Dispersion relations for the self-energy in non-commutative field theories

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We study the IR/UV connection in non-commutative $\phi^3$ theory as well as in non-commutative QED from the point of view of the dispersion relation for the self-energy. We show that, although the imaginary part of the self-energy is well behaved as the parameter of non-commutativity vanishes, the real part becomes divergent as a consequence of the high energy behavior of the dispersion integral. Some other interesting features that arise from this analysis are also briefly discussed.

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I. INTRODUCTION

In recent years, non-commutative field theories have been studied from various points of view \cite{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26}. These are theories defined on a manifold where the coordinates do not commute, rather satisfy

\[ [x^\mu, x^\nu] = i\theta^{\mu\nu} \] (1)

Here, $\theta^{\mu\nu}$ is a constant anti-symmetric tensor and, for unitarity to hold, one normally assumes that $\theta^{0i} = 0$, namely, one assumes only the space coordinates to have non-commutativity \cite{27, 28}.

In an earlier paper \cite{29}, we studied the self-energy in non-commutative QED (as well as in non-commutative $\phi^3$ theory) in $n$ dimensions and it was shown there that while the imaginary parts of the self-energy were well behaved as $\theta^{ij} \rightarrow 0$, the real parts were divergent for $n \geq 4$. This is, in fact, a puzzling behavior. Namely, if causality were to hold in non-commutative theories (QED as well as the $\phi^3$ theory), we would expect the real and the imaginary parts of the self-energy to be related through a dispersion relation and then, it is not clear how a well behaved imaginary part can lead to a divergent structure in the real part. It is with this in mind that we have undertaken a systematic study of the dispersion relation in the non-commutative $\phi^3$ theory as well as QED in $n$-dimensions and our study leads to various interesting results which we discuss in this paper.

We note that a dispersion relation, which is a statement about causality in a quantum field theory, holds for any well behaved analytic function that vanishes at infinity \cite{30, 31}. This is particularly true for the retarded self-energy which is expected to satisfy

\[ \text{Re} \Pi^{(R)}(p^0, \vec{p}) = \frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} d\omega \frac{\text{Im} \Pi^{(R)}(\omega, \vec{p})}{\omega - p^0} \] (2)

where “P” stands for the Cauchy principal value. Furthermore, using the fact that the imaginary part of the retarded self-energy is odd in the energy variable, the dispersion relation can be equivalently written as

\[ \text{Re} \Pi^{(R)}(p^0, \vec{p}) = 2\pi \text{P} \int_{0}^{\infty} d\omega \frac{\omega \text{Im} \Pi^{(R)}(\omega, \vec{p})}{\omega^2 - (p^0)^2} \] (3)

This form of the dispersion relation involves only positive values of the energy inside the integral and, therefore, is quite useful from our point of view. The reason for this is quite clear. In general, the Feynman amplitudes are not expected to satisfy a simple dispersion relation, but as we will show in section II, for positive energy, the Feynman and the retarded self-energies coincide and, consequently, satisfy (3). In the subsequent discussions, we will omit the principal value symbol for simplicity, although it is to be understood throughout.

In a non-commutative theory, the self-energy has a planar term and a non-planar term. The behavior of the planar terms is the same as in a commutative theory and, therefore, we concentrate only on the non-planar terms in the self-energy, commenting where necessary on the planar terms. It is in the non-planar terms that the nontrivial phase factors of the interaction vertices are likely to modify the complex structure of the amplitudes and, therefore, the validity of the dispersion relations is more crucial for these terms. We note that the non-planar terms are ultraviolet finite (unlike the planar ones) and, therefore, may satisfy an unsubtracted dispersion relation. Furthermore, it is also in these terms that the real parts of the amplitudes show a divergence as $\theta^{ij} \rightarrow 0$ which we would like to understand from the point of view of the dispersion relations. We assume, as is conventionally done, that $\theta^{0i} = 0$ because it is in this case that explicit calculations exist and the theories are believed to be unitary. In section II,
we evaluate the imaginary part of the Feynman self-energy in the massless non-commutative $\phi^3$ theory. We show that, for positive values of the energy, the Feynman amplitude coincides with the retarded one and use the dispersion relation to calculate the real part of the self-energy. We then show that this coincides exactly with the real part of the self-energy calculated in \[29\] proving that the dispersion relation indeed holds true in the non-commutative theory. It follows from this analysis that the divergence structure of the real part of the self-energy, as $\theta^{ij} \to 0$, arises from large values of energy, $\omega$, in the dispersion integral, giving yet another manifestation of the IR/UV mixing. In section III, we carry out a similar analysis for QED and show that the dispersion relations hold in this case as well. In section IV, we show that the real and the imaginary parts of the self-energy, both for the non-commutative $\phi^3$ theory as well as for non-commutative QED, can be expressed in closed form (without any Feynman parameter integrals) and this brings out some nice features. In section V, we discuss various other interesting features that arise from our analysis. Some mathematical details are discussed briefly in the appendix.

II. DISPERSION RELATION IN THE NON-COMMUTATIVE $\phi^3$ THEORY

In this section, we will discuss the dispersion relation for the self-energy in the massless non-commutative $\phi^3$ theory in $n$-dimensions. Since the dispersion relation for the planar part of the self-energy is well understood, we will concentrate here only on the non-planar part of the amplitude. We note that the non-planar part of the self-energy, in $n$-dimensions, has the form (apart from a factor of $\frac{\lambda^2}{4}$ where $\lambda$ represents the scalar coupling)

$$i\Pi^{(\text{non-planar})}(p) = \int \frac{d^n k}{(2\pi)^n} \frac{\epsilon^{\bar{\theta} \cdot k}}{(k^2 + i\epsilon)((k + p)^2 + i\epsilon)}$$

Here, we have defined

$$\bar{\theta}^\mu = \theta^{\mu\nu} p_\nu$$

which has only a nontrivial space component. The integral over $k^0$ can be trivially done yielding

$$\Pi^{(\text{non-planar})}(p) = \frac{1}{4(2\pi)^{n-1}} \int d^{n-1}k \frac{\cos \bar{\theta} \cdot k}{|k||k + p|} \left[ \frac{1}{p^0 + |k| + |k + p| - i\epsilon} - \frac{1}{p^0 - |k| - |k + p| + i\epsilon} \right]$$

It follows now that the imaginary part of the non-planar self-energy has the form

$$\text{Im} \Pi^{(\text{non-planar})}(p) = \frac{\pi}{4(2\pi)^{n-1}} \int d^{n-1}k \frac{\cos \bar{\theta} \cdot k}{|k||k + p|} \left( \delta(p^0 - |\vec{k}| - |\vec{k} + \vec{p}|) + \delta(p^0 + |\vec{k}| + |\vec{k} + \vec{p}|) \right)$$

There are several things to note here. First, in spite of the similarities of the non-commutative theories to thermal field theories as alluded to in \[29\], we see that the imaginary part of the self-energy only involves two delta functions as opposed to four, normally encountered in a thermal field theory \[32, 33, 34\]. This is another evidence of the fact that, in spite of the non-analyticity present in non-commutative theories, there are no additional channels of reaction (as would be the case in thermal field theories) and, therefore, no new branch cuts. Second, for positive energy, $p^0 > 0$ (which is what is needed in the dispersion relation \[31\]), only one of the delta functions contributes to the imaginary part. Without giving the mathematical details (which we discuss in the appendix), we note that, in this case, the imaginary part of the self-energy can be written as (for $p^0 > 0$)

$$\text{Im} \Pi^{(\text{non-planar})}(p) = \frac{\theta(p^2)^{\frac{n}{2} + 1}}{(2\pi)^n} \int_0^1 dx \frac{1}{|M|^{4-n}} \left( \frac{|\bar{\theta}| |M|}{2} \right)^{2 - \frac{n}{2}} J_{\frac{n}{2} - 2}(\frac{|\bar{\theta}| |M|}{2})$$

where we have defined

$$|M| = |(-x(1-x)p^2)|^\frac{1}{2}, \quad |\bar{\theta}| = (-\bar{\theta} \cdot \bar{\theta})^\frac{1}{2}$$

We note here that the imaginary part of the self-energy, for $p^0 < 0$, is also exactly the same so that the imaginary part of the Feynman amplitude in \[31\] is symmetric in $p^0$. Furthermore, from the identity satisfied by the Bessel functions \[33\],

$$\lim_{z \to 0} z^{-\nu} J_\nu(z) = \frac{1}{2\nu \Gamma(\nu + 1)}$$
it follows that the imaginary part of the self-energy is well behaved as $|\vec{\theta}| \to 0$.

Let us recall that the self-energy for the non-commutative $\phi^4$ theory, in $n$ dimensions, was evaluated earlier in [29] and the non-planar part was shown to have the form

$$
\Pi^{(\text{non-planar})}(p) = \frac{2\pi \bar{x}}{(2\pi)^n} \int_0^1 dx \frac{1}{(M^2)^{2-\bar{x}}} \left( |\vec{\theta}| M \right)^{2-\bar{x}} K_{2-\bar{x}}(|\vec{\theta}| M)
$$

(11)

Using the series representations for the Bessel functions, it can be checked that the imaginary part of this expression indeed coincides exactly with expression (8) following from the direct evaluation of the imaginary part.

So far, we have been working with the time-ordered Feynman amplitude. The dispersion relation, on the other hand, holds for the retarded amplitudes. In fact, we note from the dispersion relation in [3] that we need the imaginary part of the retarded self-energy only for positive values of energy. We show now that, for positive energy, the retarded self-energy coincides with the time-ordered Feynman self-energy that we have already evaluated. This is easily seen as follows. From the definition of the retarded and the time-ordered amplitudes, it follows that

$$
R(\phi(x_1)\phi(x_2)) = T(\phi(x_1)\phi(x_2)) - \phi(x_2)\phi(x_1)
$$

(12)

Furthermore, from the definition of the positive energy Green’s function, we know that we can write

$$
\langle 0 | \phi(x_1)\phi(x_2) | 0 \rangle = \int d^n \theta \theta(p^0) \delta(\theta^2) e^{-ip^0(x_1-x_2)} f(p)
$$

(13)

from which it follows that

$$
\langle 0 | \phi(x_1)\phi(x_2) | 0 \rangle = \int d^n \theta \theta(-p^0) \delta(\theta^2) e^{-ip^0(x_1-x_2)} f(-p)
$$

(14)

Therefore, we see that the vacuum expectation value of the second term on the right hand side in (12) has contributions only for negative values of the energy. As a result, for positive energy, this term does not contribute and the retarded and the Feynman self-energies coincide. As a practical rule for obtaining the retarded self-energy from the Feynman amplitude in (3), we note that the Feynman $i\epsilon$’s in (3) can be replaced with $p^0 \to p^0 + i\epsilon$ to yield the retarded self-energy. In this case, it follows from (3) that the imaginary part of the retarded self-energy is anti-symmetric in $p^0$, as it should be. Furthermore, the real parts of the time-ordered and the retarded self-energy are the same for all energies.

With this, we can now use the dispersion relation, Eq. (3), to determine the real part of the self-energy.

$$
\text{Re} \Pi^{(\text{non-planar})}(p) = \frac{2\pi \bar{x}}{(2\pi)^n} \int_0^\infty d\omega \frac{\omega \text{Im} \Pi^{(\text{non-planar})}(\omega, \vec{p})}{\omega^2 - (p^0)^2}
$$

$$
= \frac{2\pi \bar{x}}{(2\pi)^n} \int_0^1 dx \int_0^\infty d\omega \frac{\omega^2 - (p^0)^2}{\omega^2 - (p^0)^2} \frac{1}{M(\omega, \vec{p})^{1-n}} \left( \frac{\theta(1-x)}{\omega^2 - (p^0)^2} \right)^{2-\bar{x}} J_{2-\bar{x}}(|\vec{\theta}| M(\omega, \vec{p})^{1-n})
$$

(15)

It is worth noting here that, as $|\vec{\theta}| \to 0$, even though the combination of the Bessel functions is well behaved (see [4]), the integral diverges as (for $n > 4$)

$$
\text{Re} \Pi^{(\text{non-planar})} \sim \omega^{n-4} \sim |\vec{\theta}|^{4-n}
$$

(16)

where we have used dimensional reasoning in the last relation. Thus, even though the imaginary part of the self-energy is well behaved as $|\vec{\theta}| \to 0$, the dispersion relation induces a divergence in the real part of the amplitude which arises from the region of integration involving large values of energy. This is yet another interesting manifestation of the IR/UV mixing. We will comment more on this later in the discussions.

For the present, let us simply note that with a simple change of variables, we can write the real part of the self-energy to be

$$
\text{Re} \Pi^{(\text{non-planar})}(p) = \frac{2\pi \bar{x}}{(2\pi)^n} \left( \frac{\theta}{2} \right)^{2-\bar{x}} \int_0^1 dx (x(1-x))^{2-\bar{x}} \int_0^\infty dz \frac{1}{z^2 - p^2} z^{2-\bar{x}} J_{2-\bar{x}}(|\vec{\theta}| \sqrt{x(1-x)})
$$

$$
= \frac{2\pi \bar{x}}{(2\pi)^n} \text{Re} \int_0^1 dx \frac{1}{(M^2)^{2-\bar{x}}} \left( \frac{|\vec{\theta}| M}{2} \right)^{2-\bar{x}} K_{2-\bar{x}}(|\vec{\theta}| M)
$$

(17)

where

$$
M^2 = -x(1-x)p^2
$$

(18)

and we have used standard integrals involving Bessel functions [5]. We can now compare this real part of the amplitude determined from the dispersion relation with that from the exact amplitude in (11) calculated earlier and we see that they agree completely.
III. DISPERSION RELATION IN NON-COMMUTATIVE QED

We can now study the dispersion relation in non-commutative QED. As we have already shown in [29], the self-energy for the photon is transverse and, in a general covariant gauge, can be parameterized as

\[ \Pi_{\mu\nu} = (\eta_{\mu\nu}p^2 - p_\mu p_\nu) \Pi_1 + \frac{\bar{\theta} \theta}{\theta^2} \Pi_2 \]  

We note that \( \Pi_1, \Pi_2 \) in [29] are defined such that \( \Pi_1 = A_p, \Pi_2 = B \) of ref. [29]. Once again, we will only concentrate on the non-planar parts of the self-energy and will comment on the planar part later.

Let us first analyze the dispersion relation for \( \Pi_1^{\text{non-planar}} \) before discussing the same for \( \Pi_2^{\text{non-planar}} \). The imaginary part of \( \Pi_1^{\text{non-planar}} \) can be directly calculated from the Feynman diagram, as in the last section, and, in this case, leads to

\[
\text{Im} \Pi_1^{\text{non-planar}} = \frac{(-1)^{\frac{d}{2}}e^{\frac{d}{2}+1}\theta(p^2)}{(2\pi)^n} \int_0^1 dx \frac{1}{|M|^4-n} \left[ a_1 \left( \frac{|ar{\theta}| |M|}{2} \right)^{2-\frac{d}{2}} J_{\frac{d}{2}-2}(\bar{\theta} |M|) + 2a_2 \left( \frac{|ar{\theta}| |M|}{2} \right)^{3-\frac{d}{2}} J_{\frac{d}{2}-3}(\bar{\theta} |M|) \right]
\]  

Here, we have defined

\[
a_1 = 3 + 2(n-1)x - 4(n-2)x^2 - 2(1 - \xi)x(6x - 5),
\]
\[
a_2 = (1 - \xi) \left( 1 + 4x - 4x^2 - \frac{1 - \xi}{4} \right)
\]

with \( \xi \) representing the gauge fixing parameter. As can be easily checked using [30], this imaginary part is well behaved as \(|\bar{\theta}| \to 0\).

For positive energy, which is involved in the dispersion relation [3], as we have already shown, the Feynman and the retarded amplitudes coincide. As a result, we can evaluate the real part of \( \Pi_1^{\text{non-planar}} \) as

\[
\text{Re} \Pi_1^{\text{non-planar}} = \frac{2(-\pi)^{\frac{d}{2}}e^{\frac{d}{2}}}{(2\pi)^n} \int_0^1 dx \int_0^{\infty} d\omega \frac{\omega \theta(\omega^2 - p^2)}{\omega^2 - (p^2)^2} \frac{1}{|M(\omega, p)|^{4-n}} \times \left[ a_1 \left( \frac{|ar{\theta}| |M|}{2} \right)^{2-\frac{d}{2}} J_{\frac{d}{2}-2}(\bar{\theta} |M|) + 2a_2 \left( \frac{|ar{\theta}| |M|}{2} \right)^{3-\frac{d}{2}} J_{\frac{d}{2}-3}(\bar{\theta} |M|) \right]
\]

Once again, we see that as \(|\bar{\theta}| \to 0\), the combination of the Bessel functions is well behaved, but the integral develops a divergence coming from large values of \( \omega \) as \((n > 4)\)

\[
\text{Re} \Pi_1^{\text{non-planar}} \to \omega^{-n-4} \sim |\bar{\theta}|^{4-n}
\]

which shows the manifestation of the IR/UV mixing.

We note that there are more structures in [22] compared with [3]. Nevertheless, with a simple change of variables, we can evaluate using standard integrals of Bessel functions to obtain [3]

\[
\text{Re} \Pi_1^{\text{non-planar}} = \frac{2(-\pi)^{\frac{d}{2}}e^{\frac{d}{2}}}{(2\pi)^n} \int_0^1 dx \left[ a_1 \left( \frac{|ar{\theta}|}{2} \right)^{2-\frac{d}{2}} (x(1-x))^{\frac{d}{2}-1} \int_0^{\infty} \frac{dz}{z^{\frac{d}{2}+1}} z^{\frac{d}{2}+1} J_{\frac{d}{2}-2}(\bar{\theta} \sqrt{x(1-x)z}) + 2a_2 \left( \frac{|ar{\theta}|}{2} \right)^{3-\frac{d}{2}} (x(1-x))^{\frac{d}{2}-1} \int_0^{\infty} \frac{dz}{z^{\frac{d}{2}+1}} z^{\frac{d}{2}+1} J_{\frac{d}{2}-3}(\bar{\theta} \sqrt{x(1-x)z}) \right] = \text{Re} \left( \frac{-\pi)^{\frac{d}{2}}e^{\frac{d}{2}}}{(2\pi)^n} \int_0^1 dx \frac{1}{(M^2)^{2-\frac{d}{2}}} \left[ 2a_1 \left( \frac{|ar{\theta}| M}{2} \right)^{2-\frac{d}{2}} K_{\frac{d}{2}-2}(\bar{\theta} |M|) - 4a_2 \left( \frac{|ar{\theta}| M}{2} \right)^{3-\frac{d}{2}} K_{\frac{d}{2}-3}(\bar{\theta} |M|) \right]
\]

This can, in fact, be checked to coincide completely with the corresponding term in the real part of the exact self-energy calculated in [27].
The discussion of the dispersion relation for $\Pi_2$ proceeds in a similar manner. Therefore, without giving details, we simply note that the imaginary part of $\Pi_2^{\text{(non-planar)}}$ has the form (we note that, as pointed out in [29], $\Pi_2$ does not have a planar part)

$$\text{Im } \Pi_2^{\text{(non-planar)}} = \frac{-e^2\pi^{\frac{n+1}{2}}}{(2\pi)^n} \int_0^1 dx \left( \frac{1}{|M|^{4-n}} \right) \left( -\frac{b_1}{2} \left( \frac{\bar{\theta}|M|}{2} \right)^{1-\frac{2}{n}} J_{\frac{2}{2}-1}(\bar{\theta}|M|) + b_2 \left( \frac{\bar{\theta}|M|}{2} \right)^{2-\frac{2}{n}} J_{\frac{2}{2}}-2(\bar{\theta}|M|) \right) -2 b_3 \left( \frac{\bar{\theta}|M|}{2} \right)^{3-\frac{2}{n}} J_{\frac{2}{2}-3}(\bar{\theta}|M|) + 4 b_4 \left( \frac{\bar{\theta}|M|}{2} \right)^{4-\frac{2}{n}} J_{\frac{2}{2}}-4(\bar{\theta}|M|) \right)$$

where we have defined

$$b_1 = -4(n-2)\xi^2 x(1-x)$$
$$b_2 = 2(3-2n)\xi x(1-x) + 1 - 2x^2 + 2(1-\xi)(n-4)x$$
$$b_3 = (1-\xi) \left( 2x - (n-6) \frac{(1-\xi)}{4} \right)$$
$$b_4 = -\frac{(1-\xi)^2}{4}$$

The dispersion relation now leads to

$$\text{Re } \Pi_2^{\text{(non-planar)}} = \frac{2}{\pi} \int_0^\infty d\omega \frac{\omega \text{Im } \Pi_2^{\text{(non-planar)}}(\omega, \bar{\theta})}{\omega^2 - (p^2)^2}$$

$$= \text{Re} \frac{e^2(-\pi)^{\frac{n}{2}}}{(2\pi)^n} \int_0^1 dx \left( \frac{1}{|M|^2} \right)^{\frac{2}{n}} \left[ b_1 \left( \frac{|\bar{\theta}|M}{2} \right)^{1-\frac{2}{n}} K_{\frac{2}{2}-1}(\bar{\theta}|M) + 2 b_2 \left( \frac{|\bar{\theta}|M}{2} \right)^{2-\frac{2}{n}} K_{\frac{2}{2}}-2(\bar{\theta}|M) \right] + 4 b_3 \left( \frac{|\bar{\theta}|M}{2} \right)^{3-\frac{2}{n}} K_{\frac{2}{2}-3}(\bar{\theta}|M) + 8 b_4 \left( \frac{|\bar{\theta}|M}{2} \right)^{4-\frac{2}{n}} K_{\frac{2}{2}}-4(\bar{\theta}|M) \right]$$

(27)

This can be seen to coincide completely with the corresponding real part of the perturbative calculation in [29].

IV. CLOSED FORM EXPRESSIONS FOR REAL AND IMAGINARY PARTS OF THE SELF-ENERGY

Our discussion so far has involved imaginary and real parts of the self-energy represented in a parametric form (namely, with integrals over the Feynman parameter). In this section, we will show that the integration over the Feynman parameter can, in fact, be done exactly and that both the imaginary as well as the real parts of the self-energy have closed form expressions in terms of well known functions.

To begin with, let us consider the self-energy for the scalar theory discussed in section II. As we have seen there (see [8]), the imaginary part of the self-energy is given by

$$\text{Im } \Pi^{\text{(non-planar)}} = \frac{\theta(p^2)\pi^{\frac{n+1}{2}}}{(2\pi)^n} \int_0^1 dx \left( \frac{1}{|M|^{4-n}} \right) \left( \frac{|\bar{\theta}|M}{2} \right)^{2-\frac{2}{n}} J_{\frac{2}{2}}-2(\bar{\theta}|M|)$$

where

$$|M| = \sqrt{|(-x(1-x)p^2)|^\frac{1}{2}}$$

(29)

The $x$ integration can, in fact, be done and leads to

$$\text{Im } \Pi^{\text{(non-planar)}} = \frac{\theta(p^2)\pi^{\frac{n+1}{2}}}{(2\pi)^n} (p^2)^{\frac{n+1}{2}} \left( \frac{|\bar{\theta}|(p^2)^{\frac{1}{2}}}{2} \right)$$

(30)

The closed form evaluation of the real part of the self-energy, on the other hand, is slightly more involved. As we have seen in [17], the real part of the self-energy can be written as

$$\text{Re } \Pi^{\text{(non-planar)}} = \frac{2\pi^{\frac{n}{2}}}{(2\pi)^n} \left( \frac{|\bar{\theta}|}{2} \right)^{2-\frac{2}{n}} \int_0^1 dx (x(1-x))^{\frac{n}{2}-1} \int_0^\infty dz \frac{1}{z^2 - p^2} z^{\frac{n}{2}-1} J_{\frac{2}{2}}-2(\bar{\theta}) \sqrt{x(1-x)}$$

(31)
The $z$ integral can be done and leads to

$$\text{Re} \Pi^{\text{(non-planar)}} = \frac{2\pi^{n+1}}{(2\pi)^n} \left( \frac{1}{|\bar{\theta}|} \right)^{\frac{n-1}{2}} \int_0^\infty dz \frac{1}{z^2 - p^2} z \frac{1}{2} J_{\frac{|\bar{\theta}|}{2}} (\frac{z}{2})$$

(32)

Furthermore, the $z$ integration can also be done and the result can, in fact, be written in terms of generalized hypergeometric functions. However, this is not very useful. Instead, a more useful form is as follows.

$$\text{Re} \Pi^{\text{(non-planar)}} = \frac{\pi^{n+2}}{2(2\pi)^n} \left( \frac{1}{|\bar{\theta}|} \right)^{n-1} \cos \left( \frac{n-3\pi}{2} \right) \left[ (-1)^{\frac{n}{2}} H_{\frac{n}{2}-n} \left( \frac{|\bar{\theta}|\sqrt{p^2}}{2} \right) - N_{\frac{n}{2}-n} \left( \frac{|\bar{\theta}|\sqrt{p^2}}{2} \right) \right]$$

(33)

Here $H$ denotes the Struve function while $N$ is the Neumann function \[35\] and the principal value prescription is to be understood when $p^2 > 0$. We note that, in spite of the apparent singularity (coming from the cosine in the denominator) for $n = 2k$, this expression is, in fact, well behaved for $k > 1$ because of the identity

$$H_{\frac{n}{2}-n} (z) = N_{\frac{n}{2}-n} (z)$$

(34)

However, for $n = 4, 6$, which are the cases we are interested in (as we will see, this is what will be needed to discuss the self-energy for QED), it is nontrivial to extract the finite part from \[35\]. Therefore, in what follows, we give an alternative closed form expression for the real part of the self-energy when $n = 4, 6$.

Let us define

$$S_1 = \text{Re} \Pi^{\text{(non-planar)}}_{n=4} = \frac{2\pi^{\frac{n}{2}}}{(2\pi)^4 |\bar{\theta}|^{\frac{n}{2}}} \int_0^\infty dz \frac{1}{z^2 - p^2} z \frac{1}{2} J_{\frac{|\bar{\theta}|}{2}} (\frac{z}{2})$$

$$S_2 = \text{Re} \Pi^{\text{(non-planar)}}_{n=6} = \frac{2\pi^{\frac{n}{2}}}{(2\pi)^6 |\bar{\theta}|^{\frac{n}{2}}} \int_0^\infty dz \frac{1}{z^2 - p^2} z \frac{1}{2} J_{\frac{|\bar{\theta}|}{2}} (\frac{z}{2})$$

(35)

With the explicit representations for the Bessel functions,

$$J_{\frac{1}{2}} (z) = \sqrt{\frac{2}{\pi z}} \sin z, \quad J_{\frac{3}{2}} (z) = \sqrt{\frac{2}{\pi z}} \left( \sin z - \cos z \right)$$

(36)

the $z$ integration can be explicitly done and leads to

$$S_1 = \frac{1}{4\pi^2 |\bar{\theta}| (p^2)^{\frac{n}{2}}} \left[ \cos |\bar{\theta}| (p^2)^{\frac{n}{2}} (\sin |\bar{\theta}| (p^2)^{\frac{n}{2}} + \frac{\pi}{2}) - \sin |\bar{\theta}| (p^2)^{\frac{n}{2}} \cos |\bar{\theta}| (p^2)^{\frac{n}{2}} + \frac{\pi}{2} \right]$$

$$S_2 = \frac{1}{16\pi^3 |\bar{\theta}|^2} \left[ \frac{2}{|\bar{\theta}| (p^2)^{\frac{n}{2}}} \left( \cos |\bar{\theta}| (p^2)^{\frac{n}{2}} (\sin |\bar{\theta}| (p^2)^{\frac{n}{2}} + \frac{\pi}{2}) - \sin |\bar{\theta}| (p^2)^{\frac{n}{2}} \cos |\bar{\theta}| (p^2)^{\frac{n}{2}} + \frac{\pi}{2} \right) + \left( \cos |\bar{\theta}| (p^2)^{\frac{n}{2}} \cos |\bar{\theta}| (p^2)^{\frac{n}{2}} (\sin |\bar{\theta}| (p^2)^{\frac{n}{2}} + \frac{\pi}{2}) \right) \right]$$

(37)

Here, $ci(z), si(z)$ are the well known cosine and sine integrals defined as \[35\]

$$ci(z) = -\int_z^\infty dt \frac{\cos t}{t} = C + \ln z + \int_0^z dt \frac{\cos t - 1}{t}$$

$$si(z) = -\int_z^\infty dt \frac{\sin t}{t} = -\frac{\pi}{2} + \int_0^z dt \frac{\sin t}{t}$$

(38)

where $C$ represents Euler’s constant. From the series representation of the $ci(z), si(z)$ functions, we obtain in a simple manner the behavior of the real parts of the self-energy as $|\bar{\theta}| \to 0$, namely,

$$\lim_{|\bar{\theta}| \to 0} S_1 \to \frac{1}{8\pi^2} \left( \ln \frac{|\bar{\theta}|^2}{2} + C - 1 \right)$$

$$\lim_{|\bar{\theta}| \to 0} S_2 \to \frac{1}{16\pi^3 |\bar{\theta}|^2} \left( 1 + \frac{|\bar{\theta}|^2 p^2}{9} - \frac{|\bar{\theta}|^2 p^2}{12} (C + \ln \frac{|\bar{\theta}|^2}{2}) \right)$$

(39)
With these results, we can now discuss the closed form expressions for the real and the imaginary parts of the self-energy for QED. For simplicity, we give the results in four dimensions in the Feynman gauge where the expressions can be written in terms of the functions \( S_1, S_2 \) in (37) corresponding to the scalar self-energy in \( n = 4, 6 \) dimensions. After carrying out the \( x \) as well as the \( z \) integrations, the real and the imaginary parts of the functions \( \Pi_1, \Pi_2 \) take the forms (in the Feynman gauge)

\[
\text{Im} \Pi_1^{(\text{non-planar})} = \frac{4e^2\pi^3\theta(p^2)}{(2\pi)^4} \sqrt{\frac{\pi}{\theta(p^2)^2}} \left( J_\frac{1}{2} \frac{\theta(p^2)^\frac{1}{2}}{2} - \frac{1}{\theta(p^2)^\frac{1}{2}} J_\frac{3}{2} \frac{\theta(p^2)^\frac{3}{2}}{2} \right)
\]

\[
\text{Re} \Pi_1^{(\text{non-planar})} = \frac{e^2}{\pi^2 |\theta|^2 p^2} + 4S_1 - \frac{16\pi}{p^2} S_2
\]

\[
\text{Im} \Pi_2^{(\text{non-planar})} = \frac{2e^2\pi^3\theta(p^2)^2}{(2\pi)^4} \sqrt{\frac{\pi}{\theta(p^2)^2}} \left( -J_\frac{1}{2} \frac{\theta(p^2)^\frac{1}{2}}{2} + \frac{6}{\theta(p^2)^\frac{1}{2}} J_\frac{3}{2} \frac{\theta(p^2)^\frac{3}{2}}{2} \right)
\]

\[
\text{Re} \Pi_2^{(\text{non-planar})} = \frac{e^2}{\pi^2 |\theta|^2} - 2p^2 S_1 + 48\pi S_2
\]

(40)

From the limiting behaviors in (39), it is easy to see that the imaginary parts of \( \Pi_1, \Pi_2 \) in (40) are well behaved when \( |\theta| \to 0 \) while the real parts diverge.

V. DISCUSSION

In this paper, we have systematically studied and shown that the dispersion relations hold true for the self-energy in the non-commutative \( \phi^3 \) theory as well as in QED in any dimensions. This also explains, as a manifestation of the IR/UV mixing, how a well behaved imaginary part of the self-energy develops a divergence structure in the real part as \( |\theta| \to 0 \).

Another interesting aspect of non-commutative QED is the on-shell behavior of the self-energy. Since \( \Pi_2 \) has no planar contribution, we see from Eq. (40) that the complete structure satisfies

\[
\lim_{p^2 \to 0} \text{Im} \Pi_2^{(\text{non-planar})} = \lim_{p^2 \to 0} \text{Im} \Pi_2 = 0.
\]

(41)

Similarly, we note from Eq. (40) that, for \( p^2 > 0 \),

\[
\lim_{p^2 \to 0} \text{Im} \Pi_1^{(\text{non-planar})} = \frac{2e^2\pi^3}{(2\pi)^4} \frac{10}{6}
\]

(42)

This does not vanish independently, but for \( \Pi_1 \), there is a nontrivial planar term \( 29 \) which gives, for \( p^2 > 0 \),

\[
\text{Im} \Pi_1^{(\text{planar})} = \frac{2e^2\pi^3}{(2\pi)^4} \frac{10}{6}
\]

(43)

so that once again, we see

\[
\lim_{p^2 \to 0} \text{Im} \Pi_1 = \lim_{p^2 \to 0} \text{Im} \Pi_1^{(\text{planar})} + \lim_{p^2 \to 0} \text{Im} \Pi_1^{(\text{non-planar})} = 0
\]

(44)

These results can be understood simply as follows. As we see from the explicit forms of the imaginary parts in (40), the combination \( |\theta|(p^2)\frac{1}{2} \) comes together in the arguments so that \( p^2 \to 0 \) can also be thought of as \( |\theta| \to 0 \). However, since the integrand of the self-energy involves the factor \((1 - \cos \theta \cdot k)\), its imaginary part which behaves smoothly as \( |\theta| \to 0 \) will vanish in this limit.

The real parts of the self-energy, on the other hand, are non-vanishing in this limit. In fact, we see from Eqs. (39) and (40) that

\[
\lim_{p^2 \to 0} \text{Re} \Pi_2 = \lim_{p^2 \to 0} \text{Re} \Pi_2^{(\text{non-planar})} = \frac{e^2}{(4\pi)^2} \frac{32}{|\theta|^2}
\]

(45)

which is a gauge independent result. Furthermore, in the Feynman gauge, we have

\[
\lim_{p^2 \to 0} \text{Re} \Pi_1^{(\text{non-planar})} = \frac{2e^2}{(4\pi)^2} \left[ -\frac{10}{6} \left( 2C + \ln \frac{|\theta|^2 p^2}{4} \right) + \frac{28}{9} \right]
\]

(46)
The dispersion relation, on the other hand, leads to (in the Feynman gauge)

$$\lim_{p^2 \to 0} \Re \Pi_1^{\text{(planar)}} = -\frac{2e^2}{(4\pi)^2} \left[ -\frac{10}{6} \left( C + \frac{\ln |p^2|}{4\pi\mu^2} \right) + \frac{31}{9} \right]$$

where $\mu$ represents the renormalization scale. As a result, we see that the $\ln |p^2|$ terms cancel in the complete real part and we obtain

$$\lim_{p^2 \to 0} \Re \Pi_1 = -\frac{2e^2}{(4\pi)^2} \left[ \frac{10}{6} \left( C + \ln(\pi |\theta|^2 \mu^2) \right) + \frac{1}{3} \right]$$

There are several things to note from this. First, unlike in ordinary Yang-Mills theory, the self-energy in non-commutative QED does not exhibit any logarithmic singularity as $p^2 \to 0$. Second, even though the imaginary parts of $\Pi_1, \Pi_2$ vanish because of the $(1 - \cos \theta \cdot k)$ factors in the integrand, the real parts do not, which can be understood as a consequence of the large energy behavior of the dispersion integral.

Let us discuss this feature in some more detail. First, we note from Eq. (40) that, for a fixed $p^2$,

$$\lim_{|\theta| \to 0} \Im \Pi_2 = \lim_{|\theta| \to 0} \Im \Pi_2^{\text{(non-planar)}} \sim |\bar{\theta}|^2 (p^2)^2 \to 0$$

The dispersion relation, on the other hand, leads to

$$\lim_{|\theta| \to 0} \Re \Pi_2 = \lim_{|\theta| \to 0} \frac{2}{\pi} \int_0^\infty d\omega \frac{\omega \theta (\omega^2 - \bar{p}^2)}{\omega^2 - (p^0)^2} \Im \Pi_2(\omega, \bar{p}, |\bar{\theta}|) \sim \int_0^\infty d\omega \frac{\omega \theta (\omega^2 - \bar{p}^2)}{\omega^2 - (p^0)^2} |\bar{\theta}|^2 (\omega^2 - p^0)^2$$

Thus, we see that, although for any finite $\omega$, the integrand vanishes as $|\bar{\theta}| \to 0$, it is, in fact, strongly divergent as $\omega \to \infty$ leading to a nontrivial real part that diverges as $|\bar{\theta}|^2$ for small $|\bar{\theta}|$.

We can analyze the corresponding behavior for $\Pi_1$ as well. Here, although a new feature develops since $\Pi_1^{\text{(planar)}}$ is divergent and satisfies a subtracted dispersion relation, the analysis is completely parallel and leads to the result that the real part of $\Pi_1$ diverges logarithmically as $|\bar{\theta}| \to 0$. A similar analysis can also be carried out for non-commutative $U(N)$ theory.

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**APPENDIX A: CALCULATION OF IMAGINARY PARTS**

In this section, we will discuss briefly how the imaginary parts of the self-energy are calculated. We note that, for $p^0 > 0$, only one of the delta functions contributes and, in this case, we have

$$\Im \Pi^{\text{(non-planar)}} = \frac{\pi}{4(2\pi)^{n-1}} \int d^{n-1}k \frac{\cos \bar{\theta} \cdot k}{|k||\bar{k} + \bar{p}|} \delta(p^0 - |\bar{k}| - |\bar{k} + \bar{p}|)$$

$$= \frac{\pi}{4(2\pi)^{n-1}} \int k^{n-2}dk (\sin \theta_1)^{n-3}d\theta_1 (\sin \theta_2)^{n-4}d\theta_2 \cdots d\theta_{n-2} \frac{\cos \bar{\theta} \cdot k}{k|\bar{k} + \bar{p}|} \delta(p^0 - k - |\bar{k} + \bar{p}|)$$

Here, we have identified $k = |\bar{k}|$ as the radial component of the momentum vector. If we next identify $\bar{p}$ to lie along the $(n-1)$-th axis and $\bar{\theta}$ to lie along the $(n-2)$-th axis, we have

$$\bar{k} \cdot \bar{p} = |\bar{k}||\bar{p}| \cos \theta_1, \quad \bar{\theta} \cdot k = |\bar{\theta}| k \sin \theta_1 \cos \theta_2$$

The delta function can now be used to do the $\theta_1$ integration. In fact, the delta function determines

$$\cos \theta_1 = \frac{1}{2k|\bar{p}|} (p^2 - 2kp^0)$$

$$\Im \Pi_1^{\text{(planar)}} = -\frac{2e^2}{(4\pi)^2} \left[ -\frac{10}{6} \left( C + \frac{\ln |p^2|}{4\pi\mu^2} \right) + \frac{31}{9} \right]$$

$$\lim_{|\theta| \to 0} \Re \Pi_1 = -\frac{2e^2}{(4\pi)^2} \left[ \frac{10}{6} \left( C + \ln(\pi |\theta|^2 \mu^2) \right) + \frac{1}{3} \right]$$
and limits the range of the $k$ integration to $k_{\min} = \frac{p^0 - |\vec{p}|}{2} \leq k \leq |\vec{p}|^2 = k_{\max}$. This also shows that the imaginary part is nontrivial only for $p^2 \geq 0$. Doing the $\theta_1$ integral and shifting $k \to k - k_{\min}$, we obtain,

$$\text{Im} \Pi^{(\text{non-planar})} = \frac{\theta(p^2)\pi}{4(2\pi)^{n-1}} \int (\sin \theta_3)^{n-5} d\theta_3 \cdots d\theta_{n-2}$$

$$\times \int_0^{k_{\max} - k_{\min}} \frac{dk}{|\vec{p}|} (\sin \theta_2)^{n-4} d\theta_2 \left( \frac{(p^2(k_{\max} - k_{\min} - k)k)^{\frac{1}{2}}}{|\vec{p}|} \right)^{n-4}$$

$$\times \cos \left( \frac{\theta(p^2(k_{\max} - k_{\min} - k)k)^{\frac{1}{2}}}{|\vec{p}|} \cos \theta_2 \right)$$

(A4)

Rescaling

$$k = x(k_{\max} - k_{\min}) = x|\vec{p}|$$

(A5)

the imaginary part becomes

$$\text{Im} \Pi^{(\text{non-planar})} = \frac{\theta(p^2)\pi}{4(2\pi)^{n-1}} \int (\sin \theta_3)^{n-5} d\theta_3 \cdots d\theta_{n-2}$$

$$\times \int_0^1 dx (x(1-x)p^2)^{n-4} (\sin \theta_2)^{n-4} d\theta_2 \cos(\theta(p^2(x(1-x)p^2)\frac{1}{2}\cos \theta_2)$$

$$= \frac{\pi^{\frac{n}{2}+1}}{(2\pi)^n} \int_0^1 dx \frac{1}{|M|^{n-1}} \left( \frac{\theta(|M|)}{2} \right)^{2-\frac{n}{2}} J_{\frac{n}{2}}(\theta(|M|))$$

(A6)

where $|M|$ is defined earlier in Eq. (9). We take this opportunity to correct a typographical error in Eq. (61) of ref. [29]. The imaginary part of the modified Bessel function satisfies, for $p^2 > 0$, the relation

$$\text{Im} \frac{K_i(\theta |M|)}{(\theta |M|)^2} = (-1)^i \frac{\pi}{2} \frac{J_i(\theta |M|)}{|(\theta |M|)|},$$

(A7)

which can also be used to calculate the imaginary part of the self-energy from the result in [29].

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