A Simulation on Vertically Shaken Granular Layers

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A hybrid model of molecular dynamics and continuum mechanics is introduced to study a system of vertically shaken granular layers. Despite the simplicity the model shows pattern formation in the granular layers due to the formation of heaplets. We show from a simple analysis that the onset of pattern formation is density dependent and this result is justified by the subsequent computer simulations. Our simulations also show that the heaping process can be divided into three stages: an early stationary stage, an intermediate growing stage and a late-time saturated stage. In the early stage, the average volume of the heaplets remains almost unchanged until the system crosses over to the intermediate growing stage. The length of time that system remains in the early stage defines the onset time of the instabilities, \( k_0 \) which depends on the shaking intensity, \( \gamma \). In the growing stage, the average volume of the heaplets grows with time and can be approximated by a power law with a shaking intensity dependent growth exponent, \( z \). From our simulation, we show that the growth exponent, \( z \) depends linearly on the shaking intensity, \( \gamma \). There is a critical shaking intensity \( \gamma_c \approx 14 \), such that for shaking intensities greater than \( \gamma_c \), \( z \) cannot be properly defined. The late-time saturated stage is where most of the particles are trapped in a big heap and this big heap is in equilibrium with the surrounding granular gas.

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I. INTRODUCTION

We are interested in pattern formation in a thin layer of granular particles upon a plate. We begin by distinguishing between experiments on systems in which the plate is vertically vibrated \( \ddots \) or in which the plate is tapped \( \ddots \) or shaken. In the vertically vibrating systems, the particles dance upon the oscillating plate and the surface of the layer becomes patterned with regions characterised by different modes of oscillation or by the the same modes of oscillation but different phases. The features described as oscillons \( \ddots \) and the interface between regions \( \ddots \) are both of interest here. In the tapped or shaken system there is a long period between taps/shakes in which the plate is at rest and the particle layer relaxes in this period to a static configuration in which the surface of the layer is marked by ripples or heaplets. In other words the perturbations are discrete and are followed by periods of stasis in distinction to the vibrated system which is continuously moving.

Following the original observation of square and striped patterns by Melo et. al \( \ddots \) in the vibrating system, there have been many experimental observations \( \ddots \) and theoretical discussions \( \ddots \) for this type of pattern formation. All these works and many others have contributed greatly and also establish a good understanding of the dynamics of the pattern formation of granular layer under continuous vibration.

On the other hand in the tapped/shaken system, Duran has observed the formation of ripples \( \ddots \) and granular heaplets \( \ddots \) in a tapped granular layer. However, there is less literature available on this class of experiments, and our paper is an attempt to fill this gap.

We now further distinguish between vertically shaken granular layers and tapped granular layers. In the shaken system the plate moves up and then is withdrawn sharply to a neutral position. When the plate reaches the highest point, the particles continue their motion and then travel ballistically, falling back to the stationary plate, rebounding and then settling down and relaxing into a static configuration. The plate is subsequently moved up again and the process repeated. The lateral redistribution, which produces the patterns, takes place when the falling particles rebound from the plate and relax. On the other hand in the tapped system the plate scarcely moves but, as in the Newton’s cradle experiment, the tap is imparted impulsively to the particles which jerk off the plate, fall back, rebound and relax. Most of the lateral redistribution takes place at the jerk. However, shaken and tapped granular layers also share some common properties such as the formation of isolated granular heaplets, which coarsen with the number of taps/shakes, and the rate of coarsening of the heaplets increases as the intensity of the shocks increases. We have discussed tapped layers in a previous article \( \ddots \). The present is concerned mainly with shaken granular layers.

This article is organised as follows: first, we introduce a simple model to study shaken granular layers. The model is based on a hybrid of a molecular dynamics description and continuum modelling of a granular sheet. Then we show typical patterns produced by the model and these patterns resemble typical experimental morphologies of shaken granular layers with isolated heaplets. An analysis is performed and leads to the prediction that the depinning shaking intensity of the granular layers is density dependent, which is later justified from the simulations. Later we show from the simulation that the coarsening of the pattern comes in three stages: an early stationary stage, a growing stage and a saturated stage. In the early
stage, the layer remains flat, with small fluctuation superimposed, for a short duration \( k_0 \) which depends on the shaking intensity \( \gamma \). The early stage is characterised by a small value \( V \) (defined later), which measure the degree of heapings. Then we discuss the cross-over time from the early stage to the growing stage, which indicates the onset time \( k_0 \) of instabilities. In the growing stage, we approximate the growth of the average volume of heaplets by a power law. The growth exponent \( z \) is \( \gamma \)-dependent and the power law approximation fails when \( \gamma \) is larger than a critical value, \( \gamma_c \). Finally, the late-time saturated stage is where all the granular heaplets are merged into a big heap and the big heap is in equilibrium with a scattering of particles on the plate which can be described as a granular gas. The time of onset of saturation is dependent on the size of the system. In our simulations, the system size is small (with periodic boundary conditions), and as a result the time of onset will almost certainly be small compared to that experimentally observed in real systems.

II. MODEL

The spatial distribution of a granular layer can be described by a continuum field variable \( n(\vec{r}, t) \), where \( \vec{r} = (x, y) \) are the Cartesian coordinates parallel to the plate. The number density \( n(\vec{r}, t) \) also represents the layer thickness if we take the average diameter of grains as unity assuming no compaction throughout the whole shaking process \( \frac{\partial u}{\partial t} = \nu \nabla^2 u - g + B(h, u, \alpha). \quad (1) \)

Here \( \nu \) is a dissipative coefficient; it represents the lateral exchange of velocities when particles collide with each other and \( g \) is the acceleration due to gravity. The function \( B \) specifies how the layer bounces when it hits the plate. The bouncing term \( B \) is zero except when \( h = 0 \) and it is defined by the mapping

\[ u \rightarrow -\alpha u, \quad (2) \]

where \( \alpha \) is the coefficient of restitution. The same equations here have been used by Rothman \cite{14} and Moon et al. \cite{13} to describe the phenomena of oscillations and stripes structures in a vertically vibrated sand layer.

The coefficient of restitution \( \alpha \) is expected to depend on \( n \) because the energy dissipation is greater in regions of higher local density. However, the precise form of \( \alpha(n) \) is not known, except that it should be a monotonic decreasing function of \( n \). In fact, which we will show in the following section, it is possible to deduce roughly the functional form of \( \alpha(n) \) from experiments by measuring the depinning shaking intensities for homogeneous granular layer with different thickness. In this paper, for the sake of definiteness, \( \alpha(n) \) is chosen to be a Lorentzian, i.e.

\[ \alpha(n) = \frac{n_0^2}{n^2 + n_1^2}, \quad (3) \]

where \( n_0 \) and \( n_1 \) are parameters.

In the shake the plate moves up and is then withdrawn sharply. The particles continue in free flight, all with the same velocity, and fall under gravity until they hit the plate at rest, all with the same velocity \(-U\). They then rebound with local velocity \( u(\vec{r}, t) = \alpha(n)U \) which depends on the local density \( n(\vec{r}, t) \) because of the density dependence of the coefficient of restitution \( \alpha \). It is useful to use \( \gamma = U/U_0 \) as a dimensionless measure of the shaking intensity, here \( U_0 = 3 \text{ cm/s} \) being a very crude estimate of a typical take-off velocity of the grains when the plate is withdrawn. The purpose of \( U_0 \) is to allow us to present the data on a reasonable scale.

After the shake, the particles undergo motion as given by Eq.(1). When the particles at a point fall back to the plate again, they bounce and redistribute onto the neighbourhood. To describe this, we discretise the system into \( L \times L \) squares of side \( a \) with the square at \( \vec{r}_i \) containing \( N_i = n(\vec{r}_i, t) a^2 \) particles. Then we implement the redistribution in the following way:

1. We define an integer \( N_i \) equal to the the next integer greater than \( n(\vec{r}_i, t) \), which is the number of particles on the rebounding site.

2. We surround the rebounding site \( \vec{r}_i \) by a disc of radius \( R_{\text{max}} \), which is given by

\[ R_{\text{max}} = \xi \left( \frac{u(\vec{r}_i, t)}{U_0} \right)^2, \quad (4) \]

where \( \xi \) is a parameter which controls the distance travelled by the particles.

3. We specify a hopping angle \( \phi \) by choosing a random number in the range \([0, 2\pi]\) with equal probability.

4. We specify a hopping range \( R \) by choosing a random number in the range \([0, R_{\text{max}}]\) with equal probability.

5. The angle \( \phi \) and hopping range \( R \) together define a receiving site \( \vec{r}_{ij} \) such that \(|\vec{r}_{ij} - \vec{r}_i| = R \) and the angle between the vector \( \vec{r}_{ij} - \vec{r}_i \) and a fixed reference direction is \( \phi \).

6. The process is repeated \( N_i \) times but receiving sites may be repeated and the rebounding site itself may be a receiving site.
7. On each repetition (except the last) the density \( n(\vec{r}_i, t) \) on the rebounding site is reduced by unity and the density on the receiving site is increased by unity. On the last repetition, leftovers are redistributed.

The redistribution rules can be summarised by first defining \( N_i \) and the size of the leftovers, \( \Delta n_i \equiv n(\vec{r}_i, t) - N_i + 1 \), then repeat the following \( N_i \) times,

\[
\begin{align*}
    n(\vec{r}_j, t) &\rightarrow n(\vec{r}_j, t) - \Delta_j, \\
    n(\vec{r}_j', t) &\rightarrow n(\vec{r}_j', t) + \Delta_j,
\end{align*}
\]

for \( j = 1, 2, \ldots, N_i \),

\[
\Delta_j = \begin{cases} 
    1 & j = 1, 2, \ldots, N_i - 1 \\
    \Delta n_j & j = N_i.
\end{cases}
\]

where

\[
\frac{1}{\Delta n_j} = \frac{\gamma U}{\sqrt{\beta}}.
\]

The process conserves the number of particles in the redistribution as it should. If \( R_{max} \) is greater than system size the physical reality is not well-represented.

The mass transfer after redistribution can be describe by a continuity equation \( \partial_t n = -\nabla \cdot \vec{J} \). The constitutive current equation is given by

\[
\vec{J} = \begin{cases} 
    -\eta(\beta|\nabla n|^2 - 1)|\nabla n|, & |\nabla n| > \frac{1}{\sqrt{\beta}} \\
    \nabla n, & |\nabla n| \leq \frac{1}{\sqrt{\beta}}
\end{cases}
\]

where \( \frac{1}{\sqrt{\beta}} = \tan \theta_c \) is the critical slope of the heaplet side and \( \eta \) sets the diffusion rate. Here the angle of repose \( \theta_c \) is chosen to be \( \pi/6 \). The constitutive equation is constructed in such a way that when the gradient is greater than a critical slope, mass current will flow down hill to smooth out density fluctuation, but no mass is transferred if the gradient is less than the critical slope, because this corresponds to a stable situation.

### III. SIMULATION

The model is studied on a \( L \times L \) square lattice with periodic boundary condition. Initially the granular layer is prepared with the density \( n \) randomly distributed about an average value \( \bar{n} \). At the beginning of each shake, each site is assigned a velocity \( u(\vec{r}, t) = \alpha(n(\vec{r}, t))\gamma U_0 \). We choose the parameters \( n_0 \) and \( n_1 \) in Eq. (3) for \( \alpha(n) \) such that \( n_0^2 = 1/3 \) and \( n_1^2 = 2/3 \). The value of the lattice spacing \( a \) is set equal to the average diameter of the particles and the hopping parameter \( \xi \) is set to \( 75\sqrt{3}U_0^2/2g \). The reason we choose \( \xi \) differently from reference \([13] \) is to allow us present data on a reasonable scale. Then Eq. (3) and the equation of continuity with Eq. (5) are solved using a Runge-Kutta method. On each shaking cycle, these equations are iterated until no more lateral mass transfer occurs at each site because of the rebound and because of the local gradient on each site is less than the tangent of the angle of repose, \( \theta_c = \pi/6 \) with 1% cutoff.

IV. RESULTS

A. Morphology

Fig. 1 shows typical results of the pattern formation in two different simulations. The top panel corresponds to a lower shaking intensity \( \gamma = 4.0 \), and the lower panel corresponds to a higher shaking intensity \( \gamma = 7.0 \). Bright regions correspond to a higher density region of the granular heaplets in the granular layer. Dark regions correspond to regions of low densities of freely moving interstitial granular particles (granular gases). Both panels show an effective surface tension as suggested in reference \([13] \). The effective surface tension depends on the shaking intensity. In the lower panel a greater effective surface tensions is associated with a higher shaking intensity, and this greater surface tension produces more compact heaplets, which in agreement with the findings in reference \([13] \).

B. Depinning Shaking Intensity

In order to form heaplets in the granular layer, the shake intensity not only needs to be strong enough to lift grains off the plate but it must also be able to redistribute grains so the hopping range must greater than the lattice spacing \( a \). One can predict the shaking intensity \( \gamma_0 \) for such a depinning process from a mean field argument. Let the thickness of the granular layer be homogeneous throughout the system and be equal to \( \bar{n} \). After the first shake, the rebound velocity of the layer is \( \bar{u} = \alpha(\bar{n})\gamma U_0 \) everywhere. We can obtain the depinning shaking intensity \( \gamma_0 \) from Eq. (4) and setting \( R_{max}(\bar{n}) = a \), so that

\[
\gamma_0 = C(\bar{n}^2 + n_1^2)
\]
and
\[ C = \left( \frac{a}{\xi} \right)^{1/2} \frac{1}{n_0^2C_0}. \]

However, one might expect the relation given by Eq. (3) to be not strictly quadratic in \( \bar{n} \) when fluctuations are important. Fig. 2 shows the variation of \( \gamma_0 \) against average density \( \bar{n} \) from the simulation. The graph shows that the relation between \( \gamma_0 \) and \( \bar{n} \) is in good agreement with the prediction. From the inset of Fig. 2 shows that the exponent is only slightly less than 2 (~1.91). This result is due to the specific form for the effective coefficient of restitution \( \alpha(n) \) we chose in Eq. (3). However, this also suggests a possibility that one may construct the functional form of \( \alpha(n) \) from experiments. This can be done by noting the relation between depinning shaking density \( \gamma_0 \), which can be measured from experiments, and the corresponding average density of layer \( \bar{n} \) is given by \( \gamma_0 = (a/\xi)^{1/2} [\alpha(\bar{n})]^{-1} \). We present this as a challenge to experimentalists.

![FIG. 2: Variation of \( \gamma_0 \) for different values of \( \bar{n} \). The inset shows the log-log plot for \( \gamma_0 \) against \( \bar{n} \). The linear regression fit line for the inset gives a slope of 1.91.](image)

### C. Average volume of heaplets

We are also interested in how the size of heaplets grows in time. The relevant length scale can be obtained as the average of the thickness at site \( i \) weighted by its local volume. We know that the thickness is \( n_i \) and the local volume element of site \( i \) is \( V_i = n_i \sigma \), where site area \( \sigma \) is the square of the lattice constant, then
\[ l = \frac{\sum_i n_i V_i}{\sum_i V_i} = \frac{\sum_i n_i^2}{\sum_i n_i}. \]  

Hence the volume of the heaplet is \( V = l^3 \) and Fig. 3 and Fig. 4 shows the typical linear and log-log plots of heaplet volume \( V \) for different values of \( \gamma \) as a function of the number of shakes, \( k \).

In each plot in Fig. 3 there are three different stages and each stage corresponds to a different underlying process. In the early stage, the volume remains almost unchanged in time. In this stage, the fluctuations in the randomly distributed granular layer are not strong enough to trigger coalescence. As a result, the layer remains nearly flat and has a small value of \( V \). The second stage is the heaplet formation stage, which corresponds to the growing regions in the plots. In this stage, high density regions of the granular layer act as a kinetic energy sinks, many random hopping granular particles hop into these sinks and are trapped there. As a result of capturing particles, these high density regions grow and merge to form bigger heaplets. Finally, the late time stage is where all the the heaplets have joined into a single big heap and stop growing. There are some remaining individual particles (granular gas) hopping randomly about and occasionally encountering the big heap and getting trapped there. Occasionally particles on the big heap will leave the heap and join the surrounding granular gas. When the number of particles leaving the big heap is balanced by those entering the big heap, the system is now in an equilibrium situation which very similar to that of droplets in a supersaturated vapour. However there are very few individual particles involved in the motion across boundaries during equilibrium and therefore the fluctuations of the volume of big heap are small as observed in Fig. 3 and Fig. 4. Of course the size of the saturated heap depends on the size of the simulated system and the average number density of particles. Larger system sizes will result in a later saturation and bigger saturated heaps.

![FIG. 3: Log-Log plot of heaplet size, \( V \) against number of shakes, \( k \) for different values of \( \gamma \). From left to right the corresponding value of \( \gamma \) for each curve is 6, 8, 10, 12 and 14. Each curve is averaged over 10 different runs. Note the break-down of power-law growth in the intermediate region for the curves with \( \gamma > 14 \).](image)
D. Growth Exponents

In the growing stage, the growth exponent $z$ is $\gamma$-dependent. Fig. 4 shows the growth exponent $z$ for different values of $\gamma$. As we can see there is a trend of increasing growth exponent as the shaking intensity $\gamma$ is increased. The relation between the growth exponent and the shaking intensity is roughly linear and is given by $z \approx 0.348 \gamma - 0.644$, as shown as the straight line in Fig. 4. However, the relation breaks down when $\gamma \gtrsim 14$. The reason of the break-down of the power law approximation will be discussed in the following section.

We can understand the dependence of $z$ to $\gamma$ from the fact that larger shaking intensities give larger kinetic energy to allow the system explore greater regions of phase space. As a result, systems with larger shaking intensities are able to find stable configurations easier than systems with lower shaking intensities. Of course, the final stable configuration is a single heap which favour greater dissipation.

E. Onset Time

We also note from Fig. 3 that the onset of the growing stage occurs at different times for different $\gamma$'s. Generally the onset time $k_0$ increases as $\gamma$ increases. Fig. 5 plots $k_0$ against $\gamma$ and shows that $k_0$ is an increasing function of $\gamma$. However, for $\gamma$ greater than a critical value $\gamma \gtrsim 14$, heaplets can start forming at any time hence no proper $k_0$ can be defined. The same reason that $k_0$ loses its predictability also explains why the power law approximation fails for $\gamma > \gamma_c$. This can be clearly seen from Fig. 4. Each panel of Fig. 4 contains time series of $V$ for 10 different runs where each run has a different initial configuration but the same shaking intensity. The top panel is clear that there is a well defined onset time of instabilities at $k \approx 6$ and consistent growth after the instability starts. The irregular slow-down of the growth during the cross-over from growing stage to saturated stage is due to the formation of multiple metastable heaps and the merging process of these heaps is slow. On the other hand, curves in the lower panel do not have a well defined onset time, instabilities in each curve appear at a different time. In fact, some curves remain trapped in the metastable early stage and instabilities never appear through out the simulations.

V. CONCLUSION

We have introduced a simple model to study vertical shaken granular layers. The model shows the shaking intensity for the depinning of granular layer is related to the layer thickness and to the effective coefficient of restitution of the granular layer. The dynamics of heaplet formation in shaken granular layer can be divided into three stages: an early stationary stage ($\gamma$-independent), an intermediate growing stage ($\gamma$-dependent), and a saturated/equilibrium stage. In the growing stage, the growth exponent $z$ is an increasing function of $\gamma$. The onset time $k_0$ increases as $\gamma$ increases up to a critical value of $\gamma_c$ ($\approx 14$) where heaplet growth can start at any time dependent on details of the initial configuration.

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FIG. 6: Each panel shows the time series of $V$ for several different runs. The top panel corresponds to $\gamma = 10$. For shaking intensities $\gamma < 14$, the onset time of the instabilities is consistent for different initial configurations. The irregular slow-down of the growth in some of the curves is due to the formation of more than one heaplet. The merging process of the heaplets is slow, this is due to the strong dissipation in each heaplet and particles in heaplets are less mobile than those not in heaplets. The lower panel corresponds to a shaking intensity $\gamma$ where the scaling law fails ($\gamma = 14$). At shaking intensities $\gamma \geq 14$, one loses the predictability of the onset time of the instabilities. The early stage where the granular layer is relatively flat is metastable and the system can be trapped in this stage and then suddenly jump at an unpredictable time to the saturated stage within a few shakes. The time of the jump appears to depend sensitively on the initial conditions.