On a family of quartic graphs: Hamiltonicity, matchings and isomorphism with circulants

John Baptist Gauci
Department of Mathematics, University of Malta, Malta.

Jean Paul Zerafa
Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università di Modena e Reggio Emilia, Via Campi 213/B, Modena, Italy

Abstract

A pairing of a graph $G$ is a perfect matching of the underlying complete graph $K_G$. A graph $G$ has the PH-property if for each one of its pairings, there exists a perfect matching of $G$ such that the union of the two gives rise to a Hamiltonian cycle of $K_G$. In 2015, Alahmadi et al. proved that the only three cubic graphs having the PH-property are the complete graph $K_4$, the complete bipartite graph $K_{3,3}$, and the 3-dimensional cube $Q_3$. Most naturally, the next step is to characterise the quartic graphs that have the PH-property, and the same authors mention that there exists an infinite family of quartic graphs (which are also circulant graphs) having the PH-property. In this work we propose a class of quartic graphs on two parameters, $n$ and $k$, which we call the class of accordion graphs $A[n,k]$, and show that the quartic graphs having the PH-property mentioned by Alahmadi et al. are in fact members of this general class of accordion graphs. We also study the PH-property of this class of accordion graphs, at times considering the pairings of $G$ which are also perfect matchings of $G$. Furthermore, there is a close relationship between accordion graphs and the Cartesian product of two cycles. Motivated by a recent work by Bogdanowicz (2015), we give a complete characterisation of those accordion graphs that are circulant graphs. In fact, we show that $A[n,k]$ is not circulant if and only if both $n$ and $k$ are even, such that $k \geq 4$.

Keywords: Hamiltonian cycle, perfect matching, quartic graph, circulant graph.

Math. Subj. Class.: 05C70, 05C45, 05C60.

1 Introduction

The graphs considered in this work are undirected and simple (without loops or multiple edges). A graph $G$ having vertex set $V(G)$ of cardinality $n$ and edge set $E(G)$ has a perfect matching $M$ if there is a set of exactly $n/2$ edges of $G$ which have all the $n$ vertices of $G$ as endvertices. This tacitly implies that a necessary condition for a graph to have a perfect matching is that it has an even number of vertices. Let $k$ be a non-negative integer. A path on $k$ vertices, denoted by $P_k$, is a sequence of mutually distinct vertices $v_1, \ldots, v_k$ with corresponding edge set $\{v_iv_{i+1} : i \in [k-1]\}$ (if $k > 1$). For $k \geq 3$, a cycle of length $k$ (or a $k$-cycle), denoted by $C_k$, is a sequence of mutually distinct vertices $v_1, v_2, \ldots, v_k$ with

E-mail addresses: john-baptist.gauci@um.edu.mt (John Baptist Gauci), jeanpaul.zerafa@unimore.it (Jean Paul Zerafa)
corresponding edge set \( \{v_1v_2, \ldots, v_k^{-1}v_k, v_kv_1\} \). We denote the cycle \( C_k \) by \((v_1, \ldots, v_k)\).

A Hamiltonian cycle (path) of a graph \( G \) is a cycle (path) in \( G \) that visits all the vertices of \( G \) exactly once. We refer the reader to [5] for definitions and notation not explicitly stated here.

The two main concepts of this paper, namely Hamiltonicity and perfect matchings, have been extensively studied in literature by various authors and in various settings. The study of both together has recently been put in the limelight in [2], where the authors ask whether a perfect matching of the complete graph \( K_n \) of both together has recently been put in the limelight in \([2]\), where the authors ask whether another perfect matching \( M \) of \( G \), called a pairing of \( G \), can be extended to a Hamiltonian cycle of \( K_G \) by using only edges of \( G \). Equivalently, given a pairing \( M \) of \( G \), we say that \( M \) can be extended to a Hamiltonian cycle \( H \) of \( K_G \) if we can find another perfect matching \( N \) of \( G \) such that \( M \cup N = E(H) \), where \( E(H) \) is the corresponding edge set of the Hamiltonian cycle \( H \). By using this terminology, the authors of [2] define a graph \( G \) on an even number of vertices as having the Pairing-Hamiltonian property (or, for short, the PH-property) if every pairing \( M \) of \( G \) can be extended to a Hamiltonian cycle \( H \) of \( K_G \). On a similar flavour, the authors of [1] define the Perfect-Matching-Hamiltonian property (or, for short, the PMH-property) for a graph \( G \) if every perfect matching of \( G \) can be extended to a Hamiltonian cycle of \( K_G \), which in this case, would also be a Hamiltonian cycle of \( G \) itself. It can be easily seen that if a graph does not have the PMH-property, then it surely does not have the PH-property, although the converse is not true.

A complete characterisation of the cubic graphs having the PH-property was given in \([2]\), and thus the most obvious next step would be to characterise 4-regular graphs which have the PH-property. This endeavour proved to be more elusive, and thus far a complete characterisation of such quartic graphs remains unknown. In an attempt to advance in this direction, in Section 2, we define a class of graphs on two parameters, \( n \) and \( k \), which we call the class of accordion graphs \( A[n,k] \). This class presents a natural generalisation of the well-known antiprism graphs, and of a class of graphs which is known to have the PH-property, corresponding to \( A[n,1] \) and \( A[n,2] \), respectively. In this section we discuss some fundamental properties and characteristics of accordion graphs and, in particular, we see that accordion graphs can be drawn in a grid-like manner, which resembles a drawing of the Cartesian product of two cycles \( C_{n_1} \square C_{n_2} \), for appropriate cycle lengths \( n_1 \) and \( n_2 \). In 2015, Bogdanowicz [4] gave all possible values of \( n_1 \) and \( n_2 \) for which the graph \( C_{n_1} \square C_{n_2} \) is circulant, namely when \( \gcd(n_1, n_2) = 1 \). Due to the similarity between the two classes of graphs, in Section 5, we give a complete characterisation of which accordion graphs are circulant graphs. In Section 3 we prove that all antiprism graphs have the PMH-property but only four of them also have the PH-property. In the same section, we provide a proof that \( A[n,2] \) has the PH-property, which result, although known to be communicated to the authors of \([2]\), has no published proof. These encouraging outcomes motivate our proposal of the class of accordion graphs as a possible candidate for graphs having the PMH-property and/or the PH-property. Empirical evidence suggests that, apart from the above mentioned, there are (possibly an infinite number of) other accordion graphs which have the PMH-property, and possibly some of them even have the PH-property, but a proof for this is currently unavailable. In Section 4, by extending an argument introduced in [7], we show that we can exclude some graphs \( A[n,k] \) from this search for graphs having the PMH- and/or the PH-property. In fact, we prove that the graphs \( A[n,k] \) for which the greatest common divisor of \( n \) and \( k \) is at least 5 do not have the PMH-property. The technique used does not seem to lend itself when coming to show whether the remaining
accordion graphs have, or do not have, the PH-property or the PMH-property. We thus pose some related questions and open problems in Section 6.

2 Accordion graphs

Definition 2.1. Let \( n \) and \( k \) be integers such that \( n \geq 3 \) and \( 0 < k \leq \frac{n}{2} \). The accordion graph \( A[n, k] \) is the quartic graph with vertices \( \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\} \) such that the edge set consists of the edges

\[
\{u_iu_{i+1}, v_iv_{i+1}, u_iv_i, u_iv_{i+k} : i \in [n]\}.
\]

The edges \( u_iu_{i+1} \) and \( v_iv_{i+1} \) are called the outer-cycle edges and the inner-cycle edges, respectively, or simply the cycle edges, collectively; and the edges \( u_iv_i \) and \( u_iv_{i+k} \) are called the vertical spokes and the diagonal spokes, respectively, or simply the spokes, collectively.

For simplicity, we sometimes refer to the accordion graph \( A[n, k] \) as the accordion \( A[A[n, k]] \).

An observation that will prove to be useful in the sequel revolves around the greatest common divisor of \( n \) and \( k \), denoted by \( \gcd(n, k) \). However, before proceeding further, we give the definition of the Cartesian product of graphs. The Cartesian product \( G \square H \) of two graphs \( G \) and \( H \) is a graph whose vertex set is the Cartesian product \( V(G) \times V(H) \) of \( V(G) \) and \( V(H) \). Two vertices \( (u_i, v_j) \) and \( (u_k, v_l) \) are adjacent precisely if \( u_i = u_k \) and \( v_jv_l \in E(H) \) or \( u_iu_k \in E(G) \) and \( v_j = v_l \).

Remark 2.2. The graph obtained from \( A[n, k] \) after deleting the edges

\[
\{u_{tq}u_{tq+1}, v_{tq}v_{tq+1} : q = \gcd(n, k) \text{ and } t \in \{1, \ldots, \frac{n}{q}\}\}
\]

![Figure 1: The accordion graph \( A[10, 3] \)](image)
is isomorphic to the Cartesian product $C_{2^n} \square P_q$. This can be easily deduced by an appropriate drawing of $A[n, k]$, as shown in Figure 2 for the case when $k = 5$ and $\gcd(n, k) = 5$. Thus, any perfect matching of $C_{2^n} \square P_q$ is also a perfect matching of $A[n, k]$, although the converse is trivially not true.

![Diagram](image)

Figure 2: Two different drawings of $A[n, k]$ when $k = 5$ and $\gcd(n, k) = 5$

3 The accordion graph $A[n, k]$ when $k \leq 2$

3.1 $A[n, 1]$

As already mentioned above, the accordion graph $A[n, 1]$ is isomorphic to the widely known antiprism graph $A_n$ on $2n$ vertices. Let $M$ be a perfect matching of $A_n$. We note that, if $M$ contains at least one vertical spoke, then no diagonal spoke can be contained in $M$, and if any inner-cycle edges are in $M$, then the outer-cycle edges having the same indices must also belong to $M$. A similar argument can be made if $M$ contains diagonal spokes. Thus, for every $i, j \in [n]$, $u_i v_i \in M$ or $\{u_i u_{i+1}, v_i v_{i+1}\} \subset M$ if and only if $u_j v_{j+1} \notin M$ or $\{u_j u_{j+1}, v_j v_{j+2}\} \notin M$. This can be summarised in the following remark.

Remark 3.1. Let $M$ be a perfect matching of $A_n$. Then, $M$ is either a perfect matching of $A_n - \{u_i v_i : i \in [n]\}$ or of $A_n - \{u_i v_{i+1} : i \in [n]\}$.

Consequently, in what follows, without loss of generality, we only consider perfect matchings of $A_n$ containing spokes of the type $u_i v_i$, that is, vertical spokes, and if a perfect matching contains the edge $u_j u_{j+1}$, then it must also contain the edge $v_j v_{j+1}$.

Theorem 3.2. The antiprism $A_n$ is PMH.

Proof. Let $M$ be a perfect matching of $A_n$. Consider first the case when $M = \{u_i v_i : i \in [n]\}$. It is easy to see that $(v_1, u_1, v_2, u_2, \ldots, v_{n-1}, u_{n-1}, v_n, u_n)$ is a Hamiltonian
cycle of \(A_n\) containing \(M\). So assume that \(M\) does not consist of only vertical spokes. Without loss of generality, we can assume that \(M\) contains the edges \(u_n u_{n+1}\) and \(v_n v_{n+1}\), by Remark 3.1. We proceed by induction on \(n\). The antiprism \(A_3\) was already shown to be PMH in [1], since \(A_3\) is the line graph of the complete graph \(K_4\). So assume result is true up to \(n \geq 3\), and consider \(A_{n+1}\) and a perfect matching \(M\) of \(A_{n+1}\). Let \(M'\) be \(M \cup \{u_n v_n\} - \{u_n u_{n+1}, v_n v_{n+1}\}\). Then, \(M'\) is a perfect matching of \(A_n\), and so, by induction, there exists a Hamiltonian cycle \(H'\) of \(A_n\) which contains \(M'\). We next show that \(H'\) can be extended to a Hamiltonian cycle \(H\) of \(A_{n+1}\) containing \(M\) by considering each of the following possible induced paths in \(H\):

(i) \(u_{n-1}, u_n, v_n, v_{n-1}\) is replaced by \(u_{n-1}, u_n, u_{n+1}, v_n, v_{n-1}\) (similarly \(u_1, u_n, v_n, v_1\) is replaced by \(u_1, u_{n+1}, u_n, v_n, v_{n+1}\)),

(ii) \(u_{n-1}, v_n, u_n, u_1\) is replaced by \(u_{n-1}, v_n, v_{n+1}, u_n, u_{n+1}, u_1\) (similarly \(v_n, v_{n+1}, u_n, u_{n+1}, v_1\) is replaced by \(v_n, v_{n+1}, u_n, u_{n+1}, v_1\)), and

(iii) \(u_{n-1}, u_n, v_n, v_1\) or \(u_{n-1}, v_n, u_n, v_1\) are replaced by \(u_{n-1}, v_n, v_{n+1}, u_n, u_{n+1}, v_1\) (similarly \(v_n, v_{n+1}, u_n, u_{n+1}, v_1\) is replaced by \(v_n, v_{n+1}, u_n, u_{n+1}, v_1\)).

Consequently, \(A_{n+1}\) is PMH, proving our theorem. \(\square\)

**Theorem 3.3.** The only antiprisms having the PH-property are \(A_3, A_4, A_5\) and \(A_6\).

**Proof.** The graph \(A_3\) is PMH as already explained in Theorem 3.2, so what is left to show is that every pairing \(M\) of \(A_3\) containing some edge belonging to \(E(K_{A_3}) - E(A_3)\) (referred to as a non-edge) can be extended to a Hamiltonian cycle of \(K_{A_3}\). The pairing \(M\) can only contain one or three non-edges. It is an easy exercise to check that, in either case, \(M\) can be extended to a Hamiltonian cycle of \(K_{A_3}\) by using only edges of \(A_3\).

The graph \(A_4\) has the PH-property because it contains the cube \(Q_3\) as a spanning subgraph, and by the main theorem in [6], all hypercubes have the PH-property.

An exhaustive computer check was conducted through Wolfram Mathematica [10] to verify that all pairings of the antiprism graphs \(A_5\) and \(A_6\) can be extended to a Hamiltonian cycle of the same graphs, thus proving (by brute-force) that they both have the PH-property.

![Figure 3: A pairing in \(A_n, n \geq 7\), which is not extensible to a Hamiltonian cycle of \(K_{A_n}\).](image)

The antiprism \(A_7\) does not have the PH-property because the pairing \(M = \{u_1 v_5, u_2 v_2, u_3 v_3, u_4 v_4, u_5 v_6, u_6 v_7, u_7 v_1\}\), depicted in Figure 3, cannot be extended to a Hamiltonian cycle of \(K_{A_7}\). For, suppose not, and let \(N\) be a perfect matching of \(A_7\) such that \(M \cup N\) gives a Hamiltonian cycle of \(K_{A_7}\). Then, \(|N \cap \{u_2 u_3, u_2 v_3, v_2 v_3\}|\), and \(|N \cap \{u_3 u_4, v_3 v_4, v_3 v_4\}|\) must both be equal to 1. Consequently, \(M \cup N\) induces a \(M\)-alternating path containing the edges \(u_2 v_2, u_3 v_3, u_4 v_4\) such that its endvertices are \(x \in \{u_2, v_2\}\) and \(y \in \{u_4, v_4\}\). If \(x u_1 \in N\), then \(N \cap \{u_7 u_1, u_1 v_1, v_1 v_2\}\) is empty, implying that \(M \cup N\) induces the 4-cycle \((u_6, u_7, v_1, v_7)\), a contradiction. Therefore, \(N\) contains the edge \(v_2 v_1\).
implying that \( x = v_2 \). Consequently, \( N \) contains the edge \( u_7u_1 \) as well. However, this implies that \( N \cap \{u_6u_7, u_7v_7, v_7v_1\} \) is empty, implying that \( M \cup N \) induces the 4-cycle \( (u_5, u_6, v_7, v_6) \), a contradiction once again. Thus, \( A_7 \) does not have the PH-property.

The pairing \( M \) considered above can be easily extended to a pairing of \( A_n \), for any \( n > 7 \), and by the same argument for \( A_7 \), we conclude that \( A_n \) does not have the PH-property for every \( n \geq 7 \).

\[ \square \]

### 3.2 \( A[n, 2] \)

In [2], it is mentioned that Seongmin Ok and Thomas Perrett informed the authors that they have obtained an infinite class of 4-regular graphs having the PH-property: such a graph in this family is obtained from a cycle of length at least three, by replacing each vertex by two isolated vertices and replacing each edge by the four edges joining the corresponding pairs of vertices. More formally, the resulting graph starting from a \( n \)-cycle, for \( n \geq 3 \), has vertex set \( \{s_i, t_i : i \in [n]\} \) such that, for every \( i \in [n] \), \( s_i \) and \( t_i \) are both adjacent to \( s_{i+1} \) (mod \( n \)) and \( t_{i+1} \) (mod \( n \)). As far as we know, no proof of this can be found in literature, and in what follows we give a proof of this result. Before we proceed, we remark that the function that maps \( s_i \) to \( u_i \), and \( t_i \) to \( v_{i+1} \) (mod \( n \)), for every \( i \in [n] \), is an isomorphism between the above graph and the accordion graph \( A[n, 2] \).

**Theorem 3.4.** The accordion graph \( A[n, 2] \) has the PH-property, for every \( n \geq 3 \).

**Proof.** Let \( A'[n, 2] \) be the graph depicted in Figure 4, obtained from \( A[n, 2] \) after deleting the following set of edges: \( \{u_1u_n, v_1v_n, u_{n-1}v_1, u_nv_2\} \). We use induction on \( n \) to show that \( A'[n, 2] \) has the PH-property, for every \( n \geq 3 \). The result then follows since \( A'[n, 2] \) is a spanning subgraph of \( A[n, 2] \).

![Figure 4: The graph \( A'[n, 2] \)](image)

When \( n = 3 \), one can show by a case-by-case analysis (or using an exhaustive computer search) that the graph \( A'[3, 2] \) has the PH-property. So we assume that \( n > 3 \) and let \( M \) be a pairing of \( A'[n, 2] \), hereafter denoted by \( G \). If \( M \) consists of only vertical spokes, that is, \( M = \{u_iv_i : i \in [n]\} \), then

(i) \( \{v_1, u_1, v_3, u_3, \ldots, u_{n-1}, v_n, u_n, u_{n-2}, v_{n-2}, \ldots, u_2, v_2\} \), when \( n \) is even, or

(ii) \( \{v_1, u_1, v_3, u_3, \ldots, v_n, u_n, u_{n-1}, v_{n-1}, u_{n-3} \ldots, u_2, v_2\} \), when \( n \) is odd,

is a Hamiltonian cycle of \( K_G \) containing the pairing \( M \). So assume that \( M \neq \{u_iv_i : i \in [n]\} \). Consequently, there exists \( i \in [n] \) such that \( u_iv_i \not\in M \). Let \( \alpha = \max\{i \in [n] : u_iv_i \not\in M\} \). We note that by a parity argument, \( \alpha > 1 \). Consider the two subgraphs of \( G \) induced by \( \{u_1, v_1, \ldots, u_{\alpha-1}, v_{\alpha-1}\} \), denoted by \( G_1 \), and \( \{u_{\alpha}, v_{\alpha}, \ldots, u_n, v_n\} \), denoted by \( G_2 \). We remark that \( G_1 \) and \( G_2 \) are the two components obtained after deleting from \( G \) the set of edges \( X \), where \( X = \{u_{\alpha-1}u_{\alpha}, v_{\alpha-1}v_{\alpha}, u_{\alpha}v_{\alpha+1}\} \) if \( \alpha = n \), and \( X = \{u_{\alpha-1}u_{\alpha}, v_{\alpha-1}v_{\alpha}, u_{\alpha-2}v_{\alpha}, u_{\alpha-1}v_{\alpha+1}\} \) otherwise. We also remark that depending on the
value of $\alpha$, we have that $G_1$ is isomorphic to either $K_2, C_4$ or $A'[\alpha - 1, 2]$, and that $G_2$ is isomorphic to $K_2, C_4$ or $A'[n - (\alpha - 1), 2]$. Without loss of generality, we assume that $|V(G_1)| \geq |V(G_2)|$, implying that $3 \leq \alpha \leq n$.

Next, consider the two edges $yu_\alpha$ and $zv_\alpha$ in $M$. Since for $i > \alpha$, $u_iv_i \in M$, then $y$ and $z$ both belong to $\{u_1, v_1, \ldots, u_{\alpha-1}, v_{\alpha-1}\}$. Let $M_1 = (M \cap E(K_{G_1})) \cup \{yz\}$. One can see that $M_1$ is a pairing of $G_1$, and so, by induction, $M_1$ is contained in a Hamiltonian cycle $H_1$ of $K_{G_1}$. Consequently, $H_1$ contains a Hamiltonian path of $K_{G_1}$ with endvertices $y$ and $z$. We denote this path by $H'_1$.

When $\alpha = n$, we obtain a Hamiltonian cycle of $K_G$ containing $M$, by adding the edges $yu_\alpha, u_\alpha v_\alpha, v_\alpha z$ to $E(H'_1)$. For $\alpha \leq n - 1$, we proceed as follows. Let $M_2 = (M \cap E(G_2)) \cup \{u_\alpha v_\alpha\}$. This is clearly a pairing of $G_2$, and so, by induction, there exists a Hamiltonian cycle $H_2$ of $K_{G_2}$ containing $M_2$. Let $H'_2$ be the Hamiltonian path of $G_2$ obtained by deleting the edge $u_\alpha v_\alpha$ from $E(H_2)$. Consequently, combining $H'_1$ and $H'_2$ together with the edges $yu_\alpha$ and $zv_\alpha$, we form a Hamiltonian cycle of $K_G$ containing $M$, as required.

\section{The accordion graph $A[n, k]$ when $\gcd(n, k) \geq 5$}

The method adopted in this section follows a similar line of thought as that used in [7]. Let $q = \gcd(n, k) \geq 5$, let $p = \frac{2n}{\gcd(n, k)}$, and let $p' = \frac{p}{2}$. Consider a grid-like drawing of the accordion graph $A[n, k]$ as in Remark 2.2. For simplicity, we let the vertices $v_1, u_1, v_1+k, u_1+k, \ldots, v_1+(p'-1)k, u_1+(p'-1)k$, be referred to as $a_1, a_2, \ldots, a_p$. We define the vertices $\{b_i, c_i, d_i, e_i : i \in [p]\}$ in a similar way, where, in particular, the vertices $b_1, \ldots, e_1$, and $b_2, \ldots, e_2$ represent $v_2, \ldots, v_5$, and $u_2, \ldots, u_5$, respectively. If $\gcd(n, k) = 6$, we refer to $v_6, u_6, v_6+k, u_6+k, \ldots, v_6+(p'-1)k, u_6+(p'-1)k$, as $f_1, f_2, \ldots, f_p$, and if $\gcd(n, k) > 6$, we simply do not label all the other vertices since we are only interested in the subgraph of $A[n, k]$ generated by the vertices $\{a_i, \ldots, f_i : i \in [p]\}$. This can be seen better in Figure 5.

\begin{figure}[h]
\centering
\begin{tikzpicture}
\node at (0,0) {$a_1$}; \node at (0,-1) {$a_2$}; \node at (0,-2) {$a_3$}; \node at (0,-3) {$a_4$}; \node at (0,-4) {$a_5$}; \node at (0,-5) {$a_6$};
\node at (1,0) {$b_1$}; \node at (1,-1) {$b_2$}; \node at (1,-2) {$b_3$}; \node at (1,-3) {$b_4$}; \node at (1,-4) {$b_5$}; \node at (1,-5) {$b_6$};
\node at (2,0) {$c_1$}; \node at (2,-1) {$c_2$}; \node at (2,-2) {$c_3$}; \node at (2,-3) {$c_4$}; \node at (2,-4) {$c_5$}; \node at (2,-5) {$c_6$};
\node at (3,0) {$d_1$}; \node at (3,-1) {$d_2$}; \node at (3,-2) {$d_3$}; \node at (3,-3) {$d_4$}; \node at (3,-4) {$d_5$}; \node at (3,-5) {$d_6$};
\node at (4,0) {$e_1$}; \node at (4,-1) {$e_2$}; \node at (4,-2) {$e_3$}; \node at (4,-3) {$e_4$}; \node at (4,-4) {$e_5$}; \node at (4,-5) {$e_6$};
\node at (5,0) {$f_1$}; \node at (5,-1) {$f_2$}; \node at (5,-2) {$f_3$}; \node at (5,-3) {$f_4$}; \node at (5,-4) {$f_5$}; \node at (5,-5) {$f_6$};
\end{tikzpicture}
\caption{Edges belonging to $S$ in $A[n, k]$ when $\gcd(n, k) \geq 6$}
\end{figure}
For each $i \in [p]$, let $L_i$ and $R_i$ represent the edges $b_ie_i$ and $d_ie_i$, respectively, whilst
$L = \{L_i : i \in [p]\}$ and $R = \{R_i : i \in [p]\}$. Let $S$ denote the following set of edges:

(i) $a_ia_{i+1}$, for every even $i \in [p]$,  
(ii) $b_ib_{i+1}$ and $e_ie_{i+1}$, for every odd $i \in [p]$,  
(iii) $c_id_i$, for every $i \in [p]$, and  
(iv) in the case when $q \geq 6$, $f_if_{i+1}$, for every even $i \in [p]$.

Since $p$ is even, $A[n, k]$ has a perfect matching $M$ that contains $S$.

In [7], it was shown that $C_p \Box C_q$ is not PMH except when $p = q = 4$. In the case when $q \geq 6$, the proof utilises exclusively the set $S$ of edges described above (and adapted to $C_p \Box C_q$) to show that a perfect matching $M$ containing this set cannot be extended to a Hamiltonian cycle of $C_p \Box C_q$. Since the same set $S$ of edges can also be chosen in a perfect matching of $A[n, k]$, the proof extends naturally and hence we have the following result.

**Lemma 4.1.** $A[n, k]$ is not PMH if $\gcd(n, k) \geq 6$.

We shall now show that $A[n, k]$ is not PMH in the case when $\gcd(n, k) = 5$. We remark that, in this case, the proof in [7] cannot be extended to $A[n, k]$ because it makes use of the edges $e_ie_i$ of $C_p \Box C_q$, which are missing in $A[n, k]$. For the remaining part of this section, we shall need some results extracted from the proof of the main theorem in [7], and adapted for accordion graphs. We note that the arguments in [7] are quite elaborate and lengthy, but when adapted to our case, they remain essentially the same. Hence our decision not to reproduce them in detail here but to only give the main points in the following lemma.

**Lemma 4.2.** [7] Let $\gcd(n, k) \geq 5$. If there exists a perfect matching $M$ of $A[n, k]$ containing $S$ and another perfect matching $N$ of $A[n, k]$ such that $M \cup N$ is a Hamiltonian cycle $H$ of $A[n, k]$, then the following statements hold.

(i) $|L \cap N|$ and $|R \cap N|$ are both even and nonzero.

(ii) A maximal sequence of consecutive edges belonging to $L - N$ (or $R - N$) is of even length (consecutive edges are edges having indices which are consecutive integers taken modulo $p$, with complete residue system $\{1, \ldots, p\}$).

(iii) The edges of $L \cap N$ are partitioned into pairs of edges $\{L_\gamma, L_\gamma'\}$, where $\gamma$ is odd and $\gamma'$ is the least integer greater than $\gamma$ (taken modulo $p$) such that $L_\gamma' \in N$ (and similarly for $R \cap N$). In this case, if we start tracing the Hamiltonian cycle $H$ from $c_\gamma$ going towards $b_\gamma$, then $H$ contains a path with edges alternating in $N$ and $M$, starting from $c_\gamma$ and ending at $c_{\gamma'}$, with the internal vertices on this path being $\{b_\gamma, b_{\gamma'}\}$, if $\gamma' = \gamma + 1$, or belonging to the set $\{b_\gamma, a_{\gamma+1}, b_{\gamma+1}, \ldots, a_{\gamma'-1}, b_{\gamma'-1}, b_{\gamma'}\}$, if $\gamma' \neq \gamma + 1$. In each of these two cases we refer to such a path between $c_\gamma$ and $c_{\gamma'}$ as an $L_\gamma L_{\gamma'}$-bracket, or just a left-bracket, with $L_\gamma$ and $L_{\gamma'}$ being the upper and lower edges of the bracket, respectively. Right-brackets are defined similarly.

(iv) If $L_{\gamma'}$ is a lower edge of a left bracket, then $L_{\gamma'+1}$ belongs to $N$, and so is an upper edge of a possibly different left bracket (and similarly for right-brackets).

(v) If $L_i \notin N$, for some even $i \in [p]$, then $c_ic_{i+1} \in N$ (and similarly for edges belonging to $R - N$).
Theorem 4.4. The accordion graph $A[n, k]$ is not PMH if $\gcd(n, k) \geq 5$.

5 Accordsions and circulant graphs

For distinct integers $a$ and $b$, $\text{Ci}[2n, \{a, b\}]$ denotes the quartic circulant graph on the vertices $\{x_i : i \in [2n]\}$, such that $x_i$ is adjacent to the vertices in the set $\{x_{i+a}, x_{i-a}, x_{i+b}, x_{i-b}\}$. We say that the edges arising from these adjacencies have length $a$ and $b$, accordingly, and we also remark that operations in the indices of the vertices $x_i$ are taken modulo $2n$, with complete residue system $\{1, \ldots, 2n\}$. In [4], the necessary and sufficient conditions for a circulant graph to be isomorphic to a Cartesian product of two cycles were
given. Motivated by this, we show that there is a non-empty intersection between the class of accordions \( A[n, k] \) and the class of circulant graphs \( Ci[2n, \{a, b\}] \), although neither one is contained in the other. In particular, we here show that the only accordion graphs \( A[n, k] \) which are not circulant are those with both \( n \) and \( k \) even, such that \( k \geq 4 \). Results of a similar flavour about 4-regular circulant graphs, perfect matchings and Hamiltonicity can be found in [8].

We shall be using the following two results about circulant graphs (not necessarily quartic). Let the circulant graph on \( n' \) vertices and with \( r \) different edge lengths be denoted by \( Ci[n', \{a_1, \ldots, a_r\}] \). The first result, implied by a classical result in number theory, says that \( Ci[n', \{a_1, \ldots, a_r\}] \) has \( \gcd(n', a_1, \ldots, a_r) \) isomorphic connected components (see [3]). Consequently, in our case we have that the circulant graph \( Ci[2n, \{a, b\}] \) is connected if and only if \( \gcd(2n, a, b) = 1 \). Secondly, let \( \gcd(n', a_1, \ldots, a_r) = 1 \). Heuberger [9] showed that \( Ci[n', \{a_1, \ldots, a_r\}] \) is bipartite if and only if \( a_1, \ldots, a_r \) are odd and \( n' \) is even. Restated for our purposes we have that \( Ci[2n, \{a, b\}] \) is bipartite if and only if \( a \) and \( b \) are both odd.

**Remark 5.1.** When \( n \) and \( k \) are even, the sets \( \{u_1, v_2, u_3, v_4, \ldots, u_{n-1}, v_n\} \) and \( \{v_1, u_2, v_3, u_4, \ldots, v_{n-1}, u_n\} \) are two independent sets of vertices of \( A[n, k] \), implying that for these values of \( n \) and \( k \), \( A[n, k] \) is bipartite. On the other hand, it is easy to see that when at least one of \( n \) and \( k \) is odd, \( A[n, k] \) is not bipartite. Consequently, \( A[n, k] \) is bipartite if and only if \( n \) and \( k \) are both even.

Before proceeding, we recall that operations in the indices of the vertices of \( A[n, k] \) and \( Ci[2n, \{a, b\}] \) are taken modulo \( n \) and modulo \( 2n \), respectively.

**Lemma 5.2.** The accordion graph \( A[n, 2] \) is circulant for any even integer \( n \geq 4 \).

**Proof.** For every even integer \( n \geq 4 \), we claim that the following function \( \phi : V(A[n, 2]) \to V(Ci[2n, \{1, n-1\}]) \) defined by:

- \( \phi : u_i \mapsto x_{1+(n-1)(i-1)} \pmod{2n} \) for all \( i \in [n] \),
- \( \phi : v_1 \mapsto x_{2+n} \pmod{2n} \), and
- \( \phi : v_i \mapsto x_{2+(n-1)(i-1)} \pmod{2n} \) for \( 2 \leq i \leq n \),

is an isomorphism. We first show that the function \( \phi \) is bijective. Since \( n-1 \geq 3 \) is odd and \( \gcd(n, n-1) = 1 \), we have that \( \gcd(2n, n-1) = 1 \), and so the vertices in \( \{\phi(u_i) : 1 \leq i \leq n\} \) are mutually distinct. For the same reason, since \( 2 + n \equiv 1 + (n-1)(n-1) \pmod{2n} \), the vertex \( \phi(v_1) \) is distinct from any vertex \( \phi(u_i) \). Moreover, since \( 1 \neq n - 1 \) and the vertices \( \phi(u_2), \ldots, \phi(u_n) \) are mutually distinct, the vertices in \( \{\phi(v_j) : 2 \leq j \leq n\} \) are mutually distinct as well, and are not equal to some vertex \( \phi(u_i) \). Finally, since \( \gcd(n, n-1) = 1 \), we have that \( 2 + (n-1)(i-1) \neq 2 + n \pmod{2n} \), for any \( 2 \leq i \leq n \). Consequently, \( \phi(v_1) \) is distinct from any vertex \( \phi(v_j) \), proving that \( \phi \) is, in fact, bijective. We next show that \( \phi \) is an isomorphism. Since \( Ci[2n, \{1, n-1\}] \) has the same number of edges as \( A[n, 2] \), it suffices to show that an edge in \( A[n, 2] \) is mapped to an edge in \( Ci[2n, \{1, n-1\}] \).

We first take an edge \( u_iu_j \) from the outer-cycle of \( A[n, 2] \), for some \( i \in [n] \) and \( j \equiv i+1 \pmod{n} \), without loss of generality. Consider \( \phi(u_i)\phi(u_j) \). The length of \( \phi(u_i)\phi(u_j) \) can be calculated using \( 1 + (n-1)(j-1) - (1 + (n-1)(i-1)) \pmod{2n} \). This is equal to \( n-1 \), when \( j \neq 1 \), and to \(-n^2 + 2n - 1 \equiv -1 \pmod{2n} \), otherwise. Since in
Lemma 5.3. The length of the edge $\phi(u_i)\phi(u_j)$ belongs to $\{\pm 1, \pm (n-1)\}$, $\phi(u_i)\phi(u_j) \in E(Ci[2n, \{1, n-1\}])$. 

Next, we consider the vertical spoke $u_iv_i$, for some $i \in [n]$. The length of the edge $\phi(u_i)\phi(v_i)$ is $2 + n - (1 + (n-1)(i-1)) \equiv -(n-1) \pmod{2n}$, when $i = 1$, and $2 + (n-1)(i-1) - (1 + (n-1)(i-1)) = 1$, otherwise, implying that the length of the edge $\phi(u_i)\phi(v_i)$ belongs to $\{\pm 1, \pm (n-1)\}$ in both cases. Consequently, $\phi(u_i)\phi(v_i) \in E(Ci[2n, \{1, n-1\}])$.

We now take the diagonal spoke $u_iv_j$, for some $i \in [n]$, and for $j \equiv i + 2 \pmod{n}$. The length of the edge $\phi(u_i)\phi(v_j)$ is:

- $2 + (n-1)(j-1) - (1 + (n-1)(i-1)) = 2n - 1 \equiv -1 \pmod{2n}$, when $1 \leq i \leq n - 2$,
- $2 + n - (1 + (n-1)(i-1)) = -n^2 + 4n - 1 \equiv -1 \pmod{2n}$, when $i = n - 1$, and
- $2 + (n-1)(j-1) - (1 + (n-1)(i-1)) = -n^2 + 3n - 1 \equiv n - 1 \pmod{2n}$, when $i = n$.

In each of these three cases, the length of $\phi(u_i)\phi(v_j)$ belongs to $\{\pm 1, \pm (n-1)\}$, implying that $\phi(u_i)\phi(v_j) \in E(Ci[2n, \{1, n-1\}])$.

Finally, we take an edge $v_iv_j$ from the inner-cycle of $A[n, 2]$, for some $i \in [n]$ and $j \equiv i + 1 \pmod{n}$, without loss of generality. The length of the edge $\phi(v_i)\phi(v_j)$ is:

- $2 + n - (2 + (n-1)(i-1)) = -n^2 + 3n - 1 \equiv n - 1 \pmod{2n}$, when $j = 1$,
- $2 + (n-1)(j-1) - (2 + n) = -1$, when $j = 2$, and
- $2 + (n-1)(j-1) - (2 + (n-1)(i-1)) = n - 1$, when $3 \leq j \leq n$.

This implies that the length of the edge $\phi(v_i)\phi(v_j)$ belongs to $\{\pm 1, \pm (n-1)\}$ in both cases, and so $\phi(v_i)\phi(v_j) \in E(Ci[2n, \{1, n-1\}])$.

There are no more cases to consider, proving our result.

We next show that the only circulant accordions with both $n$ and $k$ even are the ones having $k = 2$. By Remark 5.1, this means that the only circulant bipartite accordions are the ones with $n$ even and $k = 2$.

Lemma 5.3. For $n$ and $k$ even, the accordion graph $A[n, k]$ is circulant if and only if $k = 2$.

Proof. Let $n$ and $k$ be even. By Lemma 5.2, it suffices to show that accordion graphs admitting $k \geq 4$ are not circulant. We can further assume that $n \geq 8$, since the only accordions with $n = 4$ or $6$, and $k$ even are $A[4, 2]$ and $A[6, 2]$. Suppose, for contradiction, that for $n \geq 8$, there exists an even integer $k \geq 4$ such that $A[n, k]$ is circulant. Then, by Remark 5.1 and Heuberger’s result [9], $A[n, k] \simeq Ci[2n, \{a, b\}]$ for some distinct odd integers $a$ and $b$. For simplicity, we refer to $A[n, k]$, or equivalently $Ci[2n, \{a, b\}]$, by $G$.

By the definition of quartic circulant graphs we can assume that $1 \leq a < b \leq n - 1$.

Claim. $\gcd(2n, a) = \gcd(2n, b) = 1$.

Proof of Claim. Suppose that $\gcd(2n, a) \neq 1$, for contradiction. Then, the least common multiple of $a$ and $2n$ is $2na'$, for some $a' < a$. Consequently, there exists an even integer $p$
such that \( ap = 2na' \). Moreover, since \( a \neq a' \) and \( a \) is odd, \( \frac{a}{a'} \) (or equivalently \( \frac{2n}{p} \)) is odd and at least 3, and so \( p < n \). By considering the edges in \( G \) having length \( a \), there exists a partition \( \mathcal{P} \) of the \( 2n \) vertices of \( G \) into \( \frac{2n}{p} \) sets, each inducing a \( p \)-cycle. This follows since \( \gcd(2n,a) = \frac{2n}{p} \neq 1 \). Furthermore, \( \mathcal{P} \) has an odd number of components, namely \( \gcd(2n,a) \), or equivalently \( \frac{2n}{p} \).

Since \( G \) is a connected quartic graph and \( \frac{2n}{p} > 1 \), two vertices on a particular \( p \)-cycle in \( \mathcal{P} \) are adjacent in \( G \) if and only if there is an edge of length \( a \) between them, in other words, the subgraph induced by the vertices on a \( p \)-cycle in \( \mathcal{P} \) is the \( p \)-cycle itself. Therefore, the graph contains two adjacent vertices \( x_i \) and \( x_j \) belonging to two different \( p \)-cycles from \( \mathcal{P} \). Consequently, since \( i \equiv j \pmod{2n} \), the vertices of these two \( p \)-cycles induce \( C_p \square P_2 \), where \( P_2 \) is the path on two vertices. By a similar argument to that used on \( x_i \) and \( x_j \), we deduce that \( G \) contains a spanning subgraph \( G_0 \) isomorphic to \( C_p \square P_{\frac{2n}{p}} \).

We now denote the set \( \{ u_1, u_2, \ldots, u_n \} \) of vertices on the outer-cycle of \( \mathcal{A}[n, k] \) by \( \mathcal{U} \), and the set \( \{ v_1, v_2, \ldots, v_n \} \) of vertices on the inner-cycle of \( A[n, k] \) by \( \mathcal{V} \), and claim that:

(i) given two adjacent vertices from some \( p \)-cycle in \( \mathcal{P} \), say \( x_i \) and \( x_{i+a} \), if \( x_i \) is a vertex in \( \mathcal{U} \), then \( x_{i+a} \) is a vertex in \( \mathcal{V} \), or vice-versa; and

(ii) given two adjacent vertices from two different \( p \)-cycles in \( \mathcal{P} \), say \( x_i \) and \( x_{i+b} \), we have that either both belong to \( \mathcal{U} \) or both belong to \( \mathcal{V} \).

First of all, we note that the vertices inducing a \( p \)-cycle from \( \mathcal{P} \), cannot all belong to \( \mathcal{U} \), since the latter set of vertices induces a \( n \)-cycle, and \( p < n \). Similarly, the vertices inducing a \( p \)-cycle from \( \mathcal{P} \), cannot all belong to \( \mathcal{V} \). Secondly, let \( i \in [2n] \), such that \( x_i \) is of degree 3 in \( G_0 \), and \( x_ix_{i+b} \in E(G_0) \). Consider the 4-cycle \( (x_i, x_{i+a}, x_{i+a+b}, x_{i+b}) \). Since \( n > 4 \), these four vertices cannot all belong to \( \mathcal{U} \) (or \( \mathcal{V} \)). Also, we cannot have three of them which belong to \( \mathcal{U} \) (or \( \mathcal{V} \)), because otherwise we would have \( k = 2 \), or, \( k \equiv -2 \pmod{n} \), that is, \( k = n - 2 \). Since we are assuming that \( k \geq 4 \), we must have that \( k = n - 2 \), but by Definition 2.1, \( k \) is at most \( \frac{n}{2} \), and so, \( n - 2 \leq \frac{n}{2} \), a contradiction, since \( n \geq 8 \). This means that exactly two vertices from \( (x_i, x_{i+a}, x_{i+a+b}, x_{i+b}) \) belong to \( \mathcal{U} \), and the other two belong to \( \mathcal{V} \). Without loss of generality, assume that \( x_i \) belongs to \( \mathcal{U} \).

Suppose that \( x_{i+b} \notin \mathcal{U} \), for contradiction. Then, \( \mathcal{U} \) must contain exactly one of \( x_{i+a} \) and \( x_{i+a+b} \). Suppose we have \( x_{i+a+b} \in \mathcal{U} \). Consequently, \( x_{i+a} \) and \( x_{i+b} \) belong to \( \mathcal{V} \), and so since \( \frac{2n}{p} \geq 3 \), \( x_{i+2a+b} \) and \( x_{i+a+2b} \) belong to \( \mathcal{U} \), giving rise to a 4-cycle in \( G_0 \) with three of its vertices belonging to \( \mathcal{U} \), a contradiction. Therefore, we have \( x_{i+a} \in \mathcal{U} \). This means that \( x_{i+b} \) and \( x_{i+a+b} \) both belong to \( \mathcal{V} \). Suppose further that \( x_{i+2a} \in \mathcal{V} \). Since we cannot have three vertices in a 4-cycle belonging to \( \mathcal{V} \), \( x_{i+2a+b} \in \mathcal{U} \). However, this once again gives rise to a 4-cycle in \( G_0 \) with three of its vertices belonging to \( \mathcal{U} \), a contradiction. Therefore, \( x_{i+2a} \) must belong to \( \mathcal{U} \). By repeating the same argument we get that all the vertices in the \( p \)-cycle, from \( \mathcal{P} \), containing \( x_i \) belong to \( \mathcal{U} \), a contradiction. Hence, \( x_{i+a} \notin \mathcal{U} \). Thus, neither one of \( x_{i+a} \) and \( x_{i+a+b} \) is in \( \mathcal{U} \), contradicting our initial assumption. This implies that \( x_{i+b} \in \mathcal{U} \), and that \( x_{i \pm a} \) and \( x_{i+b \pm a} \) belong to \( \mathcal{V} \). This forces all the vertices not considered so far to satisfy the two conditions in the above claim.

Thus, by Remark 2.2, \( \frac{2n}{p} = \gcd(n,k) \). This is a contradiction, since \( \frac{2n}{p} \) is odd and the greatest common divisor of two even numbers is even. Hence, \( \gcd(2n,a) = 1 \), and by a similar reasoning, \( \gcd(2n,b) = 1 \) as well. ■
This implies that \( a \) does not divide \( n \), \( b \) does not divide \( n \), and that the edges of \( G \) can be partitioned in two Hamiltonian cycles induced by the edges having length \( a \) and \( b \), respectively.

In particular, since \( a \) does not divide \( 2n \), there exists an edge of length \( a \) with both endvertices belonging to \( \{ u_i : i \in [n] \} \subset V(A[n,k]) \). Without loss of generality, assume that \( u_1u_2 \) has length \( a \), and consider the 4-cycle \( C = (u_1,u_2,v_2,v_1) \). Since the edges having length \( a \) (and similarly the edges having length \( b \)) induce a Hamiltonian cycle, and \( n > 4 \), the lengths of the edges in \( C \) cannot all be the same. Hence, the lengths of the edges \( (u_1u_2,v_2v_1,v_1u_1) \) of \( C \) can be of Type A1 := \( (a,b,b,b) \), Type A2 := \( (a,a,b,a) \), Type A3 := \( (a,a,a,b) \), Type A4 := \( (a,b,a,a) \), Type B1 := \( (a,a,b,b) \), Type B2 := \( (a,b,b,a) \), or Type B3 := \( (a,b,a,b) \). Some of these types are depicted in Figure 7.

![Figure 7: Different lengths of the edges in C](image)

If \( C \) is of Type A1, then \( a \equiv \pm 3b \pmod{2n} \). This implies that the two endvertices of an edge of length \( a \) are also endvertices of a 3-path whose edges are all of length \( b \). Also, the two endvertices of a 3-path whose edges are all of length \( b \), must be adjacent. Consider the edge \( v_2v_3 \). Since \( v_1v_2 \) and \( u_2v_2 \) have length \( b \), the edge \( v_2v_3 \) has length \( a \), and so it must belong to some 4-cycle with the other three edges of the cycle having length \( b \). We denote this 4-cycle by \( C_4(v_2v_3) \). First, assume that \( u_2u_3 \) is of length \( a \). If \( u_2v_2 \in E(C_4(v_2v_3)) \), then \( C_4(v_2v_3) \) contains \( u_2v_2+k \pmod{n} \), and consequently \( v_2+k \pmod{n} \), which is impossible, since \( k \geq 4 \). Therefore, \( C_4(v_2v_3) \) contains \( v_1v_2 \), and so \( C_4(v_2v_3) = (v_3,v_2,v_1,u_1) \), implying once again that \( k = 2 \), a contradiction. Consequently, \( u_2u_3 \) must be of length \( b \), implying that \( C_4(v_2v_3) = (v_3,v_2,u_2,u_3) \). By using the same arguments we can deduce that the outer- and inner-cycle edges, and the vertical spokes in \( G \) have lengths as shown in Figure 8.

![Figure 8: A[n,k] when C is of Type A1](image)

Since \( k \) is even, we also have \( b \equiv \pm 3a \pmod{2n} \) (see for example the 4-cycle \( (u_1,u_2,v_2+k \pmod{n},v_1+k \pmod{n}) \)). This implies that \( a \equiv \pm 9a \pmod{2n} \), that is, \( 8a \equiv 2n \pmod{2n} \), or \( 10a \equiv 2n \pmod{2n} \). Since \( \gcd(2n,a) = 1 \), the total number of vertices of \( G \) must be equal to 8 or 10, a contradiction, since \( n \geq 8 \). By using a very similar argument it can be shown that \( C \) cannot be of Type A2 (see Figure 9).
Figure 9: $A[n, k]$ when $C$ is of Type A2

So assume that $C$ is of Type A3. Then, $b \equiv \pm 3a \pmod{2n}$ and, in particular, the edge $u_2u_3$ has length $b$. Consequently, this edge must belong to some 4-cycle with the other three edges of the cycle having length $a$. We denote this 4-cycle by $C_4(u_2u_3)$. Since $u_1u_2$ and $u_2v_2$ are both of length $a$, we have the following cases:

- if $C_4(u_2u_3) = (u_3, u_2, u_1, v_{1+k})$, then $k = 2$, a contradiction,
- if $C_4(u_2u_3) = (u_3, u_2, u_1, u_n)$, then $n = 4$, a contradiction, and
- if $C_4(u_2u_3) = (u_3, u_2, v_2, v_1)$, then $u_3$ is adjacent to $v_1$. Consequently, we have that $k \equiv -2 \pmod{n}$, and as before, this implies that $n - 2 \leq \frac{n}{2}$, a contradiction, since $n \geq 8$.

Thus $C$ cannot be of Type A3, and by using a similar argument, it can be shown that $C$ cannot be of Type A4 either.

If $C$ is of Type B1 or Type B2, then we have that $2a \equiv \pm 2b \pmod{2n}$, and since $1 \leq a < b \leq n - 1$, we can further assume that $2a \equiv -2b \pmod{2n}$. Consequently, we have that $a + b = n$, and so by Lemma 5.2, $G \simeq A[n, 2]$, a contradiction. Therefore, $C$ and all other possible 4-cycles in $G$ must be of Type B3, which is impossible, because then the edges having length $a$ would induce two disjoint $n$-cycles, contradicting the fact that the edges having length $a$ induce a Hamiltonian cycle (and thus a $2n$-cycle). As a consequence, $A[n, k]$ is not circulant, contradicting our initial assumption.

Using the above two lemmas we can now prove the main result of this section.

**Theorem 5.4.** The accordion graph $A[n, k]$ is not circulant if and only if both $n$ and $k$ are even, such that $k \geq 4$.

**Proof.** By Lemma 5.3, it suffices to show that the accordion graph $A[n, k]$ is circulant if and only if either

(i) $k$ is odd, or
(ii) $k$ is even and $n$ is odd, or
(iii) $k = 2$ and $n$ is even.

**Case (i).** For $k$ odd, we claim that the function $\phi : V(A[n, k]) \rightarrow V(Ci[2n, \{2, k\}])$ defined by $\phi : u_i \mapsto x_{2i}$ and $\phi : v_i \mapsto x_{2i-k \pmod{2n}}$, where $i \in [n]$, is an isomorphism. Since $2i - k$ is odd, for every $i \in [n]$, one can deduce that the function $\phi$ is bijective. Also, $Ci[2n, \{2, k\}]$ has the same number of edges as $A[n, k]$, and thus it suffices to show that an edge in $A[n, k]$ is mapped to an edge in $Ci[2n, \{2, k\}]$. 


identify which accordions are PMH and which are not, for
3
by a computer check conducted through Wolfram Mathematica
[10]. In particular, we
gcd(n, k) ≥ 5, a complete characterisation of which accordions graphs have the PMH- or the PH-property is definitely of interest but still inaccessible. In Section 3, partial results were obtained for the cases when
gcd(n, k) ≤ 2. These are portrayed in Table 1 together with other partial results obtained
by a computer check conducted through Wolfram Mathematica [10]. In particular, we
identify which accordions are PMH and which are not, for 3 ≤ k ≤ 10 and for n ≤ 21.
We remark that some values of n and k are marked as “unknown” due to problems with
computation time and memory.

Additionally, as already remarked before, the main result in [4] gives more than just all
the possible values of n1 and n2, for which Cn1 □Cn2 is circulant. In fact, the main result
of the above paper is the following.

(a) We first take an edge u_iu_j from the outer-cycle of A[n, k], for some i ∈ [n] and
j ≡ i + 1 (mod n), without loss of generality. Consider φ(u_i)φ(u_j). The length of
φ(u_i)φ(u_j) can be calculated using 2(j − i) which is equivalent to 2 (mod 2n). Since
this belongs to {±2, ±k}, φ(u_i)φ(u_j) ∈ E(Ci[2n, {2, k}]).

(b) By a similar reasoning to that used in (a), φ(v_i)φ(v_j) is an edge in Ci[2n, {2, k}], for
any i ∈ [n] and j ≡ i + 1 (mod n), without loss of generality.

(c) We now consider the spokes. Let i ∈ [n] and j ≡ i + k (mod n). The length of
φ(u_i)φ(v_i) can be calculated using 2i − k − 2i, which is equal to −k. On the other
hand, the length of φ(u_i)φ(v_j) can be calculated using 2j − k − 2i, which is equal
to k. In both cases, the lengths obtained belong to {±2, ±k}, and so φ(u_i)φ(v_i) and
φ(u_i)φ(v_j) are edges in Ci[2n, {2, k}].

Case (ii). For k even and n odd, we claim that the following function φ : V (A[n, k]) →
V (Ci[2n, {2, n − k}]) defined by φ : u_i → x_{2i} and φ : v_i → x_{2i+n−k} (mod 2n), where
i ∈ [n], is an isomorphism. As in Case (i), the function φ is bijective, since n − k is odd.
Moreover, Ci[2n, {2, n − k}] has the same number of edges as A[n, k], and so it suffices to
show that an edge in A[n, k] is mapped to an edge in Ci[2n, {2, n − k}].

(a) By the same reasoning used in Case (i), φ(u_i)φ(u_j) and φ(v_i)φ(v_j) are edges in
Ci[2n, {2, n − k}], for i ∈ [n], and j ≡ i + 1 (mod n), without loss of generality.

(b) We now consider the spokes. Let i ∈ [n] and j ≡ i + k (mod n). The length of
φ(u_i)φ(v_i) can be calculated using 2i + n − k − 2i, which is equal to n − k. On the
other hand, the length of φ(u_i)φ(v_j) can be calculated using 2j + n − k − 2i, which
is equivalent to −(n − k) (mod 2n). In both cases, the lengths obtained belong to
{±2, ±(n − k)}, and so φ(u_i)φ(v_i) and φ(u_i)φ(v_j) are edges in Ci[2n, {2, n − k}].

Case (iii). This was proven in Lemma 5.2.

The following result follows immediately from the proof of Theorem 5.4.

Corollary 5.5. The accordion graph A[n, k] is isomorphic to the circulant graph

(i) Ci[2n, {2, k}], when k is odd, and

(ii) Ci[2n, {2, n − k}], when n is odd and k is even.

6 Concluding remarks and open problems

Despite ruling out all accordion graphs A[n, k] having gcd(n, k) ≥ 5, a complete characterisation of which accordion graphs have the PMH- or the PH-property is definitely of interest but still inaccessible. In Section 3, partial results were obtained for the cases when
gcd(n, k) ≤ 2. These are portrayed in Table 1 together with other partial results obtained
by a computer check conducted through Wolfram Mathematica [10]. In particular, we
identify which accordions are PMH and which are not, for 3 ≤ k ≤ 10 and for n ≤ 21.
We remark that some values of n and k are marked as “unknown” due to problems with
computation time and memory.
Theorem 6.1. [4] The circulant graph $C_i[\overline{n'}, \{a_1, a_2\}]$ is isomorphic to $C_{n_1} \Box C_{n_2}$ if and only if:

(i) $n' = n_1 n_2$,

(ii) $n_1 = \gcd(n', a_j)$ and $n_2 = \gcd(n', a_{3-j})$, where $j = 1$ or $j = 2$, and

(iii) $\gcd(n_1, n_2) = 1$.

In this sense, we think that it would be an interesting endeavour to give a necessary and sufficient condition for a quartic circulant graph to be isomorphic to some accordion.

References

[1] M. Abreu, J.B. Gauci, D. Labbate, G. Mazzuoccolo and J.P. Zerafa, Extending Perfect Matchings to Hamiltonian Cycles in Line Graphs, (2019), arXiv:1910.01553.

[2] A. Alahmadi, R.E.L. Aldred, A. Alkenani, R. Hijazi, P. Solé and C. Thomassen, Extending a perfect matching to a Hamiltonian cycle, Discrete Math. Theor. Comput. Sci. 17(1) (2015), 241–254.
[3] F. Boesch and R. Tindell, Circulants and their connectivities, *J. Graph Theory* 8 (1984), 487–499.

[4] Z.R. Bogdanowicz, On isomorphism between circulant and Cartesian product of 2 cycles, *Discrete Appl. Math.* 194 (2015), 160–162.

[5] R. Diestel, *Graph Theory*, Graduate Texts in Mathematics 173, Springer-Verlag, New York, 2000.

[6] J. Fink, Perfect matchings extend to Hamilton cycles in hypercubes, *J. Comb. Theory, Ser.B* 97(6) (2007), 1074–1076.

[7] J.B. Gauci and J.P. Zerafa, A note on perfect matchings and Hamiltonicity in the Cartesian product of cycles, (2020), arxiv:2005.02913.

[8] S. Herke and B. Maenhaut, Perfect 1-Factorisations of Circulants with Small Degree, *Electron. J. Combin* 20(1) (2013), #P58.

[9] C. Heuberger, On planarity and colorability of circulant graphs, *Discrete Math.* 268 (2003), 153–169.

[10] Wolfram Research, Inc., Mathematica, Version 12.1, *Wolfram Research, Inc.*, Champaign, Illinois (2020)