Nonstandard Vector Space with a Metric and Its Topological Structure

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Abstract

In this paper, we introduce the nonstandard vector space in which the concept of additive inverse element will not be taken into account. We also consider a metric defined on this nonstandard vector space. Under these settings, the conventional intuitive properties for the open and closed balls will not hold true. Therefore, four kinds of open and closed sets are proposed. Furthermore, the topologies generated by these different concepts of open and closed sets are investigated.

Keywords: Nonstandard vector space; Nonstandardly open set; Nonstandardly closed set; Metric space; Null set.

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1 Introduction

It is well-known that the topic of topological vector space is based on the vector space by referring to the monographs [1, 2, 3, 4, 5]. We can see that the set of all closed intervals cannot form a real vector space. The main reason is that there will be no additive inverse element for each closed interval in $\mathbb{R}$. Under this inspiration, we are going to propose the concept of nonstandard vector space which will not own the concept of additive inverse element and will be weaker than the (conventional) vector space in the sense of axioms.

Let $X$ be an universal set. We can define a (conventional) metric $d$ on $X$, which satisfies the following conditions:

- for any $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$;
- for any $x, y \in X$, $d(x, y) = d(y, x)$;
- for any $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

If $X$ is taken as a vector space, then we can define a norm $\| \cdot \|$ on $X$, which satisfies the following conditions:

- $\| x \|$ if and only if $x = \theta$;
- $\| \alpha x \| = |\alpha| \cdot \| x \|$ for any $x \in X$ and $\alpha \in \mathbb{F}$;
- for any $x, y \in X$, $\| x + y \| \leq \| x \| + \| y \|$.

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It is well-known that the normed space \((X, \| \cdot \|)\) is also a metric space \((X, d)\) with the metric defined by 
\[
d(x, y) = \| x - y \| \quad \text{such that} \quad d \text{ is translation-invariant and homogeneous, i.e., } d(x + a, y + a) = d(x, y) \quad \text{and} \quad d(\alpha x, \alpha y) = |\alpha| \cdot d(x, y).
\]

Conversely, we assume that the metric space \((X, d)\) is also a vector space. If \(d\) is not translation-invariant and homogeneous, then we cannot induced a normed space from \((X, d)\) by defining a suitable norm based on the metric \(d\). Of course, if \(d\) is translation-invariant and homogeneous and we define a nonnegative function \(p(x) = d(x, \theta)\), then we can show that \(p\) is indeed a norm on \(X\). Therefore, when \(X\) is taken as a vector space and the metric \(d\) is not translation-invariant and homogeneous, the metric space \((X, d)\) is weaker than the normed space \((X, \| \cdot \|)\) in the sense that the properties which hold true in \((X, \| \cdot \|)\) are also hold true in \((X, d)\) with \(d(x, y) = \| x - y \|\), and the converse is not true. When \(X\) is taken as the nonstandard vector space over \(F\), we are going to study the so-called nonstandard metric space in which \(X\) is endowed with a metric \(d\). Therefore, in the nonstandard metric space, we can perform the vector addition and scalar multiplication. In the conventional metric space, the vector addition and scalar multiplication are not allowed, unless the universal set \(X\) is taken as the vector space. We have to mention that the mathematical structures of nonstandard metric space that is based on the nonstandard vector space are completely different from that of the (conventional) metric space that is based on the (conventional) vector space. This is the main purpose of this paper.

In Sections 2 and 3, the concept of nonstandard vector space is proposed, where some interesting properties are derived in order to study the the topology generated by this kind of space. In Section 4, we introduce the concept of metric defined on the nonstandard vector space defined in Section 2. In Section 5, we provide the non-intuitive properties for the open and closed balls. In Sections 6 and 7, we propose many different concepts of nonstandard open and closed sets. Finally, in Section 8, we investigate the topologies generated by these different concepts of open and closed sets.

## 2 Nonstandard Vector Spaces

Let \(X\) be a universal set and let \(F\) be a scalar field. We assume that \(X\) is endowed with the vector addition \(x \oplus y\) and scalar multiplication \(\alpha x\) for any \(x, y \in X\) and \(\alpha \in F\). If \(X\) is also closed under the vector addition and scalar multiplication, then we call \(X\) a universal set over \(F\). In the conventional vector space over \(F\), the additive inverse element of \(x\) is denoted by \(-x\), and it can also be shown that \(-x = -1x\). Here, we shall not consider the concept of inverse element. However, for convenience, we still adopt \(-x = -1x\).

For \(x, y \in X\), the substraction \(x \ominus y\) is defined by \(x \ominus y = x \oplus (-y)\), where \(-y\) means the scalar multiplication \((-1)y\). For any \(x \in X\) and \(\alpha \in F\), we have to mention that \((-\alpha)x \neq -\alpha x\) and \(\alpha(-x) \neq -\alpha x\) in general, unless \(\alpha(\beta x) = (\alpha \beta)x\) for any \(\alpha, \beta \in F\). Here, this law will not always be assumed to be true.

**Example 2.1.** Let \(I\) be the set of all closed intervals in \(\mathbb{R}\). The vector addition is given by
\[
[a, b] \oplus [c, d] = [a + c, b + d]
\]
and the scalar multiplication is given by
\[
k[a, b] = \begin{cases} 
ka, kb & \text{if } k \geq 0 \\
kb, ka & \text{if } k < 0.
\end{cases}
\]
It is easy to see that \(I\) is not a (conventional) vector space under the above vector addition and scalar multiplication. The main reason is that the inverse element does not exist for any non-degenerated closed interval.

Since the universal set \(X\) over \(F\) can just own the vector addition and scalar multiplication, in general, the universal set \(X\) cannot own the zero element. The set 
\[
\Omega = \{x \ominus x : x \in X\}
\]
is called the *null set* of $X$. Therefore, the null set can be regarded as a kind of zero element of $X$.

**Example 2.2.** Continued from Example 2.1 for any $[a, b] \in \mathcal{I}$, we have

$$[a, b] \oplus [a, b] = [a, b] \oplus (\lnot [a, b]) = [a, b] \oplus (\lnot b, -a] = [a - b, b - a] = \lnot (b - a), b - a].$$

Therefore, we have the null set $\Omega = \{\lnot -k, k \geq 0\}$.

Now, we are in a position to define the concept of nonstandard vector space.

**Definition 2.1.** Let $X$ be a universal set over $\mathbb{F}$. We say that $X$ is a *nonstandard vector space* over $\mathbb{F}$ if and only if the following conditions are satisfied:

1. $1x = x$ for any $x \in X$;
2. $x = y$ implies $x \oplus z = y \oplus z$ and $\alpha x = \alpha y$ for any $x, y, z \in X$ and $\alpha \in \mathbb{F}$.
3. the commutative and associative laws for vector addition hold true in $X$; that is, $x \oplus y = y \oplus x$ and $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for any $x, y, z \in X$.

Let $X$ be a universal set over $\mathbb{F}$. More laws about the vector addition and scalar multiplication can be defined below.

- We say that the *distributive law for vector addition* holds true in $X$ if $\alpha(x \oplus y) = \alpha x \oplus \alpha y$ for any $x, y \in X$ and $\alpha \in \mathbb{F}$.
- We say that the *positively distributive law for vector addition* holds true in $X$ if $\alpha(x \oplus y) = \alpha x \oplus \alpha y$ for any $x, y \in X$ and $\alpha > 0$.
- We say that the *associative law for scalar multiplication* holds true in $X$ if $\alpha(\beta x) = (\alpha \beta)x$ for any $x \in X$ and $\alpha, \beta \in \mathbb{F}$.
- We say that the *associative law for positive scalar multiplication* holds true in $X$ if $\alpha(\beta x) = (\alpha \beta)x$ for any $x \in X$ and $\alpha, \beta > 0$.
- We say that the *distributive law for scalar addition* holds true in $X$ if $(\alpha + \beta)x = \alpha x \oplus \beta x$ for any $x \in X$ and $\alpha, \beta \in \mathbb{F}$.
- We say that the *distributive law for positive scalar addition* holds true in $X$ if $(\alpha + \beta)x = \alpha x \oplus \beta x$ for any $x \in X$ and $\alpha, \beta > 0$.
- We say that the *distributive law for negative scalar addition* holds true in $X$ if $(\alpha + \beta)x = \alpha x \oplus \beta x$ for any $x \in X$ and $\alpha, \beta < 0$.

We remark that if the distributive law for positive and negative scalar addition hold true in $X$, then $(\alpha + \beta)x = \alpha x \oplus \beta x$ for any $x \in X$ and $\alpha \beta > 0$.

**Example 2.3.** It is not hard to see that the distributive law for scalar addition does not hold true in $\mathcal{I}$ (the set of all closed intervals). However, the distributive law for positive and negative scalar addition hold true in $\mathcal{I}$.

Let $X$ be a nonstandard vector space over $\mathbb{F}$ with the null set $\Omega$. We say that $\theta$ is the *zero element* of $X$ if and only if $x = \theta \oplus x = x \oplus \theta$ for each $x \in X$. We write $x \oplus \Omega$ if one of the following conditions is satisfied:

1. $x = y$;
2. there exists $\omega \in \Omega$ such that $x = y \oplus \omega$ or $x \oplus \omega = y$;
(c) there exist \( \omega_1, \omega_2 \in \Omega \) such that \( x \oplus \omega_1 = y \oplus \omega_2 \).

**Remark 2.1.** Suppose that the nonstandard vector space \( X \) also contains the zero element \( \theta \). Then we can simply say that \( x \oplus y \) if and only if there exist \( \omega_1, \omega_2 \in \Omega \) such that \( x \oplus \omega_1 = y \oplus \omega_2 \) (i.e., only condition (c) is satisfied), since conditions (a) and (b) can be rewritten as condition (c) by adding \( \theta \). We also remark that if we want to discuss some properties based on \( x \oplus y \), it suffices to just consider the case of condition (c) \( x \oplus \omega_1 = y \oplus \omega_2 \), even though \( X \) does not contain the zero element \( \theta \). The reason is that the same arguments are still applicable for the cases of condition (a) or (b) when condition (c) has been shown to be valid.

For any \( x, y, z \in X \), we see that \( x \oplus z = y \oplus z \) does not imply \( x = y \), i.e., the cancellation law does not hold true. However, we have the following interesting results.

**Proposition 2.1.** Let \( X \) be a nonstandard vector space over \( \mathbb{F} \) with the null set \( \Theta \). The following statements hold true.

(i) For any \( x, y, z \in X \), if \( x \oplus z = y \oplus z \), then \( x \equiv y \).

(ii) If \( x \oplus y \in \Omega \), then \( x \equiv y \).

(iii) Suppose that \( \Omega \) is closed under the vector addition. If \( x \equiv y \), then there exists \( \omega \in \Omega \) such that \( x \oplus y \oplus \omega \in \Omega \).

**Proof.** Since \( x \oplus z = y \oplus z \), by adding \( -z \) on both sides, we obtain \( x \oplus \omega = y \oplus \omega \), where \( \omega = z \oplus \omega \in \Omega \). This proves (i). For result (ii), we have \( x \oplus (-y) = \omega_1 \) for some \( \omega_1 \in \Omega \). Therefore \( x \oplus (-y) \oplus y = \omega_1 \oplus y \) by adding \( y \) on both sides. This shows that \( x \oplus \omega_2 = \omega_1 \oplus y \) for \( \omega_1, \omega_2 \in \Omega \), which proves (ii). Now suppose that \( x \equiv y \). Then \( x \oplus \omega_1 = \omega_1 \oplus y \) for some \( \omega_1, \omega_2 \in \Omega \). By adding \( -\omega_1 \) on both sides, we obtain \( x \oplus \omega_3 = \omega_3 \oplus \omega_3 \in \Omega \), where \( \omega_3 = y \oplus y \in \Omega \). We complete the proof.

If \( X \) is a nonstandard vector space over \( \mathbb{F} \), then \( -(x \oplus y) = (-x) \oplus (-y) \) does not hold true in general. However, we have the following interesting results.

**Proposition 2.2.** Let \( X \) be a nonstandard vector space over \( \mathbb{F} \) with the null set \( \Omega \). Suppose that \( \Omega \) is closed under the vector addition. We have \( -(x_1 \oplus \cdots \oplus x_n) \equiv (\omega_1 \oplus \cdots \oplus \omega_n) \) and \( -(x \oplus y) \equiv (-x) \oplus y \).

**Proof.** Since \( \Omega \) is closed under the vector addition, we have \( x_1 \oplus \cdots \oplus x_n \oplus (-x_1) \oplus \cdots \oplus (-x_n) = \omega_1 \) for some \( \omega_1 \in \Omega \). By adding \( -(x_1 \oplus \cdots \oplus x_n) \) on both sides, we obtain 
\[
-(x_1 \oplus \cdots \oplus x_n) \oplus (x_1 \oplus \cdots \oplus x_n) \oplus (-x_1) \oplus \cdots \oplus (-x_n) = -(x_1 \oplus \cdots \oplus x_n) \oplus \omega_1,
\]
which says that \( \omega_2 \oplus (-x_1) \oplus \cdots \oplus (-x_n) = -(x_1 \oplus \cdots \oplus x_n) \oplus \omega_1 \), where \( \omega_2 = -(x_1 \oplus \cdots \oplus x_n) \oplus (x_1 \oplus \cdots \oplus x_n) \in \Omega \). Therefore, we obtain \( -(x_1) \oplus \cdots \oplus (-x_n) \equiv -(x_1 \oplus \cdots \oplus x_n) \). Similarly, we have \( x \oplus y \oplus (-x) \oplus y = \omega_3 \) for some \( \omega_3 \in \Omega \). Therefore, we have \( \omega_4 \oplus (-x) \oplus y = -(x \oplus y) \oplus \omega_3 \), which also means \( -(x \oplus y) \equiv (-x) \oplus y \). We complete the proof.

Now, we are going to introduce the concepts of generalized inverse elements. Also, the uniqueness of inverse element is, in some sense, different from that of conventional vector space.

**Definition 2.2.** Let \( X \) be a nonstandard vector space over \( \mathbb{F} \) with the null set \( \Theta \). For any \( x \in X \), we say that \( y \) is the generalized inverse of \( x \) if and only if \( x \oplus y \in \Theta \).

**Proposition 2.3.** Let \( X \) be a nonstandard vector space over \( \mathbb{F} \) with the null set \( \Theta \). For any \( x \in X \), if \( y \) and \( z \) are the generalized inverse of \( x \), then \( y \equiv z \).
Proof. Suppose that $y$ and $z$ are the generalized inverse. Then we have $x + y = x$ and $x + z = x$ for some $\omega_1, \omega_2 \in \Omega$. By adding $z$ on both sides of the first equality, we have $x + z + y = z + \omega_1$, i.e., $y + \omega_2 = z + \omega_1$. This shows that $y = z$. We complete the proof. 

Corollary 2.1. Let $X$ be a nonstandard vector space over $\mathbb{F}$ with the null set $\Omega$. For any $x \in X$, if $y$ is the generalized inverse of $x$, then $y + x = x = -1x$.

Proof. By definition, we have $x + x = x + (-x) \in \Theta$. This shows that $-x = -1x$ is the generalized inverse of $x$. The results follow from Proposition 2.3 immediately.

Let $X$ be a nonstandard vector space over $\mathbb{F}$, and let $Y$ be a subset of $X$. We say that $X$ is a subspace of $Y$ if and only if $Y$ is closed under the vector addition and scalar multiplication, i.e., $x + y \in X$ for any $x, y \in Y$ and $\alpha x \in Y$ for any $\alpha \in \mathbb{F}$.

Remark 2.2. Let $Y$ be a subspace of $X$. In the case of (conventional) vector space, if $y \in Y$, then $Y = y + Y$. However, in the case of nonstandard vector space, we just have $Y \subseteq Y$ and $Y + \omega \subseteq Y + y$, where $\omega = y + y \in \Omega$, since, for any $\omega \in Y$, we have $\omega + \omega = (\omega + y) + y \in Y + y$.

For any $x \in X$, since the distributive law $(\alpha + \beta)x = \alpha x + \beta x$ does not hold true in general as shown in Example 2.3, it says that $(\alpha x) + \cdots + \alpha x \neq (\alpha_1 + \cdots + \alpha_n)x$ in general. Therefore, we need to carefully interpret the concept of linear combination. Let $X$ be a nonstandard vector space over $\mathbb{F}$ and let $\{x_1, \ldots, x_n\}$ be a finite subset of $X$. A linear combination of $\{x_1, \ldots, x_n\}$ is an expression of the form

$$y_1 + \cdots + y_m,$$

where $y_i = \alpha_i x_k$ for some $k \in \{1, \ldots, n\}$ and the coefficients $\alpha_i \in \mathbb{F}$ for $i = 1, \ldots, m$. We allow $y_i = \alpha_i x_k$ and $y_j = \alpha_j x_k$ for the same $x_k$. In this case, we see that

$$y_i + y_j = \alpha_i x_k + \alpha_j x_k \neq (\alpha_i + \alpha_j)x_k$$

in general. For any nonempty subset $S$ of $X$, the set of all linear combinations of finite subsets of $S$ is called the span of $S$, which is also denoted by span(S). Then, it follows that $S \subseteq \text{span}(S)$.

Remark 2.3. Let $X$ be a nonstandard vector space over $\mathbb{R}$. For $x \in X$, we see that each element in span($\{x\}$) has the form of $\alpha_1 x + \cdots + \alpha_n x$ for some finite sequence $\{\alpha_1, \ldots, \alpha_n\}$ in $\mathbb{R}$. Now suppose that the distributive law for positive and negative scalar addition hold true in $X$, i.e., $(\alpha + \beta)x = \alpha x + \beta x$ for any $x \in X$ and $\alpha \beta > 0$. For example, the distributive law for positive and negative scalar addition hold true in the set $\mathbb{I}$ of all closed intervals. Let $J_0 = \{j : \alpha_j = 0\}$, $J_+ = \{j : \alpha_j > 0\}$ and $J_- = \{j : \alpha_j < 0\}$. We also write $\alpha^+ = \sum_{j \in J_+} \alpha_j$ and $\alpha^- = \sum_{j \in J_-} \alpha_j$. Then, for $J_0 \neq \emptyset$, we see that

$$\alpha_1 x + \cdots + \alpha_n x = \begin{cases} \alpha^+ x + \alpha^- x + 0 x + \cdots + 0 x & \text{if } J_+ \neq \emptyset \text{ and } J_- \neq \emptyset \\ \alpha^+ x + 0 x + \cdots + 0 x & \text{if } J_+ \neq \emptyset \text{ and } J_- = \emptyset \\ \alpha^- x + 0 x + \cdots + 0 x & \text{if } J_+ = \emptyset \text{ and } J_- \neq \emptyset \\ 0 x + \cdots + 0 x & \text{if } J_+ = \emptyset \text{ and } J_- = \emptyset \end{cases}$$

For $J_0 = \emptyset$, we can have the similar expression without $0 x + \cdots + 0 x$. We need to remark that $0 x$ is not necessarily in the null set $\Omega$. However, we have the following relations. Suppose that the distributive law $(0 + 0)x = 0 x + 0 x$ holds true in general. Then, we have $0 x = (0 + 0)x = 0 x + 0 x$ by adding $-0 x$ on both sides, we have $\omega = 0 x + \omega$, where $\omega = 0 x - 0 x \in \Omega$.

Proposition 2.4. Let $X$ be a nonstandard vector space over $\mathbb{F}$ such that the distributive law for vector addition and the associative law for scalar multiplication hold true. Then span(S) is a subspace of $X$. 


3 Decomposition

In the (conventional) vector space $X$, any element $x \in X$ can be decomposed as $x = \hat{x} + (x - \hat{x})$ for some $\hat{x} \in X$. However, under the nonstandard vector space $X$, we cannot have the decomposition $x = \hat{x} \oplus (x \ominus \hat{x})$ in general, since, in fact, $\hat{x} \oplus (x \ominus \hat{x}) = x \oplus \omega \neq x$, where $\omega = \hat{x} \ominus \hat{x}$. Now, we propose a very basic notion of decomposition.

**Definition 3.1.** Let $X$ be a nonstandard vector space over $\mathbb{F}$. Given any $x \in X$, we say that $x$ has the *null decomposition* if $x = \hat{x} + \omega$ for some $\hat{x} \in X$ and $\omega \in \Omega$. The space $X$ is said to own the null decomposition if every element $x \in X$ has the null decomposition.

**Remark 3.1.** Let $X$ be a nonstandard vector space with the null set $\Omega$. Then it is easy to see that $X \oplus \Omega \subseteq X$. Now if $X$ owns the null decomposition, then we see that $X = X \oplus \Omega$.

The following example shows that the null decomposition is automatically satisfied in the nonstandard vector space of all closed intervals.

**Example 3.1.** Let $\mathcal{I}$ be the nonstandard vector space of all closed intervals with the null set $\Omega = \{-k, k : k \geq 0\}$. For $x = [a, b] \in \mathcal{I}$, we can take $k = (b - a)/2$ and $\hat{x} = [a + k, b - k]$. In this case, $x = [a, b] = [a + k, b - k] \ominus [-k, k] = \hat{x} \ominus \omega$. Therefore, $\mathcal{I}$ owns the null decomposition.

**Definition 3.2.** Let $X$ be a nonstandard vector space over $\mathbb{F}$.

- Given a fixed $\omega_0 \in \Omega$, we say that $\Omega$ owns the *self-decomposition with respect to $\omega_0$* if every $\omega \in \Omega$ can be represented as $\omega = \omega' \oplus \omega_0$ for some $\omega' \in \Omega$.

- We say that $\Omega$ owns the *self-decomposition* if $\Omega$ owns the self-decomposition with respect to every $\omega_0 \in \Omega$.

**Example 3.2.** Let $\mathcal{I}$ be the nonstandard vector space of all closed intervals with the null set $\Omega = \{-k, k : k \geq 0\}$. Given $\omega = [-k, k]$ for $k \neq 0$, i.e., $k > 0$, we can write $k = k_1 + k_2$ with $k_1, k_2 > 0$. Then, we have

$$\omega = [-k, k] = - (k_1 + k_2), k_1 + k_2] = [-k_1, k_1] \oplus [-k_2, k_2],$$

where $\omega_1 = [-k_1, k_1], \omega_2 = [-k_2, k_2] \in \Omega$. This says that $\Omega \oplus \Omega = \Omega$ under the space $\mathcal{I}$. In other words, the null set $\Omega$ owns the self-decomposition.

4 Nonstandard Metric Spaces

Now, we are in a position to introduce the concept of the so-called nonstandard metric space.

**Definition 4.1.** Let $X$ be a nonstandard vector space over $\mathbb{F}$ with the null set $\Omega$. For the nonnegative real-valued function $d$ defined on $X \times X$, we consider the following conditions:

(i) $d(x, y) = 0$ if and only if $x \overset{\Omega}{=} y$ for all $x, y \in X$;

(i') $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$;

Different kinds of metric spaces are defined below.

- A pair $(X, d)$ is called a *pseudo-metric space on a nonstandard vector space* $X$ if and only if $d$ satisfies conditions (ii) and (iii).
• A pair \((X, d)\) is called a \textit{metric space on a nonstandard vector space} \(X\) if and only if \(d\) satisfies conditions (i'), (ii) and (iii).

• A pair \((X, d)\) is called a \textit{nonstandard metric space} if and only if \(d\) satisfies conditions (i), (ii) and (iii).

Now we consider the following conditions:

(iv) for any \(\omega_1, \omega_2 \in \Omega\) and \(x, y, z \in X\), we have
\[
d(x \oplus \omega_1, y \oplus \omega_2) \geq d(x, y), \quad d(x \oplus \omega_1, y) \geq d(x, y)\text{ and } d(x, y \oplus \omega_2) \geq d(x, y);
\]

(iv') for any \(\omega_1, \omega_2 \in \Omega\) and \(x, y, z \in X\), we have
\[
d(x \oplus \omega_1, y \oplus \omega_2) = d(x, y), \quad d(x \oplus \omega_1, y) = d(x, y)\text{ and } d(x, y \oplus \omega_2) = d(x, y).
\]

We say that \(d\) satisfies the \textit{null inequalities} if and only if condition (iv) is satisfied, and that \(d\) satisfies the \textit{null equalities} if and only if condition (iv') is satisfied. We also consider the following conditions:

(v) for any \(x, y, a, b \in X\) and any finite sequences \(\{\alpha_1, \cdots, \alpha_n\}\) and \(\{\beta_1, \cdots, \beta_m\}\) in \(\mathbb{F}\) with \(\sum_{i=1}^n \alpha_i = 0\) and \(\sum_{j=1}^m \beta_j = 0\), we have
\[
d(x \oplus \alpha_1 a \oplus \cdots \oplus \alpha_n a, y \oplus \beta_1 b \oplus \cdots \oplus \beta_m b) \geq d(x, y),
\]
\[
d(x \oplus \alpha_1 a \oplus \cdots \oplus \alpha_n a, y) \geq d(x, y)\text{ and } d(x, y \oplus \beta_1 b \oplus \cdots \oplus \beta_m b) \geq d(x, y);
\]

(v') for any \(x, y, a, b \in X\) and any finite sequences \(\{\alpha_1, \cdots, \alpha_n\}\) and \(\{\beta_1, \cdots, \beta_m\}\) in \(\mathbb{F}\) with \(\sum_{i=1}^n \alpha_i = 0\) and \(\sum_{j=1}^m \beta_j = 0\), we have
\[
d(x \oplus \alpha_1 a \oplus \cdots \oplus \alpha_n a, y \oplus \beta_1 b \oplus \cdots \oplus \beta_m b) = d(x, y),
\]
\[
d(x \oplus \alpha_1 a \oplus \cdots \oplus \alpha_n a, y) = d(x, y)\text{ and } d(x, y \oplus \beta_1 b \oplus \cdots \oplus \beta_m b) = d(x, y).
\]

We say that the metric (or pseudo-metric) \(d\) satisfies the \textit{zero-sum inequalities} if and only if condition (v) is satisfied, and that \(d\) satisfies the \textit{zero-sum equalities} if and only if condition (v') is satisfied.

Remark 4.1. For any \(\omega_1, \omega_2 \in \Omega\), since \(\omega_1 = a \oplus a\) and \(\omega_2 = b \oplus b\) for some \(a, b \in X\), it is not hard to see that condition (v) implies condition (iv), and condition (v') implies condition (iv').

Example 4.1. We consider the nonstandard vector space \(\mathcal{I}\) of all closed intervals in \(\mathbb{R}\). Let us define a nonnegative real-valued function \(d : \mathcal{I} \times \mathcal{I} \to \mathbb{R}_+\) by
\[
d([a, b], [c, d]) = |(a + b) - (c + d)|. \tag{1}
\]
Then we are going to claim that \((\mathcal{I}, d)\) is a nonstandard metric space such that the metric \(d\) satisfies the zero-sum equalities. Remark 4.1 says that the metric also satisfies the null equalities.

• We consider the closed intervals \([a, b]\) and \([c, d]\). Then we see that \(a - d \leq b - c\). Therefore, if \(b - c < 0\), then \(d([a, b], [c, d]) = |a + b - c - d| \neq 0\). Suppose that \(b - c \geq 0\) and \(d([a, b], [c, d]) = |a + b - c - d| = 0\). Then we have \(a + b = c + d\), i.e., \(a + c - d = 2c - b\). It is easy to see that \(a + c - d \leq b + d - c\) and \(2c - b \leq b + d - c\) by using the facts that \(a \leq b\), \(c \leq d\) and \(b \geq c\) (in fact, this can be understood from (2) below). Therefore we can form the two identical closed intervals \([a + c - d, b + d - c]\) and \([2c - b, b + d - c]\). Now the closed intervals \([a + c - d, b + d - c]\) and \([2c - b, b + d - c]\) can be written as
\[
[a + c - d, b + d - c] = [a, b] \oplus [c - d, d - c]\text{ and } [2c - b, b + d - c] = [c, d] \oplus [c - b, b - c]. \tag{2}
\]
Let $\omega_1 = [c - d, d - c] = (d - c) \cdot [-1, 1] \in \Omega$ and $\omega_2 = [c - b, b - c] = (b - c) \cdot [-1, 1] \in \Omega$. Therefore, from (2), we have $[a, b] \oplus \omega_1 = [c, d] \oplus \omega_2$, which shows $[a, b] \oplus \omega_1 = [c, d]$, since $\omega_1, \omega_2 \in \Omega$. Conversely, suppose that $[a, b] \oplus \omega_1 = [c, d]$. Then $[a, b] \oplus \omega_1 = [c, d] \oplus \omega_2$, where $\omega_1 = [-k_1, k_1], \omega_2 = [-k_2, k_2] \in \Omega$. Therefore, we have $[a - k_1, b + k_1] = [c - k_2, d + k_1]$, i.e., $a - k_1 = c - k_2$ and $b + k_1 = d + k_2$. Then we obtain

$$d([a, b], [c, d]) = |(a - c) + (b - d)| = |(k_1 - k_2) + (k_2 - k_1)| = 0.$$

- We have
  $$d([a, b], [c, d]) = |a + b - c - d| = |c + d - a - b| = d([c, d], [a, b]).$$

- We have
  $$d([a, b], [c, d]) = |a + b - c - d| = |(a + b - e - f) + (e + f - c - d)|
  \leq |a + b - e - f| + |e + f - c - d|
  = d([a, b], [e, f]) + d([e, f], [c, d]).$$

- For any finite sequence $\{\alpha_1, \ldots, \alpha_n\}$ in $\mathbb{R}$ with $\sum_{i=1}^n \alpha_i = 0$, we have
  $$\alpha_1 [c_1, d_1] \oplus \cdots \oplus \alpha_n [c_1, d_1] = [e_1, f_1],$$
  where
  $$e_1 = \sum_{\alpha_i \geq 0} \alpha_i c_1 + \sum_{\alpha_i < 0} \alpha_i d_1 \quad \text{and} \quad f_1 = \sum_{\alpha_i \geq 0} \alpha_i d_1 + \sum_{\alpha_i < 0} \alpha_i c_1.$$

Therefore

$$e_1 + f_1 = \sum_{\alpha_i \geq 0} \alpha_i (c_1 + d_1) + \sum_{\alpha_i < 0} \alpha_i (c_1 + d_1) = (c_1 + d_1) \sum_{i=1}^n \alpha_i = 0.$$

Similarly, for any finite sequence $\{\beta_1, \ldots, \beta_m\}$ in $\mathbb{R}$ with $\sum_{j=1}^m \beta_j = 0$, we have $\beta_1 [c_2, d_2] \oplus \cdots \oplus \beta_m [c_2, d_2] = [e_2, f_2]$ implies $e_2 + f_2 = 0$. Now, we have

$$d([a_1, b_1] \oplus \alpha_1 [c_1, d_1] \oplus \cdots \oplus \alpha_n [c_1, d_1], [a_2, b_2] \oplus \beta_1 [c_2, d_2] \oplus \cdots \oplus \beta_m [c_2, d_2])
= d([a_1, b_1] \oplus [e_1, f_1], [a_2, b_2] \oplus [e_2, f_2])
= d([a_1 + e_1, b_1 + f_1], [a_2 + e_2, b_2 + f_2])
= d([a_1 + e_1 + b_1 + f_1] - (a_2 + e_2 + b_2 + f_2)]
= |(a_1 + b_1) - (a_2 + b_2)|
= d([a_1, b_1], [a_2, b_2]).$$

This shows that $(\mathcal{I}, d)$ is also a nonstandard metric space such that the metric $d$ satisfies the zero-sum and null equalities. We also remark that $(\mathcal{I}, d)$ cannot be a metric space, since condition (i') in Definition 4.1 cannot hold true.

**Example 4.2.** Example 4.1 shows that $(\mathcal{I}, d)$ is a nonstandard metric space such that the metric $d$ satisfies the zero-sum equalities define in (1). Let $(X, d)$ be another nonstandard metric space such that the metric $d$ satisfies the zero-sum equalities. We consider the interval-valued function $F$ defined on $X$. In other words, for each $x \in X$, $F(x) = [f^L(x), f^U(x)]$ is a closed interval in $\mathbb{R}$. Then we say that $F$ is continuous at $x_0$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, x_0) < \delta$ implies $d_2(F(x), F(x_0)) < \epsilon$. We see that

$$d_2(F(x), F(x_0)) = |f^L(x) + f^U(x) - f^L(x_0) - f^U(x_0)|.$$
We also say that $F$ is continuous on $X$ if $F$ is continuous at each $x_0 \in X$. We also denote by $\mathcal{IC}(X)$ the set of all continuous functions $F : (X, d) \to (I, d_I)$ on $X$. Now the vector addition and scalar multiplication in $\mathcal{IC}(X)$ are defined by

$$(F \oplus G)(x) = F(x) \oplus G(x) \text{ and } (\alpha F)(x) = \alpha F(x)$$

for any $F, G \in \mathcal{IC}(X)$ and $\alpha \in \mathbb{R}$. Then we are going to show that $F \oplus G$ and $\alpha F$ are also in $\mathcal{IC}(X)$. Since

$$d_I((F \oplus G)(x), (F \oplus G)(x_0)) = |f^U(x) + g^U(x) + g^L(x) - f^U(x_0) - g^L(x_0) - g^U(x_0)|$$

and

$$d_I(\alpha F(x), \alpha F(x_0)) = |\alpha| \cdot |f^L(x) + f^U(x) - f^L(x_0) - f^U(x_0)|,$$

we see that $F \oplus G$ and $\alpha F$ are continuous on $X$, since $F$ and $G$ are continuous on $X$. The commutative and associative laws for vector addition hold true obviously. Therefore, we conclude that $\mathcal{IC}(X)$ is a nonstandard vector space. It is not hard to see that the null set us

$$\Omega_{\mathcal{IC}} = \{-k(x), k(x) : k(x) \geq 0 \text{ for all } x \in X\}.$$

Now we want to introduce a metric $d_{\mathcal{IC}}$ to make $(\mathcal{IC}(X), d_{\mathcal{IC}})$ as a nonstandard metric space such that the metric $d_{\mathcal{IC}}$ satisfies the zero-sum equalities. For $F, G \in \mathcal{IC}(X)$, we define

$$d_{\mathcal{IC}}(F, G) = \sup_{x \in X} d_I(F(x), G(x)).$$

We need to check four conditions.

- We have that

$$0 = d_{\mathcal{IC}}(F, G) = \sup_{x \in X} d_I(F(x), G(x))$$

implies $d_I(F(x), G(x)) = 0$ for all $x \in X$, i.e., $F(x) \supseteq G(x)$ for all $x \in X$, which also says that $F \supseteq G$ in the sense of $\Omega_{\mathcal{IC}}$.

- Since $d_I$ is symmetric, it is easy to see that $d_{\mathcal{IC}}$ is symmetric.

- We have

$$d_{\mathcal{IC}}(F, G) = \sup_{x \in X} d_I(F(x), G(x)) \leq \sup_{x \in X} [d_I(F(x), E(x)) + d_I(E(x), G(x))]$$

$$\leq \sup_{x \in X} d_I(F(x), E(x)) + \sup_{x \in X} d_I(E(x), G(x)) = d_{\mathcal{IC}}(F, E) + d_{\mathcal{IC}}(E, G).$$

- For any finite sequences $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_m\}$ in $\mathbb{R}$ with $\sum_{i=1}^{\alpha_i} = 0$ and $\sum_{j=1}^{\beta_j} = 0$ and $H_1, H_2 \in \mathcal{IC}(X)$, we see that $\alpha_1 H_1(x) \oplus \cdots \oplus \alpha_n H_1(x)$ and $\beta_1 H_2(x) \oplus \cdots \oplus \beta_m H_2(x)$ are also in $\mathcal{IC}(X)$. Therefore, we have

$$d_{\mathcal{IC}}(F \oplus \alpha_1 H_1 \oplus \cdots \oplus \alpha_n H_1, G \oplus \beta_1 H_2 \oplus \cdots \oplus \beta_m H_2)$$

$$= \sup_{x \in X} d_I(F(x) \oplus \alpha_1 H_1(x) \oplus \cdots \oplus \alpha_n H_1(x), G(x) \oplus \beta_1 H_2(x) \oplus \cdots \oplus \beta_m H_2(x))$$

$$= \sup_{x \in X} d_I(F(x), G(x)) = d_{\mathcal{IC}}(F, G).$$

Therefore, we conclude that $(\mathcal{IC}(X), d_{\mathcal{IC}})$ is indeed a nonstandard metric space such that the metric $d_{\mathcal{IC}}$ satisfies the zero-sum equalities.

**Definition 4.2.** Let $(X, d)$ be a pseudo-metric space on a nonstandard vector space $X$. We say that the pseudo-metric $d$ is translation-invariant if and only if $d(x \oplus z, y \oplus z) = d(x, y)$. We say that $d$ is absolutely homogeneous if and only if $d(\alpha x, \alpha y) = |\alpha| d(x, y)$.  


5 Open and Closed Balls

Let \((X, d)\) be a pseudo-metric space on a nonstandard vector space \(X\). Given a point \(x_0 \in X\) and a positive number \(\epsilon > 0\), the open ball about \(x_0\) is defined by
\[
B(x_0; \epsilon) = \{ x \in X : d(x, x_0) < \epsilon \},
\]
the closed ball about \(x_0\) is defined by
\[
\bar{B}(x_0; \epsilon) = \{ x \in X : d(x, x_0) \leq \epsilon \}
\]
and the sphere about \(x_0\) is defined by
\[
S(x_0; \epsilon) = \{ x \in X : d(x, x_0) = \epsilon \}.
\]
In all three cases, \(x_0\) is called the center and \(\epsilon\) the radius.

**Proposition 5.1.** Let \(X\) be a nonstandard vector space over \(\mathbb{F}\), and let \((X, d)\) be a pseudo-metric space. The following statements hold true.

(i) If \(d\) satisfies the null inequalities, then \(x \oplus \omega \in B(x_0; \epsilon)\) implies \(x \in B(x_0; \epsilon)\) for any \(\omega \in \Omega\).

(ii) If \(d\) satisfies the null equalities, then \(x \oplus \omega \in B(x_0; \epsilon)\) if and only if \(x \in B(x_0; \epsilon)\) for any \(\omega \in \Omega\).

Moreover, we have the following inclusion \(B(x_0; \epsilon) \oplus \Omega \subseteq B(x_0; \epsilon)\).

**Proof.** To prove part (i), suppose that \(x \oplus \omega \in B(x_0; \epsilon)\). According to condition (iv) of Definition 4.1, we have
\[
d(x, x_0) \leq d(x \oplus \omega, x_0) < \epsilon,
\]
which shows that \(x \in B(x_0; \epsilon)\). To prove part (ii), for \(x \in B(x_0; \epsilon)\) and \(\omega \in \Omega\), according to condition (iv') of Definition 4.1 it follows that
\[
d(x_0, x \oplus \omega) = d(x_0, x) < \epsilon,
\]
which says that \(x \oplus \omega \in B(x_0; \epsilon)\). This shows the desired inclusion, and the proof is complete. 

**Proposition 5.2.** Let \((X, d)\) be a pseudo-metric space on a nonstandard vector space \(X\). The following statements hold true.

(i) If the pseudo-metric \(d\) satisfies the null inequalities, then \(B(x_0 \oplus \omega; \epsilon) \subseteq B(x_0; \epsilon)\) for any \(\omega \in \Omega\).

(ii) If the pseudo-metric \(d\) satisfies the null equalities, then \(B(x_0 \oplus \omega; \epsilon) = B(x_0; \epsilon)\) for any \(\omega \in \Omega\).

**Proof.** If the pseudo-metric \(d\) satisfies the null inequalities, then the inclusion \(B(x_0 \oplus \omega; \epsilon) \subseteq B(x_0; \epsilon)\) follows from the inequality \(\epsilon > d(x, x_0 \oplus \omega) \geq d(x, x_0)\) immediately. Suppose that the pseudo-metric \(d\) satisfies the null equalities. Then the inclusion \(B(x_0; \epsilon) \subseteq B(x_0 \oplus \omega; \epsilon)\) follows from the equality \(\epsilon > d(x, x_0) = d(x, x_0 \oplus \omega)\) immediately. This completes the proof.

**Proposition 5.3.** Let \(X\) be a nonstandard vector space over \(\mathbb{F}\), and let \((X, d)\) be a pseudo-metric space. Suppose that \(X\) owns the null decomposition. The following statements hold true.

(i) If \(d\) satisfies the null inequalities, then \(B(x_0; \epsilon) \subseteq B(x_0; \epsilon) \oplus \Omega\).

(ii) If \(d\) satisfies the null equalities, then \(B(x_0; \epsilon) \oplus \Omega = B(x_0; \epsilon)\).
Proof. For any \( x \in B(x_0; \epsilon) \), since \( x = \bar{x} \odot \omega \) for some \( \bar{x} \in X \) and \( \omega \in \Omega \) by the null decomposition, we have
\[
d(\bar{x}, x_0) \leq d(\bar{x} \odot \omega, x_0) = d(x, x_0) < \epsilon.
\]
This says that \( \bar{x} \in B(x_0; \epsilon) \), i.e., \( x = \bar{x} \odot \omega \in B(x_0; \epsilon) \odot \Omega \). This proves part (i). Part (ii) follows from part (ii) of Proposition 5.1 and part (i). This completes the proof. 

In the (conventional) pseudo-metric space \((X, d)\), where \( X \) is taken as a (conventional) vector space. If \( d \) is translation-invariant, then we have the following equality
\[
B(x; \epsilon) \odot \hat{x} = B(x \odot \hat{x}; \epsilon).
\] (3)

However, if \( X \) is taken as the nonstandard vector space, then the intuitive observation in (3) will not hold true. The following proposition presents the exact relationship, and will be used for studying the topology induced by the nonstandard metric space.

**Proposition 5.4.** Let \( X \) be a nonstandard vector space over \( \mathbb{F} \), and let \((X, d)\) be a pseudo-metric space. The following statements hold true.

(i) If \( d \) is translation-invariant, then
\[
B(x; \epsilon) \odot \hat{x} \subseteq B(x \odot \hat{x}; \epsilon) \quad \text{for any } \omega \in \Omega.
\] (4)

and
\[
B(x; \epsilon) \odot \omega \subseteq x \odot B(\omega; \epsilon), \quad \text{where } \omega_x = x \odot x \in \Omega.
\] (5)

(ii) If \( d \) is translation-invariant and satisfies the null inequalities, then
\[
B(x; \epsilon) \odot \omega \subseteq B(x; \epsilon) \quad \text{and } B(\omega; \epsilon) \odot \hat{x} \subseteq B(\hat{x}; \epsilon) \quad \text{for any } \omega \in \Omega.
\] (6)

and
\[
B(x \odot \hat{x}; \epsilon) \odot \omega_x \subseteq B(x; \epsilon) \odot \hat{x}, \quad \text{where } \omega_x = \hat{x} \odot \hat{x}.
\] (7)

(iii) Suppose that \( d \) is translation-invariant and satisfies the null inequalities. We also assume that \( X \) owns the null decomposition and \( \Omega \) owns the self-decomposition, then we have the following equality
\[
B(x \odot \hat{x}; \epsilon) = B(x; \epsilon) \odot \hat{x} \quad \text{for any } \omega \in \Omega.
\] (8)

(iv) Suppose that \( d \) is translation-invariant and satisfies the null equalities. We also assume that \( X \) owns the null decomposition and \( \Omega \) owns the self-decomposition, then we have the following equalities
\[
B(x; \epsilon) \odot \omega = B(x; \epsilon) \quad \text{and } B(\omega; \epsilon) \odot \hat{x} = B(\hat{x}; \epsilon)
\]
for any \( \omega \in \Omega \).

**Proof.** To prove part (i), for \( y \in B(x; \epsilon) \odot \hat{x} \), we have \( y = \hat{y} \odot \hat{x} \) with \( d(\hat{y}, x) < \epsilon \). Then we can obtain
\[
d(y, x \odot \hat{x}) = d(\hat{y} \odot \hat{x}, x \odot \hat{x}) = d(\hat{y}, x) < \epsilon,
\]
which says that \( y \in B(x \odot \hat{x}; \epsilon) \). Therefore, we obtain (4). To prove (5), for \( \hat{x} \in B(x; \epsilon) \) and \( \omega_x = x \odot x \), we have \( \hat{x} \odot \omega_x = x \odot (\hat{x} \odot x) \). Then we obtain
\[
d(\hat{x} \odot x, \omega_x) = d(\hat{x} \odot x, x \odot x) = d(\hat{x} \odot (-x), x \odot (-x)) = d(\hat{x}, x) < \epsilon,
\]
which says that \( \hat{x} \odot x \in B(\omega_x; \epsilon) \). This shows that \( \hat{x} \odot \omega_x = x \odot (\hat{x} \odot x) \in x \odot B(\omega_x; \epsilon) \).

To prove part (ii), we take \( \hat{x} = \omega \in \Omega \) in (4). By part (i) of Proposition 5.2, we have
\[
B(x; \epsilon) \odot \omega \subseteq B(x \odot \omega; \epsilon) \subseteq B(x; \epsilon)
\]
for any $\omega \in \Omega$. Similarly, if we take $x = \omega$, then we have

$$B(\omega; \epsilon) \oplus \hat{x} \subseteq B(\omega \oplus \hat{x}; \epsilon) = B(\hat{x}; \epsilon).$$

Therefore, we obtain (6). To prove (7), for $y \in B(x \oplus \hat{x}; \epsilon)$, we have $d(y, x \oplus \hat{x}) < \epsilon$. Since

$$\epsilon > d(y, x \oplus \hat{x}) = d(y \oplus \hat{x}, x \oplus \hat{x} \oplus \hat{x}) = d(y \oplus \hat{x}, x \oplus \omega_{\hat{x}}) \geq d(y \oplus \hat{x}, x),$$

we see that $y \oplus \hat{x} \in B(x; \epsilon)$, where $\omega_{\hat{x}} = \hat{x} \oplus \hat{x} \in \Omega$. Since $y \oplus \omega_{\hat{x}} = \hat{x} \oplus (y \oplus \hat{x})$, it says that $y \oplus \omega_{\hat{x}} \in B(x; \epsilon) \oplus \hat{x}$, and proves (7).

To prove part (iii), for $y \in B(x \oplus \hat{x}; \epsilon)$, we have $d(y, x \oplus \hat{x}) < \epsilon$. Since $X$ owns the null decomposition, we have $y = \hat{y} \oplus \hat{\omega}_0$ for some $\hat{y} \in X$ and $\hat{\omega}_0 \in \Omega$. Since $\Omega$ owns the self-decomposition, we also have $\hat{\omega}_0 = \omega_{\hat{x}} \oplus \hat{\omega}_1$ for some $\hat{\omega}_1 \in \Omega$. Then we have $y = \hat{y} \oplus \omega_{\hat{x}} \oplus \hat{\omega}_1$ and

$$d(\hat{y} \oplus \hat{\omega}_1, x \oplus \hat{x}) \leq d(\hat{y} \oplus \hat{\omega}_1 \oplus \omega_{\hat{x}}, x \oplus \hat{x}) = d(y, x \oplus \hat{x}) < \epsilon.$$

Since

$$\epsilon > d(\hat{y} \oplus \hat{\omega}_1, x \oplus \hat{x}) = d(\hat{y} \oplus \hat{\omega}_1 \oplus \hat{x}, x \oplus \hat{x} \oplus \hat{x}) = d(\hat{y} \oplus \hat{\omega}_1 \oplus \hat{x}, x \oplus \omega_{\hat{x}}) \geq d(\hat{y} \oplus \hat{\omega}_1 \oplus \hat{x}, x),$$

it follows that $\hat{y} \oplus \hat{\omega}_1 \oplus \hat{x} \in B(x; \epsilon)$. By adding $\hat{x}$ on both sides, we have

$$y = \hat{y} \oplus \hat{\omega}_1 \oplus \omega_{\hat{x}} \in B(x; \epsilon) \oplus \hat{x};$$

that is, we have the inclusion

$$B(x \oplus \hat{x}; \epsilon) \subseteq B(x; \epsilon) \oplus \hat{x}.$$  

From (1), we obtain

$$B(x \oplus \hat{x}; \epsilon) = B(x; \epsilon) \oplus \hat{x}.$$  

To prove part (iv), we take $\hat{x} = \omega \in \Omega$ in (5). By part (ii) of Proposition 5.2, we have

$$B(x; \epsilon) \oplus \omega = B(x \oplus \omega; \epsilon) = B(x; \epsilon)$$

for any $\omega \in \Omega$. Similarly, if we take $x = \omega$, then we have

$$B(\omega; \epsilon) \oplus \hat{x} = B(\omega \oplus \hat{x}; \epsilon) = B(\hat{x}; \epsilon).$$

This completes the proof. □

**Proposition 5.5.** Let $(X, d)$ be a pseudo-metric space on a nonstandard vector space $X$. Suppose that the pseudo-metric $d$ satisfies the null equalities and $\Omega$ owns the self-decomposition with respect to $\omega_0$. Then $B(a; \epsilon) \oplus \omega_0 = B(a; \epsilon) \oplus \Omega$ for any open ball $B(a; \epsilon)$. Of course, if $\Omega$ owns the self-decomposition. Then $B(a; \epsilon) \oplus \omega = B(a; \epsilon) \oplus \Omega$ for any $\omega \in \Omega$ and open ball $B(a; \epsilon)$.

**Proof.** We see that $B(a; \epsilon) \oplus \omega_0 \subseteq B(a; \epsilon) \oplus \Omega$ for any $\omega \in \Omega$. On the other hand, for any $y \in B(a; \epsilon) \oplus \Omega$, we have $y = \hat{y} \oplus \hat{\omega}$ for some $\hat{y} \in B(a; \epsilon)$ and $\hat{\omega} \in \Omega$. Since $\Omega$ owns the self-decomposition with respect to $\omega_0$, we have $\hat{\omega} = \omega' \oplus \omega_0$ for some $\omega' \in \Omega$. Therefore, we have $y = \hat{y} \oplus \omega = \hat{y} \oplus \omega' \oplus \omega_0$. Then we obtain $d(\hat{y} \oplus \omega', a) = d(\hat{y}, a) < \epsilon$, which says that $\hat{y} \oplus \omega' \in B(a; \epsilon)$. Therefore, we obtain that $y \in B(a; \epsilon) \oplus \omega_0$. This shows that $B(a; \epsilon) \oplus \Omega = B(a; \epsilon) \oplus \omega_0$. We complete the proof. □

**Proposition 5.6.** Let $(X, d)$ be a nonstandard metric space. Suppose that the pseudo-metric $d$ satisfies the null equalities and $\Omega$ owns the self-decomposition. Then, given any $w \in \Omega$, we have $d(\omega', \omega) = 0$ for any $\omega' \in \Omega$. In other words, we have $\Omega \subseteq B(\omega; \epsilon)$ for any $\omega \in \Omega$ and $\epsilon > 0$. 


Proof. Given any \( w \in \Omega \), since \( \Omega \) owns the self-decomposition, we have \( \omega' = \omega \oplus \hat{\omega} \) for some \( \hat{\omega} \in \Omega \). Since \( d \) satisfies the null equalities, we have \( d(\omega', \omega) = d(\omega \oplus \hat{\omega}, \omega) = d(\omega, \omega) = 0 \), which shows that \( \omega' \in B(\omega; \varepsilon) \). This completes the proof. \( \blacksquare \)

Proposition 5.7. Let \((X, d)\) be a pseudo-metric space on a nonstandard vector space \( X \). Suppose that the following conditions are satisfied.

- The associative law for scalar multiplication holds true.
- The pseudo-metric \( d \) is absolutely homogeneous and satisfies the null inequalities.
- The null set \( \Omega \) is closed under the scalar multiplication and owns the self-decomposition.

Then \( \alpha B(\omega; \varepsilon) = B(\omega; |\alpha| \varepsilon) \) for any \( \omega \in \Omega \) and \( \alpha \neq 0 \).

Proof. For \( x \in B(\omega; \varepsilon) \), since \( \alpha \omega \in \Omega \), we have

\[ d(\alpha x, \omega) = d(\alpha x, \omega + \alpha \omega) = d(\alpha x, \alpha \omega) = |\alpha| d(x, \omega) < |\alpha| \varepsilon, \]

i.e., \( \alpha x \in B(\omega; |\alpha| \varepsilon) \). This shows the inclusion \( \alpha B(\omega; \varepsilon) \subseteq B(\omega; |\alpha| \varepsilon) \). On the other hand, for \( x \in B(\omega; |\alpha| \varepsilon) \), we have \( d(x, \omega) < |\alpha| \varepsilon \), i.e., \( 1/|\alpha| \cdot d(x, \omega) < \varepsilon \). Since \( d \) is absolutely homogeneous, we have \( d(x/\alpha, \omega/\alpha) < \varepsilon \). Since \( \Omega \) owns the self-decomposition, we have \( \omega/\alpha = \omega \oplus \hat{\omega} \) for some \( \hat{\omega} \in \Omega \). Therefore, we obtain \( d(x/\alpha, \omega) = d(x/\alpha, \omega \oplus \hat{\omega}) = d(x/\alpha, \omega/\alpha) < \varepsilon \), which says that \( x/\alpha \in B(\omega; \varepsilon) \). Since \( 1x = x \) and \( (\alpha \beta) x = \alpha (\beta x) \), we conclude that \( x \in \alpha B(\omega; \varepsilon) \). This completes the proof. \( \blacksquare \)

6 Nonstandardly Open Sets

Let \((X, d)\) be a pseudo-metric space on a nonstandard vector space \( X \). We are going to consider the concept of openness of subsets of \( X \).

Definition 6.1. Let \( A \) be a subset of a pseudo-metric space \((X, d)\) on a nonstandard vector space \( X \).

- A point \( x_0 \in A \) is said to be a nonstandard interior point of \( A \) if there exists \( \varepsilon > 0 \) such that \( B(x_0; \varepsilon) \subseteq A \). The collection of all interior points of \( A \) is called the nonstandard interior of \( A \) and is denoted by \( \text{int}(A) \).
- A point \( x_0 \in A \) is said to be a nonstandard type-I interior point of \( A \) if there exists \( \varepsilon > 0 \) such that \( B(x_0; \varepsilon) \oplus \Omega \subseteq A \). The collection of all nonstandard type-I interior points of \( A \) is called the nonstandard type-I interior of \( A \) and is denoted by \( \text{int}^{(i)}(A) \).
- A point \( x_0 \in A \) is said to be a nonstandard type-II interior point of \( A \) if there exists \( \varepsilon > 0 \) such that \( B(x_0; \varepsilon) \subseteq A \oplus \Omega \). The collection of all nonstandard type-II interior points of \( A \) is called the nonstandard type-II interior of \( A \) and is denoted by \( \text{int}^{(ii)}(A) \).
- A point \( x_0 \in A \) is said to be a nonstandard type-III interior point of \( A \) if there exists \( \varepsilon > 0 \) such that \( B(x_0; \varepsilon) \oplus \Omega \subseteq A \oplus \Omega \). The collection of all nonstandard type-III interior points of \( A \) is called the nonstandard type-III interior of \( A \) and is denoted by \( \text{int}^{(iii)}(A) \).

We see that the concept of interior point in Definition 6.1 is the same as the conventional definition. We also remark that if \( X \) happens to be a (conventional) vector space, then \( \Omega = \{0\} = (\text{a zero element of } X) \). In this case, the four concepts of (nonstandard) interior points coincide with the conventional definition.

Remark 6.1. Let \((X, d)\) be a pseudo-metric space on a nonstandard vector space \( X \). We have the following observations
• Since \( B(x_0; \epsilon) \subseteq A \) implies \( B(x_0; \epsilon) \oplus \Omega \subseteq A \oplus \Omega \), a nonstandard interior point is also a nonstandard type-III interior point. In other words, we have \( \operatorname{int}(A) \subseteq \operatorname{int}^{(III)}(A) \).

• If we assume that \( \Omega \oplus \Omega = \Omega \), then a nonstandard type-I interior point is also a nonstandard type-III interior point, since \( B(x_0; \epsilon) \oplus \Omega \subseteq A \) implies \( B(x_0; \epsilon) \oplus \Omega \oplus \Omega \subseteq A \oplus \Omega \). In other words, we have \( \operatorname{int}^{(I)}(A) \subseteq \operatorname{int}^{(III)}(A) \).

• Suppose that the pseudo-metric \( d \) satisfies the null equalities. From Proposition 5.1, we see that \( B(x; \epsilon) \oplus \Omega \subseteq B(x; \epsilon) \). It says that if \( x \) is a nonstandard interior point, then it is also a nonstandard type-I interior point, and if \( x \) is a nonstandard type-II interior point, then it is also a nonstandard type-III interior point. In other words, we have \( \operatorname{int}(A) \subseteq \operatorname{int}^{(I)}(A) \) and \( \operatorname{int}^{(I)}(A) \subseteq \operatorname{int}^{(III)}(A) \).

• Suppose that the pseudo-metric \( d \) satisfies the null inequalities and \( \mathcal{X} \) owns the null decomposition. From Proposition 5.3, we see that \( B(x; \epsilon) \subseteq B(x; \epsilon) \oplus \Omega \). It says that if \( x \) is a nonstandard type-I interior point, then it is also a nonstandard interior point, and if \( x \) is a nonstandard type-III interior point, then it is also a nonstandard type-II interior point. In other words, we have \( \operatorname{int}^{(I)}(A) \subseteq \operatorname{int}(A) \) and \( \operatorname{int}^{(III)}(A) \subseteq \operatorname{int}(A) \).

• Suppose that pseudo-metric \( d \) satisfies the null equalities and \( \mathcal{X} \) owns the null decomposition. Then Proposition 5.3 shows that the concepts of nonstandard interior and nonstandard type-I interior are equivalent, and the concepts of nonstandard type-II interior and nonstandard type-III interior are equivalent. In other words, we have \( \operatorname{int}(A) = \operatorname{int}^{(I)}(A) \) and \( \operatorname{int}^{(III)}(A) = \operatorname{int}^{(III)}(A) \).

**Example 6.1.** Let \( \mathcal{X} \) be the set of all closed intervals with the null set \( \Omega = \{ [−k, k] : k \geq 0 \} \). Given \( \omega = [−k, k] \) for \( k \neq 0 \), i.e., \( k > 0 \), we can write \( k = k_1 + k_2 \) with \( k_1, k_2 > 0 \). Then, we have

\[
\omega = [−k, k] = [−(k_1 + k_2), k_1 + k_2] = [−k_1, k_1] \oplus [−k_2, k_2],
\]

where \( \omega_1 = [−k_1, k_1] \), \( \omega_2 = [−k_2, k_2] \) \( \in \Omega \). This says that \( \Omega \oplus \Omega = \Omega \) under this space \( \mathcal{X} \). Therefore, the assumption in Remark 6.1 (ii) is automatically satisfied under this space \( \mathcal{X} \).

**Remark 6.2.** Although \( B(x; \epsilon) \oplus \Omega \subseteq B(x; \epsilon) \) as shown in Proposition 5.1 is satisfied under the assumption of null equalities, the set \( B(x; \epsilon) \oplus \Omega \) does not necessarily contain the center \( x \) unless \( x \) has the null decomposition (which will be shown below). Therefore, it can happen that there exists an open ball such that \( B(x; \epsilon) \oplus \Omega \) is contained in \( A \) even though the center \( x \) is not in \( A \). In this situation, we do not say that \( x \) is a nonstandard type-I interior point, since \( x \) is not in \( A \). However, if the center \( x \) has the null decomposition, then \( B(x; \epsilon) \oplus \Omega \) will contain the center \( x \), since \( x = \bar{x} \oplus \omega \) for some \( \bar{x} \in \mathcal{X} \) and \( \omega \in \Omega \) satisfying

\[
\epsilon > 0 = d(x, \bar{x}) = d(x, \bar{x} \oplus \omega) \geq d(x, \bar{x}),
\]

which says that \( \bar{x} \in B(x; \epsilon) \), i.e., \( x = \bar{x} \oplus \omega \in B(x; \epsilon) \oplus \Omega \).

Based on Remark 6.2, we can define the concepts of pseudo-interior point.

**Definition 6.2.** Let \( \mathcal{X} \) be a subset of a pseudo-metric space \( (X, d) \) on a nonstandard vector space \( \mathcal{X} \).

- A point \( x_0 \in \mathcal{X} \) is said to be a **nonstandard pseudo-interior point** of \( A \) if there exists \( \epsilon > 0 \) such that \( B(x_0; \epsilon) \subseteq A \). The collection of all interior points of \( A \) is called the **nonstandard pseudo-interior** of \( A \) and is denoted by \( \operatorname{pint}(A) \).

- A point \( x_0 \in \mathcal{X} \) is said to be a **nonstandard type-I pseudo-interior point** of \( A \) if there exists \( \epsilon > 0 \) such that \( B(x_0; \epsilon) \oplus \Omega \subseteq A \). The collection of all nonstandard type-I pseudo-interior points of \( A \) is called the **nonstandard type-I pseudo-interior** of \( A \) and is denoted by \( \operatorname{pint}^{(I)}(A) \).
A point \( x_0 \in X \) is said to be a nonstandard type-II pseudo-interior point of \( A \) if there exists \( \epsilon > 0 \) such that \( B(x_0; \epsilon) \subseteq A \oplus \Omega \). The collection of all nonstandard type-II pseudo-interior points of \( A \) is called the nonstandard type-II pseudo-interior of \( A \) and is denoted by \( \text{pint}^{(ii)}(A) \).

A point \( x_0 \in X \) is said to be a nonstandard type-III pseudo-interior point of \( A \) if there exists \( \epsilon > 0 \) such that \( B(x_0; \epsilon) \ominus \Omega \subseteq A \ominus \Omega \). The collection of all nonstandard type-III pseudo-interior points of \( A \) is called the nonstandard type-III pseudo-interior of \( A \) and is denoted by \( \text{pint}^{(iii)}(A) \).

**Remark 6.3.** Let \((X, d)\) be a pseudo-metric space on a nonstandard vector space \( X \). We have the following observations.

- Each type of nonstandard pseudo-interior points of \( A \) does not necessarily belong to \( A \). However, each type of nonstandard interior points of \( A \) is the corresponding nonstandard pseudo-interior points of \( A \). In other words, we have \( \text{int}^{(i)}(A) \subseteq \text{pint}^{(i)}(A) \), \( \text{int}^{(ii)}(A) \subseteq \text{pint}^{(ii)}(A) \) and \( \text{int}^{(iii)}(A) \subseteq \text{pint}^{(iii)}(A) \).

- If the pseudo-metric \( d \) satisfies the null inequalities and \( X \) owns the null decomposition, then, from Proposition 5.3, we have \( \text{pint}^{(i)}(A) \subseteq \text{int}(A) \subseteq A \), since \( x_0 \in B(x_0; \epsilon) \subseteq B(x_0; \epsilon) \oplus \Omega \subseteq A \).

- If \( A \ominus \Omega \subseteq A \), then we have \( \text{pint}^{(ii)}(A) \subseteq \text{int}(A) \subseteq A \).

- If \( A \ominus \Omega \subseteq A \), the pseudo-metric \( d \) satisfies the null inequalities and \( X \) owns the null decomposition, then, from Proposition 5.3, we have \( \text{pint}^{(iii)}(A) \subseteq \text{int}(A) \subseteq A \).

**Definition 6.3.** Let \( A \) be a subset of a pseudo-metric space \((X, d)\) on a nonstandard vector space \( X \). The set \( A \) is said to be nonstandardly open if \( A = \text{int}(A) \). The set \( A \) is said to be nonstandardly type-I-open if \( A = \text{int}^{(i)}(A) \). The set \( A \) is said to be nonstandardly type-II-open if \( A = \text{int}^{(ii)}(A) \). The set \( A \) is said to be nonstandardly type-III-open if \( A = \text{int}^{(iii)}(A) \). We can similarly define the nonstandardly pseudo-open, nonstandardly type-I pseudo-open, nonstandardly type-II pseudo-open and nonstandardly type-III pseudo-open set.

**Remark 6.4.** Let \( A \) be a subset of a pseudo-metric space \((X, d)\) on a nonstandard vector space \( X \). We have the following observations.

- From Remark 6.3, we see that if \( A \) is nonstandardly type-I-open, then we have \( A \subseteq \text{pint}^{(i)}(A) \).
  We can have the similar observations for the other types of openness.

- It is clear that \( \text{int}^{(i)}(A) \subseteq A \). Let \( O \) be any nonstandardly type-I-open subset of \( A \). Then \( O = \text{int}^{(i)}(O) \subseteq \text{int}^{(i)}(A) \). This says that \( \text{int}^{(i)}(A) \) is the largest nonstandardly type-I-open set contained in \( A \). Similarly, \( \text{int}(A) \) is the largest open set contained in \( A \), \( \text{int}^{(ii)}(A) \) is the largest nonstandardly type-II-open set contained in \( A \) and \( \text{int}^{(iii)}(A) \) is the largest nonstandardly type-III-open set contained in \( A \).

  For convenience, we adopt \( \emptyset \ominus \Omega = \emptyset \).

**Remark 6.5.** Let \( A \) be a subset of a pseudo-metric space \((X, d)\) on a nonstandard vector space \( X \). We consider the extreme cases of the empty set \( \emptyset \) and whole set \( X \).

(a) Since the empty set \( \emptyset \) contains no elements, it means that \( \emptyset \) is nonstandardly open and pseudo-open (we can regard the empty set as an open ball). Obviously, \( X \) is also nonstandardly open and pseudo-open, since \( x \in B \subseteq X \) for any open ball \( B \), i.e., \( X \subseteq \text{int}(X) \) and \( X \subseteq \text{pint}(X) \).

(b) Since \( \emptyset \ominus \Omega = \emptyset \subseteq \emptyset \), the empty set \( \emptyset \) is nonstandardly type-I-open and type-I-pseudo-open. Obviously, \( X \) is also nonstandardly type-I-open and type-I-pseudo-open, since \( x \in B \ominus \Omega \subseteq X \) for any open ball \( B \), i.e., \( X \subseteq \text{int}^{(i)}(X) \) and \( X \subseteq \text{pint}^{(i)}(X) \).
Proposition 6.1. Let a nonstandardly pseudo-open, or nonstandardly type-I-pseudo-open subset of a pseudo-metric space $(X, d)$ the metric $\omega$ and type-II-pseudo-open set. Indeed, for any $x \in X$ and any open ball $B$, we have $x \in B \subseteq X \subseteq X \oplus \Omega$. If we assume that $X$ owns the null decomposition, then $X$ will become a nonstandardly type-II-open and type-II-pseudo-open set. From Proposition 6.2, we also have $B \subseteq X \subseteq X \oplus \Omega$. Therefore, we obtain $a = a \oplus \omega$ for any $a \in A$. From (ii), we conclude that $a = a \oplus \omega$ for any $a \in A$.

Proposition 6.2. Let $(X, d)$ be a pseudo-metric space on a nonstandard vector space $X$. Then the following statements hold true.

(i) We have $\text{int}(i)\Omega \subseteq A$. Moreover, if $A$ is a nonstandardly type-I open subset of $X$, then $A \oplus \Omega \subseteq A$.

Proof. To prove part (i), we consider the case of nonstandardly type-I-pseudo-open. For $a \in A = \text{pint}(\Omega)(A)$, by definition, there exists $\epsilon > 0$ such that $B(a; \epsilon) \oplus \Omega \subseteq A \oplus \Omega$. From Proposition 5.2, we also have $B(a; \epsilon) \oplus \Omega \subseteq A \oplus \Omega$, which says that $a \oplus \omega \in \text{pint}(\Omega)(A) = A$. Suppose that the metric $d$ satisfies the null equalities. If $a \oplus \omega \in A = \text{pint}(\Omega)(A)$, there exists $\epsilon > 0$ such that $B(a \oplus \omega; \epsilon) \oplus \Omega \subseteq A \oplus \Omega$. By Proposition 5.2, we also see that $a \in \text{pint}(\Omega)(A) = A$. The similar arguments can also apply to the other cases of openness.

To prove part (ii), we also consider the case of nonstandardly type-II-pseudo-open. If $a \in A \oplus \omega$, then $a = a \oplus \omega$ for some $\omega \in A = \text{pint}(\Omega)(A)$. Therefore, there exists $\epsilon > 0$ such that $B(a; \epsilon) \oplus \Omega \subseteq A \oplus \Omega$. Since $B(a; \epsilon) = B(a \oplus \omega; \epsilon) = B(\hat{a}; \epsilon)$ by Proposition 5.2, we see that $B(a; \epsilon) \oplus \Omega \subseteq A \oplus \Omega$, i.e., $a \in \text{pint}(\Omega)(A) = A$. Now, for $a \in A \oplus \Omega$, we see that $a \in A \oplus \omega$ for some $\omega \in A$, which implies $a \in A$. Therefore, we obtain $A \oplus \Omega \subseteq A$. Now, for $a \oplus \omega \in A \oplus \omega \subseteq A \oplus \Omega \subseteq A$, we also have $a \oplus \omega \in A$. From (i), we obtain $a \in A$.

To prove part (iii), for $a \in A$, since $X$ owns the null decomposition, we have $a = \hat{a} \oplus \omega_0$ for some $\hat{a} \in X$ and $\omega_0 \in \Omega$. Since $a \in A = \text{pint}(\Omega)(A)$, there exists $\epsilon > 0$ such that $B(a \oplus \omega_0; \epsilon) \oplus \Omega \subseteq A \oplus \Omega$. From Proposition 5.2, we also have $B(a \oplus \omega_0; \epsilon) \oplus \Omega = B(\hat{a} \oplus \omega_1; \epsilon) \oplus \Omega \subseteq A \oplus \Omega$, i.e., $\hat{a} \in \text{pint}(\Omega)(A) = A$. This shows that $a = \hat{a} \oplus \omega_0 \in A \oplus \Omega$. Therefore, from (ii), we conclude that $A = A \oplus \Omega$. We further assume that $\Omega$ owns the self-decomposition. Then, given any $\omega \in \Omega$, we have $\omega_0 = \omega \oplus \omega_1$ for some $\omega_1 \in \Omega$. From Proposition 5.2, we also have $B(\hat{a} \oplus \omega_1; \epsilon) \oplus \Omega = B(\hat{a}; \epsilon) \oplus \Omega \subseteq A \oplus \Omega$, i.e., $\hat{a} \oplus \omega_1 \in \text{pint}(\Omega)(A) = A$. Therefore, we obtain $a = a \oplus \omega = (\hat{a} \oplus \omega_1) \oplus \omega \in A \oplus \omega$. From (ii), we conclude that $A = A \oplus \omega$ for any $\omega$. This completes the proof. □
(ii) We have $\text{int}^{(iii)}(A) \subseteq A + \Omega$. Moreover, if $A$ is a nonstandardly type-II-open subset of $X$, then $A \subseteq A + \Omega$.

(iii) Suppose that the pseudo-metric $d$ satisfies the null equalities. We have $(\text{int}^{(iii)}(A))^c + \Omega \subseteq (\text{int}^{(iii)}(A))^c$. Moreover, if $A$ is nonstandardly type-II-open subset of $X$, then $A^c + \Omega \subseteq A^c$.

(iv) Suppose that the pseudo-metric $d$ satisfies the null equalities. We have $\text{int}(A) + \Omega \subseteq A$. Moreover, if $A$ is a nonstandardly open subset of $X$, then $A + \Omega \subseteq A$.

(v) Suppose that the pseudo-metric $d$ satisfies the null inequalities and $X$ owns the null decomposition. We have $\text{int}^{(iii)}(A) \subseteq A + \Omega$. Moreover, if $A$ is a nonstandardly type-III-open subset of $X$, then $A \subseteq A + \Omega$.

Proof. To prove part (i), it suffices to prove the case of null set $\Omega$. For any $x \in \text{int}^{(i)}(A)$, there exists an open ball $B(x; \epsilon)$ such that $B(x; \epsilon) + \Omega \subseteq A$. Since $x \in B(x; \epsilon)$, we have $x + \Omega \subseteq B(x; \epsilon) + \Omega \subseteq A$. This shows that $\text{int}^{(i)}(A) + \Omega \subseteq A$.

To prove part (ii), for any $x \in \text{int}^{(iii)}(A)$, there exists an open ball $B(x; \epsilon)$ such that $B(x; \epsilon) \subseteq A + \Omega$. Then, we have $x \in A + \Omega$, since $x \in B(x; \epsilon)$. This shows that $\text{int}^{(iii)}(A) \subseteq A + \Omega$.

To prove part (iii), for any $x \in (\text{int}^{(iii)}(A))^c + \Omega$, we have $x = \hat{x} + \hat{\omega}$ for some $\hat{x} \in (\text{int}^{(iii)}(A))^c$ and $\hat{\omega} \in \Omega$. By definition, we see that $B(\hat{x}; \epsilon) \subseteq A + \Omega$ for every $\epsilon > 0$. Since $B(x; \epsilon) = B(\hat{x} + \hat{\omega}; \epsilon) = B(\hat{x}; \epsilon)$, we have $\text{int}^{(iii)}(A) \subseteq A + \Omega$.

To prove part (iv), for any $x \in \text{int}(A)$, there exists an open ball $B(x; \epsilon)$ contained in $A$. From Proposition 5.1, we have $x + \Omega \subseteq B(x; \epsilon) + \Omega \subseteq B(x; \epsilon) \subseteq A$. This shows that $\text{int}(A) + \Omega \subseteq A$.

To prove part (v), for any $x \in (\text{int}^{(iii)}(A))^c$ + $\Omega$, we have $x = \hat{x} + \hat{\omega}$ for some $\hat{x} \in (\text{int}^{(iii)}(A))^c$ and $\hat{\omega} \in \Omega$. By the arguments of (iii), we see that $B(x; \epsilon) + \Omega = B(\hat{x}; \epsilon) + \Omega \subseteq A + \Omega$ for every $\epsilon > 0$. This says that $x \notin \text{int}^{(iii)}(A) = A$, which shows that $x \in A^c$. We complete the proof.

Proposition 6.3. Let $(X, d)$ be a pseudo-metric space on a nonstandard vector space $X$. Then, the following statements hold true.

(i) Suppose that the pseudo-metric $d$ satisfies the null inequalities. If $A$ is nonstandardly open, then $A$ is nonstandardly type-III-open.

(ii) Suppose that the metric $d$ satisfies the null equalities. If $A$ is nonstandardly open, then $A$ is nonstandardly type-I-open, if $A$ is nonstandardly type-II-open, then $A$ is also nonstandardly type-III-open.

(iii) Suppose that the pseudo-metric $d$ satisfies the null inequalities and $\Omega \oplus \Omega = \Omega$. If $A$ is nonstandardly type-I-open, then $A$ is nonstandardly type-III-open.

(iv) Suppose that the pseudo-metric $d$ satisfies the null equalities. If $A$ is simultaneously nonstandardly type-I and type-II-open, then $A$ is simultaneously nonstandardly open and nonstandardly type-III-open.

(v) Suppose that $X$ own the null decomposition and the pseudo-metric $d$ satisfies the null equalities. If $A$ is simultaneously nonstandardly open and nonstandardly type-III-open, then $A$ is also nonstandardly type-I-open.
(vi) Suppose that \( \Omega \oplus \Omega = \Omega \) and the pseudo-metric \( d \) satisfies null equalities. If \( A \) is nonstandardly open, then \( A \) is nonstandardly type-I-open if and only if \( A \) is nonstandardly type-III-open.

(vii) Suppose that the space \( X \) owns the null decomposition and the pseudo-metric \( d \) satisfies the null equalities. Then \( A \) is nonstandardly open if and only if \( A \) is nonstandard type-I-open, and \( A \) is nonstandard type-II-open if and only if \( A \) is nonstandard type-III-open.

(viii) Suppose that the space \( X \) owns the null decomposition, the pseudo-metric \( d \) satisfies the null equalities and \( \Omega \oplus \Omega = \Omega \). If \( A \) is nonstandardly open or nonstandardly type-I open, then \( A \) is simultaneously open, nonstandardly type-I-open, type-II-open and type-III-open.

**Proof.** To prove part (i), from Remark 6.1 (i), we see that \( \text{int}(A) \subseteq \text{int}^{(n)}(A) \). If \( A \) is nonstandardly open, then \( A = \text{int}(A) \subseteq \text{int}^{(n)}(A) \), i.e., \( A = \text{int}^{(n)}(A) \). This shows that \( A \) is nonstandardly type-III open. Part (ii) follows from Remark 6.1 (iii) immediately. Part (iii) follows from Remark 6.1 (ii) immediately. To prove part (iv), if \( A \) is simultaneously nonstandardly type-I-open and type-II-open, then, by Proposition 6.2 (i) and (ii), we have \( B(x_0; \epsilon) \subseteq A \oplus \Omega = A \) and \( B(x_0; \epsilon) \oplus \Omega \subseteq A = A \oplus \Omega \). This proves the result. To prove part (v), if \( A \) is nonstandardly open and nonstandardly type-III-open, then \( B(x_0; \epsilon) \oplus \Omega \subseteq A \oplus \Omega \subseteq A \) by Proposition 6.2 (iii) and (iv). Part (vi) follows from (iii) and (v) immediately. Part (vii) follows from Remark 6.1 (v) immediately. Part (viii) follows from (vi) and (vii) immediately. This completes the proof.

**Proposition 6.4.** Let \( X \) be a pseudo-metric space on a nonstandard vector space \( X \) such that the pseudo-metric \( d \) satisfies the null equalities. Suppose that the space \( X \) owns the null decomposition, the pseudo-metric \( d \) satisfies the null equalities and \( \Omega \oplus \Omega = \Omega \). If \( A \) is nonstandardly open, then \( A = A \oplus \Omega \).

**Proof.** The result follows from Propositions 6.3 (viii) and 6.2 (iii) immediately.

**Proposition 6.5.** Let \((X, d)\) be a pseudo-metric space on a nonstandard vector space \( X \) such that the pseudo-metric \( d \) satisfies the null equalities, and let \( A \) be a subset of \( X \). Then we have \( \text{int}(\text{int}(A)) = \text{int}(A) \) and \( \text{int}^{(n)}(\text{int}^{(n)}(A)) = \text{int}^{(n)}(A) \). In other words, \( \text{int}(A) \) is nonstandardly open and \( \text{int}^{(n)}(A) \) is nonstandardly type-I open. If we further assume that the pseudo-metric \( d \) satisfies the null equalities, then we have \( \text{int}^{(n)}(\text{int}^{(n)}(A)) = \text{int}^{(n)}(A) \) and \( \text{int}^{(n)}(\text{int}^{(n)}(A)) = \text{int}^{(n)}(A) \). In other words, \( \text{int}^{(n)}(A) \) is nonstandardly type-II open and \( \text{int}^{(n)}(A) \) is nonstandardly type-III open.

**Proof.** We consider the case of nonstandardly type-I-open set. It will be enough to show the inclusion \( \text{int}^{(n)}(A) \subseteq \text{int}^{(n)}(\text{int}^{(n)}(A)) \). Suppose that \( x \in \text{int}^{(n)}(A) \). Then there exist \( \varepsilon > 0 \) such that \( B(x; \varepsilon) \oplus \Omega \subseteq A \). We want to claim that each element of \( B(x; \varepsilon) \oplus \Omega \) is a nonstandard type-I interior point of \( A \). We take the element \( \hat{y} = y \oplus \omega \in B(x; \varepsilon) \oplus \Omega \), where \( y \in B(x; \varepsilon) \) and \( \omega \in \Omega \), i.e., \( d(x, y) < \varepsilon \). Let \( \hat{\varepsilon} = d(x, y) \). Then we consider the open ball \( B(\hat{y}, \varepsilon - \hat{\varepsilon}) \) centered at \( \hat{y} \) with radius \( \varepsilon - \hat{\varepsilon} \). For \( z \in B(\hat{y}, \varepsilon - \hat{\varepsilon}) \), from condition (iv) in Definition 4.1, we have

\[
d(y, z) \leq d(y \oplus \omega, z) = d(\hat{y}, z) < \varepsilon - \hat{\varepsilon}.
\]

Therefore, we obtain

\[
d(x, z) \leq d(x, y) + d(y, z) = \hat{\varepsilon} + d(y, z) < \hat{\varepsilon} + \varepsilon - \hat{\varepsilon} = \varepsilon.
\]

This shows that \( z \in B(x, \varepsilon) \), i.e., \( B(\hat{y}, \varepsilon - \hat{\varepsilon}) \subseteq B(x, \varepsilon) \). Then we have

\[
B(\hat{y}, \varepsilon - \hat{\varepsilon}) \oplus \Omega \subseteq B(x; \varepsilon) \oplus \Omega \subseteq A,
\]

which says that \( \hat{y} \in \text{int}^{(n)}(A) \). This shows that \( B(x; \varepsilon) \oplus \Omega \subseteq \text{int}^{(n)}(A) \), i.e., \( x \in \text{int}^{(n)}(\text{int}^{(n)}(A)) \). Without considering the null set \( \Omega \), we can also use the above similar arguments to show \( \text{int}(A) \subseteq \text{int}(\text{int}(A)) \).
For the case of nonstandardly type-II-open set, if \( x \in \text{int}^{(II)}(A) \), there exists \( \epsilon > 0 \) such that \( B(x; \epsilon) \subseteq A \oplus \Omega \). Let \( y = B(x; \epsilon) \) and \( \tilde{\epsilon} = d(x, y) \). Therefore, we have \( y = a \oplus \omega \) for some \( a \in A \) and \( \omega \in \Omega \). We consider the open ball \( B(y, \epsilon - \tilde{\epsilon}) \). For \( z \in B(y, \epsilon - \tilde{\epsilon}) \), we have

\[
d(x, z) \leq d(x, y) + d(y, z) = \tilde{\epsilon} + d(y, z) < \epsilon - \tilde{\epsilon} = \epsilon.
\]

This shows that \( z \in B(x, \epsilon) \), i.e., \( B(y, \epsilon - \tilde{\epsilon}) \subseteq B(x, \epsilon) \subseteq A \oplus \Omega \). Now we want to claim that \( B(a; \epsilon - \tilde{\epsilon}) \subseteq B(y; \epsilon - \tilde{\epsilon}) \). Suppose that \( x \in B(a; \epsilon - \tilde{\epsilon}) \). Since the metric \( d \) satisfies the null equalities, we have

\[
d(\hat{x}, \hat{y}) = d(\hat{x}, a \oplus \omega) = d(\hat{x}, a) < \epsilon - \tilde{\epsilon},
\]

which says that \( \hat{x} \in B(\hat{y}; \epsilon - \tilde{\epsilon}) \). Therefore, we obtain \( B(a; \epsilon - \tilde{\epsilon}) \subseteq B(\hat{y}; \epsilon - \tilde{\epsilon}) \subseteq A \oplus \Omega \). This shows that \( a \in \text{int}^{(II)}(A) \), i.e., \( \hat{y} = a \oplus \omega \in \text{int}^{(II)}(A) \oplus \Omega \). Therefore, we obtain \( B(x; \epsilon) \subseteq \text{int}^{(II)}(A) \oplus \Omega \), which implies that \( x \in \text{int}^{(II)}(\text{int}^{(II)}(A)) \). Therefore, we obtain the inclusion \( \text{int}^{(II)}(A) \subseteq \text{int}^{(II)}(\text{int}^{(II)}(A)) \).

Finally, for the case of nonstandardly type-III-open set, if \( x \in \text{int}^{(III)}(A) \), there exists \( \epsilon > 0 \) such that \( B(x; \epsilon) \subseteq A \oplus \Omega \). For \( y \in B(x; \epsilon) \) and \( \omega \in \Omega \), we have \( y \oplus \omega = a \oplus \hat{\omega} \) for some \( a \in A \) and \( \hat{\omega} \in \Omega \). Let \( \epsilon = d(x, y) \). From the above arguments, we have \( B(y, \epsilon - \tilde{\epsilon}) \subseteq B(x; \epsilon) \), i.e.,

\[
B(y, \epsilon - \tilde{\epsilon}) \oplus \Omega \subseteq B(x; \epsilon) \oplus \Omega \subseteq A \oplus \Omega.
\]

We want to claim that \( B(a; \epsilon - \tilde{\epsilon}) \subseteq B(y; \epsilon - \tilde{\epsilon}) \). Suppose that \( \hat{x} \in B(a; \epsilon - \tilde{\epsilon}) \). Since the metric \( d \) satisfies the null equalities, we have

\[
d(\hat{x}, \hat{y}) = d(\hat{x}, \hat{y} \oplus \omega) = d(\hat{x}, a \oplus \hat{\omega}) = d(\hat{x}, a) < \epsilon - \tilde{\epsilon},
\]

which says that \( \hat{x} \in B(\hat{y}; \epsilon - \tilde{\epsilon}) \). Therefore, we obtain

\[
B(a; \epsilon - \tilde{\epsilon}) \oplus \Omega \subseteq B(\hat{y}; \epsilon - \tilde{\epsilon}) \oplus \Omega \subseteq A \oplus \Omega.
\]

This shows that \( a \in \text{int}^{(III)}(A) \), i.e., we have \( \hat{y} \oplus \omega = a \oplus \hat{\omega} \in \text{int}^{(III)}(A) \oplus \Omega \). Therefore, we obtain \( B(x; \epsilon) \oplus \Omega \subseteq \text{int}^{(III)}(A) \oplus \Omega \), which implies that \( x \in \text{int}^{(III)}(\text{int}^{(III)}(A)) \). Therefore, we obtain the inclusion \( \text{int}^{(III)}(A) \subseteq \text{int}^{(III)}(\text{int}^{(III)}(A)) \). This completes the proof. \( \blacksquare \)

**Proposition 6.6.** Let \((X, d)\) be a pseudo-metric space on a nonstandard vector space \(X\) such that the pseudo-metric \(d\) satisfies the null inequalities, and let \(B(x_0; \epsilon)\) be any open ball centered at \(x_0\) with radius \(\epsilon\). Then the following statements hold true.

(i) The open ball \(B(x_0; \epsilon)\) is also an nonstandardly open and nonstandardly type-III-open subset of \(X\). The result also holds true if \((X, d)\) is taken as a pseudo-metric space.

(ii) Suppose that the metric \(d\) satisfies the null equalities. Then the open ball \(B(x_0; \epsilon)\) is also a nonstandardly type-I-open subset of \(X\).

(iii) Suppose that \(X\) owns the null decomposition. Then the open ball \(B(x_0; \epsilon)\) is also a nonstandardly type-II-open subset of \(X\).

**Proof.** To prove part (i), for any \(x \in B(x_0; \epsilon)\), we have \(d(x, x_0) < \epsilon\). Let \(\tilde{\epsilon} = d(x, x_0)\). We consider the open ball \(B(x; \epsilon - \tilde{\epsilon})\) centered at \(x\) with radius \(\epsilon - \tilde{\epsilon}\). Then, for any \(\hat{x} \in B(x; \epsilon - \tilde{\epsilon})\), we have \(d(\hat{x}, x) < \epsilon - \tilde{\epsilon}\). Then we have

\[
d(\hat{x}, x_0) \leq d(\hat{x}, x) + d(x, x_0) = \tilde{\epsilon} + d(\hat{x}, x) < \epsilon + \tilde{\epsilon} = \epsilon,
\]

which means that \(\hat{x} \in B(x_0; \epsilon)\), i.e.,

\[
B(x; \epsilon - \tilde{\epsilon}) \subseteq B(x_0; \epsilon).
\]
This shows that the open ball $B(x_0; \epsilon)$ is nonstandardly open. Moreover, we also have $B(x; \epsilon - \epsilon) \oplus \Omega \subseteq B(x_0; \epsilon) \oplus \Omega$ from (11). This says that $B(x_0; \epsilon)$ is nonstandardly type-III-open.

To prove part (ii), suppose that the metric $d$ satisfies the null equalities. Then, from Proposition 5.3 and (11), we have $B(x; \epsilon - \epsilon) \oplus \Omega \subseteq B(x_0; \epsilon) \oplus \Omega \subseteq B(x_0; \epsilon)$. This says that $B(x_0; \epsilon)$ is nonstandardly type-I-open.

To prove part (iii), suppose that $X$ owns the null decomposition. From (11) and Proposition 5.3, we have $B(x; \epsilon - \epsilon) \subseteq B(x_0; \epsilon) \subseteq B(x_0; \epsilon) \oplus \Omega$. This says that the open ball $B(x_0; \epsilon)$ is also nonstandardly type-II-open. We complete the proof.

7 Nonstandardly Closed Sets

In the (conventional) metric space $(X, d)$, let $A$ be a subset of $X$. A point $x_0 \in X$ is said to be a closure point of $A$ (or limit point of $A$) if every open ball $B(x_0; \epsilon)$ centered at $x_0$ contains points of $A$. Equivalently, since $x_0 \in B(x_0; \epsilon)$, the point $x_0 \in X$ is a closure point of $A$ if $x_0 \in A$ or, for $x_0 \in A^c$, every open ball $B(x_0; \epsilon)$ centered at $x_0$ contains points of $A$. For the pseudo-metric space $(X, d)$ on a nonstandard vector space $X$, now different types of closure points are defined below.

Definition 7.1. Let $(X, d)$ be a pseudo-metric space on a nonstandard vector space $X$, and let $A$ be a subset of $X$.

- A point $x_0 \in X$ is said to be a nonstandard closure point of $A$ (or nonstandard limit point of $A$) if every open ball $B(x_0; \epsilon)$ centered at $x_0$ contains points of $A$. The collection of all closure points of $A$ is called the nonstandard closure of $A$ and is denoted by $\text{cl}(A)$.

- A point $x_0 \in X$ is said to be a nonstandard type-I closure point of $A$ (or nonstandard type-I limit point of $A$) if $x_0 \in A$, or if $x_0 \in A^c$ and, for every open ball $B(x_0; \epsilon)$ centered at $x_0$, the set $B(x_0; \epsilon) \oplus \Omega$ contains points of $A$. The collection of all nonstandard type-I closure points of $A$ is called the nonstandard type-I closure of $A$ and is denoted by $\text{cl}^{(I)}(A)$.

- A point $x_0 \in X$ is said to be a nonstandard type-II closure point of $A$ (or nonstandard type-II limit point of $A$) if $x_0 \in A$, or if $x_0 \in A^c$ and every open ball $B(x_0; \epsilon)$ centered at $x_0$ contains points of $A \oplus \Omega$. The collection of all nonstandard type-II closure points of $A$ is called the nonstandard type-II closure of $A$ and is denoted by $\text{cl}^{(II)}(A)$.

- A point $x_0 \in X$ is said to be a nonstandard type-III closure point of $A$ (or nonstandard type-III limit point of $A$) if $x_0 \in A$, or if $x_0 \in A^c$ and, for every open ball $B(x_0; \epsilon)$ centered at $x_0$, the set $B(x_0; \epsilon) \oplus \Omega$ contains points of $A \oplus \Omega$. The collection of all nonstandard type-III closure points of $A$ is called the nonstandard type-III closure of $A$ and is denoted by $\text{cl}^{(III)}(A)$.

We also remark that if $X$ happens to be a (conventional) vector space, then the four concepts of nonstandard closure point coincide with the conventional definition of closure point. The concepts proposed in Definition 7.1 are based on the following observations:

- For any $x \in A$, the set $B(x; \epsilon) \oplus \Omega$ does not necessarily contains the points of $A$ in the sense of nonstandard type-I closure point, unless $B(x; \epsilon) \subseteq B(x; \epsilon) \oplus \Omega$. In this case, we have $x \in B(x; \epsilon)$, i.e., $B(x; \epsilon)$ contains points of $A$, which implies $B(x; \epsilon) \oplus \Omega$ contains points of $A$. However, if $x \in A$, we still define $x$ as a nonstandard type-I closure point of $A$ as in the conventional case.

- For any $x \in A$, the open ball $B(x; \epsilon)$ does not necessarily contains the points of $A \oplus \Omega$ in the sense of nonstandard type-II closure point, unless $A \subseteq A \oplus \Omega$. In this case, we have $x \in B(x; \epsilon)$, i.e., $B(x; \epsilon)$ contains points of $A$, which implies $B(x; \epsilon)$ contains points of $A \oplus \Omega$. However, if $x \in A$, we still define $x$ as a nonstandard type-II closure point of $A$ as in the conventional case.
• For any \( x \in A \), the open ball \( B(x; \epsilon) \oplus \Omega \) does not necessarily contains the points of \( A \oplus \Omega \) in the sense of nonstandard type-\( \text{III} \) closure point, unless \( B(x; \epsilon) \subseteq B(x; \epsilon) \oplus \Omega \) and \( A \subseteq A \oplus \Omega \). In this case, we have \( x \in B(x; \epsilon) \), i.e., \( B(x; \epsilon) \) contains points of \( A \), which implies \( B(x; \epsilon) \oplus \Omega \) contains points of \( A \oplus \Omega \). However, if \( x \in A \), we still define \( x \) as a nonstandard type-\( \text{III} \) closure point of \( A \) as in the conventional case.

**Remark 7.2.** Let \((X, d)\) be a pseudo-metric space on a nonstandard vector space \( X \) such that the pseudo-metric \( d \) satisfies the null inequalities. We have the following observations:

• We further assume that the pseudo-metric \( d \) satisfies the null equalities. Proposition 5.1 shows the inclusion \( B(x_0; \epsilon) \oplus \Omega \subseteq B(x_0; \epsilon) \). This also says that if \( x \) is a nonstandard type-\( \text{I} \) closure point, then it is also a close point, and if \( x \) is a nonstandard type-\( \text{III} \) closure point, then it is also a nonstandard type-\( \text{III} \) closure point. Now we further assume that \( \Omega \) is closed under the vector addition, i.e., \( \Omega \oplus \Omega \subseteq \Omega \). If \( x \) is a nonstandard type-\( \text{II} \) closure point, then \( B(x; \epsilon) \) contains points of \( A \oplus \Omega \). Equivalently, by adding \( \Omega \), we see that \( B(x; \epsilon) \oplus \Omega \) contains points of \( A \oplus \Omega \oplus \Omega \subseteq A \oplus \Omega \). We conclude that \( B(x; \epsilon) \oplus \Omega \) contains points of \( A \oplus \Omega \), i.e., \( x \) is a nonstandard type-\( \text{III} \) closure point. This shows that the concepts of nonstandard type-\( \text{II} \) closure point and nonstandard type-\( \text{III} \) closure point are equivalent under the assumption that \( \Omega \) is closed under the vector addition and the metric \( d \) satisfies the null equalities.

• Suppose that \( X \) owns the null decomposition. By Proposition 5.3 we have the inclusion \( B(x_0; \epsilon) \subseteq B(x_0; \epsilon) \oplus \Omega \). This also says that if \( x \) is a closure point, then it is also a nonstandard type-\( \text{I} \) closure point, and if \( x \) is a nonstandard type-\( \text{III} \) closure point, then it is also a nonstandard type-\( \text{III} \) closure point.

• Suppose that \( X \) owns the null decomposition and the metric \( d \) satisfies the null equalities. By Proposition 5.3 the concepts of closure point and nonstandard type-\( \text{I} \) closure point are equivalent, and the concepts of nonstandard type-\( \text{II} \) closure point and nonstandard type-\( \text{III} \) closure point are equivalent.

**Remark 7.2.** Remark 6.2 says that if the center \( x \) has the null decomposition and the pseudo-metric \( d \) satisfies the null equalities, then \( B(x; \epsilon) \oplus \Omega \) will contain the center \( x \). Therefore, if \( X \) owns the null decomposition and the metric \( d \) satisfies the null equalities, then we can simply say that a point \( x_0 \in X \) is a nonstandard type-\( \text{I} \) closure point of \( A \) if, for every open ball \( B(x_0; \epsilon) \) centered at \( x_0 \), the set \( B(x_0; \epsilon) \oplus \Omega \) contains points of \( A \).

**Definition 7.2.** Let \( A \) be a subset of a pseudo-metric space \((X, d)\) on a nonstandard vector space \( X \). The set \( A \) is said to be \textit{nonstandardly closed} if and only if \( A = \text{cl}(A) \). The set \( A \) is said to be \textit{nonstandardly type-\( \text{I} \)-closed} if and only if \( A = \text{cl}^{(\text{I})}(A) \). The set \( A \) is said to be \textit{nonstandardly type-\( \text{II} \)-closed} if and only if \( A = \text{cl}^{(\text{II})}(A) \). The set \( A \) is said to be \textit{nonstandardly type-\( \text{III} \)-closed} if and only if \( A = \text{cl}^{(\text{III})}(A) \).

**Remark 7.3.** We have the following observations.

(i) It is clear that \( A \subseteq \text{cl}(A) \), \( A \subseteq \text{cl}^{(\text{I})}(A) \), \( A \subseteq \text{cl}^{(\text{II})}(A) \) and \( A \subseteq \text{cl}^{(\text{III})}(A) \).

(ii) It is also clear that the empty set and the whole space \( X \) are nonstandardly closed, nonstandardly type-\( \text{I} \)-closed, type-\( \text{II} \)-closed, and type-\( \text{III} \)-closed.

(iii) Let \( C \) be any nonstandardly type-\( \text{I} \)-closed set containing \( A \). Then \( \text{cl}^{(\text{I})}(A) \subseteq \text{cl}^{(\text{I})}(C) \). This shows that \( \text{cl}^{(\text{I})}(A) \) is the smallest nonstandardly type-\( \text{I} \)-closed set containing \( A \). Similarly, \( \text{cl}(A) \) is the smallest nonstandardly closed set containing \( A \), \( \text{cl}^{(\text{II})}(A) \) is the smallest nonstandardly type-\( \text{II} \)-closed set containing \( A \) and \( \text{cl}^{(\text{III})}(A) \) is the smallest nonstandardly type-\( \text{III} \)-closed set containing \( A \).
(iv) Let $A$ be a subset of $X$ such that, for every $x \in A^c$, $x$ is not a (resp. nonstandard type-I, type-II and type-III) closure point of $A$. In other words, by definition, if $x \in \text{cl}(A)$ (resp. $x \in \text{cl}^{(i)}(A)$) then $x \in A$, i.e., $\text{cl}(A) \subseteq A$. This says that $A$ is nonstandardly closed (resp. nonstandardly type-I-closed, type-II-closed and type-III-closed) in $X$.

**Remark 7.4.** We have the following observations.

- Let $(X, d)$ be a pseudo-metric space. A singleton set $\{x\}$ is a nonstandardly closed and nonstandardly type-III-closed set, since every ball $B(x; \varepsilon)$ contains $\{x\}$, which also implies that $B(x; \varepsilon) \cap \Omega$ contains $\{x\} \cap \Omega$.

- Let $(X, d)$ be a pseudo-metric space. If the singleton set $\{x\}$ has the null decomposition, then $\{x\}$ is also a nonstandardly type-I-closed set by Remark 6.2.

- Let $(X, d)$ be a pseudo-metric space such that the pseudo-metric $d$ satisfies the null equalities. Now, for any $\omega \in \Omega$, we have $d(x \oplus \omega, x) = d(x, x) = 0$, which says that $x \oplus \omega \in B(x; \varepsilon)$, i.e., the open ball $B(x; \varepsilon)$ contains points of $\{x\} \cap \Omega$. It says that the singleton set $\{x\}$ is a nonstandardly type-II-closed set.

**Proposition 7.1.** Let $(X, d)$ be a pseudo-metric space on a nonstandard vector space $X$ such that the pseudo-metric $d$ satisfies the null inequalities. Then, we have

\[
\text{cl}(\text{cl}(A)) = \text{cl}(A) \quad \text{and} \quad \text{cl}^{(i)}(\text{cl}^{(i)}(A)) = \text{cl}^{(i)}(A).
\]

If we further assume that the pseudo-metric $d$ satisfies the null equalities, then we have

\[
\text{cl}^{(i)}(\text{cl}^{(i)}(A)) = \text{cl}^{(i)}(A) \quad \text{and} \quad \text{cl}^{(iii)}(\text{cl}^{(iii)}(A)) = \text{cl}^{(iii)}(A).
\]

**Proof.** We consider the case of nonstandardly type-I-closed set. It suffices to show the inclusion $\text{cl}^{(i)}(\text{cl}^{(i)}(A)) \subseteq \text{cl}^{(i)}(A)$. Suppose that $x \in \text{cl}^{(i)}(\text{cl}^{(i)}(A))$. Then we want to claim that $x \in \text{cl}^{(i)}(A)$. If $x \in A$, then $x \in \text{cl}^{(i)}(A)$. Therefore, we assume $x \notin A$. By definition, for any open ball $B(x; \varepsilon)$, we have

\[
\text{cl}^{(i)}(A) \cap [B(x; \varepsilon) \oplus \Omega] \neq \emptyset. \tag{12}
\]

Suppose that $y$ is an element in the intersection of (12). Then $y = \hat{x} \oplus \omega \in \text{cl}^{(i)}(A)$ with $\omega \in \Omega$ and $\hat{x} \in B(x; \varepsilon)$, i.e., $d(x, \hat{x}) < \varepsilon$. Since $y \in \text{cl}^{(i)}(A)$, we have $y \in A$, or $y \in A^c$ with $A \cap [B(y; \varepsilon - \hat{\varepsilon}) \oplus \Omega] \neq \emptyset, \tag{13}$

where $\hat{\varepsilon} = d(x, \hat{x})$. Since $y \in B(x; \varepsilon) \oplus \Omega$, if $y \in A$, then we see that $A \cap [B(x; \varepsilon) \oplus \Omega] \neq \emptyset$, i.e., $x \in \text{cl}^{(i)}(A)$, since $x \notin A$. Now, for $y \in A^c$ and any $\hat{y} \in B(y; \varepsilon - \hat{\varepsilon})$, we have $d(\hat{y}, y) < \varepsilon - \hat{\varepsilon}$ and

\[
\begin{align*}
d(\hat{y}, x) & \leq d(\hat{y}, y) + d(y, x) = d(\hat{y}, y) + d(\hat{x} \oplus \omega, x) \\
& = d(\hat{y}, y) + d(\hat{x}, x) \\
& = d(\hat{y}, y) + \hat{\varepsilon} < \varepsilon - \hat{\varepsilon} + \hat{\varepsilon} = \varepsilon,
\end{align*}
\]

which shows that $\hat{y} \in B(x; \varepsilon)$, i.e., $B(y; \varepsilon - \hat{\varepsilon}) \oplus \Omega \subseteq B(x; \varepsilon) \oplus \Omega$. From (13), it also says that $A \cap [B(x; \varepsilon) \oplus \Omega] \neq \emptyset$, i.e., $x \in \text{cl}^{(i)}(A)$, since $x \notin A$. This shows the inclusion $\text{cl}^{(i)}(\text{cl}^{(i)}(A)) \subseteq \text{cl}^{(i)}(A)$.

For the case of nonstandardly type-II-closed set, suppose that $x \in \text{cl}^{(ii)}(\text{cl}^{(ii)}(A))$. Then we want to claim that $x \in \text{cl}^{(ii)}(A)$. As described above, we may assume that $x \notin A$. By definition, for any open ball $B(x; \varepsilon)$, we have

\[
B(x; \varepsilon) \cap [\text{cl}^{(ii)}(A) \oplus \Omega] \neq \emptyset. \tag{14}
\]

Suppose that $y$ is an element in the intersection of (14). Then $y = \hat{y} \oplus \omega \in B(x; \varepsilon)$ with $\omega \in \Omega$ and $\hat{y} \in \text{cl}^{(ii)}(A)$. Then we have $d(\hat{y}, x) \leq d(\hat{y} \oplus \omega, x) = d(y, x) < \varepsilon$. Since $\hat{y} \in \text{cl}^{(ii)}(A)$, we have $\hat{y} \in A$, or $\hat{y} \in A^c$ with

\[
B(\hat{y}; \varepsilon - \hat{\varepsilon}) \cap [A \oplus \Omega] \neq \emptyset, \tag{15}
\]
where \( \hat{\epsilon} = d(\hat{y}, x) \). If \( \hat{y} \in A \), then \( y = \hat{y} \oplus \omega \in B(x; \epsilon) \cap [A \oplus \Omega] \), i.e., \( B(x; \epsilon) \cap [A \oplus \Omega] \neq \emptyset \). This says that \( x \in \cl(i)(A) \), since \( x \notin A \). Now, for \( \hat{y} \in A^c \) and any \( \hat{x} \in B(\hat{y}; \epsilon - \hat{\epsilon}) \), we have \( d(\hat{x}, \hat{y}) < \epsilon - \hat{\epsilon} \) and

\[
d(\hat{x}, x) \leq d(\hat{x}, \hat{y}) + d(\hat{y}, x) < \epsilon - \hat{\epsilon} + \hat{\epsilon} = \epsilon,
\]

which shows that \( \hat{x} \in B(x; \epsilon) \), i.e., \( B(\hat{y}; \epsilon - \hat{\epsilon}) \subseteq B(x; \epsilon) \). From (15), it also says that \( B(x; \epsilon) \cap [A \oplus \Omega] \neq \emptyset \), i.e., \( x \in \cl(i)(A) \), since \( x \notin A \). Therefore, \( x \in \cl(i)(A) \). This shows the inclusion \( \cl(i)(\cl(i)(A)) \subseteq \cl(i)(A) \). Without considering the null set \( \Omega \), the above arguments can also show \( \cl(\cl(A)) \subseteq \cl(A) \).

Finally, for the case of nonstandardly type-III-closed set, suppose that \( x \in \cl(i)(\cl(i)(A)) \). Now we may assume that \( x \notin A \). By definition, for any open ball \( B(x; \epsilon) \), we have

\[
[B(x; \epsilon) \oplus \Omega] \cap [A \oplus \Omega] \neq \emptyset.
\]

(16)

Suppose that \( y \) is an element in the intersection of \( \Omega \). Then \( y = \hat{y} \oplus \omega_1 = \hat{x} \oplus \omega_2 \) with \( \omega_1, \omega_2 \in \Omega \), \( \hat{x} \in B(x; \epsilon) \) and \( \hat{y} \in \cl(i)(A) \). Since \( d \) satisfies null equalities, we have

\[
d(\hat{y}, x) = d(\hat{y} \oplus \omega_1, x) = d(\hat{x} \oplus \omega_2, x) = d(\hat{x}, x) < \epsilon,
\]

Since \( \hat{y} \in \cl(i)(A) \), we have \( \hat{y} \in A \), or \( \hat{y} \in A^c \) with

\[
[B(\hat{y}; \epsilon - \hat{\epsilon}) \oplus \Omega] \cap [A \oplus \Omega] \neq \emptyset,
\]

(17)

where \( \hat{\epsilon} = d(\hat{y}, x) \). If \( \hat{y} \in A \), then \( y \in [B(x; \epsilon) \oplus \Omega] \cap [A \oplus \Omega] \neq \emptyset \), i.e., \( x \in \cl(i)(A) \), since \( x \notin A \). Now, for \( \hat{y} \in A^c \) and any \( z \in B(\hat{y}; \epsilon - \hat{\epsilon}) \), we have \( d(z, \hat{y}) < \epsilon - \hat{\epsilon} \) and

\[
d(z, x) \leq d(z, \hat{y}) + d(\hat{y}, x) < \epsilon - \hat{\epsilon} + \hat{\epsilon} = \epsilon,
\]

which shows that \( z \in B(x; \epsilon) \), i.e., \( B(\hat{y}; \epsilon - \hat{\epsilon}) \oplus \Omega \subseteq B(x; \epsilon) \oplus \Omega \). From (17), it also says that \( [B(x; \epsilon) \oplus \Omega] \cap [A \oplus \Omega] \neq \emptyset \), i.e., \( x \in \cl(i)(A) \), since \( x \notin A \). Therefore, \( x \in \cl(i)(A) \). This shows the inclusion \( \cl(i)(\cl(i)(A)) \subseteq \cl(i)(A) \). We complete the proof.

Inspired by the above proposition, we may consider the different combinations of nonstandardly closed sets like \( \cl(i)(\cl(i)(A)) \), \( \cl(i)(\cl(i)(A)) \), \( \cl(i)(\cl(i)(A)) \) and so on. The following results are very useful for further discussion.

**Lemma 7.1.** Let \( X \) be a nonstandard vector space over \( \mathbb{F} \) and own the null decomposition. Suppose that \( \Omega \) owns the self-decomposition. Let \( A \) be any subset of \( X \). Then the following statements hold true.

(i) Given any fixed \( x_0 \in X \), \( x + \omega \in A \oplus \Omega \) for any \( \omega \in \Omega \) and \( x \in X \).

In particular, we have \( x + \omega \in A \oplus \Omega \) for any \( \omega \in \Omega \) and \( x \in X \). In this case, we also have \( A \subseteq A \oplus \Omega \).

(ii) Given any \( x \in X \), we have the following properties.

- \( x + \omega \in A \oplus \omega \) implies \( x \in A \oplus \Omega \) for any \( \omega \in \Omega \).
- \( x + \omega \in A \oplus \Omega \) implies \( x \in A \oplus \Omega \).

(iii) We further assume that \( \Omega \) is closed under the vector addition. Given any fixed \( x_0 \in X \), \( x + \omega \in A \oplus \Omega \) if and only if \( x \in A \oplus \Omega \) for any \( \omega \in \Omega \) and \( x \in X \). In particular, we have \( x + \omega \in A \oplus \Omega \) if and only if \( x \in A \oplus \Omega \) for any \( \omega \in \Omega \) and \( x \in X \).

**Proof.** We have \( x + \omega = a \oplus \omega \oplus x_0 \) for some \( a \in A \) and \( \omega \in \Omega \). By adding \( -x \) on both sides, we have \( a + \omega = a \oplus x + \omega_1 \oplus x_0 \), where \( \omega_1 = x \oplus x \in \Omega \). Let \( \omega_2 = \omega_1 \oplus \omega \in \Omega \). Then we obtain
\( \omega_2 = a \oplus \omega_0 \oplus x_0. \) Since \( X \) owns the null decomposition, we have \( x = \hat{x} \oplus \omega_1 \) for some \( \hat{x} \in X \) and \( \omega_3 \in \Omega. \) Then we have

\[
\begin{align*}
x &= \hat{x} + \omega_1 = \hat{x} \oplus \omega_1 \oplus \omega_2 \\
&= \hat{x} \oplus \omega_1 \oplus a \oplus x \oplus \omega_0 \oplus x_0 = \hat{x} \oplus a \oplus x \oplus \omega_5 \oplus x_0 \\
&= \hat{x} + a \oplus x \oplus \omega_6 \oplus \omega_3 \oplus x_0 \quad \text{(where} \ \omega_6 \in \Omega, \ \text{since} \ \Omega \ \text{owns the self-decomposition)} \\
x &= \{ a \oplus x \oplus \omega_6 \oplus \omega_3 \oplus x_0 \} \quad \text{where} \ \omega_7 = \omega_6 \oplus x \oplus x \oplus \Omega. 
\end{align*}
\]

This shows that \( x \oplus \omega \in A \oplus \Omega \oplus x_0 \) implies \( x \in A \oplus \Omega \oplus x_0 \) for any \( \omega \in \Omega \) and \( x \in X. \) If we further assume that \( \Omega \) is closed under the vector addition, the converse is obvious. Without considering \( x_0, \) we can also show that \( x \oplus \omega \in A \oplus \Omega \) implies \( x \in A \oplus \Omega, \) which implies \( x \in A \oplus \Omega, \) i.e., \( A \subseteq A \oplus \Omega. \) This proves (i) and (iii). The above arguments can also obtain the results (ii). We complete the proof.

**Proposition 7.2.** Let \( (X, d) \) be a pseudo-metric space on a nonstandard vector space \( X \) such that the pseudo-metric \( d \) satisfies the null inequalities. If \( X \) owns the null decomposition and \( \Omega \) owns the self-decomposition and is closed under the vector addition, then \( \text{cl}^{(\text{in})}(A) \subseteq \text{cl}^{(\text{in})}(A) \oplus \omega \) and \( \text{cl}^{(\text{in})}(A) \subseteq \text{cl}^{(\text{in})}(A) \oplus \omega \) for any \( \omega \in \Omega. \) In other words, if \( A \) is a nonstandardly type-II or type-III closed subset of \( X, \) then \( A \subseteq A \oplus \omega \) for any \( \omega \in \Omega. \)

**Proof.** Let \( a \in \text{cl}^{(\text{in})}(A). \) Since \( a \) has the null decomposition, we have \( a = a \oplus \omega_0 \) for some \( a \in X \) and \( \omega_0 \in \Omega. \) Given any \( \omega \in \Omega, \) since \( \Omega \) owns the self-decomposition, we have \( \omega_0 = \omega \oplus \omega_1 \) for some \( \omega_1 \in \Omega, \) i.e., \( a = a \oplus \omega_1 \oplus \omega \in A. \) We want to claim that \( a \oplus \omega_1 \) is also in \( \text{cl}^{(\text{in})}(A). \) By definition, we just need to consider \( a \oplus \omega_1 \in A. \) Now if \( a \in A, \) i.e., \( a = (a \oplus \omega_1) \oplus \omega, \) then \( a \oplus \omega_1 \in A \oplus \Omega \) by Lemma 7.4(ii). This says that \( a \oplus \omega_1 \in B(a \oplus \omega_1; \varepsilon) \cap (A \oplus \Omega) \neq \emptyset, \) i.e., \( a \oplus \omega_1 \in \text{cl}^{(\text{in})}(A). \) Suppose that \( a \in A. \) By definition, every open ball \( B(a; \varepsilon) \) contains points of \( A \oplus \Omega. \) We want to claim that \( B(a; \varepsilon) \subseteq B(a \oplus \omega_1; \varepsilon). \) For any \( x \in B(a; \varepsilon), \) we have \( d(x, a) < \varepsilon \) and

\[
d(a \oplus \omega_1, x) \leq d(a \oplus \omega_1, a) + d(a, x) \leq d(a \oplus \omega_1, \omega, a) + d(a, x) = d(a, a) + d(a, x) < 0 + \varepsilon = \varepsilon,
\]

which says that \( x \in B(a \oplus \omega_1; \varepsilon). \) Therefore, we conclude that \( B(a \oplus \omega_1; \varepsilon) \) contains points of \( A \oplus \Omega, \) i.e., \( a \oplus \omega_1 \in \text{cl}^{(\text{in})}(A). \) This shows that \( \text{cl}^{(\text{in})}(A) \subseteq \text{cl}^{(\text{in})}(A) \oplus \omega. \)

Let \( a \in \text{cl}^{(\text{in})}(A). \) We want to claim that \( a \oplus \omega_1 \) is also in \( \text{cl}^{(\text{in})}(A). \) Suppose that \( a \in A. \) By Proposition 7.3, we see that \( a \oplus \omega_1 \in B(a \oplus \omega_1; \varepsilon) \cap (A \oplus \Omega) \neq \emptyset, \) i.e., \( a \oplus \omega_1 \in \text{cl}^{(\text{in})}(A). \) Suppose that \( a \in A. \) By definition, every set \( B(a; \varepsilon) \) contains points of \( A \oplus \Omega. \) Since \( B(a; \varepsilon) \subseteq B(a \oplus \omega_1; \varepsilon), \) we see that every set \( B(a \oplus \omega_1; \varepsilon) \) contains points of \( A \oplus \Omega, \) i.e., \( a \oplus \omega_1 \in \text{cl}^{(\text{in})}(A). \) This shows that \( \text{cl}^{(\text{in})}(A) \subseteq \text{cl}^{(\text{in})}(A) \oplus \omega. \) ■

**Proposition 7.3.** Let \( (X, d) \) be a pseudo-metric space on a nonstandard vector space \( X. \) Then the following statements hold true.

(i) The complement of an nonstandardly open set is nonstandardly closed, and the complement of a nonstandardly closed set is nonstandardly open.

(ii) The complement of a nonstandardly type-I-closed set is nonstandardly type-I-open, and the complement of a nonstandardly type-I-open set is nonstandardly type-I-closed.

**Proof.** Let \( O \) be a nonstandardly type-I-open set and \( O^c \) be its complement. For \( x \in O = (O^c)^c, \) there is an open ball \( B(x; \varepsilon) \) such that \( B(x; \varepsilon) \oplus \Omega \subseteq O, \) i.e., \( B(x; \varepsilon) \oplus \Omega \) is disjoint from \( O^c. \) It means that \( x \) is not a nonstandard type-I closure point of \( O^c. \) By Remark 7.5(iv), we see that \( O^c \) is nonstandardly type-I-closed. Similarly, without considering the null set \( \Omega, \) we can show that \( O^c \) is a nonstandardly closed set if \( O \) is a nonstandardly open set. On the other hand, let \( A \) be a nonstandardly type-I-closed and \( A^c \) be its complement. By definition, we have \( A = \text{cl}^{(i)}(A). \)
For $x \in A^c$, $x$ is not a nonstandard type-I closure point of $A$. It says that there is an open ball $B(x; \epsilon)$ such that $B(x; \epsilon) \oplus \Omega$ is disjoint from $A$, i.e., $B(x; \epsilon) \oplus \Omega$ is contained in $A^c$. This shows that $x \in \text{int}(\Omega)(A)$, i.e., $A^c \subseteq \text{int}(\Omega)(A)$. Therefore, $A^c$ is nonstandardly type-I-open. Similarly, we can show that $A^c$ is an nonstandardly open set if $A$ is a nonstandardly closed set without considering the null set $\Omega$. ■

**Proposition 7.4.** Let $(X, d)$ be a pseudo-metric space on a nonstandard vector space $X$ such that the pseudo-metric $d$ satisfies the null equalities. Then, the following statement holds true.

(i) The complement of a nonstandardly type-II-open set is nonstandardly type-II-closed.

(ii) The complement of a nonstandardly type-III-open set is nonstandardly type-III-closed.

**Proof.** To prove part (i), we first claim $A^c \oplus \Omega \subseteq (A \oplus \Omega)^c$. For any $x \in A^c \oplus \Omega$, we have $x = \hat{x} \oplus \hat{\omega}$ for some $\hat{x} \in A^c = (\text{int}(\Omega)(A))^c$ and $\hat{\omega} \in \Omega$. By definition, we see that $B(\hat{x}; \epsilon) \not\subseteq A \oplus \Omega$ for every $\epsilon > 0$. Since $B(x; \epsilon) = B(\hat{x} \oplus \hat{\omega}; \epsilon) = B(\hat{x}; \epsilon)$ by Proposition 5.2, we also have $B(x; \epsilon) \not\subseteq A \oplus \Omega$ for every $\epsilon > 0$. This says that $x \not\in \text{int}(A \oplus \Omega)$. Now we want to show that $\text{int}(A \oplus \Omega) = A \oplus \Omega$. For $y \in A \oplus \Omega$, we have $y = y_0 \oplus \omega_0$ for some $y_0 \in A = \text{int}(\Omega)(A)$ and $\omega_0 \in \Omega$. By definition, there exists $\epsilon > 0$ such that $B(y_0; \epsilon) \subseteq A \oplus \Omega$. Therefore, we conclude that there is an open ball $B(x; \epsilon)$ which is disjoint from $A^c \oplus \Omega$, i.e., $x$ is not a nonstandard type-II closure point of $A$. By Remark 7.3 (iv), we see that $A^c$ is nonstandardly type-II-closed.

To prove part (ii), we still need to claim the inclusion $A^c \oplus \Omega \subseteq (A \oplus \Omega)^c$. For any $x \in A^c \oplus \Omega$, we have $x = \hat{x} \oplus \hat{\omega}$ for some $\hat{x} \in A = (\text{int}(\Omega)(A))^c$ and $\hat{\omega} \in \Omega$. By the arguments of (i), we see that $B(x; \epsilon) \not\subseteq A \oplus \Omega$ for every $\epsilon > 0$. This says that $x \not\in \text{int}(\Omega)(A \oplus \Omega)$. Now we want to show that $\text{int}(\Omega)(A \oplus \Omega) = A \oplus \Omega$. For $y \in A \oplus \Omega$, we have $y = y_0 \oplus \omega_0$ for some $y_0 \in A = \text{int}(\Omega)(A)$ and $\omega_0 \in \Omega$. By definition, there exists $\epsilon > 0$ such that $B(y; \epsilon) \subseteq A \oplus \Omega$. Therefore, we conclude that there is an open ball $B(x; \epsilon)$ such that the set $B(x; \epsilon) \oplus \Omega$ is disjoint from $A^c \oplus \Omega$, i.e., $x$ is not a nonstandard type-III closure point of $A^c$. By Remark 7.3 (iv), we see that $A^c$ is nonstandardly type-III-closed. ■

**Proposition 7.5.** Let $X$ be a pseudo-metric space on a nonstandard vector space $X$ such that the pseudo-metric $d$ satisfies the null equalities and $X$ owns the null decomposition, and $\Omega$ owns the self-decomposition and is closed under the vector addition. Then the following statements hold true.

(i) The complement of a nonstandardly type-II-closed set is simultaneously nonstandardly open and nonstandardly type-II-open.

(ii) The complement of a nonstandardly type-III-closed set is simultaneously nonstandardly type-I and type-III-open.

**Proof.** To prove part (i), for $x \in A^c$, $x$ is not a nonstandard type-II closure point of $A$. It says that there is an open ball $B(x; \epsilon)$ such that $B(x; \epsilon)$ is disjoint from $A \oplus \Omega$, i.e., $B(x; \epsilon)$ is contained in $(A \oplus \Omega)^c \subseteq A^c$ by Lemma 7.1 (iii). This says that $A^c$ is nonstandardly open. Since $A^c \subseteq A \oplus \Omega$ by Lemma 7.1 (iii) again, i.e., $B(x; \epsilon)$ is contained in $A^c \oplus \Omega$. This shows that $x \in \text{int}(\Omega)(A)$, i.e., $A^c$ is nonstandardly type-II-open.
To prove part (ii), for \(x \in A^c\), \(x\) is not a nonstandard type-III closure point of \(A\). It says that there is an open ball \(B(x; \epsilon)\) such that \(B(x; \epsilon) \cup \Omega\) is disjoint from \(A \cup \Omega\), i.e., \(B(x; \epsilon) \cup \Omega\) is contained in \((A \cup \Omega)^c \subseteq A^c\) by Lemma \(\text{7.3 (iii)}\). This says that \(A^c\) is nonstandardly type-I open. Since \(A^c \subseteq A^c \cup \Omega\), i.e., \(B(x; \epsilon) \cup \Omega\) is contained in \(A^c \cup \Omega\). This shows that \(x \in \text{int}^{(\Omega)}(A^c)\), i.e., \(A^c\) is nonstandardly type-III-open. This completes the proof. 

**Proposition 7.6.** Let \((X, d)\) be a pseudo-metric space on a nonstandard vector space \(X\) such that the pseudo-metric \(d\) satisfies the null equalities. If \(A\) is nonstandardly closed or type-I-closed, then \(A\) is simultaneously nonstandardly closed, type-I-closed, type-II-closed and type-III-closed.

**Proof.** Let \(A\) be a nonstandardly closed or type-I-closed set. Then Proposition \(\text{7.3}\) says that \(A^c\) is nonstandardly open or type-I-open, respectively, which is also simultaneously nonstandardly open, type-I-open, type-II-open and type-III-open by Proposition \(\text{7.3 (viii)}\). Using Propositions \(\text{7.3}\) and \(\text{7.4}\), \(A = (A^c)^c\) is simultaneously nonstandardly closed, type-I-closed, type-II-closed and type-III-closed. This completes the proof. 

**Proposition 7.7.** Let \((X, d)\) be a pseudo-metric space on a nonstandard vector space \(X\). Then the following statement holds true.

(i) A closed ball is also a nonstandardly closed and type-III-closed subset of \(X\).

(ii) Suppose that the pseudo-metric \(d\) satisfies the null inequalities. A closed ball is also a nonstandardly type-I-closed subset of \(X\).

(iii) Suppose that \(X\) owns the null decomposition and the pseudo-metric \(d\) satisfies the null equalities. A closed ball is also a nonstandardly type-II-closed subset of \(X\).

**Proof.** To prove part (i), let \(B(x_0; \epsilon)\) be a closed ball. We want to claim that its complement \(B^c(x_0; \epsilon)\) is a nonstandardly open set. For any \(x \in B^c(x_0; \epsilon)\), we have \(d(x, x_0) > \epsilon\). Let \(\hat{\epsilon} = d(x, x_0)\). We consider the open ball \(B(x, \hat{\epsilon} - \epsilon)\). Then, for any element \(\hat{x} \in B(x; \hat{\epsilon} - \epsilon)\), i.e., \(d(\hat{x}, x) < \hat{\epsilon} - \epsilon\), we have

\[
\hat{\epsilon} = d(x, x_0) \leq d(x, \hat{x}) + d(\hat{x}, x_0) < \hat{\epsilon} - \epsilon + d(\hat{x}, x_0),
\]

which implies \(d(\hat{x}, x_0) > \epsilon\), i.e., \(B(x, \hat{\epsilon} - \epsilon) \subseteq B^c(x_0; \epsilon)\). This shows that \(B^c(x_0; \epsilon)\) is a nonstandardly open set. According to Proposition \(\text{7.3 (i)}\), it says that \(B(x_0; \epsilon)\) is a nonstandardly closed set. We also have \(B(x_0; \epsilon) \cup \Omega \subseteq B^c(x_0; \epsilon) \cup \Omega\) by adding \(\Omega\) on both sides, i.e., \(B^c(x_0; \epsilon)\) is nonstandardly type-III-open. Therefore, by Proposition \(\text{7.3 (ii)}\), \(B(x_0; \epsilon)\) is a nonstandardly closed subset of \(X\).

To prove part (ii), from the above arguments, since \(d(\hat{x} + \omega_0, x_0) \geq d(\hat{x}, x_0) > \epsilon\), we have \(\hat{x} + \omega_0 \in B^c(x_0; \epsilon)\), i.e., \(B(x_0; \epsilon) \cup \Omega \subseteq B^c(x_0; \epsilon)\). This shows that \(B^c(x_0; \epsilon)\) is nonstandardly type-I-open. Therefore, by Proposition \(\text{7.3 (i)}\), \(B(x_0; \epsilon)\) is a nonstandardly type-I-closed set.

To prove part (iii), for \(\hat{x} \in B^c(x_0; \epsilon)\), since \(\hat{x}\) has the null decomposition, i.e., \(\hat{x} = x_1 + \omega_1\) for some \(\omega_1 \in \Omega\), we have \(d(x_1, x_0) = d(x_1 + \omega_1, x_0) = d(\hat{x}, x_0) > \epsilon\), which says that \(x_1 \in B^c(x_0; \epsilon)\). Therefore, we have \(\hat{x} \in B^c(x_0; \epsilon) \cup \Omega\), i.e., \(B^c(x_0; \epsilon) \cup \Omega\). Since \(B(x, \hat{\epsilon} - \epsilon) \subseteq B^c(x_0; \epsilon) \cup \Omega\) by the above arguments, we also have \(B(x, \hat{\epsilon} - \epsilon) \subseteq B^c(x_0; \epsilon) \cup \Omega\). This shows that \(B^c(x_0; \epsilon)\) is nonstandardly type-II-open. By Proposition \(\text{7.3 (ii)}\), \(B(x_0; \epsilon)\) is a nonstandardly type-II-closed set. This completes the proof.

### 8 Topological Spaces

Now, we are in a position to investigate the topological structure generated by the pseudo-metric space \((X, d)\) on a nonstandard vector space \(X\). In this case, we can obtain a nonstandard topological vector space \(X\).
Theorem 8.1. Let \((X, d)\) be a pseudo-metric space on a nonstandard vector space \(X\). We denote by \(\tau^{(i)}\) the set of all nonstandardly type-\(I\)-open subsets of \(X\). Then \((X, \tau^{(i)})\) is a topological space.

Proof. By part (b) of Remark 6.5, we see that \(\emptyset \in \tau^{(i)}\) and \(X \in \tau^{(i)}\). Let \(A = \bigcap_{i=1}^{n} A_i\), where \(A_i\) are nonstandardly type-\(I\)-open sets for all \(i = 1, \ldots, n\). For \(x \in A\), we have \(x \in A_i\) for all \(i = 1, \ldots, n\). Then there exist \(\epsilon_i\) such that \(B(x; \epsilon_i) \cup \Omega \subseteq A_i\) for all \(i = 1, \ldots, n\). Let \(\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}\). Then \(B(x; \epsilon) \cup \Omega \subseteq \bigcap_{i=1}^{n} A_i = A\). Therefore, the union \(A\) is nonstandardly type-\(I\)-open. On the other hand, we let \(\tilde{A} = \bigcup_{\delta} A_{\delta}\). Then \(x \in \tilde{A}\) implies that \(x \in A_{\delta}\) for some \(\delta\). This says that \(B(x; \epsilon) \cup \Omega \subseteq A_{\delta} \subseteq A\) for some \(\epsilon > 0\). Therefore, the union \(A\) is nonstandardly type-\(I\)-open. □

Theorem 8.2. Let \((X, d)\) be a pseudo-metric space on a nonstandard vector space \(X\). We denote by \(\tau_0\) the set of all nonstandardly open subsets of \(X\). Then \((X, \tau_0)\) is a topological space.

Proof. The empty set \(\emptyset\) and \(X\) are nonstandardly open by part (a) of Remark 6.5. The remaining proof follows from the arguments of Theorem 8.1 without considering the null set \(\Omega\). □

Let \(\tau^{(i)}\) and \(\tau^{(ii)}\) be the families of nonstandard type-\(II\) and type-\(III\) open subsets of \(X\), respectively. Then \(\tau^{(i)}\) and \(\tau^{(ii)}\) cannot be the topologies. The main reason is that the equality

\[
[(A_1 \oplus \Omega) \cap (A_2 \oplus \Omega)] = (A_1 \cap A_2) \oplus \Omega
\]

cannot hold true in general for any subsets \(A_1\) and \(A_2\) of \(X\). However, we still have some related results that will be discussed below.

Lemma 8.1. Let \(X\) be a nonstandard vector space over \(\mathbb{F}\), and let \(A_1\) and \(A_2\) be subsets of \(X\). Then we have

\[
(A_1 \cap A_2) \oplus \Omega \subseteq [(A_1 \oplus \Omega) \cap (A_2 \oplus \Omega)].
\]

If we further assume that the null set \(\Omega\) owns the self-decomposition, and any one of the following conditions is satisfied:

- \(A_2 \oplus \Omega \subseteq A_2\) and, for any \(\omega \in \Omega\), \(a \oplus \omega \in A_1 \oplus \omega\) implies \(a \in A_1\).
- \(A_1 \oplus \Omega \subseteq A_1\) and, for any \(\omega \in \Omega\), \(a \oplus \omega \in A_2 \oplus \omega\) implies \(a \in A_2\).

Then we have

\[
[(A_1 \oplus \Omega) \cap (A_2 \oplus \Omega)] = (A_1 \cap A_2) \oplus \Omega.
\]

Proof. For \(y \in (A_1 \cap A_2) \oplus \Omega\), we have \(y = a \oplus \omega\) with \(a \in A_i\) for \(i = 1, 2\) and \(\omega \in \Omega\), which also says that \(y \in [(A_1 \oplus \Omega) \cap (A_2 \oplus \Omega)]\), i.e., \((A_1 \cap A_2) \oplus \Omega \subseteq [(A_1 \oplus \Omega) \cap (A_2 \oplus \Omega)]\). Conversely, we assume that the first condition is satisfied. Let \(x \in (A_1 \oplus \Omega) \cap (A_2 \oplus \Omega)\). Then \(x = a_1 \oplus \omega_1 = a_2 \oplus \omega_2\) for some \(a_1 \in A_1\), \(a_2 \in A_2\) and \(\omega_1, \omega_2 \in \Omega\). Since \(\Omega\) owns the self-decomposition, we have \(\omega_2 = \hat{\omega}_2 \oplus \omega_1\) for some \(\hat{\omega}_2 \in \Omega\). If we write \(\hat{a}_2 = a_2 \oplus \hat{\omega}_2 \in A_2 \oplus \hat{\Omega} \subseteq A_2\), then we have \(\hat{a}_2 \in A_2\). In this case, we have

\[
\hat{a}_2 \oplus \omega_1 = a_2 \oplus \hat{\omega}_2 \oplus \omega_1 = a_2 \oplus \omega_2 = a_1 \oplus \omega_1 \in A_1 \oplus \omega_1,
\]

which shows that \(\hat{a}_2 \in A_1\). This says that \(\hat{a}_2 \in A_1 \cap A_2\), i.e., \(x = \hat{a}_2 \oplus \omega_1 \in (A_1 \cap A_2) \oplus \Omega\).

Now suppose that the second condition is satisfied. Since \(\Omega\) owns the self-decomposition, we have \(\omega_1 = \hat{\omega}_1 \oplus \omega_2\) for some \(\hat{\omega}_1 \in \Omega\). If we write \(\hat{a}_1 = a_1 \oplus \hat{\omega}_1 \in A_1 \oplus \Omega \subseteq A_1\), then we have \(\hat{a}_1 \in A_1\). In this case, we have

\[
\hat{a}_1 \oplus \omega_2 = a_1 \oplus \hat{\omega}_1 \oplus \omega_2 = a_1 \oplus \omega_1 = a_2 \oplus \omega_2 \in A_2 \oplus \omega_2,
\]

which shows that \(\hat{a}_1 \in A_2\). This says that \(\hat{a}_1 \in A_1 \cap A_2\), i.e., \(x = \hat{a}_1 \oplus \omega_2 \in (A_1 \cap A_2) \oplus \Omega\). This completes the proof. □
We denote by $\bar{\tau}^{(ii)}$ the set of all nonstandardly type-II-open subsets of $X$ such that $\emptyset \in \bar{\tau}^{(ii)}$ and, for each $\emptyset \neq A \in \bar{\tau}^{(ii)}$, the following condition is satisfied:

$$A \oplus \Omega \subseteq A \text{ and, for any } \omega \in \Omega, a \oplus \omega \in A \oplus \omega \text{ implies } a \in A.$$  

We also denote by $\bar{\tau}^{(iii)}$ the set of all nonstandardly type-III-pseudo-open subsets of $X$ such that $\emptyset \in \bar{\tau}^{(iii)}$ and, for each $\emptyset \neq A \in \bar{\tau}^{(iii)}$, the above condition $\text{(18)}$ is satisfied. Of course, we see that $\bar{\tau}^{(ii)} \subseteq \tau^{(ii)}$ and $\bar{\tau}^{(iii)} \subseteq \tau^{(iii)}$.

**Proposition 8.1.** Let $(X, d)$ be a pseudo-metric space on a nonstandard vector space $X$ such that the pseudo-metric $d$ satisfies the null inequalities, and let $B(x; \epsilon)$ be any open ball centered at $x$ with radius $\epsilon$. Then, the following statements hold true.

(i) Suppose that $X$ owns the null decomposition and the pseudo-metric $d$ satisfies the null equalities. Then $B(x_0; \epsilon) \in \bar{\tau}^{(ii)}$.

(ii) Suppose that the pseudo-metric $d$ satisfies the null equalities. Then $B(x_0; \epsilon) \in \bar{\tau}^{(iii)}$.

**Proof.** To prove part (i), from Proposition 6.6 (iii), we remain to show that $B(x; \epsilon)$ satisfies condition $\text{(18)}$. From Proposition 5.1 we immediately have $B(x; \epsilon) \oplus \Omega \subseteq B(x; \epsilon)$. Now, for $a \oplus \omega \in B(x; \epsilon) \oplus \omega$, we have $a \oplus \omega = \hat{a} \oplus \omega$ for some $\hat{a} \in B(x; \epsilon)$, i.e., $d(\hat{a}, x) < \epsilon$. Since $d$ satisfies the null equalities, we have

$$d(a, x) = d(a \oplus \omega, x) = d(\hat{a} \oplus \omega, x) = d(\hat{a}, x) < \epsilon,$$

which says that $a \in B(x; \epsilon)$. Therefore, we obtain the desired result.

To prove part (ii), from Proposition 6.6 (i), we remain to show that $B(x; \epsilon)$ satisfies condition $\text{(18)}$. The arguments of (i) are also valid to show that $B(x; \epsilon)$ satisfies condition $\text{(18)}$. We complete the proof. 

**Theorem 8.3.** Let $(X, d)$ be a pseudo-metric space on a nonstandard vector space $X$. Then $(X, \bar{\tau}^{(ii)})$ is a topological space.

**Proof.** Given $A_1, A_2 \in \bar{\tau}^{(ii)}$, we let $A = A_1 \cap A_2$. For $x \in A$, we have $x \in A_i$ for $i = 1, 2$. Then there exist $\epsilon_i$ such that $B(x; \epsilon_i) \subseteq A_i \oplus \Omega$ for all $i = 1, 2$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then $B(x; \epsilon) \subseteq B(x; \epsilon_i) \subseteq A_i \oplus \Omega$ for all $i = 1, 2$, which says that $B(x; \epsilon) \subseteq (A_1 \oplus \Omega) \cap (A_2 \oplus \Omega) = (A_1 \cap A_2) \oplus \Omega = A \oplus \Omega$ by Lemma 5.1. This shows that $A$ is nonstandardly type-II-open. We also need to show that $A$ satisfies condition $\text{(18)}$. For $x \in A \oplus \Omega$, we have $x = a \oplus \omega$ for some $a \in A$ and $\omega \in \Omega$. Since $a \in A_1 \cap A_2$, it also says that $x \in A_1 \oplus \Omega \subseteq A_1$ and $x \in A_2 \oplus \Omega \subseteq A_2$. Therefore, we have $x \in A_1 \cap A_2 = A_i$, i.e., $A \oplus \Omega \subseteq A$. On the other hand, $a \oplus \omega \in A \oplus \omega = (A_1 \cap A_2) \oplus \omega \subseteq A_1 \oplus \omega$, which says that $a \in A_1$. We can similarly have $a \oplus \omega \subseteq A_2 \oplus \omega$, which also implies $a \in A_2$. Therefore, we have $a \in A_1 \cap A_2 = A$. This says that $A$ is indeed in $\bar{\tau}^{(ii)}$. Therefore, the intersection of finitely many members of $\bar{\tau}^{(ii)}$ is a member of $\bar{\tau}^{(ii)}$. Now given $\{A_i\} \subseteq \bar{\tau}^{(ii)}$, we let $A = \bigcup A_i$. Then $x \in A$ means that $x \in A_i$ for some $i$. This says that $B(x; \epsilon) \subseteq A_i \oplus \Omega \subseteq A \oplus \Omega$ for some $\epsilon > 0$. Therefore, the union $A$ is nonstandardly type-II-open. We also need to show that $A$ satisfies condition $\text{(18)}$. For $x \in A \oplus \Omega$, we have $x = a \oplus \omega$, where $a \in A$, i.e., $a \in A_\delta$ for some $\delta$. It also says that $x \in A_\delta \oplus \Omega \subseteq A \oplus \Omega \subseteq A$. On the other hand, for $a \oplus \omega \in A \oplus \omega$, we have $a \oplus \omega = \hat{a} \oplus \omega$ for some $\hat{a} \in A$, i.e., $\hat{a} \in A_\delta$ for some $\delta$, which also implies $a \oplus \omega = \hat{a} \oplus \omega \in A_\delta \oplus \omega$. Therefore, we obtain $a \in A_\delta \subseteq A$. This shows that $A$ is indeed in $\bar{\tau}^{(ii)}$. By part (c) of Remark 6.3 we see that $\emptyset$ and $X$ are also nonstandardly type-II open subsets of $X$. It is not hard to see that $X$ satisfies condition $\text{(18)}$. This shows that $X \in \bar{\tau}^{(ii)}$. We complete the proof. 

**Theorem 8.4.** Let $(X, d)$ be a pseudo-metric space on a nonstandard vector space $X$. Then $(X, \bar{\tau}^{(ii)})$ is a topological space.
Proof. By part (d) of Remark 6.3 we see that $\emptyset, X \in \tau^{(1)}$, since $X$ satisfies condition (18). Given $A_1, A_2 \in \tau^{(1)}$, we let $A = A_1 \cap A_2$. For $x \in A$, there exist $\epsilon_i$ such that $B(x; \epsilon_i) \subseteq A_i \oplus \Omega$ for all $i = 1, 2$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then $B(x; \epsilon) \subseteq B(x; \epsilon_i) \subseteq A_i \oplus \Omega$ for all $i = 1, 2$, which says that $B(x; \epsilon) \subseteq [(A_1 \oplus \Omega) \cap (A_2 \oplus \Omega)] = (A_1 \cap A_2) \oplus \Omega = A \oplus \Omega$ by Lemma 8.2. This shows that $A$ is nonstandardly type-III open. From the arguments of Theorem 8.3 we see that $A$ satisfies condition (18). Therefore, the intersection of finitely many members of $\tau^{(1)}$ is a member of $\tau^{(1)}$. Now, given $\{A_3\} \subseteq \tau^{(1)}$, we let $A = \bigcup_i A_3$. Then $x \in A$ implies that $x \in A_3$ for some $\delta$. This says that $B(x; \epsilon) \subseteq A_3 \oplus \Omega \subseteq A \oplus \Omega$ for some $\epsilon > 0$. Therefore, the union $A$ is nonstandardly type-III open. From the arguments of Theorem 8.3 we also see that $A$ satisfies condition (18). We complete the proof.

Remark 8.1. Let $(X, d)$ be a pseudo-metric space on a nonstandard vector space $X$ such that the pseudo-metric $d$ satisfies the null equalities and $\Omega$ owns the self-decomposition. Then we have the following observations.

- From part (ii) of Proposition 6.3 we see that if $A$ is $\tau_0$-open, then $A$ is also $\tau^{(1)}$-open, and if $A$ is $\tau^{(1)}$-open, then $A$ is also $\tau^{(1)}$-open. This says that $\tau^{(1)}$ is finer than $\tau_0$ and $\tau^{(1)}$ is finer than $\tau^{(1)}$.

- Suppose that $X$ owns the null decomposition. Then part (vii) of Proposition 6.3 says that $\tau_0 = \tau^{(1)}$, $\tau^{(1)} = \tau^{(1)}$, and $\tau^{(1)} = \tau^{(1)}$.

Let $(X, d)$ be a pseudo-metric space on a nonstandard vector space $X$ such that the pseudo-metric $d$ satisfies the null equalities. We denote by $p\tau^{(1)}$ the set of all nonstandardly type-II pseudo-open subsets of $X$ and by $p\tau^{(1)}$ the set of all nonstandardly type-III pseudo-open subsets of $X$.

Lemma 8.2. Let $(X, d)$ be a pseudo-metric space on a nonstandard vector space $X$ such that the pseudo-metric $d$ satisfies the null equalities.

(i) Suppose that $A_1, A_2 \in p\tau^{(1)}$. Then, we have

$$(A_1 \cap A_2) \oplus \Omega \subseteq [(A_1 \oplus \Omega) \cap (A_2 \oplus \Omega)].$$

If we further assume that $\Omega$ owns the self-decomposition and the pseudo-metric $d$ satisfies the null equalities, then

$$[(A_1 \oplus \Omega) \cap (A_2 \oplus \Omega)] = (A_1 \cap A_2) \oplus \Omega.$$

(ii) Suppose that $A_1, A_2 \in p\tau^{(1)}$. Then we have the same results as given in (i).

Proof. The results follow immediately from Proposition 6.1 and part (ii) of Proposition 6.1.

Theorem 8.5. Let $(X, d)$ be a pseudo-metric space on a nonstandard vector space $X$ such that the pseudo-metric $d$ satisfies the null equalities and $\Omega$ owns the self-decomposition. Then $(X, p\tau^{(1)})$ is a topological space.

Proof. Given $A_1, A_2 \in p\tau^{(1)}$, we let $A = A_1 \cap A_2$. We want to show $A = \text{pint}^{(1)}(A)$. For $x \in A$, we have $x \in A_i$ for $i = 1, 2$. Then there exist $\epsilon_i$ such that $B(x; \epsilon_i) \subseteq A_i \oplus \Omega$ for all $i = 1, 2$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then $B(x; \epsilon) \subseteq B(x; \epsilon_i) \subseteq A_i \oplus \Omega$ for all $i = 1, 2$, which says that $B(x; \epsilon) \subseteq [(A_1 \oplus \Omega) \cap (A_2 \oplus \Omega)] = (A_1 \cap A_2) \oplus \Omega = A \oplus \Omega$ by Lemma 8.2. This shows that $x \in \text{pint}^{(1)}(A)$, i.e., $A \subseteq \text{pint}^{(1)}(A)$. Conversely, for $x \in \text{pint}^{(1)}(A)$, by part (ii) of Proposition 6.1 we have

$$x \in B(x; \epsilon) \subseteq A \oplus \Omega = (A_1 \ominus A_2) \oplus \Omega \subseteq A_1 \oplus \Omega \subseteq A_1.$$

We can similarly obtain $x \in A_2$, i.e., $x \in A_1 \cap A_2 = A$. This shows that $\text{pint}^{(1)}(A) \subseteq A$. Therefore, the intersection of finitely many members of $p\tau^{(1)}$ is a member of $p\tau^{(1)}$. Now, given $\{A_3\} \subseteq p\tau^{(1)}$, we let
$A = \bigcup \delta A_3$. Then $x \in A$ implies that $x \in A_\delta$ for some $\delta$. This says that $B(x; \epsilon) \subseteq A_\delta \oplus \Omega \subseteq A \oplus \Omega$ for some $\epsilon > 0$. Therefore, we obtain $A \subseteq \text{pint}^{(\Pi)}(A)$. Conversely, for $x \in \text{pint}^{(\Pi)}(A)$, we have $x \in B(x; \epsilon) \subseteq A \oplus \Omega$, i.e., $x = a \oplus \omega$ for some $a \in A$ and $\omega \in \Omega$. Then we have $a \in A_\delta$ for some $\delta$, i.e., $x = a \oplus \omega \in A_\delta \oplus \Omega \subseteq A_\delta \subseteq A$ by part (ii) of Proposition 6.1. This shows that $\text{pint}^{(\Pi)}(A) \subseteq A$, i.e., the union $A$ is indeed in $\text{pint}^{(\Pi)}(A)$. By part (c) of Remark 6.5 we also see that $\emptyset, X, A \in \text{p}^{(\Pi)}$. This completes the proof. 

**Theorem 8.6.** Let $(X, d)$ be a pseudo-metric space on a nonstandard vector space $X$ such that the pseudo-metric $d$ satisfies the null inequalities, $X$ owns the null decomposition and $\Omega$ owns the self-decomposition. Then $(X, \text{p}^{(\Pi)})$ is a topological space.

**Proof.** By part (d) of Remark 6.5 we also see that $\emptyset, X, A \in \text{p}^{(\Pi)}$. Given $A_1, A_2 \in \text{p}^{(\Pi)}$, we let $A = A_1 \cap A_2$. For $x \in A$, there exist $\epsilon_i$ such that $B(x; \epsilon_i) \oplus \Omega \subseteq A_i \oplus \Omega$ for all $i = 1, 2$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then $B(x; \epsilon) \oplus \Omega \subseteq B(x; \epsilon_i) \oplus \Omega \subseteq A_i \oplus \Omega$ for all $i = 1, 2$, which says that $B(x; \epsilon) \oplus \Omega \subseteq [(A_1 \oplus \Omega) \cap (A_2 \oplus \Omega)] = (A_1 \cap A_2) \oplus \Omega = A \oplus \Omega$ by Lemma 8.2. This shows that $x \in \text{p}^{(\Pi)}$, i.e., $A \subseteq \text{p}^{(\Pi)}$. Conversely, for $x \in \text{pint}^{(\Pi)}(A)$, by Proposition 5.3 and part (ii) of Proposition 6.1 we have

$$x \in B(x; \epsilon) \subseteq B(x; \epsilon) \oplus \Omega \subseteq A \oplus \Omega = (A_1 \cap A_2) \oplus \Omega \subseteq A_1 \oplus \Omega \subseteq A_1.$$ 

We can similarly obtain $x \in A_2$, i.e., $x \in A_1 \cap A_2 = A$. This shows that $\text{pint}^{(\Pi)}(A) \subseteq A$. Therefore, the intersection of finitely many members of $\text{p}^{(\Pi)}$ is a member of $\text{p}^{(\Pi)}$. Now, given $\{A_\delta\} \subseteq \text{p}^{(\Pi)}$, we let $A = \bigcup \delta A_\delta$. Then $x \in A$ implies that $x \in A_\delta$ for some $\delta$. This says that $B(x; \epsilon) \oplus \Omega \subseteq A_\delta \oplus \Omega \subseteq A \oplus \Omega$ for some $\epsilon > 0$. Therefore, we obtain $A \subseteq \text{pint}^{(\Pi)}(A)$. Conversely, for $x \in \text{pint}^{(\Pi)}(A)$, we have $x \in B(x; \epsilon) \subseteq B(x; \epsilon) \oplus \Omega \subseteq A \oplus \Omega$, i.e., $x = a \oplus \omega$ for some $a \in A$ and $\omega \in \Omega$. Then we have $a \in A_\delta$ for some $\delta$, i.e., $x = a \oplus \omega \in A_\delta \oplus \Omega \subseteq A_\delta \subseteq A$ by part (ii) of Proposition 6.1. This shows that $\text{pint}^{(\Pi)}(A) \subseteq A$, i.e., the union $A$ is indeed in $\text{pint}^{(\Pi)}(A)$. We complete the proof.

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