Regularization of $\delta'$ potential in general case of deformed space with minimal length

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Abstract

In general case of deformed Heisenberg algebra leading to the minimal length we present a definition of the $\delta'(x)$ potential as a linear kernel of potential energy operator in momentum representation. We find exactly the energy level and corresponding eigenfunction for $\delta'(x)$ and $\delta(x) - \delta'(x)$ potentials in deformed space with arbitrary function of deformation. The energy spectrum for different partial cases of deformation function is analysed.

Keywords: deformed Heisenberg algebra, minimal length, delta prime potential.
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1 Introduction

String theory and quantum gravity suggest the existence of minimal length as a finite lower bound to the possible resolution of length [1–3]. Kempf et al. showed that minimal length can be achieved by modifying usual canonical commutation relations [4–7]. One of the simplest deformed Heisenberg algebra in one-dimensional case is the one proposed by Kempf [4]

$$[\hat{X}, \hat{P}] = i\hbar(1 + \beta P^2), \quad (1)$$

leading to minimal length $\hbar\sqrt{\beta}$.

More general deformed algebra has the form

$$[\hat{X}, \hat{P}] = i\hbar f(\hat{P}), \quad (2)$$

where $f$ is called function of deformation, being strictly positive ($f > 0$), even function. Algebra (2) admits the following representation

$$\hat{X} = \hat{x} = i\hbar \frac{d}{dp}, \quad (3)$$

$$\hat{P} = g(p).$$

Function $g(p)$ is an odd function satisfying $\frac{dg(p)}{dp} = f(g(p))$. It is defined on finite domain $p \in [-b, b]$, with $b = g^{-1}(a)$. Here $a$ denotes the limit of momentum $P \in [-a, a]$. Note that finiteness of $b$ provides the existence of minimal uncertainty in position [8].

The study of the effect of the minimal length on systems with singular potentials or point interactions is of particular interest, since such systems are expected to have a nontrivial sensitivity to minimal length. The impact of the minimum length has been studied in the context of the following problems with singularity in
potential energy: hydrogen atom [9,16], gravitational quantum well [17,19], a particle in delta potential and double delta potential [20,21], one-dimensional Coulomb-like problem [21,23], particle in the singular inverse square potential [24,27], two-body problems with delta and Coulomb-like interactions [28].

In undeformed quantum mechanics the interest in studies of point interactions is twofold. The first reason is that point interaction is a good model of a very localized interaction connected with different structures like quantum waveguides [29,30], spectral filters [31,32], or infinitesimally thin sheets [33,34]. Another reason is that it is often possible for such systems to obtain the solution exactly.

The problem of the correct interpretation of the hamiltonian with \( \delta' \)-function in potential energy

\[
\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \kappa \delta'(x) \tag{4}
\]

has been considered in literature since the 80s of last century [35–37]. The \( \delta' \)-potential is very sensitive to a way of its regularization. From a physical point of view, this means that there is no unique one-dimensional model of the delta prime interaction described by the hamiltonian (4). In order to avoid any confusion it should be emphasized that there can be distinguished a few different approaches corresponding to \( \delta' \)-potential definition.

By the first time \( \delta' \)-interaction was considered in [35]. The hamiltonian \( H \) was defined as the one-parameter family of self-adjoint extensions of an operator

\[
H_\beta = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \kappa \varepsilon \alpha (\delta(x+\varepsilon) + \delta(x-\varepsilon)) \tag{11}
\]

acting on the domain of wavefunction with derivative to be continuous, while the wavefunction has a jump proportional to its derivative at \( x = 0 \)

\[
\psi'(-0) = \psi'(+0), \quad \psi(+0) - \psi(-0) = \beta \psi'(0), \tag{6}
\]

with \( \beta \) depending on \( \kappa \), defined in [4]. It was shown in [37] that this selfadjoint extensions correspond to the heuristic operator

\[
H_\beta = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \beta |\delta'> < \delta'|, \tag{7}
\]

with renormalized coupling. Here \( |\delta'> < \delta'| \) denotes the following operator

\[
(|\delta'> < \delta'| \psi)(x) = \delta'(x) \int \delta'(y)\psi(y)dy. \tag{8}
\]

Operator (7) is not very good to describe the \( \delta' \)-potential, thus.

In [38] it was proposed to define self-adjoint operator [4] using distribution theory for discontinuous functions and derive the following boundary conditions at the point where the interaction occurs

\[
\psi(+0) - \psi(-0) = \kappa \frac{2}{\varepsilon} (\psi(+0) + \psi(-0)) \tag{9}
\]

\[
\psi'(+0) - \psi'(-0) = -\kappa \frac{2}{\varepsilon} (\psi'(+0) + \psi'(-0)). \tag{10}
\]

In the same paper [37] the alternative definition of the problem was proposed

\[
H_\beta = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{\kappa}{\varepsilon \alpha} (\delta(x+\varepsilon) + \delta(x-\varepsilon)). \tag{11}
\]

However, Seba has proved that in the limit of \( \varepsilon \to 0 \) the interaction disappears if \( \alpha < 1/2 \), appears as a \( \delta(x) \) potential for \( \alpha = 1/2 \) and splits the system into two independent subsystems lying on the half-lines \((-\infty, 0)\) and \((0, \infty)\) if \( \alpha > 1/2 \).

The one more way of defining the Schrödinger operator with a potential \( \delta' \) is to approximate \( \delta' \) by regular potentials and then to investigate the convergence of the corresponding family of regular Schrödinger operators. This approach was firstly realized in [37] and studied in [39,40].
The aim of this paper is to show that in case of generalized uncertainty principle it is possible to introduce \( \delta'(x) \) potential in some natural way for deformed space with minimal length. Our proposal leads to the regularization of the divergent integrals associated with the energy levels of the \( \delta'(x) \) potential in one-dimensional nonrelativistic quantum mechanics.

We organize the rest of this paper as follows. In Section 2, we propose the definition of \( \delta'(x) \) potential in momentum representation in general case of deformed space with minimal length and obtain the exact relation from which the corresponding bound states energies can be extracted. In Section 3, the Schrödinger equation with \( \delta(x) - \delta'(x) \) potential in context of minimal length assumption is solved exactly. Some concluding remarks are reported in the last section.

2 \( \delta' \) potential

In general case of deformed space with minimal length Schrödinger equation in the momentum representation can be written as

\[
\frac{g^2(p)}{2m} \phi(p) + \int_{-b}^{b} U(p - p') \phi(p') dp' = E \phi(p)
\] (12)

with \( U(p - p') \) being the kernel of the potential energy operator. In undeformed space this kernel can by obtained by

\[
U(p - p') = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} V(x) \exp \left( -\frac{i}{\hbar} (p - p') x \right) dx.
\] (13)

In case of the delta prime interaction \( V(x) = \kappa \delta'(x) \) the kernel of potential energy is

\[
U(p - p') = \frac{i\kappa}{2\pi\hbar^2} (p - p').
\] (14)

We assume that in deformed space with minimal length \( U(p - p') \) is still expressed by formula (14) and write the Schrödinger equation for the delta prime potential in general case of deformed space with minimal length as

\[
(g^2(p) + q^2) \phi(p) + \frac{i\kappa m}{\pi\hbar^2} \int_{-b}^{b} (p - p') \phi(p') dp' = 0.
\] (15)

The solution of (15) can be proposed in the form

\[
\psi(p) = \frac{A p + B}{g^2(p) + q^2}.
\] (16)

Substituting (16) into (15) we obtain the following formulas

\[
A + \frac{i\kappa m}{\pi\hbar^2} B \int_{-b}^{b} \frac{dp}{g^2(p) + q^2} = 0,
\] (17)

\[
B - \frac{i\kappa m}{\pi\hbar^2} A \int_{-b}^{b} \frac{p^2 dp}{g^2(p) + q^2} = 0,
\] (18)

which yield the equation for energy spectrum

\[
1 = \alpha I_1(\varepsilon) I_2(\varepsilon)
\] (19)

where \( \alpha = \frac{\kappa^2 m^2}{\pi^2 \hbar^2} \) and

\[
I_1(\varepsilon) = b \int_{-b}^{b} \frac{dp}{g^2(p) + q^2} = \int_{-1}^{1} \frac{dy}{k^2(y) + \varepsilon^2}
\] (20)

\[
I_2(\varepsilon) = \frac{1}{b} \int_{-b}^{b} \frac{p^2 dp}{g^2(p) + q^2} = \int_{-1}^{1} \frac{y^2 dy}{k^2(y) + \varepsilon^2}
\] (21)
with \( y = \frac{p}{b} \in [-1, 1] \), \( g(p) = bk(y) \) and \( \varepsilon = \frac{q}{b} \).

It is important to note that in undeformed limit \( b \to \infty \) integral \( I_1(\varepsilon) \) diverges and Schrödinger equation (12) has no solution. In case of finite \( b \) integrals in (19) are convergent and the problem of delta prime potential is regularized in deformed space with minimal length. Both integrals \( I_1(\varepsilon) \) and \( I_2(\varepsilon) \) are positive and decreasing functions of \( \varepsilon > 0 \). This means that equation (19) has only one solution since \( \alpha \) can take only positive values.

Let us find the energy spectra of considerable problem for some special examples of deformation function.

**Example 1.**

In the simplest deformed commutation relation leading to minimal length

\[
f(P) = 1, \quad P \in [-b, b], \quad g(p) = p.
\]

Integrals \( I_1(\varepsilon) \) and \( I_2(\varepsilon) \) are

\[
I_1(\varepsilon) = \frac{2 \arctan \left( \frac{1}{\varepsilon} \right)}{\varepsilon},
\]

\[
I_2(\varepsilon) = 2 - 2\varepsilon \arctan \left( \frac{1}{\varepsilon} \right).
\]

Equation for energy spectrum reads

\[
1 = 4\alpha \arctan \left( \frac{1}{\varepsilon} \right) \left( \frac{1}{\varepsilon} - \arctan \left( \frac{1}{\varepsilon} \right) \right).
\]

**Example 2.** The next example of deformation function is the following

\[
f(p) = (1 + \beta p^2)^{3/2}, \quad a = \infty, \quad g(p) = \frac{p}{\sqrt{1 - \beta p^2}}, \quad b = \frac{1}{\sqrt{\beta}}.
\]

The needed integrals can be calculated explicitly

\[
I_1(\varepsilon) = -2 \frac{2}{1 - \varepsilon^2} + 2 \arctan \left( \frac{\sqrt{1 - \varepsilon^2}}{\varepsilon} \right),
\]

\[
I_2(\varepsilon) = 2 - \frac{2 \varepsilon \arctan \left( \frac{\sqrt{1 - \varepsilon^2}}{\varepsilon} \right)}{(1 - \varepsilon^2)^2} - \frac{2(\varepsilon^2 + 2)}{3(1 - \varepsilon^2)^2}.
\]

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**Figure 1:** Energy level of delta prime problem in deformed space with minimal length dependent on the coupling constant \( \alpha \).
Example 3. In case of Kempf’s deformation

\[ f(P) = (1 + \beta P^2), \quad a = \infty, \]  
\[ g(p) = \frac{1}{\sqrt{\beta}} \tan(\sqrt{\beta}p), \quad b = \frac{\pi}{2\sqrt{\beta}} \]  

integrals \( I_1(\varepsilon) \) and \( I_2(\varepsilon) \) can also be calculated

\[ I_1(\varepsilon) = \frac{2\pi}{\varepsilon(\pi\varepsilon + 2)}, \quad (32) \]
\[ I_2(\varepsilon) = \frac{2}{3} \frac{\varepsilon\pi^3 - 12\text{polylog}(2, \frac{\pi\varepsilon}{\pi\varepsilon - 2}) + 12\text{polylog}(2, \frac{\pi\varepsilon + 2}{\pi\varepsilon - 2})}{\pi\varepsilon(\pi^2\varepsilon^2 - 4)}. \quad (33) \]

The comparison of the dependencies of the energy \( \varepsilon \) on coupling constant \( \alpha \) is presented on Fig. 1.

3 \( \delta - \delta' \) potential

In this section let us consider more general potential in the form

\[ V(x) = -\lambda \delta(x) + \kappa \delta'(x) \]  

(34)

The kernel of the potential energy operator is the following

\[ U(p - p') = -\frac{\lambda}{2\pi\hbar} + \frac{i\kappa}{2\pi\hbar^2}(p - p'). \]  

(35)

We propose to write the Schrödinger equation for considerable problem as

Figure 2: Dependencies of the energy level of \( \delta - \delta' \) well on coupling constant \( \alpha \) for different for different values of \( \gamma \) in case of special examples of deformation function
\[
(g(p)^2 + q^2)\phi(p) - \frac{\lambda m}{\pi \hbar} \int_{-b}^{b} \phi(p') dp' + \frac{i \kappa m}{\pi \hbar^2} \int_{-b}^{b} (p - p') \phi(p') dp' = 0.
\] (36)

Similarly to previous section the solution of the Schrödinger equation is assumed to have the form

\[
\psi(p) = Ap + B g^2(p) + q^2
\] (37)

Substituting (37) into (36) we obtain the following equations

\[
A + \frac{i \kappa m}{\pi \hbar^2} B \int_{-b}^{b} \frac{dp}{g^2(p) + q^2} = 0,
\] (38)

\[
B - \frac{i \kappa m}{\pi \hbar^2} A \int_{-b}^{b} \frac{p^2 dp}{g^2(p) + q^2} - \frac{\lambda m}{\pi \hbar} B \int_{-b}^{b} \frac{dp}{g^2(p) + q^2} = 0,
\] (39)

which yields

\[
1 = \alpha I_1(\varepsilon) I_2(\varepsilon) + \gamma I_1(\varepsilon),
\] (40)

where \( \alpha = \frac{\kappa^2 m^2}{\pi \hbar^2} \), \( \gamma = \frac{\lambda m}{\pi \hbar} \) and \( I_1(\varepsilon) \) and \( I_2(\varepsilon) \) are defined in the previous section.

Remembering that \( I_1(\varepsilon) \) and \( I_2(\varepsilon) \) are positive and decreasing functions of \( \varepsilon \) and \( \alpha \) is always positive, we conclude that equation (40) has only one solution but in the case of

\[
\gamma > -\gamma_0, \quad \gamma_0 = \alpha I_2(0) > 0,
\] (41)

with \( I_2(0) \) being the maximal value of \( I_2(\varepsilon) \).

For the considered in previous section special examples of deformation function value \( I_2(0) \) is equal to 2, 4/3 and \( 4 \ln 2 - \pi^2/6 \approx 1.12765 \) correspondingly. From this results we conclude that \( \gamma_0 \) strongly depends on the choice of the function of deformation. The energy level dependent on \( \alpha \) for different values of \( \gamma \) is presented on Fig.2

4 Conclusion

In this paper we have studied the general case of deformed Heisenberg algebra leading to the minimal length. The problem of the definition of the \( \delta'(x) \) operator has been examined. We have proposed the definition of \( \delta'(x) \) as the lineal kernel of potential energy operator given by (14). Using this definition we have solved exactly 1D \( \delta'(x) \)-potential problem in the general case of deformed Heisenberg algebra leading to the minimal length. In general case we obtain that energy spectrum consists of only one energy level. In the undeformed limit the divergency in the integral \( I_1(\varepsilon) \) occurs and energy level vanishes. We also have obtained the transcendental equations on the energy spectrum in some particular cases of the deformation functions.

We have also considered the case of \( -\lambda \delta(x) + \kappa \delta'(x) \) potential. We obtain that there is one bound state for \( \kappa > -\kappa_0 \) and no bound states for \( \kappa \leq -\kappa_0 \), with \( \kappa_0 \) (which is up to notations given in (41)) depending on \( \lambda \) and choice of function of deformation. This fact can serve as a distinguishing factor for different deformation functions.

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