The $\bar{B} \to X_s \gamma$ Photon Spectrum

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Abstract

The photon energy spectrum in inclusive weak radiative $\bar{B} \to X_s \gamma$ decay is computed to order $\alpha_s^2/\beta_0$. This result is used to extract a value for the HQET parameter $\Lambda$ from the average $\langle 1 - 2E_\gamma/m_B \rangle$, and a value of the parameter $\lambda_1$ from $\langle (1 - 2E_\gamma/m_B)^2 \rangle$. An accurate measurement of $\langle 1 - 2E_\gamma/m_B \rangle$ can determine the size of the nonperturbative contributions to the $\Upsilon(1S)$ mass which cannot be absorbed into the $b$ quark pole mass.
Comparison of the measured weak radiative $B \to X_s \gamma$ decay rate with theory is an important test of the standard model. In contrast to the decay rate itself, the shape of the photon spectrum is not expected to be sensitive to new physics, but it can nevertheless provide important information. First of all, studying the photon spectrum is important for understanding how precisely the total rate can be predicted in the presence of an experimental cut on the photon energy $[1]$, which is important for a model independent interpretation of the resulting decay rate. Secondly, moments of the photon spectrum may be used to measure the heavy quark effective theory (HQET) parameters which determine the quark pole mass and kinetic energy $[2,3]$, much like the shape of the lepton energy $[4]$ or hadronic invariant mass $[5]$ spectrum in semileptonic $B \to X_c \ell \bar{\nu}$ decay. The main purpose of this paper is to present the order $\alpha_s^2 \beta_0$ piece of the two-loop correction to the photon spectrum, and to study its implications. A calculation to this order is required for a meaningful comparison of the HQET parameters extracted from $B \to X_s \gamma$ with those from other processes.

To leading order in small weak mixing angles the effective Hamiltonian is

$$H_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} V_{tb} V_{ts}^* \sum_{i=1}^{8} C_i(\mu) O_i,$$

(1)

where $G_F$ is the Fermi constant, $V_{ij}$ are elements of the Cabibbo–Kobayashi–Maskawa matrix, $C_i(\mu)$ are Wilson coefficients evaluated at a subtraction point $\mu$, and $O_i$ are the dimension six operators

$$O_1 = (\bar{c}_{L\beta} \gamma^\mu b_{L\beta})(\bar{s}_{L\alpha} \gamma_\mu c_{L\alpha}), \quad O_2 = (\bar{c}_{L\alpha} \gamma^\mu b_{L\alpha})(\bar{s}_{L\beta} \gamma_\mu c_{L\beta}),$$

$$O_3 = (\bar{s}_{L\alpha} \gamma^\mu b_{L\alpha}) \sum_q (\bar{q}_{L\beta} \gamma_\mu q_{L\beta}), \quad O_4 = (\bar{s}_{L\alpha} \gamma^\mu b_{L\beta}) \sum_q (\bar{q}_{L\beta} \gamma_\mu q_{L\alpha}),$$

$$O_5 = (\bar{s}_{L\alpha} \gamma^\mu b_{L\alpha}) \sum_q (\bar{q}_{R\beta} \gamma_\mu q_{R\beta}), \quad O_6 = (\bar{s}_{L\alpha} \gamma^\mu b_{L\beta}) \sum_q (\bar{q}_{R\beta} \gamma_\mu q_{R\alpha}),$$

$$O_7 = \frac{e}{16\pi^2} m_b \bar{s}_{L\alpha} \sigma^{\mu\nu} b_{R\alpha} F_{\mu\nu}, \quad O_8 = \frac{g}{16\pi^2} m_b \bar{s}_{L\alpha} \sigma^{\mu\nu} T_a^{\alpha\beta} b_{R\beta} G_{\mu\nu}^a.$$

(2)

In Eq. (2), $e$ is the electromagnetic coupling, $g$ is the strong coupling, $m_b$ is the $b$ quark mass, $F_{\mu\nu}$ is the electromagnetic field strength tensor, $G_{\mu\nu}^a$ is the strong interaction field strength tensor, and $T^a$ is a color $SU(3)$ generator. The sums over $q$ include $q = u, d, s, c, b$ and the subscripts $L, R$ denote left and right handed fields. The Wilson coefficients have been
calculated to next-to-leading order (NLO) \cite{6, 7}. Using $\alpha_s(m_Z) = 0.12$, and the convention that the covariant derivative is $D_\mu = \partial_\mu + igA_\mu^a T^a + ie Q A_\mu$ (where $Q$ is the fermion’s electric charge), the values we need are $C_2(m_b) = 1.13$, $C_7(m_b) = -0.306$, $C_8(m_b) = -0.168$ \cite{6}.

For the photon energy, $E_\gamma$, not too close to its maximal value, the photon spectrum $d\Gamma/dE_\gamma$ for weak radiative $B$ decay has a perturbative expansion in the strong interaction fine structure constant $\alpha_s$. It is known at order $\alpha_s$ and the main purpose of this letter is to present the order $\alpha_s^2 \beta_0$ (so-called BLM \cite{9}) contribution. It is well known that the part of the order $\alpha_s^2$ piece proportional to the one-loop beta function, $\beta_0 = 11 - 2n_f/3$ usually provides a reliable estimate of the full order $\alpha_s^2$ piece. This part of the order $\alpha_s^2$ contribution is straightforward to compute using the method of Smith and Voloshin \cite{10}.

Using the dimensionless variable\cite{11}, $x_b = 2E_\gamma/m_b$, the photon energy spectrum in $\bar{B} \to X_s\gamma$ takes the form

$$\frac{1}{\Gamma_0} \frac{d\Gamma}{dx_b} \bigg|_{x_b<1} = A_0(x_b) + \frac{\alpha_s(m_b)}{\pi} A_1(x_b) + \left( \frac{\alpha_s(m_b)}{\pi} \right)^2 \beta_0 A_2(x_b) + \ldots ,$$

where

$$\Gamma_0 = \frac{G_F^2 |V_{tb}V_{ts}|^2 \alpha_{em} C_7^2}{32\pi^4 m_b^5} ,$$

is the contribution of the tree level matrix element of $O_7$ to the $B \to X_s\gamma$ decay rate, and

$$A_p(x_b) = \sum_{i,j} a_p^{ij}(x_b) \left[ \frac{C_i(m_b) C_j(m_b)}{C_7(m_b)^2} \right] .$$

The sums over $i,j$ in Eq. (3) give the contributions of the various operators in Eq. (2) to the photon energy spectrum.

It is important to note that since the coefficients in $H_{\text{eff}}$ are known only to NLO accuracy, the BLM calculation of the $O_1 - O_8$ contribution to the photon spectrum is only meaningful away from the endpoint. At the endpoint, order $\alpha_s^2$ contributions to the matrix elements are

\footnote{Later we will introduce a dimensionless photon energy variable normalized by the $B$ meson mass, $x_B = 2E_\gamma/m_B$.}
the same order as the unknown NNLO running [where \( \alpha_s \ln(m_W/m_b) \) is counted as \( \mathcal{O}(1) \)]. Neglecting the small contribution to \( A_0 \) from \( O_1 - O_6 \) discussed in the next paragraph, at least one gluon must be in the final state to populate the spectrum for \( x_b < 1 \), so it is consistent to combine the \( \alpha_s^2 \) matrix elements with the NLO Wilson coefficients. (Strictly speaking, we should for consistency only use the \( \beta_0 \) part of the NLO running of the operators with the BLM calculation, but for simplicity we will use the full NLO result. The difference between these two approaches is small.) Thus powers of \( \alpha_s \) in Eq. (3) and elsewhere reflect the perturbation expansion of the matrix elements only, and not of the Wilson coefficients.

At zeroth order in the strong coupling, the spectrum for \( x_b < 1 \) arises from matrix elements of the four-quark operators \( O_1 - O_6 \) in Eq. (2). Of these \( O_1 \) and \( O_2 \) include two charm quarks in the final state, and therefore they contribute to the photon spectrum only for lower values of \( x_b \) than what we consider in this paper. These contributions are divergent in perturbation theory, and the divergence can be absorbed into the definition of the quark to photon fragmentation function, \( D^{q\rightarrow \gamma}(x) \), which depends on an infrared scale \( \Lambda \). \( D^{q\rightarrow \gamma}(x) \) is calculable in the leading logarithmic approximation \([11,12]\). There is some data on \( D^{q\rightarrow \gamma}(x) \), however, the experimental errors are still quite large \([13]\). This fragmentation contribution to the coefficients \( a^{ij}_0(x) \) vanishes as \( x_b \to 1 \), and it is small in the region of large \( x_b \), \( 0.65 < x_b \), which we consider in this paper.

A very important \( B \) decay background to the \( \bar{B} \to X_s \gamma \) photon spectrum is from non-leptonic \( b \to c \bar{u}d \) and \( b \to u \bar{u}d \) decays, where a massless quark in the final state radiates a photon. Such backgrounds due to the operators \((\bar{c}_L \gamma^\mu b_L)(\bar{d}_L \gamma_\mu u_L)\) and \((\bar{u}_L \gamma^\mu b_L)(\bar{d}_L \gamma_\mu u_L)\) are shown in Fig. 1 (using \( |V_{ub}/V_{cb}| = 0.1 \)). We used the Duke–Owens parameterization of the fragmentation function \([14]\), setting \( \Lambda = 1.3 \) GeV and \( Q^2 = m_b^2 \). (This value of \( \Lambda \) is motivated by a fit to the ALEPH data \([13]\).) The uncertainty of this result is sizable, since the \( \Lambda \)-dependence is large and \( m_b \) may not be large enough to justify keeping only the leading logarithms. Close to maximal \( x_b \) the resummed fragmentation function may predict too large a suppression of the photon spectrum, since the lightest exclusive final states dominate there. The background from \( b \to c \bar{u}d \) \((b \to u \bar{u}d)\) is more than 50% of the 77 contribution
FIG. 1. $B$ decay background to the photon spectrum due to the operators $(\bar{c}L\gamma^\mu b_L)(\bar{d}L\gamma^\mu u_L)$ (solid curve) and $(\bar{u}L\gamma^\mu b_L)(\bar{d}L\gamma^\mu u_L)$ (dashed curve).

Therefore, we will concentrate on the region $x_b > 0.65$; to measure the $B \to X_s\gamma$ photon spectrum at lower values of $x_b$ would not only require excluding final states with charm with very good efficiency, but also demanding a strange quark in the final state. Note that for $B \to X_d\gamma$, the fragmentation contribution from $b \to u\bar{u}d$ is larger than the short distance piece unless $x_b$ is very close to 1.

Neglecting the strange quark mass, $a_1^{88}$ is also divergent in perturbation theory. This divergence can also be absorbed into the definition of fragmentation functions. In the leading logarithmic approximation [15]

\[
\frac{a_1^{88}(x)}{\Gamma_0} = \left(\frac{4\pi}{3\alpha_{em}}\right) [D_{s\to\gamma}(x) + D_{g\to\gamma}(x)],
\]

where $D_{s\to\gamma}(x)$ and $D_{g\to\gamma}(x)$ are the strange quark to photon and gluon to photon fragmentation functions, which have large uncertainties. In the region $x_b > 0.65$, the $a_1^{88}$ contribution to the photon spectrum $(1/\Gamma_0)d\Gamma/dx_b$ is less than 0.01. Given the uncertainty in $a_1^{88}$, and its small magnitude, it does not appear useful to calculate $a_2^{88}$.

Note that these backgrounds are steeply falling functions of $x_b$, and are indeed negligible in the present CLEO region of $E_\gamma > 2.1$ GeV. The tree level contribution of the operators $O_3 - O_6$ in Eq. (2) to the photon spectrum is about a fifth of the $b \to u\bar{u}d$ background.
Experimentally, because of backgrounds, only $B \to X_s \gamma$ photons with large energies can be detected. The present experimental cut is $E_\gamma > 2.1$ GeV at CLEO [1], which corresponds to $x_b > 0.875$ with $m_b = 4.8$ GeV. In the large $x_b$ region the most important contribution to the sum in Eq. (3) come from the 77 term, with moderate corrections from the 22, 78, and 27 terms. The other contributions (88, 28, and the ones involving $O_1$ and $O_3 - O_6$) are very small, and will be neglected in this paper.

Simple analytic expressions for $a_{77}^1$ and $a_{78}^1$ are available,

$$a_{77}^1(x) = \frac{(2x^2 - 3x - 6)x + 2(x^2 - 3) \ln(1 - x)}{3(1 - x)},$$

$$a_{78}^1(x) = \frac{8}{9} \left[ \frac{4 + x^2}{4} + \frac{1 - x}{x} \ln(1 - x) \right].$$

Neglecting the small $A_0$ term in Eq. (3), we can calculate the shape of the photon spectrum away from $x = 1$ to order $\alpha_s^2\beta_0$ accuracy knowing the effective Hamiltonian to order $\alpha_s$ (NLO) only. At order $\alpha_s^2\beta_0$, we find that $a_{77}^2$ and $a_{78}^2$ are given by

$$a_{77}^2(x) = \frac{1}{18} \left[ \frac{38x^3 - 93x^2 + 6x - 36}{4(1 - x)} - \frac{6x^4 - 31x^3 + 24x^2 - 30x + 18}{2x(1 - x)} \ln(1 - x) \right.$$

$$\left. + 3(3 - x^2) \frac{3 \ln^2(1 - x) + 2L_2(x)}{2(1 - x)} \right],$$

$$a_{78}^2(x) = \frac{1}{9} \left[ \frac{19x^2 - 24x + 88}{12} - \frac{3x^3 - 12x^2 + 56x - 32}{6x} \ln(1 - x) \right.$$

$$\left. - (1 - x) \frac{3 \ln^2(1 - x) + 2L_2(x)}{x} \right],$$

where $L_2(z) = - \int_0^z dt \ln(1 - t)/t$ is the dilogarithm. The strange quark mass is neglected throughout this paper; it only enters the final results quadratically, as $m_s^2/[m_b^2(1 - x_b)]$.

The functions of $a_{22}^1$ and $a_{27}^1$ are known in the literature [16,17], and we agree analytically with those results. The order $\alpha_s^2\beta_0$ contributions, $a_{22}^2$ and $a_{27}^2$, are computed numerically. We find it most useful to present simple approximations to these functions.
\[ a_{12}^2(x) \simeq -0.0842 + 0.3333x - 0.2005x^2 + 0.0227x^3 \]
\[ + \left( \frac{m_c}{m_b} - \frac{1.4}{4.8} \right) (-0.454 + 0.061x), \]
\[ a_{22}^2(x) \simeq -0.1272 + 0.3957x - 0.3227x^2 + 0.0952x^3 - 0.0180 \ln(1 - x) \]
\[ + \left( \frac{m_c}{m_b} - \frac{1.4}{4.8} \right) [-0.155 - 0.106x + 0.106 \ln(1 - x)], \] (11)

and

\[ a_{12}^7(x) \simeq -0.1064 + 0.4950x - 0.4361x^2 + 0.0373x^3 \]
\[ + \left( \frac{m_c}{m_b} - \frac{1.4}{4.8} \right) (-1.207 + 2.901x), \]
\[ a_{22}^7(x) \simeq -0.0156 + 0.0463x + 0.3467x^2 - 0.3045x^3 + 0.0027 \ln(1 - x) \]
\[ + \left( \frac{m_c}{m_b} - \frac{1.4}{4.8} \right) [-1.523 + 2.538x - 0.448 \ln(1 - x)]. \] (12)

These approximations are accurate to within 1% in the region \( x_b > 0.6 \) for \( m_c/m_b = 1.4/4.8 \). The 27 contribution is very sensitive to \( m_c/m_b \). Changing \( m_c/m_b \) from \( 1.4/4.8 \) to \( 1.2/4.6 \) or \( 1.6/5.0 \) modifies \( a_{12}^7 \) and \( a_{22}^7 \) dramatically. The 22 contribution only changes in the previously mentioned range of \( m_c/m_b \) by \( \pm (20 - 25)\% \). The 22 contribution is also accurate to within 1% when \( m_c/m_b \) changes by \( \pm 0.03 \). However, the 27 contribution is only accurate at the 20% level when \( m_c/m_b \) changes in this range. Note that the perturbation series in \( \alpha_s \) is particularly badly behaved for the 27 contribution. Roughly 2/3 of the 22 contribution is from absorptive parts corresponding to real intermediate states.

The coefficients \( a_{ij}^2 \) are determined by calculating the order \( \alpha_s^2 n_f \) piece and making the identification, \(-2n_f/3 \rightarrow \beta_0\). There is a subtlety in applying this method to weak radiative \( B \) decay. There is a contribution of order \( \alpha_s^0 n_f \) from the tree level \( b \rightarrow s\gamma q\bar{q} \) matrix elements of \( O_3 - O_6 \), coming from Feynman diagrams where the photon couples to the bottom or strange quarks. It is not associated with a term of order \( \alpha_s^0 \beta_0 \). To avoid adding an analogous spurious order \( \alpha_s^2 \beta_0 \) contribution to \( a_{22}^7 \) and \( a_{22}^2 \), only diagrams where the photon couples to the charm quark were included in the calculation of the matrix element of \( O_2 \).

Part of the \( \bar{B} \rightarrow X_s\gamma \) matrix element of \( O_2 \) is not adequately calculated in perturbation theory. It corresponds to the process \( \bar{B} \rightarrow J/\psi X_s \) followed by the decay \( J/\psi \rightarrow \gamma + \)
FIG. 2. The sum of the 77, 22, 78, and 27 contributions to \((1/\Gamma_0)d\Gamma/dx_b\) at order \(\alpha_s\) (thick dashed curve) and \(\alpha_s^2\beta_0\) (thick solid curve). The thin curves show the 77 contribution only. The scale is the same as in Fig. 1.

(light hadrons). There will be large corrections to the part of the charm quark loop where the \(c\bar{c}\) are almost on-shell and have the same velocity. In this region there are large “Coulombic QCD corrections” that produce the \(J/\psi\) state. However, cutting this small part of the \(c\bar{c}\) phase space out of our calculation of the matrix element of \(O_2\) has a negligible effect. Hence, at the order of perturbation theory to which we are working, calculating the \(c\bar{c}\) loop while removing \(J/\psi\)’s from the data would be a consistent approximation.

The sum of the 77, 22, 78, and 27 contributions is plotted in Fig. 2 in the region \(0.65 < x_b < 0.9\) (using \(\alpha_s(m_b) = 0.22\) and \(\beta_0 = 25/3\)). For very large \(x\), other effects that we have not calculated become important. There are both nonperturbative and perturbative terms that are singular as \(x \to 1\). They sum into a shape function that modifies the spectrum in this region [8]. Unfortunately, at the present time, it is not possible to make a model independent estimate of these effects. Therefore, we do not plot the perturbation theory predictions for \(x_b > 0.9\). In the plotted region, the 22, 78, and 27 terms make a moderate correction to the dominant 77 contribution to \((1/\Gamma_0)d\Gamma/dx\), which is shown in Fig. 2 with the thin curves.

The \(b\) quark mass can be eliminated in favor of the \(B\) meson mass by a change of variables to
Using $m_b = m_B - \bar{\Lambda} + (\lambda_1 + 3\lambda_2)/(2m_b) + \ldots$, the photon spectrum becomes

$$\frac{d\Gamma}{dx_B} = \left(1 + \frac{\bar{\Lambda}}{m_B} + \ldots\right) \frac{d\Gamma}{dx_b} \bigg|_{x_b=x_B(1+\bar{\Lambda}/m_B+\ldots)}.$$  

(14)

For $x_B$ within a region of order $\Lambda_{\text{QCD}}/m_B$ of unity (its maximal value) nonperturbative effects are very important. However, for integrals of $x_B$ over a large enough range these nonperturbative effects are small.

An important integral of this type is

$$\left. \frac{(1-x_B)}{x_B>1-\delta} \right|_{x_B>1-\delta} = \frac{\int_{1-\delta}^{1} dx_B (1-x_B) \frac{d\Gamma}{dx_B}}{\int_{1-\delta}^{1} dx_B \frac{d\Gamma}{dx_B}}.$$  

(15)

The parameter $\delta = 1 - 2E_\gamma^{\text{min}}/m_B$ has to satisfy $\delta > \Lambda_{\text{QCD}}/m_B$; otherwise nonperturbative effects are not under control. It is straightforward to show that

$$\left. \frac{(1-x_B)}{x_B>1-\delta} \right|_{x_B>1-\delta} = \frac{(1-\frac{\bar{\Lambda}}{m_B}) (1-x_B) \bigg|_{x_b>1-\delta} - \frac{\bar{\Lambda}}{m_B} \delta (1-\delta) \frac{1}{\Gamma_0} \frac{d\Gamma}{dx_b} \bigg|_{x_b=1-\delta} + \ldots,}$$  

(16)

where

$$\left. (1-x_b) \right|_{x_b>1-\delta} = \int_{1-\delta}^{1} dx_b (1-x_b) \frac{1}{\Gamma_0} \frac{d\Gamma}{dx_b}.$$  

(17)

Note that all terms but the first one in Eq. (16) have perturbative expansions which begin at order $\alpha_s$. The ellipses denote contributions of order $(\Lambda_{\text{QCD}}/m_B)^3$, $\alpha_s(\Lambda_{\text{QCD}}/m_B)^2$, and $\alpha_s^2$ terms not enhanced by $\beta_0$, but it does not contain contributions of order $(\Lambda_{\text{QCD}}/m_B)^2$ or additional terms of order $\alpha_s(\Lambda_{\text{QCD}}/m_B)$. Terms in the operator product expansion

\footnote{There are actually additional contributions formally of order $\alpha_s(\Lambda_{\text{QCD}}/m_B)$ coming from the expansion of $m_c/m_b$ in the 22 and 27 terms. Although the 27 term is very sensitive to the value of $m_c/m_b$, this $\bar{\Lambda}$-dependence is negligible for $\left. (1-x_B) \right|_{x_B>1-\delta}$.

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FIG. 3. The sum of the 77, 22, 78, and 27 contributions to $\langle 1 - x_b \rangle|_{x_b > 1 - \delta}$ at order $\alpha_s$ (thick dashed curve) and $\alpha_s^2\beta_0$ (thick solid curve). The thin curves show the 77 contribution only.

proportional to $\lambda_{1,2}/m_b^2$ enter precisely in the form so that they are absorbed in $m_B$ in Eq. (16) [3]. There are also nonperturbative corrections suppressed by $(\Lambda_{\text{QCD}}/m_c)^2$ instead of $(\Lambda_{\text{QCD}}/m_b)^2$ [19]. These do not contribute to Eq. (16).

Using our results, $\langle 1 - x_b \rangle|_{x_b > 1 - \delta}$ in Eq. (17) is known to order $\alpha_s^2\beta_0$. Writing

$$\langle 1 - x_b \rangle|_{x_b > 1 - \delta} = B_0(\delta) + \frac{\alpha_s(m_b)}{\pi} B_1(\delta) + \left(\frac{\alpha_s(m_b)}{\pi}\right)^2 \beta_0 B_2(\delta) + \ldots ,$$  \hspace{1cm} (18)

$B_p$ have decompositions analogous to Eq. (5),

$$B_p(\delta) = \sum_{i \leq j} b_{ij}^p(\delta) \left[ \frac{C_i(m_b)C_j(m_b)}{C^2(m_b)} \right].$$  \hspace{1cm} (19)

Neglecting $B_0(\delta)$, Eqs. (7) and (9) yield for the dominant 77 contribution

$$b_{177}^1(\delta) = \frac{\delta}{54} \left[ -9\delta^3 + 14\delta^2 + 72\delta - 54 + 12(\delta^2 - 3\delta - 6) \ln \delta \right],$$  \hspace{1cm} (20)

$$b_{277}^2(\delta) = \frac{1}{2592} \left[ -369\delta^4 + 116\delta^3 + 1800\delta^2 - 3852\delta $$

$$+ 408\pi^2 + 12\delta(9\delta^3 + 34\delta^2 - 102\delta + 66) \ln \delta $$

$$- 216\delta(\delta^2 - 3\delta - 6) \ln^2 \delta - 144(\delta^3 - 3\delta^2 - 6\delta + 17) L_2(1 - \delta) \right].$$  \hspace{1cm} (21)

Our prediction for $\langle 1 - x_b \rangle|_{x_b > 1 - \delta}$ is shown in Fig. 3 as a function of $\delta$, both at order $\alpha_s$ and $\alpha_s^2\beta_0$. The bad behavior of the perturbation expansion would improve somewhat by evaluating the strong coupling at a smaller scale than $m_b$, such as $m_b\sqrt{\delta}$, the maximal
available invariant mass of the hadronic final state. This bad behavior may also be related
to the renormalon ambiguity \[20\] in \(\bar{\Lambda}\).

A determination of \(\bar{\Lambda}\) is straightforward using Eq. (16). The left hand side is directly
measurable, while \(\langle 1 - x_b \rangle |_{x_B > 1 - \delta}\) and \((1/\Gamma_0) d\Gamma/dx_b |_{x_B > 1 - \delta}\) in the second and third terms
on the right hand side can be read off from Figs. 3 and 2, respectively. Using the CLEO
data in the region \(E_\gamma > 2.1\) GeV \[1\], we obtain the central values \(\bar{\Lambda}_{\alpha_s^2\beta_0} \simeq 270\) MeV and
\(\bar{\Lambda}_{\alpha_s} \simeq 390\) MeV. We have indicated the order kept in the perturbation expansion to determine
\(\bar{\Lambda}\), since a value of \(\bar{\Lambda}\) extracted from data can only be used consistently in predictions valid
to the same order in \(\alpha_s\). These values are consistent with the ones obtained from a fit to
the \(\bar{B} \to X_c \ell \bar{\nu}\) lepton spectrum \[4\], and from the CLEO fit \[21\] to the \(\bar{B} \to X_c \ell \bar{\nu}\) hadron
mass distribution \[3\].

At the present time this extraction of \(\bar{\Lambda}\) has large uncertainties. The potentially most
serious one is from both nonperturbative and perturbative terms that are singular as \(x \to 1\)
and sum into a shape function that modifies the spectrum near the endpoint. A model
independent determination of these effects is not available at the present time, however,
it may be possible to address this issue using lattice QCD \[22\]. For sufficiently large \(\delta\)
these effects are not important. It has been estimated that they may be significant even if
the cut on the photon energy is lowered to around \(E_\gamma = 2\) GeV \[23,24\], but this is based
on phenomenological models. We have implicitly neglected these effects throughout our
analysis. The validity of this can be tested experimentally by checking whether the value of
\(\bar{\Lambda}\) extracted from Eq. (16) is independent of \(\delta\) in some range. This would also improve our
confidence that the total decay rate in the region \(x_B > 1 - \delta\) can be predicted in perturbative
QCD without model dependence.

The value of \(\bar{\Lambda}\) at order \(\alpha_s\) has a sizable scale dependence: lowering the scale such that
\(\alpha_s\) changes from 0.22 to 0.3 reduces the value of \(\bar{\Lambda}_{\alpha_s}\) by about 40 MeV. At order \(\alpha_s^2\beta_0\) this
scale dependence is much smaller. Uncertainties due to the unknown order \((\Lambda_{QCD}/m_B)^3\)
terms in the OPE \[24\] are largely uncorrelated to those in the analyses of the lepton energy
or hadron mass spectra in \(\bar{B} \to X_c \ell \bar{\nu}\) \[25\]. The effect of the boost from the \(B\) rest frame
FIG. 4. Prediction for \((1 - x_B)|_{x_B > 1 - \delta}\) in the upsilon expansion at order \(\epsilon\) (thick dashed curve) and \((\epsilon^2)_{BLM}\) (thick solid curve). The thin curves show the 77 contribution only.

into the \(\Upsilon(4S)\) is small for \((1 - x_B)|_{x_B > 1 - \delta}\) [23].

The upsilon expansion [24] yields parameter free predictions for \((1 - x_B)|_{x_B > 1 - \delta}\) in terms of the \(\Upsilon(1S)\) meson mass. The analog of Eq. (16) is

\[
\left|1 - x_B\right|_{x_B > 1 - \delta} = 1 - \frac{m_\Upsilon}{2m_B} \left[1 + 0.011\epsilon + 0.019(\epsilon^2)_{BLM} - \left\langle 1 - x_b\right\rangle_{x_b > (2m_B/m_\Upsilon)(1 - \delta)}\right],
\]

where \(\epsilon \equiv 1\) denotes the order in the upsilon expansion. For \(E_\gamma > 2.1\) GeV this relation gives 0.111, whereas the central value from the CLEO data is around 0.093.\(^4\) In Fig. 4 we plot the prediction for \((1 - x_B)|_{x_B > 1 - \delta}\) as a function of \(\delta\), both at order \(\epsilon\) and \((\epsilon^2)_{BLM}\). The perturbation expansion is much better behaved than the one shown in Fig. 3. The most important uncertainty in this approach is the size of nonperturbative contributions to the \(\Upsilon(1S)\) mass other than those which can be absorbed into the \(b\) quark mass. These have been neglected in Eq. (22). If the nonperturbative contribution to the \(\Upsilon(1S)\) mass, \(\Delta_\Upsilon\), were known, it could be included by replacing \(m_\Upsilon\) by \(m_\Upsilon - \Delta_\Upsilon\). For example, \(\Delta_\Upsilon = +300\) MeV

\(^4\)It is interesting to note that including the CLEO data point in the 1.9 GeV < \(E_\gamma\) < 2.1 GeV bin, the experimental central value of \((1 - x_B)\) over the region \(E_\gamma > 1.9\) GeV is 0.117, whereas the upsilon expansion predicts 0.120.
increases \((1 - x_B)\) by 21\%, so measuring \((1 - x_B)\) with such accuracy will have important implications for the physics of quarkonia as well as for \(B\) physics.

The variance of the photon energy distribution can be used to determine \(\lambda_1\) [3, 24]. The analog of Eq. (16) in this case is

\[
(1 - x_B)^2 \big|_{x_B > 1 - \delta} - \left[ (1 - x_B) \big|_{x_B > 1 - \delta} \right]^2 = -\frac{\lambda_1}{3 m_B^2} + \frac{\beta^2}{3} + \left( 1 - \frac{2 \bar{\Lambda}}{m_B} \right) \langle (1 - x_b)^2 \rangle \big|_{x_b > 1 - \delta}
\]

\[
- \frac{\bar{\Lambda}}{m_B} \delta^2 (1 - \delta) \frac{1}{\Gamma_0} \frac{d\Gamma}{dx_b \big|_{x_b = 1 - \delta}} + \ldots ,
\]

where \(\beta \approx 0.064\) is the magnitude of the velocity of the \(B\) meson in the \(\Upsilon(4S)\) rest frame, and only the leading \(\beta\)-dependence has been kept. The ellipses denote terms of order \((\Lambda_{\text{QCD}}/m_B)^3\), \(\alpha_s(\Lambda_{\text{QCD}}/m_B)^2\), and \(\alpha_s^2\) terms not enhanced by \(\beta_0\). Our prediction for \(\langle (1 - x_b)^2 \rangle \big|_{x_b > 1 - \delta}\) is shown in Fig. 5. Note that unlike the case of \((1 - x_B) \big|_{x_B > 1 - \delta}\), the effect of the boost is very important in Eq. (23). Using the CLEO data in the region \(E_\gamma > 2.1\) GeV, we obtain the central value \(\lambda_1 \simeq -0.1\) GeV\(^2\), with large experimental errors. The uncertainty in this value of \(\lambda_1\) due to \(\bar{\Lambda}\) is small. Nonperturbative effects from the cut on \(E_\gamma\) [24], and the unknown higher order contributions to Eq. (23) are expected to have a larger impact on the determination of \(\lambda_1\) than the corresponding effects have on the determination of \(\bar{\Lambda}\) from Eq. (16).
In summary, we calculated order $\alpha_s^2/\beta_0$ corrections to the shape of the photon energy spectrum in weak radiative $\bar{B} \to X_s \gamma$ decay. The dominant 77 contribution is given by simple analytic formulae in Eqs. (7) and (9). The other terms relevant in the region $x_b > 0.65$ are the 22 and 27 contributions given in Eqs. (11) and (12), and the 78 term given in Eqs. (8) and (10). The HQET parameter $\bar{\Lambda}$ can be extracted from the average $\langle 1 - 2 E_\gamma/m_B \rangle$ using Eq. (16), and it can also be used to test whether the nonperturbative contribution to the Upsilon mass is small. The CLEO data in the region $E_\gamma > 2.1$ GeV implies the central values $\bar{\Lambda}_{\alpha_s} \simeq 390$ MeV and $\bar{\Lambda}_{\alpha_s^2/\beta_0} \simeq 270$ MeV at order $\alpha_s$ and $\alpha_s^2/\beta_0$, respectively. Possible contributions to the total decay rate from physics beyond the standard model are unlikely to affect this determination of $\bar{\Lambda}$. In the future, checking the $\delta$-independence of the extracted value of $\bar{\Lambda}$, and comparing the experimental and theoretical shapes of the photon spectrum for $x_b < 0.9$ can provide a check that nonperturbative effects and backgrounds are under control. This would also improve our confidence that the total decay rate in the region $x_B > 1 - \delta$ can be predicted model independently, and used to search for signatures of new physics with better sensitivity.

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