Averaging principle of stochastic Burgers equation driven by Lévy processes

Hongge Yue\textsuperscript{a}, Yong Xu\textsuperscript{a,b}, Ruifang Wang\textsuperscript{a}, Zhe Jiao\textsuperscript{a,*}

\textsuperscript{a}School of Mathematics and Stochastics, Northwestern Polytechnical University, Xi'an, 710072, China
\textsuperscript{b}MHT Key Laboratory of Dynamics and Control of Complex Systems, Northwestern Polytechnical University, Xi'an, 710072, China

\textbf{Abstract}

We are concerned about the averaging principle for the stochastic Burgers equation with slow-fast time scale. This slow-fast system is driven by Lévy processes. Under some appropriate conditions, we show that the slow component of this system strongly converges to a limit, which is characterized by the solution of stochastic Burgers equation whose coefficients are averaged with respect to the stationary measure of the fast-varying jump-diffusion. To illustrate our theoretical result, we provide some numerical simulations.

\textbf{Keywords.} Stochastic Burgers equation, averaging principle, Lévy noise, strong convergence

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1. Introduction

The study of the averaging principle for stochastic partial differential equations with slow-fast time scale has attracted many researchers’ attention (see e.g. \cite{1–6} and the references therein). In 2009, Cerrai \cite{7} proved that an averaging principle holds for a general class of stochastic reaction–diffusion systems in any space dimension, which show that the classical Khasminskii approach for systems with a finite number of degrees of freedom can be extended to infinite-dimensional systems. Afterwards, an averaging principle for the complex Ginzburg-Landau equations, perturbed by a mixing random force on long time intervals, was established in \cite{8}. Recently, the authors in \cite{9} proved the slow component of the stochastic 2D Navier–Stokes equation converges to the solution of the corresponding averaged equation for any given initial value in the separable real Hilbert space, where the solution is a weak solution.

In this paper, we consider the initial-boundary value problem for the following one dimensional stochastic Burgers equation with slow-fast time scale

\begin{equation}
\begin{aligned}
    dX^\varepsilon_t &= \left[ \nu \Delta X^\varepsilon_t + \frac{1}{2} \frac{\partial}{\partial x} X^\varepsilon_t(x) \right] dx dt + dW^Q_1(t, \xi) + \int_{|z|<1} h_1(X^\varepsilon_t(x), z) N_1(dx, dt), \\
    dY^\varepsilon_t &= \frac{1}{2} \left[ \nabla \cdot (f(X^\varepsilon_t(x), Y^\varepsilon_t(x))) \right] dt + \int_{|z|<1} h_2(X^\varepsilon_t(x), Y^\varepsilon_t(x), z) N_2^\varepsilon(dx, dt),
\end{aligned}
\end{equation}

for $\xi \in [0, 1]$ and $t \in [0, T]$, $T < \infty$. Here, $\Delta$ is the Laplacian operator, $f_i$ and $h_i$, $i = 1, 2$, are nonlinear coefficients. The independent Poisson random measures $N_1(\cdot, \cdot)$ and $N_2^\varepsilon(\cdot, \cdot)$, are given by $N_1(dx, dt) = N_1(dx, dt) - \mu_1(dx) dt$, and $N_2^\varepsilon(dx, dt) = N_2(dx, dt) - \frac{1}{2} \mu_2(dx) dt$ respectively, where $N_i(dx, dt)$, $i = 1, 2$, is associated Poisson measure, and $\mu_i$, $i = 1, 2$, the Lévy measure satisfying $\int_{\mathbb{R}^2} (1 + z^2) \mu_i(dx) < \infty$. The $Q_t$-Wiener processes $\{W^Q_t\}_{t \geq 0}$, $i = 1, 2$ are mutually independent with values in $L^2([0, 1])$, which are also independent of $N_i(\cdot, \cdot)$. The scaling parameter $\varepsilon > 0$ is used to describe the separation of time scale between the slow variable $X^\varepsilon_t$ and the fast variable $Y^\varepsilon_t$. $\nu > 0$ is the kinematic viscosity and $c \geq 0$ is the diffusion coefficient. For the sake of simplicity, we set the coefficient $\nu$ to be 1.

*Corresponding author

Email addresses: yuehongge8038163.com (Hongge Yue), hsux3@nwpu.edu.cn (Yong Xu), wangruifang_07140163.com (Ruifang Wang), zjiao@nwpu.edu.cn (Zhe Jiao)
This model (1.1) describes Burgers turbulence in the presence of random forces. If the non-Gaussian white noise in system (1.1) is absent, the authors in [10] proved the convergence of the slow component, both in the strong sense and in the weak sense. The purpose of this paper is to deal with the case of non-Gaussian white noise. We prove that for any \( t \in [0, T] \), as \( \varepsilon \) goes to zero, the slow component \( X^\varepsilon_t \) of the system (1.1) strongly converges to \( X_t \) which is the solution of the corresponding average equation. Our proof is based on the Khasminskii argument proposed in [11]. However, Lévy noise and nonlinear external force, these two terms bring about essential difficulty in proving that the slow component is equicontinuous in some sense. Therefore, we have to find some new ideas to deal with this problem, and obtain a new high-order estimate (see in Lemma 3.2) of the slow component in order to establish the averaging principle for the system (1.1).

The outline of this paper is as follows. In Section 2, we present some notations, give some prior estimates, which play significant role in proving our main result. Section 3 gives some precise conditions for the slow-fast system. At the end of this section, we give the statement of 3.2) of the slow component in order to establish the averaging principle for the system (1.1). Section 5 presents some numerical examples illustrating the averaging principle for the system (1.1). Throughout the paper, \( C, C_i \) are as generic constants whose values may change from line to line.

2. Preliminaries

The quadruple \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) is a given stochastic basis satisfying the usual hypotheses in this paper. \( \mathbb{E}(\cdot) \) stands for expectation with respect to the probability measure \( \mathbb{P} \). Let \( D \) be a bounded domain in a Euclidean space. \( L^2(D) \) is the space of square integrable real-valued functions on \( D \). The norm in this space is denoted by \( \| \cdot \| \). If \( k \geq 0 \) is an integer, we define \( H^k(D) \) to consist of all functions in \( L^2 \) whose differentials belong to \( L^2 \) up to the order \( k \). For each \( r \geq 0 \), one can define \( H^r(D) \) by interpolation \([L^2(D), H^k(D)]_r, k \leq s \leq s = \theta k \). \( H^0(D) \) is the subspace of \( H^2(D) \) consisting of the functions vanishing on the boundary. Denote by \( H^{-1}(D) \) the dual space to \( H^0(D) \). Throughout this paper, we take \( D = (0, 1) \).

Define the bilinear operator \( B : L^2 \times H^1_0 \rightarrow H^{-1} \) by \( B(x, y) = x(\xi) \frac{\partial}{\partial y} y(\xi) \), and the trilinear operator \( b : L^2 \times H^1_0 \times L^2 \rightarrow \mathbb{R} \) by \( b(x, y, z) = \int_0^1 x(\xi) \frac{\partial y(\xi)}{\partial z} z(\xi) d\xi \). Set \( B(x) = B(x, y) \) if \( x = y \).

The Laplacian operator is given by \( AX := \Delta X = \frac{\partial^2}{\partial x^2} X \) in which \( X \) belongs to the domain \( \mathcal{D}(A) := H^2 \cap H^1_0 \). Then from [12] we know that \( A \) is the infinitesimal generator of a \( C_0 \)-contraction semigroup \( e^{tA}, t \geq 0 \), which has a regularizing effect, that is, for any \( s_1 \leq s_2 \)

\[
\|e^{tA}X\|_{H^{s_2}} \leq C \left(1 + t^{\frac{s_2 - s_1}{2}}\right) \|X\|_{H^{s_1}}, \quad X \in H^{s_1}.
\] (2.1)

The eigenfunctions of \(-A\) is given by \( e_k = \sqrt{2} \sin(k\pi \xi), k \in \mathbb{N}^+, \xi \in [0, 1] \), with the corresponding eigenvalues \( \lambda_k = k^2 \pi^2 \). For each \( \alpha \in \mathbb{R}, (-A)^\alpha \) is the power of the operator \(-A\), and \( |\cdot|_\alpha \) is the norm of \( \mathcal{D}((-A)^\alpha) \) which is equivalent to the norm of \( H^\alpha \).

The \( Q_1 \)-Wiener processes \( W_t^{Q_1} \) can be given by

\[
W_t^{Q_1} = \sum_{k=1}^{\infty} \sqrt{\alpha_k} \beta^k_t e_k, \quad t \geq 0,
\] (2.2)

where \( \alpha_k \geq 0 \), satisfying \( \sum_{k=1}^{\infty} \alpha_k < +\infty \), and \( \{\beta^k_t\}_{k \in \mathbb{N}} \) is a sequence of mutually independent standard Brownian motions. And we also assume that \( W_t^{Q_2} \) also has a similar decomposition as in (2.2).

To formulate our main result, we introduce the following assumptions.

(A1) There exist two positive constants \( L_{f_1}, L_{f_2} \), such that for any \( x_1, x_2, y_1, y_2 \in L^2 \),

\[
\|f_1(x_1, y_1)\| \leq L_{f_1}(1 + \|x_1\| + \|y_1\|),
\|f_2(x_1, y_1)\| \leq L_{f_2}(1 + \|x_1\| + \|y_1\|),
\|f_1(x_1, y_1) - f_1(x_2, y_2)\| \leq L_{f_1}(\|x_1 - x_2\| + \|y_1 - y_2\|),
\|f_2(x_1, y_1) - f_2(x_2, y_2)\| \leq L_{f_2}(\|x_1 - x_2\| + \|y_1 - y_2\|).\]
(A2) There exist constants $L_{h_1}, L_{h_2} > 0$, such that for any $\gamma \geq 1$, $\alpha \in \left[1, \frac{3}{2}\right)$, $x_1, x_2, y_1, y_2 \in L^2$,

\[
\int_{|z|<1} \|h_1(0, z)\|^{\gamma} \mu_1(dz) < \infty, \quad \int_{|z|<1} \|h_2(0, 0, z)\|^{\gamma} \mu_2(dz) < \infty,
\]

\[
\int_{|z|<1} \|h_1(x_1, z) - h_1(x_2, z)\|^{\gamma} \mu_3(dz) \leq L_{h_1} \|x_1 - x_2\|^{\gamma},
\]

\[
\int_{|z|<1} \|h_2(x_1, y_1, z) - h_2(x_2, y_2, z)\|^{\gamma} \mu_2(dz) \leq L_{h_2}(\|x_1 - x_2\|^{\gamma} + \|y_1 - y_2\|^{\gamma}),
\]

\[
\int_{|z|<1} \|h_1(x_1, z)\|_1^{\alpha} \mu_1(dz) \leq L_{h_1} (1 + |x_1|^\alpha).
\]

(A3) Let $\eta := 2\lambda_1 - L_{f_2} - L_{h_2} > 0$.

(A4) There exist constants $\alpha \in \left[1, \frac{3}{2}\right)$, $\beta \in (0, \infty)$, $\rho \in (2, \infty)$ and $\frac{2(\rho - 2)}{\rho} < 1$ such that

\[
\sum_{k=1}^{\infty} \frac{\alpha_k^2}{\lambda_k^2} < +\infty.
\]

Under the conditions (A1)-(A2), it deduces from [13] that if the initial data $(x, y) \in H^\alpha \times L^2$, there exist solutions $X^\epsilon_t \in \mathcal{D}(A)$ and $Y^\epsilon_t \in L^2([0, T]; H_0^2) \cap C([0, T]; L^2)$ satisfying

\[
\begin{cases}
X^\epsilon_t = e^{tA} x + \int_0^t e^{(t-s)A} B(X^\epsilon_s) ds + \int_0^t e^{(t-s)A} f_1(X^\epsilon_s, Y^\epsilon_s) ds + \int_0^t e^{(t-s)A} dW^1_s, \\
+ \int_0^t \int_{|z|<1} e^{(t-s)A} h_1(X^\epsilon_s, z, \eta_1) ds dz,
\end{cases}
\]

\[
\begin{cases}
Y^\epsilon_t = e^{tA} y + \frac{1}{\epsilon} \int_0^t e^{(t-s)A} f_2(X^\epsilon_s, Y^\epsilon_s) ds + \frac{1}{\epsilon^2} \int_0^t e^{(t-s)A} dW^2_s, \\
+ \int_0^t \int_{|z|<1} e^{(t-s)A} h_2(X^\epsilon_s, Y^\epsilon_s, z, \eta_2) ds dz.
\end{cases}
\]

(2.3)

For any fixed $x, y \in L^2$, we consider the following frozen equation associated with the fast component

\[
\begin{cases}
dY_t = [cAX_t + f_2(x, Y_t)] dt + dW^2_t + \int_{|z|<1} h_2(x, Y_t, z) \tilde{N}_2(t, dz), \\
Y_t(0) = Y_t(1) = 0, \quad t \in [0, T].
\end{cases}
\]

(2.4)

The assumptions above imply from Chapter 16 in [14] that there exists a unique invariant measure $\mu^\epsilon$ for (2.4).

Now we give the statement of our main result.

**Theorem 2.1.** Under the assumption (A1)-(A4), the slow component $X^\epsilon_t$ of the stochastic Burgers system with the initial data $(x, y) \in H^\alpha \times L^2$ satisfy the averaging principle

\[
\lim_{\epsilon \to 0} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \|X^\epsilon_t - \bar{X}_t\|^p \right] = 0, \quad p \geq 2
\]

where $\bar{X}_t$ is the solution of the averaged equation

\[
\begin{cases}
d\bar{X}_t = \Delta \bar{X}_t dt + \frac{1}{2} \frac{\partial}{\partial x} (\bar{X}_t)^2 dt + \tilde{f}_1(\bar{X}_t) dt + dW^1_t + \int_{|z|<1} h_1(\bar{X}_t, z) \tilde{N}_1(dt, dz), \\
\bar{X}_0 = x,
\end{cases}
\]

with $\tilde{f}_1(x) = \int_{L^2} f_1(x, y) \mu^\epsilon(dy)$.

3. Prior estimates

Before giving the proof of our main result, we need some prior estimates.

**Lemma 3.1.** Assume that the conditions (A1)-(A2) are satisfied, then there exist a positive constant $C_{q,T}$ such that for any $x \in H^\alpha$, $y \in L^2$, $q \geq 1$, $T > 0$ and $\epsilon \in (0, 1)$

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|X^\epsilon_t\|^{2q} \right] + \mathbb{E} \left[ \int_0^T \|X^\epsilon_t\|^{2q-2} |X^\epsilon_t|^1_t dt \right] \leq C_{q,T}(1 + \|x\|^{2q} + \|y\|^{2q}),
\]

(3.1)
and

\[
\sup_{0 \leq t \leq T} \mathbb{E}[\|Y_t^\varepsilon\|^{2q}] \leq C_{q,T}(1 + \|x\|^{2q} + \|y\|^{2q}),
\]

(3.2)

where \(C_{q,T}\) is independent of \(\varepsilon\).

**Proof:** According to Itô’s formula, (A1) - (A2) and the Poincaré inequality in [13], we have

\[
d\mathbb{E}[\|Y_t^\varepsilon\|^{2q}] = \frac{2q\varepsilon}{\varepsilon} \mathbb{E}\left[\|Y_t^\varepsilon\|^{2q-2} \langle AY_t^\varepsilon, Y_t^\varepsilon \rangle\right] + \frac{2q}{\varepsilon} \mathbb{E}\left[\|Y_t^\varepsilon\|^{2q-2} \langle f_2(X_t^\varepsilon, Y_t^\varepsilon), Y_t^\varepsilon \rangle\right]
\]

\[
+ \frac{q}{\varepsilon} \mathbb{E}\left[\|Y_t^\varepsilon\|^{2q-2}\text{Tr} Q_2\right] + \frac{2q(q-1)}{\varepsilon^2} \mathbb{E}\left[\|Y_t^\varepsilon\|^{2q-2}\text{Tr} Q_2\right]
\]

\[
+ \frac{1}{\varepsilon} \mathbb{E}\left[\int_{|z| < 1} \left(\|Y_t^\varepsilon + h_2(X_t^\varepsilon, Y_t^\varepsilon, z)\|^{2q} - \|Y_t^\varepsilon\|^{2q}\right)\mu_2(dz)\right]
\]

\[
- \frac{2q}{\varepsilon} \mathbb{E}\left[\int_{|z| < 1} \|Y_t^\varepsilon\|^{2q-2}(h_2(X_t^\varepsilon, Y_t^\varepsilon, z), Y_t^\varepsilon)\mu_2(dz)\right]
\]

\[
\leq \frac{2q\gamma_\lambda}{\varepsilon} \mathbb{E}[\|Y_t^\varepsilon\|^{2q}] + \frac{q}{\varepsilon} \mathbb{E}[\|Y_t^\varepsilon\|^{2q-2} \langle f_2(X_t^\varepsilon, Y_t^\varepsilon), Y_t^\varepsilon \rangle]
\]

\[
+ \frac{1}{\varepsilon} \mathbb{E}\left[\sum_{i=2}^{2q} C_{2q}^i \int_{|z| < 1} \|Y_t^\varepsilon\|^{2q-i}h_2(X_t^\varepsilon, Y_t^\varepsilon, z)\|^{i}\mu_2(dz)\right]
\]

(3.3)

From (3.3), using condition (A3), and the Young’s inequality, we deduce that there exist a positive constant \(\gamma'\) such that

\[
d\mathbb{E}[\|Y_t^\varepsilon\|^{2q}] \leq -\frac{q\gamma'}{\varepsilon} \mathbb{E}[\|Y_t^\varepsilon\|^{2q}] + \frac{C_q}{\varepsilon} \mathbb{E}[\|X_t^\varepsilon\|^{2q}] + \frac{C_2}{\varepsilon}.
\]

(3.4)

Applying the comparison theorem, we get

\[
\mathbb{E}[\|Y_t^\varepsilon\|^{2q}] \leq \|y\|^{2q} e^{-\frac{q\gamma'}{\varepsilon} t} + \frac{C_q}{\varepsilon} \int_0^t e^{-\frac{q\gamma'}{\varepsilon} (t-s)} (1 + \mathbb{E}[\|X_s^\varepsilon\|^{2q}]) ds.
\]

(3.5)

For \(X_t^\varepsilon\), by Itô’s formula, we have

\[
\mathbb{E}[\|X_t^\varepsilon\|^{2q}] := \|x\|^{2q} + \sum_{i=1}^{\frac{q}{2}} \Xi_i(t).
\]

(3.6)

where

\[
\Xi_1(t) = 2q \mathbb{E}\left[\int_0^t \|X_s^\varepsilon\|^{2q-2} \langle AX_s^\varepsilon, X_s^\varepsilon \rangle ds\right],
\]

\[
\Xi_2(t) = 2q \mathbb{E}\left[\int_0^t \|X_s^\varepsilon\|^{2q-2} \langle B(X_s^\varepsilon), X_s^\varepsilon \rangle ds\right],
\]

\[
\Xi_3(t) = 2q \mathbb{E}\left[\int_0^t \|Y_s^\varepsilon\|^{2q-2} \langle f_1(X_s^\varepsilon, Y_s^\varepsilon), X_s^\varepsilon \rangle ds\right],
\]

\[
\Xi_4(t) = q \mathbb{E}\left[\int_0^t \|X_s^\varepsilon\|^{2q-2}\text{Tr} Q_1 ds\right],
\]

\[
\Xi_5(t) = 2q(q-1) \mathbb{E}\left[\int_0^t \|X_s^\varepsilon\|^{2q-2}\text{Tr} Q_2 ds\right],
\]

\[
\Xi_6(t) = 2q \mathbb{E}\left[\int_0^t \|X_s^\varepsilon\|^{2q-2} \langle X_s^\varepsilon, dW_s^Q \rangle\right],
\]

\[
\Xi_7(t) = \mathbb{E}\left[\int_0^t \int_{|z| < 1} \left(\|X_s^\varepsilon + h_1(X_s^\varepsilon, Y_s^\varepsilon, z)\|^{2q} - \|X_s^\varepsilon\|^{2q}\right)\tilde{N}_1(ds, dz)\right],
\]

\[
\Xi_8(t) = \mathbb{E}\left[\int_0^t \int_{|z| < 1} \left(\|X_s^\varepsilon + h_1(X_s^\varepsilon, Y_s^\varepsilon, z)\|^{2q} - \|X_s^\varepsilon\|^{2q}\right)\tilde{N}_2(ds, dz)\right].
\]
\[ \mathbb{E} \left[ \int_0^t \left[ \|X_s^\varepsilon\|^{2q-2} \langle B(X_s^\varepsilon), X_s^\varepsilon \rangle \right] ds \right] \leq C_T \sup_{0 \leq t \leq T} \|X_t^\varepsilon\|^{2q-2} \left[ \int_0^t \langle B(X_s^\varepsilon), X_s^\varepsilon \rangle ds \right] \]

Note that \( \langle AX, X \rangle = -\|\nabla X\|^2 \leq -\beta |X|^2 + \gamma \|X\|^2 \), \( \beta > 0, \gamma > 0 \) (see e.g. [10]), by the linear growth conditions in (A1) and Young’s inequality, we have

\[ \mathbb{E} \left[ \sum_{i=1}^5 \sup_{0 \leq t \leq T} \mathbb{E} \Xi_i(t) \right] \leq -2q \int_0^t \|X_s^\varepsilon\|^{2q-2} \|X_s^\varepsilon\|^2 ds + C_{q,T} \mathbb{E} \left[ \int_0^T \|Y_t^\varepsilon\|^2 dt \right] + C_{q,T}. \quad (3.7) \]

On the one hand, with the help of Burkholder-Davis-Gundy inequality, it yields

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \mathbb{E} \Xi_6(t) \right] \leq C_q \mathbb{E} \left[ \left( \int_0^T \|X_t^\varepsilon\|^{4q-2} \|Q_t^\varepsilon\|^2 dt \right)^\frac{1}{2q} \right] \]

\[ \leq C_q \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} \|X_t^\varepsilon\|^{2q} \int_0^T \|X_s^\varepsilon\|^{2q-2} ds \right)^\frac{1}{2} \right] \leq C_{q,T} + C_q \int_0^T \mathbb{E} \|X_t^\varepsilon\|^{2q} dt + \frac{1}{3} \mathbb{E} \sup_{0 \leq t \leq T} \|X_t^\varepsilon\|^{2q}. \quad (3.8) \]

On the other hand, by Burkholder-Davis-Gundy type inequality for stochastic integral with respect to Poisson compensated martingale measures and Young inequality, we have

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \mathbb{E} \Xi_7(t) \right] = \mathbb{E} \left[ \sup_{0 \leq t \leq T} \int_0^T \int_{|z| < 1} \left( \|X_t^\varepsilon + h_1(X_t^\varepsilon, z)\|^{2q} - \|X_t^\varepsilon\|^{2q} \right) \mu_1(dz, ds) \right] \]

\[ \leq C \mathbb{E} \left[ \left( \int_0^T \int_{|z| < 1} \left( \|X_t^\varepsilon + h_1(X_t^\varepsilon, z)\|^{2q} - \|X_t^\varepsilon\|^{2q} \right)^2 \mu_1(dz, ds) \right)^\frac{1}{2} \right] \]

\[ \leq C \mathbb{E} \left[ \int_0^T \int_{|z| < 1} \left( \|X_t^\varepsilon\|^{4q-2} \|h_1(X_t^\varepsilon, z)\| \right)^2 + \|h_1(X_t^\varepsilon, z)\|^{2q} \mu_1(dz, ds) \right]^\frac{1}{2} \]

\[ \leq C \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} \|X_t^\varepsilon\|^{2q} \int_0^T \int_{|z| < 1} \|X_s^\varepsilon\|^{2q-2} \|h_1(X_t^\varepsilon, z)\|^2 \mu_1(dz, ds) \right)^\frac{1}{2} \right] + C \mathbb{E} \left[ \left( \int_0^T \int_{|z| < 1} \|h_1(X_t^\varepsilon, z)\|^{2q} \mu_1(dz, ds) \right)^\frac{1}{2} \right] \quad (3.9) \]

\[ \leq \frac{1}{3} \mathbb{E} \sup_{0 \leq t \leq T} \|X_t^\varepsilon\|^{2q} + C_q \mathbb{E} \left[ \int_0^T \int_{|z| < 1} \|h_1(X_t^\varepsilon, z)\|^{2q} \mu_1(dz, ds) \right] \]
Next, by Binomial theorem, (A2) and Young’s inequality, we have

\[
\begin{align*}
C & \leq C_q \sum_{i=2}^{2q} C_{q,i} \left[ \sup_{0 \leq t \leq T} \|X_t^\varepsilon\|^{2q} \right] + C_{q,T} \int_0^T \|X_t^\varepsilon\|^{2q} dt + C_{q,T}.
\end{align*}
\]

Combining the estimates (3.7) (3.8) (3.9) (3.10) and (3.5), it is easy to see that

\[
\begin{align*}
C & \leq C_q \sum_{i=2}^{2q} C_{q,i} \left[ \sup_{0 \leq t \leq T} \|X_t^\varepsilon\|^{2q} \right] + C_{q,T} \int_0^T \|X_t^\varepsilon\|^{2q} dt + C_{q,T} \int_0^T \sup_{0 \leq t \leq T} \|Y_t^\varepsilon\|^{2q} dt,
\end{align*}
\]

which also gives

\[
\sup_{0 \leq t \leq T} \|Y_t^\varepsilon\|^{2q} \leq C_{q,T} (1 + \|x\|^{2q} + \|y\|^{2q}).
\]

The proof is completed. \(\square\)

**Lemma 3.2.** Assume that the conditions (A1)-(A4) are satisfied, then there exist a positive constant \(C_{q,T,\alpha}\) such that for any \(x \in H^\alpha, y \in L^2, q > 2, \alpha \in \left[1, \frac{3}{2}\right]\) and \(\varepsilon \in (0,1)\)

\[
\begin{align*}
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t^\varepsilon|^{2q} \right] & \leq C_{q,T,\alpha,\varepsilon} (1 + |x|^{2q} + |y|^{2q}),
\end{align*}
\]

where \(C_{q,T,\alpha}\) is independent of \(\varepsilon\).

**Proof:** From Eq.(2.3), we note \(X_t^\varepsilon := \sum_{i=1}^5 J_i\), in which

\[
\begin{align*}
J_1 & = e^{tA}x, \\
J_2 & = \int_0^t e^{(t-s)A}B(X_s^\varepsilon)ds, \\
J_3 & = \int_0^t e^{(t-s)A}f_1(X_s^\varepsilon,Y_s^\varepsilon)ds, \\
J_4 & = \int_0^t e^{(t-s)A}dW_s^{Q,1}.
\end{align*}
\]
\[ J_3 = \int_0^t \int_{|z|<1} e^{(t-s)A} h_1(X_s^z, z) \tilde{N}_1(ds, dz). \]

For \( J_1 \), it is clear that
\[ |e^{tA}x|_{\alpha}^{2q} \leq |x|_{\alpha}^{2q}. \quad (3.12) \]

For \( J_2 \), according to (2.1), Lemma (3.1) and Sobolev inequalities in ([15], Lemma 2.1), we obtain
\[
\begin{align*}
E & \sup_{0 \leq t \leq T} \int_0^t e^{(t-s)A} B(X_s^x) ds \bigg|_{\alpha}^{2q} \\
\leq & \ C E \left[ \sup_{0 \leq t \leq T} \int_0^t \left( 1 + (t-s)^{-\frac{\alpha_1}{2}} \right) |B(X_s^x)|_{-\alpha_3} ds \right]^{2q} \\
\leq & \ C E \left[ \sup_{0 \leq t \leq T} \int_0^t \left( 1 + (t-s)^{-\frac{\alpha_1}{2}} \right) |X_s^x|_{\alpha_1} |X_s^x|_{\alpha_2+1} ds \right]^{2q},
\end{align*}
\]
where \( \alpha_1 + \alpha_2 + \alpha_3 > \frac{1}{2}, \alpha_3 > 0 (i = 1, 2, 3) \). Using the interpolation inequality, we have that
\[ |X_s^x|_{\alpha_1} \leq C \| X_s^x \|_{\frac{2\alpha_1}{\alpha_2}} |X_s^x|_{\alpha_2}^{\frac{\alpha_2}{\alpha_1}}, \quad (3.13) \]
for any \( 0 < \alpha_1 < \alpha \), and that
\[ |X_s^x|_{\alpha_2+1} \leq C \| X_s^x \|_{\frac{2\alpha_2+\alpha_1}{\alpha_1}} |X_s^x|_{\alpha_1}^{\frac{\alpha_1}{\alpha_2+1}}, \quad (3.14) \]
for any \( 0 < \alpha_2 + 1 < \alpha \). Let \( \alpha_1 \) and \( \alpha_2 \) be small enough such that \( 1 + \alpha_1 + \alpha_2 \in (1, \alpha) \). It follows from (3.13) and (3.14), and according to Hölder inequality, Young’s inequality, we obtain that
\[
\begin{align*}
E & \sup_{0 \leq t \leq T} \int_0^t e^{(t-s)A} B(X_s^x) ds \bigg|_{\alpha}^{2q} \\
\leq & \ C E \left[ \sup_{0 \leq t \leq T} \int_0^t \left( 1 + (t-s)^{-\frac{\alpha_1}{2}} \right) |X_s^x|^{\frac{2\alpha_1}{\alpha_2+1} \alpha_1} |X_s^x|^{\frac{\alpha_2}{\alpha_1}} ds \right]^{2q} \\
\leq & \ C_2 E \left[ \sup_{0 \leq t \leq T} \int_0^t |X_s^x|^{2q} |X_s^x|^{\frac{2\alpha_2+1}{\alpha_1}} ds \right] \\
\leq & \ C_2 T \left( 1 + \int_0^T s^{-\frac{\alpha_3}{2}} ds \right)^{2q-1} \left( \int_0^T E \| X_s^x \|^{2q} ds + \int_0^T E |X_s^x|^{2q} ds \right),
\end{align*}
\]
let \( q \) be large enough such that \( \frac{\alpha_3 + \alpha_2}{2} \cdot \frac{2q}{2q-1} < 1, \alpha \in (1, \frac{3}{2}) \). Next, let \( \alpha = 1 \), we have
\[
\begin{align*}
E & \sup_{0 \leq t \leq T} \int_0^t e^{(t-s)A} B(X_s^x) ds \bigg|_1^{2q} \\
\leq & \ E \left[ \left( \sup_{0 \leq t \leq T} \int_0^t e^{(t-s)A} B(X_s^x) ds \right|_1^{2q} \right] \\
\leq & \ E \left[ \left( \sup_{0 \leq t \leq T} \int_0^t \left( 1 + (t-s)^{-\frac{\alpha_1}{2}} \right) |B(X_s^x)|_{-\alpha_3} ds \right|_1^{2q} \right] \\
\leq & \ C_3 E \left[ \left( \sup_{0 \leq t \leq T} \int_0^t \left( 1 + (t-s)^{-\frac{\alpha_1}{2}} \right) |X_s^x| \| X_s^x \|_1 ds \right|_1^{2q} \right] \\
\leq & \ C_3 E \left[ \left( \int_0^t \left( 1 + (t-s)^{-\frac{\alpha_1}{2}} \right) ds \right)^{\frac{2q}{2q-1}} \left( \int_0^t \| X_s^x \|^{2q} \right) \left( \int_0^t \| X_s^x \|^{2q} ds \right) \right],
\end{align*}
\]
\[ \leq C_q \left( \int_0^T \left( 1 + (t-s)^{-\frac{\alpha q}{4}+1} \right) \frac{2q}{2q+1} ds \right)^{2q-1} E \left[ \sup_{0 \leq t \leq T} \|X_t^\varepsilon\|^{2q} \right] \int_0^t E[|X_s^\varepsilon|^{2q}ds] \]

\[ \leq C_{q,T} (1 + \|x\|^{2q} + \|y\|^{2q}) \int_0^T E[|X_s^\varepsilon|^{2q}ds], \]

for \( q > 2 \), where \( \alpha' \in \left( \frac{1}{2}, \frac{2}{q} \right) \).

For \( J_3 \), according to (2.1), taking \( q \) large enough such that \( \alpha \in \left[ 0, 2 - \frac{1}{q} \right) \), it follows from Lemma 3.1 that

\[ E \left[ \sup_{0 \leq t \leq T} \int_0^t e^{(t-s)^A} f_1(X_s^\varepsilon, Y_s^\varepsilon) ds \right] \leq C \] \[ \leq C E \left[ \sup_{0 \leq t \leq T} \int_0^t \left( 1 + (t-s)^{-\frac{2q}{2q+1}} \right) (1 + \|X_s^\varepsilon\| + |Y_s^\varepsilon|) ds \right] \]

\[ \leq C_{q,T} \left( 1 + \int_0^T s^{-\frac{2q}{2q+1}} ds \right)^{2q-1} E \left[ \int_0^T (1 + \|X_s^\varepsilon\|^{2q} + |Y_s^\varepsilon|^{2q}) ds \right] \quad (3.17) \]

\[ \leq C_{q,T} \left( 1 + \|x\|^{2q} + \|y\|^{2q} \right). \]

For \( J_4 \), following the same argument as in the proof of Lemma 4.1 in [7], according to (A4), and \( \alpha > 0 \) such that for any \( \alpha \in [0, \alpha] \), \( q > 1 \) and \( \varepsilon \in (0, 1) \), we have

\[ E \left[ \sup_{0 \leq t \leq T} \int_0^t e^{(t-s)^A} dW_t^2 \right] \leq C_{q,T,\alpha} \left( 1 + \|x\|^{2q} + \|y\|^{2q} \right). \]

choose \( \theta > 0 \) such that \( 2 \theta + \frac{2q(\theta-2)+\alpha(\theta+2)}{2q} < 1 \). Here, the detailed proof is omitted for the sake of brevity.

For \( J_5 \), with aid of the Kunita’s first inequality [16], Hölder’s inequality, (A2), (2.1) and (3.1), there holds

\[ E \left[ \sup_{0 \leq t \leq T} \int_0^t \int_{|z|<1} e^{(t-s)^A} h_1(X_s^\varepsilon, z) N_1(ds, dz) \right] \]

\[ \leq E \left[ \sup_{0 \leq t \leq T} \left( \int_0^t \int_{|z|<1} |e^{(t-s)^A} h_1(X_s^\varepsilon, z)|_{\alpha} N_1(ds, dz) \right)^{2q} \right] \]

\[ \leq C E \left[ \left( \int_0^T \int_{|z|<1} |e^{(t-s)^A} h_1(X_s^\varepsilon, z)|_{\alpha}^2 \mu_1(dz) ds \right)^q \right] \]

\[ + C E \left[ \left( \int_0^T \int_{|z|<1} |e^{(t-s)^A} h_1(X_s^\varepsilon, z)|_{\alpha}^{2q} \mu_1(dz) ds \right)^q \right] \]

\[ \leq C E \left[ \left( \int_0^T (1 + |X_s^\varepsilon|^2) ds \right)^q \right] + C E \left[ \int_0^T (1 + |X_s^\varepsilon|^{2q}) ds \right] \]

\[ \leq C_{q,T} \left( 1 + \int_0^T E[|X_s^\varepsilon|^{2q}] ds \right). \]

We conclude the proof by combining (3.12), (3.15), (3.16), (3.17), (3.18), (3.19) and Gronwall’s inequality.

\[ \square \]

**Lemma 3.3.** Assume that the conditions (A1)-(A4) are satisfied, then there exist a positive constant \( C_{q,T,\alpha} \) such that for any \( x \in B_\alpha \), \( y \in L^2 \), \( 0 \leq t \leq t + h \leq T \), \( \alpha \in \left[ 1, \frac{1}{2} \right) \) and \( \varepsilon \in (0, 1) \),

\[ E \left[ \|X_{t+h}^\varepsilon - X_t^\varepsilon\|^{2q} \right] \leq C_{q,T,\alpha} h^{\alpha q} \left( 1 + \|x\|^{2q} + \|y\|^{2q} \right), \]

where \( C_{q,T,\alpha} \) is independent of \( \varepsilon \).
Proof: After simple calculations, we have $X_{t+h}^\varepsilon - X_t^\varepsilon =: \sum_{i=1}^5 \Theta_i$, in which

$$\Theta_1 = (e^{Ah} - I)X_t^\varepsilon,$$
$$\Theta_2 = \int_t^{t+h} e^{(t+s-h)A} B(X_s^\varepsilon) ds,$$
$$\Theta_3 = \int_t^{t+h} e^{(t+s-h)A} f_1(X_s^\varepsilon, Y_s^\varepsilon) ds,$$
$$\Theta_4 = \int_t^{t+h} e^{(t+s-h)A} dW_s^q,$$
$$\Theta_5 = \int_t^{t+h} \int_{|z|<1} e^{(t+s-h)A} h_1(X_s^\varepsilon, z) \tilde{N}_1(ds, dz).$$

For $\Theta_1$, there exist a constant $C_\alpha > 0$ such that for any $X \in \mathcal{D}((-A)\frac{h}{2})$, $\|e^{Ah}X - X\| \leq C_\alpha h^\frac{\alpha}{2} |X|_\alpha$ (see e.g. [17]). Then using Lemma 3.2, we get

$$\mathbb{E} [\|\Theta_1\|^{2q}] \leq C_\alpha h^{\alpha q} \mathbb{E} [X_1^\alpha |X_0^\alpha|] \leq C_q, T, \alpha h^{\alpha q} (1 + |x|^{2q} + \|y\|^{2q}). \quad (3.20)$$

For $\Theta_2$, using the contractive property of the semigroup $e^t A$, Lemma 3.1 and Lemma 3.2, we obtain

$$\mathbb{E} \left[ \| \int_t^{t+h} e^{(t+s-h)A} B(X_s^\varepsilon) ds \|^{2q} \right] \leq \mathbb{E} \left[ \int_t^{t+h} \| B(X_s^\varepsilon) \|^{2q} ds \right] \leq C_q, T h^{2q}. \quad (3.21)$$

For $\Theta_3$, applying (A1) and Lemma 3.1, we get

$$\mathbb{E} \| \Theta_3 \|^{2q} \leq C_h^{2q-1} \mathbb{E} \left[ \int_t^{t+h} \| f(X_s^\varepsilon, Y_s^\varepsilon) \|^{2q} ds \right] \leq C_q, T h^{2q} (1 + \|x\|^{2q} + \|y\|^{2q}). \quad (3.22)$$

For $\Theta_4$, note that $\Theta_4$ is the centered Gaussian random variable with the variance given by $S_h = \int_0^h e^{-(h-r)A} Q_1 e^{(h-r)A^*} dr$. Then, for any $q > 1$, we get

$$\mathbb{E} \| \Theta_4 \|^{2q} \leq C_q, T [\text{Tr} (S_h)]^q = C_q \left( \sum_{k=1}^\infty \int_0^h e^{-2(h-r)\lambda_k} \alpha_k dr \right)^q \leq C_q \left( \sum_{k=1}^\infty \alpha_k \right)^q h^q. \quad (3.23)$$

For $\Theta_5$, applying Burkholder-Davis-Gundy inequality [18], Hölder’s inequality and (A2), it yields

$$\mathbb{E} \left[ \left\| \int_t^{t+h} \int_{|z|<1} e^{(t+s-h)A} h_1(X_s^\varepsilon, z) \tilde{N}_1(ds, dz) \right\|^{2q} \right] \leq C_q \mathbb{E} \left[ \left( \int_t^{t+h} \int_{|z|<1} \| e^{(t+s-h)A} h_1(X_s^\varepsilon, z) \|^{2q} \mu_1(dz) ds \right)^q \right] + C_q \mathbb{E} \left[ \int_t^{t+h} \int_{|z|<1} \| e^{(t+s-h)A} h_1(X_s^\varepsilon, z) \|^{2q} \mu_1(dz) ds \right] \leq C_q h^{q-1} \mathbb{E} \left[ \int_t^{t+h} (1 + \|X_s^\varepsilon\|^2)^q ds \right] + C_q \mathbb{E} \left[ \int_t^{t+h} (1 + \|X_s^\varepsilon\|^2)^q ds \right] \leq C_q, T h (1 + \|x\|^{2q} + \|y\|^{2q}).$$

Putting (3.20)-(3.23) together, the result follows. \qed
4. Proof of Main Result

In this section, we intend to give a complete proof for our main result, i.e. the slow component \( X^\varepsilon \) converges strongly to the solution \( \bar{X} \) of the averaged equation, as \( \varepsilon \to 0 \). To this end, we construct a stopping time, i.e., for \( \varepsilon \in (0,1) \), \( n > 0 \),

\[
\tau^\varepsilon_n = \inf \{ t \geq 0 : |X^\varepsilon_t|_1 + |\bar{X}_s|_1 \geq n \}. \tag{4.1}
\]

**Proof of Theorem 2.1.** The proof is divided into three steps.

**Step 1.** For any \( q > 2 \), we know that

\[
E \left[ \sup_{0 \leq t \leq T} \|X^\varepsilon_t - \bar{X}_t\|^{2q} \right] \leq \sup_{0 \leq t \leq T} \|X^\varepsilon_t - \bar{X}_t\|^{2q} \cdot \chi(T > \tau^\varepsilon_n) + E \left[ \sup_{0 \leq t \leq T} \|X^\varepsilon_t - \bar{X}_t\|^{2q} \right].
\]

Using the similar argument as in the proof of Lemma 3.1, Lemma 3.2 and Lemma 3.3, we also get

\[
E \left[ \sup_{0 \leq t \leq T} \|X^\varepsilon_t\|^{2q} \right] \leq C_{q,T}(1 + \|x\|^{2q} + \|y\|^{2q}).
\]

By Hölder’s inequality, Chebyshev’s inequality and Lemma 3.1, we deduce that

\[
E \left[ \sup_{0 \leq t \leq T} \|X^\varepsilon_t - \bar{X}_t\|^{2q} \cdot \chi(T > \tau^\varepsilon_n) \right] \leq \frac{C_{q,T}}{n} \left( \sup_{0 \leq t \leq T} \|X^\varepsilon_t\|^{4q} + \sup_{0 \leq t \leq T} \|\bar{X}_t\|^{4q} \right)^{\frac{q}{2}} \left( \frac{n}{\sqrt{q}} \right)^{\frac{q}{2}} \left( \frac{n}{\sqrt{q}} \right)^{\frac{q}{2}} \leq \frac{C_{q,T}}{n} (1 + \|x\|^{2q} + \|y\|^{2q}),
\]

let \( n \) large enough, it yields

\[
\lim_{\varepsilon \to 0} E \left[ \sup_{0 \leq t \leq T} \|X^\varepsilon_t - \bar{X}_t\|^{2q} \cdot \chi(T > \tau^\varepsilon_n) \right] = 0. \tag{4.3}
\]

**Step 2.** In this part, we will prove the following result:

\[
\lim_{\varepsilon \to 0} E \left[ \sup_{0 \leq t \leq T} \|X^\varepsilon_t - \bar{X}_t\|^{2q} \right] = 0, \quad q > 2.
\]

According to the definitions of \( X^\varepsilon_t \) and \( \bar{X}_t \), for any \( t \in [0,T] \), we have

\[
E \left[ \sup_{0 \leq t \leq T} \|X^\varepsilon_t - \bar{X}_t\|^{2q} \right] \leq \sum_{i=1}^{3} I_i(t), \tag{4.4}
\]

where

\[
I_1(t) := C_q E \left[ \sup_{0 \leq t \leq T} \| \int_0^t e^{(t-s)A} \left( B(X^\varepsilon_s) - B(\bar{X}_s) \right) ds \|^{2q} \right],
\]

\[
I_2(t) := C_q E \left[ \sup_{0 \leq t \leq T} \| \int_0^t e^{(t-s)A} \left( f_1(X^\varepsilon_s, Y^\varepsilon_s) - f_1(\bar{X}_s) \right) ds \|^{2q} \right],
\]

\[
I_3(t) := C_q E \left[ \sup_{0 \leq t \leq T} \| \int_0^t e^{(t-s)A} \left( h_1(X^\varepsilon_s, z) - h_1(\bar{X}_s, z) \right) \bar{N}_1(ds, dz) \|^{2q} \right].
\]

Firstly, estimate the term \( I_1(t) \), according to (2.1) and Sobolev inequalities in ([15], Lemma 2.1 ), we get

\[
E \left[ \sup_{0 \leq t \leq T} \|I_1(t)\|^{2q} \right].
\]
\[ \begin{align*}
&\leq C_q \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T \land \tau^\alpha_n} \int_0^t \left( 1 + (t - r)^{-\frac{1}{2}} \right) |B(X^\varepsilon_r) - B(X_r)| \right) dr \right]^{2q} \\
&\leq C_q \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T \land \tau^\alpha_n} \int_0^t \left( 1 + (t - r)^{-\frac{1}{2}} \right) \|X^\varepsilon_r - X_r\| (|X^\varepsilon_r|_1 + |X_r|_1) dr \right) dr \right]^{2q} \\
&\leq C_{q,n} \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T} \int_0^{t \land \tau^\alpha_n} \left( 1 + (t + \tau^\alpha_n - r)^{-\frac{1}{2}} \right) \|X^\varepsilon_r - X_r\| dr \right) dr \right]^{2q} \\
&\leq C_{q,n,T} \int_0^T \mathbb{E} \left[ \sup_{0 \leq r \leq s \land \tau^\alpha_n} \|X^\varepsilon_r - X_r\|^{2q} \right] ds.
\end{align*} \]

Secondly, estimate the term \( I_2(t) \), we know that
\[
I_2(t) \leq C_q \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T \land \tau^\alpha_n} \left( \int_0^t e^{(t-s)A} \left( f_1(X^\varepsilon_s, Y^\varepsilon_s) - f_1(X^\varepsilon_s, \hat{Y}^\varepsilon_s) \right) ds \right) \left\| Y^\varepsilon_t - \hat{Y}^\varepsilon_t \right\|^{2q} \right) dr \right]
\]
\[
+ C_q \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T \land \tau^\alpha_n} \left( \int_0^t e^{(t-s)A} \left( f_1(X^\varepsilon_s, Y^\varepsilon_s) - \hat{f}_1(X^\varepsilon_s) \right) ds \right) \left\| Y^\varepsilon_t - \hat{Y}^\varepsilon_t \right\|^{2q} \right) dr \right]
\]
\[
+ C_q \mathbb{E} \left[ \left( \sup_{0 \leq t \leq T \land \tau^\alpha_n} \left( \int_0^t e^{(t-s)A} \left( f_1(X^\varepsilon_s, \hat{Y}^\varepsilon_s) - f_1(X^\varepsilon_s) \right) ds \right) \left\| Y^\varepsilon_t - \hat{Y}^\varepsilon_t \right\|^{2q} \right) dr \right]
\]
\[
=: \sum_{j=1}^3 K_j(t),
\]

where \( s(\delta) = \lfloor \frac{\delta}{4} \rfloor \) is the nearest breakpoint preceding \( |s| \) denotes the largest integer which is no more than \( s \), and \( \hat{Y}^\varepsilon \) is an auxiliary process with the initial value \( \hat{Y}^\varepsilon_t = \hat{y}^\varepsilon = y \). For any \( t \in [k\delta, \min\{(k + 1)\delta, T]\} \), \( k \in \mathbb{N} \),
\[
\hat{Y}^\varepsilon_t = Y^\varepsilon_t + \frac{c}{\varepsilon} \int_{k\delta}^t A\hat{Y}^\varepsilon_s ds + \frac{1}{\varepsilon} \int_{k\delta}^t f_2(X^\varepsilon_s, \hat{Y}^\varepsilon_s) ds + \frac{1}{\sqrt{\varepsilon}} \int_{k\delta}^t dW^Q_s
\]
\[
+ \int_{k\delta}^t \int_{|z| < 1} h_3(X^\varepsilon_s, \hat{Y}^\varepsilon_s, z) N^2_\varepsilon(ds, dz).
\]
\[
\text{(4.6)}
\]

Applying Itô’s formula, there exist \( \eta > 0 \), such that
\[
\frac{d}{dt} \mathbb{E} \left[ \left\| Y^\varepsilon_t - \hat{Y}^\varepsilon_t \right\|^{2q} \right] = \frac{2q^2\lambda_1}{\varepsilon} \mathbb{E} \left[ \left( \left\| Y^\varepsilon_t - \hat{Y}^\varepsilon_t \right\|^{2q} - \left\| Y^\varepsilon_t - \hat{Y}^\varepsilon_t \right\|^2 \right) \right]
\]
\[
+ \frac{2q}{\varepsilon} \mathbb{E} \left[ \left( \left\| Y^\varepsilon_t - \hat{Y}^\varepsilon_t \right\|^{2q} - \left( f_2(X^\varepsilon_t, Y^\varepsilon_t) - f_2(X^\varepsilon_t, \hat{Y}^\varepsilon_t) \right) \right) \right]
\]
\[
+ \frac{1}{\varepsilon} \mathbb{E} \left[ \int_{|z| < 1} \left( \left\| Y^\varepsilon_t - \hat{Y}^\varepsilon_t + h_2(X^\varepsilon_t, Y^\varepsilon_t, z) - h_2(X^\varepsilon_t, \hat{Y}^\varepsilon_t, z) \right\|^{2q} - \left\| Y^\varepsilon_t - \hat{Y}^\varepsilon_t \right\|^{2q} \right) \right] \mu_2(dz)
\]
\[
- \frac{2q}{\varepsilon} \mathbb{E} \left[ \int_{|z| < 1} \left( \left\| Y^\varepsilon_t - \hat{Y}^\varepsilon_t \right\|^{2q} - \left( h_2(X^\varepsilon_t, Y^\varepsilon_t, z) - h_2(X^\varepsilon_t, \hat{Y}^\varepsilon_t, z) \right) \right) \right] \mu_2(dz)
\]
\[
\leq - \frac{q}{\varepsilon} \mathbb{E} \left[ \left\| Y^\varepsilon_t - \hat{Y}^\varepsilon_t \right\|^{2q} \right] + C_{q,T,n} \mathbb{E} \left[ \left\| X^\varepsilon_t - X^\varepsilon_{k\delta} \right\|^{2q} \right].
\]

The Gronwall’s inequality [19] and Lemma 3.3 yields that
\[
\mathbb{E} \left[ \left\| Y^\varepsilon_t - \hat{Y}^\varepsilon_t \right\|^{2q} \right] \leq \frac{C_{q,T,n}}{q\eta} \delta^{\alpha q} (1 + |x^\varepsilon|_2^q + \|y\|_2^q) (1 - e^{-\frac{q\eta}{2\delta}})
\]
For $K_1(t)$, using the contractive property of semigroup $e^{tA}$, (A2), Lemma 3.3 and (4.7), gives

$$K_1(t) = C_q E \left[ \sup_{0 \leq t \leq T \wedge \tau^n} \| \int_0^t e^{(t-s)A} (f_1(x_s^\varepsilon, Y_s^\varepsilon) - f_1(x_{s(\delta)}^\varepsilon, Y_{s(\delta)}^\varepsilon)) ds \|^{2q} \right]$$

$$\leq C_q E \left[ \sup_{0 \leq t \leq T \wedge \tau^n} \left\| \int_0^t \left( f_1(x_s^\varepsilon, Y_s^\varepsilon) - f_1(x_{s(\delta)}^\varepsilon, Y_{s(\delta)}^\varepsilon) \right) ds \right\|^{2q} \right]$$

$$\leq C_q E \left[ \sup_{0 \leq t \leq T \wedge \tau^n} \left( \| x_s^\varepsilon - x_{s(\delta)}^\varepsilon \|^q + \| Y_s^\varepsilon - Y_{s(\delta)}^\varepsilon \|^q \right) ds \right]$$

$$\leq C_q \int_0^T E \left[ \sup_{0 \leq t \leq \tau^n} \| x_s^\varepsilon - x_{s(\delta)}^\varepsilon \|^q \right] ds$$

For $K_2(t)$, thanks to the Lipschitz continuity of $f_1(\cdot)$, we have

$$K_2(t) = C_q E \left[ \sup_{0 \leq t \leq T \wedge \tau^n} \left\| \int_0^{T \wedge \tau^n} e^{(t \wedge \tau^n-s)A} (f_1(x_s^\varepsilon) - f_1(X_s^\varepsilon)) ds \right\|^{2q} \right]$$

$$\leq E \left[ \sup_{0 \leq t \leq T \wedge \tau^n} \left\| \int_0^t e^{(t \wedge \tau^n-s)A} (f_1(x_s^\varepsilon) - f_1(X_s^\varepsilon)) ds \right\|^{2q} \right]$$

For $K_3(t)$, set $\bar{n}_t = \lfloor \frac{t}{\delta} \rfloor$, we write

$$K_3(t) = C_q E \left[ \sum_{0 \leq t \leq T \wedge \tau^n} \| \int_0^t e^{(t-s)A} \left( f_1(x_{s(\delta)}^\varepsilon, Y_{s(\delta)}^\varepsilon) - f_1(x_{s(\delta)}^\varepsilon) \right) ds \|^{2q} \right]$$

where

$$K_3^1(t) = \sum_{k=0}^{\bar{n}_t-1} \int_{k\delta}^{(k+1)\delta} e^{(t-s)A} \left( f_1(x_{s(\delta)}^\varepsilon, Y_{s(\delta)}^\varepsilon) - f_1(x_{s(\delta)}^\varepsilon) \right) ds,$$

$$K_3^2(t) = \sum_{k=0}^{\bar{n}_t-1} \int_{k\delta}^{(k+1)\delta} e^{(t-s)A} \left( f_1(x_{s(\delta)}^\varepsilon) - f_1(x_{s(\delta)}^\varepsilon) \right) ds,$$

$$K_3^3(t) = \int_{\bar{n}_t\delta}^t e^{(t-s)A} \left( f_1(x_{s(\delta)}^\varepsilon, Y_{s(\delta)}^\varepsilon) - f_1(x_{s(\delta)}^\varepsilon) \right) ds.$$
Moreover, we also have
\[ \varepsilon \]

Then, thanks to (4.14) and (4.15), we get that for any \( \varepsilon \),

\[ E \left[ \sup_{0 \leq t \leq T \wedge \tau_n^\delta} \| K_1(t) \|^{2(2q-1)} \right] \leq C_{q,T} \left\{ 1 + E \left[ \sup_{0 \leq t \leq T} \| X_{\varepsilon_s} \|^{2(2q-1)} \right] \right\}^{\frac{1}{2}} \left\{ E \left[ \sup_{0 \leq t \leq T \wedge \tau_n^\delta} \| K_1(t) \|^{2} \right] \right\}^{\frac{1}{2}}. \tag{4.12} \]

Thanks to Lemma 3.1 and the property of \( \tilde{\gamma}_t \), we have

\[ E \left[ \sup_{0 \leq t \leq T \wedge \tau_n^\delta} \| K_1(t) \|^{2} \right] \leq E \left[ \sup_{0 \leq t \leq T \wedge \tau_n^\delta} \| K_1(t) \|^{2(2q-1)} \right] \leq C_{q,T} \left\{ 1 + E \left[ \sup_{0 \leq t \leq T} \| X_{\varepsilon_s} \|^{2(2q-1)} \right] + E \left[ \sup_{0 \leq t \leq T} \| Y_{\varepsilon_r} \|^{2(2q-1)} \right] \right\}. \tag{4.13} \]

Moreover, we also have

\[
E \left[ \sup_{0 \leq t \leq T \wedge \tau_n^\delta} \| K_1(t) \|^{2(2q-1)} \right] \leq \sum \left\{ e^{(k+1)\delta} e^{(k+1)\delta} \left[ f_1 \left( X_{\kappa_s}, \tilde{\gamma}_s \right) - f_1 \left( X_{\kappa_s} \right) \right] ds \right\}^{2} \tag{4.14} \]

\[ C \left( \frac{T}{\delta} \right)^{\frac{2}{2}} \max_{0 \leq k \leq \lceil \frac{T}{\delta} \rceil - 1} \int_{k\delta}^{(k+1)\delta} e^{(k+1)\delta} \left[ f_1 \left( X_{\kappa_s}, \tilde{\gamma}_s \right) - f_1 \left( X_{\kappa_s} \right) \right] ds \right\}^{2} \]

\[ C \left( \frac{\varepsilon}{\delta} \right)^{\frac{2}{2}} \max_{0 \leq k \leq \lceil \frac{\varepsilon}{\delta} \rceil - 1} \int_{\varepsilon}^{\max} e^{(\delta-\varepsilon)A} \left[ f_1 \left( X_{\kappa_s}, \tilde{\gamma}_s \right) - f_1 \left( X_{\kappa_s} \right) \right] ds \right\}^{2} \]

where

\[ \Phi_k(s,r) = E \left[ e^{(k+1)\delta} \left( f_1 \left( X_{\kappa_s}, \tilde{\gamma}_s \right) - f_1 \left( X_{\kappa_s} \right) \right) \right] \]

\[ e^{(\delta-\varepsilon)A} \left( f_1 \left( X_{\kappa_s}, \tilde{\gamma}_s \right) - f_1 \left( X_{\kappa_s} \right) \right), \]

here the distribution of \( \left( X_{\kappa_s}, \tilde{\gamma}_s \right) \) coincides with the distribution of \( \left( X_{\kappa_s}, Y_{\kappa_s} \right) \). Similar as the argument in [20, 21], one can verify that

\[ \Phi_k(s,r) \leq C \left[ 1 + \| X_{\kappa_s} \|^2 + \| Y_{\kappa_s} \|^2 \right] e^{-\frac{s-r}{2}} \leq C \left( 1 + \| x \|^2 + \| y \|^2 \right) e^{-\frac{s-r}{2}}. \]

Then, thanks to (4.14) and (4.15), we get that for any \( \varepsilon \in (0,1) \),

\[ E \left[ \sup_{0 \leq t \leq T \wedge \tau_n^\delta} \| K_1(t) \|^{2} \right] \leq C_{q} \left( \frac{\varepsilon}{\delta} \right)^{\frac{2}{2}} \left( 1 + \| x \|^{2q} + \| y \|^{2q} \right) \]

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Substituting \((4.16)\) and \((4.13)\) into \((4.12)\), we get

\[
E \left[ \sup_{0 \leq t \leq T \wedge \tau_n} \| K_3(t) \|^2 \right] = C_{q,T,\alpha,n} \sqrt{\frac{\varepsilon}{\delta}} \left( 1 + \| x \|^2 + \| y \|^2 \right).
\]  

(4.17)

Combining \((4.10)\), \((4.11)\) and \((4.17)\), we can conclude

\[
E \left[ \sup_{0 \leq t \leq T \wedge \tau_n} \| K_3(t) \|^2 \right] \leq C_{q,T,\alpha,n} (\delta^{\alpha q} + \delta + \sqrt{\frac{\varepsilon}{\delta}}) (1 + \| x \|^2 + \| y \|^2).
\]  

(4.18)

Thirdly, estimate the term \(I_3(t)\), by the Kunita’s first inequality, the Hölder inequality, \((A2)\) and Young’s inequality, we have

\[
I_3(t) \leq C_q E \left[ \sup_{0 \leq t \leq T \wedge \tau_n} \int_0^T \int_{|z| < 1} e^{(t-s)A} \left( h_1(X_s^x, z) - h_1(\bar{X}_s, z) \right) \bar{N}_1(ds, dz) \right]^{2q}.
\]

(4.19)

Again, substituting \((4.5)\), \((4.8)\), \((4.9)\), \((4.18)\) and \((4.19)\) into \((4.4)\), we can derive

\[
E \left[ \sup_{0 \leq t \leq T \wedge \tau_n} \| X_t^x - \bar{X}_t \|^2 \right] \leq C_{q,T,\alpha,n} (1 + |x|^2 + \| y \|^2) (\delta^{\alpha q} + \delta + \sqrt{\frac{\varepsilon}{\delta}}) E \left[ \sup_{0 \leq t \leq T \wedge \tau_n} \| X_t^x - \bar{X}_t \|^2 \right] ds.
\]

Using Gronwall’s inequality, we get

\[
E \left[ \sup_{0 \leq t \leq T \wedge \tau_n} \| X_t^x - \bar{X}_t \|^2 \right] \leq C_{q,T,\alpha,n} (1 + |x|^2 + \| y \|^2) (\delta^{\alpha q} + \delta + \sqrt{\frac{\varepsilon}{\delta}}) e^{C_{q,T,\alpha,n}}.
\]

(4.20)

Taking \(\delta = \varepsilon^{\frac{1}{2}}\), we conclude

\[
\lim_{\varepsilon \to 0} E \left[ \sup_{0 \leq t \leq T} \| X_t^x - \bar{X}_t \|^2 \cdot \chi(T \leq \tau_n) \right] = 0.
\]  

(4.20)

**Step 3.** Taking \(\delta = \varepsilon^{\frac{1}{2}}\) and letting \(\varepsilon \to 0\) first, then \(n \to \infty\), by \((4.2)\) and \((4.20)\), for \(q > 2\), we have

\[
\lim_{\varepsilon \to 0} E \left[ \sup_{0 \leq t \leq T} \| X_t^x - \bar{X}_t \|^2 \right] = 0.
\]

Applying the Jensen inequality, for \(2 \leq p \leq q\), we get

\[
E \left[ \sup_{0 \leq t \leq T} \| X_t^x - \bar{X}_t \|^p \right] \leq \left( E \left[ \sup_{0 \leq t \leq T} \| X_t^x - \bar{X}_t \|^2 \right] \right)^{\frac{p}{2}}.
\]

thus, taking \(\delta = \varepsilon^{\frac{1}{2}}\), we have

\[
\lim_{\varepsilon \to 0} E \left[ \sup_{0 \leq t \leq T} \| X_t^x - \bar{X}_t \|^p \right] = 0, \ p \geq 2.
\]  

(4.21)

This completes the proof of Theorem 2.1.
5. Numerical Simulations

In this section, we present several examples to illustrate graphically the averaging principle for system (1.1). Consider the Burgers equation with an external force

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \xi^2} + u \frac{\partial u}{\partial \xi} + F(u, v, \zeta),
\]

(5.1)

The external force \(F(u, v, \zeta)\) depends on \(u, v\) and \(\zeta\), which makes the fluid run complicatedly. As we know, the term \(v\) can be governed by a fast evolution process, and \(\zeta\) may be a stochastic process. Specifically, taking

\[
\begin{align*}
F(u, v, \zeta)dt &= -(u + v)dt + \zeta dt, \\
v_{\varepsilon}t &= -\frac{1}{\varepsilon}v_{\varepsilon}t dt + \frac{1}{\sqrt{\varepsilon}}dW_t, \\
\zeta dt &= dW_t + \int_{|z|<1} u_{\varepsilon}N(dt, dz),
\end{align*}
\]

then the equation (5.1) can be rewritten as follows with suitable initial data and Dirichlet boundary condition

\[
\begin{align*}
d\bar{u}_{\varepsilon}(\xi) &= \left[\Delta \bar{u}_{\varepsilon}(\xi) + \frac{1}{2} \varepsilon \left( \bar{u}_{\varepsilon}(\xi) + u_0(\xi) \right)^2 - \bar{u}_{\varepsilon}(\xi) \right] dt + dW_t + \int_{|z|<1} u_{\varepsilon}N(dt, dz), \\
dv_{\varepsilon} &= -\frac{1}{\varepsilon}v_{\varepsilon}dt + \frac{1}{\sqrt{\varepsilon}}dW_t, \\
v_{\varepsilon}t(0) &= u_{\varepsilon}(1) = v_{\varepsilon}(0) = v_{\varepsilon}(1) = 0, \ \xi \in [0,1],
\end{align*}
\]

(5.2)

which is a typical example of the system (1.1). It is not hard to deduce the averaged equation associated with the system (5.2)

\[
d\bar{u}(\xi) = \left[\Delta \bar{u}(\xi) + \frac{1}{2} \frac{\partial}{\partial \xi} \left( \bar{u}(\xi) \right)^2 - \bar{u}(\xi) \right] dt + dW_t + \int_{|z|<1} \bar{u}(\xi)N(dt, dz), \quad \bar{u}_{0}(\xi) = 2.
\]

(5.3)

As shown in Fig.1, the solution \(u_{\varepsilon}(\xi)\) of system (5.2) converges to the solution \(\bar{u}(\xi)\) of the averaged equation (5.3) if \(\varepsilon \to 0\) in \(L^p(\Omega, L^2(0,1))\), \(p = 3, 4\), which is in accordance with the conclusion given in Theorem 2.1.

![Figure 1](image1.png)

(a) \(p = 3\)

![Figure 1](image2.png)

(b) \(p = 4\)

Figure 1: Convergence for one dimensional stochastic Burgers equation as \(\varepsilon\) goes to zero.

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