A Distributed Implementation of Steady-State Kalman Filter

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Abstract—This article studies the distributed state estimation in sensor network, where \( m \) sensors are deployed to infer the \( n \)-dimensional state of a linear time-invariant Gaussian system. By a lossless decomposition of the optimal steady-state Kalman filter, we show that the problem of distributed estimation can be reformulated as that of the synchronization of homogeneous linear systems. Based on such decomposition, a distributed estimator is proposed, where each sensor node runs a local filter using only its own measurement, alongside with a consensus algorithm to fuse the local estimate of every node. We prove that the average of local estimates from all sensors coincides with the optimal Kalman estimate, and under certain condition on the graph Laplacian matrix and the system matrix, the covariance of local estimation error is bounded and the asymptotic error covariance is derived. As a result, the distributed estimator is stable for each single node. We further show that the proposed algorithm has a low message complexity of \( \min\{m, n\} \). Numerical examples are provided in the end to illustrate the efficiency of the proposed algorithm.

Index Terms—Consensus algorithm, distributed estimation, Kalman filter, linear system synchronization.

I. INTRODUCTION

The past decades have witnessed remarkable research interests in multisensor networked systems. As one of its important focuses, distributed estimation has been widely studied in various applications, including robot formation control, environment monitoring, and spacecraft navigation (see [1]–[5]). Compared with the centralized architecture, it provides better robustness, flexibility, and reliability.

One fundamental problem in distributed estimation is to estimate the state of a linear time-invariant (LTI) Gaussian system using multiple sensors, where the well-known Kalman filter provides the optimal solution in a centralized manner [6]. Thus, many research efforts have been devoted to the distributed implementation of the Kalman filter. For example, in an early work [7], the authors suggested a fusion algorithm for two-sensor networks, where local estimate of the first sensor is considered as a pseudomeasurement of the second one. Due to its ease of implementation, this approach has then inspired the sequential fusion in multisensor networks [8]–[10], where the multiple nodes repeatedly perform the two-sensor fusion in a sequential manner. As the result of serial operation, these algorithms require special communication topology, which should be sequentially connected as a ring/chain. Olfati-Saber [11] considered the more general network topology. The author introduced the consensus algorithms into distributed estimation, and proposed the Kalman-consensus filter, where the average consensus on local estimates was performed. Since then, various consensus-based distributed estimators have been proposed in the literature [12]–[24]. For example, instead of doing consensus on local estimates, Battistelli et al. [14] suggested achieving consensus on noisy measurements and inverse-covariance matrices, respectively. On the other hand, Battistelli and Chisci [25] found that by performing consensus on the Kullback–Leibler average of local probability density function, estimation stability is also guaranteed. They further proved that if the single-consensus step is used, this approach is reduced to the well-known covariance intersection fusion rule [26], [27]. Since the consensus-based estimators usually require multiple consensus steps during each sampling period, they generated better estimation performance.

In Fig. 1, we present the general information flow of the existing consensus-based estimation algorithms, where \( \Delta_i(k) \) is the information transmitted by the sensor \( i \) and to be fused by consensus algorithms, which could be the local estimate [11], [12], measurement [13]–[15], [28], or information matrix [25], [29]. It is noticed from the figure that the consensus/synchronization process is usually coupled with the local filter in these works, making it hard to analyze the performance of local estimates. Due to this fact, while the aforementioned algorithms are successful in distributing the fusion task over multiple nodes and providing stable local estimates, i.e., the error covariance is proved to be bounded at each sensor side, the exact calculation of error covariance can hardly be obtained. Moreover, the global optimality (namely, whether performance of the algorithm can converge to that of the centralized Kalman filter) is also difficult to be analyzed and guaranteed in some works.

It is worth noticing that in theory, the gain of the Kalman filter converges to a steady-state gain exponentially fast [30], which can be calculated offline. Moreover, in practice, a fixed gain estimator is usually implemented, which has the same asymptotic performance as the time-varying Kalman filter. Hence, this article focuses on the distributed implementation of the centralized steady-state Kalman filter. In contrast to most of the existing algorithms, we decouple the local filter from the consensus process. Such decoupling enables us to provide a new framework for designing distributed estimators by reformulating the problem of distributed state estimation into that of linear system
synchronization. We, hence, are able to leverage the methodologies from latter field to propose solutions for distributed estimation. To be specific, in the synchronization of linear systems, the dynamics of each agent is governed by an LTI system, the control input to which is generated using the local information within the neighborhood, in order to achieve asymptotic consensus on the local states of agents. Over the past years, lots of research efforts have been devoted to this area (see, e.g., [31]–[36]) by designating synchronization algorithms that can handle various network constraints. Exploiting the results therein, the distributed estimator in this work is designed through the following two phases.

1) **Local Measurement Processing**: A lossless decomposition of the steady-state Kalman filter is proposed, where each sensor node runs a local estimator based on this decomposition using solely its own measurement.

2) **Information Fusion Via Consensus**: The sensor infers the local estimates of all the others via a modified consensus algorithm designed for achieving linear system synchronization.

The contributions of this article are summarized as follows.

1) By removing assumptions regarding the eigenvalues of the system matrix, this article extends, in a nontrivial way, the results in [37], and thus develops the local filters for losslessly decomposing the Kalman filter in estimating general systems (see Lemma 3).

2) Through the decomposition of the Kalman filter, this article bridges two different fields and makes it possible to leverage a general class of algorithms designed for achieving the synchronization of linear systems to solve the problem of distributed state estimation. By doing so, we can propose stable distributed estimators under different communication constraints, such as time delay, switching topology, random link failures (see Theorem 4), etc.

3) For a certain synchronization algorithm, e.g., [31], the stability criterion of the proposed estimator is established. Moreover, in contrast to the existing literature, the covariance of the estimation error can be exactly derived by solving Lyapunov equations (see Theorems 2 and 3 and Corollary 1).

4) The designed estimator enjoys low communication cost, where the size of message sent by each sensor is \( \min\{m, n\} \), with \( m \) and \( n \) being the dimensions of the state and measurement, respectively (see Remark 6).

Some preliminary results are reported in our previous work [38], where most of the proofs are missing. This article further extends the results in [38] by computing the exact asymptotic error covariance, instead of only showing the stability of proposed algorithms. The extension to the more general random communication topology is also added. Moreover, a model reduction method is further proposed in this work to reduce the message complexity from \( \min\{m, n\} \) to \( \min\{m, n\} \).

**Notations**: For vectors \( v_i \in \mathbb{R}^{m_i} \), the vector \( [v_1^T, \ldots, v_N^T]^T \) is defined by \( \text{col}(v_1, \ldots, v_N) \). Moreover, \( A \otimes B \) indicates the Kronecker product of matrices \( A \) and \( B \). Throughout this article, we define a stochastic signal as “stable” if its covariance is bounded at any time.

The rest of this article is organized as follows. Section II introduces the preliminaries and formulates the problem of interest. A lossless decomposition of the optimal Kalman filter is given in Section III, where a model reduction approach is further proposed to reduce the system order. With the aim of realizing the optimal Kalman filter, distributed solutions for state estimation are given and analyzed in Section IV. We then discuss some extensions in Section V, and validate performance of the developed estimator through numerical examples in Section VI. Finally, Section VII concludes this article.

### II. PROBLEM FORMULATION

In this article, we consider the LTI system as follows:

\[
    x(k + 1) = Ax(k) + w(k)
\]  
(1)

where \( x(k) \in \mathbb{R}^n \) is the system state and \( w(k) \sim \mathcal{N}(0, Q) \) is the independent and identically distributed (i.i.d.) Gaussian noise with zero mean and covariance matrix \( Q \geq 0 \). The initial state \( x(0) \) is also assumed to be Gaussian with zero mean and covariance matrix \( \Sigma \geq 0 \) and is independent from the process noise \( \{w(k)\} \).

A network consisting of \( m \) sensors is monitoring the abovementioned system. The measurement from each sensor \( i \in \{1, \ldots, m\} \) is given by

\[
    y_i(k) = C_i x(k) + v_i(k)
\]  
(2)

where \( y_i(k) \in \mathbb{R} \) is the output of sensor \( i \), \( C_i \) is an \( n \)-dimensional row vector, and \( v_i(k) \in \mathbb{R} \) is the Gaussian measurement noise.

By stacking the measurement equations, one gets

\[
    y(k) = Cx(k) + v(k)
\]  
(3)

where

\[
    y(k) \triangleq \begin{bmatrix} y_1(k) \\ \vdots \\ y_m(k) \end{bmatrix}, \quad C \triangleq \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix}, \quad v(k) \triangleq \begin{bmatrix} v_1(k) \\ \vdots \\ v_m(k) \end{bmatrix}
\]  
(4)

and \( v(k) \) is the zero-mean i.i.d. Gaussian noise with covariance \( R \geq 0 \) and is independent from \( w(k) \) and \( x(0) \).

Throughout this article, we assume that \( (A, C) \) is observable. On the other hand, \( (A, C_i) \) may not necessarily be observable, i.e., a single sensor may not be able to observe the whole state space.

#### A. Preliminaries: The Centralized Kalman Filter

If all measurements are collected at a single fusion center, the centralized Kalman filter is optimal for state estimation purpose, and provides a fundamental limit for all other estimation schemes. For this reason, this part will briefly review the centralized solution given by the Kalman filter.

Let us denote by \( P(k) \) the error covariance of estimate given by the Kalman filter at time \( k \). Since \( (A, C) \) is observable, it is well-known that the error covariance will converge to the steady state [6]

\[
    P = \lim_{k \to \infty} P(k).
\]  
(5)

Since the operation of a typical sensor network lasts for an extended period of time, we assume that the Kalman filter is in the steady state, or equivalently \( \Sigma = P \), which results in a steady-state Kalman filter with fixed gain\(^2\)

\[
    K = PC^T(CPC^T + R)^{-1}.
\]  
(6)

Accordingly, the optimal Kalman estimate is computed recursively as

\[
    \hat{x}(k + 1) = A\hat{x}(k) + K(y(k + 1) - CA\hat{x}(k))
\]  
(7)

\[
    = (A - KCA)\hat{x}(k) + Ky(k + 1).
\]

It is clear that the optimal estimate (7) requires the information from all sensors. However, in a distributed framework, each sensor is only capable of communicating with immediate neighbors, rendering the centralized solution impractical. Therefore, this article is devoted to the implementation of the Kalman filter in a distributed fashion.

#### III. DECOMPOSITION OF THE KALMAN FILTER

This section, we shall provide a local decomposition of the Kalman filter (7), where the Kalman estimate can be recovered as a linear

\(^1\)The results in this article can be readily generalized to cases where the sensor outputs a vector measurement by treating each entry independently as a scalar measurement.

\(^2\)Notice that even if \( \Sigma \neq P \), the Kalman estimate converges to the steady-state Kalman filter, i.e., the steady-state estimator is asymptotically optimal.
A matrix is defined to be nonderogatory if every eigenvalue of it has geometric multiplicity 1.

A. Local Decomposition of the Kalman Filter

To locally decompose the Kalman filter, we first introduce the following lemmas, the proofs of which are given in [39].

Proposition 1: If $A$ is a nonderogatory Jordan matrix, then both $(Λ, 1)$ and $(Λ^T, 1)$ are controllable.

Lemma 1: Let $(X, p)$ be controllable, where $X \in \mathbb{R}^{n \times n}$ and $p \in \mathbb{R}^n$. For any $q \in \mathbb{R}^n$, if $X + pq^T$ and $X$ do not share any eigenvalues, then $(X + pq^T, q^T)$ is observable, or equivalently $(X^T + qp^T, q)$ is controllable.

Lemma 2: Let $(X, p)$ be controllable, where $X \in \mathbb{R}^{n \times n}$ and $p \in \mathbb{R}^n$. Denote the characteristic polynomial $X$ as $φ(s) = \det(sI - X)$. Let $Y \in \mathbb{R}^{m \times n}$ and $q \in \mathbb{R}^n$. Suppose that $φ(Y)q = 0$. Then, there exists $T \in \mathbb{R}^{mn \times n}$, which solves the following equation:

$$TX = YT, \quad Ty = q.$$ (12)

With the above-mentioned preparations, let us consider the optimal Kalman estimate in (7). For simplicity, we denote by $K$ the $j$th column of the Kalman gain $K$. Namely, $K = [K_1, \ldots, K_m]$. Accordingly, (7) can be rewritten as

$$\hat{x}(k + 1) = (A - KCA)\hat{x}(k) + \sum_{i=1}^{m} K_i y_i(k + 1).$$ (13)

Notice that $A - KCA$ is stable. It is clear that we can always find a Jordan matrix $Λ \in \mathbb{R}^{n \times n}$, such that $Λ$ is strictly stable, nonderogatory, and has the same characteristic polynomial of $A - KCA$. In view of Proposition 1, we conclude that $(Λ, 1)$ is controllable. Therefore, by Lemma 2, we can always find matrices $F$, such that the following equalities hold for $i = 1, \ldots, m$:

$$F_iΛ = (A - KCA)F_i, \quad F_i1_n = K_i.$$ (14)

Suppose each sensor $i$ performs the following local filter solely based on its own measurements:

$$\hat{ξ}_i(k + 1) = Λ\hat{ξ}_i(k) + 1_n y_i(k + 1)$$ (15)

where $\hat{ξ}_i(k)$ is the output of the local filter from sensor $i$ and $1_n \in \mathbb{R}^n$ is a vector of all ones. Then, it is proved that the optimal Kalman filter can be decomposed as a weighted sum of local estimates $\hat{ξ}_i(k)$, as stated in the following.

Lemma 3: Suppose each sensor performs the local filter (15). The optimal Kalman estimate (7) can be recovered from the local estimates $\hat{ξ}_i(k), i = 1, 2, \ldots, m$, as

$$\hat{x}(k) = \sum_{i=1}^{m} F_i\hat{ξ}_i(k).$$ (16)

where $F_i$ is defined in (14).

Proof: By multiplying both sides of the recursive (15) by $F_i$, we arrive at

$$F_i\hat{ξ}_i(k + 1) = F_iΛ\hat{ξ}_i(k) + F_i1_n y_i(k + 1).$$ (17)

Then, it follows from (14) that

$$F_i\hat{ξ}_i(k + 1) = (A - KCA)F_i\hat{ξ}_i(k) + K_i y_i(k + 1).$$ (18)

Summing up the abovementioned equation for all $i = 1, \ldots, m$ and comparing it with (13), we can conclude that (16) holds.

Notice that the equality in Lemma 3 surely holds. This means that the Kalman filter can be perfectly recovered by (16). We hence claim that (16) is a lossless decomposition of the optimal Kalman filter. To better illustrate the ideas, the information flow of the centralized Kalman filter and local decomposition (16) is shown in Fig. 2.

B. Reformulation of (15) With Stable Inputs

It is noted that the system matrix $A$ may be unstable, which implies that the covariance of measurement $y(k)$ is not necessarily bounded. As a result, we need to redesign (15), using the stable residual $z_i(k)$ as an input instead of the raw measurement $y_i(k)$. The main reason for this reformulation is to make the consensus algorithm feasible and develop stable distributed estimators, which will be further discussed in the proof of Theorem 3, as provided in the full version of this article [39].

Towards the end, notice that $(Λ, 1)$ is controllable, $Λ$ is stable, and any eigenvalue of $A_0$ is unstable. Hence, there always exists a nonzero $β \in \mathbb{R}^n$, and

$$S = Λ + 1β^T$$ (19)

such that the following hold.

1) The characteristic polynomial of $A^0$ divides $φ(s)$, where $φ(s)$ is the characteristic polynomial of $S$, and $φ(s)/\det(sI - A^0)$ has only strictly stable roots.

2) $S$ do not share eigenvalues with $Λ$. Hence, by the virtue of Lemma 1, $(S^T, β)$ is controllable.

Remark 1: Notice that by using $β$, we place the eigenvalues of $S$ to the locations, which consist of two parts: 1) the unstable ones that coincide with the eigenvalues of $A_0$; and 2) the stable ones that are freely assigned, but cannot be the eigenvalues of $Λ$. This is feasible as $(Λ, 1)$ is controllable.

Next, let us consider the filter as follows:

$$z_i(k) = y_i(k + 1) - β^T\hat{ξ}_i(k)$$

Fig. 2. Information flow of the centralized Kalman filter (left-hand side) and the local decomposition of Kalman filter (16) (right-hand side).
∀E⊂V×V ˜ is always bounded. Fig. 2 ˆ with deg ˆ e = ∈E ≜ = (G i y

The proof is presented in [39]. e are given in (19).

≥ i from all sensors. ≥ when the state dimension is less than the number of sensors, for all ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k ˜ a a in (16), we can always S(29) k 38x597]IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 68, NO. 4, APRIL 2023

Theorem 1: Consider the following system:

$$\begin{align*}
\begin{bmatrix}
\theta_1(k+1) \\
\vdots \\
\theta_n(k+1)
\end{bmatrix} &= (I_n \otimes S)
\begin{bmatrix}
\theta_1(k) \\
\vdots \\
\theta_n(k)
\end{bmatrix} + T
\begin{bmatrix}
z_1(k) \\
\vdots \\
z_m(k)
\end{bmatrix} \\
\hat{x}(k) &= H
\begin{bmatrix}
\theta_1(k) \\
\vdots \\
\theta_n(k)
\end{bmatrix}
\end{align*}$$

(27)

where

$$
T = [T_1, T_2, \ldots, T_m], \quad H = [H_1, H_2, \ldots, H_n].
$$

(28)

It holds that system (27) shares the same transfer function with (22).

Proof: The proof is presented in [39].

Therefore, by performing model reduction, we present system (27), which shares the same transfer function with (22), but with a reduced order. As proved previously, the output of (22) is the optimal Kalman estimate. As a result, (27) also has the Kalman estimate, as its output and the Kalman filter can be perfectly recovered by (27) as well. We hereby refer both (22) and (27) to lossless decomposition of the Kalman filter. Depending on the sizes of m and n, one should use a system with smaller dimension to represent the centralized Kalman filter.

IV. LOCAL IMPLEMENTATION OF THE KALMAN FILTER

From Fig. 2, it is clear that local decomposition proposed in Section III is still centralized, as a fusion center is required for calculating the weighted sum. In this section, we have provided distributed algorithms for implementing it, where each sensor node performs local filtering by using the results from Section III and global fusion by exchanging information with neighbors and running the consensus algorithm. Based on whether n is greater than m or not, different algorithms are presented to achieve a low communication complexity.

We use a weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ to model the interaction among nodes, where $\mathcal{V} = \{1, 2, \ldots, m\}$ is the set of sensors, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges, and $A = [a_{ij}]$ is the weighted adjacency matrix. It is assumed that $a_{ii} \geq 0$ and $a_{ij} = a_{ji}$ for $i, j \in \mathcal{V}$. An edge between sensors i and j is denoted by $e_{ij} \in \mathcal{E}$, indicating that these two agents can communicate directly with each other. Note that $e_{ii} \in \mathcal{E}$ if and only if $a_{ii} > 0$. By denoting the degree matrix as $D = \text{diag}(deg_1, \ldots, deg_m)$ with $deg_i = \sum_{j=1}^{m} a_{ij}$, the Laplacian matrix of $\mathcal{G}$ is defined as $L_{\mathcal{G}} = D - A$. In this article, a connected network is considered. We therefore can arrange the eigenvalues of the Laplacian matrix as $0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_m$.

A. Description of the Distributed Estimator

In light of (16), the optimal estimate fuses $\hat{x}_i(k)$ from all sensors. However, in a distributed framework, each sensor can only access the information in its neighborhood. Hence, any sensor i needs to, through the communication over network, infer $\hat{x}_i(k)$ for all $j \in \mathcal{V}$ to achieve a stable local estimate. Let us denote by $\eta_{i,j}(k)$ as the inference from sensor j on sensor i. As will be proved later in this section, $\eta_{i,j}(k)$, by running a synchronization algorithm, can track $\frac{1}{m} \hat{x}_j(k)$ with bounded error. Hence, every sensor i can make a decent inference on $\hat{x}_i(k)$.

By collecting its inference on all sensors together, each sensor i keeps a local state as follows:

$$
\eta_i(k) \triangleq \begin{bmatrix}
\eta_{i,1}(k) \\
\vdots \\
\eta_{i,m}(k)
\end{bmatrix} \in \mathbb{R}^{m \times 1}
$$

(29)
which will be updated by synchronization algorithms. Since \( \eta_i(k) \) contains the fair inference on all \( \hat{\xi}_j(k), j \in \mathcal{V} \), sensor \( i \) finally uses it to compute a stable local estimate.

To be concrete, let us define the message sent by the agent \( i \) at time \( k \) as \( \Delta_i(k) \triangleq \hat{\eta}_i(k) \in \mathbb{R}^m \), where \( \hat{\Gamma} = I_m \otimes \hat{\gamma} \) and \( \hat{\gamma} \) is a design parameter to be given. We are now ready to present the main algorithm. Suppose each node \( i \) is initialized with \( \hat{\xi}_i(0) = 0 \) and \( \eta_i(0) = 0 \). At any instant \( k > 0 \), its update is outlined in Algorithm 1, the information flow of which is shown in Fig. 3. Compared with Fig. 2, the proposed algorithm is achieved in a distributed manner.

Remark 3: Instead of transmitting the raw estimate \( \eta_i(k) \in \mathbb{R}^{mn} \), each agent sends a “coded” vector \( \Delta_i(k) \) with a smaller size \( m \).

**B. Performance Analysis**

This part is devoted to the performance analysis of Algorithm 1. We first provide the following theorem.

**Theorem 2:** With Algorithm 1, the average of fused estimates from all sensors coincides with the optimal Kalman estimate at any instant \( k \). That is,

\[
\frac{1}{m} \sum_{i=1}^{m} \hat{x}_i(k) = \hat{x}(k) \quad \forall k \geq 0. \tag{33}
\]

**Proof:** Summing (31) overall \( i = 1, 2, \ldots, m \) yields

\[
\sum_{i=1}^{m} \eta_i(k+1) = S \sum_{i=1}^{m} \eta_i(k) + \sum_{i=1}^{m} L_i z_i(k) \tag{34}
\]

where we use the fact that \( a_{ij} = a_{ji} \) for any \( i, j \in \mathcal{V} \). Comparing it with (20), it holds for any instant \( k \) and any \( j \in \mathcal{V} \) that

\[
\hat{\xi}_j(k) = \sum_{i=1}^{m} \eta_{ij}(k). \tag{35}
\]

Therefore, the following equation is satisfied at any \( k \geq 0 \):

\[
\frac{1}{m} \sum_{i=1}^{m} \hat{x}_i(k) = \sum_{i=1}^{m} F \hat{\eta}_i(k) = \sum_{i=1}^{m} \sum_{j=1}^{m} F_{ij} \eta_{ij}(k) = \sum_{j=1}^{m} \sum_{i=1}^{m} F_{ij} \hat{\xi}_j(k) = \hat{x}(k). \tag{36}
\]

This completes the proof.

On the other hand, in order to show the stability of the proposed estimator, it is also desired to prove the boundedness of error covariance. Towards this end, we introduce the following lemma, the condition of which is characterized in terms of a certain relation between the Mahler measure (the absolute product of unstable eigenvalues of \( S \)) and the graph condition number (the ratio of the maximum and minimum nonzero eigenvalues of the Laplacian matrix).

**Lemma 6:** Suppose that the product of all unstable eigenvalues of the matrix \( S \) meets the following condition:

\[
\prod_j |\lambda_j(S)| < \frac{1 + \mu_2/\mu_m}{1 - \mu_2/\mu_m} \tag{37}
\]

where \( \lambda_j(S) \) represents the \( j \)th unstable eigenvalue of \( S \). Let

\[
\Gamma = \frac{2}{\mu_2 + \mu_m} \frac{1^T P S}{1_n^T P 1_n} \in \mathbb{R}^{1 \times n} \tag{38}
\]

where \( \mu_2 \) and \( \mu_m \) are the second smallest and the largest eigenvalues of \( L_G \), respectively. Moreover, \( \mathcal{P} > 0 \) is the solution to the following modified algebraic Riccati inequality:

\[
\mathcal{P} - S^T \mathcal{P} S + (1 - \zeta^2) \frac{S^T P 1_n 1_n^T P S}{1_n^T P 1_n} > 0 \tag{39}
\]

with \( \zeta \) satisfying \( \prod_j |\lambda_j(S)| < \zeta < \frac{1 + \mu_2/\mu_m}{1 - \mu_2/\mu_m} \). Then, for any \( j \in \{2, \ldots, n\} \), it holds that

\[
\rho(S - \mu_j 1_n \Gamma) < 1. \tag{40}
\]

**Proof:** For any \( j \in \{2, \ldots, n\} \), let us denote \( \zeta_j = 1 - 2\mu_j/(\mu_2 + \mu_m) \leq \zeta \). Since \( (S, 1_n) \) is controllable, there exists some \( \mathcal{P} > 0 \), which solves (39). Together with (38), it holds that

\[
(S - \mu_j 1_n \Gamma)^T \mathcal{P} (S - \mu_j 1_n \Gamma) - \mathcal{P} = S^T \mathcal{P} S - (1 - \zeta_j^2) \frac{S^T P 1_n 1_n^T P S}{1_n^T P 1_n} - \mathcal{P} \leq S^T \mathcal{P} S - (1 - \zeta^2) \frac{S^T P 1_n 1_n^T P S}{1_n^T P 1_n} < 0. \tag{41}
\]

Hence, our proof completes.

**Remark 4:** Note that if all the eigenvalues of \( S \) lie on or outside the unit circle, You and Xie [31] proved that (40) holds if and only if (37) is satisfied. In Lemma 6, we further show that (37) is still a sufficient condition to facilitate (40), if \( S \) has stable modes.

**Remark 5:** Invoking Remark 1, each \( \lambda_j(S) \) corresponds to a root of the characteristic polynomial of \( A^n \). Thus, the condition (37) can be
rewritten using the system matrix $A^u$

$$\prod_j |r_j(A^u)| < \frac{1 + \mu_2/\mu_{\infty}}{1 - \mu_2/\mu_{\infty}}$$  \hspace{1cm} (42)$$

where $r_j(A^u)$ is a root of the characteristic polynomial of $A^u$.

We are now ready to analyze the error covariance of local estimator as follows, the proofs of which are given in [39].

**Theorem 3:** Suppose that the Mahler measure of $S$ meets condition (37), and $\Gamma$ is designed based on (38) and (39). With Algorithm 1, the error covariance of each local estimate $\hat{x}_i(k)$ is bounded at any instant $k$.

**Corollary 1:** Suppose that the Mahler measure of $S$ meets condition (37), and $\Gamma$ is designed based on (38) and (39). Let $\tilde{W}$ be the asymptotic error covariance of local estimates. Namely,

$$\tilde{W} \triangleq \lim_{k \to \infty} \text{cov}(\tilde{e}(k))$$

where $\tilde{e}(k) \triangleq \text{col}[(\tilde{x}_1(k) - x(k)), \ldots, (\tilde{x}_m(k) - x(k))]$. By using Algorithm 1, it holds that

$$\tilde{W} = \tilde{W} + (1_m 1_m^T) \otimes P$$  \hspace{1cm} (43)$$

where $\tilde{W}$ is the asymptotic covariance of the error between local and Kalman estimates and $P$ is the error covariance of the Kalman estimate, as defined in (5). Moreover, $\tilde{W}$ can be exactly calculated.

Due to the space limitations, we provide the exact calculation of $\tilde{W}$, i.e., the performance gap between our estimator and the optimal Kalman filter, in [39, Appendix G]. Moreover, as seen from the calculation, $\tilde{W}$ is purely caused by the consensus error, as will be further discussed in Section V.

Combining Theorems 2 and 3, the local estimator is stable at each sensor side. Therefore, we conclude that by applying the algorithm designed for linear system synchronization, i.e., (31), the problem of distributed state estimation is resolved.

**Remark 6:** Note that Algorithm 1 requires each agent to send out an $m$-dimensional vector $\Delta_i(k)$ at any time. Therefore, in the network with a large number of sensors, i.e., $n < m$, this solution will cause a high communication cost. To address this issue, this remark, by leveraging the reduced-order estimator (27) in Theorem 1, modifies Algorithm 1 to introduce less communication complexity. To be specific, we aim to implement the reduced order system (27) with distributed estimators. Similar as before, any agent $i$ stores its estimate on all the others in a variable $\vartheta_i(k)$, where

$$\vartheta_i(k) \triangleq \begin{bmatrix} \vartheta_{i,1}(k) \\ \vdots \\ \vartheta_{i,n}(k) \end{bmatrix} \in \mathbb{R}^{n^2}. \hspace{1cm} (44)$$

For each sensor $i$, it is initialized with $\tilde{x}_i(0) = 0$ and $\vartheta_i(0) = 0$. For the case of $n < m$, the estimation algorithm works as in Algorithm 2. Following similar arguments, the local estimator at each sensor side is proved to be stable.

Combining it with Algorithm 1, we conclude the size of message sent by each sensor at any time is $\min\{m, n\}$. Compared with the existing solutions in distributed estimation, e.g., [12]–[16], our algorithm enjoys lower message complexity.

### V. Extensions of Proposed Solutions

In the previous sections, we leveraged the linear system synchronization algorithm proposed in [31] to solve the problem of distributed state estimation. In this section, we aim to extend such a result and show that any control strategy, which can facilitate the linear system synchronization, can be modified to yield a stable distributed estimator. As a result, we bridge the fields of distributed state estimation and linear system synchronization.

#### Algorithm 2: Distributed Estimation Algorithm For Sensor $i$

1. Using the latest measurement from itself, sensor $i$ computes the local residual and update the local estimate by

$$z_i(k) = y_i(k + 1) - \beta^2 \hat{z}_i(k)$$

$$\hat{z}_i(k + 1) = S \hat{z}_i(k) + 1_n z_i(k).$$

2. Compute $\Delta_i(k) = (I_n \otimes \Gamma) \vartheta_i(k)$ such that $\Gamma$ is calculated by (38). Collect $\Delta_i(k)$ from neighbors and fuse the neighboring information with the consensus algorithm as

$$\vartheta_i(k + 1) = (I_n \otimes S) \vartheta_i(k) + T_i z_i(k)$$

$$+ (I_n \otimes 1_n) \sum_{j=1}^m a_{ij}(\Delta_j(k) - \Delta_i(k)). \hspace{1cm} (45)$$

where $T_i$ is defined in (26).

3. Update the fused estimate on system state as

$$\hat{x}_i(k + 1) = m H \vartheta_i(k + 1) \hspace{1cm} (46)$$

where $H$ is given in (28).

4. Transmit the new state $\Delta_i(k + 1)$ to neighbors.

Let us consider the synchronization of the following homogeneous LTI system:

$$\eta_i(k + 1) = \tilde{S} \eta_i(k) + \tilde{B} u_i(k) \quad \forall i \in \mathcal{V} \hspace{1cm} (47)$$

where $u_i(k)$ is the control input of the agent $i$. In the literature, a large variety of synchronization algorithms has been proposed with the following framework:

$$\omega_i(k + 1) = A \omega_i(k) + B \eta_i(k) + 1_n \Delta_i(k) = \tilde{\Gamma} \omega_i(k)$$

$$u_i(k) = \sum_{j=1}^m a_{ij} \gamma_{ij}(k)(\Delta_j(k) - \Delta_i(k)) \hspace{1cm} (48)$$

where $\omega_i(k)$ is the “hidden state”, which is necessary for the agent $i$ to yield the communication state $\Delta_i(k)$ and input $u_i(k)$, and $\tilde{\Gamma}$ refers to the control gain. Notice that (48) can be used to model the controller with memory. Moreover, $\gamma_{ij}(k) \in [0, 1]$ models the fading or lossy effect of the communication channel from agents $j$ to $i$. At every time, the agent collects the available information in its neighborhood and synthesizes its communication state and control signal via (48).

For simplicity, we denote $\mathcal{U}$ as the control strategy that can be represented by (48). Let the average of local states at time $k$ be

$$\bar{\eta}(k) = \frac{1}{m} \sum_{i=1}^m \eta_i(k).$$

The network of subsystems (47) reaches strong synchronization under $\mathcal{U}$, if the following statements hold at any time.

1) **Consistency:** The average of local states keeps consistent throughout the execution, i.e.,

$$\bar{\eta}(k + 1) = \tilde{S} \bar{\eta}(k). \hspace{1cm} (49)$$

2) **Exponential Stability:** Agents exponentially reach consensus in mean square sense, i.e., there exist $c > 0$ and $\rho \in (0, 1)$, such that

$$\mathbb{E}[||\eta_i(k) - \bar{\eta}(k)||^2] \leq c e^{\rho k} \quad \forall i \in \mathcal{V}. \hspace{1cm} (50)$$

We now review several existing strategies, which facilitate the strong synchronization and show that they can be represented by (48).
1) Let $\Delta_i(k) = \bar{F}_i \eta_i(k)$ be the communication state defined in Section IV-A. To facilitate the synchronization of homogeneous linear systems in the undirected communication topology, You and Xie [31] designed the following control law:

$$u_i(k) = \sum_{j=1}^{m} a_{ij}(\Delta_j(k) - \Delta_i(k))$$

which coincides with (48).

2) Another example is the filtered consensus protocol given in [34]. By designing the hidden state as

$$\omega_i(k) = F(q) \eta_i(k)$$

where $q$ is the unit advance operator, i.e., $q^{-1}s(k) = s(k - 1)$, and $F(z)$ is the transfer function of a square stable filter, the synchronization of linear systems is achieved by (48) under a more relaxed condition than (37), that is, $\prod_j |\lambda_j^{\infty}(S)| < \frac{1 + \sqrt{\mu_2/\mu_m}}{1 - \sqrt{\mu_2/\mu_m}}$.

3) Instead of focusing on perfect communication channels, You et al. [32] and Xu et al. [33] developed the control protocols to account for the random failure on communication links and Markovian switching topologies, respectively. By modeling the packet loss with the Bernoulli random variable $\gamma_{ij}(k) \in \{0, 1\}$, these works complemented the results in [31] and proved the mean square stability under the control strategy (48).

Notice that Algorithms 1 and 2 utilize (51) for achieving synchronization and producing stable distributed estimators. In what follows, we argue that the optimal Kalman estimate can indeed be distributively implemented using any linear system synchronization algorithms facilitating (49) and (50). To be specific, Algorithm 1 should be modified by replacing (31) with

$$\eta_i(k+1) = \tilde{S}_i \eta_i(k) + B u_i(k) + L_i z_i(k)$$

where $u_i(k)$ is generated by $U$ that facilitates (49) and (50). We then state the stability of local estimators as follows.

**Theorem 4:** Consider any algorithm $U$, which facilitates the statements (49) and (50). At any time $k$, suppose each $\gamma_{ij}(k)$ is independent of the noises $\{w(k)\}$ and $\{v(k)\}$. Then, (53) yields a stable estimator for each sensor node. Specifically, the following statements hold for any $k \geq 0$.

1) The average of local estimations from all sensor coincides with the optimal Kalman estimate.

2) The error covariance of each local estimate is bounded.

**Proof:** The proof is given in [39].

**Remark 7:** Theorem 4 assumes the independence of the communication topology and system/measurement noises. Therefore, as for the event-based synchronization algorithms, where the communication relies on the agents’ states, we cannot analyze its efficiency of solving the distributed estimation problem by directly resorting to Theorem 4. In the future work, we will continue to investigate this topic.

In contrast with the existing works, as shown in Fig. 1, this work, by using the lossless decomposition of the Kalman filter, decouples the local filter from the consensus process, as shown in Fig. 3. The decoupling enables us to leverage the rich results in linear systems synchronization to analyze the performance of local estimators, as proved in Theorem 4. Moreover, following the similar proof arguments as that of Theorem 3 and Corollary 1, we can show that with our framework, the error covariance of each local estimate actually consists of two orthogonal parts: 1) the inherent estimation error of the Kalman filter; and 2) the distance from local estimate to the Kalman filter.

**VI. NUMERICAL EXAMPLE**

In this section, we present a numerical example\(^4\) to verify the theoretical results obtained in previous sections.

Let us consider the case where four sensors cooperatively estimate the system state. The system parameters are listed as follows:

$$A = \begin{bmatrix} 0.9 & 0 \\ 0 & 1.1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}^T, \quad Q = 0.25 I_2, \quad R = 4 I_4.$$  \hspace{1cm} (54)

In this example, the number of states is smaller than that of sensors, i.e., $n < m$. We therefore choose Algorithm 2. Moreover, notice that the system is unstable and sensor 1 cannot observe the unstable state.

\(^4\) Similarly, in the case of $n < m$, one can also derive the general form of Algorithm 2 with any linear system synchronization strategy $U$.

\(^5\) More examples can be found in [39].
We set the initial state $\pi(0) \sim \mathcal{N}(0, I)$ and the initial local estimate $\hat{x}_i(0) = 0$ for each sensor $i \in \{1, 2, 3, 4\}$.

Suppose that the topology of these four sensors is a ring with weight 1 for each edge. The Laplacian matrix is thus

$$L_2 = \begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
-1 & 0 & -1 & 2
\end{bmatrix}. \tag{55}$$

It is not difficult to check that the second smallest and the largest eigenvalues of $L_2$ are $\mu_2 = 2$ and $\mu_4 = 4$, respectively. To fulfill the sufficient condition in Lemma 6, let us choose $\zeta = 0.5$.

It can be seen that the mean squared local estimation error $e_i(k)$ enters steady state and is stable after a few steps (see Fig. 4).

VII. CONCLUSION

In this article, the problem of distributed state estimation was studied for an LTI Gaussian system. We investigated both cases where $m > n$ and $m \leq n$, and proposed distributed estimators for both cases to introduce low communication costs. The local estimator was proved to be stable at each sensor side, in the sense that the covariance of estimation error was proved to be bounded and the asymptotic error covariance can also be derived. Our major merit lies in reformulating the problem of distributed estimation to that of linear system synchronization.

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