Slow light and the phase of a Bose-Einstein condensate

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We investigate the propagation of light with ultra low group velocity in a Bose-Einstein condensate where the phase is not uniform. The light is shown to couple strongly to the phase gradient of the condensate. The interaction between the light and the condensate enables us to perform a phase imprinting where the phase of the condensate is imprinted on the light. We illustrate the effect by showing how one can measure the fluctuating phase in an elongated quasicondensate.

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Superfluids such as Bose-Einstein condensates of alkali atoms \[^{1}\] are systems which show a conceptual simplicity yet provide all the intriguing qualities of an interacting many-body system. Many superfluid phenomena are strongly connected to the phase of the fluid. For instance vortices \[^{2}\] which have been created in a stirring manner similar to the rotating bucket experiment \[^{3}\], and solitons \[^{4}\], \[^{5}\] are typical examples of a condensate with a nonuniform phase. Also quantum shock waves causing topological defects in the superfluid, have been created using ultra-compressed pulses of slow light \[^{6}\]. Here we show how the properties of slow light can be used to detect the phase of the macroscopic wavefunction of the condensate nondestructively, where the slow light receives the phase of the condensate in a way similar to phase imprinting \[^{7}\], \[^{8}\]

The recent experimental demonstrations with slow light has revealed a whole new playground with interesting applications \[^{9}\], \[^{10}\]. In this paper we show the advantages with slow light when the phase of the condensate is of interest. Light, and especially the phase contrast imaging method, has been successfully used to probe the density of the condensate in-situ \[^{11}\]. In order to measure the phase, on the other hand, destructive methods have been used where the interference pattern has revealed the phase of the condensate \[^{12}\]. In this paper we show using slow light that the phase of the condensate can be imprinted on the light in a nondestructive way, making it possible to follow the phase evolution of the condensate in-situ.

The envisioned experiment is based on Electromagnetically Induced Transparency (EIT) \[^{13}\], \[^{14}\]. Suppose that the condensate is trapped in an elongated trap and illuminated by a uniform control beam. The beam controls the group velocity \(v_g\) of a second beam, the probe beam. The probe shall be a pulse with a frequency in the laboratory frame that matches exactly the atomic transition frequency between the ground state and one of the excited states coupled by the control beam. The condensate density, which can be monitored in-situ by phase-contrast microscopy \[^{15}\], is supposed to be stationary, but the phase can in certain cases fluctuate \[^{16}\], \[^{17}\]. For low group velocities the probe beam couples strongly to the condensate phase gradient which results in a phase imprinting on the light. The condensate phase can consequently be read out by analyzing the interference pattern between the initial and final probe light.

Let us start by deriving the equations of motion for the slow light and the Bose-Einstein condensate. The probe light is described by a real scalar field \(\Phi\) where we ignore the polarization, and \(\varphi\) represents the electric field in units of the vacuum noise, \(E = (\hbar/\varepsilon_0)^{1/2}\omega_0\varphi\). Here \(\omega_0\) denotes the resonance frequency of EIT. In a medium at rest, slow light is subject to the principle of least action with the Lagrangian density \[^{12}\]

\[
\mathcal{L}_L = \frac{\hbar}{2} \left( (1 + \alpha)(\partial_t \varphi)^2 - c^2 (\nabla \varphi)^2 - \alpha \omega_0^2 \varphi^2 \right)
\] (1)

where the group index \(\alpha\) corresponds to a group velocity of

\[
v_g = \frac{c}{1 + \alpha}
\] (2)

and \(\alpha\) is proportional to the density of the condensate, \(\rho\), and inversely proportional to the intensity of the control beam \[^{16}\], \[^{17}\],

\[
(1 + \alpha) \frac{\varepsilon_0 |E_0|^2}{\hbar \omega_0} = \frac{1}{2} \frac{\Omega_p^2}{\Omega_c^2} \rho.
\] (3)

Here the intensities of the probe and control fields are calibrated in terms of the Rabi frequencies \(\Omega_p\) and \(\Omega_c\), respectively. The control beam will dominate and in practice \(|\Omega_p|^2/|\Omega_c|^2\) does not exceed \(10^{-1}\) \[^{18}\]. This means the less intense the control beam is the slower the light is. The corresponding momentum density of the light which is needed when describing the coupling between the light and the condensate, is obtained from the symmetric energy-momentum tensor \[^{17}\], \[^{18}\]

\[
P = -\hbar (\partial_t \varphi) \nabla \varphi.
\] (4)

Consider for the time being the general situation with a moving condensate. The condensate with the flow \(u\)
and the probe light must now be taken as a combined dynamical system with the total Lagrangian density
\[
\mathcal{L} = \mathcal{L}_L + \mathcal{L}_M
\] (5)
where the condensate is described by the Gross-Pitaevskii Lagrangian density
\[
\mathcal{L}_M = -\rho \left( \frac{\hbar S}{2} + \frac{m}{2} \nu^2 + \frac{2m}{3} (\nabla \rho)^2 + \frac{g}{2} \rho + V \right). \tag{6}
\]
Here \(m\) is the atomic mass, \(g\) characterizes the atom-atom collisions, \(V\) denotes the external potential and \(\rho\) the condensate density. The coupling between light and matter is given by the relation \[18\]
\[m \nu = \hbar S + \alpha_0 \nu, \quad \alpha = \alpha_0 \nu\] (7)
where \(\alpha_0\) is a constant. This coupling gives the correct equation of motion
\[\partial_t \nu + \nabla (\nu \nu) = 0\] (8)
from the Euler-Lagrange equation.
Using the Lagrangian in Eq. \[10\] we derive the equation of motion for the light field. The resulting wave equation is
\[(1 + \alpha) \partial_t^2 - c^2 \nabla^2 + \alpha_0 \nu^2 + \alpha \nu \partial_t \nu \cdot \nabla + \nabla \cdot (\alpha \nu \partial_t \nu) \varphi = 0.\] (9)
In order to simplify things we assume the light field can be expressed as a propagating pulse with a single frequency \(\omega_0\) and a slowly varying amplitude. The resulting equation of motion is then of the Schrödinger type
\[i \partial_t \varphi(r) = \frac{c^2}{2(\alpha + 1) \omega_0} \nabla^2 \varphi(r) + \frac{i \alpha}{2 \alpha + 1} (2\nu \cdot \nabla \varphi(r) + (\nabla \cdot \nu) \varphi(r))\] (10)
where the coupling between light and matter is primarily described by the phase gradient of the condensate.
Let us illustrate the consequences of the coupling between the light and the phase of the condensate by using an elongated quasi-condensate where the phase can fluctuate \[14\]. Fluctuations of the density and the phase of a condensate are related to the elementary excitations. The density fluctuations are dominated by the excitations of the order of the chemical potential \(\mu\). If the condensate is very elongated but still of a 3D character, the wavelength of the density excitations are much smaller than the radial size of the condensate. The fluctuations are therefore of an ordinary 3D form and are small. Consequently the total field operator can be written as
\[\hat{\Psi}(r) = \rho_0(r)e^{i\theta(r)}\] (11)
where \(\rho_0(r)\) is the stationary density and \(\hat{\theta}(r)\) is the operator describing the phase of the condensate. This operator is given by \[14\]
\[\hat{\theta}(r) = \frac{1}{\sqrt{4\rho_0}} \sum_\nu f_\nu^\dagger \hat{a}_\nu + h.c.\] (12)
where \(\hat{a}_\nu\) is the quasi particle annihilation operator with quantum number \(\nu\) and energy \(\epsilon_\nu\). The mode functions \(f_\nu^\dagger = u_\nu + v_\nu\) are the sum of the two functions \(u_\nu\) and \(v_\nu\) which are the solutions to the Bogoliubov-deGennes equations.
In an elongated condensate the excitations are mainly of two kinds: axial excitations with \(\epsilon_\nu < \hbar \omega_\nu\) and radial excitations \(\epsilon_\nu > \hbar \omega_\nu\). The latter has a 3D character since the wavelength is typically less than the radial size of the cloud and consequently the fluctuations are small. The axial excitations on the other hand have wavelengths larger than the radial size of the cloud and have a 1D behavior. Therefore these excitations will be most important for the axial fluctuations of the phase. In order to actually calculate the phase we note that for a harmonic external potential the low energy axial modes are described by the energy \(\epsilon_\nu = \frac{1}{2} \hbar \omega_\nu \sqrt{\nu(\nu + 3)}\) and the functions
\[f_\nu^\dagger = \sqrt{(\nu + 2)(2\nu + 3)\rho_0(r)} \frac{P^{(1,1)}_\nu(z)}{4\pi(\nu + 1)R^2\epsilon_\nu}\] (13)
where \(P^{(1,1)}_\nu\) are the Jacobi polynomials, \(R\) the radial size and \(L\) the axial length of the cloud. The quasi-particle annihilation operators are now replaced by complex amplitudes \(\gamma\) and \(\gamma^*\). To reproduce the quantum statistical properties of the phase the amplitudes are sampled as random variables with a zero mean value, \(\langle \gamma_\nu \rangle = \langle \gamma^*_\nu \rangle = 0\), and bosonic number density
\[\langle |\gamma_\nu|^2 \rangle = \frac{1}{e^{\beta \gamma_\nu} - 1}\] (14)
where \(\beta\) is the inverse temperature \[14\].
If the coupling is dominated by the phase gradient, in other words we neglect the nonlinear term for the light in Eq. \[14\], it is clear that we can use the light as a weak probe which is affected by the condensate phase gradient. The condensate will necessarily be very elongated. We can therefore solve the dynamics in 1D since the low energy axial excitations will acquire a 1D character. From Eq. \[14\] it is immediately clear that the light will pick up a phase
\[\varphi(r, z) = \tilde{\varphi}(r, z)e^{i\xi(z)}\] (15)
where the phase is given by the integral
\[\xi(z) = -\frac{k_0}{v_g} \int^z dx u(x) = -\frac{\hbar k_0}{mv_g} S(z)\] (16)
with $k_0 = \frac{\hbar}{m}$ and $S(z)$ the phase of the condensate. This scenario requires that the interaction between the probe light and the condensate is turned on suddenly. This can indeed be achieved by tuning the control beam such that the probe beam is rapidly slowed down and allowed to propagate in the condensate. This results in a pulse delay typically of the order of a few micro seconds.

In summary we have shown that light with extremely low group velocity can be used to probe the phase of a condensate. The method was illustrated by imprinting the fluctuating phase of a quasicondensate onto the slow light propagating through the condensate. The phase imprinting technique also allows for studying other forms of phase gradients such as vortices and solitons. Especially solitons in two-component condensates with repulsive interaction could be detected without opening the trap. Two-component solitons are significantly more difficult to observe because the total density will always stay constant, compared to the single component condensate soliton which shows a density notch at the position of the soliton. The probe light in this case would simply pick up the phase of the soliton solution. If the light propagates during a long time in a medium with a nonuniform phase it is also possible to study scattering of the light from the condensate phase which in principle can also be used to measure the phase gradi-
ent, although this method will not necessarily reveal the condensate phase as clearly as a direct interference experiment.

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FIG. 2. (a) One possible phase at the temperature $T = 0.75T_{BEC}$. Only the lowest modes are important in the phase fluctuations. (b) The shape of the light pulse (solid black line) is only slightly altered due to the acquired phase. The grey curve shows the initial pulse shape (arb. units). The dashed line shows the density of the condensate (arb. units) (c) The resulting intensity of the light when interfering with the initial probe light (arb. units).

[1] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 71, 463 (1999).
[2] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. 83, 2498 (1999); K. W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, ibid. 84, 806 (2000); J. R. Abo-Shaeer, C. Raman, J. M. Vogels, and W. Ketterle, Science 292, 476 (2001).
[3] D. R. Tilley and J. Tilley, Superfluidity and Superconductivity, (Adam Hilger, Bristol, 1990).
[4] S. Burger, K. Bongs, S. Dettmer, W. Ertmer, K. Sengstock, A. Sanpera, G. V. Shlyapnikov, and M. Lewenstein, Phys. Rev. Lett. 83, 5198 (1999).
[5] J. Denschlag, J. E. Simsarian, D. L. Feder, Charles W. Clark, L. A. Collins, J. Cubizolles, L. Deng, E. W. Hagley, K. Helmerson, W. P. Reinhardt, S. L. Rolston, B. I. Schneider, and W. D. Phillips, Science 83, 97 (2000).
[6] Z. Dutton, M. Budde, Ch. Slowe, and L. V. Hau, Science 293, 663 (2001).
[7] L. V. Hau, S. E. Harris, Z. Dutton, and C. H. Behroozi, Nature 397, 594 (1999); Ch. Liu, Z. Dutton, C. H. Behroozi, and L. V. Hau, ibid. 409, 490 (2001).
[8] U. Leonhardt and P. Fiwnicki, Phys. Rev. Lett. 84, 822 (2000).
[9] M. R. Andrews, M.-O. Mewes, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Science 273, 84 (1996).
[10] S. Inouye, S. Gupta, T. Rosenband, A. P. Chikkatur, A. Görlitz, T. L. Gustavson, A. E. Leanhardt, D. E. Pritchard, and W. Ketterle, Phys. Rev. Lett. 87, 080402 (2001).
[11] M. O. Scully and M. Zubairy, Quantum Optics (Cambridge University Press, Cambridge, 1997).
[12] U. Leonhardt, arXiv:gr-qc/0108083.
[13] I. Carusotto, M. Artoni and G. C. La Rocca, JETP Letters 72, 289 (2000).
[14] S. Dettmer, D. Hellweg, P. Ryytty, J.J. Arlt, W. Ertmer, K. Sengstock, D.S. Petrov, G.V. Shlyapnikov, H. Kreutzmann, L. Santos, and M. Lewenstein, Phys. Rev. Lett. 87, 160406 (2001).
[15] D.S. Petrov, G.V. Shlyapnikov, and J.T.M. Walraven, Phys. Rev. Lett. 87, 050404 (2001).
[16] M. Fleischhauer and M. D. Lukin, Phys. Rev. Lett. 84, 5094 (2000).
[17] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields (Pergamon, Oxford, 1975).
[18] U. Leonhardt and P. Öhberg, arXiv:cond-mat/0110514.
[19] D.V. Fil and S.I. Shevchenko, Phys. Rev. A, 64 (2001) 013607.
[20] S. Stringari, Phys. Rev. A, 58, 2385 (1998).
[21] D.S. Petrov, M. Holzmann, and G.V. Shlyapnikov, Phys. Rev. Lett. 84, 2551 (2001).
[22] M. Artoni and I. Carusotto, arXiv:cond-mat/0206104.
[23] P. Öhberg and L. Santos, Phys. Rev. Lett. 86, 2918 (2001).