A NOTE ON LOWER BOUNDS OF MARTINGALE MEASURE DENSITIES

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Dedicated to the memory of Joe Doob

Abstract. For a given element $f \in L^1$ and a convex cone $C \subset L^\infty$, $C \cap L^\infty_+ = \{0\}$, we give necessary and sufficient conditions for the existence of an element $g \geq f$ lying in the polar of $C$. This polar is taken in $(L^\infty)^*$ and in $L^1$. In the context of mathematical finance the main result concerns the existence of martingale measures whose densities are bounded from below by a prescribed random variable.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Consider a convex cone $C \subset L^\infty = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, satisfying the condition

$$(1.1) \quad C \cap L^\infty_+ = \{0\},$$

where $L^\infty_+$ is the non-negative orthant of $L^\infty$. Typically, $C$ consists of random variables, dominated by stochastic integrals $\int_0^T H_t dS_t$ (compare [4]). Here $S = (S_t)_{0 \leq t \leq T}$ is a semimartingale, describing the stock-price process and $H = (H_t)_{0 \leq t \leq T}$ is a predictable $S$-integrable process, belonging to some class of admissible trading strategies. Assumption (1.1) is usually referred to as the no-arbitrage condition. Note that the cases of transaction costs, portfolio constraints and infinitely many assets can also be incorporated in this framework.

Furthermore, consider the polar of $C$, taken in $L^1 = L^1(\Omega, \mathcal{F}, \mathbb{P})$:

$$(1.2) \quad \{ y \in L^1 : \int_\Omega xy \, d\mathbb{P} \leq 0, \ x \in C \}.$$
For the case of a bounded process $S$, the set (1.2) is generated by densities of absolutely continuous martingale measures. In this note we discuss the following question:

(Q) Let $f \in L^1$. Under what conditions does there exist an element $g \in L^1$ in the polar of $C$ such that $g \geq f$?

In fact, this question concerns the existence of a martingale measure $Q$ whose density is bounded from below by the prescribed random variable $f$ up to a multiplicative constant $\alpha > 0$: $dQ/dP \geq \alpha f$.

Sometimes it is useful to take the polar of $C$ in $(L^\infty)^*$, the dual space of $L^\infty$; see, e.g., [3]. In our case it also appears that an easier answer to the question (Q) can be given if $g$ is allowed to lie in $(L^\infty)^*$; see Corollary 1 below and [8]. The answer to this question in precise terms is given in Corollary 2.

Our results are essentially the following. Regard $f \in L^1$ as a functional on $L^\infty$, defined by the formula

$$\langle x, f \rangle = \int_\Omega xf \, dP.$$ 

Then the existence of the desired element $g$ is equivalent to the boundedness of $f$ from above on a certain subset of the cone $C$. If $g$ is allowed to be an element of $(L^\infty)^*$, this subset may be chosen as

$$C_1 = \{ x \in C : x^- \leq 1 \text{ a.s.} \},$$

where $x^- = \max\{-x, 0\}$. If we seek $g \in L^1$, such a subset should be somewhat bigger:

$$C_V = \{ x \in C : x^- \in V \},$$

where $V$ is a neighbourhood of zero in the Mackey topology $\tau(L^\infty, L^1)$.

2. Answer to the question (Q)

We find it natural to examine the problem in a somewhat more general context. Let $(X, \tau)$ be a locally convex-solid Riesz space. This means that $X$ is a vector lattice, endowed with a topology $\tau$ whose local base consists of convex solid sets; see [1] for details. For an element $x \in X$, its positive part, negative part and absolute value are denoted by $x^+$, $x^-$ and $|x|$. The set $V \subset X$ is called solid if the conditions $x \in V$, $|y| \leq |x|$ imply that $y \in V$.

Consider a convex cone $C \subset X$, such that

$$(2.1) \quad C \cap X_+ = \{0\},$$

where $X_+ = \{ x \in X : x \geq 0 \}$. Let $V$ be a solid subset of $X$. Put

$$C_V = \{ x \in C : x^- \in V \}.$$

Using the implication

$$(2.2) \quad x \leq y \implies x^- \geq y^-,$$
it is elementary to check that

\[(2.3) \quad C_V = C \cap (V + X_+).\]

Denote by \(X^*\) the topological dual of \(X\) with the order induced by the dual cone \(X_+^* = \{ \xi \in X^* : \langle x, \xi \rangle \geq 0, \ x \in X_+ \}\). The polar of \(C\) is taken in \(X^*\):

\[C^0 = \{ \xi \in X^* : \langle x, \xi \rangle \leq 0, \ x \in C \}.\]

We use the customary notation \(\sigma(X^*, X)\) for the weak-star topology and \(|\sigma|(X, X^*)\) for the coarsest locally convex-solid topology on \(X\), compatible with the duality \(\langle X, X^* \rangle\) [1]. The polar of an arbitrary set \(A \subset X\) is defined as follows:

\[A^0 = \{ \xi \in X^* : \langle x, \xi \rangle \leq 1, \ x \in A \}.\]

**Theorem 1.** Let \((X, \tau)\) be a locally convex-solid Riesz space. Assume that there exists a \(\sigma(X^*, X)\)-compact set \(\Gamma \subset X_+^*\) such that the convex cone generated by \(\Gamma\) is \(\sigma(X^*, X)\)-dense in \(X_+^*\). Let \(C \subset X\) be a convex cone satisfying (2.1). Then for any \(f \in X^*\) the following conditions are equivalent:

(i) There exists a convex solid \(\tau\)-neighbourhood of zero \(V\) such that

\[
\sup_{x \in C_V} \langle x, f \rangle < +\infty, \quad C_V = \{ x \in C : x^- \in V \}.
\]

(ii) There exists \(g \in C^0\) such that \(g \geq f\).

**Proof.** (ii) \(\Rightarrow\) (i). Consider the convex solid \(\sigma(X^*, X)\)-neighbourhood of zero

\[V = \{ x \in X : (|x|, g - f) \leq 1 \}.
\]

Let \(x \in C_V\). Then

\[\langle x, f \rangle = \langle x, g \rangle + \langle x, f - g \rangle \leq \langle -x, g - f \rangle \leq \langle x^-, g - f \rangle \leq 1.
\]

(i) \(\Rightarrow\) (ii). Let \(\Gamma'\) be the \(\sigma(X^*, X)\)-closed convex hull of the set \(\Gamma \cup \{0\}\). Consider the \(\sigma(X^*, X)\)-compact convex set

\[\Pi = (V - X_+)^0 + \Gamma' = (V^0 \cap X_+^*) + \Gamma'.\]

and put

\[(2.4) \quad \lambda = \sup_{x \in C_V} \langle x, f \rangle.
\]

If the condition (ii) is false, we may apply the Hahn-Banach theorem [9, Chap. II, Th. 9.2] to separate the sets \(f + \lambda \Pi\) and \(C^0\) by an element \(x \in X\):

\[\sup_{\eta \in C^0} \langle x, \eta \rangle < \inf_{\zeta \in f + \lambda \Pi} \langle x, \zeta \rangle.
\]

Since \(C^0\) is a cone, we get \(\langle x, \eta \rangle \leq 0, \ \eta \in C^0\). Thus, \(x \in C^{\infty} = \text{cl} C\) by the bipolar theorem [9, Chap. IV, Th. 1.5], where \(\text{cl} C\) is the closure of \(C\) in any topology, compatible with the duality \(\langle X, X^* \rangle\), and

\[(2.5) \quad \langle x, f \rangle + \lambda \inf_{\zeta \in \Pi} \langle x, \zeta \rangle > 0.
\]
Furthennore, since \( \inf_{\zeta \in \Pi} \langle x, \zeta \rangle \leq 0 \), we conclude that \( \langle x, f \rangle > 0 \) and \( x \notin X_+ \). Indeed, for any \( \tau \)-neighbourhood of zero \( W \) take an element \( y_W \in (\mu x + W \cap V) \cap C, \mu > 0 \). If \( x^- = 0 \), then \( y_W \geq z_W \) for some \( z_W \in V \). By (2.2) and the solidness of \( V \) we have \( y_W \in V \). Thus, \( \mu x \in \cl C_V \) for any \( \mu > 0 \) and we obtain a contradiction, since \( \langle x, f \rangle > 0 \) and \( f \) must be bounded (from above) on \( \cl C_V \).

Moreover, \( \inf_{\zeta \in \Pi} \langle x, \zeta \rangle < 0 \), because otherwise \( x \) is non-negative on \( \Gamma \) and consequently on \( X^+_\ast \). In other words, \( x \in X_+ \), which we just have seen to be wrong. So, we may normalize \( x \) such that \( \inf_{\zeta \in \Pi} \langle x, \zeta \rangle = -1 \) and

\[
\langle x, f \rangle > \lambda
\]

by (2.5). Noting that \( -\Pi^0 \subset -(V - X_\ast)^\infty = \cl(V + X_+) \), we get

\[
x \in -\Pi^0 \cap \cl C \subset \cl(V + X_+) \cap \cl C \subset \cl C_V.
\]

To prove the last inclusion in (2.7) note that \( \alpha x \) is an interior point of \( V + X_\ast \) for all \( \alpha \in [0,1] \); see, e.g., [9, Chap.II]. For fixed \( 0 \leq \alpha < 1 \) let \( W \) be a \( \tau \)-neighbourhood of zero such that \( \alpha x + W \subset V + X_\ast \). Since \( \alpha x \in \cl C \), the set \((\alpha x + W) \cap C\) is non-empty. By (2.3) this means that \( \alpha x \in \cl C_V \) for each \( 0 \leq \alpha < 1 \) and therefore also for \( \alpha = 1 \).

Clearly, relations (2.6), (2.7) yield the desired contradiction to (2.4), which completes the proof. \( \square \)

The conditions of Theorem 1 are satisfied for any Banach lattice \( X \) (with the norm topology \( \tau \)) since we can take \( \Gamma = B_{X\ast} \cap X^+_\ast \), where \( B_{X\ast} \) is the unit ball of \( X^\ast \). Moreover, in this case, we can consider only one neighbourhood of zero \( V = B_X \) in condition (i). The corresponding result for the space \( L^\infty \) with the norm topology is formulated below.

**Corollary 1.** For any element \( f \in (L^\infty)^\ast \) the following conditions are equivalent:

1. \( \sup_{x \in C_1} \langle x, f \rangle < +\infty \), \( C_1 = \{ x \in C : x^- \leq 1 \text{ a.s.} \} \).
2. There exists \( g \in (L^\infty)^\ast \) such that \( g \geq f \) and \( g \in C^\infty \).

As a second example, the Mackey topology \( \tau(L^\infty, L^1) \) is locally convex-solid (see [2, Section 11]) and the set

\[
\Gamma = \{ x \in L^\infty_+ : \|x\|_{L^\infty} \leq 1 \} \subset L^1_+
\]

is \( \sigma(L^1, L^\infty) \)-compact (weakly compact in \( L^1 \)). Thus, Theorem 1 is valid for the space \( (L^\infty, \tau(L^\infty, L^1)) \). To make this result more concrete, we recall another description of the topology \( \tau(L^\infty, L^1) \).

A function \( \varphi : [0, \infty) \to [0, \infty) \) is called an \( N \)-function if it is convex and

\[
\lim_{t \to +0} \frac{\varphi(t)}{t} = 0, \quad \lim_{t \to +\infty} \frac{\varphi(t)}{t} = \infty.
\]
It follows that $\varphi$ is non-decreasing and continuous. Let $\|x\|_\varphi$ denote the Luxemburg norm (see, e.g., [6]):

$$\|x\|_\varphi = \inf\{\lambda > 0 : \int_\Omega \varphi(|x|/\lambda) \, dP \leq 1\}.$$ 

It is known that the Mackey topology $\tau(L^\infty, L^1)$ is generated by the family of Luxemburg norms $\{\| \cdot \|_\varphi : \varphi \in \Phi_N\}$, where $\Phi_N$ is the collection of all $N$-functions (see [7]).

In addition, this topology is generated by sets 

$$\mu \bigcap_{k=1}^\infty U_{\varepsilon_k}, \quad U_{\varepsilon_k} = \{x : \mathbb{P}(|x| \geq k) \leq \varepsilon_k\}, \quad k = 1, \ldots, \infty, \quad \mu > 0,$$

where $(\varepsilon_k)_{k=1}^\infty$ is any positive sequence. Indeed, for any sequence $\varepsilon_k > 0$ there exists an $N$-function $\varphi$ satisfying the conditions

$$\varphi(t) \geq \max_{1 \leq i \leq k} \{1/\varepsilon_i\}, \quad t \geq k.$$ 

If $\|x\|_\varphi \leq 1$, then

$$\mathbb{P}(|x| \geq k) = \int_{\{|x| \geq k\}} d\mathbb{P} \leq \varepsilon_k \int_{\{|x| \geq k\}} \varphi(|x|) \, d\mathbb{P} \leq \varepsilon_k.$$ 

Conversely, for any $N$-function $\varphi$ put $\varepsilon_k = k^{-2}/\varphi(k+1)$. If $x \in \bigcap_{k=1}^\infty U_{\varepsilon_k}$, then

$$\|x\|_\varphi \leq \int_{|x| < 1} \varphi(|x|) \, d\mathbb{P} + \sum_{k=1}^\infty \int_{k \leq |x| < k+1} \varphi(|x|) \, d\mathbb{P}$$

$$\leq \varphi(1) + \sum_{k=1}^\infty \varphi(k+1) \mathbb{P}\{|x| \geq k\} \leq \varphi(1) + \sum_{k=1}^\infty k^{-2}.$$ 

We collect these results in the following corollary, which gives the answer to the question (Q).

**Corollary 2.** For any element $f \in L^1$ the following conditions are equivalent:

(i) There exists a sequence $\varepsilon_k > 0$ such that

$$\sup \{\langle x, f \rangle : x \in \bigcap_{k=1}^\infty C_{\varepsilon_k}\} < \infty, \quad C_{\varepsilon_k} = \{x \in C : \mathbb{P}(x^- \geq k) \leq \varepsilon_k\}.$$ 

(ii) There exists an $N$-function $\varphi$ such that

$$\sup_{x \in C_{\varphi}} \langle x, f \rangle < \infty, \quad C_{\varphi} = \{x \in C : \|x^-\|_\varphi \leq 1\}.$$
(iii) There exists a convex solid $\tau(L^\infty, L^1)$-neighbourhood of zero $V$ such that
$$\sup_{x \in C_V} \langle x, f \rangle < +\infty, \quad C_V = \{ x \in C : x^- \in V \}.$$  
(iv) There exists $g \in L^1$ such that $g \geq f$ and $g \in C^\circ_\tau$.

The equivalence between (iii) and (iv) follows from Theorem 1. The two other equivalencies are implied by the properties of the Mackey topology $\tau(L^\infty, L^1)$, presented above.

3. Examples

Recall that $(L^\infty)^*$ may be identified with the space of all bounded finitely additive measures $\mu$ on $\mathcal{F}$ with the property that $P(A) = 0$ implies that $\mu(A) = 0$ [5]. Our first example shows that in the context of Corollary 1, in general, it is not possible to find the element $g \in (L^\infty)^*$ already in $L^1$ even if $f \in L^\infty$.

**Example 1.** Let $\Omega = [0, 1]$, suppose $\mathcal{F}$ consists of all Lebesgue measurable sets and let $P$ be the Lebesgue measure. Consider a purely finitely additive measure $\mu : \mathcal{F} \rightarrow \{0, 1\}$ such $\mu(I) = 1$ for any open interval $I \subset (0, 1)$ containing $1/2$ (see [10]). It follows that $\mu\{t - 1/2 \geq \delta\} = 0$ for all $\delta > 0$.

Put
$$C = \{ x \in L^\infty : \int_\Omega x d(P + \mu) \leq 0 \}.$$  
The element $f = 1 \in (L^\infty)^* \cap L^\infty$ is bounded on the set $C_1$, defined in Corollary 1:
$$\langle x, 1 \rangle = \int_\Omega x dP \leq -\int_\Omega x d\mu \leq 1, \quad x \in C_1,$$
and it is dominated by the element of $C^\circ \subset (L^\infty)^*$ corresponding to the measure $P + \mu$. However, $f$ is unbounded on any set $\bigcap_{k=1}^{\infty} C^{\varepsilon_k}$, defined in Corollary 2.(i).

To show this, consider a sequence $x_n \in L^\infty$, defined by the formulas
$$x_n(t) = n, \quad |t - 1/2| \geq \varepsilon_n/2, \quad x_n(t) = -n, \quad |t - 1/2| < \varepsilon_n/2, \quad n \geq 1, \ t \in [0, 1].$$  
Without loss of generality, we may assume that $\varepsilon_k > 0$ monotonically tends to 0. Evidently, $x_n \in \bigcap_{k=1}^{\infty} C^{\varepsilon_k}$:
$$\int_\Omega x_n d(P + \mu) = \int_0^1 x_n(t) dt - n = -2n\varepsilon_n \leq 0,$$
$$P(x_n^- \geq k) = 0, \quad n < k; \quad P(x_n^- \geq k) = \varepsilon_n \leq \varepsilon_k, \quad n \geq k.$$
But
$$\langle x_n, 1 \rangle = \int_0^1 x_n(t) dt = n(1 - 2\varepsilon_n) \to +\infty, \quad n \to \infty.$$
Hence, by Corollary 2, $f = 1$ cannot be dominated by any element of $C^0 \cap L^1$.

The next examples are in a more financial spirit. Note that in both of them the cone $C$ is a subspace. This is not essential: passing to $C - L_+^\infty$, the results still hold true.

**Example 2.** We consider a slight modification of an example given in [4, Remark 6.5.2]. Let $\Omega = \mathbb{N}$, the sigma-algebra $\mathcal{F}_0$ is generated by the sets $(\{2n-1, 2n\})_{n=1}^\infty$, and let $\mathcal{F} = \mathcal{F}_1$ be the power set of $\Omega$. Define the probability measure $P$ on $\mathcal{F}$ by $P(\{2n-1\}) = P(\{2n\}) = 2^{-n-1}$. Let the asset prices $(S_t)_{t=0}^1$ at times 0 and 1 be $S_0 \equiv 0$, and $S_1(2n-1) = 1$, $S_1(2n) = -2^{-n}$, $n \in \mathbb{N}$.

Let the cone $C$ be generated by the elements $\gamma (S_1 - S_0)$ in $L^\infty$, where $\gamma$ is an $\mathcal{F}_0$-measurable random variable. As usual, $\gamma$ may be interpreted as an investor’s portfolio at time $t = 0$. Then the set $C$ consists of the possible investor’s gains at time $t = 1$. Evidently, the no-arbitrage condition (1.1) is satisfied.

We claim that for any $f \in L_1^+$ the conditions of Corollaries 1 and 2 are equivalent and that there exists an element $g \geq f$, $g \in C^0 \cap L^1$, if and only if

$$\sum_{n=0}^{\infty} f(2n-1) < \infty. \tag{3.1}$$

It suffices to show that condition (3.1) implies condition (iv) of Corollary 2 and that condition (i) of Corollary 1 implies (3.1). Assume that (3.1) is satisfied and put

$$g(2n-1) = \max \{ f(2n-1), 2^{-n} f(2n) \}, \quad g(2n) = 2^n g(2n-1), \quad n \in \mathbb{N}.$$

Then $g \in L^1(P)$ and $g \geq f$. Computing the conditional expectation,

$$E_{\mathcal{F}_0} (gS_1 | \mathcal{F}_0) (2n-1) = (g(2n)S_1(2n) + g(2n-1)S_1(2n-1) )/2^n = 0,$$

we see that $g \in C^0$.

Now assume that condition (i) of Corollary 1 is satisfied. Put $\gamma(2n-1) = \gamma(2n) = 2^n$. Then $\gamma S_1 \in C_1$ and

$$\langle \gamma S_1, f \rangle = \sum_{n=1}^{\infty} (f(2n-1)/2 - 2^{-n-1} f(2n)) < +\infty.$$

Since $f \in L^1(P)$, we have $\sum_{n=1}^{\infty} 2^{-n-1} f(2n) < +\infty$ and condition (3.1) holds true.

For the cone considered in Example 2, there is no difference between the conditions of Corollaries 1 and 2 (in contrast to Example 1, which did not allow
for a financial interpretation). Below we consider a market with infinitely many assets, where these conditions are different and the following is true:

\[(f + L_1^\infty) \cap C^o = \emptyset, \quad (f + (L^\infty_\infty)^\ast) \cap C^o \neq \emptyset\]

for some \(f \in L_1^1\).

**Example 3.** Consider the same probability space \((\Omega, \mathcal{F}, P)\) as in Example 1. Let \((A_n)_{n=1}^\infty, A_n \subset [0, 1/2]\), be a sequence of independent events with probabilities \(P(A_n) = 1/2^n\). To construct such a sequence take independent random variables \(\xi_n : \Omega \mapsto \{0, 1\}\) such that \(P(\xi_n = 1) = 1/2^{n-1}\) and put

\[A_n = \{\xi_n^{-1}(1)/2 = t \in [0, 1/2] : \xi_n(t) = 1\} .\]

Furthermore, put \(b_0 = 1/2, b_n = b_{n-1} + 4^{-n}, n \geq 1\), and consider the sequence of intervals \(B_n = (b_{n-1}, b_n] \subset (1/2, 5/6]\). The sets \(B_n\) are mutually disjoint and disjoint from \(\bigcup_{n=1}^\infty A_n\). Let

\[f = \sum_{n=1}^\infty 2^n I_{B_n} + I_{[0,1/2]} + I_{[5/6,1]} .\]

Clearly, \(f \in L_1^1(P)\).

Now we introduce a countable sequence of asset price increments,

\[x_n = S^n_0 - S^n_0 = 2^n I_{B_n} - I_{A_n}, \quad n \in \mathbb{N} .\]

at times 0 and 1. We assume that the processes \((S^n_t)_{t=0}^1\) are adapted to the filtration \((\mathcal{F}_0, \mathcal{F}_1)\), where \(\mathcal{F}_1 = \mathcal{F}\) and \(\mathcal{F}_0\) is trivial. The portfolios \(\gamma^n\) are non-random, since they are assumed to be \(\mathcal{F}_0\)-measurable.

Let \(C\) be the linear subspace of \(L_\infty^\infty\) spanned (algebraically) by \(x_n\). The elements of \(C\) describe the investor’s gains, obtained by trading in a finite collection of assets. The condition \(E_P(x_n) = 0\) implies that \(C\) is disjoint from \(L_\infty^\infty \setminus \{0\}\).

Let \(z = \sum_{n \in J} \gamma^n x_n\) be any element of \(C_1\). Here \(J\) is a finite subset of \(\mathbb{N}\) and \(\gamma^n\) are some constants. By the definition of \(C_1\) we have

\[z = \sum_{n \in J} \gamma^n (2^n I_{B_n} - I_{A_n}) \geq -1, \text{ a.s.}\]

Considering this inequality on the sets \(B_n\) and \(\bigcap_{n \in J} A_n\), we get

\[-\gamma^n 2^n \leq 1, \quad \sum_{n \in J} \gamma^n \leq 1 .\]

It follows that condition (i) of Corollary 1 is satisfied:

\[
\langle z, f \rangle = \sum_{n \in J} \gamma^n (2^n \int_{B_n} f dP - \int_{A_n} f dP) \\
= \sum_{n \in J} \gamma^n (1 - 2^{-n}) \leq 1 + \sum_{n \in J} 4^{-n} \leq 4/3 .
\]
To show that condition (i) of Corollary 2 fails, consider any sequence $\varepsilon_k > 0$, $k \geq 1$, and assume that $f$ is bounded from above by a constant $\beta$ on the set $\bigcap_{k=1}^{\infty} C^{\varepsilon_k}$. Define natural numbers $m, n_1, \ldots, n_m$ as follows:

$$m > \beta + 1, \quad \sum_{i=1}^{m} \frac{1}{2^{n_i}} \leq \min\{\varepsilon_1, \ldots, \varepsilon_m\}.$$

We have

$$P(x_{n_1} + \cdots + x_{n_m} \leq -k) = 0, \quad k > m,$$

and

$$P(x_{n_1} + \cdots + x_{n_m} \leq -k) \leq P\left(\bigcup_{i=1}^{m}\{x_{n_i} \leq -1\}\right) \leq \sum_{i=1}^{m} \frac{1}{2^{n_i}} \leq \varepsilon_k, \quad k \leq m.$$

Thus $x_{n_1} + \cdots + x_{n_m} \in \bigcap_{k=1}^{\infty} C^{\varepsilon_k}$ and we obtain a contradiction:

$$\langle x_{n_1} + \cdots + x_{n_m}, f \rangle = \sum_{i=1}^{m} \left(2^{n_i} \int_{B_{n_i}} f \, dP - \int_{A_{n_i}} f \, dP\right)$$

$$= m - \sum_{i=1}^{m} 2^{-n_i} \geq m - 1 > \beta.$$

Note also that if $\nu$ is the non-negative finitely additive measure corresponding to an element $g \in C^\circ$, $g \geq f$, then

$$\nu(A_n) = \langle I_{A_n}, g \rangle = 2^n \langle I_{B_n}, g \rangle \geq 2^n \langle I_{B_n}, f \rangle = 1.$$

Hence, $\nu$ is not countably additive.

Finally, we mention that it would be interesting to determine if the relations (3.2) can hold true for the case of finitely many assets.

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