Vibration characteristics of piezoelectric functionally graded carbon nanotube-reinforced composite doubly-curved shells

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Abstract This paper presents an analytical solution for the free vibration behavior of functionally graded carbon nanotube-reinforced composite (FG-CNTRC) doubly curved shallow shells with integrated piezoelectric layers. Here, the linear distribution of electric potential across the thickness of the piezoelectric layer and five different types of carbon nanotube (CNT) distributions through the thickness direction are considered. Based on the four-variable shear deformation refined shell theory, governing equations are obtained by applying Hamilton’s principle. Navier’s solution for the shell panels with the simply supported boundary condition at all four edges is derived. Several numerical examples validate the accuracy of the presented solution. New parametric studies regarding the effects of different material properties, shell geometric parameters, and electrical boundary conditions on the free vibration responses of the hybrid panels are investigated and discussed in detail.

Key words free vibration, four-variable shear deformation refined theory, functionally graded carbon nanotube-reinforced composite (FG-CNTRC), piezoelectric material

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1 Introduction

Functionally graded carbon nanotube-reinforced composite (FG-CNTRC) materials are a new generation of composite materials in which carbon nanotubes (CNTs) are designed purposefully to grade with specific rules along with desired directions within an isotropic matrix. Thanks to the outstanding properties of FG-CNTRC and electromechanical properties of piezoelectric materials, the complicated FG-CNTRC structures with integrated piezoelectric layers (PFG-CNTRC) offer great potential for use in engineering, advanced aerospace, medicine, military, and automotive structural applications. The PFG-CNTRC structures are applied as smart structures to control the free vibration, suppress the forced vibration, decrease the deflection and stresses, delay the buckling, decrease the post-buckling deflection, flutter control,
and suppress the snap-through phenomenon. Subsequently, researchers started to investigate
the behavior of the PFG-CNTRC structures in the last few years.

Based on the three-dimensional (3D) theory of elasticity, Alibeigloo[1–5] performed the bend-
ing and free vibration analysis of PFG-CNTRC plates and cylindrical panels (CYLs) subjected
to mechanical uniform pressure, thermal load, and applied voltage field. Using a unified formu-
lation of a finite layer model based on Reissner’s mixed variational theorem, the 3D buckling
responses of the PFG-CNTRC plates under bi-axial compressive loads were investigated by
Wu and Chang[6]. In the classical lamination theory based on the Kirchhoff plate theory,
the interlaminar shear deformation is neglected. Nasihatgozar et al.[7] studied the stability of
the PFG-CNTRC cylindrical panels under the axial and biaxial loadings. Applying the Ritz
energy approach, the post-buckling behavior of PFG-CNTRC cylindrical shells subjected to
axial compression and lateral loads in the thermal environment was investigated by Ansari et
al.[8]. Using the first-order shear deformation plate theory (FSDT) with the assumption of
constant transverse shear deformation, Rafiee et al.[9] analyzed the nonlinear dynamic stability
of PFG-CNTRC plates with initial geometric imperfection subjected to a combined thermal
and electrical loadings and interaction of parametric and external resonance. Rafiee et al.[10]
also investigated the large amplitude free vibration of the immovable simply supported PFG-
CNTRC plates. Based on the FSDT and the Ritz method, Kiani[11] developed a model for
the free vibration analysis of FG-CNTRC plates with various mechanical and electrical bound-
ary conditions. Sharma et al.[12] implemented a finite element model with the nonlinear fuzzy
logic controller to perform active vibration control of the FG-CNTRC plates. Some research
papers on the analysis of PFG-CNTRC plates and shells based on the higher-order shear de-
formation plate theory are also available. Kolahchi et al.[13] presented a model based on the
refined piezoelasticity zig-zag theory for general wave propagation analysis of the piezoelectric
sandwich plate. The free vibration behavior of the PFG-CNTRC quadrilateral spherical panel
was examined by Setoodeh et al.[14]. Nguyen-Quang et al.[15] introduced an extension of the
isogeometric approach based on the non-uniform rational B-spline (NURBS) basis functions for
the dynamic response of the laminated PFG-CNTRC plates. Based on Reddy’s higher-order
shear deformation theory, Selim et al.[16] studied the impact analysis of the PFG-CNTRC plates
using the element-free IMLS-Ritz model. Song et al.[17] presented the active vibration control
of FG-CNTRC plates with surface-bonded piezoelectric actuator and sensor. Selim et al.[18]
presented a novel element-free IMLS-Ritz model for the active vibration control of the FG-
CNTRC plate with different positions of piezoelectric sensor layer and actuator layer. Zhang
et al.[19] presented an optimal shape control of PFG-CNTRC plates with the open-loop control
and the displacement feedback control gain.

Free vibration characteristics of shell structures were reported by many authors such as Tran
et al.[20], Baghlani et al.[21] for FGM cylindrical shells; Song et al.[22] for FNTRC cylindrical
shells; Zhu et al. for cylindrical nano-shell[23], and doubly curved nano-shell[24].

In the present work, the previous works of Tran et al.[25] and HUU et al.[26] for PFG-CNTRC
plates are utilized and developed wherein multilayer FG-CNTRC doubly curved panels (DCPs)
with surface-bonded piezoelectric layers are modeled using four-variable shear deformation re-
efined theory for the first time. By applying Navier’s solution, the governing equations can be
solved to obtain the vibrational responses of hybrid shell panels. Several numerical examples
validate the accuracy of the present model. The effects of material properties, geometric pa-
rameters, and electrical boundary conditions on the vibrational responses of the hybrid panels
are also investigated and discussed in detail.

2 Mathematical formulation

2.1 Geometry and material properties

Consider a laminated FG-CNTRC doubly curved panel (FG-CNTRC-DCP) with two perfect
bonded piezoelectric layers at the top and bottom surfaces, as shown in Fig. 1.
Vibration characteristics of PFG-CNTRC doubly-cured shells

Fig. 1 Schematic of laminated PFG-CNTRC doubly curved shell panel (color online)

The panel with constant principal curvatures is referred to as an orthogonal curvilinear coordinate system \((x, y, z)\). The length, width, and two radii of principal curvatures of the middle surface of the panel are denoted by \(a\), \(b\), and \(R_x\) and \(R_y\), respectively. The thickness of the core and each piezoelectric layer are also marked by \(h_c\) and \(h_p\), respectively.

As shown in Fig. 2, for each layer, five types of distributions of CNT are considered. UD represents the uniform distributions, and the other four types of functionally graded distributions of CNT are denoted by FG-A, FG-V, FG-X, and FG-O.

Fig. 2 Distribution types of FG-CNT reinforced composite layers: (a) UD layer; (b) FG-A layer; (c) FG-V layer; (d) FG-X layer; (e) FG-O layer (color online)

According to the distributions of CNTs, the CNT volume fractions \(V_{\text{CNT}}(z)\) for each FG-CNTRC layer are given as\(^{[27]}\)

\[
V_{\text{CNT}}(z) = \begin{cases} 
V^*_{\text{CNT}} & \text{for UD CNTRC}, \\
2V^*_{\text{CNT}} \frac{z - z_k}{z_{k+1} - z_k} & \text{for FG-V CNTRC}, \\
2V^*_{\text{CNT}} \frac{z_{k+1} - z}{z_{k+1} - z_k} & \text{for FG-A CNTRC}, \\
2V^*_{\text{CNT}} \left(1 - \frac{|2z - z_k - z_{k+1}|}{z_{k+1} - z_k}\right) & \text{for FG-O CNTRC}, \\
2V^*_{\text{CNT}} \left(\frac{|2z - z_k - z_{k+1}|}{z_{k+1} - z_k}\right) & \text{for FG-X CNTRC}, 
\end{cases}
\] (1)
where
\[
V^*_{\text{CNT}} = \frac{w_{\text{CNT}}}{w_{\text{CNT}}} + \frac{\eta_{\text{CNT}}}{\eta_{\text{CNT}}} - \frac{\rho_{\text{CNT}}}{\rho_{\text{CNT}}},
\]
and
\[
\mathbf{K} = \begin{bmatrix}
E_{11}(z) &= \eta_1 V_{\text{CNT}}(z) E^1_{11} + V_m(z) E^m, \\
E_{22}(z) &= \eta_2 V_{\text{CNT}}(z) E^1_{22} + V_m(z) E^m, \\
\end{bmatrix}
\]
where \(E^1_{11}, E^1_{22}, \text{ and } E^m\) are Young’s moduli of CNT and matrix; \(G_{12}^\text{CNT}\) and \(G_m\) are the shear modulus of CNT and matrix; \(\eta_1, \eta_2, \text{ and } \eta_3\) are CNT efficiency parameters; \(V_m\) and \(V_{\text{CNT}}\) are volume fractions of matrix and CNT with the relation of \(\rho_{\text{CNT}} + \rho_m = 1; \nu^\text{CNT}\) and \(\nu^m\) are Poisson’s ratios of CNT and matrix.

### 2.2 Approximation on mechanical displacement

The displacements field for doubly curved shell based on the four-variable shear deformation refined theory is given by the following equations:
\[
\begin{aligned}
u(x, y, z, t) &= \left(1 + \frac{z}{R_x}\right) v_0(x, y, t) - z \frac{\partial w_b(x, y, t)}{\partial x} - f(z) \frac{\partial w_s(x, y, t)}{\partial x}, \\
v(x, y, z, t) &= \left(1 + \frac{z}{R_y}\right) v_0(x, y, t) - z \frac{\partial w_b(x, y, t)}{\partial y} - f(z) \frac{\partial w_s(x, y, t)}{\partial y}, \\
w(x, y, z, t) &= v_0(x, y, t) + w_s(x, y, t),
\end{aligned}
\]
where \(u_0\) and \(v_0\) are the in-plane displacements in the directions of \(x\) and \(y\); \(w_b\) and \(w_s\) represent the bending and shear components of the transverse displacement, respectively. The polynomial shape function is used as \(f(z) = z^3 - \frac{3}{2} z^2 + \frac{1}{2} \left(\frac{z}{R_x}\right)^3\). The strains at any point in the shell space associated with the displacement field in Eq. (4) can be obtained by applying the fundamental kinematic relations of a 3D body in curvilinear coordinates as follows:
\[
\begin{aligned}
\varepsilon_x &= \frac{1}{1 + z/R_x} \left(\varepsilon_x^0 + z \kappa_x^0 + f(z) \kappa_x^1\right), \\
\varepsilon_y &= \frac{1}{1 + z/R_y} \left(\varepsilon_y^0 + z \kappa_y^0 + f(z) \kappa_y^1\right), \\
\gamma_{xy} &= \frac{1}{1 + z/R_x} \left(\gamma_{xy}^0 + z \kappa_{xy}^0 + f(z) \kappa_{xy}^1\right) \\
\gamma_{yz} &= \frac{1}{1 + z/R_y} \left(\gamma_{yz}^0 + z \kappa_{yz}^0 + f(z) \kappa_{yz}^1\right), \\
\gamma_{xz} &= \frac{1}{1 + z/R_z} g(z) \gamma_{xz}^0, \\
\end{aligned}
\]
2.3 Approximation on electric potential

In this work, the electric potential distribution in the transverse direction of each piezoelectric layer is approximated by the linear function as follows\textsuperscript{[31–33]}:

\[
\begin{align*}
\Phi^i(x, y, z, t) &= \left(z - \frac{h}{2}\right)\frac{1}{h_p}\phi^i(x, y, t) = Z_p^i\phi^i(x, y, t), \quad \frac{h_c}{2} \leq z \leq \frac{h_c}{2} + h_p, \\
\Phi^b(x, y, z, t) &= -\left(z + \frac{h}{2}\right)\frac{1}{h_p}\phi^b(x, y, t) = Z_p^b\phi^b(x, y, t), \quad -\frac{h_c}{2} - h_p \leq z \leq -\frac{h_c}{2}.
\end{align*}
\]

(7)

The electric field is related to electric potential by the following relation:

\[
E_i = -\nabla \Phi_i, \quad i = x, y, z,
\]

(8)

where \(\nabla\) denotes the gradient operator. Thus, the components of the electric field are obtained as

\[
\begin{align*}
(E^i_x, E^i_y, E^i_z) &= -\left(\frac{1}{1 + z/R_c}Z_p^i\phi_x, \frac{1}{1 + z/R_c}Z_p^i\phi_y, \frac{1}{h_p}\phi^i\right), \\
(E^b_x, E^b_y, E^b_z) &= -\left(\frac{1}{1 + z/R_c}Z_p^b\phi_x, \frac{1}{1 + z/R_c}Z_p^b\phi_y, \frac{1}{h_p}\phi^b\right).
\end{align*}
\]

(9)

2.4 Constitutive equations

The stress-strain relation for the \(k\)th CNT layer is expressed as\textsuperscript{[34]}

\[
\begin{pmatrix}
\sigma_{xx}^k \\
\sigma_{yy}^k \\
\tau_{xy}^k \\
\tau_{xz}^k \\
\tau_{yz}^k \\
\gamma_{xy}^k \\
\gamma_{xz}^k \\
\gamma_{yz}^k
\end{pmatrix} = \begin{pmatrix}
Q_{11}^k(z) & Q_{12}^k(z) & 0 & 0 & Q_{16}^k(z) & 0 & 0 & 0 \\
Q_{12}^k(z) & Q_{22}^k(z) & 0 & 0 & 0 & Q_{26}^k(z) & 0 & 0 \\
0 & 0 & Q_{44}^k(z) & Q_{45}^k(z) & 0 & 0 & 0 & 0 \\
0 & 0 & Q_{54}^k(z) & Q_{55}^k(z) & 0 & 0 & 0 & 0 \\
Q_{16}^k(z) & 0 & 0 & 0 & 0 & Q_{66}^k(z) & -Q_{66}^k(z) & 0 \\
Q_{26}^k(z) & 0 & 0 & 0 & 0 & 0 & Q_{66}^k(z) & -Q_{66}^k(z) \\
0 & 0 & Q_{56}^k(z) & -Q_{66}^k(z) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}\begin{pmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy} \\
\gamma_{xz} \\
\gamma_{yz}
\end{pmatrix},
\]

(10)

in which \(Q_{ij}^k(z)\) are transformed material constants expressed in terms of material constants\textsuperscript{[35]},

\[
\begin{align*}
Q_{11}^k(z) &= Q_{11}^k(z)\cos^2\theta^k + 2(Q_{12}^k(z) + 2Q_{66}^k(z))\sin^2\theta^k\theta^k + Q_{22}^k(z)\sin^4\theta^k, \\
Q_{12}^k(z) &= (Q_{11}^k(z) + Q_{22}^k(z) - 4Q_{66}^k(z))\sin^2\theta^k\cos^2\theta^k + Q_{12}^k(z)(\sin^4\theta^k + \cos^4\theta^k), \\
Q_{22}^k(z) &= Q_{11}^k(z)\sin^2\theta^k + 2(Q_{12}^k(z) + 2Q_{66}^k(z))\sin^2\theta^k\cos^2\theta^k + Q_{22}^k(z)\cos^4\theta^k, \\
Q_{16}^k(z) &= (Q_{11}^k(z) - Q_{12}^k(z) - 2Q_{66}^k(z))\sin\theta^k\cos^3\theta^k \\
&\quad + (Q_{12}^k(z) - Q_{22}^k(z) + 2Q_{66}^k(z))\sin^3\theta^k\cos\theta^k, \\
Q_{26}^k(z) &= (Q_{11}^k(z) - Q_{12}^k(z) - 2Q_{66}^k(z))\sin^3\theta^k\cos\theta^k \\
&\quad + (Q_{12}^k(z) - Q_{22}^k(z) + 2Q_{66}^k(z))\sin^3\theta^k\cos\theta^k, \\
Q_{56}^k(z) &= (Q_{11}^k(z) + Q_{22}^k(z) - 2Q_{12}^k(z) - 2Q_{66}^k(z))\sin^2\theta^k\cos^2\theta^k \\
&\quad + Q_{66}^k(z)(\sin^4\theta^k + \cos^4\theta^k), \\
Q_{54}^k(z) &= Q_{44}^k(z)\cos^2\theta^k + Q_{55}^k(z)\sin^2\theta^k, \\
Q_{45}^k(z) &= (Q_{55}^k(z) - Q_{44}^k(z))\cos\theta^k\sin\theta^k, \\
Q_{55}^k(z) &= Q_{55}^k(z)\cos^2\theta^k + Q_{44}^k(z)\sin^2\theta^k,
\end{align*}
\]

(11)

where \(Q_{ij}^k(z)\) are the plane stress-reduced stiffnesses defined in terms of the engineering constants in the material axes of the layer,

\[
\begin{align*}
Q_{11}^c(z) &= \frac{E_{11}(z)}{1 - \nu_{12}^2}, & Q_{12}^c(z) &= \frac{\nu_{12}E_{22}(z)}{1 - \nu_{12}^2}, & Q_{22}^c(z) &= \frac{E_{22}(z)}{1 - \nu_{12}^2}, \\
Q_{44}^c &= G_{23}(z), & Q_{55}^c &= G_{13}(z), & Q_{66}^c &= G_{12}(z).
\end{align*}
\]

(12)
For the $k_p$th piezoelectric layer ($k_p = \text{top, bottom}$), the stress components and electric displacement in the piezoelectric layer are given as \[ \frac{\partial \delta E}{\partial T} \] where $C_{ij}^{kp}$ are stiffness coefficients for the piezoelectric layers; $e_{ij}$ is the electromechanical coupling matrix; $p_{ij}$ is the dielectric permittivity matrix; $E$ is the electric field, and $D$ is the displacement in the piezoelectric layer.

\subsection{2.5 Governing equation}

Hamilton's principle is applied herein to obtain the equations of motion of laminated PFG-CNTRC doubly curved shell panels,

\[
\int_0^t (\delta U - \delta T) \, dt = 0,
\]

where $\delta U$ is the variation of strain energy, and $\delta K$ is the variation of kinetic energy. The expression of variation of strain energy for the PFG-CNTRC panels is

\[
\delta U = \int_0^b \int_0^a \int_{-h/2}^{h/2} \left( \sigma_{xx} \delta\varepsilon_{xx} + \sigma_{yy} \delta\varepsilon_{yy} + \sigma_{xy} \delta\varepsilon_{xy} + \tau_{xy} \delta\gamma_{xy} + \tau_{yx} \delta\gamma_{yx} + \tau_{xx} \delta\gamma_{xz} + \tau_{yy} \delta\gamma_{yz} - D_x^{ik} \delta E_x^k + D_y^{ik} \delta E_y^k - D_x^{jk} \delta E_x^j - D_y^{jk} \delta E_y^j \right) \, dz \, dx \, dy.
\]

Substituting Eq. (5), Eq. (9), Eq. (13), and Eq. (14) into Eq. (17), we can get the variation of strain energy $\delta U$ as

\[
\delta U = \int_0^b \int_0^a \left( N_x \delta\varepsilon_{xx} + N_y \delta\varepsilon_{yy} + N_{yx} \delta\varepsilon_{xy} + N_{xy} \delta\varepsilon_{yx} + M_x^{ik} \delta\kappa_{ikx} + M_y^{ik} \delta\kappa_{iky} + M_{xy}^{ik} \delta\kappa_{ikxy} + M_{yx}^{ik} \delta\kappa_{ikyx} + Q_{xx} \delta\gamma_{xx} + Q_{xx} \delta\gamma_{xx} + Q_{yy} \delta\gamma_{xx} + Q_{yy} \delta\gamma_{xx} \right) \, dz \, dx \, dy.
\]
\[
\delta T = \int_0^b \int_y^b \rho(z) (\ddot{u} \dot{w} + \ddot{v} \dot{w} + \ddot{w} \dot{w}) \left(1 + \frac{z}{R_z}\right) \left(1 + \frac{z}{R_y}\right) \, dz \, dx \\
= \delta T_1 + \delta T_2 + \delta T_3 + \delta T_4, \tag{19}
\]

where

\[
\delta T_1 = \int_0^b \int_y^b \left( \frac{1}{R_x} \left( - \left( \bar{I}_1 + \bar{I}_2 \right) \ddot{u} + \ddot{I}_2 \frac{\partial \ddot{w}_b}{\partial x} + J_2 \frac{\partial \ddot{w}_b}{\partial x} \right) + \bar{I}_1 \frac{\partial \ddot{u}}{\partial x} \right) \delta u_0 \, dx \, dy,
\]

\[
\delta T_2 = \int_0^b \int_y^b \left( \frac{1}{R_y} \left( - \left( \bar{I}_1 + \bar{I}_2 \right) \ddot{v} + \ddot{I}_2 \frac{\partial \ddot{w}_b}{\partial y} + J_2 \frac{\partial \ddot{w}_b}{\partial y} \right) + \bar{I}_1 \frac{\partial \ddot{v}}{\partial y} \right) \delta v_0 \, dx \, dy,
\]

\[
\delta T_3 = \int_0^b \int_y^b \left( - \left( \bar{I}_1 + \bar{I}_2 \right) \ddot{u} + \ddot{I}_2 \frac{\partial \ddot{w}_b}{\partial x} + J_2 \frac{\partial \ddot{w}_b}{\partial x} \right) \delta w_0 \, dx \, dy,
\]

\[
\delta T_4 = \int_0^b \int_y^b \left( - \left( \bar{I}_1 + \bar{I}_2 \right) \ddot{v} + \ddot{I}_2 \frac{\partial \ddot{w}_b}{\partial y} + J_2 \frac{\partial \ddot{w}_b}{\partial y} \right) \delta w_0 \, dx \, dy,
\]

in which \(\rho(z)\) is the mass density, and \(\bar{I}_i\) \((i = 0, 1, 2)\) and \(\bar{J}_i\) \((i = 1, 2)\) are inertia terms defined by

\[
\begin{align*}
\bar{I}_i &= i + 1 + \frac{1}{R_x} + \frac{1}{R_y} \\
\bar{I}_{i+2} &= \frac{1}{R_x} \left( \bar{I}_1 \bar{I}_2 \right) \frac{\partial \ddot{w}_b}{\partial x} \left( \bar{I}_1 \bar{I}_2 \right) \frac{\partial \ddot{w}_b}{\partial x} \\
\bar{I}_1 &= f(z) \bar{I}_1, \quad \bar{K}_1 = f^2(z) \bar{I}_1.
\end{align*}
\tag{24}
\]

The governing equations can be obtained by substituting the variation of strain energy and kinetic energy from Eq. (18) and Eq. (19) into Eq. (16) and then set the coefficients of \(\delta u_0, \delta v_0, \delta w_b, \delta w_s, \delta \phi^a, \) and \(\delta \phi^b\) to be zero,

\[
\begin{align*}
\delta u_0 : \frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} + \frac{Q_{zb}}{R_x} = \bar{I}_0 \ddot{u} + \bar{I}_1 \left( \frac{\ddot{u}}{R_x} - \frac{\partial \ddot{w}_b}{\partial x} \right) - \bar{J}_1 \frac{\partial \ddot{w}_b}{\partial x},
\delta v_0 : \frac{\partial N_y}{\partial x} + \frac{\partial N_x}{\partial y} + \frac{Q_{zb}}{R_y} = \bar{I}_0 \ddot{v} + \bar{I}_1 \left( \frac{\ddot{v}}{R_y} - \frac{\partial \ddot{w}_b}{\partial y} \right) - \bar{J}_1 \frac{\partial \ddot{w}_b}{\partial y},
\delta w_b : \frac{N_x}{R_x} + \frac{N_y}{R_y} - \frac{\partial Q_{zb}}{\partial x} - \frac{\partial Q_{zb}}{\partial y} = - \bar{I}_0 (\ddot{w}_b + \ddot{w}_s),
\delta w_s : \frac{N_x}{R_x} + \frac{N_y}{R_y} - \frac{\partial Q_{zs}}{\partial x} - \frac{\partial Q_{zs}}{\partial y} = - \bar{I}_0 (\ddot{w}_b + \ddot{w}_s),
\delta \phi^a : \frac{\partial \bar{D}_b^a}{\partial x} + \frac{\partial \bar{D}_b^a}{\partial y} + \bar{D}_z^a = 0,
\delta \phi^b : \frac{\partial \bar{D}_b^b}{\partial x} + \frac{\partial \bar{D}_b^b}{\partial y} + \bar{D}_z^b = 0.
\end{align*}
\tag{25}
\]
3 Solution procedure

In this study, two sets of simply supported boundary conditions named cross-ply (SS-1) and angle-ply (SS-2) laminates shown in Table 1 are considered.

Table 1 Two sets of simply supported boundary conditions

| Edge | Boundary condition | SS-1 | SS-2 |
|------|-------------------|------|------|
| x = 0 and x = a | u₀ = u₁ = u₂ = w₁, y = w₁, y = 0 | u₀ = u₁ = w₁ = w₁, y = u₀, y = w₁ = 0 | Nₓ = Mᵧ = Mₓ = Mᵧ = φₓ = φᵧ = 0 |
| y = 0 and y = b | u₀ = u₁ = w₁ = w₁, x = w₁, x = 0 | v₀ = w₁ = w₁ = u₁, x = w₁, x = 0 | Nᵧ = Mₓ = Mₓ = φₓ = φᵧ = 0 |

Following the Navier solution procedure, the following expansion displacements for u₀, v₀, wₛ, wᵧ, φₓ, and φᵧ are chosen to satisfy the boundary conditions given in Table 1. Here, uₘₙ, vₘₙ, wₘₙ, wᵧₘₙ, φₓₘₙ, and φᵧₘₙ are arbitrary parameters to be determined, α = \( \alpha \pi \), β = \( \beta \pi \), and \( m, n \) denote the numbers of half-waves in the x- and y-directions, respectively.

By substituting the expansion displacement functions in Table 2 into Eq. (25), the following matrix form can be obtained:

\[
\begin{pmatrix}
  m_{11} & m_{12} & m_{13} & m_{14} & 0 & 0 \\
  m_{12} & m_{22} & m_{23} & m_{24} & 0 & 0 \\
  m_{13} & m_{23} & m_{33} & m_{34} & 0 & 0 \\
  m_{14} & m_{24} & m_{34} & m_{44} & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
  \hat{u}_{mn} \\
  \hat{v}_{mn} \\
  \hat{w}_{mn}^b \\
  \hat{w}_{mn}^t \\
  \hat{\psi}_{mn} \phi^b \\
  \hat{\psi}_{mn} \phi^t
\end{pmatrix}
\]
FG-CNTRC shell panels integrated with piezoelectric layers, equation in Eq. (24), we obtain a short form of the equation for vibration characteristics of K respectively. Equation (27) can be rewritten in the short form as

\[
\begin{pmatrix}
  k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\
  k_{12} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\
  k_{13} & k_{23} & k_{33} & k_{34} & k_{35} & k_{36} \\
  k_{14} & k_{24} & k_{34} & k_{44} & k_{45} & k_{46} \\
  k_{15} & k_{25} & k_{35} & k_{45} & k_{55} & k_{56} \\
  k_{16} & k_{26} & k_{36} & k_{46} & k_{56} & k_{66}
\end{pmatrix}
\begin{pmatrix}
  u_{mn} \\
  v_{mn} \\
  w_m^b \\
  w_s^p \\
  \psi_m^t \\
  \psi_m^b
\end{pmatrix} = \begin{pmatrix}
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix},
\]

(27)

where \(m_{ij}\) and \(k_{ij}\) are given in Appendices B1 and B2 for SS-1 and SS-2 boundary conditions, respectively. Equation (27) can be rewritten in the short form as

\[
\begin{pmatrix}
  M_{uu} & 0 \\
  0 & 0
\end{pmatrix}
\begin{pmatrix}
  \ddot{u} \\
  \ddot{\phi}
\end{pmatrix} + \begin{pmatrix}
  K_{uu} & K_{u\phi} \\
  K_{u\phi} & K_{\phi\phi}
\end{pmatrix}
\begin{pmatrix}
  u \\
  \phi
\end{pmatrix} = \begin{pmatrix}
  0 \\
  0
\end{pmatrix},
\]

(28)

Table 2 Expansion displacements \((u_0, v_0, w_s, \phi^l, \phi^b)\)

| Displacement | SS-1 | Boundary condition | SS-2 |
|--------------|------|--------------------|------|
| \(u_0(x, y, t)\) | \(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn} \cos(\alpha x) \sin(\beta y)\) | \(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn} \sin(\alpha x) \cos(\beta y)\) |
| \(v_0(x, y, t)\) | \(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} v_{mn} \sin(\alpha x) \sin(\beta y)\) | \(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} v_{mn} \cos(\alpha x) \sin(\beta y)\) |
| \(w_s(x, y, t)\) | \(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_s^{mn} \sin(\alpha x) \sin(\beta y)\) | \(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_s^{mn} \cos(\alpha x) \sin(\beta y)\) |
| \(w_s(x, y, t)\) | \(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_s^{mn} \sin(\alpha x) \sin(\beta y)\) | \(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_s^{mn} \cos(\alpha x) \sin(\beta y)\) |
| \(\phi^l(x, y, t)\) | \(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_m^{mn} \sin(\alpha x) \sin(\beta y)\) | \(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_m^{mn} \cos(\alpha x) \sin(\beta y)\) |
| \(\phi^b(x, y, t)\) | \(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_m^{mn} \sin(\alpha x) \sin(\beta y)\) | \(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_m^{mn} \cos(\alpha x) \sin(\beta y)\) |

where \(M_{uu}\) is the mass matrix, \(K_{uu}\) is the elastic matrix, \(K_{u\phi}\) is the piezoelectric matrix, and \(K_{\phi\phi}\) is the permittivity matrix. Substituting the solution of the second equation into the first equation in Eq. (24), we obtain a short form of the equation for vibration characteristics of FG-CNTRC shell panels integrated with piezoelectric layers,

\[
(M_{uu}) \ddot{u} + ((K_{uu} - (K_{u\phi})(K_{\phi\phi})^{-1}(K_{\phi u}))(u) = 0.
\]

(29)

For free vibration, Eq. (29) reduces to an eigenvalue problem by setting \((u) = (\ddot{u})e^{i\omega t}\)

\[
((K_{uu}) - (K_{u\phi})(K_{\phi\phi})^{-1}(K_{\phi u}))(\ddot{u}) = 0.
\]

(30)

Equation (30) is associated with the natural frequencies of an open-circuit (Opc) PFG-CNTRC doubly curved shell panels (see Fig. 3(a)). For the closed-circuit (Clc) (see Fig. 3(b)) condition, both the upper and lower piezoelectric layers are grounded cause the electric displacement disappears from Eq. (28). Hence, we obtain the following eigenvalue problem for the Clc boundary condition:

\[
((K_{uu}) - \omega^2(M_{uu}))(\ddot{u}) = 0.
\]

(31)

The eigensolution of free vibration for PFG-CNTRC-DCP can be obtained using a general eigenvalue approach.
Fig. 3  Electrical boundary conditions: (a) closed-circuit and (b) open-circuit

4  Numerical results and discussion

In this section, several comparison studies are conducted to verify the accuracy of instant solution in predicting natural frequencies of a simply supported isotropic doubly curved shell panel and a rectangular FG-CNTRC; both of these structures are integrated with piezoelectric layers at the top and bottom surfaces. Furthermore, the effects of involved parameters such as CNT volume fractions, distribution types of CNT, and different geometric parameters on free vibration response of PFG-CNTRC doubly curved shell panel are also investigated in detail. Unless mentioned otherwise, the material properties of PFG-CNTRC doubly curved shell panels are given in Table 3.

Table 3  Material properties of piezoelectric and FG-CNT materials

| Property       | Substrate layer | Piezoelectric layer |
|----------------|-----------------|---------------------|
|                | PmPV[15]        | CNT[15]             | Al₂O₃[32] | PZT-4[32] | PZT-5A[15] |
| ρ/(kg·m⁻³)     | 1 150           | 1 400               | 3 800     | 7 500     | 7 750      |
| E₁₁/GPa        | Eₘ=2.5          | 5.64×10⁵            | 380       | –         | E = 63     |
| E₂₂/GPa        | Eₘ=2.5          | 7.08×10⁵            | 380       | –         | –          |
| G₁₂/GPa        | –               | 1 945.5             | –         | –         | G = 23.3   |
| ν₁₂            | 0.34            | 0.175               | 0.3       | –         | 0.35       |
| C₁₁/GPa        | –               | –                   | –         | 139       | –          |
| C₁₂/GPa        | –               | –                   | –         | 77.8      | –          |
| C₁₃/GPa        | –               | –                   | –         | 74.3      | –          |
| C₂₂/GPa        | –               | –                   | –         | 115       | –          |
| C₃₃/GPa        | –               | –                   | –         | 25.6      | –          |
| C₄₄/5₅/GPa     | –               | –                   | –         | 30.6      | –          |
| e₃₁/cm⁻²       | –               | –                   | –         | –         | –9.299     |
| e₃₃/cm⁻²       | –               | –                   | –         | –         | 15.1       |
| e₁₁/cm⁻²       | –               | –                   | –         | 12.7      | 12.322     |
| p₁₁/(nFm⁻¹)    | –               | –                   | –         | 6.75      | 1.53       |
| p₃₃/(nFm⁻¹)    | –               | –                   | –         | 5.9       | 1.50       |

Also, the CNT efficiency parameters ηₖ(j = 1, 2, 3) associated with a given volume fraction V*ₖCNT are[15]: η₁ = 0.137, η₂ = 1.022 for the case of V*ₖCNT = 0.12, η₁ = 0.142, η₂ = 1.626 for the case of V*ₖCNT = 0.17, and η₁ = 0.141, η₂ = 1.585 for the case of V*ₖCNT = 0.28, and in all cases η₃ = 0.7η₂.

4.1 Comparison studies

Since there are no published results for the PFG-CNTRC doubly curved shell in the open literature, two numerical examples predict the free vibration behaviors of the piezoelectric isotropic doubly curved shell panels, and the PFG-CNTRC plates are performed to validate the present approach.

4.1.1 Example 1

Firstly, the fundamental frequencies of the simply supported isotropic doubly curved shell panel with surface-bonded piezoelectric layers for different geometric parameters are presented in Table 4. The Opc and Clc of electric boundary conditions are also considered. The shell
Vibration characteristics of PFG-CNTRC doubly-cured shells

Table 4  Fundamental frequencies $\omega$ of piezoelectric isotropic DCP ($a/b = 1, R_y/b = 5$)

| $a/R_x$ | $h/a$ | $h_p/h$ | $\omega_{\text{Clc}}/\text{Hz}$ | $\omega_{\text{Opc}}/\text{Hz}$ |
|--------|--------|---------|-------------------------------|-------------------------------|
|        |        |         | Present Ref. [38] Ref. [32] | Present Ref. [38] Ref. [32] |
| −0.2   | 0.1    | 0.1     | 823.126 824.049 823.997      | 840.081 841.189 843.029      |
| 0.2    | 0.1    | 0.1     | 1 484.893 1 483.998 1 483.888 | 1 514.076 1 512.112 1 515.550 |
| 0      | 0.1    | 0.1     | 1 480.806 1 398.792 1 398.093 | 1 457.032 1 449.614 1 461.337 |
| 0.2    | 0.1    | 0.1     | 1 498.803 1 498.047 1 498.297 | 1 528.068 1 526.689 1 530.183 |
| 0.2    | 0.2    | 1       | 1 412.678 1 411.727 1 411.025 | 1 469.096 1 462.898 1 474.891 |
| 0.2    | 0.2    | 0.1     | 1 418.902 1 411.727 1 411.025 | 1 469.096 1 462.898 1 474.891 |
| 0.2    | 0.2    | 0.2     | 1 543.343 1 542.349 1 542.240 | 1 543.343 1 542.349 1 545.778 |
| 0.2    | 0.2    | 0.2     | 1 425.281 1 424.974 1 424.279 | 1 481.171 1 475.765 1 487.688 |

Panel substrate is made of alumina (Al$_2$O$_3$), and two piezoelectric layers are made of PZT-4. It should be noted that the results reported by Sayyaadi et al.[32, 38] were based on the HSDT. The excellent agreement between the results shows the accuracy of the current approach.

4.1.2 Example 2

Secondly, the free vibration of the laminated cross-ply and angle-ply PFG-CNTRC plates with simply supported boundary conditions is investigated. The fundamental frequencies are given in Table 5 for different CNT volume fractions, CNT distribution types, and electrical boundary conditions (EBC). Properties of the plate are set equal to $a = b = 0.4\text{m}$, $h = a/20$, and $h_p = h/10$. The substrate is made of a multi-laminate of armchair SWCNT, and piezoelectric layers are PZT-5A. It can be seen that the present results agree well with those by Nguyen-Quang et al.[15] based on the isogeometric approach and the HSDT. The maximum difference is only 1.41% for the case that $V_{\text{CNT}} = 0.28$, FG-X, and $[p/−45^{°}/45^{°}/−45^{°}/−45^{°}/45^{°}/−45^{°}/p]$ configuration.

4.2 Parametric studies

4.2.1 Free vibration of PFG-CNTRC doubly curved shell panels

Parametric studies are carried out to understand the effects of material properties, geometric parameters, laminate configurations, and electrical boundary conditions on the free vibration responses of PFG-CNTRC doubly curved shell panels. The substrate of the panel is made of armchair SWCNT, and two piezoelectric layers are PZT-5A. Tables 6–8 list the fundamental frequencies of the CYL, spherical panel (SPH), and hyperbolic paraboloid (HPR) with various inlet parameters. It is observed from these tables that the distribution types of CNT have a significant effect on the stiffness of the panel. In detail, the FG-O panel has the lowest value of frequencies, while the FG-X panels have the highest ones. This conclusion is compatible with the conclusion of other researchers in the related studies in the literature. In three forms of doubly-curved shell panels, the HPR shell panels have the lowest frequencies, and the SPH shell panels have the highest frequencies. These results may become from the fact that SHP has a curvature effect, while the HPR has both positive and negative curvature that neutralize the effect of each other. These tables also reveal that the Opc of electrical boundary conditions always have higher frequencies than the Clc with all other parameters.

4.2.2 Effect of CNT distribution types

Figure 4 depicts the effect of distribution types of CNT on the fundamental frequencies of cross-ply PFG-CNTRC doubly curved shell panels for different $R_x/R_y$ ratio in case of the Opc condition. It is observed that the previous conclusions regarding the CNT distribution types are
also shows that the value of the fundamental frequencies is minimum at the ratio confirmed.

\[
T_{\text{830}} \quad \text{this example are set by}
\]

with closed and open piezoelectric layers are shown in Fig. 7. The parameters of the panels in 4.2.4 Effect of electrical boundary conditions

\[
f_\omega \text{PC} \quad \text{where}
\]

This observation, maybe because of the curvature effect, is suppressed when the panels have 4.2.3 Effect of CNT volume fractions

\[
\text{FG-X} \quad \text{FG-O}
\]

Moreover, the percentage change of frequency \((f_{\text{PC}})\) of the SPH panel is shown in Fig. 5, where \(f_{\text{PC}}\) is defined as

\[
f_{\text{PC}} = \left( \frac{\omega_{\text{FG}} - \omega_{\text{UD}}}{\omega_{\text{UD}}} \right) \times 100\%.
\]

(32)

It can be seen from Fig. 5 that among four CNT distribution types, only FG-X shows the positive of \(f_{\text{PC}}\), while others show the negative of \(f_{\text{PC}}\). These results indicate that only the FG-X panel has higher stiffness than the UD panel, while UD is the simplest distribution type.

4.2.3 Effect of CNT volume fractions

Figure 6 indicates that the fundamental frequencies of the PFG-CNTRC doubly curved shell panels strongly increase with the increase of CNT volume fractions. The panels are set by \(a/h = 20, a/b = 1, R_x/a = 5, \text{FG-X in Opc condition}\) and \([p/0^\circ/90^\circ/0^\circ/p]\) of configuration.

4.2.4 Effect of electrical boundary conditions

The natural frequency of laminated cross-ply FG-CNTRC doubly curved shell panels coupled with closed and open piezoelectric layers are shown in Fig. 7. The parameters of the panels in this example are set by \(a/h = 20; a/b = 1; R_x/a = 5; \text{FG-X; } V_{\text{CNT}}^* = 28\%\). Figure 7 shows that the FG-CNTRC panels coupled with the Opc always vibrate with the higher value of frequencies compared to the FG-CNTRC panels coupled with the Clc because the Opc converts electric potential to mechanical energy while the Clc does not.

### Table 5

| \(V_{\text{CNT}}^*\) | FG type | EBC | Laminate configuration | \([p/0^\circ/90^\circ/0^\circ/p]\) | \([p/-45^\circ/45^\circ/-45^\circ/45^\circ/-45^\circ/p]\) |
|------------------|---------|-----|------------------------|-------------------------------|--------------------------------|
| 0.12             | UD Clc  | 587.093 | 563.510 | 0.61 | 662.572 | 656.538 | 0.92 |
|                  | Op Clc  | 622.030 | 627.716 | 0.91 | 692.902 | 695.085 | 0.31 |
|                  | FG-X Clc| 593.270 | 588.372 | 0.83 | 667.328 | 658.696 | 1.31 |
|                  | FG-V Clc| 585.306 | 581.714 | 0.87 | 661.675 | 655.660 | 0.93 |
|                  | FG-O Clc| 620.461 | 626.205 | 0.92 | 692.094 | 694.272 | 0.31 |
|                  | FG-O Clc| 608.947 | 578.737 | 0.38 | 657.878 | 654.510 | 0.51 |
|                  | Op Clc  | 616.286 | 623.243 | 0.13 | 688.462 | 693.196 | 0.68 |
| 0.17             | UD Clc  | 628.442 | 624.543 | 0.62 | 727.596 | 720.800 | 0.94 |
|                  | Op Clc  | 660.887 | 665.615 | 0.71 | 754.830 | 755.388 | 0.07 |
|                  | FG-X Clc| 636.977 | 631.317 | 0.90 | 753.961 | 731.781 | 1.41 |
|                  | FG-V Clc| 668.952 | 671.913 | 0.44 | 760.925 | 758.217 | 0.36 |
|                  | FG-O Clc| 665.500 | 668.427 | 0.32 | 721.527 | 718.247 | 0.46 |
|                  | Op Clc  | 653.061 | 659.687 | 1.00 | 749.042 | 752.995 | 0.52 |
| 0.28             | UD Clc  | 692.016 | 686.852 | 0.75 | 828.983 | 821.713 | 0.88 |
|                  | Op Clc  | 720.749 | 723.150 | 0.33 | 851.843 | 850.524 | 0.16 |
|                  | FG-X Clc| 704.853 | 697.260 | 0.19 | 838.282 | 826.415 | 1.44 |
|                  | Op Clc  | 733.029 | 732.991 | 0.01 | 860.872 | 855.093 | 0.68 |
|                  | FG-V Clc| 685.165 | 682.974 | 0.76 | 827.912 | 820.463 | 0.91 |
|                  | Op Clc  | 717.276 | 719.788 | 0.35 | 850.901 | 849.465 | 0.17 |
|                  | FG-O Clc| 680.340 | 677.986 | 0.35 | 821.193 | 818.750 | 0.30 |
|                  | Op Clc  | 709.665 | 714.904 | 0.73 | 844.364 | 847.767 | 0.40 |
Table 6  Fundamental frequencies $\omega$ of PFG-CNTRC-DCP ($a = b, R_x/a = 5$, and $V_{\text{CNT}}^\ast = 0.12$).

| Shell panel | FG type | EBC | Laminate configuration |
|-------------|---------|-----|------------------------|
| Cylindrical (CYL, $R_y = \infty$) | UD | Clc | 592.304 | 154.618 | 593.607 | 159.006 | 687.170 | 235.130 |
| | Clc | Opc | 626.968 | 161.568 | 628.015 | 165.064 | 717.108 | 241.601 |
| | FG-A | Clc | 567.175 | 147.982 | 589.313 | 156.416 | 687.365 | 235.617 |
| | FG-A | Opc | 604.219 | 155.591 | 624.305 | 162.746 | 717.697 | 240.706 |
| | FG-V | Clc | 571.471 | 153.409 | 594.398 | 161.032 | 685.268 | 234.439 |
| | FG-V | Opc | 607.158 | 160.109 | 628.652 | 166.879 | 714.979 | 243.326 |
| | FG-X | Clc | 630.839 | 161.458 | 599.683 | 160.097 | 721.811 | 243.326 |
| | FG-X | Opc | 663.076 | 168.121 | 633.711 | 166.108 | 721.811 | 243.326 |
| | FG-O | Clc | 550.429 | 147.529 | 587.575 | 157.968 | 682.292 | 233.388 |
| | FG-O | Opc | 588.048 | 154.803 | 622.373 | 164.072 | 712.477 | 239.897 |

| Spherical (SPH, $R_y/b = 5$) | UD | Clc | 614.685 | 227.995 | 618.604 | 239.508 | 761.658 | 405.509 |
| | Clc | Opc | 648.704 | 234.979 | 651.724 | 244.267 | 791.004 | 413.788 |
| | FG-A | Clc | 586.781 | 221.745 | 611.821 | 236.324 | 762.835 | 406.178 |
| | FG-A | Opc | 623.529 | 229.384 | 645.821 | 241.370 | 792.821 | 414.711 |
| | FG-V | Clc | 598.356 | 229.060 | 622.037 | 242.409 | 758.969 | 404.781 |
| | FG-V | Opc | 632.817 | 235.592 | 654.682 | 246.917 | 787.815 | 412.812 |
| | FG-X | Clc | 651.893 | 232.777 | 624.478 | 240.418 | 767.182 | 408.539 |
| | FG-X | Opc | 683.647 | 239.612 | 657.247 | 245.140 | 796.300 | 416.780 |
| | FG-O | Clc | 574.487 | 223.264 | 612.812 | 238.726 | 756.211 | 402.522 |
| | FG-O | Opc | 611.225 | 230.398 | 646.288 | 243.522 | 785.789 | 410.839 |

| Hyperbolic paraboloid (HPR, $R_y/b = −5$) | UD | Clc | 582.399 | 120.108 | 582.436 | 120.108 | 656.344 | 136.697 |
| | Clc | Opc | 617.059 | 127.493 | 617.099 | 127.493 | 686.410 | 143.221 |
| | FG-A | Clc | 560.869 | 115.105 | 580.953 | 119.736 | 655.459 | 136.496 |
| | FG-A | Opc | 597.416 | 122.836 | 615.839 | 127.165 | 685.614 | 143.038 |
| | FG-V | Clc | 557.348 | 114.953 | 580.372 | 119.711 | 655.459 | 136.496 |
| | FG-V | Opc | 593.608 | 122.670 | 615.246 | 127.140 | 685.614 | 143.038 |
| | FG-X | Clc | 621.407 | 128.706 | 588.566 | 121.415 | 661.040 | 137.732 |
| | FG-X | Opc | 653.585 | 135.620 | 622.840 | 128.725 | 690.861 | 144.209 |
| | FG-O | Clc | 539.908 | 110.864 | 576.336 | 118.804 | 651.711 | 135.669 |
| | FG-O | Opc | 577.610 | 118.829 | 611.398 | 126.266 | 682.029 | 142.240 |

Fig. 4  Effects of distribution types of CNT on $\omega$ of PFG-CNTRC-DCP for different $R_x/R_y$ ratio ([p/0°/90°/0°/p], $a/h = 20$, $a/b = 1$, $R_x/a = 5$, and $V_{\text{CNT}}^\ast = 0.28$) (color online).

Fig. 5  Variation of PCF of PFG-CNTRC spherical shell panel with $R_x/R_y$ ratio (color online).
Table 7  Fundamental frequencies $\omega$ of PFG-CNTRC-DCP ($a = b$, $R_y/a = 5$, and $V_{CNT}^* = 0.17$)

| Shell panel configuration | FG type | EBC   | $[p/0^\circ/p]$ | $[p/0^\circ/90^\circ/0^\circ/p]$ | $[p/(-45^\circ/45^\circ/-45^\circ)a/p]$ |
|---------------------------|---------|-------|-----------------|-----------------------------------|----------------------------------------|
|                           |         |       | $a/h = 20$     | $a/h = 100$                       | $a/h = 20$                             | $a/h = 100$                           |
| Cylindrical (CYL, $R_y = \infty$) | UD  | Clc   | 633.572 164.396 | 635.074 169.546 | 758.173 269.051 | 269.051 |
|                           |       | OpC   | 665.777 170.885 | 667.020 175.122 | 784.980 274.812 | 274.812 |
|                           | FG-A  | Clc   | 598.652 155.938 | 629.723 166.595 | 758.225 269.554 | 269.554 |
|                           |       | OpC   | 633.715 163.155 | 662.292 172.434 | 785.445 275.334 | 275.334 |
| FG-V  | Clc   | 603.036 161.691 | 635.332 171.781 | 755.967 268.334 | 268.334 |
|                           |       | OpC   | 636.713 167.967 | 667.202 177.105 | 782.551 273.892 | 273.892 |
| FG-X  | Clc   | 685.601 173.875 | 643.487 171.108 | 791.305 277.054 | 277.054 |
|                           |       | OpC   | 714.948 180.017 | 674.977 176.629 | 784.980 274.812 | 274.812 |
| FG-O  | Clc   | 575.813 154.583 | 626.931 168.188 | 758.225 269.554 | 269.554 |
|                           |       | OpC   | 611.696 161.471 | 659.343 173.815 | 778.967 272.653 | 272.653 |
| Spherical (SPH, $R_y/b = 5$) | UD  | Clc   | 656.589 240.614 | 661.155 254.209 | 849.365 469.563 | 469.563 |
|                           |       | OpC   | 688.210 247.117 | 691.855 258.431 | 875.501 476.964 | 476.964 |
| FG-A  | Clc   | 619.120 233.156 | 653.114 250.730 | 850.514 470.304 | 470.304 |
|                           |       | OpC   | 653.929 240.360 | 684.716 255.223 | 877.308 477.950 | 477.950 |
| FG-V  | Clc   | 706.953 247.435 | 669.331 255.599 | 856.600 473.443 | 473.443 |
|                           |       | OpC   | 735.893 253.754 | 699.614 259.786 | 882.553 480.800 | 480.800 |
| FG-X  | Clc   | 601.151 234.135 | 653.377 253.302 | 842.404 465.845 | 465.845 |
|                           |       | OpC   | 636.156 240.819 | 681.501 257.557 | 868.792 473.287 | 473.287 |
| Hyperbolic paraboloid (HPR, $R_y/b = -5$) | UD  | Clc   | 623.421 128.847 | 623.462 128.848 | 720.586 150.857 | 150.857 |
|                           |       | OpC   | 655.611 135.726 | 655.656 135.726 | 747.584 156.763 | 156.763 |
| FG-A  | Clc   | 592.150 121.654 | 621.209 128.289 | 719.455 150.584 | 150.584 |
|                           |       | OpC   | 626.700 128.968 | 653.684 135.224 | 746.555 156.509 | 156.509 |
| FG-V  | Clc   | 588.329 121.488 | 620.521 128.260 | 719.454 150.584 | 150.584 |
|                           |       | OpC   | 622.956 128.789 | 652.991 135.194 | 746.555 156.509 | 156.509 |
| FG-X  | Clc   | 676.025 140.546 | 631.932 130.654 | 726.872 152.244 | 152.244 |
|                           |       | OpC   | 705.298 146.873 | 663.600 137.441 | 753.602 158.098 | 158.098 |
| FG-O  | Clc   | 564.824 116.046 | 615.213 127.073 | 714.596 149.505 | 149.505 |
|                           |       | OpC   | 600.798 123.644 | 647.887 134.043 | 741.873 155.462 | 155.462 |

Fig. 6  Effect of $V_{CNT}^*$ on fundamental frequencies $\omega$ of PFG-CNTRC shell panel (color online)

Fig. 7  Effect of electrical boundary conditions on $\omega$ of PFG-CNTRC-DCP (color online)

4.2.5  Effect of curvature of shell panels

The effect of radii of curvature on the natural frequencies of the SPH, CYL, and HPR shell panels are shown in Figs. 8 to 10, respectively. It can be seen from these figures that with the
Table 8  Fundamental frequencies $\omega$ of PFG-CNTRC-DCP ($a = b$, $R_x/a = 5$, and $V_{\text{CNT}} = 0.28$)

| Shell panel | FG type | Laminate configuration |
|-------------|---------|------------------------|
|             | EBC     | $a/h = 20$              | $a/h = 100$              | $a/h = 20$              | $a/h = 100$              |
| Cylindrical | UD Clc  | 696.079 176.288       | 697.771 182.254        | 869.286 323.219        |
|             | Clc     | 645.376 165.016       | 691.385 179.245        | 869.286 323.703        |
|             | Opc     | 677.457 171.831       | 720.359 184.516        | 892.263 328.795        |
| Spherical   | UD Clc  | 717.463 250.215       | 722.558 266.355        | 987.409 570.737        |
|             | Clc     | 745.644 256.334       | 749.826 270.100        | 1 009.173 577.071      |
| Hyperbolic  | UD Clc  | 686.512 142.888       | 686.568 142.888        | 820.668 174.254        |
| paraboloid  | Clc     | 638.958 131.868       | 683.163 141.999        | 819.621 173.881        |
|             | Opc     | 670.478 138.592       | 712.050 148.236        | 842.404 178.979        |
|             | Clc     | 634.839 131.687       | 682.322 141.962        | 819.621 173.881        |
|             | Opc     | 666.077 138.397       | 711.212 148.199        | 842.404 178.979        |
|             | Clc     | 761.021 150.902       | 699.305 145.597        | 829.855 176.247        |
| Hyperbolic  | Opc     | 786.416 165.428       | 727.267 151.652        | 852.241 181.268        |
| paraboloid  | Clc     | 600.719 123.820       | 674.977 140.327        | 812.987 172.392        |
|             | Opc     | 634.061 130.896       | 704.078 146.602        | 835.950 177.524        |

Increase of $R_x/a$ ratio, the frequencies of SPH and CYL decrease while the frequencies of HPR increase. The frequencies of all three types of shell panels are approximately equal to those of the plate with the corresponding input parameters when the value of $R_x/a$ ratio reaches 20. This observation once again confirms that the opposite curvature will reduce the stiffness of the shells.

4.2.6 Effect of piezoelectric layer thickness

In order to investigate the effect of piezoelectric layer thickness on the free vibration response of the composite shell panel, the variation of percentage difference in natural frequency $\beta$ is defined as

$$\beta = \frac{\omega_{\text{with piezoelectric layer}} - \omega_{\text{without piezoelectric layer}}}{\omega_{\text{without piezoelectric layer}}} \times 100\%.$$  (33)

Figures 11, 12, and 13 present the percentage difference in natural frequency $\beta$ versus the $h_p/h$ ratios for different forms of the panel, distribution types of CNT, and CNT volume fractions. It is observed from these figures that the HPR panel has a higher value of $\beta$ than CYL and SPH panel, the panel reinforced with lower CNT volume fractions has a higher value of $\beta$, among five CNT distribution types, the FG-O panel has the highest value of $\beta$ while the FG-X has lowest one. It can be concluded that the piezoelectric effects are more effective in the case of smaller stiffness panels rather than the case of greater stiffness ones. Furthermore,
Fig. 8  Effect of radius of curvature on $\omega$ of PFG-CNTRC spherical shell panels $(a/h = 20, \ a/b = 1, \ V_{\text{CNT}} = 0.28, \ R_y = R_x, [p/0^\circ/90^\circ/0^\circ/p])$ (color online)

Fig. 9  Effect of radius of curvature on $\omega$ of PFG-CNTRC cylindrical shell panels $(a/h = 20, \ a/b = 1, \ V_{\text{CNT}} = 0.28, \ R_y = \infty, [p/0^\circ/90^\circ/0^\circ/p])$ (color online)

Fig. 10  Effect of radius of curvature on $\omega$ of PFG-CNTRC HPR shell panels $(a/h = 20, \ a/b = 1, \ V_{\text{CNT}} = 0.28, \ R_y = -R_x, [p/0^\circ/90^\circ/0^\circ/p])$ (color online)

Fig. 11  Variation of $\beta$ of the PFG-CNTRC-DCP versus $h_p/h$ ratios for electrical boundary conditions $(a/h = 20, \ a/b = 1, \ R_y/a = 5, \ FG-X, \ V_{\text{CNT}} = 0.28, \ [0^\circ/90^\circ/0^\circ/p])$ (color online)

Fig. 12  Variation of $\beta$ of spherical panel versus $h_p/h$ ratios for different CNT distribution types $(a/h = 20, \ a/b = 1, \ R_y/a = R_y/b = 5, \ V_{\text{CNT}} = 0.28, \ [p/0^\circ/90^\circ/0^\circ/p])$ (color online)

Fig. 13  Variation of $\beta$ of spherical panel versus $h_p/h$ ratios for different CNT volume fractions $(a/h = 20, \ a/b = 1, \ R_y/a = R_y/b = 5, \ FG-X, \ [p/0^\circ/90^\circ/0^\circ/p])$ (color online)
these figures also indicate that increasing the thickness of piezoelectric layers from zero to a specific value leads to a decrement of $\beta$. It can be explained by the fact that the piezoelectric material has a higher mass density and lower elastic moduli compared with the core material. After that specific value of $h_p/h$, the percentage difference in natural frequency $\beta$ increases by the increment of piezoelectric layers thickness because the electromechanical coupling effect increases with an increase in $h_p/h$ and should increase the value of $\beta$. The electromechanical coupling effect is lower than the combined effects of the increase in the mass density, and the decrease in the stiffness results in negative values of $\beta$ for values of $h_p/h$.

5 Conclusions

In this paper, an analytical solution based on the four-variable shear deformation refined theory is developed for carrying out the free vibration of the laminated functionally graded nanotube-reinforced composite doubly curved shell panels with surface-bonded piezoelectric layers. Comparison studies validate the accuracy of the model. Numerical results are provided to explore the effects of the CNT volume fraction, the CNT distribution type, the thickness of the piezoelectric layers, laminate configurations, and mechanical and electrical boundary conditions on the natural frequencies of the hybrid panels. Through the present formulation and numerical results, some conclusions can be drawn. (i) The volume fraction of CNT has substantial effects on dynamic responses of PFG-CNTRC panels. (ii) The natural frequencies of PFG-CNTRC panel with Opc electrical boundary condition are always higher than those with the Clc case when all other inlet parameters are the same. (iii) CNT reinforcements distributed close to top and bottom are more efficient than those distributed near the mid-plane for increasing the stiffness of the panels.

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Appendix A

\[
\begin{pmatrix}
N_{xx} \\
N_{xy}
\end{pmatrix} = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \left( 1 + \frac{\bar{z}}{R_y} \right) \left( \frac{\sigma_{xx}^{k}}{\tau_{xy}^{k}} \right) dz - \int_{-h/2}^{-h/2-h_p} \left( 1 + \frac{\bar{z}}{R_y} \right) \left( \frac{\sigma_{xx}^{p}}{\tau_{xy}^{p}} \right)^T dz
\]

where \( \bar{z} = \frac{z}{h_p} \).
\[
\begin{align*}
\left( N_{yy} \right)_{M_{yy}} &= \sum_{k=1}^{n} \int_{y_k}^{y_{k+1}} \left( 1 + \frac{z}{R_x} \right) \left( \frac{\sigma_{yy}}{\tau_{yy}} \right) dz - \int_{-h/2}^{-h/2-h_y} \left( 1 + \frac{z}{R_y} \right) \left( \frac{\sigma_{yy}^{\text{pie}}}{\tau_{yy}^{\text{pie}}} \right) dz \\
&\quad - \int_{h/2}^{h/2+h_y} \left( 1 + \frac{z}{R_y} \right) \left( \frac{\sigma_{yy}^{\text{pie}}}{\tau_{yy}^{\text{pie}}} \right) dz, \\
\left( M_{xy} \right)_{M_{xy}} &= \sum_{k=1}^{n} \int_{y_k}^{y_{k+1}} \left( 1 + \frac{z}{R_y} \right) \left( \frac{\sigma_{xy}}{\tau_{xy}} \right) dz - \int_{-h/2}^{-h/2-h_y} \left( 1 + \frac{z}{R_y} \right) \left( \frac{\sigma_{xy}^{\text{pie}}}{\tau_{xy}^{\text{pie}}} \right) dz \\
&\quad - \int_{h/2}^{h/2+h_y} \left( 1 + \frac{z}{R_y} \right) \left( \frac{\sigma_{xy}^{\text{pie}}}{\tau_{xy}^{\text{pie}}} \right) dz, \\
\left( M_{yx} \right)_{M_{xy}} &= \sum_{k=1}^{n} \int_{y_k}^{y_{k+1}} \left( 1 + \frac{z}{R_y} \right) \left( \frac{\sigma_{yx}}{\tau_{yx}} \right) dz - \int_{-h/2}^{-h/2-h_y} \left( 1 + \frac{z}{R_y} \right) \left( \frac{\sigma_{yx}^{\text{pie}}}{\tau_{yx}^{\text{pie}}} \right) dz \\
&\quad - \int_{h/2}^{h/2+h_y} \left( 1 + \frac{z}{R_y} \right) \left( \frac{\sigma_{yx}^{\text{pie}}}{\tau_{yx}^{\text{pie}}} \right) dz, \\
Q_{ex} &= \sum_{k=1}^{n} \int_{y_k}^{y_{k+1}} \left( 1 + \frac{z}{R_y} \right) \left( \frac{t_{ex}}{t_{ex}^{\text{pie}}} \right) dz - \int_{-h/2}^{-h/2-h_y} \left( 1 + \frac{z}{R_y} \right) \left( \frac{t_{ex}^{\text{pie}}}{t_{ex}^{\text{pie}}} \right) dz \\
&\quad - \int_{h/2}^{h/2+h_y} \left( 1 + \frac{z}{R_y} \right) \left( \frac{t_{ex}^{\text{pie}}}{t_{ex}^{\text{pie}}} \right) dz, \\
Q_{ey} &= \sum_{k=1}^{n} \int_{y_k}^{y_{k+1}} \left( 1 + \frac{z}{R_y} \right) \left( \frac{t_{ey}}{t_{ey}^{\text{pie}}} \right) dz - \int_{-h/2}^{-h/2-h_y} \left( 1 + \frac{z}{R_y} \right) \left( \frac{t_{ey}^{\text{pie}}}{t_{ey}^{\text{pie}}} \right) dz \\
&\quad - \int_{h/2}^{h/2+h_y} \left( 1 + \frac{z}{R_y} \right) \left( \frac{t_{ey}^{\text{pie}}}{t_{ey}^{\text{pie}}} \right) dz.
\end{align*}
\]

Appendix B1

\[m_{11} = -\left( \frac{I_0 + \tilde{I}_1}{R_x} + \frac{I_2}{R_x^2} \right), \quad m_{12} = 0, \quad m_{13} = \left( \frac{I_1}{R_x} + \frac{I_2}{R_x^2} \right) \alpha_m, \quad m_{14} = \left( \frac{J_1}{R_x} + \frac{J_2}{R_x^2} \right) \alpha_m, \]
\[m_{22} = -\left( \frac{I_0 + \tilde{I}_1}{R_y} + \frac{I_2}{R_y^2} \right), \quad m_{23} = \left( \frac{I_1}{R_y} + \frac{I_2}{R_y^2} \right) \beta_n, \quad m_{24} = \left( \frac{J_1}{R_y} + \frac{J_2}{R_y^2} \right) \beta_n, \]
\[m_{33} = -\left( \frac{I_0}{R_x} - \tilde{I}_1 (\alpha_m^2 + \beta_n^2) \right), \quad m_{34} = -\tilde{I}_0 - J_2 (\alpha_m^2 + \beta_n^2), \quad m_{44} = -\tilde{I}_0 - \tilde{K}_1 (\alpha_m^2 + \beta_n^2). \]
\[k_{11} = -\left( B_{11} + \tilde{B}_{11} + \tilde{D}_{11} \right) \alpha_m - \left( A_{66} + 2 \tilde{B}_{66} + \tilde{D}_{66} \right) \beta_n, \]
\[k_{12} = -\left( A_{12} + A_{66} + (B_{12} + B_{66}) \left( \frac{1}{R_x} + \frac{1}{R_y} \right) + \frac{1}{R_x R_y} (D_{12} + D_{66}) \right) \beta_n. \]
\[k_{13} = \left( \frac{A_{11}}{R_x} + \frac{A_{12}}{R_x} + \frac{1}{R_x} (\tilde{B}_{11} + \frac{B_{11}}{R_x} + \frac{B_{12}}{R_y}) \right) + \left( B_{12} + B_{66} + \tilde{B}_{66} + \frac{1}{R_x} (D_{12} + D_{66} + D_{66}) \right) \beta_n \]
\[+ \left( \frac{\tilde{B}_{11}}{R_x} + \frac{\tilde{B}_{12}}{R_x} \right) \alpha_m, \]
\[k_{14} = \left( \frac{\tilde{B}_{11}}{R_x} + \frac{\tilde{D}_{11}}{R_x} \right) \alpha_m + \left( \frac{B_{12} + B_{66} + \tilde{B}_{66} + \frac{D_{12}}{R_x} + \frac{D_{66}}{R_x} + \frac{D_{66}}{R_x}}{R_x} \right) \beta_n \]
\[+ \left( \frac{\tilde{D}_{11}}{R_x} + \frac{\tilde{B}_{11}}{R_x} + \frac{\tilde{B}_{12}}{R_x} \right) \alpha_m. \]
\[ k_{15} = -\left( A_{11}^p + \frac{B_{12}^p}{R_e} \right) \alpha_m, \quad k_{16} = -\left( A_{12}^p + \frac{B_{12}^p}{R_e} \right) \alpha_m, \]
\[ k_{22} = -\left( \tilde{A}_{12} + 2 \frac{B_{12}}{R_e} + \frac{D_{12}}{R_e} \right) \alpha_m^2 - \left( \tilde{A}_{22} + 2 \frac{B_{22}}{R_e} + \frac{D_{22}}{R_e} \right) \beta_n^2, \]
\[ k_{23} = \left( \frac{A_{12}}{R_e} + \frac{\tilde{A}_{12}}{R_y} + \frac{B_{12}^p}{R_e} + \frac{\tilde{B}_{12}^p}{R_y} \right) \beta_n + \left( B_{22} + \frac{D_{22}}{R_e} \right) \beta_n^2, \]
\[ + \left( B_{12} + B_{12}^p + \frac{B_{66}}{R_y} + \frac{1}{R_y} \left( D_{12} + D_{66} + D_{16} \right) \right) \beta_n \alpha_m, \]
\[ k_{24} = \left( \frac{A_{12}}{R_e} + \frac{\tilde{A}_{12}}{R_y} + \frac{B_{12}^p}{R_e} + \frac{\tilde{B}_{12}^p}{R_y} \right) \beta_n + \left( B_{22}^p + \frac{D_{22}^p}{R_y} \right) \beta_n^2, \]
\[ + \left( B_{12} + B_{12}^p + \frac{B_{66}}{R_y} + \frac{1}{R_y} \left( D_{12} + D_{66} + D_{16} \right) \right) \beta_n \alpha_m, \]
\[ k_{25} = -\left( A_{22}^p + \frac{B_{22}^p}{R_y} \right) \beta_n, \quad k_{26} = -\left( A_{22}^p + \frac{B_{22}^p}{R_y} \right) \beta_n, \]
\[ k_{33} = -\frac{A_{11}}{R_e} - \frac{2 B_{12}}{R_e} - \frac{A_{22}}{R_e} - \frac{2 B_{22}}{R_e} - \frac{2 B_{11}^p}{R_y} \beta_n^2 - \frac{2 B_{12}^p}{R_y} \alpha_m^2, \]
\[ - D_{11} \alpha_m - (2 D_{12} + 2 D_{66} + D_{16} + \tilde{D}_{12} \alpha_m \beta_n^2 - \tilde{D}_{22} \beta_n^4, \]
\[ k_{34} = -\frac{A_{11}}{R_e} - \frac{2 B_{12}}{R_e} - \frac{A_{22}}{R_e} - \frac{2 B_{22}}{R_e} - \frac{2 B_{11}^p}{R_y} \beta_n^2 - \frac{2 B_{12}^p}{R_y} \alpha_m^2, \]
\[ - (2 D_{11}^p + 2 D_{66}^p + D_{16}^p) \alpha_m \beta_n^2 - \tilde{D}_{11} \alpha_m^4 - \tilde{D}_{22} \beta_n^4, \]
\[ k_{35} = B_{21}^p \beta_n^2 + B_{11}^p \alpha_m^2 + \frac{A_{11}^p}{R_e} + \frac{A_{22}^p}{R_y}, \quad k_{36} = -\frac{A_{11}^p}{R_e} + \frac{A_{22}^p}{R_y} + B_{12}^p \alpha_m + B_{22}^p \beta_n, \]
\[ k_{44} = -\frac{A_{11}}{R_e} - \frac{2 B_{12}}{R_e} - \frac{A_{22}}{R_e} - \frac{2 B_{22}}{R_e} - \frac{2 B_{11}^p}{R_y} \beta_n^2 - \frac{2 B_{12}^p}{R_y} \alpha_m^2, \]
\[ - (2 D_{11}^p + 2 D_{66}^p + D_{16}^p + \tilde{D}_{16}^p) \alpha_m \beta_n^2 - \tilde{D}_{11} \alpha_m^4 - \tilde{D}_{22} \beta_n^4, \]
\[ k_{45} = \frac{A_{11}^p}{R_e} + \frac{A_{22}^p}{R_e} + (D_{11}^p + C_{11}) \alpha_m + (C_{21}^p + D_{11}^p) \beta_n, \]
\[ k_{46} = \frac{A_{11}^p}{R_e} + \frac{A_{22}^p}{R_e} + (D_{11}^p + C_{11}^p) \alpha_m + (C_{21} + D_{11}^p) \beta_n, \]
\[ k_{55} = -D_{12}^p \alpha_m^2 - D_{92} \beta_n^2 + Q^{pT}, \quad k_{56} = 0, \quad k_{66} = -D_{22}^p \alpha_m^2 - D_{92} \beta_n^2 + Q^{pT}, \]

Appendix B2

\[ m_{11} = -\left( \tilde{I}_0 + 2 \frac{\tilde{I}_1}{R_e} + \frac{\tilde{I}_2}{R_e} \right), \quad m_{12} = m_{13} = m_{14} = 0, \quad m_{22} = -\left( \tilde{I}_0 + \frac{\tilde{I}_1}{R_e} + \frac{\tilde{I}_2}{R_e} \right), \]
\[ m_{23} = m_{24} = 0, \quad m_{33} = -\tilde{I}_0 - \tilde{I}_2 (\alpha_m^2 + \beta_n^2), \quad m_{34} = -\tilde{I}_0 - \tilde{I}_2 (\alpha_m^2 + \beta_n^2), \quad m_{44} = -\tilde{I}_0 - \tilde{K}_1 (\alpha_m^2 + \beta_n^2), \]
\[ k_{11} = -\left( \tilde{A}_{11} + 2 \frac{\tilde{B}_{11}}{R_e} + \frac{\tilde{D}_{11}}{R_e} \right) \alpha_m^2 - \frac{2 \tilde{B}_{11}}{R_e} \frac{D_{11}}{R_e} - \frac{\tilde{D}_{11}}{R_e} \beta_n^2, \]
\[ k_{12} = -\left( \tilde{A}_{12} + \tilde{A}_{22} \right) \alpha_m^2 + \frac{1}{R_e} \left( \frac{1}{R_e} + \frac{1}{R_y} \right) \left( D_{12} + D_{66} \right) \beta_n \alpha_m, \]
\[ k_{13} = \left( \frac{A_{12}}{R_e} + \frac{\tilde{A}_{12}}{R_y} + \frac{B_{12}}{R_e} + \frac{\tilde{B}_{12}}{R_y} \right) + \left( 2 B_{16}^p + B_{16} + \frac{2 D_{16}}{R_e} \right) \beta_n \alpha_m^2 + \left( B_{26} + \frac{D_{26}}{R_e} \right) \beta_n^3, \]
\[ k_{14} = \left( \frac{B_{26}}{R_e} + \frac{D_{26}}{R_y} \right) \beta_n^3 + \left( \frac{2 B_{16} + B_{16} + \frac{2 D_{16}}{R_e}}{R_e} + \frac{2 D_{16}}{R_e} \alpha_m + \frac{\tilde{A}_{16}}{R_e} + \frac{\tilde{B}_{16}}{R_y} + \frac{B_{16} + \frac{D_{26}}{R_y}}{R_y} \right) \beta_n, \]
\[ k_{15} = 0, \quad k_{16} = 0, \quad k_{22} = -\left( \tilde{A}_{12} + 2 \frac{\tilde{B}_{12}}{R_y} + \frac{\tilde{D}_{12}}{R_y} \right) \alpha_m^2 - \left( \tilde{A}_{22} + 2 \frac{\tilde{B}_{22}}{R_y} + \frac{\tilde{D}_{22}}{R_y} \right) \beta_n^2, \]
\[k_{23} = \left( \frac{A_{16}}{R_c} + \frac{A_{26}}{R_y} + \frac{1}{R_y} \left( \frac{B_{26}}{R_c} + \frac{B_{16}}{R_y} \right) \right) \alpha_m + \left( \frac{\overline{D}_{16}}{R_y} + \frac{B_{16}}{R_y} \right) \alpha_m + \left( 2B_{26} + \frac{B_{16}}{R_y} \right) \alpha_m + \left( \overline{D}_{26} + \frac{2B_{26}}{R_y} \right) \beta_n \alpha_m,
\]

\[k_{24} = \left( \frac{A_{16}}{R_c} + \frac{A_{26}}{R_y} + \frac{1}{R_y} \left( \frac{B_{26}}{R_c} + \frac{B_{16}}{R_y} \right) \right) \alpha_m + \left( \frac{\overline{D}_{16}}{R_y} + \frac{B_{16}}{R_y} \right) \alpha_m + \left( 2B_{26} + \frac{B_{16}}{R_y} + \frac{1}{R_y} \left( \overline{D}_{26} + \frac{D_{26}^e}{R_y} \right) \right) \alpha_m \beta_n,
\]

\[k_{25} = 0, \quad k_{26} = 0,
\]

\[k_{33} = -\frac{\overline{A}_{11}}{R_x^2} - 2\frac{A_{12}}{R_x R_y} - \frac{\overline{A}_{22}}{R_x^2} - 2 \left( \frac{B_{12}}{R_x} + \frac{B_{22}}{R_y} \right) \beta_n^2 - 2 \left( \frac{\overline{B}_{11}}{R_x} + \frac{B_{11}}{R_y} \right) \alpha_m^2 - \frac{\overline{D}_{11} \alpha_m^4}{R_y} - 2D_{11}^e + 2D_{66}^e + D_{66}^e \alpha_m^2 \beta_n^2 - D_{22} \beta_n^4,
\]

\[k_{34} = -\frac{\overline{A}_{11}}{R_x^2} - 2\frac{A_{12}}{R_x R_y} - \frac{\overline{A}_{22}}{R_x^2} - \left( \frac{B_{12}}{R_x} + \frac{B_{22}}{R_y} \right) \beta_n^2 - \left( 2D_{12}^e + 2D_{66}^e + D_{66}^e \right) \alpha_m^2 \beta_n^2 - D_{11}^e \alpha_m^4 - D_{22} \beta_n^4,
\]

\[k_{35} = B_{21} \beta_n^2 + B_{11}^e \alpha_m^2 + \frac{A_{21}^e}{R_y} + \frac{A_{11}^e}{R_x^2} + k_{36} = \frac{A_{12}^e}{R_x} + \frac{A_{21}^e}{R_y} + B_{12} \alpha_m^2 + B_{22} \beta_n^2,
\]

\[k_{44} = -\frac{\overline{A}_{11}}{R_x^2} - 2\frac{A_{12}}{R_x R_y} - \frac{\overline{A}_{22}}{R_x^2} - \left( \frac{A_{12}}{R_x} + \frac{B_{22}^e}{R_y} \right) \beta_n^2 - \left( A_{11}^e + \frac{B_{11}^e}{R_x} + 2 \frac{B_{22}^e}{R_y} \right) \alpha_m^2 - \left( 2E_{12} + 2E_{66}^e + E_{66}^e \right) \alpha_m^2 \beta_n^2 - \left( \overline{E}_{11}^e \alpha_m^4 - \overline{E}_{22} \beta_n^4,
\]

\[k_{45} = \frac{A_{21}^e}{R_x} + \frac{A_{11}^e}{R_y} + (C_{21}^e + C_{11}^e) \alpha_m^2 + (C_{21}^e + D_{11}^e) \beta_n^2,
\]

\[k_{46} = \frac{B_{21}^e}{R_x} + \frac{B_{11}^e}{R_y} + (C_{21}^e + D_{22}^e) \alpha_m^2 + (D_{21}^e + C_{22}^e) \beta_n^2,
\]

\[k_{55} = -D_{s12}^e \beta_n^2 + D_{p13}^e \beta_n^2 + Q^e T^e, \quad k_{56} = 0, \quad k_{66} = -D_{s22}^e \alpha_m^2 - D_{s22}^e \beta_n^2 + Q^e B^e,
\]

where

\[A_{11}^e = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( 1 + \frac{z}{R_x} \right) \varepsilon_3 - \frac{\partial Z_2^e}{\partial z} \phi^d \, dz,
\]

\[A_{12}^e = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( 1 + \frac{z}{R_y} \right) \varepsilon_3 - \frac{\partial Z_2^e}{\partial z} \phi^d \, dz,
\]

\[A_{21}^e = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( 1 + \frac{z}{R_x} \right) \varepsilon_3 - \frac{\partial Z_2^e}{\partial z} \phi^d \, dz,
\]

\[A_{22}^e = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( 1 + \frac{z}{R_y} \right) \varepsilon_3 - \frac{\partial Z_2^e}{\partial z} \phi^d \, dz,
\]

\[B_{11}^e = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( 1 + \frac{z}{R_x} \right) \varepsilon_3 - \frac{\partial Z_2^e}{\partial z} \phi^d \, dz,
\]

\[B_{12}^e = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( 1 + \frac{z}{R_y} \right) \varepsilon_3 - \frac{\partial Z_2^e}{\partial z} \phi^d \, dz,
\]

\[B_{21}^e = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( 1 + \frac{z}{R_x} \right) \varepsilon_3 - \frac{\partial Z_2^e}{\partial z} \phi^d \, dz,
\]

\[B_{22}^e = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( 1 + \frac{z}{R_y} \right) \varepsilon_3 - \frac{\partial Z_2^e}{\partial z} \phi^d \, dz,
\]

\[C_{11}^e = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( 1 + \frac{z}{R_x} \right) \varepsilon_3 - \frac{\partial Z_2^e}{\partial z} \phi^d \, dz,
\]

\[C_{21}^e = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( 1 + \frac{z}{R_y} \right) \varepsilon_3 - \frac{\partial Z_2^e}{\partial z} \phi^d \, dz,
\]

\[D_{11}^e = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( 1 + \frac{z}{R_x} \right) \varepsilon_2 \frac{-1}{1 + \frac{z}{R_y}} Z_1^d \, dz,
\]

\[D_{12}^e = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( 1 + \frac{z}{R_y} \right) \varepsilon_2 \frac{-1}{1 + \frac{z}{R_y}} Z_1^d \, dz,
\]

\[D_{21}^e = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( 1 + \frac{z}{R_x} \right) \varepsilon_1 \frac{-1}{1 + \frac{z}{R_y}} Z_1^d \, dz,
\]

\[D_{22}^e = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( 1 + \frac{z}{R_y} \right) \varepsilon_1 \frac{-1}{1 + \frac{z}{R_y}} Z_1^d \, dz.
\]