A FREE BOUNDARY PROBLEM OF THE CANCER INVASION

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Abstract. This paper deals with a free boundary problem for the cancer invasion model over a one dimensional habitat in the micro-environment, in which the free boundary represents the spreading front and is caused by tumour cells and acid-mediated. In this problem it is assumed that the tumour cells spread from the given initial region, and the spreading front expands at a speed that is proportional to the tumour cell and acids’ population gradient at the front. The main objective is to realize the dynamics/variations of the healthy cells, tumour cells, acid-mediated and the free boundary. We prove a spreading-vanishing dichotomy for this model, namely the tumour cells either successfully spreads to infinity as time tends to infinite at the front, or it fails to establish and dies out in long run while the healthy cells stabilizes at a positive steady-state. The long time behavior of solution and criteria for spreading and vanishing are obtained.

1. Introduction. A variety of reaction-diffusion systems are used to describe diverse phenomena in population ecology. To consider the spreading of an invasive species in the new environment, Du and Lin [7] introduced a free boundary problem to describe the spreading of invasive species as follows:

\[
\begin{cases}
  u_t - Du_{xx} = u(a - bu), & t > 0, \quad 0 < x < h(t), \\
  h'(t) = -\mu u_x(t, h(t)), & t \geq 0, \\
  u_x(t, 0) = u(t, h(t)) = 0, & t \geq 0, \\
  h(0) = h_0, \quad u(0, x) = u_0(x) \geq 0, & 0 \leq x \leq h_0,
\end{cases}
\]

(1)

where \( u(t, x) \) represents the population density of an invasive species, \( x = h(t) \) denotes the spreading front, \( a, b, \mu \) and \( h_0 \) are positive constants. They have proved that system (1) admits a unique global solution \((u, h)\), which satisfies that as \( t \to \infty \), either \( h(t) \to \infty \) and \( u(t, x) \to a/b \), or \( h(t) \to h_\infty \leq \pi \sqrt{D/4a} \) and \( u(t, x) \to 0 \). This result revealed that the species either successfully establishes itself in the new environment (called spreading) or fails to establish and eventually vanishes (called vanishing). Moreover, they derived sharp criteria for spreading and vanishing by \( h_0 \) and \( \mu \), assuming the other parameters are fixed. Later on, Du and Guo [3] extended these results to the case of higher space dimension with radial symmetry.

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A typical model is the Gatenby-Gawlinski system, which was proposed by Jessica B. McGillen et al. ([12]) in an attempt to model the metabolism and behaviour of healthy cells, tumour cells and acid-mediated in the micro-environment. Gatenby and Gawlinski were the first to put acid-mediated invasion hypothesis into a reaction-diffusion framework (Gatenby and Gawlinski [10]). The acid-mediated invasion hypothesis is motivated by viewing a tumour as an invasive species. These models have received a great of deal of attention in the past few years, motivated in part by their widespread occurrence in models of the tumour invasion phenomena and in part by the richness of the dynamics. (see [1, 2, 14, 15]).

In this paper, we will consider the Gatenby-Gawlinski model with double free boundaries as follows:

\[
\begin{align*}
&u_t = u (1 - u - a_2 v) - d_1 uw, & t > 0, & -\infty < x < \infty, \\
v_t - D v_{xx} = r_2 v (1 - v - a_1 u) - d_2 vw, & t > 0, & g(t) < x < h(t), \\
w_t - w_{xx} = c (v - w), & t > 0, & g(t) < x < h(t), \\
h'(t) = -\mu (v_x(t, h(t)) + \tau w_x(t, h(t))), & t \geq 0, \\
g'(t) = -\mu (v_x(t, g(t)) + \tau w_x(t, g(t))), & t \geq 0, \\
u(0, x) = u_0(x), & -\infty < x < \infty, \\
v(0, x) = v_0(x), w(0, x) = w_0(x), & -h_0 \leq x \leq h_0, \\
h(0) = h_0, g(0) = -h_0.
\end{align*}
\]

(2)

where \(a_1, a_2, r_2, d_1, d_2, c, D, \mu, \tau\) and \(h_0\) are positive constants, \(u, v\) and \(w\) represent the density of healthy cells, the density of tumour cells and the concentration of extracellular lactic acid in excess of normal tissue acid concentrations, respectively. The free boundaries \(x = h(t)\) and \(x = g(t)\) show the spreading front. The initial functions \(u_0(x)\), defined on \((-\infty, \infty)\), stands for the initial population density of healthy cells in the whole space, \(v_0(x)\) defined on an initial region \([-h_0, h_0]\), signifies the initial population density of tumour cells, and \(w_0(x)\) defined on an initial region \([-h_0, h_0]\), means the initial population density of acid-mediated. It is assumed that

\[
\begin{align*}
&u_0 \in C^2([0, \infty)), \ 0 < u_0(x) < \infty \text{ in } (-\infty, \infty); \\
v_0 \in C^2([-h_0, h_0]), \ v_0(-h_0) = v_0(h_0) = 0 \text{ and } v_0(x) > 0 \text{ in } (-h_0, h_0); \\
w_0 \in C^2([-h_0, h_0]), \ w_0(-h_0) = w_0(h_0) = 0 \text{ and } w_0(x) > 0 \text{ in } (-h_0, h_0).
\end{align*}
\]

(3)

For simple, we define

\[
\begin{align*}
&f_1(u, v, w) : = u (1 - u - a_2 v) - d_1 uw; \\
f_2(u, v, w) : = r_2 v (1 - v - a_1 u) - d_2 vw; \\
f_3(u, v, w) : = c (v - w).
\end{align*}
\]

Obviously, system (1) is a one-dimensional reaction-diffusion system. We would like to show that (1.2) has a unique solution \((u, v, w, g, h)\) defined for all \(t > 0\), with \(u > 0, v > 0, w > 0, g' < 0\) and \(h' > 0\). Assume that the model parameters fulfill

\[
a_1 < 1,
\]

\[
1 - a_1 a_2 - a_1 d_1 > \frac{d_2}{r_2},
\]

\[
d_1 < 1 + \frac{d_2}{r_2} - a_2.
\]

Then, (1.2) have a state-steady solution \((u^*, v^*, w^*)\). Moreover, a spreading-vanishing dichotomy holds for (1.2), namely, as time tends to infinite either
(i) the \( v \) and \( w \) successfully establishes itself in human environment (henceforth called spreading) in the sense that \( h(t) \to \infty \), both \( u, v \) and \( w \) tend to positive constants \((u^*, v^*, w^*)\) (see Theorem 3.2).

(ii) the tumour cells \( v \) fails to establish and vanishes eventually (called vanishing), i.e., \( h(t) \to h_\infty \leq \frac{\sqrt{D}}{r_2} \) and

\[
\lim_{t \to \infty} ||v(t, \cdot)||_{C([g(t), h(t))]} = 0 \quad \lim_{t \to \infty} ||w(t, \cdot)||_{C([g(t), h(t)])} = 0.
\]

(see Theorem 3.2).

The criteria for spreading and vanishing are the following: If the initial occupying area \([-h_0, h_0]\) is beyond a critical size, i.e., \( h_0 \geq \frac{\sqrt{D}}{r_2} \), then regardless of the initial population size \((u_0, v_0)\), spreading always happens. On the other hand, if \( h_0 < \frac{\sqrt{D}}{r_2} \), then whether spreading or vanishing occurs is determined by the initial population size \((u_0, v_0, w_0)\) and the coefficient \( \mu \).

Recently, the free boundary problem has been described the dynamics of one species population \([3, 4, 5, 7, 13, 17]\). To better understand the asymptotic behavior of multiple species, Guo and Wu \([11]\) investigated the free boundary problems for Lotka-Volterra competition system. Wang, Zhao and Zhang \([20, 22, 23]\) studied the Lotka-Volterra prey-predator problems with different free boundaries conditions. The spreading-vanishing dichotomy, long time behavior of solution and criteria for spreading and vanishing have been established in there. The other related studies to free boundary problems of biological models, we refer to, for instance, \([6, 8, 9, 21]\) and references cited therein.

The organization of this paper is as follows. In Section 2, we first use a Schauder fixed point theorem to prove the local existence and apply the estimate of solution to state the uniqueness of solution to (2), and then show that it exists for all time \( t \). Section 3 is devoted to the long time behavior of \((u, v, w)\). In Section 4, we would like to provide the criteria for spreading and vanishing.

2. Existence and uniqueness of global solution. In this section, we first prove a local existence result by Schauder fixed point theorem, and then we use suitable estimate to illustrate that the solution is unique and defined for all time \( t \in (0, +\infty) \).

Define

\[
\Pi = \{a_1, a_2, r_2, d_1, d_2, c, D, \mu, h_0, \|u_0\|_\infty, \|v_0\|_{W^2_\infty(-h_0, h_0)}, \|w_0\|_{W^2_\infty(-h_0, h_0)}, \}.
\]

**Theorem 2.1.** For any given \((u_0, v_0, w_0)\) satisfying (3) and any \( \alpha \in (0, 1) \) and \( p > 3/(1-\alpha) \), there is a \( T > 0 \) such that problem (2) admits a unique solution

\[
(u, v, w, h, g) \in C^1([0, T]; L^\infty(\mathbb{R})) \times (W^{1,2}_p(D_T) \cap C^{1+\alpha/2, 1+\alpha}(\bar{D_T}))^2 \times C^{1+\alpha/2}(\bar{0, T})),
\]

where

\[
D_T = \{(t, x) \in \mathbb{R}^2 : t \in [0, T], g(t) < x < h(t)\}.
\]

Moreover

\[
\|v\|_{W^{1,2}_p(D_T)} + \|v\|_{C^{1+\alpha/2, 1+\alpha}(D_T)} + \|w\|_{W^{2,2}_p(D_T)} + \|w\|_{C^{1+\alpha/2}(\bar{0, T})} \leq C,
\]

where \( C \) and \( T \) only depend on \( \Pi \).

**Proof.** In the proof, positive constant \( C_i \) and \( C'_i \) depend on \( \Pi \). To save the space, we take \( \mu = 1 \) and \( \tau = 1 \) in this proof.
Step 1: Transformation of the solution (1). Let
\[
y = \frac{2x - g(t) - h(t)}{h(t) - g(t)}.
\]
Then (1) can be transformed into
\[
\begin{align*}
\{ u_t &= f_1(u, v, w), \quad t > 0, \quad x \in \mathbb{R}, \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R}, \\
z_1(t, y) &= u(t, \frac{1}{2}[(h(t) - g(t))y + h(t) + g(t)]), \\
z_2(t, y) &= v(t, \frac{1}{2}[(h(t) - g(t))y + h(t) + g(t)]), \\
z_3(t, y) &= w(t, \frac{1}{2}[(h(t) - g(t))y + h(t) + g(t)]).
\end{align*}
\]

Then (1) can be transformed into
\[
\begin{align*}
\begin{cases}
z_2(t) - D\rho^2(t)z_{2yy} - D\zeta(t, y)z_{2y} &= f_2(z_1, z_2, z_3), \quad t > 0, \quad |y| < 1, \\
z_3 - \rho^2(t)z_{3yy} - \zeta(t, y)z_{3y} &= f_3(z_1, z_2, z_3), \quad t > 0, \quad |y| < 1, \\
z_2(t, \pm 1) &= z_3(t, \pm 1) = 0, \quad t \geq 0, \\
z_2(0, y) &= v_0(h_0y) := z_{20}(y), \quad |y| < 1, \\
z_3(0, y) &= w_0(h_0y) := z_{30}(y), \quad |y| < 1,
\end{cases}
\end{align*}
\]
where
\[
\rho(t) = \frac{2}{h(t) - g(t)}, \quad \zeta(t, y) = \frac{h'(t) + g'(t)}{h(t) - g(t)}, \quad \frac{h'(t) - g'(t)}{h(t) - g(t)} y.
\]

We should mention that we can not transform the differential equation of \(u(t, x)\) to that \(z_1(t, y)\). If we do so, then (7) will become
\[
\begin{align*}
\begin{cases}
z_{1y} - \zeta(t, y)z_{1y} &= f_1(z_1, z_2, z_3), \quad t > 0, \quad x \in \mathbb{R}, \\
z_1(0, y) &= u_0(h_0y), \quad y \in \mathbb{R},
\end{cases}
\end{align*}
\]
which is a hyperbolic problem and may be ill-defined.

Let \(T_1 = \{1, \frac{h_0}{2(4 + |g^*| + |h^*|)}\},\) and \(h^* = -v_0(h_0),\) \(g^* = -w_0(-h_0).\) For \(0 < T \leq T_1,\) denote \(I_T = [0, T] \times [-1, 1]\) and \(D_T := D_T^1 \times D_T^2 \times D_T^3 \times D_T^4,\) where
\[
\begin{align*}
D_T^1 &= \{z_2 \in C(I_T) : z_2(0) = -z_{20}(y), \quad z_2(t, \pm 1) = 0, \quad \|z_2 - z_{20}\|_{L^\infty(I_T)} \leq 1, \}
\end{align*}
\]
\[
\begin{align*}
D_T^2 &= \{z_3 \in C(I_T) : z_3(0) = -z_{30}(y), \quad z_3(t, \pm 1) = 0, \quad \|z_3 - z_{30}\|_{L^\infty(I_T)} \leq 1, \}
\end{align*}
\]
\[
\begin{align*}
D_T^3 &= \{g \in C^1[0, T] : g(0) = -h_0, \quad g'(0) = g^*, \quad \|g' - g^*\|_{L^\infty([0, T])} \leq 1, \}
\end{align*}
\]
\[
\begin{align*}
D_T^4 &= \{h \in C^1[0, T] : h(0) = h_0, \quad h'(0) = h^*, \quad \|h' - h^*\|_{L^\infty([0, T])} \leq 1. \}
\end{align*}
\]
Clearly, \(D_T\) is a bounded and closed convex set of \([C(I_T)]^2 \times [C^1[0, T]]^2.\)
Due to the choice of $T$, we can extend $g, h$ to new functions, denoted by themselves, such that $(g, h) \in \Omega^3_{T_1} \times \Omega^4_{T_1}$, where

\[
\begin{align*}
\Omega^3_{T_1} &= \{ g \in C^1[0, T] : g(0) = -h_0, \; g'(0) = g^*, \; \|g' - g^*\|_{L^\infty([0,T_1])} \leq 2 \}, \\
\Omega^4_{T_1} &= \{ h \in C^1[0, T] : h(0) = h_0, \; h'(0) = h^*, \; \|h' - h^*\|_{L^\infty([0,T_1])} \leq 2 \}.
\end{align*}
\]

Therefore, when $(g, h) \in D^3_T \times D^4_T$, we have $(g, h) \in \Omega^3_{T_1} \times \Omega^4_{T_1}$ and

\[
|g(t) + h_0| + |h(t) - h_0| \leq \frac{h_0}{2}, \; \forall t \in [0, T_1],
\]

which implies

\[
h_0 \leq h(t) - g(t) \leq 3h_0, \; \forall t \in [0, T_1].
\]

Thus, functions $\rho(t)$ and $\zeta(t, y)$ are well defined on $[0, T_1]$.

Step 2: Existence of the solution to (7)-(9). For the given $(z_2, z_3, g, h) \in D_T$. We first set $z_2 = 0$ and $z_3 = 0$ in $[0, T] \times ((-\infty, -1] \cup [1, \infty))$ and then extend $z_2$ and $z_3$ to $[T, T_1] \times \mathbb{R}$ by setting $z_2(t, y) = z_2(T, y)$ and $z_3(t, y) = z_3(T, y)$ for all $T < t \leq T_1$. Then, taking

\[
v(t, x) = z_2(t, \frac{2x - g(t) - h(t)}{h(t) - g(t)}), \; w(t, x) = z_3(t, \frac{2x - g(t) - h(t)}{h(t) - g(t)})
\]

in (10), we get a Cauchy problem of a logistic equation. It is easy to show that this Cauchy problem (7) has a unique solution

\[
u \in C^1([0, T_1]; L^\infty(\mathbb{R})), \; \|u\|_{L^\infty([0,T_1] \times \mathbb{R})} \leq C.
\]

Thus, the function

\[
z_1(t, y) = u(t, \frac{1}{2}|h(t) - g(t)|y + h(t) + g(t))
\]

satisfies $z_1 \in L^\infty([0, T_1] \times \mathbb{R})$.

Substitute these known functions $z_1(t, y)$ into (8) and consider the following problem

\[
\begin{cases}
\ddot{z}_2 - D\rho^2(t)\ddot{z}_2 + D\zeta(t, y)\ddot{z}_2 + r_2(a_1z_1 - 1)\ddot{z}_2, \\
= d_2z_2z_3 - r_2z_2^2, \quad t > 0, |y| < 1, \\
\ddot{z}_3 - \rho^2(t)\ddot{z}_3 + \zeta(t, y)\ddot{z}_3 + c\ddot{z}_3 = c\ddot{z}_2, \quad t > 0, |y| < 1, \\
\ddot{z}_2(t, \pm 1) = \ddot{z}_3(t, \pm 1) = 0, \quad t \geq 0, \\
\ddot{z}_2(0, y) = v_0(h_0y) := z_20(y), \quad |y| < 1, \\
\ddot{z}_3(0, y) = w_0(h_0y) := z_30(y), \quad |y| < 1.
\end{cases}
\]

The problem has a unique solution $\ddot{z}_2, \ddot{z}_3$ and

\[
\begin{align*}
\|\ddot{z}_2\|_{W^{1,2}_p(D_{T_1})} + \|\ddot{z}_2\|_{C^{1+\alpha/2,1+\alpha}(D_{T_1})} + \|\ddot{z}_3\|_{W^{1,2}_p(D_{T_1})} + \|\ddot{z}_3\|_{C^{1+\alpha/2,1+\alpha}(D_{T_1})} \leq C_1.
\end{align*}
\]

Define

\[
\begin{align*}
\tilde{g}(t) &= -h_0 - \int_0^t \rho(\tau)\ddot{z}_2(\tau, -1) + \rho(\tau)\ddot{z}_3(\tau, -1)d\tau, \\
\tilde{h}(t) &= h_0 - \int_0^t \rho(\tau)\ddot{z}_2(\tau, 1) + \rho(\tau)\ddot{z}_3(\tau, 1)d\tau.
\end{align*}
\]
Then $\hat{y}(0) = -h_0$, $\hat{h}(0) = h_0$, $\hat{y}'(0) = g^*$, $\hat{h}'(0) = h^*$, and
\[
\|\hat{h}\|_{C^{1+\alpha/2}([0, T])}, \quad \|\hat{y}'\|_{C^{1+\alpha/2}([0, T])} \leq C_2
\]

For $0 < T \leq T_1$, we define a mapping $\mathfrak{F} : D_T \to [C(I_T)]^2 \times [C^1([0, T])]^2$ by
\[
\mathfrak{F}(z_2, z_3, g, h) = (\hat{z}_2, \hat{z}_3, \hat{g}, \hat{h}).
\]
Obviously, $\mathfrak{F}$ is continuous in $D_T$, and $(z_2, z_3, g, h) \in D_T$ is a fixed point of $\mathfrak{F}$ if and only if $(u, v, w, g, h)$ solves (7), (8) and (9). According to (10) and (11), we know that $\mathfrak{F}$ is compact and
\[
\|\hat{z}_2 - z_{20}\|_{L^\infty(I_T)} \leq \|\hat{z}_2 - z_{20}\|_{C^{(1+\alpha)/2, 0}(I_T)} T^{(1+\alpha)/2} \leq C_1 T^{(1+\alpha)/2}, \forall t \in [0, T]
\]
\[
\|\hat{z}_3 - z_{30}\|_{L^\infty(I_T)} \leq \|\hat{z}_3 - z_{30}\|_{C^{(1+\alpha)/2, 0}(I_T)} T^{(1+\alpha)/2} \leq C_1 T^{(1+\alpha)/2}, \forall t \in [0, T]
\]
\[
\|\hat{y}' - g^*\|_{L^\infty([0, T])} \leq \|\hat{y}' - g^*\|_{C^{\alpha/2}([0, T])} T^{\alpha/2} \leq C_2 T^{\alpha/2}, \forall (t, y) \in I_T,
\]
\[
\|\hat{h}' - h^*\|_{L^\infty([0, T])} \leq \|\hat{h}' - h^*\|_{C^{\alpha/2}([0, T])} T^{\alpha/2} \leq C_2 T^{\alpha/2}, \forall (t, y) \in I_T.
\]
Therefore, if we take
\[
T \leq \min\{T_1, (C_2)^{-2/\alpha}, (C_1)^{-2/(1+\alpha)}\},
\]
then $\mathfrak{F}$ maps $D_T$ into itself. Consequently, $\mathfrak{F}$ has at least one fixed point
\[
(z_2, z_3, g, h) \in D_T
\]
by the Schauder fixed point theorem and (7)-(9) have at least one solution
\[
(u, z_2, z_3, g, h) \in [0, T] \quad t \in [0, T].
\]
Moreover, from the above discussion we see that $(u, z_2, z_3, g, h)$ satisfies
\[
g, h \in C^{1+\frac{1+\alpha}{2}}([0, T]), \quad g'(t) < 0, \quad h'(t) > 0, \quad \forall t \in (0, T],
\]
\[
eq C^1([0, T]; L^\infty(\mathbb{R})) \quad z_2 \in (W^1,2(I_T) \cap C^{1+\frac{1+\alpha}{2}}(I_T)),
\]
\[
z_3 \in (W^1,2(I_T) \cap C^{1+\frac{1}{2}+\alpha}(I_T))
\]
Step 3: Existence of solution $(u, v, w, g, h)$ to (2). Define as before,
\[
v(t, x) = z_2(t, 2x - g(t) - h(t) \quad w(t, x) = z_3(t, 2x - g(t) - h(t) \quad h(t) - g(t))
\]
Then $(u, v, w, g, h)$ satisfies (1), and $(u, v, w)$ satisfies
\[
u \in C^1([0, T]; L^\infty(\mathbb{R})), \quad v \in (W^1,2(D_T) \cap C^{1+\frac{1+\alpha}{2}, 1+\alpha}(D_T)),
\]
\[
w \in (W^1,2(D_T) \cap C^{1+\frac{1+\alpha}{2}, 1+\alpha}(D_T))
\]
Moreover, the estimate (10) gives
\[
\|z_2\|_{C^{(1+\alpha)/2, 1+\alpha}(I_T)} \leq C_1, \quad \|z_3\|_{C^{(1+\alpha)/2, 1+\alpha}(I_T)} \leq C_1,
\]
which implies
\[
\|v\|_{C^{(1+\alpha)/2, 1+\alpha}(D_T)} \leq C'_1, \quad \|w\|_{C^{(1+\alpha)/2, 1+\alpha}(D_T)} \leq C'_1,
\]
Thus $z_{2y} \in C^\alpha(I_T)$, $z_{3y} \in C^\alpha(I_T)$, $v_x \in C^\alpha(D_T)$, $w_x \in C^\alpha(D_T)$, and
\[
\|z_{2y}\|_{L^\infty(I_T)} \leq C'_1, \quad \|z_{3y}\|_{L^\infty(I_T)} \leq C'_1, \quad \|v_x\|_{L^\infty([0, T] \times \mathbb{R})} \leq C'_1,
\]
\[
\|w_x\|_{L^\infty([0, T] \times \mathbb{R})} \leq C'_1,
\]
where we have defaulted $v(t, x) = 0$ and $w(t, x) = 0$, when $x \not\in (g(t), h(t))$. 

Recall $u_0 \in C^1_b(\mathbb{R})$, \( \|u_x\|_{L^\infty([0,T] \times \mathbb{R})} \leq C'_1 \) and \( \|w_x\|_{L^\infty([0,T] \times \mathbb{R})} \leq C'_1 \), using the continuous dependence of solutions on parameters, we can show that \( u(t, \cdot) \in C^1_b(\mathbb{R}) \) and
\[
\|u, u_x\|_{L^\infty([0,T] \times \mathbb{R})} \leq C'_2. \tag{13}
\]

Step 4: Uniqueness of solution \((u, v, w, g, h)\) to (2). In the following we prove the uniqueness. Let \((\bar{u}, \bar{v}, \bar{w}, \bar{g}, \bar{h})\) and \((\hat{u}, \hat{v}, \hat{w}, \hat{g}, \hat{h})\) be the two solutions of (2), which are defined for \( t \in [0, T] \) with \( 0 < T \ll 1 \). Let
\[
\begin{align*}
\tilde{z}_1(t, y) &= \bar{u}(t, \frac{1}{2}((\tilde{h}(t) - \hat{g}(t))y + \hat{h}(t) + \hat{g}(t))), \\
\tilde{z}_2(t, y) &= \bar{v}(t, \frac{1}{2}((\tilde{h}(t) - \hat{g}(t))y + \hat{h}(t) + \hat{g}(t))), \\
\tilde{z}_3(t, y) &= \bar{w}(t, \frac{1}{2}((\tilde{h}(t) - \hat{g}(t))y + \hat{h}(t) + \hat{g}(t))).
\end{align*}
\]
and
\[
\begin{align*}
\hat{z}_1(t, y) &= \hat{u}(t, \frac{1}{2}((\hat{h}(t) - \hat{g}(t))y + \hat{h}(t) + \hat{g}(t))), \\
\hat{z}_2(t, y) &= \hat{v}(t, \frac{1}{2}((\hat{h}(t) - \hat{g}(t))y + \hat{h}(t) + \hat{g}(t))), \\
\hat{z}_3(t, y) &= \hat{w}(t, \frac{1}{2}((\hat{h}(t) - \hat{g}(t))y + \hat{h}(t) + \hat{g}(t))).
\end{align*}
\]
Then we have
\[
\begin{align*}
\left\| \tilde{z}_2 \right\|_{W^{1,2}_p(I_T)} &\leq C_1, & \left\| \tilde{z}_3 \right\|_{W^{1,2}_p(I_T)} &\leq C_1, \\
\left\| \hat{z}_2 \right\|_{W^{1,2}_p(I_T)} &\leq C_1, & \left\| \hat{z}_3 \right\|_{W^{1,2}_p(I_T)} &\leq C_1, \\
\left\| \tilde{z}_2 \right\|_{L^\infty(I_T)} &\leq C'_1, & \left\| \tilde{z}_3 \right\|_{L^\infty(I_T)} &\leq C'_1, \\
\left\| \hat{z}_2 \right\|_{L^\infty(I_T)} &\leq C'_1, & \left\| \hat{z}_3 \right\|_{L^\infty(I_T)} &\leq C'_1, \\
\left\| \bar{u}, \bar{u}_x \right\|_{L^\infty([0,T] \times \mathbb{R})} &\leq C'_2, & \left\| \hat{u}, \hat{u}_x \right\|_{L^\infty([0,T] \times \mathbb{R})} &\leq C'_2.
\end{align*}
\tag{14}
\]
Clearly, \((\tilde{z}_2, \tilde{z}_3, \tilde{g}, \tilde{h})\) and \((\hat{z}_2, \hat{z}_3, \hat{g}, \hat{h})\) solves (8) and (9) with \( z_1 = \tilde{z}_1 \) and \( z_1 = \hat{z}_1 \), respectively. Set
\[
\begin{align*}
U &= \hat{u} - \bar{u}, & V &= \hat{v} - \bar{v}, & W &= \hat{w} - \bar{w}, & Z_1 &= \hat{z}_1 - \tilde{z}_1, \\
Z_2 &= \hat{z}_2 - \tilde{z}_2, & Z_3 &= \hat{z}_3 - \tilde{z}_3, & G &= \hat{g} - \tilde{g}, & H &= \hat{h} - \tilde{h}.
\end{align*}
\]
Then we have
\[
\begin{align*}
\left\{ \begin{array}{ll}
U_t + (\hat{u} + \bar{u} + a_2 \hat{v} + d_1 \hat{w} - 1)U = -a_2 \bar{v}V - d_1 \bar{w}W, & 0 < t \leq T, \ x \in \mathbb{R}, \\
U(0, x) = 0, & x \in \mathbb{R}.
\end{array} \right.
\tag{15}
\]
By (14), it follows from (15) and (16) that
\[
\begin{align*}
H_G(z_2, y) &= \left\{ \begin{array}{ll}
H_G(z_2, y) &= t > 0, \ |y| < 1, \\
H_G(z_2, y) &= t \geq 0, \ |y| < 1, \\
H_G(z_2, y) &= t = 0, \ |y| < 1,
\end{array} \right.
\end{align*}
\]
and
\[
\begin{align*}
H'(t) &= -\rho_2(t)z_2(t, 1) - (\rho_2(t) - \rho_1(t))\bar{z}_2(t, 1) - \rho_2(t)Z_3(y, 1), \\
G'(t) &= t > 0, \\
G'(t) &= t \leq 0, \ |y| < 1,
\end{align*}
\]
\[
g(0) = -h_0, \ h(0) = h_0.
\]
By (14), it follows from (15) and (16) that
\[
\begin{align*}
\|U\|_{L^\infty([0,T] \times \mathbb{R})} &\leq C_3T\left(\|V\|_{L^\infty([0,T] \times \mathbb{R})} + \|W\|_{L^\infty([0,T] \times \mathbb{R})}\right), \\
\|Z_2\|_{W_{1,2}^{2}(I_T)} &\leq C_4\left(\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])} + \|Z_1\|_{L^\infty(I_T)} + \|Z_3\|_{L^\infty(I_T)}\right), \\
\|Z_3\|_{W_{1,2}^{2}(I_T)} &\leq C_5\left(\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])} + \|Z_2\|_{L^\infty(I_T)}\right).
\end{align*}
\]
For the given \((t, y) \in I_T\), using the relation
\[
\begin{align*}
\hat{x}(t, y) &= \frac{1}{2}\left(\hat{h}(t) - \hat{g}(t)\right)y + \hat{h}(t) + \hat{g}(t), \\
\hat{z}(t, y) &= \frac{1}{2}\left(\hat{h}(t) - \hat{g}(t)\right)y + \hat{h}(t) + \hat{g}(t),
\end{align*}
\]
and (14), (18), by a series of calculations we can get
\[
\begin{align*}
\|Z_1(t, y)\| &\leq C_3T\left(\|V\|_{L^\infty([0,T] \times \mathbb{R})} + \|W\|_{L^\infty([0,T] \times \mathbb{R})}\right) \\
&\quad + C'\left(\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])}\right)
\end{align*}
\]
This implies
\[
\|Z_1\|_{L^\infty(I_T)} \leq C_3T\left(\|V\|_{L^\infty([0,T] \times \mathbb{R})} + \|W\|_{L^\infty([0,T] \times \mathbb{R})}\right) \\
+ C_2'\left(\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])}\right). \tag{21}
\]
Substituting (21) into (19), we get
\[
\|Z_2\|_{W_{1,2}^{2}(I_T)} \leq C_6\left(\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])} + T\left(\|V\|_{L^\infty([0,T] \times \mathbb{R})}\right) \\
+ \|W\|_{L^\infty([0,T] \times \mathbb{R})} + \|Z_3\|_{L^\infty(I_T)}\right). \tag{22}
\]
By a series of carefully analysis, we can show that
\[
\|V\|_{L^\infty([0,T] \times \mathbb{R})} \leq C_7\left(\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])}\right) + \|Z_2\|_{L^\infty(I_T)} \tag{23}
\]
and
\[
\|W\|_{L^\infty([0,T] \times \mathbb{R})} \leq C_8\left(\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])}\right) + \|Z_3\|_{L^\infty(I_T)}. \tag{24}
\]
Substituting (23) and (24) into (22) one has
\[ \|Z_2\|_{W^1_p(I_T)} \leq C_9(\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])}) + T(\|Z_2\|_{L^\infty(I_T)} + \|Z_3\|_{L^\infty(I_T)}) \] (25)

From the proof of Theorem 1.1 in [18], we have
\[ [Z_2]_{C^\frac{\alpha}{2}_T(I_T)}, [Z_3y]_{C^\frac{\alpha}{2}_T(I_T)} \leq C \|Z_2\|_{W^1_p(I_T)}, \] (26)
\[ [Z_3]_{C^\frac{\alpha}{2}_T(I_T)}, [Z_3y]_{C^\frac{\alpha}{2}_T(I_T)} \leq C \|Z_3\|_{W^1_p(I_T)} \] (27)

where \( C \) independent of \( T^{-1}, [\cdot]_{C^\frac{\alpha}{2}_T(I_T)} \) is Hölder semi-norm. Note that \( Z_2(0, y) = 0 \), for any \((t, y) \in I_T\) these hold:
\[ |Z_2(t, y)| \leq [Z_2]_{C^\frac{\alpha}{2}_T(I_T)} T^\frac{\alpha}{2} \leq C \|Z_2\|_{W^1_p(I_T)} T^\frac{\alpha}{2}, \]
which implies
\[ \|Z_2\|_{L^\infty(I_T)} \leq C \|Z_2\|_{W^1_p(I_T)} T^\frac{\alpha}{2}. \]

Similarly, we also have
\[ \|Z_3\|_{L^\infty(I_T)} \leq C \|Z_3\|_{W^1_p(I_T)} T^\frac{\alpha}{2}. \]

These combined with (20) and (25) allow us to derive
\[ \|Z_3\|_{W^1_p(I_T)} \leq C_5(\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])}) + \|Z_2\|_{W^1_p(I_T)} T^\frac{\alpha}{2}, \]
and
\[ \|Z_2\|_{W^1_p(I_T)} \leq C_9 C T(\|Z_2\|_{W^1_p(I_T)} + \|Z_3\|_{W^1_p(I_T)}) + C_9(\|G\|_{C^1([0,T])} \]
\[ + \|H\|_{C^1([0,T])}) + C_9 \|Z_3\|_{W^1_p(I_T)} T^\frac{\alpha}{2}, \]
This imply
\[ \|Z_2\|_{W^1_p(I_T)} + \|Z_3\|_{W^1_p(I_T)} \leq C_9 CT(\|Z_2\|_{W^1_p(I_T)} + \|Z_3\|_{W^1_p(I_T)}) \]
\[ + (C_5 + C_9)(\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])}) + C_5 \|Z_2\|_{W^1_p(I_T)} T^\frac{\alpha}{2}, \]
\[ + C_9 \|Z_3\|_{W^1_p(I_T)} T^\frac{\alpha}{2}, \] (28)

If \( C_5 CT + C_9 T^\frac{\alpha}{2} \leq \frac{1}{2} \) and \( C_5 CT + C_9 T^\frac{\alpha}{2} \leq \frac{1}{2}. \) Using (17), (26), (27) and (28) we can derive
\[ [G']_{C^\frac{\alpha}{2}_T([0,T])} + [H']_{C^\frac{\alpha}{2}_T([0,T])} \leq C_{10}(\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])}) \]
where \( C_{10} \) is independent of \( T^{-1}. \) Noticing \( G'(0) = G(0) = H'(0) = H(0), \) we have
\[ \|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])} = T^\frac{\alpha}{2}(1 + T)([G']_{\frac{\alpha}{2}} + [H']_{\frac{\alpha}{2}}) \]
\[ \leq 2C_{10} T^\frac{\alpha}{2}(\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])}). \]
Take \( T \) satisfying \( 2C_{10} T^\frac{\alpha}{2} < 1. \) Then \( H \equiv 0 \) and \( G \equiv 0 \) in \([0,T]. \) Consequently,
\( Z_1 \equiv Z_2 \equiv Z_3 \equiv 0. \) This implies \( \vec{v} = \vec{v}, \ w = \vec{w}, \) and \( \vec{u} = \vec{u}. \) The uniqueness is obtained and the proof is complete. \( \square \)

To show that the local solution obtained in Theorem 2.1 can be extended to the all \( t > 0, \) we need the estimates of solution. In the same way as [19, 20], we can prove the following lemma.
Lemma 2.2. The solution of the free boundary problem (2) satisfies
\[ 0 < u(x,t) \leq M_1 \quad \text{for} \quad 0 \leq t \leq T, \quad -\infty < x < \infty, \]
\[ 0 < v(x,t) \leq M_2 \quad \text{for} \quad 0 \leq t \leq T, \quad g(t) \leq x \leq h(t), \]
\[ 0 < w(x,t) \leq M_3 \quad \text{for} \quad 0 \leq t \leq T, \quad g(t) \leq x \leq h(t), \]
and
\[ 0 < h'(t) \leq M_4 \quad \text{for} \quad 0 < t \leq T, \]
\[ -M_5 \leq g'(t) < 0 \quad \text{for} \quad 0 < t \leq T, \]
where \( M_i \) is independent of \( T \) for \( i = 1, 2, 3, 4, 5 \).

Based on the above estimates, we are now in a position to prove that the solution of problem (2) is actually a global solution.

Theorem 2.3. The solution of problem (2) exists and is unique for all \( t \in (0, \infty) \).

Proof. Let \([0, T_{\text{max}}]\) be the maximal time interval in which the solution exists. By Theorem 2.1, \( T_{\text{max}} > 0 \). It remains to show that \( T_{\text{max}} = \infty \). Arguing indirectly, it is assumed that \( T_{\text{max}} < \infty \). By Lemma 2.2, there exist positive constants \( M_1, M_2, M_3, M_4 \) and \( M_5 \) independent of \( T_{\text{max}} \) such that for \( t \in [0, T_{\text{max}}) \),
\[
\begin{cases}
0 < u(t, x) \leq M_1, & -\infty < x < \infty, \\
0 < v(t, x) \leq M_2, & 0 < w(t, x) \leq M_3, \\
h_0 \leq h(t) \leq h_0 + M_4 t, & 0 < h'(t) \leq M_4, \\
g_0 - M_5 t \leq g(t) \leq g_0, & -M_5 < g'(t) \leq 0, \\
g(t) < x < h(t),
\end{cases}
\]

For fixed \( \tau \in [0, T_{\text{max}}) \), it follows from the proof of Theorem 2.1 that there a \( \tau > 0 \), depending only on \( C \) and \( M_i (i = 1, 2, 3, 4, 5) \), such that the solution of (2) with initial time \( T_{\text{max}} - \tau/2 \) can be extended uniquely to the time \( T_{\text{max}} - \tau/2 + \tau \). But this contradicts the definition of \( T_{\text{max}} \). The proof is complete. \( \square \)

Remark 1. It follows from the uniqueness of the solution to (2) and some standard compactness argument that the unique solution \((u, v, w, g, h)\) depends continuously on the parameters appearing in the problem (2). This fact will be used in later analysis.

3. Long time behavior of \((u, v, w)\). In this section, we mainly study the long time behavior of \((u, v, w)\). In view of Lemma 2.2, we see that the free boundary \( x = h(t) \) is strictly increasing function and \( x = g(t) \) is strictly decreasing function with respect to the time \( t \). Thus, either
\[ h_\infty - g_\infty < \infty \quad \text{or} \quad h_\infty - g_\infty = \infty, \]
where
\[ \lim_{t \to \infty} (h(t) - g(t)) = h_\infty - g_\infty. \]
To save the space, we take \( \tau = 1 \) in this section. We first present a very useful estimate, which can be proved similarly as [11, Lemma 3.3] and [20, Theorem 4.1].

Lemma 3.1. If \( h_\infty - g_\infty < \infty \), then there exists a constant \( K > 0 \), such that the solution \((u, v, w, g, h)\) of (2) satisfies
\[
\|v(t, \cdot)\|_{C^1_0([g(t), h(t)])} \leq K, \quad \|v(t, \cdot)\|_{C^1([g(t), h(t)])} \leq K, \quad \forall \ t > 1; \\
\lim_{t \to \infty} g'(t) = 0, \quad \lim_{t \to \infty} h'(t) = 0.
\]
Let us recall the definitions of spreading and vanishing. If \( g_\infty - h_\infty = \infty \),
\[
\liminf_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t))]} > 0, \quad \liminf_{t \to \infty} \|w(t, \cdot)\|_{C([g(t), h(t))]} > 0,
\]
then the tumour cells and acid-mediated can survive and spread to the whole domain \((-\infty, \infty)\), so we call it spreading. While if \( h_\infty - g_\infty < \infty \) and
\[
\liminf_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t))]} = 0, \quad \liminf_{t \to \infty} \|w(t, \cdot)\|_{C([g(t), h(t))]} = 0.
\]
then the tumour cells and acid-mediated will be maintained in a finite region and finally goes to extinction, thus we call this vanishing.

3.1. Vanishing case. Our first result indicates that if the tumour cells and acid-mediated cannot spread to the whole domain \((-\infty, \infty)\), then it must vanish.

**Theorem 3.2.** Suppose that (4), (5) and (6) hold. Let \((u, v, w, g, h)\) be any solution of (2). If \( h_\infty - g_\infty < \infty \), then the solution \((u, v, w, g, h)\) of (2) satisfies
\[
\lim_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t))]} = 0.
\]
(29)
\[
\lim_{t \to \infty} \|w(t, \cdot)\|_{C([g(t), h(t))]} = 0.
\]
(30)
\[
\lim_{t \to \infty} u(t, x) = 1 \quad \text{uniformly on any compact subset of } (-\infty, \infty).
\]
(31)

**Proof.** Conclusion (29) and (30) can be deduced from Lemma 2.2 and Proposition 3.1 in [16]. In the following we prove (31). By the comparison principle \(u(t, x) \leq \bar{u}(t)\) for all \(t \in [0, \infty)\) and \(x \in (-\infty, \infty)\), where
\[
\bar{u}(t) = e^t \left( e^t - 1 + \frac{1}{\|u_0\|_\infty} \right)^{-1},
\]
(32)
which is the solution of the ODE problem
\[
\bar{u}'(t) = \bar{u}(1 - \bar{u}), \quad t > 0; \quad \bar{u}(0) = \|u_0\|_\infty.
\]

Since \(\lim_{t \to \infty} \bar{u}(t) = 1\), we have that \(\limsup_{t \to \infty} u(t, x) \leq 1\) uniformly in \((-\infty, \infty)\).

On the other hand, for any given \(\varepsilon\), there exists \(T_\varepsilon > 0\) and \(l_\varepsilon > 0\), such that the function \(u(t, x)\) satisfies
\[
u_t \geq u(1 - u - \varepsilon), \quad t > T_\varepsilon, \quad -l_\varepsilon < x < l_\varepsilon
\]

For any \(L > 0\), consider the unique solution \(u\) of
\[
\begin{cases}
  u = u(1 - u - \varepsilon), & t > T_\varepsilon, \quad -L < x < L \\
  u(T_\varepsilon) = \min_{[-L, L]} u(T_\varepsilon), & t > T_\varepsilon.
\end{cases}
\]

It follows form the comparison principle that Thus \(u(t, x) \leq u(t, x)\) for \(t > T_\varepsilon\) and \(x \in [-L, L]\). Hence
\[
\liminf_{t \to \infty} \min_{[-L, L]} u(t, \cdot) \geq \lim_{t \to \infty} u(t, x) = 1 - \varepsilon.
\]
In view of the arbitrariness of the \(L\), we derive that \(\liminf_{t \to \infty} u(t, x) \geq 1 - \varepsilon\) uniformly in the compact subset of \((-\infty, \infty)\). Since \(\varepsilon > 0\) is arbitrary, it follows that \(\liminf_{t \to \infty} u(t, x) \geq 1\) uniformly in any bounded subset of \((-\infty, \infty)\). The proof is completed. \(\square\)

**Remark 2.** This theorem plays key roles in the following two aspects: (i) affirming the tumour cells and acid-mediated disappears eventually; (ii) determining the criteria for spreading and vanishing (see the following Section 5).
3.2. **Spreading case.** The following theorem shows that the tumour cells and acid-mediated successfully establishes itself in human environment, and moreover both $u$, $v$ and $w$ go to constant in the sense that $h_\infty - g_\infty = \infty$. A heterogeneous state, $(u, v, w) = (1 - (a_2 + d_1)\bar{v}, \bar{v}, \bar{v})$, where
\[
\bar{v} = \frac{1 - a_1}{1 - a_1(a_2 + d_1) + \frac{d_2}{r_2}}.
\]
We define
\[
(u^*, v^*, w^*) = (1 - (a_2 + d_1)\bar{v}, \bar{v}, \bar{v})
\]
(33)

**Theorem 3.3.** Suppose that (4), (5) and (6) holds. If $h_\infty - g_\infty = \infty$, then the solution $(u, v, w, g, h)$ of (2) satisfies
\[
\lim_{t \to \infty} u(t, x) = u^*, \quad \lim_{t \to \infty} v(t, x) = v^*, \quad \lim_{t \to \infty} w(t, x) = w^*
\]
uniformly in any compact subset of $(-\infty, \infty)$.

**Proof.** Let $M = \max\{M_1, M_2, M_3, M_4, M_5\}$, where $M_i$ is determined by Lemma 2.2, $i = 1, 2, 3, 4, 5$. We would like to construct the suitable iteration sequences to complete the proof.

**Step 1: The construction of $\bar{u}_1$.**

For any given $0 < \varepsilon << 1$, notice that $v > 0$ and $w > 0$, there exists $l_\varepsilon$ and $T_1$ such that $u$ satisfies
\[
u_t \leq u(1 + \varepsilon - u), \quad t > T_1, \quad -l_\varepsilon < x < l_\varepsilon,
\]
For any $L > 0$, consider the unique solution $\bar{u}$ of
\[
\begin{aligned}
\bar{u}_t &= \bar{u}(1 + \varepsilon - \bar{u}), \quad t > T_1, \quad -L < x < L, \\
\bar{u}(T_1) &= \max_{[-L, L]} u(T_1, x), \quad t > T_1.
\end{aligned}
\]
Applying comparison principle, we have
\[
u(t, x) \leq \bar{u}(t, x), \quad t \geq T_1, \quad -L < x < L.
\]
It arrive at
\[
\limsup_{t \to \infty} \max_{[-L, L]} u(t, \cdot) \leq \lim_{t \to \infty} \bar{u}(t, x) = 1 + \varepsilon
\]
By the arbitrariness of $\varepsilon$, this implies
\[
\limsup_{t \to \infty} u(t, x) \leq 1 := \bar{u}_1 \quad \text{uniformly on the compact subset of} \quad (\infty, \infty). \tag{34}
\]

**Step 2: The construction of $\bar{v}_1$.**

For any given $L > 0$, $0 < \delta << 1$ and $0 < \varepsilon << 1$, let $l_\varepsilon$ is given by Lemma A.2 with $d_1 = \bar{D}$, $\beta = \bar{r}_2$, $\theta = \bar{r}_2$ and $q = M$. In view of $g_\infty = \infty$ and $h_\infty = \infty$, there exist $T_2 > T_1$ such that Therefore, $v$ satisfies
\[
\begin{aligned}
v_t - Dv_{xx} &\leq \bar{r}_2 v(1 - v), \quad t \geq T_2, \quad -l_\varepsilon < x < l_\varepsilon, \\
v(t, t, \pm l_\varepsilon) &\leq M, \quad t > T_2.
\end{aligned}
\]
As $v(T_2, x) > 0$ in $[-L, L]$, by use of Lemma A.2, it deduces that
\[
\limsup_{t \to \infty} v(t, x) \leq 1 + \varepsilon \quad \text{uniformly on} \quad [-L, L].
\]
The arbitrariness of $\varepsilon$ and $L$ imply that
\[
\limsup_{t \to \infty} v(t, x) \leq 1 := \bar{v}_1 \quad \text{uniformly on the compact subset of} \quad (-\infty, \infty). \tag{35}
\]
Step 3: The construction of \( \bar{w}_1 \).
For any given \( 0 < \varepsilon < < 1 \), According to assuming, \( h_\infty - g_\infty = \infty \), there is \( T_3 > T_2 \) such that
\[
g(t) < -l_\varepsilon, \quad h(t) > l_\varepsilon, \quad \forall t > T_3, \quad -l_\varepsilon < x < l_\varepsilon,
\]
and
\[
w_t - w_{xx} = c(v - w), \quad t > T_3, \quad -l_\varepsilon < x < l_\varepsilon,\]
Since \( \bar{v}_1 > v(t, x) \) for \( t \geq T_3 \) and \(-l_\varepsilon < x < l_\varepsilon \). Applying comparison principle, we have
\[
w(t, x) \leq \bar{v}_1, \quad t \geq T_3, \quad -L < x < L.
\]
Thus
\[
\limsup_{t \to \infty} w(t, x) \leq \bar{v}_1 := \bar{w}_1 \quad \text{uniformly on the compact subset of } (-\infty, \infty). \quad (36)
\]

Step 4: The construction of \( u_1 \).
For any given \( 0 < \delta << 1 \), \( \delta > 0 \) and \( 0 < \varepsilon << 1 \), let \( l_\varepsilon > 0 \), there exists \( T_4 > T_3 \) such that
\[
v(t, x) \leq \bar{v}_1 + \delta, \quad w(t, x) \leq \bar{w}_1 + \delta, \quad \forall t > T_4, \quad -l_\varepsilon < x < l_\varepsilon.
\]
Therefore \( u \) satisfies
\[
u_t \geq u(1 - u - \bar{v}_1 - \bar{w}_1 - 2\delta), \quad t \geq T_4, \quad -l_\varepsilon < x < l_\varepsilon,
\]
For any \( L > 0 \), consider the unique solution \( \bar{u} \) of
\[
\begin{align*}
u_t &= u(1 - u - \bar{v}_1 - \bar{w}_1 - 2\delta), \quad t \geq T_4, \quad -L < x < L, \\
u(T_4) &= \min_{[-L,L]} u(T_4), \quad t \geq T_4.
\end{align*}
\]
It follows from the comparison principle that
\[
u(t, x) \leq u(t, x), \quad t \geq T_4, \quad -L < x < L.
\]
It yields \( \lim_{t \to \infty} \min_{[-L,L]} u(t, \cdot) > \lim_{t \to \infty} \bar{u}(t, x) = 1 - \bar{v}_1 - \bar{w}_1 - 2\delta \) uniformly on \([-L, L]\). Since \( \delta > 0 \) can be arbitrarily small, this implies that
\[
\lim_{t \to \infty} u(t, x) \geq u_1 := 1 - \bar{v}_1 - \bar{w}_1 \quad \text{uniformly on the compact subset of } (-\infty, \infty). \quad (37)
\]

Step 5: The construction of \( \bar{v}_1 \).
For given \( L > 0 \) and \( 0 < \delta, \varepsilon << 1 \), let \( l_\varepsilon \) be given by Lemma A.1 with \( d_1 = D \), \( \beta = r_2(1 - a_1(\bar{u}_1 + \delta) - \frac{d_2}{r_2}(\bar{w}_1 + \delta)) \) and \( \theta = r_2 \). By virtue of (34) and (36), there is \( T_5 > T_4 \) such that
\[
u(t, x) \leq \bar{u}_1 + \delta, \quad w(t, x) \leq \bar{w}_1 + \delta, \quad \forall t > T_5, \quad -l_\varepsilon < x < l_\varepsilon.
\]
Thus, \( u \) satisfies
\[
\begin{align*}
v_t - Dv_{xx} &\geq r_2 v \left( 1 - v - a_1(\bar{u}_1 + \delta) - \frac{d_2}{r_2}(\bar{w}_1 + \delta) \right), \quad t > T_5, -l_\varepsilon < x < l_\varepsilon, \\
v(t, \pm l_\varepsilon) &\geq 0,
\end{align*}
\]
We can get
\[
\liminf_{t \to \infty} v(t, x) \geq 1 - a_1 \bar{u}_1 - \frac{d_2}{r_2} \bar{w}_1 := \bar{v}_1
\]
uniformly on the compact subset of \((-\infty, \infty)\).

Step 6: The construction of \( w_1 \).
For any given $L > 0$ and $0 < \varepsilon < 1$. According to assuming, $h_{\infty} - g_{\infty} = \infty$, there is $T_6 > T_5$ such that
\[ g(t) < -L, \quad h(t) > L, \quad \forall \ t > T_6, \quad -L < x < L. \]
Since $w$ satisfies
\[ w_t - w_{xx} = c(v - w), \quad t > T_6, \quad -L < x < L, \]
Note that $v_1 \leq v(t, x)$ for $t > T_6$ and $-l_\varepsilon < x < l_\varepsilon$. By comparison principle, we know that
\[ w(t, x) \geq v_1, \quad \forall \ t > T_6, -L < x < L. \]
Thus
\[ \liminf_{t \to \infty} w(t, x) \geq \overline{w}_1 := v_1 \text{ uniformly on the compact subset of } (-\infty, \infty). \quad (39) \]

**Step 7: The construction of $\overline{u}_2$.**

For any given $0 < \varepsilon, \delta << 1$, notice that $v > 0$ and $w > 0$, these exist $l_\varepsilon$ and $T_7$, such that $u$ satisfies
\[ u_t \leq u(1 - u - a_2(v_1 - \delta) - d_1(w_1 - \delta)), \quad t > T_7, \quad -L < x < L. \]
For any $L > 0$, consider the unique solution $\overline{u}$ of
\[
\begin{cases}
\overline{u}_t &= \overline{u}(1 - \overline{u} - a_2(v_1 - \delta) - d_1(w_1 - \delta)), \quad t > T_7, \quad -L < x < L, \\
\overline{u}(T_7) &= \max_{[-L, L]} u(T_7, x), \quad t > T_7.
\end{cases}
\]

Applying comparison principle, we have
\[ u(t, x) \leq \overline{u}(t, x), \quad t \geq T_7, \quad -L < x < L. \]
It arrive at
\[ \limsup_{t \to \infty} \max_{[-L, L]} u(t, \cdot) \leq \liminf_{t \to \infty} \overline{u}(t, x) = 1 - a_2\overline{w}_1 - d_1\overline{w}_1. \]
By the arbitrariness of $\varepsilon$ and $L$, this implies
\[ \limsup_{t \to \infty} u(t, x) \leq 1 - a_2\overline{w}_1 - d_1\overline{w}_1 := \overline{u}_2 \quad (40) \]
uniformly on the compact subset of $(-\infty, \infty)$.

**Step 8: The construction of $\overline{v}_2$.**

For any given $L > 0$, $0 < \delta << 1$ and $0 < \varepsilon << 1$, let $l_\varepsilon$ is given by Lemma A.2 with $d_1 = D$, $\beta = r_2(1 - a_1(u_1 - \delta) - \frac{d_2}{r_2}(w_1 - \delta))$, $\theta = r_2$ and $q = M$. In view of
\[ g_{\infty} = \infty \text{ and } h_{\infty} = \infty, \text{ there exist } T_8 \text{ such that Therefore, } v \text{ satisfies} \]
\[
\begin{cases}
v_t - Dv_{xx} \leq r_2v \left(1 - v - a_1(u_1 - \delta) - \frac{d_2}{r_2}(w_1 - \delta)\right), \quad t \geq T_8, \quad -L < x < L, \\
v(t, \pm L) \leq M, \quad t > T_8.
\end{cases}
\]
As $v(T_8, x) > 0$ in $[-L, L]$, by use of Lemma A.1, it deduces that
\[ \limsup_{t \to \infty} v(t, x) \leq 1 - a_1\overline{w}_1 - \frac{d_2}{r_2}\overline{w}_1 + \varepsilon \text{ uniformly on } [-L, L]. \]
The arbitrariness of $\varepsilon$ and $L$ imply that
\[ \limsup_{t \to \infty} v(t, x) \leq 1 - a_1\overline{w}_1 - \frac{d_2}{r_2}\overline{w}_1 =: \overline{v}_2 \quad (41) \]
uniformly on the compact subset of $(-\infty, \infty)$.

**Step 9: The construction of $\overline{w}_2$.**
For any given $L > 0$ and $0 < \varepsilon << 1$, according to $g_\infty = \infty$ and $h_\infty = \infty$, there is $l_\varepsilon > L$ and $T_9 > T_k$ such that

$$g(t) < -l_\varepsilon, \quad h(t) > l_\varepsilon, \quad \forall \ t > T_9, \ -l_\varepsilon < x < l_\varepsilon.$$  

Since $w$ satisfies

$$w_t - w_{xx} = (v - w), \quad t > T_9, \ -l_\varepsilon < x < l_\varepsilon,$$

Note that $\bar{v}_2 \geq v(t, x)$, for $t > T_9$ and $-l_\varepsilon < x < l_\varepsilon$. By use of comparison principle, we have

$$w(t, x) \leq \bar{v}_2, \quad \forall \ t > T_9, \ -L < x < L.$$  

Thus

$$\limsup_{t \to \infty} w(t, x) \leq \bar{v}_2 := \bar{v}_2 \text{ uniformly on the compact subset of } (-\infty, \infty).$$  

**Step 10: The construction of $u_2$.**

For any given $L > 0$, $0 < \delta \ll 1$, $\delta > 0$ and $0 < \varepsilon << 1$, let $l_\varepsilon > 0$, there exists $T_{10} > T_9$ such that

$$v(t, x) \leq \bar{v}_2 + \delta, \quad w(t, x) \leq \bar{w}_2 + \delta, \quad \forall \ t > T_{10}, \ -l_\varepsilon < x < l_\varepsilon.$$  

Therefore $u$ satisfies

$$u_t \geq u(1 - u - \bar{v}_2 - \bar{w}_2 - 2\delta), \quad t \geq T_{10}, \ -l_\varepsilon < x < l_\varepsilon.$$  

For any $L > 0$, consider the unique solution $u$ of

$$\begin{cases} u = u(1 - u - \bar{v}_2 - \bar{w}_2 - 2\delta), & t \geq T_{10}, \ -L < x < L, \\ u(T_{10}) = \min_{[-L,L]} u(T_{10}, \cdot) > 0, & t \geq T_{10}. \end{cases}$$

It follows from the comparison principle that

$$u(t, x) \leq u(t, x), \quad t \geq T_{10}, \ -L < x < L.$$  

It yields $\liminf_{t \to \infty} \min_{[-L,L]} u(t, \cdot) > \lim_{t \to \infty} u(t, x) = 1 - \bar{v}_2 - \bar{w}_2 - 2\delta$ uniformly on $[-L, L]$. Since $\varepsilon > 0$ and $L$ are arbitrary, this implies that

$$\liminf_{t \to \infty} u(t, x) \geq 1 - \bar{v}_2 - \bar{w}_2 := u_2$$  

uniformly on the compact subset of $(-\infty, \infty)$.

**Step 11: The construction of $v_2$.**

For given $L > 0$ and $0 < \delta$, $\varepsilon << 1$, let $l_\varepsilon$ be given by Lemma A.1 with $d_1 = D$, $\beta = r_2(1 - a_1(\bar{u}_2 + \delta) - \frac{d_2}{r_2}(\bar{w}_2 + \delta))$ and $\theta = r_2$. By virtue of (40) and (42), there is $T_{11} > T_{10}$ such that

$$u(t, x) \leq \bar{u}_2 + \delta, \quad w(t, x) \leq \bar{w}_2 + \delta, \quad \forall \ t > T_{11}, \ -l_\varepsilon < x < l_\varepsilon.$$  

Thus, $u$ satisfies

$$\begin{cases} v_t - Dv_{xx} \geq r_2 v \left(1 - v - a_1(\bar{u}_2 + \delta) - \frac{d_2}{r_2}(\bar{w}_2 + \delta)\right), & t > T_{11}, \ -l_\varepsilon < x < l_\varepsilon, \\ v(t, \pm l_\varepsilon) \geq 0, & t > T_{11}. \end{cases}$$  

Similarly to Step 5, we can get

$$\liminf_{t \to \infty} v(t, x) \geq 1 - a_1 \bar{u}_2 - \frac{d_2}{r_2} \bar{w}_2 := v_2$$  

uniformly on the compact subset of $(-\infty, \infty)$.  

---
Step 12: The construction of \( \overline{w}_2 \). For any given \( L > 0, 0 < \varepsilon << 1 \), according to \( g_{\infty} = \infty \) and \( h_{\infty} = \infty \), there is \( l_\varepsilon > L \) and \( T_{12} > T_{11} \) such that
\[
g(t) < -l_\varepsilon, \quad h(t) > l_\varepsilon, \quad \forall \ t > T_{12}, \ -l_\varepsilon < x < l_\varepsilon.
\]
Since \( w \) satisfies
\[
w_t - w_{xx} = c(v - w), \quad t > T_{12}, \quad -l_\varepsilon < x < l_\varepsilon,
\]
Note that \( v_2 \leq v(t, x) \) for \( t > T_{12} \) and \( -l_\varepsilon < x < l_\varepsilon \). By comparison principle, we know
\[
w(t, x) \geq v_2, \quad \forall \ t > T_{12}, \ -L < x < L.
\]
Therefore
\[
\liminf_{t \to \infty} w(t, x) \geq \overline{w}_2 := v_2 \quad \text{uniformly on the compact subset of} \ (-\infty, \infty). \quad (45)
\]

Step 13: Repeating the above procedure, we can find four sequences \( \{\bar{u}_i\}, \{\underline{u}_i\}, \{\bar{v}_i\}, \{\underline{v}_i\} \) \( \{\bar{w}_i\} \text{ and } \{\underline{w}_i\} \), such that, for all \( i \),
\[
\bar{u}_i \leq \liminf_{t \to \infty} u(t, x) \leq \limsup_{t \to \infty} u(t, x) \leq \underline{u}_i,
\]
\[
\bar{v}_i \leq \liminf_{t \to \infty} v(t, x) \leq \limsup_{t \to \infty} v(t, x) \leq \underline{v}_i,
\]
\[
\bar{w}_i \leq \liminf_{t \to \infty} w(t, x) \leq \limsup_{t \to \infty} w(t, x) \leq \underline{w}_i,
\]
uniformly on the compact subset of \( (-\infty, \infty) \). Applying by (34)-(45), these sequences can be determined by the following iterative formulas:
\[
\bar{u}_{i+1} = \frac{1 - a_2 \bar{v}_i}{1 + d_1 \bar{w}_i}, \quad \underline{u}_{i+1} = \frac{1 - a_2 \underline{v}_i}{1 + d_1 \underline{w}_i},
\]
\[
\bar{v}_{i+1} = \frac{1}{r_2} (1 - a_1 \bar{u}_{i+1} - d_2 \bar{w}_i), \quad \underline{v}_{i+1} = \underline{u}_{i+1},
\]
\[
\bar{w}_{i+1} = \frac{1}{r_2} (1 - a_1 \bar{u}_{i+1} - d_2 \underline{w}_i), \quad \underline{w}_{i+1} = \underline{w}_{i+1}, \quad i \geq 1.
\]
Moreover the following monotonicity holds:
\[
\underline{u}_1 \leq \cdots \leq \underline{u}_i \leq \cdots \leq \liminf_{t \to \infty} u(t, x) \leq \limsup_{t \to \infty} u(t, x) \leq \cdots \leq \bar{u}_i \leq \cdots \leq \bar{u}_1,
\]
\[
\underline{v}_1 \leq \cdots \leq \underline{v}_i \leq \cdots \leq \liminf_{t \to \infty} v(t, x) \leq \limsup_{t \to \infty} v(t, x) \leq \cdots \leq \bar{v}_i \leq \cdots \leq \bar{v}_1,
\]
\[
\underline{w}_1 \leq \cdots \leq \underline{w}_i \leq \cdots \leq \liminf_{t \to \infty} w(t, x) \leq \limsup_{t \to \infty} w(t, x) \leq \cdots \leq \bar{w}_i \leq \cdots \leq \bar{w}_1.
\]
Since the constant sequences \( \{\bar{u}_i\}, \{\bar{v}_i\} \text{ and } \{\bar{w}_i\} \) are monotone non-increasing and bounded from the below, and the sequences \( \{\underline{u}_i\}, \{\underline{v}_i\} \text{ and } \{\underline{w}_i\} \) are monotone non-decreasing, and are bounded from the above, the limits of these sequences exists. Let us denote their limits by \( \bar{u}, \underline{u}, \bar{v}, \underline{v}, \bar{w}, \underline{w} \), respectively, as \( i \to \infty \), it is not difficult to see that
\[
\bar{u} = \frac{1 - a_2 \underline{v}}{1 + d_1 \bar{w}}, \quad \underline{u} = \frac{1 - a_2 \bar{v}}{1 + d_1 \underline{w}},
\]
\[
\bar{v} = \frac{1}{r_2} (1 - a_1 \bar{u} - d_2 \bar{w}), \quad \bar{w} = \bar{v},
\]
\[
\underline{v} = \frac{1}{r_2} (1 - a_1 \underline{u} - d_2 \underline{w}), \quad \underline{w} = \underline{v}.
This implies 
\[ \bar{u} = u = u^*, \quad \bar{v} = v = v^*, \quad \bar{w} = w = w^*. \]
The proof is complete. \( \square \)

4. The criteria governing spreading and vanishing. This section is devoted to the study of the criteria governing spreading and vanishing. We first give a comparison principle.

**Lemma 4.1.** For given \( T \in (0, \infty) \), \( \bar{g} \in C^1([0, T]), \bar{h} \in C^1([0, T]), \bar{g}(t) > 0 \) and \( \bar{h}(t) > 0 \) in \( [0, T] \). Let \( \bar{u} \in C^{1,2}(O_T) \) with \( O_T = \{(t, x) \in \mathbb{R}^2 : 0 < t \leq T, -\infty < x < \infty\} \), and \( \bar{v}, \bar{w} \in C(G_T) \cap C^{1,2}(G_T) \) with \( G_T = \{(t, x) \in \mathbb{R}^2 : 0 < t \leq T, \bar{g}(t) < x < \bar{h}(t)\} \). Assume that \((\bar{u}, \bar{v}, \bar{w}, \bar{g}, \bar{h})\) satisfies

\[
\begin{aligned}
\bar{u}_t &\geq \bar{u}(1 - \bar{u}), & 0 < t \leq T, & -\infty < x < \infty, \\
\bar{v}_t - D\bar{v}_{xx} &\geq r_2\bar{v}(1 - \bar{v}), & 0 < t \leq T, & -\bar{g}(t) < x < \bar{h}(t), \\
\bar{w}_t - D\bar{w}_{xx} &\geq -c\bar{w}, & 0 < t \leq T, & -\bar{g}(t) < x < \bar{h}(t), \\
\bar{v}(t, \bar{g}(t)) = \bar{v}(t, \bar{h}(t)) = 0, & 0 < t \leq T, \\
\bar{w}(t, \bar{g}(t)) = \bar{w}(t, \bar{h}(t)) = 0, & 0 < t \leq T, \\
\bar{h}''(t) &\geq -\bar{v}_x(t, \bar{h}(t)) - \bar{w}_x(t, \bar{h}(t)), & 0 < t \leq T, \\
\bar{g}'(t) &\leq -\bar{v}_x(t, \bar{h}(t)) - \bar{w}_x(t, \bar{h}(t)), & 0 < t \leq T,
\end{aligned}
\]

If \( \bar{h}(0) \geq h_0, \bar{g}(0) \leq -h_0, \text{ and } \bar{u}(0, x) \geq u_0(0, x) \) in \( (-\infty, \infty) \)

\[
\begin{aligned}
v_0(x) &\leq \bar{v}(0, x) \text{ on } [-h_0, h_0], & v_0(x) &\geq 0 \text{ on } [-\bar{g}(0), \bar{h}(0)], \\
w_0(x) &\leq \bar{w}(0, x) \text{ on } [-h_0, h_0], & w_0(x) &\geq 0 \text{ on } [-\bar{g}(0), \bar{h}(0)],
\end{aligned}
\]
then the solution \((u, v, w, g, h)\) of (2) satisfies

\[
\begin{aligned}
h(t) &< \bar{h}(t) \text{ on } [0, T], & g(t) &> \bar{g}(t) \text{ on } [0, T], \\
u(t, x) &\leq \bar{u}(t, x) \text{ on } \bar{G}_T, & v(t, x) &\leq \bar{v}(t, x) \text{ on } \bar{D}_T, & w(t, x) &\leq \bar{w}(t, x) \text{ on } \bar{D}_T,
\end{aligned}
\]

where \( D_T = \{(t, x) \in \mathbb{R}^2 : t \in (0, T), g(t) < x < h(t)\} \).

The proof of Lemma 4.1 is essentially same as to that of Lemma 3.1 in [19] and is hence omitted.

Consider the following problem:

\[
\begin{aligned}
-Dz'' &\geq r_2z(1 - z), & -l < x < l, \\
z(-l) &\geq z(l) = 0,
\end{aligned}
\]

(46)

where \( d, \rho, c \) and \( k \) are positive constants. It is well known that if \( l > \frac{2}{\pi} \sqrt{D/r_2} \)

then (46) have a unique positive solution which is globally asymptotically stable. Denote \( \Lambda := \pi \sqrt{D/r_2} \). Next, we shall give a necessary condition for vanishing.

**Theorem 4.2.** Suppose that (4), (5) and (6) hold. If \( h_\infty - g_\infty < \infty \), then

\[ h_\infty - g_\infty \leq \Lambda. \]

**Proof.** By Theorem 3.2, if \( h_\infty - g_\infty < \infty \), then

\[
\lim_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} = 0,
\]

\[
\lim_{t \to \infty} \|w(t, \cdot)\|_{C([g(t), h(t)])} = 0,
\]

\[
\lim_{t \to \infty} u(t, x) = 1 \text{ uniformly on the compact subset of } (-\infty, \infty).
\]
We assume $h_\infty - g_\infty > \Lambda$ to get a contradiction. For any small $\varepsilon > 0$, there exist $\tau \gg 1$ such that $h(\tau) - g(\tau) > \max \{2h_0, \Lambda\}$ and
\[
u(t, x) \geq 1 - \varepsilon, \quad \forall \ t \geq \tau, \ -g_\infty \leq x \leq h_\infty.
\]
Set $l = h(\tau) - g(\tau)$, then $l > \Lambda$. Let $\omega = \omega(t, x)$ be the positive solution of the following initial boundary value problem
\[
\begin{aligned}
\omega_t &= D\omega_{xx} + r_2 \omega (1 - \omega), \quad t > \tau, \ -l < x < l, \\
\omega(t, \pm l) &= 0, \quad t > \tau, \\
\omega(\tau, x) &= \psi(\tau, x), \quad -l \leq x \leq l.
\end{aligned}
\]
By the comparison principle,
\[
\omega(t, x) \leq v(t, x) \quad \text{for} \quad t \geq \tau, \ -l \leq x \leq l.
\]
Since $l > \frac{\tau}{2} \sqrt{\frac{D}{2r_2}}$, it is well known that $w(t, x) \to W(x)$ as $t \to \infty$ uniformly in the compact subset $(-l, l)$, where $W$ is the unique solution of
\[
\begin{aligned}
D\omega_{xx} + r_2 \omega (1 - \omega) &= 0, \quad -l < x < l, \\
\omega(\pm l) &= 0.
\end{aligned}
\]
Hence
\[
\lim_{t \to \infty} v(t, x) \geq \lim_{t \to \infty} \omega(t, x) = W(x) > 0 \quad \text{in} \quad (-l, l),
\]
This is a contradiction to $\lim_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} = 0$. \hfill \Box

**Lemma 4.3.** Suppose that (4), (5) and (6) hold and $h_0 < \Lambda/2$. Then for the problem (2), there exists $\mu^0$ depending on $u_0(x)$, $v_0(x)$ and $w_0(x)$ such that $h_\infty - g_\infty = \infty$ if $\mu > \mu^0$.

**Proof.** The idea of this proof comes from [13, Lemma 3.6]. We see from (2) that there exists a constant $\delta^* > 0$ such that
\[
v_t - dv_{xx} = r_2 v (1 - v - a_1 u) - d_2 vw \geq -\delta^* v, \quad t > 0, \ -g(t) < x < h(t),
\]
\[
w_t - dw_{xx} = c(v - w) \geq -\delta^* w, \quad t > 0, \ -g(t) < x < h(t),
\]
Consider the following auxiliary free boundary problem
\[
\begin{aligned}
\phi_t - d\phi_{xx} &= -\delta^* \phi, \quad t > 0, \ -r_2(t) < x < r_1(t), \\
\varphi_t - d\varphi_{xx} &= -\delta^* \varphi, \quad t > 0, \ -r_2(t) < x < r_1(t), \\
\phi &= 0, \quad t > 0, \ x = r_1(t), \\
\varphi &= 0, \quad t > 0, \ x = r_2(t), \\
\phi(0, x) &= v_0(x), \ \varphi(0, x) = w_0(x), \ -h_0 < x < h_0, \\
r_1(0) &= h_0, \ r_2(0) = -h_0.
\end{aligned}
\]
The comparison principle yields $r_1(t) \leq h(t)$, $r_2(t) \geq g(t)$ and $\phi(t, x) \leq v(t, x)$, $\varphi(t, x) \leq w(t, x)$ for all $t \geq 0$ and $g(t) \leq x \leq h(t)$. Similar to the proof of Lemma 3.2 in [19], there is a constant $\mu^0 > 0$ such that $r_1(2) - r_2(2) \geq \Lambda$ for all $\mu \geq \mu^0$. Hence,
\[
h_\infty - g_\infty = \lim_{t \to \infty} h^\mu(t) - \lim_{t \to \infty} g^\mu(t) \geq r_1(2) - r_2(2) \geq \Lambda, \quad \forall \mu \geq \mu^0.
\]
This together with Theorem 4.2, derives the desired result. \hfill \Box
Lemma 4.4. Suppose that (4), (5) and (6) hold and \( h_0 < \Lambda/2 \). Then there exists \( \mu_0 > 0 \) for the problem (2), depending on \( u_0(x) \), \( v_0(x) \) and \( w_0(x) \), such that \( h_\infty - g_\infty < \infty \) when \( \mu \leq \mu_0 \).

Proof. We shall construct the suitable upper solutions \((\bar{u}, \bar{v}, \bar{h}, \bar{g})\) and then apply Lemma 4.1. By the comparison principle, \( u(t, x) \leq \bar{u}(t) \) for all \( t, x \geq 0 \), where \( \bar{u}(t) \) is given by (32).

Inspired by Guo and Wu [11], we define
\[
\bar{h}(t) = h_0 \left( 1 + \delta - \frac{\delta}{2} e^{-\alpha t} \right), \quad t \geq 0,
\]
\[
\bar{g}(t) = -h_0 \left( 1 + \delta - \frac{\delta}{2} e^{-\alpha t} \right), \quad t \geq 0,
\]
\[
\bar{v}(x, t) = M_1 e^{-\alpha t} V \left( \frac{x}{\bar{h}(x)} \right), \quad \bar{g}(t) \leq x \leq \bar{h}(t),
\]
\[
\bar{w}(x, t) = M_2 e^{-\alpha t} V \left( \frac{x}{\bar{h}(x)} \right), \quad \bar{g}(t) \leq x \leq \bar{h}(t),
\]
where
\[
\delta = \frac{1}{2} \left( \frac{\Lambda}{h_0} - 1 \right), \quad V(y) = \cos \left( \frac{\pi}{2} y \right),
\]
\( \alpha, M_1 \) and \( M_2 \) are to be determined later. Since \( h_0(1 + \delta) < \Lambda/2 \), we have
\[
\alpha := \frac{1}{2} \min \left\{ \left[ \left( \frac{\pi}{2} \right)^2 \frac{d}{(1 + \delta)^2 h_0^2} - r_2 \right], \left[ \left( \frac{\pi}{2} \right)^2 \frac{1}{(1 + \delta)^2 h_0^2} - c \left( \frac{M_1}{M_2} - 1 \right) \right] \right\}.
\]
The direct computation yields
\[
\bar{v}_t - d\bar{v}_{xx} - r_2 \bar{v} \geq M_1 V e^{-\alpha t} \left[ \left( \frac{\pi}{2} \right)^2 \frac{d}{(1 + \delta)^2 h_0^2} - r_2 - \alpha \right] \geq 0,
\]
\[
\bar{w}_t - \bar{w}_{xx} - c(\bar{v} - \bar{w}) \geq M_2 V e^{-\alpha t} \left[ \left( \frac{\pi}{2} \right)^2 \frac{1}{(1 + \delta)^2 h_0^2} - c \left( \frac{M_1}{M_2} - 1 \right) - \alpha \right] \geq 0.
\]
By choosing \( M_1 \) and \( M_2 \) sufficient large, we have \( \bar{v}(x, 0) \geq v_0(x) \) and \( \bar{w}(x, 0) \geq w_0(x) \) for all \( x \in [-h_0, h_0] \). Moreover, for such fixed constants \( \delta, \alpha, M_1, M_2 \), there exists \( \mu_0 \) such that
\[
\bar{h}'(t) + \mu(\bar{v}_x(t, \bar{h}(t)) + \bar{w}_x(t, \bar{g}(t))) \geq 0, \quad \forall \mu \leq \mu_0,
\]
\[
\bar{g}'(t) + \mu(\bar{v}_x(t, \bar{h}(t)) + \bar{w}_x(t, \bar{g}(t))) \leq 0, \quad \forall \mu \leq \mu_0,
\]
By Lemma 4.1, \( \bar{h}(t) < \bar{h}(t) \) and \( g(t) > \bar{g}(t) \). Thus \( h_\infty - g_\infty \leq \bar{h}(\infty) - \bar{g}(\infty) = 2h_0(1 + \delta) < \infty \). The proof is complete.

Summarizing the above arguments we get the following theorem.

Theorem 4.5. Suppose that (4), (5) and (6) hold and \( h_0 < \Lambda/2 \). For problem (2), there exists \( \mu^* \geq \mu_0 > 0 \), depending on \( u_0(x) \), \( v_0(x) \) and \( w_0(x) \), such that \( h_\infty - g_\infty \leq \Lambda \) if \( \mu \leq \mu^* \), and \( h_\infty - g_\infty = \infty \) if \( \mu > \mu^* \).

In order to investigate the long time behaviour of the solution \((u, v, w)\) to (2), we should prove the following some lemmas.
Lemma A.1 (\cite{20}) For any given \( \varepsilon > 0 \) and \( L > 0 \), there exist \( T_\varepsilon > 0 \) and \( l_\varepsilon > \max\{L, \frac{\varepsilon}{\beta} \sqrt{d_1/\beta}\} \), such that when the continuous and non-negative functions \( w(t,x) \) satisfies
\[
\begin{align*}
  w_t - d_1 w_{xx} &\geq (\leq) w (\beta - \theta w), & t > 0, \quad -l_\varepsilon < t < l_\varepsilon, \\
  w(t,\pm l_\varepsilon) &\geq (=) 0, & t > 0
\end{align*}
\]
and \( w(0,x) > 0 \) in \((-l_\varepsilon, l_\varepsilon)\), then
\[
w(t,x) > \frac{\beta}{\theta} - \varepsilon \quad (w(t,x) < \frac{\beta}{\theta} + \varepsilon), \quad \forall \ t \geq T_\varepsilon, \quad -L \leq x \leq L.
\]
Which implies
\[
\liminf_{t \to \infty} w(t,x) > \frac{\beta}{\theta} - \varepsilon \quad \left( \limsup_{t \to \infty} w(t,x) < \frac{\beta}{\theta} + \varepsilon \right) \quad \text{uniformly on } [-L, L].
\]

Lemma A.2 (\cite{20}) Let \( q \) be a positive constant. For any given \( \varepsilon > 0 \) and \( L > 0 \), there exist \( T_\varepsilon > 0 \) and \( l_\varepsilon > \max\{L, \frac{\varepsilon}{\beta} \sqrt{d_1/\beta}\} \), such that when the continuous and non-negative functions \( z(t,x) \) satisfies
\[
\begin{align*}
  z_t - d_1 z_{xx} &\geq (\leq) z (\beta - \theta z), & t > 0, \quad -l_\varepsilon < x < l_\varepsilon, \\
  z(t,\pm l_\varepsilon) &\geq (\leq) q, & t > 0
\end{align*}
\]
and \( z(0,x) > 0 \) in \((-l_\varepsilon, l_\varepsilon)\), then
\[
z(t,x) > \frac{\beta}{\theta} - \varepsilon \quad (z(t,x) < \frac{\beta}{\theta} + \varepsilon), \quad \forall \ t \geq T_\varepsilon, \quad -L \leq x \leq L.
\]
This implies
\[
\liminf_{t \to \infty} z(t,x) > \frac{\beta}{\theta} - \varepsilon \quad \left( \limsup_{t \to \infty} z(t,x) < \frac{\beta}{\theta} + \varepsilon \right) \quad \text{uniformly on } [-L, L].
\]

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