NORMALISERS OF ABELIAN IDEALS OF A BOREL SUBALGEBRA AND 
\[ \mathbb{Z} \]-GRADINGS OF A SIMPLE LIE ALGEBRA

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ABSTRACT. Let \( g \) be a simple Lie algebra and \( \mathfrak{A}b \) the poset of all abelian ideals of a fixed Borel subalgebra of \( g \). If \( a \in \mathfrak{A}b \), then the normaliser of \( a \) is a standard parabolic subalgebra of \( g \). We give an explicit description of the normaliser for a class of abelian ideals that includes all maximal abelian ideals. We also elaborate on a relationship between abelian ideals and \( \mathbb{Z} \)-gradings of \( g \) associated with their normalisers.

INTRODUCTION

Let \( g \) be a complex simple Lie algebra with a triangular decomposition \( g = u \oplus t \oplus u^- \), where \( t \) is a fixed Cartan subalgebra and \( b = u \oplus t \) is a fixed Borel subalgebra. A subspace \( a \subset b \) is an abelian ideal if \( [b, a] \subset a \) and \( [a, a] = 0 \). Then \( a \subset u \). The general theory of abelian ideals of \( b \) is based on their relations with the so-called minuscule elements of the affine Weyl group \( \hat{W} \), which is due to D. Peterson (see Kostant’s account in [6]). The subsequent development has lead to a number of spectacular results of combinatorial and representation-theoretic nature, see e.g. [2, 3, 4, 7, 8, 10, 13, 14].

The normaliser of \( a \) in \( g \), denoted \( n_g(a) \), contains \( b \), i.e., it is a standard parabolic subalgebra of \( g \). In this note, we study the normalisers of abelian ideals using the corresponding minuscule elements of \( \hat{W} \) and \( \mathbb{Z} \)-gradings of \( g \).

Let \( \Delta \) be the root system of \( (g, t) \), \( \Delta^+ \) the set of positive roots corresponding to \( u \), \( \Pi \) the set of simple roots in \( \Delta^+ \), and \( \theta \) the highest root in \( \Delta^+ \). Then \( W \) is the Weyl group and \( g_\gamma \) is the root space for \( \gamma \in \Delta \). We write \( \mathfrak{A}b = \mathfrak{A}b(g) \) for the set of all abelian ideals of \( b \) and think of \( \mathfrak{A}b \) as poset with respect to inclusion. Since \( a \in \mathfrak{A}b \) is a sum of certain root spaces of \( u \), we often identify such an \( a \) with the corresponding subset \( I = I_a \) of \( \Delta^+ \).

Let \( \mathfrak{A}b^o \) denote the set of nonzero abelian ideals and \( \Delta^+_I \) the set of long positive roots. In [8, Sect. 2], we defined a surjective mapping \( \tau : \mathfrak{A}b^o \to \Delta^+_I \) and studied its fibres. If \( \tau(a) = \mu \), then \( \mu \in \Delta^+_I \) is called the rootlet of \( a \). Letting \( \mathfrak{A}b_\mu = \tau^{-1}(\mu) \), we get a partition of \( \mathfrak{A}b^o \) parameterised by \( \Delta^+_I \). Each fibre \( \mathfrak{A}b_\mu \) is a sub-poset of \( \mathfrak{A}b \). By [8, Sect. 3], the poset \( \mathfrak{A}b_\mu \) has a unique minimal and unique maximal element for any \( \mu \in \Delta^+_I \). These are denoted by \( a(\mu)_{\text{min}} \) and \( a(\mu)_{\text{max}} \), respectively. The corresponding sets of positive roots are \( I(\mu)_{\text{min}} \) and \( I(\mu)_{\text{max}} \). The abelian ideals of the form \( a(\mu)_{\text{min}} \) (resp. \( a(\mu)_{\text{max}} \)) will be referred to as
the root-minimal (resp. root-maximal). The set of globally maximal abelian ideals coincides with \( \{ a(\alpha)_{\max} \mid \alpha \in \Pi_l \} \), where \( \Pi_l = \Delta_l^+ \cap \Pi \) [8, Cor. 3.8].

If \( p \supset b \), then a Levi subalgebra \( l \) of \( p \) is said to be standard, if \( l \supset t \). Set \( p[\mu]_{\min} = n_g(\alpha(\mu)_{\min}) \) and \( p[\mu]_{\max} = n_g(\alpha(\mu)_{\max}) \). Write \( \Pi[\mu]_{\min} \) for the simple roots of the standard Levi subalgebra of \( p[\mu]_{\min} \), and likewise for ‘max’. Our main results are the following:

I. We explicitly describe \( \Pi[\mu]_{\min} \) for any root-minimal ideal \( a(\mu)_{\min} \). The answer is given in terms of the element \( w_\mu \in W \) that takes \( \theta \) to \( \mu \) and has minimal possible length, see Theorem 2.3. The elements \( w_\mu \) have already been considered in [8], and we also provide here new properties of them. Furthermore, if \( \theta \) is fundamental and \( \alpha_\theta \in \Pi \) is such that \( (\theta, \alpha_\theta) \neq 0 \), then \( \alpha_\theta \) is long and we prove that \( \Pi \setminus \Pi[\alpha_{\min}] \) consists of the simple roots that are adjacent to \( \alpha_\theta \) in the Dynkin diagram (Proposition 2.4).

II. We give a new characterisation of normalisers of arbitrary \( b \)-stable subspaces of \( u \) (Theorem 3.3) and then explicitly describe the normalisers of the globally maximal abelian ideals, i.e., we determine \( \Pi[\alpha]_{\max} \) for all \( \alpha \in \Pi_l \) (Theorem 3.9). This is based on a relationship between \( a(\alpha)_{\min} \) and \( a(\alpha)_{\max} \) for \( \alpha \in \Pi_l \) [10, Theorem 4.7], which allows us to retrieve information on \( \Pi[\alpha]_{\max} \) from that on \( \Pi[\alpha]_{\min} \).

III. In Section 4, we relate \( a \in \mathfrak{Ab}(g) \) to the \( \mathbb{Z} \)-grading of \( g \) corresponding to \( n_g(\alpha) \). Let \( \mathfrak{Par}(g) \) denote the set of all standard parabolic subalgebras of \( g \). By Peterson’s theorem [6], \#\( \mathfrak{Ab}(g) = 2^{rk g} \), hence the sets \( \mathfrak{Ab}(g) \) and \( \mathfrak{Par}(g) \) are equipotent. There is the natural mapping \( f_1 : \mathfrak{Ab}(g) \to \mathfrak{Par}(g) \) that takes \( a \) to \( n_g(\alpha) \). By [12], \( f_1 \) is a bijection if and only if \( g = sl_{n+1} \) or \( sp_{2n} \). Using the \( \mathbb{Z} \)-grading associated with \( p \in \mathfrak{Par}(g) \), we define here the natural mapping \( f_2 : \mathfrak{Par}(g) \to \mathfrak{Ab}(g) \) and prove that \( f_2 \) is a bijection if and only if \( g = sl_{n+1} \) or \( sp_{2n} \); furthermore, \( f_2 = f_1^{-1} \) for these two series (Theorem 4.5). We say that \( a \in \mathfrak{Ab} \) is reflexive, if \( (f_2 \circ f_1)(a) = a \). Then all abelian ideals for \( sl_{n+1} \) and \( sp_{2n} \) are reflexive. We also prove that \( a(\alpha)_{\min} \) and \( a(\alpha)_{\max} \) \( (\alpha \in \Pi_l) \) are always reflexive and characterise them in terms of the corresponding \( \mathbb{Z} \)-gradings (see Theorem 4.2 and Remark 4.6). Finally, we conjecture that the sets \( \text{Im}(f_1 \circ f_2) \) and \( \text{Im}(f_2 \circ f_1) \) are always equipotent and the maps \( f_1 \) and \( f_2 \) induce the mutually inverse bijections between them.

We refer to [1, 5] for standard results on root systems and (affine) Weyl groups.

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1. Preliminaries on minuscule elements and normalisers of abelian ideals

We equip \( \Delta^+ \) with the usual partial ordering ‘\( \preceq \)’. This means that \( \mu \preceq \nu \) if \( \nu - \mu \) is a non-negative integral linear combination of simple roots. If \( M \) is a subset of \( \Delta^+ \), then \( \min(M) \) and \( \max(M) \) are the minimal and maximal elements of \( M \) with respect to “\( \preceq \)”. 
Any $b$-stable subspace $c \subset u$ is a sum of certain root spaces in $u$, i.e., $c = \bigoplus_{\gamma \in I} g_{\gamma}$. The relation $[b, c] \subset c$ is equivalent to that $I = I_\nu$ is an upper ideal of the poset $(\Delta^+, \leq)$, i.e., if $\nu \in I$, $\gamma \in \Delta^+$, and $\nu \preceq \gamma$, then $\gamma \in I$. We mostly work in the combinatorial setting, so that a $b$-ideal $c \subset u$ is being identified with the corresponding upper ideal $I$ of $\Delta^+$. The property of being abelian additionally means that $\gamma' + \gamma'' \not\in \Delta^+$ for all $\gamma', \gamma'' \in I$.

We recall below the notion of a minuscule element of $\widehat{W}$ and their relation to abelian ideals. We have $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, the vector space $t_\mathbb{R} = V = \bigoplus_{i=1}^n \mathbb{R}\alpha_i$, the Weyl group $W$ generated by simple reflections $s_1, \ldots, s_n$, and a $W$-invariant inner product $(,)$ on $V$. Letting $\widehat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$, we extend the inner product $(,)$ on $\widehat{V}$ so that $(\delta, V) = (\lambda, V) = (\delta, \delta) = (\lambda, \lambda) = 0$ and $(\delta, \lambda) = 1$. Set $\alpha_0 = \delta - \theta$, where $\theta$ is the highest root in $\Delta^+$. Then
\[
\widehat{\Delta} = \{\Delta + k\delta \mid k \in \mathbb{Z}\} \text{ is the set of affine (real) roots;}
\]
\[
\widehat{\Delta}^+ = \Delta^+ \cup \{\Delta + k\delta \mid k \geq 1\} \text{ is the set of positive affine roots;}
\]
\[
\widehat{\Pi} = \Pi \cup \{\alpha_0\} \text{ is the corresponding set of affine simple roots;}
\]
\[
\mu^\vee = 2\mu/(\mu, \mu) \text{ is the coroot corresponding to } \mu \in \widehat{\Delta}.
\]

For each $\alpha_i \in \widehat{\Pi}$, let $s_i = s_{\alpha_i}$ denote the corresponding reflection in $GL(\widehat{V})$. That is, $s_i(x) = x - (x, \alpha_i)\alpha_i^\vee$ for any $x \in \widehat{V}$. The affine Weyl group, $\widehat{W}$, is the subgroup of $GL(\widehat{V})$ generated by the reflections $s_0, s_1, \ldots, s_n$. The extended inner product $(,)$ on $\widehat{V}$ is $\widehat{W}$-invariant. The inversion set of $w \in \widehat{W}$ is $N(w) = \{\nu \in \widehat{\Delta}^+ \mid w(\nu) \in -\widehat{\Delta}^+\}$. Note that if $w \in W \subset \widehat{W}$, then $N(w) \subset \Delta^+$.

Following Peterson, we say that $w \in \widehat{W}$ is minuscule, if $N(w) = \{ -\gamma + \delta \mid \gamma \in I_w \}$ for some $I_w \subset \Delta$. One then proves that (i) $I_w \subset \Delta^+$, (ii) $I_w$ is (the set of roots of) an abelian ideal, and (iii) the assignment $w \mapsto I_w$ yields a bijection between the minuscule elements of $\widehat{W}$ and the abelian ideals, see [6], [2, Prop. 2.8]. Conversely, if $a \in \mathfrak{nil}$ and $I = I_a$, then $w_a \in \widehat{W}$ stands for the corresponding minuscule element. Clearly, $\dim a = \#I_a = \#N(w_a)$.

Given $a \in \mathfrak{nil}^0$ and $w_a \in \widehat{W}$, the rootlet of $a$ is defined by
\[
\tau(a) = w_a(\alpha_0) + \delta = w_a(2\delta - \theta).
\]

By [8, Prop. 2.5], we have $\tau(a) \in \Delta^+_t$ and every $\mu \in \Delta^+_t$ occurs in this way.

Let $l$ be the standard Levi subalgebra of $p = n_\mathfrak{g}(a)$ and $\Pi(l) \subset \Pi$ the set of simple roots of $l$. By [9, Theorem 2.8], the set $\Pi(l)$ is determined by $w_a$ as follows:
\[
\alpha \in \Pi(l) \iff w_a(\alpha) \in \widehat{\Pi}.
\]

(Actually, this result of [9] has been proved for any $b$-stable subspace $c \subset u$ in place of $\alpha$. To this end, one also needs a more general theory of elements of $\widehat{W}$ associated with arbitrary $b$-stable subspaces of $u$ [2].)

An advantage of our situation is that, for the root-minimal abelian ideals $a = a(\mu)_{\text{min}}$, there is a simple formula for $w_a$, which allows us to describe the corresponding normaliser in terms of $\mu$. We also need the following facts:
It is known that \( \#\tau^{-1}(\mu) = 1 \) (i.e., \( a(\mu)_{\text{min}} = a(\mu)_{\text{max}} \)) if and only if \((\theta, \mu) \neq 0 \) [8, Theorem 5.1].

- \( a \) is root-minimal if and only if \( I_\alpha \subset \mathcal{H} := \{ \gamma \in \Delta^+ \mid (\gamma, \theta) \neq 0 \} \) [8, Theorem 4.3].

In what follows, it will be important to distinguish the cases whether \( \theta \) is fundamental or not, and whether \((\theta, \mu) = 0 \) or not. Recall that \( \theta \) is fundamental if and only if \( \Delta \) is not of type \( A_n \) or \( C_n \). One also has \( \#(\Pi \cap \mathcal{H}) = \begin{cases} 2 & \text{for } A_n \\ 1 & \text{for all other types} \end{cases} \). For the classical series, we use the standard notation and numbering for \( \Pi \), which seems to be the same in all sources. For instance, for \( A_n \), we have \( \alpha_i = \varepsilon_i - \varepsilon_{i+1} \) \((i = 1, \ldots, n)\), whence \( \Pi \cap \mathcal{H} = \{\alpha_1, \alpha_n\} \). For \( E_6 \), our numbering is \( 1\ldots3\ldots4\ldots5\ldots6 \); hence \( \Pi \cap \mathcal{H} = \{\alpha_6\} \).

For \( \gamma \in \Delta \) and \( \alpha \in \Pi \), \([\gamma : \alpha]\) stands for the coefficient of \( \alpha \) in the expression of \( \gamma \) via \( \Pi \).

2. Normalisers of the root-minimal abelian ideals

In this section, we describe normalisers of the root-minimal abelian ideals for all \( \mu \in \Delta_1^+ \).

There is a unique element of minimal length in \( W \) taking \( \theta \) to \( \mu \) [8, Theorem 4.1], which is denoted by \( w_\mu \). The ideal \( a(\mu)_{\text{min}} \) is completely determined by \( w_\mu \). Namely, \( w_\mu s_0 \in \hat{W} \) is the minuscule element corresponding to \( a(\mu)_{\text{min}} \) [8, Theorem 4.2]. We begin with two useful properties of the elements \( w_\mu \).

**Lemma 2.1.** If \( \beta \in \Pi \) and \((\beta, \mu) = 0 \), then \( w_\mu^{-1}(\beta) \in \Pi \) and \((w_\mu^{-1}(\beta), \theta) = 0 \).

**Proof.** It is known that \( N(w_\mu^{-1}) = \{ \gamma \in \Delta^+ \mid (\gamma, \mu^\vee) = -1 \} \) [8, Theorem 4.1(2)]. Therefore \( w_\mu^{-1}(\beta) \in \Delta^+ \). Assume that \( w_\mu^{-1}(\beta) = \gamma_1 + \gamma_2 \) is a sum of positive roots. Then \( \beta = w_\mu(\gamma_1) + w_\mu(\gamma_2) \). Without loss of generality, one may assume that \(-\nu_1 := w_\mu(\gamma_1) \) is negative. Then \( \nu_1 \in N(w_\mu^{-1}) \), hence \((-\nu_1, \mu^\vee) = 1 \). Consequently, \((\gamma_1, \theta^\vee) = 1 \). On the other hand, \( 0 = (\mu, \beta) = (\theta, \gamma_1 + \gamma_2) \) and therefore \( (\theta, \gamma_2) < 0 \), which is impossible. Thus, \( w_\mu^{-1}(\beta) \) must be simple and \((w_\mu^{-1}(\beta), \theta) = (\beta, \mu) = 0 \). \( \square \)

**Lemma 2.2.** Suppose that \( \theta \) is fundamental and \( \alpha_\theta \in \Pi \) is not orthogonal to \( \theta \). If \((\theta, \mu) > 0 \) and \( \theta \neq \mu \), then \( w_\mu^{-1}(\theta) = \theta - \alpha_\theta \); or, equivalently, \( w_\mu(a_\theta) = \mu - \theta \).

**Proof.** It is well known and easily verified that \( \alpha_\theta \) is long and \([\theta : \alpha_\theta] = 2 \) (cf. also Theorem 4.1(ii)). If \( \mu \in \mathcal{H} \setminus \{\theta\} \), then \([\mu : \alpha_\theta] = 1 \). By [11, Section 1], multiplicities of the simple reflections in any reduced expression of \( w_\mu \) are the same, and they are determined by the coefficients of \( \theta - \mu \). In particular, \( s_{\alpha_\theta} \) occurs only once, since \([\theta - \mu : \alpha_\theta] = 1 \) and \( \alpha_\theta \) is long. Moreover, the reduced expressions of \( w_\mu \) are in bijections with the “root paths” connecting \( \theta \) with \( \mu \) inside \( \Delta_1^+ \). Since \( \theta \) is fundamental, the passage \( \theta \sim s_{\alpha_\theta}(\theta) \) is the only step down from \( \theta \) inside \( \Delta_1^+ \). Hence any root path leading to \( \mu \) starts with this step. Therefore, every reduced expression of \( w_\mu \) begins with \( s_{\alpha_\theta} \), and one can write \( w_\mu = w's_{\alpha_\theta} \), where \( w' \) does not contain factors \( s_{\alpha_\theta} \). Therefore, \( w_\mu^{-1}(\theta) = s_{\alpha_\theta}w'^{-1}(\theta) = s_{\alpha_\theta}(\theta) = \theta - \alpha_\theta \). \( \square \)
Remark. This is a generalisation of [11, Lemma 4.3], where the similar assertion is proved for \( \mu = \alpha_\theta \).

Recall that \( \Pi[\mu]_{\min} \subset \Pi \) is the set of simple roots for the standard Levi subalgebra of \( \mathfrak{p}[\mu]_{\min} \). Since \( \theta \) is not fundamental if and only if \( \Delta = A_n \) or \( C_n \), the following result covers all the possibilities for \( \mu \).

**Theorem 2.3.** For any \( \mu \in \Delta^+ \), the set \( \Pi[\mu]_{\min} \) has the following description.

(i) \( \Pi[\mu]_{\min} \cap \theta^\perp = \{ w_\mu^{-1}(\beta) \mid \beta \in \Pi & (\beta, \mu) = 0 \} = \{ \alpha \in \Pi \mid w_\mu(\alpha) \in \Pi & (\alpha, \theta) = 0 \} \).

(ii) If \((\mu, \theta) = 0\), then \( \Pi[\mu]_{\min} = \{ w_\mu^{-1}(\beta) \mid \beta \in \Pi & (\beta, \mu) = 0 \} \). In particular, \( \Pi[\mu]_{\min} \subset \theta^\perp \).

(iii) Suppose that \((\mu, \theta) \neq 0\) (i.e., \( \mu \in \mathcal{H} \)) and \( \mu \neq \theta \).
   a) if \( \theta \) is fundamental, then \( \Pi[\mu]_{\min} = \{ \alpha_\theta \} \cup \{ w_\mu^{-1}(\beta) \mid \beta \in \Pi & (\beta, \mu) = 0 \} \), where \( \alpha_\theta \) is the only simple root such that \((\theta, \alpha_\theta) \neq 0\);
   b) if \( \Delta = C_n \), then there is no such long roots \( \mu \);
   c) if \( \Delta = A_n \) and \( \mu = \alpha_1 + \cdots + \alpha_i = \gamma_i \) \((i < n)\) or \( \alpha_j + \cdots + \alpha_n = \gamma_j \) \((j > 1)\), then
      \[ \Pi[\gamma_i]_{\min} = \{ \alpha_n \} \cup \{ w_\mu^{-1}(\beta) \mid \beta \in \Pi & (\beta, \gamma_i) = 0 \} = \Pi \setminus \{ \alpha_1, \alpha_i \} \text{ and} \]
      \[ \Pi[\gamma_j]_{\min} = \{ \alpha_1 \} \cup \{ w_\mu^{-1}(\beta) \mid \beta \in \Pi & (\beta, \gamma_j) = 0 \} = \Pi \setminus \{ \alpha_j, \alpha_n \} \].

(iv) If \( \mu = \theta \), then \( \Pi[\theta]_{\min} = \{ \beta \in \Pi \mid (\beta, \theta) = 0 \} \).

*Proof.* Since \( w_\mu s_0 \in \hat{W} \) is the minuscule element corresponding to \( I(\mu)_{\min} \), the general theory of normalisers of \( t \)-stable subspaces of \( \mathfrak{u} \) asserts that

\( (\text{2.1}) \quad \alpha \in \Pi[\mu]_{\min} \iff w_\mu s_0(\alpha) \in \hat{\Pi}, \)

see [9, Theorem 2.8]. Here one has to distinguish two possibilities:

1. \( w_\mu s_0(\alpha) \in \Pi; \)
2. \( w_\mu s_0(\alpha) = \alpha_0 = \delta - \theta. \)

- Suppose that \( w_\mu s_0(\alpha) = \beta \in \Pi. \) Then \( w_\mu^{-1}(\beta) = s_0(\alpha) \in \Delta. \) Hence \( s_0(\alpha) = \alpha \) and therefore \((\theta, \alpha) = 0\) and \((\beta, \mu) = (w_\mu(\alpha), w_\mu(\theta)) = 0\). Thus, if \( \alpha \in \Pi[\mu]_{\min} \) satisfies (1), then \( w_\mu(\alpha) = \beta \in \Pi \) and \((\beta, \mu) = (\theta, \alpha) = 0\).

Conversely, if \( \beta \in \Pi \) and \((\beta, \mu) = 0\), then Lemma 2.1 shows that \( \alpha := w_\mu^{-1}(\beta) \in \Pi \) and \((\alpha, \theta) = 0\). Hence (1) is satisfied for \( \mu \) and \( \alpha \).

- Suppose that \( w_\mu s_0(\alpha) = \alpha_0 = \delta - \theta. \) Then \( w_\mu^{-1}(\delta - \theta) = s_0(\alpha). \) Therefore, \( \alpha \in \Pi_t \) and \( s_0(\alpha) \neq \alpha, \) i.e., \((\alpha, \theta) \neq 0\). More precisely, \( \delta - w_\mu^{-1}(\theta) = \delta - (\theta - \alpha), \) hence \( w_\mu^{-1}(\theta) = \theta - \alpha. \) The last equality can be rewritten as \( \theta = \mu - w_\mu(\alpha). \) Therefore, \((\mu, \theta) \neq 0\) and \( \mu \neq \theta. \) Hence equality (2) can only occur for \( \mu \in \mathcal{H} \setminus \{ \theta \} \) and \( \alpha \in \mathcal{H}. \) Furthermore, if \( \theta \) is fundamental, then one must have \( \alpha = \alpha_\theta. \) By Lemma 2.2, the equality \( w_\mu^{-1}(\theta) = \theta - \alpha_\theta \) is then satisfied and we conclude that \( \alpha_\theta \in \Pi[\mu]_{\min}. \)

This proves parts (i),(ii),(iii).
Parts (iiib) is clear, and (iiic) is obtained by a direct calculation.

(iv) Here $a(\theta)_{\text{min}} = g_\theta$, and the assertion is obvious. \hfill \Box

Theorem 2.3 provides a complete description of $\Pi[\mu]_{\text{min}}$ for all $\mu \in \Delta^+$. But for some long simple roots, the assertion can be made even more precise.

**Proposition 2.4.** If $\theta$ is fundamental and $(\theta, \alpha_\theta) \neq 0$, then $\Pi[\alpha_\theta]_{\text{min}} = \{\alpha_\theta\} \cup \{\beta \in \Pi : (\beta, \alpha_\theta) = 0\}$. Therefore, $\Pi \setminus \Pi[\alpha_\theta]_{\text{min}}$ consists of the simple roots that are adjacent to $\alpha_\theta$ in the Dynkin diagram.

**Proof.** By Theorem 2.3(iii), we have $\Pi[\alpha_\theta]_{\text{min}} = \{\alpha_\theta\} \cup \{w^{-1}_\alpha(\beta) : \beta \in \Pi \setminus (\beta, \alpha_\theta) = 0\}$. Therefore, we are to prove that $w^{-1}_\alpha$ permutes the simple roots orthogonal to $\alpha_\theta$. If $\beta \in \Pi$ and $(\beta, \alpha_\theta) = 0$, then we already know that $w^{-1}_\alpha(\beta) \in \Pi$. Next, using Lemma 2.2 with $\mu = \alpha_\theta$, we obtain

$$(w^{-1}_\alpha(\beta), \alpha_\theta) = (\beta, w^{-\alpha_\theta}_\alpha(\alpha_\theta)) = (\beta, w^{-\alpha_\theta}_\alpha(\alpha_\theta)) = - (\beta, \theta).$$

Since $\beta \neq \alpha_\theta$ and $\theta$ is fundamental, this must be zero. \hfill \Box

The minuscule elements for the root-maximal abelian ideals do not admit a simple formula. Therefore, we cannot explicitly describe $p[\mu]_{\text{max}}$ for all $\mu \in \Delta^+$. However, if $\mu \in \Pi_l$, then $a(\mu)_{\text{min}}$ is closely related to $a(\mu)_{\text{max}}$, and such a situation is considered in the next section.

3. Normalisers of Some Root-Maximal Abelian Ideals

We begin with a new property of the normaliser of an arbitrary $b$-stable subspace of $u$. Let $c \subset u$ be such a subspace and $I_c$ the corresponding set of positive roots. Being a standard parabolic subalgebra, $n_g(c)$ is fully determined by the simple roots of the standard Levi subalgebra or, equivalently, by the set of simple roots $\alpha$ such that $g^{-\alpha}_\theta \not\in n_g(c)$. The following is proved in [12, Theorem 3.2].

**Theorem 3.1.** For any $b$-stable subspace $c \subset u$ and $\alpha \in \Pi$, we have

$$g^{-\alpha}_\theta \not\in n_g(c) \iff \exists \gamma \in \min(I_c) \text{ such that } \gamma - \alpha \in \Delta^+ \cup \{0\}.$$

The point of this result is that it suffices to test only the minimal roots of $I_c$. Note that if $\gamma - \alpha$ is a root, then $\gamma - \alpha \in \Delta^+ \setminus I_c$. Our new observation is that it is equally suitable to test only the maximal roots of $\Delta^+ \setminus I_c$. To this end, we first provide an auxiliary assertion.

**Lemma 3.2.** Suppose that $\mu \in \Delta^+$ and $\alpha, \bar{\alpha}$ are different simple roots. If $\mu + \alpha, \mu + \bar{\alpha} \in \Delta$, then $\mu + \alpha + \bar{\alpha} \in \Delta$. 


Theorem 3.3. Suppose that \( \mu \) automatically.

\[ \text{Proof.} \]  
The implication "\( \Rightarrow \)" is obvious.

"\( \Leftarrow \)". If \( g_{-\alpha} \notin n_{\theta}(c) \), then there is \( \mu \in \min(I_c) \) such that \( \mu - \alpha \in (\Delta^+ \setminus I_c) \cup \{0\} \).

- If \( \mu - \alpha \in \max(\Delta^+ \setminus I_c) \), then \( \gamma = \mu - \alpha \), and we are done;

- If \( \mu - \alpha \) is nonzero and not maximal in \( \Delta^+ \setminus I_c \), then there is an \( \tilde{\alpha} \in \Pi \) such that \( \mu - \alpha + \tilde{\alpha} \in \Delta^+ \setminus I_c \). Applying Lemma 3.2 to \( \mu - \alpha \) shows that \( \mu + \tilde{\alpha} \) is a root and then automatically, \( \mu + \tilde{\alpha} \in I_c \). Thus, the pair \( \{\mu - \alpha, \mu\} \) can be replaced with the "higher" pair \( \{\mu - \alpha + \tilde{\alpha}, \mu + \tilde{\alpha}\} \). Eventually, we obtain a pair whose lower root is maximal in \( \Delta^+ \setminus I_c \).

- If \( \mu = \alpha \), then \( I_c \) contains all positive roots with nonzero coefficient of \( \alpha \). Since \( \Delta^+ \setminus I_c \neq \emptyset \), there exists a \( \nu \in \Delta^+ \setminus I_c \) such that \( \nu + \alpha \) is a root, necessarily in \( I_c \). If \( \nu \notin \max(\Delta^+ \setminus I_c) \), then we can perform the induction procedure of the previous paragraph. \( \square \)

In the setting of abelian ideals, there is a special case in which \( \max(\Delta^+ \setminus I_c) \) is related to the minimal roots of another ideal.

Proposition 3.4 ([10, Theorem 4.7]). For any \( \tilde{\alpha} \in \Pi_I \), one has

\[ \gamma \in \min(I(\tilde{\alpha})_{\min}) \iff \theta - \gamma \in \max(\Delta^+ \setminus I(\tilde{\alpha})_{\max}). \]

In particular, if \( \text{rk} \Delta > 1 \) (i.e., \( I(\tilde{\alpha})_{\min} \neq \{\theta\} \)), then \( \max(\Delta^+ \setminus I(\tilde{\alpha})_{\max}) \subset H \setminus \{\theta\} \).

In the rest of this section, we only consider the abelian ideals with rootlet \( \tilde{\alpha} \in \Pi_I \). Using Theorem 3.3 and Proposition 3.4, we are going to compare the normalisers \( p[\tilde{\alpha}]_{\max} = n_{\theta}(a(\tilde{\alpha})_{\max}) \) and \( p[\tilde{\alpha}]_{\min} = n_{\theta}(a(\tilde{\alpha})_{\min}) \). We write \( S[\tilde{\alpha}]_{\max} \) and \( S[\tilde{\alpha}]_{\min} \), respectively, for the simple roots that do not belong to their standard Levi subalgebras. In other words, \( S[\tilde{\alpha}]_{\min} := \Pi \setminus \Pi[\tilde{\alpha}]_{\min} \), and likewise for ‘max’.

Theorem 3.5. For any \( \tilde{\alpha} \in \Pi \), we have \( S[\tilde{\alpha}]_{\max} \subset S[\tilde{\alpha}]_{\min} \) and thereby \( p[\tilde{\alpha}]_{\max} \supset p[\tilde{\alpha}]_{\min} \).
Proof. If \( g \neq s t_2 \), then \([u, u] \neq 0\). Hence \( a(\tilde{\alpha})_{\max} \neq u \), i.e., \( I(\tilde{\alpha})_{\max} \neq \Delta^+ \). Therefore, \( \alpha \in S[\tilde{\alpha}]_{\max} \) if and only if there exists \( \gamma \in \max(\Delta^+ \setminus I(\tilde{\alpha})_{\max}) \) such that \( \gamma + \alpha \in I(\tilde{\alpha})_{\max} \) (Theorem 3.3). Then \( \gamma \in \mathcal{H} \setminus \{\theta\} \) (Proposition 3.4) and hence \( \gamma + \alpha \in \mathcal{H} \cap I(\tilde{\alpha})_{\max} = I(\tilde{\alpha})_{\min} \) [10, Proposition 3.2]. By Proposition 3.4, we have \( \nu := \theta - \gamma \in \min(I(\tilde{\alpha})_{\min}) \) and \( \nu - \alpha = \theta - (\gamma + \alpha) \) is either a root or zero. In both cases, applying Theorem 3.1 to \( \nu \), we conclude that \( \alpha \in S[\tilde{\alpha}]_{\min} \). \[ \square \]

Actually, there is a more precise statement.

**Theorem 3.6.** Excluding the case in which \( \Delta \) is of type \( A_n \) with \( \tilde{\alpha} = \alpha_1 \) or \( \alpha_n \), we have \( S[\tilde{\alpha}]_{\max} = S[\tilde{\alpha}]_{\min} \cap \theta^\perp \).

**Proof.** 1. Suppose that \( \alpha \in S[\tilde{\alpha}]_{\max} \) and \( \gamma \in \max(\Delta^+ \setminus I(\tilde{\alpha})_{\max}) \) is such that \( \gamma + \alpha \in I(\tilde{\alpha})_{\max} \). As explained in the previous proof, we then have \( \nu = \theta - \gamma \in \min(I(\tilde{\alpha})_{\min}) \subset \mathcal{H} \) and \( \nu - \alpha \in \Delta^+ \cup \{0\} \). Consider these two possibilities for \( \nu - \alpha \).

(i) \( \nu = \alpha \). Then \( \alpha \in I(\tilde{\alpha})_{\min} \), which is only possible if \( \tilde{\alpha} = \alpha \), since \( I(\tilde{\alpha})_{\min} \subset \{\mu \in \Delta^+ | \mu \succ \tilde{\alpha}\} \) [10, Proposition 3.4]. Therefore \( \tilde{\alpha} = \alpha, \tilde{\alpha} \in \mathcal{H} \), and \( [\theta : \tilde{\alpha}] = 1 \). All this only occurs for \( \Delta \) of type \( A_n \) with \( \tilde{\alpha} = \alpha_1 \) or \( \alpha_n \).

(ii) \( \nu - \alpha \in \Delta^+ \). Then \( \nu - \alpha \in \mathcal{H} \), since \( (\nu - \alpha) + (\gamma + \alpha) = \theta \). That is both \( \nu \) and \( \nu - \alpha \) belong to \( \mathcal{H} \setminus \{\theta\} \). Hence \( (\theta, \alpha) = 0 \).

2. Conversely, assume that \( \alpha \in S[\tilde{\alpha}]_{\min} \cap \theta^\perp \). That is, \( (\theta, \alpha) = 0 \) and for some \( \nu \in \min(I(\tilde{\alpha})_{\min}) \), we have \( \nu - \alpha \in \Delta^+ \cup \{0\} \).

For \( \nu = \alpha \), we argue as in part 1(i). If \( \nu - \alpha \in \Delta^+ \), then both \( \gamma = \theta - \nu \) and \( \gamma + \alpha \) are roots, and \( \gamma \in \max(\Delta^+ \setminus I(\tilde{\alpha})_{\max}) \) in view of Proposition 3.4. Hence \( \alpha \in S[\tilde{\alpha}]_{\max} \). \[ \square \]

**Remark 3.7.** Recall that \( a(\tilde{\alpha})_{\min} = a(\tilde{\alpha})_{\max} \) if and only if \( (\tilde{\alpha}, \theta) \neq 0 \), i.e., \( \tilde{\alpha} \in \mathcal{H} \) [8, Theorem 5.1(i)]. If this is the case (and \( \Delta \neq A_n \)), then Theorem 3.6 implies that \( S[\tilde{\alpha}]_{\max} = S[\tilde{\alpha}]_{\min} \subset \theta^\perp \). In the distinguished case of \( (A_n, \alpha_1 \text{ or } \alpha_n) \), we have \( a(\alpha_1)_{\min} = a(\alpha_1)_{\max} \) and \( S[\alpha_1]_{\min} = \{\alpha_1\} \), whereas \( \Pi \cap \mathcal{H} = \{\alpha_1, \alpha_n\} \).

**Corollary 3.8.** If \( I(\tilde{\alpha})_{\min} \neq I(\tilde{\alpha})_{\max} \), then \( p[\tilde{\alpha}]_{\min} \neq p[\tilde{\alpha}]_{\max} \).

**Proof.** Since \( I(\tilde{\alpha})_{\min} \neq I(\tilde{\alpha})_{\max} \), we have \( (\tilde{\alpha}, \theta) = 0 \). Then \( \Pi[\tilde{\alpha}]_{\min} \subset \theta^\perp \) by Theorem 2.3(ii). Then \( S[\tilde{\alpha}]_{\min} \subset \Pi \cap \mathcal{H} \), and \( S[\tilde{\alpha}]_{\max} \cap \mathcal{H} = \varnothing \) in view of Theorem 3.6. That is, \( S[\tilde{\alpha}]_{\min} \neq S[\tilde{\alpha}]_{\max} \). \[ \square \]

Combining Theorems 2.3 and 3.6 yields a complete description of the normaliser for the maximal abelian ideals \( a(\tilde{\alpha})_{\max} \), which turns out to be more uniform than that for \( a(\tilde{\alpha})_{\min} \). In the rest of the section, we write \( \bar{w} \) in place of \( w_{\tilde{\alpha}} \).

**Theorem 3.9.** (i) Excluding the case in which \( \Delta \) is of type \( A_n \) with \( \tilde{\alpha} = \alpha_1 \) or \( \alpha_n \), we have

\[
\Pi[\tilde{\alpha}]_{\max} = (\Pi \cap \mathcal{H}) \biguplus \{\bar{w}^{-1}(\beta) | \beta \in \Pi \& (\beta, \tilde{\alpha}) = 0\}.
\]
(ii) In particular, if \((\theta, \tilde{\alpha}) = 0\), then \(\Pi[\tilde{\alpha}]_{\text{max}} = (\Pi \cap \mathcal{H}) \sqcup \Pi[\tilde{\alpha}]_{\text{min}}\).

(iii) In particular, if \(\theta\) is fundamental and \((\theta, \tilde{\alpha}) \neq 0\), then

\[
\Pi[\tilde{\alpha}]_{\text{max}} = \Pi[\tilde{\alpha}]_{\text{min}} = \{\tilde{\alpha}\} \sqcup \{\beta \in \Pi \mid (\beta, \tilde{\alpha}) = 0\}.
\]

Let us say that \(\beta \in \Pi\) is admissible (for \(\tilde{\alpha}\)) if \((\beta, \tilde{\alpha}) = 0\). It follows from Theorem 2.3 that an admissible root always gives rise to a simple root of the Levi subalgebra of \(\mathfrak{p}[\tilde{\alpha}]_{\text{min}}\). Furthermore, if \(\theta\) is fundamental and \((\tilde{\alpha}, \theta) \neq 0\), then \(\tilde{\alpha}\) also belongs to \(\Pi[\tilde{\alpha}]_{\text{min}}\).

**Example 3.10.** (1) \(\Delta = A_n, \tilde{\alpha} = \alpha_2\). Here \(\tilde{w} = s_1s_3 \ldots s_n\) and the admissible roots are \(\alpha_4, \ldots, \alpha_n\). One has \(\tilde{w}^{-1}(\alpha_i) = \alpha_{i-1}\) for them. Hence \(\Pi[\alpha_2]_{\text{min}} = \{\alpha_3, \alpha_4, \ldots, \alpha_{n-1}\}\) and \(S[\alpha_2]_{\text{min}} = \{\alpha_1, \alpha_2, \alpha_n\}\). Then \(S[\alpha_2]_{\text{max}} = \{\alpha_2\}\).

More generally, for \(\tilde{\alpha} = \alpha_i\) \((2 \leq i \leq n - 1)\), one obtains \(S[\alpha_i]_{\text{min}} = \{\alpha_1, \alpha_i, \alpha_n\}\) and \(S[\alpha_i]_{\text{max}} = \{\alpha_i\}\).

(2a) \(\Delta = D_4, \tilde{\alpha} = \alpha_1\). Here \(\tilde{w} = s_2s_3s_4s_2\) and the admissible roots are \(\alpha_3, \alpha_4\). One has \(\tilde{w}^{-1}(\alpha_3) = \alpha_4\) and \(\tilde{w}^{-1}(\alpha_4) = \alpha_3\). Hence \(S[\alpha_1]_{\text{min}} = \{\alpha_1, \alpha_2\}\) and \(S[\alpha_1]_{\text{max}} = \{\alpha_1\}\).

(2b) \(\Delta = D_4, \tilde{\alpha} = \alpha_2\). There is no admissible roots here, hence \(\tilde{w}\) is not really needed. Since \((\alpha_2, \theta) \neq 0\), we have \(S[\alpha_2]_{\text{min}} = S[\alpha_2]_{\text{max}} = \{\alpha_1, \alpha_3, \alpha_4\} = \Pi \setminus (\Pi \cap \mathcal{H})\).

(3) \(\Delta = C_n, \tilde{\alpha} = \alpha_n\) (the only long simple root). Here \(\tilde{w} = s_{n-1} \ldots s_2s_1\) and the admissible roots are \(\alpha_1, \ldots, \alpha_{n-2}\). One has \(\tilde{w}^{-1}(\alpha_i) = \alpha_{i+1}\) for them. Hence \(\Pi[\alpha_n]_{\text{min}} = \{\alpha_2, \alpha_3, \ldots, \alpha_{n-1}\}\) and \(S[\alpha_n]_{\text{min}} = \{\alpha_1, \alpha_n\}\). Then \(S[\alpha_n]_{\text{max}} = \{\alpha_n\}\).

(4a) \(\Delta = E_6, \tilde{\alpha} = \alpha_3\). Here \(\tilde{w} = s_6s_4s_5s_3s_1s_2s_4s_3s_6\) and the admissible roots are \(\alpha_1, \alpha_5\). One has \(\tilde{w}^{-1}(\alpha_1) = \alpha_4\) and \(\tilde{w}^{-1}(\alpha_5) = \alpha_2\). Hence \(S[\alpha_3]_{\text{min}} = \{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}\) and \(S[\alpha_3]_{\text{max}} = \{\alpha_1, \alpha_3, \alpha_5\}\).

(4b) \(\Delta = E_6, \tilde{\alpha} = \alpha_2\). Here \(\tilde{w} = s_3s_6s_4s_5s_3s_1s_2s_4s_3s_6\) and the admissible roots are \(\alpha_4, \alpha_5, \alpha_6\). One has \(\tilde{w}^{-1}(\alpha_4) = \alpha_3, \tilde{w}^{-1}(\alpha_5) = \alpha_2\) and \(\tilde{w}^{-1}(\alpha_6) = \alpha_5\). Hence \(S[\alpha_2]_{\text{min}} = \{\alpha_1, \alpha_4, \alpha_6\}\) and \(S[\alpha_2]_{\text{max}} = \{\alpha_1, \alpha_4\}\).

4. Normalisers of Abelian Ideals and \(\mathbb{Z}\)-Gradings

In this section, we elaborate on a relationship between the abelian ideals, their normalisers and the associated \(\mathbb{Z}\)-gradings. Any subset \(S \subset \Pi\) gives rise to a \(\mathbb{Z}\)-grading of \(\mathfrak{g}\). Set

\[
\deg(\alpha) = \begin{cases} 
0, & \alpha \in \Pi \setminus S \\
1, & \alpha \in S
\end{cases},
\]

and extend it to the whole of \(\Delta\) by linearity. Then the \(\mathbb{Z}\)-grading \(\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)\) is defined by the requirement that \(t \subset \mathfrak{g}(0)\) and \(\mathfrak{g}_\gamma \subset \mathfrak{g}(\deg(\gamma))\) for any \(\gamma \in \Delta\). Set \(\mathfrak{g}(\geq j) = \bigoplus_{i \geq j} \mathfrak{g}(i)\). If we wish to make the dependance on \(S\) explicit, then we write \(\mathfrak{g}(i; S)\) and \(\mathfrak{g}(j; S)\).

Let \(\mathfrak{p}\) be a standard parabolic subalgebra, \(\mathfrak{l}\) the standard Levi subalgebra of \(\mathfrak{p}\), and \(\Pi(\mathfrak{l})\) the set of simple roots of \(\mathfrak{l}\). Then \(S = S(\mathfrak{p}) = \Pi \setminus \Pi(\mathfrak{l})\) determines the \(\mathbb{Z}\)-grading associated
with \( p \), and we also write \( p = p(S) \). In this case, \( g(0; S) = 1 \), \( g(\geq 0; S) = p \), and \( g(\geq 1; S) \) is the nilradical of \( p \).

The height of a \( \mathbb{Z} \)-grading is the maximal \( i \) such that \( g(i) \neq \{0\} \). For \( S = \Pi \setminus \Pi(1) \), we also say that it is the height of \( p(S) \), denoted \( \text{ht}(p(S)) \). It is easily seen that \( \text{ht}(p(S)) = \deg(\theta) = \sum_{\alpha \in \mathcal{S}[\theta : \alpha]}\). Clearly, if \( j \geq \frac{\text{ht}(p)}{2} + 1 \), then \( g(\geq j) \) is an abelian ideal of \( b \).

**Convention.** If \( (\theta, \tilde{\alpha}) \neq 0 \), then \( I(\tilde{\alpha})_{\text{min}} = I(\tilde{\alpha})_{\text{max}} \). In this case, we omit the subscripts ‘min’ and ‘max’ from the notation for all relevant objects; that is, we merely write \( p[\tilde{\alpha}], \mathcal{S}[\tilde{\alpha}] \), etc.

**Theorem 4.1.** Suppose that \( \theta \) is fundamental, with the corresponding \( \alpha_\theta \in \Pi \).

(i) \( \mathcal{S}[\alpha_\theta] = \{ \beta \in \Pi \setminus \{ \alpha_\theta \} \mid (\beta, \alpha_\theta) \neq 0 \} \), the set of all simple roots adjacent to \( \alpha_\theta \);

(ii) \( \alpha_\theta \) is long, \( [\theta : \alpha_\theta] = 2 \), and \( \text{ht}(p[\alpha_\theta]) = 3 \);

(iii) \( a(\alpha_\theta) = g(\geq 2; \mathcal{S}[\alpha_\theta]) \).

**Proof.** (i) It is already proved in Proposition 2.4.

(ii) If \( \theta \) is fundamental, then \( (\theta, \alpha_\theta^\vee) = 1 = (\alpha_\theta, \theta^\vee) \). Hence \( \alpha_\theta \) is necessarily long. Furthermore,

\[
(\theta, \theta) = (\theta, \sum_{\alpha \in \Pi}[\theta : \alpha]\alpha) = [\theta : \alpha_\theta](\theta, \alpha_\theta) = \frac{1}{2}[\theta : \alpha_\theta](\theta, \theta).
\]

Hence \( [\theta : \alpha_\theta] = 2 \). Finally,

\[
1 = (\theta, \alpha_\theta^\vee) = 2[\theta : \alpha_\theta] - \sum_{\beta \text{ adjacent}}[\theta : \beta],
\]

where the sum ranges over the simple roots \( \beta \) adjacent to \( \alpha_\theta \) in the Dynkin diagram. Therefore, \( 3 = \sum_{\beta \text{ adjacent}}[\theta : \beta] = \text{ht}(p[\alpha_\theta]) \).

(iii) A general description of the minimal roots for all root-minimal ideals \( a(\mu)_{\text{min}} \) is provided in [8, Prop. 4.6]. In the situation with \( \mu = \alpha_\theta \), this yields

\[
\text{min}(I(\alpha_\theta)) = \{ w_{\alpha_\theta}^{-1}(\alpha_\theta + \beta_i) \mid \beta_i \in \Pi \& \beta_i \text{ is adjacent to } \alpha_\theta \}.
\]

Set \( \nu_i = w_{\alpha_\theta}^{-1}(\alpha_\theta + \beta_i) = \theta + w_{\alpha_\theta}^{-1}(\beta_i) \) and write \( \nu_i = m\alpha_\theta + \sum_j m_j \beta_j + \text{(others)} \). Then \( m = 1 \), since \( m = (\nu_i, \theta^\vee) = (\theta + w_{\alpha_\theta}^{-1}(\beta_i), \theta^\vee) = 2 - 1 = 1 \). Next, using Lemma 2.2 with \( \mu = \alpha_\theta \), we obtain

\[
(\nu_i, \alpha_\theta^\vee) = (\theta + w_{\alpha_\theta}^{-1}(\beta_i), \alpha_\theta^\vee) = 1 + (\beta_i, \alpha_\theta^\vee - \theta^\vee) = 1 - 1 = 0.
\]

On the other hand,

\[
(\nu_i, \alpha_\theta^\vee) = 2m - \sum_j m_j.
\]

Therefore, \( \sum_j m_j = 2 \) and all minimal roots belong to \( g(2; \mathcal{S}[\alpha_\theta]) \). Since \( g(\geq 2; \mathcal{S}[\alpha_\theta]) \) is an abelian ideal and \( a(\alpha_\theta) \) is maximal abelian, we must have \( g(\geq 2; \mathcal{S}[\alpha_\theta]) = a(\alpha_\theta) \).

**Theorem 4.1** is a particular case of the following general assertion.
Theorem 4.2.

(i) For any $\tilde{\alpha} \in \Pi_l$ and $n_{\tilde{\alpha}} := [\theta : \tilde{\alpha}]$, we have $\text{ht}(\mathfrak{p}[\tilde{\alpha}]_{\text{max}}) = 2n_{\tilde{\alpha}} - 1$ and $a(\tilde{\alpha})_{\text{max}} = g(\geq n_{\tilde{\alpha}}; S[\alpha]_{\text{max}})$.

(ii) If $(\tilde{\alpha}, \theta) = 0$ (and hence $S[\tilde{\alpha}]_{\text{max}} \neq S[\tilde{\alpha}]_{\text{min}}$), then $\text{ht}(\mathfrak{p}[\tilde{\alpha}]_{\text{min}}) = 2n_{\tilde{\alpha}} + 1$ and $a(\tilde{\alpha})_{\text{min}} = g(\geq n_{\tilde{\alpha}} + 1; S[\tilde{\alpha}]_{\text{min}})$.

Proof. Our proof for both parts consists of a case-by-case verification. Using explicit information on $\min(I(\tilde{\alpha})_{\text{min}})$ and $\min(I(\tilde{\alpha})_{\text{max}})$ or results of Section 3, we explicitly determine $S[\tilde{\alpha}]_{\text{min}}$ and $S[\tilde{\alpha}]_{\text{max}}$. This yields the associated $\mathbb{Z}$-gradings and height of all parabolics involved. The minimal roots of $I(\tilde{\alpha})_{\text{min}}$ can be determined with the help of [8, Prop. 4.6], whereas the minimal roots of $I(\tilde{\alpha})_{\text{max}}$ (“generators”) are indicated in [12, Tables I, III]. Then one verifies that the sets $\min(I(\tilde{\alpha})_{\text{min}})$ and $\min(I(\tilde{\alpha})_{\text{max}})$ always coincide with the set of minimal roots of $g(\geq n_{\tilde{\alpha}} + 1; S[\tilde{\alpha}]_{\text{min}})$ and $g(\geq n_{\tilde{\alpha}}; S[\tilde{\alpha}]_{\text{max}})$, respectively. □

Remark 4.3. We can directly explain the following outcome of Theorem 4.2:

If $(\tilde{\alpha}, \theta) = 0$, then $\text{ht}(\mathfrak{p}[\tilde{\alpha}]_{\text{min}}) = \text{ht}(\mathfrak{p}[\tilde{\alpha}]_{\text{max}}) + 2$.

For, by Theorem 3.9(ii), we know that $S[\tilde{\alpha}]_{\text{min}} = (\Pi \cap \mathcal{H}) \cup S[\tilde{\alpha}]_{\text{max}}$. Hence

$$\text{ht}(\mathfrak{p}[\tilde{\alpha}]_{\text{min}}) - \text{ht}(\mathfrak{p}[\tilde{\alpha}]_{\text{max}}) = \sum_{\beta \in \Pi \cap \mathcal{H}} n_{\beta}.$$ 

If $\theta$ is fundamental, then $\Pi \cap \mathcal{H} = \{\alpha_\theta\}$ and $n_{\alpha_\theta} = 2$ (Theorem 4.1(ii)). For $A_n$, we have $\Pi \cap \mathcal{H} = \{\alpha_1, \alpha_n\}$ and $n_{\alpha_1} + n_{\alpha_n} = 2$. This does not apply to $C_n$, where $(\tilde{\alpha}, \theta) \neq 0$ for the unique long simple root $\tilde{\alpha}$.

Example 4.4. If $n_{\tilde{\alpha}} = 1$, then $I(\tilde{\alpha})_{\text{max}} = \{\gamma \in \Delta^+ \mid [\gamma : \tilde{\alpha}] = 1\}$ and $\mathfrak{p}[\tilde{\alpha}]_{\text{max}}$ is the maximal parabolic subalgebra with $S[\tilde{\alpha}]_{\text{max}} = \{\tilde{\alpha}\}$. Here $\text{ht}(\mathfrak{p}[\tilde{\alpha}]_{\text{max}}) = 1$. Hence Theorem 4.2(i) is satisfied here. Furthermore, if $\theta$ is fundamental and $(\tilde{\theta}, \alpha_\theta) \neq 0$, then $\tilde{\alpha} \neq \alpha_\theta$ (because $n_{\alpha_\theta} = 2$), $(\tilde{\theta}, \tilde{\alpha}) = 0$, and $S[\tilde{\alpha}]_{\text{min}} = \{\tilde{\alpha}, \alpha_\theta\}$, see Theorem 3.9(ii). Therefore $\text{ht}(\mathfrak{p}[\tilde{\alpha}]_{\text{min}}) = 3$, and I can prove a priori that $a(\tilde{\alpha})_{\text{min}} = g(\geq 2; \{\tilde{\alpha}, \alpha_\theta\})$. (As this is not a decisive step, the proof is omitted.)

That is, in principle, there is a better proof of Theorem 4.2 if $n_{\tilde{\alpha}} = 1$ or $\tilde{\alpha} = \alpha_\theta$.

Now, we consider arbitrary abelian ideals of $\mathfrak{b}$. Let $\mathfrak{Par}(\mathfrak{g}, \mathfrak{b}) = \mathfrak{Par}(\mathfrak{g})$ be the set of all standard parabolic subalgebras of $\mathfrak{g}$. If $\mathfrak{a} \in \mathfrak{Ab}(\mathfrak{g})$, then $n_{\mathfrak{a}}(\mathfrak{a}) \in \mathfrak{Par}(\mathfrak{g})$. It is proved in [12] that the assignment $\mathfrak{a} \mapsto f_1(\mathfrak{a}) = n_{\mathfrak{a}}(\mathfrak{a})$ sets up a bijection $\mathfrak{Ab}(\mathfrak{g}) \xrightarrow{f_1} \mathfrak{Par}(\mathfrak{g})$ if and only if $\Delta$ is of type $A_n$ or $C_n$ (i.e., $\theta$ is not fundamental).

Here we extend that observation by looking at a natural mapping in the opposite direction. For $\mathfrak{p} \in \mathfrak{Par}(\mathfrak{g})$ and the associated $\mathbb{Z}$-grading, we set

$$f_2(\mathfrak{p}) = g(\geq [\text{ht}(\mathfrak{p})/2] + 1) \in \mathfrak{Ab}(\mathfrak{g}).$$

This mapping occurs implicitly in Theorem 4.2, where $\text{ht}(\mathfrak{p})$ appears to be always odd.
Theorem 4.5.

(i) If $\Delta$ is of type $A_n$ or $C_n$, then $f_2 : \mathcal{P}ar(g) \to \mathcal{A}b(g)$ is a bijection. Moreover, $f_2 = f_1^{-1}$;

(ii) If $\theta$ is fundamental, then $f_2$ is not a bijection. In fact, there is a uniform construction of two different $p_1, p_2 \in \mathcal{P}ar(g)$ such that $f_2(p_1) = f_2(p_2)$.

Proof. (i) First, we recall the (slightly modified) construction of the bijection $f_1$ for $A_n$. For $a \in \mathcal{A}b(sl_{n+1})$, let $\min(I_a) = \{\gamma_1, \ldots, \gamma_k\}$ with $\gamma_i = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_j}$, where $i_1 \leq j$. Assuming that $i_1 \leq i_2 \leq \ldots \leq i_k$, we actually obtain the restrictions

$$1 \leq i_1 < i_2 < \cdots < i_k \leq j_1 < \cdots < j_k \leq n$$

and thereby the bijection between $\mathcal{A}b(sl_{n+1})$ and the subsets of $[n] = \{1, \ldots, n\}$. Here one obtains a subset of odd (resp. even) cardinality if $i_k = j_1$ (resp. $i_k < j_1$). Moreover, if $p = n_g(a)$, then it follows from Theorem 3.3.1 that $S = S(p) = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}, \alpha_{j_1}, \ldots, \alpha_{j_k}\}$, modulo the possible coincidence of $i_k$ and $j_1$.

Suppose that $p \in \mathcal{P}ar(sl_{n+1})$ and $\#S$ is odd, $S \sim \{t_1, t_2, \ldots, t_{2k-1}\} \subset [n]$, with $t_1 < \cdots < t_{2k-1}$. Then $ht(p) = 2k - 1$ and the minimal roots of $g(\geq k; S)$ are in a bijection with the shortest intervals of $[n]$ that contain $k$ elements of $S$. Therefore, these minimal roots are

$$\gamma_1 = \alpha_{t_1} + \alpha_{t_2} + \cdots + \alpha_{t_k}$$
$$\gamma_2 = \alpha_{t_2} + \alpha_{t_3} + \cdots + \alpha_{t_{k+1}}$$
$$\cdots$$
$$\gamma_k = \alpha_{t_k} + \alpha_{t_{k+1}} + \cdots + \alpha_{t_{2k-1}}$$

and it is immediate that, for the abelian ideal $a = f_2(p)$ generated by $\gamma_1, \ldots, \gamma_k$, we have $f_1(a) = p$.

If $\#S$ is even, $S \sim \{t_1, t_2, \ldots, t_{2k}\} \subset [n]$, then $ht(p) = 2k$ and the minimal roots of $g(\geq k + 1; S)$ are

$$\gamma_1 = \alpha_{t_1} + \alpha_{t_2} + \cdots + \alpha_{t_{k+1}}$$
$$\gamma_2 = \alpha_{t_2} + \alpha_{t_3} + \cdots + \alpha_{t_{k+2}}$$
$$\cdots$$
$$\gamma_k = \alpha_{t_k} + \alpha_{t_{k+1}} + \cdots + \alpha_{t_{2k}}$$

Here again one obtains $a = f_2(p)$ such that $f_1(a) = p$.

We omit the part related to $C_n$, since it goes along the same lines, using the explicit description of $f_1$ given in [12, Theorem 3.3]. The point is that the unfolding $C_n \sim A_{2n-1}$ (see picture below) yields the identification of $\mathcal{A}b(sp_{2n})$ and $\mathcal{P}ar(sp_{2n})$ with the symmetric (with respect to the middle) subsets of $[2n-1]$, and one can use a symmetrised version of the previous argument.

(ii) Our goal is to produce two different subsets $S_1, S_2 \subset \Pi$ such that $p(S_1)$ and $p(S_2)$ give rise to the same abelian ideal. Below we use Theorem 4.1 and its proof.
As usual, $\alpha_\theta$ is the only simple root that is not orthogonal to $\theta$. Let $S_1$ be the set of all simple roots adjacent to $\alpha_\theta$ and $S_2 = S_1 \cup \{\alpha_\theta\}$. Then $p(S_1) = p[\alpha_\theta], \text{ht}(p[\alpha_\theta]) = 3$, and $a(\alpha_\theta) = g(\geq 2; S_1)$. Since $n_{\alpha_\theta} = 2$, we have $\text{ht}(p(S_2)) = 2 + \text{ht}(p[\alpha_\theta]) = 5$ and $g(\geq 3; S_2)$ is an abelian ideal. The proof of Theorem 4.1 shows that if $\nu_i \in \min(I(\alpha_\theta))$, then $[\nu_i : \alpha_\theta] = 1$ and $\sum_{\beta \in S_1}[\nu : \beta] = 2$. Hence $g_{\nu_i} \in g(3; S_2)$ and $a(\alpha_\theta) \subset g(\geq 3; S_2)$. As $a(\alpha_\theta)$ is maximal abelian, one has the equality and therefore $f_2(p(S_1)) = f_2(p(S_2))$.

**Remark 4.6** (Some speculations). Set $F = f_1 \circ f_2$ and $F = f_2 \circ f_1$. We say that $a \in \mathbb{Ab}(g)$ is reflexive, if $\tilde{F}(a) = a$; likewise, $\mathfrak{P}(g)$ is reflexive, if $F(\mathfrak{p}) = \mathfrak{p}$. It is easily seen that $F(\mathfrak{p}) \supseteq \mathfrak{p}$ for all $\mathfrak{p}$, while it can happen that $\tilde{F}(a) \not\supset a$ for some $a$ (e.g. if $g = E_6$).

For $\mathfrak{sl}_{n+1}$ and $\mathfrak{sp}_{2n}$, all abelian ideals are reflexive, whereas this is certainly not the case for the other simple types. However, Theorem 4.2 implies that the ideals $a(\tilde{\alpha})_{\text{min}}$ and $a(\tilde{\alpha})_{\text{max}}$ ($\tilde{\alpha} \in \Pi_l$) are always reflexive. It might be interesting to explicitly determine all reflexive abelian ideals.

Our calculations with $g$ up to rank 4 suggest that it also might be true that (the restrictions of) $f_1$ and $f_2$ induce the mutually inverse bijections between $\text{Im}(\tilde{F}) \subset \mathbb{Ab}(g)$ and $\text{Im}(F) \subset \mathfrak{Par}(g)$; in particular, $\#\text{Im}(F) = \#\text{Im}(\tilde{F})$. But the equality $\#\text{Im}(f_1) = \#\text{Im}(f_2)$ is false in general (e.g. for $g = so_9$).

We also conjecture that $\text{Im}(F) = \{\mathfrak{p} \mid F(\mathfrak{p}) = \mathfrak{p}\}$ and $\text{Im}(\tilde{F}) = \{a \mid \tilde{F}(a) = a\}$; in other words, $F^2 = F$ and $\tilde{F}^2 = \tilde{F}$ in the rings of endomorphisms of the finite sets $\mathfrak{Par}(g)$ and $\mathbb{Ab}(g)$, respectively.

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