2D hydrodynamical systems: invariant measures of Gaussian type

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Abstract

Gaussian measures $\mu^{\beta,\nu}$ are associated to some stochastic 2D hydro-
dynamical systems. They are of Gibbsian type and are constructed by
means of some invariant quantities of the system depending on some pa-
rameter $\beta$ (related to the 2D nature of the fluid) and the viscosity $\nu$. We
prove the existence and the uniqueness of the global flow for the stochastic
viscous system; moreover the measure $\mu^{\beta,\nu}$ is invariant for this flow and
is unique. Finally, we prove that the deterministic inviscid equation has
a $\mu^{\beta,\nu}$-stationary solution (for any $\nu > 0$).

1 Introduction

The goal in this paper is to study a class of mathematical models related to 2D
fluids. We will deal with an abstract stochastic evolution equation in a Hilbert
space of the following form

\begin{equation}
du(t) + [\nu Au(t) + B(u(t), u(t))] dt = \sqrt{Q}dw(t),
\end{equation}

where $w$ is a cylindrical Wiener process and $Q$ is a linear operator. The un-
bounded linear operator $A$ and the bilinear operator $B$ will satisfy certain prop-
erties related to 2D fluids that will be given in details in the following sections.
The coefficient $\nu \geq 0$ is the viscosity. There is an extensive literature about the
existence and uniqueness of solutions with initial data of finite energy. Its long
time behavior has also been extensively studied, including the existence and
uniqueness of invariant measures (see, e.g., [3] and the reference therein). In
the present paper, we are interested in the qualitative behavior of these invari-
ant measures. In particular, we prove the existence and uniqueness of invariant
measures of Gaussian type for the viscous case (1); moreover, this Gaussian
measure is proved to be invariant also for the deterministic and inviscid model
($\nu = 0, Q = 0$).

We point out that the Gaussian invariant measure that we consider here is
not that one considered in previous papers [1], [2], [5], [4], [6], but has a more
regular support. In particular, the support of this measure is a Sobolev space
of positive exponent.

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As far as the content of this paper, in Section 2 we introduce the operators associated to the model with their properties and the Gibbs measures \( \mu^{\beta,\nu} \). We introduce the Ornstein-Uhlenbeck equation with a suitable noise and prove that \( \mu^{\beta,\nu} \) is its unique invariant. In Section 3, we deal with the viscous stochastic case; we prove the existence and uniqueness of strong solutions and that \( \mu^{\beta,\nu} \) is its unique invariant measure. The uniqueness of the invariant measure is proved by means of Girsanov Theorem. Moreover, some ergodic properties of this measure with its rate of convergence are shown. In Section 4, we introduce a particular example, shell models of turbulence with an emphasis on the Sabra model. The coefficient \( \beta \) characterizing the measure \( \mu^{\beta,\nu} \) will be related to the coefficients \( a, b \) of the Sabra model through the condition (47). Section 5 is devoted to the deterministic inviscid model, in particular we present our results for the inviscid Sabra model with \( \beta = 1 \). For any \( \nu > 0 \) we prove the existence of a stationary process whose law at any fixed time is \( \mu^{1,\nu} \).

2 Introduction to the model and functional setting

2.1 Operators and spaces

Let \( (H, | \cdot |) \) be a real separable Hilbert space endowed with an inner product denoted by \( (\cdot, \cdot) \), and \( A \) an unbounded self-adjoint positive linear operator on \( H \) with compact resolvent. We denote by \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \) the eigenvalues of \( A \) and by \( e_1, e_2, \ldots \) a complete orthonormal system in \( H \) given by the eigenfunctions of the operator \( A \):

\[
A e_n = \lambda_n e_n
\]

We assume that \( \lim_{n \to \infty} \lambda_n = \infty \).

Let \( H_n = \text{span}\{e_1, e_2, \ldots, e_n\} \) and \( \Pi_n \) the projector operator onto \( H_n \).

For any \( \alpha \in \mathbb{R} \) we can define the power operators \( A^\alpha \) as

\[
A^\alpha x = \sum_{n=1}^{\infty} A_n^\alpha (x, e_n) e_n
\]

\[
D(A^\alpha) = \{ x = \sum_{n=1}^{\infty} x_n e_n : \sum_{n=1}^{\infty} \lambda_n^{2\alpha} x_n^2 < \infty \}.
\]

We set

\[
H^\alpha = D(A^{\alpha/2}).
\]

Each \( H^\alpha \) is a Hilbert space with scalar product \( \langle u, v \rangle_{H^\alpha} = (A^{\alpha/2} u, A^{\alpha/2} v) \). We denote by \( \| \cdot \|_{\alpha} \) the norm in \( H^\alpha \).

Let \( B : H \times H \to H^{-1} \) be a bilinear operator; we assume that there exists a positive constant \( c \) such that

\[
\| B(u, v) \|_{-1} \leq c |u||v|.
\]

We consider the finite dimensional approximation of the bilinear operator \( B \); this is the bilinear operator \( B^M \) defined as

\[
B^M(u, v) = \Pi_M B(\Pi_M u, \Pi_M v)
\]

for any \( M \in \mathbb{N} \). For each \( B^M \) we have the same estimate as (2) (with the constant \( c \) independent of \( M \)).

For any \( \nu > 0 \) and \( \beta > 0 \), let \( \mu^{\beta,\nu} \) be the Gaussian measure \( \mathcal{N}(0, \frac{1}{\nu} A^{-\beta}) \) (see, e.g., [19, 13]).
2.2 Assumptions

Besides the basic properties of the operators $A$ and $B$ given above, we present other important assumptions.

**Condition (C1):** For any $\nu > 0$, the operator $\nu A$ generates an analytic semigroup of contractions in $H$ and for any $p > 0$ there exists $c_{p,\nu} > 0$ such that

$$|A^p e^{-\nu A t} x| \leq \frac{c_{p,\nu}}{t^p} |x| \quad \forall t > 0, x \in H.$$  

**Condition (C2):** The bilinear operator $B$ satisfies the following properties:

(i) $\langle B(u, v), w \rangle = -\langle B(u, w), v \rangle$

(ii) $\langle B(u, v), v \rangle = 0$

(iii) $\exists \beta > 0$ such that $\langle B(u, u), A^{\beta} u \rangle = 0$

for any $u, v, w$ giving meaning to the above relationships.

**Condition (C3):** There exists $\alpha \in [0, \beta)$ (with $\beta$ given by (C2 iii)) such that the embedding $H^{\beta} \subset H^\alpha$ is Hilbert-Schmidt, i.e.

$$\sum_{n=1}^{\infty} \lambda_n^{\alpha-\beta} < \infty.$$  

**Condition (C4):** for $\alpha$ and $\beta$ given in (C2)-(C3), $B : H^\alpha \times H^\alpha \rightarrow H^{\beta-1}$ is a continuous operator, i.e.

$$\|B(u, v)\|_{\beta-1} \leq c \|u\|_{\alpha} \|v\|_{\alpha} \quad \forall u, v \in H^\alpha$$

Moreover, if $\alpha > 0$ we assume

$$\|B(u, v)\|_{\alpha-1} \leq c \|u\|_{\alpha} \|v\|_{\alpha} \quad \forall u \in H, v \in H^\alpha$$

**Condition (C5):** For each $n$ set $B_n(u, v) = \langle B(u, v), e_n \rangle$. Then we have

$$\int |B_n(x, x)|^2 \mu^{\beta, \nu}(dx) < \infty \quad \forall n$$

and $B_n(x, x)$ independent of $x_n$ (where $x = \sum_n x_n e_n$). Moreover

$$\lim_{M \to \infty} \sum_{n=1}^{M} \int |(B^M(x, x) - B(x, x), e_n)|^2 \mu^{\beta, \nu}(dx) = 0.$$  

**Remark 2.1** (i) We have the relationships corresponding to assumption (C2):

(i) $\langle B^M(u, v), w \rangle = -\langle B^M(u, w), v \rangle$

(ii) $\langle B^M(u, v), v \rangle = 0$

(iii) $\langle B^M(u, u), A^{\beta} u \rangle = 0$

(ii) By means of the bilinearity and of estimate (ii) we have

$$\lim_{M \to \infty} \|B^M(u, v) - B(u, v)\|_{\beta-1} = 0 \quad \forall u, v \in H^\alpha$$

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(iii) Since $\alpha \geq 0$, the inequality \[5\] implies
\[\|B(u, v)\|_{\alpha-1} \leq c\|u\|_\alpha\|v\|_\alpha \quad \forall u, v \in H^\alpha.\]
Moreover,
\[\lim_{M \to \infty} \|B^M(u, v) - B(u, v)\|_{\alpha-1} = 0 \quad \forall u, v \in H^\alpha.\]

(iv) Assumption (C3) implies that the space $H^\alpha$ has full measure $\mu^{\beta,\nu}$, i.e.
$\mu^{\beta,\nu}(H^\alpha) = 1$. However, for Gaussian measures in infinite dimensional spaces
we have $\mu^{\beta,\nu}(H^\beta) = 0$ (see, e.g., [19]).

We denote by $L^p(\mu^{\beta,\nu})$ the space of measurable functions $\phi$ defined in the
support of the measure $\mu^{\beta,\nu}$ and such that
$\int |\phi|^p d\mu^{\beta,\nu} < \infty$.

2.3 The equations
Set $Q = 2A^{1-\beta}$ in (11), that is we consider the following nonlinear stochastic
equation
\[\frac{du}{dt}(t) + \left[\nu Au(t) + B(u(t), u(t))\right]dt = \sqrt{2A^{1-\beta}} dw(t).\]
In addition we deal with the inviscid and deterministic equation
\[\frac{du}{dt}(t) + B(u(t), u(t)) = 0\]
and with the viscous linear stochastic equation
\[dz(t) + \nu Az(t) \, dt = \sqrt{2A^{1-\beta}} dw(t).\]

Relationship (ii) in Assumption (C2) implies a formal law of conservation of
energy $E(t) = \frac{1}{2}\|u(t)\|^2$ in equation (13). We recall that the energy is a conserved
quantity in the motion of incompressible inviscid fluids (13).

Relationship (iii) in Assumption (C2) implies that $S_\beta(t) = \frac{1}{2}\|u(t)\|_\beta^2$ is a
conserved quantity for equation (13), that is formally we have
\[
\frac{dS_\beta}{dt}(t) = (\dot{u}(t), A^\beta u(t)) = -(B(u(t), u(t)), A^\beta u(t)) = 0.
\]
For $\beta = 1$, $S_1$ is the enstrophy which is a conserved quantity in the motion of
2D incompressible inviscid fluids.

The Gaussian measure $\mu^{\beta,\nu} = \mathcal{N}(0, \frac{1}{2}A^{-\beta})$ can be described heuristically as
$\mu^{\beta,\nu}(du) = \frac{1}{Z} e^{-\nu S_{\beta}(u)} du^\nu$
where $Z$ is a normalization constant to make $\mu^{\beta,\nu}$ to be a probability measure.
Therefore it makes sense to see if the measure $\mu^{\beta,\nu}$, described by means of the
invariant quantity $S_{\beta}$, is a stationary statistical solution for the inviscid equation
(13). To this end, we will first prove that $\mu^{\beta,\nu}$ is a stationary measure for the
viscous and stochastic equation (12) looking for a dynamics in the space $H^\alpha$ of
full measure $\mu^{\beta,\nu}$. However, the basic stochastic case to deal with is the linear
equation (14) for which we recall well known properties (see [13]).
Proposition 2.2 Let assumptions (C1), (C2 iii) and (C3) be satisfied. Then, for any \( z(0) \in H^\alpha \) there exists a unique strong solution to equation (14) such that

\[
z \in C([0, T]; H^\alpha) \quad \mathbb{P} - \text{a.s.}
\]

The stationary process solving equation (14) is

\[
\zeta(t) = \sqrt{2} \int_{-\infty}^{t} e^{-\nu(t-s)} A \frac{1}{x^2} dw(s)
\]

and the law of \( \zeta(t) \) is \( \mu^{\beta, \nu} \) for any time \( t \).

3 Stochastic viscous models

We consider equation (12); first we prove that there exists a unique solution for any initial data in \( H^\alpha \). The solution is strong in the probabilistic sense and uniqueness is in pathwise sense. Moreover, we show that \( \mu^{\beta, \nu} \) is the unique invariant measure associated to this stochastic equation.

3.1 Strong solution

We look for dynamics in the state space \( H^\alpha \) with \( 0 \leq \alpha < \beta \) fulfilling assumptions (C1)-(C4). We consider any finite time interval \([0, T]\).

Theorem 3.1 Let assumptions (C1),(C2), (C3) and (C4) be satisfied. Then, for any \( u(0) \in H^\alpha \), there exists a unique solution \( u \) to equation (12) such that

\[
 u \in C([0, T]; H^\alpha) \quad \mathbb{P} - \text{a.s.}
\]

Moreover, the process \( u \) is a Markov process, Feller in \( H^\alpha \).

We divide the proof in three steps in the following subsections.

3.1.1 Existence of strong solutions for the viscous stochastic model

We use a well known trick to study a stochastic semilinear equation with additive noise: we set \( v = u - z \). Then

\[
\frac{dv}{dt}(t) + \nu Av(t) + B(v(t) + z(t), v(t) + z(t)) = 0
\]

with \( v(0) = u(0) - z(0) \). Set \( z(0) = 0 \).

Proposition 3.2 We consider the same assumptions as in Theorem 3.1. Let \( v(0) \in H^\alpha \). Then there exists a solution to equation (15) such that

\[
v \in C([0, T]; H^\alpha) \cap L^2(0, T; H^{1+\alpha}) \quad \mathbb{P} - \text{a.s.}
\]
Proof. We proceed pathwise. Take the scalar product of the left hand side of equation (15) with \( v \) in \( H \); we get some a priori estimates

\[
\frac{1}{2} \frac{d}{dt} |v|^2 + \nu \| v \|_1^2 = - \langle B(v + z, v + z), v \rangle \\
= - \langle B(v + z, z), v \rangle \quad \text{by (C2 ii)} \\
\leq \| B(v + z, z) \|_{-1} \| v \|_1 \\
\leq c \| v + z \| |z| \| v \|_1 \quad \text{by (2)} \\
\leq \frac{\nu}{2} \| v \|_1^2 + \frac{c_v}{2} |z|^2 \| v \|_1^2 + \frac{c_v}{2} |z|^4
\]

by Young inequality, for some positive constant \( c_v \). Henceforth, we denote by \( c_v \) a generic constant depending on \( \nu \).

Therefore

\[
\frac{d}{dt} |v|^2 + \nu \| v \|_1^2 \leq c_v |z|^2 |v|^2 + c_v |z|^4.
\]

Hence, Gronwall inequality applied to

\[
\frac{d}{dt} |v|^2 \leq c_v |z|^2 |v|^2 + c_v |z|^4
\]

gives

\[
\sup_{0 \leq t \leq T} |v(t)|^2 \leq e^{c_v T \| z \|^2_{C([0,T] ; H)}} \left( |v(0)|^2 + c_v T \| z \|^4_{C([0,T] ; H)} \right) < \infty
\]

and integrating in time \( \| v \|^2_{L^2(0,T)} \leq (\| v(0) \|^2 + T c_v) \left( \| z \|^2_{C([0,T] ; H)} \| v \|^2_{C([0,T] ; H)} + \| z \|^4_{C([0,T] ; H)} \right) < \infty.\]

Moreover, when \( \alpha \geq 0 \) we proceed in a similar way: we take the scalar product of the left hand side of equation (15) with \( A^\alpha v \) in \( H \); then

\[
\frac{1}{2} \frac{d}{dt} \| v \|_{\alpha}^2 + \nu \| v \|_{\alpha + 1}^2 = - \langle A^{-\alpha} B(v + z, v + z), A^\alpha v \rangle \\
\leq \| B(v + z, v + z) \|_{-\alpha} \| v \|_{\alpha + 1} \\
\leq c \| v + z \| |v + z| \| v \|_{\alpha + 1} \quad \text{by (5)} \\
\leq \frac{\nu}{2} \| v \|_{\alpha + 1}^2 + \frac{c_v}{2} (|v|^2 + |z|^2) \| v \|_{\alpha}^2 + \frac{c_v}{2} |z|^4_{\alpha}.
\]

This gives

\[
\frac{d}{dt} \| v \|_{\alpha}^2 + \nu \| v \|_{\alpha + 1}^2 \leq c_v \left( |v|^2 + \| z \|_{\alpha}^2 \right) \| v \|_{\alpha}^2 + c_v |z|^4_{\alpha}.
\]

Therefore, using (14) and the fact that \( \alpha \geq 0 \) we get

\[
\sup_{0 \leq t \leq T} \| v(t) \|_{\alpha}^2 \leq \| v(0) \|_{\alpha}^2 e^{c_v \int_0^T (|v(t)|^2 + \| z(t) \|_{\alpha}^2) dt} \\
+ c_v \int_0^T e^{c_v \int_0^T (|v(s)|^2 + \| z(s) \|_{\alpha}^2) ds} \| z(t) \|_{\alpha}^2 dt < \infty
\]
and integrating in time \[19\]
\[
\int_0^T \|v(s)\|^2_{H^{\alpha+1}} ds < \infty.
\]

Actually, the a priori estimates are for the Galerkin approximation \(v^M\). We define the Galerkin problem associated to \[12\]
\[
\left\{ \begin{array}{l}
 \frac{du^M}{dt}(t) + [\nu Au^M(t) + B^M(u^M(t), u^M(t))]dt = \Pi_M \sqrt{2A^{1-\beta}} \ dw(t) \\
 u^M(0) = \Pi_M x
\end{array} \right.
\]
where \(M\) is any positive integer. Similarly we have
\[
\frac{dv^M}{dt}(t) + \nu Av^M(t) + B(v^M(t) + z^M(t), v^M(t) + z^M(t)) = 0
\]
with \(z^M(t) = \Pi_M z(t)\).

The previous estimates give
\[22\] \[\sup_M \|v^M\|^2_{L^\infty(0,T;H^{\alpha})} < \infty\]
\[23\] \[\sup_M \|v^M\|^2_{L^2(0,T;H^{1+\alpha})} < \infty\]

In addition \(\frac{dv^M}{dt}\) is bounded: indeed
\[
\frac{dv^M}{dt}(t) = -\nu Av^M(t) - B(v^M(t) + z^M(t), v^M(t) + z^M(t));
\]
using \[23\]-\[22\], we have that the first term in the r.h.s belongs to the space \(L^2(0,T;H^{\alpha-1})\) and the second to the space \(C([0,T];H^{\alpha-1})\) (use \[10\]) and thus in \(L^2(0,T;H^{\alpha-1})\). Then
\[24\] \[\sup_M \left\| \frac{dv^M}{dt} \right\|^2_{L^2(0,T;H^{\alpha-1})} < \infty.\]

Since the space \(\{v : v \in L^2(0,T;H^{1+\alpha})\}\) is compactly embedded in the space \(L^2(0,T;H^{\alpha+1})\) we get that there exists a subsequence \(\{v^M_i\}\) weakly convergent to \(v\) in \(L^2(0,T;H^{1+\alpha})\), weakly-* convergent in \(L^2(0,T;H^{\alpha})\) and strongly convergent in \(L^2(0,T;H^{\alpha})\). By means of the bilinearity of \(B\), of the strong convergence result and of \[11\], we conclude that the limit \(v\) fulfills \[15\].

The fact that \(v \in C([0,T];H^{\alpha})\) comes from a result in Temam \[23\] (Lemma 1.4, page 263): if \(v \in L^2(0,T;H^{1+\alpha})\) and \(\frac{dv}{dt} \in L^2(0,T;H^{-1+\alpha})\), then \(v \in C([0,T];H^{\alpha})\).

\textbf{Remark 3.3} We can prove also the uniqueness of this solution \(v\), but we do not need it here. Anyway, the proof of uniqueness would be based on the same estimes as in the next Section \[7.1.2\].

We conclude for \(u = v + z\).

\textbf{Proposition 3.4} We consider the same assumptions as in Theorem \[3.1\]. Let \(u(0) \in H^{\alpha}\). Then there exists a solution to equation \[12\] such that
\[
u \in C([0,T];H^{\alpha}) \quad \mathbb{P}-\text{a.s.}
\]
3.1.2 Pathwise uniqueness

Now we prove that the strong solution \( u \) constructed in the previous section is pathwise unique, that is

**Proposition 3.5** We consider the same assumptions as in Theorem 3.1. Let \( u_1, u_2 \) be two solutions to equation (12) with the same initial data, defined on the same stochastic basis and with the same Wiener process. Then \( u_1 = u_2 \) P-a.s., the equality being in \( C([0,T];H^\alpha) \).

**Proof.** We proceed pathwise. Let \( u_1, u_2 \in C([0,T];H^\alpha) \) be two paths (for fixed \( \omega \) in a set of P-measure 1).

Set \( U = u_1 - u_2 \). Then \( U \in C([0,T];H^\alpha) \) and it solves an equation which is deterministic (for any path):

\[
\frac{dU}{dt} + \nu AU + B(u_1,u_1) - B(u_2,u_2) = 0; \quad U(0) = 0.
\]

(25)

First, we notice that \( U \) is more regular than the \( u_i \)'s (the noise term has disappeared and we expect more regularity as for equation (15)).

By the bilinearity of the operator \( B \), we have

\[
\frac{dU}{dt} + \nu AU + B(u_1,U) + B(U,u_2) = 0; \quad U(0) = 0.
\]

(26)

We get an a priori estimate:

\[
\frac{1}{2} \frac{d}{dt} \|U(t)\|^2_{1+\alpha} + \nu \|U(t)\|^2_{1+\alpha} \\
= - \left( A^{\alpha+1} [B(u_1(t),U(t)) + B(U(t),u_2(t))], A^{\alpha+1} U(t) \right) \\
\leq \left[ \|u_1(t)\|_{\alpha} + \|u_2(t)\|_{\alpha} \right] \|U(t)\|_{\alpha} \|U(t)\|_{1+\alpha} \text{ by (4)} \\
\leq \frac{\nu}{2} \|U(t)\|^2_{1+\alpha} + \frac{c_\nu}{2} \left[ \|u_1(t)\|^2_{\alpha} + \|u_2(t)\|^2_{\alpha} \right] \|U(t)\|^2_{\alpha}.
\]

Therefore

\[
\frac{d}{dt} \|U(t)\|^2_{\alpha} \leq c_\nu \left[ \|u_1(t)\|^2_{\alpha} + \|u_2(t)\|^2_{\alpha} \right] \|U(t)\|^2_{\alpha};
\]

from this, by Gronwall inequality follows

\[
\|U(t)\|^2_{\alpha} \leq \|U(0)\|^2_{\alpha} e^{c_\nu \int_0^t \left[ \|u_1(s)\|^2_{\alpha} + \|u_2(s)\|^2_{\alpha} \right] ds}.
\]

(27)

Finally, \( U(t) = 0 \) for all \( t \), since \( U(0) = 0 \).

**Remark 3.6** Markovianity is inherited from the Galerkin approximations.

3.1.3 Feller property

Let us denote by \( u(t;x) \) the solution of equation (12) with initial data \( x \). Define the Markov semigroup \( P_t : B_b(H^\alpha) \to B_b(H^\alpha) \) as

\[
P_t \phi(x) = \mathbb{E}[\phi(u(t;x))].
\]

This is a contraction semigroup. Moreover, it is Feller in \( H^\alpha \), that is

\[
P_t : C_b(H^\alpha) \to C_b(H^\alpha).
\]
This comes from the estimates for the pathwise uniqueness. Indeed, if \( \|x - y\|_\alpha \to 0 \) then \( \| \| \) gives

\[
\|u(t; x) - u(t; y)\|_2 \leq \|x - y\|_2 e^{\int_0^t \|u(s; x)\|_2^2 + \|u(s; y)\|_2^2} ds
\]

for \( t > 0 \) fixed. By \( \| \| \) we get a uniform estimate of \( \|u(\cdot; x)\|_{L^\infty(0, T; H^\alpha)} \) when \( \|x\|_\alpha \) is bounded, i.e.

\[
\forall R > 0 \exists C_R : \sup_{\|x\|_\alpha \leq R} \|u(\cdot; x)\|_{C([0, T]; H^\alpha)} < C_R.
\]

Hence, when \( \|x - y\|_\alpha \to 0 \) from \( \| \| \) we get \( \|u(t; x) - u(t; y)\|_\alpha \to 0 \). We conclude that \( \phi(u(t; x)) \to \phi(u(t; y)) \) for \( \phi \in C_b(H^\alpha) \) and therefore \( E[\phi(u(t; x))] \to E[\phi(u(t; y))] \) by the dominated convergence. This means that \( P_t \phi \in C_b(H^\alpha) \) for any \( t > 0 \) and \( \phi \in C_b(H^\alpha) \).

### 3.2 Invariant measure

We prove the following theorem:

**Theorem 3.7** Besides the assumptions of Theorem \( \| \| \) we consider (C5). Then, \( \mu^{\beta, \nu} \) is the unique invariant measure for equation \( \| \| \), that is

\[
\int P_t \phi \, d\mu^{\beta, \nu} = \int \phi \, d\mu^{\beta, \nu} \quad \forall \phi \in L^1(\mu^{\beta, \nu}) \text{ and } t \geq 0.
\]

First, we show that \( \mu^{\beta, \nu} \) is an invariant measure for the nonlinear equation \( \| \| \). Then we prove that this is indeed the unique invariant measure.

A consequence of this result is the following

**Corollary 3.8** Given any initial data with law \( \mu^{\beta, \nu} \), there exists a unique stationary solution of equation \( \| \| \). To prove our result, we need to introduce the Kolmogorov operator associated to the stochastic equation \( \| \| \). Let \( FC_b^\infty \) be the space of infinitely differentiable cylindrical functions bounded and with bounded derivatives; \( \phi \in FC_b^\infty \) means that there exist \( m \in \mathbb{N} \), \( \hat{\phi} \in C_b^\infty(\mathbb{R}^m) \) and multiindices \( (i_1, i_2, \ldots, i_m) \) such that

\[
\phi(x) = \hat{\phi}(x, e_{i_1}, x, e_{i_2}, \ldots, x, e_{i_m}).
\]

We set \( \frac{\partial \phi}{\partial x_i} = \frac{\partial \hat{\phi}}{\partial x_i} \) with \( x_i = (x, e_i) \). \( FC_b^\infty \) is a dense subset of \( L^p(\mu^{\beta, \nu}) \) for any \( p \geq 1 \).

We define the Kolmogorov operator first on these very regular functions \( \phi \in FC_b^\infty \) as

\[
K \phi(x) = \sum_n \left[ \lambda_n^{-\beta} \frac{\partial^2 \phi}{\partial x_n^2}(x) - B_n(x, x) \frac{\partial \phi}{\partial x_n}(x) - \nu \lambda_n x_n \frac{\partial \phi}{\partial x_n}(x) \right].
\]

We have that \( K \phi \in L^1(\mu^{\beta, \nu}) \) for any \( \phi \in FC_b^\infty \) (use that each \( B_n \in L^1(\mu^{\beta, \nu}) \) and the sums are finite).
3.2.1 Existence of the invariant measure

We know that the linear stochastic equation (14) has $\mu^{\beta,\nu}$ as unique invariant measure, that is $\mu^{\beta,\nu}$ is the unique probability measure such that

$$\int \mathbb{E}[\phi(z(t; x))] \mu^{\beta,\nu}(dx) = \int \phi(x) \mu^{\beta,\nu}(dx) \quad \forall t \geq 0, \phi \in B_b(H^\alpha)$$

(see [13, 14]). Actually we can define the latter relationship for all $\phi \in L^p(\mu^{\beta,\nu})$, given any $1 \leq p < \infty$ (see, e.g., [10, 11]).

Now, we want to show that $\mu^{\beta,\nu}$ is an invariant measure also for the nonlinear equation (12). The role of the nonlinear term $B$ is analyzed first considering the finite dimensional $B_M$ and then passing to the limit as $M \to \infty$. Here we need 6 of (C5).

First, we prove that $\mu^{\beta,\nu}$ is an infinitesimally invariant measure for equation (12) in the sense that

$$(31) \quad \int K \phi \, d\mu^{\beta,\nu} = 0 \quad \forall \phi \in FC_b^\infty.$$  

Indeed, we can write $K$ as the sum of two operators, $K = Q + L$, with domains $FC_b^\infty$ and we have the infinitesimal invariance for both these operators. We integrate by parts:

$$(32) \quad \int Q \phi \, d\mu^{\beta,\nu} \equiv \int \sum_n \left[ \lambda_n^{1-\beta} \frac{\partial^2 \phi}{\partial x_n^2}(x) - \nu \lambda_n x_n \frac{\partial \phi}{\partial x_n}(x) \right] \mu^{\beta,\nu}(dx) = 0$$

and

$$(33) \quad \int L \phi \, d\mu^{\beta,\nu} \equiv -\int \sum_n B_n(x, x) \frac{\partial \phi}{\partial x_n}(x) \mu^{\beta,\nu}(dx)$$

$$= -\nu \int \sum_n \lambda_n^\beta B_n(x, x) x_n \phi(x) \, d\mu^{\beta,\nu}(dx) = 0$$

since $B_n$ does not depend on the variable $x_n$.

Now we use an approximative criterium of Eberle [16] to show that the measure $\mu^{\beta,\nu}$ is an invariant measure for equation (12). First, with similar computations as above, we get that the Kolmogorov operator $(K, FC_b^\infty)$ is dissipative in $L^1(\mu^{\beta,\nu})$, that is

$$(34) \quad \int K \phi \, d\mu^{\beta,\nu} \leq 0 \quad \forall \phi \in FC_b^\infty$$

(see also [19]). Hence it is closable (see [22]).

With assumption 6 we can apply the results of Eberle (in particular, we use Theorem 5.2, Corollary 5.3 and (5.46) at page 226 of [16] with $p = 1$): they state that the closure $K$ on $L^1(\mu^{\beta,\nu})$ of the Kolmogorov operator $(K, FC_b^\infty)$ generates a sub-Markovian strongly continuous semigroup $T_t = e^{Kt}$. Moreover, $T_t$ is the only strongly continuous semigroup on $L^1(\mu^{\beta,\nu})$ which has generator that extends $(K, FC_b^\infty)$ (see Appendix A in [16]).
Since $FC_b^\infty$ is a core for the infinitesimal generator of $T_t$ in $L^1(\mu^{\beta,\nu})$ by density from (31), we get that

$$\int K\phi\,d\mu^{\beta,\nu} = 0 \quad \forall \phi \in D(K).$$

This is equivalent to

$$\int T_t\phi\,d\mu^{\beta,\nu} = \int \phi\,d\mu^{\beta,\nu} \quad \forall \phi \in L^1(\mu^{\beta,\nu}) \text{ and } t \geq 0.$$ (34)

Now, we go back to the semigroup $\{P_t\}$; it has been constructed by means of the unique solution $u$ of equation (12) such that $u(t,x) \in H^\alpha$ (for any $t > 0$, $x \in H^\alpha$). On the other hand, the analytical analysis of the Kolmogorov operator has led to the construction of the semigroup $\{T_t\}$; it provides a martingale solution to the stochastic equation (12) (see, e.g., [16] and references therein). By our previous results of Section 3.1 on the stochastic equation (12) we can relate these semigroups and get that the semigroup $\{P_t\}$ can be extended to $L^1(\mu^{\beta,\nu})$ (where this semigroup is exactly $\{T_t\}$).

Henceforth, we denote these semigroups in $C_b(H^\alpha)$ and $L^1(\mu^{\beta,\nu})$ with the same symbol $P_t$. Therefore (34) completes our proof. □

**Remark 3.9** Because of the invariance of the measure $\mu^{\beta,\nu}$, the contraction semigroup $P_t$ in $C_b(H^\alpha)$ can be uniquely extended to a strongly continuous contraction semigroup in $L^p(\mu^{\beta,\nu})$ also for any $p > 1$. Indeed,

$$|P_t\phi(x)|^p = |E[\phi(u(t,x))]|^p \leq E[|\phi(u(t,x))|^p] = P_t|\phi|^p(x)$$

and by the invariance of the measure $\mu^{\beta,\nu}$

$$\int |P_t\phi|^p\,d\mu^{\beta,\nu} \leq \int P_t|\phi|^p\,d\mu^{\beta,\nu} = \int |\phi|^p\,d\mu^{\beta,\nu}.$$  

Since $C_b(H^\alpha)$ is dense in $L^p(\mu^{\beta,\nu})$, we can uniquely define the semigroup on $L^p(\mu^{\beta,\nu})$ for any $p > 1$. We use the same symbol $P_t$ to denote all these semigroups.

Notice that in condition (C5) we require $\int |B_n(x,x)|^2\,d\mu^{\beta,\nu}(dx) < \infty$ for any $n$. Therefore $K : FC_b^\infty \rightarrow L^2(\mu^{\beta,\nu})$. Moreover, according to Corollary 5.3 of [16], we have that the restriction of $T_t$ to $L^2(\mu^{\beta,\nu})$ is a strongly continuous semigroup on $L^2(\mu^{\beta,\nu})$ and the generator of this semigroup again extends $(K, FC_b^\infty)$. In the sequel we will use the same symbol to denote these semigroups in both spaces $L^1(\mu^{\beta,\nu})$ and $L^2(\mu^{\beta,\nu})$.

### 3.2.2 Uniqueness of the invariant measure

Now we prove that equation (12) has at most one invariant measure. Let $R(t,x,\cdot)$ be the law of $z(t;x)$ and $P(t,x,\cdot)$ be the law of $u(t;x)$. Then any $R(t,x,\cdot)$ is equivalent to the Gibbs measure $\mu^{\beta,\nu}$ (see, e.g., [14]); we write it as $R(t,x,\cdot) \sim \mu^{\beta,\nu}$. Moreover we have that

$$\int_0^T |\sqrt{A^{\beta-1}}B(z(t),z(t))|^2\,dt < \infty \quad \mathbb{P} - a.s.$$ (35)
and
\[ \int_0^T |\sqrt{A^{\beta-1}} B(u(t), u(t))|^2 dt < \infty \quad P - a.s. \]

For this use that \(\|B(x, x)\|_{\beta-1} \leq c\|x\|^2\) from assumption (4) and that \(P \{ z \in C([0, T]; H^\alpha) \} = P \{ u \in C([0, T]; H^\alpha) \} = 1\).

According to Theorem 9.2 in [17], (35)-(36) imply that the measure \(P(t, x, \cdot)\) is equivalent to \(R(t, x, \cdot)\). On the other side \(R(t, x, \cdot) \sim R(s, y, \cdot) \sim \mu_{\beta, \nu}\), hence we get
\[ P(t, x, \cdot) \sim P(t, y, \cdot) \sim \mu_{\beta, \nu} \]
for any \(x, y \in H^\alpha\) and \(t > 0\).

Using Doob theorem (see, e.g., Theorem 4.2.1 in [14]), we deduce that \(\mu_{\beta, \nu}\) is the unique invariant measure for equation (12). Moreover, it is strongly mixing
\[ \lim_{t \to \infty} P(t, x, \Gamma) = \mu_{\beta, \nu}(\Gamma) \]
for arbitrary \(x \in H^\alpha\) and Borel set \(\Gamma\) in \(H^\alpha\).

3.2.3 Rate of convergence

Now, we consider the semigroup \(P_t\) in \(L^2(\mu_{\beta, \nu})\) (see Remark 3.9).

We recall the "Carré du champ" identity. For the reader’s convenience we give the proof (see, e.g., [10]).

**Proposition 3.10** We have
\[ \int \phi K \phi \, d\mu_{\beta, \nu} = - \int |\sqrt{A^{1-\beta}} D\phi|^2 \, d\mu_{\beta, \nu} \quad \forall \phi \in D(K). \]

**Proof.** First we take \(\phi \in FC^\infty_b\). A straightforward computation yields that
\[ K \phi^2 = 2\phi K \phi + 2 |\sqrt{A^{1-\beta}} D\phi|^2. \]

By the \(\mu_{\beta, \nu}\)-invariance, we have \(\int K \phi^2 \, d\mu_{\beta, \nu} = 0\); thus
\[ \int \phi K \phi \, d\mu_{\beta, \nu} = - \int |\sqrt{A^{1-\beta}} D\phi|^2 \, d\mu_{\beta, \nu}. \]

Now, taking \(\phi \in D(K)\), we use that \(FC^\infty_b\) is a core for \(K\); therefore there exists a sequence \(\{\phi_n\} \subset FC^\infty_b\) such that
\[ \phi_n \to \phi, K\phi_n \to K\phi \quad \text{in } L^2(\mu_{\beta, \nu}). \]

From (39) we get
\[ \int |\sqrt{A^{1-\beta}} D(\phi_n - \phi_m)|^2 \, d\mu_{\beta, \nu} \leq \int |\phi_n - \phi_m| |K(\phi_n - \phi_m)| \, d\mu_{\beta, \nu}. \]

Hence, the sequence \(\{\sqrt{A^{1-\beta}} D\phi_n\}\) is a Cauchy sequence in \(L^2(\mu_{\beta, \nu})\) and we get \(\phi = \int \phi \, d\mu_{\beta, \nu}\).

Now, given \(\phi \in L^2(\mu_{\beta, \nu})\) we set \(\overline{\phi} = \int \phi \, d\mu_{\beta, \nu}\); then we have the following theorem on the rate of convergence of \(P_t \phi\) as \(t \to \infty\).
Theorem 3.11

\[ \int |P_t \phi(x) - \overline{\phi}|^2 \mu_{\beta,\nu} (dx) \leq e^{-\lambda_1 t} \int |\phi(x) - \overline{\phi}|^2 \mu_{\beta,\nu} (dx) \]

for any \( \phi \in L^2(\mu_{\beta,\nu}) \) and \( t > 0 \).

Proof.

Let us define the space

(40)

\[ L^2_0(\mu_{\beta,\nu}) = \{ \phi \in L^2(\mu_{\beta,\nu}) : \phi = 0 \} ; \]

it is not difficult to prove that it is invariant for the semigroup \( P_t \) (see [11]).

First, let us take \( \phi \in L^2_0(\mu_{\beta,\nu}) \cap D(K) \); then \( P_t \phi \in L^2_0(\mu_{\beta,\nu}) \cap D(K) \) and by the Hille-Yosida theorem

\[ \frac{d}{dt} P_t \phi = KP_t \phi. \]

Therefore, bearing in mind (38)

\[ \frac{1}{2} \frac{d}{dt} \int |P_t \phi|^2 d\mu_{\beta,\nu} = \int P_t \phi KP_t \phi d\mu_{\beta,\nu} = - \int |\sqrt{A^{1-\beta}} D_x P_t \phi|^2 d\mu_{\beta,\nu} \]

Since a Gaussian measure fulfills the spectral gap inequality (see [7]) we have

\[ \int |\sqrt{A^{1-\beta}} D_x P_t \phi(x)|^2 \mu_{\beta,\nu} (dx) \geq \frac{\lambda_1}{2} \int |P_t \phi(x)|^2 \mu_{\beta,\nu} (dx) \]

where \( \lambda_1 > 0 \) is the first eigenvalue of the operator \( A \). By the two latter relationships we get

\[ \frac{d}{dt} \int |P_t \phi|^2 d\mu_{\beta,\nu} \leq -\lambda_1 \int |P_t \phi|^2 d\mu_{\beta,\nu}. \]

Hence, using Gronwall lemma, we have that for any \( t > 0 \)

(41)

\[ \int |P_t \phi|^2 d\mu_{\beta,\nu} \leq e^{-\lambda_1 t} \int |\phi|^2 d\mu_{\beta,\nu} \quad \forall \phi \in L^2_0(\mu_{\beta,\nu}) \cap D(K). \]

Now we take \( \phi \in D(K) \); replacing \( \phi \) with \( \phi - \overline{\phi} \) in (11), we obtain that

\[ \int |P_t \phi - \overline{\phi}|^2 d\mu_{\beta,\nu} = \int |P_t (\phi - \overline{\phi})|^2 d\mu_{\beta,\nu} \leq e^{-\lambda_1 t} \int |\phi - \overline{\phi}|^2 d\mu_{\beta,\nu}. \]

Using that \( D(K) \) is dense in \( L^2(\mu_{\beta,\nu}) \) we get the result. \( \square \)

4 An example: shell models of turbulence

Shell models of turbulence describe the evolution of complex Fourier-like components of a scalar velocity field. Here we present the details for the SABRA shell model (see [20]), but the same results hold for the GOY shell model (see [18, 21]). In recent years there has been an increasing interest in these fluid dynamical models, both for the deterministic and the stochastic case (see also [9], [4], [6], [8]). They are easier to analyze than the Navier-Stokes or Euler...
equations, but they retain many important features of the true hydrodynamical models.

Instead of dealing with complex valued unknowns we deal with the real and imaginary part of each component of the scalar velocity field (for the basic settings we follow [5]); this defines a sequence \( \{u_n\}_n \) with \( u_n \in \mathbb{R}^2 \). For \( x = (x_1, x_2) \in \mathbb{R}^2 \) we set \( |x|^2 = x_1^2 + x_2^2 \) and the scalar product in \( \mathbb{R}^2 \) is \( x \cdot y = x_1 y_1 + x_2 y_2 \).

Then, using the notations of Section 2.1, we define the basic space \( H \) as

\[
H = \{ u = (u_1, u_2, \ldots) \in (\mathbb{R}^2)^\infty : \sum_{n=1}^{\infty} |u_n|^2 < \infty \}.
\]

The basis in \( H \) in given by the sequence \( \{e_n^{(1)}, e_n^{(2)}, \ldots\} \) of elements of \( (\mathbb{R}^2)^\infty \), where

\[
e_n^{(1)} = ((0, 0), \ldots, (0, 0), (1, 0), (0, 0), \ldots)
\]

\[
e_n^{(2)} = ((0, 0), \ldots, (0, 0), (0, 1), (0, 0), \ldots)
\]

with the nonvanishing vectors in place \( n \).

The eigenvalues are

\[
\lambda_n = k_n^2 \lambda^{2n}
\]

with \( \lambda > 1 \). Hence we can take any \( \alpha < \beta \) to fulfil (C3). Inequality (3) holds with \( \epsilon_{p, \nu} = (\frac{\kappa_{\nu}}{\kappa_{p}})^\rho \).

We set \( k_n = \sqrt{\lambda_n} \). The bilinear term \( B \) is defined by means of the components \( B_n = (B_{n,1}, B_{n,2}) \) as follows (see, e.g., [5]):

\[
B_{1,1}(u, v) = ak_2[\bar{u}_{2,2}v_3,1 + u_{2,1}v_3,2]
\]

\[
B_{1,2}(u, v) = -ak_2u_3 \cdot v_3
\]

\[
B_{2,1}(u, v) = ak_3[-u_3,2v_4,1 + u_3,1v_4,2] + bk_2[-u_{1,2}v_3,1 + u_{1,1}v_3,2]
\]

\[
B_{2,2}(u, v) = -ak_3v_3 \cdot v_4 - bk_2u_1 \cdot v_3
\]

and for \( n > 2 \)

\[
B_{n,1}(u, v) = ak_{n+1}[-u_{n+1,2}v_{n+2,1} + u_{n+1,1}v_{n+2,2}]
\]

\[+ bk_n[-u_{n-1,2}v_{n-1,1} + u_{n-1,1}v_{n-1,2}]
\]

\[+ ak_{n-1}[u_{n-2,2}v_{n-1,1} + u_{n-2,1}v_{n-1,2}]
\]

\[+ bk_{n-1}[u_{n-1,2}v_{n-1,1} + u_{n-1,1}v_{n-1,2}]
\]

\[
B_{n,2}(u, v) = -ak_{n+1}[u_{n+1,1}v_{n+2,1} + u_{n+1,2}v_{n+2,2}]
\]

\[+ bk_n[u_{n-1,1}v_{n-1,1} + u_{n-1,2}v_{n-1,2}]
\]

\[+ ak_{n-1}[u_{n-2,1}v_{n-1,1} + u_{n-2,1}v_{n-1,2}]
\]

\[+ bk_{n-1}[u_{n-1,2}v_{n-1,1} + u_{n-1,1}v_{n-1,2}]
\]

where \( a \) and \( b \) are real numbers such that

\[
a + b\lambda^{2\beta} = (a + b)\lambda^{4\beta}
\]

for some \( \beta > 0 \), that is

\[
\lambda^{2\beta} = \frac{a}{a + b}
\]

(47)
(recall that $\lambda > 1$). This condition implies (C2 iii), whereas (C2 ii) holds for any real $a$ and $b$. For instance, let us check that (10) implies (C2 iii). We have

$$\sum_{n=1}^{\infty} k_n^{2\beta} B_n(u, u) \cdot u_n$$

$$= \sum_{n=1}^{\infty} k_n^{2\beta} [B_{n,1}(u, u)u_{n,1} + B_{n,2}(u, u)u_{n,2}]$$

$$= \sum_{n=1}^{\infty} [a + b\lambda^{2\beta} - (a + b)\lambda^{4\beta}] k_n^{2\beta+1} (u_{n+2} \cdot u_n) (u_{n+1,2} + u_{n+1,1}).$$

Moreover we have (see (5))

**Lemma 4.1** For any $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$

$$B : H^{\alpha_1} \times H^{\alpha_2} \rightarrow H^{-\alpha_3} \text{ with } \alpha_1 + \alpha_2 + \alpha_3 \geq 1$$

and there exists a constant $c$ (depending on $a, b, \lambda$ and the $\alpha_j$'s) such that

$$\|B(u, v)\|_{-\alpha_3} \leq c\|u\|_{\alpha_1}\|v\|_{\alpha_2} \quad \forall u \in H^{\alpha_1}, v \in H^{\alpha_2}.$$

This implies that conditions (C4) are true: (4) for any $\beta > \frac{1}{2}$; this includes the interesting physical case of $\beta = 1$ (see Section 2.3). Indeed, for the SABRA shell model

$$B^M_{n,1}(x, x) - B_{n,1}(x, x) = \begin{cases} 0 & \text{for } n \leq M - 2 \\ -a k_n (x_{M,1} x_{M,1,2} - x_{M,2} x_{M,1,1}) & \text{for } n = M - 1 \\ -a k_{n+1} (x_{M,1} x_{M,2,1} - x_{M,1} x_{M,1,1}) & \text{for } n = M - 2 \\ -b k_n (x_{M-1,1} x_{M,1,2} - x_{M-1,2} x_{M,1,1}) & \text{for } n = M \\ -b k_{n+1} (x_{M-1,1} x_{M,2,1} - x_{M-1,2} x_{M,1,1}) & \text{for } n = M + 1 \\ -b k_{n+2} (x_{M-2,1} x_{M,2,1} - x_{M-2,2} x_{M,1,1}) & \text{for } n = M + 2 \end{cases}$$

and

$$B^M_{n,2}(x, x) - B_{n,2}(x, x) = \begin{cases} 0 & \text{for } n \leq M - 2 \\ -a k_n (-x_{M,1} x_{M,1,1} - x_{M,2} x_{M,1,2}) & \text{for } n = M - 1 \\ -a k_{n+1} (-x_{M,1} x_{M,2,1} - x_{M+1,2} x_{M,2,1}) & \text{for } n = M - 2 \\ -b k_n (-x_{M-1,1} x_{M,1,2} - x_{M-1,2} x_{M,1,1}) & \text{for } n = M \end{cases}$$

Therefore

$$\sum_{n=1}^{M} |B^M_n - B_n|^2 = |B^M_{M-1} - B_{M-1}|^2 + |B^M_M - B_M|^2$$

so

$$\lim_{M \rightarrow \infty} \int \sum_{n=1}^{M} |B^M_n - B_n|^2 d\mu^{\beta, \nu} \leq \lim_{M \rightarrow \infty} \frac{8}{\nu^2} \frac{a^2}{\lambda^{2\beta}} k_M^{2-4\beta} + \frac{6^2}{\lambda^{2\beta}} k_{M+1}^{2-4\beta} + b^2 k_M^{2-4\beta} = 0.$$

This holds for $\beta > \frac{1}{2}$.

We finally point out that our results of Section 3.2 hold also in any space $L^p(\mu^{\beta, \nu})$ with $p = 1, 2, \ldots$ (see Remark 3.9). Indeed, we have

\begin{equation}
\int |B_n(x, x)|^q \mu^{\beta, \nu}(dx) < \infty \quad \forall n, q \in \mathbb{N}.
\end{equation}
5 Inviscid models

We are interested in the deterministic inviscid and unforced dynamics represented by equation (13). Here we present our results for the SABRA shell model with $\beta = 1$ (the physical relevant case) only to make simpler the exposition, but it can be generalized to the other fluid dynamic models.

Equation (13) is formally obtained from equation (12) setting $\nu = 0$ and considering a vanishing right hand side. More generally we can consider the nonlinear viscous equation

$$(49) \quad du^\varepsilon(t) + [\nu Au^\varepsilon(t) + B(u^\varepsilon(t), u^\varepsilon(t))]dt = \sqrt{\varepsilon} dw(t), \quad t > 0.$$ 

with $\varepsilon > 0$. When $\varepsilon = 0$ we get equation (13) (with $\beta = 1$). Our results of the previous sections hold true for any $\varepsilon > 0$.

The fact that the measure $\mu^{1,\nu}$ is an invariant measure for any $\varepsilon > 0$ can be easily checked. We proceed as in the previous section, but now the Kolmogorov operator associated to equation (49) is $K^\varepsilon = \varepsilon Q + L$; bearing in mind (32) and (33) we get that $\mu^{1,\nu}$ is an infinitesimal invariant measure for the operator $(K^\varepsilon, FC^\varepsilon_b)$. And for any $\varepsilon > 0$ the operator $(K^\varepsilon, FC^\varepsilon_b)$ is dissipative.

We are going to prove that when the initial data is a random variable with law $\mu^{1,\nu}$, then equation (13) has a solution which is a stationary random process, whose law at any fixed time is $\mu^{1,\nu}$.

An important property is the integrability of $B$ with respect to the measure $\mu^{1,\nu}$.

**Proposition 5.1** If $\nu > 0$, then for any $\alpha < 1$ we have

$$\int \|B(x, x)\|_p \mu^{1,\nu}(dx) < \infty$$

for any $p \in \mathbb{N}$.

**Proof.** We write the proof for $p = 2$ but it is the same for the other values of $p$, since $\mu^{1,\nu}$ is Gaussian and the $B_n$’s are second order polynomial. We have

$$\int |B_{n,1}(x, x)|^2 \mu^{1,\nu}(dx) = \int |ak_{n+1}[-x_{n+1,2}x_{n+2,1} + x_{n+1,1}x_{n+2,2}]$$

$$+ bk_n[-x_{n-1,2}x_{n+1,1} + x_{n-1,1}x_{n+1,2}]$$

$$+ (a + b)k_{n-1}[x_{n-1,2}x_{n-2,1} + x_{n-1,1}x_{n-2,2}]|^2 \mu^{1,\nu}(dx)$$

$$\leq 2 \int (a^2k_{n+1}[x_{n+1,2}^2x_{n+2,1} + x_{n+1,1}^2x_{n+2,2}]$$

$$+ b^2k_n[x_{n-1,2}^2x_{n+1,1} + x_{n-1,1}^2x_{n+1,2}]$$

$$+ (a + b)^2k_{n-1}[x_{n-1,2}^2x_{n-2,1} + x_{n-1,1}^2x_{n-2,2}]]\mu^{1,\nu}(dx)$$

$$= \frac{16}{\nu^2} \{a^2\lambda^{-4} + b^2 + (a + b)^2\lambda^4\} \lambda^{-2n}.$$
Similarly we estimate \( \int |B_n(x,x)|^2 \mu^{1,\nu}(dx) \). Therefore

\[
\int \|B(x,x)\|^2 \mu^{1,\nu}(dx) = \int \sum_{n=1}^{\infty} \lambda_n^2 |B_n(x,x)|^2 \mu^{1,\nu}(dx)
\leq c_{\nu,k_0,\lambda}(|a|^2 + |b|^2) \sum_{n=1}^{\infty} \lambda^{2n(\alpha-1)}
\]

which is finite if \( \alpha < 1 \).

\( \blacksquare \)

Here is our main result.

**Theorem 5.2** For any \( \nu > 0 \), there exists a \( \mu^{1,\nu} \)-stationary process, whose paths solve equation (13) \( P \)-a.s. In particular, the paths are in \( C^\delta(\mathbb{R};H^\alpha) \) (for any \( 0 \leq \delta < \frac{1}{2} \) and \( \alpha < 1 \)).

**Proof.** We fix \( \nu > 0 \) arbitrarily. According to Corollary \ref{cor:existence}, equation (49) has a unique \( \mu^{1,\nu} \)-stationary solution \( v \in \mathbb{E} \); this process is a strong solution and has paths in \( C([0,\infty);H^\alpha) \) a.s. (for \( \alpha < 1 \), but we always think of \( \alpha \) as much close to 1 as possible).

First, we prove that the sequence \( \{v_{\varepsilon}\}_{0 < \varepsilon \leq 1} \) is tight in \( C_{\tilde{\delta}}([0,T];H^\tilde{\alpha}) \) for any \( \tilde{\delta} \in (0,\frac{1}{2}) \) and \( \tilde{\alpha} < \alpha \).

We write equation (49) in the mild form:

\[
\bar{v}(t) = \bar{v}(t) - \int_0^t e^{-\nu \varepsilon A(t-s)} B(v(s),v(s))ds,
\]

where

\[
\bar{v}(t) = e^{-\nu \varepsilon A(t)} v(0) + \int_0^t e^{-\nu \varepsilon A(t-s)} \sqrt{2\varepsilon} dw(s)
\]

is the \( \mu^{1,\nu} \)-stationary solution of the linear equation

\[
dz^\varepsilon(t) + \nu \varepsilon Az^\varepsilon(t)dt = \sqrt{2\varepsilon} dw(t)
\]

with the initial data of law \( \mu^{1,\nu} \).

We consider the two terms in the right hand side of (50). Using the \( \mu^{1,\nu} \)-stationarity we have that for any \( 0 \leq \delta < \frac{1}{2} \) there exists a constant \( C_\delta > 0 \) such that

\[
\sup_{0 < \varepsilon \leq 1} \mathbb{E}[\|z^\varepsilon\|_{C^\delta([0,T];H^\alpha)}] \leq C_\delta.
\]

We take \( \eta \in (0,1) \) and set \( \gamma = \alpha - 2\eta \). For the convolution integral in (50) we have
\[ (52) \quad \left\| \int_0^T e^{-\nu e^{A(t-s)}} B(\varphi(s), \varphi'(s)) ds \right\|_{W^{1,p}(0,T; H^\gamma)}^p \\
\quad = \int_0^T \left( \int_0^t e^{-\nu e^{A(t-s)}} B(\varphi(s), \varphi'(s)) ds \right)^p dt + \int_0^T \left\| B(\varphi(t), \varphi'(t)) \right\|_{\gamma}^p dt \\
\quad + \int_0^T \nu \mathbb{E} \left( \int_0^t \| A e^{-\nu e^{A(t-s)}} B(\varphi(s), \varphi'(s)) \|_{\gamma}^p ds \right) dt \\
\quad \leq \int_0^T t^{p-1} \left( \int_0^t e^{-\nu e^{A(t-s)}} B(\varphi(s), \varphi'(s)) ds \right)^p dt + \int_0^T \left\| B(\varphi(t), \varphi'(t)) \right\|_{\gamma}^p dt \\
\quad + \nu \int_0^T \left( \int_0^t \| A e^{-\nu e^{A(t-s)}} B(\varphi(s), \varphi'(s)) \|_{\gamma}^p ds \right) dt \\
\quad \leq \int_0^T t^{p-1} \left( \int_0^t \left\| B(\varphi(s), \varphi'(s)) \right\|_{\gamma}^p ds \right) dt + \int_0^T \left\| B(\varphi(t), \varphi'(t)) \right\|_{\gamma}^p dt \\
\quad + \nu \int_0^T \left( \int_0^t \| A^{1-\eta} e^{-\nu e^{A(t-s)}} A B(\varphi(s), \varphi'(s)) \|_{\gamma}^p ds \right) dt \\
\quad \leq \left( \frac{1}{p} T^p + 1 \right) \int_0^T \left\| B(\varphi(t), \varphi'(t)) \right\|_{\gamma}^p dt + \nu \int_0^T \left( \int_0^t \mathbb{E} \left\| B(\varphi(s), \varphi'(s)) \right\|_{\gamma}^p ds \right) dt 
\]

For the latter integral we use Hölder inequality and get that
\[
\left( \int_0^t \mathbb{E} \left\| B(\varphi(s), \varphi'(s)) \right\|_{\gamma}^p ds \right)^{\frac{1}{p}} \leq \left( \int_0^t ds \right)^{\frac{1}{p}} \left( \int_0^t \mathbb{E} \left\| B(\varphi(s), \varphi'(s)) \right\|_{\gamma}^{p-1} ds \right)^{\frac{1}{p-1}}.
\]

Hence, for \( p > \frac{2}{\eta} - 1 \) we have
\[(53) \quad \left\| \int_0^T e^{-\nu e^{A(t-s)}} B(\varphi(s), \varphi'(s)) ds \right\|_{W^{1,p}(0,T; H^\gamma)}^p \\
\quad \leq \left( \frac{1}{p} T^p + 1 \right) \int_0^T \left\| B(\varphi(t), \varphi'(t)) \right\|_{\gamma}^p dt + \nu T^m \int_0^T \left\| B(\varphi(t), \varphi'(t)) \right\|_{\gamma}^p dt
\]

for some positive constant \( m = m_{\eta,\nu,p} \).

Integrating with respect to the measure \( \mu^{\beta,\nu} \) and using the invariance we get
\[(54) \quad \mathbb{E} \left\| \int_0^T e^{-\nu e^{A(t-s)}} B(\varphi(s), \varphi'(s)) ds \right\|_{W^{1,p}(0,T; H^\gamma)}^p \\
\quad \leq T \left( 1 + \frac{1}{p} T^p + \nu T^m \right) \int \| B(x,x) \|_{\alpha}^p \mu^{\beta,\nu}(dx)
\]

Now, we use that \( W^{1,p}(0,T) \subset C^\delta([0,T]) \) if \( 1 - \frac{1}{p} > \delta \). Then, using the previous estimates in \(50\), given any \( 0 \leq \delta < \frac{1}{2}, \; p > \frac{1}{1-\eta} \) and \( p > \frac{2}{\eta} - 1 \) we have
\[(55) \quad \sup_{0 < \epsilon \leq 1} \mathbb{E} \left\| \varphi \right\|_{C^\delta([0,T]; H^\gamma)}^p < \infty.
\]

On the other hand, the space \( C^\delta([0,T]; H^\gamma) \) is compactly embedded in \( C^\delta([0,T]; H^{\hat{\gamma}}) \) if \( \hat{\gamma} < \gamma \); this follows from the compact embedding \( H^\gamma \subset H^{\hat{\gamma}} \) and
from the Ascoli-Arzelà theorem. Because these results hold for any \( \delta \in [0, \frac{1}{2}) \) and \( \tilde{\gamma} < \gamma < \alpha < 1 \) (with \( p \) big enough, but we use (48)), we can consider any \( \tilde{\delta} < \frac{1}{2} \) and any \( \tilde{\gamma} < 1 \). The tightness follows from (55) as usual by means of Chebyshev inequality. And to simplify notation henceforth we consider the tightness in the space \( C^{\delta}([0, T]; H^\alpha) \) (\( \delta < \frac{1}{2} \) and \( \alpha < 1 \)).

By the tightness result and Prohorov theorem, the sequence of the laws of \( v^\varepsilon \) has a subsequence \( \{v^\varepsilon_n\}_{n=1}^{\infty} \) weakly convergent as \( n \to \infty \) (with \( \varepsilon_n \to 0 \)) in \( C^{\delta}([0, T]; H^\alpha) \). By Skorohod theorem, there exist a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\), a random variable \( \tilde{v} \) and a sequence \( \{\tilde{v}^\varepsilon\} \) such that law(\( \tilde{v}^\varepsilon \))=law(\( v^\varepsilon \)), law(\( \tilde{v} \))=\( \mu^1,\nu \) and \( \tilde{v}^\varepsilon \) converges to \( \tilde{v} \) a.s. in \( C^{\delta}([0, \infty); H^\alpha) \).

We now identify the equation satisfied by \( \tilde{v} \). We are going to prove that \( \tilde{\mathbb{P}} \)-almost each path solves (13).

It is enough to control the behavior of the terms with \( B \). First

\[
e^{-\nu \varepsilon A(t-s)} B(\tilde{v}^\varepsilon(s), \tilde{v}^\varepsilon(s)) - B(\tilde{v}^\varepsilon(s), \tilde{v}^\varepsilon(s))
\]

\[
e^{-\nu \varepsilon A(t-s)} [B(\tilde{v}^\varepsilon(s), \tilde{v}^\varepsilon(s)) - B(\tilde{v}^\varepsilon(s), \tilde{v}^\varepsilon(s))]
\]

\[
+ [e^{-\nu \varepsilon A(t-s)} - I] B(\tilde{v}^\varepsilon(s), \tilde{v}^\varepsilon(s)).
\]

When we consider the second addend in the mild form expression, it trivially converges to zero; but for the convergence of the first one it is enough to verify that

\[
\int_0^t \|B(\tilde{v}^\varepsilon(s), \tilde{v}^\varepsilon(s)) - B(\tilde{v}^\varepsilon(s), \tilde{v}^\varepsilon(s))\|_{\alpha-1} ds \to 0
\]
as \( \varepsilon \to 0 \); for this we use the bilinearity and the estimate (10).

Similarly we work on the time interval \([-T, 0]\) by considering the reversed-time parabolic nonlinear equation

\[
(56) \quad du^\varepsilon(t) + [-\nu \varepsilon Au^\varepsilon(t) + B(u^\varepsilon(t), u^\varepsilon(t))] dt = \sqrt{2\varepsilon} dw(t), \quad t < 0
\]

It has a unique \( \mu^1,\nu \)-stationary solution \( u^\varepsilon \); this process is a strong solution, has paths in \( C^{\delta}((\infty, 0]; H^\alpha) \). The tightness and the convergence are obtained in the same way as above.

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2D hydrodynamical systems: invariant measures of Gaussian type

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Abstract

Gaussian measures $\mu^{\beta,\nu}$ are associated to some stochastic 2D hydro-
dynamical systems. They are of Gibbsian type and are constructed by
means of some invariant quantities of the system depending on some pa-
rameter $\beta$ (related to the 2D nature of the fluid) and the viscosity $\nu$. We
prove the existence and the uniqueness of the global flow for the stochastic
viscous system; moreover the measure $\mu^{\beta,\nu}$ is invariant for this flow and
is unique. Finally, we prove that the deterministic inviscid equation has
a $\mu^{\beta,\nu}$-stationary solution (for any $\nu > 0$).

1 Introduction

The goal in this paper is to study a class of mathematical models related to 2D
fluids. We will deal with an abstract stochastic evolution equation in a Hilbert
space of the following form

$$du(t) + [\nu Au(t) + B(u(t), u(t))] dt = \sqrt{Q}dw(t),$$

where $w$ is a cylindrical Wiener process and $Q$ is a linear operator. The un-
bounded linear operator $A$ and the bilinear operator $B$ will satisfy certain prop-
erties related to 2D fluids that will be given in details in the following sections.
The coefficient $\nu \geq 0$ is the viscosity. There is an extensive literature about the
existence and uniqueness of solutions with initial data of finite energy. Its long
time behavior has also been extensively studied, including the existence and
uniqueness of invariant measures (see, e.g., [3] and the reference therein). In
the present paper, we are interested in the qualitative behavior of these invari-
ant measures. In particular, we prove the existence and uniqueness of invariant
measures of Gaussian type for the viscous case $[1]$; moreover, this Gaussian
measure is proved to be invariant also for the deterministic and inviscid model
($\nu = 0, Q = 0$).

We point out that the Gaussian invariant measure that we consider here is
not that one considered in previous papers [1, 12, 15, 2, 5], but has a more
regular support. In particular, the support of this measure is a Sobolev space
of positive exponent.

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As far as the content of this paper, in Section 2 we introduce the operators associated to the model with their properties and the Gibbs measures $\mu^{\beta,\nu}$. We introduce the Ornstein-Uhlenbeck equation with a suitable noise and prove that $\mu^{\beta,\nu}$ is its unique invariant. In Section 3, we deal with the viscous stochastic case; we prove the existence and uniqueness of strong solutions and that $\mu^{\beta,\nu}$ is its unique invariant measure. The uniqueness of the invariant measure is proved by means of Girsanov theorem. Moreover, some ergodic properties of this measure with its rate of convergence are shown. In Section 4, we introduce a particular example, shell models of turbulence with an emphasis on the Sabra model. The coefficient $\beta$ characterizing the measure $\mu^{\beta,\nu}$ will be related to the coefficients $a$ and $b$ of the Sabra model through the condition (54). Section 5 is devoted to the deterministic inviscid model, in particular we present our results for the inviscid Sabra model with $\beta = 1$. For any $\nu > 0$ we prove the existence of a stationary process whose law at any fixed time is $\mu^{1,\nu}$.

2 Introduction to the model and functional setting

2.1 Operators and spaces

Let $(H, | \cdot |)$ be a real separable Hilbert space endowed with an inner product denoted by $(\cdot, \cdot)$, and $A$ an unbounded self-adjoint positive linear operator on $H$ with compact resolvent. We denote by $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ the eigenvalues of $A$ and by $e_1, e_2, \ldots$ a complete orthonormal system in $H$ given by the eigenfunctions of the operator $A$

$$Ae_n = \lambda_n e_n$$

We assume that $\lim_{n \to \infty} \lambda_n = \infty$.

Let $H_n = \text{span}\{e_1, e_2, \ldots, e_n\}$ and $\Pi_n$ the projector operator onto $H_n$.

For any $\alpha \in \mathbb{R}$ we can define the power operators $A^{\alpha}$ as

$$A^{\alpha}x = \sum_{n=1}^{\infty} \lambda_n^{\alpha} (x, e_n)e_n \quad D(A^{\alpha}) = \{ x = \sum_{n=1}^{\infty} x_n e_n : \sum_{n=1}^{\infty} \lambda_n^{2\alpha} x_n^2 < \infty \}.$$ 

We set

$$H^{\alpha} = D(A^{\alpha/2}).$$

Each $H^{\alpha}$ is a Hilbert space with scalar product $(u, v)_{H^{\alpha}} := (A^{\alpha/2} u, A^{\alpha/2} v)$. We denote by $\| \cdot \|_{\alpha}$ the norm in $H^{\alpha}$.

Let $B : H \times H \to H^{-1}$ be a bilinear operator; we assume that there exists a positive constant $c$ such that

$$(2) \quad \| B(u, v) \|_{-1} \leq c |u| |v|.$$ 

We consider the finite dimensional approximation of the bilinear operator $B$; this is the bilinear operator $B^M$ defined as

$$B^M(u, v) = \Pi_M B(\Pi_M u, \Pi_M v)$$

for any $M \in \mathbb{N}$. For each $B^M$ we have the same estimate as (2) (with the constant $c$ independent of $M$).

For any $\nu > 0$ and $\beta > 0$, let $\mu^{\beta,\nu}$ be the Gaussian measure $\mathcal{N}(0, \frac{1}{\beta} A^{-\beta})$ (see, e.g., [19], [13]).
2.2 Assumptions

Besides the basic properties of the operators $A$ and $B$ given above, we present other important assumptions.

**Condition (C1):** For any $\nu > 0$, the operator $\nu A$ generates an analytic semigroup of contractions in $H$ and for any $p > 0$ there exists $c_{p,\nu} > 0$ such that

$$|A^p e^{-\nu A t} x| \leq \frac{c_{p,\nu}}{t^p} |x| \quad \forall t > 0, x \in H.$$  

**Condition (C2):** The bilinear operator $B$ satisfies the following properties:

(i) $\langle B(u, v), w \rangle = -\langle B(u, w), v \rangle$

(ii) $\langle B(u, v), v \rangle = 0$

(iii) $\exists \beta > 0$ such that $\langle B(u, u), A^\beta u \rangle = 0$ for any $u, v$ giving meaning to the above relationships.

**Condition (C3):** There exists $\alpha \in [0, \beta)$ (with $\beta$ given by (C2 iii)) such that the embedding $H^\beta \subset H^\alpha$ is Hilbert-Schmidt, i.e.

$$\sum_{n=1}^{\infty} \lambda_n^{\alpha-\beta} < \infty.$$  

**Condition (C4):** for $\alpha$ and $\beta$ given in (C2)-(C3), $B : H^\alpha \times H^\alpha \to H^{\beta-1}$ is a continuous operator, i.e.

$$\|B(u, v)\|_{\beta-1} \leq c \|u\|_{\alpha} \|v\|_{\alpha} \quad \forall u, v \in H^\alpha.$$  

Moreover, if $\alpha > 0$ we assume

$$\|B(u, v)\|_{\alpha-1} \leq c \|u\|_{\alpha} \|v\|_{\alpha} \quad \forall u \in H, v \in H^\alpha.$$  

**Condition (C5):** For each $n$ set $B_n(u, v) = \langle B(u, v), e_n \rangle$. Then we have

$$\int |B_n(x, x)|^2 \mu^{\beta,\nu}(dx) < \infty \quad \forall n$$

and $B_n(x, x)$ independent of $x_n$ (where $x = \sum_n x_n e_n$). Moreover, we require $\beta \leq 1$ and

$$\lim_{M \to \infty} \sum_{n=1}^{M} \int |\langle B^M(x, x) - B(x, x), e_n \rangle|^2 \mu^{\beta,\nu}(dx) = 0.$$  

**Remark 2.1 (i) We have the relationships corresponding to assumption (C2):**

(7) $\langle B^M(u, v), w \rangle = -\langle B^M(u, w), v \rangle$

(8) $\langle B^M(u, v), v \rangle = 0$

(9) $\langle B^M(u, u), A^\beta u \rangle = 0$

(ii) By means of the bilinearity and of estimate (1) we have

$$\lim_{M \to \infty} \|B^M(u, v) - B(u, v)\|_{\beta-1} = 0 \quad \forall u, v \in H^\alpha.$$
Since $\alpha \geq 0$, the inequality \(5\) implies
\[
\|B(u, v)\|_{\alpha - 1} \leq c\|u\|_\alpha\|v\|_\alpha \quad \forall u, v \in H^\alpha.
\]
Moreover,
\[
\lim_{M \to \infty} \|B_M(u, v) - B(u, v)\|_{\alpha - 1} = 0 \quad \forall u, v \in H^\alpha.
\]

Assumption (C3) implies that the space $H^\alpha$ has full measure $\mu^{\beta, \nu}$, i.e. $\mu^{\beta, \nu}(H^\beta) = 1$. However, for Gaussian measures in infinite dimensional spaces we have $\mu^{\beta, \nu}(H^\beta) = 0$ (see, e.g., [19]).

We denote by $L^p(\mu^{\beta, \nu})$ the space of measurable functions $\phi$ defined in the support of the measure $\mu^{\beta, \nu}$ and such that $\int |\phi|^p \, d\mu^{\beta, \nu} < \infty$.

2.3 The equations

Set $Q = 2A^{1-\beta}$ in (1), that is we consider the following nonlinear stochastic equation
\[
du(t) + [\nu Au(t) + B(u(t), u(t))] \, dt = \sqrt{2A^{1-\beta}} \, dw(t).
\]
In addition we deal with the inviscid and deterministic equation
\[
\frac{du}{dt}(t) + B(u(t), u(t)) = 0
\]
and with the viscous linear stochastic equation
\[
dz(t) + \nu Az(t) \, dt = \sqrt{2A^{1-\beta}} \, dw(t).
\]

Relationship (ii) in Assumption (C2) implies a formal law of conservation of energy $E(t) = \frac{1}{2}||u(t)||^2$ in equation (13). We recall that the energy is a conserved quantity in the motion of incompressible inviscid fluids.

Relationship (iii) in Assumption (C2) implies that $S_\beta(t) = \frac{1}{2}||u(t)||^2_\beta$ is a conserved quantity for equation (13), that is formally we have
\[
\frac{dS_\beta}{dt}(t) = (\dot{u}(t), A^\beta u(t)) = -(B(u(t), u(t)), A^\beta u(t)) = 0.
\]
For $\beta = 1$, $S_1$ is the enstrophy which is a conserved quantity in the motion of 2D incompressible inviscid fluids.

The Gaussian measure $\mu^{\beta, \nu} = \mathcal{N}(0, \frac{1}{2}A^{-\beta})$ can be described heuristically as
\[
\mu^{\beta, \nu}(du) = \frac{1}{Z}e^{-\nu S_\beta(u)} \, du
\]
where $Z$ is a normalization constant to make $\mu^{\beta, \nu}$ to be a probability measure. Therefore it makes sense to see if the measure $\mu^{\beta, \nu}$, described by means of the invariant quantity $S_\beta$, is a stationary statistical solution for the inviscid equation (13). To this end, we will first prove that $\mu^{\beta, \nu}$ is a stationary measure for the viscous and stochastic equation (12) looking for a dynamics in the space $H^\alpha$ of full measure $\mu^{\beta, \nu}$. However, the basic stochastic case to deal with is the linear equation (14) for which we recall well known properties (see [13]).
Proposition 2.2 Let assumptions (C1), (C2 iii) and (C3) be satisfied. Then, for any \( z(0) \in H^\alpha \) there exists a unique strong solution to equation (14) such that
\[
z \in C([0,T]; H^\alpha) \quad \mathbb{P} - a.s.
\]
The stationary process solving equation (14) is
\[
\zeta(t) = \sqrt{2} \int_{-\infty}^{t} e^{-\nu(t-s)} A \frac{\beta}{2} dw(s)
\]
and the law of \( \zeta(t) \) is \( \mu^{\beta,\nu} \) for any time \( t \).

3 Stochastic viscous models

We consider equation (12); first we prove that there exists a unique solution for any initial data in \( H^\alpha \). The solution is strong in the probabilistic sense and uniqueness is in pathwise sense. Moreover, we show that \( \mu^{\beta,\nu} \) is the unique invariant measure associated to this stochastic equation.

3.1 Strong solution

We look for dynamics in the state space \( H^\alpha \) with \( 0 \leq \alpha < \beta \) fulfilling assumptions (C1)-(C4). We consider any finite time interval \([0,T]\).

Theorem 3.1 Let assumptions (C1), (C2), (C3) and (C4) be satisfied. Then, for any \( u(0) \in H^\alpha \), there exists a unique solution \( u \) to equation (12) such that
\[
u \in C([0,T]; H^\alpha) \quad \mathbb{P} - a.s.
\]
Moreover, the process \( u \) is a Markov process, Feller in \( H^\alpha \).

We divide the proof in three steps in the following subsections.

3.1.1 Existence of strong solutions

We use a well known trick to study a stochastic semilinear equation with additive noise: we set \( v = u - z \). Then
\[
\frac{dv}{dt}(t) + \nu Av(t) + B(v(t) + z(t), v(t) + z(t)) = 0
\]
with \( v(0) = u(0) - z(0) \). Set \( z(0) = 0 \).

Proposition 3.2 We consider the same assumptions as in Theorem 3.1. Let \( v(0) \in H^\alpha \). Then there exists a solution to equation (15) such that
\[
v \in C([0,T]; H^\alpha) \cap L^2(0,T; H^{1+\alpha}) \quad \mathbb{P} - a.s.
\]
**Proof.** We proceed pathwise. Take the scalar product of the left hand side of equation (15) with \( v \) in \( H \); we get some a priori estimates

\[
\frac{1}{2} \frac{d}{dt} \|v\|^2 + \nu \|v\|_1^2 = -(B(v + z, v + z), v)
\]

\[
= - (B(v + z, z), v) \quad \text{(C2 ii)}
\]

\[
\leq \|B(v + z, z)\|_{-1} \|v\|_1
\]

\[
\leq c \|v + z\|_2 \|v\|_1 \quad \text{by (2)}
\]

\[
\leq \frac{\nu}{2} \|v\|_1^2 + \frac{\nu}{2} \|v\|_2^2 + \frac{\nu}{2} \|z\|^2
\]

by Young inequality, for some positive constant \( c \). Henceforth, we denote by \( c_\nu \) a generic constant depending on \( \nu \).

Therefore

\[
\frac{d}{dt} \|v\|_1^2 \leq c_\nu \|v\|_2^2 + c_\nu \|z\|^2.
\]

Hence, Gronwall inequality applied to

\[
\frac{d}{dt} \|v\|_1^2 \leq c_\nu \|v\|_2^2 + c_\nu \|z\|^2
\]

gives

\[
\sup_{0 \leq t \leq T} (\|v(t)\|_1^2) \leq e^{c_\nu T \|v\|_2^2} (\|v(0)\|_1^2 + c_\nu T \|v\|_1^2) < \infty
\]

and integrating in time (16)

\[
\nu \int_0^T \|v(s)\|_2^2 ds \leq \|v(0)\|^2 + T c_\nu \left( \|v\|_2^2 + \|v\|_2^2 \right)
\]

Moreover, when \( \alpha \geq 0 \) we proceed in a similar way: we take the scalar product of the left hand side of equation (15) with \( A^\alpha v \) in \( H \); then

\[
\frac{1}{2} \frac{d}{dt} \|v\|_\alpha^2 + \nu \|v\|_\alpha^2 = -(A^{\frac{1+\alpha}{2}} B(v + z, v + z), A^{\frac{1+\alpha}{2}} v)
\]

\[
\leq \|B(v + z, v + z)\|_{1+\alpha} \|v\|_{1+\alpha}
\]

\[
\leq c \|v + z\|_2 \|v\|_{1+\alpha} \quad \text{by (5)}
\]

\[
\leq \frac{\nu}{2} \|v\|_{1+\alpha}^2 + \frac{\nu}{2} \|v\|_2^2 + \frac{\nu}{2} \|z\|_\alpha^2
\]

This gives

\[
\frac{d}{dt} \|v\|_\alpha^2 + \nu \|v\|_\alpha^2 \leq c_\nu \left( \|v\|^2 + \|z\|^2 \right) \|v\|_\alpha^2 + c_\nu \|z\|_\alpha^4.
\]

Therefore, using (15) and the fact that \( \alpha \geq 0 \) we get

\[
\sup_{0 \leq t \leq T} \|v(t)\|_\alpha^2 \leq \|v(0)\|_\alpha^2 e^{c_\nu \int_0^T (\|v(t)\|^2 + \|z(t)\|^2) dt}
\]

\[
+ c_\nu \int_0^T e^{\frac{\nu}{2} \int_0^t (\|v(s)\|^2 + \|z(s)\|^2) ds} \|z(t)\|_\alpha^4 dt < \infty
\]
and integrating in time \( (19) \)

\[
\int_0^T \|v(s)\|_{H_{1+\alpha}}^2 ds < \infty.
\]

Actually, the a priori estimates are for the Galerkin approximation \( v^M \). We define the Galerkin problem associated to \((12)\)

\[
\begin{aligned}
&\left\{ \frac{du^M}{dt} + [\nu Au^M(t) + B^M(u^M(t), u^M(t))]dt = \Pi_M \sqrt{2A^{1-\beta}} dw(t) \\
u^M(0) = \Pi_M x
\end{aligned}
\]

where \( M \) is any positive integer. Similarly we have

\[
\frac{dv^M}{dt} = -\nu Av^M(t) - B(v^M(t) + z^M(t), v^M(t) + z^M(t)) = 0
\]

with \( z^M(t) = \Pi_M z(t) \).

The previous estimates give

\[
\begin{aligned}
&\sup_M \|v^M\|^2_{L^\infty(\Omega; H^\alpha)} < \infty \\
&\sup_M \|v^M\|^2_{L^2(\Omega; H^{1+\alpha})} < \infty
\end{aligned}
\]

In addition \( \frac{dv^M}{dt} \) is bounded: indeed

\[
\frac{dv^M}{dt}(t) = -\nu Av^M(t) - B(v^M(t) + z^M(t), v^M(t) + z^M(t));
\]

using \((23)-(22)\), we have that the first term in the r.h.s belongs to the space \( L^2(0, T; H^{\alpha-1}) \) and the second to the space \( C([0, T]; H^{\alpha-1}) \) (use \((10)\)) and thus in \( L^2(0, T; H^{\alpha-1}) \). Then

\[
\sup_M \left\| \frac{dv^M}{dt} \right\|^2_{L^2(\Omega; H^{\alpha-1})} < \infty.
\]

Since the space \( \{ v : v \in L^2(0, T; H^{1+\alpha}), \frac{dv}{dt} \in L^2(0, T; H^{\alpha-1}) \} \) is compactly embedded in the space \( L^2(0, T; H^\alpha) \), from \((22)-(24)\) we get that there exists a subsequence \( \{ v^M_i \} \) weakly convergent to \( v \) in \( L^2(0, T; H^{1+\alpha}) \), weakly-* convergent in \( L^\infty(0, T; H^\alpha) \) and strongly convergent in \( L^2(0, T; H^\alpha) \). By means of the bilinearity of \( B \), of the strong convergence result and of \((11)\), we conclude that the limit \( v \) fulfills \((13)\).

The fact that \( v \in C([0, T]; H^\alpha) \) comes from a result in Temam \(23\) (Lemma 1.4. page 263): if \( v \in L^2(0, T; H^{1+\alpha}) \) and \( \frac{dv}{dt} \in L^2(0, T; H^{-1+\alpha}) \), then \( v \in C([0, T]; H^\alpha) \).

\begin{remark}
We can prove also the uniqueness of this solution \( v \), but we do not need it here. Anyway, the proof of uniqueness would be based on the same estimates as in the next Section \(3.1.2\).
\end{remark}

We conclude for \( u = v + z \).

\begin{proposition}
We consider the same assumptions as in Theorem \(3.1\). Let \( u(0) \in H^\alpha \). Then there exists a solution to equation \((12)\) such that

\[
u \in C([0, T]; H^\alpha) \quad \mathbb{P} - a.s.
\end{proposition}
3.1.2 Pathwise uniqueness

Now we prove that the strong solution $u$ constructed in the previous section is pathwise unique, that is

**Proposition 3.5** We consider the same assumptions as in Theorem 3.1. Let $u_1, u_2$ be two solutions to equation (12) with the same initial data, defined on the same stochastic basis and with the same Wiener process. Then $u_1 = u_2$ P-a.s., the equality being in $C([0, T]; H^\alpha)$.

**Proof.** We proceed pathwise. Let $u_1, u_2 \in C([0, T]; H^\alpha)$ be two paths (for fixed $\omega$ in a set of $P$-measure 1).

Set $U = u_1 - u_2$. Then $U \in C([0, T]; H^\alpha)$ and it solves an equation which is deterministic (for any path):

$$
\frac{dU}{dt} + \nu AU + B(u_1, u_1) - B(u_2, u_2) = 0; \quad U(0) = 0.
$$

First, we notice that $U$ is more regular than the $u_i$’s (the noise term has disappeared and we expect more regularity as for equation (15)).

By the bilinearity of the operator $B$, we have

$$
\frac{dU}{dt} + \nu AU + B(u_1, U) + B(U, u_2) = 0; \quad U(0) = 0.
$$

We get an a priori estimate:

$$
\frac{1}{2} \frac{d}{dt} \|U(t)\|_{\alpha}^2 + \nu \|U(t)\|_{1+\alpha}^2
= - \left( \alpha \frac{d}{dt} [B(u_1(t), U(t)) + B(U(t), u_2(t))], \alpha \frac{d}{dt} U(t) \right)
\leq \left[ \|u_1(t)\|_{\alpha} + \|u_2(t)\|_{\alpha} \right] \|U(t)\|_{1+\alpha} \leq \frac{\nu}{2} \|U(t)\|_{1+\alpha}^2 + \frac{c_\nu}{2} \left[ \|u_1(t)\|_{\alpha}^2 + \|u_2(t)\|_{\alpha}^2 \right] \|U(t)\|_{1+\alpha}^2.
$$

Therefore

$$
\frac{d}{dt} \|U(t)\|_{\alpha}^2 \leq c_\nu \left[ \|u_1(t)\|_{\alpha}^2 + \|u_2(t)\|_{\alpha}^2 \right] \|U(t)\|_{1+\alpha}^2;
$$

from this, by Gronwall inequality follows

$$
\|U(t)\|_{\alpha}^2 \leq \|U(0)\|_{\alpha}^2 e^{c_\nu \int_0^t \|u_1(s)\|_{\alpha}^2 + \|u_2(s)\|_{\alpha}^2 ds}.
$$

Finally, $U(t) = 0$ for all $t$, since $U(0) = 0$.

**Remark 3.6** Markovianity is inherited from the Galerkin approximations.

3.1.3 Feller property

Let us denote by $u(t; x)$ the solution of equation (12) with initial data $x$, by $B_b(H^\alpha)$ the space of Borel bounded functions $\phi : H^\alpha \to \mathbb{R}$ and by $C_b(H^\alpha)$ its space of continuous bounded functions.

Define the Markov semigroup $P_t : B_b(H^\alpha) \to B_b(H^\alpha)$ as

$$
P_t \phi(x) = \mathbb{E}[\phi(u(t; x))].
$$
This is a contraction semigroup. Moreover, it is Feller in $H^\alpha$, that is
\[ P_t : C_b(H^\alpha) \rightarrow C_b(H^\alpha). \]
This comes from the estimates for the pathwise uniqueness. Indeed, if $\|x - y\|_\alpha \rightarrow 0$ then (27) gives
\begin{equation}
\|u(t; x) - u(t; y)\|_\alpha^2 \leq \|x - y\|_\alpha^2 e^{C_2 \int_0^t \|u(s; x)\|_2^2 + \|u(s; y)\|_2^2} ds
\end{equation}
for $t > 0$ fixed. By (20) we get a uniform estimate of $\|u(\cdot; x)\|_{L^\infty(0, T; H^\alpha)}$ when $\|x\|_\alpha$ is bounded, i.e.
\[ \forall R > 0 \exists C_R : \sup_{\|x\|_\alpha \leq R} \|u(\cdot; x)\|_{L^\infty(0, T; H^\alpha)} < C_R. \]
Hence, when $\|x - y\|_\alpha \rightarrow 0$ from (28) we get $\|u(t; x) - u(t; y)\|_\alpha \rightarrow 0$. We conclude that $\phi(u(t; x)) \rightarrow \phi(u(t; y))$ for $\phi \in C_b(H^\alpha)$ and therefore $E[\phi(u(t; x))] \rightarrow E[\phi(u(t; y))]$ by the dominated convergence. This means that $P_t \phi \in C_b(H^\alpha)$ for any $t > 0$ and $\phi \in C_b(H^\alpha)$.

3.2 Invariant measure

We prove the following theorem:

**Theorem 3.7** Besides the assumptions of Theorem 3.6 we consider (C5). Then, $\mu^{\beta, \nu}$ is the unique invariant measure for equation (12), that is
\begin{equation}
\int P_t \phi \ d\mu^{\beta, \nu} = \int \phi \ d\mu^{\beta, \nu} \quad \forall \phi \in L^1(\mu^{\beta, \nu}) \text{ and } t \geq 0.
\end{equation}

First, we show that $\mu^{\beta, \nu}$ is an invariant measure for the nonlinear equation (12). Then we prove that this is indeed the unique invariant measure.

A consequence of this result is the following

**Corollary 3.8** Given any initial data with law $\mu^{\beta, \nu}$, there exists a unique stationary solution of equation (12) whose law at any fixed time is $\mu^{\beta, \nu}$.

Finally, in Section 3.2.3 we analyse the rate of convergence of $P_t \phi$, as $t \rightarrow \infty$.

To prove our results, we need to introduce the Kolmogorov operator associated to the stochastic equation (12). Let $FC_b^{\infty}$ be the space of infinitely differentiable cylindrical functions bounded and with bounded derivatives; $\phi \in FC_b^{\infty}$ means that there exist $m \in \mathbb{N}$, $\hat{\phi} \in C^{\infty}_b(\mathbb{R}^m)$ and multiindices $(i_1, i_2, \ldots, i_m)$ such that
\[ \phi(x) = \hat{\phi}(x, e_{i_1}, \ldots, (x, e_{i_m})). \]
We set $\frac{\partial \phi}{\partial x_i} = \frac{\partial \hat{\phi}}{\partial x_i}$ with $x_i = (x, e_i)$. $FC_b^{\infty}$ is a dense subset of $L^p(\mu^{\beta, \nu})$ for any $p \geq 1$.

We define the Kolmogorov operator first on these very regular functions $\phi \in FC_b^{\infty}$ as
\begin{equation}
K \phi(x) = \sum_j \left[ \lambda_j^{-1} \phi \frac{\partial^2 \phi}{\partial x_j^2} (x) - B_j(x, x) \frac{\partial \phi}{\partial x_j} (x) - \nu \lambda_j x_j \frac{\partial \phi}{\partial x_j} (x) \right].
\end{equation}
We have that $K \phi \in L^1(\mu^{\beta, \nu})$ for any $\phi \in FC_b^{\infty}$ (use that each $B_j \in L^1(\mu^{\beta, \nu})$ and the sums are finite).
3.2.1 Existence of the invariant measure

We know that the linear stochastic equation (14) has \( \mu^{\beta,\nu} \) as unique invariant measure, that is \( \mu^{\beta,\nu} \) is the unique probability measure such that

\[
\int E[\phi(z(t;x))] \mu^{\beta,\nu}(dx) = \int \phi(x) \mu^{\beta,\nu}(dx) \quad \forall t \geq 0, \phi \in B_b(H^\alpha)
\]

(see [13, 14]). Actually we can define the latter relationship for all \( \phi \in L^p(\mu^{\beta,\nu}) \), given any \( 1 \leq p < \infty \) (see, e.g., [10, 11]).

Now, we want to show that \( \mu^{\beta,\nu} \) is an invariant measure also for the nonlinear equation (12). The role of the nonlinear term \( B \) is analyzed first considering the finite dimensional \( B^M \) and then passing to the limit as \( M \to \infty \). Here we need (C5) of [12].

First, we prove that \( \mu^{\beta,\nu} \) is an infinitesimally invariant measure for equation (12) in the sense that

\[
(31) \quad \int K \phi \, d\mu^{\beta,\nu} = 0 \quad \forall \phi \in FC_b^\infty.
\]

Indeed, we can write \( K \) as the sum of two operators, \( K = Q + L \), with domains \( FC_b^\infty \) and we have the infinitesimal invariance for both these operators. We integrate by parts:

\[
(32) \quad \int Q \phi \, d\mu^{\beta,\nu} \equiv \int \sum_j \left[ \lambda_j^{1-\beta} \frac{\partial^2 \phi}{\partial x_j^2}(x) - \nu \lambda_j x_j \frac{\partial \phi}{\partial x_j}(x) \right] \mu^{\beta,\nu}(dx) = 0
\]

and

\[
(33) \quad \int L \phi \, d\mu^{\beta,\nu} \equiv -\int \sum_j B_j(x,x) \frac{\partial \phi}{\partial x_j}(x) \mu^{\beta,\nu}(dx)
\]

\[= -\nu \int \sum_j \lambda_j^2 B_j(x,x) x_j \phi(x) \mu^{\beta,\nu}(dx) = 0 \]

since \( B_j \) does not depend on the variable \( x_j \).

With similar computations, we get that the Kolmogorov operator \( (K, FC_b^\infty) \) is dissipative in \( L^1(\mu^{\beta,\nu}) \), that is

\[
\int \phi \, K \phi \, d\mu^{\beta,\nu} \leq 0 \quad \forall \phi \in FC_b^\infty
\]

(see also [16]). Hence it is closable (see [22]).

Now we use an approximative criterium of Eberle [16] in order to show that the measure \( \mu^{\beta,\nu} \) is an invariant measure for equation (12):

**Proposition 3.9** Besides the assumptions of Theorem [17] we consider (C5). Then, the closure operator \( K \) of the Kolmogorov operator \( (K, FC_b^\infty) \) in \( L^1(\mu^{\beta,\nu}) \) generates a sub-Markovian strongly continuous semigroup \( T_t = e^{Kt} \) in \( L^1(\mu^{\beta,\nu}) \).

Moreover, \( T_t \) is the only strongly continuous semigroup on \( L^1(\mu^{\beta,\nu}) \) which has generator that extends \( (K, FC_b^\infty) \) (see Appendix A in [16]).
We postpone the proof of this result and continue our analysis. Since $F_{C_b}^\infty$ is a core for the infinitesimal generator of $T_t$ in $L^1(\mu^{\beta,\nu})$ by density from (29) we get that

$$\int K\phi \, d\mu^{\beta,\nu} = 0 \quad \forall \phi \in D(K).$$

This is equivalent to

(34) $$\int T_t\phi \, d\mu^{\beta,\nu} = \int \phi \, d\mu^{\beta,\nu} \quad \forall \phi \in L^1(\mu^{\beta,\nu}) \text{ and } t \geq 0.$$

Now, we go back to the semigroup $\{P_t\}$; it has been constructed by means of the unique solution $u$ of equation (22) such that $u(t; x) \in H^\alpha$ (for any $t > 0$, $x \in H^\alpha$). On the other hand, the analytical analysis of the Kolmogorov operator has led to the construction of the semigroup $\{T_t\}$; it provides a martingale solution to the stochastic equation (22) (see, e.g., [16] and references therein). By our previous results of Section 3.1 on the stochastic equation (12) we can relate these semigroups and get that the semigroup $\{P_t\}$ can be extended to $L^1(\mu^{\beta,\nu})$ (where this semigroup is exactly $\{T_t\}$).

Henceforth, we denote these semigroups in $C_b(H^\alpha)$ and $L^1(\mu^{\beta,\nu})$ with the same symbol $P_t$. Therefore (34) completes our proof of (29).

Now, we go back to the proof of Proposition 3.9. We refer to [16] for all the details; in particular, we use Theorem 5.2, Corollary 5.3, Lemma 5.1 and closure of the operator $(K, F_{C_b}^\infty)$ From Lumer-Phillips theorem we know that the operator $T_t$ is a core for the infinitesimal generator of $\{P_t\}$. For each $N$, we take $F \in L^1(\mu^{\beta,\nu})$ and $\varphi \in \mathbb{R}^N$ such that $\int F \, d\mu^{\beta,\nu} = \int \varphi \, d\mu^{\beta,\nu} \geq 0$.

Now, we go back to the proof of Proposition 3.9. We refer to [16] for all the details; in particular, we use Theorem 5.2, Corollary 5.3, Lemma 5.1 and (5.46) at page 226 of [16] with $p = 1$.

**Proof.** (of Proposition 3.9) From Lumer-Phillips theorem we know that the closure of the operator $(K, F_{C_b}^\infty)$ in $L^1(\mu^{\beta,\nu})$ generates a strongly continuous semigroup $T_t$ if and only if the range of $(\lambda - K, F_{C_b}^\infty)$ is dense in $L^1(\mu^{\beta,\nu})$ for some (and all) $\lambda > 0$. To prove the density result, we use an approximative criterium:

(35) $$\forall F \in L^1(\mu^{\beta,\nu}) \forall \varepsilon > 0 \quad \exists v \in F_{C_b}^\infty : \| (\lambda - K) v - F \|_{L^1(\mu^{\beta,\nu})} < \varepsilon.$$

Now, we take $F \in L^1(\mu^{\beta,\nu})$. Then there exists a sequence $\{F_N\}_{N \in \mathbb{N}}$ with $F_N \in C_b^\infty(\mathbb{R}^N)$ and

(36) $$\lim_{N \to \infty} \| F_N - F \|_{L^1(\mu^{\beta,\nu})} = 0, \quad \sup_N \| F_N \|_{C_b} < \infty.$$

On the other hand, the assumption $B_n \in L^2(\mu^{\beta,\nu})$ (for any $n$) implies that $B_N \in L^2(\mu^{\beta,\nu})$ for any $N$, and therefore there exists a sequence $\{C_N\}_{N \in \mathbb{N}}$ with $C_N \in C_b^\infty(\mathbb{R}^N) \to \mathbb{R}^N$ and

(37) $$\| B_N - C_N \|_{L^2(\mu^{\beta,\nu})} \leq \frac{1}{N}.$$

Bearing in mind (36), this implies that

(38) $$\| \Pi_N B - C_N \|_{L^2(\mu^{\beta,\nu})} \leq \| \Pi_N B - B_N \|_{L^2(\mu^{\beta,\nu})} + \| B_N - C_N \|_{L^2(\mu^{\beta,\nu})} \to 0$$
as $N \to \infty$.

For each $N$, we introduce a regularized finite dimensional Kolmogorov operator $K^N$ acting on functions $\phi \in C_b^\infty(\mathbb{R}^N)$:

$$(K^N \phi)(x) = \sum_{j=1}^N \left[ \lambda_j^{-1} \frac{\partial^2 \phi(x)}{\partial x_j^2} - C_N^j(x) \frac{\partial \phi(x)}{\partial x_j}(x) - \nu \lambda_j x_j \frac{\partial \phi(x)}{\partial x_j}(x) \right].$$
It as smooth coefficients. Therefore, given \( F^N \in C^\infty_b(\mathbb{R}^N) \) and \( \lambda > 0 \) the equation

\[
(\lambda - K^N) \phi^N = F^N
\]

has a unique solution \( \phi^N \in C^\infty_b(\mathbb{R}^N) \); moreover

\[
\lambda \| \phi^N \|_{C^0_b} \leq \| F^N \|_{C^0_b}.
\]

Further, setting \( \sqrt{A^1} D\phi^N = \sum_{j=1}^{N} \lambda_j \frac{\partial \phi^N}{\partial x_j} \), by a straightforward computation we have

\[
K(\phi^N)^2 = 2 \phi^N K\phi^N + 2|\sqrt{A^1} D\phi^N|^2.
\]

Using the infinitesimal invariance (31), we have

\[
\text{Integrating with respect to the measure } \mu \text{ we find }
\]

\[
K(\phi^N)^2 = 2 \phi^N K\phi^N + 2|\sqrt{A^1} D\phi^N|^2.
\]

Using the infinitesimal invariance (31), we have

\[
\| \sqrt{A^1} D\phi^N \|_{L^2(\mu^{\beta,\nu})}^2 = \int \phi^N K\phi^N d\mu^{\beta,\nu}
\]

\[
= \int \phi^N |K\phi^N - K N \phi^N| d\mu^{\beta,\nu} - \int \phi^N K N \phi^N d\mu^{\beta,\nu}
\]

\[
= \int \phi^N \sum_{j=1}^{N} (B_j - C_j) \frac{\partial \phi^N}{\partial x_j} d\mu^{\beta,\nu} + \int \phi^N (F^N - \lambda \phi^N) d\mu^{\beta,\nu}
\]

\[
\leq \| \phi^N \|_{C^0_b} \| \Pi_N B - C^N \|_{L^2(\mu^{\beta,\nu})} \| D\phi^N \|_{L^2(\mu^{\beta,\nu})} + \| \phi^N \|_{C^0_b} \| F^N \|_{C^0_b} + \lambda \| \phi^N \|_{C^0_b}.
\]

Using (41) and the fact that \( \lambda \frac{1}{\lambda} \| D\phi^N \|_{L^2(\mu^{\beta,\nu})} \) for \( \beta \leq 1 \), we get that there exists a constant \( C_{\beta,\lambda} > 0 \) such that

\[
\| D\phi^N \|_{L^2(\mu^{\beta,\nu})} \leq C_{\beta,\lambda} \| F^N \|_{C^0_b} \left[ 1 + \| \Pi_N B - C^N \|_{L^2(\mu^{\beta,\nu})} \right].
\]

Let us go back to (35); by (39) we have

\[
(\lambda - K) \phi^N - F = (K^N - K) \phi^N + F^N - F \equiv \sum_{j=1}^{N} (B_j - C_j) \frac{\partial \phi^N}{\partial x_j} + F^N - F.
\]

Integrating with respect to the measure \( \mu^{\beta,\nu} \) we get

\[
\| (\lambda - K) \phi^N - F \|_{L^2(\mu^{\beta,\nu})} \leq \int \left| \sum_{j=1}^{N} (B_j - C_j) \frac{\partial \phi^N}{\partial x_j} \right| d\mu^{\beta,\nu} + \| F^N - F \|_{L^2(\mu^{\beta,\nu})}
\]

\[
\leq \| \Pi_N B - C^N \|_{L^2(\mu^{\beta,\nu})} \| D\phi^N \|_{L^2(\mu^{\beta,\nu})} + \| F^N - F \|_{L^2(\mu^{\beta,\nu})}
\]

by Schwarz inequality. Using (41) we find

\[
\| (\lambda - K) \phi^N - F \|_{L^2(\mu^{\beta,\nu})} \leq C_{\beta,\lambda} \| \Pi_N B - C^N \|_{L^2(\mu^{\beta,\nu})} \| F^N \|_{C^0_b} \left[ 1 + \| \Pi_N B - C^N \|_{L^2(\mu^{\beta,\nu})} \right]
\]

\[
+ \| F^N - F \|_{L^2(\mu^{\beta,\nu})}.
\]

Bearing in mind the assumptions on the approximating terms and (38), we find 35.
Remark 3.10 Because of the invariance of the measure $\mu^{p,\nu}$, the contraction semigroup $P_t$ in $C_b(H^\alpha)$ can be uniquely extended to a strongly continuous contraction semigroup in $\mathcal{L}^p(\mu^{p,\nu})$ also for any $p > 1$. Indeed,

$$|P_t\phi(x)|^p = |E[\phi(u(t,x))]|^p \leq E[|\phi(u(t;x))|^p] = P_t|\phi|^p(x)$$

and by the invariance of the measure $\mu^{p,\nu}$

$$\int |P_t\phi|^p d\mu^{p,\nu} \leq \int P_t|\phi|^p d\mu^{p,\nu} = \int |\phi|^p d\mu^{p,\nu}.$$

Since $C_b(H^\alpha)$ is dense in $\mathcal{L}^p(\mu^{p,\nu})$, we can uniquely define the semigroup on $\mathcal{L}^p(\mu^{p,\nu})$ for any $p > 1$. We use the same symbol $P_t$ to denote all these semigroups.

Notice that in condition (C5) we require $\int |B_n(x,x)|^2 \mu^{p,\nu}(dx) < \infty$ for any $n$. Therefore $K : FC^\infty_b \to \mathcal{L}^2(\mu^{p,\nu})$. Moreover, according to Corollary 5.3 of [16], we have that the restriction of $T_t$ to $\mathcal{L}^2(\mu^{p,\nu})$ is a strongly continuous semigroup on $\mathcal{L}^2(\mu^{p,\nu})$ and the generator of this semigroup again extends $(K, FC^\infty_b)$. In the sequel we will use the same symbol to denote these semigroups in both spaces $\mathcal{L}^1(\mu^{p,\nu})$ and $\mathcal{L}^2(\mu^{p,\nu})$.

3.2.2 Uniqueness of the invariant measure

Now we prove that equation (12) has at most one invariant measure. We use the results of Section 3.1.

Let $R(t,x,\cdot)$ be the law of $z(t;x)$ and $P(t,x,\cdot)$ be the law of $u(t;x)$. Then any $R(t,x,\cdot)$ is equivalent to the Gibbs measure $\mu^{p,\nu}$ (see, e.g., [14]); we write it as $R(t,x,\cdot) \sim \mu^{p,\nu}$. Moreover we have that

$$\int_0^T |\sqrt{A^{p-1}}B(z(t),z(t))|^2 dt < \infty \quad \mathbb{P} - a.s.$$ (42)

and

$$\int_0^T |\sqrt{A^{p-1}}B(u(t),u(t))|^2 dt < \infty \quad \mathbb{P} - a.s.$$ (43)

For this use that $\|B(x,x)\|_{\beta-1} \leq c\|x\|_2^\beta$ from assumption (4) and that $\mathbb{P}\{z \in C([0,T];H^\alpha)\} = \mathbb{P}\{u \in C([0,T];H^\alpha)\} = 1$.

According to Theorem 9.2 in [17], (42)–(43) imply that the measure $P(t,x,\cdot)$ is equivalent to $R(t,x,\cdot)$. On the other side $R(t,x,\cdot) \sim R(s,y,\cdot) \sim \mu^{p,\nu}$, hence we get

$$P(t,x,\cdot) \sim P(t,y,\cdot) \sim \mu^{p,\nu}$$

for any $x,y \in H^\alpha$ and $t > 0$. Using Doob theorem (see, e.g., Theorem 4.2.1 in [14]), we deduce that there exists at most one invariant measure.

By means of the existence result of the previous section, we get that $\mu^{p,\nu}$ is the unique invariant measure for equation (12). Moreover, it is strongly mixing

$$\lim_{t \to \infty} P(t,x,\Gamma) = \mu^{p,\nu}(\Gamma)$$

for arbitrary $x \in H^\alpha$ and Borel set $\Gamma$ in $H^\alpha$.  

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3.2.3 Rate of convergence

Now, we consider the semigroup \( P_t \) in \( L^2(\mu^{\beta,\nu}) \) (see Remark 3.10).

We recall the "Carré du champ" identity. For the reader's convenience we give the proof (see, e.g., [10])

**Proposition 3.11** Besides the assumptions of Theorem 3.1 we consider (C5).

Then, we have

\[
\int \phi \, K\phi \, d\mu^{\beta,\nu} = -\int |\sqrt{A^{1-\beta}} D\phi|^2 d\mu^{\beta,\nu} \quad \forall \phi \in D(K).
\]

**Proof.** First we take \( \phi \in FC^\infty_b \). A straightforward computation yields that

\[
K\phi^2 = 2\phi \, K\phi + 2|\sqrt{A^1-\beta} D\phi|^2.
\]

By the \( \mu^{\beta,\nu} \)-infinitesimal invariance, we have

\[
\int K\phi^2 \, d\mu^{\beta,\nu} = 0; \text{ thus (46)}
\]

\[
\int \phi \, K\phi \, d\mu^{\beta,\nu} = -\int |\sqrt{A^{1-\beta}} D\phi|^2 d\mu^{\beta,\nu}.
\]

Now, taking \( \phi \in D(K) \), we use that \( FC^\infty_b \) is a core for \( K \); therefore there exists a sequence \( \{\phi_n\} \subset FC^\infty_b \) such that

\[
\phi_n \to \phi, \quad K\phi_n \to K\phi \quad \text{in } L^2(\mu^{\beta,\nu}).
\]

From (46) we get

\[
\int |\sqrt{A^{1-\beta}} D(\phi_n - \phi_m)|^2 d\mu^{\beta,\nu} \leq \int |\phi_n - \phi_m||K(\phi_n - \phi_m)|d\mu^{\beta,\nu}.
\]

Hence, the sequence \( \{\sqrt{A^{1-\beta}} D\phi_n\} \) is a Cauchy sequence in \( L^2(\mu^{\beta,\nu}) \) and we get \( \Box \).

Now, given \( \phi \in L^2(\mu^{\beta,\nu}) \) we set \( \overline{\phi} = \int \phi \, d\mu^{\beta,\nu} \); then we have the following theorem on the rate of convergence of \( P_t\phi \) as \( t \to \infty \).

**Theorem 3.12** Besides the assumptions of Theorem 3.1 we consider (C5). Then

\[
\int |P_t\phi(x) - \overline{\phi}|^2 \mu^{\beta,\nu}(dx) \leq e^{-\lambda_1 t} \int |\phi(x) - \overline{\phi}|^2 \mu^{\beta,\nu}(dx)
\]

for any \( \phi \in L^2(\mu^{\beta,\nu}) \) and \( t > 0 \).

**Proof.**

Let us define the space

\[
L^2_0(\mu^{\beta,\nu}) = \{ \phi \in L^2(\mu^{\beta,\nu}) : \overline{\phi} = 0 \};
\]

it is not difficult to prove that it is invariant for the semigroup \( P_t \) (see [11]).

First, let us take \( \phi \in L^2_0(\mu^{\beta,\nu}) \cap D(K) \); then \( P_t\phi \in L^2_0(\mu^{\beta,\nu}) \cap D(K) \) and by the Hille-Yosida theorem

\[
\frac{d}{dt} P_t\phi = KP_t\phi.
\]

Therefore, bearing in mind (15)

\[
\frac{1}{2} \frac{d}{dt} \int |P_t\phi|^2 \, d\mu^{\beta,\nu} = \int P_t\phi \, \overline{K}P_t\phi \, d\mu^{\beta,\nu} = -\int |\sqrt{A^{1-\beta}} D\phi|^2 d\mu^{\beta,\nu}
\]

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Since a Gaussian measure fulfills the spectral gap inequality (see [7]) we have
\[
\int |\sqrt{A^{1-\beta}} D_\beta \phi(x)|^2 \mu^\beta(dx) \geq \frac{\lambda_1}{2} \int |\phi(x)|^2 \mu^\beta(dx)
\]
where \( \lambda_1 > 0 \) is the first eigenvalue of the operator \( A \). By the two latter relationships we get
\[
\frac{d}{dt} \int |P_t \phi|^2 d\mu^{\beta,\nu} \leq -\lambda_1 \int |P_t \phi|^2 d\mu^{\beta,\nu}.
\]
Hence, using Gronwall lemma, we have that for any \( t > 0 \)
\[
\int |P_t \phi|^2 d\mu^{\beta,\nu} \leq e^{-\lambda_1 t} \int |\phi|^2 d\mu^{\beta,\nu} \quad \forall \phi \in L^2(\mu^{\beta,\nu}) \cap D(\mathcal{K}).
\]

Now we take \( \phi \in D(\mathcal{K}) \); replacing \( \phi \) with \( \phi - \overline{\phi} \) in (48), we obtain that
\[
\int |P_t \phi - \overline{\phi}|^2 d\mu^{\beta,\nu} = \int |P_t (\phi - \overline{\phi})|^2 d\mu^{\beta,\nu} \leq e^{-\lambda_1 t} \int |\phi - \overline{\phi}|^2 d\mu^{\beta,\nu}.
\]
Using that \( D(\mathcal{K}) \) is dense in \( L^2(\mu^{\beta,\nu}) \) we get the result.

\[\square\]

4 An example: shell models of turbulence

Shell models of turbulence describe the evolution of complex Fourier-like components of a scalar velocity field. Here we present the details for the SABRA shell model (see [20]), but the same results hold for the GOY shell model (see [18, 21]). In recent years there has been an increasing interest in these fluid dynamical models, both for the deterministic and the stochastic case (see also [9], [4], [6], [8]). They are easier to analyze than the Navier-Stokes or Euler equations, but they retain many important features of the true hydrodynamical models.

Instead of dealing with complex valued unknowns we deal with the real and imaginary part of each component of the scalar velocity field (for the basic settings we follow [5]); this defines a sequence \( \{u_n\} \) with \( u_n \in \mathbb{R}^2 \). For \( x = (x_1, x_2) \in \mathbb{R}^2 \) we set \( |x|^2 = x_1^2 + x_2^2 \) and the scalar product in \( \mathbb{R}^2 \) is \( x \cdot y = x_1 y_1 + x_2 y_2 \).

Then, using the notations of Section 2.1, we define the basic space \( H \) as
\[
H = \{u = (u_1, u_2, \ldots) \in (\mathbb{R}^2)^\infty : \sum_{n=1}^{\infty} |u_n|^2 < \infty\}.
\]

The basis in \( H \) is given by the sequence \( \{e_1^{(1)}, e_1^{(2)}, e_2^{(1)}, e_2^{(2)}, e_3^{(1)}, e_3^{(2)}, \ldots\} \) of elements of \( (\mathbb{R}^2)^\infty \), where
\[
e_1^{(1)} = ((0, 0), \ldots, (0, 0), (1, 0), (0, 0), \ldots)
\]
\[
e_2^{(2)} = ((0, 0), \ldots, (0, 0), (0, 1), (0, 0), \ldots)
\]
with the nonvanishing vectors in place \( n \).

The eigenvalues are
\[
\lambda_n = \lambda_0^{2n}
\]
with \( \lambda > 1 \). Hence we can take any \( \alpha < \beta \) to fulfil (C3). Inequality \( (3) \) holds with \( c_{\alpha \nu} = (\frac{c}{\nu})^p \).

We set \( \kappa_n = \sqrt{\lambda_n} \). The bilinear term \( B \) is defined by means of the components \( B_n = (B_{n,1}, B_{n,2}) \) as follows (see, e.g., [5]):

(49) \[
B_{1,1}(u, v) = ak_2[-u_{2,2}v_{3,1} + u_{2,1}v_{3,2}] \\
B_{1,2}(u, v) = -ak_2u_2 \cdot v_3
\]

(50) \[
B_{2,1}(u, v) = ak_3[-u_{3,2}v_{4,1} + u_{3,1}v_{4,2}] + bk_2[-u_{1,2}v_{3,1} + u_{1,1}v_{3,2}] \\
B_{2,2}(u, v) = -ak_3u_3 \cdot v_4 - bk_2u_1 \cdot v_3
\]

and for \( n > 2 \)

(51) \[
B_{n,1}(u, v) = ak_{n+1}[-u_{n+1,2}v_{n+2,1} + u_{n+1,1}v_{n+2,2}] \\
+ bk_n[-u_{n-1,2}v_{n+1,1} + u_{n-1,1}v_{n+1,2}] \\
+ ak_{n-1}[u_{n-1,2}v_{n-2,1} + u_{n-1,1}v_{n-2,2}] \\
+ bk_{n-1}[u_{n-2,2}v_{n-1,1} + u_{n-2,1}v_{n-1,2}],
\]

(52) \[
B_{n,2}(u, v) = -ak_{n+1}[u_{n+1,1}v_{n+2,2} + u_{n+1,2}v_{n+2,2}] \\
- bk_n[u_{n-1,1}v_{n-1,1} + u_{n-1,2}v_{n-1,2}] \\
- ak_{n-1}[u_{n-1,2}v_{n-2,1} - u_{n-1,2}v_{n-2,2}] \\
- bk_{n-1}[u_{n-2,2}v_{n-1,1} - u_{n-2,2}v_{n-1,2}],
\]

where \( a \) and \( b \) are real numbers such that

(53) \[
a + b\lambda^{2\beta} = (a + b)\lambda^{4\beta}
\]

for some \( \beta > 0 \), that is

(54) \[
\lambda^{2\beta} = -\frac{a}{a + b}
\]

(recall that \( \lambda > 1 \)). This condition implies (C2 iii), whereas (C2 ii) holds for any real \( a \) and \( b \). For instance, let us check that \( (53) \) implies (C2 iii). We have

\[
\sum_{n=1}^{\infty} k_n^{2\beta} B_n(u, u) \cdot u_n
\]

\[
= \sum_{n=1}^{\infty} k_n^{2\beta} [B_{n,1}(u, u)u_{n,1} + B_{n,2}(u, u)u_{n,2}]
\]

\[
= \sum_{n=1}^{\infty} (a + b\lambda^{2\beta} - (a + b)\lambda^{4\beta})\lambda^{2\beta} + 1 (u_{n+2} \cdot u_n)(u_{n+1,2} + u_{n+1,1}).
\]

Moreover we have (see [5])

**Lemma 4.1** For any \( \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \)

\[
B : H^{\alpha_1} \times H^{\alpha_2} \to H^{-\alpha_3} \text{ with } \alpha_1 + \alpha_2 + \alpha_3 \geq 1
\]

and there exists a constant \( c \) (depending on \( a, b, \lambda \) and the \( \alpha_j \)'s) such that

\[
\|B(u, v)\|_{-\alpha_3} \leq c\|u\|_{\alpha_1}\|v\|_{\alpha_2} \quad \forall u \in H^{\alpha_1}, v \in H^{\alpha_2}.
\]
This implies that conditions (C4) are true: 1 for any $\frac{\beta}{2} \leq \alpha < \beta$ and 1 for any $\alpha$.

Condition 6 holds for $\beta > \frac{1}{2}$; this includes the interesting physical case of $\beta = 1$ (see Section 2.3). Indeed, for the SABRA shell model

$$B^M_n(x, x) - B_{n,1}(x, x) = \begin{cases} 
0 & \text{for } n \leq M - 2 \\
-ak_M(x_{M,1}x_{M+1,2} - x_{M,2}x_{M+1,1}) & \text{for } n = M - 1 \\
-ak_{M+1}(x_{M+1,1}x_{M+2,2} - x_{M+1,2}x_{M+2,1}) & \text{for } n = M
\end{cases}$$

and

$$B^M_n(x, x) - B_{n,2}(x, x) = \begin{cases} 
0 & \text{for } n \leq M - 2 \\
-ak_M(-x_{M,1}x_{M+1,1} - x_{M,2}x_{M+1,2}) & \text{for } n = M - 1 \\
-ak_{M+1}(-x_{M+1,1}x_{M+2,2} + x_{M+1,2}x_{M+2,1}) & \text{for } n = M
\end{cases}$$

Therefore

$$\sum_{n=1}^{M} |B^M_n - B_n|^2 = |B^M_{M-1} - B_{M-1}|^2 + |B^M_M - B_M|^2$$

so

$$\lim_{M \to \infty} \int \sum_{n=1}^{M} |B^M_n - B_n|^2 d\mu^{\beta,\nu} \leq \lim_{M \to \infty} \frac{8}{\nu^2} \frac{a^2}{\lambda^{2\beta}} k_M^{2-4\beta} + \frac{a^2}{\lambda^{2\beta}} k_{M+1}^{2-4\beta} + b^2 k_M^{2-4\beta} = 0.$$ 

This holds for $\beta > \frac{1}{2}$.

We finally point out that our results of Section 3.2 hold also in any space $L^p(\mu^{\beta,\nu})$ with $p = 1, 2, \ldots$ (see Remark 3.10). Indeed, we have

$$\int |B_n(x, x)|^q \mu^{\beta,\nu}(dx) < \infty \quad \forall n, q \in \mathbb{N}.$$ 

5 Inviscid models

We are interested in the deterministic inviscid and unforced dynamics represented by equation (13). Here we present our results for the SABRA shell model with $\beta = 1$ (the physical relevant case) only to make simpler the exposition, but it can be generalized to the other fluid dynamic models.

Equation (13) is formally obtained from equation (12) setting $\nu = 0$ and considering a vanishing right hand side. More generally we can consider the nonlinear viscous equation

$$du^\varepsilon(t) + [\nu + A u^\varepsilon(t) + B(u^\varepsilon(t), u^\varepsilon(t)))]dt = \sqrt{2\varepsilon} dw(t), \quad t > 0.$$ 

with $\varepsilon > 0$. When $\varepsilon = 0$ we get equation (13) (with $\beta = 1$). Our results of the previous sections hold true for any $\varepsilon > 0$.

The fact that the measure $\mu^{\beta,\nu}$ is an invariant measure for any $\varepsilon > 0$ can be easily checked. We proceed as in the previous section, but now the Kolmogorov
operator associated to equation (56) is \(K^\varepsilon = \varepsilon Q + L\); bearing in mind (52) and (53) we get that \(\mu^{1,\nu}\) is an infinitesimal invariant measure for the operator \((K^\varepsilon, FC^\infty_b)\). And for any \(\varepsilon > 0\) the operator \((K^\varepsilon, FC^\infty_b)\) is dissipative.

We are going to prove that when the initial data is a random variable with law \(\mu^{1,\nu}\), then equation (13) has a solution which is a stationary random process, whose law at any fixed time is \(\mu^{1,\nu}\).

An important property is the integrability of \(B\) with respect to the measure \(\mu^{1,\nu}\).

**Proposition 5.1** If \(\nu > 0\), then for any \(\alpha < 1\) we have

\[
\int \|B(x, x)\|_p^{\nu} \mu^{1,\nu}(dx) < \infty
\]

for any \(p \in \mathbb{N}\).

**Proof.** We write the proof for \(p = 2\) but it is the same for the other values of \(p\), since \(\mu^{1,\nu}\) is Gaussian and the \(B_n\)'s are second order polynomial. We have

\[
\int |B_{n,1}(x, x)|^2 \mu^{1,\nu}(dx) = \int |ak_n x_{n+1} - x_{n+2} + x_{n+1} x_{n+2}| + (a + b)k_{n-1} x_{n+1} x_{n+2} + (a + b)^2k_n x_{n+1} x_{n+2} + \lambda^{-2n}.
\]

Similarly we estimate \(\int |B_{n,2}(x, x)|^2 \mu^{1,\nu}(dx)\). Therefore

\[
\int \|B(x, x)\|_p^\nu \mu^{1,\nu}(dx) = \int \sum_{n=1}^\infty \lambda_0^n |B_n(x, x)|^2 \mu^{1,\nu}(dx)
\]

\[
\leq c_{\nu, k_0} |a|^2 + |b|^2 \sum_{n=1}^\infty \lambda^{2n(\alpha - 1)}
\]

which is finite if \(\alpha < 1\). \(\square\)

Here is our main result.

**Theorem 5.2** For any \(\nu > 0\), there exists a \(\mu^{1,\nu}\)-stationary process, whose paths solve equation (13) \(\mathbb{P}\)-a.s. In particular, the paths are in \(C^\delta(\mathbb{R}; H^\alpha)\) for any \(0 \leq \delta < \frac{\alpha}{2}\) and \(\alpha < 1\).

**Proof.** We fix \(\nu > 0\) arbitrarily. According to Corollary 3.8, equation (56) has a unique \(\mu^{1,\nu}\)-stationary solution \(\overline{\varphi}\); this process is a strong solution and has paths in \(C(\mathbb{R}; H^\alpha)\) a.s. (for \(\alpha < 1\), but we always think of \(\alpha\) as much close to 1 as possible).
First, we prove that the sequence \( \{\varphi^r\}_{0 < r < 1} \) is tight in \( C^\delta([0,T]; H^\alpha) \) for any \( \delta \in (0, \frac{1}{2}) \) and \( \alpha < \alpha \).

We write equation (56) in the mild form:

\[
\varphi(t) = \varphi(0) + \int_0^t e^{-\nu t A(t-s)} B(\varphi(s), \varphi(s)) ds,
\]

where \( \varphi(t) = e^{-\nu t A} \varphi(0) + \int_0^t e^{-\nu t A(t-s)} \sqrt{2\xi} dw(s) \)

is the \( \mu_{1,\nu} \)-stationary solution of the linear equation

\[
d\varphi(t) + \nu \sigma \varphi(t) dt = \sqrt{2\xi} dw(t)
\]

with the initial data of \( \mu_{1,\nu} \).

We consider the two terms in the right hand side of (57). Using the \( \mu_{1,\nu} \)-stationarity we have that for any \( 0 \leq \delta < \frac{1}{4} \) there exists a constant \( C_\delta > 0 \) such that

\[
\sup_{0 < r < 1} E[\|\varphi\|_{C^\delta([0,T]; H^\alpha)}] \leq C_\delta.
\]

We take \( \eta \in (0,1) \) and set \( \gamma = \alpha - 2\eta \). For the convolution integral in (57) we have

\[
\left\| \int_0^t e^{-\nu t A(-s)} B(\varphi(s), \varphi(s)) ds \right\|_{\gamma, T, H^\alpha}^p
\]

\[
= \int_0^T \left\| \int_0^t e^{-\nu t A(t-s)} B(\varphi(s), \varphi(s)) ds \right\|_{\gamma, T, H^\alpha}^p dt
\]

\[
+ \nu \int_0^T \left\| \int_0^t e^{-\nu t A(t-s)} B(\varphi(s), \varphi(s)) ds \right\|_{\gamma, T, H^\alpha}^p dt
\]

\[
\leq \int_0^T t^{p-1} \left( \int_0^t \left\| e^{-\nu t A(t-s)} B(\varphi(s), \varphi(s)) \right\|_{\gamma, T, H^\alpha}^p ds \right) dt
\]

\[
+ \nu \int_0^T \left( \int_0^t \left\| B(\varphi(s), \varphi(s)) \right\|_{\gamma, T, H^\alpha}^p ds \right) dt
\]

\[
\leq \frac{1}{p} \left( \int_0^t \left\| B(\varphi(t), \varphi(t)) \right\|_{\gamma, T, H^\alpha}^p dt + \nu \int_0^t \left( \int_0^t \frac{\left\| B(\varphi(s), \varphi(s)) \right\|_{\gamma, T, H^\alpha}^p ds}{(t-s)^{1-\gamma}} \right) dt \right) \]

For the latter integral we use Hölder inequality and get that

\[
\left( \int_0^t \frac{\left\| B(\varphi(s), \varphi(s)) \right\|_{\gamma, T, H^\alpha}^p ds}{(t-s)^{1-\gamma}} \right) \leq \left( \int_0^t \frac{ds}{(t-s)^{1-\gamma}} \right)^{2p-2} \left( \int_0^t \frac{\left\| B(\varphi(s), \varphi(s)) \right\|_{\gamma, T, H^\alpha}^p ds}{(t-s)^{1-\gamma}} \right)^{\frac{p-2}{2}}.
\]
Hence, for $p > \frac{2}{\gamma} - 1$ we have

$$
\left\| \int_0^t e^{-\nu A(s-t)} B(\tilde{\sigma}(s), \tilde{\sigma}'(s)) ds \right\|_{W^{1,p}(0,T; H^\gamma)}^p 
\leq \left( \frac{1}{p} T^p + 1 \right) \int_0^T \left\| B(\tilde{\sigma}(t), \tilde{\sigma}'(t)) \right\|_\gamma^p dt + \nu \epsilon T^m \int_0^T \left\| B(\tilde{\sigma}(t), \tilde{\sigma}'(t)) \right\|_\gamma^p dt
$$

for some positive constant $m = m_{\eta, \nu, \rho}$.

Integrating with respect to the measure $\mu^{\beta, \nu}$ and using the invariance we get

$$
\mathbb{E} \left\| \int_0^T e^{-\nu A(s-t)} B(\tilde{\sigma}(s), \tilde{\sigma}'(s)) ds \right\|_{W^{1,p}(0,T; H^\gamma)}^p 
\leq T(1 + \frac{1}{p} T^p + \nu \epsilon T^m) \int \left\| B(x, x) \right\|_{\mu^{1, \nu}}^p (dx)
$$

Now, we use that $W^{1,p}(0,T) \subset C^\delta([0,T])$ if $1 - \frac{1}{p} > \delta$. Then, using the previous estimates in (61), given any $0 \leq \delta < \frac{1}{2}$, $p > \frac{1}{2}$ and $p > \frac{2}{\gamma} - 1$ we have

$$
\sup_{0 < \epsilon \leq 1} \mathbb{E} \left\| \| \tilde{\sigma} \|_{C^\delta([0,T]; H^\gamma)}^p \right\| < \infty.
$$

On the other hand, the space $C^\delta([0,T]; H^\gamma)$ is compactly embedded in $C^{\hat{\delta}}([0,T]; H^\gamma)$ if $\delta < \hat{\delta}$ and $\gamma < \gamma$; this follows from the compact embedding $H^\gamma \subset H^\delta$ and from the Ascoli-Arzelà theorem. Because these results hold for any $\delta \in [0, \frac{1}{2})$ and $\gamma < \gamma < \alpha < 1$ (with $p$ big enough, but we use (57), we can consider any $\delta < \frac{1}{2}$ and any $\gamma < 1$. The tightness follows from (62) as usual by means of Chebyshev inequality. And to simplify notation henceforth we consider the tightness in the space $C^{\hat{\delta}}([0,T]; H^\gamma)$ ($\delta < \frac{1}{2}$ and $\alpha < 1$).

By the tightness result and Prohorov theorem, the sequence of the laws of $\tilde{\sigma}$ has a subsequence $\{\tilde{\sigma}^n\}_{n=1}^{\infty}$ weakly convergent as $n \to \infty$ (w.r. $\tilde{\epsilon}_n \to 0$) in $C^{\hat{\delta}}([0,T]; H^\gamma)$ to some limit measure. By a diagonal argument, this holds for any $T$ and therefore the limit measure leaves in $C^\delta([0,\infty); H^\gamma)$. By Skorohod theorem, there exist a probability space $(\tilde{\Omega}, \tilde{\mathbb{P}}, \tilde{\mathbb{P}})$, a random variable $\tilde{v}$ and a sequence $\{\tilde{v}^\epsilon\}$ such that $\text{law}(\tilde{v}^\epsilon) = \text{law}(\tilde{\sigma})$, $\text{law}(\tilde{v}) = \mu^{1, \nu}$ and $\tilde{v}^\epsilon$ converges to $\tilde{v}$ a.s. in $C^{\hat{\delta}}([0,\infty); H^\gamma)$.

We now identify the equation satisfied by $\tilde{v}$. We are going to prove that $\tilde{\mathbb{P}}$-almost each path solves (19).

It is enough to control the behavior of the terms with $B$. First

$$
e^{-\nu A(t-s)} B(\tilde{v}^{\nu, \epsilon}(s), \tilde{v}^{\nu, \epsilon}(s)) - B(\tilde{v}^{\nu}(s), \tilde{v}^{\nu}(s))
= e^{-\nu A(t-s)} [B(\tilde{v}^{\nu, \epsilon}(s), \tilde{v}^{\nu, \epsilon}(s)) - B(\tilde{v}^{\nu}(s), \tilde{v}^{\nu}(s))]
+ \left[ e^{-\nu A(t-s)} - I \right] B(\tilde{v}^{\nu}(s), \tilde{v}^{\nu}(s)).
$$

When we consider the second addend in the mild form expression, it trivially converges to zero; but for the convergence of the first one it is enough to verify that

$$
\int_0^t \left\| B(\tilde{v}^{\nu, \epsilon}(s), \tilde{v}^{\nu, \epsilon}(s)) - B(\tilde{v}^{\nu}(s), \tilde{v}^{\nu}(s)) \right\|_{\alpha-1} ds \to 0
$$
as $\epsilon \to 0$; for this we use the bilinearity and the estimate (10).
Similarly we work on the time interval $[-T,0]$ by considering the reversed-time parabolic nonlinear equation

$$
\frac{du^\varepsilon(t)}{dt} + [-\nu \varepsilon Au^\varepsilon(t) + B(u^\varepsilon(t), u^\varepsilon(t))] dt = \sqrt{2\varepsilon} \, dw(t), \quad t < 0
$$

It has a unique $\mu^{1,\nu}$-stationary solution $v^\varepsilon$; this process is a strong solution, has paths in $C^\delta((-\infty,0]; H^\alpha)$. The tightness and the convergence are obtained in the same way as above.

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