THE MUON ANOMALOUS MAGNETIC MOMENT
- SIGNIFICANCE OF THE NEW MEASUREMENT

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0.1 INTRODUCTION

The beautiful experiment measuring the muon anomalous magnetic moment \((g-2)\) now in progress at Brookhaven [1] has already attained a precision that challenges our ability to calculate the expected result of the experiment. The uncertainty of the Brookhaven \((g-2)\) measurement is now about 1.5 in units of \(10^{-10} a_{\mu} \) (where \(a_{\mu} = \frac{g-2}{2}\)), with 1 year of data yet to be analyzed according to reference 1. Weak and electromagnetic contributions to \((g-2)\) can be calculated according to a well-understood theory with about 1/30'th of the uncertainty of the measurement[2], as is noted in the reference. The difference between that calculation and the measurement is 71.8 with the uncertainty of the experimental result. The difference is ascribed to hadronic effects (see [3] for a recent review).

The significance of the new result is, in my view, that it measures the hadronic contribution to the muon \((g-2)\) with a precision of the order of 1 %. The precision may increase by about an order of magnitude when the experiment is completed. There is no theory of hadrons that predicts hadronic interactions so precisely. The appropriate theory should presumably be grounded in quantum chromodynamics (QCD), which does not yet lend itself to such precise predictions. There are phenomenological arguments which relate the hadronic contribution to \((g-2)\) to other experiments and provide the basis for current estimates which range [4] from 69.2 to 72.5, each with a quoted uncertainty of the order of a percent.

The quoted uncertainties of current estimates reflect only the uncertainties of the experimental data used for phenomenological estimates and the computational uncertainties of the estimates. They do not, in my view, adequately reflect the uncertainties in the theories underlying the phenomenological estimates, as I shall explain in the remainder of this note.

0.2 DESCRIPTION OF THE HADRONIC CONTRIBUTION

In the notation of Sommerfield [5], the muon propagator \(G_{\mu \nu}\) relates to a mass function \(M\) as

\[
G_{\mu \nu}^{-1} = \gamma \Pi + M
\]

(1)

where \(\Pi_\mu = (p - eA)_\mu\), \(p\) is the muon momentum and \(A\) is the potential for a weak constant external field. The anomalous g-factor occurs as a term proportional to \(\sigma_{\mu \nu} F_{\mu \nu}\) in the mass function \(M\). That function is calculated from the equation

\[
M = m_0 + i e^2 \gamma G_{\mu \nu} \Gamma G
\]

(2)

involving the photon Green’s function \(G\) and the “dressed photon-muon vertex” \(\Gamma\).

The photon Green’s function is assumed [6] to obey a Källen-Lehman representation [7]. According to that representation each element \(G_{\mu \nu}\) may be written as a dispersion integral

\[
G_{\mu \nu}(k^2) = \frac{\delta_{\mu \nu}}{k^2} - \left[\frac{\delta_{\mu \nu} - k_\mu k_\nu}{k^2}\right] \int_0^\infty \frac{s(x)}{x + k^2 - i\epsilon} dx.
\]

(3)
The weight function \( s(x) \) is proportional to the sum over possible intermediate states produced by the electromagnetic current, according to the equation

\[
k^4 s(-k^2) \delta_{\mu \nu} = \frac{1}{3} \sum_{\alpha < 0} \langle j_\mu(0) | \alpha > \langle \alpha | j_\nu(0) | 0 > \delta^4(\alpha - k)
\]

(4)

where \( j \) is the electromagnetic current operator. The hadronic contribution to \( G \) is then given by that part of \( s(x) \) for which the states \( | \alpha > \) consist of hadrons. The matrix elements \( \langle 0 | j_\mu | \alpha > \) are the amplitudes for production of the state \( | \alpha > \) from the vacuum by the electromagnetic current. The corresponding contributions to \( s(x) \) are therefore proportional, to lowest order in the fine-structure constant, to the cross-sections for producing states \( \alpha \) in \( e^+e^- \) collisions. It would accordingly seem that such measured cross-sections could be used to calculate the hadronic contributions to \( G \).

The components \( \Gamma_{\mu} \) of the vertex function \( \Gamma \) contain terms corresponding to the production of a virtual hadron that couples to the muon. The lowest order such terms (in powers of the charge \( e \)) - the so-called “light-light scattering” terms - are of order \( e^6 \). Power counting suggests that such terms contribute to the anomalous muon moment at a level of about \( 10^{-6} \)th of the uncertainty in the Brookhaven measurement. There is, however, no theory comparable to electroweak theory for calculating these terms, as Hayakawa and Kinoshita have emphasized \[8\] in connection with their own estimates. Those estimates are not severely tested at the present level of experimental accuracy.

### 0.3 Uncertainties in Hadronic-Contribution Estimates

The Källen-Lehman representation \[7\] for the photon propagator is of the nature of a dispersion relation. The dispersion relation is a statement that \( G_{\mu,\nu}(k^2) \) is a real analytic function of the variable \( k^2 \), cut along the negative (that is, timelike) \( k^2 \) axis \[9\]. It is also, however, a statement that \( G \) is a polynomially-bounded function for large, complex \( k^2 \). I contend that the latter statement is unprovable\[10\] \[11\]. There does not, in any event, appear to be any existing proof that \( G \) obeys such a boundedness property. Any attempt to assign a numerical uncertainty to a calculation based upon a Källen-Lehman representation for the photon propagator must therefore be regarded as speculative.

There are recent estimates of the hadronic contributions to the muon g-2 that are within a few per cent of values that one would deduce from the new experimental result \[1\]. Is this near agreement evidence for the polynomial boundedness of \( G \)? I present in the next section models, by way of counter-examples, where such near agreement can occur even with functions that grow exponentially. Polynomial boundedness cannot, therefore, be deduced from the present data. Uncertainty estimates based upon dispersion-relation calculations of \( G \) must be understood as lower limits because the theories underlying the calculations are incompletely defined.
0.4 Dispersion Relations for Functions Having Exponential Growth

I present two models to show that an attempt to calculate a function by a dispersion relation, when the assumption of polynomial boundedness is incorrect, need not lead to a result that is wildly different from the correct result. The fact that dispersive calculations of the hadronic component of the muon g-2 are close to the values deduced from experiment is, therefore, not evidence that such calculations can give correct results.

Both of the models happen to involve functions related to Bessel functions. The essence of the demonstration, however, may be made with the two elementary functions natural logarithm and exponential. Consider, for this purpose the following function \( Q(z) \) of a complex variable \( z \):

\[
Q(z) = \ln(-z) + e^{-bz} \tag{5}
\]

where \( b \) is some real positive constant. \( Q(z) \) is an analytic function of the complex variable \( z \) except for logarithmic branch points at 0 and \( \infty \) and an essential singularity (of the exponential) at \( \infty \). \( Q \) is made single-valued by a cut along the positive real axis, giving \( Q \) an imaginary part equal to \(-i\pi\) as the real axis is approached from above.

The point of the models is that they involve functions that, like \( Q \), have an essential singularity and are therefore not polynomially bounded. One may nevertheless try write a dispersion relation for \( Q \) by integrating around the cut, because the imaginary part, being a constant in this case, is well-behaved. The result of the dispersion integral will be to recover only the \( \ln \) part of \( Q \). The remaining part of \( Q \), the exponential, will be negligible (compared with the logarithm) almost everywhere along the positive real axis, depending upon the size of the parameter \( b \). The erroneous assumption that \( Q \) satisfies a dispersion relation therefore makes only a small, model-dependent error almost everywhere along the positive real axis.

0.4.1 A Hankel-Function Model

In the first model I assume that the correct function that I am trying to calculate is the Hankel function \( H_0^{(1)}(z) \). The imaginary part of the Hankel function is the Neumann function \( Y_0(z) \). Both functions emulate a coulomb-like singularity at \( z = 0 \), albeit only logarithmically, so a numerical integration of a dispersion relation must be cut off at small \( z \). The function \( H_0^{(1)}(z) \) behaves for large, complex \( z \) like \( e^{iz}/\sqrt{z} \), and is not, therefore, polynomially bounded.

The model is unrealistic in the sense that the imaginary part of the “true function” is oscillatory, unlike the strictly positive cross-section data that is used as input in the photon-propagator calculations. That particular unrealistic feature is corrected in the second model.

The game, then, is to pretend that we know from experiment the imaginary part of the “correct” function. We then put the “experimental” data into an unsubtracted dispersion relation under the erroneous assumption that the “correct” function vanishes sufficiently rapidly at large \( |z| \), cutting off the integration at some small value of \( z \). We want to determine how much our answer deviates from the correct answer which is, in this case, the Bessel function \( J_0(z) \).

The answer can be determined analytically, because
1 \pi \int_{0}^{\infty} \frac{Y_0(x)dx}{x-z} = J_0(z) - \frac{2}{\pi^2} S_{-1,0}(z) \quad (6)

where the principal value of the integral is intended and \( S_{-1,0}(z) \) is a Lommel function whose explicit form has been given by Watson \([13]\). It is a solution of the inhomogeneous Bessel Equation

\[ S'' + \frac{1}{z} S' + S = \frac{1}{z^2} \quad (7) \]

subject to the condition that it behaves asymptotically like \( \frac{1}{z^2} \). It grows for small \( z \) like \( \ln^2(z) \), but rapidly becomes small as \( z \) increases. Fig. 1 compares the right hand side of Eq. 6, shown by the solid line, with the “correct” answer \( J_0(z) \), shown by the dotted line. It is apparent that the error from using misusing the dispersion integral can be made quite small if small values of \( z \) are excluded, as they are in “real life” when coulomb and infra-red effects play a role.

0.4.2 A Strictly Positive Imaginary Part

A more realistic model would require that the simulated experimental data be given by a function that is strictly positive, like a cross-section. A convenient choice is the modified Hankel function \( K_0(z) \) which approaches zero asymptotically for large \( z \) like \( \frac{\ln z}{\sqrt{z}} \) and grows logarithmically for small \( z \) like \( -\ln(z) \). For \( z \) approaching the positive real axis from above, there is a function \( f(z) \) given by

\[
f(z) = \frac{1}{\pi} \int_{0}^{\infty} K_0(x)dx\left( \frac{1}{x-z} - \frac{1}{x+z} \right) = \pi(L_0(z) - I_0(z)) + iK_0(z) \quad (8)
\]

Here \( L_0 \) is the modified Struve function and \( I_0 \) is the modified Bessel function, each of order zero.

The complex function \( f(z) \) does not satisfy a dispersion relation. This is because the integral of the function around a closed loop at large \( z \) cannot be made arbitrarily small, since its imaginary
part, $K_0(z)$, grows without bound when $\frac{\pi}{2} < \text{arg}(z) < \frac{3\pi}{2}$. The imaginary part of $f(z)$ again represents the simulated experimental data.

The game, as before, is to put the simulated experimental data into an unsubtracted dispersion relation and compare with the correct answer, given in Eq 8. The result is shown in Fig. 2 which displays the real part of the dispersion result as a solid line and the real part of $f(s)$ as a dotted line (the imaginary parts are, of course, identical). It is evident that for values of $z$ greater than about unity, the dispersion result is not greatly different from the “correct” value.

### 0.5 Conclusion

Hadronic contributions to g-2 originate, according to our present understanding, from electromagnetic excitation of virtual quark pairs from the vacuum. The task of calculating the effects of such excitations therefore falls within the realm of Quantum Chromodynamics ("QCD"). Such calculations have not yet been done. It has been tempting to suppose, as an alternative, that dispersion relations using empirical data could be used to predict the results of QCD calculations, at least for the vacuum polarization corrections to the photon propagator.

The difficulty with the supposition is that a dispersion relation involves an assumption about the behavior of a function for large values of its argument, namely, that the function is at least polynomially bounded. In the case of the photon propagator there is no way to justify that assumption.

It might be tempting to try to evade this lack of justification by reversing the burden of proof, asking: "What is the physical origin of the analogous terms in the physically relevant case? In particular, if $\text{Exp}[-z] \to \text{Exp}[-E/E_0]$, what is the meaning and value of the energy scale $E_0$?"

Because the presence of an essential singularity at infinite energy, as I have modeled with an exponential term, is purely conjectural, any mass, such as the pion mass, the QCD scale or a
spontaneously generated mass [14] is a possible candidate for the energy scale E0. One doesn’t see such terms in the usual discussions of analytic properties of amplitudes because those discussions are customarily guided by perturbation theory [13] [11], where such terms cannot occur. But we know of models where perturbation theory breaks down. [14]. And essential singularities are a common feature of solutions of differential equations such as the relativistic Schrödinger equation, cf. [16].

Proof of the existence of an essential singularity would, of course, end the discussion; dispersion relations would not provide an algorithm for calculating the hadronic contribution to vacuum polarization. That is not the intent of this note, however. The intent is merely to point out that there is, as yet, no mathematical justification for the use of dispersion relations for that purpose, and the accuracy of the resulting predictions must therefore remain uncertain.

What I have shown here is that violation of the assumption of polynomial boundedness can, in some circumstances, lead to results that are close to the correct results. There is, however, no apparent a priori way to estimate the size of the error resulting from the violation. There is consequently no way to estimate the error involved in using a dispersion relation to calculate the hadronic contribution in question. [17]

The Brown, et al. [1] measurement of the muon g-2 is therefore significant because it measures the hadronic contribution to g-2 with unprecedented precision. That measurement stands as a challenge to our understanding of the quark structure of the vacuum. [17]

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