Correlations in the Adiabatic Response of Chaotic Systems

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Adiabatic variation of the parameters of a chaotic system results in a fluctuating reaction force. In the leading order in the adiabaticity parameter, a dissipative force, that is present in classical mechanics was found to vanish in quantum mechanics. On the time scale \( t \), this force is proportional to \( I(t) \), the integral of the force-force correlation function over time \( t \). In order to understand the crossover between the classical and the quantum mechanical behavior we calculated \( I(t) \) in random matrix theory. We found that for systems belonging to the Gaussian unitary ensemble this crossover takes place at a characteristic time (proportional to the Heisenberg time) and for longer times \( I(t) \) practically vanishes, resulting in vanishing dissipation. For systems belonging to the Gaussian orthogonal ensemble \( I(t) \sim 1/t \) and there is no such characteristic time. \( I(t) \) is calculated for various models and the relation to experiment is discussed.

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Friction is usually described in statistical physics as transfer of energy to a system that consists of an infinite number of degrees of freedom in an irreversible way. The irreversibility is a result of the complexity of the motion of the many body system. A natural question is whether the coupling to a small chaotic system can exhibit dissipation resulting from its complex dynamics. Mechanisms of this nature were first introduced in the context of nuclear physics [1], and various aspects of it were discussed in the past [2,3]. In the present work we study, following Berry and Robbins (BR) [3] and Jarzynski [4], a specific model where the chaotic system is defined by the Hamiltonian \( \mathcal{H}(\mathbf{R}, \mathbf{z}) \) where \( \mathbf{z} \equiv (\mathbf{p}, \mathbf{r}) \) are the phase space coordinates of the chaotic system and \( \mathbf{R} \) is an adiabatically varying parameter. The system is chaotic for each value of \( \mathbf{R} \). The crucial feature of the system we studied in this work is that it exhibits a wide separation of time scales - the evolution of the chaotic system, characterized by the time scale \( T_{\text{fast}} \), is so rapid that it explores all of the phase space accessible to it before the parameter \( \mathbf{R} \), characterized by the time scale \( T_{\text{slow}} \), changes appreciably. The adiabaticity parameter is \( \varepsilon \sim T_{\text{fast}}/T_{\text{slow}} \). The average generalized force, that is applied by the chaotic system in question on the system that forces the adiabatic variation of \( \mathbf{R} \), is given by:

\[
\mathbf{F}(\tau_a) = -\int d\mathbf{z} \rho(\mathbf{z}, \tau_a) \partial_R \mathcal{H}(\mathbf{z}, \mathbf{R}(\tau_a)),
\]

where \( \rho(\mathbf{z}, \tau_a) \) is a normalized probability density in the fast particle phase space. If \( \mathbf{R} \) is a position space coordinate, \( \mathbf{F}(\tau_a) \) reduces to a regular force. It can be expanded in powers of the adiabaticity parameter \( \varepsilon \) as was done by BR [3] and Jarzynski [4]. The present work follows closely the formalism of BR.

Berry and Robbins [3] were able to calculate the force acting on the slow particle up to first order in \( \varepsilon \):

\[
\mathbf{F} \approx \mathbf{F}_0 + \varepsilon \mathbf{F}_1.
\]

To leading order, the force is given by the classical analogue of the Born-Oppenheimer force \( F_{0,1}(\tau_a) = -\partial_R E(\mathbf{R}) \) where \( E(\mathbf{R}) \) is chosen such that the phase space volume of the fast particle, \( \Omega(E(\mathbf{R}), \mathbf{R}) \), is constant [3]. The leading correction to \( \mathbf{F}_0 \) includes a velocity dependent force \( \mathbf{F}_1(\tau_a) \), and describes two qualitatively different forces. The first of these is geometric magnetism, that is related to the Berry phase [4]. The second one is related to deterministic friction [3,10,11]. A central question is under which conditions friction due to the velocity dependent force \( \mathbf{F}_1 \) is found.

For the exploration of the existence of friction it is sufficient to study the case where \( \mathbf{R} \) is replaced by a scalar \( X \). In this case the velocity dependent force \( \mathbf{F}_1(\tau_a) \) reduces to

\[
F_1(\tau_a) = -\dot{X} \sum E(X) \left( \partial_X \mathcal{H} \right)_0(\mathbf{R}(\tau_a)) \Big|_{E(X)},
\]

where \( \sum E(X) \equiv \partial_E \Omega(E, X) \) and:

\[
I(t) = \int_0^t C(t') dt'.
\]

The fluctuating force-force correlation function at time difference \( t' \) is:

\[
C(t') \equiv \left\langle \left( \partial_X \mathcal{H} \right)_0(\mathbf{R}(\tau_a)) \left( \partial_X \mathcal{H} \right)_0(\mathbf{R}(0)) \right\rangle_{E, X},
\]

where \( \langle \cdots \rangle_{E, X} \) denotes the microcanonical average over the energy surface with a fixed value of the parameter \( X \). The volume enclosed by this surface is an adiabatic invariant [5]. The fluctuation of the energy is \( \partial_X \mathcal{H}(\mathbf{z}, X) = \mathcal{H}(\mathbf{z}, X) - E(X) \). A finite value of the integral \( I(\infty) \) is required for friction. Within classical mechanics this integral is indeed positive. It was shown by BR that in quantum mechanics the dissipative part of the force \( \mathbf{F}_1 \) vanishes. To understand the reason for this discordance [3] was calculated and in quantum mechanics it takes the form:

\[
C(t) = \sum_{m \neq n} \left| \left\langle m \left| \partial_X \mathcal{H} \right| n \right\rangle \right|^2 \cos \left( \frac{\hbar}{2} (E_n - E_m) \right),
\]
where the $E_m$ are the eigenenergies of the chaotic system. The initial state is $n$ and the dependence on it is not important for the present work. The sum is over $m$. For this correlation function the integral $I(\infty)$ of Eq. (1) vanishes. In order to understand how the crossover between the classical and the quantum behavior occurs, it is instructive to calculate the integral of the correlation function over a finite time. Taking the classical limit $\hbar \to 0$ for any finite $t$ and then the limit $t \to \infty$ should result in a non-vanishing value of $I(\infty)$, while for any finite value of $\hbar$, $I(\infty)$ should vanish. The friction on the time scale $t$ is proportional to $I(t)$ as can easily be inferred from (1). For systems whose classical dynamics is chaotic, the energy levels are distributed according to random matrix theory (RMT) (2). The long time behavior of $I(t)$ is determined by the levels nearest to $n$, namely $n \pm 1$, as can be seen from (1). The natural question to ask is whether there is a characteristic time scale for the crossover between the quantum behavior of the integral $I(t)$ and its classical behavior. The most naive answer to this question is that the characteristic time scale is the Heisenberg time. On the other hand, one can argue that there is no time scale for this crossover at all (2). In RMT the probability for two consecutive levels to be separated by an energy difference $s$ behaves like $s^\beta$ for small spacings (3). Consequently, $\langle I_\beta(t) \rangle \sim t^{-\beta}$ for long times. Here $\langle \ldots \rangle$ denotes the RMT ensemble average, and $I_\beta$ is the integral for some $\beta$. The $\langle \ldots \rangle$ will be dropped from $C_\beta$ and $I_\beta$ for notational simplicity.

For the Gaussian orthogonal ensemble (GOE) ($\beta = 1$) one indeed finds that $I_\beta(t)$ decays like $1/t$, but for the Gaussian unitary ensemble (GUE) ($\beta = 2$) one finds that it decays like a Gaussian with a characteristic time proportional to the Heisenberg time or vanishes after the Heisenberg time depending on the parametric dependence on $X$. Why is the nature of the decay of $I_\beta(t)$ important? There is the quantum-classical discordance that has already been mentioned, and one would like to analyze the scale that is required to observe the crossover between the regimes. In addition, the model discussed in the present work is relevant for some experimental situations. Consider for example a molecular beam prepared in a classical configuration, where initially many levels are substantially populated. The beam travels in a slowly varying field (5). Consequently the internal dynamics in the molecules is in a slowly varying potential. Another example is of quantum dots where parameters are varied adiabatically like in pumping experiments, but with closed dots, so that their spectrum is discrete (6). A dramatic change in the energy absorption is predicted when a magnetic field is applied and the transition from GOE to GUE takes place.

The time over which the correlation function decays should be compared with other time scales present in the specific system studied. One such time scale is $T_2 \sim \varepsilon^{-2}$, which is the time scale for the breakdown of the first order of the multiple scale expansion in $\varepsilon$. Non-perturbative effects, such as Landau-Zener tunneling, become important on a time scale of $T_{LZ}$. In realistic experiments there is also the time scale for quantum decoherence $T_\phi$. In order to observe the classical to quantum crossover discussed in the present work ($I_\beta(t)$) should exhibit substantial decay for $t \ll \min(T_2, T_{LZ})$ and of the order of $T_\phi$. The energy absorption by the internal degrees of freedom is proportional to $I_\beta(T_\phi)$.

![FIG. 1. The integral of the correlation function for GOE. Numerical results for $N = 3$ (O), $N = 13$ (v), $N = 53$ (c) and $N = 103$ (s) are shown. Also shown is the large N approximation (Eq. (4) (line).](image1)

![FIG. 2. Similar to Fig. 1 for GUE, compared to the large N approximation (Eq. (6)].(image2)

We shall use the Hamiltonian introduced by Austin and Wilkinson (4) and model a parameter dependent system by the $N \times N$ random matrix:

$$H(X) = H_1 \cos X + H_2 \sin X,$$

where $H_{1,2}$ are $N \times N$ random matrices from the same
RMT ensemble. There are three advantages to working with $H(X)$: (a) it belongs to the same ensemble that $H_{1,2}$ belong to; (b) the derivatives of its matrix elements belong to the same ensemble; (c) the matrices $H(X)$ and $H'(X) \equiv dH(X)/dX$ are statistically independent. If we insert $H(X)$ and $H'(X)$ into Eq. [6] and then perform the ensemble average, we obtain:

$$C_\beta(t) = \left\langle \sum_{m \neq n} |H'(X)_{n,m}|^2 \cos \left[ \frac{t}{\hbar} (E_n - E_m) \right] \right\rangle,$$  (8)

where $H'(X)_{n,m} \equiv \langle n | dH(X)/dX | m \rangle$ and $\langle \ldots \rangle$ denotes RMT ensemble averaging.

The statistical independence of $H'(X)$ and $H(X)$ implies

$$C_\beta(t) = \beta \mu^2 \sum_{m \neq n} \left\langle \cos \left[ \frac{t}{\hbar} (E_n - E_m) \right] \right\rangle.$$  (9)

The fact that $H'(X)$ belongs to the same ensemble as $H(X)$, leading to $\langle |H'(X)_{n,m}|^2 \rangle = \langle |H_{n,m}|^2 \rangle = \beta \mu^2$ for $m \neq n$, was used.

We would like to make the connection between $C_\beta(t)/\beta \mu^2$ and the form factor:

$$K(t) = \int \left[ \frac{1}{\overline{p}(E)} \left\langle \rho(E + \epsilon/2\overline{p}) \rho(E - \epsilon/2\overline{p}) \right\rangle - 1 \right] \frac{1}{2\overline{p}(E)} d\epsilon,$$  (10)

where $\rho(E) = \sum_i \delta(E_i - E)$ is the density of states and $\overline{p}(E)$ is the smoothed density of states. The variable $\epsilon$ is the energy measured in units of the mean level spacing $1/\overline{p}(E)$ and $\tau = t/T_H$ is time in units of the Heisenberg time, $T_H = \hbar/\overline{p}(E)$. In what follows units where $\beta \mu^2 T_H = 1$ will be used. Eq. [10] can be written in the following form:

$$\frac{C_\beta(\tau)}{\beta \mu^2} = \int \left[ \frac{1}{\overline{p}(E)} \left\langle \rho(E + \epsilon/2\overline{p}) \rho(E - \epsilon/2\overline{p}) \right\rangle - \delta(\epsilon) \right] \times e^{i2\pi \tau \epsilon} d\epsilon.$$  (11)

Comparing the last equation with [10] one can see that:

$$C_\beta(\tau)/\beta \mu^2 = K(\tau) + \delta(\tau) - 1.$$  (12)

In this work we are mainly interested in the time integral of the correlation function [4]:

$$I_\beta(\tau) = 1/2 - \int_0^\tau d\tau' \left( 1 - K(\tau') \right).$$  (13)

In the limit $\tau \to \infty$ the RHS is just $R_2(\epsilon = 0)$, the two point spectral correlation function at zero energy separation. It vanishes as a result level repulsion.

In order to perform actual calculations we make use of the well known form factor for GOE and GUE [14]. It is standard to define $b(\tau) = 1 - K(\tau)$. For GOE (see [14] p. 137): $b(\tau) = 1 - 2\tau + \tau \ln b_+ \text{ for } \tau \leq 1$ and $b(\tau) = -1 + \tau \ln |b_+/b_-| \text{ for } \tau \geq 1$ where $b_\pm = 2\tau \pm 1$, leading to:

$$I_\beta(\tau) = \left\{ \begin{array}{ll} \frac{1}{2} - \left( \frac{\tau - \tau^2}{4} + \frac{1}{2} \left( \tau^2 - \frac{1}{4} \right) \ln \left[ 1 + \frac{1}{2\tau} \right] \right) & \tau \leq 1 \\ \frac{1}{2} - \left( \frac{1 - \tau}{2} + \frac{1}{2} \left( \tau^2 - \frac{1}{4} \right) \ln \left[ \frac{4\tau}{4\tau + 1} \right] \right) & \tau \geq 1. \end{array} \right.$$  (14)

For $\tau \to \infty$ it falls off asymptotically as

$$I_1(\tau) \sim 1/12\tau.$$  (15)

For GUE (see [14] p. 95): $b(\tau) = 1 - \tau$ for $\tau \leq 1$ and it vanishes for $\tau \geq 1$, from which one obtains:

$$I_2(\tau) = \left\{ \begin{array}{ll} 1/2 - \left[ \tau - \tau^2/2 \right] & \tau \leq 1 \\ 0 & \tau \geq 1. \end{array} \right.$$  (16)

The behavior for the Gaussian symplectic ensemble (GSE) is very similar.

In order to compare the analytical results that hold in the infinite $N$ limit with numerical data, ensembles of $N \times N$ matrices $H_1$ and $H_2$ of [6] and [8], belonging to GOE or GUE were generated numerically and the results are presented in Figs. 3 & 4.

The model [6] is very specific in its dependence on the parameter $X$. An important property of this model is the statistical independence between $H(X)$ and $H'(X)$.

Such independence holds to a good approximation for disordered systems [14]. It is reasonable to make this approximation also for RMT models of chaotic systems. The reason is that most eigenstates look random, are statistically independent of the eigenvalues and therefore for many types of perturbations the matrix elements of $H'(X)$ will look random and independent of the spectrum. Although this argument is reasonable for many types of parametric dependence it is clearly not general. For the asymptotic behavior much less is required, since the long time asymptotics is dominated by the nearest neighbor level spacing. The reason for this dominance is that if $\tau \gg 1$ the terms in the sum [6] oscillate wildly as a function of $m$, so that the important net contribution is from the terms nearest to being stationary. These are obviously $m = n \pm 1$. Introducing also the crucial assumption that that fluctuations of $\langle (H'(X))_{n,n+1} \rangle$ can be ignored, we find that in RMT:

$$C_\beta(\tau)/\beta \mu^2 \approx 2 \int_0^\infty ds \, P_\beta(s) \cos(2\pi \tau s),$$  (17)

where $s$ is the nearest neighbor level spacing in units of the mean level spacing $\Delta E = 1/\overline{p}(0)$ and $P_\beta(s)$ is the distribution of $s$. The integral of the correlation function is:

$$I_\beta(\tau) \approx \frac{1}{\pi} \int_0^\infty ds \, \frac{P_\beta(s)}{s} \sin(2\pi \tau s).$$  (18)
For the nearest neighbor level spacing distribution we use the Wigner surmise (see Eq. 202 in [13]):

\[ P_\beta(s) = c \, s^\beta \exp[-as^2] \]  

(19)

with \( a = \pi/4 \) and \( c = \pi/2 \) for GOE, and \( a = 4/\pi \) and \( c = 32/\pi^2 \) for GUE. The integral (18) can be calculated for these distributions. For GOE one finds that for large \( \tau \) it takes the form: \( I_1(\tau) \sim 1/4\pi \tau \), that is extremely close to [13]. For GUE one finds: \( I_2(\tau) = 2\tau \exp[-\pi^2\tau^2/4] \), that for large \( \tau \) is very close to [13]. Here too the behavior for GSE is very similar to that of GUE.

We found that for the RMT ensembles \( I_\beta(\tau) \) of [1] is dominated by the nearest neighbor level spacings resulting in the approximation (18) for \( I_\beta(\tau) \). Under the assumptions leading to this approximation \( I_\beta(\tau) \) can be calculated for the nearest neighbor level spacing distribution [14] with arbitrary \( \beta \), \( a \) and \( c \) even if \( H \) does not belong to an invariant Gaussian RMT ensemble. The integral of the correlation function [14] takes the form \( I_\beta(\tau) = (c/\pi a^{3/2}) I_\beta(y) \) where \( I_\beta(y) \equiv \int_0^\infty ds \, s^{\beta-1} \exp[-s^2] \sin sy \) with \( y = 2\pi\tau/\sqrt{\alpha} \). It satisfies the ordinary ordinary differential equation:

\[ \frac{d^2}{dy^2} I_\beta(y) + \frac{y}{2} \frac{d}{dy} I_\beta(y) + \frac{\beta}{2} I_\beta(y) = 0 \]  

(20)

with the boundary conditions \( I_\beta(0) = 0 \) and \( dI_\beta(0)/dy = \Gamma[(\beta+1)/2]/2 \). Making the substitution \( I_\beta(y) = f_\beta(y/\sqrt{\alpha}) \exp[-y^2/8] \) and changing to the variable \( x = y/\sqrt{2} \), one arrives at a new differential equation for \( f \) that is a well known equation, and its solutions are Parabolic Cylinder Functions [15]: \( U[1/2 - \beta, x], V[1/2 - \beta, x] \). For arbitrary \( \beta \), its solution is a linear combination of these two functions. From the initial conditions and the known behavior of \( U \) and \( V \) at 0 and \( \infty \) the asymptotic behavior of \( I_\beta(\tau) \) is found to be of two types. For \( \beta \neq 2n \) \( n = 1, 2, 3, \ldots \): \( I_\beta(\tau) \sim \tau^{-\beta} \), as \( \tau \to \infty \), while for \( \beta = 2n \) \( n = 1, 2, 3, \ldots \): \( I_\beta(\tau) \sim \exp[-\pi^2\tau^2/\alpha] \tau^{\beta-1} \), as \( \tau \to \infty \). This is precisely the type of behavior found for the RMT ensembles treated explicitly.

It was shown that for the model [1] there is a big qualitative difference between the integrals of the correlation functions of the RMT ensembles (Eqs. [14] & [16]). It was argued that the qualitative difference holds for a wide variety of models. For models where the mean level spacing is [14], \( I_\beta(\tau) \) decays like the power-law \( \tau^{-\beta} \), except for \( \beta = 2n \), which is positive even integers, for which it decays like a Gaussian with a characteristic time that is proportional to the Heisenber time. Only for even values of \( \beta \) the integral [13] can be written as an integral over an analytic function in the interval \([-\infty, \infty]\), and is dominated by a saddle point in the complex plane.

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[15] D. Cohen, Private Communication.
[16] M. Switkes, C. M. Marcus, K. Campman and A. C. Gos- sard, Science 283, 1905 (1999) and references therein.
[17] I. T. Chalker, I. V. Lerner and R. A. Smith, Phys. Rev. Lett. 77 554 (1996); J. Math. Phys. 37 5061 (1996).
[18] M. Abramowitz and A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Washington, D.C., 1964).