VECTOR VALUED FUNCTIONS NOT CONSTANT ON CONNECTED SETS OF CRITICAL POINTS

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Abstract. Whitney type examples of maps $f \in C^k(\mathbb{R}^m, \mathbb{R}^n)$ for a maximal possible real $k$, and multidimensional space-filling curves with special properties are constructed.

1. Introduction.

In 1935 Hassler Whitney [10] published his example of a $C^1$ function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ not constant on a connected set of critical points. The following theorem is the main result of this paper.

Theorem 1. For any $n,m \in \mathbb{N}$ there exist a map $p : [0,1]^m \rightarrow [0,1]^n$, contained in $C^k$ for all real $k < \frac{m}{n}$, and a connected set $E \subseteq [0,1]^m$, such that every partial derivative of $p$ of order $< \frac{m}{n}$ vanishes on $E$ and $p(E) = [0,1]^n$.

Easy corollary from the Theorem 1 is:

Theorem 2. Let $n,m,r$ be non-negative integer numbers, $m > n > r$, then there exists a map $p : \mathbb{R}^m \rightarrow \mathbb{R}^n$, contained in $C^k$ for all real $k < \frac{m-r}{n-r}$, and a connected subset $E$ of points of rank $r$ such that $p(E)$ contains an open set.

Similar to the Theorem 2 results, but without connectedness of $E$, were obtained by R.Kaufman [3], S.M.Bates [1],[2], A.Norton [6].

The following theorem, being a weak version of the Bates’ theorem [2], shows the sharpness of the $k$ in Theorems 1,2.

Theorem (Bates) Let $n,m,r$ be nonnegative integers satisfying $m > n > r$, and define $s = (m-r)/(n-r)$. If $E$ is a set of rank $r$ for $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $f \in C^s$, then $f(E)$ has Lebesgue measure zero in $\mathbb{R}^n$.

Whitney indicated in same paper [10] how to generalize his construction for higher dimensions (ie. $f \in C^k(\mathbb{R}^m, \mathbb{R}^1)$ for $m,k \in \mathbb{N}, k < m$). He wrote for the case $m = 3, k = 2$ "Let $Q$ be a cube of side 1. Let $Q_0, \ldots, Q_7$ be cubes of side $2/5$ arranged in $Q$ so that $Q_i$ is adjacent to $Q_{i-1}$". But practically this "adjacency" is not such easy to make. The problem deeply relates to the problem of construction of multidimensional space-filling curves with special properties, which is quite complicated itself. One can find the properties that fit Whitney’s example for cases $m = 3$ in Sagan’s curve $f : [0,1] \rightarrow [0,1]^3$ (see [8]) published in 1993, and for case $m = 4$ in Steinhaus’ curve $f : [0,1] \rightarrow [0,1]^4$ (see [7]). In Theorems 3,4 of this paper the author constructs the space-filling function $f : [0,1] \rightarrow [0,1]^m$ for any...

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Such a type of space-filling curves appears to be a very useful tool in solution of some problems involving Euclidian space and $C^k$ functions. One of such applications is the Whitney’s type examples constructed in this paper.

1.0.1. Preliminary results.

**Definition 1.1.** If $f : \mathbb{R}^m \to \mathbb{R}^n$, and $\lambda \in (0, 1]$, we define $\lambda$-partial derivatives $f_1^{(\lambda)}, \ldots, f_m^{(\lambda)}$ by the formula:

$$f_i^{(\lambda)}(a) = \lim_{{t \to 0}} \text{sign}(t) \frac{f(a_1, \ldots, a_{i-1}, a_i + t, a_{i+1}, \ldots, a_n) - f(a)}{|t|^\lambda}$$

for $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$. If all $\lambda$-partial derivatives are continuous on some $M \subseteq \mathbb{R}^m$ we say that $f \in C^\lambda$ on $M$.

**Definition 1.2.** For $k \in \mathbb{N}$ a function $f : M \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is a $C^k$-function (or $f \in C^k$), if $f \in C^{[k]}$ and every $[k]^{th}$ partial derivative of $f$ on $M$ is $C^{k-[k]}$-function, where $[k]$ is the integer part of $k$. If $f \in C^k$ for every $k < k_0$, we will write $f \in C^{<k_0}$.

We begin by setting $K_0^n = \{Q^n_0\}$, where $Q^n_1 = [0, 1]^n$ is the closed cube in $\mathbb{R}^n$ with side length 1.

In general, having constructed the cubes of $K_{s-1}^n$, divide each $Q^n_{1,i_1,i_2,\ldots,i_{s-1}} \subseteq K_{s-1}^n$ into $2^n$ closed cubes of side $\frac{1}{2^n}$, and let $K_s^n$ be the set of all these cubes. More precisely we will write

$K_s^n = \{Q^n_{1,i_1,i_2,\ldots,i_{s-1},i_s} : Q^n_{1,i_1,i_2,\ldots,i_{s-1},i_s} \subseteq Q^n_{1,i_1,i_2,\ldots,i_{s-1}} \subseteq K_{s-1}^n, 1 \leq i_s \leq 2^n \}$.

We also define:

- $K^n = \bigcup_{s \in \mathbb{N}} K_s^n$ (note: $K^n$ is defined for $\mathbb{R}^n$);
- $S(Q^n_{1,i_1,i_2,\ldots,i_s}) = \frac{1}{2^n}$ - the length of a side of $Q^n_{1,i_1,i_2,\ldots,i_s}$;
- $\text{meas}(\delta)$ be the Lebesgue measure of $\delta \subseteq \mathbb{R}^n$.

The main goal of this paragraph is to construct for any $n \in \mathbb{N}$ a continuous space filling curve

$$f_n : [0, 1] \xrightarrow{onto} [0, 1]^n$$

with special properties:

(1.1) if $\alpha \subseteq [0, 1]$ and for some $s \in \mathbb{N}$ $\alpha \in K_{n,s}^1$ then $f_n(\alpha) \subseteq \delta$ for some $\delta \in K_s^n$

(1.2) if $\delta \subseteq [0, 1]^n$ and for some $s \in \mathbb{N}$ $\delta \in K_s^n$ then $f_n^{-1}(\text{int}(\delta)) \subseteq \alpha$ for some $\alpha \in K_{n,s}^1$ where $\text{int}(\delta)$ is the set of interior points of $\delta$.

**Definition 1.3.** We will call a function $f_n : [0, 1] \to [0, 1]^n$ with the properties (1.1), (1.2) cubes preserving.
Note that a continuous cubes preserving function $f_n$ is a space-filling and measure preserving function with a property:

if $α \subseteq [0, 1]$ and for some $s \in \mathbb{N}$ $α \in K_{n,s}^1$, then $f_n(α) = δ$ for some $δ \in K_{n,s}^n$.

**Definition 1.4.** Extending Sagan’s definition of measure preserving function [7] to dimension $n$, we will call a function $q : [0, 1] \rightarrow [0, 1]^n$ measure preserving, if for every $P \subseteq [0, 1]^n$ $\text{meas}(q^{-1}(P)) = \text{meas}(P)$.

Once we have a cube preserving function $q : [0, 1] \rightarrow [0, 1]^n$ for some $n \in \mathbb{N}$, it allows us to build a linear order on $K_{n,s}^n$ with respect to the $q$ for $s \in \mathbb{N}$:

\[
∀δ, δ' \in K_{n,s}^n \, δ \prec_q δ' \text{ if } ∀x \in q^{-1}(\text{int}(δ)), x' \in q^{-1}(\text{int}(δ')) \, x < x'.
\]

So that we can enumerate the elements of $K_{n,s}^n$ as \( \{Q^n_{1,i_1,i_2,...,i_n} : 1 \leq i_j \leq 2^n\} \), where $Q^n_{1,i_1,i_2,...,i_n} \prec_q Q^n_{1,i_1',i_2',...i_n'}$ if $\exists j_0 \, (1 \leq j_0 \leq s)$ such that $∀j < j_0 \, i_j \geq i'_j$ and $i'_{j_0} < i_{j_0}$. Also note that $Q^n_{1,i_1,i_2,...,i_n-1,i_n} \subseteq Q^n_{1,i_1,i_2,...,i_{n-1}} \forall i_n \, (1 \leq i_n \leq 2^n)$. Thereby we have enumerated all elements of the set $K^n$. Let us designate the enumeration by $\langle K^n, \prec_q \rangle$.

Note that cubes preserving functions were already constructed for $n \leq 4$. For example:

- In case $n = 2$: the Hilbert curve $f_H : [0, 1] \rightarrow [0, 1]^2$ and the Peano curve $f_P : [0, 1] \rightarrow [0, 1]^2$.
- In case $n = 3$: the Sagan’s generalization of the Hilbert curve $f_S : [0, 1] \rightarrow [0, 1]^3$.
- In case $n = 4$: the Steinhaus Space-Filling curve $f_{SH} : [0, 1] \rightarrow [0, 1]^4$ with its coordinate functions: $ψψ, ψϕ, φψ, φϕ$, where $ψ, φ$ are the coordinate functions of the Hilbert curve.

We now will prove that one can construct a cubes preserving function for any $n \in \mathbb{N}$.

**Lemma 1.1.** Let $E_1, E_2$ be copies of $\mathbb{R}$. ∀$n \in \mathbb{N}$ there exists continuous $S_n : [0, 1]^2 \rightarrow [0, 1]^2 \subseteq E_1 \times E_2$ such that

1. If $α \in K_{n,s}^1$, then $S_n(α) \subseteq α' \times α''$, where $α' \subseteq K_{(n-1),s}^1$, $α'' \subseteq K_{s}^1$, $α' \subset E_1$, $α'' \subset E_2$.
2. If $α' \times α'' \subseteq [0, 1]^2$ such that $α' \subseteq E_1$, $α'' \subseteq E_2$, $α' \in K_{(n-1),s}^1$, $α'' \in K_{s}^1$, then $S_n^{-1}(\text{int}(α' \times α'')) \subseteq α \in K_{n,s}^n$.

**Proof.** Let us define for every $n \geq 2$ a function $S_n$ as follows:

If the interval $[0, 1]$ can be mapped continuously onto the square $[0, 1]^2$, then after partitioning $[0, 1]$ into $2^n$ congruent subintervals and $[0, 1]^2$ into $2^n$ congruent subrectangles with sides $\frac{1}{2^n}$, $\frac{1}{2^n}$, each subinterval can be mapped continuously onto one of the subrectangles.

Next, each subinterval is, in turn, partitioned into $2^n$ congruent subintervals, and each subrectangle into $2^n$ congruent subrectangles with sides $\frac{1}{2^n}$, $\frac{1}{2^n}$ and the argument is repeated. If this is carried on indefinitely, $[0, 1]$ and $[0, 1]^2$ are partitioned into $2^{ns}$ congruent replica each with sides $\frac{1}{2^n}$, $\frac{1}{2^n}$ for $s \in \mathbb{N}$. Fig. 1 shows how we divide each rectangle(square in first iteration) into subrectangles.

We need to demonstrate that the subrectangles can be arranged so that adjacent subintervals correspond to adjacent subrectangles with an edge in common, and so
Each iteration divides subrectangle into $2^n$ sub-subrectangles with sides $\frac{1}{2(n-1)}s$, $\frac{1}{2^n}$.

Figure 1. Each iteration $s$ divides subrectangle into $2^n$ sub-subrectangles with sides $\frac{1}{2(n-1)}s$, $\frac{1}{2^n}$.

that the inclusion relationships are presented, i.e. if a rectangle corresponds to an interval, then its subrectangles correspond to the subintervals of that interval.

We will use here a combination of two different methods to construct these space-filling curves:

(1) The first method is based on idea of Peano [7]. Fig. 2 shows the way of construction the curve by Peano method. For the future use, we designate this method as "P".

(2) The second method is based on idea of Hilbert [7]. Fig. 3 shows the ways of construction the curve by Hilbert method. We designate this method as "H".

Note: For both methods, the choice of which of two ways will be used depends only on the choice of the start point, and disposition of the end point follows the parity of $2^{n-1}$ and does not depend on $n$.

Figure 2. Peano Method ("P")

Figure 3. Hilbert Method ("H")

To create the next iteration curve, we will give the means of how to present each subectangle from the previous iteration (see Fig. 4, 5).

And finally in Fig. 6, 7 we indicate how this process is to be carried out for the next iteration.

**Definition 1.5.** Every $t \in [0, 1]$ is uniquely determined by a sequence of nested closed intervals (that are generated by our successive partitioning), the lengths of which shrink to 0. With this sequence corresponds a unique sequence of nested closed squares, the diagonals of which shrink into a point, and which define a unique point in $[0, 1]^2$, the image $S_n(t)$ of $t$. 
Theorem 3. For every $n \in \mathbb{N}$ there exists a continuous cubes preserving function $f_n : [0,1] \rightarrow [0,1]^n$.

Proof. By the induction on $n \in \mathbb{N}$. For $n = 1$ let $f_1$ be the identity map $I : [0,1] \rightarrow [0,1]$. For $n = 2$ let $f_2$ be the Hilbert space-filling curve. Now if we have already defined $f_{n-1} : [0,1] \rightarrow [0,1]^{n-1}$ continuous cubes preserving function, then let:

$$f_n = \left( f_{n-1} \circ \varphi \right)$$

where $\varphi, \psi$ are the coordinate functions of $S_n = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ defined as in Lemma \ref{lemma}. Then $f_n : [0,1] \rightarrow [0,1]^n$ is continuous. Let us prove that $f_n$ is cubes preserving.

Let $\alpha \subseteq [0,1]$ be an element of $K_{n,s}^1$ for some $s \in \mathbb{N}$. Then $S_n(\alpha) = \alpha' \times \alpha''$ for some $\alpha' \in K_{(n-1),s}^1$, $\alpha'' \in K_s^1$. Or in other words $\varphi(\alpha) = \alpha'$, $\psi(\alpha) = \alpha''$. It follows
that $f_{n-1} \circ \varphi(\alpha) \subseteq \delta$ for some $\delta \in K_n^{n-1}$. Consequently,

$$f_n(\alpha) = \begin{pmatrix} f_{n-1} \circ \varphi(\alpha) \\ \psi(\alpha) \end{pmatrix} \subseteq \delta \times \alpha'' \in K_n^1$$

So that the property $\mathbf{1}$ of cubes preserving function is proven.

Now let $\delta$ be an element of $K_n^m$ for some $s \in \mathbb{N}$. Then $\text{int}(\delta) = \text{int}(\delta') \times \text{int}(\alpha)$ for some $\delta' \in K_n^{s-1}$, $\alpha \in K_1^1$.

We can present the function $f_n$ in matrix form as:

$$f_n = \begin{pmatrix} f_{n-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

or $f_n = F \cdot S_n$, where $F = \begin{pmatrix} f_{n-1} & 0 \\ 0 & I \end{pmatrix}$ and $I$ is the identity map.

Then $f_n^{-1} = S_n^{-1} \circ F^{-1}$. And we can see that

$$F^{-1}(\text{int}(\delta)) \subseteq f_n^{-1}(\text{int}(\delta')) \times I^{-1}(\text{int}(\alpha)).$$

By the induction hypothesis, we have $f_n^{-1}(\text{int}(\delta')) \subseteq \alpha'$ for some $\alpha' \in K_{(n-1)s}^1$.

Recalling that $f_{n-1}$ is continuous and $\text{int}(\delta')$ is an open set, we can say that $f_n^{-1}(\text{int}(\delta')) \subseteq \text{int}(\alpha')$. And obviously $I^{-1}(\text{int}(\alpha)) = \text{int}(\alpha)$. So we can write

$$F^{-1}(\text{int}(\delta)) \subseteq \text{int}(\alpha') \times \text{int}(\alpha) = \text{int}(\alpha' \times \alpha)$$

where $\alpha' \in K_{(n-1)s}^1$, $\alpha \in K_1^1$.

Further, by the property $\mathbf{2}$ of the function $S_n$ (see Lemma $\mathbf{1}$), it follows that

$$S_n^{-1}(\text{int}(\alpha' \times \alpha)) \subseteq \alpha'' \text{ for some } \alpha'' \in K_{n-s}^1.$$  

So finally we conclude

$$f_n^{-1}(\text{int}(\delta)) = S_n^{-1}(F_n^{-1}(\text{int}(\delta))) \subseteq S_n^{-1}(\text{int}(\alpha') \times \text{int}(\alpha)) = S_n^{-1}(\text{int}(\alpha' \times \alpha)) \subseteq \alpha'' \in K_{n-s}^1.$$ 

The property $\mathbf{1}$ for the function $f_n$ is proven, and hereby the theorem $\mathbf{3}$ is proven as well.

\[\square\]

**Theorem 4.** For every $n \in \mathbb{N}$ there exists a continuous cubes preserving function

$$f_n : [0, 1] \rightarrow [0, 1]^n,$$

with the property:

if $[a, b], [b, c] \in K_1^n$, for some $s \in \mathbb{N}$, then $f_n([a, b]), f_n([b, c]) \in K_1^n$ have $(n-1)$-dimensional face of side $\frac{1}{s}$ in common, and $f_n(b)$ is a vertex of the face.

**Proof.** The proof is the same as the proof of the Theorem $\mathbf{4}$ and the property is just a property of the special functions $f_n$ constructed in the proof of the Theorem $\mathbf{3}$.

\[\square\]

2. Proof of the Theorem $\mathbf{1}$

Let us define the function $p$. Our plan is first to define $p$ on some closed set $B_0 \subseteq [0, 1]^m$ such that $p(B_0) = [0, 1]^n$. Then we will extend the definition of $p$ over all cube $[0, 1]^m$ with the property that $p$ on $[0, 1]^m$ is a $C^{\infty}$-function and every partial derivative of order $< \frac{m}{n}$ vanishes on $B_0$. Finally joining appropriate pairs of points of $B_0$ by straight line segments, each mapping by $p$ on a singular point in $[0, 1]^n$, we get a connected set $E$ such that $B_0 \subseteq E \subseteq [0, 1]^m$ and $p(E) = [0, 1]^n$. Showing that every partial derivative of order $< \frac{m}{n}$ vanishes on $E$, we will finish the proof...
of the Theorem

For the construction of function \( p \), we enumerate a set of subcubes of \([0, 1]^n\) and a set of subcubes of \([0, 1]^m\). In case of the cube \([0, 1]^n\), a set of subcubes is the \( K^m = \bigcup_{s \in \mathbb{N}} K^m_s \) with the enumeration \( \langle K^m, \prec_{f_n} \rangle \) (see (1.3)), where \( f_n \) is as in Theorem 4 for the number \( n \in \mathbb{N} \). So that for every \( s \in \mathbb{N} \), \( K^m_s = \{ Q^n_{1,i_1,i_2,\ldots,i_j,\ldots,i_s} : 1 \leq i_j \leq 2^n \} \) where \( Q^n_{i_1,i_2,\ldots,i_s} \prec_{f_n} \) \( Q^n_{i_1,i_2,\ldots,i_j,i_{j+1},\ldots,i_s} \) if \( \exists \ j_0 \ (1 \leq j_0 \leq s) \) such that \( \forall j < j_0 \ i_j \geq i'_j \) and \( i_{j_0} > i'_{j_0} \), and \( \forall i_s \ (1 \leq i_s \leq 2^n) \) \( Q^n_{1,i_1,i_2,\ldots,i_{s-1},i_s} \subset Q^n_{1,i_1,i_2,\ldots,i_{s-1}} \).

In case \([0, 1]^m\) a set of subcubes, which we designate by \( \tilde{K}^m = \bigcup_{s \in \mathbb{N}} \tilde{K}^m_s \), where \( \forall s \in \mathbb{N}, \tilde{K}^m_s = \{ \tilde{Q}^m_{1,i_1,i_2,\ldots,i_s} : 1 \leq i_j \leq i_s \} \), is constructed similarly to the standard set \( K^m = \bigcup_{s \in \mathbb{N}} K^m_s \) as follows:

- \( \tilde{Q}^m_1 = [0, 1]^m \) is the cube of side \( S_1 = 1 = \frac{1}{2^m} \);
- \( \{ \tilde{Q}^m_{i_1} : i_1 = 1, 2, \ldots, 2^m \} \) is a set of \( m \)-dimensional subcubes of \( \tilde{Q}^m_1 \) of side \( S_2 = \frac{1}{2^m}(1 - \frac{1}{2^m}) = \frac{1}{2^m}S_1(1 - \frac{1}{2^m}) = \frac{1}{2^m}S_1 < \frac{1}{2}S_1 \), such that in every corner of \( \tilde{Q}^m_1 \) there locates the only one subcube \( \tilde{Q}^m_{i_1} \) for some \( i_2 = 2 \).
- Now if the cube \( \tilde{Q}^m_{1,i_1,i_2,\ldots,i_{s-1}} \) have already been defined for some \( s \in \mathbb{N} \), then we define a set of \( m \)-dimensional subcubes \( \tilde{Q}^m_{1,i_1,i_2,\ldots,i_{s-1},i_s} \subset \tilde{Q}^m_{1,i_1,i_2,\ldots,i_{s-1}} ; i_s = 1, 2, \ldots, 2^m \), each of side \( S_s = \frac{1}{2^m} \prod_{j=1}^{s} (1 - \frac{1}{2^m}) = \frac{1}{2}S_{s-1}(1 - \frac{1}{2^m}) < \frac{1}{2}S_{s-1} \), such that in every corner of \( \tilde{Q}^m_{1,i_1,i_2,\ldots,i_{s-1}} \) there locates the only one subcube \( \tilde{Q}^m_{1,i_1,i_2,\ldots,i_{s-1},i_s} \) for some \( i_s = 2 \).

Because of similarity of the construction of the sets \( K^m \) and \( \tilde{K}^m \), the standard enumeration of the set \( K^m : \langle K^m, \prec_{f_m} \rangle \), where \( f_m \) is as in the Theorem 4 for the number \( m \in \mathbb{N} \), can be easily transferred on the set \( \tilde{K}^m \). Let us designate that enumeration by \( \langle \tilde{K}^m, \prec_{f_m} \rangle \).

The figures show for case \( m = 2 \) the enumeration \( \langle K^m, \prec_{f_m} \rangle \) on the set \( K^m \) and the corresponding enumeration \( \langle \tilde{K}^m, \prec_{f_m} \rangle \) on the set \( \tilde{K}^m \) for the second iteration. Recall that for \( m = 2 \) \( f_m = f_H \)-the Hilbert space-filling function.

Now we turn to the definition of the function \( p : [0, 1]^m \to [0, 1]^n \). We establish some conditions the function \( p \) must satisfy. By induction: \( p(\tilde{Q}^m_1) = \tilde{Q}^m_1 \), and if 
\[
p(\tilde{\delta}) = \delta \text{ for } \tilde{\delta} \in \tilde{K}^m_{(s-1)}, \quad \delta \in K^m_{(s-1)}
\]
then sets \( \tilde{\delta}^* = \{ \tilde{\gamma} \in \tilde{K}^m_{ns} : \tilde{\gamma} \subseteq \tilde{\delta} \}, \delta^* = \{ \gamma \in K^m_{ns} : \gamma \subseteq \delta \} \) each have \( 2^{nm} \) elements, and both are linear ordered by \( \prec_{f_m}, \prec_{f_n} \) respectively. So that we have an order preserving bijection between the sets \( \delta^* \) and \( \tilde{\delta}^* \). For every \( \tilde{\gamma} \in \tilde{\delta}^* \) let us write
\[
p(\tilde{\gamma}) = \gamma \in \delta^* \text{ such that if } \tilde{\gamma}, \tilde{\gamma}' \in \tilde{\delta}^*, \tilde{\gamma} \prec_{f_m} \tilde{\gamma}' \implies p(\tilde{\gamma}) \prec_{f_m} p(\tilde{\gamma}').
\]

Now let us define the set \( B_0 \subseteq \tilde{Q}^m_1 \) and \( p \upharpoonright B_0 \) such that \( p(B_0) = [0, 1]^n \). Let \( B_0 \) be a set of points \( x \in \tilde{Q}^m_1 \), each of which is uniquely determined by a sequence of nested closed \( m \)-dimensional cubes (that are generated by our successive partitioning), diagonals of which shrink into the point.
Let \( x \) be an element of \( B_0 \) and \( \{ P_i : i \in \mathbb{N} \} \) be the sequence of the \( m \)-dimensional cubes \( \tilde{Q}_1^m \) that determine the point \( x \), then \( x = \bigcap_{i \in \mathbb{N}} P_i \) and also 
\[
\bigcap_{i \in \mathbb{N}} p(P_i) \in [0, 1]^n.
\]
Let us define \( p(x) = \bigcap_{i \in \mathbb{N}} p(P_i) \). It is not difficult to see that \( p : B_0 \to [0, 1]^n \).

We have completed definition of the set \( B_0 \subseteq [0, 1]^m \) and the function \( p : B_0 \to [0, 1]^n \). Let \( p_r : B_0 \to [0, 1] : r = 1, \ldots, n \) be the coordinate function for \( p : B_0 \to [0, 1]^n \). Then \( \forall t \in B_0 \)

\[
p(t) = (p_1(t), p_2(t), \ldots, p_n(t)) \in [0, 1]^n.
\]

To construct a \( C^\infty \) extension of \( p \) over all \( [0, 1]^m \), we need to find a \( C^\infty \) extension of \( p_r \) over \( [0, 1]^m \) for every \( r (1 \leq r \leq n) \). Let us fix such a \( p_r \) for some \( r (1 \leq r \leq n) \).

To extend the function \( p_r \), we will need to introduce an intermediate function \( p_{x',x''} \). Let \( x' = (x'_1 \cdots x'_j \cdots x'_m) \), \( x'' = (x''_1 \cdots x''_j \cdots x''_m) \in \mathbb{R}^m \) be such that:

\[
\exists j' \leq m : \forall j \neq j' \quad x'_j = x''_j \text{ and } x'_{j'} \leq x''_{j'}.
\]
For such $x', x''$ let us define $L_{x', x''} = \{x \in \mathbb{R}^m : x_j = x'_j$ if $j \neq j'$ and $x'_j \leq x_j \leq x''_j\}$. And $\forall x \in L_{x', x''}$ we define

\[
p_{x', x''}(x) = (p_r(x'') - p_r(x')) \cdot g\left(\frac{x_1 - x'_1}{x''_1 - x'_1}\right) + p_r(x')
\]

where (following [3], p.6) $g : \mathbb{R} \to [0, 1]$ is a smooth map such that

- $g \upharpoonright (-\infty, 0] = 0$,
- $g \upharpoonright [1, \infty) = 1$,
- $g'(t) > 0$ for $0 < t < 1$.

Therefore $p_{x', x''} : L_{x', x''} \to [p_r(x'), p_r(x'')] \subset \mathbb{R}$. Obviously, $p_{x', x''}$ makes sense only if $p_r(x')$, $p_r(x'')$ have already been defined. Then $p_{x', x''}$ is a smooth, strictly monotone increasing bijection if $p_r(x') < p_r(x'')$, or is a constant if $p_r(x') = p_r(x'')$.

We will define $p_r$ on $[0, 1] \setminus B_0$ sequentially. As for all $x \in B_0$ $p_r(x)$ have already been defined, we begin by definition $p_r(x)$ for all $x$ that lie on some edge of some cube $\tilde{Q}^{m}_{1i_1...i_jn}$, $j \in \mathbb{N}$. Let us designate the set of such $x$ as $B_1 \subset [0, 1]^m$. For any $x \in B_1 \setminus B_0$ there exits $L_{x', x''} \ni x$, lying on the same as $x$ edge of some cube $\tilde{Q}^{m}_{1i_1...i_jn}$, and $L_{x', x''} \cap B_0 = \{x', x''\}$ because, by the construction, $B_0$ is a closed set containing all vertexes of all cubes in $\tilde{K}^m_{jn}$. For $x \in B_1 \setminus B_0$ we define $p_r(x) = p_{x', x''}(x)$.

Now by induction, let us define a set $B_l \subset [0, 1]^m$ as a set of all points $x$ lying on some $l$-dimensional face of some cube $\tilde{Q}^{m}_{1i_1...i_jn} \subset \tilde{K}^m_{jn}$, and $x \notin B_{l-1}$. Suppose that we have defined $p_r(x)$, $x \in \bigcup_{l=0}^{l-1} B_l$. If $x \in B_l \setminus B_0$, then there exists $\tilde{Q}(x) \in \tilde{K}^m_{jn}$ such that $x = (x_1 \ldots x_m) \in \tilde{Q}(x)$, $x \notin \tilde{Q}(x)_{i_jn+1...i_{j+1}n}$.

If there exists $L_{x', x''} \ni x$, where $x', x''$ satisfy (2.1) and

- $x' \in B_{l-1} \cap \tilde{Q}(x)_{i'_jn+1...i'_{j+1}n}$
- $x'' \in B_{l-1} \cap \tilde{Q}(x)_{i''jn+1...i''_{j+1}n}$

for some $i'_{j+1} < i''_{j+1}$, then $\text{int}(L_{x', x''}) \subset \tilde{Q}(x) \setminus \bigcup_{i_jn+1...i_{j+1}n=1}^{2^m} \tilde{Q}(x)_{i_jn+1...i_{j+1}n}$.

then $p_r(x) \overset{def}{=} p_{x', x''}(x)$.

Let us designate all points $x$ of the set $B_l$ for which such $L_{x', x''}$ exists as $B'_l$. One can notice that $B'_1 = B_1 \setminus B_0$.

We can suppose that $p_r(x)$ is already defined for all $x \in B'_l$. And now for every $x \in B''_l = B_l \setminus (B'_l \cup B_0)$ let us define $p_r(x) = p_{x', x''}$, where $x', x''$
\( \tilde{Q}(x) \cap B'_i \) satisfy \((2.1)\) with least possible \(j'\) required by the \((2.1)\), such that \(L_{x',x''} \ni x\) and \(\text{int}(L_{x',x''}) \subseteq \tilde{Q}(x) \setminus \bigcup_{i_{j_1+1}, \ldots, i_{j+1}=1} \tilde{Q}(i_{j_1+1}, \ldots, i_{j+1}).\)

\[
\frac{\text{diam}(p_r(\tilde{Q}_{1_{s_1}, \ldots, i_{s_n}}))}{(S_{(j(s)+1)n-1 - 2S_{(j(s)+1)n})^k} = 0.
\]

**Proof.**

\[
\lim_{j(s) \to \infty} \frac{\text{diam}(p_r(\tilde{Q}_{1_{s_1}, \ldots, i_{s_n}}))}{(S_{(j(s)+1)n-1 - 2S_{(j(s)+1)n}})^k} \leq \sqrt{n} \left(\frac{1}{2^{(j(s)+1)n-1}} \cdot \frac{1}{((j(s)+1)^n)^{j(s)+1 - 1}} \cdot \prod_{i=2} \left(1 - \frac{1}{i^2}\right)\right)^k = (2.3)
\]

\[
\sqrt{n} \cdot 2^{(n-2)k+1} \cdot \lim_{j(s) \to \infty} \frac{(j(s)+1)^n}{2^{(j(s))(n-2-k)}} \cdot \lim_{j(s) \to \infty} \frac{1}{\prod_{i=2} (1 - \frac{1}{i^2})^k} = 0
\]
which follows from the fact that the first limit in Lemma 2.3 is proven.

Lemma 2.2. If \( \{x^s; s \in \mathbb{N}\} \subseteq [0, 1]^m \setminus B_0 \), such that \( \lim_{s \to \infty} |L(x^s)| = 0 \), then \( \forall k \ (0 \leq k < \frac{m}{n}) \)
\[
\lim_{s \to \infty} \frac{p_r(L''(x^s)) - p_r(L'(x^s))}{|L(x^s)|^k} = 0.
\]

Proof.
If \( \bar{Q}(x^s) = \bar{Q}_{1i_1 \ldots i_j(s)n}^m \), then
\[
\lim_{s \to \infty} \frac{|p_r(L''(x^s)) - p_r(L'(x^s))|}{|L(x^s)|^k} \leq \lim_{j(s) \to \infty} \frac{\text{diam}(p_r(\bar{Q}_{1i_1 \ldots i_j(s)n}^m))}{(S_{(j(s)+1)n-1} - 2S_{(j(s)+1)n})^k} = 0 \text{ by Lemma 2.1}
\]
Lemma 2.2 is proven.

Lemma 2.3. Let \( x^0 = (x_1^0, x_2^0, \ldots, x_i^0, \ldots, x_m^0) \in B_0 \) and \( k < \frac{m}{n} \), then
\[
\lim_{x \to x^0} \frac{p_r(x) - p_r(x^0)}{|x_i - x_i^0|^k} = 0 \text{ for } x = (x_1, x_2, \ldots, x_i, \ldots, x_m).
\]

Proof.
To prove that, suffice it to show that the limit is equal to 0 for some arbitrary chosen sequence \( \{x^j = (x_1^j, x_2^j, \ldots, x_i^j, \ldots, x_m^j) \in B_{i_0}^r \setminus B_0\} \), also without loss of generality we can suppose that \( x_i^j > x_i^{j+1} \). Let us define \( \forall j \in \mathbb{N} \) \( j(s) \) as the lowest number such that \( x_i^j \in \tilde{Q}_{1i_1 \ldots i_j(s)n}^m \) for some \( \tilde{Q}_{1i_2 \ldots i_j(s)n}^m \in \tilde{K}_{j(s)n}^m \), then
\[
\lim_{x \to x^0} \frac{p_r(x) - p_r(x^0)}{|x_i - x_i^0|^k} \leq \lim_{j(s) \to \infty} \frac{\text{diam}(p_r(\tilde{Q}_{1i_1 \ldots i_j(s)n}^m))}{(S_{(j(s)+1)n-1} - 2S_{(j(s)+1)n})^k} = 0 \text{ by Lemma 2.1}
\]
Lemma 2.3 is proven.

Note: if \( f \in C^k \), then it is not difficult to see that \( f^{(k')} \equiv 0 \ \forall k' \in \mathbb{R}^+ \setminus \mathbb{Z}, \ k' < k \).

We will use the induction by \( i \ (1 \leq i \leq m) \).
For \( i_0 = 1, \ p_r \in C^\infty \) on \( B_1 \setminus B_0 \) and if \( x \in B_1 \setminus B_0, \ k > 0 \), then
\[
D_k p_r(x) = \frac{p_r(x'') - p_r(x')}{(x'' - x^0)_{k'} \cdot D_k g \left( x_i - x_i' \right) \cdot \frac{x_i - x_i'}{x'' - x_i'}}
\]
where \( x' = L'(x) \), \( x'' = L''(x) \), \( x_i \) is unique coordinate that is not a constant on \( L(x) \). If \( x \in B_1 \cap B_0 \), then \( D_k p_r(x) = 0 \) for every \( k < \frac{m}{n} \). For the proof see Case 2 for \( i_0 = 1 \).
Now let $i_0 \in \mathbb{N}$ ($i_0 \leq m$) be such that $p_r \upharpoonright \bigcup_{i < i_0} B_i$ is a $C^{< \frac{m}{n}}$-function and all its partial derivatives of order less than $\frac{m}{n}$ vanish on $\bigcup_{i < i_0} B_i \cap B_0$.

Let us consider some $k$ ($0 \leq k < \frac{m}{n}$), and a set $T(k) = \{0, 1, \cdots, [k], k, k - 1, \cdots, k - [k]\}$.

Let $D_t p_r(x)$ be short for $\frac{\partial^{t_1 + \cdots + t_m}}{\partial x_1^{t_1} \cdots \partial x_m^{t_m}} p_r(x_1^t \cdots x_m^t)$.

**Case 1.**
If $x \in \bigcup_{i \leq i_0} B_i' \setminus B_0$, then considering derivatives with respect to every variable that is not constant in a neighborhood of $x \in \bigcup_{i \leq i_0} B_i' \setminus B_0$, we write the following general formula

$$
D_k p_r(x) = \frac{D_{k-t} p_r(x^{n'}) - D_{k-t} p_r(x')}{(x_i^t - x_i^{n'})^t} D_t g \left( \frac{x_i - x_i^t}{x_i - x_i^{n'}} \right) + D_k p_r(x')
$$

for some $t \in T(k)$.

where $x' = L'(x)$, $x'' = L''(x)$, $x_i$ is the unique coordinate that is not a constant on $L(x)$.

$D_k p_r$ exists at point $x$ and is continuous. It follows from the choice of $i_0$ and from $g$ being a $C^\infty$-function.

We also can notice that if $t \neq 0$, then $D_k p_r(x') = 0$; and if $t$ is not integer, then $D_t g = 0$.

**Case 2.**
Let $x^0 \in \bigcup_{i \leq i_0} B_i' \cap B_0$. Without loss of generality, we can suppose that $p_r \upharpoonright \bigcup_{i \leq i_0} B_i$ is a $C^{k'}$-function, where

$$
k' = \begin{cases} 
[k] & \text{if } k \text{ is not integer} \\
 k - 1 & \text{if } k \text{ is integer},
\end{cases}
$$

and all partial derivatives of order $\leq k'$ vanish on $B_0 \cap \bigcup_{i \leq i_0} B_i'$.

Now we need to prove that

$$
D_k(x^0) = \lim_{x \to x^0} \frac{|D_{k'} p_r(x) - D_{k'} p_r(x^0)|}{|x - x^0|^{\lambda}} = 0
$$

where $x = (x_1^0, x_2^0, \cdots, x_i, \cdots, x_m^0)$ for some $i$ depending on the $D_k$,

and $\lambda = \begin{cases} 
 k - [k] & \text{if } k \text{ is not integer} \\
 1 & \text{if } k \text{ is integer}.
\end{cases}$

To prove that, suffice it to show that the limit is equal to 0 for some arbitrary chosen sequence $\{x^j = (x_1^0, x_2^0, \cdots, x_i^0, \cdots, x_m^0) \in B_i' \setminus B_0\}$, also without loss of generality we can suppose that $x_i^j > x_i^{j+1}$. Then for every point $x^j$ ($j \in \mathbb{N}$) by the definition of $L(x^j)$ and that $x^0 \in B_0$, it follows that $L(x^j)$ lies on the straight line $(x_1^0, x_2^0, \cdots, x_i^0, \cdots, x_m^0)$ and

$$
L'(x^j), L''(x^j) \in B_0 \cap \bigcup_{i \leq i_0} B_i.'
Substituting the sequence \( \{ x^j; j \in \mathbb{N} \} \) in (2.6), we write:

\[
\lim_{x^j \to x^0} \frac{|D_{k'} p_{r_j}(x^j) - D_{k'} p_{r_j}(x^0)|}{|x^j_i - x^0_i|^\lambda} \leq 0
\]

(2.8)

\[
\lim_{x^j \to x^0} \frac{|D_{k'} p_{r_j}(x^j) - D_{k'} p_{r_j}(L'(x^j))|}{|x^j_i - L'(x^j)_i|^\lambda} = \frac{D_{iL} \left( \frac{x^j_i - L'(x^j)_i}{L''(x^j)_i - L'(x^j)_i} \right) - D_{tL} \left( \frac{L''(x^j)_i - L'(x^j)_i}{L'(x^j)_i - L'(x^j)_i} \right)}{|x^j_i - L'(x^j)_i|^\lambda} = \frac{1}{(L''(x^j)_i - L'(x^j)_i)^\lambda} = 0
\]

(2.9)

\[
\lim_{x^j \to x^0} \frac{|D_{k'} p_{r_j}(x^j) - D_{k'} p_{r_j}(L'(x^j))|}{|L''(x^j)_i - L'(x^j)_i|^{i+\lambda}} \cdot \frac{D_{iL}g(y) - D_{tL}g(0)}{|y - 0|^\lambda} = 0
\]

(2.10)

where

- in (2.6): \( D_{k'} p_{r_j}(x^0) = 0 \) by our assumption on \( k' \) (see (2.5));
- in (2.8): \( D_{k'} p_{r_j}(L'(x^j)) = 0 \) if \( k' > 0 \), because \( L'(x^j) \in B_0 \cap \bigcup_{i \leq i_0} B'_i \) (when \( k' = 0 \) the limit is equal to zero by Lemma 2.4);
- in (2.9): see (2.4);
- in (2.10): \( D_{tL}g(0) = 0 \) \( \forall t \in T \);
- substitution in (2.11): \( y = \frac{x^j_i - L'(x^j)_i}{L''(x^j)_i - L'(x^j)_i} \);
- in case that there exists \( j_0 \in \mathbb{N} \) such that \( \forall j \geq j_0 \) \( L'(x^j) = x_0 \), limit of the first factor in (2.11) is bounded and limit of the second factor is equal to zero. Otherwise the second factor in (2.11) is bounded and the limit of the first factor is equal to 0 by the Lemma 2.2 if \( k' - t = 0 \) or by (2.7) and (2.5) if \( k' - t \neq 0 \).

**Case 3.**

For \( x^0 \in \bigcup_{i \leq i_0} (B_i \setminus B'_i) \) the proof is similar to **Case 1**, assuming that \( p_{r_j} \setminus (\bigcup_{i \leq i_0} B_i \cup B'_i) \) is a \( C^k \)-function.

Now it follows from the cases 1,2,3:

- if \( p_{r_j} \setminus \bigcup_{i \leq i_0} B_i \) is \( C<\frac{m}{n} \) function and all its partial derivatives of order \( < \frac{m}{n} \) vanish on \( \bigcup_{i \leq i_0} B_i \cap B_0 \), then \( p_{r_j} \setminus \bigcup_{i \leq i_0} B_i \) is \( C<\frac{m}{n} \) function and all its partial derivatives of order \( < \frac{m}{n} \) vanish on \( \bigcup_{i \leq i_0} B_i \cap B_0 \). So that by induction, we have that \( p_{r_j} \mid [0,1]^m \) is \( C<\frac{m}{n} \) function and all its partial derivatives of order \( < \frac{m}{n} \) vanish on \( B_0 \).
The arc.
If \( x(1, i_1, \ldots, i_s) = x(1, i_1, \ldots, i_s+1) = b \) is a common vertex of cubes
\[ Q^m_{1,i_1,i_2,\ldots,i_s} = f_m([a, b]), \quad Q^m_{1,i_1,i_2,\ldots,i_s+1} = f_m([a, b]) \in K^m_s \]
for some \( s \in \mathbb{N} \), and \([a, b], [b, c] \in K^1_{ns} \), then the corresponding to it vertexes
\( \tilde{x}(1, i_1, \ldots, i_s), \quad \tilde{x}(1, i_1, \ldots, i_s+1) \) of cubes \( \tilde{Q}_1^m, \tilde{Q}_1^m \in \tilde{K}_s^m \)
respectively are, by the construction of \( \tilde{K}_s^m \), elements of \( B_0 \) satisfying \((\ref{2.1})\), and, by the definition of function \( p \) on \( B_0 \),
\[ p(\tilde{x}(1, i_1, \ldots, i_s)) = p(\tilde{x}(1, i_1, \ldots, i_s+1)). \]
So that there exists the \( L_{\tilde{x}(1,i_1,\ldots,i_s)}\tilde{x}(1, i_1,\ldots,i_{s+1}) \) and from \((\ref{2.2})\)
\[ p \upharpoonright L_{\tilde{x}(1,i_1,\ldots,i_s)}\tilde{x}(1, i_1,\ldots,i_{s+1}) \text{ is a constant. And more than that, because of the fact that } \tilde{x}(1, i_1, \ldots, i_s), \tilde{x}(1, i_1, \ldots, i_{s+1}) \text{ is a } B_0 \text{ all partial derivatives of } p \text{ of order } \frac{m}{n} \text{ vanish at those points and, as follows from } (\ref{2.4}), \text{ they vanish on } \]
\[ L_{\tilde{x}(1,i_1,\ldots,i_s)}\tilde{x}(1, i_1,\ldots,i_{s+1}). \]
Let us define \( E = B_0 \cup L_{\tilde{x}(1,i_1,\ldots,i_s)}\tilde{x}(1, i_1,\ldots,i_{s+1}) \), \( (1 \leq i_j \leq 2^m, \ s \in \mathbb{N} ) \). Then \( E \subseteq [0,1]^m \), \( p(E) = [0,1]^n \), every partial derivative of \( p \) of order \( \frac{m}{n} \) vanishes on \( E \), and the connectivity of \( E \) can be shown by an argument similar to that of Whitney \((\ref{10})\).

\[
\square
\]

3. Proof of the Theorem \((\ref{2})\)

Using the result of the Theorem \((\ref{1})\) we can take a map \( p' : [0,1]^{m-r} \rightarrow [0,1]^{n-r} \) contained in \( C^k \) for all real \( k < \frac{m-r}{n-r} \), and a connected subset \( E' \) such that every partial derivative of \( p' \) of order \( \frac{m-r}{n-r} \) vanishes on \( E' \) and \( p'(E') = [0,1]^{n-r} \). Then \( p = (I, p') \) and \( E = E' \times [0,1]^r \), where \( I : [0,1]^r \rightarrow [0,1]^r \) is the identity map, satisfy the conditions of the Theorem \((\ref{2})\).

\[
\square
\]

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