Research Article

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On stochastic inverse problem of construction of stable program motion

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Abstract: One of the inverse problems of dynamics in the presence of random perturbations is considered. This is the problem of the simultaneous construction of a set of first-order Ito stochastic differential equations with a given integral manifold, and a set of comparison functions. The given manifold is stable in probability with respect to these comparison functions.

Keywords: stochastic differential equations, inverse problems, stability, integral manifold

MSC 2020: 34-xx, 60-xx

1 Introduction

The theory of inverse problems of differential systems and general methods of their solving are quite fully developed in the class of ordinary differential equations [1–11].

Note that one of the important requirements in the theory of inverse problems of differential systems, which is associated with the system's intransigence to perturbations, is the requirement of stability of the given properties of motion [5]. Therefore, the solution to the problem of stability of program motion is essential for the further development of the theory of inverse problems of differential systems and the theory of constructing systems of program motion.

In the theory of stability, possible perturbed motions of a material system are compared with unperturbed motion in relation to the corresponding values of the given kinematic indicators of motion for each moment of time. In the established formulations of stability problems, the comparison functions \( Q(y, t) \) are given. The unperturbed motion and equations of motion of the considered material system are also given. Thus, the solution to the stability problem is reduced to determine the stability conditions for a given motion of the system under consideration with respect to the given comparison functions \( Q(y, t) \). However, in many problems of stability theory, it is useful to construct the comparison functions themselves, with respect to which there is the stability of the given properties of the motion of a mechanical system.

A. S. Galiullin posed the following inverse problem of dynamics, namely, the problem of constructing a set of ordinary differential equations possessing a given integral manifold, as well as the problem of constructing a set of comparison functions with respect to which the stability of the given integral manifold takes place [1–3].

It is required to construct the set of equations of motion for the material system

\[ \dot{y} = Y(y, t), \]  

(1.1)

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according to the given law of motion
\[ \Lambda : y = \varphi(t), \quad y \in R^n, \quad \varphi \in C^1 \{ t \geq t_0 \}, \tag{1.2} \]
in the class of equations admitting the existence of a unique solution of (1.1) for the initial conditions
\[ y_{k=t_0} = \varphi(t_0), \{ t_0, \varphi(t_0) \}. \]
And, it is required to construct the set of \( n \)-dimensional vector functions \( Q(y) \), in which \( Q(y) \) are holomorphic vector functions in some \( \varepsilon \)-neighborhood \( \Lambda_\varepsilon = \{ ||y - \varphi(t)|| < \varepsilon \} \) of the integral manifold \( \Lambda(t) \) (1.2) for all \( t \geq t_0 \) such that \( \Lambda(t) \) is stable by Lyapunov in relation to their components.

The posed problem of constructing the set of ordinary differential equations and a set of comparison functions was solved in [1–3] by the Lyapunov characteristic numbers method. The solution of this problem determines a set of kinematic indicators of motion, in relation to which the given properties of the motion of the system under consideration are stable.

The set of equations of motion of the system, for which the given motion (1.2) is one of the possible, is constructed in [1–3] in the following form
\[ \dot{y} = \dot{\varphi}(t) + \Phi(y, t). \tag{1.3} \]
Here, \( \Phi(y, t) \) is some holomorphic vector function in the domain \( \Lambda_\varepsilon \) for all \( t \geq t_0 \), \( \Phi(y, t)|_{y=\varphi(t)} = 0 \). Furthermore, the equation of perturbed motion of the first approximation with respect to the vector function \( Q(y) \) is compiled. The equation of perturbed motion is reduced to a system of linear differential equations by some linear transformation. It is required that the resulting system is correct and that its characteristic numbers are positive. If the applied linear transformation is Lyapunov transformation, then the given motion (1.1) is stable with respect to the vector function \( Q(y) \).

The set of sought components of the vector functions \( Q(y) \) in [2] is determined by the following conditions:
\[ C, \dot{C} \text{ are limited, } \det C^{-1} \neq 0 \quad \text{for all } t \geq t_0. \tag{1.4} \]
Here, \( C = ||\psi^i_l||_n, \psi^i_l = \frac{\partial Q_i(y)}{\partial y^l}|_{y=\varphi(t)}, \quad i, \nu = 1, \ldots, n \), \( C(t) \) is the Lyapunov transformation [12].

Ito stochastic differential equations describe models of mechanical systems that are important in the application, which take into account the effect of external random forces, for example, the motion of an artificial Earth satellite under the action of gravitational forces and aerodynamic forces [13] or fluctuation drift of a heavy gyroscope in a gimbal [14] and many others. An example showing the importance of taking into account random perturbations is the inverse problem of the dynamics of a spacecraft flight. For example, the aerodynamic moments of a spacecraft always have random components [13] due to fluctuations in the density of the planet’s atmosphere. And, random changes in the moments of inertia cause thermoelastic vibrations of stabilizing rods and vibrations of liquids in banks, antennas, and solar panels. Analysis of the influence of random perturbations is so important that ignoring these perturbations of the spacecraft can significantly reduce its service life [15]. Inverse problems in the class of stochastic differential systems were considered in [16–18]. And, the stability in probability of the given program motion is investigated by the Lyapunov functions method in [19,20]. Let us consider the problem posed earlier in the class of ordinary differential equations [1–3] under the additional assumption of the presence of random perturbations.

### 2 Problem statement

Let the program motion
\[ \Lambda : \Lambda \equiv y - \varphi(t) = 0, \quad y \in R^n, \quad \varphi \in C^1, \quad ||\varphi|| \leq l \tag{2.1} \]
be given. It is required to construct a corresponding set of stochastic differential equations of motion of the system
\[ \dot{y} = Y(y, t) + \sigma(y, t)\xi, \quad \xi \in R^k, \tag{2.2} \]
in the class of equations admitting the existence of a unique solution of (2.2) for the initial conditions \( y|_{t=t_0} = \varphi(t_0) \). And, it is required to construct the set of \( n \)-dimensional vector functions \( Q(y) \), which are holomorphic vector functions in some \( \varepsilon \)-neighborhood \( \Lambda_\varepsilon = \{ \| y - \varphi(t) \| < \varepsilon \} \) of the integral manifold \( \Lambda(t) \) (2.1) for all \( t \geq t_0 \) such that \( \Lambda(t) \) is stable by Lyapunov in relation to the components of \( Q(t) \).

Here, \( \xi(t) = \omega(t) + \int_{t_0}^t c(y)P(t, dy) \) is a random process with independent increments. \( \omega(t) \) is a Wiener process. \( P(t, A) \) is a Poisson process as a function of \( t \) and a Poisson stochastic measure as a function of the set \( A \). \( c(y) \) is a vector function mapping \( \mathbb{R}^n \) into the space of values of the process \( \xi(t) \) for each \( t \) [21].

Following [2,16], the equation of perturbed motion of the material system, for which the given motion (2.1) is possible, is represented as

\[
\dot{\lambda} = A(\lambda; y, t) + B(\lambda; y, t)\dot{\xi}.
\]  

(2.3)

Here, \( A(\lambda; y, t) \) is a vector function and \( B(\lambda; y, t) \) is an \( n \times k \)-dimensional Erugin-type matrix [22] such that \( A(0; y, t) \equiv 0, B(0; y, t) \equiv 0 \).

**Definition 1.** [23] A function \( a(r) \) is called a function of the Khan class \( a(r) \in K \) if it is continuous and strictly increasing and satisfies the condition \( a(0) = 0 \).

**Definition 2.** [24] The program manifold (2.1) of the equation (2.2) is called \( \rho \)-stable in probability if

\[
\lim_{\rho(y_0, \Lambda(\lambda); t_0) \to 0} \mathbb{P}_0 \left\{ \sup_{t \to 0} \rho(y^b(t), \Lambda(t)) > \varepsilon \right\} = 0.
\]

**2.1 The set of vector functions \( Q(y) \) that do not explicitly depend on time**

**Theorem 1.** Let in the neighborhood \( \Lambda_\varepsilon \) there exist a Lyapunov function \( V(\lambda; y, t) \) of the integral manifold \( \Lambda \) with the properties

\[
a(\| \Lambda \|) \leq V(\lambda; y, t) \leq b(\| \Lambda \|), \quad a, b \in K,
\]  

(2.4)

\[
LV \leq -c(\| \Lambda \|), \quad c \in K.
\]

(2.5)

Then, the program motion \( \dot{\lambda} \equiv y - \varphi(t) = 0 \) of system (2.3) is asymptotically \( \rho \)-stable in probability with respect to an arbitrary \( s \)-dimensional vector function \( Q(y) \), which is continuous in the neighborhood \( \Lambda_\varepsilon \) for \( 1 \leq s \leq n \).

**Proof.** Let us consider the difference \( x = Q(y) - Q(\varphi(t)) \) by the definition of \( Q \)-stability of motion [2]. The existence of Lyapunov function \( V(\lambda; y, t) \) with properties (2.4), (2.5) by the condition of the theorem ensures the asymptotic \( \rho \)-stability in probability of the program motion \( \dot{\lambda} = 0 \) [19]

\[
\lim_{\rho(y_0, \Lambda(\lambda); t_0) \to 0} \mathbb{P}_0 \left\{ \lim_{t \to \infty} \sup_{t \to 0} \rho(y^b(t), \Lambda(t)) = 0 \right\} = 1.
\]

(2.6)

And from the continuity of the vector function \( Q(y) \) and from condition (2.6), it follows that

\[
\lim_{\rho(y_0, \Lambda(\lambda); t_0) \to 0} \mathbb{P}_0 \left\{ \lim_{t \to \infty} \sup_{t \to 0} \| Q(y(t, t_0, y_0)) - Q(\varphi(t)) \| = 0 \right\} = 1.
\]

The fulfillment of the latter gives the asymptotic stability of the motion \( \dot{\lambda} \equiv y - \varphi(t) = 0 \) of system (2.3) with respect to the vector function \( Q(y) \).
2.2 The set of vector functions \( Q(y, t) \) depending on \( y \) and \( t \)

**Theorem 2.** Let in the neighborhood \( \Lambda_e \) there exist a Lyapunov function \( V(\lambda; y, t) \) of the integral manifold \( \Lambda \) with the properties (2.4), (2.5). Then, the program motion \( \lambda = y - \varphi(t) = 0 \) of system (2.3) is asymptotically \( \rho \)-stable in probability with respect to an arbitrary continuous in \( y \) and \( t \) \( s \)-dimensional vector function \( Q(y, t) \) satisfying the condition

\[
\|x\| \leq \beta\|\lambda\|, \quad \beta \in K.
\]  

(2.7)

Here, \( x = Q(y, t) - Q(\varphi(t), t) \).

**Proof.** The existence of the function \( V(\lambda; y, t) \) with properties (2.4), (2.5) implies asymptotic \( \rho \)-stability in probability of motion \( \dot{\lambda} = y - \varphi(t) = 0 \) uniform with respect to \( \{t_0, y_0\} \) [19], that is

\[
\lim_{\rho(\lambda, t) \to 0} \mathbb{P} \sup_{t \to \infty} \rho(y^{\varphi(t)}, t) = 0 \to \infty = 1
\]

and

\[
\lim_{\rho(\lambda, t) \to 0} \mathbb{P} \sup_{t \to \infty} \|Q(y(t, t_0, y_0), t) - Q(\varphi(t), t)\| = 0 \to \infty = 1
\]

follows from (2.6) and (2.7).

Consequently, \( \lambda = y - \varphi(t) = 0 \) is asymptotically stable in probability with respect to the vector function \( Q(y, t) \).

2.3 Program motion \( \lambda(y, t) = 0 \) and a set of comparison vector functions \( Q(\lambda, t) \)

Let us define the program motion in the following form:

\[
\Lambda(t) : \dot{\lambda}(y, t) = 0.
\]

(2.8)

Here, \( \lambda \in \mathbb{R}^k \), \( y \in \mathbb{R}^n \), \( k \leq n \).

Suppose that it takes place \( \text{rang} \frac{\partial \Lambda}{\partial y} = k \) for all \( y \in \Lambda_h, t \geq t_0 \) in the neighborhood \( \Lambda_h(t) \in \mathbb{R}^n \):

\[
\Lambda_h(t) : \|\lambda(y, t)\| \leq h, \quad t \geq t_0.
\]

(2.9)

The set of equations of perturbed motion can be represented as

\[
\dot{\lambda} = A(\lambda; y, t) + B(\lambda; y, t) \dot{\xi},
\]

(2.10)

following the Erugin’s method [22]. Here, \( A(\lambda; y, t) \) is a vector function and \( B(\lambda; y, t) \) is a \( n \times k \)-dimensional Erugin-type matrix such that \( A(0; y, t) \equiv 0, B(0; y, t) \equiv 0 \).

Consider the continuous \( s \)-dimensional vector functions \( Q(\lambda, t) \) for which the inequality

\[
\|x\| \leq \beta\|\lambda\|, \quad \beta \in K
\]

holds. Here, \( x = Q(\lambda(y, t), t) - Q(0, t), 1 \leq s \leq n \).

**Theorem 3.** Suppose that in neighborhood (2.9) of the integral manifold (2.8) there exists a Lyapunov function \( V(\lambda; y, t) \) with properties (2.4) and

\[
LV \leq -c\|\lambda\|, \quad c \in K.
\]

(2.12)

Then, the integral manifold \( \Lambda(t) \) (2.8) is asymptotically stable in probability with respect to an arbitrary \( s \)-dimensional vector function \( Q(\lambda, t) \), which is continuous with respect to \( \lambda \) and \( t \) and satisfying condition (2.11) for \( 1 \leq s \leq n \).

The proof of Theorem 3 is carried out similarly to the proof of Theorem 2.
### 2.4 A set of \( n \)-dimensional vector functions of the form \( C(t)\lambda \)

Assume that the perturbed motion equation (2.3) in the first approximation has the form

\[
\dot{\lambda} = A_1(t)\lambda + A_2(\lambda, t) + B_1\dot{\xi}.
\] (2.13)

Let us consider a Lyapunov function \( V(\lambda) = (\lambda, \lambda) \) and a \( n \)-dimensional vector function \( Q(y, t) = C(t)\lambda \). Here, \( \lambda \equiv y - \varphi(t) \). In this case, \( x = Q(y) - Q(\varphi(t)) \) has the form

\[
x = C(t)\lambda.
\] (2.14)

Assume that

(i) the matrix \( A_1^T(t) + A_1(t) \) is definitely negative, and the vector function \( A_2 \) satisfies the condition \( \|A_2\| = o(\|\lambda\|) \);

(ii) the matrix

\[
C(t) = \left. \frac{\partial Q}{\partial y} \right|_{y=\varphi(t)}
\]

is continuous and limited for all \( t \geq t_0 \). (2.15)

Then, taking into account properties (i), (ii) and Theorem 2, the following theorem holds.

**Theorem 4.** Let \( A_1(t) \) and \( C(t) \) be continuous matrices such that conditions (i) and (ii) are satisfied.

Then, the motion \( \lambda \equiv y - \varphi(t) = 0 \) of the system (2.3) is stable in probability with respect to the arbitrary vector functions \( Q(y, t) = C(t)\lambda \).

**Remark.** In Theorem 4, the weaker condition (2.15) is required instead of condition (1.4).

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### References

[1] A. S. Galiullin, *Problems of construction of comparison functions in the theory of motion stability*, Differ. Equ. **10** (1974), no. 8, 1527–1529 (in Russian).

[2] A. S. Galiullin, *Methods for Solving Inverse Problems of Dynamics*, Science, Moscow, 1986 (in Russian).

[3] A. S. Galiullin, *Selected Works in Two Volumes, V. I. and V. II*, People’s Friendship University, Moscow, 2009 (in Russian).

[4] R. G. Mukharlyamov, *On construction of systems of differential equations of movement of mechanical systems*, Differ. Equ. **39** (2003), no. 3, 369–380 (in Russian).

[5] R. G. Mukharlyamov, *Reduction of dynamical equations for the systems with constraints to given structure*, J. Appl. Math. Mech. **71** (2007), no. 3, 401–410.

[6] R. G. Mukharlyamov, *Differential-algebraic equations of programmed motions of Lagrangian dynamical systems*, Mech. Solids **46** (2011), no. 4, 534–543.

[7] R. G. Mukharlyamov and M. I. Tleubergenov, *Control of system dynamics and constrain constraints stabilization*, in: V. Vishnevskiy, K. Samouylov, D. Kozirev (eds), Distributed Computer and Communication Networks, DCCN 2017, Communications in Computer and Information Science, vol. 700, Springer, Cham, pp. 431–442, DOI: https://doi.org/10.1007/978-3-319-66836-9_36.

[8] S. S. Zhumatov, *Asymptotic stability of implicit differential systems in the vicinity of program manifold*, Ukrainian Math. J. **66** (2014), no. 4, 625–632.

[9] S. S. Zhumatov, *Exponential stability of a program manifold of indirect control systems*, Ukrainian Math. J. **62** (2010), no. 6, 907–915.

[10] S. S. Zhumatov, *Stability of a program manifold of control systems with locally quadratic relations*, Ukrainian Math. J. **61** (2009), 500, DOI: https://doi.org/10.1007/s11253-009-0224-y.
[11] J. Llibre and R. Ramirez, *Inverse Problems in Ordinary Differential Equations and Applications*, Springer International Publishing, Switzerland, 2016.

[12] B. P. Demidovich, *Lectures on the Mathematical Theory of Stability*, Nauka, Moscow, 1967 (in Russian).

[13] P. Sagirov, *Stochastic Methods in the Dynamics of Satellites*, Mechanics. Periodical collection of translations of foreign articles 5 (1974), no. 6, 28–47 (in Russian).

[14] I. N. Sinitsyn, *About gyroscope fluctuations in gimbal*, News of Academy of Sciences of the USSR, Mechanics of Solids 3 (1976), no. 3, 23–31 (in Russian).

[15] V. G. Demin, *The Motion of an Artificial Satellite in an off-center Gravitational Field*, Nauka, Moscow, 1968 (in Russian).

[16] M. I. Tleubergenov, *Main inverse problem for differential system with generate diffusion*, Ukrainian Math. J. 65 (2013), no. 5, 787–792.

[17] M. I. Tleubergenov, *On the inverse stochastic reconstruction problem*, Differ. Equ. 50 (2014), no. 2, 274–278.

[18] M. I. Tleubergenov, *Stochastic inverse problem with indirect control*, Differ. Equ. 53 (2017), no. 10, 1387–1391.

[19] G. K. Vassilina and M. I. Tleubergenov, *Solution of the problem of stochastic stability of an integral manifold by the second Lyapunov method*, Ukrainian Math. J. 68 (2016), 14–28.

[20] G. K. Vassilina and M. I. Tleubergenov, *On the optimal stabilization of an integral manifold*, J. Math. Sci. 229 (2018), no. 4, 390–402.

[21] V. S. Pugachev and I. N. Sinitsyn, *Stochastic Differential Systems. Analysis and Filtering*, Wiley, Chichester, 1987.

[22] N. P. Erugin, *Construction of the entire set of systems of differential equations with a given integral curve*, J. Appl. Math. Mech. 10 (1952), no. 6, 659–670 (in Russian).

[23] N. Rush, P. Abets, and M. Lalua, *Lyapunov’s Direct Method in Stability Theory*, Mir, Moscow, 1980 (in Russian).

[24] R. Z. Khasminskii, *Stability of Stochastic Differential Equations*, Springer-Verlag, Berlin, Heidelberg, 2012.