A Model with Propagating Spinons beyond One Dimension.

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(Dated: March 22, 2022)

For the model of frustrated spin-1/2 Heisenberg magnet described in A. A. Nersesyan and A. M. Tsvelik, (Phys. Rev. B67, 024422 (2003)) we calculate correlation functions of staggered magnetization and dimerization. The model is formulated as a collection of antiferromagnetic chains weakly coupled by a frustrated interaction. The calculation done for the case of four chains demonstrates that these functions do not vanish. Since the correlation functions in question factorize into a product of correlation functions of spinon creation and annihilation operators, this constitutes a proof that spinons in this model propagate in the direction perpendicular to the chains.

I. INTRODUCTION

In [1] Alexander Nersesyan and one of the authors presented a proof of existence of fractional quantum number excitations in a model describing a certain frustrated magnet in the number of dimensions greater than one. This magnet consists of spin-1/2 antiferromagnetic Heisenberg chains weakly coupled by a frustrated antiferromagnetic interaction:

\[ H = \sum_{j,n} \left\{ J_\parallel S_{j,n} \cdot S_{j+1,n} + \sum_{\mu=\pm 1} [J_r(n, n + \mu)S_{j,n} + J_d(n, n + \mu) (S_{j+1,n} + S_{j-1,n})] \cdot S_{j,n+\mu} \right\}, \]  

(1)

where \( S_{j,n} \) are spin-1/2 operators, and \( J_\parallel \gg J_r, J_d > 0 \). The interaction pattern of this model resembles the flag of American Confederation which gave the model its name. Fractional quantum number excitations appear when \( J_r(n, n + \mu) = 2J_d(n, n + \mu) \). If this condition is fulfilled, the interaction between staggered components of the magnetization on chains \( n \) and \( n + \mu \) vanishes. The weakness of the interchain coupling allows us to employ the continuous description. In the continuum limit each Heisenberg chain is represented by the SU(2) Wess-Zumino-Novikov-Witten (WZNW) model and the relevant part of the interchain interaction is reduced to the interaction of spin currents with different chirality:

\[ H = \sum_{n=1}^{N} \left[ H_n + \gamma(n, n + \mu) \int dx (J_n \bar{J}_{n+\mu} + J_{n+\mu} \bar{J}_n) \right] \]

(2)

Here \( H_n \) is the SU(2) WZNW Hamiltonian

\[ H_n = \frac{2\pi v}{3} \int dx \left( J_n^2 + : J_n^3 : \right) \]

(3)

where \( v = \pi J_\parallel a_0 / 2 \) is the spin velocity, \( J_n^a(x) \), \( \bar{J}_n^a(x) \) are operators representing holomorphic and antiholomorphic currents belonging to the SU(2) Kac-Moody algebra. The currents with different \( n \) commute. The coupling constant \( \gamma(n, n + \mu) \sim J_r(n, n + \mu) \). Continuum limit Hamiltonian \( \hat{H} \) coincides with the one introduced by Emery, Kivelson and Zachar[3] in the context of theory of stripes. We emphasise that the interchain interaction in this frustrated model remains relevant and generates spectral gaps. Therefore this model does not belong to the class of sliding Luttinger liquid models where soliton excitations remain confined to the chains.

Though the authors of [1] have managed to proof the existence of spin \( S = 1/2 \) excitations in model \( \hat{H} \), it remained unclear whether these excitations are able to propagate in the direction transverse to the chains or remain confined. In the Confederate Flag model the staggered magnetization operator creates two non-interacting spinons (the absence of interaction follows from the fact that these spinons belong to different sectors of the Hamiltonian \( H^+, H^- \), -see below). Therefore one way to resolve the problem of propagation is to calculate correlation functions of the staggered magnetizations between different chains. If these correlation functions do not vanish, the excitations propagate. The difficulty is that such correlation functions are essentially non-perturbative objects (they remain zero in any order of perturbation theory) and to calculate them one has to somehow go beyond perturbation theory. The corresponding methods are available for a finite number of chains (namely \( N = 2, 3, 4 \) where exact solutions are available). Though the most interesting case corresponds to an infinite number of chains \( N \to \infty \), information obtained for finite \( N \) may also provide valuable insights. The calculations of the interchain correlation function of staggered magnetizations done...
For the case of two chains gave a nonvanishing answer, but one may argue that $N = 2$ is too small a number to allow even a qualitative extrapolation to the thermodynamic limit. Similar calculations done for $N = 4$ could not distinguish between intra and interchain correlation functions. In this paper we perform accurate calculations for the case of four chains $N = 4$. We find that the interchain correlation functions do not vanish.

In the continuum limit the staggered magnetization $N = (-1)^{j} S_{j}$ and the dimerization $\varepsilon = (-1)^{j}(S_{j}S_{j+1})$ operators become smooth fields. For the chain number $n$ they are expressed through matrix elements of the $S = 1/2$ primary field of the $n$-th WZNW model:

$$\hat{g}_{\sigma \sigma'}(n; x) = \delta_{\sigma \sigma'} \epsilon(n; x) + i(\sigma)_{\sigma \sigma'} N(n; x)$$  \hspace{1cm} (4)

The action for model $\mathcal{E}$ is local in $g$ and possesses the global $SU(2)\times SU(2)$ symmetry. Namely, the action remains invariant under the following transformations:

$$g_{2n} \rightarrow g_{2n} V, \quad g_{2m+1} \rightarrow V^{+} g_{2m+1},$$
$$g_{2n} \rightarrow U g_{2n}, \quad g_{2m+1} \rightarrow g_{2m+1} U^{+},$$  \hspace{1cm} (5)

where $V, U$ are coordinate-independent $SU(2)$ matrices. This symmetry dictates the following form of the two-point correlation functions:

$$\langle \langle g_{\sigma_{1} \sigma_{2}}(r, x; 2n) g_{\sigma_{3} \sigma_{4}}(0, 0; 2m) \rangle \rangle = \delta_{\sigma_{1} \sigma_{3}} \delta_{\sigma_{2} \sigma_{4}} D_{2n, 2m}(r, x)$$
$$\langle \langle g_{\sigma_{1} \sigma_{2}}(r, x; 2n) g_{\sigma_{3} \sigma_{4}}(0, 0; 2m + 1) \rangle \rangle = \delta_{\sigma_{1} \sigma_{3}} \delta_{\sigma_{2} \sigma_{4}} D_{2n, 2m+1}(r, x)$$  \hspace{1cm} (6)

where $D$ do not contain spin indices. Substituting Eq.(4) in the above equations we obtain the following important relation between correlation functions of staggered energy density and staggered magnetization:

$$\langle \langle \varepsilon_{2n}(r, x) \varepsilon_{2m}(0, 0) \rangle \rangle = \langle \langle N_{2n}^{a}(r, x) N_{2m}^{a}(0, 0) \rangle \rangle$$
$$\langle \langle \varepsilon_{2n+1}(r, x) \varepsilon_{2m+1}(0, 0) \rangle \rangle = \langle \langle N_{2n+1}^{a}(r, x) N_{2m+1}^{a}(0, 0) \rangle \rangle$$
$$\langle \langle \varepsilon_{2n}(r, x) \varepsilon_{2m+1}(0, 0) \rangle \rangle = -\langle \langle N_{2n}^{a}(r, x) N_{2m+1}^{a}(0, 0) \rangle \rangle$$  \hspace{1cm} (7)

Now let us recall the important feature of Hamiltonian $\mathcal{H}$: as was noticed in [1], it separates into a sum of two commuting parts: $H = H^{+} + H^{-}$ (such structure of the Hamiltonian was first observed in the context of Kondo lattice in [2]). Therefore eigenstates are separated into two sectors with different parity. The Hamiltonian density in the plus parity sector is

$$\mathcal{H}^{+} = \frac{2\pi v}{3} (\hat{J}_{1}^{2} : + \hat{J}_{3}^{2} : + \hat{J}_{2}^{2} : + \hat{J}_{4}^{2} : ) + (\lambda \hat{J}_{1} + \lambda' \hat{J}_{3})(\lambda \hat{J}_{2} + \lambda' \hat{J}_{4})$$  \hspace{1cm} (8)

For the purposes of this paper we find it convenient to consider different coupling constants $\lambda \neq \lambda'$ such that the difference between these coupling constants is small $(|\lambda - \lambda'| \ll \lambda)$. Physically this means that we still consider the frustrated interchain interactions, but allow their amplitudes to vary between different chains.

By using the Abelian bosonization procedure for the $SU(1, 2)$ currents one can write the Hamiltonians $H^{\pm}$ in terms of bosonic fields $\varphi, \bar{\varphi}$ living on each chain (see [1] for the details). In this representation fields $\varphi_{1,3}$ ($\bar{\varphi}_{1,3}$) interact with the fields $\varphi_{2,4}$ ($\bar{\varphi}_{2,4}$) such that

$$H = H^{+}[\varphi_{1,3}; \bar{\varphi}_{2,4}] + H^{-}[\bar{\varphi}_{1,3}; \varphi_{2,4}]$$  \hspace{1cm} (9)

The convenience of this representation is related to the fact that it allows to represent $\hat{g}(j)$ in a factorized form:

$$\hat{g}_{\sigma \sigma'} = \frac{1}{\sqrt{2}} C_{\sigma \sigma'} : \exp[-i\sqrt{2}\pi(\sigma \varphi + \sigma' \bar{\varphi})] : = C_{\sigma \sigma'} z_{\sigma} \bar{z}_{\sigma'},$$
$$C_{\sigma \sigma'} = e^{i(1-\sigma \sigma')/4},$$  \hspace{1cm} (10)

where

$$z_{\sigma} = \exp[i\sigma \sqrt{2}\varphi], \quad \bar{z}_{\sigma} = \exp[-i\sigma \sqrt{2}\bar{\varphi}], \quad (\sigma = \pm 1).$$  \hspace{1cm} (11)

Eqs. (10) select a set of potentially non-vanishing interchain correlators:

$$G_{13} = \langle \langle e^{i\sqrt{2}\varphi_{1}(1)} e^{-i\sqrt{2}\varphi_{3}(3)} \rangle \rangle, \quad \bar{G}_{13} = \langle \langle e^{i\sqrt{2}\bar{\varphi}_{1}(1)} e^{-i\sqrt{2}\bar{\varphi}_{3}(3)} \rangle \rangle$$
$$D_{12} = \langle \langle e^{i\sqrt{2}\varphi_{1}(1)} e^{i\sqrt{2}\varphi_{2}(2)} \rangle \rangle, \quad \bar{D}_{12} = \langle \langle e^{i\sqrt{2}\bar{\varphi}_{1}(1)} e^{-i\sqrt{2}\bar{\varphi}_{2}(2)} \rangle \rangle$$  \hspace{1cm} (12)
From these correlators one can obtain correlation functions of both vector N and scalar ε staggered fields:
\[
\langle \epsilon_{2n}(\tau, x)\epsilon_{2m}(0, 0) \rangle = |G_{2n, 2m}(\tau, x)|^2
\]
\[
\langle \epsilon_{2n}(\tau, x)\epsilon_{2m+1}(0, 0) \rangle = |D_{2n, 2m+1}(\tau, x)|^2
\]
(13)

According to these formulas spins on neighboring chains are oriented antiferromagnetically. We emphasise that this conclusion follows just from the symmetry arguments and the chiral decoupling of the operators and the Hamiltonian.

II. EXACT SOLUTION OF MODEL

The exact solution of model was obtained in . The spectrum includes heavy solitons and antisolitons with mass M and a light singlet Majorana fermion with mass m. The mass ratio is \(m/M \sim (\lambda-\lambda')\) and vanishes in the limit of equal couplings. This is the limit which was studied in , it corresponds to the uniform interchain interactions and periodic boundary conditions in the transverse direction. The presence of such periodicity constitutes an additional symmetry whose presence leads to vanishing of certain correlation functions. Therefore we prefer to keep \(m/M\) finite.

The two-particle scattering matrix of model is given by
\[
S(\theta) = \left( \begin{array}{cc}
S[su(2); \theta]_{\sigma_1, \sigma_2} & \delta_{\sigma, \bar{\sigma}} e^{\theta/2 - i} \\
\delta_{\sigma, \bar{\sigma}} e^{\theta/2 + i} & -1
\end{array} \right)
\]
(14)

where \(S[su(2)]\) is a 2×2 scattering matrix of the SU(2) Thirring model.

The Thermodynamic Bethe Ansatz (TBA) are
\[
F/L = -\frac{Tm}{2\pi} \int d\theta \cosh \theta \ln[1 + e^{\epsilon_1(\theta)/T}] - \frac{TM}{2\pi} \int d\theta \cosh \theta \ln[1 + e^{\epsilon_2(\theta)/T}]
\]
(15)

\[
\epsilon_n(\theta) = Ts \ast \ln[1 + e^{\epsilon_{n-1}(\theta)/T}] [1 + e^{\epsilon_{n+1}(\theta)/T}] - \delta_{n,1} m \cosh \theta - \delta_{n,2} M \cosh \theta
\]
(16)

These equations are valid for \(\lambda' \approx \lambda\).

It is worth mentioning that model is generated in the relativistic limit as the spin sector of the following fermionic model:
\[
H = \int dx \left[ \sum_{j=1}^{2} C_{j\sigma}(-\partial_x^2 - k_F^2)C_{j\sigma} + t(C_{1\sigma}^+ C_{2\sigma} + C_{2\sigma}^+ C_{1\sigma}) - gC_{j\sigma}^+ C_{k\sigma}C_{k\sigma}^+ C_{j\sigma} \right],
\]
\[
\sigma = \pm 1/2, j = 1, 2; \quad g > 0
\]
(17)

From the solution of this model obtained in it can be extracted that
\[
\frac{m}{M} = \frac{\pi}{8} \left( \frac{t}{\epsilon_F} \right)^2 \ln^2(\epsilon_F/M)
\]
\[
M = \frac{4}{\pi^2} k_F g \exp(-\pi k_F/g)
\]
(18)

which provides a direct relationship with model.

III. THE CORRELATION FUNCTIONS

We shall calculate the correlation functions using the formfactor approach. In this approach one calculates matrix elements (formfactors) of various operators using their transformation properties under various symmetries of the problem. General information about the method can be obtained from various review articles.
Let us consider operators \( \exp[i\sqrt{2\pi}\varphi_1] \) and \( \exp[i\sqrt{2\pi}\varphi_3] \). We will be interested only in the limit \( m \ll M \), therefore it is sufficient to consider the states with one soliton (antisoliton) and arbitrary number of light particles. In this case it is reasonable to suggest the following formfactor expansions:

\[
\exp[i\sqrt{2\pi}\varphi_1]|0> = Q_{\text{even}} + Q_{\text{odd}}
\]
\[
\exp[i\sqrt{2\pi}\varphi_3]|0> = Q_{\text{even}} - Q_{\text{odd}}
\]

where \( Q_{\text{even}} (Q_{\text{odd}}) \) has matrix elements with the states with of one heavy soliton and even (odd) number of light Majorana fermions. Notice that the on-chain correlation functions for both operators are the same as it must be. The problem is simplified by the fact that the Majorana fermions have a non-trivial scattering matrix only with the heavy particle. Otherwise the dependence of the matrix element on the Majorana fermion rapidities is like for the Ising model. Using the results of [3], where a systems with a similar S-matrix was studied, we obtain for the following expressions:

\[
<s(\beta); \chi_1(\beta_1), \ldots \chi_{2n}(\beta_{2n})|Q_{\text{even}} = (C^+)^{1/2}e^{\beta/4}\prod_{j=1}^{2n}\psi^-(\beta - \beta_j)\prod_{i,j}\tanh(\beta_{ij}/2)
\]

\[
<s(\beta); \chi_1(\beta_1), \ldots \chi_{2n+1}(\beta_{2n+1})|Q_{\text{odd}} = C^{-1/2}e^{\beta/4}\prod_{j=1}^{2n+1}\psi^-(\beta - \beta_j)\prod_{i,j}\tanh(\beta_{ij}/2)
\]

where \( \beta \) is the soliton’s rapidity and \( \beta_j \) stand for the rapidities of the light particles. \( C^\pm \) are normalization factors to be determined later. The function \( \psi^{(-)}(\beta) \) satisfies the following equation

\[
\psi^{(-)}(\beta)\psi^{(-)}(\beta - i\pi) = \frac{1}{1 + ie^{\beta}}
\]

and is given by

\[
\psi^{(-)}(\beta) = e^{-\beta/4}\psi^{(0)}(\beta),
\]

\[
\psi^{(0)}(\beta) = 2^{-3/4}\exp\left\{-\int_0^\infty \frac{d\omega}{\omega} \frac{2\sin^2\left[(\beta + i\pi)\omega/2\right] + \sinh^2(\pi\omega/2)}{2\sinh(\pi\omega)\cosh(\pi\omega/2)}\right\}
\]

As follows from [22], this function has the following asymptotics:

\[
\psi^{(-)}(\beta \to +\infty) \to i e^{-\beta/2}
\]
\[
\psi^{(-)}(\beta \to -\infty) \to 1
\]

The operators \( \exp[i\sqrt{2\pi}\varphi(2,4)] \) are expressed in a similar way, but with all rapidities having the opposite sign. Therefore their expansions include the function \( \psi^{(\pm)}(\beta) = \psi^{(-)}(-\beta) \) which decays at \( -\infty \) and approaches \( 1 \) at \( \beta \to +\infty \).

Let us briefly comment on expressions [20 21]. The operators under consideration have Lorentz spin \( 1/4 \). This dictates that under the Lorentz transformation \( \beta \to \beta + \theta, \beta_j \to \beta_j + \theta \) the matrix element of that operator must acquire a factor \( \exp(\theta/4) \). As we see from [20 21], this property is fulfilled. Another property is that the formfactor is multiplied on the S-matrix \( S(\beta, \beta') \) whenever two particles with rapidities \( \beta \) and \( \beta' \) are interchanged. Since \( \tanh(\beta_{ij}/2) \) is an odd function, this is obviously fulfilled for the Majorana fermions. The function \( \psi \) plays the same role for the interchange of a Majorana fermion and the heavy particle. The choice of asymptotics of functions \( \psi^{(\pm)} \) is determined by the fact that in on small distances bosonic exponents (19) become (anti)holomorphic fields.

Substituting [20 21] into Eqs. [19] and performing certain algebraic manipulations (see Appendix 1), we arrive at the following expressions for the correlation functions on the same and on different chains \( r^2 = r^2 + x^2 \), we put \( v = 1 \):

\[
G_{pq} = \langle \langle e^{i\sqrt{2\pi}\varphi(p)} e^{-i\sqrt{2\pi}\varphi(q)} \rangle \rangle = \left( \frac{\tau - ir}{\tau + ir} \right)^{1/4} C(M^4m)^{1/8} F_{pq}(r)
\]

\[
D_{pq} = \langle \langle e^{i\sqrt{2\pi}\varphi(p)} e^{i\sqrt{2\pi}\varphi(q)} \rangle \rangle = C(M^4m)^{1/8} F_{pq}(r)
\]
where $C$ is a numerical coefficient, $p, q = 1, 3$ for $G$ and $p = 1, 3; q = 2, 4$ for $D$.

$$\mathcal{F}_{11}(r) = \mathcal{F}_{33}(r) =$$

$$= \int \frac{d\beta}{e^{\beta^2}} e^{-M r} \cosh \beta \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \int d\beta \prod_{j} \left| \psi(-) (\beta - \beta_j) \right|^2 e^{-m r \cosh \beta} \prod_{i>j} \cosh(\beta_{ij}/2) \right\}$$

$$\mathcal{F}_{12}(r) = \mathcal{F}_{24}(r) =$$

$$= \int \frac{d\beta}{e^{\beta^2}} e^{-M r} \cosh \beta \exp \left\{ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int d\beta \prod_{j} \left| \psi(-) (\beta - \beta_j) \right|^2 e^{-m r \cosh \beta} \prod_{i>j} \cosh(\beta_{ij}/2) \right\}$$

For the correlation functions on neighboring chains (see Eqs. [13]) $(p = 1, 3; q = 2, 4)$ we have

$$D_{12}(r) = \int \frac{d\beta}{e^{\beta^2}} e^{-M r} \cosh \beta \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \int d\beta \prod_{j} \left| \psi(0) (\beta - \beta_j) \right|^2 e^{-m r \cosh \beta} \prod_{i>j} \cosh(\beta_{ij}/2) \right\}$$

$$D_{14}(r) = \int \frac{d\beta}{e^{\beta^2}} e^{-M r} \cosh \beta \exp \left\{ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int d\beta \prod_{j} \left| \psi(0) (\beta - \beta_j) \right|^2 e^{-m r \cosh \beta} \prod_{i>j} \cosh(\beta_{ij}/2) \right\}$$

The normalization factor $(M^3 m)^{1/8}$ in Eqs. [26] is dictated by the facts that (i) the single chain correlation functions do not vanish in the limit $m \to 0$, (ii) the scaling dimension of the operator is 1/4. In the limit $Mr >> 1, m r << 1$ the integral in $\beta$ converges at $|\beta| \sim (Mr)^{-1/2}$. Since $|\psi(0) (\beta_j)|^2 \sim \exp(-|\beta_j|/2)$, the integrals in $\beta_j$ converge even at $m r = 0$. As result we get

$$D_{12}(r) = C(M^3 m)^{1/8} K_0(Mr), \quad D_{14}(r) = C(M^3 m)^{1/8} K_0(Mr)$$

where is $C$ the numerical constant which remains underdetermined. To calculate the other asymptotics is a more complicated task. The calculations are done in Appendix 2; the result is

$$\mathcal{F}_{11}(r) = C_{11} M^{-1/4} e^{-Mr} m^{-5/8}$$

$$\mathcal{F}_{13}(r) = C_{13} m^{1/2} M^{-3/4} r^{-1/8} e^{-Mr}$$

where $C$ are again underdetermined numerical constants. From here we derive the following estimates for the singularities in the imaginary part of the staggered magnetic susceptibility (dimerization) at $s = 2M, (s^2 = \omega^2 - q^2)$:

$$\Im m \chi_{11} (s) \sim (s - 2M)^{-1/4}, \quad \Im m \chi_{12} \sim -m^{1/4}(s - 2M)^{-1/2}, \quad \Im m \chi_{13} (s) \sim m M^{-5/4} \theta(s - 2M)$$

These expressions are valid at $s - 2M >> m$ so that $\Im m \chi_{12}$ cannot really become greater than $\Im m \chi_{11}$.

**IV. CONCLUSIONS**

Looking at expressions for the interchain correlation functions, we see that they are proportional to powers of the particle masses. These masses are generated dynamically in the theory, they are exponentially small in the inverse coupling constant and therefore the interchain tunneling of the solitons is an essentially non-perturbative process. This conclusion coincides with the result for two chains [1]. The four chain case, however, introduces a new feature: the presence of the singlet particle. It is interesting that the prefactor in the interchain correlation functions contains both masses and vanishes when the mass of the singlet particle goes to zero (this occurs for periodic boundary conditions in the transverse direction; the case considered in [1]). This indicates that the singlet sector plays an important role in the interchain propagation of solitons probably providing conditions for an uninhibited tunneling. We remind the reader that the fraction of the Hilbert space occupied by single excitations grows with the number of chains [1] and these excitations will certainly play an important role in the thermodynamic limit determining interchain soliton propagation.

In conclusion we would like to emphasise a curious symmetry between correlations functions of the staggered energy density and magnetization [12] existing in Confederate Flag magnet. This symmetry units $\epsilon$ and $\mathbf{N}$ into a four-component vector giving rise to an analogy between this model and models of non-collinear magnets. As it was pointed out in [8], spinons naturally exist in disordered non-collinear magnets and it looks likely that Confederate Flag model provides a microscopic realization of that scenario.
V. ACKNOWLEDGEMENTS

We are grateful to Alexander Its for valuable discussions and references. AMT acknowledges the support from US DOE under contract number DE-AC02-98 CH 10886. FS acknowledges the support from the Institute for Strongly Correlated and Complex Systems at BNL and from INTAS grant number 00-00055.

APPENDIX 1

\[ G_{11} = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\beta e^{\beta/2} e^{-Mr \cosh \beta - Mr \sinh \beta} \prod_{j=1}^{n} d\beta_j \prod_{i<j}^{n} |\psi(-) (\beta - \beta_j)|^2 e^{-mr \cosh \beta_j - ixm \sinh \beta_j} \prod_{i<j} \tanh^2(\beta_{ij}/2) \]

\[ G_{13} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d\beta e^{\beta/2} e^{-Mr \cosh \beta - Mr \sinh \beta} \prod_{j=1}^{n} d\beta_j \prod_{i<j}^{n} |\psi(-) (\beta - \beta_j)|^2 e^{-mr \cosh \beta_j - ixm \sinh \beta_j} \prod_{i<j} \tanh^2(\beta_{ij}/2) \]

(32)

By the shift of the integration contours one obtains the following expressions:

\[ G_{pq}(\tau, x) = \left(\frac{\tau - ix}{\tau + ix}\right)^{1/4} F_{pq}(r), \quad r^2 = \tau^2 + x^2 \]

(33)

\[ F_{11}(r) = F_{33}(r) = \]

\[ \sum_{n=0}^{\infty} \frac{1}{n!} \int d\beta \prod_{j=1}^{n} d\beta_j e^{\beta/2} e^{-Mr \cosh \beta} \prod_{i<j}^{n} |\psi(-) (\beta - \beta_j)|^2 e^{-mr \cosh \beta_j} \prod_{i<j} \tanh^2(\beta_{ij}/2) \]

\[ = \int d\beta e^{\beta/2} e^{-Mr \cosh \beta} \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{n!} \int d\beta \prod_{j=1}^{n} |\psi(-) (\beta - \beta_j)|^2 e^{-mr \cosh \beta_j} \prod_{i<j} \tanh^2(\beta_{ij}/2) \right\} \]

(34)

\[ F_{13}(r) = F_{31}(r) = \]

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d\beta \prod_{j=1}^{n} d\beta_j e^{\beta/2} e^{-Mr \cosh \beta} \prod_{i<j}^{n} |\psi(-) (\beta - \beta_j)|^2 e^{-mr \cosh \beta_j} \prod_{i<j} \tanh^2(\beta_{ij}/2) \]

\[ = \int d\beta e^{\beta/2} e^{-Mr \cosh \beta} \exp \left\{ \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int d\beta \prod_{j=1}^{n} |\psi(-) (\beta - \beta_j)|^2 e^{-mr \cosh \beta_j} \prod_{i<j} \tanh^2(\beta_{ij}/2) \right\} \]

(35)

Here we have used the fact that the correlation functions are equal to the determinants of the Fredholm operators (see Appendix 2) and used Eq. (37). The correlation functions on neighboring chains are given by similar integrals:

\[ D_{pq} = \langle e^{\int \sqrt{2\beta} \varphi(1)} e^{\int \sqrt{2\beta} \varphi(2)} \rangle = F_{pq}(r) \]

(36)

\[ F_{12}(r) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\beta \prod_{j=1}^{n} d\beta_j e^{-Mr \cosh \beta} \prod_{i<j}^{n} \psi(-) (\beta - \beta_j) \psi(+) (\beta - \beta_j) e^{-mr \cosh \beta_j} \prod_{i<j} \tanh^2(\beta_{ij}/2) \]

\[ F_{14}(r) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d\beta \prod_{j=1}^{n} d\beta_j e^{-Mr \cosh \beta} \prod_{i<j}^{n} \psi(-) (\beta - \beta_j) \psi(+) (\beta - \beta_j) e^{-mr \cosh \beta_j} \prod_{i<j} \tanh^2(\beta_{ij}/2) \]

Taking into account that \( \psi^+(\beta) \psi^-(\beta) = |\psi(0)(\beta)|^2 \) and using Eq. (37) we arrive at Eqs. (28).

APPENDIX 2

In this Appendix we calculate the asymptotics of correlation functions \( F_{pq}(r) \). First, let us consider as an example the scaling of the Ising model. It is well-known that the correlation functions of the order and disorder parameter operators for this model can be expressed as follows:

\[ \langle \mu(x) \mu(0) \rangle = G_+ (mr) + G_- (mr), \]

\[ \langle \sigma(x) \sigma(0) \rangle = G_+ (mr) - G_- (mr), \]
The functions $G_\pm$ are determinants of the Fredholm operators:

$$G_\pm(\rho) = \det (I \pm K)$$

where $K$ is the integral operator with the kernel:

$$K(\beta_1, \beta_2) = e^{-\frac{1}{2}\rho(\cosh \beta_1 + \cosh \beta_2)} \frac{1}{\cosh \frac{1}{2}(\beta_1 - \beta_2)}$$

From here one can conclude that $G_\pm$ satisfy the Painlevé equation [10] and perform a rather detailed analysis of these functions. The analysis of asymptotic behavior is a simpler task however and can be performed as follows. Using the formula

$$\log(\det A) = \text{Tr}(\log A) \quad (37)$$

one obtains:

$$\log (G_\pm(\rho)) = \sum_{n=1}^{\infty} (\pm)^n \frac{1}{n} \int_{-\infty}^{\infty} d\beta_1 \cdots \int_{-\infty}^{\infty} d\beta_n e^{-\rho \sum \cosh \beta_j} \prod_{i<j} \frac{1}{\cosh \frac{1}{2}(\beta_i - \beta_j)}$$

where $\beta_{n+1} \equiv \beta_1$. Consider one of integrals:

$$\int_{-\infty}^{\infty} d\beta_1 \cdots \int_{-\infty}^{\infty} d\beta_n e^{-\rho \sum \cosh \beta_j} \prod_{i=1}^{n} \frac{1}{\cosh \frac{1}{2}(\beta_i - \beta_{i+1})} \quad (38)$$

We are interested in the asymptotics for $\rho \to 0$ where the integral diverges. Let us introduce new variables:

$$\Delta_i = \beta_i - \beta_{i+1}, \quad i = 1, \ldots, n-1,$$

$$\omega = \sum_{j=1}^{N} \cosh(\beta_j) \quad (39)$$

Then the integral becomes

$$\int_{-\infty}^{\infty} d\Delta_1 \cdots \int_{-\infty}^{\infty} d\Delta_{n-1} 2 \int_{D(\Delta)} \frac{e^{-\rho \omega}}{\sqrt{\omega^2 - D(\Delta)^2}} \prod_{j=1}^{n-1} \frac{1}{\cosh \left( \frac{\Delta_j}{2} \right)} \frac{1}{\cosh \left( \frac{\sum \Delta_j}{2} \right)}$$

where

$$D(\Delta_1, \ldots, \Delta_{n-1}) = \left( \prod e^{\beta_j} \right) \left( \prod e^{-\beta_j} \right)$$

It is clear that the integrals over $\Delta_j$ are always rapidly converging while the integral over $\omega$ diverges at $\rho \to 0$. The estimation of leading contribution is straightforward. We give the final result for $G_+$ which diverges at $\rho \to 0$:

$$\log (G_+(\rho)) \simeq - \left[ 2 \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} d\Delta_1 \cdots \int_{-\infty}^{\infty} d\Delta_{n-1} \prod_{j=1}^{n-1} \frac{1}{\cosh \left( \frac{\Delta_j}{2} \right)} \frac{1}{\cosh \left( \frac{\sum \Delta_j}{2} \right)} \right] \log(\rho) \quad (40)$$

The expression in square brackets is the anomalous dimension. It equals $\frac{1}{4}$. This value is known from many sources, but actually even direct summation of series is possible. In a similar way we obtain

$$\log (G_-(\rho)) \sim \rho^{3/4} \quad (41)$$
Now let us turn to the case considered in the present paper. 

\[ \mathcal{F}_\pm (\rho) = \]

\[ = \sum_{n=0}^{\infty} (\pm)^n \frac{1}{n!} \int_{-\infty}^{\infty} d\beta_1 \cdots \int_{-\infty}^{\infty} d\beta_n e^{-\rho \sum \cosh \beta_j} \prod_{i<j} \tanh^{2} \frac{1}{2} (\beta_i - \beta_j) \prod_{j=1}^{n} |\psi^{(-)}(\beta_j)|^2 \]

The functions \( \mathcal{F}_\pm \) are determinants of the integral operators

\[ \mathcal{F}_\pm (\rho) = \det (I \pm L) \]

where \( L \) is the integral operator with the kernel:

\[ L(\beta_1, \beta_2) = e^{-\frac{1}{2} \rho (\cosh \beta_1 + \cosh \beta_2)} \frac{1}{\cosh \frac{1}{2}(\beta_1 - \beta_2)} |\psi^{(-)}(\beta_1)\psi^{(-)}(\beta_2)| \]

So, similarly to the Ising case the logarithms of these determinants can be written as

\[ \log (\mathcal{F}_\pm (\rho)) = \]

\[ = \sum_{n=1}^{\infty} (\pm)^n \frac{1}{n} \int_{-\infty}^{\infty} d\beta_1 \cdots \int_{-\infty}^{\infty} d\beta_n e^{-\rho \sum \cosh \beta_j} \prod_{i=1}^{n} \frac{1}{\cosh^{2} \frac{1}{2} (\beta_i - \beta_{i+1})} |\psi^{(-)}(\beta_j)|^2 \]

Consider the integral

\[ \int_{-\infty}^{\infty} d\beta_1 \cdots \int_{-\infty}^{\infty} d\beta_n e^{-\rho \sum \cosh \beta_j} \prod_{i=1}^{n} \frac{1}{\cosh^{2} \frac{1}{2} (\beta_i - \beta_{i+1})} |\psi^{(-)}(\beta_j)|^2 \]

and make change of variables \([39]\). The difference with the Ising case is due to the fact that we need to express \( \beta_j \) in terms of the new variables and to substitute them into \( \psi \). We have:

\[ \beta_j = \log \left( \omega \mp \sqrt{\omega^2 - D^2(\Delta)} \right) - \log \left( e^{-\beta_j} \sum_k e^{\beta_k} \right) \]  \hspace{1cm} (42)

where the last term depends only on \( \Delta \)'s. The integral over \( \omega \) is taken between the limits \( D(\Delta) \) and \( \infty \) over two branches: one the first one one takes + sign in \([42]\), and on the second one the minus sign. The part divergent at \( \rho \to 0 \) comes from the region where \(- \log(\omega) \) is large. It is not the second branch of the integral over \( \omega \), since \(|\psi^{(-)}(\beta)\)| is rapidly decreasing as \( \beta \to +\infty \). So, only the first branch contributes, and, since \(|\psi^{(-)}(\beta)\)| rapidly approaches 1 as \( \beta \to -\infty \), we can replace all \(|\psi^{(-)}(\beta_j)\)| by 1. The result of this considerations is that the asymptotics of the integral is \( \frac{1}{8} \) of what we had in the Ising case. So,

\[ \log (\mathcal{F}_+ (\rho)) \simeq -\frac{1}{8} \log(\rho), \quad \log (\mathcal{F}_- (\rho)) \simeq \frac{3}{8} \log(\rho) \]  \hspace{1cm} (43)

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