CHROMATIC POLYNOMIALS OF SIMPLICIAL COMPLEXES

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Abstract. In this note we consider s-chromatic polynomials of finite simplicial complexes. The s-chromatic polynomials of simplicial complexes are higher dimensional analogues of chromatic polynomials for graphs.

1. Introduction

Let \( K \) be a finite simplicial complex with vertex set \( V(K) \neq \emptyset \) and let \( r \geq 1 \) and \( s \geq 1 \) be two natural numbers. A map \( \text{col} : V(K) \to \{1, 2, \ldots, r\} \) is an \((r, s)\)-coloring of \( K \) if there are no monochrome \( s \)-simplices in \( K \) [5]. We write \( \chi^s(K, r) \) for the number of \((r, s)\)-colorings of \( K \).

Definition 1.1. The \( s \)-chromatic polynomial of \( K \) is the function \( \chi^s(K, r) \) of \( r \). The \( s \)-chromatic number of \( K \), \( \text{chr}^s(K) \), is the minimal \( r \geq 1 \) with \( \chi^s(K, r) > 0 \).

The theorem below shows that \( \chi^s(K, r) \) is indeed polynomial in \( r \) for fixed \( K \) and \( s \). (By notational convention, \( [r]^i = r(r-1)\cdots(r-i+1) \) is the \( i \)th falling factorial in \( r \).)

Theorem 1.2. The \( s \)-chromatic polynomial of \( K \) is

\[
\chi^s(K, r) = \sum_{i=\text{chr}^s(K)}^{\vert V(K) \vert} S(K, i, s)[r]^i
\]

where \( S(K, i, s) \) is the number of partitions of \( V(K) \) into \( i \) blocks containing no \( s \)-simplex of \( K \).

For \( s = 1 \), an \((r, 1)\)-coloring of \( K \) is a usual graph coloring, \( \chi^1(K, r) \) is the usual chromatic polynomial, and \( \text{chr}^1(K) \) the usual chromatic number of the 1-skeleton of \( K \). In general, \( \chi^s(K, r) \) depends only on the \( s \)-skeleton of \( K \). Although the higher \( s \)-chromatic polynomials for simplicial complexes are analogues of 1-chromatic polynomials for graphs we shall shortly see that there are structural differences between the cases \( s = 1 \) and \( s > 1 \).

Figure 1 shows a triangulation MB of the Möbius band. To the left is a \((5, 1)\)- and to the right a \((2, 2)\)-coloring of MB. The chromatic polynomials and chromatic numbers \(^1\) of MB are

\[
\chi^s(MB, r) = \begin{cases} 
      r^5 - 10r^4 + 35r^3 - 50r^2 + 24r & s = 1 \\
      r^5 - 5r^3 + 5r^2 - r & s = 2 \\
      r^5 & s \geq 3 
\end{cases}
\]

\(^1\)The computations behind the examples of this note were carried out in the computer algebra system Magma [3].
1.1. Notation. We shall use the following notation throughout the paper:

- $K$: a finite simplicial complex
- $K^s$: the $s$-skeleton of $K$
- $F^s(K)$: the set of $s$-simplices of $K$
- $\#V$ or $|V|$: the number of elements in the finite set $V$
- $V(K)$: the vertex set $\bigcup K$ of $K$ and $m(K) = |V(K)|$ is the number of vertices in $K$
- $D[V]$: the complete simplicial complex of all subsets of the finite set $V$
- $[m]$: the finite set $\{1, \ldots, m\}$ of cardinality $m$
- $[r]$: the falling factorial polynomial $[r]_i = i!(r)_i$ in $r$
- $P(a, b)$: the open interval $(a, b)$ in the poset $P$

2. Three ways to the $s$-chromatic polynomial of a simplicial complex

In this section we present three different approaches to the $s$-chromatic polynomial $\chi^s(K, r)$:

- Theorem 2.5 via 1-chromatic polynomials of graphs;
- Theorem 2.25 via the Möbius function for the $s$-chromatic lattice;
- Theorem 1.2 via the simplicial $s$-Stirling numbers of the second kind.

2.1. Block-connected $s$-independent vertex partitions. Let $s \geq 1$ be a natural number.

Definition 2.1. Let $B \subset V(K)$ be a set of vertices of $K$. Then

- $B$ is $s$-independent if $B$ contains no $s$-simplex of $K$;
- $B$ is connected if $K \cap D[B]$ is a connected simplicial complex;
- the connected components of $B$ are the maximal connected subsets of $B$.

Definition 2.2. Let $P$ be a partition of $V(K)$.

- The graph $G_0(P)$ of $P$ is the simple graph whose vertices are the blocks of $P$ and with two blocks connected by an edge if their union is connected;
- The block-connected refinement $P_0$ of $P$ is the refinement whose blocks are the connected components of the blocks of $P$;
- $P$ is block-connected if the blocks of $P$ are connected (i.e., if $P = P_0$).

Lemma 2.3. Let $P$ be a partition of $V(K)$. If two different blocks of the block-connected refinement $P_0$ are connected by an edge in the graph $G_0(P_0)$ of $P$ then they lie in different blocks of $P$.

Proof. The connected components of the blocks of $P$ are maximal with respect to connectedness.

Definition 2.4. $BCP^s(K)$ is the set of all block-connected $s$-independent partitions of $V(K)$.

Recall that $\chi^1(G_0(P), r)$ is the 1-chromatic polynomial of the simple graph $G_0(P)$ of the partition $P$.

Theorem 2.5. The $s$-chromatic polynomial for $K$ is the sum

$$\chi^s(K, r) = \sum_{P \in BCP^s(K)} \chi^1(G_0(P), r)$$

of the 1-chromatic polynomials and the $s$-chromatic number of $K$ is the minimum

$$\chi^r(K) = \min_{P \in BCP^s(K)} \chi^1(G_0(P))$$

of the 1-chromatic numbers for the graphs of all the block-connected $s$-independent partitions of $V(K)$.

Proof. Let $\text{col}: V(K) \to [r]$ be an $(r, s)$-coloring of $K$. The monochrome partition $P(\text{col})$ of $V(K)$ is the $s$-independent partition whose blocks are the nonempty monochrome sets of vertices $\{\text{col} = i\}$ for $i \in [r]$. The block-connected refinement $P(\text{col})_0$ of the monochrome partition is a block-connected $s$-independent partition of $K$. The original coloring col of $K$ is also a coloring of the graph $G_0(P(\text{col})_0)$ of $P(\text{col})_0$ for, by Lemma 2.3, distinct vertices of 1-simplices of this graph have distinct colors. We have shown that any $(r, s)$-coloring col of $K$ induces an $(r, 1)$-coloring $\text{col}_0$ of the graph $G_0(P(\text{col})_0)$ of the block-connected refinement of the monochrome partition.

Let $P \in BCP^s(K)$ be a block-connected $s$-independent partition of $V(K)$ and $\text{col}_0: P \to \{1, \ldots, r\}$ an $(r, 1)$-coloring of its graph $G_0(P)$. Then $\text{col}_0$ determines a map $\text{col}: V(K) \to [r]$ that is constant on the blocks of $P$. An $s$-simplex of $K$ can not be monochrome under col as it intersects at least two different blocks of $P$ connected by an edge of $G_0(P)$. Thus col is an $(r, s)$-coloring of $K$.

These two constructions are inverses of each other.
Remark 2.6 (The minimal block-connected $s$-independent partition). Let $C_0 = \{ \{v\} \mid v \in V(K) \}$ be the block-connected $s$-independent partition of $V(K)$ whose blocks are singletons. The graph $G_0(C_0) = K^1$ is the 1-skeleton of $K$. Thus the 1-chromatic polynomial of the 1-skeleton of $K$ is always one of the polynomials in the sum of Theorem 2.5. If $K$ is 1-dimensional, $BCP^1(K)$ consists only of the partition $C_0$ and Theorem 2.5 simply says that the 1-chromatic polynomial of a simplicial complex is the 1-chromatic polynomial of its 1-skeleton.

Example 2.7 (The block-connected 2-independent partitions for $D[3]$). The 2-simplex $D[3]$ has 4 block-connected 2-independent partitions $C_0$, $\{\{1\}, \{2, 3\}\}$, $\{\{2\}, \{1, 3\}\}$, and $\{\{3\}, \{1, 3\}\}$. The graph of $C_0$ is the complete graph $K_3$, the 1-skeleton of $D[3]$. The graphs of the other three partitions are all the complete graph $K_2$. Thus the 2-chromatic polynomial of $D[3]$ is $\chi^2(D[3], r) = \chi^1(K_3, r) + 3\chi^1(K_2, r) = \left| r \right|_3 + 3 \left| r \right|_2 = \left| r \right|_2(r + 1) = r^2 - r$ and the 2-chromatic number is $\text{chr}^2(D[3]) = 2$.

Example 2.8 (A $(2, 2)$-coloring and the graph of the block-connected refinement of its monochrome partition). The picture below illustrates a $(2, 2)$-coloring of a 9-vertex triangulation of the torus

and its corresponding graph. There are 6937 block-connected partitions of the vertex set, and 3 of them has the graph shown above. The 2-chromatic polynomial is $21[r]^2 + 742[r]_3 + 3747[r]_4 + 4908[r]_5 + 2295[r]_6 + 444[r]_7 + 36[r]_8 + [r]_9 = [r]_2(r^2 + r - 1)(r^2 + r - 1)(r^2 + r - 1)$ and the 2-chromatic number is 2.

Example 2.9 (The $(r, 2)$-colorings of a simplicial complex $K$). Let $K$ be the pure 2-dimensional complex with facets $F^2(K) = \{\{1, 2, 3\}, \{2, 3, 4\}, \{4, 5, 6\}\}$.

The picture shows a $(2, 2)$-coloring of $K$ and the corresponding $(2, 1)$-coloring of the associated graph, $G_0(P_0)$, the block connected refinement of the monochrome partition $P = \{\{1, 2, 5, 6\}, \{3, 4\}\}$. Table 1 shows the graphs $G_0(P)$ for all block connected partitions $P \in BCP^2(K)$. For each graph, the table records its 1-chromatic polynomial and its 1-chromatic number. The 2-chromatic polynomial of $K$ is $\chi^2(K, 2) = 15[r]_2 + 73[r]_3 + 62[r]_4 + 15[r]_5 + [r]_6 = [r]_2(r - 1)(r + 1)(r^2 + r - 1)$ and the 2-chromatic number is $\text{chr}^2(K) = 2$.

Example 2.10 (The $(r, 2)$-colorings of the Möbius band). The set $BCP^2(MB)$ of block-connected 2-independent partitions of the triangulated Möbius band MB (Figure 1) has 36 elements. There are 5, 5, 15, 10, 1 partitions in $BCP^2(MB)$ realizing the partitions $[3, 2]$, $[3, 1, 1]$, $[2, 2, 1]$, $[2, 1, 1, 1]$, $[1, 1, 1, 1, 1]$ of the integer $|V(MB)| = 5$. All associated graphs are complete graphs. This yields the 2-chromatic polynomial $\chi^2(MB, r) = 5[r]_2 + 20[r]_3 + 10[r]_4 + [r]_5 = [r]_2(r^3 + r^2 - 4r + 1) = r^5 - 5r^3 + 5r^2 - r$ and the 2-chromatic number is $\text{chr}^2(MB) = 2$.

Remark 2.11 (The $S$-chromatic polynomial of $K$). Let $S$ be a set of connected subcomplexes of $K$. A set $B \subset V(K)$ of vertices is $S$-independent if $B$ is not a superset of any member of $S$. Let $BCP^S(K)$ be the set of
S-independent partitions of $V(K)$. An $(r,S)$-coloring is a map $V(K) \to \{1,\ldots,r\}$ such that $\#\text{col}(S) > 1$ for all $S \in S$. The number of $(r,S)$-colorings of $K$ is

$$\chi^S(K,r) = \sum_{P \in \text{BCP}^S(K)} \chi^1(G_0(P),r)$$

as one sees by an obvious generalization of Theorem 2.5. An $(r,s)$-coloring of $K$ is an $(r,S)$-coloring of $K$ where $S = F^s(K)$ is the set of $s$-simplices.

2.2. The $s$-chromatic linear program. Read [9, §10] explains how to construct a linear program with minimal value equal to the $s$-chromatic number $\text{chr}^s(K)$ of $K$.

**Definition 2.13.** $M^s(K)$ is the set of all maximal $s$-independent subsets of $V(K)$.

Let $A$ be the $(m(K) \times |M^s(K)|)$-matrix

$$A(v,M) = \begin{cases} 1 & v \in M \\ 0 & v \not\in M \end{cases}$$

recording which vertices $v \in V(K)$ belong to which maximal $s$-independent sets $M \in M^s(K)$). Now the $s$-chromatic number

$$\text{chr}^s(K) = \min\{ \sum_{M \in M^s(K)} x(M) \mid x: M^s(K) \to \{0,1\}, \forall v \in V(K): \sum_{M \in M^s(K)} A(v,M)x(M) \geq 1 \}$$

is the minimal value of the objective function $\sum_{M \in M^s(K)} x(M)$ in $|M^s(K)|$ variables $x: M^s(K) \to \{0,1\}$, taking values 0 or 1, and $m(K)$ constraints $\sum_{M \in M^s(K)} A(v,M)x(M) \geq 1$, $v \in V(K)$.

2.3. The $s$-chromatic lattice. Our approach here simply follows Rota’s classical method for computing chromatic polynomials from Möbius functions of lattices [10, §9]. We need some terminology in order to characterize the monochrome loci for colorings of $K$. Recall that $F^s(K)$ is the set of $s$-simplices of $K$.

**Definition 2.12.** Let $S \subset F^s(K)$ be a set of $s$-simplices of $K$.

- The equivalence relation $\sim$ is the smallest equivalence relation in $S$ such that $s_1 \cap s_2 \neq \emptyset \Rightarrow s_1 \sim s_2$ for all $s_1, s_2 \in S$;
- the connected components of $S$ are the equivalence classes under $\sim$;
- $\pi_0(S)$ is the set of connected components of $S$;
- $S$ is connected if it has at most one component;
- $V(S) = \bigcup S$ is the vertex set of $S$
- $\pi(S)$ is the partition of $V(K)$ whose blocks are the vertex sets of the connected components of $S$ together with the singleton blocks $\{v\}$, $v \in V(K) - V(S)$, of vertices not in any simplex in $S$;
• $S$ is closed if $S$ contains any $s$-simplex in $K$ contained in the vertex set of $S$, i.e.
\[ \{ \sigma \in F^s(K) \mid \sigma \subset V(S) \} = S \]
• the closure of $S$ is the smallest closed set of $s$-simplices containing $S$.

For instance, the empty set $S = \emptyset$ of $0$-simplices is connected with $0$ connected components. If $K = D[4]$, the set $\{\{1, 2\}, \{2, 4\}\}$ of $1$-simplices is connected while $\{\{1, 2\}, \{3, 4\}\}$ has the two components $\{\{1, 2\}\}$ and $\{\{3, 4\}\}$.

A set of $s$-simplices is closed if and only if it equals its closure. For instance in $F^2(D[5])$, the set $\{\{1, 2, 3\}, \{3, 4, 5\}\}$ is not closed because its closure is the set of all $2$-simplices in $D[5]$. The empty set of $s$-simplices, any set of just one $s$-simplex, and any set of disjoint $s$-simplices are closed.

In this picture the green set of $2$-simplices is connected and not closed, closed and not connected, closed and connected, respectively.

The partition $\pi(S)$ has $|\pi(S)| = |\pi_0(S)| + m(K) - |V(S)|$ blocks.

Lemma 2.14. Let $S$ be a set of $s$-simplices in $K$ and $S_0$ a connected component of $S$. Then $S_0$ is closed if and only if
\[ \{ \sigma \in F^s(K) \mid \sigma \subset V(S_0) \} \subset S \]

Proof. Since the condition is certainly necessary we only need to see that it is sufficient. Let $\sigma$ be an $s$-simplex in $K$ with all its vertices in $V(S_0)$. Then $\sigma$ lies in $S$ by assumption. But $\sigma$ is equivalent to all elements of the equivalence class $S_0$. Thus $\sigma \in S_0$. \hfill \qed

Lemma 2.15. Let $S$ and $T$ be sets of $s$-simplices in $K$.

1. If $S$ and $T$ are closed, so is $S \cap T$.
2. If $S$ and $T$ have closed connected components, so does $S \cap T$.

Proof. (1) Let $\sigma$ be an $s$-simplex of $K$ and suppose that $\sigma \subset V(S \cap T)$. Then $\sigma \subset V(S)$ an $\sigma \subset V(T)$ so that $\sigma \in S$ and $\sigma \in T$ as $S$ and $T$ are closed.
(2) Let $R$ be a connected component of $S \cap T$. Let $S_0$ be the connected component of $S$ containing $R$ and $T_0$ be the connected component of $T$ containing $R$. Then $R \subset S_0 \cap T_0$. Suppose that $\sigma \in F^s(K)$ is an $s$-simplex with $\sigma \subset V(R)$. Then $\sigma \subset V(S_0 \cap T_0)$ so $\sigma \in S_0 \cap T_0$ by (1) as the connected components $S_0$ and $T_0$ are assumed to be closed. In particular, $\sigma \in S \cap T$. According to Lemma 2.14, the connected component $R$ is closed. \hfill \qed

Definition 2.16. The $s$-chromatic lattice of $K$ is the set $L^s(K)$ of all subsets of $F^s(K)$ with closed connected components. $L^s(K)$ is a partially ordered by set inclusion.

The set $L^s(K)$ contains the empty set $\emptyset$ of $s$-simplices and the set $F^s(K)$ of all $s$-simplices. These two elements of $L^s(K)$ are distinct when $K$ has dimension at least $s$.

Corollary 2.17. $L^s(K)$ is a finite lattice with $\hat{0} = \emptyset$, $\hat{1} = F^s(K)$, and meet $S \cap T = S \cap T$.

Proof. If $S, T \in L^s(K)$ then $S \cap T$ is also in $L^s(K)$ by Lemma 2.15 and this is clearly the greatest lower bound of $S$ and $T$. It is now a standard result that $L^s(K)$ is a finite lattice [12, Proposition 3.3.1]. The join $S \vee T$ of $S, T \in L^s(K)$ is the intersection of all supersets $U \in L^s(K)$ of $S \cup T$. \hfill \qed

Example 2.18 (The $s$-chromatic lattice $L^s(D[m])$). The closed and connected elements of the $s$-chromatic lattice $L^s(D[m])$ of the complete simplex $D[m]$ on $m > s$ vertices are $\emptyset$ and the $\binom{m}{s}$ sets $F^s(D[k])$ of all $s$-simplices in the subcomplexes $D[k]$ for $s < k \leq m$. The map $S \rightarrow \pi(S)$ is an isomorphism between the lattice $L^s(D[m])$ and the lattice, ordered by refinement, of all partitions of the set $[m]$ into blocks of size $> s$ or $1$. The least element, $\hat{0} = (1) \cdots (m)$, is the partition with $m$ blocks and the greatest element, $\hat{1} = (1 \cdots m)$, the partition with $1$ block.
$L^s(D[m])$ is not a graded lattice [12, p 99] in general when $s \geq 2$. To see this, observe that the 2-chromatic lattices $L^2(D[3])$, $L^2(D[4])$, and $L^2(D[4])$ are graded but the lattice $L^2(D[6])$ is not graded as it contains two maximal chains
\[
\hat{0} = (1)(2)(3)(4)(5)(6) < (123)(4)(5)(6) < (1234)(5)(6) < (12345)(6) < (123456) = \hat{1}
\]
\[
\hat{0} = (1)(2)(3)(4)(5)(6) < (123)(4)(5)(6) < (123)(45)(6) < (123)(456) < (123456) = \hat{1}
\]
of unequal length. In contrast, the 1-chromatic lattice of any finite simplicial complex is always graded and even geometric [10, §9, Lemma 1].

**Remark 2.19** (The Möbius function for the s-chromatic lattices \(L^s(D[m])\)). Our discussion of the Möbius function for the lattice \(L^s(D[m])\) echoes the exposition of the Möbius function for the geometric lattice \(L^1(D[m])\) of all partitions from [12, Example 3.10.4].

Let \(w: [m] \to \mathbb{N}\) be a function that to every element of \([m]\) associates a natural number, thought of as a weight function. We write \(w = 1^t \cdot 2^s \cdots r^m\), or something similar, for the weight function \(w\) defined on the set \([m]\) of cardinality \(m = \sum_i i_j\) and mapping \(i_j\) elements to \(j\) for \(1 \leq j \leq r\). The map \(w\) extends to a map, also called \(w\), defined on the set of all nonempty subsets \(X\) of \([m]\) given by \(w(X) = \sum x \in X w(x)\). Let \(L^s_m(w)\) be the lattice of all partitions of the set \([m]\) into blocks \(X\) that are singletons or have weight \(w(X) > s\). The non-singleton blocks of the meet \(\sigma \wedge \tau\) of two partitions \(\sigma, \tau \in L^s_m(w)\) are the subsets of weight \(> s\) of the form \(S \cap T\) where \(S\) is a block in \(\sigma\) and \(T\) a block in \(\tau\). Write \(\mu^s_m(w)\) for the Möbius function of \(L^s_m(w)\).

In particular, \(L^s_m(1^m)\) is a synonym for \(L^s(D[m])\) and we are primarily interested in the Möbius function \(\mu^s_m(1^m)\) of the uniform weight \(w = 1^m\). However, the computation of this Möbius function will involve the Möbius functions of other weights as well. We shall therefore discuss the Möbius functions \(\mu^s_m(w)\) for general weight functions \(w\).

Suppose that \(\sigma \in L^s_m(w)\), \(\sigma < \widehat{1}\), is a partition of \([m]\) into singleton blocks or blocks of weight \(> s\). Let \(w(\sigma)\) be the restriction of \(w\) to the set of blocks of \(\sigma\). Thus \(w(\sigma)(X) = \sum x \in X w(x)\) for any block \(X\) of \(\sigma\). Then the interval

\[
L^s_m(w) \supset [\sigma, \widehat{1}] = L^s_{|\sigma |}(w(\sigma))
\]

so that \(\mu^s_m(w(\sigma))(\widehat{1}) = \mu^s_{|\sigma |}(w(\sigma))(\widehat{0}, \widehat{1})\). More generally, suppose that \(\sigma < \tau\) for some \(\tau \in L^s_m(w)\). Assume that the partition \(\tau\) has blocks \(\tau_j\). Let \(\sigma_j\) be the set of those blocks of \(\sigma\) that intersect the block \(\tau_j\) of \(\tau\). Let \(w(\sigma_j)\) be the restriction of \(w(\sigma)\) to \(\sigma_j\). Then the interval

\[
L^s_m(w) \supset [\sigma, \tau] = \prod_j L^s_{|\sigma_j |}(w(\sigma_j))
\]

and therefore the value of the Möbius function on the pair \((\sigma, \tau)\)

\[
\mu^s_m(w)(\sigma, \tau) = \prod_j \mu^s_{|\sigma_j |}(w(\sigma_j))(\widehat{0}, \widehat{1})
\]

by the product theorem for Möbius functions [12, Proposition 3.8.2]. We conclude that the complete Möbius functions on all the lattices \(L^s_m(w)\), are actually determined by the values \(\mu^s_m(w)(\widehat{0}, \widehat{1})\) of these Möbius functions on just \((\widehat{0}, \widehat{1})\). See Equation (2.36) for more information about these Euler characteristics.

For the following it is convenient to name the elements of the domain \([m]\) of \(w\) so that the element \(m\) carries minimal weight. Assume that \(a_m = (1 \cdots m - 1)(m)\) is an element of \(L^s_m(w)\), ie that \(w(1) + \cdots + w(m - 1) > s\). We shall determine the set of lattice elements \(x\) with \(\chi \wedge a_m = \widehat{0}\). There is only one solution to this equation with \(x \leq a_m\) and that is \(x = \widehat{0}\). As the other solutions satisfy \(x \not< a_m\), they must have a block that contains \(m\) and at least one other element. It follows that the solutions \(x \not= \widehat{0}\) are all elements of the form

\[
x = (x_1 \cdots x_t m)(\cdot \cdots \cdot) \quad \text{with} \quad \begin{cases} w(x_1) > s - w(m) \\
 s \geq w(x_1) + \cdots + w(x_t) > s - w(m) 
\end{cases} \quad t = 1
\]

where all blocks but the unique block containing \(m\) are singletons. There are \(t + 1\) elements in the block containing \(m\) where \(t\) is some number in the range \(1 \leq t \leq s\). (All the solutions \(x \not= \widehat{0}\) are atoms in the lattice \(L^s_m(w)\).) Since we are in a lattice, the Möbius function satisfies the equation [12, Corollary 3.9.3]

\[
\mu^s_m(w)(\widehat{0}, \widehat{1}) = - \sum_{x \wedge a_m = \widehat{0}} \mu^s_m(w)(x, \widehat{1})
\]

which translates to

\[
(2.20) \quad \mu^s_m(w)(\widehat{0}, \widehat{1}) = - \sum_{x \wedge a_m = \widehat{0}} \mu^s_{|x|}(w(x))(\widehat{0}, \widehat{1})
\]

\[- \sum_{1 \leq x_1 \leq m-1 \atop w(x_1) > s - w(m)} \mu^s_{m-1}(w(x_1 m) w(\cdot \cdots \cdot))(\widehat{0}, \widehat{1}) - \sum_{1 \leq t \leq s \atop s \geq w(x_1) + \cdots + w(x_t) > s - w(m)} \sum_{1 \leq x_1, \ldots, x_t \leq m-1} \mu^s_{m-t}(w(x_1 \cdots x_t m))(\cdot \cdots \cdot)(\widehat{0}, \widehat{1})
\]

This describes a recursive procedure for computing all values of the Möbius function on the weight lattices \(L^s_m(w)\).
As an illustration we compute $\mu_2^2(1^6)(\bar{0}, \bar{1})$. Using (2.20) twice gives
$$\mu_2^2(1^6)(\bar{0}, \bar{1}) = -10\mu_2^2(3111)(\bar{0}, \bar{1}) = 10(\mu_2^2(411)(\bar{0}, \bar{1}) + \mu_2^2(33)(\bar{0}, \bar{1}))$$
The lattices $L_3^2(411)$ and $L_2^2(33)$ have 4 and 2 elements, respectively, and they look like

```
  \[ \begin{array}{c}
    \mu(\bar{0}, \cdot) = 1 \\
    \mu(\bar{1}, \cdot) = -1 \\
  \end{array} \]
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\[ \begin{array}{c}
    \mu(\bar{0}, \cdot) = 1 \\
    \mu(\bar{1}, \cdot) = 1 \\
    \mu(\bar{0}, \cdot) = 1 \\
    \mu(\bar{1}, \cdot) = 1 \\
  \end{array} \]

so that $\mu_2^2(411)(\bar{0}, \bar{1}) = 1$ and $\mu_2^2(33)(\bar{0}, \bar{1}) = -1$. Therefore $\mu_2^2(1^6)(\bar{0}, \bar{1}) = 0$.

We remind the reader of the well-known fact that $\mu_m^s(w)(\bar{0}, \bar{1})$ is the reduced Euler characteristic of the open interval $L_m^s(w)(\bar{0}, \bar{1})$ between $\bar{0}$ and $\bar{1}$ in the lattice $L_m^s(w)$.

**Proposition 2.21.** [10, §6] [12, Proposition 3.8.5] Let $x < y$ be two elements in a finite poset. The value of the Möbius function on the pair $(x, y)$ is the reduced Euler characteristic of the open interval $(x, y)$.

**Proof.** Write $\mu$ be the Möbius function of $P$ and $E$ for Euler characteristic. The closed interval from $x$ to $y$ has Euler characteristic 1 since it has a smallest element. Thus

$$1 = E([x, y]) = \sum_{a, b \in [x, y]} \mu(a, b) = \sum_{a, b \in (x, y)} \mu(a, b) + \sum_{a \in [x, y]} \mu(a, y) + \sum_{b \in [x, y]} \mu(x, b) - \mu(x, y) = E((x, y)) + 0 + 0 - \mu(x, y) = E((x, y)) - \mu(x, y)$$
or $\mu(x, y) = \bar{E}((x, y))$. $\square$

For $1 \leq s \leq m + 1$ let $B(m, s)$ be the graded poset of nonempty subsets of $[m]$ of cardinality less than $s$.

**Lemma 2.22.** The reduced Euler characteristic of $B(m, s)$ is

$$\bar{E}(B(m, s)) = (-1)^s \binom{m - 1}{s - 1}, \quad 1 \leq s \leq m + 1$$

**Proof.** It is rather easy to get the recurrence relation

$$E(B(m, 2)) = m$$

$$E(B(m, s)) = E(B(m, s - 1)) + \binom{m}{s - 1} \sum_{j=1}^{s-1} (-1)^{s-1-j} \binom{s-1}{j}, \quad 2 < s < 2 + m$$

Since the sum of binomial coefficients has value $(-1)^s$, we get the recurrence relation

$$\bar{E}(B(m, 2)) = m - 1$$

$$\bar{E}(B(m, s)) = \bar{E}(B(m, s - 1)) + (-1)^s \binom{m}{s - 1}, \quad 2 < s < 2 + m$$

for the reduced Euler characteristic. The claim of the lemma follows immediately. $\square$

**Example 2.23 (Reduced Euler characteristics of the s-chromatic lattice intervals $L_m^s(w)(\bar{0}, \bar{1})$).** The reduced Euler characteristics $\mu_m^s(1^m)(\bar{0}, \bar{1}) = \bar{E}(L_m^s(1^m)(\bar{0}, \bar{1}))$, $m \geq s + 2$, for $s = 1, 2, \ldots, 8$ are

2. $-6, 24, -120, 720, -5040, 40320, -362880, 3628800, -39916800, 479001600, -6227020800, 87178291200, \ldots$

3. $-6, 0, 90, -630, 2520, 0, -113400, 1247400, -7484400, 681080400, -10216206000, 81729648000, \ldots$

4. $-10, 20, -70, 560, -4200, 25200, -138600, 924000, -8408400, 84084000, -798798000, 7399392000, \ldots$

5. $-15, 35, -70, 0, 2100, -23100, 173250, -1051050, 5255250, -15765750, -10510500, 285885600, \ldots$

6. $-21, 56, -126, 252, -924, 11088, -126126, 1093092, -76936864, 46143496, -254438184, 1492322832, \ldots$

7. $-28, 84, -210, 462, -924, 0, 42042, -630630, 6306300, -51459408, 351639288, -2118412296, 11406835440, \ldots$

8. $-36, 120, -330, 792, -1716, 3432, -12870, 205920, -3150576, 35706528, -322583976, 2460949920, \ldots$

9. $-45, 165, -495, 1287, -3003, 6435, -12870, 0, 787644, -14965236, 191222460, -1920538620, \ldots$
The first sequence, $\mu_m^1(1^m)(\emptyset, 1)$, $m \geq 2$, is the sequence $(-1)^{m-1}(m-1)!$ of reduced Euler characteristics of the lattice of partitions of $[m]$ [12, Example 3.10.4]. The second sequence, $\mu_m^2(1^m)(\emptyset, 1)$, $m \geq 3$, seems to coincide with first terms of the sequence A009014 from The On-Line Encyclopedia of Integer Sequences (OES). The remaining 6 sequences apparently do not match any sequences of the OES.

The first $s$ terms of these sequences are signed binomial coefficients. This is because the interval $(\emptyset, 1)$ in $L^s(D[m])$ is isomorphic to the opposite of the poset $B(m, m-s)$ when $s+2 \leq m \leq 2s+1$. Thus the reduced Euler characteristic

$$\mu_m^s(1^m)(\emptyset, 1) = \hat{E}(B(m, m-s)) = (-1)^{m-s} \binom{m-1}{s}, \quad s + 2 \leq m \leq 2s+1,$$

according to Lemma 2.22.

The first terms of the sequence $\mu_m^2(3^11^{m-1})(\emptyset, 1)$, $m \geq 3$, of reduced Euler characteristics of the weighted lattice intervals $L_m^2(3^11^{m-1})(\emptyset, 1)$,

$$1, 0, -6, 30, -90, 0, 2520, -22680, 113400, 0, -7484400, 97297200, -681080400, 0, 81729648000, -1389404016000, \ldots$$

 seem to coincide up to sign with first terms of the sequence A009775 from OES. The sequence of reduced Euler characteristics $\mu_m^2(3^21^{m-2})(\emptyset, 1)$, $m \geq 3$, of the lattice interval $L_m^2(3^21^{m-2})(\emptyset, 1)$,

$$2, -4, 6, 120, 720, -2520, -2520, 136080, -1360800, 7484400, 7484400, -778377600, 10897286400, -81729648000, 0, 1389404016000, \ldots$$

 apparently does not match any sequence in the OES.

Define the $s$-monochrome set of a map col: $V(K) \to [r] = \{1, \ldots, r\}$ to be the set

$$M^s(\text{col}) = \{\sigma \in F^s(K) \mid |\text{col}(\sigma)| = 1\}$$

of all monochrome $s$-simplices in $K$. The map col is an $(r, s)$-coloring of $K$ if and only if $M^s(\text{col}) = \emptyset$.

**Lemma 2.24.** The $s$-monochrome set $M^s(\text{col})$ of any map col: $V(K) \to [r]$ is an element of the $s$-chromatic lattice $L^s(K)$.

**Proof.** Let $S$ be a connected component of $M^s(\text{col})$. Since $S$ is connected, all vertices in $S$ have the same color. Let $\sigma \in F^s(K)$ be an $s$-simplex of $K$ such that $\sigma \in V(S)$. The $\sigma$ is monochrome: $\sigma \in M^s(\text{col})$. By Lemma 2.14, $S$ is closed. \qed

**Theorem 2.25.** The number of $(r, s)$-colorings of $K$ is

$$\chi^s(K, r) = \sum_{T \in L^s(K)} \mu(\emptyset, T)^{|\pi(T)|}$$

where $\mu$ the Möbius function for the $s$-chromatic lattice $L^s(K)$.

**Proof.** For any $B \in L^s(K)$, let $\chi(K, r, s, B)$ be the number of maps col: $V(K) \to [r]$ with $M^s(\text{col}) = B$. We want to determine $\chi(K, r, s, \emptyset) = \chi^s(s, K)$. For any $A \in L^s(K)$,

$$r^{|\pi(A)|} = \sum_{A \leq B} \chi(K, r, s, B)$$

because there are $r^{|\pi(A)|} r^{|m(K)|-|V(A)|} = r^{|\pi(A)|}$ maps col: $V(K) \to [r]$ with $A \leq M^s(\text{col})$. Equivalently,

$$\sum_{A \leq B} \mu(A, B)^{r^{|\pi(B)|}} = \chi(K, r, s, A)$$

by Möbius inversion [12, Proposition 3.7.1]. The statement of the theorem is the particular case of this formula where $A = \emptyset$. \qed

The defining rules for the Möbius function of the poset $L^s(K)$ [12, 3.7]

- $\mu(S, S) = 1$ for all $S \in L^s(K)$
- $\sum_{R \leq S \leq T} \mu(R, S) = 0$ when $R \lessdot T$
- $\mu(R, S) = 0$ when $R \lessdot S$

imply that $\mu(\emptyset, 0) = 1$ and $\mu(\emptyset, |\sigma|) = (-1)^{|\sigma|}$ for every $s$-simplex $\sigma \in F^s(K)$.

**Corollary 2.26.** The highest degree terms of the $s$-chromatic polynomial are

$$\chi^s(K, r) = r^{|m(K)|} - f^s_s(K) r^{|m(K)|-s} + \ldots$$

Thus the $s$-chromatic polynomial determines $f_0(K)$ and $f_s(K)$. 

Proof. The $s$-chromatic polynomial is
\[ \chi_s(K, r) = \mu(\hat{0}, \hat{0})r^{f_0(K)} + \sum_{\sigma \in F^s(K)} \mu(\hat{0}, \{\sigma\})r^{f_0(K) - s} + \ldots \]
where $\mu(\hat{0}, \hat{0}) = 1$ and $\mu(\hat{0}, \{\sigma\}) = -1$ for all $s$-simplices $\sigma$ of $K$. \qed

Example 2.27. Consider the 2-dimensional complex $K$ from Example 2.9. The 2-chromatic lattice $L^2(K)$ of $K$

\[
\begin{array}{c}
\hat{1} \\
\{1, 2, 3\} & \{1, 2, 3\} & \{2, 3, 4\} & \{4, 5, 6\} & \{3, 5, 6\}
\end{array}
\]

consists of all subsets of $F^2(K)$. The 2-chromatic polynomial is
\[ \chi^2(K, r) = r^6 - r^4 - r^4 + r^3 + r^2 - r = r^6 - 3r^4 + r^3 + 2r^2 - r \]
$K$ has $\chi^2(K, 2) = 30 (2, 2)$-colorings and $\chi^2(K, 3) = 528 (3, 2)$-colorings.

Example 2.28. The triangulation MB of the Möbius band with $f$-vector $f(MB) = (5, 10, 5)$ shown in Figure 1 has the following (reduced) 2-chromatic lattice $L^2(MB) - \{\hat{0}, \hat{1}\}$

\[
\begin{array}{c}
\{1, 3, 5\} & \{1, 3, 5\} & \{2, 4, 5\} & \{1, 2, 4\} & \{2, 4, 5\} \\
\{2, 3, 5\} & \{1, 3, 4\} & \{2, 3, 5\} & \{1, 3, 4\} & \{1, 2, 4\}
\end{array}
\]

\[
\begin{array}{c}
\{1, 3, 5\} & \{2, 3, 5\} & \{1, 3, 4\} & \{2, 4, 5\} & \{1, 2, 4\}
\end{array}
\]

and 2-chromatic polynomial
\[ \chi^2(MB, r) = r^5 - 5r^3 + 5r^2 - r \]
The lattice $L^2(MB)$ is graded but it is still not semi-modular [12, Proposition 3.3.2]: The meet and join of $a = \{\{2, 3, 5\}\}$ and $b = \{\{1, 3, 4\}\}$ are $a \wedge b = \hat{0}$ and $a \vee b = \hat{1}$. Thus $a$ and $b$ cover $a \wedge b$ but $a \vee b$ covers neither $a$ nor $b$.

Example 2.29. Let MT be Möbius’s minimal triangulation of the torus with $f$-vector $f(MT) = (7, 21, 14)$ and $P_2$ the triangulation of the projective plane with $f$-vector $f(P_2) = (1, 6, 15, 10)$ shown in Figure 2 (decorated with $(3, 2)$-colorings). The chromatic polynomials of these two simplicial complexes are
\[
\begin{align*}
\chi^1(MT, r) &= [r]_7, \\
\chi^2(MT, r) &= r^7 - 14r^5 + 21r^4 + 7r^3 - 21r^2 + 6r \\
\chi^1(P_2, r) &= [r]_6, \\
\chi^2(P_2, r) &= r^6 - 10r^4 + 15r^3 - 6r^2
\end{align*}
\]
In both cases, the 1-skeleton is the complete graph on the vertex set. The chromatic numbers are $\text{chr}^1(MT) = 7$, $\text{chr}^1(P_2) = 6$, and $\text{chr}^2(MT) = 3 = \text{chr}^2(P_2)$.

The chromatic polynomials of simple graphs (the 1-chromatic polynomials of simplicial complexes) are known to have these properties:
- The coefficients are sign-alternating [10, §7, Corollary]
- The coefficients are log-concave (Definition 2.43) in absolute value [7]
- There are no negative roots and no roots between 0 and 1 [14]
In contrast, the coefficients of the 2-chromatic polynomial
\[ \chi^2(\text{MT}, r) = r^7 - 14r^5 + 21r^4 + 7r^3 - 21r^2 + 6r = [r]_3(r + 1)(r^3 + 2r^2 - 9r + 3) \]
are not sign-alternating, not log-concave in absolute value, and the polynomial has a negative root and a root between 0 and 1.

2.4. The $s$-chromatic polynomial in falling factorial form. Theorem 1.2 provides an interpretation of the coefficients of the falling factorial $[r]_s$ in the $s$-chromatic polynomial of the simplicial complex $K$.

**Definition 2.30.** $S(K, r, s)$ is the number of partitions of $V(K)$ into $r$ $s$-independent blocks.

We think of $S(K, r, s)$ as an $s$-Stirling number of the second kind for the simplicial complex $K$. If $s > \dim(K)$, then there are no $s$-simplices in $K$ and all partitions of $V(K)$ are $s$-independent, so that $S(K, r, s)$ is the Stirling number of the second kind $S(m(K), r)$ [12, p 33]. We now explain the general relation between these simplicial Stirling numbers $S(K, r, s)$ and the usual Stirling numbers of the second kind.

Define the $s$-monochrome set of a partition $P$ of $V(K)$ to be the set
\[ M^s(P) = \{ \sigma \in F^s(K) \mid \sigma \text{ is contained in a block of } P \} \]
of all $s$-simplices entirely contained in one of the blocks of $P$. The set $M^s(P)$ is an element of the $s$-chromatic lattice $L^s(K)$ by Lemma 2.24.

**Theorem 2.31.** The number of partitions of $V(K)$ into $r$ $s$-independent blocks is
\[ S(K, r, s) = \sum_{T \in L^s(K)} \mu(\emptyset, T)S(|\pi(T)|, r) \]
where $\mu$ the Möbius function for the $s$-chromatic lattice $L^s(K)$.

**Proof.** For any $B \in L^s(K)$, let $S(K, r, s, B)$ be the number of partitions $P$ of $V(K)$ into $r$ blocks with monochrome set $M^s(P) = B$. We want to determine $S(K, r, s, \emptyset) = S(K, r, s)$. For any $A \in L^s(K)$,
\[ S(|\pi(A)|, r) = \sum_{A \leq B} S(K, r, s, B) \]
because there are $S(|\pi(A)|, r)$ partitions $P$ of $V(K)$ into $r$ blocks with $A \leq M^s(P)$. Equivalently,
\[ \sum_{A \leq B} \mu(A, B)S(|\pi(B)|, r) = S(K, r, s, A) \]
by Möbius inversion [12, Proposition 3.7.1]. The statement of the theorem is the particular case of this formula where $A = \emptyset$. \qed

**Proof of Theorem 1.2.** We simply follow the proof of the similar statement for chromatic polynomials for graphs [9, Theorem 15]. When $r \geq i$ we can get an $(r, s)$-coloring out of one of the $S(K, i, s)$ partitions of $V(K)$ into $i$ $s$-independent blocks by choosing $i$ out of the $r$ colors and assigning them to the $i$ blocks. There are \( \binom{r}{i} \) ways of
choosing the \(i\) out of \(r\) colors and \(i!\) ways of coloring \(i\) blocks in \(i\) colors. The number of \((r, s)\)-colorings of \(K\) in exactly \(i\) colors is thus

\[
S(K, i, s) \binom{r}{i} i! = S(K, i, s)[r]_i
\]

so that

\[
\chi^s(K, r) = \sum_{i=1}^{m(K)} S(K, i, s)[r]_i
\]

is the total number of \((r, s)\)-colorings of \(K\).

\[
\square
\]

**Corollary 2.32.** The reduced Euler characteristic of the open interval \((\hat{0}, \hat{1})\) in \(s\)-chromatic lattice \(L^s(K)\) is

\[
\mu(L^s(K))(\hat{0}, \hat{1}) = \sum_{i=\text{chr}^s(K)}^{m(K)} (-1)^{i-1}(i-1)!S(K, i, s)
\]

**Proof.** Equate the terms of degree 1 of the two expressions

\[
\sum_{T \in L^s(K)} \mu(\hat{0}, T)\pi(|\pi(T)|) = \sum_{i=\text{chr}^s(K)}^{m(K)} S(K, i, s)[r]_i
\]

from Theorem 2.25 and Theorem 1.2 for the \(s\)-chromatic polynomial of \(K\).

\[
\square
\]

We observe that

\[
\sum_i S(K, i, s)[r]_i = \sum_i \mu(\hat{0}, T)S(|\pi(T)|, i)[r]_i = \sum_i \mu(\hat{0}, T)S(|\pi(T)|, i)[r]_i = \sum_i \mu(\hat{0}, T)r^{|\pi(T)|}
\]

so that Theorem 2.31 implies Theorem 1.2.

The \(s\)-chromatic number of \(K\) is immediately visible with the \(s\)-chromatic polynomial in factorial form because

\[
\text{chr}^s(K) = \min\{i \mid S(K, i, s) \neq 0\}
\]

is the lowest degree of the nonzero terms. The positive integer sequence

\[
\chi^s(K, \text{chr}^s(K)), \ldots, \chi^s(K, m(K)) = 1
\]

has no internal zeros. (If there is a partition of \(V(K)\) into \(r\) blocks not containing any \(s\)-simplex of \(K\) and \(r < m(K)\), then split one of the blocks with more than one vertex into two sub-blocks to get a partition of \(V(K)\) into \(r + 1\) blocks containing no \(s\)-simplices of \(K\).)

The simplicial Stirling numbers satisfy the recurrence relations

\[
S(K, r, s) = \sum_{\emptyset \subseteq U \subseteq V(K) \setminus \{v_0\} \atop V(K) - U \text{ s-independent}} S(K \cap D[U], r - 1, s), \quad S(K, 1, s) = \begin{cases} 1 & s > \text{dim}(K) \geq 0 \\ 0 & \text{otherwise} \end{cases}
\]

To see this, fix a vertex \(v_0\) of \(K\). Let \(P\) be partition of \(V(K)\) into \(r\) \(s\)-independent subsets. Let \(U_0\) be the block containing \(v_0\). The other blocks in \(P\) form a partition \(P_0\) of \(K \cap D[V(K) - U_0]\) into \(r - 1\) \(s\)-independent subsets. The map \(P \leftrightarrow (P_0, U_0)\) is a bijection.

The familiar recurrence relation \(S(m, r) = S(m - 1, r - 1) + rS(m - 1, r)\) for Stirling numbers of the second kind does not readily apply to simplicial Stirling numbers. The closest analogue may be

\[
S(K, r, s) = S(K \cap D[V(K) - \{v_0\}], r - 1, s) + \sum_{P \in S(K \cap D[V(K) - \{v_0\}], r, s)} |\{B \in P \mid B \cup \{v_0\} \text{ is s-independent in } K\}|
\]

where \(v_0\) is a vertex of \(K\) and \(S(K \cap D[V(K) - \{v_0\}], r, s)\) is the set of partitions \(P\) of the vertex set of \(K \cap D[V(K) - \{v_0\}]\) into \(r\) \(s\)-independent subsets.

**Proposition 2.34.** Let \(K\) be a subcomplex of \(L\) and assume that \(V(K) = V(L)\).

1. \(S(K, r, s) \geq S(L, r, s)\) for all \(r\).
2. If \(S(K, r, s) = S(L, r, s)\) for some \(r\) with \(\frac{1}{3}(|V| - 1) \leq r \leq |V| - s\), then \(K^s = L^s\).
shows that \( S(1) \) can be used to compute these numbers. Table 2, column \( \sigma \), provides a means to compute these numbers. Table 2. Chromatic tables for complete simplices \( D[m] \) for \( m = 2, \ldots, 7 \)

**Proof.** (1) Let \( V \) be the vertex set of \( K \) and \( L \). Write \( S(K, r, s) \) and \( S(L, r, s) \) for the set of partitions of \( V \) into \( r \) blocks containing no \( s \)-simplex of \( K \) or \( L \), respectively. Then \( S(L, r, s) \subseteq S(K, r, s) \) for all \( r \) and \( s \). Thus \( S(L, r, s) \subseteq S(K, r, s) \).

(2) Suppose that \( \sigma \in F^s(L) - F^s(K) \) is an \( s \)-simplex of \( L \) that is not an \( s \)-simplex of \( K \). Any partition of the form 

\[
\emptyset \subseteq \tau, \quad \tau \in S(D[V - \sigma], r - 1, s),
\]

in \( S(K, r, s) \) - \( S(L, r, s) \). The set \( S(D[V - \sigma], r - 1, s) \) is nonempty when

\[
\text{chr}^s(D[V - \sigma]) = \left\lfloor \frac{|V| - s - 1}{s} \right\rfloor \leq r - 1 \leq |V| - s - 1
\]

and thus \( S(K, r, s) \) is strictly greater than \( S(L, r, s) \) when \( \frac{|V| - 1}{s} \leq r \leq |V| - s \).

**Remark 2.35** \( (S(K, r, s) \) for the complete simplex \( K = D[m] \)). For any finite set \( M \), let \( S(M, r, s) \) stand for \( S(D[M], r, s) \) (Definition 2.30), the number of partitions of the set \( M \) into \( r \) blocks containing at most \( s \) elements. Let us even write \( S(m, r, s) \) in case \( M = [m] \), \( m \geq 1, r, s \geq 0 \). Clearly, \( S(m, r, s) \) is nonzero only when \( m/s \leq r \leq m \). Also, \( S(m, r, s) = S(m, r) \) when \( r \) is among the \( s \) numbers \( m - s + 1, \ldots, m \). The recurrence relation

\[
S(m, r, s) = \sum_{j=m-s}^{m-1} \binom{m-1}{j} S(j, r - 1, s)
\]

can be used to compute these numbers. Table 2 shows \( S(m, r, s) \) for small \( m \); the number \( S(m, r, s) \) is in row \( s \) and column \( r \) in the chromatic table (Definition 2.39) for \( D[m] \). All the red numbers are usual Stirling numbers of the second kind.

According to Theorem 1.1, the numbers \( S(m, r, s) \) determine the \( s \)-chromatic polynomial in falling factorial form of the complete simplex on \( m \) vertices

\[
\chi^s(D[m], r) = \sum_{i=[m/s]}^{m} S(m, i, s)[r]_i
\]

and, according to Corollary 2.32, they also determine the reduced Euler characteristic

\[
\mu^s_m(1^m)(\hat{0}, \hat{1}) = \sum_{i=[m/s]}^{m} (-1)^{i-1}(i-1)!S(m, i, s)
\]

of the \( s \)-chromatic lattice \( L^s(D[m]) \).

More generally, if \( w: M \to \mathbb{N} \) is a function on \( M \) with natural numbers as values, let \( S(M, w, r, s) \) be the number of partitions of \( M \) into admissible blocks, where we declare a block admissible if it is a singleton or it has weight at most \( s \). (Then \( S(m, r, s) = S([m], 1^m, r, s) \) occur when \( M = [m] \) and \( w = 1^m \) places weight 1 on all elements.) Any such partition is a partition of \( M \) into blocks of weight at most \( s \), and therefore \( S(M, w, r, s) \leq S(#M, r, s) \).

In particular, \( S(M, w, r, s) \) is nonzero only when \( #M/s \leq r \leq #M \). The recurrence relation

\[
S(M, w, r, s) = \sum_{\emptyset \neq J \subseteq M - \{\text{max}(M)\}} S(J, w, j, r - 1, s)
\]

provides a means to compute these numbers.
The weighted version of Equation (2.33) for $K = D[m]$,\
\[
\sum_{\sigma \in L^*_m(w)} \mu^m_{\sigma}(w)(\hat{0}, \sigma)^{[\sigma]} = \sum_{i=[m/s]}^{m} S([m], w, i, s)[r]i,
\]
implies, by equating coefficients of first degree terms, the expression\
\[
\mu^m_{\sigma}(w)(\hat{0}, \hat{1}) = \sum_{i=[m/s]}^{m} (-1)^{i-1}(i-1)!S([m], w, i, s)
\]
for the Euler characteristic of the weighted lattice $L^*_m(w)$ from Remark 2.19.

Because any simplicial complex $K$ is a subcomplex of the complete simplex $D[m(K)]$ on its vertex set, we have\
\[
S(m(K), r) \geq S(K, r, s) \geq S(m(K), r, s), \quad 1 \leq r \leq m(K)
\]
Moreover, these inequalities are equalities for the $s$ highest values $m(K) - s + 1, \ldots, m(K)$ of $r$. Thus the $s$ terms of highest falling factorial degree in the $s$-chromatic polynomial of $K$
\[
\chi^s(K, r) = \sum_{i=0}^{m(K)-s} S(K, i, s)[r]i + \sum_{i=m(K)-s+1}^{m(K)} S(m(K), i)[r]i,
\]
are given by the $s$ Stirling numbers $S(m(K), m(K)-s+1), \ldots, S(m(K), m(K))$ of the second kind. These coefficients depend only on the size of the vertex set of $K$. We shall next show that the coefficient number $s + 1$ counted from above, $S(K, m(K) - s, s)$, informs about the number $f_s(K)$ of $s$-simplices in $K$.

**Proposition 2.38.** $S(K, m(K) - s, s) = S(m(K), m(K) - s) - f_s(K)$. If $S(K, m(K) - s, s) = S(m(K), m(K) - s)$ then $K^s = D[m(K)]^s$.

**Proof.** The only partitions of the $S(m, m-s)$ partitions of $V(K)$ into $m-s$ blocks that are not $s$-independent are those consisting of one $s$-simplex of $K$ together with singleton blocks. If $S(K, m(K) - s, s) = S(D[m(K)], m(K) - s, s)$ then $f_s(K) = f_s(D[m(K)])$ so $K^s = D[m(K)]^s$. (This is a special case of Proposition 2.34.(2).) \hfill $\square$

**Definition 2.39.** The chromatic table, $\chi(K)$, of $K$ is the $(\text{dim}(K) \times m(K))$-table with $S(K, r, s)$ in row $s$ and column $r$.

This means that row $s$ in the chromatic table lists the coefficients of the $s$-chromatic polynomial. The chromatic table of a 3-dimensional simplicial complex $K$, for instance, looks like this

\[
\begin{array}{cccccccc}
  & r = 1 & r = 2 & \cdots & r = m-3 & r = m-2 & r = m-1 & r = m \\
S(K, 1, 1) & S(K, 1, 1) & S(K, 2, 1) & \cdots & S(K, m-3, 1) & S(K, m-2, 1) & S(m, m-1) - f_1 & S(m, m) = 1 \\
S(K, 2, 2) & S(K, 1, 2) & S(K, 2, 2) & \cdots & S(K, m-3, 2) & S(m, m-2) - f_2 & S(m, m-1) & S(m, m) = 1 \\
S(K, 3, 3) & S(K, 1, 3) & S(K, 2, 3) & \cdots & S(m, m-3) - f_3 & S(m, m-2) & S(m, m-1) & S(m, m) = 1 \\
\end{array}
\]

where the red entries in row $s$ are Stirling numbers of the second kind $S(m, r)$ for $m-s+1 \leq r \leq m$, and the blue entry in row $s$ is $S(m(K), m(K) - s) - f_s(K)$.

**Example 2.40.** The chromatic tables of the 2-dimensional simplicial complexes from Examples 2.9, 2.28, and 2.29 are

\[
\chi(K) = \begin{pmatrix}
0 & 0 & 2 & 10 & 7 & 1 \\
0 & 15 & 73 & 62 & 15 & 1 \\
\end{pmatrix} \quad \chi(\text{MB}) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 5 & 20 & 10 & 1 \\
\end{pmatrix}
\]

\[
\chi(\text{MT}) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 84 & 231 & 126 & 21 & 1 \\
\end{pmatrix} \quad \chi(\text{P2}) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 45 & 55 & 15 & 1 \\
\end{pmatrix}
\]

The red entries in column $r$ are Stirling numbers $S(m, r)$ and they are independent of the row index. The blue entry in row $s$ and column $m-s$, which equals $S(m-s, s) - f_s(K)$, detects if $K$ has maximal $s$-skeleton by Proposition 3.

**Example 2.41.** Let $K = \text{AS3}$ be Altshuler’s peculiar triangulation of the 3-sphere with $f$-vector $f = (10, 45, 70, 35)$ [1]. The 1-chromatic polynomial is $\chi^1(\text{AS3}, r) = [r]_{10}$ as $K^1$ is the complete graph on 10 vertices. The chromatic table is

\[
\chi(\text{AS3}) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 26 & 4320 & 25915 & 38550 & 22152 & 5845 & 750 & 45 & 1 \\
\end{pmatrix}
\]

The blue numbers determine the $f$-vector

\[
f(\text{AS3}) = (10, S(10, 9) - \chi(\text{AS3}), S(10, 8) - \chi(\text{AS3}), S(10, 7) - \chi(\text{AS3}))
\]
The simplicial Stirling numbers for $S^3_{17,74}$

The row numbers of the first nonzero term in each row tell us that $\text{chr}^1(AS3) = 10$, $\text{chr}^2(AS3) = 4$, and $\text{chr}^3(AS3) = 2$.

**Example 2.42.** The nonconstructible, nonshellable 3-sphere $S^3_{17,74}$, $f = (17, 91, 148, 74)$, found by Lutz [8], has

as its chromatic table. Figure 3 shows a semi-logarithmic plot of the simplicial Stirling numbers $S(S^3_{17,74}, r, s)$. The triangulation $\Sigma^3_{16}$, $f = (16, 106, 180, 90)$, of the Poincaré homology 3-sphere constructed by Björner and Lutz [2, Theorem 5] has

as its chromatic table.

Observe that all the above chromatic tables have strictly log-concave rows.

**Definition 2.43.** [11] A finite sequence $a_1, a_2, \ldots, a_N$ of $N \geq 3$ nonnegative integers is strictly log-concave if $a_i a_{i+1} < a_i^2$ for $1 < i < N$ (and log-concave if $a_i a_{i+1} \leq a_i^2$).
It has been conjectured that the sequence of coefficients of the 1-chromatic polynomial of a simple graph in falling factorial form, \( r \to S(K, 1, r) \), \( \text{chr}^1(K) \leq r \leq m(K) \), is log-concave [4, Conjecture 3.11]. More generally, one may ask

**Question 2.44.** Is the finite sequence of simplicial Stirling numbers

\[
\begin{align*}
    r & \to S(K, r, s), \\
    \text{chr}^s(K) & \leq r \leq m(K),
\end{align*}
\]

log-concave for fixed \( K \) and \( s \)?

This seems to be the right question to ask as it may be true for all the chromatic polynomials of a simplicial complex and we have seen that the absolute value of the coefficients of the \( s \)-chromatic polynomial are simply not log-concave for \( s > 1 \).

Note that the Stirling numbers of the second kind, which are upper bounds for the simplicial Stirling numbers \( S(K, r, s) \) by the inequalities (2.37), are log-concave in \( r \) [11, Corollary 2].

We shall now examine Question 2.44 on two spherical boundary complexes of cyclic \( n \)-polytopes.

**Definition 2.45.** \( \partial \text{CP}(m, n) \), \( m > n \), is the \((n − 1)\)-dimensional simplicial complex on the ordered set \([m]\) with the following facets: An \( n \)-subset \( \sigma \) of \([m]\) is a facet if and only if between any two elements of \([m] − \sigma \) there is an even number of vertices in \( \sigma \).

By Gale’s Evenness Theorem [6], the simplicial complex \( \partial \text{CP}(m, n) \) triangulates the boundary of the cyclic \( n \)-polytope on \( m \) vertices. Thus \( \partial \text{CP}(m, n) \) is a simplicial \((n − 1)\)-sphere on \( m \) vertices and it is \([n/2]\)-neighborly in the sense that \( \partial \text{CP}(m, n) \) has the same \( s \)-skeleton as the full simplex on its vertex set when \( s < [n/2] \).

**Example 2.46** (Cyclic polytopes with log-concave simplicial Stirling numbers of the second kind). Let \( \partial \text{CP}(m, n) \) be the triangulated boundary of the cyclic polytope on \( m \) vertices in \( \mathbb{R}^n \). The simplicial complex \( \partial \text{CP}(m, n) \) is an \( m \)-vertex triangulation of \( S^{n−1} \). The chromatic tables of the simplicial 3-spheres \( \partial \text{CP}(m, 4) \) on \( m = 6, 7, 8, 9, 10 \) vertices are

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 21 & 47 & 15 & 1 \\
0 & 16 & 81 & 65 & 15 & 1 \\
0 & 94 & 1062 & 2523 & 1719 & 408 & 36 & 1 \\
0 & 36 & 1729 & 6471 & 6591 & 2619 & 462 & 36 & 1 \\
0 & 46 & 4445 & 25960 & 38550 & 22152 & 4225 & 680 & 45 & 1
\end{pmatrix}
\]

All rows are strictly log-concave. As \( \partial \text{CP}(m, 4)^3 = D[m]^3 \), the 1-chromatic number \( \text{chr}^1(\partial \text{CP}(m, 4)) = m \), and it is not difficult to see that the 2-chromatic number \( \text{chr}^2(\partial \text{CP}(m, 4)) = 2 \) if \( m \) is even and 3 if \( m \) is odd [5].

Right multiplication with the upper triangular matrix \( ([ij]_{1 \leq i,j \leq m(K)}) \) with \([ij] = \binom{i}{j} \) in row \( i \) and column \( j \) transforms, by Theorem 1.2, the chromatic table into the \((\dim(K) \times m(K))\)-matrix

\[
\chi(K) | [ij]_{1 \leq i,j \leq m(K)} = (\chi^s(K, i))_{1 \leq s \leq \dim(K)}
\]

with the \( m(K) \) values \( \chi^s(K, i) \), \( 1 \leq i \leq m(K) \), of the \( s \)-chromatic polynomial in row \( s \). This matrix of chromatic polynomial values appears also to have log-concave rows.

3. Chromatic uniqueness

In this section we briefly discuss to what extent simplicial complexes are determined by their chromatic polynomials. Proposition shows that the chromatic table of a simplicial complex determines its \( f \)-vector.

**Definition 3.1.** \( K \) is chromatically unique if it is determined up to isomorphism by its chromatic table.

In Lemma 3.2 below, \( K \sqcup L \) is the disjoint union and \( K \vee L \) the one-point union of \( K \) and \( L \). The proof is identical to the one for the similar statements about chromatic polynomials for simple graphs.

**Lemma 3.2.** If \( K \) and \( L \) are finite simplicial complexes then

\[
\chi^s(K \sqcup L, r) = \chi^s(K, r)\chi^s(L, r), \quad \chi^s(K \vee L, r) = \frac{\chi^s(K, r)\chi^s(L, r)}{r}
\]

for all \( r \) and all \( s \geq 0 \).

The two nonisomorphic simplicial complexes
are not chromatically unique as they have identical chromatic tables
\[
\begin{pmatrix}
0 & 0 & 2 & 10 & 7 & 1 \\
0 & 15 & 73 & 62 & 15 & 1
\end{pmatrix}
\]
by Lemma 3.2. (These two complexes are, however, PL-isomorphic.) On the other hand, Proposition 2.34.(2) immediately implies that the s-skeleton of a full simplex is chromatically unique (in a very strong sense).

**Proposition 3.3.** If $K$ has the same s-chromatic polynomial as a full simplex $D[N]$, then $K$ and $D[N]$ have isomorphic s-skeleta.

**Proof.** If $K$ and $D[N]$ have the same s-chromatic polynomial for some $s \geq 1$, then $K$ has $N$ vertices (Corollary 2.26), and, since $\chi^s(K, N-s) = \chi^s(D[N], N-s)$, the s-skeleton of $K$ is isomorphic to the s-skeleton of the full simplex on $N$ vertices (Proposition 2.34.(2)). □

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