HOMOGENEOUS UNIVERSAL H-FIELDS

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Abstract. We consider derivations \( \partial \) on Conway’s field \( \mathbb{No} \) of surreal numbers such that the ordered differential field \((\mathbb{No}, \partial)\) has constant field \( \mathbb{R} \) and is a model of the model companion of the theory of \( H \)-fields with small derivation. We show that this determines \((\mathbb{No}, \partial)\) uniquely up to isomorphism, and that this structure is absolutely homogeneous universal for models of this theory with constant field \( \mathbb{R} \).

Aschenbrenner and van den Dries [1] introduced \( H \)-fields in order to formalize some basic first-order properties of Hardy fields in their role of ordered and valued differential fields. Hardy fields containing \( \mathbb{R} \) are \( H \)-fields, and so is the system \( T \) of transseries. In [2], Aschenbrenner, van den Dries, and van der Hoeven (ADH) proved that the theory of \( H \)-fields has a model companion whose models are the \( H \)-fields that are Liouville closed, \( \omega \)-free, and newtonian. Adding to these axioms for the model companion the axiom that the derivation is small yields a complete theory \( T \) in the language \( \mathcal{L} = \{ 0, 1, +, \cdot, \leq, \preceq, \partial \} \) of ordered valued differential fields. Thus \( T \) is complete as well as model complete. Another key result from [2] is that \( T \) is a model of \( T \). See [2, Introduction] for the relevant definitions.

Using an idea from Schmeling’s thesis [11] due to van der Hoeven, Berarducci and Mantova [5] constructed so-called surreal derivations on Conway’s ordered field \( \mathbb{No} \) of surreal numbers [7]; even a “simplest” one, \( \partial_{BM} \), that makes \((\mathbb{No}, \partial_{BM})\) an \( H \)-field with small derivation and constant field \( \mathbb{R} \). They proved also that this \( H \)-field is Liouville closed. ADH [3] subsequently showed that \((\mathbb{No}, \partial_{BM})\) is a model of \( T \) that is universal with respect to \( H \)-fields with small derivation and constant field \( \mathbb{R} \): every such \( H \)-field, including each Hardy field containing \( \mathbb{R} \), can be embedded as an ordered differential field into \((\mathbb{No}, \partial_{BM})\). The purpose of this note is to point out that in the course of establishing the just-said result, [3] proves almost enough to obtain the following:

Theorem. Let \( \partial \) be any derivation on \( \mathbb{No} \) with constant field \( \mathbb{R} \) such that \((\mathbb{No}, \partial)\) is a model of \( T \). Then \((\mathbb{No}, \partial)\) is up to isomorphism the unique model of \( T \) with constant field \( \mathbb{R} \) that is absolutely homogeneous universal with respect to models of \( T \) with constant field \( \mathbb{R} \).

The uniqueness gives \((\mathbb{No}, \partial) \cong (\mathbb{No}, \partial_{BM})\). Part of the interest of the theorem lies in the circumstance that \( \partial_{BM} \) seems to take the “wrong” values on some infinite iterates of the exponential function applied to \( \omega \); for more on this, see [4, 6]. We do expect there is an optimal surreal derivation—better than \( \partial_{BM} \)—that also satisfies the hypothesis of the theorem, and thus its conclusion.

Let \( \partial \) be as in the theorem. That \((\mathbb{No}, \partial)\) is absolutely universal with respect to models of \( T \) with constant field \( \mathbb{R} \) means that every model of \( T \) with constant field

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$\mathbb{R}$ (whose universe is a set or a proper class) can be embedded in $(\mathbb{No}, \mathcal{B})$. That it is absolutely homogeneous with respect to models of $T$ with constant field $\mathbb{R}$ means that every isomorphism between substructures of $(\mathbb{No}, \mathcal{B})$ that are set-models of $T$ with constant field $\mathbb{R}$ extends to an automorphism of $(\mathbb{No}, \mathcal{B})$.

Our set theory here is von Neumann-Bernays-Gödel set theory with Global Choice (NBG), a conservative extension of ZFC in which all proper classes are in bijective correspondence with the class $\text{On}$ of all ordinals. By “set-model” (“class-model”) we mean a model whose universe is a set (a proper class). By “cardinal” we mean below “set-cardinal” and we let $\kappa$ range over cardinals.

To establish the theorem we follow the proof of [3, Theorem 3], which deals with the case $\mathcal{B} = \mathcal{B}_{\text{BM}}$. To handle arbitrary $\mathcal{B}$ we use some extra lemmas. The first one slightly extends [3, Lemma 5.3], which considers only regular $\kappa$. The proof of that lemma goes through if we replace $\kappa$ at various places by its cofinality $\text{cf}(\kappa)$.

**Lemma 1.** Let $\kappa$ be uncountable. Then the underlying ordered sets of $\mathbb{No}(\kappa)$ and $v(\mathbb{No}(\kappa))$ are $\text{cf}(\kappa)$-saturated.

**Lemma 2.** Let $L$ be a countable (one-sorted) language and $\mathbb{No}_L$ an $L$-structure with universe $\mathbb{No}$. Then there are cardinals $\kappa$ of arbitrarily large cofinality such that $\mathbb{No}(\kappa)$ is the underlying set of an elementary substructure of $\mathbb{No}_L$.

**Proof.** By Skolemizing we arrange that $\text{Th}(\mathbb{No}_L)$ has built-in Skolem functions, so any substructure of $\mathbb{No}_L$ is an elementary substructure. Let $\kappa$ be an infinite regular cardinal. We build in the usual way simultaneously by transfinite recursion a strictly increasing sequence $(\kappa_\alpha)_{\alpha < \kappa}$ of infinite cardinals and an elementary chain $(K_\alpha)_{\alpha < \kappa}$ of elementary substructures of $\mathbb{No}_L$ such that for all $\alpha < \kappa$,

1. $\mathbb{No}(\kappa_\alpha) \subseteq K_\alpha \subseteq \mathbb{No}(\kappa_{\alpha+1})$, where $K_\alpha$ denotes also its underlying set.
2. if $\alpha$ is an infinite limit ordinal, then $\kappa_\alpha = \sup_{\beta < \alpha} \kappa_\beta$ and $K_\alpha = \bigcup_{\beta < \alpha} K_\beta$.

Then $\mathbb{No}(\kappa_\infty)$ with $\kappa_\infty := \sup_{\alpha < \kappa} \kappa_\alpha$ is the underlying set of the elementary substructure $\bigcup_{\alpha < \kappa} K_\alpha$ of $\mathbb{No}_L$, and $\kappa_\infty$ has cofinality $\kappa$. \hfill $\Box$

In the next two lemmas “$H$-field” should be read as “$H$-field whose universe is a set”. We note that any embedding between $H$-fields with common constant field $\mathbb{R}$ is automatically the identity on $\mathbb{R}$. In the rest of the paper we fix a class-model $(\mathbb{No}, \mathcal{B})$ of $T$ with constant field $\mathbb{R}$. Here is the relevant analogue of the Claim in the proof of [3, Theorem 3]:

**Lemma 3.** Let $E \subseteq K$ be an extension of $\omega$-free $H$-fields with $\mathbb{R}$ as their common constant field, and let $i : E \rightarrow (\mathbb{No}, \mathcal{B})$ be an embedding. Then $i$ extends to an embedding $K \rightarrow (\mathbb{No}, \mathcal{B})$.

**Proof.** First extend $K$ to make it a model of $T$; by [2, 16.4.1 and 14.5.10] this can be done without changing its constant field. By Lemma 2 we can take an uncountable cardinal $\kappa$ such that $\text{cf}(\kappa) > \text{card}(K)$. $\mathbb{No}(\kappa)$ underlies an elementary substructure $L$ of $(\mathbb{No}, \mathcal{B})$ and $i(E) \subseteq \mathbb{No}(\kappa)$. Using Lemma 1 and [2, 16.2.3] we then extend $i$ to an embedding $K \rightarrow L$. \hfill $\Box$

**Lemma 4.** There is an $\omega$-free $H$-field with constant field $\mathbb{R}$ that embeds into every model of $T$ with constant field $\mathbb{R}$.

**Proof.** Let $F$ be the Hardy field $\mathbb{R}(x)$ (so $x > \mathbb{R}$, $x' = 1$). Then $F$ is a grounded $H$-field with constant field $\mathbb{R}$ that embeds into every model of $T$ with constant field $\mathbb{R}$.
Proof of the Theorem. Recall that \((\mathbb{N}, \mathcal{O})\) is a model of \(T\) with constant field \(\mathbb{R}\).

As to universality for set-models, let \(K\) be a set-model of \(T\) with constant field \(\mathbb{R}\). Use Lemma 4 to make \(K\) an extension of an \(\omega\)-free \(H\)-field \(E\) with an embedding \(E \to (\mathbb{N}, \mathcal{O})\), and then use Lemma 3 to extend this embedding to an embedding \(K \to (\mathbb{N}, \mathcal{O})\). As to universality for class-models, let \(K\) be a class-model of \(T\) with constant field \(\mathbb{R}\). Then \(K\) is the union of a chain \((K_{\beta})_{\beta \in \text{On}}\) of set-models of \(T\) with constant field \(\mathbb{R}\). First embed \(K_0\) into \((\mathbb{N}, \mathcal{O})\), and then use transfinite recursion, Lemma 3, and Global Choice to construct a family \((i_{\beta})_{\beta \in \text{On}}\) of embeddings \(i_{\beta} : K_\beta \to (\mathbb{N}, \mathcal{O})\), with \(i_{\beta}\) extending \(i_{\alpha}\) whenever \(\alpha < \beta\). Then the common extension of these \(i_{\beta}\) is an embedding \(K \to (\mathbb{N}, \mathcal{O})\).

For homogeneity for set-models, let \(i : E \to F\) be an isomorphism between set-models \(E, F \preceq (\mathbb{N}, \mathcal{O})\). Given any \(a \in \mathbb{N} \setminus E\) we use Lemma 3 to extend \(i\) to an isomorphism \(K \to L\) between set-models \(K, L \preceq (\mathbb{N}, \mathcal{O})\) with \(a \in K\). Likewise for \(b \in \mathbb{N} \setminus F\) we can extend \(i\) to an isomorphism \(K \to L\) between set-models \(K, L \preceq (\mathbb{N}, \mathcal{O})\) with \(b \in L\). The usual back-and-forth argument then gives an automorphism of \((\mathbb{N}, \mathcal{O})\) extending \(i\). We have now shown that \((\mathbb{N}, \mathcal{O})\) is absolutely homogeneous universal with respect to models of \(T\) with constant field \(\mathbb{R}\).

As to uniqueness, let \(M\) be any class-model of \(T\) with constant field \(\mathbb{R}\) that is absolutely homogeneous universal with respect to models of \(T\) with constant field \(\mathbb{R}\); we need to show that \(M \cong (\mathbb{N}, \mathcal{O})\). First, let a set-model \(E \preceq M\) and an isomorphism \(i : E \to F \preceq (\mathbb{N}, \mathcal{O})\) be given. To go forth, Lemma 3 allows us to extend \(i\) for any \(a \in M \setminus E\) to an isomorphism \(K \to L\) between set-models \(K \preceq M\) and \(L \preceq (\mathbb{N}, \mathcal{O})\) with \(a \in K\). To go back, let \(b \in \mathbb{N} \setminus F\), take a set-model \(L \preceq (\mathbb{N}, \mathcal{O})\) with \(F \subseteq L\) and \(b \in L\), and take an embedding \(j : L \to M\). Then \(j \circ i\) maps \(E\) isomorphically onto \(j(i(E)) \preceq M\), and so extends to an automorphism \(\sigma\) of \(M\). Then \(\sigma^{-1} \circ j\) extends \(i^{-1}\) and maps \(L\) isomorphically onto some \(K \preceq M\) with \(K \supseteq E\). Thus back-and-forth yields an isomorphism \(M \to (\mathbb{N}, \mathcal{O})\).

Alternatively (but there is really little difference with the approach above) we could have adapted familiar arguments of Jónsson [9, 10] for the existence and uniqueness (up to isomorphism) of an \(L\)-structure of inaccessible power \(\kappa\) that is \(\kappa\)-homogeneous and \(\kappa\)-universal with respect to a class of \(L\)-structures that has amalgamation and satisfies a few other simple conditions; see also Ehrlich [8]. We would in any case still need Lemmas 3 and 4 to verify those conditions.

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