Bulk viscosity of superfluid hyperon stars

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Abstract

We calculated bulk viscosity due to non-equilibrium weak processes in superfluid nucleon-hyperon matter of neutron stars. For that, the dissipative relativistic hydrodynamics, formulated in paper [1] for superfluid mixtures, was extended to the case when both nucleons and hyperons are superfluid. It was demonstrated that in the most general case (when neutrons, protons, Λ, and Σ− hyperons are superfluid), non-equilibrium weak processes generate sixteen bulk viscosity coefficients, with only three of them being independent. In addition, we corrected an inaccuracy in a widely used formula for the bulk viscosity of non-superfluid nucleon-hyperon matter.

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I. INTRODUCTION

It is well known (see, e.g., [2, 3, 4, 5]), that neutron stars can be unstable with respect to the emission of gravitational waves. The matter in a pulsating neutron star is not (even locally) in chemical equilibrium. The relaxation towards chemical equilibrium is accompanied by the dissipation of pulsation energy. This process is one of the most important dissipative processes, suppressing the growth of gravitational-wave instability. It can be described by the introduction of an effective bulk viscosity in the hydrodynamic equations (see, e.g., [6]).

The bulk viscosity due to non-equilibrium weak processes in neutron stars was calculated by a number of authors (for a review see, e.g., [7]). It strongly depends on the composition of stellar matter. In this paper we consider the matter of inner layers of neutron stars, composed of electrons, muons, neutrons, protons, Λ, and Σ− hyperons (nucleon-hyperon matter). A calculation of the bulk viscosity for nucleon-hyperon matter is complicated by the fact that baryons in such matter can be superfluid [8, 9, 10, 11].

The bulk viscosity of superfluid matter was calculated in a number of papers (see, e.g., [12, 13, 14, 15, 16, 17]). In these papers only one bulk viscosity coefficient was studied, analogous to that in non-superfluid hydrodynamics. The effects of superfluidity were taken into account only in calculating the reaction rates. However, it is known that there are several bulk viscosity coefficients in hydrodynamics of superfluid liquid [1, 6, 18, 19].

In a recent paper [1], four bulk viscosity coefficients were calculated for the matter composed of superfluid neutrons, superfluid protons and electrons. It was shown that taking into account three additional bulk viscosity coefficients results in a significant decrease of characteristic damping times of sound modes (approximately, by a factor of three).

In this paper we extend the results of Ref. [1] to the case of nucleon-hyperon matter. In particular, we demonstrate how the dissipative hydrodynamics [1] should be modified to describe a possible presence of superfluid hyperons. Next, we show that in the most general case, when baryons of any species are superfluid, non-equilibrium processes of mutual transformations of particles generate sixteen bulk viscosity coefficients, only three of them being independent. In addition, we correct an inaccuracy in the expression for the bulk viscosity of non-superfluid nucleon-hyperon matter made in Ref. [15] and spreaded widely in the literature (see, e.g., [20, 21, 22, 23, 24, 25]). We calculate the bulk viscosity correctly and compare our results with those of Ref. [15].
The paper is organized as follows. In section II we calculate the bulk viscosity of non-superfluid nucleon-hyperon matter. In section III we calculate and analyze all sixteen bulk viscosity coefficients describing dissipation in superfluid nucleon-hyperon mixtures; the relations between these coefficients are also discussed. Section IV presents a summary.

II. BULK VISCOSITY OF NON-SUPERFLUID HYPERON MATTER

In this section we derive an expression for the bulk viscosity due to non-equilibrium processes of particle transformations in a dense non-superfluid matter composed of electrons (e), muons (μ), neutrons (n), protons (p), and hyperons (Λ and Σ− hyperons). Here and below the variation δA of some physical quantity A will be defined as the difference A − A0, where A0 is the value of A in thermodynamic equilibrium (when matter is unperturbed).

The most effective weak processes in nucleon-hyperon matter are the following non-leptonic reactions [14, 15, 26, 27]

\[ n + n \leftrightarrow p + \Sigma^-, \]  
\[ n + p \leftrightarrow p + \Lambda, \]  
\[ n + n \leftrightarrow n + \Lambda, \]  
\[ n + \Lambda \leftrightarrow \Lambda + \Lambda. \]  

The full thermodynamic equilibrium implies the equilibrium with respect to these reactions,

\[ 2\mu_{n0} - \mu_{p0} - \mu_{\Sigma0} = 0, \]  
\[ \mu_{n0} - \mu_{\Lambda0} = 0. \]

Here μ\textsubscript{i0} are the chemical potentials of particle species \( i = n, p, \Lambda, \Sigma \) taken at equilibrium (correspondingly, μ\textsubscript{i} are the chemical potentials in the perturbed matter). Notice that, the equilibrium conditions for reactions (2), (3), and (4) coincide.

Leptonic reactions (e.g., direct and modified Urca processes with electrons or muons) are much slower in comparison to the reactions (1)–(4). For ‘typical’ perturbation frequencies (e.g., \( 10^3 - 10^4 \) s\(^{-1} \) for radial modes or for r-modes of rapidly rotating neutron stars) the leptonic reactions cannot influence substantially the chemical composition of perturbed matter. Hence, the main contribution to the bulk viscosity comes from the non-leptonic reactions
In addition to the processes described above, there is a fast non-leptonic reaction due to the strong interaction of baryons

\[ n + \Lambda \leftrightarrow p + \Sigma^- \]  

(7)

In accordance with Ref. \[15\] we assume that the perturbed matter is always in equilibrium with respect to this reaction,

\[ \delta \mu_{\text{fast}} \equiv \mu_n + \mu_\Lambda - \mu_p - \mu_\Sigma = 0. \]  

(8)

Let us obtain the expression for the bulk viscosity of non-superfluid nucleon-hyperon matter. For that, we consider a pulsating nucleon-hyperon matter, slightly perturbed from an equilibrium state (so that one can use the linear perturbation theory). If the reactions (1)–(4) are forbidden, then pulsations are reversible and there is no energy dissipation (notice that, the reaction (7) is open). We denote the pressure in this case by \( P_{\text{eq}} \). The presence of the reactions (1)–(4) in the pulsating matter leads to a difference between the real pressure \( P \) and \( P_{\text{eq}} \). We define the bulk viscosity \( \xi \) by the formula

\[ P - P_{\text{eq}} \equiv -\xi \text{ div}(\mathbf{u}), \]  

(9)

where \( \mathbf{u} \) is the hydrodynamic velocity of pulsations. Notice that, this definition differs from the usually accepted one (see, e.g., Ref. \[15\]). Usually, instead of \( P_{\text{eq}} \) in formula (9) it is common to substitute the pressure which would be established in the pulsating matter assuming that there is an equilibrium with respect to all the reactions (i.e. the reactions are very fast). Both these approaches are possible.

Generally, the pressure \( P \) and the other thermodynamic quantities depend on six parameters, for example, the number densities \( n_j \), where \( j = n, p, \Lambda, \Sigma, e, \mu \) (one can neglect the dependence on temperature in strongly degenerate neutron-star matter, see, e.g., Refs. \[1, 28, 29\]). However, these parameters are not all independent because in the nucleon-hyperon matter two conditions should be satisfied: the equilibrium condition (8) with respect to the reaction (7) and the condition of quasineutrality,

\[ n_p = n_e + n_\mu + n_\Sigma. \]  

(10)

Taking into account that the reactions (1)–(4) and (7) conserve the number of leptons and the leptonic processes are neglected, we obtain that the relative number densities of leptons...
\[ x_e \equiv n_e/\varepsilon_b \text{ and } x_\mu \equiv n_\mu/\varepsilon_b \] (\(\varepsilon_b \equiv n_n + n_p + n_\Lambda + n_\Sigma\) is the baryon number density) remain constant during the pulsations,

\[ \delta x_e = \delta x_\mu = 0. \] (11)

This result is valid only for non-superfluid matter and follows from the continuity equations for baryons, electrons, and muons,

\[ \frac{\partial \delta n_b}{\partial t} + \text{div}(n_b \mathbf{u}) = 0, \] (12)

\[ \frac{\partial \delta n_e}{\partial t} + \text{div}(n_e \mathbf{u}) = 0, \] (13)

\[ \frac{\partial \delta n_\mu}{\partial t} + \text{div}(n_\mu \mathbf{u}) = 0. \] (14)

If baryons of any species \(n, p, \Lambda\) and/or \(\Sigma\) are superfluid, then the continuity equation for baryons (12) should be modified (see Section 3) and Eq. (11) does not hold.

In view of Eqs. (8) and (10), pressure is a function of only four independent variables, say, \(n_b, n_H, x_e,\) and \(x_\mu\) (\(n_H \equiv n_\Lambda + n_\Sigma\) is the hyperon number density). Expanding \(P(n_b, n_H, x_e, x_\mu)\) in the Taylor series near the equilibrium state, one obtains for the variation of pressure \(\delta P\),

\[ \delta P = \frac{\partial P(n_b, n_H, x_e, x_\mu)}{\partial n_b} \delta n_b + \frac{\partial P(n_b, n_H, x_e, x_\mu)}{\partial n_H} \delta n_H, \] (15)

where we used Eq. (11). The variations \(\delta n_b\) and \(\delta n_H\) can be found from the continuity equations for baryons (12) and hyperons,

\[ \frac{\partial \delta n_H}{\partial t} + \text{div}(n_H \mathbf{u}) = \Delta \Gamma_1 + \Delta \Gamma_2 + \Delta \Gamma_3 + \Delta \Gamma_4. \] (16)

Here \(\Delta \Gamma_1, \Delta \Gamma_2, \Delta \Gamma_3,\) and \(\Delta \Gamma_4\) are the net numbers of hyperons generated in a unit volume per unit time in reactions (1), (2), (3), and (4), respectively. If deviation from the equilibrium state is small, the sources \(\Delta \Gamma_l\) \((l=1, \ldots, 4)\) can be expressed as (see, e.g., [12, 13, 14])

\[ \Delta \Gamma_l = \lambda_l \delta \mu_l, \] (17)

where \(\lambda_l\) are the ‘reaction rates’, some functions of number densities and temperature; \(\delta \mu_1 \equiv 2\mu_n - \mu_p - \mu_\Sigma, \delta \mu_2 = \delta \mu_3 = \delta \mu_4 \equiv \mu_n - \mu_\Lambda\) are the chemical potential disbalances for the reactions (1), (2), (3), and (4), respectively. Taking into account the equilibrium condition (8) for the fast reaction (7), one has: \(\delta \mu_1 = \delta \mu_2 = \delta \mu_3 = \delta \mu_4 \equiv \delta \mu.\)
Notice that, there is no source in the equation (16) owing to the fast reaction (7), because this reaction does not change the number of hyperons. The choice of another variable instead of \( n_H \) (for example, the neutron number density \( n_n \)) would make it necessary to take into account the source due to the reaction (7). In the paper of Lindblom and Owen [15] (and in the subsequent papers [20, 21, 22, 23, 24, 25]) the number density of neutrons was chosen as such variable, but the source of neutrons owing to the fast reaction (7) was neglected. This leads to an error in the expression for the bulk viscosity.

In fact, one could think that the source of neutrons due to the fast reaction (7) equals zero, because the matter, as we mentioned before, is in equilibrium with respect to this reaction. However, this is not quite true, because even small (negligible in all other situations) deviation from the equilibrium \( \delta \mu_{\text{fast}} = \mu_n + \mu_\Lambda - \mu_p - \mu_\Sigma \) multiplied by the large reaction rate \( \lambda_{\text{fast}} \) results in a finite (non-zero) source \( \Delta \Gamma_{\text{fast}} = \lambda_{\text{fast}} \delta \mu_{\text{fast}} \). This fact was emphasized by Jones [26].

Now let us assume that the perturbation of matter is periodic, so that all the thermodynamic quantities oscillate near their equilibrium values with the frequency \( \omega \). Then one finds from the continuity equations (12) and (16)

\[
\delta n_b = \frac{n_b}{i \omega} \text{div}(\mathbf{u}), \quad \quad \quad (18)
\]

\[
\delta n_H = -\frac{1}{i \omega} [n_H \text{div}(\mathbf{u}) - \lambda \delta \mu], \quad \quad \quad (19)
\]

where \( \lambda \equiv \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \). As in the case of the pressure, the chemical potential disbalance \( \delta \mu \) in Eq. (19) is a function of \( n_b, n_H, x_e, \) and \( x_\mu \). In analogy with Eq. (15) for \( \delta P \), it can be expanded near the equilibrium state and written as

\[
\delta \mu = \frac{\partial \delta \mu(n_b, n_H, x_e, x_\mu)}{\partial n_b} \delta n_b + \frac{\partial \delta \mu(n_b, n_H, x_e, x_\mu)}{\partial n_H} \delta n_H. \quad \quad \quad (20)
\]

Here we take into account that \( \delta \mu = 0 \) in the full equilibrium and that the relative number densities \( x_e \) and \( x_\mu \) do not change in the course of pulsations (see Eq. (11)).

Now, solving the system of equations (18)–(20) and substituting the expressions for \( \delta n_b \) and \( \delta n_H \) into Eq. (15), we derive

\[
P - P_{eq} = \text{div}(\mathbf{u}) \left( \frac{\partial P(n_b, x_H, x_e, x_\mu)}{\partial x_H} \frac{\lambda}{\omega^2} \frac{\partial \delta \mu(n_b, x_H, x_e, x_\mu)}{\partial n_b} \right) \times \left[ \frac{i \lambda}{\omega n_b} \frac{\partial \delta \mu(n_b, x_H, x_e, x_\mu)}{\partial x_H} + 1 \right]^{-1}. \quad \quad \quad (21)
\]
Here the independent variables are \( n_b, x_H \equiv n_H/n_b, x_e, \) and \( x_\mu. \) It is easy to express the bulk viscosity \( \xi \) from this equation. We are mainly interested in the real part of \( \xi \) because it is \( \text{Re}(\xi) \) that is responsible for the energy dissipation (see, e.g., [12]). In this sense it is probably more appropriate to define \( \text{Re}(\xi) \) as the ‘real’ bulk viscosity. It equals

\[
\text{Re}\xi = -\frac{n_b^2}{\lambda} \frac{\partial P(n_b, x_H, x_e, x_\mu)}{\partial x_H} \frac{\partial \delta \mu(n_b, x_H, x_e, x_\mu)}{\partial n_b} \left[ \frac{\partial \delta \mu(n_b, x_H, x_e, x_\mu)}{\partial x_H} \right]^{-2} \frac{1}{1 + \omega^2 \tau^2}, \tag{22}
\]

where \( \tau \equiv n_b/\lambda \left[ \frac{\partial \delta \mu(n_b, x_H, x_e, x_\mu)}{\partial x_H} \right]^{-1}. \)

For comparison, we present here the result of Lindblom and Owen [15] (notice that, these authors neglected the reactions 3 and 4, thus assuming that \( \lambda_3 = \lambda_4 = 0) \n
\[
\text{Re}\xi_L = -\frac{n_b^2}{2\lambda_1 + \lambda_2} \frac{\partial P(n_b, x_n, x_e, x_\mu)}{\partial x_n} \frac{\partial \delta \mu(n_b, x_n, x_e, x_\mu)}{\partial n_b} \left[ \frac{\partial \delta \mu(n_b, x_n, x_e, x_\mu)}{\partial x_n} \right]^{-2} \frac{1}{1 + \omega^2 \tau_L^2}, \tag{23}
\]

where \( \tau_L \equiv n_b/(2\lambda_1 + \lambda_2) \left[ \frac{\partial \delta \mu(n_b, x_n, x_e, x_\mu)}{\partial x_n} \right]^{-1}; \) \( x_n \equiv n_n/n_b. \)

The bulk viscosity (22) depends on the reaction rates \( \lambda_1, \lambda_2, \lambda_3, \) and \( \lambda_4. \) Some of them were calculated in a number of papers [14, 15, 20, 26, 27]. Different authors present different results for the rates; the discussion of advantages and disadvantages of their calculations can also be found in those papers. Unfortunately, to date, there are no strict calculations of the reaction rates. It is reasonable to think that all the rates are of the same order of magnitude.

The dependence of the bulk viscosity on the baryon number density for the temperature \( T = 3 \times 10^9 \text{ K} \) and the oscillation frequency \( \omega = 10^4 \text{ s}^{-1} \) is presented in Fig. 1. While calculating the bulk viscosity, we used the third equation of state of Glendenning [30].

The solid line illustrates our results for the bulk viscosity obtained from Eq. (22). We employed the reaction rates from Ref. [15] and, following that paper, we put the rates of the reactions [3] and [4] equal zero, \( \lambda_3 = \lambda_4 = 0. \) The dot-dashed line is the bulk viscosity calculated as described in Ref. [15] (see also formula 23). We remind that in that paper the relative number density of neutrons \( x_n \) was chosen as one of the independent variables. However, the source of neutrons due to the fast reaction (7) was erroneously neglected. As one can see, this mistake does not influence the results significantly (typically, by \( \sim (10 - 30) \% \)). By the long dashes we show the results of Ref. [14]. In Ref. [14], only one hyperon reaction was taken into account, namely the reaction (1). As in Ref. [15], the source due to the fast reaction (7) was neglected and, in addition, the equilibrium condition (8) with respect to this reaction was ignored. Moreover, the authors of Ref. [14] used the non-relativistic approximation when calculating the rate of the reaction (1). This assumption is
FIG. 1: Bulk viscosity $\xi$ versus baryon number density $n_b$ at $T = 3 \times 10^9$ K and $\omega = 10^4$ s$^{-1}$ for non-superfluid matter. Solid, long-dashed, and dot-dashed lines show our results, the results of Ref. [14] and Ref. [15], respectively. Vertical dashes indicate the thresholds for (from left to right): appearance of muons; direct Urca processes involving electrons and muons, respectively; appearance of $\Lambda$ and $\Sigma^-$ hyperons, respectively. The inset demonstrates the difference between our calculations and those of Refs. [14] and [15] in more detail.

not well justified for the baryon number densities in the range $n_b \gtrsim (0.3 - 0.6)$ fm$^{-3}$ because of the strong dependence of the reaction rates on $n_b$ (see, e.g., Ref. [15]).

III. BULK VISCOSITY OF SUPERFLUID NUCLEON-HYPERON MATTER

In this section, unless otherwise is stated, the subscripts $i$ and $k$ refer to baryons ($i, k = n, p, \Lambda, \Sigma$). The summation is assumed over repeated baryon indices $i$ and $k$. The subscript $l$ refers to leptons ($l = e, \mu$); the subscript $j$ runs over all particle species ($j = n, p, \Lambda, \Sigma, e, \mu$).

A. The relativistic hydrodynamics of superfluid nucleon-hyperon mixture

In Ref. [1] the dissipative relativistic hydrodynamics of superfluid mixtures was formulated for $npe$ matter. Here we extend this hydrodynamics to the case of superfluid nucleon-hyperon matter composed of superfluid protons, neutrons, $\Lambda$ and $\Sigma^-$ hyperons, as well as
normal electrons and muons.

The general formulae (Eqs. 26–34 of Ref. [1]) describing the relativistic hydrodynamics of superfluid mixture remain valid with the notion that now the subscripts \( i \) and \( k \) refer not only to superfluid nucleons \( (i, k = n, p) \) but also to superfluid hyperons \( (i, k = n, p, \Lambda, \Sigma) \). For instance, the continuity equations for particle species \( j \) are written as

\[
\partial_\mu j^\mu_{(j)} = 0, \tag{24}
\]

with

\[
j^\mu_{(i)} = n_i u^\mu + Y_{ik} w^\mu_{(k)}, \quad j^\mu_{(l)} = n_l u^\mu. \tag{25}\]

Energy-momentum conservation law has the form

\[
\partial_\mu T^{\mu\nu} = 0, \tag{26}
\]

where

\[
T^{\mu\nu} = (P + \varepsilon) u^\mu u^\nu + P \eta^{\mu\nu} + Y_{ik} \left[w^\mu_{(i)} w^\nu_{(k)} + \mu_i w^\mu_{(k)} u^\nu + \mu_k w^\nu_{(i)} u^\mu\right] + \tau^{\mu\nu} \tag{27}
\]

and \( \tau^{\mu\nu} \) is the dissipative correction to the energy-momentum tensor which will be specified below. It satisfies the constraint

\[
u_\mu u_\nu \tau^{\mu\nu} = 0. \tag{28}\]

The hydrodynamic equations must be supplemented by the second law of thermodynamics

\[
d\varepsilon = T dS + \mu_i d n_i + \mu_e d n_e + \mu_\mu d n_\mu + \frac{Y_{ik}}{2} d \left[w^\mu_{(i)} w_{(k)\mu}\right]. \tag{29}\]

Using the quasineutrality condition (10) and the condition of equilibrium (8) with respect to the fast reaction (7), Eq. (29) can be rewritten as

\[
d\varepsilon = T dS + \mu_n d n_b - \delta \mu d n_H - (\mu_n - \mu_p - \mu_e) d n_e - (\mu_n - \mu_p - \mu_\mu) d n_\mu + \frac{Y_{ik}}{2} d \left[w^\mu_{(i)} w_{(k)\mu}\right], \tag{30}\]

where we remind the notation \( \delta \mu \equiv \mu_n - \mu_\lambda \). In full thermodynamic equilibrium the third, fourth, and the fifth terms are zero because of Eqs. (5) and (6), and of the beta-equilibrium conditions, \( \mu_n = \mu_p + \mu_e \) and \( \mu_n = \mu_p + \mu_\mu \) (see, e.g., Refs. [12, 13]).

In Eqs. (24)–(30) \( Y_{ik} \) is a \( 4 \times 4 \) symmetric matrix which is related in the non-relativistic limit to the entrainment matrix \( \rho_{ik} \) by the equality [1, 34]

\[
Y_{ik} = \rho_{ik}/(m_i m_k), \tag{34}
\]

where \( m_i \)
is the mass of a free baryon of species $i$ (the matrix $\rho_{ik}$ is a natural generalization of the superfluid density to the case of superfluid mixtures, see, e.g., Refs. \[31, 32, 33\]). To the best of our knowledge, the matrix $Y_{ik}$ has not been calculated for a nucleon-hyperon matter. Furthermore, $\varepsilon$ is the energy density; $\mu_j$ is the relativistic chemical potential of particle species $j$; and $S$ is the entropy density. The pressure $P$ in Eq. (27) is defined in the same way as for ordinary (non-superfluid) matter \[1, 34\],

$$
P = -\varepsilon + \mu_i n_i + \mu_e n_e + \mu_\mu n_\mu + TS.
$$

(31)

Next, $\eta^{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ in Eq. (27) is the special relativistic metric; $u^\mu$ is the four-velocity of normal (non-superfluid) liquid component normalized so that $u_\mu u^\mu = -1$ (we assume that all non-superfluid components move with the same velocity $u^\mu$). The four-vectors $w^\mu_{(i)}$ satisfy the condition

$$
u u_{(i)}^\mu = 0
$$

(32)

and describe motion of superfluid components. To take into account the potentiality of superfluid motion, a four-vector $w^\mu_{(i)}$ should be expressed through some scalar functions $\phi_i$ and written as (see Ref. \[1\])

$$
w^\mu_{(i)} = \partial^\mu \phi_i - q_i A^\mu - (\mu_i + \kappa_i) u^\mu.
$$

(33)

Here the scalar $\phi_i$ is related to the wave function phase of the Cooper-pair condensate $\Phi_i$ by the equality $\nabla \phi_i = \hbar \nabla \Phi_i / 2$; $A^\mu$ is the four-potential of the electromagnetic field; $q_i$ is the electric charge of particle species $i$; $\kappa_i$ is a small dissipative correction to be determined below.

Note that one can avoid the introduction of new functions $\phi_i$ in the hydrodynamics of superfluid mixtures if one formulates the potentiality condition \[33\] in the equivalent way

$$
\partial^\nu \left[ w^\mu_{(i)} + q_i A^\mu + (\mu_i + \kappa_i) u^\mu \right] = \partial^\mu \left[ w^\nu_{(i)} + q_i A^\nu + (\mu_i + \kappa_i) u^\nu \right].
$$

(34)

Below we will use the latter formulation because it is more suitable for our purpose. In this approach, four-vectors $w^\mu_{(i)}$ are treated as independent hydrodynamic variables.

The hydrodynamics discussed above would be incomplete without an indication what we mean by a *comoving frame*, that is the frame where we measure (and define) all the
thermodynamic quantities. As was demonstrated in Ref. [1], the condition (32) dictates that the comoving is the frame where the four-velocity \( u^\mu \) equals \( u^\mu = (1, 0, 0, 0) \). In this frame, the basic thermodynamic quantities \( \varepsilon, n_j, \) and \( w_{(i)} \) (or \( \nabla\phi_i \)) are defined by (see Eqs. (25), (27), (28), and (32))

\[
\begin{align*}
\dot{j}_j^0 &= n_j, \\
\dot{j}_i &= Y_{ik} w_{(k)} = Y_{ik} \nabla\phi_k, \\
T^{00} &= \varepsilon.
\end{align*}
\]

(35) \hspace{1cm} (36) \hspace{1cm} (37)

All other thermodynamic quantities can be considered as their functions or, equivalently, the functions of \( \varepsilon, n_j, \) and \( w_{(i)}^{\mu} w_{(k)\mu} \).

In analogy with Ref. [1], from Eqs. (24), (26), and (29) one can derive the entropy generation equation, which defines the dissipative corrections \( \tau^{\mu\nu} \) and \( \kappa_i \),

\[
\partial_\mu S^\mu = -\frac{\kappa_i}{T} \partial_\mu \left[ Y_{ik} w_{(k)}^{\mu} \right] - \tau^{\mu\nu} \partial_\mu \left( \frac{u_\nu}{T} \right) + \frac{1}{T^2} \partial_\mu T + u^\mu Y_{ik} w_{(k)}^{\mu} \frac{\kappa_i}{T} \partial_\nu u_\mu.
\]

(38)

Here \( S^\mu \) is the entropy current density,

\[
S^\mu = S u^\mu - \frac{u_\nu}{T} \tau^{\mu\nu} - \frac{\kappa_i}{T} Y_{ik} w_{(k)}^{\mu},
\]

(39)

satisfying the natural constraint \( u_\mu S^\mu = -S \). The last two terms in Eq. (38) are small and can be omitted in the majority of applications (for more details, see the discussion in Ref. [1]).

Using the requirement that the entropy does not decrease, one can obtain for the dissipative corrections (neglecting the last two terms in the right-hand side of Eq. 38)

\[
\tau^{\mu\nu} = \kappa \left( H^{\mu\gamma} u^\nu + H^{\nu\gamma} u^\mu \right) \left( \partial_\tau T + T u^\delta \partial_\delta u_\gamma \right) - \eta H^{\mu\gamma} H^{\nu\delta} \left( \partial_\delta u_\gamma + \partial_\gamma u_\delta - \frac{2}{3} \eta_{\gamma\delta} \partial_\varepsilon u_\varepsilon \right) - \xi_{1i} H^{\mu\nu} \partial_\gamma \left[ Y_{ik} w_{(k)}^\gamma \right] - \xi_{2} H^{\mu\nu} \partial_\gamma v^\gamma, \tag{40}
\]

\[
\kappa_n = -\xi_{3i} \partial_\mu \left[ Y_{ik} w_{(k)}^\mu \right] - \xi_{4n} \partial_\mu u^\mu, \tag{41}
\]

\[
\kappa_A = -\xi_{5i} \partial_\mu \left[ Y_{ik} w_{(k)}^\mu \right] - \xi_{4A} \partial_\mu u^\mu, \tag{42}
\]

\[
\kappa_\Sigma = -\xi_{6i} \partial_\mu \left[ Y_{ik} w_{(k)}^\mu \right] - \xi_{4\Sigma} \partial_\mu u^\mu, \tag{43}
\]

\[
\kappa_p = -\xi_{7i} \partial_\mu \left[ Y_{ik} w_{(k)}^\mu \right] - \xi_{4p} \partial_\mu u^\mu. \tag{44}
\]
Here $\kappa$ and $\eta$ are the thermal conductivity and shear viscosity coefficients, respectively; $H^{\mu\nu} \equiv \eta^{\mu\nu} + u^{\mu} u^{\nu}$ is the projection matrix; $\xi_{1i}, \xi_{2i}, \xi_{3i}, \xi_{4i}, \xi_{5i}, \xi_{6i},$ and $\xi_{7i}$ are twenty five bulk viscosity coefficients ($i = n, p, \Lambda, \Sigma$). We would like to emphasize that the dissipative corrections (40)–(44) are incomplete in a sense that they (i) do not include various terms related to particle diffusion; (ii) neglect (typically) small terms, explicitly depending on $w^{\mu}_{(i)}$. For example, we neglect the terms of the form $w^{\mu}_{(i)} \partial_{\mu} T$ and $u^{\mu} w^{\nu}_{(i)} \partial_{\gamma} u^{\gamma}$ in the expressions for $\w_{k}$ and $\tau^{\mu\nu}$, respectively. The similar approximation is very well known in the non-relativistic theory of superfluids (see, e.g., the textbook of Landau and Lifshitz [6], §140 or Ref. [1]). An inclusion of all these dissipative terms would lead to a number of kinetic coefficients much larger than 27. Since in this paper we are mainly interested in the bulk viscosity coefficients, we restrict ourselves to a simplified form (40)–(44) of dissipative corrections.

The number of bulk viscosity coefficients can be reduced. Notice that, from the quasineutrality condition (10) and the continuity equations (24), it follows that

$$\partial_{\mu} \left[ Y_{pk} w^{\mu}_{(k)} \right] = \partial_{\mu} \left[ Y_{\Sigma k} w^{\mu}_{(k)} \right].$$

(45)

A similar condition for superfluid $npe$ matter was derived in Ref. [1] (see Eq. 41 of that reference). Owing to the condition (45), there is no need to introduce the bulk viscosity coefficients both for protons and for $\Sigma^{-}$ hyperons. For example, it is sufficient to introduce the quantity $\xi_{1\Sigma p} \equiv \xi_{1p} + \xi_{1\Sigma}$ instead of $\xi_{1p}$ and $\xi_{1\Sigma}$. Because of the same reason, we are not interested in the quantities $\w_{\Sigma}$ and $\w_{p}$ taken separately. Instead, we will introduce the sum $\w_{\Sigma p} \equiv \w_{\Sigma} + \w_{p}$ (notice that, $\w_{\Sigma p}$ is the quantity that appears in the entropy generation equation 38). As a result, the corrections $\tau^{\mu\nu}$ and $\w_{q}$ take the form

$$\tau^{\mu\nu} = -\kappa \left( H^{\mu\gamma} u^{\nu} + H^{\nu\gamma} u^{\mu} \right) \left( \partial_{\gamma} T + T u^{\delta} \partial_{\delta} u_{\gamma} \right) - \eta H^{\mu\gamma} H^{\nu\delta} \left( \partial_{\delta} u_{\gamma} + \partial_{\gamma} u_{\delta} - \frac{2}{3} \eta_{\gamma\delta} \partial_{\varepsilon} u^{\varepsilon} \right) - \xi_{1q} H^{\mu\nu} \partial_{\gamma} \left[ Y_{qk} w^{\gamma}_{(k)} \right] - \xi_{2} H^{\mu\nu} \partial_{\gamma} u^{\gamma},$$

(46)

$$\w_{n} = -\xi_{3q} \partial_{\mu} \left[ Y_{qk} w^{\mu}_{(k)} \right] - \xi_{4n} \partial_{\mu} u^{\mu},$$

(47)

$$\w_{\Lambda} = -\xi_{5q} \partial_{\mu} \left[ Y_{qk} w^{\mu}_{(k)} \right] - \xi_{4\Lambda} \partial_{\mu} u^{\mu},$$

(48)

$$\w_{\Sigma p} = -\xi_{6q} \partial_{\mu} \left[ Y_{qk} w^{\mu}_{(k)} \right] - \xi_{4\Sigma p} \partial_{\mu} u^{\mu}.$$  

(49)

Here and below the subscript $q$ takes on the values $n, \Lambda, \Sigma p$. The expression $\partial_{\mu} \left[ Y_{\Sigma pk} w^{\mu}_{(k)} \right]$ in Eqs. (40)–(49) means $\partial_{\mu} \left[ Y_{\Sigma k} w^{\mu}_{(k)} \right]$ or, equivalently, $\partial_{\mu} \left[ Y_{pk} w^{\mu}_{(k)} \right]$.  

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As follows from the above equations, we have actually *sixteen* (rather than twenty five) bulk viscosity coefficients which can contribute to the dissipation of mechanical energy in superfluid nucleon-hyperon matter. Using the Onsager symmetry principle, we obtain

\[
\xi_{3\Lambda} = \xi_{5n}, \quad \xi_{3\Sigma p} = \xi_{6n}, \quad \xi_{4n} = \xi_{1n}, \quad \xi_{5\Sigma p} = \xi_{6\Lambda}, \quad \xi_{4\Lambda} = \xi_{1\Lambda}, \quad \xi_{4\Sigma p} = \xi_{1\Sigma p}.
\]  

(50)

Thus, generally, only ten of them are independent.

**B. Calculation of the bulk viscosity coefficients for superfluid nucleon-hyperon matter**

Let us calculate these phenomenological coefficients assuming they are due to non-equilibrium processes (1)–(4). As in Sec. II, we assume that the matter is slightly perturbed out of equilibrium state and pulsates with the frequency \(\omega\). Since the deviation from the equilibrium is small the hydrodynamic equations can be linearized. Because of the same reason, the dependence of various thermodynamic quantities (e.g., the pressure \(P\)) on the scalars \(w^\mu_{(i)} w_{(k)\mu}\) can be neglected. We assume that the four-vectors \(w^\mu_{(i)}\) characterizing the superfluid flow of particle species \(i = n, p, \Lambda, \) or \(\Sigma\) equal zero in the unperturbed matter. Thus, by perturbing the system, we produce some small \(w^\mu_{(i)}\) so that the scalars \(w^\mu_{(i)} w_{(k)\mu}\) will be of the second-order smallness and can be omitted in the linear approximation.

We start from the non-dissipative relativistic hydrodynamics of superfluid nucleon-hyperon mixture. In this case the energy-momentum tensor is given by Eq. (27), where the dissipative correction \(\tau^{\mu\nu}\) should be set zero. Similarly, the expressions for \(w^\mu_{(i)}\) are given by Eq. (34) with \(\kappa_i = 0\). The non-equilibrium processes (1)–(4) lead to the appearance of the sources in the right-hand side of the continuity equations (24) which, as we will demonstrate below, generate the 'effective' dissipative corrections \(\tau^{\mu\nu}\) and \(\kappa_q\).

To calculate \(\xi_{1q}\) and \(\xi_2\) it is convenient to expand the energy-momentum tensor of the pulsating matter (27) (with \(\tau^{\mu\nu} = 0\)) in the comoving frame (where \(u^\mu = (1, 0, 0, 0)\)) near the equilibrium, as it was done in Ref. [1],

\[
T^{00} = \varepsilon_0 + \delta\varepsilon, \\
T^{0m} = T^{m0} = \mu_{i0} Y_{ik} w^m_{(k)}, \\
T^{nm} = (P_0 + \delta P) \delta_{nm}.
\]  

(51)

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Here we restrict ourselves to the linear perturbation terms. The spatial indices \( n \) and \( m \) vary over 1, 2, and 3; \( \varepsilon_0, \mu_0, \) and \( P_0 \) are the corresponding thermodynamic quantities calculated at equilibrium (in the absence of pulsations).

Now our aim is to extract from the tensor (51) various terms which are generated by the non-equilibrium processes (1)–(4) and contribute to dissipation. Then, we will write these terms in the form of a separate dissipative tensor \( \tau_{\mu\nu}^{\text{bulk}}. \)

As follows from Eq. (30), in the linear approximation \( \delta \varepsilon \) remains the same as in the absence of dissipation, \( \delta \varepsilon = \mu_n \delta n_b \) (this is because the dissipative processes (1)–(4) conserve the number of baryons, hence \( \delta n_b \) is independent of the reaction rates \( \lambda_l, l = 1, 2, 3, \) or 4). Thus, the component \( \tau_{00}^{\text{bulk}} \) of the tensor \( \tau_{\mu\nu}^{\text{bulk}} \) is zero. Similarly, \( \tau_{0m}^{\text{bulk}} = \tau_{m0}^{\text{bulk}} = 0. \) On the contrary, the variation \( \delta P \) of the pressure contains a dissipative part (we denote it by \( \delta P_{\text{dis}} \)). According to Refs. [12, 15] and Sec. 2, it is given by

\[
\delta P_{\text{dis}} = \text{Re}(\delta P),
\]

so that the tensor \( \tau_{\mu\nu}^{\text{bulk}} \) can be presented in the form (in the comoving frame)

\[
\begin{align*}
\tau_{00}^{\text{bulk}} &= 0, \\
\tau_{0m}^{\text{bulk}} &= \tau_{m0}^{\text{bulk}} = 0, \\
\tau_{nm}^{\text{bulk}} &= \delta P_{\text{dis}} \delta_{nm}.
\end{align*}
\]

Let us determine \( \delta P_{\text{dis}} \). For that purpose, we present the pressure \( P \) as a function of \( n_b, n_H, n_{\Sigma n} \equiv n_\Sigma + n_n, \) and \( y \equiv x_e/x_\mu. \) All other number densities can be expressed through \( n_b, n_H, n_{\Sigma n}, \) and \( y \) with the help of Eqs. (8) and (10). Notice that we choose \( n_{\Sigma n} \) and \( y \) instead of the variables \( x_e \) and \( x_\mu \) of Sec. 2. The variables \( x_e \) and \( x_\mu \) are less convenient here because Eq. (11) does not hold in the case of superfluid matter. Expanding the pressure \( P(n_b, n_H, n_{\Sigma n}, y) \) near the equilibrium state, we write

\[
\delta P_{\text{dis}} = \frac{\partial P}{\partial n_b} \text{Re}(\delta n_b) + \frac{\partial P}{\partial n_H} \text{Re}(\delta n_H) + \frac{\partial P}{\partial n_{\Sigma n}} \text{Re}(\delta n_{\Sigma n}) + \frac{\partial P}{\partial y} \text{Re}(\delta y).
\]

It is straightforward to show that

\[
\delta y = 0,
\]

as a consequence of the continuity equations (13) and (14) for electrons and muons. The variations of other variables, \( n_b, n_H, \) and \( n_{\Sigma n}, \) can be found from corresponding continuity equations.
equations. Using Eq. (24) and the fact, that the variations depend on time \( t \) as \( \exp(i\omega t) \), we obtain in the comoving frame

\[
i\omega \delta n_b + \text{div}(\mathbf{J}_b) = 0, \quad (56)
\]

\[
i\omega \delta n_H + \text{div}(\mathbf{J}_H) = \Delta \Gamma, \quad (57)
\]

\[
i\omega \delta n_{\Sigma n} + \text{div}(\mathbf{J}_{\Sigma n}) = -\Delta \Gamma. \quad (58)
\]

Here \( \Delta \Gamma \equiv \Delta \Gamma_1 + \Delta \Gamma_2 + \Delta \Gamma_3 + \Delta \Gamma_4 = \lambda \delta \mu \), and

\[
\delta \mu(n_b, n_H, n_{\Sigma n}) = \frac{\partial \delta \mu}{\partial n_b} \delta n_b + \frac{\partial \delta \mu}{\partial n_H} \delta n_H + \frac{\partial \delta \mu}{\partial n_{\Sigma n}} \delta n_{\Sigma n}. \quad (59)
\]

In Eqs. (56)–(58) \( \mathbf{J}_b = n_b \mathbf{u} + \sum_i Y_{ik} w_{(k)}; \mathbf{J}_H = n_H \mathbf{u} + Y_{\Sigma k} w_{(k)} + Y_{\Lambda k} w_{(k)}; \mathbf{J}_{\Sigma n} = n_{\Sigma n} \mathbf{u} + Y_{\Sigma k} w_{(k)} + Y_{nk} w_{(k)} \). \( \mathbf{u} \) and \( w_{(i)} \) are the spatial components of the four-velocity \( u^\mu \) and four-vector \( w_{\mu}^{(i)} \), respectively. The solution to the above system of equations gives

\[
\text{Re}(\delta n_b) = 0, \quad (60)
\]

\[
\text{Re}(\delta n_{\Sigma n}) = -\text{Re}(\delta n_H), \quad (61)
\]

\[
\text{Re}(\delta n_H) = k \left[ \frac{\partial \delta \mu}{\partial n_b} \text{div}(\mathbf{J}_b) + \frac{\partial \delta \mu}{\partial n_H} \text{div}(\mathbf{J}_H) + \frac{\partial \delta \mu}{\partial n_{\Sigma n}} \text{div}(\mathbf{J}_{\Sigma n}) \right], \quad (62)
\]

where we use the notations \( 1/k \equiv \lambda (\partial \delta \mu/\partial n_H - \partial \delta \mu/\partial n_{\Sigma n})^2 (1 + \omega^2 \tau^2) \); \( \tau \equiv 1/\lambda (\partial \delta \mu/\partial n_H - \partial \delta \mu/\partial n_{\Sigma n})^{-1} \). Now, using Eq. (54) for \( \delta P_{\text{dis}} \) and Eqs. (60)–(62), we can find the dissipative tensor \( \tau_{\mu\nu}^{\text{bulk}} \) in the comoving frame (see Eq. (53)). Then it can be easily rewritten in an arbitrary frame. The result is

\[
\tau_{\mu\nu}^{\text{bulk}} = H^\mu_\nu k \left( \frac{\partial P}{\partial n_H} - \frac{\partial P}{\partial n_{\Sigma n}} \right) \left[ \left( n_b \frac{\partial \delta \mu}{\partial n_b} + n_H \frac{\partial \delta \mu}{\partial n_H} + n_{\Sigma n} \frac{\partial \delta \mu}{\partial n_{\Sigma n}} \right) \partial_\gamma u^\gamma 
  + \left( \frac{\partial \delta \mu}{\partial n_b} + \frac{\partial \delta \mu}{\partial n_{\Sigma n}} \right) \partial_\gamma Y_{nk} w_{(k)}^\gamma 
  + \left( \frac{\partial \delta \mu}{\partial n_b} + \frac{\partial \delta \mu}{\partial n_H} \right) \partial_\gamma Y_{\Lambda k} w_{(k)}^\gamma \right]. \quad (63)
\]

A comparison of \( \tau_{\mu\nu}^{\text{bulk}} \) with the phenomenological dissipative tensor \( \tau_{\mu\nu} \) (see Eq. (46)) allows us to identify the bulk viscosity coefficients \( \xi_{1b}, \xi_{1\Lambda}, \xi_{1\Sigma p}, \) and \( \xi_2 \), generated by non-equilibrium processes (1)-(4).
\[\xi_{1n} = -k \left( \frac{\partial P}{\partial n_H} - \frac{\partial P}{\partial n_{\Sigma n}} \right) \left( \frac{\partial \delta \mu}{\partial n_b} + \frac{\partial \delta \mu}{\partial n_{\Sigma n}} \right), \tag{64}\]

\[\xi_{1\Lambda} = -k \left( \frac{\partial P}{\partial n_H} - \frac{\partial P}{\partial n_{\Sigma n}} \right) \left( \frac{\partial \delta \mu}{\partial n_b} + \frac{\partial \delta \mu}{\partial n_H} \right), \tag{65}\]

\[\xi_{1\Sigma p} = -k \left( \frac{\partial P}{\partial n_H} - \frac{\partial P}{\partial n_{\Sigma n}} \right) \left( 2 \frac{\partial \delta \mu}{\partial n_b} + \frac{\partial \delta \mu}{\partial n_H} + \frac{\partial \delta \mu}{\partial n_{\Sigma n}} \right), \tag{66}\]

\[\xi_2 = -k \left( \frac{\partial P}{\partial n_H} - \frac{\partial P}{\partial n_{\Sigma n}} \right) \left( \frac{\partial \delta \mu}{\partial n_b} + n_b \frac{\partial \delta \mu}{\partial n_H} + n_{\Sigma n} \frac{\partial \delta \mu}{\partial n_{\Sigma n}} \right). \tag{67}\]

It can be shown, that the expression for \(\xi_2\) formally coincides with Eq. (22) for the bulk viscosity of normal matter (however, these formulae give different numerical results because the reaction rates \(\lambda_l\) \((l = 1, \ldots, 4)\) differ for superfluid and normal matter). To prove this, we can rewrite Eq. (67) using \(n_b, x_H, x_e,\) and \(x_\mu\) as independent variables instead of \(n_b, n_H, n_{\Sigma n},\) and \(y.\) The following equalities will be helpful (\(f\) is an arbitrary function)

\[
\frac{1}{n_b} \frac{\partial f(n_b, x_H, x_e, x_\mu)}{\partial x_H} = \frac{\partial f(n_b, n_H, n_{\Sigma n}, y)}{\partial n_H} - \frac{\partial f(n_b, n_H, n_{\Sigma n}, y)}{\partial n_{\Sigma n}}, \tag{68}\]

\[
\frac{n_b}{n_b} \frac{\partial f(n_b, x_H, x_e, x_\mu)}{\partial n_b} = n_b \frac{\partial f(n_b, n_H, n_{\Sigma n}, y)}{\partial n_b} + n_H \frac{\partial f(n_b, n_H, n_{\Sigma n}, y)}{\partial n_H} + n_{\Sigma n} \frac{\partial f(n_b, n_H, n_{\Sigma n}, y)}{\partial n_{\Sigma n}}. \tag{69}\]

To calculate other bulk viscosity coefficients let us apply the same consideration to Eq. (34) for \(w_{(i)}^\mu\). As a result, we obtain (in the comoving frame) the expression for the dissipative component \(\kappa_i\) generated by nonequilibrium processes (1)–(4)

\[\kappa_i = \delta \mu_{\text{dis}} = \frac{\partial \mu_i}{\partial n_b} \text{Re}(\delta n_b) + \frac{\partial \mu_i}{\partial n_H} \text{Re}(\delta n_H) + \frac{\partial \mu_i}{\partial n_{\Sigma n}} \text{Re}(\delta n_{\Sigma n}). \tag{70}\]

Here \(\delta \mu_{\text{dis}}\) is the dissipative term in the Taylor expansion of the chemical potential \(\mu_i\) near the equilibrium state (similar to \(\delta P_{\text{dis}}\)); \(\text{Re}(\delta n_b), \text{Re}(\delta n_H),\) and \(\text{Re}(\delta n_{\Sigma n})\) are taken from Eqs. (60)–(62). In a fully covariant form, \(\kappa_q\) is given by Eqs. (47)–(49) with the bulk viscosity
coefficients

\[ \xi_{3n} = -k \left( \frac{\partial \mu_n}{\partial n_H} - \frac{\partial \mu_n}{\partial n_{\Sigma n}} \right) \left( \frac{\partial \delta \mu}{\partial \mu} + \frac{\partial \delta \mu}{\partial \mu_{\Sigma n}} \right), \]  

(71)

\[ \xi_{3\Lambda} = -k \left( \frac{\partial \mu_n}{\partial n_H} - \frac{\partial \mu_n}{\partial n_{\Sigma n}} \right) \left( \frac{\partial \delta \mu}{\partial \mu} + \frac{\partial \delta \mu}{\partial \mu_{\Sigma n}} \right), \]  

(72)

\[ \xi_{3\Sigma p} = -k \left( \frac{\partial \mu_n}{\partial n_H} - \frac{\partial \mu_n}{\partial n_{\Sigma n}} \right) \left( 2 \frac{\partial \delta \mu}{\partial \mu} + \frac{\partial \delta \mu}{\partial \mu_{\Sigma n}} \right), \]  

(73)

\[ \xi_{4n} = -k \left( \frac{\partial \mu_n}{\partial n_H} - \frac{\partial \mu_n}{\partial n_{\Sigma n}} \right) \left( \frac{\partial \delta \mu}{\partial \mu} + \frac{\partial \delta \mu}{\partial \mu_{\Sigma n}} + n_{\Sigma n} \frac{\partial \delta \mu}{\partial \mu_{\Sigma n}} \right), \]  

(74)

\[ \xi_{4\Lambda} = -k \left( \frac{\partial \mu_n}{\partial n_H} - \frac{\partial \mu_n}{\partial n_{\Sigma n}} \right) \left( \frac{\partial \delta \mu}{\partial \mu} + \frac{\partial \delta \mu}{\partial \mu_{\Sigma n}} + n_{\Sigma n} \frac{\partial \delta \mu}{\partial \mu_{\Sigma n}} \right), \]  

(75)

\[ \xi_{4\Sigma p} = -k \left[ \frac{\partial (\mu_p + \mu_{\Sigma})}{\partial n_H} - \frac{\partial (\mu_p + \mu_{\Sigma})}{\partial n_{\Sigma n}} \right] \left( \frac{\partial \delta \mu}{\partial \mu} + \frac{\partial \delta \mu}{\partial \mu_{\Sigma n}} + n_{\Sigma n} \frac{\partial \delta \mu}{\partial \mu_{\Sigma n}} \right), \]  

(76)

\[ \xi_{5n} = -k \left( \frac{\partial \mu_{\Lambda}}{\partial n_H} - \frac{\partial \mu_{\Lambda}}{\partial n_{\Sigma n}} \right) \left( \frac{\partial \delta \mu}{\partial \mu} + \frac{\partial \delta \mu}{\partial \mu_{\Sigma n}} \right), \]  

(77)

\[ \xi_{5\Lambda} = -k \left( \frac{\partial \mu_{\Lambda}}{\partial n_H} - \frac{\partial \mu_{\Lambda}}{\partial n_{\Sigma n}} \right) \left( \frac{\partial \delta \mu}{\partial \mu} + \frac{\partial \delta \mu}{\partial \mu_{\Sigma n}} \right), \]  

(78)

\[ \xi_{5\Sigma p} = -k \left( \frac{\partial \mu_{\Lambda}}{\partial n_H} - \frac{\partial \mu_{\Lambda}}{\partial n_{\Sigma n}} \right) \left( \frac{\partial \delta \mu}{\partial \mu} + \frac{\partial \delta \mu}{\partial \mu_{\Sigma n}} \right), \]  

(79)

\[ \xi_{6n} = -k \left[ \frac{\partial (\mu_p + \mu_{\Sigma})}{\partial n_H} - \frac{\partial (\mu_p + \mu_{\Sigma})}{\partial n_{\Sigma n}} \right] \left( \frac{\partial \delta \mu}{\partial \mu} + \frac{\partial \delta \mu}{\partial \mu_{\Sigma n}} \right), \]  

(80)

\[ \xi_{6\Lambda} = -k \left[ \frac{\partial (\mu_p + \mu_{\Sigma})}{\partial n_H} - \frac{\partial (\mu_p + \mu_{\Sigma})}{\partial n_{\Sigma n}} \right] \left( \frac{\partial \delta \mu}{\partial \mu} + \frac{\partial \delta \mu}{\partial \mu_{\Sigma n}} \right), \]  

(81)

\[ \xi_{6\Sigma p} = -k \left[ \frac{\partial (\mu_p + \mu_{\Sigma})}{\partial n_H} - \frac{\partial (\mu_p + \mu_{\Sigma})}{\partial n_{\Sigma n}} \right] \left( 2 \frac{\partial \delta \mu}{\partial \mu} + \frac{\partial \delta \mu}{\partial \mu_{\Sigma n}} \right). \]  

(82)

In Eqs. (71)–(82) we assumed that the thermodynamic quantities are functions of \( n_b, n_H, n_{\Sigma n}, \) and \( y. \) One can easily check that all the six Onsager relations (50) are satisfied. Moreover, it turns out that the bulk viscosities (64)–(67) and (71)–(82) obey a number of additional constraints (\( q = n, \Lambda, \Sigma p \))

\[ \xi_{6q} = \xi_{3q} + \xi_{5q}, \quad \xi_{4\Sigma p} = \xi_{4n} + \xi_{4\Lambda}, \]  

(83)

\[ \xi_{1n}^2 = \xi_{2}\xi_{3n}, \quad \xi_{1\Lambda}^2 = \xi_{2}\xi_{5\Lambda}, \quad \xi_{1\Sigma p}^2 = \xi_{2}\xi_{3\Sigma p}, \]  

(84)

\[ \xi_{1n}\xi_{1\Lambda} = \xi_{2}\xi_{6n}, \quad \xi_{1\Sigma p}\xi_{1\Lambda} = \xi_{2}\xi_{6\Sigma p}, \]  

(85)

so that we have only three independent bulk viscosity coefficients, say \( \xi_2, \xi_{1n}, \) and \( \xi_{1\Lambda}. \) All other coefficients can be expressed through these three. The coefficients \( \xi_2, \xi_{1n}, \) and \( \xi_{1\Lambda} \) are compared in Fig. 2. Because the dimensions of the coefficients \( \xi_{ii} (i = n, \Lambda) \) and \( \xi_2 \)
are different, we show the dimensionless combinations \( \xi_{1i}/\xi_2 \) as functions of the baryon number density \( n_b \).

What is the nature of the relations (83)–(85)? The relations (83) follow from the equilibrium condition (8) with respect to the fast reaction (7). This condition is valid even if we allow for the dissipative processes (1)–(4). Consequently, we can write

\[
\text{Re}(\mu_\Sigma) + \text{Re}(\mu_p) = \text{Re}(\mu_n) + \text{Re}(\mu_\Lambda) \tag{86}
\]

or, in view of Eq. (70) (we remind that \( \text{Re}(\mu_i) \equiv \mu_{\text{dis}}^i; \kappa_{\Sigma p} \equiv \kappa_\Sigma + \kappa_p \)),

\[
\kappa_{\Sigma p} = \kappa_n + \kappa_\Lambda. \tag{87}
\]

Then, substituting Eqs. (47)–(49) into Eq. (87) and equating prefactors in front of the same four-divergences, we obtain the relations (83).

It is convenient to note that the constraint (87) leads to a simple equation interrelating four-vectors \( w^\mu_{(i)} \). It holds true only if the bulk viscosities are generated by the non-equilibrium reactions. To derive this equation, we sum up the potentiality conditions (34) (with proper signs) for all baryon species. Then, using Eqs. (8) and (87), we get

\[
\partial^\nu \left[ w^\mu_{(n)} + w^\mu_{(\Lambda)} - w^\mu_{(\Sigma)} - w^\mu_{(p)} \right] = \partial^\nu \left[ w^\nu_{(n)} + w^\nu_{(\Lambda)} - w^\nu_{(\Sigma)} - w^\nu_{(p)} \right]. \tag{88}
\]
In view of this equation and Eq. (45), there are only \(4 - 2 = 2\) independent four-vectors \(w^{\mu}_{(i)}\) in the system.

Now let us explain why the bulk viscosities satisfy Eqs. (84) and (85). For that purpose, we consider the entropy generation equation. Neglecting all the dissipative processes except for the non-equilibrium reactions (1)–(4) (e.g., neglecting the thermal conductivity and shear viscosity), we can obtain from the hydrodynamics discussed in this section,

\[
T \partial_\mu S^\mu = \delta \mu \Delta \Gamma = \lambda \delta \mu^2. \tag{89}
\]

A similar expression is valid for \(npe\) matter (see Eq. (79) of Ref. [1]). The chemical potential disbalance \(\delta \mu\) is given by Eq. (59). Substituting into Eq. (59) variations \(\delta n_b, \delta n_H, \text{ and } \delta n_{\Sigma n}\) calculated from the continuity equations (56)–(58), we find

\[
\delta \mu = \frac{1}{\lambda} \left( \frac{\partial \delta \mu}{\partial n_H} - \frac{\partial \delta \mu}{\partial n_{\Sigma n}} \right) - i\omega \left[ \frac{\partial \delta \mu}{\partial n_b} \text{div}(J_b) + \frac{\partial \delta \mu}{\partial n_H} \text{div}(J_H) + \frac{\partial \delta \mu}{\partial n_{\Sigma n}} \text{div}(J_{\Sigma n}) \right]. \tag{90}
\]

It follows from Eq. (90), that for any given \(u^\mu\) it is always possible to choose four-vectors \(w^{\mu}_{(i)}\) in such a way that \(\delta \mu = 0\) at some point and in some particular moment (even if some baryon species are non-superfluid). In other words, this means that we can vanish the entropy generation rate (89) at this point.

On the other hand, the entropy generation equation in terms of the effective bulk viscosities takes the form (see Eqs. (38) and (40)–(44))

\[
T \partial_\mu S^\mu = \left\{ \xi_{1q} \partial_\mu [Y_{qk} w^{\mu}_{(k)}] + \xi_{2} \partial_\mu w^\mu \right\} \partial_\mu u^\mu + \\
\left\{ \xi_{3q} \partial_\mu [Y_{qk} w^{\mu}_{(k)}] + \xi_{4n} \partial_\mu u^\mu \right\} \partial_\mu [Y_{nk} w^{\mu}_{(k)}] + \\
\left\{ \xi_{5q} \partial_\mu [Y_{qk} w^{\mu}_{(k)}] + \xi_{4\Lambda} \partial_\mu u^\mu \right\} \partial_\mu [Y_{\Lambda k} w^{\mu}_{(k)}] + \\
\left\{ \xi_{6q} \partial_\mu [Y_{qk} w^{\mu}_{(k)}] + \xi_{4\Sigma p} \partial_\mu u^\mu \right\} \partial_\mu [Y_{\Sigma pk} w^{\mu}_{(k)}]. \tag{91}
\]

The right-hand side of this equation satisfies two conditions. First, it is a positive-definite quadratic form (entropy cannot decrease!). Second, according to Eqs. (89) and (90), one can vanish it by an appropriate choice of \(u^\mu\) and \(w^{\mu}_{(i)}\) (at some particular moment and at some point). There is a mathematical theorem that these two conditions are consistent with each other if and only if the determinant of the matrix composed of coefficients of the quadratic form vanishes. This result is independent of an actual number of superfluid baryon species. That is, the determinant will be zero in the case when all four baryon species are superfluid.
as well as in the case when some of them are normal (for example, nucleons are normal, hyperons are superfluid, or neutrons are superfluid, while other particles are normal). As a consequence, we arrive at the six additional constraints (84) and (85) on the bulk viscosity coefficients.

In this section we have assumed that baryons of all species are superfluid. However, the hydrodynamics formulated here can be easily extended to the situation when some baryon species are normal. In this case, one should vanish matrix elements $Y_{ik}$ related to these baryon species. For example, if neutrons are non-superfluid, then $Y_{nk} = Y_{kn} \equiv 0$.

IV. SUMMARY

In this paper we have analyzed the bulk viscosity due to non-equilibrium particle transformations in superfluid nucleon-hyperon matter of neutron stars. Our approach is similar to that used in Ref. [1] for the case of superfluid $npe$ matter.

Our main results are as follows:

(i) We have demonstrated that the expression for the bulk viscosity of normal (non-superfluid) nucleon-hyperon matter, widely used in the literature, is inaccurate. We have presented the correct derivation of the bulk viscosity and compared it with the results of Ref. [15]. Numerically, both formulae give almost similar results (within a few tens of percent).

(ii) We have extended the hydrodynamics of superfluid mixtures reported in Refs. [1, 34] to allow for a possible presence of superfluid hyperons. We have determined the general form of dissipative terms entering the equations of this hydrodynamics and showed that generally (when all baryon species are superfluid), it contains sixteen bulk viscosity coefficients.

(iii) We have calculated and analyzed the sixteen bulk viscosity coefficients assuming they are generated by non-equilibrium reactions (1)–(4) of particle mutual transformations. We have shown that only three of them are independent. All other 13 bulk viscosities can be expressed through these three using Eqs. (50) and (83)–(85). In addition, we have demonstrated that Eq. (67) for the bulk viscosity coefficient $\xi_2$ formally coincides with Eq. (22) for that in normal matter (however, the reaction rates $\lambda_l$ ($l = 1, \ldots, 4$) are affected by superfluidity).

Our results can be important for the studies of dynamical instabilities in pulsating superfluid neutron stars, especially for the studies of the r-mode instability. They can also be
important for modeling of the thermal evolution of pulsating neutron stars and for analyzing rotochemical and gravitochemical heating of millisecond pulsars with superfluid nucleon-hyperon cores (for non-superfluid $npe$ matter of neutron stars, this problem was considered in Refs. \[28, 35, 36, 37\]).

Let us notice, that to start such an analysis one needs to know the matrix $Y_{ik}$, which is the most important ingredient in the hydrodynamics of superfluid nucleon-hyperon mixture. To our best knowledge, it has not been considered in the literature so far. We are planning to fill this gap and present its extensive calculations in a subsequent publication.

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