1 introduction

Model considerations are always very important for the concept clarification in theoretical and experimental physics. In particular, an exact solvable model may provide much more detailed informations which are useful for a complete understanding of the nature of the phenomena. Thus, in the study of magnetic phase transitions of inhomogeneous systems, different ways of modeling the systems can be found in the literature. But to implement the inhomogeneity of real systems into a model usually make the model itself hardly to have exact solution. However, inhomogeneous systems are frequently encountered in nature, and they also play an important role in material science. For example, to develope new materials, the inhomogeneous systems formed by doping with other materials in a bulk system are often grown artificially in laboratory. Thence, it is important to construct some exact solvable models so that we can deepen our insights to the nature of such systems. The hierarchical spin models may be viewed as one of the efforts in this direction.

About one and half decades ago, spin models defined on hierarchical lattices received much attention in the literature\cite{1-9}. Hierarchical lattices are defined as the infinite limit of iterative steps of replacing each fundamental bond or cell with a given type of decoration unit, and thence they are self-similar fractal lattices. Since hierarchical lattices are highly inhomogeneous in the local connectivity and lack of translational invariance, spin models defined on hierarchical lattices may provide a very good frame in understanding the magnetic properties of inhomogeneous systems. Moreover, we can apply the Migdal-Kadanoff renormalization\cite{10,11} scheme to obtain exact solutions of the partition functions of hierarchical spin models. Among a variety of hierarchical spin models, the zero-field $q$-state Potts model defined on diamond-hierarchical lattice has been widely studied. The results revealed from the exact solutions show that the corresponding systems have some nonclassical thermodynamic features. They include that the second-order phase transition occurs at finite temperature with the cusp-behavior appearing in the specific heat at the critical point\cite{8,9,12}, the distribution of the partition function zeros owns a multifractal structure on the complex temperature plane, and the free energy near the critical point has a spatially modulated structure\cite{6,7}.

We notice that owing to the infinite iterative steps of decorations in lattice construction, there exist a well-defined thermodynamic limit for a hierarchical spin model. Thence, the degree of the inhomogeneity is completely specified for a hierarchical spin model. If we had performed any finite iterative steps of decorations in lattice constrution, it would lead to the lack of thermodynamic limit for the model. To improve this while remain the model to be exactly solvable, we may study Ising systems defined on decorated lattices.

A decorated lattice is constructed by implementing bond- or cell- decorations to a regular lattice in an iterative way. The geometric complexity of a decorated lattice then is characterized by the decoration level $n$. For a decorated lattice of the decoration level $n$, a primary bond or cell defined on the original regular lattice is replaced by an $n$-bond or cell. An $n$-bond or cell owns a complicated
inherent structure formed by $n$ iterative steps of replacing a bond or cell with some basic unit of decoration. In the limit of infinite $n$, an $n$-bond or cell essentially possesses a fractal lattice as the inherent structure. For this type of lattices, the degree of inhomogeneity in the local connectivity is proportional to the decoration level $n$ and there exists a well-defined thermodynamic limit for any value of $n$.

Triangular type decorated Ising models are examples. The analyses on these models have been carried out by Plechko[13] and Plechko and Sobolev[14], and there exists some non-classical results. In particular, the results of [14] about the specific heat of the Ising model on cell-decorated triangular lattices have revealed some unusual features of ferromagnetic phase transitions. In the curve of the specific heat versus the temperature, as $n$ increases, the critical region is narrowing and a round peak appears above the critical point. While the critical point marks the set in for the long-range ordering among different $n$-cells, the appearance of the round peak corresponds to the occurrence of the short-range ordering inside an $n$-cell. As $n$ approaches to the limit of infinity, an $n$-cell has the Sierpiński gasket as its inherent structure. Then, in approaching the limit, $n$-cells are factorized and become independent in the Ising system. Moreover, the critical point approaches the zero temperature, and the round peak temperature remains finite.

Bond-decorated and cell-decorated lattices are different ways of modelling inhomogeneous systems. As the decoration level $n$ increases, the order of ramification remains a constant for a cell-decorated triangular type lattice but it increases unboundedly for a bond-decorated square lattice. Furthermore, it was pointed out by Gefen et. al. that there is no phase transition at finite temperature for a fractal lattice with the finite order of ramification[15]. Hence, the non-universal features of phase transitions in cell-decorated Ising models may be quite different from that of bond-decorations. To obtain a more complete picture about the possible features of inhomogeneous systems, we study the diamond-type bond-decorated Ising model. Note that the diamond-type bond-decorating was also adopted in a widely studied model, the diamond-hierarchical $q$-state Potts model[1–7]. In this work, we are interested in the following questions: (i) How does the degree of the inhomogeneity in a system, characterized by the decoration level $n$, affect the nature of the phase transition? (ii) How do the characteristic features of ferromagnetic phase transitions change with the lattice structure varying from cell-decorated triangular type lattices to bond-decorated square lattices? (iii) How do the non-universal features obtained from the diamond-hierarchical Ising model[1–9] appear? In particular, can we understand better about the cusp-behavior appearing in the specific heat at the critical point instead of the well-known logarithmic divergence?

The analytic expression of the partition function of square Ising model with diamond-type bond-decorations can be obtained by referring to the exact result of two-dimensional Ising model. The exact solution of the Ising model on a square lattice was first given by Onsager[16]. Since then, other alternatives in obtaining the exact solution have been developed. Besides of the combinatorial method, which was first developed by Kac and Ward[17] and then rigorously
reformulated by Green and Hurst[18], more recently Plechko used a nonstandard method of Grassman multipliers to obtain the analytic expressions of the partition functions for a variety of cases[13,14,19−21]. Based on the frame provided by these results, in Section 2 we use the bond-renormalization scheme to construct the analytic solution of the free energy. In Section 3, we determine the critical point of an n-lattice and study how the critical temperature vary with n. The condition of the critical point is given by the occurrence of the nonanalyticity in the free energy. In Section 4, we use the solution of the free energy to calculate the internal energy and specific heat in an analytic way. Since the free energy can be expressed as the sum of those contributed by the local interactions among the sub-bonds of an n-bond and the long-range interactions among different n-bonds, we also use the same separation in the internal energy and specific heat. Thence, in general, we can obtain some insights about the nature of the phase transition of an inhomogeneous system by comparing the contributions from the local and the long-range interactions. Our results indicate that the cross over from a finite-decorated system to an infinite-decorated system may not be a smooth continuation, we then calculate the critical values of the internal energy and specific heat of an infinite-decorated system in Section 5. Finally, Section 6 is reserved for the summary and discussion.

2 free energy

We study the Ising model defined on an arbitrary n-lattice. A two-dimensional square lattice is used as backbones for the construction of an n-lattice. The simple square lattice is referred as 0-lattice, and its connecting bonds between lattice sites are named as 0-bonds. For a 0-lattice, the total site-number is denoted as \( n_s \) and the total bond-number as \( n_b \) with \( n_b = 2n_s \). Starting from a 0-bond, we obtain the corresponding n-bond as the result of replacing a bond with a diamond iteratively \( n \) times. This n-bond then is specified by the two sites connected by the 0-bond. For an n-bond, we have the total site-number \( s^{(n)} = 2(4^n + 2)/3 \) and bond number \( b^{(n)} = 4^n \). The construction of an n-bond is schematized in Fig. 1. From the primary structure of a 0-lattice, we replace each 0-bond with an n-bond to form an n-lattice. For an n-lattice, the average site- and bond-numbers per unit square are \( N_s^{(n)} = 2s^{(n)} - 3 \) and \( N_b^{(n)} = 2b^{(n)} \) respectively. Examples of \( n \)-lattices are shown in Fig. 2.

The general form of the partition function for the Ising model defined on an \( n \)-lattice reads

\[
Z^{(n)} = \sum_{\{\sigma\}} \prod_{\langle i,j \rangle} \exp (\eta \sigma_i \sigma_j),
\]

where the sum is over all bond-connected pairs \( \langle i, j \rangle \) of the \( n \)-lattice, and the Ising spin takes two possible values \( \sigma_i = \pm 1 \). Here we consider uniform ferromagnetic couplings characterized by the coupling strength \( J \), and the dimensionless coupling parameter \( \eta \) is defined as \( \eta = J/\kappa_B T \).
To implement the bond-renormalization scheme in the evaluation of the partition function, we rewrite Eq. (1) as

\[
Z^{(n)} = 2^{n_k} (s^{(n)} - 2) \sum_{\{\sigma\}} \prod_{(\mu, \nu)} B^{(n)}_{(\mu, \nu)},
\]

(2)

where the Boltzmann \(n\)-factor, \(B^{(n)}_{(\mu, \nu)}\), is the contribution to the partition function from the Boltzmann factors associated with the \(n\)-bond specified by the two sites, \(\mu\) and \(\nu\), the sum is over the spins defined only on the lattice sites of the corresponding 0-lattice, and the front factor is added to compensate the normalization factor added in \(B^{(n)}_{(\mu, \nu)}\). Note that the number of the decorated sites of an \(n\)-bond is \(s^{(n)} - 2\), and the Ising spins on the decorated sites are referred as the inner-spins which couple only to the spins on the same \(n\)-bond.

Thence, the \(B^{(n)}_{(\mu, \nu)}\) of Eq. (2) is defined as the result of taking the sum over the inner-spins for the product of all Boltzmann factors associated with the \(n\)-bond,

\[
B^{(n)}_{(\mu, \nu)} = \left( \frac{1}{2} \right)^{(s^{(n)}-2)} \sum_{\{\sigma_k\}_{k=1, \ldots, (s^{(n)}-2)}} \exp \left[ \eta \left( \sigma_\mu \sigma_1 + \sigma_1 \sigma_2 + \ldots + \sigma_{s^{(n)}-2} \sigma_\nu \right) \right],
\]

(3)

where the sum is over the \((s^{(n)}-2)\) inner-spins, and the front factor is added for the normalization of the sum.

To obtain the explicit form of \(B^{(n)}_{(\mu, \nu)}\) for an arbitrary \(n\), first we express the Boltzmann 0-factor, which takes the form of \(\exp(\eta \sigma_\mu \sigma_\nu)\), as

\[
B^{(0)}_{(\mu, \nu)} = (\cosh \eta) + \sigma_\mu \sigma_\nu (\sinh \eta).
\]

(4)

According to the definition of Eq. (3), the Boltzmann 1-factor is given as

\[
B^{(1)}_{(\mu, \nu)} = \left( \frac{1}{2} \right)^2 \sum_{\sigma_1, \sigma_2} \exp \left[ \eta \left( \sigma_\mu \sigma_1 + \sigma_\mu \sigma_2 + \sigma_1 \sigma_\nu + \sigma_2 \sigma_\nu \right) \right].
\]

(5)

Using Eq. (4) for \(\exp(\eta \sigma_\mu \sigma_\nu)\), after some algebraic manipulations we obtain the result of Eq. (5) as

\[
B^{(1)}_{(\mu, \nu)} = \left[ \exp(\eta^{(1)}) \right] \left[ \cosh \left( \eta^{(1)} \right) + \sigma_\mu \sigma_\nu \sinh \left( \eta^{(1)} \right) \right],
\]

(6)

with

\[
\exp \left( \eta^{(1)} \right) = \cosh (2\eta).
\]

(7)

By continuing such construction, we can express Eq. (3) as

\[
B^{(n)}_{(\mu, \nu)} = R^{(n)}(\eta) \left[ \cosh \left( \eta^{(n)} \right) + \sigma_\mu \sigma_\nu \sinh \left( \eta^{(n)} \right) \right]
\]

(8)
for \( n \geq 1 \), where the function \( R^{(n)}(\eta) \) is given by
\[
R^{(n)}(\eta) = \prod_{k=1}^{n} \left[ \exp \left( \eta^{(k)} \right) \right]^{4^{n-k}},
\]
and \( \eta^{(k)} \) is the image of \( \eta^{(k-1)} \) under the renormalization map defined as
\[
\exp \left( \eta^{(k)} \right) = \cosh \left( 2\eta^{(k-1)} \right)
\]
with the initial condition, \( \eta^{(0)} = \eta \) for \( 1 \leq k \leq n \).

For the Ising system defined on a single \( n \)-bond, the partition function of Eq. (2) has \( n_b = 1 \) and it reduces to
\[
Z^{(n)}_{b} = 2^{s(n)} \left( \frac{1}{2} \right)^{2} \sum_{\sigma_{\mu}, \sigma_{\nu}} B^{(n)}_{(\mu, \nu)},
\]
After the substitution of Eq. (8) for \( B^{(n)}_{(\mu, \nu)} \), this yields
\[
Z^{(n)}_{b} = 2^{s(n)} R^{(n)}(\eta) \cosh \left( \eta^{(n)} \right).
\]
Then, the corresponding free energy per bond per \( k_B T \), is given as
\[
f^{(n)}_{b} = -\frac{1}{b^{(n)}} \ln Z^{(n)}_{b},
\]
where \( b^{(n)} \) is the total bond number contained in an \( n \)-bond. This, up to a constant, yields
\[
f^{(n)}_{b} = -\frac{1}{4^n} \ln \left[ \cosh \left( \eta^{(n)} \right) \right] - \sum_{k=1}^{n} \frac{\eta^{(k)}}{4^k}.
\]
Note that \( f^{(n)}_{b} \) in the limit of infinite \( n \) is exactly the free energy density of the diamond-hierarchical Ising model.

Using the result of Eq. (12) for \( Z^{(n)}_{b} \), we can express Eq. (2) in terms of \( Z^{(n)}_{b} \) as
\[
Z^{(n)} = 2^{n_s} \left( Z^{(n)}_{b} \right)^{n_b} Q^{(n)},
\]
where \( Q^{(n)} \), referred as the reduced partition function, takes the form of
\[
Q^{(n)} = \left( \frac{1}{2} \right)^{n_s} \sum_{\{\sigma\}} \left\{ \prod_{\langle \mu, \nu \rangle} \left[ 1 + \sigma_{\mu} \sigma_{\nu} \tanh \eta^{(n)} \right] \right\}.
\]
The reduced partition function of an \( n \)-lattice given by Eq. (16) has exactly the same form as that of simple square lattice when the variable \( \eta^{(n)} \) is changed
to $\eta$. Thence, the corresponding free energy can be easily written down according to the formal expression of exact solution of simple square lattice provided by Refs. [13, 16, 19, 21, 22]. Up to a constant, we can express the free energy per bond per $k_B T$ as

$$f^{(n)} = f_b^{(n)} + f_{bb}^{(n)},$$

(17)

where $f_b^{(n)}$ is given by Eq. (14), and $f_{bb}^{(n)}$ is the contribution from $Q^{(n)}$ and it is given as

$$f_{bb}^{(n)} = \frac{-1}{2N_b^{(n)}} \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta \frac{2\pi}{2\pi} \ln \left[ A_0^{(n)} - A_1^{(n)} (\cos \theta + \cos \phi) \right],$$

(18)

with $A_0^{(n)}$ and $A_1^{(n)}$ defined as

$$A_0^{(n)} = \frac{\cosh^2 (2\eta^{(n)})}{\cosh^4 (\eta^{(n)})},$$

(19)

$$A_1^{(n)} = \frac{2 \sinh (\eta^{(n)})}{\cosh^3 (\eta^{(n)})}.$$  

(20)

Since $f_b^{(n)}$ is the free energy density of independent $n$-bonds, $f_{bb}^{(n)}$ can be identified as the contribution from the interactions among different $n$-bonds. Thus, we refer $f_b^{(n)}$ as the contribution from the local interactions and $f_{bb}^{(n)}$ as that from the long-range interactions.

To simplify the expression of $f_{bb}^{(n)}$, we change the variables, $\omega_1 = (\theta + \phi)/2$ and $\omega_2 = (\theta - \phi)/2$, to rewrite Eq. (18) as

$$f_{bb}^{(n)} = -\frac{1}{2N_b^{(n)}} \ln A_0^{(n)} - \frac{1}{N_b^{(n)}} \int_0^{\pi/2} \frac{d\omega_2}{\pi} \int_0^{\pi} \frac{d\omega_1}{\pi} \ln \left( 1 - \kappa^{(n)} \cos \omega_1 \cos \omega_2 \right),$$

(21)

with

$$\kappa^{(n)} = \frac{2A_1^{(n)}}{A_0^{(n)}}.$$  

(22)

Then, the integral over $\omega_1$ can be easily performed and the result is

$$f_{bb}^{(n)} = -\frac{1}{2N_b^{(n)}} \ln A_0^{(n)} - \frac{1}{N_b^{(n)}} \int_0^{\pi/2} \frac{d\omega_2}{\pi} \ln \left[ \frac{1}{2} \left( 1 + \sqrt{1 - \kappa^{(n)} \sin^2 \omega} \right) \right].$$

(23)

### 3 critical point

The free energy density, $f_{bb}^{(n)}$, is not completely analytic for the physical region of temperature. The non-analyticity arises from the second derivative of $f_{bb}^{(n)}$ with respect to temperature at the point $\kappa^{(n)} = 1$ for the physical region $0 \leq \kappa^{(n)} \leq 1$. This implies that for the Ising model on an $n$-lattice, the long-range interactions
among different $n$-bonds are responsible for the occurrence of the Ising phase transition at the finite temperature, and the critical point is determined by the condition $\kappa^{(n)} = 1$. Thus, we can obtain the bulk critical temperature of the ferromagnetic phase transition from the solution of the condition,

$$A_0^{(n)} - 2A_1^{(n)} = 0.$$  \hspace{1cm} (24)

Here, for convenience, we use the notation, $\equiv$, to denote the equivalence established only at the critical temperature. With the definitions of $A_0^{(n)}$ and $A_1^{(n)}$ given by Eqs. (19) and (20), we can rewrite the critical condition as the familiar form, $\sinh 2\eta^{(n)} \equiv 1$, or as

$$\exp \left( \eta^{(n)} \right) \equiv h^{(0)}$$  \hspace{1cm} (25)

with

$$h^{(0)} = \sqrt{1 + \sqrt{2}}$$  \hspace{1cm} (26)

for an $n$-lattice.

As the consequence of the map of Eq. (10), Eq. (25) yields the critical value of the $\exp \left( \eta^{(n-1)} \right)$ variable as

$$\exp \left( \eta^{(n-1)} \right) \equiv h^{(1)}$$  \hspace{1cm} (27)

for $n \geq 1$ with

$$h^{(1)} = \left( h^{(0)} + \sqrt{(h^{(0)})^2 - 1} \right)^{1/2}.$$  \hspace{1cm} (28)

By applying the recursion relation of Eq. (10) continuously, we can establish the result,

$$\exp \left( \eta^{(n-k)} \right) \equiv h^{(k)}$$  \hspace{1cm} (29)

for $1 \leq k \leq n$, where $h^{(k)}$ is determined by the recursion relation

$$h^{(k)} = \left( h^{(k-1)} + \sqrt{(h^{(k-1)})^2 - 1} \right)^{1/2}.$$  \hspace{1cm} (30)

By taking $k = n$ in Eq. (29) we obtain the reduced critical temperature of the Ising transition as

$$\left( \frac{k_B T_c}{J} \right)_n = \left[ \ln \left( h^{(n)} \right) \right]^{-1}$$  \hspace{1cm} (31)

for an arbitrary $n$-lattice. For $n = 0$, this gives the well-known result, $k_B T_c / J = 2.269185$. As $n$ approaches $\infty$, the $h^{(n)}$ value can be determined by the fixed points of the map of Eq. (30). In fact, the $h^{(\infty)}$ value is the repellor of the map of Eq. (30), and this repellor locates at $h^{(\infty)} = 1.839287$. The corresponding critical temperature is $(k_B T_c / J)_{n \to \infty} = 1.641018$. However, the nature of
the phase transition at the repellor remains to be answered, and this will be discussed in Section 5 where the case of infinite decoration-level is discussed.

The numerical values of \((k_B T_c / J)_n\) for some different \(n\) values are given in Table 1, and the result as a function of \(n\) is shown in Fig. 3 with the solid curve given by the fitting function \((k_B T_c / J)_n = 1.6410 + (0.6281) \exp \left[-(0.5857) n\right]\). This result indicates that the critical temperature decreases exponentially toward the limiting value 1.6410 as \(n\) increases. On the other hand, it has been shown\([14]\) that for the cell-decorated triangular Ising model the critical temperature as a function of \(n\) is given by \((k_B T_c / J)_n = 4 / \ln \left\{ 4n - 4 \ln \left[ (n + 1) / 2 \right] + 1 \right\}\) for \(n \geq 1\), and this leads to \((k_B T_c / J)_n = 4 / \ln (4n)\) as \(n \to \infty\). Thus, the decreasing rate of the critical temperature toward the limit of infinite decoration level is more faster for the bond-decorated Ising model in comparing with the cell-decorated Ising model.

4 specific heat

In this section, we follow the standard procedure, for example as given by Ref. \([22]\), to calculate the specific heat for an \(n\)-lattice. The analytic result is expressed in terms of the complete elliptic integrals of the first and second kinds. The contributions to the specific heat contain those from the local and the long-range interactions, and the comparison between these two is given quantitatively. The critical behaviors of the specific heats of different \(n\)-lattices are also discussed. In particular, we compare the results with the cell-decorated case.

Using the free energy density \(f^{(n)}\) of an \(n\)-lattice given by Eq. (17), we define the dimensionless internal energy per bond as

\[
\epsilon^{(n)} = \frac{\partial f^{(n)}}{\partial \eta}.
\]  
(32)

Due to the separable contributions to the free energy density, we can write the internal energy density as the sum of two parts,

\[
\epsilon^{(n)} = \epsilon_b^{(n)} + \epsilon_{bb}^{(n)},
\]  
(33)

with \(\epsilon_b^{(n)} = \partial f_b^{(n)} / \partial \eta\) and \(\epsilon_{bb}^{(n)} = \partial f_{bb}^{(n)} / \partial \eta\). By using \(f_b^{(n)}\) and \(f_{bb}^{(n)}\) of Eqs. (14) and (23), we can perform the direct differentiation to obtain the corresponding internal energy density as

\[
\epsilon_b^{(n)} = - \left( \frac{\tanh \left( \eta^{(n)} \right)}{4^n} \right) \left( \frac{\partial \eta^{(n)}}{\partial \eta} \right) - \sum_{k=1}^{n} \left[ \left( \frac{1}{4^k} \right) \left( \frac{\partial \eta^{(k)}}{\partial \eta} \right) \right],
\]  
(34)

and

\[
\epsilon_{bb}^{(n)} = - \frac{1}{2 N_b^{(n)}} \left[ \left( \frac{\partial}{\partial \eta} \ln A_1^{(n)} \right) - \left( \frac{2}{\pi} \right) \left( \frac{\partial}{\partial \eta} \ln \kappa^{(n)} \right) K \left( \kappa^{(n)} \right) \right],
\]  
(35)
where the explicit forms of the functions, $\partial \ln A_1^{(n)}/\partial \eta$ and $\partial \ln \kappa^{(n)}/\partial \eta$, can be obtained by direct differentiations of Eqs. (20) and (22),

$$
\frac{\partial}{\partial \eta} \ln A_1^{(n)} = \left( \frac{\cosh (\eta^{(n)})}{\sinh (\eta^{(n)})} - \frac{3 \sinh (\eta^{(n)})}{\cosh (\eta^{(n)})} \right) \left( \frac{\partial \eta^{(n)}}{\partial \eta} \right),
$$

(36)

and

$$
\frac{\partial}{\partial \eta} \ln \kappa^{(n)} = \left( \frac{2 \cosh (2\eta^{(n)})}{\sinh (2\eta^{(n)})} - \frac{4 \sinh (2\eta^{(n)})}{\cosh (2\eta^{(n)})} \right) \left( \frac{\partial \eta^{(n)}}{\partial \eta} \right),
$$

(37)

with

$$
\frac{\partial \eta^{(k)}}{\partial \eta} = 2^k \left\{ \prod_{j=1}^{k} \left[ \tanh \left( 2\eta^{(j-1)} \right) \right] \right\}
$$

(38)

for $1 \leq k \leq n$ obtained from the map of Eq. (10), and $K\left( \kappa^{(n)} \right)$ is the complete elliptic integral of the first kind defined as

$$
K\left( \kappa^{(n)} \right) = \int_0^{\pi/2} \frac{d\omega}{(1 - \kappa^{(n)} \sin^2 \omega) \omega^{1/2}},
$$

(39)

We notice that $K\left( \kappa^{(n)} \right)$ is singular at the critical point $\kappa^{(n)} = 1$. As $\kappa^{(n)} \to 1$, it can be shown[22] that

$$
K\left( \kappa^{(n)} \right) \bigg|_{T \to T_c} = - \ln \left| \frac{T - T_c}{T_c} \right|.
$$

(40)

This logarithmic divergence appears in the specific heat, and it becomes the signature of the Ising phase transition in two-dimensions. But, by substituting the critical condition, $\sinh 2\eta^{(n)} \equiv 1$, into Eq. (37), we obtain

$$
\frac{\partial}{\partial \eta} \ln \kappa^{(n)} \equiv 0.
$$

(41)

As a consequence of Eq. (41), we see that $\epsilon_{bb}^{(n)}$ of Eq. (35) is a continuous function of temperature $T$ even at the critical point, $T = T_c$.

To have a qualitative understanding about the variations of $\epsilon_b^{(n)}$ and $\epsilon_{bb}^{(n)}$ with respect to the decoration-level $n$, we compute the critical values of $\epsilon_b^{(n)}$ and $\epsilon_{bb}^{(n)}$ for different $n$ values. By using the critical condition of Eq. (29) for an $n$-lattice, we obtain the critical values of Eqs. (36) and (38) as

$$
\frac{\partial}{\partial \eta} \ln A_1^{(n)} \equiv \left( 4 - 2\sqrt{2} \right) \left( \frac{\partial \eta^{(n)}}{\partial \eta} \right),
$$

(42)

and

$$
\frac{\partial \eta^{(k)}}{\partial \eta} \equiv 2^k \left\{ \prod_{j=n-k+1}^{n} \left[ \left( \frac{h(j)}{h(j)} \right)^4 - 1 \right] \right\}
$$

(43)
for $1 \leq k \leq n$. Here and hereafter, we use the subscript $c$ to denote that the quantity in the parenthesis is evaluated at the critical point of an $n$-lattice. Then, using Eqs. (41) and (42) we can express the critical value of $\epsilon_b^{(n)}$ and $\epsilon_{bb}^{(n)}$ given by Eqs. (34) and (35) as

$$\epsilon_{b}^{(n)} = -\frac{1}{4n} \left( \frac{\sqrt{2}}{2+\sqrt{2}} \right) \left( \frac{\partial\eta(n)}{\partial\eta} \right)_c - \sum_{k=1}^{n} \left[ \left( \frac{1}{4^k} \right) \left( \frac{\partial\eta(k)}{\partial\eta} \right)_c \right],$$

and

$$\epsilon_{bb}^{(n)} = -\frac{1}{4^{n+1}} \left( 4 - 2\sqrt{2} \right) \left( \frac{\partial\eta(n)}{\partial\eta} \right)_c,$$

with $(\partial\eta(k)/\partial\eta)_c$ given by Eq. (43).

The numerical results of Eqs. (44) and (45) for some decoration levels $n$ are given in Table 1, and the corresponding curves are shown in Fig. 4. These results indicate that the local correlations among the $b^{(n)}$ bonds of a given $n$-bond are enhanced and the long-range correlations among different $n$-bonds are reduced as $n$ increases. For $n \geq 6$, the long-range correlations are almost absent, and the system is very close to that composing of independent $n$-bonds.

In calculating the specific heat $C_V^{(n)}$ of an $n$-lattice, we use the dimensionless quantity, $c^{(n)} = C_V^{(n)}/k_B$, defined as

$$c^{(n)} = -\eta^2 \frac{\partial\epsilon^{(n)}}{\partial\eta}.$$

Similar to the internal energy density, we can express $c^{(n)}$ as the sum of two parts,

$$c^{(n)} = c_b^{(n)} + c_{bb}^{(n)},$$

with $c_b^{(n)} = -\eta^2 \partial\epsilon_b^{(n)}/\partial\eta$ and $c_{bb}^{(n)} = -\eta^2 \partial\epsilon_{bb}^{(n)}/\partial\eta$.

For the direct differentiation of $\epsilon_b^{(n)}$ of Eq. (34) with respect to $\eta$, we obtain

$$\frac{c_b^{(n)}}{\eta^2} = \frac{1}{4^n} \left[ \left( \frac{\partial\eta(n)/\partial\eta}{\cosh(\eta(n))} \right)^2 + \left( \tanh(\eta(n)) \right) \left( \frac{\partial^2\eta(n)/\partial\eta^2}{\partial\eta^2} \right) \right] + \sum_{k=1}^{n} \left[ \left( \frac{1}{4^k} \right) \left( \frac{\partial^2\eta(k)/\partial\eta^2}{\partial\eta^2} \right) \right],$$

where the function, $(\partial\eta(k)/\partial\eta)_c$, is given by Eq. (38), and its further derivative yields

$$\frac{\partial^2\eta(k)/\partial\eta^2}{\partial\eta^2} = \left( \frac{\partial\eta(k)/\partial\eta}{\partial\eta^2} \right) \left\{ \sum_{i=1}^{k} \left[ \left( \frac{2^{i+1}}{\sinh(4\eta(i-1))} \right) \prod_{j=1}^{i-1} \tanh \left( 2\eta(j-1) \right) \right] \right\},$$

for $k \geq 1$. Note that in Eq. (49) the product for the index $j$ in the last parenthesis exists only for $i \geq 2$. 

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On the other hand, the direct differentiation of \( \epsilon_{bb}^{(n)} \) of Eq. (35) with respect to \( \eta \) yields

\[
\frac{\epsilon_{bb}^{(n)}}{\eta^2} = \frac{1}{2N_b^{(n)}} \left\{ \left( \frac{\partial^2}{\partial \eta^2} \ln A_1^{(n)} \right) - \frac{2}{\pi} \left[ P_0(\kappa^{(n)}) K(\kappa^{(n)}) + Q_0(\kappa^{(n)}) E(\kappa^{(n)}) \right] \right\},
\]

(50)

where the functions, \( P_0(\kappa^{(n)}) \) and \( Q_0(\kappa^{(n)}) \), are defined as

\[
P_0(\kappa^{(n)}) = \left( \frac{\partial^2}{\partial \eta^2} \ln \kappa^{(n)} \right) - \left( \frac{\partial}{\partial \eta} \ln \kappa^{(n)} \right)^2,
\]

(51)

and

\[
Q_0(\kappa^{(n)}) = \left( \frac{1}{1 - (\kappa^{(n)})^2} \right) \left( \frac{\partial}{\partial \eta} \ln \kappa^{(n)} \right)^2,
\]

(52)

and \( E(\kappa^{(n)}) \) is referred as the complete elliptic integral of the second kind and defined as

\[
E(\kappa^{(n)}) = \int_0^{\pi/2} d\omega \left( 1 - \kappa^{(n)2} \sin^2 \omega \right)^{1/2}.
\]

(53)

Note that in obtaining Eq. (50) we have used the well-known formula[22],

\[
\frac{\partial K(\kappa^{(n)})}{\partial \kappa^{(n)}} = \kappa^{(n)} \left[ \frac{1}{1 - (\kappa^{(n)})^2} \right] E(\kappa^{(n)}) - K(\kappa^{(n)}) \right].
\]

(54)

We also notice that by further differentiating the results of Eqs. (36) and (37) with respect to \( \eta \) we obtain

\[
\frac{\partial^2}{\partial \eta^2} \ln A_1^{(n)} = -P_1(\eta^{(n)}) \left( \frac{\partial \eta^{(n)}}{\partial \eta} \right)^2 + Q_1(\eta^{(n)}) \left( \frac{\partial^2 \eta^{(n)}}{\partial \eta^2} \right),
\]

(55)

and

\[
\frac{\partial^2}{\partial \eta^2} \ln \kappa^{(n)} = -P_2(\eta^{(n)}) \left( \frac{\partial \eta^{(n)}}{\partial \eta} \right)^2 + Q_2(\eta^{(n)}) \left( \frac{\partial^2 \eta^{(n)}}{\partial \eta^2} \right),
\]

(56)

with

\[
P_1(\eta^{(n)}) = \frac{1}{\sinh^2(\eta^{(n)})} + \frac{3}{\cosh^2(\eta^{(n)})},
\]

(57)

\[
Q_1(\eta^{(n)}) = \cosh(\eta^{(n)}) \frac{3 \sinh(\eta^{(n)})}{\sinh(\eta^{(n)})} \cosh(\eta^{(n)})
\]

(58)

\[
P_2(\eta^{(n)}) = \frac{4}{\sinh^2(2\eta^{(n)})} + \frac{8}{\cosh^2(2\eta^{(n)})},
\]

(59)

and

\[
Q_2(\eta^{(n)}) = \frac{2 \cosh(2\eta^{(n)})}{\sinh(2\eta^{(n)})} - \frac{4 \sinh(2\eta^{(n)})}{\cosh(2\eta^{(n)})}.
\]

(60)
The values of \( c^{(n)}_{b} \) and \( c^{(n)}_{bb} \) of Eqs. (48) and (50) can be calculated by using the explicit forms of Eqs. (36-38), (49), (55), and (56). The numerical results as a function of reduced temperature \( k_B T / J \) for some \( n \) values are shown in Fig. 5. The peak temperature of \( c^{(n)}_{b} \), denoted by \((k_B T_m / J)_n\), marks the occurrence of the local ordering in an \( n \)-bond, and the numerical values of \((k_B T_m / J)_n\) for some decoration levels \( n \) are given in Table 1. The plot of \((k_B T_m / J)_n\) versus \( n \) is also shown in the lower part of Fig. 3 with the solid curve given by the fitting function \((k_B T_m / J)_n = 1.6410 - (0.8063) \exp \left[ - (0.7144) n \right]\). Thus, because of the enhancement of the local correlation for the increase of \( n \), \((k_B T_m / J)_n\) increases as \( n \) increases. Moreover, the peak temperature of \( c^{(n)}_{bb} \) is the critical temperature \((k_B T_c / J)_n\) which signifies the occurrence of the long-range ordering among the Ising spins defined on the primary sites, and the behavior of \((k_B T_c / J)_n\) versus \( n \) is already given in the upper part of Fig. 3. The results of Fig. 3 indicate that the local ordering occurs at the temperature below the critical temperature, and this is exactly opposite to the case of cell-decorated triangular Ising model[14].

The singular behavior of \( c^{(n)} \) arises from \( c^{(n)}_{bb} \) of Eq. (50) in the form of the logarithmic divergence given by Eq. (40). Thus, we can express the singular behavior of \( c^{(n)} \) as

\[
c^{(n)} = A^{(n)}_{\sin g} \ln \left| \frac{T - T_c}{T_c} \right| .
\]

(61)

where \( A^{(n)}_{\sin g} \), referred as the critical amplitude, is given as

\[
A^{(n)}_{\sin g} = - \frac{\eta_c^2}{N_b^{(n)} \pi} \left[ P_0 \left( \kappa^{(n)} \right) \right]_c ,
\]

(62)

with \( P_0 \left( \kappa^{(n)} \right) \) given by Eq. (51). By using Eq. (41), we can rewrite the expression of \( A^{(n)}_{\sin g} \) as

\[
A^{(n)}_{\sin g} = - \frac{\eta_c^2}{N_b^{(n)} \pi} \left( \frac{\partial^2}{\partial \eta^2} \ln \kappa^{(n)} \right)_c .
\]

(63)

By taking the critical value of Eq. (56) with \([Q_2 \left( \eta^{(n)} \right)]_c = 0\) and \([P_2 \left( \eta^{(n)} \right)]_c = 8\), we can use the result of Eq. (43) to obtain

\[
\frac{\partial^2}{\partial \eta^2} \ln \kappa^{(n)} \leq - \frac{A^{n+2}}{2} \left\{ \prod_{k=1}^{n} \left[ \frac{\left( h^{(k)} \right)^4 - 1}{\left( h^{(k)} \right)^4 + 1} \right] \right\}^2 .
\]

(64)

Thus, using the reduced critical temperature given by Eq. (31) we can express Eq. (63) as

\[
A^{(n)}_{\sin g} = \frac{4}{\pi} \left[ \ln h^{(n)} \right]^2 \left\{ \prod_{k=1}^{n} \left[ \frac{\left( h^{(k)} \right)^4 - 1}{\left( h^{(k)} \right)^4 + 1} \right] \right\}^2 .
\]

(65)

The numerical values of Eq. (65) for some \( n \) values are given in Table 1. The plot of \( A^{(n)}_{\sin g} \) versus \( n \) is also given in Fig. 6 where the solid curve is given.
by the fitting function $A_{\text{sin}}^{(n)} = (0.2473) \exp[-(0.3018)n]$. Since the critical amplitude associated with the logarithmic singularity characterizes the width of the critical region, our results indicate that as increasing $n$ the critical region is narrowing down to the vanish limit. This characteristic feature in the specific heat can also be seen in the model on anisotropic regular lattices[13, 14, 20, 22]. We also notice that for the cell-decorated triangular Ising model the critical amplitude behaves as $A_{\text{sin}}^{(n)} \sim 3^{-n} (n \ln n)^2$ for very large $n$ [14]. Thus, as the decoration level $n$ increases the critical amplitude approaches zero less quickly for the bond-decorated model in comparing with the cell-decorated model.

5 infinite decoration-level

In this section, we discuss the critical properties of the Ising model on an $n$-lattice in the limit of infinite $n$. First we demonstrate the absence of Ising phase transition. This feature is caused mainly by the fragmentation of $n$-bonds. Then, we calculate the critical values of the internal energy and specific heat. The specific heat curve near the critical point becomes a cusp, and the exact critical value is given.

The map of Eq. (10) has three fixed points, $\eta_f$, given by the solutions of

$$\eta_f = \ln \left[ \cosh \left(2\eta_f\right)\right].$$

(66)

Among the three fixed points, two are attractors locating at $\eta_f^{a1} = 0$ and $\eta_f^{a2} = \infty$ respectively, and one is a repellor locating at $\eta_f = 0.609378$. For the limit of infinite decoration-level, the repellor is the critical point of the bulk system. Since the critical point is a repellor, the critical behavior of the system may be different from that obtained by extrapolating the results of finite-decorated systems to the limit of infinite $n$. To show this, first we give the plots of $\eta^{(n)}$ versus $\eta$ for $n = 0, 2, 4, \text{ and } 6$ in Fig. 7. The results of Fig. 7 indicate that for the limit of infinite $n$ the possible $\eta^{(n)}$ values are restricted to the fixed points of Eq. (66): $\eta^{(n)} = \eta_f^{a1}$ for $\eta < \eta_f^r$, $\eta^{(n)} = \eta_f^r$ for $\eta = \eta_f^r$, and $\eta^{(n)} = \eta_f^{a2}$ for $\eta > \eta_f^r$. This leads to the result that the value of $\kappa^{(n)}$ defined by Eq. (22) has only two possible values,

$$\kappa^{(\infty)} = 0$$

for $\eta \leqslant \eta_f^r$, and

$$\kappa^{(\infty)} = 2\sqrt{\left[ \exp(-2\eta_f^r) \right] - \left[ \exp\left(-4\eta_f^r\right) \right]}$$

(68)

for $\eta = \eta_f^r$. The latter gives the numerical value $\kappa^{(\infty)} = 0.912622$. Thence, the logarithmic divergence of the Ising transition, caused by the divergence of the elliptic integral $K(\kappa^{(n)})$ of Eq. (39) at the point $\kappa^{(n)} = 1$, is absent for the infinite $n$ limit. This indicates that the cross over from a finite-decorated system to an infinite-decorated system may not be a smooth continuation.
To exhibit the critical properties of the infinite-decorated system more explicitly, we then calculate the critical values of the internal energy and specific heat. Since the critical point of the infinite-decorated system is the repellor of the map of Eq. (10), we have the following identities for the derivatives evaluated at the critical point $\eta = \eta^r_f$:

$$ \frac{\partial \eta^{(i+k)}}{\partial \eta^{(i)}} = \frac{\partial \eta^{(k)}}{\partial \eta}, \quad (69) $$

and

$$ \frac{\partial^2 \eta^{(i+k)}}{\partial \eta^{(i)}^2} = \frac{\partial^2 \eta^{(k)}}{\partial \eta^2}. \quad (70) $$

Here, we use the notation, $\ast$, to denote the equivalence established at the repellor. On the other hand, by successively applying the chain rule we have

$$ \frac{\partial \eta^{(k)}}{\partial \eta} = k \prod_{i=1}^{k} \frac{\partial \eta^{(i)}}{\partial \eta^{(i-1)}}, \quad (71) $$

Then, as a result of Eq. (69) we obtain

$$ \frac{\partial \eta^{(k)}}{\partial \eta} \ast (d^{(1)})^k, \quad (72) $$

with $d^{(1)} = (\partial \eta^{(1)}/\partial \eta)_{\eta=\eta^r_f} = 1.678574$.

For the second derivative, $\partial^2 \eta^{(k)}/\partial \eta^2$, we first perform the direct differentiation of Eq. (71) and then use the chain rule to obtain

$$ \frac{\partial^2 \eta^{(k)}}{\partial \eta^2} = \left( \frac{\partial^2 \eta^{(1)}}{\partial \eta^2} \right) \left[ D^k_2 (\eta) \right] + \left( \frac{\partial \eta^{(1)}}{\partial \eta} \right)^2 \left( \frac{\partial^2 \eta^{(2)}}{\partial \eta^{(1)}^2} \right) \left[ D^k_3 (\eta) \right] + \cdots + [D^k_{k-1} (\eta)]^2 \left( \frac{\partial^2 \eta^{(k)}}{\partial \eta^{(k-1)}^2} \right), \quad (73) $$

with

$$ D^k_a (\eta) = \prod_{i=a}^{k} \frac{\partial \eta^{(i)}}{\partial \eta^{(i-1)}}, \quad (74) $$

for $a \leq k$. Then, by using the identities of Eqs. (69) and (70) we can express Eq. (73) as

$$ \frac{\partial^2 \eta^{(k)}}{\partial \eta^2} \ast d^{(2)} \left[ \left( d^{(1)} \right)^{k-1} + \left( d^{(1)} \right)^k + \cdots + \left( d^{(1)} \right)^{2(k-1)} \right], \quad (75) $$

which yields

$$ \frac{\partial^2 \eta^{(k)}}{\partial \eta^2} \ast d^{(2)} \left( d^{(1)} \right)^{k-1} \left( \frac{(d^{(1)})^k - 1}{d^{(1)} - 1} \right), \quad (76) $$

with $d^{(2)} = (\partial^2 \eta^{(1)}/\partial \eta^2)_{\eta=\eta^r_f} = 1.182391$. 

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To calculate the internal energy density and specific heat at the repellor, we take \( \eta = \eta_f \), Eq. (72) for \((\partial \eta^{(n)}/\partial \eta)_{\eta=\eta_f}\), and Eq. (76) for \((\partial^2 \eta^{(n)}/\partial \eta^2)_{\eta=\eta_f}\). Then, for the internal energy density of Eqs. (34) and (35) we obtain the numerical results as

\[
\epsilon_b^{(n)} = -(0.543689) \left(a^{(1)}\right)^n - a^{(1)} \left(\frac{1 -(a^{(1)})^n}{1 -(a^{(1)})^{n}}\right),
\]

and

\[
\epsilon_{bb}^{(n)} = -(0.414742) \left(a^{(1)}\right)^n,
\]

with \(a^{(1)} = d^{(1)}/4 < 1\). Similarly, for the specific heat of Eqs. (48) and (50) at the repellor we obtain the numerical results as

\[
c_b^{(n)} = (0.261574) \left(a^{(2)}\right)^n + (0.201894) g^{(1)} \left(\frac{1 -(a^{(2)})^n}{1 -(a^{(2)})^{n}}\right) \left(a^{(2)}\right)^n + (0.371342) g^{(1)} \left[g^{(2)} - g^{(3)}\right],
\]

and

\[
c_{bb}^{(n)} = -(0.115082) \left(a^{(2)}\right)^n + (0.154011) g^{(1)} \left(\frac{1 -(a^{(2)})^n}{1 -(a^{(2)})^{n}}\right) \left(a^{(2)}\right)^n,
\]

with \(a^{(2)} = (d^{(1)}/4 < 1, g^{(1)} = d^{(2)}/(d^{(1)^2} - d^{(1)}), g^{(2)} = a^{(2)} \left(1 -(a^{(2)})^n\right)/(1 -(a^{(2)})^{n})\), and \(g^{(3)} = a^{(1)} \left(1 -(a^{(1)})^n\right)/(1 -(a^{(1)})^{n})\).

For the limit of infinite decoration-level, we have \((a^{(1)})^n \to 0, (a^{(2)})^n \to 0,\) and \(1/(d^{(1)})^n \to 0\) as \(n \to \infty\). Then, in this limit Eqs. (77) and (79) reduce to

\[
\epsilon_b^{(\infty)} = -(a^{(1)}) \left(\frac{1 -(a^{(1)})^n}{1 -(a^{(1)})^{n}}\right) = -0.723079,
\]

and

\[
c_b^{(\infty)} = (0.371342) g^{(1)} \left[\left(a^{(2)} \left(\frac{1 -(a^{(2)})^n}{1 -(a^{(2)})^{n}}\right) - (a^{(1)} \left(\frac{1 -(a^{(1)})^n}{1 -(a^{(1)})^{n}}\right)\right)\right] = 0.639852,
\]

while the long-range parts vanish, \(\epsilon_{bb}^{(\infty)} = 0\) and \(c_{bb}^{(\infty)} = 0\).

From these results we may conclude that the system of an \(n\)-lattice reduces to that composing of independent \(n\)-bonds as \(n \to \infty\). In this limit, our results also show that the critical specific heat becomes a cusp with the height \(c^{(\infty)} = 0.639852\) at the critical point. Thus, the corresponding critical exponent of the specific heat, \(\alpha\), is the same as that of the diamond-hierarchical Ising model, \(\alpha = -0.67652\) [9, 23].
We use the Ising model defined on square lattices with diamond-type bond-decorations to study the ferromagnetic phase transition of an inhomogeneous system. In this model, the decoration level $n$ associated with a decorated lattice can be used as a parameter to characterize the degree of the inhomogeneity of a system. Based on the exact solutions of the two-dimensional Ising model, we employ the bond-renormalization scheme to obtain the analytic form of the free energy. The free energy and its derived quantities, the internal energy and specific heat, are expressed as the sum of those contributed by the local interactions among the sub-bonds of an $n$-bond and the long-range interactions among different $n$-bonds. Then, the change in the nature of the phase transition as $n$ increases can be viewed as the reflection of the competition between the local and the long-range interactions.

In general, as $n$ increases, the long-range interaction is reduced and the local interaction is enhanced. The logarithmic singularity of the specific heat at the critical point is caused by the long-range interaction. Along with the reduction of the long-range interaction as $n$ gets larger, we have the critical temperature lowering down and the width of the critical region becoming narrower. By fitting the data numerically, we obtain the critical temperature changes with the decorination level $n$ as:

$$\left(\frac{k_B T_c}{J}\right)_n = 1.6410 + (0.6281) \exp \left[-(0.5857) n\right]$$

Moreover, because of the enhancement of the local interactions, the occurring temperature of the local ordering increases with the form:

$$\left(\frac{k_B T_m}{J}\right)_n = 1.6410 - (0.1081) \exp \left[-(0.4098) n\right]$$

The cross over from a finite-decorated system to an infinite-decorated system is not a smooth continuation. For the infinite decoration-level, the critical temperature $k_B T_c/J$ is $1.64101(8)$ which is exactly the location of the repellor of the renormalization map of the variable $\eta^{(n)}$. In this limit, the system of an $n$-lattice reduces to that of independent $n$-bonds, and the specific heat becomes a cusp with the value 0.63985(0) at the critical point instead of the logarithmic divergence.

It is interesting to compare the results we obtain with those from the cell-decorated triangular Ising model given in Ref. [14]. The critical region decreases with increasing $n$ for both cases. But, there exists some different features as the followings: (i) The decreasing speed of the critical amplitude of the specific heat with increasing $n$ is slower for the bond-decoration case in comparing with that of the cell-decoration. (ii) For the case of bond-decorations, the occurring temperature of the local ordering $\left(\frac{k_B T_m}{J}\right)_n$ of an $n$-lattice is smaller than the critical temperature $\left(\frac{k_B T_c}{J}\right)_n$ and the distance between $\left(\frac{k_B T_m}{J}\right)_n$ and $\left(\frac{k_B T_c}{J}\right)_n$ decreases with increasing $n$. However, for the case of cell-decorations, $\left(\frac{k_B T_m}{J}\right)_n$ is larger than $\left(\frac{k_B T_c}{J}\right)_n$ and $\left(\frac{k_B T_m}{J}\right)_n$ is freezed for $n \geq 5$. (iii) There is a finite peak arising to the right of the critical point for the specific
heat due to the local ordering in the cell-decoration case, but such a peak is not observed in the bond-decoration case.

We may use the idea of the order of ramification to explain the differences described in the above. To characterize the geometric difference between bond-decorated and cell-decorated lattices, we introduce the quantities, $R_1$ and $R_2$, with $R_1$ for the cutting-bond number of isolating an $n$-bond or $n$-cell and $R_2$ for the average of the total bond number contained in an $n$-bond or $n$-cell. We notice that the $R_1$ value signifies the ability of establishing the long-range correlations among different $n$-bonds or $n$-cells and the interactions among $R_2$ bonds are referred as the local interactions.

For the cell-decorated triangular Ising model with decoration-level $n$, we have $R_1 = 12$ and $R_2 = 3^{n+1}$. Because of $R_1 \ll R_2$ for $n \geq 3$, the long-range correlation is very hard to establish, and this leads to the higher temperature for the occurrence of the local ordering in comparing with that of the long-range ordering. For sufficiently large $n$ with $R_1/R_2 \simeq 0$, an $n$-cell almost become a closed subsystem, and thence a round peak, contributed by the subsystem, appears above the critical point in the curve of the specific heat.

For the diamond-type bond-decorated Ising model discussed in this work, we have $R_1 = 6 \cdot 2^n$ and $R_2 = 4^n$. The values of $R_1$ and $R_2$ are comparable for $n \leq 5$, and the long-range effect is almost absent for $n \geq 6$. Thence, the round peak in the specific heat, which appears in the case of cell-decorations, is absent. Moreover, the coordination numbers of the decorated sites in an $n$-bond are highly inhomogeneous, and this is mainly responsible for the lower occurring temperature of the local ordering in comparing with that of the long-range ordering.

Thus, the non-universal features of phase transition for the diamond-type bond-decorated Ising model is quite different from that for the cell-decorated triangular Ising model.

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Table 1. The numerical results of the critical temperature \((k_B T_c/J)_n\), the occurring temperature of the local ordering \((k_B T_m/J)_n\), the critical values of the internal energy density \(\langle \epsilon^{(n)} \rangle_c\), the internal energy density due to the local interactions \(\langle \epsilon^{(n)}_b \rangle_c\), and the internal energy density due to the long-range interactions \(\langle \epsilon^{(n)}_bb \rangle_c\), and the critical amplitude at the logarithmic divergence in the specific heat \(A_{\text{sin}g}^{(n)}\).

| n  | \((kT_c/J)_n\) | \((kT_m/J)_n\) | \(\langle \epsilon^{(n)} \rangle_c\) | \(\langle \epsilon^{(n)}_b \rangle_c\) | \(\langle \epsilon^{(n)}_bb \rangle_c\) | \(A_{\text{sin}g}^{(n)}\) |
|----|----------------|----------------|----------------|----------------|----------------|----------------|
| 0  | 2.269185       | 0.833520       | -0.707107      | -0.414214      | -0.292893      | 0.247269       |
| 1  | 1.982072       | 1.255390       | -0.653281      | -0.541190      | -0.112085      | 0.189850       |
| 2  | 1.833652       | 1.449949       | -0.568949      | -0.614280      | -0.044674      | 0.140957       |
| 3  | 1.752268       | 1.540593       | -0.575920      | -0.657720      | -0.018202      | 0.102495       |
| 4  | 1.706095       | 1.586440       | -0.691315      | -0.683810      | -0.007508      | 0.073587       |
| 5  | 1.679371       | 1.610705       | -0.702627      | -0.699510      | -0.003119      | 0.052430       |
| 6  | 1.663721       | 1.623980       | -0.710252      | -0.708950      | -0.001301      | 0.037184       |
| 7  | 1.654492       | 1.631345       | -0.715166      | -0.714626      | -0.000544      | 0.026299       |
| 8  | 1.649027       | 1.635501       | -0.718250      | -0.718022      | -0.000228      | 0.018570       |
| 9  | 1.645783       | 1.637855       | -0.720154      | -0.720055      | -0.54965(-5)   | 0.013100       |
| 10 | 1.643854       | 1.639191       | -0.721316      | -0.721280      | -4.00447(-5)   | 0.009235       |
| 11 | 1.642707       | 1.639962       | -0.722020      | -0.722003      | -1.67971(-5)   | 0.006509       |
| 12 | 1.642024       | 1.640400       | -0.722444      | -0.722433      | -7.04693(-6)   | 0.004586       |
| 13 | 1.641617       | 1.640664       | -0.722699      | -0.722697      | -2.95673(-6)   | 0.003231       |
| 14 | 1.641375       | 1.640809       | -0.722852      | -0.722849      | -1.24066(-6)   | 0.002276       |
| 15 | 1.641231       | 1.640893       | -0.722943      | -0.722939      | -5.20605(-7)   | 0.001604       |
| 16 | 1.641145       | 1.640946       | -0.722998      | -0.723000      | -2.18461(-7)   | 0.001130       |
| 17 | 1.641093       | 1.640975       | -0.723030      | -0.723030      | -9.1674(-8)    | 7.95678(-4)    |
| 18 | 1.641063       | 1.640993       | -0.723050      | -0.723050      | -3.84699(-8)   | 5.60485(-4)    |
| 19 | 1.641045       | 1.641003       | -0.723061      | -0.723060      | -1.61435(-8)   | 3.9481(-4)     |
| 20 | 1.641034       | 1.641009       | -0.723068      | -0.723070      | -6.7745(-9)    | 2.78106(-4)    |
Figure 1: Examples of $n$-bonds specified by the two lattice sites $\mu$ and $\nu$ for (a) $n = 0$, (b) $n = 1$, and (c) $n = 2$. Note that the Ising spins except $\sigma_\mu$ and $\sigma_\nu$ are referred as the inner-spins.

Figure 2: Examples of $n$-lattices for (a) $n = 0$, (b) $n = 1$, and (c) $n = 2$. 
Figure 3: The critical temperature \((k_B T_c/J)_n\) and the peak temperature of the specific heat \((k_B T_m/J)_n\) caused by the local interactions of an \(n\)-lattice denoted by black dots and black triangles respectively. The continuous curves are given by the fitting functions as 
\[
(k_B T_c/J)_n = 1.6410 + (0.6281) \exp[-(0.5857) n]
\]
and 
\[
(k_B T_m/J)_n = 1.6410 - (0.8063) \exp[-(0.7144) n]
\] respectively.
Figure 4: The critical values of $\varepsilon^{(n)}$ (triangles), $\epsilon_b^{(n)}$ (black dots), and $\epsilon_{bb}^{(n)}$ (squares) as a function of decoration-level $n$. 
Figure 5: The numerical results of $c_b^{(n)}$ (with round peaks) and $c_{bb}^{(n)}$ (with sharp peaks) as a function of reduced temperature $k_B T/J$ for $n = 2, 4, \text{and } 6$. 
Figure 6: The critical amplitude $A_{\sin g}^{(n)}$ of the specific heat as a function of decoration-level $n$ with the solid curve given by the fitting function $A_{\sin g}^{(n)} = (0.2473) \exp [- (0.3018) n]$. 
Figure 7: The variable $\eta^{(n)}$ as a function of $\eta$ for $n = 0, 2, 4, \text{and } 6$. Note that the vertical line corresponds to the location of the repellor of the map $\eta^{(n-1)} \rightarrow \eta^{(n)}$. 
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