Covariance Estimation under One-bit Quantization

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Abstract

We consider the classical problem of estimating the covariance matrix of a subgaussian distribution from i.i.d. samples in the novel context of coarse quantization, i.e., instead of having full knowledge of the samples, they are quantized to one or two bits per entry. This problem occurs naturally in signal processing applications. We introduce new estimators in two different quantization scenarios and derive non-asymptotic estimation error bounds in terms of the operator norm. In the first scenario we consider a simple, scale-invariant one-bit quantizer and derive an estimation result for the correlation matrix of a centered Gaussian distribution. In the second scenario, we add random dithering to the quantizer. In this case we can accurately estimate the full covariance matrix of a general subgaussian distribution by collecting two bits per entry of each sample. In both scenarios, our bounds apply to masked covariance estimation. We demonstrate the near-optimality of our error bounds by deriving corresponding (minimax) lower bounds and using numerical simulations.

1 Introduction

Estimating covariance matrices of high-dimensional probability distributions from a finite number of samples is a core problem in multivariate statistics with numerous applications in, e.g., financial mathematics, pattern recognition, signal processing, and signal transmission [14, 25, 34, 35]. In the classical problem setup, the task is to estimate the covariance matrix

\[ \Sigma = \mathbb{E}(XX^T) \in \mathbb{R}^{p \times p} \]

of a mean-zero random vector \( X \in \mathbb{R}^p \) using \( n \) i.i.d. samples \( X^1, \ldots, X^n \sim X \). The sample covariance matrix

\[ \hat{\Sigma}_n = \frac{1}{n} \sum_{k=1}^n X^k (X^k)^T \]

is a natural estimator as it converges a.s. to \( \Sigma \) for \( n \to \infty \). From a practical perspective, this asymptotic result is of limited use: it provides no information on the number of samples needed to guarantee \( \| \hat{\Sigma}_n - \Sigma \| < \varepsilon \), for a desired \( \varepsilon > 0 \), where the error is measured in an appropriate norm (the most common choices being the operator and Frobenius norms). In the last two decades, numerous works on the non-asymptotic analysis of covariance estimation showed that reliable approximation of \( \Sigma \) by \( \hat{\Sigma}_n \) becomes feasible for subgaussian distributions if \( n \gtrsim p \). For instance, if \( X \) has a Gaussian distribution, then it is well known (see e.g. [48]) that with probability at least \( 1 - 2e^{-t} \)

\[ \| \hat{\Sigma}_n - \Sigma \| \lesssim \| \Sigma \| \left( \sqrt{\frac{p+t}{n}} + \frac{p+t}{n} \right), \]

where \( a \lesssim b \) denotes \( a \leq Cb \), for a certain absolute constant \( C > 0 \). Similar error bounds can be obtained for heavy-tailed distributions using more sophisticated estimators [1, 2, 11, 33, 38, 44, 47]. The sufficient number of samples for reliable estimation can be further reduced under suitable priors on \( \Sigma \), such as sparsity, low-rankness, or Toeplitz-structure, see e.g. [6, 10, 12, 30, 31, 36]. In several concrete applications, the assumption that one has direct access to the samples \( X^k \) may be unrealistic. Especially in applications related to signal processing, samples are collected via sensors and hence need
Theorem 1. There exist absolute constants $c_1, c_2 > 0$ such that the following holds. Let $X \sim \mathcal{N}(0, \Sigma)$ with $\Sigma_{i,i} = 1$, for $i \in [p]$, and $X^1, \ldots, X^n \overset{i.i.d.}{\sim} X$. Let $M \in \{0, 1\}^{p \times p}$ be a fixed symmetric mask. If $t \geq 0$ and $n \geq c_1 \log^2(p)(\log(p)+t)$, then with probability at least $1 - 2e^{-c_2 t}$

$$
\left\| M \circ \bar{\Sigma}_n - M \circ \Sigma \right\| \lesssim \sigma(M \circ A) \sqrt{\frac{\log(p) + t}{n}} + \max \left\{ \|M \circ A\|, \|M \circ \Sigma\| \right\} \frac{\log(p) + t}{n},
$$

1.1 One-bit quantization

In the first quantization model, we have access to one-bit quantized samples $\text{sign}(X^k) \in \{-1, 1\}$, for $k \in [n]$. Here and in the rest of this work, standard functions are applied entry-wise to matrices. In this setting, the quantizer is scale-invariant, i.e., $\text{sign}(z) = \text{sign}(Dz)$ for any diagonal matrix $D \in \mathbb{R}^{p \times p}$ with strictly positive entries and $z \in \mathbb{R}^p$, and as a result we can only hope to recover the correlation matrix of the distribution. Hence, we assume in this scenario that $X \sim \mathcal{N}(0, \Sigma)$, where $\Sigma$ has ones on its diagonal.

As an estimator for $\Sigma$ we consider

$$
\bar{\Sigma}_n = \sin \left( \frac{\pi}{2n} \sum_{k=1}^n \text{sign}(X^k)\text{sign}(X^k)^T \right).
$$

Its specific form is motivated by Grothendieck’s identity (see, e.g., [48, Lemma 3.6.6]), also known as the “arcsin-law” in the engineering literature [26, 46], which implies that

$$
E \left( \text{sign}(X^k)\text{sign}(X^k)^T \right) = \frac{2}{\pi} \arcsin(\Sigma).
$$

Combined with the strong law of large numbers and the continuity of the sine function, this immediately shows that $\bar{\Sigma}_n$ is a consistent estimator of $\Sigma$.

Our first main result is a non-asymptotic error bound for masked estimation with $\bar{\Sigma}_n$.

Theorem 1. There exist absolute constants $c_1, c_2 > 0$ such that the following holds. Let $X \sim \mathcal{N}(0, \Sigma)$ with $\Sigma_{i,i} = 1$, for $i \in [p]$, and $X^1, \ldots, X^n \overset{i.i.d.}{\sim} X$. Let $M \in \{0, 1\}^{p \times p}$ be a fixed symmetric mask. If $t \geq 0$ and $n \geq c_1 \log^2(p)(\log(p)+t)$, then with probability at least $1 - 2e^{-c_2 t}$

$$
\left\| M \circ \bar{\Sigma}_n - M \circ \Sigma \right\| \lesssim \sigma(M \circ A) \sqrt{\frac{\log(p) + t}{n}} + \max \left\{ \|M \circ A\|, \|M \circ \Sigma\| \right\} \frac{\log(p) + t}{n},
$$
where
\[ A = \cos(\arcsin(\Sigma)), \quad \Gamma = \mathbb{E} \left( \text{sign}(X^k) \text{sign}(X^k)^T \right) = \frac{2}{\pi} \arcsin(\Sigma), \]
and \( \sigma(Z) \) is defined for symmetric \( Z \) by
\[ \sigma(Z)^2 := Z \odot \Gamma - (Z \odot \Gamma)^2 = \frac{2}{\pi} Z \odot \arcsin(\Sigma) - \frac{4}{\pi^2} (Z \odot \arcsin(\Sigma))^2. \]

The first term on the right hand side of (6), featuring the factor \( \|\sigma(M \odot A)\| \), is rather uncommon in view of existing covariance estimation results, cf. (2), but not a proof artefact: in Proposition 14 we derive a lower bound for the expected squared error, which verifies that this term is indeed the leading error term. Its appearance suggests that the estimator may outperform the sample covariance matrix in cases where the coordinates of \( X \) are very strongly correlated. Indeed, since \( A_{i,j} = (1 - \Sigma^2_{i,j})^{1/2} \), the first term completely vanishes in the extreme case of full correlation (\( \Sigma = 1 \)) and the second term, in this case equal to \( \frac{\log(p) + \ell}{n} \), becomes dominant. In the numerical experiments in Section 5 we verify this intuition. Finally, let us note that our estimator achieves the minimax rate up to the factor \( \log(p) \) per vector entry.

We prove Theorem 1 by writing \( M \odot \Sigma_n - M \odot \Sigma \) in the form
\[ f \left( \frac{1}{n} \sum_{k=1}^{n} W_k \right) - f \left( \mathbb{E} \left( \frac{1}{n} \sum_{k=1}^{n} W_k \right) \right), \tag{7} \]
where \( W_k \in \mathbb{R}^{d \times p} \) are independent random matrices and \( f \) is real analytic. By applying a Taylor series expansion, the latter can be written as
\[ \sum_{\ell=0}^{\infty} C_{\ell} \odot \left( \frac{1}{n} \sum_{k=1}^{n} (W_k - \mathbb{E}W_k) \right)^{\odot \ell}, \]
where \( Z^{\odot \ell} = Z \odot \cdots \odot Z \) denotes the \( \ell \)-th Hadamard power of a matrix \( Z \). The main technical work is to estimate the terms of this series and we refer to the start of Section 2 for a more detailed proof sketch. Along the way we develop several tools that will be useful to estimate the operator norm of (7) for other real analytic \( f \) and random matrices \( W_k \) than the ones considered here.

**Remark 2.** The estimator \( M \odot \Sigma_n \) may not be positive semidefinite. Indeed, this is already true in the unmasked case (\( M = I \)) as applying the sine function element-wise to a matrix may fail to preserve positive semidefiniteness [43]. In practice, it may therefore be preferable to use \( P_{PSD}(M \odot \Sigma_n) \) as an alternative estimator, where \( P_{PSD} \) denotes the projection onto the positive semidefinite cone in terms of the operator norm. This positive semidefinite projection can easily be computed by performing an SVD, see [9, Section 8.1.1] for details. Since any convex projection is \( 1 \)-Lipschitz, Theorem 1 yields the same estimate for the error \( \|P_{PSD}(M \odot \Sigma_n) - \Sigma\| \) of this alternative estimator.

### 1.2 Quantization with dithering

In the quantization model studied above, we were only able to estimate correlation matrices of Gaussian distributions. In our second quantization model, we aim to estimate the full covariance matrix of any subgaussian distribution. To achieve this, we introduce dithering in the one-bit quantizer, i.e., we add random noise with a suitably chosen distribution to the samples before quantizing them. The insight that dithering can substantially improve the reconstruction from quantized observations goes back to the work of Roberts [41] in the engineering literature (see also [23, 24]). In the context of one-bit compressed sensing, the effect of dithering was recently rigorously analyzed in [5, 19, 20, 29, 32].

To be able to reconstruct the full covariance matrix we collect two bits per entry of each sample vector and use independent uniformly distributed dithers when quantizing. Concretely, we have access to the quantized sample vectors
\[ \text{sign}(X^k + \tau^k), \text{sign}(X^k + \bar{\tau}^k), \quad k = 1, \ldots, n, \]
where the dithering vectors $\tau^1, \tau^1, \ldots, \tau^n$ are independent and uniformly distributed in $[-\lambda, \lambda]^p$, with $\lambda > 0$ to be specified later. Using these quantized observations, we construct the estimator

$$
\hat{\Sigma}_{n}^{\text{dith}} = \frac{1}{2} \hat{\Sigma}'_n + \frac{1}{2} (\hat{\Sigma}'_n)^T
$$

where

$$
\hat{\Sigma}'_n = \frac{\lambda^2}{n} \sum_{k=1}^{n} \text{sign}(X^k + \tau^k) \text{sign}(X^k + \tau^k)^T.
$$

Our second main result quantifies the performance of $\hat{\Sigma}_{n}^{\text{dith}}$ for masked covariance estimation of the full covariance matrix. Below we write $a \lesssim_K b$ to abbreviate $a \leq C_K b$ for a constant $C_K$ depending only on $K$.

**Theorem 3.** Let $X$ be a mean-zero, $K$-subgaussian vector with covariance matrix $\Sigma$. Let $X^1, \ldots, X^n \overset{i.i.d.}{\sim} X$. Let $M \in [0,1]^{p \times p}$ be a fixed symmetric mask. If $\lambda^2 \gtrsim_K \log(n) \|\Sigma\|_\infty$, then with probability at least $1 - e^{-t}$,

$$
\left\| M \odot \hat{\Sigma}_{n}^{\text{dith}} - M \odot \Sigma \right\| \lesssim_K \|M\|_{1 \to 2} (\log(n) \|\Sigma\|_1^{1/2} + \lambda^2) \sqrt{\frac{\log(p) + t}{n}} + \lambda^2 \|M\| \frac{\log(p) + t}{n}.
$$

In particular, if $\lambda^2 \gtrsim_K \log(n) \|\Sigma\|_\infty$

$$
\left\| M \odot \hat{\Sigma}_{n}^{\text{dith}} - M \odot \Sigma \right\| \lesssim_K \log(n) \|M\|_{1 \to 2} \sqrt{\frac{\|\Sigma\| \log(p) + t}{n}} + \log(n) \|M\| \frac{\log(p) + t}{n}.
$$

A remark analogous to Remark 2 can be made in the context of Theorem 3: $M \odot \hat{\Sigma}_{n}^{\text{dith}}$ is not positive semidefinite in general and hence it may be preferable in practice to use $P_{\text{PSD}}(M \odot \hat{\Sigma}_{n}^{\text{dith}})$ as an estimator. Theorem 3 immediately yields performance guarantees for this alternative estimator.

Surprisingly, the error bound (10) matches the minimax rate up to logarithmic factors (see Appendix A) and is of the same form (up to different logarithmic factors) as the best known estimate for the masked sample covariance matrix in (3), even though the sample covariance matrix requires direct access to the “unquantized” samples $X^k$. Nevertheless, the performance of our estimator based on two-bit samples can be significantly worse than the performance of the sample covariance matrix, as can be seen more clearly in the ‘unmasked’ case (i.e., $M = 1$). Indeed, in [33] it was shown that if the samples $X^k$ are Gaussian, then

$$
\mathbb{E}[\hat{\Sigma}_{n} - \Sigma] \simeq \sqrt{\frac{\|\Sigma\| \text{Tr}(\Sigma)}{n}} + \frac{\text{Tr}(\Sigma)}{n},
$$

whereas (10) yields

$$
\mathbb{E}[\hat{\Sigma}_{n}^{\text{dith}} - \Sigma] \lesssim \log(n) \sqrt{\frac{p\|\Sigma\| \|\Sigma\|_\infty \log(p)}{n}} + \log(n) \frac{p\|\Sigma\|_\infty \log(p)}{n}
$$

via tail integration. The latter estimate is clearly worse as $\text{Tr}(\Sigma) \leq p\|\Sigma\|_\infty$. In our numerical experiments we show that this difference is not a proof artefact, but rather a result of the distortion produced by the coarse quantization of the samples: whereas $\hat{\Sigma}_{n}$ and $\hat{\Sigma}_{n}^{\text{dith}}$ perform similarly for covariance matrices with a constant diagonal, $\hat{\Sigma}_{n}$ is observed to perform significantly better in situations where $\text{Tr}(\Sigma) \ll p\|\Sigma\|_\infty$.

The first part of the proof of Theorem 3 is to control the bias of the estimator $M \odot \hat{\Sigma}_{n}^{\text{dith}}$ using the tuning parameter $\lambda$. The proof can afterwards be finished using the matrix Bernstein inequality. This proof strategy easily carries over to heavier-tailed random vectors. Nevertheless, the number of samples then increases since $\lambda$ has to be chosen larger. For instance, if $X$ is a $K$-subexponential random vector, one would already need $\lambda^2 \gtrsim_K \log(n)^2 \|\Sigma\|_\infty$. The dependence of $\lambda$ on $n$, both in the latter statement and Theorem 3, is not an artifact of proof and observable in numerical experiments, see Section 5.
1.3 Organization and notation

Section 2 is devoted to the proof of Theorem 1. In Section 3 we develop a corresponding lower bound for the estimation error. In Section 4 we prove Theorem 3. We conclude with numerical experiments in Section 5.

A discussion on minimax bounds in the quantized setting is deferred to the Appendix.

We write $|n| = \{1, \ldots, n\}$ for $n \in \mathbb{N}$. We use the notation $a \lesssim b$ (resp. $\gtrsim a$) to abbreviate $a \leq C_n b$ (resp. $C_n b$), for a constant $C_n > 0$ depending only on $a$. Similarly, we write $a \lesssim b$ if $a \leq C b$ for an absolute constant $C > 0$. We write $a \simeq b$ if both $a \lesssim b$ and $b \lesssim a$ hold (with possibly different implicit constants). Whenever we use absolute constants $c, C > 0$, their values may vary from line to line. We denote the all ones-matrix by $1 \in \mathbb{R}^{p \times p}$. We let scalar-valued functions act component-wise on vectors and matrices. In particular,

$$[\text{sign}(x)]_i = \begin{cases} 1 & \text{if } x_i \geq 0 \\ -1 & \text{if } x_i < 0, \end{cases}$$

for all $x \in \mathbb{R}^p$ and $i \in [p]$. The only exception is $|Z| := (Z^T Z)^{\frac{1}{2}}$, for a matrix $Z \in \mathbb{R}^{p_1 \times p_2}$. For symmetric $W, Z \in \mathbb{R}^{p \times p}$ we write $W \preceq Z$ if $Z - W$ is positive semidefinite. We will use that

$$\|\mathbb{E}Z\|^2 \leq \mathbb{E}|Z|^2 \quad \text{for any } Z \in \mathbb{R}^{p_1 \times p_2},$$

which is immediate from $0 \leq \mathbb{E}|Z - \mathbb{E}Z|^2$, and refer to it as Kadison’s inequality (as it is a special case of Kadison’s inequality in noncommutative probability theory).

For $Z \in \mathbb{R}^{p \times p}$, we denote the operator norm (maximum singular value) by $\|Z\| = \sup_{u \in \mathbb{R}^{p \times 1}} \|Zu\|_2$, the max norm by $\|Z\|_\infty = \max_{i,j} |Z_{i,j}|$, and the maximum column norm by $\|Z\|_{1 \rightarrow 2} = \max_{j \in [p]} \|z_j\|_2$ where $z_j$ denotes the $j$-th column of $Z$. We denote the Hadamard (i.e., entry-wise) product of two matrices by $\odot$ and define

$$Z^{\odot k} = Z \odot \cdots \odot Z, \quad \ell\text{-times}.$$

The diag-operator, when applied to a matrix, extracts the diagonal as a vector; when applied to a vector, it outputs the corresponding diagonal matrix. The subgaussian ($\psi_2$-) and subexponential ($\psi_1$-) norms of a random variable $X$ are defined by

$$\|X\|_{\psi_2} = \inf \left\{ t > 0 : \mathbb{E} \left( \exp \left( \frac{X^2}{2t^2} \right) \right) \leq 2 \right\} \quad \text{and} \quad \|X\|_{\psi_1} = \inf \left\{ t > 0 : \mathbb{E} \left( \exp \left( \frac{|X|}{t} \right) \right) \leq 2 \right\}.$$

A mean-zero random vector $X$ in $\mathbb{R}^p$ is called $K$-subgaussian if

$$\|\langle X, x \rangle\|_{\psi_2} \leq K(\mathbb{E}(X,x)^2)^{1/2} \quad \text{for all } x \in \mathbb{R}^p.$$

2 Proof of Theorem 1

Throughout our exposition, we write $Y_{i,j} = \text{sign}(X_i)\text{sign}(X_j)$, $Y_{i,j}^k = \text{sign}(X_i^k)\text{sign}(X_j^k)$, $Y = \text{sign}(X)$, and $Y^k = \text{sign}(X^k)$ so that the matrices $YY^T$ and $Y^k(Y^k)^T$ have entries $Y_{i,j}$ and $Y_{i,j}^k$. By using the Taylor series expansion

$$\sin(x) = \sin(a) + \cos(a)(x - a) - \frac{\sin(a)}{2}(x - a)^2 - \frac{\cos(a)}{6}(x - a)^3 + \cdots,$$

for $x = \frac{\pi}{2n}\sum_{k=1}^n Y_{i,j}^k$ and $a = \frac{\pi}{2n}\mathbb{E}Y_{i,j} = \arcsin(\Sigma_{i,j})$ (see (5) for the latter identity), we find

$$\begin{align*}
(\Sigma_n)_{i,j} - \Sigma_{i,j} &= \cos \left( \frac{\pi}{2n}\mathbb{E}Y_{i,j} \right) \left( \frac{\pi}{2n} \sum_{k=1}^n (Y_{i,j}^k - \mathbb{E}Y_{i,j}) \right) - \frac{\sin \left( \frac{\pi}{2n}\mathbb{E}Y_{i,j} \right)}{2} \left( \frac{\pi}{2n} \sum_{k=1}^n (Y_{i,j}^k - \mathbb{E}Y_{i,j}) \right)^2 \\
&\quad - \frac{\cos \left( \frac{\pi}{2n}\mathbb{E}Y_{i,j} \right)}{6} \left( \frac{\pi}{2n} \sum_{k=1}^n (Y_{i,j}^k - \mathbb{E}Y_{i,j}) \right)^3 + \cdots.
\end{align*}$$

(12)
First note that \( \cos \left( \frac{\pi}{n} \Xi_{i,j} \right) = \cos(\text{arcsin}(\Sigma_{i,j})) = \sqrt{1 - \Sigma_{i,j}^2} = A_{i,j} \) and likewise \( \sin \left( \frac{\pi}{n} \Xi_{i,j} \right) = \Sigma_{i,j} \), for all \( i, j \in [p] \). Let us define the random matrix \( B \in \mathbb{R}^{p \times p} \) with entries \( B_{i,j} = \frac{\pi}{n} \sum_k (Y_{i,j}^k - E Y_{i,j}) \), for all \( i, j \in [p] \), and note that

\[
B = \frac{\pi}{2n} \sum_{k=1}^n B_k, \quad \text{where} \quad B_k = Y^k(Y^k)^T - E (Y^k(Y^k)^T).
\] (13)

With this notation, the Taylor expansion (12) yields

\[
\hat{\Sigma}_n - \Sigma = A \odot B - \Sigma \odot \frac{1}{2} B^\odot 2 - A \odot B^\odot 3 + \cdots
\]

\[
= \sum_{\ell=0}^\infty (-1)^\ell \left( A \odot B - \frac{1}{2(\ell + 1)} \Sigma \odot B^\odot 2 \right) \odot \frac{1}{(2\ell + 1)!} B^\odot 2\ell.
\] (14)

Before delving into the formal proof of Theorem 1, let us first sketch the steps that we will take. Each term in the series expansion (14) has the form of a ‘Hadamard chaos’

\[
C \odot B^\odot \theta = \sum_{k_1, \ldots, k_\theta = 1}^n C \odot B_{k_1} \odot \cdots \odot B_{k_\theta}.
\] (15)

We separately consider the cases \( \theta > \log(p) \) and \( \theta \leq \log(p) \). In the first case, it suffices to make the trivial estimate \( \|C \odot B^\odot \theta\| \leq \|C\| \|B^\odot \theta\|_{1 \to 2} \) (see Lemma 4) and to estimate the right hand side via Bernstein’s inequality. In the harder case \( \theta \leq \log(p) \) we will first show, using Lemma 5, that (15) can be expressed as a sum of terms of the form

\[
\sum_{k_1 \neq \ldots \neq k_{\theta'}} C' \odot B_{k_1} \odot \cdots \odot B_{k_{\theta'}},
\] (16)

where \( \theta' \leq \theta \) and \( k_1 \neq \cdots \neq k_{\theta'} \) means that we only sum terms for which all indices are different. Next, we show that terms of this form can be decoupled, i.e., we can replace \( B_{k_1}, \ldots, B_{k_{\theta'}} \) by copies \( B_{k_1}^{(1)}, \ldots, B_{k_{\theta'}}^{(\theta')} \) that are independent of each other (Lemma 6). The resulting decoupled ‘Hadamard chaoses’ can be estimated by iteratively applying the matrix Bernstein inequality (Lemma 7). By using these steps and doing careful bookkeeping we arrive at a suitable estimate in Lemma 10.

Let us now formally collect all ingredients for the proof. The first is a tool to control the operator norm of Hadamard products of matrices. It is a simple consequence of Schur’s product theorem [28, Eq. (3.7.11)].

**Lemma 4.** Let \( A, B \in \mathbb{R}^{p \times p} \), let \( A \) be symmetric and define \( C = |A| = (A^T A)^{\frac{1}{2}} \). Then

\[
\|A \odot B\| \leq \left( \max_{i \in [p]} C_{ii} \right) \|B\| = \|A\|_{1 \to 2} \|B\|.
\]

If \( A \) is in addition positive semidefinite, then

\[
\|A \odot B\| \leq \left( \max_{i \in [p]} A_{ii} \right) \|B\|.
\]

The next observation allows to reduce the problem of estimating a Hadamard chaos to estimating terms of the form (16).

**Lemma 5.** Let \( \Gamma = \frac{2}{n} \text{arcsin}(\Sigma) \) and define

\[
\Phi = 1 - \Gamma^\odot 2 \quad \text{and} \quad \Psi = -2\Gamma.
\]

Define the sequences \( (\Phi_n)_{n \geq 1} \) and \( (\Psi_n)_{n \geq 1} \) recursively by setting \( \Phi_1 = 0, \Psi_1 = 1, \) and

\[
\Phi_n = \Phi \odot \Psi_{n-1}, \quad \Psi_n = \Phi_{n-1} + \Psi \odot \Psi_{n-1},
\]

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for \( n \geq 2 \). Then, for \( B_k \) as defined in (13) and any \( n \geq 1 \),
\[
B_k^\otimes n = \Phi_n + \Psi_n \odot B_k.
\]
Moreover, for any symmetric \( C \in \mathbb{R}^{p \times p} \),
\[
\max \{ \| \Phi_n \odot C \|, \| \Psi_n \odot C \| \} \leq \sqrt{2 \| \Phi \|_\infty^2 (n-1) \| C \|} \leq 4^{n-1} \| C \|.
\]

**Proof.** Note that \( \Phi_2 = \Phi \) and \( \Psi_2 = \Psi \) and thus, by (5),
\[
B_k^\otimes 2 = (Y_k(Y_k)^T)^{\otimes 2} - 2Y_k(Y_k)^T \odot (E(Y_k(Y_k)^T) + (E(Y_k(Y_k)^T))^\odot 2
\]
\[= 1 - 4(1 + \pi \arcsin(\Sigma)) \odot B_k - 4 \pi \arcsin(\Sigma)^\odot 2 = \Phi + \Psi \odot B_k,
\]
i.e., the asserted formula holds for \( n = 2 \). Furthermore, if it holds for \( n - 1 \), then
\[
B_k^\otimes n = B_k^\otimes (n-1) \odot B_k = [\Phi_{n-1} + \Psi_{n-1} \odot B_k] \odot B_k = \Phi_{n-1} \odot B_k + \Psi_{n-1} \odot [\Phi + \Psi \odot B_k]
\]
\[= \Phi_n + \Psi_n \odot B_k.
\]
The result therefore follows by induction. To prove the final statement, we write \( c_n = \| \Psi_n \odot C \| \), so in particular \( c_1 = \| C \| \). Since \( \Gamma \) is the covariance matrix of \( \text{sign}(X) \) (see (5)), it is positive semidefinite. Lemma 4 together with \( 2x \leq (1 + x)^2 \), for \( x \in \mathbb{R} \), therefore yields
\[
\| \Psi_2 \odot C \| = 2\| \Gamma \odot C \| \leq 2\| \Gamma \|_\infty \| C \| \leq (1 + \| \Gamma \|_\infty)^2 \| C \|.
\]
Moreover, for any \( n \geq 3 \), by the triangle inequality and Lemma 4
\[
c_n = \| \Psi_n \odot C \| = \| \Phi \odot \Psi_{n-2} \odot C + \Psi \odot \Psi_{n-1} \odot C \|
\leq \| \Phi \odot \Psi_{n-2} \odot C \| + \| \Gamma^{\otimes 2} \odot \Psi_{n-2} \odot C \| + 2 \| \Gamma \odot \Psi_{n-1} \odot C \|
\leq (1 + \| \Phi \|_\infty^2) c_{n-2} + 2 \| \Gamma \|_\infty c_{n-1} \leq (1 + \| \Gamma \|_\infty^2) \max \{ c_{n-2}, c_{n-1} \}.
\]
By induction, it immediately follows that, for any \( n \geq 1 \),
\[
c_n \leq (1 + \| \Gamma \|_\infty)^{2(n-1)} \| C \|.
\]
As a consequence, for any \( n \geq 1 \),
\[
\| \Phi_n \odot C \| = \| \Phi \odot \Psi_{n-1} \odot C \| \leq (1 + \| \Gamma \|_\infty^2) c_{n-1} \leq (1 + \| \Gamma \|_\infty^2) (1 + \| \Gamma \|_\infty)^{2(n-2)} \| C \|
\leq (1 + \| \Gamma \|_\infty)^{2(n-1)} \| C \|,
\]
which concludes the proof.

Thanks to Lemma 5, we can reduce the terms in (15) to a sum of terms of the form (16), in which all indices are different. This allows us to make use of the following decoupling inequality. Let us note that a similar decoupling result, where the summation on the right hand side of (17) is restricted to terms with different indices, is a consequence of a general result of de la Peña (see [15, Theorem 2]). Our proof is a straightforward generalization of the proof of the well-known decoupling inequality for a second order chaos, due to Bourgain and Tzafirri [8].

**Lemma 6.** Fix \( C \in \mathbb{R}^{p \times p} \) and let \( B_1, \ldots, B_n \) be independent, mean-zero random matrices in \( \mathbb{R}^{p \times p} \). Let \( F \colon \mathbb{R}^{p \times p} \to \mathbb{R} \) be a convex function, let \( \theta \geq 2 \), and let \( (B_k^{(1)})_{k=1}^n, \ldots, (B_k^{(\theta)})_{k=1}^n \) be independent and identically distributed with \( (B_k)_k \). Then, for any \( J \subset [n] \) with \( |J| \geq \theta \),
\[
\mathbb{E} F \left( \sum_{k_1 \neq \ldots \neq k_0 \in J} C \odot B_{k_1} \odot \cdots \odot B_{k_0} \right) \leq \mathbb{E} F \left( C_\theta \sum_{k_1, \ldots, k_0 \in J} C \odot B_{k_1}^{(1)} \odot \cdots \odot B_{k_0}^{(\theta)} \right),
\]
where the sum on the left-hand side is over all tuples consisting of \( \theta \) indices which all differ from each other and \( C_\theta = \theta! \Pi_{k=1}^\theta \left( \frac{k}{\theta-1} \right) \leq \theta! e^\theta \).
Proof. We show the claim via induction. For $\theta = 1$, it trivially holds, for any $J \subset [n]$ and $C_1 = 1$. Let us now assume that the claim holds for $\theta - 1$ and any $J \subset [n]$. We show that the claim holds as well for $\theta$. Fix any $J = \{j_1, \ldots, j_{|J|}\} \subset [n]$. We introduce $|J|$ i.i.d. Bernoulli random variables $\delta = (\delta_{j_1}, \ldots, \delta_{j_{|J|}})$ with $\mathbb{P}[\delta_{j_i} = 1] = 1 - \mathbb{P}[\delta_{j_i} = 0] = c$ and choose $c$ such that, for any $k_1 \neq \cdots \neq k_\theta \in J$, $\mathbb{E}(\delta_{k_1} \cdots \delta_{k_{\theta-1}}(1 - \delta_{k_\theta})) = (1-c)^{\theta-1}$ is maximal, i.e., we set $c = \frac{\theta-1}{\theta}$. By Jensen’s inequality and Fubini’s theorem,

$$
\mathbb{E} \left( \sum_{k_1 \neq \cdots \neq k_{\theta} \in J} C \circ B_{k_1} \cdots \circ B_{k_{\theta}} \right) = \mathbb{E}_B \mathbb{E} \left( \theta \left( \frac{\theta}{\theta-1} \right)^{\theta-1} \sum_{k_1 \neq \cdots \neq k_{\theta} \in J} \mathbb{E}_\delta \left( \delta_{k_1} \cdots \delta_{k_{\theta-1}}(1 - \delta_{k_\theta}) \right) \cdot C \circ \bigcirc_{i=1}^\theta B_{k_i} \right) \\
\leq \mathbb{E}_B \mathbb{E}_B \mathbb{E} \left( \theta \left( \frac{\theta}{\theta-1} \right)^{\theta-1} \sum_{k_1 \neq \cdots \neq k_{\theta-1} \in J} \sum_{k_\theta \in J \setminus J_{k_\theta}} \mathbb{C} \circ \bigcirc_{i=1}^{\theta-1} B_{k_i} \circ B_{(k_\theta)}^{(i)} \right),
$$

where we denote $J_\delta = \{j \in J : \delta_j = 1\}$. Hence, there must exist a realization $\delta^* \equiv \delta$ such that

$$
\mathbb{E} \left( \sum_{k_1 \neq \cdots \neq k_{\theta} \in J} C \circ B_{k_1} \circ \cdots \circ B_{k_{\theta}} \right) \\
\leq \mathbb{E}_{B^{(1)}, \ldots, B^{(\theta)}} F \left( \theta \left( \frac{\theta}{\theta-1} \right)^{\theta-1} \sum_{k_1 \neq \cdots \neq k_{\theta-1} \in J} \sum_{k_\theta \in J \setminus J_{k_\theta}} \mathbb{C} \circ \bigcirc_{i=1}^{\theta-1} B_{k_i} \circ B_{(k_\theta)}^{(i)} \right).
$$

with $I := J_\delta$. Applying the induction hypothesis, we find

$$
\mathbb{E} \left( \sum_{k_1 \neq \cdots \neq k_{\theta} \in J} C \circ B_{k_1} \circ \cdots \circ B_{k_{\theta}} \right) \\
\leq \mathbb{E}_{B^{(1)}, \ldots, B^{(\theta)}} \mathbb{E}_B \left( \theta \left( \frac{\theta}{\theta-1} \right)^{\theta-1} \sum_{k_1 \neq \cdots \neq k_{\theta-1} \in J} \sum_{k_\theta \in J \setminus J_{k_\theta}} \mathbb{C} \circ \bigcirc_{i=1}^{\theta-1} B_{k_i} \circ B_{(k_\theta)}^{(i)} \right) \\
\leq \mathbb{E}_{B^{(1)}, \ldots, B^{(\theta)}} \left( C_\theta \sum_{k_1, \ldots, k_{\theta-1} \in J} \sum_{k_\theta \in J \setminus J_{k_\theta}} \mathbb{C} \circ \bigcirc_{1}^{\theta-1} B_{k_i}^{(i)} + \sum_{k_\theta \in J} \mathbb{C} \circ \bigcirc_{1}^{\theta-1} B_{k_i}^{(i)} \circ \mathbb{E}_{B^{(\theta-1)}} B_{k_{\theta-1}}^{(\theta-1)} \right) \\
\leq \mathbb{E} \left( C_\theta \sum_{k_1, \ldots, k_{\theta} \in J} \mathbb{C} \circ \bigcirc_{1}^{\theta} B_{k_1}^{(1)} \cdots \circ B_{k_{\theta}}^{(\theta)} \right),
$$

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where we use the induction hypothesis in the second inequality to replace \( B_{k_1}, \ldots, B_{k_{\theta-1}} \) by \( B_{k_1}^{(1)}, \ldots, B_{k_{\theta-1}}^{(1)} \) and then apply \( \theta \) consecutive steps of adding terms \( \mathbb{E}B_{k_i}^{(i)} = 0 \) and applying Jensen’s inequality. The estimate of \( C_{\theta} \) is a consequence of \((1 + \frac{1}{k})^k \leq c\) being monotonically increasing with limit \( e \). \( \blacksquare \)

The decoupled terms can be estimated by the following lemma. Recall that \( \sigma(Z) \) is defined for symmetric \( Z \) by

\[
\sigma(Z)^2 := Z^2 \circ \Gamma - (Z \circ \Gamma)^2.
\]

**Lemma 7.** Let \( X \sim \mathcal{N}(0, \Sigma) \) and \( X_1, \ldots, X_n \) i.i.d. \( X \). Define \((B_k)_{k=1}^n \) as in (13) and let \((B^{(1)}_k)_{k=1}^n, \ldots, (B^{(\theta)}_k)_{k=1}^n \) be independent and identically distributed with \((B_k)_{k=1}^n \). There exists an absolute constant \( c > 0 \) such that for any \( \theta \geq 1 \), any symmetric \( C \in \mathbb{R}^{p \times p} \), and any \( q \geq \log(p) \),

\[
\left( \mathbb{E} \left\| \sum_{k=1}^n C \otimes B_k^{(1)} \otimes \cdots \otimes B_k^{(\theta)} \right\| \right)^{1/q} \leq c^q(\sqrt{\theta n} + q)^{\theta-1}(\sqrt{\theta n} \|C\| + \|C\|).
\]

To prove Lemma 7 we will make use of the following \( L^p \)-version of the matrix Bernstein inequality [45], see [17, Theorem 6.2], and the basic observation in Lemma 9 below.

**Theorem 8.** Let \( 2 \leq q < \infty \). If \((\Xi_k)_{k=1}^n \) is a sequence of independent, mean-zero random matrices in \( \mathbb{R}^{d_1 \times d_2} \), then

\[
\left( \mathbb{E} \left\| \sum_{k=1}^n \Xi_k \right\|^q \right)^{1/q} \leq C_{q,d} \max \left\{ \left( \mathbb{E} \left\| \sum_{k=1}^n \Xi_k \right\|^2 \right)^{1/2}, \left( \mathbb{E} \left\| \sum_{k=1}^n \Xi_k^T \right\|^2 \right)^{1/2}, 2C_{q,d} \left( \mathbb{E} \max_{1 \leq k \leq n} \|\Xi_k\|^q \right)^{\frac{1}{q}} \right\},
\]

where \( d = \max\{d_1, d_2\} \) and \( C_{q,d} \leq 2^{q/2} e(1 + \sqrt{2}) \sqrt{\max\{q, \log d\}} \).

We will combine Theorem 8 with the following observation.

**Lemma 9.** For any symmetric \( Z \in \mathbb{R}^{p \times p} \), \( \|\sigma(Z)\| \leq \sqrt{2}\|Z\| \).

**Proof.** By the triangle inequality,

\[
\|\sigma(Z)\|^2 \leq \|Z^2 \circ \Gamma\| + \|(Z \circ \Gamma)^2\| = \|Z^2 \circ \Gamma\| + \|Z \circ \Gamma\|^2.
\]

Since \( \Gamma = \mathbb{E} (\text{sign}(X) \text{sign}(X)^T) \) is positive semidefinite, Lemma 4 immediately yields

\[
\|\sigma(Z)\|^2 \leq (\|\Gamma\|_\infty + \|\Gamma\|^2_\infty) \|Z\|^2 \leq 2 \|Z\|^2.
\]

\( \blacksquare \)

**Proof of Lemma 7.** Recall \( Y = \text{sign}(X) \), \( Y^k = \text{sign}(X^k) \), and \( B_k = Y^k(Y^k)^T - \mathbb{E}(Y^k(Y^k)^T) \). Let us first note that Theorem 8 implies that, for any symmetric \( C_1, \ldots, C_n \in \mathbb{R}^{p \times p} \) and \( q \geq \log(p) \),

\[
\left( \mathbb{E} \left\| \sum_{k=1}^n C_k \otimes B_k \right\|^q \right)^{\frac{1}{q}} \leq \sqrt{\theta n} \left( \max_{k \in [n]} \|C_k\| \right) + q \left( \max_{k \in [n]} \|C_k\| \right).
\]

Indeed, since \( C_k \otimes Y^k(Y^k)^T = \text{diag}(Y^k)C_k \text{diag}(Y^k) \), the triangle inequality, sub-multiplicativity of the operator norm, and \( \|\text{diag}(Y^k)\| = 1 \) yield

\[
\|C_k \otimes B_k\| = \|\text{diag}(Y^k) \cdot C_k \cdot \text{diag}(Y^k) - \mathbb{E}(\text{diag}(Y^k) \cdot C_k \cdot \text{diag}(Y^k))\| \leq 2 \max_{k \in [n]} \|C_k\|
\]
and
\[
\left\| \sum_{k=1}^{n} E(\mathbf{C}_k \circ \mathbf{B}_k)^2 \right\| \leq n \max_{k \in [n]} \left\| E(\mathbf{C}_k \circ \mathbf{B}_k)^2 \right\|
\]
\[
= n \max_{k \in [n]} \left\| E(\mathbf{C}_k \circ Y^{k}(Y^{k})^T)^2 - (E(\mathbf{C}_k \circ Y^{k}(Y^{k})^T))^2 \right\|
\]
\[
= n \max_{k \in [n]} \left\| C_k^2 \circ E(Y^{k}(Y^{k})^T) - (C_k \circ E(Y^{k}(Y^{k})^T))^2 \right\| = n \max_{k \in [n]} \|\sigma(\mathbf{C}_k)^2\|. 
\]

In particular, by Theorem 8 the result holds for \( \theta = 1 \).

Assume now that the asserted inequality holds for \( \theta - 1 \). Applying (18) to the inner expectation in
\[
\left( E \left\| \sum_{k_1, \ldots, k_{\theta-1}} \mathbf{C} \circ \mathbf{B}_{k_1}^{(1)} \circ \cdots \circ \mathbf{B}_{k_{\theta-1}}^{(\theta-1)} \right\|^q \right)^{1/q} = \left( E \left\| \sum_{k_1, \ldots, k_{\theta-1}} \mathbf{C} \circ \mathbf{B}_{k_1}^{(1)} \circ \cdots \circ \mathbf{B}_{k_{\theta-1}}^{(\theta-1)} \right\|^q \right)^{1/q},
\]
where
\[
\mathbf{C} = \sum_{k_1, \ldots, k_{\theta-1}} \mathbf{C} \circ \mathbf{B}_{k_1}^{(1)} \circ \cdots \circ \mathbf{B}_{k_{\theta-1}}^{(\theta-1)},
\]
and using that \( \|\sigma(\mathbf{C})^2\| \leq 2 \|\mathbf{C}\|^2 \) by Lemma 9, we immediately find, for \( q \geq \log(p) \),
\[
\left( E \left\| \sum_{k_1, \ldots, k_{\theta-1}} \mathbf{C} \circ \mathbf{B}_{k_1}^{(1)} \circ \cdots \circ \mathbf{B}_{k_{\theta-1}}^{(\theta-1)} \right\|^q \right)^{1/q} \leq C_\theta \left( E \left\| \sum_{k_1, \ldots, k_{\theta-1}} \mathbf{C} \circ \mathbf{B}_{k_1}^{(1)} \circ \cdots \circ \mathbf{B}_{k_{\theta-1}}^{(\theta-1)} \right\|^q \right)^{1/q}.
\]

Applying the induction hypothesis to the right hand side yields the claim.

Combining the above results, we can control the moments of the ‘Hadamard chaos’ in (15), for \( \theta \leq \log(p) \).

**Lemma 10.** Let \( \mathbf{B} = \frac{\pi}{2n} \sum_{k=1}^{n} \mathbf{B}_k \), with \( \mathbf{B}_k \) as defined in (13). For any \( 2 \leq \theta \leq \log(p) \) \( \leq q \leq \frac{n}{\log^2(p)} \) and any symmetric \( \mathbf{C} \in \mathbb{R}^{p \times p} \),
\[
\frac{1}{\theta !} \left( E \| \mathbf{C} \circ \mathbf{B}^{\circ \theta} \|^q \right)^{1/q} \leq \left( \frac{C\theta}{n} \right)^{\theta/2} \| \mathbf{C} \|.
\]

**Proof.** Throughout, \( c \) will denote an absolute constant that may change from line to line. We decompose \( \mathbf{C} \circ \mathbf{B}^{\circ \theta} \) into terms with all different indices, i.e.,
\[
\frac{1}{\theta !} \mathbf{C} \circ \mathbf{B}^{\circ \theta} = \frac{1}{\theta !} \left( \frac{\pi}{2n} \right)^\theta \sum_{m=0}^{\theta-2} \sum_{m'=1}^{\theta-m} \sum_{\theta-m \geq j_1 \geq \cdots \geq j_{m'} \geq 2} \sum_{k_1 \neq \cdots \neq k_{m+m'}} \left( \sum_{k_1 \neq \cdots \neq k_{m+m'}} \mathbf{C} \circ \mathbf{B}_{k_1} \circ \cdots \circ \mathbf{B}_{k_{m+m'}} \right)
\]
\[
+ \frac{1}{\theta !} \left( \frac{\pi}{2n} \right)^\theta \sum_{k_1 \neq \cdots \neq k_{m}} \mathbf{C} \circ \mathbf{B}_{k_1} \circ \cdots \circ \mathbf{B}_{k_{m}} =: \alpha + \beta.
\]

Note that the index \( m \) tracks the number of \( \mathbf{B}_k \) with power 1 and \( m' \) tracks the number of \( \mathbf{B}_k \) with power at least 2. By Lemmas 6 and 7,
\[
\left( E \| \beta \|^{\theta} \right)^{1/q} = \frac{1}{\theta !} \left( \frac{\pi}{2n} \right)^\theta \left( E \left\| \sum_{k_1 \neq \cdots \neq k_{m}} \mathbf{C} \circ \mathbf{B}_{k_1} \circ \cdots \circ \mathbf{B}_{k_{m}} \right\|^{\theta} \right)^{\frac{1}{\theta}} \leq c^\theta \left( \left( \frac{C\theta}{n} \right)^{\frac{\theta}{2}} + \left( \frac{C\theta}{n} \right)^{\theta} \right) \| \mathbf{C} \|.
\]

It remains to estimate \( \left( E \| \alpha \|^{\theta} \right)^{1/q} \). We use
\[
\sum_{m=0}^{\theta-2} \sum_{m'=1}^{\theta-m} \sum_{\theta-m \geq j_1 \geq \cdots \geq j_{m'} \geq 2} \frac{1}{j_1 + \cdots + j_{m'} = \theta - m} \leq P(\theta) \leq c^{\sqrt{\theta}},
\]
where
where the second inequality is a standard estimate (see [3]) for the partition function

\[ P(\theta) = \left| \{(j_1, j_2, \ldots, j_n) : n \in \mathbb{N}, j_1, \ldots, j_n \in \mathbb{N}, j_1 \leq j_2 \leq \cdots \leq j_n : j_1 + \cdots + j_n = \theta \} \right|, \]

which counts the number of ways a natural number can be partitioned into a sum of natural numbers. This estimate implies

\[
(\mathbb{E} \Vert \alpha \Vert^q)^{1/q} \leq c^{\theta} \frac{1}{n^{1/\theta}} \cdot e^{c\sqrt{q}} \cdot \max_{m, m', (j_1, \ldots, j_m)} \left( \frac{\theta!}{j_1! \cdots j_m! m!} \left( \mathbb{E} \left\| \sum_{k_1 \neq \cdots \neq k_{m+m'}} C \odot \bigodot B_{k_i} \odot \bigodot B_{j_{k_{m+m'}}} \right\|^q \right)^{1/q} \right).
\]

Applying Lemma 5 and recalling that \( j_i \geq 2 \), for all \( i \in [m'] \), we obtain

\[
(\mathbb{E} \Vert \alpha \Vert^q)^{1/q} \leq c^{\theta} \max_{m, m', (j_1, \ldots, j_m)} \frac{1}{2^{m'} m!} \left( \mathbb{E} \left\| \sum_{k_1 \neq \cdots \neq k_{m+m'}} C \odot \bigodot B_{k_i} \odot \bigodot (\Phi_{j_{k_i}} + \Psi_{j_{k_i}} + B_{k_{m+m'}}) \right\|^q \right)^{1/q}.
\]

For \( s \in [m'] \), let us define

\[ C^{s, m'}_{\Phi, \Psi} := \left\{ \bigodot_{i \in I} \Phi_{j_{k_i}} \odot \bigodot_{i \in I} \Psi_{j_{k_i}} : I \subset [m'], |I| \leq s \right\}. \]

Note that \( |C^{s, m'}_{\Phi, \Psi}| = (m')^s \) and, by Lemma 5,

\[
\|C_s \odot D\| \leq (1 + \|\Gamma\|_{\infty})^{2(\theta - m - m')} \|D\| \leq 2^\theta \|D\|,
\]

for all \( s \in [m'] \), \( C_s \in C^{s, m'}_{\Phi, \Psi} \), and symmetric \( D \in \mathbb{R}^{p \times p} \). We thus obtain with Lemmas 6 and 7 that

\[
(\mathbb{E} \Vert \alpha \Vert^q)^{1/q} \leq c^{\theta} \max_{m, m', (j_1, \ldots, j_m)} \left( \frac{1}{2^{m'} m!} \left( \mathbb{E} \left\| \sum_{k_1 \neq \cdots \neq k_{m+m'}} (C \odot C_s) \odot \bigodot B_{k_i} \right\|^{m+s} \right)^{1/q} \right) \]

\[
\leq c^{\theta} \theta \max_{m, m', s, C_s, (j_1, \ldots, j_m)} \left( \frac{1}{m!} \left( m' \left( \frac{n - (m + s)!}{(n - (m + m'))!} \right) (m + s)! e^{m+s} \right) \right) \]

\[
\leq c^{\theta} \max_{m, m', s, C_s, (j_1, \ldots, j_m)} \left( \frac{1}{m!} \left( m' \left( \frac{n - (m + s)!}{(n - (m + m'))!} \right) (m + s)! e^{m+s} \left( \sqrt{q} n + q \right)^{m+s} \|C \odot C_s\| \right) \]

\[
\leq c^{\theta} \max_{m, m', s, C_s, (j_1, \ldots, j_m)} \left( \frac{1}{m!} \left( m' \left( \frac{n - (m + s)!}{(n - (m + m'))!} \right) (m + s)! \left( \sqrt{q} n + q \right)^{m+s} \|C\| \right) ,
\]

(21)
where in the final step we used $\theta \leq e^\theta$ and (20). Since

$$\frac{(n - (m + s))!}{(n - (m + m'))!} (m + s)! = \frac{(m + m')! (m + m')}{(m + s)!},$$

we find using the well-known estimates $k! \simeq k^{k+\frac{1}{2}} e^{-k}$ and $\left(\frac{k}{\ell}\right)^{\ell} \leq \left(\frac{\ell}{k}\right)^{\ell}$ that

$$\frac{1}{m!} \left(\frac{m'}{s}\right) \frac{(n - (m + s))!}{(n - (m + m'))!} (m + s)! \leq e^\theta \frac{1}{m^m} \left(\frac{m'}{s}\right)^s \left(\frac{m + m'}{m + s}\right)^{m + m'} \left(\frac{n}{m + s}\right)^{m + s} \leq e^\theta \frac{1}{m^m} \left(\frac{m'}{s}\right)^s (m + s)^{m - s} n^{m - s} \leq e^\theta \left(\frac{m'}{m}\right)^s \sqrt{n^{2m'}},$$

where we repeatedly used $s, m', m \leq \theta$. Applying this in (21) yields

$$(\mathbb{E}[\alpha^q]^{1/q}) \leq \frac{e^\theta}{n^q} \max_{n, m', s} \left(\frac{\sqrt{m + q} m'}{n}\right)^s \sqrt{n^{2m'}} \mathbb{E}[\alpha^q] \leq e^\theta \left(\frac{q}{n} + \frac{q}{n}\right)^q \|C\|,$$

as $m + 2m' \leq \theta \leq \log(p)$ and $n \geq q \log^2(p) \geq q(m'^2)$.

To convert the $L^q$-bound in Lemma 10 into a tail estimate, we use the following immediate consequence of Markov’s inequality.

**Lemma 11.** Assume $0 \leq q_0 < q_1 \leq \infty$. Let $\xi$ be a random variable satisfying

$$(\mathbb{E}[|\xi|^q])^{1/q} \leq a \sqrt{q} + bq$$

for certain $a, b > 0$ and all $q_0 \leq q \leq q_1$. Then,

$$\mathbb{P}[|\xi| \geq 2e \max\{a\sqrt{t}, bt\}] \leq e^{-t}$$

for all $q_0 \leq t \leq q_1$.

**Proof.** Let $s > 0$ be such that

$$2\max\{a\sqrt{q_0}, b q_0\} \leq s \leq 2\max\{a\sqrt{q_1}, b q_1\},$$

then

$$q := \min\left\{\frac{s^2}{4a^2}, \frac{s}{2b}\right\}$$

satisfies $q_0 \leq q \leq q_1$. By Markov’s inequality,

$$\mathbb{P}[|\xi| \geq es] \leq \frac{\mathbb{E}[|\xi|^q]^{1/q}}{(es)^q} \leq \left(\frac{a \sqrt{q} + bq}{es}\right)^q \leq \left(\frac{1}{e}\right)^q = \exp\left(\min\left\{\frac{s^2}{4a^2}, \frac{s}{2b}\right\}\right).$$

Setting $s = 2\max\{a\sqrt{t}, bt\}$ yields the result.

Finally, we will use the following simple consequence of Bernstein’s inequality. We include a proof for the sake of completeness.

**Lemma 12.** There exist absolute constants $c_1, c_2 > 0$ such that the following holds. Suppose that $n \geq c_2 \log(p)$. Let $X \sim \mathcal{N}(0, \Sigma)$ with $\Sigma_{\ell,i} = 1$ for all $i \in [p]$ and let $X^1, \ldots, X^n \stackrel{i.i.d.}{\sim} X$. Define $B_k$ as in (13). Then

$$\mathbb{P}\left[\left\|\frac{1}{n} \sum_{k=1}^n B_k\right\|_\infty \geq \sqrt{\frac{c_1 \log(p)}{n} + t}\right] \leq 2e^{-c_2nt}.$$
Proof. Recall the notation \( Y_{i,j}^k = \text{sign}(X_i^k)\text{sign}(X_j^k) \) and note that \((B_k)_{i,j} = Y_{i,j}^k - EY_{i,j}^k\) since \(|Y_{i,j}^k - EY_{i,j}^k| \leq 2\) for all \(i,j,k\), the bound is trivial for \(t \geq 4\). Using Bernstein’s inequality for bounded random variables (see, e.g., [48, Theorem 2.8.4]) and \(|Y_{i,j}^k - EY_{i,j}^k| \leq 2\), we find, for any \(u \leq 3\),

\[
P\left( \frac{1}{n} \sum_{k=1}^{n} Y_{i,j}^k - EY_{i,j}^k \geq u \right) \leq 2e^{-d_4 \min\left\{\frac{u^2}{\sigma_{i,j}^2}, \frac{n}{\sigma_{i,j}}\right\}} \leq 2e^{-d_2 n \min\{u^2, n\}} \leq 2e^{-d_3 n u^2},
\]

as

\[
\sigma_{i,j}^2 := \sum_{k=1}^{n} \text{E}(Y_{i,j}^k - EY_{i,j}^k)^2 = \sum_{k=1}^{n} \left( \text{E}(Y_{i,j}^k)^2 - (EY_{i,j}^k)^2 \right) = n \cdot \left( 1 - \left( \frac{2}{\pi} \arcsin(\Sigma_{i,j}) \right)^2 \right) \leq n.
\]

For any given \(t < 4\) and \(n \geq d_4 \log(p)\) (with \(d_4\) chosen such that \(d_3d_4 > 2\)), we set \(u = \sqrt{d_4 \frac{\log(p)}{n} + t} \leq 3\) and apply a union bound to obtain

\[
P\left( \frac{1}{n} \sum_{k=1}^{n} B_k \right) \geq \sqrt{d_4 \frac{\log(p)}{n} + t} \leq 2p e^{-d_3 d_4 \log(p) - d_3 n t} \leq 2e^{-d_3 n t}.
\]

We can now complete the proof of Theorem 1.

**Proof of Theorem 1.** Recall that \(A = \cos(\arcsin(\Sigma))\) and \(\Gamma = \text{E} \left( \text{sign}(X)\text{sign}(X)^T \right) = \frac{2}{n} \arcsin(\Sigma)\). Let us, for the sake of clarity, consider the case \(M = 1\). The general case follows by replacing \(A\) and \(\Sigma\) by \(M \odot A\) and \(M \odot \Sigma\) throughout the proof. We will moreover assume throughout that \(\lceil \log(p) \rceil \geq 2\), the remaining case is similar (and easier).

By (14) and the triangle inequality,

\[
\left\| \tilde{\Sigma}_n - \Sigma \right\| \leq \alpha + \beta + \gamma,
\]

where

\[
\alpha = \left\| A \odot B - \frac{1}{2} \Sigma \odot B^{\otimes 2} \right\|
\]

\[
\beta = \sum_{\ell = 1}^{\left\lfloor \frac{1}{4} \log(p) \right\rfloor} \frac{1}{(2\ell + 1)!} \left\| A \odot B^{\otimes (2\ell + 1)} \right\| + \sum_{\ell = 1}^{\left\lceil \frac{1}{4} \log(p) \right\rceil} \frac{1}{(2\ell + 2)!} \left\| \Sigma \odot B^{\otimes (2\ell + 2)} \right\|
\]

\[
\gamma = \sum_{\ell = \left\lceil \frac{1}{4} \log(p) \right\rceil}^{\infty} \frac{1}{(2\ell + 1)!} \left\| A \odot B^{\otimes (2\ell + 1)} \right\| + \sum_{\ell = \left\lceil \frac{1}{4} \log(p) \right\rceil}^{\infty} \frac{1}{(2\ell + 2)!} \left\| \Sigma \odot B^{\otimes (2\ell + 2)} \right\|
\]

We start by estimating \(\alpha\). Write \(B = \frac{1}{2n} \sum_{k=1}^{n} B_k\) and recall from Lemma 5 that \(B_k^{\otimes 2} = \Phi + \Psi \odot B_k\) with \(\Phi = \Phi_2 = (1 - \Gamma^{\otimes 2})\) and \(\Psi = \Psi_2 = -2\Gamma\). This yields

\[
A \odot B - \frac{1}{2} (\Sigma \odot B^{\otimes 2}) = \frac{\pi^2}{2n} \sum_{k=1}^{n} A \odot B_k - \frac{\pi^2}{8n^2} \sum_{k=1}^{n} \Sigma \odot B_k^{\otimes 2} - \frac{\pi^2}{8n^2} \sum_{k \neq \ell} \Sigma \odot B_k \odot B_\ell
\]

\[
= \frac{\pi^2}{2n} \sum_{k=1}^{n} A \odot B_k - \frac{\pi^2}{8n^2} \sum_{k=1}^{n} \Sigma \odot \Phi \odot B_k - \frac{\pi^2}{8n^2} \sum_{k \neq \ell} \Sigma \odot B_k \odot B_\ell
\]

Using Lemma 6 and Lemma 7, we obtain for all \(\log(p) \leq q \leq n\),

\[
(\log q)^{1/q} \leq \frac{\pi^2}{8n} \left\| \Sigma \odot \Phi \right\| + \frac{\pi^2}{2n} \left( \text{E} \left( \sum_{k=1}^{n} A \odot B_k \right)^q \right)^{1/q} + \frac{\pi^2}{8n^2} \left( \text{E} \left( \sum_{k=1}^{n} \Sigma \odot \Psi \odot B_k \right)^q \right)^{1/q}
\]

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\[ + \frac{\pi^2}{8n^2} \left( \mathbb{E} \left\| \sum_{k \neq \ell} \Sigma \circ B_k \circ B_\ell \right\|^q \right)^{1/q} \]

\[ \lesssim \frac{1}{n} \left\| \Sigma \circ \Phi \right\| + \frac{1}{n} \left( \sqrt{\gamma_n} \left\| \sigma(A) \right\| + q \left\| A \right\| \right) + \frac{1}{n^2} \left( \sqrt{\gamma_n} \left\| \sigma(\Sigma) \right\| + q \left\| \Sigma \right\| \right) \]

\[ + \frac{1}{n^2} \left( \sqrt{\gamma_n} + q \right) \sqrt{\gamma_n} \left\| \sigma(\Sigma) \right\| \]

\[ \lesssim \sqrt{\frac{q}{n}} \left\| \sigma(A) \right\| + \frac{q}{n} \max \left\{ \left\| A \right\| , \left\| \Sigma \right\| \right\} \]

where we used in the last inequality that \( q \leq n \), \( \left\| \sigma(\Sigma) \right\|^2 \leq 2 \left\| \Sigma \right\|^2 \) and \( \left\| \sigma(\Sigma \circ \Psi) \right\|^2 \leq 2 \left\| \Sigma \circ \Psi \right\|^2 \) by Lemma 9 and that \( \left\| \Sigma \circ \Phi \right\| , \left\| \Sigma \circ \Psi \right\| \leq 4 \left\| \Sigma \right\| \) by Lemma 5.

Let us now estimate \( \beta \). By the triangle inequality and Lemma 10, we obtain for all \( \log(p) \leq q \leq \frac{n}{\log^2(p)} \)

\[ \left( \mathbb{E} \beta^q \right)^{1/q} \leq \sum_{\ell = 1}^{\left\lfloor \frac{1}{4} \log(p) \right\rfloor} \frac{1}{(2 \ell + 1)!} \left( \mathbb{E} \left\| A \circ B^{\otimes(2\ell+1)} \right\|^q \right)^{1/q} + \sum_{\ell = 1}^{\left\lfloor \frac{1}{2} \log(p) \right\rfloor} \frac{1}{(2 \ell + 2)!} \left( \mathbb{E} \left\| \Sigma \circ B^{\otimes(2\ell+2)} \right\|^q \right)^{1/q} \]

\[ \leq \sum_{\ell = 1}^{\left\lfloor \frac{1}{4} \log(p) \right\rfloor} \left( \frac{cq}{n} \right)^{\frac{(2\ell+1)}{2}} \left\| A \right\| + \sum_{\ell = 1}^{\left\lfloor \frac{1}{2} \log(p) \right\rfloor} \left( \frac{cq}{n} \right)^{\frac{(2\ell+2)}{2}} \left\| \Sigma \right\| \]

\[ \leq \left( \frac{cq}{n} \right)^{3/2} \left\| A \right\| \sum_{\ell = 1}^{\infty} \left( \frac{cq}{n} \right)^{\ell-1} + \left( \frac{cq}{n} \right)^2 \left\| \Sigma \right\| \sum_{\ell = 1}^{\infty} \left( \frac{cq}{n} \right)^{\ell-1} \]

(24)

and hence

\[ \left( \mathbb{E} \beta^q \right)^{1/q} \lesssim \frac{q}{n} \max \left\{ \left\| A \right\| , \left\| \Sigma \right\| \right\}. \]

In summary, if \( \log(p) \leq q \leq \frac{n}{c \log^2(p)} \), then

\[ \left( \mathbb{E} (\alpha + \beta)^q \right)^{1/q} \lesssim \sqrt{\frac{q}{n}} \left\| \sigma(A) \right\| + \frac{q}{n} \max \left\{ \left\| A \right\| , \left\| \Sigma \right\| \right\}. \]

Lemma 11 yields, for all \( n \geq c \log^2(p)(\log(p) + u) \),

\[ \mathbb{P} \left[ \alpha + \beta \geq 2eC \left( \sqrt{\frac{\log(p) + u}{n}} \left\| \sigma(A) \right\| + \frac{\log(p) + u}{n} \max \left\{ \left\| A \right\| , \left\| \Sigma \right\| \right\} \right) \right] \leq e^{-u + \log(p)}. \]

It remains to estimate \( \gamma \). Using Lemma 4 and \( \| Z \|_{1 \rightarrow 2} \leq \sqrt{p} \| Z \|_{\infty} \) for \( Z \in \mathbb{R}^{p \times p} \), we find

\[ \gamma = \sum_{\ell = \left\lfloor \frac{1}{4} \log(p) \right\rfloor}^{\infty} \frac{1}{(2 \ell + 1)!} \left\| A \circ B^{\otimes(2\ell+1)} \right\| + \sum_{\ell = \left\lfloor \frac{1}{2} \log(p) \right\rfloor}^{\infty} \frac{1}{(2 \ell + 2)!} \left\| \Sigma \circ B^{\otimes(2\ell+2)} \right\| \]

\[ \leq \max \{ \left\| A \right\| , \left\| \Sigma \right\| \} \sum_{k = 2 \left\lfloor \frac{1}{4} \log(p) \right\rfloor}^{\infty} \frac{1}{k!} \left\| B \right\|_{1 \rightarrow 2} \]

\[ \leq \max \{ \left\| A \right\| , \left\| \Sigma \right\| \} \sum_{k = \left\lfloor \log(p) \right\rfloor}^{\infty} \frac{\sqrt{p}}{k!} \left\| B \right\|_{\infty} \leq \max \{ \left\| A \right\| , \left\| \Sigma \right\| \} \sum_{k = \left\lfloor \log(p) \right\rfloor}^{\infty} \frac{e^{k/2}}{k!} \left\| B \right\|_{\infty}^{k/2}, \]

(25)

as \( \sqrt{p} \leq e^{k/2} \) for \( k \geq \log(p) \). Recall that with probability at least \( 1 - 2e^{-c_2u} \)

\[ \left\| B \right\|_{\infty} \leq \sqrt{c_1 \frac{\log(p) + u}{n}}, \]

see Lemma 12. Upon this event,

\[ \gamma \leq \max \{ \left\| A \right\| , \left\| \Sigma \right\| \} \sum_{k = \left\lfloor \log(p) \right\rfloor}^{\infty} \frac{1}{k!} \left( \frac{ce_1 (\log(p) + u)}{n} \right)^{k/2} \leq \max \{ \left\| A \right\| , \left\| \Sigma \right\| \} \frac{\log(p) + u}{n}, \]

since \( \left\lfloor \log(p) \right\rfloor \geq 2 \) and \( n \geq \log(p) + u \). Combining the estimates for \( \alpha + \beta \) and \( \gamma \) yields the assertion. \( \blacksquare \)
3 A lower bound

By similar techniques as in Section 2, we can derive a lower bound on the expected squared error. The lower bound agrees with the upper bound in Theorem 1 in its dominant term.

In the proof we will use the following special case of [16, Proposition 5.29]. We include a short proof for the sake of completeness. Recall that \(|Z| = (Z^TZ)^{1/2}|.

**Lemma 13.** Let \(d_1, d_2 \in \mathbb{N}\). The map \(W \mapsto \|(E|W|^2)^{1/2}\) defines a norm on the space of random matrices in \(\mathbb{R}^{d_1 \times d_2}\). In particular, the map \(Z \mapsto \|\sigma(Z)\|\) on the symmetric matrices in \(\mathbb{R}^{p \times p}\) satisfies the (reverse) triangle inequality.

**Proof.** We only prove the triangle inequality. The remaining norm properties are immediate. For any \(W, Z \in \mathbb{R}^{d_1 \times d_2}\) and \(t > 0\), \(|tW - \frac{1}{t}Z|^2 \geq 0\) and hence

\[|W + Z|^2 \leq (1 + t^2)|W|^2 + (1 + t^{-2})|Z|^2.\]

Taking expectations and norms and using the triangle inequality, we obtain

\[\|\mathrm{E}W + Z\|^2 \leq (\|\mathrm{E}W\|^2 + \|\mathrm{E}|W|^2|^2)^{1/2}.\]

Minimizing the right hand side, we find

\[\|\mathrm{E}|W|^2 + Z\|^2 \leq (\|\mathrm{E}|W|^2\|^2 + \|\mathrm{E}|W|^2|^2)^{1/2},\]

which shows that the triangle inequality holds.

The final statement is an immediate consequence, as

\[\|\sigma(Z)\| = \|\mathrm{E}|Z \circ B_1|^2\|^{1/2},\]

with \(B_1\) defined as in (13).

We are now ready to prove the lower bound.

**Proposition 14.** There exist constants \(c_1, c_2, c_3 > 0\) such that the following holds. If \(n \geq \log^3(p)\), \(X \sim \mathcal{N}(0, \Sigma)\) with \(\Sigma_{ii} = 1\), for \(i \in [p]\), \(X^1, ..., X^p\) i.i.d. \(X\), and \(M \in [0, 1]^{p \times p}\) is a symmetric mask, then

\[\left(\mathrm{E}\left\|M \circ \Sigma_n - M \circ \Sigma\right\|^2\right)^{1/2} \geq \frac{c_2}{\sqrt{n}}\|\sigma(M \circ A)\| + \frac{c_2}{n}\|M \circ \Sigma \circ (1 - \Gamma^{\circ 2})\|\]

\[+ \frac{c_2}{n}\|\sigma(M \circ \Sigma)^2 \circ \Gamma\|^1/2 - \max\{\|A\|, \|\Sigma\\}\left(\frac{c_3\log(p)}{n}\right)^{3/2}.\]

**Proof.** As in the proof of Theorem 1 restrict ourselves for convenience to the case where \(M = 1\) and \([\log(p)] \geq 3\). The adaption to the general case is straightforward. By the (reverse) triangle inequality, we can write

\[\left(\mathrm{E}\left\|\Sigma_n - \Sigma\right\|^2\right)^{1/2} \geq (\mathrm{E}|\alpha|^2)^{1/2} - (\mathrm{E}|\beta|^2)^{1/2} - (\mathrm{E}|\gamma|^2)^{1/2},\]

with \(\alpha, \beta, \gamma\) as defined in (22). We start by estimating \((\mathrm{E}|\alpha|^2)^{1/2}\). Taking expectations in (23), we find

\[\mathrm{E}\left(A \circ B - \frac{1}{2}\Sigma \circ B^{\circ 2}\right)^2\]

\[= \mathrm{E}\left(\frac{\pi}{2n} \sum_{k=1}^n A - \frac{\pi}{4n} \Sigma \circ \Psi \right) \circ B_k - \frac{\pi^2}{8n} \Sigma \circ \Phi - \frac{\pi^2}{8n^2} \sum_{k \neq \ell} \Sigma \circ B_k \circ B_{\ell}\]

\[= \mathrm{E}\left(\frac{\pi}{2n} \sum_{k=1}^n A - \frac{\pi}{4n} \Sigma \circ \Psi \right) \circ B_k + \left(\frac{\pi^2}{8n} \Sigma \circ \Phi\right)^2 + \mathrm{E}\left(\frac{\pi^2}{8n^2} \sum_{k \neq \ell} \Sigma \circ B_k \circ B_{\ell}\right)^2,\]
where we use that the expectation of all “cross-terms” in the expansion of the square are zero as \( B_k \) and \( B_\ell \) are independent for all \( k \neq \ell \) and \( \mathbb{E}B_k = 0 \) for all \( k \in [n] \). Since all the terms are positive semidefinite,

\[
\mathbb{E} a^2 \geq \| \mathbb{E} \left( A \odot B - \frac{1}{2} \Sigma \odot B^\otimes 2 \right) \|^2 \\
\simeq \max \left\{ \| \mathbb{E} \left( \frac{\pi}{2n} \sum_{k=1}^n (A - \frac{\pi}{4n} \Sigma \odot \Psi) \odot B_k \right) \|^2, \| \left( \frac{\pi^2}{8n} \Sigma \odot \Phi \right)^2 \|, \| \mathbb{E} \left( \frac{\pi^2}{8n^2} \sum_{k \neq \ell} \Sigma \odot B_k \odot B_\ell \right) \|^2 \right\}.
\]

Recall that \( B_k = Y^k(Y^k)^T - \mathbb{E} (Y^k(Y^k)^T) = Y^k(Y^k)^T - \Gamma \). Since

\[
C \odot YY^T = \text{diag}(Y) \cdot C \cdot \text{diag}(Y)
\]

for all \( C \in \mathbb{R}^{p \times p} \), and \( \text{diag}(Y)^2 = \text{Id} \), we find

\[
\mathbb{E} \left( \frac{\pi}{2n} \sum_{k=1}^n (A - \frac{\pi}{4n} \Sigma \odot \Psi) \odot B_k \right)^2 = \frac{\pi^2}{4n} \mathbb{E} \left( \left( A - \frac{\pi}{4n} \Sigma \odot \Psi \right) \odot B_1 \right)^2
\]

\[
= \frac{\pi^2}{4n} \left( \left( A - \frac{\pi}{4n} \Sigma \odot \Psi \right)^2 \odot \Gamma - \left( \left( A - \frac{\pi}{4n} \Sigma \odot \Psi \right) \odot \Gamma \right)^2 \right)
\]

\[
= \frac{\pi^2}{4n} \sigma \left( A - \frac{\pi}{4n} \Sigma \odot \Psi \right)^2.
\]

Using Lemma 13 we obtain

\[
\left\| \left( \mathbb{E} \left( \frac{\pi}{2n} \sum_{k=1}^n (A - \frac{\pi}{4n} \Sigma \odot \Psi) \odot B_k \right) \right)^2 \right\|^{1/2} = \frac{\pi}{\sqrt{n}} \| \sigma \left( A - \frac{\pi}{4n} \Sigma \odot \Psi \right) \|
\]

\[
\geq \frac{\pi}{\sqrt{n}} \| \sigma(A) \| - \frac{\pi^2}{4n^{3/2}} \| \sigma(\Sigma \odot \Psi) \|
\]

\[
\geq \frac{\pi}{\sqrt{n}} \| \sigma(A) \| - \frac{\pi^2}{2n^{3/2}} \| \Sigma \|.
\]

Moreover, by independence of \( B_k \) and \( B_\ell \), for \( k \neq \ell \), and \( \mathbb{E}B_k = 0 \), for all \( k \in [n] \),

\[
\mathbb{E} \left( \frac{\pi^2}{8n^2} \sum_{k \neq \ell} \Sigma \odot B_k \odot B_\ell \right)^2 = \frac{\pi^4}{64n^4} \sum_{k_1 \neq \ell_1} \sum_{k_2 \neq \ell_2} \mathbb{E} \left( (\Sigma \odot B_{k_1} \odot B_{\ell_1})(\Sigma \odot B_{k_2} \odot B_{\ell_2}) \right)
\]

\[
= \frac{\pi^4}{32n^4} \sum_{k \neq \ell} \mathbb{E} \left( (\Sigma \odot B_k \odot B_\ell)(\Sigma \odot B_k \odot B_\ell) \right)
\]

\[
= \frac{\pi^4n(n-1)}{64n^4} \mathbb{E}(\Sigma \odot B_1 \odot B_2)(\Sigma \odot B_1 \odot B_2)
\]

\[
= \frac{\pi^4n(n-1)}{64n^4} (\Sigma^2 \odot \Gamma^2 - (\Sigma \odot \Gamma)^2 \odot \Gamma)
\]

\[
= \frac{\pi^4n(n-1)}{64n^4} \sigma(\Sigma)^2 \odot \Gamma.
\]

In the penultimate step we used that (27) implies

\[
\mathbb{E} \left( (C \odot B_k)(D \odot B_k) \right) = \mathbb{E}(CD) \odot \Gamma - (\mathbb{E}C \odot \Gamma)(\mathbb{E}D \odot \Gamma)
\]

for any random matrices \( C, D \in \mathbb{R}^{p \times p} \) that are independent of \( B_k \), so in particular

\[
\mathbb{E} \left( (\Sigma \odot B_1 \odot B_2)(\Sigma \odot B_1 \odot B_2) \right) = \mathbb{E}(\Sigma \odot B_1)^2 \odot \Gamma - (\mathbb{E}(\Sigma \odot B_1) \odot \Gamma)^2
\]

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\[ \Sigma^2 \odot \Gamma^\odot^2 - (\Sigma \odot \Gamma)^2 \odot \Gamma. \]

To estimate the remaining terms in (26), note that
\[
(\mathbb{E} \gamma^2)^{1/2} \leq (\mathbb{E} \gamma^2 \log(p))^{1/2} \leq \max\{\|A\|, \|\Sigma\|\} \left( \frac{c \log(p)}{n} \right)^{3/2}
\]
if \( n \gtrsim \log^2(p) \) by (24). Moreover, by (25),
\[
(\mathbb{E} \gamma^2)^{1/2} \leq \max\{\|A\|, \|\Sigma\|\} \sum_{k=\lceil \log(p) \rceil}^{\infty} \frac{e^{k/2}}{k!} (\mathbb{E} \|B\|^{2k})^{1/2}.
\]

Using that \( k \geq \log(p) \) and applying Bernstein's inequality,
\[
(\mathbb{E} \|B\|^{2k})^{1/2} = \frac{\pi}{2} \left( \mathbb{E} \max_{1 \leq i,j \leq p} \left| \frac{1}{n} \sum_{m=1}^{n} (Y_{i,j}^m - \mathbb{E}[Y_{i,j}^m]) \right|^{2k} \right)^{1/2k} \leq \frac{c \pi}{2} \max_{1 \leq i,j \leq p} \left( \mathbb{E} \left| \frac{1}{n} \sum_{m=1}^{n} (Y_{i,j}^m - \mathbb{E}[Y_{i,j}^m]) \right|^{2k} \right)^{1/2k} \lesssim \sqrt{\frac{k}{n} + \frac{k}{n}}.
\]

Using this estimate together with \( k! \geq k^k e^{-k} \), we obtain
\[
(\mathbb{E} \gamma^2)^{1/2} \leq \max\{\|A\|, \|\Sigma\|\} \sum_{k=\lceil \log(p) \rceil}^{\infty} \frac{e^{k/2}}{k!} \left( \frac{k^k e^{-k}}{n^{k/2}} + \frac{k^k}{n^k} \right) \leq \max\{\|A\|, \|\Sigma\|\} \sum_{k=\lceil \log(p) \rceil}^{\infty} \left( \frac{C_2}{n} \right)^{k/2} \lesssim \max\{\|A\|, \|\Sigma\|\} \left( \frac{C_2}{n} \right)^{3/2},
\]
as \( \lceil \log(p) \rceil \geq 3 \). This completes the proof.

\section{4 Proof of Theorem 3}

To prove Theorem 3, we first estimate the bias of \( \tilde{\Sigma}' \), see Lemma 17. Its proof relies on Lemmas 15 and 16. The remainder of the proof is a relatively straightforward application of the matrix Bernstein inequality (Theorem 8).

\begin{lemma}
Let \( a, b \in \mathbb{R} \), \( \lambda > \max\{|a|, |b|\} \) and let \( \sigma, \sigma' \) be independent and uniformly distributed in \([-\lambda, \lambda] \). Then,
\[ \mathbb{E}(\text{sign}(a + \sigma)\text{sign}(b + \sigma')) = \frac{ab}{\lambda^2}. \]
\end{lemma}

\begin{proof}
By independence of \( \sigma \) and \( \sigma' \),
\[ \mathbb{E}(\text{sign}(a + \sigma)\text{sign}(b + \sigma')) = \mathbb{P}(\text{sign}(a + \sigma) = \text{sign}(b + \sigma')) - \mathbb{P}(\text{sign}(a + \sigma) \neq \text{sign}(b + \sigma')) \]
\[ = 2 \mathbb{P}(\text{sign}(a + \sigma) = \text{sign}(b + \sigma')) - 1 \]
\[ = 2 \left( \mathbb{P}(\sigma > -a, \sigma' > -b) + \mathbb{P}(\sigma < -a, \sigma' < -b) \right) - 1 \]
\[ = 2 \left( \frac{1}{2\lambda^2}(\lambda + a)(\lambda + b) + \frac{1}{2\lambda^2}(\lambda - a)(\lambda - b) \right) - 1 \]
\[ = \frac{1}{2\lambda^2} (2\lambda^2 + 2ab) - 1 = \frac{ab}{\lambda^2}. \]
This completes the proof.
\end{proof}
In the proof of the following lemma we use that for any subgaussian random variable $W$ and any $\mu > 0$,

$$
\mathbb{E}(|W|1_{\{|W| > \mu\}}) = \int_0^\infty \mathbb{P}(|W|1_{\{|W| > \mu\}} > t) \, dt = \mu \mathbb{P}(|W| > \mu) + \int_\mu^\infty \mathbb{P}(|W| > t) \, dt
$$

$$
\leq \mu e^{-\mu^2/\|W\|_2^2} + \frac{\|W\|_2^2}{\mu} e^{-\mu^2/\|W\|_2^2},
$$

(28)

where the final inequality follows from the well-known estimate

$$
\int_u^\infty e^{-t^2/2} \, dt \leq \frac{1}{u} e^{-u^2/2}, \quad u > 0.
$$

Similarly, for any subexponential random variable $Z$,

$$
\mathbb{E}(|Z|1_{\{|Z| > \mu\}}) \lesssim \mu e^{-\mu/\|Z\|_{\psi_1}} + \|Z\|_{\psi_1} e^{-\mu/\|Z\|_{\psi_1}}.
$$

(29)

**Lemma 16.** Let $U,V$ be subgaussian random variables, let $\lambda > 0$, and let $\sigma, \sigma'$ be independent, uniformly distributed in $[-\lambda, \lambda]$ and independent of $U$ and $V$. Then,

$$
|\mathbb{E}(\lambda^2 \text{sign}(U + \sigma)\text{sign}(V + \sigma')) - \mathbb{E}(UV)| \lesssim (\lambda^2 + \theta_{U,V}^2)e^{-\lambda^2/2\theta_{U,V}^2},
$$

where $\theta_{U,V} = \max\{\|U\|_{\psi_2}, \|V\|_{\psi_2}\}$.

**Proof.** Since $U$ and $V$ are independent of $\sigma$ and $\sigma'$, Lemma 15 yields

$$
\mathbb{E}_{\sigma,\sigma'}(\lambda^2 \text{sign}(U + \sigma)\text{sign}(V + \sigma')) 1_{\{|U| \leq \lambda, |V| \leq \lambda\}} = UV1_{\{|U| \leq \lambda, |V| \leq \lambda\}}.
$$

Hence,

$$
\mathbb{E}(\lambda^2 \text{sign}(U + \sigma)\text{sign}(V + \sigma')) - \mathbb{E}(UV) = \mathbb{E}((\lambda^2 \text{sign}(U + \sigma)\text{sign}(V + \sigma') - UV) 1_{\{|U| > \lambda, |V| > \lambda\}}).
$$

(30)

Since $U,V$ are subgaussian,

$$
|\mathbb{E}(\lambda^2 \text{sign}(U + \sigma)\text{sign}(V + \sigma') 1_{\{|U| > \lambda, |V| > \lambda\}})| \leq \lambda^2 (\mathbb{P}(|U| > \lambda) + \mathbb{P}(|V| > \lambda)) \leq 2\lambda^2 e^{-\lambda^2/2\theta_{U,V}^2}.
$$

(31)

Moreover,

$$
|\mathbb{E}(UV 1_{\{|U| > \lambda, |V| > \lambda\}})| \leq \mathbb{E}(|UV| 1_{\{|U| > \lambda\}}) + \mathbb{E}(|UV| 1_{\{|V| > \lambda\}}).
$$

(32)

By (28) and (29)

$$
\mathbb{E}(|UV| 1_{\{|U| > \lambda\}}) = \mathbb{E}(|UV|(1_{\{|U| > \lambda, |V| > \lambda\}} + 1_{\{|U| > \lambda, |V| \leq \lambda\}})) \leq \mathbb{E}(|UV| 1_{\{|U| > \lambda, |V| > \lambda\}}) + \lambda \mathbb{E}(|U| 1_{\{|U| > \lambda\}})
$$

$$
\leq (\lambda^2 + \|UV\|_{\psi_1}) e^{-\lambda^2/\|UV\|_{\psi_1}} + \lambda \left(\lambda e^{-\lambda^2/\|U\|_{\psi_2}^2} + \frac{\|U\|_{\psi_2}^2}{\lambda} e^{-\lambda^2/\|U\|_{\psi_2}^2}\right)
$$

$$
\leq 2(\lambda^2 + \theta_{U,V}^2) e^{-\lambda^2/2\theta_{U,V}^2},
$$

(33)

where we have used $\|UV\|_{\psi_1} \leq \|U\|_{\psi_2} \|V\|_{\psi_2}$. The claim follows by using (31)-(33) in (30). \qed

**Lemma 17.** There exists constants $c_1, c_2 > 0$ depending only on $K$ such that the following holds. Let $X$ be a mean-zero, $K$-subgaussian vector with covariance matrix $\mathbb{E}(XX^T) = \Sigma$. Let $\lambda > 0$ and let $Y = \text{sign}(X + \tau)$ and $\bar{Y} = \text{sign}(X + \bar{\tau})$, where $\tau, \bar{\tau}$ are independent and uniformly distributed in $[-\lambda, \lambda]^p$ and independent of $X$. Then,

$$
\|\lambda^2 \mathbb{E}(YY^T) - \Sigma\|_\infty \leq c_1(\lambda^2 + \|\Sigma\|_\infty) e^{-c_2\lambda^2/\|\Sigma\|_\infty}
$$

and

$$
\|\lambda^2 \mathbb{E}(YY^T) - (\Sigma - \text{diag}(\Sigma) + \lambda^2 \text{Id})\|_\infty \leq c_1(\lambda^2 + \|\Sigma\|_\infty) e^{-c_2\lambda^2/\|\Sigma\|_\infty}.
$$

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Proof. Since $X$ is $K$-subgaussian, for any $\ell \in [p]$,
\[
\|X_\ell\|_{v_2} = \|\langle X, e_\ell \rangle\|_{v_2} \leq K(E(X, e_\ell)^2)^{1/2} = K\Sigma_\ell^{1/2} \leq K\|\Sigma\|_{\infty}^{1/2}.
\]
Lemma 16 applied for $U = X_i$ and $V = X_j$ yields
\[
|E(\lambda^2\text{sign}(X_i + \tau_i)\text{sign}(X_j + \bar{\tau}_j)) - \Sigma_{i,j}| \lesssim (\lambda^2 + K^2\|\Sigma\|_{\infty})e^{-c_1\lambda^2/K^2\|\Sigma\|_{\infty}}
\]
for all $i, j \in [p]$ and
\[
|E(\lambda^2\text{sign}(X_i + \tau_i)\text{sign}(X_j + \tau_j)) - \Sigma_{i,j}| \lesssim (\lambda^2 + K^2\|\Sigma\|_{\infty})e^{-c_1\lambda^2/K^2\|\Sigma\|_{\infty}}
\]
whenever $i \neq j$. These two observations immediately imply the two statements. 

We are now ready to prove the main result of this section.

Proof of Theorem 3. Recall the definitions of $\Sigma_n^{\dith}$ and $\tilde{\Sigma}_n'$ in (8) and (9). Clearly,
\[
\|M \odot \Sigma_n^{\dith} - M \odot \Sigma\| \leq \|M \odot \tilde{\Sigma}_n' - M \odot \Sigma\|
\]
and
\[
\|M \odot \tilde{\Sigma}_n' - M \odot \Sigma\| \leq \|M \odot \tilde{\Sigma}_n' - M \odot E\tilde{\Sigma}_n'\| + \|M \odot (E\tilde{\Sigma}_n' - \Sigma)\|.
\]
By noting that $M$ has only nonnegative entries and applying Lemma 17, we obtain
\[
\|M \odot (E\tilde{\Sigma}_n' - \Sigma)\| = \sup_{\nu \in \mathbb{R}^p : \|\nu\|_2 \leq 1} \left| \sum_{i,j=1}^p M_{i,j} \left( E\tilde{\Sigma}_n' - \Sigma \right)_{i,j} \nu_i \nu_j \right|
\leq \sup_{\nu \in \mathbb{R}^p : \|\nu\|_2 \leq 1} \left| \sum_{i,j=1}^p M_{i,j} \left( E\tilde{\Sigma}_n' - \Sigma \right)_{i,j} \right| |\nu|\|\nu||
\leq \|M\| \|E\tilde{\Sigma}_n' - \Sigma\|_{\infty} = \|M\| |\lambda^2 E (Y\bar{Y}^T) - \Sigma\|_{\infty}
\lesssim K\|M\| (\lambda^2 + \|\Sigma\|_{\infty})e^{-c_1\lambda^2/K^2\|\Sigma\|_{\infty}} \lesssim K\frac{\lambda^2\|M\|}{n}
\]
provided that $\lambda^2 \geq \frac{1}{(1/2)}\|\Sigma\|_{\infty}\log(n)$. Let us write $Y_k = \text{sign}(X_k + \tau_k)$ and $\bar{Y}_k = \text{sign}(X_k + \bar{\tau}_k)$. We estimate the first term in (34) by defining random matrices
\[
\Xi_k = \frac{\lambda^2}{n}M \odot (Y_k\bar{Y}_k^T - E(Y_k\bar{Y}_k^T)) \quad \text{so that} \quad M \odot \tilde{\Sigma}_n' - M \odot E\tilde{\Sigma}_n' = \sum_{k=1}^n \Xi_k.
\]
For any $1 \leq k \leq n$,
\[
\|\Xi_k\| = \frac{\lambda^2}{n}\|M \odot (Y_k\bar{Y}_k^T - E(Y_k\bar{Y}_k^T))\| \leq \frac{2\lambda^2\|M\|}{n}
\]
since $M \odot Y_k\bar{Y}_k^T = \text{diag}(Y_k) M \text{diag}(\bar{Y}_k)$ and $\|\text{diag}(Y_k)\| = 1$. Moreover, using that $(Y, \bar{Y})$ and $(\bar{Y}, Y)$ are identically distributed, we get (recall that $|Z| = (Z^TZ)^{1/2}$)
\[
\left\| \left( \sum_{k=1}^n E|\Xi_k|^2 \right)^{1/2} \right\| = \frac{\lambda^2}{\sqrt{n}} \left\| E|M \odot YY^T|^2 - |M \odot E(YY^T)|^2 \right\|^{1/2}
= \frac{\lambda^2}{\sqrt{n}} \left\| M^2 \odot E(YY^T) - |M \odot E(YY^T)|^2 \right\|^{1/2}
= \frac{\lambda^2}{\sqrt{n}} \left\| M^2 \odot E(YY^T) - |M \odot E(YY^T)|^2 \right\|^{1/2}
\]
as, with $\text{diag}(Y)^2 = \text{Id}$,

$$|M \circ YY^T|^2 = \text{diag}(Y) M \text{diag}(Y) M = M^2 \circ YY^T.$$  

Interchanging the roles of $Y_k$ and $\tilde{Y}_k$ yields

$$\left\| \left( \sum_{k=1}^n E(\tilde{Y}_k^T)^2 \right)^{1/2} \right\| = \frac{\lambda^2}{\sqrt{n}} \left\| M^2 \circ E(YY^T) - |M \circ E(YY^T)|^2 \right\|^{1/2}. $$

Since $|M \circ E(YY^T)|^2 \leq E|M \circ YY^T|^2$ by Kadison’s inequality (11), we find

$$\frac{\lambda^2}{\sqrt{n}} \left\| M^2 \circ E(YY^T) - |M \circ E(YY^T)|^2 \right\|^{1/2} \leq \frac{2\lambda^2}{\sqrt{n}} \left\| M^2 \circ E(YY^T) \right\|^{1/2} \leq \frac{2\lambda}{\sqrt{n}} \left( \left\| M^2 \circ (\lambda^2 E (YY^T) - (\Sigma - \text{diag}(\Sigma) + \lambda^2 \text{Id})) \right\| + \left\| M^2 \circ (\Sigma - \text{diag}(\Sigma) + \lambda^2 \text{Id}) \right\| \right)^{1/2}$$

By the same reasoning as in (35) together with Lemma 17

$$\left\| M^2 \circ (\lambda^2 E (YY^T) - (\Sigma - \text{diag}(\Sigma) + \lambda^2 \text{Id})) \right\| \leq \left\| M^2 \right\| \left\| \lambda^2 E (YY^T) - (\Sigma - \text{diag}(\Sigma) + \lambda^2 \text{Id}) \right\|_\infty \lesssim \frac{\lambda^2}{\sqrt{n}} \left\| M \right\| \left( \lambda + \left\| \Sigma \right\|_{\infty} \right) e^{-c_1 \lambda^2/\|\Sigma\|_\infty}.$$  

Hence,

$$\frac{\lambda^2}{\sqrt{n}} \left\| M^2 \circ E(YY^T) - |M \circ E(YY^T)|^2 \right\|^{1/2} \lesssim \frac{\lambda}{\sqrt{n}} \left\| M \right\| \left( \lambda + \left\| \Sigma \right\|_{\infty} \right) e^{-c_1 \lambda^2/\|\Sigma\|_\infty} + \frac{\lambda}{\sqrt{n}} \left\| M \right\| \left( \lambda + \left\| \Sigma \right\|_{\infty} \right) e^{-c_1 \lambda^2/\|\Sigma\|_\infty}.$$  

By Lemma 4, we find

$$\left\| M^2 \circ (\Sigma - \text{diag}(\Sigma) + \lambda^2 \text{Id}) \right\| \leq \left\| M^2 \circ \Sigma \right\| + \left\| M^2 \circ \text{Id} \circ \Sigma \right\| + \lambda^2 \left\| M^2 \circ \text{Id} \right\| \leq \left\| M \right\|^3 \left( 2 \left\| \Sigma \right\| + \lambda^2 \right).$$

In summary,

$$\max \left\{ \left\| \left( \sum_{k=1}^n E(\tilde{Y}_k^T)^2 \right)^{1/2} \right\|, \left\| \left( \sum_{k=1}^n E(\tilde{Y}_k)^2 \right)^{1/2} \right\| \right\} \lesssim K \frac{1}{\sqrt{n}} \left( \left\| M \right\|^3 \left( 2 \left\| \Sigma \right\| + \lambda^2 \right) + \lambda \left\| M \right\| \left( \lambda + \left\| \Sigma \right\|_{\infty} \right) e^{-c_1 \lambda^2/\|\Sigma\|_\infty} \right)$$

provided that $\lambda^2 \geq 2\left\| \Sigma \right\|_{\infty} \log(n)/c_1$. By Theorem 8, (36), and (37) we find that for any $q \geq 2\log(p)$

$$\left( \mathbb{E} \left\| M \circ \tilde{\Sigma}'_n - M \circ E \left[ \tilde{\Sigma}'_n \right] \right\|^q \right)^{1/q} \lesssim K \frac{\left\| M \right\|^3 \left( 2 \left\| \Sigma \right\| + \lambda^2 \right) + \lambda \left\| M \right\| \left( \lambda + \left\| \Sigma \right\|_{\infty} \right) e^{-c_1 \lambda^2/\|\Sigma\|_\infty}}{\sqrt{n}} + \frac{\lambda^2 \left\| M \right\|}{n}.$$  

The result now follows immediately from Lemma 11.

5 Numerical experiments

Let us compare our theoretical prediction of the behavior of the two estimators based on quantized samples to their actual performance in numerical experiments. In the case of quantization without dithering, we will
use the estimator $\tilde{\Sigma}_n$ in (4). In the case of quantization with dithering, we will use the positive semidefinite projection of the estimator in (8). With some abuse of notation, we will denote it by the same symbol, i.e., from here on on

$$\tilde{\Sigma}_n^\text{dith} = P_{\text{PSD}} \left( \frac{1}{2} \left( \tilde{\Sigma}'_n + (\tilde{\Sigma}'_n)^T \right) \right), \quad (38)$$

where $\tilde{\Sigma}'_n$ was defined in (9). The sample mean $\hat{\Sigma}_n$ in (1), which is computed using knowledge of the ‘unquantized’ samples, will serve as a benchmark. In all our experiments we use i.i.d. samples $X_1, ..., X_n$ from a Gaussian distribution with mean zero and covariance matrix $\Sigma$. All experiments are averaged over 100 realizations of the samples $X_1, ..., X_n$. Moreover, we always tune the dithering parameter $\lambda$ via grid-search on $(0, 4 \| \Sigma \|_\infty)$. We leave it as an open problem for future research to find a suitable $\lambda$ when no prior knowledge of $\| \Sigma \|_\infty$ is available.

5.1 Comparison of all estimators

In the first experiment, we fix $n = 200$ and estimate for $p \in [5, 30]$ the covariance matrix $\Sigma \in \mathbb{R}^{p \times p}$ with all proposed estimators from samples $X_1, ..., X_n$. We construct $\Sigma$ as having ones on its diagonal and all remaining entries equal to 0.2. This special form allows fair comparison since it fulfills the more restrictive theoretical assumptions for the estimator $\tilde{\Sigma}_n$.

Figure 1a shows the approximation error of all considered estimators in operator norm when varying the ambient dimension $p$. Notably, the estimator $\tilde{\Sigma}_n$, which uses quantized samples without dithering, performs almost as good as the sample mean $\hat{\Sigma}_n$, even though the latter estimator uses the full, undistorted samples. Moreover, the estimator $\tilde{\Sigma}_n^\text{dith}$ performs similar but slightly worse. This is not surprising, as it does not exploit that $\Sigma$ has an all-ones diagonal. The outcome shows that coarse quantization hardly causes loss in estimation quality. Let us emphasize that a proper choice of $\lambda$ is essential for the performance of $\tilde{\Sigma}_n^\text{dith}$, see Figure 1b. Figure 2 furthermore suggests that the optimal choice of $\lambda$ is influenced by the number of samples $n$, supporting the dependence of $\lambda$ on $\log(n)$ in Theorem 3.

![Figure 1](image1.png)

**Figure 1:** The left plot depicts the average estimation errors in operator norm, for $n = 200$ and $p$ varying from 5 to 30. The dithered estimator here uses the $\lambda \in (0, 4 \| \Sigma \|_\infty)$ that is optimized via grid-search. The right plot depicts the average estimation error in operator norm, for $n = 200$, $p = 5$, and $\lambda$ varying from 0 to $4 \| \Sigma \|_\infty$. Although they are not affected by changes in $\lambda$, the sample mean and $\tilde{\Sigma}_n$ are depicted for reference.

5.2 One-bit estimator – influence of correlation

In our second experiment we compare the sample mean $\hat{\Sigma}_n$ to the estimator $\tilde{\Sigma}_n$. Theorem 1 suggests that the performance of $\tilde{\Sigma}_n$ heavily depends on the correlations between the different entries of $X$ (via the off-
diagonal entries of $A$, see the discussion following Theorem 1). To illustrate this numerically, we choose three different ground-truths $\Sigma \in \mathbb{R}^{p \times p}$ having ones on the diagonal and being constant $c$ on all remaining entries: one with medium correlation ($c = 0.5$), one with high correlation ($c = 0.9$), and one with very high correlation ($c = 0.99$). We fix the ambient dimension as $p = 20$ and vary the number of samples $X_1, \ldots, X_n$ from $n = 10$ to $n = 300$. As Figure 3a illustrates, the estimator $\hat{\Sigma}_n$ indeed outperforms sample mean if the correlation is high. This is surprising, as the sample mean uses the full, rather than the quantized samples. Let us emphasize, though, that the sample mean will outperform $\hat{\Sigma}_n$ in many other scenarios. In particular, $\hat{\Sigma}_n$ is only a suitable estimator if $\text{diag}(\Sigma) = \text{Id}$.

Figure 2: Optimal choices of $\lambda$ for various numbers of samples. Here $p = 5$ is fixed.

Figure 3: The left plot depicts the average estimation errors of $\hat{\Sigma}_n$ and $\hat{\Sigma}_n$ in operator norm, for $p = 20$, $n$ varying from 10 to 300 and three different choices of the ground-truth $\Sigma$ with ones on the diagonal and off-diagonal entries equal to $c = 0.5$, $c = 0.9$, and $c = 0.99$. The right plot depicts the average estimation errors of $\hat{\Sigma}_n$ and $\hat{\Sigma}_n^{\text{dith}}$ in operator norm, for $n = 200$, $p$ varying from 5 to 30 and two different choices of the ground-truth $\Sigma$.

5.3 Two-bit estimator – influence of diagonal

In our last experiment we verify that there is a performance gap between the sample mean $\hat{\Sigma}_n$ and dithered two-bit estimator $\hat{\Sigma}_n^{\text{dith}}$ for covariance matrices $\Sigma$ with $\text{Tr}(\Sigma) \ll p \|\Sigma\|_{\infty}$, see Theorem 3 and the subsequent...
discussion. To this end, we compare the reconstruction of $\Sigma$ having ones on the diagonal and entries 0.2 everywhere else ($\text{Tr}(\Sigma) = p \|\Sigma\|_\infty$) with the reconstruction of $\Sigma'$ being the same as $\Sigma$ apart from $\Sigma'_{1,1} = 10$ ($\text{Tr}(\Sigma) \ll p \|\Sigma\|_\infty$). We fix the number of samples as $n = 200$ and vary $p \in [5, 30]$. Figure 3b shows a considerable increase of the gap in reconstruction accuracy in the second case.

A Appendix: minimax lower bounds

We conclude by deriving minimax lower bounds for the quantized covariance estimation setting. In case of quantization without dithering, a minimax lower bound is directly implied by the results in [11]. We will therefore only consider quantization with dithering. The main observation is the following lemma, which allows to transfer known bounds for estimation from ‘unquantized’ samples from a Gaussian distribution to their one-bit quantized counterparts. We use the following notation. For a Gaussian distribution with zero mean and covariance matrix $\Sigma \in \mathbb{R}^{p \times p}$, we denote the corresponding probability distribution by $\mathbb{P}_\Sigma$ and its density by $\phi_\Sigma$, i.e.,

$$\phi_\Sigma(z) = \frac{1}{\sqrt{(2\pi)^p \det(\Sigma)}} e^{-\frac{1}{2}z^T \Sigma^{-1} z}.$$

If $X \sim \mathcal{N}(0, \Sigma)$, $\lambda > 0$, $\tau \sim \text{Unif}([\lambda, \lambda]^p)$, and $Y = \text{sign}(X + \tau)$, then we denote the probability distribution of $Y$ by $Q_\Sigma$ and use $q_\Sigma$ to denote the associated probability mass function.

**Lemma 18.** For any two covariance matrices $\Sigma, \Sigma' \in \mathbb{R}^{p \times p}$,

$$\sum_{y \in \{-1,1\}^p} \min \{q_\Sigma(y), q_{\Sigma'}(y)\} \geq \int_{\mathbb{R}^p} \min \{\phi_\Sigma(x), \phi_{\Sigma'}(x)\} \, dx.$$

**Proof.** Since $\min\{a, b\} = \frac{1}{2}(a + b) - \frac{1}{2}|a - b|$ for all $a, b \in \mathbb{R}$, it suffices to show that

$$\sum_{y \in \{-1,1\}^p} |q_\Sigma(y) - q_{\Sigma'}(y)| \leq \int_{\mathbb{R}^p} |\phi_\Sigma(z) - \phi_{\Sigma'}(z)| \, dz.$$

Observe that

$$q_\Sigma(y) = \int_{[-\lambda, \lambda]^p} \left(\frac{1}{2\lambda}\right)^p \int_{O_\Sigma^y} \phi_\Sigma(z) \, dz \, d\tau,$$

where $O_\Sigma^y = \{z \in \mathbb{R}^p : \text{sign}(z + \tau) = y\}$. Consequently,

$$\sum_{y \in \{-1,1\}^p} |q_\Sigma(y) - q_{\Sigma'}(y)| = \sum_{y \in \{-1,1\}^p} \int_{[-\lambda, \lambda]^p} \left(\frac{1}{2\lambda}\right)^p \int_{O_\Sigma^y} \phi_\Sigma(z) \, dz \, d\tau - \int_{[-\lambda, \lambda]^p} \left(\frac{1}{2\lambda}\right)^p \int_{O_\Sigma^y} \phi_{\Sigma'}(z) \, dz \, d\tau$$

$$\leq \sum_{y \in \{-1,1\}^p} \int_{[-\lambda, \lambda]^p} \left(\frac{1}{2\lambda}\right)^p \int_{O_\Sigma^y} |\phi_\Sigma(z) - \phi_{\Sigma'}(z)| \, dz \, d\tau$$

$$= \int_{[-\lambda, \lambda]^p} \left(\frac{1}{2\lambda}\right)^p \int_{\mathbb{R}^p} |\phi_\Sigma(z) - \phi_{\Sigma'}(z)| \, dz \, d\tau$$

$$= \int_{\mathbb{R}^p} |\phi_\Sigma(z) - \phi_{\Sigma'}(z)| \, dz.$$

Using Lemma 18, we can easily derive a minimax lower bound by following the steps in [11, Section 3.3.1]. Assume for simplicity that $p$ is even and consider the set of covariance matrices

$$\mathcal{F}^* = \left\{ \Sigma_\theta = \text{Id} + \frac{1}{2\sqrt{n}p} \sum_{k=1}^{2} \theta_k M_k : \theta \in \{0, 1\}^\frac{p}{2} \right\} \subset \left\{ \Sigma \in \mathbb{R}^{p \times p} : \Sigma_{i,i} = 1, \text{ for all } i \in [p] \right\},$$

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where the matrices $M_k \in \mathbb{R}^{p \times p}$ have entries

$$(M_k)_{i,j} = \chi\{i=k \text{ and } k+1 \leq j \leq p, \text{ or } j=k \text{ and } k+1 \leq i \leq p\}.$$ 

Note that [11, Lemma 5] applies to $F^*$, for $n > p^{1+2\alpha}$ where $\alpha > 0$ may be chosen arbitrarily small. Using Lemma 18 we can now extend [11, Lemma 6] from $X \sim \mathcal{N}(0, \Sigma)$ to $Y$ as defined in the beginning of this section to obtain the following result.

**Theorem 19.** Assume $n > p$. Then, for some absolute constant $c > 0$,

$$\inf_{\Sigma} \sup_{\Sigma \in F^*} \mathbb{E} \left\| \hat{\Sigma} - \Sigma \right\|^2 \geq \frac{c}{n}.$$

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