Abstract

We present an scheme to generate entangled state between two traveling modes for fixed number of photon, which is based on beam splitter transformation and conditional zero photon counters.

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It has long realized that the striking nature of entanglement lies at the heart of study of the fundamental issues in quantum mechanics as witnessed by Einstein-Podolsky-Rosen paper [1], Bell’ inequality [2] and its subsequent experimental verification [3] [4]. The recent surge of interest and progress in quantum information theory allows one to take a more positive view of entanglement and to regard it as an essential resource for many ingenious applications such as quantum teleportation [5] and quantum cryptography [6]. These applications rely on the ability to engineer and manipulate entangled states in a controlled way. So far, the generation and manipulation of entangled states have been demonstrated with ions in a ion trap [7] and with atoms in Cavity QED [8]. Recently, quantum engineering of light received extensively attention, since phenomena such as teleportation [9] and quantum dense coding [10] found their implementation in the quantum domain. Different schemes have been proposed to generate any superposition of vacuum and one photon state [11, 12]. A conditional scheme based on beam splitters and conditional zero counters has also been suggested to produce any quantum superposition of traveling waves [13]. More recently, it has been shown that single photon entangled states can be used to implement teleportation of superposition states of zero and one photon states [14, 15] and to test quantum nonlocality [14]. The scheme for teleportation of traveling wave states, which is a coherent superposition of N Fock state

$$\Psi = \sum_{n=0}^{N} C_n |n >,$$

is proposed [16]. As a first step of teleportation, we need to prepare entangled states

$$\Psi = \frac{1}{\sqrt{N+1}} \sum_{n=0}^{N} |n >_1 |N - n >_2,$$
which are entangled states of two traveling modes for fixed number $N$ of photons. Such state are also of potential interest in the context of phase sensitivity in two mode interferometer, where they should allow measurements at the Heisenberg uncertainty limit [18]. Recently it is also shown that such states allow the sub-diffraction limited lithography [19].

In this paper, we propose a generation of Ref [13] procedure for generation of an arbitrary two modes entangled state for fixed number of photons, which can be expressed as

$$\Psi = \sum_{n=0}^{N} C_n |n>_1 |N-n>_2 . \quad (3)$$

Using the definition of Fock state, Eq.(3) can be expressed as

$$\Psi = \sum_{n=0}^{N} \frac{C_n}{\sqrt{n!(N-n)!}} a^{\dagger n} b^{\dagger N-n} |0> . \quad (4)$$

Here $|0>$ is vacuum state of two traveling modes. Similar to Ref [13], we rewrite the Eq.(3) as

$$\Psi = (a^{\dagger} - \beta N b^{\dagger})(a^{\dagger} - \beta_{N-1} b^{\dagger}) \cdots (a^{\dagger} - \beta_1 b^{\dagger})|0> . \quad (5)$$

Here $\beta_1 \cdots \beta_N$ are the $N$ (complex) roots of the characteristic polynomial

$$\sum_{n=0}^{N} C_n \frac{\beta^n}{\sqrt{n!(N-n)!}} = 0 . \quad (6)$$

The relations

$$a^{\dagger} - \beta b^{\dagger} = \sqrt{1+|\beta|^2} B(\theta, \varphi) a^{\dagger} B^{\dagger}(\theta, \varphi) ;$$

$$B(\theta, \varphi) = \exp(\theta e^{-i\varphi} a b - \theta e^{i\varphi} a^{\dagger} b^{\dagger}) ;$$

$$\cos \theta = \frac{1}{\sqrt{1+|\beta|^2}} ;$$

$$e^{i\varphi} = \frac{\beta}{|\beta|} ;$$

are used. The quantum state

$$\Psi = \left( \prod_{i=1}^{N} \sqrt{1+|\beta_i|^2} \right) B(\theta_N, \varphi_N) a^{\dagger} B^{\dagger}(\theta_N, \varphi_N) \cdots B(\theta_2, \varphi_2) a^{\dagger} B^{\dagger}(\theta_2, \varphi_2) B(\theta_1, \varphi_1) a^{\dagger} |0> \quad (7)$$

is obtained. Hence, any entangled quantum state of the form (3) can be generated from the one-photon state by this method, which bases on a succession of alternate single photon addition and beam splitter transformations. These are determined by the roots of the characteristic polynomial (6).

In what follows, we give a scheme to generate quantum states of the form (7), based on the conditional zero photon counters. In Ref [17] it was shown that a mode prepared in an arbitrary state $|\Phi>$ is mixed at the beam splitter with a single photon Fock state. A zero photon measurement is performed in one of the output channels of the beam splitter. Then the quantum state of the mode in the other output channel collapses to

$$|0>$$
Y|φ with Y = Ra†T^na. R is the reflectance and T is the transmittance of the beam splitter. The implementation of the present scheme is outlined in Fig.1. Following Ref. [13], the quantum state, which is generated if no photon is detected in each of the N − 1 conditional output measurement, is given by

$$\Psi \sim B(\theta'_N, \varphi'_N)a^\dagger T^naB(\theta'_{N-1}, \varphi'_{N-1})a^\dagger T^na \cdots \\ \times B(\theta'_3, \varphi'_3)a^\dagger T^naB(\theta'_2, \varphi'_2)a^\dagger T^naB(\theta'_1, \varphi'_1)a^\dagger|0> \ .$$

(8)

In order to bring Eq.(8) into the form of Eq.(7), we write Eq.(8) as

$$\Psi \sim B_Na^\dagger B_N^TB_{N-1}a^\dagger B_{N-1}^TT^{-na}B_N^\dagger \times B_NT^naB_{N-1}T^naB_{N-2}a^\dagger B_{N-2}^TT^{-na}B_N^\dagger \cdots \\ \times B_NT^naB_{N-1}T^na \cdots B_2T^na B_1a^\dagger B_1^TT^{-na}B_2^\dagger \cdots T^{-na}B_{N-1}^TT^{-na}B_N^\dagger|0> \ .$$

(9)

The abbreviation $B_i = B(\theta'_1, \varphi'_1)$ is used. By means of the Hausdorff formula, we obtain

$$B_NT^naB_{N-1}T^na \cdots T^naB_{N-i}a^\dagger B_{N-i}^TT^{-na} \cdots T^{-na}B_{N-1}^TT^{-na}B_N^\dagger = A_i a^\dagger - B_i b^\dagger \ \ B_NT^naB_{N-1}T^na \cdots T^naB_{N-i}b^\dagger B_{N-i}^TT^{-na} \cdots T^{-na}B_{N-1}^TT^{-na}B_N^\dagger = C_i a^\dagger - D_i b^\dagger \ .$$

(10)

The abbreviations $A_j, B_j, C_j$ and $D_j$ are

$$A_{j+1} = (T \cos \theta'_{N-j-1}A_j - \sin \theta'_{N-j-1}e^{i\varphi'_{N-j-1}}C_j) ; \\ B_{j+1} = (T \cos \theta'_{N-j-1}B_j - \sin \theta'_{N-j-1}e^{i\varphi'_{N-j-1}}D_j) ; \\ C_{j+1} = (\cos \theta'_{N-j-1}C_j + T \sin \theta'_{N-j-1}e^{-i\varphi'_{N-j-1}}A_j) ; \\ D_{j+1} = (\cos \theta'_{N-j-1}D_j - T \sin \theta'_{N-j-1}e^{-i\varphi'_{N-j-1}}B_j) ; \ \\ A_0 = \cos \theta'_N ; \\ B_0 = \sin \theta'_N e^{i\varphi'_N} ; \\ C_0 = \sin \theta'_N e^{-i\varphi'_N} ; \\ D_0 = \cos \theta'_N .$$

Thus, we obtain

$$\Psi \sim B(\alpha_N, \beta_N)a^\dagger B^\dagger(\alpha_N, \beta_N) \cdots B(\alpha_2, \beta_2)a^\dagger B^\dagger(\alpha_2, \beta_2)B(\alpha_1, \beta_1)a^\dagger|0> \ .$$

(11)

The abbreviations are the following:

$$\alpha_N = \theta'_N ; \ \ \ \ \ \beta_N = \varphi'_N ; \ \ \ \ \ \cos \alpha_{N-i} = \frac{|A_i|}{\sqrt{|A_i|^2 + |B_i|^2}} ; \ \ \ \ \ \ e^{i\beta_{N-i}} = \frac{B_i|A_i|}{|B_i|A_i} .$$

Comparing Eq.(11) and Eq.(7), we find that these two equations become identical, if the parameters are chosen to

$$\alpha_j = \theta_j ; \ \ \ \ \ \beta_j = \varphi_j .$$

(12)

The solution of the Eq.(12) always exits and we can indeed prepare any desired entangled state of form Eq.(3).
In the following, we calculate the probability \( P \) of generating a desired state (3). Obviously this probability is determined by the requirement that all the \( N - 1 \) detectors do not register photons. It can be given by

\[
P = P(N - 1, 0|1, 0; 2, 0; \cdots; N - 2, 0) \cdots P(2, 0|1, 0)P(1, 0)
\]

Here \( P(k, 0|1, 0; 2, 0; \cdots; k - 1, 0) \) is the probability that the \( k \)-th detector does not register photons under the condition, that the detectors \( D_1, \cdots, D_{k-1} \) have also not registered photons. Starting from

\[
P(1, 0) = ||YB_1|1 > ||^2 \quad \quad Y = Ra^\dagger Tm\quad \quad \quad (14)
\]

Here \( \langle \psi | \psi \rangle = \sqrt{< \psi || \psi >} \). We derive

\[
P(k, 0|1, 0; 2, 0; \cdots; k - 1, 0) = \frac{||YB_kYB_{k-1} \cdots YB_1|1 > ||^2}{||YB_{k-1} \cdots YB_1|1 > || \cdots ||YB_1|0 > ||} \quad \quad (15)
\]

\[
P = P^2_{N-1} \prod_{k=1}^{N-2} (P_k)^{k+3-N} \quad \quad (16)
\]

\[
P = \frac{||YB_kYB_{k-1} \cdots YB_1|1 > ||}{||YB_{k-1} \cdots YB_1|1 > || \cdots ||YB_1|0 > ||} \quad \quad (17)
\]

\[
P = \frac{||YB_kYB_{k-1} \cdots YB_1|1 > ||}{||YB_{k-1} \cdots YB_1|1 > || \cdots ||YB_1|0 > ||} \quad \quad (18)
\]

\[
(P_k/R_{1k})^2 = \sum_{j_1+\cdots+j_k=0} \frac{(1+j_1+\cdots+j_k)!}{A_{k+1,1} \cdots A_{k+1,1} B_{k+1,1} \cdots B_{k+1,1}^2} \quad \quad (19)
\]

Where \( j_i \in \{0; 1\} \) and \( A_{i,j} \) and \( B_{i,j} \) are determined by:

\[
A_{k,j+1} = (T \cos \theta_{k-j-1}^j - \sin \theta_{k-j-1}^j e^{i\varphi_{k,j-1}})C_{k,j};
\]

\[
B_{k,j+1} = (T \cos \theta_{k-j-1}^j - \sin \theta_{k-j-1}^j e^{i\varphi_{k,j-1}})D_{k,j};
\]

\[
C_{k,j+1} = (\cos \theta_{k-j-1}^j + T \sin \theta_{k-j-1}^j e^{-i\varphi_{k,j-1}})A_{k,j};
\]

\[
D_{k,j+1} = (\cos \theta_{k-j-1}^j + T \sin \theta_{k-j-1}^j e^{-i\varphi_{k,j-1}})B_{k,j}.
\]

The initial conditions are \( A_{k,0} = D_{k,0} = 1 \) and \( B_{k,0} = C_{k,0} = 0 \).

As an example, we consider the generation of maximally entangled four-photon state

\[
\Psi = \frac{1}{\sqrt{2}}(|0 > |4 > -|4 > |0 >).
\]

We now give a simplified scheme. Consider the experiment, which is shown schematically in Fig.2. A pair of photons from the independent single photon source is incident on a symmetric beam splitter. Behind the Beam Splitter, the state become an two-photon path entangled state \( \Psi_1 = \frac{1}{\sqrt{2}}(|0, 2 > -|2, 0 >) \). In order to generate four-photon path entangled states, the spatial separated photons are incident on two symmetric beam splitters \( BS_2 \) and \( BS_3 \), respectively. The second input port of each of these beam splitters is assumed to be single photon state produced by single photon source. After the beam splitter \( BS_2 \) and \( BS_4 \) transformations, the two auxiliary photons are measured and the outcome is accepted only when no photon was detected by two photon detector \( D_1 \) and \( D_2 \). Thus, the state is projected into
$\Psi_2 = \frac{1}{\sqrt{2}}(|1, 3 > -|3, 1 >)$. Then this output state is taken as the input to the $BS_4$. Thus, we can obtain the four-photon path entangled state $|\Psi_3 >= \frac{1}{\sqrt{2}}(|0, 4 > -|4, 0 >)$ with probability of success 1/16, which is slightly more than the probability of success (3/64) for the scheme presented in Ref [20].

In summary, we have suggested a feasible scheme to generate entangled state between two traveling modes for a fixed number of photons, which is based on beam splitter transformation and conditional zero-photon counters. We also give a simplified scheme to generate entangled four-photon quantum states with probability of success 1/16. This is slightly more than the probability of success (3/64), which is achievable with the scheme presented in Ref [20]. One of the difficulties of our scheme in respect to an experimental demonstration is the availability of photon number sources. Another difficulty consists in the requirement on the sensitivity of the detectors. These detectors should be capable of distinguishing between no photon, one photon or more photons.

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Figure Captions

**Figure 1.** The schematic is shown to generate entangled state between two traveling modes for fixed number of photon. $BS_i(i = 1, \cdots, N)$ denotes the beam splitters and $D_i$ are photon number detectors. $M$ denotes the mirror.

**Figure 2.** This figure shows the simplified scheme of entangled 4-photon quantum state generation. $BS_i$ denotes the beam splitters and $D_i$ are photon number detectors. $M$ denotes the mirror.