STATIONARY SCATTERING THEORY FOR REPULSIVE HAMILTONIANS

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Abstract. In the present paper we discuss stationary scattering theory for repulsive Hamiltonians. We show the existence and completeness of stationary wave operators and unitarity of the scattering matrix. Moreover we completely characterize asymptotic behaviors of generalized eigenfunctions with minimal growth in terms of the scattering matrix. In our argument the radiation condition bounds for limiting resolvents play major roles. In fact, it is used to construct the stationary wave operators.

Keywords: repulsive Hamiltonians, stationary scattering theory, scattering matrix.

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1. INTRODUCTION

The purpose of this paper is to establish stationary scattering theory for repulsive Hamiltonians given by

\[ H_\alpha = \frac{1}{2} p^2 - \frac{1}{2} |x|^\alpha + q(x) \text{ on } L^2(\mathbb{R}^d), \]

where \( 0 < \alpha < 2 \), \( p = -i \partial \) and \( q \) is a perturbation. We assume that \( q \) is a real-valued function belonging to \( L^2_{\text{loc}}(\mathbb{R}^d) \) and decays at infinity at a rate dependent on the parameter \( \alpha \). We will give a more precise condition on \( q \) in Section 2.

It is well-known that the spectrum of \( H_\alpha \) with short-range perturbation \( q \) is purely absolutely continuous. Moreover by [4] it was proved that the wave operators

\[ W^{\pm} = \text{s-lim}_{t \to \pm \infty} e^{itH_\alpha} e^{-itH_{\alpha,0}}, \]

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exist and are complete. Here we denote the free repulsive Hamiltonian as $H_{\alpha,0}$, that is

$$H_{\alpha,0} := \frac{1}{2}p^2 - \frac{1}{2}|x|^\alpha.$$  

However stationary scattering theory is not yet established for the repulsive Hamiltonians even for short-range perturbations, as far as the author knows. In the paper, we deal with several topics on stationary scattering theory. The first one is existence and completeness of the stationary wave operators $F^\pm$. To construct the stationary wave operators we use the radiation condition bounds stated as Corollary 2.7 below and employ the schemes of [6, 12, 16]. The second one is unitarity of the scattering matrix $S(\cdot)$. The last one is asymptotic behaviors of minimal generalized eigenfunctions (see Theorem 2.14). Here we say minimal in the sense that the growth order at infinity is minimal. We obtain a characterization of them by outgoing/incoming spherical waves. We note these topics are not dealt with in [4]. In this sense our results are new, and this is a novelty of the paper.

In this paper we discuss only the case of $0 < \alpha < 2$, although the case of $\alpha = 2$ is included in [4]. When $\alpha = 2$ the classical particles scatter with exponential order, although when $0 < \alpha < 2$ they scatter with polynomial order. Then our escape function, which plays an important role in the study of repulsive Hamiltonians (cf. (2.2)), is defined by using a logarithmic function, see [13, 14]. Thus when $\alpha = 2$, we need a more stronger result than Corollary 2.7 to construct the stationary wave operators.

In Section 2 we state our setting and results. We introduce our basic setting, for example, definition of escape function and an assumption on $q$, and spectral theory, which is a refinement of the results of our previous papers, and state our main results on stationary scattering theory. We are going to give the proofs in later sections. In Sections 3 we discuss properties of stationary wave operators, and in Section 4 we investigate a characterization of minimal generalized eigenfunctions.

2. Setting and results

2.1. Basic setting. We choose a smooth cut-off function $\chi$ which satisfies

$$\chi(s) = \begin{cases} 1 & \text{for } s \leq 1, \\ 0 & \text{for } s \geq 2, \end{cases} \quad \frac{d}{ds}\chi = \chi' \leq 0. \quad (2.1)$$

Throughout the paper, we fix the function $\chi$. By using the function $\chi$ we introduce the function $r \in C^\infty(\mathbb{R}^d)$, which is a modification of $|x|$ on a neighborhood of the origin, by

$$r = r(x) = \chi(|x|) + (1 - \chi(|x|))|x|.$$  

Now our escape function $f \in C^\infty(\mathbb{R}^d)$ is given as follows:

$$f(x) = \frac{r^{1-\alpha/2} - 1}{1 - \alpha/2} + 1. \quad (2.2)$$

Such a choice of $f$ is based on the scattering order of classical particles subject to the repulsive electric field (see [13, 14]). We note that $f \geq 1$ on $\mathbb{R}^d$.

Throughout the paper we assume the following condition.

**Condition 2.1.** The perturbation $q$ is a real-valued function and belongs to $C^1(\mathbb{R}^d)$. Moreover there exist $\rho, C_k > 0$ for $k = 0, 1$ such that

$$|\partial^k q| \leq C_k f^{-1-k-\rho}.$$
Under Condition 2.1 it follows by the Faris–Lavine theorem (cf. [22]) that $H_\alpha$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$. We denote its self-adjoint extension by the same letter for simplicity.

Next we introduce the Agmon–Hörmander spaces associated with the function $f$. We let $F(S)$ be the sharp characteristic function of a general subset $S \subseteq \mathbb{R}^d$, and set

$$F_n = F(\{ x \in \mathbb{R}^d \mid 2^n \leq f(x) < 2^{n+1} \}) \quad \text{for } n \in \mathbb{N}_0.$$  

Then define the Agmon–Hörmander spaces $\mathcal{B}$, $\mathcal{B}^*$ and $\mathcal{B}^*_0$ as

$$\mathcal{B} = \left\{ \psi \in L^2_{\text{loc}}(\mathbb{R}^d) \mid \| \psi \|_{\mathcal{B}} := \sum_{n \in \mathbb{N}_0} 2^{n/2} \| F_n \psi \|_2 < \infty \right\},$$  

$$\mathcal{B}^* = \left\{ \psi \in L^2_{\text{loc}}(\mathbb{R}^d) \mid \| \psi \|_{\mathcal{B}^*} := \sup_{n \in \mathbb{N}_0} 2^{-n/2} \| F_n \psi \|_2 < \infty \right\},$$  

$$\mathcal{B}^*_0 = \left\{ \psi \in \mathcal{B}^* \mid \lim_{n \to \infty} 2^{-n/2} \| F_n \psi \|_2 = 0 \right\}.$$  

Note that $\mathcal{B}$ is a Banach space with respect to the norm $\| \cdot \|_{\mathcal{B}}$, and $\mathcal{B}^*$ and $\mathcal{B}^*_0$ are Banach spaces with respect to the same norm $\| \cdot \|_{\mathcal{B}^*}$. Note also that, if we introduce the $f$-weighted $L^2$-spaces of order $s \in \mathbb{R}$ as

$$L^2_s = f^{-s}L^2,$$

for any $s > 1/2$ the following inclusion relations hold:

$$L^2_s \subset B \subset L^2_{1/2} \subset L^2 \subset L^2_{-1/2} \subset B^*_0 \subset B^* \subset L^2_{-s}.$$  

We introduce differential operators $\partial^r$ and $\partial^f$ as

$$\partial^r = (\partial r) \partial, \quad \partial^f = (\partial f) \partial = r^{-\alpha/2}(\partial r) \partial,$$

respectively, and then we define a ‘conjugate operator’ $A$ as

$$A = \Re p^f = \frac{1}{2} \left( (p^f)^* + p^f \right), \quad p^f = -i \partial^f.$$  

We note that (cf. [13]) $A$ is self-adjoint operator with domain $\{ \psi \in L^2 \mid A\psi \in L^2 \}$, and has expressions

$$A = p^f - \frac{1}{2}(\Delta f) = (p^f)^* + \frac{1}{2}(\Delta f).$$

We also note $A$ is different from the standard conjugate operator, cf. [4, 20]. In fact, the commutator $[H, iA]$ has only weak positivity decaying at infinity, see [13]. We denote the resolvent of $H_\alpha$ for $z \in \mathbb{C} \setminus \mathbb{R}$ by $R(z)$, i.e.,

$$R(z) = (H_\alpha - z)^{-1}.$$  

Let us introduce the function $\theta \in C^\infty(\mathbb{R}^d)$ by

$$\theta(\lambda, x) = r^{1+\alpha/2}/(1 + \alpha/2) + \lambda f.$$  

Note that the function $\theta$ is an approximate solution to the eikonal equation

$$\frac{1}{2} |\partial^r \theta(\lambda, x)|^2 - \frac{1}{2} |x|^\alpha + q - \lambda = 0,$$

in the sense that for $2/3 < \alpha < 2$ the quantity of the left-hand side tends to 0 faster than $f^{-1}$ as $f \to \infty$. More precisely, the function $\theta$ satisfies

$$\frac{1}{2} |\partial^r \theta(\lambda, x)|^2 - \frac{1}{2} |x|^\alpha + q - \lambda = O(f^{-1-\min\{\rho, (3\alpha/2-1)/(1-\alpha/2)\}}).$$  

(2.5)
Remark 2.2. (1) When \( \alpha = 1, 2 \), the functions \( \theta_1, \theta_2 \in C^\infty(\mathbb{R}^d) \) satisfying
\[
\theta_1(\lambda, x) = \frac{2}{3}(r + 2\lambda)^{3/2} \quad \text{for} \quad r > 1 - 2\lambda, \\
\theta_2(\lambda, x) = \frac{1}{2}(r^2 + 2\lambda)^{1/2} + \lambda \log \left\{ \frac{1}{2} + \frac{1}{2}(r^2 + 2\lambda)^{1/2} \right\} \quad \text{for} \quad r^2 > 1 - 2\lambda,
\]
respectively, solve the eikonal equation of the free case for sufficiently large \( r \). In particular, their leading terms coincide with \( \theta(\lambda, x) \).

(2) We constructed \( \theta \) by the following simple approximation.
\[
\theta(\lambda, x) \sim \int (r^\alpha + 2\lambda)^{1/2}(dr) \sim \int (r^{\alpha/2} + \lambda r^{-\alpha/2} + O(r^{-3\alpha/2}))(dr) dx.
\]

Thus by adding some lower order terms to \( \theta(\lambda, x) \), we can improve the order of the right-hand side of (2.5).

In the following we assume that \( 2/3 < \alpha < 2 \). However our results hold for all \( \alpha \in (0, 2) \) by retaking \( \theta \) appropriately, as stated in the above remark.

2.2. Results of our previous papers. Before stating our main results, let us recall several results of \([13, 14]\). Because we use the radiation condition bounds for limiting resolvents of the forms of \([14]\) and Sommerfeld's uniqueness theorem to construct the stationary wave operators and the scattering matrix, respectively. However, as for the radiation condition bounds we need slightly stronger estimates than those of \([14]\) even for the short–range case. Thus we need to refine the results except for Rellich’s theorem and limiting absorption principle bounds.

In this section we state improved results. We prove only Theorem 2.5 stated below, since if we get Theorem 2.5 other result can be proved by quite similar way to \([3, 14]\). The proof of Theorem 2.5 is given in Appendix A.

The first result is the absence of \( \mathcal{B}_0^* \)-eigenfunctions, which is called Rellich’s theorem. Since the condition on \( q \) of this paper is stronger than that of \([13]\), we have the following theorem.

Theorem 2.3. Let \( \lambda \in \mathbb{R} \). Suppose a function \( \phi \in \mathcal{B}_0^* \) satisfies
\[
(H_\alpha - \lambda)\phi = 0
\]
in the distributional sense. Then \( \phi = 0 \) on \( \mathbb{R}^d \).

We set
\[
\ell_{jk} = |\partial f|^2 \delta_{jk} - (\partial_j f)(\partial_k f), \tag{2.6}
\]
where \( \delta_{jk} \) is Kronecker’s delta. For any compact interval \( I \subseteq \mathbb{R} \) we introduce
\[
I_\pm = \{ z = \lambda \pm i\Gamma \mid \lambda \in I, \quad \Gamma \in (0, 1) \},
\]
respectively. We also use the notation \( \langle T \rangle_\psi = \langle \psi, T\psi \rangle \) for a general linear operator \( T \). The following limiting absorption principle bounds (LAP bounds) for the resolvent \( R(z) \) also hold in the setting of this paper.

Theorem 2.4. There exists \( C > 0 \) such that for any \( \psi \in \mathcal{B} \) and \( z \in I_\pm \)
\[
\|R(z)\psi\|_{\mathcal{B}^*} + \|p^f R(z)\psi\|_{\mathcal{B}^*} + \langle p_j f^{-1} \ell_{jk} p_k \rangle^{1/2}_{R(z)\psi} + \|r^{-\alpha} p^2 R(z)\psi\|_{\mathcal{B}^*} \leq C \|\psi\|_{\mathcal{B}}.
\]

Using the function \( \chi \) of (2.1), we define smooth cut-off functions \( \chi_m, \tilde{\chi}_m, \chi_{m,n} \in C^\infty(\mathbb{R}^d) \) for \( m, n \in \mathbb{N}_0 \) as
\[
\chi_m = \chi(f/2^m), \quad \tilde{\chi}_m = 1 - \chi_m, \quad \chi_{m,n} = \chi_m \chi_n. \tag{2.7}
\]
We choose and fix large $m \in \mathbb{N}$ so that on $\text{supp} \, \tilde{x}_m$
\[ 2 \text{Re} \, z - 2q_0 + r^\alpha > 1, \quad r = |x|, \]
where $z \in I_\pm$ and
\[ q_0 = q + \frac{1}{8} r^\alpha (\Delta f)^2 + \frac{\alpha}{4} r^{\alpha/2 - 1} (\Delta f) + \frac{1}{4} r^\alpha (\partial^f \Delta f) - \frac{\alpha}{4} r^{-2}. \]
We set an asymptotic complex phase $\alpha$ by
\[ a = a_2 = \tilde{x}_m \left[ r^{-\alpha/2} \sqrt{2(z - q_0) + r^\alpha \pm \frac{\alpha}{4} r^{-\alpha/2 - 1} \pm \frac{\alpha}{4} \frac{z - q_0}{z - q_0 + r^\alpha} r^{-\alpha/2 - 1} \right] \]
for $z \in I_\pm$. Here we choose the branch of square root as $\text{Re} \sqrt{\lambda} > 0$ for $s \in \mathbb{C} \setminus (-\infty, 0]$. Let
\[ \beta_c = \min\{ \rho + 1/(1 - \alpha/2), 1 + \alpha/(1 - \alpha/2) \}. \]
Then we have refined radiation condition bounds for complex spectral parameters.

**Theorem 2.5.** For all $\beta \in (0, \beta_c)$, there exists $C > 0$ such that for any $\psi \in f^{-\beta} \mathcal{B}$ and $z \in I_\pm$
\[ \| f^{\beta}(A \mp a)R(z)\psi \|_{\mathcal{B}^*} + \langle p_J f^{2\beta - 1} \ell_{jk} p_k \rangle R(z)^{1/2} \psi \leq C \| f^{\beta} \psi \|_{\mathcal{B}}, \]
respectively.

Let us state several applications of Theorem 2.4 and Theorem 2.5. The first one is the limiting absorption principle.

**Corollary 2.6.** For any $s > 1/2$ and $\omega \in (0, \beta_c) \cap (0, \min\{ s - 1/2, 1 \})$ there exists $C > 0$ such that for any $z, z' \in I_\pm$ or $z, z' \in I_-$
\[ \| R(z) - R(z') \|_{\mathcal{L}(\mathcal{H}_+, \mathcal{H}_{-})} \leq C|z - z'|^{\omega}, \]
\[ \| r^{-\alpha/2} p\{ R(z) - R(z') \} \|_{\mathcal{L}(\mathcal{H}_+, \mathcal{H}_{-})} \leq C|z - z'|^{\omega}. \]
In particular, for any $\lambda \in \mathbb{R}$, there exist uniform limits
\[ \lim_{(0,1) \ni \Gamma \searrow 0} R(\lambda \pm i\Gamma), \lim_{(0,1) \ni \Gamma \searrow 0} r^{-\alpha/2} pR(\lambda \pm i\Gamma), \]
in the norm topology of $\mathcal{L}(\mathcal{H}_+, \mathcal{H}_{-})$. We denote these limits by $R(\lambda \pm i0), r^{-\alpha/2} pR(\lambda \pm i0)$, respectively. These limiting resolvents belong to $\mathcal{L}(\mathcal{B}, \mathcal{B}^*)$.

The second one is the radiation condition bounds for real spectral parameters, which follows from Theorem 2.5 and Corollary 2.6. We set
\[ a_\pm := \lim_{z \to \lambda \pm i0} a_z, \quad \lambda \in I. \]

**Corollary 2.7.** Let $\lambda \in I$. Then for all $\beta \in (0, \beta_c)$, there exists $C > 0$ such that for any $\psi \in f^{-\beta} \mathcal{B}$
\[ \| f^{\beta}(A \mp a_\pm)R(\lambda \pm i0)\psi \|_{\mathcal{B}^*} + \langle p_J f^{2\beta - 1} \ell_{jk} p_k \rangle R(\lambda \pm i0)^{1/2} \psi \leq C \| f^{\beta} \psi \|_{\mathcal{B}}, \]
respectively.

The last one is Sommerfeld's uniqueness theorem.

**Corollary 2.8.** Let $\lambda \in \mathbb{R}, \phi \in f^{-\beta} \mathcal{B}^*$ and $\psi \in f^{-\beta} \mathcal{B}$ with $\beta \in (0, \beta_c)$. Then $\phi = R(\lambda \pm i0)\psi$ hold if and only if both of the following conditions hold:
\[ (i) \quad (H_\alpha - \lambda) \phi = \psi \text{ in the distributional sense}, \]
\[ (ii) \quad (A \mp a_\pm) \phi \in f^{-\beta} \mathcal{B}^*_0, \]
respectively.
2.3. Main results. We introduce the operators $\mathcal{F}^\pm(\lambda, f)$ which map from $C^\infty_0(\mathbb{R}^d)$ to $L^2(\mathbb{S}^{d-1})$ by
\[
(\mathcal{F}^\pm(\lambda, f)\psi)(\omega) = \frac{1}{\sqrt{2\pi}} \exp\left\{ \pm \frac{\pi i}{4} \left( \frac{d-\alpha/2-3}{1+\alpha/2} \right) \right\} r^{(d+\alpha/2-1)/2} e^{\mp i\theta(\lambda, \omega)} R(\lambda \pm i0)\psi(\pm \omega),
\]
respectively, where $\psi \in C^\infty_0(\mathbb{R}^d)$ and $\omega \in \mathbb{S}^{d-1}$.

**Theorem 2.9.** $\mathcal{F}^\pm(\lambda, f)$ are bounded operators from $C^\infty_0(\mathbb{R}^d) (\subseteq \mathcal{B})$ to $L^2(\mathbb{S}^{d-1})$, and for any $\psi \in C^\infty_0(\mathbb{R}^d)$ there exist limits
\[
\lim_{f \to \infty} \mathcal{F}^\pm(\lambda, f)\psi = \mathcal{F}^\pm(\lambda)\psi \quad \text{in} \quad L^2(\mathbb{S}^{d-1}).
\]
Moreover it holds that
\[
\frac{1}{2\pi} \langle R(\lambda + i0)\psi - R(\lambda - i0)\psi, \psi \rangle = \|\mathcal{F}^\pm(\lambda)\psi\|^2_{L^2(\mathbb{S}^{d-1})}.
\]

We note, by (2.11), the operators $\mathcal{F}^\pm(\lambda)$ are extended to bounded operators from $\mathcal{B}$ into $L^2(\mathbb{S}^{d-1})$, and satisfy $\|\mathcal{F}^+(\lambda)\psi\| = \|\mathcal{F}^-(\lambda)\psi\|$ for any $\psi \in \mathcal{B}$. We also note that $\mathcal{F}^\pm(\lambda)$ are continuous in $\lambda \in \mathbb{R}$. This follows from the continuity of $R(\lambda \pm i0)$ in $\lambda \in \mathbb{R}$, (2.11) and (3.14) stated below.

We introduce the spaces
\[
\mathcal{H} = L^2(\mathbb{R}^d), \quad \tilde{\mathcal{H}} = L^2(\mathbb{R}; d\lambda; L^2(\mathbb{S}^{d-1})),
\]
and define the operators $\mathcal{F}^\pm : \mathcal{B} \to C(\mathbb{R}; L^2(\mathbb{S}^{d-1}))$ as
\[
(\mathcal{F}^\pm \psi)(\lambda) = \mathcal{F}^\pm(\lambda)\psi, \quad \psi \in \mathcal{B},
\]
respectively.

**Proposition 2.10.** The operators $\mathcal{F}^\pm$ defined as mappings $\mathcal{B} \to C(\mathbb{R}; L^2(\mathbb{S}^{d-1}))$ by (2.12) extend uniquely to isometries $\mathcal{H} \to \tilde{\mathcal{H}}$. These operators satisfy $\mathcal{F}^\pm H_\alpha \subseteq M_\lambda \mathcal{F}^\pm$.

We call the operator $\mathcal{F}^\pm : \mathcal{H} \to \tilde{\mathcal{H}}$ stationary wave operator. Existence of the stationary wave operators follows from Proposition 2.10. Since Ran $\mathcal{F}^\pm(\lambda)$ are dense in $L^2(\mathbb{S}^{d-1})$ (see (2.16) below), by the density argument we can see that $\mathcal{F}^\pm$ are surjection, see e.g. [1]. Therefore we obtain the completeness of the stationary wave operators.

**Theorem 2.11.** The operators $\mathcal{F}^\pm$ are unitary, and satisfy
\[
\mathcal{F}^\pm H_\alpha = M_\lambda \mathcal{F}^\pm.
\]

Let us introduce the functions $\phi^\pm_\lambda[v]$ for $v \in L^2(\mathbb{S}^{d-1})$ by
\[
\phi^\pm_\lambda[v](f, \omega) = \frac{1}{\sqrt{2\pi}} \exp\left\{ \pm \frac{\pi i}{4} \left( \frac{d+\alpha/2-1}{1+\alpha/2} \right) \right\} r^{-(d+\alpha/2-1)/2} e^{\pm i\theta(\lambda, \omega)} v(\pm \omega),
\]
respectively. We may call these functions outgoing/incoming approximate generalized eigenfunctions. In fact for $v \in C^\infty(\mathbb{S}^{d-1})$ we can see that
\[
\psi^\pm_\lambda[v] := (H_\alpha - \lambda)\phi^\pm_\lambda[v] \in \mathcal{B},
\]
cf. (3.11). The adjoints of $\mathcal{F}^\pm(\lambda)$:
\[
\mathcal{F}^\pm(\lambda)^* \in \mathcal{L}(L^2(\mathbb{S}^{d-1}), \mathcal{B}^*),
\]
which are called the stationary wave matrices, are characterized by $\phi^\pm_\lambda$ and $\psi^\pm_\lambda$ as follows.
Proposition 2.12. Let \( v \in C^\infty(S^{d-1}) \), and let \( \phi_\lambda^\pm[v] \) and \( \psi_\lambda^\pm[v] \) be given by (2.13) and (2.14), respectively. Then

\[
\mathcal{F}^\pm(\lambda)^* v = \phi_\lambda^\pm[v] - R(\lambda \mp i0)\psi_\lambda^\pm[v] \quad (\in B^*) ,
\]

respectively.

\( \mathcal{F}^\pm(\lambda)^* \) are also called eigenoperators. In fact, by Proposition 2.12 and the density argument we can see that

\[
(H_\alpha - \lambda)\mathcal{F}^\pm(\lambda)^* v = 0 \quad \text{for any } v \in L^2(S^{d-1}).
\]

By Sommerfeld’s uniqueness theorem stated as Corollary 2.8, we have

\[
\phi_\lambda^\pm[v] - R(\lambda \pm i0)\psi_\lambda^\pm[v] = 0 \quad \text{for } v \in C^\infty(S^{d-1}).
\]

We can deduce from this equality that

\[
v = \pm 2\pi i\mathcal{F}^\pm(\lambda)\psi_\lambda^\pm[v] \quad \text{for } v \in C^\infty(S^{d-1}),
\]

and then we have

\[
C^\infty(S^{d-1}) \subseteq \text{Ran.}\mathcal{F}^\pm(\lambda) \subseteq L^2(S^{d-1}).
\]

Therefore we can define the scattering matrix \( S(\lambda) \) as satisfying for \( \psi \in B^* \)

\[
\mathcal{F}^+(\lambda)\psi = S(\lambda)\mathcal{F}^-(\lambda)\psi.
\]

Then by Theorem 2.9 we can see that the scattering matrix is extended to an unitary operator.

Proposition 2.13. \( S(\lambda) \) defined by (2.17) is extended to a unitary operator on \( L^2(S^{d-1}) \) and is strongly continuous in \( \lambda \in \mathbb{R} \).

Finally, we obtain a characterization of the \( B^* \)-eigenfunctions in terms of \( \phi_\lambda^\pm \) similar to [16]. Let us introduce the set of minimal generalized eigenfunctions.

\[ E_\lambda := \{ \phi \in B^* \mid (H_\alpha - \lambda)\phi = 0 \text{ in the distributional sense.} \}. \]

Theorem 2.14. For any fixed \( \lambda \) in \( \mathbb{R} \) the following assertions hold.

(i) For any one of \( \xi_\pm \in L^2(S^{d-1}) \) or \( \phi \in E_\lambda \) the two other quantities in \( \{\xi_+, \xi_-, \phi\} \) uniquely exist such that

\[
\phi - \phi_\lambda^+(\xi_+) - \phi_\lambda^-(\xi_-) \in B_0^*. \quad (2.18)
\]

(ii) For the quantities \( \{\xi_+, \xi_-, \phi\} \) satisfying (2.18), the following relations hold.

\[
\phi = \mathcal{F}^+(\lambda)^*\xi_+, \quad \xi_- = S(\lambda)\xi_-, \quad (2.19)
\]

\[
\xi_\pm = \pm \frac{1}{2} c_\pm \lim_{R \to \infty} \frac{1}{R} \int_R^{2R} r^{(d+\alpha/2-1)/2} e^{\pm i\theta}(A \pm a_0)\phi \, df, \quad (2.20)
\]

where \( c_\pm = \sqrt{2\pi} \exp\{ \pm \frac{\pi}{4} \frac{(d+\alpha/2-1)}{1+\alpha/2} \} \) and \( a_0 = r^{-\alpha/2} \sqrt{2\lambda - 2\theta_0 + r^\alpha} \). In particular the wave matrices \( \mathcal{F}^\pm(\lambda)^* \) give one-to-one correspondences between the spaces \( L^2(S^{d-1}) \) and \( E_\lambda \).

(iii) The operators \( \mathcal{F}^\pm(\lambda) : B \to L^2(S^{d-1}) \) are surjections.

There are many literature on scattering theory for the Laplacian with decaying potentials. We refer e.g. [5, 6, 7, 9, 10, 12, 16, 18, 19, 23]. However there seems to be no literature on stationary scattering problem for repulsive Hamiltonians even for short-range perturbation, although time-dependent scattering problem for that is well studied cf. e.g. [4, 11, 21]. In this sense, our results are new. Moreover
since we use Agmon–Hörmander spaces, which are used only for the Laplacian with decaying potentials, so far, cf. [2, 17], our results have sharp form.

To prove our main results we use the schemes of [6, 12, 16] as mentioned above. Since considered Hamiltonians in this paper are different from theirs, we can not apply their schemes directly. Thus we improve that by using escape function (2.2) and an approximate solution (2.4) to the eikonal equation.

We have already obtained similar results to [15, 13, 14] in [3] for the Stark Hamiltonians. Thus by considering the results of [16] and this paper, we can expect that stationary scattering theory can be established for the Stark Hamiltonians.

3. Wave operator

In this section we discuss on the stationary wave operators $\mathcal{F}^\pm$. In Section 3.1 we prove Theorems 2.9 by employing Isozaki’s approach, cf. [12, 6]. Proposition 2.10 will be proved in Section 3.2.

3.1. Stationary state. To prove Theorem 2.9 let us introduce the following lemmas.

**Lemma 3.1.** Let $\psi \in C_0^\infty(\mathbb{R}^d)$ and $\phi = R(\lambda \pm i0)\psi$. Then

$$
\langle R(\lambda + i0)\psi - R(\lambda - i0)\psi, \psi \rangle = i \lim_{f \to \infty} \int_{f(x) = f} |\phi|^2 dS_{f \to f}. 
$$

**Lemma 3.2.** Let $\psi \in C_0^\infty(\mathbb{R}^d)$. Then there exists a weak limit

$$
\text{w-lim}_{f \to \infty} \mathcal{F}^\pm(\lambda, f)\psi \equiv \mathcal{F}^\pm(\lambda)\psi \quad \text{in } L^2(S^{d-1}).
$$

**Lemma 3.3.** Let $\psi \in C_0^\infty(\mathbb{R}^d)$. Then there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ satisfying $f_n \to \infty$ as $n \to \infty$ such that $\mathcal{F}^\pm(\lambda, f_n)\psi$ tends to $\mathcal{F}^\pm(\lambda)\psi$ in $L^2(S^{d-1})$ as $n \to \infty$.

By using the function (2.4), we define the differential operators $\mathcal{D}_f^\pm$, $\mathcal{D}_j^\pm$, $\mathcal{D}_\omega^\pm$ and $\mathcal{D}_\alpha^\pm$ by

$$
\mathcal{D}_j^\pm = \partial_j + \frac{d\alpha/2 - 1}{2} \pm i \frac{\partial \theta}{\partial x_j}(\lambda, x), \quad j = 1, \ldots, d,
$$

$$
\mathcal{D}^\pm = (\mathcal{D}_1^\pm, \ldots, \mathcal{D}_d^\pm),
$$

$$
\mathcal{D}_f^\pm = r^{-\alpha/2} \frac{r}{\partial r} \mathcal{D}_j^\pm,
$$

$$
\mathcal{D}_\omega^\pm = \partial_x - \tilde{\omega} \partial_{|x|} + i \left( \frac{\partial \theta}{\partial r}(\lambda, x) - \frac{\partial \theta}{\partial |x|}(\lambda, x) \right),
$$

respectively. On the region $\{x \in \mathbb{R}^d \mid r(x) \geq 2\}$, $\mathcal{D}_f^\pm$ is expressed as

$$
\mathcal{D}_f^\pm = r^{-\alpha} \partial_f + \frac{d\alpha/2 - 1}{2} r^{-\alpha/2 - 1} \pm i r^{-\alpha/2} \frac{\partial \theta}{\partial |x|}(\lambda, x),
$$

and these operators satisfy the identity

$$
\sum_{j=1}^d (\mathcal{D}_j^\pm)^* \mathcal{D}_j^\pm = (r^{\alpha/2} \mathcal{D}_f^\pm)^* r^{\alpha/2} \mathcal{D}_f^\pm + (\mathcal{D}_\omega^\pm)^* \mathcal{D}_\omega^\pm = (\mathcal{D}_f^\pm)^* r^\alpha \mathcal{D}_f^\pm + (\mathcal{D}_\omega^\pm)^* \mathcal{D}_\omega^\pm, \quad (3.3)
$$

where

$$
(\mathcal{D}_j^\pm)^* = -\partial_j + \frac{d\alpha/2 - 1}{2} \pm i \frac{\partial \theta}{\partial x_j}(\lambda, x),
$$

$$
(\mathcal{D}_f^\pm)^* = -\partial_f r^{-\alpha} - \frac{d\alpha/2 - 1}{2} r^{-\alpha/2 - 1} \pm i r^{-\alpha/2} \frac{\partial \theta}{\partial |x|}(\lambda, x),
$$

$$
(\mathcal{D}_\omega^\pm)^* = -\partial_x + \partial_{|x|} + (d-1) r^{-1} \tilde{\omega} \pm i \left( \frac{\partial \theta}{\partial x}(\lambda, x) - \frac{\partial \theta}{\partial |x|}(\lambda, x) \right).$$

Lemma 3.4. Let \( D_2 \) noting \( 2 < \alpha < 2 \) it follows from Corollary 2.7 that for any \( \psi \in \mathcal{B} \) and \( \beta \in [0, \beta_c) \)
\[
f^\beta D_2^\dagger R(\lambda \pm i0)\psi \in \mathcal{B}^*. \tag{3.4}
\]
We prove Lemmas 3.1–3.3 and Theorem 2.9 only for the upper sign. Thus in the following we consider only for \( D_2^+, D^+, \bar{D}_2^+, \mathcal{F}(\lambda, f) \) and \( \mathcal{F}(\lambda) \), and then, for notational simplicity, we omit the superscript from these operators such as \( \mathcal{D}_j \).

First we see the following property of \( \phi = R(\lambda \pm i0)\psi \) for \( \psi \in C_0^\infty(\mathbb{R}^d) \).

**Lemma 3.4.** Let \( \psi \in C_0^\infty(\mathbb{R}^d) \) and \( \phi = R(\lambda \pm i0)\psi \). Then
\[
\lim_{f \to \infty} \int_{f(x) = f} \left( \mathcal{D}_f \phi \right) \tilde{\phi} - \phi \left( \mathcal{D}_f \phi \right) dS_{f=j} = 0.
\]

**Proof.** Let \( \psi \) and \( \phi \) be as in the assertion, and \( \tilde{f} \geq 2 \). Noting the expressions of \( \mathcal{D}_f \) and \( \mathcal{D}_f^* \) we can compute as
\[
\frac{d}{df} \int_{f(x) = f} \left( \mathcal{D}_f \phi \right) \tilde{\phi} dS_{f=j} = \int_{f(x) = f} r^\alpha |D_f \phi|^2 dS_{f=j} - \left( d + \alpha/2 - 1 \right) \int_{f(x) = f} r^\alpha/2 \left( \mathcal{D}_f \phi \right) \tilde{\phi} dS_{f=j}
\]
\[
+ \int_{f(x) = f} \left( -D_f r^\alpha D_f \phi \right) \tilde{\phi} dS_{f=j}.
\]
We can see by Corollary 2.7 that the first and the second terms belong to \( L^1((2, \infty)) \). As for the third term, let us further compute the factor \( -D_f r^\alpha D_f \phi \). By a straightforward calculation we have
\[
\sum_{j=1}^d \mathcal{D}_j^* \mathcal{D}_j \phi - 2i \sum_{j=1}^d \frac{\partial \phi}{\partial x_j} \mathcal{D}_j \phi
\]
\[
= 2\psi - \left\{ \left| \frac{\partial \phi}{\partial x} (\lambda, x) \right|^2 - r^\alpha + 2q - 2\lambda + \frac{(d+\alpha/2-1)(d-\alpha/2-3)}{4\lambda} \right\} \phi
\]
\[
+ i \left\{ (\Delta \theta)(\lambda, x) - (d + \alpha/2 - 1)r^{-1} \frac{\partial}{\partial x} (\lambda, x) \right\} \phi. \tag{3.5}
\]
Therefore by using (3.3), (3.5) and the relation \( \mathcal{D} = \mathcal{D}_\omega + \bar{\mathcal{D}}^\dagger r^\alpha/2 \mathcal{D}_f \), we can obtain
\[
-D_f r^\alpha D_f \phi = \mathcal{D}_\omega \mathcal{D}_f \phi - 2i r^\alpha D_f \phi \left( -2i \mathcal{D}_f \phi - 2\psi
\right.
\]
\[
+ \left\{ \left| \frac{\partial \phi}{\partial x} (\lambda, x) \right|^2 - r^\alpha + 2q - 2\lambda + \frac{(d+\alpha/2-1)(d-\alpha/2-3)}{4\lambda} \right\} \phi
\]
\[
- i \left\{ (\Delta \theta)(\lambda, x) - (d + \alpha/2 - 1)r^{-1} \frac{\partial}{\partial x} (\lambda, x) \right\} \phi.
\]
Hence by Theorem 2.4, Corollary 2.7, (2.4), (2.5) and (3.4) we have
\[
\frac{d}{df} \int_{f(x) = f} \left( \mathcal{D}_f \phi \right) \tilde{\phi} dS_{f=j} = \int_{f(x) = f} \left( \mathcal{D}_\omega \mathcal{D}_f \phi \right) \tilde{\phi} dS_{f=j} - 2ir^\alpha \int_{f(x) = f} \left( \mathcal{D}_f \phi \right) \tilde{\phi} dS_{f=j} + G_1(f), \tag{3.6}
\]
where \( G_1 \) is a certain function which satisfying \( \int_{f_2}^\infty |G_1(s)| ds < \infty \), and \( \tilde{r} \geq 2 \) is a solution of
\[
\tilde{f} = (\tilde{r}^{1-\alpha/2} - 1)/(1 - \alpha/2) + 1.
\]
Since we can see by integration by parts

\[ \int_{f(x)=\tilde{f}} (\mathcal{D}_\omega \mathcal{D}_\omega \phi) \tilde{\phi} \, dS_{f=\tilde{f}} = \int_{f(x)=\tilde{f}} |\mathcal{D}_\omega \phi|^2 \, dS_{f=\tilde{f}}, \]

we obtain from (3.6) that

\[
\frac{d}{df} \int_{f(x)=\tilde{f}} (\mathcal{D}_f \phi) \tilde{\phi} - \phi(\mathcal{D}_f \phi) \, dS_{f=\tilde{f}} = -2i\tilde{\alpha} \int_{f(x)=r_n} (\mathcal{D}_f u) \tilde{u} - u(\mathcal{D}_f u) \, dS_{f(x)=r_n} + G(\tilde{f}),
\]

where \( G = G_1 - \overline{G_1} \). Now, if we let

\[ u(\tilde{f}) = e^{2i(\tilde{r}^{1+\alpha}/2)/(1+\alpha/2)} v(\tilde{f}), \quad v(\tilde{f}) = \int_{f(x)=\tilde{f}} (\mathcal{D}_f \phi) \tilde{\phi} - \phi(\mathcal{D}_f \phi) \, dS_{f=\tilde{f}}, \]

we have

\[ \frac{d}{df} u(\tilde{f}) = e^{2i(\tilde{r}^{1+\alpha}/2)/(1+\alpha/2)} G(\tilde{f}). \]

The solution of this differential equation is given by

\[ u(\tilde{f}) = u(2) + \int_2^{\tilde{f}} e^{2i(\tilde{r}^{1+\alpha}/2)/(1+\alpha/2)} G(\tilde{f}) \, d\tilde{f}. \]

Since \( G(\tilde{f}) \in L^1((2, \infty)) \), there exists a limit \( \lim_{\tilde{f} \to \infty} u(\tilde{f}) \). On the other hand, by Theorem 2.4, Corollary 2.7 and the Cauchy-Schwarz inequality, we have

\[ \int_2^\infty |v(\tilde{f})| \, d\tilde{f} < \infty. \]

This implies that \( \liminf_{\tilde{f} \to \infty} |v(\tilde{f})| = \liminf_{\tilde{f} \to \infty} |u(\tilde{f})| = 0 \). Therefore we have \( \lim_{\tilde{f} \to \infty} u(\tilde{f}) = 0 \), or \( \lim_{\tilde{f} \to \infty} v(\tilde{f}) = 0 \). Hence we are done. \( \square \)

Proof of Lemma 3.1. Let \( \tilde{f} \geq 2 \), and then take \( \tilde{r} \geq 2 \) which solves the equation

\[ \tilde{r} = (\tilde{r}^{1-\alpha/2} - 1)/(1 - \alpha/2) + 1. \]

Since \( (H_\alpha - \lambda) \phi = \psi \), in the distributional sense, we can compute by Green’s formula as

\[
\int_{|x|\leq \tilde{r}} \phi \overline{\psi} - \overline{\phi} \psi \, dx = \frac{1}{2} \int_{f(x)=\tilde{f}} (r^{-\alpha} \partial_r \phi) \overline{\phi} - \phi(r^{-\alpha} \partial_r \overline{\phi}) \, dS_{f=\tilde{f}} = \frac{1}{2} \int_{f(x)=\tilde{f}} (\mathcal{D}_f \phi) \overline{\phi} - \phi(\mathcal{D}_f \overline{\phi}) \, dS_{f=\tilde{f}} + i \int_{f(x)=\tilde{f}} r^{-\alpha/2} (\partial_{\overline{\phi}} (\mathcal{D}_f \phi)) \overline{\phi} \, dS_{f=\tilde{f}}. \tag{3.7}
\]

Clearly, the left-hand side of (3.7) tends to the left-hand side of (3.1) as \( \tilde{f} \to \infty \). The first term on the right-hand side converges to 0 by Lemma 3.4. In addition, noting (2.4), it holds that

\[ \lim_{\tilde{f} \to \infty} i \int_{f(x)=\tilde{f}} r^{-\alpha/2} (\partial_{\overline{\phi}} (\mathcal{D}_f \phi)) \overline{\phi} \, dS_{f=\tilde{f}} = i \lim_{\tilde{f} \to \infty} \int_{f(x)=\tilde{f}} |\phi|^2 \, dS_{f=\tilde{f}}. \]

Hence by taking the limit \( \tilde{f} \to \infty \) in the both side of (3.7) we have the assertion. \( \square \)
Let us recall the operator $\mathcal{F}^\pm(\lambda, f)$ which is defined by

$$
(\mathcal{F}^\pm(\lambda, f)\psi)(\omega) = \frac{1}{\sqrt{2\pi}} e^{\mp \frac{\pi i}{4}(\frac{d-\alpha/2-3}{1+\alpha/2})} r^{(d+\alpha/2-1)/2} e^{\pm i\theta(\lambda, \omega)} (R(\lambda \pm i0)\psi)(\pm \cdot \omega),
$$

where $\psi \in C^\infty_0(\mathbb{R}^d)$ and $\omega \in S^{d-1}$. Then we have by Lemma 3.1

$$
\frac{1}{2m} (R(\lambda + i0)\psi - R(\lambda - i0)\psi, \psi) = \lim_{f \to \infty} \|\mathcal{F}^\pm(\lambda, f)\psi\|^2_{L^2(S^{d-1})}.
$$

(3.8)

Note that $\|\mathcal{F}^\pm(\lambda, f)\psi\|_{L^2(S^{d-1})}$ is uniformly bounded in $f$.

**Proof of Lemma 3.2.** Let $\psi \in C^\infty_0(\mathbb{R}^d)$. It suffices to show the existence of the limit

$$
\lim_{f \to \infty} \langle v, \mathcal{F}(\lambda, f)\psi \rangle_{L^2(S^{d-1})} \quad \text{for} \quad v \in C^\infty(S^{d-1}).
$$

Let $v(\omega) \in C^\infty(S^{d-1})$, and define the function $u$ by

$$
u = r^{-(d+\alpha/2-1)/2} e^{i\theta(\lambda, x)} v(\omega), \quad \omega \in S^{d-1}.
$$

(3.9)

In addition we let

$$
g(\lambda, x) = (H_\alpha - \lambda) u.
$$

(3.10)

Then we have if $f > 2$

$$
g(\lambda, x) = e^{i\theta(\lambda, x)} \left\{ \left[ \frac{\lambda^2}{2} r^{-\alpha} + q + \frac{1}{8}(d + \alpha/2 - 1)(d - \alpha/2 + 1)r^{-2}\right] v(\omega) - i\frac{\alpha}{2} r^{-\alpha/2-1}(\Delta_{S^{d-1}} v)(\omega) \right\} r^{-(d+\alpha/2-1)/2},
$$

(3.11)

where $\Delta_{S^{d-1}}$ is the Laplace–Beltrami operator on $S^{d-1}$. In particular, we have $g(\lambda, \cdot) \in B$. We have also by straightforward calculations that for some $\delta > 0$

$$
u \in B^*, \quad \mathcal{D} u \in L^2_{2+\delta}, \quad \mathcal{D} f u = 0 \quad \text{if} \quad f > 2.
$$

(3.12)

Now we let $\phi = R(\lambda + i0)\psi$, and then by using Green’s formula we have

$$
\frac{1}{2} \int_{|x| < \tilde{f}} \{(\Delta u)\tilde{\phi} - u(\Delta \tilde{\phi})\} \, dx = \frac{1}{2} \int_{f(x) = \tilde{f}} \{(\mathcal{D}_f u)\tilde{\phi} - u(\mathcal{D}_f \tilde{\phi})\} \, dS_{f = \tilde{f}} + i \int_{f(x) = \tilde{f}} r^{-\alpha/2} \frac{\partial \theta}{\partial |x|}(\lambda, x) u\tilde{\phi} \, dS_{f = \tilde{f}}.
$$

(3.13)

The left-hand side of (3.13) converges to $\langle u, \psi \rangle - \langle g, R(\lambda + i0)\psi \rangle$ as $\tilde{f} \to \infty$. By noting (3.12) and

$$
\int_{f(x) = \tilde{f}} u(\mathcal{D}_f \mathcal{D} \omega \phi) \, dS_{f = \tilde{f}} = \int_{f(x) = \tilde{f}} (\mathcal{D}_f \mathcal{D} \omega \phi) \, dS_{f = \tilde{f}} \in L^1((2, \infty)),
$$

we can see by a similar argument of the proof of Lemma 3.4 that

$$
\int_{f(x) = \tilde{f}} \{(\mathcal{D}_f u)\tilde{\phi} - u(\mathcal{D}_f \tilde{\phi})\} \, dS_{f = \tilde{f}} \to 0 \quad \text{as} \quad \tilde{f} \to \infty.
$$

Moreover, we can also see that

$$
\lim_{f \to \infty} i \int_{f(x) = \tilde{f}} r^{-\alpha/2} \frac{\partial \theta}{\partial |x|}(\lambda, x) u\tilde{\phi} \, dS_{f = \tilde{f}} = \lim_{f \to \infty} i \int_{f(x) = \tilde{f}} u\tilde{\phi} \, dS_{f = \tilde{f}} = \lim_{f \to \infty} c^\alpha \langle v, \mathcal{F}(\lambda, f)\psi \rangle_{L^2(S^{d-1})},
$$

where $c^\alpha$ is the norm of $v$.
where \( c_\alpha = \sqrt{2\pi} \exp \left\{ \frac{\pi}{4} \left( \frac{d+\alpha/2-1}{1+\alpha/2} \right) \right\} \). Therefore by taking the limit \( \tilde{f} \to \infty \) in the both side of (3.13), we have
\[
\langle u, \psi \rangle - \langle g, R(\lambda + i0)\psi \rangle = \lim_{f \to \infty} c_\alpha \langle v, \mathcal{F}(\lambda, f)\psi \rangle_{L^2(\mathbb{R}^d)}.
\] (3.14)

Hence we have the assertion. 

By Corollary 2.7 we have for any \( \beta \in (0, \beta_c) \) and for some \( \delta > 0 \)
\[
\int_2^\infty \int_{|x|=\tilde{f}} |Df\phi|^2 dS_{f=\tilde{f}} \leq \beta \int_2^\infty \int_{|x|=\tilde{f}} |Df\phi|^2 dS_{f=\tilde{f}} < \infty.
\]

Thus there exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \) tending to infinity such that
\[
f_n^{2\beta-\delta} \int_{|x|=f_n} |Df\phi|^2 dS_{f=f_n} \to 0 \quad \text{as} \quad n \to \infty.
\] (3.15)

In particular, if we take \( \beta = 1/(1-\alpha/2) < \beta_c \) and \( \delta = \alpha/(1-\alpha/2) \), we have
\[
\int_{|x|=f_n} |Df\phi|^2 dS_{f=f_n} = o(f_n^{-2}).
\] (3.16)

Let us fix a sequence \( \{f_n\}_{n \in \mathbb{N}} \) satisfying (3.15). In the proof of Lemma 3.3, we use the following estimate.

**Proposition 3.5.** Let \( \psi \in C_0^\infty(\mathbb{R}^d) \) and \( v \in C^1(\mathbb{R}^d) \), and assume
\[
\sup_{f \geq 2} \|v(f, \cdot)\|_{L^2(\mathbb{R}^d)} < \infty.
\]

Then there exists \( \varepsilon > 0 \) such that for \( 2 \leq f_m < f_n \)
\[
\left| \langle \mathcal{F}(\lambda, f_m)\psi - \mathcal{F}(\lambda, f_n)\psi, v \rangle_{L^2(\mathbb{R}^d)} \right|
\leq C(f_m) \left( \left( \sup_{f \geq 2} \|v(f, \cdot)\|_{L^2(\mathbb{R}^d)} \right)^2 + \left( \int_{f_m}^\infty f^{-1-\varepsilon} \|Df\|_{L^2(\mathbb{R}^d)} \, df \right)^{1/2} \right),
\]

where \( C(f_m) \) is a constant which is independent of \( v \) and tends to 0 as \( m \to \infty \).

**Proof.** We show the assertion only for \( v \in C^\infty(\mathbb{R}^d) \). Let \( v \in C^\infty(\mathbb{R}^d) \). Then we set the function \( u \) and \( g \) similarly to (3.9) and (3.10), respectively. Then we have by Green’s formula and (3.12)
\[
\begin{align*}
\int_{r_m < |x| < r_n} (\bar{u}\psi - g\phi) \, dx + \frac{1}{2} \int_{f(x)=f_n} u(Df\phi) \, dS_{f=f_n} \\
- \frac{1}{2} \int_{f(x)=f_m} u(Df\phi) \, dS_{f=f_m}
= i \int_{f(x)=f_n} r^{-\alpha/2} \frac{\partial}{\partial |x|} (\lambda, x) \bar{u}\phi \, dS_{f=f_n} - i \int_{f(x)=f_m} r^{-\alpha/2} \frac{\partial}{\partial |x|} (\lambda, x) \bar{u}\phi \, dS_{f=f_m}.
\end{align*}
\] (3.17)

By noting the definition \( \mathcal{F}(\lambda, f) \), we can estimate the right-hand side of (3.17) as
\[
\begin{align*}
&\left| i \int_{f(x)=f_n} r^{-\alpha/2} \frac{\partial}{\partial |x|} (\lambda, x) \bar{u}\phi \, dS_{f=f_n} - i \int_{f(x)=f_m} r^{-\alpha/2} \frac{\partial}{\partial |x|} (\lambda, x) \bar{u}\phi \, dS_{f=f_m} \right| \\
&\geq \sqrt{2\pi} \langle \langle \mathcal{F}(\lambda, f_n)\psi - \mathcal{F}(\lambda, f_m)\psi, v \rangle_{L^2(\mathbb{R}^d)} \rangle \\
&- C_r_m^\alpha \left( \sup_{n \geq m} \|\mathcal{F}(\lambda, f_n)\psi\|_{L^2(\mathbb{R}^d)} \right) \left( \sup_{f \geq 2} \|v(f, \cdot)\|_{L^2(\mathbb{R}^d)} \right).
\end{align*}
\] (3.18)
As for the second and the third terms on the left-hand side of (3.17), we can bound by the Cauchy–Schwarz inequality and (3.16) as follows.

\[
\frac{1}{2} \left| \int_{f(x)=f_n} u(\mathcal{D}_f \psi) \, dS_{f=f_n} - \int_{f(x)=f_m} u(\mathcal{D}_f \psi) \, dS_{f=f_m} \right| \\
\leq C f_m^{-1} \left( \sup_{f \geq 2} \|v(f, \cdot)\|_{L^2(\mathbb{S}^{d-1})} \right).
\]

(3.19)

By the Cauchy-Schwarz inequality we have for some \( \varepsilon_1 \in (0, 1) \)

\[
\left| \int_{r_m < |x| < r_n} u\bar{\psi} \, dx \right| \\
\leq \left( \int_{|x| > r_m} r^{-1-2\varepsilon_1} |v|^2 r^{-(d-1)} \, dx \right)^{1/2} \left( \int_{|x| > r_m} r^{1-\alpha/2+2\varepsilon_1} |\psi|^2 \, dx \right)^{1/2} \] (3.20)

\[
\leq C_{\varepsilon_1} r_m^{-\varepsilon_1/2} \left( \sup_{f \geq 2} \|v(f, \cdot)\|_{L^2(\mathbb{S}^{d-1})} \right) \left( \int_{|x| > r_m} r^{1-\alpha/2+2\varepsilon_1} |\psi|^2 \, dx \right)^{1/2} .
\]

Note that the last factor of the right-hand side is finite, since \( \psi \in C_0^\infty(\mathbb{R}^d) \). By using (3.11), we can estimate

\[
\left| \int_{r_m < |x| < r_n} g\bar{\phi} \, dx \right| \\
\leq C \int_{|x| > r_m} f^{1-\min\{\rho, (3\alpha/2-1)/(1-\alpha/2)\}} r^{-(d+\alpha/2-1)/2} |v| |\phi| \, dx \\
+ C \int_{r_m < |x| < r_n} r^{-2} (\Delta_{\mathbb{S}^{d-1}} v) r^{-(d+\alpha/2-1)/2} e^{i\theta(\lambda, x)} \bar{\phi} \, dx .
\]

(3.21)

Take \( \varepsilon_2 \in (0, \min\{\rho, (3\alpha/2-1)/(1-\alpha/2)\}) \). By using the Cauchy–Schwarz inequality

we can estimate the first term on the right-hand side of (3.21) as

\[
C \int_{|x| > r_m} f^{1-\min\{\rho, (3\alpha/2-1)/(1-\alpha/2)\}} r^{-(d+\alpha/2-1)/2} |v| |\phi| \, dx \\
\leq C \left( \sup_{f \geq 2} \|v(f, \cdot)\|_{L^2(\mathbb{S}^{d-1})} \right) \left( \sup_{f \geq 2} \|\mathcal{F}(\lambda, f)\psi\|_{L^2(\mathbb{S}^{d-1})} \right) \int_{f_m}^\infty f^{1-\varepsilon_2} \, df \]

(3.22)

\[
\leq C_{\varepsilon_2} f_m^{-\varepsilon_2/2} \left( \sup_{f \geq 2} \|v(f, \cdot)\|_{L^2(\mathbb{S}^{d-1})} \right) \left( \sup_{f \geq 2} \|\mathcal{F}(\lambda, f)\psi\|_{L^2(\mathbb{S}^{d-1})} \right) .
\]

To evaluate the second term on the right-hand side of (3.21), we note that \( r^{-2} \Delta_{\mathbb{S}^{d-1}} \)

has the following expression, on the region \( \{ x \mid r(x) \geq 2 \} \),

\[
r^{-2} \Delta_{\mathbb{S}^{d-1}} = \sum_{j=1}^d (\partial_j - \frac{r}{r \partial_r})^2 = \mathcal{L}_w.
\]

Thus we can compute by integration by parts

\[
\int_{r_m < |x| < r_n} r^{-2} (\Delta_{\mathbb{S}^{d-1}} v) r^{-(d+\alpha/2-1)/2} e^{i\theta(\lambda, x)} \bar{\phi} \, dx \\
= - \int_{r_m < |x| < r_n} (\mathcal{L}_w v) r^{-(d+\alpha/2-1)/2} e^{i\theta(\lambda, x)} (\mathcal{D}_\omega \phi) \, dx.
\]

We also note that since

\[
|\mathcal{D}_\omega \phi|^2 = |\partial \phi - \frac{r}{r^{\alpha/2}} \partial' \phi|^2 = r^{\alpha} \left( |p \phi|^2 - |p' \phi|^2 \right) \quad \text{on} \quad \{ x \mid r(x) \geq 2 \},
\]
it holds that
\[
\int_{f(x)>2} |\partial_x \phi|^2 \, dx \leq \langle p_j f^\alpha \ell_{jk \phi} \rangle \phi.
\]
Therefore by the Cauchy–Schwarz inequality we have
\[
\left| \int_{r_m<|x|<r_n} r^{-2} (\Delta_{\mathbb{S}^{d-1}} v) r^{-(d+\alpha/2-1)/2} e^{i\lambda(x)} \phi \, dx \right|
\leq C f_m^{-\varepsilon_3/2} \left( \int_{f(x)>2} f^{-1-\varepsilon_3} |\partial_x v(f, \cdot)|^2 \| \lambda, f \|^2_{L^2(\mathbb{S}^{d-1})} \, df \right)^{-1/2} \langle p_j f^{1+2\varepsilon_3} r^\alpha \ell_{jk \phi} \rangle \phi,
\]
\[\text{(3.23)}\]
where \( \varepsilon_3 > 0 \). If we take \( \varepsilon_3 \in (0, \min\{\rho, (\alpha/2)/(1 - \alpha/2)\}) \), we can see by Corollary 2.7 the last factor of the right-hand side of (3.23) is finite. Hence by (3.17)–(3.23) the assertion follows.

Proof of Lemma 3.3. Since \( \sup_{f_m \geq 2} \| \mathcal{F}(\lambda, f_m) \|_{L^2(\mathbb{S}^{d-1})} < \infty \) and
\[
\int_{f_m} f^{-1-\varepsilon} \| \partial_x \mathcal{F}(\lambda, f_m) \psi \|_{L^2(\mathbb{S}^{d-1})}^2 \, df \leq \int_{f(x)>2} f^{-1-\varepsilon} |\partial_x \mathcal{F}(\lambda + i0) \psi|^2 \, dx
\]
\[
\leq \langle p_j f^{-1-\varepsilon} r^\alpha \ell_{jk \phi} \rangle \mathcal{F}(\lambda + i0) \psi < \infty;
\]
for any \( \varepsilon > 0 \), we can apply Proposition 3.5 to \( v = \mathcal{F}(\lambda, f_m) \psi \). Then we have
\[
\left| \langle \mathcal{F}(\lambda, f_m) \psi - \mathcal{F}(\lambda, f_n) \psi, \mathcal{F}(\lambda, f_m) \psi \rangle_{L^2(\mathbb{S}^{d-1})} \right|
\leq C(f_m) \left( \sup_{f_m \geq 2} \| \mathcal{F}(\lambda, f_m) \psi \|_{L^2(\mathbb{S}^{d-1})} \right) + \langle p_j f^{-1-\varepsilon} r^\alpha \ell_{jk \phi} \rangle \mathcal{F}(\lambda + i0) \psi \}
\]
Since \( \mathcal{F}(\lambda, f_n) \psi \) converges weakly to \( \mathcal{F}(\lambda) \psi \) in \( L^2(\mathbb{S}^{d-1}) \), by taking the limit \( n \to \infty \) we have
\[
\left| \mathcal{F}(\lambda, f_m) \psi \right|_{L^2(\mathbb{S}^{d-1})}^2 - \langle \mathcal{F}(\lambda) \psi, \mathcal{F}(\lambda, f_m) \psi \rangle_{L^2(\mathbb{S}^{d-1})}
\leq C(f_m) \left( \sup_{f_m \geq 2} \| \mathcal{F}(\lambda, f_m) \psi \|_{L^2(\mathbb{S}^{d-1})} \right) + \langle p_j f^{-1-\varepsilon} r^\alpha \ell_{jk \phi} \rangle \mathcal{F}(\lambda + i0) \psi \}
\]
Then by taking the limit \( m \to \infty \) we have
\[
\lim_{m \to \infty} \| \mathcal{F}(\lambda, f_m) \psi \|^2_{L^2(\mathbb{S}^{d-1})} = \| \mathcal{F}(\lambda) \psi \|^2_{L^2(\mathbb{S}^{d-1})}.
\]
Hence by (3.24) and (3.2) we can see that
\[
\| \mathcal{F}(\lambda, f_m) \psi - \mathcal{F}(\lambda) \psi \|^2_{L^2(\mathbb{S}^{d-1})} = \| \mathcal{F}(\lambda, f_m) \psi \|^2_{L^2(\mathbb{S}^{d-1})} + \| \mathcal{F}(\lambda) \psi \|^2_{L^2(\mathbb{S}^{d-1})}
\]
\[
- 2 \text{Re} \langle \mathcal{F}(\lambda, f_m) \psi, \mathcal{F}(\lambda) \psi \rangle_{L^2(\mathbb{S}^{d-1})} \to 0 \quad \text{as} \quad m \to \infty.
\]
This implies the assertion.

Now let us prove Theorem 2.9.

Proof of Theorem 2.9. Since \( \mathcal{F}(\lambda, f) \psi \) converges weakly to \( \mathcal{F}(\lambda) \psi \) in \( L^2(\mathbb{S}^{d-1}) \), we have
\[
\| \mathcal{F}(\lambda) \psi \|^2_{L^2(\mathbb{S}^{d-1})} \leq \liminf_{f \to \infty} \| \mathcal{F}(\lambda, f) \psi \|^2_{L^2(\mathbb{S}^{d-1})},
\]
\[\text{(3.25)}\]
On the other hand, by using the sequence \( \{f_n\}_{n \in \mathbb{N}} \) appearing in Lemma 3.3 we can see that
\[
\liminf_{f \to \infty} \|\mathcal{F}(\lambda, f)\psi\|_{L^2(S^{d-1})} \leq \lim_{n \to \infty} \|\mathcal{F}(\lambda, f_n)\psi\|_{L^2(S^{d-1})} = \|\mathcal{F}(\lambda)\psi\|_{L^2(S^{d-1})}. \tag{3.26}
\]
Thus by (3.25), (3.26) and (3.8) we have
\[
\|\mathcal{F}(\lambda)\psi\|_{L^2(S^{d-1})} = \liminf_{f \to \infty} \|\mathcal{F}(\lambda, f)\psi\|_{L^2(S^{d-1})} = \lim_{f \to \infty} \|\mathcal{F}(\lambda, f)\psi\|_{L^2(S^{d-1})}.
\]
By using this equality we can conclude (2.10). The equation (2.11) follows immediately from (2.10) and (3.8).

\[\square\]

3.2. Stationary wave operators. First we give expressions formulae of \( \mathcal{F}^\pm(\lambda) \).

**Proposition 3.6.** For any \( \psi \in \mathcal{B} \) the vectors \( \mathcal{F}^\pm(\lambda)\psi \) have expressions:
\[
\mathcal{F}^\pm(\lambda)\psi = \lim_{R \to \infty} R^{-1} \int_R^{2R} \mathcal{F}^\pm(\lambda, f)\psi\,df,
\tag{3.27}
\]
respectively, in \( L^2(S^{d-1}) \). Moreover \( R^{-1} \int_R^{2R} \mathcal{F}^\pm(\lambda, f)\psi\,df \) converge locally uniformly in \( \lambda \in \mathbb{R} \).

**Proof.** First we prove (3.27) for \( \psi \in \mathcal{B} \). We note that (3.27) for \( \psi \in C_0^\infty(\mathbb{R}^d) \) obviously holds by Theorem 2.9. Moreover the left-hand side is continuous for \( \psi \in \mathcal{B} \). Thus it suffices to verify the existence and continuity of the right-hand side of (3.27) for \( \psi \in \mathcal{B} \). To do this we show the following estimate by similar way to [16]: For any \( \psi \in \mathcal{B} \)
\[
\sup_{R > 2} \left\| \frac{1}{R} \int_R^{2R} \mathcal{F}^\pm(\lambda, f)\psi\,df \right\|_{L^2(S^{d-1})} \leq C \|R(\lambda \pm i0)\psi\|_{B^*}. \tag{3.28}
\]

For any \( R > 2 \) we choose \( n \in \mathbb{N} \) satisfying \( 2^n \leq 2R < 2^{n+1} \). Then by using the Cauchy–Schwarz inequality we have
\[
\left\| \frac{1}{R} \int_R^{2R} \mathcal{F}^\pm(\lambda, f)\psi\,df \right\|_{L^2(S^{d-1})}^2 \leq \frac{1}{R} \int_R^{2R} \|\mathcal{F}^\pm(\lambda, f)\psi\|_{L^2(S^{d-1})}^2\,df \leq \frac{1}{2\pi R} \int_{2^{n-1}}^{2^{n+1}} r^d d\Omega \left\| \mathcal{F}^\pm(\lambda \pm i0)\psi\right\|_{L^2(S^{d-1})}^2\,df \leq \frac{3}{2\pi} \|R(\lambda \pm i0)\psi\|_{B^*}^2.
\]
By taking supremum in \( R > 2 \) we have (3.28) and then, we conclude that (3.27) holds for any \( \psi \in \mathcal{B} \).

Next we prove the latter assertion. We note that we may assume that \( \psi \in C_0^\infty(\mathbb{R}^d) \). By the Cauchy–Schwarz inequality we have
\[
\left\| \mathcal{F}^\pm(\lambda)\psi - \frac{1}{R_1} \int_{R_1}^{2R_1} \mathcal{F}^\pm(\lambda, f_1)\psi\,df_1 \right\|_{L^2(S^{d-1})}^2 \leq \lim_{R_1 \to \infty} \frac{1}{R_1} \int_{R_1}^{2R_1} \int_{R_2}^{2R_2} \|\mathcal{F}^\pm(\lambda, f_2)\psi - \mathcal{F}^\pm(\lambda, f_1)\psi\|_{L^2(S^{d-1})}^2\,df_2\,df_1. \tag{3.29}
\]
We write
\[
\|\mathcal{F}^\pm(\lambda, f_2)\psi - \mathcal{F}^\pm(\lambda, f_1)\psi\|_{L^2(S^{d-1})}^2
\]
\[
= \|\mathcal{F}^\pm(\lambda, f_2)\psi\|^2_{L^2(\mathbb{S}^{d-1})} - \|\mathcal{F}^\pm(\lambda, f_1)\psi\|^2_{L^2(\mathbb{S}^{d-1})} + 2\,\text{Re}(\mathcal{F}^+(\lambda, f_1)\psi - \mathcal{F}^+(\lambda, f_2)\psi, \mathcal{F}^+(\lambda, f_1)\psi)_{L^2(\mathbb{S}^{d-1})}
\]
and substitute this into the right-hand side of (3.29), then we can obtain
\[
\left\|\mathcal{F}^\pm(\lambda)\psi - \frac{1}{R_1} \int_{R_1}^{2R_1} \mathcal{F}^\pm(\lambda, f_1)\psi \, df_1\right\|^2_{L^2(\mathbb{S}^{d-1})} \leq \left(1 + \frac{1}{R_1} \int_{R_1}^{2R_1} \|\mathcal{F}^\pm(\lambda, f_1)\psi\|^2_{L^2(\mathbb{S}^{d-1})} \, df_1\right) + 2\frac{2}{R_1} \int_{R_1}^{2R_1} \text{Re}(\mathcal{F}^+(\lambda, f_1)\psi - \mathcal{F}^+(\lambda)\psi, \mathcal{F}^+(\lambda, f_1)\psi)_{L^2(\mathbb{S}^{d-1})} \, df_1.
\]
By the proofs of Lemmas 3.1 and 3.4, the second term on the right-hand side tends to \(-\|\mathcal{F}^+(\lambda)\psi\|^2_{L^2(\mathbb{S}^{d-1})}\) locally uniformly in \(\lambda \in \mathbb{R}\). Similarly, by the proof of Lemma 3.3, the third term tends to 0 locally uniformly in \(\lambda \in \mathbb{R}\). Hence we have the assertion. \(\square\)

Proof of Proposition 2.10. By (2.11) and Stone's formula we can compute as, for any \(\psi \in \mathcal{B}\),
\[
\|\mathcal{F}^\pm\psi\|^2_{\tilde{H}} = \int_{\mathbb{R}} \|\mathcal{F}^\pm(\lambda)\psi\|^2_{L^2(\mathbb{S}^{d-1})} \, d\lambda = \int_{\mathbb{R}} \frac{d(E(\lambda)\psi, \psi)}{d\lambda} \, d\lambda = \|\psi\|^2_{\tilde{H}},
\]
where \(E(\cdot)\) is the spectral projection of \(H_\alpha\). Thus \(\mathcal{F}^\pm\) extend to isometries \(\mathcal{H} \rightarrow \tilde{\mathcal{H}}\).
To verify \(\mathcal{F}^\pm H_\alpha \subseteq M_\lambda \mathcal{F}^\pm\) it suffices to show that
\[
\mathcal{F}^\pm(H_\alpha - i)^{-1}\psi = (M_\lambda - i)^{-1}\mathcal{F}^\pm\psi \quad \text{for any } \psi \in \mathcal{B}. \quad \tag{3.30}
\]
By using the expressions (3.27), the resolvent equations
\[
R(\lambda \pm i0)R(i) = (\lambda - i)^{-1}R(\lambda \pm i0) - (\lambda - i)^{-1}R(i)
\]
and the Cauchy–Schwarz inequality, we obtain
\[
\mathcal{F}^\pm(\lambda)R(i)\psi = \lim_{R \rightarrow \infty} \frac{1}{R} \int_{R}^{2R} \mathcal{F}^\pm(\lambda, f)R(i)\psi \, df = (\lambda - i)^{-1}\mathcal{F}^\pm(\lambda)\psi.
\]
Thus we have (3.30). \(\square\)

4. Wave matrix and scattering matrix

In this section we prove Proposition 2.12 and Theorem 2.14.

Proof of Proposition 2.12. We consider only for the upper sign. It immediately follows from (3.14) that for any \(\psi \in \mathcal{B}\)
\[
\langle \phi^+_\lambda[v], \psi \rangle - \langle \psi^+_\lambda[v], R(\lambda + i0)\psi \rangle = \langle \nu, \mathcal{F}^+(\lambda)\psi \rangle_{L^2(\mathbb{S}^{d-1})}.
\]
This implies (2.15). \(\square\)

To show Theorem 2.14, we introduce and prove several lemmas. First we prove uniqueness.
Lemma 4.1. Suppose $\xi_\pm \in L^2(S^{d-1})$ and $\phi \in E_\lambda$ satisfy
\begin{equation}
\phi - \phi_+^\lambda [\xi_+] - \phi_-^\lambda [\xi_-] \in B^*_0.
\end{equation}
Then we have
\begin{equation}
||\xi_+||^2 + ||\xi_-||^2 = \lim_{n \to \infty} 2^{-n} \int_{2^n < f < 2^{n+1}} 2\pi |\phi|^2 \, dx,
\end{equation}
\begin{equation}
||\xi_+|| = ||\xi_-||.
\end{equation}
In particular, the two quantities in $\{\xi_+, \xi_-, \phi\}$ satisfying (4.1) uniquely determined by the other one of them.

Proof. First we prove (4.2). We can compute the right-hand side of (4.2) as
\[
\lim_{n \to \infty} 2^{-n} \int_{2^n < f < 2^{n+1}} 2\pi |\phi|^2 \, dx
\]
\[
= \lim_{n \to \infty} 2^{-n} \int_{2^n < f < 2^{n+1}} 2\pi |\phi_+^\lambda [\xi_+] + \phi_-^\lambda [\xi_-]|^2 \, dx
\]
\[
= ||\xi_+||^2 + ||\xi_-||^2 + 2\text{Re} \lim_{n \to \infty} 2^{-n} \int_{2^n}^{2^{n+1}} e^{-\frac{n}{2} (\frac{d + \alpha}{d + \alpha})} e^{2i\theta} \, df \int_{S^{d-1}} \xi_+(\omega) \bar{\xi}_-(\bar{\omega}) \, d\omega.
\]
Since $e^{2i\theta} = (2i(e^\alpha + \lambda))^{-1} \frac{d}{df} e^{2i\theta}$, by integrating by parts we can see that the last term vanishes. Hence we have (4.2).

Next we prove (4.3). Noting that $AR(i) \in L(B^*)$ and $A\phi = (\lambda - i)AR(i)\phi \in B^*$, we obtain
\[
0 = \lim_{n \to \infty} \langle i[H_\alpha, \chi_n]\phi \rangle
\]
\[
= \lim_{n \to \infty} \langle A\chi_n'\phi \rangle
\]
\[
= \lim_{n \to \infty} \langle A\phi, \chi_n' (\phi_+^\lambda [\xi_+] + \phi_-^\lambda [\xi_-]) \rangle
\]
\[
= \lim_{n \to \infty} \langle \phi, \chi_n' (A\phi_+^\lambda [\xi_+] + A\phi_-^\lambda [\xi_-]) \rangle
\]
\[
= \lim_{n \to \infty} \langle \phi, \chi_n' (\phi_+^\lambda [\xi_+] - \phi_-^\lambda [\xi_-]) \rangle
\]
\[
= \frac{1}{2\pi} (||\xi_+||^2 - ||\xi_-||^2).
\]
This implies (4.3).

Finally, the uniqueness statement follows from (4.2), (4.3), linearity of $\phi_+^\lambda$ and Rellich’s theorem, or the absence of $B^*_0$-eigenfunctions for $H_\alpha$. □

Next we construct $\phi \in E_\lambda$ from $\xi_\pm \in C^\infty(S^{d-1}).$

Lemma 4.2. Let $\lambda \in \mathbb{R}$. For any $\xi_- \in C^\infty(S^{d-1})$ we define $\phi \in E_\lambda$ and $\xi_+ \in L^2(S^{d-1})$ by
\[
\phi = \phi_-^\lambda [\xi_-] - R(\lambda + i0)\psi_-^\lambda [\xi_-], \quad \xi_+ = -2\pi i \mathcal{F}^+ (\lambda) \psi_-^\lambda [\xi_-].
\]
Then (2.18) and (2.19) hold for $\{\xi_+, \xi_-, \phi\}$.

Proof. By (2.15) and (2.16), we can see (2.19) holds. Moreover we can obtain (2.18) by the following calculation.
\[
\phi - \phi_+^\lambda [\xi_+] - \phi_-^\lambda [\xi_-]
\]
\[
= 2\pi i \phi_+^\lambda [\mathcal{F}^+ (\lambda) \psi_-^\lambda [\xi_-]] - R(\lambda + i0)\psi_-^\lambda [\xi_-]
\]
\[
\begin{aligned}
&= \sqrt{2\pi e^{-\frac{\pi}{4}(\frac{d+\alpha/2-1}{1+\alpha/2})}} r^{-(d+\alpha-1)/2} e^{i\theta} \mathcal{F}^+(\lambda)\psi_\chi^-[\xi_-] - R(\lambda + i0)\psi_\chi^-[\xi_-] \\
&= \sqrt{2\pi e^{-\frac{\pi}{4}(\frac{d-\alpha/2-3}{1+\alpha/2})}} r^{-(d+\alpha/2-1)/2} e^{i\theta} (\mathcal{F}^+(\lambda)\psi_\chi^-[\xi_-] - \mathcal{F}^+(\lambda, f)\psi_\chi^-[\xi_-]) .
\end{aligned}
\]

We can obtain a similar result for \( \xi_+ \in C^\infty(S^{d-1}) \) given first.

**Lemma 4.3.** Let \( \lambda \in \mathbb{R} \). For any \( \xi_+ \in C^\infty(S^{d-1}) \) we define \( \phi \in E_\lambda \) and \( \xi_- \in L^2(S^{d-1}) \) by

\[
\phi = \phi^+_\xi[\xi_+] - R(\lambda - i0)\psi^+ \xi_+], \quad \xi_- = 2\pi i\mathcal{F}^-(\lambda)\psi^+ \xi_+].
\]

Then (2.18) and (2.19) hold for \( \{\xi_+, \xi_-, \phi\} \).

Let us construct \( \xi_\pm \in L^2(S^{d-1}) \) from \( \phi \in E_\lambda \).

**Lemma 4.4.** Let \( \lambda \in \mathbb{R} \). For any \( \phi \in E_\lambda \) there exist \( \xi_\pm \in L^2(S^{d-1}) \) such that (2.19) holds.

**Proof.** We use similar scheme to [16, 24]. By the definition of \( S(\lambda) \), to verify (2.19) it suffices to show that there exists \( \xi \in L^2(S^{d-1}) \) such that \( \phi = \mathcal{F}^+(\lambda)\xi \) holds.

We take and fix a function \( \eta \in C_0^\infty(\mathbb{R}) \) which satisfies \( \eta(t) = t \) in neighborhood of \( t = \lambda \). We define \( \phi_\pm \in B^* \) and \( \xi_n \in L^2(S^{d-1}), n \in \mathbb{N} \) for fixed large \( m \in \mathbb{N} \) by

\[
\phi_\pm = \frac{1}{2a_0} \chi_n(A \pm a_0)\phi, \quad \xi_n = 2\pi i\mathcal{F}^+(\lambda)\chi_n(\eta(H) - \lambda)\phi_+, \quad a_0 = t^{-\alpha/2}2^{\lambda/2} - 2^{\lambda/2} + t\alpha.
\]

First we see \( \xi_n \) is bounded uniformly in \( n \in \mathbb{N} \). By commuting \( \chi_n \) and \( \eta(H) \) and noting \( \mathcal{F}^+(\lambda)(\eta(H) - \lambda) = 0 \), we have for \( v \in C^\infty(S^{d-1}), \|v\|_{L^2(S^{d-1})} = 1 \),

\[
\langle v, \xi_n \rangle_{L^2(S^{d-1})} = -2\pi i\langle \mathcal{F}^+(\lambda)^*v, [\chi_n, \eta(H)]\phi_+ \rangle_{B^* \times B}.
\]

By the Helffer–Sjöstrand formula (cf. [8])

\[
\eta(H) = \int_\mathbb{C} R(z) \, d\mu(z); \quad d\mu(z) = -(2\pi i)^{-1}\partial_z\hat{\eta}(z) \, dz \, d\bar{z},
\]

where \( \partial_z = \frac{1}{2}(\partial_x + i\partial_y) \) for \( z = x + iy \) and \( \hat{\eta} \) is a almost analytic extension of \( \eta \), we have the expression

\[
[\chi_n, \eta(H)]\phi_+ = i(A\chi_n' + i|\partial f|^2\chi_n'')\eta'(H)\phi_+ + \frac{1}{2} \int_\mathbb{C} R(z)|\partial f|^2\chi_n'' R(z)\phi_+ \, d\mu(z) + \int_\mathbb{C} R(z) ([H, iA]_{\chi_n'} + z|\partial f|^2\chi_n') R(z)\phi_+ \, d\mu(z).
\]

Then we can write

\[
[\chi_n, \eta(H)]\phi_+ = i(A\chi_n' + i|\partial f|^2\chi_n'')\eta'(H)\phi_+ + \frac{1}{2} \int_\mathbb{C} R(z)|\partial f|^2\chi_n'' R(z)\phi_+ \, d\mu(z) + \int_\mathbb{C} R(z) ([H, iA]_{\chi_n'} + z|\partial f|^2\chi_n') R(z)\phi_+ \, d\mu(z).
\]

Since \([H, iA]_{\chi_n'}\) can be expressed as, cf. [14, Lemma 2.2],

\[
[H, iA]_{\chi_n'} = \text{Re} \langle \gamma_1 H \rangle + A\gamma_2 A + \gamma_3,
\]

where \( \gamma_j, j = 1, 2, 3 \) are certain real-valued functions which satisfying \( \text{supp} \gamma_j \subseteq \text{supp} \chi_n' \) and \( |\gamma_j| = O(f^{-2}) \), we can see the second and the third terms are bounded in \( B \) uniformly in \( n \in \mathbb{N} \). Thus we can estimate as

\[
|\langle v, \xi_n \rangle_{L^2(S^{d-1})}| \leq C_1 (\|A\mathcal{F}^+(\lambda)^*v\|_{B^*} + \|\mathcal{F}^+(\lambda)^*v\|_{B^*}) \leq C_2.
\]
Therefore the sequence \( \{ \xi_n \}_{n \in \mathbb{N}} \subset L^2(\mathbb{S}^{d-1}) \) is bounded. Let us choose a weakly convergent subsequence of \( \{ \xi_n \}_{n \in \mathbb{N}} \) and denote its weak limit by \( \xi \). By changing notation, we may assume \( w^\ast \lim_{n \to \infty} \xi_n = \xi \in L^2(\mathbb{S}^{d-1}) \). For this \( \xi \), we show \( \mathcal{F}^+(\lambda)^\ast \xi = \phi \). We introduce the function \( \tilde{\eta}(t) := (\eta(t) - \lambda)(t - \lambda)^{-1} \) and compute

\[
\mathcal{F}^+(\lambda)^\ast \xi = w^\ast-B^\ast \lim_{n \to \infty} 2\pi i \mathcal{F}^+(\lambda)^\ast \mathcal{F}^-(\lambda) \chi_n(\eta(H) - \lambda) \phi_+
\]

By Corollary 2.8 and (4.4), we have

\[
\mathcal{F}^-(\lambda)^\ast \xi = -i w^\ast-B^\ast \lim_{n \to \infty} \int_{\mathbb{C}} R(z) R(\lambda + i0) A \chi_n R(z) \, d\mu(z) \phi_+
\]

By noting \( \phi - \phi_+ = \chi_m \phi - \phi_- \), we can see by similar argument that

\[
w^\ast-B^\ast \lim_{n \to \infty} R(\lambda - i0) [\chi_n, \eta(\mathcal{H})](\phi - \phi_+) = 0.
\]

Hence we are done. \( \square \)

**Proof of Theorem 2.14.** First we let \( \xi_- \in L^2(\mathbb{S}^{d-1}) \) and choose a sequence \( \{ \xi_{-n} \} \subset C^\infty(\mathbb{S}^{d-1}) \) such that \( \xi_{-n} \) converges to \( \xi_- \) in \( L^2(\mathbb{S}^{d-1}) \) as \( n \to \infty \). Then by Lemma 4.2 we have

\[
\mathcal{F}^-(\lambda)^\ast \phi_\chi^{\pm}[S(\lambda) \xi_{-n}] - \phi_\chi^{\pm}[ \xi_{-n} \chi_n \in B_0^*.
\]

Since \( \mathcal{F}^-(\lambda)^\ast, \phi_\chi^{\pm}[\cdot] \) and \( S(\lambda) \) are continuous, we can obtain by taking the limit \( n \to \infty \)

\[
\mathcal{F}^-(\lambda)^\ast \phi_\chi^{\pm}[S(\lambda) \xi_-] - \phi_\chi^{\pm}[ \xi_- \in B_0^*.
\]

Thus it is shown that (2.18) and (2.19) hold when \( \xi_- \) is given first. By Lemma 4.3 and similar argument, we can see that these hold when \( \xi_+ \) is given first. By Lemmas 4.1 and 4.4 we can conclude (i).

To verify (ii), let us show (2.20). We have by straightforward calculations

\[
(A \mp a_\pm) \phi_\chi^{\pm}[\xi] \in B_0^*
\]

for any \( \xi \in C^\infty(\mathbb{S}^{d-1}) \), and then we can deduce from (2.15) that

\[
(A \pm a_\pm) \mathcal{F}^\pm(\lambda)^\ast \xi \equiv 2a_\pm \phi_\chi^{\pm}[\xi] = (A \mp a_\pm) \phi_\chi^{\pm}[\xi] - (A \pm a_\pm) R(\lambda \mp i0) \phi_\chi^{\pm}[\xi] \in B_0^*.
\]
Therefore by the definition of $\phi^\pm_\alpha[\cdot]$ we obtain, for any $\xi \in C^\infty(\mathbb{S}^{d-1})$,

$$
\xi = \pm \frac{1}{2} c_\pm \lim_{R \to \infty} \int_R^{2R} r^{(d+\alpha/2-1)/2} e^{\mp i\varphi} \left[ \frac{1}{a_\pm} (A \pm a_\pm), \mathcal{F}^\pm(\lambda)^* \xi \right] (f, \pm) \, df
$$

$$
= \pm \frac{1}{2} c_\pm \lim_{R \to \infty} \int_R^{2R} r^{(d+\alpha/2-1)/2} e^{\mp i\varphi} \left[ (A \pm a_0), \mathcal{F}^\pm(\lambda)^* \xi \right] (f, \pm) \, df,
$$

(4.5)

where $c_\pm = \sqrt{2\pi} \exp \left\{ \pm \pi i \left( \frac{d+\alpha/2-1}{1+\alpha/2} \right) \right\}$. Since

$$(A \pm a_\pm), \mathcal{F}^\pm(\lambda)^* = (\lambda - i) \{(A \pm a_\pm) R(i)\}, \mathcal{F}^\pm(\lambda)^* \in \mathcal{L}(L^2(\mathbb{S}^{d-1}), B^*)$$

we can see that (4.5) holds for all $\xi \in L^2(\mathbb{S}^{d-1})$. Hence we have shown (ii).

By (2.19) and Lemma 4.1 we have

$$
\|\xi\|^2_{L^2(\mathbb{S}^{d-1})} \leq \pi \|\mathcal{F}^\pm(\lambda)^* \xi\|^2_{B^*}.
$$

In particular, $\text{Ker} \mathcal{F}^\pm(\lambda)^* = \{0\}$ and $\text{Ran} \mathcal{F}^\pm(\lambda)^*(= \mathcal{E}_\lambda)$ are closed in $B^*$. Then by closed range theorem (cf. e.g. [25]), the range of $\mathcal{F}^\pm(\lambda) : B \to L^2(\mathbb{S}^{d-1})$ coincides with $L^2(\mathbb{S}^{d-1})$, respectively. Hence we are done. \( \square \)

**Appendix A. Proof of Theorem 2.5**

First we note that the following lemma holds.

**Lemma A.1.** Let $z \in I^{\pm}$.

1. There exists $C > 0$ such that for any $z \in I^{\pm}$ and $x \in \text{supp} \chi_{m+1}$

$$
|a| \leq C, \quad \pm \text{Im} a \geq \frac{\alpha}{2} r^{-\alpha/2-1} - Cr^{-3\alpha/2-1},
$$

(A.1)

respectively.

2. One can rewrite $H_\alpha - z$ on $\text{supp} \chi_{m+1}$ as

$$
H_\alpha - z = \frac{1}{2} (A \pm a) r^\alpha (A \mp a) + \frac{1}{2} p_j r^\alpha \ell_{jk} p_k + q_2,
$$

(A.2)

where $q_2$ is a certain complex-valued function which satisfies

$$
\chi_{m+1} q_2 = O\left( r^{-1} f^{-1} - \min\{\rho, (3\alpha/2)/(1-\alpha/2)\} \right).
$$

(A.3)

**Proof.** The bounds in (A.1) clearly hold by the definition of (2.8). Thus we discuss on (2). By noting the expressions (2.3) and (2.6), we can compute, on $\text{supp} \chi_{m+1}$, as

$$
H_\alpha - z = \frac{1}{2} (p^f)^* r^\alpha p^f + \frac{1}{2} p_j r^\alpha \ell_{jk} p_k - \frac{1}{2} r^\alpha + q - z
$$

$$
= \frac{1}{2} A r^\alpha A + \frac{1}{2} p_j r^\alpha \ell_{jk} p_k - \frac{1}{2} r^\alpha + q + \frac{1}{8} r^\alpha (\Delta f)^2 + \frac{1}{8} r^\alpha / 2 (\Delta f)
$$

$$
+ \frac{1}{2} r^\alpha (\partial^f \Delta f) - z
$$

$$
= \frac{1}{2} (A \pm a) r^\alpha (A \mp a) + \frac{1}{2} p_j r^\alpha \ell_{jk} p_k \pm \frac{1}{2} (p^f r^\alpha a) + \frac{1}{2} r^\alpha a^2
$$

$$
- \frac{1}{2} r^\alpha + q_0 + \frac{\alpha}{4} r^{-2} - z.
$$

Therefore by letting

$$
q_2 = \pm \frac{1}{2} (p^f r^\alpha a) + \frac{1}{2} r^\alpha a^2 - \frac{1}{2} r^\alpha + q_0 + \frac{\alpha}{4} r^{-2} - z,
$$

we have the expression (A.2). In addition, we can compute $q_2$ on $\text{supp} \chi_{m+1}$ as

$$
q_2 = \pm \frac{1}{2} \partial^f \left\{ r^{\alpha/2} \sqrt{2z - 2q_0 + r^\alpha} \pm \frac{i\alpha}{2} r^{\alpha/2-1} \mp \frac{i\alpha}{2} \frac{z - q_0}{2z - 2q_0 + r^\alpha} r^{\alpha/2-1} \right\}
$$

$$
+ \frac{1}{2} \left\{ \sqrt{2z - 2q_0 + r^\alpha} \pm \frac{i\alpha}{2} r^{-1} + \frac{i\alpha}{2} \frac{z - q_0}{2z - 2q_0 + r^\alpha} r^{-1} \right\}^2 - \frac{1}{2} r^\alpha + q_0 - z + \frac{\alpha}{4} r^{-2}
$$
The last expression combined with Condition 2.1 leads us to the estimate (A.3). □

We introduce a weight function

\[ \Theta = \tilde{x}_{m+2}^{\gamma} \theta^{2\beta}; \quad \theta = \int_0^{f/2^\nu} (1 + s)^{-1-\delta} ds = [1 - (1 + f/2^\nu)^{-\delta}]/\delta, \quad (A.4) \]

where \( \beta, \delta > 0 \) and \( \nu \in \mathbb{N}_0 \). If we denote the derivatives of \( \theta \) in \( f \) by primes, we have

\[ \theta' = (1 + f/2^\nu)^{-1-\delta}/2^\nu, \quad \theta'' = -(1 + \delta)(1 + f/2^\nu)^{-2-\delta}/2^{2\nu}. \]

We note that on \( \text{supp} \Theta \) it holds that

\[ \partial_i \partial_j f = r^{-\alpha/2-1} \delta_{ij} - (1 + \alpha/2)r^{\alpha/2-1}(\partial_i f)(\partial_j f), \quad (A.5) \]

and we also note the function \( \theta \) has the following properties.

**Lemma A.2.** Fix any \( \delta > 0 \) in (A.4). Then there exist \( c, C, C_k > 0, \ k = 2, 3, \ldots \) such that for any \( k = 2, 3, \ldots \) and uniformly in \( \nu \in \mathbb{N}_0 \)

\[ c/2^\nu \leq \theta \leq \min\{C, f/2^\nu\}, \]

\[ c(\min\{2^\nu, f\})^\delta f^{-1-\delta} \theta \leq \theta' \leq f^{-1} \theta, \]

\[ 0 \leq (-1)^{k-1} \theta^{(k)} \leq C_k f^{-k} \theta. \]

We omit the proof. We only refer [15, 14] for the details. The following lemma is a key to prove Theorem 2.5.

**Lemma A.3.** Let \( \beta \in (0, 1 + \alpha/(1 - \alpha/2)) \). Fix any \( \delta \in (0, \alpha/(1 - \alpha/2)) \) in (A.4).
Then there exist \( c, C > 0 \) such that uniformly in \( z \in I \) and \( \nu \in \mathbb{N}_0 \), as quadratic forms on \( \mathcal{D}(H) \),

\[ \text{Im}((A \mp a)^* \Theta(H_\alpha - z)) \]

\[ \geq c(A \mp a)^* \tilde{x}_{m+2}^{\gamma} \theta^{2\beta-1}(A \mp a) + c f^{-1} \Theta \varepsilon_{jk} p_k \]

\[ - C f^{-1-2 \min\{\rho+1/(1-\alpha/2), 1+2 \alpha/(1-\alpha/2)\}+\delta} \theta^{2\beta} + \text{Re}(\gamma \theta^{2\beta}(H_\alpha - z)), \]

where \( \gamma = \gamma_{z, \nu} \) is a certain function satisfying \( |\gamma| \leq C r^{-2-\alpha} f^{-1-2\rho+\delta} \).

**Proof.** Fix \( \beta \in (0, 1 + \alpha/(1 - \alpha/2)) \) and \( \delta \in (0, \alpha/(1 - \alpha/2)) \) in the assertion. By the expression (A.2) we can write

\[ 2 \text{Im}((A \mp a)^* \Theta(H_\alpha - z)) \]

\[ \geq \text{Im}((A \mp a)^* \Theta(A \pm a) r^{\alpha}(A \mp a)) + \text{Im}((A \mp a)^* \Theta p_j r^{\alpha} \varepsilon_{jk} p_k) \]

\[ + 2 \text{Im}((A \mp a)^* \Theta q_2), \quad (A.6) \]
Let us estimate each term. By the bounds (A.1), we can estimate the first term of (A.6) as
\[
\text{Im}(A \mp a)^* \Theta (A \pm a) r^\alpha (A \mp a) = \text{Im}(A \mp a)^* \Theta r^\alpha (A \mp a) \pm (A \mp a)^* \Theta (\text{Im} a) r^\alpha (A \mp a) \\
\geq \beta (A \mp a)^* \bar{\chi}_{m+2} \theta^{2\beta-1} (A \mp a) - \frac{\alpha}{2} (A \mp a)^* \Theta r^{\alpha/2-1} (A \mp a) \\
+ \frac{\alpha}{2} (A \mp a)^* \Theta r^{\alpha/2-1} (A \mp a) - C_1 (A \mp a)^* \Theta r^{\alpha/2-1} (A \mp a).
\]
(\text{A.7)}

As for the second term of (A.6) we first decompose into four terms as follows.
\[
\text{Im}(A \mp a)^* \Theta p_j r^\alpha \ell_{jkp} = \text{Im}(A \mp a)^* p_j \Theta r^\alpha \ell_{jkp} \\
= \text{Im}(p_j A r^\alpha \ell_{jkp}) + \text{Im}([A, p_j] \Theta r^\alpha \ell_{jkp}) \\
\mp p_j (\text{Im} a^*) \Theta r^\alpha \ell_{jkp} \mp \text{Im}([a^*, p_j] \Theta r^\alpha \ell_{jkp}).
\]
\text{(A.8)}

We further compute and estimate each term of (A.8). Noting the equality (A.5), we can compute the first and second terms as
\[
\text{Im}(p_j A r^\alpha \ell_{jkp}) + \text{Im}([A, p_j] \Theta r^\alpha \ell_{jkp}) \\
= \frac{1}{2} p_j \{ A r^\alpha \ell_{jk} - \ell_{jk} r^\alpha A \} p_k + \text{Re}(p_j (\partial_j \partial_j f) \Theta r^\alpha \ell_{jkp}) \\
= - \beta p_j \bar{\chi}_{m+2} \theta^{2\beta-1} \ell_{jkp} - \frac{1}{2} p_j \bar{\chi}'_{m+2} \theta^{2\beta} \ell_{jkp} + p_j r^{\alpha/2-1} \Theta \ell_{jkp}.
\]
(\text{A.9)}

To estimate the fourth term of (A.8), we use Condition 2.1 and the Cauchy-Schwarz inequality. For any small \( \varepsilon_1 > 0 \) we have
\[
\mp \text{Im}([a^*, p_j] \Theta r^\alpha \ell_{jkp}) \\
= \mp \text{Re}((\partial_j a)^* \Theta r^\alpha \ell_{jkp}) \\
\geq -\varepsilon_1 p_j f^{-1-\delta} \Theta \ell_{jkp} - C_3 \varepsilon_1^{-1} (\partial_j a)^* \Theta f^{1+\delta} r^{2\alpha} \ell_{jk}(\partial_k a) \\
\geq -\varepsilon_1 p_j r^{\alpha/2-1} \Theta \ell_{jkp} - C_4 \varepsilon_1^{-1} f^{-1-2\rho} r^{-2} \Theta.
\]
\text{(A.11)}

We substitute the bounds (A.9), (A.10) and (A.11) into (A.8), and then we obtain
\[
\text{Im}((A \mp a)^* \Theta p_j r^\alpha \ell_{jkp}) \\
\geq (1 + \alpha/2 - \varepsilon_1) p_j r^{\alpha/2-1} \Theta \ell_{jkp} - \beta p_j \bar{\chi}_{m+2} \theta^{2\beta-1} \ell_{jkp} \\
- C_2 p_j r^{-\alpha/2-1} \Theta \ell_{jkp} - \frac{1}{2} p_j \bar{\chi}'_{m+2} \theta^{2\beta} \ell_{jkp} - C_4 \varepsilon_1^{-1} f^{-1-2\rho} r^{-2} \Theta.
\]
(\text{A.12)}

Let us fix \( \varepsilon_1 \in (0, (1 + \alpha/2) - (1 - \alpha/2)\beta) \). Then we have
\[
\text{Im}((A \mp a)^* \Theta p_j r^\alpha \ell_{jkp}) \geq c_1 p_j f^{-1} \Theta \ell_{jkp} - C_5 Q.
\]
\text{(A.12)}

where
\[
Q = r^{-2} f^{-1-2\min(\rho, (3\alpha/2)/(1-\alpha/2)) + \delta} \theta^{2\beta} + p_j f^{-1-2\rho + \delta} r^{-2-\alpha} \theta^{2\beta} \delta \ell_{jkp}.
\]

Let \( \varepsilon_2 > 0 \). Using the Cauchy-Schwarz inequality and (A.3), we can estimate the third term of (A.6) by
\[
2 \text{Im}((A \mp a)^* \Theta q) _2 \geq -\varepsilon_2 (A \mp a)^* f^{-1-\delta} \Theta (A \mp a) - C_6 \varepsilon_2^{-1} f^{1+\delta} |q_2|^2 \Theta \\
\geq -\varepsilon_2 (A \mp a)^* f^{-1-\delta} \Theta (A \mp a) - C_7 \varepsilon_2^{-1} Q.
\]
\text{(A.13)}
By combining (A.6), (A.7), (A.12) and (A.13) we have
\[
2 \text{Im}((a \mp a)^*\Theta(H_a - z))
\geq \beta(a \mp a)^*\chi_{m+2}\theta^{2a-1}(a \mp a) - \varepsilon_2(a \mp a)^*f^{-1-\delta}\Theta(a \mp a)
- C_1(a \mp a)^*\Theta r^{-1/2-1}(a \mp a) + c_1p_jf^{-1}\Theta\ell j k p_k - (C_5 + C_7\varepsilon_2^{-1})Q.
\]
By taking \(\varepsilon_2 > 0\) small enough, we can obtain
\[
2 \text{Im}((a \mp a)^*\Theta(H_a - z))
\geq \varepsilon_2(a \mp a)^*\chi_{m+2}\theta^{2a-1}(a \mp a) + c_1p_jf^{-1}\Theta\ell j k p_k - C_8 Q. \tag{A.14}
\]
Finally we estimate the remainder term \(Q\) as
\[
Q \leq C_{9\nu}^{-1}f^{-1-2\min\{\rho/(2a/(1-\alpha/2))\}+\delta}\rho^{2\beta} + \text{Re} \left(f^{-1-2\rho+\delta}f^{-2-\alpha}\theta^{2\beta}(H_a - z)\right). \tag{A.15}
\]
By substituting (A.15) into (A.14) we can obtain the desired bounds. \(\square\)

**Proof of Theorem 2.5.** We consider only for the upper sign for simplicity. If \(\delta = 0\), the bounds (2.9) follow immediately from Theorem 2.4. We let \(\beta \in (0, \beta_c)\), and take any \(\delta \in (0, 2\min\{\rho+1/(1-\alpha/2), 1+2\alpha/(1-\alpha/2)\}-2\beta)\cap(0, \alpha/(1-\alpha/2)), \psi \in f^{-\beta}B\) and \(z \in I_+\). Then by Lemma A.3, the Cauchy-Schwarz inequality, Theorem 2.4 and Lemma A.2
\[
\|\left(\frac{\chi_{m+2}\theta^{1/2}(a - a)R(z)\psi}{\beta}\right)^{1/2} + \left\langle p_jf^{-1}\Theta\ell j k p_k\right\rangle_{R(z)\psi}
\leq C_1 \left[\left\|\Theta^{1/2}(a - a)R(z)\psi\right\|_{B^*}\left\|\beta^{1/2}\right\|_{B^*}^2 + \left\|f^{-1-2\min\{\rho+1/(1-\alpha/2), 1+2\alpha/(1-\alpha/2)\}-\delta/2}\theta^{1/2}R(z)\psi\right\|_B^2 \right]
+ \left\|f^{-1-2\rho+1/(1-\alpha/2)-\delta}/2\theta^{1/2}R(z)\psi\right\| \left\|f^{-1-2\rho+1/(1-\alpha/2)-\delta}/2\theta^{1/2}\psi\right\|_B^2 \right]
\leq C_2\left[\left\|\chi_{m+2}\theta^{1/2}(a - a)R(z)\psi\right\|_{B^*}\left\|\beta^{1/2}\right\|_{B^*}^2 + \left\|f^{1/2}\psi\right\|_B^2 \right]. \tag{A.16}
\]
By commuting \(R(z)\) and powers of \(f\), we can see \(f^{1/2}(a - a)R(z)\psi \in B^*\) for each \(z \in I_+.\) Thus the right-hand side of (A.16) is finite, and then it follows that
\[
2^{2\beta\nu}\left\|\chi_{m+2}\theta^{1/2}(a - a)R(z)\psi\right\|_B^2 + 2^{2\beta\nu}\left\langle p_jf^{-1}\Theta\ell j k p_k\right\rangle_{R(z)\psi}
\leq C_2\left[\left\|\chi_{m+2}\theta^{1/2}(a - a)R(z)\psi\right\|_{B^*}\left\|\beta^{1/2}\right\|_{B^*}^2 + \left\|f^{1/2}\psi\right\|_B^2 \right]. \tag{A.17}
\]
In the first term on the left-hand side of (A.17), we restrict the integral region to \(\{\nu \leq f < 2^{\nu+1}\}\) and take supremum in \(\nu \in \mathbb{N}\). Then we obtain
\[
c_1\left\|\chi_{m+2}\theta^{1/2}(a - a)R(z)\psi\right\|_B^2
\leq C_2\left[\left\|\chi_{m+2}\theta^{1/2}(a - a)R(z)\psi\right\|_{B^*}\left\|\beta^{1/2}\right\|_{B^*}^2 + \left\|f^{1/2}\psi\right\|_B^2 \right].
\]
By the Cauchy-Schwarz inequality we can deduce
\[
\left\|\chi_{m+2}\theta^{1/2}(a - a)R(z)\psi\right\|_B^2 \leq C_3\left\|f^{1/2}\psi\right\|_B^2. \tag{A.18}
\]
As for the second term on the left-hand side of (A.17), we apply the bound (A.18) and take limit \(\nu \to \infty\), and then we obtain by the Lebesgue’s monotone convergence theorem and the concavity of \(\theta\)
\[
\left\langle p_j\chi_{m+2}\theta^{2a-1}\ell j k p_k\right\rangle_{R(z)\psi}^{1/2} \leq C_4\left\|f^{1/2}\psi\right\|_B. \tag{A.19}
\]
Taking into account Theorem 2.4, we can replace the cut-off $\bar{\chi}_{m+2}$ of the bounds (A.18) and (A.19) to 1. Hence we are done.

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