\( \eta'(\eta) \to \gamma \gamma: \text{A Tale of Two Anomalies} \)

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Abstract

The radiative decays of the pseudoscalar mesons \( P = (\pi^0, \eta, \eta') \) are of special interest as they provide an experimental window on the electromagnetic and colour chiral \( U(1)_A \) anomalies. \( \pi^0 \to \gamma \gamma \) is well described by the electromagnetic \( U(1)_A \) anomaly under the assumption that \( \pi^0 \) is a Goldstone boson for spontaneously broken chiral symmetry, but the analogous results for \( \eta(\eta') \to \gamma \gamma \) are complicated by the colour \( U(1)_A \) anomaly in QCD. This paper reviews the theory of \( \eta(\eta') \to \gamma \gamma \) decays, emphasising the role of the colour \( U(1)_A \) anomaly and the renormalisation group. The relation to the Witten-Veneziano mass formula for the \( \eta \) and the QCD topological susceptibility is derived. The implications for the phenomenological analysis of \( \eta(\eta') \to \gamma \gamma \) decays are reviewed and a proposal is made for a comprehensive re-analysis of existing data on \( \eta(\eta') \to \gamma \gamma \).

1. Principal results

The radiative decays of the pseudoscalar mesons \( P \to \gamma \gamma \), where \( P = (\pi^0, \eta, \eta') \) have long been of special interest because of their intimate relation with the chiral (axial) \( U(1)_A \) anomaly. The decay \( \pi^0 \to \gamma \gamma \) is of course the textbook example of the phenomenological importance of the electromagnetic contribution to the \( U(1)_A \) anomaly. It has played an important role both in understanding the theory of anomalies and in pinning down the quantum numbers of the quarks and number of colours \( N_c \) in the early days of QCD. The decay \( \eta' \to \gamma \gamma \) is doubly interesting, since it also involves in an essential way the colour contribution to the \( U(1)_A \) anomaly.

An immediate consequence of the colour \( U(1)_A \) anomaly in QCD is that even in the limit of massless quarks, the \( \eta' \), unlike the \( \pi^0 \) and \( \eta \), is not a Goldstone boson of spontaneously broken chiral symmetry. The success of the theoretical prediction relating the decay amplitude for \( \pi^0 \to \gamma \gamma \) to the electromagnetic \( U(1)_A \) anomaly is based on the pion being an approximate Goldstone boson in the massless limit. The complete theoretical analysis can then be abbreviated in ‘soundbite’ form by writing the familiar PCAC relation:

\[
\bar{\psi}^a \gamma_\mu am0 P_{\mu} \psi \to f_{\pi} m_{\pi}^2 \pi \tag{1}
\]

where on the r.h.s. \( \pi \) is a phenomenological field for the \( \pi^0 \).

The corresponding analysis for the \( \eta' \) is complicated first by the importance of nontrivial \( \eta - \eta' \) mixing due to the explicit SU(3) breaking induced by \( m_s > m_u = m_d \) and second, by the presence of the colour contribution to the \( U(1)_A \) anomaly. Nevertheless, the analogous analysis can be pushed through using the anomalous Ward identities and, subject to reasonable dynamical assumptions, results in the following generalised PCAC relation for the \( \eta \) and \( \eta' \):

\[
\bar{\psi}^a \gamma_\mu P_{\mu} \eta \to f_{\pi} m_{\pi}^2 \eta \beta + 2 n_f G \delta_{ab} \tag{2}
\]

where \( \eta^2 = (\eta, \eta') \) and \( a = 0, 8 \). The new term on the r.h.s. reflects the presence of the anomaly. Very roughly, what happens is that due to the anomaly, the flavour singlet pseudoscalar mixes with the gluonic operator \( G^\alpha G_{\alpha \mu} \).

The conventional PCAC relation applies only to the fictitious state before mixing. Rearranging this into a part involving the physical \( \eta' \) therefore leaves a residual ‘glue’ contribution, represented by \( G \). However, great care must be taken with this picture and especially with the interpretation of eq.(2).

In particular, \( G \) is not to be understood necessarily as the lightest \( 0^+ \) glueball, and the relation (2) is only valid as an operator relation for insertions into zero-momentum Green functions. These points will be explained carefully in due course.

In this paper we review the special features that arise due to the gluonic contribution to the \( U(1)_A \) anomaly when PCAC methods (which include chiral Lagrangians) are used to analyse radiative pseudoscalar meson decays in the flavour singlet channel. In [1], we presented an analysis of \( \eta \to \gamma \gamma \) decay in the chiral limit of QCD, taking into account the gluonic anomaly and the associated anomalous scaling implied by the renormalisation group. Here, we summarise the results of a more recent analysis [2] extending this to QCD with massive quarks, incorporating \( \eta - \eta' \) mixing. In particular, we show how a combination of the radiative decay formula and a generalisation of the Witten–Veneziano mass formula for the \( \eta' \) could be used to measure the gluon topological susceptibility \( \chi(0) \) in full QCD with massive quarks.

Our main result is summarised in the formulae:

\[
f^{\alpha a} g_{\gamma \gamma} + 2 n_f A_G G_{\alpha \gamma} \delta_{ab} = d_{\pi m} \frac{a}{\pi} \tag{3}
\]

which describes the radiative decays, and

\[
f^{\alpha a} (m^2)^a d_{T \bar{T} \beta} = (2 n_f) \hat{A}_0 \delta_{ab} \delta_{\beta 0} - 2 d_{\pi m} \alpha T^m \begin{pmatrix} m_u (\bar{u} u) & 0 & 0 \\ 0 & m_d (\bar{d} d) & 0 \\ 0 & 0 & m_s (\bar{s} s) \end{pmatrix} \tag{4}
\]

which defines the decay constants appearing in (3) through a modification of the Dashen, Gell-Mann Oakes Renner [3,4]

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1 Hereafter simply referred to as the Dashen formula.
formula to include the gluon contribution to the \( U_A(1) \) anomaly.

In these formulae, \( \eta^* \) denotes the neutral pseudoscalars \( \pi^0, \eta, \eta' \). The (diagonal) mass matrix is \( (m^2)_{ab} \) and \( g_{\eta^*\gamma\gamma} \) is the appropriate coupling, defined as usual from the decay amplitude by \( \langle \gamma\gamma | \eta^* \rangle = -i g_{\eta^*\gamma\gamma} e_p e_q (p_1\epsilon^0(p_2) \epsilon^0(p_2)) \). The constant \( a_{em}^2 \) is the coefficient of the electromagnetic contribution to the axial current anomaly:

\[
\hat{e}_F \delta_{ab} m^2_{\pi} \phi_0^b + 2 \eta_i \delta_{ab} Q + a_{em}^2 \frac{Z}{8\pi} F_{\mu\nu} F_{\mu\nu}.
\]

Here, \( j_P^a \) is the axial current, \( \phi_0^b = q^a \gamma^5 T^a q \) is the quark pseudoscalar and \( Q = \frac{2}{8\pi} \text{tr} G_{\mu\nu} G^{\mu\nu} \) is the gluon topological charge. \( m^a \) are the quark masses (see Eq. (16)), \( a = 0, 3, 8 \) is the flavour index, \( T^{3,8} \) are SU(3) generators and \( T^0 = 1 \). The \( d \)-symbols are defined by \( \left\{ T^a, T^b \right\} = \frac{1}{2} \delta_{ab} T^8 \). Since this includes the flavour singlet \( U_A(1) \) generator, they are only symmetric on the two indices. For \( n_f = 3 \), the explicit values are \( d_{00} = d_{03} = d_{08} = 2, d_{30} = d_{88} = 1/3, d_{33} = d_{88} = -d_{88} = 1/\sqrt{3} \).

These formulae become clearer if we immediately make the approximation \( m_c, m_d \ll m_s \). The formulae for the \( \pi^0 \) decouple:

\[
f_{\pi} g_{\pi\gamma\gamma} = a_{em}^2 \frac{Z}{\pi} \tag{6}
\]

where \( a_{em} = \frac{1}{2} N_c \) and

\[
f_{\pi}^2 m_{\pi}^2 = -(n_c \langle u\bar{u} \rangle + n_d \langle d\bar{d} \rangle) \tag{7}
\]

leaving the following new results [2] for the \( \eta \) and \( \eta' \):

\[
\begin{align*}
f_{\eta}^0 g_{\eta\gamma\gamma} + f_{\eta'}^0 g_{\eta'\gamma\gamma} + 6 A g_{\pi\gamma\gamma} &= a_{em}^2 \frac{Z}{\pi} \tag{8} \\
f_{\eta}^0 g_{\eta\gamma\gamma} + f_{\eta'}^0 g_{\eta'\gamma\gamma} &= a_{em}^2 \frac{Z}{\pi} \tag{8}
\end{align*}
\]

where \( a_{em} = \frac{4}{\pi} N_c \) and \( f_{\eta}^0 = \frac{1}{3\sqrt{3}} N_c \), together with the generalised Dashen formulae:

\[
\begin{align*}
(f_{\eta}^0)^2 m_{\eta}^2 + (f_{\eta'}^0)^2 m_{\eta'}^2 &= -4 m_{\langle ss \rangle} + 36 A, \\
(f_{\eta}^0)^2 m_{\eta}^2 + f_{\eta'}^0 f_{\eta}^0 m_{\eta'}^2 &= \frac{4}{3} m_{\langle ss \rangle}, \\
(f_{\eta'}^0)^2 m_{\eta'}^2 + f_{\eta}^0 f_{\eta'}^0 m_{\eta}^2 &= \frac{4}{3} m_{\langle ss \rangle}.
\end{align*}
\]

The decay constants \( f_{\pi}^\eta \) in (3) are defined so as to satisfy the relation (4). It is crucial to recognise that in general they are not the couplings of the pseudoscalar mesons to the axial current [1]. In the flavour singlet sector, such a definition would give a RG non-invariant decay constant which would not coincide with those in the correct decay formula (3). In contrast, all the quantities in the formulae (3), (4) are separately RG invariant [2,5]. The proof is not immediately obvious, and depends on the RGEs for the various Green functions and vertices defining the terms in (3), (4) being evaluated on-shell or at zero-momentum.

As we have seen, since flavour SU(2) symmetry is almost exact, the relations for \( \pi^0 \) decouple and are simply the standard ones with \( f_\pi^\pi \) identified as \( f_\pi \). In the octet-singlet sector, however, there is mixing and the decay constants form a \( 2 \times 2 \) matrix:

\[
f_{\gamma} = \begin{pmatrix} f_{\gamma\gamma} & f_{\gamma\eta} \\ f_{\gamma\eta'} & f_{\gamma\eta'} \end{pmatrix}.
\]

The four components are independent. In particular, for broken SU(3), there is no reason to express \( f_{\gamma} \) as a diagonal matrix times an orthogonal \( \eta - \eta' \) mixing matrix, which would give just three parameters. Several convenient parametrisations may be made, e.g. involving two constants and two mixing angles, but this does not reflect any special dynamics. We return to this point when we discuss phenomenology in Section 3.

The novelty of our results of course lies in the extra terms arising in (3) and (4) due to the gluonic contribution to the \( U_A(1) \) anomaly. The coefficient \( A \) is the non-perturbative number which specifies the topological susceptibility in full QCD with massive dynamical quarks. The topological susceptibility is defined as

\[
\chi(0) = \int d^4x \langle 0 | T(x) Q(0) | 0 \rangle.
\]

The anomalous chiral Ward identities determine its dependence on the quark masses and condensates up to an undetermined parameter, viz.

\[
\chi(0) = -A \left( 1 - A \sum_q \frac{1}{m_q \langle q\bar{q} \rangle} \right)^{-1}.
\]

Notice how this satisfies the well-known result that \( \chi(0) \) vanishes if any quark mass is set to zero.

The modified flavour singlet Dashen formula is in fact a generalisation of the Witten–Veneziano mass formula for the \( \eta' \). Here, however, we do not impose the leading order in \( 1/N_c \) approximation that produces the Witten–Veneziano formula. Recall that this states

\[
m_{\eta'}^2 + m_{\eta}^2 - 2m_K^2 = -\frac{6}{f_\pi^2} \chi(0)_{YM}.
\]

To recover (13) from our result (see the first of Eqs (9)) the condensate is replaced using the standard Dashen formulae \( f_{\eta}^2 m_{\eta}^2 = \frac{4}{3} f_{\eta'}^2 m_{\eta'}^2 = -\frac{1}{4} m_{\langle ss \rangle} \) and the singlet decay constants are approximated by \( f_{\eta'} \simeq \sqrt{6} f_{\pi} \) and \( f_{\eta} \simeq 0 \). The identification of the large \( N_c \) limit of the coefficient \( A \) with the non-zero topological susceptibility of pure Yang-Mills theory follows from large \( N_c \) counting rules and is explained in Ref. [2]. Lattice evaluations [6] (see also Ref. [7]) in the quenched approximation find the following value,

\[
\chi(0)_{YM} = -(180 \text{ MeV})^4.
\]

A similar result has also been obtained using QCD spectral sum rules [8].

The final element in (3) is the extra ‘coupling’ \( g_{G\gamma\gamma} \) in the flavour singlet decay formula, which arises because even in the chiral limit the \( \eta' \) is not a Goldstone boson because of the gluonic \( U_A(1) \) anomaly. A priori, this is not the coupling of a physical particle, although (suitably normalised) it could be modelled as the coupling of the lightest predominantly glueball state mixing with \( \eta' \). However, this interpretation would probably stretch the basic dynamical assumptions underlying (3) too far, and is
not necessary either in deriving or interpreting the formula. In fact, the $g_{G\gamma}$ term arises simply because in addition to the electromagnetic anomaly, the divergence of the axial current contains both the quark pseudoscalar $\phi_5^a$ and the gluonic anomaly $Q$. Diagonalising the propagator matrix for these operators isolates the $\eta$ and $\eta'$ poles, whose couplings to $\gamma\gamma$ give the usual terms $g_{\gamma\gamma\gamma}$ and $g_{\gamma\gamma\gamma}$. However, the remaining operator (which we call $G$) also couples to $\gamma\gamma$ and therefore also contributes to the decay formula, whether or not we assume that its propagator is dominated by a 'glueball' pole. We emphasise again that there is no need whatsoever to make any assumption about the spectrum of the $G$ propagator in deriving the decay formula.

The presence of the coupling $g_{G\gamma}$ in (8) however removes any immediate predictivity from the $\eta' \to \gamma\gamma$ decay formula. In order to push the theory further, we therefore need to make additional dynamical assumptions. The obvious possibility is to explore whether the phenomenologically successful OZI rule can be applied in this context. (The OZI rule is of course closely related (but not identical [9]) to the $1/N_c$ expansion, which is discussed in this context in Ref. [1].) In fact, as we now argue, the OZI rule does provide some justification for believing that the extra coupling $g_{G\gamma}$ is indeed small. The theoretical argument is based on the fact that $g_{G\gamma}$ is both OZI suppressed and renormalisation group (RG) invariant [1]. Since violations of the OZI rule are associated with the $U_4(1)$ anomaly, it is a plausible conjecture that we can identify OZI-violating quantities by their dependence on the anomalous dimension associated with the non-trivial renormalisation of $\rho_{\eta}$ due to the anomaly. In this way, RG non-invariance can be used as a flag to indicate those quantities expected to show large OZI violations. If this conjecture is correct, then we would expect the OZI rule to be reasonably good for the RG invariant $g_{G\gamma}$, which would therefore be suppressed relative to $g_{\gamma\gamma\gamma}$. (An important exception to this argument is of course the $\eta$ mass itself, which although obviously RG invariant is not zero in the chiral limit as it would be in the OZI limit of QCD.) Notice that this conjecture has been applied already with some success to the 'proton spin' problem in polarised deep inelastic scattering [10].

If it is indeed a good dynamical approximation to assume $g_{G\gamma}$ is small compared to $g_{\eta\gamma\gamma}$, we could combine Eqs (8) and (9) to give a measurement of the non-perturbative coefficient $A(\chi(0))$. To see this, assume that the physical quantities $m_\eta$, $m_{\eta'}$, $g_{G\gamma}$ and $g_{\eta\gamma\gamma}$ are all known and (temporarily) neglect $g_{G\gamma}$. Clearly, the two purely octet formulae can be used to find $f_{\eta}$ and $f_{\eta'}$ if both $g_{G\gamma}$ and $g_{\eta\gamma\gamma}$ are known. The off-diagonal Dassen formula then expresses $f_{\eta}$ in terms of $f_{\eta'}$. This leaves the two single singlet formulae involving the still-undetermined decay constant $f_{\eta'}$, the topological susceptibility coefficient $A$, and the coupling $g_{G\gamma}$. The result follows immediately. If we neglect $g_{G\gamma}$, we can find $f_{\eta}$ from the singlet decay formula and thus determine $A$ from the remaining flavour singlet Dassen formula. This is the generalisation of the Witten–Veneziano formula. Determining $A$ in this way would of course be an important link between the phenomenology of $\eta'$ decays and the important subject of gluon topology in QCD [5].

However, the issue of the magnitude of $g_{G\gamma}$ is ultimately an experimental question. It is therefore crucial that phenomenological analyses of the data on $\eta$ and $\eta'$ decays start from the complete formulae (8), (9) and do a best fit involving the full set of parameters. Only then will we really know whether or not the extra anomaly-induced coupling $g_{G\gamma}$ is small.

We return to this issue in Section 3. First, we provide a PCAC-based proof of the key formulae (3),(4). Carefully used, these methods permit a very quick derivation. However, there are many subtleties in the analysis which are easily missed using these conventional techniques. For this reason, we prefer the more field-theoretic approach explained in Refs [1,2] (see also [5]). This makes very clear exactly what dynamical assumptions must be made and where they enter the argument. The price is working with a formalism which, although elegant and easy to use, is unfamiliar to many phenomenologists. We present this field-theoretic derivation in Section 5, although readers interested only in the phenomenological results will find the discussion in Sections 1–4 complete and self-contained.

2. $U_4(1)$ PCAC

Consider first QCD by itself without the coupling to electromagnetism. The axial anomaly is

$$\delta^a f_5^a = M_{ab} \phi_b^0 + 2 \eta Q \delta_{ab}. \quad (15)$$

The notation is defined in Ref. [2]. The quark mass matrix is written as $m^a T^a$,

$$\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \end{pmatrix} = m^0 I + m^3 I^3 + m^8 I^8. \quad (16)$$

The condensates are written as

$$\left( \begin{array}{c} \langle \bar{u}u \rangle \\ \langle \bar{d}d \rangle \\ \langle \bar{s}s \rangle \end{array} \right) = \frac{1}{3} \left( \begin{array}{c} \phi_0^0 \\ \phi_0^3 \\ \phi_0^8 \end{array} \right) = \frac{1}{3} \left( \begin{array}{c} 1 \\ 3 \end{array} \right) \left( \begin{array}{c} \phi_0^0 \\ \phi_0^3 \\ \phi_0^8 \end{array} \right) = \frac{1}{3} \left( \begin{array}{c} 1 \\ 3 \end{array} \right) \left( \begin{array}{c} 1 \\ 3 \end{array} \right) \left( \begin{array}{c} 1 \\ 3 \end{array} \right) \left( \begin{array}{c} 1 \\ 3 \end{array} \right) = \frac{1}{3}. \quad (17)$$

where $\langle \phi_0^0 \rangle$ is the VEV $\langle qT^a q \rangle$. Then

$$M_{ab} = d_{abc} m^c, \quad \Phi_{ab} = d_{abc} \langle \phi_0^c \rangle. \quad (18)$$

An essential approximation in the application of PCAC methods is to use identities valid at zero momentum and make certain smoothness assumptions (see Section 5 for a careful discussion) to extrapolate to the mass-shell of the physical states. We therefore use the zero-momentum chiral Ward identities. For the two-point Green functions of the relevant operators these are [2]

$$2 \eta \langle Q \phi \rangle \delta_{ab} + M_{ab} \langle \phi \rangle \delta_{ab} = 0, \quad (19)$$

which imply

$$M_{ab} M_{cd} \langle \phi_5^a \phi_5^d \rangle = - (M^2)_{ab} + (2 \eta)^2 \langle Q \phi \rangle \delta_{ab}. \quad (20)$$

We also need the result for the general form of the topological susceptibility (see Eq. (12)):
\( \chi(0) \equiv \langle Q Q \rangle = -\frac{A}{1 - (2\eta)^2 A M \Phi} \) \hspace{1cm} (21)

Although the pseudoscalar operators \( \phi_5^2 \) and \( Q \) couple to the physical states \( \eta' = (\pi^0, \eta, \eta') \), it is more convenient to redefine linear combinations such that the resulting propagator matrix is diagonal and properly normalised. So we define operators \( \eta^2 \) and \( G \) such that

\[
\begin{pmatrix}
\langle Q Q \rangle & \langle Q \phi_5^2 \rangle \\
\langle \phi_5^2 Q \rangle & \langle \phi_5^4 \rangle
\end{pmatrix} \rightarrow \begin{pmatrix}
-G & 0 \\
0 & \langle \eta^2 \eta \rangle
\end{pmatrix}.
\]

This is achieved by

\[
G = Q - \langle Q \phi_5^2 \rangle (\langle \phi_5 \phi_5 \rangle)^{-1} \phi_5^2
\]
which reduces at zero momentum to

\[
G = Q + 2\eta A \Phi \phi_5^4 \phi_5^4
\]
and we define

\[
\eta^2 = f^{T \phi_5} \phi_5 \phi_5.
\]

With this choice, the \( \langle G G \rangle \) propagator at zero momentum is

\[
\langle G G \rangle = -A
\]
and we impose the normalisation

\[
\langle \eta^2 \eta \rangle = -\frac{1}{k^2 - m_{\eta'}^2} \delta^{\eta\eta'}.
\]

This implies that the constants \( f_{\eta'\eta} \) in (25), which are simply the decay constants, must satisfy the (Dashen) identity

\[
f^{\eta'\eta} m_{\eta'}^2 f^{T \phi_5} = \Phi_{aa}(\langle \phi_5 \phi_5 \rangle)^{-1} \Phi_{ab}
\]
\[
= -A M \Phi \delta_{ab} + (2\eta)^2 A \delta_{ab} \delta_{\eta 0}.
\]

The last line follows from the Ward identities (20) and (21). In terms of these new operators, the anomaly equation (14) at zero momentum is:

\[
\partial^\mu J_{\mu}^{a} \rightarrow f^{\eta'\eta} m_{\eta'}^2 \eta^\eta + 2\eta G \delta_{a 0}.
\]

The notation \( \rightarrow \) is to emphasise that the identity is only true for insertions of the operators into zero-momentum Green functions and matrix elements. At non-zero momentum, other operators appear on the r.h.s. (In particular, we can not take the on-shell matrix elements between the vacuum and the \( |\eta'\rangle \) and use \( \langle 0|G|\eta'\rangle = 0 \) to conclude that the decay constants \( f_{\eta'\eta} \) can be identified as usual with the matrix elements \( \langle 0|\partial^\mu J_{\mu}^{a}|\eta'\rangle \). As we have repeatedly emphasised, the anomaly removes the familiar correspondence between the decay constants and the current matrix elements.)

Now recall how conventional PCAC is applied to the calculation of \( \pi^0 \rightarrow \gamma \gamma \). The pion decay constant is defined as the coupling of the pion to the axial current

\[
\langle 0|J_{\mu 5}^{a} |\pi\rangle = i k_{\mu} f_{\pi} \Rightarrow \langle 0|\partial^\mu J_{\mu 5}^{a} |\pi\rangle = f_{\pi} m_{\pi}^2
\]
and satisfies the usual Dashen formula The next step is to define a ‘phenomenological pion field’ \( \pi \) by

\[
\partial^\mu J_{\mu 5}^{a} \rightarrow f_{\pi} m_{\pi}^2 \pi.
\]

To include electromagnetism, the full anomaly equation is extended as in (5) to include the \( F_{\mu \nu} \bar{F}_{\mu \nu} \) contribution. Using (29) we have

\[
\begin{aligned}
\langle \gamma \gamma \rangle J_{\mu 5}^{a} |0\rangle &= f_{\pi} m_{\pi}^2 \langle \gamma \gamma |\pi|0\rangle + \frac{e_{\eta'}}{8\pi} \langle \gamma \gamma |F_{\mu \nu} \bar{F}_{\mu \nu}|0\rangle \\
&= f_{\pi} m_{\pi}^2 \langle \pi \rangle \langle \pi \rangle + \frac{e_{\eta'}}{8\pi} \langle \gamma \gamma |F_{\mu \nu} \bar{F}_{\mu \nu}|0\rangle
\end{aligned}
\]
where \( \langle \pi \pi \rangle \) is the pion propagator \(-1/(k^2 - m_{\pi}^2)\). At zero momentum, the l.h.s. vanishes because of the explicit \( k_{\mu} \) factor and the absence of massless poles. We therefore find,

\[
f_{\pi} g_{\gamma'\gamma'} = \frac{a_{\eta'}^2}{\pi}.
\]

In the full theory including the flavour singlet sector and the gluonic anomaly, we find a similar result. The ‘phenomenological fields’ are defined by (29) where the decay constants satisfy the generalised Dashen formula (28). Notice again that they are not simply related to the couplings to the axial current as in (31) for the flavour non-singlet. We therefore find:

\[
\langle \gamma \gamma |J_{\mu 5}^{a} |0\rangle = f_{\pi} m_{\pi}^2 \langle \gamma \gamma |\eta' \rangle |0\rangle + 2\eta f_{\eta} \langle \gamma \gamma |G |0\rangle \delta_{a 0}
\]
\[
+ \frac{e_{\eta'}}{8\pi} \langle \gamma \gamma |F_{\mu \nu} \bar{F}_{\mu \nu}|0\rangle,
\]
using the fact that the propagators are diagonal in the basis \( \eta', G \). Using the explicit expressions (26),(27) for the Green functions, we find in this case:

\[
f_{\pi} g_{\gamma'\gamma'} = 2\eta A g_{\eta'\eta} \delta_{a 0} = \frac{a_{\eta'}^2}{\pi}
\]
where the extra coupling \( g_{\eta'\eta} \) is defined through (34). This completes the derivation. It is evidently a straightforward generalisation of conventional PCAC with the necessary modification of the usual formulæ to take account of the extra gluonic contribution to the axial anomaly in the flavour non-singlet channel, the key point being the identification of the operators \( \eta' \) and \( G \) in (29).

3. Phenomenology

In this section, we discuss critically the way in which data on \( \eta' \rightarrow \gamma \gamma \) decays (and closely related processes such as \( \eta(1400) \rightarrow \mu^+ \mu^- \)) are analysed in the phenomenological literature\(^2\). The existing analyses are based on formulæ in which the impact of the colour \( U(1) \) anomaly has not been correctly taken into account. We therefore propose a comprehensive re-analysis of the data based on the formulæ (8),(9) derived above.

The two-photon decay widths are given by

\(^2\) Here, and in Section 4, we have only cited the few papers which have been most influential in the preparation of this article. This is not intended to be a comprehensive review of the diverse and interesting literature on the phenomenology of \( \eta \) and \( \eta' \) physics.
\[ \Gamma(\eta'(\eta) \to \gamma \gamma) = \frac{m_{\eta'(\eta)}^2}{64\pi} |g_{\eta'(\eta)\gamma\gamma}|^2. \]  

(36)

The current experimental data, quoted in the Particle Data Group tables [11] are

\[ \Gamma(\eta \to \gamma \gamma) = 0.510 \pm 0.026 \text{ keV} \]  

(37)

arising principally from the 1988 Crystal Ball [13] and 1990 ASP [14] results on \( e^+e^- \to e^+e^-\eta \) (here, we follow the note in the 1994 PDG compilation [12] and use only the two-photon \( \eta \) production data), and

\[ \Gamma(\eta' \to \gamma \gamma) = 4.28 \pm 0.19 \text{ keV} \]  

(38)

dominated by the 1998 L3 data [15] on the two-photon formation of \( \eta' \) in \( e^+e^- \to e^+e^-\pi^+\pi^-\gamma \).

For the purposes of comparing our theoretical results with the standard phenomenology, we focus on the very thorough and complete discussion in Ref. [16], updated in [17]. These authors have studied not only the two-photon decays of \( \eta' \) and \( \eta \) but also the related radiative vector-meson decays \( \eta'(\eta) \to V\gamma \), where \( V = (\rho, \omega, \phi) \). They also analyse \( \psi \to \eta'(\eta)\gamma \) decays.

There are two main points of difference between our theory and the analysis in these papers (and all other phenomenological treatments of which we are aware, including the notes on the \( \eta' \) and \( \eta \) decay constants and two-photon decay formulae in the Particle Data Group tables [11]). The first is that the term involving the coupling \( g_{\eta'\gamma\gamma} \) in the decay formula (8) is neglected. The second is that the decay constants entering the decay formulae are assumed to be given (just as for the pion) by the matrix elements of the axial current. Further, in [16] it is assumed that the decay constant matrix (10) has just three independent components, although this is generalised to four in the second paper [17].

To make this concrete, let us define quantities \( f^{aa} \) by

\[ \langle 0|\bar{J}^{a\mu}_b|\eta'\rangle = f^{aa}_b m_{\eta'}. \]  

(39)

The decay formulae used in [16,17] are then simply

\[ f^{aa}_b g_{\eta'\gamma\gamma} = d_{\mu\nu}^{aa} \frac{a^2}{\pi} \]  

(40)

in our notation. The most obvious problem with Eq. (40) is that it is not consistent with the renormalisation group. Since the singlet axial current \( J_5^{a\mu} \) is not RG invariant because of the anomaly (see Section 5) neither are the \( f^{aa}_b \). This is recognised in [16,17] but the associated running of the ‘decay constants’ \( f^{aa}_b \) is assumed there to be negligible. However, this is not the main problem, which is that the RG non-invariance of \( f^{aa}_b \) is a signal that the basic decay formula (40) is wrong – as we have seen above, it must be modified to include the coupling \( g_{\eta'\gamma\gamma} \).

The interesting question for phenomenology to address is whether \( g_{\eta'\gamma\gamma} \) is in fact small, as suggested by the OZI-based discussion above. The success of the programme carried through in [16,17] suggests that this may well be true. However, the only real way to settle the question is to repeat the analysis using the correct formulae (8) and to fit the data with the full set of decay constants \( f^{aa}_b \) (subject to the Dashen constraints (9)) and the extra coupling \( g_{\eta'\gamma\gamma} \). The actual value of \( g_{\eta'\gamma\gamma} \) may also be of relevance in other related processes (see below).

The second main problem stems from the mistaken identification of the relevant decay constants with the axial current matrix elements (39). It is clear that the anomaly equation may be used to relate the (non RG invariant) matrix elements \( \langle 0|Q|\eta'\rangle \) of the topological charge density to the ‘decay constants’ \( f^{aa}_b \). Specifically, if we neglect \( m_q, m_t, \) we have in our notation (c.f. [16,17])

\[ \langle 0|Q|\eta'\rangle = \frac{1}{6} \left( f^{0\gamma}_b m_{2\gamma} + \sqrt{3} f^{8\gamma}_b m_{8\gamma} \right). \]  

(41)

This expresses the ratio of the coupling of \( Q = \frac{2}{4\pi} \text{tr} G_{\mu\nu} \) to \( \eta' \) and \( \eta \) in terms of the \( f^{aa}_b \). It has been pointed out in [18] that the matrix elements \( \langle 0|Q|\eta'(\eta)\rangle \) enter into the formulae for the radiative decays \( \psi \to \eta'(\eta)\gamma \). The current data [11] is

\[ \Gamma(\psi \to \eta\gamma)/\Gamma_{\text{total}} = (0.86 \pm 0.08) \times 10^{-3} \]  

(42)

and

\[ \Gamma(\psi \to \eta'\gamma)/\Gamma_{\text{total}} = (4.31 \pm 0.30) \times 10^{-3} \]  

(43)

where

\[ \Gamma_{\text{total}} = (87 \pm 5) \text{ keV}. \]  

(44)

Studying these decays therefore gives information on the ‘glue content’ of the \( \eta' \) and \( \eta \). However, in Refs [16, 17] the formula (41) is used in conjunction with the erroneous (40) to relate the \( \langle 0|Q|\eta'(\eta)\rangle \) to the decay constants appearing in \( \eta'(\eta) \to \gamma \gamma \). This is formally incorrect, and in any case it must surely be inconsistent to use the naive decay formula (40), which is derived on the basis that both \( \eta' \) and \( \eta \) are true pseudo-Goldstone bosons, to estimate their anomaly-induced glue content. In truth, as originally pointed out in [1], the physical, RG invariant, decay constants \( f^{aa}_b \) appearing in the true decay formulae for the \( \eta' \) and \( \eta \) are in principle quite independent of the quantities \( f^{aa}_b \) related to the axial current matrix elements.

A final comment on the theory concerns \( \eta - \eta' \) mixing and the decay constants \( f^{aa}_b \). For some time, it was usual to describe \( \eta'(\eta) \to \gamma\gamma \) decays in terms of two decay constants \( f^0 \) and \( f^8 \) and a single \( \eta - \eta' \) mixing angle defined in terms of the SU(3) eigenstates by

\[ |\eta'\rangle = \cos \delta |\eta_0\rangle + \sin \delta |\eta_8\rangle \]

\[ |\eta\rangle = - \sin \delta |\eta_0\rangle + \cos \delta |\eta_8\rangle. \]  

(45)

More recently [17,19–22] etc., it has been realised that the \( f^{aa}_b \) decay constant matrix (10) should instead have four independent components, which are conventionally parametrised in terms of the two \( f^{0,8} \) and \( f^{0,8} \) angles. In the version favoured by [17], these would be defined through

\[ \begin{pmatrix} f^{0\eta'} & f^{0\eta} \\ f^{8\eta'} & f^{8\eta} \end{pmatrix} = \begin{pmatrix} f^0 \cos \delta_{\eta'} & -f^0 \sin \delta_{\eta'} \\ f^8 \sin \delta_{\eta'} & f^8 \cos \delta_{\eta'} \end{pmatrix}. \]  

(46)

Results for \( \delta_{\eta'} \) and \( \delta_{\eta} \) derived using (40) are quoted in Ref. [17], where the difference between the two angles is claimed to be an ‘energy dependence’ of the \( \eta - \eta' \) mixing angle, with one being evaluated on the \( \eta' \) mass-shell and the other on the \( \eta \) mass-shell. To us, however, it seems that this interpretation has no consequence. It is clearly correct to use four
parameters to describe the decay constant matrix \( f_{\alpha\beta} \) and a parametrisation in terms of two constants and two angles is as good as any other, but we see no reason to superpose the notion of an ‘energy-dependent mixing angle’ on this simple formalism.

The analysis of \( \eta'(\eta) \to \gamma\gamma \) decays presented here can clearly be extended to study \( \eta'(\eta) \to V\gamma \), where \( V \) is a flavour-singlet vector meson \( \rho,\omega,\phi \), using the familiar assumptions of vector meson dominance. This allows us to extract the couplings \( g_{\eta'(\eta)V\gamma} \) from the 3-point Green functions \( \langle 0 | J_{\mu}^{\rho} J_{\nu}^{\omega} J_{\rho}^{\phi} | 0 \rangle \) by relating them to the appropriate electromagnetic anomaly coefficients in exactly the same way as for the two-photon decays, then assuming pole-dominance for the vector current. This is explained in detail in Ref. [16]. However, exactly the same comments apply. The true decay formulae for \( P \to V\gamma \) involve extra \( g_{G\gamma V} \) couplings in the singlet sector (the full set of formulae analogous to those quoted in Ref. [16] can be derived using the method described in Section 2). These are omitted in Ref. [16]. Again, the apparently reasonable agreement with data [11] found there suggests that these new couplings may be small, but this should be checked by repeating the analysis including the \( g_{G\gamma V} \) and finding the best fit to data. It would also be interesting to make a detailed comparison with the quark model/OZI analysis of Ref. [23].

Another obvious extension is to the decays \( \eta'(\eta) \to \pi^+\pi^-\eta \), related to the 4-point Green functions \( \langle 0 | J_{\mu}^{\rho} J_{\nu}^{\omega} J_{\rho}^{\phi} J_{\mu}^{\pi} | 0 \rangle \). This Green function has an AAAV box anomaly in direct analogy to the AVV triangle anomaly considered above. Clearly this can be analysed simply using the \( U_A(1) \) PCAC theoretical methods described here. On the experimental side, while \( \eta \to \pi^+\pi^-\eta \) has been observed with decay width [11]

\[
\Gamma(\eta \to \pi^+\pi^-\eta) = 0.056 \pm 0.005 \text{ keV}
\]

the most recent L3 results [15] find no evidence for the non-resonant \( \eta' \to \pi^+\pi^-\gamma \) decay in a channel dominated by the rho: \( \eta' \to \rho\gamma \), \( \rho \to \pi^+\pi^- \). This is in contradiction with earlier claims [24–26] that the non-resonant process had been seen.

To conclude this discussion of radiative \( \eta' \) and \( \eta \) decays, it will be clear that a re-analysis of the experimental data using the decay and Dashen formulae (8), (9) should be performed to verify whether or not \( g_{G\gamma V} \) is negligibly small or, if not, to measure it. We would then have a theoretically reliable determination of all the parameters playing a role in this sector. This would be free of any additional prior theoretical input based on either OZI [1,2] or the large \( N_c \) expansion in the chiral Lagrangian framework [21,22,27–29]. At the same time, it should be recognised that the ‘glue content’ \( \langle 0 | Q^{\eta'}(\eta') | 0 \rangle \) of the \( \eta' \) and \( \eta \) are not expressible in terms of the true, RG-invariant decay constants \( f_{\alpha\beta} \) and the relations (41) should not be used in the analysis of \( \psi \to \eta'(\eta)\gamma \) data.

### 4. Related experiments

We now mention briefly a few other experiments where the theoretical techniques described in this paper can be applied.

The first is the behaviour of the polarised photon structure function \( g_1^\gamma \), considered as a function of the ‘target photon’ momentum \( p \). Standard analysis of the two-photon DIS process \( e^+e^- \to e^+e^-X \) reduces the problem of finding the first moment of \( g_1^\gamma \) to a non-perturbative evaluation of the AVV three-point Green function \( \langle 0 | J_{\mu}^{\rho} J_{\mu}^{\omega} J_{\rho}^{\phi} | 0 \rangle \). This is of course precisely the problem in evaluating \( \eta'(\eta) \to \gamma\gamma \), except that we now want to study a range of off-shell photon momenta. This has been carried out in Refs [30,31]. A sum rule for \( g_1^\gamma \) has been presented which fixes the \( p \to 0 \) and asymptotically large \( p \) behaviour of the first moment of \( g_1^\gamma \), in terms of the electromagnetic and colour \( U_A(1) \) anomaly coefficients. For intermediate \( p \to m_\pi \), the behaviour is shown to be sensitive to the non-perturbative realisation of chiral symmetry breaking. Detailed results are given in Refs [30,31].

The theoretical techniques developed here are equally applicable to the flavour-singlet contribution to the first moment of the polarised proton structure function \( g_1^\rho \), the famous ‘proton spin’ problem. This is shown [32] to be related to a \( U_A(1) \) Goldberger-Treiman relation, which is evaluated easily by the techniques of Section 2 and which involves an analogous new coupling \( g_{G\gamma N} \). The OZI and RG conditions in this case are however rather different [33–36].

More recently, it has been proposed [37,38] that the theoretical formalism of Section 2 may also provide a way of understanding recent data from the CLAS collaboration [39,40] on the photo- and electro-production of \( \phi \) mesons. The idea here is that one important contribution to the photoproduction process \( \gamma N \to \phi N \) may be modelled by ‘glue’ exchange, which in turn could be related to the \( g_{G\gamma N} \) coupling in the \( U_A(1) \) Goldberger-Treiman formula and the \( g_{G\gamma \rho} \) coupling extracted from \( \eta'(\eta) \to \phi\eta \) as discussed above. This is an interesting proposal which certainly deserves to be pursued.

Another closely related process is \( \eta'(\eta) \) photoproduction. Both \( \gamma p \to \eta p \) [41] and \( \gamma p \to \eta p \) [42,43] have been studied in low energy experiments close to threshold. A theoretical discussion of these reactions, based on an effective action incorporating the \( U_A(1) \) PCAC formalism of Section 2, may be found in Ref. [44]. For a recent general review of \( \eta' \) physics along these lines, emphasising the importance of the \( U_A(1) \) anomaly and gluonic degrees of freedom and incorporating an extensive survey of experiments, see also [45] and references therein.

We should also make a special mention here of the new WASA 4x detector [46] at CELSIUS, which will provide an important facility for precision \( \eta \) and \( \eta' \) decays. This is of course the stimulus for the present Workshop.

Finally, and somewhat remarkably, the process \( \eta'(\eta) \to \gamma\gamma \) plays an important role in calculations of the strong-interaction contributions to the anomalous magnetic moment \( g_\mu - 2 \) of the muon. This is of great current interest because of the discrepancy between the most recent experiments [47,48] and theory (see e.g. [49] for a recent compilation of sub-process contributions to \( g_\mu - 2 \) and the suggestion that this could be a signal of non-standard model physics (e.g. [50]). The important contribution for our purposes is the \( O(\alpha^2) \) correction arising from the Feynman diagram in which a QCD sub-diagram is linked to the muon line by three virtual photons, effectively a
hadronic contribution to light-by-light scattering. The dominant light pseudoscalar intermediate state therefore involves back-to-back $P \to \gamma\gamma$ vertices, where of course the photons are off-shell. This is fully discussed in Ref. [51] where it is recognised that the conventional theoretical analysis is inadequate and a proper treatment incorporating the gluonic anomaly along the lines discussed here is really required. In its absence, Ref. [51] relies on using phenomenological models and inputs from experiment. The result is that the total hadronic light-by-light scattering contribution to the muon $g_{\mu} - 2$ is $-79.2(15.4) \times 10^{-11}$, to be compared with the current experiment versus theory discrepancy in $a_\mu = \frac{1}{2}(g_{\mu} - 2)$ of $\delta a_\mu^{\exp} - \delta a_\mu^{\text{SM}} = 333(173) \times 10^{-11}$ [49].

5. Theory

In this section we derive the $\eta’(\eta) \to \gamma\gamma$ decay formula using the method of operator Green functions and generalised 1PI vertices developed in Refs [1,2] etc. We present the discussion in three parts – first the anomalous chiral Ward identities for the Green functions of the relevant composite operators and the 1PI vertices, then the derivation of the decay formulae in terms of 1PI vertices (which are essentially the couplings $g_{\eta’(\eta),\gamma\gamma}$, and finally the renormalisation group equations. We then comment briefly on alternative methods involving chiral Lagrangians.

5.1. Chiral ward identities

The anomalous chiral Ward identities for QCD with massive quarks have been written down in the form used here in the review [5]. We refer to this article for more complete derivations and restrict ourselves here to the most essential identities. For the moment, we omit the electromagnetic contribution to the anomaly.

The composite operators involved in the Green functions and 1PI vertices studied here are the currents and pseudoscalar operators $J_{\mu3}^\alpha, Q, \phi_5^\alpha$ and the corresponding scalar $\phi^\alpha$. We use the compact notation for the quark masses and condensates introduced at the start of Section 2.

The expressions for these operators given in Section 1 define the bare operators. The renormalised composite operators are defined as follows:

\[ J_{\mu3}^0 = Z_{J_{\mu3}} J_{\mu3}^B, \quad J_{\mu3}^{\neq 0} = J_{\mu3}^{\neq B}, \]

\[ Q = Q_B - \frac{1}{2M_f}(1 - Z)\theta^0 J_{\mu3}^B, \]

\[ \phi_5^0 = Z_{\phi_5} \phi_5^B, \quad \phi^{\neq 0} = Z_{\phi^{\neq}} \phi^{\neq B}, \]  

(48)

where $Z_\phi$ is the inverse of the mass renormalisation, $Z_\theta = Z_{\theta B}$. The non-trivial renormalisation of $J_{\mu3}^\alpha$ means that its matrix elements scale with an anomalous dimension $\gamma$ related to $Z$. This occurs because $J_{\mu3}^\alpha$ is not a conserved current, due to the anomaly $Q$. Notice in particular the mixing of the operator $Q$ with $\partial \phi_5^B$ under renormalisation. As explained in [52], this leaves the combination $\partial \phi_5^B - 2\eta\phi_5^B$ appearing in the anomaly equation invariant under renormalisation.

The Green functions for these operators are constructed by functional differentiation from the generating functional $W[V\alpha, 0, S_5^\alpha, S_5^\alpha]$, where $V\alpha, \phi_5^\alpha, S_5^\alpha$ are the sources for the composite operators $J_{\mu3}^\alpha, Q, \phi_5^\alpha$ respectively. For example, the Green function $i(0iT Q(x) Q(y))$ is given by $\frac{i}{\partial^2 \phi_5^B/\partial \phi_5^B}$, which we abbreviate as $W_{\phi_5^B}$. This compact notation is perhaps unfamiliar, but is very convenient for manipulating complicated expressions involving the chiral Ward identities.

In this functional formalism, the anomalous chiral Ward identity is

\[ \partial_\mu W_{\phi_5^B} = 2\eta \delta_{\phi_5} W_\phi + M_\phi W_{S_5^\phi} \]

\[ - d_{\phi_5} S_5^\phi W_{S_5^\phi} + d_{\phi_5} S_5^\phi W_{S_5^\phi}. \]  

(49)

This can be compared with Eq. (15). Notice the presence of the source variation terms on the r.h.s. This immediately give the identities for the 2-point Green functions:

\[ ik_{\mu} W_{\phi_5^B} = 2\eta \delta_{\phi_5} W_\phi - M_\phi W_{S_5^\phi} = 0, \]

\[ ik_{\mu} W_{\phi_5^B} - 2\eta \delta_{\phi_5} W_\phi - M_\phi W_{S_5^\phi} = 0, \]

\[ ik_{\mu} W_{\phi_5^B} - 2\eta \delta_{\phi_5} W_\phi - M_\phi W_{S_5^\phi} = 0. \]  

(50)

Combining these individual equations, we find the familiar identity

\[ k_{\mu} k_{\nu} W_{\phi_5^B} - M_\phi W_{\phi_5^B} = W_{S_5^\phi} \]  

(51)

where $S_5^\phi$ is the source for the current divergence operator $D^\nu = 2\eta \delta_{\phi_5} Q + M_\phi \phi_5^\alpha$.

For our purposes, we really only need the zero-momentum chiral Ward identities. Clearly, assuming there is no massless boson coupling to the $U_A(1)$ current, these are just

\[ 2\eta \delta_{\phi_5} W_\phi + M_\phi W_{S_5^\phi} = 0, \]

\[ 2\eta \delta_{\phi_5} W_\phi + M_\phi W_{S_5^\phi} + M_{\phi_5} = 0. \]  

(52)

which implies the following identity for the topological susceptibility,

\[ (2\eta)^2 \chi(0) = M_{\phi_5} W_{S_5^\phi}, M_{\phi_5} = (M\phi)(0). \]  

(53)

The 1PI vertices used below are defined as functional derivatives of a second generating functional (effective action) $\Gamma$, constructed from $W$ by a partial Legendre transform with respect to the fields $Q, \phi_5^\alpha$ and $\phi_5^\alpha$ (not the currents $J_{\mu3}^\alpha$). The resulting vertices are ‘1PI’ w.r.t. the propagators for these composite operators only. This separates off the particle poles in these propagators, and gives the closest identification of the field theoretic vertices with the physical couplings such as $g_{\eta’(\eta),\gamma\gamma}$ [5].

The basic anomalous chiral Ward identity for $\Gamma$ follows immediately from that for $W$:

\[ \partial_\mu \Gamma_{\phi_5^B} = 2\eta \delta_{\phi_5} Q + M_\phi \phi_5^\alpha \]

\[ - d_{\phi_5} \phi_5^\alpha \Gamma_{\phi_5^B} + d_{\phi_5} \phi_5^\alpha \Gamma_{\phi_5^B}. \]  

(54)

and other identities follow simply by functional differentiation. In particular, for the two-point vertices, we find the following identities
\[\newcommand{\kappa}{\kappa}  
\begin{align*}
  i\kappa^\mu \Gamma^a_{\nu\rho} + \Phi_{a\mu} \Gamma^a_{\kappa\rho} &= 0, \\
  i\kappa^\mu \Gamma^a_{\nu\rho} - 2\eta^\kappa \delta_{a\rho} + \Phi_{a\mu} \Gamma^a_{\nu\rho} &= 0, \\
  i\kappa^\mu \Gamma^a_{\nu\rho} + \Phi_{a\mu} \Gamma^a_{\kappa\rho} - M_{ab} &= 0
\end{align*}
from which follows
\[ \kappa^\mu \Gamma^a_{\nu\rho} + M_{ab} \Phi_{a\mu} = \Phi_{a\mu} \Gamma^a_{\kappa\rho} \Phi_{a\mu}. \] 
At zero momentum, these reduce to
\[ \Phi_{a\mu} \Gamma^a_{\nu\rho} - 2\eta^\kappa \delta_{a\rho} = 0, \]
\[ \Phi_{a\mu} \Gamma^a_{\kappa\rho} - M_{ab} = 0 \]
which together imply
\[ \Phi_{a\mu} \Gamma^a_{\kappa\rho} \Phi_{a\mu} = -(M\Phi)_{ab}. \] 

The fact that the topological susceptibility is zero for vanishing quark mass can be seen immediately from Eq. (53). One of the simplest ways to derive the precise form (12) or (21) is in fact to use an identity involving \( \Gamma \). The two-point vertices are simply the inverse of the two-point Green functions (propagators), so in the pseudoscalar sector we have the matrix inversion formula:
\[ \Gamma_{QQ} = -\left( W_{00} - W_{0S} \left( W_{SS} \right)^{-1} W_{S0} \right)^{-1} \]
\[ = -\left( W_{00} - W_{0S} M_{ab} (M W_{SS} M)^{-1} M_{ab} W_{S0} \right)^{-1} \] (59)
and using the identities (52) and (53) this implies
\[ \Gamma_{QQ}^{-1} = -[1 - (2\eta)^2 (M\Phi)^{-1}]^{-1} \] (60) 
al all zero momentum. Inverting this relation gives the important result for the topological susceptibility,
\[ \chi = -\Gamma_{QQ}^{-1} \left( 1 - (2\eta)^2 \Gamma_{QQ}^{-1} (M\Phi)^{-1} \right)^{-1}. \] (61) 
Substituting the explicit expression for \( (M\Phi)^{-1} \) (which is easily found from the definitions above), viz.
\[ (M\Phi)^{-1} = \left( \frac{1}{2\eta^2} \right) \sum \frac{1}{m_q[\bar{q}q]} \] (62) 
we see that (61) reproduces the general form (12) where we can now identify the (mass-independent) non-perturbative coefficient as
\[ A = \Gamma_{QQ}^{-1}. \] (63) 
We have already exploited these formulae in Section 2.

5.2. \( \eta(\eta) \rightarrow \gamma\gamma \) from 1PI vertices

We are now ready to present the derivation of the decay formula (3) and generalised Dashen formula (4). The technique relies on the identification of the couplings \( g \eta^a \gamma_i \) with the zero-momentum limit of the appropriate 1PI vertex functions introduced above.

The starting point is the Ward identity (54) extended to include the electromagnetic contribution to the anomaly for the axial current:
\[ \partial_\mu \Gamma^a_{\nu\rho} = 2\eta^\mu \delta_{a\rho} + \partial^a \phi(\nu) + M_{ac} \Phi_{c\mu} \]
\[ - d_{abc} \Phi^b \Gamma^c_{\kappa\rho} + d_{abc} \Phi^b \Phi_{c\mu} \Gamma^c_{\kappa\rho}. \] (64)

\( Q_m(A) \) is just shorthand notation for \( \frac{e^2}{8\pi} F_{\mu\nu} \), where \( F_{\mu\nu} \) is the field strength for the electromagnetic field \( A_\mu \). (Since we are working only to leading order in \( x \), it is not necessary to consider \( Q_m \) as an independent composite operator with non-trivial renormalisation.)

Differentiating twice w.r.t. the field \( A_\mu \), evaluating at the VEVs, and taking the Fourier transform, we find
\[ i\kappa^\mu \Gamma^a_{\nu\rho} + M_{ab} \Phi_{a\mu} = -\partial^a \frac{\chi}{\pi} \]
\[ \Rightarrow \Phi_{a\mu} \Gamma^a_{\kappa\rho} \Phi_{a\mu} \] (65)

where \( p_1, p_2 \) are the momenta of the photons. Notice that the mass term in (64) does not contribute explicitly to this formula. From its definition as 1PI w.r.t. the pseudoscalar fields, the vertex \( \Gamma^a_{\nu\rho} \) has no pole at \( k^2 = 0 \) (even in the chiral limit) so the first term vanishes at zero momentum \( k \), leaving simply
\[ \Phi_{a\mu} \Gamma^a_{\nu\rho} \Phi_{a\mu} \] (66)

(To simplify notation, it will be convenient from now on to define vertices \( \Gamma \) with the kinematical factors removed, e.g.
\[ \Gamma_{\phi\phi\Gamma} = -\Gamma_{\phi\phi\Gamma} \left( \frac{\chi}{\pi} \right). \]

The first step in converting (66) to the decay formula (3) is to identify the physical states \( \eta \). These appear as poles in the propagator matrix for the four pseudoscalar operators \( Q, \phi_5 \) \((a = 0, 3, 8)\). To isolate these poles, we diagonalise the propagator matrix in this sector then normalise the three operators coupling to the physical states.

We therefore define the operator
\[ G = Q - W_{0S} (W_{SS} W_{ab})^{-1} \phi_5 \]
(67)
so that by construction the propagators \( G \phi_5 \) all vanish. (Notice that integrations over repeated spacetime arguments are implied in this condensed notation.) Then define operators
\[ \eta^a = C_{ab} \phi_5 \]
(68)
such that the propagator matrix
\[ \langle \eta^a \eta^b \rangle = \sum S^a S^b \]
\[ = \left( \begin{array}{ccc}
    -\frac{1}{k^2-m_\eta^2} & 0 & 0 \\
    0 & \frac{1}{k^2-m_\eta^2} & 0 \\
    0 & 0 & \frac{1}{k^2-m_\eta^2}
  \end{array} \right) \] (69)

where \( S^a \) are the sources for the operators \( \eta^a \).

This change of variable affects the partial functional derivatives in \( \Gamma_{\phi\phi\Gamma} \) in (66), which involves \( \frac{\partial}{\partial \phi_5} \) at fixed \( Q \). In terms of the new variables \( G, \eta^a \) we have
\[ \left. \frac{\delta}{\delta \phi_5} \right|_Q = \left. \frac{\delta}{\delta \phi_5} \right|_Q \frac{\delta}{\delta \eta^a} + \frac{\delta G}{\delta \phi_5} \frac{\delta}{\delta \eta^a} \]
\[ = C_{T \eta^a} \frac{\delta}{\delta \eta^a} - (W_{SS} W_{ab})^{-1} W_{S0} \frac{\delta}{\delta G}. \] (70)
The decay formula therefore becomes
\[ \Phi_{ab} C^{Tb} \hat{\Gamma}_{\pi^\alpha, A^\nu} - \Phi_{ab}(W_{S_1 S_0})^{-1} W_{S_0} \hat{\Gamma}_{G, A^\nu} = d_a^\nu \frac{Z}{\pi}. \]  
(71)

The decay constants are identified as
\[ f^{\pi \nu} = \Phi_{ab} C^{Tb}. \]  
(72)

In terms of the propagators, we can write (from Eq. (68))
\[ f^{\pi \nu}(W_{S_0 S_0})^{-1} f^{Tb} = \Phi_{ac}(W_{S_0 S_0})^{-1} \Phi_{db} \]  
(73)

and so at zero momentum
\[ f^{\pi \nu} m_{\pi}^2 f^{Tb} = \Phi_{ac}(W_{S_0 S_0})^{-1} \Phi_{db} \]  
(74)
as quoted in (28).

The remaining steps in finding (3) and (4) are an exercise in manipulating the zero-momentum Ward identities. First note that combining the two identities in (52) gives
\[ M_{ac} W_{S_0 S_0} M_{db} = -(M \Phi)_{ac} + (2n) \chi^0(0) \delta_{ac} \delta_{db} \]  
(75)
whose \( a, b \) component is just (53). Note that \( (M \Phi)_{ab} \) is symmetric. Also define \( I_{\Phi} = \delta_{ac} \delta_{ab} \). Then we can write
\[ \Phi_{ac}(W_{S_0 S_0})^{-1} \Phi_{db} = (M \Phi)_{ac} (M_{W_{S_0 S_0}} M_{cd} W_{S_0 S_0})^{-1} M_{de} W_{S_0 S_0} \]
\[ = -2nI_{\Phi} M_{de} W_{S_0 S_0} \]
\[ + (2n)^2 \chi^0(0) \delta_{ac} \delta_{db} \]
\[ = 2n \chi^0(0)(1 - (2n)^2 \chi^0(0)(M \Phi)^{-1} \delta_{ac} \delta_{db} \]
\[ = -2n \gamma M_{de} W_{S_0 S_0} \]  
(76)
where in the final step we have used the identification (61).

Similarly,
\[ \Phi_{ac}(W_{S_0 S_0})^{-1} \Phi_{db} = (M \Phi)_{ac} (M_{W_{S_0 S_0}} M_{cd} W_{S_0 S_0})^{-1} (M \Phi)_{db} \]
\[ = (M \Phi)(- (M \Phi) + (2n)^2) \]
\[ \times \chi^0(0) I_{\Phi} \delta_{ac} \delta_{db} \]
\[ = -(M \Phi)_{ab} + (2n)^2 \gamma M_{de} W_{S_0 S_0} \]  
(77)

This establishes the required results. Substituting (76) and (77) into (71) and (74) we find the decay formula
\[ f^{\pi \nu} \hat{\Gamma}_{\pi^\alpha, A^\nu} + 2n \gamma M_{de} W_{S_0 S_0} = d_a^\nu \frac{Z}{\pi} \]  
(78)
where the decay constants satisfy
\[ f^{\pi \nu} m_{\pi}^2 f^{Tb} = -(M \Phi)_{ab} + (2n)^2 \gamma M_{de} W_{S_0 S_0} \]  
(79)

The final step is to identify the 1PI vertices with the couplings defined in Section 1, viz.
\[ \hat{\Gamma}_{\pi^\alpha, A^\nu} = \gamma_{A^\nu} \]  
(80)

and similarly for \( \gamma_{G^\nu} \). It is at this point that the central dynamical assumption is made. In fact, Eqs (78) and (79) are exact identities, following simply from the definitions and the zero-momentum chiral Ward identities. To make contact with the radiative decays of the physical particles, we must assume in particular that the 1PI vertex evaluated at \( k = 0 \) accurately approximates the physical coupling \(^3\), which is defined on mass-shell. This requires that \( \Gamma_{\pi^\alpha, A^\nu} \) has only a weak momentum dependence in the range \( 0 \leq k^2 \leq m_{\pi}^2 \). This is reasonable, since it is defined to be the 1PI with pole dominance – the assumption that the dominant particle poles in the pseudo-Goldstone bosons increases. The hope here, in common with all attempts to include the \( \eta' \) in the framework of PCAC (including chiral Lagrangians with \( 1/N_c \) effects included [21,22,27–29]), is that the approximation remains sufficiently good at the mass of the \( \eta' \).

5.3. Renormalisation group
It is important to determine the renormalisation group behaviour of all the quantities appearing in these formulae. Recall, for example, that the RG behaviour was a crucial factor in the conjecture that \( G_{\gamma \gamma} \) may be small in the leading OZI approximation. In general, the RG equations play a key role in understanding the physics of the \( U_A(1) \) channel. We therefore include here a brief and rather novel discussion of the RGs for the relevant Green functions and 1PI vertices in the functional formalism. The essential results were first given in Ref. [1], but are generalised here to include SU(3) breaking and \( \eta - \eta' \) mixing. The results are a straightforward extension of Refs [1,5] but were not explicitly written down in [2].

The fundamental RGE for the generating functional \( W \) in pure QCD follows immediately from the definitions (48) of the renormalised composite operators. It is:
\[ DW = \gamma \left( \gamma_{\pi^\alpha} - \frac{1}{2n} \partial \theta \right) W_{\gamma_{\pi^\alpha}} \]
\[ + \gamma_{\delta} \left( S_{\delta} W_{S_{\delta}} + S_{S} W_{S} \right) \]  
(81)

where
\[ D = \left( \mu \frac{e}{Q^2} + \beta \frac{\partial}{\partial \beta} - \gamma_{m} \sum_q m_q \frac{\partial}{\partial m_q} \right)_{V, \Phi, S, S}. \]

The notation + \( \cdots \) (which is suppressed in the following equations) refers to the additional terms of \( O(k^2) \) and \( O(k^4) \) which are required to produce the contact term contributions to the RGs for \( n \)-point Green functions of composite operators. (This notation is omitted in the following equations, but it should be remembered that it is implicit.) These terms are discussed fully in Refs [53,1], but will be omitted here for simplicity. They vanish at zero momentum so do not directly affect the derivation of the decay formulae, but do have important implications for the validity of PCAC extrapolations from zero-momentum to on-shell quantities.

The RGs for Green functions are found simply by differentiating Eq. (81) with respect to the sources. Simplifying the results using the chiral Ward identities (50), we find a complete set of RGs for the 2-point functions. These are:

3 The assumption that the 1PI vertices as defined here can be identified at all with the decay couplings of the physical particles rests on pole dominance – the assumption that the dominant particle poles in the pseudo-Goldstone propagator matrix are indeed those of the \( \eta' \) (see Eq. (69)).
\[ DW^\nu_{\alpha} = (\gamma \delta_{\alpha \beta} + \gamma \delta_{0 \beta}) W^\nu_{\alpha} \]
\[ DW^\nu_{\alpha} = (\gamma \delta_{\alpha \beta} + \gamma \delta_{0 \beta}) W^\nu_{\alpha} + \gamma \frac{1}{2n_f} M_{0\beta} W^\nu_{\alpha}; \]
\[ DW^\nu_{\alpha} = (\gamma \delta_{\alpha \beta} + \gamma \delta_{0 \beta}) W^\nu_{\alpha} + \gamma \frac{1}{2n_f} M_{0\beta} W^\nu_{\alpha}; \]
\[ DW^\nu_{\alpha} = (\gamma \delta_{\alpha \beta} + \gamma \delta_{0 \beta}) W^\nu_{\alpha} + \gamma \frac{1}{2n_f} (M_{0k} W^\nu_{\alpha}; + \Phi_{0\beta}). \]
\[ DW^\nu_{\alpha} = (\gamma \delta_{\alpha \beta} + \gamma \delta_{0 \beta}) W^\nu_{\alpha} + \gamma \frac{1}{2n_f} (M_{0k} W^\nu_{\alpha}; + \Phi_{0\beta}). \]

The pattern of cancellations which ensures the consistency of these equations with the chiral Ward identities is quite intricate, but may readily be checked.

At zero momentum, we can immediately use the second of Eqs (52) to write the above RGE for \( W \) as
\[ DW^\nu_{\alpha} = 2\gamma W_{\alpha} - 2\gamma \frac{1}{(2n_f)} (M_{0k} W^\nu_{\alpha}; + \Phi_{0\beta}) \]

using (53). This shows that the zero-momentum topological susceptibility is a RG invariant,
\[ D\chi(0) = 0 \]
and thus
\[ DA = 0 \]

where \( A \) is the non-perturbative parameter in Eq. (12), which enters into the final decay and Dashen formulae.

Next, we need the RGE for the generating functional of the 1PI vertices. This follows immediately from its definition and the RGE (81) for \( W \):
\[ \tilde{D} \Gamma = \gamma \left( V^\nu_{\alpha} - \frac{1}{2n_f} \Gamma Q \partial_{\alpha} \right) \Gamma^\nu_{\alpha} - \gamma \phi \left( \phi^2 \Gamma \phi - \phi^2 \Gamma \phi \right) + \ldots \]

where \( \tilde{D} = \left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \beta} - \gamma_m \sum_q m_q \frac{\partial}{\partial m_q} \right) V^{\phi \phi \phi \phi} \).

The RGEs for the 1PI vertices are found by differentiation, and using the Ward identities (55) to simplify the results, we find for the pseudoscalar sector:
\[ \partial \Gamma Q \partial \phi = -2\gamma \Gamma Q \partial \phi + 2\gamma \frac{1}{2n_f} \left[ \Phi_{0k} \Gamma Q \partial \phi \right]; \]
\[ \partial \Gamma Q \partial \phi = -\gamma (\gamma \phi) \Gamma Q \partial \phi \]
\[ + \gamma \frac{1}{2n_f} \left[ \Phi_{0k} \Gamma Q \partial \phi \right] \]
\[ \partial \Gamma Q \partial \phi \phi = -2\gamma \phi \Gamma Q \partial \phi \phi \]
\[ + \gamma \frac{1}{2n_f} \left[ \Phi_{0k} \Gamma Q \partial \phi \phi \right]. \]

Here, \( D = \tilde{D} + \gamma \phi \left( \phi^2 \right) \frac{\partial}{\partial \phi} \). As explained in Ref. [1], this is identical to the RG operator \( D \) defined above (acting on \( W \)) when the sources are set to zero and the fields to their VEVs.

These RGEs play two roles in the discussion. First, they are used as consistency checks on the various formulae we derive. Second, according to our conjecture, they provide the clue to identifying quantities which are likely to show violations of the OZI rule and those for which we may reasonably expect the OZI limit to be a good approximation. This is because we can identify quantities which will be particularly sensitive to the \( U_A(1) \) anomaly as those which have RGEs involving the anomalous dimension \( \gamma \).

We now derive the RGEs for the Green functions and 1PI vertices involved in the various expressions related to the \( \eta'(\eta) \rightarrow \gamma \gamma \) amplitude. To do this, we first need to include the electromagnetic fields and their anomalous dimensions. As already noted in Section 5.2, the anomalous dimension for the composite operator \( Q_{em}(A) \) entering the anomaly equation is of \( O(\alpha^2) \) so can be neglected at the order at which we are working. We denote the anomalous dimension corresponding to the usual electromagnetic field renormalisation by \( \gamma_A \).

The RGEs for the 1PI vertices \( \Gamma_{Q,A} \) and \( \Gamma_{Q,A} \) are easily found by differentiating Eq. (86) and simplifying using the Ward identities. We find,
\[ (D + 2\gamma_A) \Gamma_{Q,A} = -\gamma \Gamma_{Q,A} + \gamma \frac{1}{2n_f} \Gamma_{Q,em} \left[ \phi \right] \Gamma_{Q,A} \]
\[ + \gamma \frac{1}{2n_f} \left[ \Phi_{0k} \Gamma_{Q,em} \right] \]
\[ \Gamma_{Q,em} \Gamma_{Q,A} \]

and,
\[ (D + 2\gamma_A) \Gamma_{Q,A} = -\gamma \Gamma_{Q,A} + \gamma \frac{1}{2n_f} \Gamma_{Q,em} \left[ \phi \right] \Gamma_{Q,A} \]
\[ + \gamma \frac{1}{2n_f} \left[ \Phi_{0k} \Gamma_{Q,em} \right] \]
\[ \Gamma_{Q,em} \Gamma_{Q,A} \]

These are very similar to the corresponding equations in Ref. [1], with the obvious inclusion of the mass term and SU(3) breaking in the VEVs and flavour mixing.

These expressions simplify remarkably at \( k^2 = 0 \). Using the zero-momentum chiral Ward identities (57) for \( \Gamma_{Q,em} \) and \( \Gamma_{Q,em} \), together with (66), we find
\[ (D + 2\gamma_A) \Gamma_{Q,A} \bigg|_{k=0} = 0, \]
\[ (D + 2\gamma_A) \Gamma_{Q,A} \bigg|_{k=0} = -\gamma \Gamma_{Q,A} \bigg|_{k=0} = 0. \]

From the latter, we immediately have
\[ (D + 2\gamma_A) \Gamma_{Q,A} \bigg|_{k=0} = 0. \]

verifying the RG consistency of the basic identity (66).

It only remains to rewrite these results in terms of the 1PI vertices for \( \eta' \) and \( G \). First, recall the identification of the decay constants:
This confirms that the true decay constants \( f^{\alpha} \) are RG invariant. Contrast this with the current matrix element definition \( \langle 0 | J^{\mu}_a | \eta^\alpha \rangle = i k^{\mu} f^{\alpha} \), for which

\[
D f^{\alpha} = \gamma_0 d f^{\alpha}.
\]

Now, since \( \Gamma_{G,4V} | \eta^\alpha \rangle \equiv \Gamma_{Q,4V} | \phi^\alpha \rangle \), we immediately deduce from Eq. (90) above that

\[
(D + 2 \gamma_A) g_{\eta^\alpha} = 0.
\]

Finally, from Eq. (70), we have

\[
\Phi_{ab} \Gamma_{\phi^\alpha A^V} = f^{ab} \Gamma_{\phi^\alpha A^V} - \Phi_{ab} (W_{S_i S_i})^{ab}_i W_{S_i S_i} \Gamma_{G,4V}.
\]

Combining (90) with the RG identities (82), we then find after further use of the chiral Ward identity (52) that indeed

\[
(D + 2 \gamma_A) \Gamma_{\eta^\alpha A^V} |_{k=0} = 0
\]

at zero momentum, i.e.

\[
(D + 2 \gamma_A) g_{\eta^\alpha} = 0
\]

as promised. In fact, if we had included the contact terms in the RG equations throughout, as in Ref. [1], we would have found at this point that the coupling \( g_{\eta^\alpha \gamma^\gamma}(k^2) \) is actually not RG invariant for all \( k \). However, it was found in [1] by keeping careful track of the contact terms that it is also RG invariant on-shell. This is an important point – it is a necessary condition for the dynamical assumption that the on-shell couplings may be well approximated by their zero-momentum values, essential to the PCAC method, to be valid.

This completes our survey of the RG properties of the radiative \( \eta^\gamma(\eta) \rightarrow \gamma \gamma \) decay formulae confirming that, as stated in Section 1, all the quantities appearing in the formulae are RG invariant. In particular, this confirms the identification of \( f^{\alpha} \) as the true, physical decay constant.

5.4. Chiral lagrangians, OZI and 1/\( N_c \)

An alternative to the approach presented here is the popular method of chiral Lagrangians, so we include a few comments on their relation. Chiral Lagrangians are models of low-energy QCD in which the basic fields are chosen to parametrise the coset manifold \( G/H \) (for a chiral symmetry breaking pattern \( G \rightarrow H \)) and thus lie in one-to-one correspondence with the Goldstone bosons. The dynamics, which is determined by the isometry group of this coset manifold, is therefore arranged from the outset to satisfy the (zero-momentum) chiral Ward identities. The great advantage of chiral Lagrangians is that they provide a systematic way of going beyond leading order in a low-momentum expansion, higher order terms being developed by the loop expansion in this non-renormalisable QFT [54].

It is, however, important not to forget that chiral Lagrangians are simply models of QCD. They implement the chiral Ward identities in a particularly elegant, geometric way but they still implicitly assume the same dynamical approximations of pole dominance (in selecting the most relevant low-energy states) and smoothness of momentum extrapolations that are explicit in the actual QCD treatment in terms of operator Green functions.

This is especially important when chiral Lagrangians are extended [21,22,28–29] to the non-linear \( U(3) \times U(3)/U(3) \) models incorporating the \( \eta^\gamma \), which is of course not a Goldstone boson because of the anomaly. The dynamics of these models is therefore not entirely constrained by the geometry of the coset space but must be implemented in part by hand if they are to be accurate representations of true QCD. The most promising systematic approach is to use the 1/\( N_c \) expansion, since at leading order in 1/\( N_c \) the \( \eta^\gamma \) becomes a true Goldstone boson (because the anomaly is sub-leading). However, as we have emphasised, the leading 1/\( N_c \) (or the OZI\( ^4 \)) approximation, while a good approximation for some quantities, is completely invalid for others. In our presentation, we have pursued the consequences of the anomalous chiral Ward identities as far as possible without making extra dynamical assumptions, introducing these only at the end to make contact with the physically observed couplings and decay constants. In particular, we have used the renormalisation group as a guide to which quantities we expect to have a smooth 1/\( N_c \) perturbation expansion and which violate 1/\( N_c \) or OZI significantly at leading order.

It would therefore be of considerable interest to make a detailed comparison of the 1/\( N_c \) chiral Lagrangian predictions [21,22,28,29] with those made here (and also for the closely related analysis of the \( U(1) \) Goldberger-Treiman relation [33–36] and its link with the ‘proton spin’ structure function \( g_1^p \)). Since the fundamental anomalous symmetry and dynamical assumptions should be the same, it would be interesting to see how these are realised in these two, in principle equivalent, approaches.

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4 In the text, we have preferred to refer to the OZI approximation, rather than 1/\( N_c \). The OZI limit is precisely defined [9] as the truncation of full QCD in which non-planar and quark-loop diagrams are retained, but diagrams in which the external currents are attached to distinct quark loops, so that there are purely gluonic intermediate states, are omitted. (This last fact makes the connection with the familiar phenomenological form of the OZI or Zweig rule.) This is a more accurate approximation to full QCD than either the leading 1/\( N_c \) limit (\( N_c \rightarrow \infty \) at fixed \( n_f \)), the quenched approximation (\( n_f \rightarrow 0 \) at fixed \( N_c \)), or the leading topological expansion (\( N_c \rightarrow \infty \) at fixed \( n_f / N_c \)). In the OZI or leading 1/\( N_c \) limits, the \( U(1) \) anomaly is absent, there is an extra Goldstone boson, and there is no meson-gluon mixing.

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6. Epilogue

In this paper, we have reviewed the theory and phenomenology of the radiative $\eta'(\eta) \rightarrow \gamma\gamma$ decays, together with closely related processes such as $\eta'(\eta) \rightarrow V\gamma$, $\eta'(\eta) \rightarrow \pi\pi\gamma$, $\psi \rightarrow \eta'(\eta)\gamma$, etc. The theory of these decays is indeed a tale of two anomalies: first, the electromagnetic $U_A(1)$ anomaly, which was spectacularly successful historically in explaining the otherwise mysterious $\pi \rightarrow \gamma\gamma$ decay; second, the gluonic $U_A(1)$ anomaly, which makes the physics of the flavour-singlet $0^-$ channel in QCD so subtle and interesting.

Indeed, it is the role of the gluonic $U_A(1)$ anomaly that makes the $\eta'$ and its decays worth studying. It is therefore disappointing that this new physics is so often obscured by phenomenological analyses which try to fit the data into the straightjacket of decay formulae written down in naive analogy with $\pi \rightarrow \gamma\gamma$, without taking the implications of the gluonic anomaly fully into account. The purpose of this paper is to urge a fresh phenomenological look at $\eta'$ physics, treating both the electromagnetic and colour $U_A(1)$ anomalies in a complete and theoretically self-consistent manner.

The goal, going beyond mere confirmation of the well-established physics of pseudo-Goldstone bosons and their interactions, is to gain new phenomenological insight into the rich and fascinating subject of gluon topology in QCD.

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Note added in proof

Since the paper was prepared, the authors of ref. [51] have identified in their original analysis and now quote [55] $\rightarrow 8.9(15.4) \times 10^{-11}$ for the total hadronic light-by-light scattering contribution to the mu on $\rho_\mu - 2$, with a corresponding reduction in the experiment versus theory discrepancy.

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