ON THE COLOUR CONFINEMENT AND THE MINIMAL SURFACE

Sadatika Furui
School of Science and Engineering, Teikyo University, Utsunomiya, 320-8551, Japan
E-mail: furui@dream.ics.teikyo-u.ac.jp

Bilal Masud
Centre for High Energy Physics, Punjab University, Lahore, 54590 Pakistan
E-mail: bilalmasud@usa.net

In the analysis of the energy of the four-quark system obtained in the SU(2) lattice Monte Carlo, the f-model in which the transition potential is expressed in the form

\[ f = f_c e^{-k_A b_s A - k_P \sqrt{b_s} P} \]

where A is the area and P is the perimeter of the Wilson loop, was successful in the case of simple configurations of the four quarks. In the case of tetrahedral geometry, an estimation of the minimal surface whose contours run the positions of the four quarks is necessary. We show that the regular surface approximation whose area can be calculated analytically, is a good approximation for evaluating the minimal surface. The numerical value of the coefficient \( k_A b_s \) is close to \( 2 \text{fm}^{-2} \) which is the density of the \( Z_2 \) vortex in the SU(2) lattice Monte Carlo.

One of the most significant manifestations of the colour confinement in QCD is the area law. In 1994, Helsinki group measured the four-quark energies in quenched SU(2) lattice Monte Carlo at \( \beta = 2.4 \) and 2.5 on \( 16^4 \times 32 \) lattice. Quarks are infinitely heavy, but via rearrangement of the flux tubes, the two two-quark systems interact with each other and energy spectra of various kinds of four-quark configurations were measured. The energies were analysed by a method similar to the resonating group method based on Hamiltonian QCD. In this theory, the transition potential between different 4-quark configurations is expressed by

\[ V = \begin{pmatrix} v_{13} + v_{24} & (f/N_c)V_{AB} \\ (f/N_c)V_{AB} & v_{14} + v_{23} \end{pmatrix} \]

where

\[ f = f_c e^{-k_A b_s A - k_P \sqrt{b_s} P} \]

Here, \( b_s \) is defined from the string tension, A is the area and P is the perimeter of the minimal surface. The effective string action of QCD is expected to have the Nambu-Goto area term and Jacobian that comes from the field variables to the string variables. The long-range dynamics is dominated by the area term, and we include corrections by the perimeter dependent term.
We consider minimal surface spanned by four fixed points that makes a tetrahedron. If the conformality is ignored, a surface defined by four fixed points in $\mathbb{R}^3$ can be achieved by taking the surface spanned by straight lines $X_u$ and $X_v$, where $X(u,v)$ is defined as

$$X(u,v) = u(1-v)r_{41} + v(1-u)r_{42} + uv r_{43} \quad (\text{regular 2})$$

or

$$X(u,v) = u(1-v)r_{14} + v(1-u)r_{34} + uv r_{24} \quad (\text{regular 1})$$

where the $r_{ij} = r_i - r_j$ is the vector between position of quarks. Here 12 are on the bottom and 34 are on the top, 13 are on the left and 24 are on the right.

The surface spanned by straight lines is called the regular surface, and its area is given as

$$\text{area} = \int_0^1 du \int_0^1 dv |X_u \times X_v|$$

which can be expressed as an analytical function of parameters $r$ and $d$.

Since the regular surface is not the minimal surface, we performed variation such that the mean curvature becomes as small as possible. The ansatz that we adopt is,

$$X'(u,v) = X - tm(u,v)k$$

$$m(u,v) = uv(1-u)(1-v)[1 + cuv(1-u)(1-v)]H_n(u,v)$$

where

$$H_n(u,v) = -2X_u \cdot X_v (X_u \times X_v) \cdot X_{uv} \quad (8)$$

| c          | t            | area  |
|------------|--------------|-------|
| 0          | 0            | 1.2808|
| 0          | 0.48364      | 1.2794|
| -5.97149   | 0.49226      | 1.2793|

Table 1: The area calculated by variation. ($r = d = 1$)

As shown in the Table 1, and the Figure 1,2, the difference between the regular surface and the variationally obtained non-regular surface is small. Examples of other terahedra were also calculated.

If one relaxes the boundary condition, the minimal surface for 4 fixed points can be obtained by the conformal mapping. In the study of a string whose ends run with the light velocity, a minimal surface whose pair of opposite boundaries are straight lines but the other pairs are free, which is called the 'Gergonne’s surface', was studied. The conformal mapping which yields the
Figure 1: The regular surface. The area is 1.2808. 
Figure 2: The 'minimal' surface with four straight line boundary condition. The area is 1.2793.

Gergonne surface of a tetrahedron is given by 

\[ x = -u, \quad y = \cos u \sinh v/(\cos(\pi/4) \sinh(\pi/4)), \quad z = \sin u \sinh v/(\sin(\pi/4) \sinh(\pi/4)) \],

whose 3d-plot is shown in Fig.3. Due to the curved boundary, the area of this surface 13.807 is larger than that of the corresponding regular2 surface, which is 11.062. The area of the regular1 surface, which has different boundaries is 8.761.

When one relaxes all the four boundaries, the conformal mapping in the lowest non-trivial order 

\[ x = uv/3, \quad y = -v + (3u^2 v - v^3)/108, \quad z = u + (u^3 - 3uv^2)/108, \]

yields the minimal surface. An example shown in Fig. 4 has the area 49.4, which is larger than that of regular surface $36 \times 1.2808 = 46.12$.

In the effective string theory, the short distance dynamics is not defined by the area term alone, but the difference of the area of regular1 and regular2 surface allows to judge which type of fluxtube breaking is preferred.

| $\beta$ | $k_A$ | $b_a a^2$ | $a(\text{fm})$ | $k_A b_a (\text{fm}^{-2})$ | $k_P$ |
|---------|-------|------------|----------------|---------------------------|-------|
| 2.4     | 0.296(11) | 0.0724 | 0.12 | 1.5 | 0.080(2) | FGM |
| 2.4     | 0.38(4) | 0.0709 | 0.1194(9) | 1.89 | 0.087(10) | Pennanen |
| 2.5     | 0.73(8) | 0.0373 | 0.0866(9) | 3.63 | 0.008(13) | Pennanen |

Table 2: Parameters of the f-model.

The f-model for the four-quark system of simple configurations indicates that the perimeter contribution reduces as $\beta$ becomes large. The parameters used in the f-model are summarized in Table 2. The value of $k_A b_a$ is about

---

The text continues with more mathematical and theoretical content, including discussions on area calculations, boundary conditions, and effective string theory applications.
Figure 3: The minimal surface with two straight line boundary condition. The area is 13.81. \((r = \frac{3\pi}{2}, d = 2)\)

Figure 4: The minimal surface with curved boundary condition. The area is 49.4. \((r = d = 6)\)

\(2f m^{-2}\) which is close to the density of the \(Z_2\) centre vortex in the SU(2). Since in the change of the topology of flux tubes, piercing of a thick centre vortex is expected to play a role, this numerical coincidence is suggestive, but we need further investigation on the role of the centre vortex in the confinement.

S.F. is grateful to Dr. Polykarpov for the information on the string theory and the minimal surface.

References

1. A.M. Green, C. Michael and M. Sainio, Z. Phys. C67, 291 (1995); hep-lat/9404004; A.M. Green, J. Lukkarinen, P. Pennanen, C. Michael and S. Furui, Nucl. Phys. B(Proc. Suppl.) 42, 249 (1995).
2. S. Furui, A.M. Green and B. Masud, Nucl. Phys. A582, 682 (1995); S. Furui and B. Masud, Confinement III proc., June 1998, Jefferson Lab. hep-lat/9809079.
3. P. Pennanen, Phys. Rev. D55, 3958 (1997).
4. Yu. M. Makeenko and A. A. Migdal, Phys. Lett. B97, 253 (1980).
5. K. Langfeld, H. Reinhardt and O. Tennert, Phys. Lett. B419, 317 (1998).
6. P.W. Stephenson, Nucl. Phys. B539, 577 (1999).
7. T.G. Kovacs and E.T. Tomboulian, Phys. Lett. B463, 104 (1999).
8. M.P. do Carmo, Differential Geometry of Curves and Surfaces, (Prentice-Hall New Jersey), 1976; Differentialgeometrie von Kurven und Flächen, (Vieweg, Braunschweig), 1983, p.149.
9. B.L.G. Bakker, M.N. Chernodub, M.I. Polykarpov and N.C.J. Schoonderwoerd, private communication