The Cohomology and Laplacians of Weighted Hypergraphs and Applications

Shiquan Ren, Chengyuan Wu, Jie Wu

Abstract. Weighted hypergraphs are generalizations of weighted graphs and weighted simplicial complexes. Since 1990’s, the weighted Laplacians of weighted graphs and weighted simplicial complexes have been studied. In this paper, we study the weighted Laplacians and the weighted embedded cohomology of weighted hypergraphs. Generalizing the Hodge theorem on the Laplacian of simplicial complexes, we prove that the kernels of certain weighted Laplacians are isomorphic to the weighted embedded cohomology of weighted hypergraphs. As by-products, we discuss the weighted embedded cohomology and weighted Laplacians of joins of weighted hypergraphs. Finally, we give some (potential) applications in the data-driven symptom-based diagnosis of diseases.

1 Introduction

The graph Laplacian is a self-adjoint operator on graphs defined by the adjacency relations of the vertices (cf. [6, Section 1.2]). In 1847, the graph Laplacian was firstly investigated by G. Kirchhoff [19] in the study of electrical networks. Since 1970’s, the spectrum of the graph Laplacian has been extensively investigated (cf. [1, 2, 6, 10]). In 1996, weights on vertices as well as the weighted graph Laplacian was studied by F.R.K. Chung and R.P. Langlands [8]. In 2000, the eigenvalues of the weighted graph Laplacian was studied by F.R.K. Chung and K. Oden [9] and some isoperimetric inequalities were proved.

Hypergraphs (cf. [3]) are higher-dimensional generalizations of graphs. In a graph, an edge is a segment joining two vertices, hence is of dimension 1. While in a hypergraph, an n-dimensional hyperedge (or simply an n-hyperedge) is a set of n + 1 vertices. We use $\mathcal{H}$ to denote a hypergraph. Then $\mathcal{H}$ is a collection of certain hyperedges. The dimension of $\mathcal{H}$ is the maximal dimension of its hyperedges. In 1983, the graph Laplacian was generalized to the Laplacian of hypergraphs by F.R.K. Chung [5]. Later in 1996, some applications in number theory of the Laplacian of hypergraphs was found by K. Feng and W.C.W. Li [14]. Moreover, in 1992, some cohomology groups of hypergraphs with mod 2 coefficients was constructed by F.R.K. Chung and R.L. Graham [7].
(Abstract) simplicial complexes (cf. [16]) can be regarded as complete hypergraphs such that all the faces of hyperedges are still hyperedges. Precisely, if for any hyperedge $\sigma \in \mathcal{H}$ and any nonempty subset $\tau \subseteq \sigma$, we always have $\tau \in \mathcal{H}$, then $\mathcal{H}$ is a simplicial complex. In this case, the hyperedges are called simplices. A simplicial complex $\mathcal{K}$ has an associated chain complex $C_n(\mathcal{K})$, $n \geq 0$, with boundary maps $\partial_n : C_{n+1}(\mathcal{K}) \to C_n(\mathcal{K})$ such that $\partial_{n+1} \partial_n = 0$. One can generalize the graph Laplacian to higher dimensions and construct the Laplacian of simplicial complexes (cf. [13], [12, p. 4314], [18, p. 304])

$$L_n = \partial_{n+1} \partial^*_n + \partial^*_n \partial_n. \quad (1.1)$$

Here $\partial^*_n$ (resp. $\partial^*_{n+1}$) is the dual operator of $\partial_n$ (resp. $\partial_{n+1}$) with respect to certain inner product on each $C_*(\mathcal{K})$, $* \geq 0$. In 1944, a discrete version of the Hodge theorem for $L_n$ was proved by Eckmann [13] (cf. [12, Theorem 3.3], [18, Theorem 2.2]). In 2002, the spectrum of the Laplacian $L_n$ was investigated by A.M. Duval and V. Reiner [12].

Weighted simplicial complexes are simplicial complexes equipped with certain weight functions on the simplices. In 1990, by twisting the boundary maps using the weights, R.J. MacG. Dawson [11] studied the homology of weighted simplicial complexes. In 2013, by twisting the boundary maps in the Laplacians (1.1) using the weights, and considering the cohomology, D. Horak and J. Jost [17, 18] studied the weighted Laplacians of weighted simplicial complexes. Recently, the weight functions on simplicial complexes were generalized to inner products on cochain complexes by C. Wu, S. Ren, J. Wu and K. Xia [28]. The properties, classifications and applications of weighted (co)homology and weighted Laplacians of weighted simplicial complexes were studied (cf. [27, 28]).

In this paper, we generalize the weighted (co)homology and the weighted Laplacian studied in [11, 17, 18, 27, 28] from weighted simplicial complexes to weighted hypergraphs. With the help of the cochain complexes constructed in [15], the embedded homology of hypergraphs constructed in [4], and the weight functions constructed in [28], we construct the weighted embedded cohomology of weighted hypergraphs in Subsection 3.1. And with the help of the weighted Laplacians constructed in [17, 18, 28], we construct the weighted infimum Laplacians and weighted supremum Laplacians of weighted hypergraphs in Subsection 3.2. As a generalization of the Hodge theorem for $L_n$ (cf. [13, 18, Theorem 2.2]), the main result of this paper is a Hodge-type theorem for weighted hypergraphs, given in Theorem 3.4:

[Main Result (Theorem 3.4)]. For any weight $\phi$ on a hypergraph $\mathcal{H}$, both the kernel of the $\phi$-weighted infimum Laplacian and the kernel of the $\phi$-weighted supremum Laplacian are isomorphic to the $\phi$-weighted embedded cohomology of $\mathcal{H}$. Moreover, each cohomology class of the $\phi$-weighted cohomology can be represented by a unique cochain in the cohomology class such that the cochain is in the kernel of the Laplacians.

The paper is organized as follows. In Section 2 we discuss the cohomology and the Laplacian of cochain complexes as preparations for weighted hypergraphs. In Section 3 we prove the main
result and give some corollaries. In Section 4 we discuss some examples. In Section 5 as a by-product of the Main Result, we prove Theorem 5.1 for the weighted cohomology of joins of weighted hypergraphs. And in Section 6 we give some applications in the data-driven symptom-based diagnosis of diseases.

Throughout this paper, we assume that hypergraphs (resp. simplicial complexes) have finitely many hypredges (resp. simplices), unless otherwise specified. Since in computer science, people are mostly interested in the finite situations, the assumption is well-accepted.

2 Cohomology and Laplacian of Cochain Complexes

In this section, we discuss the cohomology and the Laplacian of cochain complexes. In Subsection 2.1 we introduce the embedded cohomology of graded subgroups of cochain complexes. In Subsection 2.2 we discuss the Laplacian of cochain complexes as well as its restrictions to sub-cochain complexes. In Subsection 2.3 we discuss the Hodge decomposition and the Hodge theorem for the embedded cohomology of graded subgroups of cochain complexes.

2.1 The Embedded Cohomology of Graded Subgroups of Cochain Complexes

Let $C^n, n = 0, 1, 2, \ldots$, be a sequence of abelian groups equipped with a sequence of group homomorphisms (called coboundary maps)

$$0 \xrightarrow{\delta_{n-1}} C^0 \xrightarrow{\delta_0} C^1 \xrightarrow{\delta_1} C^2 \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_{n-1}} C^n \xrightarrow{\delta_n} C^{n+1} \xrightarrow{\delta_{n+1}} \cdots$$

(2.1)

with $\delta_n \circ \delta_{n-1} = 0$ for each $n \geq 0$. Such a sequence (2.1) of abelian groups and homomorphisms is called a cochain complex (cf. [16, p. 191], [20, p. 8]). Both the intersection and the direct sum of a family of cochain complexes are still cochain complexes. For each $n \geq 0$, the $n$-th cohomology of the cochain complex (2.1) is defined as the quotient group (cf. [16 p. 191], [20 p. 8])

$$H^n(\{C^*, \delta_*\}) = \text{Ker}\delta_n / \text{Im}\delta_{n-1}.$$  

For simplicity, $H^n(\{C^*, \delta_*\})$ is also denoted as $H^n(C^*)$. For each $n \geq 0$, let $D^n$ be a subgroup of $C^n$. We have a graded subgroup $D^*$ of $C^*$. In particular, if for each $n \geq 1$, $\delta_{n-1}D^{n-1} \subseteq D^n$, then we call the sequence $\{D^*, \delta_*\}$ a sub-cochain complex of $\{C^*, \delta_*\}$. Given a sub-cochain complex of a cochain complex, the sequence of the quotient groups, equipped with the respective quotient maps, is still a cochain complex.

**Definition 1.** Given a graded subgroup $D^*$ embedded in a cochain complex $C^*$, the **infimum cochain complex** $\text{Inf}^n(D^*, C^*)$ of the sequence $\{D^*, C^*\}$ is the cochain complex

$$\text{Inf}^n(D^*, C^*) = \sum \{C'^n \mid C'^n \text{ is a sub-cochain complex of } C^* \text{ and } C'^n \subseteq D^n\}.$$
It follows immediately from Definition 1 that if $D^*$ is a sub-cochain complex, then $\text{Inf}^*(D^*, C^*) = D^*$.

**Proposition 2.1.** Given a cochain complex $C^*$ and a graded subgroup $D^*$, the infimum cochain complex is given by

$$\text{Inf}^n(D^*, C^*) = D^n \cap \delta_n^{-1}(D^{n+1}).$$

**(2.2)**

**Proof.** The proof is similar to the proof of [4, Proposition 2.1]. The idea was originally given in [15, Section 3.3] for a chain complex constructed from digraphs. 

**Definition 2.** Given a graded subgroup $D^*$ embedded in a cochain complex $C^*$, the supremum cochain complex $\text{Sup}^*(D^*, C^*)$ of the sequence $\{D^*, C^*\}$ is the cochain complex

$$\text{Sup}^n(D^*, C^*) = \bigcap \{C'^n \mid C'^* \text{ is a sub-cochain complex of } C^* \text{ and } D^n \subseteq C'^n\}.$$ 

It follows immediately from Definition 2 that if $D^*$ is a sub-cochain complex, then $\text{Sup}^n(D^*, C^*) = D^*$.

**Proposition 2.2.** Given a cochain complex $C^*$ and a graded subgroup $D^*$, the supremum cochain complex is given by

$$\text{Sup}^n(D^*, C^*) = D^n + \delta_{n-1}D^{n-1}.$$ 

**(2.3)**

**Proof.** The proof is similar to the proof of [4, Proposition 2.2].

**Remark 1:** In (2.3), when $n = 0$, we let $\delta_{-1} = 0$ and $D^{-1} = 0$.

It follows from Proposition 2.1 that the cohomology of the infimum chain complex is

$$H^n(\text{Inf}^*(D^*, C^*)) = \text{Ker}(\delta_n|_{D^n \cap \delta_n^{-1}(D^{n+1})}) / \text{Im}(\delta_{n-1}|_{D^{n-1} \cap \delta_{n-1}^{-1}(D^n)})$$

$$= \text{Ker}(\delta_n|_{D^n}) / (D^n \cap \delta_{n-1}D^{n-1}).$$

Moreover, we have the next proposition.

**Proposition 2.3.** The cohomology groups of $\text{Inf}^*(D^*, C^*)$ and the cohomology groups of $\text{Sup}^*(D^*, C^*)$ are isomorphic, as graded abelian groups.

**Proof.** The proof is similar to the proof of [4, Proposition 2.4].

### 2.2 The Laplacian of Cochain Complexes

In the remaining part of Section 2, we suppose in addition that the cochain complex $C^*$ is a graded Hermitian space equipped with a Hermitian product $\langle \ , \ \rangle$. That is, for each $n \geq 0$, $C^n$
is a Hermitian space with a Hermitian product $\langle , \rangle_{C^n}$. Let $\delta_n^*$ be the adjoint operator of $\delta_n$ such that for any cochains $w \in C^n$ and $w' \in C^{n+1}$,

$$\langle \delta_n w, w' \rangle_{C^{n+1}} = \langle w, \delta_n^* w' \rangle_{C^n}.$$  

The Laplacian is given by (cf. [13], [18, p. 304])

$$\Delta_n = \delta_n^* \delta_n + \delta_n - 1 \delta_n^* - 1.$$  

(2.4)

Lemma 2.4. The Laplacian $\Delta_n$ is a self-adjoint and linear operator from $C^n$ to itself.

Proof. Since both $\delta_{n-1}$ and $\delta_n$ are linear, it follows that $\Delta_n$ is linear. Let $\omega, \omega' \in C^n$. Then

$$\langle \Delta_n \omega, \omega' \rangle = \langle \delta_n^* \delta_n \omega, \omega' \rangle + \langle \delta_n - 1 \delta_n^* - 1 \omega, \omega' \rangle = \langle \omega, \delta_n^* \delta_n \omega' \rangle + \langle \omega, \delta_n - 1 \delta_n^* - 1 \omega' \rangle = \langle \omega, \Delta_n \omega' \rangle.$$  

Hence $\Delta_n$ is self-adjoint. □

In the following part of this section, we assume that each $\Delta_n$ is a compact bounded linear operator. The following lemma follows from the Spectral Theorem (cf. [21, p. 107 and Theorem 3.1 on p. 443]).

Lemma 2.5. There exists an orthonormal basis $\{e_j \mid j \in J\}$ of $C^n$, where $J$ is an index set, such that each $e_j$ is an eigenvector of $\Delta_n$.

A cochain $\omega \in C^n$ is called harmonic if $\Delta_n \omega = 0$, i.e. $\omega \in \text{Ker} \Delta_n$. We notice that for any $\omega \in C^n$,

$$\langle \Delta_n \omega, \omega' \rangle_{C^n} = \langle \delta_n^* \delta_n \omega, \delta_n \omega' \rangle_{C^n} + \langle \delta_n - 1 \delta_n^* - 1 \omega, \delta_n^* - 1 \omega' \rangle_{C^n} = \langle \delta_n \omega, \delta_n \omega' \rangle_{C^{n+1}} + \langle \delta_n^* - 1 \omega, \delta_n^* - 1 \omega' \rangle_{C^{n-1}}.$$  

Hence $\omega$ is harmonic if and only if

$$\delta_n \omega = \delta_n^* - 1 \omega = 0.$$  

(2.5)

The spectrum of $\Delta_n$ is denoted as

$$\text{Spec}(\Delta_n) = \{\lambda_i \mid i \in \mathcal{I}\},$$

where $\mathcal{I}$ is an index set. The corresponding eigenspaces are denoted as

$$V(\lambda_i), i \in \mathcal{I}.$$  

Let $\tilde{C}^*$ be a sub-cochain complex of $C^*$. We denote the restriction of $\delta_n$ to $\tilde{C}^*$ as $\delta_n \mid_{\tilde{C}^*}$. Then we have an induced Laplacian

$$\Delta_n \mid_{\tilde{C}^*} = \delta_n^* \mid_{\tilde{C}^*} \delta_n \mid_{\tilde{C}^*} + \delta_n - 1 \mid_{\tilde{C}^*} \delta_n^* - 1 \mid_{\tilde{C}^*}.$$  

(2.6)
Here $\Delta_n |_{\tilde{C}^*}$ is a self-adjoint linear operator from $\tilde{C}^n$ to itself. The subspace $\tilde{C}^n$ of $C^n$ is $\Delta_n$-invariant. Hence $\Delta_n |_{\tilde{C}^*}$ is the restriction of $\Delta_n$ to $\tilde{C}^n$. It follows that

$$\text{Ker}(\Delta_n |_{\tilde{C}^*}) = \text{Ker} \Delta_n \cap \tilde{C}^n.$$  \hspace{1cm} (2.7)

Moreover, with the help of Lemma 2.5, the spectrum of $\Delta_n |_{\tilde{C}^*}$ is

$$\text{Spec}(\Delta_n |_{\tilde{C}^*}) = \{ \lambda_i | \dim(V(\lambda_i) \cap \tilde{C}^n) \geq 1, i \in I \}.$$  

And for each $\lambda_i$, the corresponding eigenspace of $\Delta_n |_{\tilde{C}^*}$ is

$$V(\lambda_i) \cap \tilde{C}^n.$$  

Let $\tilde{C}^*$ be $\text{Inf}^*(D^*, C^*)$ and $\text{Sup}^*(D^*, C^*)$ respectively. Then (2.7) implies

$$\text{Ker}(\Delta_n |_{\text{Inf}^*(D^*, C^*)}) = \text{Ker} \Delta_n \cap \text{Inf}^*(D^*, C^*),$$  \hspace{1cm} (2.8)

$$\text{Ker}(\Delta_n |_{\text{Sup}^*(D^*, C^*)}) = \text{Ker} \Delta_n \cap \text{Sup}^*(D^*, C^*).$$  \hspace{1cm} (2.9)

Moreover, the spectrum of $\Delta_n |_{\text{Inf}^*(D^*, C^*)}$ is

$$\{ \lambda_i | \dim(V(\lambda_i) \cap \text{Inf}^*(D^*, C^*)) \geq 1, i \in I \}$$

and the spectrum of $\Delta_n |_{\text{Inf}^*(D^*, C^*)}$ is

$$\{ \lambda_i | \dim(V(\lambda_i) \cap \text{Sup}^*(D^*, C^*)) \geq 1, i \in I \}.$$  

### 2.3 A Hodge-type Theorem for the Embedded Cohomology of Graded Subgroups of Cochain Complexes

We notice that the proof of [12, Theorem 3.3], [18, Theorem 2.2] and [28, Theorem 5.8] apply for any cochain complexes. Hence by an analogous proof of [12, Theorem 3.3], [18, Theorem 2.2] or [28, Theorem 5.8], we obtain the next two lemmas. We give the proofs for completeness.

**Lemma 2.6** (Hodge decomposition of cochain complexes). We have the isomorphism of Hermitian spaces

$$C^n \cong \text{Ker}(\Delta_n) \oplus \delta_n^* C^{n+1} \oplus \delta_{n-1} C^{n-1}.$$  \hspace{1cm} (2.10)

**Proof.** Let $\omega_1 \in \text{Ker}(\Delta_n)$, $\omega_2 \in C^{n+1}$ and $\omega_3 \in C^{n-1}$. Then

$$\langle \delta_n^* \omega_2, \delta_{n-1} \omega_3 \rangle_{C^n} = \langle w_2, \delta_n \delta_{n-1} \omega_3 \rangle_{C^{n+1}} = 0.$$  

And with the help of (2.5),

$$\langle \omega_1, \delta_n^* \omega_2 \rangle_{C^n} = \langle \delta_n \omega_1, \omega_2 \rangle_{C^{n+1}} = 0,$$

$$\langle \omega_1, \delta_{n-1} \omega_3 \rangle_{C^n} = \langle \delta_{n-1}^* \omega_1, \omega_3 \rangle_{C^{n-1}} = 0.$$
Hence the spaces $\text{Ker}(\Delta_n)$, $\delta^*_n C^{n+1}$, and $\delta_{n-1} C^{n-1}$ are orthogonal to each other. To prove (2.10), we assume to the contrary that (2.10) does not hold. Then

$$\text{Ker}(\Delta_n) \oplus \delta^*_n C^{n+1} \oplus \delta_{n-1} C^{n-1} \subset C^n.$$  

Hence there exists a nonzero cochain $\omega_0 \in C^n$ such that $\omega_0$ is orthogonal to $\text{Ker}(\Delta_n) \oplus \delta^*_n C^{n+1} \oplus \delta_{n-1} C^{n-1}$. By using that $\omega_0$ is orthogonal to $\delta^*_n C^{n+1}$ and $\delta_{n-1} C^{n-1}$, we obtain

\[
0 = \langle \omega_0, \delta^*_n \delta_n \omega_0 \rangle_{C^n} = \langle \delta_n \omega_0, \delta_n \omega_0 \rangle_{C^{n+1}},
\]

\[
0 = \langle \omega_0, \delta_{n-1} \delta^*_{n-1} \omega_0 \rangle_{C^n} = \langle \delta^*_{n-1} \omega_0, \delta^*_{n-1} \omega_0 \rangle_{C^{n-1}},
\]

which implies $\delta_n \omega_0 = \delta^*_{n-1} \omega_0 = 0$. Hence $\omega_0$ is harmonic. By using that $\omega_0$ is orthogonal to $\text{Ker}\Delta_n$, we have $\omega_0 = 0$. This contradicts with our assumption that $\omega_0$ is nonzero. Therefore, we have (2.10).

Remark 2: Lemma 2.6 is a generalization of [24, Theorem 4.18] to cochain complexes.

**Lemma 2.7** (Hodge theorem of cochain complexes). We have the isomorphism of complex vector spaces

$$\text{Ker}(\Delta_n) \cong H^n(\{C^*, \delta_s\}).$$  

Moreover, in the isomorphism (2.11), each cohomology class in $H^n(\{C^*, \delta_s\})$ can be represented by a unique harmonic cochain of the class in $\text{Ker}(\Delta_n)$.

Proof. Note that

$$\text{Ker}\delta_n \cap \text{Ker}\delta^*_n \subseteq \text{Ker}\delta_n^* \delta_n \cap \text{Ker}\delta_{n-1} \delta^*_{n-1} \subseteq \text{Ker}\Delta_n.$$  

(2.12)

On the other hand, let $f \in \text{Ker}\Delta_n$. It follows from (2.5) that $\delta_n f = \delta^*_{n-1} f = 0$, which implies $f \in \text{Ker}\delta_n \cap \text{Ker}\delta^*_{n-1}$. Hence

$$\text{Ker}\Delta_n \subseteq \text{Ker}\delta_n \cap \text{Ker}\delta^*_{n-1}.$$  

(2.13)

With the help of (2.12) and (2.13), we have

$$\text{Ker}\Delta_n = \text{Ker}\delta_n \cap \text{Ker}\delta^*_{n-1} = \text{Ker}\delta_n \cap \text{Im}(\delta_{n-1})^\perp \cong H^n(\{C^*, \delta_s\}).$$

The last isomorphism follows from that as complex vector spaces, $\text{Ker}\delta_n \cap \text{Im}(\delta_{n-1})^\perp$ is isomorphic to $\text{Ker}\delta_n / \text{Im}(\delta_{n-1})$. Hence (2.11) follows. The second assertion follows straightforwardly.

Remark 3: Lemma 2.7 is a generalization of [24, Theorem 4.16] to cochain complexes.
The next proposition is obtained from Lemma 2.7.

**Proposition 2.8.** Let $D^\ast$ be a graded complex vector subspace of $C^\ast$. Then we have the isomorphism of complex vector spaces

$$
\text{Ker}(\Delta_n |_{\text{Inf}^n(D^\ast, C^\ast)}) \cong \text{Ker}(\Delta_n |_{\text{Sup}^n(D^\ast, C^\ast)}) \\
\cong H^n(\text{Inf}^n(D^\ast, C^\ast)).
$$

Moreover, each cohomology class can be represented by a unique harmonic cochain of the class.

**Proof.** Applying Lemma 2.7 to the cochain complexes $\text{Inf}^n(D^\ast, C^\ast)$ and $\text{Sup}^n(D^\ast, C^\ast)$ respectively, we obtain

$$
\text{Ker}(\Delta_n |_{\text{Inf}^n(D^\ast, C^\ast)}) \cong H^n(\text{Inf}^n(D^\ast, C^\ast)), \\
\text{Ker}(\Delta_n |_{\text{Sup}^n(D^\ast, C^\ast)}) \cong H^n(\text{Sup}^n(D^\ast, C^\ast)).
$$

With the help of Proposition 2.3 we obtain Proposition 2.8.

The next corollary is a consequence of Proposition 2.8.

**Corollary 2.9.** For any cochain complex $C'^\ast$ such that

$$
\text{Inf}^n(D^\ast, C^\ast) \subseteq C'^n \subseteq \text{Sup}^n(D^\ast, C^\ast),
$$

we have the isomorphism of complex vector spaces

$$
(\text{Ker}\Delta_n) \cap C'^n \cong H^n(\text{Inf}^n(D^\ast, C^\ast)).
$$

**Proof.** By (2.3), (2.9) and Proposition 2.8 as complex vector spaces, both $(\text{Ker}\Delta_n) \cap \text{Inf}^n(D^\ast, C^\ast)$ and $(\text{Ker}\Delta_n) \cap \text{Sup}^n(D^\ast, C^\ast)$ are isomorphic to $H^n(\text{Inf}^n(D^\ast, C^\ast))$. On the other hand, by (2.14),

$$
(\text{Ker}\Delta_n) \cap \text{Inf}^n(D^\ast, C^\ast) \subseteq (\text{Ker}\Delta_n) \cap C'^n \subseteq (\text{Ker}\Delta_n) \cap \text{Sup}^n(D^\ast, C^\ast).
$$

Therefore, as complex vector spaces, $(\text{Ker}\Delta_n) \cap C'^n$ is also isomorphic to $H^n(\text{Inf}^n(D^\ast, C^\ast))$.

The next corollary is a consequence of Lemma 2.6 and Proposition 2.8.

**Corollary 2.10.** Suppose each $C^n$ is finite dimensional. Let $\lambda_i$, $i \in \mathcal{I}$, be the spectrum of $\Delta_n$. Then the dimension of $\text{Sup}^n(D^\ast, C^\ast)$ is greater than the dimension of $\text{Inf}^n(D^\ast, C^\ast)$ with the difference

$$
\sum_{\lambda_i \neq 0} \left( \dim(V(\lambda_i) \cap \text{Sup}^n(D^\ast, C^\ast)) - \dim(V(\lambda_i) \cap \text{Inf}^n(D^\ast, C^\ast)) \right),
$$

or equivalently,

$$
\dim \delta_n^\ast \text{Sup}^{n+1}(D^\ast, C^\ast) + \dim \delta_{n-1}^\ast \text{Sup}^{n-1}(D^\ast, C^\ast) \\
- \dim \delta_n^\ast \text{Inf}^{n+1}(D^\ast, C^\ast) - \dim \delta_{n-1}^\ast \text{Inf}^{n-1}(D^\ast, C^\ast).
$$
Proof. We notice that $\Delta_n$ is a self-adjoint linear operator. Thus $C^n$ is the direct sum of the eigenspaces of $\Delta_n$, i.e.

$$C^n = \oplus_{\lambda_i} V(\lambda_i).$$

Consequently,

$$\text{Sup}^n(D^*, C^*) = \oplus_{\lambda_i} V(\lambda_i) \cap \text{Sup}^n(D^*, C^*),$$

$$\text{Inf}^n(D^*, C^*) = \oplus_{\lambda_i} V(\lambda_i) \cap \text{Inf}^n(D^*, C^*).$$

(2.17) \hspace{1cm} (2.18)

On the other hand, it follows from Proposition 2.8 that

$$V(0) \cap \text{Sup}^n(D^*, C^*) = V(0) \cap \text{Inf}^n(D^*, C^*).$$

(2.19)

Therefore, (2.15) follows from (2.17), (2.18), and (2.19). Moreover, (2.16) follows from Lemma 2.6 and Proposition 2.8. \qed

3 Cohomology and Laplacians of Weighted Hypergraphs

In this section, we define weighted hypergraphs as well as the embedded cohomology of weighted hypergraphs. We consider some Laplacian operators on weighted hypergraphs. In Subsection 3.1, we discuss the embedded cohomology of weighted hypergraphs. In Subsection 3.2, we discuss the Laplacians of weighted hypergraphs. In Subsection 3.3, we give a Hodge-type theorem for weighted hypergraphs.

In the remaining part of this paper, we let $\mathcal{H}$ be a hypergraph. For any hypergraph $\mathcal{H}$ (resp. simplicial complex $\mathcal{K}$) and any $n \geq 0$, we let $\mathcal{H}_n$ (resp. $\mathcal{K}_n$) be the collection of all the $n$-hyperedges of $\mathcal{H}$ (resp. the collection of all the $n$-simplices of $\mathcal{K}$).

3.1 Weighted Hypergraphs and Their Embedded Cohomology

The lower-associated complex $\delta \mathcal{H}$ of $\mathcal{H}$ is the largest simplicial complex that can be embedded in $\mathcal{H}$ (cf. [26, Section 2.1]). It consists of the simplices

$$\delta \mathcal{H} = \{ \eta \in \mathcal{H} \mid \text{for any } \tau \subseteq \eta, \tau \in \mathcal{H} \}.$$

The associated complex $\Delta \mathcal{H}$ of $\mathcal{H}$ is the smallest simplicial complex that $\mathcal{H}$ can be embedded in (cf. [25]). It consists of the simplices (cf. [31, Section 3.1], [26, Section 2.1])

$$\Delta \mathcal{H} = \{ \eta \subseteq \tau \mid \tau \in \mathcal{H} \}.$$

Let $R$ be a commutative ring with unit. Let $R(\mathcal{H})$ be the free $R$-module generated by all the hyperedges of $\mathcal{H}$. Let $G$ be an $R$-module. The weights on $\mathcal{H}$ are defined in the next definition.
Definition 3. A weight on $\mathcal{H}$ is a bilinear map $\phi : R(\mathcal{H}) \times R(\mathcal{H}) \rightarrow R$ such that

$$\phi(d_i\sigma, d_jd_i\sigma)\phi(\sigma, d_i\sigma)g = \phi(d_j\sigma, d_jd_i\sigma)\phi(\sigma, d_j\sigma)g$$

for any $\sigma \in \mathcal{H}$, any $g \in G$ and any $j < i$. Here $d_i\sigma$ (resp. $d_j\sigma$, $d_jd_i\sigma$) is regarded as the zero element $0 \in R(\mathcal{H})$ (resp. $d_j\sigma \notin \mathcal{H}$, $d_jd_i\sigma \notin \mathcal{H}$).

Remark 4: Definition 3 is a generalization of [28, Definition 2.1] from (weighted) simplicial complexes to (weighted) hypergraphs. In particular, if $\mathcal{H}$ is a simplicial complex, then Definition 3 is equivalent to the weights on simplicial complexes given in [28, Definition 2.1].

Suppose $\phi$ is a weight on $\mathcal{H}$. Then $\phi$ extends to be a weight $\phi^{\Delta}$ on the simplicial complex $\Delta \mathcal{H}$. For any distinct simplices $\tau, \tau' \in \Delta \mathcal{H}$, if both $\tau$ and $\tau'$ are hyperedges of $\mathcal{H}$, then we let

$$\phi^{\Delta}(\tau, \tau') = \phi(\tau, \tau').$$

Otherwise, we let

$$\phi^{\Delta}(\tau, \tau') = 0.$$
By [28, Definition 4.5], the **weighted coboundary maps** $\delta_n$ are the dual maps of $\partial_n$, $n \geq 0$. We have a cochain complex

$$\{G(\Delta \mathcal{H})^*, \delta_*\}.$$ \hfill (3.2)

Moreover, we let

$$G(\delta \mathcal{H})^* = \text{Hom}_G(G((\delta \mathcal{H})_*); G).$$

We let $\delta_n |_{\delta \mathcal{H}}$ be the dual operator of $\partial_n |_{\delta \mathcal{H}}$. Then $\delta_n |_{\delta \mathcal{H}}$ is the restriction of $\delta_n$ to $G(\delta \mathcal{H})^*$. We have a cochain complex

$$\{G(\delta \mathcal{H})^*, \delta_* |_{\delta \mathcal{H}}\},$$

which is a sub-cochain complex of (3.2). Furthermore, we consider

$$G(\mathcal{H})^* = \text{Hom}_G(G(\mathcal{H}_*); G).$$

Then as graded groups and graded subgroups,

$$G(\delta \mathcal{H})^* \subseteq \text{Inf}^*(G(\mathcal{H})^*, G(\Delta \mathcal{H})^*) \subseteq G(\mathcal{H})^*$$

$$\subseteq \text{Sup}^*(G(\mathcal{H})^*, G(\Delta \mathcal{H})^*) \subseteq G(\Delta \mathcal{H})^*. \hfill (3.3)$$

And as cochain complexes and sub-cochain complexes,

$$G(\delta \mathcal{H})^* \subseteq \text{Inf}^*(G(\mathcal{H})^*, G(\Delta \mathcal{H})^*)$$

$$\subseteq \text{Sup}^*(G(\mathcal{H})^*, G(\Delta \mathcal{H})^*) \subseteq G(\Delta \mathcal{H})^*. \hfill (3.4)$$

Finally, by Proposition [28, Proposition 2.3], we define the $n$-dimensional $\phi$-**weighted embedded cohomology** of $\mathcal{H}$ as

$$H^n(\mathcal{H}, \phi; G) = H^n(\text{Inf}^*(G(\mathcal{H})^*, G(\Delta \mathcal{H})^*), \delta_* |_{\text{Inf}^*(G(\mathcal{H})^*, G(\Delta \mathcal{H})^*)})$$

$$\cong H^n(\text{Sup}^*(G(\mathcal{H})^*, G(\Delta \mathcal{H})^*), \delta_* |_{\text{Sup}^*(G(\mathcal{H})^*, G(\Delta \mathcal{H})^*)}). \hfill (3.5)$$

**Remark 5:** There are some particular cases of the $\phi$-weighted embedded cohomology:

(i). If $\mathcal{H}$ is a simplicial complex, then (3.5) is the same as the weighted cohomology of simplicial complexes defined in [28, Definition 4.7];

(ii). If $\phi$ in Definition [4.5] takes the constant value 1, then $\mathcal{H}$ is the usual hypergraph. And analogous to the embedded homology discussed in [4], $H^n(\mathcal{H}, \phi; G)$ is the usual embedded cohomology of hypergraphs.
3.2 Weighted Hypergraphs, their Laplacians, and Spectrums

In the remaining part of this section, we suppose in addition that both \( R \) and \( G \) are the complex numbers \( \mathbb{C} \).

By Definition 3 and the proof of [28, Proposition 3.1], a weight \( \phi \) on \( \mathcal{H} \) must satisfy

\[
\phi(d_i \sigma, d_j d_i \sigma) \phi(\sigma, d_i \sigma) = \phi(d_j \sigma, d_j d_i \sigma) \phi(\sigma, d_j \sigma)
\]

for any \( \sigma \in \mathcal{H} \) and any \( j < i \). Let \( \langle \cdot, \cdot \rangle \) be the canonical Hermitian product given by

\[
\langle \sum_i a_i \sigma_i, \sum_j b_j \tau_j \rangle = \sum_{a_i = \tau_j} a_i \bar{b}_j.
\]

Here \( \bar{b}_j \) is the conjugate complex number of \( b_j \). Let \( \Delta_* \) be the Laplacians of the cochain complex \( (\mathbb{C}(\mathcal{H})^*, \delta_*) \). The next definition follows from (3.4).

**Definition 4.** Let \( n \) be a nonnegative integer.

(i). We call \( \Delta_n \) the \( n \)-th \( \phi \)-weighted associated Laplacian of \( \mathcal{H} \) and denote it as \( \Delta_n^{(\Delta \mathcal{H}, \phi)} \).

It is the \( \phi \)-weighted Laplacian of the associated complex \( \Delta \mathcal{H} \) of \( \mathcal{H} \);

(ii). We call \( \Delta_n |_{\mathbb{C}(\delta \mathcal{H})^*} \), the restriction of \( \Delta_n \) to \( \mathbb{C}(\delta \mathcal{H})^* \), the \( n \)-th \( \phi \)-weighted lower-associated Laplacian of \( \mathcal{H} \) and denote it as \( \Delta_n^{(\delta \mathcal{H}, \phi)} \). It is the \( \phi \)-weighted Laplacian of the lower-associated complex \( \delta \mathcal{H} \) of \( \mathcal{H} \);

(iii). We call \( \Delta_n |_{\text{Inf}^* (\mathbb{C}(\mathcal{H})^*, \mathbb{C}(\Delta \mathcal{H})^*)} \) the \( n \)-th \( \phi \)-weighted infimum Laplacian of \( \mathcal{H} \) and denote it as \( \Delta_n^{(\text{Inf} \mathcal{H}, \phi)} \);

(iv). We call \( \Delta_n |_{\text{Sup}^* (\mathbb{C}(\mathcal{H})^*, \mathbb{C}(\Delta \mathcal{H})^*)} \) the \( n \)-th \( \phi \)-weighted supremum Laplacian of \( \mathcal{H} \) and denote it as \( \Delta_n^{(\text{Sup} \mathcal{H}, \phi)} \).

Since \( \mathcal{H} \) is assumed to be finite, each of the Laplacians \( \Delta_n^{(\Delta \mathcal{H}, \phi)}, \Delta_n^{(\delta \mathcal{H}, \phi)}, \Delta_n^{(\text{Inf} \mathcal{H}, \phi)} \) and \( \Delta_n^{(\text{Sup} \mathcal{H}, \phi)} \) is a self-adjoint and compact bounded linear operator. With the help of Subsection [2.2], the next proposition is an observation of the Laplacians defined in Definition 4.

**Proposition 3.1.** For any nonnegative integer \( n \), the spectra of \( \Delta_n^{(\delta \mathcal{H}, \phi)}, \Delta_n^{(\text{Inf} \mathcal{H}, \phi)} \) and \( \Delta_n^{(\text{Sup} \mathcal{H}, \phi)} \) are all subsets of the spectrum of \( \Delta_n^{(\Delta \mathcal{H}, \phi)} \). The eigenspaces of \( \Delta_n^{(\delta \mathcal{H}, \phi)}, \Delta_n^{(\text{Inf} \mathcal{H}, \phi)} \) and \( \Delta_n^{(\text{Sup} \mathcal{H}, \phi)} \) are the intersections of the eigenspaces of \( \Delta_n^{(\Delta \mathcal{H}, \phi)} \) with \( \mathbb{C}(\delta \mathcal{H})^* \), \( \text{Inf}^* (\mathbb{C}(\mathcal{H})^*, \mathbb{C}(\Delta \mathcal{H})^*) \) and \( \text{Sup}^* (\mathbb{C}(\mathcal{H})^*, \mathbb{C}(\Delta \mathcal{H})^*) \) respectively.

With the help of our assumption that \( \mathcal{H} \) is finite, the next proposition follows from Lemma 2.5.

**Proposition 3.2.** There exists an orthonormal basis \( \{e_j \mid j \in J\} \) of \( \mathbb{C}(\Delta \mathcal{H})^n \) (resp. \( \mathbb{C}(\delta \mathcal{H})^n \), \( \text{Inf}^* (\mathbb{C}(\mathcal{H})^*, \mathbb{C}(\Delta \mathcal{H})^*) \) and \( \text{Sup}^* (\mathbb{C}(\mathcal{H})^*, \mathbb{C}(\Delta \mathcal{H})^*) \) ), where \( J \) is a finite index set, such that each \( e_j \) is an eigenvector of \( \Delta_n^{(\Delta \mathcal{H}, \phi)} \) (resp. \( \Delta_n^{(\delta \mathcal{H}, \phi)}, \Delta_n^{(\text{Inf} \mathcal{H}, \phi)} \) and \( \Delta_n^{(\text{Sup} \mathcal{H}, \phi)} \)).
Remark 6: Suppose we do not assume that \( H \) is finite. In order to get Proposition 3.2, we need to ensure that each of the Laplacians \( \Delta_n^{(H, \phi)} \), \( \Delta_n^{(\delta H, \phi)} \), \( \Delta_n^{(\text{Inf} H, \phi)} \) and \( \Delta_n^{(\text{Sup} H, \phi)} \) is a self-adjoint and compact bounded linear operator. We impose the additional condition:

\[ (\ast). \text{for each } n \geq 0, \delta_n \text{ is a compact bounded linear operator}. \]

By [21, p. 416-417], (\ast) implies that \( \Delta_n \) is a compact bounded linear operator. Thus Lemma 2.5 can be applied and Proposition 3.2 can be obtained.

3.3 A Hodge-type Theorem for Weighted Hypergraphs

The next theorem follows from Lemma 2.6.

**Theorem 3.3** (Hodge-type decomposition for weighted hypergraphs). We have the isomorphisms of Hermitian spaces

\[
\begin{align*}
\mathbb{C}(\Delta H)^n & \cong \ker(\Delta_n^{(H, \phi)}) \oplus \delta_n^*(\mathbb{C}(\Delta H)^{n+1}) \oplus \delta_n(\mathbb{C}(\Delta H)^{n-1}), \\
\mathbb{C}(\delta H)^n & \cong \ker(\Delta_n^{(\delta H, \phi)}) \oplus \delta_n^*(\mathbb{C}(\delta H)^{n+1}) \oplus \delta_n(\mathbb{C}(\delta H)^{n-1}), \\
\text{Inf}^n(\mathbb{C}(H)^*, \mathbb{C}(\Delta H)^*) & \cong \ker(\Delta_n^{(\text{Inf} H, \phi)}) \oplus \delta_n^*(\text{Inf}^{n+1}(\mathbb{C}(H)^*, \mathbb{C}(\Delta H)^*)) \\
& \quad \oplus \delta_n(\text{Inf}^{n-1}(\mathbb{C}(H)^*, \mathbb{C}(\Delta H)^*)), \\
\text{Sup}^n(\mathbb{C}(H)^*, \mathbb{C}(\Delta H)^*) & \cong \ker(\Delta_n^{(\text{Sup} H, \phi)}) \oplus \delta_n^*(\text{Sup}^{n+1}(\mathbb{C}(H)^*, \mathbb{C}(\Delta H)^*)) \\
& \quad \oplus \delta_n(\text{Sup}^{n-1}(\mathbb{C}(H)^*, \mathbb{C}(\Delta H)^*)).
\end{align*}
\]

The next theorem follows from Proposition 2.8 and (3.5).

**Theorem 3.4** (Hodge-type theorem for weighted hypergraphs). For any nonnegative integer \( n \), we have the isomorphism of complex vector spaces

\[ \ker(\Delta_n^{(\text{Inf} H, \phi)}) \cong \ker(\Delta_n^{(\text{Sup} H, \phi)}) \cong H^n(H, \phi). \tag{3.7} \]

Here the coefficients of the cohomology are the complex numbers \( \mathbb{C} \). Moreover, in the isomorphism (3.7), each cohomology class in \( H^n(H, \phi) \) can be represented by a unique harmonic cochain of the class in \( \ker(\Delta_n^{(\text{Inf} H, \phi)}) \), and also can be represented by a unique harmonic cochain of the class in \( \ker(\Delta_n^{(\text{Sup} H, \phi)}) \).

We can apply Corollary 2.9 and Corollary 2.10 to the weighted hypergraphs. We omit the exact statements since the statements can be obtained from Corollary 2.9 and Corollary 2.10 by minor modifications. The next two corollaries also follow from Theorem 3.4.

**Corollary 3.5.** For any nonnegative integer \( n \),

\[ \dim H^n(\delta H, \phi) \leq \dim H^n(H, \phi) \leq \dim H^n(\Delta H, \phi). \]

Here the coefficients of the cohomology are the complex numbers \( \mathbb{C} \).
Proof. It follows from Proposition 3.1 that

\[ \text{Ker}(\Delta_n^{(\delta H, \phi)}) \subseteq \text{Ker}(\Delta_n^{(\inf H, \phi)}) \subseteq \text{Ker}(\Delta_n^{(H, \phi)}). \]  

(3.8)

On the other hand, by \[28\text{, Theorem 5.8}], we have the isomorphisms of complex vector spaces

\[ H^n(\delta H, \phi) \cong \text{Ker}(\Delta_n^{(\delta H, \phi)}), \]  

(3.9)

\[ H^n(\Delta H, \phi) \cong \text{Ker}(\Delta_n^{(\delta H, \phi)}). \]  

(3.10)

By substituting Theorem 3.4, (3.9) and (3.10) into (3.8), and counting the dimensions, the assertion follows.

Corollary 3.6. Suppose the hypergraph \( H \) is not a simplicial complex. Suppose \( \phi \) takes the constant value 1. Then for any \( n \geq 0 \), there exists a nonzero eigenvalue \( \lambda \) such that at least one of the following holds

(i). \( \lambda \) is an eigenvalue of \( \Delta_n^{(\text{Sup} H, \phi)} \) but not an eigenvalue of \( \Delta_n^{(\text{Inf} H, \phi)} \);

(ii). \( \lambda \) is an eigenvalue of both \( \Delta_n^{(\text{Sup} H, \phi)} \) and \( \Delta_n^{(\text{Inf} H, \phi)} \). The dimension of the corresponding eigenspace of \( \Delta_n^{(\text{Sup} H, \phi)} \) is strictly greater than the dimension of the corresponding eigenspace of \( \Delta_n^{(\text{Inf} H, \phi)} \).

Proof. Since \( H \) is not a simplicial complex and \( \phi \) takes the constant value 1, it follows that \( \mathbb{C}(H)^* \) are not cochain complexes. Hence by (3.3), we have

\[ \text{Inf}^n(\mathbb{C}(H)^*, \mathbb{C}(H)^*) \subseteq \text{Sup}^n(\mathbb{C}(H)^*, \mathbb{C}(H)^*). \]  

(3.11)

On the other hand, it follows from Proposition 3.2 that

\[ \text{Inf}^n(\mathbb{C}(H)^*, \mathbb{C}(H)^*) = \text{Ker}(\Delta_n^{(\text{Inf} H, \phi)}) \oplus (\oplus_{\lambda \neq 0} U(\lambda)), \]  

(3.12)

\[ \text{Sup}^n(\mathbb{C}(H)^*, \mathbb{C}(H)^*) = \text{Ker}(\Delta_n^{(\text{Sup} H, \phi)}) \oplus (\oplus_{\lambda' \neq 0} W(\lambda')). \]  

(3.13)

Here in (3.12), \( \lambda \) runs over all the nonzero eigenvalues of \( \Delta_n^{(\text{Inf} H, \phi)} \) and \( U(\lambda) \) is the corresponding eigenspace of \( \lambda \); while in (3.13), \( \lambda' \) runs over all the nonzero eigenvalues of \( \Delta_n^{(\text{Sup} H, \phi)} \) and \( W(\lambda') \) is the corresponding eigenspace of \( \lambda' \). By Theorem 3.4, (3.11), (3.12) and (3.13), it follows that

\[ \oplus_{\lambda \neq 0} U(\lambda) \subsetneq \oplus_{\lambda' \neq 0} W(\lambda'). \]

The assertion follows.

Remark 7: Corollary 3.6 characterizes the Laplacians of non-simplicial hypergraphs, with trivial weights.
4 Some Examples and a Classification of Weight Functions

In this section, we give some examples to illustrate the impact of the weight function \( \phi \) on the \( \phi \)-weighted embedded cohomology and the \( \phi \)-weighted Laplacians. We give a classification of the weight functions \( \phi \) satisfying certain conditions on hypergraphs in Proposition 4.3. Throughout this section, we let \( \mathcal{H} \) be a hypergraph. We let \( w: \mathcal{H} \rightarrow \mathbb{R}^+ \subseteq \mathbb{C} \) be an evaluation function with positive real values on the hyperedges of \( \mathcal{H} \).

4.1 The First Example

The first family of weight functions on \( \mathcal{H} \) is described in the next example.

Example 4.1. For any \( \tau, \tau' \in \mathcal{H} \), let
\[
\phi(\tau, \tau') = C \cdot \frac{w(\tau)}{w(\tau')}. \tag{4.1}
\]
Here \( C \) is a constant positive real number which does not depend on the choice of \( \tau \) and \( \tau' \). We extend \( \phi \) bilinearly over \( \mathbb{C} \). It is straightforward to verify that \( \phi \) is a weight on \( \mathcal{H} \).

Remark 8: Particularly, let \( \mathcal{H} \) be a simplicial complex in Example 4.1. We obtain the weighted simplicial complex model studied by D. Horak and J. Jost in [17, 18].

Definition 5. Let \( \{v_0, \ldots, v_{n+1}\} \) be an \((n+1)\)-hyperedge of a hypergraph \( \mathcal{H} \). Suppose for some \( 0 \leq k \leq n \), \( \{v_0, \ldots, \hat{v}_k, \ldots, v_{n+1}\} \) is an \( n \)-hyperedge of \( \mathcal{H} \). The boundary of the oriented face \( \{v_0, \ldots, v_{n+1}\} \) is
\[
\partial_n(\{v_0, \ldots, v_{n+1}\}) = \sum_{i=0}^{n+1} (-1)^i \{v_0, \ldots, \hat{v}_i, \ldots, v_{n+1}\}.
\]
The sign of \( \{v_0, \ldots, \hat{v}_k, \ldots, v_{n+1}\} \) in the boundary of \( \{v_0, \ldots, v_{n+1}\} \) is
\[
\text{sgn}(\{v_0, \ldots, \hat{v}_k, \ldots, v_{n+1}\}, \partial_n(\{v_0, \ldots, v_{n+1}\})) = (-1)^k.
\]

We consider a weight \( \phi \) given in (4.1) on a hypergraph \( \mathcal{H} \). The \( \phi \)-weighted Laplacians of \( \mathcal{H} \) can be calculated as follows.

(i) For any \( f \in \text{Hom}(\mathbb{C}((\Delta \mathcal{H})_n); \mathbb{C}) \) and any simplex \( \{v_0, \ldots, v_n\} \in \Delta \mathcal{H} \), the \( n \)-th \( \phi \)-
weighted associated Laplacian $\Delta_n^{(\Delta_H, \phi)}$ of $\mathcal{H}$ is given as follows:

$$
(\Delta_n^{(\Delta_H, \phi)} f)(\{v_0, \ldots, v_n\})
= \sum_{\sigma \in (\Delta_H)_{n+1}} \frac{w(\sigma)}{w(\{v_0, \ldots, v_n\})} f(\{v_0, \ldots, v_n\})
+ \sum_{\tau \in (\Delta_H)_n, \tau \neq v_0, \ldots, v_n, \tau \subseteq \sigma} \frac{w(\sigma)}{w(\{v_0, \ldots, v_n\})} \text{sgn}(v_0, \ldots, v_n, \partial \sigma) \times \text{sgn}(\tau, \partial \sigma) f(\tau)
+ \sum_{\eta \in (\Delta_H)_{n-1}, \eta \subseteq v_0, \ldots, v_n} \frac{w(\eta)}{w(\{v_0, \ldots, v_n\})} f(\{v_0, \ldots, v_n\})
+ \sum_{\theta \in (\Delta_H)_n, \{v_0, \ldots, v_n \} \cap \theta = \eta} \frac{w(\theta)}{w(\eta)} \text{sgn}(\eta, \partial \{v_0, \ldots, v_n\}) \times \text{sgn}(\eta, \partial \theta) f(\theta).
$$

The equation (4.2) follows from [18, Section 2]. By the discrete Hodge theorem of weighted simplicial complexes (cf. [28, Theorem 5.8]), we have the isomorphism of complex vector spaces

$$\text{Ker}(\Delta_n^{(\Delta_H, \phi)}) \cong H^n(\Delta_H, \phi; \mathbb{C}).$$

(ii). For any $f \in \text{Hom}(\mathcal{C}(\delta \mathcal{H})_n; \mathbb{C})$, and any simplex $\{v_0, \ldots, v_n\} \in \delta \mathcal{H}$, the $n$-th $\phi$-weighted lower-associated Laplacian $\Delta_n^{(\delta \mathcal{H}, \phi)}$ of $\mathcal{H}$ is given as follows:

$$
(\Delta_n^{(\delta \mathcal{H}, \phi)} f)(\{v_0, \ldots, v_n\})
= \sum_{\sigma \in (\delta \mathcal{H})_{n+1}} \frac{w(\sigma)}{w(\{v_0, \ldots, v_n\})} f(\{v_0, \ldots, v_n\})
+ \sum_{\tau \in (\delta \mathcal{H})_n, \tau \neq v_0, \ldots, v_n, \tau \subseteq \sigma} \frac{w(\sigma)}{w(\{v_0, \ldots, v_n\})} \text{sgn}(v_0, \ldots, v_n, \partial \sigma) \times \text{sgn}(\tau, \partial \sigma) f(\tau)
+ \sum_{\eta \in (\delta \mathcal{H})_{n-1}, \eta \subseteq v_0, \ldots, v_n} \frac{w(\eta)}{w(\{v_0, \ldots, v_n\})} f(\{v_0, \ldots, v_n\})
+ \sum_{\theta \in (\delta \mathcal{H})_n, \{v_0, \ldots, v_n \} \cap \theta = \eta} \frac{w(\theta)}{w(\eta)} \text{sgn}(\eta, \partial \{v_0, \ldots, v_n\}) \times \text{sgn}(\eta, \partial \theta) f(\theta).
$$

The equation (4.3) also follows from [18, Section 2]. Also by the discrete Hodge theorem of weighted simplicial complexes (cf. [28, Theorem 5.8]), we have the isomorphism of complex vector spaces
vector spaces

\[ \text{Ker}(\Delta_n^{(\delta H, \phi)}) \cong H^n(\delta H, \phi; \mathbb{C}). \]

(iii). Let \( f \in \text{Hom}(\text{Inf}_n(\mathbb{C}(\mathcal{H}_*), \mathbb{C}((\Delta \mathcal{H})_*)); \mathbb{C}) \). Let \( \omega \in \text{Inf}_n(\mathbb{C}(\mathcal{H}_*), \mathbb{C}((\Delta \mathcal{H})_*)). \) By [4, Proposition 2.1],

\[ \text{Inf}_n(\mathbb{C}(\mathcal{H}_*), \mathbb{C}((\Delta \mathcal{H})_*)) = \mathbb{C}(\mathcal{H}_n) \cap \partial_{n-1} \mathbb{C}(\mathcal{H}_{n-1}). \]

Hence we may write \( \omega = \sum_i a_i \sigma_i \) where \( a_i \in \mathbb{C}, \sigma_i \in \mathcal{H}_n, \) and \( \sum_i a_i \partial_n \sigma_i \in \mathbb{C}(\mathcal{H}_{n-1}). \) The \( n \)-th \( \phi \)-weighted infimum Laplacian \( \Delta_n^{(\text{Inf} \mathcal{H}, \phi)} \) of \( \mathcal{H} \) is given as follows:

\[ (\Delta_n^{(\text{Inf} \mathcal{H}, \phi)} f)(\omega) = \sum_i a_i (\Delta_n^{(\Delta \mathcal{H}, \phi)} f)(\sigma_i). \]

(iv). Let \( f \in \text{Hom}(\text{Sup}_n(\mathbb{C}(\mathcal{H}_*), \mathbb{C}((\Delta \mathcal{H})_*)); \mathbb{C}) \). Let \( \omega \in \text{Sup}_n(\mathbb{C}(\mathcal{H}_*), \mathbb{C}((\Delta \mathcal{H})_*)). \) By [4, Proposition 2.2],

\[ \text{Sup}_n(\mathbb{C}(\mathcal{H}_*), \mathbb{C}((\Delta \mathcal{H})_*)) = \mathbb{C}(\mathcal{H}_n) + \partial_{n+1} \mathbb{C}(\mathcal{H}_{n+1}). \]

Hence we may write

\[ \omega = \sum_i a_i \sigma_i + \sum_j b_j \sum_{k=0}^{n+1} (-1)^k \partial_{n+1}^k \tau_j \tag{4.4} \]

where \( a_i, b_j \in \mathbb{C}, \sigma_i \in \mathcal{H}_n, \tau_j \in \mathcal{H}_{n+1}, \) and

\[ \sum_j b_j \sum_{k=0}^{n+1} (-1)^k \partial_{n+1}^k \tau_j \in \mathbb{C}(\mathcal{H}_n). \tag{4.5} \]

Here in (4.4) and (4.5), we write

\[ \partial_{n+1} = \sum_{k=0}^{n+1} (-1)^k \partial_{n+1}^k, \]

and we use \( \partial_{n+1}^k \) to denote the face map by deleting the \( k \)-th vertex. The \( n \)-th \( \phi \)-weighted supremum Laplacian \( \Delta_n^{(\text{Sup} \mathcal{H}, \phi)} \) of \( \mathcal{H} \) is given as follows:

\[ (\Delta_n^{(\text{Sup} \mathcal{H}, \phi)} f)(\omega) = \sum_i a_i (\Delta_n^{(\Delta \mathcal{H}, \phi)} f)(\sigma_i) + \sum_j b_j \sum_{k=0}^{n+1} (\Delta_n^{(\Delta \mathcal{H}, \phi)} f)(\partial_{n+1}^k \tau_j). \]

By Theorem 3.4 we have the isomorphism of complex vector spaces

\[ \text{Ker}(\Delta_n^{(\text{Inf} \mathcal{H}, \phi)}) \cong \text{Ker}(\Delta_n^{(\text{Sup} \mathcal{H}, \phi)}) \cong H^n(\mathcal{H}, \phi; \mathbb{C}). \]
4.2 The Second Example and a Classification

The second family of weight functions on \( \mathcal{H} \) is described in the next example.

**Example 4.2.** Let \( g : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R} \) be a map such that

\[
g(x, y)g(y, z) = g(x, w)g(w, z)
\]

for any \( x, y, z, w \in \mathbb{R}^+ \). For any \( \tau, \tau' \in \mathcal{H} \), let

\[
\phi(\tau, \tau') = g(w(\tau), w(\tau')).
\]

We extend \( \phi \) bilinearly over \( \mathbb{C} \). It is straightforward to verify that \( \phi \) is a weight on \( \mathcal{H} \). Let \( A, B \subset \mathbb{R}^+ \) be proper subsets of \( \mathbb{R}^+ \) such that \( A \cup B = \mathbb{R}^+ \). Suppose in addition, \( g(x, y) = 0 \) if either \( x \in A \) or \( y \in B \). Then we call the weight \( \phi \) given by (4.6) a semi-trivial weight.

**Remark 9:** Particularly, let \( \mathcal{H} \) be a simplicial complex in Example 4.2. We obtain the semi-trivial weight on simplicial complexes defined in [28, Section 3].

We consider a weight \( \phi \) given in (4.6) on a hypergraph \( \mathcal{H} \). Analogous to the proof of [28, Proposition 3.5], the next proposition classifies all the weights on \( \mathcal{H} \) given by (4.6).

**Proposition 4.3.** A weight on \( \mathcal{H} \) given by (4.6) is either a semi-trivial weight or a weight given by (4.1).

5 Joins of Weighted Hypergraphs and the Weighted Embedded Cohomology

The Laplacian of joins of weighted simplicial complexes has been studied by D. Horak and J. Jost in [18, Section 6.2]. We generalize the definition of joins of weighted simplicial complexes and study the weighted Laplacians of joins of weighted hypergraphs. As by-products of Theorem 3.4 we prove Theorem 5.1 for the weighted embedded cohomology of joins of hypergraphs. Throughout this section, we suppose that both \( R \) and \( G \) are the complex numbers \( \mathbb{C} \).

For any positive integer \( m \), let \( [m] = \{1, 2, \ldots, m\} \) denote a set of \( m \) vertices. Let \( \mathcal{H}[m] = \{ \mathcal{H} \mid V(\mathcal{H}) \subseteq [m] \} \) denote the collection of all the hypergraphs whose sets of vertices are subsets of \( [m] \). Let \( m \) and \( m' \) be positive integers. Let \( \mathcal{H} \in \mathcal{H}[m] \) and \( \mathcal{H}' \in \mathcal{H}[m'] \). We define the join of \( \mathcal{H} \) and \( \mathcal{H}' \) as the hypergraph

\[
\mathcal{H} \ast \mathcal{H}' = \{ \sigma \sqcup \sigma' \mid \sigma \in \mathcal{H}, \sigma' \in \mathcal{H}' \},
\]

where \( \sqcup \) denotes the disjoint union. Suppose \( \phi \) is a weight on \( \mathcal{H} \) and \( \phi' \) is a weight on \( \mathcal{H}' \). We define the weighted join of \((\mathcal{H}, \phi)\) and \((\mathcal{H}', \phi')\) as the weighted hypergraph

\[
(\mathcal{H}, \phi) \ast (\mathcal{H}', \phi') = (\mathcal{H} \ast \mathcal{H}', \phi \ast \phi'),
\]
where $\phi \ast \phi'$ is a weight on $H \ast H'$ given by

$$(\phi \ast \phi')(\sigma \sqcup \sigma', \tau \sqcup \tau') = \phi(\sigma, \tau)\phi'(\sigma', \tau').$$

(5.2)

Here $\sigma, \tau \in H$ and $\sigma', \tau' \in H'$. By (5.1) and the definitions of associated complexes and lower-associated complexes, the associated complexes and the lower-associated complexes of the joins of hypergraphs are

$$\Delta(H \ast H') = \Delta H \ast \Delta H',$$

$$\delta(H \ast H') = \delta H \ast \delta H'.$$

Hence by [12, p. 4325] or [18, p. 326],

$$C(\Delta(H \ast H'))^0 = C(\Delta H)^0 \oplus C(\Delta H')^0;$$

$$C(\Delta(H \ast H'))^n = \bigoplus_{n_1+n_2+1=n} C(\Delta H)^{n_1} \otimes C(\Delta H')^{n_2}, \ n \geq 1.$$  (5.3)

Moreover, the graded subgroups $C(H \ast H')^*$ of the joins of hypergraphs can be written as

$$C(H \ast H')^0 = C(H)^0 \oplus C(H')^0;$$

$$C(H \ast H')^n = \bigoplus_{n_1+n_2+1=n} C(H)^{n_1} \otimes C(H')^{n_2}, \ n \geq 1.$$  (5.4)

Let $p_i : C(\Delta H)^i \rightarrow H^i((\Delta H); \mathbb{C})$ and $p'_j : C(\Delta H')^j \rightarrow H^j((\Delta H'); \mathbb{C})$ be the canonical quotient maps induced from the definition of cohomology. The next theorem follows (5.3) - (5.11).

**Theorem 5.1.** As complex vector spaces,

$$H^n(H \ast H', \phi \ast \phi'; \mathbb{C})$$

$$\cong \sum_{i+j=n} \left( p_i(C(H)^i) \otimes H^j(H', \phi'; \mathbb{C}) + H^i(H, \phi; \mathbb{C}) \otimes (p'_j(C(H')^j)) \right).$$

(5.7)

Before proving Theorem 5.1 firstly we prove the following lemma.

**Lemma 5.2.** For $n \geq 1$,

(a). the weighted coboundary map $\delta_n$ is a graded derivation

$$\delta_n(\omega \otimes \omega') = \delta_i \omega \otimes \omega' + (-1)^i \omega \otimes \delta_j \omega'$$

for any $\omega \in C(H)^i$ and any $\omega' \in C(H')^j$ with $i+j=n$;

(b). the adjoint map $\delta^*_n$ of the weighted coboundary map is a graded derivation

$$\delta^*_n(\omega \otimes \omega') = \delta^*_i \omega \otimes \omega' + (-1)^i \omega \otimes \delta^*_j \omega'$$

for any $\omega \in C(H)^i$ and any $\omega' \in C(H')^j$ with $i'+j' = n+1$.  

19
Proof. (a). We notice that the derivation property of the coboundary map $\delta_\ast$ of simplicial complexes given in [18, p. 326] also applies to general cochain complexes. We consider the cochain complex $\{C(\Delta(H \ast H')), \delta_\ast\}$ (cf. Subsection 3.1) where $\delta_\ast$ are the weighted coboundary maps. Restricting $\delta_\ast$ to $C(H \ast H')^\ast$, we obtain (5.8).

(b). For any $\omega \otimes \omega'$ and $\eta \otimes \eta'$, where $\omega, \eta \in C(\Delta H)^i$ and $\omega', \eta' \in C(\Delta H')^j$, $i + j = n$, we have

$$
\langle \delta_\ast^\ast(\omega \otimes \omega'), \eta \otimes \eta' \rangle = \langle \omega \otimes \omega', \delta_n(\eta \otimes \eta') \rangle
= \langle \omega, \delta_n(\eta) \rangle \langle \omega', \eta' \rangle + (-1)^i \langle \omega, \delta_j(\eta) \rangle \langle \omega', \eta' \rangle
= \langle \delta_\ast^\ast \omega, \eta \rangle \langle \omega', \eta' \rangle + (-1)^i \langle \omega, \delta_\ast^\ast \omega', \eta' \rangle
= \langle \delta_\ast^\ast \omega \otimes \omega' + (-1)^i \omega \otimes \delta_\ast^\ast \omega', \eta \otimes \eta' \rangle.
$$

Here we use $\langle \omega, \eta \rangle$ to denote $\langle \omega, \eta \rangle_{C(\Delta H)}^i$ if $i' = i$ and to denote 0 otherwise. Hence by choosing $\omega, \eta \in C(H)^i$ and choosing $\omega', \eta' \in C(H')^j$, (5.9) follows. An analogous statement of (5.9) is given in [12, p. 4326] without proofs. 

It follows from (5.3) - (5.8) that the infimum cochain complex and the supremum cochain complex of the join $H \ast H'$ can be written respectively as

$$
\text{Inf}^n\left(C(H \ast H')^\ast, C(\Delta(H \ast H'))^\ast\right)
= \left( \bigoplus_{i+j=n} C(H)^i \otimes C(H')^j \right) \cap \delta_n^{-1}\left( \bigoplus_{i+j+1=n+1} C(H)^i \otimes C(H')^j \right)
= \left( \bigoplus_{i+j=n} C(H)^i \otimes C(H')^j \right)
\cap \left( \bigoplus_{i'+j'+1=n+1} \delta^{-1}_{i'-1}C(H)^i' \otimes C(H')^{j'} \oplus C(H)^i' \otimes \delta^{-1}_{j'-1}G(H')^{j'} \right)
= \bigoplus_{i+j=n} C(H)^i \otimes \text{Inf}^i\left(C(H')^*, C(\Delta H')^\ast\right) \oplus \text{Inf}^j\left(C(H)^*, C(\Delta H)^\ast\right) \otimes C(H')^j; \quad (5.10)
$$

and

$$
\text{Sup}^n\left(C(H \ast H')^\ast, C(\Delta(H \ast H'))^\ast\right)
= \left( \bigoplus_{i+j=n} C(H)^i \otimes C(H')^j \right) + \delta_{n-1}\left( \bigoplus_{i'+j'+1=n-1} C(H)^i' \otimes C(H')^{j'} \right)
= \left( \bigoplus_{i+j=n} C(H)^i \otimes C(H')^j \right)
+ \left( \bigoplus_{i'+j'+1=n-1} \delta_{i'-1}C(H)^i' \otimes C(H')^{j'} \oplus C(H)^i' \otimes \delta_{j'-1}C(H')^{j'} \right)
= \bigoplus_{i+j=n} C(H)^i \otimes \text{Sup}^i\left(C(H')^*, C(\Delta H')^\ast\right) \oplus \text{Sup}^j\left(C(H)^*, C(\Delta H)^\ast\right) \otimes C(H')^j. \quad (5.11)
$$

Secondly, we compute the kernels of the weighted Laplacians in the next proposition.
Proposition 5.3. As complex vector spaces,

\[ \text{Ker}(\Delta_n^{(\inf(H \ast H'), \phi \ast \phi')}) \]

\[ = \sum_{i+j=n} (\text{Ker} \Delta_i^{(\Delta(H, \phi) \cap C(H)^i)} \otimes \text{Ker} \Delta_j^{(\inf H', \phi')} \]

\[ + \text{Ker} \Delta_i^{(\inf H', \phi')} \otimes (\text{Ker} \Delta_j^{(\Delta H', \phi')} \cap C(H')^j); \quad (5.12) \]

and

\[ \text{Ker}(\Delta_n^{(\sup(H \ast H'), \phi \ast \phi')}) \]

\[ = \sum_{i+j=n} (\text{Ker} \Delta_i^{(\Delta(H, \phi) \cap C(H)^i)} \otimes \text{Ker} \Delta_j^{(\sup H', \phi') \}

\[ + \text{Ker} \Delta_i^{(\sup H', \phi')} \otimes (\text{Ker} \Delta_j^{(\Delta H', \phi')} \cap C(H')^j). \quad (5.13) \]

Proof. Let

\[ \omega \in \inf^n \left( (C(H \ast H')^*, C(\Delta(H \ast H'))^*) \right) \]

(resp. \[ \omega \in \sup^n \left( (C(H \ast H')^*, C(\Delta(H \ast H'))^*) \right) \].

Then we can write

\[ \omega = \sum_{i+j=n} \eta_i \otimes \omega'_j + \omega_i \otimes \eta'_j, \quad (5.14) \]

where

\[ \eta_i \in C(H)^i, \]

\[ \eta'_j \in C(H')^j, \]

\[ \omega_i \in \inf^i \left( (C(H)^*, C(\Delta H)^*) \right), \]

\[ \omega'_j \in \inf^j \left( (C(H')^*, C(\Delta H')^*) \right) \]

(resp. \[ \eta_i \in C(H)^i, \]

\[ \eta'_j \in C(H')^j, \]

\[ \omega_i \in \sup^i \left( (C(H)^*, C(\Delta H)^*) \right), \]

\[ \omega'_j \in \sup^j \left( (C(H')^*, C(\Delta H')^*) \right) \].

With the help of Lemma 5.2 and (5.14), the weighted infimum Laplacians and the weighted supremum Laplacians of joins of hypergraphs can be calculated as

\[ \Delta_n \omega = \sum_{i+j=n} \delta_n^*(\delta_n \otimes \omega'_j) + \delta_{n-1}^*(\delta_{n-1} \otimes \omega'_j) \]

\[ + \delta_n^*(\delta_n \otimes \eta'_j) + \delta_{n-1}^*(\delta_{n-1} \otimes \eta'_j) \]

\[ = \sum_{i+j=n} \Delta_i \eta_i \otimes \omega'_j + \eta_i \otimes \Delta_j \omega'_j + \Delta_i \omega_i \otimes \eta'_j + \omega_i \otimes \Delta_j \eta'_j, \quad (5.15) \]
The calculation of (5.15) is analogous to [18, Equation (6.9)] and [12, Equation (4.6)].

Let $\omega \in \text{Ker}(\Delta_n^{(\text{Inf}(\mathcal{H}^*\mathcal{H}'), \phi^*\phi')})$. With the help of (5.14) and (5.15), it follows that the right-hand side of (5.14) is in the sum

$$
\sum_{i+j=n} \left( \text{Ker}\Delta_i^{(\Delta \mathcal{H}, \phi)} \cap \mathcal{C}(\mathcal{H})^i \right) \otimes \left( \text{Ker}\Delta_j^{(\Delta \mathcal{H}', \phi')} \cap \text{Inf}^j(\mathcal{C}(\mathcal{H}'), \mathcal{C}(\Delta \mathcal{H}')) \right)
$$

$$
+ \left( \text{Ker}\Delta_i^{(\Delta \mathcal{H}, \phi)} \cap \text{Inf}^i(\mathcal{C}(\mathcal{H}), \mathcal{C}(\Delta \mathcal{H})) \right) \otimes \left( \text{Ker}\Delta_j^{(\Delta \mathcal{H}', \phi')} \cap \mathcal{C}(\mathcal{H}')^j \right). \tag{5.16}
$$

Conversely, for any sum in (5.16), it follows from (5.14) and (5.15) that $\omega \in \text{Ker}(\Delta_n^{(\text{Inf}(\mathcal{H}^*\mathcal{H}'), \phi^*\phi')})$. Consequently, with the help of Definition 4 (iii), we obtain (5.12).

Similar to the proof of (5.12), with the help of Definition 4 (iv), (5.14) and (5.15), we obtain (5.13).

Finally, we prove Theorem 5.1 with the help of Proposition 5.3.

**Proof of Theorem 5.1.** It follows from that Theorem 3.4 and Proposition 5.3 that

$$
H^n(\mathcal{H} \ast \mathcal{H}', \phi \ast \phi'; \mathbb{C}) 
\cong \sum_{i+j=n} \left( \text{Ker}\Delta_i^{(\Delta \mathcal{H}, \phi)} \cap \mathcal{C}(\mathcal{H})^i \right) \otimes H^j(\mathcal{H}', \phi'; \mathbb{C})
$$

$$
+ H^i(\mathcal{H}, \phi; \mathbb{C}) \otimes \left( \text{Ker}\Delta_j^{(\Delta \mathcal{H}', \phi')} \cap \mathcal{C}(\mathcal{H}')^j \right).
$$

Moreover, it follows from the Hodge theorem of weighted simplicial complexes (cf. [28, Theorem 6.9]), or alternatively, substituting $\mathcal{H}$ with $\Delta \mathcal{H}$ and $\Delta \mathcal{H}'$ in Theorem 3.3 that

$$
\text{Ker}\Delta_i^{(\Delta \mathcal{H}, \phi)} \cap \mathcal{C}(\mathcal{H})^i = p_i(\mathcal{C}(\mathcal{H})^i), \tag{5.17}
$$

$$
\text{Ker}\Delta_j^{(\Delta \mathcal{H}', \phi')} \cap \mathcal{C}(\mathcal{H}')^j = p_j'(\mathcal{C}(\mathcal{H}')^j). \tag{5.18}
$$

By substituting (5.17) and (5.18) to (5.7), we obtain Theorem 5.1.

6 Applications in the Data-driven Symptom-based Diagnosis of Diseases

In this section, we give some applications of the Hodge-type theorem of weighted hypergraphs in the data-driven symptom-based diagnosis of diseases.

Hypergraphs and related structures have significant applications in the computer-assisted disease diagnosis (cf. [22, 23, 30]). Recently, M. Liu, J. Zhang, P. Yap and D. Shen [23] used hypergraphs as the mathematical model to help to diagnose the Alzheimer’s disease; Y. Zhu, X. Zhu, M. Kim, D. Kaufer and G. Wu [30] used dynamic hypergraphs to assist computers to diagnose neuro-diseases; and M. Liu, Y. Gao, P. Yap and D. Shen [22] used multi-hypergraphs, which can be regarded as integer-valued weighted hypergraphs, to analyze the data in biomedical and health informatics.
In practice, the weighted hypergraph model can be constructed as follows. Each disease has a symptom-list. A patient has some of the symptoms in the symptom-list, and the aim is to diagnose whether the patient has the disease or not. For example, diabetes mellitus has the following symptom-list: polyuria, polydipsia, polyphagia, weight loss, a history of blurred vision, itchiness, peripheral neuropathy, recurrent vaginal infections, fatigue, etc. The symptom-list of diabetes mellitus is a summary of the symptoms of all the patients suffering from diabetes mellitus. A specific patient may have some of the symptoms, and do not have other symptoms. If diabetes mellitus is diagnosed through the symptoms, then it is important to analyze the complexity of the symptom-patient system.

Let each vertex $v$ represent a possible symptom of a disease. Then the vertex set $V$ is the collection of all possible symptoms of the disease. For a nonempty subset $\sigma$ of $V$, we assign $\sigma$ as a hyperedge if there exists certain patients of the disease such that each of the patients has $\sigma$ exactly as his symptoms. We assign a weight $w(\sigma)$ to $\sigma$ as the percentage/probability of the patients having $\sigma$ exactly as their symptoms. Then $0 < w(\sigma) \leq 1$. For any such $\sigma, \sigma'$, let

$$\phi(\tau, \tau') = \frac{w(\sigma)}{w(\sigma')}.$$ 

With the help of Example 4.1 we obtain a weighted hypergraph $(H, \phi)$. 

By computing the kernels of the $\phi$-weighted associated Laplacian, the $\phi$-weighted lower-associated Laplacian, and either the $\phi$-weighted infimum Laplacian or the $\phi$-weighted supremum Laplacian, we obtain $H^*(\Delta H, \phi; \mathbb{C})$, $H^*(\delta H, \phi; \mathbb{C})$ and $H^*(H, \phi; \mathbb{C})$ respectively. Since the cohomology groups can reflect the complexity of weighted hypergraphs, we have the following prospective method to measure the accuracy of data-driven symptom-based diagnosis of diseases:

**Prospective method.** The smaller that the dimensions of the kernels of the $\phi$-weighted associated Laplacian, the $\phi$-weighted lower-associated Laplacian, and either the $\phi$-weighted infimum Laplacian or the $\phi$-weighted supremum Laplacian are, the easier and the more accurate to make a symptom-based diagnosis.

Moreover, joins of weighted hypergraphs have potential applications in the the data-driven symptom-based diagnosis of multiple diseases. Suppose two diseases $A$ and $A'$ are independent random events, and the weighted hypergraph models for the symptom-patient systems of $A$ and $A'$ are $(H, \phi)$ and $(H', \phi')$ respectively. In order to measure the accuracy of the data-driven symptom-based simultaneous diagnosis of $A$ and $A'$, we can consider the join $(H \ast H', \phi \ast \phi')$. The set of vertices of $H \ast H'$ is the symptom-list of $A$ and $A'$ (if a clinical symptom belongs to both $A$ and $A'$, then we regard the clinical symptom as two mathematical symptoms, one is a symptom of $A$, and the other is a symptom of $A'$). Each hyperedge $\sigma \sqcup \sigma'$ of $H \ast H'$ is a disjoint union of symptoms $\sigma$ of $A$ and symptoms $\sigma'$ of $A'$. The probability that $\sigma \sqcup \sigma'$ happens as the exact symptoms of the multiple diseases $A$ and $A'$ equals to the product of the probability that $\sigma$ happens as the exact symptoms of $A$ and the probability that $\sigma'$ happens as the exact
symptoms of $A'$. Hence with the help of Example \ref{example4.1} and \ref{example5.2}, the weight on $H \ast H'$ is $\phi \ast \phi'$. Therefore, $(H \ast H', \phi \ast \phi')$ is a model for the data-driven symptom-based diagnosis of multiple diseases $A$ and $A'$.

Acknowledgement. The project was supported in part by the Singapore Ministry of Education research grant (AcRF Tier 1 WBS No. R-146-000-222-112). The first author was supported by Guangdong Ocean University. The second author was supported in part by the President’s Graduate Fellowship of National University of Singapore. The third author was supported by a grant (No. 11329101) of NSFC of China. The authors would like to express their gratitude to Professor Stephane Bressan for his discussions and suggestions about the applications in computer science.

References

[1] W.N. Anderson and T.D. Morley, *Eigenvalues of the Laplacian of a graph*. Univ. of Maryland Tech. Report, **TR-71-45** (1971); Linear and Multilinear Algebra **18** (1985), 141-145.

[2] A. Banerjee and J. Jost, *On the spectrum of the normalized graph Laplacian*. Linear Algebra Appl. **428** (2008), 3015-3022.

[3] C. Berge, *Graphs and hypergraphs*. North-Holland Mathematical Library, Amsterdam, 1973.

[4] S. Bressan, J. Li, S. Ren and J. Wu, *The embedded homology of hypergraphs and applications*. Asian J. Math. accepted (2018). https://arxiv.org/abs/1610.00890.

[5] F.R.K. Chung, *The Laplacian of a hypergraph*. DIMACS Ser. in Discrete Math. Theoret. Comput. Sci. **10** (1983), 21-36.

[6] F.R.K. Chung, *Spectral Graph Theory*. CBMS Reg. Conf. Ser. in Math. **92**, Amer. Math. Soc., Providence, RI, 1997.

[7] F.R.K. Chung and R.L. Graham, *Cohomological aspects of hypergraphs*. Trans. Amer. Math. Soc. **334** (1) (1992), 365-388.

[8] F.R.K. Chung and R.P. Langlands, *A combinatorial Laplacian with vertex weights*. J. Combinatorial Theory, series A, **75** (1996), 316-327.

[9] F.R.K. Chung and K. Oden, *Weighted graph Laplacians and isoperimetric inequalities*. Pacific J. Math. **192** (2) (2000), 257-273.

[10] D.M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs. Theory and Applications*. 3rd ed., Johann Ambrosius Barthm Heidelberg, 1995.
[11] R.J. MacG. Dawson, *Homology of weighted simplicial complexes*. Cahiers de Topologie et Géométrie Différentielle Catégoriques 31 (3) (1990), 229-243.

[12] A.M. Duval and V. Reiner, *Shifted simplicial complexes are Laplacian integral*. Trans. Amer. Math. Soc. 354 (2002), 4313-4344.

[13] B. Eckmann, *Harmonische funktionen und randwertaufgaben in einem komplex*. Comment. Math. Helv. 17 (1) (1944), 240-255.

[14] K. Feng and W.C.W. Li, *Spectra of hypergraphs and applications*. J. Number Theory 60 (1996), 1-22.

[15] A. Grigor’yan, Y. Lin, Y. Muranov and S.T. Yau, *Homologies of path complexes and digraphs*. arXiv (2012). [http://arxiv.org/abs/1207.2834](http://arxiv.org/abs/1207.2834)

[16] A. Hatcher, *Algebraic topology*. Cambridge University Press, Cambridge, 2001.

[17] D. Horak and J. Jost, *Interlacing inequalities for eigenvalues of discrete Laplace operators*. Ann. Global Anal. Geom. 43 (2) (2013), 177-207.

[18] D. Horak and J. Jost, *Spectra of combinatorial Laplace operators on simplicial complexes*. Adv. Math. 244 (2013), 303-336.

[19] G. Kirchhoff, *Über die Auflösung der Gleichungen auf welche man bei der Untersuchen der linearen Vertheilung galvanischer Ströme gefühlt wird*. Ann. der Phys. und Chem. 72 (1847), 495-508.

[20] A.I. Kostrikin and R. Shafarevich, *Homological Algebra*. Algebra V, Encyclopaedia of Mathematical Sciences 38, Springer-Verlag Berlin-Heidelberg, 1994.

[21] S. Lang, *Real and functional analysis*. 3rd edition. Graduate Textbooks in Mathematics 142, Springer, New York, 1993.

[22] M. Liu, Y. Gao, P. Yap and D. Shen, *Multi-hypergraph learning for incomplete multi-modality data*. IEEE Journal of Biomedical and Health Informatics (2017).

[23] M. Liu, J. Zhang, P. Yap and D. Shen, *View-aligned hypergraph learning for Alzheimer’s disease diagnosis with incomplete multi-modality data*. Med. Image Anal. 36 (2017), 123-134.

[24] S. Morita, *Geometry of differential forms*. Translations of Mathematical Monographs 201, American Mathematical Society, 2001.

[25] A.D. Parks and S.L. Lipscomb, *Homology and hypergraph acyclicity: a combinatorial invariant for hypergraphs*. Naval Surface Warfare Center, 1991.
[26] S. Ren, C. Wu and J. Wu, *Evolutions of hypergraphs and their embedded homology*. arXiv (2018). https://arxiv.org/abs/1804.07132.

[27] S. Ren, C. Wu and J. Wu, *Weighted persistent homology*. Rocky Mountain J. Math. accepted (2018). https://arxiv.org/abs/1804.07132.

[28] C. Wu, S. Ren, J. Wu and K. Xia, *Weighted cohomology and weighted Laplacian*. arXiv (2018). https://arxiv.org/abs/1804.06990.

[29] D. Zhou, J. Huang and B. Schölkopf, *Learning with hypergraphs: Clustering, classification, and embedding*. Proc. Adv. Neural Inf. Process. Syst. (2006), 1601-1608.

[30] Y. Zhu, X. Zhu, M. Kim, D. Kaufer and G. Wu, *A novel dynamic hypergraph inference framework for computer assisted diagnosis of neuro-diseases*. Information Processing in Medical Imaging (2017), 158-169.

[31] J.Y. Zien, M.D. Schlag and P.K. Chan, *Multilevel spectral hypergraph partitioning with arbitrary vertex sizes*. IEEE Trans. Comput.-Aided Des. Integr. Circuits Syst. 18 (9) (1999), 1389-1399.

Shiquan Ren  
Address: *a* School of Mathematics and Computer Science, Guangdong Ocean University. Haida Road 1, Zhanjiang 524088, China. *b* School of Computing, National University of Singapore. 117417, Singapore.  
e-mail: sren@u.nus.edu

Chengyuan Wu  
Address: Department of Mathematics, National University of Singapore. 119076, Singapore.  
e-mail: wuchengyuan@u.nus.edu

Jie Wu  
Address: Department of Mathematics, National University of Singapore. 119076, Singapore.  
e-mail: matwuj@nus.edu.sg