Extended jordanian twists for Lie algebras

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Abstract

Jordanian quantizations of Lie algebras are studied using the factorizable twists. For a restricted Borel subalgebras $B^\vee$ of $sl(N)$ the explicit expressions are obtained for the twist element $\mathcal{F}$, universal $\mathcal{R}$-matrix and the corresponding canonical element $\cT$. It is shown that the twisted Hopf algebra $U_\mathcal{F}(B^\vee)$ is self dual. The cohomological properties of the involved Lie bialgebras are studied to justify the existence of a contraction from the Dinfeld-Jimbo quantization to the jordanian one. The construction of the twist is generalized to a certain type of inhomogenous Lie algebras.

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1 Introduction

The thorough formulation of the theory of quantum groups by Drinfeld [1] includes two types of Hopf algebras: triangular (with the universal $R$-matrix satisfying the relation $R_{21}R = 1$) and quasitriangular (with $R_{21}R \neq 1$). Deformations of universal enveloping of simple Lie algebras initiated by the quantum inverse scattering method and discovered by Drinfeld and Jimbo [1, 2] belong to the latter class. In the framework of the deformation quantization theory [3] these quantum algebras correspond to Lie bialgebras with classical $r$-matrix

$$r_{DJ} = \sum_{i=1}^{k} t_{ij} H_i \otimes H_j + \sum_{\alpha \in \Phi_+} E_\alpha \otimes E_{-\alpha},$$

where $k$ is the rank, $t_{ij}$ is the inverse Cartan matrix, and $\Phi_+$ is the set of positive roots. This $r_{DJ}$ is one of the multitude of solutions to the classical Yang-Baxter equation. The detailed classification of solutions was performed for simple Lie algebras in [4]. Only for some of these classical $r$-matrices the corresponding quantum $R$-matrices are known explicitly.

Although the existence of quantization for any Lie bialgebra is now proved [3], the explicit knowledge of $R$-matrix as an algebraic element $R$ or a matrix in some irreducible representations is required in the FRT approach [4] and in variety of applications of quantum groups. One can mention the universal $R$-matrix of the quantum algebra $U_q(sl(2))$ [1] which is a building block for the universal $R$-matrices for other simple Lie and Kac-Moody algebras. As about triangular quantum groups and twisting [4, 8], the well known example is the jordanian quantization of $sl(2)$ or, more exactly, of its Borel subalgebra $B_+ (\{h, x|[h, x] = 2x\})$ with $r = h \otimes x - x \otimes h = h \wedge x$ [4] and the triangular $R$-matrix $R = F_{21}F^{-1}$ defined by the twisting element [3, 11]

$$F = \exp\{\frac{1}{2} h \otimes \ln(1 + 2\xi x)\}. \quad (1)$$

This quantum algebra $U_\xi(sl(2))$ also found numerous applications from the deformed Heisenberg $XXX$-spin chain to the quantum Minkowski space (see e.g. [11]) and in few other cases [12, 13].
In the present paper we propose different extensions of this twist element. The suggested construction implies the existence (in the universal enveloping algebra to be deformed) of a subalgebra $L$ with special properties of multiplication. This is a solvable subalgebra with at least four generators. All simple Lie algebras except $sl(2)$ contain such $L$ and in any of them a deformation induced by twist of $L$ can be performed. In particular we study a jordanian deformation of $U(sl(N))$, reaching a closed form of deformed compositions lacking in [9]. Using the notion of factorizable twist [14] we prove that the element $F \in U(sl(N)) \otimes^2$, 

$$F = \exp\{2\xi \sum_{i=2}^{N-1} E_{ii} \otimes E_{iN}e^{-\sigma}\} \exp\{H \otimes \sigma\},$$  

(2) where $x = E_{1N}, H = E_{11} - E_{NN}, \sigma = \frac{1}{2}\ln(1+2\xi x)$, satisfies the twist equation. Hence, it defines a triangular deformation of $U(sl(N))$. In such Hopf algebras deformed by jordanian twist the subset of Cartan generators $\{E_{ii} - E_{jj}\}$ with $i < j; \ i, j \neq 1, N$ remains untouched. Hence there is a possibility to perform additional multiparametric deformation using Reshetikhin twist [15]. The main ingredients of the quantum group theory [1] are constructed: the universal $R$-matrix, the dual Hopf algebra (quantized function algebra on $SL(N)$), the universal $T$-matrix (canonical element) for the subalgebra which induces the twist of $U(sl(N))$ and the self-duality of $L$. Cohomological interpretation of the interrelation between the Drinfeld-Jimbo (or standard) quantum algebra $U_q(sl(N))$ and the jordanian (or non-standard) one $U_\xi(sl(N))$ is discussed. The real form and the corresponding quantum linear space are given. We present also further generalization in which the subalgebra $L$ is substituted by a certain type of inhomogeneous Lie algebras.

The connection of the Drinfeld-Jimbo deformation [1, 2] with the jordanian deformation was already pointed out in [9]. The similarity transformation of the classical matrix $r_{DJ}$ performed by the operator $\exp(\xi adE_{1N})$ (with the highest root generator $E_{1N}$) turns $r_{DJ}$ into the sum $r_{DJ} + \xi r_j$ [9]. Hence, 

$$r_j = -\xi \left(H_{1N} \wedge E_{1N} + 2 \sum_{k=2}^{N-1} E_{1k} \wedge E_{kN}\right),$$  

(3)
is a classical $r$-matrix too, which defines corresponding deformation. A singular contraction of the quantum Manin plane $xy = qyx$ of $\mathcal{U}_q(sl(2))$ with the mentioned above transformation in the fundamental representation $M = 1 + \theta \rho(E_{1N})$, $\theta = \xi(1 - q)^{-1}$ results in the jordanian plane $x'y' = y'x' + \xi y'^2$ of $\mathcal{U}_\xi(sl(2))$ \[10\]. Later, this singular contraction in the fundamental representation of $sl(3)$ and $sl(N)$ was used in many papers (cf \[13, 17\] and references therein). Let us point out that in our formulas we do not refer to any particular representation of deformed algebras.

The paper is organized as follows. After reminding briefly the basic material on twisting of Hopf algebras (Sec.2), we construct an extended jordanian twist $\mathcal{F}$ for four generator Lie algebra and apply it to twist the universal enveloping algebra $\mathcal{U}(sl(N))$ (Sec.3). The next Section contains cohomological explanation of the connection between the Drinfeld-Jimbo and jordanian quantization. The main objects of the theory of quantum groups are constructed in Sec.5. Further generalization of the extended jordanian twist to a special class of inhomogenous Lie algebras and possible research topics are given in Sec.6 and in the Conclusion.

### 2 Twisting of Hopf algebras

A Hopf algebra $\mathcal{A}(m, \Delta, \epsilon, S)$ with multiplication $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, coproduct $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, counit $\epsilon: \mathcal{A} \to C$, and antipode $S: \mathcal{A} \to \mathcal{A}$ (see definitions in Refs.\[1, 8, 18\]) can be transformed \[7\] with an invertible element $\mathcal{F} \in \mathcal{A} \otimes \mathcal{A}$, $\mathcal{F} = \sum f_i^{(1)} \otimes f_i^{(2)}$ into a twisted one $\mathcal{A}_t(m, \Delta_t, \epsilon, S_t)$. This Hopf algebra $\mathcal{A}_t$ has the same multiplication and counit maps but the twisted coproduct and antipode

$$\Delta_t(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1}, \quad S_t(a) = vS(a)v^{-1}, \quad v = \sum f_i^{(1)}S(f_i^{(2)}), \quad a \in \mathcal{A}. $$

The twisting element has to satisfy the identities

$$ (\epsilon \otimes \text{id})(\mathcal{F}) = (\text{id} \otimes \epsilon)(\mathcal{F}) = 1, \quad (4) $$

$$ \mathcal{F}_{12}(\Delta \otimes \text{id})(\mathcal{F}) = \mathcal{F}_{23}(\text{id} \otimes \Delta)(\mathcal{F}), \quad (5) $$

3
where the first one is just a normalizing condition and follows from the second relation modulo a non-zero scalar factor.

A quasitriangular Hopf algebra $\mathcal{A}(m, \Delta, \epsilon, S, R)$ has additionally an element $R \in \mathcal{A} \otimes \mathcal{A}$ (a universal $R$-matrix) satisfying

$$
(\Delta \otimes \text{id})(R) = R_{13} R_{23}, \quad (\text{id} \otimes \Delta)(R) = R_{13} R_{12}.
$$

(6)

The coproduct $\Delta$ and its opposite $\Delta^{\text{op}}$ are related by the similarity transformation (twisting) with $R$

$$
\Delta^{\text{op}}(a) = R \Delta(a) R^{-1}, \quad a \in \mathcal{A},
$$

and in this case the relation (5) is just the Yang-Baxter equation.

A twisted quasitriangular quantum algebra $\mathcal{A}_t(m, \Delta_t, \epsilon, S_t, R_t)$ has the twisted universal $R$-matrix

$$
R_t = \tau(F) R F^{-1},
$$

(7)

where $\tau$ means permutation of the tensor factors: $\tau(f \otimes g) = (g \otimes f), \tau(F) = F_{21}$.

Although, in principle, the possibility to quantize an arbitrary Lie bialgebra has been proved [4], an explicit formulation of Hopf operations remains a nontrivial task. In particular, the knowledge of explicit form of the twisting cocycle is a rare case even for classical universal enveloping algebras, despite of advanced Drinfeld’s theory [8]. Most of such explicitly known twisting elements have the factorization property with respect to comultiplication (cf. (6))

$$
(\Delta \otimes \text{id})(F) = F_{23} F_{13} \quad \text{or} \quad (\Delta \otimes \text{id})(F) = F_{13} F_{23},
$$

and similar property involving $(\text{id} \otimes \Delta)$. To satisfy the twist equation, these identities are combined with additional requirement $F_{12} F_{23} = F_{23} F_{12}$ or the Yang-Baxter equation on $F$ [14, 15].

An important subclass of factorizable twists consists of elements satisfying the following equations

$$
(\Delta \otimes \text{id})(F) = F_{13} F_{23},
$$

(8)
(id ⊗ Δt)(F) = F_{12}F_{13}. \quad (9)

It is easy to see that the universal $R$-matrix $R$ satisfies these equations, for $Δ_t = Δ^{op}$.

Another well developed case is the jordanian twist of $sl(2)$ with $F$ \cite{10}. Due to the fact that the Cartan element $h$ is primitive in $sl(2): Δ(h) = h ⊗ 1 + 1 ⊗ h$, and $σ$ is primitive in the jordanian $U_t(sl(2)): Δ_t(σ) = σ ⊗ 1 + 1 ⊗ σ$, one gets

\[
(Δ ⊗ id)e^{h⊗σ} = e^{h⊗1⊗σ} e^{1⊗h⊗σ},
\]

\[
(id ⊗ Δ_t)e^{h⊗σ} = e^{h⊗σ⊗1} e^{h⊗1⊗σ}.
\]

It will be shown in the next Sec.3 that the element $F$ \cite{2} also satisfies the factorization equations (8),(9) and can be used to twist the universal enveloping algebra of $sl(N)$.

Let us mention, that the composition of appropriate twists can be defined $F = F_2F_1$. The element $F_1$ has to satisfy the twist equation with the coproduct of the original Hopf algebra, while $F_2$ must be its solution for $Δ_{t_1}$ of the intermediate Hopf algebra twisted by $F_1$. In particular, if $F$ is a solution to the twist equation (3) then $F^{-1}$ satisfies this equation with $Δ → Δ_t$.

3 Factorizable twists

Now we shall propose a new factorizable twist similar to (1) and defined on the abstract set of generators.

Let $L$ be a four dimensional Lie algebra with generators $\{H, A, B, E\}$ containing $B_+$ and representable in a form of semidirect sum of one dimensional space $V_H$ with basic element $H$ and a Heisenberg subalgebra $H(A, B, E): L = V_H \triangleright H$:

\[
[H, E] = 2E, \quad [H, A] = \alpha A, \quad [H, B] = \beta B, \quad \alpha + \beta = 2, \quad [E, A] = [E, B] = 0, \quad [A, B] = \gamma E. \quad (10)
\]

Extending the twist deformation $U_t(B_+)$ performed by

\[
Φ = \exp(\frac{1}{2}H \otimes \ln(1 + γE)) = e^{H⊗σ}
\]
to the universal enveloping $\mathcal{U}(L)$ one gets the twisted algebra $\mathcal{U}_\Phi(L)$ . It retains the initial multiplication defined by (10) while its coproduct $\Delta_\Phi = \Phi \Delta \Phi^{-1}$ becomes noncocommutative:

\[
\begin{align*}
\Delta_\Phi(H) &= H \otimes e^{-2\sigma} + 1 \otimes H, \\
\Delta_\Phi(A) &= A \otimes e^{\alpha \sigma} + 1 \otimes A, \\
\Delta_\Phi(B) &= B \otimes e^{\beta \sigma} + 1 \otimes B, \\
\Delta_\Phi(E) &= E \otimes e^{2\sigma} + 1 \otimes E,
\end{align*}
\]

(11)

We shall show that the algebra $\mathcal{U}(L)$ allows a more complicated twist deformation containing $\Phi$ as a factor.

**Proposition.** The element

\[
\mathcal{F} = \Phi \Phi_1 = \exp(H \otimes \sigma) \exp(A \otimes B e^{-2\sigma})
\]

(12)

is a twist for $\mathcal{U}(L)$ .

**Proof.** We shall show that $\mathcal{F} = \Phi \Phi_1$ belongs to the subclass defined by the equations (8,9). The equation (8) is obviously true: $H$ and $A$ are the primitive elements and $B$ commutes with $\sigma$ in $\mathcal{U}(L)$. To check the second equation (9) let us consider the coproducts $\Delta_\mathcal{F}(\sigma)$ and $\Delta_\mathcal{F}(B)$ . It is known that in twisted (by $\Phi$ ) universal enveloping of Borel subalgebra the element $\sigma$ is primitive [10]. The element $\sigma$ commutes not only with $B$ but also with $A$, so $\sigma$ remains primitive with respect to $\Delta_\mathcal{F}$. Using the properties of ”roots” $\alpha - 2 = -\beta$ the twisted coproduct of $B$ can be written in the following form

\[
\begin{align*}
\Delta_\mathcal{F}(B) &= \exp \left( \text{ad} \left( A \otimes B e^{-\beta \sigma} \right) \right) \circ \exp \left( \text{ad} \left( H \otimes \sigma \right) \right) \circ (B \otimes 1 + 1 \otimes B) = \\
&= \exp \left( \text{ad} \left( A \otimes B e^{-\beta \sigma} \right) \right) \circ (B \otimes e^{\beta \sigma} + 1 \otimes B) \\
&= \exp \left( \text{ad} \left( A \otimes B e^{-\beta \sigma} \right) \right) \circ (B \otimes e^{\beta \sigma}) + 1 \otimes B.
\end{align*}
\]

From (11) one can see that $(\text{ad}_A)^2 \circ B = 0$. So the obtained expression can be simplified,

\[
\Delta_\mathcal{F}(B) = B \otimes e^{\beta \sigma} + (1 + [A, B]) \otimes B = B \otimes e^{\beta \sigma} + e^{2\sigma} \otimes B.
\]

Now using the coproduct

\[
\Delta_\mathcal{F}(B e^{-2\sigma}) = B e^{-2\sigma} \otimes e^{-\alpha \sigma} + 1 \otimes B e^{-2\sigma}
\]
one can easily see that
\[ \exp (\text{ad} (H \otimes 1 \otimes \sigma)) \circ (A \otimes B e^{-2\sigma} \otimes e^{-\alpha \sigma}) = A \otimes B e^{-2\sigma} \otimes 1. \]

The latter garanties the validity of the equation (9) for the twisting element \( \mathcal{F} \).

The deformed algebra \( \mathcal{U}_\mathcal{F} (\mathbf{L}) \) has initial commutation relations generated by (10) and twisted coproducts:
\[
\begin{align*}
\Delta_\mathcal{F} (H) &= H \otimes e^{-2\sigma} + 1 \otimes H - 2A \otimes B e^{(\alpha-4)\sigma}, \\
\Delta_\mathcal{F} (A) &= A \otimes e^{-\beta \sigma} + 1 \otimes A, \\
\Delta_\mathcal{F} (B) &= B \otimes e^{\beta \sigma} + e^{2\sigma} \otimes B, \\
\Delta_\mathcal{F} (E) &= E \otimes e^{2\sigma} + 1 \otimes E,
\end{align*}
\]

(13)

Let us rewrite the twist element \( \mathcal{F} \) in the reverse order:
\[ \mathcal{F} = \tilde{\Phi}_1 \Phi = \exp (A \otimes B e^{-\beta \sigma}) \exp (H \otimes \sigma) \]

(14)

Now we know that both \( \mathcal{F} \) and \( \Phi \) are twists for \( \mathcal{U} (\mathbf{L}) \) and both satisfy the equations (8),(9). Hence \( \tilde{\Phi}_1 \) is also a twist element with respect to the algebra \( \mathcal{U}_\Phi (\mathbf{L}) \). Using the coalgebra relations (11) it is easy to check that \( \tilde{\Phi}_1 \) satisfies the general twist equation (12),
\[ (\tilde{\Phi}_1)_{12} (\Delta_\Phi \otimes \text{id}) \tilde{\Phi}_1 = (\tilde{\Phi}_1)_{23} (\text{id} \otimes \Delta_\Phi) \tilde{\Phi}_1. \]

Note that contrary to the properties of \( \mathcal{F} \) and \( \Phi \) this twist \( \tilde{\Phi}_1 \) does not belong to the subclass of factorizable twists defined by the equations (8),(9).

Subalgebras of the type \( \mathbf{L} \) exist in a large class of Lie algebras. They can also be found in any simple Lie algebra of rank greater than 1. Such simple algebras contain at least one pair of roots \( \lambda_1 \) and \( \lambda_2 \) such that \( \lambda_3 = \lambda_1 + \lambda_2 \) is also a root. The corresponding generators \( X_1, X_2, X_3 \) together with the Cartan element \( H_3 \) dual to the root \( \lambda_3 \) form the subalgebra equivalent to \( \mathbf{L} \). As we have shown above such subalgebra can be twisted with the element \( \mathcal{F} \) and the corresponding deformation can be extended to the whole algebra \( \mathcal{U} \) and its twisted version \( \mathcal{U}_\mathcal{F} \) can be thus constructed.

We shall demonstrate the deformations generated by these twists in case of simple algebras of series \( A_{N-1} \). For our purposes it will be convenient to use the canonical
basis of $gl(N)$ for the compositions of $U(sl(N))$

$$[E_{ik}, E_{lm}] = \delta_{kl} E_{im} - \delta_{im} E_{lk}, \quad i, k, l, m = 1, \ldots N.$$  \hfill (15)

The Cartan elements of $U(sl(N))$ will be fixed as $H_{ik} = E_{ii} - E_{kk}$.

Let $H \in L$ be identified with the Cartan element dual to the highest root of $sl(N)$, this root will be denoted by $\lambda_H$. Collecting all the pairs of roots with the property $\lambda_H = \lambda_1 + \lambda_2$ one can get the multiparametric twist of the type $F$ with

$$H = H_{1N}, \quad E = E_{1N},$$

$$A = \sum_k \left( a^{1k} E_{1k} + a^{kN} E_{kN} \right),$$

$$B = \sum_k \left( b^{1k} E_{1k} + b^{kN} E_{kN} \right),$$

$$\{ \ k = 2, \ldots, N - 1; \ a^{mn}, b^{nm} \in \mathbb{C} \ \},$$

Here it is convenient to put $\gamma = 2\xi$,

$$\sigma(E) = \frac{1}{2} \ln (1 + 2\xi E).$$

In these terms the consistency condition would take the form

$$[A, B] = e^{2\sigma} - 1 = 2\xi E$$

and the only nontrivial commutator of $\sigma$ with the basic elements of $L$ is

$$[H, \sigma] = 1 - e^{-2\sigma}. \hfill (19)$$

According to the Proposition the element

$$F = \Phi \Phi_1 = \exp(H \otimes \sigma) \exp(A \otimes Be^{-2\sigma})$$

is a twist of $U(sl(N))$. Using the particular properties of $L$ one can apply the Cambell-Hausdorf formula to rewrite the twisting element in the following form

$$F = \exp(A \otimes Be^{-\sigma}) \exp(H \otimes \sigma)$$

$$= \exp \left( H \otimes \sigma + A \otimes B \sigma e^{-2\sigma} (1 - e^{-\sigma})^{-1} \right). \hfill (21)$$
Note. Any number of factors of the type $\Phi_1$ can appear in the expression (20):

$$
F = \Phi \prod_j \Phi_j = \exp(H \otimes \sigma) \prod_j \exp(A_j \otimes B_j e^{-2\sigma})
$$

(22)

with $A_j$ and $B_j$ as in (17) and (16) and subject to the additional conditions

$$
[A_{j_1}, A_{j_2}] = [B_{j_1}, B_{j_2}] = 0,
$$

while the correlation equation (18) takes the form

$$
[A_j, B_k] = \delta_{jk} \left( e^{2\sigma} - 1 \right).
$$

(23)

Using the twist (20) with the sole factor $\Phi_1$ one gets the maximal number of free parameters – the relation (18) imposes the only condition on the coefficients $a$’s and $b$’s,

$$
\sum_{k=2}^{N-1} \left( a^{1k} b^{kN} - a^{kN} b^{1k} \right) = 2\xi.
$$

(24)

On the contrary, supplying $F$ with the maximal number $(N-2)$ of factors $\Phi_j$ one gets the $(N-2)^2$ conditions (23). In particular one can satisfy $(N-2)(N-3)$ of these conditions using the basic relations (15) and the specific choice of $A_j$ and $B_j$ (one root $\lambda_j$ for each factor $\Phi_j$):

$$
A_j = a^{1j} E_{1j} + a^{iN} E_{jN}, \quad B_j = \left( b^{1j} E_{1j} + b^{iN} E_{jN} \right),
$$

(with no summation on $j$).

(25)

Here the essential relations rest

$$
a^{1j} b^{jN} - a^{jN} b^{1j} = 2\xi, \quad \{ j = 2, ..., N-1 \}.
$$

(26)

Equation (24) (as well as (26)) shows that it is natural to renormalize the element $A$ (or the elements $A_j$) putting

$$
A = 2\xi \tilde{A}
$$

so that

$$
[\tilde{A}, B] = E.
$$
In these notations the twisting elements

\[ F = \exp(H \otimes \sigma) \exp(2\xi \tilde{A} \otimes Be^{-2\sigma}), \] (27)

\[ F = \exp(H \otimes \sigma) \prod_j \exp(2\xi \tilde{A}_j \otimes B_j e^{-2\sigma}) \] (28)

have the trivial limit \( \lim_{\xi \to 0} F = 1 \). So does the universal \( R \)-matrix \( (R = F_{21}F^{-1}) \) and one can easily write down the corresponding classical \( r \)-matrices

\[ r = -\left( H \wedge E + 2\tilde{A} \wedge B \right) \] (29)

or

\[ r = -\left( H \wedge E + 2 \sum \tilde{A}_j \wedge B_j \right). \] (30)

Their form clearly indicates that twisting by \( F \) corresponds to the quantization of the self-dual Lie bialgebra \( (\mathbf{L}, \mathbf{L}^*) \approx \mathbf{L} \) just as in the case of the jordanian twist of \( \mathbf{B}(1) \) \[ \[ \text{[11, 12]. The same is true for the twisted Hopf algebra} \mathcal{U}_F(\mathbf{B}^\vee), \text{it is self-dual. We shall discuss this property in the next Section and prove it in the Section 5 where the canonical element will be constructed.} \]

For the special case of \( \mathcal{U}(sl(N)) \) according to the Proposition the following form of twisting element \( F \) can be chosen

\[ F = \exp \left( H_{1N} \otimes \sigma \right) \prod_{j=2}^{N-1} \exp \left( 2\xi E_{1j} \otimes E_{jN} e^{-2\sigma} \right). \]

This twist of \( \mathcal{U}(sl(N)) \) is generated by the twist of \( \mathcal{U}(\mathbf{L}) \) (here \( \mathbf{L} \) is the restricted Borel subalgebra \( \mathbf{B}^\vee \) of \( sl(N) \) with the basic elements \( \{ H_{1N}, E_{1N}, E_{1j}, E_{jN} \}_{j=2,...,N-1} \) leading to the Hopf algebra \( \mathcal{U}_f(\mathbf{B}^\vee) \) with the initial commutation relations (as in (31)), the twisted coproducts

\[ \Delta_F H_{1N} = H_{1N} \otimes e^{-2\sigma} + 1 \otimes H_{1N} - 4\xi \sum_{j=2}^{N-1} E_{1j} \otimes E_{jN} e^{-3\sigma}, \]

\[ \Delta_F E_{1i} = E_{1i} \otimes e^{-\sigma} + 1 \otimes E_{1i}, \]

\[ \Delta_F E_{iN} = E_{iN} \otimes e^{\sigma} + e^{2\sigma} \otimes E_{iN}, \]

\[ \Delta_F E_{1N} = E_{1N} \otimes e^{2\sigma} + 1 \otimes E_{1N}, \] (31)
antipodes

\[ S_F(\sigma) = -\sigma, \quad S_F(E_{1i}) = -E_{1i}e^{\sigma}, \]
\[ S_F(E_{iN}) = -E_{iN}e^{-3\sigma}, \quad S_F(E_{1N}) = -E_{1N}e^{-2\sigma}, \]
\[ S_F(H_{1N}) = -H_{1N}e^{2\sigma} - 4\xi \sum_{j=2}^{N-1} E_{1j}E_{jN} \]

and the universal \( R \)-matrix of the form

\[ R = F_{21}F^{-1} \]
\[ = \Pi_j \exp (2\xi E_{jN}e^{-\sigma} \otimes E_{1j}) \exp (\sigma \otimes H_{1N}) \exp (-H_{1N} \otimes \sigma) \Pi_j \exp (-2\xi E_{1j} \otimes E_{jN}e^{-\sigma}). \]

The coproducts and antipodes for other elements of \( U_\xi(sl(N)) \) can be calculated using the standard formulas. The obtained expressions are rather cumbersome. Thus, for example, in the case of \( U_\xi(sl(3)) \) the coproduct of \( E_{32} \) looks like

\[ \Delta_F E_{32} = E_{32} \otimes e^{-\sigma} + 1 \otimes E_{32} \]
\[ + \xi H_{13} \otimes E_{12}e^{-2\sigma} + 2\xi E_{12} \otimes H_{23}e^{-\sigma} \]
\[ -\xi H_{13}E_{12} \otimes (e^{-\sigma} - e^{-3\sigma}) \]
\[ -4\xi^2 E_{12} \otimes E_{23}E_{12}e^{-3\sigma} \]
\[ -4\xi^2 E_{12}^2 \otimes E_{23}e^{-4\sigma}. \]

Twisting the coproducts is acting by the exponential of the adjoint operator defined on the tensor product \( U(sl(N)) \otimes U(sl(N)) \). One can check that this operator is nilpotent and all the twisted coproducts can be expressed through the finite number of its powers.

4 Connections between standard and jordanian deformations

It is well known that some sorts of jordanian deformations can be treated as limiting structures for certain sequences of standard quantizations \[8, 9, 10, 11, 12, 13\]. As will be shown below this is due to the specific properties of Lie bialgebras involved in the quantizations. These properties are more transparent when formulated for quantum groups rather than for quantum algebras. For this reason in the current Section we use the dual picture to treat Lie bialgebraic characteristics.
The generators of the standard (FRT-deformed) quantum group $\text{Fun}_h(SL(N))$ ($h = \ln q$) will be described by the entries of the $N \times N$-matrix $T$. Let $T$ be subject to the similarity transformation with the matrix

$$M = 1 + \frac{\xi}{q-1} \rho(E_{1N})$$

(for the generators the canonical coproduct ($\Delta T = T \otimes T$) is conserved). As far as $q \neq 1$ the transformed quantum group $\text{Fun}_{h;\xi}(SL(N))$ is equivalent to the original one. Compare the corresponding Lie bialgebras: $\left(g, g^*_0\right) = (sl(N), (sl(N))^*)$ and $\left(g, g^*_\xi\right)$. Here the Lie algebra $g = sl(N)$ is not changed, the transformation $T \rightarrow MTM^{-1}$ does not touch the canonical coproduct for the generators of the Hopf algebra $\text{Fun}_h(SL(N))$. Only the second Lie multiplication $\left(\mu^*_0 : V_{g^*} \wedge V_{g^*} \rightarrow V_{g^*}\right)$ changes:

$$\mu^*_0 \rightarrow \mu^*_\xi.$$ 

The structure of the similarity transformation shows that the new Lie product decomposes as:

$$\mu^*_\xi = \mu^*_0 + \xi \mu'.$$

The component $\mu'$ is fixed by the commutation relations that can be extracted from the transformed $RTT = TTR$ equations. For this purpose one has to change the coordinate functions of $SL(N)$ arranged in matrix $T$ for the exponential ones $T = \exp(\epsilon Y)$ and also change the parameters $h \mapsto \epsilon h$, $\xi \mapsto \epsilon \xi$. Tending $\epsilon$ to zero one gets both summands in (35). The second one of them looks as follows:

$$\begin{align*}
\mu'(Y_{ik}, Y_{ij}) &= 2\delta_{ik}Y_{Nj}, \text{ for } k, j < N; \quad i > 1, \\
\mu'(Y_{ij}, Y_{IN}) &= -2\delta_{ij}Y_{Nj}, \text{ for } j < N; \quad i, l > 1, \\
\mu'(Y_{ij}, Y_{1N}) &= -\delta_{j1}Y_{i1} - \delta_{iN}Y_{Nj}, \text{ for } j < N; \quad i > 1, \\
\mu'(Y_{1i}, Y_{IN}) &= -Y_{1i}, \text{ for } i > 1, \\
\mu'(Y_{1N}, Y_{kN}) &= Y_{kN}, \text{ for } k < N, \\
\mu'(Y_{11}, Y_{1N}) &= \mu'(Y_{1N}, Y_{NN}) = -(Y_{11} - Y_{NN}), \\
\mu'(Y_{1i}, Y_{1k}) &= \delta_{i1}Y_{Nk}, \text{ for } k, i < N, \\
\mu'(Y_{1N}, Y_{kN}) &= -\delta_{kN}Y_{i1}, \text{ for } k, i > 1, \\
\mu'(Y_{i1}, Y_{kN}) &= \delta_{i1}Y_{k1} - \delta_{kN}Y_{Ni} - 2\delta_{ik}(Y_{11} - Y_{NN}), \text{ for } i < N; \quad k > 1.
\end{align*}$$

(36)
Here for simplicity of exposition we use the canonical $gl(N)$-basis. One can check that this deforming function $\mu'$ not only defines the infinitesimal deformation of $\mu_{h,0}^*$ but is itself a Lie multiplication.

Consider the decomposition (33) as a deformation equation for the original dual Lie algebra $g_{h,0}^*$. Its main property is that $\mu'$ does not depend on $h$ or $\xi$. So the transformed law has the form

$$\mu_{h,\xi}^* = \mu_{h,0}^* + \mu_{0,\xi}^*.$$  \hspace{1cm} (38)

This means that $\mu_{h,\xi}^*$ is a Lie multiplication deformed in the first order. Both summands are Lie maps and at the same time can be considered as deforming functions of each other. As a result both deforming functions are Hochschild 2-cocycles for the corresponding Lie algebras ($g_{0,\xi}^*$ with the multiplication $\mu_{0,\xi}^*$ and $g_{h,0}^*$ defined by $\mu_{h,0}^*$)

$$\mu_{h,0}^* \in Z^2 \left( g_{0,\xi}^*, g_{0,\xi}^* \right),$$

$$\mu_{0,\xi}^* \in Z^2 \left( g_{h,0}^*, g_{h,0}^* \right).$$

The equivalence of the algebraic structures in $Fun_{h,\xi} (SL(N))$ and $Fun_h (SL(N))$ (for $h \neq 0$) signifies that $\mu_{0,\xi}^*$ is in fact a coboundary,

$$\mu_{0,\xi}^* \in B^2 \left( g_{h,0}^*, g_{h,0}^* \right).$$

On the contrary, the composition $\mu_{h,0}^*$ corresponds to a nontrivial cohomology class

$$\mu_{h,0}^* \in H^2 \left( g_{h,\xi}^*, g_{0,\xi}^* \right),$$

the deformation of $\mu_{0,\xi}^*$ by $\mu_{h,0}^*$ is essential [19].

Notice that the multiplication maps here have certain cohomological properties also with respect to cochain complex $C$ of maps $C^n : \wedge^n V_g \to V_g \wedge V_g$, where the $g$-module is chosen to be $\wedge V_g$ with the canonically extended adjoint action on it. The dualization of spaces $V_g \leftrightarrow V_g^*$ converts the map $\mu^*$ into the chain $\mu^* \in C^1 (g, g \wedge g)$. As it was mentioned above the initial coproduct for the generators of $Fun_h (SL(N))$ rests unchanged under the transformation. All the Lie algebras $g_{h,\xi}^*$ are dual to one and the same $g = sl(N)$. Thus both $\mu_{h,0}^*$ and $\mu_{0,\xi}^*$ are 1-cocycles for the complex $C$.  

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This set of characteristics necessarily indicates that the classical \( r \)-matrix of \( \mathcal{U}_{\hbar \xi}(sl(N)) \approx (Fun_{\hbar \xi}(SL(N)))^* \) must also exhibit this decomposition property:

\[
\begin{align*}
\mathcal{R}_{\hbar \xi} &= \mathcal{R}_{\hbar 0} + \mathcal{R}_{0 \xi} = \frac{k}{N} \left( \sum_{k=1}^{N-1} k(N-k) H_{k,k+1} \otimes H_{k,k+1} \right) \\
&\quad + \sum_{k<l}(N-l)k(H_{k,l+1} \otimes H_{l,l+1} + H_{l,l+1} \otimes H_{k,k+1}) \\
&\quad + 2\hbar \sum_{k<l}(E_{lk} \otimes E_{kl}) \\
&\quad - \xi H_{1N} \wedge E_{1N} - 2\xi \sum_{k=2}^{N-1} E_{1k} \wedge E_{kN}.
\end{align*}
\]

(39)

In the limit \( \hbar \to 0 \) one gets the element

\[
\lim_{\hbar \to 0} \mathcal{R}_{\hbar \xi} = \mathcal{R}_{0 \xi} = -\xi \left( H_{1N} \wedge E_{1N} + 2 \sum_{k=2}^{N-1} E_{1k} \wedge E_{kN} \right).
\]

(40)

that coincides with \( r \)-matrix that can be obtained from \( \mathcal{R} \) presented above (33). So the jordanian quantum group \( Fun_{0 \xi}(SL(N)) \) has the same \( R \)-matrix as the twisted algebra \( \mathcal{U}_\mathcal{F}(sl(N)) \) (with \( \mathcal{F} \) as in (28) and \( \tilde{A}_j, B_j \) as in (27),(26)).

The \( r \)-matrices (33) and (10) are known for a long time. In [9] \( r_{0 \xi} \) was obtained by applying \( \text{ad}_{E_{1N}} \) to the canonical antisymmetric \( r_\wedge = \sum_{i<j} E_{ij} \wedge E_{ji} \). It was stressed that \( r_{0 \xi} \) lay in the boundary of the dense set of orbits of standard quantizations induced by \( r_\wedge \). The \( r \)-matrix (40) was also obtained in the discussion of conformal algebra deformations [20].

The \( r \)-matrix \( r_{0 \xi} \) is the element of the space \( B^\vee \wedge B^\vee \). Its structure suggests the renumeration of the basic elements of \( B^\vee \); we shall describe the corresponding basis as the set

\[
\{P_\alpha, X_\beta\}_{\alpha,\beta=1,\ldots,N-1} \quad \text{with} \quad \begin{cases} P_1 = E_{1N}, \quad P_i = E_{iN}; \\
X_1 = H_{1N}, \quad X_j = 2E_{1j}; \end{cases} \quad i, j = 2, \ldots, N - 1.
\]

In these notations \( r_{0 \xi} \) takes the form

\[
r_{0 \xi} = -\xi X_\alpha \wedge P_\alpha.
\]

The basic exponential coordinate functions \( \{Y_{1N}, Y_{iN}, Y_{11} - Y_{NN}, Y_{ii}\} \) are chosen so that they are canonically dual to those of \( \{P_\alpha, X_\beta\} \). Let us apply the homomorphism

\[
r_{0 \xi} : Y \to -\xi X_\alpha \wedge \langle P_\alpha, Y \rangle
\]

(41)
to the Lie algebra $(B^\vee)^*$ described by the last six compositions $\mu'$ (see (37)). As a result we shall get the Lie algebra $B^\vee$. The significant fact is that (41) is an isomorphism, that is $B^\vee \approx (B^\vee)^*$. The twist $F$ induces the self-dual Lie bialgebra $(B^\vee, B^\vee)$.

It is useful to compare this situation with that of a classical double of dual Lie algebras $(g, g^*)$. There the composition law of the double can also be presented as a sum of two multiplications with independent linear parameters. But in that case both summands are cohomologically nontrivial. What is more important – such parametrization (and subdivision) can not be performed in only one algebra of a Lie bialgebra $(g, g^*)$ corresponding to a classical double. In fact these are the Lie bialgebras that can be parametrized in that case so that their arbitrary linear combination is again a Lie bialgebra [21]. When a Lie bialgebra is nontrivially decomposed (that is the decomposition goes parallel in both dual algebras) the $r$-matrix for a linear combination of Lie bialgebras doesn’t inherit the decomposition property.

To clarify the contraction properties of $Fun_{h,\xi}(SL(N))$ let us consider the 1-parameter subvariety $\{g_{h,1-h}\}$ of Lie algebras $g^*_{h,\xi}$ (putting $\xi = 1 - h$ in (38)). Each dual pair $(sl(N), g^*_{h,1-h})$ is a Lie bialgebra and thus is quantizable [3]. The result is the set $A_{s,h}$ of deformation quantizations parametrized by $h$ and the deformation parameter $s$. This set can be considered smooth in the sense compatible with the formal series topology [22] – close Lie bialgebras give rise to close deformations. The 1-dimensional boundaries $A_{0,h}$ and $A_{s,0}$ of $A_{s,h}$ are formed respectively by the quantizations of $(sl(N), g^*_{0,0})$ (the standard Lie bialgebra) and $(sl(N), g^*_{0,1})$ (the jordanian one). Each internal point in $A_{s,h}$ can be connected with a boundary by a smooth parametric curve $a(u)$. One can choose it so that it starts in $A_{0,h}$ and ends in $A_{s,0}$. So a jordanian Hopf algebra obtained by twisting deformation can be also treated as a limit point of a smooth 1-dimensional subvariety $a(u)$. This does not necessarily mean that this limit is a faithful contraction – it may be impossible to attribute the curve $a(u)$ to an orbit of some Hopf algebra in $A$. This is just what happens when the transformation $M$ is applied to $Fun_{h,0}(SL(N))$. For every positive $h$ fixed the subset $\{Fun_{h,\xi}(SL(N))\}$ is in the $GL(N^2)$-orbit of the corresponding $Fun_{h,0}(SL(N))$. To attain the points $Fun_{0,\xi}(SL(N))$ one must tend $h$ to zero and this can be done only by crossing the set of orbits referring to inequivalent
Hopf algebras. These specific interrelations of different types of quantizations where noted in [9]. It was demonstrated for the case of $sl(N)$ that the standard deformation $Fun_{h,0}(SL(N))$ can be accompanied by a smooth transformation of a jordanian deformation so that the latter reaches the orbit of $Fun_{h,0}(SL(N))$. Applying the operator $M$ to an element of the set $\{Fun_{h,0}(SL(N))\}$ one gets an intersection point of an orbit and of a curve parametrized by $\xi$.

One of the principle conclusions is that the possibility to obtain the jordanian deformation $Fun_{0;\xi}(SL(N))$ as a limiting transformation of the standard quantum group $Fun_{h,0}(SL(N))$ (and on the dual list to get the twisted $q$-algebra $U_{F}(sl(N))$ as a limit of the variety of standardly quantized algebras $U_{q}(sl(N))$) is provided by the fact that the 1-cocycle $\mu_{0;\xi}^{*} \in Z^{1}(sl(N), sl(N) \wedge sl(N))$ (that characterizes the Lie bialgebra for $U_{F}(sl(N))$) is at the same time the 2-coboundary $\mu_{0;\xi}^{*} \in B^{2}(g_{h,0}^{*}, g_{h,0}^{*})$ the Lie algebra $g_{h,0}^{*}$ being the standard dual of $sl(N)$.

5 Canonical element and jordanian quantum space

The set $\{P_{\alpha}, X_{\beta}\}$ forms the basis appropriate to deal with the Lie bialgebras $(L, L^{*})$. To study the properties of $R$-matrix $R$ and the canonical element $T$ it is reasonable to perform the corresponding rearrangement of basis for the whole Hopf algebra $U_{\xi}(B^{\vee})$. We shall consider the set

$$\{z_{k}\}_{k=1,...,2(N-1)} = \{x_{\alpha}, \pi_{\beta}\}_{\alpha,\beta=1,...,N-1}$$  \hspace{1cm} (42)

as the generators of $U_{\xi}(B^{\vee})$ with

$$x_{1} = H_{1N}, \hspace{0.5cm} x_{i} = 2E_{1i}, \hspace{1cm} \pi_{1} = \frac{1}{4\xi} \sigma = \frac{1}{2\xi} \ln (1 + 2\xi E_{1N}), \hspace{0.5cm} \pi_{i} = E_{iN}e^{-2\sigma}.$$  \hspace{1cm}

The basis will be formed by the set of ordered monomials:

$$\left\{z_{\vec{k}}\right\}_{\vec{k}=(\vec{m},\vec{n})=\{m_{1},...,m_{N-1},n_{1},...,n_{N-1}\}} = \left\{x_{1}^{m_{1}} \cdots x_{N-1}^{m_{N-1}} \pi_{1}^{n_{1}} \cdots \pi_{N-1}^{n_{N-1}} \right\}.$$  \hspace{1cm} (43)

In these terms the $R$-matrix (33) can be rewritten as

$$R = \prod_{\alpha=1,...,N-1}^{<} \exp(\pi_{\alpha} \otimes \xi x_{\alpha}) \prod_{\alpha=1,...,N-1}^{>} \exp(-\xi x_{\alpha} \otimes \pi_{\alpha}),$$  \hspace{1cm} (44)
We shall use the standard Hopf algebra homomorphism $\mathcal{R}: \mathcal{A}^* \to \mathcal{A}_-$ where in our case $\mathcal{A}$ is the twisted algebra $\mathcal{U}_\xi (\mathcal{B}^\vee)$ and "−" indicates the opposite multiplication. It would be appropriate to consider $\mathcal{R}$ as belonging to $\mathcal{A}_- \otimes \mathcal{A}$ with the decomposition

$$\mathcal{R} = \sum R^k_l y_k \otimes z_l. \quad (45)$$

It is implied that the basic monomials $y_k \in \mathcal{A}_-$ contain the same sequences of generators $z_k$ as the corresponding basic elements $z_k \in \mathcal{A}$ (see (43)) but the multiplication that glue them is opposite to that of $\mathcal{A}$. Let $\{z_k\} = \{x^\alpha, \pi^\beta\}$ be the canonical dual basis of $\mathcal{A}^*$, $\langle z_k, z_l \rangle = \delta^k_l$. The morphism $\mathcal{R}$ can be defined by its values on the basic elements:

$$\mathcal{R} (z_k) = \sum R^k_l y_l \quad (46)$$

Let us extract the first terms of the decomposition (45) for the $\mathcal{R}$-matrix (44)

$$\mathcal{R} = 1 \otimes 1 + R^{kl} z_k \otimes z_l + \ldots \quad (47)$$

(Note that in such a presentation the second term is not proportional to the classical $r$-matrix, the generators $z_l$ do not form a Lie algebra.) The terms written explicitly in (47) are the only ones containing the first powers of generators. Thus the images $\mathcal{R} (z_k)$ are the linear combinations of the generators $z_l$. In our case the matrix $\{R^{kl}\}$ is invertible,

$$R = \xi \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \Rightarrow \begin{cases} \mathcal{R} (x^\alpha) = \xi \pi^\alpha \\ \mathcal{R} (\pi^\alpha) = -\xi x^\alpha \end{cases}$$

$$R^{-1} = -\frac{1}{\xi^2} R \Rightarrow \begin{cases} \mathcal{R}^{-1} (x^\alpha) = -\frac{1}{\xi} \pi^\alpha \\ \mathcal{R}^{-1} (\pi^\alpha) = \frac{1}{\xi} x^\alpha \end{cases}.$$ 

Reversing the formula (46) we get the expression for the elements of the dual basis in terms of generators $z^k_{|k=1,\ldots,N-1}$:

$$z_k = \sum R^k_l \mathcal{R}^{-1} (y_l) = \sum R^k_l \left( (R^{-1})_{k1} z^{k_1} \right)^{l_1} \ldots \left( (R^{-1})_{k_{2(N-1)}2(N-1)} z^{k_{2(N-1)}} \right)^{l_{2(N-1)}}. \quad (48)$$
The basic decomposition for the $R$-matrix (44) can be written explicitly,

$$ R = \sum (\xi_{\vec{n}})(\xi_{\vec{m}}) x_1^{n_1} \ldots x_{N-1}^{n_{N-1}} (\pi_1)^{m_1} \ldots (\pi_{N-1})^{m_{N-1}} \otimes 
\otimes x_1^{m_1} \ldots x_{N-1}^{m_{N-1}} (\pi_1)^{n_1} \ldots (\pi_{N-1})^{n_{N-1}}, 
|\vec{n}| = n_1 + \ldots + n_{N-1}; \quad \vec{n}! = n_1! n_2! \ldots n_{N-1}!, $$

Here we used the inclusion $R \in A_- \otimes A$ and the fact that all the generators $\pi_\alpha$ commute. The structure of $R$-morphism is clearly seen here. It states the one-to-one correspondence between the basic monoms of $A^*$ and $A_-$. This evidently signifies that the Hopf algebras $A^*$ and $A_-$ are equivalent. One must also take into account that in our case $A_-$ is the twisted universal enveloping algebra $U_{\xi}(B^\vee)$ with the opposite product. Hence it is isomorphic to its multiplicatively inverse. The result is

$$ A^* \approx A_- \approx A $$

The Hopf algebra $U_{\xi}(B^\vee)$ is self-dual.

The structure constants $R_{\vec{k}\vec{l}}$ presented in the decomposition (49) can be substituted in the expression (48) to fix explicitly the form of the dual basis. Hence the canonical element $T$ is completely defined

$$ T = \sum_{\vec{k} = (\vec{m}, \vec{n})} z_{\vec{k}} \otimes z_{\vec{k}} = \sum_{\min} (\pi_1)^{n_1} \ldots (\pi_{N-1})^{n_{N-1}} (x_1)^{m_1} \ldots (x_{N-1})^{m_{N-1}} \otimes 
\otimes x_1^{m_1} \ldots x_{N-1}^{m_{N-1}} \pi_1^{n_1} \ldots \pi_{N-1}^{n_{N-1}}. $$

We can recollect this expansion into the ordered product using the following property of the $T$-matrix: $(id \otimes S)(T) = T^{-1}$.

$$ T^{-1} = \sum_{(\vec{m}, \vec{n})} \frac{1}{\min} (\pi_1)^{n_1} \ldots (\pi_{N-1})^{n_{N-1}} (x_1)^{m_1} \ldots (x_{N-1})^{m_{N-1}} \otimes 
\otimes (S(\pi_{N-1}))^{n_{N-1}} \ldots (S(\pi_1))^{n_1} (S(x_1))^{m_1} \ldots (S(x_{N-1}))^{m_{N-1}}. $$

The antipodes used here can be easily found using the expressions (32) given in Sec.3:

$$ S(\pi_1) = -\pi_1, \quad S(\pi_i) = -\pi_i e^{2\xi \pi_1}, $$

$$ S(x_1) = -x_1 e^{2\xi \pi_1} - 4\xi \sum x_i \pi_i e^{2\xi \pi_1}, \quad S(x_i) = -x_i e^{\xi \pi_1}. $$

Note that the homomorphic image in $A^*$ of the abelian subalgebra generated by elements $\{\pi_\alpha\}$ is itself a commutative subalgebra. This enables us to write the final
expression for the canonical element

$$\mathcal{T} = \prod_\alpha \exp(-x^\alpha \otimes S(x^\alpha)) \prod_\alpha \exp(-\pi^\alpha \otimes S(\pi^\alpha)).$$  \hspace{1cm} (50)

The corresponding constructions for jordanian deformations of the Lie superalgebra $sl(M|N)$ can be found in [23].

Let us present a real form for $\mathcal{U}_F(sl(N))$. We focus first on the subalgebra $B^\vee$ in the general setting of the previous section and with the basis $\{z_k\}$ (see (42)). The anti-algebraic anti-linear transformation given on the generators by

$$\theta(x^\alpha) = -x^\alpha, \quad \theta(\pi^\alpha) = \pi^\alpha$$

respects the classical comultiplication and defines a real form on $\mathcal{U}(L)$. At the same time, the twisting element $F$ turns into $F^{-1}$. Henceforth, $\theta$ is a real form (cohomomorphic and anti-homomorphic) for the twisted algebra $\mathcal{U}_F(L)$ as well. Turning to the specific case of $sl(N)$, the question is whether $\theta$ can be extended from the subalgebra $B^\vee$ to the entire $sl(N)$. This is possible, and the corresponding transformation is

$$\theta(E_{ij}) = -E_{ij}, \quad i, j < N \text{ or } i, j = N; \quad \theta(E_{kN}) = E_{kN}, \quad \theta(E_{Nk}) = E_{Nk}, \quad k < N.$$ 

It is evident that $\theta$ is a Lie algebra anti-automorphism. The real form for $N = 2$ case of the jordanian $\mathcal{U}_\xi(sl(2))$ was given in [24].

Twisting of a symmetry Hopf algebra $\mathcal{A}$ of a manifold $\mathcal{M}$ induces deformation of its whole geometry, so that the notion of symmetry is preserved in the framework of the non-commutative geometry. Such deformation includes that of function algebras (vector bundles, $\ast$-structure, and so on) expressing new objects in terms of the untwisted ones by explicit formulas involving twisting 2-cocycle $F$. Here we present, as an application of the developed jordanian-type quantization of $sl(N)$, the corresponding noncommutative space $\mathcal{M}_F$. We deduce commutation relations for generators of $\mathcal{M}_F$, and the differential calculus. The basic formula connecting multiplications in $\mathcal{A}$-modules $\mathcal{M}_F$ and $\mathcal{M}$ (the twisted and the untwisted ones) is

$$f \ast g = \mathcal{F}^{-1}_{(1)}(f) \cdot \mathcal{F}^{-1}_{(2)}(g), \quad f, g \in \mathcal{M}.$$  \hspace{1cm} (51)
The star stands for the new product on $\mathcal{M}_F$ defined through the old one $\cdot$ and the element $F$. If $\mathcal{M}$ is classical, the twisting cocycle is represented by a bidifferential operator according to the correspondent representation of $F$. Thus $\mathcal{M}_F$ and $\mathcal{M}$ coincide as linear spaces but they are endowed with different algebraic structures. The transformation is performed in such a way that the symmetry property $h(f \cdot g) = h_1(f) \cdot h_2(g)$, $h \in \mathcal{A}$, $f, g \in \mathcal{M}$, is inherited by the twisted algebra $\mathcal{A}_F$.

Let $x^a$, $a = 1, \ldots, N$, be the generators of $\mathcal{M}_F$. To evaluate commutation relations among the generators, it is sufficient to retain only the following terms:

$$F = 1 \otimes 1 + \xi (x^1 \partial_1 - x^N \partial_N) \otimes x^1 \partial_N + 2\xi \sum_{k=2}^{N-1} x^1 \partial_k \otimes x^k \partial_N + \ldots,$$

with the rest of the series vanishing. Resolving formula (51) (twisting is an invertible operation) we come to

$$F_{(1)}(x^\mu) \ast F_{(2)}(x^\nu) = x^\mu \cdot x^\nu = x^\nu \cdot x^\mu = F_{(1)}(x^\nu) \ast F_{(2)}(x^\mu).$$

This gives (commutators are understood in terms of the twisted product $\ast$)

$$[x^1, x^N] = \xi x^N \ast x^N, \quad [x^i, x^k] = 0,$$

$$[x^1, x^k] = 2\xi x^k \ast x^N, \quad [x^k, x^N] = 0.$$

Hereafter (in this Section) the small Latin indices run from 2 to $N - 1$. Similarly, for the contravariant entities $p_\mu$ we have

$$[p_1, p_N] = \xi p_1 \ast p_1, \quad [p_i, p_k] = 0,$$

$$[p_k, p_N] = 2\xi p_1 \ast p_k, \quad [p_1, p_k] = 0.$$

Let us note that after the quantization the bases $\{p_\mu\}$ and $\{x^\mu\}$ are no longer conjugate. The invariant canonical element turns out to be $x^\mu \cdot p_\mu = x^\mu \ast p_\mu + \xi x^N \ast p_1$. Non-trivial cross-relations between coordinates and momenta are expressed by

$$[p_N, x^1] = \xi (p_N \ast x^N + 2 \sum_{k=2}^{N-1} p_k \ast x^k + p_1 \ast x^1 + \xi p_1 \ast x^N).$$
\[ [p_1, x^1] = -\xi p_1 \ast x^N, \quad [p_k, x^k] = -2\xi p_1 \ast x^N, \quad [p_N, x^N] = -\xi p_1 \ast x^N. \]

Partial derivatives \( \partial_\mu \) satisfies the same identities as \( p_\mu \), whereas the cross-relations are modified accordingly:

\[ [\partial_N, x^1] = \xi (x^N \ast \partial_N + 2 \sum_{k=2}^{N-1} x^k \ast \partial_k + x^1 \ast \partial_1 + \xi x^N \ast \partial_1) \]
\[ [\partial_1, x^1] = 1 - \xi x^N \ast \partial_1, \quad [\partial_k, x^k] = 1 + 2\xi x^N \ast \partial_1, \quad [\partial_N, x^N] = 1 - \xi x^N \ast \partial_N. \]

6 Group cocycles and twisting

To generalize the construction of Sec.3 let us arrange the generators of \( B^\vee \) into the two sets \((H, A_j)\) and \((E, B_j)\) spanning two mutually complement Lie subalgebras. We denote them \( H \) and \( H^* \), respectively, regarding as dual linear spaces. Subalgebra \( H \) acts on \( H^* \), thus endowing \( B^\vee \) with the semidirect sum \( L = H \triangleright H^* \) structure. In this section we establish the cohomological properties of the previous constructions in terms of the Lie algebra \( H \) and its formal Lie group \( G = \exp H \).

Let \( H_\mu \) be basic elements of a Lie algebra \( H \) and \( X^\nu \) be their conjugate. Commutation relations in \( H \) are specified by the structure constants \( C^\sigma_{\mu\nu} \):

\[ [H_\mu, H_\nu] = C^\sigma_{\mu\nu} H_\sigma. \quad (52) \]

Suppose a left action of \( H \) on \( H^* \)

\[ [H_\mu, X^\nu] = -L^\nu_{\mu\sigma} X^\sigma \quad (53) \]

which enables us to build the semidirect sum \( L = H \triangleright H^* \) where \( H^* \) is assumed to be an abelian subalgebra. The element

\[ r = X^\nu \otimes H_\nu - H_\nu \otimes X^\nu \in L \wedge L \quad (54) \]

is a solution to the classical Yang-Baxter equation if and only if

\[ C^\sigma_{\mu\nu} = L^\sigma_{\mu\nu} - L^\sigma_{\nu\mu}. \quad (55) \]
The structure constants $L_{\mu\nu}^\sigma$ define also a left action of $H$ on itself according to the rule

$$H_\mu \triangleright H_\nu = L_{\mu\nu}^\sigma H_\sigma.$$  

Equality (55) implies that the following quasi-associativity property holds

$$(H_\mu \triangleright H_\nu) \triangleright H_\sigma - (H_\nu \triangleright H_\mu) \triangleright H_\sigma = H_\mu \triangleright (H_\nu \triangleright H_\sigma) - H_\nu \triangleright (H_\mu \triangleright H_\sigma).$$  

(56)

Conversely, if a bilinear pairing $\triangleright$ on $H$ satisfies this condition, the skew-symmetric operation

$$[H_\mu, H_\nu] = (H_\mu \triangleright H_\nu) - (H_\nu \triangleright H_\mu)$$

(57)

fulfils the Jackobi identity, and $\triangleright$ becomes a left representation of the Lie algebra $H$ equipped with Lie bracket (57) on itself.

Lie algebra action $\triangleright$ induces an action of the Lie group $G$ turning $H$ into the left $G$-module. Consider now a 1-cocycle $\varphi$ on the group $G$ with values in $H$ [25]. This means that $\varphi$ obeys the equation

$$\varphi(xy) = \varphi(y) + y^{-1} \triangleright \varphi(x), \quad x, y \in G.$$  

(58)

Lie algebra 1-cocycle $\partial \varphi$ is in one-to-one correspondence with $\varphi$, being its derivative taken at the group identity $\varphi(x)$. It satisfies the equation (cf. with (57)) $\partial \varphi([H_\mu, H_\nu]) = H_\mu \triangleright \partial \varphi(H_\nu) - H_\nu \triangleright \partial \varphi(H_\mu)$. Suppose the linear mapping $\partial \varphi$ to be nondegenerate. Then the identity map $id: H \rightarrow H$ is a 1-cocycle with respect to the new action defined as $(\partial \varphi)^{-1} \circ \triangleright \circ \partial \varphi$. Thus, nondegenerate 1-cocycles of Lie algebras are in one-to-one correspondence with bilinear quasi-associative, in the sense of (54), operations on $H$. Only non-degenerate cocycles are suitable for our purposes, so we will think of them as of identity maps, and all the freedom will be encoded in the choice of action $\triangleright$. Note that a 1-coboundary normalized to $id$ implies the existence of the right unity $H_e$ that is

$$H_\mu \triangleright H_e = H_\mu.$$  

22
The group cocycle in terms of Lie algebra coordinates $\xi^\mu$ in a neighbourhood of the identity reads
\[
\varphi^\mu(\xi) = \left( e^{-L(\xi)} - 1 \right)^\mu \xi^\nu
\]
and the coboundary can be written as
\[
\varphi^\mu(\xi) = (1 - e^{-L(\xi)})^\mu \xi^\nu, \quad H_\nu \xi^\nu = H_e.
\]
Consider the semidirect sum $L = H \triangleright H^*$ with the Lie bracket given by (52) and (53) such that the condition (55) holds. Since $\partial \varphi = id$ is non-degenerate, the function $\varphi$ is invertible in a neighbourhood of the identity in $H$. Its inverse $\psi$ as well as $\varphi$ itself are treated as columns whose components are formal series in coordinate functions generating $U(H^*)$.

**Theorem 1** The element $F = \exp(H_\nu \otimes \psi^\nu(X))$ satisfies the twist equation.

**Proof.** The element $\exp(H_\nu \otimes \psi^\nu(X))$ satisfies the identity (8). If we prove the second identity (9), the theorem will be stated. Denote $\tilde{X}^\mu = \psi^\mu(X)$ and evaluate $\Delta_F(X)$:
\[
\Delta_F(X^\mu) = \exp(H \otimes \tilde{X})(X^\mu \otimes 1 + 1 \otimes X^\mu) \exp(-H \otimes \tilde{X})
\]
\[
= X^\nu \otimes (e^{-L(\tilde{X})})^\mu_{\nu} + 1 \otimes X^\mu.
\] (59)
The map $\Delta_F(h) = F \Delta(h) F^{-1}$ is an algebra homomorphism $U(L) \rightarrow U(L)^{\otimes 2}$. Henceforth, (53) entails the equation
\[
\varphi(\Delta_F(\tilde{X})) = e^{-L(1 \otimes \tilde{X})} \varphi(\tilde{X} \otimes 1) + \varphi(1 \otimes \tilde{X}).
\] (60)
Since $\varphi$ is invertible as a map of $H$ on $H$, we find $\Delta_F(\tilde{X}^\mu) = \mathcal{D}^\mu(\tilde{X} \otimes 1, 1 \otimes \tilde{X})$ where $\mathcal{D}^\mu(\xi_1, \xi_2)$ is the Campbell-Hausdorf series. This yields (60) and therefore the twist equation (5) for $\exp(H_\nu \otimes \psi^\nu(X))$ is valid. •

Now we can evaluate the twisted coproducts in terms of new generators $\tilde{X}^\mu$. A straightforward calculations show that
\[
\Delta_F(H_\mu) = H_\nu \otimes g(\tilde{X})^\nu_{\mu} + 1 \otimes H_\mu,
\] (61)
where $g(\xi)$ is a map $H \rightarrow H$ to be found. Imposing coassociativity conditions we find that function $g$ realizes a left group action on $H$ which is generated by a Lie algebra representation. To evaluate this action let us perform the following Lie algebra isomorphism $H_\mu \rightarrow H_\mu$, $X_\mu \rightarrow \xi X_\mu$. The specific form of the classical $r$-matrix allows us to consider $\xi$ as the deformation parameter. Taking into account $\frac{d}{d\xi} \tilde{X}_\mu(0) = X_\mu$, $\tilde{X}_\mu(0) = 0$ and calculating $\frac{d}{d\xi} F \Delta(H_\nu) F^{-1}|_{\xi=0}$ we find

$$\frac{d}{d\xi} \Delta_F(H_\nu)|_{\xi=0} = [H_\sigma \otimes X^\sigma, H_\nu \otimes 1 + 1 \otimes H_\nu] = C_{\sigma\nu}^\mu H_\mu \otimes X^\sigma + L_{\nu\sigma}^\mu H_\mu \otimes X^\sigma = L_{\sigma\nu}^\mu H_\mu \otimes X^\sigma.$$

Performing this for the coproduct (61) and comparing the results we find $g(\tilde{X}) = e^{L(\tilde{X})}$. Thus the coproduct on generators $H_\mu, \tilde{X}_\nu$ reads

$$\Delta_F(\tilde{X}_\mu) = D_\mu(\tilde{X} \otimes 1, 1 \otimes \tilde{X}), \quad \Delta_F(H_\mu) = H_\nu \otimes (e^{L(\tilde{X})})^\nu_\mu + 1 \otimes H_\mu.$$  

(62)

Using these relations it is easy to find also the antipodes,

$$S_F(\tilde{X}_\mu) = -\tilde{X}_\mu, \quad S_F(H_\mu) = -H_\nu (e^{-L(\tilde{X})})^\nu_\mu.$$  

(63)

Expressing $\tilde{X}_\mu$ through $X_\nu$ we can evaluate the twisted antipode on the classical generators as well.

## 7 Conclusion

The triangular deformation of the universal enveloping algebra of $sl(N)$ started already by Gerstenhaber et al [9], was realized in this paper as a twisting with explicit form of the twist element $F$ (extended jordon twist). The Hopf subalgebra of the type $U_F(B^\nu)$ generated by the twist is self-dual. The twisted coproduct of the $sl(N)$ generators can be expressed as finite sums of classical generators and a function $\sigma$ of the highest root vector primitive with respect to the twisted coproduct. The commutation relations of the quantum space generators were defined using the twist $F$ action on commutative coordinates. The cohomological properties of the involved Lie bialgebra
permit to explain the connection of the Drinfeld-Jimbo (standard) quantization with this twisting.

The explicit expression of the twist $\mathcal{F}$ gives rise to a possibility to evaluate the Clebsch-Gordan coefficients of the twisted $sl(N)$ in terms of the original CGC and the entries of the matrix $F = (\rho_1 \otimes \rho_2)\mathcal{F}$, as well as to get the relations among the FTR-approach generators $L^{(\pm)}$ of the twisted algebra and the generators of the original algebras. It can be used also to construct the quantum double.

The construction of the extended jordanian twist was generalized to certain class of inhomogenuous Lie algebras, using properties of the Campbell-Hausdorff series. Further generalizations, in particular to Lie superalgebras, twisting of the corresponding Yangians and new integrable models, twist elements for other boundary solutions to the classical Yang-Baxter equation are under study.

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