NEWTON POLYGONS OF \( L \)-FUNCTIONS OF POLYNOMIALS

\( x^d + ax^{d-1} \) WITH \( p \equiv -1 \mod d \)

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Abstract. For prime \( p \equiv -1 \mod d \) and \( q \) a power of \( p \), we obtain the slopes of the \( q \)-adic Newton polygons of \( L \)-functions of \( x^d + ax^{d-1} \in \mathbb{F}_q[x] \) with respect to finite characters \( \chi \) when \( p \) is larger than an explicit bound depending only on \( d \) and \( \log_q p \). The main tools are Dwork’s trace formula and Zhu’s rigid transform theorem.

1. Main results

Let \( q = p^h \) be a power of the rational prime number \( p \). Let \( v \) be the normalized valuation on \( \mathbb{Q}_p \) with \( v(p) = 1 \). For a polynomial \( f(x) \in \mathbb{F}_q[x] \), let \( \hat{f} \in \mathbb{Z}_q[x] \) be its Teichmüller lifting. For a finite character \( \chi : \mathbb{Z}_p \to \mathbb{C}^\times \) of order \( p^m \chi \), define the \( L \)-function

\[
L^*(f, \chi, t) = \exp \left( \sum_{m=1}^{\infty} S_m^*(f, \chi) \frac{t^m}{m} \right),
\]

where \( S_m^*(f, \chi) \) is the exponential sum

\[
S_m^*(f, \chi) = \sum_{x \in \mu_{q^m-1}} \chi(\text{Tr}_{\mathbb{Q}_q/\mathbb{Q}_p} \hat{f}(x))
\]

and \( \mu_n \) is the group of \( n \)-th roots of unity. Then \( L^*(f, \chi, t) \) is a polynomial of degree \( p^m - 1 \) by Adolphson-Sperber [AS] and Liu-Wei [LWe]. We denote \( \text{NP}_q(f, \chi, t) \) the \( q \)-adic Newton polygon of \( L^*(f, \chi, t) \).

We fix a character \( \Psi_1 : \mathbb{Z}_p \to \mathbb{C}^\times \) of order \( p \), and denote \( L^*(f, t) = L^*(f, \Psi_1, t) \) and \( \text{NP}_q(f, t) = \text{NP}_q(f, \Psi_1, t) \). When \( p \equiv 1 \mod d \), it is well-known that \( \text{NP}_q(f, t) \) coincides the Hodge polygon with slopes \( \{i/d : 0 \leq i \leq d-1\} \).

Let \( a \) be a nonzero element in \( \mathbb{F}_q \). For \( f(x) = x^d + ax^s (s < d) \), Liu-Niu and Zhu obtained the slopes of \( \text{NP}_q(f, t) \) for \( p \) large enough under certain conditions in [LN2 Theorem 1.10] and [Z2], but these conditions are not so easy to check. For \( f(x) = x^d + ax \), Zhu, Liu-Niu and Ouyang-J. Yang obtained the slopes in [Z2 Theorem 1.1], [LN1 Theorem 1.10] and [OY Theorem 1.1], see also R. Yang [Y §1 Theorem] for earlier results.

In [DWX], Davis-Wan-Xiao gave a result on the behavior of the slopes of \( \text{NP}_q(f, \chi, t) \) when the order of \( \chi \) is large enough. In this way for \( p \) sufficiently large, they can obtain the slopes of \( \text{NP}_q(f, \chi, t) \) based on the slopes of \( \text{NP}_q(f, \chi_0, t) \) with \( \chi_0 \) a character of order \( p^2 \). In [N], Niu gave a lower bound of the Newton polygon \( \text{NP}_q(f, \chi, t) \). In [OY Theorem 4.3], Ouyang–Yang showed that if the Newton polygon of \( L^*(f, t) \) is sufficiently close to its Hodge polygon, the slopes of \( \text{NP}_q(f, \chi, t) \) for \( \chi \) in general follow from the slopes of \( \text{NP}_q(f, t) \). As a consequence

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they obtained the slopes of \( NP_q(x^d + ax, \chi, t) \) when \( p \) is bigger than an explicit bound depending only on \( d \) and \( h \).

Our main results are the following two theorems.

**Theorem 1.** Let \( f(x) = x^d + ax^{d-1} \) be a polynomial in \( \mathbb{F}_q[x] \) with \( a \neq 0 \). Let \( N(d) = \frac{d^2 + 3}{4} \) for \( q = p \) and \( \frac{d^2}{2} \) for general \( q \). If \( p \equiv -1 \mod d \) and \( p > N(d) \), the \( q \)-adic Newton polygon of \( L^*(f, t) \) has slopes

\[
\{ w_0, w_1, \ldots, w_{d-1} \},
\]

where

\[
w_i = \begin{cases} \frac{(p+1)i}{d(p-1)}, & \text{if } i < \frac{d}{2}; \\ \frac{(p+1)i-d}{d(p-1)}, & \text{if } i = \frac{d}{2}; \\ \frac{(p+1)i-2d}{d(p-1)}, & \text{if } i > \frac{d}{2}. \end{cases}
\]

**Remark.** (1) For general \( p \), write \( p_i = dk_i + r_i \) with \( 1 \leq i, r_i \leq d - 1 \). If \( r_i > s \) for any \( 1 \leq i \leq s \), then one can decide that the first \( s + 1 \) slopes of \( NP_q(f, t) \) are \( \{0, \frac{k_i+1}{p-1}, \ldots, \frac{k_s+1}{p-1} \} \) by our method for sufficiently large \( p \). For the rest of slopes, one needs to calculate the determinants of submatrices of “Vandermonde style” matrices.

(2) The slopes in our case coincide Zhu’s result in \([Z2]\).

**Theorem 2.** Assume \( f(x) \) and \( N(d) \) as above. For any non-trivial finite character \( \chi \), if \( p \equiv -1 \mod d \) and \( p > \max\{N(d), \frac{b(d-1)}{4d} + 1 \} \), the \( q \)-adic Newton polygon of \( L^*(f, \chi, t) \) has slopes

\[
\{ p^{1-mx}(i + w_j) : 0 \leq i \leq p^{m-1} - 1, 0 \leq j \leq d - 1 \}.
\]

2. **Preliminaries**

2.1. **Dwork’s trace formula.** We will recall Dwork’s work for \( f(x) = x^d + ax^{d-1} \). For general \( f \), one can see \([OY] \) \S 2.

Let \( \gamma \in \mathbb{Q}_p(\mu_p) \) be a root of the Artin-Hasse exponential series

\[
E(t) = \exp\left( \sum_{m=0}^{\infty} p^{-m} t^m \right)
\]

such that \( v(\gamma) = \frac{1}{p-1} \). Fix a \( \gamma^{1/d} \in \bar{\mathbb{Q}}_p \). Let

\[
\theta(t) = E(\gamma t) = \sum_{m=0}^{\infty} \gamma_m t^m
\]

be Dwork’s splitting function. Then \( v(\gamma_m) \geq m/(p-1) \), and \( \gamma_m = \gamma^m / m! \) for \( 0 \leq m \leq p - 1 \). Let

\[
F(x) = \theta(x^d)\theta(ax^{d-1}) = \sum_{i=0}^{\infty} F_i x^i,
\]

then

\[
F_i = \sum_{d m + (d-1)n = i} \gamma_m \gamma_n a^n.
\]

One can see \( m + n \geq i/d \) and \( v(F_i) \geq \frac{i}{d(p-1)} \).
Set $A_1 = (F_{p^j - j}^{\gamma(j-\ell)/d})_{i,j \geq 0}$. This is a nuclear matrix over $\mathbb{Q}_q(\gamma^{1/d})$ with
\[ v(F_{p^j - j}^{\gamma(j-\ell)/d}) \geq \frac{p^j - j}{d(p - 1)} + \frac{j - \ell}{d(p - 1)} = \frac{i}{d}. \]
We extend the Frobenius $\varphi$ to $\mathbb{Q}_q(\gamma^{1/d})$ with $\varphi(\gamma^{1/d}) = \gamma^{1/d}$.

**Theorem 3** (Dwork). Let $A_h = A_1 \varphi(A_1) \cdots \varphi^{h-1}(A_1)$. Then
\[ L^*(f, t) = \frac{\det \varphi^{-1}(I - tA_h)}{\det \varphi^{-1}(I - tqA_h)}. \]

2.2. **Zhu’s rigid transformation theorem.** Let $U_1 = (u_{ij})_{i,j \geq 0}$ be a nuclear matrix over $\mathbb{Q}_q(\gamma^{1/d})$. Then the Fredholm determinant $\det(I - tU_1)$ is well defined and $p$-adic entire (see [S]). Write
\[ \det(I - tU_1) = c_0 + c_1 t + c_2 t^2 + \cdots. \]
For $0 \leq t_1 < t_2 < \cdots < t_s$, denote by $U_1(t_1, \ldots, t_s)$ the principal sub-matrix consisting of $(t_i, t_j)$-entries of $U_1$ for $1 \leq i, j \leq s$. In particular, denote $U_1[s] = U_1(0, 1, \ldots, s - 1)$. Then we have $c_0 = 1$ and for $s \geq 1$,
\[ c_s = (-1)^s \sum_{0 \leq t_1 < t_2 < \cdots < t_s} \det U_1(t_1, t_2, \ldots, t_s). \]
Let $U_h = U_1 \varphi(U_1) \cdots \varphi^{h-1}(U_1)$. Write
\[ \det(I - tU_h) = C_0 + C_1 t + C_2 t^2 + \cdots. \]

**Theorem 4.** (See [Z1] Theorem 5.3.) Suppose $(\beta_s)_{s \geq 0}$ is a strictly increasing sequence such that
\[ \beta_i \leq v(a_{ij}) \text{ and } \lim_{s \to +\infty} \beta_s = +\infty. \]
If
\[ \sum_{s < i} \beta_s \leq v(\det U_1[i]) \leq \frac{\beta_i - \beta_{i-1}}{2} + \sum_{s < i} \beta_s \]
holds for every $1 \leq i \leq k$, then $v(C_i) = hv(\det U_1[i])$ for $1 \leq i \leq k$ and
\[ \text{NP}_q(\det(I - tA_h[k])) = \text{NP}_p(\det(I - tA_1[k])). \]

3. **Slopes of the Newton polygon of** $L^*(f, \chi, t)$

From now on, we assume $p \equiv -1 \mod d$ and write $p = dk - 1$.

3.1. **The case** $\chi = \Psi_1$.

**Lemma 5.** Let $M(s) = (a_{ij})_{1 \leq i, j \leq s}$ be an $s \times s$ matrix with entries
\[ a_{ij} = \frac{a^{i+j}}{(ki - i - j)(i + j)!}. \]
Then $v(\det M(s)) = 0$ for $1 \leq s \leq d - 1$.

**Proof.** Denote $x[0] = 1$ and $x[n] := x(x - 1) \cdots (x - n + 1)$ for $n \geq 1$. Then $x[n]$ is a polynomial of $x$ of degree $n$ and $\{ (x + j)[t] : 0 \leq t \leq j - 1 \}$ is a basis of the space of polynomials of degree $\leq j - 1$. Thus we can write
\[ ((k - 1)x - 1)[j - 1] = c_0(j) + \sum_{t=1}^{j-1} c_t(j) \cdot (x + j)[t]. \]
Let $x = -j$, we get
\[ c_0(j) = ((k-1)(-j)-1)[j-1] = ((1-k)j - 1)[j-1]. \]
For any $1 \leq u \leq j - 1$, 
\[ 1 \leq (k-1)j + u < kj \leq k(d-1) \leq p. \]
Hence $p \nmid (1-k)j - u$ and $v(c_0(j)) = 0$. 
Let $D = \text{diag}\{a, a^2, \ldots, a^s\}$ and $M' = (a'_{ij})_{1 \leq i,j \leq s}$ with $a'_{ij} = a_j a^{-i-j}$, then 
\[ M(s) = DM'D. \]  
Let $a''_{ij} := (ki - i - 1)!(i+s)!a'_{ij}$. Then 
\[ a''_{ij} = (ki - i - 1)[j-1] \cdot (i+s)[s-j] \]
\[ = \sum_{t=0}^{j-1} c_t(j) \cdot (i+j)[t] \cdot (i+s)[s-j], \]
\[ = \sum_{t=0}^{j-1} c_t(j) \cdot (i+s)[s-j+t] \]
\[ = \sum_{t=1}^{j} (i+s)[s-t] \cdot c_{j-t}(j). \]
Define $c_{j-t}(j) := 0$ for $j < t$. Write $M'' = (a''_{ij})_{1 \leq i,j \leq s}$, $M_1 = ((i+s)[s-t])_{1 \leq i,t \leq s}$ and $M_2 = (c_{j-t}(j))_{1 \leq i,t \leq s}$. Then 
\[ M'' = M_1 M_2. \]  
Write 
\[ x[n] = \sum_{t=0}^{n} c'_t(n)x^t, \]
then $c'_n(n) = 1$ and 
\[ (i+s)[s-j] = \sum_{t=0}^{s-j} c'_t(s-j)(i+s)^t. \]
Define $c'_t(n) := 0$ for $t > n$. Write $M_{11} = ((i+s)^{t-1})_{1 \leq i,t \leq s}$ and $M_{12} = (c'_{t-1}(s-t))_{1 \leq i,t \leq s}$. Then 
\[ M_1 = M_{11} M_{12}. \]  
Notice that $M_{11}$ is a Vandermonde matrix with determinant $\det M_{11} = \prod_{i=1}^{s} t^{s-t}$. One can also easily find 
\[ \det M_{12} = (-1)^{[s/2]} \quad \text{and} \quad \det M_2 = \prod_{i=1}^{s} c_0(i). \]  
Now by $[3]$, $(4)$, $(5)$ and $(6)$, 
\[ \det M(s) = a^{s(s+1)}(-1)^{[s/2]} \prod_{i=1}^{s} \frac{i^{s-i} c_0(i)}{(ki-i-1)! (i+s)!}. \]
Hence $v(\det M(s)) = 0$. \hfill \(\square\)

Denote $O(x)$ a number in $\mathbb{Q}_p$ with valuation $\geq v(x)$ for $x \in \mathbb{Q}_p$. 
Lemma 6. (i) For $i + j < d$, $F_{pi-j} = \gamma^{ki}(a_{ij} + O(\gamma))$.  
(ii) For $i + j \geq d$, $v(F_{pi-j}) = ki - 1$ and 

$$F_{pi-(d-i)} = \frac{\gamma^{ki-1}(1 + O(\gamma))}{(ki - 1)!}.$$ 

Proof. Let 

$$m = \begin{cases} ki - i - j, & \text{if } j < d - i; \\ ki - i - j + d - 1, & \text{if } j \geq d - i, \end{cases}$$ 

$$n = \begin{cases} i + j, & \text{if } j < d - i; \\ i + j - d, & \text{if } j \geq d - i. \end{cases}$$ 

Then $pi - j = dm + (d - 1)n$ and $0 \leq n \leq d - 1$. This lemma follows from 

$$F_{pi-j} = \sum_{l \geq 0} \gamma^{m-(d-1)l} \gamma^{n+dl} a^{n+dl} = \gamma^{m} \gamma^{n} a^{n}(1 + O(\gamma)) = \frac{\gamma^{m+n} a^{n}}{m! n!} (1 + O(\gamma)).$$

Proposition 7. For $1 \leq s \leq d - 1$, the valuation of $\det A_{s+1} = w_{0} + w_{1} + \cdots + w_{s}$.

Proof. Note that the first row of $A_{1}$ is $(1, 0, 0, \ldots)$. Let $A$ be the matrix by deleting the first row and column of $A_{s+1}$. Then $\det A_{s+1} = \det A$.

Let $D_{1} = \text{diag}(\gamma^{0/d}, \gamma^{1/d}, \ldots, \gamma^{s/d})$, $D_{2} = \text{diag}(\gamma^{1-k}, \gamma^{2k-1}, \ldots, \gamma^{(d-1)k-1})$ and $B[s] = (\gamma^{1-k} F_{pi-j})_{1 \leq i, j \leq s}$. Then $A = D_{1}^{-1} D_{2} B[s] D_{1}$. It suffices to compute $v(\det B[s])$.

Note that for $s = d - 1$, 

$$B[d-1] = \begin{pmatrix} \gamma a_{11} + O(\gamma^2) \cdots & \gamma a_{1,d-2} + O(\gamma^2) & \frac{1+O(\gamma)}{(k-1)!} \\ \vdots & \ddots & b_{2,d-1} \\ \gamma a_{d-2,1} + O(\gamma^2) & \cdots & \frac{1+O(\gamma)}{(k-1)!} b_{d-1,2} \cdots b_{d-1,d-1} \end{pmatrix}$$

with $v(b_{ij}) = 0$. If $1 \leq s \leq \frac{d-1}{2}$, then

$$B[s] = \begin{pmatrix} \gamma a_{11} + O(\gamma^2) \cdots & \gamma a_{1,s} + O(\gamma^2) \\ \vdots & \ddots & \vdots \\ \gamma a_{s,1} + O(\gamma^2) \cdots & \gamma a_{ss} + O(\gamma^2) \end{pmatrix}$$

has determinant 

$$\det B[s] = \gamma^{s} (\det M(s) + O(\gamma)).$$

The valuation of $\det B[s]$ is $sv(\gamma)$.

If $\frac{d}{2} \leq s \leq d - 1$, then

$$B[s] = \begin{pmatrix} B[d-1 - s] & P_{1} \\ P_{2} & Q \end{pmatrix}.$$ 

The valuation of any entry of $B[d-1 - s], P_{1}, P_{2}$ is $v(\gamma)$ and

$$Q \equiv \begin{pmatrix} 0 & \frac{1}{(k-1)!} \\ \frac{1}{(d-1)k-1} & \ast \end{pmatrix} \mod \gamma.$$
Thus $Q$ is invertible over the ring of integers of $\mathbb{Q}_p(\gamma)$. The determinant
\[
\det B[s] = \det Q \det(B[d-1-s] - P_1 Q^{-1} P_2) = \det Q \det B[d-1-s](1 + O(\gamma))
\]
has valuation $(d-1-s)\nu(\gamma)$.

Finally, $A = D_1^{-1} D_2 B[s] D_1$ has valuation
\[
\left( \sum_{i=1}^s (ki - 1) + \min\{s, d-1-s\} \right) \nu(\gamma) = w_0 + w_1 + \cdots + w_s. 
\]
\[ \square \]

**Proof of Theorem 4.** For $1 \leq s \leq d-1$, we have
\[
v(\det A_1[s+1]) = \sum_{i \leq s} w_i
\]
\[
= \begin{cases} 
\frac{s(s+1)}{2d} + \frac{s(s+1)}{2d}, & \text{if } s \leq (d-1)/2; \\
\frac{s(s+1)}{2d} + \frac{(d-s)(d-s-1)}{d(p-1)}, & \text{if } s \geq d/2;
\end{cases}
\]
\[
\leq \frac{s(s+1)}{2d} + \frac{d^2 - 1}{4d(p-1)}.
\]
If $p > \frac{d^2 + 3}{4}$, then $\frac{d^2 - 1}{4d(p-1)} < 1/d$. For $0 \leq t_0 < t_1 < \cdots < t_s$, assume $t_s \neq s$. Since
\[
v(F_{p^i-j\gamma}^{(j-i)/d}) \geq i/d,
\]
we have
\[
v(\det A_1[t_0, \ldots, t_s]) \geq \frac{s^2 + s + 2}{2d} > v(\det A_1[s+1]).
\]
Thus $v(C_{s+1}) = v(\det A_1[s+1]) = \sum_{i \leq s} w_i$ and \{\{w_0, w_1, \ldots, w_{d-1}\} are slopes of $NP_p(\det(I - t A_1))$.

If moreover $p > \frac{d^2}{2}$, then $p \geq \frac{d^2+1}{2}$ and $\frac{d^2-1}{4d(p-1)} \leq \frac{1}{2d}$. Choose $\beta_i = i/d$ in Theorem 4 we have
\[
v(C_{s+1}) = h(w_0 + w_1 + \cdots + w_s)
\]
and
\[
NP_p(\det(I - t A_h[d])) = NP_p(\det(I - t A_1[d])).
\]
Thus $w_0, w_1, \ldots, w_{d-1}$ are $q$-adic slopes of $NP_q(\det^{\psi^{-1}}(I - t A_h))$.

By Theorem 3
\[
\det^{\psi^{-1}}(I - t A_h) = L^*(f, t) \det^{\psi^{-1}}(I - t q A_h).
\]
Since the valuation of any entry of $A_h$ is $\geq 0$, the $q$-adic slopes of $\det^{\psi^{-1}}(I - t A_h)$ are $\geq 0$ and the $q$-adic slopes of $\det^{\psi^{-1}}(I - t q A_h)$ are $\geq 1$. Thus any $q$-adic slope of $\det^{\psi^{-1}}(I - t A_h)$ less than 1 must be a $q$-adic slope of $L^*(f, t)$. But $L^*(f, t)$ has degree $d$, hence $w_0, \ldots, w_{d-1}$ are all slopes of $L^*(f, t)$. \[ \square \]

### 3.2. The case for general $\chi$.

Let $f(x) \in \mathbb{F}_q[x]$ be a polynomial with degree $d$. Assume $p \nmid d$. Let $NP(f, x)$ be the piecewise linear function whose graph is the $q$-adic Newton polygon of $\det(I - t A_h)$. Let $HP(f, x)$ be the piecewise linear function whose graph is the polygon with vertices
\[
(k, \frac{k(k-1)}{2d}), \quad k = 0, 1, 2, \ldots.
\]
Then \( \text{NP}(f, x) \geq \text{HP}(f, x) \) (cf. [LW] [OY]). Set
\[
\text{gap}(f) = \max_{x \geq 0} \{\text{NP}(f, x) - \text{HP}(f, x)\}.
\]

**Theorem 8.** (See [OY] Theorem 4.3.) Let \( 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{d-1} < 1 \) denote the slopes of the \( q \)-adic Newton polygon of \( L^*(f, t) \). If \( \text{gap}(f) < 1/h \), then the \( q \)-adic Newton polygon of \( L^*(f, \chi, t) \) has slopes
\[
\{p^{1-m}(i + \alpha_j) : 0 \leq i \leq p^{m-1}-1, 0 \leq j \leq d-1 \}
\]
for any non-trivial finite character \( \chi \).

**Proof of Theorem 8.** The slopes of \( \text{NP}(f, x) \) are
\[
\{i + w_j : i \geq 0, 0 \leq j \leq d-1 \}.
\]
Notice that
\[
\sum_{i=0}^{d-1} w_i = \sum_{i=0}^{d-1} \frac{i}{d},
\]

\[
\text{NP}(f, x) - \text{HP}(f, x) \text{ is a periodic function with period } d. \text{ For } 0 \leq k < d, \]
\[
\text{NP}(f, x) - \text{HP}(f, x) \leq \sum_{i \leq (d-1)/2} \frac{2i}{d(p-1)} \leq \frac{d^2 - 1}{4d(p-1)}.
\]
If \( p > \frac{b(2d-1)}{4d} + 1 \), then \( \text{gap}(f) < 1/h \) and this concludes the proof. \( \square \)

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