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LIMITING MOTION FOR THE PARABOLIC GINZBURG-LANDAU EQUATION WITH INFINITE ENERGY DATA

DELPHINE CÔTE AND RAPHAËL CÔTE

Abstract. We study a class of solutions to the parabolic Ginzburg-Landau equation in dimension 2 or higher, with ill-prepared infinite energy initial data. We show that, asymptotically, the vorticity evolves according to motion by mean curvature in Brakke’s weak formulation. Then, we prove that in the plane, point vortices do not move in the original time scale. These results extend the works of Bethuel, Orlandi and Smets [8, 9] to infinite energy data; they allow to consider point vortices on a lattice (in dimension 2), or filament vortices of infinite length (in dimension 3).

1. Introduction

1.1. Setting and motivation of the problem. We consider the parabolic Ginzburg-Landau equation for complex functions $u_\varepsilon : \mathbb{R}^d \times (0, +\infty) \to \mathbb{C}$

(PGL$_\varepsilon$)

\[
\begin{aligned}
\partial_t u_\varepsilon - \Delta u_\varepsilon &= \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) & \quad & \text{on } \mathbb{R}^d \times (0, +\infty), \\
u_\varepsilon(x, 0) &= u_\varepsilon^0(x) & & \quad \text{for } x \in \mathbb{R}^d,
\end{aligned}
\]

in dimension $d \geq 2$, and its associated energy

\[
E_\varepsilon(w) = \int_{\mathbb{R}^d} e_\varepsilon(w)(x) dx = \int_{\mathbb{R}^d} \left( \frac{|\nabla w(x)|^2}{2} + V_\varepsilon(w(x)) \right) dx \quad \text{for } w : \mathbb{R}^d \to \mathbb{C},
\]

where $V_\varepsilon$ denotes the non-convex double well potential and $e_\varepsilon$ is the energy density:

\[
V_\varepsilon(w) = \frac{(1 - |w|^2)^2}{4\varepsilon^2}, \quad \text{and} \quad e_\varepsilon(w) = \frac{|\nabla w|^2}{2} + V_\varepsilon(w).
\]

It is a classical result that an initial data $u^0 \in L^\infty \cap H^1$ yields a global in time solution $u(t) \in \mathcal{C}([0, +\infty), L^\infty \cap H^1)$. The Ginzburg-Landau equation (PGL$_\varepsilon$) in the plane ($d = 2$) admits, for $\ell \in \mathbb{Z}^*$, vortex solutions of the form

\[
\Psi(x, t) = \Psi(x) = U_\ell(r) \exp(i \ell \theta), \quad \text{with} \quad U_\ell(0) = 0 \quad \text{and} \quad U_\ell(+\infty) = 1,
\]

where $(r, \theta)$ corresponds to the polar coordinates in $\mathbb{R}^2$ (by scaling, (PGL$_\varepsilon$) admits stationary vortex solutions as well). Such functions $\Psi$ define complex planar vector fields whose zeros are called vortices (of order $\ell$, also called $\ell$-vortices). Vortices solutions arise naturally in Physics applications, and it is an important question to study the asymptotic analysis, as the parameter $\varepsilon$ goes to zero, of solutions to (PGL$_\varepsilon$).

We must stress out the fact that a single vortex does not belong to $H^1(\mathbb{R}^d)$ (for $d = 2$, $|\nabla \Psi|(r, \theta) \sim r^{-d/2}$). To overcome this problem, an easy way out is to consider configurations of multiple vortices where the sum of degrees of the vortices is equal to zero. In that case, the initial data belongs to the

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space $L^\infty \cap H^1$, and one talks about well-prepared data, see for example Jerrard and Soner [21], Lin [24], Sander and Serfaty [30], and Spirn [34].

One way to relax this condition was done in the seminal works by Bethuel, Orlandi and Smets [8,9,10,11]: they consider (PGL$\epsilon$) with $u^\epsilon : \mathbb{R}^d \times [0, +\infty) \to \mathbb{C}$ and assume that the initial data $u^0_\epsilon$ is in the energy space and verifies the bound

\[ E_\epsilon(u^0_\epsilon) \leq M_0 |\ln \epsilon| \]

where $M_0$ is a fixed positive constant. Observe that this condition encompasses large data, and almost gets rid of any well preparedness assumption. The only limitation can be seen as follows in dimension $d = 2$ (where vortices are points): \(1.2\) allows general sum of vortices, which are balanced by adding “vortices at infinity” (where the center of the vortices goes to spatial infinity as $\epsilon \to 0$): in that case, for each $\epsilon > 0$, the initial data is of finite energy, but the limiting configuration can be any configuration of finitely many vortices.

The main emphasis of [8], valid in any dimension $d \geq 2$, is placed on the asymptotic limits of the Radon measures $\mu_\epsilon$ defined on $\mathbb{R}^d \times [0, +\infty)$, and their time slices $\mu^t_\epsilon$ defined on $\mathbb{R}^d \times \{t\}$, by

\[ \mu_\epsilon(x, t) = \frac{e^\epsilon(u_\epsilon)(x, t)}{|\ln \epsilon|} \, dx \, dt, \quad \text{and} \quad \mu^t_\epsilon(x) = \frac{e^\epsilon(u_\epsilon)(x, t)}{|\ln \epsilon|} \, dx, \]

so that $\mu_\epsilon = \mu^t_\epsilon \, dt$.

The bound on the energy gives that, up to a subsequence $\epsilon_m \to 0$, there exists a Radon measure $\mu_* = \mu_*^t \, dt$ defined on $\mathbb{R}^d \times [0, +\infty)$ such that

\[ \mu_\epsilon \rightharpoonup \mu_*, \quad \text{and} \quad \mu^t_\epsilon \rightharpoonup \mu_*^t \]

as measures on $\mathbb{R}^d \times [0, +\infty)$ and $\mathbb{R}^d \times \{t\}$ for all $t > 0$, respectively (see [8 Lemma 1] and [20]).

The purpose of [8] is to describe the properties of the measures $\mu^t_\epsilon$: the main result is that asymptotically, the vorticity (concentrated) part of $\mu^t_\epsilon$ evolves according to motion by mean curvature in Brakke’s weak formulation.

In dimension $d = 2$, though, the vorticity part of $\mu^t_\epsilon$ is supported on a finite set of points (the vortices). One can actually compute that the energy of a $t$-vortex is roughly $\pi t^2 |\ln \epsilon|$: the above bound \(1.2\) implies that only a finite number of vortices can be created (at most $M_0/\pi$). However the mean curvature flow for discrete points is trivial, they do not move. Therefore, in order to see the vortices evolve, one needs to consider a different regime, where time is adequately rescaled by a factor $|\ln \epsilon|$. This is done by Bethuel, Orlandi and Smets in [9,10,11]: they describe completely the asymptotics, and analyze precisely the dissipation times where collision or splitting of vortices occur. Again, the only assumption is the bound \(1.2\) on the initial data $u^0_\epsilon$ (and thus $u^0_\epsilon$ is in the energy space).

Our goal in this paper is to extend the results in [8], by relaxing the global energy bound \(1.2\) to a local one. More precisely, we say that a function $f : [0, +\infty) \to [0, +\infty)$ has mild growth if $f$ is non-decreasing and that for all $R > 0$, there exists a constant $C(R)$ such that

\[ \forall x, y \geq 0, \quad f(x + y) \leq C(R) f(2x) e^{\gamma \frac{|y|}{R^2}}. \]

Observe that if $0 < \gamma < 2$ and $C > 0$, then $f(x) = e^{C e^{\gamma x^2}}$ has mild growth.

We study families of solutions to (PGL$\epsilon$) whose initial data $u^0_\epsilon$ satisfy the following assumptions, for some function $f$ with mild growth, independent of $\epsilon$:

\[ (H_\epsilon(f)) \]

\[ \left\{ \begin{array}{l}
 u^0_\epsilon \in L^\infty(\mathbb{R}^d), \\
 \forall x \in \mathbb{R}^d, \quad |\int_{B(x, 1)} e^\epsilon(u^0_\epsilon)(y) \, dy| \leq f(|x||\ln \epsilon|).
 \end{array} \right. \]

Observe that [15, Theorem 2] shows the existence of a unique solution $u_\epsilon \in \mathcal{C}_b((0, +\infty), L^\infty(\mathbb{R}^d))$ to (PGL$\epsilon$) (globally well defined for positive times), with initial data $u^0_\epsilon$ in the sense that it satisfies
Theorem 1.1. We start with the description of the vorticity set and the decomposition of the asymptotic energy $\mu$. In all the following, we consider a family of solutions of (PGL $\epsilon$) of dimension $d$, show that it evolves under the mean curvature flow in a weak formulation. Finally we show that in $\Sigma$, our main results are the following. We define the limiting energy $\mu^\ast$ and construct the vorticity set $\Sigma^\ast$ (and prove some regularity properties). Then we consider the concentrated energy $\nu^\ast$ on $\Sigma^\ast$ and show that it evolves under the mean curvature flow in a weak formulation. Finally we show that in dimension $d = 2$, $\Sigma^\ast$ is made of a discrete set of points which do not move.

In all the following, we consider a family of solutions of (PGL) $(u_\epsilon)_{\epsilon \in (0,1/2)}$ such that their initial conditions $u_0^\ast$ satisfy $H(f)$ for some function $f$ with mild growth. We start with the description of the vorticity set and the decomposition of the asymptotic energy density.

**Theorem 1.1.** There exist a sequence $\epsilon_n \to 0$, a subset $\Sigma_\mu$ of $\mathbb{R}^d \times (0, +\infty)$, and a smooth real-valued function $\Phi$, defined on $\mathbb{R}^d \times (0, +\infty)$ such that the following properties hold.

1. $\Sigma_\mu$ is closed in $\mathbb{R}^d \times (0, +\infty)$ and for any compact subset $K \subset \mathbb{R}^d \times (0, +\infty) \setminus \Sigma_\mu$,
   \[
   |u_\epsilon(x,t)| \to 1 \text{ uniformly on } K \text{ as } n \to +\infty.
   \]

2. For any $t > 0$ and $x \in \mathbb{R}^d$, $\Sigma_\mu^t = \Sigma_\mu \cap \mathbb{R}^d \times \{t\}$ verifies
   \[
   \mathcal{H}^{d-2}(\Sigma_\mu^t \cap B(x, 1)) < +\infty,
   \]
   (and the bound is uniform in $(x, t)$ on compact sets of $\mathbb{R}^d \times (0, +\infty)$).

3. The function $\Phi$, verifies the heat equation on $\mathbb{R}^d \times (0, +\infty)$.

4. For each $t > 0$, the measure $\mu_\epsilon^t = \lim_{n \to +\infty} \mu_\epsilon^t$ can be decomposed as
   \[
   \mu_\epsilon^t = |\nabla \Phi|^2(t) \mathcal{H}^d + \Theta_\epsilon(t) \mathcal{H}^{d-2} \Sigma_\mu^t
   \]
   where $x \mapsto \Theta_\epsilon(x, t) := \lim_{r \to 0} \frac{\mu_\epsilon^t(B(x, r))}{\alpha_{d-2} r^{d-2}}$ is bounded on bounded sets of $\mathbb{R}^d \times (0, +\infty)$.

5. For almost every $t > 0$, the set $\Sigma_\mu^t$ is $(d-2)$-rectifiable and $\Theta_\epsilon(x, t) > 0$ for $\mathcal{H}^{d-2}$ a.e. $x \in \Sigma_\mu^t$.

In view of the decomposition (1.5), $\mu_\epsilon^t$ can be split into two parts: a diffuse part $|\nabla \Phi|^2 \mathcal{H}^d$, and a concentrated part

\[
\nu_\epsilon^t := \Theta_\epsilon(x, t) \mathcal{H}^{d-2} \Sigma_\mu^t.
\]

Observe that both measures $\mu_\epsilon^t$ and $\nu_\epsilon^t$ are $\sigma$-finite (due to (2)). By (3), the diffuse part is governed by the heat equation. Our next theorem focuses on the evolution of the concentrated part $\nu_\epsilon^t$.

**Theorem 1.2.** The family $(\nu_\epsilon^t)_{t > 0}$ is a mean curvature flow in the sense of Brakke (see Section 4.1 for definitions).

For our last result, we focus on dimension $d = 2$, where vortices are points. We show that these vortex points do not move in the original time scale.

\[
\forall t > 0, \quad u_\epsilon(t) = G(t) * u_\epsilon^0 + \int_0^t G(t-s) * \left( \frac{1}{\epsilon^2} u_\epsilon(s)(1 - |u_\epsilon(s)|^2) \right) ds,
\]

where $G(x, t) = \frac{1}{(4\pi)^d} e^{-x^2/4t}$ is the heat kernel. The crucial point is that $(H(f))$ is a property which propagates along time, if one allows $f$ to depend on time. We also refer to [2] where the Ginzburg-Landau functional is studied under a local energy bound in $|\ln |$ (and a $\Gamma$-convergence result is obtained).

1.2. **Statements of the main results.** Let us emphasize that the analysis here is done in the original time scale and not in the accelerated time scale (which is relevant for dimension $d = 2$ only).

Our main results are the following. We define the limiting energy $\mu^\ast$ and construct the vorticity set $\Sigma^\ast$ (and prove some regularity properties). Then we consider the concentrated energy $\nu^\ast$ on $\Sigma^\ast$ and show that it evolves under the mean curvature flow in a weak formulation. Finally we show that in dimension $d = 2$, $\Sigma^\ast$ is made of a discrete set of points which do not move.

In all the following, we consider a family of solutions of (PGL) $(u_\epsilon)_{\epsilon \in (0,1/2)}$ such that their initial conditions $u_0^\ast$ satisfy $H(f)$ for some function $f$ with mild growth.

We start with the description of the vorticity set and the decomposition of the asymptotic energy density.
Theorem 1.3. Let $d = 2$. There exist a discrete set of $\mathbb{R}^2$, which we can enumerate $\{b_i : i \in \mathbb{N}\}$ and that contains all the $\Sigma^t_\mu$ for $t > 0$: $\Sigma^t_\mu \subset \{b_i : i \in \mathbb{N}\}$. Also, $v_\epsilon(t) = \sum_{i=1}^{+\infty} \sigma_i(t) \delta_{b_i}$, where the functions $\sigma_i(t)$ are non-increasing.

Local bounds on the energy, i.e. assumption (H_1(f)), make the set of admissible initial data more natural. We can consider general vortex configurations in dimension 2, without adding a vortex at infinity to balance it (so as to make the total sum of the vortices' degrees equal to 0 for each $\epsilon > 0$); important physically relevant examples encompass point vortices on an infinite lattice (dimension 2) or general vortex filament families (with possibly infinite length) in dimension 3. Another striking difference between the global bound (1.2) and (H_1(f)) is the following. In dimension $d \geq 3$, (1.2) implies that after some finite time, the vorticity vanishes, that is $\Sigma^t_\mu \subset \mathbb{R}^d \times [0, T]$ for some $T$ depending on $M_0$ (see [8 Proposition 3]). This is now longer the case under (H_1(f)), which we believe is a more physically accurate phenomenon.

The assumption that $u^\epsilon \in L^\infty(\mathbb{R}^d)$ seems technical (because it comes without bounds in term of $\epsilon$), but uneasy to get rid of: the main reason being the lack of a suitable local well posedness in the space of functions with locally finite energy (in particular, lack of uniqueness of the solution): we refer to the works by Ginibre and Velo [17] and [15] (besides [15], on which we rely, in the $L^\infty$ setting). In the same spirit, the growth condition on $f$ seems almost optimal, as it is required for the monotonicity formula – which we use heavily – to make sense.

These results are an extension of the works of Bethuel, Orlandi and Smets [8,9], and the proofs are strongly inspired by these: Theorems 1.1 and 1.2 by Theorems A and B in [8] and Theorem 1.3 by Theorem 3.1 in [9]. Our main contribution will be to systematically improve their estimates, in order to solve the new problems raised by our dealing with infinite energy solutions of (PGL), especially to make sense of a monotonicity property, which is at the heart of the proofs in [8,9]. We will also need to derive pointwise estimates on $u_\epsilon$ and $L^2$ space time estimates on $\partial_t u_\epsilon$: in the finite energy setting, it appears as the flux of the energy, but this is no longer the case in our context. A leitmotiv of this paper is that, although many of the bounds in [8,9] are global in time and/or space, their arguments are in fact local in nature, and so can be adapted under the hypothesis (H_1(f)). In the proofs, we will focus on the differences brought by our change of context, and only sketch the arguments when they are similar to that of [8,9].

A natural question is now to concentrate on dimension $d = 2$ and to study the dynamics of vortices in the accelerated time frame, as it is done in [9,10,11]. We believe that the arguments in these works could be extended under the hypothesis (H_1(f)). However one has to make a meaningful sense of the limiting equation, (a pseudo gradient flow of the Kirchoff-Onsager functional involved), as it is not obviously well posed for a countable infinite number of points. We leave these perspectives to subsequent research.

1.3. Organization of the proofs and notations. This paper is organized as follows. In Section 2, we study (PGL) and prove our main PDE tool, namely the clearing-out (stated in Theorem 2.1). In Section 3, we define the limiting measure and the vorticity set $\Sigma_\mu$: we prove in particular regularity properties of $\Sigma_\mu$ and complete the proof of Theorem 1.1. In Section 4, we show Theorem 1.2 that is, the singular part $v_\epsilon^*$ follows the mean curvature flow in Brakke’s weak formulation. Finally, in Section 5, we consider the dimension $d = 2$ and prove Theorem 1.3.

In the inequalities, the implicit constants $C$ can change from line to line and, unless stated otherwise, can depend on the space dimension $d$ and on the function $f$ (given in (1.4)), while the other dependencies will be specified: for example $C(R)$ (where $R > 0$ is a real number) can stand short for
a bound of the type \((1 + R)^d f(2R)\). The functions \(C\) will always be continuous, and non decreasing in their arguments valued in \([0, +\infty)\).

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2. PDE Analysis of \((PGL_\epsilon)\)

2.1. Statement of the main results on \((PGL_\epsilon)\). In this section, we work on \((PGL_\epsilon)\), that is with smooth solutions \(u_\epsilon\), where the parameter \(\epsilon\), although small, is strictly positive. We derive a number of properties on \(u_\epsilon\), which enter directly in the proof of the clearing-out Theorem 3.14 at the limit \(\epsilon \to 0\). Heuristically, the clearing-out means that if there is not enough energy in some region of space, then at a later time, vortices can not be created in that region.

Let us first state the main results which will be proved in this section.

2.1.1. Clearing-out and annihilation for vorticity. The two main ingredients in the proof of Theorem 3.14 are a clearing-out theorem for vorticity \((\epsilon > 0)\), as well as some precise pointwise energy bounds. Throughout this section, we suppose that \(0 < \epsilon < 1\). We define the vorticity set \(\mathcal{V}_\epsilon\) as

\[
\mathcal{V}_\epsilon = \left\{ (x, t) \in \mathbb{R}^d \times (0, +\infty) : |u_\epsilon(x, t)| \leq \frac{1}{2} \right\}.
\]

Here is the precise statement.

**Theorem 2.1.** Let \(0 < \epsilon < 1/2\), \(u_0^\sigma \in L^\infty(\mathbb{R}^d)\) and \(u_\epsilon\) be the associated solution of \((PGL_\epsilon)\). Let \(\sigma > 0\) be given. There exists \(\eta_1 = \eta_1(\sigma) > 0\) depending only on the dimension \(d\) and on \(\sigma\) such that if

\[
\int_{\mathbb{R}^d} e_\epsilon(u_0^\sigma)(x) \exp\left(-\frac{|x|^2}{4}\right) dx \leq \eta_1|\ln \epsilon|,
\]

then

\[
|u_\epsilon(0, 1)| \geq 1 - \sigma.
\]

Notice that we only assume that \(u_0^\sigma \in L^\infty(\mathbb{R}^d)\), whereas in [8, Theorem 1] the assumption was \(E_\epsilon(u_\epsilon) < +\infty\); of course this latter bound will not be available for Theorem 1.1. Observe that \(L^\infty\) prevents us to use a density argument, and even more so as we are interested in non zero degree initial data. Also, the assumption (2.1) is not enough by itself to ensure existence and uniqueness of the solution to \((PGL_\epsilon)\), but \(L^\infty\) is suitable (see [15]). Nonetheless the proof follows closely that of [8, Part I], and we will only emphasize the differences.

The proof of Theorem 2.1 requires a number of tools, in particular

- the monotonicity formula, first derived by Struwe [36] in the case of the heat-flow for harmonic maps
- a localizing property for the energy inspired by Lin and Rivière [27]
- refined Jacobian estimates due to Jerrard and Soner [23]
- techniques first developed for the stationary equation (for example [4, 5, 6]).

Equation \((PGL_\epsilon)\) has standard scaling properties. If \(u_\epsilon\) is a solution to \((PGL_\epsilon)\), then for \(R > 0\) the function \((x, t) \mapsto u_\epsilon(Rx, R^2t)\) is a solution to \((PGL)(R^{-1}x, R^{-2}t)\), to which we may then apply Theorem 2.1. As an immediate consequence of Theorem 2.1 and scaling, we have the following result.

**Proposition 2.2.** Let \(T > 0\), \(x_T \in \mathbb{R}^d\), and set \(z_T = (x_T, T)\). Let \(u_0^\sigma \in L^\infty(\mathbb{R}^d)\) and \(u_\epsilon\) be the associated solution of \((PGL_\epsilon)\). Let \(R > \sqrt{2\epsilon}\). Assume moreover

\[
\frac{1}{R^{d-2}} \int_{\mathbb{R}^d} e_\epsilon(u_\epsilon)(x, T) \exp\left(-\frac{|x - x_T|^2}{4R^2}\right) dx \leq \eta_1(\sigma)|\ln \epsilon|,
\]
then

$$|u_\varepsilon(x_T, T + R^2)| \geq 1 - \sigma.$$  

The condition in Proposition 2.2 involves an integral on the whole space $\mathbb{R}^d$. In some situations, it will be convenient to integrate on finite domains. Here is how one should localize in space the conditions.

**Proposition 2.3.** Let $u_\varepsilon$ be a solution of [PGL] satisfying the initial data $[H_1(f)]$. Let $\sigma > 0$ be given. Let $T > 0$, $x_T \in \mathbb{R}^d$, and $R \in [\sqrt{2\varepsilon}, 1]$. There exists a positive continuous function $\lambda$ defined on $\mathbb{R}^d \times (0, +\infty)$ such that

$$\bar{\eta}(x_T, T, R) := \frac{1}{R^{d-2}|\ln \varepsilon|} \int_{B(x_T, \lambda(x_T, T)R)} e_\varepsilon(u_\varepsilon)(x, T) \, dx \leq \frac{\eta_1(\lambda)}{2},$$

then

$$|u_\varepsilon(x, t)| \geq 1 - \sigma \quad \text{for } t \in [T + T_0, T + T_1] \text{ and } x \in B\left(x_T, \frac{R}{2}\right),$$

where $T_0 = \max\left(2\varepsilon, (\frac{2\varepsilon}{\eta_1(\sigma)})^{\frac{d}{d-2}}R^2\right)$ ($T_0 = 2\varepsilon$ in dimension $d = 2$), and $T_1 = R^2$.

Furthermore, $\lambda \geq 1$ and for all $x \in \mathbb{R}^d$, $t \to \lambda(x, t)$ is non increasing on $(0, 1]$, non decreasing on $[1, +\infty)$, and there exist a constant $C = C(\sigma)$ such that

$$\forall T \in (0, 1/2], \quad \frac{1}{C} \sqrt{|\ln T| + \ln f(2|x_0|)} \leq \lambda(x_0, T) \leq C \sqrt{|\ln T| + \ln f(2|x_0|)}.$$

**Remark 2.4.** Recall that in dimension $d \geq 3$, a bound on the initial energy on the whole space [1.2] implies that in finite time, the vorticity vanishes (i.e., $\Sigma^\mu_i \subset \mathbb{R}^d \times [0, T]$). It is an easy consequence of the monotonicity formula combined with Theorem 2.1 ($E_{w,\varepsilon}(x, 2t, \sqrt{t}) \to 0$ uniformly in $x$, see (2.17) for definition).

In the case of a local bound on the energy ($H_1(f)$), this result does not persist, because the monotonicity formula does not imply the vanishing of $E_{w,\varepsilon}$ for large times. This is one striking difference with the finite energy case.

2.1.2. **Improved pointwise energy bounds.** The following result reminds of a result of Chen and Struwe [14] developed in the context of the heat flow for harmonic maps.

**Theorem 2.5.** Let $u_\varepsilon$ be a solution of [PGL] whose initial data satisfies $[H_1(f)]$. Let $B(x_0, R)$ be a ball in $\mathbb{R}^d$ and $T > 0, \Delta T > 0$ be given. Consider the cylinder

$$\Lambda = B(x_0, R) \times [T, T + \Delta T].$$

There exist two constants $0 < \sigma \leq \frac{1}{2}$ and $\beta > 0$ depending only on $d$ such that the following holds. Assume that

$$|u_\varepsilon| \geq 1 - \sigma \text{ on } \Lambda.$$

Then

$$e_\varepsilon(u_\varepsilon)(x, t) \leq C(\Lambda) \int_\Lambda e_\varepsilon(u_\varepsilon),$$

for any $(x, t) \in \Lambda_2 := B\left(x_0, \frac{R}{2}\right) \times [T + \frac{\Delta T}{4}, T + \Delta T]$. Moreover,

$$e_\varepsilon(u_\varepsilon) = |\nabla \Phi_e|^2 + \kappa_\varepsilon \quad \text{in } \Lambda_2,$$

where the functions $\Phi_e$ and $\kappa_\varepsilon$ are defined on $\Lambda_2$, and verify

$$\frac{\partial \Phi_e}{\partial t} - \Delta \Phi_e = 0 \quad \text{in } \Lambda_2,$$

$$\|\kappa_\varepsilon\|_{L^\infty(\Lambda_2)} \leq C(\Lambda) e^\beta, \quad \|\nabla \Phi_e\|_{L^\infty(\Lambda_2)} \leq C(\Lambda) |\ln \varepsilon|. $$
On $\Lambda$ one can write $u_\epsilon = \rho_\epsilon e^{i\varphi_\epsilon}$, where $\varphi_\epsilon$ is smooth (and $\rho_\epsilon = |u_\epsilon|$), and we have the bound
\begin{equation}
\|\nabla \varphi_\epsilon - \nabla \Phi_\epsilon\|_{L^\infty(\Lambda)} \leq C(\Lambda)\epsilon^\beta.
\end{equation}

Combining Proposition 2.3 and Theorem 2.5 we obtain the following immediate consequence.

**Proposition 2.6.** Let $u_\epsilon$ be a solution of \((\text{PGL}_\epsilon)\) whose initial data satisfies \((H_\epsilon(f))\). There exist an absolute constant $\eta_2 > 0$ and a positive function $\lambda$ defined on $\mathbb{R}^d \times (0, +\infty)$ such that, if for $x \in \mathbb{R}^d$, $t > 0$ and $r \in [\sqrt{2}\epsilon, 1]$, we have
\[\int_{B(x, \lambda(x,t)r)} e_\epsilon(u_\epsilon) \leq \eta_2 r^{d-2} |\ln r|,\]
then
\[e_\epsilon(u_\epsilon) = |\nabla \Phi_\epsilon|^2 + \kappa_\epsilon\]
in $\Lambda_\epsilon(x, t, r) := B(x, \frac{\epsilon}{4}) \times [t + \frac{15}{10} r^2, t + r^2]$, where $\Phi_\epsilon$ and $\kappa_\epsilon$ are as in Theorem 2.5. In particular,
\[\mu_\epsilon = \frac{e_\epsilon(u_\epsilon)}{|\ln r|} \leq C(t, r) \text{ on } \Lambda_\epsilon(x, t, r).
\](The constant $\eta_2$ is actually defined as $\eta_2 = \eta_1(\sigma)$ where $\sigma$ is the constant in Theorem 2.5 and $\eta_1$ is the function defined in Proposition 2.3.)

2.1.3. **Identifying the sources of non compactness.** We identified in the previous arguments a possible source of non compactness, due to oscillations in the phase. But this analysis was carried out on the complement of the vorticity set. Now $u_\epsilon$ is likely to vanish on $\mathcal{V}_\epsilon$, which leads to a new contribution to the energy: however, this new contribution does not correspond to a source of non compactness, as it is stated in the following theorem.

**Theorem 2.7.** Let $u_\epsilon$ be a solution of \((\text{PGL}_\epsilon)\) whose initial data satisfies \((H_\epsilon(f))\). Let $K \subset \mathbb{R}^d \times (0, +\infty)$ be any compact set. There exist a real-valued function $\Phi_\epsilon$ and a complex-valued function $w_\epsilon$, both defined on a neighborhood of $K$, such that
\begin{enumerate}
\item $u_\epsilon = w_\epsilon \exp(i\Phi_\epsilon)$ on $K$,
\item $\Phi_\epsilon$ verifies the heat equation on $K$,
\item $|\nabla \Phi_\epsilon(x, t)| \leq C(K)\sqrt{\ln r}$ for all $(x, t) \in K$,
\item $\|\nabla w_\epsilon\|_{L^p(K)} \leq C(p, K)$, for any $1 \leq p < \frac{d+1}{d}$.
\end{enumerate}
Here, $C(K)$ and $C(p, K)$ are constants depending only on $K$, and $p, K$ respectively.

The proof relies on the refined Jacobian estimates of [21].

We stress out the fact that Theorem 2.7 provides an exact splitting of the energy in two different modes, that is the topological mode (the energy related to $w_\epsilon$), and the linear mode (the energy of $\Phi_\epsilon$): in some sense, the lack of compactness is completely locked in $\Phi_\epsilon$.

The remainder of this section is to provide proofs for the results described above, which will be done in section 2.4; we need some preliminary considerations before.

2.2. **Pointwise estimates.** In this section, we provide pointwise parabolic estimates for $u_\epsilon$ solution of \((\text{PGL}_\epsilon)\), which rely ultimately on a supersolution argument, i.e a variant of the maximum principle.

**Proposition 2.8.** Let $u_\epsilon^0 \in L^\infty(\mathbb{R}^d)$ and $u_\epsilon$ be the associated solution of \((\text{PGL}_\epsilon)\). Then for all $t > 0$, $u_\epsilon(t)$, $\nabla u_\epsilon(t)$ and $\partial_t u_\epsilon(t)$ belong to $L^\infty(\mathbb{R}^d)$. More precisely, there exists a (universal) constant $K_0 > 0$ such that for all $t > \epsilon^2$ and $x \in \mathbb{R}^d$,
\begin{equation}
|u_\epsilon(x, t)| \leq 2, \quad |\nabla u_\epsilon(x, t)| \leq \frac{K_0}{\epsilon}, \quad |\partial_t u_\epsilon(x, t)| \leq \frac{K_0}{\epsilon^2}.
\end{equation}
Also, for all $t > 0$ and $x \in \mathbb{R}^d$,
\begin{equation}
|u_\epsilon(x, t)| \leq \max(1, \|u_\epsilon^0\|_{L^\infty}) \quad \text{and} \quad |\nabla u_\epsilon(x, t)| \leq C \left( \sqrt{t} + \frac{1}{\sqrt{t}} \right) \max(1, \|u_\epsilon^0\|_{L^\infty})^3.
\end{equation}

\textbf{Remark 2.9.} We emphasize that, past the time layer $t \geq \epsilon^2$, $\|u_\epsilon(t)\|_{L^\infty}$ is bounded independently of $u_\epsilon^0$.

\textbf{Proof.} We make a change of variable, setting
\[ v(x, t) = u_\epsilon(\epsilon x, \epsilon^2 t), \]
so that the function $v$ satisfies
\begin{equation}
\partial_\sigma v - \Delta v = v(1 - |v|^2) \quad \text{on } \mathbb{R}^d \times [0, +\infty).
\end{equation}
We have to prove that, for $\sigma \geq 1$ and $x \in \mathbb{R}^d$, 
\[ |v(x, t)| \leq 2, \quad |\nabla v(x, t)| \leq K_0, \quad |\partial_\sigma v(x, t)| \leq K_0, \]
and that for $t > 0$ and $x \in \mathbb{R}^d$, 
\[ |v(x, t)| \leq \max(\|u_\epsilon^0\|_{L^\infty}, 1). \]
Recall that $v \in \mathcal{C}_0((0, +\infty), L^\infty(\mathbb{R}^d))$, and that lim sup_{t \to 0}, $\|v(t)\|_{L^\infty} \leq \|u_\epsilon^0\|_{L^\infty}$ (see [15]). We begin with the $L^\infty$ estimates for $v$. Set 
\[ \sigma(x, t) = |v(x, t)|^2 - 1. \]
Multiplying equation (2.8) by $U$, we are led to the equation for $\sigma$,
\begin{equation}
\partial_\sigma \sigma - \Delta \sigma + 2|v|^2 + 2\sigma(1 + \sigma) = 0.
\end{equation}
Consider next the EDO
\begin{equation}
y'(t) + 2y(t)(y(t) + 1) = 0,
\end{equation}
and notice that (2.10) admits the explicit solution defined for $t > 0$ by
\begin{equation}
y_{t_0}(t) = \frac{\exp(-(t - t_0)/2)}{1 - \exp(-(t - t_0)/2)}, \quad \text{with} \quad t_0 = 2\ln \left( 1 - \frac{1}{\max(1, \|u_\epsilon^0\|_{L^\infty})^2} \right),
\end{equation}
so that $y_{t_0}(0) = \max(1, \|u_\epsilon^0\|_{L^\infty})^2 - 1$ and as consequence
\[ \sup_{x \in \mathbb{R}^d} \sigma(x, 0) \leq y_{t_0}(0). \]
We claim that
\begin{equation}
\forall t > 0, \forall x \in \mathbb{R}^d, \quad \sigma(x, t) \leq y_{t_0}(t).
\end{equation}
Indeed, set $\tilde{\sigma}(x, t) = y_{t_0}(t)$. Then
\begin{equation}
\partial_\sigma \tilde{\sigma} - \Delta \tilde{\sigma} + 2\tilde{\sigma}(1 + \tilde{\sigma}) = 0,
\end{equation}
and therefore by (2.9),
\begin{equation}
\partial_\sigma (\tilde{\sigma} - \sigma) - \Delta(\tilde{\sigma} - \sigma) + 2(\tilde{\sigma} - \sigma)(1 + \tilde{\sigma} + \sigma) \geq 0.
\end{equation}
Note that $1 + \tilde{\sigma} + \sigma = |v|^2 + \tilde{\sigma} \geq 0$ and $\tilde{\sigma}(0) - \sigma(0) > 0$. The maximum principle implies that
\[ \forall t > 0, \forall x \in \mathbb{R}^d, \quad \tilde{\sigma}(x, t) - \sigma(x, t) \geq 0, \]
which proves the claim (2.12). Then observe that $t_0 < 0$ and that $y_0$ is decreasing on $(0, +\infty)$, so that
\[ \forall t > 0, \forall x \in \mathbb{R}^d, \quad \sigma(t, x) \leq y_{t_0}(t) \leq y_0(t). \]
Observe that the first bound give $|v(x, t)| \leq \max(1, \|u_\epsilon^0\|_{L^\infty})$ for all $t > 0$ and $x \in \mathbb{R}^d$, i.e the first part of (2.7). Also for $t \geq 1$ and $x \in \mathbb{R}^d$, 
\[ |v(x, t)| \leq \sqrt{1 + y_0(1)} \leq 2. \]
We next turn to the space and time derivatives. The Duhamel formula yields
\[ \nabla v(x, t) = (\nabla G(t) * u_\epsilon^0(\epsilon))(x) + \int_0^t \left( (\nabla G)(t - s) * (v(s)(1 - |v(s)|^2)) \right)(x)ds. \]
Recall that \(\|\nabla_x G(t)\|_{L^1} \leq C/\sqrt{t}\). Also, due to (2.7), for some \(C > 0\) independent of \(x, t\) and \(\epsilon\), there hold
\[
\forall t \geq 0, \quad \|v(t)(1 - |v(t)|^2)\|_{L^\infty} \leq C \max(1, \|u_\epsilon^0\|_{L^\infty})^3.
\]
Hence
\[
\|\nabla_x v(t)\|_{L^1} \leq \|\nabla G(t)\|_{L^1} \|u_\epsilon^0\|_{L^\infty} + \int_0^t \|\nabla G(t-s)\|_{L^1} \|v(s)(1 - |v(s)|^2)\|_{L^\infty} ds
\]
\[
\leq \frac{C}{\sqrt{t}} \|u_\epsilon^0\|_{L^\infty} + C \max(1, \|u_\epsilon^0\|_{L^\infty})^3 \int_0^t \frac{ds}{\sqrt{t-s}}
\]
\[
\leq C \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) \max(1, \|u_\epsilon^0\|_{L^\infty})^3,
\]
which is the second part of (2.7). We now focus on (2.6). Since \(|v(x, t)| \leq \sqrt{1 + y_0(1/2)}\) for \(t \geq 1/2\), there exists \(K_1 \geq 1\) (independent of \(\epsilon\)) such that
\[
(2.15) \quad \forall t \geq 1/2, \quad \forall x \in \mathbb{R}^d, \quad |v(x, t)|^3 + |v(x, t)| \leq K_1.
\]
Let \(t \geq 1\). Now, differentiating in space the Duhamel formula between times \(t - 1/2 \geq 1/2\) and \(t\) gives
\[
\nabla v(t) = (\nabla G)(1/2) * v(t - 1/2) + \int_{t-1/2}^t (\nabla G)(t-s) * (v(s)(1 - |v(s)|^2)) ds,
\]
where \(G(x, t) = \frac{1}{(4\pi t)^{d/2}} e^{-x^2/4t}\) is the heat kernel. Also, as \(t - 1/2 \geq 1/2\), there holds \(\|v(t - 1/2)\|_{L^\infty} \leq 2\) and (2.15) for all \(s \in [t-1/2, t]\): hence
\[
\|\nabla v(t)\|_{L^\infty} \leq \|\nabla G(1/2)\|_{L^1} \|v(t - 1/2)\|_{L^\infty}
\]
\[
+ \int_0^t \|\nabla G(t-s)\|_{L^1} \|v(s)(1 - |v(s)|^2)\|_{L^\infty} ds ds
\]
\[
\leq \frac{C}{\sqrt{2}} + CK_1 \int_{t-1/2}^t \frac{ds}{\sqrt{t-s}} \leq CK_1.
\]
Similarly, we can differentiate the Duhamel formula twice:
\[
\nabla^2 v(t) = (\nabla^2 G)(1/2) * v(t - 1/2) + \int_{t-1/2}^t (\nabla G)(t-s) * \nabla((v(s)(1 - |v(s)|^2))) ds,
\]
Using that \(|\nabla(v(s, x)(1 - |v(s, x)|^2))| \leq CK_1|\nabla v(s, x)|\) and \(\|\nabla^2 G(t)\| \leq C/t\), we can differentiate once
\[
\|\nabla^2 v(t)\|_{L^\infty} \leq \|\nabla^2 G(1/2)\|_{L^1} \|v(t - 1/2)\|_{L^\infty}
\]
\[
+ \int_{t-1/2}^t \|\nabla G(t-s)\|_{L^1} \|\nabla(v(s)(1 - |v(s)|^2))\|_{L^\infty} ds ds
\]
\[
\leq C \sqrt{2} + \int_{t-1/2}^t \frac{CK_1}{\sqrt{t-s}} ds \leq CK_1.
\]
Finally,
\[
|\partial_{\epsilon} v| = |\Delta v + v(1 - |v|^2)| \leq \|\nabla^2 v\| + K_1 \leq CK_1.
\]
We have the following variant of Proposition 2.8.

**Proposition 2.10.** Let \(u_\epsilon^0 \in L^\infty(\mathbb{R}^d)\) and \(u_\epsilon\) be the associated solution of \((\text{PGL}_\epsilon_u)\). Assume that for some constants \(C_0 \geq 1\), \(C_1 \geq 0\) and \(C_2 \geq 0\),
\[
\forall x \in \mathbb{R}^d, \quad |u_\epsilon^0(x)| \leq C_0, \quad \|\nabla u_\epsilon^0(x)\| \leq \frac{C_1}{\epsilon}, \quad \|\nabla^2 u_\epsilon^0(x)\| \leq \frac{C_2}{\epsilon^2}.
\]
Then for any $t > 0$ and $x \in \mathbb{R}^d$, we have
\[ |u_\epsilon(x, t)| \leq C_0, \quad |\nabla u_\epsilon(x, t)| \leq \frac{C}{\epsilon}, \quad |\partial_t u_\epsilon(x, t)| \leq \frac{C}{\epsilon^2}, \]
where $C$ depends only on $C_0$, $C_1$ and $C_2$.

Proposition 2.10 provides an upper bound for $|u_\epsilon|$. The next lemma provides a local lower bound on $|u_\epsilon|$, when we know it is away from zero on some region. Since we have to deal with parabolic problems, it is natural to consider parabolic cylinders of the type
\[
(2.16) \quad \Lambda_\alpha(x_0, T, R, \Delta T) = B(x_0, \alpha R) \times [T + (1 - \alpha^2)\Delta T, T + \Delta T].
\]
Sometimes, it will be convenient to choose $\Delta T = R$ and write $\Lambda_\alpha(x_0, T, R)$. Finally if there is no ambiguity, we will simply write $\Lambda_\alpha$, and even $\Lambda$ if $\alpha = 1$.

Lemma 2.11 ([3]). Let $u_\epsilon$ be a solution of (PGL$_\alpha$) whose initial data satisfies (H$_1$($f$)). Let $x_0 \in \mathbb{R}^d$, $R > 0$, $T > 0$ and $\Delta T > 0$ be given. Assume that
\[
|u_\epsilon| \geq \frac{1}{2} \quad \text{on } \Lambda(x_0, T, R, \Delta T),
\]
then
\[ 1 - |u_\epsilon| \leq C(\alpha, \Lambda) \epsilon^2 \left( \|\nabla \phi_\epsilon\|_{L^\infty(\Lambda)} + |\ln \epsilon| \right) \quad \text{on } \Lambda_\alpha,
\]
where $\phi_\epsilon$ is defined on $\Lambda$, up to a multiple of $2\pi$, by $u_\epsilon = |u_\epsilon| \exp(i\phi_\epsilon)$.

Proof. We refer to [3] Lemma 1.1, p. 52. \hfill \square

2.3. The monotonicity formula and some consequences. In this section, we provide various tools which will be required in the proof of Theorem 2.1.

2.3.1. The monotonicity formula. For $(x_*, t_*) \in \mathbb{R}^d \times [0, +\infty)$ we set
\[ z_* = (x_*, t_*).
\]
For $t_0 > 0$ and $0 < R \leq \sqrt{t_0}$ we defined the weighted energy, scaled and time shifted, by
\[
(2.17) \quad E_w(u_\epsilon, z_*, R) = E_w(z_*, R) := \frac{1}{R^{d-2}} \int_{\mathbb{R}^d} \epsilon_*(u_\epsilon)(x, t_* - R^2) \exp \left( -\frac{|x - x_*|^2}{4R^2} \right) \, dx.
\]
(We may drop the $\epsilon$ or the $u_\epsilon$ when there is no ambiguity). We stress out that in the integral defining $E_w$, we introduced a time shift $\delta t = -R^2$. Also it will be convenient to use the multiplier
\[
(2.18) \quad \Xi(u_\epsilon, z_*)(x, t) = \frac{1}{4|t - t_*|} \left( (x - x_*) \cdot \nabla u_\epsilon(x, t) + 2(t - t_*) \partial_t u_\epsilon(x, t) \right)^2.
\]

The following monotonicity formula was first derived and used by Struwe [36] in his study of the heat flow for harmonic maps.

Proposition 2.12. Let $u_\epsilon$ be a solution of (PGL$_\alpha$) whose initial data satisfies (H$_1$($f$)). We have, for $0 < r < \sqrt{t_*}$,
\[
\frac{dE_w}{dR}(z_*, r) = \frac{1}{R^{d-1}} \int_{\mathbb{R}^d} \frac{1}{2r^2} \left( (x - x_*) \cdot \nabla u_\epsilon(x, t_* - r^2) - 2r^2 \partial_t u_\epsilon(x, t_* - r^2) \right)^2 \exp \left( -\frac{|x - x_*|^2}{4r^2} \right) \, dx
\]
\[
+ \frac{1}{R^{d-1}} \int_{\mathbb{R}^d} 2\epsilon_*(u_\epsilon)(x, t_* - r^2) \exp \left( -\frac{|x - x_*|^2}{4r^2} \right) \, dx
\]
\[
= \frac{(4\pi)^{d/2}}{r} \int_{\mathbb{R}^{d+1}} 2|t - t_*| \Xi(z_*)(x, t) G(x - x_*, t - t_*) dxd\delta_{t_* - r^2}(t)
\]
\[
+ (4\pi)^{d/2} r \int_{\mathbb{R}^{d+1}} 2\epsilon_*(u_\epsilon)(x, t) G(x - x_*, t - t_*) dxd\delta_{t_* - r^2}(t),
\]
where $G(x, t)$ denotes the heat kernel

\[
G(t, x) = \begin{cases} 
\frac{1}{(4\pi t)^{d/2}} \exp \left( -\frac{|x|^2}{4t} \right) & \text{for } t > 0, \\
0 & \text{for } t \leq 0.
\end{cases}
\]

In particular,

\[
dE_w(m, r) \geq 0.
\]

As a consequence, $R \to E_w(m, R)$ can be extended to a non-decreasing, continuous function of $R$ on $[0, \sqrt{T_e}]$, with $E_w(\infty, 0) = 0$.

\textbf{Proof.} For $0 < R < \sqrt{T_e}$, the map $(R, x) \to e_u(x, t, t - R^2)$ is smooth (due to parabolic regularization), and satisfies domination bounds due to (2.6) and the Gaussian weight. Therefore $R \to E_w(m, R)$ is smooth on $(0, \sqrt{T_e})$ and we can perform the same computations as in Proposition 2.1 in [8]; integrations by parts are allowed for the same reasons. This proves formula (2.19), and the monotonicity property follows immediately. It remains to study the continuity at the endpoints.

For the limit $R \to 0$, the bounds (2.7) show that $|e_u(x, t, t - R^2)| \leq C(t, ||u^0||_L^\infty) / \varepsilon^2$ uniformly for $R \leq t / 2$, so that in that range

\[
E_w(m, R) = R^2 \int e_u(x, t, t - R^2) \frac{1}{R^d} \exp \left( -\frac{|x - x^2|^2}{4R^2} \right) dx 
\]

\[
\leq R^2(4\pi)^{d/2}C(t, ||u^0||_L^\infty) / \varepsilon^2 \rightarrow 0 \text{ as } R \to 0.
\]

For the derivative term, we use the Duhamel formula:

\[
\nabla u_\varepsilon(t) = G(t) * \nabla u^0 + \int_0^t \nabla G(t - s) * (u_\varepsilon(1 - |u_\varepsilon|^2))(s) ds = G(t) * \nabla u^0 + D(x, t).
\]

The Duhamel term $D(x, t)$ is harmless, indeed

\[
|D(x, t)| \leq \int_0^t \|\nabla G(t - s)||u_\varepsilon(1 - |u_\varepsilon|^2)||_L^\infty ds 
\]

\[
\leq C \max(1, ||u^0||_L^\infty)^3 \int_0^t \frac{ds}{\sqrt{t - s}} \leq C \max(1, ||u^0||_L^\infty)^3 \sqrt{t}.
\]

Therefore

\[
\int |D(x, t, t - R^2)|^2 \exp \left( -\frac{|x - x^1|^2}{4R^2} \right) dx \leq C(t - R^2) \rightarrow 0 \text{ as } R \to \sqrt{T_e}.
\]

The linear term requires to recall Claim 13 of [15]. Due to assumption (H1(f)), for any $\alpha > 0$, $\nabla u_\varepsilon \in L^2(e^{-[a]x^2} dx)$. Indeed, as $f$ has mild growth, using (1.4) with $R = \sqrt{2/\alpha}$, we see that, $f(x) \leq C(\sqrt{2/\alpha})e^{ax^2/2}$; denoting $Q_k$ the cube of size 1 centered at $k \in \mathbb{Z}^d$, it can be covered by at most $C(d)$ balls of radius and center in $Q_k$, so that

\[
\int |\nabla u_\varepsilon^0(x)|^2 e^{-[a]x^2} dx = \sum_{k \in \mathbb{Z}^d} \int_{Q_k} |\nabla u_\varepsilon^0|^2 e^{-[a]x^2} dx \leq \sum_{k \in \mathbb{Z}^d} e^{-\alpha(k - 1)^2} \int_{Q_k} |\nabla u_\varepsilon^0|^2 dx 
\]

\[
\leq \sum_{k \in \mathbb{Z}^d} e^{-\alpha|k|^2 + 2\alpha|k|C(d)|f(|k| + 1)|\ln \varepsilon |} \leq C(\alpha, d)|\ln \varepsilon |.
\]
As a consequence, we can apply Lebesgue’s dominated convergence theorem and conclude that
$$\beta$$
(where \(\tau, \phi(x) = \phi(x - h)\)), and satisfies
$$\|\tau_h \nabla u_{\epsilon}^0 - \nabla u_{\epsilon}^0\|_{L^2(e^{-|x|^2} dx)} \to 0 \quad \text{as} \ h \to 0$$

(15, Claim 13), we infer from \(\nabla u_{\epsilon}^0 \in L^2(e^{-|x|^2} dx)\) that for \(\beta > 2 \alpha\),

$$\|\tau_h \nabla u_{\epsilon}^0 - \nabla u_{\epsilon}^0\|_{L^2(e^{-|x|^2} dx)} \to 0 \quad \text{as} \ h \to 0$$



Let
$$\text{Bounds on the energy.}$$



2.3.2. Boundedness

We consider the case \(\forall \epsilon > 0, \\text{satisfies } H_1(\phi)\), and hence, summing up, we proved that \(R \mapsto E_w(z, R)\) is (left-)continuous at \(R = t_\epsilon\).

Choose \(\beta = 1/(8 t_\epsilon)\) and \(\alpha = 1/(17 t_\epsilon)\). Then it follows, using Cauchy-Schwarz inequality, that

$$\int (G(t_\epsilon - R)) \nabla u_{\epsilon}^0(x) \exp\left(-\frac{|x - x_\epsilon|^2}{4R^2}\right) - \int \nabla u_{\epsilon}^0(x) \exp\left(-\frac{|x - x_\epsilon|^2}{4t_\epsilon}\right) dx$$

$$\leq \int G(t_\epsilon - R) \nabla u_{\epsilon}^0(x) - \nabla u_{\epsilon}^0(x)$$

$$\times \left[ (|G(t_\epsilon - R) \nabla u_{\epsilon}^0(x) + \nabla u_{\epsilon}^0(x)) \exp\left(-\frac{|x - x_\epsilon|^2}{4R^2}\right) \right] dx$$

$$\leq \|G(t_\epsilon - R) \nabla u_{\epsilon}^0(x) - \nabla u_{\epsilon}^0(x)\|_{L^2(e^{-|x|^2} dx)}$$

$$\times \|G(t_\epsilon - R) \nabla u_{\epsilon}^0(x) - \nabla u_{\epsilon}^0(x)\|_{L^2(e^{-|x|^2} dx)} + o(1)$$

$$\to 0 \quad \text{as} \ R \to \sqrt{t_\epsilon}.$$
Also there exists a constant $C(d)$ such that any cube $Q_k$ is covered by $C(d)R^d$ balls of radius 1. Therefore,

$$
\int e_\epsilon(v_\epsilon(y)) e^{-\frac{|x-y|^2}{R^2}} dy \leq \sum_{k \in \mathbb{Z}^d} \exp \left(-\frac{(R|k| - R\sqrt{2})^2}{R^2}\right) \int_{Q_k} e_\epsilon(v_\epsilon(y)) dy
$$

$$
\leq C(d)R^d |\ln \epsilon| \sum_{k \in \mathbb{Z}^d} e^{-((|k| - \sqrt{2})f(|x| + R|k| + 1))}
$$

$$
\leq C(d,R)C(2R)|\ln \epsilon| f(|x|) \sum_{k \in \mathbb{Z}^d} e^{-((|k| - \sqrt{2}f(1/2)\epsilon)^2 + \frac{1}{2}(|k| + 1)^2)}
$$

$$
\leq C(R)|\ln \epsilon| f(2|x|).
$$

(We used (1.4) with parameter $2R$ on the 3rd line). This is the first result claimed. For the second, we clearly have

$$
\int e_\epsilon(w_\epsilon(y)) \exp \left(-\frac{|x-y|^2}{R^2}\right) d\epsilon y \geq \frac{1}{e} \int_{B(x,R)} e_\epsilon(w_\epsilon(y)) dy.
$$

This means that the energy on $B(x,R)$ is at most $\epsilon f(|x|)|\ln \epsilon|$. If $R \gg 1$, then this is enough and we get the result with $g(x) = \epsilon f(x)$. If $R \ll 1$, then $B(y,1)$ can be covered by at most $C(d)/R^d$ balls of radius $R$ and with center at distance at most $|y| + 1$ from 0, so that for all $y \in \mathbb{R}^d$,

$$
\int_{B(x,1)} e_\epsilon(w_\epsilon(y)) dy \leq \frac{C(d)}{R^d} \epsilon f(|x| + 1)|\ln \epsilon|.
$$

The first consequence of the monotonicity formula is that $(H_1)$ is a condition which propagates in time in the following way.

**Proposition 2.14.** Let $u_\epsilon$ be a solution of (PGL$_\epsilon$) whose the initial data satisfies $(H_1(f))$. Then for any $T > 0$, $(x, t) \to u_\epsilon(x, t) + t$ is still a solution of (PGL$_\epsilon$), whose initial condition satisfies $H_1(g_\epsilon)$ where $g_\epsilon(x) = C(T) f(2|x|)$. More precisely there holds

$$
(2.23) \quad \forall \epsilon, t, R > 0, \forall x \in \mathbb{R}^d, \quad \int_{B(x,R)} e_\epsilon(u_\epsilon(y,t)) dy \leq C(t,R) f(2|x|)|\ln \epsilon|.
$$

**Proof.** We apply the monotonicity formula at the point $(x, t + \frac{R^2}{4})$ between $\frac{R}{2}$ and $\sqrt{t + \frac{R^2}{4}}$, we get

$$
\left(\frac{2}{R}\right)^{d-2} \int_{\mathbb{R}^d} e_\epsilon(u_\epsilon(y,t)) \exp \left(-\frac{|x-y|^2}{R^2}\right) dy \leq \frac{2^{d-2}}{(4t + R^2)^{\frac{d+2}{2}}} \int_{\mathbb{R}^d} e_\epsilon(u_\epsilon(y,t)) \exp \left(-\frac{|x-y|^2}{t + \frac{R^2}{4}}\right) dy.
$$

Hence, due to Lemma 2.13

$$
\int_{B(x,R)} e_\epsilon(u_\epsilon(y,t)) dy \leq e \int_{\mathbb{R}^d} e_\epsilon(u_\epsilon(y,t)) \exp \left(-\frac{|x-y|^2}{R^2}\right) dy
$$

$$
\leq \left(\frac{R^2}{4t + R^2}\right)^{\frac{d+2}{2}} C(d, t + \frac{R^2}{4}) f(2|x|)|\ln \epsilon| \leq C(t,R) f(2|x|)|\ln \epsilon|.
$$

As an immediate consequence, we infer an upper bound on the energy on compact sets. The bound (2.24) below will be very useful in order to prove Theorem 2.7 in the same way as in [8].

**Corollary 2.15.** Let $u_\epsilon$ be a solution of (PGL$_\epsilon$) satisfying the initial data $(H_1(f))$. Then for any compact $K \subset \mathbb{R}^d \times [0, +\infty)$, we have

$$
(2.24) \quad \int_K e_\epsilon(u_\epsilon(x,t)) dx dt \leq C(K)|\ln \epsilon|.
$$
Proof. Being compact, \( K \) is bounded and we can assume that for some large \( T > 0 \), \( K \subset B(0, \sqrt{T}) \times [0, T] \). Therefore,
\[
\int_K e_x(u_\epsilon)(x, t)dxdt \leq \int_{B(0, \sqrt{T}) \times [0, T]} e_x(u_\epsilon)(x, t)dxdt
\]
\[
\leq \int_0^T C(d, \sqrt{T})|\ln \epsilon|dt \leq C(T)|\ln \epsilon|.
\]
Whence (2.24). \( \square \)

2.3.3. Space-time estimates. One crucial lack in relaxing (1.2) to \((H_1(f))\) is that the energy no longer provides a bound on \(\|\partial_t u_\epsilon\|_{L^2} \leq E_\epsilon(u_\epsilon)\). To remedy this, we make use of the \(\Xi\) multiplier.

**Lemma 2.16.** Let \( u_\epsilon \) be a solution of \((\text{PGL}_\epsilon)\) whose initial data satisfies \((H_1(f))\). For any \( z_\epsilon = (x_\epsilon, t_\epsilon) \in \mathbb{R}^d \times [0, +\infty) \), the following equality holds, for \( R_\epsilon = \sqrt{T_\epsilon} \).

\[
\int_{\mathbb{R}^d \times [0, T]} (V_\epsilon(u_\epsilon) + \Xi(u_\epsilon, z_\epsilon))(x, t) G(x - x_\epsilon, t - t_\epsilon) \, dx \, dt
\]
\[
= \frac{1}{(4\pi)^{d/2}T_\epsilon^{d/2}} \int_{\mathbb{R}^d \times [0]} e_x(u)(x, 0) \exp\left(-\frac{|x - x_\epsilon|^2}{4t_\epsilon}\right) \, dx = E_\epsilon(z_\epsilon, R_\epsilon),
\]
where \( \Xi \) is defined in (2.18).

**Proof.** Integrating equality (2.19) from 0 to \( R_\epsilon \) (recall that \( E_\epsilon(z^*, 0) = 0 \)), we obtain

\[
(4\pi)^{d/2}E_\epsilon(z_\epsilon, R_\epsilon) = \int_0^{R_\epsilon} 2rdr \int_{\mathbb{R}^d \times [t_\epsilon - r^2]} V_\epsilon(u_\epsilon(x, t))G(x - x_\epsilon, t - t_\epsilon) \, dx
\]
\[
+ \int_0^{R_\epsilon} 2rdr \int_{\mathbb{R}^d \times [t_\epsilon - r^2]} \frac{1}{4r^2} \left[ (x - x_\epsilon) \cdot \nabla u_\epsilon - 2r^2 \partial_t u_\epsilon \right]^2 G(x - x_\epsilon, t - t_\epsilon) \, dx.
\]

Expressing the integral on the right-hand side of (2.26) in the variable \( t = t_\epsilon - r^2 \) (so that \( dt = -2rdr \) yields

\[
(4\pi)^{d/2}E_\epsilon(z_\epsilon, R_\epsilon) = \int_0^{t_\epsilon} dt \int_{\mathbb{R}^d \times [t]} V_\epsilon(u_\epsilon(x, t))G(x - x_\epsilon, t - t_\epsilon) \, dx
\]
\[
- \int_0^{t_\epsilon} 2rdr \int_{\mathbb{R}^d \times [t]} \frac{1}{4|t - t_\epsilon|} \left( (x - x_\epsilon) \cdot \nabla u - 2r^2 \partial_t u_\epsilon \right)^2 G(x - x_\epsilon, t - t_\epsilon) \, dx. \square
\]

**Proposition 2.17.** Let \( u_\epsilon \) be a solution of \((\text{PGL}_\epsilon)\) whose initial data satisfies \((H_1(f))\). For any compact \( K \subset \mathbb{R}^d \times [0, +\infty) \), there exist a constant \( C(K) \) such that

\[
\int_K |\partial_t u_\epsilon(x, t)|^2 \, dx \, dt \leq C(K)|\ln \epsilon|.
\]

**Proof.** It suffices to prove the bound on the compacts \( K_T = B(0, \sqrt{T}) \times [0, T] \) for all \( T \geq 1 \). Let \( t_\epsilon = 2T \) and \( x_\epsilon = 0 \), then for \( (x, t) \in B(0, \sqrt{T}) \times [0, T] \), we have \( t - t_\epsilon \geq T \geq |x - x_\epsilon|^2 \) so that

\[
G(x - x_\epsilon, t - t_\epsilon) \geq \frac{1}{(4\pi(t - t_\epsilon))^{d/2}} \exp\left(-\frac{|x - x_\epsilon|^2}{4(t - t_\epsilon)^2}\right) \geq e^{-1/4} \frac{(4\pi T)^{d/2}}{(4\pi(t - t_\epsilon))^{d/2}}
\]
and

\[
|\partial_t u_\epsilon(x, t)|^2 \leq \frac{1}{2T} \frac{1}{4|t - t_\epsilon|} (2(t - t_\epsilon) |\partial_t u_\epsilon(x, t)|)^2
\]
\[
\leq \frac{1}{T} \Xi(u_\epsilon, z_\epsilon)(x, t) + \frac{2}{t - t_\epsilon} ((x - x_\epsilon) \cdot \nabla u_\epsilon(x, t))^2
\]
\[
\leq \frac{1}{T} \Xi(u_\epsilon, z_\epsilon)(x, t) + |\nabla u_\epsilon(x, t)|^2.
\]
Therefore, using $V_\epsilon(u_\epsilon) \geq 0$, (2.25) and (2.24), we get
\[
\int_0^T \int_{B(0,v_T)} |\mathcal{E}_\epsilon u_\epsilon(x,t)|^2 dx \leq C \int \mathcal{E}(u_\epsilon^\ast, z_\epsilon^\ast)(x,t)G(x-x_\ast, t-t_\ast)dx dt
\]
\[
+ \int_{B(0,v_T)} |\nabla u_\epsilon(x,t)|^2 dx dt
\]
\[
\leq C(T) \int \mathcal{E}(u_\epsilon^\ast, z_\epsilon^\ast)(x,t)G(x-x_\ast, t-t_\ast)dx dt
\]
\[
+ \int_{B(0,v_T)} e_\epsilon(u_\epsilon)(x,t)dx dt
\]
\[
\leq C(T) \mathcal{E}_w(z_\epsilon^\ast, \sqrt{t_\epsilon^\ast}) + C(t_\ast)|\ln \epsilon| \leq C(T)|\ln \epsilon|.
\]
\[\square\]

2.3.4. Localizing the energy. In some of the proofs of the main results, it will be convenient to work on bounded domains for fixed time slices. But since the integral in the definition of $\mathcal{E}_w$ is computed on the whole space, we will have to use two kinds of localization methods.

The first one results from the monotonicity formula.

**Proposition 2.18.** Let $u_\epsilon$ be a solution of $\text{(PGL}_w\text{)}$ whose initial data satisfies $\text{(H}_1\text{)(f)}$.

Let $T > 0$, $x_\ast \in \mathbb{R}^d$ and $r, \lambda > 0$. There holds
\[\tag{2.28} \frac{1}{r^{d-2}} \int_{\mathbb{R}^d} e_\epsilon(u_\epsilon)(x,T) \exp\left(-\frac{|x-x_T|^2}{4r^2}\right) \leq \frac{1}{r^{d-2}} \int_{B(x_T, \lambda r)} e_\epsilon(u_\epsilon)(x,T) \exp\left(-\frac{\lambda^2}{8}\right) \exp\left(-\frac{|x-x_T|^2}{8r^2}\right)\]
\[
+ \frac{C(T + r^2)}{(T + r^2)^{d+2}} \exp\left(-\frac{\lambda^2}{8}\right) f(2|x_T|)|\ln \epsilon|.
\]

**Proof.** We split the integral between $B(x_T, \lambda r)$ and its complement. Now for $x$ such that $|x-x_T| \geq \lambda r$, we have
\[
\exp\left(-\frac{|x-x_T|^2}{8r^2}\right) \leq \exp\left(-\frac{\lambda^2}{8}\right) \exp\left(-\frac{|x-x_T|^2}{8r^2}\right).
\]

Therefore
\[
\int_{\mathbb{R}^d} e_\epsilon(u_\epsilon)(x,T) \exp\left(-\frac{|x-x_T|^2}{4r^2}\right) dx
\]
\[
\leq \int_{B(x_T, \lambda r)} e_\epsilon(u_\epsilon)(x,T) dx + e^{-\lambda^2/8} \int_{\mathbb{R}^d} e_\epsilon(u_\epsilon)(x,T) \exp\left(-\frac{|x-x_T|^2}{8r^2}\right) dx.
\]

Now, we apply the monotonicity formula to the second integral term of the right hand side, at the point $(x_T, T + 2r^2)$, and between $\sqrt{2}r$ and $\sqrt{T + 2r^2}$. We get
\[
\frac{1}{r^{d-2}} \int_{\mathbb{R}^d} e_\epsilon(u_\epsilon)(x,T) \exp\left(-\frac{|x-x_T|^2}{8r^2}\right) dx
\]
\[
\leq \frac{1}{(T + 2r^2)^{d+2}} \int_{\mathbb{R}^d} e_\epsilon(u_\epsilon^0)(x) \exp\left(-\frac{|x-x_T|^2}{4(T + 2r^2)}\right) dx
\]
\[
\leq \frac{C(T + 2r^2)}{(T + r^2)^{d+2}} f(2|x_T|)|\ln \epsilon|,
\]
and the conclusion follows. 

The second localization method, inspired by Lin and Rivièrè [27], is based on a Pohozaev type inequality.
Proposition 2.19 ([3] Proposition 2.4). Let \( u_\epsilon \) be a solution of \((\text{PGL})\) whose initial data satisfies \((\text{H}_1(f))\), and \( \Xi \) as in \((2.18)\).

Let \( 0 < t < T \). The following inequality holds, for any \( x_T \in \mathbb{R}^d \):

\[
\int_{\mathbb{R}^d} e_\epsilon(u_\epsilon)(x,t) \frac{|x-x_T|^2}{4(T-t)} \exp \left( -\frac{|x-x_T|^2}{4(T-t)} \right) \, dx
\]
\[
\leq \frac{d}{2} \int_{\mathbb{R}^d} e_\epsilon(u_\epsilon)(x,t) \exp \left( -\frac{|x-x_T|^2}{4(t-t)} \right) \, dx
\]
\[
+ \int_{\mathbb{R}^d} (V_\epsilon(u_\epsilon) + 3\Xi(u_\epsilon, z_T)) \exp \left( -\frac{|x-x_T|^2}{4(T-t)} \right) \, dx.
\]

As a consequence,

\[
\int_{\mathbb{R}^d \times \{ t \}} e_\epsilon(u_\epsilon) \exp \left( -\frac{|x-x_T|^2}{4(T-t)} \right) \, dx \leq \int_{B(x_T, r_T) \times \{ t \}} e_\epsilon(u_\epsilon) \exp \left( -\frac{|x-x_T|^2}{4(T-t)} \right) \, dx
\]
\[
+ \frac{2}{d} \int_{\mathbb{R}^d \times \{ t \}} (V_\epsilon(u) + 3\Xi(u, z_T)) \exp \left( -\frac{|x-x_T|^2}{4(T-t)} \right) \, dx
\]

where \( r_T = 2\sqrt{d(T-t)} \).

2.4. End of the proof of clearing-out. In this paragraph, we complete the proofs of Theorems 2.1, 2.5 and 2.7.

Outline of the proof of Theorem 2.7 The proof follows word for word that of Theorem 1 in [3] Sections 3, p. 67-99. Indeed, either calculations are made on bounded domains \( K \times [T_0, T_1] \) (where \( K \) is a compact of \( \mathbb{R}^d \) and \( T_1 > T_0 > 0 \), on which we have the same kind of bounds

- for the energy: for any \( t \in [T_0, T_1] \):
  \[
  \int_K |e_\epsilon(u_\epsilon)|(x,t) \, dx \leq C(K, T_1)|\ln \epsilon| \tag{obtained in (2.23)}
  \]

- for the kinetic energy, in \( L^2(dx, dt) \):
  \[
  \int_{K \times [0, T_1]} |\partial_t u_\epsilon|(x,t) \, dx \leq C(K, T_1)|\ln \epsilon| \tag{obtained in (2.27)}
  \]

- pointwise: \( |u_\epsilon| + \epsilon|\nabla u_\epsilon| + \epsilon^2|\partial_t u_\epsilon|(x,t) \leq C \) for \( (x,t) \in K \times [\epsilon^2, T_1] \) (obtained in (2.6) and useful as soon as \( \epsilon \) is so small that \( T_1 > \epsilon^2 \)),

or we multiply the energy by a weight of the form \( \exp \left( -\frac{|x|^2}{R^2} \right) \), for which we have the monotonicity formula and bounds (2.28).

For the convenience of the reader, we remind the main steps of the argument. We therefore consider a family of solutions \((u_\epsilon)_\epsilon\) to \((\text{PGL})\) such that for all \( \epsilon \in (0, \frac{1}{2}] \), \( u_\epsilon \in L^\infty(\mathbb{R}^d) \) and

\[
E_{\mu, \epsilon}(0,1,1) \leq \eta |\ln \epsilon|,
\]

where \( \eta > 0 \) is a small constant to be determined at the end of the proof.

Let \( r_\epsilon = 1 - \sqrt{\frac{\sigma}{2K}\epsilon} \) and \( t_\epsilon = 1 - r_\epsilon^2 \geq 0 \). As \( |\partial_t u_\epsilon| \leq K/\epsilon^2 \) (in view of (2.6)),

\[
|1 - |u_\epsilon(0,1)|| \leq |1 - |u_\epsilon(0,t_\epsilon)|| + \frac{K}{\epsilon^2}(1 - t_\epsilon) \leq |1 - |u_\epsilon(0,t_\epsilon)|| + \frac{\sigma}{2} \tag{2.29}
\]

Now, as \( |\nabla x u_\epsilon| \leq K/\epsilon \) (again (2.6)), one easily deduces (see [5] Lemma 3.3 p.458) that

\[
|1 - |u_\epsilon(0,t_\epsilon)|| \leq C \left( \frac{1}{\epsilon^d} \int_{B(0,\epsilon)} (1 - |u_\epsilon(x, t_\epsilon)|)^2 \, dx \right)^{\frac{1}{d+2}} \tag{2.30}
\]
This bound can be related to the weighted energy as follows:

\[(2.31) \quad \frac{1}{r^d} \int_{\mathbb{R}^d} (1 - |u_\epsilon(x, t_\epsilon)|^2) \, dx \leq \frac{C}{r^{d-2}} \int_{B(0, r)} V_\epsilon(u_\epsilon)(x, t_\epsilon) \exp\left(-\frac{|x|^2}{r^2}\right) \, dx \leq E_{w, \epsilon}((0, 1), r_\epsilon).
\]

Let \( \delta \in (0, 1/2) \) to be fixed at the end of the proof. By considering the variations of the weighted energy on time intervals of the form \([1 - \delta^{2k}, 1 - \delta^{2(k+1)}]\) for \(0 \leq k \leq \lceil \ln \epsilon \rceil / 4\) (so that \(1 - \delta^{2k} \leq 1 - r_\epsilon^{-2}\) and \(E_{w, \epsilon}((0, 1), r_\epsilon) \leq E_{w, \epsilon}((0, 1), \delta^{4k})\) by monotonicity), at least one of these intervals (say for \(k_0\)) shows a decay less than \(8\eta |\ln \delta|\) (we gained a \(|\ln \epsilon|\) factor). One can actually perform a time shift and rescaling (as \(k \leq |\ln \epsilon|/4\)), so that we can furthermore assume \(k_0 = 0\):

\[(2.32) \quad E_{w, \epsilon}((0, 1), 1) - E_{w, \epsilon}((0, 1), \delta) \leq 8\eta |\ln \delta|, \quad E_{w, \epsilon}((0, 1), r_\epsilon) \leq E_{w, \epsilon}((0, 1), 1).
\]

Gathering \((2.29), (2.30), \) and \((2.31),\)

\[|1 - |u_\epsilon|\| \leq \frac{\sigma}{2} + CE_{w, \epsilon}((0, 1), 1) \frac{1}{\epsilon^4}.
\]

The crux of the argument is therefore to bound \(E_{w, \epsilon}((0, 1), 1)\) solely in terms of \(\eta\) and more precisely,

\[(2.33) \quad E_{w, \epsilon}((0, 1), 1) \leq C(\delta) \sqrt{\eta}.
\]

To prove \((2.33),\) the starting point is the observation that

\[|u_\epsilon|^2 |\nabla u_\epsilon|^2 = |u_\epsilon \cdot \nabla u_\epsilon|^2 + |u_\epsilon|^2 |\nabla u_\epsilon|^2,
\]

(which can be derived easily writing \(u_\epsilon = |u_\epsilon| e^{i\phi}.\)) As \(|\nabla u_\epsilon| \leq C/\epsilon\) from \((2.6),\)

\[(1 - |u_\epsilon|^2) |\nabla u_\epsilon|^2 \leq \frac{1}{2} |\nabla u_\epsilon|^2 + CV_e(u_\epsilon),
\]

and we get the pointwise bound

\[e_\epsilon(u_\epsilon) = 2|u_\epsilon|^2 |\nabla u_\epsilon|^2 + CV_e(u_\epsilon) + |u_\epsilon \cdot \nabla u_\epsilon|^2.
\]

It is not so hard to obtain improved bounds on \(|u_\epsilon|\) and its derivative. Indeed, write the parabolic equation for \(1 - |u_\epsilon|^2:\) using \((2.25)\) (in particular to treat the \(\partial_t u_\epsilon\) terms) and an averaging in time argument, one can infer that the set of times \(t\) such that

\[\int_{B(0, 1)} (|\nabla u_\epsilon|^2 + V_e(u_\epsilon))(x, t) \, dx \leq C(\delta) \sqrt{\eta} (E_{w, \epsilon}((0, 1), 1) + 1)
\]

is of large relative measure in \([1 - 4\delta^2, 1 - \delta^2]\) (for some explicit \(C(\delta)\)).

It follows that the crucial term to estimate is \(u_\epsilon \cdot \nabla u_\epsilon\). For this term, one has the following Hodge-de Rham decomposition

\[u_\epsilon \cdot \nabla u_\epsilon = d\phi(t) + d^*\psi(t) + \xi(t) \quad \text{on} \quad B(0, 3/2) \times \{t\}
\]

where \(d\) is the exterior derivative on \(\mathbb{R}^d\) (\(d^*\) is its adjoint).

It is constructed as follows: define the Jacobian \(J u_\epsilon := d(u_\epsilon \wedge d u_\epsilon),\) and let \(\psi(t)\) be such that

\[-\Delta \psi(t) = \chi J u_\epsilon
\]

where \(\chi\) is a cut-off function on \(B(0, 2).\) Observe that \(d^* d \psi(t)\) is closed on \(B(0, 3/2) \times \{t\};\) invoking the Poincaré lemma, there exists \(\xi(t)\) such that \(d \xi(t) = d^* d \psi(t),\) \(d^* \xi(t) = 0.\) Then \(\phi(t)\) is obtained by invoking once again the Poincaré lemma.

Direct elliptic estimates show that \(\|\xi(t)\|_{L^2(B(0, 3/2))} \leq C\|\nabla \psi(t)\|_{L^2(B(0, 2))}.\)

Noticing that \(u_\epsilon \wedge \partial_t u_\epsilon = -d^*(u_\epsilon \cdot \nabla u_\epsilon),\) we have the elliptic equation for \(\phi(t):\) for any small \(\delta > 0,
\]

\[-\Delta + \frac{\chi}{2\delta^2} \phi(t) = u_\epsilon \wedge \left(\frac{\chi}{2\delta^2} \cdot \nabla u_\epsilon - \partial_t u_\epsilon\right) - (d^* \psi(t) + \xi(t)) \cdot \frac{\chi}{2\delta^2}.
\]

One can obtain weighted elliptic estimates for the operator

\[-\Delta + \frac{\chi}{2\delta^2} = -\exp\left(\frac{|x|^2}{4\delta^2}\right) \nabla \cdot \left(\exp\left(\frac{|x|^2}{4\delta^2}\right) \nabla\right)
\]
and from there obtain the bound
\[ \int |\nabla \phi(t)|^2 \exp\left( -\frac{|x|^2}{4\delta^2} \right) dx \leq C \delta^d E_w((0,1), \delta) + C(\delta) \left( R(t) + \sqrt{R(t)} E_w((0,1), \delta) \right), \]
where \( R(t) \) only involves \( \xi(t) \) (already bounded), \( \Xi(u_\epsilon) \) and \( V_\epsilon(u_\epsilon) \) (taken care of by (2.25)) and \( \psi(t) \):
\[
R(t) = \int_{\mathbb{R}^d} (\Xi(u_\epsilon, (0,1))(x, t) + V_\epsilon(u_\epsilon)) \exp\left( -\frac{|x|^2}{4\delta^2} \right) dx
+ \int_{B(0,3/2)} |\nabla \psi(x, t)|^2 + |\xi(x, t)|^2 \exp\left( -\frac{|x|^2}{4\delta^2} \right) dx.
\]

The remaining task is to estimate \( \psi(t) \), which is the most involved. A first ingredient is a refined estimate on the Jacobian due to Jerrard and Soner [23]: there exist \( \beta, C > 0 \) such that for any smooth \( w \) and test function \( \varphi \),
\[
\left| \int_{\mathbb{R}^d} (Jw, \varphi) dx \right| \leq \frac{C}{|\ln \epsilon|} \| \varphi \|_{L^\infty} \int_{\text{Supp} \varphi} e_{\epsilon}(w) dx
+ C \epsilon^\beta \| \varphi \|_{W^{1,\infty}} \left( 1 + \int_{\text{Supp} \varphi} e_{\epsilon}(w) dx \right) (1 + J^d(\text{Supp} \varphi)^2).
\]
(Observe the \( |\ln \epsilon| \) gain). A second ingredient is to compare \( \psi \) with the solution \( \bar{\psi} \) to the analogous heat equation
\[
\partial_t \bar{\psi} - \Delta \psi(t) = \chi J u_\epsilon.
\]
One can get bounds on \( \bar{\psi} \) by the use of the monotonicity formula on \( u_\epsilon \), and then relate to \( \psi \) by treating \( \partial_t \bar{\psi} \) as a perturbation. After an averaging in time argument, one can choose a right time slice \( t \in [1 - 4\delta^2, 1 - \delta^2] \) such that, using the bounds from (2.25) and (2.27),
\[
\frac{1}{(1-t)^{d/2}} \int_{\mathbb{R}^d} (\Xi(u_\epsilon, (0,1))(x, t) + V_\epsilon(u_\epsilon))(x, t) \exp\left( -\frac{|x|^2}{4(1-t)} \right) dx \leq C \frac{|\ln(1-t)|}{t^2} |\ln \epsilon|
\]
(a bound suitable for \( \psi(t) \) and \( \xi(t) \)) and
\[
\int_{B(0,2)} |\nabla \psi(t)|^2 dx \leq C(\delta) \epsilon^{1/6} E_w((0,1), \delta)
+ C(\delta)(E_w((0,1), \delta) + 1) \int_{\mathbb{R}^d} V_\epsilon(u_\epsilon)(t) \exp\left( -\frac{|x|^2}{4\delta^2} \right) dx.
\]
Combining all the above (and the monotonicity formula), one can choose \( \delta > 0 \) small enough (independent of \( \eta \) or \( u_\epsilon \)) so that
\[
E_w((0,1), \delta) \leq \frac{1}{2} E_w((0,1), 1) + C \sqrt{\eta}.
\]
Then using the first part of (2.32), this proves (2.33) and the proof is complete. \( \square \)

From Theorem 2.1 Proposition 2.2 follows immediately. We now provide a proof of Proposition 2.3 which also is a consequence of Theorem 2.1

Proof of Proposition 2.3: Let \( x_0 \) be any given point in \( B \left( x_T, \frac{R}{2} \right) \). The crux of the argument is the following claim.

Claim 2.20. We can find \( \lambda(T) > 0 \) such that, for every \( \sqrt{T_0} < r < \sqrt{T_1} \),
\[
\frac{1}{r^{d-2}} \int_{\mathbb{R}^d} e_{\epsilon}(u_\epsilon)(x, T) \exp\left( -\frac{|x-x_0|^2}{4r^2} \right) dx \leq \eta_1 |\ln \epsilon|,
\]
provided that $\bar{\eta} \leq \frac{\eta_1}{2}$ (recall that $\lambda$ enters in the definition of $\bar{\eta}$).

To prove the claim, we use (2.28). Let $\lambda > 0$ and $\sqrt{T_0} < r < \sqrt{T_1} = R$, we have

$$\frac{1}{r^{d-2}} \int_{B(x_0,r)} e_\epsilon(u_\epsilon)(x,T) \exp\left(-\frac{|x-x_0|^2}{4r^2}\right) \, dx \leq \frac{1}{r^{d-2}} \int_{B(x_0,r)} e_\epsilon(u_\epsilon)(x,T) \, dx$$

\[+ \left(\frac{C(T+2r^2)}{(T+2r^2)^{d/2}}\right) \exp\left(-\frac{\lambda^2}{8}\right) f(2|x_0|) \ln \epsilon.\]

As $r \leq 1$, it follows that

$$\frac{C(T+2r^2)}{(T+2r^2)^{d/2}} \leq \frac{C(T)}{T^{d/2}}.$$  

We first choose a function $\lambda_0$ continuous such that for $T > 0$, $\lambda_0(x_0,T) \geq 1$ is so large that

$$\frac{C(T)}{T^{d/2}} \exp\left(-\frac{\lambda_0^2(T)}{8}\right) f(2|x_0|) \leq \frac{\eta_1}{2}.$$

Finally define

$$\lambda(x_0,T) = \begin{cases} 1 + \sup_{r \in [T,1]} x \in B(x_0,r) 2\lambda_0(x,t) & \text{if } T \leq 1, \\ 1 + \sup_{r \in [1,T]} x \in B(x_0,r) 2\lambda_0(x,t) & \text{if } T \geq 1. \end{cases}$$

Recall that $C(T)$ is bounded (by $C(1/2)$) for $T \in [0,1/2]$, so that all this can be done under the additional hypothesis that for some constant $C = C(\sigma)$,

$$\forall T \in (0,1/2], \quad \frac{1}{C} \sqrt{|\ln T| + \ln f(2|x_0|)} \leq \lambda(x_0,T) \leq C \sqrt{|\ln T| + \ln f(2|x_0|)}.$$  

Then the function $\lambda$ is positive, continuous and satisfies the conditions of Proposition 2.3. Furthermore, since $x_0$ belongs to $B(x_T,R/2)$ and $r < R < 1$, it follows that

$$B(x_0, \lambda_0(x_0,T)r) \subset B(x_T, \lambda(x_T,T)r).$$

Therefore,

$$\frac{1}{r^{d-2}} \int_{B(x_0,\lambda_0(T)r)} e_\epsilon(u_\epsilon)(x,T) \, dx \leq \left(\frac{R}{r}\right)^{d-2} \frac{1}{R^{d-2}} \int_{B(x_T,\lambda(T)r)} e_\epsilon(u_\epsilon)(x,T) \, dx$$

\[\leq \left(\frac{R}{r}\right)^{d-2} \bar{\eta} |\ln \epsilon| \leq \left(\frac{R}{\sqrt{T_0}}\right)^{d-2} \bar{\epsilon} |\ln \epsilon|.

Choosing $T_0 = \left(\frac{2}{\eta_1}\right)^{\frac{d-2}{2}} R^2$, we obtain

$$\frac{1}{r^{d-2}} \int_{B(x_0,\lambda_0(T)r)} e_\epsilon(u_\epsilon)(x,T) \, dx \leq \frac{\eta_1}{2} |\ln \epsilon|.$$  

We finally combine (2.35), (2.36) and (2.37), and the claim is proved. The conclusion then follows from Proposition 2.2. \[\square\]

The proofs of both Theorems 2.5 and 2.7 are now exactly the same as in [8] (see sections 5.2 to 5.4) since computations are made on compact domains $\Omega \subset \mathbb{R}^d \times (0, +\infty)$, where the bounds (2.23), (2.27) and (2.6) hold. As in the proof of Theorem 2.1 (and explained at its beginning), these are the only bounds which are made use of, along with the monotonicity formula and the identity (2.25). For the convenience of the reader, we remind a few elements of the proofs.
Outline of the proof of Theorem 2.5 One writes \( u_\epsilon = \| u_\epsilon \| e^{i\varphi_\epsilon} \), and starts with the equation for the phase \( \varphi_\epsilon \). As \( |u_\epsilon| \geq 1/2 \), it is a uniformly parabolic equation. After performing a spatial cut-off in order to get rid of boundary conditions on \( \Lambda \), one splits \( \varphi_\epsilon = \varphi_{\epsilon,0} + \varphi_{\epsilon,1} \), where \( \varphi_{\epsilon,0} \) is a solution to the linear heat equation (which turns out to be the desired phase \( \Phi \)) and \( \varphi_{\epsilon,1} \) is a solution to a nonlinear equation with 0 initial data.

\( \varphi_{\epsilon,0} \) admits improved bounds due to parabolic regularity

\[
\| \nabla \varphi_{\epsilon,0} \|^2_{L^2(L^p(\Lambda_{3/4}))} + \| \nabla \varphi_{\epsilon,0} \|^2_{L^\infty(\Lambda_{3/4})} \leq C(\Lambda) \int_{\Lambda} e_\epsilon(u_\epsilon).
\]

\( \varphi_{\epsilon,1} \) is shown to be essentially a perturbation (via a fixed point argument), and so \( \varphi_\epsilon \) enjoys integrability bounds in \( L^q Q(\Lambda_{3/4}) \) for some \( q > 2 \) (depending on \( d \)).

Plugging this information in the equation for \( |u_\epsilon| \), one infers that

\[
e_\epsilon(u_\epsilon) \leq \frac{1}{2} |\nabla \varphi_{\epsilon,0}|^2 + \kappa_\epsilon,
\]

where \( \int_{\Lambda_{3/4}} \kappa_\epsilon \leq C(\Lambda) e^{\alpha} \), for some \( \alpha \) depending only on \( d \). A key ingredient is provided by a result by Chen and Struwe [14] where the bound (2.3) is proved under an extra assumption of small energy (see also [3] Proposition 5.1). Together with (2.38) and a scaling argument, one can relax the small energy assumption, and prove (2.3).

Finally, inserting (2.3) in the equation for \( |u_\epsilon| \), Lemma 2.11 allows to improve the estimates to

\[
|\nabla |u_\epsilon| + V_\epsilon(u_\epsilon) \leq C(\Lambda) e^{\alpha},
\]

and from there, obtain \( L^\infty \) bounds on \( \varphi_{\epsilon,1} \): this yields the bounds (2.4) and (2.5), and completes the proof of Theorem 2.5

Outline of the proof of Theorem 2.7 It suffices to prove the estimates on any cylinder of the form \( \Lambda = B \times [T_0, T_1] \) where \( B \) is a ball and \( T_1 > T_0 > 0 \). Up to increasing slightly the cylinder (and due to (2.24)), one can assume that

\[
\int_\Lambda e_\epsilon(u_\epsilon) dx dt + \int_{\partial \Lambda} e_\epsilon(u_\epsilon) d\sigma(x, t) \leq C(\Lambda) |\ln \epsilon|.
\]

Denote now \( \delta \) and \( \delta^* \) the (space time) exterior derivatives on \( \mathbb{R}^{d+1} \). The main step is a Hodge-de Rham type decomposition with estimates:

\[
u_\epsilon \wedge \delta u_\epsilon = \delta \Phi + \delta^* \Psi + \zeta,
\]

where a key feature is to prove that in addition to the expected bound

\[
\| \nabla \Phi \|_{L^2(\Lambda)} + \| \nabla \Psi \|_{L^2(\Lambda)} \leq C(\Lambda) |\ln \epsilon|,
\]

one also has that for any \( p \in \left[ 1, 1 + \frac{1}{d} \right] \),

\[
\| \nabla \Psi \|_{L^p(\Lambda)} \leq C(p, \Lambda), \quad \text{and} \quad \| \zeta \|_{L^p(\Lambda)} \leq C(p, \Lambda) \sqrt{\epsilon}.
\]

In other words, \( \Phi \) completely accounts for the lack of compactness.

The argument for (2.40) is as follows: one starts with the usual Hodge-de Rham decomposition on \( \partial \Lambda \) (which is simply connected: here we use that \( (d + 1) - 1 \geq 2 \) so that

\[
u_\epsilon \wedge \delta u_\epsilon = d_{\partial \Lambda} \Phi_{\partial \Lambda} + d_{\partial \Lambda}^* \Psi_{\partial \Lambda}.
\]

Then on \( \partial \Lambda, -\Delta_{\partial \Lambda} \Psi_{\partial \Lambda} = 2d_{\partial \Lambda} u_\epsilon \): the Jerrard and Soner estimate on the Jacobian (2.34) gives the gain

\[
\| J_{\partial \Lambda} u_\epsilon \|_{C^0(\partial \Lambda)} \lesssim C(\Lambda),
\]

from where a similar gain is derived on \( \Psi_{\partial \Lambda} \).

Let \( \Phi_0 \) be the harmonic extension of \( \Phi_{\partial \Lambda} \) on \( \Lambda \): we gauge away by considering on \( \Lambda \nabla_\epsilon := u_\epsilon e^{-i\Phi_0} \).
One now considers on $\Lambda$ the Hodge-de Rham decomposition $v_\epsilon \wedge \delta v_\epsilon = \delta \Phi_1 + \delta^* \Psi$, so that
\[ u_\epsilon = \delta \Phi_1 + \delta^* \Psi + \delta \Phi_0 + (1 - |u_\epsilon|^2) \delta \Phi_0. \]
Define $\zeta := (1 - |u_\epsilon|^2) \delta \Phi_0$; it satisfies (2.41) using the bound on the energy (2.39), and $\zeta_{|_{\partial \Lambda}}$ as well.

For $\Psi$, one has again $-\Delta \Psi = 2/w_\epsilon$, now with boundary conditions involving $\Psi_{|_{\partial \Lambda}}$ and $\zeta_{|_{\partial \Lambda}}$. A Jacobian estimate similar to (2.34) (and proven in [7] Proposition II.1) thus yields the bound (2.41) on $\Psi$. Finally defining $\Phi := \Phi_0 + \Phi_1$ completes the decomposition.

With (2.40) and (2.41) at hand, we now define $\Phi_\epsilon$ by writing $\Phi = \Phi_\epsilon + \Phi_2$, where
\[ \partial_t \Phi_\epsilon - \Delta \Phi_\epsilon = \partial_t \Phi - \Delta \Phi \quad \text{on } \Lambda, \]
with the boundary condition $\Phi_2 = 0$ on $B \times \{0\} \cup \partial B \times (T_0, T_1)$. In particular, $\Phi_\epsilon$ solves the (homogeneous) heat equation on $\Lambda$.

From parabolic estimates,
\[ \|\nabla \Phi_\epsilon\|_{L^\infty(\Lambda)} \leq C \|\nabla_x \Phi\|_{L^2(B \times \{T_0\} \cup \partial B \times (T_0, T_1))} \leq C(\Lambda) \ln \epsilon, \]
which is the estimate (3). It remains to bound $w_\epsilon := u_\epsilon e^{-\Phi_\epsilon}$ in $W^{1,p}(\Lambda)$. First we separate between
\[ A_\Lambda = \{(x, t) \in \Lambda : |w_\epsilon(x, t)| \leq \epsilon^{1/4}\} \quad \text{and} \quad B_\Lambda = \Lambda \setminus A_\Lambda. \]
On $B_\Lambda$, the rough estimate $|\nabla w_\epsilon(x, t)| \leq C/\epsilon$ inherited from (2.6) yields
\[
\|\nabla w_\epsilon\|_{L^p(B_\Lambda)} \leq C(\Lambda) \int_{\Lambda} \left(1 - |w_\epsilon(x, t)|^2\right)^2 \frac{dxdt}{\epsilon^{1/2}} \leq C(\Lambda) \epsilon^{2-p-1/2} \int_{\Lambda} \left(1 - |w_\epsilon(x, t)|^2\right)^2 \frac{dxdt}{\epsilon^{2}} \leq C(\Lambda) \epsilon^{2\rho - 1/2} \ln \epsilon \to 0
\]
(observe that $|u_\epsilon| = |w_\epsilon|$ and $\rho < \frac{d + 1}{d} \leq \frac{3}{2}$). We now work on $A_\Lambda$. Then (with the same computation as in the proof of Theorem 2.1), we have
\[
|\nabla w_\epsilon|^2 \leq 2|w_\epsilon|^2 |\nabla w_\epsilon|^2 = 2|w_\epsilon|^2 |\nabla|w_\epsilon|^2 + 2|\nabla (|u_\epsilon|^2)|^2 + 2|w_\epsilon \wedge \nabla w_\epsilon|^2.
\]
It follows from (2.41) (writing $w_\epsilon$ in terms of $u_\epsilon$) that $\|w_\epsilon \wedge \nabla w_\epsilon\|_{L^p(\Lambda)} \leq C(\Lambda)$. Then one actually bounds $\|\nabla(|u_\epsilon|^2)\|_{L^p(\Lambda)}$; for this, one writes the (uniformly) parabolic equation for $\rho = |u_\epsilon|^2$, from which we get
\[
\int |\nabla \rho|^2 \chi \leq \int (1 - \rho) |\nabla u_\epsilon|^2 \chi + \int (|\nabla \rho| |\nabla \chi| + \partial_t \rho |\chi|) (1 - \rho),
\]
where $\chi$ is a suitable non negative cut-off function, $\chi = 1$ on $\Lambda$). To bound the right-hand side, we split again between $A_\Lambda$ and $B_\Lambda$; $B_\Lambda$ has small measure (arguing as before) and on $A_\Lambda$ one uses smallness of $|1 - \rho|$. Gathering all the above yields a final bound
\[
\|\nabla(|u_\epsilon|^2)\|_{L^p(\Lambda)} \leq C(\Lambda) \epsilon^{1/4 - \rho/8} \ln \epsilon \to 0.
\]
This gives estimate (4).

\[ \square \]

3. Description of the Limiting Measure and Concentration Set

Let $u_\epsilon$ be a solution of (PGL) satisfying the initial data $(I_1(f))$.

Our goal in this section is to study the asymptotic limit, as $\epsilon \to 0$, of the Radon measures $\mu_\epsilon$ defined on $\mathbb{R}^d \times [0, +\infty)$ by
\[ \mu_\epsilon(x, t) = \frac{e_\epsilon(u_\epsilon)(x, t)}{|\ln \epsilon|} \, dxdt. \]
To that purpose, we will study their time slices $\mu_\epsilon^t$ defined on $\mathbb{R}^d \times \{t\}$ by
\[ \mu_\epsilon^t(x) = \frac{e_\epsilon(u_\epsilon)(x, t)}{|\ln \epsilon|} \, dx. \]
When $\varepsilon \to 0$, these measures converge to $\mu_*$ and $\mu_*^t$ respectively (up to a subsequence), as it is shown in the next paragraph.

Then we will study the evolution of $\mu_*^t$, and show that it follows Brakke’s weak formulation of the mean curvature flow. For this, we will of course heavily rely on the properties of $\mu_*^t$ obtained in the previous section.

### 3.1. Absolute continuity with respect to time of the limiting measure.
According to inequality (3.2), we have for all $R > 0$ and $T > 0$,

$$
\int_{B(0,R) \times [0,T]} d\mu_*(x,t) \leq C(T,R).
$$

The bound (3.1) yields a limiting measure via a diagonal extraction argument. This can also be done simultaneously for the time sliced measures: more precisely, following the proof in Brakke [13] word for word, we have the following.

**Theorem 3.1.** There exist a sequence $\varepsilon_m \to 0$, a Radon measure $\mu_*$ defined on $\mathbb{R}^d \times [0, +\infty)$, bounded on compact sets and, for each $t \geq 0$, a Radon measure $\mu_*^t$ on $\mathbb{R}^d \times \{t\}$ such that:

$$
\mu_*^t \to \mu_* \quad \text{as} \quad m \to \infty, \quad \text{and for each} \ t \geq 0, \quad \mu_*^t \to \mu_*^t \quad \text{as} \quad m \to \infty.
$$

Moreover, the $(\mu_*^t)$, enjoy the bound

$$
\forall x \in \mathbb{R}^d, \forall R > 0, \forall t > 0, \quad \mu_*^t(B(x,R)) \leq C(T,R) \chi(|x|).
$$

and

$$
\mu_* = \mu_*^t dt.
$$

For the proof of Theorem 3.1 we will need a few classical identities for the evolution of $\mu_*^t$.

Before we dive into the study of the singular measure $\mu_*^t$, let us state two useful following identities.

**Lemma 3.2.** Let $u^0_\varepsilon \in L^\infty(\mathbb{R}^d)$ and $u_\varepsilon$ be the associated solution of (2.23). Then, for all $\chi \in \mathcal{D}(\mathbb{R}^d)$ and for all $t \geq 0$, we have

$$
\frac{d}{dt} \int_{\mathbb{R}^d} \chi(x) d\mu_*^t = - \int_{\mathbb{R}^d} \chi(x) \frac{\partial u_\varepsilon}{\partial \varepsilon}^2 \frac{dx}{|\ln \varepsilon|} + \int_{\mathbb{R}^d} \nabla \chi(x) \frac{\partial_i u_\varepsilon}{\partial \varepsilon} \nabla u_\varepsilon \frac{dx}{|\ln \varepsilon|}.
$$

We usually choose $\chi \geq 0$, and this choice makes the first term of the right-hand side non negative. To handle the second term, we provide another identity which involves the stress-energy tensor.

**Lemma 3.3.** Let $\vec{X} \in \mathcal{D}(\mathbb{R}^d, \mathbb{R}^d)$. Then for all $t \geq 0$,

$$
\frac{1}{|\ln \varepsilon|} \int_{\mathbb{R}^d \times \{t\}} (e_\varepsilon(u_\varepsilon))_{\hat{e}i} - \partial_i u_\varepsilon \partial_j u_\varepsilon \hat{e}_j X_i = - \int_{\mathbb{R}^d \times \{t\}} \vec{X} \cdot \frac{\partial_i u_\varepsilon}{\partial \varepsilon} \nabla u_\varepsilon \frac{dx}{|\ln \varepsilon|}.
$$

(Here we use Einstein’s convention of implicit summation over repeated indices; $\delta_{ij}$ is the Kronecker symbol which equals 1 if $i = j$ and 0 otherwise.)

The proof of Lemma 3.3 is given in [9], and involves the stress-energy matrix $A_\varepsilon$ given by

$$
A_\varepsilon = A_\varepsilon(u_\varepsilon) := e_\varepsilon(u_\varepsilon) \text{Id} - \nabla u_\varepsilon \otimes \nabla u_\varepsilon = T(u_\varepsilon) + V_\varepsilon(u_\varepsilon) \text{Id},
$$

where the matrix $T(u)$ and the potential $V_\varepsilon$ are given by

$$
T(u) = \frac{1}{2} |\nabla u|^2 \text{Id} - \nabla u \otimes \nabla u, \quad V_\varepsilon(u) = \frac{(1 - |u|^2)^2}{4\varepsilon^2}.
$$

Combining Lemma 3.2 and Lemma 3.3 with the choice $\vec{X} = \nabla \chi$, we get rid of the time derivative of the right hand side of (3.4). More precisely

**Lemma 3.4.**

$$
\frac{\partial}{\partial t} \int_{\mathbb{R}^d} \chi(x) d\mu_*^t = - \int_{\mathbb{R}^d \times \{t\}} \chi(x) \frac{\partial_i u_\varepsilon}{\partial \varepsilon}^2 \frac{dx}{|\ln \varepsilon|} + \int_{\mathbb{R}^d \times \{t\}} \left( \frac{\partial^2 u_\varepsilon}{\partial \varepsilon^2} \nabla u_\varepsilon \cdot \nabla u_\varepsilon - \Delta e_\varepsilon(u_\varepsilon) \right) \frac{dx}{|\ln \varepsilon|}.
$$
Proof of Theorem \[3.7\] It boils down to the following claim.

Claim 3.5. Let \( T > 0 \). We consider a sequence \( e_m \to 0 \). Then there exists a subsequence \( e_{\sigma(m)} \) such that for every \( s \in [0, T] \),

\[
\mu_{e_{\sigma(m)}}^s \to \mu_s^i \text{ as } m \to \infty, \text{ in the sense of measures.}
\]

Indeed let us proof Theorem \[3.1\] assuming this claim holds. With \( T_n = n \), the claim let us dispose of sequences \( e_{n,m} \to 0 \) as \( m \to +\infty \) such that \( \mu_{e_{n,m}}^s \) enjoys the desired convergence on \( s \in [0, T_n] \). By an argument of diagonal extraction, we deduce that there exist a sequence \( \mu_{e_0} \) on \( \mathbb{R}^d \times \{s\} \) such that for every \( s \geq 0 \), \( \mu_{e_0}^s \to \mu_s^i \) as \( m \to \infty \). (3.3) follows from (2.23), and from there, the time variable absolute continuity \( \mu_s^i = \mu_s^i dt \).

We now prove the claim. We use the following lemma, which is an easy variant of Helly’s selection principle.

Lemma 3.6. Let \( I \) be an at most countable set, and let \( \{f_i^m\}_{m \in \mathbb{N}, i \in I} \) be a collection of real-valued functions defined on some interval \((a, b)\). Assume that for each \( i \in I \), the family \( \{f_i^m\}_{m \in \mathbb{N}} \) is equibounded and satisfies the following semi-decreasing property

\[
\text{For all } \delta > 0, \text{ there exist } \tau > 0 \text{ and } m_i \in \mathbb{N} \text{ such that, if } s_1, s_2 \in (a, b) \text{ and } s_2 - \tau \leq s_1 \leq s_2, \text{ then for all } m \leq m_i, \quad f_i^m(s_2) \leq f_i^m(s_1) + \delta.
\]

Then there exist a subsequence \( \sigma(m) \) and a family \( \{f_i^m\}_{i \in I} \) of real-valued functions on \((a, b)\) such that for all \( s \in (a, b) \) and \( i \in I \),

\[
f_{\sigma(m)}^i(s) \to f_i^i(s).
\]

Let \( \{\chi_i \}_{i \in I} \) be a countable family of compactly supported non-negative smooth functions on \( \mathbb{R}^d \); assume that for all \( i \in I \), \( 0 \leq \chi_i \leq 1 \), and that \( \text{Span}(\chi_i)_{i \in I} \) is dense in \( \mathcal{C}_c^0(\mathbb{R}^d) \). Let \( m_0 \) be such that if \( m \geq m_0 \), then \( \epsilon_m \leq \frac{1}{2} \). We define for \( m \in \mathbb{N} \), \( i \in I \) the function \( f_i^m \) defined on \([0, T]\) by

\[
f_i^m(s) = \int_{\mathbb{R}^d} \chi_i d\mu_{\epsilon_m}^s.
\]

Step 1. We first show that \( \{f_i^m\}_{m \geq m_0} \) satisfies (3.8).

Let \( i \in I \). Recalling Lemma 3.4, we have

\[
\frac{d}{ds} \int_{\mathbb{R}^d} \chi_i d\mu_{\epsilon_m}^s = -\int_{\mathbb{R}^d} \chi_i(x) \frac{\partial u_m}{|\ln \epsilon_m|} \, dx + \frac{1}{|\ln \epsilon_m|} \int_{\mathbb{R}^d} (D^2 \chi_i \nabla u_m \nabla u_m - \Delta \chi_i e_{\epsilon_m}(u_m)) \, dx.
\]

Therefore

\[
\frac{d}{ds} \int_{\mathbb{R}^d} \chi_i d\mu_{\epsilon_m}^s \leq \frac{1}{|\ln \epsilon|} \int_{\mathbb{R}^d} (D^2 \chi_i \nabla u_m \nabla u_m - \Delta \chi_i e_{\epsilon_m}(u_m)) \, dx.
\]

Let \( R > 0 \) such that \( \text{Supp}(\chi_i) \subset B(0, R) \). We have, for \( s \in [0, T] \)

\[
\frac{d}{ds} f_i^m(s) = \frac{d}{ds} \int_{\mathbb{R}^d} \chi_i d\mu_{\epsilon_m}^s \leq 3\|\chi_i\|_{q_2} \int_{B(0, R) \times \{s\}} \frac{e_{\epsilon_m}(u_m)}{|\ln \epsilon_m|} \, dx \leq 3\|\chi_i\|_{q_2} C(T, R), \quad (3.9)
\]

by Proposition 2.15.

Let \( \delta > 0 \). We set \( \tau = \frac{\delta}{3\|\chi_i\|_{q_2} C(T, R)}. \)
If \( s_1, s_2 \in [0, T] \) and \( s_2 - \tau \leq s_1 \leq s_2 \), the inequality (3.9) leads to
\[
f_m^i(s_2) - f_m^i(s_1) \leq 3\| \chi_1 \|_{L^2}(s_2 - s_1)C(T, R) \leq (s_2 - s_1)\frac{\delta}{\tau} \leq \delta.
\]
Therefore
\[
\forall m \leq m_0, \quad f_m^i(s_2) \leq f_m^i(s_1) + \delta.
\]
Now, we prove that \( \forall i \in I \), the family \((f_m^i)_{m \geq m_0}\) is equibounded.
Let \( i \in I \). Let \( R > 0 \) such that \( \text{Supp}(\chi_i) \subset B(0, R) \) and \( t \in [0, T] \). We have
\[
\int_{\mathbb{R}^d} \chi_i d\mu_m^i \leq \int_{B(0, R)} d\mu_m^i \leq C(T, R).
\]
Therefore,
\[
\forall m \geq m_0, \quad \|f_m^i\|_{L^\infty} \leq C(T, R),
\]
i.e. the family \((f_m^i)_{m \geq m_0}\) is equibounded. According to Lemma [3.6], there exist a subsequence \( \sigma(m) \) and for all \( i \in I \), a Radon measure \( f^i \) such that for all \( i \in I \), and \( s \in (a, b) \)
\[
f_{\sigma(m)}^i(s) \to f^i(s).
\]
Hence, for all \( s \in [0, T] \), \( \mu_{\epsilon(n)}^i(\chi_i) \) converges as \( m \to +\infty \).

**Step 2.**
Let \( s_0 \in [0, T] \) be arbitrary but fixed, and \( \chi \in C_\infty^0(\mathbb{R}^d) \). Let us show that \((\mu_{\epsilon(n)}^0(\chi))_{m \in \mathbb{N}}\) is a Cauchy sequence in \( \mathbb{R} \).
Let \( \alpha > 0 \). Since \( \text{Span}(\chi_i) \) is dense in \( C_\infty^0(\mathbb{R}^d) \), there exist a finite subset \( J \subset I \), and real numbers \((\lambda_j)_{j \in J}\) such that
\[
\left\| \sum_{j \in J} \lambda_j \chi_j - \chi \right\|_{L^\infty} \leq \frac{\alpha}{3C(T, R)}.
\]
Let \( m \geq k \) be two integers, we have
\[
|\mu_{\epsilon(n)}^0(\chi) - \mu_{\epsilon(n)}^0(\chi)| \leq \left| \mu_{\epsilon(n)}^0(\chi) - \sum_{j \in J} \lambda_j \mu_{\epsilon(n)}^0(\chi_j) \right| + \left| \sum_{j \in J} \lambda_j (\mu_{\epsilon(n)}^0(\chi_j) - \mu_{\epsilon(n)}^0(\chi_j)) \right|
\]
\[
+ \left| \sum_{j \in J} \lambda_j (\mu_{\epsilon(n)}^0(\chi_j) - \mu_{\epsilon(n)}^0(\chi_j)) \right|.
\]
Recall that
\[
\mu_{\epsilon(n)}^0(\chi) - \sum_{j \in J} \lambda_j \mu_{\epsilon(n)}^0(\chi_j) = \int_{\mathbb{R}^d} (\chi - \sum_{j \in J} \lambda_j \chi_j) d\mu_{\epsilon(n)}^0.
\]
Let \( R > 0 \) such that \( \text{Supp}(\chi) \subset B(0, R) \) and for all \( j \in J \), \( \text{Supp}(\chi_j) \subset B(0, R) \). We can bound
\[
\left| \mu_{\epsilon(n)}^0(\chi) - \sum_{j \in J} \lambda_j \mu_{\epsilon(n)}^0(\chi_j) \right| \leq \left\| \chi - \sum_{j \in J} \lambda_j \chi_j \right\|_{L^\infty} \int_{B(0, R)} d\mu_{\epsilon(n)}^0
\]
\[
\leq \left\| \chi - \sum_{j \in J} \lambda_j \chi_j \right\|_{L^\infty} C(T, R).
\]
Therefore we have
\[
\left| \mu_{\epsilon(n)}^0(\chi) - \sum_{j \in J} \lambda_j \mu_{\epsilon(n)}^0(\chi_j) \right| \leq \frac{\alpha}{3}.
\]
and
\[
\left| \mu_{\epsilon r(k)}^0(\chi) - \sum_{j \in J} \lambda_j \mu_{\epsilon r(k)}^0(\chi_j) \right| \leq \frac{\alpha}{3}.
\]
Since the sequence \( \mu_{\epsilon r(m)}^0(\chi) \) converges for all \( j \in J \), we have, for \( k \) large enough,
\[
\left| \sum_{j \in J} (\mu_{\epsilon r(m)}^0(\chi_j) - \mu_{\epsilon r(k)}^0(\chi_j)) \right| \leq \frac{\alpha}{3}.
\]
Therefore, for \( k \) large enough,
\[
|\mu_{\epsilon r(m)}^0(\chi) - \mu_{\epsilon r(k)}^0(\chi)| \leq \alpha.
\]
This proves that the family \( (\mu_{\epsilon r(m)}^0(\chi))_{m \in \mathbb{N}} \) is a Cauchy sequence in \( \mathbb{R} \), thus it converges. This determines the measure \( \mu_{\epsilon r}^0 \) and establishes the convergence for \( s = s_0 \). Since \( s_0 \) was arbitrary, the conclusion of Theorem 3.1 follows.

In the following proofs, we will only work on the sequence \( (u_{\epsilon r})_{m \in \mathbb{N}} \) selected by Theorem 3.1 but for simplicity, we will use our notation for the whole family \( (u_{\epsilon r})_{\in (0,1)} \).

### 3.2. The monotonicity formula on \( \mu_{\epsilon r}^0 \)

The following result transfers the monotonicity formula on \( \mu_{\epsilon r}^0 \) to its singular limit \( \mu_{\epsilon r}^s \). In the case where the initial energy on the whole space is bounded by \( M_0|\ln \varepsilon| \), it is just an easy consequence of the monotonicity formula for \( u_{\epsilon r} \). In our case, with our initial condition \( H_0(f) \), it requires some more elaborate computations.

**Proposition 3.7.** For each \( t > 0 \), \( x \in \mathbb{R}^d \) and \( 0 < r \leq \sqrt{t} \), we have
\[
\int_{\mathbb{R}^d} \exp \left( -\frac{|x-y|^2}{4r^2} \right) d\mu_{\epsilon r}^{t-r^2}(y) \to \int_{\mathbb{R}^d} \exp \left( -\frac{|x-y|^2}{4r^2} \right) d\mu_{\epsilon r}^{t-r^2}(y) \quad \text{as} \quad \varepsilon \to 0.
\]

**Proof of Proposition 3.7:** Let \( t > 0 \), \( x \in \mathbb{R}^d \) and \( 0 < r \leq \sqrt{t} \) fixed. Since \( \mu_{\epsilon r}^{t-r^2} \to \mu_{\epsilon r}^{t-r^2} \) as measures, we have for all \( \phi \in C_c^\infty(\mathbb{R}^d) \)
\[
\int_{\mathbb{R}^d} \phi(y) \exp \left( -\frac{|y-x|^2}{8r^2} \right) d\mu_{\epsilon r}^{t-r^2}(y) \to \int_{\mathbb{R}^d} \phi(y) \exp \left( -\frac{|y-x|^2}{8r^2} \right) d\mu_{\epsilon r}^{t-r^2}(y).
\]
For \( x \in \mathbb{R}^d \), \( t > 0 \), and \( 0 < r \leq \sqrt{t} \) fixed, the measures \( \exp \left( -\frac{|y-x|^2}{8r^2} \right) d\mu_{\epsilon r}^{t-r^2}(y) \) are bounded independently of \( \varepsilon \): indeed, we apply the monotonicity formula at the point \( (x, t+r^2) \) between \( \sqrt{2r} \) and \( \sqrt{t+r^2} \).
\[
\int_{\mathbb{R}^d} e_\varepsilon(u_{\epsilon r})(y, t-r^2) \exp \left( -\frac{|y-x|^2}{8r^2} \right) dy \leq \left( \frac{2r^2}{t+r^2} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e_\varepsilon(u^0)(y) \exp \left( -\frac{|y-x|^2}{4(t+r^2)} \right) dy \leq C(t, r)|\ln \varepsilon|,
\]
by Lemma 2.13. Therefore,
\[
\int_{\mathbb{R}^d} \exp \left( -\frac{|y-x|^2}{8r^2} \right) d\mu_{\epsilon r}^{t-r^2}(y) \leq C(t, r).
\]
Hence, the measures \( \exp \left( -\frac{|y-x|^2}{8r^2} \right) d\mu_{\epsilon r}^{t-r^2}(y) \) are bounded independently of \( \varepsilon \). On the other hand, they converge in measure to \( \exp \left( -\frac{|y-x|^2}{8r^2} \right) d\mu_{\epsilon r}^{t-r^2}(y) \). As a consequence, the measure
\[ \exp\left(-\frac{|y-x|^2}{8r^2}\right) d\mu^{t-r}_x(y) \text{ is bounded on } \mathbb{R}^d \text{ and for all } \phi \in \mathcal{C}_b^\infty(\mathbb{R}^d), \text{ there holds} \]
\[ \int_{\mathbb{R}^d} \phi(y) \exp\left(-\frac{|y-x|^2}{8r^2}\right) d\mu^{t-r}_x(y) \to \int_{\mathbb{R}^d} \phi(y) \exp\left(-\frac{|y-x|^2}{8r^2}\right) d\mu^{t-r}_x(y). \]

In particular, for \( \phi(y) = \exp\left(-\frac{|y-x|^2}{8r^2}\right), \) we have
\[ \int_{\mathbb{R}^d} \exp\left(-\frac{|y-x|^2}{4r^2}\right) d\mu^{t-r}_x(y) \to \int_{\mathbb{R}^d} \exp\left(-\frac{|y-x|^2}{4r^2}\right) d\mu^{t-r}_x(y). \]

Then, we can get an analogous of the monotonicity formula on \( \mu^{t}_x. \)

**Proposition 3.8.** For each \( t > 0 \) and \( x \in \mathbb{R}^d, \) the function \( \mathcal{E}_\mu((x,t), \cdot) \) defined on \( (0, +\infty) \) by
\[ r \mapsto \mathcal{E}_\mu((x,t), r) = \frac{1}{r^{d-2}} \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{4r^2}\right) d\mu^{t-r}_x(y) \]
is non-decreasing for \( 0 < r \leq \sqrt{T}. \)

**Proof.** It is a direct consequence of Proposition 3.7 and of the monotonicity formula for \( u. \)

Arguing as for (2.22), we thus get
\[ \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{4r^2}\right) d\mu^{t}_x(y) \leq \frac{R^{d-2}}{(t + R^2)^{\frac{d-2}{2}}} \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{4(t + R^2)}\right) d\mu^{0}_x(y). \]

From there, we can get the following energy bound.

**Proposition 3.9.** For all \( x \in \mathbb{R}^d \) and \( T > 0, \) we have
\[ \forall t \in (0, T], \; s \in [0, T], \; \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{4t}\right) d\mu^{t}_x \leq C(T). \]

**Proof.** Due to the monotonicity formula (2.21) and Lemma 2.13,
\[ \int_{\mathbb{R}^d} \exp\left(-\frac{|y-x|^2}{4t}\right) e_x(u)_s(y) d y \leq \frac{t^{\frac{d-2}{2}}}{(t + s)^{\frac{d-2}{2}}} \int_{\mathbb{R}^d} \exp\left(-\frac{|y-x|^2}{4(s + t)}\right) e_x(u)_s(y) d y \leq C(t)|\ln t|. \]

Thus,
\[ \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{4t}\right) d\mu^{t}_x \leq C(T). \]

Letting \( \epsilon \to 0, \) the conclusion follows by Proposition 3.7.

The remainder of this section is devoted to prove Theorem 1.1.

### 3.3. Densities and concentration set.

In order to analyse geometric properties of the measures \( \mu_x \) and \( \mu^{t}_x, \) a key point is the concept of densities. For a given Radon measure \( \nu \) on \( \mathbb{R}^d, \) we have the classical definition:

**Definition 3.10.** For \( m \in \mathbb{N}, \) the \( m \)-dimensional lower density of \( \nu \) at the point \( x \) is defined by
\[ \Theta_{\nu,m}(x) = \liminf_{r \to 0} \frac{\nu(B(x,r))}{\omega_m r^m}, \]
where \( \omega_m \) denotes the volume of the unit ball \( B^m. \) Similarly, the \( m \)-dimensional upper density \( \Theta^*_\nu(x) \) is given by
\[ \Theta^*_\nu(x) = \limsup_{r \to 0} \frac{\nu(B(x,r))}{\omega_m r^m}. \]
When both quantities coincide, $\nu$ admits a $m$-dimensional density $\Theta_m(\nu, x)$ at the point $x$, defined as the common value.

Since the energy measure is expected to concentrate on $(d-2)$-dimensional sets, our main efforts will be devoted to the study of the density $\Theta_{s,d-2}(\mu^s, \cdot)$. Invoking the monotonicity formula once more, we have

**Lemma 3.11.** For all $x \in \mathbb{R}^d$ and for all $t > 0$,

$$\Theta_{s,d-2}(\mu^s, x) \leq \Theta^{*}_{d-2}(\mu^s, x) \leq \frac{C(|x|, t)}{t^{\frac{d+2}{2}}} < +\infty.$$

**Proof.** Only the middle inequality needs explanations. It is a consequence the bounds on $\mu^s$ derived from Lemma 2.13 and Proposition 2.2. Indeed fix $t > 0$ and $x \in \mathbb{R}^d$; for $r > 0$, there holds

$$\frac{\mu^s(B(x, r))}{r^{d-2}} = \frac{1}{r^{d-2}} \int_{B(x, r)} d\mu^s \leq \frac{e^{1/4}}{r^{d-2}} \int \exp \left( -\frac{|x-y|^2}{4r^2} \right) d\mu^s \leq \lim_{r \to 0} \frac{e^{1/4}}{r^{d-2}} \int \exp \left( -\frac{|x-y|^2}{4r^2} \right) d\mu^s.

Now, for any $\epsilon > 0$, the mononicity formula yields

$$\frac{e^{1/4}}{r^{d-2}} \int \exp \left( -\frac{|x-y|^2}{4r^2} \right) d\mu^s \leq \frac{e^{1/4}}{(t + r)^{d-2}} \int \exp \left( -\frac{|x-y|^2}{4(t + r)} \right) \frac{\epsilon(t) \nu^s(y)}{|\ln \epsilon|} dy \leq \frac{C(t + r^2)}{(t + r)^{\frac{d+2}{2}}} f(2|x|).$$

As this does not depend on $\epsilon$, we infer that

$$\frac{\mu^s(B(x, r))}{r^{d-2}} \leq \frac{C(t + r^2)}{(t + r)^{\frac{d+2}{2}}} f(2|x|).$$

Taking the lim sup in $r \to 0$, we get

$$\Theta^{*}_{d-2}(\mu^s, x) \leq \frac{C(t)}{t^{\frac{d+2}{2}}} f(2|x|). \quad \Box$$

The previous lemma provides an upper bound. For regularity properties (of the concentration set) it is well known that lower bounds play a key role. However, it seems difficult to work with $\Theta_{s,d-2}(\mu^s, \cdot)$ directly (since the equation depends on time); instead, we will first consider parabolic densities (which involve space-time measures), whose definition is recalled below, and which is more natural in view of the monotonicity for $\mu_s$ (Proposition 3.8).

**Definition 3.12.** Let $\nu$ be a Radon measure on $\mathbb{R}^d \times [0, +\infty)$ such that $\nu = \nu^r dt$. For $t > 0$ and $m \in \mathbb{N}$, the parabolic $m$-dimensional lower density of $\nu$ at the point $(x, t)$ is defined by

$$\Theta_{s,m}^p(\nu, (x, t)) = \liminf_{r \to 0} \frac{1}{r^m} \int_{\mathbb{R}^d} \exp \left( -\frac{|x-y|^2}{4r^2} \right) d\nu^{r-t}(y).$$

The parabolic upper density and parabolic density are defined accordingly, and denoted respectively $\Theta^{p*}_{s,m}$ and $\Theta^{p}_{s,m}$.

It clearly follows from the monotonicity formula that for $\nu = \mu_s$ and $m = d-2$, the limit in definition 3.12 is decreasing, so that $\Theta^{p}_{s,d-2}(\mu^s, (x, t))$ exists everywhere in $\mathbb{R}^d \times (0, +\infty)$. Another consequence, is that the parabolic measure dominates the $(d-2)$-dimensional density for $\mu_s$ (see Proposition 3.18 for a precise statement). Motivated by this fact, we define

$$\Sigma_{\mu} := \{(x, t) \in \mathbb{R}^d \times (0, +\infty) \text{ such that } \Theta^{p}_{s,d-2}(\mu^s, (x, t)) > 0 \},$$

where $\Sigma_{\mu}$ is the set of points $(x, t)$ where the parabolic measure is positive.
and for $t > 0$,

$$
(3.14) \quad \Sigma_\mu^t := \Sigma_\mu \cap (\mathbb{R}^d \times \{t\}).
$$

We now state the properties we need on $\Sigma_\mu$ to conclude the proof of Theorem 1.1 and postpone their proofs to the end of this subsection.

### 3.4. Properties and regularity of $\Sigma_\mu$.

#### 3.4.1. Diffuse part of $\mu_t^i$ outside of the vorticity.

Let us first state an important consequence of the analysis carried out above in Section 2.

**Theorem 3.13.** There exist an absolute constant $\eta_2 > 0$ and a positive continuous function $\lambda$ defined on $(0, +\infty)$ such that, if for $x \in \mathbb{R}^d$, $t > 0$ and $r > 0$ we have

$$
(3.15) \quad \mu_t^i(B(x, \lambda(t)r)) < \eta_2 r^{d-2},
$$

then for every $s \in [t + \frac{15}{16} r^2, t + r^2]$, $\mu_t^i$ is absolutely continuous with respect to the Lebesgue measure on the ball $B(x, \frac{1}{4} r)$. More precisely

$$
\mu_t^i = |\nabla \Phi_t|^2 dx \text{ on } B \left( x, \frac{1}{4} r \right),
$$

where $\Phi_t$ satisfies the heat equation in $\Lambda_\frac{1}{4} = B(x, \frac{1}{4} r) \times [t + \frac{15}{16} r^2, t + r^2]$.

Notice that the constant $\eta_2$ and the function $\lambda$ are the same as in Proposition 2.6 of the previous section. Notice also that $\mu_t = |\nabla \Phi_t|^2 dx dt$ on $\Lambda_\frac{1}{4}$, and that $|\nabla \Phi_t|^2$ is a smooth function.

**Proof.** We briefly sketch the proof of Theorem 3.13, which is a direct consequence of Theorems 2.1 and 2.5 of the previous section.

Let $x \in \mathbb{R}^d$, $t > 0$, $r > 0$ be fixed such that (3.15) is verified, then for $\epsilon$ small enough, we have

$$
\int_{B(x, \lambda(t)r)} e_\epsilon(u_\epsilon) < \eta_2 r^{d-2} |\ln \epsilon|,
$$

so that we may invoke Proposition 2.6. This yields

$$
e_\epsilon(u_\epsilon) = |\nabla \Phi_\epsilon|^2 + \kappa_\epsilon \quad \text{in } \Lambda_\frac{1}{4},
$$

where $\Phi_\epsilon$ verifies the heat equation in $\Lambda_\frac{1}{2}$ and we have the bounds

$$
|\nabla \Phi_\epsilon|^2 \leq C(\Lambda) |\ln \epsilon|, \quad \text{and} \quad |\kappa_\epsilon| \leq C(\Lambda) \epsilon^\beta \quad \text{in } \Lambda_\frac{1}{4}.
$$

Extracting possibly a further subsequence, we may assume that

$$
\frac{\Phi_\epsilon}{\sqrt{|\ln \epsilon|}} \to \Phi_t \quad \text{uniformly on } \Lambda_\frac{1}{4}.\n$$

Since $\Phi_\epsilon$ verifies the heat equation, it follows that for every $k \in \mathbb{N}$,

$$
\frac{\Phi_\epsilon}{\sqrt{|\ln \epsilon|}} \to \Phi_t \quad \text{in } C^k(\Lambda_\frac{1}{4}),
$$

and $\Phi_t$ verifies the heat equation on $\Lambda_\frac{1}{4}$. On the other hand,

$$
\kappa_\epsilon \to 0 \quad \text{uniformly on } \Lambda_\frac{1}{4},
$$

so that

$$
e_\epsilon(u_\epsilon) \to |\nabla \Phi_t|^2 \quad \text{uniformly on } \Lambda_\frac{1}{4}. \quad \square$$
3.4.2. Clearing-out. Following Brakke [13] and Ilmanen [19], the main tool in the study of geometric properties of $\Sigma_\mu$ is the following result.

**Theorem 3.14 (Clearing-out).** There exists a positive continuous function $\eta_3$ defined on $\mathbb{R}^d \times (0, +\infty)$ such that, for any $(x, t) \in \mathbb{R}^d \times (0, +\infty)$ and any $0 < r < \sqrt{t}$, if
\[
\mathcal{E}_\mu^f((x, t), r) := \frac{1}{r^{d-2}} \int_{\mathbb{R}^d} \exp(-\frac{|x-y|^2}{4r^2}) \, d\mu_{^t-r}^s(y) \leq \eta_3(x, t - r^2),
\]
then
\[(x, t) \notin \Sigma_\mu.\]

An immediate corollary is

**Corollary 3.15.** For any $(x, t) \in \Sigma_\mu$, we have
\[
\Theta^\sharp_{d-2}(\mu_+, (x, t)) \not\supseteq \eta_3(x, t).
\]

The remainder of this paragraph is devoted to the proof of Theorem 3.14 which is essentially a consequence of Theorem 3.13. We first need two preliminary lemmas.

**Lemma 3.16.** Let $x \in \Sigma_\mu^i$ and $0 < r < \sqrt{t}$. Then, we have
\[
r^{2-d} \mu_{^t-r}^s(B(x, \lambda(x, t - r^2)r)) \geq \eta_2,
\]
where $\eta_2$ is the constant in Theorem 3.13.

**Proof.** Indeed, assume by contradiction that
\[
r^{2-d} \mu_{^t-r}^s(B(x, \lambda(x, t - r^2)r)) < \eta_2.
\]
Then, by Theorem 3.13, for every $\tau \in [t - \frac{1}{16}r^2, t]$
\[
\mu_{^\tau} = |\nabla \Phi_{\tau}|^2 \, dx \quad \text{on } B\left(x, \frac{r}{4}\right),
\]
where $\Phi_{\tau}$ is smooth. We are going to show that
\[
s^{2-d} \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{4s^2}\right) \, d\mu_{^t-s}^s \to 0 \quad \text{as } s \to 0.
\]
Indeed, we write and compute
\[
s^{2-d} \int_{B(x, \frac{r}{4})} \exp\left(-\frac{|x-y|^2}{4s^2}\right) \, d\mu_{^t-s}^s \leq s^{2-d} \|\nabla \Phi_{\tau}\|_{L^\infty(B(x, \frac{r}{4}))} \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{4s^2}\right) \, dx \leq K \|\nabla \Phi_{\tau}\|_{L^\infty(B(x, \frac{r}{4}))} s^2 \to 0 \quad \text{as } s \to 0.
\]

On the other hand,
\[
s^{2-d} \int_{\mathbb{R}^d \setminus B(x, \frac{r}{4})} \exp\left(-\frac{|x-y|^2}{4s^2}\right) \, d\mu_{^t-s}^s \to 0 \quad \text{as } s \to 0.
\]
Indeed,
\[
s^{2-d} \int_{\mathbb{R}^d \setminus B(x, \frac{r}{4})} \exp\left(-\frac{|x-y|^2}{4s^2}\right) \, d\mu_{^t-s}^s = s^{2-d} \int_{\mathbb{R}^d \setminus B(x, \frac{r}{4})} \left(\exp\left(-\frac{|x-y|^2}{8s^2}\right)\right)^2 \, d\mu_{^t-s}^s
\leq s^{2-d} \int_{\mathbb{R}^d \setminus B(x, \frac{r}{4})} \exp\left(-\frac{r^2}{8 \times 64s^2}\right) \exp\left(-\frac{|x-y|^2}{8s^2}\right) \, d\mu_{^t-s}^s
\leq s^{2-d} \exp\left(-\frac{r^2}{512s^2}\right) \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{8s^2}\right) \, d\mu_{^t-s}^s
\leq \left(\frac{2}{t + s^2}\right)^{\frac{d-2}{2}} \exp\left(-\frac{r^2}{512s^2}\right) \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{4(t + s^2)}\right) \, d\mu_0
\]
by the monotonicity formula at \((x, t + s^2)\) between \(\sqrt{2}s\) and \(\sqrt{t + s^2}\) (we can assume \(s < \sqrt{t}\)).

Since

\[
\int_{\mathbb{R}^d} \exp \left( - \frac{|x - y|^2}{4(t + s^2)} \right) d\mu^0_s \leq C(t)
\]

by inequality \((3.12)\), we have

\[
s^{2-d} \int_{\mathbb{R}^d \setminus B(x, \frac{s}{2})} \exp \left( - \frac{|x - y|^2}{4s^2} \right) d\mu_{s^{-2}} - \exp \left( - \frac{r^2}{512s^2} \right) \rightarrow 0 \quad \text{as} \quad s \rightarrow 0.
\]

Combining \((3.18)\) and \((3.19)\), \((3.17)\) follows and hence \(\Theta^p_{d-2}(\mu_s, (x, t)) = 0\), i.e. \((x, t) \notin \Sigma_m\), which is a contradiction. \(\square\)

**Lemma 3.17.** The function \((x, t) \rightarrow \Theta^p_{d-2}(\mu_s, (x, t))\) is upper semi-continuous on the set \(\mathbb{R}^d \times (0, +\infty)\).

**Proof.** Let \((x, t) \in \mathbb{R}^d \times (0, +\infty)\), and let \((x_m, t_m)_{m \in \mathbb{N}}\) be a sequence such that \((x_m, t_m) \rightarrow (x, t)\). We are going to show that

\[
\limsup_{m \to +\infty} \Theta^p_{d-2}(\mu_{s^2}(x_m, t_m)) \leq \Theta^p_{d-2}(\mu_s, (x, t)).
\]

Let \(0 < r < \frac{1}{2} \sqrt{t}\) be fixed for the moment. For \(m\) sufficiently large, let \(r_m = \sqrt{r^2 + t_m - t}\), so that \(t - r^2 = t_m - r_m^2\). By the monotonicity formula, we have

\[
\Theta^p_{d-2}(\mu_s, (x_m, t_m)) \leq \frac{1}{r_m^{d-2}} \int_{\mathbb{R}^d} \exp \left( - \frac{|y - x_m|^2}{4r_m^2} \right) d\mu^0_{s^{-2}}(y)
\]

\[
\leq \frac{1}{r_m^{d-2}} \int_{\mathbb{R}^d} \exp \left( - \frac{|y - x_m|^2}{4r^2} \right) d\mu^0_{s^{-2}}(y).
\]

We assert that, as \(m \to +\infty\),

\[
\frac{1}{r_m^{d-2}} \int_{\mathbb{R}^d} \exp \left( - \frac{|y - x_m|^2}{4r^2} \right) d\mu^0_{s^{-2}}(y) \to \frac{1}{r^{d-2}} \int_{\mathbb{R}^d} \exp \left( - \frac{|y - x|^2}{4r^2} \right) d\mu^0_{s^{-2}}(y).
\]

Indeed, since the sequence \((x_m)_m\) is bounded by a constant \(M > 0\), we write

\[
\frac{1}{r_m^{d-2}} \int_{\mathbb{R}^d} \exp \left( - \frac{|y - x_m|^2}{4r^2} \right) d\mu^0_{s^{-2}}(y) = \frac{1}{r_m^{d-2}} \int_{B(0, 2M)} \exp \left( - \frac{|y - x_m|^2}{4r^2} \right) d\mu^0_{s^{-2}}(y)
\]

\[
+ \frac{1}{r_m^{d-2}} \int_{B(0, 2M)^c} \exp \left( - \frac{|y - x_m|^2}{4r^2} \right) d\mu^0_{s^{-2}}(y).
\]

Passing to the limit in the integral on \(B(0, 2M)\) does not raise any difficulty. For the second integral, we have

\[
|y - x_m| > |y| - |x_m| \geq |y| - M \geq \frac{1}{2} |y|,
\]

so, if \(y \in \mathbb{R}^d \setminus B(0, 2M)\),

\[
\exp \left( - \frac{|y - x_m|^2}{4r^2} \right) d\mu^0_{s^{-2}}(y) \leq \exp \left( - \frac{|y|^2}{16r^2} \right) d\mu^0_{s^{-2}}(y),
\]

and by the monotonicity formula at point \((0, t + r^2)\) between \(r_1 = 2r\) and \(r_2 = \sqrt{t + 3r^2}\),

\[
\int_{\mathbb{R}^d} \exp \left( - \frac{|y|^2}{16r^2} \right) d\mu^0_{s^{-2}}(y) \leq \left( \frac{2r}{\sqrt{t + 3r^2}} \right)^{d-2} \int_{\mathbb{R}^d} \exp \left( - \frac{|y|^2}{4(t + 3r^2)} \right) d\mu^0_s(y) \leq C(t),
\]

by \((3.12)\). We then get the result as a consequence of Lebesgue's dominated convergence Theorem. So if we let \(m \to +\infty\), we obtain

\[
\limsup_{m \to +\infty} \Theta^p_{d-2}(\mu_{s^2}(x_m, t_m)) \leq \frac{1}{r^{d-2}} \int_{\mathbb{R}^d} \exp \left( - \frac{|y - x|^2}{4r^2} \right) d\mu^0_{s^{-2}}(y).
\]
Now, let \( r \to 0 \), and (3.20) follows. \( \square \)

**Proposition 3.18.** There exists an explicit constant \( K \) such that for all \( x \in \mathbb{R}^d \) and \( t > 0 \),

\[
\Theta_{d-2}^p(\mu_+(x, t)) \geq K \Theta_{d-2}^*(\mu_+^t, x).
\]

In particular,

\[
\Theta_{d-2}^*(\mu_+^t, x) \equiv 0 \quad \text{on } \mathbb{R}^d \setminus \Sigma_{\mu}^t.
\]

**Proof.** Let \((x, t) \in \mathbb{R}^d \times (0, +\infty)\) be given. Let \( 0 < r < t \) be fixed for the moment. We write, for every \( 0 < s < \sqrt{t} \),

\[
\frac{1}{r^{d-2}} \mu_+^t(B(x, r)) \leq \exp\left(\frac{1}{4} \right) \frac{1}{r^{d-2}} \int_{\mathbb{R}^d} \exp\left(\frac{-|y-x|^2}{4r^2}\right) d\mu_+^t(y)
\]

\[
\leq \exp\left(\frac{1}{4} \right) \left( \frac{1}{(r^2 + s^2)^{\frac{d-2}{2}}} \int_{\mathbb{R}^d} \exp\left(\frac{-|y-x|^2}{4(r^2 + s^2)}\right) d\mu_+^{t-r}(y),
\]

where we have used the monotonicity formula at the point \((x, t + r^2)\) between \( r \) and \( \sqrt{r^2 + s^2} \) for the last inequality. Next, we choose \( s = \sqrt{t} \). This yields

\[
\frac{1}{r^{d-2}} \mu_+^t(B(x, r)) \leq \exp\left(\frac{1}{4} \right) \left( \frac{1}{(r^2 + r)^{\frac{d-2}{2}}} \int_{\mathbb{R}^d} \exp\left(\frac{-|y-x|^2}{4(r^2 + r)}\right) d\mu_+^{t-r}(y).
\]

In the last integral, we decompose

\[
\mathbb{R}^d = B(x, 1) \cup (\mathbb{R}^d \setminus B(x, 1)).
\]

On \( B(x, 1) \), observe that

\[
\exp\left(\frac{-|y-x|^2}{4(r^2 + r)}\right) \leq K \exp\left(\frac{-|y-x|^2}{4r}\right),
\]

for some absolute constant \( K \). On the other hand, on \( \mathbb{R}^d \setminus B(x, 1) \), we have

\[
\frac{1}{(r^2 + r)^{\frac{d-2}{2}}} \int_{\mathbb{R}^d \setminus B(x, 1)} \exp\left(\frac{-|x-y|^2}{4(r^2 + r)}\right) d\mu_+^{t-r} \leq \frac{C(t)}{t^{\frac{d-2}{2}}} \exp\left(\frac{-1}{8(r^2 + r)}\right).
\]

Indeed,

\[
\frac{1}{(r^2 + r)^{\frac{d-2}{2}}} \int_{\mathbb{R}^d \setminus B(x, 1)} \exp\left(\frac{-|x-y|^2}{4(r^2 + r)}\right) d\mu_+^{t-r}
\]

\[
\leq \exp\left(\frac{-1}{8(r^2 + r)}\right) \left( \frac{1}{(r^2 + r)^{\frac{d-2}{2}}} \int_{\mathbb{R}^d} \exp\left(\frac{-|x-y|^2}{8(r^2 + r)}\right) d\mu_+^{t-r}(y)
\]

\[
\leq \exp\left(\frac{2}{t}\right) \exp\left(\frac{-1}{8(r^2 + r)}\right) \int_{\mathbb{R}^d} \exp\left(\frac{-|x-y|^2}{4(2r^2 + r + t)}\right) d\mu_0(y)
\]

\[
(3.25)
\]

\[
(3.26)
\]

where we have used the monotonicity formula at point \((x, 2r^2 + r + t)\) between \( \sqrt{2(r^2 + r)} \) and \( \sqrt{2r^2 + r + t} \) for inequality (3.25). Then, (3.26) and Lemma 2.13 lead to inequality (3.23). Going back to (3.22), we infer

\[
\frac{1}{r^{d-2}} \mu_+^t(B(x, r)) \leq \frac{K}{r^{\frac{d-2}{2}}} \int_{\mathbb{R}^d} \exp\left(\frac{-|x-y|^2}{4r}\right) d\mu_+^{t-r} + \frac{C(t)}{t^{\frac{d-2}{2}}} \exp\left(\frac{-1}{8(r^2 + r)}\right).
\]

Letting \( r \) go to zero, the conclusion follows. \( \square \)

We can now complete the proof of Theorem 3.14.
Proof of Theorem 3.14. Let \((x, t) \in \mathbb{R}^d \times (0, +\infty)\) and \(0 < r < \sqrt{t}\). We have

\[
  r^{2-d}\mu_{\lambda}^{\frac{r^2}{2}}(B(x, \lambda(x, t - r^2)r)) \leq \exp\left(\frac{\lambda^2(x, t - r^2)}{4}\right) \mathcal{E}(x, t, r).
\]

Consider therefore the function

\[
  \eta_3(x, s) = \exp\left(-\frac{\lambda^2(x, s)}{4}\right) \eta_2,
\]

and assume next that, for some \(0 < r < \sqrt{t}\),

\[
  \mathcal{E}(x, t, r) \leq \eta_3(x, t - r^2).
\]

Then, by (3.27),

\[
  r^{2-d}\mu_{\lambda}^{\frac{r^2}{2}}(B(x, \lambda(x, t - r^2)r)) \leq \eta_2
\]

and the conclusion follows by Lemma 3.16. \(\square\)

3.4.3. Consequences: regularity of \(\Sigma_\mu\) and decomposition of \(\mu_s\). At this stage, we are in position to derive the following conclusions, without invoking any further property of the equation (PGL).

**Proposition 3.19.**

1. The set \(\Sigma_\mu\) is closed in \(\mathbb{R}^d \times (0, +\infty)\).
2. Let \(x \in \mathbb{R}^d\). For \(t \in (0, 1/2]\), then

\[
  \mathcal{H}^{d-2}(B(x, 1) \cap \Sigma_\mu^t) \leq C(|x|) \ln t^{\frac{d-2}{2}}.
\]

For \(t \geq 1/2\), we have

\[
  \mathcal{H}^{d-2}(B(x, 1) \cap \Sigma_\mu^t) \leq C(|x|, t).
\]

3. For any \(t > 0\), the measure \(\mu^t_s\) can be decomposed as

\[
  \mu^t_s = g(x, t) \mathcal{H}^{d-2} + \Theta_s(x, t) \mathcal{H}^{d-2}(\Sigma_\mu^t),
\]

where \(g\) is some smooth function defined on \(\mathbb{R}^d \times (0, +\infty) \setminus \Sigma_\mu\) and \(\Theta_s\) verifies the bound

\[
  \Theta_s(x, t) \leq C(|x|, t) \frac{1}{t^{\frac{d-2}{2}}}.\]

The bound (3.28) will actually be relevant in the proof of Theorem 1.3 for \(d = 2\), it does not depend on small times.

The function \(\Theta_s\) is the Radon-Nikodym derivative of \(\mu^t_s, \Sigma_\mu^t\) with respect to \(\mathcal{H}^{d-2}\); at this stage we may just assert that it lies between the lower and upper densities.

Concerning \(g\), it can be locally defined as \(|\nabla \Phi_t|^2\) for some smooth \(\Phi_t\) verifying the heat equation. The function \(\Phi_t\) is however not yet globally defined.

**Proof.** (1) In view of Corollary 3.15, we have

\[
  \Sigma_\mu = \{(x, t) \in \mathbb{R}^d \times (0, +\infty) : \Theta_{d-2}^\mu(x, t) \geq \eta_3(x, t)\}.
\]

Since \(\eta_3(\cdot)\) is continuous and since \(\Theta_{d-2}^\mu(\mu_s, \cdot)\) is upper semi-continuous by Lemma 3.17, we deduce that \(\Theta_{d-2}^\mu(\mu_s, \cdot) - \eta_3(\cdot)\) is upper semi-continuous as well and the conclusion follows.

(2) We argue slightly differently for small and large values of \(t\). Let us start with \(t \leq 1/2\), where we seek better estimates.

Let \(0 < \delta < \sqrt{t}/2\). Consider a standard covering of \(B(x, 1)\) such that

\[
  B(x, 1) \subset \bigcup_{j \in J} B(x_j, \lambda(x, t)\delta), \quad \text{and} \quad B\left(x_i, \lambda(x, t)\frac{\delta}{2}\right) \cap B\left(x_j, \lambda(x, t)\frac{\delta}{2}\right) = \emptyset \quad \text{for} \ i \neq j.
\]
Recall from Proposition 2.3 where the function was first defined that $\lambda$ is continuous and has slow variations on $B(x, 1) \times (0, 1]$ in the sense of (2.2). It follows that there exists a function $K$ continuous on $[0, +\infty)$ such that

\begin{equation}
\forall (y, \tau) \in B(x, 1) \times [t/2, t], \quad \lambda(y, \tau) \leq (K(|x|) - 1)\lambda(x, t).
\end{equation}

(Observe that actually, if $f$ has at most exponential growth, $K$ can be chosen to be a constant independent of $x$). Define

$$I_\delta = \left\{ i \in J : B(x_i, \lambda(x_i, t)\delta) \cap B(x, 1) \cap \Sigma_\mu^t \neq \emptyset \right\}.$$

For $i \in I_\delta$, there exists some $y_i \in \Sigma_\mu^t \cap B(x_i, \lambda(x_i, t)\delta)$. Hence, by Lemma 3.11

$$\mu_i^{1-\delta^2}(B(y_i, \lambda(y_i, t - \delta^2)\delta)) > \eta_2 \delta^{d-2}.$$

Due to (3.29) and as $t - \delta^2 \geq t/2, B(y_i, \lambda(y_i, t - \delta^2)\delta) \subset B(x_i, K(|x|)\lambda(x_i, t)\delta)$ and

\begin{equation}
\mu_i^{1-\delta^2}(B(x_i, K(|x|)\lambda(x_i, t)\delta)) > \eta_2 \delta^{d-2}.
\end{equation}

If $\delta$ is so small that $K(|x|)\lambda(x, t)\delta < 1$, the balls $B(x_i, K(|x|)\lambda(x_i, t)\delta)$ are included in $B(x, 2)$; since the balls $B \left( x_i, (\lambda(x_i, t)\delta/2) \right)$ are disjoint, the balls $B(x_i, K(|x|)\lambda(x_i, t)\delta)$ cover at most $C(d)K(|x|)^d$ times the ball $B(x, 2)$. Therefore, using inequality (3.3),

$$\delta^{2-d}\sum_{i \in I_\delta} \mu_i^{1-\delta^2}(B(x_i, K(|x|)\lambda(x_i, t)\delta)) \leq C(d)\delta^{2-d}K(|x|)^d \mu_i^{1-\delta^2}(B(x, 2))$$

$$\leq C(|x|)^d\delta^{2-d}f(2|x|) \leq C(|x|)\delta^{2-d}.$$

With (3.30), this implies that

$$\eta_2 \operatorname{Card} I_\delta < C(|x|)\delta^{2-d}.$$

Now by definition, we have

$$\mathcal{H}^{d-2}(B(x, 1) \cap \Sigma_\mu^t) \leq \lim_{\delta \to 0} \operatorname{Card} I_\delta (\lambda(x, t)\delta)^{d-2} \leq \lambda(x, t)^{d-2}C(|x|) < +\infty,$$

which yields the desired bound: indeed, observe that for all $a, b \geq 0$, $\sqrt{a + b} \leq \sqrt{1 + a} \sqrt{1 + b}$ so that

$$\lambda(x, t) \leq C \sqrt{\ln t} + \ln f(2x) \leq C \sqrt{1 + \ln t} \sqrt{1 + \ln f(2x)} \leq C(|x|)\sqrt{\ln t}.$$

If we now consider the case $t \geq 1/2$, we can argue in the same fashion, except now $\lambda$ has no longer slow variations, but is still continuous. Therefore, we can replace (3.29) with

\begin{equation}
\forall (y, \tau) \in B(x, 1) \times [t/2, t], \quad \lambda(y, \tau) \leq (K(|x|, t) - 1)\lambda(x, t),
\end{equation}

for some function $K$ continuous on $[0, +\infty) \times [1, +\infty)$. The remainder of the argument is unchanged, and yields a bound $C(|x|, t)$ where we have no control in $t$.

(3) As $\Sigma_\mu^t$ is closed, we have the decomposition

$$\mu_i^t = \mu_i^t \lambda(\Sigma_\mu^t) + \mu_i^t \Sigma_\mu^t.$$

On the one hand, on $\Sigma_\mu^t$, and due to the bound (2), we can use the Radon-Nikodym theorem, which provides a function $\Theta \in L^1_{\text{loc}}(\mathcal{H}^{d-2}(\mathbb{R}^d))$ such that

$$\mu_i^t \Sigma_\mu^t = \Theta \mathcal{H}^{d-2} \Sigma_\mu^t.$$

Due to Lemma 3.11 we also have the bound

$$\Theta(x, t) \leq C(|x|, t)\frac{t^{d-2}}{t^{d-2}}.$$
On the other hand, for \( x_0 \notin \Sigma_{\mu}^\epsilon, \Theta_{d-2}(\mu_\epsilon(x,t)) \to 0 \) as \((x_0,t_0) \to (x,t)\) by Lemma 3.17. From there, and using the bound (3.21), there exists \( \delta > 0 \) small enough such that
\[
\mu_\epsilon^{1-\delta^2}(B(x, \lambda(x, t - \delta^2) \delta)) \leq \eta_2 \delta^{d-2}.
\]
Then Theorem 3.13 applies and shows that
\[
\mu_\epsilon = |\nabla \Phi_\epsilon(x,s)|^2 dx \quad \text{on} \quad B(x_0, \delta_0/4) \times [t - \delta^2/16, t],
\]
where \( \Phi_\epsilon \) solves the heat equation and is smooth: this allows to define \( g \) locally.

3.4.4. Lower bounds for \( \Theta_{s,d-2}(\mu_\epsilon) \). As already mentioned, lower bounds for \( \Theta_{s,d-2} \) play an important role for regularity issues: however, up to now we have only lower bounds for \( \Theta_{d-2} \) (see Corollary 3.15). The next result provides the reverse inequality to (3.21).

**Proposition 3.20.** There exists a continuous function \( \kappa : \mathbb{R}^d \times (0, +\infty) \to (0, +\infty) \), such that for almost every \( t > 0 \), the following inequality holds
\[
\Theta_{s,d-2}(\mu_\epsilon^t, x) \geq \kappa(x, t) \Theta_{d-2}^p(\mu_\epsilon(x, t))
\]
for \( \mathcal{H}^{d-2} \) almost every \( x \in \mathbb{R}^d \).

Combining Corollary 3.15 and Proposition 3.20, we are led to

**Corollary 3.21.** For almost every \( t > 0 \),
\[
\Theta_{s,d-2}(\mu_\epsilon^t, x) \geq \kappa(x, t) \eta_3(x, t) \quad \text{for} \quad \mathcal{H}^{d-2} \text{ a.e. } x \in \mathbb{R}^d.
\]

Our goal in this subsection is to prove Proposition 3.20, but we need some preliminary tools first. Our starting point is the estimate for the time derivative \( \partial_t u_\epsilon \) provided by Proposition 4.2 namely
\[
\forall T > 0, \forall R > 0, \quad \frac{1}{\ln \epsilon} \int_{B(0,R) \times [0,T]} |\partial_t u_\epsilon|^2 \leq C(T,R).
\]

Therefore, by a diagonal extraction argument, we may assume that there exists some non negative Radon measure \( \omega_\epsilon \) defined on \( \mathbb{R}^d \times [0, +\infty) \) such that
\[
\frac{1}{\ln \epsilon} |\partial_t u_\epsilon|^2 \to \omega_\epsilon \quad \text{as measures},
\]
so that \( \omega_\epsilon(B(0,R) \times [0, T]) \leq C(T,R) \).

In (3.3), it is known at this point that \( \Sigma_{\mu} \subset \mathbb{R}^d \times (0, T + 1) \) for some constant \( T_f \). In our case, we don't have the same result, and since we want to work on compact domains in \( \mathbb{R}^d \times [0, +\infty) \), we will fix \( T > 0 \), and then intersect our sets of \( \mathbb{R}^d \) with balls \( B(0,R) \).

**Step 1.** We fix \( T > 0 \).

We want to prove that for almost \( t \in (0, T] \), inequality (3.32) holds for every \( x \in \mathbb{R}^d \).

We define, for \( \ell \in \mathbb{N}_+ \) and \( q > 0 \) to be fixed later, the set
\[
A_{\ell}(\omega_\epsilon) = \{(x, t) \in \mathbb{R}^d \times (0, T] : \limsup_{\ell \to 0} \frac{1}{r^q} \int_{B(x,t) \times [t-r^2,t]} \omega_\epsilon > 1\}
\]
and its intersection with \( B(0,R) \)
\[
A_{\ell}(\omega_\epsilon^R) = \{(x, t) \in A_{\ell}(\omega_\epsilon) : |x| \leq R\}.
\]

The following shows that \( A_{\ell}(\omega_\epsilon) \) is small in some appropriate sense.

**Lemma 3.22.** For each \( \ell \in \mathbb{N}_+ \) and \( R > 0 \),
\[
\mathcal{H}_p^d(A_{\ell}(\omega_\epsilon^R)) < +\infty,
\]
where \( \mathcal{H}_p^d \) denotes the \( q \)-dimensional Hausdorff measure with respect to the parabolic distance
\[
d_p((x,t),(x',t')) = \max(|x-x'|,|t-t'|^{1/2}).
\]
Proof. Let \( 0 < \delta \leq 1 \) be given, and fixed for the moment. For \((x, t) \in A_t(\omega^p)\), there exists \( r = r(x, t) < \delta \) such that

\[
\int_{B(x,r) \times [t-r^2, t]} \omega_x \geq r^q.
\]

If we denote

\[
\Gamma^p_t(x, t, r(x, t)) = B(x, \ell r(x, t)) \times [t - r(x, t)^2, t],
\]

clearly, the reunion of the parabolic balls \( \bigcup_{(x, t) \in A_t(\omega^p)} \Gamma^p_t(x, t, r(x, t)) \) covers \( A_t(\omega^p) \). Notice that \( \text{diam}(\Gamma^p_t) < 2\ell r \). Since \( A_t(\omega^p) \subset B(0, R) \times [0, T] \), we may apply the Besicovitch covering theorem (see for example [16]). There exists an integer \( m(\ell, d) \) depending only on \( d \) and \( \ell \), and there exists a sub-covering of the form

\[
A_t(\omega^p) \subset \bigcup_{i=1}^{m(\ell, d)} \bigcup_{j \in J^p_i} \Gamma^p_t(x_j, t_j, r_j(x_j, t_j)),
\]

where for fixed \( i \), the sets \( \Gamma_j = \Gamma^p_t(x_j, t_j, r_j(x_j, t_j)) \) are disjoint. Consequently, it follows that for each \( i \in \llbracket 1, m(\ell, n) \rrbracket \),

\[
\sum_{j \in J^p_i} r(x_j, t_j)^q \leq \sum_{j \in J^p_i} \int_{\Gamma_j} \omega_x \leq \int_{B(0, R+\ell) \times [0, T]} \omega_x \leq C(T, R, \ell).
\]

Therefore,

\[
\sum_{j=1}^{m(\ell, d)} \sum_{j \in J^p_i} \text{diam}(\Gamma_j)^q \leq m(\ell, d) \ell^q C(T, R, \ell).
\]

Observe that the constant on the right hand side is independent of \( \delta \). Hence, letting \( \delta \to 0 \), we obtain

\[
\mathcal{H}^d_p(A_t(\omega^p)) \leq \limsup_{\delta \to 0} \left( \sum_{j=1}^{m(\ell, d)} \sum_{j \in J^p_i} \text{diam}(\Gamma_j)^q \right) \leq m(\ell, d) \ell^q C(T, R, \ell),
\]

and the proof is complete. \( \square \)

We fix \( q = d - \frac{3}{2} \). This choice has no specific geometrical meaning, but is convenient (as the following shows).

**Corollary 3.23.** We have

\[
\mathcal{H}^{d-1}
\left( \bigcup_{t \in \mathbb{N}} A_t(\omega_x) \right) = 0.
\]

Hence, for almost every \( t > 0 \),

\[
\mathcal{H}^{d-2}
\left( \bigcup_{t \in \mathbb{N}} A_t^p(\omega_x) \right) = 0,
\]

where \( A_t^p(\omega_x) = A_t(\omega_x) \cap (\mathbb{R}^d \times \{ t \}) \).

**Proof.** Since, by Lemma 3.22, \( \mathcal{H}^{d-\frac{3}{2}}_p(A_t(\omega^p)) < \infty \), it follows that

\[
\mathcal{H}^{d-1}_p(A_t(\omega^p)) = 0.
\]

On the other hand, parabolic balls are smaller than euclidian balls of the same radius, so that the parabolic Hausdorff measure dominates the euclidian Hausdorff measure. It follows that

\[
\mathcal{H}^{d-1}
\left( \bigcup_{t \in \mathbb{N}} \bigcup_{R \in \mathbb{N}} A_t^p(\omega_x) \right) = 0.
\]
Lemma 3.24. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Then, for $(x, t) \in \Omega_\omega$, 

$$
\lim_{r \to 0} \left( \frac{1}{r^{d-2}} \int_{\mathbb{R}^d} \chi \left( \frac{y-x}{r} \right) d\mu^r(y) - \frac{1}{r} \int_{\mathbb{R}^d} \chi \left( \frac{y-x}{r} \right) d\mu^{r-\ell}(y) \right) = 0.
$$

Proof. We need to go back to the level of the function $u^\ell$. For $0 < r < \sqrt{t}$, by Lemma 3.2, we have 

$$
\frac{1}{r} \int_{\mathbb{R}^d} \chi \left( \frac{y-x}{r} \right) d\mu^r(y) = \int_{\mathbb{R}^d} \frac{\partial u^\ell}{|\ln \epsilon|} \chi \left( \frac{y-x}{r} \right) d\mu^r(y).
$$

(3.34) 

Let $t \in \mathbb{N}^*$ such that $\text{Supp}(\chi) \subset B(\ell)$. We set $\Lambda = B(x, \ell \ell) \times [t-r^2, t]$, and estimate the last term in the previous identity by the Cauchy-Schwarz inequality, 

$$
\frac{1}{r|\ln \epsilon|} \left| \int_{\Lambda} \partial u^\ell \nabla u^\ell \cdot \nabla \chi \left( \frac{y-x}{r} \right) \right| \leq \left( \int_{\Lambda} \left| \partial u^\ell \right|^2 \right)^{\frac{1}{2}} \left( \int_{\Lambda} \left| \nabla \chi \right|^2 \right)^{\frac{1}{2}} \| \nabla \chi \|_{\infty}.
$$

(3.35) 

We now let $\epsilon \to 0$ in (3.35), therefore obtaining the inequality for measures 

$$
\frac{1}{r^{d-2}} \int_{\mathbb{R}^d} \chi \left( \frac{y-x}{r} \right) (d\mu^r - d\mu^{r-\ell})(y) \leq \left( \frac{1}{r^{d-2}} \int_{\mathbb{R}^d} \chi \left( \frac{y-x}{r} \right) \right)^{\frac{1}{2}} \| \chi \|_{\ell^1}.
$$

Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times [0, +\infty))$ such that $\psi = 1$ in $\Lambda$, and $\psi = 0$ out of $B(x, 2\ell \ell) \times [t-2r^2, t]$. We have 

$$
\frac{1}{r^d} \int_{B(x, 2\ell \ell) \times [t-2r^2, t]} e^\chi \left( \frac{y-x}{r} \right) d\mu^r(y) 
\leq \frac{1}{r^d} \int_{[t-2r^2, t]} e^{-\frac{|x-y|}{\sqrt{2}} \frac{d}{r^2}} \int_{B(x, 2\ell \ell)} e^{-\frac{|x-y|}{\sqrt{2}} \frac{d}{r^2}} \int_{B(x, 2\ell \ell)} e^{-\frac{|x-y|}{\sqrt{2}} \frac{d}{r^2}} dy ds
$$

(3.36) 

where we used the monotonicity formula applied at point $(x, s + 4\ell^2 r^2)$ between $2\ell r$ and $\sqrt{s + 4\ell^2 r^2}$. From there, we infer 

$$
\frac{1}{r^d} \int_{B(x, 2\ell \ell) \times [t-2r^2, t]} e^\chi \left( \frac{y-x}{r} \right) d\mu^r(y) \leq \frac{C(t + 4\ell^2 r^2)}{t^{\frac{d}{2}}}
$$

(3.37) 

Letting $\epsilon \to 0$, and then $r \to 0$, we get 

$$
\lim_{r \to 0} \frac{1}{r^d} \int_{B(x, 2\ell \ell) \times [t-2r^2, t]} e^\chi \left( \frac{y-x}{r} \right) d\mu^r(y) \leq \frac{C(t, \ell)}{t^{\frac{d}{2}}}.
$$

Since $\bigcup_{\ell \in \mathbb{N}^*} A^\ell_t(\omega_\ast) = A_\ell(\omega_\ast)$, we obtain that 

$$
\mathcal{H}^{d-1} \left( \bigcup_{\ell \in \mathbb{N}^*} A_\ell(\omega_\ast) \right) = 0,
$$

and the proof is complete. □
Recall that
\begin{equation}
\text{if } (x, t) \in \Omega_{\omega}, \text{ then } \limsup_{r \to 0} \frac{1}{r^{d-2}} \int_{B(x, t) \times [t-r^2, t]} \omega_s \leq 1.
\end{equation}

Now we have
\begin{align*}
\frac{1}{r^{d-2}} \int_{\Lambda} \omega_s + \left( \frac{1}{r^{d-2}} \int_{\Lambda} \omega_s \right)^{\frac{1}{2}} \left( \frac{1}{r^d} \int_{\Lambda} d\mu_s \right)^{\frac{1}{2}} \\
\leq \left( \frac{1}{r^{d-2}} \int_{\Lambda} \omega_s \right)^{\frac{1}{2}} + r^\frac{1}{2} \left( \frac{1}{r^{d-2}} \int_{\Lambda} \omega_s \right)^{\frac{1}{2}} \left( \frac{1}{r^d} \int_{\Lambda} \psi d\mu_s \right)^{\frac{1}{2}}.
\end{align*}

According to (3.36), and letting $r \to 0$, we deduce that
\begin{equation}
\int_{\Lambda} \omega_s + \left( \frac{1}{r^d} \int_{\Lambda} d\mu_s \right)^{\frac{1}{2}} \to 0,
\end{equation}
and the proof is complete. \qed

In Lemma 3.24 we assumed that $\chi$ has compact support. The following shows that the result still holds for $\chi(x) = \exp\left(-\frac{|x|^2}{4}\right)$, which is of special interest in view of the monotonicity formula.

**Corollary 3.25.** We have, for any $(x, t) \in \Omega_{\omega}$,
\begin{equation}
\lim_{r \to 0} \frac{1}{r^{d-2}} \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{4r^2}\right) d\mu_s(y) - \frac{1}{r^{d-2}} \int_{\mathbb{R}^d} \exp\left(-\frac{|x|^2}{4r^2}\right) d\mu_s(y) = 0.
\end{equation}
In particular, for $(x, t) \in \Sigma_{\omega} \cap \Omega_{\omega}$, the following limit exists and verifies the inequality
\begin{equation}
\lim_{r \to 0} \frac{1}{r^{d-2}} \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{4r^2}\right) d\mu_s(y) \geq \eta_3(x, t).
\end{equation}

**Proof.** Let $\zeta$ be a smooth cut-off function such that $0 \leq \zeta \leq 1$, $\zeta = 1$ on $B(1)$ and $\zeta = 0$ outside $B(2)$. For $\ell > 0$, consider the function $\zeta_\ell$ defined by $\zeta_\ell(y) = \zeta(y/\ell)$, and denote $\chi_\ell$ the function defined on $\mathbb{R}^d$ by
\begin{equation}
\chi_\ell(y) = \exp\left(-\frac{|y|^2}{4}\right) \zeta_\ell(y).
\end{equation}
We apply Lemma 3.24 to $\chi_\ell$, so that
\begin{equation}
\lim_{r \to 0} \left( \frac{1}{r^{d-2}} \int_{\mathbb{R}^d} \chi_\ell\left(\frac{y-x}{r}\right) d\mu_s(y) - \frac{1}{r^{d-2}} \int_{\mathbb{R}^d} \chi_\ell\left(\frac{y-x}{r}\right) d\mu_s(y) \right) = 0.
\end{equation}
On the other hand, we claim that, for $r < \frac{1}{2} \sqrt{\ell}$ and every $s \in [t-r^2, t]$,
\begin{equation}
\frac{1}{r^{d-2}} \int_{\mathbb{R}^d} \left( \exp\left(-\frac{|y-x|^2}{4r^2}\right) - \chi_\ell\left(\frac{y-x}{r}\right) \right) d\mu_s(y) \leq \frac{C(t)}{r^{d-2}} \exp\left(-\frac{\ell^2}{8}\right).
\end{equation}
(The implicit constant does not depend on $\ell$). Indeed, notice first that
\begin{align*}
\exp\left(-\frac{|y-x|^2}{4r^2}\right) - \chi_\ell\left(\frac{y-x}{r}\right) &= \exp\left(-\frac{|y-x|^2}{4r^2}\right) \left(1 - \zeta_\ell\left(\frac{y-x}{r}\right)\right) \\
&\leq \exp\left(-\frac{|y-x|^2}{8r^2}\right) \exp\left(-\frac{\ell^2}{8}\right).
\end{align*}
Second, due to the monotonicity formula, we have
\[
\frac{1}{(\sqrt{2r})^{d-2}} \int_{\mathbb{R}^d} \exp \left( -\frac{|y-x|^2}{8r^2} \right) d\mu^t(y) \leq \frac{1}{(s + 2r)^{\frac{d+2}{2}}} \int_{\mathbb{R}^d} \exp \left( -\frac{|y-x|^2}{4(s + 2r)^2} \right) d\mu^0(y) \leq \frac{C(s + 2r^2)}{(s + 2r)^{\frac{d+2}{2}}} f(2|x|),
\]
and as \(2s = t \geq \max(s, 2r^2)\), the claim \((3.38)\) follows. Observe that the right hand side of \((3.38)\) does not depend on \(r\), for \(r < \frac{1}{2} \sqrt{t}\). Combining \((3.37)\) and \((3.38)\), we conclude that
\[
\limsup_{r \to 0} \left( \frac{1}{r^{d-2}} \int_{\mathbb{R}^d} \exp \left( -\frac{|x-y|^2}{4r^2} \right) (d\mu^t - d\mu^{t-r})(y) \right) \leq \frac{C(t)}{t^{\frac{d+2}{2}}} f(2|x|) \exp \left( -\frac{t^2}{8} \right).
\]
Letting \(\ell \to +\infty\), the conclusion follows. \(\square\)

We are now in position to prove Proposition 3.20.

**Proof of Proposition 3.20.** For \((x, t) \in \Omega_\omega\), define
\[
\check{\Theta}_{d-2}(\mu^t, x) = \lim_{r \to 0} \frac{1}{r^{d-2}} \int_{\mathbb{R}^d} \exp \left( -\frac{|x-y|^2}{4r^2} \right) d\mu^t(y).
\]
In view of Corollary 3.25, \(\check{\Theta}_{d-2}(\mu^t, x)\) exists on \(\Omega_\omega\) and \((3.39)\)
\[
\check{\Theta}_{d-2}(\mu^t, x) = \Theta_{d-2}^p(\mu^t, (x, t)).
\]
If \((x, t) \notin \Sigma_\mu\), then \(\Theta_{d-2}^p(\mu^t, (x, t)) = 0\) so that \((3.32)\) is obviously verified. Therefore, we assume in the following that \((x, t) \in \Sigma_\mu \cap \Omega_\omega\). Arguing as for the claim in Corollary 3.25, we obtain
\[
\frac{1}{(\ell r)^{d-2}} \int_{B(x, \ell r)} \frac{1}{r^{d-2}} \int_{\mathbb{R}^d} \exp \left( -\frac{|x-y|^2}{4r^2} \right) d\mu^t - K C(t) \frac{t^{\frac{d+2}{2}}}{t^{\frac{d}{2}}} f(2|x|) \exp \left( -\frac{t^2}{8} \right).
\]
Hence, letting \(r \to 0\), and by \((3.39)\),
\[
\Theta_{d-2}(\mu^t, x) \geq \frac{K}{\ell^{d-2}} \left( \Theta_{d-2}^p(\mu^t, (x, t)) - \ell^{d-2} \exp \left( -\frac{t^2}{8} \right) C(t) \frac{t^{\frac{d+2}{2}}}{t^{\frac{d}{2}}} f(2|x|) \right).
\]
In order to obtain \((3.32)\), we invoke the fact that on \(\Sigma_\mu\), \(\Theta_{d-2}^p \geq \eta_3(t)\). We choose \(\ell = \ell(x, t)\) sufficiently large so that
\[
K \ell^{d-2} \exp \left( -\frac{t^2}{8} \right) C(t) \frac{t^{\frac{d+2}{2}}}{t^{\frac{d}{2}}} f(2|x|) \leq \frac{1}{2} \eta_3(x, t) \leq \frac{1}{2} \Theta_{d-2}^p(\mu^t, (x, t)).
\]
Going back to \((3.40)\), with this choice of \(\ell\), we obtain
\[
\Theta_{d-2}(\mu^t, x) \geq \frac{K}{2\ell^{d-2}} \Theta_{d-2}^p(\mu^t, (x, t))
\]
so that inequality \((3.32)\) is true for all \((x, t) \in \Omega_\omega\). Choose \(\kappa(x, t) = \frac{K}{2 \ell^d(x, t)^{d-2}}\), then Corollary 3.23 allows to conclude. \(\square\)

3.4.5. \(\Sigma^t_\mu\) is \((d - 2)\)-rectifiable. Now, we have to deal with the diffuse part, and different kinds of arguments could then lead to regularity for \(\Sigma^t_\mu\). One way is to prove the existence of the density \(\Theta_{d-2}\) and then to invoke Preiss' regularity Theorem [28]. More precisely, we have:

**Proposition 3.26.** For almost every \(t > 0\),
\[
\Theta_{d-2}(\mu^t, x) = \Theta_{d-2}^t(\mu^t, x) \geq \eta(x, t) := \kappa(x, t) \eta_3(x, t) \quad \mathcal{H}^{d-2} \text{-a.e } x \in \Sigma^t_\mu.
\]
As a consequence, for almost every \(t > 0\), the set \(\Sigma^t_\mu\) is \((d - 2)\)-rectifiable.
Proof. The proof is as in [8, Proof of Proposition 8, p.155]. For the convenience of the reader, we provide a brief sketch.

Fix \((x, t) \in \Omega_\omega\). Denote \(L^\infty_c([0, +\infty), \mathbb{R})\) the set of bounded, real valued functions with compact support; for \(r > 0\) and \(g \in L^\infty([0, +\infty), \mathbb{R})\), we define

\[
I_r(g) = \frac{1}{r^{d-2}} \int g \left( \frac{|y-x|}{r} \right) d\mu_r(y).
\]

Consider the set

\[
F = \left\{ g \in L^\infty([0, +\infty), \mathbb{R}) : I(g) := \lim_{r \to 0} I_r(g) \text{ exists and is finite} \right\}.
\]

Then \(F\) is a vector space, and the equality of upper and lower \((d-2)\)-dimensional densities is equivalent to \(I_{[0,1]} \in F\).

Observe that \(F\) is stable by scaling \(g \mapsto g_\alpha\) where \(g_\alpha(x) = g(\alpha x)\) for \(\alpha > 0\) and that

\[
I(g_\alpha) = I(g) \alpha^{\frac{d-2}{2}}.
\]

The starting point is Corollary \(3.25\). It shows that \(x \mapsto e^{-ax^2}\) belongs to \(F\). By repetitively differentiating \((3.41)\) with respect to \(\alpha\), one can then infer that for any \(k \in \mathbb{N}\), \(\psi_k : x \mapsto x^k e^{-x^2}\) belongs to \(F\).

Denote \(V_m = \text{Span}(\psi_k, k = 0, \ldots, m)\), and \(P_m\) the \(L^2\) projection on \(V_m\).

Using the Hermite polynomials, one can prove that for \(f \in \mathcal{C}^2_c(\mathbb{R})\),

\[
\|f - P_m(f)\|_{L^1} \leq \frac{C}{\sqrt{m}} (\|f''\|_{L^2} + \|x^2 f\|_{L^2}).
\]

From there, if \(g \in \mathcal{C}^2_c([0, +\infty))\) satisfies \(g'(0) = 0\), we can symmetrize it into \(\bar{g} \in \mathcal{C}^2_c(\mathbb{R})\), and an approximation argument shows that \(g \in F\).

Then one approximates \(I_{[0,1]}\) by the sequence of functions \(g_n \in \mathcal{C}^2_c([0, +\infty))\) defined as follows.

Let \(\chi \in \mathcal{C}^\infty(\mathbb{R})\) be non increasing, and such that \(\chi(x) = 1\) for \(x \leq -1/2\) and \(\chi(x) = 0\) for \(x \geq 0\); we set for \(x \geq 0\), \(g_n(x) = \chi(n(x - 1))\). This allows to prove that \(I_{[0,1]} \in F\), as desired.

Finally, we invoke Corollary \(3.21\) for the lower bound.

\(\square\)

3.5. Convergence in modulus outside \(\Sigma_{\mu}\).

End of the proof of Theorem \(1.1\). We can now complete the proof of Theorem \(1.1\). We gather the results obtained in Propositions \(3.19\) and \(3.26\). Only two points are left to clarify.

The first point is that the function \(\Phi_{e}\) which appears in Theorem \(1.1\) is global, whereas up to now the function \(\Phi_{e}\) constructed in Theorem \(3.13\) was only locally defined. Let us sketch how to overcome this problem, we refer to [8, Part II, Section 4, p. 131] for more details.

Fix \((x_0, t_0) \notin \Sigma_{\mu}\). Recall that \(\Phi_{e}\) is defined on a neighborhood \(\Lambda(x_0, t_0)\) of \((x_0, t_0)\) as the limit of

\[
\Phi_{e} = \frac{1}{\sqrt{|\ln e|}} \Phi_{e}.
\]

Now denote \(\tilde{\Phi}_{e,m}\) the phase provided by Theorem \(2.7\) on the compact \(K_m = B(0, m) \times [1/m, m]\). Up to changing \(\tilde{\Phi}_{e,m}\) by a constant, we can also assume that

\[
\forall e > 0, \forall m, \quad \tilde{\Phi}_{e,m}(x_0, t_0) = \Phi_{e}(x_0, t_0).
\]

The bounds given in Theorem \(2.7\) (and parabolic regularization) show the existence of \(\Phi_{e}\) such that

\[
\frac{\tilde{\Phi}_{e,m}}{\sqrt{|\ln e|}} \to \Phi_{m} \quad \text{in } K_{m-1},
\]

so that \(\Phi_{m}\) also satisfies the heat equation on \(K_{m-1}\).
On the other hand, using the bounds in Theorem 2.5 (and also in the remark that follows) on the gradient, one can prove that
\[
\left\| \frac{\nabla \Phi_{\epsilon, m}}{\sqrt{|\ln \epsilon|}} - \frac{\hat{\nabla} \Phi_{\epsilon}}{\sqrt{|\ln \epsilon|}} \right\|_{L^p(\Lambda(x_0, t_0)_{\epsilon})} \to 0 \quad \text{as} \ \epsilon \to 0.
\]

Therefore, for \( m \) so large that \( \Lambda(x_0, t_0)_{\epsilon} \subset K_{m-1} \),
\[
\nabla \Phi_{\epsilon, m} = \nabla \Phi_{\epsilon} \quad \text{on} \ \Lambda(x_0, t_0)_{\epsilon}.
\]

It follows from there and (3.42) that for \( m \) large enough, all the \( \Phi_m \) coincide on \( \Lambda(x_0, t_0)_{\epsilon} \). By analyticity, for all \( m_0 \) large enough, the \( (\Phi_m)_{m \geq m_0} \) coincide on \( K_{m_0-1} \). Letting \( m_0 \to +\infty \), we denote \( \Phi_* \) their common value. By construction, it satisfies all the requirements of Theorem 1.1.

The second and last remaining point is to show that \( |u_\epsilon(x, t)| \to 1 \) uniformly on every compact subset \( K \subset \mathbb{R}^d \times (0, +\infty) \setminus \Sigma_{\mu} \). As \( \Sigma_{\mu} \) is closed, up to doing finite reunions, it suffices to prove it for a family of cylinders (which cover \( \mathbb{R}^d \times (0, +\infty) \setminus \Sigma_{\mu} \)). Therefore, we can furthermore assume that there exist \( \delta > 0, K_1 \subset \mathbb{R}^d \) convex and \( K_2 \subset (0, +\infty) \) a compact interval such that
\[
\{(x, t) : d((x, t), K) \leq \delta \} \subset K_1 \times K_2 \subset \mathbb{R}^d \times (0, +\infty) \setminus \Sigma_{\mu}.
\]

Denote \( \mathcal{X} = K_1 \times K_2 \). We have, for every \( t \in K_2 \),
\[
\int_{K_1} \frac{e_\epsilon (u_\epsilon(x, t))}{|\ln \epsilon|} \, dx = \int_{K_1} |\nabla \Phi_*|^2 \, d\mathcal{X} \leq C(\mathcal{X}).
\]

Hence
\[
\int_{K_1} (1 - |u_\epsilon|^2)^2(x, t) \, dx \leq C(\mathcal{X}) (\ln \epsilon),
\]
and
\[
\int_{K_1} |\nabla u_\epsilon|^2(x, t) \, dx \leq C(\mathcal{X}) |\ln \epsilon|.
\]

Let
\[
\mathcal{A} = \left\{(x, t) \in K : \forall \epsilon > 0, \ \ 1 - |u_\epsilon(x, t)|^2 \geq \kappa_1(\epsilon) \right\},
\]
where \( \kappa_1(\epsilon) > 0 \) will be determined later.
We want to show that \( \mathcal{A} \) is empty, and argue by contradiction. If not, let \( (x, t) \in \mathcal{A} \). We try to find \( \gamma(\epsilon) \) such that
\[
\forall y \in B(x, \gamma(\epsilon)), \quad |1 - |u_\epsilon(y)|^2| \geq \frac{\kappa_1(\epsilon)}{2}.
\]

We have
\[
|u_\epsilon(x, t) - u_\epsilon(y, t)| \leq \int_y^x |\nabla u_\epsilon(t)| \leq \|\nabla u_\epsilon(t)\|_{L^2} \sqrt{|y - x|} \leq C(\mathcal{X}) \sqrt{|\ln \epsilon|} \sqrt{\gamma(\epsilon)},
\]
and
\[
|u_\epsilon(y, t)|^2 - 1 = (|u_\epsilon(y, t)| - 1)(|u_\epsilon(y, t)| + 1) \leq 2(|u_\epsilon(y, t)| - 1) \leq 2 \left( |u_\epsilon(x, t) - u_\epsilon(y, t)| + |u_\epsilon(x, t)| - 1 \right) \leq 2 \left( C(\mathcal{X}) \sqrt{|\ln \epsilon|} \sqrt{\gamma} + |u_\epsilon(x, t)| - 1 \right).
\]

Since
\[
1 - |u_\epsilon(x, t)| \geq \frac{\kappa_1(\epsilon)}{1 + |u_\epsilon(x, t)|} \geq \frac{1}{2} \kappa_1(\epsilon),
\]
we deduce that
\[
|u_\epsilon(y, t)|^2 - 1 \leq 2C(\mathcal{X}) \sqrt{|\ln \epsilon|} \sqrt{\gamma} - \kappa_1(\epsilon).
\]
So we choose \( \gamma(\epsilon) = \left( \frac{\kappa_1(\epsilon)}{4C(\mathcal{K})} \right)^2 \frac{1}{|\ln \epsilon|} \). Then, assuming that \( B(x, \gamma) \subset K_1 \), we have

\[
\int_{B(x, \gamma)} (1 - |u_\tau(y, t)|^2) d\gamma \geq \int_{B(x, \gamma)} \left( \frac{\kappa_1(\epsilon)}{2} \right)^2 \geq \kappa_1(\epsilon)^2 \frac{\omega_d}{4} \gamma(\epsilon)^d \\
\geq \kappa_1(\epsilon)^{2d+2} \frac{1}{|\ln \epsilon|^d} \frac{\omega_d}{4} C(\mathcal{K})^d.
\]

Thus,

\[[3.43] C(\mathcal{K})^2 |\ln \epsilon| \geq \kappa_1(\epsilon)^{2d+2} \frac{1}{|\ln \epsilon|^d} \frac{\omega_d}{4} C(\mathcal{K})^d.
\]

This inequality leads us to define

\[ \kappa_1(\epsilon) = 2 \left( \frac{4^{d+2} C(\mathcal{K})^{d+1}}{\omega_d} \epsilon^2 |\ln \epsilon|^{d+1} \right)^{\frac{1}{d+2}} = 8 \sqrt{\frac{C(\mathcal{K})}{\omega_d} \epsilon \sqrt{|\ln \epsilon|}}. \]

Observe that in this case, \( \gamma(\epsilon) \to 0 \) as \( \epsilon \to 0 \).

Let \( \epsilon_0 > 0 \) be so small that for \( \epsilon \in (0, \epsilon_0) \), \( \gamma(\epsilon) < \delta \). Then \( B(x, \gamma(\epsilon)) \subset K_1 \), so that the computation [3.43] above holds: this is a contradiction. This proves that for \( \epsilon < \epsilon_0 \), the set \( A \) is empty. Arguing similarly, we obtain that the set

\[ B = \{ (x, t) \in K : \forall \epsilon > 0, \quad 1 - |u_\tau(x, t)|^2 \leq -\kappa_1(\epsilon) \} \]

is also empty. Therefore, for \( \epsilon < \epsilon_0 \),

\[ \|1 - |u_\tau|^2\|_{L^\infty(K)} \leq \kappa_1(\epsilon) \leq C(\mathcal{K}) \epsilon \sqrt{|\ln \epsilon|}, \]

which means that \( |u_\tau(x, t)| \to 1 \) uniformly on \( K \).

The proof of Theorem 1.1 is complete. \( \square \)

4. Evolution of the limiting density

Our goal in this section is to provide a proof for Theorem 1.2.

4.1. Mean curvature flows. Let us start by recalling some definitions on the mean curvature flow.

4.1.1. The classical notion. Let \( \Sigma \) be a smooth compact manifold of dimension \( n \), and \( \gamma_0 : \Sigma \to \mathbb{R}^d \) (\( n \gg k \)) be a smooth embedding, so that \( \Sigma^0 = \gamma_0(\Sigma) \) is a smooth \( k \)-dimensional submanifold of \( \mathbb{R}^d \).

The mean curvature vector at the point \( x \) of \( \Sigma^0 \) is the vector of the orthogonal space \((T_\Sigma \Sigma^0)^\perp\) given by

\[[4.1] \hat{H}_{\Sigma^0}(x) = -\frac{d-k}{a=1} \left( \sum_{j=1}^k \tau_j \cdot \frac{\partial \gamma^a}{\partial \tau_j} \right) \gamma^a = -\sum_{a=1}^{d-k} (\text{div}_{T_\Sigma \Sigma^0} \gamma^a) \gamma^a, \]

where \((\tau_1, \ldots, \tau_k)\) is an orthonormal moving frame on \( T_\Sigma \Sigma^0 \), \((v_1, \ldots, v^{d-k})\) is an orthonormal moving frame on \((T_\Sigma \Sigma^0)^\perp\), and \(\text{div}_{T_\Sigma \Sigma^0}\) denotes the tangential divergence at point \( x \). The integral formulation of [4.1] is given by

\[[4.2] \int_{\Sigma^0} \text{div}_{T_\Sigma \Sigma^0} \vec{X} d\mathcal{H}^k = -\int_{\Sigma^0} \hat{H}_{\Sigma^0} \cdot \vec{X} d\mathcal{H}^k, \]

for all \( \vec{X} \in \mathcal{D}(\mathbb{R}^d, \mathbb{R}^d) \).

Next, we introduce a time dependence, and consider a smooth family \( \{\gamma_t\}_{t \in I} \) of smooth embeddings of \( \Sigma \) in \( \mathbb{R}^d \), where \( I \) denotes some open interval containing 0. We set \( \Sigma^t = \gamma_t(\Sigma) \). If \( \chi \) is a smooth compactly supported function on \( \mathbb{R}^d \), a standard computation shows that
(4.3) \[ \frac{d}{dt} \int_\Sigma \chi(x) d\mathcal{H}^k = \int_\Sigma -\chi(x) \vec{H}_\Sigma(x) \cdot \vec{Y}(x) d\mathcal{H}^k, \]

where \( \vec{Y}(x) = \frac{d}{ds} \gamma_i(y_t^{-1}(x)) \) is the velocity vector at point \( x \), and \( P \) denotes the orthogonal projection on \((T_x, \Sigma)^\perp \).

The family \((\Sigma^t)_{t \in I}\) is moved by mean curvature in the classical sense if and only if, for all \( m \in \Sigma \) and \( t \in I \),

(4.4) \[ \frac{d}{dt} \gamma_i(m) = \vec{H}_\Sigma(y_t(m)). \]

In particular, if \((\Sigma^t)_{t \in I}\) is moved by mean curvature, (4.3) becomes

(4.5) \[ \frac{d}{dt} \int_\Sigma \chi(x) d\mathcal{H}^k = -\int_\Sigma \chi(x)|\vec{H}_\Sigma(x)|^2 d\mathcal{H}^k + \int_\Sigma \nabla\chi(x) \cdot \vec{H}_\Sigma(x) d\mathcal{H}^k. \]

Now, \( \chi \) can be chosen arbitrarily, so that (4.5) is actually equivalent to (4.4).

4.1.2. Brakke’s flows. In the attempt to extend (4.4) or (4.5) to a larger class of (less regular) objects, and in particular to extend the flow past singularities, Brakke [13] relaxed equality in (4.5), and considered instead sub-solutions, i.e. verifying the inequality

(4.6) \[ \frac{d}{dt} \int_\Sigma \chi(x) d\mathcal{H}^k \leq -\int_\Sigma \chi(x)|\vec{H}_\Sigma(x)|^2 d\mathcal{H}^k + \int_\Sigma \nabla\chi(x) \cdot \vec{H}_\Sigma(x) d\mathcal{H}^k, \]

for all non-negative \( \chi \in \mathcal{D}(\mathbb{R}^d) \). Following Brakke [13], we are thus going to extend (4.6) to less regular objects than smooth embedded manifolds. Actually, we adopt the point of view of Ilmanen [19], which is slightly different from Brakke’s original one.

Recall that a Radon measure \( \nu \) on \( \mathbb{R}^d \) is said to be \( k \)-rectifiable if there exists a \( k \)-rectifiable set \( \Sigma \), and a density function \( \Theta \in L^1_{loc}(\mathcal{H}^k \setminus \Sigma) \) such that

\[ \nu = \Theta \mathcal{H}^k \setminus \Sigma. \]

Since \( \Sigma \) is rectifiable, for \( \mathcal{H}^k \)-a.e. \( x \in \Sigma \), there exist a unique tangent space \( T_x \Sigma \) belonging to the Grassmanian \( G_{n,k} \). The distributional first variation of \( \nu \) is the vector-valued distribution \( \delta \nu \) defined by

(4.7) \[ \delta \nu(\vec{X}) = \int_\Sigma \text{div}_{T_x \Sigma} \vec{X} d\nu \quad \text{for all } \vec{X} \in D(\mathbb{R}^d, \mathbb{R}^d). \]

In the case when the measure \( |\delta \nu| \) is absolutely continuous with respect to \( \nu \), we say that \( \nu \) has a first variation and we may write

\[ \delta \nu = \vec{H} \nu, \]

where \( \vec{H} \) is the Radon-Nikodym derivative of \( \delta \nu \) with respect to \( \nu \). In this setting, formula (4.7) becomes

(4.8) \[ \int_\Sigma \text{div}_{T_x \Sigma} \vec{X} d\nu = -\int_\Sigma \vec{H} \cdot \vec{X} d\nu. \]

We are now in position to give the precise definition of a Brakke flow. Let \((\nu_t)_{t \geq 0}\) be a family of Radon measures on \( \mathbb{R}^d \). For \( \chi \in C^2_c(\mathbb{R}^d, [0, +\infty)) \), we define

(4.9) \[ \bar{D}_t \nu^\phi(x) = \limsup_{t \to t_0} \frac{\nu^\phi(x) - \nu^\phi_0(x)}{t - t_0}. \]
Proposition 4.2. For all \( T \in \mathbb{R} \), the energy (this will be handled thanks to Theorem 1.1). Second, since (4.12) involves the quadratic term \( \sigma \), and for all \( t \),

\[
(4.10) \quad \hat{D}_t \nu^t(\chi) \leq \mathcal{B}(\nu^t, \chi),
\]

for every \( \chi \in \mathcal{D}(\mathbb{R}^d, [0, +\infty)) \) and for all \( t > 0 \).

4.2. Mean curvature flow in Brakke’s formulation for \( \nu^t \).

4.2.1. Relating \( \nu^t \) to mean curvature flow. The starting point of the analysis is the formal analogy of equality (4.5), namely

\[
\frac{d}{dt} \int_{\Sigma^t} \chi(x) d\mathcal{H}^k = -\int_{\Sigma^t} \chi(x) |\bar{H}(x)|^2 d\mathcal{H}^k + \int_{\Sigma^t} \nabla \chi(x) \cdot \bar{H}(x) d\mathcal{H}^k,
\]

with the evolution of local energies for (4.10) (see equation (3.5))

\[
(4.11) \quad \frac{d}{dt} \int_{\mathbb{R}^d} \chi(x) d\mu^t_e = -\int_{\mathbb{R}^d \times \{t\}} \chi(x) \frac{|\partial_t u_x|^2}{|\ln \epsilon|} dx + \int_{\mathbb{R}^d \times \{t\}} \nabla \chi(x) \left(\frac{-\partial_t u_x \cdot \nabla u_x}{|\ln \epsilon|}\right) dx.
\]

We already know that as \( \epsilon \to 0 \), \( \mu^t_e \to \mu^t \). Therefore, the comparison of the two formulas suggests, at least formally, that in the limit

\[
(4.12) \quad \omega^t_e := \frac{|\partial_t u_x|^2}{|\ln \epsilon|} dx \to |\bar{H}|^2 d\mu^t,
\]

and

\[
(4.13) \quad \sigma^t_e := \frac{-\partial_t u_x \cdot \nabla u_x}{|\ln \epsilon|} dx \to \bar{H} d\mu^t.
\]

Actually, this is a little over optimistic for two reasons. First, we have to deal with the diffuse part of the energy (this will be handled thanks to Theorem 1.1). Second, since (4.12) involves the quadratic term \( |\bar{H}|^2 \), only lower semi-continuity can be expected at first sight.

4.2.2. Convergence of \( \sigma^t_e \) and decomposition of the limit. In this section, we use the following estimate of the time derivative \( \partial_t u_x \), which requires some calculations, whereas it was straightforward in [8].

Proposition 4.2. For all \( T > 0 \) and \( R > 0 \), there holds

\[
\frac{1}{|\ln \epsilon|} \int_{B(0,R) \times [0,T]} |\partial_t u_x|^2 \leq C(T,R).
\]

Proof. By Lemma 3.4, we have for all \( \chi \in C^\infty_c(\mathbb{R}^d) \),

\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^d} \chi(x) d\mu^t_e = -\int_{\mathbb{R}^d \times \{t\}} \chi(x) \frac{|\partial_t u_x|^2}{|\ln \epsilon|} dx + \int_{\mathbb{R}^d \times \{t\}} \frac{D^2 \chi \nabla u_x \cdot \nabla u_x - \Delta \chi e_x(u_x)}{|\ln \epsilon|} dx.
\]
So the measure \( \sigma \) is bounded in \( L^\infty(\mathbb{R}^d) \) and \( \mu \) is locally bounded on \( \mathbb{R}^d \times [0, T] \) uniformly in \( \varepsilon > 0 \). Hence, passing to a further subsequence, we may assume that \( \sigma \rightarrow \sigma_* \) as measures. The Radon-Nikodym derivative of \( \sigma \) with respect to \( \mu \) verifies

\[
\frac{d|\sigma|}{d\mu} \leq \frac{|\partial_t u_\varepsilon|}{e_\varepsilon(u_\varepsilon)}.
\]

Consider the measure \( \sigma_\varepsilon = \sigma'_\varepsilon dt \) defined on \( \mathbb{R}^d \times [0, +\infty) \). By Cauchy-Schwarz inequality with \( (2.23) \) and \( (2.27) \), for every \( T > 0 \), \( \sigma_\varepsilon \) is locally bounded on \( \mathbb{R}^d \times [0, T] \) uniformly in \( \varepsilon > 0 \). Hence, passing to a further subsequence, we may assume that \( \sigma_\varepsilon \rightarrow \sigma_* \) as measures. The Radon-Nikodym derivative of \( \sigma_\varepsilon \) with respect to \( \mu_\varepsilon \) verifies

\[
\frac{d|\sigma_\varepsilon|}{d\mu_\varepsilon} \leq \frac{|\partial_t u_\varepsilon|}{e_\varepsilon(u_\varepsilon)}.
\]

Now, let \( T > 0 \) and \( R > 0 \) be fixed. We have

\[
\frac{1}{|\ln \varepsilon|} \int_{B(R)} [\partial_t u_\varepsilon]^2 \leq \int_{\mathbb{R}^d} \chi(x) [\partial_t u_\varepsilon]^2 \frac{1}{|\ln \varepsilon|} dx \leq C(T, R).
\]

Let's deal with the two last terms. Due to \( (H_2(F)) \), we have

\[
\int_{\mathbb{R}^d} \chi(x) d\mu_\varepsilon^0 \leq C(R).
\]

Finally,

\[
\int_{\mathbb{R}^d} \chi(x) d\mu_\varepsilon^r \geq 0.
\]

So

\[
\frac{1}{|\ln \varepsilon|} \int_{B(R)} [\partial_t u_\varepsilon]^2 \leq \int_{\mathbb{R}^d} \chi(x) [\partial_t u_\varepsilon]^2 \frac{1}{|\ln \varepsilon|} dx \leq C(T, R). \quad \square
\]

So that \( \frac{d|\sigma_\varepsilon|}{d\mu_\varepsilon} \) is bounded in \( L^2(B(0, R) \times [0, T], d\mu_\varepsilon) \) uniformly in \( \varepsilon > 0 \). Since it is true for all \( T > 0 \) and \( R > 0 \), it follows that \( \sigma_* \) is absolutely continuous with respect to \( \mu_\varepsilon \) (see \([1]\) Remark 2.2) for further details. Therefore, we may write

\[
\sigma_* = \tilde{h} \mu'_\varepsilon dt,
\]

where \( \tilde{h} \in L^1(\mathbb{R}^d \times [0, T], \mu'_\varepsilon dt) \) and satisfies the bound

\[
||\tilde{h}||_{L^1(\mathbb{R}^d \times [0, T], \mu'_\varepsilon dt)} \leq C(T, R).
\]

Arguing as in Theorem\([1.1]\) and its proof, we infer

**Lemma 4.3.** The measure \( \sigma_* \) can be decomposed as \( \sigma_* = \sigma'_t dt \), where for a.e. \( t > 0 \),

\[
\sigma'_t = -\partial_t \Phi_\varepsilon \cdot \nabla \Phi_\varepsilon dx + \tilde{h}_* v_\varepsilon.
\]
4.2.3. Mean curvature of $\nu_\epsilon^t$. The next step will be to identify the restriction of $\tilde{h}$ on $\Sigma_\epsilon^t$ with the mean curvature defined by (4.8), that is:

**Proposition 4.4.** For $t > 0$ a.e., $\nu_\epsilon^t$ has a first variation and

$$\delta \nu_\epsilon^t = \tilde{h} \nu_\epsilon^t,$$

i.e. $\tilde{h}$ is the mean curvature of $\nu_\epsilon^t$.

**Proof.** Notice that we already know by Theorem 1.1 that $\nu_\epsilon^t$ is $(d-2)$-rectifiable for a.e. $t > 0$. The starting point is formula (3.5). Indeed, let $X \in \mathcal{D}(\mathbb{R}^d, \mathbb{R}^d)$. Then for all $t > 0$,

$$\int_{\mathbb{R}^d} \frac{1}{|\ln \epsilon|} \left( e_\epsilon(u_\epsilon) \delta_{ij} - \frac{\partial u_\epsilon}{\partial x_i} \frac{\partial u_\epsilon}{\partial x_j} \right) \frac{\partial X_i}{\partial x_j} dx = - \int_{\mathbb{R}^d} \tilde{X} \cdot \sigma_\epsilon^t.$$  

(4.17)

Formula (4.17) is already very close to (4.8), in particular the right hand side. In order to handle the diffuse energy, as well as to interpret the left-hand side as a tangential divergence, we need to analyse the weak-limit of the stress-energy density tensor

$$\alpha_\epsilon^t := A_\epsilon dx = \left( \text{Id} - \frac{\nabla u_\epsilon \otimes \nabla u_\epsilon}{e_\epsilon(u_\epsilon)} \right) d\mu_\epsilon^t.$$

(Recall that the stress-energy matrix $A_\epsilon$ was defined in (3.6).) Clearly, $|\alpha_\epsilon^t| \leq K d\mu_\epsilon^t$, and we may assume that

$$\alpha_\epsilon^t \rightharpoonup \alpha_*^t := A d\mu_*^t,$$

where $A$ is a $d \times d$ symmetric matrix. Since the symmetric matrix $\nabla u_\epsilon \otimes \nabla u_\epsilon$ is non-negative, we have

$$A \leq \text{Id}.$$  

On the other hand,

$$\text{Tr}(e_\epsilon(u_\epsilon) \text{Id} - \nabla u_\epsilon \otimes \nabla u_\epsilon) = (d-2)e_\epsilon(u_\epsilon) + d V_\epsilon(u_\epsilon).$$

Therefore, as the trace is a linear operation, we may take the limit $\epsilon \to 0$ and obtain

$$\text{Tr}(A) = (d-2) + d \frac{d V_\epsilon}{d\mu_*^t},$$

(4.18)

where the non-negative measure $V_\epsilon$ is the limit (up to a possibly further subsequence) of $V_\epsilon(u_\epsilon)/|\ln \epsilon|$. Going to the limit in (4.17), and using the decomposition in Theorem 1.1, we obtain for $t \geq 0$ a.e.,

$$\int_{\mathbb{R}^d} A_\epsilon^t \frac{\partial X_i}{\partial x_j} d\nu_\epsilon^t + \int_{\mathbb{R}^d} \left( \frac{|\nabla \Phi_*|^2}{2} \delta_{ij} - \frac{\partial \Phi_*}{\partial x_i} \frac{\partial \Phi_*}{\partial x_j} \right) \frac{\partial X_i}{\partial x_j} dx

= - \int_{\mathbb{R}^d} \tilde{X} \cdot \tilde{h} d\nu_\epsilon^t - \int_{\mathbb{R}^d} \tilde{X} \cdot \nabla \Phi_* \partial_t \Phi_* dx.$$  

(4.19)

On the other hand, $\Phi_*$ verifies the heat equation

$$\frac{\partial \Phi_*}{\partial t} - \Delta \Phi_* = 0.$$  

(4.20)

Multiplying (4.20) by $\tilde{X} \cdot \nabla \Phi_*$, we obtain

$$\int_{\mathbb{R}^d} \frac{\partial \Phi_*}{\partial t} \frac{\partial \Phi_*}{\partial x_j} dx = - \int_{\mathbb{R}^d} \tilde{X} \cdot \nabla \Phi_* \partial_t \Phi_* dx.$$  

(4.21)

Combining (4.19) and (4.21) we have therefore proved

**Lemma 4.5.** For $t > 0$ a.e., and for every $\tilde{X} \in \mathcal{D}(\mathbb{R}^d, \mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} A_\epsilon^t \frac{\partial X_i}{\partial x_j} d\nu_\epsilon^t = \int_{\mathbb{R}^d} \tilde{X} \cdot \tilde{h} d\nu_\epsilon^t.$$  

(4.22)
Recall that we already know that $\Sigma^t_\mu$ is rectifiable for a.e. $t \geq 0$. Comparing (4.22) with (4.8) in order to identify $\tilde{h}$ with the mean curvature of $\nu^t$, we merely have to prove that the matrix $A$ corresponds to the orthogonal projection $P$ onto the tangent space $T_x \Sigma^t_\mu$. We first have

**Lemma 4.6** ([8] Lemma 6). For $t > 0$ a.e.,

$$A(x) \left( \int_{T_x \Sigma^t_\mu} \nabla \chi(y) d\mathcal{H}^{d-2}(y) \right) = 0 \quad \text{for } \mathcal{H}^{d-2} \text{-a.e. } x \in \Sigma^t_\mu,$$

and for all $\vec{X} \in \mathcal{D}(\mathbb{R}^d, \mathbb{R})$.

A straightforward consequence is

**Corollary 4.7.** For $t$ and $x$ as in Lemma 4.6,

$$\left( T_x \Sigma^t_\mu \right) ^\perp \subseteq \ker A(x).$$

With a little more elementary linear algebra, we further deduce

**Corollary 4.8** ([8, Corollary 4]). For $t$ and $x$ as in Lemma 4.6 $A = P$ is the orthogonal projection onto the tangent space $T_x \Sigma^t_\mu$.

Gathering (4.22) and Corollary 4.8 proves the proposition. \hfill \Box

**Remark 4.9.** Corollaries 4.7 and 4.8 have many important consequences.

1. Using (4.18), we deduce that $\frac{dV_\nu}{d\nu} = 0$, i.e. there is only kinetic energy in the limit.

2. Let $(\tau_1, \ldots, \tau_n)$ be an orthonormal frame such that $T_x \Sigma^t_\mu = \text{Span}(\tau_3, \ldots, \tau_n)$. In view of the expression of the stress-energy tensor in these coordinates, we infer that the energy concentrates in the $(\tau_1, \tau_2)$ plane (i.e. $(T_x \Sigma^t_\mu)^\perp$) and uniformly with respect to the direction.

In particular, since $\sigma^t_s$ is colinear to $\nabla u_s$, this suggests strongly that $\tilde{h}$ is perpendicular to $T_x \Sigma^t_\mu$. Such an argument is made rigorous in [11, Proposition 6.2].

### 4.2.4. Semi-continuity of $\omega^t_s$.

It solely remains to show (4.10) to complete the proof of Theorem 1.2. We now prove that for a.e. $t \geq 0$,

$$\liminf_{\varepsilon \to 0} \int_{\mathbb{R}^d \times \{t\}} \chi \frac{|\nabla u_\varepsilon|^2}{|\ln \varepsilon|} \geq \int_{\mathbb{R}^d \times \{t\}} \chi |\tilde{h}|^2 d\nu_s + \int_{\mathbb{R}^d \times \{t\}} \chi |\partial_t \Phi_s|^2 d\lambda.

We recast the problem in the framework of Young measures, which turns out to be an appropriate concept to analyse the energies of the oscillations. In this direction, denote $p_\varepsilon = -\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$, and consider the measure (defined on $\mathbb{R}^d \times \mathbb{R}^{2d}$)

$$\tilde{\omega}_s^t = \delta_{p_\varepsilon(s)} \frac{|\partial_t u_\varepsilon - p_\varepsilon|^2}{|\ln \varepsilon|} d\lambda.$$

Extracting possibly a further subsequence, we may assume that $\tilde{\omega}_s^t d\lambda \to \tilde{\omega}_s$ as measures. Arguing as in Theorem 1.1 and its proof once more,

**Lemma 4.10.** The measure $\tilde{\omega}_s$ decomposes as $\tilde{\omega}_s = \tilde{\omega}_s^t d\lambda$, and for $t \geq 0$ a.e.

$$\tilde{\omega}_s^t = \Pi^t_\varepsilon(p) |\partial_t \Phi_s|^2 d\lambda + \mathcal{M}_s,$$

where $\Pi^t_\varepsilon(p)$ is a measure on $\mathbb{R}^{2d}$ (with support on the unit ball) and $\mathcal{M}_s = \tilde{\omega}_s^t \Sigma^t_\mu$. Moreover, $\Pi^t_\varepsilon(p)(\mathbb{R}^{2d}) = 1$.

The main ingredient that we will borrow directly from the analysis by Ambrosio and Soner [11] can be formulated as follows.
Proposition 4.11 ([1] Section 6). For \( t \geq 0 \) a.e., and every \( \chi \in \mathcal{D}(\mathbb{R}^d) \),

\[
\int_{\mathbb{R}^d \times \mathbb{R}^m} \chi(x) \mathcal{M}^*_\chi(x, p) \geq \int_{\mathbb{R}^d} |\hat{h}|^2 \, dv.
\]

At this stage, we are in position to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. In view of [1] Theorem 4.4, it suffices to establish the integral version of (4.10). Let \( 0 < T_0 < T_1 \). We integrate (4.11) on \([T_0, T_1]\), and let \( \epsilon \) go to zero. Combining the results of Lemma 4.3, Proposition 4.4, Lemma 4.10, Proposition 4.11, Theorem 1.1, we obtain

\[
\int_{\mathbb{R}^d \times \{T_1\}} \chi \left| \nabla \Phi \right|^2 \, dx - \int_{\mathbb{R}^d \times \{T_0\}} \chi \left| \nabla \Phi \right|^2 \, dx 
\leq - \int_{\mathbb{R}^d \times \{T_0, T_1\}} \chi |\hat{h}|^2 \, dv + \int_{\mathbb{R}^d \times \{T_0, T_1\}} \nabla \chi P(\hat{h}) \, dv
\]

(4.24)

\[
- \int_{\mathbb{R}^d \times \{T_0, T_1\}} \chi |\partial_\nu \Phi|^2 \, dx \, dt + \int_{\mathbb{R}^d \times \{T_0, T_1\}} \nabla \chi \nabla \Phi \partial_t \Phi.
\]

Since \( \Phi \) verifies the heat equation, we have the identity

(4.25) \[
\int_{\mathbb{R}^d \times \{T_1\}} \chi \left| \nabla \Phi \right|^2 \, dx - \int_{\mathbb{R}^d \times \{T_0\}} \chi \left| \nabla \Phi \right|^2 \, dx 
= \int_{\mathbb{R}^d \times \{T_0, T_1\}} \chi |\partial_\nu \Phi|^2 \, dx \, dt + \int_{\mathbb{R}^d \times \{T_0, T_1\}} \nabla \chi \nabla \Phi \partial_t \Phi.
\]

Combining (4.24) and (4.25) we obtain

\[
\nu^{T_1}_\epsilon(\chi) - \nu^{T_0}_\epsilon(\chi) \leq - \int_{\mathbb{R}^d \times \{T_0, T_1\}} \chi |\hat{h}|^2 \, dv + \int_{\mathbb{R}^d \times \{T_0, T_1\}} \nabla \chi P(\hat{h}) \, dv.
\]

As mentioned above, this integral formulation implies (4.10), under suitable assumptions which are fulfilled here, namely rectifiability of \( \Sigma^\epsilon \), lower bounds on the density \( \Theta \), and orthogonality of the mean curvature \( \hat{h} \) with \( (T, \Sigma^\epsilon) \). The proof of Theorem 1.2 is complete. \( \square \)

5. Point vortices in two space dimensions

Our goal in this last section is to prove Theorem 1.3. We start with the derivation of some pointwise estimates, which actually hold in any dimension \( d \geq 2 \), and refine the estimates of Theorem 2.5. Indeed, we need a careful analysis on the set where \( |u_\epsilon| \) is far from zero. For this purpose, we consider, for \( T > 0, \Delta T > 0, R > 0 \) given, the cylinder

\[
\Lambda = B(x_0, R) \times [T, T + \Delta T] \subset \mathbb{R}^2 \times [0, +\infty),
\]

and we assume that for some constant \( 0 < \sigma < \frac{1}{2} \),

\[
(5.1) \quad |u_\epsilon| \geq 1 - \sigma \quad \text{on} \; \Lambda.
\]

In particular, we may write

\[
u_\epsilon = \rho_\epsilon \exp(i \phi_\epsilon) \quad \text{on} \; \Lambda,
\]

where \( \rho_\epsilon = |u_\epsilon| \) and where \( \phi_\epsilon \) is a smooth real-valued map on \( \Lambda \). Set

\[
\Lambda_\alpha = B(x_0, \alpha R) \times [T + (1 - \alpha^2) \Delta T, T + \Delta T].
\]

The following higher-order regularity for \( u_\epsilon \) holds.
Theorem 5.1. Assume that (5.1) holds. There exist constants $0 < \sigma_0 \leq \frac{1}{2}$ and $0 < \alpha, \beta < 1$ depending only on the dimension $d$, such that if $\sigma < \sigma_0$, then

\begin{align*}
(5.2) & \quad \|\nabla \Phi_\varepsilon\|_{L^\infty(\Lambda_1)} \leq C(\Lambda) \sqrt{|\ln \varepsilon|}, \\
(5.3) & \quad \|1 - \rho_\varepsilon\|_{L^\infty(\Lambda_1)} \leq C(\Lambda) \varepsilon^2 \left(1 + \|\nabla \Phi_\varepsilon\|_{L^\infty(\Lambda_1)}^2\right), \\
(5.4) & \quad \|\partial_t \rho_\varepsilon\|_{W^{1,\infty}(\Lambda_1)} + \|\nabla \rho_\varepsilon\|_{W^{1,\infty}(\Lambda_1)} \leq C(\Lambda) \varepsilon^\beta.
\end{align*}

In addition, there exists a real-valued function $\Phi_\varepsilon$ defined on $\Lambda_1$, and satisfying the heat equation, such that

\begin{align*}
(5.5) & \quad \|\partial_t \Phi_\varepsilon - \partial_t \Phi_\varepsilon\|_{W^{1,\infty}(\Lambda_1)} + \|\nabla \Phi_\varepsilon - \nabla \Phi_\varepsilon\|_{W^{1,\infty}(\Lambda_1)} \leq C(\Lambda) \varepsilon^\beta.
\end{align*}

Proof. The proof is similar to the one given in [9 Theorem 2.1] since we have on $\Lambda$ the same bounds on the energy $\int_\Lambda (|\partial_t u_\varepsilon|^2 + e_\varepsilon(u_\varepsilon)) dx \leq C(\Lambda) |\ln \varepsilon|$. It relies in particular (and improves) on Theorem 2.5.

From now on and throughout the rest of this section, we work in dimension $d = 2$. In order to prove Theorem 1.3, let us first recall Theorem 1.1. In dimension 2, it asserts that $\Sigma^\mu_\varepsilon$ has locally finite $\mathcal{H}^0$ measure, i.e. $\Sigma^\mu_\varepsilon$ is a discrete set. In particular, we can write

$$
\Sigma^\mu_\varepsilon = \{b_i(t) : i \in I\}, \quad \text{where } I \text{ is finite or } I = \mathbb{N}.
$$

Also, $\Sigma_\mu = \bigcup_{t > 0} \Sigma^\mu_\varepsilon$ is a closed set in $\mathbb{R}^2 \times (0, +\infty)$ and $|u_\varepsilon| \rightarrow 1$ locally uniformly on $\mathbb{R}^2 \times (0, +\infty) \setminus \Sigma_\mu$. Moreover, a.e. $t \geq 0$,

$$
\mu^\varepsilon_i = \frac{\|\nabla \Phi_\varepsilon\|^2}{2} (\cdot, t) dx + \nu^\varepsilon_i, \quad \text{where } \nu^\varepsilon_i = \sum_{i \in I} \sigma_i(t) \delta_{b_i(t)},
$$

where the function $\Phi_\varepsilon$ satisfies the heat equation on $\mathbb{R}^2 \times (0, +\infty)$.

Off the singular set $\Sigma^\mu_\varepsilon$, the main contribution to the time derivative $\partial_t u_\varepsilon$ stems from the phase $\Phi_\varepsilon$. In this direction, the following proposition is motivated by Lemma 3.2.

Proposition 5.2. We have, as $\varepsilon \rightarrow 0$,

$$
\frac{|\partial_t u_\varepsilon|^2}{|\ln \varepsilon|} \rightarrow |\partial_t \Phi_\varepsilon|^2 \quad \text{in } \mathcal{H}_0^0(\mathbb{R}^2 \times (0, +\infty) \setminus \Sigma_\mu),
$$

$$
\frac{\partial_t u_\varepsilon \cdot \nabla u_\varepsilon}{|\ln \varepsilon|} \rightarrow \partial_t \Phi_\varepsilon \cdot \nabla \Phi_\varepsilon \quad \text{in } \mathcal{H}_0^0(\mathbb{R}^2 \times (0, +\infty) \setminus \Sigma_\mu).
$$

Proof. As $\Sigma_\mu$ is open, it suffices to show uniform convergence on small cylinders (which cover $\mathbb{R}^2 \times (0, +\infty) \setminus \Sigma_\mu$). This is then an immediate consequence of Theorem 5.1 which shows that

$$
\partial_t u_\varepsilon = i \exp(i \phi_\varepsilon) \partial_t \Phi_\varepsilon + O(\varepsilon^\beta), \quad \text{and} \quad \nabla u_\varepsilon = i \exp(i \phi_\varepsilon) \nabla \Phi_\varepsilon + O(\varepsilon^\beta).
$$

We now need to establish some asymptotics for the measures

$$
\frac{|\partial_t u_\varepsilon|^2}{|\ln \varepsilon|} \, dx \, dt \quad \text{and} \quad \frac{\partial_t u_\varepsilon \cdot \nabla u_\varepsilon}{|\ln \varepsilon|} \, dx \, dt.
$$

For the first one, it suffices to have the inequality

\begin{align*}
(5.6) & \quad \liminf_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^2 \times (0, +\infty)} \frac{|\partial_t u_\varepsilon|^2}{|\ln \varepsilon|} \chi(x) \, dx \, dt \geq \int_{\mathbb{R}^2 \times (0, +\infty)} |\partial_t \Phi_\varepsilon|^2 \chi(x) \, dx \, dt,
\end{align*}

where $\chi$ is a test function.
which is a straightforward consequence of Proposition 5.2. For the second one, we need to be a little bit more careful. We have

**Lemma 5.3.** Extracting possibly a further subsequence,

\[
\sigma_\varepsilon := -\frac{\partial_t u_\varepsilon \cdot \nabla u_\varepsilon}{|\ln \varepsilon|} \, dx \, dt \rightarrow \sigma_* := -\partial_t \Phi_* \cdot \nabla \Phi_* \, dx \, dt + h \, \nu_*,
\]

weakly as measures on \( \mathbb{R}^2 \times [0, +\infty) \), where \( \nu_* = \nu'_* \, dt = \mu_* \cdot \Sigma_\mu \) and \( h \in L^2(\nu_*) \).

**Proof.** Let \( R > 0 \), and \( T > 0 \). We have

\[
(5.8) \quad \int_{\overline{B(0,R) \times [0,T]}} \frac{d \sigma_\varepsilon}{|\ln \varepsilon|} \leq \left( \int_{\overline{B(0,R) \times [0,T]}} \frac{|\partial_t u_\varepsilon|^2}{|\ln \varepsilon|} \right)^{\frac{1}{2}} \left( \int_{\overline{B(0,R) \times [0,T]}} \frac{|
abla u_\varepsilon|^2}{|\ln \varepsilon|} \right)^{\frac{1}{2}} \leq C(T,R).
\]

In view of (5.8), and by an argument of diagonal extraction, we see that there exists a Radon measure, defined on \( \mathbb{R}^2 \times [0, +\infty) \), bounded on compacts, such that, up to a subsequence,

\[
\sigma_\varepsilon \rightarrow \sigma_* \quad \text{as measures in} \quad \mathbb{R}^2 \times [0, +\infty).
\]

We claim that \( \sigma_* \) is absolutely continuous with respect to \( \mu_* \). In order to prove this, we compute the Radon-Nikodym derivative of \( \sigma_\varepsilon \) with respect to \( \mu_\varepsilon \), obtaining

\[
(5.9) \quad \frac{d \sigma_\varepsilon}{d \mu_\varepsilon} \leq \frac{|\partial_t u_\varepsilon| \cdot |\nabla u_\varepsilon|}{e_\varepsilon(u_\varepsilon)},
\]

and therefore for all \( T > 0 \) and \( R > 0 \),

\[
(5.10) \quad \int_{\overline{B(0,R) \times [0,T]}} \left| \frac{d \sigma_\varepsilon}{d \mu_\varepsilon} \right|^2 \leq \int_{\overline{B(0,R) \times [0,T]}} \frac{|\partial_t u_\varepsilon|^2}{|\ln \varepsilon|} \, dx \, dt \leq C(T,R).
\]

Invoking a result of Reshetnyak [29] as in [9], the claim is proved.

It follows from Proposition 5.2 that on \( \mathbb{R}^2 \times [0, +\infty) \setminus \Sigma_\mu \), \( \sigma_* = -\partial_t \Phi_* \cdot \nabla \Phi_* \, dx \, dt \) and the conclusion follows. \( \square \)

In the same spirit, we have

**Lemma 5.4.** Extracting possibly a further subsequence,

\[
A_\varepsilon \quad \frac{d \sigma_\varepsilon}{d \mu_\varepsilon} \rightarrow A_* =: T(\Phi_*) \, dx \, dt + B \, \nu_*
\]

weakly as measures on \( \mathbb{R}^2 \times [0, +\infty) \), where \( T \) is defined in (3.7) and \( B \in L^\infty(\nu_*) \).

The proof is identical to the proof of Lemma 5.3. The next result expresses the fact that the points have "zero mean curvature".

**Proposition 5.5.** The vector \( h \) and the matrix \( B \) given above are identically equal to zero.

**Proof.** We rely again on equation (3.5). Let \( \vec{X} \in \mathcal{D}(\mathbb{R}^2, \mathbb{R}^2) \) be a smooth, time independent, vector field, we integrate (3.5) on the time interval \([T_1, T_2]\) for \( 0 < T_1 < T_2 \). Taking the limit \( \varepsilon \rightarrow 0 \), we get

\[
\int_{\mathbb{R}^2 \times [T_1, T_2]} (A_\varepsilon)_{ij} \partial_j X_i = \int_{\mathbb{R}^2 \times [T_1, T_2]} \vec{X} \cdot \sigma_\varepsilon.
\]

From (5.11), and as \( \Phi_* \) verifies the heat equation, we infer

\[
\int_{\mathbb{R}^2 \times [T_1, T_2]} B_{ij} \partial_j X_i \, d \nu_* = \int_{\mathbb{R}^2 \times [T_1, T_2]} \vec{X} \cdot h \, d \nu_*,
\]
Therefore letting $T_1, T_2 \to t$, we infer that a.e $t \geq 0$,
\[
\int_{\mathbb{R}^2} B_i(t) \partial_i X_i dx = \int_{\mathbb{R}^2} \vec{X} \cdot h d v_*^t.
\]
Now the support of $v_*^t$ is a discrete set: by using cut-off vector fields, we see that $h = 0$, and from there, $B = 0$. \hfill \square

\textbf{Proof of Theorem 1.3}. We claim that for any function $\chi \geq 0$ compactly supported on $\mathbb{R}^2$, we have for a.e. $t > 0$,
\[
(5.12) \quad \frac{d}{dt} \int_{\mathbb{R}^2} \chi d v_*^t \leq 0.
\]
Indeed, passing to the limit in \eqref{3.4} and using \eqref{5.6}, Lemma 5.3, Lemma 5.4 and Proposition 5.5, we obtain
\[
(5.13) \quad \frac{d}{dt} \int_{\mathbb{R}^2} \frac{|\nabla \Phi_*|^2}{2} \chi dx + \frac{d}{dt} \int_{\mathbb{R}^2} \chi d v_*^t \leq -\int_{\mathbb{R}^2} (|\partial_t \Phi_*|^2 \chi - \partial_t \Phi_* \cdot \nabla \Phi_* \cdot \nabla \chi) dx.
\]
On the other hand, since $\Phi_*$ solves the heat equation, we have
\[
\frac{d}{dt} \int_{\mathbb{R}^2} \frac{|\nabla \Phi_*|^2}{2} \chi dx = \int_{\mathbb{R}^2} \nabla (\partial_t \Phi_*) \cdot \nabla \Phi_* \chi = -\int_{\mathbb{R}^2} (|\partial_t \Phi_*|^2 \chi - \partial_t \Phi_* \cdot \nabla \Phi_* \cdot \nabla \chi),
\]
so that \eqref{5.12} follows. We deduce that
\[
(5.14) \quad v_*^t \leq v_*^0 \quad \text{for any } 0 < t_0 \leq t_1.
\]
It then follows as an easy consequence that for $0 < t_0 \leq t_1$, $\Sigma_*^{t_1} \subset \Sigma_*^{t_0}$. Let $\Sigma_*^0 := \bigcup_{k \geq 1} \Sigma_*^{1/k}$, then $\Sigma_*^0$ is countable, and we can enumerate it $\Sigma_*^0 = \{ b_i : i \in \mathbb{N} \}$. Then for all $t > 0$, there exist an injective function $\rho(t) : \mathbb{N} \to \mathbb{R}$ such that $b_i(t) = b_{\rho(i)}(t)$, and (up to permuting the $(b_i(t))$,) we can assume without loss of generality that $\rho(t, i) = i$. This means that the $b_i(t)$ do not move.

Fix $x \in \mathbb{R}^2$. As Card($\Sigma_*^0 \cap B(x, 1)$) is uniformly bounded as $t \to 0$ due to \eqref{3.28} for $d = 2$, we infer that $\Sigma_*^0 \cap B(x, 1)$ is a finite set. As a consequence, $\Sigma_*^0$ is a discrete set.

Finally, the other consequence of \eqref{5.14} is that the $\sigma_*^i(t)$ are non increasing. \hfill \square

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