NON-SINGULAR SPACETIMES WITH A NEGATIVE COSMOLOGICAL CONSTANT: III. STATIONARY SOLUTIONS WITH MATTER FIELDS

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Abstract. We construct infinite-dimensional families of non-singular stationary space times, solutions of Yang-Mills-Higgs-Einstein-Maxwell-Chern-Simons-dilaton-scalar field equations with a negative cosmological constant. The families include an infinite-dimensional family of solutions with the usual AdS conformal structure at conformal infinity.

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1. Introduction

There is currently considerable interest in the literature in space-times with a negative cosmological constant. This is fueled on one hand by studies of the AdS-CFT conjecture and of the implications thereof. On the other hand, these solutions are interesting because of a rich dynamical morphology: existence of periodic or quasi-periodic solutions, and of instabilities. All
this leads naturally to the question of existence of stationary solutions of the Einstein equations with \( \Lambda < 0 \), with or without sources, and of properties thereof. Several families of such solutions have been recently constructed numerically \[6, 11, 23, 27, 33, 36\]; cf. also \[5\] for a rigorous construction of static solutions of the Einstein-Vlasov equations with \( \Lambda < 0 \).

In an accompanying paper \[18\] two of us (PTC and ED) have constructed an infinite dimensional family of non-singular static space times, solutions of the Einstein-Maxwell equations with a negative cosmological constant. These families include an infinite-dimensional family of solutions with the usual AdS conformal structure at conformal infinity. The object of this work is to generalise the construction there to obtain a similar large family of singularity-free stationary solutions of the Yang-Mills-Higgs-Einstein-Maxwell-dilaton-Chern-Simons-scalar field equations, including a class of boson-star solutions with stationary metric but periodic complex scalar field. We also show that our methods can be used to obtain solutions of a class of \( f(R) \)-theories.

Note that existence of such solutions of the Einstein-Maxwell-Yang-Mills-dilaton field equations with the Kaluza-Klein value of the coupling constant is a special case of the results in \[4\].

More precisely, we construct strictly stationary solutions of the Einstein-matter field equations with a negative cosmological constant and with a smooth conformal boundary for a large class of matter models. Here we say that a space-time \((\mathcal{M}, g)\) is strictly stationary if there exists on \(\mathcal{M}\) a Killing vector field which is timelike everywhere. Such a solution is defined to be non-degenerate if a certain operator associated with the linearisation of the field equations is an isomorphism, cf. Section 3 for a precise definition.

An example of a non-degenerate solution is anti-de Sitter space-time. Our solutions are constructed using an implicit function theorem near a non-degenerate vacuum metric \((\mathcal{M}, \tilde{g})\). We also construct solutions with a time-periodic complex scalar field accompanied by time-independent metric and Maxwell fields. The solutions are uniquely determined by certain freely prescribable coefficients in the asymptotic expansion of the metric, of the Yang-Mills or Maxwell fields, of the dilaton field and of the scalar fields. Here uniqueness is guaranteed in a neighborhood of the metric \(g\). In this way we obtain infinite dimensional families of solutions with, if desired, the same conformal structure at infinity as the initial static vacuum metric \(\tilde{g}\).

By switching-off some free data at the conformal boundary, or setting to zero one of the coupling constants, one can obtain non-trivial solutions of the Einstein-Yang-Mills equations, or Einstein-scalar field equations, or static Einstein-Maxwell-Chern-Simons-dilaton solutions, etc. In particular we establish rigorously existence of Einstein-Yang-Mills solutions in near-AdS configurations, as constructed numerically in \[11, 13\], and in fact we provide a much larger family of such solutions.

The method is a conceptually-straightforward repetition of the arguments in \[17, 18\], so that our presentation will be suitably sketchy: we will only provide details at places which require technical or calculational changes.

Our hypothesis of strict stationarity excludes black hole solutions. The extension of our analysis to black holes will be discussed elsewhere \[19\].
A similar construction works near any non-degenerate stationary solutions of the equations under consideration, provided that the linearisations of the matter equations lead to isomorphisms. This last property appears to require a case-by-case analysis of the solutions at hand.

2. **Stationary Einstein-Maxwell-Chern-Simons-dilaton-scalar field equations in \( n + 1 \) dimensions**

We consider the Einstein equations for a metric
\[
g = g_{\mu\nu}dx^\mu dx^\nu
\]
in space-time dimension \( n + 1 \), \( n \geq 3 \),
\[
\text{(2.1)} \quad \text{Ric}(g) - \frac{\text{Tr}_g \text{Ric}(g)}{2} g + \Lambda g = 8\pi G T,
\]
where \( T \) is the energy-momentum tensor of matter fields. A constant rescaling of \( g \) allows one to normalise a negative cosmological constant to
\[
\text{(2.2)} \quad \Lambda = -\frac{n(n-1)}{2},
\]
and we will often use this normalisation. The space-time manifold \( \mathcal{M} \) will be taken of the form \( \mathbb{R} \times M \), with the \( \mathbb{R} \) coordinate running along the orbits of a Killing vector field which is timelike everywhere.

In the Einstein-Maxwell-Chern-Simons-dilaton-scalar field case we have \[12, 30\]
\[
\text{(2.3)} \quad T_{\alpha\beta} = \frac{1}{8\pi G} \left[ \frac{1}{2} \partial_\alpha \phi \partial_\beta \phi + 2W(\phi)F_{\alpha\mu}F^{\beta\mu} - g_{\alpha\beta} \left( \frac{1}{4} (\nabla \phi)^2 + \frac{W(\phi)}{2} |F|^2 + \frac{1}{2} \mathcal{V}(\phi) \right) \right],
\]
with action
\[
\text{(2.4)} \quad S = \int d^{n+1}x \sqrt{-\det g} \left[ R(g) - 2\Lambda - W(\phi)|F|^2 - \frac{1}{2} (\nabla \phi)^2 - \mathcal{V}(\phi) \right] + S_{CS},
\]
where \( R(g) \) is the Ricci scalar of the metric \( g \) and where, in odd space-time dimensions, \( S_{CS} \) is the Abelian Chern-Simons action:
\[
\text{(2.5)} \quad S_{CS} = \begin{cases} 
0, & n \text{ is odd;} \\
\frac{\lambda}{16\pi G} \int A \wedge F \wedge \cdots \wedge F, & n = 2k,
\end{cases}
\]
for a constant \( \lambda \in \mathbb{R} \). We will assume that \( W \) and \( \mathcal{V} \) are smooth functions, and require
\[
\text{(2.6)} \quad W(0) = 1, \quad \mathcal{V}(0) = 0
\]
(note that this differs from the conventions of the accompanying paper \[18\], where \( \phi \equiv 0 \) and where the normalisation \( W \equiv 1/2 \) has been used).

We can view \( \phi \) as taking values in a Euclidean \( \mathbb{R}^{N+1} \) for some \( N \geq 0 \), with the first component \( \phi^1 \) corresponding to the dilaton field, and with \( W \) depending only upon \( \phi^1 \). Then the remaining components \( (\phi^2, \ldots, \phi^{N+1}) \) of \( \phi \) describe \( N \) minimally-coupled scalar fields, possibly interacting with each other through the potential \( \mathcal{V} \) which might or might not depend upon \( \phi^1 \).
Taking $N = 0$, $\mathcal{V} \equiv 0$, $\phi = 2u$, $W(u) = e^{-2au}$ for a constant $a \in \mathbb{R}$, and setting the Chern-Simons coupling constant $\lambda$ to zero one obtains the usual Einstein-Maxwell-dilaton equations with action \[25\]

\[\begin{align*}
S &= \frac{1}{16\pi G} \int d^{n+1}x \sqrt{-\det g} \left[ R - 2\Lambda - e^{-2au}|F|^2 - 2(\nabla u)^2 \right].
\end{align*}\]

Similarly, we can view $F$ as taking values in a Euclidean $\mathbb{R}^{N_1}$ for some $N_1 \geq 0$, in which case we obtain a collection of Abelian Yang-Mills fields $F_{\mu \nu}^B dx^\mu \wedge dx^\nu$, $B = 1, \ldots, N_1$. The Chern-Simons action (2.5) can then be replaced by

\[\begin{align*}
S_{\text{CS}} &= \begin{cases} 
0, & n \text{ is odd}, \\
\frac{1}{16\pi G} \lambda_{BB_1 \ldots B_k} \int A^B \wedge F^{B_1} \wedge \cdots \wedge F^{B_k}, & n = 2k,
\end{cases}
\end{align*}\]

for a set of constants $\lambda_{BB_1 \ldots B_k}$, totally symmetric in the last $k$ indices.

Our analysis extends to general Yang-Mills-Higgs-dilaton-Chern-Simons fields in the obvious way, by replacing $\partial \phi$ by a gauge-covariant derivative. This is addressed in Section 5.3 below.

3. Definitions, notations and conventions

Our definitions and conventions are identical to those in [17, Section 2]. In particular $\rho$ denotes a non-negative smooth function which has nowhere vanishing gradient near $\partial M$ and vanishes precisely at $\partial M$.

Recall that the linearisation of the Ricci tensor in dimension $(n + 1)$ equals \[8, Equations (1.180a)-(1.180b), p. 64\]

\[\begin{align*}
\frac{1}{2} (\Delta L h_{ij} - 2\delta^i \delta h - Dd(tr h)),
\end{align*}\]

or in index notation

\[\begin{align*}
\frac{1}{2} (\Delta_L h_{ij} + D_i D^k h_{kj} + D_j D^k h_{ki} - D_i D_j h^k_k),
\end{align*}\]

where $\Delta_L h$ is the Lichnerowicz Laplacian acting on the symmetric two-tensor field $h$, defined as \[8, \S 1.143\]

\[\Delta_L h_{ij} = -D^k D_i h_{kj} + R_{ik} h_{j}^{\ k} + R_{jk} h_{i}^{\ k} - 2R_{ikj} h^{kl}.
\]

An explicit form of $\Delta_L$ for a metric of the form (4.1) below can be read-off from the formulae in [17, Appendix A].

We will say that a metric $g$ is non-degenerate if $\Delta_L + 2n$ has no $L^2$-kernel. Large classes of non-degenerate Einstein metrics are described in [1,2,4,32].

4. Method

We seek to construct Lorentzian metrics $g$ in any space-dimension $n \geq 3$, with Killing vector $X = \partial/\partial t$. In adapted coordinates those metrics can be written as

\[\begin{align*}
g &= -V^2(dt + \theta_i dx^i)^2 + g_{ij} dx^i dx^j, \\
\partial_t V &= \partial_i \theta = \partial_i g = 0.
\end{align*}\]

\[\text{(4.2)}\]
Let us denote by $\varphi = (\varphi^a)$ all matter fields, where the index $a$ runs over some index set $\{1, \ldots, N_m\}$, $1 \leq N_m < \infty$. The $\varphi^a$'s will be required to satisfy
\begin{equation}
\partial_t \varphi = 0
\end{equation}
in the coordinate system of (4.1), except in Section 5.4 respectively Section 6.1, where time-periodic matter field configurations are considered with static, respectively stationary, metrics. In the case of the action (2.4) we thus have $\varphi = (A_\mu, \phi^a)$, but the overall argument applies to more general systems as long as the energy-momentum tensor is at least quadratic in the fields.

Consider the Einstein equations (2.1), in space-time dimension $n + 1$, with a cosmological constant $\Lambda$. We impose (4.1)-(4.3), and assume that the energy-momentum tensor $T$ does not depend upon more than one derivative of $g$. We further suppose that
\begin{equation}
\text{whenever the matter field equations are satisfied we have } \nabla_\mu T^{\mu\nu} = 0,
\end{equation}
regardless of whether or not the metric $g$ satisfies (2.1). (We use the symbol $\nabla$ to denote the covariant derivative of the metric $g$.)

In order to obtain an elliptic system of equations for $(V, \theta, g)$ we replace (2.1) by
\begin{equation}
R(g)_{\mu\nu} - \frac{R(g)}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} + \nabla_\mu \Omega_\nu + \nabla_\nu \Omega_\mu - \nabla_\alpha \Omega_{\alpha\nu} g_{\mu\nu} = 8\pi G T_{\mu\nu},
\end{equation}
where
\begin{equation}
\Omega_\nu := g_{\nu\mu} g^{\alpha\beta}(\Gamma^\mu_{\alpha\beta}(g) - \Gamma^\mu_{\alpha\beta}(\tilde{g}))
\end{equation}
(cf. Appendix B below). Assume that (4.5)-(4.6) can be solved for $(V, \theta, g, \varphi)$. The Bianchi identity and (4.4) imply that
\begin{equation}
\nabla_\mu (\nabla_\nu \Omega_\nu + \nabla_\nu \Omega_\mu - \nabla_\alpha \Omega_{\alpha\mu}) = 0.
\end{equation}
We show in Appendix A that this equation implies $\Omega \equiv 0$ whenever $|\Omega|_g = o(\rho^{-1})$ (as will be the case for our solutions), and consequently we will obtain the desired solution of the original equations.

For the Einstein-scalar field equations this is the end of the story, provided we can construct solutions of the system (4.5)-(4.6). This will be done using the implicit function theorem around the solution $\phi \equiv 0, \ g = \tilde{g}$. Now, the associated linearised equations are rather complicated, in particular the question of invertibility of the linearisation of the modified Einstein equations is not a trivial issue. This is solved in [3, 11, 17, 18] by the following
artefact, which we apply again here: It is well known that the Einstein tensor for a Riemannian metric
\[ g = V^2 dt^2 + g \]
on \( S^1 \times M \) coincides with the Einstein tensor of \( g = -V^2 dt^2 + g \). This implies that the isomorphism property of the linearised operator for the “harmonically reduced Riemannian Einstein equations”, at a static solution, carries over to the Lorentzian equations; compare [17, Section 3 and Appendix A]. Hence, our hypothesis of non-degeneracy of \( g \) together with the implicit function theorem can be used to obtain solutions of the Lorentzian equations, provided that suitable isomorphism theorems can be established for the matter equations. This will be done for the equations at hand in Sections 4 and 5 below.

In cases involving Maxwell fields there will arise an issue related to gauge freedom for Maxwell fields which will be addressed in a somewhat similar manner to the addition of the \( \Omega \)-terms to the Einstein tensor: To render (4.5) well posed we will add to it a “gauge-fixing” term \( \sigma F \), which will have to be shown to vanish. For definiteness we consider the equations resulting from (2.4):
\[(4.9) \nabla^\mu (W(\phi)(\nabla_\nu A_\mu - \nabla_\mu A_\nu)) \]
\[ \equiv -\nabla^\mu (W \nabla_\mu A_\nu) - WR^\alpha_\nu A_\alpha + W' \nabla^\mu \phi \nabla_\nu A_\mu + W \nabla_\nu \nabla^\mu A_\mu \]
\[ = -\lambda \epsilon_{\mu_1...\mu_2k} F^{\mu_1\mu_2} ... F^{\mu_{2k-1}\mu_{2k}}, \]
and note that the divergence of the rightmost term above vanishes. We will show that we can solve the equation obtained by setting \( \sigma F \) to zero in (4.9):
\[(4.10) \]
\[ -\nabla^\mu (W \nabla_\mu A_\nu) - WR^\alpha_\nu A_\alpha + W' \nabla^\mu \phi \nabla_\nu A_\mu \]
\[ = -\lambda \epsilon_{\mu_1...\mu_2k} F^{\mu_1\mu_2} ... F^{\mu_{2k-1}\mu_{2k}}. \]
Equivalently,
\[(4.11) \]
\[ \nabla^\mu (W(\phi)(\nabla_\nu A_\mu - \nabla_\mu A_\nu)) \]
\[ = -W \nabla_\nu \nabla^\mu A_\mu - \lambda \epsilon_{\mu_1...\mu_2k} F^{\mu_1\mu_2} ... F^{\mu_{2k-1}\mu_{2k}}. \]
Since the divergence of the left-hand side vanishes, we obtain
\[(4.12) \]
\[ \nabla^\nu (W \nabla_\nu \nabla^\mu A_\mu) = 0. \]
It follows e.g. from [18, Theorem 3.3] that if \( \nabla^\mu A_\mu \to 0 \) then \( \nabla^\mu A_\mu = 0 \), so that the solution of (4.11) solves (4.9). This implies that the energy-momentum tensor is divergence-free, and we conclude as in the case without Maxwell fields.

For some purposes it is convenient to replace (4.6) with its equivalent form
\[(4.13) \]
\[ R_{\alpha\beta} = T_{\alpha\beta} - \nabla_\alpha \Omega_\beta - \nabla_\beta \Omega_\alpha + \frac{2\Lambda - g^{\mu\nu} T_{\mu\nu}}{n - 1} g_{\alpha\beta}. \]
We note that the linearisation at \( \tilde{g} \) of the \( \Omega \)-contribution above is
\[(4.14) \]
\[ \tilde{g}^{\mu\nu}(\nabla_\alpha \nabla_\mu h_{\beta\nu} + \nabla_\beta \nabla_\mu h_{\alpha\nu}) - \nabla_\alpha \nabla_\beta (\tilde{g}^{\mu\nu} h_{\mu\nu}), \]
which cancels exactly the non-$\Delta_L$ terms in \((3.1)\).

5. Static metrics

In this section we present the construction of static solutions of the equations at hand. Strictly speaking, the results in this section are a special case of those in Section 4 but it appears instructive to present them separately, taking into account that the analysis here is computationally less demanding than the general case.

Assuming staticity, in adapted coordinates the metric $g$ becomes

\begin{equation}
    g = -V^2 dt^2 + g, \quad \partial_t V = 0 = \partial_t g.
\end{equation}

Equations \((2.1)\) lead to the following set of equations, where we denote by $D$ the covariant derivative of $g$ (recall that $\nabla$ is the covariant derivative of $g$), and where $R_{ij}$ is the Ricci tensor of $g$:

\begin{equation}
    R_{ij} = V^{-1} D_i D_j V + \frac{2\Lambda}{n-1} g_{ij} + 8\pi G \left( T_{ij} - \frac{\text{Tr} T}{n-1} g_{ij} \right),
\end{equation}

\begin{equation}
    V D^i D_i V = 8\pi G V^2 \left( T_{\alpha\beta} N^\alpha N^\beta + \text{Tr} g T - \frac{1}{n-1} g_{ij} \right) - V^2 \frac{2\Lambda}{n-1}, \quad T_{0i} = 0,
\end{equation}

where $N^\alpha \partial_\alpha$ is the $g$-unit timelike normal to the level sets of $t$. Choosing $\Lambda$ as in \((2.2)\), taking into account the Maxwell equations and the scalar field equations, together with

\begin{equation}
    \partial_t F_{\mu\nu} = 0 = \partial_t \phi,
\end{equation}

one is led to the system

\begin{equation}
\begin{cases}
    V(-\Delta g V + nV) = -2W(\phi)F_{0i}F_0^i + \frac{V^2}{n-1}(\nabla(\phi) - W(\phi)|F|^2), \\
    R_{ij} + n g - V^{-1} D_i D_j V = \frac{1}{2} \partial_\alpha \partial_\beta \phi + 2F_{\alpha i} F^\alpha_j W(\phi) + \frac{g_{ij}}{n-1}(\nabla(\phi) - W|F|^2), \\
    V^{\frac{1}{2}} \text{det} g \partial_\mu (V^{\frac{1}{2}} \text{det} g W(\phi) F^\mu) + B_{CS} = 0, \\
    V^{\frac{1}{2}} \text{det} g \partial_\mu (V^{\frac{1}{2}} \text{det} g g^{ij} \partial_j \phi) - W'|\phi)|F|^2 - \nabla' (\phi) = 0, \\
    W(\phi) F_{0j} F_j^0 = 0,
\end{cases}
\end{equation}

where $W'$ and $\nabla'$ are understood as differentials of $W$ and $\nabla$ when $\phi$ is $\mathbb{R}^{N+1}$ valued with $N \geq 1$, and where the Chern-Simons source-term $B_{CS}'$ is given by

\begin{equation}
    B_{CS}' = \begin{cases}
        0, & n \text{ odd}; \\
        -\frac{\lambda}{2^{k+2}} \epsilon^{\alpha_1 \beta_1 \cdots \alpha_k \beta_k} F_{\alpha_1 \beta_1} \cdots F_{\alpha_k \beta_k}, & n = 2k.
    \end{cases}
\end{equation}

5.1. Purely electric configurations. One way of satisfying the last equation in \((5.5)\) is to assume a purely electric Maxwell field:

\begin{equation}
    F = d(U dt), \quad \partial_t U = 0.
\end{equation}

(Purely magnetic configurations will be considered in Section 5.2 below, while configurations with both electric and magnetic fields can be obtained.
by applying duality rotations to the Maxwell field at the end of the construction.) Equation (5.7) leads to the following form of (5.5):

\[
(5.8) \quad \begin{cases}
V(\Delta g V + nV) = -2(n-2)W(\phi)|dU|_g^2 + \frac{1}{n-1}V^2 \mathcal{V}'(\phi), \\
\text{Ric}(g) + n\mathcal{V} - V^{-1}\text{Hess}_g V = \frac{1}{2}d\phi \otimes d\phi + \frac{1}{n-1}\mathcal{V}'(\phi)g \\
+ 2W(\phi)V^{-2}\left(-dU \otimes dU + \frac{1}{n-1}|dU|^2_g\right), \\
\text{div}_g(V^{-1}W(\phi)DU) = 0, \\
V^{-1}\text{div}_g(VD\phi) + 2V^{-2}|dU|^2_g\mathcal{W}'(\phi) - \mathcal{V}'(\phi) = 0
\end{cases}
\]

(the Chern-Simons term drops out because the purely spatial components of \(F\) vanish).

When \(\phi \equiv 0\) the \(U\)-equation coincides with that in [18], thus Theorem 3.3 there with \(s = -1\) applies. By continuity it still applies for all fields \(\phi\) which are sufficiently small in \(C^{k+1,\alpha}\), with any \(\epsilon > 0\). This motivates us to seek again solutions with \(U\) of the form

\[
(5.9) \quad U = \hat{U} + O(\rho),
\]

where \(\hat{U}\) is smooth-up-to-boundary on \(\overline{M}\). (Here two comments are in order: First, the key information is contained in the function \(\hat{U}|_{\partial M}\) defined on the boundary, but it is useful to invoke a function \(\hat{U}\) defined on \(\overline{M}\), which avoids the issue of considering extensions to \(M\) of functions defined on \(\partial M\). Next, the uniqueness part of our analysis below implies that solutions with \(\hat{U}|_{\partial M} = c\) for some constant \(c \in \mathbb{R}\) lead to configurations with \(U \equiv c\), hence trivial Maxwell fields. In other words, in (5.9) only \(\hat{U}\) modulo constants matters as far as physically relevant fields are concerned. Nevertheless, different \(c\)'s lead indeed to different fields \(U\).)

We will seek a solution such that

\[
(5.10) \quad \phi \to 0 \text{ as } \rho \to 0.
\]

Note that with the choice (5.9) the coefficient \(2V^{-2}|dU|^2_g\) appearing in the \(\phi\)-equation will be \(O(\rho^4)\). Assuming (5.10) we must have \(\mathcal{V}'(0) = 0\), so that in the scalar case the indicial exponents for the \(\phi\)-equation (cf., e.g., [32]) will be solutions of the equation

\[
(5.11) \quad \sigma(\sigma - n) - \mathcal{V}''(0) = 0 \iff \sigma = \frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \mathcal{V}''(0)}.
\]

When \(\phi\) is \(\mathbb{R}^{N+1}\) valued, with \(N \geq 1\), we obtain a collection of indicial exponents, with \(\mathcal{V}''(0)\) in (5.11) replaced by the eigenvalues of \(\mathcal{V}''(0)\). Now, to ensure useful properties of the operator associated with the equation for \(\phi\) we need all \(\sigma_\pm\) to be real with \(\sigma_+ \neq \sigma_-\), which leads to the condition

\[
(5.12) \quad -\frac{n^2}{4} < \mathcal{V}''(0),
\]

understood as a matrix inequality for the Hessian of \(\mathcal{V}\) at \(\phi = 0\) when \(N \geq 1\). After diagonalising \(\mathcal{V}''(0)\), each component in the diagonalising basis of the solutions will have the asymptotic behaviour

\[
(5.13) \quad \phi = \hat{\phi}\rho^{\sigma_+} + o(\rho^{\sigma_-}).
\]
possibly with $\hat{\phi}|_{\partial M} \equiv 0$, and note that we have
\begin{equation}
\phi \rightarrow \rho \rightarrow 0 \text{ and } \hat{\phi}|_{\partial M} \neq 0 \quad \Rightarrow \quad (\sigma_- > 0 \Leftrightarrow \mathcal{V}''(0) < 0).
\end{equation}

The properties of solutions of the $\phi$-equation depend now upon whether or not $W'(0) = 0$.

Let us first assume that $W'(0) = 0$ and that $\mathcal{V}''(0)$ is not an $L^2$-eigenvalue of the operator $\Delta_\delta$ (see Remark 6.2 below).

By Theorem D.1, Appendix D with $s = 1$ the operator $\phi \mapsto \hat{\phi} V^{-1} \text{div}_g (V D \phi) - V''(0) \phi$

that arises by linearising the $\phi$-equation is an isomorphism from $C^{k+2,\alpha}\delta$ to $C^{k,\alpha}\delta$ for $\delta \in (\sigma_-, \sigma_+)^0$.

We will then have a non-trivial solution $\phi \neq 0$ tending to zero when $\rho \rightarrow 0$ if and only if $\hat{\phi}|_{\partial M} \neq 0$. Since the case $\mathcal{V}''(0) \geq 0$ leads then to solutions which do not tend to zero at the conformal boundary, the hypothesis that $\mathcal{V}''(0) \geq 0$ and $W'(0) = 0$ leaves us with Maxwell matter fields only.

Assuming that $\mathcal{V}''(0) < 0$ and $W'(0) = 0$, it remains to check that the source terms in the remaining equations are compatible with the isomorphism ranges of the relevant operators. For this it is convenient to rewrite the $\mathcal{V}$-equation as
\begin{equation}
-\Delta_g V + (n + \frac{1}{n-1} \mathcal{V}(\phi)) V = -\frac{2(n-2)}{n-1} V^{-1} W(\phi)|dU|^2_g.
\end{equation}

Since the coefficient $\frac{1}{n-1} \mathcal{V}$ goes to zero at the boundary when $\phi$ does, we obtain the same indicial exponents as when $\phi \equiv 0$, and thus again an isomorphism for $\phi$ small enough. No new conditions arise from the remaining equations either.

Summarising, for $\hat{\phi}|_{\partial M} \neq 0$ in view of (5.12) we must have
\begin{equation}
-n^2 < 4 \mathcal{V}''(0) < 0.
\end{equation}

If $\phi$ has more than one component then the above inequalities apply with $\mathcal{V}''(0)$ replaced by the relevant eigenvalue of the Hessian of $\mathcal{V}(0)$ (in this case it is convenient to work in a diagonalising basis).

If $W'(0) \neq 0$ the situation is different, as then the scalar fields $\phi$ are driven both by the term $2V^{-2}|dU|^2_g W'(0)$ and by $\hat{\phi}|_{\partial M}$. First, if $\hat{U}|_{\partial M} \equiv 0$, then $U \equiv 0$, and if $\hat{\phi}|_{\partial M}$ vanishes as well, then $\phi \equiv 0$, and we are in vacuum.

On the other hand, if $\hat{\phi}|_{\partial M} \equiv 0$ and $\hat{U}|_{\partial M} \neq 0$ then we have non-trivial solutions with the following asymptotic behaviour:
\begin{equation}
\phi = \begin{cases}
O(\rho^{\sigma_+}) & \text{if } \sigma_+ < 4; \\
O(\rho^4 \ln \rho) & \text{if } \sigma_+ = 4; \\
O(\rho^4) & \text{if } \sigma_+ > 4.
\end{cases}
\end{equation}

An analysis similar to that of the case $W'(0) = 0$, taking into account that $\sigma_+ > n/2$ under (5.12), shows that non-trivial $\phi$’s tending to zero at the boundary will be obtained when $\hat{U}|_{\partial M} \neq 0$ if, in the one-component scalar field case,
\begin{equation}
\begin{cases}
-n^2 < 4 \mathcal{V}''(0) < 0, \quad \text{or} \\
\mathcal{V}''(0) \geq 0 \quad \text{and } \hat{\phi}|_{\partial M} \equiv 0.
\end{cases}
\end{equation}
The question of stability of the solutions with $\mathcal{V}''(0) < 0$ is unclear (compare [36], and there are in fact hints that some solutions with $\mathcal{V}''(0) < 0$ might be stable [9]), but this is irrelevant from the point of view of the question existence of static or stationary solutions, which is our main interest in this work.

One can now proceed exactly as in [18] to obtain the following: Consider the field equations associated with the action (2.4) for time-independent fields, with

\begin{equation}
W(0) = 1, \quad \mathcal{V}(0) = 0 = \mathcal{V}'(0), \quad \mathcal{V}''(0) > -\frac{n^2}{4}.
\end{equation}

Let

$$\mathring{\mathfrak{g}} = -\mathring{V}^2 dt^2 + \mathring{\mathfrak{g}}$$

be smoothly conformally compactifiable at $\mathbb{R} \times \partial M$ and satisfy the vacuum Einstein equations with a negative cosmological constant. Then:

**Proposition 5.1.** Suppose that $(S^1 \times M, \mathring{\mathcal{V}}^2 dt^2 + \mathring{\mathfrak{g}})$ is nondegenerate, and that $\mathcal{V}''(0)$ is not an $L^2$-eigenvalue of $\Delta \mathring{\mathfrak{g}}$ (cf. Remark 6.2 below). Under (5.19), assume that

1. $\mathcal{V}''(0) < 0$ with $\mathring{U}$ and $\mathring{\phi}$ which are smooth functions on $\overline{M}$ sufficiently close to zero, or
2. $\mathring{\phi} \equiv 0$ and $\mathring{U}$ is a smooth function on $\overline{M}$ sufficiently close to zero.

Then there exists a static solution of the equations with $U$ as in (5.9) and $\phi$ as in (5.13). The solutions are uniquely determined, within the class of static solutions belonging to some neighbourhood of $\mathring{\mathfrak{g}}$, by $\mathring{U}|_{\partial M}$ and $\mathring{\phi}|_{\partial M}$, with all fields having a polyhomogeneous expansion at $\partial M$. \hfill \square

If $\mathring{\phi}|_{\partial M} \equiv 0$ and $\mathcal{V}'(0) = 0$ then $\phi \equiv 0$, so that we obtain the solutions of Einstein-Maxwell equations already constructed in [18].

The reader will find some more information about the asymptotics of the fields in Section 7.

We emphasise that uniqueness is in the gauges implicitly defined above; for instance, two solutions with $U$-fields differing by a constant are considered distinct, even though they define of course the same Maxwell field $F_{\mu\nu}$. Furthermore, uniqueness is up-to diffeomorphisms which are the identity at the boundary in any case. It is conceivable that uniqueness holds for arbitrary diffeomorphisms, but this does not follow from our arguments.

### 5.2. Purely magnetic configurations.

Another way to satisfy the last equation in (5.5) is to consider purely magnetic fields, i.e.

\begin{equation}
F = d(A_i dx^i), \quad \partial_i A_i = 0.
\end{equation}

This implies $B^i_{\text{CS}} = 0$ and leads to the following matter equations,

\begin{equation}
\begin{cases}
D_j(VWg^{jk}g^{il}(A_{l,k} - A_{k,i})) = 0, \\
V^{-1}D_j(Vg^{ij}\partial_j\phi - 2W''(\phi)(A_{j,i} - A_{i,j})g^{ik}g^{jl}A_{l,k} - \mathcal{V}'(\phi) = 0, \\
B^0_{\text{CS}} = 0,
\end{cases}
\end{equation}
with the Einstein equations taking now the form
\begin{equation}
(5.22)
\begin{cases}
V(\Delta_g V + nV) = \frac{V^2}{2}\left(\gamma - 2W W''(\phi)(A_{j,i} - A_{i,j})g^{ik}g^{jl}A_{l,k}\right),
R_{ij} + n g_{ij} - V^{-1}D_i D_j V = \frac{1}{2}(\partial_i \phi)(\partial_j \phi) + 2W(A_{k,i} - A_{i,k})(A_{l,j} - A_{j,l})g^{kl} + \frac{g_{ij}}{n-1}(\gamma - 2WW''(\phi)(A_{j,i} - A_{i,j})g^{ik}g^{jl}A_{l,k}).
\end{cases}
\end{equation}
To satisfy the last equation of (5.21) one might as well assume that the Chern-Simons coupling constant \( \lambda \) vanishes. The first line of (5.21) can be rewritten, after introducing \( \sigma_F \) as in (4.12), in the form
\begin{equation}
(5.23)
0 = D_j(VW g^{ik}g^{jl}(D_k A_l)) - D^k(VW) D^l A_k - VWR^i_{\ell} A^\ell - VWD^i \left(\sigma_F - V^{-1}A_k D^k V\right).
\end{equation}
If we develop (5.23) and drop \( \sigma_F \), the operator acting on \( A \) becomes
\[ VWB(A)_i + VD^k W(D_k A_i - D_i A_k) + WA^k D_i D_k V - VWR_{ik} A^k, \]
where \( B \) is the operator of Lemma A.3 in [17]. The operator
\[ \mathcal{P} := B + (V^{-1}DD V - \text{Ric}(g)) \]
appears as part of the \((n + 1)\)-dimensional Riemannian Hodge Laplacian \( D_\theta^* D_\theta + \text{Ric}(g) \) [17]. Recall that, in coordinates, the characteristic indices for \( D_\theta^* D_\theta + \text{Ric}(g) \sim D_\theta^* D_\theta - n \) belong to \([-1, n-1]\) in the normal direction and to \([0, n-2]\) in the tangential one (cf., e.g. [4 Section 2.3]). We deduce from Theorem C(c) and Corollary 7.4 of [32] that if there are no harmonic forms in \( L^2 \) for \((S^1 \times M, \tilde{g})\), then \( \mathcal{P} \) will be an isomorphism from \( C_{k,1+\alpha}^k \) to \( C_{\alpha}^k \) for
\begin{equation}
(5.24)
\left|\delta - \frac{n}{2}\right| < \frac{n}{2} - 1.
\end{equation}
Let us denote by
\[ (\rho, x^\alpha) \]
local coordinates near \( \partial M \). From what has been said one expects solutions to take the form
\begin{equation}
(5.25)
A = (\hat{A}_\rho \rho^{-1} + O(1)) d\rho + (\hat{A}_a + O(\rho)) dx^a,
\end{equation}
with smooth-up-to-boundary functions \( \hat{A}_i \). As discussed in detail at the end of Section 4 we will have \( \sigma_F \equiv 0 \) if and only if \( \nabla^\mu A_\mu \) tends to zero as \( \rho \) tends to zero. Now
\begin{equation}
(5.26)
\nabla^\mu A_\mu = \frac{\partial_i (g^{ij} \sqrt{\text{det} g} V A_j)}{\sqrt{\text{det} g} V} = \partial_\rho (\rho^{1-n} A_\rho) \rho^{1+n} + O(\rho).
\end{equation}
This will be satisfied if and only if \( \hat{A}_\rho |_{\partial M} \equiv 0 \), without any restrictions on \( \hat{A}_a |_{\partial M} dx^a \). We conclude that:

**Proposition 5.2.** Under the hypotheses of Proposition 5.1 suppose moreover that the Einstein metric \((S^1 \times M, \tilde{V}^2 dt^2 + \hat{g})\) has no harmonic one-forms which are in \( L^2 \). Then the conclusions of Proposition 5.1 hold with \( U \) replaced by \( A_\rho dx^\rho \) of the form (5.25) with \( A_\rho |_{\partial M} \equiv 0 \), so that \( U |_{\partial M} \) is replaced by \( \hat{A}_a |_{\partial M} dx^a \). \( \square \)
As discussed in Appendix C, it follows from [15] that there are no $L^2$ harmonic one-forms on $S^1 \times M$ equipped with the Riemannian counterpart of the AdS metric $\hat{g}$, so the same is true for nearby metrics.

Further remarks concerning asymptotics and total energy are to be found in Section 7 below.

5.3. **Yang-Mills-Higgs fields.** The analysis so far readily generalises to Yang-Mills-Higgs-Chern-Simons fields. Here one often assumes that the Lie algebra $\mathfrak{g}$ of the structure group $G$ admits a positive-definite scalar product, but this is not needed in our considerations.

We denote by $A = A_\mu dx^\mu$ the Yang-Mills connection, with $A_\mu$ taking values in $G$, and by $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ its curvature:

$$F_{\mu
u} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

The scalar fields $\phi$ are allowed to be coupled to $A$ in the usual way, with derivatives involving $\phi$ replaced by gauge-covariant derivatives

$$\partial_\mu \phi \mapsto \partial_\mu \phi + T(A_\mu) \phi,$$

where $T$ is the linear map determined by the relevant representation; e.g., if $\phi$ are sections of a bundle associated to the adjoint representation, then $T(A_\mu) \phi = [A_\mu, \phi]$.

We suppose that the $G$-valued current vector $j^\nu$ appearing in the Yang-Mills equations,

$$D_\mu F^{\mu\nu} := \nabla_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = j^\nu,$$

is at least quadratic in all the fields, and satisfies the obvious compatibility conditions arising from

$$D_\nu D_\mu F^{\mu\nu} = D_\nu D_\mu [F_{\mu\nu}, F^{\mu\nu}] = 0 \quad \Rightarrow \quad D_\nu j^\nu = 0.$$

Equivalently

$$\nabla_\nu (j^\nu - [A_\mu, F^{\mu\nu}]) = 0.$$

More precisely, we will need that

$$(5.31) \quad D_\nu j^\nu = 0 \quad \text{whenever the field equations for } \phi \text{ are satisfied.}$$

This will be automatically satisfied by currents arising from gauge-invariant Lagrange functions, and by the current arising from Chern-Simons terms [16], whether Abelian or non-Abelian.

Let us, first, assume that the Yang-Mills principal bundle, say $P(G)$, is trivial, so that $F$ is globally defined as a two-form. The case where $A = U dt$, $\partial_t U = 0$, where $U$ is $G$-valued, works exactly as in the Maxwell case, leading immediately to an obvious Yang-Mills equivalent of Proposition 5.1 whenever $P(G) = G \times M$.

A purely-magnetic Yang-Mills potential, $A_0 \equiv 0$, $\partial_t A_i \equiv 0$, does not require much more work. Under our conditions, (5.21)-(5.22) are only modified by terms which are at least quadratic in the fields and which are lower order in terms of derivatives. Such terms, when small enough in relevant norms, do not affect the argument: One can view $F$ as a collection of several electric fields, introduce a vector-valued gauge-source function $\sigma_F$ (one such
function for each component of $F$), and conclude as before. In other words, assuming (5.27)-(5.29) and (5.32), we have established the Yang-Mills equivalent of Proposition 5.2 for trivial $G$-bundles. This establishes, for small $A$, existence of the solutions constructed numerically in [11], and in fact provides a much larger family of such solutions. (An even larger family of such solutions results from Theorem 6.1 below with $\tilde{\psi}|_{\partial M} = 0 = W'(0).$)

Summarising, we have proved:

**Proposition 5.3**. The conclusions of Propositions 5.1 and 5.2 hold when Maxwell and scalar fields are replaced by Yang-Mills and Higgs fields on a trivial gauge bundle.

Non-trivial bundles can be handled by introducing a suitably regular background Yang-Mills connection $\hat{A}$. This leads to a globally defined $\mathfrak{g}$-valued one-form $A - \hat{A}$, and a corresponding globally defined $\mathfrak{g}$-valued $\sigma_F$ function

$$\sigma_F := \hat{\nabla}^\mu (A_\mu - \hat{A}_\mu) := \nabla^\mu (A_\mu - \hat{A}_\mu) + [\hat{A}^\mu, A_\mu - \hat{A}_\mu].$$

The existence argument goes through if one moreover assumes that

1. there are no covariantly-constant Higgs fields $\phi$ which are in $L^2(S^1 \times M)$, and that
2. there are no $\mathfrak{g}$-valued harmonic forms which are in $L^2(S^1 \times M)$.

### 5.4. Time-periodic scalar fields.

Let us allow complex-valued $\phi$’s, and assume that (5.33)

$$V(\phi) = G_V(|\phi|^2) \quad \text{and} \quad W(\phi) = G_W(|\phi|^2)$$

for some differentiable functions $G_V$ and $G_W$, with the term $(\nabla \phi)^2$ in the action replaced by $\nabla^\alpha \tilde{\phi} \nabla_\alpha \phi$, where $\tilde{\phi}$ is the complex conjugate of $\phi$. Considering, as in [29], a time dependent field of the form

$$\phi(t, x) = e^{i\omega t}\hat{\psi}(x), \quad \text{with } \omega, \psi(x) \in \mathbb{R},$$

(5.34) leads to

$$\hat{\psi} + \left\{ \begin{array}{l}
V(-\Delta_{\phi} + n V) = -\frac{1}{2} \omega^2 \psi^2 - 2 G_W F_0 F_0^i + \frac{V^2}{n-1}(G_{\psi} - G_W F_{\alpha\beta} F^{\alpha\beta}), \\
R_{ij} + n g_{ij} - V^{-1} D_i D_j V = \frac{1}{2} \partial_i \hat{\psi} \partial_j \hat{\psi} + 2 F_\alpha F_j^\alpha G_W \\
\frac{1}{V \sqrt{\det g}} \partial_j (V \sqrt{\det g} G_W F^{j\nu}) + B^\nu_{CS} = 0, \\
\frac{1}{V \sqrt{\det g}} \partial_i (V \sqrt{\det g} g^{ij} \partial_j \hat{\psi}) + V^{-2} \omega^2 \psi - (G_W' |F|^2 + G_{\psi}') \psi = 0, \\
G_W F_{0j} F^j_0 = 0.
\end{array} \right.$$  

We see that the indicial exponents for the system remain unchanged, so that the existence and uniqueness theory with $\omega = 0$, presented above, applies without changes for all sufficiently small $\omega \in \mathbb{R}$:

**Proposition 5.4**. The conclusions of Propositions 5.1 and 5.2 hold for all sufficiently small $\omega \in \mathbb{R}$ where $\phi$ takes the form (5.34) with $\hat{\psi} = e^{i\omega t}\hat{\psi}$, and where $\hat{\psi}$ is smooth up-to-boundary.  

\[\Box\]
Recall that in [6] similar Einstein-scalar field solutions have been constructed with, however, $\mathcal{V} \equiv 0$. Our condition $\mathcal{V}'' < 0$ is evaded there by letting $-\omega^2$ be an eigenvalue of the operator $\psi \mapsto V D_i (V D^i \psi)$. It would be of interest to provide a proof of existence of such solutions using our methods; compare [10] for the spherically symmetric asymptotically flat case. We plan to return to this in a near future.

The discussion of the finiteness of energy of the resulting field configurations, to be found in Section 7, is identical to the $\omega = 0$ case.

5.5. $f(R)$ theories. It is well known that $f(R)$ theories can be reduced to the Einstein-scalar field equations with a specific potential. The object of this section is to show that there exists a class of functions $f$ for which our method applies, providing static or stationary asymptotically AdS solutions. More precisely, we consider $f(R)$-vacuum theories as described in [22, Section 2.3] (compare [28, Equation 32]), except for interchanging $g$ and $\tilde{g}$.

We assume that the space-dimension $n$ equals three. There one starts with the action

$$S = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-\tilde{g}} f(\tilde{R}),$$

and defines $F := f'(\tilde{R})$, assuming $F > 0$. As reported in [22], a conformal transformation $g := F \tilde{g}$, brings the action to the Einstein-scalar field form

$$S_E = \int d^4 x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \mathcal{V}(\phi) \right],$$

where

$$\phi := \frac{\sqrt{3/2}}{\kappa} \ln F, \quad \mathcal{V}(\phi) = \frac{F \tilde{R} - f(\tilde{R})}{2\kappa^2 F^2}.$$  

The question then arises, whether there exist functions $f$ which satisfy the hypotheses on $\mathcal{V}$ set forth in Section 5.1.

To ensure that we can invert $f'$ to obtain $\tilde{R} = f'^{-1}(F)$ we assume $f'' \neq 0$.

Using the definitions of $F$ and $\phi$ gives

$$\mathcal{V}(\phi) = \frac{e^{-2\sqrt{3\kappa} \phi}}{2\kappa^2} \left( e^{\sqrt{3\kappa} \phi} f'^{-1}(e^{\sqrt{3\kappa} \phi}) - f(f'^{-1}(e^{\sqrt{3\kappa} \phi})) \right).$$

The first and second derivatives of $\mathcal{V}(\phi)$ are given by

$$\mathcal{V}'(\phi) = \frac{\sqrt{2/3}}{\kappa} \left( e^{-2\sqrt{3\kappa} \phi} f(f'^{-1}(e^{\sqrt{3\kappa} \phi})) - \frac{1}{2} e^{-\sqrt{3\kappa} \phi} f'(f'^{-1}(e^{\sqrt{3\kappa} \phi})) \right),$$

$$\mathcal{V}''(\phi) = \frac{1}{3} \left( e^{-\sqrt{3\kappa} \phi} f'(f'^{-1}(e^{\sqrt{3\kappa} \phi})) - 4e^{-2\sqrt{3\kappa} \phi} f(f'^{-1}(e^{\sqrt{3\kappa} \phi})) + \frac{(f'^{-1})'(e^{\sqrt{3\kappa} \phi})}{f''(f'^{-1}(e^{\sqrt{2/3\kappa} \phi}))} \right).$$
Evaluating at \( \phi = 0 \) gives
\[
\mathcal{V}(0) = \frac{1}{2\kappa^2} \left( f'^{-1}(1) - f(f'^{-1}(1)) \right),
\]
\[
\mathcal{V}'(0) = \frac{\sqrt{2/3}}{\kappa} \left( f(f'^{-1}(1)) - \frac{1}{2} f'^{-1}(1) \right),
\]
\[
\mathcal{V}''(0) = \frac{1}{3} \left( f'^{-1}(1) - 4f(f'^{-1}(1)) + \frac{1}{f''(f'^{-1}(1))} \right).
\]

Set \( a := f'^{-1}(1) \). To obtain a negative cosmological constant in the action for \( g \) we need \( \mathcal{V}'(0) < 0 \) and therefore \( f(a) > a \). The condition \( \mathcal{V}'(0) = 0 \) leads to \( f(a) = a/2 \), hence \( a < 0 \).

Summarising, we require
\[
(5.40) \quad f' > 0, \quad f'' \neq 0, \quad f'^{-1}(1) < 0, \quad f(f'^{-1}(1)) = f'^{-1}(1)/2.
\]

To provide an example of function \( f \) which satisfies our requirements, we let
\[
f(\tilde{R}) = d\tilde{R} + c\tilde{R}^{\alpha+1} + e,
\]
with constants \( c, d \) and \( e \). This gives
\[
(5.41) \quad F = d + c(\alpha + 1)\tilde{R}^\alpha \iff \left( \tilde{R}(F) \right)^\alpha = \frac{F - d}{c(\alpha + 1)}.
\]
We consider \( d < 1, c < 0, \alpha = 1, 3, 5, \ldots, \text{and} \)
\[
e = \frac{\alpha(1 - 2d) - 1}{2(\alpha + 1)} \sqrt[\alpha]{\frac{1 - d}{c(\alpha + 1)}},
\]
where \( \sqrt[\alpha]{\cdot} \) denotes the real root. Then \( a = \sqrt[\alpha]{\frac{1 - d}{c(\alpha + 1)}} < 0, \mathcal{V}'(0) = 0 \) and \( \mathcal{V}'(0) < 0 \).

Additionally our method requires \( \mathcal{V}''(0) > -n^2/4 = 9/4 \) (recall that \( n = 3 \) in [22]) and we need \( \mathcal{V}''(0) < 0 \) to obtain nontrivial solutions for \( \phi \) without Maxwell fields. These conditions are fulfilled if
\[
(5.42) \quad -\sqrt[\alpha]{\frac{1 - d}{c(\alpha + 1)}} + \frac{\sqrt[\alpha]{\frac{1 - d}{c(\alpha + 1)}}^{1-\alpha}}{c(\alpha + 1)} + \frac{27}{4} \equiv \frac{a}{(1 - d)\alpha} - a + \frac{27}{4} > 0
\]
and \( d > (\alpha - 1)/\alpha \). Equation (5.42) can be satisfied by choosing either \( c \) large enough (as the l.h.s is of the form \( K/\sqrt{\alpha} \) for some constant \( K \) independent of \( c \), or \( \alpha \) large enough (as it is of the form \( \sqrt[\alpha]{K_1/(\alpha + 1)(1 - 1/(\alpha K_2) + 27/4) \) for positive constants \( K_{1,2} \) independent of \( \alpha \)).

6. Stationary metrics

We return to the metric (4.1)-(4.2). Let \( e^\mu \) be the coframe \( e^\mu := dt + \theta, e^i := dx^i \). The corresponding components \( R_{\mu\nu} \) of the Ricci tensor of \( g \) read (see, e.g., [21])
\[
(6.1) \quad \begin{cases}
R_{00} = V\Delta_\rho V + \frac{1}{4}\lambda_0^2, \\
R_{ij} = \text{Ric}(g)_{ij} - V^{-1}\text{Hess}_g V_{ij} + \frac{1}{4\alpha^2}\lambda_{ik}\lambda_j^k, \\
R_{0j} = -\frac{1}{2}V^{-1}D_i(V\lambda^i_j),
\end{cases}
\]
where

\[ \lambda_{ij} = -V^2(\partial_i \theta_j - \partial_j \theta_i). \]

For a general energy-momentum tensor \( T \) the equations are

\[
\begin{align*}
V(-\Delta_g V + nV) &= \frac{1}{4} |\lambda_g|^2 - 8\pi G \left( T_{00} + \frac{\nu^2}{n-1} \text{Tr}_g T \right), \\
R_{ij} + n g_{ij} - V^{-1} D_i D_j V &= \frac{1}{2V^2} \lambda_{ik} \lambda^k_j + 8\pi G (\theta_i \theta_j T_{00} - \theta_i T_{0j} - \theta_j T_{0i}) \\
&\quad + T_{ij} - \frac{g_{ij}}{n-1} \text{Tr}_g T, \\
D^j(V \lambda_{ij}) &= 16\pi G V (T_{0i} - \theta_i T_{00}).
\end{align*}
\]

When \( T \) is given by (2.3) we have

\[ \text{Tr}_g T = -\frac{1}{8\pi G} \left( \frac{n-3}{2} |W(\phi)||F|^2 + \frac{n-1}{4} |D\phi|^2 + \frac{n+1}{2} \varphi(\phi) \right). \]

Assuming moreover \( \partial_i \phi = 0 \) we obtain

\[
\begin{align*}
V(-\Delta_g V + nV) &= \frac{1}{4} |\lambda_g|^2 - 2WF_0 F_0' + \frac{V^2}{n-1} (\varphi' - W|F|^2), \\
R_{ij} + n g_{ij} - V^{-1} D_i D_j V &= \frac{1}{2V^2} \lambda_{ik} \lambda^k_j + \frac{1}{2} (\partial_i \phi)(\partial_j \phi) + 2WF_0 F_0' \\
&\quad + \frac{g_{ij}}{n-1} (\varphi' - W|F|^2) \\
&\quad - 2WF_0 (F^k_i \theta_i + F^k_i \theta_j - F^k_i \theta_j), \\
D^j(V \lambda_{ij}) &= 4VW F_0 (F^j_i \theta_i - F^j_i \theta_j).
\end{align*}
\]

The matter equations remain formally unchanged, as compared to (5.5), when written in the form

\[
\begin{align*}
\frac{1}{\sqrt{\text{det} g}} \partial_\nu (V \sqrt{\text{det} g} W^{\mu\nu}) + B_{\text{CS}}^\nu &= 0, \\
\frac{1}{\sqrt{\text{det} g}} \partial_\nu (V \sqrt{\text{det} g} g^{ij} \partial_\nu \phi) - W'(\phi)|F|^2 - \varphi'(\phi) &= 0,
\end{align*}
\]

with \( B_{\text{CS}}^\nu \) as in (5.5); indeed, the theta-dependent terms are hidden in \( |F|^2 \) and \( F^{\mu\nu} \). Letting \( F = d(U dt + A_i dx^i) \) and \( \partial_i U = \partial_t A = 0 \) we have

\[
\begin{align*}
D_j (V W g^{ik} g^{jl} (A_{l,k} - A_{k,l} + \theta_k U_l - \theta_l U_k)) + V B_{\text{CS}}^l &= 0, \\
D_j (V W g^{ik} (V^{-1} U_k + g^m_{i\lambda} \theta_{m \lambda} (\theta_{m,k} U_m + A_{k,m} - A_{m,k}))) + V B_{\text{CS}}^l &= 0, \\
V^{-1} D_i (V g^{ij} \partial_j \phi) - W'(\phi)|F|^2 - \varphi'(\phi) &= 0,
\end{align*}
\]

where

\[ |F|^2 = 2 \left[ (A_{j,i} + A_{i,j}) g^{ik} g^{jl} (A_{l,k} - 2U_{j,k} \theta_l) + |\nabla U|^2 g_{ij} (\theta^2_g - V^{-2}) - (U_i \theta^2_j) \right]. \]

Let \( \epsilon > 0 \). For \( \theta \) small in \( C^{\epsilon,\alpha}_g \)-norm the last two operators in (6.3), acting on \( \phi \) and \( U \), are close in norm to the operators considered in Section 4 and therefore isomorphisms as discussed there. Hence, in the implicit function argument we can choose \( \hat{U} \) and \( \hat{\phi} \) as in Section 5. It remains to consider the \( A \)-equation. We set

\[
\sigma_F = \nabla_\mu A^{\mu} = \frac{1}{\sqrt{\text{det} g}} \partial_\mu (\sqrt{\text{det} g} g^{\mu\nu} A_{\nu}) = V^{-1} D_i (V (A^i - U \theta^i)).
\]
Then the first line of (6.5) can be rewritten as
\[ (6.7) \quad D_j(VW g^{ik} g^{il} (D_k A_l - D_l A_k + \theta_k U_{il} - \theta_l U_{ik})) + VB^i_{CS} \]
\[ = D_j(VW g^{ik} g^{il} (D_k A_l + \theta_k U_{il} - \theta_l U_{ik})) \]
\[ - D^k(VW) D^i A_k - VW D_j D^i A^j + VB^i_{CS} \]
\[ = D_j(VW g^{ik} g^{il} (D_k A_l + \theta_k U_{il} - \theta_l U_{ik})) - D^k(VW) D^i A_k \]
\[ - VW R^i \ell A^\ell - VW D^i \left( \sigma_F - V^{-1} A^k D_k V + V^{-1} D_k (UV \theta^k) \right) + VB^i_{CS}. \]

The resulting linear operator acting on $A$ coincides with the one in (6.23), so that the discussion there applies. Inserting the asymptotic expansions for $V$ and $g$ into (6.6) gives
\[ (6.8) \quad \sigma_F = \nabla^\mu A_\mu = \rho^{1+n} \rho \partial_\mu (\rho^{1-n} (A_\mu - \theta_\mu) U) + O(\rho) \]
and, as $U = O(1)$, setting $\hat{A}_\rho|_\partial M \equiv 0$ guarantees $\sigma_F \equiv 0$, as discussed at the end of Section 4.

The rest of the proof is an application of the implicit function theorem, we sketch the details. We work with
\[ (V, g, \hat{V}, \hat{g}, \hat{h}) \]
close to $(\hat{V}, \hat{g})$. Keeping in mind that
\[ \lambda = -V^2 d\theta, W(\phi) = 1 + O(\phi) \text{ and } \mathcal{V}'(\phi) = \mathcal{V}''(0) + O(\phi^2), \]
the system obtained after the addition of the $\Omega$-terms as in (6.6) is of the form:
\[ (6.9) \quad \left\{ \begin{array}{ll}
\Omega(v, h, \theta) - q_1[v, h, \theta, A, U, \phi] = 0, \\
\mathcal{P}(A) - d\sigma_F - q_2[v, h, \theta, A, U, \phi] = 0, \\
V D_j(V^{-1} D^i U) - q_3[v, h, \theta, A, U, \phi] = 0, \\
V^{-1} D_i(V D^j \phi) - \mathcal{V}''(0) \phi - q_4[v, h, \theta, A, U, \phi] = 0,
\end{array} \right. \]
where the $q_i$’s are at least quadratic in their arguments and their first derivatives, and where $\Omega(v, h, \theta)$ corresponds to the operators $(l, L, \mathcal{L})$ of [17] Corollaries 3.2 and 3.3, which are the three components of the operator $\frac{1}{2} \Delta_L + n$, with $(l, L)$ and $\mathcal{L}$ being isomorphisms. For $s \in \mathbb{R}$ we define, as in [18], the operators
\[ T_s = V^{-s} D_i(V^s D^i \cdot). \]

We consider the modified system (6.9) with the Maxwell gauge term $d\sigma_F$ added:
\[ (6.10) \quad \left\{ \begin{array}{ll}
\Omega(v, h, \theta) - q_1[v, h, \theta, A, U, \phi] = 0, \\
\mathcal{P}(A) - q_2[v, h, \theta, A, U, \phi] = 0, \\
T_{-1}(U) - q_3[v, h, \theta, A, U, \phi] = 0, \\
(\mathcal{T}_1 - \mathcal{V}''(0))(\phi) - q_4[v, h, \theta, A, U, \phi] = 0.
\end{array} \right. \]

A solution, close to zero, of the elliptic system (6.10), with prescribed behavior at infinity, can be constructed in the following way: Let us define
\[ \mathcal{X} = (v, h, \theta, U, A, \phi), \]
we want to solve an equation of the form
\[ \mathcal{F}(\mathcal{X}) = 0, \]
with $\mathcal{X} \sim \hat{X}$ at large distance, for a prescribed small
\[
\hat{X} = (\hat{v}, \hat{h}, \hat{\theta}, \hat{U}, \hat{A}, \hat{\phi})
\]
(some of the components vanishing if desired). We let $\mathcal{X} = \hat{X} + X$ and define
\[
\mathcal{F}(\hat{X}, X) := \mathcal{F}(\hat{X} + X).
\]
We have $\mathcal{F}(0, 0) = 0$, with the linearisation
\[
D_X \mathcal{F}(0, 0) = \text{diag}(\Omega, \mathcal{P}, \mathcal{T}_{1,-1}, (\mathcal{T}_1 - \mathcal{V}'(0)))
\]
being an isomorphism. By the implicit function theorem, for all $\hat{X}$ small, there exist a small $X$, depending smoothly on $\hat{X}$, such that $\mathcal{X}$ is a solution of (6.10).

We have already explained why this solution solves the desired original equations.

For completeness we provide examples of functional spaces where the preceding procedure involving $\mathcal{F}$ applies. Taking into account the weights needed so that each of the operators involved is an isomorphism, a natural space for $X$ is, without indicating the tensor character of the relevant bundles,

\[
(6.11) \quad \mathcal{E}^{k+2} := C_{-1+s}^{k+2,\alpha} \times C_{1+s}^{k+2,\alpha} + C_{1+s}^{k+2,\alpha} \times C_{1+s}^{k+2,\alpha} + C_{1+s}^{k+2,\alpha} \times C_{1+s}^{k+2,\alpha},
\]

where $s$ is greater than and close to zero. The space for $\hat{X}|_{\partial M}$ is

\[
\rho^{-1}C^{k+2,\alpha} \times \rho^0 C^{k+2,\alpha} \times \rho^0 C^{k+2,\alpha} \times \rho^0 C^{k+2,\alpha} \times \rho^0 C^{k+2,\alpha} \times \rho^0 C^{k+2,\alpha}.
\]

The tensor fields in $C^{k+2,\alpha}(\partial M)$, can then be extended away from $\partial M$ to smooth tensor fields on $\hat{M}$ in any convenient way, keeping in mind the conditions

\[
(6.12) \quad \hat{A}_a|_{\partial M} = \hat{\theta}_a|_{\partial M} = \hat{h}_{\rho \rho}|_{\partial M} = 0.
\]

The reader can check that with the spaces chosen above, both $\mathcal{F}(\hat{X}, \cdot)$ and $D_X \mathcal{F}(0, 0)$ map $\mathcal{E}^{k+2}$ to $\mathcal{E}^k$.

We have thus proved:

**Theorem 6.1.** Suppose that the Einstein metric $(S^1 \times M, \hat{V}^2 dt^2 + \hat{g})$ is non-degenerate, has no harmonic one-forms which are in $L^2$, and $\mathcal{V}'(0)$ is not an $L^2$-eigenvalue of the operator $\Delta_{\hat{g}}$. Assume that

\[
W(0) = 1, \quad \mathcal{V}'(0) = 0 = \mathcal{V}''(0), \quad \mathcal{V}''(0) > -n^2/4,
\]

(1) and $\mathcal{V}''(0) < 0$ with $\hat{\theta}_a|_{\partial M} dx^a$, $\hat{U}|_{\partial M}$, $\hat{A}_a|_{\partial M} dx^a$, and $\hat{\phi}|_{\partial M}$ which are sufficiently small smooth fields on $\partial M$, or

(2) $\hat{\phi} \equiv 0$ and $\hat{\theta}_a|_{\partial M} dx^a$, $\hat{U}|_{\partial M}$, and $\hat{A}_a|_{\partial M} dx^a$ are sufficiently small smooth fields on $\partial M$.

Then there exist a solution of the Einstein-Maxwell-dilaton-scalar fields-Chern-Simons equations, or of the Yang-Mills-Higgs-Chern-Simons-dilaton equations with a trivial principal bundle, so that near $\partial M$ we have

\[
(6.13) \quad g \to \rho \to \hat{g}, \quad V \to \rho \to \hat{V}, \quad U \to \rho \to \hat{U}, \quad A \to \rho \to \hat{A}_a dx^a, \quad \theta \to \rho \to \hat{\theta},
\]

with all convergences in $\hat{g}$-norm. The hypothesis of non-existence of harmonic $L^2$-one-forms is not needed if $\hat{A}_a|_{\partial M} dx^a \equiv 0 \equiv \hat{U}|_{\partial M}$, in which case the Maxwell field or the Yang-Mills field are identically zero. \(\square\)
Remark 6.2. Let us comment on the kernel conditions above.

First, we show in Appendix C that the condition of non-existence of $L^2$-harmonic forms is satisfied near anti-de Sitter space-time in any case.

Next, it has been shown by Lee [31, Theorem A] that there are no $L^2$-eigenvalues of $\Delta_{\tilde{g}}$ when the Yamabe invariant of the conformal infinity is positive, in particular near anti-de Sitter space-time. Furthermore, and quite generally, $\mathcal{V}''(0) = 0$ is never an eigenvalue; this is, essentially, a consequence of the maximum principle. Finally, again quite generally, the $L^2$-spectrum of $-\Delta_{\tilde{g}}$ for asymptotically hyperbolic manifolds is $[n^2/4, +\infty]$ together with possibly a finite set of eigenvalues, with finite multiplicity, between 0 and $n^2/4$ [20] (compare [34]), so our non-eigenvalue condition is true except for at most a finite number of values of $\mathcal{V}''(0) \in (-n^2/4, 0)$ for all asymptotic geometries. □

Remark 6.3. Some comments on properties of the solutions are in order.

The case $(\tilde{v}|_{\partial M}, \tilde{h}_{ab}|_{\partial M}dx^a dx^b, \tilde{\theta}_a|_{\partial M}dx^a) = 0$ leads to a solution with the usual AdS conformal boundary when $\tilde{g}$ is taken to be the AdS metric.

The case $\tilde{\theta}_a|_{\partial M}dx^a = \tilde{U}|_{\partial M} = \tilde{\Lambda}_a|_{\partial M}dx^a = \tilde{\phi}|_{\partial M} = 0$ with $|\tilde{v}|_{\partial M} + |\tilde{h}_{ab}|_{\partial M}dx^a dx^b| \neq 0$ leads to the non-trivial vacuum configurations constructed in [17].

Note that $\tilde{\theta}_a|_{\partial M}dx^a \equiv 0$ but $\tilde{U} \neq 0$ and $\tilde{\Lambda} \neq 0$ might lead to a solution with $\theta \neq 0$ because the off-diagonal terms of the Maxwell energy-momentum tensor will drive a non-zero $\theta$.

If we choose $\tilde{\Lambda}_a|_{\partial M}dx^a \equiv 0$, $\tilde{U}|_{\partial M} \equiv 0$ and $\tilde{\theta}_a|_{\partial M}dx^a \equiv 0$ we obtain, from uniqueness of solutions, static solutions with scalar fields.

The vanishing of $\phi|_{\partial M}$ will lead to $\phi = 0$ everywhere only if $W'(0) = 0$, since otherwise the equation for $\phi$ is non-homogeneous. □

The discussion of Section 5.5 applies in the stationary case without changes, so that Theorem 6.1 also leads to solutions of $f(R)$ theories for the class of functions $f$ described there.

6.1. Time-periodic scalar fields. Consider time-periodic scalar fields of the form

$$(6.15) \quad \phi(t,x) = e^{i\omega t}\psi(x), \quad \omega \in \mathbb{R},$$

as done in the static case in section 5.4, but where now $\psi(x)$ is allowed to be complex. Assume for simplicity that all Maxwell fields are Abelian. Using the notation (6.38), and adapting the action as before gives for the Einstein
equations (6.16)

\[
\begin{aligned}
V(-\Delta_g V + nV) &= \frac{1}{4}|\lambda|^2 - \frac{1}{2}\omega^2|\psi|^2 - 2G_W F_0 F_0^* + \frac{V^2}{n-1}(G_\gamma - G_W |F|^2), \\
R_{ij} + n g_{ij} - V^{-1} D_i D_j V &= \frac{1}{2V^2} \lambda_{ik} \lambda_{kj} + \frac{1}{2} \Re(\partial_j \psi \partial_j \bar{\psi}) \\
&\quad + \frac{1}{2}(\theta_j \omega^2|\psi|^2 - \omega \partial_j \Im(\bar{\psi} \partial_j \psi) - \omega \partial_j \Im(\bar{\psi} \partial_i \psi)) \\
&\quad + 2G_W F_{\alpha j} F_j^\alpha + \frac{g_{ij}}{n-1}(G_\gamma - G_W |F|^2) \\
&\quad - 2G_W F_{ikl}(F_j^k \theta_i + F_i^k \theta_j - F_{0k} \theta_{ij}), \\
V^{-1} D_i (V \lambda_{ij}) &= 4G_W F_{ij}(F_i^j - F_0^j \theta_i) + \omega \Im(\bar{\psi} \partial_i \psi) - \theta \omega^2|\psi|^2,
\end{aligned}
\]

where \(\Im\) denotes taking the imaginary part. The matter equations are then (6.17)

\[
\begin{aligned}
D_j(V G_W g^{jk} g^{il}(A_{i,k} - A_{k,l} + \theta_k U_{i,l} - \theta_l U_{i,k}) + VB^\text{CS}_0 = 0, \\
D_j(V G_W g^{jk}(-V^{-2} U_{i,k} + g^{lm} \theta_1(\theta_m U_{i,k} - \theta_k U_{m,l} + A_{k,m} - A_{m,k}))) \\
&\quad + V B^\text{CS}_0 = 0, \\
V^{-1} D_i(V g^{ij} \partial_j \psi) - (G_{W'}/|F|^2 + G_{W'}) \psi \\
&\quad + (V^{-2} - \theta_1 \theta_k) \omega^2 \psi + i \omega(\theta^j \partial_j \psi + V^{-1} D_j(V \theta^j \psi)) = 0.
\end{aligned}
\]

It should be clear that all our previous arguments apply to this system of equations, leading to

**Proposition 6.4.** Assuming \(6.33\), the conclusions of Theorem 6.1 concerning the Einstein-Maxwell-Chern-Simons-dilaton-scalar fields hold for all sufficiently small \(\omega \in \mathbb{R}\) and \(\bar{\psi}|_{\partial M}\) when \(\phi\) takes the form \(6.15\) with \(\hat{\phi} = e^{i\omega t} \bar{\psi}\), where \(\hat{\psi}\) is smooth up-to-boundary. \(\square\)

7. Asymptotics and Energy

Whatever follows applies to the static, or stationary, or time-periodic solutions constructed above.

All our solutions have a polyhomogeneous expansion at \(\partial M\), that is, expansions in terms of integer powers of \(\ln \rho\) and of suitable powers of \(\rho\), as determined by all indicial exponents. It is straightforward but tedious to obtain a detailed description of the asymptotic behavior by inserting polyhomogeneous expansions in the equations and comparing coefficients.

We emphasise that non-integer indicial exponents \(\sigma_{\pm}\) for the scalar fields (cf. \(5.11\)) will introduce non-integer powers of \(\rho\) in asymptotic expansions for small \(\rho\) of all fields involved unless some miraculous cancellations occur. Logarithms of \(\rho\) are expected in the expansion for generic solutions regardless of whether or not \(\sigma_{\pm}\) are in \(\mathbb{Z}\).

We have already described the leading-order behaviour of the matter fields. Concerning the metric, the following holds: In the vacuum case, from what has been said it follows that the metrics under consideration have the following asymptotic behavior

\[
\rho V - \hat{V} = O(\rho^2), \quad \theta_i - \hat{\theta}_i = O(\rho), \quad \rho^2 g_{ij} - \hat{g}_{ij} = O(\rho^2),
\]
where all the fields at the left-hand sides have smooth limits at \( \rho = 0 \) in local coordinates near the boundary, and where we have extended the fields \( \hat{V}, \hat{\theta}, \) and \( \hat{g} \) from the conformal boundary to the interior by requiring them to be \( \rho \)-independent in some arbitrarily chosen coordinate system \((\rho, x^A)\) near the boundary.

In the presence of matter, we have to look at the error terms arising from the energy-momentum tensor. On constant-\( t \) slices the matter energy-density \( T_{00} \) reads

\[
T_{00} = \frac{1}{16\pi G} \left[ \frac{1}{2} \left( V^{-2} \partial_0 \phi \partial_0 \phi + |d\phi|^2 \right) + \mathcal{V} + W \left( 4V^{-2} F_\mu F^\mu + |F|^2 \right) \right].
\]

As in [18], the total energy content of non-trivial Maxwell or Yang-Mills fields will be finite only in space-dimensions \( n = 3 \) and \( n = 4 \). If \( \hat{\phi} |_{\partial M} \neq 0 \) the \( \phi \)-contribution to \( \rho \) behaves as \( \rho^{2\sigma_-} \), which will lead to a finite total energy of the scalar field if and only if

\[
\text{(7.2)} \quad -n^2 < 4\mathcal{V}''(0) < -n^2 + 1.
\]

If \( \hat{\phi} |_{\partial M} \equiv 0 \) then either \( \phi \equiv 0 \) when there is no Maxwell field and so the space-time is vacuum, or there is a Maxwell field in which case the \( \phi \)-contribution to the energy is

\[
\text{(7.3)} \quad \begin{cases} O(\rho^{2\sigma_+}) & \text{if } \sigma_+ < 4; \\ O(\rho^8 \ln \rho) & \text{if } \sigma_+ = 4, \end{cases}
\]

which gives a finite integral in either case.

This behavior of the energy-momentum tensor will affect the asymptotics only if 1) \( 0 < \sigma_- < 1 \), in which case the relative corrections of the metric components as above will be of order \( O(\rho^{2\sigma_-}) \) in place of \( O(\rho^2) \), e.g.

\[
V = \rho^{-1} \hat{V} (1 + O(\rho^{2\sigma_-})),
\]

e tc.; or if 2) \( \sigma_- = 1 \), which would lead to relative corrections \( O(\rho^{2\sigma_-} \ln \rho) \).

We note that the requirement of a well-defined Hamiltonian mass of the metric reads [20], taking the previous constraint \( 0 < \sigma_- < 1 \) into account,

\[
\text{(7.4)} \quad \min(2, 2\sigma_-) > n/2,
\]

which is satisfied by our solutions with non-trivial scalar field with \( \hat{\phi} \rho^{2\sigma_-} \) asymptotic behaviour and with \( \hat{\phi} \neq 0 \) if and only if

\[
\text{(7.5)} \quad -n^2 < 4\mathcal{V}''(0) < -\frac{3n^2}{4},
\]

with \( n = 3 \). On the other hand, all solutions with \( \hat{\phi} \equiv 0 \) have well defined and finite Hamiltonian mass.

The reader will note that, in all our solutions, only the scalar fields can possibly affect the leading-order asymptotics \((7.1)\) of the metric.
APPENDIX A. The $\Omega$ equation

Consider (4.7):

$$(\mathcal{g} P \Omega)_\mu := \nabla^\mu (\nabla_\mu \Omega_\nu + \nabla_\nu \Omega_\mu - \nabla^\alpha \Omega_\alpha g_{\mu \nu}) = 0.\tag{A.1}$$

We have

$$-(\mathcal{g} P \Omega)^\nu = \nabla_\mu (\nabla^\mu \Omega^\nu + \nabla^\nu \Omega^\mu) - \nabla^\nu \nabla_\mu \Omega_\alpha$$

$$= \nabla_\mu (\nabla^\mu \Omega^\nu - \nabla^\nu \Omega^\mu + 2 \nabla^\nu \Omega^\mu) - \nabla^\nu \nabla_\mu \Omega_\alpha$$

$$= \nabla_\mu (\nabla^\mu \Omega^\nu - \nabla^\nu \Omega^\mu) + \nabla^\nu \nabla_\mu \Omega_\mu + 2 \nabla^\nu \Omega^\mu$$

$$= \frac{1}{\sqrt{|\det g|}} \partial_\mu (\sqrt{|\det g|} g^{\mu \alpha} \partial_\alpha (\partial_\beta \Omega_\beta - \partial_\beta \Omega_\alpha))$$

$$+ g^{\nu \alpha} \partial_\alpha \left[ \frac{1}{\sqrt{|\det g|}} \partial_\mu (\sqrt{|\det g|} g^{\mu \beta} \Omega_\beta) \right] + 2 \nabla^\nu \Omega^\mu.\tag{A.2}$$

In the static case, with a time-independent $\Omega_\nu$ and an Einstein $(n + 1)$-dimensional metric $g$ normalised so that $R_{\mu \nu} = -ng_{\mu \nu}$, this reads

$$\nabla_\mu (\nabla^\mu \Omega_0 + \nabla_0 \Omega^\mu) = g_{\mu 0} \nabla_\mu (\nabla^\mu \Omega_0 + \nabla_0 \Omega^\mu)$$

$$= g_{\mu 0} \frac{1}{\sqrt{|\det g|}} \partial_\mu (\sqrt{|\det g|} g^{ij} g_{\mu 0} \partial_j \Omega_0) - 2n \Omega_0$$

$$= \frac{V}{\sqrt{|\det g|}} \partial_\mu (V^{-1} \sqrt{|\det g|} g^{ij} \partial_j \Omega_0) - 2n \Omega_0$$

$$= V D_i (V^{-1} D^i \Omega_0) - 2n \Omega_0,\tag{A.3}$$

$$\nabla_\mu (\nabla^\mu \Omega_k + \nabla_k \Omega^\mu) =$$

$$\frac{1}{\sqrt{|\det g|}} g_{\mu \ell} \partial_\mu (\sqrt{|\det g|} g^{ij} g^{\ell m} (\partial_j \Omega_m - \partial_m \Omega_j))$$

$$+ \partial_k \left[ \frac{1}{\sqrt{|\det g|}} \partial_\mu (\sqrt{|\det g|} g^{ij} \Omega_j) \right] - 2n \Omega_k.\tag{A.4}$$

One notices that $\mathcal{g} P$ coincides with its Riemannian analogue $\mathcal{g} P$ when mapping covectors to covectors:

$$\mathcal{g} P = \mathcal{g} P.\tag{A.6}$$

In particular if

$$\nabla_\mu (\nabla^\mu \Omega^\nu + \nabla^\nu \Omega^\mu - g^{\mu \nu} \nabla^\alpha \Omega_\alpha) = 0$$

in the Lorentzian metric, then the same remains true in the Riemannian one.

Consider a one-form $\Omega$ which is in $L^2$ and satisfies

$$\mathcal{g} P \Omega_\nu \equiv \nabla_\mu \nabla^\mu \Omega_\nu + R^\mu_\nu \Omega_\mu = 0.\tag{A.5}$$

Multiplying (A.5) by $-g^{\mu \nu} \Omega_\mu$, and integrating by parts over $S^1 \times M$ using $g$ everywhere one finds

$$0 = \int_{S^1 \times M} \nabla_\mu \Omega_\nu \nabla^\mu \Omega^\nu + n \Omega^\alpha \Omega_\alpha \equiv \int_{S^1 \times M} |\nabla \Omega|^2 + n |\Omega|^2,$$

where the vanishing of the boundary term follows from elliptic regularity and density arguments. Thus $\Omega_\mu \equiv 0$; equivalently, the $L^2$ kernel of $\mathcal{g} P$ is trivial.
As \( P \) is an elliptic, formally self-adjoint operator, geometric in the sense of [32], it is an isomorphism from \( C_{k+2,\alpha}^\delta \) to \( C_{\delta}^{k,\alpha} \) by the results in this last reference when 
\[
\delta - \frac{n}{2} < \frac{n}{2} + 1.
\]

Equivalently, in local coordinates, the indicial exponents belong to \( \{-2,n\} \).

In particular any \( C_{0,\alpha-1+\epsilon} \) one-form, with \( \epsilon > 0 \), in the kernel of \( P \) is trivial.

In our applications, we will have \( \Omega \) in the kernel with \( \Omega \in C_{k+1,\alpha}^{1} \) where \( s \) is positive close to zero when \( (\hat{v},\hat{h}) = 0 \) and \( s = 0 \) if \( (\hat{v},\hat{h}) \neq 0 \), which guarantees the vanishing of \( \Omega \).

The usual perturbation arguments show that the above conclusions will remain true for all sufficiently small \( \theta \).

The alert reader will have noted that the above discussion goes through for all negative-definite Ricci tensors such that \( R_{0i} = 0 \).

**Appendix B. The Christoffels**

For further reference, and to get insight into (4.7), we compute the Christoffels symbols of a stationary metric. The computations will be made for a Lorentzian metric but the change \( V \rightarrow iv \), where \( i^2 = -1 \) gives the results for a Riemannian metric.

We recall the form of the metric considered here:

\[
\begin{align*}
g &= -V^2(dt + \theta_i dx^i)^2 + g_{ij} dx^i dx^j, \\
\partial_i V &= \partial_i \theta = \partial_i g = 0.
\end{align*}
\]

Its inverse is

\[
\begin{align*}
g^{\#} &= g^{\mu\nu} \partial_\mu \partial_\nu = -(V^{-2} - g^{ij} \theta_i \theta_j) \partial_0^2 - 2g^{ij} \theta_i \partial_0 \partial_j + g^{ij} \partial_i \partial_j \\
&= -V^{-2} \partial_0^2 + g^{ij} (\partial_i - \theta_i \partial_0)(\partial_j - \theta_j \partial_0).
\end{align*}
\]

The Christoffel symbols are then

\[
\begin{align*}
\Gamma^0_{00} &= -\theta^2 V D_i V, \\
\Gamma^k_{00} &= V D^k V, \\
\Gamma^0_{0j} &= -(|\theta|^2 - V^{-2}) V D_j V + \frac{1}{2} \theta^k \partial_k (V^2 \theta_j) - \partial_k (V^2 \theta_j), \\
\Gamma^0_{ij} &= -\frac{1}{2} (|\theta|^2 - V^{-2}) [\partial_j (V^2 \theta_i) + \partial_i (V^2 \theta_j)] - \theta^k \Gamma^0_{ij} (g - V^2 \theta \otimes \theta), \\
\Gamma^k_{i0} &= \theta^k V D_i V - \frac{1}{2} g^{kj} [\partial_j (V^2 \theta_i) - \partial_j (V^2 \theta_i)], \\
\Gamma^k_{ij} &= \frac{1}{2} \theta^k [\partial_i (V^2 \theta_j) + \partial_j (V^2 \theta_i)] + g^{kl} \Gamma^l_{ij} (g - V^2 \theta \otimes \theta),
\end{align*}
\]

where, as usual, \( g^{ij} \) is inverse to \( g_{ij} \) and where

\[
\Gamma_{ijk} = \frac{1}{2} (g_{jk,i} + g_{ji,k} - g_{jk,i}).
\]
Appendix C. AdS, $\mathbb{H}^n$ and a Quotient

Let us consider the Poincaré ball model of the hyperbolic space $\mathbb{H}^n$. The hyperbolic space is then the unit ball of $\mathbb{R}^n$ endowed with the hyperbolic metric

$$\hat{g} = \rho^{-2} \delta,$$

where $\delta$ is the Euclidean metric and

$$\rho(x) = \frac{1}{2}(1 - |x|^2).$$

We define

$$\hat{V} = \rho^{-1} - 1.$$

A model for $(n + 1)$-dimensional hyperbolic space is $\mathbb{R} \times \mathbb{H}^n$, equipped with the warped product metric

$$\hat{g} = \hat{V}^2 dt^2 + \hat{g},$$

de we thus write as usual

$$\mathbb{H}^{n+1} = \mathbb{R} \times \mathbb{H}^n.$$

Anti-de Sitter space is the same manifold with the Lorentzian metric

$$\hat{g} = -\hat{V}^2 dt^2 + \hat{g}.$$

Let $\Gamma = \mathbb{Z} \subset \mathbb{R}$ be a discrete subgroup of isometries of the $\mathbb{R}$ factor of $\mathbb{H}^{n+1}$. Then we can write

$$\Gamma \backslash \mathbb{H}^{n+1} = S^1 \hat{V}^2 \times \mathbb{H}^n.$$

Recall that the limit set $\Lambda(\Gamma)$ (see eg. [35], §12.1 page 573) is a subset of the sphere at infinity of $\mathbb{H}^{n+1}$ consisting of the union of the limits of all the orbits. We will show that $\Lambda(\Gamma)$ consists of two points at infinity. It then follows from [15, Theorem C] that our quotient has no $L^2$ harmonic one-forms.

To justify our claim about $\Lambda(\Gamma)$, consider the half space model of $\mathbb{H}^{n+1}$ with $(\vec{x}, y) \in \mathbb{R}^n \times (0, \infty)$ endowed with the metric $ds^2 = y^{-2}((d\vec{x})^2 + dy^2)$. Let $\mathcal{H}$ be the half sphere $|\vec{x}|^2 + y^2 = 1$ with $y > 0$. It is well known that the totally geodesic hypersurface $\mathcal{H}$ with the induced metric $h$ is a model of the hyperbolic space $\mathbb{H}^n$. We reparameterise $\mathbb{H}^{n+1}$ with $(\vec{u}, \sqrt{1 - |\vec{u}|^2}) \in \mathcal{H}$ and $t \in \mathbb{R}$ by

$$(\vec{x}, y) = e^t(\vec{u}, \sqrt{1 - |\vec{u}|^2}).$$

In the coordinate system $(t, \vec{u}) \in \mathbb{R} \times \mathbb{B}^n(0, 1)$, the metric becomes

$$ds^2 = (1 - |\vec{u}|^2)^{-1} dt^2 + h.$$
Appendix D. An isomorphism on functions

Let $s$ and $\lambda$ be real numbers. We will need an isomorphism property for the following operator acting on functions:

$$\sigma \mapsto (T_s + \lambda)\sigma := V^{-s}\nabla^i(V^{s}\nabla_i\sigma) + \lambda\sigma = \nabla^i\nabla_i\sigma + sV^{-1}\nabla^iV\nabla_i\sigma + \lambda\sigma.$$  

We note that, whenever $V^2d\varphi^2 + g$ is a static asymptotically hyperbolic metric on $S^1 \times M$, it holds that

$$V^{-2}|dV|^2 \to 1 \text{ and } V^{-1}\nabla^iV \to n$$  
as the conformal boundary is approached.

**Theorem D.1.** Let $(M, g)$ be an $n$-dimensional Riemannian manifold with an asymptotically hyperbolic metric with $V > 0$ and assume that (D.1) holds. Let $s \neq 1 - n$ and $\lambda < \left(\frac{s+n-1}{2}\right)^2$ suppose that

$$\left|\delta - \frac{s+n-1}{2}\right| < \sqrt{\left(\frac{s+n-1}{2}\right)^2 - \lambda}.$$  

If $T_s + \lambda := V^{s/2}(T_s + \lambda)V^{-s/2}$ has no $L^2$-kernel, then $T_s + \lambda$ is an isomorphism from $C_{\delta}^{k+2,\alpha}(M)$ to $C_{\delta}^{k,\alpha}(M)$.

**Proof.** The proof is an easy adaptation of the one of Theorem 3.3 in [18]. We sketch the steps. We set $\sigma = V^{-\frac{s}{2}}f$, thus

$$T_s\sigma = V^{-\frac{s}{2}}\left[V^{s}\nabla_i,f - \frac{s}{2}\left(\frac{s}{2} - 1\right)V^{-2}|dV|^2 + V^{-1}\nabla^iV\nabla_iV\right]f =: V^{-\frac{s}{2}}T_s f.$$  

By assumption $V^{-2}|dV|^2 \to 1$ and $V^{-1}\nabla^iV \to n$ at the conformal boundary, leading to the following indicial exponents for $T_s + \lambda$:

$$\delta = \frac{n-1}{2} \pm \sqrt{\left(\frac{s+n-1}{2}\right)^2 - \lambda}.$$  

The calculation immediately after Lemma 3.4 of [18] shows that the operator $T_s + \lambda$ satisfies condition (1.4) of [32],

$$\|u\|_{L^2} \leq C\|(T_s + \lambda)u\|_{L^2},$$  

for smooth $u$ compactly supported in a sufficiently small open set $U \subset M$ such that $\overline{U}$ is a neighborhood of $\partial M$, with

$$C^{-1} = \frac{(s+n-1)^2}{4} - \lambda.$$  

We conclude using [32], Theorem C(c), keeping in mind that $V^{-\frac{s}{2}}f$ is in $C_{\delta}^{k,\alpha}(M)$ iff $f \in C_{\delta}^{k,\alpha}(M)$, and our hypothesis that there is no $L^2$-kernel. □

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