ON THE NOTION OF GEOMETRIC REALIZATION

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To Borya Feigin on his 50th birthday

Abstract. We explain why geometric realization commutes with Cartesian products and why the geometric realization of a simplicial set (resp. cyclic set) is equipped with an action of the group of orientation preserving homeomorphisms of the segment $[0,1]$ (resp. the circle).

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The next simplest curve is a circle. Even so simple a figure as this has given rise to so many and such profound investigations that they could constitute a course by themselves.

D. Hilbert and S. Cohn-Vossen, Geometry and the imagination (Anschauliche Geometrie), ch. 1, page 1.

In this note there are no theorems, its only goal is to clarify the notion of geometric realization for simplicial sets \cite{14,13,6,7} and cyclic sets \cite{4,2,11}. We reformulate the definitions so that the following facts become obvious:

(i) geometric realization commutes with finite projective limits (e.g., with Cartesian products);

(ii) the geometric realization of a simplicial set (resp. cyclic set) is equipped with an action of the group of orientation preserving homeomorphisms of the segment $I := [0,1]$ (resp. the circle $S^1$).

In the traditional approach \cite{14,13,6,7,4,2,11} these statements are theorems, and understanding their proofs requires some efforts.

Example. To a small category $C$ there corresponds a simplicial set $NC$ (the nerve of $C$) and a cyclic set $N_{\text{cyc}} C$ (the cyclic nerve). It follows from our formula \eqref{1.1} that a point of the geometric realization $|NC|$ is a piecewise constant functor $I \to C$. Here it is assumed that the category structure on $I$ comes from the standard order on $I$, and the definition of piecewise constant functor is explained in \cite{14}. In \cite{6} we give a similar description of $|N_{\text{cyc}} C|$ in which $I$ is replaced by $S^1$.
Conventions. The set of non-negative integers is denoted by $\mathbb{Z}_+$. Unless stated otherwise, an ordered set $I$ is always equipped with the category structure such that the number of morphisms from $i \in I$ to $j \in I$ equals 1 if $i \leq j$ and 0 otherwise.

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Having written this article I learned that a very similar approach had been developed by A. Besser [1] and D. Grayson [8]. It is used in [8] to treat not only products of simplicial sets but also Quillen’s edgewise subdivision.

1. Simplicial sets

Put $I := [0, 1]$. This is an ordered set and therefore a category.

Recall that a simplicial set is a functor $X : \Delta^\circ \to \text{Sets}$, where $\Delta$ is the category whose objects are the ordered sets $[n] := \{0, \ldots, n\}$, $n \in \mathbb{Z}_+$, and whose morphisms are nondecreasing maps. One can extend such $X$ to a functor $\Delta_{\text{big}}^\circ \to \text{Sets}$, where $\Delta_{\text{big}}$ is the category of non-empty finite linearly ordered sets. Such an extension is unique up to unique isomorphism.

We define the geometric realization of a simplicial set $X$ to be the set

$$|X| := \lim_{\to} X(\pi_0(I \setminus F)),$$

where $F$ runs through the set of all finite subsets of $I$ and $\pi_0(I \setminus F)$ is equipped with the natural order (so $\pi_0(I \setminus F) \in \Delta_{\text{big}}$ is a quotient of $I \setminus F$). At this point $|X|$ is viewed merely as a set, the definition of the topology on $|X|$ will be recalled later. One can also rewrite (1.1) as

$$|X| := X(I),$$

where $I$ is the pro-object of $\Delta_{\text{big}}$ which is the projective limit of the objects $\pi_0(I \setminus F) \in \Delta_{\text{big}}$ over all finite subsets $F \subset I$.

It immediately follows from the definition that geometric realization commutes with finite projective limits (e.g., the map $|X \times Y| \to |X| \times |Y|$ is bijective) and that the group $\text{Aut} I$ of orientation preserving homeomorphisms of $I$ acts on $|X|$.

Example. The geometric realization $|[n]|$ of $[n]$ (i.e., of the functor $\Delta^\circ \to \text{Sets}$ represented by $[n]$) is the set of piecewise constant nondecreasing functions $f : I \to [n]$ modulo the following equivalence relation: $f_1 \sim f_2$ if $f_1$ and $f_2$ are equal outside a finite set. We identify $|[n]|$ with the standard simplex

$$\text{Sim}^n := \{(x_1, \ldots, x_n) \in I^n | x_1 \leq \ldots \leq x_n \}$$

(a function $f : I \to [n]$ such that $f(x) = i$ for $x_i < x < x_{i+1}$ is identified with $(x_1, \ldots, x_n) \in \Delta^n$; here we assume that $x_0 := 0, x_{n+1} := n$).
Recall that the nerve $NC$ of a small category $C$ is defined by $NC(\delta) := \text{Funct}(\delta, C)$ for $\delta \in \Delta_{\text{big}}$, where $\text{Funct}(\delta, C)$ is the set of functors $\delta \to C$. So $|NC|$ is the set of piecewise constant functors $I \to C$, more precisely

$$(1.4) \quad |NC| := \lim_{F} \text{Funct}(\pi_{0}(I \setminus F), C).$$

To pass to the usual definition of $|X|$ use the canonical representation of a functor $X : \Delta^{o} \to \text{Sets}$ as an inductive limit of representable ones, namely

$$(1.5) \quad X = \lim_{\Delta/X} \Phi,$$

where $\Delta/X$ is the category of pairs consisting of $\delta \in \Delta$ and a morphism $\xi : \delta \to X$ (i.e., an element $\xi \in X(\delta)$), and $\Phi$ is the functor from $\Delta/X$ to the category of simplicial sets that sends $(\delta, \xi)$ to $\delta$. As geometric realization commutes with inductive limits and the functor $n \mapsto |[n]|$ identifies with the usual “standard simplex” functor $\text{Sim} : \Delta \to \text{Sets}$ we get the usual formula

$$(1.6) \quad |X| = \lim_{\Delta/X} |\Phi|, \quad |\Phi|(\delta, \xi) := \text{standard simplex } |\delta|,$$

which can be rewritten as

$$(1.7) \quad |X| = X \times_{\Delta} \text{Sim},$$

where $X \times_{\Delta} \text{Sim}$ denotes the coend of the bifunctor $X \times \text{Sim} : \Delta^{o} \times \Delta \to \text{Sets}$ (which is a kind of “tensor product of the right $\Delta$-module $X$ and the left $\Delta$-module $\text{Sim}$”, see §IX.6 of [12] or Appendix B.7 of [11] for more details on the $\times_{\Delta}$ bifunctor).

One uses the topology on the standard simplices and either (1.6) or (1.7) to define the standard topology on the geometric realization, then one checks that geometric realization considered as a functor from the category of simplicial sets to that of compactly generated spaces commutes with finite projective limits; see [6, 7, 13] for more details.

One can describe the topology on $|X|$ using the following metric. To define it first choose a measure $\mu$ on $I$ such that the measure of every point is zero and the measure of every non-empty open set is non-zero. For a finite $F \subset I$ denote by $\mu_{F}$ the measure on $\pi_{0}(I \setminus F)$ induced by $\mu$. If $u, v \in X(\pi_{0}(I \setminus F))$ define the distance $d_{\mu}(u, v)$ to be the minimum of $\mu_{F}(\pi_{0}(I \setminus F) \setminus A)$ for all subsets $A \subset \pi_{0}(I \setminus F)$ such that the images of $u$ and $v$ in $X(A)$ are equal. If $F' \supset F$ and $u', v'$ are the images of $u, v$ in $X(\pi_{0}(I \setminus F))$ then $d_{\mu}(u', v') = d_{\mu}(u, v)$, so we get a well defined metric $d_{\mu}$ on $|X|$. It is easy to see that $d_{\mu}$ is continuous for the standard topology of $|X|$. It follows that this topology is Hausdorff. It also follows that if $X$ is finite the standard topology coincides with the one defined by the metric $d_{\mu}$. For any $X$ the topological space $|X|$ is the direct limit of the geometric realizations of the finite subsets of $X$. 


Remarks. (i) Already [14] and [6] suggest that in the theory of simplicial sets one should use the realization $[1.3]$ of the standard simplex rather than the “baricentric” realization $y_0 + \ldots + y_n = 1, y_i \geq 0$. This is also natural in view of the theory of iterated integrals [10].

(ii) Here is a way to think of the pro-object $I$ from $[1.2]$. Consider the following category $\nabla$. The objects of $\nabla$ are finite linearly ordered sets $J$ of order $\geq 2$ (so the minimal element $0 = 0_J \in J$ does not equal the maximal element $1 = 1_J \in J$). The morphisms of $\nabla$ are nondecreasing maps $f : J \to J'$, $J, J' \in \text{Ob} \nabla$, such that $f(0_J) = 0_{J'}$ and $f(1_J) = 1_{J'}$. It is essentially explained in §III.1.1 of [6] that $\nabla$ is antiequivalent to $\Delta$. More precisely, the functor $\Delta_{\text{big}}^o \to \nabla$ that sends $I \in \Delta_{\text{big}}$ to the set $I^* := \text{Hom}_{\Delta_{\text{big}}}(I, \{0, 1\})$ equipped with the natural (“pointwise”) order is an equivalence. To see this first notice that it has a left adjoint (namely the functor $\nabla \to \Delta_{\text{big}}^o$ that sends $J \in \nabla$ to the set $J^* := \text{Hom}_{\nabla}(J, \{0, 1\})$ equipped with the natural order) and then show that for all $I \in \text{Ob} \Delta_{\text{big}}$ and $J \in \text{Ob} \nabla$ the adjunction morphisms $I \to I^*$ and $J \to J^*$ are isomorphisms.

The functor $I \mapsto I^*$ induces an antiequivalence between the category of pro-objects of $\Delta_{\text{big}}$ and that of ind-objects of $\nabla$. The latter category identifies with the category $\nabla_{\text{finite}}$ of all linearly ordered sets $J$ having a minimal element $0 = 0_J \in J$ and a maximal element $1 = 1_J \in J$ such that $0 \neq 1$ (the morphisms of this category are nondecreasing maps such that $0 \mapsto 0$ and $1 \mapsto 1$). So one can consider $I = [0, 1]$ as an ind-object of $\nabla$. It is easy to see that the pro-object $I$ from $[1.2]$ corresponds to $I = [0, 1]$ viewed as an ind-object of $\nabla$. Therefore (1.2) implies Theorem III.1.3 of [6], which says that the group of automorphisms of the geometric realization functor from the category of simplicial sets to that of sets equals the group of automorphisms of the ordered set $I$ (which is the same as the group of orientation preserving homeomorphisms of $I$).

(iii) Formula (1.2) is natural in view of the general theory of fiber functors on a topos developed in §6.8 of [9] (especially cf. §6.8.6 - §6.8.7). We need only the case of a topos of the form $\hat{C} :=$ the category of presheaves of sets on a small category $C$ (i.e., $\hat{C}$ is the category of functors $C^\circ \to \text{Sets}$). According to [9], a functor $\hat{C} \to \text{Sets}$ is said to be a fiber functor if it commutes with finite projective limits and arbitrary inductive limits. Let $\text{Fib}(\hat{C})$ denote the full subcategory of the category of functors $\hat{C} \to \text{Sets}$ that consists of fiber functors. It is closed under inductive limits, so the functor $F : C^\circ \to \text{Fib}(\hat{C})$ defined by $(Fc)(X) := X(c), c \in C$, $X \in \hat{C}$, canonically extends to a functor $(\text{Pro-}\ C)^o = \text{Ind-}\ C^\circ \to \text{Fib}(\hat{C})$, where Pro-$C$ (resp. Ind-$C$) denotes the category of pro-objects (resp. ind-objects). According to §6.8 of [9], the functor $(\text{Pro-}\ C)^o \to \text{Fib}(\hat{C})$ is an equivalence. As explained in [9], the quasi-inverse functor is constructed as follows. Let $\Phi : \hat{C} \to \text{Sets}$ be a fiber functor. Let $\mathcal{I}$ be the category of pairs $(c, \xi), c \in C^\circ, \xi \in \Phi(c)$. According to [9], $\mathcal{I}$ is a filtering category, and the functor $\mathcal{I} \to C^\circ$ defines the desired
pro-object of \( C \). To show that \( \mathcal{I} \) is filtering the authors of \[9\] first notice that the category \( \mathcal{I} \) of pairs \((c, \xi), c \in \mathcal{C}^0, \xi \in \Phi(c)\), is filtering (because \( \Phi \) commutes with finite projective limits) and then deduce from this that \( \mathcal{I} \) is also filtering using the fact that \( \Phi \) commutes with direct limits (this fact and the analog of formula \((1.5)\) with \( \Delta \) replaced by \( C \) imply that every object of \( \mathcal{I} \) can be mapped to an object of \( \mathcal{I} \)).

2. \( \mathbb{Z}_+ \)-categories and the category \( \Lambda \)

In the theory of cyclic sets the role of \( I = [0,1] \) and \([n] \in \Delta \) is played by certain \( \mathbb{Z}_+ \)-categories.

We define a \( \mathbb{Z}_+ \)-category to be a category \( C \) equipped with a morphism of monoids \( \mathbb{Z}_+ \rightarrow \text{End} \ id_C \) (the operation in \( \mathbb{Z}_+ \) is addition). In other words, each object \( c \in C \) should be equipped with an endomorphism \( 1_c \) so that \( f 1_{c_1} = 1_{c_2} f \) for every morphism \( f : c_1 \rightarrow c_2 \) in \( C \). If \( C, C' \) are \( \mathbb{Z}_+ \)-categories then a \( \mathbb{Z}_+ \)-functor \( C \rightarrow C' \) is a functor \( \Phi : C \rightarrow C' \) such that \( \Phi(1_c) = 1_{\Phi(c)} \) for all \( c \in C \). A \( \mathbb{Z}_+ \)-isomorphism is a \( \mathbb{Z}_+ \)-functor which is an isomorphism. A full subcategory of a \( \mathbb{Z}_+ \)-category is a \( \mathbb{Z}_+ \)-category.

Examples. In this work we will work only with the following examples of \( \mathbb{Z}_+ \)-categories. All of them belong to the class described in Proposition 2.1:

1. Unless stated otherwise, we will always equip \( S^1 := \mathbb{R}/\mathbb{Z} \) with the following structure of \( \mathbb{Z}_+ \)-category: morphisms from \( c \in S^1 \) to \( c' \in S^1 \) are homotopy classes of “oriented” paths \( I \rightarrow S^1 \) (i.e., of those paths which can be represented as a composition \( I \overset{f}{\rightarrow} \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} = S^1 \) with \( f \) nondecreasing), and the composition of morphisms is usual (so our category is a subcategory of the fundamental groupoid of \( S^1 \), and \( 1_x \) is the homotopy class of a degree 1 path from \( x \in S^1 \) to \( x \)).

The category \( S^1 \) is generated by morphisms \( xy, x \neq y \), where \( xy \) is the homotopy class of the shortest oriented path from \( x \in S^1 \) to \( y \in S^1 \). If \( x \neq y \) then \( yx xy = 1_x \), and if \( x, y, z \in S^1 \) are distinct points in the correct cyclic order then \( yz x = xz \). This is a complete set of relations between the generators \( xy \) and \( 1_x \).

2. For every \( n \in \mathbb{Z}_+ \) embed \([n] := \{0, \ldots, n\} \) into \( S^1 \) by \( k \rightarrow k/(n + 1) \) mod \( \mathbb{Z} \). The set \([n]\) equipped with the \( \mathbb{Z}_+ \)-category structure induced from \( S^1 \) will be denoted by \([n]_{\text{cyc}} \).

3. For every non-empty finite \( F \subset S^1 \) there is a unique \( \mathbb{Z}_+ \)-category \( C \) equipped with a \( \mathbb{Z}_+ \)-functor \( \pi_F : S^1 \setminus F \rightarrow C \) whose restriction to any system of representatives \( R \) of connected components of \( S^1 \setminus F \) is an isomorphism (\( R \) is considered as a full subcategory of \( S^1 \)). We will denote \( C \) simply by \( \pi_0(S^1 \setminus F) \). Of course, \( \pi_0(S^1 \setminus F) \) is (non-canonically) \( \mathbb{Z}_+ \)-isomorphic to \([n]_{\text{cyc}} \), \( n = \text{Card} F - 1 \). If \( F \subset F' \) there is a unique \( \mathbb{Z}_+ \)-functor \( \Phi : \pi_0(S^1 \setminus F') \rightarrow \pi_0(S^1 \setminus F) \) such that \( \Phi \pi_{F'} = \pi_F \).

4. Given a small category \( \mathcal{A} \) we will define a \( \mathbb{Z}_+ \)-category \( \mathcal{A}_{\text{cyc}} \) with the same set of objects (in fact, we will use this construction only for categories
that come from linearly ordered sets, and it is not clear if it is reasonable in general). By definition, \( \mathcal{A}_{\text{cyc}} \) is generated by \( \mathcal{A} \) and new generators \( f^* \) corresponding to \( f \in \text{Mor} \mathcal{A} \). More precisely, if \( f \) is a morphism \( c_1 \to c_2 \) then \( f^* \) is a morphism \( c_2 \to c_1 \) (intuitively, \( f^* = 1 f^{-1} = f^{-1} 1 \)). Defining relations in \( \mathcal{A}_{\text{cyc}} \): \((gf)^* g = f^* \), \((gf)^* = g^* f^* \). \( \mathcal{A}_{\text{cyc}} \) becomes a \( \mathbb{Z}_+ \)-category if one defines \( 1_c : c \to c \) by \( 1_c := (1_c)^* \). To see this notice that for every \( \mathcal{A} \)-morphism \( f : c_1 \to c_2 \) one has \( f 1_{c_1} = f f^* f = 1_{c_2} f \) and \( f^* 1_{c_2} = f^* f^* = 1_{c_1} f^* \) (because \( f^* f = (f 1_{c_1})^* f = (1_{c_1})^* = 1_{c_1} \) and \( f f^* = f (1_{c_2} f)^* = (1_{c_2})^* = 1_{c_2} \)). Clearly \( \mathcal{A} \to \mathcal{A}_{\text{cyc}} \) is a functor from the category of small categories to that of small \( \mathbb{Z}_+ \)-categories.

If \( \mathcal{A} \) is the ordered set \( [n], n \in \mathbb{Z}_+ \), considered as a category then \( \mathcal{A}_{\text{cyc}} \) identifies with the \( \mathbb{Z}_+ \)-category \( [n]_{\text{cyc}} \) from Example 2. If \( F \subset I := [0, 1] \) is a finite set containing \( \{0, 1\} \) and \( \mathcal{A} \) is the ordered set \( \pi_0 (I \setminus F) \) considered as a category then \( \mathcal{A}_{\text{cyc}} \) identifies with the \( \mathbb{Z}_+ \)-category \( \pi_0 (S^1 \setminus \bar{F}) \) from Example 3, where \( \bar{F} \subset S^1 = \mathbb{R}/\mathbb{Z} \) is the image of \( F \). If \( \mathcal{A} \) is the ordered set \( [0, 1] \) then \( \mathcal{A}_{\text{cyc}} \) identifies with \( S^1 \).

**Proposition 2.1.** The following properties of a small \( \mathbb{Z}_+ \)-category \( C \) are equivalent:

(i) \( C \) is \( \mathbb{Z}_+ \)-isomorphic to \( \mathcal{A}_{\text{cyc}} \) for some linearly ordered set \( \mathcal{A} \);

(ii) every non-empty finite full subcategory of \( C \) is \( \mathbb{Z}_+ \)-isomorphic to \( [n]_{\text{cyc}} \) for some \( n \in \mathbb{Z}_+ \).

(iii) every full subcategory of \( C \) with \( n \) objects, \( n \in \{1, 2\} \), is \( \mathbb{Z}_+ \)-isomorphic to \( [n-1]_{\text{cyc}} \).

**Proof.** It suffices to show that \((iii) \Rightarrow (i)\). For every \( x, y \in C \) there exists \( f : x \to y \) such that every \( g : x \to y \) is uniquely representable as \( g = f 1^n_x \), \( n \in \mathbb{Z}_+ \). This \( f \) (which is clearly unique) will be denoted by \( f_{xy} \). Now fix \( c \in C \). For every \( x, y \in C \) one has \( f_{xy} f_{cx} = f_{cy} 1^n_x \) for some \( m = m(x, y) \in \mathbb{Z}_+ \). Clearly \( m(x, y) + m(y, z) = m(x, z) \) and \( m(x, y) + m(y, x) = 1 \). Let \( \mathcal{A} \subset C \) be the subcategory with \( \text{Ob} \mathcal{A} = \text{Ob} C \) whose morphisms are the \( f_{xy} \)'s corresponding to those \( x, y \) for which \( m(x, y) = 0 \). Then \( \mathcal{A} \) is a linearly ordered set and \( \mathcal{A}_{\text{cyc}} = C \). \( \square \)

Let \( \Lambda_{\text{huge}} \) be the category whose objects are small \( \mathbb{Z}_+ \)-categories satisfying the conditions of Proposition 2.1 and whose morphisms are \( \mathbb{Z}_+ \)-functors. Let \( \Lambda \) (resp. \( \Lambda_{\text{big}} \)) be the full subcategory of \( \Lambda_{\text{huge}} \) formed by the \( \mathbb{Z}_+ \)-categories \( [n]_{\text{cyc}}, n \in \mathbb{Z}_+ \) (resp. by \( \mathbb{Z}_+ \)-categories which are \( \mathbb{Z}_+ \)-isomorphic to \( [n]_{\text{cyc}} \) for some \( n \in \mathbb{Z}_+ \)).

The following remarks are used only at the end of §3.

**Remarks.** (i) A. Connes \[3\] showed that \( \Lambda^0 \) is equivalent to \( \Lambda \) (see also Proposition 6.1.11 of [11]). A more understandable proof of this equivalence was given by Elmendorf \[5\]. Here is a modification of it based on an idea of D. Arinkin. If \( C_1, C_2 \) are \( \mathbb{Z}_+ \)-categories and \( C_1 \) is small then \( \mathbb{Z}_+ \)-\( \text{Funct}(C_1, C_2) \) (i.e., the full subcategory of \( \mathbb{Z}_+ \)-functors in the category of functors \( C_1 \to C_2 \)) has an obvious structure of \( \mathbb{Z}_+ \)-category. So we get
a functor \((\Lambda_{\text{huge}})^* \to \Lambda_{\text{huge}}\) defined by \(C \mapsto C^* := Z_+\text{-}\text{Funct}(C, [0]_{\text{cyc}})\), where \([0]_{\text{cyc}}\) is the \(Z_+\text{-}\text{category}\) from Example 2 (it has a single object 0, and \(\text{End} \ 0 = Z_+\)). As \(([0]_{\text{cyc}})^o = [0]_{\text{cyc}}\) one has \(Z_+\text{-}\text{Funct}(C_1, C_2^*) = (Z_+\text{-}\text{Funct}(C_2, C_1))^o\), so one gets a canonical \(Z_+\text{-}\text{functor} \ F_C : C \to C^{**}\). Finally, if \(C = [n]_{\text{cyc}}\) then \(C^*\) is \(Z_+\)-isomorphic to \([n]_{\text{cyc}}\) and \(F_C\) is an isomorphism. So the functor \((\Lambda_{\text{huge}})^* \to \Lambda_{\text{huge}}\) induces an equivalence \((\Lambda_{\text{big}})^* \to \Lambda_{\text{big}}\).

(ii) It is easy to see that in the situation of Example 3 the \(Z_+\text{-}\text{category} \ F^*\) defined in Remark (i) canonically identifies with \(\pi_0(S^1 \setminus F)\). Here is an abstract way to identify them. Let \(Z_+\text{-}\text{tors}\) denote the \(Z_+\text{-}\text{category}\) of \(Z_+\text{-}\text{torsors, i.e.,} \ Z_+\text{-}\text{sets isomorphic to} \ Z_+\). For \(C \in \Lambda_{\text{huge}}\) one has the \(Z_+\text{-}\text{bifunctor} \ C^o \times C \to Z_+\text{-}\text{tors}\) defined by \((c, c') \mapsto \text{Mor}(c, c')\). Composing it with the unique \(Z_+\text{-}\text{functor} \ Z_+\text{-}\text{tors} \to [0]_{\text{cyc}}\) one gets a \(Z_+\text{-}\text{functor} \ \Phi = \Phi_C : C \to C^*\). One also has the \(Z_+\text{-}\text{functor} \ \Phi' = (\Phi_C^o)^o : C \to (\Phi_C^o)^o = C^*\) (which is usually different from \(\Phi_C\)). In the situation of Example 3 the compositions \(S^1 \setminus F \hookrightarrow S^1 \xrightarrow{\Phi} (S^1)^* \to F^*\) and \(S^1 \setminus F \hookrightarrow S^1 \xrightarrow{\Phi'} (S^1)^* \to F^*\) are equal. This \(Z_+\text{-}\text{functor}\) can be taken as \(\pi_F\) (see Example 3).

(iii) The category of ind-objects of \(\Lambda_{\text{big}}\) (or of \(\Lambda\)) identifies with \(\Lambda_{\text{huge}}\). The category of pro-objects of \(\Lambda_{\text{big}}\) can be identified with \((\Lambda_{\text{huge}})^o\) using the equivalence \((\Lambda_{\text{big}})^o \sim \to \Lambda_{\text{big}}\) from Remark (i). It gives a class of equivalences \(\Lambda^o \sim \to \Lambda\) each of which are related by a unique isomorphism.

3. **Cyclic sets**

In \([2][3][4]\) we defined the categories \(\Lambda_{\text{huge}} \supset \Lambda_{\text{big}} \supset \Lambda\). According to A. Connes, a **cyclic set** is a functor \(X : \Lambda^o \to \text{Sets}\). One can extend such \(X\) to a functor \(\Lambda_{\text{big}}^o \to \text{Sets}\), and the extension is unique up to unique isomorphism. The functor \(C \mapsto C_{\text{cyc}}\) from Example 4 of \([2]\) can be considered as a functor \(\Delta_{\text{big}} \to \Lambda_{\text{big}}\). Using this functor one can consider any cyclic set as a simplicial set.

The **geometric realization** \(|X|\) of a cyclic set \(X\) is defined in \([3][7][13]\) to be the geometric realization of \(X\) considered as a simplicial set. By \([3][11]\), this means that \(|X|\) is the direct limit of \(X((\pi_0(I \setminus F))_{\text{cyc}}),\) where \(F\) runs through the set of all finite subsets of \(I := [0, 1]\). We will get the same answer if \(F\) runs only through the set of finite subsets of \(I\) containing 0 and 1. For such \(F\) one has \((\pi_0(I \setminus F))_{\text{cyc}} = \pi_0(S^1 \setminus \overset{\rightarrow}{F}),\) where \(\overset{\rightarrow}{F} \subset S^1 = \mathbb{R}/\mathbb{Z}\) is the image of \(F\) (see Example 4 of \([2]\)). So

\begin{equation}
|X| := \varinjlim_{\overset{\rightarrow}{F}} X((\pi_0(S^1 \setminus F)),
\end{equation}

where \(F\) runs through the set of all non-empty finite subsets of \(S^1\) and \(\pi_0(S^1 \setminus F)\) is equipped with the \(Z_+\text{-}\text{category structure}\) from Example 3 of \([2]\). The reader may prefer to rewrite \([3][11]\) as

\begin{equation}
|X| := X(S^1),
\end{equation}
where $S^1$ is the pro-object of $\Lambda_{\text{big}}$ which is the projective limit of the objects

$$\pi_0(S^1 \setminus F) \in \Lambda_{\text{big}}$$

for all finite subsets $F \subset S^1$.

Let us recall that the cyclic nerve $N_{\text{cyc}} C$ of a small category $C$ is defined by $N_{\text{cyc}} C(\lambda) = \text{Funct}(\lambda, C)$ for $\lambda \in \Lambda_{\text{big}}$, where $\text{Funct}(\lambda, C)$ is the set of functors $\lambda \to C$. So $|N_{\text{cyc}} C|$ is the set of piecewise constant functors $S^1 \to C$, more precisely

$$|N_{\text{cyc}} C| := \varprojlim_F \text{Funct}(\pi_0(S^1 \setminus F), C).$$

If $X$ is a cyclic set then $|X|$ is the geometric realization of $X$ considered as a simplicial set, so it is equipped with a topology. On the other hand, formula (3.1) or (3.2) makes it clear that the group $\text{Aut} S^1$ of orientation preserving homeomorphisms of $S^1$ acts on $|X|$. In particular, the group of rotations $SO(2)$ acts.

We claim that the action of $\text{Aut} S^1$ on $|X|$ is continuous if $\text{Aut} S^1$ is equipped with the compact-open topology. To prove this, note that similarly to (1.7) and (1.6) one has canonical homeomorphisms

$$|X| = X \times_\Lambda \Psi,$$

$$|X| = \varinjlim_{\Lambda/X} \Psi_X,$$

where $\Psi : \Lambda \to \{\text{Spaces}\}$, $\Psi(\lambda) :=$ the geometric realization of the functor represented by $\lambda$, and $\Psi_X$ is the restriction of $\Psi$ to $\Lambda/X$. Using the homeomorphism (3.4) or (3.5) one reduces the question to the particular case where $X$ is the cyclic set $[n]_{\text{cyc}}$ (i.e., the functor $\Lambda^o \to \text{Sets}$ represented by $[n]_{\text{cyc}}$). In this case one can use the following explicit description of the geometric realization $|[n]_{\text{cyc}}|$. 

**The geometric realization of the cyclic set $[n]_{\text{cyc}}$.** We claim that there are canonical bijections

$$|[n]_{\text{cyc}}| \xrightarrow{\sim} \text{Funct}_{\mathbb{Z}^+}([n]_{\text{cyc}}, S^1) \xrightarrow{\sim} Y/\mathbb{Z},$$

where $\text{Funct}_{\mathbb{Z}^+}([n]_{\text{cyc}}, S^1)$ stands for the set of $\mathbb{Z}^+$-functors $[n]_{\text{cyc}} \to S^1$ and $Y$ is the set of non-decreasing maps $f : \mathbb{Z} \to \mathbb{R}$ such that $f(i + n + 1) = f(i)$ for all $i \in \mathbb{Z}$ (the $\mathbb{Z}$-action on $Y$ is induced by the action of $\mathbb{Z}$ on $\mathbb{R}$ by translations). The second bijection in (3.6) is clear. The first one is constructed as follows. By (3.1),

$$|[n]_{\text{cyc}}| := \varinjlim_F \text{Funct}_{\mathbb{Z}^+}((\pi_0(S^1 \setminus F), [n]_{\text{cyc}}).$$

By Remark (ii) from [2] $\pi_0(S^1 \setminus F) = F^*$, so after fixing a $\mathbb{Z}^+$-isomorphism $([n]_{\text{cyc}})^* \xrightarrow{\sim} [n]_{\text{cyc}}$, the set $\text{Funct}_{\mathbb{Z}^+}((\pi_0(S^1 \setminus F), [n]_{\text{cyc}})$ identifies with

$$\text{Funct}_{\mathbb{Z}^+}([n]_{\text{cyc}}, F)$$
and the set $|[n]_{\text{cyc}}|$ identifies with
\[
\lim_{\longrightarrow} \text{Funct}_{\mathbb{Z}_+}([n]_{\text{cyc}}, F) = \text{Funct}_{\mathbb{Z}_+}([n]_{\text{cyc}}, S^1).
\]

It is easy to check that the composition of the bijections (3.6) is a homeomorphism (if $Y$ is equipped with the natural topology). Note that

$$\text{Aut } S^1 = G/\mathbb{Z},$$

where $G$ is the group of orientation-preserving homeomorphisms $f : \mathbb{R} \sim \rightarrow \mathbb{R}$ such that $f(x + 1) = f(x) + 1$. So $\text{Aut } S^1$ acts on $Y/\mathbb{Z}$. This action is continuous, and the homeomorphism $|[n]_{\text{cyc}}| \sim \rightarrow Y/\mathbb{Z}$ is $(\text{Aut } S^1)$-equivariant. So the action of $\text{Aut } S^1$ on $|[n]_{\text{cyc}}|$ is continuous.

**Remark.** The space $|[n]_{\text{cyc}}| = Y/\mathbb{Z}$ is homeomorphic to the product of $S^1$ and the $n$-dimensional simplex.

**Warning.** Associating to a $\mathbb{Z}_+$-functor $\Phi : [n]_{\text{cyc}} \rightarrow S^1$ the sequence $\Phi(0), \ldots, \Phi(n)$ one gets a map
\[
|[n]_{\text{cyc}}| = \text{Funct}_{\mathbb{Z}_+}([n]_{\text{cyc}}, S^1) \rightarrow (S^1)^{n+1}.
\]

Unless $n = 0$, it is not injective (contrary to what I wrote in the previous versions of this article including the journal version). More precisely, the preimage of a point of the diagonally embedded $S^1 \subset (S^1)^{n+1}$ has cardinality $n + 1$. (All other preimages have cardinalities $\leq 1$.)

**Remarks.** (i) By Remark (iii) at the end of §2 the pro-object $S^1$ from (3.2) can be considered as an object of $\Lambda_{\text{huge}}$. This object canonically identifies with $S^1$ (e.g., one can use Remark (iii) from §2).

(ii) The group of automorphisms of the geometric realization functor from the category of cyclic sets to that of sets equals $\text{Aut } S^1$. This follows, e.g., from the previous remark.

**References**

[1] A. Besser, A simple approach to geometric realization of simplicial and cyclic sets. E-print available at [http://www.math.bgu.ac.il/~besser](http://www.math.bgu.ac.il/~besser)

[2] D. Burghelea, Z. Fiedorowicz, Cyclic homology and algebraic $K$-theory of spaces. II. Topology, vol. 25 (1986), no. 3, 303–317.

[3] A. Connes, Cohomologie cyclique et foncteurs $\text{Ext}^n$. C. R. Acad. Sci. Paris Sér. I Math., vol. 296 (1983), no. 23, 953–958.

[4] W. G. Dwyer, M. J. Hopkins, D. M. Kan, The homotopy theory of cyclic sets. Trans. Amer. Math. Soc., vol. 291 (1985), no. 1, 281–289.

[5] A. D. Elmendorf, A simple formula for cyclic duality. Proc. Amer. Math. Soc., vol. 118 (1993), no. 3, 709–711.

[6] P. Gabriel, M. Zisman, Calculus of fractions and homotopy theory. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35 Springer-Verlag, New York, 1967. Russian translation: “Mir”, Moscow, 1971.

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1The mistake was noticed by M. Kapranov.
[7] P. Goerss, J.F. Jardine, Simplicial homotopy theory. Progress in Mathematics, 174. Birkhäuser Verlag, Basel, 1999.

[8] D. Grayson, Algebraic $K$-theory, Lecture notes available at [http://www.math.uiuc.edu/~dan/Courses/2003/Spring/416/GraysonKtheory.ps](http://www.math.uiuc.edu/~dan/Courses/2003/Spring/416/GraysonKtheory.ps)

[9] A. Grothendieck et J. L. Verdier, Topos. In: SGA 4, tome 1, Lecture Notes in Mathematics, vol. 269, 299–519. Springer-Verlag, 1972.

[10] R. M. Hain, Iterated integrals and homotopy periods. Mem. Amer. Math. Soc., vol. 47 (1984), no. 291.

[11] J-L. Loday, Cyclic homology. Grundlehren der Mathematischen Wissenschaften, 301. Springer-Verlag, Berlin, 1998.

[12] S. Mac Lane, Categories for the working mathematician. Graduate Texts in Mathematics, 5. Springer-Verlag, 1998.

[13] J-P. May, Simplicial objects in algebraic topology. Van Nostrand, Princeton - Toronto - London, 1967. University of Chicago Press, 1992.

[14] J. Milnor, The geometric realization of a semi-simplicial complex. Ann. of Math. (2), vol. 65 (1957), no. 2, 357–362.

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