ENDOMORPHISMS OF POWER SERIES FIELDS AND RESIDUE FIELDS OF FARGUES-FONTAINE CURVES

KIRAN S. KEDLAYA AND MICHAEL TEMKIN

Abstract. We show that for $k$ a perfect field of characteristic $p$, there exist endomorphisms of the completed algebraic closure of $k((t))$ which are not bijective. As a corollary, we resolve a question of Fargues and Fontaine by showing that for $p$ a prime and $\mathbb{C}_p$ a completed algebraic closure of $\mathbb{Q}_p$, there exist closed points of the Fargues-Fontaine curve associated to $\mathbb{C}_p$ whose residue fields are not (even abstractly) isomorphic to $\mathbb{C}_p$ as topological fields.

1. Introduction

In this short note, we address the following question. By an analytic field, we will always mean a field complete with respect to a nonarchimedean multiplicative absolute value (assumed to be real-valued and written multiplicatively); by default, we always allow the trivial absolute value.

Question 1.1. Let $K$ be an analytic field. Let $k$ be a trivially valued subfield of $K$. Is every continuous $k$-linear homomorphism from $K$ to itself which induces automorphisms of residue fields and value groups necessarily surjective (and hence an automorphism)?

We will view Question 1.1 as a collection of distinct cases indexed by the choice of $K, k$. For example, one has affirmative answers in the following cases:

- when $K$ is trivially valued, discretely valued, or more generally spherically complete (Proposition 3.1);
- when $\text{char}(k) = 0$ and $K$ is the completed algebraic closure of a power series field over $k$ (Remark 3.3);

whereas one has negative answers in the following cases:

- in certain cases in characteristic 0 (Example 3.2);
- when $\text{char}(k) > 0$ and $K$ is the completed perfect closure of a power series field over $k$ (see [9]).

Hereafter, fix a prime number $p$. Our main result is a negative answer to Question 1.1 when $\text{char}(k) = p$ and $K$ is the completed algebraic closure of a power series field over $k$.

Theorem 1.2. Let $K$ be a completed algebraic closure of $k((t))$ for some field $k$ of characteristic $p$. Then there exists a continuous $k$-linear homomorphism $\tau : K \to K$ which is not an isomorphism.
The proof depends on a calculation using completed modules of Kähler differentials of analytic fields, as recently studied by the second author [14]. We develop here the bare minimum of this subject needed for the proof of Theorem 1.2; a more detailed treatment of completed differentials between analytic fields will be given by the second author elsewhere.

Theorem 1.2 was prompted by an application to a foundational question of $p$-adic Hodge theory, specifically in the perfectoid correspondence (commonly known as tilting) between nonarchimedean fields in mixed and equal characteristics (generalizing the field of norms correspondence of Fontaine and Wintenberger). A nonarchimedean field $K$ of residue characteristic $p$ is perfectoid if it is not discretely valued and the Frobenius automorphism on $\sigma_K/(p)$ is surjective. Given such a field, let $K^b$ be the inverse limit of $K$ under the $p$-power map; one then shows that $K^b$ naturally carries the structure of a perfectoid (and hence perfect) nonarchimedean field of equal characteristic $p$ and that there is a canonical isomorphism between the absolute Galois groups of $K$ and $K^b$ [8, 10, 12]. The functor $K \mapsto K^b$ is not fully faithful, even on fields of characteristic 0; for instance, one can construct many algebraic extensions of $\mathbb{Q}_p$ whose completions map to the completed perfect closure of a power series field over $\mathbb{F}_p$ (e.g., the cyclotomic extension $\mathbb{Q}_p(\mu_{p^{\infty}})$ and the Kummer extension $\mathbb{Q}_p(p^{1/p^{\infty}})$). However, Fargues and Fontaine have asked [3, Remark 2.24] (see also [4]) whether this can happen for a completed algebraic closure of $\mathbb{Q}_p$, and using Theorem 1.2 we are able to answer this question.

**Theorem 1.3.** Let $\mathbb{C}_p$ be a completed algebraic closure of $\mathbb{Q}_p$. Then there exists a perfectoid field $K$ which is not isomorphic to $\mathbb{C}_p$ as a topological field, but for which there exists an isomorphism $K^b \cong \mathbb{C}_p^b$.

This result admits the following geometric interpretation. For each perfectoid field $K$, Fargues and Fontaine define an associated scheme $X_K$ which is a “complete curve” (i.e., a regular one-dimensional noetherian scheme equipped with a surjection of its Picard group onto $\mathbb{Z}$) in terms of which $p$-adic Hodge theory over $K$ can be simply formulated. Theorem 1.3 implies that for $K = \mathbb{C}_p$, there exists a closed point of $X_K$ whose residue field is not isomorphic to $\mathbb{C}_p$.

We conclude this introduction by pointing out that after we prepared our proof of Theorem 1.2, we learned that this statement is a special case of a result of Matignon and Reversat [11, Théorème 2]. However, since our proof of the special case is somewhat simpler than the more general argument of Matignon–Reversat, we have elected to retain the proof here.

### 2. Analytic fields and completed differentials

As a technical input into the proof of Theorem 1.2, we review some basic properties of analytic fields and completed differentials.

**Definition 2.1.** By an analytic field, we will mean a field equipped with a multiplicative nonarchimedean absolute value with respect to which the field is complete. By default, we allow the trivial absolute value. When we consider an extension $L/K$ of analytic fields, we require that the absolute value on $L$ restricts to the absolute value on $K$.

**Definition 2.2.** We say that an extension $L/K$ of analytic fields is primitive if there exists $t \in L^\times$ such that $K(t)$ is dense in $L$; we will write $L = \widehat{K(t)}$ if we need to indicate the choice of $t$. 
With \( t \) given, the extension \( \hat{K}(t)/K \) corresponds to a point in the projective line over \( K \) in the category of Berkovich nonarchimedean analytic spaces \[2\]. Without \( t \) given, the points associated to \( L/K \) are all of the same type 1–4 in Berkovich’s classification \[2, (1.4.4)\]; we thus classify \( L/K \) accordingly. Write
\[
E_{L/K} = \dim_{\mathbb{Q}}(|L^x|/|K^x|) \otimes_{\mathbb{Z}} \mathbb{Q}, \quad F_{L/K} = \text{trdeg}_{\kappa(K)} \kappa(L),
\]
where \( \kappa(*) \) denotes the residue field of \(* \); these are determined by the type of \( L/K \) as follows.

| Type of \( L/K \) | \( E_{L/K} \) | \( F_{L/K} \) |
|-------------------|-------------|-------------|
| 1                 | 0           | 0           |
| 2                 | 0           | 1           |
| 3                 | 1           | 0           |
| 4                 | 0           | 0           |

In all cases we have \( E_{L/K} + F_{L/K} \leq 1 \), as per Abhyankar’s inequality (e.g., see [13, Lemma 2.1.2]). However, types 1 and 4 cannot be distinguished using \( E_{L/K} \) and \( F_{L/K} \) alone: one must instead observe that \( L/K \) is of type 1 if and only if \( L \) embeds into the completed algebraic closure of \( K \).

In order to better distinguish between primitive extensions of types 1 and 4, we will use completed modules of differentials.

**Definition 2.3.** Let \( L/K \) be an extension of analytic fields. As described in [14, §4], the module \( \Omega_{L/K} \) admits a maximal seminorm \( \| \cdot \| \) (the Kähler seminorm) with respect to which \( d_{L/K} : L \to \Omega_{L/K} \) is nonexpanding. Let \( \hat{\Omega}_{L/K} \) denote the completion of \( \Omega_{L/K} \) with respect to \( \| \cdot \| \); it receives an induced derivation \( \hat{d}_{L/K} : L \to \hat{\Omega}_{L/K} \).

**Lemma 2.4.** Let \( L/K \) be a primitive extension, and choose \( t \in L \) such that \( K(t) \) is dense in \( L \).

(a) The module \( \hat{\Omega}_{L/K} \) is generated over \( L \) by the single element \( \hat{d}_{L/K}(t) \).

(b) The equality \( \hat{\Omega}_{L/K} = 0 \) holds if and only if the separable closure of \( K \) in \( L \) is dense. (Note that this condition implies that \( L/K \) is of type 1, and conversely whenever \( \text{char}(K) = 0 \).)

**Proof.** Since \( \Omega_{K(t)/K} \) is generated by \( d_{L/K}(t) \), (a) is obvious.

Let \( l \) be the separable integral closure of \( K \) in \( L \). If \( l \) is dense in \( L \) (which forces \( L/K \) to be of type 1), then \( \Omega_{l/K} = 0 \) and so \( \hat{\Omega}_{l/K} = 0 \). This proves the inverse implication in (b).

Suppose that \( L/K \) is not of type 1. Let \( K' \) be a completed algebraic closure of \( K \) and put \( L' = l \otimes_K K' \); then the natural map \( \Omega_{L/K} \hat{\otimes}_LL' \to \Omega_{L'/K'} \) sends \( \hat{d}_{L/K}(t) \otimes 1 \) to \( \hat{d}_{L'/K'}(t) \). The latter is nonzero by [3, Theorem 2.3.2(i)], so \( \hat{d}_{L/K}(t) \neq 0 \).

It remains to consider the case when \( L/K \) is of type 1 but \( l \) is not dense in \( L \). (Note that this last step is not needed for the proof of Theorem 1.2 so the uninterested reader can skip it.) Observe that any separable extension of \( \hat{l} \) is the closure of a separable extension of \( l \). Since \( l \) is separably closed in \( L \), we obtain that \( \hat{l} \) is separably closed in \( L \) too. It suffices to show that \( \hat{d}_{\hat{l}/K}(t) \neq 0 \), so after replacing \( K \) by \( \hat{l} \) we can assume that \( K = l \).

Fix an embedding of \( L \) into \( K' \). Let \( G \) be the group of continuous automorphisms of \( K' \) fixing \( K \); this group is naturally identified with the absolute Galois group of \( K \). The subgroup \( H \) fixing \( L \) is closed in \( G \), and hence is the absolute Galois group of some separable
extension $L_0$ of $K$. If char $K = 0$, then the Ax-Sen theorem \cite{1} applied to both $L_0$ and $L$ implies that $(K')^H = \hat{L}_0 = L$, but this contradicts our previous assumption that $K = l \neq L$.

We must then have char $K = p > 0$. By Ax-Sen again, we have $(K')^H = \hat{L}_0^{1/p^\infty} = \hat{L}^{1/p^\infty}$. If $L_0 \neq K$, we may choose a separable irreducible polynomial $P \in K[T]$ of degree $> 1$ with a root in $L_0$; by Krasner’s lemma, $P$ has a root $x$ in $L^{1/p^n}$ for some sufficiently large $n$.

But then $x^{p^n} \in L$ generates a nontrivial separable extension of $K$, again contradicting our assumption that $K = l \neq L$. We conclude that $L_0 = K$ and so $t \in K^{1/p^\infty} \setminus K$.

Choose $a_0 = 0, a_1, \ldots \in K$ such that the sequence $r_n = |t - a_1/p^n|$ converges to zero. Then $L$ is the completion of its subalgebra $\bigcup_n k\{r_0^{-1}t, r_n^{p^n}(t^{p^n} - a_n)\}$; in particular, $k[t]$ is dense in $L$. Consider the Banach ring $A := L \hat{\otimes}_K L$ provided with the tensor product norm $\|\cdot\|$ and note that the ideal $J = \text{Ker}(A \to L)$ is generated by $T := 1 \otimes t - t \otimes 1$.

We claim that $\|T^{p^n}\| \leq r^{p^n}$. Indeed, since $|t^{p^n} - a_n| = r^{p^n}_n$, we have that $\|1 \otimes t^{p^n} - a_n\| \leq r^{p^n}_n$ and $\|t^{p^n} \otimes 1 - a_n\| \leq r^{p^n}_n$. (Note, for the sake of completeness, that $T$ is quasi-nilpotent, i.e. its spectral norm vanishes, and hence $L$ is the uniform completion of $A$, i.e. the completion with respect to the spectral seminorm. This is a topological extension of the classical fact that $T$ is nilpotent and $L$ is the reduction of $A$ when $L/K$ is finite and purely inseparable.)

By \cite{14} Remark 4.3.4(ii)], there is an isomorphism $J/J^2 \sim \hat{\Omega}_{L/K}$ that takes $T$ to $\hat{d}_{L/K}(t)$. Thus, we should only show that $T \neq aT^2$ in $A$. Assume, to the contrary, that $T = aT^2$ and set $s = \|a\|$. Then, $\|T\| = \|a^{p^n-1}T^{p^n}\| \leq s^{p^n-1}r^{p^n}_n$ for any $n$. Hence $\|T\| = 0$. Thus $\hat{L} \hat{\otimes}_K L = L$ and since $L \otimes_K L$ embeds into $L \hat{\otimes}_K L$ by \cite{3} 3.2.1(4)], we obtain a contradiction. \qed

3. Proofs and examples

We now settle the questions raised in the introduction.

Proposition 3.1. Question \cite{17} admits an affirmative answer if $K$ is spherically complete.

Proof. Let $\tau : K \to K$ be a homomorphism as in Question \cite{17} Suppose by way of contradiction that there exists $x \in K$ with $x \notin \tau(K)$. Since $K$ is spherically complete, the set of possible valuations of $x - \tau(y)$ for $y \in K$ has a least element. If $y$ realizes this valuation, then by the matching of value groups, we can find $y' \in K$ such that $\tau(y')$ and $x - \tau(y)$ have the same valuation; by the matching of residue fields, we can further choose $y'$ such that $(x - \tau(y))/\tau(y')$ maps to 1 in $k$. But then $x - \tau(y + y')$ has smaller valuation than $x - \tau(y)$, a contradiction. \qed

Example 3.2. Let $k$ be an analytic field whose absolute value is nondiscrete, and choose a sequence $x_1, x_2, \ldots \in k^\times$ such that $|x_i| < 1$ and $\lim_n |x_1 \cdots x_n| > 0$. (For a more concrete example, take $k$ to be a completed algebraic closure of $\mathbb{C}((t))$ and take $x_n = t^{2^{-n}}$.) Let $K$ be the completion of $k(t_1, t_2, \ldots)$ for the Gauss valuation (i.e., the valuation of a nonzero polynomial is the maximum valuation of its coefficients); then $K$ admits a unique valuation-preserving endomorphism $\tau$ fixing $k$ and taking $t_n$ to $t_n - x_n t_{n+1}$ for each $n$. We will show that the image of $\tau$ does not contain $t_1$, and hence $\tau$ is not an isomorphism.

Suppose to the contrary that there exists $y \in K$ with $\tau(y) = t_1$. By hypothesis, there exists some $\lambda \in k$ such that $|\lambda| < |x_1 \cdots x_n|$ for all $n$. We may then choose $y' \in K_0(t_1, t_2, \ldots, t_n)$ for some positive integer $n$ in such a way that $|y - y'| < |\lambda|$. Put $y'' = t_1 + x_1 t_2 + \cdots + x_1 \cdots x_n t_{n+1}$; then $\tau(y'') = t_1 - x_1 \cdots x_{n+1} t_{n+2}$, so $|y'' - y| = |\tau(y'' - y)| = |x_1 \cdots x_{n+1}| > |\lambda|$. Hence
\[ |y'' - y'| = |x_1 \cdots x_{n+1}|, \text{ but } y'' - y' \text{ equals } x_1 \cdots x_n t_{n+1} \text{ plus an element of } k(t_1, \ldots, t_n) \text{ and so cannot have valuation less than } |x_1 \cdots x_n|. \] This yields the desired contradiction.

**Remark 3.3.** Let \( k \) be a field of characteristic 0. For each positive integer \( n \), the derivation \( \frac{d}{dt} \) on \( k((t)) \) extends to the derivation \( \partial_n = n^{-1}t^{1/n-1} \frac{d}{dt} \) on \( k((t^{1/n})) \) satisfying \( |\partial_n f| \leq |t|^{-1} |f| \) for any \( f \in k((t^{1/n})) \). Let \( K \) be a completed algebraic closure of \( k((t)) \); by Puiseux’s theorem, \( K \) is the completion of \( \bigcup_{n=1}^{\infty} \overline{k((t^{1/n}))} \) for \( \overline{k} \) the algebraic closure of \( k \) in \( K \), so the derivation \( \frac{d}{dt} \) extends uniquely to a continuous derivation on \( K \). Consequently, \( \hat{\Omega}_{K/k} \) is generated by \( d_{K/k}(x) \) for any \( x \in K - \overline{k} \). For any \( k \)-linear automorphism \( \tau \) of \( K \), let \( L \) be the completion of \( \tau(K)(t) \) within \( K \); taking \( x = \tau(t) \) in the previous discussion shows that \( \hat{\Omega}_{L/\tau(K)} = 0 \). By Lemma 2.4(b) we conclude that \( L/\tau(K) \) is of type 1. Since \( \tau(K) \) is algebraically closed, it follows that \( L = \tau(K) \) and hence \( \tau \) is an isomorphism.

**Proof of Theorem 1.2** Choose a sequence \( \{d_i\}_{i=1}^{\infty} \) of positive integers in such a way that:

(a) \( d_i \) is not divisible by \( p \);

(b) \( \lim_{i \to \infty} (d_{i+1} - pd_i) = \infty \); and

(c) the sequence \( \{p^{-i}d_i\}_{i=1}^{\infty} \) is strictly increasing (for large \( i \), this follows from (b)) and bounded.

For a concrete example, take

\[ d_i := 1 + pi + p^2 (i - 1) + p^3 (i - 2) + \cdots + p^i. \]

Choose a sequence \( \{c_i\}_{i=1}^{\infty} \) of elements of \( k \) such that each field \( \mathbb{F}_p(c_i) \) is finite, but the field \( \mathbb{F}_p(c_1, c_2, \ldots) \) is infinite (in fact any \( c_i \neq 0 \) will do, but this assumption shortens the argument). Set

\[ \alpha_n := \sum_{i=0}^{n} c_i t^{p^{-i}d_i} \in K \]

and

\[ r_n := |\alpha_{n+1} - \alpha_n| = |t|^{p^{-n-1}d_{n+1}}; \]

by construction, \( \{r_n\}_{n=1}^{\infty} \) is a strictly decreasing sequence with nonzero limit. Consider the Berkovich affine line \( \mathbb{A}_k((t)) \) with coordinate \( x \) and let \( E_n \) be the closed disc of radius \( r_n \) centered at \( \alpha_n \). The intersection of the \( E_n \) does not contain any element of any finite extension of \( k((t)) \), so it consists of a single point \( z \) of type 4. (Otherwise, by [7, Lemma 10.1, Corollary 11.9] the generalized power series \( x = \sum_{i=0}^{\infty} c_i t^{p^{-i}d_i} \) would be algebraic over \( k(t) \), hence over \( \mathbb{F}_p(t) \) because the minimal polynomial must be invariant under coefficientwise automorphisms, hence over \( \mathbb{F}_p(t) \); but then [7, Corollary 11.9] would force the \( c_i \) to belong to a finite extension of \( \mathbb{F}_p \).) The completed residue field \( L = \mathcal{H}(z) \) of this point is a primitive extension of \( k((t)) \) topologically generated by \( x \), and the conditions \( z \in E_n \) mean that \( x^{p^n} - \alpha_n^{p^n} = r_n^{p^n} \) for each \( n \). Since \( \hat{\Omega}_{L/k} \) is nonexpanding, we have \( \|\hat{\alpha}_n^{p^n}\| \leq r_n^{p^n} \).

Furthermore, in the expression \( \alpha_n^{p^n} = \sum_{i=0}^{n} t^{p^{-i}d_i} \) only the term \( t^{d_n} \) is not a \( p \)-th power, so

\[ \hat{\Omega}_{L/k}(\alpha_n^{p^n}) = \hat{\Omega}_{L/k}(t^{d_n}) = d_n t^{d_n-1} \hat{\Omega}_{L/k}(t) \]

and hence (since \( d_n \) is not divisible by \( p \))

\[ \|\hat{\Omega}_{L/k}(t)\| \leq r_n^{p^n} |t|^{1-d_n} = |t|^{p^{-d_n+1}-d_n+1}. \]
Since this holds for all \( n \), we conclude that \( \| \hat{a}_{L/k}(t) \| = 0. \)

By the previous paragraph, \( \hat{a}_{L/k}(t) = 0 \) and hence \( \hat{a}_{L/k((x))}(t) = 0. \) By Lemma 2.4 \( L/k((x)) \) is a primitive extension of type 1; the inclusion \( k((x)) \to L \) thus induces an isomorphism of completed algebraic closures. That is, \( t \) belongs to the completed algebraic closure of \( k((x)) \), but \( x \) does not belong to the completed algebraic closure of \( k((t)) \). If we write \( K' \) for a completed algebraic closure of \( k((x)) \), we then have a strict inclusion \( K \to K' \); composing this with an identification \( K' \cong K \) yields the desired endomorphism. \( \square \)

**Proof of Theorem 1.2:** We use [8, Theorem 1.5.6] as our blanket reference concerning the perfectoid correspondence. By [8, Example 1.3.5], there is an algebraic extension of \( \mathbb{Q}_p \) whose completion is perfectoid with tilt isomorphic to the completed perfect closure of \( \mathbb{F}_p((t)) \); hence \( \mathbb{C}_p \) is perfectoid and \( \mathbb{C}_p^b \) is isomorphic to the completed algebraic closure of \( \mathbb{F}_p((t)) \). By Theorem 1.2, there exists an endomorphism \( \tau : \mathbb{C}_p^b \to \mathbb{C}_p^b \) which is not surjective; this corresponds to a morphism \( \mathbb{C}_p \to K \) of perfectoid fields which is not surjective either. In particular, the integral closure of \( \mathbb{Q}_p \) in \( K \) is not dense, so \( K \) cannot admit any isomorphism to \( \mathbb{C}_p \) in the category of topological fields. \( \square \)

**References**

[1] J. Ax, Zeros of polynomials over local fields—the Galois action, *J. Alg.* 15 (1970), 417–428.
[2] V. Berkovich, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*, Surveys and Monographs 33, Amer. Math. Soc., Providence, 1990.
[3] A. Cohen, M. Temkin, and D. Trushin, Morphisms of Berkovich curves and the different function, *Adv. Math.* 303 (2016), 800–858.
[4] L. Fargues and J.-M. Fontaine, Courbes et fibrés vectoriels en théorie de Hodge \( p \)-adique, in preparation; draft (September 2015) available at [http://webusers.imj-prg.fr/~laurent.fargues/](http://webusers.imj-prg.fr/~laurent.fargues/).
[5] L. Fargues and J.-M. Fontaine, Vector bundles on curves and \( p \)-adic Hodge theory, in *Automorphic Forms and Galois Representations, Volume 1*, London Math. Soc. Lect. Note Ser. 414, Cambridge Univ. Press, 2014.
[6] L. Gruson, Théorie de Fredholm \( p \)-adique, *Bull. Soc. Math. France* 94 (1966), 67–95.
[7] K.S. Kedlaya, On the algebraicity of generalized power series, *Beitr. Algebra Geom.* (2016).
[8] K.S. Kedlaya, New methods for \( (\phi, \Gamma) \)-modules, *Res. Math. Sci.* 2:20 (2015).
[9] K.S. Kedlaya, Automorphisms of perfect power series rings, [arXiv:1602.09051v1](http://arxiv.org/abs/1602.09051) (2016).
[10] K.S. Kedlaya and R. Liu, Relative \( p \)-adic Hodge theory, I: Foundations, *Astérisque* 371 (2015), 239 pages.
[11] M. Matignon and M. Reversat, Sous-corps fermés d’un corps valué, *J. Algebra* 90 (1984), 491–515.
[12] P. Scholze, Perfectoid spaces, *Publ. Math. IHÉS* 116 (2012), 245–313.
[13] M. Temkin, Stable modifications of relative curves, *J. Alg. Geom.* 19 (2010), 603–677.
[14] M. Temkin, Metrization of differential pluriforms on Berkovich analytic spaces, *Nonarchimedean and Tropical Geometry*, Simons Symposia, Springer, 2016, 195–285.