Wall-crossing for zero-dimensional sheaves and Hilbert schemes of points on Calabi–Yau 4-folds

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Abstract

Gross–Joyce–Tanaka \cite{GJT} proposed a wall-crossing conjecture for Calabi–Yau fourfolds. Assuming that it holds, we prove the conjecture of Cao–Kool \cite{Cao-Kool} for 0-dimensional DT4 invariants on projective Calabi–Yau 4-folds and then compute virtual fundamental classes of Hilbert schemes of points. As a results we are able to express generating series of invariants in terms of universal power-series. On $\mathbb{C}^4$, Nekrasov proposed invariants with a conjectured closed form \cite{Nekrasov}. Assuming wall-crossing, we prove an analogue of his formula for compact CY 4-folds. Finally, we notice a relationship to corresponding generating series for Quot-schemes on elliptic surfaces which are also governed by a wall-crossing formula. This leads to Segre–Verlinde correspondence for Calabi–Yau fourfolds. We will study the relation further in the sequence to this paper.

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1
1 Introduction

Enumerating coherent sheaves satisfying different conditions has seen a major development since 1990’s following the construction of virtual fundamental cycles (VFC) in \[ \text{VFC} \]. Even in recent days, there are many new results computing virtual counts of coherent sheaves on surfaces and 3-folds. To extend these ideas to 4-folds, a new approach was pioneered by Borisov–Joyce \[ \text{B-J} \] (using derived differential geometry) and Oh-Thomas \[ \text{O-T} \]. They constructed new virtual fundamental classes for moduli spaces of sheaves on Calabi–Yau 4-folds, which unlike their predecessors obtained using Behrend–Fantechi \[ \text{B-F} \], are not canonically determined by their deformation and obstruction theory.

The additional input needed to use the machinery of \[ \text{B-J} \] and \[ \text{O-T} \] is a choice of orientation on the moduli space, existence of which was proven in \[ \text{C-K} \] for projective Calabi–Yau 4-folds and by the author \[ \text{C-Q} \] for quasi-projective ones. It should be noted that this leads to complications that require new technology.

In this paper, we focus on the compact case. Some partial results in this setting have been obtained by Cao–Qu \[ \text{C-Q} \] extending Cao–Kool \[ \text{C-K} \], but the majority of computations have been done for local 4-folds. Because for now the construction of the deformation invariant VFC requires the CY condition even for ideal sheaves of points, one is unable to use algebraic cobordism to reduce everything to toric computations as in \[ \text{C-K} \]. Instead, we will use the conjectural wall-crossing along the lines of Gross–Joyce–Tanaka \[ \text{G-J-T} \].

Parallel results using toric computations have been obtained by Martijn Kool and Jørgen Rennemo \[ \text{K-R} \] on \[ \mathbb{C}^4 \] in the process of writing this paper. Unlike the Donaldson–Thomas invariants of Hilbert schemes of points for 3-folds, where this would describe the corresponding invariants of all compact 3-folds, there is currently no relation between the local and compact setting for Calabi–Yau 4-folds. Moreover, when gluing copies of \[ \mathbb{C}^4 \] to larger toric Calabi–Yau 4-folds, there is no indication whether the signs used in \[ \text{B-J} \] glue in the sense of Cao–Kool \[ \text{C-K} \].

Wall-crossing conjecture for Calabi–Yau 4-folds

We use the term Calabi–Yau 4-fold for a projective 4-fold \( X \) with a trivial canonical bundle \( K_X \) and \( H^2(O_X) = 0 \), because in the case \( H^2(O_X) \neq 0 \) all invariants counting sheaves with positive rank have been proved to vanish by Kiem–Park \[ \text{K-P} \]. We also always assume that \( X \) is connected as all our results can be easily generalized to multiple components.

We use \( G^0(X) \) to denote the Grothendieck group of vector bundles. Let \( \alpha \in G^0(X) \) and \( M_{\alpha,L} \) be a projective moduli scheme of perfect complexes with fixed determinant \( L \) in class \( \alpha \) satisfying some stability condition, then Oh–Thomas \[ \text{O-T} \], Borisov–Joyce \[ \text{B-J} \] give us a VFC \( [M_{\alpha,L}]_{\text{vir}} \in A_{\frac{1}{2}}(\chi(O_X) - \frac{1}{2} \chi(\alpha,\alpha)(M_{\alpha,L}, \mathbb{Z}[2^{-1}])) \) when \( \chi(\alpha,\alpha) \) is even. We will always take its image in \( H_{\frac{1}{2}}(\chi(O_X) - \chi(\alpha,\alpha)(M_{\alpha,L}, \mathbb{Q})) \). The fraction \( \frac{1}{2} \) is the result of taking “half of the obstruction theory”. The degree \( \chi(O_X) - \chi(\alpha,\alpha) \) was interpreted in Borisov–Joyce \[ \text{B-J} \] as

\[ \text{See also the work of Cao–Leung which constructed the class in some simple cases} \[ \text{C-L} \]. \]
the real virtual dimension of $M_{\alpha,L}$.

These fundamental classes were conjectured to satisfy universal wall-crossing formulae by Gross–Joyce–Tanaka \[39\]. Motivated by this, we formulate a wall-crossing conjecture for Joyce–Song stable pairs as in Joyce–Song \[49\] (see Conjecture \[2.19\] also \[39, Def. 4.3\]) and use it to study the virtual fundamental classes of Hilbert schemes of points. Our goal is two fold:

- Reduce existing conjectures in literature to the wall-crossing conjecture. These are the Conjecture \[1.1\] of Cao–Kool and \[1.10\] conjectured by Nekrasov for $\mathbb{C}^4$.
- Study new invariants and their symmetries. Along these lines we especially obtain a relation of universal generating series between CY fourfolds and elliptic surfaces, consequence of which is for example the Segre–Verlinde correspondence for DT$_4$ invariants. This all follows from our main result in Theorem \[4.1\] which gives an explicit description of the virtual fundamental class in the homology of $C_X^\infty = \text{Map}_{\mathcal{C}_0}(X, BU \times \mathbb{Z})$, where $\text{Map}_{\mathcal{C}_0}(-,-)$ denotes the mapping space of topological spaces.

In this paper, we choose to focus more on the particular generating series of interesting invariants and their symmetries over the more general result mentioned in the second point. We expect that this will motivate new research directions.

**Tautological insertions** One natural insertion on $\text{Hilb}^n(X)$, considered for surfaces by Götsche \[33\] and by Cao–Kool \[14\], Cao–Qu \[18\] and Nekrasov \[67\] for Calabi–Yau 4-folds is the top Chern class $c_n(L^{[n]})$ of the vector bundle

$$L^{[n]} = \pi_2^*(\mathcal{F}_n \otimes \pi^*X(L)),$$

where $X \xrightarrow{\pi_X} X \times \text{Hilb}^n(X) \xrightarrow{\pi_2} \text{Hilb}^n(X)$ are the projections and $\mathcal{O} \to \mathcal{F}_n$ is the universal complex on $\text{Hilb}^n(X)$.

For the generating series of invariants

$$I(L; q) = 1 + \sum_{n>0} I_n(L)q^n = 1 + \sum_{n>0} \int_{[\text{Hilb}^n(X)]^{\text{vir}}} c_n(L^{[n]})q^n$$

Cao–Kool \[14\] conjecture the following:

**Conjecture 1.1** (Cao–Kool \[14\]). Let $X$ be a projective Calabi–Yau 4-fold and $L$ a line bundle on $X$ then

$$I(L; q) = M(-q)\int_X c_1(L) \cdot c_3(X)$$

for some choice of orientations. Here $M(q) = \prod_{i=1}^{\infty} (1 - (q^i)^{-1})$ is the Mac-Mahon function. \[\dagger\]In the toric setting.
Cao–Qu [18] prove that if \( L = \mathcal{O}(D) \) for a smooth connected divisor \( D \subset X \), then this conjecture holds. Using Conjecture 2.19 we reduce Conjecture 1.1 for any line bundle \( L \) to their result. It turns out that the correct orientations of Conjecture 1.1 are obtained by fixing canonical orientations for \( K \)-theory classes of \( \mathcal{O}_X \) and \( \mathcal{O}_x \), where \( x \in X \) is a \( \mathbb{C} \)-point, and using the compatibility under sums in the sense of [18, Thm. 1.15] to extend these.

We call these orientations point-canonical. They are precisely the orientations of Cao–Qu [18] for any smooth divisor \( D \) such that \( D \cdot c_3(X) \neq 0 \). In particular, they are induced by the canonical orientations of Behrend–Fantechi [6] under the virtual pullback of [18, Prop. 1.6] with a plus sign.

To work with general invariants that follow, it will be useful to define the universal transformation on power-series:

\[
U(f(q)) = \prod_{n>0} \prod_{k=1}^{n} f(-e^{2\pi i k/n} q)^{-n}.
\]

**Segre series of all ranks**  For a surface \( S \) and a line bundle \( L \rightarrow S \) the Segre series

\[
R(S, L; q) = \int_{\text{Hilb}^n(S)} s_{2n}(L^{[n]}) q^n
\]

appeared in Tyurin [78] in relation to Donaldson invariants. Its precise form was conjecture by Lehn [56]. This was proven by Marian–Oprea–Pandharipande [63] for K3 surfaces and the general case in [65]. One can replace \( L \) in [111] by any class \( \alpha \in G^0(X) \). For any rank, these invariants have been considered by Marian–Oprea–Pandharipande [64], because of their relation to Verlinde numbers and strange duality. The invariants counting sheaves on Calabi–Yau fourfolds are meant to be a complex analog of Donaldson invariants for surfaces. Therefore, we study analogous questions in our case. A more general relation between elliptic surfaces and Calabi–Yau fourfolds is given in (1.14).

Let \( \vec{\alpha} = (\alpha_1, \ldots, \alpha_N) \) for \( \alpha_1, \ldots, \alpha_N \in G^0(X) \) and \( \vec{y} = (y_1, \ldots, y_N) \). We define the generalized DT4-Segre series for Calabi–Yau 4-folds by

\[
R(\vec{\alpha}, \vec{y}; q) = 1 + \sum_{n>0} q^n \int_{[\text{Hilb}^n(X)]^{\text{vir}}} s_{t_1}(\alpha_1^{[n]}) \cdots s_{t_N}(\alpha_N^{[n]}).
\]

The corresponding series for virtual fundamental classes of Quot-schemes on surfaces were studied by Oprea–Pandharipande [72]. When \( N = 1 \), we will write

\[
R(\alpha; q) = 1 + \sum_{n>0} q^n \int_{[\text{Hilb}^n(X)]^{\text{vir}}} s_n(\alpha^{[n]}).
\]
To express their generating series, recall that Fuss-Catalan numbers are given by

\[ C_{n,a} = \frac{1}{an+1} \left(\binom{an+1}{n}\right). \]  \hspace{1cm} (1.5)

They were defined by Fuss \[29\]† and appeared also in the work of Oprea–Pandharipande \[72\]. We denote their generating series by

\[ B_a(q) = \sum_{n \geq 0} C_{n,a} q^n. \]  \hspace{1cm} (1.6)

**Theorem 1.2.** Let \( \alpha, \alpha_1, \ldots, \alpha_N \in G^0(X) \), \( a = \text{rk}(\alpha), a_i = \text{rk}(\alpha_i) \), then assuming Conjecture 2.19 we have

\[ R(\vec{\alpha}, \vec{t}; q) = U \left[ \prod_{i=1}^N (1 + t_i z)^{c_1(\alpha_i)} c_3(X) \right] \]

for point-canonical orientations. Here \( z \) is the unique solution to

\[ z(1 + t_1 z)^{a_1} \cdots (1 + t_N z)^{a_N} = q. \]

Moreover, in the same setting we have the explicit expressions

\[ R(\alpha; q) = \begin{cases} 
U \left[ B_{a+1}(-q)^{c_1(\alpha)-c_3(X)} \right] & \text{for } a \geq 0 \\
U \left[ B_{-a}(q)^{c_1(\alpha)-c_3(X)} \right] & \text{for } a < 0
\end{cases}. \]  \hspace{1cm} (1.7)

As a Corollary, we obtain

\[ R(L; q) = U \left[ \frac{1 + \sqrt{1 + 4q}}{2} \right]^{c_1(L)-c_3(X)}. \]

**Nekrasov genus** K-theoretic invariants for Calabi–Yau 3-folds using twisted virtual structure sheaves were introduced by Nekrasov–Okounkov \[69\] to study the correspondence between DT \(_3\) invariants and curve-counting in CY five-folds. For CY four-folds Oh–Thomas \[71\] define twisted virtual structure sheaves \( \hat{\mathcal{O}}_{\text{vir}} \) on \( M_{\alpha,L} \).

For any \( A \in G^0(\text{Hilb}^n(X)) \) one can compute the twisted virtual Euler characteristic

\[ \hat{\chi}_{\text{vir}}(\text{Hilb}^n(X), A) = \chi(\text{Hilb}^n(X), \hat{\mathcal{O}}_{\text{vir}} \otimes A). \]  \hspace{1cm} (1.8)

Let

\[ \mathcal{N}_{\mu}(\alpha^{[n]}) = \Lambda_{\mu-1}^\ast(\alpha^{[n]}) \otimes \det^{-\frac{1}{2}}(\alpha^{[n]} \cdot y^{-1}), \]

where one should think of \( y \) as the equivariant parameter for the \( \mathbb{C}^\ast \) action on \( \mathcal{O}_X \), then we

†They were rigorously studied in \[24, 36, 74, 75, 72\].
define for any \( \alpha_1, \ldots, \alpha_N \in G^0(X) \)

\[
K(\vec{\alpha}, \vec{y}; q) = 1 + \sum_{n \geq 0} \hat{\chi}_{\text{vir}}(\text{Hilb}^n(X), \mathcal{N}_{y_1}(\alpha_1^{[n]}) \otimes \cdots \otimes \mathcal{N}_{y_N}(\alpha_N^{[n]})) \cdot (1.9)
\]

Segre series from \( \S 1 \) can be obtained as a classical limit of these invariants (see Proposition 5.5). We again use \( K(\alpha, y; q) \) to denote the \( N=1 \) case.

The case \( \alpha = L \) for a line bundle \( L \) was introduced by Nekrasov \[67, \S 4.2.4\], Nekrasov–Piazzalunga \[70\] and were further studied by Cao–Kool–Monavari \[15\] in relation to the DT/PT correspondence.

Equivariant version of these invariants can be defined for any toric CY 4-fold using localization of Oh–Thomas \[71, \S 7\] and the action of the 3-dimensional torus \( T = \{ t \in (\mathbb{C}^*)^4 : t_1t_2t_3t_4 = 1 \} \). Nekrasov \[67\] conjectured the following formula for \( \mathbb{C}^4 \):

\[
K_{\text{Nek}}(L_y, t; q) = \text{Exp}\left[ \chi(\mathbb{C}^4, q(\mathcal{O}(\mathbb{C}) - \mathcal{O}(\mathbb{C}^{-1})) \bigg( L_{y_1}^{\frac{1}{2}}y_2 - L_{y_2}^{\frac{1}{2}}y_1 \bigg) \right), (1.10)
\]

where \( \chi \) denotes the equivariant Euler characteristic on \( \mathbb{C}^4 \), \( L_y \) is the line bundle on \( \mathbb{C}^4 \) with weight \( y \) of the \( \mathbb{C}^* \) action and \( \text{Exp}[f(y, q)] = \exp\left[ \sum_{n>0} f(y^n, q^n) \right] \) is the plethystic exponential of a power-series \( f \). For \( X = \text{Tot}_{\mathbb{P}^1}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}) \) a similar conjecture was given by Cao–Kool–Monavari \[15, \text{Conjecture 0.16}\].

Replacing \( \mathbb{C}^4 \) with a compact \( X \) in (1.10), we obtain a prediction for compact Calabi–Yau 4-folds:

\[
K(L, y; q) = \text{Exp}\left[ \chi(X, q(\mathcal{O}(X) - \mathcal{O}(\mathbb{C}^{-1})) \bigg( L_{y_1}^{\frac{1}{2}}y_2 - L_{y_2}^{\frac{1}{2}}y_1 \bigg) \right) \right). (1.11)
\]

We thank Noah Arbesfeld for pointing out this direct relation to our previous version of the formula.

**Theorem 1.3.** If Conjecture 2.19 holds, then for all \( \alpha_1, \ldots, \alpha_N \) with \( a_i = \text{rk}(\alpha_i) \), \( \sum_i a_i = 2b + 1 \) and point-canonical orientations, we have

\[
K(\vec{\alpha}, \vec{y}; q) = \prod_{i=1}^N U \left[ \frac{(y_i - 1)^2u}{(y_i - u)^2} \right]^{\frac{1}{2}c_1(\alpha_i)c_3(X)}, \quad \text{where} \quad q = \frac{(u - 1)u^b}{\prod_{j=1}^N (y^2_j - y^{-2}_j)u^{a_i}}.
\]

In particular, (1.11) holds when \( N = 1 \), \( \alpha_1 = \alpha \), \( y_1 = y \), \( a_1 = 1 \).

During the writing of this paper, Martijn Kool and Jørgen Rennemo \[54\] announced a proof of (1.10), which in particular proves also the equivariant version of Conjecture 1.1 on \( \mathbb{C}^4 \). This gives further motivation to our belief that an argument replacing that of algebraic cobordism in \[59, 58\] should exist for DT4 invariants.
The invariants considered in the previous paragraph are special to 4-folds. Here we address K-theoretic invariants which are the analog of Verlinde series for surfaces.

Verlinde series for a smooth projective surface $S$ is given by

$$V(S, L; q) = 1 + \sum_{n>0} \chi(\text{Hilb}^n(S), L(n) \otimes E^r) q^n$$

(1.12)

where $(-)^{(n)}$ is the pull-back from symmetric product, $E = \det(\mathcal{O}_S^{[n]})$ and were defined and studied by Ellingsrud–Göttsche–Lehn [23]. By Göttsche [34, Rem. 5.3] it follows that when $r = \text{rk}(\alpha) - 1$ and $L = \det(\alpha)$ they can be expressed as

$$V(S, L; q) = V(S, \alpha; q) = 1 + \sum_{n>0} \chi(\text{Hilb}^n(S), \det(\alpha^n)) q^n.$$ 

Their virtual analogs were studied in [32] and [4].

Using this as an inspiration together with the relation to higher rank Nekrasov genus (see Remark 5.12), we define square root DT Verlinde series

$$V_{1/2}(\alpha; q) = 1 + \sum_{n>0} \hat{\chi}_{\text{vir}}(\text{Hilb}^n(X), \det_{1/2}(L_{\alpha}^{[n]} \otimes E^{r+1}) q^n,$$

where $L_{\alpha} = \det(\alpha)$, $E = \det(O_X^{[n]})$ and the DT Verlinde series for CY 4-folds:

$$V(\alpha; q) = 1 + \sum_{n>0} \hat{\chi}_{\text{vir}}(\text{Hilb}^n(X), \det(\alpha^{[n]} \otimes E^{1/2})) .$$

These are related as one would expect: $V(\alpha; q) = (V_{1/2}(\alpha; q))^2$ (see also Remark 5.12 for a relation to Nekrasov genus).

One of the most notable properties of the Verlinde and Segre series on surfaces motivated by strange duality, was proposed by Johnson [43]. An explicit formulation was given by Marian–Oprea–Pandharipande [65] as a change of variables $z = f(q), w = g(q)$ giving

$$V(S, \alpha; z) = R(S, -\alpha; w).$$

(1.13)

For Quot-schemes of $n$ points $\text{Quot}_S(\mathbb{C}^N, n)$ on surfaces, it was observed by Arbesfeld et al [4] that this correspondence takes the simple form $R(\alpha; q) = S(\alpha; (-1)^N q)$. We obtain the following:

**Theorem 1.4.** If Conjecture 2.19 holds, then for any $\alpha$, we have

$$V(\alpha; q) = R(\alpha; -q).$$

We obtain this result by computation, which leads to us asking the following.
Question 1.5. Is there a geometric interpretation of the Segre–Verlinde correspondence for CY 4-folds similar to the one of [43]?

4D-2D-1D correspondence  We discuss now a stronger correlation between invariants on surfaces and CY 4-folds. To a reader familiar with the results of [72] and [4], this will be a generalization of some of the above results including the Segre–Verlinde correspondence.

Let $S$ be a surface, then $\text{Hilb}^n(S)$ carries a virtual fundamental class $\left[\text{Hilb}^n(S)\right]^{\text{vir}} = \left[\text{Hilb}^n(S)\right] \cap \left((K_{\text{Hilb}}^n(S))^\vee\right)$ constructed in [63, Lem. 1]. Let $f, h$ be multiplicative genera (see §4.2). If $S$ is elliptic, then there is a universal series $A(q)$ depending on $f, h$ and $a = \text{rk}(\beta)$, such that

$$1 + \sum_{n>0} q^n \int_{\text{Hilb}^n(S)} f(\beta[n]) h(T_{\text{Hilb}^n(S)}^\text{vir}) = A(c_1(\beta) \cdot c_1(S)),$$

where $T_{\text{Y}}^\text{vir}$ denotes the virtual tangent bundle of $Y$ with given obstruction theory. Assuming Conjecture 2.19, we show that for point-canonical orientations, $\alpha \in G^0(X)$ with $\text{rk}(\alpha) = a$ and $h(z) = g(z)g(-z)$ we get

$$1 + \sum_{n>0} q^n \int_{\text{Hilb}^n(X)} f(\alpha[n]) g(T_{\text{Hilb}^n(X)}^\text{vir}) = U \left(A(q)^{c_1(\alpha) \cdot c_3(X)}\right). \tag{1.14}$$

We consider a single series $f$ for the purpose of the introduction. For general statement see Theorem 5.15. Taking $g$ to be trivial and $f$ to be the Segre class, we relate Theorem 1 to [72, Thm. 3, Thm. 6, Thm. 14].

Let $f$ be a multiplicative genus which we view as a map to $G^0(-) \otimes \mathbb{Q}$ instead of $H^*(-, \mathbb{Q})$, then we find for $\alpha, \beta$ as above that there is a universal series $B(q)$ depending on $a$ and $f$ such that

$$1 + \sum_{n>0} q^n \int_{\text{Hilb}^n(X)} f(\alpha[n]) \otimes E^2 = U \left[B(q)^{c_1(\alpha) \cdot c_3(X)}\right].$$

This relates Theorem 1.4 to [4, Thm. 13]. Moreover, [4], [72] reduce integrals over $[\text{Hilb}^n(S)]^{\text{vir}}$ to integrals over the symmetric products $C^{[n]}$ of smooth canonical curves (when they exist). This leads to a natural question to ask.

Question 1.6. What is the geometric interpretation of the universal transformation $U$ and the correspondence between the universal series for CY 4-folds and elliptic surfaces or elliptic curves?

A less geometric answer is given in Proposition 5.24 where we recover the universal generating series of Arbesfeld et al [4] for elliptic surfaces from an explicit expression for its virtual fundamental class obtained using the same wall-crossing methods.
In our future work \cite{future_work}, we pursue this question further by studying (virtual) fundamental classes of Quot-schemes \([\text{Quot}_Y(F, n)]^\text{vir}\), where \(F \to Y\) is a torsion-free sheaf and \(Y\) is a curve, surface or a Calabi–Yau fourfold to obtain a unification of these three theories.

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## 2 Vertex algebras in algebraic geometry and topology

In this section, we give a quick recollection of the vertex algebra construction in algebraic geometry of Joyce \cite{joyce} formulating it in a more algebraic topological language. We then set up the language for integrating insertions over classes in the associated Lie algebras obtained as quotients.

### 2.1 Vertex algebras

Let us recall first the definition of vertex algebras focusing on graded super-lattice vertex algebra. For background literature, we recommend \cite{borcherds, joyce, mackay, mazur, sugiyama, witten}, with Borcherds \cite{borcherds} being most concise.

**Definition 2.1.** A \(\mathbb{Z}\)-graded vertex algebra over a field \(\mathbb{Q}\) is a collection of data \((V_*, T, |0\rangle, Y)\), where \(V_*\) is a \(\mathbb{Z}\)-graded vector space, \(T : V_* \to V_* + 2\) is graded linear, \(|0\rangle \in V_0\), \(Y : V_* \to \text{End}(V_*)[[z, z^{-1}]]\) is graded linear after setting \(\text{deg}(z) = -2\), satisfying the following: Let \(u, v, w \in V_*\), then

i. We always have \(Y(u, z)v \in V_*(\langle z \rangle)\),

ii. \(Y(|0\rangle, z)v = v\),

iii. \(Y(v, z)|0\rangle = e^{zT}v\),

iv. Let \(\delta(z) = \sum_{n \in \mathbb{Z}} z^n \in \mathbb{Q}[z, z^{-1}]\)

\[
\begin{align*}
\delta^{-1}(\frac{z_1 - z_0}{z_2})Y(Y(u, z_0)v, z_2)w &= z_0^{-1}\delta(\frac{z_1 - z_2}{z_0})Y(u, z_1)Y(v, z_2)w \\
&\quad - (-1)^{\text{deg}(u)\text{deg}(v)}z_0^{-1}\delta(\frac{z_2 - z_1}{z_0})Y(v, z_2)Y(u, z_1)w.
\end{align*}
\]
By Borcherds \cite{11}, the graded vector-space \( V_{*+2}/T(V_*) \) carries a graded Lie algebra structure determined by
\[
[u,v] = z^{-1}Y(u,z)v.
\] (2.1)

Let \( A^\pm \) be abelian groups and \( \chi^\pm : A^\pm \times A^\pm \to \mathbb{Z} \) be symmetric, resp. anti-symmetric bi-additive maps. Let us denote \( h^\pm = A^\pm \otimes_{\mathbb{Z}} \mathbb{Q} \) and fix a basis of \( B^\pm \) of \( h^\pm \). For \((A^+, \chi^+)\) and a choice of a group 2-cocycle \( \epsilon : A^+ \times A^+ \to \mathbb{Z}_2 \) satisfying
\[
\epsilon_{\alpha,\beta} = (-1)^{\chi^+(\alpha,\beta)+\chi^+(\alpha,\alpha)\chi^+(\beta,\beta)}\epsilon_{\beta,\alpha}, \quad \forall \, \alpha, \beta \in A^+
\]
\[
\epsilon_{\alpha,0} = \epsilon_{0,\alpha} = 1, \quad \epsilon_{\alpha,\beta}\epsilon_{\alpha+\beta,\gamma} = \epsilon_{\beta,\gamma}\epsilon_{\alpha,\beta+\gamma},
\] (2.2)

(the second line is precisely the condition for the map to be a group 2-cocycle) there is a natural graded vertex algebra on
\[
\mathbb{Q}[A^+] \otimes_{\mathbb{Q}} \text{Sym}_{\mathbb{Q}} \llbracket u_{v,i}, v \in B^+, i > 0 \rrbracket \cong \mathbb{Q}[A^+] \otimes_{\mathbb{Q}} \text{Sym}(h^+ \otimes t^2\mathbb{Q}[t^2]),
\] (2.3)

where the isomorphism takes \( u_{v,i} \mapsto v \otimes t^{2i} \) and \( t \) is of degree 1. This vertex algebra is called the \emph{generalized lattice vertex algebra} (see \cite[§6.4]{57}, \cite[§5.4]{51}). For given \((A^-, \chi^-)\), Abe \cite{1} describes a natural \( \mathbb{Z} \)-graded vertex algebra on
\[
\Lambda_{\mathbb{Q}}[u_{v,i}, v \in B^-, i > 0] \cong \Lambda(h^- \otimes t\mathbb{Q}[t^2]),
\] (2.4)

where the isomorphism is given by \( u_{v,i} \mapsto v \otimes t^{2i-1} \) and \( t \) is of degree 1. Suppose we have vertex algebras \((V_*, T_V, |0\rangle_V, Y_V)\) and \((W_*, T_W, |0\rangle_W, Y_W)\), then there is a graded Vertex algebra on their tensor product, with state to field correspondence
\[
Y_{V \otimes W_*}(v \otimes w, z)(u \otimes t) = (-1)^{\deg(u)\deg(w)}Y_{V_*}(v, z)u Y_{W_*}(w, z)t.
\]
The super lattice vertex algebra for \((A^+ \oplus A^-, \chi^*)\) is then given by the tensor product of (2.3) and (2.4).

From the definition of the super-lattice vertex algebra \((V_*, T, |0\rangle, Y)\) associated to \((A_+ \oplus A_-, \chi^*)\) we can easily deduce the fields on the generators of:
\[
V_* \cong \mathbb{Q}[A_+] \otimes_{\mathbb{Q}} \text{SSym}_{\mathbb{Q}}[u_{v,i}, v \in B, i > 0]
\]
\[
\cong \mathbb{Q}[A_+] \otimes_{\mathbb{Q}} \text{Sym}_{\mathbb{Q}}(A_+ \otimes_{\mathbb{Z}} t^2\mathbb{Q}[t^2]) \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}(A_- \otimes_{\mathbb{Z}} t\mathbb{Q}[t^2]).
\]

Let \( \alpha \in A_+ \), such that \( \alpha = \sum_{v \in B_+} \alpha_v v \). We use \( e^\alpha \) to denote the corresponding element in
For $K \in V_*$, we have

$$Y(e^0 \otimes u_1, z)e^\beta \otimes K = e^\beta \otimes \left\{ \sum_{k>0} u_{v,k} \cdot K z^{k-1} \right\},$$

$$+ \sum_{k>0} \sum_{w \in B} k \chi^*(v, w) \frac{dK}{du_{w,k}} z^{-k-1} + \chi^*(v, \beta) z^{-1} \right\},$$

$$Y(e^\alpha \otimes 1, z)e^\beta \otimes K = \epsilon_{\alpha, \beta} z \chi(\alpha, v)dK_{uv,k}z^k \right\} K. \quad (2.5)$$

Note that by the reconstruction lemma [57, Thm. 5.7.1], Ben-Zvi [26, Thm. 4.4.1] and Kac [51, Thm. 4.5] these formulae are sufficient for determining all fields.

2.2 Axioms of vertex algebras on homology

For a higher stack $S$, we denote by $H_*(S) = H_*(S^{top})$, $H^*(S) = H^*(S^{top})$ its Betti (co)homology as in Joyce [46], Gross [37]. Note that we will always treat $H_*(T, \mathbb{Q})$ as a direct sum and $H^*(T, \mathbb{Q})$ as a product over all degrees. Following May–Ponto [66, §24.1] define the topological K-theory of $S$ to be

$$K^0(S) = [S^{top}, BU \times \mathbb{Z}],$$

where $[X, Y] = \pi_0(Map_{C0}(X, Y))$. For any $E$ in $L_{pe}(S)$ there is a unique map $\phi_E : S \to \text{Perf}_C$ in $\text{Ho}(\text{HSt})$. Using Blanc [8, §4.1], this gives

$$[E] : S^{top} \to BU \times \mathbb{Z}.$$

in $\text{Ho}(\text{Top})$. We then have a well defined map assigning to each perfect complex $E$ its class $[E] \in K^0(S)$.

The cohomology of $BU \times \mathbb{Z}$ is given by

$$H^*(BU \times \mathbb{Z}) \cong \mathbb{Q}[\mathbb{Z}] \otimes_\mathbb{Q} \mathbb{Q}[[\beta_1, \beta_2, \ldots]],$$

where $\beta_i = ch_i(\mathcal{U})$ and $\mathcal{U}$ is the universal K-theory class. Similarly to [46], we define $ch_i(E) = [E]^n(\beta_i)$ and the Chern classes by the Newton identities for symmetric polynomials:

$$\sum_{n \geq 0} c_n(E)q^n = \exp\left[\sum_{n=1}^\infty (-1)^{n+1}(n-1)!ch_n(E)q^n\right]. \quad (2.6)$$

As $BU \times \mathbb{Z}$ is a ring space [66, §4.1], the set $K^0(S)$ carries a natural ring structure. Moreover, by similar arguments as in [66, §4.1], one also has a map $(-)^\vee : BU \times \mathbb{Z} \to BU \times \mathbb{Z}$ inducing a map $(-)^\vee : K^0(S) \to K^0(S)$. When $S$ is replaced with a compact CW-complex $X$, this
becomes the standard K-theory $K^0(X)$ and $(-)^\vee$ corresponds to taking duals.

**Definition 2.2 (Joyce [46]).** Let $(\mathcal{A}, K(\mathcal{A}), \mathcal{M}, \Phi, \mu, \Theta, \epsilon)$ be data satisfying:

- $\mathcal{A}$ is an abelian category or derived category.
- Let $K_0(\mathcal{A}) \to K(\mathcal{A})$ be a map of abelian groups. For each $E \in \text{Ob}(\mathcal{A})$ denote $[E] \in K(\mathcal{A})$ the image of its class.
- $\mathcal{M}$ a moduli stack of objects in $\mathcal{A}$ with an action $\Phi : [*/\mathbb{G}_m] \times \mathcal{M} \to \mathcal{M}$ corresponding to multiplication by $\lambda \text{id}$ of $\text{Aut}(E)$ for any $E \in \text{Ob}(\mathcal{A})$ and a map $\mu : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ corresponding to direct sum.
- For each $\alpha \in K(\mathcal{A})$, $\mathcal{M}_\alpha$ is an open and closed substack of objects $[[E]] = \alpha$.
- $\Theta \in L_{pe}(\mathcal{M} \times \mathcal{M})$ satisfying $\sigma^*(\Theta) \cong \Theta^\vee[2n]$ for some $n \in \mathbb{Z}$ where $\sigma : \mathcal{M} \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ interchanges factors and \[(\mu \times \text{id}_{\mathcal{M}})^*(\Theta) \cong \pi_{13}^*(\Theta) \oplus \pi_{23}^*(\Theta), \quad (\text{id}_{\mathcal{M}} \times \mu)^*(\Theta) \cong \pi_{12}^*(\Theta) \oplus \pi_{13}^*(\Theta) \quad \text{and} \quad (\Phi \times \text{id}_{\mathcal{M}})^*(\Theta) \cong \mathcal{V}_1 \otimes \Theta, \quad (\text{id}_{\mathcal{M}} \times \Phi)^*(\Theta) \cong \mathcal{V}_1^* \otimes \Theta, \] (2.7)

where $\mathcal{V}_1$ is the universal line bundle on $[*/\mathbb{G}_m]$. One also writes $\Theta_{\alpha,\beta} = \Theta|_{\mathcal{M}_\alpha \times \mathcal{M}_\beta}$ and $\chi(\alpha, \beta) = \text{rk}(\Theta_{\alpha,\beta})$, where $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ is a bi-additive symmetric form.

- A group 2-cocycle $\epsilon : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}_2$ satisfying (2.2) with respect to $\chi^+ = \chi$.

Let $\hat{H}_*(\mathcal{M})$ be the homology with shifted grading given by $\hat{H}_n(\mathcal{M}_\alpha) = H_{n-\chi(\alpha,\alpha)}(\mathcal{M}_\alpha)$ for each $\alpha \in K(X)$, then using the above data one constructs a vertex algebra $(\hat{H}_*(\mathcal{M}), |0\rangle, e^{zT}, Y)$ over the $\mathbb{Q}$ vector space $\hat{H}_*(\mathcal{M})$. It is defined by:

- $|0\rangle = 0_*(\ast)$, where $0 : \ast \to \mathcal{M}$ is the inclusion of the zero object,
- $T(u) = \Phi_*(t \boxtimes u)$ for all $u \in \hat{H}_*(\mathcal{M})$ where $t \in H_2([*/\mathbb{G}_m]) = H_2(\mathbb{C}P^\infty)$ is the generator of homology given by inclusion $\mathbb{C}P^1 \subset \mathbb{C}P^\infty$.
- The state to field correspondence $Y$ is given by
  \[ Y(u, z)v = \epsilon_{\alpha,\beta}(-1)^{\alpha \chi(\beta,\beta)}z^{\alpha(\alpha,\beta)}\mu^\text{top}_\alpha(e^{zT} \otimes \text{id})(u \boxtimes v) \cap \gamma_{z-1}(\Theta_{\alpha,\beta}). \]
  for all $u \in \hat{H}_b(\mathcal{M}_\alpha)$, $v \in \hat{H}_b(\mathcal{M}_\beta)$.

The following definition is familiar to experts and can be extracted from a more general formula for generalized complex cohomology theories in Gross [38, Prop. 5.3.8].
Definition 2.3. Let \((\mathcal{C}, \mu, 0)\) be an \(H\)-space with a \(\mathbb{CP}^\infty\) action \(\Phi : \mathbb{CP}^\infty \times \mathcal{C} \to \mathcal{C}\) which is an \(H\)-map. Let \(\theta \in K^0(\mathcal{C}) = [\mathcal{C}, BU \times \mathbb{Z}]\) be a \(K\)-theory class satisfying \(\sigma^*(\theta) = \theta^\vee\) and

\[
(\mu \times \text{id}_\mathcal{C})^*(\theta) = \pi_{13}^*\theta + \pi_{23}^*\theta, \quad (\text{id}_\mathcal{C} \times \mu)^*(\theta) = \pi_{12}^*\theta + \pi_{13}^*\theta,
\]

\[
(\Phi \times \text{id}_\mathcal{C})^*(\theta) = V_1 \boxtimes \theta, \quad (\text{id}_\mathcal{C} \times \Phi)^*(\theta) = V_1^* \boxtimes \theta,
\]

where \(V_1 \to \mathbb{CP}^\infty\) is the universal line bundle.

Let \(\pi_0(\mathcal{C}) \to K\) be a morphism of commutative monoids. Denote \(\mathcal{C}_\alpha\) to be the open and closed subset of \(\mathcal{C}\) which is the union of connected components of \(\mathcal{C}\) mapped to \(\alpha \in K\). We write again \(\theta_{\alpha, \beta} = \theta|_{\mathcal{C}_\alpha \times \mathcal{C}_\beta}\), and \(\chi(\alpha, \beta) = \text{rk}(\theta_{\alpha, \beta})\) must be a symmetric bi-additive form on \(K\). Let \(\epsilon : K \times K \to \{-1, 1\}\) satisfying (2.2) be a group 2-cocycle and \(\hat{H}_a(C_\alpha) = H_{a-\chi(\alpha, \alpha)}(C_\alpha)\). Then we denote by \((\hat{H}_s(\mathcal{C}), [0], e^{zT}, Y)\) the vertex algebra on the graded \(\mathbb{Q}\)-vector space \(\hat{H}_s(\mathcal{C})\) defined for the data \((\mathcal{C}, K(\mathcal{C}), \Phi, \mu, 0, \theta, \epsilon)\) by

- \([0] = 0_{\epsilon(*)}\) and \(T(u) = \Phi_*(t \boxtimes u)\) as before
- the state to field correspondence \(Y\) is given by

\[
Y(u, z)v = \epsilon_{\alpha, \beta}(-1)^{\alpha\beta}z^\chi(\alpha, \beta)\mu_*(e^{zT} \boxtimes \text{id})\left((u \boxtimes v) \cap c_{z-1}(\theta_{\alpha, \beta})\right),
\]

for all \(u \in \hat{H}_a(C_\alpha), v \in \hat{H}_b(C_\beta)\).

Remark 2.4. We can assign to \((\mathcal{A}, K(\mathcal{A}), \mathcal{M}, \Phi, \mu, \Theta, \epsilon)\) the data

\[
(M^{\text{top}}, C_0(\mathcal{A}), \Phi^{\text{top}}, \mu^{\text{top}}, 0^{\text{top}}, \theta, \epsilon)
\]

from Definition 2.3 where \(C_0(\mathcal{A}) \subset K(\mathcal{A})\) is the cone of all \([E] \in K(\mathcal{A})\), \(\Phi^{\text{top}}, \mu^{\text{top}}, 0^{\text{top}}\) are maps in \(\text{Ho}(\text{Top})\) and \(\theta := [\Theta]\). The two vertex algebras obtained on \(\hat{H}_s(\mathcal{M})\) are clearly the same.

The wall-crossing formulae in Joyce [48], Gross–Joyce–Tanaka [39] are expressed in terms of a Lie algebra defined by Borcherds [11]. Let \((\hat{H}_s(\mathcal{C}), [0], e^{zT}, Y)\) be the vertex algebra from Definition 2.3 and define

\[
\Pi_{s+2} : \hat{H}_{s+2}(\mathcal{C}) \longrightarrow \hat{H}_s(\mathcal{C}) = \hat{H}_{s+2}(\mathcal{C})/T(\hat{H}_s(\mathcal{C})),
\]

then, by (2.1), this has a natural Lie algebra structure.

2.3 Insertions

To compute invariants using the homology classes of Conjecture 2.19 we need to consider elements in the dual of \(\hat{H}_0(\mathcal{M})\) or \(\hat{H}_0(N_{a,n})\) (see Definition 2.12). We do so in the algebraic topological language, as it is more general and is closer to the computations that follow.
Definition 2.5. Let $(\mathcal{C}, K(\mathcal{C}), \Phi, \mu, 0, \theta, \epsilon)$ be the data as in Definition 2.3 then a weight 0 insertion is a cohomology class $I \in H^*(\mathcal{C})$ satisfying

$$\Phi^*(I) = 1 \boxtimes I.$$ 

Lemma 2.6. Let $I \in H^*(\mathcal{C})$ be a weight 0 insertion, then $I_m \in H^m(\mathcal{C})$ induces a well defined map $\hat{I}_{-2+\chi(\alpha, \alpha)+m} : \hat{H}_{-2+\chi(\alpha, \alpha)+m}(\mathcal{C}_0) \to \mathbb{Q}$ for all $m \geq 0$.

Proof. Suppose that we have $V, V' \in H_m(\mathcal{C})$ such that $V - V' = D(W)$ for $W \in H_{m-2}(\mathcal{C})$. As $D(W) = \Phi_*(t \boxtimes W)$, using the push-pull formula in (co)homology we see

$$D(W) \cap I_m = \Phi_*(t \boxtimes W) \cap I_m = \Phi_*(t \boxtimes W \cap \Phi^*(I_m))$$

$$= \Phi_*((t \boxtimes W) \cap (1 \boxtimes I_m)) = \Phi_*(t \boxtimes (W \cap I_m)) = 0.$$

Integrating cohomology class 1 \in H_0(\mathcal{C}) on both sides shows that $I_m(V - V') = 0$. \hfill \square

Let $[M] \in \hat{H}_m(\mathcal{C})$ and $I$ a weight zero insertion. Then we will use the notation

$$\int_{[M]} I = \hat{I}_{-2+\chi(\alpha, \alpha)+m}([M]). \quad (2.8)$$

Example 2.7. Suppose that $\mathcal{J} \in K^0(\mathcal{C} \times \mathcal{C})$ satisfies

$$(\Phi \times \text{id}_{\mathcal{C}})^*(\mathcal{J}) = V_1^* \boxtimes \mathcal{J}, \quad (\text{id}_{\mathcal{C}} \times \Phi)^*(\mathcal{J}) = V_1 \boxtimes \mathcal{J},$$

then $\mathcal{I} = \Delta^*(\mathcal{J})$ satisfies $\Phi^*(\mathcal{I}) = 1 \boxtimes \mathcal{I}$. In particular if $p(x_1, x_2, \ldots)$ is a power series in infinitely many variables then $I = p(\chi_1(\mathcal{I}), \chi_2(\mathcal{I}), \ldots)$ is a weight zero insertion.

Often times insertions behave well under direct sums. In the algebraic setting the following definition has been stated more generally in [30, Definition 2.11].

Definition 2.8. Let $(\hat{H}_*(\mathcal{C}), [0), e^{zT}, Y)$ be a vertex algebra for the data in Definition 2.3. Let $F \to \mathcal{C} \times \mathcal{C}$ be a vector bundle satisfying

$$(\mu \times \text{id}_{\mathcal{C}})^*(F) \cong \pi_{13}^*(F) \oplus \pi_{23}^*(F), \quad (\text{id}_{\mathcal{C}} \times \mu)^*(F) \cong \pi_{12}^*(F) \oplus \pi_{13}^*(F)$$

$$(\Phi \times \text{id}_{\mathcal{C}})^*(F) \cong V_1^* \boxtimes F, \quad (\text{id}_{\mathcal{C}} \times \Phi)^*(F) \cong V_1 \boxtimes F, \quad (2.9)$$

such that $\xi(\alpha, \beta) := \text{rk}(F|_{\mathcal{C}_\alpha \times \mathcal{C}_\beta})$ is constant for all $\alpha, \beta \in K(\mathcal{C})$. Then define the $F$-twisted vertex algebra associated to $(\hat{H}_*(\mathcal{C}), [0), e^{zT}, Y)$ to be the vertex algebra given by Definition 2.3 for the data $(\mathcal{C}, K(\mathcal{C}), \Phi, \mu, 0, \theta^F, \epsilon^F)$, where

$$\theta^F = \theta + \llbracket F^* \rrbracket + \llbracket \sigma^*(F) \rrbracket,$$

$$\epsilon^F_{\alpha, \beta} = (-1)^{\xi(\alpha, \beta)} \epsilon_{\alpha, \beta} \quad \forall \alpha, \beta \in K(\mathcal{C}).$$

We denote this vertex algebra by $(\hat{H}_*(\mathcal{C}), [0), e^{zT}, Y^F)$. 

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One can then conclude by the same arguments as in the proof of [39, Thm. 2.12] (see Joyce [46]) that:

**Proposition 2.9.** In the situation of Definition 2.8 let $E = \Delta^*(F)$ and consider the morphism of graded $\mathbb{Q}$-vector spaces $(-) \cap c_\xi(E) : \tilde{H}_s(\mathcal{C}) \to \tilde{H}_s(\mathcal{C})$, such that on $\tilde{H}_s(\mathcal{C})$ it acts by $u \mapsto u \cap c_{\xi_{(\alpha,\alpha)}}(E)$. Then it induces a morphism of vertex algebras

$$(-) \cap c_\xi(E) : (\tilde{H}_s(\mathcal{C}), [0], e^{zT}, Y) \longrightarrow (\tilde{H}_s(\mathcal{C}), [0], e^{zT}, Y^F). \tag{2.10}$$

Moreover, let $[-,-]$ be the Lie bracket on $\tilde{H}_s(\mathcal{C})$ and $[-,-]^F$ the Lie bracket on $\tilde{H}_s(\mathcal{C}) = \tilde{H}_{s+2}(\mathcal{C})/T(\tilde{H}_s(\mathcal{C}))$. Then (2.10) induces a well-defined map of Lie algebras

$$(-) \cap c_\xi(E) : (\tilde{H}_s(\mathcal{C}), [-, -]) \longrightarrow (\tilde{H}_s(\mathcal{C}), [-, -]^F).$$

**Point-canonical orientations** Let $\mathcal{M}_X$ be the $\infty$-stack of perfect complexes on $X$ as in Toén–Vaquié [77]. Let $\mathcal{E}$ in $L_{\text{per}}(X \times \mathcal{M}_X)$ be the universal perfect complex and $\mathcal{E}^\text{xt} = \text{Hom}_{\mathcal{M}_X}(\mathcal{E}^\vee \otimes \mathcal{E})$, then there is a natural isomorphism

$$\mathcal{E}^\text{xt} \cong \mathcal{E}^\text{xt}^\vee [-4]. \tag{2.11}$$

The sheaf of sections of the orientation $\mathbb{Z}_2$-bundle $O^\omega \to \mathcal{M}_X$ consists of local trivializations $o : \text{det}(\mathcal{E}^\text{xt}) \to \mathcal{O}$ satisfying

$$o \otimes o = i^\omega : \text{det}(\mathcal{E}^\text{xt}) \otimes \text{det}(\mathcal{E}^\text{xt}) \to \mathcal{O},$$

where $i^\omega : \text{det}(\mathcal{E}^\text{xt}) \otimes \text{det}(\mathcal{E}^\text{xt}) \to \mathcal{O}$ is the result of taking determinants of (2.11). For any quasi-projective CY 4-fold $X$ this $\mathbb{Z}_2$-bundle is trivializable and Oh–Thomas [71] and Borisov–Joyce [12] use this to construct virtual fundamental classes depending on the choice of trivializations which are called orientations. More can be said about the orientations and their compatibility under direct sums. For this recall also that using the notation $\mathcal{C}_X = \text{Map}_{\mathcal{C}^0}(X_{\text{an}}, BU \times \mathbb{Z})$, there is a natural map $\Gamma : \mathcal{M}_X^{\text{top}} \to \mathcal{C}_X$ induced by $X_{\text{an}} \times \mathcal{M}_X^{\text{top}} \to (\text{Perf}_{\mathcal{C}^{\text{top}}}) = BU \times \mathbb{Z}.$

**Theorem 2.10** (Cao–Gross–Joyce [13]). Let $X$ be a projective Calabi–Yau 4-fold, then there exists a natural trivializable $\mathbb{Z}_2$-bundle $O^{dg} \to \mathcal{C}_X$ with a natural isomorphism

$$\Gamma^*(O^{dg}) \cong (O^\omega)^{\text{top}}.$$

Let $C_\alpha$ be the connected component $\alpha \in \pi_0(\mathcal{C}_X) = K^0(X)$, $\mathcal{M}_\alpha$ such that $\mathcal{M}_\alpha^{\text{top}} = \Gamma^{-1}(C_\alpha)$, $O^{dg}_\alpha = O^{dg}|_{C_\alpha}$, $O^\omega_\alpha = O^\omega|_{\mathcal{M}_\alpha}$, then there are natural isomorphisms

$$\phi^{dg}_{\alpha,\beta} : O^{dg}_\alpha \otimes O^{dg}_\beta \to \mu^*_\alpha \beta(O^{dg}_{\alpha+\beta}), \quad \phi^\omega_\alpha \beta : O^\omega_\alpha \otimes O^\omega_\beta \to \mu^*_\alpha \beta(O^\omega_{\alpha+\beta}),$$

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Let $o^dg_\alpha$ be a choice of trivialization of $O^d_\alpha$ for all $\alpha \in K^0(X)$, then define $\sigma^\alpha_\beta = \Gamma^*(o^dg_\beta)$ the trivializations of $O^\omega_\alpha$. There exist unique signs $\epsilon_{\alpha,\beta}$ for all $\alpha, \beta \in K^0(X)$ satisfying (2.2), such that

$$\phi^\omega_{\alpha,\beta}(o^\omega_\alpha \boxtimes o^\omega_\beta) = \epsilon_{\alpha,\beta} \phi^\omega_{\alpha+\beta}$$

if one chooses $o^dg_0$ such that $\phi^dg_0(o^dg_0 \boxtimes o^dg_0) = o^dg_0$.

Joyce–Tanaka–Upmeier [50, Thm. 2.27] describe a method for extending orientations from generators of the K-theory group (see also [10, Thm. 5.4] for compactly supported K-theory specialized to Calabi–Yau fourfolds). For our purposes, we will have a preferred set of generators.

Let us now fix orientations $o_\alpha$ for $\alpha = N[O_X] + np$. For this we set $p$ to be the K-theory class of a sky-scraper sheaf at some point $x \in X$. Let $M_p$ denote the moduli scheme of sheaves of class $p$. There is an isomorphism $M_p = X$ and Cao–Leung [17, Proposition 7.17]) showed that

$$[M_p]^{vir} = \pm \text{Pd}(c_3(X)) \in H_2(X),$$

(2.12)

where $\text{Pd}(-)$ denotes the Poincare dual. There is a natural map $m_p : M_p \to M_X$ giving $\Gamma \circ m_{p}^{\text{top}} : M_{p}^{\text{top}} \to \mathcal{C}_p$. Similarly, the point moduli space $\{O_X\} = *$ comes with a natural map $i_{O_X} : \{O_X\} \to M_X$ and carries a natural virtual fundamental class $1 \in H_0(\{O_X\})$. We will denote these choices of orientations $o^\text{can}_p$ and $o^\text{can}_{[O_X]}$ respectively. By the construction, these determine canonical orientations for all $\alpha = N[O_X] + np$.

### 2.4 Vertex algebras over pairs

In this section, a Calabi–Yau fourfold $(X, \Omega)$ must additionally satisfy $H^2(O_X) = 0$. We also use the notation $\chi = \chi(O_X)$. Let us now construct the vertex algebra on the auxiliary category of pairs and its topological analog.

**Definition 2.12.** Let $X$ be a Calabi–Yau fourfold and $\mathcal{A} = \text{Coh}(X)$. Fix a choice of an ample divisor $H$ and let $\tau$ denote the Gieseker stability condition with respect to $H$. Then let $\mathcal{A}_q$ be a full abelian subcategory of $\mathcal{A}$ with objects the zero sheaf and $\tau$-semistable sheaves with reduced Hilbert polynomial $q$. Define $\mathcal{B}_q$ to be the abelian category of triples $(E, V, \phi)$, where $E \in \text{Ob}(\mathcal{A}_q)$, $V \in \text{Vect}_\mathbb{C}$ and $\phi : V \otimes O_X(-n) \to E$. The morphisms are pairs $(f, g) : (E, V, \phi) \to (E', V', \phi')$ where $f : E \to E'$ and $g : V \to V'$ satisfy $\phi' \circ g \otimes \text{id}_{O_X(-n)} = f \circ \phi$. The moduli stack $\mathcal{N}_q$ of $\mathcal{B}_q$ is Artin by [49, Lem. 13.2]. Moreover, consider the full exact subcategory $\mathcal{B}_{q,n}$ of objects $(E, V, \phi)$, such that $H^i(E(n)) = 0$ for $i > 0$ and the corresponding open substack $\mathcal{N}_{q,n}$, where the openness follows from [41, Thm. 12.8].
Let
\[ \lambda : K_0(A) \rightarrow K^0(X) \] (2.13)
be induced by the usual comparison map. We define \( C(A_{q,n}) \subset K_0(A) \) to be the cone of \([E]\) for non-zero \( E \in \text{Ob}(A_q)\) such that \((E, 0, 0) \in \text{Ob}(B_{q,n})\). Let \( C_0(B_{q,n}) = (C(A_{q,n}) \sqcup \{0\}) \times (\mathbb{N} \sqcup \{0\})\), then for all \((\alpha, d) \in C_0(B_{q,n})\) define \( \mathcal{N}^{\alpha,d}_{q,n} \) as follows:

- If \((\alpha, d) \in C(A_{q,n}) \times \mathbb{N}\) then \( \mathcal{N}^{\alpha,d}_{q,n} \) is the total space of a vector bundle \( \pi_{\alpha,d} : \pi_{2,\ast}(\mathcal{F}_{q,n}^\alpha) \boxtimes \mathcal{V}_d^\ast \rightarrow \mathcal{M}^\alpha_{q,n} \times \ast / \text{GL}(d, \mathbb{C})\). Here
  \[ \mathcal{F}_{q,n}^\alpha = \pi_1^\ast(O_X(n)) \otimes \mathcal{E}_{q,n}^\alpha, \] (2.14)
where \( \mathcal{E}_{q,n}^\alpha \) is the universal sheaf over \( X \times \mathcal{M}^\alpha_{q,n} \) and \( \mathcal{M}^\alpha_{q,n} \) the moduli stack of \( \tau \)-semistable objects \( E \) with \( p_E = q \) and \([E] = \alpha\). We also use \( \mathcal{V}_d \) to denote the universal vector bundle of rank \( d \),

- \( \mathcal{N}^{\alpha,0}_{q,n} = \mathcal{M}^\alpha_{q,n}, \mathcal{N}^{0,d}_{q,n} = \ast / \text{GL}(d, \mathbb{C}) \) and \( \mathcal{N}^{0,0}_{q,n} = \ast \).

Then we have
\[ \mathcal{N}_{q,n} = \bigsqcup_{(\alpha,d) \in C_0(B_{q,n})} \mathcal{N}^{\alpha,d}_{q,n}. \]

We now describe the remaining ingredients needed in Definition 2.2. For a perfect complex/K-theory/cohomology class \( \kappa \) on \( Z_i \times Z_j \), we use the notation \((\kappa)_{i,j}\) to denote \( \pi_{i,j}^\ast(\kappa) \), where
\[ \pi_{i,j} : \prod_{k \in I} Z_k \rightarrow Z_i \times Z_j \]
is a projection to the \( i, j \) components.

**Definition 2.13.** We have a natural action \( \Phi_{\mathcal{N}_{q,n}} : \ast / \mathbb{G}_m \times \mathcal{N}^{\alpha,d}_{q,n} \rightarrow \mathcal{N}^{\alpha,d}_{q,n} \) which is a lift of the diagonal \([\ast / \mathbb{G}_m]\) action on the base \( \mathcal{M}^\alpha_{q,n} \times \ast / \text{GL}(d, \mathbb{C}) \) to the total space. We define the map of monoids
\[ K(\Omega) : C_0(B_{q,n}) \xrightarrow{(\lambda, id)} K^0(X) \times \mathbb{Z}. \] (2.15)
Let \( \Theta_{\alpha,\beta} = \text{Hom}_{\mathcal{M}^\alpha_{q,n} \times \mathcal{M}^\beta_{q,n}}(\mathcal{E}^\alpha_{q,n}, \mathcal{E}^\beta_{q,n})^{\vee} \). Let \( \mathcal{F}^\alpha_{q,n} \) be as in (2.14). We define \( \Theta_{(\alpha_1,d_1),(\alpha_2,d_2)} \in L_{pe}(\mathcal{N}^{\alpha_1,d_1}_{q,n} \times \mathcal{N}^{\alpha_2,d_2}_{q,n}) \) for all \((\alpha_i, d_i) \in C_0(\mathcal{N}_{q,n})\) by
\[ \Theta_{(\alpha_1,d_1),(\alpha_2,d_2)} = (\pi_{\alpha_1,d_1} \times \pi_{\alpha_2,d_2})^\ast \left\{ (\Theta_{1,\ast})_{1,3} \otimes \chi \left( (\mathcal{V}_{d_1} \boxtimes \mathcal{V}_{d_2}^\ast)^{\boxtimes 2} \right)_{2,4} \right. \left. \oplus \left( \mathcal{V}_{d_1} \boxtimes \pi_{2,\ast}(\mathcal{F}^\alpha_{q,n})^{\vee} \right)_{2,3} \right. \left. \oplus \left( \pi_{2,\ast}(\mathcal{F}^\alpha_{q,n}) \boxtimes \mathcal{V}_{d_2}^\ast \right)_{1,4} \right\}. \] (2.16)

The perfect complex \( \Theta_{pa}^{pa} \) on \( \mathcal{N}_{q,n} \times \mathcal{N}_{q,n} \) is defined to have the restriction to \( \mathcal{N}^{\alpha_1,d_1}_{q,n} \times \mathcal{N}^{\alpha_2,d_2}_{q,n} \).
given by \[\chi^{\text{pa}}((\alpha, d_1), (\alpha, d_2)) = \text{rk}(\Theta^{\text{pa}}_{\alpha, d_1, \alpha, d_2})\]
\[= \chi(\alpha, \alpha_2) + \chi(d_1 d_2 - d_1(\chi(\alpha_2(n)) - d_2(\chi(\alpha_1(n)))].\]

The signs \(\epsilon_{\alpha, d_1, \alpha, d_2}^{\text{pa}}\) are defined by:
\[\epsilon_{\alpha, d_1, \alpha, d_2}^{\text{pa}} = \epsilon_{\alpha, d_1}(\chi_{\alpha} - d_1(\chi_{\alpha_2(n)}) + \chi_{\alpha_2} - d_2(\chi_{\alpha_1(n)}).\]

where \(\epsilon_{\alpha, d_1, \alpha, d_2}^{\text{pa}}\) is from Theorem 2.10.

We show that the conditions of Definition 2.2 are satisfied (only) in K-theory, i.e.
\[(\mathcal{N}_{q,n})^{\text{top}}, C_0(\mathcal{B}_{q,n}), \mu^{\text{top}}, \mu^{\text{top}}, \theta, [\Theta^{\text{pa}}], \epsilon^{\text{pa}}]\]

satisfy assumptions of Definition 2.9.

**Lemma 2.14.** The data \((\mathcal{N}_{q,n})^{\text{top}}, C_0(\mathcal{B}_{q,n}), \mu^{\text{top}}, \mu^{\text{top}}, \theta, [\Theta^{\text{pa}}], \epsilon^{\text{pa}})\) satisfies the conditions in 2.3. Denote by \((\hat{H}(\mathcal{N}_{q,n}), [0], e^{\epsilon T}, Y)\) the corresponding vertex algebra.

**Proof.** To show that \([\Theta^{\text{pa}}]\) satisfies \(\sigma^{*}([\Theta^{\text{pa}}]) = [\Theta^{\text{pa}}]\) we note that
\[\sigma^{*}_{\alpha_1, \alpha_2}((\Theta_{\alpha_2, \alpha_1}) \cong (\Theta_{\alpha_1, \alpha_2})^{-1}[\alpha],\]
\[\sigma^{*}_{\alpha, d_1, \alpha, d_2}(V_{d_2} \boxtimes V_{d_2}^{*})_{2,4} = (V_{d_1} \boxtimes V_{d_2}^{*})_{2,4} = (V_{d_1} \boxtimes V_{d_2}^{*})_{2,4},\]
\[\sigma^{*}_{\alpha, d_1, \alpha, d_2}(V_{d_1} \boxtimes \pi_{2*}(\mathcal{F}_{q,n}^{\text{top}})(\chi_{\alpha(d)}^{\text{top}}) \boxtimes V_{d_2}^{*})_{2,4} = (\pi_{2*}(\mathcal{F}_{q,n}^{\text{top}})(\chi_{\alpha(d)}) \boxtimes V_{d_2}^{*})_{2,4}.\]

The rest of the properties for \([\Theta^{\text{pa}}]\) follow immediately, because \(V_{d}^{*}\) and \(\pi_{2*}(\mathcal{F}_{q,n}^{\text{top}})\) are weight 1 (see Joyce [46]) with respect to the \([*/G_m]\) action and they are additive under sums. The signs \(\epsilon_{\alpha, d_1, \alpha, d_2}^{\text{pa}}\) satisfy (2.2) for \(\chi^{\text{pa}}\) because the map \(\tau : K(A_{q,n}) \times \mathbb{Z} \to K(X)\) given by \(\tau(\alpha, d) = \lambda(\alpha) - d[O_X(\alpha)]\) is a group homomorphism satisfying \(\chi \circ (\tau \times \tau) = \chi^{\text{pa}}\). The latter statement uses that \(\chi(O_X) = \chi\).

We use the map \(\Sigma : \mathcal{N}_{q,n} \to \mathcal{M}_X \times \text{Perf}_\mathbb{C}\) where \(\mathcal{M}_X = \mathcal{M}_{L^X}\) is the higher moduli stack of perfect complexes on \(X\) as in [37] §3.5. For each \((\alpha, d) \in C_0(\mathcal{N}_{q,n})\) the restriction \(\Sigma_{(\alpha, d)} = \Sigma_{\mathcal{N}_{q,n}}\) can be expressed as \(\Sigma_{(\alpha, d)} = (\iota_{q,n}^{\alpha} \times \iota_{d}) \circ \pi_{\alpha, d}\), where \(\iota_{q,n}^{\alpha} : \mathcal{M}_{q,n}^{\alpha} \to \mathcal{M}_X\) and \(\iota_{d} : [*/\text{GL}(d, \mathbb{C})] \to \text{Perf}_\mathbb{C}\) are the inclusions. As \(\pi_{\alpha, d} : \mathcal{N}_{q,n}^{\alpha} \to \mathcal{M}_{q,n}^{\alpha} \times [*/\text{GL}(d, \mathbb{C})]\) is an \(A^1\)-homotopy equivalence we do not lose any information.

While there is an explicit description of \(H_*(\mathcal{M}_X)\) (see Gross [37]) in terms of the semi-topological K-theory groups \(K_{*}^{	ext{s}}(X)\) of Friedlander–Walker [28], we will not use it because these can be complicated for general Calabi–Yau fourfolds. Instead we transfer the problem into completely topological setting using
\[\Omega = (\Gamma \times \text{id}) \circ \Sigma^{\text{top}} : \mathcal{N}_{q,n}^{\text{top}} \to \mathcal{M}_X^{\text{top}} \times BU \times \mathbb{Z} \to \mathcal{C}_X \times BU \times \mathbb{Z}.\]

(2.19)
where $\Gamma$ is constructed as follows: Let $u : X \times \mathcal{M}_X \to \text{Perf}_C$ be the canonical map (using that $\mathcal{M}_X$ is the mapping stack between $X$ and $\text{Perf}_C$). Then $\Gamma$ is obtained from $u^{\text{top}}$.

This will induce a morphism of vertex algebras when using the correct data on $\mathcal{P}_X := \mathcal{C}_X \times BU \times \mathbb{Z}$. Denote by $\mathcal{U}$ and $\mathcal{E}$ the universal K-theory classes on $BU \times \mathbb{Z}$, respectively $X \times \mathcal{C}_X$. We will also use the notation $\mathfrak{f}_n = \pi_1^*([\mathcal{O}_X(n)]) \cdot \mathcal{E}$.

**Definition 2.15.** Define $\theta_P \in K^0(\mathcal{P}_X \times \mathcal{P}_X)$ by

$$
\theta_P = (\theta_C)_{1,3} + \chi(\mathcal{U} \boxtimes \mathcal{U}^\vee)_{2,4} - (\mathcal{U} \boxtimes \pi_2^*(\mathfrak{f}_n)^\vee)_{2,3} - (\pi_2^*(\mathfrak{f}_n) \boxtimes \mathcal{U}^\vee)_{1,4},
$$

where $\theta_C = \pi_{2,3}^*(\pi_{1,2}^*(\mathfrak{E}) \cdot \pi_{1,3}^*(\mathcal{E})^\vee)$.

Let $\Phi_P$ be given by the diagonal action on $\mathcal{C}_X \times (BU \times \mathbb{Z})$. We use the natural H-space structure $(\mathcal{P}_X, \mu, 0)$. Choosing $K(\mathcal{P}_X) = K^0(X) \times \mathbb{Z}$ we set for all $(\alpha_i, d_i) \in K(X) \times \mathbb{Z}$:

$$
\bar{\chi}((\alpha_1, d_1), (\alpha_2, d_2)) = \chi(\alpha_1 + \alpha_2) + d_1\chi(\alpha_1(n)) - d_2\chi(\alpha_2(n)),
$$

$$
\bar{\tilde{\epsilon}}(\alpha_1, d_1), (\alpha_2, d_2) = \epsilon_{\alpha_1} - d_1[\mathcal{O}_X(-n)]_2 - d_2[\mathcal{O}_X(-n)],
$$

(2.20)

where $\epsilon$ is from Theorem 2.10. We construct the vertex algebra $(\hat{\mathcal{H}}_*(\mathcal{P}_X), [0], e^{zT}, Y)$.

**Proposition 2.16.** The map $\Omega_* : H_*(\mathcal{N}_{q,n}) \to H_*(\mathcal{P}_X)$ induces a morphism of graded vertex algebras $(\hat{\mathcal{H}}_*(\mathcal{N}_{q,n}), [0], e^{zT}, Y) \to (\hat{\mathcal{H}}_*(\mathcal{P}_X), [0], e^{zT}, Y)$. It gives a morphism of Lie algebras

$$
\Omega_* : (\hat{\mathcal{H}}_*(\mathcal{N}_{q,n}), [-, -]) \to (\hat{\mathcal{H}}_*(\mathcal{P}_X), [-, -]).
$$

**Proof.** We use this opportunity to check that conditions of Definition 2.3 are satisfied. Using arguments from the proof of Lemma 2.14 and Gross [38, Prop. 5.3.12], we reduce it to showing that $\sigma^*(\theta_C) = \theta_C^\vee$. Recall that we have the natural homotopy theoretic group completion $\gamma : \mathcal{V}_X \to \mathcal{C}_X$. Using universality of the group-completion proved by Caruso–Cohen–May–Taylor [19, Proposition 1.2], we restrict it to showing

$$
\sigma^*(\gamma^*(\theta_C)) = \gamma^*(\theta_C)^\vee.
$$

(2.21)

Two compact families $K, L \to \mathcal{V}_X$ correspond to two families of vector bundles $V_K, V_L$ which we can assume to be smooth along $X$ so we choose partial connections $\nabla_{V_K}, \nabla_{V_L}$ in the $X$ direction for both of them. The pullback of the class $\gamma^*(\theta_C)$ to $K \times L$ is the index of the family of operators $(\partial + \bar{\partial}^*)^\vee_{V_L \boxtimes V_L} : \Gamma_\infty(A^{0, \text{even}} \otimes V_L^* \otimes V_K) \to \Gamma_\infty(A^{0, \text{even}} \otimes V_K^* \otimes V_L)$. Using Serre duality, we have the formula $\text{ind}(\partial + \bar{\partial}^*)^\vee_{V_L \boxtimes V_K} = \text{ind}(\partial + \bar{\partial}^*)^\vee_{V_L \boxtimes V_L}$, which is precisely (2.21) by the family index theorem [3, §3.1].

\footnote{Using the left-multiplication on $U(n)$ by $U(1)$ we get the action of $\mathbb{C}P^n$ on $BU(n)$. Taking a union over all $n$ we get a $\mathbb{C}P^n$ action on $\bigsqcup_n BU(n)$. As $\bigsqcup_n BU(n) \to BU \times \mathbb{Z}$ is a homotopy theoretic group completion, using [19, Proposition 1.2] we can extend to an action on $BU \times \mathbb{Z}$}
To show that $\Omega$, induces a morphism of vertex algebras we note that in $\text{Ho} (\text{Top})$, $\Omega : (\mathcal{N}_{q,n})^\text{op} \to \mathcal{P}_X$ is a morphism of monoids with $\mathbb{C} \mathbb{P}^\infty$ action. The pullback $\Omega^\ast (\theta_\mathcal{P})$ is equal to $[\Theta^\text{pa}]$ by construction and arguments in the proofs of [37, Prop. 5.12, Lem. 6.2]. By considering the action of $\Omega$ on connected components, we get precisely $K(\Omega) : C_0(\mathcal{B}_{q,n}) \to K^0(X) \times \mathbb{Z}$ from (2.15) which satisfies

$$\hat{\chi} \circ (K(\Omega) \times K(\Omega)) = \chi^\text{pa} : C_0(\mathcal{B}_{q,n}) \times C_0(\mathcal{B}_{q,n}) \longrightarrow \mathbb{Z},$$

for the same choices of $\epsilon_{\alpha, \beta}$ in (2.17) and (2.20). Therefore $\Omega_* : \hat{H}_s(\mathcal{N}_{q,n}) \to \hat{H}_s(\mathcal{P}_X)$ is a degree 0 graded morphism compatible with the vertex algebra structure.

**Remark 2.17.** We will only restrict to the case when $n=0$, as we will be working with 0-dimensional sheaves only in the following sections.

### 2.5 Wall-crossing conjecture for Calabi–Yau fourfolds

In this subsection, we conjecture the wall-crossing formula for Joyce–Song stable pairs, following Joyce–Song [49, §5.4], Joyce [48]. For the abelian category of coherent sheaves the conjecture has been stated by Gross–Joyce–Tanaka [39, Conj. 4.11]. Before this, we recall some background from [46].

Consider an Artin stack $\mathcal{M}_{\partial, 0} = \mathcal{M}_0$, where $\mathcal{M}$ as in Definition [2.2]. Using rigidification as in Abramovich–Olsson–Vistoli [2] and Romagny [73], one defines $\mathcal{M}_{\partial, 0}^\text{pl} = \mathcal{M}_{\partial, 0}/[\ast/\mathbb{G}_m]$. One can define a shifted grading on $\hat{H}_s(M_{\partial, 0}^\text{pl}) = H_{s+2-\chi(\alpha, \alpha)}(M_{\partial, 0}^\text{pl})$ such that the projection $\Pi_{\text{pl}} : \mathcal{M}_{\partial, 0} \to \mathcal{M}_{\partial, 0}^\text{pl}$ induces a map of graded $\mathbb{Q}$-vector spaces, Joyce [46] proves that it factors $\hat{H}_{s+2}(\mathcal{M}_{\partial, 0}) \xrightarrow{\Pi} \hat{H}_s(\mathcal{M}_{\partial, 0}) \xrightarrow{\hat{H}} \hat{H}_{s+2}(\mathcal{M}_{\partial, 0}^\text{pl})$, such that $\hat{H}_0(\mathcal{M}_{\partial, 0}) \to \hat{H}_0(\mathcal{M}_{\partial, 0}^\text{pl})$ is an isomorphism. If $\tau$ is a stability condition on $\mathcal{A}$ from [47, Def. 4.1] and $0 \neq \alpha \in K(\mathcal{A})$, then let $M^\text{st}_\alpha(\tau)$ denote the moduli scheme of $\tau$-stable objects in class $\alpha$ and $M^\text{st}_\alpha(\tau) \subset M^\text{st}_\alpha(\tau)$ the finite type stacks of $\tau$-stable and $\tau$-semistable objects. There is a natural open embedding $i^\text{st}_\alpha : M^\text{st}_\alpha(\tau) \hookrightarrow M^\text{pl}_\alpha(\tau)$. In particular, if $[M^\text{st}_\alpha(\tau)]^\text{vir} \in H_s(M^\text{st}_\alpha(\tau))$ is defined, then we write $[M^\text{st}_\alpha(\tau)]^\text{vir} = i^\text{st}_\alpha (\tau) \in H_s(M^\text{pl}_\alpha(\tau))$.

Let now $X$ be a CY fourfold, $\mathcal{A} = \text{Coh}(X)$ and $\tau$ a Gieseker stability, then in the case that $M^\text{st}_\alpha(\tau) = \text{CH}_\alpha(\tau)$, Oh–Thomas [71] and Borisov–Joyce [12] construct virtual fundamental classes $[M^\text{st}_\alpha(\tau)]^\text{vir} \in H_{\chi - \chi(\alpha, \alpha)}(M^\text{st}_\alpha(\tau))$, where $\chi = \chi(\mathcal{O}_X)$ again. Thus we have $[M^\text{st}_\alpha(\tau)]^\text{vir} \in H_s(M^\text{pl}_\alpha(\tau))$. We lift it to an element $(\hat{H}_0)^{-1}([M^\text{st}_\alpha(\tau)]^\text{vir})$ which we also denote by $[M^\text{st}_\alpha(\tau)]^\text{vir}$.

For $\mathcal{A} = \text{Coh}(X)$ we now fix the data from Definition [2.2].

**Definition 2.18.** Define $(\mathcal{A}, K(\mathcal{A}), \Phi, \mu, \Theta, \epsilon)$ as follows:

- $\lambda : K_0(\mathcal{A}) \xrightarrow{\lambda} K^0(X) =: K(\mathcal{A})$, $\Theta$ from Definition 2.13.
• For \( \alpha, \beta \in K(A) \) define \( \epsilon_{\alpha, \beta} = \epsilon_{\lambda(\alpha), \lambda(\beta)} \) using the orientations from Theorem 2.10

Moreover, use the fixed orientations above to construct Oh–Thomas/ Borisov–Joyce classes \( [M_{\alpha}^{st}(\tau)]_{\text{vir}} \in H_*(M_{\alpha}^{st}(\tau)) \) for all \( \alpha, \tau \), such that \( M_{\alpha}^{st}(\tau) = M_{\alpha}^{ss}(\tau) \). Let \( \tau_{\text{pa}} \) denote the Joyce–Song stability condition on pairs (see Joyce–Song [49, §5.4]), then [12], [71] still give us \( [M_{(\alpha,1)}^{st}(\tau_{\text{pa}})]_{\text{vir}} \in H_{2-\chi_{\text{pa}}((\alpha,1), (\alpha,1))}(M_{(\alpha,1)}^{st}(\tau_{\text{pa}})) \). The chosen orientation is again used to determine orientation of \( [M_{(\alpha,1)}^{st}(\tau_{\text{pa}})]_{\text{vir}} \) under the inclusion \( M_{(\alpha,1)}^{st} \to M_X \).

**Conjecture 2.19.** Let \( \tau \) be a Gieseker stability, then there are unique classes \( [M_{\alpha}^{ss}(\tau)]_{\text{inv}} \in \hat{H}_0(M_{\alpha}) \) for all \( \alpha \in K(A) \) satisfying:

i. If \( M_{\alpha}^{ss}(\tau) = M_{\alpha}^{st}(\tau) \) then \( [M_{\alpha}^{ss}(\tau)]_{\text{inv}} = [M_{\alpha}^{st}(\tau)]_{\text{vir}} \).

ii. If \( \tilde{\tau} \) is another Gieseker stability condition then these classes satisfy the wall-crossing formula [39, eq. (4.1)] in \( \hat{H}_*(M) \). If \( M_{\beta}^{ss}(\tau) = M_{\beta}^{ss}(\tilde{\tau}) \) then \( [M_{\beta}^{ss}(\tau)]_{\text{inv}} = [M_{\beta}^{ss}(\tilde{\tau})]_{\text{inv}} \).

iii. We have the formula in \( \hat{H}_*(N_{\alpha,n}) \):

\[
[M_{(\alpha,1)}^{st}(\tau_{\text{pa}})]_{\text{vir}} = \sum_{n \geq 1, \alpha_1, \ldots, \alpha_n \in C(A) \atop \alpha_1 + \cdots + \alpha_n = \alpha, \tau(\alpha) = \tau(\alpha_i)} \frac{(-1)^n}{n!} \left[ \cdots \left[ [M_{(0,1)}]_{\text{inv}}, [M_{\alpha_1}^{ss}(\tau)]_{\text{inv}}, \ldots, [M_{\alpha_n}^{ss}(\tau)]_{\text{inv}} \right] \right],
\]

where \( [M_{(0,1)}]_{\text{inv}} \in \hat{H}_0(N_{1,0}) \cong \mathbb{Z} \) is the generator determined by orientation on \( C(O_X) \).

One can compute well-defined invariants using \( [M_{\alpha}^{ss}(\tau)]_{\text{inv}} \) and weight 0 insertions from 2.5. We will use the map \( \Omega_* : H_*(N_q) \to H_*(P_X) \) and give explicit formulae for \( [M_{\alpha}^{ss}(\tau)]_{\text{inv}} \) in the cases we study.

### 2.6 Explicit vertex algebra of topological pairs

We give here an explicit description of the vertex algebra \( (\hat{H}_*(P_X), \{0\}, e^{zT}, Y) \) which will apply some of the work of Joyce [49] and Gross [37]. We also set some notations, conventions and write down some useful identities. In the following, \( X \) is a connected smooth projective variety of dimension \( n \) unless specified.

**Definition 2.20.** Let us write \((0,1)\) for the generator of \( \mathbb{Z} \) in \( \mathbb{K}^0(X) \oplus \mathbb{Z} \). Let \( ch : \mathbb{K}^0(X) \otimes \mathbb{Q} \oplus K^1(X) \otimes \mathbb{Q} \to H^*(X) \) be the Chern character. For each \( 0 \leq i \leq 2n \) choose a subset \( B_i \subset \mathbb{K}^i(X) \otimes \mathbb{Q} \oplus K^1(X) \otimes \mathbb{Q} \) such that \( ch(B_i) \) is a basis of \( H^i(X) \). We take \( B_0 = \{[O_X] \} \) and \( B_{2n} = \{p\} \). Then we write \( B = \bigsqcup_i B_i \) and \( \mathbb{B} = B \cup \{(0,1)\} \). Let \( K_*(X) \) denote the topological K-homology of \( X \). Let \( ch^\vee : K_*(X) \otimes \mathbb{Q} \to H_*(X) \) be defined by commutativity of the following:

\[
\begin{array}{ccc}
K_*(X) \otimes \mathbb{Q} \otimes K^*(X) \otimes \mathbb{Q} & \xrightarrow{ch^\vee \otimes ch} & H_*(X) \otimes H^*(X) \\
\downarrow & & \downarrow \\
\mathbb{Q} & \xrightarrow{id} & \mathbb{Q}
\end{array}
\]
then choose $B^\vee \subset K_*(X) \otimes \mathbb{Q}$ such that $B^\vee$ is a dual basis of $B$, we also write $B^\vee = B^\vee \cup \{(0,1)\}$, where $(0,1)$ is the natural generator of $Z$ in $K_0(X) \oplus K_1(X) \oplus \mathbb{Z}$. The dual of $\sigma \in B$ will be denoted by $\sigma^\vee \in B^\vee$. For each $\sigma \in B$, $(\alpha, d) \in K^0(X) \times \mathbb{Z}$ and $i \geq 0$ we define

$$e^{(\alpha, d)} \otimes \mu_{\sigma, i} = ch_i(\mathcal{E}(\alpha, d)/\sigma^\vee),$$

(2.22)

using the slant product $K^i(Y \times Z) \otimes K_j(Y) \to K^{i-j}(Z)$. We have a natural inclusion $i_{C, P} : \mathcal{C}_X \to \mathcal{P}_X : (x, 1, 0) \in \mathcal{C}_X \times BU \times \mathbb{Z}$, so we identify $H_*(\mathcal{C}_X)$ with the image of $(i_{C, P}, s)$, which in turn corresponds to $H_*(\mathcal{C}_X) \boxtimes 1 \subset H_*(\mathcal{C}_X) \boxtimes H_*(BU \times \mathbb{Z}) = H_*(\mathcal{P}_X)$. The universal K-theory class $\mathcal{E}_P$ on $(X \sqcup *) \times (\mathcal{P}_X)$ restricts to $\mathcal{E} \boxtimes 1$ on $(X \times \mathcal{C}_X) \times BU \times \mathbb{Z}$ and $1 \boxtimes 1$ on $* \times \mathcal{C}_X \times (BU \times \mathbb{Z})$.

The next proposition follows by the arguments in the proof of Gross [37, Thm. 4.15] and the remark below it.

**Proposition 2.21.** The cohomology ring $H^*(\mathcal{P}_X)$ is generated by

$$\{e^{(\alpha, d)} \otimes \mu_{\sigma, i} \}_{(\alpha, d) \in K^0(X) \times \mathbb{Z}, \sigma \in B, i \geq 1}. $$

Moreover, there is a natural isomorphism of rings

$$H^*(\mathcal{P}_X) \cong \mathbb{Q}[K^0(X) \oplus \mathbb{Z}] \otimes_{\mathbb{Q}} \mathrm{SSym}_\mathbb{Q}[\mu_{\sigma, i}, \sigma \in B, i > 0].$$

(2.23)

From now on, when we compute explicitly with $H^*(\mathcal{P}_X)$, we replace it using this isomorphism. The dual of (2.23) gives us an isomorphism

$$H_*(\mathcal{P}_X) \cong \mathbb{Q}[K^0(X) \times \mathbb{Z}] \otimes_{\mathbb{Q}} \mathrm{SSym}_\mathbb{Q}[u_{\sigma, i}, \sigma \in B, i > 0],$$

(2.24)

where we use the normalization

$$e^{(\alpha, d)} \otimes \prod_{\sigma \in B, i \geq 1} \mu_{\sigma, i}^{m_{\sigma, i}} \left(e^{(\beta, e)} \otimes \prod_{\tau \in B, j \geq 1} u_{\tau, j}^{n_{\tau, j}}\right) = \left\{ \begin{array}{ll}
\prod_{i \geq 1} m_{\sigma, i}! 
\tau = (\beta, e), \mu_{\sigma, i} = n_{\tau, j} \times \prod_{\sigma \in B, i \geq 1} (\sigma - 1)^{m_{\sigma, i}}
0 \quad \text{otherwise.}
\end{array} \right.$$  

(2.25)

We will be using the following simple result in computations later.

**Lemma 2.22.** Let $f(tx_\sigma, t^2x_\sigma, \ldots)$ be a power-series, then for any set of coefficients $a_{\sigma, j}$ we have

$$e^{(\alpha, d)} \otimes \exp \left( \sum_{j > 0} \sum_{\tau \in B} a_{\tau, j} t^{j} \mu_{\tau, j} q^j \right) \left(e^{(\beta, e)} \otimes f(\mu_{\sigma, 1}, \mu_{\sigma, 2}, \ldots) \right)$$

$$= \delta_{\alpha, \beta} \delta_{d, e} f(a_{\sigma, 1} q, a_{\sigma, 2} q^2, \ldots, \frac{a_{\sigma, k}}{(k - 1)!} q^k, \ldots).$$
Proof. Notice that acting with \( e(\alpha, d) \otimes \prod_{\sigma \in \mathbb{B}} \mu_{\sigma,i}^{m_{\sigma,i}} \) corresponds to acting with
\[
\delta_{\alpha, \beta} \delta_{d,e} \prod_{\sigma \in \mathbb{B}} \left( \frac{1}{(i-1)!} \frac{d}{du_{\sigma,i}} \right)^{m_{\sigma,i}}
\]
and then evaluating at \( u_{\sigma,i} = 0 \). As a result we obtain
\[
e(\alpha, d) \otimes \exp \left( \sum_{i \geq 1} a_{\sigma,j} \frac{d}{du_{\sigma,j}} q^j \right) \left( e(\beta,e) \otimes f(u_{\sigma,1}, u_{\sigma,2}, \ldots) \right) \big|_{u_{\sigma,i}=0}
= \delta_{\alpha, \beta} \delta_{d,e} f(a_{\sigma,1} q, a_{\sigma,2} q^2, \ldots, \frac{a_{\sigma,k}}{(k-1)!} q^k, \ldots),
\]
by a standard computation. \( \square \)

When \( \sigma = (v, 0) \) or \( \sigma = (0, 1) \) we will shorten the notation to
\[
\mu_{\sigma,i} = \mu_{v,i}, \quad u_{\sigma,i} = u_{v,i} \quad \text{or} \quad \mu_{\sigma,i} = \beta_i, \quad u_{\sigma,i} = b_i.
\]
Setting \( \beta_i = 0, b_i = 0 \) and only considering \( \mathbb{Q}[K^0(X)] \subset \mathbb{Q}[K^0(X) \oplus \mathbb{Z}] \) gives us the (co)homology of \( H^*(C_X), H_*(C_X) \) up to a canonical isomorphism. Using this notation we can write
\[
\text{ch}(E_\alpha) = \sum_{v \in B} \text{ch}(v) \mathbb{E} \left( \sum_{i \geq 0} e^{v} \otimes \mu_{v,i} \right). \quad (2.26)
\]

Let \( X \) now be a CY fourfold. The following theorem is the topological version of [37, Thm. 1.1], [46, Thm. **] extending it also to pairs.

**Proposition 2.23.** Let \( \mathbb{Q}[K^0(X) \times \mathbb{Z}] \otimes \mathbb{Q} \text{Sym}_{\mathbb{Q}}[u_{\sigma,i}, \sigma \in \mathbb{B}, i > 0] \) be the generalized super-lattice vertex algebra associated to \( ((K^0(X) \oplus \mathbb{Z}) \oplus K^1(X), \tilde{\chi}^\bullet) \), where \( \tilde{\chi}^\bullet = \tilde{\chi} \oplus \chi^- \) for \( \tilde{\chi} \) from (2.20) and
\[
\chi^- : K^1(X) \times K^1(X) \longrightarrow \mathbb{Z}, \quad \chi^-(\alpha, \beta) = \int_X \text{ch}(\alpha)^\vee \text{ch}(\beta) \text{Td}(X). \quad (2.27)
\]

Then the isomorphism (2.24) induces a graded isomorphism of vertex algebras
\[
\hat{H}_*(\mathcal{P}_X) \cong \mathbb{Q}[K^0(X) \times \mathbb{Z}] \otimes \mathbb{Q} \text{Sym}_{\mathbb{Q}}[u_{\sigma,i}, \sigma \in \mathbb{B}, i > 0],
\]
if the same signs \( \tilde{\epsilon}_{(\alpha,d),(\beta,e)} \) from (2.20) are used for constructing the vertex algebras on both
sides. On the right hand side the degrees are given by

\[
\deg \left( e^{(\alpha,d)} \otimes \prod_{\sigma \in B_{\text{even}} \sqcup \{(0,1)\}} u^{m_{\sigma,i}} \otimes \prod_{v \in B_{\text{odd}}} u^{m_{v,j}} \right)
\]

\[
= \sum_{\sigma \in B_{\text{even}} \sqcup \{(0,1)\}} m_{\sigma,i} 2i + \sum_{v \in B_{\text{odd}}} m_{v,j} (2j - 1) - \tilde{\chi}(\alpha,d), (\alpha,d).
\]

**Proof.** The proof is nearly identical to \[37, \text{Thm. 1.1}\], \[46, \text{Thm. **}\]. We need an explicit expression for chk(θP) replacing Proposition \[37, \text{Prop. 5.2}\] and a similar result in \[46, \text{Thm. **}\] for quivers. This is given in Lemma 4.6.

Before we move on to the applications, let us write down some identities we will need later on. From now on, we always fix a point-canonical orientation of Definition 2.11, the associated signs of ǫ_{\alpha,\beta} of Theorem 2.10 and the corresponding ˜{\epsilon}_{(\alpha,d), (\beta,e)} from (2.20).

**Lemma 2.24.** Consider the vertex algebra \(\hat{H}^*(P_X), |0\rangle, e^zT, Y\), then

i. \(\text{rk}(e^{(\alpha,d)/\sigma^\vee}) = (\alpha,d)(\sigma^\vee)\)

ii. Let \(v_1, \ldots, v_k \in B_{\text{even}}\) and \(i_1, \ldots, i_k \geq 1\), then

\[
T(e^{(\alpha,d)} \otimes u_{v_1,i_1} \cdots u_{v_k,i_k}) = e^{(\alpha,d)} \sum_{\sigma \in B_{\text{even}} \sqcup \{(0,1)\}} (\alpha,d)(\sigma^\vee) u_{\sigma,1} u_{v_1,i_1} \cdots u_{v_k,i_k} + \sum_{l=1}^k \tilde{\epsilon}_{iu_1,i_1} \cdots \tilde{\epsilon}_{iu_l,i_l} u_{v_1,i_1} \cdots u_{v_l,i_l+1} u_{v_l,i_l+1} \cdots u_{v_k,i_k},
\]

iii. For all \(k, l, M, N \geq 0\) we have \(\tilde{\epsilon}_{(kp,N), (lp,M)} = (-1)^{Mk}\).

**Proof.** i. To see this, we use functoriality of the slant product:

\[
\text{rk}(e^{(\alpha,d)/\sigma^\vee}) = \text{rk}(i_{c,b}^*(e^{(\alpha,d)/\sigma^\vee}))
\]

\[
= \text{rk}((\text{id} \times i_{c,b})^*(e^{(\alpha,d)/\sigma^\vee}) = 1 \boxtimes (\alpha,d)/\sigma^\vee = (\alpha,d)(\sigma^\vee),
\]

where \(i_{c,b}\) is an inclusion of a point into \(P_{(\alpha,d)}\). The second statement is a generalization of \[37, \text{Lemma 5.5}\] using i. A similar formula has been shown in \[46\] for quivers. The last statement follows from \[10, \text{Thm. 5.5}\] together with Definition 2.11 and (2.20).

We will often avoid specifying the connected component where the (co)homology class sits by simply omitting \(e^{(\alpha,d)}, e^\alpha\) where it is obvious.

## 3 Cao–Kool conjecture

After reformulating Conjecture 1.1 in terms of the vertex algebra of pairs, we compute (assuming Conjecture 2.19) the virtual fundamental classes of semistable 0-dimensional
sheaves viewed as elements of $H_*(C_X)$ by wall-crossing in the vertex algebra of Definition 3.4 and using the result of Cao–Qu [18, Theorem 1.2]. By wall-crossing back we prove Conjecture 1.1.

3.1 L-twisted vertex algebras

For a Calabi–Yau fourfold $X$ let $\text{Hilb}^n(X)$ be the Hilbert scheme of $n$ points and $[\text{Hilb}^n(X)]^\text{vir} \in H_{2n}(\text{Hilb}^n(X))$ the virtual fundamental class defined by Oh-Thomas [71, Thm. 4.6] using the orientations in Definition 2.11. We consider the vector bundle $L[n] \to \text{Hilb}^n(X)$ given by (1.1). The real rank of $L[n]$ is $2n$, so Cao–Kool [14] define

$$I_n(L) = \int_{[\text{Hilb}^n(X)]^\text{vir}} c_n(L[n]).$$

(3.1)

The proof of Conjecture 1.1 will be given at the end of subsection 3.2 in the following form.

**Theorem 3.1.** Let $X$ be a smooth projective Calabi–Yau fourfold for which Conjecture 2.19 holds, and $L$ a line bundle on $X$. Then

$$I(L; q) = 1 + \sum_{n=1}^\infty I_n(L)q^n = M(-q)^{c_1(L)\cdot c_3(X)}$$

for the point-canonical orientations of Definition 2.11.

For the invariants $I_n(L)$ this is equivalent to

$$I_n(L) = \sum_{k \geq 1} d_k(n) I(L)^k,$$

where

$$d_k(n) = \sum_{\sum n_i = n} \frac{1}{k!} \prod_{i=1}^k (-1)^{n_i} \frac{n_i!}{i^2}, \quad I(L) = c_1(L) \cdot c_3(X).$$

(3.2)

Let us interpret this in the language of §2.5. Take $A_q = A_0$ to be the abelian category of sheaves with 0-dimensional support. Let $B_q = B_0$ be the corresponding category of pairs from Definition 2.12 and $N_0$ its moduli stack with $n = 0$. We have the identification

$$\text{Hilb}^n(X) = N^\text{st}_{(np,1)}(\tau^{\text{pa}}),$$

noting that $p(F) = 1$ for any zero-dimensional sheaf $F$. This gives us

$$[\text{Hilb}^n(X)]^\text{vir} = [N^\text{st}_{(np,1)}(\tau^{\text{pa}})]^\text{vir}$$

by part i. of Conjecture 2.19.

As $\text{Hilb}^n(X)$ carries a universal family $\mathcal{S}_n \to X \times \text{Hilb}^n(X)$, there exists a natural lift
of the open embedding \( \iota_n^\text{pl} : \text{Hilb}^n(X) \to \mathcal{N}_0^\text{pl} \)

\[ \xymatrix{ \text{Hilb}^n(X) \ar[r]^-{\iota_n^\text{pl}} & \mathcal{N}_0^\text{pl} } \]

We use \( \iota_n \) to express (3.1) in terms of insertions on \( \mathcal{N}_0^\text{pl} \).

**Definition 3.2.** Using the notation from Definition 2.13 for all \((np, d) \in C_0(B_0)\) we will write \( \mathcal{N}_{n,d} = \mathcal{N}_{np,d} \). Then define \( \mathcal{L}_{d_1,d_2}^{[n_1,n_2]} \to \mathcal{N}_{n_1,d_1} \times \mathcal{N}_{n_2,d_2} \) by

\[ \mathcal{L}_{d_1,d_2}^{[n_1,n_2]} = (\pi_{n_1,d_1} \times \pi_{n_2,d_2})^* \left( Y^*_d \boxtimes \pi_2^* \left( \pi_X^*(L) \otimes \mathcal{E}_0 \right) \right)_{2,3}, \]

where \( \mathcal{E}_0 \) is the universal sheaf on \( \mathcal{M}_0 \) the moduli stack of \( \mathcal{A}_0 \). It is a vector bundle of rank \( d_1n_2 \). We define

\[ \mathcal{L}^{[-,-]}|_{\mathcal{N}_{n_1,d_1} \times \mathcal{N}_{n_2,d_2}} = \mathcal{L}_{d_1,d_2}^{[n_1,n_2]}. \]

Set \( \mathcal{L} = \Delta^*(\mathcal{L}^{[-,-]}) \). From Example 2.7 we know that \( \iota_n(\mathcal{L}) \) is a weight 0 insertion and from definition it follows that \( \iota_n^* (\mathcal{L}) = L[n] \). Using this together with \([\text{Hilb}^n(X)]_{\text{vir}} = (\Pi^! \circ \iota_n)_* \left( [\text{Hilb}^n(X)]_{\text{vir}} \right)\), we see that

\[ I_n(L) = \int_{[\text{Hilb}^n(X)]_{\text{vir}}} c_n(\mathcal{L}) . \quad (3.4) \]

The following is clear from the construction.

**Lemma 3.3.** The vector bundle \( \mathcal{L}^{[-,-]} \to \mathcal{N}_0 \times \mathcal{N}_0 \) satisfies the conditions of Definition 2.8. Let \((\hat{H}_s(\mathcal{N}_0), [0], e^{\tau T}, Y^L)\) be the \( \mathcal{L}^{[-,-]} \)-twisted vertex algebra, \((\hat{H}_s(\mathcal{N}_0), [-,-]^L)\) the associated Lie algebra. By Proposition 2.9 we have the morphism \((-) \cap c_{\text{cop}}(\mathcal{L}) : (\hat{H}_s(\mathcal{N}_0), [-,-]) \to (\hat{H}_s(\mathcal{N}_0), [-,-]^L) \).

We construct its topological counterpart.

**Definition 3.4.** Define the data \((\mathcal{P}_X, K(\mathcal{P}_X), \Phi_{\mathcal{P}}, \mu_\mathcal{P}, 0, \theta_{\mathcal{P}}^L, \tilde{\chi}^L)\) as follows:

- \( K(\mathcal{P}_X) = K^0(X) \times \mathbb{Z} \).

- Set \( \mathcal{L} = \pi_{2,*}(\pi_X^*(L) \otimes \mathcal{E}) \in K^0(\mathcal{L}_X) \). Then on \( \mathcal{P}_X \times \mathcal{P}_X \) we define

\[ \theta_{\mathcal{P}}^L = (\theta)_{1,3} + \chi(\mathcal{U} \boxtimes \mathcal{U}^\vee)_{2,4} - \left( \mathcal{U} \boxtimes (\pi_{2,*}(\mathcal{E}) - \mathcal{L}) \right)_{2,3} - \left( (\pi_{2,*}(\mathcal{E}) - \mathcal{L}) \boxtimes \mathcal{U}^\vee \right)_{1,4} . \]

- The symmetric form \( \tilde{\chi}^L : (K^0(X) \times \mathbb{Z}) \times (K^0(X) \times \mathbb{Z}) \to \mathbb{Z} \) is given by

\[ \tilde{\chi}^L((\alpha, d), (\beta, e)) = \text{rk} \left( \theta_{\alpha,d,\beta,e}^L \right) = \chi(\alpha, \beta) + \chi de - d(\chi(\beta) - \chi(\beta \cdot L)) - e(\chi(\alpha) - \chi(\alpha \cdot L)) . \quad (3.5) \]
The signs are defined by

$$
\ell^L_{(\alpha,d),(\beta,e)} = (-1)^{d\chi(L,\beta)}\ell_{(\alpha,d),(\beta,e)},
$$

in terms of $\ell_{\alpha,\beta}$ from (2.20).

We denote by $(\hat{H}_s(P_X), [0], e^{T}, Y^L)$ the vertex algebra associated to this data and $(\hat{H}_s(P_X), [-,-]^L)$ the corresponding Lie algebra.

We are unable to use Proposition 2.9 directly because $\mathfrak{V} \otimes \mathfrak{L}$ is not a vector bundle. However, one can easily show the following result similar to Proposition 2.23 and Proposition 2.16.

**Proposition 3.5.** Let $\mathbb{Q}[K^0(X) \times \mathbb{Z}] \otimes _{\mathbb{Q}} \text{Sym}_{\mathbb{Q}}[u_{\sigma,i}, \sigma \in \mathbb{B}, i > 0]$ be the generalized super-lattice vertex algebra associated to $((K^0(X) \otimes \mathbb{Z}) \otimes K^1(X), (\tilde{\chi}^L)^*)$, where $(\tilde{\chi}^L)^* = \tilde{\chi}^L \oplus \chi^-$ for $\tilde{\chi}^L$ from (3.5) and $\chi^-$ from (2.27). The isomorphism (2.28) induces an isomorphism of graded vertex algebras

$$
\tilde{H}_s(P_X) \cong \mathbb{Q}[K^0(X) \times \mathbb{Z}] \otimes _{\mathbb{Q}} \text{Sym}_{\mathbb{Q}}[u_{\sigma,i}, \sigma \in \mathbb{B}, i > 0],
$$

if the same signs $\ell^L_{(\alpha,d),(\beta,e)}$ from (3.6) are used for constructing the vertex algebras on both sides. On the right hand side the degrees are given by

$$
\deg(\ell^{(\alpha,d)} \otimes \prod_{\sigma \in \mathbb{B}_{\text{even}}, i \{0,1\}, i > 0} u_{\sigma,i}^{m_{\sigma,i}} \otimes \prod_{\nu \in \mathbb{B}_{\text{odd}}, j > 0} u_{\nu,j}^{m_{\nu,j}})
$$

$$
= \sum_{\sigma \in \mathbb{B}_{\text{even}}, i \{0,1\}, i > 0} m_{\sigma,i}2i + \sum_{\nu \in \mathbb{B}_{\text{odd}}, j > 0} m_{\nu,j}(2j - 1) - \chi^L((\alpha,d), (\alpha,d)).
$$

The map $\Omega_+ : H_+(N_0) \rightarrow H_+(P_X)$ induces a morphism of graded vertex algebras $(\hat{H}_s(N_0), [0], e^{T}, Y^L) \rightarrow (\hat{H}_s(P_X), [0], e^{T}, Y^L)$ and of graded Lie algebras

$$
\Omega_+ : (\hat{H}_s(N_0), [-,-]^L) \rightarrow (\hat{H}_s(P_X), [-,-]^L).
$$

**Proof.** Using Lemma 1.6 for $\alpha = [L]$, we see that

$$
\text{ch}_k(\mathfrak{V} \otimes \mathfrak{L}) = \sum_{\nu \in \mathbb{B}_{\text{even}}, j \geq i+k} (-1)^j\chi(L,v)\delta_k \otimes \mu_{v,k}.
$$

Using $\chi(L,v) = \chi(v \cdot L)$, one can prove the first part of the theorem by following the proof of [37, Thm. 1.1] or [46, Thm. **]. To show the second part, note that $(\Omega \times \Omega)^*(\mathfrak{V} \otimes \mathfrak{L}) = \mathfrak{L}^{[-,-]}$ and

$$
\xi^L((n_1 p, d_1), (n_2 p, d_2)) := \text{rk}\left(\mathcal{L}_{d_1,d_2}^{n_1,n_2}\right) = d_1 n_2 = d_1 \chi(n_2 p \cdot L).
$$
The statement then follows from Definition 2.8 and Definition 3.4 by the same arguments as in the proof of Proposition 2.16.

This completes the following diagram of morphisms of Lie algebras:

\[
\begin{array}{c}
\hat{H}_*(N_q), [-, -] \xrightarrow{\Omega} \hat{H}_*(P_X, [-, -]) \\
\cap_{\text{top}(L)} \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
isomorphism for all $\alpha \in K^0(X)$

$$H_2(C_\alpha) \cong H^{even}(X) \oplus \Lambda^2 H^{odd}(X). \quad (3.11)$$

We will for now work with the following assumption:

$$H^1(X, \mathcal{O}_X) = 0 = H^1(X, \mathbb{Q}). \quad (3.12)$$

This will be dropped in Remark 4.7, before we compute any integrals on $[\text{Hilb}^n(X)]^{\text{vir}}$.

**Lemma 3.6.** If (3.12) holds, the image of $(\Omega_{np})_* : H_2(N_{(np,0)}) \to H_2(C_{np})$ is contained in $H^6(X) \oplus H^8(X) \text{ under the isomorphism (3.11)}$.

**Proof.** We show that $\Omega^*_{(np,0)}(e^{np} \otimes \mu_{v,1}) = 0$ whenever $v \notin B_6 \cup B_8$. Then for any class $U \in H_4(N_{(np,0)})$ we get

$$e^{np} \otimes \mu_{v,1}((\Omega_{np})_*(U)) = \Omega^*_{np}(e^{np} \otimes \mu_{v,1})(U) = 0$$

for $v \in B_{even} \setminus (B_6 \cup B_8)$ and

$$e^{np} \otimes \mu_{v,1,1}\mu_{w,1}((\Omega_{np})_*(U)) = \Omega^*_{np}(e^{np} \otimes \mu_{v,1,1}\mu_{w,1})(U) = 0$$

for $v, w \in B_{odd}$. The conclusion then follows from (2.26).

The K-theory class $[\mathcal{E}_{np}]$ of the universal sheaf of points on $N_{(np,0)}$ is given by $(\text{id}_X \times \Omega_{np})^*(\mathcal{E}_{np})$. Then from (2.26) we see

$$\text{ch}(\mathcal{E}_{np}) = \sum_{\chi(v) \in B_{i \geq 0}} v \boxtimes \Omega^*_{np}(e^{np} \otimes \mu_{v,i}). \quad (3.13)$$

We also know that $\text{ch}_i(\mathcal{E}_{np}) = 0$ for $i < 4$ by dimension arguments. By assumption H and Poincaré duality, we have $H^7(X) = 0$. We thus only need to consider $v \in B_j$ for $j < 6$. Then from looking at (3.13) we see $v \boxtimes \Omega^*_{np}(e^{np} \otimes \mu_{v,1}) = 0$ because it is in degree $2 + j < 8$ or $1 + j < 8$ and $B$ is a basis. Therefore $\Omega^*_{np}(e^{np} \otimes \mu_{v,1}) = 0$.

Notice that we can write

$$\mathcal{M}_{np} = e^{(np,0)} \otimes 1 \cdot \mathcal{N}_{np} + \mathbb{Q}T(e^{(np,0)} \otimes 1). \quad (3.14)$$

**Proposition 3.7.** Assuming (3.12), there is a unique $\mathcal{N}_{np}$ from (3.14), such that for some $a_v(n) \in \mathbb{Q}$, $v \in B_6$ we have

$$\mathcal{N}_{np} = \sum_{v \in B_6} a_v(n) u_{v,1}.$$
Proof. As \( \mathcal{M}_{np} = (\Omega_{np})_* (|\mathcal{M}_{np}|_{\text{inv}}) \), by Lemma 3.6 we have

\[
\mathcal{N}_{np} = \sum_{v \in B_0 \cup B_8} a_v(n) u_{v,1}.
\]

From Lemma 2.21 we see that \( T(e^{np} \otimes 1) = e^{np} \otimes n u_{p,1} \). Therefore, we get \( H^8(X) = T(H_0(C_{np})) \) which concludes the proof. \( \square \)

Let \( \text{Amp}(X) \subset H^2(X) \) be the image of the ample cone under the natural map \( A^1(X, \mathbb{Q}) \rightarrow H^2(X, \mathbb{Q}) \). Let us choose \( B_2 \) such that its elements are \( c_1(L) \) for very ample line bundles \( L \). This is possible: We assumed \( H^2(\mathcal{O}_X) = 0 \) for \( X \) a CY fourfold, so \( H^2(X) = H^{1,1}(X) \). Thus every element in \( H^2(X) \) is obtained as \( \frac{a}{n} [D] \) for an algebraic divisor \( D \subset X \) and \( m, n \in \mathbb{Z} \). On the other hand \( [D] + n[H] \) is very ample if \( n \gg 0 \) and \( H \) very ample so \( [D] = ([D] + n[H]) - n[H] \), where both terms are very ample.

Using the Poincaré pairing on \( H^2(X) \times H^0(X) \rightarrow \mathbb{Q} \) we choose a basis \( B_6 \) of \( H^0(X) \) which is dual to \( B_2 \).

Lemma 3.8. For each line bundle \( L \) such that \( c_1(L) \in B_2 \) and \( c_1(L) \cdot c_3(X) \neq 0 \), there exist unique orientations \( o_n(L) \) on \( \text{Hilb}^n(X) \) such that Conjecture 1.4 holds for \( L \).

Proof. If \( L \) is very ample, since \( \dim(X) > 1 \), Bertini’s theorem \([41, \text{Thm. 8.18}]\) tells us that there exists a smooth connected divisor \( D \) such that \( L = \mathcal{O}_X(D) \). The lemma then follows from

Theorem 3.9 (Cao–Qu \([18, \text{Thm. 1.2}]\)). Conjecture 1.4 holds for any \( X \) and \( L \cong \mathcal{O}_X(D) \) for a smooth connected divisor \( D \).

The uniqueness of \( o_n(L) \) in the case \( c_1(L) \cdot c_3(X) \neq 0 \) follows because changing orientations changes the sign as \( \text{Hilb}^n(X) \) is connected for all \( n \) and \( X \) by Hartshorne \([40]\). \( \square \)

Let us denote \( o_n \) the orientations on \( \text{Hilb}^n(X) \) induced by the point-canonical orientations. We will see that the orientations \( o_n(L) = o_n \) for all \( L \) with \( c_1(L) \cdot c_3(X) \neq 0 \).

Theorem 3.10. If Conjecture 2.19 holds for \( X \) together with (3.12), then the following is true:

i. For all \( L \) from Lemma 3.8 with \( c_1(L) \cdot c_3(X) \neq 0 \) the orientation \( o_n(L) \) coincide with the ones obtained from the point-canonical orientations in Definition 2.11.

ii. Let \( \mathcal{N}(q) = \sum_{n>0} e^{np} \otimes \mathcal{N}_{np} q^n \) be the generating series, then we can express its exponential as

\[
\exp(\mathcal{N}(q)) = M(e^q) \left( \sum_{v \in B_6} c_3(X)_v q^{u_v,1} \right),
\]

where \( c_3(X)_v = c_3(X)(\text{ch}(v^\vee)) \). Equivalently, we can write this as

\[
\mathcal{N}_{np} = \sum_{l|n} \frac{n}{l^2} \sum_{v \in B_6} c_3(X)_v q^{u_v,1}.
\]
Proof. We prove the theorem by induction on $n$. We begin by giving an explicit formula for the brackets in (3.10). Using (2.5) together with Lemma 2.24 ii., we have:

\[
Y^L(e^{(mp,1)} \otimes 1, z)(e^{(np,0)} \otimes \mathcal{N}_{np})
= (-1)^n e^{(m+n)p,1} \exp \left[ \sum_{j > 0} \frac{b_j}{j} z^j \right] \left[ 1 - z^{-1} \sum_{v \in B_{even}} \chi^L((mp,1),(v,0)) \frac{d}{dv} \right] \mathcal{N}_{np},
\]

where we used that $\mathcal{N}_{np}$ is linear in $u_{v,1}$. Using (3.5) together with Proposition 3.7, we get the following after taking $[z^{-1}(-)]$ of the last formula:

\[
[e^{(mp,1)} \otimes 1, e^{(np,0)} \otimes \mathcal{N}_{np}]^L = -(-1)^n e^{(m+n)p,1} \otimes \sum_{v \in B_0} \int_X c_1(L) \text{ch}(v) a_v(n).
\] (3.17)

Let $L_1$ be such that $c_1(L_1) \in B_2$ and $v_1 \in B_6$ its dual. For now let us not fix the orientation $o_p = o_{can}^p$, but fix $o[O_{X}] = o_{can}^X$ and use the rest of Definition 2.11.

For $n = 1$, we can choose $o_p$ so that $o(1) = o_1(L_1)$, then $I_1(L_1) = -I(L_1)$. Using (3.17) together with (3.6) and (3.2), we get $\int_X c_1(L_1) \text{ch}(v_1) a_{v_1}(1) = I(L_1)$. Therefore $a_{v_1}(1) = c_3(X)_{v_1}$. Suppose that $L_2$ is a line bundle with $c_1(L_2) \in B_2$ different from $c_1(L_1)$ and $I(L_i) \neq 0$ for $i = 1, 2$. If $o_1(L_2) = -o_1(L_1)$, then $a_{v_2}(1) = -c_3(X)_{v_2}$ for $v_2 \in B_6$ the dual of $c_1(L_2)$ and this contradicts Lemma 3.8. For any $A, B \in \mathbb{Z}_{>0}$ we know that $L = L_1^A \otimes L_2^B$ is very ample, then from (3.17) we get

\[
-\left[ AI(L_1) + BI(L_2) \right] = I(L) = -AI(L_1) + BI(L_2)
\]

which can not be true for all $A, B$. For any $v \in B_6$, we then have $a_{v_1}(1) = c_3(X)_{v_1}$.

This shows i. and ii. for $n = 1$ except $o_p = o_{can}^p$. Let us now assume that i. and ii. hold for all $1 \leq k \leq n$ except $o_p = o_{can}^p$. If $o_{n+1}(L_1) = -o(n + 1)$ then $I_{n+1}(L_1) = -[q]^{n+1} [M(1)c_1(L_1)c_3(X)]$. Using the assumption together with (3.17), we get using the notation of (3.2)

\[
\sum_{k>1,n_{1},...,n_{k}>0 \atop n_{1}+...+n_{k}=n+1} \frac{(-1)^k}{k!} \left[ \ldots [e^{(0,1)} \otimes 1, \mathcal{M}_{n_1p}]^{L_1}, \ldots, \mathcal{M}_{nkp}]^{L_1} \right]^{L_1} = \sum_{k>1} d_k(n+1)I(L_1)^k.
\]

Subtracting this from $I_{n+1}(L_1)$ and using (3.2) expresses $a_v(n+1)$ as

\[
a_v(n+1) = -d_1(n+1)I(L_1) - 2d_2(n+1)I(L_1)^2 - \cdots - 2d_{n+1}(n+1)I(L_1)^{n+1},
\]

*Note that, we assume that $I_{n+1}(L_1) \neq 0$. We can do this, because otherwise we would obtain contradiction in the same way using $I(L_1) \neq 0$. 

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Let \( L = L^\otimes N \) for \( N > 0 \), then wall-crossing and using (3.17) with (3.10) gives
\[
I_{n+1}(L) = \sum_{k=2}^{n+1} d_k(n+1)(NI(L_1))^k - d_1(n+1)NI(L_1) - 2N \sum_{k=2}^{n+1} d_k(n+1)I(L_1)^k.
\]

By comparing the coefficients of different powers of \( N \) with \( \pm I_{n+1}(L) \), we obtain a contradiction. This also shows \( o_{n+1}(L_1) = o(n+1) = o_{n+1}(L_2) \) for any \( L_2 \) with \( I(L_2) \neq 0 \). Assuming ii. holds for coefficients \( k < n \) and using (3.8), (3.2) gives us
\[
(-1)^{n+1}a_{v_1}(n+1) = \sum_{k \geq 1} d_k(n+1)I(L_1)^k - \sum_{k>1} d_k(n+1)I(L_1)^k
\]
\[
= (-1)^{n+1} \sum_{l|n+1} \frac{n+1}{l^2} c_3(X)_{v_1}.
\]

This holds for all \( L_i \in B_2 \) with their duals \( v_i \in B_6 \), so part ii. follows as we obtain (3.16).

To finish the proof of part i., we only need to show that \( o_p = o_p^{\text{can}} \). For this, choose \( L \) such that \( I(L) \neq 0 \). Using Lemma 4.6, we see \( c_1(\mathcal{L}) = \sum_{v \in B_{\text{even}}} \chi(L^\vee, v)\mu_{v,1} \). Using \( X = M_p \) and (2.12), we see that \( \int_{[M_p]} \chi_L \) is equal to
\[
\int_{\mathcal{M}_p} \sum_{v \in B_{\text{even}}} \chi(L^\vee, v)\mu_{v,1}, \quad \mathcal{M}_p = \sum_{v \in B_6} c_3(X)_v u_{v,1} + c_p u_{p,1},
\]
which gives \( I(L) + c_p \). As \( c_p \) does not depend on \( L \) it has to be 0. Therefore for the invariants to coincide, we need \( o_p = o_p^{\text{can}} \).

**Remark 3.11.** Changing orientation \( o_p \mapsto -o_p \) changes \( o_{np} \mapsto (-1)^n o_{np} \), so if the classes \( [\mathcal{M}_{np}]_{\text{inv}} \) were constructed using Borisov–Joyce [12] or Oh–Thomas [71], then we would get
\[
\mathcal{M}_{np} = (-1)^n \sum_{l|n} \frac{n}{l^2} \sum_{v \in \Lambda} c_3(X)_v u_{v,1}
\]
However, as these are obtained indirectly through wall-crossing, we should check this is satisfied. Choosing \( o_p \) such that \( o(1) = -o_1(L_1) \) in the proof of Theorem 3.10 does indeed give this formula. Similarly, switching to \( -o_p^{\text{can}} \) does not change the result as it should not.

The following is shown just for completeness, as we will prove a much more general statement for tautological insertions using any K-theory class in \( \S 5.1 \).

**Proof of Theorem 3.1** Using (3.17), (3.16) and (3.10) we obtain for any line bundle \( L \) that (3.2) holds.
4 Virtual classes of Hilbert schemes of points and invariants

In this section, we use the result of Theorem 3.10. One could think of $\text{e}^{\text{np}} \otimes N_{np} \in H_2(\mathcal{C}_X)$ and $\mathcal{H}_n \in H_0(\mathcal{P}_X)$ as explicit invariants already. We use wall-crossing from (3.9) to compute $\mathcal{H}_n$ and then consider insertions, which can be expressed in the form $\exp \left[ F(\mu_{v,k}) \right]$, where $F(\mu_{v,k})$ is linear in $\mu_{v,k}$. After obtaining a general formula for the corresponding invariants, we apply it to multiplicative genera of tautological classes and virtual tangent bundles showing that they fit into this class. Thus we obtain an explicit expression for these, which will be used in the following section to compute new invariants.

4.1 Virtual fundamental cycle of Hilbert schemes

The following could be viewed as the main result of this chapter.

**Theorem 4.1.** If Conjecture 2.7 and (3.12) hold, then the generating series $\mathcal{H}(q) = 1 + \sum_{n>0} \frac{\mathcal{H}_n}{n!} q^n$ for point-canonical orientations is given by

$$\mathcal{H}(q) = \exp \left[ \sum_{n>0} \sum_{l, v \in B_6} (-1)^n \frac{n}{n!} c_3(X) v [z^n] \left\{ U_v(z) \exp \left[ \sum_{j>0} \frac{n j v}{j} z^j \right] \right\} q^n \right], \quad (4.1)$$

where we fix the notation $y_j = u_{p,j}$ and $U_v(z) = \sum_{k>0} u_{v,k} z^k$.

**Remark 4.2.** Notice that the only $u_{\sigma,k}$ that appear in (4.1) are for $\sigma = (v,0)$, $v \in B_6 \cup B_8 =: B_{6,8}$. We may therefore assume $K^1(X) = 0$ from now on when (3.12) holds (see also Remark 3.17 for the general case). As there is no contribution of $b_j$, this is the unique representation of $\mathcal{H}_n$ without terms with $b_j$ as can be seen from Lemma 2.24. Using (3.3), we have a class $\tilde{\mathcal{H}}_n = \Omega_\ast \circ \iota_{n,v} \left( [\text{Hilb}^n(X)]^{\text{virt}} \right)$ which satisfies $\Pi_0(\tilde{\mathcal{H}}_n) = \mathcal{H}_n$. There will also be no terms containing $b_j$ in $\tilde{\mathcal{H}}_n$, thus $[q^n] (\mathcal{H}(q))$ describe this canonical lift.

**Proof.** We begin by using the reconstruction lemma for vertex algebras to write the field $Y(e^{\text{np}} \otimes u_{v,1}) = Y(u_{v,1}, z) Y(e^{\text{np}} \otimes 1, z) :$, where $: -$ : denotes the normal ordered product for fields of vertex algebras (see [57, §3.8], [26, Def. 2.2.2], [51, §3.1]) which acts on $e^{\text{mp}} \otimes U$ as

$$Y(u_{v,1}, z) Y(e^{\text{np}} \otimes 1, z) : (e^{\text{mp}} \otimes U) =$$

$$(-1)^n z^{-n} e^{((n+m)p,1)} \otimes \left( \sum_{k>0} u_{v,k} z^{k-1} \right) \exp \left[ \sum_{i>0} \frac{n y_i}{i} z^i \right] \exp \left[ - n \sum_{i>0} \frac{d}{d u_{[i],j}} z^{-i} \right]$$

$$\exp \left[ \sum_{i>0} \frac{d}{d b_j} z^i \right] + \exp \left[ \sum_{i>0} \frac{n y_i}{i} z^i \right] \exp \left[ - n \sum_{i>0} \frac{d}{d u_{[j],i}} z^{-i} \right]$$

$$\exp \left[ \sum_{i>0} \frac{d}{d b_j} z^i \right] \left[ \tilde{\chi}((v,0),(mp,1)) z^1 + \sum_{k>0, \sigma \in \mathbb{B}} k \tilde{\chi}((v,0),\sigma) \frac{d}{d u_{\sigma,k}} z^{k-1} \right] U \quad (4.2)$$

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Where we used \((\ref{2.20})\) to get \(\tilde{\chi}((np,0),\sigma) = n\) if \(\sigma = ([\mathcal{O}_X],0)\), \(\tilde{\chi}((np,0),\sigma) = -n\) if 
\(\sigma = (0,1)\) and 0 otherwise together with part iii. of Lemma \((\ref{2.24})\).

We claim that for any \(r > 0, n_1,\ldots,n_r > 0\) and \(\sum_{i=1}^r n_i = n\), we have the following :

\[
[e^{(np, 0)} \otimes N(n_1 p), e^{(n_2 p, 0)} \otimes N(n_2 p), \ldots, e^{(n_r p, 0)} \otimes N(n_r p), e^{(0, 1)} \otimes 1] \ldots
\]

\[
= e^{(np, 1)} \otimes \prod_{i=1}^r \mathcal{H}_{n_i}
\]

in \(\tilde{H}(\mathcal{P}_X)\), where

\[
\mathcal{H}_n = \sum_{l|n, v \in B_6} (-1)^n \frac{n}{l^2} c_3(X)_v[z^n] \left\{ U_v(z) \exp \left[ \sum_{j > 0} \frac{n_j y_j}{j} z^j \right] \right\}.
\]

We show this by induction on \(r\), where for \(r = 0\) it is obvious. Assuming that it holds for \(r - 1\), we need to compute

\[
[e^{(np, 0)} \otimes N(n_1 p), e^{((n-n_1)p, 1)} \otimes \prod_{i=2}^r \mathcal{H}_{n_i}]
\]

\[
= [z^{-1}] \left\{ Y(u_{v, 1}, z) Y(e^{(np, 0)} \otimes 1, z) : e^{((n-n_1)p, 1)} \otimes \prod_{i=2}^r \mathcal{H}_{n_i} \right\}
\]

From Remark \((\ref{4.2})\), we see that we can replace all \(\exp \left[ n \sum_{i > 0} \frac{d}{d\tilde{v}_i} \right] \) and \(\exp \left[ -n \sum \frac{d}{d\tilde{v}_i} [\mathcal{O}_X]^1, i] \right] \) by \(1\) in \((\ref{4.2})\). The second term under the curly bracket in \((\ref{4.2})\) vanishes, because it contains \(\tilde{\chi}((v, 0), (mp, 1)) z^{-1} = \chi(v, mp) z^{-1} - \chi(v) z^{-1}\) where the result is zero for degree reasons and because \(td_1(X) = 0\). In the term with \(\sum_{k > 0, \sigma \in S} k \tilde{\chi}((v, 0), \sigma) \frac{d}{d\tilde{u}_k} z^{-k-1}\) the sum can be taken over all \(\sigma = (w, 0), w \in B_{6, 8}\) by Remark \((\ref{4.2})\) so it vanishes because \(\chi(v, w) = 0\) whenever \(v, w \in B_{6, 8}\). We are therefore left with

\[
[z^{-1}] \left\{ (-1)^n z^{-n_1} e^{(np, 1)} \otimes \left( \sum_{k > 0} u_{v, k} z^k \right) \exp \left[ \sum_{i > 0} \frac{n_1 y_i}{i} z^i \right] \right\} \prod_{i=2}^r \mathcal{H}_{n_i}
\]

\[
= (-1)^n e^{(np, 1)} [z^{-1}] \left\{ U_v(z) \exp \left[ \sum_{j > 0} \frac{n_1 y_j}{j} z^j \right] \right\} \prod_{i=2}^r \mathcal{H}_{n_i}.
\]

Multiplying with the coefficients \(\sum_{l|n} \frac{n}{l^2} \sum_{v \in B_6} c_3(X)_v \) of \(\mu_{v, 1}\) in \((\ref{3.10})\) and summing over all \(v \in B_6\), we obtain the result as we are able to rewrite \((\ref{3.9})\) by reordering the terms keeping track of the signs as

\[
\mathcal{M}_n = \sum_{\substack{k \geq 1, n_1, \ldots, n_k \\
\sum_{n_i = n}} \frac{1}{k!} [e^{(np, 0)} \otimes N(n_1 p), \ldots, e^{(nk_p, 0)} \otimes N(n_k p), e^{(0, 1)} \otimes 1] \ldots]
\]

\(\square\)
We now describe a general formula for integrating topological insertions over $\mathcal{H}_n$ which will be applied in the following to examples.

**Proposition 4.3.** Let $\mathcal{A} \subset K^0(X)\setminus\{0\}$ be a finite subset. For each $\alpha \in \mathcal{A}$ let us have some exponential generating series

$$A_{\alpha}(z, p) = \sum_{n \geq 0} a_{\alpha}(n, p) \frac{z^n}{n!},$$

where $p = (p_1, p_2, \ldots)$ are additional variables and $\alpha(n, p) \in \mathbb{Q}[p]$, s.t. $a_{\alpha}(0, 0) = 0$. If $I_n \in H^*(\text{Hilb}^n(X))$ is such that $I_n = (\Omega \circ \iota_n)^*(T)$ for a weight 0 insertion $T \in H^*(\mathcal{P}_X)$, where

$$\int_{\mathcal{H}_n} T = \int_{\mathcal{H}_n} e^{(np,1)} \otimes \exp \left[ \sum_{\alpha \in \mathcal{A}} \sum_{k \geq 0} a_{\alpha}(k, p) \chi(\alpha^\vee, v) \mu_{v,k} \right],$$

then the generating series $\text{Inv}(q) = 1 + \sum_{n > 0} \int_{[\text{Hilb}^n(X)]_{vir}} I_n q^n$ is given by

$$\prod_{\alpha \in \mathcal{A}} \exp \left\{ \sum_{n > 0} (-1)^n \sum_{l | n} \frac{n}{l^2} \left[ z^l d z \left( A_{\alpha}(z, p) - A_{\alpha}(0, p) \right) \right] \right\} c_1(\alpha) c_3(X).$$

(4.3)

**Proof.** Using Lemma 2.22 to act on the homology classes $\mathcal{H}_n$ from Theorem 4.1, we obtain

$$\text{Inv}(q) = \exp \left[ \sum_{n > 0} \sum_{l | n} \frac{n}{l^2} c_3(X)_v [z^n] \left\{ \sum_{k > 0} \sum_{\alpha \in \mathcal{A}} \chi(\alpha^\vee, v) a_{\alpha}(k, p) \frac{z^k}{(k-1)!} \right\} \right] \cdot \exp \left[ \sum_{j \geq 0} \sum_{\alpha \in \mathcal{A}} \frac{\text{rk}(\alpha) a_{\alpha}(j, p)}{j!} \frac{z^j}{j!} \right],$$

which can be seen to be equal to (4.3). \qed

We get the following simple consequence of the above results.

**Corollary 4.4.** With the notation and assumptions from Proposition 4.3 it follows that $\text{Inv}(q)$ depends only on $c_1(\alpha) \cdot c_3(X)$ and $\text{rk}(\alpha)$ for all $\alpha \in \mathcal{A}$. For more general insertions, the invariants only depend on $\int_X c_3(X) \cdot (-) : H^2(X) \rightarrow \mathbb{Z}$.

**Remark 4.5.** For the classes $[\mathcal{M}_{np}]_{inv} \in \hat{H}_2(\mathcal{N}_X)$, we did not find any interesting non-zero invariants of the form $\int_{[\mathcal{M}_{np}]_{inv}} \nu$ for some weight 0 insertion $\nu$ on $\mathcal{N}_X$. We already know that $\mathcal{L}|_{\mathcal{N}_{0,0}} = 0$. On the other hand, if one takes $I_0(\alpha) = \pi_2^*(\pi_X^*(\alpha) \otimes \mathcal{E}_0)$ on $\mathcal{M}_0$ for any class $\alpha \in G^0(X)$ (see \S 4.11), we can consider its topological counterpart $\mathcal{L}_{\text{wt}=1}(\alpha) = \pi_2^*(\pi_X^*(\alpha) \otimes \mathcal{E})$.
on $\mathcal{C}_X$ which has weight 1. Then denoting $\nu = p(ch_1(\mathcal{X}_{wt=1}(\alpha)), ch_2(\mathcal{X}_{wt=1}(\alpha)), \ldots)$, the integral $\int_{[\mathcal{M}_{n,p}]} \nu$ is not well defined as it does not satisfy Lemma 2.6.

Moreover, consider the complex $\mathbb{E}_0 = \pi_*(\text{Hom}_{\mathcal{M}_0}(\mathcal{E}_0, \mathcal{E}_0))$ on $\mathcal{M}_0$, then this will be weight zero. However, taking $\nu = p(ch_1(\mathcal{E}_0), ch_2(\mathcal{E}_0), \ldots)$ we get

$$\int_{[\mathcal{M}_{n,p}]} \nu = \int_{e(0,np) \oplus \mathcal{M}_{n,p}} p(ch_1(\Delta^*(\theta)), ch_2(\Delta^*(\theta), \ldots),$$

which can be shown to be always zero.

### 4.2 Multiplicative genera as insertions

The main examples we want to address are multiplicative genera of tautological classes below.

For a scheme $S$, let $G^0(S)$ and $G_0(S)$ denote its Grothendieck groups of vector bundles and coherent sheaves respectively. We have the map $\lambda : G^0(S) \to K^0(S)$ which we often neglect to write, i.e. $\lambda(\alpha) = \alpha$. We have the Chern-character $ch : G^0(S) \to A^*(S, \mathbb{Q})$ which under the natural maps to $K^0(S)$ and $H^*(S, \mathbb{Q})$ corresponds to the topological Chern-character $ch : K^0(S) \to H^{even}(X, \mathbb{Q})$.

Let $f(p,z) = \sum_{n \geq 0} f_n(p) z^n \in \mathbb{Q}[p][[z]]$ be a formal power-series in formal power-series of additional variables $p = (p_1, \ldots, p_k)$ with $f(0,0) = 1$, then a multiplicative genus $\mathcal{G}_f(p,\cdot)$ of Hirzebruch [42, §4] associated to $f$ is a group homomorphism

$$\mathcal{G}_f : G^0(X) \longrightarrow (A^*(X, \mathbb{Q})[p]_1,$$

where $(A^*(X, \mathbb{Q})[p])_1$ denotes the multiplicative group of the ring $A^*(X, \mathbb{Q})[p]$ containing power-series with constant term in $p$ and $A^0(X, \mathbb{Q})$ being 1. For each vector bundle $E \to X$ of $\text{rk}(E) = a$ there is by using the splitting principles a unique factorization $c(E) = \prod_{i=1}^a (1 + x^i)$, where $x^i \in A^1(X, \mathbb{Q})$. Then $\mathcal{G}_f$ is given by

$$\mathcal{G}_f(E) = \prod_{i=1}^a f(p, x_i).$$

Define $\Lambda^*_i : G^0(S) \to \left(G^0(S)[t]\right)_1$ to be a group homomorphism acting on each vector bundle $E$ by

$$[E] \mapsto \sum_{i=0}^\infty [\Lambda^i E] (-t)^i,$$

where $\left(G^0(S)[t]\right)_1$ denotes the power-series in $t$ with constant term $[\mathcal{O}_X] \in G^0(S)$, $G_0(S)$ is a group under the addition and $\left(G^0(S)[t]\right)_1$ under the multiplication induced by the tensor product. We also have $\text{Sym}^*_i : G^0(S) \to \left(G^0(S)[t]\right)_1$ under the multiplication induced by the tensor product. We also have $\text{Sym}_i^*(\alpha) = \left(\Lambda^i_*(\alpha)\right)^{-1}$.
for all $\alpha \in G^0(S)$. On $\text{Hilb}^n(X)$, we will consider the classes
\[ \alpha[n] = \pi_2^*(\pi_X^*(\alpha) \otimes \mathcal{F}_n), \quad \alpha \in G^0(X), \quad T_n^{\text{vir}} = \text{Hom}_{\text{Hilb}^n(X)}(\mathcal{I}_n, \mathcal{I}_n)[1], \quad (4.5) \]
where $\mathcal{I}_n = (\mathcal{O}_X \to \mathcal{F}_n)$ is the universal complex on $\text{Hilb}^n(X)$ and $(-)_0$ denotes the trace-less part. The corresponding topological analogs are
\[ \mathfrak{T}(\alpha) = \mathfrak{U}^\vee \otimes \pi_2^*(\pi_X^*(\alpha) \otimes \mathcal{E}) \quad \text{and} \quad -\theta_P^\vee \in K^0(\mathcal{P}_X \times \mathcal{P}_X). \]

Lemma 4.6. In $H^*(\mathcal{P}_X \times \mathcal{P}_X)$ the following holds for all $\alpha \in K^0(X)$:
\[ \text{ch}_k(\mathfrak{T}(\alpha)) = \sum_{v \in B_{\text{even}}} (-1)^l \chi(\alpha^\vee, v)\beta_i \otimes \mu_{v,j}, \]
\[ \text{ch}_k(\theta_P) = \sum_{\sigma, \tau \in B \setminus B_{\text{odd}}} (-1)^l \tilde{\chi}(\sigma, \tau)\mu_{\sigma,i} \otimes \mu_{\tau,j} + \sum_{v, w \in B_{\text{odd}}} (-1)^{l+1} \chi^-(v, w)\mu_{v,i} \otimes \mu_{w,j}. \]

We also have the identity
\[ \text{ch}(T_n^{\text{vir}}) = -(\Omega \circ \iota_n)^*(\text{ch}(\Delta^*\theta_P^\vee)) + 2 \]
in $H^*(\text{Hilb}^n(X))$.

Proof. Using Atiyah–Hirzebruch–Riemann–Roch [22], we get
\[ \text{ch}_i(\pi_2^*(\pi_X^*(\alpha) \otimes \mathcal{E})) = \sum_{v \in B_{\text{even}}} \int_X \text{ch}(\alpha)\text{ch}(v)\text{Td}(X) \otimes \mu_{v,i} = \sum_{v \in B_{\text{even}}} \chi(\alpha^\vee, v)\mu_{v,j}. \]

Taking a product with $\text{ch}(\mathfrak{U}^\vee)$ and using $\beta_j = \text{ch}_j(\mathfrak{U})$, we get
\[ \text{ch}_j(\mathfrak{T}(\alpha)) = \sum_{v \in B_{\text{even}}} (-1)^l \chi(\alpha^\vee, v)\beta_l \otimes \mu_{v,k}. \quad (4.6) \]

A similar explicit computation leads to the second formula. Let us therefore address the final statement.

Let $\mathcal{P} : \text{Hilb}^n(X) \to \mathcal{M}_X$ map $[\mathcal{O}_X \to \mathcal{F}]$ to $[\mathcal{O}_X[1] \oplus \mathcal{F}]$ and $\text{Ext}_n = \text{Hom}_{\text{Hilb}^n(X)}(\mathcal{I}_n, \mathcal{I}_n)$. We have the following $A^1$-homotopy commutative diagram:
\[ \begin{array}{ccc}
\text{Hilb}^n(X) & \xrightarrow{\mathcal{P}} & \mathcal{M}_X \\
\downarrow{\text{Ext}_n} & & \downarrow{\text{Ext}_n} \\
\text{Perf}_C & & \\
\end{array} \quad (4.7) \]

where $\text{Ext}_n$ are the maps associated to the perfect complexes of the same name and $\text{Ext} = \text{Hom}_{\mathcal{M}_X}(\mathcal{U}, \mathcal{U})$ for the universal perfect complex $\mathcal{U} \to X \times \mathcal{M}_X$. From Definition
we easily deduce \( \iota_n^* \circ \Delta_*(\Theta^{pa}) = \mathcal{P}^* \mathcal{E} \text{xt}^\vee \). Taking topological realization of (4.7), we obtain that

\[
[\mathcal{E} \text{xt}_n] = (\mathcal{P}^{top})^* [\mathcal{E} \text{xt}] = \iota_n^* [\Delta^* (\Theta^{pa})]^\vee = (\Omega \circ \iota_n)^* (\Delta^* (\theta_p))^\vee.
\]

Finally, we use \( \text{rk}((\mathcal{E} \text{xt}_n)_0) = \text{rk}(\mathcal{E} \text{xt}_n) - 2 \).

**Remark 4.7.** We explain now how to drop the condition (3.12). Going through the arguments in Theorem 3.10 and Theorem 4.1 without the assumption (3.12), one can check that under the projection \( \Pi_{even} : \hat{H}_s(P_X) \to \hat{H}_{even}(P_X) \) we still obtain the same results. This is sufficient for us, because we never integrate odd cohomology classes, except when integrating polynomials in \( \text{ch}_k(T^\text{vir}_n) \), but as the only terms \( \mu_{v,k} \) contained in \( \text{ch}_k(T^\text{vir}) \) for \( v \in B_{odd} \) are given for \( v \in B_7 \), each such integral will contain a factor of \( \chi^{-}(v,w) = 0 \) for \( v, w \in B_7 \). Using \( \Delta_{23} \) we may therefore now assume \( K^1(X) = 0 \) in general.

To simplify notation, we will not write \( P \) unless necessary and use \( f(\alpha^{[n]}) \) and \( f(T^\text{vir}_n) \) instead of the full \( \mathcal{G}_f(-) \).

**Lemma 4.8.** Let \( f \) be an invertible power-series, then

\[
\int_{[\text{Hilb}^n(X)]^\text{vir}} f(\alpha^{[n]}) = \int_{\mathcal{M}_n} \exp \left[ \sum_{k \geq 0 \atop v \in B_{6,8}} a_v(k) \chi(\alpha^\vee, v) \mu_{v,k} \right], \quad \text{where}
\]

\[
A_v(z) = \sum_{k \geq 0} a_v(k) \frac{z^k}{k!} = \log(f(z)),
\]

\[
\int_{[\text{Hilb}^n(X)]^\text{vir}} f(T^\text{vir}_n) = \int_{\mathcal{M}_n} \exp \left[ \sum_{k \geq 0 \atop v \in B_{6,8}} a_{[O_X]}(k) \chi(v) \mu_{v,k} \right], \quad \text{where}
\]

\[
A_{[O_X]}(z) = \sum_{k \geq 0} a_{[O_X]}(k) \frac{z^k}{k!} = \log(f(z)f(-z)).
\]

**Proof.** We show that in the action of \( \text{ch}_k(\theta_p^\vee) \) from Lemma 4.6 on \( \mathcal{M}_n \) only terms linear in \( \mu_{v,k}, k > 0 \) have non-trivial contributions. In Remark 4.4 we set \( K^1(X) = 0 \), thus we only need to look at \( \sum_{i+j=k \atop \sigma, \tau \in \mathcal{B} \setminus B_{odd}} (-1)^i \tilde{\chi}(\sigma, \tau) \mu_{\sigma,i} \boxtimes \mu_{\tau,j} \) and we claim it reduces to the action by

\[
- \sum_{v \in B_{6,8}} (1 + (-1)^k) \chi(v) \mu_{v,k} = -(1 + (-1)^k) \mu_{p,k}
\]

(4.8)

For \( i, j > 0 \) if \( \sigma = (0,1) \) or \( \tau = (0,1) \), then due to Remark 4.2 this term vanishes. If \( \sigma = (v,0), \tau = (w,0) \) then \( v, w \in B_{6,8} \) and \( \tilde{\chi}(\sigma, \tau) = \chi(v,w) = 0 \). So consider the case \( i = 0 \), then \( j = k > 0 \). If \( \sigma = (v,0), \tau = (w,0) \) or \( \tau = (0,1) \) then the term is again 0, because \( \mu_{v,0} = np(v^\vee) = 0 \) unless \( v = p \) in which case \( \chi(v,w) = 0 \) for each \( w \in B_{6,8} \). However, if \( \sigma = (0,1), \tau = (v,0) \), then \( \mu_{\sigma,0} = 1 \) and \( \tilde{\chi}((0,1),(v,0)) = -\chi(v) \). If \( j = 0 \), then the same applies, thus the statement follows.
Let \( E \) be a vector bundle with \( c(E) = \prod_{i=1}^a (1 + x_i) \), then we write
\[
f(E) = \prod_{i=1}^a f(x_i) = \exp \left[ \sum_{n>0} g_n \sum_{i=1}^a \frac{x_i^n}{n!} \right],
\]
where \( \sum_{n>0} \frac{g_n}{n!} x^n = \log (f(x)) \) and \( \sum_{i=1}^a \frac{x_i^n}{n!} = \chi_i(E) \). This extends to any class \( \alpha \in G^a(\text{Hilb}^n(X), \mathbb{Q}) \), so we get after using Remark 4.2, (4.8) and Lemma (4.6) that
\[
\int_{\text{Hilb}^n(X)} \chi_1(T^\text{vir}_n), \chi_2(T^\text{vir}_n), \ldots = \int_{\mathbb{R}_n} \tilde{p}_n(\mu_{p,1}, \mu_{p,2}, \ldots),
\]
where we use \( \mu_{p,0} = n \) and \( \text{rk}(T^\text{vir}_n) = 2n \). From this we immediately see \( A_\alpha(z) = \log(f(z)) \) and \( A[\mathcal{O}_X](z) = \log(f(z)f(-z)) \).

As an immediate Corollary of (4.8), we obtain the following:

**Corollary 4.9.** For each \( n \) let \( p_n(x_1 t, x_2 t^2, \ldots) \) be a formal power-series in infinitely many variables, then
\[
\int_{\text{Hilb}^n(X)} p_n(\chi_1(T^\text{vir}_n), \chi_2(T^\text{vir}_n), \ldots) = 0.
\]

**Proof.** We use
\[
\int_{\text{Hilb}^n(X)} p_n(\chi_1(T^\text{vir}_n), \chi_2(T^\text{vir}_n), \ldots) = \int_{\mathbb{R}_n} \tilde{p}_n(\mu_{p,1}, \mu_{p,2}, \ldots),
\]
where we use some new formal power-series \( \tilde{p}(x_1 t, x_2 t^2, \ldots) \) given by (4.8). Because each term in (4.1) contains at least one factor of the form \( \mu_{v,k} \) for \( v \in B_6, k > 0 \), the above integral is zero by (2.25).

**Definition 4.10.** Let us define the universal transformation \( U \) of formal power-series \( U : (R[t])_1 \to (R[t])_1 \) by
\[
f(t) \mapsto \prod_{n>0} \prod_{k=1}^n f(-e^{2\pi i k/n} t)^n,
\]
for any ring \( R \). Moreover, we will use the notation
\[
\{f\}(t) = f(t)f(-t).
\]

In fact, \( U \) is a well-defined bijection. To see this, note: \( \log \left( \prod_{k=1}^n f(-e^{2\pi i k/n} t)^n \right) = \sum_{m=0}^n n^2 f_{mn} t^m \) by Knuth [52, eq. (13), p. 89]. Therefore \( \prod_{k=1}^n f(-e^{2\pi i k/n} t)^n = 1 + O(t^n) \).
This is precisely the condition necessary for the infinite product to be well-defined. Moreover, it maps integer valued power-series back to integer-valued ones. To construct an inverse, we can take the logarithm of (4.9) to get

\[ \sum_{n>0} \sum_{m=0}^{\infty} n^2 f_{nm} q^{nm} = \sum_{n>0} \sum_{m=0}^{\infty} n^2 f_n q^n \]

where \( \log(f(q)) = \sum_{n>0} f_n q^n \). This corresponds to acting with a diagonal invertible matrix on the coefficients \( f_n \), so we have an inverse.

**Example 4.11.** Acting with \( U^{-1} \) on the MacMahon function \( M(-q) \), we obtain \( \frac{1}{1+q} \).

We will need later the following generalization of the Lagrange inversion theorem:

**Lemma 4.12.** Let \( Q(t) \in R[[t]] \) (with a non-vanishing constant term) and \( g_i(x) \) for \( i = 1, \ldots, N \) be the different solutions to

\[ (g_i(x))^N = xQ(g_i(x)) \quad (4.10) \]

then for any formal power-series \( \phi(t) \), \( \phi(0) = 0 \) we have

\[ \sum_{k=1}^{N} \phi(g_i(x)) = \sum_{n>0} \frac{1}{n} [t^{nN-1}] (\phi(t)Q(t)^n)x^n. \]

**Proof.** The usual Lagrange formula (see e.g. Gessel [31, Thm. 2.1.1]) tells us that for \( h(x) = xQ(h(x)) \), we have \( [x^n]\phi(h(x)) = \frac{1}{n}[t^n-1]\phi(t)Q(t)^n \) for \( n > 0 \). Taking the unique Newton–Puiseux series satisfying \( g(x^{1/N}) = x^{1/N} Q^{1/N} (g(x^{1/N})) \) for a fixed \( N \)'th root of \( Q \), we can write by Weierstrass preparation theorem together with the Newton–Puiseux theorem (see e.g. [45, Chap. 3.2, Chap. 5.1, ] every solution of (4.10) by \( g_k(x) = g(e^{2\pi k/N}x^{1/N}) \). We obtain

\[ \sum_{k=1}^{N} \phi(g_k(x)) = \sum_{k=1}^{N} \sum_{n>0} \frac{1}{n} [t^{nN-1}] (\phi(t)Q^{1/N}(t)) \left( e^{2\pi i k/N} x^{1/N} \right)^n \]

\[ = \sum_{n>0} \frac{1}{n} [t^{nN-1}] (\phi(t)Q^n(t)) x^n. \]

We prove now the main result that we will use throughout the next section.

**Proposition 4.13.** Let \( f_0(p, \cdot), f_1(p, \cdot), \ldots, f_M(p, \cdot) \) be power-series with \( f(0, 0) = 1 \), then define

\[ \text{Inv}(\vec{f}, \vec{\alpha}, q) = 1 + \sum_{n>0} \int_{[\text{Hilb}^n(X)]_\text{vir}} f_0(T^n_{\text{vir}})f_1(\alpha_1^{[n]}) \cdots f_M(\alpha_M^{[n]}) q^n, \]

where \( (\vec{\cdot}) \) is meant to represent a vector, and we omit the additional variables. Then setting
\[ \text{rk}(\alpha_i) = a_i, \text{ we have} \]

\[ \text{Inv}(\tilde{f}, \tilde{a}, q) = U \left\{ \prod_{i=1}^{M} f_i(H(q)) \right\}^{c_1(\alpha_i) - c_3(X)}, \tag{4.11} \]

where \( H(q) \) is the unique solution for

\[ q = \frac{H(q)}{\prod_{j=1}^{M} f_j''(H(q)) f_0(H(q))}. \]

**Proof.** Combining Lemma 4.8 with Proposition 4.3, we obtain

\[ \text{Inv}(q) = \prod_{i=1}^{M} \exp \left\{ \sum_{n>0} \sum_{l|n} \frac{n^2}{l^2} (-1)^n [z^n-1] \left[ \frac{d}{dz} \left( \log f_i(z) - \log f_i(0) \right) \right] \prod_{j=1}^{M} f_j(z)^{a_j n} [f_0(z)]^{n} q^n \right\}^{c_1(\alpha_i) - c_3(X)} \]

setting \( \phi = \log(f_i) - \log(f_i(0)) \), \( Q = \prod_{i=1}^{M} f_i''(f_0) \) and using Lemma 4.12, this gives

\[ \prod_{i=1}^{M} \exp \left\{ \sum_{n>0} \sum_{l|n} \frac{n^2}{l^2} [t^n] \log f_i(H(t)) - \log f_i(0) \right\}^{c_1(\alpha_i) - c_3(X)} \]

\[ = \prod_{i=1}^{M} \exp \left\{ \sum_{n>0} \sum_{k=1}^{n} [t^n] \log f_i(H(t)) - \log f_i(0) \right\}^{c_1(\alpha_i) - c_3(X)} \]

\[ = \prod_{i=1}^{M} \prod_{n>0} \prod_{k=1}^{n} \left[ f_i(H(-e^{2\pi i k} n q)) \right]^{nc_1(\alpha_i) - c_3(X)} = \prod_{i=1}^{M} \left[ f_i(H(q)) \right]^{c_1(\alpha_i) - c_3(X)}, \]

where \( H(q) \) is the solutions of \( 4.11 \). \( \square \)

### 5 New invariants

We define and compute many new invariants using the formula derived in the previous section. These include tautological series, virtual Verlinde numbers and Nekrasov genera. We study their symmetries and their relation to lower-dimensional geometries. We obtain an explicit correspondence between virtual Donaldson invariants on elliptic surfaces and DT4 invariants on projective Calabi–Yau fourfolds via the universal \( U \) transformation. Note that the Segre–Verlinde correspondence among the results that follow could be traced back to already existing results of Oprea–Pandharipande [72] and Arbesfeld–Johnson–Lim–Oprea–Pandharipande [4] using Theorem 5.15, but as we worked these out independently we prefer to present them so. The final section is dedicated to wall-crossing for quot-schemes of elliptic surfaces and curves.
5.1 Segre series

Setting $f_0 = 1$ and $f_i = (1 + t_i x)^{-1}$ in Proposition 4.13, we obtain the generalized DT$_4$-Segre series

$$R(\tilde{\alpha}, \tilde{\ell}; q) = 1 + \sum_{n > 0} q^n \int_{\text{Hilb}^n(X)^{\text{vir}}} s_{t_1}^{[n]}(\alpha_1^{[n]}) \cdots s_{t_M}^{[n]}(\alpha_M^{[n]}).$$

Recall our notation for generating series of Fuss-Catalan numbers from (1.6).

Theorem 5.1. Let $\alpha_1, \ldots, \alpha_M \in G^0(X)$, $a = \text{rk}(\alpha)$, then assuming Conjecture 2.19 for point-canonical orientations we have

$$R(\tilde{\alpha}, \tilde{\ell}; q) = U \left[ (1 + t_1 z)^{c_1(\alpha_1) - c_3(X)} \cdots (1 + t_M z)^{c_1(\alpha_M) - c_3(X)} \right],$$

where $z$ is the solution to $z(1 + t_1 z)^a \cdots (1 + t_M z)^a = q$. Moreover, we have the explicit expression:

$$R(\alpha; q) = \begin{cases} 
U \left[ B_{a+1}(-q)^{c_1(\alpha) - c_3(X)} \right] & \text{for } a \geq 0 \\
U \left[ B_{-a}(q)^{c_1(\alpha) - c_3(X)} \right] & \text{for } a < 0
\end{cases}$$

(5.1)

Proof. The first statement follows immediately from Proposition 4.13. Specializing to the DT$_4$ Segre series

$$R(\alpha; q) = \int_{\text{Hilb}^n(X)^{\text{vir}}} s_n(\alpha^{[n]}),$$

we obtain for $a = \text{rk}(\alpha)$

$$R(\alpha; q) = U \left[ (1 + z)^{c_1(\alpha) - c_3(X)} \right], \quad \text{where } q = z(1 + z)^a.$$

The theorem then follows from the following lemma.

Lemma 5.2. Let $y$ be the solutions of $y(y + 1)^a = q$ for $a > 0$, then

$$\frac{1}{1 + y} = \begin{cases} 
B_{a+1}(-q) & \text{for } a \geq 0 \\
B_{-a}(q)^{-1} & \text{for } a < 0
\end{cases}$$

(5.2)

Proof. We use Lemma 1.12. For $a \geq 0$ we change variables $z = 1/(1 + y)$ this implies $g(z) := (1 - z)/z^{a+1} = q$. Then the statement follows from

$$[(z - 1)^{n-1} \left( \frac{z - 1}{g(z)} \right)^n = [(z - 1)^{n-1} \frac{(-1)^n (1 + (z - 1))^{(a+1)n}}{n} = (-1)^n \frac{(a + 1)n}{n - 1} = (-1)^n \frac{(a + 1)n + 1}{n} = (-1)^n C_{n,a},$$

where we used the notation from (1.3). When $a < 0$, then change variables by $(u + 1) =
\[(z+1)^{-1}\] to get \(z = -u(u+1)^{-1}\) and thus \(-u(u+1)^{-a-1} = q\). Then \((1+z)^{-1} = (1+u) = \mathcal{B}_a(q)^{-1}\) by the above.

Using this, we obtain (5.1).

### 5.2 K-theoretic insertions

In this section, we use the Oh–Thomas Riemann–Roch formula for their twisted virtual structure sheaf.

**Theorem 5.3** (Oh–Thomas [71, Thm. 6.1]). Let \(M\) be projective with a fixed choice of orientation, \(V \in G^0(M)\), then

\[\hat{\chi}^{\text{vir}}(V) = \int_{|M|^{\text{vir}}} \sqrt{Td(E)ch(V)}.\]  \hspace{1cm} (5.3)

**Proof.** This is just Theorem [71, Theorem 6.1] stated in terms of \(\hat{\chi}^{\text{vir}}\) and using the notation of (1.8).

Recall that \(Td = \mathcal{G}_f\) for \(f(x) = x - e^{-x}\) which satisfies

\[\{\sqrt{f}\}(x) = \frac{x}{e^x - e^{-x}}.\]  \hspace{1cm} (5.4)

An immediate consequence of Corollary 4.9 is

**Corollary 5.4.** For all \(n > 0\), \(\hat{\chi}^{\text{vir}}(\text{Hilb}^n(X)) = 0\).

Nekrasov genus gives us refinements of invariants considered in §5.1 as

\[K_n(\vec{\alpha}, \vec{y}) = \hat{\chi}^{\text{vir}}(\mathcal{N}_1(\alpha_1^{[n]}) \cdots \mathcal{N}_M(\alpha_M^{[n]})), \quad K(\vec{\alpha}, \vec{y}; q) = \sum_{n > 0} K_n(\vec{\alpha}, \vec{y})q^n.\]

It is given by its series \(\mathcal{N}_q(x) = (y_1^{1/e^x} - y^{-1} e^x)\). Note that \(\mathcal{N}_q(0) \neq 1\), but we can write it as \(\mathcal{N}_q(x) = (1 - y^{-1}e^x)e^{-\frac{x}{2}}y^{\frac{x}{2}}\) and simply keep track of \(y^{\frac{x}{2}}\) separately. DT\(_4\) Segre series of (5.1) can be obtained as a classical limit of these invariants. Explicitly this means the following:

**Proposition 5.5.** For any \(\alpha_1, \ldots, \alpha_M \in G^0(X)\) with \(a_j = \text{rk}(\alpha_j)\) and \(i_j\), such that \(\sum_j i_j = n\), we have

\[\lim_{y_1 \to 1^+} \cdots \lim_{y_M \to 1^+} (1 - y_1^{i_1}a_1) \cdots (1 - y_M^{i_M}a_M) K_n(\vec{\alpha}, \vec{y}) = (-1)^n \int_{[\text{Hilb}^n(X)]^{\text{vir}}} c_{i_1}(\alpha_1^{[n]}) \cdots c_{i_M}(\alpha_M^{[n]}).\]
Proof. We conclude it from a more general result for any scheme $S$.

Define $A^> = \bigoplus_{i>0} A^i(S, Q)$ and $A^\leq = A^*(S, Q)/A^>$. Let $\gamma \in G^0(S)$, $a = \rk(\gamma)$ and $k \geq 0$, then we claim that in $A^\leq$ the following holds:

$$L_k(\gamma) := \lim_{y \rightarrow 1^+} (1 - y^{-1})^{k-a} \left[ \text{ch}\left( A_y^{-1} \gamma \cdot \det^{-1/2} \gamma \cdot y^{-1} \right) \right] = (-1)^k c_k(\gamma).$$

Let $\gamma = [E] - [F]$ and $c(E) = \prod_{i=1}^e (1 - x_i)$, $c(F) = \prod_{i=1}^f (1 - z_i)$, then in $A^\leq$, we have

$$L_i(E) = \lim_{\lambda \rightarrow 0^+} \left[ (1 - e^{-\lambda})^{l-e} \prod_{i=1}^e(e^{-\frac{\lambda}{2}} - e^{-\frac{\lambda}{2} + \frac{x_i}{2}}) \right]$$

$$= \lim_{\lambda \rightarrow 0^+} \left[ (\lambda - O(\lambda^2))^{l-e} \prod_{i=1}^e((\lambda - x_i) + O((\lambda - x_i)^3)) \right] = [\lambda^{-e}] \prod_{i=1}^e (1 - \lambda^{-1} x_i)$$

$$= (-1)^e c_i([E]).$$

Similarly, we obtain in $A^\leq$

$$L_m(-[F]) = \lim_{\lambda \rightarrow 0^+} \left[ (1 - e^{-\lambda})^{m-f} \prod_{j=1}^f(e^{-\frac{\lambda}{2}} - e^{-\frac{\lambda}{2} + \frac{z_j}{2}}) \right]$$

$$= \lim_{\lambda \rightarrow 0^+} \left[ (\lambda - O(\lambda^2))^{l-e} \prod_{i=1}^e((\lambda - x_i) + O((\lambda - x_i)^3)) \right] = [\lambda^{-m}] \prod_{i=1}^f (1 - \lambda^{-1} x_i)$$

$$= (-1)^m c_m(-[F]).$$

We combine these two to obtain $L_k(\gamma) = \sum_{i+m=k} L_i(E).L_m(-[F]) = (-1)^k c_k(\gamma)$, where both equalities are true only in $A^\leq$.

To conclude the proof, we apply this statement to each $N_i^{\bullet} \alpha_i^{[n]}$ separately. Then using $\sum_i i_j = n$ we see from Theorem 5.3 that we are integrating $c_i(\alpha_i^{[n]}) \cdots c_{i_M}(\alpha_M^{[n]}) \sqrt{Td_0(T^\vir)}.$

Only the case where $\sum_j \rk(\alpha_j) = 2b + 1$ is interesting, because one then obtains integer invariants assuming that $c_i(\alpha_i)$ are divisible by two. Moreover, we mostly focus on $M = 1$ and $\rk(\alpha_1) = 1$ which is motivated by the work of Nekrasov 68, Nekrasov–Piazzalunga 70 and Cao–Kool–Monavari 15.

**Theorem 5.6.** If Conjecture 2.19 holds, then for all $\alpha_1, \ldots, \alpha_M$ with $\alpha_i = \rk(\alpha_i)$, $\sum_i a_i = 2b + 1$ and point-canonical orientations, we have

$$K(\alpha, \gamma; q) = \prod_{i=1}^M U \left[ \frac{(y_i - 1)^2 u}{(y_i - u)^2} \right] \chi_{c_i(\alpha_i) - c_3(X)}, \quad \text{where} \quad q = \frac{(u - 1)^b}{\prod_{i=1}^M (y_i^{2} - y_i^{2} u)^{a_i}}$$

When $M = 1$, $\alpha_1 = \alpha$, $y_1 = y$, $a_1 = 1$, then

$$K(\alpha, y; q) = \exp \left[ \chi \left( X, q \cdot \frac{(TY^2 - T^*X)(\alpha^{1/2} y^{1/2} - \alpha^{1/2} y^{-1/2})}{(1 - q \alpha^{1/2} y^{1/2})(1 - q \alpha^{1/2} y^{-1/2})} \right) \right].$$
where \( \text{Exp}[f(y, q)] = \exp \left[ \sum_{n>0} \frac{f_n(y, q)}{n} \right] \). In particular, the coefficients of \( K(\vec{\alpha}, \vec{y}; q) \) lie in \( \mathbb{Z}[y_1^{\pm \frac{1}{2}}, \ldots, y_M^{\pm \frac{1}{2}}] \) if \( c(\alpha_1) \in H_2(X, \mathbb{Z}) \).

**Proof.** Using Proposition 4.13 together with (5.4) and Theorem 5.3, we obtain

\[
K(\vec{\alpha}, \vec{y}; q) = \prod_{i=1}^M U \left[ \frac{y_i^{\frac{1}{2}} - y_i^{-\frac{1}{2}}}{y_i^{\frac{1}{2}} e^{-\frac{z_i}{2}} - y_i^{-\frac{1}{2}} e^{\frac{z_i}{2}}} c_1(\alpha_i) c_3(X) \right], \quad \text{where } q = \frac{e^{\frac{z_i}{2}} - e^{-\frac{z_i}{2}}}{\prod_{i=1}^M (y_i^{\frac{1}{2}} e^{-\frac{z_i}{2}} - y_i^{-\frac{1}{2}} e^{\frac{z_i}{2}})},
\]

setting \( u = e^z \) and using

\[
\sqrt{\frac{(1 - uy_i^{-1})^2}{(1 - y_i^{-1})^2 u}} = \frac{(y_i^{\frac{1}{2}} u^{\frac{1}{2}} - y_i^{-\frac{1}{2}} u^{-\frac{1}{2}})}{y_i^{\frac{1}{2}} - y_i^{-\frac{1}{2}}},
\]

we obtain

\[
K(\vec{\alpha}, \vec{y}; q) = \prod_{i=1}^M U \left[ \frac{(y_i - 1)^2 u}{(y_i - u)^2} \right]^{\frac{1}{2} c_1(\alpha_1) c_3(X)}, \quad \text{where } q = \frac{(u - 1) u^b}{\prod_{j=1}^M (y_j^{\frac{1}{2}} - y_j^{-\frac{1}{2}} u)^{a_j}}.
\]

When \( M = 1 \) and \( a_1 = 1 \) then this gives

\[
u = \frac{1 + q y^{\frac{1}{2}}}{1 + q y^{-\frac{1}{2}}},
\]

which after plugging into the above formula gives rise to

\[
K(\alpha, y; q) = U \left[ (1 + q y^{\frac{1}{2}}) (1 + q y^{-\frac{1}{2}}) \right]^{\frac{1}{2} c_1(\alpha) c_3(X)}
= \sqrt{M(q y^{\frac{1}{2}}) M(q y^{-\frac{1}{2}})}
= \text{Exp} \left[ \frac{q y^{\frac{1}{2}}}{(1 - q y^{\frac{1}{2}})^2} - \frac{q y^{-\frac{1}{2}}}{(1 - q y^{-\frac{1}{2}})^2} \right]^{\frac{1}{2} c_1(\alpha) c_3(X)}
= \text{Exp} \left[ \chi \left( X, q \frac{(T X - T^* X)(\alpha \frac{1}{2} y^{\frac{1}{2}} - \alpha^{-\frac{1}{2}} y^{-\frac{1}{2}})}{(1 - q\alpha \frac{1}{2} y^{\frac{1}{2}})(1 - q\alpha^{-\frac{1}{2}} y^{-\frac{1}{2}})} \right) \right],
\]

where the second equality uses \( M(q) = \text{Exp} \left( \frac{q}{(1 - q y^{\frac{1}{2}})} \right) \) and the last equality uses Grothendieck–Riemann–Roch.

The following remark is the result of the search for the correct replacement for the \( \chi_y \)-genus and elliptic genus of Fantechi–Göttsche, it was motivated by Cao–Kool–Monavari [15, Remark 1.19] to answer what the correct generalization of the above invariants should be. The authors of loc cit. tried the \( \chi_{y^*} \)-genus, we explain why this is not the right choice.

**Remark 5.7.** On a real manifold \( M \), a natural generalization of the \( \hat{A} \) genus is the universal
elliptic genus which can be computed as
\[
W(M, V, q) = \int_{[M]} \hat{A}(X) \prod_{k \geq 1} \text{ch} \left( \text{Sym}_{q^k}^k(TM \otimes \mathbb{C}) \right) (1 - q^k)^{2 \dim(M)}.
\]

The $\chi_y$-genus is however defined only for complex manifold, as it needs the additional complex structure. This motivates:

**Definition 5.8.** We define the $DT$ Witten-genus of $M^\text{st}_\alpha(\tau)$ by
\[
W\left(M^\text{st}_\alpha(\tau), V, q\right) = \chi\left( \hat{O}_{\text{vir}} \otimes \bigotimes_{k \geq 1} \text{Sym}_{q^k}^k(E - \text{rk}(E)) \otimes V\right).
\]

**Example 5.9.** Let $M$ be a moduli scheme with a perfect obstruction theory $F^* \to \Lambda_M$ as in Behrend–Fantechi [6], then [71, 17, 21] consider the $-2$-shifted cotangent bundle 3-term obstruction theory $E^* = F^* \oplus (F^* \vee [2]) \to \Lambda_M$. In this situation, Oh–Thomas [71, §8] show
\[
\hat{O}_{\text{vir}} = O_{\text{vir}} \sqrt{K_{\text{vir}}},
\]
where $O_{\text{vir}}$ is the virtual structure sheaf of Fantechi–Göttsche [24], $K_{\text{vir}} = \text{det}(F^*)$ and the square root is taken in $\mathcal{O}^0(M, \mathbb{Z}[2^{-1}])$, where it always exists (see Oh–Thomas [71, Lemma 2.1]). The term on the right hand side is in fact the *twisted virtual structure sheaf* $\hat{O}_{\text{vir}}^{\text{NO}}$ of Nekrasov–Okounkov [69]. If $\text{rk}(F^*) = 0$, i.e. virtual dimension of $M$ is 0, then
\[
W(M, V, q) = \chi\left( \hat{O}_{\text{vir}}^{\text{NO}} \otimes \bigotimes_{k \geq 1} \text{Sym}_{q^k}^k(F^* \oplus (F^*) \vee) \otimes V\right),
\]
which is the *virtual chiral elliptic genus* of Fasola–Monavari–Ricolfi [25] motivated by the work of physicists Benini–Bonelli–Poggi–Tanzini [7]. As the assumption on rank is a bit silly, one should really work equivariantly, and we plan to return to this question as we expect to relate the recent work of Kool–Rennemo [54] with the work of Fasola–Monavari–Ricolfi [25] by dimensional reduction as in [16], [54], where it is considered only the $\hat{A}$-genus.

For Hilbert schemes, the correct object to study which generalizes Nekrasov’s genus is the *Nekrasov–Witten genus*
\[
W(\text{Hilb}^n(X), N_y(L[i]), q)
\]. Using Proposition 4.13 the corresponding generating series can be expressed as
\[
1 + \sum_{n \geq 0} z^n W(\text{Hilb}^n(X), N_y(L[i]), q) = U \left[ \frac{(y_i - 1)^2 u}{(y_i - u)^2} \right],
\]
where
\[
z = \frac{u - 1}{y^2 - y \frac{1}{2} u} \prod_{k > 0} \frac{(1 - q^k u)(1 - q^k u^{-1})}{(1 - q^k)^2}.
\]
5.3 Untwisted K-theoretic invariants

We propose a version of DT$_4$ Verlinde numbers for Calabi–Yau fourfolds as higher dimensional analogues of Verlinde numbers for surfaces studied in [23, 64, 35]. After computing generating series for these invariants, we obtain a simple Segre–Verlinde correspondence.

In the spirit of Calabi–Yau fourfolds, they require an additional twist by a square-root of a tautological determinant line bundle.

**Definition 5.10.** Let $E = \det(O_X^{[n]})$, then the **untwisted virtual structure sheaf** is defined by

$$\mathcal{O}^{\text{vir}} = \hat{\mathcal{O}}^{\text{vir}} \otimes E^{-\frac{1}{2}}.$$

We define the **untwisted virtual characteristic**

$$\chi^{\text{vir}}(\text{Hilb}^n(X), A) = \chi^{\text{vir}}(\text{Hilb}^n(X), E^{\frac{1}{2}} \otimes A) = \int_{[\text{Hilb}^n(X)]^{\text{vir}}} \sqrt{Td(T^{\text{vir}}_n)} \text{ch}(E^{\frac{1}{2}}) \text{ch}(A).$$

Clearly, this changes $A_{[O_X]}(z) = z/(e^z - e^{-z})$ from Lemma 4.8 to $A_{[O_X]}(z) = z/(1 - e^{-z})$.

**Definition 5.11.** Let $X$ be a Calabi–Yau fourfold, then its **square root DT$_4$ Verlinde series** are defined for all $\alpha \in G^0(X)$ by

$$V^{\frac{1}{2}}(\alpha; q) = 1 + \sum_{n>0} V^{\frac{1}{2}}_n(\alpha) q^n = 1 + \sum_{n>0} \chi^{\text{vir}}(\text{Hilb}^n(X), \det(L^{[n]}_{\alpha}) \otimes E^a) q^n,$$

where $L_{\alpha} = \det(\alpha)$, $a = \text{rk}(\alpha)$. The DT$_4$ Verlinde series is defined by

$$V(\alpha; q) = 1 + \sum_{n>0} V_n(\alpha) q^n = 1 + \sum_{n>0} \chi^{\text{vir}}(\text{Hilb}^n(X), \det(\alpha^{[n]}) q^n.$$

**Remark 5.12.** Just for the purpose of this remark, let us define **negative square root Verlinde series** by

$$V^{-\frac{1}{2}}(\alpha; q) = 1 + \sum_{n>0} V^{-\frac{1}{2}}_n(\alpha) q^n = 1 + \sum_{n>0} \chi^{\text{vir}}(\text{Hilb}^n(X), \det^{-\frac{1}{2}}(L^{[n]}_{\alpha}) \otimes E^{-a}) q^n,$$

for each $\alpha \in G^0(X)$, where $a = \text{rk}(\alpha)$.

1. When $\alpha = [V]$ is a vector bundle of rank $a$, one can show that

$$[y^{\frac{1}{2}(2a+1)}] \left( K_n(L_{\alpha} \oplus E^\oplus 2a, y) \right) = V^{\frac{1}{2}}_n(V).$$

2. From the expression $K(L, y; q) = \sqrt{M(qg^{\frac{1}{2}})M(qg^{-\frac{1}{2}})^{-1(\alpha)}}$, we obtain that Nekrasov generating series decouples into the positive and negative square-root Ver-
linde series:
\[ K(L, y; q) = V^\frac{1}{2}(\mu(L)y^{-1}; q)V^{-\frac{1}{2}}(\mu(L)y^{-1}; q), \]
where \( \mu(L) = L - O_X \), as it can be written as a product of series only with positive or negative powers of \( y^\frac{1}{2} \). Thus
\[ V^\pm \frac{1}{2}(\mu(L); q) = M(q)^\frac{1}{2}c_1(L)\cdot c_3(X). \]

3. By applying Proposition 4.13, one can show that
\[ V(\alpha; q) = (V^\frac{1}{2}(\alpha; q))^2. \]

**Theorem 5.13.** Assuming Conjecture 2.19 holds, we have the following Segre–Verlinde correspondence for any choice of orientations on \( \text{Hilb}^n(X) \):
\[ V(\alpha; q) = R(\alpha; -q). \]

**Proof.** From Proposition 4.13 together with (5.4) and Definition 5.10 we see after setting \( a = \text{rk}(\alpha) \) that
\[ V(\alpha; q) = U(e^z)^{c_1(\alpha)\cdot c_3(X)}, \quad \text{where} \quad q = \frac{(1 - e^{-z})}{e^{az}}. \]
Changing variables to \( t = e^z - 1 \) we obtain
\[ V(\alpha; q) = U(1 + t)^{c_1(\alpha)\cdot c_3(X)}, \quad \text{where} \quad q = t(t + 1)^{-(a+1)}. \]
We therefore see from Lemma 5.2 that
\[ V(\alpha; q) = \begin{cases} U(\mathcal{B}_{a+1}(q)\cdot c_3(X)) & \text{for } a \geq 0 \\ U(\mathcal{B}_{-a}(-q)\cdot c_3(X)) & \text{for } a < 0 \end{cases}. \]
(5.5)
Comparing with (5.1) concludes the proof. \(\Box\)

We can also study the series:
\[ Z(\alpha, k; q) = 1 + \sum_{n>0} q^n \chi^\text{vir}(\wedge^{k_1} a^{[n]}_1 \otimes \ldots \otimes \wedge^{k_M} a^{[n]}_M). \]
We show that they give rise to interesting formulae. This was motivated by investigating the rationality question as studied in [4] and their example [4, Ex. 7]

**Example 5.14.** For \( \alpha \in G^0(X) \), take the series \( Z(\alpha; q) = \sum_{n>0} \chi^\text{vir}(\alpha^{[n]}) \), then it can be expressed as
\[ Z(\alpha; q) = \frac{\partial}{\partial y} Z(\alpha, y; q)|_{y=0}, \quad \text{where} \quad Z(\alpha, y; q) = 1 + \sum_{n>0} \chi^\text{vir}(\Lambda^\bullet_y a^{[n]}). \]
Using Proposition 4.13 we have

\[ Z(\alpha, y; q) = \left[ \prod_{n>0} \prod_{k=1}^{n} \frac{1 + ye^{z(-\omega_n^k q)}}{1 + y} \right] c_1(\alpha) \cdot c_3(X), \quad \text{where} \quad q = \frac{1 - e^{-z}}{(1 + ye^{z})^a}. \]

After changing variables \(1 + u = e^{z}\), this gives

\[ Z(\alpha, y; q) = \left[ \prod_{n>0} \prod_{k=1}^{n} \frac{1 + y(1 + u(-\omega_n^k q))}{1 + y} \right] c_1(\alpha) \cdot c_3(X), \quad \text{where} \quad q = \frac{u}{(1 + u)(1 + y + yu)^a}. \]

Acting with \(\partial/\partial y\) on the last formula, using that the terms under the product are equal to 1 for \(y = 0\) and that the derivative \((\partial/\partial y)u\) exist we obtain from a product rule for infinite products

\[ Z(\alpha; q) = c_1(\alpha) \cdot c_3(X) \sum_{n>0} \sum_{k=1}^{n} u(-\omega_n^k q) \quad \text{where} \quad u = \frac{q}{1 - q} \]

We can write this as

\[ Z(\alpha; q) = c_1(\alpha) \cdot c_3(X) \sum_{n>0} \frac{(-q)^n}{1 - (-q)^n} = c_1(\alpha) \cdot c_3(X) S(-q), \]

where \(S(q)\) is the Lambert series as considered by Lambert \[55\].

### 5.4 4D-2D-1D correspondence

We obtain a one-to-one correspondence between invariants on compact CY fourfolds and elliptic surfaces.

Recall from \[63\, \text{Lem. 1}\] that the virtual obstruction theory on \(\text{Quot}_S(\mathbb{C}^N, n)\) is given by

\[ F = \left( \tau_{[0,1]} \text{Hom}_{\text{Quot}_S(\mathbb{C}, n)}(I, \mathcal{F}) \right)^\vee, \quad (5.6) \]

where \(I = (\mathcal{O} \rightarrow \mathcal{F})\) is the universal complex on \(\text{Quot}_S(\mathbb{C}, n)\). When \(N = 1\), we have \(\text{Quot}_S(\mathbb{C}^1, n) = \text{Hilb}^n(S)\) and the virtual fundamental classes get identified by Oprea–Pandharipande \[72\, \text{eq. (31)}\] with \(\text{Hilb}^n(S)_{\text{vir}} = \left[\text{Hilb}^n(S) \cap c_n(K_{\text{Hilb}}^n(S)^\vee)\right]\).

using that \(\text{Hilb}^n(S)\) is smooth. Here we also use \(\chi_{\text{vir}}(-)\) to denote the virtual Euler characteristic of Fantechi–Göttsche \[24\].

**Theorem 5.15.** Let \(X\) be a Calabi–Yau fourfold, \(S\) an elliptic surface. Let \(f_1, \ldots, f_M, g\) be power-series, \(\alpha_{Y,1}, \ldots, \alpha_{Y,M}\) in \(G^0(Y)\) for \(Y = X, S\) and \(\text{rk}(\alpha_{Y,i}) = a_i\), then there exist
universal series $A_1, \ldots, A_M$ depending on $f_i, \{g\}$ and $a_i$ such that

$$1 + \sum_{n>0} q^n \int_{[\text{Hilb}^n(S)]^\text{vir}} f_1(\alpha^{[n]}_{S,1}) \cdots f_M(\alpha^{[n]}_{S,M}) \{g\} (T^\text{vir}_{\text{Hilb}^n(S)}) = \prod_{i=1}^M A_i^{c_1(\alpha_{S,i})-c_1(S)},$$

$$1 + \sum_{n>0} q^n \int_{[\text{Hilb}^n(X)]^\text{vir}} f_1(\alpha^{[n]}_{X,1}) \cdots f_M(\alpha^{[n]}_{X,M}) g(T^\text{vir}_n) = \prod_{i=1}^M U(A_i)^{c_1(\alpha_{X,i})-c_3(X)}.$$

Moreover, there are universal generating series $B_i$ depending on $f_i, a_i$, such that

$$1 + \sum_{n \in \mathbb{Z}} q^n \chi^\text{vir} f_1(\alpha^{[n]}_{S,1}) \cdots f_M(\alpha^{[n]}_{S,M}) = \prod_{i=1}^M B_i^{c_1(\alpha_{S,i})-c_1(S)},$$

$$1 + \sum_{n>0} q^n \chi^\text{vir} f_1(\alpha^{[n]}_{X,1}) \cdots f_M(\alpha^{[n]}_{X,M}) = \prod_{i=1}^M U(B_i)^{c_1(\alpha_{X,i})-c_3(X)}.$$

where we abuse the notation by thinking of $\mathcal{G}_i$, as mapping to $G^0(-) \otimes \mathbb{Q}$.

**Proof.** Arbesfeld–Johnson–Lim–Oprea–Pandharipande [4] prove general formulae for generating series

$$\sum_{n \in \mathbb{Z}} q^n \int_{[\text{Quot}_S(\mathbb{C}^N,\beta,n)]^\text{vir}} f_1(\alpha^{[n]}_1) \cdots f_M(\alpha^{[n]}_M) h(T^\text{vir}_{\text{Hilb}^n(S)}).$$

When $\beta = 0, N = 1$ and $K^2 = 0$ the results of [4, §2.2 & Eq. (14)] imply

$$1 + \sum_{n>0} q^n \int_{[\text{Hilb}^n(S)]^\text{vir}} f_1(\alpha^{[n]}_1) \cdots f_M(\alpha^{[n]}_M) h(T^\text{vir}_{\text{Hilb}^n(S)}) = \prod_{i=1}^M \left[ \frac{f_i(H(q))}{f_i(0)} \right]^{c_1(\alpha_i)-c_1(X)},$$

where $q = \frac{H}{\prod f_i^\text{vir}(H) h(H)}$.

Replacing $h$ with $\{g\}$, and comparing to the result of Proposition [4,13] we obtain the first two formulae.

Using (5.4), we see that $\left[ \sqrt{T_d} (T^\text{vir}_{\text{Hilb}^n(S)}) \right]^{E^\frac{1}{2}}$ contributes

$$\frac{x}{1 + e^{-x}}$$

to the variable change above. This corresponds precisely to the Todd-genus $T_d(T^\text{vir}_{\text{Hilb}^n(S)}) = \frac{x}{1 + e^{-x}} (T^\text{vir}_{\text{Hilb}^n(S)}).$ The second result for elliptic surface $S$ then follows from the virtual Riemann–Roch of Fantechi–Göttsche [24] together with definition of $\chi^\text{vir}(-)$ in §5.2. \hfill \square

**Remark 5.16.** By the work of and Oprea–Pandharipande [72, Lem. 34] there is a relation between integrals over $[\text{Quot}_C(\mathbb{C}^N, n)]$ and $[\text{Quot}_S(\mathbb{C}^N, n)]^\text{vir}$, where the former is a smooth moduli space of dimension $nN$ and $C$ is a smooth anti-canonical curve in $S$ (if it exists).
When $\alpha \in K^0(S)$ and $\Theta$ is the theta divisor on $C$, we obtain

$$1 + \sum_{n>0} \int_{C^{[n]}} f((\alpha|_C)^{[n]}_i) g(TC^{[n]} + (\Theta^{[n]})^\vee - \Theta^{[n]})$$

It is interesting that this gives a precise relation between the generating series of three sets of virtual invariants in 3 different dimensions. We will unify these results by applying similar arguments to the ones in §3 and §4 to extend the results of Arbesfeld et al [4], [61] and [72] in the author’s future work [9], where we replace $C_N$ by a general torsion free sheaf $E$.

In the Calabi–Yau case, we will assume $E$ additionally to be rigid and stable to construct virtual fundamental classes.

Using that $[\text{Hilb}^1(X)]^{\text{vir}} = \text{Pd}(c_3(x))$ together with Theorem 5.3 and that we have natural isomorphisms $\Lambda^i(TX|_x) \cong \text{Ext}^i(O_x, O_x)$ which hold in a family one can show:

**Corollary 5.17.** All of the results of this section hold mod $q^2$.

### 5.5 4D-2D correspondence explained by wall-crossing

Virtual fundamental classes of Quot-schemes on surfaces have been used by Marian–Oprea–Pandharipande [63] to prove Lehm’s conjecture [56] for the generating series of tautological invariants on Hilbert schemes of points. More recently their virtual fundamental classes $[\text{Quot}_S(C^N, \beta, n)]^{\text{vir}}$ were studied by Arbesfeld et al [4], Lim [61], Johnson–Oprea–Pandharipande [44] and Oprea–Pandharipande [72]. Our goal here is to recover the formulae when $\beta = 0$ for an elliptic surface $S$ to explain the relationship in Theorem 5.15. We only need one ingredient for this. Similarly, as in the case of Calabi–Yau 4-folds let us denote

$$I(L, q) = 1 + \sum_{n>0} q^n \int_{[\text{Quot}_S(C^1, n)]^{\text{vir}}} c_n(L^{[n]})$$

Knowing these invariants, we will be able to determine $[\text{Quot}_S(C^N, n)]^{\text{vir}}$ as an element in $H_{nN}(P_S)$ similarly to what we obtained for four-folds. For this we will need a different definition of the the moduli stack of pairs. For simplicity, we assume that $b_1(S) = 0$, but we then drop this requirement in Remark 5.23. As we are recovering known results using different methods this section should be viewed as purely philosophical. In the sequel [9], we are going to obtain the entire information about virtual fundamental classes $[\text{Quot}_S(E, n)]^{\text{vir}}$ for any surface and $E$ a torsion-free sheaf using these methods. These results will be new.

Let us for now set up the general framework for $[\text{Quot}_S(C^N, n)]^{\text{vir}}$ for any smooth projective surface $S$.

**Definition 5.18.** • We consider this time the abelian category $B_N$ of triples $(E, V, \phi)$, where $\phi : V \otimes C^N \otimes O_S \to F$ and $F$ is a zero-dimensional sheaf.
• The moduli stack $N^N$ is constructed as in Definition 2.12, except in the first bullet point we take the total space of $\pi_{np,d} : \pi_2^*(O_X)^\oplus N \boxtimes E_{np} \boxtimes V_d^\vee \to M_{np} \times [*/\text{GL}(d, \mathbb{C})]$.

• We define $\Theta^{N,pa}$ by

$$\Theta^{N,pa}_{(n_1 p, d_1), (n_2 p, d_2)} = (\pi_{n_1 p, d_1} \times \pi_{n_2 p, d_2})^* \left\{ (\Theta_{n_1 p, n_2 p})_{1,3} \oplus \left( (V_{d_1})^\oplus N \boxtimes \pi_2^*(E_{n_2 p})^\vee \right)_{2,3} [1] \right\} \quad (5.7)$$

with $\Theta_{n_1 p, n_2 p} = \text{Hom}_{M_{n_1 p} \times M_{n_2 p}} (E_{n_1 p}, E_{n_2 p})^\vee$ and the form $\chi^{N,pa}((n_1 p, d_1), (n_2 p, d_2)) = \text{rk}(\Theta^{N,pa}_{(n_1 p, d_1), (n_2 p, d_2)}) = -N\text{d}_1\text{d}_2$

The rest of the data has obvious modification, which we do not mention here.

Note that working with surfaces the correct vertex algebra structure requires the symmetrization of $\Theta^{pa}$, thus the correct data is

$$((\Lambda^p)^{\text{top}}, \mathbb{Z} \times \mathbb{Z}, \mu^{\text{top}}_{N_0}, \mu^{\text{top}}_{N_1}, 0^{\text{top}}, [\Theta^{N,pa} + \sigma^*(\Theta^{N,pa})^\vee], \epsilon^N) \quad (5.8)$$

where $\epsilon^N_{(n_1 p, d_1), (n_2 p, d_2)} = (-1)^{N\text{d}_1\text{d}_2}$. We have again a universal family $\mathbb{C}^N \otimes O_{S \times \text{Quot}_S(\mathbb{C}^N, n)} \to F$ giving us

$$\text{Quot}_S(\mathbb{C}^N, n) \xrightarrow{\epsilon_{n,N}} N^N_0 \xrightarrow{\text{Ipl}} (N^N_0)^{\text{pl}}$$

and $[\text{Quot}_S(\mathbb{C}^N, n)]_{\text{vir}} \in H_*(N^N_0)$. Notice, that there is an obvious modification of the Joyce–Song stability $\tau^N_\text{pa}$, such that $\mathbb{C}^N \otimes O_X \xrightarrow{\phi} F$ is $\tau^N_\text{pa}$-stable if and only if $\phi$ is surjective. Therefore, we again obtain

$$\text{Quot}_S(\mathbb{C}^N, n) = N^N_{(np, 1)}(\tau^N_\text{pa}), \quad [\text{Quot}_S(\mathbb{C}^N, n)]_{\text{vir}} = [N^N_{(np, 1)}(\tau^N_\text{pa})]_{\text{vir}}.$$

Once the work of Joyce is complete, the following conjecture will be a consequence of a more general theorem after proving that some axioms are satisfied.

**Conjecture 5.19.** For any smooth projective surface $S$, in $\hat{H}_*(N^N_0)$ we have for all $n, N$

$$[\text{Quot}_S(\mathbb{C}^N, n)]_{\text{vir}} = \sum_{k>0, n_1, \ldots, n_k} \frac{(-1)^k}{k!} \left[ \ldots \left[ [N_{(0,1)}]_{\text{inv}}, [M^s_{n_1 p}]_{\text{inv}}, \ldots \right], [M^s_{n_k p}]_{\text{inv}} \right]$$

for some $[M^s_{n_k p}]_{\text{inv}} \in \hat{H}_2(N^N_0)$.

We again construct the vertex algebra on topological pairs and the $L$-twisted vertex algebra.
Definition 5.20. Define the data \((P_S, K(P_S), \Phi_{P_S}, \mu_{P_S}, 0, \theta_{P_S}^L, \tilde{\chi}^N)\), \((P_S, K(P_S), \Phi_{P_S}, \mu_{P_S}, 0, \theta_{P_S}, \tilde{\chi}^N)\) as follows:

- \(K(P_S) = K^0(S) \times \mathbb{Z}\).
- Set \(\mathcal{L} = \pi_{2*}(\pi_S^*(L) \otimes \mathcal{E}) \in K^0(C_S)\). Then on \(P_S \times P_S\) we define \(\theta_{N,ob} = (\theta)_{1,3} - N\left(1 \otimes \pi_{2*}(\mathcal{E})\right)_{2,3}\), where \(\theta = \pi_{2*}(\pi_{1,2}^*(\mathcal{E}) \cdot \pi_{1,3}^*(\mathcal{E})^\vee)\) and

\[
\theta_{P_S, N}^{L} = \theta_{N,ob} +\sigma^*(\theta_{N,ob})^\vee
\]

\[
\theta_{P_S, N}^{L} = \theta_{P_S} + N\left(1 \otimes \mathcal{L}^\vee\right)_{2,3} + \left(\mathcal{L} \otimes \mathcal{L}^\vee\right)_{1,4},
\]

- The symmetric forms \(\tilde{\chi} : (K^0(S) \times \mathbb{Z}) \times (K^0(S) \times \mathbb{Z}) \to \mathbb{Z}\), \(\tilde{\chi}^L : (K^0(S) \times \mathbb{Z}) \times (K^0(S) \times \mathbb{Z}) \to \mathbb{Z}\) are given by

\[
\tilde{\chi}\left((\alpha, d), (\beta, e)\right) = \chi(\alpha, \beta) + \chi(\beta, \alpha) - dN\chi(\beta) - eN\chi(\alpha),
\]

\[
\tilde{\chi}^L\left((\alpha, d), (\beta, e)\right) = \chi(\alpha, \beta) - dN\chi(\beta - \chi(\beta \cdot L)) - eN\chi(\alpha - \chi(\alpha \cdot L)).
\]

\[
(5.9)
\]

- The signs are defined by \(\tilde{\epsilon}_{(\alpha,d),(\beta,e)} = (-1)^{\chi(\alpha,\beta) + nd\chi(\beta)}\) and \(\tilde{\epsilon}_{(\alpha,d),(\beta,e)}^L = (-1)^{\chi(\alpha,\beta) + nd\chi(\beta)}\).

We denote by \((\hat{H}_*(P_S), [0], e^T, Y_N)\), resp. \((\hat{H}_*(P_S), [0], e^T, Y_N^L)\) the vertex algebras associated to this data and \((H_*(P_S), [-,-]_N)\), resp. \((H_*(P_X), [-,-]_N^L)\) the corresponding Lie algebras. We now consider the map

\[
\Omega^N = (\Gamma \times \text{id}) \circ (\Sigma_N)^{\text{top}} : (N_0^N)^{\text{top}} \to \mathcal{M}^{\text{top}}_X \times BU \times \mathbb{Z} \to \mathcal{C}_X \times BU \times \mathbb{Z},
\]

where \(\Sigma_N\) maps \([E, V, \phi]\) to \([E, V \otimes O_S]\).

Let \(\mathbb{B} = B \sqcup \{(0,1)\}\), where \(B = \bigsqcup_{i=1}^4 B_i\), ch(B_i) basis of \(H^i(S)\) with \(B_0 = \{[O_S]\}\), \(B_4 = \{p\}\). Combining all the ideas we used for fourfolds, we can state the following:

Proposition 5.21. Let \(\mathbb{Q}[K^0(S) \times \mathbb{Z}] \otimes_{\mathbb{Q}} \text{Sym}_{\mathbb{Q}}[u_{\sigma,i}, \sigma \in \mathbb{B}, i > 0]\) be the generalized superlattice vertex algebra associated to \(((K^0(S) \oplus \mathbb{Z}) \oplus K^1(S), (\tilde{\chi})^*), \text{ resp. } ((K^0(S) \oplus \mathbb{Z}) \oplus K^1(S), (\tilde{\chi})^*),\) where \((\tilde{\chi})^* = \tilde{\chi} \oplus \tilde{\chi}^-, (\tilde{\chi}^L)^* = \tilde{\chi}^L \oplus \tilde{\chi}^-\) and

\[
\chi^- : K^1(S) \times K^1(S) \to \mathbb{Z},
\]

\[
\chi^-(\alpha, \beta) = \int_S \text{ch}(\alpha) \text{ch}(\beta) \text{Td}(S) - \int_S \text{ch}(\beta) \text{ch}(\alpha) \text{Td}(S).
\]

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The isomorphism (2.24) induces an isomorphism of graded vertex algebras for all $N$:

$$
H_s(P_S) \cong \mathbb{Q}[K^0(S) \times \mathbb{Z}] \otimes_{\mathbb{Q}} \text{SSym}_{\mathbb{Q}}[u_{\sigma,i}, \sigma \in \mathbb{B}, i > 0],
$$

$$
H_s(P_S) \cong \mathbb{Q}[K^0(S) \times \mathbb{Z}] \otimes_{\mathbb{Q}} \text{SSym}_{\mathbb{Q}}[u_{\sigma,i}, \sigma \in \mathbb{B}, i > 0].
$$

The map $(\Omega^N)_*$ : $H_*(N^N_0) \rightarrow H_*(P_S)$ induces morphisms of graded vertex algebras $(H_*(N^N_0), [0], e^{zT}, Y_N) \rightarrow (H_*(P_S), [0], e^{zT}, Y_N)$, $(H_*(N^N_0), [0], e^{zT}, Y^L_N) \rightarrow (H_*(P_S), [0], e^{zT}, Y^L_N)$ and of graded Lie algebras

$$
\tilde{\Omega}_*: (H_*(N^N_0), [-,-], N) \rightarrow (H_*(P_S), [-,-], N),
$$

$$
\tilde{\Omega}_*: (H_*(N^N_0), [-,-], N) \rightarrow (H_*(P_S), [-,-], N).
$$

The following result replaces Theorem 3.10 and it is noticeably simpler due to canonical orientations. We use the notation

$\mathcal{Q}_{N,n} = \tilde{\Omega}^N_*(\text{Quot}_S(\mathbb{C}^N, n)_{\text{vir}})$, and $\mathcal{M}_{np} = \tilde{\Omega}^N_*(\text{[M}_{np}\text{]}_{\text{vir}}).

**Lemma 5.22.** Let $S$ be a smooth projective surface with $b_1(S) = 0$. If Conjecture 5.19 holds, then

$\mathcal{M}_{np} = e^{(np, 1)} \otimes 1 \cdot \mathcal{M}_{np} + \mathcal{Q}T(e^{(np, 1)} \otimes 1),$

where for the series $\mathcal{N}(q) = \sum_{n>0} \mathcal{M}_{np} q^n$ we have

$$
\exp(\mathcal{N}(q)) = \left(1 - e^q \right) \left(\sum_{v \in B_2} c_1(S)_v u_v, 1\right).
$$

If $S$ is moreover elliptic, we have

$$
1 + \sum_{n>0} \frac{\mathcal{Q}_{N,n}}{e^{(np, 1)} q^n} = \exp \left[ \sum_{n>0} \left(\sum_{v \in B_2} c_1(u_v) U_v(z) \exp \left[ \sum_{k>0} \frac{ny_k}{k} z^k \right] \right) q^n \right]. \quad (5.11)
$$

**Proof.** We have $\text{Quot}_S(\mathbb{C}^1, n)_{\text{vir}} \cap c_n(L^{[n]}) = \text{[Hilb}^n(S)] \cap c_n(\langle (K_{\text{Hilb}}(S))^\vee \rangle) \cap c_n(L^{[n]}) = (-1)^n \text{[Hilb}^n(S)] \cap c_{2n}(R_S^{[n]} \oplus L^{[n]})$ for an algebraic line bundles $L \rightarrow S$. Then by [64, eq. (18)], we see

$$
I(L, q) = \left(\frac{1}{1-q}\right) c_1(L) c_1(X).
$$

Using that $H^2(X) = H^{1,1}(X)$ because of $b_1(X) = 0$, we have $\text{ch}(B_2) \subset H^{1,1}(X)$ and therefore the above result for algebraic line bundles is sufficient. We obtain by similar computation as in the proof of Theorem 3.10

$$
[e^{(mp, 1)} \otimes 1, e^{(np, 0)} \otimes N_{np}]^L = (-1)^n e^{(m+n)p, 1} \sum_{v \in B_2} \int_X c_1(L) \text{ch}(v) a_v(n).
$$

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By similar but simpler arguments as in the proof of Theorem 3.10, we obtain \( \mathcal{N}(np) = \frac{1}{n} \sum_{v \in B_2} c_1(S)v^{u_v,1} \). Then an analogous argument as in the proof of Theorem 4.1 leads to (5.11), where we are using \( c_1^f(S) = 0 \).

**Remark 5.23.** Going through the above computation without the assumption \( b_1(S) = 0 \), we need two modifications. Firstly, we would use the result of Oprea–Pandharipande [72, Cor. 15] instead of [63, eq. (18)] in the proof of Lemma 5.22 to avoid the issue of purely topological line bundles. One can moreover check that under the projection \( \Pi_{\text{even}} : \check{H}_*(\mathcal{P}_S) \to \check{H}_{\text{even}}(\mathcal{P}_S) \) we still obtain the same results for an elliptic surface. This is sufficient for us, because we never integrate odd cohomology classes, except when integrating polynomials in \( \text{ch}_k(T_{\text{vir}}) \), but as the only terms \( \mu_{v,k} \) for \( v \in B_{3} \) are given for \( v \in B_{3} \), each such integral will contain a factor of \( \chi^-(v, w) = 0 \) for \( v, w \in B_{3} \).

As a consequence, we then obtain the following result which could also be extracted from Arbesfeld et al [4] for a smooth projective elliptic surface.

**Proposition 5.24.** Let \( S \) be a smooth projective elliptic surface and \( f_0(p, \cdot), f_1(p, \cdot), \ldots, f_m(p, \cdot) \) be power-series with \( f(0, 0) = 1 \), then define

\[
\text{Inv}_N(\vec{f}, \vec{a}, q) = 1 + \sum_{n>0} \int_{[\text{Quot}_S(C, n)]^\text{vir}} f_0(T^\text{vir}) f_1(\alpha_1^n) \cdots f_m(\alpha_m^n) q^n.
\]

Setting \( \text{rk}(\alpha_j) = a_j \), we have

\[
\text{Inv}_N(\vec{f}, \vec{a}, q) = \left[ \prod_{j=1}^N \prod_{i=1}^m f_i(H_j(q)) \right]^{c_1(\alpha_j)-c_1(S)},
\]

where \( H_j(q) \), \( j = 1, \ldots, N \) are the different solutions for

\[
q = \frac{H_j^N}{\prod_{i=1}^m f_i^n(H_j) f_0^n(H_j)}.
\]

**Proof.** We can show again

\[
\int_{[\text{Quot}_S(C, n)]^\text{vir}} f_0(T^\text{vir}) f_1(\alpha_1^n) \cdots f_m(\alpha_m^n)
\]

\[
= \int_{\check{Z}_{N,n}} \exp \left[ \sum_{k>0} \sum_{v \in B_{2,4}} \alpha_{v}^{k}(k) \chi(a^{v}, v) \mu_{v,k} + N b_k \chi(v) \mu_{v,k} \right],
\]

where \( \sum_{k>0} \frac{a_{v}^{k}(k)}{k^k} q^k = \log(f_i(q)) \) and \( \sum_{k>0} b_k (k) q^k = \log(f_0(q)) \). The rest then follows from Lemma 2.22 and 4.12 by a similar computation as in §4.

**Remark 5.25.** For an elliptic curve \( C \) the quot-scheme \( \text{Quot}_C(C, n) \) carries the obstruction theory \( \mathcal{F} = \left( \tau_{[0,1]} \text{Hom}_{\text{Quot}_C(C, n)}(\mathcal{I}, \mathcal{F}) \right)^{\vee} \) constructed by Marian–Oprea [62] which is
just a vector bundle of rank $nN$, therefore the construction of the vertex algebra is identical and the same computation applies. We leave it to the reader to check using [72, Thm. 3] that under the projection $\Pi_{\text{even}} : \check{H}_*(\mathcal{P}_C) \to \check{H}_{\text{even}}(\mathcal{P}_C)$ the generating series $\sum_{n>0} \frac{(2\pi)^n}{\ln(n)}$ is given by

$$\exp \left[ - \sum_{n>0} \frac{(-1)^n}{n} [z^{nN-1}] \left\{ U_{\mathcal{O}_C}(z) \exp \left[ \sum_{k>0} \frac{ny_k}{k} z^k \right] \right\} q^n \right].$$

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