Shortest Trajectories and Reversibility
in Boolean Automata Networks

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1 Introduction

§1 Intuition and motivation. It seems that often, when a Boolean Automata Network (BAN) \( \mathcal{N} \) can make a global change, it can do it rather quickly, i.e. with few local changes. More precisely, when it is possible to reach a certain specific configuration \( y \in \mathbb{B}^n = \{0, 1\}^n \) of \( \mathcal{N} \), starting from an initial configuration \( x \in \mathbb{B}^n \) of \( \mathcal{N} \), then it seems that the following is often the case. To make the global change \( x \rightsquigarrow y \), only a small number of local changes need to be made, i.e. only a small number of automata need to ‘move’ (change states): something (polynomially) comparable to the size \( n \in \mathbb{N} \) of the network \( \mathcal{N} \) i.e. to the number \( n \) of automata in \( \mathcal{N} \) and to the total number of different conceivable automata moves away from \( x \). To check this conjecture and specify the meaning of “often” in this context, we take interest here in “long trajectories”.

§2 Long trajectories. To qualify as long, a trajectory must switch some automata state values back and forth between 0 and 1. In a trajectory that isn’t long, every automaton \( i \in V = \{1, \ldots, n\} \) of the BAN \( \mathcal{N} \) either doesn’t move at all, or only moves once. The whole length of a trajectory that isn’t long (the number of automata moves it involves) is no greater than the total number \( n \) of automata in \( \mathcal{N} \).

§3 Long shortest trajectories. We are interested in the case where to get from a configuration \( x \in \mathbb{B}^n \) to a configuration \( y \in \mathbb{B}^n \), there is no shorter way than to have some automata moving back and forth. In other terms, all shortest trajectories from \( x \) to \( y \) are long. In such cases, we will say that to get from \( x \) to \( y \) requires “reversibility”.

§4 Moves. Formally, in this Boolean context, a move of an automaton \( i \in V \) is the transition of its actual state, the state \( x_i \in \mathbb{B} = \{0, 1\} \) that \( i \) has in configuration \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{B}^n \), to the only other different state \( i \) can take, namely \( ¬x_i \). So a move of \( i \in V \) is either \( x_i = 0 \rightsquigarrow ¬x_i = 1 \), or \( x_i = 1 \rightsquigarrow ¬x_i = 0 \). The first (resp. second) kind of move is represented by the value +1 (resp. -1). Generally, we write \( \nabla x_i = ¬\nabla x_i = ¬x_i \iff x_i \in \mathbb{S} = \{-1, +1\} \). \( \nabla x_i \) (resp. \( -\nabla x_i \)) represents \( i \)'s move away from (resp. towards) \( x \). If ever automaton \( i \in V \) has the capacity to make a move in configuration \( x \in \mathbb{B}^n \), then this move necessarily is the move represented by \( \nabla x_i \in \mathbb{S} \).

§5 Signs and Boolean values. We introduce function \( \mathbb{SB} : \mathbb{S} \rightarrow \mathbb{B} \) defined by \( \mathbb{SB}(s) = \frac{1+s}{2} \) so that for any configuration \( x \in \mathbb{B}^n \), \( \mathbb{SB}(¬\nabla x_i) = x_i \) equals the state of automaton \( i \) in configuration \( x \). And \( \mathbb{SB}^{-1} = \mathbb{BS} : \mathbb{B} \rightarrow \mathbb{S} \) so that \( \mathbb{BS}(x_i) = 2x_i - 1 = -\nabla x_i \) is \( i \)'s move towards \( x \).

§6 BANs (Boolean Automata Networks). It remains the question: In \( x \in \mathbb{B}^n \), can automaton \( i \) make move \( \nabla x_i \) or can it not? Precisely, this is determined by the definition of the BAN. A BAN \( \mathcal{N} \) is a set of local transition functions : \( \mathcal{N} = \{ f_i : \mathbb{B}^n \rightarrow \mathbb{B}, \ i \in V \} \), one for each automaton in the set \( V = [1, n] \) of all the BAN’s automata.
§7 **Instability and stability.** In configuration \( x \), the automaton \( i \in V \) can make move \( \nabla x_i \) if and only if the following holds: \( f_i(x) = \neg x_i \) and equivalently \( \nabla x_i = f_i(x) - x_i \). In this case, \( i \) is said to be **unstable in configuration** \( x \). The set of automata that are unstable in \( x \) (ready to make a move in \( x \)) is \( U(x) = \{ i \in V : f_i(x) \neq x_i \} \). If automaton \( i \in V \) cannot make move \( \nabla x_i \) in configuration \( x \), i.e. if \( f_i(x) - x_i = 0 \), then \( i \) is said to be **stable in** \( x \). The set of automata that are stable in \( x \) is \( S(x) = \{ i \in V : x_i = f_i(x) \} \).

§8 **Signature of punctual influences.** Let \( i, j \in V \) be two different automata of the same BAN \( \mathcal{N} \). We define the sign of the influence that automaton \( j \) has on automaton \( i \) in configuration \( x \in \mathbb{B}^n \) as follows (where configuration \( \overline{x}_j = (x_1, \ldots, x_{j-1}, \neg x_j, x_{j+1}, \ldots) \) is exactly the same as configuration \( x \) except for component \( j \) : \( \overline{x}_j = \neg x_j \) and \( \forall k \neq j, \overline{x}_k = x_k \).

\[
\forall x \in \mathbb{B}^n, \forall i \neq j \in V, \text{sign}(x, j, i) = \frac{(f_i(\overline{x}_j) - f_i(x)) \cdot (\overline{x}_j - x_j)}{\nabla x_j} \in \mathbb{S} \cup \{0\}, \text{ i.e.}
\]

\[
\begin{cases}
0 & \text{if } i \in U(x) \cap U(\overline{x}_j) \text{ because then : } f_i(x) = \neg x_i = \neg \overline{x}_j = f_i(\overline{x}_j) \\
-\nabla x_i \cdot \nabla x_j & \text{if } i \in U(x) \cap S(\overline{x}_j) \text{ because then : } f_i(x) = \neg x_i \text{ and } f_i(\overline{x}_j) = \overline{x}_i = \neg x_i \\
\nabla x_i \cdot \nabla x_j & \text{if } i \in S(x) \cap U(\overline{x}_j) \text{ because then : } f_i(x) = x_i \text{ and } f_i(\overline{x}_j) = \neg \overline{x}_j = \neg x_i \\
0 & \text{if } i \in S(x) \cap S(\overline{x}_j) \text{ because then : } f_i(x) = x_i = \overline{x}_i = f_i(\overline{x}_j)
\end{cases}
\]

When \( j = i \), we define the sign of the influence automaton \( i \) has on itself as follows.

\[
\forall x \in \mathbb{B}^n, \forall i \in V, \text{sign}(x, i, i) = \frac{(f_i(\overline{x}_i) - f_i(x)) \cdot (\overline{x}_i - x_i)}{\nabla x_i}, \text{ i.e. :}
\]

\[
\begin{cases}
-\nabla x_i \cdot \nabla x_i = -1 & \text{if } i \in U(x) \cap U(\overline{x}_i) \text{ because then : } f_i(x) = \neg x_i \neq x_i = f_i(\overline{x}_i) \\
0 & \text{if } i \in U(x) \cap S(\overline{x}_i) \text{ because then : } f_i(x) = \neg x_i = f_i(\overline{x}_i) \\
0 & \text{if } i \in S(x) \cap U(\overline{x}_i) \text{ because then : } f_i(x) = x_i = \overline{x}_i = f_i(\overline{x}_i) \\
\nabla x_i \cdot \nabla x_i = +1 & \text{if } i \in S(x) \cap S(\overline{x}_i) \text{ because then : } f_i(x) = x_i \neq \overline{x}_i = f_i(\overline{x}_i)
\end{cases}
\]

§9 **Monotone functions and Monotone BANs.** A Boolean function \( f : \mathbb{B}^n \to \mathbb{B} \) is said to be monotone when the following holds. In the conjunctive normal expression of \( f(x) \), a literal \( x_j \) either only appears unnegated, or it only appears negated. In a BAN \( \mathcal{N} \), the monotony of \( f_j \) is equivalent to : 

\[
\forall x, y \in \mathbb{B}^n, \text{sign}(x, i, j) \neq 0 \land \text{sign}(y, i, j) \neq 0 \implies \text{sign}(x, i, j) = \text{sign}(y, i, j).
\]

In that case, we write \( \text{sign}(i, j) = \text{sign}(x, i, j) \forall x \in \mathbb{B}^n \) s.t. \( \text{sign}(x, i, j) \neq 0 \). A monotone BAN is one in which all local transition functions \( f_i \) are monotone.

§10 **Interaction Graph.** The interaction graph of a BAN \( \mathcal{N} \) is the digraph \( G = (V, A) \) where \( A = \{(j, i) \in V \times V : \exists x \in \mathbb{B}^n, \text{sign}(x, j, i) \neq 0\} \). We say that a BAN \( \mathcal{N} \) is (strongly) connected if its interaction graph \( G \) is (strongly) connected.

§11 **Path signs.** In a monotone BAN, the sign of a path of \( G \) is the product of the signs of the arcs it is comprised of.
§12 Monotone and contradictory paths. In $G$, if there are no paths from automaton $j$ to automaton $i$, we let $\text{sign}^*(j, i) = 0$. Otherwise, if all paths from automaton $j$ to automaton $i$ have the same sign $s$, then we let $\text{sign}^*(j, i) = s$. And if there exists a negative path as well as a positive path from $j$ to $i$, we say that these two paths are contradictory paths. In particular, the path covering a negative cycle once, and the path covering it twice are contradictory paths.

§13 Nice Networks and totally positive ones. A nice BAN is a BAN for which $\text{sign}^*(i, j)$ is defined $\forall i, j \in V$. In other terms, a nice BAN is a monotone BAN without contradictory paths, and in particular without negative cycles. A totally positive BAN is a monotone (and nice) BAN in which $\forall i, j \in V : \text{sign}(i, j) = +1$.

§14 An important remark on the meaning of automata state values. In BANs, the values 0 and 1 of automata states are often assumed to represent two opposite state values, the same for all automata, so that when state $x_i$ of automaton $i$ equals 0, and state $x_j$ of automaton $j$ also equals 0, then the equality $x_i = x_j$ is taken to mean that the same thing is happening to $i$ and $j$. Often, $x_i = x_j = 0$ is taken to mean that $i$ and $j$ are both “inactive”, as opposed to “active” as they would be if they were in state 1. In reality, in a BAN, 0 and 1 are just labels. $x_i = 0$ could just as well mean that $i$ is “open” rather than “closed”, while for the automaton $j$ next door, $x_j = 0$ would mean that $j$ is “blue” rather than “black”. Thus, $x_i = x_j = 0$ only is meaningful in the sense of $x_i = 0$ and $x_j = 0$, not in the sense of $x_i = x_j$. There are no other relations between $x_i = 0$ and $x_j = 0$ than the ones ensuing from the definitions of the BAN’s local transitions functions. And since the values 0 and 1 of automata states are just names of automata state values, this means that we can exchange state 0 of automaton $i$ with state 1, for instance. Replacing (a) every $f_j(x)$ by $f_j(\overline{x})$ and then (b) $f_i(x)$ by $\overline{f_i}(x)$ yields different a formalisation of the exact same network. In this new formalisation, all arcs $(j, i) \in A$ and $(i, j) \in A, j \in V$ have changed signs.

§16 It was proven in [1] that any nice strongly connected BAN can be re-formulated as a totally positive BAN.

§16 Neighbourhoods. The in- (resp. out-) neighbourhood of an automaton $i \in V$ is the set $V_{\rightarrow i} = \{ j \in V : (j, i) \in A \}$ (resp. the set $V_{\rightarrow i} = \{ j \in V : (i, j) \in A \}$).

§17 Neighbour Inputs and Straight functions. In a monotone BAN, $\forall j \in V_{\rightarrow i}, \text{sign}(j, i)$ is defined. We write $x_{j \rightarrow i} = \text{SB}(\text{sign}(j, i) \cdot \text{BS}(x_j)) = \text{SB}( - \text{sign}(j, i) \cdot \nabla x_j ) \in E$ to denote the Boolean value incoming automaton $j$ that automaton $i$ computes with in configuration $x$, i.e. the input that $i$ receives in $x$ from $j$:

$$x_{j \rightarrow i} = \begin{cases} f_i(x) = \overline{x_i} & \text{if } i \in U(x) \text{ and } \text{sign}(x, j, i) = \text{sign}(j, i) \\ f_i(x) = x_i & \text{if } i \in S(x) \text{ and } \text{sign}(x, j, i) = \text{sign}(j, i) \\ \overline{f_i}(x) = x_i & \text{if } i \in U(x) \text{ and } \text{sign}(x, j, i) = 0 \\ \overline{f_i}(x) = \overline{x_i} & \text{if } i \in S(x) \text{ and } \text{sign}(x, j, i) = 0 \end{cases}$$

When $j \notin V_{\rightarrow i}$, we let $x_{j \rightarrow i} = x_j$. With $x_{j \rightarrow i}$ thus defined for all $x$, we define the “straight” local transition function $g_i$ associated to any local transition function $f_i$ of the BAN:

$$\forall x \in E^n, g_i(x_{1 \rightarrow i}, \ldots x_{n \rightarrow i}) = f_i(x).$$

The interest in $g_i$ is that its conjunctive normal form contains no negation. The negations of $f_i$ are all already taken into account by changing $x_j$ for
§18 **(Asynchronous) Trajectories.** A trajectory from \(x = (x_1, \ldots, x_n) \in \mathbb{E}^n\) to \(y = (y_1, \ldots, y_n) \in \mathbb{E}^n\) is a sequel of configurations \((x(t))_{t \in T}\) such that \(x(0) = x, x(T) = y\), and \(\forall t < T\), configurations \(x(t)\) and \(x(t+1)\) are related as follows: \(x(t+1) = \overline{x(t)}\) for some automaton \(i \in U(x(t)) = U(t)\) that is unstable at time step \(t\). We denote this automaton by \(i = \nu(t)\). Thus:
\[\forall t \in [0, T], \; x(t+1) = \overline{x(t)^{\nu(t)}}.\]

§19 **Recurrent configurations.** Let \(x \in \mathbb{E}^n\) be a configuration of the BAN \(\mathcal{N}\). Consider the set of all configurations \(y \in \mathbb{E}^n\) such that there is a trajectory from \(x\) to \(y\). If all configurations \(y\) in this set are such that there also is a trajectory back from \(y\) to \(x\), then we call \(x\) a recurrent configuration (\(y\) necessarily is one too). Stable configurations \(x \in \mathbb{E}^n\) s.t. \(S(x) = V\) are special kinds of recurrent configurations.

§21 From now on, we consider an arbitrary trajectory \(\mathcal{T} = (x(t))_{t \in T}\) from \(x = x(0)\) to \(y = x(T)\), where \(T = [0, T]\) is the set of all time steps of the trajectory and \(T \in \mathbb{N}\) is the last of them.

2 **Causality (version 0.1)**

§21 **Causality.** Consider time step \(t\) of \(\mathcal{T}\) at which move \(\nabla x(t)_i\) is made by automaton \(i = \nu(t) \in U(t) = U(x(t))\). We want to pinpoint what “caused” or “unlocked” the possibility of making this move. There are two cases:

1. Move \(\nabla x(t)_i\) was possible ever since the beginning of \(\mathcal{T}\) : \(\forall t' < t, \; i \in U(t')\) and \(\nu(t') \neq i\). In this case, \(t\) is said to be a root step, and move \(\nabla x(t)_i\) is a root move. A root move happening at a root step has no cause.

2. There exists a step \(t' < t\) at which move \(\nabla x(t)_i\) wasn’t yet possible : \(i \in S(t')\) was stable and in state \(x(t')_i = x(t)_i\). We denote by:
\[\tau(t) = \max\{t' < t : \forall s \in \llbracket t', t \rrbracket, \; i \in U(s) \cap S(t') \text{ and } \nu(s) \neq i\}\]
the most recent time step at which \(i\) was stable before \(t\). At time step \(\tau(t)\) a move \(\nabla x(\tau(t))_j\) was made by a certain automaton \(j = \nu(\tau(t))\). Right after that, \(i\) became unstable. And it remained unstable and unmoved until the time step \(t\) at which it made move \(\nabla x(t)_i = \nabla x(\tau(t))_i = \nabla x(s), \forall s \in \llbracket \tau(t), t \rrbracket\). Automaton \(j\)’s move \(\nabla x(\tau(t))_j\) at time \(\tau(t)\) is said to be the cause of automaton \(i\)’s move \(\nabla x(t)_i\) at time \(t\).

**Lemma 1.**
Let \(\mathcal{T} = (x(t))_{t \in \mathbb{T}}\) be a trajectory of a BAN \(\mathcal{N} = \{f_i : \mathbb{E}^n \rightarrow \mathbb{E}, \; i \in V\}\). Let \(t_1 < t_2 < T\) be two time steps of \(\mathcal{T}\). Let \(j = \nu(t_1)\) and \(i = \nu(t_2)\).
If \(t_1 = \tau(t_2)\), then \(\text{sign}(x(t_1)_j, j, i) = \nabla x(t_1)_j \cdot \nabla x(t_1)_i = \nabla x(t_1)_j \cdot \nabla x(t_2)_i\).

**Proof:** 
\[\overline{x(t_1)}^j = x(t_1 + 1) \text{ and } i \in S(x(t_1)) \cap U(x(t_1 + 1)).\]

§22 **Causality branches.** \(\forall t \in \mathbb{T}\), we let \(\tau^1(t) = \tau(t)\), and \(\forall q \in \mathbb{N}, \; \tau^{q+1}(t) = \tau(\tau^q(t))\) if it exists. It doesn’t if \(\tau^q(t)\) is a root step. And if \(p = \max\{q \in \mathbb{N} : \tau^q(t) \text{ exists}\}\), then we denote by \(\tau^p(t) = \tau^p(t)\) this root step. A \(\tau\)-branch of trajectory \(\mathcal{T}\) is a sequel of \(q \in \mathbb{N}\) time steps \((\tau^{\tau^{-p}(t)})_{p \leq q}\) ending with time step \(t < T\).

The following lemma relates path signs in \(G\) to moves made along \(\tau\)-branches.
Lemma 2. Same conditions as Lemma 1
If \( \exists q \in \mathbb{N} : t_1 = \tau^q(t_2) \), then there is a path of length \( p \) in \( G \) from \( j \) to \( i \) (which has sign \( \nabla x(t_1) \cdot \nabla x(t_2) \)) if \( N \) is monotone). If \( j = i \), this path is a cycle and all automata that move on the same branch \( (\tau^q(t_2))_{p \leq q} \) between \( t_1 \) and \( t_2 \) are strongly connected in \( G \).

Proof: By induction on \( q \). If \( q = 1 \), then it follows from Lemma 1. If \( q > 1 \), then there exists \( t \in [t_1, t_2] : t_1 = \tau(t) \) and \( t = \tau^{q-1}(t_2) \). By the induction hypothesis, there exists a path of length \( q-1 \) (which has sign \( \nabla x(t_1) \cdot \nabla x(t_2) \)) if \( N \) is monotone) from \( k = \nu(t) \) to \( i = \nu(t_2) \) in \( G \). There also exists an arc \( (j, k) \in A \) (of sign \( \nabla x(t_1) \cdot \nabla x(t_2) \)). The concatenation of this path and this arc defines a path from \( j \) to \( i \) of length \( q + 1 \) (and sign \( \nabla x(t_2) \cdot \nabla x(t_1) \)). \( \square \)

The next result is a direct consequence of Lemma 2

Lemma 3.
Let \( T = (x(t))_{t \in \mathbb{T}} \) be a trajectory of a BAN \( \mathcal{N} = \{ f_i : \mathbb{B}^n \rightarrow \mathbb{B}, i \in V \} \). If there exists an automaton moving up and down along a \( \tau \)-branch of \( T \), then \( \mathcal{N} \) is not nice : either it is non-monotone, or there is a negative cycle in \( G \) involving \( i \).

Proof: The existence of an automaton moving up and down along a \( \tau \)-branch of \( T \) is equivalent to the following. \( \exists p \in \mathbb{N}, t_1, t_2 \in \mathbb{T} : t_2 = \tau^p(t_1) \) and \( \nu(t_1) = \nu(t_2) = i \) and \( \nabla x(t_1)i = -\nabla x(t_2)i \). \( \square \)

Causality trees. A \( \tau \)-tree is a set of \( \tau \)-branches sharing their smallest time step. The smallest time step of a maximal \( \tau \)-tree is a root step. There are as many maximal \( \tau \)-trees as there are root moves, so no more than \( |U(0)| \leq n \).

The following result is a second consequence of Lemma 2

Lemma 4. Same conditions as Lemma 1
If there exists an automaton \( i \) moving up and down on the same \( \tau \)-tree of \( T \), then \( \mathcal{N} \) is not nice : either it is non-monotone or there are contradictory paths in \( G \) (from \( \nu(t_0) \) to \( i \) where \( t_0 \) is the smallest time-step of the \( \tau \)-tree).

Proof: The existence of an automaton \( i \) moving up and down on a \( \tau \)-tree of \( T \) is equivalent to the following. \( \exists p, q \in \mathbb{N}, t_0, t_1, t_2 \in \mathbb{T} : t_0 = \tau^p(t_1) = \tau^q(t_2) \) and \( \nu(t_1) = \nu(t_2) = i \) and \( \nabla x(t_1)i = -\nabla x(t_2)i \). \( \square \)

Lemma 5. Same conditions as Lemma 1
If \( \mathcal{N} \) is a nice BAN and if an automaton makes twice the same move on a \( \tau \)-branch of \( T \), then there are at least two \( \tau \)-trees in \( T \) and a positive cycle in \( G \).

Proof: If automaton \( i \) goes up twice on the same branch (making move \( \nabla_i \)), then, by Lemma 2, it belongs to a cycle which must be positive because of \( \mathcal{N} \)’s niceness. And \( i \) must go back down once (making move \( -\nabla_i \)) between every time it goes up. The niceness of \( \mathcal{N} \) and Lemma 3 imply that this must happen on another tree. \( \square \)

If \( \mathcal{N} \) is totally positive, then there are only two types of trees : (i) trees comprised of \( \nabla x(t)_i = 1 \) moves, i.e. trees on which automata in state 0 move to state 1 and (ii) trees comprised of \( \nabla x(t)_i = 0 \) moves, i.e. trees on which automata in state 1 move to state 0.
$G_{\tau}$ and the anti-graph of the $\tau$ function. By definition of $\tau$, $\forall t \in \mathbb{T}$ there is at most one $t' \in \mathbb{T}$ satisfying $t' = \tau(t)$. We can therefore consider the graph of function $\tau^{-1}$ a.k.a. anti-graph of function $\tau$. This is the digraph with node set $\mathbb{T}$ and arc set $\{(\tau(t), t, t \in \mathbb{T})\}$. Let us call strongly acyclic a digraph whose undirected version is acyclic. The anti-graph of $\tau$ is strongly acyclic. A $\tau$-tree is a connected component of the anti-graph of $\tau$. A $\tau$-branch is a linear sub-graph of the anti-graph of $\tau$. We can also naturally define $G_\tau = (V, A_\tau)$ where $A_\tau = \{ (\nu(\tau(t)), \nu(t)), t \in \mathbb{T} \}$. By Lemma 4, $G_\tau$ is a subgraph of the interaction graph $G$.

3 Hamiltonian shortest trajectories

Hamiltonian shortest trajectories. A Hamiltonian shortest trajectory of a BAN $\mathcal{N}$, is a shortest trajectory $\mathcal{T}$ of $\mathcal{N}$ that goes from one of its configurations $x$ to another $y$ by going through all other configurations of $\mathcal{N}$. A Hamiltonian shortest trajectory of a BAN of size $n$ has length $T = 2^n$.

Proposition 1. If a BAN has a Hamiltonian shortest trajectory then it is not a nice BAN.

Proof: Let $\mathcal{T} = (x(t))_{t \in \mathbb{T}}$ be a Hamiltonian shortest trajectory of $\mathcal{N}$. Let $i = \nu(0)$ be the first automaton to make a move along $\mathcal{T}$. If $\exists j \in U(\mathcal{T}) \setminus \{i\}$, then this $j$ can make a move in configuration $x(0)$ and $x(0) \rightarrow x(0)^j$ is a transition that $\mathcal{N}$ can make, which shortcuts $\mathcal{T}$. Indeed, since $\mathcal{T}$ is Hamiltonian, $\exists t \in \mathbb{T}$ such that $x(t) = x(0)^j$. And thus the transition $x(0) \rightarrow x(0)^j = x(t)$ takes $\mathcal{N}$ straight from $x(0)$ to $x(t)$ without going through configuration $x(1) = x(0)^j \neq x(0)^j = x(t)$. But $\mathcal{T}$ cannot be shortcut if $\mathcal{T}$ is a shortest trajectory. Thus $U(\mathcal{N}) = \{i\}$. $\mathcal{T}$ has a unique root, and thereby a unique $\tau$-tree. Proposition 1 results from this, from Lemma 4, and from the fact that along a Hamiltonian trajectory (which has length $2^n > n$) there necessarily are automata moving up and down.

This settles the case of the longest type of long shortest trajectories: they only exist in non-nice BANs. Still, nice BANs can have long (non-Hamiltonian) shortest trajectories as the following example shows.

Example 1. $\mathcal{N} = \{f_i, i \in V = \{1, \ldots, 5\}\}$ where:

\[
\begin{align*}
  f_1 &= x_4 \land x_5 \\
  f_2 &= x_1 \lor x_2 \\
  f_3 &= (x_1 \lor x_2) \land x_4 \\
  f_4 &= x_3 \\
  f_5 &= x_1 \lor x_3 \lor x_4
\end{align*}
\]

Let $x = (x_1, \ldots, x_5) = (1, 0, 1, 1, 0)$ and $y = (0, 1, 0, 0, 0)$. To get from configuration $x$ to configuration $y$, $\mathcal{N}$ has to move automaton 1 up twice and down once. There is no other way:

$x = x(0) = (1, 0, 1, 1, 0) \xrightarrow{x_4} x(1) = (0, 0, 1, 1, 0) \xrightarrow{x_5} x(2) = (0, 0, 0, 1, 0) \xrightarrow{x_3} \ldots$ $\ x(3) = (0, 0, 0, 1, 1) \xrightarrow{x_5} x(4) = (1, 0, 0, 1, 1) \xrightarrow{x_3} x(5) = (1, 1, 0, 1, 1) \xrightarrow{x_3} \ldots$ $\ x(6) = (1, 1, 0, 0, 1) \xrightarrow{x_5} x(7) = (0, 1, 0, 0, 1) \xrightarrow{x_5} x(8) = (0, 1, 0, 0, 0) = y$

Remarkably, $\mathcal{N}$ is a totally positive BAN, and $y$ is a stable configuration ($S(y) = V$).
4 Causality (versions 0.2 and 0.3)

Assume move $\nabla x(t_1)_j$ is made by $j$ at time $t_1 = \tau(t_2)$, thereby unlocking the possibility of move $\nabla x(t_2)_i$ to be made by $i$ at time $t_1 + 1$. Assume move $\nabla x(t_1)_j$ is undone before move $\nabla x(t_2)_i$ is done. In other terms, assume move $-\nabla x(t_1)_j$ is done at some time $t_3 \in [t_1, t_2]$. Then, for the interval of time $[t_3, t_2]$, move $\nabla x(t_2)_i$ remains possible despite $j$ no longer being in the state that unlocked the possibility of this move. The notion of $\tau$-causality is not such a good notion of causality because of this. It allows for a caused effect to remain caused even when its cause is no longer effective. In particular, with $\tau$-causality, a move causing nothing can be made on a shortest trajectory, and later undone. It isn’t clear why such a move would ever be made in the first place. $\tau$-causality does not help understand why such a move couldn’t be skipped to make the trajectory shorter. Let us call target move any move $\nabla x_i$ where $i \in HD(x, y) = \{i \in V : x_i \neq y_i\}$. And let us call target-move-causing move a move that leads to a target move in a causality branch. We would like a notion of causality that allows us to say that shortest trajectories make no other moves that target moves and target-move-causing moves. Then, we could say that there are as many causality branches as there are causality leaves (branch endpoints), and as many of those as there are target moves, i.e. $|HD(x, y)| \leq n$.

A second notion of causality, $\kappa$-causality, can be proposed both as an alternative and a complementary to $\tau$-causality. In particular, this new version pinpoints the reason why, although $\tau$-causing nothing, a move may still be indispensable because it $\kappa$-causes something that is indispensable.

Causality 0.2. Let $t \in \mathbb{I}$ and $i = \nu(t) \in V$. $\nabla x(t_1)_i$ is the move made at time step $t_1$ by $i$. $\kappa(t)$ is the set of time steps at which are made the moves causing (favouring) move $\nabla x(t_1)_i$ in the following sense :

$$t \in \kappa(t_1) \iff \forall s \in [t, t_1], \nu(s) \neq j \text{ and } i \in U(t_1) \cap S(x(t_1)_j).$$

Notably, all lemmas above given in terms of $\tau$-causality translate straightforwardly in terms of $\kappa$-causality because Lemma 1 does.

With this definition, any move of a trajectory $\mathcal{T}$ that $\kappa$-causes nothing can be skipped.

The drawback of this definition is that a move can have several immediate causes, so it does not yield acyclic causality trees.

Causality 0.3. A third version of causality considers that anything is a cause if it is a cause either by the first version of causality or by the second version.

In the next section, we concentrate on monotone BANs and take a different point of view on trajectories.
§30 **Source Automata.** In the sequel, we are going to assume that \(\forall i \in V : \text{deg}^-(i) > 0\). The reason is the following. For the sake of keeping notations and developments simple, we want to avoid having a source automaton \(i\) whose local transition function is constant. Unlike with automata that have in-neighbours, the potential for a source automaton \(i\) to change states is not carried by any automaton of \(V\). Nor is it transmitted through any arc of \(A\). Let \(i \in V\) be an arbitrary source automaton and let \(f_i : x \in \mathbb{B}^n \rightarrow b\) be its local transition function. Automaton \(i\) can change states only when its current state is \(\neg b\). So on an arbitrary trajectory \(T = (x(t))_{t \in \mathbb{T}}\), automaton \(i\) moves if and only if \(x(0)_i = \neg b\). If that is the case, \(i\) moves at most once. And when it does, it doesn’t inherit its new state from any automaton of \(V\). Its local transition function is the sole responsible actor of the change. Here, we disregard BANs where such automata exist. Importantly, all targeted results will apply nonetheless to these BANs because they are equivalent to BANs in which each source automaton \(i\) is replaced and represented by two automata \(i_1\) and \(i_2\) where \(f_{i_1}(x) = f_{i_2}(x) = x_i\) and where initially in \(T\), \(i_1\) is in state \(x(0)_{i_1} = f_i(x) = b\) and \(i_2\) is in state \(x(0)_{i_2} = x(0)_i\).

In BANs without real source automata, we call (positive) source automaton any automaton \(i \in V : V_{\rightarrow i} = \{i\} \land \text{sign}(i, i) = +1\). And we call source loop the cycle defined by the single positive arc \((i, i) \in A\). Source automata never change states.

We let \(V^* \subset V\) be the set of non-source automata. Those are the only ones that possibly move along \(T : V^* \supset \bigcup_{t \in \mathbb{T}} U(t) \supset \{\nu(t) : t \in \mathbb{T}\}\).

§31 **Potentials.** We are now going to consider couples \((t, i) \in \mathbb{P} = \mathbb{T} \times V\). Couple \(\pi = (t, i)\) is called the potential carried by automaton \(i\) at time \(t\). \(\pi\) is taken to represent the fact that automaton \(i\) is in state \(x(t)_i \in \mathbb{B}\) at time \(t\). And the idea of this section is to trace back time in order to find and relate all anterior potentials \((t', j), t' < t\) that have participated in \((t, i)\).

§32 **Original potential.** Original, or initial potentials are potentials of the form \((0, i)\) for some \(i \in V\).

§33 **Equality among potentials.** We define a binary reflexive and transitive relation = on \(\mathbb{P}\) to signify that (a) as long as an automaton \(i\) is not moved, the potential it carries is the same, and (b) two different automata cannot carry the same potential, even at different times:

\[
(t, i) = (t', j) \iff i = j \text{ and } \forall s \in \llbracket t, t' \rrbracket : i \neq \nu(s).
\]

§34 **Transmission and inheritance.** We define the another binary relation on \(\mathbb{P}\) denoted \(\times\). Let \(\pi_1 = (t_1, j)\) and \(\pi_2 = (t_2, j)\) be two potentials. \(\pi_2 \times \pi_2\) means that \(\pi_2\) is inherited from \(\pi_1\) and \(j\) transmitted his potential to \(i\) in the following sense: \(\forall t \in \mathbb{T}, \forall j \in V, \forall i = \nu(t) \in V,\)

\[
(t, j) \times (t + 1, i) \iff \text{sign}(j, i) = -\nabla x(t)_j \cdot \nabla x(t)_i = \nabla x(t)_j \cdot \nabla x(t + 1)_i
\]

\(\times\) this \(j\)-move \(\times\) this \(i\)-move
\(\times\) towards \(x(t)\) \(\times\) away from \(x(t)\).

\[
\not\equiv x(t)_j \rightarrow_i = \text{SB}(- \text{sign}(j, i)) \nabla x(t)_j = \text{SB}(- \nabla x(t + 1)_i)
\]

\[
\not\equiv x(t)_j \rightarrow_i = x(t + 1)_i.
\]

\(\equiv\) \(i\) being in this state \(\equiv\) \(i\) being in this one.
By the remark made at the end of §36, we have the third implication below:

\[
\begin{align*}
i = \nu(t) & \implies i \in U(t) \\
& \implies f_i(x(t)) = \neg x_i(t) = x_i(t+1) \\
& \implies \exists j \in V_{\rightarrow i} : x_{j \rightarrow i}(t) = x_i(t+1) \iff \exists j \in V_{\rightarrow i} : \langle t, j \rangle \bowtie \langle t+1, i \rangle.
\end{align*}
\]

Thus, no automaton can change states without inheriting.

To define a relation of inheritance, we could require much stronger conditions than those implied by this definition of $\bowtie$. For instance, let us call 0-prime implicant (resp. 1-prime implicant) of $f : \mathbb{B}^n \to \mathbb{B}$ a prime implicant of $\neg f(x)$ (resp. of $f(x)$). For $\langle t+1, i \rangle$ to inherit from $\langle t, j \rangle$, we could require that $x(t)_{j \rightarrow i}$ be involved in a $x(t+1)_{i'}$-prime implicant of $f_i$. This new, stronger requirement would imply the looser one of §35. But for now, we keep the looser version of §35 because it is lighter to manipulate.

$\bowtie^*$ denotes the transitive and reflexive closure of $\bowtie$. When $\pi_1 \bowtie^* \pi_2$ holds, we also say that $\pi_2$ inherits from $\pi_1$.

**Lemma 6.** \(\forall i, j \in V \text{ and } \forall t, t' \in T, \text{ if } \langle t_1, j \rangle \bowtie^* \langle t_2, j \rangle \text{ holds, then there is a path of sign } \nabla x(t_1)_j \cdot \nabla x(t_2)_i \text{ in } G \text{ from } j \text{ to } i.\)

Proof: By induction on the length of the lineage, using the definition of $\bowtie$ in §35 and the definition of equality amongst potentials in §34. □

**Potential representatives.** For any potential $\pi$ and any time $t$, we define

\[
\mathcal{R}^*(\pi, t) = \{ j \in V : \pi \bowtie^* \langle j, t \rangle \}
\]

as the set of automata that represent or carry $\pi$ a time $t$. When $\pi = \langle 0, i \rangle$ is an original potential, we rather write this set $\mathcal{R}^*_i(t)$:

\[
\mathcal{R}^*_i(t) = \mathcal{R}^*(\langle 0, i \rangle, t).
\]

**Potential Charge.** For any automaton $i \in V$ and any time step $t$,

\[
\begin{align*}
\mathcal{P}(i, t) &= \{ \pi : \pi \bowtie \langle t, i \rangle \} \\
\mathcal{P}^*(i, t) &= \{ \pi : \pi \bowtie^* \langle t, i \rangle \} = \{ \pi : i \in \mathcal{R}^*(\pi, t) \}
\end{align*}
\]

denotes the set of potential carried by $i$ at time $t$, and

\[
\begin{align*}
\mathcal{P}_0(i, t) &= \{ \langle 0, j \rangle \in \mathcal{P}(i, t) \} \\
\mathcal{P}^*_0(i, t) &= \{ \langle 0, j \rangle \in \mathcal{P}^*(i, t) \} = \{ \langle 0, j \rangle : i \in \mathcal{R}^*_j(t) \}
\end{align*}
\]

denotes the set of original potential carried by $i$ at time $t$, and

\[
\mathcal{P}^*_0(t) = \bigcup_{i \in V} \mathcal{P}^*_0(i, t)
\]

denotes the set of original potential still represented in the BAN at time $t$.

**Lemma 7.** From the first time $i \in V$ is updated onwards, $i$ carries potential:

\[
i = \nu(t) \implies \forall s > t : \mathcal{P}(i, s) \neq \emptyset.
\]

Proof: §36. □
Note that an automaton \( i \) can inherit the same potential \( \pi \) several times, at different time steps. In Example 1, automaton 1 inherits potential \((0,5)\) at times steps 1 and 7 : \( 1 \in R^*_5(1) \cap R^*_5(7) \). Indeed we have : \( \langle 0,5 \rangle \bowtie \langle 1,1 \rangle \) and \( \langle 0,5 \rangle \bowtie \langle 1,1 \rangle \bowtie \langle 2,3 \rangle \bowtie \langle 6,4 \rangle \bowtie \langle 7,1 \rangle \).

**Lemma 8.** In a nice BAN, an automaton \( i \in V \) always makes the same move at times it inherits the same potential \( \pi : \)
\[
x_i(t_1) \neq x_i(t_2) \implies i \notin R^*(\pi, t_1) \cap R^*(\pi, t_2).
\]

Equivalently :
\[
x(t_1)_i \neq x(t_2)_i \implies \mathcal{P}^*(i, t_1) \cap \mathcal{P}^*(i, t_2) = \emptyset.
\]

Proof: Let \( \pi = \langle t, j \rangle \). By definition of \( R^* \), \( i \in R^*(\pi, t_1) \cap R^*(\pi, t_2) \) implies \( \pi \bowtie (t_1, i) \bowtie \pi \bowtie (t_2, i) \). By Lemma 6 this implies there is a path of sign \( \nabla x(t), j, \nabla x(t_1) \rangle \) from \( j \) to \( i \), as well as a path of sign \( \nabla x(t), j, \nabla x(t_2) \rangle \) in \( G \). The BAN being nice, these paths must have the same sign so \( \nabla x(t_1) \rangle = \nabla x(t_2) \rangle \) must hold. \( \square \)

Let us also note that several potentials can be transmitted to the same automaton at once. In Example 1 because \( f_1(x) = x_4 \land x_5 \), this happens to automaton 1 every time it moves up to state 1 from state 0. For instance, at time \( t = 4 \), it inherits the potentials that automata 4 and 5 were carrying at time \( t = 3 : \langle 3,4 \rangle \bowtie \langle 4,1 \rangle \) and \( \langle 3,5 \rangle \bowtie \langle 4,1 \rangle \).

An automaton can also inherit a potential several times at once through different lineages. This is the case with automaton 5 and potential \((0,5)\) in Example 1. Indeed \( (0,5) \bowtie \langle 1,1 \rangle \bowtie \langle 2,3 \rangle \bowtie \langle 8,5 \rangle \) and \( (0,5) \bowtie \langle 1,1 \rangle \bowtie \langle 2,3 \rangle \bowtie \langle 6,4 \rangle \bowtie \langle 8,5 \rangle \).

**Lemma 9.** Each time an automaton moves, it inherits potential that it never represented in the past. Formally, \( \forall t \in \mathbb{T}, i \bowtie (t-1) \) satisfies :
\[
\mathcal{P}(i, t) \setminus \bigcup_{t' < t} \mathcal{P}(i, t') \neq \emptyset.
\]

Proof: Because of the equality relation among potentials, we can concentrate on time steps at which \( i \) is updated. Let \( T_i^+ = \{ t' < t : i \bowtie (t' - 1) \wedge x(t')_i = x(t)_i \} \) (resp. \( T_i^- = \{ t' < t : i \bowtie (t' - 1) \wedge x(t')_i = \neg x(t)_i \} \) be the set of dates before \( t \) at which \( i \) moves away from (resp. towards) state \( x_i(t) \). Let \( t^+ = \max T_i^+ \). Since \( x_i(t^+) = x_i(t) \) and \( i \bowtie (t^+ - 1) = \bowtie (t - 1) \) and \( t^+ < t \), there must be a time step \( t^- \in T_i^- \cap \lbrack t^+, t \rbrack \). Indeed, at both time steps \( t^+ - 1 \) and \( t - 1 \), \( i \) moves away from state \( x_i \). There must be a time step in between at which \( i \) moves towards \( x_i \). And actually, this time step necessarily is \( t^- = \max T_i^- - 1 \). Let \( W_i^+ = \{ j \in V_{i'} : x_j(t^- - 1) = x_i \} \) be the set of neighbours of \( i \) that favour the move \( i \) makes at time \( t^- - 1 \) towards \( x_i \). If none of those neighbours change states between \( t^- \) and \( t \), then \( i \) has no incentive to move again. But it does at time \( t - 1 \). So one of those neighbours \( j \in W_i^- \) must have moved back so that \( x_j(t^- - 1) = \neg x_i \). Then, \( \pi = (t - 1, j) \bowtie (t, i) \). And since \( \pi \) is born between \( t^- \) and \( t \), it cannot have been inherited by \( i \) any time before \( t \). \( \square \)

Lemma 9 is going to be very useful because it allows to match injectively each move of automaton \( i \) on \( T \) to a path of \( G \) corresponding to the lineage of a brand new potential it inherits then. If the BAN contains no other cycles than source loops, then this limits the number of times an automaton needs be updated because it limits the number of times the automaton can move. Precisely, in this case, Lemma 9 implies that no automaton \( i \) can move a
number of times that is greater than the length of the longest loopless path
going through $i$ (recall that source automata don’t move at all).

If there are real cycles in $G$ (not positive source loops) however, Lemma 9 does not limit the total number of times an automaton moves along a shortest trajectory.

§44 Loosing potential. A potential $\pi$ can cease to be represented in the BAN: $R^*(\pi, t) = \emptyset$ for some $t$. When that happens, the potential is obviously lost for good: $R^*(\pi, t) = \emptyset \implies \forall t' \geq t, R^*(\pi, t') = \emptyset$. This happens when at some point, there is only one automaton representing potential $\pi$, and this automaton is updated before it gets a chance to transmit its potential to one of its out-neighbours. In Example 4 at time step 0, automaton 1 is updated and the potential $\langle 0, 1 \rangle$ is lost for good.

§45 Survivor potential. We call survivor potential any potential $\pi$ that is represented in the destination configuration $y$: $R^*(\pi, T) \neq \emptyset$. Unless $(\nu(0), \nu(0)) \in A$ is a negative loop over the first automaton that moves in $T$, the potential $\langle 0, \nu(0) \rangle$ is lost, so among the original potentials, there are no more than $n - 1$ survivors.

**Lemma 10.** On a shortest trajectory $T$, at each time step, only survivor potential is represented and transmitted.

Proof: Let $t < T$ be the last time step before $T$ where non-survivor potential is inherited. Note that $T - t = 1$ is not possible unless $\nu(t - 1) = \nu(T - 1)$. A shortest trajectory doesn’t update twice the same automaton in a row. Generally, let $i = \nu(t - 1)$. There necessarily exists $t - 1 < t' - 1 < T$ s.t. $i = \nu(t' - 1)$. Otherwise, $i$ carries the same non-survivor potential until the end which contradicts the definition of survivor potential. Since no non-survivor potential is transmitted after $t - 1$, no automaton inherits the non-survivor potential $i$ is carrying at time $t$. In other terms, $i$ could just as well not have changed states and inherited this potential, it wouldn’t have bothered any other automaton that was supposed to change states after $t$. A shortest trajectory wouldn’t update $i$. □

§46 Updates and Effective updates (moves). From the fact that once lost, an (original) potential can never be recovered, it is tempting to derive that recurrent configurations cannot lose (original) potential. However, what (original) potential is present in a configuration depends on the trajectory that lead to the configuration and not just on this configuration itself. Here, we need to emphasise the difference between $i$ making move $\nabla x(t)_i$ from configuration $x(t)$, and $i \in V$ merely having its state updated in configuration $x(t)$. When $i \in U(t)$, the update of $i$’s state is equivalent to (/immediately results in) $i$ making move $\nabla x(t)_i$. When $i \in S(t)$, the update of $i$’s state is ineffective (in the sense that it does not result in $i$ moving), although still possible. The difference this latter kind of update makes is therefore not appreciable in the value of $i$’s state. It is, however, in terms of the potential $i$ carries.
Example 2. $\mathcal{N} = \{f_i, \ i \in V = \{1, \ldots, 4\}\}$ where:

$$
\begin{align*}
  f_1 &= x_1 \\
  f_2 &= x_2 \\
  f_3 &= x_1 \\
  f_4 &= (x_1 \wedge x_3) \lor x_2
\end{align*}
$$

Consider the following series of updates, the first two of which define a trajectory $T$ from $x = x(0) = (1, 1, 0, 0)$ to fix point $y = x(2) = (1, 1, 1, 1)$:

$$
x(0) = (1, 1, 0, 0) \xrightarrow{1} x(1) = (1, 1, 0, 1) \xrightarrow{3} x(2) = (1, 1, 1, 1) \xrightarrow{4} x(3) = (1, 1, 1, 1).
$$

Consider also the following series of updates, the first and only two of which also define a trajectory $T'$ from configuration $x = z(0)$ to configuration $y = z(2)$:

$$
z(0) = (1, 1, 0, 0) \xrightarrow{3} z(1) = (1, 1, 1, 0) \xrightarrow{4} z(2) = (1, 1, 1, 1) = x(2).
$$

At the end of both series, all sets $R^*_i(t)$ have become definitely stable. The trajectory $T$ embedded in the first series, however, carries potential $(0, 3)$ along longer than is needed.

Example 2 proves that a trajectory might reach a recurrent configuration before it loses the possibility to loose potential. It remains the following question: What configurations other than recurrent configurations are there that have no ability to loose potential? In other terms: Having lost the possibility to loose potential, can $\mathcal{N}$ still move far enough away from a configuration and reach a point beyond which it can never get back to this configuration?

Let us extend our notations to allow for time steps at which no moves are made but some ineffective updates are made.

**Updates and Streamlines.** We have been defining trajectories $T = (x(t))_{t \in T}$ as series of configurations starting in an initial configuration $x = x(0)$ and ending in a final target configuration $x(T) = y$. We could just as well have defined them with a couple $(x, (\nu(t)))_{t \in T}$ comprised of the initial configuration $x$ and the series of automata $\nu(t)$ moving (being effectively updated) at each time step $t \in T$ to get from $x(t)$ to $x(t + 1)$. With the same kind of definition we can introduce formally *series of automata updates* aka *streamlines* : $(x, (\lambda(t)))_{t \in T}$ where $\forall t \in T, \lambda(t) \in V$. At time $t + 1$ of a streamline $\Lambda = (x, (\lambda(t)))_{t \in T}$, $\mathcal{N}$ is in the configuration $x(t + 1)$, where $\forall j \neq \lambda(t) : x(t + 1)_j = x(t)_j$ and automaton $i = \lambda(t)$ is in state $x(t + 1)_i = f_i(x(t)) = \begin{cases} x(t)_i & \text{if } i \in S(t) \\
-x(t)_i & \text{if } i \in U(t). \end{cases}$

The transmission relation $\bowtie$ extends naturally to streamlines with the following definition: $\forall t \in T, \forall j, i = \lambda(t) \in V, (t, j) \bowtie (t + 1, i) \iff \text{sign}(j, i) = \nabla x(t)_j \cdot \nabla x(t + 1)_i$.

From now on we assume this more general definition of $\bowtie$ (without loss of anything said before using the original definition).

**Beyond.** Consider all streamlines whose first $T$ steps are identical to those
of trajectory $T$. Among all these streamlines, consider those that are long enough to have reached a point $\hat{T} \geq T$ where the set of original survivor potential is stable (i.e. super). This must happen because, a streamline cannot indefinitely loose original potential. It must stop loosing original potential before it has none left.

§49 **Super survivor potential.** When $T$ leads to a recurrent configuration $y$, we call super survivor potential any survivor potential that lingers after *any* series of additional updates in $y$. In Example 2, among the set $\{(0,1), (0,2), (0,3), (0,4)\}$, of original potentials of $T$, three are survivor potentials of $T : (0,1), (0,2)$ and $(0,3)$, while only two are super survivors of $T : (0,1)$ and $(0,2)$.

In a recurrent configuration, super-survivor potential can be carried by automata with loops over them that have the ability of keeping their charge when they are updated. But if it is not, then super-survivor potential must be represented by at least two different automata.

By Lemma 10, from the very beginning, it is never useful to move an automaton that is on the verge of inheriting non-surviving potential. We would like to know if the same holds as well for non-super surviving potential:

**Conjecture 1.** *(If the only cycles in $G$ are source loops, then) from an arbitrary configuration $x$ to a recurrent configuration $y$, there is a shortest trajectory in which at each time step, only super survivor potential is transmitted.*

If Conjecture 1 isn’t true, then we would like to understand why. In other terms we would like to understand what is survivor potential that is not super survivor potential, what is the need for it, and where does it lie?

Let us point out that there can be several trajectories $T$ of the kind mentioned in Conjecture 1 leading to a recurrent configuration $y$. An example is the case of isolated cycles. The recurrent configurations of a cycle eventually only carry around 1 super-survivor potential. If there is any automaton to move in $x = x(0)$ to get to $y$, i.e. if $x \neq y$, and if the cycle is positive (resp. negative), then any one of the automata that are already in their target state (resp. any one of the $n$ automata) can serve as placeholders for the single original potential destined to survive in $y$ and beyond.

§50 **Attractor.** In the sequel, an attractor is a maximal set of recurrent configurations with trajectories going to an back each configuration in this set.

We take interest in what we refer to as shortest trajectories between configurations $x$ and attractors $A \subset \mathbb{E}^n$, a.k.a. “configuration-attractor shortest trajectories”. What we mean by this is shortest trajectories from $x$ to any of the configurations $y \in A$. Since the trajectories we consider are shortest trajectories, this implies $y$ to be among the configurations of $A$ that are the closest to $x$.

§51 **Depths and grounds.** In the sequel, given a set of automata $W_0$ that we call the *grounds* of $G$, we define level $d$ of $G$ as the set $W_d \subset V$ of automata whose longest path to nodes of $W_0$ have length $d$. The depth of automata is given by function $\omega : \forall i \in W_d, \omega(i) = d$. Starting from an arbitrary configuration $x$, if nodes of lower depth $d > 0$ can be moved before nodes of greater depth, and if all non-source nodes can be moved, then the original potential initially carried on the grounds is be survivor potential, spreading
The proof of the next lemma is very similar to proofs given in [2].

**Lemma 11** (Single paths and cycles). If $G$ is a single directed path, then:

1. The shortest trajectories between any two configurations have length at most $O(n^2)$.
2. The shortest trajectories between any configurations and any attractor have length at most $n$.
3. On configuration-attractor shortest trajectories, only one potential is survivor potential: the original potential carried on the grounds by the source automaton (the attractor is a stable configuration). And each automaton whose original state differs from that of the source automaton is updated exactly once. Other automata are not updated.

If $G$ is a positive cycle, then:

4. All the same holds. In Item 3, on the grounds, “source automaton” must be replaced by “any automaton whose initial state is already equal to its final state”.

If $G$ is a negative cycle then:

5. Again, Items 1, 2 and 3 still hold. In Item 3 “source automaton” must be replaced by “any automaton”.

6. From an arbitrary configuration, all recurrent configurations can be reached in at most $2n$ steps.

Proof: Item 3 is immediate. Item 2 is the immediate consequence of Item 3. Item 1 follows from the following. There are at most $n$ original potential that survives. None of them can get inherited more than once by the same automaton. Each of them can be inherited by at most $n$ automata. Item 4 comes from the fact that positive cycles behave just like single paths except for the following difference. In a positive cycle, any automaton can be the one that imposes its original potential to the others. And actually, if two automata are connected by a positive path, then eventually it will not matter which of them is chosen. A positive cycle has two alternatives. A single path has only one. Other than that both types of structures behave exactly the same way on shortest trajectories.

All cases of Lemma 11 can be seen in terms of dropping original potential down a single path. In all cases, no potential that eventually disappears needs ever be transmitted. And informally, if an original potential travels all around the path, then either that is because this potential has ridden the whole BAN from all other original potential. Or this potential and others have been cycling unnecessarily around the cycle. There are only two cases in which the cyclic nature of the BAN’s structure really counts: in the case of a positive cycle, in the choice mentioned in the proof above, and in the case of a negative cycle, in the cyclic attractor.

**Important Remark**: This example suggests that on the way to a recurrent configuration, a potential has no need to go twice through a path that transmits it without transformation.

**Lemma 12** (Acyclic $G$). If $G$ is acyclic (except for the source loops), then shortest trajectories to attractors have length at most $n$. And in this case, there is only one attractor which is a stable configuration.

Proof: Define the grounds as the set of source automata, and 

□
In the sequel, $\nabla_i = -\nabla y_i$ denotes a move towards $y$, the ultimate move that $i$ must have made (or at least not un-made) on $T$.

(Dis)Favourable Potential. First, we say that a potential $\pi = \langle t, j \rangle$ is favourable (resp. disfavourable) to automaton $i$ or to move $\nabla_i$ if all paths from $j$ to $i$ have $\text{sign}^{*}(j, i) = \nabla x(t)_j \cdot \nabla_i$ (resp. $\text{sign}^{*}(j, i) = -\nabla x(t)_j \cdot \nabla_i$). Nice BANs are the ones where $\forall \pi, \forall i, \pi$ is either favourable or disfavourable to $i$. A potential $\pi$ that is neither favourable nor disfavourable to $i$ is said to be undecided to $i$. Nice BANs have no undecided potential.

Lemma 13. In $y$, an automaton $i$ only represents survivor potential that is favourable to $i$.

Proof: By definition of $\times$ and of favourable. \hfill $\square$

(Dis)Favourable Neighbours. We introduce the following two sets relative to trajectory $T$:

$$
A_T^+ = \{(j, i) \in A : \text{sign}(j, i) = +\nabla_j \nabla_i\}
$$

$$
A_T^- = \{(j, i) \in A : \text{sign}(j, i) = -\nabla_j \nabla_i\}.
$$

In-neighbours $j : (i, j) \in A_T^+$ (resp. $\in A_T^-$) are called favourable (resp. disfavourable) in-neighbours of $i$. They having already (resp. them having not yet) made move $\nabla_j$ and already being in state $y_j$ (resp. still being in state $\neg y_j$) can only favour $i$ in making move $\nabla_i$. The following Lemma is thus about the most favourable conditions for $i$ to make move $\nabla_i$.

Lemma 14. $\forall i \in V, \forall z \in \mathbb{B}^n : \begin{cases} z_j = y_j, & \forall j : (i, j) \in A_T^+ \\ z_j = \neg y_j, & \forall j : (i, j) \in A_T^- \end{cases} \implies i \in U(z)$.

Note that a disfavourable neighbour acts favourably when it is in state $\neg y_k$. And a favourable neighbour acts disfavourable when it is not in state $y_j$.

Cycles and disfavours. Note also that it is not possible to have cycles in $G$ comprised solely of arcs in $A_T^+$. And a cycle comprised of an even (resp. odd) number of arcs in $A_T^-$ is necessarily a positive (resp. a negative) cycle .

Lemma 15. Assume that $\forall i \in V, i$ either favours all his out-neighbours, or disfavours them all. Then, $T$ (a shortest trajectory) has length at most $n$.

Proof: As long as there are unstable automata of the first kind (that favour all their out-neighbours) that are in state $\neg y_i$, those can be moved without adding any disfavour to any move $\nabla_j$ that still needs to be made. When we get to the point where all unstable automata that need to move are of the second kind (disfavouring all their out-neighbours), all their out-neighbours necessarily already are in state $y_j$. \hfill $\square$

Lemma 16. If $A_T^- = \emptyset$ and $T$ is a shortest trajectory, then $T$ (a shortest trajectory) has length at most $n$. And in this case, if $T$ is a shortest trajectory to an attractor, the attractor is a stable configuration, one of at most two if $G$ is strongly connected (eg if $G$ is a positive cycle).

Proof: No automaton $i$ has disfavourable neighbours. No move $\nabla_j$ can disfavour $\nabla_i$. And no move $\neg \nabla_j$ will favour any move $\nabla_i$. So only moves $\nabla_i$ need be made. And all $\nabla_i$ moves can be made.

The second part of Lemma 16 comes from the fact that once all automata $i$ have made move $\nabla_i$, none of them can make move $\neg \nabla_i$ since none of them has in-neighbours favourable to that move. \hfill $\square$
Favour Graph. We define graph $H^T = (V, A^H)$ where $A^H = A^T_+ \cup \{(i, j) : (j, i) \in A^T_\downarrow\}$.

Lemma 17. If $H^T$ is acyclic except for loops, and if no automata $i$ are initially already in their destination state $y_i$, except possibly the source automata, then $T$, a shortest trajectory from $x$ to $y$ has length less than $n$.

Proof: We let the grounds of $H^T$ be the set of source automata. We let $\mu : V \to [1, n]$ be an injective function satisfying $\forall (j, i) \in A^H, i \neq j, \mu(j) < \mu(i)$ and $\forall i, j \in V, \omega(j) < \omega(i) \implies \mu(j) < \mu(i)$. This function exists because $H^T$ is acyclic. We show that $\forall t, i = \mu(t) \in U(t)$ by induction on $t$, by showing that the conditions of Lemma 14 are maintained. Thus we can let $\nu = \mu^{-1}$ so that at each time step $t < n$ the non-source automaton $\nu(t)$ is updated.

Lemma 18. If $H^T$ is acyclic except for loops, and if $y$ is stable configuration $T$, a shortest trajectory from $x$ to $y$ has length less than $n$.

Proof: We apply the same process as in the proof of Lemma 17: we define the source automata to be the grounds, and move automata from shallowest to deepest. However, in this case does the process does not guarantee maintaining the ideal conditions of Lemma 14 because some disfavourable in-neighbours $k$ of $i$ might already be in state $y_k = x_k$. Lemma 18 is proven by comparing the inputs of $i$ on an arbitrary configuration $x(t)$ of $T$, and the inputs of $i$ in the stable configuration $y$. In both $x(t)$ and $y$, all favourable in-neighbours of $i$ are in their favourable state (in $x(t)$ this is because of the order according to which we are moving automata). The difference between the two situations is that in $x(t)$, not all disfavourable neighbours of $i$ are in their disfavourable state. Thus, if $f_i(y) = y_i = \neg x_i(t)$ (and it is), then $f_i(x(t)) = y_i = \neg x_i(t)$ and $i$ can indeed be moved in that configuration.

In Lemma 18, $H^T$ is acyclic. $G$ isn’t necessarily. The source automata of $H^T$ aren’t necessarily source automata of $G$. Something must be keeping them in their state $y_i$. Since they are source automata of $H^T$, all their in-neighbours distinct from themselves are disfavourable. Thus, there must be a positive loop over each one of them.

Lemma 19. If $H^T$ is acyclic except for loops (eg if $G$ is a negative cycle), then $T$ a shortest trajectory from $x$ to an attractor $A$ has length no greater than $n$.

Proof: Here, we apply the same process as before: we define the source automata of $H^T$ to be the grounds, and move automata from shallowest to deepest. However, here again, we cannot guarantee the ideal conditions of Lemma 14 as in Lemma 17. Nor can we rely on the stability of the destination configuration $y$ as in Lemma 18. Instead we are going to exploit the following flexibility we have. $T$ needs to end on a configuration $y$ of $A$, and any configuration $y \in A$ will do. We will exploit this flexibility by allowing ourselves to change our target configuration $y$ along the way. More precisely, we initialise our target to a certain $y = y(0) \in A$. Then, at each time step we update our target if we know of a closer $y(t) \in A$ belonging to the same attractor and that can be reached from $x(t)$. The idea is to maintain at each time step $t$, the property that the automaton $i = \nu(t)$ that we are considering for update has all its favourable in-neighbours $j$ in state $y_j(t)$. So we want to ensure that (i) $\forall t, x = x(0)$ can indeed reach $x(t)$, and (ii) either $HD(x(t), y(t)) > HD(x(t + 1), y(t + 1))$, or no move is made at time $t$ in which case $HD(x(t), y(t)) = HD(x(t + 1), y(t + 1))$ and step $t$ of the process doesn’t count as a step of the trajectory.
By definition, the source automata $i$ of $H^T$ have no favourable in-neighbours $j \neq i$, and thus no favourable in-neighbours that aren’t already in state $y(0)_j$.

Let $t$ be an arbitrary time step of the process where $i = \nu(t)$ is considered. There are three cases:

1. $i \in U(t)$ and $x(t)_i = y(t)_i$. In this case, $i$ is not moved we move on to $t + 1$ and to automaton $\nu(t + 1)$.

2. $i \in U(t)$ and $x(t)_i \neq y(t)_i$. In this case, $i$ is updated to state $y_i$. We let $x(t + 1) = \overline{x(t)}$ be the configuration that is reached from $x(t)$ by moving $i$, and we maintain the target $y(t + 1) = y(t)$, so that $H \overline{D}(x(t + 1), y(t + 1)) = H \overline{D}(x(t), y(t)) - 1$.

3. $i \in S(t)$ and $x(t)_i \neq y(t)_i$. In this case, we cannot move $i$ to get closer to our current target. So we let $x(t + 1) = x(t)$ and change our target to $y(t + 1) = \overline{y(t)}$ because of the following reasons. In $y(t)$, $i$ must be unstable ($i \in U(y(t))$). Indeed, in $y(t)$, all automata $j$ are in state $y(t)_j$. In particular, all in-neighbours $j$ of $i$ favouring $i$ being in state $y(t)_i$ are in state $y(t)_j$. They also already are in $x(t)$. As far as $i$ is concerned, the difference between $x(t)$ and $y(t)$ is that in $x(t)$ there are less disfavourable in-neighbours presently disfavouring $i$ being in state $y(t)_i$. Yet $i$ still cannot move to state $y(t)_i : f_i(x(t)) = x(t)_i = \overline{y_i}(t)$. This implies that in $y(t)$, the situation is even less favourable and $i$ cannot maintain state $y(t)_i : f_i(y(t)) = x(t)_i = \overline{y_i}(t)$. Thus $y(t + 1)$ is the configuration that is reached from $y(t)$ by moving the unstable automaton $i$. It is also a recurrent configuration belonging to the same attractor as the one $y(t)$ belongs to. And we have: $H \overline{D}(x(t + 1), y(t + 1)) = H \overline{D}(x(t), y(t)) - 1$.

The next lemma is redundant. Note that a nice BAN only has stable configurations as attractors since it contains no negative cycles.

**Lemma 20.** If $\mathcal{N}$ is nice, $H^T$ is acyclic except for strongly connected components comprised solely of arcs of $A^T_-$ then $T$ a shortest trajectory from $x$ to a stable configuration $y$ has length no greater than $n$.

Proof: Let $H^T_\overline{-}$ be the reduced version of $H^T$, identical to $H^T$ except that all strongly connected components (SCCs) have been reduced to a single node $c \in V$ with a loop over it. In this case, the loop can be considered as a favouring loop : $(c, c) \in A^T_\overline{-}$. The sets $A^T_+$ and $A^T_\overline{-}$ can be defined straightforwardly so that $A^H = A^T_+ \cup A^T_\overline{-}$. Indeed, because $\mathcal{N}$ is nice, and because of the assumption on the arcs of SCCs, the following holds. If $j$ and $k$ are two in-neighbours (resp. out-neighbours) of $i$ belonging to the same SCC of $H^T$, then they either both are favourable to $i$ or they are both disfavourable (resp. $i$ is either favourable to both or disfavourable to both). Thus, we can still order nodes of $H^T_\overline{-}$ as in the proofs of the previous lemma in order to move from shallowest to deepest. To extend this order to nodes of $H^T$, we use Lemma [16] for nodes of SCCs of $H^T$. This way, the automata belonging to source SCCs of $H^T$ are moved under the same conditions as the isolated SCCs are in the proof of Lemma [16]. Next come the automata that have in-neighbours in these source-SCCs. For them, everything happens as in the proof of Lemma [19]. And when comes the time to update a non-source SCC, then again our extended ordering of automata guarantees we can make the targeted moves as in the proof of Lemma [16] because the in- neighbourhood of the SCC is consistently favourable to every automaton in it.
Dropping potential. All proofs above about shortest trajectory lengths are based on a notion of depth and consist in “dropping” original survivor potential from the grounds to increasingly deep levels and “trampling” over the other potential causing its loss.

**Lemma 21.** If \( \mathcal{N} \) is nice, then \( T \), a shortest trajectory from \( x \) to a stable configuration \( y \) (only possible kind of attractor in a nice BAN) has length no greater than \( n^2 \).

**Proof:** In a nice BAN, there is only one direction an automaton \( i \) can move to favour a specific move \( \nabla_j \) of another automaton : \( \nabla_i \) is sign\( ^*(i, j) = \nabla_i \cdot \nabla_j \), and \(-\nabla_i \) is sign\( ^*(i, j) = -\nabla_i \cdot \nabla_j \). If \( T \) is a shortest trajectory, then on \( T \), every move that is made must lead to making at least one definite target move \( \nabla_j \) that will not be cancelled. That is to say the following. Let \( i = \nu(t) \) be an automaton that moves at time \( t \). Then among all automata that inherit \( \pi = \langle t + 1, i \rangle \) at a later time, some will never change states again after having made the move by which they inherited \( \pi \). Indeed, if all moved and lost \( \pi \) then \( i \) didn’t need to be moved at time \( t \) unless it was its very last move.

Thus, in a nice BAN, an automaton does not move more often then there are definite target moves to be made. \( \square \)

The cases of Lemma 21 that were not already covered by the previous lemmas – i.e. the cases of nice BANs that might have long shortest trajectories – are the cases in which the following holds : neither \( H^T \) nor \( G \) is acyclic, some SCCs of \( G \) contain arcs of \( A^T \), and some SCCs of \( H^T \) contain arcs in \( A^H \setminus A^T \).

I actually conjecture that shortest trajectories to attractors of nice BANs have length no greater than \( \mathcal{O}(n) \). They can be long but still, only a limited number of automata are updated several times, and those that are updated several times actually are updated only twice. In any case, insight is that Lemma 8 limits the number of times a potential is used by/goes through a specific automaton. Besides, an automaton only moves when it has new potential to transmit. Even less than that, it only moves when it is being inputted potential that is inconsistent with the one it is actually carrying.

At most, in the destination configuration \( y \), every automaton represents a different original potential. But this situation is impossible and situations that look like it are unlikely, especially since in a nice BAN, we expect this to mean that the consistent potential would have to be divided into isolated items benefiting only to one unique automaton.

The proof of Lemma 21 shows that in the case of a nice BAN \( T \) has at most \( n \) non-mutually-interfering tasks to carry out, namely to bring each of the \( n \) original potential to its single destination. And then, at most each of the \( n \) automata is involved no more than once in each of these \( n \) tasks. Note that to allow a task to move each automaton once is to allow it a lot because if \( n \) automata are moved in order to settle one automaton in its definitive state, then where is all the other original potential stocked in the meantime? We conjecture that a particularity of nice BANs is that looking at the furthest an original potential has travelled is informative because potential does not recede in them.

**Conjecture 2.** If \( \mathcal{N} \) is nice, then \( T \), a shortest trajectory from \( x \) to a stable configuration \( y \) has length no greater than \( \mathcal{O}(n) \).

The following example proves that monotone BANs that are not nice can have shortest trajectories that are significantly longer than \( n^2 \).
Example 3. \( \mathcal{N} = \{ f_i, \ i \in V = W \uplus W' \} \) where \( W = \{0, \ldots, \frac{n}{2} - 1\} \), \( W' = \{0', \ldots, \frac{n}{2} - 1'\} \) and :

\[
\begin{align*}
  f'_0 &= \neg x'0' \\
  f_0 &= x_0 \\
  f'_i &= \neg x_i \land \bigwedge_{j < i} x_j \land \bigwedge_{j < i} \neg x_j' \\
  f_i &= x_i \land \bigwedge_{j < i} \neg x_j
\end{align*}
\]

For any configuration \( x \in \mathbb{B}^n \) of this BAN where \( \forall i' \in W', \ x_{i'} = 0, \) we denote this configuration as follows : \( x = \sum_{i \in W} x_i \cdot 2^i \). It can be checked that starting from configuration \( [0] \), there is only one way to get automaton \( i \) to take state 1. First, get all automata \( j < i \) to take state 1. Then, thanks to this, make automaton \( i' \) take state 1. Once this is done, make all automata \( j < i \) go back to state 0 in order to allow \( i \) to finally make its move.

Let \( \alpha(i) \) be the minimal number of steps required to get from configuration \( x = [0] \) to configuration \( \pi^i = [2^i] \). The following can be checked :

\[
\begin{align*}
  \alpha(0) &= 3 \\
  \alpha(i) &= 3 + i + \sum_{j < i} \alpha(j) = 2\alpha(i - 1) + 1 = 2^{i+2} - 1 \\
  \alpha\left(\frac{n}{2}\right) &= 4 \cdot \sqrt{2^0} - 1
\end{align*}
\]

Let \( \beta(i) \) be the minimal number of steps required to get from configuration \( x = [0] \) to configuration \( \pi^{(j \leq i)} = [2^{i+1} - 1] \). The following can be checked :

\[
\begin{align*}
  \beta(0) &= 3 \\
  \beta(i) &= \alpha(i) + \beta(i - 1) = 2^{i+3} - (i + 5) \\
  \beta\left(\frac{n}{2}\right) &= 8 \cdot \sqrt{2^0} - (\frac{n}{2} + 5)
\end{align*}
\]

The BAN of this Example 3 is built as counter. It can be checked that to get from configuration \( x = [0] \) to configuration \( y = \pi^W = [2^\frac{n}{2}+1 - 1] \), it has no other choice than to go through each configuration of the \( 2^\frac{n}{2}+1 - 1 \) configurations \( [m] \), \( m \in [0, 2^\frac{n}{2}+1 - 1] \) through an essentially recursive process which is fed by the negative loop on automaton 0’ (this loop acts as a “stock” of potential that it can emit when needed) and hinged on the contradictory paths between each \( j < i \) and \( i \).

Our intuition that in nice BANs, the length of the longest shortest trajectories are much more limited than they are other BANs is based on the following. As Example 3 shows, in a BAN that is not nice, to make a particular automaton \( i \) take a certain state \( y_i \), it can require to implement a recursive process that takes an number of steps that is exponential in the size of the BAN, and exponential in the length of the longest path to \( i \) in \( G \). As mentioned at the beginning of this document, nice BANs can be reformulated as totally positive BANs. And in a totally positive BAN, it is never the case that moving an automaton \( j \) to state \( \neg y_i \) can help automaton \( i \) to take state \( y_i \). As a consequence, it never can take more than \( n \) steps to push automaton \( i \) to the desired state. In addition, nothing is really reversible in a nice BAN in the sense that no quantity of potential that is lost may be recovered. Providing \( i \) with the right potential \( \pi \) means transmitting this potential through a path \( P \) of \( G \). While this is done, all other potential initially lying on \( P \) have to be stocked and saved somewhere else in the BAN, or else they disappear for good.
Below is a list of remaining questions and remarks.

– What lengths have the shortest trajectories of totally positive BANs (that are not strongly connected)?
– Non-monotone BANs can have very long (Hamiltonian even) shortest trajectories, but can we tell by their definitions that they don’t?
– The long shortest trajectories of nice BANs seem to be “fragile” trajectories in the sense that there are many ways to leave the attraction basin of their target for good, and only a very specific order of moves that will ensure to stay in them. These trajectories are thereby not shortest trajectories in the sense that they shortcut longer trajectories. They seem to be the only trajectories (or at least one of the few) that have those particular endpoints. Can this remark be formalised and formally supported?
– Can we say more about the relation that inheritance (resp. original potential) entertains with \( \tau \)-branches (resp. \( \tau \)-roots)?

6 The non-monotone effect

In this section, we concentrate on nice, strongly connected BANs. In §6 we mentioned that these BANs can be reformulated as totally positive BANs. So in this section, we concentrate on totally positive, strongly connected BANs. Let us call them very nice BANs.

We still are considering a shortest trajectory \( T = (x(t))_{t \in \mathbb{T}} \) from \( x = x(0) \) to \( y = x(T) \).

Two types of Potential. \( \forall t \in \mathbb{T} \), we let \( 1 = \{ \pi = \langle t, i \rangle \in \mathbb{P} : x_i(t) = 1 \} \) and \( 0 = \{ \pi = \langle t, i \rangle \in \mathbb{P} : x_i(t) = 0 \} \).

The non-monotone effect (NME). We say that \( T \) involves a non-monotone effect if \( \exists i, j \in V \) such that there is a path from \( i \) to \( j \) comprised solely of arcs of \( A_{+1} \), and a path from \( i \) to \( j \) comprised solely of arcs of \( A_{-1} \).

Conjecture 3. Without a non-monotone effect, (i) the number of times an automaton moves on the same branch is limited somehow by the number of original ((super) survivor) potentials. (iii) Only an automaton involved in a non-monotone effect, with an outgoing arc from \( A_{+1} \) and an outgoing arc from \( A_{-1} \), could possibly need to move because of another reason than transmitting different original potential (indeed, an automaton \( i \) which disfavours (resp. favours) no automaton affects consistently each automaton that it affects: if \( i \) takes its target state before (resp. after) any other automaton does, by doing so, it will not disfavour (resp. fail to favour) any other automaton taking its own target state, unless its out-neighbours are not yet ready to take their target state and have other inconsistent potential to transmit before, in which case, we can say that \( i \) is just “following the movement”, as opposed to obeying constraints that weigh on itself).

Implementation of the NME. Now let us consider again any kind of BAN (nice, monotone or not). Having an NME always requires the presence of cycles. But in combination to this condition, there are several distinct notable ways to realise the NME in a BAN:

1. “First degree non-monotony”, i.e. a XOR in some local transition functions, i.e. the CNF of some \( f_i \) depending both on \( x_j \) and on \( \neg x_j \) for some \( j \);
2. (i) “General/relaxed non-monotony”, i.e. the presence of contradictory paths in \( G \), in combination to (ii) an appropriate sequentialisation of automata.
moves, \textit{i.e.} in combination to some non synchronicity (cf monotone BAN in Example\textsuperscript{3});

3. (i) “In situ inconsistent potential” (in a totally positive BAN, this is equivalent to \textit{not} starting in configuration $x = 000000\ldots$, nor in configuration $x = 11111\ldots$), combined to (ii) an appropriate sequentialisation of automata moves, and to (iii) well situated $\land$ and $\lor$ connectors – all of that combined in order to do mimic what a XOR or its monotone rewriting $(a \lor \neg b) \land (\neg a \lor b)$ do (cf the totally positive BAN of Example\textsuperscript{1}).

\section*{Effects of the NME.} I conjecture that the NME is actually involved in:
- The possibility of exploiting a lot of reversibility, and having very long (Hamiltonian) trajectories in non-monotone BANs;
- The (very rare) need for and possibility of lots of reversibility and/or recursion in (monotone) BANs;
- The (very rare) sensitivity of (monotone) BANs to synchronicity (it seems that positive cycles of even lengths and negative cycles with odd lengths have some importance for both synchronism-sensitivity\textsuperscript{3} and the NME, which would actually make a lot of sense because the NME was originally supposed to help finish characterise synchronism-sensitive BANs);
- Other weird and rare stuff in monotone BANs.

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