SEGAL-BARGMANN TRANSFORM AND PALEY-WIENER
THEOREMS ON MOTION GROUPS

SUPARNA SEN

Abstract. We study the Segal-Bargmann transform on a motion group $\mathbb{R}^n \ltimes K$, where $K$ is a compact subgroup of $SO(n)$. A characterization of the Poisson integrals associated to the Laplacian on $\mathbb{R}^n \ltimes K$ is given. We also establish a Paley-Wiener type theorem using the complexified representations.

MSC 2000 : Primary 22E30; Secondary 22E45.

Keywords : Segal-Bargmann transform, Poisson integrals, Paley-Wiener theorems.

1. Introduction

The Segal-Bargmann transform, also called the coherent state transform, was developed independently in the early 1960’s by Segal in the infinite-dimensional context of scalar quantum field theories and by Bargmann in the finite-dimensional context of quantum mechanics on $\mathbb{R}^n$. We consider the following equivalent form of Bargmann’s original result.

A function $f \in L^2(\mathbb{R}^n)$ admits a factorization $f(x) = g \ast p_t(x)$ where $g \in L^2(\mathbb{R}^n)$ and $p_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$ (the heat kernel on $\mathbb{R}^n$) if and only if $f$ extends as an entire function to $\mathbb{C}^n$ and we have

$$\frac{1}{(2\pi t)^{n/2}} \int_{\mathbb{C}^n} |f(z)|^2 e^{-\frac{|y|^2}{4t}} dxdy < \infty \quad (z = x + iy).$$

In this case we also have

$$\|g\|_2^2 = \frac{1}{(2\pi t)^{n/2}} \int_{\mathbb{C}^n} |f(z)|^2 e^{-\frac{|y|^2}{4t}} dxdy.$$

The mapping $g \rightarrow g \ast p_t$ is called the Segal-Bargmann transform and the above says that the Segal-Bargmann transform is a unitary map from $L^2(\mathbb{R}^n)$ onto $O(\mathbb{C}^n)$.

The author was supported by Shyama Prasad Mukherjee Fellowship from Council of Scientific and Industrial Research, India.
\[ \bigcap L^2(\mathbb{C}^n, \mu), \] where \( d\mu(z) = \frac{1}{(2\pi t)^{n/2}} e^{-|w|^2/2t} \, dw \) and \( \mathcal{O}(\mathbb{C}^n) \) denotes the space of entire functions on \( \mathbb{C}^n \).

In the paper [4], B. C. Hall introduced a generalization of the Segal-Bargmann transform on a compact Lie group. If \( K \) is such a group, this coherent state transform maps \( L^2(K) \) isometrically onto the space of holomorphic functions in \( L^2(G, \mu_t) \), where \( G \) is the complexification of \( K \) and \( \mu_t \) is an appropriate heat kernel measure on \( G \). The generalized coherent state transform is defined in terms of the heat kernel on the compact group \( K \) and its analytic continuation to the complex group \( G \). Similar results have been proved by various authors. See [12], [6], [5], [8] and [7].

Next, consider the following result on \( \mathbb{R} \) due to Paley and Wiener. A function \( f \in L^2(\mathbb{R}) \) admits a holomorphic extension to the strip \( \{ x + iy : |y| < t \} \) such that

\[
\sup_{|y| \leq s} \int_{\mathbb{R}} |f(x + iy)|^2 \, dx < \infty \quad \forall \ s < t
\]

if and only if

\[
\int_{\mathbb{R}} e^{2s|\xi|} |\tilde{f}(\xi)|^2 \, d\xi < \infty \quad \forall \ s < t
\]

(1.1)

where \( \tilde{f} \) denotes the Fourier transform of \( f \).

The condition (1.1) is the same as

\[
\int_{\mathbb{R}} |e^{2s\Delta^{1/2}} f(\xi)|^2 \, d\xi < \infty \quad \forall s < t
\]

where \( \Delta \) is the Laplacian on \( \mathbb{R} \). This point of view was explored by R. Goodman in Theorem 2.1 of [2].

The condition (1.1) also equals

\[
\int_{\mathbb{R}} |e^{i(x+iy)\xi}|^2 |\tilde{f}(\xi)|^2 \, d\xi < \infty \quad \forall y < t.
\]

Here \( \xi \mapsto e^{i(x+iy)\xi} \) may be seen as the complexification of the parameters of the unitary irreducible representations \( \xi \mapsto e^{ix\xi} \) of \( \mathbb{R} \). This point of view also was further developed by R. Goodman (see Theorem 3.1 from [3]). Similar results were established for the Euclidean motion group \( M(2) \) of the plane \( \mathbb{R}^2 \) in [11]. Aim of this paper is to prove corresponding results in the context of general motion groups.
The plan of this paper is as follows: In the following section we recall the representation theory and Plancherel theorem of the motion group \( M \). We also describe the Laplacian on \( M \). In the next section we prove the unitarity of the Segal-Bargmann transform on \( M \) and we study generalized Segal-Bargmann transform which is an analogue of Theorem 8 and Theorem 10 in [4]. The fourth section is devoted to a study of Poisson integrals on \( M \) via a Gutzmer-type formula on \( M \) which is proved by using a Gutzmer formula for compact Lie groups established by Lassalle in 1978 (see [9]). This section is modelled after the work of Goodman [2]. In the final section we prove another characterization of functions extending holomorphically to the complexification of \( M \) which is an analogue of Theorem 3.1 of [3].

2. Preliminaries

Let \( K \) be a compact, connected Lie group which acts as a linear group on a finite dimensional real vector space \( V \). Let \( M \) be the semidirect product of \( V \) and \( K \) with the group law

\[
(x_1, k_1) \cdot (x_2, k_2) = (x_1 + k_1 x_2, k_1 k_2) \text{ where } x_1, x_2 \in V; k_1, k_2 \in K.
\]

\( M \) is called the motion group. Since \( K \) is compact, there exists a \( K \)-invariant inner product on \( V \). Hence, we can assume that \( K \) is a connected subgroup of \( SO(n) \), where \( n = \dim V \). When \( K = \{1\} \), \( M = V \cong \mathbb{R}^n \) and if \( K = SO(n) \), \( M \) is the Euclidean motion group. Henceforth we shall identify \( V \) with \( \mathbb{R}^n \) and \( K \) with a subgroup of \( SO(n) \).

The group \( M \) may be identified with a matrix subgroup of \( GL(n+1, \mathbb{R}) \) via the map

\[
(x, k) \mapsto \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix}
\]

where \( x \in \mathbb{R}^n \) and \( k \in K \subseteq SO(n) \).

We normalize the Haar measure \( dm \) on \( M \) such that \( dm = dx dk \), where \( dx = (2\pi)^{-\frac{n}{2}} dx_1 dx_2 \cdots dx_n \) and \( dk \) is the normalized Haar measure on \( K \). Let \( \mathcal{H} = L^2(K) \) be the Hilbert space of all square integrable functions on \( K \). Denote by \( \langle \cdot, \cdot \rangle \) the Euclidean
inner product on $\mathbb{R}^n$. Let $\hat{V}$ be the dual space of $V$. Then we can identify $\hat{V}$ with $\mathbb{R}^n$ so that $K$ acts on $\hat{V}$ naturally by $\langle k \cdot \xi, x \rangle = (\xi, k^{-1} \cdot x)$ where $\xi \in \hat{V}$, $x \in V$, $k \in K$.

For any $\xi \in \hat{V}$ let $U^\xi$ denote the induced representation of $M$ by the unitary representation $x \mapsto e^{i<\xi, x>}$ of $V$. Then for $F \in H$ and $(x, k) \in M$,

$$U^\xi(x, k)F(u) = e^{i<x, u \cdot \xi>}F(k^{-1}u).$$

The representation $U^\xi$ is not irreducible. Any irreducible unitary representation of $M$ is, however, contained in $U^\xi$ for some $\xi \in \hat{V}$ as an irreducible component.

Let $K_\xi$ be the isotropy subgroup of $\xi \in \hat{V}$ i.e. $K_\xi = \{k \in K : k \cdot \xi = \xi\}$. Consider $\sigma \in \widehat{K_\xi}$, the unitary dual of $K_\xi$. Denote by $\chi_\sigma$, $d_\sigma$ and $\sigma_{ij}$ the character, degree and matrix coefficients of $\sigma$ respectively. Let $R$ be the right regular representation of $K$. Define

$$P^\sigma = d_\sigma \int_{K_\xi} \overline{\chi_\sigma(w)}R_wdw$$

and

$$P^\sigma_\gamma = d_\sigma \int_{K_\xi} \overline{\sigma_\gamma(w)}R_wdw$$

where $dw$ is the normalized Haar measure on $K_\xi$. Then $P^\sigma$ and $P^\sigma_\gamma$ are both orthogonal projections of $H$. Let $H^\sigma = P^\sigma H$ and $H^\sigma_\gamma = P^\sigma_\gamma H$. The subspaces $H^\sigma_\gamma$ are invariant under $U^\xi$ for $1 \leq \gamma \leq d_\sigma$ and the representations of $M$ induced on $H^\sigma_\gamma$ under $U^\xi$ are equivalent for all $1 \leq \gamma \leq d_\sigma$. We fix one of them and denote it by $U^\xi,\sigma$.

Two representations $U^\xi,\sigma$ and $U^{\xi',\sigma'}$ are equivalent if and only if there exists an element $k \in K$ such that $\xi = k \cdot \xi'$ and $\sigma'$ is equivalent to $\sigma^k$ where $\sigma^k(w) = \sigma(kwk^{-1})$ for $w \in K_\xi$.

The Mackey theory [10] shows that under certain conditions on $K$ (for details refer to Section 6.6 of [1]), each $U^\xi,\sigma$ is irreducible and every infinite dimensional irreducible unitary representation is equivalent to one of $U^\xi,\sigma$ for some $\xi \in \mathbb{R}^n$ and $\sigma \in \widehat{K_\xi}$. Since $H = \bigoplus_{\sigma \in \widehat{K_\xi}} H^\sigma$ and $H^\sigma = \bigoplus_{\gamma=1}^{d_\sigma} H^\sigma_\gamma$, we have

$$U^\xi \cong \bigoplus_{\sigma \in \widehat{K_\xi}} d_\sigma U^\xi,\sigma.$$
For any $f \in L^1(M)$ define the Fourier transform of $f$ by

$$\hat{f}(\xi, \sigma) = \int_M f(m) U_m^{\xi,\sigma} \, dm.$$ 

Then the Plancherel formula gives

$$\int_M |f(m)|^2 \, dm = \sum_{\sigma \in \hat{K}_\xi} d_\sigma \int_{\mathbb{R}^n} \| \hat{f}(\xi, \sigma) \|_{HS}^2 \, d\xi$$

where $\| \cdot \|_{HS}$ is the Hilbert-Schmidt norm of an operator. We will be working with the generalized Fourier transform defined by

$$\hat{f}(\xi) = \int_M f(m) U_m^{\xi} \, dm.$$ 

Then we also have

$$\int_M |f(m)|^2 \, dm = \int_{\mathbb{R}^n} \| \hat{f}(\xi) \|_{HS}^2 \, d\xi.$$ 

Let $\mathfrak{k}$ and $\mathfrak{m}$ be the Lie algebras of $K$ and $M$ respectively. Then

$$\mathfrak{m} = \left\{ \begin{pmatrix} K & X \\ 0 & 0 \end{pmatrix} : X \in \mathbb{R}^n, \ K \in \mathfrak{k} \right\}.$$ 

Let $K_1, K_2, \cdots, K_N$ be a basis of $\mathfrak{k}$ and $X_1, X_2, \cdots, X_n$ be a Lie algebra basis of $\mathbb{R}^n$. Define

$$M_i = \begin{pmatrix} K_i & 0 \\ 0 & 0 \end{pmatrix} \text{ for } 1 \leq i \leq N$$

$$= \begin{pmatrix} 0 & X_i \\ 0 & 0 \end{pmatrix} \text{ for } N + 1 \leq i \leq N + n.$$ 

Then it is easy to see that $\{M_1, M_2, \cdots, M_{N+n}\}$ forms a basis for $\mathfrak{m}$. The Laplacian $\Delta_M = \Delta$ is defined by

$$\Delta = -(M_1^2 + M_2^2 + \cdots + M_{N+n}^2).$$ 

A simple computation using the fact $K \subseteq SO(n)$ shows that

$$\Delta = -\Delta_{\mathbb{R}^n} - \Delta_K$$
where $\Delta_{\mathbb{R}^n}$ and $\Delta_K$ are the Laplacians on $\mathbb{R}^n$ and $K$ respectively given by $\Delta_{\mathbb{R}^n} = X_1^2 + X_2^2 + \cdots + X_n^2$ and $\Delta_K = K_1^2 + K_2^2 + \cdots + K_N^2$.

3. **Segal-Bargmann transform and its generalisation**

Since $\Delta_{\mathbb{R}^n}$ and $\Delta_K$ commute, it follows that the heat kernel $\psi_t$ associated to $\Delta$ is given by the product of the heat kernels $p_t$ on $\mathbb{R}^n$ and $q_t$ on $K$. In other words

$$\psi_t(x, k) = p_t(x)q_t(k) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} \sum_{\pi \in \hat{K}} d_\pi e^{-\frac{\lambda_\pi t}{t}} \chi_\pi(k).$$

Here, for each unitary, irreducible representation $\pi$ of $K$, $d_\pi$ is the degree of $\pi$, $\lambda_\pi$ is such that $\pi(\Delta_K) = -\lambda_\pi I$ and $\chi_\pi(k) = tr(\pi(k))$ is the character of $\pi$.

Denote by $G$ the complexification of $K$. Let $\kappa_t$ be the fundamental solution at the identity of the following equation on $G$:

$$\frac{du}{dt} = \frac{1}{4} \Delta_G u,$$

where $\Delta_G$ is the Laplacian on $G$. It should be noted that $\kappa_t$ is the real, positive heat kernel on $G$ which is not the same as the analytic continuation of $q_t$ on $K$.

Let $\mathcal{H}(\mathbb{C}^n \times G)$ be the Hilbert space of holomorphic functions on $\mathbb{C}^n \times G$ which are square integrable with respect to $\mu \otimes \nu(z, g)$ where

$$d\mu(z) = \frac{1}{(2\pi t)^{\frac{n}{2}}} e^{-\frac{|y|^2}{4t}} dxdy \text{ on } \mathbb{C}^n$$

and

$$d\nu(g) = \int_K \kappa_t(xg) dx \text{ on } G.$$

Then we have the following theorem:

**Theorem 3.1.** If $f \in L^2(M)$, then $f \ast \psi_t$ extends holomorphically to $\mathbb{C}^n \times G$. Moreover, the map $C_t : f \mapsto f \ast \psi_t$ is a unitary map from $L^2(M)$ onto $\mathcal{H}(\mathbb{C}^n \times G)$.

**Proof.** Let $f \in L^2(M)$. Expanding $f$ in the $K$–variable using the Peter-Weyl theorem we obtain

$$f(x, k) = \sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} f_{ij}^\pi(x) \phi_{ij}^\pi(k)$$
where for each $\pi \in \hat{K}$, $d_\pi$ is the degree of $\pi$, $\phi^\pi_{ij}$’s are the matrix coefficients of $\pi$ and 

$$f^\pi_{ij}(x) = \int_K f(x, k) \overline{\phi^\pi_{ij}(k)} dk.$$ 

Here, the convergence is understood in the $L^2$-sense. Moreover, by the universal property of the complexification of a compact Lie group (see Section 3 of [4]), all the representations of $K$, and hence all the matrix entries, extend to $G$ holomorphically.

Since $\psi_t$ is $K$-invariant (as a function on $\mathbb{R}^n$) a simple computation shows that

$$f \ast \psi_t(x, k) = \sum_{\pi \in \hat{K}} d_\pi e^{-\frac{\lambda_\pi t}{2}} \sum_{i,j=1}^{d_\pi} f^\pi_{ij} \ast p_t(x) \phi^\pi_{ij}(k).$$

It is easily seen that $f^\pi_{ij} \in L^2(\mathbb{R}^n)$ for every $\pi \in \hat{K}$ and $1 \leq i, j \leq d_\pi$. Hence $f^\pi_{ij} \ast p_t$ extends to a holomorphic function on $\mathbb{C}^n$ and by the unitarity of the Segal-Bargmann transform in $\mathbb{R}^n$ we have

$$\int_{\mathbb{C}^n} |f^\pi_{ij} \ast p_t(z)|^2 \mu(y) dxdy = \int_{\mathbb{R}^n} |f^\pi_{ij}(x)|^2 dx. \quad (3.1)$$

The analytic continuation of $f \ast \psi_t$ to $\mathbb{C}^n \times G$ is given by

$$f \ast \psi_t(z, g) = \sum_{\pi \in \hat{K}} d_\pi e^{-\frac{\lambda_\pi t}{2}} \sum_{i,j=1}^{d_\pi} f^\pi_{ij} \ast p_t(z) \phi^\pi_{ij}(g).$$

We claim that the above series converges uniformly on compact subsets of $\mathbb{C}^n \times G$ so that $f \ast \psi_t$ extends to a holomorphic function on $\mathbb{C}^n \times G$. We know from Section 4, Proposition 1 of [4] that the holomorphic extension of the heat kernel $q_t$ on $K$ is given by

$$q_t(g) = \sum_{\pi \in \hat{K}} d_\pi e^{-\frac{\lambda_\pi t}{2}} \chi_\pi(g).$$

For each $g \in G$, define the function $q^g_t(k) = q_t(gk)$. Then $q^g_t$ is a smooth function on $K$ and is given by

$$q^g_t(k) = \sum_{\pi \in \hat{K}} d_\pi e^{-\frac{\lambda_\pi t}{2}} \chi_\pi(gk)$$

$$= \sum_{\pi \in \hat{K}} d_\pi e^{-\frac{\lambda_\pi t}{2}} \sum_{i,j=1}^{d_\pi} \phi^\pi_{ij}(g) \phi^\pi_{ji}(k).$$
Since $q_t^g$ is a smooth function on $K$, we have for each $g \in G$,

$$
(3.2) \quad \int_K |q_t^g(k)|^2dk = \sum_{\pi \in \hat{K}} d_\pi e^{-\frac{\lambda t}{2}} \sum_{i,j=1}^{d_\pi} |\phi_{ij}^\pi(g)|^2 < \infty.
$$

Let $L$ be a compact set in $\mathbb{C}^n \times G$. For $(z, g) \in L$ we have,

$$
(3.3) \quad |f * \psi_t(z, g)| \leq \sum_{\pi \in \hat{K}} d_\pi e^{-\frac{\lambda t}{2}} \sum_{i,j=1}^{d_\pi} |f_{ij}^\pi * p_t(z)||\phi_{ij}^\pi(g)|.
$$

By the Fourier inversion

$$
f_{ij}^\pi * p_t(z) = \int_{\mathbb{R}^n} \hat{f}_{ij}^\pi(\xi)e^{-t|\xi|^2}e^{i\xi \cdot (x+iy)}d\xi
$$

where $z = x + iy \in \mathbb{C}^n$ and $\hat{f}_{ij}^\pi$ is the Fourier transform of $f_{ij}^\pi$. Hence, if $z$ varies in a compact subset of $\mathbb{C}^n$, we have

$$
|f_{ij}^\pi * p_t(z)| \leq \|f_{ij}^\pi\|_2 \int_{\mathbb{R}^n} e^{-2(t|\xi|^2+y\cdot \xi)}d\xi \leq C\|f_{ij}^\pi\|_2.
$$

Using the above in (3.3) and applying Cauchy-Schwarz inequality we get

$$
|f * \psi_t(z, g)| \leq C \sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \|f_{ij}^\pi\|_2 e^{-\frac{\lambda t}{2}} |\phi_{ij}^\pi(g)|
$$

$$
\leq C \left( \sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \int_{\mathbb{R}^n} |f_{ij}^\pi(x)|^2dx \right)^{\frac{1}{2}} \left( \sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} e^{-\lambda t} |\phi_{ij}^\pi(g)|^2 \right)^{\frac{1}{2}}.
$$

Noting that $\|f\|_2^2 = \sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \int_{\mathbb{R}^n} |f_{ij}^\pi(x)|^2dx$ and $q_t$ is a smooth function on $G$ we prove the claim using (3.2). Applying Theorem 2 in [4] we get

$$
\int_G |f * \psi_t(z, g)|^2d\nu(g) = \sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} |f_{ij}^\pi * p_t(z)|^2.
$$

Integrating the above against $\mu(y)dxdy$ on $\mathbb{C}^n$ and using (3.1) we obtain the isometry of $C_t$

$$
\int_{\mathbb{C}^n} \int_G |f * \psi_t(z, g)|^2\mu(y)dxdydv(g) = \|f\|_2^2.
$$
To prove that the map $C_t$ is surjective it suffices to prove that the range of $C_t$ is dense in $\mathcal{H}(\mathbb{C}^n \times G)$. For this, consider functions of the form $f(x,k) = h_1(x)h_2(k) \in L^2(M)$ where $h_1 \in L^2(\mathbb{R}^n)$, $h_2 \in L^2(K)$. Then a simple computation shows that

$$f * \psi_t(z,g) = h_1 * p_t(z)h_2 * q_t(g) \text{ for } (z,g) \in \mathbb{C}^n \times G.$$ 

Suppose $F \in \mathcal{H}(\mathbb{C}^n \times G)$ be such that

$$\int_{\mathbb{C}^n \times G} F(z,g)h_1 * p_t(z)h_2 * q_t(g)\mu(y)dxdy\nu(g) = 0 \quad (3.4)$$

\forall h_1 \in L^2(\mathbb{R}^n) \text{ and } \forall h_2 \in L^2(K). \text{ From (3.4) we have}

$$\int_G \left( \int_{\mathbb{C}^n} F(z,g)\overline{h_1 * p_t(z)d\mu(z)} \right) \overline{h_2 * q_t(g)d\nu(g)} = 0,$$

which by Theorem 2 of [4] implies that

$$\int_{\mathbb{C}^n} F(z,g)h_1 * p_t(z)d\mu(z) = 0.$$ 

Finally, an application of the surjectivity of Segal-Bargmann transform on $\mathbb{R}^n$ shows that $F \equiv 0$. Hence the proof. \qed

In [4] Brian C. Hall proved the following generalizations of the Segal-Bargmann transforms for $\mathbb{R}^n$ and compact Lie groups:

**Theorem 3.2.**

1. Let $\mu$ be any measurable function on $\mathbb{R}^n$ such that
   - $\mu$ is strictly positive and locally bounded away from zero,
   - $\forall x \in \mathbb{R}^n$, $\sigma(x) = \int_{\mathbb{R}^n} e^{2x \cdot y} \mu(y)dy < \infty$.

Define, for $z \in \mathbb{C}^n$

$$\psi(z) = \int_{\mathbb{R}^n} \frac{e^{ia(y)}}{\sqrt{\sigma(y)}} e^{-iy \cdot z} dy,$$

where $a$ is a real valued measurable function on $\mathbb{R}^n$. Then the mapping $C_\psi : L^2(\mathbb{R}^n) \rightarrow \mathcal{O}(\mathbb{C}^n)$ defined by

$$C_\psi(z) = \int_{\mathbb{R}^n} f(x)\psi(z-x)dx$$

is an isometric isomorphism of $L^2(\mathbb{R}^n)$ onto $\mathcal{O}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, dx\mu(y)dy)$. 
(II) Let $K$ be a compact Lie group and $G$ be its complexification. Let $\nu$ be a measure on $G$ such that

- $\nu$ is bi-$K$-invariant,
- $\nu$ is given by a positive density which is locally bounded away from zero,
- For each irreducible representation $\pi$ of $K$, analytically continued to $G$,

$$\delta(\pi) = \frac{1}{\dim V_\pi} \int_G \|\pi(g^{-1})\|_2^2 d\nu(g) < \infty.$$ 

Define $\tau(g) = \sum_{\pi \in \hat{K}} \frac{d_\pi}{\sqrt{\delta(\pi)}} \text{Tr}(\pi(g^{-1})U_\pi)$ where $g \in G$ and $U_\pi$'s are arbitrary unitary matrices. Then the mapping

$$C_\tau f(g) = \int_K f(k)\tau(k^{-1}g)dk$$

is an isometric isomorphism of $L^2(K)$ onto $\mathcal{O}(G) \cap L^2(G,d\nu(w))$.

A similar result holds for $M$. Let $\mu$ be any real-valued $K$-invariant function on $\mathbb{R}^n$ such that it satisfies the conditions of Theorem 3.2 (I). Define, for $z \in \mathbb{C}^n$

$$\psi(z) = \int_{\mathbb{R}^n} \frac{e^{ia(y)}}{\sqrt{\sigma(y)}} e^{-iy.z} dy,$$

where $a$ is a real valued measurable $K$-invariant function on $\mathbb{R}^n$. Next, let $\nu$, $\delta$ and $\tau$ be as in Theorem 3.2 (II). Also define $\phi(z,g) = \psi(z)\tau(g)$ for $z \in \mathbb{C}^n$, $g \in G$. It is easy to see that $\phi(z,w)$ is a holomorphic function on $\mathbb{C}^n \times G$. Then it is easy to prove the following analogue of Theorem 3.2 for $M$.

**Theorem 3.3.** The mapping

$$C_\phi f(z,g) = \int_M f(\xi,k)\phi((\xi,k)^{-1}(z,g))d\xi dk$$

is an isometric isomorphism of $L^2(M)$ onto

$$\mathcal{O}(\mathbb{C}^n \times G) \cap L^2(\mathbb{C}^n \times G, \mu(y)dxdy\nu(g)).$$
4. Gutzmer’s formula and Poisson Integrals

In this section first we briefly recall Gutzmer’s formula on compact, connected Lie groups given by Lassalle in [9]. Let \( k \) and \( g \) be the Lie algebras of a compact, connected Lie group \( K \) and its complexification \( G \). Then we can write \( g = k + p \) where \( p = ik \) and any element \( g \in G \) can be written in the form \( g = k \exp(iH) \) for some \( k \in K \), \( H \in k \). If \( h \) is a maximal, abelian subalgebra of \( k \) and \( a = ih \) then every element of \( p \) is conjugate under \( K \) to an element of \( a \). Thus each \( g \in G \) can be written (non-uniquely) in the form \( g = k_1 \exp(iH) k_2 \) for \( k_1, k_2 \in K \) and \( H \in h \). If \( k_1 \exp(iH_1) k_1^\prime = k_2 \exp(iH_2) k_2^\prime \), then there exists \( w \in W \), the Weyl group with respect to \( h \), such that \( H_1 = w \cdot H_2 \) where \( \cdot \) denotes the action of the Weyl group on \( h \). Since \( K \) is compact, there exists an Ad-\( K \)-invariant inner product on \( k \), and hence on \( g \). Let \( | \cdot | \) denote the norm with respect to the said inner product. Then we have the following Gutzmer’s formula by Lassalle.

**Theorem 4.1.** Let \( f \) be holomorphic in \( K \exp(i\Omega_r)K \subseteq G \) where \( \Omega_r = \{ H \in k : |H| < r \} \). Then we have

\[
\int_K \int_K |f(k_1 \exp iH k_2)|^2 dk_1 dk_2 = \sum_{\pi \in \hat{K}} \|\hat{f}(\pi)\|^2_{HS} \chi_{\pi}(\exp 2iH)
\]

where \( H \in \Omega_r \) and \( \hat{f}(\pi) \) is the operator-valued Fourier transform of \( f \) at \( \pi \) defined by \( \hat{f}(\pi) = \int_K f(k) \pi(k^{-1}) dk \).

For the proof of above see [9]. We prove a Gutzmer-type result on \( M \) using Lassalle’s theorem above. Define \( \Omega_{t,r} = \{(z,g) \in \mathbb{C}^n \times G : |Imz| < t, |H| < r \} \) where \( g = k_1 \exp iH k_2, k_1, k_2 \in K, H \in h \). Notice that the domain \( \Omega_{t,r} \) is well defined since \( | \cdot | \) is invariant under the Weyl group action.

**Lemma 4.2.** Let \( f \in L^2(M) \) extend holomorphically to the domain \( \Omega_{t,r} \) and

\[
\sup_{\{|y|<s, |H|<q\}} \int_{\mathbb{R}^n} \int_K \int_K |f(x + iy, k_1 \exp (iH) k_2)|^2 dk_1 dk_2 dx < \infty
\]
∀ s < t and q < r. Then

\[ \int_{\mathbb{R}^n} \int_{K} \int_{K} |f(x + iy, k_1 \exp (iH)k_2)|^2 dk_1 dk_2 dx \]

\[ = \sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \left( \int_{\mathbb{R}^n} |\tilde{f}_{ij}^{\pi}(\xi)|^2 e^{-2\xi \cdot y} d\xi \right) \chi_\pi(\exp 2iH) \]

provided |y| < t and |H| < r. Conversely, if

\[ \sup \{ |y| < s, |H| < q \} \sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \left( \int_{\mathbb{R}^n} |\tilde{f}_{ij}^{\pi}(\xi)|^2 e^{-2\xi \cdot y} d\xi \right) \chi_\pi(\exp 2iH) < \infty \forall s < t \text{ and } q < r \]

then f extends holomorphically to the domain \( \Omega_{t,r} \) and

\[ \sup \{ |y| < s, |H| < q \} \int_{\mathbb{R}^n} \int_{K} \int_{K} |f(x + iy, k_1 \exp (iH)k_2)|^2 dk_1 dk_2 dx < \infty \]

∀ s < t and q < r.

Proof. Notice that \( f_{ij}^{\pi}(x) = \int_{K} f(x, k) \overline{\phi_{ij}^{\pi}(k)} dk \). It follows that \( f_{ij}^{\pi} \) has a holomorphic extension to \( \{ z \in \mathbb{C}^n : |Imz| < t \} \) and

\[ \sup_{|y| < s} \int_{\mathbb{R}^n} |f_{ij}^{\pi}(x + iy)|^2 dx < \infty \forall s < t. \]

Consequently,

\[ \int_{\mathbb{R}^n} |f_{ij}^{\pi}(x + iy)|^2 dx = \int_{\mathbb{R}^n} |\tilde{f}_{ij}^{\pi}(\xi)|^2 e^{-2\xi \cdot y} d\xi \text{ for } |y| < s \forall s < t. \]

Now, for each fixed \( z \in \mathbb{C}^n \) with \( |Imz| < s \) the function \( g \to f(z, g) \) is holomorphic in the domain \( \{ g \in G : |H| < r \text{ where } g = k_1 \exp iHk_2, k_1, k_2 \in K, H \in \mathbb{H} \} \) for every \( s < t \) and \( q < r \) and so admits a holomorphic Fourier series (as in [4])

\[ f(z, g) = \sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} a_{ij}^{\pi}(z) \phi_{ij}^{\pi}(g). \]
It follows that \( a_{ij}^\pi(z) = f_{ij}^\pi(z) \) for every \( \pi \in \hat{K} \) and \( 1 \leq i, j \leq d_\pi \). Hence by using Theorem 4.1 we have for \((z, g) \in \Omega_{t,r},\)

\[
\int_K \int_K |f(x + iy, k_1 \exp iH, k_2)|^2 dk_1 dk_2 = \sum_{\pi \in \hat{K}} \|\hat{f}_\pi(z)\|_{HS}^2 \chi_\pi(\exp 2iH)
\]

\[
= \sum_{\pi \in \hat{K}} \sum_{i,j=1}^{d_\pi} |f_{ij}^\pi(z)|^2 \chi_\pi(\exp 2iH)
\]

where \( f_z(g) = f(z, g) \). Integrating over \( \mathbb{R}^n \) we get

\[
\int_{\mathbb{R}^n} \int_K \int_K |f(x + iy, k_1 \exp iH, k_2)|^2 dk_1 dk_2 dx = \sum_{\pi \in \hat{K}} \sum_{i,j=1}^{d_\pi} \int_{\mathbb{R}^n} |f_{ij}^\pi(x + iy)|^2 dx \chi_\pi(\exp 2iH)
\]

\[
= \sum_{\pi \in \hat{K}} \sum_{i,j=1}^{d_\pi} \left( \int_{\mathbb{R}^n} |\hat{f}_{ij}^\pi(\xi)|^2 e^{-2\xi \cdot y} d\xi \right) \chi_\pi(\exp 2iH).
\]

Hence the first part of the lemma is proved. Converse can also be proved similarly.

Recall that the Laplacian \( \Delta \) on \( M \) is given by \( \Delta = -\Delta_{\mathbb{R}^n} - \Delta_K \). If \( f \in L^2(M) \) it is easy to see that

\[
e^{-t\Delta^{1/2}} f(x, k) = \sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \left( \int_{\mathbb{R}^n} e^{-t(\|\xi\|^2 + \lambda_\pi)} \hat{f}_{ij}^\pi(\xi) e^{i\xi \cdot y} d\xi \right) \phi_{ij}^\pi(k).
\]

We have the following (almost) characterization of the Poisson integrals. Let \( \Omega_{t,r} \) denote the domain defined in Lemma 4.2.

**Theorem 4.3.** Let \( f \in L^2(M) \). Then there exists a constant \( N \) such that \( g = e^{-t\Delta^{1/2}} f \) extends to a holomorphic function on the domain \( \Omega_{t,r} \) and

\[
\sup_{\left\{ |y| < \frac{r}{\sqrt{2}}, |H| \leq \frac{2t^2}{N} \right\}} \int_{\mathbb{R}^n} \int_K \int_K |g(x + iy, k_1 \exp iH, k_2)|^2 dk_1 dk_2 dx < \infty.
\]
Conversely, there exists a fixed constant $C$ such that whenever $g$ is a holomorphic function on $\Omega_t$, then
\[
\sup_{\{|y|<s, \ |H|<\frac{2s}{\sqrt{2}}\}} \int_{\mathbb{R}^n} \int_{K} |g(x + iy, k_1 \exp (iH)k_2)|^2dk_1dk_2dx < \infty \text{ for } s < t,
\]
then $\forall \ s < t, \ \exists f \in L^2(M)$ such that $e^{-s\Delta\frac{1}{2}}f = g$.

Proof. We know that, if $f \in L^2(M)$ then
\[
g(x, k) = e^{-t\Delta\frac{1}{2}}f(x, k) = \sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \left( \int_{\mathbb{R}^n} e^{-t(|\xi|^2 + \lambda_\pi)} f_i^\pi_j(\xi) e^{i\xi \cdot y} d\xi \right) \phi_i^\pi_j(k).
\]

Also, $g(x, k) = \sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} g_i^\pi_j(x) \phi_i^\pi_j(k)$ with $g_i^\pi_j(\xi) = f_i^\pi_j(\xi) e^{-t(|\xi|^2 + \lambda_\pi)^\frac{1}{2}}$. From Lemma 6 and 7 of [4], we know that there exist constants $a, M$ such that $\lambda_\pi \geq a|\mu|^2$ and 
\[
|\chi_\pi(\exp iY)| \leq d_\pi e^{M|\mu||\mu|} \text{ where } \mu \text{ is the highest weight of } \pi.
\]
Hence we have
\[
|\chi_\pi(\exp 2iH)| \leq d_\pi e^{2M|H||\mu|} \leq d_\pi e^{N|H|\sqrt{\lambda_\pi}}
\]
where $N = \frac{2M}{\sqrt{a}}$. If $s \leq \frac{t}{\sqrt{2}}$ it is easy to show that
\[
\sup_{\{\xi \in \mathbb{R}^n, \ |\lambda_\pi| \geq 0\}} e^{-2t(|\xi|^2 + \lambda_\pi)^\frac{1}{2}} e^{2t|\xi||y|} e^{N\sqrt{\lambda_\pi}|s|} \leq C < \infty \text{ for } |y| \leq \frac{t}{\sqrt{2}}.
\]
It follows that
\[
\sup_{\{|y|<\frac{t}{\sqrt{2}}, \ |H|\leq\frac{2\sqrt{2}t}{\sqrt{a}}\}} \sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \left( \int_{\mathbb{R}^n} |g_i^\pi_j(\xi)|^2 e^{-2\xi \cdot y} d\xi \right) e^{N\sqrt{\lambda_\pi}|H|} < \infty.
\]
So we have
\[
\sup_{\{|y|<\frac{t}{\sqrt{2}}, \ |H|\leq\frac{2\sqrt{2}t}{\sqrt{a}}\}} \sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \left( \int_{\mathbb{R}^n} |g_i^\pi_j(\xi)|^2 e^{-2\xi \cdot y} d\xi \right) \chi_\pi(\exp 2iH) < \infty.
\]
Hence by Lemma 4.2 we have proved one part of the theorem.

To prove the converse, we first show that there exist constants $A, C$ such that
\[
(4.1) \quad \int_{|H|=r} \chi_\pi(\exp 2iH) d\sigma_\pi(H) \geq d_\pi A e^{Cr\sqrt{\lambda_\pi}}
\]
where \( d\sigma_r(H) \) is the normalized surface measure on the sphere \( \{ H \in \mathfrak{h} : |H| = r \} \subseteq \mathbb{R}^m \) where \( m = \dim \mathfrak{h} \). If \( H \in \mathfrak{a} \), then there exists a non-singular matrix \( Q \) and pure-imaginary valued linear forms \( \nu_1, \nu_2, \cdots, \nu_{d_\pi} \) on \( \mathfrak{a} \) such that

\[
Q\pi(H)Q^{-1} = \text{diag}(\nu_1(H), \nu_2(H), \cdots, \nu_{d_\pi}(H))
\]

where \( \text{diag}(a_1, a_2, \cdots, a_k) \) denotes \( k \times k \) order diagonal matrix with diagonal entries \( a_1, a_2, \cdots, a_k \). Now, \( \nu(H) = i\langle \nu, H \rangle \) where \( \nu \) is a weight of \( \pi \). Then

\[
\exp(2iQ\pi(H)Q^{-1}) = Q \exp(2i\pi(H))Q^{-1} = \text{diag}(e^{2i\nu_1(H)}, e^{2i\nu_2(H)}, \cdots, e^{2i\nu_{d_\pi}(H)}).
\]

Hence

\[
\chi_\pi(\exp 2iH) = Tr(Q \exp(2i\pi(H))Q^{-1})
\]

\[
= e^{-2\langle \nu_1, H \rangle} + e^{-2\langle \nu_2, H \rangle} + \cdots + e^{-2\langle \nu_{d_\pi}, H \rangle}
\]

\[
\geq e^{-2\langle \mu, H \rangle}
\]

where \( \mu \) is the highest weight corresponding to \( \pi \). Integrating the above over \( |H| = r \) we get

\[
\int_{|H|=r} \chi_\pi(\exp 2iH)d\sigma_r(H) \geq \int_{|H|=r} e^{-2\langle \mu, H \rangle}d\sigma_r(H)
\]

\[
= J_{\frac{m}{2}-1}(2ir|\mu|)
\]

\[
= \frac{(2ir|\mu|)^{\frac{m}{2}-1}}{(2ir|\mu|)^{\frac{m}{2}-1}}
\]

\[
\geq Be^{r|\mu|}
\]

where \( J_{\frac{m}{2}-1} \) is the Bessel function of order \( \frac{m}{2} - 1 \). By Weyl’s dimension formula we know that \( d_\pi \) can be written as a polynomial in \( \mu \) and \( \lambda_\pi \approx |\mu|^2 \). Hence we have

\[
\int_{|H|=r} \chi_\pi(\exp 2iH)d\sigma_r(H) \geq A d_\pi e^{Cr\sqrt{\lambda_\pi}}
\]

for some \( C \). Consider the domain \( \Omega_{t, \frac{2s}{t}} \) for this \( C \). Let \( g \) be a holomorphic function on \( \Omega_{t, \frac{2s}{t}} \) and

\[
\sup_{\{ |y| < s, |H| < \frac{2s}{t} \}} \int_{\mathbb{R}^n} \int_{K} |g(x + iy, k_1 \exp(\langle iH, k_2 \rangle)|^2 dk_1dk_2dx < \infty \text{ for } s < t.
\]
By Lemma 4.2 we have
\[
\sup_{\{ |y| < s, \ |H| < \frac{2s}{\sqrt{t}} \}} \sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \left( \int_{\mathbb{R}^n} |\tilde{g}_{ij}^\pi(\xi)|^2 e^{-2\xi \cdot y} d\xi \right) \chi_n(\exp 2iH) < \infty \ \forall \ s < t.
\]

Integrating the above over \(|H| = r = \frac{2s}{\sqrt{r}}\) and \(|y| = s < t\) we obtain
\[
\sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \left( \int_{\mathbb{R}^n} |\tilde{g}_{ij}^\pi(\xi)|^2 \frac{J_{\frac{n}{2}-1}(2is|\xi|)}{(2is|\xi|)^{\frac{n}{2}-1}} d\xi \right) \int_{|H|=r} \chi_n(\exp 2iH)d\sigma_r(H) < \infty.
\]

Noting that \(\frac{J_{\frac{n}{2}-1}(2is|\xi|)}{(2is|\xi|)^{\frac{n}{2}-1}} \sim e^{2s|\xi|}\) for large \(|\xi|\) and using (4.1) we obtain
\[
\sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \int_{\mathbb{R}^n} |\tilde{g}_{ij}^\pi(\xi)|^2 e^{2s|\xi|} e^{2s\sqrt{\lambda_\pi}} d\xi < \infty \text{ for } s < t.
\]

This surely implies that
\[
\sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \int_{\mathbb{R}^n} |\tilde{g}_{ij}^\pi(\xi)|^2 e^{2s(|\xi|^2 + \lambda_\pi)} d\xi < \infty \text{ for } s < t.
\]

Defining \(\tilde{f}_{ij}^\pi(\xi)\) by \(\tilde{f}_{ij}^\pi(\xi) = \tilde{g}_{ij}^\pi(\xi)e^{s(|\xi|^2 + \lambda_\pi)}\frac{1}{2}\) we obtain
\[
f(x, k) = \sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} f_{ij}^\pi(x) \phi_{ij}^\pi(k) \in L^2(M)
\]
and \(g = e^{-s\Delta^\frac{1}{2}} f\).

\[\square\]

5. Complexified representations and Paley-Wiener type theorems

Recall the representations \(U^\xi\) and the generalized Fourier transform \(\hat{f}(\xi)\) from the introduction where
\[
\hat{f}(\xi) = \int_M f(m)U^\xi_m dm.
\]

For \((x, k) \in M\) and matrix coefficients \(\phi_{ij}^\pi\) of \(\pi\) we have
\[
\left( U^\xi_{(x,k)} \phi_{ij}^\pi \right)(u) = e^{i(x,u,\xi)} \phi_{ij}^\pi(k^{-1}u).
\]
This action of $U^\xi_{(z,g)}$ on $\phi^\pi_{ij}$ can clearly be analytically continued to $\mathbb{C}^n \times G$ and we obtain

$$
\left( U^\xi_{(z,g)} \phi^\pi_{ij} \right)(u) = e^{i(x,u \cdot \xi)} e^{-(y,u \cdot \xi)} \phi^\pi_{ij}(e^{-iH}k^{-1}u)
$$

where $(z,g) \in \mathbb{C}^n \times G$ and $z = x + iy \in \mathbb{C}^n$ and $g = ke^{iH} \in G$.

We also note that the action of $K \subseteq SO(n)$ on $\mathbb{R}^n$ naturally extends to an action of $G \subseteq SO(n, \mathbb{C})$ on $\mathbb{C}^n$. Then we have the following theorem:

**Theorem 5.1.** Let $f \in L^2(M)$. Then $f$ extends holomorphically to $\mathbb{C}^n \times G$ with

$$
\int_{|y|=r} \int_K \int_{\mathbb{R}^n} |f(e^{-iH}(x+iy), e^{-iH}k)|^2 dxdkd\mu_r(y) < \infty \quad \forall \ H \in k
$$

(where $\mu_r$ is the normalized surface area measure on the sphere $\{|y|=r\} \subseteq \mathbb{R}^n$) iff

$$
\int_{\mathbb{R}^n} \int_{|y|=r} \|U^\xi_{(z,g)} \hat{f}(\xi)\|^2_{HS} d\mu_r(y) d\xi < \infty
$$

where $z = x + iy \in \mathbb{C}^n$, $g = ke^{iH} \in G$. In this case we also have

$$
\int_{\mathbb{R}^n} \int_{|y|=r} \|U^\xi_{(z,g)} \hat{f}(\xi)\|^2_{HS} d\mu_r(y) d\xi = \int_{|y|=r} \int_K \int_{\mathbb{R}^n} |f(e^{-iH}(x+iy), e^{-iH}k)|^2 dxdkd\mu_r(y).
$$

We know that any $f \in L^2(M)$ can be expanded in the $K$ variable using the Peter Weyl theorem to obtain

$$
f(x, k) = \sum_{\pi \in \hat{K}} d_\pi \sum_{i,j=1}^{d_\pi} f^\pi_{ij}(x) \overline{\phi^\pi_{ij}(k)}
$$

(5.1)

where for each $\pi \in \hat{K}$, $d_\pi$ is the degree of $\pi$, $\phi^\pi_{ij}$’s are the matrix coefficients of $\pi$ and $f^\pi_{ij}(x) = \int_K f(x, k) \phi^\pi_{ij}(k) dk$.

Now, for $F \in L^2(\mathbb{R}^n)$, consider the decomposition of the function $k \mapsto F(k \cdot x)$ in terms of the irreducible unitary representations of $K$ given by

$$
F(k \cdot x) = \sum_{\lambda \in \hat{K}} d_\lambda \sum_{l,m=1}^{d_\lambda} F^l_m(x) \phi^\lambda_{lm}(k)
$$
where \( F_{\lambda m}^{lm}(x) = \int_{K} F(k \cdot x) \phi_{lm}^{\lambda}(k) dk \). Putting \( k = e \), the identity element of \( K \), we obtain

\[
F(x) = \sum_{\lambda \in \hat{K}} d_{\lambda} \sum_{l=1}^{d_{\lambda}} F_{\lambda m}^{lm}(x).
\]

Then it is easy to see that for \( u \in K \),

\[
(5.2) \quad F_{\lambda m}^{ll}(u \cdot x) = \sum_{m=1}^{d_{\lambda}} F_{\lambda m}^{lm}(x) \phi_{lm}^{\lambda}(u).
\]

It also follows that the Euclidean Fourier transform \( \hat{F}_{\lambda m}^{lm} \) of \( F_{\lambda m}^{lm} \) satisfies

\[
(5.3) \quad F_{\lambda m}^{ll}(u \cdot x) = \sum_{m=1}^{d_{\lambda}} \phi_{lm}^{\lambda}(u) \hat{F}_{\lambda m}^{lm}(x) \quad \forall u \in K.
\]

From the above and the fact that \( f_{\pi ij}^{\pi} \in L^{2}(\mathbb{R}^{n}) \) for every \( \pi \in \hat{K} \) and \( 1 \leq i, j \leq d_{\pi} \) it follows that any \( f \in L^{2}(M) \) can be written as

\[
f(x, k) = \sum_{\pi \in \hat{K}} d_{\pi} \sum_{\lambda \in \hat{K}} d_{\lambda} \sum_{i,j=1}^{d_{\pi}} \sum_{l=1}^{d_{\lambda}} (f_{\pi ij}^{\pi})_{ll}(x) \phi_{\pi ij}^{\lambda}(k).
\]

We need the following lemma to prove Theorem 5.1:

**Lemma 5.2.** For fixed \( \pi, \lambda \in \hat{K} \), the theorem is true for functions of the form

\[
f(x, k) = \sum_{i,j=1}^{d_{\pi}} \sum_{l=1}^{d_{\lambda}} (f_{\pi ij}^{\pi})_{ll}(x) \phi_{\pi ij}^{\lambda}(k)
\]

where for simplicity we write \((f_{\pi ij}^{\pi})_{ll}\) as \( f_{\pi ij}^{ll} \).

**Proof.** For \( \xi \in \mathbb{R}^{n}, u \in K, \gamma \in \hat{K} \) and \( 1 \leq p, q \leq d_{\gamma} \) we have

\[
\left( \hat{f}(\xi) \overline{\phi_{pq}^{\gamma}} \right)(u) = \int_{\mathbb{R}^{n}} \int_{K} \sum_{i,j=1}^{d_{\pi}} \sum_{l=1}^{d_{\lambda}} f_{\pi ij}^{ll}(x) \phi_{\pi ij}^{\lambda}(k) e^{i(x,u\cdot\xi)} \overline{\phi_{pq}^{\gamma}(k)} dk dx
\]

\[
= \sum_{i,j=1}^{d_{\pi}} \sum_{l=1}^{d_{\lambda}} \overline{f_{\pi ij}^{ll}(u \cdot \xi)} \sum_{l=1}^{d_{\lambda}} \phi_{\pi ij}^{\lambda}(u^{-1}) (\phi_{\pi ij}^{\lambda}, \overline{\phi_{pq}^{\gamma}})_{L^{2}(K)}
\]

\[
= \frac{\delta_{\pi \gamma}}{d_{\pi}} \sum_{i=1}^{d_{\pi}} \sum_{l,m=1}^{d_{\lambda}} f_{\pi ij}^{lm}(\xi) \phi_{lm}^{\lambda}(u) \overline{\phi_{pq}^{\gamma}(u)}
\]
by (5.3) and Schur’s orthogonality relations where $\delta_{\pi\gamma}$ is the Kronecker delta in the sense of equivalence of unitary representations. Then we have

$$\left( U_{(x+iy,ke^{iH})} \hat{f}(\xi) \phi_{pq} \right)(u) = \frac{\delta_{\gamma\pi}}{d_\pi} e^{i(x+iy,u\cdot\eta)} \sum_{\lambda} \sum_{l,m=1}^{d_\lambda} \widetilde{f}_{lp}^m(\xi) \phi_{lm}^\lambda(e^{-iH}k^{-1}u) \phi_{pq}^\pi(u^{-1}ke^{iH}).$$

Hence

$$\| U_{(x+iy,ke^{iH})} \hat{f}(\xi) \|_{HS}^2 = \frac{1}{d_\pi} \sum_{p,q=1}^{d_\pi} \int_K e^{-2(y,u\cdot\eta)} \left| \sum_{\lambda} \sum_{l,m=1}^{d_\lambda} \widetilde{f}_{lp}^m(\xi) \phi_{lm}^\lambda(e^{-iH}k^{-1}u) \phi_{pq}^\pi(u^{-1}ke^{iH}) \right|^2 du.$$

Integrating the above over $|y|=r$, we obtain

$$\int_{|y|=r} \| U_{(x+iy,ke^{iH})} \hat{f}(\xi) \|_{HS}^2 d\mu_r(y) = \frac{1}{d_\pi (2r|\xi|)^{\frac{n}{2}-1}} \sum_{p,q=1}^{d_\pi} \sum_{l,m=1}^{d_\lambda} \int_K \left| \sum_{\lambda} \sum_{l,m=1}^{d_\lambda} \widetilde{f}_{lp}^m(\xi) \phi_{lm}^\lambda(e^{-iH}u) \phi_{pq}^\pi(u^{-1}ke^{iH}) \right|^2 du,$$

where $J_{\frac{n}{2}-1}$ is the Bessel function of order $\frac{n}{2} - 1$ and $\mu_r$ is the normalized surface area measure on the sphere $\{|y|=r\} \subset \mathbb{R}^n$.

Let $\mathcal{H}_\pi$ be the Hilbert space on which $\pi(k)$ acts for every $k \in K$ and $e_1, e_2, \cdots, e_{d_\pi}$ be a basis of $\mathcal{H}_\pi$. Then, for any $c_i, 1 \leq i \leq d_\pi$,

$$\sum_{q=1}^{d_\pi} \sum_{i=1}^{d_\pi} c_i \phi_{pi}^\pi(u^{-1}ke^{iH})^2 = \sum_{q=1}^{d_\pi} \sum_{i=1}^{d_\pi} c_i \phi_{pi}^\pi(u^{-1}e^{iH}) \sum_{a=1}^{d_\pi} c_a \phi_{qa}^\pi(u^{-1}e^{iH})$$

$$= \sum_{i,a=1}^{d_\pi} c_i c_a \sum_{q=1}^{d_\pi} \langle \pi(u^{-1}e^{iH})e_i, e_q \rangle \langle e_q, \pi(u^{-1}e^{iH})e_a \rangle$$

$$= \sum_{i,a=1}^{d_\pi} c_i c_a \langle \pi(u^{-1})\pi(e^{iH})e_i, e_a \rangle$$

$$= \sum_{i=1}^{d_\pi} \sum_{q=1}^{d_\pi} c_i \phi_{pi}^\pi(e^{iH})^2.$$
Hence from (5.4) we get that

\[
\sum_{q=1}^{d_\pi} \sum_{i=1}^{d_\pi} \sum_{l,m=1}^{d_\lambda} \overline{f_{lp}^m}(\xi) \phi_{lm}^\lambda(e^{-iH} u) \phi_{qj}^\pi(u^{-1} e^{iH}) = \sum_{q=1}^{d_\pi} \sum_{i=1}^{d_\pi} \sum_{l,m=1}^{d_\lambda} \overline{f_{lp}^m}(\xi) \phi_{lm}^\lambda(e^{-iH} u) \phi_{qj}^\pi(e^{iH})^2.
\]

So, we have obtained an expression for one part of Lemma 5.2. Now, looking at the other part, we have

\[
f(u^{-1} \cdot x, u^{-1} k^{-1}) = \sum_{i,j=1}^{d_\pi} \sum_{l,m=1}^{d_\lambda} f_{ij}^{lm}(x) \phi_{lm}^\lambda(u^{-1}) \phi_{ji}^\pi(k u).
\]

So, if \( f \) is holomorphic on \( \mathbb{C}^n \times G \), for \( z = x + iy \) we get

\[
f(e^{-iH} u^{-1} \cdot z, e^{-iH} u^{-1} k^{-1}) = \sum_{i,j,q=1}^{d_\pi} \sum_{l,m=1}^{d_\lambda} f_{ij}^{lm}(z) \phi_{lm}^\lambda(e^{-iH} u^{-1}) \phi_{jq}^\pi(k) \phi_{qj}^\pi(ue^{iH}).
\]

Again, by Schur's orthogonality relations and similar reasoning as before, we have

\[
\int_K \left| f(e^{-iH} u^{-1} \cdot z, e^{-iH} u^{-1} k^{-1}) \right|^2 dk = \frac{1}{d_\pi} \sum_{j,q=1}^{d_\pi} \sum_{i=1}^{d_\pi} \sum_{l,m=1}^{d_\lambda} f_{ij}^{lm}(z) \phi_{lm}^\lambda(e^{-iH} u^{-1}) \phi_{qj}^\pi(ue^{iH})
\]

\[
= \frac{1}{d_\pi} \sum_{j,q=1}^{d_\pi} \sum_{i=1}^{d_\pi} \sum_{l,m=1}^{d_\lambda} f_{ij}^{lm}(z) \phi_{lm}^\lambda(e^{-iH} u^{-1}) \phi_{qj}^\pi(e^{iH})^2.
\]
Hence, by invariance of Haar measure, we have

\[
\int_{\mathbb{R}^n} \int_K \int_K \left| f(e^{-iH}u^{-1} \cdot z, e^{-iH}u^{-1}k^{-1}) \right|^2 dk du dx
\]

\[
= \frac{1}{dn} \sum_{j,q=1}^{dn} \int_{\mathbb{R}^n} \int_K \int_K \left| f_{ij}^{lm}(z) \phi_{ip}^{\lambda}(e^{-iH}) \phi_{pm}^{\lambda}(u^{-1}) \phi_{qi}^{\pi}(e^{iH}) \right|^2 du dx
\]

\[
= \frac{1}{dn} \sum_{j,q=1}^{dn} \int_{\mathbb{R}^n} \int_K \int_K \left| f_{ij}^{lm}(x + iy) \phi_{ip}^{\lambda}(e^{-iH}) \phi_{qi}^{\pi}(e^{iH}) \right|^2 dx
\]

\[
= \frac{1}{dn} \sum_{j,q=1}^{dn} \int_{\mathbb{R}^n} \int_K \int_K \left| f_{ij}^{lm}(\xi) \phi_{ip}^{\lambda}(e^{-iH}) \phi_{qi}^{\pi}(e^{iH}) \right|^2 e^{-2(y,\xi)} d\xi.
\]

Now by the invariance of Lebesgue measure under the $K$-action on $\mathbb{R}^n$ we get that

\[
\int_{|y|=r} \int_{\mathbb{R}^n} \int_K \int_K \left| f(e^{-iH}u^{-1} \cdot z, e^{-iH}u^{-1}k^{-1}) \right|^2 dk du dx d\mu_r(y)
\]

\[
= \int_{|y|=r} \int_{\mathbb{R}^n} \int_K \left| f(e^{-iH} \cdot z, e^{-iH}k) \right|^2 dk dx d\mu_r(y).
\]

Hence the lemma follows from (3.5).

\[\square\]

**Proof of Theorem 5.1.**

To prove the theorem, it is enough to prove the orthogonality of the components

\[
f_{\pi}^{\lambda}(x, k) = \sum_{i,j=1}^{dn} \sum_{l,m=1}^{dn} f_{ij}^{lm}(x) \phi_{ip}^{\lambda}(k). \]

For $\pi, \lambda, \tau, \nu \in \hat{K}$, we have

\[
\left\langle \hat{U}_{(x+iy,ke^{iH})}^{\xi} \hat{\pi}^{\xi}(\xi), \hat{U}_{(x+iy,ke^{iH})}^{\xi} \hat{\pi}^{\nu}(\xi) \right\rangle_{HS}
\]

\[
= \sum_{\gamma \in \hat{K}} d_{\gamma} \sum_{p,q=1}^{dn} \int_K \delta_{\gamma,\pi} e^{i(x+iy, u \cdot \xi)} \sum_{i,j=1}^{dn} \sum_{l,m=1}^{dn} f_{ij}^{lm}(\xi) \phi_{ip}^{\lambda}(e^{-iH}k^{-1}u) \phi_{qi}^{\pi}(u^{-1}ke^{iH})
\]

\[
\frac{d_{\gamma}}{d_{\tau}} e^{i(x+iy, u \cdot \xi)} \sum_{a,b,c=1}^{dn} f_{ab}^{bc}(\xi) \phi_{bc}^{\nu}(e^{-iH}k^{-1}u) \phi_{qa}^{\tau}(u^{-1}ke^{iH}) du
\]

\[
= 0 \text{ if } \pi \neq \tau.
\]
Assume $\pi \cong \tau$. Then
\[
\int_{|y|=r} \left\langle U_{(x+i_y, k e^{i\theta})} F^\lambda(\xi), U_{(x+i_y, k e^{i\theta})} F^\nu(\xi) \right\rangle_{HS} d\mu_\tau(y)
\]
\[
= \frac{1}{d_\pi} \frac{J_{\frac{n}{2}-1}(2ir|\xi|)}{(2ir|\xi|)\frac{n}{2}-1} \sum_{a,i,p=1}^{d_\lambda} \sum_{l,m=1}^{d_\lambda} \sum_{b,c=1}^{d_\nu} \int_{K} \left( \sum_{q=1}^{d_\nu} \phi_{q}^\pi(u^{-1} e^{i\nu} \phi_{q}^\nu(u^{-1} e^{i\nu}) \right) \phi_{lm}^\lambda(e^{-i\nu} u) \phi_{bc}^\nu(e^{-i\nu} u) du
\]
\[
= \frac{1}{d_\pi} \frac{J_{\frac{n}{2}-1}(2ir|\xi|)}{(2ir|\xi|)\frac{n}{2}-1} \sum_{a,i,p=1}^{d_\lambda} \sum_{l,m=1}^{d_\lambda} \sum_{b,c=1}^{d_\nu} \int_{K} \phi_{lm}^\lambda(u) \phi_{bc}^\nu(u) du
\]
\[
= 0 \text{ if } \lambda \not\cong \nu.
\]

On the other hand, we have
\[
\int_{K} \phi_{ij}^\nu(e^{-i\nu} k) \phi_{bk}^\nu(e^{-i\nu} k) du
\]
\[
= 0 \text{ if } \nu \not\cong \tau.
\]

Assume $\pi \cong \tau$. Then we get
\[
\int_{K} \int_{K} \phi_{ij}^\nu(e^{-i\nu} u^{-1} \cdot z, e^{-i\nu} u^{-1} k^{-1}) f_{ij}^{\lambda}(e^{-i\nu} u^{-1} \cdot z, e^{-i\nu} u^{-1} k^{-1}) du dk
\]
\[
= \sum_{i,j,q=1}^{d_\nu} \sum_{l,m=1}^{d_\nu} \sum_{a,b,p=1}^{d_\nu} \sum_{s,t=1}^{d_\nu} \int_{K} \phi_{q}^\nu(u e^{i\nu} k) \phi_{p}^\nu(u e^{i\nu} k) \int_{K} \phi_{q}^\nu(k) \phi_{p}^\nu(k) dk
\]
\[
= 0 \text{ if } \nu \not\cong \tau.
\]

This finishes the proof.
It is easy to see that
\[ \left( \int_{\mathbb{R}^n} \left\| U_{(z,g)}^\xi \hat{f}(\xi) \right\|_{HS}^2 d\xi \right) = \sum_{\sigma \in \hat{K}_\xi} \left( \int_{\mathbb{R}^n} \left\| U_{(z,g)}^{\xi,\sigma} \hat{f}(\xi,\sigma) \right\|_{HS}^2 d\xi \right). \]

Hence we have the following corollary:

**Corollary 5.3.** For \( f \in L^2(M) \), \( f \) extends holomorphically to \( \mathbb{C}^n \times G \) with
\[ \int_{|y|=r} \int_{K} \int_{\mathbb{R}^n} |f(e^{-iH}(x+iy), e^{-iH}k)|^2 dxdkd\mu_r(y) < \infty \]
(\text{where} \( \mu_r \) is the normalized surface area measure on the sphere \( \{|y|=r\} \subseteq \mathbb{R}^n \)) \iff
\[ \sum_{\sigma \in \hat{K}_\xi} \int_{\mathbb{R}^n} \int_{|y|=r} \left\| U_{(z,g)}^{\xi,\sigma} \hat{f}(\xi,\sigma) \right\|_{HS}^2 d\mu_r(y) d\xi < \infty \]

where \( z = x + iy \in \mathbb{C}^n \), \( g \in G \) and we also have
\[ \sum_{\sigma \in \hat{K}_\xi} \int_{\mathbb{R}^n} \int_{|y|=r} \left\| U_{(z,g)}^{\xi,\sigma} \hat{f}(\xi,\sigma) \right\|_{HS}^2 d\mu_r(y) d\xi = \int_{|y|=r} \int_{K} \int_{\mathbb{R}^n} |f(e^{-iH}(x+iy), e^{-iH}k)|^2 dxdkd\mu_r(y). \]

**Acknowledgement.** The author wishes to thank Dr. E. K. Narayanan for his encouragement and for the many useful discussions during the course of this work.

**References**

1. G. B. Folland, *A course in abstract harmonic analysis*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, (1995).
2. R. W. Goodman, *Analytic and entire vectors for representations of Lie groups*, Trans. Amer. Math. Soc., 143 (1969), 55–76.
3. R. W. Goodman, *Complex Fourier analysis on a nilpotent Lie group*, Trans. Amer. Math. Soc., 160 (1971), 373–391.
4. B. C. Hall, *The Segal-Bargmann “coherent state” transform for compact Lie groups*, J. Funct. Anal., 122 (1994) no. 1, 103–151.
5. B. C. Hall, W. Lewkeeratiyutkul, *Holomorphic Sobolev spaces and the generalized Segal-Bargmann transform*, J. Funct. Anal., 217 (2004) no. 1, 192–220.
[6] B. C. Hall, J. J. Mitchell, *The Segal-Bargmann transform for non compact symmetric spaces of the complex type*, J. Funct. Anal., 227 (2005) no. 2, 338–371.

[7] B. Krötz, G. Ólafsson, R. J. Stanton, *The image of the heat kernel transform on Riemannian symmetric spaces of the non compact type*, Int. Math. Res. Not., (2005) no. 22, 1307–1329.

[8] B. Krötz, S. Thangavelu, Y. Xu, *The heat kernel transform for the Heisenberg group*, J. Funct. Anal., 225 (2005), no. 2, 301–336.

[9] M. Lassalle, *Series de Laurent des fonctions holomorphes dans la complexification d’un espace symétrique compact*, Ann. Sci. École Norm. Sup., 11 (1978), 167–210.

[10] G. W. Mackey, *Infinite-dimensional group representations*, Bull. Amer. Math. Soc., 69 (1963), 628–686.

[11] E. K. Narayanan, S. Sen, *Segal-Bargmann transform and Paley-Wiener theorems on M(2)*, to appear in Proc. Indian Acad. Sci. Math. Sci.

[12] M. B. Stenzel, *The Segal-Bargmann transform on a symmetric space of compact type*, J. Funct. Anal., 165 (1999) no. 1, 44–58.

[13] M. Sugiura, *Fourier series of smooth functions on compact Lie groups*, Osaka J. Math., 8 (1971), 33–47.

Department of Mathematics, Indian Institute of Science, Bangalore - 560012, India.

E-mail address: suparna@math.iisc.ernet.in.