Hyperplanes in Configurations, decompositions, and
Pascal Triangle of Configurations

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Abstract

An elegant procedure which characterizes a decomposition of some class of binomial configurations into two other, resembling a definition of Pascal’s Triangle, was given in [4]. In essence, this construction was already presented in [10]. We show that such a procedure is a result of fixing in configurations in some class $\mathcal{K}$ suitable hyperplanes which both: are in this class, and deleting such a hyperplane results in a configuration in this class. By a way of example we show two more (added to that of [4]) natural classes of such configurations, discuss some other, and propose some open questions that seem also natural in this context.

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Introduction

On one hand, “Pascal Triangle” is a term which is known to all mathematicians: it characterizes an arrangement of binomial coefficients in a form of a ‘pyramid’ such that each item is the sum of items placed immediately above it. In another view: the sum of each neighbour items in a row equals to the item which is their common neighbour (in the row below). Clearly, binomial coefficients are simply values of a two-argument function $b(n, k)$ defined on nonnegative integers ($n = 0, 1, \ldots, k = 0, \ldots, n$) and nothing ‘magic’ is in the pyramid defined above. It is a visual presentation of recursive equation which these coefficients satisfy. Clearly, the sequences of boundary values $b(n, 0)$ and $b(n, n)$ uniquely determine then the function $b$. Nevertheless the recurrence in question is extremely simple...

Quite recently, Gabor Gévay in [4] noted that there is family of point-line configurations which can be arranged in such a pyramid, with a suitably defined “sum” of the configurations in question. Or: each (nontrivial, non-boundary) configuration in this family can be decomposed into two other members of this family. In essence, this decomposition (even in a more general form) was presented also earlier in [10, Representation 2.12]; the class in question consists of configurations which generalize Desargues configuration considered as schemes of mutual perspectives between several simplexes. On other hand, such systems of (geometrical) perspectives can be found even in the classical book of Veblen and Young [15] (Gévay quotes also
explicitly Danzer and Cayley) and its combinatorial schemes are special instances of so called binomial graphs, investigated in the context of association schemes (cf. e.g. [5]), and associated incidence structures. Combinatorial schemes characterizing these configurations can be found already in [6] and [3]. So:

the subject was known, but its regular nature was not known – was not stated explicitly until [4].

But then it appeared that the “sum” of two configurations is not a well defined operation that depends solely on the summands, and the associated decomposition is, in fact, associated with a choice of a hyperplane in the decomposed configuration. After that become clear (we present these observation in Section 2, Theorem 2.1 and equation (9)) there appeared that there are other natural known classes of configurations that can be arranged into respective triangles. These are, in particular, so called combinatorial Veronesians (defined originally in [11], without any connections with studying hyperplanes in configurations). In Section 3 we discuss some of the classes which appear within this theory.

1 Notations, standard constructions

1.1 Elementary combinatorics

There are well known formulas concerning binomial coefficients, frequently referred to as “Pascal Triangle of Binomials”. To be more precise, these formulas correspond to the arrangement of the binomial coefficients in a pyramid with consecutive rows:

\[ \left( \left( \binom{n}{k} : k = 0, \ldots, n \right) : n = 0, 1, 2, \ldots \right). \]

Then the corresponding recursive formula is the following

\[ \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}; \quad (1) \]

equation (1) yields immediately next two:

\[ \binom{n}{k} - \binom{n-1}{k} = \binom{n-1}{k-1}, \text{ and} \]
\[ \binom{n}{k} - \binom{n-1}{k-1} = \binom{n-1}{k}. \quad (3) \]

For purposes of our next investigations it will be more convenient to arrange binomial coefficients into a (infinite) matrix:

\[ [B(m,k) : m, k = 0, 1, \ldots], \]

where

\[ B(m,k) = \binom{m+k}{m}; \quad (4) \]

clearly, \( B(m,k) = B(k,m) \); the fundamental recursive formula for the binomial coefficients takes the form

\[ B(m,k) = B(m,k - 1) + B(m - 1, k). \quad (5) \]
1.2 Rudiments of geometry of configurations

We say that a structure \( \mathfrak{R} = \langle U, \mathcal{L}, \mathbb{1} \rangle \) with \( \mathbb{1} \subset U \times \mathcal{L} \) is a \((\nu, \beta, \kappa)\)-configuration if \( \mathfrak{R} \) is a partial linear space (i.e. \( a, b \parallel A, B \) yields \( a = b \) or \( A = B \)) such that \( |U| = \nu \), \( |\mathcal{L}| = \beta \), exactly \( \rho \) elements of \( \mathcal{L} \) are in the relation \( \mathbb{1} \) with \( a \in U \), for each \( a \in U \), and exactly \( \kappa \) elements of \( U \) are in the relation \( \mathbb{1} \) with \( A \in \mathcal{L} \), for each \( A \in \mathcal{L} \).

Let \( \mathfrak{R} \) be a configuration as above, then the following equation (a specialized form of the so called fundament equation of partial linear spaces) holds

\[
\nu \cdot \rho = \beta \cdot \kappa. \tag{6}
\]

The elements of \( U \) are called \textit{points} of \( \mathfrak{R} \), the elements of \( \mathcal{L} \) are called \textit{lines} of \( \mathfrak{R} \), and the relation \( \mathbb{1} \) is the \textit{incidence}. The numbers \( \rho \) and \( \kappa \) are referred to as \textit{point rank} and \textit{line size/rank} resp. It is a folklore, that every configuration as above with \( \kappa \geq 2 \) is isomorphic to a configuration, whose lines are sets of points, and the incidence is the standard membership relation \( \in \). If this will not cause a confusion (as it may happen in particular examples) we shall frequently assume that the incidence of \( \mathfrak{R} \) is the membership relation.

A subset \( \mathcal{H} \) of the set of points of \( \mathfrak{R} \) is called a \textit{hyperplane} of \( \mathfrak{R} \) when

- \( \mathcal{H} \) is a \textit{subspace} of \( \mathfrak{R} \), i.e. if the conditions \( a, b \parallel A \in \mathcal{L} \) and \( a, b \in \mathcal{H} \), \( a \neq b \) yield \( x \in \mathcal{H} \) for every \( x \) such that \( x \parallel A \),

and

- each line of \( \mathfrak{R} \) crosses \( \mathcal{H} \), i.e. for each \( A \in \mathcal{L} \) there is \( x \in \mathcal{H} \) such that \( x \parallel A \).

Let \( \mathcal{H} \) be a hyperplane of \( \mathfrak{R} \). Then, for each line \( A \) of \( \mathfrak{R} \) either there is a unique \( x \in \mathcal{H} \) with \( x \parallel A \) (we write \( x = A^\infty \) in that case) or every point incident with \( A \) belongs to \( \mathcal{H} \); the set of such lines will be denoted by \( \mathcal{L}[\mathcal{H}] \). Clearly,

\[
\mathfrak{R} \upharpoonright \mathcal{H} := \langle \mathcal{H}, \mathcal{L}[\mathcal{H}], \mathbb{1} \cap (\mathcal{L}[\mathcal{H}] \times (\mathcal{L}[\mathcal{H}])) \rangle
\]
is a partial linear space; quite frequently in the sequel we shall make no distinction between \( \mathcal{H} \) and \( \mathfrak{R} \upharpoonright \mathcal{H} \). Clearly, the set \( U \) of all the points of \( \mathfrak{R} \) is a hyperplane of \( \mathfrak{R} \). In what follows we shall assume that a hyperplane means a \textit{proper} (i.e. \( \mathcal{H} \neq U \)) subspace that satisfies suitable conditions.

Given a hyperplane \( \mathcal{H} \) of \( \mathfrak{R} \) we define the \textit{reduct}

\[
\mathfrak{R} \setminus \mathcal{H} := \langle U \setminus \mathcal{H}, \mathcal{L} \setminus \mathcal{L}[\mathcal{H}], \mathbb{1} \cap ((U \setminus \mathcal{H}) \times (\mathcal{L} \setminus \mathcal{L}[\mathcal{H}])) \rangle;
\]

if \( \kappa \geq 3 \) then \( \mathfrak{R} \setminus \mathcal{H} \) is a partial linear space with all the lines of size (rank) \( \kappa - 1 \).

Let us write, for symmetry, \( \mathfrak{R}_1 = \mathfrak{R} \setminus \mathcal{H} \) and \( \mathfrak{R}_2 = \mathfrak{R} \upharpoonright \mathcal{H} \). Recall, that we have a function \( \infty \) from the lines of \( \mathfrak{R}_1 \) into the points of \( \mathfrak{R}_2 \). Let us try to “reverse” this decomposition:

\textbf{Construction 1.1.} Let \( \mathfrak{R}_i = \langle U_i, \mathcal{L}_i, \mathbb{1}_i \rangle \) be a partial linear space for \( i = 1, 2 \). Assume that \( U_1 \cap U_2 = \emptyset = \mathcal{L}_1 \cap \mathcal{L}_2 \). Let \( \infty : \mathcal{L}_1 \rightarrow U_2 \) be a map such that the following holds

\[
\text{if } U_1 \ni x \parallel A, B \in \mathcal{L}_1 \text{ and } A^\infty = B^\infty \text{ then } A = B.
\]

We define

\[
U := U_1 \cup U_2,
\]
Finally, we set 
\[ L := K \].

It is evident that \( L \) is a partial linear space.

**Proposition 1.2.** Let \( \mathcal{R} = \mathcal{R}_1 \times_\infty \mathcal{R}_2 \) with \( \mathcal{R}_i \) as in \([1,2]\). Then \( U_2 \) is a hyperplane in \( \mathcal{R} \) and \( L_2 = L[U_2] \).

Proof. It suffices to state directly that if \( A \in L \) then either \( A \in L_2 \) and then \( x \parallel A \) gives \( x \in U_2 \), or \( A \in L_1 \) and then \( x \parallel A \) yields \( x \in U_1 \) or \( U_2 \ni x = \infty(A) \).

The construction of the type \([1,2]\) is quite frequent in geometry. One particular case let us mention below:

**Note 1.3.** Let \( \mathcal{R}_1 = (U_1, L_1, I_1, ||_1) \) be a partial linear space with parallelism of lines; we write \([A]||_1\) for the equivalence class of \( A \in L_1 \) w.r.t. the relation \( ||_1 \) (i.e. simply for the direction of \( A \)). Suppose that there is a formula \( \Phi \) in the language of \( \mathcal{R}_1 \) such that the relation

\[ \{([A_1]||_1, [A_2]||_1, [A_3]||_1) : (A_1, A_2, A_3) \in L_1^3, \Phi(A_1, A_2, A_3) \} \]

is a ternary equivalence relation on the set \( \left(L_1/\parallel_1\right)^3 \) (cf. \([14]\)); let \( L_2 \) be the set of its equivalence classes, and \( \mathcal{R}_2 = \langle L_2/\parallel_1, L_2, \in \rangle \). With \( A_\infty = [A]||_1 \) for \( A \in L_1 \) we obtain the structure \( \mathcal{R} = \mathcal{R}_1 \times_\infty \mathcal{R}_2 \) which is called, in that context, the closure of an affine structure \( \mathcal{R}_1 \).

In particular cases of this construction, practically, the structures \( \mathcal{R}_1 \) and \( \mathcal{R} \) are given, and we search for an appropriate formula \( \Phi \) (see \([1]\): affine completion, \([2]\), \([7]\)).

Other examples of this construction will appear in the next Section.

### 1.3 Dualization

Let \( \mathcal{R} = \langle U, L, I \rangle \) be an incidence structure; we call the structure

\[ \mathcal{R}^0 = \langle L, U, I^{-1} \rangle \]

the dual of \( \mathcal{R} \). It is evident that \( \mathcal{R}^0 \) is a partial linear space whenever \( \mathcal{R} \) is so. In particular

if \( \mathcal{R} \) is a \((\nu_\rho, \beta_\kappa)\)-configuration then \( \mathcal{R}^0 \) is a \((\beta_\kappa, \nu_\rho)\)-configuration.

**Proposition 1.4.** Let \( \mathcal{H} \) be a hyperplane of a partial linear space \( \mathcal{R} = \langle U, L \rangle \) such that the induced correspondence \( \infty \) is bijective. Then \( L \setminus L[\mathcal{H}] \) is a hyperplane of \( \mathcal{R}^0 \).

Proof. Let \( L_1, L_2 \in L \setminus L[\mathcal{H}] \). Assume that \( L_1, L_2 \parallel x \in U \) and \( L \ni L \parallel^{-1} x \). Suppose that \( L \notin L \setminus L[\mathcal{H}] \), then \( L \in L[\mathcal{H}] \) and, consequently, \( x \in \mathcal{H} \). This gives \( x = \infty(L_1) = \infty(L_2) \); we have \( L_1 = L_2 \) then. This proves that \( L[\mathcal{H}] \) is a subspace of \( \mathcal{R}^0 \).

Let \( L \) be an arbitrary line of \( \mathcal{R}^0 \), then \( L \in U \). If \( L \notin \mathcal{H} \) then each line of \( \mathcal{R} \) (each point of \( \mathcal{R}^0 \)) that passes through \( L \) is in \( L \setminus L[\mathcal{H}] \). If \( L \in \mathcal{H} \) then \( \infty^{-1}(L) \parallel^{-1} L \). This suffices for the proof.
Standard examples show that the condition ∞ is bijective assumed in [1.4] cannot be removed. Indeed, the plane in a projective 3-space \( \mathbb{P} \) is a hyperplane, but the family of lines of the resulting affine 3-space is not even a subspace of \( \mathbb{P}^3 \). However, [1.4] appears useful when we deal with (binomial) configurations. Proposition 1.4 can be easily (re)formulated in a more 'constructive' fashion:

**Corollary 1.5.** Let \( \mathcal{K}_i \) be configurations as in [1.4] with a suitable map ∞ defined. Assume that ∞ is a bijection and \( \mathcal{K} = \mathcal{K}_1 \times_\infty \mathcal{K}_2 \). Then

\[
\mathcal{K}^0 = \mathcal{K}_2^0 \times_\infty \mathcal{K}_1^0
\]  

(8)

## 2 Binomial configurations

### 2.1 Generalities

The main subject of this section consists in investigations on the family of binomial configurations i.e. of configurations of the type \( \binom{k+m-1}{k} \binom{k+m-1}{m} \) for some positive integers \( k, m \). It is easily seen that each parameters of this form satisfy \( \Box \).

Let us write \( \mathcal{B}(k,m) \) for the class of all \( \binom{k+m-1}{k} \binom{k+m-1}{m} \)-configurations.

**Theorem 2.1.** Let \( \mathcal{K} \in \mathcal{B}(k,m) \) and let \( \mathcal{H} \) be a hyperplane of \( \mathcal{K} \). Assume that

(i) \( \mathcal{H} \) is a configuration (in this case this means simply that \( \mathcal{K} \mid \mathcal{H} \) has constant point rank), and

(ii) \( \mathcal{K} \setminus \mathcal{H} \) is a binomial configuration.

Then

(iii) \( \mathcal{H} \) is a binomial configuration, more precisely: \( \mathcal{K}_2 = \mathcal{K} \setminus \mathcal{H} \in \mathcal{B}(k-1,m) \);

(iv) \( \mathcal{K}_1 = \mathcal{K} \setminus \mathcal{H} \in \mathcal{B}(k,m-1) \);

(v) there is a 1-1 correspondence \( \infty \): lines of \( \mathcal{K}_1 \longrightarrow \) points of \( \mathcal{K}_2 \) such that \( \mathcal{K} = \mathcal{K}_1 \times_\infty \mathcal{K}_2 \).

**Proof.** Recall that, right from the definition, the points of \( \mathcal{K} \) have rank \( k \), and the lines of \( \mathcal{K} \) have size \( m \). Set \( n = m + k - 1 \).

Then, from the definition we get immediately that the points of \( \mathcal{K}_1 \) are all of the same rank \( k \) and the lines are all of the size \( m - 1 \), so, in accordance with [11], \( \mathcal{K}_1 \) is a \( \binom{k+m-1}{k} \binom{k+m-1}{m-1} \)-configuration, which justifies [15]. The number of points in \( \mathcal{K}_1 \) is \( \binom{n-1}{k-1} \) and the number of points of \( \mathcal{K} \) is \( \binom{n}{k} \): from the Pascalian equations the number of points of \( \mathcal{K}_2 \) is \( \binom{n}{k} - \binom{n-1}{k-1} = \binom{n-1}{k-1} \). Similarly we compute the number of lines of \( \mathcal{K}_2 \); it equals to \( \binom{n}{m} - \binom{n-1}{m-1} = \binom{n-1}{m-1} \). The size of the lines in \( \mathcal{H} \) is \( m \); from assumption [12] and [13] applied to \( \mathcal{K}_2 \) we get that the point rank in \( \mathcal{K}_2 \) equals to \( k - 1 \). So, \( \mathcal{K}_2 \) is a \( \binom{k-1+m-1}{k-1} \binom{k-1+m-1}{m-1} \)-configuration. This justifies [16]. Finally, since each point in \( \mathcal{K}_2 \) has its rank on one less than in \( \mathcal{K} \) we get that through each one of these points there passes exactly one line of \( \mathcal{K}_1 \), so \( \infty \) is a bijection, as required in [17].
Informally speaking, (9) gives a decomposition
\[ \mathcal{B}(k, m) = \mathcal{B}(k, m - 1) \times_{\infty} \mathcal{B}(k - 1, m), \]
which resembles reverent Pascalian equation (8). But note, that the “operation" \( \times_{\infty} \) is not commutative, and it depends essentially on the parameter \( \infty \).

**Remark 2.2.** Not every hyperplane of a binomial configuration is a (binomial) configuration. Indeed, it suffices to have a look on hyperplanes in binomial partial Steiner triple systems, either in a more general approach of [5] or in a more particular case of [13] and note that in the Desargues configuration a line accomplished with a point not joinable with any point on this line is a hyperplane, it contains three points of rank 3 and one point of rank 0 so, it is not a configuration.

**Remark 2.3.** Let us consider the smallest sensible and possible case: \( \mathcal{B}(2, 3) \times \mathcal{B}(3, 2) = \mathcal{B}(3, 3) \). If \( \mathcal{R} \in \mathcal{B}(3, 3) \) then \( \mathcal{R} \) is a \( \left( \binom{3+3-1}{3}, \binom{3+3-1}{3} \right) = (103, 103) \)-configuration: one of ten possible. If \( \mathcal{R}_1 \) is a \( \left( \binom{3+2-1}{3}, \binom{3+2-1}{2} \right) = (43, 62) \)-configuration then it is the complete graph \( K_4 \). If \( \mathcal{R}_2 \) is a \( \left( \binom{2+3-1}{2}, \binom{2+3-1}{3} \right) = (62, 43) \)-configuration then it is simply the Pasch-Veblen configuration \( \mathfrak{U} \). It was shown in [8] that there are exactly six maps \( \infty \) which yield pair wise non isomorphic configurations \( K_1 \times_{\infty} \mathfrak{U} \).

\[ \text{So, there are binomial configurations } \mathcal{R}_1, \mathcal{R}_2 \text{ and bijections } \infty', \infty'' : \text{lines of } \mathcal{R}_1 \rightarrow \text{points of } \mathcal{R}_2 \text{ such that } \mathcal{R}_1 \times_{\infty'} \mathcal{R}_2 \neq \mathcal{R}_1 \times_{\infty''} \mathcal{R}_2. \text{ Consequently, the symbol } \times \text{ is not a well defined operation, without the argument } \infty \text{ defined explicitly.} \]

**Remark 2.4.** Let \( \mathcal{R}_1, \mathcal{R}_2 \) be binomial configurations, let a map \( \infty : \text{lines of } \mathcal{R}_1 \rightarrow \text{points of } \mathcal{R}_2 \) be a bijection.

From assumption, \( \mathcal{R}_i \in \mathcal{B}(k_i, m_i) \) for some integers \( k_i, m_i, i = 1, 2 \). Moreover, the two numbers: of lines of \( \mathcal{R}_1 \) and of points of \( \mathcal{R}_2 \) coincide. This means that \( \binom{k_1+m_1-1}{m_1} = \binom{k_2+m_2-1}{k_2} \). Then \( k_1 + m_1 - 1 = k_2 + m_2 - 1 \) and one of the following holds:

(i) either \( m_1 = m_2 - 1 \) – in this case \( k_2 = k_1 - 1 \) and \( \mathcal{R} = \mathcal{R}_1 \times_{\infty} \mathcal{R}_2 \) is a binomial configuration,

(ii) or \( m_1 = k_2 \) and then \( k_1 = m_2 \). Consider e.g. the case \( k_1 = m_1 = k_2 = m_2 = 3 \), then \( \mathcal{R}_1 \) are \( (103, 103) \)-configurations. But then \( \mathcal{R} = \mathcal{R}_1 \times_{\infty} \mathcal{R}_2 \) has 20 points and 20 lines. Ten lines have size 3, and ten have size 4. So, in this case \( \mathcal{R} \) is not even a configuration.

This shows that a ‘sum’ of two binomial configurations, even determined by constructing ‘improper points’, may be not a binomial configuration.

In the next Section we present two remarkable families of binomial configurations which yield families indexed by positive integers and which yield “a Pascal Triangle”.

## 3 Examples

### 3.1 Example: the family of combinatorial Grassmannians

For an integer \( k \) and a set \( X \) we write \( \mathcal{V}_k(X) \) for the family of \( k \)-subsets of \( X \). Nowadays the notation \( \binom{X}{k} \) instead of \( \mathcal{V}_k(X) \) becomes widely used. We prefer, however, not to mix integers and sets.
Let \( A \in \mathcal{B}(k, m) \); then the points of \( A \) can be identified with the \( k \)-subsets of a fixed \( n \)-element set \( X \), where \( n = m + k - 1 \). Let us identify the lines of \( A \) with the elements of \( \mathcal{P}_m(X) \) and define

\[
a \mid A : \iff a \in \mathcal{P}_k(X) \land A \in \mathcal{P}_m(X) \land |a \cap A| = 1.
\]

(10)

Suppose that \( a \neq b \) and \( a, b \mid A \) with \( \mid \) defined by (10). Then \( a \cap b = X \setminus A \) and therefore \( A \) is uniquely determined by its two points \( a \) and \( b \). So, the structure

\[
\mathfrak{G}(k, m) := \langle \mathcal{P}_k(X), \mathcal{P}_m(X), \mid \rangle
\]

is a partial linear space. It is not too hard to verify that it is a configuration with the lines of size \( m \) and the points of rank \( k \), so \( \mathfrak{G}(k, m) \in \mathcal{B}(k, m) \).

In practice, the above presentation is not so easy to handle with and not too intuitive.

(i) There is a one-to-one correspondence between the elements of \( \mathcal{P}_m(X) \) and the elements of \( \mathcal{P}_{k-1}(X) \): indeed, \( n = m + (k - 1) \) so, the boolean complementation \( \mathfrak{z} \) is a bijection in question. Then we see that the pair of maps \( (\text{id}, \mathfrak{z}) \) maps \( \mathfrak{G}(k, m) \) onto the structure \( \langle \mathcal{P}_k(X), \mathcal{P}_{k-1}(X), \supset \rangle \), which coincides with the combinatorial Grassmannian \( G_{k_0}(X) \) defined in [10].

(ii) Analogously, there is a one-to-one correspondence between the elements of \( \mathcal{P}_{m-1}(X) \) and the elements of \( \mathcal{P}_k(X) \); set \( k_0 = m - 1 \), then \( (\mathfrak{z}, \text{id}) \) maps \( \mathfrak{G}(m, k) \) onto the structure \( \langle \mathcal{P}_{k_0}(X), \mathcal{P}_{k_0+1}(X), \subset \rangle \), which coincides with the combinatorial Grassmannian \( G_{k_0}(X) \) defined in [10].

Let us concentrate upon the presentation given in [10], let us drop out the superfluous index 0 and let \( \mathfrak{A} = \mathcal{G}_k(X), |X| = n \); remember that \( \mathcal{G}_k(X) \in \mathcal{B}(n - k, k + 1) \). We write \( \mathcal{G}_k(n) \) for the type of \( \mathcal{G}_k(X) \) where \( |X| = n \).

Let us fix an element \( i \in X \), then \( \mathcal{P}_k(X) \) is the disjoint union \( \mathcal{P}_k(X) = X_1 \cup X_2 \), where \( X_1 = \{a \in \mathcal{P}_k(X) : i \in a\} \) and \( X_2 = \{a \in \mathcal{P}_k(X) : i \notin a\} = \mathcal{P}_k(X \setminus \{i\}) \). The following is easily seen:

(i) \( A \setminus \{i\} \) is a hyperplane of \( \mathfrak{A} \), \( \mathfrak{A}_2 := \mathfrak{A} \setminus \{\{i\} \}
\]

(ii) \( \mathfrak{A}_1 = \mathfrak{A} \setminus \{\{i\} \} \), with the point-set \( \mathfrak{X}_1 \), is isomorphic under the map \( \mathfrak{X}_1 \ni a \mapsto a \setminus \{i\} \in \mathcal{P}_{k-1}(X \setminus \{i\}) \) to the structure \( \mathcal{G}_{k-1}(X \setminus \{i\}) \).

(iii) Let \( A \) be a line of \( \mathfrak{A}_1 \), so \( A \in \mathcal{P}_{k+1}(X) \) where \( i \in A \). Then \( A \setminus \{i\} \in \mathcal{P}_k(A) \cap X_2 \), so \( A^\infty = A \setminus \{i\} \).

In view of the above and [2.1] we get that

Proposition 3.1. If \( i \in X \) is arbitrary then

\[
\mathcal{G}_k(X) \cong \mathcal{G}_{k-1}(X \setminus \{i\}) \rtimes_{\infty} \mathcal{G}_k(X \setminus \{i\})
\]

(11)

with \( \infty \) defined by (10) above.

In numerical symbols we can write:

\[
\mathcal{G}_k(n) = \mathcal{G}_{k-1}(n - 1) \rtimes_{\infty} \mathcal{G}_k(n - 1).
\]
This decomposition was studied in many details in [1], it was also noticed in [10, Representation 2.12]. While expressed in terms of \( \mathfrak{S}(k, m) \) it assumes the form

\[
\mathfrak{S}(k_0, m_0) = \mathfrak{S}(k_0, m_0 - 1) \times_\infty \mathfrak{S}(k_0 - 1, m_0),
\]

where \( k_0 = n - k, m_0 = k + 1 \).

### 3.2 Example: the family of combinatorial Veronesians

Let \( X \) be an \( m \)-element set; we write \( \eta_k(X) \) for the \( k \)-element multisets with the elements in \( X \). In naive words, a multiset is a ‘set’ whose elements belong to \( X \), and each one of them can occur several times. Formally, it is a function \( f \) defined on \( X \) with values in the set of natural numbers (with zero); this function ‘counts’ how many times given item from \( X \) occurs in \( f \). It is a convenient way to symbolize such a function \( f \) in the form \( f = \prod_{x \in X} x^{f(x)} \) (with the natural relations like \( x^i x^j = x^{i+j} \), \( x^0 = 1 \), \( 1x = x \), etc...). Then the cardinality of \( f \) is \( |f| = \sum_{x \in X} f(x) \).

We write \( \text{supp}(f) = \{ x \in X : f(x) > 0 \} \); clearly, \( |f| = \sum_{x \in \text{supp}(f)} f(x) \).

Let us write \( \bigcup_{i=0}^{k-1} \eta_i(X) =: \eta_{<k}(X) \). On the set \( \eta_k(X) \times \eta_{<k}(X) \) we define the incidence relation \( | \) by the formula:

\[
e \mid f : \iff f = e \cdot x^{k-|e|} \quad \text{for some } x \in X.
\]

The structure

\[
V_k(m) = \langle \eta_k(X), \eta_{<k}(X), | \rangle
\]

is called a **combinatorial Veronesean**; the class of combinatorial Veronesians was introduced in [11]. It was proved that \( V_k(m) \) is a partial linear space with the points of rank \( k \) and the lines of size \( m \); the formulas counting the cardinality of \( \eta_k(X) \) and of \( \eta_{<k}(X) \) are known in the elementary combinatorics; summing up we get that \( V_k(m) \in \mathcal{B}(k, m) \).

Let us fix \( a \in X \) and define \( \mathcal{X}_2 = \{ f \in \eta_k(X) : a \in \text{supp}(f) \} \) and \( \mathcal{X}_1 = \{ f \in \eta_k(X) : a \notin \text{supp}(f) \} \); then \( \eta_k(X) \) is the disjoint union \( \mathcal{X}_1 \cup \mathcal{X}_2 \).

(i) It is seen that the map \( \eta_{k-1}(X) \ni f \mapsto f \cdot a^1 \in \mathcal{X}_2 \) is a bijection. Suppose that \( f' a^1, f'' a^1 \mid e \) where \( e \in \eta_{<k}(X) \). Then \( a \in \text{supp}(e) \) and \( f', f'' \mid e \notin \eta_{<k-1}(X) \). Finally, \( a \in \text{supp}(f) \) for every \( f \) with \( f \mid e \), which yields that \( \mathcal{X}_2 \) is a subspace of \( V_k(X) \); as we noted, it is isomorphic to \( V_{k-1}(X) \).

(ii) Let \( e \in \eta_{<k}(X) \) be a line of \( V_k(m) \). If \( a \in \text{supp}(e) \) then \( f \in \mathcal{X}_2 \) for every \( f \) with \( f \mid e \). If \( a \notin \text{supp}(e) \) then \( e^\infty = e \cdot a^{k-|e|} \) is the unique element incident with \( e \) which belongs to \( \mathcal{X}_2 \).

(iii) Evidently, the points in \( \mathcal{X}_1 \) can be considered as the points of \( V_k(X \setminus \{a\}) \). Let \( e \in \eta_{<k}(X \setminus \{a\}) \) be a line of \( V_k(X \setminus \{a\}) \); then \( e^\infty = e \cdot a^{k-|e|} \mid e \) is well defined.

(iv) In particular, the above yields that \( \mathcal{X}_2 \) is a hyperplane of \( V_k(m) \).

Summing up, we obtain

**Proposition 3.2.** Let \( a \in X \) be arbitrary.

\[
V_k(X) = V_k(X \setminus \{a\}) \times_\infty V_{k-1}(X),
\]

where \( \times_\infty \) is defined by (iii) above.
In (numerical) symbols we can express this fact by

\[ V_k(m) = V_k(m-1) \times V_{k-1}(m). \]

As a consequence of [11] Cor. 4.8, Thm. 4.5, \( V_k(m) \) is a combinatorial Grassmannian only for \( k = 2 \) or \( m = 2 \) so, Grassmannians and Veronesians are essentially distinct families.

### 3.3 Example: the family of dual combinatorial Veronesians

In Subsections 3.1 and 3.2 we have found decompositions of the scheme \( B(k, m) = B(k, m-1) \times B(k-1, m) \). Clearly, \( B(m, k) \) are dual to \( B(k, m) \); therefore, in view of [1.5] one can expect that each of these decompositions determines a decomposition of the scheme

\[ B(m, k) = B(k, m)^0 = B(k-1, m)^0 \times B(k, m-1)^0 = B(m, k-1) \times B(m-1, k) \]

In case of combinatorial Grassmannians the dualization procedure does not yield any new family of configurations:

**Fact 3.3.** Let \( n = |X| \) for a set \( X \). Then \( G_k(X)^0 \cong G_{n-k}(X) \).

However, the dual Veronesians yield another, third family: if \( V_k(m)^0 \) is isomorphic to a combinatorial Grassmannian then either \( k = 2 \) or \( m = 2 \); if it is isomorphic to a combinatorial Veronesian then \( k = 2 \), or \( m = 2 \), or \( k = 3 \) = \( m \). Even \( V_k(k)^0 \cong V_k(k) \) is not valid for \( k > 3 \) (see [11] Thm.’s 4.14, 4.15)!

Let us adopt notation of Subsection 3.2 and let \( \mathfrak{A} = (U, \mathcal{L}) = V_k(X) \); let us remind that \( \mathcal{X}_2 = \{ f \in \eta_k(X) : a \in \text{supp}(f) \} \) is a hyperplane of \( \mathfrak{A} \) and then \( \mathcal{L}[\mathcal{X}_2] = \{ e \in \eta_{k-1}(X) : a \in \text{supp}(e) \} =: \mathcal{L}_2 \). Consequently, \( \mathcal{L}_1 := \mathcal{L} \setminus \mathcal{L}_2 = \eta_{k-1}(X \setminus \{ a \}) \) is a hyperplane of \( \mathfrak{A}^0 \); set \( \mathcal{X}_1 := U \setminus \mathcal{X}_2 = \eta_k(X \setminus \{ a \}) \). Consider a line \( f \in \eta_k(X) \) of \( \langle \mathcal{L}_2, \mathcal{X}_2 \rangle \); then \( a \in \text{supp}(f) \); let \( dg(a, f) \) be the greatest integer \( s \) such that \( f = a^sg \) for a multiset \( g \). We associate with such an \( f \) the point \( f^\infty = \frac{f}{dg(a, f)} \in \mathcal{L}_1 \); it is seen that we obtain

**Proposition 3.4.** \( V_k(m)^0 = \langle \mathcal{L}_2, \mathcal{X}_2, |^{-1} \rangle \times_{\infty} \langle \mathcal{L}_1, \mathcal{X}_1, |^{-1} \rangle \cong V_{k-1}(m)^0 \times_{\infty} V_k(m-1)^0 \).

With the symbols \( V^{m}_{n}(k) = V_k(m)^0 \in B(m, k) \) we arrive to

\[ V^{m}_{n}(k) = V^{m}_{n}(k-1) \times_{\infty} V^{m}_{n-1}(k) \]

Consequently, following [1.5] we can explicitly characterize the Pascal Triangle of Configurations consisting of dual combinatorial Veronesians.

### 4 Comments and problems

We have shown three families \( \mathcal{K} \) of configurations \( \mathfrak{K}(m, k) : m, k = 1, 2 \ldots \) such that the formula \( \mathfrak{K}(m, k) = \mathfrak{K}(m, k-1) \times_{\infty, m,k} \mathfrak{K}(m-1, k) \) is valid for all \( m, k \) and suitable maps \( \infty_{m,k} \). One can expect that there are more such families: the point is to find a suitable family

\[ \infty_{m,k} : \mathfrak{K}_{m-1}(m + k - 2) \rightarrow \mathfrak{K}_{k-1}(m + k - 2) : m, k = 1, 2, \ldots \]
It is seen how huge variety of binomial partial triple systems can be obtained via ‘completing’ complete graphs (see [8]): one can expect that our procedure produces much more required configurations (cf. Problem 4.1).

However, one essential question appears: which of them can be realized in a Desarguesian projective space: we call them projective then. It is known that all the combinatorial Grassmannians are projective. It is also known that (practically all) combinatorial Veronesians are not projective (only $V_3(3)$ and $V_k(2)$, $V_2(m)$ are realizable). Similarly, dual of combinatorial Veronesians are also not projective (besides the exceptions indicated before), [11, Thm.’s 6.9, 6.10].

The statement like if $\mathcal{R}_1$ and $\mathcal{R}_2$ are realizable then $\mathcal{R}_1 \times \mathcal{R}_2$, if it is a (binomial) configuration then is realizable as well is false, in general. It suffices to present $V_3(4)$ as the “sum” of projectively realizable structures $V_3(3)$ and $V_2(4)$. So, a natural question arises

**Problem 4.1.** Assume that $\mathcal{R}_1$ and $\mathcal{R}_2$ are projective (binomial) configurations which satisfy corresponding ‘recursive equation’

$$\mathcal{R}_1 \in B(k, m - 1) \text{ and } \mathcal{R}_2 \in B(k - 1, m) \text{ for some } k, m \geq 2.$$ (14)

Then there is a bijection $\infty$: lines of $\mathcal{R}_1 \rightarrow$ points of $\mathcal{R}_2$ so as $\mathcal{R}_1 \times \infty \mathcal{R}_2 \in B(k, m)$. This observation enables us to construct ‘Pascal Triangle of Configurations’ from, practically, arbitrary boundary sequences of configurations, considering arbitrary $\infty$’s.

For which maps $\infty$ (is there necessarily at least one) the structure $\mathcal{R}_1 \times \infty \mathcal{R}_2$ is projective?  

Note that “boundary” sequences $B(2, k)$ and $B(k, 2)$ are known: $B(2, k) = \{K_{k+1}\}$ and $B(k, 2) = \{K_{k+1}\}$, and these two sequences consist of projective configurations.

So, considering configurations decomposed with the following schemes

$$B(3, k) = B(3, k - 1) \times B(2, k) = B(3, k - 1) \times K_k,$$

$$B(k, 3) = B(k, 2) \times B(k - 1, 3) = K_{k+1} \times B(k - 1, 3).$$

the real problem lies in the classification/choice of bijections $\infty$!

In particular, there are known binomial partial Steiner triple systems not in the families $V_i(?)$ nor among $G_i(?)$, and nor among $V_i(?)^o$ which are projective, for example, so called quasi-Grassmannians of [12]. Each such structure $\mathcal{R}_n$ has parameters as the corresponding $G_2(n)$. So, there arises a very particular, but intriguing

**Problem 4.2.** Is there a map $\infty$ such that the structure $\mathcal{R}_{n-1} \times \infty G_2(n - 1)$ (which has the parameters of $G_2(n)$) is realizable in a Desarguesian projective space?  

Addendum

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