SOLVABILITY OF THE OPERATOR RICCATI EQUATION IN THE FESHBACH CASE

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Abstract. We consider a bounded block operator matrix of the form

\[ L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \]

where the main-diagonal entries \(A\) and \(D\) are self-adjoint operators on Hilbert spaces \(\mathcal{H}_A\) and \(\mathcal{H}_D\), respectively; the coupling \(B\) maps \(\mathcal{H}_A\) to \(\mathcal{H}_D\), and \(C\) is an operator from \(\mathcal{H}_D\) to \(\mathcal{H}_A\). It is assumed that the spectrum \(\sigma_n(D)\) of \(D\) is absolutely continuous and uniform, being presented by a single band \([\alpha, \beta] \subset \mathbb{R}\), \(\alpha < \beta\), and the spectrum \(\sigma(A)\) is embedded into \(\sigma(D)\), that is, \(\sigma_\alpha(A) \subset (\alpha, \beta)\). In its spectral representation, the entry \(D\) reads as the operator of multiplication by the independent variable \(\lambda \in (a, b)\). One more assumption is that both the couplings \(B\) and \(C\) are defined via operator-valued functions of \(\lambda \in (\alpha, \beta)\) that are real analytic on \((\alpha, \beta)\) and admit analytic continuation onto some domain in \(\mathbb{C}\). This allows one to perform a complex deformation of \(L\). The latter involves, in particular, the replacement of the original entry \(D\) with the operators of multiplication by the complex variable \(\lambda\) running through piecewise smooth Jordan contours obtained from the interval \((\alpha, \beta)\) by continuous transformations. We formulate conditions under which there are bounded solutions to the operator Riccati equations associated with the complexly deformed block operator matrix \(L\); in such a case the deformed operator matrix \(\tilde{L}\) admits a block diagonalization. The same conditions also ensure the Markus-Matsaev-type factorization of the Schur complement \(M_\lambda(z) = A - \lambda - B(D - z)^{-1}C\) analytically continued onto the unphysical sheet(s) of the complex plane adjacent to the band \([\alpha, \beta]\). We prove that the operator roots of the continued Schur complement \(M_\lambda\) are explicitly expressed through the respective solutions to the deformed Riccati equations.

1. Introduction

Assume that \(L\) is a bounded linear operator on a Hilbert space \(\mathcal{H}\). Suppose \(\mathcal{H}\) is decomposed into the orthogonal sum

\[ \mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_D \tag{1.1} \]

of two subspaces \(\mathcal{H}_A\) and \(\mathcal{H}_D\). Then, with respect to the decomposition (1.1), the operator \(L\) reads as a \(2 \times 2\) block matrix,

\[ L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1.2} \]

where the main-diagonal entries \(A\) and \(D\) are operators respectively on \(\mathcal{H}_A\) and \(\mathcal{H}_D\); the coupling \(B\) maps \(\mathcal{H}_D\) to \(\mathcal{H}_A\), and \(C\) is an operator from \(\mathcal{H}_D\) to \(\mathcal{H}_A\). The relations

\[ XA - DX + XBX = C, \quad X : \mathcal{H}_A \rightarrow \mathcal{H}_D, \tag{1.3} \]

\[ YD - AY + YCY = B, \quad Y : \mathcal{H}_D \rightarrow \mathcal{H}_A, \tag{1.4} \]

are called the (pair of dual) operator Riccati equations associated with the block operator matrix \(L\).

It is well known (see, e.g., [4], [25]) that a bounded operator \(X\) from \(\mathcal{H}_A\) to \(\mathcal{H}_D\) is a solution to the Riccati equation (1.3) if and only if the graph \(\mathcal{G}(X)\) of \(X\),

\[ \mathcal{G}(X) := \{x \oplus Xx \mid x \in \mathcal{H}_A\}, \tag{1.5} \]

is an invariant subspace of \(L\). Similarly, a bounded operator \(Y : \mathcal{H}_D \rightarrow \mathcal{H}_A\) is a solution to the Riccati equation (1.4) if and only if the graph subspace

\[ \mathcal{G}(Y) = \{Yy \oplus y \mid y \in \mathcal{H}_D\} \tag{1.6} \]
is invariant under $L$. Thus, the problem of existence and uniqueness of solutions to the Riccati equations turns out to be an important issue in various sections of mathematics and physics involving the study of invariant subspaces of a linear operator. Among them one may place the long-standing problem of obtaining optimal bounds on variation of a spectral subspace of a self-adjoint operator under an additive perturbation that still has only partial solutions (see, e.g., the articles, in chronological order, [20] [6] [34] [33], and references therein). It is the possibility to construct reducing subspaces of a quantum-mechanical Hamiltonian in terms of solutions to operator Riccati equations that lies behind the celebrated Okubo [31] and Foldy-Wouthoysen [15] transforms. Operator Riccati equations and invariant graph subspaces are also closely related to the factorization problem [26] for operator pencils with resolvent-like dependence on the spectral parameter (see [1] [2] [3] [23] [28]).

Most of the known results on the solvability of the operator Riccati equations (1.3) / (1.4) concern the case where the spectra $\sigma_A$ and $\sigma_D$ of the main-diagonal entries are disjoint, that is,

$$d := \text{dist}(\sigma_A, \sigma_D) > 0,$$

and the corresponding block operator matrix $L$ is self-adjoint. The total list of works touching the problem of the existence of solutions to (1.3) / (1.4) associated with a self-adjoint $L$ is rather extensive and here we mention only a very few of the related publications: [2] [3] [4] [6] [20] [25] [28] [29] [33]. In the case of a self-adjoint $L$, for certain mutual positions of the (disjoint) spectral sets $\sigma_A$ and $\sigma_D$, even some sharp conditions on $B$ (and $C = B^*$) ensuring the solvability of (1.3) / (1.4) are available. These particular spectral situations correspond to the mutual positions where one of the of the spectral sets $\sigma_A$ and $\sigma_D$ is completely embedded into a finite or infinite spectral gap of the other set (see [21] [22]). The optimal solvability conditions are accompanied by sharp norm bounds on the solution $X$ that follow from the relevant estimates in the subspace perturbation problem known as the Davis-Kahan tan $\Theta$ theorem [13] and the $a \text{ priori}$ tan $\Theta$ theorem [7] [30]. Best available sufficient condition for the existence of a bounded solution $X$ to (1.3) and best (but still not optimal) norm estimate on $X$ under the single spectral assumption (1.7) follow from the main result of [33] (cf. [8]). A number of the existence results for (1.3) and estimates on the solution $X$ under the condition (1.7) in the case of a $J$-self-adjoint block operator matrix $L$ may be found in [10] [11] (also see [35] [36]). Furthermore, we refer to [5] concerning the existence results for (1.3) with disjoint $\sigma_A$ and $\sigma_D$ in the case where one of the entries $A$ and $D$ is a normal operator. Finally, in the generic non-self-adjoint case, an existence result for the Riccati equation (1.3) under condition (1.7) has been obtained in [23], based the concept of quadratic numerical range.

In paper [27] that treats the case of self-adjoint $L$, the assumption (1.7) is replaced by the hypothesis that the spectrum of one of the main-diagonal entries $A$ and $D$ is at least partly embedded into the absolutely continuous spectrum of the other one, say

$$\sigma_A \cap \sigma_D^{ac} \neq \emptyset,$$

where $\sigma_D^{ac}$ denotes the absolutely continuous spectrum of $D$. Following the quantum-mechanical terminology, we call the spectral disposition (1.8) the Feshbach case since for infinitesimally small $B \neq 0$ (and $C = B^*$) the eigenvalues of $A$ embedded into $\sigma_D^{ac}$ generically transform into Feshbach resonances [14].

Conditions on the entry $B$ (and, hence, on the entry $C = B^*$) in [27] are chosen such that the Schur complement

$$M_A(z) = A - z - B(D - z)^{-1}C, \quad z \in \mathbb{C} \setminus \sigma_D,$$

considered as an operator-valued function of $z$, admits analytic continuation through bands of $\sigma_D^{ac}$ onto certain adjacent domains lying already on unphysical sheets of the complex plane. It was found in [27] that the continued Schur complement (1.9) admits a factorization of the Markus-Matsaev type [26] and, thus, it possesses a family of operator roots. The spectrum of an operator root of $M_A$, along with a part of the usual spectrum of $L$, may possibly include a number of resonances of $L$. In [18] the results of [27] were generalized to some unbounded self-adjoint $L$ with unbounded $B$ and in [19] even to some unbounded non-self-adjoint $L$. Recently, in [9], the factorization approach of [27] allowed us to prove...
the existence of bounded solutions to the operator Riccati equation (1.3) associated with a $J$-self-adjoint block matrix $L$ of the form (1.2) in the case where $\sigma_\alpha \subset \sigma_D^{ac}$.

In the present work we consider the case where the entries $A$ and $D$ are self-adjoint, with $D$ being given in the spectral representation. Thus, finally we even adopt the hypothesis that $D$ is simply the operator of multiplication by an independent variable. Moreover, in order to ensure the maximal clarity, we then restrict ourselves to the case where all the spectrum of $D$ is absolutely continuous and uniform, being presented by a single band, that is, $\sigma_D = \sigma_D^{ac} = [\alpha, \beta]$, $-\infty < \alpha < \beta < \infty$, and $\sigma_\alpha \subset (\alpha, \beta)$. Therefore, the operator $L$ we study, is in fact nothing but an extension of one of the two celebrated Friedrichs models in [16], namely the $2 \times 2$ operator matrix model discussed in [16] Section 6. Furthermore, we assume that the entries $B$ and $C$ are defined via operator-valued functions $b(\lambda)$ and $c(\lambda)$ of $\lambda \in (\alpha, \beta)$ that are both real analytic and admit analytic continuation onto some domain $\mathcal{D} \subset \mathbb{C}$ (see Section 4 for details). This allows one to perform a complex deformation of $L$. The latter involves, in particular, the replacement of the original entry $D$ with the operators $D_\gamma$ of multiplication by the complex variable $\gamma$ running through piecewise smooth Jordan contours $\Gamma$ obtained from the interval $(\alpha, \beta)$ by a continuous transformation. It is assumed that during such a transformation the end points $\alpha$ and $\beta$ are fixed and $\Gamma \setminus \{\alpha, \beta\} \subset \mathcal{D}$. For the complexly deformed operators $B$ and $C$ we use the respective notations $B_\gamma$ and $C_\gamma$. Notice that, in case of momentum space few-body Hamiltonians, the approach we apply to $L$ is well known under the name of contour deformation method (see, e.g., [17] and references therein). One of variants of this method that reduces the deformation of the absolutely continuous spectrum just to its rotation in $\mathbb{C}$ about the threshold points is the celebrated complex scaling, used both in momentum and coordinate representations (see [12, 24, 32]).

The complex deformation of $L$ leads to the complexly deformed associated Riccati equations (1.3) / (1.4) with the same entry $A$ but with $B$, $C$, and $D$ replaced by the corresponding complexly deformed $B_\gamma$, $C_\gamma$, and $D_\gamma$. The complexly deformed main-diagonal entry $D_\gamma$ is a normal operator whose spectrum $\sigma_{D_\gamma} = \Gamma$ may be made disjoint with $\sigma_\alpha$ by a relevant choice of the contour $\Gamma$. Then one simply applies to the deformed Riccati equations the approach of [5] that works under the assumption of spectral disjointness (1.7) and that we already mentioned above. In particular, we prove that the operator roots of the continued Schur complement (1.9) are explicitly expressed through the solutions $X_\gamma$ to the complexly deformed Riccati equation (1.3).

The article is organized as follows. In Section 2 we collect the necessary information on the existence and properties of solutions to the Riccati equations (1.3) / (1.4) with special attention to the case where at least one of the entries $A$ and $D$ is a normal operator. In Section 3 we present a version of results of [27] adapted to the case $\sigma_\alpha \subset \sigma_D^{ac}$ under consideration. However, unlike in [27], we do not require that $C = B^*$. Among other things, the section contains conditions ensuring the existence of operator roots for the analytically continued Schur complement (1.9). Finally, in Section 4 we introduce an extension of the Friedrichs’ $2 \times 2$ operator matrix model from [16] Section 6] and consider its variant admitting a complex deformation. Assuming the existence of a piecewise smooth Jordan contour $\Gamma$ such that $\Gamma \setminus \{\alpha, \beta\} \subset \mathcal{D} \cap \mathbb{C}^\pm$ and the norms $\|B_\gamma\|_{E_{D_\gamma}}$ and $\|C_\gamma\|_{E_{D_\gamma}}$ of the deformed entries $B_\gamma$ and $C_\gamma$ with respect to the spectral measure of the normal operator $D_\gamma$ satisfy the condition

$$\sqrt{\|B_\gamma\|_{E_{D_\gamma}} \|C_\gamma\|_{E_{D_\gamma}}} < \frac{1}{2} \text{dist}(\sigma_\alpha, \Gamma),$$

we prove the existence of bounded solutions $X_\gamma$ and $Y_\gamma$ to the complexly deformed Riccati equations (1.3) and (1.4) (see Theorem 4.9). The operator roots of the analytically continued Schur complement (1.9) are nothing but the operators $Z_\gamma = A + B_\gamma X_\gamma$. Under (1.10) these operators depend only on the sign $s = \pm$ in the half-plane $\mathbb{C}^s$ superscript but not on the (form of the) contour $\Gamma \subset \mathcal{D} \cap (\mathbb{C}^s \cup \mathbb{R})$ itself. The solutions $X_\gamma$ and $Y_\gamma$ possess the property $\|X_\gamma Y_\gamma\| < 1$ which guarantees the block diagonalizability of the complexly deformed operator matrix $L_\gamma$ (see Corollary 4.10).

\footnote{See Definition 2.1 below for the norm of a bounded operator with respect to the spectral measure of a normal operator.}
The following notations are used throughout the paper. By a subspace of a Hilbert space we always understand a closed linear subset. The identity operator is denoted by $I$. The Banach space of bounded linear operators from a Hilbert space $\mathcal{L}$ to a Hilbert space $\mathcal{M}$ is denoted by $\mathcal{B}(\mathcal{L},\mathcal{M})$ and by $\mathcal{B}(\mathcal{L})$ if $\mathcal{L} = \mathcal{M}$. By $\sigma$, we denote the spectrum of an operator $S \in \mathcal{B}(\mathcal{M})$. The notation $E_T(\delta)$ is used for the spectral projection of a normal operator $T$ associated with a Borel set $\delta \subset \mathbb{C}$. In the particular case where $T$ is self-adjoint, $\delta \subset \mathbb{R}$. By $\overline{\delta}$ we denote the closure of an arbitrary $\delta \subset \mathbb{C}$. By $\overline{\delta}_r(\delta)$, $r > 0$, we denote the open $r$-neighbourhood of $\delta$ in $\mathbb{C}$, i.e. $\overline{\delta}_r(\delta) = \{z \in \mathbb{C} | \text{dist}(z, \delta) < r\}$. By $\mathbb{C}^+$ and $\mathbb{C}^-$ we understand respectively the upper and lower half-planes of the complex plane $\mathbb{C}$ (with excluded real axis), that is, $\mathbb{C}^\pm = \{z \in \mathbb{C} | \pm \text{Im} z > 0\}$.

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2. Preliminaries

Assume that the bounded operators $X \in \mathcal{B}(\delta_A, \delta_B)$ and $Y \in \mathcal{B}(\delta_B, \delta_A)$ are solutions to the operator Riccati equations (1.3) and (1.4), respectively. The operators

$$Z_A = A + BX,$$
$$Z_B = D + CY,$$  

and

$$\tilde{Z}_A = A - YC,$$
$$\tilde{Z}_B = D - XB,$$  

play an outstanding role in the spectral theory of the block operator matrices of the form (1.2) and related operator pencils. This concerns, in particular, the Schur complements $M_A$ and $M_B$ corresponding to the matrix $L$, $M_A(z)$ is given by (1.9) and

$$M_B(z) = D - z - C(A - z)^{-1}B, \quad z \in \mathbb{C} \setminus \sigma_A. \quad (2.5)$$

One easily verifies by inspection that the following identities hold:

$$M_A(z) = W_A(z)(Z_A - z), \quad z \in \mathbb{C} \setminus \sigma_B, \quad \text{and} \quad M_B(z) = W_D(z)(Z_B - z), \quad z \in \mathbb{C} \setminus \sigma_A, \quad (2.6)$$

where $Z_A$ and $Z_B$ are the operators (2.1) and (2.2), respectively; the entries $W_A$ and $W_D$ are explicitly given by

$$W_A(z) = I - B(D - z)^{-1}X \quad \text{and} \quad W_D(z) = I - C(A - z)^{-1}Y. \quad (2.7)$$

Similarly,

$$M_A(z) = (\tilde{Z}_A - z)\tilde{W}_A(z), \quad z \in \mathbb{C} \setminus \sigma_B, \quad \text{and} \quad M_B(z) = (\tilde{Z}_B - z)\tilde{W}_D(z), \quad z \in \mathbb{C} \setminus \sigma_A \quad (2.8)$$

where $\tilde{Z}_A$ and $\tilde{Z}_B$ are defined by (2.3) and (2.4), and

$$\tilde{W}_A(z) = I + Y(D - z)^{-1}C, \quad \tilde{W}_D(z) = I + X(A - z)^{-1}B. \quad (2.9)$$

If it so happened that

$$1 \notin \text{spec}(XY) \quad \text{and, equivalently,} \quad 1 \notin \text{spec}(YX), \quad (2.10)$$

the off-diagonal block operator matrix

$$Q = \begin{pmatrix} 0 & Y \\ X & 0 \end{pmatrix} \quad (2.11)$$
composed of the solutions $X$ and $Y$ allows one to perform similarity transformations of the operator $L$ into block diagonal operator matrices formed either of the operators (2.1), (2.2) or operators (2.3), (2.4). Namely,

$$L = (I + Q) \begin{pmatrix} Z_A & 0 \\ 0 & Z_B \end{pmatrix} (I + Q)^{-1} = (I - Q)^{-1} \begin{pmatrix} \tilde{Z}_A & 0 \\ 0 & \tilde{Z}_B \end{pmatrix} (I - Q),$$

(2.12)

Under condition (2.10), from (2.12) it follows that the operators $\tilde{Z}_A$ and $\tilde{Z}_B$ as well as the operators $Z_B$ and $\tilde{Z}_B$ are pairwise similar to each other. More precisely,

$$\tilde{Z}_A = (I - XY)Z_A(I - XY)^{-1},$$

(2.13)

$$\tilde{Z}_B = (I - XY)Z_B(I - XY)^{-1}.$$  

(2.14)

If the operator $D$ is normal and the spectra of $D$ and $Z_A$ are disjoint, the solution $X$ admits the following integral representation (see [5] for the proof and definition of the integral over the spectral measure involved):

$$X = \int_{\sigma_D} E_D(d\mu)C(Z_A - \mu)^{-1},$$

(2.15)

where $E_D$ is the spectral measure of $D$. This representation, written in the form

$$X = \int_{\sigma_D} E_D(d\mu)C(A + BX - \mu)^{-1},$$

(2.16)

may be treated as one more equation for determining $X$. Similarly, the disjointness of the spectra of $D$ and $\tilde{Z}_A = A - YC$ yields an “integral equation” for $Y$,

$$Y = -\int_{\sigma_D} (A - YC - \mu)^{-1}BE_D(d\mu).$$

(2.17)

Notice that (2.15) allows one to rewrite the function $W_A(z)$ from (2.7) in the form

$$W_A(z) = I - \int_{\sigma_D} B\bar{E}(d\mu)C1 \frac{1}{\mu - z}(Z_A - \mu)^{-1}.$$  

(2.18)

The paper [27] introduced the concept of the norm of a bounded operator with respect to the spectral measure of a given self-adjoint operator. In [5] this concept was extended to the spectral measure associated with a given normal operator. The operator norm with respect to a spectral measure proved to be a useful tool in the study of the operator Sylvester and Riccati equations (see [4] and [5] for details). We recall this concept bearing in mind its application to equations (2.16) and (2.17).

**Definition 2.1.** Let $S \in \mathcal{B}(\mathcal{H}_A, \mathcal{H}_B)$ be a bounded operator between the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, and let the operator $D \in \mathcal{B}(\mathcal{H}_B)$ be normal. Introduce the quantity

$$\|S\|_{E_D} = \left(\sup_{\delta_j} \sum_j \|S^*E_D(\delta_j)S\|\right)^{1/2},$$

(2.19)

where the supremum is taken over finite (or countable) sets of mutually disjoint Borel subsets $\delta_j$ of the spectrum $\sigma_D$ of the normal operator $D$, $\delta_j \cap \delta_k = \emptyset$, if $j \neq k$. The number $\|S\|_{E_D}$ is called the norm of $S$ with respect to the spectral measure $dE_D(z)$ or simply $E_D$-norm of $S$. For $T \in \mathcal{B}(\mathcal{H}_B, \mathcal{H}_A)$ the $E$-norm $\|T\|_{E_D}$ is defined by $\|T\|_{E_D} := \|T^*\|_{E_D}$.

**Remark 2.2.** Clearly, Definition 2.1 implies

$$\|S\| \leq \|S\|_{E_D} \quad \text{and} \quad \|T\| \leq \|T\|_{E_D}.$$  

(2.20)

In the case where both the operators $A$ and $D$ are normal one is able to prove the existence of fixed points for the mappings on the right-hand sides of (2.16) and (2.17) provided that the operators $B$ and $C$ satisfy certain smallness conditions involving the $E_D$-norms of $B$ or $C$ (see [5]; for earlier results with self-adjoint $A$ and/or $D$ see [2, 4, 29]).
Remark 2.3. For the special case of $B = 0$, the Riccati equation (1.3) turns into a linear equation

$$XA - DX = C, \quad X \in \mathcal{B}(\mathcal{S}_A, \mathcal{S}_D),$$

(2.21)
called the Sylvester equation. Similarly, for $C = 0$, the Riccati equation (1.4) turns into the Sylvester equation

$$YD - AY = B, \quad Y \in \mathcal{B}(\mathcal{S}_D, \mathcal{S}_A).$$

(2.22)

If the entry $D$ is a normal operator and $\sigma_A \cap \sigma_D = \emptyset$, the unique bounded solutions $X$ and $Y$ to (2.21) and (2.22) are given, respectively, by

$$X = \int_{\sigma_D} E_D(d\mu)C(D - \mu)^{-1} \quad \text{and} \quad Y = -\int_{\sigma_D} (A - \mu)^{-1}BE_D(d\mu)$$

(2.23)
(cf. (2.15) and (2.17); see [5, Theorem 4.5]).

The following statement is a particular case of [5, Theorem 5.7].

Theorem 2.4. Let both operators $A \in \mathcal{B}(\mathcal{S}_A)$ and $D \in \mathcal{B}(\mathcal{S}_D)$ in (1.3) be normal. Assume that $0 \neq B \in \mathcal{B}(\mathcal{S}_A, \mathcal{S}_D)$ and

$$d = \text{dist}(\sigma_A, \sigma_D) > 0.$$ 

(2.24)

Also assume that the operator $C \in \mathcal{B}(\mathcal{S}_D, \mathcal{S}_A)$ has a finite $E_D$–norm and

$$\sqrt{\|B\|\|C\|_{E_D}} < \frac{d}{2}.$$ 

(2.25)

Then the Riccati equation (1.3) has a unique solution $X$ in the ball

$$\{ T \in \mathcal{B}(\mathcal{S}_A, \mathcal{S}_D) \mid \|T\| < \|B\|^{-1} \left( d - \sqrt{\|B\|\|C\|_{E_D}} \right) \}.$$ 

(2.26)

Moreover, the solution $X$ has a finite $E_D$–norm that satisfies the bound

$$\|X\|_{E_D} \leq \frac{1}{\|B\|} \left( \frac{d}{2} - \sqrt{\frac{d^2}{4} - \|B\|\|C\|_{E_D}} \right).$$ 

(2.27)

A similar statement concerns the Riccati equation (1.4).

Theorem 2.5. Let both the operators $A \in \mathcal{B}(\mathcal{S}_A)$ and $D \in \mathcal{B}(\mathcal{S}_D)$ in (1.4) be normal. Assume that $0 \neq C \in \mathcal{B}(\mathcal{S}_D, \mathcal{S}_A)$ and condition (2.24) holds. Assume in addition that the operator $B \in \mathcal{B}(\mathcal{S}_A, \mathcal{S}_D)$ has a finite $E_D$–norm and

$$\sqrt{\|B\|_{E_D}\|C\|_{E_D}} < \frac{d}{2}.$$ 

(2.28)

Then the Riccati equation (1.4) has a unique solution $Y$ in the ball

$$\{ S \in \mathcal{B}(\mathcal{S}_D, \mathcal{S}_A) \mid \|S\| < \|C\|^{-1} \left( d - \sqrt{\|B\|_{E_D}\|C\|_{E_D}} \right) \}.$$ 

(2.29)

Moreover, the solution $Y$ has a finite $E_D$–norm that satisfies the bound

$$\|Y\|_{E_D} \leq \frac{1}{\|C\|} \left( \frac{d}{2} - \sqrt{\frac{d^2}{4} - \|B\|_{E_D}\|C\|_{E_D}} \right).$$ 

(2.30)

Corollary 2.6. Assume condition

$$\sqrt{\|B\|_{E_D}\|C\|_{E_D}} < \frac{d}{2}.$$ 

(2.31)

Under this condition the following inequalities hold:

$$\|X\|_{E_D}\|Y\|_{E_D} \leq \frac{\|B\|_{E_D}\|C\|_{E_D}}{d^2/4} < 1.$$ 

(2.32)
Proof. Notice that due to (2.20) the bound (2.31) implies both the estimates (2.25) and (2.28). Hence, the existence of solutions \( X \) and \( Y \) satisfying the corresponding bounds (2.27) and (2.30) is ensured. For the right-hand sides of these bounds we have

\[
\frac{1}{\|B\|} \left( \frac{d}{2} - \sqrt{\frac{d^2}{4} - \|B\|\|C\|_{E_D}} \right) = \frac{\|C\|_{E_D}}{\frac{d}{2} + \sqrt{\frac{d^2}{4} - \|B\|\|C\|_{E_D}}} < \frac{\|C\|_{E_D}}{d/2}, \tag{2.33}
\]

and

\[
\frac{1}{\|C\|} \left( \frac{d}{2} - \sqrt{\frac{d^2}{4} - \|B\|\|D\|_{E_D}} \right) = \frac{\|B\|_{E_D}}{\frac{d}{2} + \sqrt{\frac{d^2}{4} - \|B\|\|D\|_{E_D}}} < \frac{\|B\|_{E_D}}{d/2}. \tag{2.34}
\]

Then (2.27) and (2.30) together with (2.31) imply (2.32). \( \square \)

Remark 2.7. Taking into account (2.20), from the bound (2.32) it follows that the products \( XY \) are strict contractions \( \|XY\| < 1 \) and \( \|YX\| < 1 \), which means that under the condition (2.31) the block operator matrix (1.2) is block diagonalizable in any of the two forms (2.12).

Now consider the operators \( Z_A \) and \( Z_D \) built according (2.1) and (2.2) of the corresponding unique solutions \( X \) and \( Y \) referred to in Theorems 2.4 and 2.5. In particular, under the conditions of Theorem 2.4 the operator-valued function \( W_d(z) \), introduced in (2.7), is boundedly invertible and holomorphic in \( z \) at least on the open \( d/2 \)-neighborhood \( \mathcal{O}_{d/2}(\sigma_d) \) of the set \( \sigma_d \). By (2.27) this neighborhood contains the whole spectrum of \( Z_A \). Analogously, under the hypothesis of Theorem 2.5 the operator-valued function \( W_d(z) \) is boundedly invertible and holomorphic in \( z \) at least on the open \( d/2 \)-neighborhood \( \mathcal{O}_{d/2}(\sigma_d) \) of the set \( \sigma_d \) that contains the whole spectrum of \( Z_D \). In such a case, the factorization (2.6) yields that \( Z_A \) and \( Z_D \) are nothing but the \( \textit{operator roots} \) of the Schur complements \( M_a(z) \) and \( M_d(z) \) in the sense of Markus-Matsaev [26]. From (2.6) it follows that \( \text{spec}(M_a) \cap \mathcal{O}_{d/2}(\sigma_d) = \text{spec}(Z_A) \) and \( \text{spec}(M_d) \cap \mathcal{O}_{d/2}(\sigma_d) = \text{spec}(Z_D) \).

The same consideration is relevant to the operators \( \tilde{Z}_A \) and \( \tilde{Z}_D \) defined respectively by (2.3) and (2.4) provided that \( X \) and \( Y \) are again the unique solutions mentioned in Theorems 2.4 and 2.5. In the sense of the factorizations (2.8), these operators (2.3) and (2.4) may be named the \textit{left operator roots} of the Schur complements \( M_A \) and \( M_D \), respectively (cf. [19] Theorem 4.1)).

3. FACTORIZATION OF ONE OF THE SCHUR COMPLEMENTS IN THE FESHBACH CASE

From now on we assume that the entries \( A \) and \( D \) are self-adjoint operators. It is also supposed that the spectra of \( A \) and \( D \) overlap. More precisely, we want to consider the situation where at least a part of the spectrum of \( A \) lies on the absolutely continuous spectrum of \( D \). There are examples (see [4], Remark 3.9 and Lemma 3.10]) which show that, in such a spectral situation, (conventional) solutions to the associated Riccati equations may not exist at all.

Nevertheless, one can think of the Markus-Matsaev factorization [26] and operator roots of the analytically continued Schur complements. This idea has been fist elaborated in [27] for self-adjoint block operator matrices \( L \) involving bounded off-diagonal entries. Later on, the approach of [27] has been extended in [18] and [19] to some unbounded off-diagonal entries in the respective cases of self-adjoint and non-self-adjoint \( L \).

In order to recall the idea of the approach [27], let us rewrite the Schur complement (1.9) in terms of the spectral measure \( E_{\mu} \) of the self-adjoint operator \( D \):

\[
M_a(z) = A - z - B \int_{\mathcal{O}_{d}} E_{\mu}(d\mu) \frac{1}{\mu - z} C \tag{3.1}
\]

\[
= A - z - \int_{\mathbb{R}} B dE_{\mu}(\mu) C \frac{1}{\mu - z}, \tag{3.2}
\]

where \( E_{\mu}(\mu) = E_{\mu}((-\infty, \mu)) \) is the spectral function of \( D \). In [27] it is assumed (for \( C = B' \)) that the entries \( D \) and \( B \) are such that the operator-valued function \( M_a(z) \) admits analytic continuation in \( z \).
Figure 1. An example of the spectral situation considered in [27] with a self-adjoint operator $D$ having three disjoint intervals of the absolutely continuous spectrum $[\mu_1^{(j)}, \mu_2^{(j)}]$, $j = 1, 2, 3$. In [27] it is assumed that $C = B^*$ and the entries $D$ and $B$ are such that the Schur complement $M_A(z)$ admits analytic continuation in $z$ through the intervals $[\mu_1^{(j)}, \mu_2^{(j)}] \subset \sigma_{ac}^D$ into certain domains $\mathcal{D}_j^\pm$ (lying already on the unphysical sheets of the Riemann surface of $M_A$).

Through the appropriate segments of $\sigma_{ac}^D$ to certain domains located on the so-called unphysical sheets of the $z$ plane (see Figure 1; this figure is borrowed from [27], to which we also refer for the concept of unphysical sheet).

To make the presentation as clear as possible, we reduce the consideration to the case where all the spectrum of $D$ consists of a single finite interval of the absolutely continuous spectrum, that is,

$$\sigma_D = \sigma_{ac}^D = \Delta,$$

where

$$\Delta = (\alpha, \beta) \quad \text{for some} \quad \alpha, \beta \in \mathbb{R}, \quad \alpha < \beta,$$

and

$$\sigma_A \subset \Delta.$$ (3.5)

Main hypothesis (see [27]) is that the $\mathcal{B}(\mathcal{H}_A)$-valued function

$$K(\mu) := BE^D(\mu)C, \quad \mu \in \mathbb{R},$$ (3.6)

is real analytic on the interval $\Delta$ and admits analytic continuation from $\Delta$ onto a domain $\mathcal{D} \subset \mathbb{C}$. This assumption entails the inclusion $\Delta \subset \mathcal{D}$. By $\mathcal{D}^-$ and $\mathcal{D}^+$ we will denote the parts of the domain $\mathcal{D}$ lying respectively in the lower and upper half-planes of $\mathbb{C}$, $\mathcal{D}^\pm = \mathcal{D} \cap \mathbb{C}^\pm$ (see Figure 2). The derivative $K'(\mu) = \frac{d}{d\alpha}K(\mu), \mu \in \mathcal{D}$, is allowed to be weakly singular at the points $\alpha$ and $\beta$, namely,

$$||K'(\mu)|| \leq c|\mu - \alpha|^{-\nu} \quad \text{for any } \mu \neq \alpha \text{ lying in an open neighborhood of } \alpha \text{ with } \mathcal{D},$$ (3.7a)

$$||K'(\mu)|| \leq c|\mu - \beta|^{-\nu} \quad \text{for any } \mu \neq \beta \text{ lying in an open neighborhood of } \beta \text{ in } \mathcal{D},$$ (3.7b)

\footnote{Notice that in the self-adjoint case with $C = B^*$, the domains $\mathcal{D}^-$ and $\mathcal{D}^+$ are necessarily symmetric with respect to the real axis, $\mathcal{D}^+ = (\mathcal{D}^-)^*$, and $K(\mu^*) = K(\mu)^*$.}
where \( c \) and \( \nu \) are some real constants, \( c > 0 \) and \( 0 \leq \nu < 1 \). All the above allows one to rewrite (3.1) in the form

\[
M_A(z) = A - z - \int_{\alpha}^{\beta} d\mu \frac{K'(\mu)}{\mu - z},
\]

(3.8)

where the integral term is well defined and holomorphic for \( z \in \mathbb{C} \setminus \Delta \).

Suppose that \( \Gamma^- \) is a piecewise smooth Jordan contour having the end points \( \alpha \) and \( \beta \) and, except for these points, lying totally in \( \mathcal{D}^- \). Similarly, the notation \( \Gamma^+ \) is used for a piecewise smooth Jordan contour having the end points \( \alpha \) and \( \beta \) and, except for \( \alpha \) and \( \beta \), lying completely in \( \mathcal{D}^+ \). For \( \Gamma = \Gamma^- \) or \( \Gamma = \Gamma^+ \), by \( \Omega(\Gamma) \) we denote the domain lying inside the closed curve formed by the interval \( \Delta \) and contour \( \Gamma \). Thus, \( \Omega(\Gamma^\pm) \subset \mathcal{D}^\pm \).

The integral term on the right-hand side of (3.8) is a Cauchy type integral. Then it is elementary to prove that, under the assumptions adopted in this section, the function \( M_A(z) \) admits analytic continuation in \( z \) across the interval \((a, b)\) both from the bottom up and from the top down. After such a continuation one arrives to the sheet(s) of the Riemann surface of the function \( M_A \) that differs from the original sheet of the spectral parameter plane. For \( z \in \Omega(\Gamma) \) the corresponding continuation of \( M_A \) is given by

\[
M_A(z, \Gamma) := A - z - \int_{\alpha}^{\beta} d\mu \frac{K'(\mu)}{\mu - z}, \quad \Gamma = \Gamma^\pm.
\]

(3.9)

We note that as a function of \( z \in \mathbb{C} \setminus \Gamma^\pm \), the mapping \( M_A(\cdot, \Gamma^\pm) \) possesses the property (see [27, Lemma 2.1])

\[
M_A(z, \Gamma^s) = \begin{cases} 
M_A(z), & z \in \mathbb{C} \setminus \Omega(\Gamma^s), \\
M_A(z) + 2\pi i s K'(z), & z \in \Omega(\Gamma^s),
\end{cases} \quad s = \pm 1.
\]

(3.10)

Here and hereafter we identify the number \( s = +1 \) or \( s = -1 \) in a superscript or subscript with the corresponding sign in \( \pm \), that is, e.g., \( \Gamma^{s+1} \equiv \Gamma^+ \) and \( \Gamma^{-1} \equiv \Gamma^- \).

Now let us introduce the equation

\[
Z_A = A - \int_{\Gamma} d\mu K'(\mu)(Z_A - \mu)^{-1}, \quad \Gamma = \Gamma^\pm, \quad \Omega(\Gamma^\pm) \subset \mathcal{D}^\pm.
\]

(3.11)
that makes sense, of course, provided the spectrum \( \sigma_{\sigma} \) of the unknown \( Z_A \in \mathcal{B}(\Omega_\Gamma) \) does not intersect the integration contour \( \Gamma \), i.e. if \( \sigma_{\sigma} \cap \Gamma = \emptyset \). Also, let us introduce the quantity

\[
\mathcal{V}_K(\Gamma) := \int_\Gamma |d\mu| \|K'(\mu)\|
\]

that we call the variation of the operator-valued function \( K \) in (3.6) along the contour \( \Gamma \), and let

\[
d(\Gamma) := \text{dist}(\sigma_{\sigma}, \Gamma).
\]

(3.13)

Applying to (3.11) Banach’s Fixed Point Theorem results in (cf. [27, Theorem 3.1]):

**Theorem 3.1.** Let \( \Gamma^s, s = \pm 1 \), be a piecewise smooth Jordan contour having the end points \( \alpha, \beta \) and being such that \( \Omega(\Gamma^\pm) \subset \mathcal{D}^\pm \). Assume that

\[
\mathcal{V}_K(\Gamma^s) < \frac{1}{4} d(\Gamma^s)^2.
\]

(3.14)

Then the equation (3.11) has a solution \( Z^s_A \) of the form

\[
Z^s_A = A + T^s
\]

(3.15)

with

\[
\|T^s\| \leq r(\Gamma^s),
\]

(3.16)

where

\[
r(\Gamma^s) = \frac{d(\Gamma^s)}{2} - \sqrt{\frac{d(\Gamma^s)^2}{4} - \mathcal{V}_K(\Gamma^s)}.
\]

(3.17)

The solution \( Z^s_A \) of the form (3.15) is unique in the closed ball in \( \mathcal{B}(\mathcal{H}_A) \) centered at zero and having the radius \( d(\Gamma^s) - \sqrt{\mathcal{V}_K(\Gamma^s)} \).

**Lemma 3.2.** For a fixed value of \( s \), \( s = \pm 1 \), the unique solution \( Z^s_A \) of the form (3.15) referred to in Theorem 3.1 is the same for all the piecewise smooth Jordan contours \( \Gamma^s \) having the end points \( \alpha, \beta \) and being such that

\[
\Omega(\Gamma^s) \subset \mathcal{D}^s \quad \text{and} \quad \mathcal{V}_K(\Gamma^s) < \frac{1}{4} d(\Gamma^s)^2.
\]

(3.18)

Moreover, the following norm bound holds

\[
\|Z^s_A - A\| \leq r_0(K)
\]

(3.19)

with

\[
r_0(K) := \inf_{\Gamma^s: \omega(\Gamma^s) > 0} r(\Gamma^s)
\]

(3.20)

where the infimum is taken over all piecewise smooth Jordan contours \( \Gamma^s \) satisfying (3.18), the quantity \( r(\Gamma^s) \) is given by (3.17), and

\[
\omega(\Gamma^s) = d_0^2(\Gamma^s) - 4 \mathcal{V}_K(\Gamma^s).
\]

(3.21)

The proof of Lemma 3.2 almost literally repeats the proof of Corollary 3.4 in [27]. Thus, we omit it, too, as well as the proof of the following

**Corollary 3.3.** The spectrum of \( Z^s_A \) lies in the closed complex \( r_0(K) \)-neighborhood \( \overline{\sigma_{r_0(K)}(\sigma_A)} \) of the spectrum \( \sigma_A \) of the operator \( A \).

We conclude the section by presenting a factorization result for \( M_A(\cdot, \Gamma^\pm) \). We again skip proof since it follows exactly the same line as the proof of Theorem 4.1 in [27].

**Lemma 3.4.** Assume that the hypothesis of Theorem 3.1 is satisfied and let \( Z^s_A, s = \pm 1 \), be the unique solution to (3.11) referred to in that theorem. Then for \( z \in \mathbb{C} \setminus \Gamma^s \) the operator-valued function \( M_A(z, \Gamma^s) \) admits the following factorization:

\[
M_A(z, \Gamma^s) = W_A(z, \Gamma^s)(Z^s_A - z),
\]

(3.22)
where $W_A(z, \Gamma^g)$ is given by

$$W_A(z, \Gamma^g) = I - \int_{\Gamma^g} d\mu K'(\mu) \frac{1}{\mu - z}(Z^g - \mu)^{-1}. \quad \text{(3.23)}$$

The operator $W_A(z, \Gamma^g)$ is bounded, that is, $W_A(z, \Gamma^g) \in \mathcal{B}(\mathcal{D}_A)$, whenever $z \in \mathbb{C} \setminus \Gamma^g$. Moreover, for $\text{dist}(z, \sigma_A) \leq d(\Gamma_I)/2$ this operator is boundedly invertible and

$$\|W_A(z, \Gamma^g)\|^{-1} \leq \frac{1}{1 - \frac{\gamma_k(\Gamma_I)}{d(\Gamma_I)^2/4}} < \infty. \quad \text{(3.24)}$$

Note that finiteness of the bound (3.24) follows from the assumption (3.18). Having compared (3.23) with (2.18), one may view the factorization result (3.23) as a direct analog of the factorization (2.6).

4. Complex deformation of the block operator matrix and solvability of the deformed Riccati equation

In this section by $L$ we will understand an extension of one the two celebrated Friedrichs models in [16], namely the one discussed in [16, Section 6]. We assume that $L$ is a $2 \times 2$ block operator matrix of the form (1.2) where, from the very beginning, the entry $D$ is given in the spectral representation and, thus, it reads as the operator of multiplication by the independent variable. That is,

$$(Df_\alpha)(\lambda) = \lambda f_\alpha(\lambda), \quad f_\alpha \in \mathcal{D}_D = L^2(\Delta \to \mathfrak{h}) \quad (\Delta = (\alpha, \beta) \subset \mathbb{R}), \quad \text{(4.1)}$$

where $\mathfrak{h}$ is an auxiliary Hilbert space and $L^2(\Delta \to \mathfrak{h})$ is formed by functions $f_\alpha$ that map $\Delta$ to $\mathfrak{h}$ and are such that the $\mathfrak{h}$-norm $\|f(\lambda)\|_\mathfrak{h}$ is Lebesgue measurable and square-integrable over $\Delta$,

$$\|f\|_{\mathcal{D}_D} := \left( \int_\alpha^\beta d\lambda \|f(\lambda)\|_\mathfrak{h}^2 \right)^{1/2} < \infty. \quad \text{(4.2)}$$

The inner product in $\mathcal{D}_D = L^2(\Delta \to \mathfrak{h})$ is defined by

$$\langle f, g \rangle_{\mathcal{D}_D} = \int_\alpha^\beta d\lambda \langle f(\lambda), g(\lambda) \rangle_\mathfrak{h}, \quad f, g \in \mathcal{D}_D, \quad \text{(4.3)}$$

where $\langle \cdot, \cdot \rangle_\mathfrak{h}$ stands for the inner product in $\mathfrak{h}$.

The above definition of $D$ means that all its spectrum consists of the single branch of the absolutely continuous spectrum that uniformly covers the interval $[\alpha, \beta]$. We make no specification of the entry $A$ except for that it is self-adjoint and its spectrum is embedded into the interior of $\sigma_A$, i.e. the inclusion (3.5) holds, $\sigma_A \subset (\alpha, \beta)$. Necessarily, the coupling operator $C : \mathcal{D}_A \to \mathcal{D}_D$ acts as follows:

$$(Cf_\lambda)(\lambda) = c(\lambda)f_\lambda, \quad f_\lambda \in \mathcal{D}_A, \quad \text{(4.4)}$$

with some $\mathcal{B}(\mathcal{D}_A, \mathfrak{h})$-valued (for a.e. $\lambda \in \Delta$) function $c(\lambda)$. Also we assume that

$$Bf_D = \int_\Delta d\lambda \ b(\lambda) f_\lambda(\lambda), \quad \text{(4.5)}$$

with a $\mathcal{B}(\mathfrak{h}, \mathcal{D}_D)$-valued function $b(\lambda), \lambda \in (\alpha, \beta)$.

Surely, if $L$ is self-adjoint then $C = B^*$ and, necessarily, $c(\lambda) = b(\lambda)^*$ for a.e. $\lambda \in \Delta$. Notice that a self-adjoint block operator matrix $L$ of the form (1.2) with the entries $D$ and $B$ given by (4.1) and (4.5), respectively, and $C = B^*$, but with the entry $A$ only having point spectrum was discussed in [27, Section 8].

Main assumption of the present section is the following hypothesis.

**Hypothesis 4.1.** Assume that the operator-valued functions $b : \Delta \to \mathcal{B}(\mathfrak{h}, \mathcal{D}_A)$ and $c : \Delta \to \mathcal{B}(\mathcal{D}_A, \mathfrak{h})$ are real analytic on the interval $\Delta = (\alpha, \beta) \subset \mathbb{R}$ and admit analytic continuation from $\Delta$ onto a domain
\( \mathcal{D} \subset \mathbb{C}, \mathcal{D} \supset \Delta \) (and \( \mathcal{D} \not\supset \{\alpha, \beta\} \)). Let \( \mathcal{D}^- = \mathcal{D} \cap \mathbb{C}^- \) and \( \mathcal{D}^+ = \mathcal{D} \cap \mathbb{C}^+ \). Also assume that the following bounds hold:

\[
\|b(\mu)\|_{\mathcal{B}(\mathfrak{h}, \mathfrak{s}_\delta)} \leq c|\mu - \alpha|^{-\nu_b} \quad \text{and} \quad \|c(\mu)\|_{\mathcal{B}(\mathfrak{s}_\delta, \mathfrak{h})} \leq c|\mu - \alpha|^{-\nu_c}
\]

\( \alpha \neq \beta \) lying in some open complex neighborhood of \( \alpha \) in \( \mathcal{D} \),

\[
\|b(\mu)\|_{\mathcal{B}(\mathfrak{h}, \mathfrak{s}_\delta)} \leq c|\mu - \beta|^{-\nu_b} \quad \text{and} \quad \|c(\mu)\|_{\mathcal{B}(\mathfrak{s}_\delta, \mathfrak{h})} \leq c|\mu - \beta|^{-\nu_c}
\]

\( \alpha \neq \beta \) lying in some open complex neighborhood of \( \beta \) in \( \mathcal{D} \),

where \( c, \nu_b, \) and \( \nu_c \) are some constants, \( c > 0 \) and \( 0 \leq \nu_b < 1/2, 0 \leq \nu_c < 1/2 \).

Under Hypothesis \([4.1]\) let us consider various piecewise smooth Jordan contours \( \Gamma \) with fixed real end points \( \alpha \) and \( \beta \), \( \alpha < \beta \), obtained by continuous deformation from the interval \( \Delta = (\alpha, \beta) \) and lying either completely in \( \mathcal{D}^- \cup \Delta \) or completely in \( \mathcal{D}^+ \cup \Delta \). With every such a contour \( \Gamma \) we associate the Hilbert space \( \mathcal{H}_{\Delta^+} := L^2(\Gamma \to \mathfrak{h}) \) formed by functions \( f_{\Delta^+} : \Gamma \to \mathfrak{h} \) that are square-integrable with respect to the Lebesgue measure \( |d\lambda| \) on \( \Gamma \), that is, the inner product in \( \mathcal{H}_{\Delta^+} \) is defined by

\[
\langle f, g \rangle_{\mathcal{H}_{\Delta^+}} = \int_{\Gamma} |d\lambda| \langle f(\lambda), g(\lambda) \rangle_{\mathfrak{h}}, \quad (4.7)
\]

Surely, the Hilbert space \( \mathcal{H}_{\Delta} \) is a particular case of \( \mathcal{H}_{\Delta^+} \) for \( \Gamma = \Delta \). Then one introduces the following family of block operator matrices:

\[
L_{\Gamma} = \begin{pmatrix} A & B_{\Gamma} \\ C_{\Gamma} & D_{\Gamma} \end{pmatrix}, \quad (4.8)
\]

where \( D_{\Gamma} \) is the operator of multiplication by the independent variable \( \lambda \in \Gamma \) in \( L^2(\Gamma \to \mathfrak{h}) \), i.e.

\[
(D_{\Gamma} f_{\Delta^+})(\lambda) = \lambda f_{\Delta^+}(\lambda), \quad f_{\Delta^+} \in \mathcal{H}_{\Delta^+}.
\]

the entry \( C_{\Gamma} : \mathcal{H}_{\Delta^+} \to \mathcal{H}_{\Delta\Gamma} \) is defined as

\[
(C_{\Gamma} f_{\Delta^+})(\lambda) = c(\lambda)f_{\Delta^+}, \quad f_{\Delta^+} \in \mathcal{H}_{\Delta^+}, \quad \lambda \in \Gamma,
\]

and the entry \( B_{\Gamma} : \mathcal{H}_{\Delta\Gamma} \to \mathcal{H}_{\Delta^+} \) as

\[
B_{\Gamma} f_{\Delta^+} = \int_{\Gamma} d\lambda \ b(\lambda)f_{\Delta^+}(\lambda), \quad f_{\Delta^+} \in \mathcal{H}_{\Delta^+}.
\]

**Remark 4.2.** Since the entry \( D_{\Gamma} \), defined by \((4.9)\), is the operator of multiplication by the independent variable, it is normal, i.e. it satisfies \( D^*_\Gamma D_{\Gamma} = D_{\Gamma} D^*_\Gamma \). The spectrum \( \sigma_{D_{\Gamma}} \) of \( D_{\Gamma} \) is absolutely continuous and occupies the closure \( \overline{\Gamma} = \Gamma \cup \{\alpha, \beta\} \) of the contour \( \Gamma \). The spectral measure \( E_{D_{\Gamma}} \) of the operator \( D_{\Gamma} \) is given by

\[
(E_{D_{\Gamma}}(\delta)f)(\lambda) = \chi_\delta(\lambda)f(\lambda), \quad \lambda \in \Gamma, \quad f \in \mathcal{H}_{\Delta^+} = L^2(\Gamma \to \mathfrak{h}),
\]

where \( \delta \) is an arbitrary Borel subset of \( \overline{\Gamma} \) and \( \chi_\delta \) denotes the characteristic function of \( \delta \): \( \chi_\delta(\lambda) = 1 \) if \( \lambda \in \delta \) and \( \chi_\delta(\lambda) = 0 \) if \( \lambda \in \overline{\Gamma} \setminus \delta \).

**Remark 4.3.** Unlike the spectra of \( A \) and (original) \( D \), the spectra of \( A \) and \( D_{\Gamma} \) are disjoint,

\[
d(\Gamma) = \text{dist}(\sigma_A, \sigma_{D_{\Gamma}}) = \text{dist}(\sigma_A, \Gamma) > 0,
\]

whenever \( \Gamma \cap \Delta = \emptyset \).

We interpret \( L_{\Gamma} \) defined by \((4.8)\), \((4.11)\) as the result of the *complex deformation* of the original operator \( L_{\Delta} = L_{\Delta_{\Gamma}} \) corresponding to \( \Gamma = \Delta \). Note that only the main-diagonal entry \( D \) is varied while the other main-diagonal entry \( A \) remains unchanged. Similarly, the operator Riccati equations

\[
XA - D_{\Gamma} X + X B_{\Gamma} Y = C_{\Gamma}, \quad X \in \mathcal{B}(\mathfrak{s}_\alpha, \mathfrak{s}_{D_{\Gamma}}),
\]

\[
YD_{\Gamma} - A Y + Y C_{\Gamma} = B_{\Gamma}, \quad Y \in \mathcal{B}(\mathfrak{s}_{D_{\Gamma}}, \mathfrak{s}_\alpha)
\]

associated with the block operator matrix \( L_{\Gamma} \) are called the *complexly deformed* Riccati equations. The deformation is viewed as the one with respect to the original Riccati equations \((4.14), (4.15)\) for \( \Gamma = \Delta \).
As in Section [3] for $\Gamma = \Gamma^- \subset \mathcal{D}^-$ or $\Gamma = \Gamma^+ \subset \mathcal{D}^+$, we again denote by $\Omega(\Gamma)$ the domain lying inside the closed curve formed by the interval $\Delta$ and contour $\Gamma$. Surely $\Omega(\Gamma^\pm) \subset \mathcal{D}^\pm$. When talking on the operator $L_t$, by resonances one understands the part of the point spectrum $\sigma_p(L_t)$ of $L_t$ lying in $\Omega(\Gamma)$. The next lemma shows that the resonances lying in the intersection of the domains $\Omega(\Gamma^s)$ for several various contours $\Gamma^s$ with the same $l$ are common for the respective operators $L_{l_1}$. This property is nothing but the analog of the independence of resonances on the scaling parameter in the standard complex scaling approach (see, e.g., [32, Section XIII.10] and references therein).

**Lemma 4.4.** Let the assumptions of Hypothesis [4.1] hold. Assume that $\Gamma^s_1$ and $\Gamma^s_2$, $s = \pm 1$, are piecewise smooth Jordan contours with the end points $\alpha$ and $\beta$, obtained by continuous deformation from $\Delta$ and lying completely in $\mathcal{D}^s \cup \Delta$. Let $L_{l_{1_{j}}} \; j = 1, 2$, be operators defined for the respective contours $\Gamma^s_j$ by (4.8)–(4.11). Then $z \in \Omega(\Gamma^s_1) \cap \Omega(\Gamma^s_2)$ belongs to the point spectrum $\sigma_p(L_{l_1})$ of $L_{l_1}$ if and only if $z \in \sigma_p(L_{l_2})$.

Furthermore, if $z \in \sigma_p(L_{l_{1_1}}) \cap (\mathbb{C} \setminus \Omega(\Gamma^s_1))$ then $\lambda \in \sigma_p(L)$, where $L = L_\lambda$ is the original non-deformed operator block matrix (4.8) with $\Gamma = \Delta$.

**Proof.** Suppose that

$$z \in \sigma_p(L_{l_{1_1}}) \quad \text{and} \quad z \in \Omega(\Gamma_1) \cap \Omega(\Gamma_2), \quad (4.16)$$

where $\Gamma_1 = \Gamma^s_1$ and $\Gamma_2 = \Gamma^s_2$ for a certain value of $l$, $s = \pm 1$. Let $f \neq 0$ be an eigenvector of $L_{l_{1_1}}$ belonging to the eigenvalue $z$, $f = (f_1, f_2)$ with $f_1 \in \mathcal{D}_\alpha$ and $f_2 \in \mathcal{D}_\beta$. Equation $L_{l_{1_1}} f = z f$ is equivalent to

$$(A - z) f_1 + B_{l_{1_1}} f_2 = 0, \quad (4.17)$$

$$(C_{l_{1_1}} f_1 + (D_{l_{1_1}} - z) f_2 = 0. \quad (4.18)$$

Taking into account (4.9) and (4.10), from (4.18) we obtain for $\lambda \in \Gamma_1$ (and automatically $\lambda \neq z$)

$$f_\lambda(\lambda) = \frac{1}{\lambda - z} c(\lambda) f_\alpha. \quad (4.19)$$

Initially, the formula (4.19) only works for $\lambda \in \Gamma_1$. But, except for the point $z$, this formula may be used to make an extension of $f_\lambda$ through the whole domain where the $\mathcal{B}(\mathcal{D}_\alpha, 0)$-valued function $c$ is defined and analytic. Then, under Hypothesis [4.1] (which is assumed) the extended $\mathcal{B}$-valued function (4.19) is well defined and analytic in $\lambda \in \mathcal{D} = \mathcal{D}^- \cup \mathcal{D}^+ \cup \Delta$ except for the point $z$. Moreover, the following equality holds

$$c(\lambda) f_\alpha + (\lambda - z) f_\lambda(\lambda) = 0 \quad \text{for any } \lambda \in \mathcal{D} \setminus \{z\}. \quad (4.20)$$

At the same time, in view of (4.19) the term $B_{l_{1_2}} f_\lambda$ on the left-hand side of (4.18) reads

$$B_{l_{1_2}} f_\lambda = \int_{\Gamma_1} d\mu \frac{b(\mu) c(\mu)}{\mu - z} f_\alpha. \quad (4.21)$$

Since the function under the integration sign on the right-hand side of (4.21) is holomorphic in $\mu \in \mathcal{D} \setminus \{z\}$, the contour $\Gamma_1$ may be replaced, with no change in the integral value, by any other piecewise continuous Jordan contour $\Gamma \subset \mathcal{D}^s$ obtained from $\Gamma_1$ by continuous deformation without crossing the point $z$. In particular, since by the assumption $z \in \Omega(\Gamma_1) \cap \Omega(\Gamma_2)$, the contour $\Gamma_2$ may be chosen for such a purpose and then the equality (4.17) arises with $\Gamma_1$ replaced by $\Gamma_2$. Furthermore, restricting (4.20) to $\lambda \in \Gamma_2$ results in equality (4.18) rewritten for $L_{l_{2_2}}$.

Interchanging the roles of $\Gamma_1$ and $\Gamma_2$ proves the converse implication and, thus, completes the proof of the first statement of the lemma.

The remaining statement is proven in the same way by the observation that for $z \in \sigma_p(L_{l_{1_1}}) \cap (\mathbb{C} \setminus \Omega(\Gamma_1))$ one can equivalently replace in (4.21) integration over $L_{l_{1_1}}$ by integration over $\Delta$. In its turn, the equality (4.20) is also reduced to $\lambda \in \Delta$. Thus we conclude that, in this case, $z \in \sigma_p(L_{l_{1_1}})$ implies $z \in \sigma_p(L)$, completing the whole proof. \hfill \Box
Now let us consider the Schur complement
\[ M_{\alpha, \Gamma}(z) := A - z - B_\Gamma(D_\Gamma - z)^{-1}C_\Gamma \]
\[ = A - z - \int_{\sigma_{D_\Gamma}} B_\Gamma E_{D_\Gamma}(d\mu) C_\Gamma \frac{1}{\mu - z}, \quad z \notin \sigma_{D_\Gamma} = \Gamma, \]
(4.22)
corresponding to the block operator matrix \( L_\Gamma \). It is straightforward to see that in the case under consideration
\[ B_\Gamma E_{D_\Gamma}(d\mu) C_\Gamma = b(\mu)c(\mu) d\mu \]
(4.24)
and, thus,
\[ M_{\alpha, \Gamma}(z) := A - z - \int_{\Gamma} d\mu \frac{b(\mu)c(\mu)}{\mu - z}, \quad z \in \mathbb{C} \setminus \Gamma, \]
(4.25)
Furthermore, the corresponding operator-valued function \( K \) defined on \( \mathbb{R} \) by (3.6), for \( \mu \in \bar{\Omega} \) reads as
\[ K(\mu) = \int_{\alpha}^{\beta} d\lambda \ b(\lambda)c(\lambda), \quad \alpha \leq \mu \leq \beta. \]
(4.26)
Under the Hypothesis 4.1 the function \( K \) admits an explicit analytic continuation onto the domain \( \mathcal{D} \) simply by the formula
\[ K(\mu) = \int_{\gamma(\alpha, \mu)} d\lambda \ b(\lambda)c(\lambda), \quad \mu \in \mathcal{D}, \]
(4.27)
where \( \gamma(\alpha, \mu) \) stands for arbitrary piecewise Jordan contour having the ends \( \alpha, \mu \) and lying, except for the end point \( \alpha \), completely in \( \mathcal{D} \). Therefore, the derivative \( K'(\lambda) \) is nothing but
\[ K'(\lambda) = b(\lambda)c(\lambda), \quad \lambda \in \mathcal{D}. \]
(4.28)
This means that the Schur complement \( M_{\alpha, \Gamma}(\cdot) \) corresponding to \( L_\Gamma \) merely simply coincides with the function \( M_{\alpha}(\cdot, \Gamma) \) defined in (3.9),
\[ M_{\alpha, \Gamma}(z) = M_{\alpha}(z, \Gamma), \quad z \in \mathbb{C} \setminus \Gamma. \]
(4.29)

Remark 4.5. It is worth noting that the analyticity of the operator-valued function \( K' \) does not imply the analyticity of \( b \) and \( c \) (and, thus, in general it does not suggests the opportunity to perform the complex transformation of \( L \) of the type we did in this section). This is seen from the two following elementary examples.

Example 4.6. Let \( \alpha = -1, \beta = 1, \) and \( \delta_{\alpha} = \delta = \mathbb{C} \). Suppose that \( b(\lambda) = \lambda - i \) and \( c(\lambda) = \frac{1}{\lambda - 1}, \lambda \in (-1, 1) \). Clearly, the product \( b(\lambda)c(\lambda) \equiv 1 \) admits analytic continuation from the interval \((-1, 1)\) to the whole complex plane \( \mathbb{C} \) while the function \( c \) isn’t.

Example 4.7. Let \( \alpha, \beta, \delta_{\alpha}, \) and \( \delta \) be as in Example 4.6. Suppose that \( b(\lambda) = c(\lambda) = |\lambda|, \lambda \in (-1, 1) \). Clearly, the product \( b(\lambda)c(\lambda) \equiv \lambda^2 \) admits analytic continuation from the interval \((-1, 1)\) to the whole complex plane \( \mathbb{C} \) while none of the functions \( b \) and \( c \) is real analytic at the point \( \lambda = 0 \).

Now we notice that, due to (4.12), the \( \|\cdot\|_{E_{D_\Gamma}} \)-norms of the operators \( B_\Gamma \) and \( C_\Gamma \) (see Definition 2.1) read as
\[ \|B_\Gamma\|_{E_{D_\Gamma}} = \left( \int_{\Gamma} |d\mu| \|b(\mu)b(\mu)^*\|_{\mathcal{B}(\delta)} \right)^{1/2} = \left( \int_{\Gamma} |d\mu| \|b(\mu)\|^2_{\mathcal{B}(\delta, \delta_{\alpha})} \right)^{1/2}, \]
(4.30)
\[ \|C_\Gamma\|_{E_{D_\Gamma}} = \left( \int_{\Gamma} |d\mu| \|c(\mu)c(\mu)^*\|_{\mathcal{B}(\delta_{\alpha})} \right)^{1/2} = \left( \int_{\Gamma} |d\mu| \|c(\mu)\|^2_{\mathcal{B}(\delta_{\alpha}, \delta)} \right)^{1/2}. \]
(4.31)
In the case under consideration, from (4.28) it follows that the quantity \( \mathcal{V}(\Gamma) \), the variation (3.12) of the \( \mathcal{B}(\delta) \)-valued function \( K \) along \( \Gamma \), is explicitly written as
\[ \mathcal{V}(\Gamma) = \int_{\Gamma} |d\mu| \|b(\mu)c(\mu)\|. \]
(4.32)
Together with (4.30) and (4.31), this yields the bound
\[ \mathcal{Y}_s^\prime (\Gamma) = \int_{\Gamma} |d\mu| \langle b(\mu), c(\mu) \rangle \leq \|B_t\|_{E_{D_t}} \|C_t\|_{E_{D_t}}. \] (4.33)

It is convenient to combine Hypothesis 4.1 with our further assumptions in the form of one more hypothesis.

**Hypothesis 4.8.** Assume Hypothesis 3.1 Assume in addition that there exist piecewise smooth Jordan contours \( \Gamma^- \) and/or \( \Gamma^+ \) with the end points \( \alpha \) and \( \beta \) such that \( \Gamma^s \subset \mathcal{D}^s \cup \Delta \) and
\[ \|B_t\|_{E_{D_t}} \|C_t\|_{E_{D_t}} < \frac{d(\Gamma)^2}{4} \] with \( \Gamma = \Gamma^s \) for both \( s = \pm 1 \). (4.34)

In the following, the curves \( \Gamma^s, s = \pm 1 \), referred to in Hypothesis 4.8 are called admissible contours.

Under Hypothesis 4.8, both deformed Riccati equations (4.14) and (4.15) have their respective bounded solutions \( X_t \) and \( Y_t \). Also the integral equation (3.11) has the unique solution \( Z^\alpha_s, s = \pm 1 \), referred to in Theorem 3.1 and this solution is directly related to the operator \( X_t \) associated with \( \Gamma \subset \mathcal{D}^s \).

**Theorem 4.9.** Assume Hypothesis 4.8 Let \( \Gamma = \Gamma^s, s = \pm 1 \), be a Jordan contour from this hypothesis and set \( d = d(\Gamma) \). Then the deformed operator Riccati equations (4.14) and (4.15) have the respective solutions \( X_t \in \mathcal{B}(\mathcal{D}_{D_t}, \mathcal{D}_{D_t}) \) and \( Y_t \in \mathcal{B}(\mathcal{D}_{D_t}, \mathcal{D}_{D_t}) \) with the following properties:

- for \( B_t \neq 0 \) the operator \( X_t \) is a unique solution to (4.14) in the ball
\[ \left\{ T \in \mathcal{B}(\mathcal{D}_{D_t}, \mathcal{D}_{D_t}) \mid \|T\| < \|B_t\|_{E_{D_t}}^{-1} \left( d - \sqrt{\|B_t\|_{E_{D_t}} \|C_t\|_{E_{D_t}}} \right) \right\}. \] (4.35)

The solution \( X_t \) has a finite \( E_{D_t} \)-norm satisfying the bound
\[ \|X_t\|_{E_{D_t}} \leq \frac{1}{\|B_t\|_{E_{D_t}}} \left( \frac{d}{2} - \sqrt{\frac{d^2}{4} - \|B_t\|_{E_{D_t}} \|C_t\|_{E_{D_t}}} \right). \] (4.36)

- for \( C_t \neq 0 \) the operator \( Y_t \) is a unique solution to (4.15) in the ball
\[ \left\{ T \in \mathcal{B}(\mathcal{D}_{D_t}, \mathcal{D}_{D_t}) \mid \|T\| < \|C_t\|_{E_{D_t}}^{-1} \left( d - \sqrt{\|B_t\|_{E_{D_t}} \|C_t\|_{E_{D_t}}} \right) \right\}. \] (4.37)

The solution \( Y_t \) has a finite \( E_{D_t} \)-norm satisfying the bound
\[ \|Y_t\|_{E_{D_t}} \leq \frac{1}{\|C_t\|_{E_{D_t}}} \left( \frac{d}{2} - \sqrt{\frac{d^2}{4} - \|B_t\|_{E_{D_t}} \|C_t\|_{E_{D_t}}} \right). \] (4.38)

Furthermore, the solution \( Z^\alpha_s \) of (3.11) referred to in Theorem 3.1 reads as
\[ Z^\alpha_s = A + B_t X_t, \quad s = \pm 1, \] (4.39)
and is independent of the Jordan contour \( \Gamma \subset \mathcal{D}^s \cup \Delta \) satisfying the assumptions of Hypothesis 4.8.

**Proof.** The hypothesis includes the bound (4.34) which by (2.20) implies both (2.25) and (2.28). Then the statements concerning the solutions \( X_t \) and \( Y_t \) to the Riccati equations (4.14) and (4.15) follow from Theorem 2.4 and (2.5), respectively.

In the case under consideration, the transformed Riccati equation (2.16) for \( X_t \) is as follows:
\[ X_t = \int_{\Gamma} E_{D_t} (d\mu) C_t (A + B_t X_t - \mu)^{-1}. \] (4.40)

Notice that by (4.36) we have
\[ \|B_t X_t\| \leq \|B\|_{E_{D_t}} \|X\|_{E_{D_t}} < d(\Gamma)/2. \] (4.41)
Taking to account that \( A \) is a self-adjoint operator, this implies
\[ \|(A + B_t X_t - \mu)^{-1}\| \leq \frac{1}{\text{dist}(\mu, \sigma_A)} - \|B_t X_t\| < \frac{2}{d(\Gamma)} \] for any \( \mu \in \Gamma \), (4.42)
which means that the integral on the right-hand side of (4.40) is well defined. From (4.40) one concludes that \( Z_{A,f} := A + B_1X_1 \) satisfies the equation

\[
Z_{A,f} = A - \int_{\Gamma} d\mu B_1 E_{D_f}(d\mu) C_{f}(Z_{A,f} - \mu)^{-1}.
\]  

(4.43)

In view of (4.24), (4.27), and (4.28) this equation is merely the equation (3.11). Applying Theorem 3.1 then yields \( Z^s = Z_{A,f} \) which is just (4.39). The independence of \( Z^s \) on contours \( \Gamma \subseteq \mathcal{O} \) satisfying the assumptions of Hypothesis 4.8 is proven by Lemma 3.2.

Corollary 4.10. Under Hypothesis 4.8 the partially deformed block operator matrix \( L_f \) defined by (4.8)–(4.11) for \( \Gamma = \Gamma^s \subseteq \mathcal{O} \), \( s = \pm 1 \), admits the block diagonalization (2.12) in terms of the unique solutions \( X_1 \) and \( Y_1 \) referred to in Theorem 4.9. In particular,

\[
L_f = (I + Q_f) \begin{pmatrix} Z^s & 0 \\ 0 & Z_{D_f} \end{pmatrix} (I + Q_f)^{-1},
\]

(4.44)

where \( Z^s \) is the operator (4.39), \( Q_f \) is given by

\[
Q_f = \begin{pmatrix} 0 & Y_1 \\ X_1 & 0 \end{pmatrix}, \text{ and } Z_{D_f} = D_f + C_f Y_1.
\]

(4.45)

Proof. By the hypothesis, the condition (4.34) holds. Then the statement is proven by applying first Corollary 2.6 and then Remark 2.7.

Remark 4.11. It is worth noting that, because of (4.12), the solution \( X_1 \), \( s = \pm 1 \), represents an operator from \( \mathcal{H} \) to \( \mathcal{H}_{D_f} = L^2(\Gamma \to \mathfrak{h}) \) whose action is given by

\[
(X_1, f_\lambda)(\lambda) = \mathfrak{r}(\lambda)f_\lambda, \quad f_\lambda \in \mathcal{H}, \quad \lambda \in \Gamma^s, \quad s = \pm 1,
\]

(4.46)

where the \( \mathcal{B}(\mathcal{H}, \mathfrak{h}) \)-valued function \( \mathfrak{r} \) of the complex variable \( \lambda \in \mathcal{O} \backslash \sigma_{Z^s} \) reads

\[
\mathfrak{r}(\lambda) := \mathfrak{c}(\lambda)(Z^s - \lambda)^{-1}, \quad \lambda \in \mathcal{O} \backslash \sigma_{Z^s}.
\]

(4.47)

Similarly, the operator \( Y_1 \), for \( \Gamma = \Gamma^s \) with fixed \( s = \pm 1 \), may be presented as

\[
Y_1 f_{D_f} = \int_{\Gamma} d\mu \mathfrak{v}(\mu) f_{D_f}(\mu),
\]

(4.48)

where \( f_{D_f} \in \mathcal{H}_{D_f} \) and \( \mathfrak{v} : \Gamma \to \mathcal{B}(\mathfrak{h}, \mathcal{H}) \) is the operator-valued function given by

\[
\mathfrak{v}(\lambda) = -(\bar{Z}^s - \lambda)^{-1}b(\lambda), \quad \lambda \in \mathcal{O} \backslash \sigma_{\bar{Z}^s} \quad (= \mathcal{O} \backslash \sigma_{Z^s}),
\]

(4.49)

with \( \bar{Z}^s = A - Y_1C_1 \). By (2.13) the operators \( \bar{Z}^s \) and \( Z^s \) are similar to each other,

\[
\bar{Z}^s = (I - Y_1X_1)Z^s(I - Y_1X_1)^{-1}.
\]

(4.50)

This means that, like \( Z^s \), the operator \( \bar{Z}^s \) does not depend on a Jordan contour \( \Gamma \subseteq \mathcal{O} \) satisfying the assumptions of Hypothesis 4.8. The functions \( \mathfrak{r} \) and \( \mathfrak{v} \) work simultaneously for all the piecewise Jordan contours \( \Gamma \) with \( \Omega(\Gamma^s) \subseteq \mathcal{O} \) such that \( \sigma_{\bar{Z}^s} \subseteq \Omega(\Gamma^s) \).

Remark 4.12. By (4.44) the spectrum of the operator \( L_f \) is nothing but the union of the spectra of \( Z^s \) and \( Z_{D_f} \):

\[
\sigma_{L_f} = \sigma_{Z^s} \cup \sigma_{Z_{D_f}}, \quad \Gamma = \Gamma^s, \quad s = \pm 1.
\]

(4.51)

Under the hypothesis of Theorem 4.9 we have the bound (4.41) for the product \( B_iX_i \). Similarly, by (4.38) one obtains

\[
\|C_1 Y_1 \| \leq \|C_1 \|_{d_{D_f}} \| Y_1 \|_{d_{D_f}} < d(\Gamma)/2.
\]

(4.52)

Since the operator \( A \) is self-adjoint and \( D_f \) is normal, from (4.41) and (4.52) it respectively follows that \( \sigma_{Z^s} \subseteq \mathcal{O}_{(\Gamma^s)/2}(\sigma_{A}) \) and \( \sigma_{Z_{D_f}} \subseteq \mathcal{O}_{(\Gamma^s)/2}(\sigma_{D_f}) \). Hence, the spectra \( \sigma_{Z^s} \) and \( \sigma_{Z_{D_f}} \) are disjoint.

\[
\text{dist}(\sigma_{Z^s}, \sigma_{Z_{D_f}}) > 0.
\]

(4.53)
Recall that the set $\sigma_{Z_{\nu}}$ depends on $l$ but does not depend on (the form of) the contour $\Gamma^e$ satisfying (3.18) (see Theorem 4.9, cf. Lemma 3.2).

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