PSI-MORPHISMS

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Abstract. We extend to ring morphisms the recent work of Mohamed Khalifa on PSI-extensions.

1. Introduction

In [5], Mohamed Khalifa introduced and studied the so-called prime submodule ideal ring extensions (for short, PSI-extensions). An extension $A \subseteq B$ of integral domains is a PSI extension if every $A$-prime ideal of $B$ is prime in $B$. Here a proper ideal $I$ of $B$ is $A$-prime if $ab \in I$ with $a \in A$ and $b \in B$ implies that $a \in I$ or $b \in I$.

The next theorem summarizes a good deal of the results in [5]. Specifically, for (i)-(iii), (iv)-(v), (vi), (vii), (viii), (ix), (x), (xi), (xii) see results (2.2), (2.3), (2.4), (2.7), (2.9), (2.11), (2.14), (3.2), (3.5) in [5] respectively.

Theorem 1.1. (Khalifa) Let $A \subseteq B \subseteq C$ be extensions of domains and $X$ an indeterminate.

(i) If $A \subseteq C$ is a PSI-extension, then so is $B \subseteq C$.

(ii) For every multiplicative set $S$ of $A$, $A \subseteq AS$ is a PSI-extension.

(iii) If $A \subseteq B$ is a PSI-extension, then so is $A/(Q \cap A) \subseteq B/Q$ for each prime ideal $Q$ of $B$.

(iv) If $A \subseteq B$ is a PSI-extension, then so is $AS \subseteq BS$ for each multiplicative set $S$ of $A$.

(v) $A \subseteq B$ is a PSI-extension iff so is $AP \subseteq BP$ for each maximal ideal $P$ of $A$.

(vi) $A \subseteq B$ is a PSI-extension iff it is an INC-extension and $\Omega(P)$ is prime in $B$ or $\Omega(P) = B$ for every prime ideal $P$ of $A$. Here $\Omega(P) = PBP \cap B$.

(vii) If $A \subseteq B$ have the same prime ideals, then $A \subseteq B$ is a PSI-extension.

(viii) If $A \subseteq B$ is an integral PSI-extension and $B \subseteq C$ is a PSI-extension, then $A \subseteq C$ is a PSI-extension.

(ix) Suppose that $A$ is integrally closed in $B$ and let $\Gamma$ be the set of all intermediate rings between $A$ and $B$. Then $A \subseteq B$ is a PSI-pair, i.e. $A \subseteq T$ is a PSI-extension for each $T \in \Gamma$ iff all residue field extensions of $A \subseteq T$ are algebraic for each $T \in \Gamma$.

(x) If $D$ is a Prüfer domain with quotient field $K$, then $D + XK[[X]] \subseteq K[[X]]$ is a PSI-pair.

(xi) $A[X] \subseteq B[X]$ is a PSI-extension iff $A \subseteq B$ is a PSI-extension and all residue field extensions of $A \subseteq B$ are trivial.

(xii) Suppose that $A$ is a quasi-local integrally closed domain. Then $A[[X]] \subseteq T[[X]]$ is a PSI-extension for each overring $T$ of $A$ iff $A$ is a valuation domain.

The purpose of this paper is to extend a part of the facts in Theorem 1.1 to the more flexible setup of commutative ring morphisms. By ring we always mean a

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2 commutative unitary ring and our ring morphisms are unitary. Our notation and terminology is standard like in [4].

We give our key definition which extends the concept of PSI-extension of [5] to ring morphisms.

**Definition 1.2.** Let \( u : A \to B \) be a ring morphism.

(i) A proper ideal \( I \) of \( B \) is said to be \( A \)-prime if \( u(a)b \in I \) with \( a \in A \) and \( b \in B \) implies that \( u(a) \in I \) or \( b \in I \). It follows easily that \( u^{-1}(I) \) is a prime ideal of \( A \).

(ii) Say that \( u \) is a PSI-morphism if every \( A \)-prime ideal of \( B \) is prime in \( B \). We say that a ring extension \( C \subseteq D \) is a PSI-extension if the inclusion map \( C \hookrightarrow D \) is a PSI-morphism.

In Section 2, we study the PSI-morphisms. We prove that the PSI-morphisms are those morphisms whose fiber rings are either zero or fields (Theorem 2.1). An immediate consequence is that the spectral map of a PSI-morphism is injective (Corollary 2.3). Some examples of PSI-extensions are given in Examples 2.6 and 2.7. Our Proposition 2.8 extends parts (ii) and (vii) of Theorem 1.1 and also shows that an epimorphism is a PSI-morphism. It is easy to identify the minimal ring extensions which are PSI (Proposition 2.9). The class of PSI-morphisms is closed under map composition (Theorem 2.10), thus extending parts (i) and (viii) of Theorem 1.1.

In Section 3, we study a special type of PSI-morphism. Call a ring morphism a strong PSI-morphism if its fiber maps are isomorphisms. An epimorphism is a strong PSI-morphism (Proposition 3.2) but not conversely (Example 3.6). A PSI-morphism is strong iff its residue field extensions are trivial (Theorem 3.4). Examples of strong PSI-morphisms using Nagata idealization rings are provided by Proposition 3.5. In Theorem 3.7, we show that the class of strong PSI-morphisms is stable under base extension and derive that a ring morphism is strong PSI iff the corresponding polynomial morphism is a PSI-morphism, thus extending part (xi) of Theorem 1.1. Consequently, we prove that a finite strong PSI-morphism is surjective (Corollary 3.9) and a finite type strong PSI-morphism is an epimorphism (Theorem 3.9). The class of (strong) PSI-morphisms is closed under map composition (Theorem 3.13) and inductive limits (Proposition 3.10). If \( A \subseteq B \) is a ring extension, we can talk about the greatest intermediate ring \( A \subseteq C \subseteq B \) which is strong PSI over \( A \) (Proposition 3.10). An extension of rings \( A \subseteq B \) sharing an ideal \( I \) is a (strong) PSI-extension iff so is \( A/I \subseteq B/I \) (Proposition 3.19). Consequently, \( A[[X]] \subseteq B[[X]] \) is a PSI-extension when \( A \subseteq B \) is ring extension such that some maximal ideal \( M \) of \( B \) is contained in \( A \) (Proposition 3.20), thus extending part (xii) of Theorem 1.1.

2. PSI-morphisms

We recall some standard notation.

**Notation 2.1.** Let \( u : A \to B \) be a ring morphism. Recall that if \( P \) is a prime ideal of \( A \), then fiber ring of \( u \) at \( P \) is the ring \( k_A(P) \otimes_A B \), where \( k_A(P) = A_P/PA_P \) is the residue field of \( A \) at \( P \). We have the canonical injective map \( Spec(k_A(P) \otimes_A B) \to Spec(B) \) whose image is the set of prime ideals of \( B \) lying over \( P \) in \( A \). So \( k_A(P) \otimes_A B \) is nonzero iff there is some prime ideal of \( B \) lying over \( P \). When \( v : C \to D \) is a canonical ring morphism, we shall write \( C = D \) to mean...
that $v$ is an isomorphism. For instance, if $Q \in \text{Spec}(B)$ and $P = u^{-1}(Q)$, then
\[ k_A(P) = k_A(P) \otimes_A B = k_B(Q) \]
means that the canonical maps
\[ k_A(P) \to k_A(P) \otimes_A B \to k_B(Q) \]
are isomorphisms.

We start with two simple results about $A$-primes.

**Proposition 2.2.** Let $u : A \to B$ be a ring morphism, $I$ a proper ideal of $B$ and $P = u^{-1}(I)$. The following are equivalent.

(i) $I$ is $A$-prime.

(ii) $B/I$ is a torsion-free $A/P$-module.

(iii) $I$ is a contracted ideal via the ring morphism $B \to k_A(P) \otimes_A B$.

**Proof.** As observed in Definition 1.2, $P$ is a prime ideal. The equivalence (i) $\iff$ (ii) follows easily from definitions. Since $k_A(P)$ is the quotient field of $A/P$, (ii) holds iff the canonical map $B/I \to k_A(P) \otimes_A B/I$ is injective iff (iii) holds. \(\square\)

**Corollary 2.3.** Let $u : A \to B$ be a ring morphism and $P$ a prime ideal of $A$. The following are equivalent.

(i) All $A$-prime ideals of $B$ lying over $P$ are prime in $B$.

(ii) The fiber ring $k_A(P) \otimes_A B$ is a field.

**Proof.** Note that every ideal of $k_A(P) \otimes_A B$ is extended from $B$. So, by Proposition 2.2 (i) holds iff every ideal of $k_A(P) \otimes_A B$ is prime iff (ii) holds. \(\square\)

We give our key tool in dealing with PSI-morphisms.

**Theorem 2.4.** For a ring morphism $u : A \to B$, the following are equivalent:

(i) $u$ is a PSI-morphism.

(ii) For every $P \in \text{Spec}(A)$ the fiber $k_A(P) \otimes_A B$ is a field or the zero ring.

(iii) $u(A) \subseteq B$ is a PSI-extension.

**Proof.** (i) $\iff$ (ii) is covered by Corollary 2.3. (i) $\iff$ (iii) follows observing that $u$ and the inclusion morphism $u(A) \hookrightarrow B$ have the same nonzero fiber rings. \(\square\)

**Corollary 2.5.** Let $u : A \to B$ be a PSI-morphism. Then

(i) The spectral map $\alpha : \text{Spec}(B) \to \text{Spec}(A)$ is injective.

(ii) If $P \in \text{Im}(\alpha)$, then
\[ \alpha^{-1}(P) = \{b \in B \mid u(s)b \in PB \text{ for some } s \in A - P\} := Q \]
and $k_A(P) \otimes_A B = k_B(Q)$.

(iii) If $M \in \text{Max}(A)$, then $MB \in \text{Max}(B)$ or $MB = B$.

(iv) If $A$ is a field, then so is $B$.

**Proof.** (i) and (ii). Let $P \in \text{Im}(\alpha)$. Then $k(P) \otimes_A B$ is a field (cf. Theorem 2.4), so $Q = \alpha^{-1}(P)$ is the kernel of the morphism $B \to k(P) \otimes_A B$. A short computation shows that $Q$ has the indicated value. The canonical map $k_A(P) \otimes_A B \to k_B(Q)$ is an isomorphism since the second ring is obtained from the first one by factorization and localization. (iii) If $MB \neq B$, then $k(M) \otimes_A B = B/MB$ is a field (cf. Theorem 2.4), so $MB$ is a maximal ideal of $B$. (iv) follows from (iii). \(\square\)
Example 2.6. Let \( A \) be a one-dimensional domain and \( B \) an overring of \( A \). Then \( A \subset B \) is a PSI-extension iff \( MB \in \text{Max}(B) \) or \( MB = B \) for all \( M \in \text{Max}(A) \), cf. part (iii) of Corollary 2.5.

Consider the particular case \( A = \mathbb{Z}[\sqrt{d}], B = \mathbb{Z}[(1 + \sqrt{d})/2] \) where \( d \in \mathbb{Z}, \sqrt{d} \notin \mathbb{Q} \) and \( d \) is one modulo 4. Then \( A \subset B \) is a PSI-extension iff \( d \) is five modulo 8. Indeed, if \( d \) is one modulo 8, then the fiber ring \( k_A(2,1 + \sqrt{d}) \otimes_A B \) is isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). On the other hand, if \( d \) is one modulo 8, then every maximal ideal of \( A \) extends to a maximal ideal of \( B \). See also Proposition 2.8.

Example 2.7. If \( K \) is a field, then \( K[[X]] \subseteq K[[X]] \) is a PSI-extension, because \( K[[X]][K(X)] = K((X)) \) and \( K \otimes_K [K(X)] = K \). Meanwhile, \( A = \mathbb{Z}[X] \subseteq \mathbb{Z}[[X]] = B \) is not a PSI-extension because

\[
k_A(X - 6) \otimes_A B \simeq \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[[X]]/(X - 6) \simeq \mathbb{Q}_2 \otimes \mathbb{Q}_3
\]

where \( \mathbb{Q}_p \) is the \( p \)-adic number field.

Let \( u : A \to B \) be a ring morphism, \( v : B \otimes_A B \to B \) the canonical ring morphism induced by \( u \) and \( I = \ker(v) \). Recall that module of differentials \( \Omega_{B/A} \) is the \( B \)-module \( I/I^2 \). Recall also that \( u \) is called an epimorphism if the canonical map \( v \) is an isomorphism, that is, \( I = 0 \). Thus \( \Omega_{B/A} = 0 \) if \( u \) is an epimorphism.

Proposition 2.8. A ring morphism \( u : A \to B \) is a PSI-morphism in each case below.

(i) Every prime ideal of \( B \) is extended from \( A \).

(ii) \( u \) is an epimorphism.

(iii) \( u \) is surjective.

(iv) \( u \) is the canonical map from \( A \) to a fraction ring of \( A \).

Proof. We use Theorem 2.4. Let \( P \) be a prime ideal of \( A \). Suppose that (i) holds. Since the hypothesis of (i) is stable under factorization and localization, it is satisfied by \( k_A(P) \to k_A(P) \otimes_A B \), so \( k_A(P) \otimes_A B \) is a field. If (ii) holds, then \( k_A(P) = k_A(P) \otimes_A B \), cf. [7, Lemme 1.0' and Corollaire 1.3]. The other assertions follow from (i).

Note that a proper field extension \( K \hookrightarrow \mathbb{L} \) is a PSI-extension but not an epimorphism.

Next we extend Example 2.6. Note \( \mathbb{Z}[\sqrt{d}] \) is an index two subring of \( \mathbb{Z}[(1 + \sqrt{d})/2] \). Recall that a proper ring extension \( A \subset B \) is minimal [2] if there is no proper intermediate ring \( A \subset C \subset B \). In this case, there exists a maximal ideal called the crucial ideal of \( A \subset B \) such that \( A_P = B_P \) for each \( P \in \text{Spec}(A) - \{M\} \), cf. [2, Théorème 2.2]. Clearly \( A_M \subset B_M \).

Proposition 2.9. Let \( A \subset B \) be a minimal ring extension with crucial ideal \( M \). Then \( A \subset B \) is a PSI-extension iff \( A \subset B \) is not finite or \( A \subset B \) is finite and \( M \) is a maximal ideal of \( B \).

Proof. The assertion is a consequence of the following remarks. If \( A \subset B \) is not finite, then \( A \hookrightarrow B \) is a flat epimorphism cf. [2, Theorem 2.2]; so \( A \hookrightarrow B \) is a PSI-extension cf. Proposition 2.8. Suppose that \( A \subset B \) is finite. Then \( MB \) is a maximal ideal of \( B \) iff so is \( M \) cf. [8, Theorem 3.3].

Our next result extends parts (i) and (viii) of Theorem 1.1.
Theorem 2.10. Suppose that \( u : A \to B \) and \( v : B \to C \) are ring morphisms.

(i) If \( u \) and \( v \) are PSI-morphisms, then so is \( vu \).

(ii) If \( vu \) is a PSI-morphism, then so is \( v \).

Proof. (i) Let \( M \) be a prime ideal of \( C \), \( Q = v^{-1}(M) \), \( P = u^{-1}(Q) \). As \( u \) and \( v \) are PSI-morphisms, we have \( k_A(P) \otimes_A B = k_B(Q) \) and \( k_B(Q) \otimes_B C = k_C(M) \), cf. Corollary 2.5.

Corollary 2.11. Let \( u : A \to B \) be a ring morphism.

(i) If \( u \) is a PSI-morphism, then so is \( A/I \to B/J \) for all ideals \( I, J \) of \( A, B \) respectively such that \( u(I) \subseteq J \).

(ii) If \( u \) is a PSI-morphism, then so is \( A_S \to B_T \) for all multiplicative sets \( S, T \) of \( A, B \) respectively such that \( u(S) \subseteq T \).

(iii) \( u \) is a PSI-morphism iff so is \( A/P \to B/PB \) for each \( P \in \text{Spec}(A) \).

(iv) \( u \) is a PSI-morphism iff so is \( A_M \to B_M \) for each \( M \in \text{Max}(A) \).

Proof. (i) By Proposition 2.8 and Theorem 2.10 we get that \( A \to B/J \) is a PSI-morphism, hence so is \( A/I \to B/J \). A similar argument works for (ii). Assertion (iii) follows from Theorem 2.4 and the fact that the fiber rings of \( u \) is the union of all fiber rings of \( A/P \to B/PB \) for \( P \in \text{Spec}(A) \). A similar argument works for (iv).

3. Strong PSI-morphisms

In this section we study a special type of PSI-morphism. The motivation for doing that comes from the fact that the class of PSI-morphisms is not closed under base extension. A simple example is the field PSI-extension \( \mathbb{R} \subset \mathbb{C} \) whose polynomial extension \( \mathbb{R}[X] \subset \mathbb{C}[X] \) is not PSI, cf. part (xi) of Theorem 1.1.

Definition 3.1. A ring morphism \( u : A \to B \) is said to be a strong PSI-morphism if \( k_A(P) = k_A(P) \otimes_A B \) for all \( P \in \text{Spec}(A) \) with \( k_A(P) \otimes_A B \neq 0 \).

Proposition 3.2. An epimorphism is a strong PSI-morphism.

Proof. Use the proof of part (ii) of Proposition 2.8.

The converse of Proposition 3.2 is not true as shown by Example 3.3.

Example 3.3. It is easy to check directly that the diagonal map \( \mathbb{Z} \to \mathbb{Z}[1/2] \times \mathbb{Z}_2 \) is a strong PSI-morphism. More generally, if \( A \) is a domain and \( b \) a nonzero nonunit of \( A \), then the diagonal map \( A \to A[1/b] \times A/bA \) is an epimorphism \([4] \) page 3-09, so it is a strong PSI-morphism, cf. Proposition 3.2.
The strong PSI-morphisms are the PSI-morphisms with trivial residue field extensions.

**Theorem 3.4.** For a ring morphism \( u : A \to B \) the following are equivalent.

(i) \( u \) is a strong PSI-morphism.

(ii) \( u \) is a PSI-morphism and \( k_A(u^{-1}(Q)) = k_B(Q) \) for each \( Q \in \text{Spec}(B) \).

(iii) \( k_A(P) \otimes_A B \) is a \( k_A(P) \)-vector space of dimension \( \leq 1 \) for each \( P \in \text{Spec}(A) \).

**Proof.** Use Definition 3.1, Theorem 2.4 and \([5, \text{Lemma 3.1}]\). \( \square \)

**Proposition 3.5.** Let \( A \) be a ring, \( M \) an \( A \)-module and \( A(+)M \) the Nagata idealization ring of \( M \). The canonical map \( A \to A(+)M \) is a strong PSI-morphism iff \( k_A(P) \otimes_A M = 0 \) for all \( P \in \text{Spec}(A) \).

**Proof.** We have \( k_A(P) \otimes_A A(+)M \simeq k_A(P) \otimes (k_A(P) \otimes_A M) \) as \( k_A(P) \)-vector spaces, so Theorem 3.4 applies. \( \square \)

We exhibit a strong PSI-morphism which is not an epimorphism.

**Example 3.6.** Let \( u \) be the canonical map \( Z \to Z(+)Q/Z := B \). As \( K \otimes Z Q/Z = 0 \) for \( K = Q \) or \( Z_p \) with \( p \) prime, it follows that \( u \) is a strong PSI-morphism, cf. Proposition 3.5. On the other hand we have

\[
B \otimes Z B \simeq Z \oplus Q/Z \oplus Q/Z \not\simeq Z \oplus Q/Z = B
\]
as Abelian groups, so \( u \) is not an epimorphism. Writing \( Q/Z \) as an inductive limit of the cyclic groups \( Z_m \), it can be shown that \( \Omega_{B/Z} \) null, so the kernel of the canonical map \( B \otimes Z B \to B \) is a nonzero idempotent nilideal, cf. [7, Proposition 1.5]. Note also that \( B \) is not \( Z \)-flat because it has nonzero torsion. We were not able to find an example of a strong PSI-morphism with nonzero module of differentials.

In the light of Theorem 3.4, part (xi) of Theorem 1.1 shows (in our terminology) that a domain extension \( A \subseteq B \) is strong PSI iff the polynomial extension \( A[X] \subseteq B[X] \) is a PSI-extension. We extend this result by showing that the class of strong PSI-morphisms is stable under base extension.

**Theorem 3.7.** For a ring morphism \( u : A \to B \) the following are equivalent.

(i) \( u \) is a strong PSI-morphism.

(ii) \( u \otimes_A C : C \to C \otimes_A B \) is a strong PSI-morphism for each ring morphism \( v : A \to C \).

(iii) \( u \otimes_A C : C \to B \otimes_A C \) is a PSI-morphism for each ring morphism \( A \to C \).

(iv) \( u \otimes_A A[X] : A[X] \to B[X] \) is a PSI-morphism.

**Proof.** (i) \( \Rightarrow \) (ii) Set \( D = C \otimes_A B \). Let \( M \in \text{Spec}(C) \) such that \( k_C(M) \otimes_C D \) is nonzero and let \( P = v^{-1}(M) \). Note that \( k_A(P) \otimes_A B \) is nonzero, so \( k_A(P) = k(A(P)) \otimes_A B \). We have

\[
k_C(M) \otimes_C D = k_C(M) \otimes_C C \otimes_A B = k_C(M) \otimes_A B = \\
ek_C(M) \otimes k_A(P) \otimes_A B = k_C(M) \otimes k_A(P) k_A(P) = k_C(M).
\]
The implications (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) are clear.

(iv) \( \Rightarrow \) (i) Let \( P \in \text{Spec}(A) \) such that \( k_A(P) \otimes_A B \) is nonzero. As \( A[X] \to B[X] \) is a PSI-morphism, we get that \( A \to B \) and \( k_A(P)[X] \to k_B(Q)[X] \) are PSI-morphisms, cf. parts (i-ii) of Corollary 2.11. Thus \( k_A(P) \otimes_A B = k_B(Q) = k_A(P) \) by Theorem 2.4 and [5, Lemma 3.1]. \( \square \)
If we add some finiteness condition, a strong PSI-morphism gets a particular form.

**Corollary 3.8.** A finite strong PSI-morphism is surjective.

*Proof.* It suffices to consider the case of an injective finite strong PSI-morphism \( A \twoheadrightarrow B \). Localizing (Theorem 3.7), we may assume that \( A \) is local with maximal ideal \( M \). As \( A \twoheadrightarrow B \) is a strong PSI-morphism, we get \( A/M = B/MB \), hence \( B = A + MB \), thus \( A = B \) by Nakayama’s Lemma. □

**Theorem 3.9.** A finite type strong PSI-morphism is an epimorphism.

*Proof.* Let \( u : A \to B \) be a finite type strong PSI-morphism, \( v : B \otimes_A B \to B \) the canonical ring morphism induced by \( u \) and \( I = ker(v) \). Then \( I \) is a finitely generated ideal of \( B \otimes_A B \), so \( \Omega_{B/A} = I/I^2 \) is a finitely generated \( B \)-module, cf. [1] Lemme 1.7]. Let \( Q \in \text{Max}(B) \) and \( P = u^{-1}(Q) \). Since \( u \) is a strong PSI-morphism, we get \( k_A(P) = k_A(P) \otimes_A B = k_B(Q) \), so

\[
\Omega_{B/A} \otimes_B k_B(Q) \simeq \Omega_{B/A} \otimes_A k_A(P) \simeq \Omega_{k_A(P)/k_A(P)} = 0
\]

cf. [1] Lemme 1.13]. By Nakayama’s Lemma, we get that \( \Omega_{B/A} \) is null because it is a finitely generated \( B \)-module. By [7] Proposition 1.5], we obtain that \( u \) is an epimorphism. □

For further use, we give the following result.

**Proposition 3.10.** Let \( \mathcal{P} \) be one of the following three conditions: PSI-morphism, strong PSI-morphism, epimorphism. An inductive limit of \( \mathcal{P} \)-morphisms is a \( \mathcal{P} \)-morphism.

*Proof.* Let \( u : A \to B \) be the limit of the inductive system of ring morphisms \( \{ u_i : A_i \to B_i \}_{i \in I} \). Let \( Q \) be a prime ideal of \( B \) and \( P_i \) the inverse image of \( Q \) in \( A_i, B_i, A \) respectively. If all \( u_i \)'s are PSI-morphisms, then \( k_{A_i}(P_i) \otimes_A B_i = k_{B_i}(Q_i) \), so taking the limit we get \( k_A(P) \otimes_A B = k_B(Q) \). Hence \( u \) is a PSI-morphism. A similar argument works in the other two cases. □

**Corollary 3.11.** An inductive limit of finite type strong PSI-morphisms is an epimorphism.

*Proof.* Combine Theorem 3.9 and Proposition 3.10. □

In particular, the morphism of Example 3.6 cannot be written as an inductive limit of finite type strong PSI-morphisms.

**Remark 3.12.** We remark the following extension of part (ix) of Theorem 1.1. Let \( A \subseteq B \) be a ring extension and let \( \Gamma \) denote the set of all rings between \( A \) and \( B \). Suppose that every \( C \in \Gamma \) is integrally closed in \( B \). By [6] Theorem 5.2, page 47, \( A \subseteq C \) is a flat epimorphism (hence a strong PSI-extension) for each \( C \in \Gamma \).

The next result is a strong PSI-morphism variant of Theorem 2.10.

**Theorem 3.13.** Suppose that \( u : A \to B \) and \( v : B \to C \) are ring morphisms.

(i) If \( u \) and \( v \) are strong PSI-morphisms, then so is \( vu \).

(ii) If \( vu \) is a strong PSI-morphism, then so is \( v \).

*Proof.* Let \( M \) be a prime ideal of \( C, Q = v^{-1}(M) \), \( P = u^{-1}(Q) \). We have the residue field extensions \( k_A(P) \to k_B(Q) \to k_C(M) \). So \( k_A(P) = k_C(M) \) if \( k_A(P) = k_B(Q) \) and \( k_B(Q) = k_C(M) \). Apply Theorem 3.4 and Theorem 2.10 □
Proposition 3.19. If $u : A \to B$ and $v : B \to C$ are strong PSI-morphisms, then so is $A \to B \otimes_A C$.

Proof. Combine Theorems 3.7 and 3.13.

Corollary 3.15. Let $u : A \to B$, $v : B \to C$, $t : C \to D$, $w : B \to D$ be ring morphisms such that $wu = tv$ and $D = w(B)t(C)$. If $u$ and $v$ are strong PSI-morphisms, then so is $A \to D$.

Proof. By Corollary 3.14, $A \to B \otimes_A C$ is a strong PSI-morphism. As the canonical map $B \otimes_A C \to D$ is surjective, we are done.

The next result is in the spirit of [7, Proposition 3.4].

Proposition 3.16. Let $u : A \to B$ be a ring morphisms. There exists a greatest subring $C$ of $B$ containing $u(A)$ such that $u : A \to C$ is a strong PSI-morphism. Call $C$ the strong PSI-closure of $A$ in $B$.

Proof. $C$ is the directed union of the subrings $D$ of $B$ containing $u(A)$ such that $A \to D$ is a strong PSI-morphism, cf. Corollary 3.15. Apply Proposition 3.10.

Corollary 3.17. Let $A \subseteq B$ be an extension of domains where $A$ a Prüfer domain with quotient field $K$. Then the PSI-closure of $A$ in $B$ is $K \cap B$.

Proof. Let $C$ denote the PSI-closure of $A$ in $B$. It is clear that $C \subseteq K \cap B$. On the other hand, since $A$ is a Prüfer domain, $A \subseteq D$ is an epimorphism cf. [6, Theorem 3.13, page 37] and [3, Corollary 6.5.19], hence a strong PSI-extension, for each overring $D$ of $A$. Thus $C = K \cap B$.

Example 3.18. For a field $K$, the PSI-closure of $K[X]$ in $K[[X]]$ is contained in $K(X) \cap K[[X]] = K[X]_{(X)}$, cf. Corollary 3.17.

We close our paper by giving two more constructions of PSI-morphisms (Propositions 3.19 and 3.22).

Proposition 3.19. Let $A \subseteq B$ be ring extension and $I$ a common ideal of $A$ and $B$. Then $A \subseteq B$ is a PSI-extension (resp. strong PSI-extension) if $A/I \subseteq B/I$ is a PSI-extension (resp. strong PSI-extension).

Proof. The ”only if part” follows from Theorem 3.7. For the ”if part”, let $P \in \text{Spec}(A)$. Suppose that $P \not\supseteq I$ and pick $f \in I - P$. Then $fB \subseteq A$, so $A_P = B_P$, hence $k_A(P) = k_{A/P}(A_B)$. To complete the proof, it suffices to see that when $P \supseteq I$, we have $k_{A/I}(P/I) = k_A(P)$ and $k_{A/I}(P/I) \otimes_{A/I} B/I = k_A(P) \otimes_A B$.

Corollary 3.20. A ring extension $A \subseteq B$ is a PSI-morphism (resp. strong PSI-morphism) if $A + XB[X] \subseteq B[X]$ is a PSI-morphism (resp. strong PSI-morphism).

We give a power series application of Proposition 3.19 which extends part (xii) of Theorem 1.1.

Proposition 3.21. Let $A \subseteq B$ be ring extension such that some maximal ideal $M$ of $B$ is contained in $A$. Then $A[[X]] \subseteq B[[X]]$ is a PSI-extension.

Proof. As $M[[X]]$ is a common ideal of $A[[X]]$ and $B[[X]]$, it suffices to show that $(A/M)[[X]] \subseteq (B/M)[[X]]$ is a PSI-morphism, cf. Proposition 3.19. In other words, it suffices to show that $A[[X]] \subseteq B[[X]]$ is a PSI-morphism when $A$ is a domain and
B is a field. Since the prime ideals of \(B[[X]]\) are 0 and \(XB[[X]]\), it suffices to see that

\[
k_{A[[X]]}(0) \otimes A[[X]] B[[X]] = B[[X]][1/X] = B((X))
\]

and

\[
k_{A[[X]]}(XA[[X]]) \otimes A[[X]] B[[X]] = B[[X]]/(X) = B.
\]

\(\square\)

**Proposition 3.22.** Let \(u : A \rightarrow B, v : A \rightarrow C\) be ring morphisms and \(\alpha, \beta\) the images in \(\text{Spec}(A)\) of \(\text{Spec}(B), \text{Spec}(C)\) respectively. Let \(P\) be one of the following three conditions: PSI-morphism, strong PSI-morphism, epimorphism. Then the diagonal map \(w : A \rightarrow B \times C\) is \(P\)-morphism iff \(u, v\) are \(P\) and \(\alpha \cap \beta = \emptyset\).

**Proof.** The assertion is a consequence of the following three remarks. If \(w\) is a (strong) PSI-morphism, compose it with the canonical projections to get that \(u, v\) are (strong) PSI-morphism, cf. Theorem 2.10. For \(P \in \text{Spec}(A)\), if

\[
k_A(P) \otimes_A B \times C = k_A(P) \otimes_A B \times k_A(P) \otimes_A C
\]

is a field, then \(k_A(P) \otimes_A B = 0\) or \(k_A(P) \otimes_A C = 0\). Note that \(B \otimes_A C = 0\), otherwise for any prime ideal \(P\) of \(B \otimes_A C\), the preimage of \(P\) in \(A\) belongs to \(\alpha \cap \beta\). Thus \((B \times C) \otimes_A (B \times C) = (B \times C)\).

\(\square\)

For instance, the diagonal map \(Q[[X]] \rightarrow R((X)) \times Q\) is a PSI-morphism.

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