A FIELD THEORY OF KNOTTED SOLITONS

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Abstract. The conjecture that the elementary fermions are knotted flux tubes permits the construction of a phenomenology that is not accessible from the standard electroweak theory. In order to carry these ideas further we have attempted to formulate the elements of a field theory in which local $SU(2) \times U(1)$, the symmetry group of standard electroweak theory, is combined with global $SU_q(2)$, the symmetry group of knotted solitons.
1 Introduction.

We have previously discussed the phenomenology of \( q \)-solitons.\(^{1,2,3} \) We should now like to incorporate the \( q \)-soliton into a field theory that agrees closely with the standard point particle electroweak theory and at the same time replaces the point particles by knotted solitons. To do this we retain both local \( SU(2) \times U(1) \), the symmetry group of standard electroweak, and global \( SU_q(2) \), the symmetry group of the \( q \)-solitons. This is done by retaining the formal structure of standard electroweak but at the same time expanding the quantum fields in a new class of normal modes determined by the \( SU_q(2) \) algebra. Then all quantum vector fields will lie in \( SU(2) \times U(1) \) Lie algebra as well as in the disjoint \( SU_q(2) \) algebra.

2 Lagrangians.

The standard Lagrangian density is

\[
\mathcal{L} = -\frac{1}{4}(G^{\mu\nu}G_{\mu\nu} + H^{\mu\nu}H_{\mu\nu}) + i[\bar{L} \nabla L + \bar{R} \nabla R] + (\nabla \phi)^+ \cdot \nabla \phi - V(\phi^+ \phi) - \frac{m}{\rho_0}(L\phi R + \bar{R}\bar{\phi}^+ L) \tag{2.1}
\]

where

\[
G^{\mu\nu} = \partial^\mu \vec{W}^\nu - \partial^\nu \vec{W}^\mu - g \vec{W}^\mu \times \vec{W}^\nu, \quad R = \frac{1}{2} (1 + \gamma_5) \psi
\]

\[
H^{\mu\nu} = \partial^\mu W_o^\nu - \partial^\nu W_o^\mu, \quad L = \frac{1}{2} (1 - \gamma_5) \psi
\]

\[
\nabla^\mu = \partial^\mu + igW_o^\mu \cdot \vec{t} + igW_o^\mu \cdot t_o
\]

The changes that we make amount to primarily multiplying the usual normal modes of \( \vec{W}_\mu, \phi, \) and \( \psi \) by suitably chosen \( D^j_{m,m'}(a, \bar{a}, b, \bar{b}) \), the elements of the irreducible representations of \( SU_q(2) \). This step is analogous to appending the spin states to \( e^{i\vec{p} \vec{x}} \) and is determined by the symmetries of the algebra of \( (a, \bar{a}, b, \bar{b}) \). We shall also introduce small modifications of (2.1).
After the changes are made the new symmetry of the Lagrangian density will be

\[ \text{[local } SU(2) \times U(1)] \otimes \text{[global } U(1)] \]

and the new Lagrangian will be invariant under the new symmetry group.

\section{3 The Irreducible Representations of $SU_q(2)$.}

The $(2j + 1)$-dimensional unitary irreducible representations of $SU_q(2)$ are

\[ D^j_{mm'}(a, \bar{a}, b, \bar{b}) = \Delta^j_{mm'} \sum_{s,t} \left\langle \frac{n_+}{s} \middle| \frac{n_-}{t} \right\rangle_1 q_1^{(n_+ + 1 - s)(-1)^t} \delta(s + t, n'_+) a^s b^{n_+ - s} \bar{b}^t \bar{a}^{n_- - t} \]  

(3.1)

where

\[ n_\pm = j \pm m \quad \left\langle \frac{n}{s} \right\rangle_1 = \frac{\langle n \rangle_1!}{\langle s \rangle_1!\langle n - s \rangle_1!} \quad \langle n \rangle_1 = \frac{q_1^{2n} - 1}{q_1^2 - 1} \]  

(3.2)

and

\[ \Delta^j_{mm'} = \left[ \frac{\langle n'_+ \rangle_1! \langle n'_- \rangle_1!}{\langle n_+ \rangle_1! \langle n_- \rangle_1!} \right]^{1/2} \quad q_1 = q^{-1} \]  

(3.3)

Here the arguments $(a, b, \bar{a}, \bar{b})$ obey the following algebra:

\[ ab = qba \quad a\bar{a} + b\bar{b} = 1 \quad b\bar{b} = \bar{b}b \]

\[ \bar{a}b = q\bar{b}a \quad \bar{a}a + q_1^2 \bar{b}b = 1 \]  

(A)

In the limit $q = 1$, where $\langle n \rangle_1 \to n$ and the arguments commute, the $D^j_{mm'}$ become the matrix elements of the irreducible representations of $SU(2)$.

Remarks:

Every term of (3.1) contains a product of non-commuting factors that may be reduced (after dropping numerical factors) to the form

\[ a^{n_a} b^{n_b} \bar{b}^{n_\bar{b}} \bar{a}^{n_{\bar{a}}} \]  

(3.4)

where $n_a = s$ and $n_{\bar{a}} = n_- - t$.

But the factor $\delta(s + t, n'_+)$ appearing in (3.1) implies

\[ n'_+ = n_a + (n_- - n_{\bar{a}}) \]  

(3.5)
or
\[ n_a - n_a = m + m' \] (3.6)

Note that (3.6) holds for all terms of (3.1) and all representations (j).

By a similar argument one notes that
\[ nb - nb = m - m' \] (3.7)
again holding for all terms of (3.1) and all representations.

**State Space:**

The \(D_{mm'}^j\) take on numerical values when allowed to operate on a state space that may be defined as follows:

Since \([b, \bar{b}] = 0\), \(b\) and \(\bar{b}\) have common eigenstates. Let \(|0\rangle\) be the ground state and let
\[ b|0\rangle = \beta|0\rangle \] (3.8)
\[ \bar{b}|0\rangle = \beta^*|0\rangle \] (3.9)

From the algebra one finds
\[ \bar{b}b|n\rangle = E_n|n\rangle \] (3.10)
where
\[ E_n = q^{2n}|eta|^2 \] (3.11)
and
\[ |n\rangle \sim \hat{a}^n|0\rangle \] (3.12)

Here \(\bar{b}b\) is Hermitian with real eigenvalues and orthogonal eigenstates.

**4 Invariance of the Algebra (A).**

The algebra (A) is invariant under the following transformations
\[ a' = e^{i\phi}a \]
\[ \bar{a}' = e^{-i\phi}\bar{a} \] (4.1)
\[ b' = e^{i\phi b} \]
\[ \bar{b}' = e^{-i\phi b} \]  

The gauge transformations induced on the irreducible representations are then

\[ U_a D^j_{mm'} = e^{i\varphi_a(n_a-n_{\bar{a}})} D^j_{mm'} \]  
\[ U_b D^j_{mm'} = e^{i\varphi_b(n_b-n_{\bar{b}})} D^j_{mm'} \]  

since (3.6) and (3.7) hold for all terms of \( D^j_{mm'} \). Define with the aid of (3.6) and (3.7)

\[ Q_a \equiv k(n_a - n_{\bar{a}}) \equiv k(m + m') \]  
\[ Q_b \equiv k(n_b - n_{\bar{b}}) \equiv k(m - m') \]  

Then (4.1) and (4.2) induce the following gauge transformations on the irreducible representations:

\[ U_a D^j_{mm'} = e^{iQ_a\varphi_a/k} D^j_{mm'} \]  
\[ U_b D^j_{mm'} = e^{iQ_b\varphi_b/k} D^j_{mm'} \]  

According to (4.5) and (4.6) both \( Q_a \) and \( Q_b \) reverse sign as one goes from \( D^j_{mm'} \) to \( \bar{D}^j_{mm'} \). Since we shall associate \( D^j_{mm'} \) and \( \bar{D}^j_{mm'} \) with particles and antiparticles, \( Q_a \) and \( Q_b \) reverse sign as one passes to the antiparticle.

5 The Knot Representation.

Since \( SU_q(2) \) underlies the description of knots, we shall make this connection explicit by labelling a knot with \( D^{N/2}_{w/2-1/2} \) where \( (N, w, r) \) are the number of crossings, the writhe, and the rotation of the knot. The minimum value of \( N \) is 3, and the corresponding knot is a trefoil. There are four possible trefoils and they are characterized by \( (w, r) = (-3, 2), (3, 2), (3, -2), (-3, -2) \).

We base our work on the physical conjecture that the elementary fermions are knotted flux tubes. More precisely we associate each of the four families of elementary fermions (leptons, neutrinos, up quarks, down quarks) with one of the four trefoils and each of the three members of a family with the three lowest states of excitation of its trefoil. Thus the
electron, muon, and tau are the three lowest states of the lepton soliton. Then the fermionic families are also labelled by \( D_{mm'}^{3/2} = D_{m+1}^{3/2} \) and we shall relate \( Q_a \) and \( Q_b \) as defined in (4.5) and (4.6), to the electric charge and hypercharge of the corresponding family. In fact we may match the 4 trefoils with the 4 families in Table 1 if and only if \( k = -1/3 \) in (4.5).

Table 1.

| \((w, r)\) | \(D_{m+1}^{3/2}\) | \(Q_a\) | Family |
|-----------|-----------------|--------|--------|
| (-3, 2)   | \(D_{-1}^{3/2}\) | 0      | \((\nu_e, \nu_\mu, \nu_\tau)\) |
| (3, 2)    | \(D_{1}^{3/2}\)  | -1     | \((e, \mu, \tau)\)       |
| (3, -2)   | \(D_{-3}^{3/2}\) | -1/3   | \((dsb)\)                 |
| (-3, -2)  | \(D_{1}^{3/2}\)  | 2/3    | \((uct)\)                 |

Here

\[ Q_a = -\frac{1}{3}(m + m') \quad (5.1) \]

Note that \( m \) distinguishes between the two members of a doublet while \( m' \) labels either the light \((e, \nu)\) doublets or the heavy \((u, d)\) doublet. To complete the correspondence we write \( D_{mm'}^j \) in terms of the conventional quantum numbers that label the point particles of standard theory. Since all fermions have \( t = 1/2 \) and \( t_3 = \pm 1/2 \) while the trefoils have \( N = 3 \) and \( w = \pm 3 \) we may set

\[ t = \frac{N}{6} \quad (5.2) \]
\[ t_3 = -\frac{w}{6} \quad (5.3) \]

or

\[ D_{mm'}^j = D_{w/3+t_{31}}^{N/2} \quad (5.4a) \]

\[ = D_{3t_3 m'}^{3t_3 m'} \quad (5.4b) \]

But

\[ Q_a = -\frac{1}{3}(m + m') \]
\[ = t_3 - \frac{m'}{3} \quad (5.5) \]
Since \( Q_a \) is the electric charge, we also have according to the standard theory:

\[
Q_a = t_3 + t_0
\]

so that by (5.5)

\[
t_0 = -\frac{m'}{3}
\]

and

\[
Q_b = t_3 - t_0
\]

Hence the two charges \( Q_a \) and \( Q_b \) stemming from the gauge invariance of the internal algebra \((A)\) are simply related to the electric charge and hypercharge appearing in standard theory. We also note that \( Q_a = 0 \) implies \( n_a = n_{\bar{a}} \). Then \( a \) and \( \bar{a} \) may be eliminated from (3.4) since

\[
a^n \bar{a}^n = \prod_{s=0}^{n-1} (1 - q^{2s}b\bar{b})
\]

Therefore neutral states (neutrinos and neutral bosons) lie entirely in the \((b, \bar{b})\) subalgebra.

Since these transformations on the internal algebra are assumed to be global, they do not give rise to new fields. Rather they simply define the charge and hypercharge of the fermionic sources of the vector fields.

Having found the explicit representations \( D_{mn'}^{3/2} \) of the fermions we now turn to the representations of the vector bosons. Since the vector bosons are responsible for pair production, we shall represent them by ditrefoils with \( N = 6 \). Then by (5.2) and (5.4a)

\[
t = \frac{N}{6} = 1
\]

\[
j = \frac{N}{2} = 3
\]

We now find Table 2

|   | \( t \) | \( t_3 \) | \( t_0 \) | \( D_{mn'}^{3} \) |
|---|---|---|---|---|
| \( W^+ \) | 1 | 1 | 0 | \( D_{30}^{3} \) |
| \( W^- \) | 1 | -1 | 0 | \( D_{30}^{3} \) |
| \( W^3 \) | 1 | 0 | 0 | \( D_{00}^{3} \) |
where \( t_3 \) and \( t_0 \) are taken from standard electroweak theory. Here we have used
\[
m = -3t_3 \tag{5.11}
\]
\[
m' = -3t_0 \tag{5.12}
\]
that follow from (5.4) and (5.7).

Since \( W^0 \) in the standard theory arises from coupling to \( U(1) \) it does not belong to the isotopic spin multiplet, \( t = 1 \), and therefore its representation cannot be fixed by \( t_3 \) and \( t_0 \).

Since \( Q = -\frac{1}{3}(m + m') \) represents the electric charge, all that can be said about \( W^0 \) at this point is that \( m + m' = 0 \). \( W^0 \) can then be represented by any \( D_{mm'}^i \), where \( m + m' = 0 \), except \( D_{00}^3 \), which is excluded unless \( W^0 \) is otherwise distinguished from \( W^3 \).

Dropping numerical factors, the explicit monomials that label the four fermionic solitons are as follows:
\[
D_{\nu}^{3/2} = \bar{b}^3, \quad D_{\ell}^{3/2} = a^3, \quad D_{u}^{3/2} = \bar{a}^2 \bar{b}, \quad D_{d}^{3/2} = ab^2 \tag{5.13}
\]
The corresponding forms labelling the charged vectors are
\[
D_3^- = a^3 b^3, \quad D_3^+ = \bar{b}^3 \bar{a}^3 \tag{5.14a}
\]
while \( D_3^3 \) and \( D_0^3 \), labelling the neutral vectors are polynomials lying in the \( (b, \bar{b}) \) subalgebra.

One possibility is
\[
D_3^3 = D_{00}^3, \quad D_0^3 = D_{-11}^3 \tag{5.14b}
\]

6 The Normal Modes of the Quantum Fields.

In the proposed formalism the normal modes of the standard electroweak vector fields are to be modified by multiplying the generators \( (\tilde{t}, t_o) \) of the Lie algebra of \( SU(2) \times U(1) \) by the \( D_\alpha^3(a, \bar{a}, b, \bar{b}), \alpha = (+, -, 3, 0) \) as follows:
\[
\tilde{t} \rightarrow \tilde{\tau} = (c_+ t_+ D_+^3, c_- t_- D_-^3, c_3 t_3 D_3^3) \tag{6.1a}
\]
\[
t_0 \rightarrow \tau_0 = t_0 D_0^3 \tag{6.1b}
\]
In the fundamental representation
\[
(\tilde{\tau}, \tau_0) = \begin{pmatrix} 0 & c_+ D_+^3 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ c_- D_-^3 & 0 \end{pmatrix}, \quad \begin{pmatrix} c_3 D_3^3 & 0 \\ 0 & -c_3 D_3^3 \end{pmatrix}, \quad \begin{pmatrix} c_0 D_0^3 & 0 \\ 0 & c_0 D_0^3 \end{pmatrix} \tag{6.2}
\]
where the $D^3_\alpha (\alpha = +, -, 3, 0)$ are given by (5.14). The numerically valued matrices $(\vec{t}, t_0)$ are in this way replaced by the operator valued $(\vec{\tau}, \tau_0)$.

For the fermions and Higgs-like fields one replaces the numerically valued 2-rowed spinors of isotopic spin $SU(2)$ by the following operator valued spinors

\[
\begin{pmatrix}
D^3_{\nu}/2 & 0 \\
0 & D^3_{\ell}/2
\end{pmatrix},
\begin{pmatrix}
D^3_{u}/2 & 0 \\
0 & D^3_d/2
\end{pmatrix}
\]

\[= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D^3_{\nu}/2, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D^3_{\ell}/2, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D^3_{u}/2, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} D^3_d/2 \quad (6.3)
\]

where $D^3_r$ is an abbreviation for the irreducible representation associated with the $r$th soliton and where $r = (\nu, \ell, u, d)$. We also introduce $\psi_{Ari}$

\[
\psi_{1ri} = \psi_{1r} D_r (a\bar{a}bb) |i\rangle \quad A = 1 \quad r = \nu, \ell \quad i = 1, 2, 3
\]

\[
\psi_{2ri} = \psi_{2r} D_r (a\bar{a}bb) |i\rangle \quad A = 2 \quad r = u, d \quad i = 1, 2, 3 \quad (6.4)
\]

where the lepton and neutrino solitons are combined into one isotopic spinor, $\psi_{1ri}$, and the up and down quarks into a second isotopic spinor, $\psi_{2ri}$ and where $i$ runs over the three states of the soliton. Then

\[
\psi_1 = \begin{pmatrix} \psi_{\nu} D^3_{\nu}/2 |i\rangle \\ \psi_{\ell} D^3_{\ell}/2 |i\rangle \end{pmatrix} \quad \text{and} \quad \psi_2 = \begin{pmatrix} \psi_{u} D^3_{u}/2 |i\rangle \\ \psi_{d} D^3_{d}/2 |i\rangle \end{pmatrix} \quad (6.5)
\]

We introduce the 4 bosons by defining

\[
\mathcal{W}_{rs} = ig \vec{W} \vec{\tau}_{rs} + ig' W^0(\tau_0)_{rs} \quad (6.6)
\]

where

\[
(\tau_\pm)_{rs} = c_\pm (t_\pm)_{rs} D^3_\pm
\]

\[
(\tau_3)_{rs} = c_3 (t_3)_{rs} D^3_3
\]

\[
(\tau_0)_{rs} = c_0 (t_0)_{rs} D^3_0 \quad (6.7)
\]

and $(\vec{W}, W^0)$ are numerically valued and replace the components of the standard boson field and $\mathcal{W}$ lies in the internal algebra. The numerical coefficients $(c_\pm, c_3, c_0)$ in (6.7) are to be
considered functions of $q$ and $\beta$, the parameters of the model. They are fixed by the relative masses of the vectors and will be so determined in Section 10.

Here $(D_3^1, D_3^2, D_0^3)$ are given by (3.1) and (5.14). The covariant derivative is now

$$\nabla_{rs} = \delta_{rs} \partial + W_{rs}$$

(6.8)

The corresponding field strengths are

$$W_{\mu\lambda} = [\nabla_\mu, \nabla_\lambda]$$

(6.9)

We shall introduce the direct boson-fermion interactions as follows:

$$(\bar{\psi}_A ri \nabla_{rs} U_A(\psi_A)_{s'i'} A = 1, 2 \quad (r, s) = (\nu, \ell) \text{ or } (u, d) \text{ and } i, i' = 1, 2, 3$$

(6.10)

where $A = 1$ labels the $(\nu, \ell)$ doublet and $A = 2$ labels the quark doublet $(u, d)$ according to (6.4).

The form of $U_1$ is restricted by the universal Fermi interaction, while $U_2$ replaces the Kobayashi-Maskawa matrix. Here $U_2$ “rotates” the initial state $(\psi_A)_{s'i'}$.

### 7 The $\tau$-Commutators.

The total field strength is by (6.9) and (6.6)

$$W_{\mu\lambda} = [\nabla_\mu, \nabla_\lambda]$$

(7.1)

$$= ig(\partial_\mu \bar{W}_\lambda - \partial_\lambda \bar{W}_\mu)\tau_f + ig'(\partial_\mu W_\lambda^0 - \partial_\lambda W_\mu^0)\tau_0 - g^2 W_\mu^m W_\lambda^k [\tau_m, \tau_k]$$

In extracting the $\tau_k$-independent fields from the Tr $\mathcal{W}^{\mu\lambda} W_{\mu\lambda}$ invariant one needs to compute commutators of the $\tau_k$. By (6.7)

$$[\tau_k, \tau_\ell] = c_k c_\ell [t_k D_k, t_\ell D_\ell]$$

(7.2)

where

$$t_k = (t_+, t_-, t_3)$$

or

$$t_k = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(7.3)
and

\[ D_k = (D_+, D_-, D_3) \quad (7.4) \]

where by (5.14)

\[ D_+ = D^2_{30}/N_+ = \bar{b}^3 \bar{a}^3 \quad (7.5) \]
\[ D_- = D^3_{30}/N_- = a^3 b^3 \quad (7.6) \]
\[ D_3 = D^3_{00} = f_3(\bar{b} \bar{b}) \quad (7.7) \]

where the \( N_\pm \) are \( c \)-number normalizing factors and \( f_3 \) is a function of \( \bar{b} \bar{b} \) as determined by (3.1). Note

\[ \bar{D}_+ = D_- \quad \text{and} \quad \bar{D}_3 = D_3 \quad (7.8) \]

One has the familiar

\[ [t_+, t_3] = -2t_+ \]
\[ [t_-, t_3] = 2t_- \quad (7.9) \]
\[ [t_+, t_-] = t_3 \]

and

\[ [t_i, t_j] = c^k_{ij} t_k \quad (i, j, k) = (+, -, 3) \quad (7.10) \]

Similarly one has

\[ [D_+, D_3] = \hat{c}^+_{{+3}} D_+ \quad (7.11) \]
\[ [D_-, D_3] = \hat{c}^-_{-{3}} D_- \quad (7.12) \]
\[ [D_+, D_-] = \hat{c}^3_{+\!-} D_3 \quad (7.13) \]

where

\[ \hat{c}^+_{{+3}} = f_3(q^6_1 \bar{b} \bar{b}) - f_3(\bar{b} \bar{b}) \quad (7.14) \]

By (7.8) and (7.11)

\[ \hat{c}^-_{-{3}} = -\hat{c}^+_{{+3}} \quad (7.15) \]

and by (7.5)-(7.7)

\[ \hat{c}^3_{+\!-} = (\bar{b} \bar{b})^3[\bar{a}^3 a^3 - q^{18} a^3 \bar{a}^3]/f_3(\bar{b} \bar{b}) \quad (7.16) \]
Here
\[
a^3a^3 = \prod_{t=1}^{3}(1 - q_{1}^{2t}\bar{b}b)
\] (7.17)
\[
a^3\bar{a}^3 = \prod_{t=0}^{2}(1 - q_{2}^{2t}\bar{b}b)
\] (7.18)

It follows from these equations and from (7.14)-(7.16) that all \(c_{ij}^{k}\) are functions of \(\bar{b}b\) and mutually commute.

We now return to (7.2). Since \(t_k\) and \(D_{\ell}\) commute, we have
\[
[t_k, t_\ell] = c_{k\ell}(t_k, t_\ell)[D_k D_\ell + t_\ell t_k[D_k, D_\ell]]
\] (7.19)

By (7.3)
\[
t_+t_3 = -t_+ \quad t_3t_+ = t_+
\] (7.20)
\[
t_-t_3 = t_- \quad t_3t_- = -t_-
\] (7.21)
\[
t_+t_- = \frac{1}{2}t_3 + \frac{1}{2} \quad t_-t_+ = -\frac{1}{2}t_3 + \frac{1}{2}
\] (7.22)

By (7.5)-(7.7)
\[
D_+D_3 = \bar{b}^3a^3 f_3(\bar{b}b) = f_3(q^6\bar{b}b)D_+
\] (7.24)
\[
D_-D_3 = a^3\bar{b}^3 f_3(\bar{b}b) = f_3(q^6\bar{b}b)D_-
\] (7.25)
\[
D_+D_- = (\bar{b}^3a^3)(a^3\bar{b}^3) = (\bar{b}b)^3(a^3a^3) = f_+ - (\bar{b}b)D_3
\] (7.26)
\[
D_-D_+ = (a^3\bar{b}^3)(\bar{b}^3\bar{a}^3) = (\bar{b}b)^3q^{18}(a^3\bar{a}^3) = f_+ + (\bar{b}b)D_3
\] (7.27)

Here \(f_{\pm}\) and \(f_3\) are all functions of \(\bar{b}b\) and therefore commute since \(\bar{a}^3a^3, a^3\bar{a}^3, \) and \(D_3\) are all functions of \(\bar{b}b\).

The parts of (7.19) are then by (7.9), and by (7.20)-(7.27),
\[
[t_k, t_\ell] = \gamma_{k\ell}^{s}t_s \quad t_k t_\ell = \gamma_{k\ell}^{s}t_s + \gamma_{k\ell}
\] (7.28)
\[
[D_k, D_\ell] = \hat{c}_{k\ell}^{s}D_s \quad D_k D_\ell = \hat{c}_{k\ell}^{s}D_s
\] (7.29)

where
\[
\gamma_{k\ell} = \frac{1}{2}\delta(k, \pm)\delta(\ell, \mp)
\] (7.30)
The coefficients $c^s_{k\ell}$ and $\gamma^s_{k\ell}$ are numerically valued while $\hat{c}^s_{k\ell}$ and $\hat{\gamma}^s_{k\ell}$ are functions of $\bar{b}\bar{b}$.

By (7.19) and (7.28)-(7.30), one now has

\[
[\tau_k, \tau_\ell] = c_k c_\ell \left( [c^s_{k\ell} t_s] (\hat{\gamma}^s_{k\ell} D_s) + (\gamma^s_{\ell k} t_s + \gamma_{\ell k}) (\hat{c}^s_{k\ell} D_s) \right)
\]

or

\[
[\tau_k, \tau_\ell] = \frac{c_k c_\ell}{c_s} C^s_{k\ell} \tau_s + \frac{c_k c_\ell}{c_s} \gamma_{\ell k} \hat{c}^s_{k\ell} D_s \]

where

\[
C^s_{k\ell} = c^s_{k\ell} \hat{\gamma}^s_{k\ell} + \gamma_{\ell k} \hat{c}^s_{k\ell}
\]

The structure coefficients of the algebra (7.32) and (7.33) depend on the numerically valued $c^s_{k\ell}$ and $\gamma^s_{\ell k}$ but also on the $\hat{c}^s_{k\ell}$ and $\hat{\gamma}^s_{k\ell}$ that are functions of $\bar{b}\bar{b}$ and are therefore also numerically valued when allowed to operate on the states $|n\rangle$ of the $q$-oscillator.

8 The Field Invariant $\langle 0 | Tr \ W^{\mu\lambda} W_{\mu\lambda} | 0 \rangle$.

Here the trace is taken on only the matrix part that is dependent on the $t$. By (7.1) the non-Abelian contribution to the field strength is

\[
W_{\mu\lambda} = ig(\partial_\mu W^s_\lambda - \partial_\lambda W^s_\mu) \tau_s - g^2 W^m_\mu W^\ell_\lambda [\tau_m, \tau_\ell]
\]

or

\[
W_{\mu\lambda} = ig(\partial_\mu W^s_\lambda - \partial_\lambda W^s_\mu) \tau_s - g^2 [c_m c_\ell c^s_{\ell m} C^s_{m\ell} \tau_s + c_m c_\ell c_{\ell m} \hat{c}^s_{m\ell} D_s] W^m_\mu W^\ell_\lambda
\]

By (7.32), or

\[
W_{\mu\lambda} = W^s_{\mu\lambda} \tau_s + \hat{W}^s_{\mu\lambda} D_s
\]

where

\[
W^s_{\mu\lambda} = ig(\partial_\mu W^s_\lambda - \partial_\lambda W^s_\mu) - g^2 c_m c_\ell c^s_{\ell m} C^s_{m\ell} W^m_\mu W^\ell_\lambda
\]

\[
\hat{W}^s_{\mu\lambda} = -\frac{1}{2} g^2 c_m c_\ell \delta^{(\ell, \pm)} \delta(m, \mp) \hat{c}^s_{m\ell} W^m_\mu W^\ell_\lambda
\]

by (7.30). Then

\[
W_{\mu\lambda} W^{\mu\lambda} = W^s_{\mu\lambda} W^{\tau\mu\lambda\tau_s} \tau_r + \hat{W}^s_{\mu\lambda} \hat{W}^{\tau\mu\lambda} D_s D_r
\]

\[
\quad + W^s_{\mu\lambda} \tau_s \cdot \hat{W}^{\tau\mu\lambda} D_s + \hat{W}^s_{\mu\lambda} D_s \cdot W^{\tau\mu\lambda} \tau_r
\]
In order to reduce the invariant $\text{Tr}(0|\mathcal{W}_{\mu\lambda}\mathcal{W}^{\mu\lambda}|0)$ we next consider the expectation value of the expression (8.6) on the state $|0\rangle$

$$
\langle 0|\mathcal{W}_{\mu\lambda}\mathcal{W}^{\mu\lambda}|0\rangle = \langle 0|W^s_{\mu\lambda}W^{r\mu\lambda}\tau_s\tau_r + \hat{W}^s_{\mu\lambda}\hat{W}^{r\mu\lambda}\mathcal{D}_s\mathcal{D}_r

+ (W^s_{\mu\lambda}\tau_s) \cdot (W^{r\mu\lambda}\mathcal{D}_r) + (\hat{W}^s_{\mu\lambda}\mathcal{D}_s) \cdot (W^{r\mu\lambda}\tau_r)|0\rangle
$$

(8.7)

where $\mathcal{W}_{\mu\lambda}$ is given by (8.3)

$$
= \sum_n \left[ \langle 0|W^s_{\mu\lambda}W^{r\mu\lambda}|n\rangle \langle n|\tau_s\tau_r|0\rangle + \langle 0|\hat{W}^s_{\mu\lambda}\hat{W}^{r\mu\lambda}|n\rangle \langle n|\mathcal{D}_s\mathcal{D}_r|0\rangle \right]

+ \sum_n \left[ \langle 0|W^s_{\mu\lambda}\hat{W}^{r\mu\lambda}|n\rangle \langle n|\tau_s\mathcal{D}_r|0\rangle + \langle 0|\hat{W}^s_{\mu\lambda}W^{r\mu\lambda}|n\rangle \langle n|\mathcal{D}_r\tau_s|0\rangle \right]
$$

(8.8)

where $|n\rangle$ are the states of the $q$-oscillator.

Since $W^s_{\mu\lambda}$ and $\hat{W}^s_{\mu\lambda}$ depend on the algebra only through $\bar{b}b$, they have no off-diagonal elements in $n$. Then

$$
\langle 0|\mathcal{W}_{\mu\lambda}\mathcal{W}^{\mu\lambda}|0\rangle = \langle 0|W^s_{\mu\lambda}W^{r\mu\lambda}|0\rangle \langle 0|\tau_s\tau_r|0\rangle + \langle 0|\hat{W}^s_{\mu\lambda}\hat{W}^{r\mu\lambda}|0\rangle \langle 0|\mathcal{D}_s\mathcal{D}_r|0\rangle

+ \langle 0|W^s_{\mu\lambda}\hat{W}^{r\mu\lambda}|0\rangle \langle 0|\tau_s\mathcal{D}_r|0\rangle + \langle 0|\hat{W}^s_{\mu\lambda}W^{r\mu\lambda}|0\rangle \langle 0|\mathcal{D}_r\tau_s|0\rangle
$$

(8.9)

To continue the reduction of the field invariant $I = \langle 0|\text{Tr} \mathcal{W}_{\mu\lambda}\mathcal{W}^{\mu\lambda}|0\rangle$ we next compute

$$
\langle 0|\text{Tr} \tau_s\tau_r|0\rangle = c_sc_r\langle 0|\text{Tr} t_st_r|0\rangle \mathcal{D}_s\mathcal{D}_r|0\rangle

= c_sc_r\langle 0|\text{Tr} t_st_r|0\rangle \mathcal{D}_s\mathcal{D}_r|0\rangle
$$

(8.10)

since the trace is taken only on matrices dependent on the $t$. Then by (7.3)

$$
\text{Tr} t_st_r = \delta(s, \pm)\delta(r, \mp) + 2\delta(s, 3)\delta(r, 3)
$$

(8.11)

One also finds by (7.5)-(7.7)

$$
\langle 0|\mathcal{D}_s\mathcal{D}_r|0\rangle = [\delta(s, \pm)\delta(r, \mp) + \delta(s, 3)\delta(r, 3)]\langle 0|\bar{\mathcal{D}}_s\mathcal{D}_r|0\rangle
$$

(8.12)

while

$$
\langle 0|\text{Tr} \tau_s\mathcal{D}_r|0\rangle = \langle 0|\text{Tr} \mathcal{D}_r\tau_s|0\rangle = 0
$$

(8.13)

Then the field invariant reduces to the following expression

$$
I = \sum_{s,r= (+, -)} \langle 0|A_{sr}sW^s_{\mu\lambda}W^{\mu\lambda r} + 2\hat{W}^s_{\mu\lambda}\hat{W}^{r\mu\lambda}\mathcal{D}_s\mathcal{D}_r|0\rangle
$$

(8.14)
where

\[ A_{sr} = c_s c_r \, \text{Tr} \, t_s t_r \]  \hspace{1cm} (8.15)

Here \( \langle 0 | \mathcal{D}_s \mathcal{D}_r | 0 \rangle = 0 \) unless \( \mathcal{D}_s = \bar{\mathcal{D}}_r \) and

\[ \langle 0 | \bar{\mathcal{D}}_+ \mathcal{D}_+ | 0 \rangle = \langle 0 | a^3 b^3 \bar{a}^3 \bar{b}^3 | 0 \rangle = q^{18} \langle 0 | (\bar{bb})^3 a^3 \bar{a}^3 | 0 \rangle \]  \hspace{1cm} (8.16)

\[ \langle 0 | \bar{\mathcal{D}}_- \mathcal{D}_- | 0 \rangle = \langle 0 | \bar{b}^3 \bar{a}^3 a^3 b^3 | 0 \rangle = \langle 0 | \bar{bb} a^3 \bar{a}^3 | 0 \rangle \]  \hspace{1cm} (8.17)

\[ \langle 0 | \bar{\mathcal{D}}_3 \mathcal{D}_3 | 0 \rangle = |f(\bar{bb})|^2 \]  \hspace{1cm} (8.18)

These matrix elements are all functions of \( \bar{bb} \), while \( W^{s}_{\mu\lambda} \) and \( \hat{W}^{s}_{\mu\lambda} \) in (8.14) are given by (8.4) and (8.5) respectively.

The expression \( W^{s}_{\mu\lambda} \) is of the same form as in the standard theory but the structure coefficients differ from those of the SU(2) algebra because they depend on \( \bar{bb} \). Since (8.14) is evaluated on the state \( |0\rangle \) all expressions of the form \( F(\bar{bb}) \) become \( F(|\beta|^2) \). Therefore the structure constants \( C^{s}_{mt}(\bar{bb}) \) buried in \( W^{s}_{\mu\lambda} \) and in turn appearing in (8.14) become \( C^{s}_{mt}(|\beta|^2) \).

Then the final reduced form of \( \langle 0 | \text{Tr} \mathcal{W}_{\mu\lambda} \mathcal{W}^{\mu\lambda} | 0 \rangle \) will have one part \( (W^{s}_{\mu\lambda} W^{s}_{\mu\lambda}) \) essentially the same as the standard theory, but with structure constants \( C^{s}_{mt} \) depending on \( |\beta|^2 \). There is also a second part \( (\hat{W}^{s}_{\mu\lambda} \hat{W}^{s}_{\mu\lambda}) \) given in (8.5) depending on \( \hat{c}^{s}_{mt} \) which is (by (7.14)-(7.16)) dependent on \( q \) and \( \beta \). The sum of these two parts is multiplied by \( \langle 0 | \bar{\mathcal{D}}_s \mathcal{D}_s | 0 \rangle \), again a function of \( q \) and \( \beta \) as shown in (8.16)- (8.18) and (7.17) and (7.18). The functions \( c_s(q, \beta) \) will be given in Section 10.

9 Gauge Invariance.

The new gauge group is generated by the following unitary transformations:

\[ S = S \otimes s \]  \hspace{1cm} (9.1)

where

\[ S \in \text{local}[SU(2) \otimes U(1)] \]  \hspace{1cm} (9.2)

and

\[ s \in \text{global} \, U_a(1) \otimes U_b(1) \]  \hspace{1cm} (9.3)
or

\[ S = e^{i\vec{u}(x)}e^{i\theta_0(x)} \]  

(9.4)

and

\[ s = e^{iQ_a\theta_a}e^{iQ_b\theta_b} \]  

(9.5)

where \( \theta_a \) and \( \theta_b \) are independent of \( x \). Then

\[
\mathcal{S}\psi_1 = S \begin{pmatrix} D_\nu \\ D_\ell \end{pmatrix} = S \otimes s \begin{pmatrix} D_\nu \\ D_\ell \end{pmatrix} 
\]

(9.6)

\[
= S \begin{pmatrix} D'_\nu \\ D'_\ell \end{pmatrix} 
\]

(9.7)

\[
= e^{i\vec{u}(x)}e^{i\theta_0(x)} \begin{pmatrix} D'_\nu \\ D'_\ell \end{pmatrix} 
\]

(9.8)

where

\[ D'_k = e^{iQ_a(k)\theta_a}e^{iQ_b(k)\theta_b}D_k \quad k = (\nu, \ell) \]  

(9.9)

with similar equations for the quark solitons. Eq. (9.5) describes the action of \( s \) on irreducible representations only. In general however \( s \) denotes the action of the gauge transformations (4.1) and (4.2) on the entire Lagrangian including the interaction (6.10).

In (6.10) therefore

\[
sU_2\psi_2 = U'_2\psi'_2 
\]

\[
= U'_2 \begin{pmatrix} D'_u \\ D'_d \end{pmatrix} 
\]

The interaction terms will transform as

\[
(\bar{\psi}_A)' \nabla'(U_A\psi_A)' = \bar{\psi}_A\mathcal{S} \nabla'(U_A\psi_A) 
\]

(9.10)

Since \( \mathcal{S} \) is unitary

\[
= \bar{\psi}_A\mathcal{S}^{-1} \nabla'(SU_A\psi_A) 
\]

(9.11)

Then the interaction terms are invariant if

\[
\nabla' = \mathcal{S} \nabla\mathcal{S}^{-1} 
\]

(9.12)
and by (6.8)

\[ \mathcal{W}' = S \mathcal{W}S^{-1} + S \beta S^{-1} \]

\[ = (Ss) \mathcal{W}(s^{-1}s^{-1}) + S \beta S^{-1} \quad (9.13) \]

since \( s \) is global. The field strengths, given by (6.9), transform as follows:

\[ \mathcal{W}_{\mu\lambda}' = (\nabla_{\mu}', \nabla_{\lambda}') \]

\[ = S(\nabla_{\mu}, \nabla_{\lambda})S^{-1} \]

\[ = SW_{\mu\lambda}S^{-1} \quad (9.14) \]

\[ = SW_{\mu\lambda} S^{-1} \quad (9.15) \]

and the (standard non-Abelian) field invariant will transform as

\[ \text{Tr} \left( \mathcal{W}_{\mu\lambda} \mathcal{W}^{\mu\lambda} \right)' = \text{Tr} S \mathcal{W}_{\mu\lambda} \mathcal{W}^{\mu\lambda} S^{-1} \quad (9.17) \]

where the trace is on the \( t \) matrices. Then

\[ \text{Tr} S(\mathcal{W}_{\mu\lambda} \mathcal{W}^{\mu\lambda})S^{-1} = \text{Tr} sS(\mathcal{W}_{\mu\lambda} \mathcal{W}^{\mu\lambda})S^{-1}S^{-1} \quad (9.18) \]

\[ = s \cdot \text{Tr} S(\mathcal{W}_{\mu\lambda} \mathcal{W}^{\mu\lambda})S^{-1} \cdot s^{-1} \]

\[ = s \cdot \text{Tr} \mathcal{W}_{\mu\lambda} \mathcal{W}^{\mu\lambda} \cdot s^{-1} \quad (9.19) \]

since the matrix elements of \( t_k \) are numerically valued and therefore the matrix elements of \( S(t) \) commute with the matrix elements of \( \mathcal{W}^{\mu\lambda}(t) \).

To reduce (9.20) let us take the trace of (8.6) as follows:

\[ \text{Tr} \mathcal{W}_{\mu\lambda} \mathcal{W}^{\mu\lambda} = W^{m}_{\mu\lambda} W^{p\mu\lambda} (\text{Tr} \tau_{m} \tau_{p}) D_{m} D_{p} + 2 \hat{W}^{m}_{\mu\lambda} \hat{W}^{p\mu\lambda} D_{m} D_{p} \quad (9.21) \]

Then

\[ S(\text{Tr} \mathcal{W}_{\mu\lambda} \mathcal{W}^{\mu\lambda})S^{-1} = W^{m}_{\mu\lambda} W^{p\mu\lambda} (\text{Tr} t_{m} t_{p}) s D_{m} D_{p} s^{-1} + 2 \hat{W}^{m}_{\mu\lambda} \hat{W}^{p\mu\lambda} s D_{m} D_{p} s^{-1} \quad (9.22) \]

By (7.20)-(7.23) the first term on the right side of (9.21) vanishes unless

\[ (m, p) = (\pm, \mp) \quad \text{or} \quad (m, p) = (3, 3) \]

Hence the first term vanishes unless

\[ s D_{m} D_{p} s^{-1} = s D_{\pm} D_{\mp} s^{-1} \quad (9.23) \]
or
\[ sD_mD_p s^{-1} = sD_3D_3 s^{-1} \] (9.24)

But \( D_3D_3 \) as well as \( D_\pm D_\mp \) carries zero \( Q_a \) and \( Q_b \) charge and is therefore invariant under \( s \) transformations. The second term is invariant for the same reason since \( \hat{W}_{\mu\lambda} m \hat{W}^\mu\lambda p \) vanishes by (8.5) unless \( (m, p) = (3, 3) \). Therefore

\[ S(\text{Tr } \mathcal{W}_{\mu\lambda} \mathcal{W}^{\mu\lambda}) S^{-1} = \text{Tr } \mathcal{W}_{\mu\lambda} \mathcal{W}^{\mu\lambda} \] (9.25)
i.e. the standard field invariant remains invariant when constructed out of the modified vector potential and transformed according to the new gauge group.

10 The Higgs Sector.

(a) The Vector Masses.

The neutral couplings in the knot model are by (6.6)
\[ ig \mathcal{W}_3 \tau_3 + ig_o W_o \tau_o \] (10.1)

Introducing the physical fields \( (A \text{ and } Z) \) in the standard way we have
\[ W_o = A \cos \theta - Z \sin \theta \]
\[ W_3 = A \sin \theta + Z \cos \theta \] (10.2)

Then (10.1) becomes
\[ A \mathcal{A} + Z \mathcal{Z} \] (10.3)

where
\[ \mathcal{A} = i(g \tau_3 \sin \theta + g_o \tau_o \cos \theta) \] (10.4)
\[ \mathcal{Z} = i(g \tau_3 \cos \theta - g_o \tau_o \sin \theta) \] (10.5)

Since there is no interaction between photons and neutrinos one may write, by (10.3) and (6.3)
\[ (\bar{D}_\nu^{3/2} 0) \mathcal{A} \begin{pmatrix} D_\nu^{3/2} \\ 0 \end{pmatrix} = \bar{D}_\nu^{3/2} A_{\gamma 11} D_\nu^{3/2} = 0 \] (10.6)
According to (10.4) the preceding equation is satisfied by

\[ g(\tau_3) \frac{1}{2} \sin \theta + g_o(\tau_o) \frac{1}{2} \cos \theta = 0 \]  

(10.7)

and by (10.5)

\[ Z = ig \frac{1}{\cos \theta} (\tau_3) \]  

(10.8)

Then the covariant derivative of a neutral state is

\[ \nabla_\mu = \partial_\mu + ig \left[ W_+ \tau_+ + W_- \tau_- + \frac{Z \tau_3}{\cos \theta} \right] \]  

(10.9)

Denote the neutral Higgs scalar by

\[ \phi = \rho D_\nu \langle 0 \rangle \]  

(10.10)

where \( D_\nu \) is defined in (5.13) and is the neutral trefoil, namely (-3,2), carrying the representation \( D^{3/2}_{-1/2} \).

We now replace the kinetic energy term of the neutral Higgs of the standard model by

\[
\frac{1}{2} \text{Tr}(\nabla_\mu \bar{\varphi} \nabla^\mu \varphi) = \frac{1}{2} \text{Tr}(0|\bar{D}_\nu) \\
\times \left[ \partial_\mu \rho \partial^\mu \rho + g^2 \rho^2 \left[ W_+^\mu W_+ + W_-^\mu W_- + \frac{Z^\mu Z_\mu}{\cos^2 \theta} \tau_3 \tau_3 \right] \right] D_\nu \langle 0 \rangle
\]

\[ = I \partial_\mu \rho \partial^\mu \rho + g^2 \rho^2 \left[ I_{++} W_+^\mu W_+ + I_{--} W_-^\mu W_- + \frac{I_{33}}{\cos^2 \theta} Z^\mu Z_\mu \right] \]  

(10.11)

where

\[ I = \frac{1}{2} \text{Tr}(0|\bar{D}_\nu D_\nu) \]  

(10.13)

To agree with the masses predicted by the standard theory (10.12) must be reduced to the following

\[ \partial_\mu \bar{\rho} \partial^\mu \rho + g^2 \rho^2 \left[ W_+^\mu W_+ + W_-^\mu W_- + \frac{1}{\cos^2 \theta} Z^\mu Z_\mu \right] \]  

(10.14)
where
\[ \tilde{\rho} = \rho^{1/2} \] (10.15)

To achieve this reduction we impose the following relations:
\[ \frac{I_{kk}}{I} = 1 \quad k = (+, -, 3) \] (10.16)

or
\[ \frac{\text{Tr}\langle 0|D_\nu(\bar{\tau}_k\tau_k)D_\nu|0 \rangle}{\text{Tr}\langle 0|D_\nu D_\nu|0 \rangle} = 1 \quad k = (+, -, 3) \] (10.17)

By (6.7) and (10.17) we have
\[ |c_k|^{-2} = \frac{\langle 0|D_\nu(D_kD_k)D_\nu|0 \rangle}{\langle 0|D_\nu D_\nu|0 \rangle} \quad k = (+, -, 3) \] (10.18)

In (6.7) the coefficients \((c_\pm, c_3)\) were introduced as arbitrary functions. They are now fixed by (10.18) as definite functions of \(q\) and \(\beta\). Here the \(D_k\) are given by (5.14).

One finds by (5.13) and (5.14)
\[ |c_-|^2 = \frac{\langle 0|b^3(\bar{b}^3\bar{a}^3)(a^3b^3)\bar{b}^3|0 \rangle}{\langle 0|b^3b^3|0 \rangle} \] (10.19)
\[ = |\beta|^6\langle 0|\bar{a}^3\bar{a}^3|0 \rangle \] (10.20)
\[ = |\beta|^6 \prod_{t=1}^3(1 - q_1^{2t}|\beta|^2) \] (10.21)
\[ |c_+|^2 = |\beta|^6\langle 0|a^3\bar{a}^3|0 \rangle \] (10.22)
\[ = |\beta|^6 \prod_{t=0}^2(1 - q^{2t}|\beta|^2) \] (10.23)
\[ |c_3|^2 = \frac{\langle 0|b^3D_3D_3\bar{b}^3|0 \rangle}{\langle 0|b^3b^3|0 \rangle} \] (10.24)
\[ = \langle 0|D_3D_3|0 \rangle \] (10.25)
\[ = \langle 0|[f(\bar{b}b)]^2|0 \rangle \] (10.26)
\[ = [f(|\beta|^2)]^2 \] (10.26)

where
\[ f(\bar{b}b) = D_3^{\infty} \] (10.27)

Here \(D_3^{\infty}\) is the polynomial computed from (3.1).

\(c_o\) is not determined by (10.18) but by (10.7).
One finds by (10.7)

\[
\frac{c_2}{c_3} = -\frac{\langle 0|D^3_{30}|0\rangle}{\langle 0|D^3_{111}|0\rangle}
\]  

(10.28)

where the Weinberg relation

\[
\tan \theta = \frac{g_o}{g}
\]  

(10.29)

has been assumed.

(b) The Fermion Masses.

In the mass term of (2.1) one has

\[\bar{L}\phi R + \bar{R}\phi L\]  

(10.30)

In the present model this term would be invariant if \(L\) is chosen to be an external left doublet \(\otimes\) an internal \(SU_q(2)\) trefoil, and \(R\) is chosen to be an external right singlet \(\otimes\) internal \(SU_q(2)\) singlet, while the Higgs \(\phi\) is chosen to be an external doublet \(\otimes\) internal trefoil adjoint to the left trefoil. One would then find for the mass of the \(n^{th}\) fermion as an excited state of the \((w, r)\) soliton the following:

\[
m_n(w, r) = \rho(w, r)\langle n|\bar{D}^{3/2}_{w+1}D^{3/2}_{r+1}|n\rangle
\]  

(10.31)

Here \(\bar{D}\) represents the internal state of the soliton and \(D\) the internal state of a Higgs particle. This speculative expression is discussed in the first two references.

11 Remarks.

We have not studied the perturbative formulation of the theory in higher orders and therefore do not know whether standard renormalization procedures can be adapted to the different structure coefficients of this formalism, or whether different procedures for dealing with the higher order corrections are required. The theory is gauge invariant.

The purpose of the present work is to provide a field theoretic basis for our earlier phenomenology. The main motivation remains the possibility of providing a substructure for standard theory by replacing point particles by solitons.
The possibility that the elementary particles are knots has been raised by many authors. A particular model related to the Skyrme soliton has been described by Fadeev and Niemi.\textsuperscript{4} The present work should be regarded as a model independent description of the elementary particles as knots in the context of electroweak theory.

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