I. INTRODUCTION

Since most applications of quantum information rest upon the subtle properties of multipartite quantum systems, the qualification and quantification of multipartite entanglement is a central task of quantum information theory. Whereas the bipartite case is for finite as well as for certain infinite dimensional systems well understood, many questions are still open in the multipartite setting [1].

The set of Gaussian states still plays a major role in current experiments dealing with continuous quantum variables, as it comprises those states that are processed in most experiments. This, and the mathematical simplicity of those states, which can be fully characterized by the finite set of first and second moments, are the reason why mainly Gaussian states have been investigated in the context of continuous-variable (CV) quantum information [2].

Regarding the entanglement properties of Gaussian states, it has been shown that, as in the finite dimensional case a separable state has positive partial transpose and that there exist entangled states with positive partial transpose [3]. However, for party A possessing 1 mode and party B arbitrary many, it has been shown that partial transposition leads to a necessary and sufficient condition for separability [4]. For the general bipartite case, i.e., when both parties possess an arbitrary number of modes efficiently testable necessary and sufficient conditions of separability have been derived [4, 5]. In contrast to the case of finite dimensional Hilbert spaces, the question of which states can be distilled to pure entanglement has been solved for bipartite Gaussian states. In fact, it was shown [6] that a bipartite Gaussian state is distillable iff its partial transpose is not positive semidefinite [7]. In Refs. [6, 7] the problem of manipulation of Gaussian states has been studied. In particular, in [10, 11] the most general operations transforming Gaussian states to Gaussian states were studied. These operations are called Gaussian operations. In [11] it has been proven that it is not possible to distill Gaussian states using Gaussian operations (see also [9, 10]).

The knowledge about entanglement in the multipartite setting is still far from complete, although a large number of (mostly) partial results has been obtained. The generation of pure multipartite entangled Gaussian states was discussed in [12]. A classification of multipartite entanglement classes of arbitrary three-mode Gaussian states have been presented in [13]. Practical criteria for the certification of genuine multipartite entanglement were derived in [14]. A general solution to the multipartite separability problem in the Gaussian case was provided by the Gaussian entanglement witnesses and related semi-definite programs studied in [15]. A large number of quantum optical experiments demonstrating multi-mode entanglement in increasingly large systems [15–21], culminating in 10,000-mode (time-bin) entanglement reported in [22]. Moreover, several standard entanglement measures have been adapted to the Gaussian setting (as, e.g., robustness [11], obtainable from a semi-definite program as described in [5]) or Gaussian localizable entanglement [23] and notions such as GHZ-like states [24], maximal entanglement (as quantified by bipartite entanglement) [25], monogamy of entanglement [26] have been specialized to the Gaussian setting.

Despite these advances, the study of multipartite entanglement is still in an early stage. One method to gain more insight into the entanglement properties of multipartite states is to investigate their interconvertibility. An important fine-grained classification of multipartite entangled states sorts them according to convertibility by local unitaries, leading to the notion of local unitary (LU) equivalence [27, 30]. Clearly, two LU equivalent states possess the same amount of entanglement and are equivalent as a non-local resource. LU equivalence leads to a very detailed classification of multipartite states with a continuum of inequivalent classes. A more coarse-
grained (and therefore often more insightful) picture emerges if a larger class of transformations is allowed. Especially useful for entanglement classification is to allow for non-trace-preserving operations ([partial] measurements) and classical communication between parties which leads to the set of stochastic local operations and classical communication (SLOCC) [31]. SLOCC play an important role in entanglement theory [32, 38]. SLOCC-convertibility gives rise to fewer equivalence classes than LU-equivalence and in some cases only finitely many [32–34] SLOCC-classes exist.

For Gaussian states, it is reasonable to consider convertibility under Gaussian operations. Conversion (of mixed states) under trace-preserving local Gaussian operations (LOG; not necessarily unitary) was investigated for the two-mode case in [39] and for the general bi- and tripartite setting in [40, 41, 42], while transformation under trace-nonpreserving local Gaussian operations has been investigated in [42] for pure bipartite states. The equivalence of Gaussian states under Gaussian local unitaries (GLU) was studied for the (mixed) bipartite setting in [43] and for more parties in [26, 41, 44, 15]. In [45] and [44] standard forms for “generic” n-mode mixed and pure states were introduced. The case of pure three-mode states has been studied in detail in [24]. There, it is shown (for “generic” pure Gaussian states) that the GLU equivalence classes are characterized by three positive numbers (related to local purities) and a simple standard form was derived.

The aim of this paper is to derive a standard form for arbitrary Gaussian states which has the properties that (i) every state can be transformed into its standard form via Gaussian local unitaries, (ii) it is unique, and (iii) it can be easily computed. Due to these properties, the solution to the Gaussian LU-equivalence problem follows easily. We then focus on pure Gaussian three-mode states and show that any such state is characterized by the three local purities. The standard form of those states is used to investigate the manipulation of those states using GLOCC. We show that the completely symmetric states, which are sometimes referred to as maximally entangled states, cannot be used to obtain an arbitrary state via GLOCC.

The remainder of the paper is organized as follows. In Sec. II we briefly review the basic concepts and results on Gaussian states needed later. In Sec. III we present a standard form for arbitrary (pure and mixed) n-mode Gaussian states, where all modes are spatially separated, and derive the necessary and sufficient conditions for Gaussian state to be equivalent under Gaussian local unitaries (GLUs). As we will show, this criterion can efficiently be applied, since it only involves the computation of the singular value decomposition of $2 \times 2$ matrices, independently of the system size. We will then demonstrate our methods by considering first the simplest case of two modes, where we show that our standard form coincides with the one presented in [46, 47]. In Sec. III B we investigate the different GLU-equivalence classes of three–mode Gaussian states. We show that any pure three–mode Gaussian state is GLU–equivalent to a state with no correlations between the $X$ and $P$ quadratures and that an arbitrary three-mode pure Gaussian state is (up to GLU) uniquely characterized via the three local purities, i.e., by the bipartite entanglement between each single mode and the remaining two modes. This reproduces the results of [24] but shows that they apply to all three-mode states (not only a subset of generic states). In order to obtain more insight into the entanglement properties of three–mode states, we consider in Sec. IV the more general set of Gaussian local operations assisted by classical communication (GLOCC). In particular, we show that it is not possible to obtain from the symmetric Gaussian pure three-mode states (which are sometimes referred to as maximally entangled states or continuous–variables analogs of both GHZ and W states (“CV GHZ/W–states”, for short [12, 24, 26]), all pure three-mode state via GLOCC. This implies that those states are not, as the two–mode squeezed states are in the bipartite case, sufficient to obtain deterministically any other state via local Gaussian operations (and thus not a Gaussian analog of the maximally entangled set introduced in [37]). In contrast, we finally present a class of states from which, in particular, all symmetric states can be obtained via GLOCC. Hence, this class of states might be called more entangled than the symmetric one.

## II. PRELIMINARIES

We summarize here some results concerning Gaussian states and introduce our notation. We consider systems composed of $n$ modes, i.e., $n$ distinguishable infinite dimensional subsystems, each with Hilbert space $\mathcal{H} = \mathcal{L}(\mathbb{R})$. To each mode $k = 1, \ldots, n$ belong two canonical observables $X_k, P_k$ which obey the commutation relation $[X_k, P_k] = i$. Defining $R_{2k-1} = X_k, R_{2k} = P_k$, these relations are summarized as $[R_i, R_m] = -iJ_{im}$, using the antisymmetric $2n \times 2n$ matrix

$$J = \oplus_{k=1}^n J_1, \quad J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where here, and in the following $\oplus$ denotes the direct sum. Let us denote the unitary displacement operator by

$$D(x) = e^{i \sum_k (q_k X_k + p_k P_k)} = e^{ix \cdot R},$$

where $x = (q_1, p_1, \ldots, q_n, p_n) \in \mathbb{R}^{2n}$. Using this notation, the characteristic function of a state $\rho$ is defined as

$$\chi_\rho(x) = \text{tr}[\rho D(x)].$$

Gaussian states are those states for which $\chi$ is a Gaussian multivariate function of the phase space coordinates, $x$ [48], i.e.,

$$\chi_\rho(x) = e^{-\frac{1}{4} x^T \gamma x - i d^T x}.$$
Here, $\gamma$ is a real, symmetric, strictly positive $2n \times 2n$ matrix, the covariance matrix (CM), and $d \in \mathbb{R}^{2n}$ is a real vector, the displacement. A Gaussian state is completely determined by $\gamma$ and $d$. Note that both $\gamma$ and $d$ are directly measurable quantities, as their elements $\gamma_{kl}$ and $d_k$ are determined by the expectation values and variances of the operators $\hat{R}_k$, via

$$d_k = \text{tr}(\rho \hat{R}_k),$$

$$\gamma_{kl} = 2\text{Re}\{\text{tr}[\rho(\hat{R}_k - d_k)(\hat{R}_l - d_l)]\}. \quad (6)$$

The displacement of a (known) state can always be adjusted to $d = 0$ by a sequence of local unitary operations applied to individual modes [49]. Thus, the first moments are irrelevant for both the study of GLU–equivalence classes and the entanglement contained in the state and will therefore be set to zero.

Not all real, symmetric, positive matrices $\gamma$ correspond to the CM of a physical state, they also have to satisfy the uncertainty principle. There are several equivalent ways to characterize valid CMs, which are all useful in the following. Before we summarize them in Lemma 1, let us recall that a (real) linear transformation $S$ on phase space is called symplectic if it preserves $J$, i.e., if $SJS^T = J$ holds. The group of real symplectic $2n \times 2n$ matrices is denoted by $Sp_{2n}(\mathbb{R})$. Let us now state the conditions for a matrix to be a valid CM.

**Lemma 1. (Covariance Matrices)**

A real, symmetric and positive $2n \times 2n$ matrix, $\gamma$, is the CM of a physical state iff one of the following equivalent conditions holds

$$\gamma + J\gamma^{-1}J \geq 0, \quad (7a)$$

$$\gamma - iJ \geq 0, \quad (7b)$$

$$\gamma = S^T(D \oplus D)S, \quad (7c)$$

for $S$ symplectic and $D \geq 1$ diagonal. The CM $\gamma$ describes a pure state iff equality holds in Eq. (7a) or, equivalently, iff $D = 1$ in Eq. (7c), i.e., iff det $\gamma = 1$.

The proofs of these statements can be found in [3, 48, 50] respectively. As an example of a valid CM, let us recall that the CM of an arbitrary pure two-mode states (1 $\times$ 1 case), $\gamma$, can be written as [51]

$$\gamma = (S_1 \oplus S_2) \begin{pmatrix} \cosh r \mathbb{1} & \sinh r \sigma_z \\ \sinh r \sigma_z & \cosh r \mathbb{1} \end{pmatrix} (S_1^T \oplus S_2^T). \quad (8)$$

Here and in the following, $S_{1,2}$ are local symplectic matrices, $r \geq 0$, and $\sigma_x, \sigma_y, \sigma_z$ denote the Pauli operators. The parameter $r$ contains all information about the entanglement of the state, whereas $S_1$ and $S_2$ contain information about local squeezing [52]. An example of a pure state would be the two–mode squeezed state, whose CM is given by Eq. (8) with $S_1 = S_2 = \mathbb{1}$.

Whenever we consider a bipartite splitting of the state (n modes at one side and $m$ modes at the other, which we call $n \times m$ case in the following) we might write the CM in the index–free block form

$$\gamma = \begin{pmatrix} A & B \\ C^T & D \end{pmatrix}. \quad (9)$$

Here $A$, $B$ and $C$ are $2n \times 2n$, $2m \times 2n$, and $2n \times 2m$ matrices, respectively. Note that $A$ ($B$) is the CM corresponding to the reduced state of the first (second) system, respectively. The correlations between both systems are described by the matrix $C$, which vanishes for product states.

Since we are interested in Gaussian local unitary equivalence classes in this paper, we also review here how the CM $\gamma$ (and the displacement $d$) of a Gaussian state $\rho$ change under the evolution of a Gaussian unitary operator $U$. As can be easily verified, a unitary operator transforms any Gaussian state into a Gaussian state (i.e., describes a Gaussian operation) iff there exists a symplectic matrix $S$ and a real vector $r \in \mathbb{R}^{2n}$, such that $U^\dagger RU = SR + r$. Discarding the irrelevant displacement, the CM transforms according to [53]

$$\gamma' = S\gamma S^T. \quad (10)$$

The most general $S \in Sp_{2n}(\mathbb{R})$ can be written as $S = O_1 DO_2$, where $O_1, O_2$ are real orthogonal and symplectic matrices and $D = \text{diag}(r_1, \ldots, r_n, 1/r_1, \ldots, 1/r_n)$, with $r_i \in \mathbb{R}^+$ [54]; for $D = 1$, $S$ is called a passive operation, otherwise it is called active. Apart from describing Gaussian unitary operations, symplectic matrices can also be used to derive a simple normal form (Williamson normal form) for arbitrary CM, see Eq. (7c). The eigenvalues $d_i$ of $D$ are called the symplectic eigenvalues of $\gamma$ and are $\geq 1$. They are related to the purity of the corresponding Gaussian state, $\rho$, since $\text{tr}(\rho^2)$ is given by

$$\text{tr}(\rho^2) = |\gamma|^{-1/2} = \prod_{i=1}^n d_i^{-1}, \quad (11)$$

where here and in the following, $| \cdot |$ denotes the determinant. This can be easily verified by noting that $|\gamma| = |\gamma|J = |S^{-1} \oplus_{i=1}^n d_i \mathbb{1} (S^T)^{-1}J| = |\oplus_{i=1}^n d_i \mathbb{1}| = \prod d_i^2$. The purity can be utilized to quantify the entanglement contained in pure states. For instance, the quantity

$$P(|\Psi\rangle) = \text{tr}(\rho_{\text{red}}^2)^{-2}, \quad (12)$$

where $\rho_{\text{red}}$ denotes the reduced density operator of either system $A$ or $B$ of the pure state $|\Psi\rangle$, increases the more entangled $|\Psi\rangle$ is. Using the block form of the CM, $\gamma$ (see Eq. (9)), $P(|\Psi\rangle)$ is given by $|A| = |B|$.  



III. GLU-EQUVALENCE AND STANDARD FORM

We consider an arbitrary \( n \)-mode Gaussian state (pure or mixed), with CM \( \gamma \) and assume a partition of one mode per site. We first derive a standard form of \( \gamma \), \( S(\gamma) \), which we show to be unique, easily computable and to which each CM can be mapped via GLU. Two states are called GLU-equivalent if their density matrices can be transformed into each other by Gaussian local unitaries. Thus two Gaussian states with CM \( \gamma \) resp. \( \Gamma \) are GLU-equivalent if their CMs can be transformed into each other by a local symplectic transformation. Due to the fact that the standard form, which we introduce here, is unique it easily follows that two Gaussian states are GLU-equivalent iff their standard forms coincide.

We denote in the following by \( \gamma_{jk} \) the \( 2 \times 2 \) matrix describing the covariances between mode \( j \) and \( k \). As mentioned before any \( 2 \times 2 \) real symplectic matrix can be written as \( O_1 \text{diag}(r,1/r)O_2 \), with \( r \in \mathbb{R} \) and \( O_1 \) real orthogonal. The standard form is reached in two steps. First, we apply to each mode \( j \) the active GLU that symplectically diagonalizes \( \gamma_{jj}, \) i.e., \( S(\gamma)_{jj} = \lambda_j \mathbb{1} \). This leaves still the freedom to apply local passive operations \( S_j \) to each mode \( j \), which are given by \( O_j \in SO(2) \).

In the second step, we fix the \( O_j = \exp(ia_j \sigma_y) \) by considering the off-diagonal blocks \( \gamma_{jk}, j < k \) in turn (row by row, from left to right). First consider \( \gamma_{12} \) and determine its singular values; if they both are zero, continue with the next block; if they are non-zero but degenerate, then \( \gamma_{12} \), obeying \( \gamma_{12}^T \gamma_{12} \propto \mathbb{1} \) and \( |\gamma_{12}| > 0 \), is proportional to a real special orthogonal matrix, \( O \) which we write without loss of generality as \( O = e^{ia_1 \sigma_y} \in SO(2) \).

We fix \( a_2 = a + a_1 \) (with \( a_1 \) being determined subsequently); if they are non-degenerate and add to zero, then \( \gamma_{12} \propto \sigma_x O \), with \( O = e^{ia_2 \sigma_y} \in SO(2) \) and we fix \( a_2 = a - a_1 \) (we refer to the two cases that \( \gamma_{ij} \) is orthogonal as "degenerate"): otherwise, we fix both \( a_1, a_2 \) such that \( O_1, O_2 \) are the unique matrices in \( SO(2) \) such that \( O_1 \gamma_{12} O_2^T = \text{diag}(d_{12}, d_{12}) \), with \( d_{12} \geq |d_{12}| \). In all four cases \( S(\gamma)_{12} \) is diagonal. Now treat \( \gamma_{13} \) (and then all subsequent \( \gamma_{jk} \)) in the same manner. If \( a_j \) has already been determined in a previous step, then for non-degenerate singular values of \( \gamma_{jk} \) we fix \( a_k \) by diagonalizing \( \gamma_{jk}^T \gamma_{jk} \). In this manner, all \( a_j \) will be uniquely determined except in the case that (for some \( j \)) all \( \gamma_{jk} \) are zero (in which case the mode \( j \) factorizes and we set \( a_j = 0 \)) or that for each \( j \) there is exactly one non-vanishing degenerate \( \gamma_{jk} \) (in this case we set the undetermined \( a_j = 0 \)). Any \( n \)-mode CM is transformed to its standard form \( S(\gamma) \) by applying the \( n \) local active and \( n \) local passive unitaries as described above and we have

**Theorem 2** (Criterion for GLU-Equivalence). Any CM \( \gamma \) can be transformed into its standard form, \( S(\gamma) \), by Gaussian local unitaries. Two CMs \( \gamma \) and \( \Gamma \) are GLU-equivalent if and only if \( S(\gamma) = S(\Gamma) \).

Note that this criterion for GLU-equivalence is valid for both, mixed and pure states. Let us mention here that an essentially identical form for \( n \)-partite \( n \)-mode Gaussian states was introduced in [42] and that the \( n \times n \times n \) case was discussed in [41]. However, the question whether this is a unique standard form (which is essential for Theorem 2) was only discussed for generic states. Let us close this discussion with a remark on the relation of LU- and GLU-equivalence before using the GLU-criteria to derive the different GLU-classes of 2-mode and 3-mode states.

When studying GLU-equivalence, we restrict the allowed operations to a very small subset of all local unitaries. Hence, in general, two LU-equivalent states are not GLU-equivalent. However, for pure Gaussian states in a number of relevant cases the two notions coincide. Note that, in particular, if two pure states are LU-equivalent, then the Schmidt coefficients of these states across any bipartition must be the same. If we can show that the GLU-classes of Gaussian states are uniquely characterized by their Schmidt coefficients across all bipartitions, then it follows that for those Gaussian states LU-equivalence implies GLU-equivalence. This is actually the case for pure bipartite Gaussian states, as implied by the results of [42]: every pure \( n \times n \) Gaussian state \( |\psi\rangle \) is GLU-equivalent to \( \min(n,m) \)-mode squeezed states \( |\psi_{\text{sms}}(r_j)\rangle \) with squeezing parameters \( r_1 \geq r_2 \geq \ldots \geq r_{\min(n,m)} \geq 0 \), which fixes the Schmidt coefficients \( \lambda_1, \ldots, \lambda_n = \prod_{j=1}^n \frac{\tanh d_{lj}}{\cosh r_j} \) where \( l_j \in \mathbb{N} \). Thus, if two pure bipartite Gaussian states are LU-equivalent they have the same standard form \( \otimes_{j=1}^n |\psi_{\text{sms}}(r_j)\rangle \) and therefore are also GLU-equivalent. As we show in Subsec. III B below, the same implication also holds for pure \( 1 \times 1 \times 1 \) Gaussian states.

### A. 1 × 1 case

Let us first consider the simplest case of two mode Gaussian states. First we apply active transformations to map the reduced states, \( \gamma_{ii} \) to thermal states, \( \lambda_i \mathbb{1} \). Since the state is pure the reduced states must be identical, i.e., \( \lambda_1 = \lambda_2 = \lambda \). According to the algorithm above we apply next the orthogonal matrices, \( O_1, O_2 \) such that \( O_1 \gamma_{12} O_2^T = D \), where \( D \) is diagonal. Thus, the standard form, \( S(\gamma) \) is

\[
S(\gamma) = \begin{pmatrix} \lambda \mathbb{1} & D \\ D & \lambda \mathbb{1} \end{pmatrix}.
\]

Next, we show that the standard form introduced here coincides in the case of pure two-mode states with the form [42]

\[
\gamma = \begin{pmatrix} \cosh(r) \mathbb{1} & \sinh(r) \sigma_z \\ \sinh(r) \sigma_z & \cosh(r) \mathbb{1} \end{pmatrix},
\]

with the squeezing parameter \( r \). Note that due to condition \( \gamma J \gamma \geq J \), we have \( \lambda \geq 1 \). Imposing now the condition that \( \gamma \) corresponds to a pure state, i.e., \( \gamma J \gamma = J \).
we find $\lambda^2 \mathbb{1} + \tilde{D}D = \mathbb{1}$ and $\lambda(J, D) = 0$, where here and in the following $\{A, B\} = AB + BA$ denotes the anticommutator between any operators $A$ and $B$ and $\tilde{D} = \sigma_x D\sigma_x$. Since $\lambda \geq 1$ must be fulfilled by any CM, it must hold that $\{J, D\} = JD + D\tilde{J} = 0$, which implies that $D = \lambda_2 \mathbb{x}_2$ for some real $\lambda$. Due to the first condition we get then $\lambda^2 - \lambda^2 = 1$, which implies that we can choose $\lambda = \sinh(r)$ and $\lambda = \cosh(r)$, for some $r \in \mathbb{R}$. Thus the standard form coincides with Eq. (14).

B. $1 \times 1 \times 1$ case

In this section we identify the different GLU–classes of 3–mode Gaussian states. First we explicitly provide the general standard form of Theorem 2 for the three–mode case. Then we show that it considerably simplifies for pure states and prove an exhaustive parameterization of the pure three–mode states.

1. Standard form: Mixed states

In this section we derive the standard form for an arbitrary $1 \times 1 \times 1$ Gaussian state. It is convenient to introduce (index–free) notation for the nine $2 \times 2$ blocks of $\gamma$ by defining the matrix $\gamma$ as

$$\gamma = \begin{pmatrix} A & K & L \\ K^T & B & M \\ L^T & M^T & C \end{pmatrix}. \tag{15}$$

The basis chosen here will be called mode–ordered, as indices referring to the same mode ($A, B$ or $C$) are grouped. Sometimes the quadrature–ordered basis is used. This is a permutation in which first all the indices referring to $X$–quadratures appear, followed by those referring to $P$.

As before, we first choose the active transformations to map the reduced states into thermal states. Using the same notation as before, the real orthogonal matrices $O_i$, for $i = 1, 2, 3$ are then used to map the off–diagonal matrices into diagonal matrices. In case the singular values of all off–diagonal blocks are non–degenerate, we use $O_1$ and $O_2$ to map $K$ into a diagonal matrix with sorted entries in the diagonal, i.e., $O_1KO_2^T \equiv \text{diag}(d_{12}^+, d_{12}^-)$, with $d_{12}^+ \geq |d_{12}^-|$. $O_3$ is used to map $L$ into the form $OD_{13}$, for diagonal $D_{13}$ and some matrix $O \in SO(2)$. Thus, the standard form is given by

$$\gamma_s = \begin{pmatrix} \lambda_1 \mathbb{1} & D_{12} & OD_{13} \\ D_{12}^T & \lambda_2 \mathbb{1} & M \\ OD_{13}^T & M^T & \lambda_3 \mathbb{1} \end{pmatrix}. \tag{16}$$

where $D_{12}$ and $D_{13}$ are diagonal and $O \in SO(2)$. Hence, the number of free parameters in Eq. (16) is 12. In case of degeneracy, more of the off–diagonal blocks can be made diagonal, as explained above. Due to Theorem 2 we know that two states are GLU–equivalent iff their standard forms (Eq. (16)) coincide.

2. Standard form: Pure states

If we specialize to pure states, the CM must fulfill additional constraints and the number of free parameters is greatly reduced. We then have $\gamma J \gamma = J$, i.e., $\gamma$ is a symplectic matrix. Taking into account that $\gamma$ is symmetric, we have $\gamma = SS^T$ for a symplectic matrix $S = ODO^T$. The number of real parameters describing a pure $n$–mode state is therefore $n^2 + n$. Since the GLU, i.e., the local (single–mode) symplectic operations are parameterized by $3n$ parameters, one would expect a $n^2 - 2n$–parameter standard form. Hence, for the three mode Gaussian states considered here, one would expect three free parameters. In order to derive the parameterization we first show in the following theorem, that pure three–mode Gaussian states are of a particularly simple form.

**Theorem 3** ($1 \times 1 \times 1$ pure state $xp$ block diagonal). Any pure $1 \times 1 \times 1$ Gaussian state is GLU–equivalent to a state, whose CM, $\gamma$, as given in Eq. (16) has the property that all the submatrices $A, B, C, K, L, M$ are diagonal. I.e., in the $xp$–ordered basis we have

$$\gamma_s = \gamma_x \oplus \gamma_x^{-1}, \text{ where} \tag{17}$$

$$\gamma_x = \begin{pmatrix} \lambda_1 & d_{12}^+ & d_{13}^+ \\ d_{12}^- & \lambda_2 & d_{23}^+ \\ d_{13}^- & d_{23}^- & \lambda_3 \end{pmatrix}, \tag{18}$$

with $\lambda_i$ denoting the local purities and $d_{ij}^+ \in \mathbb{R}$.

**Proof.** In Appendix A we show that the necessary condition for $\gamma$ to correspond to a pure state, $\gamma J \gamma = J$, implies that all submatrices, $K, L, M$ have to be diagonal. This implies that pure three–mode states can always be brought into a form in which correlations exist only among the $X$–quadratures and among the $P$–quadratures, respectively. That is, the CM is $xp$–blockdiagonal in the standard form i.e., $\gamma = \gamma_x \oplus \gamma_x (in the xp–ordered basis). Using then that the state is pure, which implies the condition $J\gamma J^T \gamma = \mathbb{1}$, where $J = [0_n, -\mathbb{1}_n; \mathbb{1}_n, 0_m]$ is $J$ in the $xp$–basis, and $0_n, (\mathbb{1}_n)$ denote the $n \times n$ zero (identity) matrix, respectively, it is easy to see that for pure states $\gamma_p = \gamma_x^{-1}$, which proves the statement.

Since the positive real and symmetric matrix $\gamma_x$, can always be written as $\gamma_x = ODO^T$ for $O$ orthogonal and $D$ real and diagonal, six free parameters are required to characterize $\gamma_x$. Since in the standard form both $\gamma_x$ and $\gamma_p$ must have the same diagonal elements, this yields three constraining equations, leaving 3 parameters characterizing the equivalence classes. We derive in the next section the conditions on those parameters to correspond to a valid CM of a pure state.
3. Parameterization of pure $1 \times 1 \times 1$ states

As we have just seen, an arbitrary pure 3-mode state can be written as

$$\gamma = \begin{pmatrix} \lambda_1 \mathbb{1} & D_{12} & D_{13} \\ D_{12} & \lambda_2 \mathbb{1} & D_{23} \\ D_{13} & D_{23} & \lambda_3 \mathbb{1} \end{pmatrix},$$  \hspace{1cm} (19)

where $D_{ij}$ is diagonal. Due to the condition $\gamma \geq iJ$ [see Eq. (17b)], we have $\lambda_i \geq 1 \forall i$.

In this section we derive the conditions for $\gamma$ corresponding to a pure state and show that the CM can be fully parameterized by the three local-mixedness parameters $\lambda_j$. Recall that $\gamma$ is pure iff $\gamma \geq 0$ and $\gamma J \gamma = J$.

We first derive the necessary and sufficient conditions for a matrix $\gamma$, as given in Eq. (19) with $\lambda_i \geq 1$ to fulfill $\gamma J \gamma = J$ (see Lemma 4). After that, we derive the condition for such a matrix to be positive (see Lemma 6).

**Lemma 4.** A matrix $\gamma$, as given in Eq. (17) with $\lambda_i \geq 1$, fulfills $\gamma J \gamma = J$ iff the entries of the diagonal matrices $D_{ij} = \text{diag}(d_{ij}^+,d_{ij}^-)$ are given (up to GLUs) by

$$d_{ij}^\pm = \frac{1}{4\sqrt{\lambda_i\lambda_j}} (\sqrt{a_{ij}} \pm \sqrt{b_{ij}}),$$  \hspace{1cm} (20)

with

$$a_{ij} = [(\lambda_i - \lambda_j)^2 - (\lambda_k - 1)^2][(\lambda_i - \lambda_j)^2 - (\lambda_k + 1)^2]$$

$$b_{ij} = [(\lambda_i + \lambda_j)^2 - (\lambda_k - 1)^2][(\lambda_i + \lambda_j)^2 - (\lambda_k + 1)^2],$$

where $i \neq j$ and $k \neq i,j$ refers to the third index.

**Proof.** It is straightforward to show that the condition $\gamma J \gamma = J$ is equivalent to the following set of equations,

$$\lambda_1^2 + |D_{12}| + |D_{13}| = 1$$  \hspace{1cm} (21a)

$$\lambda_2^2 + |D_{12}| + |D_{23}| = 1$$  \hspace{1cm} (21b)

$$\lambda_3^2 + |D_{13}| + |D_{23}| = 1$$  \hspace{1cm} (21c)

$$\lambda_1 D_{12} + \lambda_3 D_{12} + \tilde{D}_{13} \circ D_{23} = 0$$  \hspace{1cm} (21d)

$$\lambda_1 D_{13} + \lambda_3 D_{13} + \tilde{D}_{12} \circ D_{23} = 0$$  \hspace{1cm} (21e)

$$\lambda_2 D_{13} + \lambda_3 D_{23} + \tilde{D}_{12} \circ D_{13} = 0$$  \hspace{1cm} (21f)

where $\circ$ denotes the componentwise multiplication (Hadamard product). Here, we used the notation $D_{ij} = \text{diag}(d_{ij}^+,d_{ij}^-)$, $\tilde{D} = \sigma_x D \sigma_x$ and that $DJ = J\tilde{D}$, (i.e., $\tilde{D} = -JJD$) for any diagonal matrix $D$ and therefore $DJD = |D|J$. Note that if $D = \text{diag}(a,b)$, then $\tilde{D} = \text{diag}(b,a)$. In Appendix B we show that those conditions (together with $\lambda_i \geq 1$) are satisfied iff the entries of the diagonal matrices $D_{ij} = \text{diag}(d_{ij}^+,d_{ij}^-)$ are given (up to GLUs) by $d_{ij}^\pm$ as given in the lemma. \(\square\)

Note that in [24] it has been stated that a generic state can be written as in Eq. (19), with the entries of the diagonal matrices given in Eq. (20). However, we are aiming here for a complete characterization of three-mode pure states. As we prove below, the results of [24] hold for all pure three-mode Gaussian states.

Clearly $a_{ij}, b_{ij}$ must be positive in order to obtain a real CM. This leads to the (mutually exclusive) conditions $|\lambda_i - \lambda_j| \leq \lambda_k - 1 \forall (ijk)$ or $|\lambda_i - \lambda_j| \leq \lambda_k + 1 \forall (ijk)$. We show now that only the first condition is compatible with the positivity of the reduced CM (at modes $(ij)$).

To see that, note that for pure three-mode states it follows from Eqs. (21a-21c) that for all $(ijk)$:

$$\lambda_k^2 = \lambda_i^2 + \lambda_j^2 + 2\lambda_i \lambda_j \lambda_k - |D_{ij}| - 1 = (\lambda_i + \lambda_j + 1)^2 - 2(\lambda_i + \lambda_j + \lambda_k) |D_{ij}| + 1.$$  \hspace{1cm} (22)

The last term in this expression is strictly negative since due to the fact that the CM of the modes $i,j$ has to be positive, we have $\lambda_i \lambda_j \geq \pm |D_{ij}|$, which implies that $\lambda_k < \lambda_i + \lambda_j + 1$. Thus, the conditions

$$\lambda_i + 1 \leq \lambda_j + \lambda_k \forall (ijk).$$  \hspace{1cm} (22)

are the necessary and sufficient conditions for a valid pure CM $\gamma$ to be real. Note that if $\lambda_i \geq \lambda_j, \lambda_k$, the conditions in Eq. (22) are equivalent to the condition $\lambda_i \leq -1 + \lambda_j + \lambda_k$. For later reference, we also note the simple expression for $|D_{ij}|$ in terms of the $\lambda$’s:

$$|D_{ij}| = \frac{1}{2} (\lambda_k^3 + 1 - \lambda_i^2 - \lambda_j^2).$$  \hspace{1cm} (23)

It remains to impose the condition that $\gamma \geq 0$. For this, we use the following Lemma (Schur’s complement), which is proven for instance in [3].

**Lemma 5.** (Positivity of self-adjoint matrices)

A self-adjoint matrix

$$M = \begin{pmatrix} A & C \\ C^\dagger & B \end{pmatrix},$$  \hspace{1cm} (24)

with $B > 0$ is positive if and only if

$$A - \frac{1}{B} C^\dagger B \geq 0.$$  \hspace{1cm} (25)

Using this lemma we show that any CM $\gamma$ as given in Eq. (19) is positive in case the condition (22) is satisfied, as stated in the following lemma.

**Lemma 6.** The symmetric matrix $\gamma$, as given in Eq. (19) with $\lambda_k \geq 1$, for $k \in \{1,2,3\}$ is positive semidefinite if Eq. (22) holds.

**Proof.** Since $\gamma = \gamma_x \oplus \gamma_z^{-1}$ (see Theorem 3), we have $\gamma > 0$ iff $\gamma_x > 0$. Using now Lemma 5 and the fact that $\lambda_k > 0$, we know that the $3 \times 3$ matrix $\gamma_x$ is positive iff the $2 \times 2$ matrix

$$Y = \begin{pmatrix} \lambda_1 & d_{12}^+ \\ d_{12}^- & \lambda_2 \end{pmatrix} - \frac{1}{\lambda_3} \begin{pmatrix} d_{13}^+ & d_{13}^- \\ d_{23}^+ & d_{23}^- \end{pmatrix} > 0.$$  \hspace{1cm} (26)

Note that $Y > 0$ iff $|Y| > 0$ and $\text{tr}(Y) > 0$. Using that $\lambda_k \geq 1$ for all $k$, tedious, but elementary calculations (see Appendix C) show that both expressions are positive if the condition (22) holds. \(\square\)
Combining Lemma 4 and Lemma 6 we obtain the following theorem.

**Theorem 7.** Any CM of a pure 3–mode Gaussian state can be written (up to GLUs) as

$$
\gamma = \begin{pmatrix} 
\lambda_1 \mathbb{I} & D_{12} & D_{13} \\
D_{12} & \lambda_2 \mathbb{I} & D_{23} \\
D_{13} & D_{23} & \lambda_3 \mathbb{I} 
\end{pmatrix},
$$

where $D_{ij} = \text{diag}(d_{ij}^+, d_{ij}^-)$, with $d_{ij}^\pm$ given in Eq. (24).

Thus, the non–local properties of any pure three-mode Gaussian state are completely characterized by the local mixedness parameters, $\lambda_i$, i.e., by the bipartite entanglement shared between each mode with the other two. Recalling our discussion of LU– and GLU–equivalence at the beginning of this Section, we see that (like the pure bipartite Gaussian states) also the pure $1 \times 1 \times 1$ Gaussian states are completely characterized by their Schmidt coefficients across the (three) different bipartitions (which are in one-to-one relation with the $\lambda_i$). Therefore, those states are LU–equivalent if they are GLU–equivalent and Theorem 7 also characterizes the LU–classes of pure three-mode Gaussian states.

### 4. Some special cases

Let us briefly consider two special cases, namely the one where one of the off-diagonal matrices, say $\gamma_{ij}$, is (a) not invertible or (b) proportional to $\mathbb{I}$.

Case (a): The condition $|D_{ij}| = 0$ together with Eq. (23) implies $\lambda_k^2 = \lambda_i^2 + \lambda_j^2 - 1$ and inserting it in Eq. (20) we find that $d_{ij}^+ = 0$ and $d_{ij}^- = \sqrt{(\lambda_i^2 - 1)(\lambda_j^2 - 1)(\lambda_i \lambda_j)^{-1/2}}$. Note that $d_{ij}^+ \neq 0$ (as are $|D_{ik}|, |D_{jk}|$) unless $\lambda_i$ or $\lambda_j$ equals 1, in which case the respective mode factorizes and the remaining two would be in a two-mode squeezed state.

Case (b): $D_{ij} \propto \mathbb{I}$ is only possible if $b_{ij} = 0$, which implies that $\lambda_k = \lambda_i + \lambda_j - 1$ (i.e., in particular, $\lambda_k \geq \lambda_i, \lambda_j$). Clearly, then $a_{ik} - a_{ij}(i) = \lambda_{ij}(i) - 1$ and thus $a_{ik} = a_{jk} = 0$. Hence, the remaining two off-diagonal blocks are both proportional to $\sigma_z$. It also holds that if $D_{ij} \propto \sigma_z$, which implies that $a_{ij} = 0$ (which fixes $\lambda_k = 1 + |\lambda_i - \lambda_j|$), that one of the remaining two off-diagonal blocks is degenerate (and the other $\propto \sigma_z$): If $\lambda_i \geq \lambda_j$ then $b_{jk} = 0$ and $a_{ik} = 0$ otherwise reversed. As we see below, these states can all be generated by letting a beam splitter couple one mode of a two-mode system in a two-mode squeezed vacuum state with a third mode in the vacuum.

Another interesting special case are the fully permutation symmetric states $14, 24$, for which the three local mixednesses are identical, i.e., $\lambda_i = \lambda \forall i$. We denote the CM of a symmetric state in standard form by $\gamma_{\text{sym}}(\lambda)$. These states were sometimes called maximally entangled $24, 26$ due to their extremal entanglement properties reminiscent of their qubit analogs $34$. For these states the matrices $D_{ij}$ are given by $\text{diag}(d_{ij}^+, d_{ij}^-)$ with

$$
d^{\pm} = \frac{1}{4\lambda}(\lambda^2 - 1) \pm \sqrt{9\lambda^4 - 10\lambda^2 + 1}.
$$

In Sec. V.C we investigate which states can be obtained from symmetric states via Gaussian Local Operations assisted by Classical Communication (GLOCC).

### IV. GENERATION OF THREE-MODE PURE STATES

Let us briefly remark on the generation of pure three-mode states. In $58$ a general state-generation scheme for this case is presented. There, a two-mode squeezed state (of modes 1 and 2, with squeezing parameter $r$) is coupled to mode 3 (in the vacuum state) by a sequence of three beam splitters (BS) acting on modes (13), (23), and (13) respectively. The transmissivities of the third BS is fixed while those of the first two are adjusted such as to produce the desired local purities.

Note that in the special case in which one of the off-diagonal matrices is degenerate (case (b) above), a simplified scheme suffices: Letting a beam splitter with transmissivity $\cos^2 \theta \in [0, 1]$ act on part of a two-mode squeezed vacuum (with squeezing parameter $s \geq 0$) and a vacuum mode allows to generate all states with degenerate CM: If $\lambda_1$ is the largest local mixedness, then

$$
\gamma(s, \theta) = B(\theta)[\gamma_{\text{tms}}(s) \oplus \mathbb{I}]B(\theta)^T,
$$

where

$$
B(\theta) = \mathbb{I} \oplus \begin{pmatrix} \cos \theta \mathbb{I}_2 & \sin \theta \mathbb{I}_2 \\ -\sin \theta \mathbb{I}_2 & \cos \theta \mathbb{I}_2 \end{pmatrix}
$$

and $\gamma(s, \theta)$ then has the three local purities $\lambda_1 = \cosh s, \lambda_2 = \sinh^2 \theta + \cosh^2 s \lambda_3 = \cosh^2 s \sinh^2 \theta$, satisfying the characteristic equation $\lambda_1 + 1 = \lambda_2 + \lambda_3$ of Case (b) above. And since for any given $\lambda_2, \lambda_3 \geq 1$ there is a pair $(s, \theta) \in \mathbb{R}^+ \times [0, 2\pi]$ such that the above equations hold, we can generate all degenerate states this way. Since these states are obtained from a two-mode squeezed state by distributing it via a beam splitter among two parties, we also refer to them as distributed two-mode squeezed states.

In order to see how the different GLU classes relate to each other we now extend the set of operations from Gaussian local unitaries to Gaussian (stochastic) local operations with classical communication (GLOCC). In particular, this will allow us to investigate whether the GHZ/W states are maximally entangled also in the sense that they allow to prepare any other Gaussian state via GLOCC (in the same way as, e.g., the Bell state does for two qubits or certain families of states do in the pure multi-qubit setting $57$).
V. GAUSSIAN LOCAL OPERATIONS

LU equivalence leads to a very detailed classification of multipartite states with a continuum of inequivalent classes. A more coarse-grained picture emerges if interconvertibility of states under a larger class of transformations, stochastic local operations and classical communication (SLOCC) \([31]\) is studied. SLOCC plays an important role in entanglement theory \([32, 34]\). Two states are said to by SLOCC-equivalent if there is a non-zero probability to convert them into each other. Due to the stochastic interconvertibility of all pure bipartite states of equal Schmidt rank \([34]\) there are \(d - 1\) different kinds of bipartite (pure state) entanglement of \(d\)-dimensional systems. In contrast, in the tripartite case, even for three qubits two inequivalent classes have been identified that are not connected by SLOCC transformations \([34]\).

Also in the Gaussian setting, GLU operations can be extended by allowing for local (generalized) measurements, namely adjoining additional modes (in a pure state) and then performing (partial) Gaussian measurements. However, Gaussian SLOCC have not been investigated since the only Gaussian operators with a bounded inverse are the Gaussian unitaries \([59]\). Instead, we are interested here in the convertibility of states under Gaussian LOCC (GLOCC). In light of the complicated structure of general LOCC transformations \([60]\) the Gaussian case is remarkably simple: all Gaussian operations can be characterized via the Choi-Jamiolkowski (CJ) isomorphism by an equivalent Gaussian state \([10, 11, 61]\). When acting on a Gaussian state with known CM, all such transformations can be implemented deterministically by teleporting through that state \([11]\). While teleportation is probabilistic (yielding a random displacement), this can be computed from the measurement outcome and the involved CMs and can then be undone by local unitaries. In particular, this implies, that a finite number of communication rounds is enough to implement any GLOCC. Note that the inverse of a GLOCC is not Gaussian, hence GLOCC does not induce an equivalence relation among Gaussian states but rather gives rise to a partial ordering (see \([42]\) for the bipartite case).

Gaussian operations mapping pure states to pure states (“pure operations”) are characterized by a pure CJ-CM \(\Gamma\) and pure operations on a single mode are characterized by a pure \(1 \times 1\) CM, i.e., by one GLU-invariant parameter \(r\) (two-mode squeezing) and two sets of three local parameters (each characterizing a single-mode Gaussian unitary), which describe local unitary pre- and post-processing of the state. Following the treatment in \([42]\) for the bipartite case we can easily obtain expressions for the output CM of a three-mode state after a general three-mode GLOCC.

A. General GLOCC on three-mode systems

The most general Gaussian operation transforming a three mode Gaussian state into another, corresponds to a six mode CJ-CM \(\Gamma = [\Gamma_1, \Gamma_{12}; \Gamma_{13}^{T}, \Gamma_2]\). Here, the index 2 (1) denotes the three input (output) modes respectively. According to \([11]\) the output CM \(\gamma'\) is related to the input CM \(\gamma\) by

\[
\gamma' = \Gamma_1 - \Gamma_{12}(\Gamma_2 + \Lambda \gamma \Lambda^{-1})^{-1} \Gamma_{12}^T,
\]

where \(\Lambda = \oplus_{x=1}^{3} \sigma_x\). For ease of notation we denote the three diagonal blocks of the input-CM \(\gamma\) by \(A_x\), \(x = 1, 2, 3\) and use the convention that indices \((x, y, z)\) in a single equation refer to distinct modes. In the case of pure LOCC transformations, the CM \(\Gamma\) is block diagonal, i.e. \(\Gamma = \oplus_{x=1}^{3} \Gamma_x\) with

\[
\Gamma_x = \begin{pmatrix} \Gamma_{1x} & \Gamma_{12x} \\ \Gamma_{2x} & \Gamma_{2x} \end{pmatrix} \equiv (S_{1x} \oplus S_{2x}) \gamma(r_x)(S_{1x} \oplus S_{2x})^T.
\]

Using that \(S_{1x}\) only describes local unitary postprocessing (which is irrelevant for GLU-invariant properties) we can without loss of generality take \(S_{1x} = I\). We write the Euler decomposition \([54]\) of \(S_{2x} = O_{1x} \sigma_x Q_{2x}\) where \(O_{1x}, O_{2x}\) are in \(SO(2)\) and \(Q_x = \text{diag}(q_x, q_x^{-1})\) with \(q_x > 0\) is a single-mode squeezing transformation. Since the effect of \(O_{2x}\) can be undone by local unitary postprocessing, we set \(O_{2x} = I\) and obtain

\[
\Gamma_{1x} = \cosh r_x I,
\]

\[
\Gamma_{2x} = \cosh r_x O_{1x} Q_x^2 O_{1x}^T,
\]

\[
\Gamma_{12x} = \sinh r_x \sigma_y Q_x O_{1x}^T.
\]

To make the ensuing expressions shorter, we will from now on use the notation \(\cosh r \equiv \cosh r_x\) and \(\sinh r \equiv \sinh r_x\). Using the Schur-complement formula for the inverse of a symmetric matrix, this allows to write the reduced CMs of the output state \(\gamma'\) at mode \(x\) in compact form as

\[
(\gamma')_{xx} = \Gamma_{1x} - \Gamma_{12x}(\Gamma_{1x} + A_x - T_x)^{-1} \Gamma_{12x}^T,
\]

where we have introduced the auxiliary matrices

\[
T_x = (D_{zy} + D_{zx}) R_x^{-1} \begin{pmatrix} D_{zy} \\ D_{zx} \end{pmatrix},
\]

\[
R_x = \begin{pmatrix} \Gamma_{zy} + A_y & D_{yz} \\ D_{zy} & \Gamma_{zz} + A_z \end{pmatrix}.
\]

Note that the identity operation corresponds to the limiting case of infinitely squeezed CJ-CM \(\Gamma\) (i.e., \(r \to \infty\) and \(O_1 = X = I\)). Hence, the case of not operating on mode \(x\) corresponds to taking the limit \(r_x \to \infty\) in the above expressions. Since the expressions in terms of the 9 pure GLOCC and 3 CM parameters is rather long and intransparent, we split the general three-mode
GLOCC into a sequence of three single-mode transformations. Sometimes we will focus on a much simpler family of transformations [which we refer to as (local) TMS filtering], namely those, where \( Q_x = Q_y = 1 \forall x \) (i.e., no unitary pre-processing), leaving only the three two-mode squeezing parameters free. In the bipartite case, GLOCC of that form are know to suffice to perform all possible transformations between GLU equivalence classes.

Here, our aim is not the complete analysis of the GLOCC-transformations of three-mode pure Gaussian states but only an illustration of the usefulness of the GLU classification and standard form. In particular, we use the standard form derived in the previous section to study which pure three-mode states, can be transformed to pure two-mode Gaussian states by the CJ isomorphism [which we refer to as (local) TMS filters] namely those, where \( \gamma_{\text{sym}}(\lambda) \) that, in contrast, allows to reach all symmetric states.

### B. GLOCC-transformation of \( 1 \times 1 \times 1 \) states

As we have seen before, a three-mode pure Gaussian state is completely characterized by its local mixedness parameters, \( \lambda_i \). Therefore, we simply write \( (\lambda_1, \lambda_2, \lambda_3) \) when referring to the CM given in Eqs. (19,20). Here we derive a compact prescription of how the CM of a Gaussian state changes under single-mode GLOCC and, in particular, give expressions for the matrices determining the local mixedness parameters \( \lambda_i \).

Let us denote the \( 1 \times 2 \) CM of the three-mode Gaussian input state \( (\lambda_1, \lambda_2, \lambda_3) \) by

\[
\gamma = \begin{pmatrix} A & K \\ K^T & R \end{pmatrix},
\]

where \( A = \lambda_1 \mathbb{1}, R = \begin{pmatrix} A_j & D_{jk} \\ D_{kj} & A_k \end{pmatrix}, \) with \( A_i = \lambda_i \mathbb{1}, \) and \( K = (D_{ij} D_{ik}) \), where the block \( A \) refers to the mode \( i \) to be acted upon.

As mentioned above, Gaussian completely positive maps (CPMs) acting on a single mode and mapping pure states to pure states are in one-to-one correspondence to pure two-mode Gaussian states by the CJ isomorphism [11,12,13]: they are GLU-equivalent to a two-mode squeezed state and can therefore be completely characterized by the \( 1 \times 1 \) CM \( \Gamma \) [see Eq. (8)]

\[
\Gamma = \begin{pmatrix} \Gamma_1 & \Gamma_{12} \\ \Gamma_{12}^T & \Gamma_2 \end{pmatrix} \equiv (S_1 \oplus S_2) \gamma(r)(S_1 \oplus S_2)^T,
\]

with symplectic \( S_1, S_2 \). As discussed in the previous section, we can without loss of generality choose \( S_1 = \mathbb{1} \) and \( S_2 = O_f \mathbb{I}^{-1} X^{-1} \) with \( X = \text{diag}(x, x^{-1}) \).

If the CPM corresponding to \( \Gamma \) acts on mode \( i \) of the state with CM \( \gamma \) of Eq. (40), it is transformed to \( \gamma' \) with

\[
\begin{align*}
A' &= \Gamma_1 - \Gamma_{12}(\Gamma_2 + \sigma_z A \sigma_z)^{-1}\Gamma_{12}^T, \\
R' &= R - K^T \sigma_z(\Gamma_2 + \sigma_z A \sigma_z)^{-1} \sigma_z K, \\
K' &= \Gamma_{12}(\Gamma_2 + \sigma_z A \sigma_z)^{-1} \sigma_z K.
\end{align*}
\]

To characterize the output state only the three \( 2 \times 2 \) diagonal blocks of \( \Gamma' \) are of interest. We have:

\[
\begin{align*}
A'_l &= \text{chr} 1 - \text{sh}^2 r(\text{chr} 1 + \lambda_l X^2)^{-1}, \\
A'_j &= \lambda_j \mathbb{1} - T_j(\text{chr} 1 + \lambda_j X^2)^{-1}T_j^T, \\
A'_k &= \lambda_k \mathbb{1} - K_k(\text{chr} 1 + \lambda_k X^2)^{-1}K_k^T,
\end{align*}
\]

where \( T_l = D_{jl} O_1 X \) for \( l = j, k \). Clearly, up to GLU the final state depends only on the parameters \( r, x, \phi \), where \( O_1 = e^{i\phi} \mathbb{1} \). Note that these expressions could be obtained from Eq. (50) in the limit \( r_y, r_z \to \infty \).

We now use the GLOCC formalism to explore the entanglement properties of certain families of pure three-mode Gaussian states, in particular the symmetric states \( \gamma_{\text{sym}}(\lambda) \). For large \( \lambda \) these are highly entangled states and they have been referred to as as maximally entangled continuous-variable states. We show, however, in the next subsection that in contrast to what one might expect, it is not possible to prepare by GLOCC an arbitrary pure three-mode Gaussian state from a state \( \gamma_{\text{sym}}(\lambda) \) no matter how large \( \lambda \). In contrast, we study in the final subsection a different family, and show that it allows to prepare, in particular, all symmetric states.

### C. Symmetric initial states

We show now that it is not possible to reach an arbitrary three-mode entangled state via GLOCC from a symmetric three-mode entangled state. To this end, we first show that a state can be generated from a symmetric state \( \gamma_{\text{sym}} \) by a single-mode GLOCC if and only if it can be generated (possibly from some other symmetric initial state) via a single-mode TMS operation [i.e., \( X = O_1 = \mathbb{1} \) in Eqs. (45,47)] and that any state \( (\lambda_1, \lambda_2, \lambda_3) \) with \( \lambda \geq \lambda_2 \geq \lambda_1 \) can be generated in this way. Then we show that starting with a state \( (\lambda_1', \lambda_2', \lambda_2') \) a general measurement on the second mode allows only to reach states with \( |D_{12}| = (\lambda_2'^2 - \lambda_2'^2 - \lambda_1'^2 + 1)/2 \leq 0 \). After that, we show that performing a general GLOCC on the third mode cannot change the sign of this determinant. Consequently, a pure three-mode Gaussian state with \( |D_{12}| > 0 \) cannot be prepared by general GLOCC starting from an (arbitrary) symmetric Gaussian state.

In order to show that it is not in general the case that the sign of the determinants of the off-diagonal blocks...
cannot be changed via GLOCC, we present in the subsequent subsection a class of states with one positive and two negative determinants, from which states with three negative determinants can be obtained via GLOCC.

Let us first show that from a symmetric state with parameter \( \lambda \) and operating on mode 1 only, we obtain \((\lambda_1', \lambda_2', \lambda_3')\) with \( \lambda \geq \lambda_2' \geq \lambda_1' \). Then we show that any ratio \( \lambda_2'/\lambda_1' \geq 1 \) can be obtained by a suitable TMS-operation and suitable choice of the initial \( \lambda \).

From Eq. (45), we see that \( \lambda_1' \) does not depend on \( O_1 \) and takes a global maximum for \( x = 1 \). Since a TMS-operation yields
\[
\lambda_1' = \frac{\lambda \text{chr} + 1}{\lambda + \text{chr}},
\]
which can take all values in \([1, \lambda]\) restricting to these operations does not constrain \( \lambda_1' \).

With Eq. (46), one readily checks that \( \lambda_2' \) is minimal \( \lambda \) for \( O_1 = 1 \). Thus \( \phi \neq 0 \) only increases the ratio \( \lambda_2'/\lambda_1' \). Since as we see below all such ratios \( \geq 1 \) can be obtained by TMS operations, we can set \( \phi = 0 \). Looking now at \( \lambda_2'/\lambda_1' \) for the case \( \phi = 0 \), we easily see that it is \( \geq 1 \) \( \lambda \).

Note that this is expected since the GLOCC (a partial measurement) is performed at mode 1 and thus our lack of knowledge about the local state there is less than at the unmeasured modes.

To complete the proof we have to show that all such ratios can be achieved by TMS operations. The parameter \( \lambda_2' \) after such a GLOCC is
\[
\lambda_2' = \left[ \frac{\lambda^2 (\text{chr}^2 r + 1) + \frac{3}{4} \lambda^4 + 6 \lambda^2 - 1}{\lambda + \text{chr}} \right]^{1/2}.
\]

That \( \lambda_2' \geq \lambda_1' \) is easily seen using that \( \lambda \geq 1 \), which implies that the first term in the numerator is larger than or equal to \( \lambda^2 \text{chr}^2 + 1 \) and the second term is larger than or equal to \( 2 \lambda \text{chr} \).

To see that all such pairs \((\lambda_1', \lambda_2')\) can be achieved by suitable choice of the initial parameter \( \lambda \) and operation-parameter \( r \) we can invert Eqs. (45,46) to find \( r, \lambda \) as a function of the target values \( \lambda_1' \) and \( f \geq 0 \) which determines the ratio \( \lambda_2'/\lambda_1' \) via
\[
(\frac{\lambda_2'}{\lambda_1'})^2 = 1 + f.
\]

We find
\[
\lambda = \left[ \frac{(3 + 4f)\lambda_1^2 + 1}{6 \lambda_1'} + \sqrt{\left( \frac{(3 + 4f)\lambda_1^2 + 1}{6 \lambda_1'} \right)^2 + \frac{1}{3}} \right],
\]
\[
\text{chr} = \frac{\lambda_1' - 1}{\lambda - \lambda_1'}.
\]

One readily checks that the values of \( \lambda, r \) are in the admissible range \((\lambda \geq 1, r \geq 0)\) for all valid target values \( \lambda_1' \geq 1, f \geq 0 \), which proves the statement.

However, it is not possible to obtain all pure three-mode Gaussian states from a symmetric initial state, not even by the most general GLOCC. This follows from the fact that the symmetric states all have the property that the three off-diagonal matrices \( {D}_{ij} \) all have non-positive determinants \(|{D}_{ij}| \leq 0 \). As we show in the following Lemma, this is a property that cannot be changed by GLOCC. However, we have already encountered states [such as the distributed two-mode squeezed states \( \gamma(s, \theta) \)], cf. Eq. (29), which have one non-positive determinant. These, therefore, cannot be reached by GLOCC from the symmetric states.

**Lemma 8.** It is impossible with GLOCC to transform a pure three-mode Gaussian state with three non-positive determinants \(|{D}_{ij}| \leq 0 \) into a state with at least one (strictly) positive determinant.

**Proof.** We consider an arbitrary initial state with \(|{D}_{ij}| \leq 0 \) for all \((ij)\), i.e., cf. Eq. (29), \( \lambda_1^2 - \lambda_2^2 - \lambda_3^2 + 1 \leq 0 \forall (ij) \) and apply an arbitrary measurement on the \( k \)th mode. Without loss of generality, we choose \((ijk) = (123)\). Similarly to before, we obtain for the matrices in the diagonal of \( \gamma \)
\[
A_1' = \lambda_1 \mathbb{1} - T_1(\text{chr} \mathbb{1} + \lambda_3 X^2)^{-1} T_1^T
\]
\[
A_2' = \lambda_2 \mathbb{1} - T_2(\text{chr} \mathbb{1} + \lambda_3 X^2)^{-1} T_2^T,
\]
\[
A_3' = \text{chr} \mathbb{1} - \text{shr}^2(\text{chr} \mathbb{1} + \lambda_3 X^2)^{-1}
\]
with \( T_1 = D_{13} O_1 X \) and \( T_2 = D_{23} O_1 X \). Again, we consider the term \( C_3 = \lambda_1^2 - \lambda_2^2 - \lambda_3^2 + 1 \), which now yields a more lengthy expression
\[
C_3 = \frac{c_x A + B s_x \cos(2\phi) \text{chr} + C}{4 \lambda_3 \lambda_5^2 + 4 \lambda_7 r + c_z \lambda_3 \text{chr}}
\]
\[
A = \lambda_1^2 - 2 (\lambda_2^2 + \lambda_3^2 + 1) \lambda_3^2 + \lambda_3^4 + (\lambda_2^2 - 1)^2
\]
\[
- 2 (\lambda_2^2 - 3) \lambda_3^2,
\]
\[
B = [(\lambda_1 - \lambda_2 - \lambda_3 - 1) (\lambda_1 - \lambda_2 - \lambda_3 + 1) \times (\lambda_1 + \lambda_2 - \lambda_3 - 1) (\lambda_1 + \lambda_2 - \lambda_3 + 1) \times (\lambda_1 - \lambda_2 + \lambda_3 - 1) (\lambda_1 - \lambda_2 + \lambda_3 + 1) \times (\lambda_1 + \lambda_2 + \lambda_3 - 1) (\lambda_1 + \lambda_2 + \lambda_3 + 1)]^2,
\]
\[
C = 4 \lambda_3 (\text{chr}^2 + 1) (\lambda_5^2 - \lambda_7^2 + \lambda_3^2),
\]
\[
c_x = x^2 + x^{-2},
\]
\[
s_x = x^2 - x^{-2}.
\]

Since the denominator, \( \text{chr} \), and \( B \) are positive \( \lambda \), to maximize this expression, \( \cos 2\phi \) should have maximal modulus and the same sign as \( s_x \); i.e., without loss of generality we can take \( x > 1 \) and \( \phi = 0 \). Note also, that \( C < 0 \) by assumption since the state \( (\lambda_1, \lambda_2, \lambda_3) \) has \(|D_{12}| < 0 \). To show finally that the whole numerator is always negative, we consider the expression for the other two determinants \(|{D}_{ij}| \), namely \( C_2 = \lambda_5^2 - \lambda_7^2 - \lambda_3^2 + 1 \)
and $C_1 = \lambda_1^2 - \lambda_2^2 - \lambda_3^2 + 1$, which are

$$C_1 = \frac{(\lambda_3^2 - \lambda_2^2 - \lambda_3^2 + 1) x^2 \sinh^2 x}{(\lambda_3 + x^2 \cosh)(\cosh + \lambda_3 x^2)}, \quad (60)$$

$$C_2 = \frac{(\lambda_3^2 - \lambda_1^2 - \lambda_3^2 + 1) x^2 \sinh^2 x}{(\lambda_3 + x^2 \cosh)(\cosh + \lambda_3 x^2)}, \quad (61)$$

i.e., up to a positive factor they are given by the input determinants $|D_{23}|$ and $|D_{13}|$, which thus do not change sign and remain non-positive. Clearly $C_1 + C_2 + C_3 = 3 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 \leq 0$. This relation must hold for all choices of $x$ and $r$. Now consider the limit $x \to \infty$, for which both $C_1, C_2 \to 0$, and

$$C_3 \to \frac{A + B}{4\lambda_3^2}, \quad (62)$$

which must therefore be $\leq 0$. Hence $A + B \leq 0$, therefore $(Ae_x + B \lambda_x)\cosh + C \leq (A + B)\cosh + C \leq 0$ which shows that all three determinants remain non-positive. This proves that a single-mode GLOCC does not allow to transform a state with only non-positive off-diagonal determinants $|D_{ij}| < 0$ into a final state with at least one positive determinant. Since any pure GLOCC operation is represented by a product CJ-state and can therefore be decomposed in a sequence of three single-mode operations, we have shown even the most general pure GLOCC cannot achieve this.

Thus, in particular, we have shown that from a symmetric state $\gamma_{\text{sym}}(\lambda)$, which has $|D_{ij}| = - (\lambda^2 - 1)/2 \leq 0$, it is impossible to obtain via arbitrary GLOCC any state with $|D_{ij}| > 0$ for some $(ij)$.

**D. Initial states with positive determinant**

To show that these signs are not a GLOCC-invariant, and that, in fact, a positive determinant $|D_{ij}| > 0$ can always be made negative by GLOCC we prove the following

**Lemma 9.** Given a pure three-mode Gaussian state with one positive determinant $|D_{ij}| > 0$, there exists a GLOCC to transform it into a state with three negative determinants.

**Proof.** Recall that for pure three-mode states there is at most one positive determinant, see, e.g., Eq. (21a). Assume, without loss of generality, that $|D_{12}| > 0$. From Eqs. (60)(61) it is clear that to change the sign of $|D_{12}|$ we must perform a GLOCC at mode 3. The determinant after a general one-mode GLOCC is given by Eq. (54). Now consider the case $\phi = 0$ and the limits $x \to \infty$ and $x \to 0$. As before, the limit $x \to \infty$ proves that $A + B \leq 0$, cf. Eq. (62). For $x \to 0$, we obtain

$$C_3(x \to 0) = \frac{A - B}{4\lambda_3^2}. \quad (63)$$

Since $B > 0$ it follows that $C_3(x \to 0) < 0$, i.e., for sufficiently small $x$ all three determinants $C_i$ are negative.

Let us, as an example consider the distributed two-mode squeezed states with CM $\gamma(s, \theta)$, discussed in Section XV. They are obtained by passing part of a two-mode squeezed state $\gamma(s)$ through a beam splitter with transmissivity $t = \cos^2 \theta$, see, e.g., Eq. (29). These states have one off-diagonal block proportional to the identity, say $D_{23} = - \sin \theta \cos \theta (\cosh - 1)\mathbb{1}$, i.e., with positive determinant. The other two off-diagonal blocks are proportional to $\sigma_z$, i.e., $D_{12} = \cos \theta \sinh \sigma_z$ and $D_{13} = - \sin \theta \sinh \sigma_z$. When performing a GLOCC characterized by two-mode squeezing parameter $r$ and local squeezing $x$, i.e., $\Gamma = [\mathbb{1} \otimes \text{diag}(x, x^{-1})] \gamma(r) [\mathbb{1} \otimes \text{diag}(x, x^{-1})]$ on mode 1 (the one with the largest local mixedness), we obtain from Eq. (33):

$$D_{23}' = D_{23} - D_{12} \left[ \text{ch} \left( \frac{x^2}{2} 0 0 \right) + \text{ch} \right]^{-1} D_{13}, \quad (64)$$

therefore

$$d_{23}^{1'} = - \frac{\sin(2\theta)\sinh^2(s/2)(x^2\cosh - 1)}{\cosh + x^2\cosh}, \quad (65)$$

i.e., for $x^2 < \cosh$ or $x^2 > \cosh$ one of the two coefficients is negative (while the other is positive), yielding $|D_{23}'| < 0$ for all $x \notin \sqrt{1/\cosh, \cosh}$. Since the signs of $|D_{12}|$ and $|D_{13}|$ do not change, we have transformed the initial state with sign($|D_{12}|$) = 1, sign($|D_{13}|$) = 1, and sign($|D_{23}'|$) = 1 to a state with all signs negative.

In fact, we can even obtain all symmetric states starting from a distributed two-mode squeezed initial state. Let us consider the simple one-parameter family of degenerate states with CM $\gamma = \gamma(s, \theta = \pi/4)$. Clearly, in this case the two smaller of the three parameters are identical, i.e., our initial state is $\lambda_1 = \cosh$ and $\lambda_2 = \lambda_3 = (\cosh + 1)/2$. Then it suffices to perform a suitable measurement at mode 1 to obtain $\lambda_1' = \lambda_2' = \lambda_3'$. Moreover, by choosing $s$ large enough, it is possible to obtain all symmetric states this way. To see this, we use again Eqs. (43)(44) for an operation characterized by $(r, x, \phi = 0)$. Then it is straightforward to see that by taking

$$x^2 + x^{-2} = \frac{\cosh^2 s}{2} \left( 3 \cosh^2 \cosh + \cosh^2 r - \cosh - 3 \right) \frac{2\cosh}{2\cosh}, \quad (66)$$

we can prepare the symmetric state $\gamma_{\text{sym}}(\lambda')$ with

$$\lambda' = \frac{4\cosh^2 s} {6(\cosh^2 r - 1) \cosh - 2\cosh^2 r + \cosh(2s) + 1}. \quad (67)$$

Note that the right-hand side of Eq. (63) is $\geq 2$ for all choices of $s, r$, thus there is always $x \geq 1$ corresponding to the desired value. Considering the limits $s \to \infty$ and $s \to 0$, we see that $\lambda' \to \cosh^2 r$ and $\lambda' \to 1$, respectively. Consequently, for any target state $\gamma_{\text{sym}}(\lambda')$ there
exist \(s, r \geq 0\) and \(x > 0\) such that the symmetric state \(\lambda\) can be prepared from the degenerate state \(\gamma(s, \pi/4)\) by a single-mode GLOCC with parameters \((r, x, 0)\). Thus, the one-parameter family \(\{\gamma(s, \pi/4) : s \geq 0\}\) is “more strongly entangled” than \(\{\gamma_{\text{sym}}(\lambda) : \lambda \geq 1\}\) in the sense that the latter can be obtained from the former by deterministic GLOCC but the reverse is not possible. We leave as an open question whether all pure three-mode Gaussian states can be obtained from \(\{\gamma(s, \pi/4) : s \geq 0\}\) by GLOCC. Since a TMS operation acting on the first mode allows to arbitrarily reduce the parameter \(s \geq 0\) (without changing \(\theta\)), a positive answer would imply that there is a single (unnormalizable) pure three-mode state from which all others can be obtained by GLOCC.

Let us finally remark on the entanglement properties associated with the appearance of a positive determinant (say \(|D_{12}| = (\lambda_3^2 + 1 - \lambda_2^2 - \lambda_1^2)/2 > 0\)): First, it means that \(\lambda_1, \lambda_2, \lambda_3\) are too small (relative to \(\lambda_3\), i.e., there is too little entanglement available between the modes (12): most (or in the case of the distributed two-mode squeezed states: all) of the mixedness at these modes arises from the entanglement with mode 3. Since a two-mode Gaussian state is necessarily separable if the off-diagonal block of its CM has non-negative determinant, we see that in that case there is no residual entanglement between modes (23). As we have seen this strong concentration of entanglement into one mode cannot be generated by GLOCC. On the other hand, we have seen that a GLOCC on mode 1 allows to induce residual entanglement between the modes (23) (e.g., by generating a symmetric state) even if their reduced state was separable initially. For the special case \(\theta = \pi/4\) and for a suitably chosen Gaussian operation (essentially, for sufficiently large \(x\) and \(r\)), one can readily check, using the partial-transpose separability criterion (e.g., in the simple form for Gaussian 1 \(\times 1\) states given in [12]), that all three reduced CMs are entangled.

VI. CONCLUSIONS

We presented an easily computable necessary and sufficient condition for Gaussian LU-equivalence for an arbitrary number of modes and derived a standard from for pure three-mode Gaussian states. This showed, in particular, that the entanglement properties of an arbitrary pure three-mode Gaussian state are completely characterized by three bipartite entanglement measures, namely the local purities. This also shows that for pure three-mode Gaussian states LU-equivalence implies GLU-equivalence.

In order to gain more insight into the relation among the GLU-classes, we investigated the more general set of GLOCC operations. We provided simple expressions for GLOCC-transformations between different GLU-classes. For the pure three-mode states we showed that they can be divided in two classes (according to whether the sign of the largest determinant \(|D_{ij}|\) is positive or not) such that no GLOCC can transform a state from the second class to the first. In particular, this shows that the set of symmetric states (GHZ/W states) does not suffice to generate an arbitrary state via GLOCC. Among the states unreachable from the symmetric states we identified a family which, in contrast, allows to prepare all symmetric states.

There are many questions concerning the GLOCC-interconvertibility of pure multipartite Gaussian states that remain to be addressed: Is there a “maximally entangled” family in the sense that all other states can be obtained from it by GLOCC? Is there a majorization relation governing which states can be GLOCC-transformed into another? Are there mutually inaccessible subsets of GLU-classes similar to the W and GHZ classes for three qubits? Can the observed restrictions on Gaussian state transformations be lifted if several copies of the states are considered? Are there examples in which general (i.e., non-Gaussian) local unitaries allow the transformation between two pure Gaussian states that are not in the same GLU-class or does LU-equivalence of Gaussian states always imply their GLU-equivalence? Answers to these questions might lead to a better understanding of the structure and qualitative features of pure Gaussian entanglement and be of practical use regarding which states are the most versatile in terms of state generation.

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Appendix A: Pure 1 \(\times 1\) \(\times 1\) states: Standard Form

We show here that the condition \(\gamma J \gamma = J\) implies that any three-mode CM \(\gamma\) is \(xp\)-blockdiagonal (see Theorem 3). Let \(\gamma\) as given in Eq. (15) denote the standard form of the CM. In particular, \(K = D_{12}\) is diagonal. However, instead of choosing \(O_3\) such that \(L = OD_{13}\), for some orthogonal matrix \(O\) and some diagonal matrix \(D_{13}\), we chose here without loss of generality \(O_3\) such that \(L\) has upper-triangular form (47). The necessary condition for \(\gamma\) to correspond to a pure state, \(\gamma J \gamma = J\), is equivalent
to the following set of equations,
\begin{align}
1 &= \lambda_2^2 + |D_{12}| + |L|, \\
1 &= \lambda_3^2 + |D_{12}| + |M|, \\
1 &= \lambda_4^2 + |L| + |M|, \\
0 &= \lambda_3JD_{12} + \lambda_2D_{12}J + LJM^T, \\
0 &= \lambda_3JL + \lambda_3LJ + D_{12}JM^T, \\
0 &= \lambda_2JM + \lambda_3MJ + D_{12}^T JL, \\
\end{align}
(A1a) \hspace{2cm} (A1b) \hspace{2cm} (A1c) \hspace{2cm} (A1d) \hspace{2cm} (A1e) \hspace{2cm} (A1f)

Note that \( \lambda_i \geq 1 \) (in particular \( \lambda_i \neq 0 \)) for all \( i \), must hold for any CM (see e.g. condition Eq. (7)). Let us use the notation \( \pi_{ij} = X_{ij} \) for \( X \in \{K, L, M\} \). Writing Eqs. (A1d, A1f) elementwise we obtain
\begin{align}
0 &= l_2m_{21}, \\
0 &= l_{12}m_{12} - l_1m_{21}, \\
0 &= \lambda_1k_2 + \lambda_2k_1 - l_2m_{21} + l_1m_{22}, \\
0 &= \lambda_1k_1 + \lambda_2k_2 - l_2m_{11}, \\
0 &= \lambda_1l_2 + \lambda_2l_2, \\
0 &= \lambda_3l_2 - k_1m_{21}, \\
0 &= \lambda_2l_2 + \lambda_3l_1 + k_1m_{22}, \\
0 &= \lambda_1l_1 + \lambda_3l_2 + k_2m_{11}, \\
0 &= \lambda_2m_{21} - \lambda_3m_{12}, \\
0 &= k_1l_2 - \lambda_2m_{12} + \lambda_3m_{11}, \\
0 &= k_2l_1 - \lambda_2m_{21} + \lambda_3m_{22}, \\
0 &= k_2l_2 - \lambda_2m_{12} - \lambda_3m_{21}. \\
\end{align}
(A2a) \hspace{2cm} (A2b) \hspace{2cm} (A2c)

We show that the above equations imply that \( l_{12} = m_{12} = m_{21} = 0 \), i.e., also \( L \) and \( M \) are diagonal. We first discuss the case \( l_2 \neq 0 \). Then the first of Eqs. (A2a) implies \( m_{21} = 0 \). Consequently, the second equation of Eqs. (A2a) implies \( l_{12} = 0 \) and the second of Eqs. (A2b) yields \( m_{21} = 0 \). If, instead, \( l_2 = 0 \), we have that \( |L| = 0 \) and we can wlog \( L = 0 \) because \( l_{12} = 0 \). Now consider \( L \neq 0 \), i.e., \( l_1 \neq 0 \). Then the second of Eqs. (A2a) yields \( m_{12} = 0 \), the first of Eqs. (A2c) implies \( m_{21} = 0 \). If, finally, \( L = 0 \), then \( k_1 + k_2 = 0 \) and \( m_1 + m_2 = 0 \) [Eqs. (A2a, A2c)], respectively, and then by the last two of Eqs. (A2a) \( K = 0 \) or \( M = 0 \), i.e., mode 1 or mode 3 factorizes. In either case, both \( L \) and \( M \) are diagonal and therefore \( \gamma \) is \( xp \)-blockdiagonal.

**Appendix B: Proof of Lemma 4**

Here we present the detail of the proof of Lemma 4. In particular, we derive the conditions under which \( \gamma \) as given in Eq. (19) obeys the necessary condition \( \gamma J \gamma = J \). In order to increase the readability of the appendix, we restate the equivalent conditions given in Eq. (21a):
\begin{align}
\lambda_1^2 + |D_{12}| + |D_{13}| = 1, \\
\lambda_2^2 + |D_{12}| + |D_{23}| = 1, \\
\lambda_3^2 + |D_{13}| + |D_{23}| = 1, \\
\lambda_1D_{12} + \lambda_2D_{13} + D_{12} \circ D_{23} = 0, \\
\lambda_1D_{13} + \lambda_3D_{12} + D_{13} \circ D_{23} = 0, \\
\lambda_2D_{23} + \lambda_3D_{23} + D_{23} \circ D_{13} = 0. \\
\end{align}
(B1a) \hspace{2cm} (B1b) \hspace{2cm} (B1c) \hspace{2cm} (B1d) \hspace{2cm} (B1e) \hspace{2cm} (B1f)

As before, \( \circ \) denotes the componentwise multiplication (Hadamard product). Here, we use the index-free notation \( D_{12} = \text{diag}(a, b) \), \( D_{13} = \text{diag}(c, d) \), and \( D_{23} = \text{diag}(e, f) \), and that \( DJ = JD \) (i.e., \( D = -JDJ \)) for any diagonal matrix \( D \) and therefore \( DJD = |D|J \). Note that if \( D = \text{diag}(x, y) \), then \( JD = \text{diag}(y, x) \). In order to solve those equations we distinguish between the following two cases

i) at least one of the diagonal matrices, \( D_{ij} \) is not invertible and

ii) none of the determinants vanishes.

Let us first consider the case (i). Since we do not impose any order on the \( \lambda_i \) we assume without loss of generality that \( e = 0 \). It is then straightforward to verify that the solution to Eq. (B1) is given by
\begin{align}
\lambda_1 &= \sqrt{-1 + \lambda_2^2 + \lambda_3^2}, \\
an &= (-1)^{k_1} \sqrt{\lambda_2(1 + \lambda_2^2)/\sqrt{\lambda_1}}, \\
b &= -(-1)^{k_1} \sqrt{-1/\lambda_2 + \lambda_2/\lambda_1}, \\
c &= (-1)^{k_2} \sqrt{\lambda_3(1 + \lambda_3^2)/\lambda_1}, \\
d &= -(-1)^{k_2} \sqrt{-1/\lambda_3 + \lambda_3/\lambda_1}, \\
e &= 0, \\
f &= (-1)^{k_1+k_2} \sqrt{(\lambda_2^2 - 1)(\lambda_3^2 - 1)/\sqrt{\lambda_2\lambda_3}}, \\
\end{align}
(B2a) \hspace{2cm} (B2b) \hspace{2cm} (B2c) \hspace{2cm} (B2d) \hspace{2cm} (B2e) \hspace{2cm} (B2f)

where \( k_1, k_2 \in \{0, 1\} \). Now, it is easy to see that the four solutions for the different values of \( k_1, k_2 \) are GLU-equivalent by choosing \( O = (-1)^{k_1} I \oplus I \oplus (-1)^{k_2+k_2} I \). Thus, we chose without loss of generality \( k_1 = k_2 = 0 \). Given the expressions of the entries of the diagonal matrices \( D_{ij} \) [see Eq. (20)] it is straightforward to check that \( a^2 = (d_{i1}^2)^2, b^2 = (d_{i2}^2)^2, c^2 = (d_{i3}^2)^2, d^2 = (d_{i4}^2)^2, e^2 = 0 = (d_{i5}^2)^2, \) and \( f^2 = (d_{i6}^2)^2 \). Moreover, it is easy to see that \( |D_{ij}| = d_{i1}^2 d_{i2}^2, \) for all three matrices. Thus, the expressions coincide up to a (independent) global phase for the matrices \( D_1, D_2 \) (the sign of \( D_3 \) is thereby fixed). Let us denote these signs by \( k_1, k_2, k_3 \) respectively. Clearly, \( d_{i2}^2 \geq 0 \), which implies, since \( b \leq 0 \), that \( -(-1)^{k_1} = -1 \). Similarly, it is easy to see that \( -(1)^{k_2} = -1 \) and \( -(1)^{k_3} = 1 \) (which has to coincide with \( -(1)^{k_1+k_2} \)). Thus, the orthogonal matrix \( -\sigma_x \oplus -\sigma_x \oplus \sigma_x \) (corresponding to a GLU) sorts the entries in the diagonal matrices and applies the right signs to map \( \gamma \) into...
Note that two of these determinants are non–positive. 

Let us now consider the more involved case (ii). First note that due to Eq. (B1) the following relations hold

\[ ab = |D_{12}| = 1/2(1 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2), \]
\[ cd = |D_{13}| = 1/2(1 - \lambda_1^2 + \lambda_2^2 - \lambda_3^2), \]
\[ ef = |D_{23}| = 1/2(1 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2). \]

Note that two of these determinants are non–positive. More precisely, if \( \lambda_i \geq \lambda_k, \lambda_l \) then \( |D_{ik}| \leq 0 \) and \( |D_d| \leq 0 \). Let us now define \( x_1 = \frac{1}{4\lambda_2\lambda_3|D_{13}|^2} \), \( x_2 = \lambda_2^6 + (-1 + \lambda_2^2)^2 \lambda_3^2 - 2(1 + \lambda_1^2)\lambda_3^4 + \lambda_3^6 - \lambda_1^2(2 + 2\lambda_1^2 + \lambda_3^2) + \lambda_3^2 \left( -1 + \lambda_1^2 \right)^2 + 4(1 + \lambda_1^2)\lambda_3^2 - \lambda_3^4 \).

The solution to Eq. (B5) is then given by

\[ f = 2 \frac{y_1x}{y_2 + y_3x^2}, \]  
\[ e = \frac{D_{23}}{f}, \]
\[ d = \frac{\sqrt{a}}{\sqrt{e}x + \lambda_2x^2}, \]  
\[ c = \frac{|D_{13}|}{d}, \]
\[ b = xd, \]  
\[ a = |D_{12}|. \]

Here we have used \( x = (-1)^k \left( \frac{1}{\sqrt{2}} \sqrt{x_1(x_2 + (-1)^l\sqrt{w})} \right), \)

\[ w = (1 + \lambda_1 - \lambda_2 - \lambda_3)(1 + \lambda_1 - \lambda_2 - \lambda_3) \times \]
\[-(1 + \lambda_1 + \lambda_2 - \lambda_3)(1 + \lambda_1 + \lambda_2 - \lambda_3) \times \]
\[-(1 + \lambda_1 - \lambda_2 + \lambda_3)(1 + \lambda_1 - \lambda_2 + \lambda_3) \times \]
\[-(1 + \lambda_1 + \lambda_2 + \lambda_3)(1 + \lambda_1 + \lambda_2 + \lambda_3) \times \]
\[ (\lambda_2^2 - \lambda_3^2)^2, \]
\[ y_1 = (\lambda_2^6 - \lambda_3^6)|D_{23}|, \]  
\[ y_2 = -2\lambda_3|D_{12}|, \]  
\[ y_3 = 2\lambda_2|D_{13}|, \]

and \( k, l \in \{0, 1\}. \)

Note that the denominator of \( x_1 \) is non–vanishing since \( D_{13} \) is invertible. Note further that the denominator of \( f \) vanishes (for \( \lambda_i \geq 1 \)) iff either (a) \( \lambda_1 = \sqrt{1 - \lambda_2^2 + \lambda_3^2} \) or (b) \( \lambda_j = \sqrt{1 - \lambda_1^2 + \lambda_3^2} \), for \( j \neq k \), or (c) \( \lambda_i = \sqrt{-1 + \lambda_2^2 + \lambda_3^2} \) or (d) \( \lambda_3 = \lambda_3 \). The cases (a–c) cannot occur here, since in those cases one of the determinants \( D_{ij} \) vanishes. Let us now first consider the case \( \lambda_2 \neq \lambda_3 \) (for case (d), \( \lambda_2 = \lambda_3 \) a similar argument applies).

Note that \( \gamma \) is real only if \( w \geq 0 \). Let \( \lambda_i \geq \lambda_k, \lambda_j \), for mutually different values of \( i, j, k \in \{1, 2, 3\} \) denote the largest value, then it can be easily seen that \( w \geq 0 \) iff either \( \lambda_i \leq \lambda_j + \lambda_k - 1 \) or \( \lambda_i \geq \lambda_j + \lambda_k + 1 \). As shown in the main text, the second choice is excluded due to the positivity of \( \gamma \).

Note further that all values of \( k, l \) lead to a solution. Those equalities have been derived as follows. First we use the conditions \( \lambda_1^2 + |D_{12}| + |D_{13}| = 1, \lambda_2^2 + |D_{12}| + |D_{23}| = 1, \lambda_3^2 + |D_{13}| + |D_{23}| = 1 \) to compute \( a, c, e \) as functions of the other parameters. As can be easily seen, the conditions given in Eq. (B1) imply that \( bc(f \lambda_2 + e\lambda_3) - ad(e\lambda_2 + f\lambda_3) = 0 \), which implies that \( b = xd \), where \( x \) is a function which depends only on \( f, x, \) and \( \lambda_1 \). Using then that \( -de - a\lambda_1 - b\lambda_2 = 0 \), we compute \( d \) as a function of \( f, x, \) and \( \lambda_1 \). Next we compute \( c \) as a function of \( f, x, \) and \( \lambda_1 \) by using that \( -be - c\lambda_1 - d\lambda_3 = 0 \). Thus, we have all variables as functions of \( f, x, \) and \( \lambda_1 \). Using then the condition \( (\gamma J_\gamma - J)_2 = 0 \) we derive \( f = y_1x/(y_2 + y_3x^2) \). The equation \( (\gamma J_\gamma - J)_2 = 0 \) allows us then to compute \( x \) as given above. Note that we obtain two solutions for \( d \), namely \( \pm d \) for \( d \) given in Eq. (B5c).

However, changing the sign of \( d \) amounts to changing the signs of \( c, a, b \) (cf. Eqs. (B5d, e)).

As shown in the following section appendix, the necessary condition that \( \gamma \geq 0 \) is equivalent to the condition given in Eq. (B2).

Note that this implies that given the three local purities \( \lambda_i \) (or equivalently the bipartite entanglement shared in the three splittings \( ijk \)), the state is uniquely determined. The reason for that is that \( \lambda_1 = \sqrt{-1 + \lambda_3^2 + \lambda_3^2} \) iff \( e = 0 \) [also in case (ii)] and therefore, knowing the parameters \( \lambda_i \) implies that we also know which of the two cases the state belongs to. Thus, an arbitrary state is uniquely determined (up to GLUs) by the bipartite entanglement.

**Appendix C: Positivity of \( \gamma(\lambda_1, \lambda_2, \lambda_3) \)**

To see that the conditions

\[ \lambda_i + \lambda_j \geq \lambda_k + 1 \forall (ijk) \]

(cf. Eq. (22)) imply positivity of the CM \( \gamma = \gamma(\lambda_1, \lambda_2, \lambda_3) \) we proceed as follows: \( \gamma \) is by construction \( x_\rho \)-blockdiagonal and since it has been constructed to satisfy the purity condition \( J_\gamma = J \), it follows that
\[ \gamma_p = \gamma_{-1}, \] hence positivity of \( \gamma \) implies positivity of \( \gamma_p \). Using the Schur complement \( [1] \), positivity of \( \gamma_p \) is, as \( \lambda_3 > 0 \), equivalent to positivity of the \( 2 \times 2 \) matrix \( Y \)

\[ Y = \begin{pmatrix} \lambda_1 & d_{12}^T \\ d_{12} & \lambda_2 \end{pmatrix} - \left( \begin{pmatrix} d_{13}^T \\ d_{23} \end{pmatrix} \right) \frac{1}{\lambda_3} \left( \begin{pmatrix} d_{13}^T \\ d_{23}^T \end{pmatrix} \right), \quad (C2) \]

which is equivalent to the two conditions

\[ \text{tr}Y \geq 0, \quad \det Y \geq 0. \quad (C3) \quad \text{(C4)} \]

The trace is found to be

\[ \frac{\lambda_1 + \lambda_2}{8\lambda_1\lambda_2\lambda_3^2} (K_1 - 1 - \sqrt{w_1}) , \]

where we have introduced

\[ K_1 = - \sum_i \lambda_i^4 + 2 \sum_i (\lambda_j^2 \lambda_k^2 + \lambda_i^2), \quad (C5) \]

\[ w_1 = \sum_i \left(\frac{\lambda_i^2 (\lambda_j^2 - \lambda_k^2)}{\lambda_i^2 + 2\sum_i (\lambda_j^2 \lambda_k^2 + \lambda_i^2)} \right), \quad (C6) \]

It follows directly from Eq. \( (C1) \) that \( w_1 \geq 0 \). It is tedious, but straightforward to show that

\[ \det Y = \frac{\text{tr}Y}{\lambda_1 + \lambda_2}. \quad (C7) \]

Thus we see that both conditions Eqs. \( (C3,C4) \) hold and therefore \( \gamma \geq 0 \) if

\[ K_1 - 1 \geq \sqrt{w_1}. \quad (C8) \]

To see that \( K_1 - 1 \geq 0 \) we write it as a sum of positive terms (using that the conditions given in Eq. \( (22) \) are satisfied):

\[ K_1 - 1 = \frac{1}{4} \left( (\lambda_3 - 1)^2 - (\lambda_2 - \lambda_1)^2 \right) \left[ (\lambda_1 + \lambda_2)^2 - (\lambda_3 + 1)^2 \right] + \]
\[ + \left[ \frac{\lambda_1^2 - (\lambda_1 - \lambda_2 - 1)^2}{(\lambda_1 + \lambda_2)^2} \right] \left[ (\lambda_1 + \lambda_2 - 1)^2 - \lambda_3^2 \right] + \]
\[ + \left[ \frac{\lambda_2^2 - (\lambda_1 - \lambda_2 + 1)^2}{(\lambda_1 + \lambda_2)^2} \right] \left[ (\lambda_1 + \lambda_2 + 1)^2 - \lambda_3^2 \right] + \]
\[ + \sum_i \lambda_i^2 \left( -\lambda_i + \lambda_j + \lambda_k \right) + \lambda_i(2\lambda_i - 1) + 2\Pi_i \lambda_i , \]

where \( (jk) \) in \( \sum_i (\lambda_j + \lambda_k) \) refer in each term to the two indices distinct from \( i \). Now the the remaining condition \( K_1 - 1 \geq \sqrt{w_1} \) can be checked for the squares of both sides and we find it trivially satisfied:

\[ (K_1 - 1)^2 - w_1 = 6 \lambda_1^2 \lambda_2^2 \lambda_3^2 \geq 0. \]

Therefore both \( \det D \geq 0 \) and \( \text{tr}D \geq 0 \) and consequently \( \gamma(\lambda_1, \lambda_2, \lambda_3) \geq 0 \) whenever the \( \lambda \)'s satisfy Eq. \( (22) \).
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[39] However, one might note that the Gaussian operator $O_{\gamma}^{T}$ diagonalizes $\gamma_{12}^{T} \gamma_{12}$, and $O_{2}$ diagonalizes $\gamma_{12}^{T} \gamma_{12}$.

[40] Note that $D_{ij} \propto \lambda$ implies that the reduced state of modes $i$ and $j$ has a positive semidefinite partial transpose and is therefore separable.

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[42] However, one might note that the Gaussian operator $G_{-c} = e^{-c(x^{2} + p^{2})/2}$, $c > 0$ has an “inverse” $G_{-c} = e^{-c(x^{2} + p^{2})/2}$ which is unbounded, but well-defined on the image of $G_{-c}$ (i.e., on sufficiently fast decaying states). This might be used for the Gaussian SLOCC equivalence of, e.g., two-mode squeezed states with different squeezing parameters $r < r'$ (which can be “stochastically” mapped to each other by are related by $G_{x,c}$ with $c' = \sqrt{\text{tanh} \frac{r'}{r}}$). However, such unbounded filtering operations have no clear physical implementation and we do not pursue this further here.

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[47] $\lambda_{x}$ is periodic with the angle $\phi$ with period $\pi$ and monotonically increases in the interval $[0, \pi]$. We have (with $c_{x} = x^{2} + x^{-2}$, $s_{x} = x^{2} - x^{-2}$):

$$\lambda_{x}^{2} \lambda_{x}^{2} = \frac{\lambda^{2}(1 + c_{x}^{2}) + \frac{3}{2} \frac{\lambda^{2}}{s_{x}} \frac{\lambda^{2}}{s_{x}} \sqrt{\lambda^{4} - 10 \lambda^{2} + 1} + 1}{\lambda^{2} c_{x}^{2} + \lambda_{x}^{2} c_{x}}.$$ 

The difference of numerator and denominator is $\lambda^{2} - 1 + (\lambda^{2} - 1)(\lambda^{2} - 1)(\lambda^{2} - 1)(\lambda^{2} - 1)$, which is positive since $c_{x} \geq |s_{x}|$ and $(\lambda^{2} + 1) > 9 \lambda^{4} - 10 \lambda^{2} + 1$.

[48] Note that $B$ is equivalent to the square root of $w_{1}$ given in Appendix C, which is necessarily positive.

[49] Note that this is always possible for any $L$; as we will see in the case considered, both conventions coincide.

[50] By choosing an appropriate orthogonal transformation at mode 3.

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