The minimum number of 4-cliques in a graph with triangle-free complement

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Abstract

Let $f(n, 4, 3)$ be minimum number of 4−cliques in a graph of order $n$ with independence number 2. We show that

$$f(n, 4, 3) \leq \frac{1}{200}n^4 + O(n^3).$$

We also show if a graph of order $n$ has independence number 2 and is close to regular then it has at least

$$\frac{1}{200}n^4 + o(n^4)$$

4-cliques.

1 Notation and conventions

Our notation and terminology are standard (see, e.g. [1]). In particular, all graphs are assumed to be defined on the vertex set $[n] = \{1, 2, ... n\}$. For any two adjacent vertices $i$ and $j$ we write $i \sim j$. We write $i \sim j$ if $i$ and $j$ are distinct nonadjacent vertices. Given a vertex $i$, $N_i$ denotes the set of its neighbors, $d_i$ denotes its degree and $t_i$ denotes the number of triangles containing it.
Given a graph \( G \), \( t_3(G) \) is the number of its triangles; \( t_4(G) \) is the number of its 4-cliques; \( t'_3(G) \) is the number of all induced subgraphs of order 3 and size 2; \( t''_3(G) \) is the number of all induced subgraphs of order 3 and size 1, and \( t'_4(G) \) is the number of all induced subgraphs of order 4 and size 5 (i.e. isomorphic to \( K_4 \) with one edge removed).

2 Introduction

Write \( f(n, 4, 3) \) for the function
\[
f(n, 4, 3) = \min\{t_4(G) : v(G) = n, \overline{G} \text{ is triangle-free}\}.
\]

This function is a particular case of a more general function introduced by Erdős in \[2\]. In \[3\] we proved that
\[
f(n, 4, 3) \leq \frac{1}{200} n^4 + o(n^4).
\]

It happens that this estimate is tight under additional assumptions. We show that, in fact,
\[
f(n, 4, 3) \leq \frac{1}{200} n^4 + O(n^3).
\]

Moreover, let \( G \) be a graph of order \( n \) such that \( \overline{G} \) is triangle-free. If \( G \) is almost regular, i.e., if
\[
\sum_{i \in [n]} \left| d_i - \frac{2m}{n} \right| = o(n^2), \tag{1}
\]
then we show that
\[
t_4(G) \geq \frac{1}{200} n^4 + o(n^4).
\]

3 Main results

Write \( C_5[K_p] \) for the lexicographic product of the 5-cycle with the complete graph on \( p \) vertices. Recall that the vertex set of \( G[K_p] \) is \( v(G) \times v(K_p) \) and


\((i, x) \sim (j, y)\) iff \(i \sim j\) or \(x = y\). Observe that the complement of \(C_5[K_p]\) is triangle-free and

\[
t_4(C_5[K_p]) \geq 5 \left( \binom{2p}{4} - \binom{p}{4} \right) = \frac{25}{8} p^4 - \frac{35}{4} p^3 + \frac{55}{8} p^2 - \frac{5}{4} p.
\]

Obviously, \(f(n, 4, 3)\) is nondecreasing with respect to \(n\). Let \(5p\) be the smallest multiple of 5 which is not smaller than \(n\). We have

\[
f(n, 4, 3) \leq 5 \left( \binom{2p}{4} - \binom{p}{4} \right) = \frac{25}{8} p^4 - \frac{35}{4} p^3 + \frac{55}{8} p^2 - \frac{5}{4} p
\]

\[
\leq \frac{25}{8} \left( \frac{n+4}{5} \right)^4 - \frac{35}{4} \left( \frac{n+4}{5} \right)^3 + \frac{55}{8} \left( \frac{n+4}{5} \right)^2 - \frac{5}{4} \left( \frac{n+4}{5} \right)
\]

\[
= \frac{1}{200} n^4 + \frac{1}{100} n^3 - \frac{17}{200} n^2 - \frac{13}{100} n + \frac{1}{5}.
\]

Our goal to the end of the paper is to prove the following assertion.

**Theorem 1** If a graph \(G\) of order \(n\) with independence number 2 satisfies \(Q\) then

\[t_4(G) \geq \frac{1}{200} n^4 + o(n^4).
\]

The proof consists of the following steps:

a) some lemmas establishing properties of graphs with triangle-free complement;

b) deduction of a lower bound on \(t_4(G)\) as a function of various graph parameters;

c) reduction of the bound to a function of vertex degrees, \(v(G)\) and \(e(G)\);

d) replacing the vertex degrees by their mean;

e) minimizing the bound with respect to \(e(G)\).

**Lemma 1** For any graph \(G = G(n, m)\)

\[
\sum_{i \sim j} d_i d_j \geq \frac{4m^3}{n^2}.
\]
Proof Obviously, we can assume that $G$ contains no isolated vertices. On the one hand, we have

$$2 \sum_{i\sim j} \frac{1}{\sqrt{d_i d_j}} \leq \sum_{i\sim j} \frac{1}{d_i} + \frac{1}{d_j} = \sum_{i\in [n]} \frac{d_i}{d_i} = n.$$ 

On the other hand, by the Cauchy-Schwarz inequality,

$$\left( \sum_{i\sim j} \sqrt{d_i d_j} \right) \left( \sum_{i\sim j} \frac{1}{\sqrt{d_i d_j}} \right) \geq m^2.$$ 

Therefore,

$$\sum_{i\sim j} \sqrt{d_i d_j} \geq \frac{2m^2}{n}.$$ 

Hence, by the Cauchy inequality,

$$\sum_{i\sim j} d_i d_j \geq \frac{1}{m} \left( \sum_{i\sim j} \sqrt{d_i d_j} \right)^2 \geq \frac{4m^3}{n^2}$$

completing the proof. 

Let $G = G(n, m)$ be a graph with no independent set on 3 vertices (i.e., $\overline{G}$ is triangle-free). We shall prove a series of short lemmas which follow from this assumption.

Lemma 2

$$6t_3(G) = n^3 - 3n^2 + 2n + \sum_{i\in [n]} 3d_i^2 - 3d_i n + 3d_i.$$ 

Proof Indeed, this is an expanded form of a well known identity. Observe that

$$t_3'(G) + t_3''(G) = \frac{1}{2} \sum_{i\in [n]} d_i (n - 1 - d_i).$$

Hence,

$$6t_3(G) = 6 \binom{n}{3} - 6 (t_3'(G) + t_3''(G))$$

$$= n^3 - 3n^2 + 2n - 3 \sum_{i\in [n]} d_i (n - 1 - d_i)$$

$$= n^3 - 3n^2 + 2n + \sum_{i\in [n]} 3d_i^2 - 3nd_i + 3d_i. \quad (2)$$
Lemma 3

\[ 2t_3(G) + t'_3(G) = (n - 2) m - \binom{n}{3}. \]  \hspace{1cm} (3)

Proof Indeed, trivially,

\[ \binom{n}{3} = t_3(G) + t'_3(G) + t''_3(G). \]

On the other hand,

\[ (n - 2) m = 3t_3(G) + 2t'_3(G) + t''_3(G). \]

Subtracting the last two identities, we obtain (3). \(\square\)

Lemma 4

\[ 8t_4(G) + 2t'_4(G) = \sum_{i \in [n]} (d_i - 2) t_i - \sum_{i \in [n]} \binom{d_i}{3}. \]  \hspace{1cm} (4)

Proof Applying lemma 3 to \(G[N_i]\) for any \(i \in [n]\), and summing over all vertices, we obtain (4). \(\square\)

Lemma 5

\[ t'_4(G) \leq \sum_{i \neq j} \binom{|N_i \cap N_j|}{2}. \]

Proof Obviously,

\[ t'_4(G) = \sum_{i \neq j} e([N_i \cap N_j]) \leq \sum_{i \neq j} \binom{|N_i \cap N_j|}{2}. \] \(\square\)
Lemma 6 For any two nonadjacent vertices $i$ and $j$

\[ |N_i \cap N_j| = d_i + d_j - n + 2. \]

**Proof** For every $k$, if $k \not\sim j$ and $k \not\sim i$ then $\{i, j, k\}$ is a triangle in $\overline{G}$. Therefore,

\[ n - 2 = |N_i \cup N_j| = d_i + d_j - |N_i \cap N_j|. \]

\[ \square \]

Lemma 7

\[ \sum_{i \in [n]} d_it_i = \sum_{i \sim j} d_id_j - \frac{1}{2} \sum_{i \in [n]} d_i^2 - \frac{1}{2} \sum_{i \not\sim j} (d_i + d_j) |N_i \cap N_j|. \]

**Proof** Let $i$ be any vertex. Obviously,

\[ \sum_{j \sim i} d_j = d_i + 2t_i + \sum_{j \not\sim i} |N_i \cap N_j| \]

hence, multiplying both sides by $d_i$, we get

\[ d_i \sum_{j \sim i} d_j = d_i^2 + 2d_it_i + d_i \sum_{j \not\sim i} |N_i \cap N_j| \]

and summing over all vertices, we obtain

\[ 2 \sum_{i \sim j} d_id_j = \sum_{i \in [n]} d_i^2 + 2 \sum_{i \in [n]} d_it_i + \sum_{j \not\sim i} (d_i + d_j) |N_i \cap N_j|. \]

\[ \square \]

**Proof of Theorem**

By lemma 4 and lemma 5

\[ 8t_4 (G) = -2t'_4 (G) + \sum_{i \in [n]} (d_i - 2)t_i - \sum_{i \in [n]} \left( \frac{d_i}{3} \right) \]

\[ \geq -2 \sum_{i \not\sim j} \left( \frac{|N_i \cap N_j|}{2} \right) - 6t_3 (G) + \sum_{i \in [n]} d_it_i - \sum_{i \in [n]} \left( \frac{d_i}{3} \right). \]
Hence, by lemma \(6\) and lemma \(7\)

\[
8t_4(G) \geq - \sum_{i \sim j} (d_i + d_j - n + 2) (d_i + d_j - n + 1) - 6t_3(G) - \sum_{i \in [n]} \binom{d_i}{3} \\
+ \sum_{i \sim j} d_id_j - \frac{1}{2} \sum_{i \in [n]} d_i^2 - \frac{1}{2} \sum_{i \sim j} (d_i + d_j) (d_i + d_j - n + 2) \\
= \sum_{i \sim j} d_id_j - \frac{1}{2} \sum_{i \sim j} (d_i + d_j - n + 2) (3d_i + 3d_j - 2n + 2) \\
- \frac{1}{6} \sum_{i \in [n]} (d_i^3 - 3d_i^2 + 2d_i) - \frac{1}{2} \sum_{i \in [n]} d_i^2 - 6t_3(G).
\]

Therefore, we have

\[
8t_4(G) \geq \sum_{i \sim j} d_id_j - \frac{1}{2} \sum_{i \sim j} (d_i + d_j - n + 2) (3d_i + 3d_j - 2n + 2) - \frac{1}{6} \sum_{i \in [n]} (d_i^3 + 2d_i) - 6t_3(G). \tag{5}
\]

We also find that

\[
\sum_{i \sim j} (d_i + d_j - n + 2) (3d_i + 3d_j - 2n + 2) \\
= \sum_{i \sim j} (3d_i^2 + 6d_id_j - 5nd_i + 8d_i + 3d_j^2 - 5nd_j + 8d_j + 2n^2 - 6n + 4) \\
= \sum_{i \sim j} 6d_id_j + \sum_{i} (3d_i^2 (n - 1 - d_i) - 5nd_i (n - 1 - d_i) + 8d_i (n - 1 - d_i)) + \\
+ \sum_{i \in [n]} (n^2 - 3n + 2) (n - 1 - d_i) \\
= \sum_{i \sim j} 6d_id_j + \sum_{i \in [n]} (8nd_i^2 - 3d_i^3 - 11d_i^2 - 6n^2d_i + 16nd_i - 10d_i) \\
+ n^4 - 4n^3 + 5n^2 - 2n.
\]

Applying this equality to \(5\) and afterwards bounding \(6t_3\) by lemma \(2\), we
get

\[8t_4 (G) \geq \sum_{i \sim j} d_i d_j - 3 \sum_{i \sim j} d_i d_j - \frac{1}{2} \left( n^4 - 4n^3 + 5n^2 - 2n \right) - \frac{1}{6} \sum_i (d_i^3 + 2d_i) \]

\[- 6t_3 + \sum_i \left( \frac{3}{2} d_i^3 - 4n d_i^2 + 3n^2 d_i + \frac{11}{2} d_i^2 - 8n d_i + 5d_i \right) \]

\[= \sum_{i \sim j} d_i d_j - 3 \sum_{i \sim j} d_i d_j - \frac{1}{2} n^4 + n^3 + \frac{1}{2} n^2 - n + \]

\[+ \sum_i \left( \frac{4}{3} d_i^3 - 4n d_i^2 + 3n^2 d_i + \frac{5}{2} d_i^2 - 5n d_i + \frac{5}{3} d_i \right)\]

On the other hand, note that

\[\sum_{i \sim j} d_i d_j - 3 \sum_{i \sim j} d_i d_j = 4 \sum_{i \sim j} d_i d_j - \frac{3}{2} \left( \sum_{i \in [n]} d_i \right)^2 + \frac{3}{2} \sum_{i \in [n]} d_i^2 \]

\[= 4 \sum_{i \sim j} d_i d_j - 6m^2 + \frac{3}{2} \sum_{i \in [n]} d_i^2 \]

Thus,

\[8t_4 (G) \geq 4 \sum_{i \sim j} d_i d_j - 6m^2 - \frac{1}{2} n^4 + n^3 + \frac{1}{2} n^2 - n + \]

\[+ \sum_{i \in [n]} \left( \frac{4}{3} d_i^3 - 4n d_i^2 + 3n^2 d_i + 4d_i^2 - 5n d_i + \frac{5}{3} d_i \right)\]

Dropping the low order terms, we see that

\[8t_4 \geq 4 \sum_{i \sim j} d_i d_j - 6m^2 - \frac{1}{2} n^4 + \sum_{i \in [n]} \left( \frac{4}{3} d_i^3 - 4n d_i^2 + 3n^2 d_i \right) + O (n^3) .\]

Due to (I), we find that

\[8t_4 \geq 4m \frac{4m^2}{n^2} - 6m^2 - \frac{1}{2} n^4 + \frac{4}{3} \frac{8m^3}{n^2} - 4n \frac{4m^2}{n} + 6n^2 m + O (n^3) \]

\[= \frac{80 m^3}{3 n^2} - 22m^2 + 6mn^2 - \frac{1}{2} n^4 + O (n^3) .\]

Since the expression

\[\frac{80 m^3}{3 n^2} - 22m^2 + 6mn^2\]
attains its minimum at

\[ m = \frac{3}{10} n^2 + o(n^2), \]

the desired inequality follows.

\[ \square \]

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