A Decomposition Method for Solving Multicommodity Network Equilibrium

MINH N. BÙI
North Carolina State University, Department of Mathematics, Raleigh, NC 27695-8205, USA
mnbui@ncsu.edu

Abstract. We consider the numerical aspect of the multicommodity network equilibrium problem proposed by Rockafellar in 1995. Our method relies on the flexible monotone operator splitting framework recently proposed by Combettes and Eckstein.

1 Problem formulation

Rockafellar proposed in [13] the important multicommodity network equilibrium model (see (1.6) in Problem 2) and studied some of its properties. In the present paper, we devise a flexible numerical method for solving this problem based on the asynchronous block-iterative decomposition framework of [6].

The following notion of a network from [12, Section 1A] plays a central role in our problem.

Definition 1 A network consists of nonempty finite sets \( \mathcal{N} \) and \( \mathcal{A} \) — whose elements are called nodes and arcs, respectively — and a mapping \( \vartheta: \mathcal{A} \to \mathcal{N} \times \mathcal{N}: j \mapsto (\vartheta_1(j), \vartheta_2(j)) \) such that, for every \( j \in \mathcal{A} \), \( \vartheta_1(j) \neq \vartheta_2(j) \). We call \( \vartheta_1(j) \) and \( \vartheta_2(j) \) the initial node and the terminal node of arc \( j \), respectively. In addition, we set

\[
\begin{align*}
\mathcal{A}^+(i) &= \{ j \in \mathcal{A} \mid \text{node } i \text{ is the initial node of arc } j \} \\
\mathcal{A}^-(i) &= \{ j \in \mathcal{A} \mid \text{node } i \text{ is the terminal node of arc } j \}.
\end{align*}
\]

(1.1)

Recall that, given a Euclidean space \( \mathcal{G} \) with scalar product \( \langle \cdot, \cdot \rangle \), an operator \( A: \mathcal{G} \to 2^\mathcal{G} \) is maximally monotone if

\[
\forall (x, x^*) \in \mathcal{G} \times \mathcal{G}, \quad (x, x^*) \in \text{gra } A \quad \iff \quad \forall (y, y^*) \in \text{gra } A \quad \langle x - y, x^* - y^* \rangle \geq 0,
\]

(1.2)

where \( \text{gra } A = \{(x, x^*) \in \mathcal{G} \times \mathcal{G} \mid x^* \in Ax\} \) is the graph of \( A \). (The reader is referred to [2] for background and complements on monotone operator theory and convex analysis.) The problem of interest is the following.

Problem 2 Under consideration is a network \( (\mathcal{N}, \mathcal{A}, \vartheta) \), together with a nonempty finite set \( \mathcal{C} \) of commodities transiting on the network. Equip \( \mathcal{H} = \mathbb{R}^\mathcal{C} \) with the scalar product \( \langle \cdot, \cdot \rangle \) and let us introduce the spaces

\[
\begin{align*}
\mathcal{X} &= \{ x = (x_j)_{j \in \mathcal{A}} \mid (\forall j \in \mathcal{A}) \quad x_j = (\xi_{j,k})_{k \in \mathcal{C}} \in \mathcal{H} \} \\
\mathcal{V} &= \{ v^* = (v^*_i)_{i \in \mathcal{N}} \mid (\forall i \in \mathcal{N}) \quad v^*_i = (\nu^*_{i,k})_{k \in \mathcal{C}} \in \mathcal{H} \}.
\end{align*}
\]

(1.3)
An element $x \in \mathcal{X}$ is called a flow on the network, where $\xi_{j,k}$ is the flux of commodity $k$ on arc $j$. The divergence of a flow $x \in \mathcal{X}$ at node $i$ is
\[ \text{div}_i x = \sum_{j \in \delta^+ (i)} x_j - \sum_{j \in \delta^- (i)} x_j. \] (1.4)
We refer to an element $v^* \in \mathcal{V}$ as a potential on the network, where $\nu^*_{i,k}$ is the potential of commodity $k$ at node $i$. Given $v^* \in \mathcal{V}$ and $j \in \delta$, the tension (or potential difference) across arc $j$ relative to the potential $v^*$ is
\[ \Delta_j v^* = v^*_{\delta_2 (j)} - v^*_{\delta_1 (j)}. \] (1.5)
For every $j \in \delta$, the flow-tension relation on arc $j$ is modeled by the sum $Q_j + R_j$ of maximally monotone operators $Q_j : \mathcal{H} \to 2^\mathcal{H}$ and $R_j : \mathcal{H} \to 2^\mathcal{H}$. Further, for every $i \in \mathcal{N}$, the divergence-potential relation at node $i$ is modeled by a maximally monotone operator $S_i : \mathcal{H} \to 2^\mathcal{H}$. The task is to find a flow $\mathbf{x} \in \mathcal{X}$ and a potential $\mathbf{v}^* \in \mathcal{V}$ such that
\[ \begin{cases} \left( \forall j \in \delta \right) \Delta_j \mathbf{v}^* = Q_j \mathbf{x}_j + R_j \mathbf{x}_j \\ \left( \forall i \in \mathcal{N} \right) \text{div}_i \mathbf{x} = S_i^{-1} \mathbf{v}_i \end{cases} \] (1.6)
under the assumption that (1.6) has a solution.

**Remark 3** The pertinence of Problem 2 is demonstrated in [12, Chapter 8] and [13], where it is shown to capture formulations arising in areas such as traffic assignment, hydraulic networks, and price equilibrium.

## 2 A block-iterative decomposition method

**Notation.** Throughout, $\mathcal{G}$ is a Euclidean space. Let $A : \mathcal{G} \to 2^\mathcal{G}$ be maximally monotone and let $x \in \mathcal{G}$. Then, in terms of the variable $p \in \mathcal{G}$, the inclusion $x \in p + Ap$ has a unique solution, which is denoted by $J_A x$. The operator $J_A : \mathcal{G} \to \mathcal{G} : x \mapsto J_A x$ is called the resolvent of $A$.

Our algorithm (see (2.2) in Proposition 4) is derived from [6, Algorithm 12] and it thus inherits the following attractive features from the framework of [6]:

- No additional assumption, such as Lipschitz continuity or cocoercivity, is imposed on the underlying operators.
- Algorithm (2.2) achieves full splitting in the sense that the operators $(Q_j)_{j \in \delta}$, $(R_j)_{j \in \delta}$, and $(S_i)_{i \in \mathcal{N}}$ are activated independently via their resolvents.
- Algorithm (2.2) is block-iterative, that is, at iteration $n$, only blocks $(Q_j)_{j \in \delta_n}$, $(R_j)_{j \in \delta_n}$, and $(S_i)_{i \in \mathcal{N}_n}$ of operators need to be activated. To guarantee convergence of the iterates, the mild sweeping condition (2.1) needs to be fulfilled.

We shall denote elements in $\mathcal{X}$ and $\mathcal{V}$ by bold letters, e.g., $q_n = (q_{j,n})_{j \in \delta}$ and $s^*_n = (s^*_{i,n})_{i \in \mathcal{N}}$.

**Proposition 4** Consider the setting of Problem 2. Let $T \in \mathbb{N}$, let $(\delta_n)_{n \in \mathbb{N}}$ be nonempty subsets of $\delta$, and let $(\mathcal{N}_n)_{n \in \mathbb{N}}$ be nonempty subsets of $\mathcal{N}$ such that $\delta_0 = \delta$, $\mathcal{N}_0 = \mathcal{N}$, and
\[ (\forall n \in \mathbb{N}) \bigcup_{k=n}^{n+T} \delta_k = \delta \quad \text{and} \quad (\forall n \in \mathbb{N}) \bigcup_{k=n}^{n+T} \mathcal{N}_k = \mathcal{N}. \] (2.1)
Let \((\lambda_n)_{n \in \mathbb{N}}\) be a sequence in \([0, 2]\) such that \(\inf_{n \in \mathbb{N}} \lambda_n > 0\) and \(\sup_{n \in \mathbb{N}} \lambda_n < 2\). Moreover, for every \(j \in \mathcal{A}\) and every \(i \in \mathcal{N}\), let \((x_{j,0}, x_{j,0}^*, v_{i,0}^*) \in \mathcal{H}^3\) and \((\gamma_j, \mu_j, \sigma_i) \in [0, +\infty[^3\). Iterate

\[
\begin{align*}
\text{for } n = 0, 1, \ldots \\
\text{for every } j \in \mathcal{A}_n \\
& l_{j,n}^* = x_{j,n}^* - \Delta_j v_{i,n}^* \\
& q_{j,n} = \bar{\gamma}_j q_j(x_{j,n} - \gamma_j l_{j,n}) \\
& q_{j,n}^* = \gamma_j^{-1}(x_{j,n} - q_{j,n}) - l_{j,n}^* \\
& r_{j,n} = \mu_j r_j(x_{j,n} + \mu_j x_{j,n}^*) \\
& r_{j,n}^* = x_{j,n}^* + \mu_j^{-1}(x_{j,n} - r_{j,n}) \\
\end{align*}
\]

for every \(j \in \mathcal{A} \setminus \mathcal{A}_n\)

\[
\begin{align*}
& q_{j,n} = q_{j,n-1}^*; q_{j,n}^* = q_{j,n-1}^*; r_{j,n} = r_{j,n-1}; r_{j,n}^* = r_{j,n-1}^* \\
& \text{for every } i \in \mathcal{N}_n \\
& l_{i,n} = \text{div}_i x_{i,n} \\
& s_{i,n} = \Delta_i s_i(l_{i,n} + \sigma_i v_{i,n}^*) \\
& s_{i,n}^* = \Delta_i^{-1}(l_{i,n} - s_{i,n}) \\
& t_{i,n} = s_{i,n} - \text{div}_i q_{i,n} \\
\end{align*}
\]

for every \(i \in \mathcal{N} \setminus \mathcal{N}_n\)

\[
\begin{align*}
& s_{i,n} = s_{i,n-1}^*; s_{i,n}^* = s_{i,n-1}^* \\
& t_{i,n} = s_{i,n} - \text{div}_i q_{i,n} \\
\end{align*}
\]

for every \(j \in \mathcal{A}\)

\[
\begin{align*}
& t_{j,n}^* = q_{j,n}^* + r_{j,n}^* - \Delta_j s_{j,n}^* \\
& u_{j,n} = r_{j,n} - q_{j,n} \\
& \tau_n = \sum_{j \in \mathcal{A}} (\|t_{j,n}^*\|^2 + \|u_{j,n}\|)^2 + \sum_{i \in \mathcal{N}} \|t_{i,n}\|^2 \\
& \text{if } \tau_n > 0 \\
& \theta_n = \lambda_n \max\{\pi_n, 0\}/\tau_n \\
& \text{else} \\
& \theta_n = 0 \\
& \text{for every } j \in \mathcal{A} \\
& x_{j,n+1} = x_{j,n} - \theta_n t_{j,n}^* \\
& x_{j,n+1}^* = x_{j,n+1} - \theta_n u_{j,n} \\
& \text{for every } i \in \mathcal{N} \\
& v_{i,n+1}^* = v_{i,n}^* - \theta_n t_{i,n} \\
\end{align*}
\]

Then \((x_{j,n}, (v_{i,n}^*)_{i \in \mathcal{N}}, n_{n \in \mathbb{N}}\) converges to a solution to (1.6).

**Proof.** Let us consider the multivariate monotone inclusion problem

\[
\begin{align*}
\text{find } \overline{\mathbf{x}} \in \mathcal{X}, \overline{\mathbf{x}}^* \in \mathcal{X}, \text{ and } \overline{\mathbf{v}}^* \in \mathcal{V} \text{ such that} \\
\left\{ \begin{array}{ll}
(\forall j \in \mathcal{A}) & \Delta_j \overline{\mathbf{x}}^* - \overline{\mathbf{x}}^*_j \in Q_j \overline{\mathbf{x}}_j \text{ and } \overline{\mathbf{x}}_j \in R_j^{-1} \overline{\mathbf{x}}_j^* \\
(\forall i \in \mathcal{N}) & \text{div}_i \overline{\mathbf{x}} \in S_i^{-1} \overline{\mathbf{x}}^*_i. 
\end{array} \right. \\
\end{align*}
\]

Then

\[
\begin{align*}
(\forall \overline{\mathbf{x}} \in \mathcal{X})(\forall \overline{\mathbf{v}}^* \in \mathcal{V}) \quad (\overline{\mathbf{x}}, \overline{\mathbf{v}}^*) \text{ solves (1.6)} \\
\iff (\exists \overline{\mathbf{v}}^* \in \mathcal{X}) \\
\left\{ \begin{array}{ll}
(\forall j \in \mathcal{A}) & \Delta_j \overline{\mathbf{v}}^* \in Q_j \overline{\mathbf{x}}_j + \overline{\mathbf{x}}^*_j \text{ and } \overline{\mathbf{x}}^*_j \in R_j \overline{\mathbf{x}}_j \\
(\forall i \in \mathcal{N}) & \text{div}_i \overline{\mathbf{x}} \in S_i^{-1} \overline{\mathbf{x}}^*_i. 
\end{array} \right. \\
\end{align*}
\]
We deduce from (2.4) that
\[(\exists \mathbf{x}' \in \mathcal{X}) (\mathbf{x}, \mathbf{x}', \mathbf{p}') \text{ solves } (2.3). \] (2.4)

Therefore, since (1.6) has a solution, so does (2.3). Next, define
\[(\forall i \in \mathcal{N})(\forall j \in \mathcal{A}) \quad \varepsilon_{i,j} = \begin{cases} 1, & \text{if node } i \text{ is the initial node of arc } j; \\ -1, & \text{if node } i \text{ is the terminal node of arc } j; \\ 0, & \text{otherwise.} \end{cases} \] (2.5)

It results from (1.4) and (1.1) that
\[(\forall x \in \mathcal{X})(\forall i \in \mathcal{N}) \quad \text{div}_i x = \sum_{j \in \mathcal{A}} \varepsilon_{i,j} x_j, \] (2.6)
and from (1.5) that
\[(\forall v^* \in \mathcal{V})(\forall j \in \mathcal{A}) \quad \Delta_j v^* = -\sum_{i \in \mathcal{N}} \varepsilon_{i,j} v^*_i. \] (2.7)

We now verify that (2.3) is a special case of [6, Problem 1] with the setting
\[
\begin{align*}
\mathcal{H} & = \mathcal{G}_k = \mathcal{H} \\
\mathcal{A}_j & = Q_j \\
\mathcal{B}_k & = \begin{cases} R_k, & \text{if } k \in \mathcal{A}; \\
S_k, & \text{if } k \in \mathcal{N} \end{cases} \\
\mathcal{L}_{k,j} & = \begin{cases} \text{Id}, & \text{if } k = j; \\
0, & \text{if } k \in \mathcal{A} \text{ and } k \neq j; \\
\varepsilon_{k,j} \text{Id}, & \text{if } k \in \mathcal{N}. \end{cases}
\end{align*}
\] (2.8)

We deduce from (2.6) that
\[
(\forall x \in \mathcal{X})(\forall k \in \mathcal{K}) \quad \sum_{j \in \mathcal{A}} \mathcal{L}_{k,j} x_j = \begin{cases} x_k, & \text{if } k \in \mathcal{A}; \\
\sum_{j \in \mathcal{A}} \varepsilon_{k,j} x_j, & \text{if } k \in \mathcal{N} \end{cases} \\
= \begin{cases} x_k, & \text{if } k \in \mathcal{A}; \\
\text{div}_k x, & \text{if } k \in \mathcal{N}, \end{cases} \quad \text{(2.9)}
\]
and from (2.7) that
\[
(\forall x^* \in \mathcal{X})(\forall v^* \in \mathcal{V})(\forall j \in \mathcal{I}) \quad \sum_{k \in \mathcal{A}} \mathcal{L}_{k,j} x^*_k + \sum_{k \in \mathcal{K}} \mathcal{L}_{k,j} v^*_k = x^*_j + \sum_{k \in \mathcal{N}} \varepsilon_{k,j} v^*_k = x_j^* - \Delta_j v^*. \quad (2.10)
\]

Hence, in the setting of (2.8), (2.3) is an instantiation of [6, Problem 1] and (2.2) is a realization of [6, Algorithm 12], where (\forall n \in \mathbb{N}) \; I_n = \mathcal{A}_n \text{ and } K_n = \mathcal{A}_n \cup \mathcal{N}_n. \text{ Thus, upon letting}
\[
(\forall n \in \mathbb{N}) \quad x_n = (x_{j,n})_{j \in \mathcal{A}}, \quad x^*_n = (x^*_{j,n})_{j \in \mathcal{A}}, \quad \text{and } v^*_n = (v^*_i)_{i \in \mathcal{N}}, \quad \text{(2.11)}
\]
we infer from [6, Theorem 13] that \((x_n, x^*_n, v^*_n)_{n \in \mathbb{N}}\) converges to a solution \((\mathbf{x}, \mathbf{x}', \mathbf{p}')\) to (2.3). Consequently, (2.4) asserts that \((\mathbf{x}, \mathbf{p}')\) solves (1.6). \(\square\)
Remark 5 Some comments are in order.

(i) One might be tempting to consider (1.6) as a special case of [6, Problem 1] with the setting

\[
I = \mathcal{A}, \quad K = \mathcal{N}, \quad \text{and} \quad (\forall j \in I)(\forall k \in K) \begin{cases}
\mathcal{H}_j = \mathcal{G}_k = \mathcal{H} \\
\mathcal{A}_j = Q_j + R_j \\
B_k = S_k \\
z^*_j = r_k = 0 \\
L_{k,j} = \varepsilon_{k,j} \text{Id},
\end{cases}
\tag{2.12}
\]

where \((\varepsilon_{i,j})_{i \in \mathcal{N}, j \in \mathcal{A}}\) are defined in (2.5), and then specialize [6, Algorithm 12] to (2.12). However, this approach necessitates the computation of the resolvents of the operators \((Q_j + R_j)_{j \in \mathcal{A}}\), which cannot be expressed in terms of the resolvents of \((Q_j)_{j \in \mathcal{A}}\) and \((R_j)_{j \in \mathcal{A}}\) in general (see Examples 6 and 7).

(ii) Algorithm (2.2) of Proposition 4 requires to evaluate the resolvents of the operators \((Q_j)_{j \in \mathcal{A}}\), \((R_j)_{j \in \mathcal{A}}\), and \((S_i)_{i \in \mathcal{N}}\). Illustrations of such calculations in some special cases of Problem 2 encountered in the literature are provided in Examples 6, 7, and 9–12.

(iii) Alternate algorithms [5, 7, 11] can also be used to solve (2.3) and, in turn, (1.6). Nevertheless, there are certain restrictions on the resulting algorithms. For example, the method of [5] must activate all the operators \((Q_j)_{j \in \mathcal{A}}, (R_j)_{j \in \mathcal{A}}, \) and \((S_i)_{i \in \mathcal{N}}\) at every iteration, while the frameworks of [7, 11] do not allow for deterministic selections of the blocks \((Q_j)_{j \in \mathcal{A}_n}, (R_j)_{j \in \mathcal{A}_n}\), and \((S_i)_{i \in \mathcal{A}_n}\). Finally, the algorithm resulted from [7] involves the inversion of a linear operator acting on \(\mathbb{R}^{MN}\), where \(N = \text{card } \mathcal{G}\) and \(M = 2 \text{card } \mathcal{A} + \text{card } \mathcal{N}\), which may not be favorable in large-scale problems, e.g., [8].

Notation. Before proceeding further, let us recall some basic notion of convex analysis (see [2] for details). Let \(\varphi: \mathcal{G} \to [\langle 0, +\infty]\) be proper, lower semicontinuous, and convex. The subdifferential of \(\varphi\) is the maximally monotone operator \(\partial \varphi: \mathcal{G} \to 2^\mathcal{G}: x \mapsto \{x^* \in \mathcal{G} | (\forall y \in \mathcal{G}) \langle y - x, x^* \rangle + \varphi(x) \leq \varphi(y)\}\). For every \(x \in \mathcal{G}\), the unique minimizer of \(\varphi + (1/2)\|x - \mathcal{C}\|^2\) is denoted by \(\text{prox}_{\varphi}x\). Let \(C\) be a nonempty closed convex subset of \(\mathcal{G}\). The indicator function of \(C\) is the proper lower semicontinuous convex function

\[
\iota_C: \mathcal{G} \to [0, +\infty]: x \mapsto \begin{cases} 
0, & \text{if } x \in C; \\
+\infty, & \text{otherwise},
\end{cases}
\tag{2.13}
\]

the normal cone operator of \(C\) is \(N_C = \partial \iota_C\), and the projector onto \(C\) is \(\text{proj}_C = \text{prox}_{\iota_C}\).

Example 6 (Separable multicommodity flows) Consider the setting of Problem 2 and suppose, in addition, that the following are satisfied:

[a] For every \(j \in \mathcal{A}\), \(c_j: \mathbb{R} \to \mathbb{R}^2\) is maximally monotone, \(C_j\) is a nonempty closed convex subset of \(\mathcal{H}\), and

\[
Q_j: \mathcal{H} \to 2^\mathcal{H}: x_j = (\xi_{j,k})_{k \in \mathcal{G}} \mapsto \left(c_j \left(\sum_{k \in \mathcal{G}} \xi_{j,k}\right)\right)_{k \in \mathcal{G}} \quad \text{and} \quad R_j = N_{C_j}.
\tag{2.14}
\]

[b] For every \(i \in \mathcal{N}\), \(s_i \in \mathcal{H}\) and

\[
S^{-1}_i: \mathcal{H} \to 2^\mathcal{H}: v^*_i \mapsto \{s_i\}.
\tag{2.15}
\]
Then (1.6) reduces to the separable multicommodity flow problem; see, e.g., [3, Section 8.3] and the references listed in [3, Section 8.9]. Take \( j \in A, i \in N, \) and \( \gamma \in [0, +\infty[. \) We have \( J_{\gamma R_j} = \text{proj}_{C_j} \) and \( J_{\gamma S_i} = s_i. \) To compute \( J_{\gamma Q_j}, \) define \( L: H \to \mathbb{R}: (\xi_k)_{k \in \mathcal{E}} \mapsto \sum_{k \in \mathcal{E}} \xi_k \) and set \( N = \text{card} \mathcal{G}. \) Then \( L^*: \mathbb{R} \to H: \xi \mapsto (\xi_k)_{k \in \mathcal{E}} \) and, therefore, \( L \circ L^* = N \text{Id}. \) At the same time, by (2.14), \( Q_j = L^* \circ c_j \circ L_j. \) Thus, we derive from [2, Proposition 23.25(iii)] that

\[
(\forall x_j = (\xi_{j,k})_{k \in \mathcal{E}} \in \mathcal{H}) \quad J_{\gamma Q_j} x_j = x_j + \frac{1}{N} \left( J_{N \gamma c_j} (L x_j) - L x_j \right)_{k \in \mathcal{E}} = (\xi_{j,k} + \eta)_{k \in \mathcal{E}},
\]

where \( \eta = \left( J_{N \gamma c_j} \left( \sum_{k \in \mathcal{E}} \xi_{j,k} \right) - \sum_{k \in \mathcal{E}} \xi_{j,k} \right) / N. \) (2.16)

**Example 7** The separable multicommodity flow problem with arc capacity constraints (see, e.g., [3, Section 8.3]) is an instantiation of Example 6 with, for every \( j \in A, c_j = \partial (\phi_j + \iota_{\Omega_j}), \) where \( \phi_j: \mathbb{R} \to ]-\infty, +\infty[ \) is a proper lower semicontinuous convex function and \( \Omega_j \) is a nonempty closed interval in \( \mathbb{R} \) such that \( \Omega_j \cap \text{dom} \phi_j \neq \emptyset. \) In this setting, it follows from [2, Example 23.3 and Proposition 24.47] that

\[
(\forall j \in A) \left( \forall \gamma \in ]0, +\infty[ \right) \quad J_{\gamma c_j} = \text{prox}_{\gamma (\phi_j + \iota_{\Omega_j})} = \text{proj}_{\Omega_j} \circ \text{prox}_{\gamma \phi_j}.
\]

**Remark 8** Consider the standard traffic assignment problem, that is, the special case of Example 6 where \( \forall j \in A \) \( C_j = [0, +\infty[^d. \)

(i) In [1, Example 4.4], this problem was solved by an application of the forward-backward method [1, Theorem 2.8], where it is further assumed that, for every \( j \in A, \) \( \text{dom} \ c_j = \mathbb{R} \) and \( c_j \) is Lipschitzian. However, some common operators found in the literature of traffic assignment [4] do not fulfill this requirement; their resolvents are provided in Examples 9–12.

(ii) The method of [9], which is an application of the Douglas–Rachford algorithm [10], requires to compute the projectors onto polyhedral sets of the form

\[
\left\{ (\xi_j)_{j \in A} \in [0, +\infty[^d \left| (\forall i \in N) \sum_{j \in A} \xi_{i,j} \xi_j = \delta_i \right\}, \right.
\]

where \( (\xi_{i,j})_{i \in N, j \in A} \) are defined in (2.5). This results in solving a subproblem at every iteration because there is no closed-form expression for such projectors.

**Example 9 (Bureau of Public Roads capacity operator)** Let \( (\alpha, \varphi, \theta, p) \in ]0, +\infty[^4 \) and define

\[
c: \mathbb{R} \to \mathbb{R}: \xi \mapsto \begin{cases} \theta \left( 1 + \alpha \left( \frac{\xi}{\theta} \right)^p \right), & \text{if } \xi \geq 0; \\
\theta, & \text{if } \xi < 0. \end{cases}
\]

(2.19)

In addition, let \( \gamma \in ]0, +\infty[ \) and \( \xi \in \mathbb{R}. \) Then the following hold:

(i) Suppose that \( \xi \geq \gamma \theta. \) Then, in terms of the variable \( s \in \mathbb{R}, \) the equation

\[
\frac{\alpha \gamma \theta}{\theta^p} s^p + s + \gamma \theta - \xi = 0
\]

(2.20)

has a unique solution \( \overline{s} \) and \( J_{\gamma \theta} \xi = \overline{s}. \)
Suppose that $\xi < \gamma \theta$. Then $J_{\gamma \xi} \xi = \xi - \gamma \theta$.

Example 10 (Logarithmic capacity operator) Let $\omega \in ]0, +\infty[$, let $\theta \in ]0, +\infty[$, and define

$$c : \mathbb{R} \to 2^\mathbb{R} : \xi \mapsto \begin{cases} \theta + \ln \frac{\omega}{\omega - \xi}, & \text{if } \xi < \omega; \\ \emptyset, & \text{if } \xi \geq \omega. \end{cases}$$

Then

$$\forall \gamma \in ]0, +\infty[ \forall \xi \in \mathbb{R} \quad J_{\gamma \xi} \xi = \omega - \gamma W(\omega \gamma^{-1} \exp(\theta - \xi / \gamma + \omega / \gamma)),$$

where $W$ is the Lambert W-function, that is, the inverse of $[-1, +\infty[ \to [-1/e, +\infty[ : \xi \mapsto \xi \exp(\xi)$.

Example 11 (Traffic Research Corporation capacity operator) Let $(\alpha, \beta, \delta, \omega) \in ]0, +\infty[^4$ and define

$$c : \mathbb{R} \to \mathbb{R} : \xi \mapsto \delta + \alpha (\xi - \omega) + \sqrt{\alpha^2 (\xi - \omega)^2 + \beta}.$$

Then

$$\forall \gamma \in ]0, +\infty[ \forall \xi \in \mathbb{R} \quad J_{\gamma \xi} \xi = \frac{-\sqrt{\gamma^2 \alpha^2 (\xi - \gamma \delta - \omega)^2 + (2 \gamma \alpha + 1) ^2 \beta + \gamma \alpha (\xi - \gamma \delta + \omega) + \xi - \gamma \delta}}{2 \gamma \alpha + 1}.$$  

Example 12 Let $\alpha \in ]1, +\infty[$, let $\theta \in ]0, +\infty[$, let $p \in ]0, +\infty[$, and define

$$c : \mathbb{R} \to \mathbb{R} : \xi \mapsto \theta \alpha^p \xi.$$

Then

$$\forall \gamma \in ]0, +\infty[ \forall \xi \in \mathbb{R} \quad J_{\gamma \xi} \xi = \xi - \frac{W(\gamma \theta \alpha^p \xi \ln \alpha)}{p \ln \alpha}.$$  

Acknowledgments. This work is a part of the author's Ph.D. dissertation and it was supported by the National Science Foundation under grant CCF-1715671. The author thanks his Ph.D. advisor P. L. Combettes for his guidance during this work.

References

[1] H. Attouch, L. M. Briceño-Arias, and P. L. Combettes, A parallel splitting method for coupled monotone inclusions, SIAM J. Control Optim., vol. 48, pp. 3246–3270, 2010.

[2] H. H. Bauschke and P. L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, 2nd ed. Springer, New York, 2017.

[3] D. P. Bertsekas, Network Optimization: Continuous and Discrete Models. Athena Scientific, Belmont, MA, 1998.

[4] D. Branston, Link capacity functions: A review, Transpn. Res., vol. 10, pp. 223–236, 1976.

[5] P. L. Combettes, Systems of structured monotone inclusions: Duality, algorithms, and applications, SIAM J. Optim., vol. 23, pp. 2420–2447, 2013.
[6] P. L. Combettes and J. Eckstein, Asynchronous block-iterative primal-dual decomposition methods for monotone inclusions, *Math. Program.*, vol. B168, pp. 645–672, 2018.

[7] P. L. Combettes and J.-C. Pesquet, Stochastic quasi-Fejér block-coordinate fixed point iterations with random sweeping, *SIAM J. Optim.*, vol. 25, pp. 1221–1248, 2015.

[8] M. Florian and S. Nguyen, An application and validation of equilibrium trip assignment methods, *Trans. Sci.*, vol. 10, pp. 374–390, 1976.

[9] M. Fukushima, The primal Douglas–Rachford splitting algorithm for a class of monotone mappings with application to the traffic equilibrium problem, *Math. Program.*, vol. 72, pp. 1–15, 1996.

[10] P. L. Lions and B. Mercier, Splitting algorithms for the sum of two nonlinear operators, *SIAM J. Numer. Anal.*, vol. 16, pp. 964–979, 1979.

[11] J.-C. Pesquet and A. Repetti, A class of randomized primal-dual algorithms for distributed optimization, *J. Nonlinear Convex Anal.*, vol. 16, pp. 2453–2490, 2015.

[12] R. T. Rockafellar, *Network Flows and Monotropic Optimization*. Wiley, New York, 1984.

[13] R. T. Rockafellar, Monotone relations and network equilibrium, in: *Variational Inequalities and Network Equilibrium Problems*, (F. Giannessi and A. Maugeri, eds.), pp. 271–288. Plenum Press, New York, 1995.