The isomorphism problem for Coxeter groups

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Abstract. By a recent result obtained by R. Howlett and the author considerable progress has been made towards a complete solution of the isomorphism problem for Coxeter groups. In this paper we give a survey on the isomorphism problem and explain in particular how the result mentioned above reduces it to its ‘reflection preserving’ version. Furthermore we describe recent developments concerning the solution of the latter.

1 Introduction

Coxeter groups are important in several mathematical areas. It is therefore a bit surprising that the isomorphism problem for those groups does not seem to have been considered before the late 1990’s. They only earlier reference known to the author where this problem has been asked is [17]. The first major contributions to it are [16] and [10]. In [16] a rigidity result is proved for a certain class of Coxeter groups. Rigidity means that the Coxeter generating sets are all conjugate. In [10] diagram twists have been introduced. Those provide non-trivial examples of non-rigid Coxeter groups. The question about which Coxeter systems are rigid arises naturally as well as the more general question about the isomorphism problem for Coxeter groups.

The purpose of the present paper is to give a survey about what is known at present about the isomorphism problem. The main motivation for writing this survey is provided by a recent result obtained by the author in collaboration with Bob Howlett. This result reduces the isomorphism problem to its ‘reflection-preserving version’. For the solution of the latter there is a conjecture stated in [10]. Considerable progress towards a proof of this conjecture was made in [38] and by recent work of Pierre-Emmanuel Caprace in [13] there is reasonable hope that this conjecture will be proved in the near future. Due to these facts there is now a clear picture of what the solution of the isomorphism problem should look like. In fact,
at present there is a solution if one assumes that there are no irreducible spherical residues of rank 3. We state two conjectures in Section 5. The first one is known to be true for all Coxeter systems having no $H_3$-subsystems; the second is a refinement of Conjecture 8.1 in [10] already mentioned. Under the assumption that both conjectures are true, we give an algorithm for the solution of the isomorphism problem.

**Two versions of the isomorphism problem.** Let $W$ be a group and let $S \subseteq W$ be a set of involutions. Then $M(S)$ denotes the square matrix $(o(ss'))_{s,s' \in S}$ where $o(w)$ denotes the order of an element $w \in W$. The matrix $M(S)$ is called the type of $S$. As the elements of $S$ are involutions we have the following.

1. For all $s, s' \in S$ we have $o(ss') \in \mathbb{N} \cup \{\infty\}$;
2. for all $s \neq s' \in S$ we have $o(ss') = o(s's) \geq 2$;
3. for all $s \in S$ we have $o(ss) = 1$.

Hence, the matrix $M(S)$ is a symmetric square matrix with entries in the set $\mathbb{N} \cup \{\infty\}$ where all entries on the main diagonal are equal to one and all remaining entries are strictly greater than one. Such a matrix is called a Coxeter matrix over $S$.

Let $(W,S)$ be as above. We call $(W,S)$ a Coxeter system (of type $M(S)$) if $\langle S \rangle = W$ and if the relations $((ss')^o(ss')) = 1_W$ hold for all $s,s' \in S$ provide a presentation of $W$. For a given Coxeter matrix $M = (M_{ij})_{i,j \in I}$ over a set $I$, we define the Coxeter group of type $M$ by setting $W(M) := \langle I \mid (ij)^{m_{ij}} = 1_{i,j \in I} \rangle$. It is a basic fact that the pair $(W(M),I)$ is a Coxeter system of type $M$ (i.e. that $o(ij) = m_{ij}$ in $W(M)$ for all $i,j \in I$).

In this paper we will consider the isomorphism problem for finitely generated Coxeter groups. Thus, if we talk about a Coxeter system $(W,S)$ or a Coxeter matrix $M$ over $I$ it is always understood that the sets $S$ and $I$ are finite. Here are two versions of the isomorphism problem for Coxeter groups.

**Problem 1** Given two Coxeter matrices $M$ and $M'$, decide whether the groups $W(M)$ and $W(M')$ are isomorphic.

**Problem 2** Given two Coxeter matrices $M$ and $M'$, find all isomorphisms from $W(M)$ onto $W(M')$.

At first sight, Problem 1 seems to be a more natural question than Problem 2. The latter is just a more general version of the first. Roughly speaking, the solution of Problem 2 is equivalent to the solution of Problem 1 and a description of the automorphism group of $W(M)$ for any Coxeter matrix $M$. This is in fact the main motivation to consider Problem 2. It turns out that for certain Coxeter matrices $M$ a good understanding of the automorphism group of the group $W(M)$ is only possible if a solution of Problem 2 is available for all Coxeter matrices $M'$.

**Content.** In Section 2 we recall some definitions, fix notation and mention some basic facts concerning Coxeter groups. In Section 3 we will consider the rigidity problem for Coxeter groups. This is an interesting special case of the isomorphism problem. In this section we will provide examples of non-rigid Coxeter systems which will play an important role later. Section 4 is devoted to explaining the results obtained in [23] and [27] and how these results reduce the isomorphism problem to its ‘reflection-preserving version’ which will then be treated in Section 5. In Section 6 we explain an algorithm to solve the isomorphism problem under the assumption that Conjectures 1 and 2 of Section 5 hold. Finally, in Section 7 we will make some remarks on the automorphism groups of Coxeter groups.
Remark: It was mentioned above that there is no contribution to the isomorphism problem for Coxeter groups before the late 1990’s. Since then, however, there are several publications concerning this subject. For instance, Problem 1 has been solved completely in the case where $M$ is assumed to be even (i.e. no odd entries) by P. Bahls and M. Mihalik (see [34] and the references given there).

In this survey paper we do not attempt to give a systematic description of all contributions to the isomorphism problem for Coxeter groups. We mention results (or consequences of them) whenever it will be convenient. However, we try to include all references on the subject in the bibliography. Thus, quite a few references will be mentioned only there.

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2 Preliminaries

Coxeter diagrams. With a Coxeter matrix $M = (m_{ij})_{i,j \in I}$ we associate its diagram $\Gamma(M)$. It is the edge-labelled graph $(I, E(M))$, where the edge-set is $E(M) := \{\{i, j\} \mid m_{ij} \geq 3\}$ and where an edge $\{i, j\} \in E(M)$ has the label $m_{ij}$. We do not distinguish between a Coxeter matrix and its diagram since they carry the same information. We call a Coxeter matrix irreducible if its associated Coxeter diagram is connected. An irreducible component of $M$ is a subset $J$ of $I$, which is a connected component of the diagram. A Coxeter matrix $M$ is called spherical if $W(M)$ is finite. The irreducible spherical Coxeter diagrams have been classified by H.S.M. Coxeter in [18]: we will use the Bourbaki notation for denoting them with the exception that we denote rank 2 diagrams for the dihedral groups of order $2n$ by $I_2(n)$. Thus we have the four series $A_n, C_n = B_n, D_n$ and $I_2(n)$ and the 6 exceptional diagrams $E_6, E_7, E_8, F_4, H_3$ and $H_4$.

An isomorphism from a Coxeter diagram $M = (m_{ij})_{i,j \in I}$ onto a Coxeter diagram $M' = (m'_{ij})_{i,j \in I'}$ is a graph isomorphism which preserves the edge-labels.

Let $M = (m_{ij})_{i,j \in I}$ be a Coxeter matrix over $I$ and let $J$ be a subset of $I$. Then we put $M_J := (m_{jk})_{j,k \in J}$ and $J^\perp := \{k \in I \mid m_{kj} = 2 \text{ for all } j \in J\}$.

A Coxeter matrix $M$ is called right-angled if all edge-labels of $\Gamma(M)$ are infinite; it is called 2-spherical if there are no infinities; it is called even if there are no odd labels and it is called of large type if the diagram is a complete graph (hence if there are no 2’s in $M$).

Coxeter systems. Let $(W,S)$ be a Coxeter system. The set of its reflections is defined to be the set $S^W := \{ws_1w^{-1} \mid s \in S \text{ and } w \in W\}$. The length of $w \in W$ is the length of a shortest product of elements in $S$ representing $w$; it is denoted by $l(w)$. We call $(W,S)$ right-angled, 2-spherical, even or of large type if this is the case for $M(S)$.

We list some facts about Coxeter systems which are important in the sequel. Facts 1 and 2 are basic and can be found in any standard reference on Coxeter groups (see [9] or [29]): Fact 3 is a non-trivial exercise in [9] but it follows also from the fact that the Davis-complex of a Coxeter system is CAT(0); Fact 4 is contained in [44]; Fact 5 can be shown by considering the geometric representation and Fact 6 is just an easy consequence of the definition of a Coxeter system.

1. If $J \subseteq S$, then $(\langle J \rangle, J)$ is a Coxeter system.
2. Let $J \subseteq S$ and $l : W \to \mathbb{N}$ be the length function of $(W, S)$. Then the following are equivalent:
   a) $(\langle J \rangle, J)$ is finite;
   b) there is an element $\rho_J$ such that $l(\rho_J) > l(x)$ for all $\rho_J \neq x \in \langle J \rangle$.
Moreover, if these two conditions are satisfied, then $\rho_J^2 = 1_W$.
3. If $X \leq W$ is a finite subgroup, then there exist $w \in W$ and $J \subseteq S$ such that $X^w \leq \langle J \rangle$ and such that $J$ is a spherical subset of $S$ (i.e. $\langle J \rangle$ finite).
4. Let $r \in W$ be an involution. Then there exist $w \in W$ and $J \subseteq S$ such that $J$ is spherical, $w\rho_Jw^{-1} = r$ and such that $\rho_J$ is central in $\langle J \rangle$.
5. Suppose that $J$ is a spherical subset of $S$ such that $\rho_J$ is central in $\langle J \rangle$.
   Then the normalizer of $\langle J \rangle$ in $W$ and the centralizer of $\rho_J$ in $W$ coincide.
6. Let $(W, S)$ be a Coxeter system. Then each permutation $\pi$ of $S$ which is an automorphism of $M(S)$ extends uniquely to an automorphism $\gamma_\pi$ of $W$.

Let $(W, S)$ be a Coxeter system. By Fact 6 we can identify the stabilizer of $S$ in $\text{Aut}(W)$ with the group of automorphisms of $M(S)$; this subgroup will be denoted by $\Gamma_S(W)$ and its elements are called the graph-automorphisms of $(W, S)$. The group $\Gamma_S$ has trivial intersection with the group $\text{Inn}(W)$ of inner automorphisms. An automorphism of $W$ will be called inner-by-graph if it can be written as a product of an inner automorphism and a graph-automorphism.

### 3 Rigidity

Let $G$ be a group and $R \subseteq G$ a set of involutions. Recall that the Coxeter matrix $M(R)$ is called the type of $R$; the set $R$ is called universal if $(\langle R \rangle, R)$ is a Coxeter system; it is called a Coxeter generating set of $G$ if it is universal and $G = \langle R \rangle$.

A Coxeter matrix $M$ is called rigid if for each Coxeter generating set $R$ of $W(M)$ the Coxeter diagrams $M(R)$ and $M$ are isomorphic. It is called strongly rigid if any two Coxeter generating sets of $W(M)$ are conjugate in $W(M)$.

Clearly, strong rigidity implies rigidity. If a Coxeter diagram is (strongly) rigid, then we call the corresponding Coxeter group and Coxeter system (strongly) rigid as well.

If one can show that the Coxeter diagram $M$ of Problem 1 is rigid, then this problem is trivially solved. The answer is just that the Coxeter diagram $M'$ has to be isomorphic to $M$.

Similarly, if one can show that the Coxeter diagram $M$ is strongly rigid, then Problem 2 is solved. An isomorphism onto $W(M')$ exists if and only if $M'$ and $M$ are isomorphic. Moreover, the automorphism group of $W(M)$ is just the semi-direct product of the group of inner automorphisms with the group of graph-automorphisms of $W(M)$; in other words: all automorphisms of $W$ are inner-by-graph.

There are several interesting classes of Coxeter systems which are not rigid. Before describing them we present some positive results. The first is due to D. Radcliffe [43].

**Theorem 3.1** Right-angled Coxeter systems are rigid.

Although we fixed the convention that all Coxeter systems in this paper are by definition of finite rank it is appropriate to mention that the theorem above has been generalized to right-angled Coxeter systems of arbitrary rank by A. Castella (see [15]). The next result about strong rigidity is the result of R. Charney and M. Davis already mentioned in the introduction (see [16]).
Theorem 3.2 Let \((W, S)\) be a Coxeter system. If \(W\) is capable of acting effectively, properly and cocompactly on some contractible manifold, then \((W, S)\) is strongly rigid. In particular, Coxeter groups of affine and compact hyperbolic type are strongly rigid.

The next result is very recent. An important step towards a proof of it was already made in [28]; in the version presented here it is a consequence of the main results in [14] and [23].

Theorem 3.3 Suppose that \((W, S)\) is irreducible, non-spherical and 2-spherical, then \((W, S)\) is strongly rigid.

In the following we describe two ways to manipulate the generating set of a given Coxeter system in order to produce a new one whose type is possibly non-isomorphic to the type of the original one. It is conjectured (and known to be true in a lot of special cases) that Coxeter systems are rigid up to these manipulations.

Pseudo-Transpositions. Let \(k \geq 1\) be a natural number and put \(n := 2(2k+1)\). We consider the dihedral group \(W\) of order \(2n\) as the group of isometries preserving a regular \(n\)-gon in the euclidian plane. Let \(s, t \in W\) be two reflections whose axes intersect in an angle \(\frac{\pi}{n}\) and let \(\rho\) be the central symmetry. Then it is easily seen that \(\{s, t\}\) and \(\{s, tst, \rho\}\) are both Coxeter generating sets for \(W\) of type \(I_2(n)\) and \(I_2(2k+1) \times A_1\) respectively. Thus, the dihedral group of order \(2n\) is a non-rigid Coxeter group because it has two Coxeter generating sets of different types. This example is of course trivial and a bit cheating because one of the two Coxeter matrices is not irreducible. However, it can be used to produce more general examples by taking direct products or free products. In [27] pseudo-transpositions have been introduced in order to describe the general feature.

Let \((W, S)\) be a Coxeter system and let \(\tau \in S\). We call \(\tau\) a pseudo-transposition if the following holds.

PT1 There is a unique \(t \in S\) such that \(o(\tau t) = 2(2k+1)\) for some natural number \(k \geq 1\).

PT2 For all \(s \in S \setminus \{\tau, t\}\) one has \(o(\tau s) \in \{2, \infty\}\) and if \(o(\tau) = 2\), then \(o(st) = 2\) as well.

The following is an easy observation about pseudo-transpositions.

Lemma 3.4 Let \((W, S)\) be a Coxeter system, let \(\tau \in S\) be a pseudo-transposition of \((W, S)\) and let \(t \in S\) be as in the definition above. Then \(S \setminus \{\tau\} \cup \{\tau t, \rho(\tau, t)\}\) is a Coxeter generating set of \(W\).

There is also another kind of pseudo-transpositions for Coxeter systems based on the fact that the Coxeter groups \(W(C_n)\) and \(W(D_n \times A_1)\) are isomorphic for odd \(n\). They yield also non-isomorphic Coxeter generating sets in a similar way. We refer to [27] for the details.

Let \((W, S)\) be a Coxeter system, let \(\tau \in S\) be a pseudo-transposition and let \(R\) be the ‘new’ Coxeter generating set as described in the lemma above. Then we call the Coxeter system \((W, R)\) an elementary reduction of \((W, S)\). A Coxeter system \((W, S')\) will be called a reduction of \((W, S)\) if it can be obtained from \((W, S)\) by a sequence of elementary reductions. Finally, we call \((W, S)\) reduced, if there are no pseudo-transpositions. It is easy to see that each Coxeter system has a reduced reduction.

Given a Coxeter diagram \(M\) over a set \(I\), then a Coxeter diagram \(M'\) over \(I'\) is called an elementary reduction of \(M\) if there is an elementary reduction of the
Coxeter system \((W(M), I)\) whose type is isomorphic to \(M'\); we call \(M'\) a reduction of \(M\) if \(M'\) can be obtained from \(M\) by a sequence of elementary reductions and we call \(M\) reduced if the system \((W(M), I)\) has no pseudo-transpositions.

Clearly, any rigid Coxeter system has to be reduced in view of Lemma 3.1 above. The following result is due to M. Mihalik [34] and is based on earlier work of P. Bahls [1]; it states that the converse is true for even Coxeter systems.

**Theorem 3.5** An even Coxeter system is rigid if and only if there is no pseudo-transposition.

Note that this result generalizes Theorem 3.1

**Twistings.** In this subsection we describe twistings as they were introduced in [10] and we give some further definitions concerning them.

Let \((W, S)\) be a Coxeter system and let \(J, K \subseteq S\). We call the pair \((J, K)\) an \(S\)-admissible pair if the following holds:

AD1 \(J\) is a spherical subset of \(S\) and \(K \cap (J \cup J^\perp) = \emptyset\).

AD2 For all \(k \in K\) and \(l \in L := S \setminus (J \cup J^\perp \cup K)\) the order of \(kl\) is infinite.

An \(S\)-admissible pair \((J, K)\) is called trivial if \(K\) or \(L\) are empty. For an \(S\)-admissible pair \((J, K)\) we put \(T_{(J,K)}(S) := J \cup J^\perp \cup K \cup \{\rho_I \rho_J | I \in L\}\).

The following lemma is not too difficult to prove (see [10]).

**Lemma 3.6** Let \((W, S)\) be a Coxeter system and let \((J, K)\) be a \(S\)-admissible pair. Then \(T_{(J,K)}(S)\) is a Coxeter generating set of \(W\) which is contained in \(SW\).

Let \((W, S), (J, K)\) and \(S' := T_{(J,K)}(S)\) be as in the previous lemma. If \(\rho_J\) is central in \(J\), then it is easily verified that \(M(S)\) is isomorphic to \(M(S')\). If \(\rho_J\) is not central in \(J\), then \(M(S)\) is not isomorphic to \(M(S')\) in the generic case. The following example of such a situation was given in [37].

**Example:** Let \((W, S)\) be a Coxeter system such that \(S = \{s_1, s_2, s_3, s_4\}\) and such that \(o(s_1 s_2) = o(s_2 s_3) = o(s_3 s_4) = 3\) and \(o(s_1 s_3) = o(s_1 s_4) = o(s_2 s_4) = \infty\). We put \(J := \{s_2, s_3\}\) and \(K := \{s_1\}\). It follows that

\[S' := T_{(J,K)}(S) := \{s'_1 := s_1, s'_2 := s_2, s'_3 := s_3, s'_4 := s_2 s_3 s_4 s_2 s_3 s_2\}\]

and that \(o(s'_1 s'_2) = o(s'_3 s'_4) = o(s'_1 s'_3) = 3\) and \(o(s'_1 s'_4) = o(s'_3 s'_4) = \infty\). Thus \(M(S)\) and \(M(S')\) are not isomorphic.

Let \(S, R\) be Coxeter generating sets of a group \(W\); we call \(R\) a twist of \(S\) if there is a \(S\)-admissible pair \((J, K)\) such that \(R = \rho_I T_{(J,K)}(S)\). It is readily verified that \(R\) is a twist of \(S\) if and only if \(S\) is a twist of \(R\) and that \(SW = RW\) in this case. A Coxeter generating set \(S\) is called twist-rigid if there are no non-trivial \(S\)-admissible pairs; i.e. if there are no twists of \(S\) which are not conjugate to \(S\) in \(W\).

Let \(M\) be a Coxeter matrix over \(I\). A Coxeter matrix \(M'\) is called a twist of \(M\) if there is a twist \(I'\) of \(I\) in the Coxeter system \((W(M), I)\) such that \(M(I')\) is isomorphic with \(M'\). As before one verifies that \(M'\) is a twist of \(M\) if and only if \(M\) is a twist of \(M'\).

We close this section with a result about strong rigidity for Coxeter groups. Obviously, if \((W, S)\) is a strongly rigid Coxeter system, then \(S\) has to be twist-rigid. The following theorem provides the converse under the additional assumption that all Coxeter generating sets \(R\) of \(W\) are contained in \(SW\). In view of Corollary 4.2 below there are ‘a lot of examples’ where this assumption holds.
Theorem 3.7 Let $M$ be a non-spherical, irreducible Coxeter diagram over $I$ such that there is no subdiagram of type $H_3$. Suppose that $I$ is a twist-rigid subset of $W(M)$ and that all Coxeter generating sets of $W(M)$ are contained in $I^{W(M)}$. Then $M$ is strongly rigid.

This theorem was first proved in the large-type case ($m_{ij} > 2$ for all $i, j$) in [38]; the result as it is stated above has been obtained recently by P.-E. Caprace [13].

4 The reduction to the restricted isomorphism problem

The restricted isomorphism problems for Coxeter groups are the following:

Problem 3: Given a Coxeter system $(W, S)$ and a Coxeter matrix $M$, decide whether there is a Coxeter generating set $R \subseteq S$ of $W$ such that $M(R) = M$.

Problem 4: Given a Coxeter system $(W, S)$ and a Coxeter matrix $M$, find all Coxeter generating sets $R \subseteq S$ of $W$ with $M(R) = M$.

In [27] Problems 1 and 2 of the introduction have been reduced to Problems 3 and 4 respectively. This reduction is based on the results on the finite continuation of a reflection in a Coxeter group, which have been obtained in [23]. The purpose of this section is to describe the results obtained in both references. The original motivation for the investigations in [23] was to find a tool to characterize reflections in abstract Coxeter groups. We first provide some examples, where an abstract Coxeter group does not determine ‘its set of reflections’.

We have already seen examples, where an abstract Coxeter group has different Coxeter generating sets yielding different sets of reflections. If $(W, S)$ is not reduced and if $R$ is an elementary reduction of $S$, then $S^W \not\subseteq R^W$ and $R^W \not\subseteq S^W$. We will now obtain further examples by producing automorphisms of Coxeter groups which do not preserve reflections. There are two kinds of such automorphisms, namely $s$-transvections and $J$-local automorphisms.

$s$-Transvections. Let $(W, S)$ be a Coxeter system and let $s \in S$. We define the odd connected component of $s$ in the diagram $\Gamma(S)$ to be the set of all elements $t \in S$ for which there is a path from $s$ to $t$ such that all its edge-labels are odd. We denote the odd component of $s$ by $\text{odd}(s)$ and we put

$$\text{eodd}(s) := \text{odd}(s) \cup \{t \in S \mid o(tt') \neq \infty \text{ for some } t' \in \text{odd}(s)\}.$$ 

Let $J_s$ denote the irreducible component of $\text{eodd}(s)$ which contains $s$ and let $K_s$ denote the union of all spherical irreducible components of $\text{eodd}(s)$ which do not contain $s$.

Let $z$ be an element in the center of $\langle K_s \rangle$. We define the mapping $\theta_{s,z} : S \to W$ by setting $\theta_{s,z}(t) = tz$ if $t \in \text{odd}(s)$ and by setting $\theta_{s,z}(t) = t$ for the remaining $t \in S$. One readily verifies that this mapping extends to an involutory automorphism of $W$ and that $sz$ is not contained in $S^W$. Hence $\theta_{s,z}(S)$ is a Coxeter generating set of $W$ providing a different set of reflections.

The involutory automorphism described above is called an $s$-transvection of the Coxeter system $(W, S)$. In fact, the definition of an $s$-transvection given in [27] is slightly more general. This is due to particular instances which might arise when there are subsystems of type $C_3$. Due to these instances the formal definition of an $s$-transvection is somewhat involved and will be omitted here. Nevertheless, we give an example of such a $C_3$-transvection because - unlike for the other kinds of automorphisms - it is not an ‘obvious automorphism easily seen from the diagram’.

Example Let \((W, S)\) be a Coxeter system where \(S = \{s, t, t', c\}\) such that \(o(st) = o(st') = 3, o(ct) = o(ct') = 4, o(sc) = 2\) and \(o(tt') = \infty\). Define \(\theta : S \to W\) by setting \(\theta(c) := c, \theta(s) := sc, \theta(t) := stcst\) and \(\theta(t') := st'cst'\). One verifies that \(\theta\) extends uniquely to an involutory automorphism of \(W\).

\[J\text{-local automorphisms.}\] Let \((W, S)\) be a Coxeter system. A subset \(J\) of \(S\) is called a graph factor of \((W, S)\) if \(J\) is spherical and if for all \(t \in S \setminus J\) either \(tj = jt\) for all \(j \in J\) or \(o(tj) = \infty\) for all \(j \in J\).

Let \(J\) be a graph factor of \((W, S)\) and let \(\alpha\) be an automorphism of \(\langle J\rangle\). Then it is readily verified that there is a unique automorphism of \(W\) stabilizing the subgroup \(\langle J\rangle\), inducing \(\alpha\) on it and inducing the identity on \(S \setminus J\). We call such an automorphism a \(J\)-local automorphism.

This observation can be used to produce non-reflection preserving automorphisms. There are lots of examples of finite Coxeter groups, having automorphisms which are not reflection preserving. Obvious examples are the elementary abelian 2-groups. A particularly interesting example is of course the exceptional automorphism of Sym(6) which is the Coxeter group of type \(A_5\).

The finite continuation of a reflection. Let \((W, S)\) be a Coxeter system. As \(S\) is supposed to be finite and as each finite subgroup of \(W\) is conjugate to a subgroup of some spherical standard parabolic subgroup it follows that there is an upper bound for the order of any finite subgroup of \(W\). This implies that there is for any subgroup \(X\) of \(W\) a unique maximal normal finite subgroup of \(X\) which we denote by \(\Omega_{\text{fin}}(X)\).

Let \(r \in W\) be an involution of \(W\); by the result of Richardson mentioned in Section 2 (Fact 4) we know that \(r\) is conjugate to some \(\rho_J\) for some spherical subset \(J\) of \(S\) and such that \(\rho_J\) is central in \(\langle J\rangle\). Now one knows that \(N_W(\langle J\rangle) = C_W(\rho_J)\) (Fact 5) and hence \(\langle J\rangle\) is contained in \(\Omega_{\text{fin}}(C_W(\rho_J))\). These considerations show that \(r\) must be a reflection if \(\Omega_{\text{fin}}(C_W(r)) = \langle r\rangle\). Hence we have found a handy criterion which ensures that a given involution of an abstract Coxeter group is a reflection for any Coxeter generating set of that group.

This idea was the starting point for the results obtained in [23]. It soon turned out that it is more convenient to work with the finite continuation \(\text{FC}(r)\) rather than with the group \(\Omega_{\text{fin}}(C_W(r))\). This is defined to be the intersection of all maximal finite subgroups of \(W\) containing \(r\). The main result of [23] is the following theorem. Its proof is based on a careful analysis of the centralizer of a reflection which had been described in detail in [11].

**Theorem 4.1** Let \((W, S)\) be a Coxeter system and let \(s \in S\). Then \(\text{FC}(s)\) is known. Moreover, if \(\text{FC}(s) = \langle s\rangle\), then \(s\) is a reflection for each Coxeter generating set of \(W\).

The description of \(\text{FC}(s)\) may become complicated if there are subsystems of type \(C_3\) or \(D_4\). If this is not the case, one can describe \(\text{FC}(s)\) by means of the subsets \(J_s\) and \(K_s\) defined in the paragraph on \(s\)-transvections as follows.

**Corollary 4.2** Let \((W, S)\) be a Coxeter system and suppose that there is no subsystem of type \(C_3\) or \(D_4\). Let \(s \in S\). If \(J_s\) is spherical, then \(\text{FC}(s) = \langle J_s \cup K_s\rangle\); in the remaining cases one has \(\text{FC}(s) = \langle \{s\} \cup K_s\rangle\). In particular, if \(K_s = \emptyset\) and \(J_s\) is non-spherical, then \(s\) is a reflection for each Coxeter generating set of \(W\).
The reduction theorem. Let \((W, S)\) be a Coxeter system. We call \(s \in S\) FC-centered if \(FC(s) = (J)\) for some \(J \subseteq S\). A fundamental reflection might not be FC-centered if there are subsystems of type \(C_3\) or \(D_4\). Moreover, the group of automorphisms of \(W\) which stabilize the subset \(S^W\) is denoted by \(\text{Ref}_S(W)\). We are now able to state the main result of [27].

**Theorem 4.3** Let \((W, S)\) be a reduced Coxeter system. For each FC-centered \(s \in S\), let \(T_s\) denote the group of all \(s\)-transvections of \((W, S)\). For each graph factor \(J \subseteq S\) let \(L_J\) denote the group of all \(J\)-local automorphisms of \((W, S)\). Let \(\Sigma\) be the subgroup of \(\text{Aut}(W)\) which is generated by all \(T_s\) and all \(L_J\), where \(s\) runs through the FC-centered elements of \(S\) and \(J\) runs through the set of graph factors of \((W, S)\). Let \(\bar{\Sigma}\) be the subgroup of \(\text{Aut}(W)\) which stabilizes \(FC(s)\) for all \(s \in S\). Then we have the following:

a) The group \(\bar{\Sigma}\) is finite and \(\Sigma \leq \bar{\Sigma}\). In particular, \(\Sigma\) is a finite subgroup of \(\text{Aut}(W)\).

b) Given a reduced Coxeter system \((W', S')\) and an isomorphism \(\alpha : W \rightarrow W'\), then there exists \(\sigma \in \Sigma\) such that \(\alpha(\sigma(S)) \subseteq S'^W\).

c) The group \(\Sigma\) (and hence also the group \(\bar{\Sigma}\)) is a finite supplement of \(\text{Ref}_S(W)\) in \(\text{Aut}(W)\).

Part b) of the theorem above says in particular, that if \((W, S)\) and \((W', S')\) are Coxeter systems which are both reduced and if there is an isomorphism from \(W\) onto \(W'\), then there is also an isomorphism between them which maps \(S^W\) onto \(S'^W\). This yields the reduction of Problem 1 to Problem 3 for reduced Coxeter systems. Moreover, given any reduced Coxeter system \((W, S)\), then its group of automorphism can be written as \(\Sigma \text{Ref}_S(W)\), hence Problem 2 is reduced to Problem 4 for reduced Coxeter systems.

### 5 The restricted isomorphism problem

In view of the reduction result described in the previous section it suffices to solve Problems 3 and 4 in order to solve Problems 1 and 2 respectively. Thus we are led to the following question.

**Question:** Let \((W, S)\) be a Coxeter system and let \(R \subseteq S^W\) be a Coxeter generating set of \(W\). What can be said about \(R\)?

We have to consider Coxeter generating sets whose elements are reflections in a given Coxeter system. The following is a first observation which can be shown by using the geometric representation of a Coxeter group.

**Lemma 5.1** Let \((W, S)\) be a Coxeter system, let \(R \subseteq S^W\) be a Coxeter generating set of \(W\) and let \(X \subseteq R\) be such that \((X)\) is finite. Then there exists a subset \(J\) of \(S\) and an element \(w \in W\) such that \((X)^w = (J)\). In particular, if \(r, r' \in R\) are such that \(o(r, r') = n \neq \infty\), then there exist \(s, s' \in S\) such that \(o(ss') = n\) and such that the subgroups \(\langle r, r' \rangle\) and \(\langle s, s' \rangle\) are conjugate.

Let \((W, S)\) be a Coxeter system and let \(R \subseteq S^W\) be a Coxeter generating set. We call \(R\) sharp-angled with respect to \(S\) if for any two reflections \(r, r' \in R\) there exists \(w \in W\) such that \(\{r, r'\}^w \subseteq S\).

Let \(W\) be the dihedral group of order \(2n\) for some natural number \(n \geq 2\). We consider \(W\) as the group of automorphisms of the regular \(n\)-gon in the euclidean plane. Let \(S = \{s, t\}\), where \(s\) and \(t\) are reflections whose axes intersect in an
angle \( \frac{\pi}{n} \). Given \( r \neq r' \in S^W \), then \( \{r, r'\} \) is sharp-angled with respect to \( S \) if the reflection axes of \( r \) and \( r' \) intersect in an angle \( \frac{\pi}{n} \).

**Angle-deformations.** Let \((W, S)\) be a Coxeter system, let \( s \neq t \in S \) be such that \( st \) has finite order, let \( x \in \langle s, t \rangle \) be such that \( \langle s, xtx^{-1} \rangle = \langle s, t \rangle \) and put \( Y := S \setminus \{\{s, t\} \cup \{s, t^\perp\}\} \). Let \( Y_x \) be the set of all \( y \in Y \) for which there exists a sequence \( y_1, \ldots, y_k = y \) in \( Y \) such that \( o(sy_1), o(y_1y_2), \ldots, o(y_{k-1}y_k) \) are finite and define \( Y_t \) analogously. We define the mapping \( \delta_x : S \to W \) by setting \( \delta_x(r) := r \) if \( r \in S \setminus (Y_t \cup \{t\}) \) and \( \delta_x(r) = xrx^{-1} \) in the remaining cases. The following is easy to verify.

**Lemma 5.2** If \( Y_s \cap Y_t = \emptyset \) then \( \delta_x \) extends uniquely to an automorphism of \( W \) which stabilizes the set \( S^W \).

If \( \{s, xtx^{-1}\} \) is not sharp-angled with respect to \( \{s, t\} \) and if \( \delta_x \) is as above, then \( \delta_x(S) \) is not sharp-angled with respect to \( S \). We therefore call the automorphisms of the lemma above *angle-deformations*.

The following result can be obtained by using rigidity of Fuchsian Coxeter groups in a similar way as it was done in [38].

**Proposition 5.3** Let \((W, S)\) be a Coxeter system and suppose that there is no 3-subset \( J \) of \( S \) such that \( M(J) = H_3 \). Let \( \Delta \) be the group generated by all angle deformations of \((W, S)\). Given a Coxeter generating set \( R \subseteq S^W \), then there exists \( \delta \in \Delta \) such that \( \delta(R) \) is sharp-angled with respect to \( S \).

In view of the previous proposition the following conjecture is known to be true for Coxeter systems having no subsystem of type \( H_3 \).

**Conjecture 1:** Let \((W, S)\) be a Coxeter system and \( R \subseteq S^W \) be a Coxeter generating set. Then there exists an automorphism \( \alpha \) of \( W \) such that \( \alpha(S^W) = S^W \) and such that \( \alpha(R) \) is sharp-angled with respect to \( S \).

**Twist-equivalence.** Let \((W, S)\) be a Coxeter system and let \( R \subseteq S^W \) be a Coxeter generating set of \( W \). Recall that \( R' \subseteq S^W \) is called a twist of \( R \) if there is an \( R \)-admissible pair \((J, K)\) such that \( R' = T_{(J, K)}(R) \). Moreover, \( R' \) is a twist of \( R \) if and only if \( R \) is a twist of \( R' \). By taking the transitive closure we obtain an equivalence relation on the set of the Coxeter generating sets contained in \( S^W \) which is called *twist-equivalence*.

If \( R' \) is a twist of \( R \subseteq S^W \), then \( R' \subseteq R^W \) and \( R' \) is sharp-angled with respect to \( R \). Hence, if \( R' \) is twist-equivalent with \( R \subseteq S^W \), then \( R' \subseteq R^W \) and \( R' \) is sharp-angled with respect to \( R \). There is some evidence that the converse is also true. This is the content of the conjecture below. This conjecture is a refinement of Conjecture 8.1 in [10].

**Conjecture 2:** Let \((W, S)\) be a Coxeter system and \( R \subseteq S^W \) a Coxeter generating set of \( W \) which is sharp-angled with respect to \( S \). Then \( R \) is twist-equivalent to \( S \).

At present, the following two theorems are known by recent work of P.-E. Caprace. The first improves earlier results obtained in [10], and [38].

**Theorem 5.4** Conjecture 2 holds for all Coxeter systems which do not contain an irreducible spherical subsystem of rank 3.

**Theorem 5.5** If \((W, S)\) is a Coxeter system such that \( M(J^\perp) \) is 2-spherical for each spherical subset \( J \) of \( S \), Conjecture 2 holds for \((W, S)\).
The main tool to prove Conjecture 2 in the references above is known to the experts as ‘Kac Conjugacy Theorem for root bases’. This theorem is proved in [31] for affine and compact hyperbolic groups. A proof for all Coxeter groups is given in [28].

6 The solution of Problem 1

Let $M$ be a Coxeter diagram over a set $I$. Recall that $M'$ is called a twist of $M$ if there is a twist $I'$ of $I \subseteq W(M)$ such that $M(I')$ is isomorphic to $M'$. Again, $M'$ is a twist of $M$ if and only if $M$ is a twist of $M'$ and by taking the transitive closure we obtain an equivalence relation on the set of Coxeter matrices which is called twist-equivalence as well.

The following lemma is easy to prove.

**Lemma 6.1** Let $(W, S)$ be a Coxeter system and let $M$ be a Coxeter matrix. Then the following are equivalent.

a) There exists a Coxeter generating set $R \subseteq S^W$ such that $M(R)$ is isomorphic to $M$ and such that $R$ is twist-equivalent to $S$.
b) The matrices $M(S)$ and $M$ are twist-equivalent.

Using the previous lemma one obtains the following theorem, which yields the solution of Problem 3.

**Theorem 6.2** Let $(W, S)$ and $(W', S')$ be Coxeter systems and suppose that Conjectures 1 and 2 hold for $(W, S)$. Then the following are equivalent.

a) $M(S)$ and $M(S')$ are twist-equivalent.
b) There exists an isomorphism $\alpha : W' \to W$ such that $\alpha(S') \subseteq S^W$.

We recall that a Coxeter system $(W, S)$ is reduced if the set $S$ contains no pseudo-transposition, that there is a natural notion of a Coxeter system or a Coxeter matrix to be a reduction of another and that it is always possible to produce a reduced reduction of a Coxeter system or Coxeter matrix by an easy algorithm. Now the previous theorem and Theorem 4.3 yield the following.

**Theorem 6.3** Let $M$ and $M'$ be irreducible Coxeter matrices of rank at least 3 and let $(W, S)$ be a Coxeter system of type $M$. If Conjectures 1 and 2 hold for $(W, S)$, then the following are equivalent.

a) The groups $W(M)$ and $W(M')$ are isomorphic.
b) If $M_1$ is a reduced reduction of $M$ and if $M'_1$ is a reduced reduction of $M'$, then $M_1$ and $M'_1$ are twist equivalent.

In view of Theorem 5.4 and Proposition 5.3 we have the following corollary.

**Corollary 6.4** Let $M$ and $M'$ be Coxeter matrices and suppose that $M$ has no subdiagram of type $A_3$, $C_3$ or $H_3$, then the following are equivalent:

a) The groups $W(M)$ and $W(M')$ are isomorphic.
b) If $M_1$ is a reduced reduction of $M$ and if $M'_1$ is a reduced reduction of $M'$, then $M_1$ and $M'_1$ are twist equivalent.

7 On automorphisms of Coxeter groups

The previous section shows that there is—under the hypothesis that Conjectures 1 and 2 are true—a satisfactory solution of Problem 1. Unfortunately, we cannot offer a satisfactory description of the automorphism groups of Coxeter groups under the same assumptions which would yield a solution of Problem 2 as well.
fact, the author has serious doubts whether such a handy description exists in the general case. Nevertheless there are several natural subgroups of the automorphism group of a Coxeter group which are quite well understood. In most of the ‘interesting’ cases, the understanding of these subgroups suffices to understand the group of automorphisms as a whole. Our discussion will be restricted to those subgroups. Before going more into the details we would like to mention that the automorphism group of a Coxeter group which are quite well understood. In most of the interest-

groups of Coxeter groups had been determined in various special cases.

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1. A presentation of the automorphism groups of right-angled Coxeter groups was given in [36]. This work was based on the results obtained in [45] and the latter is a far reaching generalization of the result in [30].

2. The automorphism groups of 2-spherical Coxeter groups are ‘trivial’ (i.e. all automorphisms are inner-by-graph) if there is no direct factor which is spherical. This result was accomplished in [13] and [23]. A ‘virtual’ result in this direction has been obtained already in [28] and the main tool developed there was used again in [14].

3. The automorphism groups of several classes of Coxeter groups which are ‘almost spherical’ have been described in [19], [20], [21] and [22]. In [22] a complete description of the automorphism groups of the irreducible spherical Coxeter groups is given.

Given an abstract Coxeter group $W$, then there is always a Coxeter generating set $S \subseteq W$ such that $(W, S)$ is reduced. Thus, there is no loss of generality if we consider only reduced Coxeter systems in this section. Let $(W, S)$ be a reduced Coxeter system. We define the following subgroups:

1. $\text{Ref}_S(W) := \{ \alpha \in \text{Aut}(W) \mid \alpha(S^W) = S^W \}$,
2. $\text{Ang}_S(W) := \{ \alpha \in \text{Ref}_S(W) \mid \alpha(S) \text{ sharp-angled with respect to } S \}$,
3. $\tilde{\Sigma}_S(W) := \{ \alpha \in \text{Aut}(W) \mid \alpha(FC(s)) = FC(s) \text{ for all } s \in S \}$
4. $\Gamma_S(W) := \{ \alpha \in \text{Aut}(W) \mid \alpha(S) = S \}$

In view of Theorem 1.3 we have $\text{Aut}(W) = \tilde{\Sigma}_S(W) \text{Ref}_S(W)$ and the group $\tilde{\Sigma}_S(W)$ is a finite group. Thus, there is a finite supplement of $\text{Ref}_S(W)$ in $\text{Aut}(W)$. There is the natural question about minimal supplements (or even complements) of $\text{Ref}_S(W)$ in $\text{Aut}(W)$. The example of the Coxeter group of type $A_1^k$ shows that there are not always complements. However, a careful analysis of several special cases provides some evidence for the following conjecture.

**Conjecture 3:** Let $(W, S)$ be a reduced Coxeter system. Then there exists a subgroup $\Omega \leq \tilde{\Sigma}_S(W)$ such that $\Pi := \Omega \cap \text{Ref}_S(W) \leq \Gamma_S(W)$ and such that $\Omega$ is a supplement of $\text{Ref}_S(W)$ in $\text{Aut}(W)$. Moreover, there is a normal 2-subgroup $U$ of $\Omega$ and a complement of $L$ of $U$ in $\Omega$ such that $L = L_1 \times L_2 \times \ldots L_k$ where $L_i$ is isomorphic to $\text{GL}(n_i, 2)$ for some natural number $n_i$ for $1 \leq i \leq k$ and $\Pi \cap L_i$ is just the set of permutation matrices.

There is a canonical candidate for the choice of the group $\Omega$ and based on this choice the validity of the conjecture is not difficult to see in several special cases. However, the arguments become somewhat involved in the general case.

**Reflection-preserving automorphisms.** As $\text{Ref}_S(W)$ has a finite supplement, a big part of $\text{Aut}(W)$ is understood if $\text{Ref}_S(W)$ is understood. A first observation is that $\text{Ang}_S(W)$ is a normal subgroup of finite index in $\text{Ref}_S(W)$ and therefore a similar remark holds for $\text{Ang}_S(W)$. We do not know whether $\text{Ang}_S(W)$ always has a finite supplement in $\text{Ref}_S(W)$ but we believe that there are examples where this is not the case. If there is no $H_3$-subdiagram, then the group $\text{Ref}_S(W)$
is generated by the angle-deformations of \((W, S)\) and \(\text{Ang}_S(W)\). We expect this to be true in general with a suitable definition of angle-deformations in the case where there are \(H_3\)-subdiagrams.

In the following we will consider the group \(\text{Ang}_S(W)\). Let

\[ R := \{ R \subseteq S^W \mid R \text{ sharp-angled Coxeter generating set of } W \text{ with respect to } S \} \]

and call two elements \(R \neq R'\) in \(R\) adjacent if one is a twist of the other. This yields a graph which we call \(C\). Conjecture 2 is equivalent to the statement that the graph \(C\) is connected.

We consider first the special case where \(M(S)\) is even in which case Conjectures 1 and 2 are known to be true. If \(M(S)\) is even, there is for each neighbor \(R\) of \(S\) in the graph \(C\) a canonical involution \(\theta_R\) in \(\text{Ang}_S(W)\) which switches \(S\) and \(R\). Setting \(X := \langle \theta_R \mid R \text{ neighbor of } S \rangle\), one verifies that \(C\) is the Cayley graph of \(X\) with respect to this generator set and that \(\Gamma_S\) is a complement of \(X\) in \(\text{Ang}_S(W)\). It is probably possible to generalize the arguments given in \[36\] in order to give a presentation of the group \(\text{Ang}_S(W)\). The key ingredient of such a generalization would be the observation that the group \(\text{Ang}_S(W)\) is something like a ‘generalized Coxeter group’ as it is in the right-angled case.

Let’s consider the general case under the assumption that Conjecture 2 holds. The situation becomes more complicated. The graph \(C\) is no longer the Cayley graph of a group but of a groupoid. We do not go into the details here. But it is worth mentioning that a similar situation occurs if one is interested in the normalizer of a parabolic subgroup in a Coxeter group. These normalizers had been described in \[8\] and \[12\] in a satisfactory way. The key observation in \[12\] is that they are finite index subgroups of a groupoid which one might call a Coxeter groupoid in view of its properties which are quite similar to those of Coxeter groups. We believe, that a presentation of \(\text{Ang}_S(W)\) can be given by using analogous ideas. It would be based on the observation that the graph \(C\) is the Cayley-graph of a generalized Coxeter groupoid of which \(\text{Ang}_S(W)\) is a subgroup of finite index. However, a concrete description of such a presentation might become rather involved.

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