Flavour models for TM$_1$ lepton mixing

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Abstract

We present a framework for lepton flavour models such that the first column of the lepton mixing matrix is $(2, -1, -1)^T/\sqrt{6}$. We show that the flavour symmetry group adequate for this purpose is $S_4$. Our models are based on a vacuum alignment that can be obtained in a supersymmetric framework.

1 Introduction

The recent experimental discovery that the lepton mixing angle $\theta_{13}$ is nonzero [1, 2, 3, 4] has rendered outdated quite a few previous phenomenological Ansätze. Notably, the tri-bimaximal mixing (TBM) Ansatz [5] cannot stand in the face of the evidence for both a nonzero $\theta_{13}$ and a non-maximal atmospheric mixing angle $\theta_{23}$ (the latter evidence is still disputable [6, 7, 8]). The stage is thus set for searches for

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alternative models and Ansätze. One interesting possibility is the embedding of a \( \mu - \tau \) interchange in a generalised CP symmetry [9,10,11]. This allows for a nonzero \( \theta_{13} \) but keeps \( \theta_{23} \) maximal. Another possibility consists in substituting the stringent TBM Ansatz by a relaxed version of it, in which either only the first column or only the second column of the lepton mixing matrix is assumed to take its TBM form; these possibilities have been named TM\(_1\) and TM\(_2\), respectively, in ref. [14]. There are many other possibilities, like for instance various models featuring ‘texture’ zeroes in the lepton mass matrices, the Ansatz of lepton mixing ‘anarchy’ [15], and models based on various flavour symmetry groups like \( A_4 \) (e.g. refs. [16,17]), \( S_4 \) (e.g. ref. [12]), \( \Delta(27) \) (e.g. refs. [18,19]), and so on (for a recent review, see ref. [20]).

The problem with many Ansätze is grounding them on well-defined field-theoretical models, which might hope to render those Ansätze stable under renormalisation. In particular, this has already been achieved for TM\(_2\) [21]. A model for TM\(_1\) based on the strategy of ‘sequential dominance’ has been presented in ref. [22]. It is the purpose of this paper to suggest a different framework for models featuring TM\(_1\).

The plan of the paper is as follows. In section 2 we define TM\(_1\) and review its phenomenological merits and predictions. In section 3 we present our framework for TM\(_1\) models by assuming a specific vacuum alignment in \( S_4 \)-based models. In section 4 we justify the vacuum alignment used in the previous section in the context of supersymmetric versions of our models. Section 5 summarises our achievements. In appendix A we make a brief review of the group \( S_4 \), its irreducible representations, and the tensor products thereof. Appendix B considers the constraints on the neutrino mass spectrum ensuing from some of our models.

2 TM\(_1\)

TM\(_1\) is defined to be the situation where the first column of the lepton mixing matrix \( U \) is

\[
u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}.
\]

This has also been done recently in ref. [23], where a justification of the TM\(_1\) Ansatz through a particular breaking of an \( A_4 \) flavour symmetry has also been attempted.
In the standard parametrisation of $U$,

$$
U = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
    -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
    s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix} P,
$$

(2)

where $c_i = \cos \theta_i$ and $s_i = \sin \theta_i$ for $i = 12, 13, 23$ and $P$ is a $3 \times 3$ diagonal unitary matrix, the diagonal elements of which are the ‘Majorana phases’. Since in TM $|U_{e1}|^2 = c_{12}^2 c_{13}^2 = 2/3$,

$$
s_{12}^2 = 1 - \frac{2}{3c_{13}^2} = \frac{1 - 3s_{13}^2}{3 - 3s_{13}^2}, \quad \text{hence} \quad c_{12} = \frac{2}{3 - 3s_{13}^2}.
$$

(3)

Moreover, since $1/6 = |U_{\mu 1}|^2 = |U_{\tau 1}|^2$,

$$
(c_{23}^2 - s_{23}^2) (s_{12}^2 - c_{12}^2) + 4c_{23}s_{23}c_{12}s_{12}s_{13}\cos \delta = 0.
$$

(4)

Inserting into eq. 4 the values of $c_{12}$ and $s_{12}$ in eqs. 2,

$$
(c_{23}^2 - s_{23}^2) (1 - 5s_{13}^2) + 4\sqrt{2 (1 - 3s_{13}^2)} c_{23}s_{23}s_{13} \cos \delta = 0.
$$

(5)

Thus,

$$
\cos \delta = \frac{(1 - 5s_{13}^2)(2s_{23} - 1)}{4s_{13}s_{23} \sqrt{2 (1 - 3s_{13}^2) (1 - s_{23}^2)}}.
$$

(6)

Equation (3) predicts $s_{12}^2$ as a function of $s_{13}^2$. Equation (6) predicts $\cos \delta$ as a function of $s_{13}^2$ and $s_{23}^2$.

We use the phenomenological data of ref. 7. According to that paper, the best-fit values of $s_{13}^2$ and $s_{23}^2$ are approximately 0.024 and 0.390, respectively. Therefore, eqs. (3) and (6) may be approximated by

$$
s_{12}^2 \approx 0.317 - 0.700 (s_{13}^2 - 0.024),
$$

(7)

$$
\cos \delta \approx -0.470 + 11.7 (s_{13}^2 - 0.024) + 4.49 (s_{23}^2 - 0.390),
$$

(8)

respectively. The prediction for $s_{12}^2$ in eq. 7 agrees very well with experiment and is, moreover, almost independent of the precise value of $s_{13}^2$. Equation (6) predicts $\cos \delta$ to be negative as long as $\theta_{23}$ is in

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3See also ref. 23. A related analysis can be found in ref. 24.
4The papers of refs. 6, 8 provide other phenomenological fits to the data. The TM Ansatz is just as good for those fits. Notice, however, that $\cos \delta$ changes sign if $s_{23}^2$ is allowed to be above 0.5.
5It is remarkable that even though, experimentally, the relative error in $s_{12}^2$ is approximately half the one in $s_{13}^2$, the TM Ansatz allows one to predict $s_{12}^2$ from $s_{13}^2$ and not the converse.
the first octant, but the prediction for the exact value of \( \cos \delta \) is much less precise than the one for \( s_{12}^2 \). Using the 1σ intervals of ref. [7],

\[
s_{13}^2 \in [0.0216, 0.0266], \quad s_{23}^2 \in [0.365, 0.410]
\]

\[\Rightarrow \cos \delta \in [-0.622, -0.359], \quad (9)\]

while, using the 3σ intervals,

\[
s_{13}^2 \in [0.0169, 0.0313], \quad s_{23}^2 \in [0.331, 0.637]
\]

\[\Rightarrow \cos \delta \in [-0.918, 0.505]. \quad (10)\]

3 The models

3.1 The general framework

We now discuss a theoretical framework that leads to TM\(_1\). In our framework we assume that there are only three light neutrinos and that they are Majorana particles. The charged-lepton mixing matrix is supposed to be diagonalised by the unitary matrix

\[
U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{pmatrix}, \quad (11)
\]

where \( \omega = \exp (i2\pi/3) \). Then, the lepton mixing matrix is

\[
U = U_\omega U_\nu, \quad (12)
\]

where \( U_\nu \) is the unitary matrix that diagonalises the effective light-neutrino Majorana mass matrix \( M_\nu \):

\[
U_\nu^T M_\nu U_\nu = \text{diag} (m_1, m_2, m_3) \equiv D_\nu, \quad (13)
\]

where the \( m_j \) \((j = 1, 2, 3)\) are non-negative real. The symmetric matrix \( M_\nu \) is supposed to have an eigenvector \((0, 1, 1)^T\). The most general symmetric matrix with that feature may be written in the form

\[
M_\nu = \begin{pmatrix}
a + 2b & f & -f \\
f & a - b & d \\
-f & d & a - b
\end{pmatrix}. \quad (14)
\]

Then,

\[
U_\nu = \begin{pmatrix}
0 & ce^{i\beta} & se^{i\beta} \\
r & rs & -rc \\
r & -rs & rc
\end{pmatrix} P, \quad (15)
\]
where \( r = 2^{-1/2} \), \( c = \cos \sigma \), \( s = \sin \sigma \), and \( P = \text{diag}(e^{i\psi_1}, e^{i\psi_2}, e^{i\psi_3}) \). Therefore, from eqs. (12), (11), and (15),

\[
U = \begin{pmatrix}
\sqrt{2/3} & c e^{i\beta} / \sqrt{3} & s e^{i\beta} / \sqrt{3} \\
-\sqrt{1/6} & c e^{i\beta} / \sqrt{3} + i s / \sqrt{2} & s e^{i\beta} / \sqrt{3} - i c / \sqrt{2} \\
-\sqrt{1/6} & c e^{i\beta} / \sqrt{3} - i s / \sqrt{2} & s e^{i\beta} / \sqrt{3} + i c / \sqrt{2}
\end{pmatrix} P,
\]

which clearly satisfies the definition of \( \text{TM}_1 \) in eq. (1). Comparing with the standard parametrisation in eq. (2), one obtains

\[
s^2 = 3 s_{13}^2, \quad \text{and} \quad cs \sin \beta = \frac{\sqrt{3}}{2\sqrt{2}} c_{13}^2 (c_{23}^2 - s_{23}^2). \tag{17a}
\]

At this stage it is useful to define the unitary matrices

\[
O_1 = \begin{pmatrix} 0 & 1 & 0 \\ r & 0 & -r \\ r & 0 & r \end{pmatrix}, \quad \text{and} \quad O_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c e^{i\beta} & s e^{i\beta} \\ 0 & -s & c \end{pmatrix}, \tag{18a}
\]

which are such that \( U_\nu = O_1 O_2 P \). Equation (13) may now be rewritten

\[
O_1^T M_\nu O_1 = O_2^* \text{ diag } (\mu_1, \mu_2, \mu_3) \ O_2^T, \tag{19a}
\]

where \( \mu_j = m_j e^{-i2\psi_j} \). Explicitly computing each matrix element on both sides of eq. (21), one obtains

\[
a - b + d = \mu_1, \tag{22a}
\]
\[
a + 2b = e^{-i2\beta} (c^2 \mu_2 + s^2 \mu_3), \tag{22b}
\]
\[
a - b - d = s^2 \mu_2 + c^2 \mu_3, \tag{22c}
\]
\[
\sqrt{2} f = e^{-i\beta} cs (\mu_2 - \mu_3). \tag{22d}
\]

One concludes from eq. (22d) that \( f \neq 0 \) is mandatory, lest either the matrix \( O_2 \) is trivial, i.e. \( cs = 0 \), or the neutrinos \( \nu_2 \) and \( \nu_3 \) are degenerate; both situations would contradict the phenomenology, cf. eq. (17a).\footnote{In the most obvious situation, \( f = 0 \) would lead to \( s_{13} = 0 \) and, thus, to TBM. Indeed, our framework is simply an extension of the framework of most models predicting TBM, with the crucial difference that those models assume \( f = 0 \) while we want \( f \) to be nonzero.}
3.2 Implementation of the framework with the group $S_4$

We want the charged-lepton mass matrix to be diagonalised by $U_\omega$. This materialises if that mass matrix is of the form

$$
\begin{pmatrix}
\chi_1 & \chi_2 & \chi_3 \\
\chi_3 & \chi_1 & \chi_2 \\
\chi_2 & \chi_3 & \chi_1
\end{pmatrix},
$$

(23)

where $\chi_{1,2,3}$ are complex numbers which must be all different lest the charged leptons are massless. In order to obtain the matrix (23) there must be a cyclic symmetry $D_{L1} \rightarrow D_{L2} \rightarrow D_{L3} \rightarrow D_{L1}$ among the three leptonic gauge-$SU(2)$ doublets $D_{L1,2,3}$. Therefore, the matrix

$$
G_3 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
$$

(24)

must represent one of the generators of the flavour symmetry group in the representation to which the $D_{Lj}$ belong.

The neutrino mass matrix in eq. (14) is symmetric under a transformation through the matrix

$$
F_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{pmatrix},
$$

(25)
i.e. $F_3 M_\nu F_3 = M_\nu$. Therefore, the matrix $F_3$ should represent another generator of the flavour symmetry group in the representation to which the $D_{Lj}$ belong.

The matrices $G_3$ and $F_3$ together generate the irreducible representation $3_1$ of the group $S_4$, cf. eq. (A3). Therefore, $S_4$ is the appropriate lepton flavour symmetry group for a model predicting TM$_1$. The group $S_4$ and its irreducible representations are reviewed in appendix A.

We shall implement our models in a supersymmetric framework, which is convenient in order to obtain the desired alignment of vacuum expectation values (VEVs). We allow for couplings of dimension higher than four, adequately suppressed by as many powers as needed of a

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7Unfortunately, $\chi_{1,2,3}$ must be severely finetuned in order for the charged-lepton masses to be as hierarchical as observed experimentally. For instance, one possibility is that $\chi_1 = \chi_2 = \chi_3$ to very high precision—this hypothesis is known in the literature as ‘flavour democracy’. We note, however, that this finetuning is just as much of a problem in our framework as in many other flavour models for lepton mixing that do not rely on additional mechanisms like the Froggatt–Nielsen paradigm [25]. Here we offer no solution to this conundrum.
high-energy (cutoff) scale $\Lambda$. We place both the $D_{Lj}$ and the charged-lepton gauge-$SU(2)$ singlets $\ell_R j$ in representations $3_1$ of the flavour symmetry group $S_4$. The superpotential includes the gauge- and $S_4$-invariants

$$\frac{1}{\Lambda} H D_L \ell_R (y_1 T_1 + y_2 T_2 + y_S S), \quad \text{(26)}$$

where $H$ is a gauge-$SU(2)$ doublet which is invariant under the flavour symmetry $S_4$ and $y_{1,2,S}$ are coupling constants. The superfields $T_1$, $T_2$, and $S$, collectively known as ‘familons’ or ‘flavons’, are gauge singlets in distinct $S_4$ representations: $T_1$ is a $3_1$, $T_2$ is a $3_2$, and $S$ is a $1_1$, i.e. invariant under $S_4$. In order to avoid undesirable terms, we use an auxiliary Abelian symmetry, which may be either a $U(1)$ or a $\mathbb{Z}_N$ with sufficiently large $N$. Let $c(f)$ denote the charge of the generic field $f$ under this $U(1)_f$ or $\mathbb{Z}_N$ symmetry. It follows from eq. (26) that $c(T_1) = c(T_2) = c(S)$ and that $c(H) + c(D_L) + c(\ell_R) = -c(S)$. Also note that $c(T_1) \neq 0$, else a term with no flavon (and no cutoff suppression) should also be present in eq. (26). Let $\langle 0 | T_1 | 0 \rangle = (v_{1x}, v_{1y}, v_{1z})$, $\langle 0 | T_2 | 0 \rangle = (v_{2x}, v_{2y}, v_{2z})$, and $\langle 0 | S | 0 \rangle = v_S$ denote the VEVs of the neutral-scalar components of these superfields, then the charged-lepton mass matrix is

$$
\begin{pmatrix}
y S v_S & y_1 v_{1y} + y_2 v_{2y} & y_1 v_{1y} - y_2 v_{2y} \\
y_1 v_{1z} - y_2 v_{2z} & y S v_S & y_1 v_{1x} + y_2 v_{2x} \\
y_1 v_{1y} + y_2 v_{2y} & y_1 v_{1x} - y_2 v_{2x} & y S v_S
\end{pmatrix}.
$$

This is of the form (23) if

$$v_{1x} = v_{1y} = v_{1z} \equiv v_1 \quad \text{and} \quad v_{2x} = v_{2y} = v_{2z} \equiv v_2. \quad \text{(28)}$$

These are precisely the conditions for the breaking of the $S_4$ symmetry in the charged-lepton sector to preserve the $\mathbb{Z}_3$ symmetry generated by $G_3$ alone,

$$G_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad \text{(29)}$$

Note that the superfields $T_1$, $T_2$, and $S$ are all needed in order that the $\chi_j$ in eq. (23) are all non-vanishing and different.

We give Majorana masses to the neutrinos through the superpotential terms

$$\frac{1}{\Lambda} D_L \Delta D_L (z_S S + z_D \bar{D} + z_T \bar{T}), \quad \text{(30)}$$

where $\Delta$ is an $SU(2)$ triplet. The neutrino mass matrix is generated via the VEV of the neutral component of $\Delta$. In eq. (30), $S$ is $S_4$-invariant, $\bar{D}$ is a $2$ of $S_4$, $T$ is a $3_1$ of $S_4$, and $z_{S,D,T}$ are their respective coupling constants. Equation (30) is the most general $S_4$-invariant since the
symmetric part of the product of the two $3_1$ of $S_4$ contains precisely an invariant, a $2$, and a $3_1$ of $S_4$, cf. eq. (A3). We introduce another auxiliary symmetry, $U(1)_{\nu}$ Let $q(f)$ denote the charge under $U(1)_{\nu}$ of a generic superfield $f$. Evidently all the flavons in eq. (30) must have the same nonzero $U(1)_{\nu}$ charge, i.e. $q(\bar{S}) = q(D) = q(T)$, and that constitutes a serious constraint on the alignment mechanisms. Let $\langle 0|\bar{S}|0 \rangle = \bar{v}_S$, $\langle 0|\bar{D}|0 \rangle = (\bar{v}_{Dp}, \bar{v}_{Dq})$, and $\langle 0|T|0 \rangle = (\bar{v}_{Tx}, \bar{v}_{Ty}, \bar{v}_{Tz})$, then

\[
(M_{\nu})_{11} = z_S \bar{v}_S + z_D (\bar{v}_{Dp} + \bar{v}_{Dq}), \quad (M_{\nu})_{22} = z_S \bar{v}_S + z_D (\omega \bar{v}_{Dp} + \omega^2 \bar{v}_{Dq}), \quad (M_{\nu})_{33} = z_S \bar{v}_S + z_D (\omega^2 \bar{v}_{Dp} + \omega \bar{v}_{Dq}),
\]

and $(M_{\nu})_{12} = z_T \bar{v}_{Tz}$, $(M_{\nu})_{13} = z_T \bar{v}_{Ty}$, and $(M_{\nu})_{23} = z_T \bar{v}_{Tz}$. One sees that $M_{\nu}$ is of the desired form in eq. (14) if $\bar{v}_{Dp} = \bar{v}_{Dq} \equiv \bar{v}_D$ and $\bar{v}_{Ty} = -\bar{v}_{Tz}$; this is precisely the alignment of VEVs that preserves the $\mathbb{Z}_2$ subgroup of $S_4$ generated, in the representation $3_1$, by $F_3$:

\[
F_3 \begin{pmatrix} k \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} k \\ 1 \\ -1 \end{pmatrix}, \quad F_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.
\]

In the notation of the previous subsection, we have $a = z_S \bar{v}_S$, $b = z_D \bar{v}_D$, $d = z_T \bar{v}_{Tz}$, and $f = z_T \bar{v}_{Tz}$. Particular cases of interest occur when either $a$, $b$, or $d$ vanish; this may be because some VEV vanishes or—in the cases of $\bar{S}$ and $\bar{D}$—because one may altogether avoid introducing those superfields in a particular model. Those particular cases of interest are dealt with in appendix B.

4 Alignment

In this section we provide an existence proof of the alignments of VEVs required in the previous section. In this proof we assume a supersymmetric implementation of the models with an $R$-symmetry

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8In principle it is possible to combine the two auxiliary symmetries $U(1)_{\ell}$ and $U(1)_{\nu}$ into a single $U(1)$, but that is beyond the scope of the present existence proof. We note that by using different auxiliary symmetries it is always possible to separate the charged-lepton and neutrino sectors, since one of them involves the superpotential terms in eq. (26) while the other one involves the terms in eq. (30).

9If $q(\bar{S})$ was zero, terms without any flavon field would have to be added to eq. (30).

10We should not, though, omit both $\bar{S}$ and $\bar{D}$, because that would lead to $a = b = 0$, which cannot fit the experimental data. See appendix B for details.

11In a full supersymmetric construction, the relevant Yukawa terms require the use of both two gauge-$SU(2)$ doublets and two gauge-$SU(2)$ triplets, see refs. [26, 27].
under which the Standard Model fermions have R-charge +1\(^{12}\). We rely mostly on F-terms for selecting the alignment directions\(^{12}\), even though, at least for some of the aligned directions, we expect it to be possible to simplify the models by employing also D-term alignment, as done for instance in refs. \[31\] \[32\].

Before discussing specific alignments it is useful to clarify some general points:

1. With the F-term alignment employed, a set of specific directions is obtained for the minima. As is often the case in this type of models, physically distinct directions are present; we argue that Nature either happened to choose the phenomenologically viable direction out of a discrete number of degenerate choices, or that some unspecified soft supersymmetry-breaking terms lift the degeneracy, as argued in ref. \[28\]. We must therefore guarantee that the desired direction is part of a discrete set of directions. When discussing specific alignments we shall highlight these degenerate directions.

2. The F-term alignment fixes directions for the VEVs but does not fix their absolute value. As in refs. \[28\] \[32\], we assume that the magnitude of the VEVs is nonzero and that it is stabilised at some finite value; this can be achieved through soft supersymmetry-breaking terms including squared masses that become negative, as argued in ref. \[32\].

4.1 Charged-lepton sector

In this subsection we want to explain the alignment of eqs. (28). We consider an interaction

\[ P_T = T^0 (a_1 S T_1 + a_2 T_1 T_1 + a_3 T_2 T_2 + a_4 T_1 T_2), \tag{33} \]

where \( T^0 \) is a ‘driving field’ or ‘alignment field’ (as a matter of fact, it is a set of three superfields) which transforms as \( 3_1 \) under \( S_4 \), has R-charge +2, and has auxiliary charge \( c (T^0) = -2 c (S) \). The \( a_{1,2,3,4} \) are coupling constants. Taking the derivative of \( P_T \) with respect to

\(^{12}\)We recall that all the allowed superpotential terms must have R-charge +2.

\(^{13}\)A similar implementation of an analogous alignment was discussed in refs. \[28\] \[29\] \[30\]. Here, however, we impose the restriction that the terms in the superpotential responsible for the alignment must be renormalisable. Non-renormalisable terms are used only in the Yukawa couplings by means of Froggatt–Nielsen \[25\] messenger fields with the appropriate gauge representations.
the three components of $T^0$, one obtains the $F$-terms

\[
\frac{\partial F_T}{\partial T_1} = a_1 v_S v_{1x} + 2a_2 v_{1y} v_{1z} + 2a_3 v_{2y} v_{2z} + a_4 (v_{1y} v_{2z} - v_{1z} v_{2y}),
\]

(34a)

\[
\frac{\partial F_T}{\partial T_2} = a_1 v_S v_{1y} + 2a_2 v_{1z} v_{1x} + 2a_3 v_{2z} v_{2x} + a_4 (v_{1z} v_{2x} - v_{1x} v_{2z}),
\]

(34b)

\[
\frac{\partial F_T}{\partial T_3} = a_1 v_S v_{1z} + 2a_2 v_{1x} v_{1y} + 2a_3 v_{2x} v_{2y} + a_4 (v_{1x} v_{2y} - v_{1y} v_{2x}).
\]

(34c)

In order to minimise the potential we must set all three $F$-terms in eqs. (34) to zero. It is clear that there is a solution featuring eq. (28), with

\[
a_1 v_S v_1 + 2a_2 v_1^2 + 2a_3 v_2^2 = 0.
\]

(35)

Here we have an instance of degenerate directions for the VEVs as mentioned previously. Namely, it would also be possible to choose both $(v_{1x}, v_{1y}, v_{1z})$ and $(v_{2x}, v_{2y}, v_{2z})$ to lie in the $(1, -1, 1)$ direction, provided

\[
a_1 v_S v_{1x} - 2a_2 v_{1x}^2 - 2a_3 v_{2x}^2 = 0.
\]

(36)

That solution would not lead to the charged-lepton mass matrix being diagonalised by $U_{\nu}$.

An alternative possibility consists in adding to the theory, either instead of or in addition to $T^0$, an alignment field $D^0$ which is a doublet of $S_4$, has $R$-charge +2, and has $c(D^0) = -2c(S)$. We then have an interaction

\[
P_D = D^0 (a_5 T_1 T_1 + a_6 T_2 T_2 + a_7 T_1 T_2).
\]

(37)

The ensuing minimisation equations are

\[
0 = \frac{\partial P_D}{\partial D_2} = a_5 (v_{1x}^2 + \omega^2 v_{1y}^2 + \omega v_{1z}^2) + a_6 (v_{2x}^2 + \omega^2 v_{2y}^2 + \omega v_{2z}^2)
\]

\[+ a_7 (v_{1x} v_{2x} + \omega^2 v_{1y} v_{2y} + \omega v_{1z} v_{2z}),
\]

(38a)

\[
0 = \frac{\partial P_D}{\partial D_1} = a_5 (v_{1x}^2 + \omega^2 v_{1y}^2 + \omega v_{1z}^2) + a_6 (v_{2x}^2 + \omega^2 v_{2y}^2 + \omega^2 v_{2z}^2)
\]

\[+ a_7 (v_{1x} v_{2x} + \omega v_{1y} v_{2y} + \omega^2 v_{1z} v_{2z}).
\]

(38b)

These equations are identically satisfied by the desired alignment in eq. (28), but they display the same type of ambiguity relative to the sign of the components already mentioned for the $T^0$ alignment.

\footnote{We note that the direction $(1, -1, 1)$ is not related to the direction $(1, 1, 1)$ through an $S_4$ transformation, since the matrix $\text{diag}(1, -1, 1)$ does not belong to $S_4$. Correspondingly, eqs. (35) and (36) are distinct.}
If the two alignment methods (with $T^0$ and $D^0$) are used together, then, for arbitrary values for the parameters $a_{1-4}$ one can only have the alignment of the desired type—up to the sign ambiguity above and to related sign ambiguities like $(1, 1, -1)$ or $(-1, 1, 1)$. Indeed, once we specify $a_{1-4}$ and solve eqs. (34) by fixing the relative magnitudes $v_1$ and $v_2$ through eq. (35), it would take extreme finetuning for a solution of eqs. (38), which depends on $a_{5-7}$, to be consistent with that specific solution of eqs. (34), so in general we will be left only with the solutions that do not depend on $a_{5-7}$, viz. with eq. (28). But in fact one of the alignment terms is enough. Taking e.g. eqs. (38) and expanding to $v_1 (1, 1 + dy, 1 + dz)$ for $T_1$ and $v_2 (1, 1 + dy', 1 + dz')$ for $T_2$ with infinitesimal perturbations, one can verify that there are no remaining continuous flat directions around the minimum.

Noting that the alignment solutions always have two insertions of the flavons, a simple specific realisation of the auxiliary symmetry is a $Z_3$ with $c(\ell_R) = 2$ and $c(T_1) = 1$. In this case, subleading terms could only appear in eq. (26) and in the alignment terms with three additional flavon insertions.

4.2 Neutrino sector

We want the VEV of $\bar{T}$ to be aligned in the $(k, 1, -1)$ direction, where $k = -d/f$. According to appendix B.1, a model with $d = 0$ necessitates large neutrino masses which may or may not conflict with the cosmological bound. So it is not clear at present whether $k = 0$ is possible or should be avoided. Anyway, we want to have a $3_1$ aligned in the $(k, 1, -1)$ direction in order to play the role of $\bar{T}$ in eq. (30).

In order to do this we first introduce the driving fields in table 1, where $q(\chi)$ and $q(\theta)$ are generic $U(1)_\nu$ charges. Then there is a term $D^0 |\chi\rangle$ in the superpotential, and this term leads, through equations similar to eqs. (38), to $\langle 0 |\chi| 0 \rangle = v_\chi (1, 1, 1)$. There is also a term $T^0 |\theta\rangle$ in the superpotential. If $\langle 0 |\theta| 0 \rangle = (v_{\theta x}, v_{\theta y}, v_{\theta z})$, that term leads to $v_{\theta y} v_{\theta z} = v_{\theta z} v_{\theta x} = v_{\theta x} v_{\theta y} = 0$; we choose $\langle 0 |\theta| 0 \rangle$ to lie in the direction $v_\theta (1, 0, 0)$.

We next introduce further driving fields as specified in table 2.

---

15 A bound on $|k|$ may be obtained as follows. From eqs. (22),

$$0 = \sqrt{2} c s e^{i\alpha} k (\mu_2 - \mu_3) + (\mu_1 - s^2 \mu_2 - c^2 \mu_3).$$

Therefore,

$$\frac{|s^2 m_2 - c^2 m_3| - m_1}{\sqrt{2} c s (m_2 + m_3)} \leq |k| \leq \frac{s^2 m_2 + c^2 m_3 + m_1}{\sqrt{2} c s |m_2 - m_3|}.$$
| Field   | $\chi$ | $\theta$ | $D^{\prime}$ | $T^{\prime}$ |
|---------|--------|----------|--------------|--------------|
| $S_1$   | $\mathbf{3}_2$ | $\mathbf{3}_1$ | 2            | $\mathbf{3}_1$ |
| $U(1)_{\nu}$ | $q(\chi)$ | $q(\theta)$ | $-2q(\chi)$ | $-2q(\theta)$ |
| $U(1)_R$ | 0      | 0        | 2            | 2            |

Table 1: Initial driving fields of the solution for the neutrino sector.

| Field | $s_1$ | $\vartheta$ | $S_1^0$ | $S_2^0$ |
|-------|-------|-------------|---------|---------|
| $S_1$ | $1_1$ | $\mathbf{3}_1$ | $1_1$ | $1_2$ |
| $U(1)_{\nu}$ | $q(\vartheta)$ | $q(\vartheta)$ | $-q(\vartheta)$ | $-q(\vartheta)$ |
| $U(1)_R$ | 0 | 0 | 2 | 2 |

Table 2: Further driving fields of the solution for the neutrino sector.

These fields allow for terms $S_1^0 \vartheta \chi$ and $S_2^0 \vartheta \chi$ which force the VEV of the triplet $\vartheta$ to be orthogonal to the VEVs of both $\theta$ and $\chi$. In this way we obtain $\langle 0 | \vartheta | 0 \rangle = v_\vartheta (0, 1, -1)$.

Finally, we assume $-2q(\vartheta)$ to be equal to the $U(1)_\nu$ charge of $D_L \Delta D_L$. Then,

$$\tilde{S} = b_1 s_1 s_1 + b_2 (\vartheta \vartheta)_{1_1}, \quad \tilde{D} = (\vartheta \vartheta)_{2}, \quad \tilde{T} = b_3 s_1 \vartheta + b_4 (\vartheta \vartheta)_{3_1} \quad (41)$$

($b_{1-4}$ are coefficients) are precisely the flavons that we need in eq. (30), appearing at the same order in flavon insertions. Indeed, with $\vartheta = v_\vartheta (0, 1, -1)$, we have $(\vartheta \vartheta)_{2} = v_\vartheta^2 (-1, -1)$ and $(\vartheta \vartheta)_{3_1} = -2v_\vartheta^2 (1, 0, 0)$, according to eq. (A5).

A simple example for an auxiliary symmetry realising the assignments listed in the tables is a $\mathbb{Z}_8$ with $q(D_L) = 0$, $q(\Delta) = 6$, and $q(\vartheta) = 1$. We may also choose $q(\theta) = 3$ and $q(\chi) = 4$. Then $q(S_1^0) = 4$, $q(S_2^0) = 3$, $q(D^{\prime}) = 0$, and $q(T^{\prime}) = 2$. If we assume that there are no messengers enabling non-renormalisable alignment terms—see ref. [17]—then the concern is with the mass terms, and in this specific case all the subleading contributions to eq. (41) appear with at least two additional fields—for instance, one may add $(\chi \chi)_{1_1}$ to any of the combinations in eq. (41). Those subleading contributions have a suppression by at least two extra powers of $\Lambda$ and, provided $\Lambda$ is much larger than the VEVs of the flavons, they are expected to be negligible.

On the other hand, if non-renormalisable alignment terms are allowed, then with this $\mathbb{Z}_8$ there are problematic alignment terms which appear with one additional field insertion. In order to disallow them one may use a symmetry $\mathbb{Z}_{14}$ instead of $\mathbb{Z}_8$; there are then several possibilities for the charges $q(\theta)$ and $q(\chi)—one possibility is $q(\Delta) = 12$. 


\[ q(\vartheta) = 1, \quad q(\theta) = 3, \quad \text{and} \quad q(\chi) = 7, \quad \text{and then} \quad q(S^0_0) = 10, \quad q(S^0_2) = 6, \quad q(D^0) = 0, \quad \text{and} \quad q(T^0) = 8. \]

5 Conclusions

In this paper we have considered the phenomenological consequences of the TM1 Ansatz in light of the recent experimental data on a nonzero reactor mixing angle and on a non-maximal atmospheric mixing angle. We have provided an explicit framework, based on a lepton flavour symmetry group \( S_4 \), for models with TM1 mixing. Confronting the predictions of some of those models with the experimental data may rule out or constrain these particular cases. We have investigated how the VEVs of the required \( S_4 \) multiplets can be aligned consistently.

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A The group \( S_4 \)

The group \( S_4 \) is the group of permutations of four objects \( o_1, o_2, o_3, o_4 \). It is generated by two permutations, \( f : o_1 \leftrightarrow o_2 \) and \( g : o_2 \rightarrow o_3 \rightarrow o_4 \rightarrow o_2 \). Those permutations satisfy

\[ f^2 = g^3 = (fg)^4 = e, \quad \text{(A1)} \]

where \( e \) is the identity permutation. The group \( S_4 \) has order \( 4! = 24 \) and five inequivalent irreducible representations (‘irreps’): the triplets \( 3_1 \) and \( 3_2 \), the doublet \( 2 \), and the singlets \( 1_1 \) and \( 1_2 \). The \( 1_1 \) is the trivial representation. The \( 1_2 \) makes \( f \rightarrow -1, \quad g \rightarrow +1 \). We choose a
basis for the doublet such that
\[
2: \quad f \to F_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g \to G_2 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad (A2)
\]
where \( \omega = \exp(i2\pi/3) \). Notice that this representation is not faithful, since \((F_2G_2)^2\) already is the unit matrix. For the \(3_1\) we choose a basis such that
\[
3_1: \quad f \to F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad g \to G_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (A3)
\]
Notice that \(1_2 = \det 3_1\). The \(3_2 = 3_1 \otimes 1_2\), therefore
\[
3_2: \quad f \to -F_3, \quad g \to G_3. \quad (A4)
\]
The irreps \(3_1\) and \(3_2\) are faithful. Notice that the matrices of the \(3_2\) have determinant +1, therefore \(S_4\) is isomorphic to a subgroup of \(SO(3)\). That subgroup is the symmetry group of the cube or of the regular octahedron.

Let \((x, y, z)\) and \((x', y', z')\) be two identical triplets of \(S_4\), either both of them \(3_1\) or both of them \(3_2\). Then,
\[
\begin{align*}
&\begin{pmatrix} yz' + y'z' \\ zx' + xz' \\ xy' + yx' \end{pmatrix}, \quad \begin{pmatrix} yz' - y'z' \\ zx' - xz' \\ xy' - yx' \end{pmatrix}, \\
&\begin{pmatrix} xx' + \omega^2yy' + \omega zz' \\ xx' + \omega yy' + \omega^2 zz' \end{pmatrix}, \quad xx' + yy' + zz' \quad (A5)
\end{align*}
\]
are, respectively, a \(3_1\), a \(3_2\), a \(2\), and a \(1_1\). If, on the other hand, \((x, y, z)\) is a \(3_1\) and \((x', y', z')\) a \(3_2\) of \(S_4\), then
\[
\begin{align*}
&\begin{pmatrix} yz' - y'z' \\ zx' - xz' \\ xy' - yx' \end{pmatrix}, \quad \begin{pmatrix} yz' + y'z' \\ zx' + xz' \\ xy' + yx' \end{pmatrix}, \\
&\begin{pmatrix} xx' + \omega^2yy' + \omega zz' \\ -xx' - \omega yy' - \omega^2 zz' \end{pmatrix}, \quad xx' + yy' + zz' \quad (A6)
\end{align*}
\]
are a \(3_1\), a \(3_2\), a \(2\), and a \(1_2\), respectively.

Let \((x, y, z)\) be a \(3_1\) and \((p, q)\) a \(2\) of \(S_4\). Then,
\[
\begin{align*}
&\begin{pmatrix} x(p + q) \\ y(\omega p + \omega^2 q) \\ z(\omega^2 p + \omega q) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x(p - q) \\ y(\omega p - \omega^2 q) \\ z(\omega^2 p - \omega q) \end{pmatrix} \quad (A7)
\end{align*}
\]
\[\text{The 2 is a faithful representation of the } S_3 \text{ subgroup of } S_4 \text{ formed by the permutations of } s_{2,3,4}.\]
are a $3_1$ and a $3_2$, respectively. If, however, the $(x, y, z)$ were a $3_2$, then the multiplets in eq. (A7) would be a $3_2$ and a $3_1$, respectively.

Let $(p, q)$ and $(p', q')$ be two $2$ of $S_4$, then

$$\left(\begin{array}{c}qq' \\ pp'\end{array}\right), \quad pq' + qp', \quad pq' - qp'$$  \hspace{1cm} (A8)

are a $2$, a $1_1$, and a $1_2$, respectively.

If $t$ transforms as a $1_2$ of $S_4$ and $(x, y, z)$ is either a $3_1$ or a $3_2$, then $(tx, ty, tz)$ will correspondingly be either a $3_2$ or a $3_1$, respectively. If $(p, q)$ is a $2$, then $(tp, -tq)$ is also a $2$.

### B Possibilities with one vanishing parameter

In this appendix we investigate the cases where either $a$, $b$, or $d$ vanish. In practice we shall have to deal with equations of the form

$$\mu_1 + p\mu_2 + q\mu_3 = 0,$$  \hspace{1cm} (B1)

where $p$ and $q$ are complex numbers with known moduli, $|p|^2 \equiv P$ and $|q|^2 \equiv Q$. Unfortunately, eq. (B1) is not very well defined because the $\mu_j = m_j e^{-i\psi_j}$ contain unknown Majorana phases $\psi_j$; moreover, the phases of $p$ and $q$ are in some cases also ambiguous. Equation (B1) states that the three complex numbers $\mu_1$, $p\mu_2$, and $q\mu_3$ form a triangle in the complex plane. We make use of the fact that, if three complex numbers $c_1, c_2, c_3$ form a triangle in the complex plane, then the moduli of those numbers must satisfy the inequality

$$|c_1|^4 + |c_2|^4 + |c_3|^4 - 2|c_1c_2|^2 - 2|c_1c_3|^2 - 2|c_2c_3|^2 \leq 0.$$  \hspace{1cm} (B2)

This allows us to extract a useful inequality from eq. (B1):

$$\lambda \equiv m_1^4 + P^2m_2^4 + Q^2m_3^4 - 2Pm_1^2m_2^2 - 2Qm_1^2m_3^2 - 2PQm_2^2m_3^2 \leq 0.$$  \hspace{1cm} (B3)

As we shall see, this inequality allows us to dismiss some of the models and to constrain the neutrino masses in the remaining ones.

If the neutrino mass hierarchy is normal, then

$$m_2^2 = m_1^2 + \Delta m_{\text{sol}}^2, \quad m_3^2 = m_1^2 + \Delta m_{\text{atm}}^2,$$  \hspace{1cm} (B4)

where $\Delta m_{\text{sol}}^2 \approx 7.5 \times 10^{-5}$ eV$^2$ and $\Delta m_{\text{atm}}^2 \approx 2.4 \times 10^{-3}$ eV$^2$ are, respectively, the solar and atmospheric neutrino mass-squared differences. On the other hand, if the neutrino mass hierarchy is inverted, then

$$m_1^2 = m_2^2 + \Delta m_{\text{atm}}^2, \quad m_2^2 = m_3^2 + \Delta m_{\text{atm}}^2 + \Delta m_{\text{sol}}^2.$$  \hspace{1cm} (B5)
We know that
\[ \epsilon \equiv \frac{\Delta m^2_{\text{sol}}}{\Delta m^2_{\text{atm}}} \] (B6)
is very small, \(0.0294 \leq \epsilon \leq 0.0335\) \[7\]. Another small quantity, of the same order of magnitude as \(\epsilon\), is \(s^2\), which, as follows from eq. [17], satisfies \(0.0648 \leq s^2 \leq 0.0798\) at the 1\(\sigma\) level. A third small quantity is
\[ X \equiv 4e^2s^2\sin^2\beta = \frac{3}{2}c^4_{13}(1 - 2s^2_{23})^2, \] (B7)which phenomenologically may be as large as 0.18 but may conceivably be much smaller than that, or even zero \[6, 7, 8\].

In the case of a normal neutrino mass spectrum, \(m_1\) is the smallest neutrino mass; let then \(f_1 \equiv m_1^2/\Delta m^2_{\text{atm}}\). In that case, from eqs. [B3], [B4], and [B6],
\[ \frac{\lambda_{\text{normal}}}{(\Delta m^2_{\text{atm}})^2} = f_1^2 (1 + P^2 + Q^2 - 2P - 2Q - 2PQ) \]
\[ + 2f_1 [Q (Q - 1 - P) + \epsilon P (P - 1 - Q)] \]
\[ + (Q - \epsilon P)^2 \leq 0. \] (B8)

In the case of an inverted neutrino mass spectrum, \(m_3\) is the smallest neutrino mass; let in that case \(f_3 \equiv m_3^2/\Delta m^2_{\text{atm}}\). Then, from eqs. [B3], [B5], and [B6],
\[ \frac{\lambda_{\text{inverted}}}{(\Delta m^2_{\text{atm}})^2} = f_3^2 (1 + P^2 + Q^2 - 2P - 2Q - 2PQ) \]
\[ + 2f_3 [1 - P - Q + (1 + \epsilon) P (P - 1 - Q)] \]
\[ + (1 - P - \epsilon P)^2 \leq 0. \] (B9)

**B.1 The case \(d = 0\)**

We start with the simplest case, namely \(d = 0\). Then, from eqs. [22],
\[ \mu_1 = s^2\mu_2 + c^2\mu_3. \] (B10)
This means that \(P = s^4\) and \(Q = c^4\). Then, \(1 + P^2 + Q^2 - 2P - 2Q - 2PQ = 0\). One quickly finds that an inverted neutrino mass spectrum is not possible in this case, while a normal neutrino mass spectrum is possible provided
\[ f_1 \geq \frac{(c^4 - \epsilon s^4)^2}{4e^2s^2(c^2 + \epsilon s^2)} \approx \frac{1}{12s^2_{13}}. \] (B11)
This inequality corresponds approximately to the area above the line in fig. [1] (where the best-fit value for \(\epsilon\) was used).
Therefore, if the neutrino mass hierarchy is normal,

\[ m_1^2 \geq \frac{(c^4 \Delta m_{\text{atm}}^2 - s^4 \Delta m_{\text{sol}}^2)^2}{4c^2 s^2 (c^2 \Delta m_{\text{atm}}^2 + s^2 \Delta m_{\text{sol}}^2)} \approx \frac{c^4}{4 s^2} \Delta m_{\text{atm}}^2 \approx 0.00718 \text{ eV}^2, \]

or \( m_1 \geq 0.085 \text{ eV} \); this indicates an almost-degenerate neutrino mass spectrum. Then the sum of the light-neutrino masses is

\[ m_1 + m_2 + m_3 \geq \left( \sqrt{0.00718} + \sqrt{0.00720} + \sqrt{0.00959} \right) \text{ eV} \approx 0.268 \text{ eV}. \]

This is a value which violates a recent cosmological bound [34] but fits nicely with a different one [35].

**B.2 The case \( a = 0 \)**

We now consider the case \( a = 0 \). It then follows from eqs. (22) that

\[ \mu_1 + \left( s^2 + c^2 e^{-i2\beta} \right) \mu_2 + \left( c^2 + s^2 e^{-i2\beta} \right) \mu_3 = 0. \]

Hence, in this case

\[ P = Q = 1 - X \]

*B.2 The case \( a = 0 \)**

We now consider the case \( a = 0 \). It then follows from eqs. (22) that

\[ \mu_1 + \left( s^2 + c^2 e^{-i2\beta} \right) \mu_2 + \left( c^2 + s^2 e^{-i2\beta} \right) \mu_3 = 0. \]

Hence, in this case

\[ P = Q = 1 - X \]
Figure 2: The minimum value for $m_1^2/\Delta m_{\text{atm}}^2$ in the case $a = 0$ as a function of $X = (3/2) \cos^4(\theta_{13}) \cos^2(2\theta_{23})$, for the best-fit values of $\epsilon$ and $s_{13}^2$.

and

$$\frac{-\lambda_{\text{normal}}}{(\Delta m_{\text{atm}}^2)^2} = f_1^2 (3 - 4X) + 2f_1 P (1 + \epsilon)$$
$$-P^2 (1 - \epsilon)^2 \geq 0,$$  \hspace{1cm} (B16a)

$$\frac{-\lambda_{\text{inverted}}}{(\Delta m_{\text{atm}}^2)^2} = f_3^2 (3 - 4X) + 2f_3 (2 - 3X + \epsilon - \epsilon X)$$
$$-(X - \epsilon + \epsilon X)^2 \geq 0.$$  \hspace{1cm} (B16b)

These inequalities yield

$$f_1 \geq \frac{1 - X}{3 - 4X} \left[ 2\sqrt{1 + \epsilon^2} (1 - X) - \epsilon (1 - 2X) - 1 - \epsilon \right],$$  \hspace{1cm} (B17a)

$$f_3 \geq \frac{1 - X}{3 - 4X} \left[ 2\sqrt{1 + \epsilon^2} (1 - X) + \epsilon (1 - 2X) - 2 - \epsilon \right.$$  
$$+ \frac{X}{1 - X} \right],$$  \hspace{1cm} (B17b)

which are valid for the normal- and inverted-hierarchy cases, respectively. So, in the case of a normal hierarchy $m_1^2 \gtrsim \Delta m_{\text{atm}}^2/3$ while in the case of an inverted hierarchy $m_3^2 \gtrsim (X - \epsilon)^2 \Delta m_{\text{atm}}^2/4$ may be considerably smaller ($m_3$ may even vanish if it happens that $X - \epsilon + \epsilon X = 0$). This can be seen in figs. 2 and 3, respectively.

In this case both neutrino mass spectra are allowed.
Figure 3: The minimum value for $m_3^2/\Delta m_{\text{atm}}^2$ in the case $a = 0$ as a function of $X = (3/2) \cos^4(\theta_{13}) \cos^2(2\theta_{23})$, for the best-fit values of $\epsilon$ and $s_{13}^2$.

B.3 The case $b = 0$

Equations (22) in this case lead to

$$\mu_1 + \left( s^2 - 2c^2e^{-i2\beta} \right) \mu_2 + \left( c^2 - 2s^2e^{-i2\beta} \right) \mu_3 = 0. \quad (B18)$$

Therefore,

$$P = 4 - 12s^2 + 9s^4 + 2X$$
$$= (2 - 3s^2)^2 + 2X, \quad (B19a)$$

$$Q = 1 - 6s^2 + 9s^4 + 2X$$
$$= (1 - 3s^2)^2 + 2X. \quad (B19b)$$

The inequalities (B8) and (B9) then read

$$8Xf_1^2 + 4f_1 \left\{ (1 - 3s^2)(2 - 3s^2) \left[ 1 - 3s^2 - \epsilon(2 - 3s^2) \right] \right\}$$
$$+ 2X \left[ 2 - 3s^2 - \epsilon(1 - 3s^2) \right]$$
$$- \left[ (1 - 3s^2)^2 + 2X - \epsilon(2 - 3s^2)^2 - 2\epsilon X \right]^2 \geq 0, (B20a)$$

$$8Xf_3^2 - 4f_3 \left\{ (1 - 3s^2)(2 - 3s^2) \left[ (1 - 3s^2) + \epsilon(2 - 3s^2) \right] \right\}$$
$$+ 2X \left[ \epsilon - 3s^2 - 3\epsilon s^2 \right]$$
$$- \left[ -3 - 4\epsilon + (1 + \epsilon)(12s^2 - 9s^4 - 2X) \right]^2 \geq 0, (B20b)$$
Figure 4: The minimum value for $m_1^2/\Delta m^2_{\text{atm}}$ in the case $b = 0$ as a function of $X = (3/2) \cos^2(\theta_{13}) \cos^2(2\theta_{23})$, for the best-fit values of $\epsilon$ and $s^2_{13}$.

respectively. The corresponding lower bounds on $f_1$ and $f_3$ are depicted in figs. 4 and 5, respectively. One sees that a normal mass spectrum is allowed but that, unless $X \gtrsim 0.05$, an inverted mass spectrum leads to much too high neutrino masses, which violate the cosmological bound.

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Figure 5: The minimum value for $m_3^2/\Delta m_{3\text{atm}}^2$ in the case $b = 0$ as a function of $X = (3/2) \cos^4(\theta_{13}) \cos^2(2\theta_{23})$, for the best-fit values of $\epsilon$ and $s_{13}^2$.

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