The Energy-Momentum Tensor in Noncommutative Gauge Field Models

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Abstract. We discuss the different possibilities of constructing the various energy-momentum tensors for noncommutative gauge field models. We use Jackiw’s method in order to get symmetric and gauge invariant stress tensors—at least for commutative gauge field theories. The noncommutative counterparts are analyzed with the same methods. The issues for the noncommutative cases are worked out.

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1 Introduction

In a previous paper [1] we have analyzed the energy-momentum tensor on noncommutative spaces and we have found that the dilation symmetry is broken due to the presence of the deformation parameter $\theta^{\mu\nu}$ characterizing the noncommutative geometry by

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}. \quad (1)$$

The existence of a constant, fixed antisymmetric tensor field $\theta^{\mu\nu}$ clearly also breaks the Lorentz symmetry [3], [4], [5] if $\theta^{\mu\nu}$ does not have a tensorial transformation behaviour with respect to Lorentz transformations. This situation resembles in some sense the axial gauge in gauge field models. There, the presence of the constant, fixed gauge ‘direction’ $n^\mu$ breaks the Lorentz invariance, too [4].

In particular, the occurrence of $\theta^{\mu\nu}$ in noncommutative quantum field models induces that the corresponding energy-momentum tensor needs neither be symmetric (for massless models), nor traceless.

The aim of the present work is the investigation of the construction of the energy-momentum tensor in massless and commutative gauge field models and their noncommutative counterparts, in order to work out the different aspects of the stress tensor for both cases.

Generally, the usual Noether procedure for the construction of the canonical energy-momentum tensor in the worst case needs an improvement procedure and the Belinfante trick [7], [8], [9] in order to get a symmetric and traceless stress tensor. However, due to the idea of Jackiw [9], [10], [11], there is a more direct method to get the correct stress tensor by combining the Noether procedure translations with field dependent gauge transformations.

The paper is organized as follows. In section 2 we demonstrate the power of Jackiw’s recipe for the construction of the energy-momentum tensor for a general $U(N)$ non-Abelian gauge field model. Both quantities, the canonical stress tensor and the symmetric one will be discussed in defining appropriate Ward-identity operators [12] for the description of infinitesimal translations. Additionally, the commutative Yang-Mills model is also characterized by the gauge symmetry implying the existence of a conserved gauge current.

Section 3 is devoted to the discussion of the energy-momentum tensor for noncommutative non-Abelian gauge field models. The methods used for the commutative case can be applied without any difficulties also to the noncommutative counterpart. The results obtained in section 3 are very similar to the corresponding output of the commutative case. However, there is a severe difference: The stress tensors are no longer locally conserved due to the fact that cyclic rotation of Moyal products occurring in the stress tensors is locally impossible. After all, the symmetric version of the energy-momentum tensor is locally covariantly conserved with respect to the covariant
derivative of the gauge symmetry. In sections 4 and 5 we discuss the influence of the so called Seiberg-Witten map on the construction of gauge invariant actions and the corresponding stress tensors. In section 4 we treat the noncommutative non-Abelian field model obtained via the Seiberg-Witten map to lowest order in $\theta^{\mu\nu}$ [13]. Section 5 discusses the special case of the $\theta$-deformed Maxwell theory ($U(1)$). The construction of the symmetric energy-momentum tensor confirms a recent result obtained by Kruglov [14]. All investigations are done in the classical approximation without radiative corrections.

2 Energy-Momentum Tensor in Ordinary Yang-Mills Theory

In order to demonstrate the various possible constructions (canonical form, Belinfante procedure, construction modulo a gauge transformation) of the energy-momentum tensor [1, 7, 8, 9, 10], let us start with a commutative Yang-Mills model, where the gauge field is matrix-valued, $A_\mu = A_\mu^a X^a$, $X^a$ being the corresponding generators of the gauge group $U(N)$.

The corresponding infinitesimal gauge transformation is given by

$$\delta_\lambda A_\mu = \partial_\mu \lambda - i [A_\mu, \lambda] =: D_\mu \lambda,$$

implying that the non-Abelian field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu]$$

transforms covariantly,

$$\delta_\lambda F_{\mu\nu} = i [\lambda, F_{\mu\nu}].$$

Therefore, the gauge invariant non-Abelian action is given by

$$\Gamma_{\text{inv}}[A] = -\frac{1}{4} \int d^4x \, tr (F_{\mu\nu} F^{\mu\nu}) =: -\frac{1}{4} \int d^4x \, tr F^2$$

The equation of motion for the gauge field is

$$\frac{\delta}{\delta A_\nu} \Gamma_{\text{inv}}[A] = D_\mu F^{\mu\nu} = 0.$$

The symmetry transformation [2] may be expressed by the global Ward-identity (WI)-operator

$$W^G(\lambda) = \int d^4x \, tr D_\mu \lambda(x) \, \frac{\delta}{\delta A_\mu(x)}.$$

2
Gauge invariance is stated through $W^G(\lambda)\Gamma_{inv} = 0$. This implies the following local identity for the gauge symmetry,

$$
\frac{\delta}{\delta \lambda(x)} W^G(\lambda)\Gamma_{inv} = 0.
$$

(8)

This defines the locally conserved current for the gauge symmetry,

$$
J^\mu_G = -i[A_\rho, F^{\rho\mu}], \quad -\partial_\mu J^\mu_G = i\partial_\mu [A_\rho, F^{\rho\mu}] = 0.
$$

(9)

By direct computation and by use of the equation of motion one easily verifies (9). Now we want to discuss the infinitesimal translation described by the following global WI-operator,

$$
W^T_\mu = \int d^4x \text{tr} \partial_\mu A_\rho(x) \frac{\delta}{\delta A_\rho(x)}.
$$

(10)

By applying the WI-operator (10) to the gauge invariant action one gets the canonical energy-momentum tensor due to translational invariance,

$$
W^T_\mu \Gamma_{inv} = \int d^4x \text{tr} \left( \partial^\rho \left( \frac{1}{2} \{ F_{\rho\nu}, \partial_\mu A_\nu \} - \frac{1}{4} g_{\rho\mu} F^2 \right) \right)
$$

$$
= - \int d^4x \partial^\rho T^c_{\rho\mu} = 0.
$$

(11)

Thus, the canonical energy-momentum tensor is defined as

$$
T^c_{\rho\mu} := -\text{tr} \left( \frac{1}{2} \{ F_{\rho\nu}, \partial_\mu A_\nu \} - \frac{1}{4} g_{\rho\mu} F^2 \right).
$$

(12)

It is simple to show that $T^c_{\rho\mu}$ is locally conserved by using the equation of motion. However, $T^c_{\rho\mu}$ is not gauge invariant, not traceless and not symmetric in ($\rho, \mu$). In order to obtain a symmetric stress tensor one has two possibilities [7], [8]. Here we follow the method proposed originally by R. Jackiw [10] in using an alternative representation for infinitesimal translations. Modulo a field dependent gauge transformation a possible description of translations is given by

$$
W^F_\mu = \int d^4x \text{tr} F_{\mu\nu}(x) \frac{\delta}{\delta A_\nu(x)},
$$

(13)

leading to

$$
W^F_\mu \Gamma_{inv} = \int d^4x \text{tr} \left( \partial^\rho \left( \frac{1}{2} \{ F_{\rho\nu}, F_{\mu\nu}^\rho \} - \frac{1}{4} g_{\rho\mu} F^2 \right) \right)
$$

$$
= - \int d^4x \partial^\rho T^s_{\rho\mu} = 0,
$$

(14)
where $T^s_{\rho\mu}$ is gauge invariant, symmetric and traceless,

$$T^s_{\rho\mu} := -tr\left(\frac{1}{2}\{F_{\rho\nu}, F_{\mu}^\nu\} - \frac{1}{4}g_{\rho\mu}F^2\right). \quad (15)$$

One observes that the Jackiw construction unifies the Belinfante and improvement procedure.

Using the splitting

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] = \partial_\mu A_\nu - D_\nu A_\mu, \quad (16)$$

one gets for the canonical tensor

$$T^c_{\rho\mu} = -tr\left(\frac{1}{2}\{F_{\rho\nu}, F_{\mu}^\nu + D_\nu A_\mu\} - \frac{1}{4}g_{\rho\mu}F^2\right), \quad (17)$$

implying that the difference between the canonical tensor and the symmetric one becomes

$$T^c_{\rho\mu} - T^s_{\rho\mu} = -\frac{1}{2}tr\{F_{\rho\nu}, D_\nu A_\mu\}. \quad (18)$$

Due to the fact that the WI-operator of the translation is represented by

$$W^T_\mu = W^F_\mu + W^G_\mu = \int d^4x tr\left(F_{\mu\nu}(x)\frac{\delta}{\delta A_\nu(x)} + D_\nu A_\mu(x)\frac{\delta}{\delta A_\nu(x)}\right), \quad (19)$$

the field dependent gauge transformation corresponds to the difference $T^c_{\rho\mu} - T^s_{\rho\mu}$,

$$-W^G_\mu \Gamma_{\text{inv}} = -\int d^4x tr\left(D_\nu A_\mu(x)\frac{\delta \Gamma_{\text{inv}}}{\delta A_\nu(x)}\right)$$

$$= -\int d^4x tr\partial_\rho\left(\frac{1}{2}\{F_{\rho\nu}, D_\nu A_\mu\}\right). \quad (20)$$

This is easily checked by explicit calculation with the use of partial integration,

$$-W^G_\mu \Gamma_{\text{inv}} = -\int d^4x trD_\nu A_\mu D_\rho F^{\rho\nu}$$

$$= -\int d^4x tr\left(\partial_\rho\left(\frac{1}{2}\{F_{\rho\nu}, D_\nu A_\mu\}\right) - D_\rho D_\nu A_\mu F^{\rho\nu}\right). \quad (21)$$

With the antisymmetry of $F^{\rho\nu}$ the second term is easily shown to vanish. This is very similar to the construction of the stress tensor of the Maxwell theory \[15\]. Another interesting comment has to be made. If we omit the $tr$ symbol in the definition of the energy momentum tensor $T^s_{\rho\mu}$,

$$T^{ss}_{\rho\mu} := -\left(\frac{1}{2}\{F_{\rho\nu}, F_{\mu}^\nu\} - \frac{1}{4}g_{\rho\mu}F^2\right), \quad (22)$$
we get an object which is (due to the equation of motion and the Bianchi identity) covariantly conserved,
\[ D^\rho T^s_{\rho \mu} = 0. \] (23)

In discussing the noncommutative counterpart we will find that a similar ‘covariant conservation’ is also valid there. With (23) one finds
\[ \partial^\rho T^s_{\rho \mu} = i [A^\rho, T^s_{\rho \mu}], \] (24)

which is consistent with
\[ W^\mu \Gamma_{inv} = - \int d^4x \text{tr}(\partial^\rho T^s_{\rho \mu}) = 0. \] (25)

### 3 Energy-Momentum Tensor in Noncommutative Yang-Mills Theory

It is now straightforward to discuss also the noncommutative structure in the spirit of the considerations done in the previous section. In noncommutative gauge field models one has to replace all field products by \( \star \)-products \[2\] and one introduces the noncommutative matrix valued gauge field \( \hat{A} \). The corresponding gauge invariant action is therefore
\[ \hat{\Gamma}_{inv}[\hat{A}] = - \frac{1}{4} \int d^4x \text{tr}(\hat{F}_{\mu \nu} \star \hat{F}^{\mu \nu}), \] (26)

with the noncommutative field strength
\[ \hat{F}_{\mu \nu} = \partial_{\mu} \hat{A}_{\nu} - \partial_{\nu} \hat{A}_{\mu} - i [\hat{A}_{\mu}, \hat{A}_{\nu}]_M. \] (27)

Here the Moyal commutator is given by
\[ [\hat{A}_{\mu}, \hat{A}_{\nu}]_M := \hat{A}_{\mu} \star \hat{A}_{\nu} - \hat{A}_{\nu} \star \hat{A}_{\mu}, \] (28)

using the \( \star \)-product,
\[ A(x) \star B(x) = e^{\frac{i}{2} \eta^{\mu \nu} \partial_\mu \delta_\eta_\nu} A(x + \xi) B(x + \eta)|_{\xi = \eta = 0}. \] (29)

The infinitesimal gauge transformation is defined as
\[ \delta_{\hat{\lambda}} \hat{A}_{\mu} = \partial_{\mu} \hat{\lambda} - i [\hat{A}_{\mu}, \hat{\lambda}]_M =: \hat{D}_{\mu} \star \hat{\lambda}, \] (30)

where \( \hat{\lambda} \) is the noncommutative counterpart of \( \lambda \) of equation \[2\]. The equation of motion for the gauge field is then
\[ \frac{\delta}{\delta \hat{A}_{\mu}} \hat{\Gamma}_{inv}[\hat{A}] = \hat{D}_{\rho} \star \hat{F}^{\rho \nu} = 0, \] (31)
and for the locally conserved gauge current we get
\[ \hat{j}_G^\mu = -i[\hat{A}_\mu, \hat{F}^\rho_\mu]_M. \]  \hfill (32)

At the level of noncommutative gauge field models one can perform the same steps as in the previous section. With
\[ \hat{W}_\mu^T \hat{\Gamma}^\text{inv} = \frac{1}{2} \int d^4x \text{tr} \left( \partial_\mu \hat{A}_\nu(x) * \frac{\delta \hat{\Gamma}^\text{inv}}{\delta \hat{A}_\nu(x)} + \frac{\delta \hat{\Gamma}^\text{inv}}{\delta \hat{A}_\nu(x)} * \partial_\mu \hat{A}_\nu(x) \right) \]
we find
\[ \hat{T}_c^{\rho\mu} = -\left( \frac{1}{2} \{ \hat{F}^\rho_\mu, \partial_\mu \hat{A}_\nu \}_M - \frac{1}{4} g_{\rho\mu} \hat{F}^{\alpha\beta} * \hat{F}^{\alpha\beta} \right). \]  \hfill (33)

Analogously we have
\[ \hat{W}_\mu^F \hat{\Gamma}^\text{inv} = \frac{1}{2} \int d^4x \text{tr} \left( \hat{F}^\mu_\nu(x) * \frac{\delta \hat{\Gamma}^\text{inv}}{\delta \hat{A}_\nu(x)} + \frac{\delta \hat{\Gamma}^\text{inv}}{\delta \hat{A}_\nu(x)} * \hat{F}^\mu_\nu(x) \right) \]
we find
\[ \hat{T}_s^{\rho\mu} := -\left( \frac{1}{2} \{ \hat{F}^\rho_\mu, \partial_\mu \hat{F}^{\nu}_\nu \}_M - \frac{1}{4} g_{\rho\mu} \hat{F}^{\alpha\beta} * \hat{F}^{\alpha\beta} \right). \]  \hfill (34)

Here \( \{ , \}_M \) represents the Moyal anti-commutator in the sense of (28). Note that in order to define ‘local’ quantities, \( \int d^4x \text{tr} \) —the integration over space-time and the trace over the colour indices—cannot be separated in noncommutative geometry. After all, (36) is symmetric, traceless and transforms covariantly with respect to (30),(9). It is also locally covariantly conserved,
\[ \hat{D}^\rho \times \hat{T}_s^{\rho\mu} = 0. \]  \hfill (37)

This is shown with the help of the equation of motion and the noncommutative analogon to the Bianchi-identity. We find that the energy-momentum tensors are not locally conserved\(^1\),
\[ \partial^\rho \hat{T}_s^{\rho\mu} \neq 0 \neq \partial^\rho \hat{T}_c^{\rho\mu}, \]  \hfill (38)
which is already known from the works [11], [13].

\(^1\) Even a ‘formal’ definition of the tensors including the \( \text{tr} \)-symbol does not help, since the trace of a Moyal commutator is not vanishing locally.
4 Energy-Momentum Tensor via Seiberg-Witten Map

Another possibility to formulate noncommutative gauge field models is based on the fact that one can use the so-called Seiberg-Witten (SW-) map [13, 16, 17]. This map ensures the gauge equivalence between an ordinary gauge field and its noncommutative counterpart. It implies that the noncommutative gauge field $\hat{A}_\mu$ and also $\hat{F}_{\mu\nu}$ can be expanded in a series in the deformation parameter $\theta^{\mu\nu}$ of the noncommutative space-time geometry, with coefficients depending on the ordinary gauge field. This section discusses the translation invariance of the SW-expansion of the noncommutative $U(N)$-Yang Mills (NCYM-) theory. The starting point is equation (26).

\[ \hat{\Gamma}_{inv}[\hat{A}] = -\frac{1}{4} \int d^4x \, tr \left( \hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu} \right). \]  

The SW-map to lowest order in $\theta^{\mu\nu}$ for the noncommutative gauge field is

\[ \hat{A}_\mu = A_\mu - \frac{1}{4} \theta^{\rho\sigma} \left\{ A_\rho, \partial_\sigma A_\mu + F_{\sigma\mu} \right\}, \]  

implying the following field strength expansion [13]

\[ \hat{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{4} \theta^{\rho\sigma} \left( 2\{ F_{\mu\rho}, F_{\nu\sigma} \} - \{ A_\rho, D_\sigma F_{\mu\nu} + \partial_\sigma F_{\mu\nu} \} \right). \]  

For the ordinary (commutative, Lie-algebra valued) field $A_\mu$ and field strength $F_{\mu\nu}$ the corresponding gauge transformations are given by (2) and (4), respectively. Expanding the $\star$-product in (26) we have [17, 18]

\[ \Gamma^\theta_{inv}[A] = \int d^4x \, tr \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \theta^{\alpha\beta} (F_{\mu\alpha} F_{\nu\beta} F^{\mu\nu} - \frac{1}{4} F_{\alpha\beta} F^2) + \mathcal{O}(\theta^2) \right) + \mathcal{O}(\theta^2). \]  

The corresponding equation of motion is

\[ -D_\rho \Pi^{\rho\nu} = D_\rho \left( F^{\rho\nu} - \frac{1}{4} (\theta^{\rho\nu} F_{\alpha\beta} F^{\alpha\beta} + \{ F^{\rho\nu}, \theta^{\alpha\beta} F_{\alpha\beta} \} \right) \]
\[ + \frac{1}{2} \left( \{ \theta^{\rho\nu} F_{\alpha\beta}, F^{\rho\alpha} \} - \{ \theta^{\rho\nu} F_{\alpha\beta}, F^{\rho\alpha} \} + \{ F^{\rho\nu}, \theta^{\alpha\beta} F^\beta_{\beta} \} \right) \]
\[ = 0. \]  

The quantity $\Pi^{\rho\nu}$ is antisymmetric,

\[ \Pi^{\rho\nu} = -\Pi^{\nu\rho} = \frac{\partial \mathcal{L}^\theta_{inv}}{\partial (\partial_\rho A_\nu)}. \]
and $\Pi^\rho$ is the canonical momentum. The analogous calculations as in section 1 give now

$$W^\mu_\rho T^\rho_{\Gamma_{\text{inv}}} := \int d^4 x \text{tr} \partial_\rho A_\nu(x) \frac{\delta T^\rho_{\Gamma_{\text{inv}}}}{\delta A_\nu(x)}$$

$$= \int d^4 x \text{tr} \left( \partial^\rho \left( \frac{1}{2} \left\{ - \Pi_{\rho \nu}, \partial_\mu A_\nu \right\} + g_{\rho \mu} \mathcal{L}_{\text{inv}}^\theta \right) \right)$$

$$= - \int d^4 x \partial^\rho T^c_{\rho \mu} = 0. \quad (45)$$

Thus, the canonical energy-momentum tensor becomes

$$T^c_{\rho \mu} := \text{tr} \left( \frac{1}{2} \left\{ \Pi_{\rho \nu}, \partial_\mu A_\nu \right\} - g_{\rho \mu} \mathcal{L}_{\text{inv}}^\theta \right). \quad (46)$$

Similarly, one gets

$$W^\mu_\rho T^\rho_{\Gamma_{\text{inv}}} := \int d^4 x \text{tr} F_{\mu \nu}(x) \frac{\delta T^\rho_{\Gamma_{\text{inv}}}}{\delta A_\nu(x)}$$

$$= \int d^4 x \text{tr} \left( \partial^\rho \left( \frac{1}{2} \left\{ - \Pi_{\rho \nu}, F_{\mu \nu} \right\} + g_{\rho \mu} \mathcal{L}_{\text{inv}}^\theta \right) \right)$$

$$= - \int d^4 x \partial^\rho T^s_{\rho \mu} = 0, \quad (47)$$

implying the following definition,

$$T^s_{\rho \mu} := \text{tr} \left( \frac{1}{2} \left\{ \Pi_{\rho \nu}, F_{\mu \nu} \right\} - g_{\rho \mu} \mathcal{L}_{\text{inv}}^\theta \right). \quad (48)$$

Both currents (46) and (48) are locally conserved,

$$\partial^\rho T^c_{\rho \mu} = \partial^\rho T^s_{\rho \mu} = 0, \quad (49)$$

and they are related by a Belinfante like procedure

$$T^s_{\rho \mu} = T^c_{\rho \mu} + \text{tr} \left( D^\nu(A_\mu \Pi_{\rho \nu}) \right)$$

$$= T^c_{\rho \mu} + \partial^\nu \chi_{[\nu \rho \mu]} \quad (50)$$

One observes that both versions of the energy-momentum tensor are neither symmetric nor traceless. This is due to the fact that the Lorentz invariance and the dilation symmetry are no longer maintained \[11\]. However, one has to stress that $T^s_{\rho \mu}$ is invariant with respect to infinitesimal gauge transformations \[2\].
5 The $U(1)$-case: $\theta$-deformed Maxwell theory

The most simple, but still interesting, case of a $\theta$-expanded gauge theory is the $U(1)$-NCYM-theory, the $\theta$-deformed Maxwell theory (without sources). One just replaces in the expressions derived in the previous section the matrix-valued $U(N)$ gauge field $A^a X^a$ by the ordinary photon field. Omitting the trace symbols we get

$$
\Gamma_{\text{inv}}^\theta[A] = \int d^4x \left( - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \theta^{\alpha\beta} (F_{\mu\alpha} F_{\nu\beta} F^{\mu\nu} - \frac{1}{4} F_{\alpha\beta} F^2) \right) + \mathcal{O}(\theta^2)
$$

$$
= \int d^4x \mathcal{L}_{\text{inv}}^\theta + \mathcal{O}(\theta^2).
$$

Again, with the canonical momentum,

$$
\Pi^\rho = \frac{\partial \mathcal{L}_{\text{inv}}^\theta}{\partial (\partial_\rho A_\nu)} = - F^{\rho\nu} + \frac{1}{4} \theta^{\rho\nu} F^{\alpha\beta} + \frac{1}{2} F^{\rho\nu} \theta^{\alpha\beta} F_{\alpha\beta}
$$

$$
- (\theta^{\nu\beta} F^\rho_{\alpha\beta} - \theta^{\beta\nu} F^\rho_{\alpha\beta}) - F^\rho_{\alpha} \theta^{\alpha\beta} F^\nu_{\beta},
$$

we find the equation of motion [14],

$$
\partial_\rho \Pi^\rho = 0.
$$

The stress tensors read

$$
T_{\rho\mu}^{c,\theta} = \Pi_{\rho\nu} \partial_\mu A^\nu - g_{\rho\mu} \mathcal{L}_{\text{inv}}^\theta,
$$

$$
T_{\rho\mu}^{s,\theta} = \Pi_{\rho\nu} F^\nu_{\mu} - g_{\rho\mu} \mathcal{L}_{\text{inv}}^\theta.
$$

Explicitly we have for $T_{\rho\mu}^{s,\theta}$

$$
T_{\rho\mu}^{s,\theta} = - g_{\rho\mu} \mathcal{L}_{\text{inv}}^\theta - F_{\mu\nu} F^\nu_{\rho} (1 - \frac{1}{2} \theta^{\alpha\beta} F_{\alpha\beta}) + \frac{1}{4} F_{\mu\nu} \theta^\alpha_{\rho} F^2
$$

$$
+ F_{\mu\nu} \theta^{\rho\beta} F_{\alpha\beta} F^{\nu\alpha} - (F_{\mu\alpha} F^\rho_{\nu\alpha} + F_{\rho\alpha} F_{\mu\nu}) F^\alpha_{\beta} \theta^{\nu\beta}.
$$

The latter equation confirms Kruglov’s result [14]. One has to stress that $T_{\rho\mu}^{s,\theta}$ is not symmetric and not traceless. Additionally, it is remarkable that in expanding the tensor (30) for the $U(1)$-case one gets

$$
\hat{T}_{\rho\mu}^{s,\theta} |_{\mathcal{O}(\theta)} = - g_{\rho\mu} \mathcal{L}_{\text{inv}}^\theta - F_{\mu\nu} F^\nu_{\rho} (1 - \frac{1}{2} \theta^{\alpha\beta} F_{\alpha\beta})
$$

$$
- (F_{\mu\alpha} F^\rho_{\nu\alpha} + F_{\rho\alpha} F_{\mu\nu}) F^\alpha_{\beta} \theta^{\nu\beta} + \theta^{\alpha\beta} \partial_\beta (A_\alpha F^\rho_{\mu} F^\nu_{\rho})
$$

$$
\neq T_{\rho\mu}^{s,\theta}
$$

We observe that (ignoring the total derivative) the nonsymmetric parts of $T_{\rho\mu}^{s,\theta}$ do not appear in the expansion of $\hat{T}_{\rho\mu}^{s,\theta}$. Moreover, these are exactly the terms where $\theta^{\nu\beta}$ carries a free index $\rho$. For $T_{\rho\mu}^{c,\theta}$ and $\hat{T}_{\rho\mu}^{c,\theta}$, we get the analogous result.

Thus we find that the calculation of the energy-momentum tensor does not commute with the Seiberg-Witten expansion of fields and Moyal products.
6 Conclusion

For noncommutative gauge field models we have studied (at the classical level) the construction of the various energy-momentum tensors in order to describe translation invariance of different noncommutative gauge field theories. Due to the presence of the deformation parameter $\theta^{\mu\nu}$ (as a constant, antisymmetric, fixed tensor) Lorentz and dilation invariances are manifestly broken, entailing that the corresponding stress tensors are not symmetric and not traceless. The obtained results may be the basis for the discussion of broken Lorentz and dilation symmetry.

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