THE TODA LATTICE IS SUPER–INTEGRABLE

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ABSTRACT

We prove that the classical, non–periodic Toda lattice is super–integrable. In other words, we show that it possesses $2N – 1$ independent constants of motion, where $N$ is the number of degrees of freedom. The main ingredient of the proof is the use of some special action–angle coordinates introduced by Moser to solve the equations of motion.

Mathematics Subject Classification: 37K10, 37J35, 70H06

Key words: Toda Lattice, super–integrable systems, Poisson brackets.

1 Introduction

The Toda lattice is arguably the most fundamental and basic of all finite dimensional Hamiltonian integrable systems. It has various intriguing connections with other parts of mathematics and physics.

The Hamiltonian of the Toda lattice is given by

$$H(q_1, \ldots, q_N, p_1, \ldots, p_N) = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}}.$$  (1)

This type of Hamiltonian was considered first by Morikazu Toda [1]. Equation (1) is known as the classical, finite, non–periodic Toda lattice to distinguish the system from the many and various other versions, e.g., the relativistic, quantum, infinite, periodic etc. The integrability of the system was established in 1974 independently by Flaschka [2], Hénon [3] and Manakov [4]. The original Toda lattice can be viewed as a discrete version of the Korteweg–de Vries equation. It is called a lattice as in atomic lattice since interatomic interaction was studied. This system also appears in Cosmology. It appears also in the work of Seiberg and Witten on supersymmetric Yang–Mills theories and it has applications in analog computing and numerical computation of eigenvalues. But the Toda lattice is mainly a theoretical mathematical model which is important due to the rich mathematical structure encoded in it.

Hamilton’s equations become
\[
\dot{q}_j = p_j \quad \quad \dot{p}_j = e^{q_{j-1} - q_j} - e^{q_j - q_{j+1}}.
\]

The system is integrable. One can find a set of independent functions \( \{H_1, \ldots, H_N\} \) which are constants of motion for Hamilton’s equations. To determine the constants of motion, one uses Flaschka’s transformation:

\[
a_i = \frac{1}{2} e^{\frac{1}{2}(q_i - q_{i+1})}, \quad b_i = -\frac{1}{2} p_i. \quad (2)
\]

Then

\[
\dot{a}_i = a_i (b_{i+1} - b_i), \quad \dot{b}_i = 2 (a_i^2 - a_{i-1}^2). \quad (3)
\]

These equations can be written as a Lax pair \( \dot{L} = [B, L] \), where \( L \) is the Jacobi matrix

\[
L = \begin{pmatrix}
b_1 & a_1 & 0 & \cdots & \cdots & 0 \\
an_1 & b_2 & a_2 & \cdots & \cdots & \\
0 & a_2 & b_3 & \cdots & \cdots & \\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\
0 & \cdots & \cdots & a_{N-1} & b_N
\end{pmatrix},
\]

and

\[
B = \begin{pmatrix}
0 & a_1 & 0 & \cdots & \cdots & 0 \\
-a_1 & 0 & a_2 & \cdots & \cdots & \\
0 & -a_2 & 0 & \cdots & \cdots & \\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\
0 & \cdots & \cdots & -a_{N-1} & 0
\end{pmatrix}.
\]

This is an example of an isospectral deformation; the entries of \( L \) vary over time but the eigenvalues remain constant. It follows that the functions \( H_j = \frac{1}{j} \text{tr} L^j \) are constants of motion. The mappings \( 2 \) is a projection from \( \mathbb{R}^{2N} \) to \( \mathbb{R}^{2N-1} \) and it is clearly not one–to–one. Replacing the vector \( \mathbf{q} \) by \( \mathbf{q} + c \) gives the same image.

The supper–integrability of this type of systems should be expected due to their dispersive asymptotic behavior. However, the construction of integrals is not typically a trivial task. In the case of the open Toda lattice, asymptotically the particles become free as time goes to infinity with asymptotic momenta being the eigenvalues of the Lax matrix. Therefore, the system behaves asymptotically like a system of free particles which is super–integrable.

The super–integrability of the Toda lattice for \( N = 2 \) was established in [5]. The additional integral in [5] was obtained using Noether’s theorem. We give the formula for the additional integral.
In the case of two degrees of freedom the potential is simply
\[ V(q_1, q_2) = e^{q_1 - q_2}, \]
and the procedure of Noether produces the following three integrals:
\[ H_1 = -\frac{1}{2}(p_1 + p_2), \quad J_1 = (p_1 - p_2)^2 + 4e^{q_1 - q_2}, \]
\[ I_1 = \frac{p_1 - p_2 + \sqrt{J_1}}{p_1 - p_2 - \sqrt{J_1}} \exp \left( \sqrt{J_1} \frac{q_1 + q_2}{p_1 + p_2} \right). \] (4)
Note that \( H = H_1^2 + \frac{1}{4}J_1 \) and that the function \( G = \frac{q_1 + q_2}{p_1 + p_2} \) which appears in the exponent of \( I_1 \) is a time function, i.e., it satisfies \( \{G, H\} = 1 \). We note that the integral \( I_1 \) remains an integral if we add a real constant to \( q_1 + q_2 \), an observation that will be important later on. The existence of the integral \( I_1 \) shows that the two degrees of freedom Toda lattice is superintegrable with three integrals of motion \( \{H_1, J_1, I_1\} \). As we will see, the complicated integral \( I_1 \) has a simple expression if one uses Moser’s coordinates.

2 Moser’s solution of the Toda lattice

Moser’s beautiful solution of the open Toda lattice uses the Weyl function \( f(\lambda) \) and an old (19th century) method of Stieltjes which connects the continued fraction of \( f(\lambda) \) with its partial fraction expansion. The key ingredient is the map which takes the \((a, b)\) phase space of tridiagonal Jacobi matrices to a new space of variables \((\lambda_i, r_i)\) where \( \lambda_i \) is an eigenvalue of the Jacobi matrix and \( r_i \) is related to the residue of some rational functions that appear in the solution of the equations. We present a brief outline of Moser’s construction.

Moser in [6] introduced the resolvent
\[ R(\lambda) = (\lambda I - L)^{-1}, \]
and defined the Weyl function
\[ f(\lambda) = (R(\lambda)e_N, e_N), \]
where \( e_N = (0, 0, \ldots, 0, 1) \).
The function \( f(\lambda) \) has a simple pole at \( \lambda = \lambda_i \). For the purpose of this paper, and this is also observed by Moser, we define \( f \) by the formula
\[ f(\lambda) = \left( \sum_{i=1}^{N} \frac{r_i^2}{\lambda - \lambda_i} \right) / \left( \sum_{i=1}^{N} r_i^2 \right). \] (5)
The differential equations in the variables \((\lambda, r)\) take a particularly simple form:
\[ \dot{\lambda}_i = 0 \]
\[ \dot{r}_i = -\lambda_i r_i. \] (6)
These equations show that \((\lambda_i, \log r_i)\) are action-angle variables for the Toda lattice.
The variables $a_i^2$, $b_i$ may be expressed as rational functions of $\lambda_i$ and $r_i$ using a continued fraction expansion of $f(\lambda)$ which dates back to Stieltjes. Since the computation of the continued fraction from the partial fraction expansion is a rational process the solution is expressed as a rational function of the variables $(\lambda_i, r_i)$. The procedure is as follows:

The $R_{NN}$ element of the resolvent, as defined previously, takes the following continued fraction representation:

$$f(\lambda) = \frac{1}{\lambda - b_N - \frac{a_{N-1}^2}{\lambda - b_{N-1} - \frac{a_{N-2}^2}{\lambda - b_{N-2} - \ldots}}}. \tag{7}$$

The function $f(\lambda)$ has $N$ simple poles at the eigenvalues of the Lax pair matrix $L$. Therefore, its partial fraction expansion has the form:

$$f(\lambda) = \left(\sum_{i=1}^{N} \frac{r_i^2}{\lambda - \lambda_i}\right) / \left(\sum_{i=1}^{N} r_i^2\right), \tag{8}$$

where the residue of $f(\lambda)$ at $\lambda = \lambda_i$ is $r_i^2 / (\sum_{i=1}^{N} r_i^2)$. Stieltjes described a procedure that allows one to express $a_i$ and $b_i$ in terms of $\lambda_1, \ldots, \lambda_N$ and $r_1, \ldots, r_N$. We briefly describe the method. We expand the partial fraction expansion of $f(\lambda)$ as given in (8) in a series of powers of $\frac{1}{\lambda}$. We obtain,

$$f(\lambda) = \left[\sum_{j=0}^{\infty} \sum_{i=1}^{N} \frac{r_i^2 \lambda_j^i}{\lambda^{j+1}}\right] / \left(\sum_{i=1}^{N} r_i^2\right).$$

The coefficient of $\lambda^{j+1}$ is denoted by $c_j$ and equals,

$$c_j = \left(\sum_{i=1}^{N} r_i^2 \lambda_j^i\right) / \left(\sum_{i=1}^{N} r_i^2\right), \quad j = 0, 1, \ldots.$$ 

The formulas of Stieltjes involve certain $i \times i$ determinants which we now define,

$$A_i = \begin{vmatrix} c_0 & c_1 & \ldots & c_{i-1} \\ c_1 & c_2 & \ldots & c_i \\ \vdots & \vdots & \ddots & \vdots \\ c_{i-1} & c_i & \ldots & c_{2i-2} \end{vmatrix}, \quad B_i = \begin{vmatrix} c_1 & c_2 & \ldots & c_i \\ c_2 & c_3 & \ldots & c_{i+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_i & c_{i+1} & \ldots & c_{2i-1} \end{vmatrix}. $$

The formulas that give the relation between the variables $(a, b)$ and $(r, \lambda)$ are,

$$a_{N-i}^2 = \frac{A_{i-1}A_{i+1}}{A_i^2}, \quad i = 1, \ldots, N - 1$$

$$b_{N+1-i} = \frac{A_iB_{i-2}}{A_{i-1}B_{i-1}} + \frac{A_{i-1}B_i}{A_iB_{i-1}}, \quad i = 1, \ldots, N$$

where $A_0 = 1, B_0 = 1, B_{-1} = 0$.

For example, in the case $N = 2$
\[ A_1 = c_0, \quad A_2 = c_0 c_2 - c_1^2, \quad B_1 = c_1, \quad B_2 = c_1 c_3 - c_2^2 \]

and therefore
\[ a_1^2 = A_2, \quad b_1 = \frac{A_2}{B_1} + \frac{B_2}{A_2 B_1}, \quad b_2 = B_1. \]

Thus,
\[
a_1^2 = \frac{r_1^2 r_2^2 (\lambda_2 - \lambda_1)^2}{(r_1^2 + r_2^2)^2}, \\
b_1 = \frac{r_1^2 \lambda_2 + r_2^2 \lambda_1}{r_1^2 + r_2^2}, \\
b_2 = \frac{r_1^2 \lambda_1 + r_2^2 \lambda_2}{r_1^2 + r_2^2}.  
\]

One can check that the differential equations \( \dot{r}_i = -r_i \lambda_i \), for \( i = 1, 2 \) correspond via transformation (9) to the \( A_2 \) Toda equations
\[
\dot{a}_1 = a_1 (b_2 - b_1) \\
\dot{b}_1 = 2a_1^2 \\
\dot{b}_2 = -2a_1^2 .
\]

As Moser notes, it is not too hard to obtain explicit expressions for \( N = 3 \) but the general case is quite complicated. With the exception of \( N = 2 \) the \( a_i \) are not rational functions of \( (\lambda_i, r_i) \) but the \( a_2^2 \) are. In general one can express (at least in theory) the functions \( a_i, b_i \) in terms of \( r_i, \lambda_i \). Again, the function is not one–to–one. In fact, if one replaces the vector \( \mathbf{r} \) with \( \alpha \mathbf{r} \) the result is the same.

Finally, we comment on the Poisson brackets in the new coordinates. The multi–hamiltonian structure of the Toda lattice was developed in [7] and [8] using master symmetries. The analogous results in \((\lambda, r)\) coordinates are due to Feybusovich and Gekhtman [9]. The Poisson brackets project onto some rational brackets in the space of Weyl functions and in particular, the Lie–Poisson bracket of the Toda lattice corresponds to the Atiyah–Hitchin bracket [10].

In general, one constructs a sequence of Poisson brackets on the space \((\lambda_i, r_i)\) whose image under the inverse spectral transform corresponds to the standard Toda hierarchy. A rational function of the form \( \frac{q(\lambda)}{p(\lambda)} \) is determined uniquely by the distinct eigenvalues of \( p(\lambda), \lambda_1, \ldots, \lambda_n \) and values of \( q \) at these roots. The residue is equal to \( \frac{q(\lambda_i)}{p'(\lambda_i)} \) and therefore we may choose

\[ \lambda_1, \ldots, \lambda_n, q(\lambda_1), \ldots, q(\lambda_n) \]

as global coordinates on the space of rational functions (of the form \( \frac{q}{p} \) with \( p \) having simple roots and \( q, p \) coprime). We have to remark that the image of the Moser map is a much larger set.

The \( k \)th Poisson bracket is defined by
\[
\{ \lambda_i, q(\lambda_i) \} = -\lambda_i^k q(\lambda_i) \\
\{ q(\lambda_i), q(\lambda_j) \} = \{ \lambda_i, \lambda_j \} = 0 .
\]
The initial Poisson bracket under the Moser map is given explicitly by

\[
\{\lambda_i, \lambda_j\} = 0 \\
\{r_i, r_j\} = 0 \\
\{\lambda_i, r_j\} = \delta_{ij} r_j, \quad i, j = 1, \ldots, N. \tag{10}
\]

Similarly, the quadratic Toda bracket, corresponds to a bracket with only non–zero terms \(\{\lambda_i, r_i\} = \lambda_i r_i\).

The Hamiltonian function in the new coordinates is \(H_2 = \frac{1}{2} \sum_{i=1}^{N} \lambda_i^2\). In other words, taking \(H_2\) as the Hamiltonian and using bracket (10) gives equations (6).

3 The Toda lattice is super–integrable

We now come to the main result of this paper. We define

\[
I_j = \left(\frac{r_j}{r_{j+1}}\right)^2 e^{F_{j,j+1}}, \quad j = 1, \ldots, N - 1, \tag{11}
\]

where

\[
F_{j,j+1} = \frac{2(\lambda_j - \lambda_{j+1})}{H_1} \ln \left(\prod_{i=1}^{N} r_i \right).
\]

It is easily shown, using equation (6) that \(\frac{dI_j}{dt} = 0\), for \(j = 1, \ldots, N - 1\) and thus the functions \(I_j\) are constants of motion.

The functions \(H_i = \lambda_1^i + \lambda_2^i + \ldots + \lambda_N^i\) and \(I_j, \quad i = 1, \ldots, N, \quad j = 1, \ldots, N - 1\) are functionally independent. In fact, the Jacobian \((2N - 1) \times 2N\) matrix of the functions \(H_i\) and \(I_j\) has a \((2N - 1) \times (2N - 1)\) subdeterminant, \(d_{N+1}\), which is obtained by deleting the \((N+1)\)-column and is not identically zero. A simple calculation gives

\[
d_{N+1} = -2^{N-1} N \frac{r_1^2}{r_{N-2}^2 r_{N-1}^3 r_N^3} \frac{\lambda_1}{H_1} e^{F_{1,N}} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j).
\]

Since the eigenvalues of real Jacobi matrices are distinct, the functions \(H_i\) and \(I_j\) are independent. We summarize the results in the following:

**Theorem 1** The Toda lattice with \(N\) degrees of freedom possesses \(2N - 1\) independent constants of motion, \(H_i, \quad i = 1, \ldots, N, \quad I_j, \quad j = 1, \ldots, N - 1\), and is therefore super–integrable.

**Remark 1**

It is clear that the functions \(H_n, \quad n = 1, \ldots, N\) are in involution. Moreover it can be shown that \(\{I_i, I_j\} = 0, \quad i, j = 1, \ldots, N - 1\). In addition, for \(n = 1, \ldots, N, \quad j = 1, \ldots, N - 1\)

\[
\{H_n, I_j\} = 2c_n \frac{(\lambda_j - \lambda_{j+1})}{H_1} E_j I_j,
\]

where \(c_n = 1\) for \(n = 2, \ldots, N, \quad c_1 = \frac{N}{N-2}\) and
\[ E_j = \sum \lambda_i^{n-1} - \left( \sum \lambda_i \right) w(n-2) - 2\lambda_{j+1}\lambda_j w(n-3). \]

The sums are taken over all \( i \) from 1 to \( N \) where \( i \neq j, j + 1 \) The function \( w(n) \) symbolizes the full homogeneous polynomial in \( \lambda_j \) and \( \lambda_{j+1} \) that have total weight equal to \( n \). For instance, \( w(0) = 0, n \in \mathbb{Z}^+ \), \( w(1) = \lambda_j + \lambda_{j+1} \), \( w(2) = \lambda_j^2 + \lambda_{j+1}^2 + \lambda_j\lambda_{j+1} \), etc.

One can, of course, use the quadratic Toda bracket in \((\lambda_i, r_i)\) coordinates. We must then take \( \text{Tr}L = \lambda_1 + \ldots + \lambda_N \) as Hamiltonian. However, in this bracket the \( H_i, I_j \) do not form a finite dimensional algebra.

**Remark 2**

We clearly have \( \{H_2, I_j\} = 0, j = 1, \ldots, N-1 \), since \( H_2 \) is the Hamiltonian and the functions \( I_j \) are constants of motion.

We define the sets \( S_1 = \{H_1, \ldots, H_N\} \) and \( S_2 = \{H_2, I_1, \ldots, I_{N-1}\} \). Then if \( f, g \in S_1 \Rightarrow \{f, g\} = 0 \) and if \( f, g \in S_2 \Rightarrow \{f, g\} = 0 \). In other words the sets \( S_1 \) and \( S_2 \) are both maximal sets of integrals in involution. We therefore have two different sets demonstrating the complete integrability of the Toda lattice.

**Remark 3**

We finally would like to comment on how the integrals \( I_j \) were guessed: The complicated integral (4) at the end of the introduction is quite simple in Moser’s coordinates. For example, \( \sqrt{J_1} \) is simply equal to \( 2(\lambda_2 - \lambda_1) \) and the expression

\[ \frac{p_1 - p_2 + \sqrt{J_1}}{p_1 - p_2 - \sqrt{J_1}} \]

reduces to \( -\left( \frac{r_1}{r_2} \right)^2 \). The exponent is simplified as follows. On the one hand,

\[ (q_1 + q_2) = p_1 + p_2 = -2(b_1 + b_2) = -2(\lambda_1 + \lambda_2). \]

On the other hand from \( \dot{r}_i = -r_i \lambda_i \) one obtains that \( \ln r_1 = -\lambda_i \). Therefore, the function \( q_1 + q_2 \) and \( 2 \ln r_1 r_2 \) differ only by a real constant. In other words, up to a constant, the exponent is simply

\[ \frac{2(\lambda_1 - \lambda_2)}{\lambda_1 + \lambda_2} \ln(r_1 r_2), \]

and up to a sign difference, the integral (4) is precisely the same as the one in (11).

**Acknowledgements:** The idea to use Moser’s coordinates is due to A. P. Veselov. The authors would like to thank him for suggesting this approach.

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