On the Noether number of $p$-groups

Kálmán Cziszter *
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
Reáltanoda u. 13 – 15, 1053 Budapest, Hungary

Abstract
A group of order $p^n$ ($p$ prime) has an indecomposable polynomial invariant of degree at least $p^{n-1}$ if and only if the group has a cyclic subgroup of index at most $p$ or it is isomorphic to the elementary abelian group of order 8 or the Heisenberg group of order 27.

Keywords: polynomial invariants, degree bounds, zero-sum sequences

1 Introduction

Let $G$ be a finite group and $V$ a $G$-module over a field $F$ of characteristic not dividing the group order $|G|$. The Noether-number $\beta(G, V)$ is the maximal degree in a minimal generating set of the ring of polynomial invariants $F[V]^G$. It is known that $\beta(G, V) \leq |G|$ (see \cite{10, 9, 8}). Even more, it was observed that $\beta(G) := \sup_V \beta(G, V)$ (where $V$ runs over all $G$-modules over the base field $F$) is typically much less than $|G|$. For an algebraically closed base field of characteristic zero it was proved in \cite{17} that $\beta(G) = |G|$ holds only if $G$ is cyclic. Then it turned out that $\beta(G) \leq \frac{1}{4}|G|$ for any non-cyclic group $G$ (see \cite{7} and \cite{19}). Moreover $\beta(G) \geq \frac{1}{2}|G|$ holds if and only if $G$ has a cyclic subgroup of index at most two, with the exception of four particular groups of small order (see \cite{2} Theorem 1.1)). Recently some asymptotic extensions of this result were given in \cite{14}. Our goal in the present article is to establish the following strengthening of this kind of results for the class of $p$-groups:

Theorem 1. If $G$ is a finite $p$-group for a prime $p$ and the characteristic of the base field $F$ is zero or greater than $p$ then the inequality

$$\beta(G) \geq \frac{1}{p}|G|$$

\hfill (1)

*Partially supported by the National Research, Development and Innovation Office (NKFIH) grants PD113138, ERC HU 15 118286, K115799 and K119934.
holds if and only if $G$ has a cyclic subgroup of index at most $p$ or $G$ is the elementary abelian group $C_2 \times C_2 \times C_2$ or the Heisenberg group of order 27.

The proof of Theorem 1 will be reduced to the study of a single critical case, the Heisenberg group $H_p$, which is the extraspecial group of order $p^3$ and exponent $p$ for an odd prime $p$. We prove about this the following result:

**Theorem 2.** For any prime $p \geq 5$ and base field $\mathbb{F}$ of characteristic 0 or greater than $p$ we have $\beta(H_p) < p^2$.

The paper is organised as follows. Section 2 contains some technical results on zero-sum sequences over abelian groups that will be needed later. In Section 3 we reduce the proof of Theorem 1 to that of Theorem 2. Then in Section 4 we explain the main invariant theoretic idea behind the proof of Theorem 2 which is also applicable in a more general setting. The proof itself of Theorem 2 will then be carried out in full detail in Section 5. Finally, Section 6 completes our argument by showing that for the case $p = 3$ we have $\beta(H_3) = 9$ in any non-modular characteristic.

## 2 Some preliminaries on zero-sum sequences

We follow here in our notations and terminology the usage fixed in [5]. Let $A$ be an abelian group noted additively. By a sequence $S$ over a subset $A_0 \subseteq A$ we mean a multiset of elements of $A_0$. They form a free commutative monoid with respect to concatenation, denoted by $S \cdot T$, and unit element the empty sequence $\emptyset$; this has to be distinguished from 0, the zero element of $A$. The sequence $a \cdot a \cdots a$, obtained by the $k$-fold repetition of an element $a \in A$, is denoted by $a^{[k]}$; this has to be distinguished from the product $ka \in A$. The multiplicity of an element $a \in A$ in a sequence $S$ is denoted by $v_a(S)$. We also write $a \in S$ to indicate that $v_a(S) > 0$. We say that $T$ is a subsequence of $S$, and write $T \mid S$, if there is a sequence $R$ such that $S = T \cdot R$. In this case we also write $R = S \cdot T^{[-1]}$. The length of a sequence, denoted by $|S|$, can be expressed as $\sum_{a \in A} v_a(S)$, whereas the sum of a sequence $S = a_1 \cdots a_n$ is $\sigma(S) := a_1 + \ldots + a_n \in A$ and by convention we set $\sigma(\emptyset) = 0$. We say that $S$ is a zero-sum sequence if $\sigma(S) = 0$.

The relevance of zero-sum sequences for our topic is due to the fact that for an abelian group $A$ the Noether number $\beta(A)$ coincides with the Davenport constant $D(A)$, which is defined as the maximal length of a zero-sum sequence over $A$ not containing any non-empty, proper zero-sum subsequence (see e.g. [5 Chapter 5]). Its value for $p$-groups is given by the following formula [12]
Theorem 5.5.9:  
\[ D(C_{p^{n_1}} \times \cdots \times C_{p^{n_r}}) = \sum_{i=1}^{r} (p^{n_i} - 1) + 1. \] (2)

A variant of this notion is the \( k \)th Davenport constant \( D_k(A) \) defined for any \( k \geq 1 \) as the maximal length of a zero-sum sequence \( S \) that cannot be factored as the concatenation \( S = S_1 \cdots S_{k+1} \) of non-empty zero-sum sequences \( S_i \) over \( A \). Its numerical value is much less known (for some recent results see [10]; we shall only need the fact that according to [12, Theorem 6.1.5.2]:
\[ D_k(C_p \times C_p) = kp + p - 1. \] (3)

The following consequence of the definition of \( D_k(A) \) will also be used:

**Lemma 3** ([12], Lemma 6.1.2). Any sequence \( S \) over an abelian group \( A \) of length at least \( D_k(A) \) factors as \( S = S_1 \cdots S_k \cdot R \) with some non-empty zero-sum sequences \( S_i \).

We define for any sequence \( S \) over \( A \) the set of all partial sums of \( S \) as \( \Sigma(S) := \{ \sigma(T) : \emptyset \neq T \mid S \} \). If \( 0 \notin \Sigma(S) \) then \( S \) is called zero-sum free. The next result could also be deduced from the Cauchy-Davenport theorem (see [12, Corollary 5.2.8.1]) but we provide here an elementary proof for the reader’s convenience:

**Lemma 4.** Let \( p \) be a prime. Then for any sequence \( S \) over \( C_p \setminus \{0\} \) we have \( |\Sigma(S)| \geq \min\{p, |S|\} \).

**Proof.** We use induction on the length of \( S \). For \( |S| = 0 \) the claim is trivial. Otherwise consider a sequence \( S \cdot a \) where the claim holds for \( S \). We have \( \Sigma(S \cdot a) = \Sigma(S) \cup \{a\} \cup (a + \Sigma(S)) \), where \( a + \Sigma(S) := \{a + s : s \in \Sigma(S)\} \). Then either \( |\Sigma(S \cdot a)| \geq |\Sigma(S)| + 1 \), or else \( a \in \Sigma(S) \) and \( a + \Sigma(S) = \Sigma(S) \), that is when \( \Sigma(S) \) is a subgroup of \( C_p \) containing \( a \). But since \( C_p \) has only two subgroups and by assumption \( \Sigma(S) \ni a \neq 0 \), this means that \( \Sigma(S) = C_p \). \( \square \)

**Lemma 5** ([12], Theorem 5.1.10.1). A sequence \( S \) over \( C_p \) (p prime) of length \( |S| = p - 1 \) is zero-sum free if and only if \( S = a^{p-1} \) for some \( a \in C_p \setminus \{0\} \).

**Lemma 6** ([12], Proposition 5.7.7.1). Let \( p \) be a prime and \( S \) be a sequence over \( C_p \times C_p \) of length \( |S| \geq 3p - 2 \). Then \( S \) has a zero-sum subsequence \( X \mid S \) of length \( p \) or \( 2p \).

We close this section with a technical result. Its motivation and relevance will become apparent through its application in the proof of Proposition 14. For any function \( \pi \) defined on \( A \) and any sequence \( S \) over \( A \) we will write \( \pi(S) \) for the sequence obtained from \( S \) by applying \( \pi \) element-wise.
Lemma 7. Let $S$ be a sequence over $C_p$ of length $|S| \geq p^2 - 1$. If we have $v_0(S) \geq p + 1$ then $S = S_1 \cdots S_\ell \cdot R$, where each $S_i$ is a non-empty zero-sum sequence and $\ell \geq 2p - 1$.

Proof. Let $\ell$ denote the maximal integer such that $S = S_1 \cdots S_\ell \cdot R$ for some non-empty zero-sum sequences $S_i$. Then each $S_i$ is irreducible, hence $|S_i| \leq p$ and $R$ is zero-sum free, hence $|R| \leq p - 1$. Assuming that $\ell \leq 2p - 2$ we get

$$p^2 - 1 \leq |S| \leq v_0(S) + (\ell - v_0(S))p + p - 1 \leq (p - 1)(2p + 1 - v_0(S))$$

whence $v_0(S) \leq p$ follows, in contradiction with our assumption. □

Proposition 8. Let $A = C_p \times C_p$ for some prime $p \geq 5$ and $\pi : A \to C_p$ the projection onto the first component. If $S$ is a sequence over $A$ with $|S| \geq p^2 - 1$ and $v_0(\pi(S)) \leq p$ then for any given subsequence $T \mid S$ of length $|T| \leq p - 1$ there is a a factorisation $S = S_1 \cdots S_{p-1} \cdot R$, where each $S_i$ is a non-empty zero-sum sequence over $A$, while $T \mid S \cdot (S_1 \cdot S_2)\{p\}$ and $\Sigma(\pi(S_1)) = C_p$.

Proof. Let $S^* \mid S \cdot T\{p\}$ be the maximal subsequence such that $0 \notin \pi(S^*)$. Then by assumption $|S^*| \geq |S| - 2p + 1 \geq 3p$, as $p \geq 5$, so there is a zero-sum subsequence $X \mid S^*$ of length $p$ or $2p$ by Lemma 6. We have two cases:

(i) If $|X| = 2p$ then $X = S_1 \cdot S_2$ for some non-empty zero-sum sequences $S_1, S_2$ such that $|S_1| \geq p$ and $|S_2| \leq p$, as $D(C_p \times C_p) = 2p - 1$ by (2).

(ii) If $|X| = p$ then we can take $S_1 := X$. Then we have $|S \cdot (S_1 \cdot T)\{p\}| \geq |S| - 2p + 1 \geq 3p$, so again by Lemma 6 we find a non-empty zero-sum sequence $S_2 \mid S \cdot (S_1 \cdot T)\{p\}$ of length $|S_2| \leq p$ as above.

In both cases $T \mid S \cdot (S_1 \cdot S_2)\{p\}$ and $|S_1 \cdot S_2| \leq 2p$ by construction. Consequently $|S \cdot (S_1 \cdot S_2)\{p\}| \geq |S| - 2p \geq p^2 - 2p - 1 = D_{p-3}(C_p \times C_p)$, hence by Lemma 3 we have a factorisation $S \cdot (S_1 \cdot S_2)\{p\} = S_3 \cdots S_{p-1} \cdot R$ with non-empty zero-sum sequences $S_i$ for each $i \geq 3$. Finally, in both cases we had $|S_1| \geq p$ and $0 \notin \pi(S_1)$, hence $|\Sigma(\pi(S_1))| = p$ by Lemma 4. □

3 Reduction of Theorem 1 to Theorem 2

Our main tool here will be the $k$th Noether number $\beta_k(G, V)$ which is defined for any $k \geq 1$ as the greatest integer $d$ such that some invariant of degree $d$ exists which is not contained in the ideal of $\mathbb{F}[V]^G$ generated by the products of at least $k + 1$ invariants of positive degree. This notion was introduced in [1] Section 2 with the goal of estimating the ordinary Noether number from information on its composition factors. This was made possible by [2, Lemma 1.4] according to which for any normal subgroup $N \triangleleft G$ we have:

$$\beta(G, V) \leq \beta_{\beta(G/N)}(N, V).$$ (4)
Proof of Theorem 1 (assuming Theorem 2). The “if” part follows from \[17, Proposition 5.1\] which states that \( \beta(C) \leq \beta(G) \) for any subgroup \( C \leq G \).

So if \( C \) is cyclic of index at most \( p \) then \( \beta(G) \geq \beta(C) = |C|/|G:C| \geq \frac{1}{p}|G| \), Moreover \( \beta(C_3^3) = 4 \) by \([2]\) and \( \beta(H_3) \geq 9 \) by Proposition \([17]\) below.

The “only if” part for \( p = 2 \) follows from \([2, Theorem 1.1]\) so for the rest we may assume that \( p \geq 3 \). Let \( G \) be a group of order \( p^n \) for which \([1]\) holds. If \( G \) is non-cyclic then it has a normal subgroup \( N \cong C_p \times C_p \) by \([1, Lemma 1.4]\). We claim that \( G/N \) must be cyclic. For otherwise by applying \([1, Lemma 1.4]\) to the factor group \( G/N \) we find a subgroup \( K \) such that \( N \triangleleft K \triangleleft G \) and \( K/N \cong C_p \times C_p \). But then we get using \([4]\) and \([3]\) that

\[
\beta(K) \leq \beta(C_p \times C_p)(C_p \times C_p) = p(2p - 1) + p - 1 = 2p^2 - 2 < p^3 = \frac{1}{p}|K|.
\]

As \( \beta(G)/|G| \leq \beta(K)/|K| \) by \([2, Lemma 1.2]\) we get a contradiction with \([1]\).

Now let \( g \in G \) be such that \( gN \) generates \( G/N \cong C_{p^{n-2}} \). Then \( g^{p^{n-2}} \in N \) has order \( p \) or \( 1 \). In the first case \( \langle g \rangle \) has index \( p \) in \( G \) and we are done. In the other case \( \langle g \rangle \cap N = \{1\} \) hence \( G \cong N \times \langle g \rangle \). If \( g \) acts trivially on \( N \) then \( G \) contains a subgroup \( H \cong C_p \times C_p \times C_p \) for which we have \( \beta(H) = 3p - 2 \) by \([2]\) hence \( \beta(G)/|G| \leq \beta(H)/|H| < 3/p^2 \leq 1/p \), as \( p \geq 3 \), a contradiction. This shows that \( g \) must act non-trivially on \( C_p \times C_p \). It is well known that \( \text{Aut}(C_p \times C_p) = \text{GL}(2, p) \) has order \( (p^2 - 1)(p^2 - p) \), so its Sylow \( p \)-subgroup must have order \( p \) and it is isomorphic to \( C_p \). Therefore \( g^p \) must act trivially on \( N \), so if \( n \geq 4 \) then \( g^p \neq 1 \) and the subgroup \( \langle N, g^p \rangle \) is isomorphic to \( C_p \times C_p \times C_p \), but this was excluded before. The only case which remains open is that \( n = 3 \) and \( G \cong (C_p \times C_p) \times C_p \), where the factor group \( C_p \) acts non-trivially on \( C_p \times C_p \). This is the Heisenberg group denoted by \( H_p \). By Theorem \([2]\) we have \( \beta(H_p) < p^2 \) for all \( p > 3 \) under our assumption on the characteristic of the base field \( F \). So among the Heisenberg groups the inequality \([1]\) can only hold for \( H_3 \).

\[\blacksquare\]

Remark 9. The precise value of the Noether number is already known for all the \( p \)-groups which satisfy \([1]\) according to Theorem \([1]\). As the Theorem states, equality holds in \([1]\) for \( C_2^3 \) and \( H_3 \). For the rest, the groups of order \( p^n \) which have a cyclic subgroup of index \( p \) were classified by Burnside (see e.g. \([1, Theorem 1.2]\)) as follows:

(i) If \( G \) is abelian, then either \( G \) is cyclic with \( \beta(G) = p^n \) or \( G = C_{p^{n-1}} \times C_p \) in which case it has \( \beta(G) = p^{n-1} + p - 1 \) by \([2]\).

(ii) If \( G \) is non-abelian and \( p > 2 \) then \( G \) is isomorphic to the modular group \( M_{p^n} \cong C_{p^{n-1}} \times C_p \). We have \( \beta(M_{p^n}) = p^{n-1} + p - 1 \) by \([3, Remark 10.4]\).
(iii) if $G$ is non-abelian and $p = 2$ then $G$ is the dihedral group $D_{2n}$ or the semi-dihedral group $SD_{2n}$ or the generalised quaternion group $Q_{2n}$. We have $\beta(Q_{2n}) = 2^{n-1} + 2$ and $\beta(D_{2n}) = \beta(SD_{2n}) = 2^{n-1} + 1$ by [3, Theorem 10.3].

Altogether these results imply that for any non-cyclic $p$-group $G$ we have

$$\beta(G) \leq \frac{1}{p}|G| + p$$

and this inequality is sharp only for the case $p = 2$.

**Remark 10.** The notion of the Davenport constant $D(G)$, originally defined only for abelian groups as in Section 2, was extended to any finite group $G$ in [11, 13]. For the conjectural connection between the Noether number and this generalisation of the Davenport constant see [5, Section 5.1] and [6].

### 4 Invariant theoretic lemmas

Let us fix here some notations related to invariant rings. For any vector space $V$ over a field $\mathbb{F}$ we denote its coordinate ring by $\mathbb{F}[V]$. We say that a group $G$ has a left action on $V$, or that $V$ is a $G$-module, if a group homomorphism $\rho : G \rightarrow GL(V)$ is given and we abbreviate $\rho(g)(v)$ by writing $g \cdot v$ for any $g \in G$ and $v \in V$. By setting $f^g(v) := f(g \cdot v)$ for any $f \in \mathbb{F}[V]$ we obtain a right action of $G$ on $\mathbb{F}[V]$. The ring of polynomial invariants is defined as $\mathbb{F}[V]^G := \{ f \in \mathbb{F}[V] : f^g = f \text{ for all } g \in G \}$. If the ring $\mathbb{F}[V]^N$ is already known for some normal subgroup $N \triangleleft G$ then $\mathbb{F}[V]^G$ as a vector space is spanned by its elements of the form $\tau^G_N(m)$, where $m$ runs over the set of all monomials and $\tau^G_N : \mathbb{F}[V]^N \rightarrow \mathbb{F}[V]^G$ is the $\mathbb{F}[V]^G$-module epimorphism defined as

$$\tau^G_N(m) = \frac{1}{|G/N|} \sum_{g \in G} m^g$$

(see e.g. [15, Chapter 2.2]). When $N$ is trivial this definition amounts to the Reynolds operator $\tau := \tau^G_{\{1\}}$. Given any character $\chi \in \hat{G} := \text{Hom}(G, \mathbb{F}^\times)$ the set $\mathbb{F}[V]^{G,\chi} := \{ f \in \mathbb{F}[V] : f^g = \chi(g)f \}$ constitutes the $\mathbb{F}[V]^{G}$-module of $G$-semi-invariants of weight $\chi$. If the restriction of $\chi$ to $N$ is trivial, i.e. when $\chi \in \hat{G}/\hat{N}$, then these semi-invariants can be obtained by the projection map $\tau_\chi : \mathbb{F}[V]^N \rightarrow \mathbb{F}[V]^{G,\chi}$ defined with the analogous formula

$$\tau_\chi(u) = \frac{1}{|G/N|} \sum_{g \in G/N} \chi^{-1}(g)u^g.$$
set $\mathbb{F}[V]^G_\circ := \bigoplus_{d \geq 1} \mathbb{F}[V]^G_d$ is a maximal ideal in $\mathbb{F}[V]^G$, while $\mathbb{F}[V]^G_+ \mathbb{F}[V]$, the ideal of $\mathbb{F}[V]$ generated by all $G$-invariant polynomials of positive degree, is the so called Hilbert-ideal. This ideal will be our main object of interest since, as observed in [4 Section 3], the graded factor ring $\mathbb{F}[V]/\mathbb{F}[V]^G_+ \mathbb{F}[V]$ is finite dimensional and its top degree, denoted by $b(G, V)$, yields an upper bound on the Noether number by an easy argument using the Reynolds operator:

$$\beta(G, V) \leq b(G, V) + 1. \tag{6}$$

It is well known that $\beta(G, V)$ is unchanged when we extend the base field so we will assume throughout this paper that $\mathbb{F}$ is algebraically closed.

**Lemma 11.** Let $G$ be a finite group with a normal subgroup $N$ such that $G/N$ is abelian. Let $W$ be a $G$-module over $\mathbb{F}$ and assume that $|G| \in \mathbb{F}^\times$. Then $(\mathbb{F}[W]^N)^k \subseteq \mathbb{F}[W]^G_+ \mathbb{F}[W]$ for any $k \geq D(G/N)$.

**Proof.** $\mathbb{F}[W]^N$ regarded as a $G/N$-module has the direct sum decomposition $\bigoplus_{\chi \in \hat{G}/N} \mathbb{F}[W]^{G/N}_\chi$. (Here we used both our assumptions on $\mathbb{F}$.) This means that any element $u \in \mathbb{F}[W]^N_+$ can be written as a sum $u = \sum_{\chi \in \hat{G}/N} \tau_\chi(u)$. Now for any $k \geq 1$ and $u_1, \ldots, u_k \in \mathbb{F}[W]^N_+$ we have

$$\prod_{i=1}^k u_i = \prod_{i=1}^k \left( \sum_{\chi \in \hat{G}/N} \tau_\chi(u_i) \right) = \sum_{\chi_1, \ldots, \chi_k \in \hat{G}/N} \tau_{\chi_1}(u_1) \cdots \tau_{\chi_k}(u_k). \tag{7}$$

The term $\tau_{\chi_1}(u_1) \cdots \tau_{\chi_k}(u_k)$ belongs to the ideal $\mathbb{F}[W]^G_+ \mathbb{F}[W]$ whenever the sequence $(\chi_1, \ldots, \chi_k)$ over $\hat{G}/N \cong G/N$ contains a non-empty zero-sum subsequence. But this holds for every term on the right of (7) as $k \geq D(G/N)$. \qed

**Lemma 12.** If in Lemma 11 the factor group $G/N \cong C_p$ is cyclic of prime order then for any $g \in G/N$ and any elements $u_1, \ldots, u_{p-1} \in \mathbb{F}[W]^N_+$ we have the relation:

$$u_1 \cdots u_{p-1} - u_1^g u_2^g u_3 \cdots u_{p-1} \in \mathbb{F}[W]^G_+ \mathbb{F}[W]. \tag{8}$$

**Proof.** Observe that in (7) with $k = p - 1$ the weight sequence $(\chi_1, \ldots, \chi_{p-1})$ over $\hat{C}_p$ is zero-sum free if and only if $\chi_1 = \ldots = \chi_{p-1}$ and $\chi_1$ is non-trivial (by Lemma 3). As a result we get:

$$u_1 \cdots u_{p-1} \in \sum_{\chi \in \hat{C}_p \setminus \{1\}} \tau_\chi(u_1) \cdots \tau_\chi(u_{p-1}) + \mathbb{F}[W]^G_+ \mathbb{F}[W].$$

Replacing here $u_1$ and $u_2$ with $u_1^g$ and $u_2^g$, respectively, and observing that by definition we have $\tau_\chi(u^g) = \chi(g) \tau(u)$ for any $u \in \mathbb{F}[W]^N$ we infer that $u_1^g u_2^g u_3 \cdots u_{p-1}$ must belong to the same residue class modulo the ideal $\mathbb{F}[W]^G_+ \mathbb{F}[W]$ to which $u_1 \cdots u_{p-1}$ does belong. This proves our claim. \qed
5 The Heisenberg group $H_p$

The Heisenberg group $H_p = \langle a, b \rangle$ can be defined by the presentation:

$$a^p = b^p = c^p = 1 \quad [a, b] = c \quad [a, c] = [b, c] = 1$$

(9)

where $[a, b]$ denotes the commutator $a^{-1}b^{-1}ab$. The subgroups $A := \langle a, c \rangle$ and $B := \langle b, c \rangle$ are normal and isomorphic to $C_p \times C_p$. The Frattini-subgroup, the center and the derived subgroup of $H_p$ all coincide with $\langle c \rangle$, so that $H_p$ is extraspecial. In particular $H_p/\langle c \rangle$ is also isomorphic to $C_p \times C_p$. Taking into account only the subgroup structure of $H_p$ the best upper bound that we can give about its Noether number by means of (4) and (3) is the following:

$$\beta(H_p) \leq \beta(\beta(C_p) \times C_p) = p^2 + p - 1.$$  (10)

Our goal in this section will be to enhance this estimate by analysing more closely the invariant rings of $H_p$.

Let $\mathbb{F}$ be an algebraically closed field with $\text{char}(\mathbb{F}) \neq p$, so that there is a primitive $p$-th root of unity $\omega \in \mathbb{F}$ that will be regarded as fixed throughout this paper. The irreducible $H_p$-modules over $\mathbb{F}$ are then of two types:

(i) Composing any group homomorphism $\rho \in \text{Hom}(C_p \times C_p, \mathbb{F}^\times)$ with the canonical surjection $H_p \to H_p/\langle c \rangle \cong C_p \times C_p$ yields $p^2$ non-isomorphic 1-dimensional irreducible representations of $H_p$.

(ii) For each primitive $p$-th root of unity $\omega^i \in \mathbb{F}$, where $i = 1, \ldots, p-1$, take the induced representation $V_{\omega^i} := \text{Ind}_{A}^{H_p} \langle v \rangle$, where $\langle v \rangle$ is a 1-dimensional left $A$-module such that $a \cdot v = v$ and $c \cdot v = \omega^i v$. In the basis $\{v, b \cdot v, \ldots, b^{p-1} \cdot v\}$ this representation is then given in terms of matrices in the following form, with $I_p$ the $p \times p$ identity matrix:

$$a \mapsto \begin{pmatrix} 1 & & & \\ \omega^i & & & \\ & \ddots & & \\ & & \omega^{i(p-1)} & \end{pmatrix} \quad b \mapsto \begin{pmatrix} 0 & \cdots & \cdots & 1 \\ 1 & & & \\ \vdots & \ddots & & \\ 1 & & & 0 \end{pmatrix} \quad c \mapsto \omega^i I_p. \quad (11)$$

Each $V_{\omega^i}$ is irreducible by Mackey’s criterion (see e.g. [13]) and for $\omega^i \neq \omega^{i'}$ it is easily seen (e.g. from the matrix corresponding to $c$) that $V_{\omega^i}$ and $V_{\omega^{i'}}$ are non-isomorphic as $G$-modules.

Adding the squares of the dimensions of the above irreducible $H_p$-modules we get $p^2 \cdot 1 + (p-1)p^2 = p^3 = |H_p|$, so that no other irreducible $H_p$-modules exist. As a result an arbitrary $H_p$-module $W$ over $\mathbb{F}$ has the canonical direct sum decomposition

$$W = U \oplus V_1 \oplus \ldots \oplus V_{p-1} \quad (12)$$
where \( U \) consists only of 1-dimensional irreducible representations of \( H_p \) with \( \langle c \rangle \) in their kernel, while each \( V_i \) is an isotypic \( H_p \)-module consisting of the direct sum of \( n_i \geq 0 \) isomorphic copies of the irreducible representation \( V_{\omega^i} \):

\[
V_i = \underbrace{V_{\omega^i} \oplus \ldots \oplus V_{\omega^i}}_{n_i \text{ times}}. \tag{13}
\]

Next we recall how does the action of \( G \) on \( W \) extend to the coordinate ring \( F[W] \). When speaking of a coordinate ring \( F[V_{\omega^i}] = F[x_{i,0}, \ldots, x_{i,p-1}] \) we always tacitly assume that the variables \( x_{i,k} \) form a dual basis of the basis used at (11). By our convention from Section 4, \( H_p \) acts from the right on the variables, i.e. \( x^g(v) = x(g \cdot v) \) for all \( g \in H_p \), so we can rewrite (11) as:

\[
x_{i,k}^b = x_{i,(k-1)\text{mod}p} \quad x_{i,k}^a = \omega^{ik}x_{i,k} \quad x_{i,k}^c = \omega^ix_{i,k}. \tag{14}
\]

(Here, by some abuse of notation, we identified the integers \( k = 0, 1, \ldots, p-1 \) occurring as indexes with the modulo \( p \) residue classes they represent.) This shows that the action of the subgroup \( A \) on a variable \( x_{i,k} \) is completely determined by the modulo \( p \) residue classes of the exponents \( ik \) and \( i \) of \( \omega \) in (14): we will call \( \phi(x_{i,k}) := (ik, i) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \) the weight of the variable \( x_{i,k} \). We shall also refer to the projections \( \phi_a(x_{i,k}) = ik \) and \( \phi_c(x_{i,k}) = i \).

With this notation it is immediate from (14) that for any \( n \in \mathbb{Z} \) and \( x = x_{i,k} \)

\[
\phi_a(x^{bn}) = \phi_a(x) - n \phi_c(x) \quad \text{and} \quad \phi_c(x^{bn}) = \phi_c(x) \tag{15}
\]

where the subtraction and multiplication with \( n \) is understood in \( \mathbb{Z}/p\mathbb{Z} \). This implies the observation, which will be used frequently later on, that for any variable \( x \) with \( \phi_c(x) \neq 0 \) and any arbitrarily given \( w \in \mathbb{Z}/p\mathbb{Z} \) there is always an element \( g \in \langle b \rangle \) such that \( \phi_a(x^g) = w \). Our discussion also shows that for a variable \( y \in F[W] \) we have \( \phi_c(y) = 0 \) if and only if \( y \in F[U] \), and otherwise the value \( \phi_c(y) = i \) determines the isotypic \( H_p \)-module \( V_i \) such that \( y \in F[V_i] \).

Any monomial \( u \in F[W] \) is an \( A \)-eigenvector, too, hence we can associate a weight \( \phi(u) := (j, i) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \) to it so that \( u^a = \omega^j u \) and \( u^c = \omega^i u \). Obviously then \( \phi(uv) = \phi(u) + \phi(v) \) for any monomials \( u, v \). If \( u = y_1 \cdots y_n \) for some variables \( y_i \in F[W] \), with repetitions allowed, then we can form the sequence \( \Phi(u) := \phi(y_1) \cdots \phi(y_n) \) over \( A \), which will be called the weight sequence of \( u \). Obviously \( \phi(u) = \sigma(\Phi(u)) = \phi(y_1) + \cdots + \phi(y_n) \) with the notations of Section 2. Observe that a monomial \( u \) is \( A \)-invariant if and only if \( \phi(u) = 0 \), that is if \( \Phi(u) \) is a zero-sum sequence over \( A \). Finally, we set \( \Phi_a(u) := (\phi_a(y_1), \ldots, \phi_a(y_n)) \) and \( \Phi_c(u) := (\phi_c(y_1), \ldots, \phi_c(y_n)) \).

**Definition 13.** We call two monomials \( u, v \in F[W] \) homologous, denoted by \( u \sim v \), if \( \deg(u) = \deg(v) = d \) and \( u = \prod_{i=1}^d y_{ni} \) while \( v = \prod_{i=1}^d y_{ni}^a \) for some variables \( y_n \in F[W] \) (with repetitions allowed) and group elements \( g_n \in \langle b \rangle \).
Proposition 14. Let \( p \geq 5 \). If \( u \in \mathbb{F}[W] \) is a monomial with \( \deg(u) \geq p^2 - 1 \), \( v_0(\Phi_c(u)) \leq p \) and \( v \mid u \) is a monomial such that \( \deg(v) \leq p \) and \( 0 \not\in \Phi_c(v) \) then for any homologous monomial \( v' \sim v \) there is a homologous monomial \( u' \sim u \) such that \( v' \mid u' \) and \( u' - u \in \mathbb{F}[W]_+^{\leq} \mathbb{F}[W] \).

Proof. We use induction on the degree \( d := \deg(v) = \deg(v') \). If \( d = 0 \) then \( v = v' = 1 \), so we are done by taking \( u' = u \). Suppose now that the claim holds for some \( d \leq p - 1 \). It suffices to prove that for any given divisor \( xv \mid u \), where \( x \) is a variable, \( \deg(v) = d \), \( 0 \not\in \Phi_c(xv) \), and for any \( v' \sim v \) and \( g \in \langle b \rangle \) a monomial \( u'' \sim u \) exists such that \( x^g v' \mid u'' \) and \( u'' - u \in \mathbb{F}[W]_+^{\leq} \mathbb{F}[W] \).

By the inductive hypothesis we already have a monomial \( u' \sim u \) such that \( v' \mid u' \) and \( u' - u \in \mathbb{F}[W]_+^{\leq} \mathbb{F}[W] \). As \( u'/v' \sim u/v \) and \( x \) divides \( u/v \) there is a \( t \in \langle b \rangle \) such that \( x^t \) divides \( u'/v' \). By applying Proposition 8 to the weight sequences \( S := \Phi(u') \), \( T := \Phi(v') \) we obtain a factorisation \( u' = u_1 \cdots u_p - u_p \) such that \( u_i \in \mathbb{F}[W]_+^d \) for all \( i = 1, \ldots, p-1 \), \( u_p \in \mathbb{F}[W] \). \( v' \) divides \( u'/u_1 u_2 \) and \( \Sigma(\Phi_c(u_1)) = \mathbb{Z}/p\mathbb{Z} \). We have two cases:

i) If \( x^t \mid u_1 \) (or similarly if \( x^t \mid u_2 \)) then take \( u'' := u_1^{-1} u_2^{k-1} u_3 \cdots u_{p-1} u_p \).
We have \( x^g v' \mid u'' \) and \( u'' - u' \in \mathbb{F}[W]_+^{\leq} \mathbb{F}[W] \) by Lemma 12.

ii) Otherwise \( x^t \mid u_k \) for some \( k > 2 \). By our assumption on \( \Sigma(\Phi_c(u_1)) \) there is a divisor \( w \mid u_1 \) with \( \phi_c(w) = -\phi_c(x^t) \). As \( \phi_c(x^t) = \phi_c(x) \notin 0 \) there is an \( h \in \langle b \rangle \) for which \( \phi_c(w^h) = -\phi_c(x^t) \). Then for \( \hat{u} := u_1^h u_2^{-h} u_3 \cdots u_{p-1} \) we have \( \hat{u} \sim u \) and \( \hat{u} - u \in \mathbb{F}[W]_+^{\leq} \mathbb{F}[W] \) by Lemma 12. Take the factorisation \( \hat{u} = \hat{u}_1 \cdots \hat{u}_p \) where \( \hat{u}_1 = w^h x^t, \hat{u}_2 = u_2^{-h}, \hat{u}_k = (u_k/x^t)(u_1/h^h) \) and \( \hat{u}_i = u_i \) for the rest. By construction \( \hat{u}_i \in \mathbb{F}[W]_+^d \) for all \( i \leq p-1 \), \( v' \) divides \( \hat{u}_i/\hat{u}_1 u_2 \) and \( x^t \mid \hat{u}_1 \), so this factorisation of \( \hat{u} \) fails under case i) and we are done. \( \square \)

We need some further notations. The decomposition (12) induces an isomorphism \( \mathbb{F}[W] \cong \mathbb{F}[U] \otimes \mathbb{F}[V_1] \otimes \cdots \otimes \mathbb{F}[V_{p-1}] \) which in turn yields for any monomial \( m \in \mathbb{F}[W] \) a factorisation \( m = m_0 m_1 \cdots m_{p-1} \) such that \( m_0 \in \mathbb{F}[U] \) and \( m_i \in \mathbb{F}[V_i] \) for all \( i \). Then for each \( i \) the decomposition (13) gives the identifications \( \mathbb{F}[V_i] = \bigotimes_{j=1}^{n_i} \mathbb{F}[V_{j}] = \mathbb{F}[x_{i,k}^{(j)} : k = 0, \ldots, p-1; j = 1, \ldots, n_i] \), where we set \( x_{i,k}^{(j)} := 1 \otimes \cdots \otimes x_{i,k} \otimes \cdots \otimes 1 \), i.e. the variable \( x_{i,k} \) introduced at (13) is placed in the \( j \)th tensor factor. So for any monomial \( m_i \in \mathbb{F}[V_i] \) we have a factorisation \( m_i = m_i^{(1)} \cdots m_i^{(n_i)} \) where each monomial \( m_i^{(j)} \) depends only on the set of variables \( \{ x_{i,k}^{(j)} : k = 0, 1, \ldots, p-1 \} \). Observe finally that if \( u, v \in \mathbb{F}[V_1 \oplus \cdots \oplus V_{p-1}] \) are homologous, \( u \sim v \), and if only if \( \deg(u^{(j)}_i) = \deg(v^{(j)}_i) \) for all \( i = 1, \ldots, p-1 \) and \( j = 1, \ldots, n_i \).
We shall also need the polarisation operators defined for any polynomial \( f \in \mathbb{F}[W] \) by the formula

\[
\Delta_i^{s,t}(f) := \sum_{k=0}^{p-1} x_{i,k}^{(t)} \phi_{i,k}^{(s)} f
\]

where \( \phi_{i,k}^{(s)} \) denotes partial derivation with respect to the variable \( x_{i,k}^{(s)} \). All polarisation operations \( \Delta := \Delta_i^{s,t} \) are degree preserving, \( \deg(\Delta(f)) = \deg(f) \), and \( G \)-equivariant, i.e. \( \Delta(f^g) = \Delta(f)^g \). Therefore by the Leibniz \[
\Delta(\mathbb{F}[W]^C_+ \mathbb{F}[W]) \subset \mathbb{F}[W]^C_+ \mathbb{F}[W] \quad \text{and} \quad
\Delta(\mathbb{F}[W]^C_+ \mathbb{F}[W]_+) \subset \mathbb{F}[W]^C_+ \mathbb{F}[W]_+.
\]

**Proposition 15.** Let \( p \geq 5 \) and assume that \( \text{char(\mathbb{F})} \) is 0 or greater than \( p \). If a monomial \( m \in \mathbb{F}[W] \) has \( \deg(m) \geq p^2 - 1 \) then \( m \in \mathbb{F}[W]^C_+ \mathbb{F}[W] \).

**Proof.** Consider the factorisation \( m = m_0 m_1 \cdots m_{p-1} \) derived from (12) as described above. Observe that for the weight sequence \( S = (\Phi(m)) \) we have \( v_0(\Phi_c(m)) = \deg(m_0) \). So if \( \deg(m_0) \geq p + 1 \) then \( m \in (\mathbb{F}[W]^C_+)^{2p-1} \mathbb{F}[W] \) by Lemma 1 and we are done, as \( D(G/\langle c \rangle) = D(C_p \times C_p) = 2p - 1 \) by (2) hence \( (\mathbb{F}[W]^C_+)^{2p-1} \subset \mathbb{F}[W]^C_+ \mathbb{F}[W] \) by Lemma 1.

It remains that \( \deg(m_0) \leq p \). Then we must have \( \deg(m_i) \geq p \) for some \( i \geq 1 \), say \( i = 1 \), as otherwise \( \deg(m) \leq \deg(m_0) + (p-1)^2 \leq p^2 - p + 1 \) would follow. Take the factorisation \( m_1 = m_1^{(1)} \cdots m_1^{(n_1)} \) corresponding to the direct decomposition (13). We proceed by induction on \( \mu(m) := \max_{i=1}^{n_1} \deg(m_1^{(j)}) \).

Assume first that \( \mu(m) \geq p \). This means that \( \deg(m_1^{(j)}) \geq p \) for some \( j \), say \( j = 1 \). Now let \( v \) be an arbitrary divisor of \( m_1^{(1)} \) with degree \( \deg(v) = p \) and let \( v' = \prod_{\gamma \in (0)} x^\gamma \) for some variable \( x \in \mathbb{F}[V_0^{(1)}] \). Then \( v' \) is \( b \)-invariant by construction. Moreover by (15) we have \( \phi_c(v') = p\phi_c(x) = 0 \) and \( \phi_a(v') = p\phi_a(x) - (1 + 2 + \cdots + p - 1)\phi_a(x) = 0 \), and consequently \( v' \) is \( G \)-invariant. Now as \( v' \sim v \), we can find by Proposition 14 a monomial \( m' \sim m \) such that \( m - m' \in \mathbb{F}[W]^G_+ \mathbb{F}[W] \) and \( v' | m' \). But then \( m' \in \mathbb{F}[W]^G_+ \mathbb{F}[W] \) and we are done for this case.

Now let \( \mu(m) < p \). As \( \deg(m_1^{(j)}) \geq p \), we can take a divisor \( v | m_1 \) such that \( v = v^{(i)} v^{(j)} \) for some indices \( i \neq j \leq n_1 \) where we have \( \deg(v^{(i)}) = \mu(m) \) and \( \deg(v^{(j)}) = 1 \). Then the monomial \( v' = (x_{1,1}^{(i)})^{(\mu(m))} x_{1,1}^{(j)} \) is homologous with this \( v \) and consequently, by Proposition 14, a monomial \( m' \in \mathbb{F}[W] \) exists such that \( m - m' \in \mathbb{F}[W]^G_+ \mathbb{F}[W] \) and \( v' | m' \). Our claim will now follow by proving that \( m' \in \mathbb{F}[W]^G_+ \mathbb{F}[W] \).

11
To this end observe that for the monomial \( \tilde{m} := x_{1,1}^{(i)} m' / x_{1,1}^{(j)} \) we have \( \mu(\tilde{m}) = \mu(m) + 1 \), hence by the induction hypothesis \( \tilde{m} \in \mathbb{F} [W]_+ G \mathbb{F} [W] \) already holds. Moreover \( \Delta_1^{(j)}(\tilde{m}) = (\mu(m) + 1)m' \) by construction, hence \( (\mu(m) + 1)m' \in \mathbb{F} [W]_+ G \mathbb{F} [W] \) by (17) and we are finished because by our assumption on \( \mathbb{F} \) we are allowed to divide by \( \mu(m) + 1 \leq p < \text{char}(\mathbb{F}) \). \hfill \( \blacksquare \)

**Proof of Theorem 2.** From Proposition 15 we see that \( \mathbb{F} [W] \) as a module over \( \mathbb{F} [W]^G \) is generated by elements of degree at most \( p^2 - 2 \). Equivalently, for the top degree in the factor ring \( \mathbb{F} [W] / \mathbb{F} [W]^G \mathbb{F} [W] \) we have the estimate \( b(G, W) \leq p^2 - 2 \), whence by (6) we conclude that \( \beta(G, W) \leq p^2 - 1 \). \hfill \( \blacksquare \)

## 6 The case \( p = 3 \)

**Proposition 16.** Consider \( V = V_\omega \) for a primitive third root of unity \( \omega \in \mathbb{F} \) as given by (11). Then \( \beta(H_3, V) \geq 9 \).

**Proof.** Let \( \mathbb{F} [V] = \mathbb{F} [x, y, z] \) with the variables conforming our conventions. \( \mathbb{F} [V]^{H_3} \) is spanned by the elements \( \tau(m) := \tau_A(m) = \frac{1}{6} (m + m^3 + m^3) \) where \( m \) is any \( A \)-invariant monomial. An easy argument shows that \( xyz, x^3, y^3, z^3 \) are the only irreducible \( A \)-invariant monomials. Then by enumerating all \( A \)-invariant monomials of degree at most 8 we see that they have degree 3 or 6 so that for \( d \leq 8 \) we have \( \mathbb{F} [V]^{H_3} = R_d \), where \( R := \mathbb{F} [xyz, \tau(x^3), \tau(x^3 y^3)] \). Now if we assume that \( \beta(H_3, V) \leq 8 \) then \( \mathbb{F} [V]^{H_3} = R \) follows. Observe however that all the generators of \( R \) are symmetric polynomials, so that \( R \subseteq \mathbb{F} [V]^{S_3} \). On the other hand \( \tau(x^3 y^3) \notin \mathbb{F} [V]^{H_3} \) is not a symmetric polynomial, whence \( \tau(x^3 y^3) \notin R \). This is a contradiction which proves that \( \beta(H_3, V) \geq 9 \). \hfill \( \blacksquare \)

The upper bound on \( \beta(H_3) \) will be obtained by an argument very similar to Propositions 8, 13 and 15 but since there are many different details, too, we preferred to give a self-contained treatment of this case here:

**Proposition 17.** If \( \text{char}(\mathbb{F}) \neq 3 \) then \( \beta(H_3) \leq 9 \).

**Proof.** Suppose that \( \beta(H_3, W) \geq 10 \) holds for a \( H_3 \)-module \( W \). Then there is a monomial \( m \in \mathbb{F} [W]^A \) with \( \deg(m) \geq 10 \) such that \( m \notin \mathbb{F} [W]_+^G \mathbb{F} [W]_+ \) (as otherwise for any \( d \geq 10 \) the space \( \mathbb{F} [W]_+^G \) spanned by the elements \( \tau(m) \) would be contained in \( (\mathbb{F} [W]_+^G)^2 \). Let \( S = \Phi_c(m) \), identify \( c \) with \( \mathbb{Z} / 3 \mathbb{Z} \) and let \( d_i = v_i(S) \) for \( i \in \mathbb{Z} / 3 \mathbb{Z} = \{ 0, 1, 2 \} \). Recall that we have the factorisation \( m = m_0 m_1 m_2 \) corresponding to the direct decomposition \( W = U \oplus V_1 \oplus V_2 \), so that \( \deg(m_i) = d_i \) for \( i = 0, 1, 2 \). We may assume by symmetry that \( d_1 \geq d_2 \).

**A. We claim that** \( d_1 \geq 5 \).
$S$ is a zero-sum sequence over $\mathbb{Z}/3\mathbb{Z}$ and this is only possible if $d_1 - d_2 \equiv 0 \mod 3$. So let $d_1 - d_2 = 3k$ for some integer $k \geq 0$. Denoting by $\ell(S)$ the maximum number of non-empty zero-sum sequences into which $S$ can be factored, we have $\ell(S) = d_0 + d_3 + d_4 \leq 5$, as otherwise by Lemma 11 applied with $N = c$ we get $m \in (F[W]_+^n)_0 \subseteq F[W]_+^n F[W]_+$, since $H_3/N \cong C_3 \times C_3$ and $D(C_3^2) = 5$ by (2). On the other hand $|S| = d_0 + d_1 + d_2 \geq 10$. Subtracting from this inequality the previous one yields $d_1 - k \geq 5$, whence the claim.

B. For any $w \mid m_1$ with $\deg(w) \leq 2$ there is a factorisation $m = u_1 u_2 u_3$ with $ui \in \mathbb{F}[W]_+^3$ such that $w \mid u_3$ and $y \mid u_1$ for some variable $y \mid m_1$.

As $\deg(m/w) \geq 8 = D_2(C_3^2)$ there is a factorisation $m/w = w_1 w_2$ with $w_1, w_2 \in \mathbb{F}[W]_+^3$. Setting $u_3 = w w_1$ enforces $u_3 \in \mathbb{F}[W]_+^3$. Here $\deg(u_3) \leq D(A) = 5$ as otherwise $u_3 \in (\mathbb{F}[W]_+^n)^2$ and $m \in (\mathbb{F}[W]_+^n)^n \subseteq \mathbb{F}[W]_+^n \mathbb{F}[W]_+$ by Lemma 11 a contradiction. Therefore we cannot have $m_1 \mid u_3$, for then by A. we have $5 \leq \deg(m_1) \leq \deg(u_3) \leq 5$, so that $m_1 = u_3$ and $\Phi_c(m_1) = 1$ by 11, contradicting the assumption that $\Phi(u_3)$ is a zero-sum sequence over $A$. As a result there is a variable $y \mid m_1$ not dividing $u_3$, whence the claim.

C. For any divisor $v \mid m_1$ with $\deg(v) \leq 3$ and any monomial $v' \sim v$ there is a monomial $m' \in \mathbb{F}[W]$ such that $v' \mid m'$ and $m - m' \in \mathbb{F}[W]_+^n \mathbb{F}[W]_+$.

Let $v = xw$ and $v' = x^y w'$ where $\deg(w) \leq 2$, $w' \sim w$ and $y$ \in \langle b \rangle$. By induction on $\deg(v)$ assume that we already have a monomial $m'' \sim m$ such that $m'' - m \in \mathbb{F}[W]_+^n \mathbb{F}[W]_+$ and $x^t w' \mid m''$ for some $t \in \langle b \rangle$. According to B. there are factorisations $m'' = u_1 u_2 u_3$ with $u_1 \in \mathbb{F}[W]_+^3$ such that $w' \mid u_3$ and $y \mid u_1$ for some variable $y \in \mathbb{F}[V_1]$. We have two cases: i) If we can take $y = x^t$ in one of these factorisations then for $m' := u_1^{-t+q} u_2^{-t-q} u_3 \sim m''$ we have $m' - m'' \in \mathbb{F}[W]_+^n \mathbb{F}[W]_+$ by Lemma 12 so we are done as $v' \mid m'$. ii) Otherwise necessarily $x^t w' \mid u_3$ and $y \neq x^t$. Still however, there is an $h \in \langle b \rangle$ such that $\phi(y^h) = \phi(x^t)$ hence for $\tilde{m} := u_1^{-t+q} u_2^{-t-q} u_3 \sim m''$ we have $m - \tilde{m} \in \mathbb{F}[W]_+^n \mathbb{F}[W]_+$ by Lemma 12 and we obtain a factorisation $\tilde{m} = u_1 u_2 u_3$ falling under case i) by setting $\tilde{u}_1 = x^t u_1^h / y^h$, $\tilde{u}_2 = u_2^{-h}$, $\tilde{u}_3 = y^h u_3 / x^t$, so we are done again.

D. Now we proceed as in the proof of Proposition 15. For the sake of simplicity from now on we rename our variables so that $\mathbb{F}[V_1] = \mathbb{F}[x_1, y_1, z_1 : i = 1, \ldots, n_1]$. Moreover we abbreviate $\Delta_{i,t}$ as $\Delta_{s,t}$.

1) If we have $\deg(m_1) \geq 3$ for some $1 \leq i \leq n_1$ then we can apply C. with $v' := x_i y_i z_i \in \mathbb{F}[W]_+^3$, concluding that $m \in \mathbb{F}[W]_+^n \mathbb{F}[W]_+$, a contradiction.

2) Otherwise if $\deg(m_1) = 2$ for some $i$ then still there is a $j \neq i$ such that $\deg(m_1) \geq 1$. After an application of C. we may assume that $m$ is divisible by $x_j^2 x_j$. But then $m = \frac{1}{3} \Delta_{i,t}(\tilde{m})$ for the monomial $\tilde{m} := m x_i / x_j$ which falls under case 1) hence $m \in \mathbb{F}[W]_+^n \mathbb{F}[W]_+$ by 17, a contradiction.

3) Finally, if $\deg(m_1) = \ldots = \deg(m_1^{n_1}) = 1$ then after an application
of C. we may assume that $x_1y_2z_3 \mid m$. Now consider the relation:

$$\Delta^{1,2}(x_1y_1z_3) + \Delta^{2,3}(x_1y_2z_2) + \Delta^{3,1}(x_3y_2z_3) = 3x_1y_2z_3 + \tau(x_3y_2z_1) \quad (18)$$

After multiplying (18) with $m' := m/x_1y_2z_3$ we get on the left hand side

$$\Delta^{1,2}(my_1/y_2) + \Delta^{2,3}(mz_2/z_3) + \Delta^{3,1}(mx_3/x_1) \in F[W]_+ \oplus F[W]_+$$

by (17), as all the three monomials occurring here fall under case 2), and on the right hand side

$$\tau(x_3y_2z_1)m' \in F[W]_+ \oplus F[W]_+$$

whence $3m = 3x_1y_2z_3m' \in F[W]_+ \oplus F[W]_+$ follows. This contradiction completes our proof.

Now comparing Proposition [10] and [17] immediately gives:

**Corollary 18.** If $\text{char}(F) \neq 3$ then $\beta(H_3) = 9$.

**Remark 19.** It would be interesting to know if Theorem [2] also extends to the whole non-modular case, i.e. for any field $F$ whose characteristic does not divide $|G|$, just as it is the case for $p = 3$ by the above result.

**Acknowledgements**

The author is grateful to Máté Domokos for many valuable comments on the manuscript of this paper. He also thanks the anonymous referee for many suggestions to improve the presentation of this material.

**References**

[1] Y. Berkovich. *Groups of Prime Power Order*, volume I of de Gruyter Expositions in Mathematics. de Gruyter, Berlin, New York, 2008.

[2] K. Cziszter, M. Domokos. *Groups with large Noether bound*, Ann. de l’Institut Fourier 64:(3) pp. 909-944. (2014)

[3] K. Cziszter, M. Domokos. *The Noether number for the groups with a cyclic subgroup of index two*, Journal of Algebra 399: pp. 546-560. (2014)

[4] K. Cziszter, M. Domokos. *On the generalised Davenport constant and the Noether number*, Central European Journal of Mathematics 11:(9) pp. 1605-1615. (2013)

[5] K. Cziszter, M. Domokos, and A. Geroldinger. *The interplay of invariant theory with multiplicative ideal theory and with arithmetic combinatorics*, in: Scott T. Chapman, M. Fontana, A. Geroldinger, B. Olberding (Eds.), *Multiplicative Ideal Theory and Factorization Theory*, Springer-Verlag, 2016, pp. 43-95.
[6] K. Cziszter, M. Domokos and I. Szöllősi. *The Noether numbers and the Davenport constants of the groups of order less than 32*, arXiv:1702.02997.

[7] M. Domokos, P. Hegedűs. *Noether’s bound for polynomial invariants of finite groups* Arch. Math. (Basel) 74 (2000), no. 3, pp. 161-167.

[8] P. Fleischmann. *The Noether bound in invariant theory of finite groups*. Ad. Math., 156(1):23-32, 2000.

[9] J. Fogarty. *On Noether’s bound for polynomial invariants of a finite group*, Electron. Res. Announc. Amer. Math. Soc. 7 (2001), 57.

[10] M. Freeze, W. A. Schmid. *Remarks on a generalization of the Davenport constant* Discrete Math. 310, Issue 23, 6 December 2010, pp. 3373-3389

[11] A. Geroldinger and D. J. Grynkiewicz. *The large Davenport constant I: Groups with a cyclic index 2 subgroup*, J. Pure Appl. Algebra 217 (2013), 863-885.

[12] A. Geroldinger, F. Halter-Koch. *Non-unique factorizations. Algebraic, combinatorial and analytic theory*. Monographs and Textbooks in Pure and Applied Mathematics, Chapman & Hall/CRC, 2006.

[13] D. J. Grynkiewicz. *The large Davenport constant II: General upper bounds*, J. Pure Appl. Algebra 217 (2013), 222-2246.

[14] P. Hegedűs, A. Maróti and L. Pyber. *Finite groups with large Noether number are almost cyclic*, arXiv:1706.08290

[15] M. D. Neusel, L. Smith. *Invariant Theory of Finite Groups*. Mathematical Surveys and Monographs 94. Providence, R.I.: American Mathematical Society, 2002

[16] E. Noether. *Der Endlichkeitsssatz der Invarianten endlicher Gruppen*. Math. Ann., 77:89-92, 1916

[17] B. J. Schmid. *Finite groups and invariant theory*. In Malliavin M. P., editor, *Topics in invariant theory*, number 1478 in Lecture Notes in Mathematics, pages 35–66. Springer, 1989-90.

[18] J. P. Serre. *Representations linéaires des groupes finis*. Hermann, Paris, 1998.

[19] M. Sezer. *Sharpening the generalized Noether bound in the invariant theory of finite groups*. J. Algebra 254 (2002), no. 2, 252263.