Stability and Solution of the Time-Dependent Bondi-Parker Flow

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ABSTRACT
Bondi (1952) and Parker (1958) derived a steady-state solution for Bernouilli’s equation in spherical symmetry around a point mass for two cases, respectively, an inward accretion flow and an outward wind. Left unanswered were the stability of the steady-state solution, the solution itself of time-dependent flows, whether the time-dependent flows would evolve to the steady-state, and under what conditions a transonic flow would develop. In a Hamiltonian description, we find that the steady state solution is equivalent to the Lagrangian implying that time-dependent flows evolve to the steady state. We find that the second variation is definite in sign for isothermal and adiabatic flows, implying at least linear stability. We solve the partial differential equation for the time-dependent flow as an initial-value problem and find that a transonic flow develops under a wide range of realistic initial conditions. We present some examples of time-dependent solutions.

Key words: hydrodynamics; stars: winds, outflows, mass loss, formation

1 INTRODUCTION
Spherical accretion onto a constant point mass (Bondi 1952) has found application in astronomy from stars to supermassive black holes and anywhere the self-gravity of the gas is insignificant compared to the gravity of the point mass. The simplicity of the model combined with the non-trivial solution that includes a transonic critical point, has proven both useful and interesting. The same equations but with outward velocities (Parker 1958) has found application in solar and stellar winds and anywhere acceleration occurs as a result of a pressure-density gradient maintained by a gravitational field. The acceleration in the Parker wind is also the same as occurs through a rocket nozzle with an exponential shape.

The Bondi-Parker (BP) flow is described by a combination of the continuity equation with Bernouilli’s equation, the latter a partial differential equation (PDE) for velocity and density as functions of time and position. Assuming the time derivative is zero results in a single ordinary differential equation (ODE) for the steady state with separable variables. The transcendental equation resulting from integration is easily solved, for example with a Newton-Raphson technique or in terms of the Lambert W function (Cranmer 2004). The constant of integration along with the branches of a quadratic term results in a family of steady-state trajectories with either subsonic, transonic, or supersonic velocities.

There have been many interesting variations of the BP flow. For example, the introduction of shock discontinuities links different trajectories in the steady-state family (McCrea 1956). A non-isothermal equation of state results in multiple critical points (Kopp & Holzer 1976). Accounting for the self-gravity of the gas results in a similarity solution either as a function of time alone (Shu 1977) or time and radius (Dhang et al. 2016) depending on the equation of state and the initial conditions.

The stability of the steady-state solution has previously been addressed with finite-difference methods (Balazs 1972; Stellingwerf & Buff 1978; Garlick 1979; Velli 1994; Del Zanna et al. 1998). These studies agree that the transonic flow is stable, but disagree about the stability of the subsonic and supersonic flows based on differences in numerical methods and choice of boundary conditions.

In contrast, a Hamiltonian description of the flow determines the evolution and stability independently of numerical methods and boundary conditions. We find that the steady-state solution is equivalent to the Lagrangian for the flow and is thus the critical function for the first variation of the functional of the flow. This condition implies that time-dependent flows evolve to the steady-state/ We find that the second variation is definite in sign for isothermal and
adiabatic equations of state. This implies that these time-dependent flows are at least linearly stable.

The choice of trajectory in the steady-state family that results from the evolution is determined by the initial velocities. We use the method of characteristics to write the PDE for the time-dependent BP flow as a pair of coupled ODEs for velocity and position both as functions of time. These may be solved numerically as an initial value problem (IVP), for example with a Runge-Kutta technique. These solutions specify the final trajectory for any initial conditions.

Examples suggest that the time-dependent solutions evolve to a transonic flow from all initial values that lie within the region bounded above and below by the inward and outward steady-state transonic trajectories. Since this region extends asymptotically from velocities with absolute values from zero to infinity, a wide range of initial velocities results in a transonic flow.

2 THE STEADY STATE BONDI-PARKER FLOW

Following Bondi (1952) and Parker (1958) we derive the steady-state solution. If the gas is isothermal so that \( \partial \rho / \partial x = a^2 \) for sound speed \( a \), and the gravitational force is that of a point of constant mass, \( M \), then the Euler equation in spherical symmetry is,

\[
\frac{\partial \tilde{u}}{\partial t} = -\tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} - \frac{\partial \tilde{\rho}}{\partial \tilde{x}} + \frac{GM}{\chi^2} \tag{2.1}
\]

where the tilde indicates a variable with dimensional units. This can be written in non-dimensional form with the definitions,

\[
\tilde{x} = \frac{GM}{a^2} x, \quad \tilde{u} = au, \quad \tilde{\rho} = \tilde{\rho}_1 \rho, \quad \text{and} \quad \tilde{t} = \left( \frac{GM}{a^3} \right) t, \tag{2.2}
\]

where \( \tilde{\rho}_1 \) is an arbitrary density. With these substitutions, equation 2.1 is,

\[
\frac{\partial \tilde{u}}{\partial \tilde{t}} = -\tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} - \frac{\partial \tilde{\rho}}{\partial \tilde{x}} + \frac{1}{\tilde{x}^2}. \tag{2.3}
\]

The density may be eliminated with the help of the non-dimensional continuity equation,

\[
\tilde{\rho} = \chi x^{-2} \tilde{u}^{-1} \tag{2.4}
\]

where \( \chi \) is the accretion rate. Then,

\[
\frac{\partial \tilde{u}}{\partial \tilde{t}} = \left( \frac{1}{\tilde{u}} - \tilde{u} \right) \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{2}{\tilde{x}} - \frac{1}{\tilde{x}^2}. \tag{2.5}
\]

In steady state the time derivative on the left-hand side is zero, and the variables can be separated and integrated,

\[
\dot{\mathcal{L}} = \log |\tilde{u}| - \frac{1}{2} \tilde{u}^2 + 2 \log |\tilde{x}| + \frac{1}{\tilde{x}}. \tag{2.6}
\]

In non-dimensional units, the constant of integration \( \mathcal{L} = \log \dot{\mathcal{L}} \) is equivalent to the energy and with a simple non-linear scaling to the the mass accretion rate. This transcendental equation can be solved either numerically, for example with a Newton-Raphson technique, or using the Lambert W function (Cranmer 2004),

\[
\tilde{u} = \frac{1}{2} \left[ -W \left( -\exp \left( \frac{1}{2} \tilde{x}^2 - 2 \tilde{x} \right) \right) \right]^{1/2} \tag{2.7}
\]

or

\[
\tilde{x} = -2W \left[ \pm \frac{1}{2} \exp \left( \frac{1}{2} \tilde{u}^2 - \ln |\tilde{u}| + \mathcal{L} \right) \right]. \tag{2.8}
\]

The family of solutions depends on the value of the integration constant, \( \mathcal{L} \), and upon the branches of the quadratic term or the Lambert W function. Representative solutions are plotted in figure 1. Of particular interest are the two transonic solutions, the Bondi accretion flow and the Parker wind, both with \( \mathcal{L} = \mathcal{L}_C = 3/2 - 2 \log 2 \) that cross at the Bondi-Parker critical point, \(( \tilde{x}, \tilde{u} \)) = \(( \frac{1}{2}, 1) \) or \(( \tilde{x}, \tilde{u} \)) = \(( GM/(2a^2), a) \). Also shown are subsonic flows with \( \mathcal{L} < \mathcal{L}_C \), and supersonic flows with \( \mathcal{L} > \mathcal{L}_C \). The solutions that are discontinuous in position derive from a different branch and are not accretion flows or winds. The region of the plot that they occupy is sometimes called the forbidden region. A more complete discussion is found in Cranmer (2004) and Holzer & Axford (1970).

3 HAMILTONIAN DESCRIPTION

Hamilton’s principle for a conservative system is,

\[
\delta J = \delta \int_{t_1}^{t_2} (K - V) dt = 0 \tag{3.1}
\]

where the difference of the kinetic and potential energies is the Lagrangian, \( K - V = \mathcal{L} \). This may be generalized for a fluid with internal energy, \( U \) (Herivel 1955),

\[
\mathcal{L} = K - V - U. \tag{3.2}
\]

Figure 1. Representative trajectories of the steady-state BP flow (equation 2.6) in non-dimensional units. The velocities are shown as absolute values. The two transonic trajectories with \( \mathcal{L} = \mathcal{L}_C \) are shown in pink. Green lines show two subsonic trajectories with \( \mathcal{L} = \mathcal{L}_C - 2 \) and \( \mathcal{L} = \mathcal{L}_C - 1 \) and two supersonic trajectories with \( \mathcal{L} = \mathcal{L}_C + 1 \) and \( \mathcal{L} = \mathcal{L}_C + 2 \). The trajectories that are discontinuous across position are shown in yellow for the same set of constants, \( \mathcal{L} \) but represent different branches.
In non-dimensional units, the Lagrangian for the BP flow is the same as the steady-state equation, 2.6, where the non-dimensional terms, \((\log |u| + 2 \log |x|)\), that derive from the dimensional pressure term in equation 2.1, are the internal energy. If we define the functional gradient as the directional derivative,

\[
\nabla J[x] = \frac{d}{dx} J[x + \epsilon \eta] \bigg|_{\epsilon=0},
\]

for any variation, \(\eta\), and constant, \(\epsilon\), then the necessary condition of Euler-Lagrange for \(x\) to be a critical function of \(J[x]\) is that the first variation is zero

\[
\langle \nabla J[x] \rangle = \frac{\partial}{\partial x} \mathcal{L}(x, \dot{x}, \ddot{x}) - \frac{d}{dt} \left( \frac{\partial}{\partial \dot{x}} \mathcal{L}(x, \dot{x}, \ddot{x}) \right)
\]

where the velocity \(u = \dot{x}\).

Since the right side is equivalent to the equation of motion 2.5 as can be verified by substitution, then \(\langle \nabla J[x] \rangle = 0\), and the steady-state solution, \(\mathcal{L}\), is the solution of stationary action toward which the time-dependent solutions evolve.

This description is similar to the simple physics problem in which the Lagrangian determines the parabolic trajectory of a particle in a gravitational field and the exact trajectory is determined from the end points as a boundary value problem (BVP) or alternatively from the initial velocities as an IVP. However, the Lagrangian for the BP flow includes the internal energy of the fluid as a function of the density-pressure gradient which is constrained by the conservation equation. Therefore, it is not always possible to find an energy conserving trajectory between two arbitrary velocities. Considered as a BVP, not all boundary conditions are consistent with a steady-state solution.

As in Lagrangian mechanics, the energy serves as a Lyapunov function for the stability. The definiteness in sign of the second variation evaluated at the steady state implies at least linear stability (Arnold, V.I. 1965, 1989). Equivalently, the integrand of the second variation,

\[
\eta^2 \frac{\partial^2 \mathcal{L}}{\partial x^2} + 2\eta \frac{\partial^2 \mathcal{L}}{\partial x \partial \dot{x}} + \eta^2 \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^2}
\]

must be definite in sign for every nonzero variation \(\eta\). Since the middle term with mixed partial derivatives is zero, and

\[
\frac{\partial^2 \mathcal{L}}{\partial x^2} = -2 \left( \frac{1}{x^2} + \frac{1}{x^3} \right)
\]

\[
\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^2} = \left( 1 + \frac{1}{x^2} \right)
\]

this is easily verified for the range \(x > 0\) allowed for BP flows. This implies that the steady-state solution, including all the subsonic, transonic, and supersonic trajectories in the family, is at least linearly stable.

The existence of a steady-state solution, a critical point in Lagrangian mechanics, is required for stability. An arbitrary IVP can evolve to the steady-state, but the boundary conditions in a BVP must be consistent with the conservation of energy to allow a stable solution. This requirement itself does not determine the steady-state solution or its stability.

For example, the derivation of the Euler-Lagrange equation assumes that the functional to be maximized or minimized has values equal to two specified endpoints. In between, the solution depends on the functional.

### 4 THE TIME DEPENDENT BONDI-PARKER FLOW

To solve for the time-dependent BP flow, equation 2.5 we use the method of characteristics to write the PDE as a pair of coupled first-order ODE’s that may be solved as an initial value problem. Equation 2.5 written as,

\[
\frac{d[u(t,x(t))]}{dt} = \frac{\partial u}{\partial t} + \left( u - \frac{1}{u} \right) \frac{\partial u}{\partial x} = \frac{2}{x} \frac{1}{x^2}.
\]

and compared with the identity,

\[
\frac{du}{dt} = \frac{2}{x} \frac{1}{x^2}
\]

suggests the pair of coupled ODE’s,

\[
\frac{dx}{dt} = u - \frac{1}{u}
\]

\[
\frac{du}{dt} = \frac{2}{x^2}
\]

to be solved with initial values, \(u(t_x) = f(x)\) at \(t = 0\). Alternatively, these may be derived from the Lagrange-Charpit (LC) equations for the PDE 2.5,

\[
\frac{dt}{\tau} = \frac{dx}{u - u^{-1}} = \frac{du}{2x^{-1} - x^{-2}}
\]

The LC equations also yield a third ODE that is equivalent to the steady-state equation 2.6.

Following the method of characteristics, we parameterize the ODE’s with functions \(t = g(\tau, s)\) and \(x = h(\tau, s)\) to obtain the following set of ODE’s with their initial conditions

\[
\frac{dt}{d\tau} = 1 \quad \text{with} \quad t(0, s) = 0,
\]

\[
\frac{dx}{d\tau} = u - \frac{1}{u} \quad \text{with} \quad x(0, s) = s,
\]

\[
\frac{du}{d\tau} = \frac{2}{x} \frac{1}{x^2} \quad \text{with} \quad u(0, s) = f(s)
\]

Here \((\tau, s)\) are the initial values for the trajectories, \(x(t), u(t)\). This set of coupled ODE’s may be solved numerically, for example with a Runge-Kutta technique for \(u(\tau, s)\) To complete the solution for \(u(t_x), s\), we need the inverse functions, \(t = g(\tau, s)\) and \(x = h(\tau, s)\). From equation 4.6, \(\tau\) is identical to \(t\). From a set of solutions of equations 4.7 and 4.8 we determine \(x = h(\tau, s)\) to obtain the solution \(u(t, h(\tau, s)) = u(t, x)\).
Example solutions suggest the proposition that a transonic flow develops if any of the initial velocities are within the forbidden region. Because the forbidden region extends asymptotically over $0 < |u| < \infty$, there is a wide range of initial values that will result in a transonic flow. Any constant nonzero initial velocities will cross a boundary of the forbidden region if the range in position is large enough. Stated another way, an initial flow has to be constructed in a special way to avoid the forbidden regions. The velocities of the subsonic and supersonic steady-state trajectories have this property.

While generalizing from examples is short of a mathematical proof, this proposition seems plausible on physical grounds. Consider an inflow with a subsonic velocity in the forbidden region with $x < \chi_C$. This flow will be accelerated to approximate free fall, $a \sim 1/\sqrt{x}$, by the gravitational force. Equation 2.5 constrains the velocity of the steady-state solution to the sound speed at the critical point, and the flow will evolve until this condition is met. The critical point thus acts as an outer boundary condition for the supersonic region of the inflow. The critical point is also an inner boundary condition for the subsonic region of the flow, $x > \chi_C$. Here the flow is able to adjust to the transonic solution by increasing the density and pressure to slow the flow until its transonic point occurs at the critical point, and both terms on the right-hand side of equation 2.5 are zero. This description also applies to the wind with appropriate modifications.

The transonic critical point effectively provides a natural boundary condition for the inward and outward transonic flows. In the case of a stellar wind for example, if an arbitrary radius is chosen for the base of the wind, then the velocity at the base is uniquely determined by the requirement that the wind pass through the transonic critical point. In the case of an astrophysical subsonic or supersonic flow, while any single point in the $(x, u)$ plane (figure 1) uniquely determines one steady-state trajectory, this would have to be determined outside of the BP flow, for example by supposing that conditions in the interstellar medium set a constant velocity at a particular radius. This velocity would need to be outside the forbidden region, assuming the proposition above.

### 5.1 Transonic solutions

Figures 2 through 4 show the development of a transonic accretion flow starting from constant initial velocities equal to -0.2, -1.0, and -2.0, respectively, for $0.005 < x < 2.0$. (Solutions of equations 4.7 and 4.8 are not defined for initial values of $x = 0$ or $u = 0$.) This evolution is similar to what might be found in a test of a numerical hydrodynamic simulation evolving to the steady-state starting from a constant velocity. In figure 4 a shock develops, indicated by multi-valued velocities, as the velocities outside the critical point decrease to subsonic and are impacted by the supersonic initial velocities at larger radii. The post-shock velocities evolve to the transonic trajectory.

In the method of characteristics, the ODE for the velocity $u(t, x(t))$ is integrated along lines of $x(t)$. These lines or characteristics are also shown in the figures. The point where the characteristics, $x(t)$, first cross indicates the formation of a shock.

Starting from the same but positive initial velocities, the flows evolve to the outward steady-state transonic trajectory, the Parker wind. Figure 5 shows the time evolution for initial values $u(x) = +0.15$ and $0.2 < x < 2.0$. A shock develops in the outer region as the outward velocities become supersonic and impact the subsonic initial velocities further out. Plots of the evolution from initial velocities, $u(x) = 1.0$ and 2.0 look as expected from figure 5 and are not shown.

The characteristics for the transonic solutions in figures 2 through 5 have both inward and outward motion. The location of the point on the $x$-axis, $t = 0$, that separates the inward and outward traveling characteristics is the point where the initial values cross the boundary of a forbidden region. For the case shown in figure 3 with initial velocities equal to the sound speed, the point of separation is the critical point.

### 5.2 Subsonic and supersonic solutions

To find initial values that are everywhere outside the forbidden regions, we start with the subsonic and supersonic steady-state trajectories.

Figure 6 shows two examples of supersonic accretion. The first begins with initial values that follow the steady-state trajectory with $\mathcal{L} = \mathcal{L}_C + 2$ inside the critical point and transition to the steady-state trajectory with $\mathcal{L} = \mathcal{L}_C + 1$ outside the critical point by means of a Gaussian. The second example begins with initial values that transition in the reverse sense. In both cases, these time-dependent flows evolve asymptotically to follow the steady-state trajectory used to set the initial velocities in the outer part of the accretion flow. While difficult to imagine as an astrophysical flow, the equations may also be solved for the same initial conditions but changing the initial direction of the flow to a wind. These solutions evolve to follow the initial steady-state trajectory in the inner part of their initial winds. The evolution of these winds is as expected from figure 6 and is not shown.

The same calculations can be done for subsonic accretion and winds. Figure 7 shows a subsonic wind sometimes called a Parker breeze. Both time-dependent wind solutions evolve asymptotically to follow the initial steady-state trajectories in the outer part of the wind. In the case of subsonic accretion, figure 8, the flows evolve to the initial steady-state trajectories in the inner region rather than the outer region.

As a final example, figure 9 shows the evolution of a subsonic accretion flow with a sinusoidal perturbation of amplitude 0.05 and period 0.25 imposed on the steady state between $x = 0.25$ and $x = 8.0$. (If a finite amplitude perturbation is continued too close to the origin then some initial velocities would be within the inner forbidden region and the flow would be swept into approximate free fall allowing the entire
flow to become transonic.) The continued oscillation of the solution shown in the figure is expected because the density structure in subsonic accretion flows is approximately hydrostatic and the PDE has no damping. This flow is Lyapunov stable in the sense that the velocities stay within some range of the initial flow set by the amplitude of the initial perturbations.

6 THE BAROTROPIC EQUATION OF STATE

Since the the Parker wind and subsonic accretion both have approximately hydrostatic density profiles inside the transonic critical point, comparison with the stability of hydrostatic equilibrium suggests that the BP flow should be stable with an adiabatic equation of state (EOS), $\gamma = 5/3$, but not with a radiation dominated EOS with $\gamma = 4/3$. The understanding developed for the isothermal BP flow allows not with a radiation dominated EOS with $\gamma = 5/3$, but with an adiabatic equation of state (EOS), $P \propto \rho^\gamma$. With a barotropic EOS, the non-dimensional Euler equation equivalent to equation 2.3 is,

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} - \rho^{\gamma-1} \frac{\partial \rho}{\partial x} - \frac{1}{x^2}.$$  \hfill (6.1)

Following Bondi (1952), the steady-state solution,

$$\frac{1}{2} u^2 + \left( \frac{1}{\gamma - 1} \right)^{\gamma-1} \frac{\rho^{\gamma-1}}{\gamma - 1} - \frac{1}{x} = 0,$$  \hfill (6.2)

can be written in separable variables similar to equation 2.6 by scaling the velocity $u$ by the local sound speed. Substituting $v = u/a = u \rho^{(\gamma - 1)/2}$ into equation 6 along with the continuity equation, results in

$$F(v) - \lambda^{2v-1} G(x) = 0$$  \hfill (6.3)

where

$$F(v) = \frac{1}{2} v^{4/\gamma + 1} + \frac{1}{\gamma - 1} v^{-2(\gamma-1)/\gamma}$$  \hfill (6.4)

and

$$G(x) = \frac{1}{\gamma - 1} \lambda^{4v-1} + \lambda^{3v - 1}.$$  \hfill (6.5)

The two terms of the second variation, equivalent to equations 3.6 and 3.7, are then,

$$\frac{\partial^2 G(x)}{\partial x^2} = \left( \frac{2\gamma - 6}{\gamma + 1} \right) \left( \frac{3\gamma - 5}{\gamma + 1} \right) \lambda^{2v-1} + \left( \frac{4}{\gamma + 1} \right) \left( \frac{3\gamma - 5}{\gamma + 1} \right) \lambda^{2v-1}$$  \hfill (6.6)

and

$$\frac{\partial^2 F(v)}{\partial x^2} = \left( \frac{2}{\gamma + 1} \right) \left( \frac{3 - \gamma}{\gamma + 1} \right) v^{-2(\gamma-1)/\gamma} + \left( \frac{2\gamma - 2}{\gamma + 1} \right) \left( \frac{3\gamma + 1}{\gamma + 1} \right) v^{-2(\gamma-1)/\gamma}.$$  \hfill (6.7)

The second variation is definite in sign for $\gamma = 5/3$, owing to the factors of $(3\gamma - 5)$, implying that an adiabatic BP flow is stable. Since the factor containing the parameter, $\lambda$, that determines whether the flow is subsonic, transonic, or supersonic, is also multiplied by this zero, all the trajectories of the adiabatic BP flow are stable. For $\gamma = 4/3$ and most other values, the second variation is indefinite in sign. This does necessarily imply that these flows are unstable, only that the stability cannot be determined by this Hamiltonian description.

The method of characteristics can also be used to solve for a time-dependent barotropic BP flow. From the PDE 6.1, the two ODEs equivalent to equations 4.3 and 4.4 can be written for example as,

$$\frac{dx}{dt} = u - \rho^{\gamma-1} \frac{1}{u},$$  \hfill (6.8)

and

$$\frac{du}{dt} = \rho^{\gamma-1} \frac{2}{x} - \frac{1}{x^2},$$  \hfill (6.9)

where $\rho(x, u)$ is given by the continuity equation 2.4. Similar to the isothermal case, these equations may be solved parametrically for $u(t, x(t))$. Examples are best left to specific applications.

7 ASTROPHYSICAL IMPLICATIONS

The steady-state BP flow is particularly useful as a context to study local phenomena in accretion or winds that involve gas pressure, shocks, or fronts. In the transonic flows, the critical point along with the conservation of energy completely specifies the steady-state solution without the need for initial values or boundary conditions. The subsonic and supersonic steady-state flows, lacking this constraint, require at least one boundary condition, depending on the method of solution, and both the inner and outer boundaries are problematic. At the inner boundary, spherical convergence leads to unrealistically high densities. At the outer boundary, these flows require an asymptotic approach to zero or infinite velocity. These are all artificial conditions. Since any other boundary conditions lead to a transonic trajectory, this is the most likely astrophysical application.

8 CONCLUSIONS

The Hamiltonian description of the Bondi-Parker flow provides a simple and definitive method for determining the evolution of time-dependent flows and the stability of the steady state. The method of characteristics allows a simple solution for the partial differential equation describing the time-dependent flows as an initial value problem. These methods provide answers to several questions about the stability and evolution of the flows that were unexplained in Bondi (1954) and Parker (1958). In particular, time-dependent flows evolve to the steady state; the steady-state solution for isothermal and adiabatic equations of state, including all subsonic, transonic, and supersonic trajectories is at least linearly stable; and a transonic flow develops under a wide range of realistic initial conditions.
**Figure 2.** Left: Time evolution of an accretion flow with initial values $u(x) = -0.2$ and $0.005 < x < 2.0$ plotted on top of the steady state trajectories from figure 1. The time-dependent solution is shown in blue for a sequence of times with the initial values in red. Velocities are plotted as their absolute values. The flow evolves asymptotically to the steady-state transonic trajectory, the Bondi accretion flow. Right: Characteristics for the solution. Only the characteristics originating in the range $(0.005 < x < 0.6)$ are shown.

**Figure 3.** Time evolution of an accretion flow with initial values $u(x) = -1.0$ and $0.005 < x < 2.0$ in the same format as figure 2.

**Figure 4.** Time evolution of an accretion flow with initial values $u(x) = -2.0$ and $0.005 < x < 2.0$ in the same format as figure 2.

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Figure 5. Left: Time evolution of a wind with initial values $u(x) = +0.15$ and $0.2 < x < 2.0$ plotted in the same format as figure 2.

Figure 6. Time evolution of a supersonic accretion flow in the same format as figure 2. The initial values are derived as a transition between two steady-state solutions as explained in the text. The trajectories evolve asymptotically to the steady-state solution in the outer region.

Figure 7. Time evolution of a subsonic wind (Parker breeze) in the same format as figure 2. The initial values are derived as a transition between two steady-state solutions as explained in the text. The trajectories evolve asymptotically to the steady-state solution in the outer region.

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Figure 8. Time evolution of a subsonic accretion flow in the same format as figure 2. The initial values are derived as a transition between two steady-state solutions as explained in the text. The trajectories evolve asymptotically to the steady-state solution in the inner region.

Figure 9. Left: Time evolution of a subsonic accretion flow in the same format as figure 2. The initial values follow a steady-state solution with a sinusoidal perturbation as described in the text. Right: characteristics for the solution. These indicate that the flow will not evolve out of the subsonic region.