LOWER BOUNDS FOR THE SIMPLEXITY OF THE N-CUBE

ALEXEY GLAZYRIN

ABSTRACT. In this paper we prove the new asymptotic lower bound for the minimal number of simplices in simplicial dissections of n-dimensional cubes.

1. Introduction

This work is devoted to some properties of dissections of convex polytopes into simplices with vertices in vertices of the polytope. From now on by a dissection we mean representation of a polytope as a union of non-overlapping (i.e. their interiors do not intersect) simplices. In case each two simplices of a dissection intersect by their common face we’ll call such dissection a triangulation. Obviously for planar case each dissection is a triangulation. However for higher dimensions that’s not true.

One of the most important problems concerning triangulations is a problem of finding a minimal triangulation for a given polytope, i.e. triangulation with a minimal number of simplices. For a polygon the number of triangles in a triangulation is always equal to \( v - 2 \), where \( v \) is a number of vertices of a polygon. The situation is very different even for the three-dimensional case. Three-dimensional cube can be triangulated into six or into five tetrahedra.

In the next section of this work we consider dissections of prismoids and prove some properties for them. By prismoids we mean \( n \)-dimensional polytopes all vertices of which lie in two parallel \((n-1)\)-dimensional hyperplanes. For instance, the set of prismoids contains cubes, prisms, 0/1-polytopes (i.e. polytopes all Cartesian coordinates of which are 0 or 1, see [10]). In the third section we’ll show how these properties can be used for finding lower bounds for the simplexity of the \( n \)-dimensional cube. In the last section we’ll prove a new asymptotic lower bound for the simplexity of the \( n \)-cubes.

We use the following notations: \( \text{dis}(n) \) is a minimal number of simplices in a dissection of the \( n \)-dimensional cube, \( \text{triang}(n) \) is a minimal number of simplices in a triangulation of the \( n \)-dimensional cube, \( \rho(n) \) is a maximal determinant of a 0/1-matrices. Obviously \( \text{triang}(n) \geq \text{dis}(n) \). In our work all the lower bounds will be given for dissections. Hence they are all true for triangulations too.

There is an obvious lower bound for \( \text{dis}(n) \):

\[
\text{dis}(n) \geq \frac{n!}{\rho(n)}.
\]

The maximal volume of a simplex with vertices in the vertices of the 0/1-cube is not greater than \( \frac{\rho(n)}{n!} \), therefore we immediately achieve this bound. An upper bound for \( \rho(n) \)
can be easily obtained by some matrix transformations and Hadamard’s inequality (more precisely for instance here — [10], the generalization of this inequality will be proved in the last section of the paper):

**Lemma 1.** \( \rho(n) \leq 2 \left( \frac{\sqrt{n} + 1}{2} \right)^{n+1} \)

Hence the following bound is true

**Theorem 1.**

\[
dis(n) \geq \frac{n!}{2 \left( \frac{\sqrt{n} + 1}{2} \right)^{n+1}} = E(n)
\]

Better bounds can be achieved by using other volumes instead of the Euclidean volume. The following bound was proved by W.D Smith in [8] by means of hyperbolic volume and was the best asymptotic bound up to the moment.

\[
dis(n) \geq H(n) \geq \frac{1}{2} 6^{\frac{n}{2}} (n+1)^{-\frac{n+1}{2}} n!
\]

\[
\lim_{n \to \infty} \left( \frac{H(n)}{E(n)} \right)^{\frac{1}{n}} = A > 1.2615
\]

In the table below lower bounds for minimal numbers of simplices in triangulations and dissections are shown up to dimension eight. Sign “=” is used when a number is known exactly.

| n | triang(n) | dis(n) |
|---|---|---|
| 3 | 5 | 5 |
| 4 | =16 (Cottle [3], Sallee [7], ’82) | =16 (H.) |
| 5 | =67 (Hughes & Anderson [5], ’96) | =61 (H.) |
| 6 | =308 (H. & A.) | 270 (H. & A.) |
| 7 | =1493 (H. & A.) | 1175 (H. & A.) |
| 8 | 5522 (Hughes [4], ’96) | 5522 (H.) |

One can also consider triangulations and dissections with additional vertices. Some bounds for simplicial covers and triangulations with additional vertices were obtained in the paper of Bliss and Su [2], ’03.

Smith’s method [8] is convenient for dissections with additional vertices. We deal only with triangulations and dissections with vertices in vertices of a cube. Hence our result is not a total improvement of bounds achieved by Smith.

Upper bounds for triang(n) can be obtained by constructing explicit examples. The best bound for the moment is \( O(0.816^n n!) \) (Orden and Santos [6], ’03)

Quite extensive surveys of the papers connected to the minimal simplicial dissections and triangulations of n-cubes can be found in the works [2], [8] mentioned above.
2. Triangulation of prismoids

Let all the vertices of an $n$-dimensional polytope $P \in \mathbb{R}^n$ lie in two parallel $(n-1)$-dimensional hyperplanes, i.e. $P$ is an $n$-dimensional prismoid. Without loss of generality we can consider hyperplanes $x_1 = 0$ and $x_1 = 1$ (no following statements depend on the distance between hyperplanes). Assume also that we have a dissection $\Delta$ of a polytope $P$ into $n$-dimensional simplices. All the vertices of simplices are vertices of the polytope.

Define $S_i$ as a set of all simplices of $\Delta$ with $i$ vertices in $x_1 = 0$ and $(n+1-i)$ vertices in $x_1 = 1$.

Denote $\Delta_i = \{T \in \Delta | \text{exactly } i \text{ vertices of } T \text{ lie in } x_1 = 0\}$. So $\Delta_i = S_i \cap \Delta$. Denote by $q_i$ the number of simplices in $\Delta_i$, by $T^j_i - j$-th simplex of the set $\Delta_i$, and by $V(T^j_i) -$ its $n$-dimensional volume (from now on we talk about the Euclidean volume).

Let for this prismoid $P$ and its simplicial dissection $\Delta$ $V(i)$ be a total volume of simplices in $\Delta_i$, i.e. $V(i) = \sum_{j=1}^{q_i} V(T^j_i)$.

**Theorem 2.** $V(i)$ is a function of $P$ and does not depend on $\Delta$, $1 \leq i \leq n$.

**Remark.** $P$ can be a prismoid with respect to two different pairs of parallel hyperplanes. In this theorem we mean that the pair is fixed.

Consider $T^j_i$ and its intersection $M_t$ with a hyperplane $x_1 = t$, where $t \in [0, 1]$. Let us prove the following lemma:

**Lemma 2.** $(n-1)$-dimensional volume $S(M_t) = c^j_i(1-t)^{i-1}t^{n-i}$, where $c^j_i$ is some constant not depending on $t$.

**Proof.** Let $A$ be a convex hull of $i$ vertices of the simplex $T^j_i$ from the hyperplane $x_1 = 0$ and let $B$ be a convex hull of $(n+1-i)$ vertices of $T^j_i$ from the hyperplane $x_1 = 1$. We’ll show now that $M_t = \{(1-t)A + tB\}$ (here by $\{\}$ we mean Minkowski sum of these two sets). Note that any point $Z$ of the intersection we consider divides some segment $XY$ with ratio $t : (1-t)$, where $X \in A$ and $Y \in B$. Thus $Z = (1-t)X + tY$ and it is obvious that all the points $Z$ that can be expressed this way lie in the intersection $M_t$.

Let $A_j, 1 \leq j \leq i$, be a $j$-th vertex of the simplex $A$ and let $B_k, 1 \leq k \leq n+1-i$, be a $k$-th vertex of the simplex $B$. Notice that all the vectors $\overrightarrow{A_1A_j}$ ($1 < j \leq i$) and $\overrightarrow{B_1B_k}$ ($1 < k \leq n+1-i$) are linearly independent (over $\mathbb{R}$) in total (because in other case vectors $\overrightarrow{A_1A_j}, \overrightarrow{A_1B_1}, \overrightarrow{B_1B_k}$ are linearly dependent and consequently $\overrightarrow{A_1A_j}, \overrightarrow{A_1B_k}$ are linearly dependent which contradicts to the fact that $A_j, B_k$ are vertices of the $n$-dimensional simplex). Let $O$ be a point of intersection of $A_1B_1$ and the hyperplane $x_1 = t$. Now we scale $M_t$ about $O$ with a coefficient $\frac{1}{1-t}$ along the vectors $\overrightarrow{A_1A_j}$ and with a coefficient $\frac{1}{t}$ along the vectors $\overrightarrow{B_1B_k}$. After this transformation $M_t$ will change to a figure congruent to $\{A+B\}$. Because of the linear independence of $\overrightarrow{A_1A_j}, \overrightarrow{B_1B_k}$ we achieve that $S(M_t) = t^{n-i}(1-t)^{i-1}S(A+B)$,
where $S$ is an $(n - 1)$-dimensional volume. We take $S(A + B)$ as $c^j_i$ and the lemma is proved. \hfill \Box

We'll use the following lemma:

**Lemma 3.** For each $m \in \mathbb{N}$ polynomials $P_i = t^i(1 - t)^{m-i}$, $0 \leq i \leq m$ (Bernstein basis polynomials \[1\]), are linearly independent over $\mathbb{R}$.

Now consider any simplicial dissection $\Delta$ of the polytope $P$. Define $c_i(\Delta) = \sum c^j_n(\Delta)$.

**Lemma 4.** All $c_i$ do not depend on $\Delta$ and are determined only by the polytope $P$.

**Proof.** Suppose we have two dissections $\Delta_1$ and $\Delta_2$. Let us prove that $c_i(\Delta_1) = c_i(\Delta_2)$ for all $0 \leq i \leq n$. Define $S(t)$ as an $(n - 1)$-dimensional volume of intersection of a hyperplane $x_1 = t$ with $P$. Then we achieve that $c_1(\Delta_1)P_{n-1} + \ldots + c_n(\Delta_1)P_0 \equiv S(t)$ (here $P_i$ are Bernstein polynomials for $m = n - 1$). Analogously $c_1(\Delta_2)P_{n-1} + \ldots + c_n(\Delta_2)P_0 \equiv S(t)$.

Hence $(c_1(\Delta_1) - c_1(\Delta_2))P_{n-1} + \ldots + (c_n(\Delta_1) - c_n(\Delta_2))P_0 \equiv 0$. By lemma \[8\] $P_0, \ldots, P_{n-1}$ are linearly independent. Thus $c_i(\Delta_1) = c_i(\Delta_2)$. \hfill \Box

Express $V(T^j_i)$ in terms of $c^j_i$. Using a volume of an $(n - 1)$-dimensional section of the simplex by a hyperplane $x_1 = t$ we obtain \[9\]

$$V(T^j_i) = \int_0^1 c^j_i t^{n-i}(1-t)^{i-1}dt = c^j_i B(n - i + 1, i) =$$

$$= c^j_i \frac{\Gamma(n - i + 1)\Gamma(i)}{\Gamma(n + 1)} = c^j_i \frac{(n - i)!(i - 1)!}{n!} = \frac{c^j_i}{n \binom{n-1}{i-1}}.

Thus we achieve that

$$\sum_{j=1}^{q_i} V(T^j_i) = \frac{\sum_{j=1}^{q_i} c^j_i}{n \binom{n-1}{i-1}} = \frac{c_i}{n \binom{n-1}{i-1}}.

Denote the right part of this equation by $V(i)$ and the theorem is proved.

**Corollary 1.** If all conditions of theorem \[2\] hold and $S(t) \equiv \text{const}$ then $V(i) = \frac{1}{n} V(P)$. Here $S(t)$ is an $(n - 1)$-dimensional volume of a section of $P$ by a hyperplane $x_1 = t$.

**Proof.** Suppose $S(t) \equiv S_0$. Then $c_1P_{n-1} + \ldots + c_nP_0 \equiv S_0$. Notice that if $\beta_i = S_0 \binom{n-1}{i-1}$ then $\beta_1P_{n-1} + \ldots + \beta_nP_0 = S_0 \binom{n-1}{0} t^0(1-t)^{n-1} + \ldots + S_0 \binom{n-1}{n-1} t^{n-1}(1-t)^0 \equiv S_0$. Similarly to the proof of theorem \[2\] using the idea of the linear independence of $P_i$, $i = 0, 1, \ldots, n - 1$ we obtain that $c_i = \beta_i$.

$$V(i) = \frac{c_i}{n \binom{n-1}{i-1}} = \frac{\beta_i}{n \binom{n-1}{i-1}} = \frac{S_0}{n}.

\text{Here } B \text{ and } \Gamma \text{ are standard Euler functions.}$$
Because of the equality \( V(P) = \int_0^1 S(t)dt = S_0 \) the corollary is proved. \( \square \)

Notice that this corollary works for all prisms and particularly for cubes. We'll use that for the following section.

3. Lower bounds for the simplicity of cubes

3.1. The general construction for the lower bound of the simplicity of \( n \)-cube.

Consider any \( n \)-dimensional \( 0/1 \)-simplex \( T \) and suppose that its vertices \( A_1, \ldots, A_{n+1} \) have coordinates \( A_1(a_{1,1}, \ldots, a_{1,n}), \ldots, A_{n+1}(a_{n+1,1}, \ldots, a_{n+1,n}) \). Notice that the Euclidean volume of \( T \) is equal to \( \frac{1}{n!} \) multiplied by the absolute value of the determinant of the following matrix:

\[
M(T) = \begin{pmatrix}
1 & a_{1,1} & \cdots & a_{1,n} \\
1 & a_{2,1} & \cdots & a_{2,n} \\
& \ddots & \ddots & \ddots \\
1 & a_{n+1,1} & \cdots & a_{n+1,n}
\end{pmatrix}
\]

Denote \( j \)-th column of this matrix by \( b_{j-1} \) (for instance, \( b_0 \) is the first column consisting of \( (n + 1) \)'s) and \( \|b_j\|^2 \) by \( i_j \) (here we mean Euclidean norm, i.e. \( i_j \) is just a number of 1’s in a column). Then we define functions \( V_{k,m}(T) = V(T) \) if \( i_k = m \) and \( V_{k,m}(T) = 0 \) if \( i_k \neq m \).

The next proposition obviously follows from corollary.

**Proposition 1.** For each dissection \( \Delta \) of the \( n \)-dimensional cube and for all \( 1 \leq k, m \leq n \)

\[
\sum_{T \in \Delta} V_{k,m}(T) = \frac{1}{n}.
\]

Now take any \( n \times n \) matrix of coefficients \( \alpha_{k,m} \), s.t. \( \sum \alpha_{k,m} = n \). Then by the proposition we have

\[
\sum_{T \in \Delta} \sum_{1 \leq k, m \leq n} \alpha_{k,m} V_{k,m}(T) = \sum_{1 \leq k, m \leq n} \alpha_{k,m} \sum_{T \in \Delta} V_{k,m}(T) = \sum_{1 \leq k, m \leq n} \frac{\alpha_{k,m}}{n} = 1
\]

Denote \( V^\alpha(T) = \sum_{1 \leq k, m \leq n} \alpha_{k,m} V_{k,m}(T) \). Then \( \sum_{T \in \Delta} V^\alpha(T) = 1 \) and \( dis(n) \geq \frac{1}{\max_{\alpha} V^\alpha} \). So in order to get the best bound we must find

\[
G = \min_{\alpha} \max_{\alpha \in IT} V^\alpha(T),
\]

which is a problem of linear programming with respect to \( \alpha \).

We can simplify our problem. We need to consider only \( \alpha \) symmetric to coordinate permutations and reflections swapping hyperplanes \( x_j = 0 \) and \( x_j = 1 \) (that can be easily proved by symmetrization of \( \alpha \) with respect to cube symmetries). So \( \alpha_{k_1,m} = \alpha_{k_2,m} \) and
\[ \alpha_{k,m} = \alpha_{k,n+1-m} \]. From now on we use notations \( \alpha_m = \alpha_{k,m} \) with the conditions \( \sum \alpha_m = 1 \) and \( \alpha_m = \alpha_{n+1-m} \) for all \( m \).

By quite exhaustive case analysis for \( n = 5, 6 \) we were able to find all linear constraints but the lower bounds \( \text{trianglin}(5) = 60, \text{trianglin}(6) = 240 \) obtained by our linear program are smaller than known bounds for the number of simplices in dissections. Nevertheless this method allows us to find new asymptotic lower bounds.

3.2. New asymptotic lower bound. Let us prove a generalization of lemma 1:

**Lemma 5.** \( (\det M)^2 \leq (n+1)^{1-n}(\prod_{j=1}^{n} i_j)(\prod_{j=1}^{n} (n+1-i_j)) \).

**Proof.** For each column make a transformation \( \phi_j : b_j \mapsto b'_j = b_j - \frac{i_j}{n+1} b_0 \). After all transformations \( \det M \) won’t change. So

\[
\|b'_j\|^2 = i_j \frac{(n+1-i_j)^2}{(n+1)^2} + (n+1-i_j) \frac{i_j^2}{(n+1)^2} = \frac{i_j(n+1-i_j)}{n+1}.
\]

By Hadamard’s inequality

\[ |\det M| \leq \sqrt{n+1} \prod_{j=1}^{n} \|b'_j\| = (n+1)^{1-\frac{1}{2}} \sqrt{\left(\prod_{j=1}^{n} i_j\right)\left(\prod_{j=1}^{n} (n+1-i_j)\right)}. \]

Hence the lemma is proved. \( \square \)

Let us set \( \alpha_i = c - \frac{1}{2} \ln(i(n+1-i)) \), where \( c \) is such that \( \sum \alpha_i = 1 \), i.e. \( c = \frac{1}{n}(1+\ln(n!)) \).

\[
V^\alpha(T) = \frac{1}{n!} |\det M| \sum_{j=1}^{n} \alpha_{ij} = \frac{1}{n!} |\det M| \sum_{j=1}^{n} \left(c - \frac{1}{2} \ln(i_j(n+1-i_j))\right) =
\]

\[
= \frac{1}{n!} |\det M|(1+\ln(n!)) - \frac{1}{2} \ln((\prod_{j=1}^{n} i_j)(\prod_{j=1}^{n} (n+1-i_j))).
\]

Then we use the inequality on \( \det M \) from lemma 3 (we are interested only in positive weighted volumes, for them inequality is correct):

\[
V^\alpha(T) \leq \frac{1}{n!} \sqrt{(n+1)^{1-n}(\prod_{j=1}^{n} i_j)(\prod_{j=1}^{n} (n+1-i_j))(1+\ln(n!)) - \frac{1}{2} \ln((\prod_{j=1}^{n} i_j)(\prod_{j=1}^{n} (n+1-i_j)))}. \]

Denote \( \frac{1}{2} \ln((\prod_{j=1}^{n} i_j)(\prod_{j=1}^{n} (n+1-i_j))) \) by \( t \). Then this inequality can be rewritten in the following form:

\[
V^\alpha(T) \leq \frac{1}{n!} (n+1)^{\frac{1}{2} - n} h(t),
\]
where \( h(t) = e^t(1 + \ln(n!) - t) \). Let us find a maximal value of this function. \( h'(t) = e^t(\ln(n!) - t) \), so \( f(t) \) reaches its maximum for \( t = \ln(n!) \) and \( \max h(t) = n! \) Thus

\[
V^\alpha(T) \leq \frac{1}{n!} \left( n + 1 \right)^{\frac{n-1}{2}} n! = (n + 1)^{\frac{n-1}{2}}
\]

and we have proved the following theorem

**Theorem 3.** For any natural \( n \)

\[
dis(n) \geq (n + 1)^{\frac{n-1}{2}} =: F(n)
\]

This bound gives an obvious asymptotic improvement with respect to the Euclidean bound

\[
\lim_{n \to \infty} \left( \frac{F(n)}{E(n)} \right)^{\frac{1}{n}} = \lim_{n \to \infty} \left( \frac{n^{\frac{n}{2}}}{n^\frac{n}{2} \left( \frac{2}{e} \right)^n} \right)^{\frac{1}{n}} = \frac{e}{2} = 1.359140914
\]

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Moscow State University, Leninskie Gory, 119992 Moscow GSP-2, Russia
E-mail address: Alexey.Glazyrin@gmail.com