3 \) | | | |
| \( \mathfrak{g}_{5.3} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \) | \( B_3 \) | | | |
| \( \mathfrak{g}_{5.4} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | \( G_2 \) | | | |
| \( \mathfrak{g}_{5.5} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \) | \( B_3^{(1)} \) | | | |
| \( \mathfrak{g}_{5.6} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | \( B_3 \) | | | |
| \( \mathfrak{g}_{6.1} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | \( A_4 \) | | | |
| \( \mathfrak{g}_{6.2} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | \( B_2 \times A_2 \) | | | |
| \( \mathfrak{g}_{6.3} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | \( A_4^{(1)} \) | | | |
| \( \mathfrak{g}_{6.4} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | \( B_3 \) | | | |
| \( \mathfrak{g}_{6.5} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | \( C_3 \) | | | |
| \( \mathfrak{g}_{6.6} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | \( H_2^{(3)} \) | \( 32 \) | | |
| \( \mathfrak{g}_{6.7} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | | | | |
| \( \mathfrak{g}_{6.8} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | | | | |
| \( \mathfrak{g}_{6.9} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | | | | |
| \( \mathfrak{g}_{6.10} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | | | | |
| \( \mathfrak{g}_{6.11} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | | | | |
| \( \mathfrak{g}_{6.12} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | | | | |
| \( \mathfrak{g}_{6.13} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | | | | |
| \( \mathfrak{g}_{6.14} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | | | | |
| \( \mathfrak{g}_{6.15} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | | | | |
| \( \mathfrak{g}_{6.16} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | | | | |
| \( \mathfrak{g}_{6.17} \) | \( \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \) | | | | |
| algebra  | GCM        | Finite | Affine | Indefinite Hyperbolic | Indefinite Not Hyperbolic |
|---------|------------|--------|--------|-----------------------|---------------------------|
| $\mathfrak{g}_6.18$ | $(\begin{array}{cc} 2 & -3 \\ -1 & 2 \end{array})$ |        |        | $G_2$                 |                           |
| $\mathfrak{g}_6.19$ | $(\begin{array}{cc} 2 & -4 \\ -2 & 2 \end{array})$ |        |        | (4,2)                 |                           |
| $\mathfrak{g}_6.20$ | $(\begin{array}{cc} 2 & -3 \\ -3 & 2 \end{array})$ |        |        | (3,3)                 |                           |
| $\mathfrak{g}_7.0.1^+$ | $(\begin{array}{cc} 2 & -5 \\ -5 & 2 \end{array})$ |        |        | (5,5)                 |                           |
| $\mathfrak{g}_7.0.2^+$ | ditto |        |        | ditto                 |                           |
| $\mathfrak{g}_7.0.3^+$ | ditto |        |        | ditto                 |                           |
| $\mathfrak{g}_7.0.4(\lambda)^+$ | $(\begin{array}{cc} 2 & -4 \\ -4 & 2 \end{array})$ |        |        | (4,4)                 |                           |
| $\mathfrak{g}_7.0.5^+$ | ditto |        |        | ditto                 |                           |
| $\mathfrak{g}_7.0.6^+$ | $(\begin{array}{cc} 2 & -3 \\ -3 & 2 \end{array})$ |        |        | (3,3)                 |                           |
| $\mathfrak{g}_7.0.7^+$ | ditto |        |        | ditto                 |                           |
| $\mathfrak{g}_7.0.8^+$ | $(\begin{array}{cc} 2 & -3 \ -3 \\ -2 & 2 \ -3 \\ -3 & -3 & 2 \end{array})$ |        |        | $H^{(3)}_{123}$ (123) |                           |
| $\mathfrak{g}_7.1.01(i)^+$ | $(\begin{array}{cc} 2 & 0 \ 0 & -4 \\ -1 & 2 \ -1 & 1 \\ -1 & -1 & 2 \end{array})$ |        |        |                        |                           |
| $\mathfrak{g}_7.1.01(ii)^+$ | ditto |        |        | ditto                 |                           |
| $\mathfrak{g}_7.1.02^+$ | $(\begin{array}{cc} 2 & -2 \\ -5 & 2 \end{array})$ |        |        | (3,2)                 |                           |
| $\mathfrak{g}_7.1.03^+$ | $(\begin{array}{cc} 2 & -2 \\ -2 & 2 \end{array})$ |        |        | (3,2)                 |                           |
| $\mathfrak{g}_7.1.1(i_{i})\lambda \neq 0$ | $(\begin{array}{cc} 2 & -5 \\ -2 & 2 \end{array})$ |        |        | (5,3)                 |                           |
| $\mathfrak{g}_7.1.1(i_{ii})\lambda = 0$ | $(\begin{array}{cc} 2 & -5 \\ -4 & -2 \end{array})$ |        |        | (5,2)                 |                           |
| $\mathfrak{g}_7.1.1(i_{i})\lambda \neq 0$ | $(\begin{array}{cc} 2 & -5 \\ -1 & 2 \end{array})$ |        |        | (5,1)                 |                           |
| $\mathfrak{g}_7.1.1(i_{ii})$ | $(\begin{array}{cc} 2 & -4 \\ -2 & 2 \\ -3 & 2 \end{array})$ |        |        | (4,3)                 |                           |
| $\mathfrak{g}_7.1.1(i_{ii})$ | $(\begin{array}{cc} 2 & -2 \\ -3 & 2 \end{array})$ |        |        | (3,2)                 |                           |
| $\mathfrak{g}_7.1.1(v)$ | $(\begin{array}{cc} 2 & 0 \ 0 & -4 \\ -2 & 2 \ -1 & 1 \\ -1 & -1 & 2 \end{array})$ |        |        | √                     |                           |
| $\mathfrak{g}_7.1.1(vi)$ | $(\begin{array}{cc} 2 & 0 \ 0 & -4 \\ -2 & 2 \ -3 & 2 \\ -3 & 2 \ -1 & 1 \ -1 & 1 & 2 \end{array})$ |        |        | √                     |                           |
| $\mathfrak{g}_7.1.1(vii)$ | $(\begin{array}{cc} 2 & -3 \ -3 \\ -3 & 2 \ 0 \\ -1 & 0 & 2 \end{array})$ |        |        | √                     |                           |
| $\mathfrak{g}_7.1.2(i_{i})^+$ | $(\begin{array}{cc} 3 & -2 \\ -3 & 2 \end{array})$ |        |        | √                     |                           |
| $\mathfrak{g}_7.1.2(ii)$ | ditto |        |        | ditto                 |                           |
| $\mathfrak{g}_7.1.2(iii)$ | ditto |        |        | ditto                 |                           |
| $\mathfrak{g}_7.1.2(iv)$ | ditto |        |        | ditto                 |                           |
| algebra          | GCM | Finite  | Affine  | Indefinite Hyperbolic | Indefinite Not Hyperbolic |
|------------------|-----|---------|---------|-----------------------|----------------------------|
| $g_7.1.3(i)_A$‡  |     |         |         |                       |                            |
| $g_7.1.3(ii)_A$‡ |     |         |         |                       |                            |
| $g_7.1.3(iii)_A$‡|     |         |         |                       |                            |
| $g_7.1.3(iv)_A$‡ |     |         |         |                       |                            |
| $g_7.1.3(v)_A$‡  |     |         |         |                       |                            |
| $g_7.1.4$        |     |         |         |                       |                            |
| $g_7.1.5$        |     |         |         |                       |                            |
| $g_7.1.6$        |     |         |         |                       |                            |
| $g_7.1.7$        |     |         |         |                       |                            |
| $g_7.1.8$        |     |         |         |                       |                            |
| $g_7.1.9$        |     |         |         |                       |                            |
| $g_7.1.10$       |     |         |         |                       |                            |
| $g_7.1.11$       |     |         |         |                       |                            |
| $g_7.1.12$       |     |         |         |                       |                            |
| $g_7.1.13$       |     |         |         |                       |                            |
| $g_7.1.14$       |     |         |         |                       |                            |
| $g_7.1.15$       |     |         |         |                       |                            |
| $g_7.1.16$       |     |         |         |                       |                            |
| $g_7.1.17$       |     |         |         |                       |                            |
| $g_7.1.18$       |     |         |         |                       |                            |
| $g_7.1.19$       |     |         |         |                       |                            |
| $g_7.1.20$       |     |         |         |                       |                            |
| $g_7.1.21$       |     |         |         |                       |                            |
| Algebra | CCM |
|---------|-----|
| $A_1$   | $D_4(0)$ |
| $H_3(0)$ | $D_4(0)$ |
| $H_2(0)$ | $A_1(0)$ |
| $H_1(0)$ | $D_4(0)$ |
| $H_0(0)$ | $A_1(0)$ |
| $H_{10}(0)$ | $D_4(0)$ |
| $H_{30}(0)$ | $A_1(0)$ |
| $H_{50}(0)$ | $D_4(0)$ |

TABLE 2, continued

ON NULL LIE ALGEBRAS
| Table 2. continued |
|---------------------|
| Field (G2)          | Hyperbolic | Indefinite |
| H(3) (106)          | (3)        | √          |
| H(5) (38)           | (3)        | √          |
| H(7) (131)          | (3)        | √          |
| H(10) (107)         | (3)        | √          |
| H(14) (28)          | (3)        | √          |
| H(16) (29)          | (3)        | √          |
| H(18) (32)          | (3)        | √          |
| H(20) (164)         | (3)        | √          |
| H(22) (32)          | (3)        | √          |

Note: The table continues on the next page.
Table 2. continued

| algebra | GCM | Finite | Affine | Indefinite Hyperbolic | Indefinite Not Hyperbolic |
|---------|-----|--------|--------|-----------------------|--------------------------|
| $\mathfrak{g}_7.1(i_{1\text{A}})$ | \[
\begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & 0 & 2 \\
-1 & 0 & 2
\end{pmatrix}
\] | $A_2^{(1)}$ | $D_4$ | $G_2^{(1)}$ | $B_3 \times A_2$ | $A_5$ |
| $\mathfrak{g}_7.1(i_{1\text{III}})$ | \[
\begin{pmatrix}
2 & -3 & -1 \\
-1 & 0 & 2 \\
2 & -1 & 2 \\
0 & -1 & 2
\end{pmatrix}
\] | $D_4^{(3)}$ | $G_2^{(1)}$ | $B_3$ | $H_1^{(3)} (1)$ | |
| $\mathfrak{g}_7.2$ | \[
\begin{pmatrix}
2 & -3 & -1 \\
-1 & 0 & 2 \\
2 & -1 & 2 \\
0 & -1 & 2
\end{pmatrix}
\] | $G_2^{(1)}$ | $D_3^{(2)}$ | $B_3$ | $H_1^{(3)} (1)$ | |
| $\mathfrak{g}_7.3$ | \[
\begin{pmatrix}
-2 & -2 & 0 \\
0 & 0 & -2 \\
0 & -1 & 0 \\
-1 & 2 & 0
\end{pmatrix}
\] | $B_4$ | $F_4$ | $C_4$ | $B_3^{(1)}$ | |
| $\mathfrak{g}_7.4$ | \[
\begin{pmatrix}
2 & -1 & -1 \\
-2 & 0 & 0 \\
0 & 1 & -1 \\
0 & -2 & 0
\end{pmatrix}
\] | $D_4$ | $F_4$ | $A_3^{(1)}$ | $A_3^{(1) \times A_2}$ | |
| $\mathfrak{g}_7.5$ | \[
\begin{pmatrix}
-2 & -2 & 0 \\
-2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}
\] | $B_3 \times A_2$ | $A_3^{(1)}$ | $A_3^{(1) \times A_2}$ | $A_5$ | |
| $\mathfrak{g}_7.6$ | \[
\begin{pmatrix}
-2 & -2 & 0 \\
-2 & 0 & 0 \\
-2 & 0 & 0 \\
-2 & 0 & 0
\end{pmatrix}
\] | $B_3 \times A_2$ | $A_3^{(1) \times A_2}$ | $A_5$ | | |
Table 2. continued

| Algebra | GCM | Finite | Affine | Indefinite Hyperbolic | Indefinite Not Hyperbolic |
|---------|-----|--------|--------|-----------------------|--------------------------|
| g7.3.20 | \[\begin{pmatrix} 2 & -2 & -2 \\ -1 & 0 & 0 \\ -1 & 0 & 2 \end{pmatrix}\] | \[C_2(1)\] | | | |
| g7.3.21 | \[\begin{pmatrix} 2 & -2 & -1 \\ -1 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}\] | \[A_4^{(2)}\] | | | |
| g7.3.22 | \[\begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}\] | \[C_3\] | | | |
| g7.3.23 | \[\begin{pmatrix} 2 & -2 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}\] | | | | |
| g7.3.24 | \[\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 2 & -1 & 2 & 1 \end{pmatrix}\] | | | \[H_{96}^{(3)} \ (103)\] | |
| g7.4.1 | \[\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 \\ 2 & -2 & 0 & 0 \end{pmatrix}\] | \[A_4\] | | | |
| g7.4.2 | \[\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 \\ 2 & -1 & 0 & 0 \end{pmatrix}\] | \[D_4\] | | | |
| g7.4.3 | \[\begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}\] | \[A_2 \times A_3\] | | | |
| g7.4.4 | \[\begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}\] | \[A_2 \times A_2 \times A_2\] | | | |