ON A QUESTION OF O’GRADY ABOUT MODIFIED DIAGONALS

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Abstract. Let $X$ be an abelian variety of dimension $g$. In a recent preprint O’Grady defines modified diagonal classes $\Gamma^m$ on $X^m$ and he conjectures that the class of $\Gamma^m$ in the Chow ring of $X^m$ is torsion for $m \geq 2g + 1$. We prove a generalization of this conjecture.

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1. Throughout this note, $R$ is a Dedekind ring and $S$ is a connected scheme that is smooth and of finite type over $R$, which is the base scheme over which we work.

Let $\pi: Y \to S$ be a smooth projective $S$-scheme, $a: S \to Y$ a section of $\pi$, and $m \geq 1$ an integer. By $Y^m$ we mean the product of $m$ copies of $Y$ relative to $S$. For $j \in \{1, \ldots, m\}$ let $pr_j: Y^m \to Y$ be the projection onto the $j$th factor and let $\hat{pr}_j: Y^m \to Y^{m-1}$ be the projection that contracts the $j$th factor.

For $I \subset \{1, \ldots, m\}$, let $\delta_I: Y \to Y^m$ be the morphism defined by the property that $pr_j \circ \delta_I$ is the identity on $Y$ if $i \in I$ and is the “constant map” $a \circ \pi$ if $i \notin I$. Following O’Grady [3] we define an algebraic cycle $\Gamma^m(Y, a)$ on $Y^m$ by

$$\Gamma^m(Y, a) = \sum_{\emptyset \neq I \subset \{1, \ldots, m\}} (-1)^{|I|} \cdot \delta_{I,*}(Y).$$

The goal of this note is to prove the following result, which proves Conjecture 0.3 of [3], generalized to the relative setting.

2. Theorem. — Let $X \to S$ be an abelian scheme of relative dimension $g$. For any section $a \in X(S)$ the class of $\Gamma^m(X, a)$ in $\text{CH}(X^m)$ is torsion if $m \geq 2g + 1$.

3. Remark. Still with $X$ an abelian scheme over $S$, let $e \in X(S)$ be the zero section. Translation over $a$ gives an isomorphism $t_a: X \sim \to X$ over $S$. The class of $\Gamma^m(X, a)$ is the push-forward of the class of $\Gamma^m(X, e)$ under $t^m_a: X^m \sim \to X^m$. Hence it suffices to prove the theorem taking $e$ as section.

4. Lemma. — Let $X/S$ be an abelian scheme of relative dimension $g$ and write $\Gamma^m = \Gamma^m(X, e)$. For $n \in \mathbb{Z}$ let $\text{mult}_X(n): X \to X$ be the endomorphism given by multiplication by $n$.

(i) For $n \in \mathbb{Z}$ we have $\text{mult}_{X^m}(n)_*[\Gamma^m] = n^{2g} \cdot [\Gamma^m]$.

(ii) For $j \in \{1, \ldots, m\}$ let $\hat{pr}_j: X^m \to X^{m-1}$ be the projection map contracting the $j$th factor. Then $\hat{pr}_{j,*}[\Gamma^m] = 0$.

Proof. For (i) we use that, with notation as in point 1, and taking $a = e$ as section, $\text{mult}_X(n) \circ \delta_I$ is the same as $\delta_I \circ \text{mult}_X(n)$ and that $\text{mult}_X(n)$ is finite flat of degree $n^{2g}$.

For (ii), consider a non-empty subset $I \subset \{1, \ldots, m\}$. If $I = \{j\}$ then $\hat{pr}_j \circ \delta_I: X \to X^{m-1}$ is constant, so that $\hat{pr}_{j,*}\delta_I(X) = 0$. Let $\mathcal{J}$ be the set of non-empty subsets of $\{1, \ldots, m\}$ different...
from \{j\}. Then \( \mathcal{I} = \mathcal{I}_0 \coprod \mathcal{I}_1 \) where \( I \in \mathcal{I}_0 \) if \( j \notin I \) and \( I \in \mathcal{I}_1 \) if \( j \in I \). The map \( \beta: \mathcal{I}_0 \to \mathcal{I}_1 \) given by \( I \mapsto I \cup \{j\} \) is a bijection. Further, for \( I \in \mathcal{I}_0 \) we have \( \hat{p}_j \circ \delta_I = \hat{p}_j \circ \delta_{\beta(I)} \) and \( |\beta(I)| = |I| + 1 \). In the calculation of \( \hat{p}_{r_1, \ldots, r_k}[\Gamma^m] \), the terms corresponding to \( I \) and \( \beta(I) \) therefore cancel, and this gives the assertion. \( \square \)

5. Let \( \text{Mot}_S \) be the category of Chow motives over \( S \) with respect to graded correspondences; see for instance [2], Section 1. If \( Y \) is a smooth projective \( S \)-scheme we write \( h(Y) \) for its Chow motive. In \( \text{Mot}_S \) we have a tensor product with \( h(Y) \otimes h(Z) = h(Y \times_S Z) \).

Let \( 1_S = h(S) \) be the unit motive and \( 1(n) \) the \( n \)th Tate twist. If \( M \) is a motive, we write \( M(n) = M \otimes 1(n) \). The Chow groups (with \( \mathbb{Q} \)-coefficients) of a motive \( M \) are defined by \( \text{CH}^i(M)_\mathbb{Q} = \text{Hom}_{\text{Mot}_S}(1(-i), M) \).

If \( f: Y \to Z \) is a morphism of smooth projective \( S \)-schemes we have induced morphisms \( f^*: h(Z) \to h(Y) \) and, assuming \( Y \) and \( Z \) are connected, \( f_*: h(Y) \to h(Z)(d) \), where \( d = \text{dim}(Z/S) - \text{dim}(Y/S) \).

6. Let \( X/S \) be an abelian scheme of relative dimension \( g \). As proven by Deninger and Murre in [2] (generalizing results of Beauville [1] over a field) we have a canonical decomposition \( h(X) = \bigoplus_{i=0}^{2g} h^i(X) \) in \( \text{Mot}_S \) that is stable under all endomorphisms \( \text{mult}_X(n)_* \), and such that \( \text{mult}_X(n)_* \) is multiplication by \( n^{2g-i} \) on \( h^i(X) \). For \( m \geq 1 \) this induces a decomposition

\[
h(X^m) = \bigoplus_{i=(i_1, \ldots, i_m)} m h^{i_1}(X),
\]

where the sum runs over the elements \( i \in \{0, \ldots, 2g\}^m \). Under this decomposition we have

\[
h^\nu(X^m) = \bigoplus_{|i|=\nu} m h^{i_1}(X),
\]

where the sum runs over the \( m \)-tuples \( i = (i_1, \ldots, i_m) \) in \( \{0, \ldots, 2g\}^m \) with \( |i| = i_1 + \cdots + i_m \) equal to \( \nu \).

If \( \pi: X \to S \) is the structural morphism, \( \pi_*: h(X) \to h(S)(-g) = 1(-g) \) is an isomorphism on \( h^{2g}(X) \) and is zero on \( \bigoplus_{i=0}^{2g-1} h^i(X) \).

7. Lemma. — Notation as above. If there is an index \( \nu \) such that \( i_\nu = 2g \) then the component of \( \bigotimes_{j=1}^{m} h^{i_j}(X) \) in \( \text{CH}(\bigotimes_{j=1}^{m} h^{i_j}(X))_\mathbb{Q} \) is zero.

Proof. Assume \( i_\nu = 2g \). Consider the projection \( \hat{p}_{r_\nu}: X^m \to X^{m-1} \). By the fact stated just before the lemma, the induced map \( \hat{p}_{r_{\nu,*}}: h(X^m) \to h(X^{m-1})(-g) \) restricts to an isomorphism of \( \bigotimes_{j=1}^{m} h^{i_j}(X) \) with a sub-motive of \( h(X^{m-1})(-g) \). The assertion therefore follows from Lemma 4(ii). \( \square \)

8. Proof of the Theorem. Write \( \Gamma^m = \Gamma^m(X, e) \). By Lemma 4(i) we have

\[
[\Gamma^m] \in \text{CH}(h^{2g(m-1)}(X^m))_\mathbb{Q} \subset \text{CH}(X^m)_\mathbb{Q},
\]

and (6.1) gives

\[
\text{CH}(h^{2g(m-1)}(X^m))_\mathbb{Q} = \bigoplus_{|i|=2g(m-1)} \text{CH}(\bigotimes_{j=1}^{m} h^{i_j}(X))_\mathbb{Q}.
\]
The assumption that $m \geq 2g + 1$ implies that for every $\mathbf{i} = (i_1, \ldots, i_m) \in \{0, \ldots, 2g\}^m$ with $|\mathbf{i}| = 2g(m - 1)$ there is an index $\nu$ with $i_\nu = 2g$ and by Lemma 7 we are done. \hfill \Box

References

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