On the Bloch–Kato conjecture for $GSp(4)$

David Loeffler
Sarah Livia Zerbes
ABSTRACT. We prove an explicit reciprocity law for the Euler system attached to the spin motive of a genus 2 Siegel modular form. As consequences, we obtain one inclusion of the Iwasawa Main Conjecture for such motives, and the Bloch–Kato conjecture in analytic rank 0 for their critical twists.

_Todas las artes tienen en común el esfuerzo de dominar la materia, de reordenar el caos._

—P. MAURENSIEG, Teoría de las sombras
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Introduction

1. Aims of this paper

Euler systems are one of the most powerful tools for controlling the cohomology groups of global Galois representations, and hence for proving cases of the two interrelated conjectures linking these groups to values of $L$-functions: the Bloch–Kato conjecture and the Main Conjecture of Iwasawa Theory. More precisely, it follows from work of Kolyvagin, Kato and Rubin that if there exists an Euler system for some Galois representation $V$, and if the bottom class of this Euler system is non-zero, then we obtain a bound on the Selmer group of $V$. So, in order to make progress on the above conjectures, we need to first construct an Euler system for $V$, and then to prove a formula (an explicit reciprocity law) relating the localisation of this Euler system at $p$ to the values of $L$-functions.

The goal of this paper is to carry out this program for the 4-dimensional spin Galois representations arising from Siegel modular forms for the group $\text{Sp}_4(\mathbb{Z})$ (of sufficiently regular weights). This builds on earlier work carried out in the paper [LSZ17] together with Chris Skinner, where we constructed an Euler system for these spin Galois representations. At the time that paper was written, we faced severe conceptual difficulties in proving an explicit reciprocity law for the Euler system; so we could not rule out the possibility that the entire Euler system was zero, and the arithmetic applications given in op.cit. were conditional on assuming that the Euler system satisfied an explicit reciprocity law of the expected form.

The main result of this paper is a proof of the missing explicit reciprocity law. This paper can be seen as a sequel to the paper [LPSZ19] with Skinner and Vincent Pilloni, in which we constructed a $p$-adic $L$-function interpolating critical values of the spin $L$-functions of an automorphic representation of $\text{GSp}_4$, using Piatetski–Shapiro’s integral formula for the spin $L$-function [PS97], Harris’ interpretation of this integral in terms of coherent cohomology [Har04], and Pilloni’s results on $p$-adic interpolation of coherent cohomology (“higher Hida theory”) [Pil17].

As a consequence of the explicit reciprocity law, we obtain one inclusion of the Iwasawa Main Conjecture for the spin Galois representation, and the Bloch–Kato conjecture for the analytic rank 0 twists of this Galois representation.

2. Main results of the paper

In order to state the results a little more precisely, we need to introduce some notation. Let $\Pi$ be a non-endoscopic, non-CAP, globally generic automorphic representation of $G(\mathbb{A}_f)$, of weights $(k_1, k_2) = (r_1 + 3, r_2 + 3)$ with $r_1, r_2 \geq 0$, and write $W_\Pi$ for the associated spin Galois representation. We suppose $\Pi$ is unramified and Borel-ordinary at $p$. Let $(q, r)$ be integers with $0 \leq q \leq r_2$ and $0 \leq r \leq r_1 - r_2$.

In [LSZ17] we defined cohomology classes

$$z^{[\Pi,q,r]}_{M,m} \in H^1_f(Q(\mu_{Mp^m}), W_\Pi(-q)),$$

for $m \geq 0$ and $M \geq 1$ not divisible by $p$ or the primes of ramification of $\Pi$, satisfying Euler-system norm compatibility conditions as $M$ and $m$ vary. These classes depend on choices of local data $(\nu, \Phi)$ at the bad places. The first main result of this paper is to evaluate the image of the $M = 1, m = 0$ class under the Bloch–Kato logarithm map

$$\log_{\text{BK}} : H^1_f(Q_p, W_\Pi(-q)) \to (\text{Fil}^1 \text{D}_{\text{dR}}(W_\Pi))^\ast,$$

relating these to non-critical $p$-adic $L$-values:

**Theorem A.** Assume that $r_2 \geq 1$. Then, for a suitable choice of element $\nu \in \text{Fil}^1 \text{D}_{\text{dR}}(W_\Pi)$, we have

$$\langle \nu, z^{[\Pi,q,r]}_{1,0} \rangle_{\text{D}_{\text{crys}}(W_\Pi)} = (\star) \times \mathcal{L}_p(\Pi, 1 - r_2 + q, r) \cdot \tilde{Z}(w, \Phi)$$

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for an explicit non-zero factor ($\ast$). Here, $\mathcal{L}_p(\Pi, j_1, j_2)$ denotes the 2-variable spin $p$-adic $L$-function constructed in [LPSZ19], and $\tilde{Z}(w, \Phi)$ is a product of local zeta integrals (which is non-zero for a suitable choice of $(w, \Phi)$).

The proof of this theorem occupies the majority of the paper. Note that we do not require $\Pi$ to have level 1 here. (The assumption that $\Pi$ be Borel-ordinary can also be relaxed to the weaker condition of Klingen-ordinarity; the stronger ordinarity condition is needed to construct classes $\tilde{z}_{M,m}$ for all $m$ without introducing unbounded denominators, but only the $m = 0$ class appears in the theorem. Klingen-ordinarity is vital, however, since without it we cannot even define the $p$-adic $L$-function.)

**Remark.** The proof of Theorem A relies on an assertion concerning the rigid cohomology of strata in the special fibre of a GSp$_4$ Shimura variety, formulated as Conjecture [10.2.3] below. A proof of this statement has been found by Lan and Skinner and will appear in a forthcoming paper.

Our second main result is a considerable strengthening of Theorem A, under far more restrictive hypotheses. We now assume that $\Pi$ satisfies the conditions of Theorem A, and also the following extra conditions:

- $\Pi$ has level one (i.e. $\Pi_\ell$ is unramified for all finite primes $\ell$);
- $r_1 - r_2 \geq 6$;
- for some (and hence every) $G_Q$-stable lattice $T$ in $W^*_\Pi$, and every Dirichlet character $\chi$ of prime-to-$p$ conductor, Rubin’s “big image” condition $\text{Hyp}(\mathbb{Q}(\mu_{p^\infty}), T(\chi))$ holds (cf. [LSZ17 Assumption 11.1.2]).

The condition $r_1 - r_2 \geq 6$ implies that the $p$-adic $L$-function factors as the product of two copies of a single-variable $p$-adic $L$-function $\mathcal{L}_p(\Pi, j)$.

**Theorem B.** There exists an Euler system for $W^*_\Pi(-1 - r_2)$, whose image under the Perrin-Riou cyclotomic regulator map is the $p$-adic $L$-function $\mathcal{L}_p(\Pi, j)$.

Note that this result relies on Theorem A not only for $\Pi$ itself, but also for all the classical specialisations of a $p$-adic family passing through $\Pi$. From Theorem B we readily obtain the following two arithmetic applications. The first gives one inclusion in the Iwasawa main conjecture for $W^*_\Pi$, up to inverting $p$:

**Theorem C.** Let $V = W^*_\Pi(-1 - r_2)$, and denote by $\tilde{H}^1_{\text{Iw}}(\mathbb{Q}_p, V)$ the Nekovář Selmer complex, with the unramified local conditions at $\ell \neq p$ and the Greenberg-type local condition at $p$ determined by the Klingen-ordinarity of $\Pi$. Assume that the above conditions are satisfied. Then the module $\tilde{H}^2_{\text{Iw}}(\mathbb{Q}(\mu_{p^\infty}), V)$ is torsion over the Iwasawa algebra, and its characteristic ideal divides the $p$-adic $L$-function $\mathcal{L}_p(\Pi, j)$.

Note that this is a divisibility of ideals in $\Lambda_L(\mathbb{Z}_p^\times)$ where $\Gamma \cong \mathbb{Z}_p^\times$ and $L$ is a finite extension of $\mathbb{Q}_p$. The module $\tilde{H}^2_{\text{Iw}}$ can also be interpreted more classically as the base-extension to $L$ of the Pontryagin dual of a Selmer group attached to a representation of cofinite type over $\mathbb{Z}_p$, linking up with more classical formulations of an Iwasawa main conjecture; see Proposition [18.1.2] below.

Our second application is to the Bloch–Kato conjecture:

**Theorem D.** Assume that the above conditions are satisfied. Let $0 \leq j \leq r_1 - r_2$, and let $\rho$ be a finite-order character of $\mathbb{Z}_p^\times$. If $L(\Pi \otimes \rho, \frac{1 - r_1 + r_2}{2} + j) \neq 0$, then $H^j_L(\mathbb{Q}, V(-j - \rho)) = 0$.

**Future plans.** In future work, we hope to relax the conditions on the weight and tame level of $\Pi$, and to consider deformations towards singular (non-cohomological) weights. This will have applications to the Birch–Swinnerton-Dyer conjecture for modular abelian surfaces.

More generally, the strategy that we developed for the proof of the explicit reciprocity law should be applicable to many other cases where an Euler system has been constructed, but where the relevant $L$-values cannot be expressed purely in terms of degree zero coherent cohomology, as they can in previously-studied cases such as $\text{GL}_2 \times \text{GL}_2$. The case of the Euler system attached to the Asai representation of quadratic Hilbert modular forms is currently being studied by one of our PhD students.

### 3. Strategy

We outline the overall strategy used in the proofs of Theorems A and B.
3.1. Strategy for Theorem A.

(1) Using equivariance properties of the Lemma–Flach classes as the test data \((w, \Phi)\) vary, we show that it suffices to prove the theorem for \((w, \Phi)\) which have a certain specific type at \(p\) (“Klingen-type test data”). For these Klingen test data at \(p\), the left-hand side of Theorem A can be expressed as a pairing \(\langle 6.2 \rangle\) between a de Rham cohomology class \(\eta_{\text{dR}}\) of Klingen parahoric level which is an ordinary eigenvector for the Hecke operator \(U'_{\text{dR}}\), and the logarithm of an \(\ell\)-adic class which is the pushforward of a pair of \(\text{GL}_2\) Eisenstein classes along a certain “twisted” embedding \(\mathfrak{A}\) of Shimura varieties \(Y_{H, A} \hookrightarrow Y_{G, K_l}\). (This embedding is also used in the definition of the \(p\)-adic \(L\)-function \(L_p(\Pi)\) in [LPSZ19].)

(2) We express the pairing \(\langle 6.2 \rangle\) using the “Nekovář–Nizioł finite-polynomial cohomology” of BLZ16 (a variant of the syntomic cohomology introduced in [NN16]). This gives a formalism of Abel–Jacobi maps, allowing us to write \(\langle 6.2 \rangle\) as a cup-product between the pushforward of the syntomic \(\text{GL}_2 \times \text{GL}_2\) Eisenstein class and a class \(\eta_{\text{NN-fp}}\) which is a lifting of \(\eta_{\text{dR}}\) to Nekovář–Nizioł fp-cohomology; see (7.8). By a new comparison result due to Ertl–Yamada [FY19], this is equivalent to a pairing in log–rigid finite-polynomial cohomology (c.f. Proposition 10.4.2).

(3) Using the Eigenspace Vanishing Conjecture 10.2.3, we show that the pairing factors through a pairing in the rigid fp-cohomology of the \(p\)-rank \(\geq 1\) locus \(Y_{G, K_l}^{\geq 1}\), which only “sees” the restriction of the Eisenstein class to the ordinary locus \(Y_{H, A}^{\text{ord}} \subseteq Y_{H, A}\) (Theorem 10.4.6). This allows us to use the explicit description, due to Bannai–Kings, of the syntomic Eisenstein classes for \(\text{GL}_2\) over the ordinary locus, in terms of non-classical \(p\)-adic Eisenstein series.

(4) To actually compute the pairing of Theorem 10.4.6 and relate it to \(p\)-adic \(L\)-functions, we need an explicit description of the lifting \(\eta_{\text{NN-fp}} \to \cdots\) in terms of classes in the coherent cohomology of GL2 over the ordinary locus, in terms of non-classical \(p\)-adic Eisenstein series.

(5) We now use an identity relating Hecke operators on \(G\) and on \(H\) (Proposition 12.5.1) to simplify the cohomology pairing until we are left with only two terms. Both can be recognised as special values at \(j = 0\) of \(p\)-adic measures \(\mathcal{L}_1(j)\) and \(\mathcal{L}_2(j)\), which are very similar, but \(a\) priori not quite identical, to the \(p\)-adic \(L\)-function of [LPSZ19] – the difference lies in the choice of local data at \(p\). By a local zeta-integral computation, we show that at critical values the measure \(\mathcal{L}_1\) has the same interpolating property as the \(p\)-adic \(L\)-function, while the measure \(\mathcal{L}_2\) is identically 0. So the regulator is given by the value of \(\mathcal{L}_1\) at \(j = 0\), and this corresponds to a non-critical value of the \(p\)-adic \(L\)-function. This completes the proof of Theorem A.

Remark.

- The first glimpse of the Poznań spectral sequence is [BK10, Proposition A.16], which represents elements of the first syntomic cohomology group of a smooth pair in terms of classes in coherent cohomology.
- The Hecke operator identity of Proposition 12.5.1 is an analogue in the present setting of an identity of Hecke operators for \(\text{GL}_2 \times \text{GL}_2\) which occurs in the proofs of regulator formulae for Rankin–Selberg convolutions; see the proof of [KLZ20, Lemma 6.4.6].
- The idea of (coherent) cohomology with partial compact support was discovered independently by Pilloni [Pill20].

3.2. Strategy for Theorem B. In order to deduce Theorem B from Theorem A, we use variation in a \(p\)-adic family. We use \(p\)-adic families of “Siegel type” – one-parameter families in which we vary \((r_1, r_2)\) \(p\)-adically while keeping the difference \(r_1 - r_2\) fixed.

If we knew that the \(p\)-adic \(L\)-function of [LPSZ19] extended to Siegel-type families, and that there existed a \(p\)-adic Eichler–Shimura isomorphism for such families, interpolating the period isomorphisms
for the middle steps of the Hodge filtration at each classical specialisation (analogous to the results of Ohta \cite{Oht95} and Andreatta–Iovita–Stevens for $GL_2$ \cite{AIS15}), then Theorem B would be a virtually immediate consequence of Theorem A (we sketch the argument in Section \ref{section:main-proof}). However, these ingredients do not seem to be available yet for $GSp_4$; both statements seem to be accessible for Klingen-type families (with $r_1$ varying but $r_2$ fixed), but the case of Siegel-type families is less clear.

Instead, we use an alternative argument, relying on the existence of a $p$-adic $L$-function for functorial liftings to $GL_4$ of Siegel-type families, a refinement of the results of \cite{DJR18}. (Details of this will appear in forthcoming work.) A careful analysis of the relation between this new “Betti” $p$-adic $L$-function for the family, and the “coherent” $p$-adic $L$-function of \cite{LPSZ19} for its classical specialisations, leads to the conclusion that the image of the Euler system for $\Pi$ under the Perrin-Riou regulator must be a scalar multiple of the $p$-adic $L$-function.

What remains to be proven is that this scalar factor is not zero. We show that if the ratio of periods giving this scalar factor degenerates to 0, then this happens not only for the Euler system class over the cyclotomic extension $\mathbb{Q}(\mu_p)$, but simultaneously for the classes over $\mathbb{Q}(\mu_{Mp})$ for all auxiliary conductors $M$. This gives an Euler system satisfying a stronger-than-expected local condition at $p$, and an result due to Mazur and Rubin shows that in fact no such Euler system can exist, contradicting our assumption. This completes the proof of Theorem B.

4. Acknowledgements

We would like to express our sincere gratitude to John Coates, Henri Darmon and Gert Schneider for their interest and constant encouragement in us writing this paper. We are also very grateful to Fabrizio Andreatta, Antonio Cauchi, Veronika Ertl, Elmar Große-Klönn, Kiran Kedlaya, Kai-Wen Lan, Chris Lazda, Bernhard Le Stum, Vincent Pilloni, Joaquin Rodrigues, Christopher Skinner, Chris Williams and Kazuki Yamada for many helpful discussions.

One of the key ingredients for proving the main results of this paper is Pilloni’s Higher Hida Theory, which we learnt about during the workshop Motives, Galois Representations and Cohomology Around the Langlands Program at the IAS in November 2017. We would like to express our gratitude to the organizers for the invitation.

We discovered a new spectral sequence which is crucial for the proof of the explicit reciprocity law while attending Gregorz Banaszak’s birthday conference in Poznań in September 2018. We are very grateful to the organizers for the invitation to such an inspiring event.

Part of the work on this paper was carried out while we were visiting the Bernoulli Centre, Princeton University, the Morningside Centre and the Isaac Newton Institute. We would like to thank these institutions for their hospitality.

5. Conventions

In this paper, $p$ is a prime. As in \cite{LSZ17} \S2, $G$ denotes the symplectic group $GSp_4$, $P_S$ and $P_K$ denote its standard Siegel and Klingen parabolic subgroups, and $H$ denotes the subgroup $GL_2 \times_{GL_1} GL_2$.  

4
Step 1: The problem, and a first reduction

6. Euler systems for Siegel automorphic representations

Here we briefly recall the Galois cohomology classes constructed in [LSZ17] and formulate the problem we are trying to solve, which is to evaluate the images of these classes under the Bloch–Kato logarithm at $p$. We then explain a reduction step (the first of many), expressing these quantities as cup-products in the variant of finite-polynomial cohomology for $\mathbb{Q}_p$-varieties introduced in [NN16] and [BLZ16].

6.1. Automorphic representations.

**Definition 6.1.1.** Let $(\Pi^H, \Pi^W)$ be a pair of non-endoscopic, non-CAP automorphic representations of $G(\mathbb{A}_\mathbb{Q})$ with the same finite part $\Pi_f$, as in [LSZ17], with $\Pi^W$ globally generic, discrete series at $\infty$ of weight $(k_1, k_2) = (r_1 + 3, r_2 + 3)$ for some integers $r_1 \geq r_2 \geq 0$.

**Note 6.1.2.** Recall that $\Pi'_f$ denotes the “arithmetically normalised” twist $\Pi'_f = \Pi_f \otimes \| \cdot \|^{-(r_1+r_2)/2}$, which is definable over a number field $E$.

**6.2. Hecke parameters at $p$.** Let $p$ be a prime such that $\Pi_f$ is unramified at $p$.

**Definition 6.2.1.** We write $\alpha, \beta, \gamma, \delta$ for the Hecke parameters of $\Pi'_p$, and $P_p(X)$ for the polynomial $(1 - \alpha X) \ldots (1 - \delta X)$.

The Hecke parameters are algebraic integers in $\mathbb{F}$, and are well-defined up to the action of the Weyl group. Extending $E$ if necessary, we may assume that they lie in $E$ itself. They all have complex absolute value $p^{w/2}$, where $w := r_1 + r_2 + 3$, and they satisfy $\alpha \beta = \beta \gamma = p^w \chi_{\Pi}(p)$, where $\chi_{\Pi}(p)$ is a root of unity.

**Note 6.2.2.** The polynomial $P_p(X)$ is consistent with the notation of Theorem 10.1.3 of [LSZ17], and in particular the local $L$-factor is given by

$$L(\Pi_p, s - \frac{w}{2}) = P_p(p^{-s})^{-1} = \left[ \left( 1 - \frac{p^w}{p^s} \right) \ldots \right]^{-1}.$$  

Note, however, that the Hecke parameters here are not quite the same as the $(\alpha, \beta, \gamma, \delta)$ in Proposition 3.2, which are the Hecke parameters of a different twist of $\Pi_p$.

We shall fix an embedding $E \hookrightarrow L \subset \overline{\mathbb{Q}}_p$, where $L$ is a finite extension of $\mathbb{Q}_p$, and let $v_p$ be the valuation on $L$ such that $v_p(p) = 1$. If we order $(\alpha, \beta, \gamma, \delta)$ in such a way that $v_p(\alpha) \leq \ldots \leq v_p(\delta)$ (which is always possible using the action of the Weyl group), then we have the valuation estimates

$$v_p(\alpha) \geq 0, \quad v_p(\alpha \beta) \geq r_2 + 1.$$  

**Remark 6.2.3.** These inequalities correspond to the fact that the Newton polygon of the $p$-adic Galois representation associated to $\Pi$ lies on or above the Hodge polygon; see Proposition 6.7.1 below.

**Definition 6.2.4.** We say $\Pi$ is Siegel ordinary at $p$ if $v_p(\alpha) = 0$, and Klingen ordinary at $p$ if $v_p(\alpha \beta) = r_2 + 1$ (and Borel ordinary if it is both Siegel and Klingen ordinary).

**Lemma 6.2.5.** If $\Pi$ is Klingen-ordinary at $p$, then none of $(\alpha, \beta, \gamma, \delta)$ has the form $p^n \zeta$ with $n \in \mathbb{Z}$ and $\zeta$ a root of unity. (In other words, Assumption 11.1.1 of [LSZ17] is satisfied.)

**Proof.** Since all of the Hecke parameters are Weil numbers of weight $w$, it follows that if one of the parameters has this form, then $w$ must be even and $n = w/2$. In particular, this parameter has $p$-adic valuation $w/2$. However, if $\Pi$ is Klingen-ordinary then $\alpha, \beta$ have valuations at most $r_2 + 1 \leq \frac{w+1}{2}$, and $\gamma, \delta$ have valuations at least $r_1 + 2 \geq \frac{w+1}{2}$, so none can have valuation $(r_1 + r_2 + 3)/2$.  

□
6.3. Shimura varieties.

**Definition 6.3.1.** For $U \subset G(\mathbf{A}_f)$ a sufficiently small level, and $K$ a field of characteristic 0, let $Y_G(U)_K$ denote the base-extension to $K$ of the canonical $\mathbf{Q}$-model of the level $U$ Shimura variety for $G$. We denote by $Y_{G,K}$ the pro-variety $\varprojlim_U Y_G(U)_K$.

**Definition 6.3.2.** For each algebraic representation $V$ of $G$, let $V$ denote the $G(\mathbf{A}_f)$-equivariant relative Chow motive over $Y_{G,Q}$ associated to $V$ via Ancona’s functor, as in [LSZ17] §6.2.

Any relative Chow motive over $Y_G(U)_\mathbf{Q}$ gives rise to an object of Voevodsky’s triangulated category of geometric motives over $\mathbf{Q}$ (via pushforward along the structure map $Y_G(U)_\mathbf{Q} \to \text{Spec} \mathbf{Q}$). Hence we can make sense of motivic cohomology $H^r_{\text{mot}}(Y_G(U)_\mathbf{Q}, V)$. We use the same symbol $V$ for the $p$-adic étale realisation of this motive, which is a locally constant étale sheaf of $\mathbf{Q}_p$-vector spaces on $Y_G(U)_\mathbf{Q}$ (with a natural extension to the canonical integral model $Y_G(U)_{\mathbf{Z}[1/N]}$ if $U$ is unramified outside $N$).

**Remark 6.3.3.** (“Liebmann’s trick”). Explicitly, suppose that $V$ is a direct factor of $W^{\otimes n}(m)$, where $W$ is the defining representation of $G$. Then $H^r_{\text{mot}}(Y_G(U)_\mathbf{Q}, V)$ is a direct summand of $H^{r+n}(A^n, \mathbf{Q}(m))$, where $A$ is the universal abelian scheme over $Y_{G}$. We have $H^r_{\text{mot}}(Y_{G}, V_{\text{mot}}) = e_V \cdot H^{r+n}_{\text{mot}}(A^n, \mathbf{Q}(m))$

for some projector $e_V$.

### 6.4. Galois representations and modular parametrisations.

Taking $V = V(r_1, r_2; r_1 + r_2)$ in the notation of [LPSZ19], the $\Pi'_p$-isotypic part of $H^3_{\text{ét}, c}(Y_G(U)_{\mathbf{Q}}, V) \otimes \mathbf{Q}_p$, $L$ is isomorphic to the sum of $\dim (\Pi')$ copies of a 4-dimensional $L$-linear Galois representation $W_{\Pi}$ (uniquely determined up to isomorphism), whose semisimplification is the representation $\rho_{\Pi,p}$ associated to $\Pi$ [LSZ17 §§10.1-10.2].

This is characterised by the relation

$$\det (1 - X \rho_{\Pi,p} (\text{Frob}_{\ell}^{-1})) = P_{\ell}(X)$$

detected for good primes $\ell$, where Frobenius is an arithmetic Frobenius at $\ell$.

We shall fix a choice of representation $W_{\Pi}$ in this isomorphism class, as follows. We have assumed that $\Pi$ is globally generic. So $\Pi$ is a fortiori everywhere locally generic: that is, for any nontrivial additive character $\psi$ of $\mathbf{A}/\mathbf{Q}$, there exists a unique space of $C$-valued functions on $G(\mathbf{A}_f)$ which transforms via $\psi$ under left-translation by $N(\mathbf{A}_f)$ and which is isomorphic to $\Pi$ as a $G(\mathbf{A}_f)$-representation. We denote this space by $W(\Pi)$, and $W(\Pi)_E$ the subspace of Whittaker functions which are defined over $E$ in the sense of [LPSZ19] Definition 10.2. This gives a canonical model of $\Pi$ as an $E$-linear representation, so we can define $W(\Pi')_F$ for any extension $F$ of $E$ by base-extension.

**Definition 6.4.1.** With the above notations, we set

$$W_{\Pi} = \text{Hom}_{L[G(\mathbf{A}_f)]} \left( W(\Pi'), L, H^3_{\text{ét},c}(Y_G, \mathbf{Q}, V)_L \right).$$

This is a 4-dimensional $L$-linear representation of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ which is a canonical representative of the isomorphism class $\rho_{\Pi,p}$. We therefore obtain a canonical isomorphism

$$W(\Pi')_L = \text{Hom}_{\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})} \left( W_{\Pi}, H^3_{\text{ét},c}(Y_G, \mathbf{Q}, V)_L, \Pi' \right).$$

**Remark 6.4.2.** One can give a slightly more down-to-earth definition of $W_{\Pi}$ using the newvector theory of [RS07] and [Oka19]. This allows us to choose a level $U(\Pi)$ such that $W(\Pi')^{U(\Pi)}$ is one-dimensional and has a canonical basis vector $w^0$ normalised such that $w^0(1) = 1$. Then evaluating at $w^0$ identifies $W_{\Pi}$ with the $U(\Pi)$-invariants in $H^3_{\text{ét}, c}(Y_G, \mathbf{Q}, V)_L[\Pi']$. In particular, if $\Pi$ has level 1, then we can take $U(\Pi) = G(\mathbf{Z})$; this special case will be needed when we consider variation in $p$-adic families in §17 below.

There is a canonical duality of $G(\mathbf{A}_f) \times \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-representations

$$\langle \cdot , \cdot \rangle_G : \left( H^3_{\text{ét},c}(Y_G, \mathbf{Q}, V) \right) \times \left( H^3_{\text{ét},c}(Y_G, \mathbf{Q}, V^\vee(3)) \right) \rightarrow \mathbb{C}$$

given at level $U$ by $\text{vol}(U) \cdot \langle \cdot , \cdot \rangle_{Y_G(U)}$, where $\langle \cdot , \cdot \rangle_{Y_G(U)}$ is the Poincaré duality pairing on the cohomology of $Y_G(U)$, and “vol” denotes volume with respect to some fixed choice of Haar measure on $G(\mathbf{A}_f)$. Via this pairing, we can interpret elements of $W(\Pi')$ as homomorphisms of Galois representations $H^3_{\text{ét}}(Y_G, \mathbf{Q}, V^\vee(3)) \rightarrow W_{\Pi}^*$, i.e. as modular parametrisations of $W_{\Pi}$ in the sense of [LSZ17] §10.4.
6.5. The Lemma–Eisenstein map. We recall the following branching law for representations of $G$ restricted to $H = \text{GL}_2 \times_{\text{GL}_1} \text{GL}_2$.

**Proposition 6.5.1.** Let $q, r$ be integers with $0 \leq q \leq r_2$ and $0 \leq r \leq r_1 - r_2$. We set

$$\langle t_1, t_2 \rangle = (r_1 - q - r, r_2 - q + r),$$

so that $t_1, t_2 \geq 0$. Then we have

$$V|_H = \bigoplus_{q, r} \left( \text{Sym}^{t_1} \boxtimes \text{Sym}^{t_2} \right) \otimes \det^q,$$

**Remark 6.5.2.** As $(q, r)$ vary, the pair $(t_1, t_2)$ takes all values satisfying $r_1 - r_2 \leq t_1 + t_2 \leq r_1 + r_2$, $|t_1 - t_2| \leq r_1 - r_2$ and $t_1 + t_2 = r_1 + r_2 \mod 2$ (cf. [LPSZ19, Proposition 6.4]). The advantage of the $(q, r)$ parametrisation is that the bounds are easier to write down.

**Notation 6.5.3.** We let $S_{(0)}(A_2^2 \times A_2^2)$ denote the space of $\mathbb{Q}$-valued Schwartz functions on $A_2^2 \times A_2^2$, with $\text{GL}_2(A_2) \times \text{GL}_2(A_2)$ acting by right-translation, satisfying the following vanishing property: if $c = 0$, then $\Phi((0, 0) \times -)$ vanishes identically, and if $d = 0$, then $\Phi(- \times (0, 0))$ vanishes identically.

In [LSZ17, §8.3], we defined a map

$$\mathcal{L}_{\text{et}}^Q(a, r) : S_{(0)}(A_2^2 \times A_2^2) \otimes \mathcal{H}(G(A_1)) \to H^4_{\text{mot}}(Y_G, \mathbb{Q}, V^r(3 - q))$$

satisfying a certain equivariance property, where $\mathcal{H}(\cdot)$ denotes the Hecke algebra with $\mathbb{Q}$-coefficients. Denote by $\mathcal{L}_{\text{et}}^Q(a, r)$ the composite of this map with the étale realisation.

**Note 6.5.4.** Since the $L$-packet of $\Pi'_1$ does not contribute to cohomology in degrees $\neq 3$, we can project this into the $\Pi''_2$ isotypical component and apply the étale Abel-Jacobi map (cf. [LSZ17, §10.3]) to obtain a homomorphism

$$\mathcal{L}_{\text{et}}^Q(a, r) : S_{(0)}(A_2^2 \times A_2^2) \otimes \mathcal{H}(G(A_1)) \to H^1(Q, H^1_{\text{et}}(J_G, \mathbb{Q}, V^r(3 - q))[\Pi''_2]).$$

Choosing a vector $w \in W(\Pi''_2)$ and applying the pairing $\langle - , - \rangle_G$, we obtain a homomorphism

$$\langle w, \mathcal{L}_{\text{et}}^Q(a, r) \rangle_G : S_{(0)}(A_2^2 \times A_2^2) \otimes H(G(A_1)) \to H^1(Q, W^*_{\Pi''_2}(-q)).$$

For our purposes, it is simpler to work with the bilinear form corresponding to $\mathcal{L}_{\text{et}}^Q(a, r)$ under Frobenius reciprocity as in [LSZ17, §3.9]:

**Definition 6.5.5.** Let

$$z^{[\Pi, q, r]} : W(\Pi''_2) \times S_{(0)}(A_2^2 \times A_2^2) \to H^1(Q, W^*_{\Pi''_2}(-q)) \otimes \| \det \|^q$$

be the unique $H(A_1)$-equivariant map such that

$$\langle w, \mathcal{L}_{\text{et}}^Q(a, r) \rangle_G = z^{[\Pi, q, r]}(\xi, w, \Phi)$$

for all $W \in W(\Pi''_2)$, $\Phi \in S_{(0)}(A_2^2 \times A_2^2)$, and $\xi \in \mathcal{H}(G(A_1))$.

Unravelling the notation we have the following:

**Proposition 6.5.6.** Suppose that $w \in W(\Pi''_2)$. Then for any open compact $U \subset G(A_1)$ such that $U$ fixes $w$ and $V = U \cap H(A_1)$ fixes $\Phi$, we have

$$z^{[\Pi, q, r]}(w, \Phi) = \text{vol}(V) \cdot \langle w, \iota_v^{[t_1, t_2]} \cdot \text{Eis}_{\text{et}, \Phi}^{[t_1, t_2]} \rangle_{Y_G(U)}.$$

Here $\iota_v^{[t_1, t_2]}$ denotes pushforward along $Y_H(V) \to Y_G(U)$, and $\text{Eis}_{\text{et}, \Phi}^{[t_1, t_2]}$ denotes the étale realisation of the motivic Eisenstein class $\text{Eis}_{\text{mot}, \Phi}^{[t_1, t_2]}$ (c.f. [KLZ20, §4.1]).

**Note 6.5.7.** Note that this shows that $z^{[\Pi, q, r]}$ is independent of the choice of Haar measure on $G$ (although not on $H$).
6.6. Zeta-integrals. We shall now make the dependence on \( w \) and \( \Phi \) a little more precise, as follows. We choose a pair of \( E \)-valued finite-order characters \( \chi = (\chi_1, \chi_2) \) of \( \mathbb{A}_f^\times / \mathbb{Q}_{>0}^\times \), with \( \chi_1 \chi_2 = \chi_N \). Let \( \mathcal{S}_{(0)}(\mathbb{A}_f^2 \times \mathbb{A}_f^2 ; \chi) \) denote the \((\chi_1, \chi_2)\)-eigenspace for the action of \( \mathbb{Z}^\times \otimes \mathbb{Z}^\times \) on \( \mathcal{S}_{(0)}(\mathbb{A}_f^2 \times \mathbb{A}_f^2) \), and similarly without \((0)\).

**Note 6.6.1.** If \( \Phi \in \mathcal{S}_{(0)}(\mathbb{A}_f^2 \times \mathbb{A}_f^2 ; \chi) \), then \( z^{[\Pi,q,r]}(w, \Phi) \) vanishes for trivial reasons unless \( (-1)^{q+r} \) is equal to the common value \( (-1)^r \chi_1(1) = (-1)^2 \chi_2(1) \).

In the Appendix (see Equation (19.1)), we use the leading terms of local zeta-integrals to define, for each prime \( \ell \), a non-zero bilinear map

\[
\tilde{Z}_\ell : \mathcal{W}(\Pi^\ell_1) \otimes \mathcal{S} \left( \mathbb{Q}_f^2 \times \mathbb{Q}_f^2 ; \chi^{-1}_\ell \right) \to E,
\]

which is \( H(\mathbb{Q}_f) \)-equivariant up to a twist by \( | \det |^{-q} \). If \( \ell \) is unramified and \((w_\ell, \Phi_\ell)\) are the spherical vectors, then \( \tilde{Z}_\ell(w_\ell, \Phi_\ell) = 1 \), so we may define a global bilinear map

\[
\tilde{Z} : \mathcal{W}(\Pi^\ell_1) \otimes \mathcal{S} \left( \mathbb{A}_f^2 \times \mathbb{A}_f^2 ; \chi^{-1}_\ell \right) \to E
\]

as the tensor product of the \( \tilde{Z}_\ell \) over all (finite) primes \( \ell \).

**Theorem 6.6.2.** Suppose that the following condition is satisfied:

- If \( r_1 = r_2 = q \), then for every ramified prime \( \ell \) such that \( L(\Pi_\ell, s) \) has a pole at \( s = -\frac{1}{2} \) and \( \Pi_\ell \) is not of Sally–Tadic type IIIa or IVa, the local characters \( \chi_{1,\ell} \) and \( \chi_{2,\ell} \) are both non-trivial.

Then there exists a class

\[
z^{[\Pi,q,r]}_{\text{can}}(\chi) \in H^1(\mathbb{Q}_f, W^\star_\Pi(-q))
\]

such that for all \((w, \Phi) \in \mathcal{W}(\Pi^\ell_1) \times \mathcal{S}_{(0)}(\mathbb{A}_f^2 \times \mathbb{A}_f^2 ; \chi^{-1}) \), we have

\[
z^{[\Pi,q,r]}(w, \Phi) = z^{[\Pi,q,r]}_{\text{can}}(\chi) \cdot \tilde{Z}(w, \Phi).
\]

Note that we do not require the \( \chi_i \) to be ramified at \( \ell \), only that they are not identically \( 1 \) on \( \mathbb{Q}_f^\times \). We are most interested in the case where \((\chi_1, \chi_2) = (\chi_N, \text{id})\), and in this case we omit \( \chi \) from the notation and write simply \( z^{[\Pi,q,r]}_{\text{can}} \).

As the proof of Theorem 6.6.2 requires methods rather far removed from the main body of this paper, it will be postponed until the appendix; see Section 19 below. Note also that the proof relies on forthcoming work in preparation by Emory and Takeda if the character \( \chi_N \) is non-trivial.

**Remark 6.6.3.** The theorem shows that although the Euler system classes of \( \text{LSZ17} \) depend on choices of local data at the bad primes, these choices are essentially unimportant, since different choices of local data give the same cohomology class up to scaling factors (modulo some minor issues in the case \( r_1 = r_2 = q \)). However, the dependence on the choice of \( r \) and \( \chi \) is much less clear. We expect, of course, that the classes for different choices of these parameters are still proportional to each other, and we shall prove this in many cases later in this paper as a consequence of our global results on Selmer groups; but this proportionality seems to be a rather deep result, and does not admit a purely local, representation-theoretic proof.

6.7. Exponential maps and regulators. Recall the following result:

**Proposition 6.7.1.** The representation \( W_{\text{H}}|_{G_{\mathbb{Q}_p}} \) is crystalline. The eigenvalues of \( \varphi \) on \( D_{\text{cris}}(W_\Pi) \) are the Hecke parameters \( \{\alpha, \beta, \gamma, \delta\} \) of Section 6.3, and the jumps in its Hodge filtration are at \( \{0, r_2 + 1, r_1 + 2, r_1 + r_2 + 3\} \).

**Lemma 6.7.2.** For all \( 0 \leq q \leq r_2 \), we have the following:

1. The operators \( 1 - \varphi \) and \( 1 - p \varphi \) are bijective on \( D_{\text{cris}}(W^\star_\Pi(-q)) \).
2. The Bloch–Kato \( H^1_f, H^3_f \) and \( H^3_{\text{ss}} \) subspaces of \( H^1(\mathbb{Q}_p, W_{\text{H}}^\star(-q)) \) coincide.
3. The Bloch–Kato exponential map

\[
\exp : \frac{D_{\text{DR}}(W_{\text{H}})}{\text{Fil}^q D_{\text{DR}}(W_{\text{H}})} \to H^1_f(\mathbb{Q}_p, W_{\text{H}}^\star(-q))
\]

is an isomorphism.
Proposition 6.8.2. Let \( w \) map to a order to allow variation in \( N \cap \text{denote its unique lifting to } \mathbb{Z}[\pi] \). Letting \( \log \) denote the inverse of the Bloch–Kato exponential, we may define
\[
\log \left( z^{\Pi,q,r}_{\text{can}} \right) \in \frac{\mathbb{D}_{\text{dR}}(W^\nu_{\Pi})}{\text{Fil}^0 \mathbb{D}_{\text{dR}}(W^\nu_{\Pi})} = (\text{Fil}^1 \mathbb{D}_{\text{dR}}(W^\nu_{\Pi}))^*.
\]
Note that the target of this map is 3-dimensional (and independent of \( q \) in this range).

Assumption 6.7.3. We assume henceforth that \( \Pi \) is Klingen-ordinary at \( p \).

It follows that there is a distinguished pair of Hecke parameters \((\alpha, \beta)\) of minimal valuation, corresponding to a 2-dimensional subspace
\[
\mathcal{N} = \ker \left( 1 - \frac{q}{p} \right) \subset \mathcal{D}_{\text{cris}}(W^\nu_{\Pi}).
\]
Note 6.7.4. From weak admissibility, we see that \( \mathcal{N} \cap \text{Fil}^1 \) must have dimension exactly 1, and that it surjects onto the 1-dimensional graded piece \( \text{Fil}^1 / \text{Fil}^2 \).

Definition 6.7.5. Let \( \nu \) be a basis of the 1-dimensional \( L \)-vector space \( \frac{\mathbb{F}^1 \mathbb{D}_{\text{dR}}(W^\nu_{\Pi})}{\text{Fil}^2 \mathbb{D}_{\text{dR}}(W^\nu_{\Pi})} \), and let \( \nu_{\text{dR}} \) denote its unique lifting to \( \mathcal{N} \cap \text{Fil}^1 \).

We can now formulate the key problem treated in this paper:

**Problem:** For \((w, \Phi) \in \mathcal{W}(\Pi)L \times \mathcal{S}_0(\mathbb{A}_L^2 \times \mathbb{A}^2, \chi^{-1})L\), compute the quantity
\[
\text{Reg}_{\nu}^{\Pi,q,r}(w, \Phi) = \left< \nu_{\text{dR}}, \log \left( z^{\Pi,q,r}_{\text{can}}(w, \Phi) \right) \right>_{\mathcal{D}_{\text{cris}}(W^\nu_{\Pi})} \in L.
\]

Remark 6.6.6. If the hypotheses of Theorem 6.6.2 hold (for \( \chi = (\chi_1, 1) \)), we can formulate this independently of \((w, \Phi)\); if we define \( \text{Reg}_{\nu,\text{can}}^{\Pi,q,r} := \left< \nu_{\text{dR}}, \log \left( z^{\Pi,q,r}_{\text{can}} \right) \right>_{\mathcal{D}_{\text{cris}}(W^\nu_{\Pi})} \), then we have
\[
\text{Reg}_{\nu}^{\Pi,q,r}(w, \Phi) = \bar{Z}(w, \Phi) \cdot \text{Reg}_{\nu,\text{can}}^{\Pi,q,r} \quad \forall \ (w, \Phi),
\]
so it suffices to determine the single constant \( \text{Reg}_{\nu,\text{can}}^{\Pi,q,r} \).

6.8. Periods and \( p \)-adic \( L \)-functions. We shall relate the regulator \( \text{Reg}_{\nu}^{\Pi,q,r} \) to the \( p \)-adic \( L \)-functions of \( \text{[LPSZ19]} \); so let us briefly recall the construction of \textit{op.cit.} (and slightly refine it by paying closer attention to the periods involved).

Recall that we have chosen a basis vector \( \nu \) of the 1-dimensional \( L \)-vector space \( \frac{\mathbb{F}^1 \mathbb{D}_{\text{dR}}(W^\nu_{\Pi})}{\text{Fil}^2 \mathbb{D}_{\text{dR}}(W^\nu_{\Pi})} \). This space is canonically the base-extension to \( L \) of an \( E \)-vector space, namely \( \text{Hom}_{E[G(A_l)]}(W(\Pi)', H^2(\Pi')) \), where \( H^2(\Pi') \) denotes the unique copy of \( \Pi' \) inside a coherent \( H^2 \) of \( Y_{G,E} \), as in \( \text{[LPSZ19]} \) \( \text{§5.2} \).

Definition 6.8.1. Let \( \Omega_p(\Pi, \nu) \) be any element of \( L^\times \) such that \( \nu_{\text{alg}} := \Omega_p(\Pi, \nu)^{-1} \cdot \nu \) is \( E \)-rational.

We can, of course, choose \( \nu \) such that \( \Omega_p(\Pi, \nu) = 1 \), but it is convenient to allow more general \( \nu \) in order to allow variation in \( p \)-adic families later in this paper.

**Proposition 6.8.2.** Let \( \tau \) be the minimal \( K_\infty \)-type of the discrete series representation \( \Pi^W \). Then there is a constant \( \Omega_\infty(\Pi, \nu) \in \mathbb{C}^\times \) such that the composite map
\[
\begin{align*}
\mathbb{W}(\Pi)_C^\nu \ni w_{\text{alg}} & \rightarrow H^2(\Pi)_C \rightarrow \text{Hom}_{K_\infty}(\tau, \Pi^W) \rightarrow \text{Hom}_{K_\infty}(\mathbb{C}, \mathbb{W}(\Pi)_C)
\end{align*}
\]
maps \( w \) to \( \Omega^W(\Pi, \nu)^{-1} \cdot w \oplus w_{\infty} \cdot w_{\infty} \), where \( w_{\infty} \in \text{Hom}_{K_\infty}(\nu, \mathbb{W}(\Pi)^W) \) is the vector of standard Whittaker functions at \( \infty \) (cf. \( \text{[LPSZ19]} \) \( \text{§10.2} \)).
Here the map $H^2(\Pi) \to \text{Hom}_{K,\nu}(\tau, \Pi^W)$ is the comparison isomorphism of Harris and Su relating coherent cohomology to automorphic forms, and the final arrow is given by the global Whittaker transform $\Pi^W \to W(\Pi^W)$. Since all of these maps are $G(A)\backslash G$-equivariant bijections, and $\Pi_1$ is irreducible, it is clear that the composite is multiplication by a nonzero scalar.

**Remark 6.8.3.** The quantities $\Omega_p(\Pi, \nu)$ and $\Omega_\infty(\Pi, \nu)$ are each only determined modulo $E^\times$, but the ratio $\Omega_p(\Pi, \nu)^{-1} \otimes \Omega_\infty(\Pi, \nu) \in L \otimes_E C$ is well-defined (once $\nu$ is chosen).

**Theorem 6.8.4.** Suppose $r_2 \geq 1$. Then there exists a $p$-adic measure $L_{p, \nu}(\Pi) \in \Lambda_p(\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times)$ whose evaluation at $(a_1 + p_{1}, a_2 + p_{2})$, for $a_i$ integers with $0 \leq a_1, a_2 \leq r_1 - r_2$ and $p_i$ finite-order characters such that
\[(a_1 + a_2) p_1 (1 - p_2) = -1,
\]
satisfies
\[
\frac{\mathcal{L}_{p, \nu}(\Pi, a_1 + p_{1}, a_2 + p_2)}{\Omega_p(\Pi, \nu)} = R_p(\Pi, a_1) R_p(\Pi, a_2) \cdot \frac{\Lambda(\Pi \otimes p_1, 1 - p_1 + r_2 + a_1) \Lambda(\Pi \otimes p_2, 1 - p_2 + a_2)}{\Omega_\infty(\Pi, \nu)}.
\]

See [LPSZ19] Proposition 10.3 for further details, including the definition of the factors $R_p(\Pi, p, a)$. Here $\Lambda(\Pi \otimes p, s)$ denotes the completed $L$-function (including its Archimedean $\Gamma$-factors). Note that the interpolating property was only proved in *op.cit.* under the assumption that $a_1 \geq a_2$, but by comparing interpolating properties at points with $a_1 = a_2$, one sees easily that $L_{p, \nu}(\Pi, j_1, j_2) = L_{p, \nu}(\Pi, j_1, j_2)$, so this condition can be removed.

We can now give a precise statement of the theorem we shall prove:

**Theorem 6.8.5 (Theorem [A]).** For any $q, r$ with $0 \leq q \leq r_2$, $0 \leq r \leq r_1 - r_2$, and $q + r = r_2 \text{ mod } 2$, we have
\[
(6.1) \quad \text{Reg}^{[\nu, q, r]}_{\nu}(w, \Phi) = \frac{(-2)^q (1 - 1)^{q} - (r_2 - q)!}{\varepsilon_p^*} \cdot \mathcal{E}_{p, \nu}(\Pi, r_2 + 1 + r) \cdot \mathcal{L}_{p, \nu}(\Pi, -1 - r_2 + q, r) \cdot \tilde{Z}(w, \Phi),
\]
where $\varepsilon_p(\Pi, n) := \left(1 - \frac{\omega}{n^2} \right) \left(1 - \frac{\omega}{n^2} \right) \left(1 - \frac{\gamma}{n^2} \right)$ (which is nonzero for all $n \in \mathbb{Z}$ by Lemma [6.2.3]).

**Note 6.8.6.**
(a) The factor $\mathcal{E}_{p}(\Pi, n)$ agrees (up to a sign) with $R_p(\Pi, \text{id}, -1 - r_2 + n)$; that is, the Euler factors relating $\mathcal{L}_{p, \nu}$ to the regulator $\text{Reg}^{[\nu, q, r]}_{\nu}$ in the “geometric” range are formally the same as those relating it to complex $L$-values in the “critical” range.
(b) If the hypotheses of Theorem 6.8.2 hold (for $\kappa = (\nu, 1)$), the dependence on $(w, \Phi)$ cancels out and we can write the conclusion simply as
\[
\text{Reg}^{[\nu, q, r]}_{\nu, \text{can}} = \frac{(-2)^q (1 - 1)^{q} - (r_2 - q)!}{\varepsilon_p^*} \cdot \mathcal{L}_{p, \nu}(\Pi, -1 - r_2 + q, r).
\]
(c) If $r_1 - r_2 > 0$ or Hypothesis 10.5 of [LPSZ19] holds, then $\mathcal{L}_{p, \nu}(\Pi, j_1, j_2)$ factors as a product of a function of $j_1$ and a function of $j_2$. However, our proof of the theorem will not directly “see” this finer decomposition.

### 6.9. Test data at $p$.

**Proposition 6.9.1.** Let $w_{p, 0} \in W(\Pi^p)$ and $\Phi_{p, 0} \in S(Q_p^2 \times Q_p^2)$ be a choice of test data at $p$ such that $\tilde{Z}_p(w_{p, 0}, \Phi_{p, 0}) \neq 0$. Then Eq. (6.1) holds for all test data $(w, \Phi)$ if and only if it holds for $(w^p w_{p, 0}, \Phi^p \Phi_{p, 0})$ for all prime-to-$p$ test data $(w^p, \Phi^p)$.

**Proof.** This follows from the fact that $\Pi_1$ is unramified and tempered, so all poles of $L(\Pi_{p, s})$ have real part 0, and in particular there is no pole at $s = -1/2$; so the local analogue of Theorem 6.6.2 at $p$ holds unconditionally (see [LSZ17] §3.8). □

**Definition 6.9.2.**
- Let $\gamma_p \in G(\mathbb{Z}_p)$ be any matrix whose first column is $(1, 1, 0, 0)^T$.
- Let $w_{p, \text{KL}} \in W(\Pi^p)$ denote the normalised $U^p_{2, \text{KL}}$-eigenvector at level $\text{KL}(p)$ defined in [20.1] below.
- Let $\Phi_{\text{crit}}$ be the Schwartz function on $Q_p^2$ defined in Section 15.5 and $\Phi_{p, \text{KL}} = \Phi_{\text{crit}} \times \Phi_{\text{crit}}$. 


We refer to the pair \((\gamma_p, w_{p,Kl}, \Phi_{p,Kl})\) as Klingen test data.

We shall evaluate \(\tilde{Z}_p(\gamma_p, w_{p,Kl}, \Phi_{p,Kl})\) in Section 20.2.4 below; the result is
\[
\tilde{Z}_p(\gamma_p, w_{p,Kl}, \Phi_{p,Kl}) = \frac{\mathcal{E}(\Pi, q)\mathcal{E}(\Pi, r_2 + 1 + r)}{(1 - \frac{\gamma}{p^{1+\nu}})(1 - \frac{\delta}{p^{1+\nu}})}.
\]

In particular, it is nonzero; so it suffices to prove Theorem 6.8.5 for test data which is of Klingen type at \(p\). For data of this form, the right-hand side of (6.1) can be written as
\[
\frac{(-2)^{(1-r)^{1+\nu}(r_2 - q)}(r_2 - q)!}{(1 - \frac{\gamma}{p^{1+\nu}})(1 - \frac{\delta}{p^{1+\nu}})} \tilde{Z}_p(w_p, \Phi) \cdot \mathcal{L}_{p,v}(\Pi, -1 - r_2 + q, r).
\]

The left-hand side of Eq. (6.1) can be written explicitly using Proposition 6.5.6. Let us choose an open compact \(U_p\) such \(U_p\) fixes \(w_p\) and \(U_p \cap H(A) = \mathcal{V}_p\) fixes \(\Phi_p\).

**Notation 6.9.3.** Write \(Y_{G,Kl,Q}\) for the \(G\)-Shimura variety of level \(U_p\) \(Kl(p)\), and \(Y_{H,\Delta, Q}\) for the \(H\)-Shimura variety of level \(U_p\) \(Kl(\Delta)\), where
\[
K_{p,\Delta} = \left\{ h \in H(Z_p) : h = \begin{pmatrix} x & \ast \\ \ast & \ast \end{pmatrix} \quad (\text{mod } p) \text{ for some } x \right\}.
\]
Write \(X_{G,Kl,Q}\) and \(X_{H,\Delta, Q}\) for toroidal compactifications, where the rational polyhedral cone decompositions are chosen as in [LPSZ19, §2.2.4].

**Remark 6.9.4.** We will define integral models over \(Z_p\) of these Shimura in Section 10.

We have \(\gamma_p^{-1}K_p,\Delta \gamma_p \subset Kl(p)\), so as in [LPSZ19, §4.1], \(\gamma_p\) gives a finite morphism of compactified Shimura varieties \(i_{\Delta} : X_{H,\Delta, Q} \to X_{G,Kl,Q}\), which restricts to a closed embedding \(Y_{H,\Delta, Q} \to Y_{G,Kl,Q}\) of the uncompactified Shimura varieties. Hence there is a pushforward map \(i_{\Delta,*}^{[1,1]}\) on étale cohomology; and we have
\[
\langle \log z^{[1,1]} (w_p, \Phi), \nu_{\text{dR}} \rangle_{U_p,\text{dR}2} = \text{vol} (V) \left\langle \left( \log \circ \text{pr} \Psi^\vee \circ i_{\Delta,*}^{[1,1]} \right)(\text{Eis}_{\text{et}, \Phi}), \eta \right\rangle_{Y_{G,U_p}},
\]
where \(\eta\) denotes the \(U_p\)-eigenvector \(\nu(w_p, Kl(p))\), and \(\eta_{\text{dR}} = \nu_{\text{dR}}(w_p, Kl(p))\) its lifting to \(\text{Fil}^1 \cap \ker[(1 - \varphi/\alpha)(1 - \varphi/\beta)]\) as in Definition 6.7.5.

We now derive a corresponding formula for the -adic \(L\)-function. Given \(\Phi_p\), the construction of [LPSZ19, §7.4] gives a 2-parameter \(-\)adic family of Eisenstein series on \(H\), which we denote simply by \(\mathcal{E}(\Phi)\). Then the \(-\)adic interpolation theory of \text{op.cit.} allows us to make sense of \(i_{\Delta,*}(\mathcal{E}(\Phi))\) as a measure taking values in \(H^1\) of the \(-\)rank \(\geq 1\) locus, and hence to define a measure
\[
\langle \eta, i_{\Delta,*}(\mathcal{E}(\Phi)) \rangle_{X_{G,Kl}^2} \in \Lambda_L(Z_p^+ \times Z_p^+).
\]
This cup product depends on the choice of the level group \(U_p\), but this can be eliminated by renormalising by vol\(V\). From the construction of the \(-\)adic \(L\)-function, we have the following explicit formula:

**Proposition 6.9.5.** For \((q, r)\) as above, the value of the measure
\[
\text{vol}(V) \cdot \left\langle i_{\Delta,*}(\mathcal{E}(\Phi)), \eta \right\rangle_{X_{G,Kl}^2}
\]
at \((-1 - r_2 + q, r)\) is \(\mathcal{L}_{p,v}(\Pi, -1 - r_2 + q, r) \cdot \tilde{Z}_p(w_p, \Phi).
\]

Summarising the above discussion, we have the following:

**Proposition 6.9.6.** The formula of (6.1) is equivalent to the following assertion:

For all prime-to-\(p\) levels \(U_p\), all \(\Phi_p\) stable under \(U_p \cap H\), and all \(\eta \in H^2(\Pi_1 U_p Kl(p)[U_2,Kl(p) = q^2 p^{1+\nu}]\), we have
\[
\left\langle \log \circ \text{pr} \Psi^\vee \circ i_{\Delta,*}^{[1,1]}(\text{Eis}_{\text{et}, \Phi}), \eta \right\rangle_{Y_{G,Kl}} = \frac{(-2)^{y}(1-r)^{1+\nu}(r_2 - q)!}{(1 - \frac{1}{p^{1+\nu}})(1 - \frac{\delta}{p^{1+\nu}})} \cdot \left\langle i_{\Delta,*}(\mathcal{E}(\Phi)), \eta \right\rangle_{X_{G,Kl}^2}.
\]

It is this formula we shall actually prove.
7. Finite-polynomial cohomology and Abel–Jacobi maps

We briefly recall some geometric formalism from [NN16] and [BLZ16], which we shall use to give formulae for the Abel–Jacobi map of étale cohomology. In this section we shall only consider varieties over \( \mathbb{Q}_p \); integral models (over \( \mathbb{Z}_p \)) will enter the picture later, when we start to make computations.

7.1. P-adic Hodge theory. We recall some constructions from p-adic Hodge theory and Galois cohomology; see [NN16 §2D] and [BLZ16 §1] for further details.

7.1.1. Filtered modules. Let \( \mathbb{Q}_p \) denote the maximal unramified extension of \( \mathbb{Q}_p \).

Definition 7.1.1. A filtered \((\varphi, N, G_{\mathbb{Q}_p})\)-module is a finite-dimensional \( \mathbb{Q}_p \)-vector space \( D \) equipped with the following structures:

- an \( \mathbb{Q}_p \)-semilinear Frobenius \( \varphi \);
- an \( \mathbb{Q}_p \)-linear monodromy operator \( N \) satisfying \( N \varphi = p \varphi N \);
- an \( \mathbb{Q}_p \)-semilinear action of \( G_{\mathbb{Q}_p} \) commuting with \( \varphi \) and \( N \), such that every \( v \in D \) is fixed by some open subgroup;
- a decreasing \( \mathbb{Q}_p \)-linear filtration \( \text{Fil}^* \) on \( D \).

We write \( D_{\text{st}} := D^{G_{\mathbb{Q}_p}} \) and \( D_{\text{cris}} := D^{1(G_{\mathbb{Q}_p}, N = 0)} \).

Fontaine’s functor \( D_{\text{pst}} \) gives an equivalence of categories between potentially semistable \( p \)-adic representations of \( G_{\mathbb{Q}_p} \) and the subcategory of \textit{weakly admissible} filtered \((\varphi, N, G_{\mathbb{Q}_p})\)-modules. If \( D = D_{\text{pst}}(V) \), then we have \( D_{\text{st}} = D_{\text{st}}(V) \), \( D_{\text{cris}} = D_{\text{cris}}(V) \), and \( D_{\text{dr}} = D_{\text{dr}}(V) \) (hence the notation).

Notation 7.1.2. For \( n \in \mathbb{Z} \), let \( \mathbb{Q}_p^{ur}(n) \) denote the filtered \((\varphi, N, G_{\mathbb{Q}_p})\)-module whose underlying vector space is \( \mathbb{Q}_p^{ur} \), with \( N = 0 \) and the \( G_{\mathbb{Q}_p} \)-action being the obvious one, but taking \( \varphi = 1 \) where \( \sigma \) is the native arithmetic Frobenius of \( \mathbb{Q}_p^{ur} \), and the filtration concentrated in degree \( -n \).

Clearly we have \( \mathbb{Q}_p^{ur}(n) = D_{\text{pst}}(\mathbb{Q}_p(n)) \), by identifying \( 1 \in \mathbb{Q}_p^{ur} \) with the basis vector \( t^{-n} \otimes e_n \in B_{\text{cris}} \otimes \mathbb{Q}_p(n) \).

7.1.2. The semistable \( P \)-complex.

Definition 7.1.3. Let \( P \in \mathbb{Q}_p[[t]] \) be a polynomial with constant term 1, and \( D \) a filtered \((\varphi, N, G_{\mathbb{Q}_p})\)-module. Define \( H^i_{\text{st}, P}(Q_p, D) \) to be the \( i \)-th cohomology group of the complex

\[
C_{\text{st}, P}(D) := \left[ D_{\text{st}} \xrightarrow{D_{\text{st}} \oplus D_{\text{st}} \oplus \text{Fil}^0} D_{\text{st}} \right],
\]

where the maps are given by

\[
x \mapsto (P(\varphi)x, N x, x \mod \text{Fil}^0) \quad \text{and} \quad (u, v, w) \mapsto Nu - P(p \varphi)v.
\]

If \( P(t) = 1 - t \), then we omit it and write simply \( H^i_{\text{st}}(Q_p, D) \) etc.

Note 7.1.4. More generally, it will sometimes be convenient to extend the definitions to the case when \( P \) is a polynomial in \( R[[t]] \), where \( R \) is a commutative \( \mathbb{Q}_p \)-subalgebra of the \( G_{\mathbb{Q}_p} \)-endomorphism algebra of \( V \).

If \( P \mid Q \) then we have a natural map of complexes \( C_{\text{st}, P}(D) \to C_{\text{st}, Q}(D) \) which is the identity in degree 0. There are also products

\[
C^i_{\text{st}, P}(Q_p, D) \otimes C^j_{\text{st}, Q}(Q_p, E) \to C^{i+j}_{\text{st}, P \circ Q}(Q_p, D \otimes E),
\]

well-defined up to homotopy, where \( P \circ Q \) is the convolution product (the polynomial whose roots are the pairwise products of those of \( P \) and \( Q \)).

7.1.3. Galois cohomology. If \( V \) is a potentially semistable \( G_{\mathbb{Q}_p} \)-representation, then \( C_{\text{st}}(D_{\text{pst}}(V)) \) is the \( G_{\mathbb{Q}_p} \)-invariants of a complex of \( G_{\mathbb{Q}_p} \)-modules that is quasi-isomorphic to \( V \). This gives rise to boundary maps

\[
H^i_{\text{st}}(Q_p, D_{\text{pst}}(V)) \to H^i(Q_p, V),
\]

which are isomorphisms for \( i = 0 \) and injective for \( i = 1 \).

Definition 7.1.5. The semistable Bloch–Kato exponential is the map \( \text{exp}_{\text{st}, V} : H^1_{\text{st}}(Q_p, D_{\text{pst}}(V)) \to H^1(Q_p, V) \) given by (7.1) for \( i = 1 \). Its image is the Bloch–Kato subspace \( H^1_{g}(Q_p, V) \).
This terminology is justified by the fact that the composition
\[
\frac{D_{\text{dR}}(V)}{\text{Fil}^0 D_{\text{dR}}(V)} \longrightarrow H^1_{\text{st}}(Q_p, D_{\text{pst}}(V)) \stackrel{\exp_{\text{st}, V}}{\to} H^1_{\text{st}}(Q_p, V)
\]
is the usual Bloch–Kato exponential map $\exp_V$, with image $H^1_{\text{et}}(Q_p, V) \subseteq H^1_{\text{st}}(Q_p, V)$.

**Notation 7.1.6.** We say a filtered $(\varphi, N, G_{Q_p})$-module $D$ is convenient if it is crystalline (i.e. $D_{\text{cris}} = D_{\text{dR}}$) and $1 - \varphi$ and $1 - p\varphi$ are bijective on $D_{\text{cris}}$. We say a crystalline $G_{Q_p}$-representation $V$ is convenient if $D = D_{\text{pst}}(V)$ is convenient.

**Note 7.1.7.** A filtered $(\varphi, N, G_{Q_p})$-module $D$ is convenient if and only if so is $D^*(1)$.

If $D$ is convenient, then $H^i_{\text{st}}(Q_p, D) = 0$ for $i \neq 1$, and the natural map $D_{\text{dR}}/\text{Fil}^0 D_{\text{dR}} \to H^1_{\text{st}}(Q_p, D)$ is an isomorphism. In particular, for a convenient Galois representation $V$ we have $H^i_{\text{et}}(Q_p, V) = H^1_{\text{et}}(Q_p, V)$ and $\exp_{\text{st}, V}$ is identified with $\exp_V$.

7.1.4. Traces and duality.

**Lemma 7.1.8.** If $P(1/p) \neq 0$ then there is a canonical map
\[
\text{tr}_{st, P} : H^1_{\text{st}, P}(Q_p, Q_p^{nr}(1)) \to Q_p
\]
given by mapping $(x, y, z) \in Z^1(C_{st, P}(Q_p^{nr}(1)))$ to $z - P(z/P)^{-1}x$. If $P(1) \neq 0$ this map is an isomorphism.

If $P$ and $Q$ are two polynomials with $Q(1/p) \neq 0$, then the trace maps for $P$ and $Q$ are compatible with the change-of-polynomial maps.

**Proof.** Immediate from the definitions. \hfill \Box

**Corollary 7.1.9.** Suppose $D$ is convenient. Then, for any $P$ satisfying the conditions of the lemma, the pairing given by
\[
H^0_{\text{st}, P}(Q_p, D) \times H^1_{\text{st}}(Q_p, D^*(1)) \to H^1_{\text{st}, P}(Q_p, Q_p^{nr}(1)) \stackrel{\text{tr}_{st, P}}{\longrightarrow} Q_p
\]
is the restriction to $H^0_{\text{st}, P}(Q_p, D)$ of the natural duality pairing $\langle \text{Fil}^0 D_{\text{dR}} \rangle \times \langle \text{Fil}^1 D_{\text{dR}}(1) \rangle \to Q_p$.

7.2. Nekovář–Nizioł cohomology. Let $X$ be any $Q_p$-variety, and let $n \in \mathbb{Z}$. Then Nekovář–Nizioł [NN16] define $R^i_{\text{NN-syn}, c}(X, n)$ and $R^i_{\text{NN-syn}, c}(X, n)$. This cohomology theory is a Bloch–Ogus theory (Appendix B in op. cit.), so it has all of the good functorial properties one expects, such as cup-products, pullbacks, pushforward maps, etc.

More generally, we can define groups $R^i_{\text{NN-fp}, c}(X, n, P)$ and $R^i_{\text{NN-fp}, c}(X, n, P)$ for any polynomial $P$ as above, with the case $P(t) = 1 - t$ recovering the theory of [NN16]; see [BLZ16] for this generalisation.

By construction, these cohomology theories satisfy the following descent spectral sequence:

**Proposition 7.2.1.** There exists a spectral sequence
\[
\begin{equation}
\label{eq:nn-E2}
\text{NN}E_2^{ij} = H^i_{\text{st}, P}(Q_p, D_{\text{st}}(X_{Q_p^{nr}}(n))) \Rightarrow H^{i+j}_{\text{NN-fp}, c}(X, n, P),
\end{equation}
\]
compatible with cup-products and change-of-polynomial maps (and similarly for the compactly-supported variant).

If $X$ is smooth of pure dimension $d$, then the étale cohomology of $X_{Q_p^{nr}}$ vanishes in degrees $> 2d$, and there is a $G_{Q_p}$-equivariant trace map $H^{2d}_{\text{et}, c}(X_{Q_p^{nr}}, Q_p(d + 1)) \to Q_p(1)$; so the edge map of this spectral sequence, combined with Lemma [7.1.8], gives a canonical trace map
\[
\text{tr}_{\text{NN-fp}, c, P} : H^{d+1}_{\text{NN-fp}, c}(X, d + 1, P) \to H^1_{\text{et}, P}(Q_p, Q_p^{nr}(1)) \to Q_p,
\]
for any polynomial $P$ such that $P(1/p) \neq 0$.

**Theorem 7.2.2.** For all $r \geq 0$, there is a natural map
\[
\text{comp} : H^r_{\text{NN-syn}, c}(X, n) \to H^r_{\text{et}}(X, Q_p(n))
\]
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which is functorial in $X$ and fits into the commutative diagram

\[
\begin{array}{ccc}
H^i_{\Mot}(X, n) & \xrightarrow{\text{res}} & H^i_{\et}(X, Q_p(n)) \\
\downarrow \text{comp} & & \downarrow \text{comp} \\
H^i_{\NN}(X, n) & \xrightarrow{\text{syn}} & H^i_{\et}(X, Q_p(n));
\end{array}
\]

and there is a morphism of spectral sequences $E^{i,j}_{\NN} \to E^{i,j}_{\et}$, compatible with comp on the abutment, which is given on the $E_2$ page by the maps $[\cdot, 1]$. Here, $E^{i,j}_{\et}$ denotes the Hochschild–Serre spectral sequence.

Let $X$ be a smooth equidimensional $Q_p$-variety of dimension $d$, as before. Recall the following definition:

**Definition 7.3.1.** A class in $H^i_{\Mot}(X, n)$ is said to be homologically trivial if it is in the kernel of the edge map

\[H^i_{\Mot}(X, n) \to H^0(Q_p, H^{i-1}_{\et}(X_{\Qp}, Q_p(n)))\]

induced by the Hochschild–Serre spectral sequence. We denote this kernel by $H^{i}_{\Mot}(X, n)_0$.

Since $G_{\Qp}$ has cohomological dimension 2, the spectral sequence gives a natural map, the étale Abel–Jacobi map,

\[\AJ_{\et}: H^i_{\Mot}(X, n)_0 \to H^i(Q_p, H^{i-1}_{\et}(X_{\Qp}, Q_p(n))).\]

**Note 7.3.2.** From Theorem 3.2.2, we have

\[\AJ_{\et} = \exp_{\et} \circ \AJ_{\syn},\]

where

\[\AJ_{\syn}: H^i_{\Mot}(X, n)_0 \to H^i_{\et}(Q_p, D_{\pst}(H^{i-1}_{\et}(X_{\Qp}, Q_p(n))))\]

is the map given by the spectral sequence $E^{i,j}_{\NN}$.

In particular, the map $\AJ_{\et}$ takes values in $H^i_{\et}(Q_p, H^{i-1}_{\et}(X_{\Qp}, Q_p(n)))$ (c.f. [NN16, Theorem B]).

Let $W$ be a subspace of $H^{2d+1-i}_{\et}(X_{\Qp}, Q_p(n))^* = H^{2d+1-i}_{\et}(X_{\Qp}, Q_p(d + 1 - n))$ of $G_{\Qp}$-representation, and suppose $W$ is convenient. Then $W^* (1)$ is naturally a quotient of $H^{2d+1-i}_{\et}(X_{\Qp}, Q_p(n))$. Moreover, the natural map $\frac{D_{\mot}(W^*(1))}{D_{\mot}(W^*(1))} \to H^{1}_{\et}(Q_p, W^*(1))$ is an isomorphism; we write $\log_{W^*(1)}(\AJ_{\et})$ for its inverse.

**Notation 7.3.3.** Write $\AJ_{W^*(1)}$ for the morphism

\[\AJ_{W^*(1)} := \log_{W^*(1)}(\AJ_{\et}) \circ \AJ_{\et}: H^i_{\Mot}(X, n)_0 \to \frac{D_{\mot}(W^*(1))}{D_{\mot}(W^*(1))} \cong Q_p\]

The canonical pairing

\[\langle \cdot, \cdot \rangle_{\et, W}: D_{\et}(W) \times D_{\et}(W^*(1)) \to D_{\et}(Q_p(1)) \cong Q_p\]

identifies the target of $\AJ_{W^*(1)}$ with the dual of $\Fil^0 D_{\et}(W)$.

**Proposition 7.3.4.** Let $\eta \in \Fil^0 D_{\cris}(W)$, and let $P$ be a polynomial with constant term 1 such that $P(\varphi)(\eta) = 0$ and $P(1/p) \neq 0$.

Then, for any $x \in H^i_{\Mot}(X, Q_p(n))$, we have

\[\langle \AJ_{W^*(1)}(x), \eta \rangle_{\et, W} = \langle \resyn(x), \tilde{\eta} \rangle_{\NN, \et, X, P}\]

where $\tilde{\eta}$ is any class in $H^{2d+1-i}_{\NN}(X, d + 1 - n, P)$ whose image in $H^0_{\et, P}(Q_p, H^{2d+1-i}_{\et}(X_{\Qp}, Q_p(d + 1 - n)))$ is $\eta$. 

\[\square\]
Proof. Since the syntomic descent spectral sequence is compatible with products, we have
\[ \langle \tilde{t}_{\text{syn}}(y), \tilde{y} \rangle_{\text{NN},p} = \text{tr}_{\text{st},p} \left( A \tilde{t}_{\text{syn}}(x) \cup y \right). \]
Since \( \eta \in H^0_{\text{st},p}(\mathbb{Q}_p, W) \), this pairing factors through the projection of \( A \tilde{t}_{\text{syn}}(x) \) to \( H^1_{\text{st}}(\mathbb{Q}_p, \mathcal{D}_{\text{pst}}(W^+(1))) \), which is by construction \( A \tilde{t}_{\text{syn}}(x) \). By Corollary 7.19, the pairing between \( H^1_{\text{st}}(\mathbb{Q}_p, \mathcal{D}_{\text{pst}}(W^+(1))) \) and \( H^0_{\text{st},p}(\mathbb{Q}_p, \mathcal{D}_{\text{pst}}(W)) \) is simply the de Rham duality pairing. \( \square \)

7.4. Pushforward and pullback. We can now use the functorial properties of \( \text{NN},p \) cohomology to compute the right-hand side. More precisely, let \( \iota : Z \hookrightarrow X \) be a finite morphism of smooth \( K \)-varieties, of codimension \( c \). Then there are pushforward maps (c.f. [Deg08])
\[ H^{i-2c}_{\text{mot}}(Z, r - c) \rightarrow H^i_{\text{mot}}(X, r) \]
and similarly for \( H^{i}_{\text{NN},p} \) and \( H^i_{\text{NN},c} \); and these are compatible with the maps \( r_{\text{syn}}, r_{\text{st}}, \) and \( r \), and comp appearing in the diagram of Theorem 7.2.2.

Proposition 7.4.1. For \( z \in H^{i-2c}_{\text{mot}}(Z, r - c) \), we have
\[ \text{tr}_{\text{NN},p,X}(\iota_*(z) \cup \tilde{w}) = \text{tr}_{\text{NN},p,Z}(z \cup \iota^*(\tilde{w})). \]
Proof. This follows from the adjunction formula relating pushforward and pullback. \( \square \)

7.5. Coefficients. If \( X = Y_G(U) \) as in Section 6.3 then we can use Liebermann’s trick 6.3.3 to define cohomology with coefficients in algebraic representations \( V \) and to obtain versions of the spectral sequence 7.2 and of Theorem 7.2.2 with coefficients. In particular, the composition of the cup product and 7.3 defines a pairing
\[ \langle \cdot, \cdot \rangle_{\text{NN},p,X,p} : H^n_{\text{NN},p}(X,V,r) \times H^{n+1}_{\text{NN},p,c}(X,V^\vee,d + 1 - r,P) \rightarrow \mathbb{Q}_p \]
for any \( P \) with \( P(1/p) \neq 0 \).

For cohomology with coefficients, the formalism of pushforward and pullback maps works as follows: suppose that we have a closed immersion of PEL Shimura varieties \( i : Y_H(U') \hookrightarrow Y_G(U) \) of codimension \( d \), for some reductive group \( H \) and \( U' = U \cap H(A_f) \). Assume that the closed immersion extends to the toroidal compactifications. Let \( W \) be a direct summand of \( V_H \). Then we obtain
\begin{align*}
(7.4) & \quad (i_U^W)_* : H^n_{\text{NN}}(Y_H(U'), W, r) \rightarrow H^{n+2d}_{\text{NN}}(Y_G(U), V, r + c), \\
(7.5) & \quad (i_U^W)^* : H^n_{\text{NN},c}(Y_G(U), V, r) \rightarrow H^n_{\text{NN},c}(Y_H(U'), W, r)
\end{align*}
for all \( r \in \mathbb{Z} \).

Moreover, these maps are adjoint with respect to these pairings: suppose that \( W \) is a direct summand of \( V_H \), so \( W' \) is a direct summand of \( (V^\vee)_H \). Then for all \( x \in H^i_{\text{NN},p}(Y_H(U'), W, r - c) \) and \( y \in H^{i+1}_{\text{NN},p,c}(X,V^\vee,d + 1 - r,P) \), we have
\[ \langle \langle i_U^W \rangle_*(x), y \rangle_{\text{NN},p,Y_U(U'), P} = \langle x, \langle i_U^W \rangle^*(y) \rangle_{\text{NN},p,Y_H(U'), P}. \]

7.6. Reduction: Step 1. Write \( D \) for the boundary divisor \( X_{Kl,q} - Y_{Kl,q} \).

Definition 7.6.1. Denote by \( \mathcal{H}_Q \) the \( GL_2(A_f) \)-equivariant relative Chow motive over \( Y_H \) associated to the standard representation of \( GL_2 \).

Lemma 7.6.2. Let \( t_1, t_2 \) be as in Proposition 6.5.1 and let \( d \geq 0 \). Then we have pushforward and pullback maps
\begin{align*}
(7.6) & \quad (i_\Delta^{t_1,t_2})_* : H^n_{\text{NN},p}(Y_{H,\Delta}, \text{Sym}^{t_1} \mathcal{H}^\vee \boxtimes \text{Sym}^{t_2} \mathcal{H}^\vee, d) \rightarrow H^{n+2}_{\text{NN},c}(Y_{G,Kl}, V^\vee, d + 1 - q), \\
(7.7) & \quad (i_\Delta^{t_1,t_2})^* : H^n_{\text{NN},c}(Y_{G,Kl}, V, r + q) \rightarrow H^n_{\text{NN},c}(Y_{H,\Delta}, \text{Sym}^{t_1} \mathcal{H} \boxtimes \text{Sym}^{t_2} \mathcal{H}, d).
\end{align*}
Proof. This is an instance of (7.4) and (7.5). \( \square \)

Notation 7.6.3. Write \( \text{Eis}^{t_1,t_2}_{\text{mot}, \Phi} \) for the image of \( \text{Eis}^{t_1,t_2}_{\text{mot}, \Phi} \) under \( r_{\text{NN},p} \).

As in Section 6.3 let \( (w, \Phi) \) be the product of some arbitrary test data \( (w^p, \Phi^p) \) away from \( p \) and the Klingen test data at \( p \). Shrinking \( K^p \) if necessary, we may assume that \( K^p \) fixes \( w^p \), and \( K_H^p \) fixes \( \Phi^p \).

Notation 7.6.4. Write \( \eta_{\text{IR}} \) for \( \eta^{ur} \in \text{Fil}^1 H^3_{\text{dR}}(X_{G,Kl}(-D), V)[\Pi]. \)
Note 7.6.5. Observe that we can consider $\eta_{dR}$ as an element $\eta_{dR,q} \in \text{Fil}^{1+q} H^3_{\text{dR}}(\ldots)$, for any $0 \leq q \leq r_2$.

Lemma 7.6.6. Let $0 \leq n \leq r_2$. Then there exists a unique lift $\eta_{\text{NN-fp},q,-D}$ of $\eta_{dR,q}$ to the group $H^3_{\text{NN-fp}}(X_G,Kl,-D),1+q;P_{1+r_2})[\Pi_1]$. Here, $P_1+q(T) = \left(1 - \frac{q^{1+q}T}{\alpha}\right)\left(1 - \frac{q^{1+q}T}{\beta}\right)$.

Proof. By definition, we have $P(\varphi) \eta_{dR} = 0$, so it follows from the definition of fp-cohomology that it will lift to a class $\eta_{\text{NN-fp},-D} \in H^3_{\text{NN-fp}}(X_G,Kl,-D),1+q;P)[\Pi_1]$. The lift is unique since $\Pi_1$ does not contribute to cohomology in degrees $\neq 3$.

Notation 7.6.7. Write $\eta_{dR,q,-D}$ for the image of $\eta_{\text{NN-fp},q,-D}$ in $H^3_{\text{dR}}(X_G,Kl,-D),V,1+q;P)[\Pi]$, so $\eta_{dR,q,-D}$ maps to $\eta_{dR,q}$ under the extension-by-0.

We can now make the first reduction of (6.2):

1st reduction: We have

$$\langle \log \circ pr_{\text{pr},\text{syn}} \circ \Delta_* \eta_{dR} \rangle_{Y_G,Kl} = \langle \Delta_* \eta_{\text{NN-fp},q,-D} \rangle_{\text{NN-fp},Y_G,Kl}.$$ 

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Step 2: Reduction to a pairing on $Y^\ord_{H,\Delta}$

8. Cohomology with partial support

We recall some basic formalism regarding cohomology of sheaves on rigid spaces, following [LS07] and [GK00], and define variants with “partial compact support”. Let $K$ be a finite extension of $\mathbb{Q}_p$, with residue field $k$ and ring of integers $\mathcal{O}$.

8.1. Frames and tubes. Recall that a frame denotes the data of a triple $(X \hookrightarrow Y \hookrightarrow \mathfrak{P})$, where $X$ and $Y$ are $k$-varieties, $\mathfrak{P}$ is a formal $\mathcal{O}$-scheme, $X \hookrightarrow Y$ is an open immersion, and $Y \hookrightarrow \mathfrak{P}$ is a closed immersion of $Y$ into $\mathfrak{P}$, necessarily factoring through the special fibre $\mathfrak{P}_0$ ([LS07] Definition 3.1.5).

Note 8.1.1. We shall always assume $\mathfrak{P}$ is an admissible formal scheme, and thus in particular quasi-compact (this is automatically satisfied if $\mathfrak{P}$ is the $p$-adic completion of a finite-type flat $\mathcal{O}$-scheme).

Definition 8.1.2. The frame $(X \hookrightarrow Y \hookrightarrow \mathfrak{P})$ is said to be smooth if $\mathfrak{P}$ is smooth over $\mathcal{O}_K$ in a neighbourhood of $X$ (Definition 3.3.5 of op.cit.); it is said to be proper if $Y$ is proper over $k$ (Definition 3.3.10).

The theory is typically only well-behaved for smooth proper frames; note that this does not imply that $X$ itself is either smooth or proper.

If $(X \hookrightarrow Y \hookrightarrow \mathfrak{P})$ is a frame, then the tube $][X]$ is an open rigid-analytic subvariety of the analytic generic fibre $\mathfrak{P}_K$. We shall henceforth omit the subscript $\mathfrak{P}$ if it is clear from context. If $X$ is affine and open in $\mathfrak{P}_0$, then $][X]$ is affinoid; it follows that if $X$ is an open subvariety of $\mathfrak{P}_0$, then $][X]$ is quasi-compact.

If $X$ is not assumed to be open in $\mathfrak{P}_0$, then $][X]$ is no longer quasi-compact. However, it can be written as an increasing union of quasi-compacta, the closed tubes $][X][\lambda$ of radius $\lambda < 1$ (which are well-defined if $\lambda > |\pi_K|$).

8.2. Sections with support. Let $V$ be any rigid analytic space over $K$, and $T$ an admissible open subset of $V$. Then we have a short exact sequence of exact functors on the category of abelian sheaves on $V$,

$$0 \to \Gamma_T^1 \to \text{id} \to j^*_{V-T} \to 0,$$

and a left-exact sequence of left-exact functors

$$0 \to \Gamma_{V-T} \to \text{id} \to h_*h^{-1},$$

where $h$ is the inclusion $T \hookrightarrow V$. The second sequence is also exact on the right on injective sheaves, and thus gives an exact triangle of right-derived functors.

Recall that if $X \hookrightarrow Y \hookrightarrow \mathfrak{P}$ is a proper smooth frame, then the rigid cohomology of $X$ (with and without compact supports) is defined by

$$\mathrm{RIG}_{\text{rig}}(X) := \mathrm{R}^i \left( [Y], j^*_{\mathfrak{P}|Y}, \Omega^*_{\mathfrak{P}|Y} \right), \quad \mathrm{RIG}_{\text{rig},c}(X) := \mathrm{R}^i \left( [Y], R^\bullet_{\mathfrak{P}|Y}, \Omega^*_{\mathfrak{P}|Y} \right).$$

See e.g. [Ber97] or [LS07] Chapter 5 for further details.

Proposition 8.2.1. We have $\Gamma_{V-T} \circ j^*_{V-T} = j^*_{V-T}$ and $j^*_{V-T} \circ \Gamma_{V-T} = \Gamma_{V-T}$.

Proof. By definition of $j^*_{V-T}$, we have $h^{-1} \circ j^*_{V-T} = 0$ and hence $h_*h^{-1} \circ j^*_{V-T} = 0$. Similarly, $h^{-1} \circ \Gamma_{V-T} = 0$ and hence $\Gamma_T^1 \circ \Gamma_{V-T} = h_*h^{-1} \circ \Gamma_{V-T} = 0$. The results now follow from the above exact sequences. \qed
The functors $\Gamma$ and $\Gamma^\dagger$ are related by the following formula. Let us say that $T' \subseteq T$ is an interior subset if $\{T, V - T'\}$ is an admissible covering of $V$ (i.e. $V - T'$ is a strict neighbourhood of $V - T$). Then we have

$$\Gamma^\dagger_T(F) = \lim_{T' \subseteq T} \Gamma_T(F),$$

where the limit is over interior subsets $T' \subseteq T$. In particular, if $T$ and $U$ are both admissible open, then there is a natural inclusion $\Gamma^\dagger_T(F) \subseteq \Gamma_T(F)$ as subsheaves of $F$, but this is not an equality (except in the trivial case when $\{T, V - T\}$ is an admissible covering that disconnects $V$). It seems reasonable to describe $\Gamma^\dagger_T(F)$ as the sections strictly supported in $T$.

It is important to note that if $Z \hookrightarrow \mathfrak{P}$ is a formal embedding with $\mathfrak{P}$ proper, and we take $V = \mathfrak{P}_K$ and $T = |Z|$, then the closed tubes $|Z|_\lambda$ of radius $\lambda < 1$ are cofinal among interior subsets of $T$, and also among quasi-compact subsets of $T$. So $\Gamma^\dagger_{|Z|}(F)$ is precisely the sections of $F$ supported in a quasicompact subset of $T$.

### 8.3. Partial compact supports
We shall consider the following setting. We suppose we are given a formal embedding $Y \hookrightarrow \mathfrak{P}$, with $Y$ and $\mathfrak{P}$ proper, and an open subvariety $U \subseteq Y$ with complement $Z = Y - U$, so that $U \hookrightarrow Y \hookrightarrow \mathfrak{P}$ is a proper frame. Let $V \subseteq Z$ be a closed subvariety, and set $W = Z - V$, as in Figure 1. We want to attach a meaning to cohomology of $U$ with compact support “towards $V$” or “towards $W$”.

**Remark 8.3.1.** Our treatment is strongly motivated by [DI87, §4.2], where such a theory is developed for de Rham cohomology in characteristic 0, assuming that $Z$ and $W$ are smooth normal-crossing divisors, which is the main case of interest. See also [Mie09] for étale cohomology, [LS07] for Hyodo–Kato cohomology, and [BD18] for Hodge cohomology of varieties over $C$.

**Proposition 8.3.2.** Let $F$ be an abelian sheaf on $|Y|$, and let $V'$ be any closed subvariety of $Y$ such that $Z = V \cup V'$.

(a) We have canonical isomorphisms

$$j^\dagger_{|Y-V'|} \Gamma_{|Y-V'|} F = j^\dagger_{|Y-Z|} \Gamma_{|Y-Z|} F \quad \text{and} \quad \Gamma_{|Y-V'|} j^\dagger_{|Y-V'|} F = \Gamma_{|Y-Z|} j^\dagger_{|Y-V'|} F.$$

(b) There is a natural map

$$j^\dagger_{|Y-V'|} \Gamma_{|Y-V'|} F \to \Gamma_{|Y-V'|} j^\dagger_{|Y-V'|} F,$$

which is an isomorphism away from $|V \cap V'|$.

**Proof.** (a) As in Prop 5.1.11 of [LS07], since $|V|$ and $|V'|$ admissibly cover $|Z|$ we have $j^\dagger_{|Y-V'|} \circ j^\dagger_{|Y-V|} = j^\dagger_{|Y-Z|}$. So, using Proposition 8.2.1 we have

$$\Gamma_{|Y-V'|} \circ j^\dagger_{|Y-V'|} \Gamma_{|Y-V'|} = j^\dagger_{|Y-V'|} \circ \Gamma_{|Y-V|} = j^\dagger_{|Y-Z|} \circ \Gamma_{|Y-V|}.$$

Similarly, we have $\Gamma_{|Y-V'|} \circ j^\dagger_{|Y-V'|} \Gamma_{|Y-V|} = \Gamma_{|Y-Z|}$ and hence

$$\Gamma_{|Y-V'|} \circ j^\dagger_{|Y-V'|} \Gamma_{|Y-V|} = \Gamma_{|Y-Z|} \circ j^\dagger_{|Y-V|}.$$

(b) It suffices to show that the composite map $j^\dagger_{|Y-V'|} \Gamma_{|Y-V'|} F \to j^\dagger_{|Y-V'|} \Gamma_{|Y-V'|} F \to h^* h^{-1} (j^\dagger_{|Y-V'|} F) = h^* h^{-1} (\Gamma_{|Y-V'|} F)$ is zero. However, this map factors through $h^* h^{-1} (\Gamma_{|Y-V'|} F)$ which is the zero sheaf.

The pair $\{Y - V, Y - V\}$ is an open covering of $Y - (V \cap V')$ as a $k$-variety, so their tubes admissibly cover $|Y - (V \cap V')|$. It is clear that the above map is an isomorphism after restriction to either $|Y - V|$ or $|Y - V'|$, so we obtain an isomorphism of sheaves over $|Y - (V \cap V')|$.

**Definition 8.3.3.** Let $F$ be an abelian sheaf on $|Y|$. We define cohomology with compact support towards $V$ (recall $V$ is closed in $Z$) by

$$R^i \Gamma^\dagger_{|V|}(F) = R^i \Gamma^\dagger_{|Y-V|}(F).$$
We define cohomology with compact support towards $W$ (recall $W$ is open in $Z$) by

$$R\Gamma_{\text{cV}}([U], \mathcal{F}) = R\Gamma\left([Y], R\Gamma_{Y-Z}[j^!_{Y-V}] \mathcal{F}\right).$$

Note that this notation is a priori ambiguous, since if both $V$ and $W$ are closed in $Y$, we have two candidate definitions of $R\Gamma_{\text{cV}}(-)$; but in fact the two candidate definitions agree, since if we start from the first definition we have

$$R\Gamma_{\text{cV}}([U], \mathcal{F}) := R\Gamma\left([Y], j^!_{Y-Z} R\Gamma_{Y-V} \mathcal{F}\right)$$

(by part (a) of the proposition)

$$= R\Gamma\left([Y], j^!_{Y-W} R\Gamma_{Y-V} \mathcal{F}\right)$$

(by part (b) of the proposition)

$$= R\Gamma\left([Y], R\Gamma_{Y-V} j^!_{Y-W} \mathcal{F}\right)$$

(by part (a) of the proposition)

which is the second definition of $R\Gamma_{\text{cV}}([U], \mathcal{F})$. (In particular, this applies when one of $V$ and $W$ is empty, and we conclude that cohomology with compact support towards $\emptyset$, or towards all of $Z$, has the expected meaning.)

**Proposition 8.3.4.** We have exact triangles

$$R\Gamma_{\text{cV}}([U], \mathcal{F}) \rightarrow R\Gamma\left([Y], j^!_{Y-V} \mathcal{F}\right) \rightarrow R\Gamma\left([Z], j^!_{W} (\mathcal{F}|_{Z})\right) \rightarrow [+1]$$

and

$$R\Gamma\left([Y], \Gamma^1_{Z}[R\Gamma_{Y-V}] \mathcal{F}\right) \rightarrow R\Gamma\left([Y], R\Gamma_{Y-V} \mathcal{F}\right) \rightarrow R\Gamma_{\text{cV}}([U], \mathcal{F}) \rightarrow [+1].$$

**Proof.** By definition we have an exact triangle of complexes of sheaves on $Y$

$$R\Gamma_{Z}[j^!_{Y-V}] \mathcal{F} \rightarrow j^!_{Z-V} \mathcal{F} \rightarrow Rh^{-1}(j^!_{Y-V} \mathcal{F}) \rightarrow [+1],$$

where $h : Z \hookrightarrow Y$ is the inclusion map. However, since $(Y - V) \cap Z = W$, we have $h^{-1}(j^!_{Y-V}) = j^!_{Z-V}(h^{-1} \mathcal{F})$, by Corollary 5.1.15 of [LS07]. Applying the (triangulated) functor $R\Gamma([-], -)$ gives the first triangle. The second is obtained similarly. $\square$

Let us note some “naturality” properties of the construction. Firstly, if we fix $Y$ and $Z$, and let $J \supseteq J'$ be two subvarieties of $Z$, then we have natural maps $R\Gamma_{\text{cV}}([U], \mathcal{F}) \rightarrow R\Gamma_{\text{cV}}([U], \mathcal{F})$ if:

- $J$ and $J'$ are both open,
- $J$ and $J'$ are both closed,
- $J$ is closed and $J'$ is open (using Proposition 8.3.2(b) with $V = J$ and $V' = Z - J'$).

(We do not consider the case when $J$ is open and $J'$ is closed, since we will not need it here.) Secondly, for coherent sheaves we have cup-products

$$R\Gamma_{\text{cV}}([U], \mathcal{F}) \otimes R\Gamma_{\text{cW}}([U], \mathcal{G}) \rightarrow R\Gamma_{\text{cZ}}([U], \mathcal{F} \otimes \mathcal{G}),$$

and these are compatible with the exact triangles of Proposition 8.3.4. Finally, we have the following compatibility with respect to morphisms of frames:

**Proposition 8.3.5.** Let $u : \mathfrak{P}' \rightarrow \mathfrak{P}$ be a morphism of formal schemes over $\mathcal{O}$, and define $Y', V', W'$ as the preimages of $Y, V, W$ etc. Then pullback along $u$ induces canonical maps

$$u^* : R\Gamma_{\text{cV}}([U], \mathcal{F}) \rightarrow R\Gamma_{\text{cV}}([U'], \mathcal{F}')$$

and

$$u^* : R\Gamma_{\text{cW}}([U], \mathcal{F}) \rightarrow R\Gamma_{\text{cW}}([U'], \mathcal{F}'),$$

compatible with the exact triangles of Proposition 8.3.4.

**Proof.** This is immediate from the compatibility of $j^!$ and $\Gamma_{\text{cV}}$ with pullback. $\square$

**Remark 8.3.6.** If we start with a variety $Y$ and two closed subvarieties $A, B$, and put $Z = A \cup B$, then we are interpreting $R\Gamma([Y], j^!_{Y-B} \Gamma_{Y-A}[F]$ as cohomology with compact support towards the closed subvariety $A$ of $Z$, and the subly different group $R\Gamma([Y], \Gamma_{Y-A}[j^!_{Y-B} \mathcal{F})$ (with the order of the functors interchanged) as cohomology with compact support towards the open subvariety $A - (A \cap B)$ of $Z$. These agree if $A \cap B = \emptyset$, but they are genuinely different otherwise (as the special case $A = B$)
shows). We shall show in Section 8.6 below that they give the same result for the cohomology of the de Rham complex when A and B intersect transversely.

8.4. Interpretation via dagger spaces. We recall from [GK00] the category of dagger spaces over K. Note that if \( \mathfrak{P} \) is a proper (admissible) \( \mathcal{O} \)-scheme, and \( X \) is a locally closed subvariety of \( \mathfrak{P}_0 \), then there is a natural structure of a dagger space on the tube \( |X| \); we denote this dagger space by \( |X|^\dagger \), and similarly \( [X]^\dagger \) for the tubes of radius \( \lambda < 1 \).

8.4.1. Non-compact support. Essentially by definition, if \( X \rightarrow Y \rightarrow \mathfrak{P} \) is a proper smooth frame, and \( V \) any strict neighbourhood of \( |X| \) in \( |Y| \), then any coherent sheaf \( \mathcal{F} \) on \( V \) defines a coherent sheaf on \( |X|^\dagger \), and we have

\[
R\Gamma(|X|^\dagger, \mathcal{F}) = R\Gamma \left( V, j^{|X|^\dagger}_* \mathcal{F} \right)
\]

(and similarly for hypercohomology of complexes of coherent sheaves).

8.4.2. Compact support. There is also a concept of compactly-supported cohomology for coherent sheaves on dagger spaces: see [GK00] \S 4.3. We will need the following computation:

**Proposition 8.4.1.** Let \( \mathfrak{P} \) be a proper admissible formal \( \mathcal{O} \)-scheme, and \( W \) a locally closed subvariety of \( \mathfrak{P}_0 \). Write \( W = X \cap Z \) with \( X \) open and \( Z \) closed. Then we have

\[
R\Gamma_c([W]^\dagger, \mathcal{F}) = R\Gamma \left( \mathfrak{P}_K, \Gamma^{|Z||X|\dagger}_\lambda \mathcal{F} \right).
\]

**Proof.** We have \( \bigcup_\lambda \Gamma^{|Z||X|\dagger}_\lambda \mathcal{F} = \lim_\lambda \Gamma[Z, \Gamma^{|X|\dagger}_\lambda \mathcal{F}], \) since \( [W] = [Z] \cap |X| \). Applying this to an injective resolution of \( \mathcal{F} \) gives the result, since \( R\Gamma_c([W], \mathcal{F}) = \lim_\lambda R\Gamma \left( \mathfrak{P}_K, \Gamma^{|W|\dagger}_\lambda \mathcal{F} \right) \). \( \square \)

These results allow the triangles of Proposition 8.3.4 to be written in the following more convenient form. Let \( (U, V, W) \) be as above, and denote the dagger space tubes of these by \( U, V, W \). Then there are exact triangles

\[
\begin{align*}
R\Gamma_c(W, \mathcal{F}) & \rightarrow R\Gamma_c(U \cup W, \mathcal{F}) \rightarrow R\Gamma_c(U, \mathcal{F}) \rightarrow [+1] \\
R\Gamma_c(U, \mathcal{F}) & \rightarrow R\Gamma(U \cup W, \mathcal{F}) \rightarrow R\Gamma(W, \mathcal{F}) \rightarrow [+1].
\end{align*}
\]

8.4.3. Duality. There is a form of Serre duality for affinoid dagger spaces \( X \): by choosing an embedding \( X \hookrightarrow \mathcal{P} \) with \( \mathcal{P} \) a smooth affinoid, we can define a *dualising complex* \( \omega_X \), which is a perfect complex of \( \mathcal{O}_X \)-modules, and for any coherent \( \mathcal{F} \), we have a perfect duality of Hausdorff topological vector spaces

\[
H^i(X, \mathcal{F}) \times H^{-i}(X, \mathcal{R}\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X)) \rightarrow K,
\]

with both sides being 0 if \( i \neq 0 \).

**Remark 8.4.2.** This is stated in [GK00] for smooth affinoids, in which case we have of course \( \omega_X = \Omega^1_{X/K} \) where \( d = \dim X \), but the proof is valid in the generality above. See [GK98] \S 7.1] for more details. \( \diamond \)

**Remark 8.4.3.** This Serre duality does not seem to extend straightforwardly to non-affinoid spaces. If \( X \) is smooth and quasi-compact, and \( \{X_i\}_{i \in I} \) is a finite affinoid covering, then we can form Cech complexes representing \( R\Gamma(X, \mathcal{F}) \) and \( R\Gamma_c(X, \mathcal{F}^| \otimes \omega_X) \) with respect to this covering. These are complexes of complete locally-convex \( K \)-vector spaces which are term-wise dual to one another, so we obtain natural pairings between the cohomology groups. However, it is not clear if the differentials in these complexes are strict; so one does not know if these pairings are perfect dualities of topological vector spaces (or even if the induced topologies on the cohomology groups are Hausdorff). \( \diamond \)

8.5. Finiteness and Poincaré duality.

8.5.1. Non-compact support.

**Theorem 8.5.1** (Grosse-Klönne). If \( X \) is a dagger space of the form \( U - V \), where \( U \) is smooth and quasicompact, and \( V \subseteq U \) is a quasicompact open subset, then the cohomology groups \( H^i_{\text{dR}}(X) := H^i(\mathcal{X}, \Omega^1_{\mathcal{X/K}}) \) are finite-dimensional over \( K \) for all \( i \).

**Proof.** This is (a special case of) the main theorem of [GK02]. \( \square \)

**Note 8.5.2.** This implies finite-dimensionality of rigid cohomology, since for a proper smooth frame \( X \hookrightarrow Y \hookrightarrow \mathfrak{P} \), the dagger space \( X = |X|^\dagger \) satisfies the hypotheses of Theorem 8.5.1 and we have \( H^i_{\text{rig}}(X) = H^i_{\text{dR}}(X) \). \( \diamond \)
8.5.2. **Compact support.** There is also a compactly-supported analogue of this result, and a Poincaré duality theorem; these are straightforward consequence of results of Grosse-Klönne, but curiously do not seem to be explicitly written down in the literature:

**Theorem 8.5.3.** Let \( X \) be a smooth dagger space of the form \( U - V \) with \( U, V \) quasicompact, as in Theorem 8.5.1 of pure dimension \( d \). Then \( \Gamma_{dR, c}\)(\( X \)) is also finite-dimensional for all \( i \), and we have perfect pairings of finite-dimensional vector spaces

\[
H^i_{dR}(X) \times H^{2d-i}_{dR, c}(X) \to K \quad \text{for } 0 \leq i \leq 2d.
\]

**Proof.** The case of affinoid \( X \) is treated in Theorem 4.9 and remark 4.10 of [GK00]. The case of \( X \) quasicompact follows readily from this, using the Cech spectral sequence associated to a finite covering of \( X \) by affinoids (since we know that the Cech complex consists of finite-dimensional vector spaces, there are no topological issues to worry about).

We now consider the general case. We can write \( X \) as a countable increasing union \( \{X_n\}_{n \in \mathbb{N}} \) of quasicompacta. Then we have

\[
H^i_{dR}(X) = \lim_n H^i_{dR}(X_n), \quad H^{2d-i}_{dR, c}(X) = \lim_n H^{2d-i}_{dR, c}(X_n).
\]

Since the terms in the two limits are dual to each other, and we know that \( H^i_{dR}(X) \) is finite-dimensional, it follows that \( H^{2d-i}_{dR, c}(X) \) is also finite-dimensional and that Poincaré duality holds for \( X \). \( \square \)

8.5.3. **Partial compact support.**

**Corollary 8.5.4.** If we are given varieties \( U, V, W \subseteq \mathfrak{P}_0 \) as in Section 8.3 with \( U \) open in \( \mathfrak{P}_0 \) and \( \mathfrak{P}_K \) smooth, then the cohomology groups of the complexes

\[
\Gamma_{dR, c V}(U) := \Gamma_{c V}(U, \Omega^*) \quad \text{and} \quad \Gamma_{dR, c W}(U) := \Gamma_{c W}(U, \Omega^*)
\]

are finite-dimensional for all \( i \), and there are perfect pairings

\[
H^i_{dR, c V}(U) \times H^{2d-i}_{dR, c W}(U) \to K.
\]

**Proof.** Rewriting the exact triangles of Proposition 8.3.4 in terms of dagger spaces using Proposition 8.4.1 as explained above, and taking \( F \) to be the rigid-analytic de Rham complex, we have long exact sequences

\[
\cdots \to H^i_{dR, c W}(U) \to H^i_{dR}(U \cup W) \to H^i_{dR}(W) \to \cdots
\]

and

\[
\cdots \leftarrow H^{2d-i}_{dR, c V}(U) \leftarrow H^{2d-i}_{dR}(U \cup W) \leftarrow H^{2d-i}_{dR, c}(W) \leftarrow \cdots.
\]

Moreover, there are compatible pairings between the groups in the first row and their neighbours in the second row. By Theorems 8.5.1 and 8.5.3 the middle and right groups on each row are finite-dimensional and the pairings between them are perfect. By induction on \( i \) we deduce that the groups in the left-hand column are also finite-dimensional and in perfect duality, as required. \( \square \)

8.5.4. **A “logarithmic” variant.** Sadly the above setting is still not quite general enough, and we shall need to consider yet another possibility. Suppose we have an proper admissible formal \( \mathcal{O}_K \)-scheme \( \mathfrak{P} \), a proper closed subvariety \( Y \to \mathfrak{P}_0 \), and a decomposition \( Y = U \cup V \cup W \) as above. We also suppose that \( \mathcal{D} \subseteq \mathfrak{P} \) is a simple normal crossing divisor relative to Spf \( \mathcal{O}_K \), which intersects transversely with \( U \) and \( W \). We write \( \mathcal{P} \) for the dagger space generic fibre of \( \mathfrak{P} \), and \( U, V, W \) for the dagger tubes of \( U, V \) and \( W \) respectively.

**Notation 8.5.5.** Write \( \Gamma_{dR, c V}(\mathcal{U}(\mathcal{D})) \), resp. \( \Gamma_{dR, c W}(\mathcal{U}(\mathcal{D})) \), for the hypercohomology of \( \mathcal{U} \) with compact support towards \( V \) (resp. \( W \)) of the logarithmic de Rham complex \( \Omega^*_{\mathcal{P}}(\mathcal{D}) \). Similarly, we write \( \Gamma_{dR, c V}(\mathcal{U}(\langle -\rangle)) \) for the hypercohomology of the “minus-log” complex \( \Omega^*_{\mathcal{P}}(\langle -\rangle) := \Omega^*_{\mathcal{P}}(\langle -\rangle) \).

**Proposition 8.5.6.** We have perfect pairings of finite-dimensional \( K \)-vector spaces

\[
H^i_{dR, c V}(\mathcal{U}, \langle -\rangle) \times H^{2d-i}_{dR, c W}(\mathcal{U}, \langle \mathcal{D} \rangle) \to K,
\]

and

\[
H^i_{dR, c V}(\mathcal{U}, \langle \mathcal{D} \rangle) \times H^{2d-i}_{dR, c W}(\mathcal{U}, \langle -\rangle) \to K.
\]
**Proof.** By the same long exact sequence argument as above, it suffices to prove the proposition in the special case $\mathcal{W} = \emptyset$, i.e. that

$$H^i_{\text{dR},c}(\mathcal{U}, \langle -D \rangle) \times H^{2d-i}_{\text{dR}}(\mathcal{U}, \langle D \rangle) \to K$$

and

$$H^i_{\text{dR}}(\mathcal{U}, \langle -D \rangle) \times H^{2d-i}_{\text{dR},c}(\mathcal{U}, \langle D \rangle) \to K$$

are perfect pairings of finite-dimensional spaces. We prove the former; the argument for the latter is identical with the role of compact and non-compact support interchanged.

Let $\mathcal{D}^{(n)}$ denote the disjoint union of the $n$-fold intersections of components of $\mathcal{D}$, and $\mathcal{E}^{(n)} : \mathcal{D}^{(n)} \to \mathcal{P}$ the natural map. The logarithmic de Rham complex $\Omega^*_{\mathcal{P}}(\mathcal{D})$ has an increasing filtration, whose $n$-th graded piece is $\mathcal{E}^{(n)}_{\mathcal{P}}(\mathcal{D}^{(n)})$. Similarly, the complex $\Omega^*_{\mathcal{P}}(-D)$ has a decreasing filtration, with the same graded pieces; and the logarithmic duality pairing

$$\Omega_{\mathcal{P}}^-(\mathcal{D}) \otimes \Omega_{\mathcal{P}}^d(-D) \to \Omega_{\mathcal{P}}^d(-D) = \omega_{\mathcal{P}},$$

where $\omega_{\mathcal{P}}$ is the dualizing sheaf, is compatible with these filtrations, and the pairing it induces on the $n$-th graded piece is the usual (non-logarithmic) duality pairing on each of the $n$-fold intersections.

So we have spectral sequences

$$E^{ij}_1 = H^i(\mathcal{U}, \mathcal{E}^{(i)}_{\mathcal{D}^{(i)}}) \Rightarrow H^{i+j}_{\text{dR},c}(\mathcal{U}(-D))$$

and

$$E^{-i,j}_1 = H^{-2i}(\mathcal{U}, \mathcal{E}^{(i)}_{\mathcal{D}^{(i)}}) \Rightarrow H^{-i+j}_{\text{dR},c}(\mathcal{U}(D)).$$

We have $H^i(\mathcal{U}, \mathcal{E}^{(i)}_{\mathcal{D}^{(i)}}) = H^i_{\text{dR},c}(\mathcal{U}(i))$, where $\mathcal{U}(i) = \mathcal{E}^{(i)}_{\mathcal{D}^{(i)}}^{-1}(\mathcal{U})$, and similarly without compact support. So by Theorem 8.5.3 the spaces $E^{ij}_1$ and $E^{-i,j}_1$ are finite-dimensional for all $i, j$ (and zero outside a bounded region), and the pairing $E^{ij}_1 \times E^{-i,j}_1 \to E^{k,2d-k}_1 \cong K$ induced by (8.1) is perfect. Hence the limits of the two spectral sequences are also finite-dimensional and in perfect duality, as required. □

### 8.6. The transversal case

Although we shall not use it in the remainder of the paper, it would be remiss not to point out the following consistency property of the above constructions. For simplicity, we suppose that $\mathcal{P}$ is smooth and proper over $\mathcal{O}$, and $Y = \mathcal{P}_0$. Let $A, B$ be two closed subvarieties of $Y$, let $U = Y - A - B$, and let $A^0 = A - (A \cap B)$ and $B^0 = B - (A \cap B)$.

**Proposition 8.6.1.** If $A, B$, and $A \cap B$ are smooth, and $\text{codim}_Y(A \cap B) = \text{codim}_Y(A) + \text{codim}_Y(B)$, then there are isomorphisms

$$H^i_{\text{dR},c-A}(U) \cong H^i_{\text{dR},c-A^0}(U)$$

for all $i$.

**Proof.** Consider the following $3 \times 3$ grid, in which each row and column is an exact triangle:

\[
\begin{array}{ccc}
? & \to & R\Gamma_{\text{rig},B}(Y) \\
\downarrow & & \downarrow \\
R\Gamma_{\text{rig},c}(Y) & \to & R\Gamma_{\text{rig}}(Y) \\
\downarrow & & \downarrow \\
R\Gamma_{\text{rig},c-A^c}(U) & \to & R\Gamma_{\text{rig}}(Y - B)
\end{array}
\]

where the term marked ‘?’ is $R\Gamma_{\mathcal{P}_K,\mathcal{O}}(\mathcal{P}_K, \mathcal{P}_{\mathcal{P}}) = R\Gamma_{\mathcal{P}_{\mathcal{P}}}(\mathcal{P}_K \cap \mathcal{P}_{\mathcal{P}})$. Our smoothness assumptions imply that there is a Gysin isomorphism $R\Gamma_{\text{rig},B}(Y) = R\Gamma_{\text{rig}}(B)[-2c]$ where $c = \text{codim}_Y(B)$, and similarly that $R\Gamma_{\text{rig},A^c}(A^c) = R\Gamma_{\text{rig}}(A^c)[2c]$ where $c = \text{codim}_Y(A^c)$. Moreover, the map $R\Gamma_{\text{rig},B}(Y) \to R\Gamma_{\text{rig},A^c}(A^c)$ is identified, via the Gysin isomorphisms, with the obvious restriction map $R\Gamma_{\text{rig}}(B) \to R\Gamma_{\text{rig}}(A^c)$ (shifted by $-2c$). Note that this compatibility of Gysin morphisms is far from being merely formal, but rather is a basic case of the “excess intersection formula” of Dégilde, see [Dégilde08, Proposition 4.10]. So the group ‘?’ has to be isomorphic to the mapping fibre of this map, which is simply $R\Gamma_{\text{rig}}(B^c)[-2c]$.

We now consider the ‘dual diagram’, obtained by first applying the functor $R\text{Hom}(-, K[-2d])$ to this diagram, and then reflecting in the off-diagonal. One sees that each term in the dual diagram for $(A, B)$ is isomorphic to the corresponding term in the original diagram for $(B, A)$. After a little book-keeping,
one also sees that these isomorphisms are compatible with the arrows in the two diagrams. Hence we deduce an isomorphism in the remaining corner also, namely
\[ R\Gamma_{\text{dr},c-A^\ell}(U)[1] \cong R\text{Hom}(R\Gamma_{\text{dr},c-B^\ell}(U)[1], K[-2d]). \]
But we have seen that the dual of \( R\Gamma_{\text{dr},c-B^\ell}(U)[1]-2d \) is \( R\Gamma_{\text{dr},c-A}(U)[1] \).

9. Log-rigid syntomic and fp-cohomology

In order to make use of the formalism of Section 7 for actual computations, we shall need to replace Nekovář–Nizioł cohomology (which has excellent functorial properties, but is inexplicit) with a more explicitly computable theory.

9.1. The semistable case. Let \( \pi \) be a uniformizer of \( K \), and write \( O_K^\pi \) for the scheme \( \text{Spec} O_K \) with the canonical log structure, given by the chart \( 1 \mapsto \pi \). For \( F \) the maximal unramified subfield of \( K \), write \( O_F^\pi \) for the scheme \( \text{Spec} O_F \) with the ‘hollow’ log structure given by \( 1 \mapsto 0 \). Denote by \( k \) the residue field of \( O_K \), and write \( k^0 \) for the scheme \( \text{Spec} k \), again with the log structure given by \( 1 \mapsto 0 \).

9.1.1. Log-rigid syntomic cohomology. Let \( X \) be a strictly semistable log scheme over \( O_K^\pi \), and \( D \subseteq X \) a closed subscheme with complement \( U \), such that \( (U, X) \) is a strictly semistable log scheme with boundary over \( O_K^\pi \) in the sense of [EY18 Definition 3.3].

We can then consider three complexes associated to \( X \) and \( D \):

- the rigid Hyodo–Kato cohomology \( R\Gamma_{\text{rig}}^{\HK}(X_0(D_0)) \), which is a complex of \( F \)-vector spaces with an \( F \)-semilinear Frobenius \( \varphi \) and an \( F \)-linear monodromy operator \( N \), satisfying \( N\varphi = p\varphi N \).

(Its definition involves a rather intricate limiting process over collections of liftings of open subsets of \( X_0 \) to characteristic zero, since it is not generally possible to find a global lifting of \( X_0 \) compatible with Frobenius.)

- the log-rigid cohomology \( R\Gamma_{\text{rig}}^{\HK}(X_0(D_0)/\mathcal{O}^\pi_K) \), which is a complex of \( K \)-vector spaces, quasi-isomorphic to the de Rham cohomology of the dagger space \( \mathcal{X} = \set{X_0}_{\mathcal{X}_K} \) with log poles along \( D_K \).

- the Deligne–de Rham cohomology \( R\Gamma_{\text{dr}}^D(U_K) \), which is a complex of \( K \)-vector spaces with a filtration \( \text{Fil}^r \) (the Hodge filtration). If \( X_K \) is proper, it is quasi-isomorphic to \( R\Gamma(X_K, \Omega^{\bullet}_{X_K}(D_K)) \) with the filtration defined by truncation.

Note 9.1.1. These complexes are related by morphisms in the derived category of \( K \)-vector spaces (cf. Equation 3.11 of op.cit.)

\[ R\Gamma_{\text{rig}}^{\HK}(X_0(D_0)) \otimes_F K \xrightarrow{\varphi_{rig}} R\Gamma_{\text{rig}}(X_0(D_0)/\mathcal{O}^\pi_K) \xrightarrow{\text{sp}} R\Gamma_{\text{dr}}^D(U_K). \]

The morphism \( \varphi_{rig} \) is a quasi-isomorphism, and sp is also a quasi-isomorphism if \( X \) is proper.

\[ \varphi_{rig} := p^r \varphi. \]

For \( r \geq 0 \), Ertl–Yamada [EY18 Definition 3.4] define a log-rigid syntomic cohomology:

**Definition 9.1.2.** Define \( R\Gamma_{\text{rig-syn}}(X(D), r, \pi) \) to be the homotopy limit of the diagram

\[
\begin{array}{ccc}
\Gamma_{\text{rig}}^D(U_K) & \xrightarrow{\varphi_{rig}} & \Gamma_{\text{rig}}^{\HK}(X_0(D_0)) \\
\downarrow{\text{sp}} & & \downarrow{1 - \varphi_{rig}} \\
\Gamma_{\text{rig}}(X_0(D_0)/\mathcal{O}^\pi_K) & \xrightarrow{N} & \Gamma_{\text{rig}}^{\HK}(X_0(D_0)) \\
\end{array}
\]

Here \( \varphi_{rig} := p^r \varphi \).

**Note 9.1.3.** There is no properness assumption on \( X \), so if we start with a strictly semistable log-scheme \( U \), we can always simply take \( X = U \) and \( D = \emptyset \) in the above construction.

However, it is important to allow more general \( X \) in order to prove the following theorem:

**Theorem 9.1.4.** Suppose \( U = X - D \) as above, with \( X \) proper. Then for all \( r \geq 0 \), there exist canonical quasi-isomorphisms

\[ R\Gamma_{\text{rig-syn}}(U, r, \pi) \cong R\Gamma_{\text{rig-syn}}(X(D), r, \pi) \cong R\Gamma_{\text{NN-syn}}(U_K, r). \]

**Proof.** This is [EY18 Corollary 4.2].
CONVENTION. Following op.cit., we shall call a semistable $O_K$-scheme $U$ compactifiable if it can be written as $X - D$ for some $X$ and $D$ as above. Thus log-rigid syntomic cohomology agrees with Nekovář–Nizioł syntomic cohomology for compactifiable semistable schemes.

9.1.2. Compactly supported log-rigid syntomic and fp-cohomology. Let $(X, D)$ be a strictly semistable log-scheme with boundary, as before; we do now assume that $X$ is proper. In [EY19], Ertl and Yamada define

- rigid Hyodo-Kato cohomology with compact support, $R\Gamma_{\text{rig}}^{\text{HK}}(X_0(-D_0))$,
- log-rigid cohomology with compact support, $R\Gamma_{\text{rig}}(X_0(-D_0)/O_K^\times)$.

Again, the former is $F$-linear and equipped with Frobenius and monodromy operators, and the latter is $K$-linear.

**Proposition 9.1.5.** Let $d = \dim(X)$. Then there exist canonical isomorphisms

\[
\begin{align*}
R\Gamma_{\text{rig}}^{\text{HK}}(X_0(-D_0)) & \xrightarrow{\varphi} R\Gamma_{\text{rig}}^{\text{HK}}(X_0(D_0))^*[-2d] \\
R\Gamma_{\text{rig}}(X_0(-D_0)/O_K^\times) & \xrightarrow{\varphi} R\Gamma_{\text{rig}}(X_0(D_0)/O_K^\times)^*[-2d].
\end{align*}
\]

**Proof.** See [EY19] Theorem 4.1. □

**Note 9.1.6.** The morphism $[0.1]$ is compatible with $\varphi$ and $N$, if we define the Frobenius $\Phi$ and monodromy on the right-hand side as $p^d \cdot (\varphi^{-1})^\vee$ and $-N^\vee$. ●

**Remark 9.1.7.** The Frobenius on $R\Gamma_{\text{rig}}^{\text{HK}}(X_0(D_0))$ admits an inverse in the derived category, although it is not necessarily invertible at the level of complexes; explicitly, we can replace the complex computing $R\Gamma_{\text{rig}}^{\text{HK}}(X_0(D_0))$ with its “perfection”, as in [Bes12] §4.

We also have the complex $R\Gamma_{\text{dir},c}^\text{D}(U_K) = R\Gamma(X_K, \Omega^{\bullet}_{X,K/K}(-D_K))$ computing compactly-supported de Rham cohomology of $U_K$, with its truncation filtration; and there are maps in the derived category relating these three complexes, as before. We define log-rigid syntomic cohomology with compact support as follows:

**Definition 9.1.8.** We define $R\Gamma_{\text{rig-syn}}(X(-D), r, \pi)$ as the homotopy limit of the diagram analogous to Definition 9.1.3 with the three complexes replaced by their $\langle -D \rangle$ versions.

**Definition 9.1.9.** Replacing $1 - p^{-r}\varphi$ by $P(p^{-r}\varphi)$, for some $P \in \mathbb{Q}[t]$ with constant coefficient 1, we obtain log-rigid fp-cohomology with compact support, which we denote by $R\Gamma_{\text{rig-fp}}(X(-D), r, \pi; P)$ and $R\Gamma_{\text{rig-fp},c}(X(-D), r, \pi; P)$ respectively.

The following result is a consequence of the results in op.cit.

**Theorem 9.1.10.** For all $r \geq 0$, there exists a canonical isomorphism

$R\Gamma_{\text{rig-fp}}(X(-D), r, \pi; P) \cong R\Gamma_{\text{NN-fp},c}(U_K, r; P)$.

Moreover, this isomorphism is compatible with pullback.

The following result (c.f. [Bes12] §4) will be useful for the construction of an ‘extension-by-0’ map (c.f. Proposition 9.2.11):

**Proposition 9.1.11.** The complex defined in 9.1.3 is quasi-isomorphic to the homotopy limit of the following diagram, shifted by $[-2d]$:

\[
\begin{array}{ccc}
\left( \text{Fil}^{d-r} R\Gamma_{\text{dir}}^\text{D}(U_K) \right)^* & \xrightarrow{\Phi_r} & \left( R\Gamma_{\text{rig}}^{\text{HK}}(X_0(D_0)) \right)^* \\
\left( (\delta^\vee)^{-1} \right) & \left( (\delta^\vee)^{-1} \right) & \left( (\delta^\vee)^{-1} \right) \\
\left( R\Gamma_{\text{rig}}(X_0(D_0)/O_K^\times) \right)^* & \xrightarrow{N^\vee} & \left( R\Gamma_{\text{rig}}^{\text{HK}}(X_0(D_0)) \right)^* \\
\end{array}
\]

**Proof.** Immediate from Note 9.1.16. □

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9.1.3. Pairings. Let \((X, \tilde{X})\) be as above.

**Proposition 9.1.12.** We have cup products

\[
\begin{align*}
R_{\text{NN-syn}}^i(X_K, r) \times R_{\text{NN-fp,c}}^j(X_K, s; P) &\longrightarrow R_{\text{NN-fp,c}}^{i+j}(X, r+s; P), \\
R_{\text{trig-syn}}^i(X, r, \pi) \times R_{\text{trig-fp,c}}^j(X, s, \pi; P) &\longrightarrow R_{\text{trig-fp,c}}^{i+j}(X, r+s, \pi; P)
\end{align*}
\]

which are compatible under the isomorphisms in Theorems 9.1.4 and 9.1.10.

**Proof.** The proof for the Nekovář–Niziol cohomology is given in [BLZ10]. The proof for log-rigid fp-cohomology is analogous. The compatibility follows from [FY19].

**Corollary 9.1.13.** When \(i + j = 2d + 1, r + s = d + 1\), and \(P(\zeta/p) \neq 0\) for all \(\zeta \in \mu_{[F, \mathbb{Q}_p]}\), then we get \(K\)-valued pairings, denoted \((\ , \ )_{\text{NN-fp}, X_K}, (\ , \ )_{\text{trig-fp}, X}\), respectively.

9.2. Cohomology theories with coefficients for smooth schemes.

9.2.1. Rigid syntomic/fp-cohomology. Let \(X\) be a smooth \(O_K\)-scheme with generic fibre \(X_K\) and special fibre \(X_0\); we assume that \(X_K\) is proper. Let \(D\) be a divisor in \(X\).

Denote by \(\tilde{X}\) the dagger space tube of \(X_0\) in \(X^n_{\text{an}}\).

**Definition 9.2.1.** An overconvergent filtered \(F\)-isocrystal on \((X, X_K)\) consists of the following data:

- an overconvergent \(F\)-isocrystal \(\mathcal{F}_{\text{rig}}\) on \(X_0\);
- an algebraic vector bundle \(\mathcal{F}_{\text{dR}}\) on the variety \(X_K\), endowed with a connection with logarithmic singularities along \(X_K - X\), and with a filtration satisfying Griffiths transversality;
- an isomorphism of rigid-analytic vector bundles over the dagger space \(X\), compatible with connections,

\[
\mathcal{F}_{\text{dR}}|_{\tilde{X}} \cong \mathcal{F}_{\text{rig}, X},
\]

where \(\mathcal{F}_{\text{rig}, X}\) is the realisation of \(\mathcal{F}_{\text{rig}}\) corresponding to the lifting \(X\) of \(X_0\).

**Definition 9.2.2.** Define

\[
R_{\text{rig}}(X_0(D_0), \mathcal{F}_{\text{rig}}) = R^i(X, \mathcal{F}_{\text{rig}, X} \otimes \Omega^i_{D_{/K}}(D_K)).
\]

**Note 9.2.3.**

1. In the case of trivial coefficients, we recover the complex \(R_{\text{dR}}(X(D))\) (c.f. Notation 8.5.5).
2. We equip \(X\) with the log structure associated to \(X_0\). Then we have \(R_{\text{rig}}(X_0(D_0), \mathcal{F}_{\text{rig}}) = R_{\text{rig}}(X_0(D_0), \mathcal{F}_{\text{rig}})\), and \(N = 0\).
3. There exists a specialisation map

\[
\text{sp}: R_{\text{dR}}(X_0(D_0), \mathcal{F}_{\text{dR}}) \longrightarrow R_{\text{rig}}(X_0(D_0), \mathcal{F}_{\text{rig}}).
\]

**Definition 9.2.4.** Let \(r \in \mathbb{Z}\), and let \(P \in \mathbb{Q}_p[t]\) have constant coefficient 1. Following Besser [Bes12], we define the rigid fp-cohomology of \(X(D)\) with coefficients \(\mathcal{F}\), twist \(r\) and polynomial \(P\) as the homotopy limit of the diagram

\[
\begin{align*}
\text{Fil}^r \ R_{\text{dR}}^i(X_K(D_K), \mathcal{F}_{\text{dR}}) &\longrightarrow R_{\text{rig}}(X_0(D_0), \mathcal{F}_{\text{rig}}) \quad \text{sp}^r \\
R_{\text{rig}}(X_0(D_0), \mathcal{F}_{\text{rig}}) &\longrightarrow R_{\text{rig}}(X_0(D_0), \mathcal{F}_{\text{rig}})
\end{align*}
\]

where the unlabelled arrow is base-extension to \(K\). We denote it by \(R_{\text{rig-fp}}(X(D), \mathcal{F}, r; P)\). When \(P(t) = 1 - t\), then we call it rigid syntomic cohomology, denoted by \(R_{\text{rig-syn}}(X(D), \mathcal{F}, r)\).

**Notation 9.2.5.** We shall write \(R_{\text{rig-fp}}(X(D), r; P)\) if \(\mathcal{F}\) is the trivial isocrystal.

**Note 9.2.6.**

1. We have \(R_{\text{rig-fp}}(X(D), \mathcal{F}, r; P) = R_{\text{rig-fp}}(X(D), \mathcal{F}(r), 0; P)\) where \(\mathcal{F}(r)\) is the \(r\)-th Tate twist of \(\mathcal{F}\).
2. If \(F = K\), the middle arrow is the identity, and the zigzag diagram collapses to the mapping fibre of the map

\[
\begin{align*}
\text{Fil}^r \ R_{\text{dR}}^i(X_K(D_K), \mathcal{F}_{\text{dR}}) \quad &\longrightarrow R_{\text{rig}}(X_0(D_0), \mathcal{F}_{\text{rig}}) \\
&\quad \text{sp}^r \longrightarrow R_{\text{rig}}(X_0(D_0), \mathcal{F}_{\text{rig}}).
\end{align*}
\]
(3) If $X$ is equipped with the log structure associated to $X_0$, then by Note 9.2.3 (2) we have a natural map
\[
\delta: R\Gamma_{\text{rig-fp}}(X(D), \mathcal{F}, r; P) \to R\Gamma_{\text{rig-fp}}(X(D), \mathcal{F}, r; P).
\]

Lemma 9.2.7. Let $X$ be a strictly semistable proper log scheme over $O_K^+$, and let $D \subset X$ be a closed subscheme with complement $U$. Suppose that $(U, X)$ is a strictly semistable log scheme, and let $Z$ be a smooth open subscheme of $X$. We then have a restriction map
\[
\text{res}_Z: R\Gamma_{\text{rig-fp}}(X(D), \mathcal{F}, r; P) \to R\Gamma_{\text{rig-fp}}(Z(D), \mathcal{F}, r; P).
\]

**Proof.** Consequence of Note 9.2.6, together with the restriction map on $\text{rig-fp}$ cohomology. □

9.2.2. Rigid fp-cohomology with compact support. We continue under the assumptions of Section 9.2.1.

**Notation 9.2.8.** Write $\text{cosp} : R\Gamma_{\text{rig-c}}(X_0(-D_0), \mathcal{F}_{\text{rig}}) \to R\Gamma_{\text{dR}}^D(X_K, \mathcal{F}_{\text{dR}})$ for the specialisation map.

**Definition 9.2.9.** Let $r \geq 0$, and let $Q \in \mathbb{Q}_{\geq 0}$ have constant coefficient 1. Define the rigid fp-cohomology with compact support of $X$ with coefficients $\mathcal{F}$, twist $r$ and polynomial $Q$, as the homotopy limit of the zigzag diagram
\[
\begin{array}{ccc}
\text{Fil}^r R\Gamma_{\text{dR}}^D(X_K, \mathcal{F}_{\text{dR}}) & \overset{\text{comp}}{\longrightarrow} & R\Gamma_{\text{rig},c}(X_0(-D_0), \mathcal{F}_{\text{rig}}) \\
R\Gamma_{\text{dR}}^D(X_K, \mathcal{F}_{\text{dR}}) & \overset{Q(\varphi_r)}{\longrightarrow} & R\Gamma_{\text{rig},c}(X_0(-D_0), \mathcal{F}_{\text{rig}})^*.
\end{array}
\]

We denote it by $R\Gamma_{\text{rig-fp},c}(X(-D), \mathcal{F}, r; Q)$.

We have the following analogue of Proposition 9.1.11 for rigid fp-cohomology with compact support:

**Lemma 9.2.10.** The complex $\Phi^{-}$ is quasi-isomorphic to the homotopy limit of the following diagram, shifted by $[-2d]$:
\[
\begin{array}{ccc}
\text{Fil}^{d-r} (R\Gamma_{\text{dR}}^D(X_K), \mathcal{F}_{\text{dR}})^* & \overset{\Phi}{\longrightarrow} & R\Gamma_{\text{rig}} X_0(D_0), \mathcal{F}_{\text{rig}})^* \\
R\Gamma_{\text{dR}}^D(X_K, \mathcal{F}_{\text{dR}})^* & \overset{Q(\varphi_r)}{\longrightarrow} & R\Gamma_{\text{rig}} X_0(D_0), \mathcal{F}_{\text{rig}})^*.
\end{array}
\]

Here $\Phi = (\varphi_r)^{-1}$.

**Proof.** See Bes12 §4. □

**Proposition 9.2.11.** Let $X$ be a strictly semistable proper log scheme over $O_K^+$, and let $D \subset X$ be a closed subscheme with complement $U$. Suppose that $(U, X)$ is a strictly semistable log scheme, and let $Z$ be a smooth open subscheme of $X$. Then in the derived category, we have an extension-by-0 morphism
\[
R\Gamma_{\text{rig-fp},c}(Z(-D), r; Q) \to R\Gamma_{\text{rig-fp}}(X(-D), r; Q).
\]

**Proof.** Clear from Proposition 9.1.11 and Lemma 9.2.10. Here, the morphism
\[
R\Gamma_{\text{rig}}(Z_0(D_0))^* \to R\Gamma_{\text{HK}}(X_0(D_0))^*
\]

is given by the composition of $\mathcal{O}_{\mathcal{D}^0}$ with the dual of the natural restriction map
\[
R\Gamma_{\text{rig}}(X_0(D_0)) \to R\Gamma_{\text{rig}}(Z_0(D_0)).
\]

**Remark 9.2.12.** There should clearly be a version of Proposition 9.2.11 with coefficients. □

**Proposition 9.2.13.** For $r, s \geq 0$, we have a cap product
\[
R\Gamma_{\text{rig-fp},c}(X(D), \mathcal{F}, r; P) \times R\Gamma_{\text{rig-fp},c}(X(-D), \mathcal{G}, r+s; Q) \to R\Gamma_{\text{rig-fp},c}(X, \mathcal{F} \otimes \mathcal{G}, r+s; P \ast Q).
\]

**Proof.** See Bes12 §2. □
Lemma 9.2.14. If $X$ is connected of dimension $d$ and $Q(\zeta/p) \neq 0$ for all $\zeta \in \mu_{[F:Q_p]}$, then there is a canonical isomorphism

$$\text{tr}_{fp,X} : H^{2d+1}_{rig-fp,c}(X, d+1; Q) \cong K.$$ 

It is given explicitly by mapping $(x, y) \in H^d_{dr,c}(X_K) \otimes H^d_{rig,c}(X_0)$ to $\text{tr}_{dr,X}(x) - Q(\frac{x}{y})^{-1} \text{tr}_{X,Y}(y)$.

Remark 9.2.15. The factor $Q(\varphi/p)$ is included to make the isomorphism compatible with change of $Q$.

Corollary 9.2.16. Assume that $(P \times Q)(\zeta/p) \neq 0$ for all $\zeta \in \mu_{[F:Q_p]}$. When $i + j = 2d + 1$, $\mathcal{G} = \mathcal{F}^\vee$ and $r + s = d + 1$, then we get a pairing denoted $(\cdot, \cdot)_{rig-fp,X}$. The restriction map and the extentension-by-$0$ are adjoint with respect to this pairing.

9.2.3. Gros fp-cohomology. In Section 12.2 we will need a variant of rigid fp-cohomology which is less refined, but more convenient for computations. Recall that we have

$$R\Gamma_{rig}(X(\pm D_0), \mathcal{F}_{rig})_K = R\Gamma(\mathcal{X}, \mathcal{F}_{rig} \otimes \Omega^*_\mathcal{X}/K(\pm D_K)) = R\Gamma(\mathcal{X}, \mathcal{F}_{dr}|_{\mathcal{X}} \otimes \Omega^*_\mathcal{X}/K(\pm D_K)),$$

where $\mathcal{X}$ is the tube of $X_0$ in $X_K^{rig}$.

Definition 9.2.17. For $r \geq 0$, we define the truncated rigid cohomology, denoted by $R\Gamma_{dr}(X(\pm D), \mathcal{F}, r)$, to be the cohomology of the subcomplex $(\text{Fil}^{r-}\mathcal{F}_{dr})|_{\mathcal{X}} \otimes \Omega^*_\mathcal{X}/K(\pm D)$, and similarly with compact support.

Note 9.2.18. We obtain “filtered” specialisation and co特殊isation maps

$$\text{sp} : \text{Fil}^r R\Gamma_{dr}(X_K(\pm D_K), \mathcal{F}_{dr}) \to R\Gamma_{dr}(X(\pm D), \mathcal{F}, r)_K,$$

$$\text{cosp} : R\Gamma_{dr,c}(\mathcal{X}(\pm D), \mathcal{F}, r) \to \text{Fil}^r R\Gamma_{dr,c}(X_K(\pm D_K), \mathcal{F}_{dr}),$$

compatible with the usual specialisation and co特殊isation maps on the non-filtered complexes.

Remark 9.2.19. The inclusion of $(\text{Fil}^{r-}\mathcal{F}_{dr})|_{\mathcal{X}} \otimes \Omega^*_\mathcal{X}/K(\pm D)$ into the full de Rham complex gives maps

$$\iota : R\Gamma_{dr}(X(\pm D), \mathcal{F}, r) \to R\Gamma_{rig}(X_0(\pm D_0), \mathcal{F}),$$

and similarly with compact support; but the maps induced by $\iota$ on cohomology are not necessarily either injective or surjective, and the groups $\tilde{H}^r_{dr}$ and $\tilde{H}^r_{dr,c}$ may not even be finite-dimensional over $K$.

Definition 9.2.20.

(a) Define the Gros fp-cohomology of $X(\pm D)$ with coefficients $\mathcal{F}$, twist $r$ and polynomial $P$ to be the cohomology of the complex $R\Gamma_{rig-fp}(X(\pm D), \mathcal{F}, r; P)$ which is the homotopy limit of the diagram

$$R\Gamma_{dr}(\mathcal{X}(\pm D), \mathcal{F}, r)_K \xleftarrow{\iota} R\Gamma_{rig}(X_0(\pm D_0), \mathcal{F}_{rig}) \xrightarrow{P(\varphi)} R\Gamma_{rig}(X_0(\pm D_0), \mathcal{F}_{rig})$$

where the unlabelled arrow is base-extension.

(b) Similarly, define the Gros fp-cohomology with compact support of $X(\mp D)$ with coefficients $\mathcal{G}$, twist $s$ and polynomial $Q$ to be the homotopy limit $R\Gamma_{rig-fp,c}(X(\mp D), \mathcal{G}_{rig}, s; P)$ of the diagram

$$R\Gamma_{dr,c}(\mathcal{X}(\mp D), \mathcal{G}, s)_K \xleftarrow{\iota} R\Gamma_{rig,c}(X_0(\mp D_0), \mathcal{G}_{rig}) \xrightarrow{Q(\varphi)} R\Gamma_{rig,c}(X_0(\mp D_0), \mathcal{G}_{rig}).$$

Note 9.2.21. As before, if $F = K$ then the middle arrow is the identity map and both diagrams can be simplified to mapping fibres: in this case we have

$$R\Gamma_{rig-fp}(X(\pm D), \mathcal{F}, r; P) = MF \left[ R\Gamma_{dr}(\mathcal{X}(\pm D), \mathcal{F}_{dr}, r)_K \xrightarrow{P(\varphi)} R\Gamma_{rig}(X_0(\pm D_0), \mathcal{F}_{rig})_K \right],$$

$$R\Gamma_{rig-fp,c}(X(\mp D), \mathcal{G}, s; Q) = MF \left[ R\Gamma_{dr,c}(\mathcal{X}(\mp D), \mathcal{G}_{dr}, s)_K \xrightarrow{Q(\varphi)} R\Gamma_{rig,c}(X_0(\mp D_0), \mathcal{G}_{rig})_K \right].$$

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Remark 9.2.22. Comparing the diagrams of Definition 9.2.20 with (9.5) and (9.6), we see that the filtered specialisation map (c.f. Note 9.2.18) on the de Rham cohomology gives a map
\[ \gamma^* : \Omega^{\gamma}_{\text{rig-fp}}(X, F_{\pm D}, r; P) \to \Omega^{\gamma}_{\text{rig-fp}}(X, \mathcal{F}, r; P). \]
Similarly, the filtered cospecialisation induces a map
\[ \gamma_* : \Omega^{\gamma}_{\text{rig-fp}, c}(X_{\pm D}, G, s; Q) \to \Omega^{\gamma}_{\text{rig-fp}, c}(X, F \otimes G, r + s; P \star Q). \]
We also have cup-products
\[ \Omega^{\gamma}_{\text{rig-fp}}(X, F, r; P) \times \Omega^{\gamma}_{\text{rig-fp}, c}(X_{\pm D}, G, s; Q) \to \Omega^{\gamma}_{\text{rig-fp}, c}(X, F \otimes G, r + s; P \star Q), \]
related to those in the (non-Gros) rigid fp-cohomology (c.f. Proposition 9.2.13) by the adjunction formula
\[ \gamma_*(\gamma^*(x) \cup y) = x \cup \gamma_*(y). \]
In particular, \( \gamma^* \) and \( \gamma_* \) are transposes of each other with respect to the pairing induced by the trace map on the degree 2\( d + 1 \) cohomology. Moreover, the pairing is compatible with the maps in Note 9.2.21.

Notation 9.2.23. We denote the pairing by \( \langle \cdot, \cdot \rangle_{\text{rig-fp}, X} \).

9.2.4. Partial compact support. We can similarly define Gros fp-cohomology with partial compact support: let \( \mathfrak{M} \) be a formal \( \mathcal{O}_K \)-scheme with special fibre \( \mathfrak{M}_0 \), and let we are given varieties \( U, V, W \subseteq \mathfrak{M}_0 \) as in Section 8.3, with \( U \) open in \( \mathfrak{M}_0 \) and \( \mathfrak{M}_K \) smooth. For \( s \geq 0 \), denote by \( \Omega^{\gamma}_{\text{rig-fp}, c}(U, G, s) \) the hypercohomology of the \( s \)-th filtration subcomplex of the de Rham complex of \( \mathcal{G} \).

Definition 9.2.24. Let \( Q \in \mathbb{Q}_p[t] \) be a polynomial with constant coefficient 1. Define the Gros fp-cohomology of \( U \) with compact support towards \( W \), coefficients \( \mathcal{G} \), twist \( s \) and polynomial \( Q \) as the mapping fibre
\[ \Omega^{\gamma}_{\text{rig-fp}, c}(U, G, s; Q) = \text{MF} \left[ \Omega^{\gamma}_{\text{rig-fp}, c}(U, G, s) \xrightarrow{Q(\varphi) \circ \iota} \Omega^{\gamma}_{\text{rig-fp}, c}(U, G) \right], \]
where \( \iota \) denotes the natural map
\[ \Omega^{\gamma}_{\text{rig-fp}, c}(U, G, s) \longrightarrow \Omega^{\gamma}_{\text{rig-fp}, c}(U, G). \]
10. Geometry of Siegel threefolds

10.1. The Klingen-level Siegel threefold.

Definition 10.1.1. Let $Y_{G,Kl}$ be the canonical $\mathbb{Z}_p$-model of the Siegel 3-fold of level $K^p \times Kl(p)$, for some (sufficiently small) tame level $K^p$. Let $X_{G,Kl}$ be a toroidal compactification of $Y_{Kl}$ (with some suitably chosen cone-decomposition $\Sigma$), and $X_{G,Kl}^{min}$ the minimal compactification. Write $D$ for the boundary divisor of the toroidal compactification.

We similarly write $Y_G$ for the canonical $\mathbb{Z}_p$-model of the Siegel 3-fold of level $K^p \times G(\mathbb{Z}_p)$, and $X_G$ for its toroidal compactification.

Remark 10.1.2. The scheme $Y_{G,Kl}$ has an interpretation as a moduli space of abelian surfaces with some prime-to-$p$ level structure and a subgroup $C$ of order $p$. The moduli interpretation of $X_{G,Kl}$ parametrises semiabelian schemes with some appropriate degeneration data at the boundary (depending on $\Sigma$).

Theorem 10.1.3. The scheme $Y_{G,Kl}$ is semistable over $\mathbb{Z}_p$. Its special fibre can be written as a disjoint union

$Y_{G,Kl,0} = Y_{G,Kl,0}^m \sqcup Y_{G,Kl,0}^e \sqcup Y_{G,Kl,0}^p$

of loci where $C$ is étale-loCALLY isomorphic to either $\mu_p$, the constant group scheme $\mathbb{Z}/p$, or $\alpha_p$ respectively.

The space $Y_{G,Kl,0}^m$ is 2-dimensional, and coincides with the non-smooth locus of $Y_{G,Kl,0}$. Both $Y_{G,Kl,0}^m$ and $Y_{G,Kl,0}^e$ are open and smooth, and the closure of either of these strata is its union with $Y_{G,Kl,0}^p$.

Proof. See Theorem 3 of [Tilouine:2017]. (Note that our notations are slightly different from Tilouine’s: he uses the notation $X_P(p)^m$ for the closure $Y_{G,Kl,0}^m$ of the multiplicative locus, rather than the multiplicative locus alone, and similarly for $X_P(p)^e$.)

The proof of the following result was suggested to us by Kai-Wen Lan:

Proposition 10.1.4. The pair $(X_{G,Kl}, Y_{G,Kl})$ is strictly semistable with boundary in the sense of [Eyiyama:2018].

Proof. We need to check that the union of $X_{G,Kl} - Y_{G,Kl}$ and $X_{G,Kl,0}$ is a strict normal crossing divisor. By [Liu-Shen:2018] Corollary 2.1.7, the toroidal compactification is étale locally a fibre product of an affine toroidal embedding and some scheme. This shows that $X_{G,Kl}$ is semistable when $Y_{G,Kl}$ is. It remains to show that $X_{G,Kl}$ is strictly semistable, not just semistable.

Let $\{Z_i\}_{i \in I}$ denote the irreducible components of $Y_{G,Kl,0}$. Let $W_i$ denote the closure of $Z_i$ in $X_{G,Kl}$ for each $i \in I$. For each $J \subseteq I$, we write $Z_J = \bigcap_{i \in J} Z_i$. Since $Y_{G,Kl}$ is strict semistable, we know that $Z_J$ is regular, for each $J \subseteq I$.

It follows from [Liu-Shen:2018] Proposition 2.3.12 that for each $i \in I$, $Z_i$ is a well-positioned subscheme of $Y_{G,Kl,0}$, and its closure $W_i$ in $X_{G,Kl}$ is an irreducible component of the special fibre $X_{G,Kl,0}$ of $X_{G,Kl}$. Let $W_J = \bigcap_{i \in J} W_i$, for each $J \subseteq I$.

By [Liu-Shen:2018] Lemma 2.2.3, $Z_J$ is also well-positioned, for each $J \subseteq I$. By Corollary 2.1.7 again, $W_J$ is also the closure of $Z_J$ in $X_{G,Kl}$, which is the toroidal compactification of $Z_J$ and satisfies Theorem 2.3.2(7) in op.cit., in particular. Then the same argument as for Proposition 2.3.13 in op.cit. shows that (when $\Sigma$ is smooth) all boundary components of $W_J$ are regular as $Z_J$ is.

Since these boundary components are exactly the intersections of $W_J$ with (multiple) irreducible components of the boundary of $X_{G,Kl}$, it follows that the union of $X_{G,Kl} - Y_{G,Kl}$ and the special fibre of $Y_{Kl}$ is a strict semistable divisor, as desired (c.f Stacks Project Tag 0BLA).

Notation 10.1.5.
- For $\ast \in \{m,e\}$, write $X_{G,Kl,0}^{\ast}$ for the closure of $Y_{G,Kl,0}^{\ast}$ in $X_{Kl}$.
- Let $X_{G,Kl,0} = X_{G,Kl,0}^m \cap X_{G,Kl,0}^e$.

Definition 10.1.6. We consider the following closed subschemes of the multiplicative component $X_{G,Kl,0}^m$.
- the locus $X_a = X_{G,Kl,0}^m$ just mentioned, where $C$ is $\alpha_p$;
- the locus $X_b$ where $C/b/C$ is bi-connected;
- the boundary $\partial X$ (the intersection of $X^m$ with the toroidal boundary divisor).

Notation 10.1.7. Following [Pilloni:2017] §9–10, let us write

$X_{G,Kl}^{= 2} := X^m - X^a - X^b$, $X_{G,Kl}^{= 1} := X^b - (X^a \cap X^b)$, $X_{G,Kl}^{= 0} := X^a$.

The notations $X_{G,Kl}^{\geq r}$ and $X_{G,Kl}^{< r}$ have the obvious meanings.
Warning. These subsets do not define a stratification (since $X_{G,Kl}^{=1}$ is not dense in $X_{G,Kl}$), so the notation is slightly abusive, but it is convenient.

**Notation 10.1.8.** We shall more often denote the open subvariety $X_{G,Kl}^{=2}$ by $X_{G,Kl}^{ord}$, since it is the ordinary locus of $X^m$ (although not of $X_{G,Kl}$). As before, we use calligraphic letters for the tubes of these subvarieties considered as dagger spaces.

**Notation 10.1.9.** If $E$ is a coherent sheaf on $X_{G,Kl}^{=1}$, write $\Gamma_{c,0} \left( X_{G,Kl}^{ord}, E \right)$ for the cohomology of $X_{G,Kl}^{ord}$ with coefficients in $E$, compactly supported away from $X_{G,Kl}^{=1}$.

**Note 10.1.10.** By definition, $\Gamma_{c,0} \left( X_{G,Kl}^{=1}, E \right)$ fits into an exact triangle

$$\Gamma_{c} \left( X_{G,Kl}^{=1}, E \right) \to \Gamma_{c} \left( X_{G,Kl}^{=1}, E \right) \to \Gamma_{c,0} \left( X_{G,Kl}^{ord}, E \right) \to [+1].$$

10.2. De Rham cohomology with coefficients. Now let us suppose $V$ is (the canonical extension of) the automorphic vector bundle associated to a $G$-representation $V$, so that $V$ is equipped with a connection $\nabla$ having logarithmic poles along the toroidal boundary divisor $D$.

**Proposition 10.2.1.** For any integer $i$, any $G$-representation $V$, and either choice of signs, there are perfect pairings of finite-dimensional vector spaces

$$H^i_{dR} \left( X_{G,Kl}^{=1}(\pm D), V \right) \times H^{i-1}_{dR,c} \left( X_{G,Kl}^{=1}(\mp D), V \right) \to \mathbb{Q}_p,$$

$$H^i_{dR,c} \left( X_{G,Kl}^{ord}(\pm D), V \right) \times H^{i-1}_{dR,c} \left( X_{G,Kl}^{ord}(\mp D), V \right) \to \mathbb{Q}_p.$$  

Moreover, for these pairings the transpose of the restriction map

$$H^i_{dR,c} \left( X_{G,Kl}^{=1}(\pm D), - \right) \to H^i_{dR,c} \left( X_{G,Kl}^{ord}(\pm D), - \right)$$

is the extension-by-0 map $H^i_{dR,c} \left( X_{G,Kl}^{ord}(\pm D), - \right) \to H_{dR} \left( X_{G,Kl}^{=1}(\pm D), - \right)$.

**Proof.** Let $A$ denote the universal abelian variety over $Y_{G,Kl}$. For any $m \geq 0$, and suitable choices of the toroidal boundary data $\Sigma$, the fibre product $A^m$ can be extended to a smooth $(2m+3)$-dimensional projective variety $K_m$ (the $m$-th Kuga–Sato variety) lying over $X_{G,Kl}$, such that the preimage of the boundary $D = X_{G,Kl} - Y_{G,Kl}$ is a normal-crossing divisor $\partial K_m$. We let $K_m$ denote the associated dagger space.

Then $V \otimes \Omega^m(D)$ is quasi-isomorphic to a direct summand of $R\tau_* \left( \Omega^m_{K_m}(\partial) \right) [2m]$, for some $m$ depending on $V$; and $V \otimes \Omega^m(-D)$ identifies with the dual direct summand of $R\tau_* \left( \Omega^m_{K_m}(-\partial) \right) [2m]$. So the result follows from Proposition 8.5.6 applied to $K_m$. 

**Corollary 10.2.2.**

(a) The groups $H^i_{dR,c} \left( X_{G,Kl}^{ord}(D), V \right)$ and $H^i_{dR,c} \left( X_{G,Kl}^{ord}(-D), V \right)$ are finite-dimensional for all $i$.

(b) The groups $H^i_{dR,c} \left( X_{G,Kl}^{ord}(\partial), V \right)$ and $H^i_{dR,c} \left( X_{G,Kl}^{ord}(\partial), V \right)$ are finite-dimensional for all $i$.

(c) There are perfect pairings

$$H^{i-1}_{dR,c} \left( X_{G,Kl}^{ord}(D), V \right) \times H^i_{dR,c} \left( X_{G,Kl}^{ord}(D), V^{\vee} \right) \to \mathbb{Q}_p$$

and

$$H^{i-1}_{dR,c} \left( X_{G,Kl}^{ord}(D), V \right) \times H^i_{dR,c} \left( X_{G,Kl}^{ord}(D), V^{\vee} \right) \to \mathbb{Q}_p.$$ 

**Proof.** This follows from Proposition 10.2.1 and the long exact sequences relating $c0$ and $c1$ cohomology established above.

We will need to assume the following conjecture, which will be proven in forthcoming work of Lan and Skinner:

**Conjecture 10.2.3** (Eigenspace Vanishing Conjecture). Let $\Pi$ be an automorphic representation contributing to $H_{dR}^2 \left( X_{G,Kl}, [V] \right)$, for some automorphic vector bundle $V$, and suppose that $\Pi_p$ is a generic spherical representation. Write $\{ \Pi \}$ for generalised $\Pi$-eigenspace for the spherical Hecke algebra of $K_p$. 

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(a) We have
\[ H^i_{\text{rig}}(X_{G,\text{KL},0}^m, \mathcal{V})\{\Pi\} = H^i_{\text{rig}}(X_{G,\text{KL},0}^e, \mathcal{V})\{\Pi\} = 0 \]
for all \( i \neq 3 \).

(b) For \( \bullet \in \{m, e\} \), \( H^3_{\text{rig}}(X_{G,\text{KL},0}^\bullet, \mathcal{V})\{\Pi\} \) is 8-dimensional.

(c) If \( X_{G,\text{KL},0}^e \) is any of the closed subvarieties \( X_{G,\text{KL},0}^m \) or \( X_{G,\text{KL},0}^e \) of \( X_{G,\text{KL},0}^m \), then \( H^i_{\text{rig}}(X_{G,\text{KL},0}^e, \mathcal{V})\{\Pi\} \) should vanish in all degrees.

If this conjecture holds, then we obtain the following consequence:

**Proposition 10.2.4.** We have an isomorphism
\[ H^3_{\text{rig}}(X_{G,\text{KL},0}(D_0)/\mathcal{O}_{K}^0, \mathcal{V})\{\Pi\} \cong H^3_{\text{rig}}(X_{G,\text{KL},0}(D_0), \mathcal{V})\{\Pi\} \oplus H^3_{\text{rig}}(X_{G,\text{KL},0}(D_0), \mathcal{V})\{\Pi\}. \]

**Proof.** By [Til06] Theorem 3], \( X_{K,0} \) is the union of the irreducible components \( X_{G,\text{KL},0}^m \) and \( X_{G,\text{KL},0}^e \). We deduce from [GK02] Theorem C that for \( \bullet \in \{m, e\} \), the canonical map
\[ H^3_{\text{DR}}\left(\left|X_{G,\text{KL},0}(D_0)\right|, \mathcal{V}\right) \rightarrow H^3_{\text{DR}}\left(\left|X_{G,\text{KL},0}(D_0)\right|, \mathcal{V}\right) \]
is an isomorphism. We hence deduce from Conjecture [10.2.3 (a)] that \( H^3_{\text{DR}}\left(\left|X_{G,\text{KL},0}(D_0)\right|, \mathcal{V}\right) = 0 \) unless \( i = 3 \).

Now we have the Čech spectral sequence (c.f. [GK07])
\[ E_1^{pq} = H^q_{\text{DR}}(Y^{p+1}(D_0), \mathcal{V}) \Rightarrow H^{p+q}_{\text{DR}}(X_{G,\text{KL}}(D), \mathcal{V}), \]
where
\[ Y^{p+1} = \begin{cases} X_{G,\text{KL},0}^m \cup X_{G,\text{KL},0}^e & \text{if } p = 0, \\ X_{G,\text{KL},0}^a & \text{if } p = 1. \end{cases} \]
We deduce from Conjecture [10.2.3] that
\[ H^3_{\text{DR}}(X_{G,\text{KL}}(D), \mathcal{V})\{\Pi\} \cong H^3_{\text{DR}}(\left|X_{G,\text{KL},0}^m(D_0)\right|, \mathcal{V})\{\Pi\} \oplus H^3_{\text{DR}}(\left|X_{G,\text{KL},0}^e(D_0)\right|, \mathcal{V})\{\Pi\}, \]
from which we deduce the required result. \( \square \)

**10.3. The map \( \iota \).** Let \( Y_H \) be the Shimura variety of \( H \) of level \( K_H^0 \times H(\mathbb{Z}_p) \), where \( K_H^0 = K^p \cap H(A_H^p) \), and write \( X_H \) for its toroidal compactification. We assume that the compactifications of \( Y_H \) and \( Y_G \) are compatible under the natural embedding \( Y_H \to Y_G \) (c.f. [LPSZ19 §2.4.1]).

**Definition 10.3.1.** Let \( X_{H,\Delta} \) be the covering of \( X_H \) parametrising choices of finite flat \( \mathbb{Z}_p \)-subgroup-scheme \( C \subset (E_1 \oplus E_2)[p] \) of order \( p \) which project nontrivially into both \( E_i \).

**Lemma 10.3.2.** There is a finite map \( \iota_\Delta : X_{H,\Delta} \to X_{\text{KL}} \) lying over the map \( X_H \to X_G \), and the preimage of \( X_{G,\text{KL}}^{\geq 1} \) is the open subscheme \( X_{H,\Delta}^{\geq 1} \) where \( C \) is multiplicative.

**Proof.** See [LPSZ19 §4.1]. \( \square \)

**Remark 10.3.3.** The map \( \iota_\Delta \) induces the map defined in Section 7.6 on the generic fibres. \( \diamond \)

We therefore obtain a finite map of dagger spaces \( \iota_\Delta : X_{H,\Delta}^{\text{ord}} \to X_{G,\text{KL}}^{\geq 1} \) (whose image is contained in \( X_{G,\text{KL}}^{\text{ord}} \)).

**Proposition 10.3.4.** For any locally free coherent sheaf \( \mathcal{E} \) on \( X_{G,\text{KL}}^{\geq 1} \), the natural pullback map
\[ \iota_\Delta^*: R\Gamma_c(X_{G,\text{KL}}^{\geq 1}, \mathcal{E}) \to R\Gamma_c(X_{H,\Delta}^{\text{ord}}, \iota_\Delta^* \mathcal{E}) \]
factors through \( R\Gamma_c(X_{K,1}^{\text{ord}}, \mathcal{E}) \).

**Proof.** This is an instance of Proposition 8.3.5. \( \square \)
10.4. The reduction: Step 2.

**Notation 10.4.1.** Write \( \eta_{\text{rig-fp}, -D} \) for the image of \( \eta_{\text{NN-fp}, -D} \) in \( H^3_{\text{rig-fp}, c}(X_{G,k}(-D), V, 1 + q; P) \).

**Proposition 10.4.2.** The pairing \( \langle 7.8 \rangle \) is equal to
\[
\left\langle \text{Eis}_{\text{rig-syn}, \Phi}^{[t_1, t_2]} \left( \iota_{\Delta}^{[t_1, t_2]} \right)^* \left( \eta_{\text{rig-fp}, -D} \right) \right\rangle_{\text{trig-fp}, Y_{\Delta}}.
\]

**Proof.** We have
\[
\left\langle \iota_{\Delta}^{[t_1, t_2]} \left( \text{Eis}_{\text{syn}, \Delta}^{[t_1, t_2]} \right), \eta_{\text{NN-fp}, q, -D} \right\rangle_{\text{NN-fp}, Y_{G, k_i}} = \left\langle \text{Eis}_{\text{syn}, \Delta}^{[t_1, t_2]} \left( \eta_{\text{NN-fp}, q, -D} \right) \right\rangle_{\text{NN-fp}, Y_{H, \Delta}} = \left\langle \iota_{\Delta}^{[t_1, t_2]} \left( \text{Eis}_{\text{syn}, \Delta}^{[t_1, t_2]} \right)^* \left( \eta_{\text{rig-fp}, -D} \right) \right\rangle_{\text{trig-fp}, Y_{\Delta}}.
\]

Here
- the first equality uses adjunction between pushforward and pullback;
- the second one follows from Theorems 9.1.3 and 9.1.10.

We now apply Proposition 9.2.11 to the case where \( X = X_{G,k} \), \( D \) is the toroidal boundary (so \( U = Y_{G,k} \)) and \( Z = X_{G,k}^{G_{k,i}} \). We obtain an extension by zero map
\[
H^3_{\text{rig-fp}, c}(X_{G,k}^{G_{k,i}}(-D), V, n, P) \to H^3_{\text{rig-fp}, c}(X_{G,k}(-D), V, n, P).
\]

**Proposition 10.4.3.** The extension-by-0 map induces an isomorphism on the \( \Pi'_i \)-eigenspace.

**Proof.** Follows from Conjecture 10.2.3.

**Corollary 10.4.4.** There is a unique class
\[
\eta_{\text{rig-fp}, q, -D} \in H^3_{\text{rig-fp}, c}(X_{G,k}^{G_{k,i}}(-D), V, 1 + q, P)[\Pi'_i]
\]
which is in the \( \Pi_i \)-eigenspace for the prime-to-\( p \) Hecke operators and maps to \( \eta_{\text{dir}, q, -D} \) under extension-by-0.

**Notation 10.4.5.** Write \( \text{Eis}_{\text{rig-syn}, \Phi, \text{ord}}^{[t_1, t_2]} \) for the image of \( \text{Eis}_{\text{rig-syn}, \Phi}^{[t_1, t_2]} \) in \( H^2_{\text{rig-fp}, c}(Y_{\text{ord}}, \text{Sym}^{t_1} \mathcal{H} \boxtimes \text{Sym}^{t_2} \mathcal{H}, 2) \) under the restriction map \( \text{res}_{Y_{H, \Delta}} \).

2nd reduction

**Theorem 10.4.6.** We have
\[
\left\langle \text{Eis}_{\text{rig-syn}, \Phi}^{[t_1, t_2]} \left( \iota_{\Delta}^{[t_1, t_2]} \right)^* \left( \eta_{\text{rig-fp}, q, -D} \right) \right\rangle_{\text{rig-fp}, Y_{\Delta}} = \left\langle \text{Eis}_{\text{rig-syn}, \Phi, \text{ord}}^{[t_1, t_2]} \left( \iota_{\Delta}^{[t_1, t_2]} \right)^* \left( \eta_{\text{rig-fp}, q, -D} \right) \right\rangle_{\text{rig-fp}, Y_{\text{ord}}^{\Phi, \text{ord}} \Delta}.
\]

**Proof.** The first equality follows from Corollary 9.2.16. The second equality is immediate from Proposition 10.3.3.
Step 3: Reduction to a pairing in coherent cohomology

11. Coherent cohomology and automorphic forms for \( G \)

11.1. Coefficient sheaves. As in [LPSZ19] §5.2, the pair \((r_1, r_2)\) determines algebraic representations \( L_i \) of the Siegel Levi \( M_{Si} \), for \( 0 \leq i \leq 3 \), all with central character \( \operatorname{diag}(x, \ldots, x) \mapsto x^{r_1+r_2} \); and hence vector bundles \( L_i = [L_i]_{\text{can}} \) on \( X_{K, Q} \) for any sufficiently small level \( K \) (the canonical extensions of the corresponding vector bundles over \( Y_{K, Q} \)).

Notation 11.1.1. For convenience, we re-number these vector bundles by setting \( N^i = L_{3-i} \), and \( N^0 = L_{3} - \), the corresponding vector bundles (so that \( N^0 \) is \( \Omega(D) \) if \( r_1 = r_2 = 0 \)).

Note 11.1.2. The cohomology of these bundles, and their subcanonical analogues \( [N^i]_{\text{sub}} = N^i(-D) \), is canonically independent of the toroidal boundary data, and hence the direct limits

\[
\lim_K H^\ast(X_{K, Q}, N^i) \quad \text{and} \quad \lim_K H^\ast(X_{K, Q}, N^i(-D))
\]

are (left) \( G(A_f) \)-representations. Our normalisations are such that an element \( \operatorname{diag}(x, \ldots, x) \in Z_G(A_f) \) with \( x \in Q_{>0} \) acts on these as multiplication by \( x^{r_1+r_2} \). We know that for each \( 0 \leq i \leq 3 \), the \( \operatorname{GSp}_4(A_f) \)-representation \( \lim_K H^{3-i}(X_{K, Q}, N^i) \otimes Q_p \) and its cuspidal counterpart both contain a unique direct summand isomorphic to \( \Pi_i \); if \( j \neq 3-i \), then the \( \Pi_j \)-generalised eigenspaces for the spherical Hecke operators in \( H^i(X_{K, Q}, N^i) \) and \( H^i(X_{K, Q}, N^i(-D)) \) are zero.

11.2. Classical Klingen-level Hecke operators. Taking the level at \( p \) to be the Klingen parahoric \( \operatorname{Kl}(p) \), we obtain an action of the local Hecke algebra \( Z[G(Q_p) \neq \kappa(p)] \) on the cohomology of the sheaves \( N^i \).

Definition 11.2.1. We define the following operators:

\[
U_{Kl,0} = p^{-(r_1+r_2)} \cdot [\operatorname{Kl}(p) \operatorname{diag}(p, p, p, p) \operatorname{Kl}(p)] \\
U_{Kl,1} = [\operatorname{Kl}(p) \operatorname{diag}(p, p, 1, 1) \operatorname{Kl}(p)] \\
U_{Kl,2} = p^{r_2} \cdot [\operatorname{Kl}(p) \operatorname{diag}(p^2, p, p, 1) \operatorname{Kl}(p)]
\]

Remark 11.2.2. The powers of \( p \) are chosen so that these operators are minimally integrally normalised – that is, they are minimal such that the relevant cohomological correspondences extend to the integral model of \( X_{Kl} \) over \( Z_p \). In particular, all their eigenvalues are \( p \)-adically integral, which recovers the estimates on the valuations of the Hecke parameters of \( \Pi \) given above. Of course, the eigenvalues of \( U_{Kl,0} \) are roots of unity, and we shall generally use the more familiar alternative notation \( \langle p \rangle \) for \( U_{Kl,0} \).

Note 11.2.3. The operators \( \{ \langle p \rangle, U_{Kl,1}, U_{Kl,2} \} \) generate a commutative subalgebra of the Hecke algebra, and \( \{ \langle p \rangle, U_{Kl,1}', U_{Kl,2}' \} \) generate another commutative subalgebra. Moreover, Serre duality interchanges these two subalgebras: more precisely, the transpose with respect to Serre duality of \( U_{Kl,1} \) is \( \langle p \rangle^{-1} U_{Kl,1}' \), and the transpose of \( U_{Kl,2} \) is \( \langle p \rangle^{-2} U_{Kl,2}' \).

11.3. Restriction to the rank \( \geq 1 \) locus. Recall that \( X_{Kl} \) parametrises pairs \( (A, C) \), where \( A \) is a semi-abelian surface with some prime-to-\( p \) level structure and degeneration data at the cusps, and \( C \) is a cyclic subgroup of order \( p \). The Fargues degree \( \deg C \) is thus a function

\[
\deg : X_{Kl}(C_p) \rightarrow [0, 1],
\]

with degree 1 corresponding to the locus where \( C \) is multiplicative.

The images of \((A, C)\) under the correspondence \( U_{Kl,r}'' \), for \( r = 1, 2 \), correspond to pairs \((A', C')\), where \( \phi : A \rightarrow A' \) is an isogeny (of some specific type depending on \( r \)) whose kernel contains \( C \), and \( C' \) is a cyclic subgroup of \( A'[p] \) such that \( \phi'(C') = C \). This implies that \( \deg C' \leq \deg C \); so \( U_{Kl,r}'' \) restricts to a correspondence \( X_{G, Kl}(C_p)[0, 1] \rightarrow X_{G, Kl}(C_p)[0, 1] \). This implies that there is a well-defined action of \( U_{Kl,r}'' \)
on the compactly-supported cohomology $R\Gamma_c(X_{G,Kl}^{≥1},\mathcal{N}^i)$ for any $i$, compatible with the extension-by-zero map to $R\Gamma(X_{G,Kl},\mathcal{N}^i)$.

**Proposition 11.3.1.** All slopes of $U'_{Kl,2}$ on $R\Gamma_c(X_{G,Kl}^{≥1},\mathcal{N}^i(-D))$ are $≥ (r_1 - r_2 + 1)$ if $i = 2$, and $≥ (r_1 + r_2 + 3)$ if $i = 3$.

**Proof.** See [Pil17].

11.4. The ordinary locus and the operator $Z'$. Inside $X_{G,Kl}^{ord}$ we have the ordinary locus $X_{G,Kl}^{ord}$ parametrising $(A,C)$ where $A$ is ordinary and $C$ multiplicative. The correspondences $U'_{Kl,1}$ and $U'_{Kl,2}$ described above both act on $R\Gamma_c(X_{G,Kl}^{ord},\mathcal{N}^i)$, since ordinarity is an isogeny invariant. However, over the ordinary locus there is an additional structure: we have a decomposition

$$U'_{Kl,1} = Z' + \Phi$$

as a sum of two simpler correspondences:

- The correspondence $\Phi$ is actually a morphism: it is the map $(A,C) \mapsto (A,\hat{A}[p],C')$ mod $\hat{A}[p]$] where $\hat{A}$ is the formal group of $A$, and $C'$ is the unique subgroup of $A[p^2]$ such that $pC' = C$. This is a lifting of the Frobenius map on the special fibre.

- The correspondence $Z'$ parametrises isogenies $(A,C) \mapsto (A/J, C')$, where $J \cap \hat{A}[p] = C$, and $C'$ is the unique multiplicative subgroup of $A'$ whose image under the dual isogeny is $C$.

These are related to classical correspondences at Iwahori level (since we can also see $X_{G,Kl}^{ord}$ as a dagger subvariety of the Iwahori-level Shimura variety, via the canonical-subgroup map): in the Iwahori-level Hecke algebra, $Z'$ corresponds to $\text{diag}(1,p,1,p)$, and $\Phi$ to $(1,1,p,p)$.

**Remark 11.4.1.** For the sheaf $\mathcal{N}^i$, this is the minimal integral normalisation of $Z'$ (but this is no longer the case on $\mathcal{N}^i$ for $i \neq 1$). We have not attempted to give an optimal normalisation for the operator $\Phi$, since this will not play such a major role in our theory.

**Note 11.4.2.** If $U'_{1w,2}$ denotes the restriction of $U'_{Kl,2}$, then this operator commutes with both $Z'$ and $\Phi$, and we have the identity

$$Z' \circ \Phi = p^{1/r_2+1}U'_{1w,2}.$$ 

**Convention.** Since the operators $U'_{Kl,1}$ and $U'_{1w,2}$ are compatible under pullback, there seems to be no harm in dropping the subscript and using the notation $U'_{X}$ for both.

11.5. Duality and vanishing for coherent cohomology.

**Proposition 11.5.1.** Both $X_{G,Kl}^{ord}$ and $X_{G,Kl}^{min}$ are smooth, and their images in the minimal compactification $X_{G,Kl}^{min}$ are affine.

**Proof.** The smoothness of $X_{G,Kl,0}^{ord}$ is immediate from that of $X_{G,Kl,0}^{min}$. It is easily seen that the space $X_{G,Kl,0}^{ord}$ maps isomorphically to the $p$-rank 1 locus in prime-to-$p$ level (cf. proof of Lemma 10.5.2.2 in [Pil17]), and the smoothness of this image is established in the course of the proof of Lemma 6.4.2 of op.cit.. For the second statement, see the proof of Theorem 11.2.1 of [Pil17].

**Notation 11.5.2.** Let $\pi : X_{G,Kl} \to X_{G,Kl}^{min}$ be the projection map. For the rest of this section, let $\mathcal{E} = [W]$ be the canonical extension to $X_{G,Kl}$ of an automorphic vector bundle attached to a $P_{SI}$-representation $W$, and $\mathcal{E}' = [W' \otimes L(3,3,0)]$, so that the Serre dual of $\mathcal{E}$ is $\mathcal{E}'(-D)$ and vice versa.

**Proposition 11.5.3.** Let $U \subseteq X_{G,Kl}^{min}$ be affinoid, and let $\bar{U} = \pi^{-1}(U) \subseteq X_{G,Kl}$. Then

(a) We have $H^i(\bar{U},\mathcal{E}(-D)) = 0$ for $i \neq 0$.

(b) We have $H^2(\bar{U},\mathcal{E}') = 0$ for $i \neq 3$.

(c) There is a perfect pairing of Hausdorff locally convex spaces

$$H^0(\bar{U},\mathcal{E}(-D)) \times H^2(\bar{U},\mathcal{E}') \to \mathbb{Q}_p.$$ 

**Proof.** Note that if $\bar{U}$ is affinoid, then this is an instance of the Serre duality results for affinoid dagger spaces proved in [GK98 §7.2]. So we shall aim to reduce to this case, using the fact that $R^i\pi_*(\mathcal{E}(-D)) = 0$ for all $i > 0$ by [Lan17, Theorem 8.6].
For part (a), Lan’s vanishing result shows that \( H^i(\bar{U}, E(-D)) = H^i(U, \pi_*(E(-D))) \). Since \( U \) is affinoid and \( \pi_*(E(-D)) \) is coherent, this vanishes for all \( i > 0 \) as required.

For parts (b) and (c), we argue as follows. Since \( \bar{U} \) is smooth and \( E' \) locally free, we have

\[
\text{RHom}_{\mathcal{O}(\bar{U})}(E', \omega_{\bar{U}}) = \text{Hom}_{\mathcal{O}(\bar{U})}(E', \omega_{\bar{U}}[3]) = E(-D)[3].
\]

So we have

\[
H^i_0(\bar{U}, E') = H^i_0(U, R\pi_*(E')) = H^{i-3}(U, R\text{Hom}(R\pi_*(E(-D)), \omega_U)) = H^{i-3}(U, R\text{Hom}(\pi_*(E(-D)), \omega_U)) = \text{Hom}_{cts}(H^{3-i}(U, \pi_*(E(-D))), Q_p) = \text{Hom}_{cts}\left(H^{3-i}(U, E(D)), Q_p\right).
\]

This clearly implies (c), and (b) follows from this together with (a).

\[
\text{Corollary 11.5.4. For } E \text{ as above, we have } H^i(X_{G, Kl}^{\geq 1}, E(-D)) = 0 \text{ for } i \notin \{0, 1\}, \text{ and } H^i_0(X_{G, Kl}^{\geq 1}, E) = 0 \text{ for } i \notin \{2, 3\}.
\]

\[
\text{Proof}. \text{ There are two affinoids } U_1, U_2 \text{ in } X_{G, Kl}^{\min} \text{ such that } \pi^{-1}(U_1) \text{ and } \pi^{-1}(U_2) \text{ cover } X_{G, Kl}^{\geq 1}. \text{ By the previous proposition, we see that } H^\bullet(X_{G, Kl}^{\geq 1}, E(-D)) \text{ is computed by a } \check{C}\text{ech complex concentrated in degrees } 0 \text{ and } 1. \text{ Similarly, the compactly-supported cohomology is supported by a “homological” } \check{C}\text{ech complex concentrated in degrees } 2 \text{ and } 3.
\]

\[
\text{Corollary 11.5.5. For } E' \text{ as above, we have } H^i_0(X_{G, Kl}^{\geq 1}, E') = 0 \text{ unless } i \in \{2, 3\}.
\]

\[
\text{Proof}. \text{ By definition, we have an exact triangle}
\]

\[
R\Gamma_! (X_{G, Kl}^{\geq 1}, E') \rightarrow R\Gamma_! (X_{G, Kl}^{\geq 1}, E') \rightarrow R\Gamma_0(X_{G, Kl}^{\geq 1}, E') \rightarrow [1].
\]

We claim that \( H^2_0(X_{G, Kl}^{\geq 1}, E') \) is concentrated in degree 3. The image of \( X_{G, Kl}^{\geq 1} \) in the minimal compactification is the locus where the Hasse invariant has positive valuation. It is thus naturally covered by an increasing sequence of affinoids \( U_i \) (given by requiring the valuation of a lift of Hasse to be \( \geq r_i \), for some sequence of positive rationals \( r_i \rightarrow 0 \)), and \( R\Gamma_! (X_{G, Kl}^{\geq 1}, E') = \varinjlim \text{R}_U R\Gamma_! (\pi^{-1}(U_i), E') \), which vanishes outside degree 3 by the proposition above. It now follows from the mapping triangle that \( R\Gamma_0 \) is supported in degrees \( \{2, 3\} \).

\[
\text{Remark 11.5.6. It seems highly likely that } R\Gamma_0(X_{G, Kl}^{\geq 1}, E') \text{ vanishes in degree } 3 \text{ as well, but this is not easy to check. It is equivalent to showing that } H^2_0(X_{G, Kl}^{\geq 1}, E') \text{ is surjective. If we knew that Serre duality held for } X_{G, Kl}^{\geq 1} \text{ this would be obvious, since the dual map } H^0(X_{G, Kl}^{\geq 1}, E(-D)) \rightarrow H^0(X_{G, Kl}^{\geq 1}, E(-D)) \text{ is clearly injective; but we do not know this, since neither } X_{G, Kl}^{\geq 1} \text{ nor its image in } X_{G, Kl}^{\min} \text{ is affinoid.}
\]

11.6. Coherent \( H^2 \) eigenclasses from \( II \). The input we need from higher Coleman theory is the following. We fix an automorphic representation \( II \) which is cohomological with coefficients in \( V(r_1, r_2; r_1 + r_2) \), and unramified and Klingen-ordinary at \( p \), as before; and we choose a vector \( \eta^{\text{alg}}_{D} \in H^2(X_{G, Kl}, \mathcal{A}^3(-D)) \) which is stable under \( K(p) \) and lies in the ordinary eigenspace for \( U_2' \).

\[
\text{Remark 11.6.1. If } (\alpha, \beta, \gamma, \delta) \text{ are the Hecke parameters of } II_\rho \text{, ordered such that } v_p(\alpha) \leq \ldots \leq v_p(\delta) \text{ and normalised such that } v_p(\alpha) \geq 0, v_p(\alpha\beta) \geq r_2 + 1, \text{ then the ordinarity condition is that } v_p(\alpha\beta) \text{ should be exactly } r_2 + 1, \text{ and the ordinary } U_2' \text{ eigenvalue is the } p\text{-adic unit } \lambda = \frac{\alpha\beta}{p^{r_2+1}}.
\]

\[
\text{Note 11.6.2. The operator } U_{G, 1} \text{ acts on } \eta \text{ as multiplication by } \alpha + \beta \text{ (which may or may not be a } p\text{-adic unit).}
\]

\[
\text{Proposition 11.6.3. There exists a unique class } \eta_{\text{coh., } -D}^{\geq 1} \in H^2_0(X_{G, Kl}, \mathcal{A}^3(-D)) \text{ with the following two properties:}
\]

1. \( U_{G, 2} \) acts on \( \eta_{\text{coh., } -D}^{\geq 1} \) as multiplication by \( \frac{\alpha\beta}{p^{r_2+1}} \).
2. The image of \( \eta_{\text{coh., } -D}^{\geq 1} \) under the extension-by-zero map is \( \eta^{\text{alg}}_{D} \).

This class enjoys the following additional properties:
(3) The operator $U'_{K_{1,1}}$ acts on $\eta^{\geq 1}_{\text{coh}, -D}$ as multiplication by $\alpha + \beta$.
(4) The spherical Hecke algebra acts via the system of eigenvalues associated to $\Pi'$.

**Proof.** Result from [Pil20]. \hfill \Box

**Definition 11.6.4.** Let $\eta^{\geq 1}_{\text{coh}}$ be the image of $\eta^{\geq 1}_{\text{coh}, -D}$ under the natural map

$$H^c_2 \left( \mathcal{X}^{\geq 1}_{G, K_{1,1}}, L_2(-D) \right) \to H^2 \left( \mathcal{X}^{\geq 1}_{G, K_{1,1}}, L_2 \right).$$

This enjoys analogues of properties (1)--(4) (mutatis mutandis).

**Definition 11.6.5.** Let $\eta^{\text{ord}}_{\text{coh}, -D}$ be the image of $\eta^{\geq 1}_{\text{coh}, -D}$ under the restriction map to $H^2_{\text{id}} \left( \mathcal{X}^{\geq 1}_{G, K_{1,1}}, N^1(-D) \right)$.

For future use, we define $\eta^{\text{ord}}_{\text{coh}}$ to be the image of $\eta^{\text{ord}}_{\text{coh}, -D}$ in $H^2_{\text{id}} \left( \mathcal{X}^{\geq 1}_{G, K_{1,1}}, N^1 \right)$.

**Proposition 11.6.6.** The class $P(\Phi) \cdot \eta^{\text{ord}}_{\text{coh}, -D}$ lies in the kernel of $Z'$, where $P(T)$ denotes the quadratic polynomial $(1 - \frac{T}{\alpha})(1 - \frac{T}{\beta})$.

**Proof.** We know that $\eta^{\text{ord}}_{\text{coh}, -D}$ is an eigenvector for $U'_{\beta}$ with eigenvalue $\alpha \beta / p^{r_2 + 1}$, and for $Z' + \Phi$ with eigenvalue $\alpha + \beta$. Using the identity (II.1) the result follows formally. \hfill \Box

**Remark 11.6.7.** This result is somewhat analogous to a step in the proof of the regulator formula for Rankin–Selberg convolutions, where one shows that the image of an eigenform under a suitable quadratic polynomial in Frobenius is $p$-depleted, i.e. in the kernel of the operator $U_p$. As we shall see later, the kernel of $Z'$ is the appropriate analogue in our context of $p$-depletedness. \hfill \diamond

---

**12. fp-cohomology and coherent fp-pairs for $G$**

**12.1. The dual BGG complex.**

**Definition 12.1.1.** Define the dual BGG complex associated to $\mathcal{V}$ to be

$$\text{BGG}(\mathcal{V}) : N^0 \xrightarrow{\nabla^0} N^1 \xrightarrow{\nabla^1} N^2 \xrightarrow{\nabla^2} N^3,$$

where the differentials are given by certain homogeneous differential operators of degrees $r_2 + 1$, $r_1 - r_2 + 1$ and $r_2 + 1$, respectively (c.f. [Til12] §7).

We equip it with the following filtration:

$$\mathcal{F}il^n \text{BGG}(\mathcal{V}) = \begin{cases} N^0 \to N^1 \to N^2 \to N^3 & \text{if } n \leq 0 \\
0 \to N^1 \to N^2 \to N^3 & \text{if } 1 \leq n \leq r_2 + 1 \\
0 \to 0 \to N^2 \to N^3 & \text{if } r_2 + 2 \leq n \leq r_1 + 2 \\
0 \to 0 \to 0 \to N^3 & \text{if } r_1 + 3 \leq n \leq r_1 + r_2 + 3 \\
 & \text{if } r_1 + r_2 + 4 \leq n. \end{cases}$$

We define $\text{BGG}_c(\mathcal{V})$ to be the subcomplex with $N^3$ replaced by $N^3(-D)$.

**Proposition 12.1.2.** The dual BGG complex $\text{BGG}(\mathcal{V})$ is a direct summand of the logarithmic de Rham complex $\text{DR}(\mathcal{V}) \cong \mathcal{V} \otimes \Omega^\bullet(D)$ (in the category of abelian sheaves over $\mathcal{X}_{G, K_{1,1}} \textbf{Q}_p$). The inclusion is Hecke equivariant, and the projection map is a quasi-isomorphism of filtered complexes. The same holds for $\text{BGG}_c(\mathcal{V})$ and $\text{DR}_c(\mathcal{V})$.

**Proof.** See [Til12] §7 for the statement for $\text{BGG}(\mathcal{V})$. For the version with compact support, see [LP18] §5.4. \hfill \Box

**Definition 12.1.3.** For $i, j \geq 0$, $n \in \mathbb{Z}$ we define

$$\mathcal{E}^{i,j}_{c} \left( \mathcal{X}^{\geq 1}_{G, K_{1,1}}, \text{BGG}_c(\mathcal{V}), n \right) = H^c_i \left( \mathcal{X}^{\geq 1}_{G, K_{1,1}}, \mathcal{F}il^n \mathcal{N}^j(-D) \right), \quad \mathcal{E}^{i,j}_{\text{ord}} \left( \mathcal{X}^{\geq 1}_{G, K_{1,1}}, \text{BGG}(\mathcal{V}), n \right) = H^c_i \left( \mathcal{X}^{\geq 1}_{G, K_{1,1}}, \mathcal{F}il^n \mathcal{N}^j \right),$$

$$\mathcal{E}^{i,j}_{c} \left( \mathcal{X}^{\geq 1}_{G, K_{1,1}}, \text{BGG}_c(\mathcal{V}), n \right) = H^c_i \left( \mathcal{X}^{\geq 1}_{G, K_{1,1}}, \mathcal{F}il^n \mathcal{N}^j \right), \quad \mathcal{E}^{i,j}_{\text{ord}} \left( \mathcal{X}^{\geq 1}_{G, K_{1,1}}, \text{BGG}(\mathcal{V}), n \right) = H^c_i \left( \mathcal{X}^{\geq 1}_{G, K_{1,1}}, \mathcal{F}il^n \mathcal{N}^j \right).$$
Note 12.1.4. If \( n \geq 1 \), which is the case which will interest us, we have
\[
\mathcal{C}^{i,j}(X_{G,Kl}^{21}, BGG_{c}(V), n) = \mathcal{C}^{i,j}(X_{G,Kl}^{ord}, BGG_{c}(V), n) = 0
\]
unless \( i \in \{1, 2, 3 \} \) and \( j \in \{2, 3\} \), since \( \mathfrak{d}^{i,n} \mathcal{N}^{i} \) is zero unless \( 1 \leq i \leq 3 \), and the functors \( H^{j}_{c}(X_{G,Kl}^{21}, -) \) and \( H^{j}_{c}(x_{G,Kl}^{ord}, -) \) vanish on canonical vector bundles unless \( j \in \{2, 3\} \) by Corollary 11.5.5.

For \( \mathcal{C}^{i,j}(X_{G,Kl}^{21}, BGG_{c}(V)(-D), n) \) we have a slightly weaker result: the non-zero terms are in degrees \( 1 \leq i \leq 3, 1 \leq j \leq 3 \), since it is obvious that \( H_{c}^{0}(X_{G,Kl}^{21}, -) \) vanishes for any locally free sheaf.

Note 12.1.5. Definition 12.1.3 also makes sense when we replace \( BGG_{c}(V) \) by \( D_{r_{i}}(V) \) and \( \mathcal{N}^{i} \) by \( V \otimes \Omega^{1}_{G} \). By Proposition 12.1.6, we obtain natural maps from the ‘BGG-version’ of the groups to the respective ‘de Rham’ versions.

Proposition 12.1.6. We have first-quadrant cohomological spectral sequences, starting at the \( E_{1} \) page (with differentials on the \( E_{1} \) page given by \( \nabla \)):
\[
\begin{align*}
\mathcal{C}^{i,j}(X_{G,Kl}^{21}, BGG_{c}(V), n) & \Rightarrow \tilde{H}^{i+j}_{dR,c}(X_{G,Kl}^{21}(-D), V, n), \\
\mathcal{C}^{i,j}(X_{G,Kl}^{21}, BGG_{c}(V), n) & \Rightarrow \tilde{H}^{i+j}_{dR,c}(X_{G,Kl}^{21}, V, n), \\
\mathcal{C}^{i,j}(X_{G,Kl}^{ord}, BGG_{c}(V), n) & \Rightarrow \tilde{H}^{i+j}_{dR,c}(X_{G,Kl}^{ord}, V, n),
\end{align*}
\]

which are compatible under the restriction map \( res^{ord} \). If \( n \geq 1 \), all three spectral sequences degenerate at \( E_{3} \). Similarly, for the unfiltered complexes we have Frölicher spectral sequences
\[
\begin{align*}
\mathcal{C}^{i,j}(X_{G,Kl}^{21}, BGG_{c}(V)) & \Rightarrow H^{i+j}_{dR}(X_{G,Kl}^{21}, (-D_{0}), V), \\
\mathcal{C}^{i,j}(X_{G,Kl}^{21}, BGG_{c}(V)) & \Rightarrow H^{i+j}_{dR}(X_{G,Kl}^{21}, V, n), \\
\mathcal{C}^{i,j}(X_{G,Kl}^{ord}, BGG_{c}(V)) & \Rightarrow H^{i+j}_{dR}X_{G,Kl}^{ord}(V).
\end{align*}
\]

Proof. In each case, the spectral sequence arises as one of the spectral sequences associated to a suitable double complex computing \( \tilde{H}^{*} \). The degeneration follows from the fact that the \( E_{1} \) terms are zero outside \( 1 \leq i \leq 3 \).

Notation 12.1.7. We denote the \( E_{2} \) pages of these spectral sequences by \( \mathcal{H}^{i,j}_{c}(\ldots), \) so \( \mathcal{H}^{i,j}_{c}(\ldots) \) is the \( i \)-th cohomology of the complex \( \mathcal{C}^{*}_{c}(\ldots) \).

Corollary 12.1.8. Let \( - \leq q \leq r_{2} \). Then the edge maps at (1, 2) of the spectral sequences (12.2) and (12.3) are isomorphisms
\[
\begin{align*}
\alpha_{G,rig,c} : \tilde{H}^{i}_{dR,c}(X_{G,Kl}^{21}, V, 1 + q) & \cong \mathcal{H}^{i}_{c}(X_{G,Kl}^{21}, BGG_{c}(V), 1 + q) \cong H^{i}_{c}(X_{G,Kl}^{21}, \Lambda^{1}_{c})^{\nabla = 0}, \\
\alpha_{G,rig,c} : \tilde{H}^{i}_{dR,c}(X_{G,Kl}^{21}, V, 1 + q) & \cong \mathcal{H}^{i}_{c}(X_{G,Kl}^{ord}, BGG_{c}(V), 1 + q) \cong H^{i}_{c}(X_{G,Kl}^{ord}, \Lambda^{1}_{c})^{\nabla = 0}.
\end{align*}
\]

The spectral sequence \( \mathcal{C}^{i,j}(X_{G,Kl}^{21}, BGG_{c}(V)(-D), 1 + q) \) gives an exact sequence
\[
\begin{align*}
0 & \rightarrow H_{c}^{1}(X_{G,Kl}^{21}, \Lambda^{2}_{c}(-D))^{\nabla = 0} \rightarrow \tilde{H}^{3}_{dR,c}(X_{G,Kl}^{21}(-D), V, 1 + q) \\
& \quad \quad \quad \quad \quad \rightarrow H_{c}^{2}(X_{G,Kl}^{21}, \Lambda^{1}_{c}(-D))^{\nabla = 0} \rightarrow H_{c}^{1}(X_{G,Kl}^{21}, \Lambda^{3}_{c}(-D))^{\nabla = 0}.
\end{align*}
\]

Proof. For the first two formulae, we know that both of the relevant spectral sequences have \( E_{1}^{i,j} = 0 \) unless \( i \geq 1 \) and \( j \geq 2 \), so \( E_{i}^{3,-i} = 0 \) for \( i \neq 1 \), and \( E_{2}^{12} = E_{1}^{12} = \ker(E_{1}^{12} \rightarrow E_{1}^{22}) \).

For the two spectral sequences over \( X_{G,Kl}^{21} \), the results of Corollary 12.1.8 can be sharpened enormously by taking into account the action of the Hecke operator \( U_{j} \). Recall that the coherent cohomology groups (both with and without \(-D\)) have slope decompositions for the action of \( U_{j} \), so the slope 0 subspace is finite-dimensional and there exists an idempotent projector \( e(U_{j}^{2}) \) projecting onto this subspace.
Moreover, the operator $U'_2$, and hence the slope 0 projector, are compatible with the morphisms in the spectral sequence.

**Proposition 12.1.9.** If $n \geq 1$ and $r_1 - r_2$ is sufficiently large, we have

$$e(U'_2) \cdot \mathcal{E}_c^{i,j}(\mathcal{X}^\geq 1_{G,Kl},\text{BGG}_c(V), n) = e(U'_2) \cdot \mathcal{E}_c^{i,j}(\mathcal{X}^\geq 1_{G,Kl},\text{BGG}(V), n) = 0$$

for $i \neq 2$.

**Proof.** This is a consequence of the slope estimates of Proposition 11.3.1. ∎

**Corollary 12.1.10.** For any $0 \leq q \leq r_2$, Eqs. (12.8a) and (12.7a) give isomorphisms

$$(12.8a) \quad e(U'_2) \cdot \tilde{H}^3_{dR,c}(\mathcal{X}^\geq 1_{G,Kl}(-D),\mathcal{V}, 1 + q) \cong e(U'_2) \cdot H^3_{c}(\mathcal{X}^\geq 1_{G,Kl},\mathcal{N}^1(-D)), $$

$$(12.8b) \quad e(U'_2) \cdot \tilde{H}^3_{dR,c}(\mathcal{X}^\geq 1_{G,Kl},\mathcal{V}, 1 + q) \cong e(U'_2) \cdot H^3_{c}(\mathcal{X}^\geq 1_{G,Kl},\mathcal{N}^1). $$

**Remark 12.1.11.** These isomorphisms are clearly compatible under the “forget $-D$” maps on both sides.

**Definition 12.1.12.** Let

$$\eta^\geq 1_{rig,-D} \in \tilde{H}^3_{dR,c}(\mathcal{X}^\geq 1_{G,Kl}(-D),\mathcal{V}, 1 + q)$$

denote the preimage of the class $\eta^\geq 1_{coh,-D}$ of Proposition 11.6.3 under the isomorphism of (12.8a); and let

$$\eta^\geq 1_{rig,-D} = \iota \left( \eta^\geq 1_{rig,q,-D} \right) \in \tilde{H}^3_{dR,c}(\mathcal{X}^\geq 1_{G,Kl},0(-D_0),\mathcal{V}),$$

where $\iota$ denotes the map on cohomology induced by the inclusion of complexes

$$\iota : \text{Fil}^g \text{BGG}_c(V) \hookrightarrow \text{BGG}_c(V).$$

**Note 12.1.13.**

1. The elements $\eta^\geq 1_{rig,-D}$ and $\eta^\geq 1_{rig,0,-D}$ are necessarily in the $\Pi'_t$-eigenspace for the prime-to-$p$ Hecke operators, since these operators commute with the ordinary projector $e(U'_2)$.
2. The space $H^3_{dR,c}(\mathcal{X}^\geq 1_{G,Kl},0(-D_0),\mathcal{V})$ computes the compactly-supported rigid cohomology of $Y_{G,Kl,0}$ (with coefficients in the overconvergent F-isocrystal $\mathcal{V}$), and it has a Frobenius map $\varphi$. ∎

12.1.1. **Properties of $\eta^\geq 1_{rig,-D}$.**

**Proposition 12.1.14.** Let $P(T) = \left(1 - \frac{T}{\alpha} \right) \left(1 - \frac{T}{\beta} \right)$, where $\alpha, \beta$ are the Hecke parameters of $\Pi'_t$ at $p$ (normalised as in Section 6.3 above). Then we have

$$P(\varphi) \cdot \eta^\geq 1_{rig,-D} = 0.$$ 

**Proof.** We shall approach this problem via a series of reductions.

Firstly, we consider the chain of morphisms

$$H^3_{dR,c}(\mathcal{X}^\geq 1_{G,Kl},0(-D_0),\mathcal{V}) \longrightarrow H^3_{dR,c}(\mathcal{X}^\geq 1_{G,Kl}(-D),\mathcal{V}) \longrightarrow H^3_{dR,c}(\mathcal{X}^{ord}_{G,Kl},\mathcal{V}),$$

where the first map is restriction from $\mathcal{X}^\geq 1_{G,Kl}$ to $\mathcal{X}^{ord}_{G,Kl}$, and the second is induced by the inclusion of sheaves $\mathcal{V}(-D) \hookrightarrow \mathcal{V}$. We claim that both of these maps are injective on the $\Pi'_t$-eigenspace. Their kernels are expressed as rigid cohomology groups of other strata in $\mathcal{X}_{G,Kl}$: the first is a quotient of $H^3_{dR,c}(\mathcal{X}^\geq 1_{G,Kl},0(-D_0),\mathcal{V})$, and the second can be expressed in terms of the boundary cohomology of the compactified Kuga–Sato variety. So we conclude that $P(\varphi) \cdot \eta^\geq 1_{rig,-D} = 0$ if and only if $P(\varphi) \cdot \eta^{ord}_{rig} = 0$.

The advantage of this reformulation is that the action of Frobenius on $H^3_{dR,c}(\mathcal{X}^{ord}_{G,Kl},\mathcal{V})$ can be described explicitly: it is given by the Hecke correspondence $\Phi$, since this is a lifting of the Frobenius on
the special fibre. Moreover, \( \eta_{\text{ord}}^{\text{rig}} \) has an alternative description via the Frölicher spectral sequence. The chain of morphisms above has analogues in filtered rigid cohomology, and in coherent cohomology:

\[
\begin{array}{cccc}
\eta_{\text{coh}, -D}^{\geq 1} & \rightarrow & \eta_{\text{coh}, -D}^{\text{ord}} & \rightarrow & \eta_{\text{coh}}^{\text{ord}} \\
\cap & & \cap & & \cap \\
H_c^2(\mathcal{X}_{G, Kl}^{\geq 1}, \mathcal{N}^1(\mathcal{D})) & \rightarrow & H_c^{\text{ord}}(\mathcal{X}_{G, Kl}^{\geq 1}, \mathcal{N}^1(\mathcal{D})) & \rightarrow & H_c^{\text{ord}}(\mathcal{X}_{G, Kl}^{\geq 1}, \mathcal{N}^1), \\
\cap & & \cap & & \cap \\
H_{\text{rig}, c}(\mathcal{X}_{G, Kl, 0}^{\geq 1}, V, 1 + q) & \rightarrow & \tilde{H}_{dR, c}^i(\mathcal{X}_{G, Kl}^{\text{ord}}, V, 1 + q) & \rightarrow & H_{dR, c}^i(\mathcal{X}_{G, Kl}^{\text{ord}}, V, 1 + q), \\
\cap & & \cap & & \cap \\
H_{\text{rig}, c}(\mathcal{X}_{G, Kl, 0}^{\geq 1}, V) & \rightarrow & H_{dR, c}^i(\mathcal{X}_{G, Kl}^{\text{ord}}, V) & \rightarrow & H_{dR, c}^i(\mathcal{X}_{G, Kl}^{\text{ord}}, V), \\
\cup & & \cup & & \cup \\
\eta_{\text{rig}, -D}^{\geq 1} & \rightarrow & \eta_{\text{rig}}^{\text{ord}} & \rightarrow & \eta_{\text{rig}}^{\text{ord}}
\end{array}
\]

We know that \( Z' \cdot P(\Phi) \cdot \eta_{\text{rig}, -D}^{\text{ord}} = 0 \), from which it follows that \( Z' \cdot P(\Phi) \cdot \eta_{\text{coh}}^{\text{ord}} = 0 \) and hence \( Z' \cdot P(\Phi) \cdot \eta_{\text{rig}}^{\text{ord}} = 0 \). However, since \( Z' \cdot \Phi = U_2' \), the operator \( Z' \) is surjective restricted to any finite-slope \( U_2' \)-eigenspace in \( H_{\text{rig}, c}^3(\mathcal{X}_{G, Kl}^{\text{ord}}, V) \); since any such eigenspace is finite-dimensional, it is also injective, and thus we may conclude \( P(\Phi) \cdot \eta_{\text{rig}}^{\text{ord}} = 0 \).

\[\square\]

**Corollary 12.1.15.** The class \( \eta_{\text{rig}, -D}^{\geq 1} \) maps to \( \eta_{\text{rig}, -D}^{\text{ord}} \) under the cospecialisation map.

**Proof.** Note that the image of the class under the cospecialisation map lands in \( \text{Fil}^{1+q} \) by construction. By Note 6.7.4, it is hence sufficient to show that the unfiltered class \( \eta_{\text{rig}, -D}^{\geq 1} \in H_{\text{rig}, c}^3(\mathcal{X}_{G, Kl, 0}^{\geq 1}(\mathcal{D}), V) \) satisfies \( P(\varphi) = 0 \). But this is true by Proposition 12.1.14.

\[\square\]

**12.2. Gros fp-cohomology and the Poznań spectral sequence.**

**12.2.1. Gros fp-cohomology.** Let \( P \in 1 + TQ_0[T] \) be a polynomial with constant term 1. Recall the definition of Gros fp-cohomology given in Definition 9.2.20 above. In our present context this becomes:

**Definition 12.2.1.** Define the Gros fp-cohomology of \( V \) over \( \mathcal{X}_{G, Kl}^{\text{ord}} \) with \( \mathcal{O} \)-support, twist \( n \) and polynomial \( P \), denoted \( H_{\text{rig}, \mathcal{O}, c}^i(\mathcal{X}_{G, Kl}^{\text{ord}}, V, n; P) \), to be the cohomology of the mapping fibre of the diagram

\[
(12.9)
\]

\[\cdots \rightarrow \text{R} \Gamma_{dR, \mathcal{O}}(\mathcal{X}_{G, Kl}^{\text{ord}}, V, n) \rightarrow P(\varphi/p^n)_{\mathcal{O}} \text{R} \Gamma_{dR, \mathcal{O}}(\mathcal{X}_{G, Kl}^{\text{ord}}, V),
\]

where \( \iota \) is the map on cohomology induced by the inclusion \( \mathcal{F} \text{il}^n N^\bullet \hookrightarrow N^\bullet \). We denote the analogous group formed using the sheaves \( N^\bullet(-D) \) instead of \( N^\bullet \) by \( H_{\text{rig}, \mathcal{O}, c}^i(\mathcal{X}_{G, Kl}^{\text{ord}}, V(-D), n; P) \).

**Lemma 12.2.2.** For all \( i \geq 0 \), we have a surjective map

\[
H_{\text{rig}, \mathcal{O}, c}^i(\mathcal{X}_{G, Kl}^{\text{ord}}, V, n; P) \twoheadrightarrow \text{R} \Gamma_{dR, \mathcal{O}}(\mathcal{X}_{G, Kl}^{\text{ord}}, V, n) P(\varphi/p^n)_{\mathcal{O}} = 0
\]

**Proof.** Clear from the long exact sequence associated to the mapping fibre.

\[\square\]

**12.2.2. The Poznań spectral sequence.** The definition of Gros fp-cohomology (Definition 12.2.1) also makes sense for cohomology of \( \mathcal{X}_{G, Kl}^{\geq 1} \) with compact support. However, in the setting of \( \mathcal{X}_{G, Kl}^{\geq 1} \) we have an additional structure, since we have a lifting of the Frobenius map to the cohomology of the individual sheaves \( N^i \). This allows us to study Gros fp-cohomology via a spectral sequence, as follows.

**Definition 12.2.3.** We define \( \mathcal{E}_{\mathcal{O}, c}^i(\mathcal{X}_{G, Kl}^{\text{ord}}, \text{BGG}(V), n; P) \) to be the mapping fibre of the morphism of complexes

\[
\mathcal{E}_{\mathcal{O}, c}^i(\mathcal{X}_{G, Kl}^{\text{ord}}, \text{BGG}(V), n) P(\varphi/p^n)_{\mathcal{O}} \rightarrow \mathcal{E}_{\mathcal{O}, c}^i(\mathcal{X}_{G, Kl}^{\text{ord}}, \text{BGG}(V)).
\]
Thus

\[ \varphi_{i,j}^{i',j'}(\mathcal{A}_{G,Kl}^{ord}, \text{BGG}(V), n; P) = H^i_{c0}(\mathcal{A}_{G,Kl}^{ord}, \mathcal{F}i^n N^i) \oplus H^j_{c0}(\mathcal{A}_{G,Kl}^{ord}, N^{i-1}) \]

with the differentials being \((x, y) \mapsto (\nabla x, P(\varphi/p^n)(x) - \nabla y)\). We shall only use this for \(n \geq 1\), in which case one sees easily that this group is zero unless \(j \in \{2, 3\}\) and \(1 \leq i \leq 4\), and the \(i = 0\) terms vanish if \(n \geq 1\).

**Proposition 12.2.4.** There is a first-quadrant spectral sequence, the Pozna\'n spectral sequence, with

\[ P^s E_1^{ij} = \varphi_{i,j}^{i',j'}(\mathcal{A}_{G,Kl}^{ord}, \text{BGG}(V), n; P). \]

The spectral sequence degenerates at \(E_3\), and its abutment is the Gros fp-cohomology \([12.9]\).

**Proof.** Choose double complexes computing \(R\Gamma_{dr,c0}(\mathcal{A}_{G,Kl}^{ord}, V, n)\) and \(R\Gamma_{dr,c0}(\mathcal{A}_{G,Kl}^{ord}, V, n)\) respectively, in such a way that \(P(\varphi/p^n)c0\) extends to a map of double complexes. Then \(H^\bullet_{rig-fp,c0}(Y_{G,Kl}^{ord}, \mathcal{E}, n, P)\) is computed by the total complex of the associated mapping fibre, i.e. by the total complex of a triple complex. The Pozna\'n spectral sequence is one of the spectral sequences associated to this triple complex.

\[ \square \]

12.2.3. Coherent fp-pairs.

**Definition 12.2.5.** (a) We define a coherent fp-pair of degree \((i, j)\), twist \(n\) and \(c0\)-support to be an element of

\[ \mathcal{F}^{i,j}_{fp,c0}(\mathcal{A}_{G,Kl}^{ord}, \text{BGG}(V), n; P) := \ker \left( \varphi_{c0}^{i,j}(\mathcal{A}_{G,Kl}^{ord}, \text{BGG}(V), n; P) \rightarrow \varphi_{c0}^{i+1,j}(\mathcal{A}_{G,Kl}^{ord}, \text{BGG}(V), n; P) \right), \]

i.e. a pair of elements

\[(x, y) \in H^i_{c0}(\mathcal{A}_{G,Kl}^{ord}, \mathcal{F}i^n N^i) \oplus H^j_{c0}(\mathcal{A}_{G,Kl}^{ord}, N^{i-1}) \]

which satisfy \(\nabla(x) = 0\), \(\nabla(y) = P(p^{-n}\varphi)(x)\), where \(\iota\) is the map on cohomology induced by the inclusion \(\mathcal{F}i^n N^i \hookrightarrow N^i\).

(b) We define the group of coherent fp-classes of degree \((i, j)\), to be the \(E_2^{ij}\)-term of the Pozna\'n spectral sequence, so it is the quotient of the group of coherent fp-pairs by the subgroup of pairs of the form

\[(x, y) = (\nabla(u), P(p^{-n}\varphi)(u) - \nabla(v)) \]

for some \((u, v) \in \varphi_{c0}^{i-1,j}(\mathcal{A}_{G,Kl}^{ord}, \text{BGG}(V), n; P)\). We denote this quotient by \(H^{i,j}_{fp,c0}(\mathcal{A}_{G,Kl}^{ord}, \text{BGG}(V), n; P)\).

**Lemma 12.2.6.** For any \(i\) and \(n\) there is a long exact sequence

\[
\ldots \rightarrow H^i_{fp,c0}(\mathcal{A}_{G,Kl}^{ord}, \text{BGG}(V), n; P) \rightarrow H^i_{c0}(\mathcal{A}_{G,Kl}^{ord}, \text{BGG}(V), n; P) \rightarrow P(p^{-n}\varphi)_{c0}^{i,j}(\mathcal{A}_{G,Kl}^{ord}, \text{BGG}(V)) \rightarrow H^{i+1,j}_{fp,c0}(\ldots) \rightarrow \]

\[
H^j_{c0}(\mathcal{A}_{G,Kl}^{ord}, \mathcal{F}i^n N^i) \big\| \nabla H^j_{c0}(\mathcal{A}_{G,Kl}^{ord}, \mathcal{F}i^n N^i) \big\| \nabla H^j_{c0}(\mathcal{A}_{G,Kl}^{ord}, N^{i-1}) \big\| \nabla H^j_{c0}(\mathcal{A}_{G,Kl}^{ord}, N^{i-1}) \big\| \nabla H^j_{c0}(\mathcal{A}_{G,Kl}^{ord}, N^{i-1}) \big\| \nabla H^j_{c0}(\mathcal{A}_{G,Kl}^{ord}, N^{i-1}) \big\|
\]

**Proof.** This is the long exact sequence associated to the mapping fibre \([12.2.3]\).

\[ \square \]

**Corollary 12.2.7.** If \(0 \leq q \leq r_2\), then the spectral sequence gives rise to an isomorphism

\[ \alpha_{G,rig-fp,c0} : \mathcal{F}^{1,2}_{fp,c0}(\mathcal{A}_{G,Kl}^{ord}, \text{BGG}(V), 1 + q; P) \rightarrow H^{1,2}_{fp,c0}(\mathcal{A}_{G,Kl}^{ord}, \text{BGG}(V), 1 + q; P) \rightarrow H^{2,3}_{fp,c0}(\mathcal{A}_{G,Kl}^{ord}, \text{BGG}(V), 1 + q; P). \]

**Proof.** Immediate from the fact that \(P^s E_2^{ij}\) is supported in the range \(i \geq 1, j \geq 2\).

\[ \square \]

**Note 12.2.8.** Replacing \(\text{BGG}(V)\) by \(\text{DR}(V)\), we obtain an isomorphism

\[
(12.10) \quad H^{1,2}_{fp,c0}(\mathcal{A}_{G,Kl}^{ord}, \text{DR}(V), 1 + q; P) \rightarrow H^{1,2}_{fp,c0}(\mathcal{A}_{G,Kl}^{ord}, \text{DR}(V), 1 + q; P)
\]

which is compatible with the natural map

\[ H^{1,2}_{fp,c0}(\mathcal{A}_{G,Kl}^{ord}, \text{BGG}(V), 1 + q; P) \rightarrow H^{1,2}_{fp,c0}(\mathcal{A}_{G,Kl}^{ord}, \text{DR}(V), 1 + q; P) \]

arising from Proposition \([12.1.2]\) (c.f. Note \([12.1.5]\)).
12.2.4. Comparison of spectral sequences. There is a crucial compatibility between the edge maps of the Poznań spectral sequence and the Frölicher spectral sequence for (truncated) rigid cohomology:

**Proposition 12.2.9.** If $0 \leq q \leq r_2$, then we have a commutative diagram

$$
\begin{array}{ccc}
\tilde{H}^3_{\text{rig-fp},c_0}(\lambda_{G,Kl}^{\text{ord}}, \mathcal{V}, 1 + q; P) & \xrightarrow{\alpha_{G,\text{rig-fp},c_0}} & \tilde{H}^3_{\text{dR},c_0}(\lambda_{G,Kl}^{\text{ord}}, \mathcal{V}, 1 + q)(p^{(r_1+1)}\varphi)\otimes=0 \\
H^{1,2}_{\text{fp},c_0}(\lambda_{G,Kl}^{\text{ord}}, \text{BGG}(\mathcal{V}), 1 + q; P) & \xrightarrow{\alpha_{G,\text{rig},c_0}} & H^{1,2}_{c_0}(\lambda_{G,Kl}^{\text{ord}}, \text{BGG}(\mathcal{V}), 1 + q)(p^{(r_1+1)}\varphi)\otimes=0
\end{array}
$$

Here, the horizontal arrows are the surjections of Lemmas 12.2.7 and 12.2.8 and the vertical isomorphisms are given by Corollaries 12.2.7 and 12.1.8.

**Proof.** Clear from the construction. \(\square\)

12.3. Coherent fp-pairs from $\eta$. From now on, let $0 \leq q \leq r_2$, and let

$$
P_q(T) = (1 - p^{q+1}T/\alpha)(1 - p^{q+1}T/\beta).
$$

**Proposition 12.3.1.** There exists a unique class

$$
\tilde{\eta}_{\text{rig-fp},q,-D}^{\geq 1} \in \tilde{H}^3_{\text{rig-fp},c}(\lambda_{G,Kl}^{\geq 1}(\mathcal{D}), \mathcal{V}, 1 + q; P_q)
$$

which lies in the $\Pi_{\ell}$-eigenspace for the spherical Hecke operators, and whose image in $\tilde{H}^3_{\text{dR},c}(\lambda_{G,Kl}^{\geq 1}(\mathcal{D}), \mathcal{V}, 1 + q)$ is $\tilde{\eta}_{\text{rig-q,-D}}^{\geq 1}$.

**Note 12.3.2.** Note that the class $\tilde{\eta}_{\text{rig-fp},q,-D}^{\geq 1}$, and the group in which it lies, are actually independent of $q$ in this range (but the different values of $q$ correspond, formally, to different twists of the motive). \(\diamondsuit\)

**Proof.** From Proposition 12.1.4 we know that

$$
\tilde{\eta}_{\text{rig-q,-D}}^{\geq 1} \in \tilde{H}^3_{\text{dR},c}(\lambda_{G,Kl}^{\geq 1}(\mathcal{D}), \mathcal{V}, 1 + q)(p^{(r_1+1)}\varphi)\otimes=0.
$$

It follows that $\tilde{\eta}_{\text{rig-fp},q,-D}^{\geq 1}$ is in the image of the map

$$
\tilde{H}^3_{\text{rig-fp},c}(\lambda_{G,Kl}^{\geq 1}(\mathcal{D}), \mathcal{V}, 1 + q; P_q) \to \tilde{H}^3_{\text{dR},c}(\lambda_{G,Kl}^{\geq 1}(\mathcal{D}), \mathcal{V}, 1 + q),
$$

and the kernel of this map is a quotient of $H^2_{\text{rig-fp},c}(\lambda_{G,Kl,0}^{\geq 1}(\mathcal{D}_0), \mathcal{V})$, which has zero $\Pi_{\ell}$-eigenspace, so there is a unique $\Pi_{\ell}$-equivariant lift of $\tilde{\eta}_{\text{rig-q,-D}}^{\geq 1}$ to a class in $\tilde{H}^3_{\text{rig-fp},c}(\lambda_{G,Kl}^{\geq 1}(\mathcal{D}), \mathcal{V}, 1 + q; P_q)$. \(\square\)

**Corollary 12.3.3.** The class $\tilde{\eta}_{\text{rig-fp},q,-D}^{\geq 1}$ is sent to $\eta_{\text{rig-fp},q,-D}^{\geq 1}$ under the corestriction map.

**Definition 12.3.4.** Define

$$
\tilde{\eta}_{\text{rig-fp},q} \in \tilde{H}^3_{\text{rig-fp},c_0}(\lambda_{G,Kl}^{\text{ord}}, \mathcal{V}, 1 + q; P_q)
$$

to be the image of $\tilde{\eta}_{\text{rig-fp},q,-D}^{\geq 1}$ under restriction to $\lambda_{G,Kl}^{\text{ord}}$ and forgetting $-D$.

**Notation 12.3.5.** Write $\text{Eis}_{\text{rig-syn},2,\text{ord}}^{[t_1,t_2]}$ for the image of $\text{Eis}_{\text{rig-syn},2,\text{ord}}^{[t_1,t_2]}$ in $\tilde{H}^2_{\text{rig-syn}}(\lambda_{H,\Delta,2})$ under the specialisation map defined in Remark 9.2.22.

By Remark 9.2.22 we obtain the following result:

**Corollary 12.3.6.** Then

$$
\left\langle \text{Eis}_{\text{rig-syn},2,\text{ord}}^{[t_1,t_2]}(t_{\Delta}^{\text{rig-fp},c_0}(\tilde{\eta}_{\text{rig-fp},q,-D}^{\geq 1})), \tilde{\eta}_{\text{rig-fp},q} \right\rangle_{\text{rig-fp,Y}_{\Delta}^{\text{ord}}}^{\text{rig-fp,Y}_{\Delta}^{\text{ord}}}
= \left\langle \text{Eis}_{\text{rig-syn},2,\text{ord}}^{[t_1,t_2]}(t_{\Delta}^{\text{rig-fp},c_0}(\tilde{\eta}_{\text{rig-fp},q,-D}^{\geq 1})), \tilde{\eta}_{\text{rig-fp},q}^{\geq 1} \right\rangle_{\text{rig-fp,Y}_{\Delta}^{\text{ord}}}^{\text{rig-fp,Y}_{\Delta}^{\text{ord}}}.
$$

In Section 16 we will evaluate this pairing in terms of coherent cohomology.
Note 12.3.7. By Corollary 12.2.7, we have an isomorphism
\[ \tilde{H}^2_{\text{rig}, q, c}(\mathcal{X}_{G, K^1}, V, 1 + q; P_q) \cong H^1_{\text{dR}, c}(\mathcal{X}_{G, K^1}, BGG(V), 1 + q; P_q). \]
Thus \( \tilde{\eta}_{\text{rig}, q}^{\text{ord}} \) is represented by a uniquely-determined coherent fp-pair, which clearly has the form \( (\eta_{\text{coh}, q}^{\text{ord}}, \zeta) \) for some class \( \zeta \in H^2_{\text{dR}}(\mathcal{X}_{G, K^1}, \mathcal{N}^0) \) such that \( P_{1+q}(\Phi_{1+q}) \eta_{\text{coh}}^{\text{ord}} = \nabla \zeta. \)

\[ (12.11) \]

Note 12.3.8. Equation (12.11) does not determine \( \zeta \) uniquely: it is only unique modulo
\[ H^2_{\text{dR}, c}(\mathcal{X}_{G, K^1}, \mathcal{N}^0). \]
However, we know that this group has zero \( \Pi'_p \)-eigenspace, so there is a unique \( \Pi'_p \)-isotypical lifting. \( \diamond \)

The following proposition is crucial for the evaluation of the regulator:

Proposition 12.3.9. The element \( \zeta \) has the following properties:
- \( Z' \cdot \zeta = 0 \),
- \( U_2' \cdot \zeta = \lambda \zeta \), where \( \lambda = \frac{a \beta}{p^2 + p + 1} \) is the unit eigenvalue of \( U_2' \) on \( \Pi'_p \).

**Proof.** We know that \( P(\Phi) \eta_{\text{coh}, q}^{\text{ord}} \) is in the kernel of the operators \( Z' \) and \( (U_2' - \lambda) \). Hence \( (U_2' - \lambda) \cdot \zeta \) and \( Z' \cdot \zeta \) lie in the group \( H^2_{\text{dR}}(\mathcal{X}_{G, K^1}, \mathcal{N}^0); V = 0 \). Moreover, they are both in the \( \Pi'_p \)-eigenspace for the spherical Hecke algebra. Since this eigenspace is zero, we conclude that \( \zeta \) is in the kernel of \( Z' \) and of \( (U_2' - \lambda) \). \( \square \)

### 12.4. Lifting to the de Rham sheaves.

**Definition 12.4.1.**
- Define \( \tilde{\zeta} \) to be the image of \( \zeta \) in \( H^2_{\text{dR}}(\mathcal{X}_{G, K^1}, V \otimes \Omega^0_G(D)) \).
- For \( 0 \leq q \leq r_2 \), define \( \tilde{\eta}_{\text{coh}, q, -D} \) to be the image of \( \eta_{\text{coh}, q, -D} \) under the composition of maps:
  \[ H^2_{\text{dR}}(\mathcal{X}_{G, K^1}, \mathcal{N}^1(-D)) \xrightarrow{\text{incl}} H^2_{\text{dR}}(\mathcal{X}_{G, K^1}, \mathcal{N}^1(-D)) \]
  where the first map is given by the inclusion of complexes in Proposition 12.1.2 and the second map is induced from the natural inclusion of sheaves.
- Write \( \tilde{\eta}_{\text{coh}, q}^{\text{ord}} \) for the image of \( \tilde{\eta}_{\text{coh}, q, -D}^{\text{ord}} \) in \( H^2_{\text{dR}}(\mathcal{X}_{G, K^1}, \mathcal{N}^1(-D)) \).

**Lemma 12.4.2.** The class \( \tilde{\eta}_{\text{coh}, q, -D}^{\text{ord}} \) maps to \( \eta_{\text{coh}, q, -D}^{\text{ord}} \) under the natural map induced from the projection:
\[ \mathcal{F}il^2 V \otimes \Omega^1_G(-D) \rightarrow \mathcal{N}^1. \]

**Proof.** It is immediate from Proposition 12.1.2 that the image of \( \mathcal{N}^1 \) in the full de Rham complex is contained in \( \mathcal{F}il^2 V \otimes \Omega^1_G \). In order to prove the result, it is hence sufficient to show that composition of the inclusion and the projection is the identity on \( \mathcal{N}^1 \). But this follows from the results in [FC90] Ch. VI, §6]. \( \square \)

The following two lemmas are direct consequences of the corresponding results for \( \eta_{\text{coh}, -D}^{\text{ord}} \) and \( \zeta \) (Propositions 11.6.3 and 12.3.9), using the the inclusion of the dual BGG complex into the de Rham complex is Hecke equivariant.

**Lemma 12.4.3.**
- The operator \( U_{K^1, 2} \) acts on \( \tilde{\eta}_{\text{coh}, q, -D}^{\text{ord}} \) as multiplication by \( \frac{a \beta}{p^2 + p + 1} \).
- The operator \( U_{K^1, 1} \) acts on \( \tilde{\eta}_{\text{coh}, q, -D}^{\text{ord}} \) as multiplication by \( \alpha + \beta \).
- The spherical Hecke algebra acts via the system of eigenvalues associated to \( \Pi' \).

**Lemma 12.4.4.** The element \( \tilde{\zeta} \) has the following properties:
- \( Z' \cdot \tilde{\zeta} = 0 \),
- \( U_2' \cdot \tilde{\zeta} = \lambda \tilde{\zeta} \), where \( \lambda = \frac{a \beta}{p^2 + p + 1} \) is the unit eigenvalue of \( U_2' \) on \( \Pi'_p \).
Proposition 12.4.5. The classes $\tilde{\zeta}$ and $\tilde{\tau}_{coh}^{ord}$ satisfy
\[
\nabla_{\tilde{\tau}_{coh}^{ord}} = 0 \quad \text{and} \quad P_{q}(\Phi_{1+q}) \tilde{\tau}_{coh}^{ord} = \nabla_{\tilde{\zeta}}
\]
and hence give rise to a class in $\mathcal{H}^{1,2}_{fr,coh}(X^{ord}_{G,K} \cdot \text{DR}(V), 1 + \eta; P_{q})$. Moreover, this class maps to $\tilde{\tau}_{coh}^{ord}$ under the isomorphism $\textit{(12.10)}$.

Proof. Immediate.

12.5. A Hecke operator identity. The reason why we care about Proposition $12.3.9$ is the following result, comparing constructions on $G$ and on $H$. Recall the embedding
\[
\iota_{\Delta} : \mathcal{X}^{ord}_{H,\Delta} \to \mathcal{X}^{ord}_{G,K}
\]
constructed in Section $10.3$ and recall that its image is also closed in $\mathcal{X}^{2,1}_{G,K}$.

Proposition 12.5.1. We have the following identity of correspondences $\mathcal{X}^{ord}_{H,\Delta} \cong \mathcal{X}^{ord}_{G,K}$:
\[
U_{p}^{\prime} \circ \iota_{\Delta} \circ (U_{p} \boxtimes U_{p}) = p(p)Z^{\prime} \circ \iota_{\Delta}.
\]

Note 12.5.2. Correspondences act contravariantly on cohomology, so this means that
\[
(U_{p} \boxtimes U_{p}) \circ \iota_{\Delta} \circ U_{p}^{\prime} = \iota_{\Delta} \circ p(p)Z^{\prime}
\]
as maps $H^{*}(\mathcal{X}^{ord}_{G,K}) \to H^{*}(\mathcal{X}^{ord}_{H,\Delta})$.

Proof. Since $\mathcal{Y}^{ord}_{H,\Delta}$ is open in $\mathcal{Y}^{ord}_{G,K}$, it suffices to prove the identity over this open subset.

We recall the moduli-space description of the varieties and correspondences involved. A point of $\mathcal{Y}^{ord}_{H,\Delta}$ (over some $p$-adic field $L$) corresponds to a triple $(E_{1}, E_{2}, \alpha)$, where $E_{i}$ are elliptic curves over $L$ with good ordinary reduction, and $\alpha$ is an isomorphism $E_{1}[p] \overset{\sim}{\to} E_{2}[p]$. The operator $U_{p} \boxtimes U_{p}$ maps $(E_{1}, E_{2}, \alpha)$ to the formal sum $\sum_{j_{1}, j_{2}}(E_{1}/j_{1}, E_{2}/j_{2}, \alpha)$ where $J_{i}$ vary over cyclic $p$-subgroups of $E_{i}$ distinct from $E_{i}[p]$, and $\alpha$ is the ensuing isomorphism
\[
\overline{E_{1}/j_{1}[p]} \overset{\sim}{\to} E_{1}[p] \overset{\alpha}{\to} E_{2}[p] \overset{\sim}{\to} \overline{E_{2}/j_{2}[p]}.
\]

Concretely, if $e_{1}, f_{1}$ denotes a choice of basis of $T_{p}E_{1}$, and $e_{2}, f_{2}$ of $T_{p}E_{2}$, giving isomorphisms $E_{1}[p^{\infty}] \cong (Q_{p}/Z_{p})^{2}$, and we assume that $e_{1}$ and $2f$ span the Tate modules of the formal groups $T_{p}E_{i}$, then $J_{1}$ has to be one of the groups $\langle \ell^{1+a_{1}} \rangle$ for $0 \leq a_{1} \leq p - 1$, and similarly $J_{2}$. We can and do assume that $\alpha(e_{1}) = e_{2}$.

Meanwhile, points of $\mathcal{X}^{ord}_{G,K}$ correspond to pairs $(A, C)$ where $A$ is an abelian surface and $C \subset \hat{A}[p]$ is a cyclic $p$-subgroup (again with some prime-to-$p$ level structure being ignored). The map $\iota_{\Delta}$ maps $(E_{1}, E_{2}, \alpha)$ to $(E_{1} \oplus E_{2}, C)$ where $C \subset (E_{1} \oplus E_{2})[p]$ is the subgroup of points of the form $(x, \alpha(x))$.

Finally, the Hecke correspondences $Z^{\prime}$, $U_{p}^{\prime}$ and $(p)$ are given as follows. Let $P = (A, C)$ be a point of $\mathcal{X}^{ord}_{G,K}$.

- The correspondence $Z^{\prime}$ is given by
  \[
  (A, C) \mapsto \sum_{J} \sum_{\hat{C}} (A/J, \hat{C} \mod J),
  \]
  where $J$ varies over isotropic $(p, p)$-subgroups such that $J \cap \hat{A}[p] = C$, and $\hat{C}$ varies over cyclic $p^{2}$-subgroups of $A/J[p]$ such that $p\hat{C} = C$. (Note that there are $p$ choices of $J$, and $p$ choices of the subgroup $\hat{C}$, so this is a correspondence of degree $p^{2}$.)

- For the correspondence $U_{p}^{\prime}$, let $J_{0}$ be the subgroup $(p^{-1} C \cap \hat{A}) + C^{\perp}$; this has invariants $(p^{2}, p, p)$ and is isotropic in the sense that $pJ_{0}$ and $J_{0}[p]$ are orthogonal complements inside $A[p]$. Then $U_{p}^{\prime}$ is given by
  \[
  (A, C) \mapsto \sum_{\hat{C}} (A/J_{0}, (p^{-1}\hat{C} \cap \hat{A}) \mod J_{0}),
  \]
  where $\hat{C}$ again varies over liftings of $C$ to a cyclic $p^{2}$-subgroup of $\hat{A}$.

- The correspondence $(p)$ sends $(A, C)$ to itself, but acts on the prime-to-$p$ level structure by multiplying it by $p$. 

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We now consider composing these operations. We choose a point $P = (E_1, E_2, \alpha)$ and fix coordinates on the $E_i$, as above. Let $(A, C) = \iota_1(P) = (E_1 \oplus E_2, (\frac{\alpha_1 + \alpha_2}{p}))$; and let $(A', C') = \iota_2(P')$ where $P'$ is one of the points in the 0-cycle $(U_p, U_p) \cdot P$, corresponding to a choice of $a_1, a_2 \in \mathbb{Z}/p$; thus we have $A' = A/(f_1', f_2')$, where $f_1' = f_1 \cdot a_1 \cdot p, f_2' = f_2 \cdot a_2 \cdot p$. Thus $(e_1, e_2, f_1', f_2')$ form a basis of $T_pA'$, regarded as a lattice in $V = T_pA \otimes \mathbb{Q}_p$ (strictly containing $T_pA$ itself), and $C'$ is the image of $C$, generated by $\frac{\alpha_1 + \alpha_2}{p}$ as a subgroup of $V/T_pA'$.

We now compute the canonical $(p^2, p, p)$-subgroup $J_0$ of $A'$: it is uniquely determined by $pJ_0 = C = \langle \frac{\alpha_1 + \alpha_2}{p} \rangle$ and $J_0[p] = C^2 = \langle \frac{\alpha_2}{p}, \frac{\alpha_1 - \alpha_2}{p} \rangle$, from which we easily compute that $J_0$ is generated by $\langle \frac{\alpha_1 + \alpha_2}{p}, \frac{e_1}{p}, \frac{e_2}{p} \rangle$ as a subgroup of $A'[p^\infty] = V/T_pA'$. Note that this subgroup contains the image of $A[p]$.

Thus the isogeny $A \to A' / J_0$ is the composite of multiplication by $p$ on $A$ (which gives the factor $(p)$) and quotient by the subgroup $K = \langle \frac{\alpha_1 + \alpha_2}{p}, \frac{e_1}{p}, \frac{e_2}{p} \rangle$. So the image of $(A', C')$ under $U_2'$ is given by $\sum_{C'}(A/K, C' \mod K)$, where $C'$ varies over multiplicative $p^2$-subgroups of $A$ lifting $C$. Note that this is the same as the inner sum of $Z' \cdot (A, C)$ when we take the subgroup $J$ to be our $K$.

To conclude the proof, it suffices to note that as $(a_1, a_2)$ vary, the subgroup $K$ hits every one of the groups $J$ in the outer sum defining $Z' \cdot (A, C)$, and each such $J$ occurs $p$ times (since $K$ only depends on $a_1 - a_2 \mod p$).

Proposition 12.5.1 has the following immediate consequence, which will be crucial in the regulator evaluation (c.f. Section 16):

**Corollary 12.5.3.** Let $z \in H_2^\mathrm{rig}(X_{\Delta}^{\mathrm{ord}}, \mathbb{V} \otimes \Omega_{G}^n)$, and assume that $z \in \ker(Z')$ and $z$ is a $U'_2$-eigenvector with non-zero eigenvalue. Then

$$i^*_\Delta(z) \in \ker(U_p \otimes U_p).$$

**Note 12.5.4.** By Lemma 12.4.4 it hence follows that $i^*_\Delta(z) \in \ker(U_p \otimes U_p)$.

## 13. fp-cohomology and coherent fp-pairs for $H$

### 13.1. The Poznań spectral sequence.

Similar considerations apply to $X_{H, \Delta}^{\mathrm{ord}}$, with or without compact support. Let $W$ be an algebraic representation of $H$, and write $W$ for the corresponding coherent sheaf on $X_{\Delta}$. Let $Q \in Q_p[t]$ have constant coefficient 1, and let $n \geq 0$. We can then consider the Gros-fp cohomology

$$\tilde{H}^*_\mathrm{rig-fp, *}(X_{H, \Delta}^{\mathrm{ord}}(\Diamond), W; n; Q),$$

where $* \in \{\emptyset, c\}$ and $\Diamond \in \{\emptyset, \Diamond_{\Delta}\}$ (c.f. Definition 12.2.1).

Recall that if $Q(p^{-1}) \neq 0$, we define the trace map

$$\tilde{H}^5_{\mathrm{rig-fp, c}}(X_{H, \Delta}^{\mathrm{ord}}, Q_p, 3; Q) \to Q_p$$

as $\frac{1}{Q(p^{-1})}$ times the trace map on rigid cohomology.

**Remark 13.1.1.** As usual, the factor $\frac{1}{Q(p^{-1})}$ serves to make the trace maps compatible with the natural maps of complexes $\tilde{H}_{\mathrm{rig-fp, c}}((\cdot); Q) \to \tilde{H}_{\mathrm{rig-fp, c}}((\cdot); Q')$ for polynomials $Q \mid Q'$. (This map acts as $(Q'/Q)(p^{-n}\varphi)$ on the rigid complex, with $n = 3$; but $\varphi = p^2$ on the top-degree cohomology, hence $Q(p^{-1})$ is the correct normalising factor.)

**Definition 13.1.2** (c.f. Definition 12.2.3). For $*, n \geq 0, * \in \{\emptyset, c\}$ and $\Diamond \in \{\emptyset, \Diamond_{\Delta}\}$, we define the complex

$$C_{*, n}^j(X_{H, \Delta}^{\mathrm{ord}}(\Diamond), W; n; Q)$$

with terms

$$C_{*, n}^j(X_{H, \Delta}^{\mathrm{ord}}(\Diamond), W; n; Q) = H_j^i(X_{H, \Delta}^{\mathrm{ord}}(\mathcal{F}^i \otimes W \otimes \Omega^{i-1}(\Diamond))) \oplus H_j^i(X_{H, \Delta}^{\mathrm{ord}}(\mathcal{W} \otimes \Omega^{i-1}(\Diamond)))$$

and differentials

$$(x, y) \mapsto \langle \nabla x, Q(\varphi_H/p^n)\varphi(x) - \nabla y \rangle.$$
We define the group of coherent fp-classes, denoted $H^{i,j}_{\mathrm{rig-fp}}(\mathcal{X}_{H,n}^0(\mathcal{O}), W, n; Q)$, analogously to Definition \[12.2.3\].

**Corollary 13.1.4.** The Pozna\'n spectral sequence gives rise to isomorphisms

$$\alpha_\Delta : H^{i,0}(\mathcal{X}_{H,n}^0(\mathcal{O}), W, n; Q) \cong \tilde{H}^i_{\mathrm{rig-fp}}(\mathcal{X}_{H,n}^0(\mathcal{O}), W, n; Q),$$

$$\alpha_{\Delta, c} : H^{i,2}_{c}(\mathcal{X}_{H,n}^0(\mathcal{O}), W, n; Q) \cong \tilde{H}^{i+2}_{\mathrm{rig-fp,c}}(\mathcal{X}_{H,n}^0(\mathcal{O}), W, n; Q).$$

**Proof.** Easy computation, using that since $\mathcal{X}_{\Delta}^0$ is affinoid, we have

$$H^i(\mathcal{X}_{H,n}^0(\mathcal{O}), W) = 0 \quad \text{for} \quad i \neq 0,$$

$$H^i_c(\mathcal{X}_{H,n}^0(\mathcal{O}), W) = 0 \quad \text{for} \quad i \neq 2.$$  

(Note that this holds for both $\mathcal{O} = \varnothing$ and $\mathcal{O} = -\Delta_{GL_2}$, in contrast to the situation for $G$.)

**Note 13.1.5.** In particular, if $n \geq 3$ we have

$$H^{3,2}_c(\mathcal{X}_{H,n}^0(-\Delta), Q_p, n; Q) \cong \tilde{H}^5_{\mathrm{rig-fp,c}}(\mathcal{X}_{H,n}^0(-\Delta), Q_p, n; Q).$$

The Frobenius operator $\varphi_H$ acts on $\tilde{H}^5_{\mathrm{rig-fp,c}}(\mathcal{X}_{H,n}^0(-\Delta), Q_p, n; Q)$ as multiplication by $p^2$. 

**Note 13.1.6.** Similarly, let $U$ be an algebraic representation of $GL_2$, and write $\mathcal{U}$ for the corresponding coherent sheaf on $\mathcal{X}_{GL_2}$. Let $\mathcal{O} \in \{\varnothing, -\Delta_{GL_2}\}$. Then the Pozna\'n spectral sequence gives rise to an isomorphism

$$o_{GL_2} : H^{i,0}(\mathcal{X}_{GL_2}^0(\mathcal{O}), U, n; Q) \cong \tilde{H}^i_{\mathrm{rig-fp}}(\mathcal{X}_{GL_2}^0(\mathcal{O}), U, n; Q).$$

**13.2. Compatibility with cup products.**

**Lemma 13.2.1.** Using the same formalism as \[Bes12\] \[2\], we can construct a cup product

$$H^{i,0}(\mathcal{X}_{H,n}^0(-\Delta), W, m; P) \times H^{j,2}(\mathcal{X}_{H,n}^0(W^\vee, n; Q) \to H^{i+j,2}(\mathcal{X}_{H,n}^0(-\Delta), Q_p, m + n; P \times Q)$$

which is compatible under the isomorphisms from Corollary \[13.1.4\] with the cup product in Gros-fp cohomology.

**Proof.** Standard check. 

**Note 13.2.2.** If $m + n \geq 3$ and $i + j = 3$, then we obtain a pairing

$$H^{1,0}(\mathcal{X}_{H,n}^0(-\Delta), W, m; P) \times H^{1,2}(\mathcal{X}_{H,n}^0(W^\vee, n; Q) \to Q_p.$$ 

**14. Coherent versus de Rham pullbacks**

**14.1. Algebraic representations of $G$ and $H$.** We can identify the representation $\text{Sym}^k$ of $GL_2$ with the space of polynomial functions on $GL_2$ satisfying

$$f \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = a^k f(g).$$

If $v$ and $w$ are the functions $(x, y) \mapsto x$ and $(x, y) \mapsto y$, then $\{v^{k-i}w^i : 0 \leq i \leq k\}$ is the standard basis of $\text{Sym}^k$, with $v^k$ being the highest-weight vector. Note that if $X_{21}$ denotes the generator $(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix})$ of the Lie algebra, then we have

$$(X_{21})^i \cdot v^k = \frac{k!}{(k-i)!} v^{k-i}w^i.$$ 

Now let us return to the setting where $V_G = V_G(r_1, r_2; r_1 + r_2)$ for some $r_1 \geq r_2 \geq 0$, and $V_H = V_H(t_1, t_2; t_1 + t_2)$, where $(t_1, t_2) = (r_1 - q - r, r_2 - q + r)$ for some $0 \leq q \leq r_2$, $0 \leq r \leq r_1$ as per our running conventions.

Since the representation $V_H(t_1, t_2; t_1 + t_2)$ of $H$ is the exterior product $\text{Sym}^{t_1} \otimes \text{Sym}^{t_2}$, we thus have a weight-vector basis $\{v^{t_1-i}w^i \otimes v^{t_2-i}w^i : 0 \leq i_n \leq t_n\}$ of this representation, realised as a space of $\mathcal{N}_H$-invariant functions on $H$. 

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We can similarly model $V_G(r_1, r_2)$ as the space of $f \in O(N_B \setminus G)$ which transform via the character $\lambda(r_1, r_2; r_1 + r_2)$ under left-translation by $T$. The standard basis vectors $v_1, \ldots, v_4$ of the 4-dimensional representation $V(1,0)$ thus correspond to the functions sending $g \in G$ to the four entries of its first row. A choice of highest-weight vector $w$ of $V(1,1)$ is given by $g \mapsto |g_{21}/g_{22}|$, and the vector $w' = Z \cdot w$ is $g \mapsto |g_{21}/g_{22}|^2$.

In [LSZ16] §4.3 we described a specific choice of morphism

$$br^{[q,r]} : V_G \otimes \det^q \to V_G$$

given by mapping the highest-weight vector $v^{t_1} \boxtimes v^{t_2}$ of $V_H$ to the vector $v^{[q,r]} \in V_G$ (denoted $v^{[a,b,q,r]}$ in op.cit.) defined by

$$w^{r_2 - q} \cdot (w')^q \cdot v_1^{r_1 - r_2 - r} \cdot v_2^r$$

where the products are taken in $O(N_B \setminus G)$ (the “Cartan product” construction).

**Note 14.1.1.** It is important to note that the Lie algebra $\mathfrak{g}$ acts on $O(G)$ by derivations, so for $X \in \mathfrak{g}$ we have the Leibniz rule

$$X^n \cdot (f_1 \times \cdots \times f_m) = \sum_{u_1 + \cdots + u_m = n} \binom{n}{u_1, \ldots, u_m} (X^{u_1} \cdot f_1) \cdots (X^{u_m} \cdot f_m).$$

In particular, $X^n \cdot f^n = n!(X \cdot f)^n$.

**Lemma 14.1.2.** Consider the vector $v^{t_1-t}w^t \boxtimes v^{t_2} \in V_H$, where $t = r_2 - q$. The image of this vector under $br^{[q,r]}$ is in $\ker(X_{12}^{n+1}) - \ker(X_{12}^n)$, where $X_{12} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \in \mathfrak{g}$ and $n = 2r_2 - q + r$. We have

$$br^{[q,r]}(v^{t_1-t}w^t \boxtimes v^{t_2}) = \frac{1}{\binom{t_1}{t}} (w'')^q v_1^{r_1-r_2-r} v_2^r \mod \ker(X_{12}^{n+1}).$$

where $w'' = X_{41} \cdot w = (g \mapsto |g_{21}/g_{22}|)$ spans the $(1,-1)$ weight space of $V_G(1,1)$.

**Proof.** We have $v^{t_1-t}w^t \boxtimes v^{t_2} = \binom{t_1-t}{t_1} X_{41}^t \cdot v^{t_1} \boxtimes v^{t_2}$ (identifying $X_{41}$ with an element of $\mathfrak{h} \subseteq \mathfrak{g}$). So we have

$$br^{[q,r]}(v^{t_1-t}w^t \boxtimes v^{t_2}) = \binom{\alpha}{\alpha} X_{41}^t \cdot v^{[q,r]}.$$

We now compute how $X_{41}$ acts on the four vectors used in the definition of $v^{[q,r]}$: it maps $v_1$ to $v_4$ and kills the other $v_i$; it sends $w$ to $w''$, and it kills $w'$ and $w''$. So $X_{41}^t \cdot v^{[q,r]}$ is a sum of terms of the form

$$w^{t-\alpha} \cdot (w'')^q \cdot v_1^{r_1-r-\beta} \cdot v_2^r \cdot v_4^\beta$$

where $\alpha + \beta = t$; and the term for $\alpha = t, \beta = 0$ has coefficient $t$.

We now consider how $X_{12}$ acts on this element. One checks that $X_{12}$ acts on $V_G(1,0)$ by $v_2 \mapsto v_1 \mapsto 0, v_4 \mapsto -v_3 \mapsto 0$; and on $V_G(1,1)$ by $w'' \mapsto -w', \ w' \mapsto -2w, \ w \mapsto 0$ where $w_+ = X_{12} \cdot w = (g \mapsto |g_{21}/g_{22}|)$ spans the $(1,-1)$ weight space of $V(1,1)$. It follows that the number of applications of $X_{12}$ needed to kill the above element is exactly $2n + q + \beta + r + 1$. Since $\alpha + \beta$ is fixed, the last term to be annihilated is the one for $\alpha = t, \beta = 0$.

We now consider the image of $br^{[q,r]}(v^{t_1-t}w^t \boxtimes v^{t_2})$ in the graded pieces of the $P_{31}$-stable filtration on $V_G$ given by eigenspaces for $Z(M_{31})$ as in [LPSZ19] Definition 6.1. Note that we have $br^{[t_1,t_2]}(v^{t_1-t_2}w^t \boxtimes v^{t_2}) \in \operatorname{Fil}^{i+1} V_G$. Moreover, although the representation $Gr'i V_G$ is far from being irreducible, it is semi-simple and has a unique direct summand of highest highest weight, isomorphic to $W_G(r_1, -r_2; r_1 + r_2)$.

Since $M_{31} \cap \operatorname{Sp}_4$ is isomorphic to $\operatorname{GL}_2$, via $(A_{-}) \mapsto A$, we can identify $W_G(r_1, -r_2; r_1 + r_2)$ with the representation $\operatorname{Sym}^{r_1+r_2} \otimes \det^{-r_2}$ of $\operatorname{GL}_2$, so it has a canonical basis $v_i^{(r_1+r_2-1)} w^t$ for $0 \leq i \leq r_1 + r_2$. We normalise the projection $Gr'i V_G \to W_G(r_1, -r_2; r_1 + r_2)$ to send $v_i^{r_1-r_2} w^{r_2} \mapsto v_i^{r_1+r_2}$.

**Proposition 14.1.3.** The image of $br^{[q,r]}(v^{t_1-t}w^t \boxtimes v^{t_2}) \in \operatorname{Fil}^{i+1} V_G$ under projection to $W_G(r_1, -r_2; r_1 + r_2)$ is given by

$$\frac{(-2)^q}{\binom{t_1}{t_1}} v_i^{r_1+r_2-n} w^n, \ n = 2r_2 - q + r.$$
Proof. Letting $X = X_{12}$ for brevity, and recalling that $n = q + r + 2t$, we have

$$X^n \cdot (w')^t (w')^q v_1^{b-r} v_2^c = \frac{n!}{(2t)!q!r!} (X^{2t} \cdot (w')^t) \times (X^q \cdot (w')^r) \times v_1^{r_1 - r_2} \times (X^r \cdot v_2^c)$$

by the Leibniz rule, with all other terms being 0. Since $X^2 w' = 0$, we have $X^q \cdot (w')^q = q! (X \cdot w')^q = (-2)^q q! (w_--)$, and similarly $X^r \cdot v_2^c = r! v_1^c$. The term $X^{2t} \cdot (w')^t$ is a little more fiddly to evaluate; we conclude that

$$X^{2t} \cdot (w')^t = \frac{2t}{2, \ldots, 2} (X^2 \cdot w')^t = \frac{(2t)!}{2^t} (2w--)^t,$$

so the conclusion is that

$$X^{2t+q+r} \cdot (w')^t (w')^q v_1^{b-r} v_2^c = (-2)^q n! v_1^{r_1 - r_2} (w--)^r.$$

On the other hand, the unique vector in the standard basis of $W_G(r_1, -r_2; r_4 + r_2)$ having the same weight as $v_1^{r_1 - r_2} w^{n}$ is $v_1^{r_1 - r_2} w^{n}$, whose image under $X_1^n$ is $n! v_1^{r_1 + r_2}$. Hence the factor $(-2)^q$.

Let us now perform a similar computation for $v_1^{q} \otimes v_2^{q} w^t \in V_H$. The image of this in $V_G$ is clearly $\frac{(t_2 - t_1)!}{t_2!} X_{32}^{t_2} \cdot v_1^{q,t}$ and we compute that this is equal to

$$\frac{1}{(t_1)!} (w--)^t v_1^{b-r} (w')^q v_2^c$$

plus other terms killed by lower powers of $X_{12}$. Acting on this by $X_1^{q+r}$ gives $(-2)^q (q + r)! v_1^{r_1 - r_2} (w--)^r$, so its image in $W_G(r_1, -r_2; r_4 + r_2)$ has to be

$$\frac{(-2)^q}{(t_1)!} v_1^{r_1 + r_2 - m} w^m, m = q + r.$$

Remark 14.1.4. Compare [LSZ17, Theorem 9.6.4]. With the benefit of hindsight, one can observe that it would have been better to define $v_1^{q,r}$ to be $\frac{1}{(-2)^q}$ times its present definition; this would simultaneously kill the error terms $(-2)^q$ both here and in op.cit..

14.2. Unit-root splittings. Now let us consider the following construction. Our choice of embedding $V_H \otimes \text{det}^q \hookrightarrow V_G$ is strictly compatible with the filtrations, and hence gives rise to a pushforward map

$$H^0 \left( \mathcal{X}_{H, \Delta}^{\text{ord}} \right)^{\text{Fil}^m V_H} \otimes \Omega_{H}(-D) \to H^1 \left( \mathcal{X}_{G, \text{Kt}}^{\geq 1} \right)^{\text{Fil}^m V_G} \otimes \Omega_{G}^1(-D)$$

for any $m \leq n$, and dually a pullback map

$$H^2 \left( \mathcal{X}_{G, \text{Kt}}^{\geq 1} \right)^{\text{Fil}^m V_G} \otimes \Omega_{G}^1 \to H^2 \left( \mathcal{X}_{H, \Delta}^{\text{ord}} \right)^{\text{Fil}^m V_H} \otimes \Omega_{H}^1$$

(we have identified $V_H$ and $V_G$ with their own duals, up to twisting).

Remark 14.2.1. More precisely, a priori we have two slightly different versions of the pushforward map. One such map (the one which is “natural” from the point of view of de Rham cohomology) arises from tensoring the short exact sequence of sheaves on $X_G$

$$0 \to \Omega^3_{X_G} \to \Omega^3_{X_G}(\log X_H) \to \iota_*(\Omega^3_{X_H}) \to 0$$

with $\text{Fil}^n V_G(-D)$. However, from the point of view of coherent sheaves it is natural to consider instead the sequence of line bundles

$$0 \to \Omega^3_{X_G} \to \Omega^3_{X_G}(\log X_H) \to \iota_*(\Omega^3_{X_H}) \to 0$$

and tensor with $\text{Fil}^n V_G(-D) \otimes \Omega^3_{X_G} \otimes (\Omega^3_{X_G})^\vee$. The two constructions are compatible via a map

$$\iota^* (\Omega^3_{X_G} \otimes (\Omega^3_{X_G})^\vee) \otimes \Omega^3_{X_H} \longrightarrow \Omega^3_{X_H}$$

defined by dualising the natural map $\iota^* (\Omega^3_{X_G}) \longrightarrow \Omega^3_{X_H}$.

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We shall be interested in the pushforward map in the form
\[ H^0 \left( \mathcal{X}_{H,\Delta}^{\text{ord}}, \frac{V_H}{\text{Fil}^{q+1} V_H} \otimes \Omega_H^{1}(-D) \right) \to H^1 \left( \mathcal{X}_{G,Kl}^{\geq 1}, \frac{V_G}{\text{Fil}^{q+1} V_G} \otimes \Omega_G^{2}(-D) \right). \]

The sheaf on the right-hand side was denoted by \([\hat{L}_1]\) in §6 of [LPSZ19], and its cohomology was termed “automorphic nearly-coherent cohomology”. We can expand this to the following diagram:

\[
\begin{array}{ccc}
H^0 \left( \mathcal{X}_{H,\Delta}^{\text{ord}}, \frac{V_H}{\text{Fil}^{q+1} V_H} \otimes \Omega_H^{1}(-D) \right) & \to & H^1 \left( \mathcal{X}_{G,Kl}^{\geq 1}, \frac{V_G}{\text{Fil}^{q+1} V_G} \otimes \Omega_G^{2}(-D) \right) \\
\uparrow & & \uparrow \\
H^0 \left( \mathcal{X}_{H,\Delta}^{\text{ord}}, \text{Gr}^{r_1-q} V_H \otimes \Omega_H^{1}(-D) \right) & \to & H^1 \left( \mathcal{X}_{G,Kl}^{\geq 1}, \text{Gr}^{r_1} V_G \otimes \Omega_G^{2}(-D) \right) \\
\downarrow & & \downarrow \\
H^1 \left( \mathcal{X}_{G,Kl}^{\geq 1}, \mathcal{N}^2(-D) \right) & \to & H^1 \left( \mathcal{X}_{G,Kl}^{\geq 1}, \hat{\mathcal{G}}(-D) \right)
\end{array}
\]

The content of Proposition 14.1.3 is to express the lower diagonal arrow on the direct summands \(\omega_{H,t}^{(t_1-t_2)}\) and \(\omega_{H,t}^{(t_1-t_2)}\) of \(\text{Gr}^{r_1-q} V_H\) as a multiple of the “standard” pushforward maps from these spaces to \(\mathcal{N}^1\) considered in [LPSZ19] §4.6.

We now pass to the \(p\)-adic completion (i.e. we replace the dagger spaces \(\mathcal{X}_{H,\Delta}^{\text{ord}}\) and \(\mathcal{X}_{G,Kl}^{\text{ord}}\) with their underlying rigid-analytic spaces, which amounts to forgetting overconvergence).

**Notation 14.2.2.** We denote these spaces by \(\mathcal{X}_{G,Kl}^{\geq 1}\) and \(\mathcal{X}_{H,\Delta}^{\text{ord}}\).

Then we have the following diagram:

\[
\begin{array}{ccc}
H^0 \left( \mathcal{X}_{H,\Delta}^{\text{ord}}, \frac{V_H}{\text{Fil}^{q+1} V_H} \otimes \Omega_H^{1}(-D) \right) & \to & H^1 \left( \mathcal{X}_{G,Kl}^{\geq 1}, \frac{V_G}{\text{Fil}^{q+1} V_G} \otimes \Omega_G^{2}(-D) \right) \\
\uparrow & & \uparrow \\
H^0 \left( \mathcal{X}_{H,\Delta}^{\text{ord}}, \text{Gr}^{r_1-q} V_H \otimes \Omega_H^{1}(-D) \right) & \to & H^1 \left( \mathcal{X}_{G,Kl}^{\geq 1}, \text{Gr}^{r_1} V_G \otimes \Omega_G^{2}(-D) \right) \\
\downarrow & & \downarrow \\
H^1 \left( \mathcal{X}_{G,Kl}^{\geq 1}, \mathcal{N}^2(-D) \right) & \to & H^1 \left( \mathcal{X}_{G,Kl}^{\geq 1}, \hat{\mathcal{G}}(-D) \right)
\end{array}
\]

Here \(\hat{\mathcal{G}} = \mathcal{G}(3 + r_1, 1 - r_2)\) is the Banach sheaf introduced in [PH17]. The dashed arrow on the right is given by [LPSZ19] Corollary 6.15, while the dashed arrow on the left is the unit-root splitting of the Hodge filtration.

**Proposition 14.2.3.** The two maps
\[
H^0 \left( \mathcal{X}_{H,\Delta}^{\text{ord}}, \frac{V_H}{\text{Fil}^{q+1} V_H} \otimes \Omega_H^{1}(-D) \right) \to H^1 \left( \mathcal{X}_{G,Kl}^{\geq 1}, \hat{\mathcal{G}}(-D) \right),
\]
given by composing either of the two dashed arrows with the remaining maps in the diagram, coincide.

**Proof.** This follows from the argument of Theorem 6.16 of [LPSZ19]; see Remark 6.18 of op.cit. (In op.cit. the cotangent sheaf \(\Omega_H^{1} \cong \omega_{(2,0)} \oplus \omega_{(0,2)}\) was replaced by the conormal sheaf \(\ker(\iota^* \Omega_G^{1} \to \Omega_H^{1}) \cong \omega_{(1,1)}\), but this makes no difference to the argument.)

We can now summarize the computations of this section in the following corollary:

**Corollary 14.2.4.** Let \(\eta \in H^2(\mathcal{X}_{G,Kl}^{\geq 1}, \mathcal{N})\) be a class which is ordinary for the \(U_{2,Kl}\) operator, and let \(\bar{\eta}\) be its unique ordinary lifting to \(H^2(\mathcal{X}_{G,Kl}^{\geq 1}, \text{Fil}^{q} V_G \otimes \Omega_G^{1})\).

Then the linear functional on \(H^0 \left( \mathcal{X}_{H,\Delta}^{\text{ord}}, \frac{V_H}{\text{Fil}^{q+1} V_H} \otimes \Omega_H^{1}(-D) \right)\) given by pairing with \((\iota_{\Delta}^{[t_1,t_2]}\iota^*)(\bar{\eta})\) factors through the composite of restriction to \(\mathcal{X}_{H,\Delta}^{\text{ord}}\) (forgetting overconvergence) and the unit-root splitting into \(\text{Gr}^{r_1-q} V_H\); and it is given on \(H^0 \left( \mathcal{X}_{H,\Delta}^{\text{ord}}, \omega^{(t_1-t_2)} \otimes \Omega_H^{1}(-D) \right)\), where \(t = r_2 - q\), by the formula
\[
(-2)^q \left( \begin{array}{c} t_1 \\ t_2 \end{array} \right) (\eta_{p-\text{adic}}(-), \eta).
\]
where $i^\text{p-adic}_*$ denotes the pushforward map for $p$-adic modular forms defined in [LPSZ19] §4. There is an analogous formula on $\omega^{(t_1, t_2 - 2t)}$ with the factor $(-2)^t \binom{t}{7}$.

**Remark 14.2.5.** We will apply Corollary [14.2.4] later to the element $\eta_{\text{coh}, q}^{\text{ord}}$. ∎
Step 4: Computation of the regulator

15. Eisenstein series and Eisenstein classes

15.1. Hecke operators for $GL_2$. Let $k \in \mathbb{Z}$. Then we define the space of modular forms for $GL_2$ of weight $k$, denoted $M_k$, as a $GL_2(\mathbb{A})$-module, normalised such that $(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix})$, for $A \in \mathbb{Q}^+$ acts as $A^{k-2}$. This means that the double-coset operator $\left[ \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right]$ on the $\{ (a, b) \mod N \}$ invariants coincides with the classical $T_\ell$ (resp. $U_\ell$) if $\ell \nmid N$ (resp. $\ell | N$).

Remark 15.1.1. These are the same normalisations as [LPSZ19] §7.1 and §7.2.

Let $\varpi_p$ be $p$ at the place $p$, and 1 elsewhere. Then we consider the operators on $M_k$ given by

- $U_p = \sum_{i=0}^{k-1} \left( \begin{smallmatrix} \varpi_p \\ 0 \end{smallmatrix} \right)$,
- $(p) = p^{2-k} \left( \begin{smallmatrix} \varpi_p \\ 0 \end{smallmatrix} \right)$,
- $\varphi = p^{1-k} \left( \begin{smallmatrix} 0 \\ \varpi_p \end{smallmatrix} \right)$.

Note 15.1.2. (1) The first two operators preserve the space of forms of level $K_0(p^n)$ or $K_1(p^n)$, for any $n \geq 1$.

(2) The operator $\varphi$ does not preserve these forms, but sends level $p^n$ to level $p^{n+1}$.

(3) The operator $(p)$ commutes with both $U_p$ and $\varphi$, and we have $U_p \circ \varphi = (p) \circ U_p$.

Remark 15.1.3. Calling this operator “$\varphi$” is a bit abusive since the action of $GL_2(\mathbb{A})$ is linear (not semilinear). However, this operator agrees with the Frobenius on the forms that are defined over $\mathbb{Q}_p$ with respect to our $\mathbb{Q}$-model of the Shimura variety.

We shall also need to consider $p$-adic modular forms of weight $k \in \mathbb{Z}$.

Definition 15.1.4. Letting $\mathcal{Y}_0(p)$ denote the (open) modular curve of level $K^p K_0(p)$, for some prime-to-$p$ level $K^p$, we define

$$\mathcal{M}_k(K^p) = H^0 \left( \mathcal{Y}_0(p)_{ord}, \omega(k; k-2) \right)$$

where $\mathcal{Y}_0(p)_{ord}$ is the ordinary locus as a dagger space.

Note 15.1.5. For $k \geq 0$, the the differential operator

$$\Theta : \mathcal{M}_{k-1}(K^p) \longrightarrow \mathcal{M}_{k+2}(K^p)$$

twists the action of Hecke operators by the $(k+1)$-st power of the norm character. In particular, we have the relations

$$U_p \circ \Theta = p^{k+1} \Theta \circ U_p \quad \text{and} \quad \varphi \circ \Theta = p^{-1-k} \Theta \circ \varphi.$$

15.2. Eisenstein series. In [LPSZ19] §7.1 we defined real-analytic Eisenstein series $E^{(r,s)}(-, s)$ for $r \geq 1$ and $\Phi \in S(\mathbb{A}^2)$. We define $F^{k+2}_s$ by setting $r = k + 2$ and $s = -k/2$. This is a holomorphic modular form if $k \geq 1$, or if $k = 0$ and $\Phi(0,0) = 0$; its $q$-expansion is given by

$$a_n \left( F^{k+2}_\Phi \right) = \sum_{u,v \in \mathbb{Q} \atop u/v = n} u^{k+1} \text{sgn}(u) \Phi'(u,v) \quad \text{for } n > 0,$$

where $\Phi'(u,v) = \int_{A} \Phi(u,x) e^{2\pi i x v} dx$.

Remark 15.2.1. This $F^{k+2}_\Phi$ is almost the same as the $F^{k+2}_\phi$ in [LSZ17] Theorem 7.2.2]; the difference is that we have changed our normalisations for the central characters.

We will be particularly interested in the cases when $\Phi'_p$ is one of the following:
Note 15.2.2. If we transport the operators $U_p, \varphi, (p)$ over to $S(Q_p^2)$ compatibly with $\Phi \mapsto F_{\Phi}^{k+2}$, we have $U_p \cdot \Phi_{\text{dep}} = 0$. Moreover, if $\Phi'(x, y) = \text{ch}(A)$ for some open compact $A \subseteq Q_p^2$, then we have

$$(\varphi \cdot \Phi)' = \text{ch}((1, p) \cdot A), \quad (p) \cdot \Phi)' = p^{k+1} \text{ch}((p^{-1}, p) \cdot A).$$

In particular, this shows that $(1 - p) \Phi_{\text{sph}} = \Phi_{\text{crit}}$, as well as $(1 - p^{-1}) \Phi_{\text{crit}} = \Phi_{\text{dep}}$; and consequently that $F_{\Phi_{\text{crit}}}^{k+2}$ is in the $U_p = p^{k+1}$ eigenspace and $F_{\Phi_{\text{crit}}}^{k+2}$ in the $U_p = 0$ eigenspace, for any prime-to-$p$ Schwartz function $\Phi'$ (hence the terminology).

Note 15.2.3. The Eisenstein series $F_{\Phi_{\text{crit}}}^{k+2}$ is $p$-adically cuspidal, and hence so is $F_{\Phi_{\text{crit}}}^{k+2}$ (since the operator $(1 - p^{-1}) \varphi$ will preserve $p$-adic cuspforms).

As in [LPSZ19 §7.3], if $\Phi_p = \Phi_{\text{dep}}$ or $\Phi_{\text{crit}}$, we can construct a $p$-adic modular form

$$E_{\Phi}^k \in H^0(X_0(p)_{\text{ord}}, \omega^{-k})$$

of weight $-k$, such that $\theta^{k+1}(E_{\Phi}^k) = F_{\Phi}^{k+2}$. Clearly the $q$-expansion of this form must be given by

$$a_0 + \sum_{n > 0} \sum_{u = u(n)} v^{-1-k} \text{sgn}(u) \Phi'(u, v);$$

and this form is $p$-adically cuspidal if $\Phi_p = \Phi_{\text{dep}}$ (see Theorem 7.6 of op.cit.).

15.3. Eisenstein classes.

Notation 15.3.1. Denote by $Y$ the infinite level modular curve.

Write $\mathcal{H}$ for the sheaf corresponding to the defining representation of $GL_2$ on a modular curve.

Theorem 15.3.2 (Beilinson). Let $k \geq 1$. There is a $GL_2(A_f)$-equivariant map

$$S(A_f^\Gamma, Q) \to H^1_{\text{mot}}(Y, \text{Sym}^k \mathcal{H}, 1 + k), \quad \Phi \mapsto \text{Eis}^{k+2, \Phi},$$

the motivic Eisenstein symbol, with the following property: the pullback of the de Rham realization $r_{\text{dr}}(\text{Eis}^{k+2, \Phi})$ to the upper half-plane is the $H^k$-valued differential 1-form

$$-F^{(k+2)}_{\Phi}(\tau)(2\pi i)^{k+2}(2\pi i d\tau),$$

where $F^{(k+2)}_{\Phi}$ is the Eisenstein series defined by

$$F^{(k+2)}_{\Phi}(\tau) = \frac{(k + 1)!}{(-2\pi i)^{k+2}} \sum_{x, y \in \mathbb{Q}} \frac{\hat{\phi}(x, y)}{(x \tau + y)^{k+2}}.$$

Proof. See [Bei86].

Notation 15.3.3. Let $\Phi^{(p)} \in S((A_f^{(p)})^2, Q)$.

- Write $\text{Eis}^{k+2, \Phi^{(p)}, \text{crit}}_{\text{mot}} \in H^1_{\text{mot}}(Y_0(p)Q_p, \text{Sym}^k \mathcal{H}, 1 + k)$ for the syntomic realisation of the class $\text{Eis}^{k+2, \Phi^{(p)}, \text{crit}}_{\text{mot}}$, and denote by $\text{Eis}^{k+2, \Phi^{(p)}, \text{crit}}_{\text{trig-symp}}$ its image under the isomorphism in Theorem 7.1.4.
- Write $\text{Eis}^{k+2, \Phi^{(p)}, \text{crit}}_{\text{trig-symp}, \text{ord}}$ for the restriction of $\text{Eis}^{k+2, \Phi^{(p)}, \text{crit}}_{\text{trig-symp}}$ to $Y_0(p)_{\text{ord}}$.
- Write $\text{Eis}^{k+2, \Phi^{(p)}, \text{crit}}_{\text{trig-symp}, \text{ord}}$ for the image of $\text{Eis}^{k+2, \Phi^{(p)}, \text{crit}}_{\text{trig-symp}, \text{ord}}$ in Gros syntomic cohomology.

All of the above depend $GL_2(A_f^{(p)})$-equivariantly on $\Phi^{(p)}$.  

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Remark 15.3.4. Let $\Phi = (\Phi^{(p)}_{\text{crit}}, \Phi^{(p)}_{\text{ord}})$. Then

$$\widetilde{\text{Eis}}_{\text{rig-syn,ord}} = \text{Eis}_{\text{rig-syn,ord}}$$

15.4. Reduction to a $p$-adically cuspidal Eisenstein class. Let $V_H$ be as defined in Section 14.1. Recall that by Remark 9.2.22, we have a pairing, denoted $\langle \ , \rangle_{\text{rig-fp, } \mathcal{X}^{\text{ord}}}$.

$$\widetilde{H}^1_{\text{rig-fp}}(\mathcal{X}^{\text{ord}}_H, V_H, 1 + q; P_q) \times \widetilde{H}^2_{\text{rig-syn}}(\mathcal{X}^{\text{ord}}_H, V_H; 2 + t_1 + t_2) \longrightarrow \mathbb{Q}_p.$$

Aim. Recall from Corollary 12.3.6 that want to compute the quantity

$$(15.1) \quad \left\langle (t_1, t_2) \cdot (\eta_{\text{rig-fp,q,-}D})^{\text{ord}}, \text{Eis}_{\text{rig-syn,ord}} \cup \text{Eis}_{\text{rig-syn,ord}} \right\rangle_{\text{rig-fp, } \mathcal{X}^{\text{ord}}},$$

in terms of coherent cohomology.

The main tool for the evaluation is the Poznań spectral sequence constructed in Propositions 12.2.4 and 13.1.3. However, we only have explicit representatives (see Proposition 12.4.5) of $(t_1, t_2) \cdot (\eta_{\text{rig-fp,q,-}D})^{\text{ord}}$ after replacing $\eta_{\text{rig-fp,q,-}D}$ by its image $\eta_{\text{rig-fp,q}} \in \widetilde{H}^3_{\text{rig-fp, c}}(\mathcal{X}^{\text{ord}}_{\text{KL}}, V, 1 + q; P_q)$. In order to be able to evaluate (15.1), we therefore need to replace the Eisenstein class by a version which is $p$-adically cuspidal.

Since rig-fp cohomology is compatible with change of polynomial, we have a natural map

$$\tilde{H}^1_{\text{rig-fp}}(\mathcal{Y}_0(p)^{\text{ord}}, \text{Sym}^k \mathcal{H}, 1 + k; \text{const} 1) \longrightarrow \tilde{H}^1_{\text{rig-syn}}(\mathcal{Y}_0(p)^{\text{ord}}, \text{Sym}^k \mathcal{H}, 1 + k).$$

Lemma 15.4.1. The class $\text{Eis}_{\text{rig-syn,ord}}$ is in the image of $\tilde{H}^1_{\text{rig-fp}}(\mathcal{Y}_0(p)^{\text{ord}}, \text{Sym}^k \mathcal{H}, 1 + k; \text{const} 1)$. In other words, we can lift it to an element $\text{Eis}_{\text{rig-fp, const,1,ord}} \in \tilde{H}^1_{\text{rig-fp}}(\mathcal{Y}_0(p)^{\text{ord}}, \text{Sym}^k \mathcal{H}, 1 + k; \text{const} 1)$.

Proof. This is just the statement that the critical-slope Eisenstein series is integrable over the ordinary locus.

Note 15.4.2. The class $\text{Eis}_{\text{rig-fp, const,1,ord}}$ is not in the image of $\tilde{H}^1_{\text{rig-fp}}(\mathcal{X}_0(p)^{\text{ord}}(-D), \text{Sym}^k \mathcal{H}, 1 + k; \text{const} 1)$ – the “degree 1 part” of our fp-pair is cuspidal, but the “degree 0 part” is not – but the constant term of the degree 0 part gets annihilated by $(1 - p^{k+1}(p)_\text{GL}_2)$, which corresponds to $1 - V_p$ on $q$-expansions in weight $-k$.

Lemma 15.4.3. The image of $\text{Eis}_{\text{rig-fp, const,1,ord}}$ under the natural map

$$\tilde{H}^1_{\text{rig-fp}}(\mathcal{Y}_0(p)^{\text{ord}}, \text{Sym}^k \mathcal{H}, 1 + k; \text{const} 1) \longrightarrow \tilde{H}^1_{\text{rig-fp}}(\mathcal{Y}_0(p)^{\text{ord}}, \text{Sym}^k \mathcal{H}, 1 + k; 1 - p^{k+1}(p)_{\text{GL}_2}^{-1} t)$$

lifts to a class

$$\text{Eis}_{\text{rig-fp, cusp,ord}} \in \tilde{H}^1_{\text{rig-fp}}(\mathcal{X}_0(p)^{\text{ord}}(-D)_{\text{GL}_2}, \text{Sym}^k \mathcal{H}, 1 + k; (1 - p^{k+1}(p)_{\text{GL}_2}^{-1} t)).$$

Proof. Immediate from Note 15.4.2.

Remark 15.4.4. These constructions are summarized by the following diagram (where we leave out the coefficients for reasons of space):

$$\begin{array}{ccc}
\tilde{H}^1_{\text{rig-syn}}(\mathcal{Y}_0(p)^{\text{ord}}) & \longrightarrow & \tilde{H}^1_{\text{rig-fp}}(\mathcal{Y}_0(p)^{\text{ord}}; \text{const} 1) \\
\downarrow & & \downarrow \\
\tilde{H}^1_{\text{rig-syn}}(\mathcal{Y}_0(p)^{\text{ord}}) & & \tilde{H}^1_{\text{rig-fp}}(\mathcal{X}_0(p)^{\text{ord}}(-D)_{\text{GL}_2}; 1 - p^{k+1}(p)^{-1} t) \\
\downarrow & & \downarrow \\
\tilde{H}^1_{\text{rig-fp}}(\mathcal{Y}_0(p)^{\text{ord}}(1 - t)(1 - p^{k+1}(p)_{\text{GL}_2}^{-1} t)) & & \tilde{H}^1_{\text{rig-fp}}(\mathcal{Y}_0(p)^{\text{ord}}; 1 - p^{k+1}(p)_{\text{GL}_2}^{-1} t) \\
\end{array}$$

Here, the diagonal arrows are given by the formalism for change of polynomial in fp-cohomology. We refer to this as the **herb-chopper diagram**.
15.5. The $\text{GL}_2$-Eisenstein class as a coherent fp-pair. We want to find representatives of the image of the class $\widetilde{\text{Eis}}_{\text{rig-fp},\text{cusp,ord}}^{k+2,\Phi(p),\text{crit}}$ under the map

$$\alpha_{\text{GL}_2}^{-1} : \tilde{H}_{\text{rig-fp}}^1(\mathcal{X}_{0}(p)_{\text{ord}}(-\mathcal{D}_{\text{GL}_2}), \text{Sym}^k\mathcal{H}, 1 + k; 1 - p^{k+1}(p)_{\text{GL}_2}^{-1}) \to H_{\text{rig-symp}}^1(\mathcal{X}_{0}(p)_{\text{ord}}(-\mathcal{D}_{\text{GL}_2}), \text{Sym}^k\mathcal{H}, 1 + k; 1 - p^{k+1}(p)_{\text{GL}_2}^{-1})$$

constructed in Section 13 (c.f. Note 13.1.6).

Notation 15.5.1. Denote by $\tilde{\omega}$ and $\tilde{\psi}$ the basis of sections $\tilde{\omega}$ and $\tilde{\psi}$ of $\mathcal{H}$ over the Igusa tower, as constructed in [KLZ17] §4.5.

Proposition 15.5.2. The class $\widetilde{\text{Eis}}_{\text{rig-fp},\text{cusp,ord}}^{k+2,\Phi(p),\text{crit}}$ is represented by the pair $\left(\left(k,\Phi^{(p)}_0,\xi,1,\Phi^{(p)}_1\right), \left(k,\Phi^{(p)}_0,\xi,1,\Phi^{(p)}_1\right)\right)$, where

$$\epsilon_0^{k,\Phi^{(p)}} = \sum_{j=0}^{k} \frac{(-1)^j k!}{(k-j)!} \ell_k^{k-j} E_{\Phi^{(p)}_0}^{k-j}, \quad \epsilon_1^{k,\Phi^{(p)}} = \ell_k^{k+2} \Phi^{p,\text{crit}} \cdot \ell_k \cdot \xi \cdot \epsilon_1,$$

where $\xi$ is as defined in [KLZ20] §4.5.

Proof. [BK10] Theorem 5.11] describes the coherent fp-pair representing the class $\widetilde{Eis}_{\text{rig-fp},\text{cusp,ord}}^{k+2,\Phi(p),\text{crit}}$, and it is easy to check that the modifications of the Eisenstein class described in Section 15.4 give the claimed result.

Lemma 15.5.3. We have

$$U_p \left(k,\Phi^{(p)}_0\right) = 0 \quad \text{and} \quad U_p \left(k,\Phi^{(p)}_1\right) = p^{k-1} \epsilon_1^{k,\Phi^{(p)}}.$$

Proof. Clear from Note 15.2.2 and from the fact that $\varphi^{-1}(w) = w$ and $\varphi^{-1}(\xi) = p^{-2}\xi$ (c.f. [KLZ20] §5.4).

15.6. The Eisenstein class for $H$ as a coherent fp-pair.

Lemma 15.6.1. Let $\Phi^{(p)}_1, \Phi^{(p)}_2 \in \mathcal{S}((\mathcal{A}_p^p)^2)$. Then the image of

$$\left(Eis_{\text{rig-fp},\text{cusp,ord}}^{t_1+2,\Phi^{(p)}_1,\text{crit}}, Eis_{\text{rig-fp},\text{cusp,ord}}^{t_2+2,\Phi^{(p)}_2,\text{crit}}\right)$$

under the isomorphism $\alpha_{\Delta}^{-1}$ (c.f. Corollary 13.1.3) is represented by the coherent fp-pair

$$\left(\alpha_{t_1,t_2,\Phi^{(p)}_1,\Phi^{(p)}_2}, \alpha_{t_1,t_2,\Phi^{(p)}_1,\Phi^{(p)}_2}\right) \in H^{2,1}(\lambda_{\text{ord}}, \mathcal{V}_H, 2 + t_1 + t_2; Q),$$

where $Q(g) = 1 - p^{t_1+t_2}(p)_{H}^{-1}$ and

$$\alpha_{t_1,t_2,\Phi^{(p)}_1,\Phi^{(p)}_2}^{(p)} = \epsilon_0^{t_1,\Phi^{(p)}_1} \cup \epsilon_1^{t_2,\Phi^{(p)}_2} + p^{t_1+1}(p)_{\text{GL}_2}^{-1} \left(\epsilon_1^{t_1,\Phi^{(p)}_1} \cup \epsilon_2^{t_2,\Phi^{(p)}_2}\right),$$

$$\alpha_{t_1,t_2,\Phi^{(p)}_1,\Phi^{(p)}_2}^{(p)} = \epsilon_0^{t_1,\Phi^{(p)}_1} \cup \epsilon_1^{t_2,\Phi^{(p)}_2}.$$

Here, $\epsilon_{\ell}^{t_1,\Phi^{(p)}}, \Phi^{(p)}$ is as defined in Lemma 15.5.2, and we write $\langle \rho \rangle_H$ for $\{\rho\} \boxtimes \{\rho\}$.

Proof. Clear from the cup product formalism and Lemma 13.2.1.

16. Evaluation of the pairing

16.1. Reduction to coherent cohomology. We will now evaluate the pairing (15.1).

By the herb–chopper diagram (or rather: by the analogous diagram for $\lambda_{\text{ord}}^{\text{sym}}$) and the compatibility of the pairings under change of polynomial, (15.1) is equal to

$$\left\langle \left(\xi_{t_1,t_2}\right)^{\text{ord}}, Eis_{\text{rig-fp},\text{cusp,ord}}^{t_1+2,\Phi^{(p)}_1,\text{crit}} \right\rangle \left\langle Eis_{\text{rig-fp},\text{cusp,ord}}^{t_1+2,\Phi^{(p)}_1,\text{crit}}, Eis_{\text{rig-fp},\text{cusp,ord}}^{t_2+2,\Phi^{(p)}_2,\text{crit}} \right\rangle$$

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Lemma 16.1. The pairing \([16.1]\) is equal to

\[
\left(\left( \left( \frac{[t_1, t_2]}{\Delta} \right)^* \right)^* (\zeta, \eta_{\text{coh}, q}) \right)_{\text{coh-fo}, X_{\text{fp}}^q}\ ,
\]

where \((\zeta, \eta_{\text{coh}, q})\) is as defined in Proposition \([12.4]\).

**Proof.** Immediate from Proposition \([12.4.5]\) and Lemmas \([13.2.1]\) and \([15.6.1]\). \(\square\)

To evaluate this pairing, we use Besser's formalism for computing the cup product, as explained in Section \(13.2.2\). Let

\[
a(x, y) = p^{t_1 + t_2 + 2}(p)_H^{-1} y
\]

and

\[
b(x, y) = \frac{P_q \left( p^{t_1 + t_2 + 2}(p)_H^{-1} y \right) - p^{t_1 + t_2 + 2}(p)_H^{-1} y P_q(x)}{1 - p^{t_1 + t_2 + 2}(p)_H y},
\]

so we have

\[
P_q * Q(x, y) = a(x, y) P_q(x) + b(x, y) \left( 1 - p^{t_1 + t_2 + 2}(p)_H^{-1} y \right).
\]

Then \((16.2)\) is equal to

\[
a(\varphi^*_{H, 1} \otimes 1, 1 \otimes \varphi^*_H) \left( (\Delta)^* (\zeta) \cup \alpha_2^{t_1, t_2, \Phi^{(p)}, \Phi^{(p)}} \right) + b(\varphi^*_{H, 1} \otimes 1, 1 \otimes \varphi^*_H) \left( (\Delta)^* (\zeta) \cup \alpha_2^{t_1, t_2, \Phi^{(p)}, \Phi^{(p)}} \right).
\]

**Proposition 16.1.2.** We have

\[
(16.2) = b(\varphi^*_{H, 1} \otimes 1, 1 \otimes \varphi^*_H) \left( (\Delta)^* (\zeta) \cup \alpha_2^{t_1, t_2, \Phi^{(p)}, \Phi^{(p)}} \right).
\]

**Proof.** We need to show that

\[
(\Delta)^* (\zeta) \cup \alpha_2^{t_1, t_2, \Phi^{(p)}, \Phi^{(p)}} = 0.
\]

Now \(U_p \boxtimes U_p \circ \varphi^*_H = (p)_H\), so

\[
(16.3) \quad (U_p \boxtimes U_p) \left( (\Delta)^* (\zeta) \cup \alpha_2^{t_1, t_2, \Phi^{(p)}, \Phi^{(p)}} \right) = (U_p \boxtimes U_p)(\Delta)^* (\zeta) \cup \varphi^*_H \alpha_2^{t_1, t_2, \Phi^{(p)}, \Phi^{(p)}} .
\]

But \((U_p \boxtimes U_p)(\Delta)^* (\zeta) = 0\) by Lemma \([12.4.4]\) and Proposition Corollary \([12.5.3]\) and hence \((16.3)\) is zero. Now by Note \([13.1.5]\), the operator \(U_p \boxtimes U_p\) acts as multiplication by \(p^{-2}\) on \(H^{t_1, t_2}(X_{\Delta}^{t_1, t_2}(D_{\Delta}), Q_p, s; P*)Q\) and hence is invertible. This finishes the proof. \(\square\)

Write \(P(x) = 1 + c_1 x + c_2 x^2\); by definition, we have \(c_2 = (\alpha \beta)^{-1}\) and \(c_1 = -\frac{\alpha + \beta}{\alpha \beta}\). Then

\[
b(x, y) = 1 - c_2 p^{t_1 + t_2 + 2}(p)_H^{-1} x y^2.
\]

We now identify \(\varphi^*_H\) with \(p^{-1} \varphi_H\).

**Corollary 16.1.3.** We have

\[
(16.2) = \left( (\Delta)^* (\zeta) \cup \alpha_2^{t_1, t_2, \Phi^{(p)}, \Phi^{(p)}} \right) - c_2 p^{t_1 + t_2}(p)_H^{-1} \varphi^*_H \left( (\Delta)^* (\zeta) \cup \alpha_2^{t_1, t_2, \Phi^{(p)}, \Phi^{(p)}} \right)
\]

\[
(16.3) = \left( \left( \left( \frac{[t_1, t_2]}{\Delta} \right)^* \right)^* (\zeta, \eta_{\text{coh}, q}) \right)_{\text{coh-fo}, X_{\text{fp}}^q}\ ,
\]

\[
- c_2 p^{t_1 + t_2}(p)_H^{-1} \varphi^*_H \left( (\Delta)^* (\zeta) \cup \alpha_2^{t_1, t_2, \Phi^{(p)}, \Phi^{(p)}} \right)
\]

\[
+ p^{t_1} \left( (\Delta)^* (\zeta) \cup \left( (p)_H^{-1} \varphi^*_H \left( (\Delta)^* (\zeta) \cup \alpha_2^{t_1, t_2, \Phi^{(p)}, \Phi^{(p)}} \right) \right) + p^{t_2} \left( (\Delta)^* (\zeta) \cup \left( (p)_H^{-1} \varphi^*_H \left( (\Delta)^* (\zeta) \cup \alpha_2^{t_1, t_2, \Phi^{(p)}, \Phi^{(p)}} \right) \right) \right)
\]

We will see that this expression simplifies.
Lemma 16.1.4. We have
\[ \varphi_H \left( (t_{t_1,t_2})^* \varphi_H^* (\psi_0^\text{ord} \cup \left( \epsilon_0 \Phi_1^\text{ord} \sqcup e_1 \Phi_2^\text{ord} \right) \right) = 0 \]
\[ \varphi_H^* \left( (t_{t_1,t_2})^* \varphi_H^* (\psi_0^\text{ord} \cup \left( \epsilon_0 \Phi_1^\text{ord} \sqcup e_1 \Phi_2^\text{ord} \right) \right) = 0 \]

Proof. Apply \((U_p \oplus U_p)^2\) and use Note 13.1.5 and the fact that \(U_p \left( \epsilon_0 \Phi_1^\text{ord} \right) = 0\) by Lemma 15.5.3.

We hence deduce the following formula for the pairing:

Proposition 16.1.5. We have
\[
\langle t_{t_1,t_2}^* \psi_0^\text{ord}, \epsilon_1 \epsilon_2 \Phi_2^\text{ord} \rangle = \langle t_{t_1,t_2}^* \psi_0^\text{ord}, \epsilon_1 \epsilon_2 \Phi_2^\text{ord} \rangle
\]

We now apply Corollary 14.2.4.

**Note 16.1.6.** For \(0 \leq \ell \leq t_1 + t_2\). A basis of \(\text{Gr}^\ell V_H\) is given by
\[ \{ v^{t_1-i_1} w^{i_1} \times v^{t_2-i_2} w^{i_2} : 0 \leq i_1 \leq t_n, i_1 + i_2 = t_1 + t_2 - \ell \}. \]

\[ \diamond \]

**Lemma 16.1.7.** The image of \(\epsilon_0 \Phi_1^\text{ord} \sqcup \epsilon_1 \epsilon_2 \Phi_2^\text{ord}\) under projection to \(\text{Gr}^{t_1 - q} V_H\) is given by
\[
(1 - 2)^q \frac{1}{r!} \times \frac{t_1!}{(r_1 - r_2 - q)!} \times \langle \psi_0^\text{ord}, \epsilon_1 \epsilon_2 \Phi_2^\text{ord} \rangle \left( \psi_0^\text{ord}, \epsilon_1 \epsilon_2 \Phi_2^\text{ord} \right)
\]

Proof. The basis vectors with non-trivial coefficients of \(\epsilon_0 \epsilon_1 \epsilon_2 \Phi_2^\text{ord}\) are of the form
\[ v^{t_1-i_1} w^{i_1} \times v^{t_2-i_2} w^{i_2} \quad 0 \leq i \leq t_1. \]

By Note 16.1.6 this will project non-trivially to \(\text{Gr}^{t_1 - q} V_H\) if and only if
\[ i_1 = t_1 - (r_1 - q) = r. \]

\[ \square \]

We analogously prove the following result:

**Lemma 16.1.8.** The image of \(\langle \psi_0^\text{ord}, \epsilon_1 \epsilon_2 \Phi_2^\text{ord} \rangle\) in \(\text{Gr}^{t_1 - q} V_H\) is given by
\[
(1 - 2)^q \frac{1}{r!} \times \frac{t_1!}{(r_1 - r_2 - q)!} \times \langle \psi_0^\text{ord}, \epsilon_1 \epsilon_2 \Phi_2^\text{ord} \rangle \left( \psi_0^\text{ord}, \epsilon_1 \epsilon_2 \Phi_2^\text{ord} \right)
\]

**Proposition 16.1.9.** We have
\[
(t_{t_1,t_2})^* \left( \psi_0^\text{ord} \cup \left( \epsilon_0 \Phi_1^\text{ord} \sqcup e_1 \Phi_2^\text{ord} \right) \right) = (1 - 2)^q \frac{1}{r!} \times \frac{t_1!}{(r_1 - r_2 - q)!} \times \left\langle \psi_0^\text{ord}, \epsilon_1 \epsilon_2 \Phi_2^\text{ord} \right\rangle
\]

and
\[
(t_{t_1,t_2})^* \left( \psi_0^\text{ord} \cup \left( \epsilon_0 \Phi_1^\text{ord} \sqcup e_1 \Phi_2^\text{ord} \right) \right) = (1 - 2)^q \frac{1}{r!} \times \frac{t_1!}{(r_1 - r_2 - q)!} \times \left\langle \psi_0^\text{ord}, \epsilon_1 \epsilon_2 \Phi_2^\text{ord} \right\rangle
\]
16.2. Families of Eisenstein series. We now interpret the cup-products of Proposition 4.9 in terms of the 2-parameter $p$-adic family of Eisenstein series studied in [LPSZ19], and a 1-parameter “critical” variant.

Proposition 16.2.1. If $\Phi^{(p)} \in S(A_p, (\chi^{(p)})^{-1})$, then the $p$-adic Eisenstein series $E_{\Phi^{(p)}, \text{dep}}^{(1)}$ is the specialisation at $(\kappa_1, \kappa_2) = (0, -1 - k)$ of a 2-parameter family of Eisenstein series $\mathcal{E}^{\Phi^{(p)}}(\kappa_1, \kappa_2)$. This family has $q$-expansion

$$\sum_{u,v \in (Z_p)^2, u,v > 0} \text{sgn}(u)u^{\kappa_1}v^{\kappa_2}(\Phi^{(p)})'((u,v)q^{uv}),$$

and its specialisation at $(a + b + \nu, \nu)$, for integers $a,b \geq 0$ and finite-order characters $\nu$ of $\mathbb{Z}_p^\times$, is the $p$-adic modular form associated to the algebraic nearly-holomorphic modular form

$$(g, \tau) \mapsto \nu(\det g) \cdot E^{(a+b+1, \Phi_{\text{crit}, \nu}, \nu)}(g, \tau; \chi^{(p)}(\mu - 1) - \frac{b - a + 1}{2}).$$

Here, $\Phi_{\text{dep}, \mu, \nu}$ is defined as in [LPSZ19] §7.3.

**Proof.** See [LPSZ19] Theorem 7.6. □

We can also put critical-slope Eisenstein series into 1-parameter $p$-adic families:

Proposition 16.2.2. Let $\ell \geq 0$. Then there exists a 1-parameter family of Eisenstein series $\mathcal{E}^{\Phi^{(p)}}(\ell, \kappa)$ with $q$-expansion

$$\sum_{u \in \mathbb{Z}_p, v \in (Z_p)^2, u,v > 0} \text{sgn}(u)u^{\ell}v^{\kappa}(\Phi^{(p)})((u,v)q^{uv}).$$

(Here, we underline the parameter which does not vary in a $p$-adic family.) Its specialisation at $a + \nu$, for an integer $a \geq 0$ and a finite-order character $\nu$ of $\mathbb{Z}_p^\times$, is the $p$-adic modular form associated to the algebraic nearly-holomorphic modular form

$$(g, \tau) \mapsto \nu(\det g) \cdot E^{(\ell + b + 1, \Phi_{\text{crit}, \nu}, \nu)}(g, \tau; \chi^{(p)}(\mu - 1) - \frac{b - \ell + 1}{2}),$$

where $\Phi_{\text{crit}, \nu}(x,y) = \text{ch}(\mathbb{Z}_p \times \mathbb{Z}_p^\times)(x,y) \cdot \nu(y)$.

We can now restate Proposition 4.9 in the following form:

Proposition 16.2.3. Let us define

$$\mathcal{L}_1 := \left\langle \text{ord}_{\text{coh}, \kappa} p^{-a} \text{ord}_{\text{coh}, a} \left(\mathcal{E}^{\Phi^{(p)}}(r', -1 - q') \boxtimes \mathcal{E}^{\Phi^{(p)}}(q' + r + 1, 0)\right)\right\rangle,$$

$$\mathcal{L}_2 := \left\langle \text{ord}_{\text{coh}, \kappa} p^{-a} \text{ord}_{\text{coh}, a} \left(\mathcal{E}^{\Phi^{(p)}}(q' + r' + 1, 0) \boxtimes \mathcal{E}^{\Phi^{(p)}}(r', -1 - q')\right)\right\rangle,$$

where $q' = r_2 - q$ and $r' = r_1 - r_2 - r$ (so $q', r' \geq 0$). Then the cup-product (15.1) is equal to

$$\left(1 - \frac{\eta}{\mu + \eta}\right) \left(1 - \frac{\delta}{\mu + \eta}\right) \times (\mathcal{L}_1 + p^{r_1 + r_2 - 2\ell} \mathcal{L}_2).$$

**Proof.** In the above notation, the two terms appearing in (16.1) are

$$\left(\epsilon^{(t_1, t_2)}_{\Delta} \cap \text{ord}_{\text{coh}, a}\right) \cup \left(\epsilon^{(t_1, t_2)}_{\Delta} \cap \text{ord}_{\text{coh}, a}\right) = (\epsilon^{t_2} - q)(-2)(r_2 - \eta)! \times \mathcal{L}_1,$$

$$\left(\epsilon^{(t_1, t_2)}_{\Delta} \cap \text{ord}_{\text{coh}, a}\right) \cup \left(\epsilon^{(t_1, t_2)}_{\Delta} \cap \text{ord}_{\text{coh}, a}\right) = (\epsilon^{t_2} - q)(-2)(r_2 - \eta)! \times \mathcal{L}_2.$$

The normalisation of the trace map on finite-polynomial cohomology gives rise to a factor $P_q(p^{r_1 + t_2 + 1} \chi_n(p))$, and using the relation $\alpha \delta = \beta \gamma = p^{r_1 + t_2 + 3} \chi_{\Pi}(p)$, we deduce that

$$P_q(p^{r_1 + t_2 + 1} \chi_n(p)) = \left(1 - \frac{\gamma}{\mu + \eta}\right) \left(1 - \frac{\delta}{\mu + \eta}\right).$$

We will see shortly that $\mathcal{L}_2$ is in fact zero, and that $\mathcal{L}_1$ coincides with a non-critical $p$-adic $L$-value. We first make a preliminary reduction.

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Proposition 16.3.2. We have

\[ L_1 = -\left\langle \eta_{\text{coh}}, \iota_*^{p-\text{adic}} \left( E^{\Phi_1(p)}(r_1 - q + 1, r) \boxtimes E^{\Phi_2(p)}(0, -1 - q' - r) \right) \right\rangle. \]

and similarly

\[ L_2 = (-1)^{r_1 - r_2 + 1} \left\langle \eta_{\text{coh}}, \iota_*^{p-\text{adic}} \left( (p)^{-1} \varphi_{G\ell_2} E^{\Phi_1(p)}(0, -1 - q' - r') \boxtimes E^{\Phi_2(p)}(r_1 - q + 1, r') \right) \right\rangle. \]

Proof. Both of these statements follow from the general fact that

\[ \langle \eta_{\text{coh}}, \iota_*^{p-\text{adic}} [F \boxtimes \theta(G) + \theta(F) \boxtimes G] = 0 \]

for any nearly-overconvergent $p$-adic modular forms $F, G$ whose weights sum to $r_1 - r_2$. (This, in turn, follows from the fact that $F \boxtimes \theta(G) + \theta(F) \boxtimes G$ is the projection to a graded piece of the Hodge filtration of an overconvergent vector-valued form in the image of $\nabla$, which must pair to 0 with $\eta_{\text{coh}}^{p-\text{adic}}$ since $\nabla(\eta_{\text{coh}}) = 0$).

16.3. Evaluation of $L_1$. We shall now evaluate $L_1$. We shall do this by interpreting this value as the specialisation at the trivial character of a measure on $Z_p^r$, whose values at certain other characters (corresponding to critical $L$-values) can be compared with the $p$-adic $L$-function of $[LPSZ19]$.

Definition 16.3.1. Define an element of $\Lambda(Z_p^r \times Z_p^s)$ by

\[ L_1(j_1, j_2) := \left\langle \eta_{\text{coh}}, \iota_*^{p-\text{adic}} \left( E^{\Phi_1(p)}(r_1 - r_2 - j_1, j_2) \boxtimes E^{\Phi_2(p)}(0, j_1 - j_2) \right) \right\rangle. \]

Proposition 16.3.2. We have

\[ L_1(j_1, j_2) = \left\langle \eta_{\text{coh}}, \iota_*^{p-\text{adic}} \left( E^{\Phi_1(p)}(r_1 - r_2 - j_1, j_2) \boxtimes E^{\Phi_2(p)}(0, j_1 - j_2) \right) \right\rangle \]

(without the underline).

Note that the Eisenstein series in the second formula is exactly the $E^\Phi(p)$ appearing in Proposition 6.9.5 (compare the choices of parameters above with the formulae of $[LPSZ19] \S7.4$).

Proof. It suffices to prove that these two measures agree after specialising at $(j_1, j_2) = (a_1 + p_1, a_2 + \rho_2)$ with $p_1$ finite-order characters and $r_1 - r_2 \geq a_1 \geq a_2 \geq 0$. In this range, both sides of the claimed formula reduce to cup-products in classical coherent cohomology; and as in $[LPSZ19]$, they can be written as Euler products of local integrals at each place, with the factors at all primes except possibly $p$ being identical. The computation of $[20.3]$ shows that the factors at $p$ are also equal (despite the slightly different choice of test data). Thus the two measures are equal. \( \square \)

Substituting $(j_1, j_2) = (-1 - r_2 + q, r)$, and applying Proposition 6.9.5, we conclude that

\[ L_1 = -\frac{\tilde{Z}(w, \Phi(p))}{\text{vol}(V)} L_{p,r}(1, -1 - r_2 + q, r). \]

16.4. Vanishing of $L_2$. In order to show that $L_2$ is identically zero, we will use a similar deformation argument. Let us write

\[ L_2(j_1, j_2) = \left\langle \eta_{\text{coh}}, \iota_*^{p-\text{adic}} \left( (p)^{-1} \varphi_{G\ell_2} E^{\Phi_1(p)}(0, j_1 - j_2) \boxtimes E^{\Phi_2(p)}(r_1 - r_2 - j_1, j_2) \right) \right\rangle. \]

Again, if we let $j_1, j_2 = (a_1 + p_1, a_2 + \rho_2)$ with $r_1 - r_2 \geq a_1 \geq a_2 \geq 0$ and $p_1$ of finite order, we obtain a cup-product in classical coherent cohomology; and the value $L_2$ above corresponds (up to a sign) to specialising at $(j_1, j_2) = (-1 - r_2 + q, r')$. However, for all of the specialisations corresponding to critical values, the term at $p$ in the resulting product is 0, again by the computations in $[20.3]$ So the measure $L_2(j_1, j_2)$ is identically 0, and hence so is its special value $L_2$. 

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16.5. Conclusion of the proof.

**Proof (of Theorem 6.8.5).** The computation in this chapter shows that

\[
\left\langle \left( \alpha_{t_1.t_2}, \phi_{p_1}^{(p)}, \phi_{p_2}^{(p)} \right), \left( t_{\Delta} \right)^* (\zeta_{\text{coh},q}) \right\rangle_{\text{coh-fp}, \chi_{G,\Delta}} = \frac{(-1)^{r_2-q+1} (2)^q (r_2 - q)!}{\left( 1 - \frac{q}{p+r} \right) \left( 1 - \frac{4}{p+r} \right)} \times \left\langle \eta_{t,\Delta,*} (E(\Phi^p)) \right\rangle_{\chi_{G,\Delta}}
\]

in the notation of Proposition 6.9.6:

\[
\left\langle \left( \alpha_{t_1.t_2}, \phi_{p_1}^{(p)}, \phi_{p_2}^{(p)} \right), \left( t_{\Delta} \right)^* (\zeta_{\text{coh},q}) \right\rangle_{\text{coh-fp}, \chi_{G,\Delta}} = \left\langle \text{Eis}_{\text{rig-fp}, \text{cusp,ord}} \cup \text{Eis}_{\text{rig-fp}, \text{cusp,ord}^*} \left( t_{\Delta} \right)^* (\eta_{\text{ord}}_{\text{rig-fp},q}) \right\rangle_{\text{rig-fp}, \chi_{G,\Delta}} \quad \text{by (16.1)}
\]

However,

\[
\left\langle \text{Eis}_{\text{rig-syn}, \Phi_{\text{ord}}} \left( t_{\Delta} \right)^* (\eta_{\text{rig-fp},q,-D}) \right\rangle_{\text{rig-fp}, \chi_{G,\Delta}} \quad \text{by Cor. 12.3.6}
\]

\[
\left\langle \text{Eis}_{\text{rig-syn}, \Phi_{\text{ord}}} \left( t_{\Delta} \right)^* (\eta_{\text{rig-fp},q,-D}) \right\rangle_{\text{rig-fp}, \chi_{G,\Delta}} \quad \text{by Thm. 10.4.6}
\]

\[
\left\langle \text{Eis}_{\text{rig-syn}, \Phi_{\text{ord}}} \left( t_{\Delta} \right)^* (\eta_{\text{rig-fp},q,-D}) \right\rangle_{\text{rig-fp}, \chi_{G,\Delta}} \quad \text{by Prop. 10.4.2}
\]

So we have proved the equality of the two sides of (6.2); and Proposition 6.9.6 shows that this assertion is equivalent to Theorem 6.8.5. \qed
Step 5: Deformation to critical values

17. Hida families

We will now change our focus slightly: rather than working with a single, fixed automorphic representation $\Pi$, we shall consider $p$-adic families of these objects. In order to avoid fiddly issues involving choices of test vectors at ramified primes, we shall suppose for simplicity that $\Pi$ has level 1 from here onwards (i.e. that $\Pi_\ell$ is unramified for all finite primes $\ell$). Note that this implies that $r_1 - r_2$ is even, and that the central character $\chi_s$ is trivial.

17.1. Families of Galois representations.

Notation 17.1.1. Let $W$ denote the $p$-adic weight space $\operatorname{Hom}(\mathbf{Z}_p^\times, \mathbf{G}_{m,L}^{\operatorname{rig}})$ (the analytification of the formal scheme $\operatorname{Spf} \mathcal{O}_L[[\mathbf{Z}_p^\times]]$). For $\epsilon \in \{\pm 1\}$ we write $W^\epsilon$ for the union of components classifying characters with $\kappa(-1) = \epsilon$, so $W = W^+ \sqcup W^-$. 

Definition 17.1.2. Let $U$ be an affinoid disc in $W$ containing 0. By a Siegel-type Hida family $\Pi$ over $U$ of tame level 1 (passing through weight $(r_1, r_2)$), we shall mean the following data:

- for each $n \in U \cap \mathbf{Z}_{\geq 0}$, a cuspidal automorphic representation $\Pi(n)$ of $\mathbf{GSp}_4$ which is globally generic, cohomological at $\infty$ with coefficients in $V(r_1 + n, r_2 + n)$, and has level 1;
- for each such $n$, an embedding of the coefficient field of $\Pi(n)$ into $L$, with respect to which $\Pi(n)$ is Siegel-ordinary at $p$;
- a collection of rigid-analytic functions $t_{1,\ell}, t_{2,\ell} \in \mathcal{O}(U)$, for $i = 1, 2$ and $\ell \neq p$, such that for each $n \in U \cap \mathbf{Z}_{\geq 0}$, the values of $t_{1,\ell}$ and $t_{2,\ell}$ at $n$ are the eigenvalues of the spherical Hecke operators $\operatorname{diag}(\ell, 1, 1)$ and $p^{-(r_2+n)} \operatorname{diag}(\ell^2, \ell, \ell, 1)$ on the arithmetic twist $\Pi(n)$;
- rigid-analytic functions $u_{1,p}, u_{2,p} \in \mathcal{O}(U)$ for $i = 1, 2$, with $u_{1,p}$ taking $p$-adic unit values, such that for all $n \in U \cap \mathbf{Z}_{\geq 0}$, we can write the Hecke parameters of $\Pi(n)$ as $(\alpha_n, \beta_n, \gamma_n, \delta_n)$ with

$$u_{1,p}(n) = \alpha_n, \quad u_{2,p}(n) = \frac{\beta_n + \gamma_n}{p^{(r_2+n)}}.$$

The following theorem is fundamental:

Theorem 17.1.3 (Tilouine–Urban). For any $\Pi$ which satisfies the conditions of $\S 6.2$ and is unramified and Siegel-ordinary at $p$, there exists a disc $U \subset W$ around 0, and an ordinary family of eigensystems $\Pi$ over $U$, such that $\Pi(0) = \Pi$.

Remark 17.1.4. Note that Klingen-ordinarity is not needed for this theorem, nor for the constructions below, until Corollary $\S 17.3.1$. However, Siegel-ordinarity is fundamental here (whereas it plays no role in the main body of the paper).

The computations of op.cit. also give rise to a natural $\mathcal{O}(U)$-module with an action of Hecke operators, which interpolate the $\Pi(n)$-eigenspace in Betti cohomology of level $G(\hat{\mathbf{Z}})$ (with coefficients varying with $n$). One can equally work with étale cohomology, to obtain the following:

Theorem 17.1.5. In the situation of Theorem 17.1.3 after possibly shrinking $U$, there exists a free rank 4 $\mathcal{O}(U)$-module $W_{\Pi}$, whose fibre at $n \in U \cap \mathbf{Z}_{\geq 0}$ is canonically isomorphic to the Galois representation $W_{\Pi(n)}$.

Note 17.1.6. More precisely, the fibre at $n$ of $W_{\Pi}$ is canonically identified with the subspace of étale cohomology of level $G(\hat{\mathbf{Z}}(p)) \times \operatorname{Si}(p)$ on which the prime-to-$p$ Hecke operators act via the eigensystem of $\Pi_\ell(n)$ and $U_{1,\ell}$ acts as $\alpha_n = u_{1,p}(n)$. This, in turn, is canonically identified with the $\Pi_\ell(n)$-eigenspace at prime-to-$p$ level via the map

$$\Pr_{\alpha_n}^*: H^3_{\text{et}}(G(\hat{\mathbf{Z}})[\Pi_\ell]) \hookrightarrow H^3_{\text{et}}(G(\hat{\mathbf{Z}}(p)) \times \operatorname{Si}(p))[\Pi_\ell] \xrightarrow{\rho_n} H^3_{\text{et}}(G(\hat{\mathbf{Z}}(p)) \times \operatorname{Si}(p))[\Pi_\ell, U_{1,\ell}] = \alpha_n$$
where \( p_n \) denotes the Hecke operator \((1 - \frac{\delta}{\alpha})(1 - \frac{\gamma}{\alpha})(1 - \frac{\delta}{\alpha})\). (Here we have written \( H^3_{et}(\mathbb{Z}) \) as a shorthand for \( H^3_{et}(\mathbb{Z}, V) \) where \( V \) is the appropriate étale coefficient sheaf).

### 17.2. Two-variable Euler system classes

We now construct families of Euler system classes taking values in \( W^\dagger \).

**Notation 17.2.1.** Write

\[
\mathcal{E}_{Si}(\Pi, q, r) := \left( 1 - \frac{p^n}{\alpha} \right) \left( 1 - \frac{\beta}{\gamma} \right) \left( 1 - \frac{\beta}{\gamma} \right) \left( 1 - \frac{\delta}{\gamma} \right) \left( 1 - \frac{\delta}{\gamma} \right),
\]

and similarly \( \mathcal{E}_{Si}(\Pi(n), q, r) \) for \( n \geq 0 \) (with \( r_2 \) in the last two factors replaced by \( r_2 + n \)).

**Theorem 17.2.2.** Let \( 0 \leq r \leq r_1 - r_2 \) be a given integer, and let \( c_1, c_2 > 1 \) be integers coprime to \( 6pN \). Then there exists a class \( c_1, c_2 \in W_{1b}(\mathbb{Q}, W^\dagger_{1b}) \) with the following property: for each \((n, q) \) with \( n \in U \cap \mathbb{Z}_{\geq 0} \) and \( 0 \leq q \leq r_2 + n \), we have

\[
mom_{n,q}(c_1, c_2) = C_{n, q} \cdot \mathcal{E}_{Si}(\Pi(n), q, r),
\]

where \( C_{n, q} \) denotes the quantity

\[
\left( c_1^2 - c_2^{(t_1+1)} \right) \left( c_2^2 - c_2^{(t_2+1)} \right) \frac{\mathcal{E}_{Si}(\Pi(n), q, r)}{(-2)^2},
\]

**Proof.** It follows from the results of [LSZ17] that there exists a cohomology class interpolating the projections of Lemma–Flach elements \( \mathcal{L} \mathcal{E}_{et}(\Phi \otimes \xi) \) to the \( U'_{et} \)-ordinary part of cohomology at level \( K_G^p \times \text{Si}(p) \), for any prime-to-\( p \) level \( K_G^p \), up to modifying by factors depending on \( (c_1, c_2) \) in order to kill the denominators. Here \( \Phi \) and \( \xi \) are products of arbitrary test data away from \( p \) with certain specific test data at \( p \) determined by the construction.

If we choose the prime-to-\( p \) parts of \( \xi \) and \( \Phi \) to be the spherical test data, then \( \mathcal{L} \mathcal{E}_{et}(\Phi \otimes \xi) \) is invariant under the group \( G(\mathbb{Z}) \times \text{Si}(p) \), and its moments are given by

\[
\left( c\text{-factor} \right) \cdot (-2)^{-q} \left( 1 - \frac{p^n}{\alpha} \right)^{\mathcal{E}_{Si}(\Pi(n), q, r)} \left( w_{p,\text{ph}} \times w_{p,\text{Si}}, \Phi_{p,\text{ph}} \times \Phi_{p,\text{Si}} \right),
\]

where \( w_{p,\text{ph}} \) is the image of the spherical Whittaker vector of \( \Pi_p(n) \) under \( \text{Pr}_{1b}^* \), and \( \Phi_{p,\text{ph}} = \text{ch}(p^{n} Z_p \times Z_p^\times)^2 \). The cohomology class in the above formula is the product of \( \mathcal{E}_{Si}(\Pi(n), q, r) \) and a local zeta-integral \( \mathcal{Z}_p(\Pi_p(n), \Phi_{p,\text{ph}}, \Phi_{p,\text{Si}}) \), which is evaluated in Proposition 20.2.2 below; after rescaling the test data to remove a harmless factor of \( \frac{1}{(p+1)^{\gamma}} \), we obtain the formula stated. \( \square \)

### 17.3. Two-variable motivic p-adic L-functions

We recall the following description of the Galois representation \( W_{1b} \). Let \( \kappa_U : Z_p^\times \to \mathcal{O}(U)^\times \) be the canonical character over \( U \) (specialising to \( x \mapsto x^a \) at each \( n \in U \cap \mathbb{Z} \)).

**Theorem 17.3.1 (Urban).** After possibly shrinking \( U \), the module \( W_{1b} \) has a 3-step increasing filtration stable under \( G_{Q_p} \), with graded pieces of ranks \((1, 2, 1)\): we can write

\[
0 = W_{1b} \subset F_3 W_{1b} \subset F_2 W_{1b} \subset F_1 W_{1b} = W_{1b}
\]

in which \( F_n \) is free of rank \( n \) as an \( \mathcal{O}(U) \)-module and is a direct summand of \( W_{1b} \), and the subquotients

\[
\frac{F_1 W_{1b}}{F_2 W_{1b}} \otimes \chi_{1b}, \quad \frac{W_{1b}}{F_2 W_{1b}} \otimes \chi_{2b}
\]

are all crystalline as \( \mathcal{O}(U) \)-linear representations.

More precisely, the graded pieces have the following description:

- \( F_1 \) is unramified, with geometric Frobenius acting as multiplication by \( u_{1, p} \in \mathcal{O}(U)^\times \).
- \( (\mathcal{F}_3/\mathcal{F}_1)(\chi_{2b}^{(r_2+1)}) \) has constant Hodge–Tate weights \((0, -r_1 + r_2 - 1)\), and the trace of Frobenius on \( \mathcal{D}_{\text{cris}}(\mathcal{F}_3/\mathcal{F}_1)(\chi_{2b}^{(r_2+1)}) \) is \( u_{2, p} \).
- \( (W_{1b}/\mathcal{F}_3)(\chi_{2b}^{(r_2+1)}) \) is unramified with geometric Frobenius acting as \( \chi(p) u_{1, p}^{-1} \).

**Proof.** The fact that such filtrations exist “pointwise”, on the fibre at \( n \) for each \( n \in U \cap \mathbb{Z}_{\geq 0} \), is due to Urban [Urb05]. Since we know that the Galois representations interpolate over \( U \), the existence of an \( \mathcal{O}(U) \)-linear filtration follows from the finite generation of local Galois cohomology groups for \( \mathcal{O}(U) \)-linear representations. \( \square \)
Proposition 17.3.2. After possibly shrinking $U$, the projection of the Iwasawa cohomology class $c_1,c_2 \in \mathbb{Z}_{\text{Iw}}$ to $\mathbb{W}_{\mathbb{H}}/\mathcal{F}_1^{3}\mathbb{W}_{\mathbb{H}}$ is zero.

Proof. This follows from the corresponding vanishing result in the fibre at a given $n \in U \cap \mathbb{Z}_{\geq 0}$, which is [LPSZ19] Proposition 11.2.2.

We can thus regard log$_p \left( c_1,c_2 \mathbb{Z}_{\text{Iw}} \right)$ as an element of the module

$$H^1_{\text{Iw}} \left( \mathcal{Q}_p^{(\mu_p=\infty)}, \mathcal{F}_1^{3}\mathbb{W}_{\mathbb{H}}/\mathcal{F}_1^{3}\mathbb{W}_{\mathbb{H}} \right) \cong H^1_{\text{Iw}} \left( \mathcal{Q}_p^{(\mu_p=\infty)}, \mathcal{F}_1^{3}\mathbb{W}_{\mathbb{H}}/\mathcal{F}_1^{3}\mathbb{W}_{\mathbb{H}} \otimes \chi_{\text{cy}}^{(\kappa_U+r_2+1)} \right)$$

where the isomorphism comes from the canonical twisting map (the twist is convenient because we land in a representation with constant Hodge–Tate weights, and also matches up better with our normalisation for analytic $p$-adic $L$-functions). Perrin-Riou’s regulator $\mathcal{L}^{\text{PR}}$ gives a canonical map from this module to $H(\mathbb{Z}_p) \otimes D^* = \mathcal{O}(W) \otimes D^*$, where

$$D \coloneqq \mathcal{D}_{\text{cris}} \left( (\mathcal{F}_3 \mathbb{W}_{\mathbb{H}}/\mathcal{F}_1 \mathbb{W}_{\mathbb{H}}) \otimes \chi_{\text{cy}}^{(\kappa_U+r_2+1)} \right).$$

Let us now assume that the Hecke parameters of $\pi = \pi(0)$ satisfy $\beta \neq \gamma$. After possibly shrinking $U$ even further, we can arrange that $\beta_n \neq \gamma_n$ for every $n \in U \cap \mathbb{Z}_{\geq 0}$, and that there is a rank 1 direct summand $\mathcal{D}_\beta$ of $\mathcal{D}$, stable under $\varphi$, whose specialisation at any $n$ is canonically identified with the $\varphi = \beta_n$ eigenspace of $\mathcal{D}_{\text{cris}}(\mathbb{W}_{\pi(n)}/\mathcal{F}_1)$.

Definition 17.3.3. Let $z$ be a basis of the free rank 1 $\mathcal{O}(U)$-module $\mathcal{D}_\beta$. We shall set

$$c_1,c_2 \mathcal{L}^{\text{mot},r} \left( \Pi \right) := \left( \mathcal{L}_\beta, \mathcal{L}^{\text{PR}} \left( c_1,c_2 \mathbb{Z}_{\text{Iw}} \right) \right) \in \mathcal{O}(U \times W),$$

which we consider as a “two-variable motivic $p$-adic L-function”.

The dependence on $(c_1,c_2)$ is mild: the element of Frac $\mathcal{O}(U \times W)$ given by

$$\mathcal{L}^{\text{mot},r} \left( \Pi \right) := \left( c_1,c_2 \mathcal{L}^{\text{mot},r} \left( \Pi \right) \right) \left( c_1 - c_2^{(1-r')} \right) \left( c_2 - c_2^{(j+1-r')} \right)$$

is independent of $c_1,c_2$, where $j$ is the canonical character $\mathbb{Z}_p \rightarrow \mathcal{O}(W)^\times$ (which we think of as a “coordinate” on $W$) and $r' = r_1 - r_2 - r$. This can be seen as a meromorphic function on $U \times W$, with poles along the lines $j = r + 1$ and $j = r' + 1$.

Proposition 17.3.4. For $n \in U \cap \mathbb{Z}_{\geq 0}$, there exists a unique vector $\nu_2(n) \in \text{Fil}^1 \mathcal{D}_{\text{cris}}(\mathbb{W}_{\pi(n)})$ whose image in $\mathcal{D}_{\text{cris}}(\mathbb{W}_{\pi(n)}/\mathcal{F}_1)$ coincides with the specialisation of $\mathcal{L}_2$ at $n$. This vector is annihilated by $(1 - \frac{\beta}{\alpha})(1 - \frac{\gamma}{\alpha})$.

Proof. Since $\mathcal{F}_1 \mathbb{W}_{\pi(n)}$ has Hodge–Tate weight 0, the subspace $\mathcal{D}_{\text{cris}}(\mathcal{F}_1 \mathbb{W}_{\pi(n)})$ of $\mathcal{D}_{\text{cris}}(\mathbb{W}_{\pi(n)})$ (which is simply the $\varphi = \alpha_n$ eigenspace) has zero intersection with $\mathcal{F}_1$. Since $\mathcal{F}_1$ is 3-dimensional, we conclude that it maps isomorphically to $\mathcal{D}_{\text{cris}}(\mathbb{W}_{\pi(n)}/\mathcal{F}_1)$; so the image of $\mathcal{L}_2$ in $\mathcal{D}_{\text{cris}}(\mathbb{W}_{\pi(n)}/\mathcal{F}_1)$ has a unique lifting to $\mathcal{F}_1$. On the other hand, since the specialisation of $\mathcal{L}_2$ is annihilated by $(1 - \frac{\beta}{\alpha})$, and $\mathcal{F}_1$ is annihilated by $(1 - \frac{\gamma}{\alpha})$, we see that this lifting must be annihilated by the given quadratic polynomial.

Notation 17.3.5. We let $\Sigma_{\text{crit}}$ and $\Sigma_{\text{geom}}$ denote the subsets of $U \times W$ given by

$$\Sigma_{\text{crit}} = \{ (n,j) : n \in U \cap \mathbb{Z}_{\geq 0}, j \in \mathbb{Z}, 0 \leq j \leq r_1 - r_2 \},$$

and

$$\Sigma_{\text{geom}} = \{ (n,j) : n \in U \cap \mathbb{Z}_{\geq 0}, j \in \mathbb{Z}, -1 - r_2 \leq j \leq -1 \}.$$

Proposition 17.3.6. For any $(n,j) \in \Sigma_{\text{geom}}$, the value of $\mathcal{L}^{\text{mot},r} \left( \Pi \right)$ at $(n,j)$ is given by

$$\mathcal{L}^{\text{mot},r} \left( \Pi, n,j \right) = \frac{(-1)^{j+n-q}}{(-2)^q(r_2+n-q)!} \frac{\mathcal{E}(\Pi(n),q) \mathcal{E}(\Pi(n),1+r_2+r)}{1 - \frac{\beta_n}{\beta} \frac{\gamma_n}{\gamma}} \left( \nu_2(n), \log_{\text{BK}} \left( \frac{\kappa_\text{can} \mathcal{L}_{\text{can}} \mathcal{L}}{1 + \frac{\beta_n}{\beta} \frac{\gamma_n}{\gamma}} \right) \right),$$

where $q = j + 1 + r_2 + n$. 

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Proof. This follows from the interpolation formulae relating the Perrin-Riou regulator to the Bloch–Kato logarithm. These formulae include a twist by \((1 - p^j)(1 - p^{1-j})^{-1}\), with \(\varphi\) acting as \(p^{2+1+n}\beta_n^{j-1}\), so we have

\[
\mathcal{L}_{\mu}^{\text{mot}}(\Pi, n, j) = \frac{(-1)^{r_2+n}q^{-1} - q^{-1}}{(r_2 + n - 1)!} \cdot \left(1 - \frac{p^j}{q} \right) \cdot \left(1 - \frac{\beta_n}{p^{2+1+n}} \right) \cdot \left(\frac{\nu_p(n), \log_{BK} \left( \text{mom}_{n,q} \frac{[\Pi r]}{\mathbf{Iwp}}(\Phi) \right) \right).
\]

Combining this with Theorem 17.2 gives the result.

Note 17.3.7. The parity constraint of Note 6.6 implies \(\mathcal{L}_{\mu}^{\text{mot},r}(\Pi)\) is supported on \(U \times \mathcal{W}^{(-1)^{r-1}}\). 

Proposition 17.3.8. There exists an element \(\mathcal{E}_r(\Pi) \in \mathcal{O}(U)\) whose value at \(n \in U \cap \mathbb{Z}_{\geq 0}\) is

\[
\left(1 - \frac{p^{2+1+n}r}{\beta_n} \right) \left(1 - \frac{p^{2+1+n}r}{\gamma_n} \right)
\]

Proof. Clear from the fact that \(p^{-n}\beta_n\) and \(p^{-n}\gamma_n\) are analytic functions on \(U\).

Notation 17.3.9. We define \(\mathcal{L}_{p,\mathbf{Z}}^{\text{mot},r}(\Pi) := \mathcal{E}_r(\Pi) \cdot \mathcal{L}_{p,\mathbf{Z}}^{\text{mot}}(\Pi) \in \mathcal{O}(U \times \mathcal{W}^{(-1)^{r+1}})\).

Remark 17.3.10. This is an cowardly definition. We should really have defined a 3-parameter or even 4-parameter family of zeta elements with both \(q\) and \(r\) varying, and shown directly that it recovered the above element after specialisation, with the Euler factor \(\mathcal{E}_r\) arising naturally from a comparison between elements at Siegel and Iwahori level. See forthcoming work of Rockwood for a more satisfying treatment of this point.

We now reimpose the assumption that \(\Pi\) be Klingen-ordinary, and we suppose that \(\beta\) is the unique Hecke parameter of minimal possible valuation \(r_2 + 1\). We can now compare the motivic \(p\)-adic \(L\)-function with the analytic \(p\)-adic \(L\)-function.

We can now re-state Theorem 6.8.5 in the following form:

Corollary 17.3.11. For all \((n, j) \in \Sigma_{\text{geom}}\), we have

\[
\mathcal{L}_{\mu}^{\text{mot}}(\Pi, n, j) = \mathcal{L}_{p,\mathbf{Z}}^{\text{mot}}(\Pi(n), j, r).
\]

17.4 Conjectures on Eichler–Shimura isomorphisms.

Conjecture 17.4.1. Let \(\Pi\) be a Siegel-type Hida family over \(\mathcal{O}(U)\) through \((r_1, r_2)\), which is also Borel-ordinary. Then:

(A) There exists a rank 1 free \(\mathcal{O}(U)\)-module \(H^1(\Pi)\), whose fibre at \(n \in U \cap \mathbb{Z}_{\geq 0}\) is canonically identified with the direct summand of \(H^1(Y_{\mathbf{G}}(K_1(M, N) \cap \mathbf{Iwp}(p)), \mathcal{N}_n^2(-D)) [\Pi_{r}^j(n)]\) which is ordinary for the Hecke operators

\[
U_{2, \mathbf{Iwp}} := \left[ \mathbf{Iwp}(p) \text{diag}(\bar{p}, p, \bar{p}, p) \right] \quad \text{and} \quad Z := \left[ \mathbf{Iwp}(p) \text{diag}(\bar{p}, p, \bar{p}, p) \mathbf{Iwp}(p) \right].
\]

(B) There exist a pushforward map sending families of \(p\)-adic modular forms for \(H\) to elements of \(H^1(\Pi)\), compatible via specialisation with the pushforward maps on classical modular forms.

(C) There is an isomorphism of \(\mathcal{O}(U)\)-modules \(D^* \cong H^1(\Pi)\), interpolating the comparison isomorphisms of \(p\)-adic Hodge theory.

A proof of part (A) of this conjecture has already been announced by Pilloni, and will appear in forthcoming work. Part (B), which is an analogue for Siegel-type families of the pushforwards constructed for Klingen-type families in [LPSZT19], should also be accessible.

These two parts of the conjecture would suffice to define a 3-variable analytic \(p\)-adic \(L\)-function

\[
\mathcal{L}_{\mu}(\Pi) \in \mathcal{O}(U \times \mathcal{W} \times \mathcal{W}),
\]

where \(\mu\) is any basis of \(H^1(\Pi)^*\), whose restriction to \(\{n\} \times \mathcal{W} \times \mathcal{W}\) coincides with \(\mathcal{L}_{\mu}(\Pi(n))\) for each \(n \in U \cap \mathbb{Z}_{\geq 0}\).

If part (C) holds, then we can arrange that \(\mu\) is the image of \(\lambda\). Then Corollary 17.3.11 would assert the equality of two analytic functions on \(U \times \mathcal{W}\) at every point \((n, q)\) in a Zariski-dense set; hence these functions would agree everywhere. Specialising to \(n = 0\), we would then obtain the strongest possible form of an explicit reciprocity law, namely the following:

Conjecture 17.4.2. We have the following equality of rigid-analytic functions of \(j \in \mathcal{W}^{(-1)^{r+1}}\):

\[
\mathcal{L}_{\mu}^{\text{mot},r}(\Pi, j) = \mathcal{L}_{\mu}(\Pi, j, r).
\]
17.5. Comparison with a $GL_4$ $p$-adic $L$-function. In order to work around our ignorance of
Conjecture[17.4.1] we shall make use of the functorial transfer to $GL_4$. This allows one to make use of
a somewhat different toolset (based on Betti rather than coherent cohomology).

Notation 17.5.1. We write $\Theta$ for the functorial transfer of $\Pi \otimes \big| \cdot \big|^{-(r_1-r_2-1)/2}$ to $GL_4(\A)$, so that $\Theta$
is an isobaric automorphic representation of $GL_4$ satisfying

$$L(\Theta, s) = L(\Pi, \frac{-r_1+r_2}{2} + s).$$

The choice of twist implies that the critical values of $L(\Theta, s)$ are at the integers $0 \leq s \leq r_1 - r_2$, matching our normalisation for $p$-adic $L$-functions. Note that since $\Pi$ is assumed to be non-CAP and non-endoscopic, the representation $\Theta$ is in fact cuspidal. The compatibility of local and global transfers at $\infty$ implies that $\Theta$ is cohomological (with infinity-type determined by $(r_1, r_2)$); and the compatibility at finite places implies that $\Theta$ has level 1, and is ordinary at $p$.

Definition 17.5.2. For each sign $\epsilon \in \{\pm 1\}$, we write $H^4_{\text{B},\epsilon}(\Theta)_F$ for the eigenspace inside the compactly-supported Betti cohomology of the infinite-level symmetric space for $GL_4$ (with coefficients in the local system of $E$-vector spaces determined by $(r_1, r_2)$) which is $\Theta_\epsilon$-isotypical for the $GL_4(\A_F)$ action, and on which complex conjugation acts as $\epsilon$.

It follows from the Eichler–Shimura–Matsushima isomorphism, together with strong multiplicity one for $GL_4$, that each of the two spaces $H^4_{\text{B},\epsilon}(\Theta)_F$ is isomorphic to a single copy of $\Theta_\epsilon$. In particular, for each choice of $\epsilon$, the $GL_4(\Z_\epsilon)$-invariants of $H^4_{\text{B},\epsilon}(\Theta)_F$ are one-dimensional. We denote this space of invariants by $W^\epsilon(\Theta)_E$, and its base-extension to $L$ by $W^\epsilon(\Theta)_L$.

Definition 17.5.3. We let $\tau = (\tau^+, \tau^-)$ be a pair of $L$-bases of the spaces $W^\epsilon(\Theta)_L$, for each choice of sign.

Having chosen $\tau$, the construction of [DJR18] shows that for each sign $\epsilon$ we can find constants $\Omega_p(\Theta, \tau^r) \in L^x/E^x$, and $\Omega_\infty(\Theta, \tau^r) \in C^x/E^x$, such that the following proposition holds:

Proposition 17.5.4. There exists a measure $\mathcal{L}_{p,\tau}(\Theta) \in \Lambda_L(\Z_p^x)$ such that for all $0 \leq a \leq r_1 - r_2$ we have

$$\frac{\mathcal{L}_{p,\tau}(\Theta, a + \rho)}{\Omega_p(\Theta, \tau^r)} = R_p(\Theta, \rho, a) \cdot \frac{\Lambda(\Theta \otimes \rho, a)}{\Omega_\infty(\Theta, \tau^r)}$$

where $\epsilon = (-1)^a(1, 1)$, and $R_p(\Theta, \rho, a)$ is a product of Euler factors and Gauss sums at $p$.

Remark 17.5.5. As with the $GSp_4$ $p$-adic $L$-function defined above, the quantity $\Omega_p(\Theta, \tau^r)^{-1} \otimes \Omega_\infty(\Theta, \tau^r) \in L \otimes_E C$ is uniquely determined by $\tau$, although the individual factors are only determined modulo $E^x$, so the measure $\mathcal{L}_{p,\tau}(\Theta)$ depends only on $\tau$.

By comparing the interpolating properties of the $p$-adic $L$-functions, we obtain the following:

Corollary 17.5.6. Suppose that $\mathcal{L}_{p,\nu}(\Pi)$ is not identically 0 (which is automatic if $r_1 - r_2 > 0$). Then there is an isomorphism of $L$-vector spaces

$$l_{\Theta} : W^+(\Theta)_L \otimes W^-(\Theta)_L \cong \Gr^1 D_{\text{LR}}(W_{\Pi})$$

with the following property: if $\nu$ is the image of $\tau^+ \otimes \tau^-$, then we have

$$\mathcal{L}_{p,\nu}(\Pi)(j_1, j_2) = \mathcal{L}_{p,\tau}(\Theta)(j_1) \cdot \mathcal{L}_{p,\tau}(\Theta)(j_2)$$

for all $(j_1, j_2) \in W \times W$ with $j_1 + j_2$ odd.

Note that this isomorphism matches up the $E$-structure $W^+(\Theta)_E \otimes W^-(\Theta)_E$ with the $E$-rational structure on the right-hand side determined by de Rham cohomology, although we shall not use this fact.

17.6. Variation in families for $GL_4$. This discussion applies identically with $\Pi$ replaced by any of the other specialisations $\Pi(n)$ of Siegel-type family through $\Pi$ discussed above, and we have the following statement:

Proposition 17.6.1. After possibly shrinking $U$, we can find free rank 1 $\mathcal{O}(U)$ modules $W(\Theta)^\epsilon$ for each sign $\epsilon$, whose specialisation at $n \in U \cap \Z_{\geq 0}$ is canonically identified with $W(\Theta(n))^\epsilon$.

The following proposition is considerably deeper, but will be established in forthcoming work:
Theorem 17.6.2. Let $\tau = (\tau^+, \tau^-)$ be $O(U)$-bases of the modules $W(\Theta)^r$. Then there exists a bounded rigid-analytic function $\mathcal{L}_{p,\tau}(\Theta): U \times W \to L$ with the following property: for every $n \in U \cap \mathbb{Z}_{\geq 0}$, the restriction of $\mathcal{L}_{p,\tau}(\Theta)$ to $\{n\} \times W$ is $\mathcal{L}_{p,\tau(n)}(\Theta(n))$, where $\tau(n)$ is the specialisation of $\tau$ at $n$.

The proof of this theorem will appear in forthcoming work of the present authors with Barrera, Dimitrov and Williams (or some subset of the above).

17.7. The reciprocity law. We now carry out a rather delicate comparison argument. We choose a $\tau$, giving us a 2-variable analytic $p$-adic $L$-function; and we choose a $\underline{\nu}$ and a value of $r$, giving a 2-variable motivic one. For technical reasons we shall suppose that $r_1 - r_2 > 0$, and take $r \in \{0, \ldots, r_1 - r_2\}$ such that $r \neq \frac{-1}{2}\underline{\nu}$.

Notation 17.7.1. Define $\mathcal{L}^{[r]}_{p,\tau}(\Theta) \in O(U \times W)$ by

$$\mathcal{L}^{[r]}_{p,\tau}(\Theta, u, j) = \left\{ \begin{array}{ll}
\mathcal{L}_{p,\tau}(\Theta, u, r) \cdot \mathcal{L}_{p,\tau}(\Theta, u, j) & j \in W^{(-1)^r+1}, \\
0 & j \in W^{(-1)^r}.
\end{array} \right.$$ 

So Corollary [17.3.1] tells us that for all $(n, j) \in \Sigma_{\text{geom}}$, we have

$$\mathcal{L}^{\text{mot,}[r]}_{p,\tau}(\Theta, u, j) = B(n) \cdot \mathcal{L}^{[r]}_{p,\tau}(\Theta, u, j),$$

where $B(n) \in L^\times$ is the constant such that

$$B(n)l_{\alpha(n)}(\tau(n)^+ \otimes \tau(n)^-) = \underline{\nu}(n).$$

Lemma 17.7.2. The function on $U \times W \times W$ defined by

$$C(u, j, j') := \mathcal{L}^{[r]}_{p,\tau}(\Theta, u, j) \cdot \mathcal{L}^{\text{mot,}[r]}_{p,\tau}(\Theta, u, j') - \mathcal{L}^{[r]}_{p,\tau}(\Theta, u, j') \cdot \mathcal{L}^{\text{mot,}[r]}_{p,\tau}(\Theta, u, j).$$

is identically zero.

**Proof.** From Corollary [17.3.1] we know that $C(u, j, j')$ vanishes at all triples $(n, j, j')$ such that both $(n, j)$ and $(n, j')$ are in $\Sigma_{\text{geom}}$. Such triples are clearly Zariski-dense, so the result follows. □

Proposition 17.7.3. There is a non-zero meromorphic function $D \in \text{Frac } O(U)$ (independent of the $W$ variable) such that we have

$$\mathcal{L}^{\text{mot,}[r]}_{p,\tau}(\Theta, u, j) = D(u) \cdot \mathcal{L}^{[r]}_{p,\tau}(\Theta, u, j).$$

Moreover, $D$ has no pole at any $n \in U \cap \mathbb{Z}_{\geq 0}$.

**Proof.** Let $s \in \{0, \ldots, r_1 - r_2\}$ with $s \neq \frac{-1}{2}\underline{\nu}$, and let $\rho$ be a finite-order character of $\mathbb{Z}_p^\times$, such that $(-1)^s \rho(-1) \neq (-1)^r$. (If $r_1 - r_2$ is $\geq 4$ then we can assume $\rho$ is trivial.) We shall substitute $j' = s + \rho$ into the identity $C(u, j, j') = 0$. Unravelling the notations, we find that

$$L^{[r]}_{p,\tau}(\Theta, u, s + \rho) = L_{p,\tau}(\Theta, u, r)L_{p,\tau}(\Theta, u, s + \rho).$$

Both factors on the right-hand side are non-vanishing at $u = n$ for any $n \in U \cap \mathbb{Z}_{\geq 0}$, since they correspond to non-central critical values of the complex $L$-function, which are non-zero by the convergence of the Euler product. So this function is a non-zero-divisor in $O(U)$; and dividing the identity $C(u, j, s + \rho) = 0$ by this function, we obtain

$$\mathcal{L}^{\text{mot,}[r]}_{p,\tau}(\Theta, u, j) = D(u) \cdot \mathcal{L}^{[r]}_{p,\tau}(\Theta, u, j),$$

$$D(u) := \frac{L^{\text{mot,}[r]}_{p,\tau}(\Theta, u, s + \rho)}{L_{p,\tau}(\Theta, u, r)L_{p,\tau}(\Theta, u, s + \rho)}.$$

□

Proposition 17.7.4. For all but finitely many integers $n \in U \cap \mathbb{Z}_{\geq 0}$, the following holds: there exists an integer $j$ with $j = r + 1 \mod 2$ such that $(n, j) \in \Sigma_{\text{geom}}$ and $\mathcal{L}_{p,\tau(n)}(\Theta(n), j) \neq 0$.

**Proof.** Assume the contrary. Then there exists an infinite sequence of integers $n_k \in U \cap \mathbb{Z}_{\geq 0}$ such that the function $\mathcal{L}_{p,\tau}(\Theta)$ vanishes at $(n_k, j)$ for all $j$ such that $j = r + 1 \mod 2$ and $(n, j) \in \Sigma_{\text{geom}}$. In particular, if we fix a $j \leq -1$ congruent to $r + 1 \mod 2$, then $\mathcal{L}_{p,\tau}(\Theta)$ vanishes at $(n_k, j)$ for all sufficiently large $k$, and since the sequence $(n_k)$ is Zariski-dense in $U$, it follows that $\mathcal{L}_{p,\tau}(\Theta, u, j)$ vanishes for all $u \in U$. Since this holds for all $j \leq -1$ of the appropriate parity, we conclude that $\mathcal{L}_{p,\tau}(\Theta)$ has to be identically 0 on $U \times W^{(-1)^r}$. This is a contradiction, since its values at $(n, j + \rho)$ with $0 \leq j \leq r_1 - r_2$...
and ρ a finite-order character are critical values of the complex L-function multiplied by explicit non-zero factors, and if j ≠ \frac{-1-c_2}{2} these values are not central or near-central, so they are non-zero by the convergence of the Euler product. □

Corollary 17.7.5. For any n ∈ U ∩ Z_{≥0}, one of the following two possibilities occurs:

- \( \mathcal{L}^{\text{mot.}, [r]}(\Pi(n)) \) is a non-zero scalar multiple of the analytic p-adic L-function \( \mathcal{L}_n(\Pi(n), -, r) \).
- \( \mathcal{L}^{\text{mot.}, [r]}(\Pi(n)) \) is identically 0.

Moreover, for all but finitely many n, the first possibility occurs and the scalar multiple is the constant \( B(n) \) of Eq. (17.1), so we have

\[ \mathcal{L}^{\text{mot.}, [r]}(\Pi(n), j) = \mathcal{L}_{p, \nu(n)}(\Pi(n), j, r) \]

as an identity of rigid-analytic functions of \( j \in W^{(-1)^{r+1}} \).

**Proof.** Since the function D of Proposition 17.7.3 is finite at any positive integer n, it must either be zero there, or an element of \( L^n \), and the result of the proposition gives the two cases stated. However, if n satisfies the condition of Proposition 17.7.4 then Eq. (17.1) shows that D(n) must equal the constant \( B(n) \), and in particular is non-zero; and by that proposition we know that this case occurs for all but finitely many n. □

Remark 17.7.6. There are two “bad” cases which could possibly occur for some \( n \): either \( D(n) = 0 \), in which case the motive p-adic L-function of \( \Pi(n) \) vanishes identically; or \( D(n) \neq 0 \) but \( B(n) \neq D(n) \), in which case the motive p-adic L-function is still a non-zero multiple of the analytic one, but the “wrong” multiple. The first case is disastrous for applications, while the second is only a minor irritant. However, since both cases occur for only finitely many n, we can shrink U to assume that neither case occurs except possibly for \( n = 0 \).

We have so far been quite agnostic about the value of r; we assumed only that it was non-central. We now consider varying r. Note that the meromorphic function \( D(u) \) must be independent of r, since the constants \( B(n) \) are independent of r. So we may conclude that the function

\[ \frac{\mathcal{L}^{\text{mot.}, [r]}(\Pi)(u, j)}{\mathcal{L}_{p, \nu}^{\text{mot}}(\Theta, u, r)} \]

is also independent of r, being equal to \( D(u) \cdot \mathcal{L}_{p, \nu}^{\text{mot}}(\Theta, u, j) \).

17.8. Proof of Theorem B. We note the following theorem:

**Theorem 17.8.1.** There exists a collection of classes

\[ c_1, c_2 \in \mathbb{Z}_{\text{Euler, } M} \in H^1(\mathbb{Q}(\mu_{MP^\infty}), W^n_{\overline{\Pi}}) \]

for every \( M \geq 1 \) coprime to \( pc_1c_2 \), satisfying the Euler system norm compatibility relations as M varies, with the \( M = 1 \) case being the class \( c_1, c_2 \in \mathbb{Z}_{\text{Euler, } M} \) above.

**Proof.** This follows from the results of [LSZ17] in the same way as the \( M = 1 \) case covered in Theorem 17.2.3. □

**Notation 17.8.2.** We let \( c_M \) be the image of \( c_1, c_2 \in \mathbb{Z}_{\text{Euler, } M} \) under the Soulé twist map

\[ H^1(\mathbb{Q}(\mu_{MP^\infty}), W^n_{\overline{\Pi}}) \to H^1(\mathbb{Q}(\mu_{MP^\infty}), W^n_{\Pi}(-1 - r_2 - \kappa_U)) \]

The following result follows easily from the integrality of the original Lemma–Flach classes:

**Lemma 17.8.3.** If \( \Omega^+(U) \) is the subring of functions of supremum norm \( \leq 1 \) in \( \mathcal{O}(U) \), then there exists a \( G_{\mathbb{Q}} \)-stable \( \Omega^+(U) \)-lattice \( \mathcal{T} \subseteq \mathbb{Z}_{\Pi}(-1 - r_2 - \kappa_U) \) independent of \( M \) such that all these classes take values in \( H^1(Q(\mu_{MP^\infty}), \mathcal{T}) \).

If \( D(0) \neq 0 \), then it is a small step from here to Theorem B. The chief difficulty is that we cannot rule out the possibility of \( D(0) \) vanishing, so we shall perform a delicate argument with “leading terms”.

\[ \text{Note that since } \Pi \text{ has tame level } 1, r_1 - r_2 \text{ must be even, and since we have assumed it is not zero, it is } \geq 2 \text{. If we allow general tame levels, then this argument becomes more delicate in the case } r_1 - r_2 = 1 \text{; we need to invoke the non-vanishing of GL}_4 \text{ L-functions along the abcissa of convergence (the "prime number theorem" for GL}_4 \text{ L-functions) due to Jacquet and Shalika.} \]
Notation 17.8.4. Let \( u \) denote a generator of the principal ideal of \( \mathcal{O}^+(U) \) corresponding to the point \( 0 \in U \).

Definition 17.8.5. For \( M \geq 0 \), let \( h(M) \) be the largest integer \( n \) such that
\[
c_M \in u^n \cdot H^1(\mathbb{Q}(\mu_{M^p\infty}), T),
\]
and let \( h = \inf_M h(M) \), where the infimum is over \( M \geq 1 \) coprime to \( pc_1c_2 \).

The Euler system norm-compatibilities imply that \( h(M) \leq h(1) \) for all \( M \), and \( h(1) \) is finite, since \( c_{1,c_2,z_{tw,M}} \) is not zero. From Proposition 17.7.3, we have
\[
h \leq h(1) \leq v_u(D)
\]
where \( v_u \) denotes the \( u \)-adic valuation on \( \mathcal{O}(U) \).

Proposition 17.8.6. There exists a collection of classes \( c_M^{(h)} \in H^1(\mathbb{Q}(\mu_{M^p\infty}), T) \) satisfying the Euler-system norm relations, such that we have
\[
c_M = u^h \cdot c_M^{(h)}
\]
for all \( M \). Moreover, there is some \( M \) such that \( c_M \) has non-zero image in \( H^1(\mathbb{Q}(\mu_{M^p\infty}), T) \), where \( T \) denotes the lattice \( T/uT \subset \mathbb{W}_\mathbb{F} \).

Proof. Let us write temporarily \( M = H^1_{tw}(\mathbb{Q}(\mu_{M^p\infty}), T) \) for some \( M \). We note that \( M/uM \) injects into \( H^1_{tw}(\mathbb{Q}(\mu_{M^p\infty}), T/uT) \), which is the Iwasawa cohomology of a finite-rank free \( \mathbb{Z}_p \)-linear representation and is therefore \( p \)-torsion-free. Thus the fact that \( c_{1,c_2,z_{tw,M}} \) is divisible by \( u^h \) in \( M/[1/p] \) implies that it is in fact divisible by \( u^h \) in \( M \). Moreover, it is even uniquely divisible by \( u^h \), since the \( u^h \)-torsion of \( M \) is a subquotient of \( H^1_{tw}(\mathbb{Q}(\mu_{M^p\infty}), T/u^hT) \) which is zero by standard properties of Iwasawa cohomology. Hence \( c_M^{(h)} \) is well-defined. Since multiplication by \( u^h \) is injective, and the \( c_M \) for varying \( M \) satisfy the Euler-system norm relations, so do the \( c_M^{(h)} \).

This argument also shows that \( c_M^{(h)} \) has non-zero image in \( H^1(\mathbb{Q}(\mu_{M^p\infty}), T) \) if and only if \( h(M) = h \). Since this does occur for some \( M \) by the definition of \( h \), the final claim follows.

Proposition 17.8.7. Assume that \( h < v_u(D) \). Then we have
\[
\text{loc}_p(c_M^{(h)} \mod u) \in H^1_{tw}(\mathbb{Q}(\mu_{M^p\infty}) \otimes \mathbb{Q}_p, \text{Fil}^2 T)
\]
for all \( M \).

Proof. It suffices to show that for every \( M \), the class \( c_M^{(h)} \mod u \) lies in the kernel of the Perrin-Riou regulator map for \( \text{Fil}^1 T/\text{Fil}^2 T \), since the kernel of this map is zero by Lemma 6.2.5.

Repeating the construction of the previous sections with the additional tame level \( M \), we obtain an “equivariant” motivic \( p \)-adic \( L \)-function \( L_{\phi,\Sigma}^{\text{mot},[r]}(\mathbb{H}, M) \) over \( U \times \mathcal{W} \), taking values in the group ring of \( (\mathbb{Z}/M\mathbb{Z})^\times \). For each character \( \chi \) of \( (\mathbb{Z}/M\mathbb{Z})^\times \), the \( \chi \)-isotypical projection of this object interpolates values of the \( L \)-function of the twisted representation \( \mathbb{H}(\nu) \otimes \chi \) in the geometric range \( \Sigma_{\text{geom}} \).

On the other hand, the GL\( 4 \) construction extends straightforwardly to an equivariant version of the analytic \( p \)-adic \( L \)-function, \( L_{\phi,\Sigma}^{[r]}(\mathbb{Q}, M) \). Both of these objects depend on the same choices of periods \( \nu, \Sigma \) as the non-equivariant \( L \)-functions of the previous section.

Hence we can run the argument of Proposition 17.7.3 to obtain a relation between the motivic and analytic equivariant \( p \)-adic \( L \)-functions, and the function \( D(u) \) that appears must be the same for all \( M \), since it is characterised by agreeing with the numbers \( B(n) \) of 17.1 for almost all \( n \), and these numbers are independent of \( M \).

From this and the definition of \( c_M^{(h)} \), we have
\[
\langle \mu_\beta, L_{\phi,\Sigma}^{\text{PR}}(c_M^{(h)} \mod u) \rangle = ((u^{-h} D)(0)) \cdot L_{\phi,\Sigma}^{[r]}(\mathbb{Q}, M).
\]
So if \( (u^{-h} D)(0) = 0 \), we can conclude that \( c_M^{(h)} \mod u \) lies in the kernel of the regulator for all \( M \) as required.

Corollary 17.8.8. If the “big image” assumption Hyp(\( \mathbb{Q}(\mu_{p^\infty}), \Sigma \)) of [Rub00] is satisfied for every Dirichlet-character twist of \( T \), then we have \( h = v_u(D) \).
since c_M(b) is non-zero for some M, its projection to some character component χ must also be non-zero, so it gives a non-zero Kolyvagin system for T(χ), contradicting Mazur and Rubin’s result.

**Theorem 17.8.9 (Theorem 18.3).** Let Π be an automorphic representation which satisfies our running hypotheses, and has tame level 1, is Borel-ordinary at p, and satisfies the “big image” condition of LSZ17 Assumption 11.1.2. Suppose also that r_1 - r_2 ≥ 6.

Then for any choice of basis τ = (τ⁺, τ⁻) as above, there exists an Euler system for W_Π^r_2(−1 − r_2) with the following property: for all M, the localisation of the class at p lands in Fil^1; and the image of the bottom class in this Euler system under the Perrin-Riou regulator is L_{p, r}(Θ).

**Proof.** The above argument shows that for each r we can construct an Euler system whose regulator is (c_r^(-1) - c_r^1 - r')(c_r^2 - c_r^1 + 1 - r)\mathcal{L}_{p, r}(Θ, r)\mathcal{L}_{p, r}(Θ, j) on W^{r_2} and 0 on W^{r_2 - 1}.

Over the −1 component of weight space, we note that the factors

\[(c_r^1 - c_r^1 + r')(c_r^2 - c_r^1 + 1 - r)\mathcal{L}_{p, r}(Θ, r)\]

for r = 0 and r = 2 between them generate the unit ideal of \mathcal{O}(W^{−1}), so we can take a suitable linear combination to obtain an Euler system with the desired regulator L_{p, r}(Θ, j). Similarly, over the other sign component, we use r = 1 and r = 3, unless r_1 - r_2 = 6, in which case we can use r = 1 and r = 5.

18. Applications

Throughout this section, we let Π be a non-endoscopic, non-CAP automorphic representation of G(A_1) of weights \(r_1, r_2, r_3\) with \(r_2 ≥ 1\) and \(r_1 - r_2 ≥ 6\). Assume that Π has tame level 1, and that it is Borel ordinary at p.

18.1. Selmer groups over \(Q_∞\). Let \(Q_∞ = Q(\mu_{p∞})\). For simplicity we write \(V = W_Π^r_2(−1 − r_2)\) in this section. (Note that this conflicts with our earlier use of \(V\) for an algebraic \(G\)-representation, but that usage will not recur here.)

**Definition 18.1.1.** Let \(\widetilde{\mathcal{H}}_ι^r_2(Q_∞, V)\) denote the Nekovář Selmer complex, with the unramified local conditions at \(Λ\) \(≠ p\), and at \(p\) the Greenberg-type local condition determined by Fil^2 W_Π^r_2.

This is a perfect complex of \(\mathcal{A}_L(Z_p^∞)\)-modules. Its cohomology groups are zero for \(i \notin \{1, 2\}\), and we have

\[\widetilde{\mathcal{H}}_ι^r_2(Q_∞, V) = \ker \left( H_ι^r_2(Q_∞, V) \to H_ι^r_2(Q_{p∞}, V/\text{Fil}^2) \right).\]

The degree 2 cohomology is related to classical \(p\)-torsion Selmer groups via Pontryagin duality:

**Proposition 18.1.2.** If \(T\) denotes a choice of lattice in \(V\), and \((-)^{V}\) denotes Pontryagin dual, then we have a canonical isomorphism of \(\mathcal{A}_L(Z_p^∞)\)-modules

\[\widetilde{\mathcal{H}}_ι^r_2(Q_∞, V) = \left( \lim_ι H_ι^r_2(Q(\mu_{p∞}), T^{V}(1 + j)) \right)^{V} (j) \otimes L,\]

for any integer \(0 ≤ j ≤ r_1 - r_2\).

We can now state our main theorem in Iwasawa-theoretic form:

**Theorem 18.1.3.** The module \(\widetilde{\mathcal{H}}_ι^r_2(Q_∞, V)\) is torsion over \(\mathcal{A}_L(Z_p^∞)\) and its characteristic ideal divides the \(p\)-adic \(L\)-function \(L_{p, r}(Θ)\). Moreover, we have \(\widetilde{\mathcal{H}}_ι^r_2(Q_∞, V) = 0\).

**Proof.** This is proved in Theorem 11.3.2 of [LSZ17] with the motivic \(p\)-adic \(L\)-function (for some specific choice of \(r\)) in place of \(L_{p, r}(Θ)\). Applying the same argument with the Euler system emerging from Theorem 17.8.9 we obtain the result stated.

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18.2. Selmer groups over $\mathbb{Q}$. By a standard descent argument (using the fact that no exceptional-zero phenomena arise because of Lemma 6.2.5), we deduce the following:

**Theorem 18.2.1.** Let $0 \leq j \leq r_1 - r_2$, and let $\rho$ be a finite-order character of $\mathbb{Z}_p^\times$. If $L(\Pi \otimes \rho, \frac{1-r_1+r_2}{2} + j) \neq 0$, then $H^1_f(\mathbb{Q}, V(-j - \rho)) = 0$.

This establishes the analytic rank 0 case of the Bloch–Kato conjecture for all critical values of the $L$-function of $\Pi$.

**Note 18.2.2.** The hypothesis $L(\Pi \otimes \rho, \frac{1-r_1+r_2}{2} + j) \neq 0$ is automatic if $j \neq \frac{r_1-r_2}{2}$.  

\[ \Diamond \]
Appendices

19. Uniqueness of local periods

The aim of this result is to prove the following somewhat delicate uniqueness result in smooth representation theory. This will be used to understand the dependence of our Euler system classes on the choice of test data at the bad primes, but the statement and its proof are purely local.

19.1. Setup. Let $F$ be a nonarchimedean local field and $| \cdot |$ the norm on $F$, normalised such that $|q| = \frac{1}{q}$, where $q$ is the order of the residue field. Let $\psi$ be a non-trivial additive character $F \rightarrow \mathbb{C}^\times$. We let $\pi$ be a generic irreducible smooth representation of $G(F)$, and we let $W(\pi)$ be its Whittaker model with respect to $\psi$. We fix a pair of characters $(\chi_1, \chi_2)$ of $F^\times$, with $\chi_1 \chi_2 = \chi_\pi$; we shall assume these characters are unitary.

Zeta integrals. In [LPSZ19] [§8.2], we defined a local zeta-integral $Z(w, \Phi, s_1, s_2)$, for $\Phi \in S(F^2 \times F^2)$ and $w \in W(\pi)$. In Theorem 8.8 op.cit. we showed (as a consequence of the computations of [RW17], [RW18]) that the minimal common denominator of the values of this zeta-integral is given by $L(\pi \otimes \chi_2, s_1 - s_2 + \frac{1}{2})$. Accordingly, the quotient

\[
Z(w, \Phi, s_1, s_2) = \lim_{(\xi_1, \xi_2) \rightarrow (s_1, s_2)} \frac{Z(w, \Phi, \xi_1, \xi_2)}{L(\pi \otimes \chi_2^{-1}, \xi_1 - \xi_2 + \frac{1}{2}) L(\pi, \xi_1 + \xi_2 - \frac{1}{2})}
\]

is a well-defined and non-zero map $W(\pi) \times S(F^2 \times F^2) \rightarrow \mathbb{C}$, for every $(s_1, s_2) \in \mathbb{C}^2$. This map is $H(F)$-equivariant up to a twist by the norm character.

Principal series of $H$. We consider the following families of induced representations of $H(F)$, depending on $(s_1, s_2) \in \mathbb{C}^2$:

\[
I_{s_1, s_2} := \text{Ind}( |\cdot|^{\frac{1}{2}-s_1} \chi_1^{-1}, |\cdot|^{\frac{1}{2}-s_2} \chi_2^{-1}) \otimes \text{Ind}( |\cdot|^{\frac{1}{2}-s_1} \chi_2^{-1}, |\cdot|^{s_2 - \frac{1}{2}})
\]

\[
I'_{s_1, s_2} := \text{Ind}( |\cdot|^{s_1 - \frac{1}{2}}, |\cdot|^{\frac{1}{2} - s_1} \chi_1^{-1}) \otimes \text{Ind}( |\cdot|^{s_2 - \frac{1}{2}}, |\cdot|^{\frac{1}{2} - s_2} \chi_2^{-1})
\]

where $\text{Ind}$ denotes normalised induction. We are interested in the behaviour of these at a point $(s_1, s_2) \in \mathbb{C}^2$ with $\Re(s_i) \leq 0$. This condition implies that $I_{s_1, s_2}$ and $I'_{s_1, s_2}$ are both irreducible unless $L(\chi_1, 2s_1)L(\chi_2, 2s_2) = \infty$. In this case $I_{s_1, s_2}$ is reducible but indecomposable, and its unique irreducible subrepresentation is generic; while $I'_{s_1, s_2}$ has the same Jordan–Hölder factors in the opposite order, so its unique irreducible quotient is generic.

Siegel sections. We can define two maps from $S(F^2)$ to smooth $\mathbb{C}(q^s)$-valued functions on $GL_2(F)$: the map $\Phi \mapsto f^\Phi(g, \chi, s)$ of [LPSZ19] [§8.1]; and the map $\Phi \mapsto F^\Phi(g, \chi, s) := \chi^{-1}(\det g) f^\Phi(g, \chi^{-1}, 1-s)$, where $\Phi(x, y) = \int_{F^2} \Phi(u, v) \psi(xv - yu) \, du \, dv$ is the Fourier transform. Both maps are $H(\mathcal{A}_F)$-equivariant up to a twist by the norm character. Taking tensor products, we obtain maps from $S(F^2 \times F^2)$ to functions on $H(F)$, denoted $\Phi \mapsto f^\Phi(h, s_1, s_2)$ and $\Phi \mapsto F^\Phi(h, s_1, s_2)$. One checks that for any $\Phi$, the function $f^\Phi$ is a meromorphic section of $I'_{s_1, s_2}$, and $F^\Phi$ is a meromorphic section of $I_{s_1, s_2}$.

If $(s_1, s_2)$ is not a reducibility point, then $F^\Phi$ and $f^\Phi$ are regular at $(s_1, s_2)$ and define surjective maps from $S(F^2 \times F^2)$ to $I'_{s_1, s_2}$ and $I'_{s_1, s_2}$ respectively. If $(s_1, s_2)$ is a reducibility point (still with $\Re(s_i) \leq 0$ as above), then $F^\Phi$ is still regular and surjective onto $I_{s_1, s_2}$, but $f^\Phi$ can have poles; and one checks that the leading terms of the $f^\Phi$ span the unique irreducible subrepresentation of $I'_{s_1, s_2}$.

We can now state the main theorems of this section:

Theorem 19.1.1. For any pair $(s_1, s_2)$ with real parts $\leq 0$, the map $(w, \Phi) \mapsto Z(w, \Phi, s_1, s_2)$ factors through $\Phi \mapsto F^\Phi(-, s_1, s_2) \in I_{s_1, s_2}$, and defines a non-zero element of the space $\text{Hom}_{H(F)}(\pi \otimes I_{s_1, s_2}, \mathbb{C})$.

Theorem 19.1.2. Suppose that one of the following conditions holds:
• We have \( L(\chi_1, 2s_1)L(\chi_2, 2s_2) \neq \infty; \)
• We have \( L(\pi, s_1 + s_2 - \frac{1}{2}) \neq \infty; \)
• \( \pi \) is of Sally–Tadic type IIIa or IVa.

Then we have \( \dim \Hom_{H(F)(\pi \otimes I_{s_1, s_2}, C)} = 1, \) and \( \tilde{Z} \) is a basis of this space. Moreover, for any non-zero \( H \)-subrepresentation \( V \) of \( I_{s_1, s_2}, \) we have \( \dim \Hom_{H(F)(\pi \otimes V, C)} = 1 \) spanned by the restriction of \( \tilde{Z}. \)

19.2. Proof of Theorem 19.1.1. As in Theorem 8.8 of [LPSZ19], renormalising Novodvorsky’s Whittaker integral by the factor \( L(\pi \otimes \chi^{-1}_2, s)^{-1} \) and evaluating at \( s = s_1 - s_2 + \frac{1}{2} \) gives a canonical nonzero intertwining map from the Whittaker model \( \sW(\sW) \) to the Bessel model \( \sB(\sW), \) with respect to a specific character of the Bessel subgroup depending on the \( \chi_1 \) and \( s_i. \)

If \( B_w \in \sB(\sW) \) is the image of \( w \in \sW(\sW), \) and \( \alpha \in C, \) we define

\[
\tilde{z}_{w, \alpha}(h) := \int_{Q^\alpha} B_w \left( \begin{pmatrix} x & \cdot \\ 0 & 1 \end{pmatrix} h \right) |x|^{\alpha - 2 + s_1 + s_2} d^\infty x,
\]

which is a meromorphic section of the 1-parameter family of representations \( J_{\pi, \alpha} \), where

\[
J_{\pi, \alpha} := \Ind(z_{w, \alpha}, F(s_1 + \frac{\alpha}{2}, s_2 + \frac{\alpha}{2})�\Ind(z_{w, \alpha}, F(s_1 - \frac{\alpha}{2}, s_2 - \frac{\alpha}{2})).
\]

is the dual of the representation \( \tilde{J}_{\pi, \alpha} \) of the previous section.

Proposition 19.2.1. We have

\[
\tilde{Z}(w, \Phi, s_1, s_2) = \lim_{\alpha \to 0} \frac{1}{L(\pi, s_1 + s_2 - \frac{1}{2})} \langle z_{w, \alpha}, f^\Phi(s_1 + \frac{\alpha}{2}, s_2 + \frac{\alpha}{2}) \rangle.
\]

In particular, the limit on the right exists for all \( (w, \Phi), \) and is non-zero for some \( (w, \Phi). \)

Proof. This is a restatement of [LPSZ19] Proposition 8.4.

Since the sections \( f^\Phi \) may have poles at \( \alpha = 0, \) it is more convenient to work with the \( F^\Phi. \) Let \( J_{s_1, s_2} \) denote the dual of \( I_{s_1, s_2}, \) and let

\[
M_{\alpha} : J_{s_1 + \frac{\alpha}{2}, s_2 + \frac{\alpha}{2}} \to J_{s_1 + \frac{\alpha}{2}, s_2 + \frac{\alpha}{2}}
\]

denote the intertwining operator (normalised to be regular and nonzero at \( \alpha = 0, \) although not necessarily an isomorphism there). Then \( M_{\alpha} \) is invertible for generic \( \alpha, \) and it follows from Propositions 3.1.5 and 3.2.3 of [LSZ17] that

\[
\tilde{Z}(w, \Phi, s_1, s_2) = C \cdot \lim_{\alpha \to 0} \frac{1}{L(\pi, s_1 + s_2 - \frac{1}{2})} \langle M^{-1}_{\alpha}(z_{w, \alpha}), F^\Phi(s_1 + \frac{\alpha}{2}, s_2 + \frac{\alpha}{2}) \rangle
\]

for \( C \) a nonzero scalar. Since the \( F^\Phi \) are regular at \( (s_1, s_2) \) and their specialisations there surject onto \( I_{s_1, s_2}, \) it follows that the limit

\[
\tilde{z}_w := \lim_{\alpha \to 0} \frac{1}{L(\pi, s_1 + s_2 - \frac{1}{2})} M^{-1}_{\alpha}(z_{w, \alpha}) \in J_{s_1, s_2}
\]

must exist for every \( w, \) and we have

\[
\tilde{Z}(w, \Phi, s_1, s_2) = \langle \tilde{z}_w, F^\Phi(s_1, s_2) \rangle.
\]

This proves Theorem 19.1.1.

19.3. Proof of Theorem 19.1.2. Let \( J = J_{s_1, s_2} \). By duality, we can regard \( \tilde{Z} \) as a non-zero \( H(F) \)-homomorphism \( \pi \to J. \) So we need to show that under the hypotheses of the theorem, if \( K \not\subseteq J \) is any \( H(F) \)-stable subspace, then the image of \( \tilde{Z} \) in \( J/K \) is non-zero and spans \( \Hom_{H(F)(\pi, J/K)} \).

Theorem 19.3.1 (Waldspurger, Emory–Takeda). For any irreducible smooth representations \( \pi \) of \( G(F) \) and \( \tau \) of \( H(F), \) we have

\[
\dim \Hom_{H(F)(\pi, \tau)} \leq 1.
\]

Proof. This will be proved in forthcoming work of Emory and Takeda (building on the case of trivial central characters, which is due to Waldspurger [Wal12].)

Note that this is a family of \( H(F) \)-representations over \( C(q^\alpha), \) not just over \( C(q^{\alpha/2}), \) since we can write it in the slightly less symmetric form \( \Ind(| \cdot |^{\alpha - 2}, \cdot, \cdot |^{\alpha - 2 + \alpha} \chi_1) \Ind(| \cdot |^{\alpha - 2}, \cdot, \cdot |^{\alpha - 2 + \alpha} \chi_1) \).
This proves Theorem 19.1.1 if \( J \) is irreducible, or equivalently, if \( L(\chi_1, 2s_1)L(\chi_2, 2s_2) \) is finite. The remaining cases are more intricate. There is a filtration of \( J \) by \( H \)-stable subspaces

\[
J \supseteq J_0 \supseteq J_{00},
\]

where \( J_{00} \) is either zero or one-dimensional, \( J/J_0 \) is irreducible and generic, and \( J_0/J_{00} \) is a direct sum of representations of the form “(generic) \( \boxtimes (1 \text{-dimensional}) \)” or “(1-dimensional) \( \boxtimes (\text{generic}) \).” By a theorem of Piatetski-Shapiro [PS97, Theorem 4.3], we have \( \text{Hom}(\pi \otimes J_{00}) = 0 \).

**Proposition 19.3.2** (Rösner–Weissauer).

(i) The choice of split Bessel model determines a factorisation

\[
L(\pi, s) = L^1(\pi, s) \cdot L^2(\pi, s)
\]

as a product of two Euler factors, where \( L^1(\pi, \alpha + s_1 + s_2 - \frac{1}{2}) \) is the minimal common denominator of the functions \( z_{w,\alpha}(1) \) for \( w \in W(\pi) \), and \( L^2(\pi, s) \) (the “subregular factor”) lies in the fractional ideal generated by \( L(\chi_1, s_1 - s_2 + \frac{1}{2} + s) \).

(ii) If the subregular factor does not have a pole at \( s = s_1 + s_2 - \frac{1}{2} \), then we have

\[
\text{Hom}_{H(F)}(\pi, J_0/J_{00}) = 0.
\]

(iii) If \( \pi \) is of Sally-Tadic type IIIa or IVa, then the subregular factor is trivial.

**Proof.** Part (i) is the factorisation of the \( \mathcal{L} \)-factor considered in [RW17], as the product of a factor arising from the asymptotics of the Bessel model \( \mathcal{B}(\pi) \) along the torus \( \text{diag}(x, x, 1, 1) \), and a second (the subregular factor) arising from the poles of the Siegel sections \( f^\# \).

The subregular poles of generic representations are classified in [RW18], where they are studied via certain auxiliary linear functionals on \( \pi \) (termed “(\( H_+ \), \( \rho \))-functionals”) which transform by a scalar under one of the subgroups \( \{(\theta, z), \star\} \) and \( \{(\theta, z^*), \star\} \) of \( H(F) \). These correspond, via Frobenius reciprocity, to elements of \( \text{Hom}_{H(F)}(\pi, J_0/J_{00}) \) in our notation.

It is shown in §3 of op.cit. that the leading term of the zeta integral at a subregular pole gives rise to an \( (H_+, \rho) \)-functional; and §4 of op.cit. shows that in fact all nonzero \( (H_+, \rho) \)-functionals arise in this way. So if \( L^2(\pi, s) \) does not have a pole at \( s_1 + s_2 - \frac{1}{2} \), then \( \text{Hom}_{H(F)}(\pi, J_0/J_{00}) = 0 \), which is (ii). The explicit classification of subregular poles also gives part (iii) (see Table 2 of op.cit.).

We can now prove Theorem 19.1.2. If \( s_1 + s_2 - \frac{1}{2} \) is not a pole of \( L(\pi, s) \), then it is certainly not a subregular pole. So every non-generic composition factor \( \sigma \) of \( J \) satisfies \( \text{Hom}_{H(F)}(\pi, \sigma) = 0 \). If \( K \) is a proper \( H \)-stable subspace of \( J \), then we must have \( K \subseteq J_0 \), and it follows from the above that \( \text{Hom}(\pi, K) = \text{Hom}(\pi, J_0/K) = 0 \); hence the natural maps

\[
\text{Hom}_{H(F)}(\pi, J) \to \text{Hom}_{H(F)}(\pi, J/K) \to \text{Hom}_{H(F)}(\pi, J_0)
\]

are both injective. However, \( \tilde{Z} \) is a non-zero element of the first space, and (since \( J/J_0 \) is irreducible) Theorem 19.3.1 shows that the third space has dimension \( \leq 1 \). So we can conclude that for any choice of \( K \), the middle space in this sequence is 1-dimensional and is spanned by the image of \( \tilde{Z} \).

### 19.4. Tempered representations.

**Proposition 19.4.1.** If \( \pi \) is a generic and tempered irreducible representation of \( G(F) \), then all poles of its \( \mathcal{L} \)-factor have real parts in the set \( \{0, -\frac{1}{2}, -\frac{3}{2}\} \); and the case \( -\frac{3}{2} \) occurs if and only if \( \pi \) is an unramified twist of the Steinberg representation of \( G(F) \) (hence of type IVa). In particular, if \( \pi \) is an unramified tempered representation, then all poles of its \( \mathcal{L} \)-factor are purely imaginary.

**Proof.** This is easily verified from Tables A.1 and A.8 of [RS07], which list the \( \mathcal{L} \)-factors for all non-supercuspidal representations and identify which are generic and/or tempered. (The tables do not include supercuspidal representations, but for such representations the \( \mathcal{L} \)-factor is identically 1 so the claim is immediate.)

**Corollary 19.4.2.** If \( s_1 = -\frac{t_1}{2} \), \( s_2 = -\frac{t_2}{2} \) for integers \( t_1, t_2 \geq 0 \), and \( \pi \) is generic, then the hypotheses of Theorem 19.1.1 are automatically satisfied, except possibly if \( \pi \) is ramified and \( (t_1, t_2) = (0, 0) \).
19.5. Adelic results. We now give a semi-local variant on the above results. Let $\Pi_\ell$ be the common finite part of a pair $(\Pi^H, \Pi^W)$ of automorphic representation of $G(A)$ satisfying the running hypotheses of this paper, so in particular $\Pi_\ell = \otimes_\ell \Pi_\ell$ with every $\Pi_\ell$ being generic and tempered.

We shall apply the results of the previous sections to the $G(\mathbb{Q})$-representation $\Pi_\ell$ for every $\ell$, taking $(s_1, s_2) = (-\frac{r_1}{2}, -\frac{r_2}{2})$ in the notation of the main body of the paper. Tensoring together the corresponding local maps, we obtain a bilinear form

$$\tilde{Z} = \prod_\ell \tilde{Z}_\ell : \mathcal{W}(\Pi_\ell) \otimes S(A_\ell^2 \times A_\ell^2) \to \mathbb{C},$$

factoring via $\Phi \mapsto F_\Phi$ to give an $H(A_\ell)$-equivariant map

$$\mathcal{W}(\Pi_\ell) \otimes I(s_1, s_2) \to \mathbb{C},$$

where $I(s_1, s_2)$ is the product of the local representations above for each $\ell$. It follows readily from Theorem 19.1.2 that the space of such $H(A_\ell)$-equivariant homomorphisms is 1-dimensional, and $\tilde{Z}$ is a basis vector, as long as there is no prime $\ell$ such that $L(\Pi_\ell, s)$ has a subregular pole at $s = s_1 + s_2 - \frac{1}{2}$.

Finally, we remark that if $(I(s_1, s_2))_{(0)}$ denotes the image in $I(s_1, s_2)$ of the functions vanishing along $(0, 0) \times A_\ell^2$ and $A_\ell^2 \times (0, 0)$, then $(I(s_1, s_2))_{(0)}$ is a sum of subrepresentations of the form $I_1, 0 \otimes I_2, 0 \otimes \otimes_\ell \xi_{(\ell_1, \ell_2)} I_\ell$, where $I_\ell$ is the local representation at $\ell$, and $I_{1, 0}$ and $I_{2, 0}$ are nonzero subrepresentations at some primes $\ell_1, \ell_2$. It now follows from the last assertion of Theorem 19.1.2 that

$$\text{Hom}_{H(A_\ell)}(\mathcal{W}(\Pi_\ell) \otimes (I(s_1, s_2))_{(0)}, \mathbb{C})$$

is one-dimensional and spanned by (the restriction of) $\tilde{Z}$.

Since we showed in sections 7 and 8 of [LSZ17] that the Lemma–Eisenstein map, restricted to Schwartz functions of character $\chi$ factors via $\Phi \mapsto F_\Phi$, this proves Theorem 6.6.2.

20. Explicit formulae at unramified primes

We now evaluate the linear functionals $\tilde{Z}(w, \Phi, s_1, s_2)$ of (19.1) explicitly, for some specific choices of the test data. We shall let the local field $F$ be $\mathbb{Q}_p$, and we shall take for $\pi$ the local factor $\Pi_p$ of a globally generic cuspidal automorphic representation at an unramified prime, as in Section 6.2.

As in that section, $r_1 \geq r_2$ are the weights of the algebraic representation for which $\Pi$ is cohomological, and we write $(\alpha, \beta, \gamma, \delta)$ for the Hecke parameters of $\pi' = \pi \otimes | - (r_1 + r_2)|$. The temperedness of $\pi$ is thus equivalent to the condition that $\alpha, \beta, \gamma, \delta$ all have complex absolute value $p^{(r_1 + r_2 + 3)/2}$.

20.1. Bases of eigenspaces at parahoric levels. Let $w^{\text{ph}}$ be the spherical Whittaker function of $\pi$, normalised such that $w^{\text{ph}}(1) = 1$ (which is always possible). We are interested in describing Hecke operators which will map $w^{\text{ph}}$ to normalised generators of the eigenspaces at the various parahoric levels, where “normalised” again means these Whittaker functions take the value 1 at the identity.

We shall write $U_{1, \text{Si}}$ for the double coset operator $p^{(r_1 + r_2)/2}[\text{Si}(p) \text{ diag}(p, p, 1, 1) \text{ Si}(p)]$ on $\pi^{\text{Si}(p)}$, and $U_{2, \text{Kl}}$ for $p^{r_1}[\text{Kl}(p) \text{ diag}(p^2, p, p, 1) \text{ Kl}(p)]$ acting on $\pi^{\text{Kl}(p)}$. These correspond to the normalisations used in the main text for double coset operators on $\pi'$, so as before the eigenvalues of $U_{1, \text{Si}}$ are $\{\alpha, \beta, \gamma, \delta\}$ and those of $U_{2, \text{Kl}}$ are $\{\frac{\alpha \beta}{p^{r_2 + r_2}}, \ldots\}$.

**Lemma 20.1.1.**

(1) The normalised generator of the $U_{1, \text{Si}} = \alpha$ eigenspace of $\pi^{\text{Si}(p)}$ is given by

$$w_{\alpha}^{\text{Si}} = \left(1 - \frac{\beta}{U_{1, \text{Si}}}\right)\left(1 - \frac{\gamma}{U_{1, \text{Si}}}\right)\left(1 - \frac{\delta}{U_{1, \text{Si}}}\right)w^{\text{ph}}.$$

(2) If $\alpha + \gamma \neq 0$, the normalised generator of the $U_{2, \text{Kl}} = \frac{\alpha \beta}{p^{r_2 + r_2}}$ eigenspace of $\pi^{\text{Kl}(p)}$ is given by

$$w_{\alpha \beta}^{\text{Kl}} = \frac{1}{1 + \frac{\alpha \gamma}{p^{r_2 + r_2}U_{2, \text{Kl}}}}\left(1 - \frac{\beta \gamma}{p^{r_2 + r_2}U_{2, \text{Kl}}}\right)\left(1 - \frac{\alpha \gamma}{p^{r_2 + r_2}U_{2, \text{Kl}}}\right)\left(1 - \frac{\beta \delta}{p^{r_2 + r_2}U_{2, \text{Kl}}}\right)w^{\text{ph}}.$$

(3) The normalised generator of the $(U_{1, 1w} = \alpha, U_{2, 1w} = \frac{\alpha \beta}{p^{r_2 + r_2}})$ eigenspace in $\pi^{\text{1w}(p)}$ is given by

$$w_{\alpha \beta}^{\text{1w}} = \left(1 - \frac{\alpha \gamma}{p^{r_2 + r_2}U_{2, 1w}}\right)w_{\alpha}^{\text{Si}} = \left(1 - \frac{\beta}{U_{1, 1w}}\right)w_{\alpha \beta}^{\text{Kl}}.$$
In each case, it is obvious that the given vector lies in the relevant eigenspace, and the content of the lemma is that it takes the value 1 at the identity. This follows by tedious computations from the Casselman–Shalika formula giving the values of $w_{\text{sph}}^0$ on any diagonal element; an explicit form of this formula in the $\text{GSp}_4$ case can be found as Equation 7.3 in [RS07].

**Proposition 20.1.2.** If $\alpha + \gamma \neq 0$, then the image of $w_{\alpha, \beta}^{\text{KL}}$ under the trace map $\varphi \mapsto \sum_{\gamma \in \text{G}(\mathbb{Z})/\text{Kl}(\mathbb{P})} \gamma \cdot \varphi$ is given by

$$\text{Tr} \left( w_{\alpha, \beta}^{\text{KL}} \right) = p^3 \left( 1 - \frac{\gamma}{p^2} \right) \left( 1 - \frac{\delta}{p^3} \right) \left( 1 - \frac{\beta}{p^2} \right) w_{\text{sph}}^0.$$  

**Proof.** This follows from an extremely tedious explicit computation; rather than the Whittaker model, one fixes an ordering of the Hecke parameters, giving a choice of model of $\pi$ as an induction from the Borel subgroup. The Klingen-invariants then have an explicit basis given by coset representatives for $B(\mathbb{Z})/G(\mathbb{Z})/\text{Kl}(\mathbb{P})$. A lengthy double-coset computation gives the matrix of $\text{U}_2, \text{Kl}$ in this basis; and the trace map in this basis is explicit, so the result follows from a routine computation. \hfill $\square$

**Remark 20.1.3.** A formula for $\text{Tr} \left( w_{\alpha, \beta}^{\text{KL}} \right)$ is stated without proof in [GT05]. However, their formula differs from ours, having terms of the form $\left( 1 - \frac{2}{p^2} \right)$ rather than $\left( 1 - \frac{2}{p^2} \right)$. We believe the formula stated above to be the correct one. \hfill $\diamond$

To link up with the zeta-integral computations of [LPSZ19], we also need to consider eigenvectors for the “transpose” Hecke operator $U_2^T \cdot \text{Kl}$, $p^{-1/2} \text{diag}(1, p, p, p^2)$. Given the above computations, it seems natural to consider the vector

$$w_{\alpha, \beta}^{\text{KL}, T} := \frac{1}{(1 + \frac{\gamma}{\alpha})} \left( 1 - \frac{\beta \gamma}{p^3 + 1} \right) \left( 1 - \frac{\alpha \gamma}{p^3 + 1} \right) \left( 1 - \frac{\beta \delta}{p^2 + 1} \right) w_{\text{sph}}^0.$$  

By dualizing the previous computation, we see that $\text{Tr} \left( w_{\alpha, \beta}^{\text{KL}, T} \right) = \text{Tr} \left( w_{\alpha, \beta}^{\text{KL}} \right)$.

We briefly summarize how this relates to the computations of *op.cit.*. Let $w = r_1 + r_2 + 3$, and let $\Lambda$ be the unramified character of $T(\mathbb{Q}_p)$ given by $\chi_1 \times \chi_2 \times \rho$ in the notation of [RS07] §2.2, where $\rho(p) = p^{-w/2 \alpha}$, $\chi_1(p) = \gamma/\alpha$, $\chi_2(p) = \beta/\alpha$. Then $\text{Ind}^{G(\mathbb{Q}_p)}_{\text{B}(\mathbb{Q}_p)}(\Lambda)$ gives an explicit model of $\pi$; and identifying the Klingen Levi $\text{M}_{\text{KL}}$ with $\text{GL}_2 \times \text{GL}_1$ as in [LPSZ19] §8.4, we can therefrom write $\pi = \text{Ind}^{\text{G}(\mathbb{Q}_p)}_{\text{B}(\mathbb{Q}_p)}(\tau \boxtimes \theta)$, where $\theta = \chi_1$ and $\tau$ is the unramified principal series $\rho \chi_2 \times \rho$ of $\text{GL}_2$.

In *op.cit.* we considered the diagram of maps

$$
\begin{array}{ccc}
\pi & \longrightarrow & \mathcal{W}(\pi) \\
\downarrow & & \downarrow \\
\tau & \longrightarrow & \mathcal{W}(\tau)
\end{array}
$$

Here the horizontal arrows are the canonical intertwining maps from the induced representations to their Whittaker models, and the vertical arrow is given by restriction of functions in the induced representation from $G(\mathbb{Q}_p)$ to $\text{GL}_2(\mathbb{Q}_p) \subset \text{M}_{\text{KL}}(\mathbb{Q}_p)$ (note that this is only $\text{GL}_2(\mathbb{Q}_p)$-equivariant up to a twist by a power of $|\det|$).

In *op.cit.* we considered a vector $\phi_1 \in \pi^{\text{KL}(\mathbb{P})}$, characterized by the property of being supported on $B(\mathbb{Q}_p) \cdot \text{Kl}(\mathbb{P})$ and taking the value $p^3$ at the identity. This maps to $p^3 \xi \in \tau$, where $\xi$ is the spherical function of $\tau$ satisfying $\xi(1) = 1$; and via the Casselman–Shalika formula, we have $W_\xi(1) = \left( 1 - \frac{\beta}{p^2} \right)$.

**Lemma 20.1.4.** The image of $\phi_1$ in $\mathcal{W}(\pi)$ is $(1 - \frac{\beta}{p^2}) w_{\alpha, \beta}^{\text{KL}, T}$.

**Proof.** Since both vectors are $U_2^T \cdot \text{Kl}$-eigenvectors with the same eigenvalue, it suffices to check that they both have the same trace down to spherical level. By construction $\phi_1$ has trace $p^3 \phi_{\text{sph}}$ where $\phi_{\text{sph}}$ is the normalised spherical function of $\phi$; and, by the Casselman–Shalika formula for $\text{GSp}_4$, the image of $p^3 \phi_{\text{sph}}$ in $\mathcal{W}(\pi)$ is $p^3 \left( 1 - \frac{\beta}{p^2} \right) \left( 1 - \frac{\gamma}{p^3} \right) \left( 1 - \frac{\delta}{p^3} \right) \left( 1 - \frac{\beta}{p^3} \right) w_{\text{sph}}^0$. This agrees with the formula we have computed above for $\text{Tr} \left( w_{\alpha, \beta}^{\text{KL}, T} \right)$. \hfill $\square$

**Remark 20.1.5.** So the “correct” basis of the ordinary eigenspace for $U_2^T \cdot \text{Kl}$ at level $\text{Kl}(\mathbb{P})$ is clearly $w_{\alpha, \beta}^{\text{KL}, T}$, which has the effect of renormalising the $\text{GL}_2$ Whittaker function $W_\xi(1)$ to take the value 1 at the identity. \hfill $\diamond$
20.2. Particular values of the zeta integral. Let us now choose characters $\chi_1, \chi_2$ of $\mathbb{Q}_p^\times$ such that $\chi_1 \chi_2 = \chi_1$; we shall suppose (for now) that the $\chi_i$ are unramified. We shall also choose integers $(q, r)$ with $0 \leq q \leq r_2$ and $0 \leq r \leq r_1 - r_2$, and consider the zeta integral at $(s_1, s_2) = (-\frac{r}{2}, -\frac{q}{2})$, where $(t_1, t_2) = (r_1 - q - r, r_2 - q + r)$. 

Note 20.2.1. We then have $L(\Pi, s_1 + s_2 - \frac{1}{2}) = L(\Pi, \frac{(r_1 + r_2 + 3)}{2} + q + 1) = \frac{1}{(1 - \frac{\alpha}{p^{r_1}}) \ldots (1 - \frac{\delta}{p^{r_2}})}$, and similarly $L(\Pi, s_1 - s_2 + \frac{1}{2}) = L(\Pi, \frac{(r_1 + r_2 + 3)}{2} + r + r_2 + 2) = \frac{1}{(1 - \frac{\alpha}{p^{r_1+2}}) \ldots (1 - \frac{\delta}{p^{r_2+2}})}$.

20.2.1. Spherical test data. We let $\Phi_{sph} = \text{ch}(Z_2^p \times Z_2^p)$, and we let $w_{sph}$ be the normalised spherical Whittaker function as above. Then, as we have already noted, we have $\tilde{Z}(w_{sph}, \Phi_{sph}) = 1$.

20.2.2. Siegel parahoric test data. Recall that $\text{Si}(p)$ denotes the Siegel parahoric modulo $p$. We choose a Hecke parameter $\alpha$ and consider the vector $w_{\alpha} \in \mathbb{W}(\pi)^{\text{Si}(p)}$ of Lemma 20.1.1 and we let $\Phi_{\text{Si}} = \text{ch}((pZ_p \times Z_p^p)^2)$.

Proposition 20.2.2. We have

$$
\tilde{Z}(w_{\alpha}, \Phi_{\text{Si}}) = \frac{1}{(p + 1)^2} \left(1 - \frac{\beta}{p^{r_1}}\right) \left(1 - \frac{\gamma}{p^{r_2}}\right) \left(1 - \frac{\delta}{p^{r_2+2}}\right) \left(1 - \frac{\delta}{p^{r_2+2+2}}\right) \left(1 - \chi_2(p)p^{r_2+1+\gamma}\right) \left(1 - \chi_2(p)p^{r_2+1+\delta}\right).
$$

Proof. We use the Bessel-model description of $\tilde{Z}(w, \Phi, s_1, s_2)$ given in Proposition 19.2.1. Note that the choice of Bessel model used depends on the value of $r$ (but is independent of $q$). The Schwartz function $\Phi_{\text{Si}}$ is chosen so that $f^{\Phi_{\text{Si}}}$ is supported on the coset $B_H(\mathbb{Q}_p) \cdot \text{Iw}_H$, where $\text{Iw}_H = \text{Si}(p) \cap H(\mathbb{Q}_p)$ is the upper-triangular Iwahori subgroup of $H$, and its value at the identity is 1. So for any $\text{Si}(p)$-invariant (or just $\text{Iw}_H$-invariant) $w$, we have

$$
\tilde{Z}(w, \Phi_{\text{Si}}) = \lim_{\xi \to 0} \frac{1}{L(\pi, s_1 + s_2 - \frac{1}{2} + \xi)} \int_{B_H \backslash H} z_{w,0}(h)f^\Phi(h, s_1 + \frac{1}{2}, s_2 + \frac{1}{2})\, dh
$$

$$
= \frac{\tilde{Z}(w, 0)}{(p + 1)^2 L(\pi, s_1 + s_2 - \frac{1}{2})}.
$$

(Note that the denominator is finite, since $\pi$ is tempered and $\Re(s_i) \leq 0$.) Since $w$ is by assumption $\text{Si}(p)$-invariant, we have $z_{w,0}(1) = F_w(p^{-1})$, where $F_w(X)$ is the rational function

$$
F_w(X) = \sum_{n \in \mathbb{Z}} p^{n(r_1 + r_2 + 6) / 2} B_w\left(\begin{pmatrix} p^n & \alpha \\ 0 & 1 \end{pmatrix}\right) X^n.
$$

It is easy to see that for any $\text{Si}(p)$-invariant $w$ we have

$$
F_w(X) = F_w(0) + X F_{\text{w}^{(0)}}(X),
$$

so in particular $F_{w^{(0)}}(X)$ is a constant multiple of $1/(1 - \alpha X)$. We can determine the constant by comparing with the spherical Whittaker vector $w_{sph}$: by Proposition 3.5.6(b) of [LSZ17], we have

$$
F_{w^{(0)}}(X) = \left(1 - \chi_1(p)p^{r_1+1-r}X\right) \left(1 - \chi_2(p)p^{r_2+1+r}X\right) \left(1 - \chi_2(p)p^{r_2+1+\gamma}X\right) \left(1 - \chi_2(p)p^{r_2+1+\delta}X\right).
$$

and an explicit computation shows that we have

$$
F_{w^{(0)}}(X) = \left(1 - \chi_1(p)p^{r_1+1-r}X\right) \left(1 - \chi_2(p)p^{r_2+1+\gamma}X\right) \left(1 - \chi_2(p)p^{r_2+1+\delta}X\right).
$$

Substituting this into the formula $\tilde{Z}(w_{\alpha}, \Phi_{\text{Si}}) = \frac{1}{(p + 1)^2} L(\Pi, s_1 + s_2 - \frac{1}{2})^{-1} \cdot F_{w^{(0)}}(p^{-1-q})$ gives the result.

20.2.3. Klingen test data: the general formula. We now consider the case of test vectors of Klingen parahoric level. This computation is largely worked out in [LPSZ19] §8.4, but without making explicit the normalisation of the vector $w \in \mathbb{W}(\Pi)$ used, so we shall tease out this detail.

We shall assume that our Hecke parameters are ordered as $(\alpha, \beta, \gamma, \delta)$ such that $\gamma / \alpha = \delta / \beta$ not equal to $-1$, so that the vector $w_{\alpha,\beta}^{\text{Kl}} \in \mathbb{W}(\pi)^{\text{Kl}(p)}$ of [20.1] is defined. As Proposition 5.6 of

\footnote{This will be automatically satisfied if $\Pi$ is Klingen-ordinary at $p$, since this ratio then has $p$-adic valuation $r_2 + 2 > 0$.}
Proposition 20.2.3. Let $p_{\text{crit}}$ be the syntomic regulator, and consider the slightly more general integral

$$Z(\gamma \cdot w^{\text{KL}}, \Phi_1 \times \Phi_2) = \int_{Z_\mathbb{Q}} \mathcal{E}(\pi, m) \mathcal{E}(\pi \times \chi_2^{-1}, r_2 + 1 + r) W^{\Phi_i}((x_1), -\frac{(r_1-2-r)}{2}) W^{\Phi_2}((x_1), -\frac{(r_2-\rho+r)}{2}) \theta(x) d^s x.$$

Remark 20.2.4. Here we are assuming that the integrand has no pole at the relevant value of $(s_1, s_2)$, which can only happen if $r = (r_1 - r_2 + 1)/2$ and $\mathcal{E}(\pi \times \chi_2^{-1}, r_2 + 1 + r)$ vanishes.

Proof. This is a special case of Proposition 8.14 of [LPSZ19]. Since we are assuming $\pi$ and the $\chi_i$ to be unramified, the epsilon-factor term in $\text{op.cit.}$ is 1; and the ratio of $L$-factors gives the two $\mathcal{E}(\pi, -)$ terms. Moreover, as shown above, our renormalisation of the Klingen test vectors is precisely the one which scales the (non-normalised) GL$_2$ Whittaker function $W_\xi$ of $\text{op.cit.}$ to its normalised equivalent $w_\tau^{\text{ph}}$.

20.2.4. Particular cases. We define the following Schwartz functions on $Q^2_\mathbb{Q}$:

- $\Phi_{\text{dep}} = \text{ch}(Z_\pi^p \times Z_\pi^p)$,
- $\Phi_{\text{crit}} = \text{ch}(Z_\pi^p \times Z_\pi^p)$.

These will correspond to holomorphic Eisenstein series that are respectively $p$-depleted, ordinary, or critical-slope (hence the notation). We let $\Phi_2$ denote the preimage of $\Phi'_1$ under the inverse Fourier transform (in the second variable only); these are a little messy to write down explicitly.

Then we have the following formulae, assuming $n \geq 0$ and $\chi$ unramified:

- $W^{\Phi_{\text{dep}}}(\binom{p^n}{1}, s) = 1$ if $n = 0$, and zero otherwise.
- $W^{\Phi_{\text{crit}}}(\binom{p^n}{1}, s) = p^{-ns}$.

Accordingly, for $m \gg 0$ the integral of Proposition 20.2.3 is given by

$$\mathcal{E}(\pi, m) \mathcal{E}(\pi \times \chi_2^{-1}, r_2 + 1 + r) = \left\{ \begin{array}{ll}
1, & \text{if } (\Phi_1, \Phi_2) = (\Phi_{\text{dep}}, \Phi_{\text{crit}}), (\Phi_{\text{crit}}, \Phi_{\text{dep}}) \text{ or } (\Phi_{\text{dep}}, \Phi_{\text{dep}});
(1 - \frac{\gamma}{p^{r+q}}) \left(1 - \frac{\delta}{p^{r+q}}\right)^{-1}, & \text{if } (\Phi_1, \Phi_2) = (\Phi_{\text{crit}}, \Phi_{\text{crit}}).
\end{array} \right.$$
is helpful to also consider it as an instance of the GSp4 × GL2 zeta-integral of op.cit. for π × σ, where σ is taken to be the representation
\[
σ := I(|\cdot|^{s_2-1/2},|\cdot|^{1/2-s_2} r χ_2^{-1})
\]
(subject to an appropriate definition of the Whittaker model of σ in the reducible case, as in the footnote to Proposition 8.14 of op.cit.). The first interpretation shows that the fractional ideal generated by the values of the renormalised zeta-integral
\[
\tilde{Z}(\ldots) = \frac{Z(\ldots)}{L(π × ρ, s_1 + s_2 - \frac{1}{2})} L(π × ρ χ_2^{-1}, s_1 - s_2 + \frac{1}{2})
\]
is the unit ideal of \(C[\ell^{±1}, ℓ^{±1}]\). The second interpretation shows that if \(Φ'(0, 0) = 0\), then for \(m \gg 0\) we have the special-value formula
\[
\tilde{Z}(γ, \omega_m^{KL}, Φ_1 × Φ_2, s_1, s_2; χ, ρ) = \frac{1}{L(τ × σ × θ, s_1)L(τ × σ, 1 - s_1)e(τ × σ, s_1)} \int_{Q_p^x} u_τ^{\text{sp}}((x_1, s)) W^{Φ_1}(x_1; χ_1, s_1)W^{Φ_2}(x_1; χ_2, s_2) \frac{θ(x)ρ(x)}{|x|} d^x x.
\]
In this more general setting, the test functions we shall use are of the form
\[
Φ'_{\text{dep}, μ, ν}(x, y) = \text{ch}(Z_p^x × Z_p^y) • μ(x)ν(y), \quad Φ'_{\text{crit}, μ, ν}(x, y) = \text{ch}(Z_p × Z_p^x) • ν(y),
\]
for finite-order characters μ, ν, with \(χ|Z_p^x = μ^{-1}ν\) (taking μ to be trivial in the case of \(Φ'_{\text{crit}, μ, ν}\), so this condition becomes simply \(χ|Z_p^x = ν\)). Note that \(Φ'_{\text{dep}, μ, ν}(x, y)\) is the same function considered in [LPSZ19] Definition 7.5].

We have
\[
W^{Φ_{\text{dep}, μ, ν}}((x_1); χ, s) = \begin{cases} μ(x)ν(−1) & \text{if } x ∈ Z_p^x \\ 0 & \text{otherwise} \end{cases}
\]
and
\[
W^{Φ_{\text{crit}, μ, ν}}((x_1); χ, s) = \begin{cases} |x|^sν(−1) & \text{if } x ∈ Z_p \\ 0 & \text{otherwise}. \end{cases}
\]
Thus, for test data \(Φ_{\text{dep}, μ_1, ν_1} × Φ_{\text{dep}, μ_2, ν_2}\), assuming \(μ_1 ν_1 = μ_2 ν_2 = 1\) and \(ρ = ν_1 ν_2\), the torus Whittaker integral is simply the integral of the constant function 1 over \(Z_p^x\), so it is 1. Similarly, for test data of the form \(Φ_{\text{dep}, μ_1, ν_1} × Φ_{\text{crit}, ν_2}\) we again obtain that the integral is 1. This equality of zeta integrals for “dep × dep” and “dep × crit” data is the input needed in [16.3]

On the other hand, the Whittaker function of \((p)^{-1}Φ_{\text{crit}, ν}\) is zero at \((\frac{1}{2}, 0)\) unless \(x ∈ pZ_p\), so if we consider test data of the form \((p)^{-1}Φ_{\text{dep}, μ, ν} × Φ_{\text{dep}, μ, ν}\), then we are integrating the product of a function supported on \(pZ_p\) and another supported on \(Z_p^x\). Hence the zeta integral is 0, as required for [16.4].
21. Variants of $\eta$

| Notation | Cohomology group | Definition |
|----------|-----------------|-----------|
| $\eta_{\text{dR}}$ | $\text{Fil}^1 D_{\text{dR}}(W_{\Pi})$ | §7.6 |
| $\eta_{\text{dR},q}$ | $\text{Fil}^{1+q} D_{\text{dR}}(W_{\Pi})$ | §7.6 |
| $\eta_{\text{dR},q,-D}$ | $H^1_{\text{dR}}(X_{\text{KL}}(D), V, 1 + q)$ | §7.6 |
| $\eta_{\text{NN-fp},q,-D}$ | $H^3_{\text{NN-fp}}(X_{\text{KL}}(-D), V, 1 + q, P)$ | §7.6 |
| $\eta_{\text{rig-fp},q,-D}$ | $H^3_{\text{rig-fp}}(X_{\text{KL}}(-D), V, 1 + q, P)$ | §10.4 |
| $\eta_{\text{rig-fp},q,-D}^{\geq 1}$ | $H^3_{\text{rig-fp}}(X_{\text{KL}}^{\geq 1}(-D), V, 1 + q, P)$ | §10.4 |
| $\eta_{\text{rig},q,-D}$ | $H^2(X_{\text{KL}}, N^1(-D))$ | §11.6 |
| $\eta_{\text{coh},q,-D}$ | $H^2_0(X_{\text{KL}}, N^1(-D))$ | §11.6 |
| $\eta_{\text{ord},q,-D}$ | $H^2_0(X_{\text{KL}}, N^1)$ | §11.6 |
| $\tilde{\eta}_{\text{dR},q,-D}$ | $\tilde{H}^3_{\text{dR},e}(X_{\text{KL}}(-D), V, 1 + q)$ | §12.1 |
| $\tilde{\eta}_{\text{rig-fp},q,-D}^{\geq 1}$ | $\tilde{H}^3_{\text{rig-fp},e}(X_{\text{KL}}^{\geq 1}(-D), V, 1 + q; P)$ | §12.1 |
| $\tilde{\eta}_{\text{rig},q,-D}^{\geq 1}$ | $\tilde{H}^3_{\text{rig},e}(X_{\text{KL}}^{\geq 1}(-D), V, 1 + q)$ | §12.1 |
| $\tilde{\eta}_{\text{rig-fp},q}$ | $\tilde{H}^3_{\text{rig-fp},e}(X_{\text{KL}}^{\geq 1}, V, 1 + q; P)$ | §12.3 |
| $\tilde{\eta}_{\text{coh},q,-D}^{\geq 1}$ | $\tilde{H}^2_0(X_{\text{KL}}^{\geq 1}, \otimes V^0 \otimes \Omega^1_{\text{KL}}(-D))$ | §12.4 |
| $\tilde{\eta}_{\text{coh},q}$ | $\tilde{H}^2_0(X_{\text{KL}}^{\geq 1}, \otimes V^0 \otimes \Omega^1_{\text{KL}})$ | §12.4 |

22. P-adic L-functions

| Function | Domain | Defined in |
|----------|--------|-----------|
| $L_{p,\nu}^{\text{mot},r}(\Pi)$ | $W \times W$ | Theorem 6.8.1 |
| $L_{1,2}^{\text{mot},r}(\Pi)$ | $W$ | Proposition 16.2.3 |
| $c_1, c_2 L_{p,\nu}^{\text{mot},r}(\Pi)$ | $U \times W$ | Definition 17.5.3 |
| $L_{p,\nu}^{\text{mot},r}(\Pi)$ | $U \times W$ | Notation 17.5.3 |
| $L_{p,\nu}(\Pi)$ (conjectural) | $U \times W \times W$ | Section 17.4 |
| $L_{p,\nu}(\Theta)$ | $W$ | Proposition 17.5.4 |
| $L_{p,\nu}(\Theta)$ | $U \times W$ | Theorem 17.6.2 |
