Introducing Quarks in Confining Strings via the Fermionic Wilson Loop

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Abstract

In the world line representation of the fermionic effective action for QCD the interaction between Fermions and the gauge field is contained in the fermionic Wilson loop, namely the Wilson loop for a spin-half particle. It is argued that a string representation of the fermionic Wilson loop can provide a link connecting QCD with a dual description of a meson as a quark and an antiquark connected by a string. This is illustrated by obtaining such a representation in compact $U(1)$ gauge theories. The resulting description contains information about the interaction of the spin of the quark with the world sheet degrees of freedom. Such interactions may be of importance in the realization of chiral symmetry in the string picture of QCD, and for delineating the possible presence of a world sheet supersymmetry in QCD strings.

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1 Introduction

A very intuitive explanation of the observed confinement of quarks in hadrons is provided by the idea that hadrons are made of quarks which are connected by strings [1, 2]. Two approximations to QCD that suggest such a string picture of hadrons are the large $N$ expansion [3, 4], where $N$ is the number of colors carried by the quarks, and the strong coupling expansion in the lattice regularization of QCD [5, 6, 7]. If we combine the insight from these two approximations then a very attractive and a plausible model of a meson as a quark and an antiquark connected by an elementary string is suggested. Such a description of QCD seems to imply that strings with quarks as endpoints are the dual variables to gluons and quarks. Assuming that such a duality does exist then the non-perturbative phenomena of QCD, like confinement and chiral symmetry breaking, should map into simple, calculable, perturbative phenomena of the dual theory. On the same account, phenomena that have a simple perturbative description in QCD, like deep inelastic scattering and the asymptotic freedom, could appear as non-perturbative phenomena in the dual description. Thus, one can hope that QCD together with its dual description in terms of strings will allows us to obtain a qualitative and, an approximate, quantitative description of all the features of strong interactions. Unfortunately, a quantitative transcription of this desired duality has remained elusive. The problems involved have been reviewed by Polchinski [10], and their possible resolution in view of the recent theoretical developments have been summarized by Polyakov [11, 12, 13].

Most investigations of a string description of QCD do not include the quark degrees of freedom. There are, of course, very good reasons for this neglect of fermionic degrees of freedom: We are far from obtaining a string representation for pure QCD and the addition of fermionic degrees of freedom may make an already intractable problem even more difficult. Also, one of the salient features of QCD which is captured by the strings, namely, confinement, can be studied without introducing any fermionic degrees of freedom via the string representation of Wilson loop [5].

Yet, as has been emphasized in ref. [14], some of the most detailed evidence we have for the string picture of hadrons comes from combining the ideas of dual resonance models with chiral symmetry [14, 15] and one cannot talk meaningfully about chiral symmetry without introducing quark degrees of freedom. Thus, it is a possibility that the inclusion of fermionic degrees of freedom may facilitate rather than hinder the task of obtaining a string
description of hadrons. More specifically, useful clues in developing a string representation for mesons might be obtained by considering the nature of pion and rho [10]. One would like to known, how does a string representation of mesons distinguish between a pion, which is a pseudo-scalar, approximate Nambu-Goldstone boson, and a rho which is a massive spin one particle [10, 17].

The simplest implementation of the string picture of mesons would be to consider models of strings whose end points carry spin and flavor quantum numbers of a quark. While it is known how to assign flavor quantum number to the end point of a string via Chan-Paton factors, there does not seem to be any simple manner in which spin degrees of freedom can be assigned to the end-points of an open string [18, 19]. Alternatively, one could explore a model in which quarks are described as a point particle on which an open string terminates. This description essentially appears in the strong coupling expansion, whether one calculates the expectation value of a Wilson loop or when one tries to diagonalize the lattice QCD Hamiltonian in the strong coupling approximation [8, 9]. The Wilson loop can be interpreted as the phase factor associated with the closed world line of a scalar test particle in the presence of an external gauge field [5, 20]. This suggests that one way of introducing the quark degrees of freedom in the string description of QCD is to look for a string representation of the fermionic Wilson loop (FWL), namely the phase factor associated with a closed world line of a spin-half particle in the presence of an external gauge field.

One can construct a fermionic Wilson loop starting directly with the Lagrangian of a spin-half particle [21, 22, 23, 24], an application of such an approach for studying studying chiral anomalies can be found in [27, 28]. This approach has many advantages, the symmetries are transparent and one can introduce flavor quantum numbers of the spin-half particle under consideration. The FWL also appears naturally in the world line formalism of the fermionic Effective action [29, 30], and it is this approach that will be used in the present preliminary investigation. The advantage of the latter approach is, as we will see in the next section, that it allows us to express, in the large N-limit, all the mesonic Green’s functions in terms of the expectation value of the FWL. From the world line point of view this approach gives us directly the appropriate vertex function for a pion or a rho.

Independent of our motivation of introducing quark degrees of freedom in the string description of meson, it has been suggested in ref. [11] that even for obtaining the string representation of pure QCD the appropriate object
is not the Wilson loop but the fermionic Wilson loop\(^1\).

Writing the mesonic propagators in terms of FWL leads us to the heart of the problem: How to convert the expectation value of a FWL over the gauge fields into a sum over surfaces whose boundary is the loop under consideration. In spite of much progress made in the case of supersymmetric Yang-Mill theories [32, 33], this problem remain unsolved for QCD. Still one can get some useful intuition by considering a string representation of theories where the mechanism of confinement is well understood. In this paper we will use Polyakov’s analysis of the string representation of Wilson loops in compact \(U(1)\) gauge theory as our starting point [26]. The string representation of Wilson loops in compact \(U(1)\) gauge theories can be be extended to the FWL, and as discussed in section 3, it does lead to a representation for “mesons” in which the string ends on the word line of quarks. Further, we will see in such a representation there is an interaction between the spin degrees of freedom and the string world sheet mediated by a Kalb-Ramond field [34].

In the last section of the paper we will discuss what can be abstracted from the present analysis of FWL in compact \(U(1)\) gauge theories to the case of interest which is QCD in the large \(N\)-limit. We will also outline the course of a future investigation of the question as to how the chiral symmetry is realized in a string representation and of the possibly related question of the existence of some form of world sheet supersymmetry for QCD strings [11, 35].

### 2 Mesons and the Fermionic Wilson Loop

Consider the Euclidean partition function of an \(SU(N)\) gauge theory in the presence of an external source with the quantum numbers of a meson, say, a pion, and with quarks in the fundamental representation,

\[
Z[J] = \int D\psi D\bar{\psi} DA \exp \left\{ - \int_x \left( \frac{1}{4g^2} \text{Tr} F_{\mu\nu} F_{\mu\nu} + \bar{\psi} (P - A - J\gamma_5)\psi \right) \right\}
\]  

\(F_{\mu\nu}\) is the field strength tensor for the \(SU(N)\) gauge theory, \(\psi\) is the quark field, with flavor and color indices suppressed, \(P = -i\partial\), \(A\) is the matrix valued gauge field, \(J\) is an external classical source, and the Feynman slash

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\(^1\)In refs. [11, 29] the fermionic Wilson loop is referred to as the SuperWilson loop, we have chosen to call it the fermionic Wilson loop in order to distinguish it from the generalization of the Wilson loop to supersymmetric gauge theories [31].
notation has been used. Fermionic degrees of freedom can be formally integrated to give a functional integral over gauge fields,

$$Z[J] = \int DA \exp \left\{ -\int_x \frac{1}{4g^2} \text{Tr} F_{\mu\nu} F^{\mu\nu} \right\} \exp \{-W[A, J]\},$$  \hspace{1cm} (2)$$

$$-W[A, J] = \text{Tr} \ln[P - A - J\gamma_5].$$  \hspace{1cm} (3)

Thus $Z[J]$ can be written as a functional integral over gauge fields alone

$$Z[J] = Z[0] < \exp\{-W[A, J]\}>_A,$$  \hspace{1cm} (4)$$

$$< \exp\{-W[A, J]\}>_A = \frac{1}{Z[0]} \int DA \exp \left\{ -\int_x \frac{1}{4g^2} \text{Tr} F_{\mu\nu} F^{\mu\nu} \right\} \times \exp\{-W[A, J]\}.$$  \hspace{1cm} (5)

We remind ourselves of an important simplification that takes place in the large $N$ limit \[36, 37\], namely

$$< \exp\{-W[A, J]\}>_A = \exp\{-W[A^m, J]\}$$  \hspace{1cm} (6)

where $A^m$ is the so-called “master field”. Thus one can write the pion propagator as

$$\Delta_\pi(x - y) = \left( \frac{\partial}{\partial J(x)} \frac{\partial}{\partial J(y)} \right)_{J=0} \ln Z[J]$$

$$= \left( \frac{\partial}{\partial J(x)} \frac{\partial}{\partial J(y)} \right)_{J=0} (-W[A^m, J]).$$  \hspace{1cm} (7)

This is of course the restatement of the familiar property of the large $N$ QCD \[4\], that the dominant diagrams are those with only a single quark loop which forms the boundary of the diagram.

The fermionic effective action, $W[A, J]$, has a natural representation in terms of a world line path-integral of a Fermion\[2\]. In what follows we will make use of the world line representation for $W[J, A]$ as developed by D’Hoker and Gagné \[30\]. First we note that one loop diagrams involving even power of $\gamma_5$ are generated by the real part of the fermionic effective action while the diagrams involving the odd powers of $\gamma_5$ are generated by the imaginary part

\[2\] The world line formalism has been reviewed in \[38\], where additional references can be found (see also \[31\] and references there in).
of the fermionic effective action \[39\]. Thus it is both natural and convenient to consider separately the world line representation of the real and the imaginary part of \(W[A, J]\). The amplitude in which we are interested, namely the pion propagator, involves two insertions of \(\gamma_5\), and thus is generated by the real part of the fermionic effective action which has the following world line representation

\[
W_R[A, J] = \frac{1}{4} \int_0^\infty \frac{dT}{T} \mathcal{N} \int_{PBC} DX \int_{APBC} D\psi \text{Tr} \mathcal{P} \\
\times \exp \left\{ - \int_0^T d\tau \left( \dot{x}^2 + \frac{1}{2} \psi_A \dot{\psi}_A + i \mathcal{E} \psi_\mu \psi_5 \partial_\mu J + \frac{1}{2} \mathcal{E} J^2 \right) \right\} \\
\times W[A; C],
\]

where the fermionic Wilson loop, \(W[A; C]\), is given by

\[
W[A; C] = \exp \left\{ i \int_0^T d\tau \left( \dot{x} \cdot A - \frac{1}{2} \mathcal{E} \psi_\mu F_{\mu\nu} \psi^\nu \right) \right\},
\]

with \(\tau\) parametrizing the closed curve \(C\), \(\mathcal{E}\) is positive constant which can be though of as a constant einbein in the proper time gauge. In what follows we will work in the proper time gauge and scale the parameter \(\tau\) so that \(\mathcal{E}\) equals to identity. In Eq. (8) the trace over the gamma matrices has been replaced by path integral over anti-commuting variables, \(\psi_A\), and \(\text{Tr}_c\) is the trace over color degrees of freedom. The index \(A\) run over values one to six. When \(A\) takes values from 1 to 4 then it is denoted by \(\psi_\mu\). The world line anti-commutating variables \(\psi_A\) are related to the six gamma matrices introduced by Mehta [40]. For a detailed derivation of the path-integral representation of the fermionic effective action, Eq. (8), the reader is referred to ref. [30].

The interaction between spin-half particle and the gauge field \(A_\mu\) is contained in the FWL, Eq. (9). The FWL is distinguished from the Wilson loop,

\[
W[A; C] = \exp \left\{ i \oint_C d\tau \dot{x} \cdot A \right\},
\]

by the presence of the additional term \(\psi_\mu F_{\mu\nu} \psi^\nu\) and is related to it via the area derivative [29, 37],

\[
\exp \left\{ - \frac{i}{2} \oint d\tau \psi_\mu(\tau) \psi_\nu(\tau) \frac{\delta}{\delta \sigma_{\mu\nu}} \right\} W[C] = W[C].
\]
From the above discussion we see that to obtain the desired string representation for mesons we need to express the expectation value of the FWL over the gauge fields as a sum over surfaces whose boundary is the loop \( C \).

It is interesting to note that FWL can be written in terms of a superfield \([23, 28, 29]\)

\[
X_\mu(\tau, \theta) = x_\mu(\tau) + \theta \psi_\mu(\tau),
\]

where \( \theta \) is the anti-commuting coordinate which together with \( \tau \) parametrizes superspace. Using the covariant derivative on superspace,

\[
D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial \tau},
\]

one can write FWL as

\[
\mathcal{W}[C] = \exp \left\{ \int d\tau d\theta DX_\mu(\tau, \theta) A_\mu[X] \right\}
\]

where the gauge field is function of the superfield,

\[
A_\mu[X] = A_\mu(x_\mu(\tau) + \theta \psi_\mu(\tau)) = A_\mu[x(\tau)] + \theta \psi_\mu(\tau) \partial_\nu A_\mu[x(\tau)].
\]

The FWL is invariant under following local supersymmetric transformation

\[
\delta x_\mu = \epsilon(\tau) \psi_\mu,
\]

\[
\delta \psi_\mu = -\epsilon(\tau) \dot{x}_\mu,
\]

where \( \epsilon \) is an arbitrary infinitesimal Grassmann variable. The reparametrization invariance and the supersymmetry of the FWL have been studied in ref. \[29\]. These symmetries hint that corresponding string representation for the expectation value of the FWL may have a world sheet supersymmetry, and the world sheet supersymmetry should reduce to the world line supersymmetry on the boundary. We will comment further on this possibility in the last section of the paper.

3 String Representation of Fermionic Wilson Loop in Compact \( U(1) \) Gauge Theories

In order to understand what additional information is contained in a string representation of the FWL, but not knowing the string representation for
QCD, we look for a tractable mode as an alternative. Compact $U(1)$ in four dimensions, with an ultra-violet cutoff, is a confining theory \cite{7, 41}. A string representation for this model was derived in ref. \cite{26}, and thanks to the simple relation between the Wilson loop and the FWL, Eq. (11), we will be able to use this result to obtain a string representation for the FWL.

We first summarize the results of ref. \cite{26}. The expectation value of the Wilson loop, $<W[A; C]>_A$, in a compact $U(1)$ gauge theory can be written as a sum over surfaces, $\Sigma_c$, binding the curve $C$,

$$<W[A; C]>_A = \sum_{\Sigma_c} \exp \{-S_{cs}[\Sigma_c]\}. \quad (17)$$

The confining string action, $S_{cs}[\Sigma_c]$, is given in terms of Kalb-Ramond field, $B_{\mu\nu}$, and can be written for large Wilson loops as

$$\exp \{-S_{cs}[\Sigma_c]\} = \frac{1}{Z} \int DB_{\mu\nu} \exp \left\{-S_{KR}[B] + i \int_{\Sigma_c} d\sigma_{\mu\nu} B_{\mu\nu}\right\}, \quad (18)$$

$$S_{KR}[B] = \int_x \left\{ \frac{1}{4e^2} B_{\mu\nu}^2 + f(dB) \right\}, \quad (19)$$

$$f(dB) = \frac{1}{4\Lambda^2} (\partial_\mu B_{\nu\alpha} + \partial_\nu B_{\alpha\mu} + \partial_\alpha B_{\mu\nu})^2. \quad (20)$$

The antisymmetric tensor field $B_{\mu\nu}$ describes both the “photons” and the magnetic monopoles which are present in compact $U(1)$ gauge theories, $\Lambda$ is an ultra-violet cutoff. The continuum limit of the confining string has been investigated in ref. \cite{12, 13}.

Since the field $B_{\mu\nu}$ appears quadratically in Eq. (18) it can be integrated out to obtain an non-local action for the confining string,

$$S_{cs}[\Sigma_c] = \int_{x,y} \left\{ T_{\mu\nu}(x) \Lambda^2 G(x - y) T_{\mu\nu}(y) + 2e^2 \partial_\sigma T_{\mu\nu} G(x - y) \partial_\sigma T_{\mu\nu} \right\} \quad (21)$$

where $T_{\mu\nu}$ is defined in terms of the area element on the world sheet, $X_{\mu\nu}$, by

$$T_{\mu\nu}(x) = \frac{1}{2} \int_\sigma X_{\mu\nu}(\sigma) \delta(x - x(\sigma)) \quad (22)$$

and

$$X_{\mu\nu} = \epsilon^{ab} \frac{\partial x_\mu}{\partial \sigma_a} \frac{\partial x_\nu}{\partial \sigma_b} \quad (23)$$

where $G(x - y)$ is the Yukawa Green’s function \cite{42}. 7
The non-local string action can be approximated by a local action by expanding Yukawa Green’s function in inverse powers of the mass of the Kalb-Ramond field \[26, 42\]

\[
S_{cs}[\Sigma] = c_1 \Lambda^2 \int_{\sigma} \sqrt{g} - c_2 \frac{\Lambda^2}{m^2} \int_{\sigma} \sqrt{g} (\partial_\mu t_{\mu})^2 + c_3 m^2 \int_{C} d\tau \left( \frac{dx_\mu}{d\tau} \frac{dx_\mu}{d\tau} \right)^{\frac{1}{2}} \tag{24}
\]

\(c_1, c_2\) and \(c_3\) are dimensionless constants depending on the regularization of the world sheet, \(m\) is the mass of the Kalb-Ramond field, and \(t_{\mu\nu} = \frac{1}{\sqrt{g}} X_{\mu\nu}\).

Now we are in the position to obtain the string representation for the FWL. Applying the relationship between the Wilson loop and the FWL as given by Eq. (11) to the Eq. (18), lead to:

\[
\langle \mathcal{W}[C] \rangle_A = \frac{1}{Z_{KR}} \int DB \exp \{-S_{KR}[B]\} \times \exp \left\{ i \int_{\Sigma_C} B_{\mu\nu} d\sigma_{\mu\nu} - i \oint_{C} d\tau \psi_\mu \psi_\nu \right\}, \tag{25}
\]

where the action for the Kalb-Ramond field, \(S_{KR}[B]\), is given by Eq. (18). As an independent check, in the appendix we outline a derivation of the above representation, for the case of three dimensional compact \(U(1)\) gauge theory, that does not use Eq. (11).

The string representation of the FWL is distinguished by an additional boundary term involving the interaction between the spin degrees of freedom and the Kalb-Ramond field. Again, as in the case of the Wilson loop, we can integrate out the Kalb-Ramond field to obtain a non-local string action for the FWL,

\[
S[\Sigma_C, C] = \int_{x,y} \left\{ (T_{\mu\nu}(x) - S_{\mu\nu}(x)) D_{\mu\nu,\lambda\rho}(x - y)(T_{\lambda\rho}(y) - S_{\lambda\rho}(y)) \right\}, \tag{26}
\]

\[
S_{\mu\nu}(x) = \oint_{C} d\tau \delta(x - y(\tau))\psi_\mu(\tau)\psi_\nu(\tau),
\]

where \(D_{\mu\nu,\lambda\rho}(x - y)\) is the propagator for the Kalb-Ramond field. Expanding the above action, and substituting the expressions for \(T_{\mu\nu}\) and \(S_{\mu\nu}\) leads to

\[
S[\Sigma_C, C] = \int_{\sigma_1, \sigma_2} d\sigma_{\mu\nu}(\sigma_1) D_{\mu\nu,\lambda\rho}(x(\sigma_1) - y(\sigma_2)) d\sigma_{\lambda\rho}(\sigma_2)
+ 2 \oint_{\tau} d\tau \psi_\mu(\tau)\psi_\nu(\tau) \int_{\sigma} D_{\mu\nu,\lambda\rho}(x(\tau) - y(\sigma)) d\sigma_{\lambda\rho}(\sigma)
+ \oint_{\tau_1, \tau_2} d\tau_1 d\tau_2 \psi_\mu(\tau_1)\psi_\nu(\tau_1) D_{\mu\nu,\lambda\rho}(x(\tau_1) - y(\tau_2)) \psi_\lambda(\tau_2)\psi_\rho(\tau_2). \tag{27}
\]
The first term in the above equation is the non-local form of the confining string action for the Wilson loop. The next two terms are specific to the FWL. The last term is purely a boundary term involving only Fermions. A similar term would appear even in a theory which has no strings, for e.g QED \[44\].

Of particular interest is the second term of Eq. (27). It represents a non-local interaction between the Fermion at the boundary and the string world sheet. To make the nature of this interaction more transparent, we approximate the above non-local action by a local action using the derivative expansion for the propagator. Keeping only the leading terms, this leads to

$$S_{jf}[\Sigma_C, C] = S_{cs}[\Sigma_c] + c_4 \Lambda^2 \int_C d\tau \psi_\mu(\tau) \psi_\nu(\tau) t_{\mu\nu}(\tau),$$

(28)

where $S_{cs}[\Sigma_c]$ is the action for the confining string for the Wilson loop, Eq. (24), and $t_{\mu\nu}(\tau)$ is the value of the area element density at the boundary of the surface. The leading effect of spin-string interaction is to correlate the spin of the “quark” with the area-element of the world sheet at the boundary. Since the world sheet in our model has no fermionic degrees of freedom, the spin-string interaction at the boundary breaks the world line supersymmetry of the FWL \[3\]. Our real interest of course is in the spin-string interaction for QCD. Even though we do not know the QCD string action, it is reasonable to expect that a term similar to the one we have delineated above should be present (though the nature and the number of world sheet degrees of freedom has to be different from those in the confining strings of the compact $U(1)$ gauge theories \[10\]). Specifically, in the string representation of the FWL in QCD the quark-string interaction should take place on the boundary. If the interaction between quark degrees of freedom and the world sheet variables is confined to the boundary, we are lead again to the question that was raised in the introduction, namely, how does the string communicate the relative spin orientation that distinguishes a pion from a rho. For a boundary interaction at the one end of an open string to influence the interaction at the other end would require a long distance correlation between the string variables on the world sheet. One would expect such a correlation to correspond to a massless mode in the interior theory. It is tempting to identify such a massless mode with a Nambu-Goldstone boson, this would be merely a tautology except that it would imply that the two ways of looking at the pion-rho mass

\[3\]

I would like to thank V. P. Nair for suggesting me to check this possibility.
difference, one as the consequence of a hyperfine splitting, and other as the result of the spontaneous breaking of chiral symmetry are related to each other in the string representation of QCD.

4 Discussion

Two points emerge from the previous sections that are independent of the model considered. Firstly, the FWL allows us to introduce the required quark degrees of freedom in the string representation of QCD. Secondly, the string representation of mesons via the FWL contains useful information about how the spin degrees of freedom of a quark and an antiquark interacts with the string that connects them.

One of the motivation for studying the FWL is that its string representation might shed some light on how chiral symmetry is realized in the dual picture of QCD. The FWL highlights one issue regarding chiral symmetry breaking. If the right way of introducing quarks in the string representation of QCD is indeed via the FWL then the dynamics responsible for chiral symmetry breaking must be located on the boundary of the string world sheet and not in the interior. As noted at the end of the last section such a mechanism also seems to suggest an intimate relationship between the spin-spin interaction, or the hyperfine splitting, in mesons and the existence of the Nambu-Goldstone bosons.

The string representation of the FWL and the resulting spin-string interaction should be of even more importance if the (unknown) QCD string action has some form of world sheet supersymmetry. An evidence for the world sheet supersymmetry emerges from the work of ref. [35], where it is shown that the string representation for QCD in two dimensions in the large $N$ limit, the t’ Hooft model, has a rigid supersymmetry. Thus the t’ Hooft model model offers an interesting laboratory for calculating and understanding the dynamics of the FWL. It is also a good place to test whether one can use the string representation of the FWL to calculate the spectrum of the theory. I hope to carry out these investigations in the near future.
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Appendix

The aim of this appendix is to outline the derivation of the string representation the fermionic Wilson loop in three dimensional compact $U(1)$ gauge theory. The calculation is essentially a repetition of the calculation for the Wilson loop in ref. [26] (see also ref. [45] for some of the details of the Wilson loop calculation.) The expectation value of the fermionic Wilson loop can be written as

$$<W[C]> = <W>_o<W>_m$$

(29)

where the first factor is the gaussian integral,

$$<W>_o=\frac{1}{Z[0]} \int DA \exp \left\{ -\frac{1}{4e^2} \int_x (\partial_\mu A_\nu - \partial_\nu A_\mu) \right\} W,$$

(30)

and the second factor, $<W>_m$, is the contribution from the magnetic monopoles. fermionic Wilson loop, $\mathcal{W}$, is given by Eq. (11). A monopole located at point $x$ gives following contribution to fermionic Wilson loop,

$$\mathcal{W}^1_m = \exp(-S_m) \exp \left\{ i(\frac{1}{2}\eta[x; C] - \chi[x; C]) \right\},$$

(31)

$$\eta[x; C] = \frac{\partial}{\partial x_\lambda} \int_{\Sigma_C} d\sigma_\lambda \frac{1}{|x - y(\sigma)|},$$

(32)

$$\chi[x; C] = \frac{1}{4} \epsilon_{\mu\nu\lambda} \frac{\partial}{\partial x_\lambda} \int d\tau \psi_\mu(\tau)\psi_\nu(\tau) \frac{1}{|x - y(\tau)|},$$

(33)
where $S_m$ is the action for a single magnetic monopole. Summing over all monopoles in the dilute gas approximation \[7, 41\] leads to

$$<W>_m = \frac{1}{Z_m} \int D\varphi \exp\left\{-e^2 \int_x \left[\frac{1}{2} (\partial \varphi)^2 + m^2 (1 - \cos(\varphi - \frac{1}{2} \eta[x; C] + \chi[x; C]))\right]\right\}$$  \hspace{1cm} (34)$$

where $m^2 = \exp \left(-\frac{\text{const}}{e^2}\right)$. Now consider the following functional integral

$$W_F[C] = \frac{1}{Z} \int DBD\phi \exp\{-\Gamma[B, \phi; \Sigma_C, C]\},$$  \hspace{1cm} (35)$$

where the effective action is given by

$$\Gamma = \int_x \left(\frac{1}{4e^2} B_{\mu\nu}^2 - i\epsilon_{\mu\nu\lambda} \partial_\lambda \phi B_{\mu\nu} + e^2 m^2 (1 - \cos \phi)\right)$$

$$+ 2\pi i \int_{\Sigma_C} B_{\mu\nu}(\sigma)d\sigma_{\mu\nu} - 2\pi i \oint_C d\tau \psi_\mu(\tau)\psi_\nu(\tau) B_{\mu\nu}(\tau).$$  \hspace{1cm} (36)$$

Since the integration over field $B_{\mu\nu}$ is Gaussian we solve for the field $B^{c}_{\mu\nu}$ that minimizes the effective action $\Gamma$,

$$B^{c}_{\mu\nu} = ie^2\{\epsilon_{\mu\nu\lambda}\partial_\lambda \phi - 2\pi \int_{\Sigma_C} \delta(x-y(\sigma))d\sigma_{\mu\nu} + 2\pi \oint_C \psi_\mu(\tau)\psi_\nu(\tau)\delta(x-y(\tau))d\tau\}.$$  \hspace{1cm} (37)$$

Further using the following identity for the derivative of $\eta(x)$,

$$4\pi \int_{\Sigma_C} \delta(x-y(\sigma))d\sigma_{\mu\nu} = -\epsilon_{\mu\nu\lambda}\partial_\lambda \eta(x) - f_{\mu\nu}(x),$$  \hspace{1cm} (38)$$

where,

$$f_{\mu\nu}(x) = \partial_\mu a_\nu - \partial_\nu a_\mu$$  \hspace{1cm} (39)$$

and

$$a_\mu(x) = \oint_C dy_\mu \frac{1}{|x-y|},$$  \hspace{1cm} (40)$$

and using a similar identity for the derivative of $\chi(x)$,

$$2\pi \oint_C d\tau \delta(x-y(\tau))\psi_\mu(\tau)\psi_\nu(\tau) = -\epsilon_{\mu\nu\lambda}\partial_\lambda \chi(x) - f^{s}_{\mu\nu}(x),$$  \hspace{1cm} (41)$$

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where,

\[ f_{\mu\nu}^s(x) = \partial_\mu b_\nu - \partial_\nu b_\mu \quad (42) \]

\[ b_\mu(x) = \frac{1}{2} \oint_C \psi_\mu(\tau)\psi(\lambda)(\tau) \frac{\partial}{\partial y_\lambda} \left( \frac{1}{|x - y(\tau)|} \right) \quad (43) \]

we get

\[ B_{\mu\nu}^c(x) = i e^2 \left( \epsilon_{\mu\nu\lambda} \partial_\lambda (\phi + \frac{\eta}{2} - \chi) + (\partial_\mu A_\nu^c - \partial_\nu A_\mu^c) \right) \quad (44) \]

where

\[ A_\mu^c(x) = \frac{1}{2} \oint_C d\tau \left( \frac{dy_\mu}{d\tau} \frac{1}{|x - y(\tau)|} - \psi_\mu(\tau)\psi(\lambda)(\tau) \frac{\partial}{\partial y_\lambda} \left( \frac{1}{|x - y(\tau)|} \right) \right) . \quad (45) \]

We notice that \( A_\mu^c \) solves the integrand of the Gaussian integral, and that \( B_{\mu\nu}^c \) has contribution both from magnetic monopoles and from the gauge field. Thus the functional integral over \( B_{\mu\nu} \) reduces \( W_F[C] \), Eq. (35), to the product \( \langle W \rangle_o \langle W \rangle_m \), once we identify the field \( \phi = \phi + \frac{\eta}{2} - \chi \).

References

[1] G. ’t Hooft, Nucl. Phys. B190 (1981) 455; “Topological Aspects of Quantum Chromodynamics” hep-th/9812204.

[2] S. Mandelstam, Phys. Rep. C23 (1976) 245.

[3] G. ’t Hooft, Nucl. Phys. B72 (1974) 461.

[4] E. Witten, Nucl. Phy. B160 (1979) 57.

[5] K. Wilson, Phys. Rev. D10 (1974) 2445.

[6] J. B. Kogut and L. Susskind, Phys. Rev.D11 (1975) 395.

[7] A. M. Polyakov, Gauge Fields and Strings, Harwood Academic Publisher, Chur (1987).

[8] J. B. Kogut, Rev. Mod. Phys 51 (1979) 659.

[9] J. B. Kogut, Rev. Mod. Phys 55 (1983) 775.
[10] J. Polchinski, “Strings and QCD” hep-th/9210045.

[11] A. M. Polyakov, “String Theory and Quark Confinement” hep-th/9711002.

[12] A. M. Polyakov, “The wall of the cave” hep-th/9809057.

[13] A. M. Polyakov, “String theory as a universal language” hep-th/0006132.

[14] M. Ademollo, G. Veneziano, and S. Weinberg, Phys. Rev. Lett. 22 (1969) 83.

[15] C. Lovelace, Phys. Lett. B34 (1971) 500.

[16] A. Casher and L. Susskind, Phys.Rev. D9 (1974), 436; R.C. Brower and L. Susskind, Phys.Rev. D7 (1973), 1032; L. Susskind, A. Casher, and J. Kogut, Phys.Rev. D8 (1973), 4448; J. Kogut and L. Susskind, Phys.Rep. 8C (1973), 75; A. Casher, S.H. Noskowicz and L. Susskind, Nucl.Phys. B32 (1971), 75.

[17] D. C. Lewellen, Nucl. Phys. B392 (1993) 137; hep-th/9110026.

[18] M. B. Green, J. H. Schwarz, and E. Witten, “Superstring Theory ”, In two volumes, Cambridge University Press, Cambridge (1987).

[19] J. Polchinski, “String Theory”, In two volumes, Cambridge University Press, Cambridge (1998).

[20] R. P. Feynman, R. B. Leighton, and M. Sands, “The Feynman Lectures on Physics”, Vol. 3, Addison-Wesley, Reading, Massachusetts (1966).

[21] F. A. Berezin and M. S. Marinov, JETP Lett. 21 (1975) 320; Ann. Phys. 104 (1977) 336.

[22] L. Brink, S. Deser, B. Zumino, P. Di Vecchia and P. Howe, Phys. Lett. B64 (1976) 435.

[23] L. Brink, P. Di Vecchia and P. Howe, Nucl. Phys. B118 (1977) 76.
[24] A. Barducci, R. Casalbuoni and L. Lusanna, *Nuovo Cimento* A35 (1976) 377.

[25] M. Henneaux and C. Teitelboim, *Ann. Phys. (N. Y.)* 104 (1977) 127.

[26] A. M. Polyakov, *Nucl. Phys.* B486 (1997) 23 hep-th/9607049.

[27] L. Alvarez-Gaumé, *Commun. Math. Phys.* 90 (1983) 161.

[28] D. Friedan and P. Windey, *Nucl. Phys.* B235 (1984) 395.

[29] A. Migdal, *Prog. Theor. Phys. Suppl.* 131 (1998) 269, hep-th/9610126.

[30] E. D’Hoker and D. Gagné, *Nucl.Phys.* B467 (1996) 272, hep-th/9508131.

[31] R. L. Karp and F. Mansouri, “Supersymmetric Wilson Lines and Loops, and Super Non-Abelian Stokes Theorem”, hep-th/0002085.

[32] J. M. Maldacena, “The Large N Limit of Superconformal Field Theories and Super-gravity,” hep-th/9711200.

[33] J. Maldacena, *Phys. Rev. Lett.* 80 (1998) 4859, hep-th/9803002.

[34] M. Kalb and P. Ramond, *Phy. Rev.* D9 (1974) 2273.

[35] P. Hořava, *Nucl.Phys.* B463 (1996) 238, hep-th/9507060; P. Hořava, *JHEP* 9901 (1999) 016, hep-th/9811028.

[36] E. Witten in, *Recent Development in Gauge Theories*, 1979 Cargese Lectures, edited by G. ’t Hooft et al. Plenum (1980).

[37] Y. Makeenko, *Large N-Gauge Theories*, hep-th/0001047.

[38] C. Schubert, *Axial Vector Amplitudes, Second Order Fermions, and Standard Model Photon Neutrino Processes*, hep-ph/9905525.

[39] L. Alvarez-Gaumé and E. Witten, *Nucl. Phys.* B234 (1983) 269.

[40] M.R. Mehta, *Phys. Rev. Lett.* 65 (1990) 1983; M.R. Mehta, *Phys. Lett.* 274B (1992) 53.
[41] A. Polyakov, *Nucl. Phys.* **B120** (1977) 82.

[42] M. C. Diamantini, F. Quevedo and C. A. Trugenberger, *Phys. Lett.* **B396** (1997) 115, [hep-th/9612103](http://arxiv.org/abs/hep-th/9612103).

[43] M. C. Diamantini, H. Kleinert, and C. A. Trugenberger, *Phys.Lett.* **B457** (1999) 87, [hep-th/9903208](http://arxiv.org/abs/hep-th/9903208).

[44] C. Alexandrou, R. Rosenfelder, and A. W. Schreiber “*Non-Perturbative Mass Renormalization in Quenched QED from the Worldline Variational Approach*”, hep-th/0003253.

[45] D. Antonov and D. Ebert, *Confining Properties of Abelian(-Projected) Theories*, [hep-th/9812112](http://arxiv.org/abs/hep-th/9812112).