Invariant Kähler potentials and symplectic reduction

P. Heinzner & B. Stratmann

Abstract

For a proper Hamiltonian action of a Lie group $G$ on a Kähler manifold $(X, \omega)$ with momentum map $\mu$ we show that the symplectic reduction $\mu^{-1}(0)/G$ is a normal complex space. Every point in $\mu^{-1}(0)$ has a $G$-stable open neighborhood on which $\omega$ and $\mu$ are given by a $G$-invariant Kähler potential. This is used to show that $\mu^{-1}(0)/G$ is a Kähler space. Furthermore we examine the existence of potentials away from $\mu^{-1}(0)$ with both positive and negative results.

Let $(X, \omega)$ be a Kähler manifold and $G$ a closed Lie subgroup of the group $\text{Iso}_O(X)$ of holomorphic isometries of the Kähler structure. Since $G$ is closed in $\text{Iso}_O(X)$, the $G$-action is proper. We will assume that the action of $G$ on $X$ is Hamiltonian, meaning that a momentum map $\mu : X \to \text{Lie}(G)^*$ is given. This is a $G$-equivariant map with $\text{Lie}(G)^*$ endowed with the coadjoint action such that for each $\xi \in \text{Lie}(G)$ the corresponding component function $\mu_\xi$ satisfies $d\mu_\xi = i\tilde{\xi}\omega$. We will refer to $(X, G, \omega, \mu)$ as a Hamiltonian Kähler manifold.

The symplectic reduction of $X$ is defined as the quotient $M/G$ where $M = \mu^{-1}(0)$. On $M$ we have the sheaf $\mathcal{O}_M = \mathcal{O}_X|_M$ where $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on $X$. Sections of $\mathcal{O}_M$ are given by functions on $M$ which are restrictions of holomorphic functions and we call them holomorphic functions on $M$. Let $p : M \to M/G$ denote the quotient map. We have the direct image sheaf $p_*\mathcal{O}_M$ on $M/G$ and the corresponding sheaf $p_*\mathcal{O}_M^G = (p_*\mathcal{O}_M)^G$ of $G$-invariant sections, denoted $\mathcal{O}_{M/G}$. In other words we associate to an open subset $Q \subset M/G$ the algebra $\mathcal{O}_{M/G}(p^{-1}(Q))$ of invariant holomorphic functions on $M$, these are $G$-invariant continuous functions on $p^{-1}(Q)$ that admit for each orbit $G \cdot x \subset p^{-1}(Q)$ a neighborhood in $X$ to which $f$ extends as a holomorphic function, in fact it extends as a $G$-invariant holomorphic function on a $G$-stable neighborhood, as the proof will show. Our first result is the following.

Theorem 1. $(M/G, \mathcal{O}_{M/G})$ is a reduced normal complex space.

If $0 \in \text{Lie}(G)^*$ is a regular value of $\mu$, then the theorem follows almost directly from the construction of the symplectic reduction $M/G$. One just has to verify that the natural almost complex structure on the quotient is integrable. In

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the case where $G$ is a semisimple Lie group the result has been obtained in the PhD thesis of M. Ammon [Amm97] and has been extended to Lie groups which are contained in some complex Lie group in the PhD thesis of A. Kurt-
dere [Kur09]. In Theorem 1 we allow $G$ to be any Lie group. Furthermore we
are able to realize $\mathcal{M}/G$ locally more explicit as a quotient $\mathcal{U}/G$ for some suit-
able $G$-stable neighborhood $\mathcal{U}$ in $X$. This quotient $\mathcal{U}/G$ is an analytic Hilbert
quotient, i.e. two points are identified if they cannot be separated by invariant
holomorphic functions. The quotient is endowed with the structure sheaf $\mathcal{O}_{\mathcal{U}/G}$
deﬁned by $\mathcal{O}_{\mathcal{U}/G}(V) = \mathcal{O}_\mathcal{U}(\pi^{-1}(V))$ for every open set $V \subset \mathcal{U}/G$ (see Deﬁni-
tion 21).

The quotient $p : \mathcal{M} \to \mathcal{M}/G$ is universal in the sense that any $G$-invariant holomorphic map factorizes over $\mathcal{M}/G$.

**Theorem 2.** For every $x \in \mathcal{M}$ there exists a $G$-stable open neighborhood $U$
such that the analytic Hilbert quotient $\pi : \mathcal{U} \to \mathcal{U}/G$ realizes $(\mathcal{U}/G, \mathcal{O}_{\mathcal{U}/G})$
as a reduced normal complex space and the inclusion $i_U : \mathcal{M}_U = \mathcal{M} \cap U \hookrightarrow U$
induces a biholomorphic map $\varphi_U : \mathcal{M}_U/G \to \mathcal{U}/G$ such that the following
diagram commutes.

\[
\begin{array}{ccc}
\mathcal{M}_U & \xrightarrow{i_U} & U \\
\downarrow p & & \downarrow \pi \\
\mathcal{M}_U/G & \xleftarrow{\varphi_U} & \mathcal{U}/G \\
\end{array}
\]

For a $G$-invariant smooth strictly plurisubharmonic function $\rho$ one can define a
Hamiltonian Kähler structure by

$$\omega = \dd^c \rho \quad \text{and} \quad \mu^v = -i_v \dd^c \rho \quad \text{for each} \quad v \in \text{Lie}(G)$$

where $\tilde{v}$ is the induced vector field on $X$ and $\mu^v(x) = \mu(x)(v)$ with the real
operator $\dd^c = \frac{1}{2}(\partial - \bar{\partial})$. In this case we will say that $\mu$ is given by a $G$-invariant
potential $\rho$ (Deﬁnition 36). In general there is no hope to obtain a global
potential for $\mu$. Our second result says that locally around each point in $\mathcal{M}$ this
is the case.

**Theorem 3.** For every point $x \in \mathcal{M}$ there exists an open $G$-stable neighborhood $Y$
of $x$ in $X$ and a $G$-invariant potential $\rho : Y \to \mathbb{R}$ for which $\omega|_Y$ and $\mu|_Y$ are
given by $\rho$.

The manifold $X$ has a stratification by $G$-orbit type. Points $x_1$ and $x_2$ belong
to the same stratum if their $G$-orbits are $G$-equivariantly diffeomorphic, i.e. if
their isotropy groups are conjugate in $G$.

The stratification on $X$ induces a stratification on $\mathcal{M}$. The intersection of $\mathcal{M}$
with a stratum gives a local submanifold. This stratification pushes down to
a stratification of the quotient $\mathcal{M}/G$. It is not diﬃcult to see that the Kähler
form $\omega$ pulled back to $\mathcal{M}$ induces a Kähler form $\tilde{\omega}$ on each stratum in $\mathcal{M}/G$.
We will show that the smooth Kähler forms on the strata extend uniquely to a Kähler structure on the complex space \( M/G \). By definition a Kähler structure on a complex space \( Q \) is given by an open covering \( Q = \bigcup Q_\alpha \) and strictly plurisubharmonic continuous functions \( \rho_\alpha : Q_\alpha \to \mathbb{R} \) such that on \( Q_\alpha \cap Q_\beta \) the difference \( \rho_\alpha - \rho_\beta \) is locally the real part of a holomorphic function (Definition 55).

Corollary 4. There is a unique Kähler structure \( \bar{\omega} \) on \( M/G \) such that \( p^* \bar{\omega} = i^*_M \omega \) with \( i_M : M \to X \) denoting the inclusion. The continuous strictly plurisubharmonic functions \( \bar{\rho}_\alpha \) defining \( \bar{\omega} \) are smooth restricted to each stratum.

More precisely, for every point \( y \in M/G \) there is an open \( G \)-stable neighborhood \( U \) of \( p^{-1}(y) \) in \( X \) and a \( G \)-invariant Kähler potential \( \bar{\rho} \) of \( \omega \) on \( U \) such that the restriction \( \bar{\rho} |_{M \cap U} \) induces a continuous function \( \bar{\rho} : p(M \cap U) \to \mathbb{R} \) which is a Kähler potential for \( \bar{\omega} \) on each non-singular stratum of \( M/G \), i.e. \( \bar{\omega} = dd^c \bar{\rho} \).

Existence of local \( G \)-invariant potentials can also be shown in \( G \)-stable neighborhoods of totally real orbits under an additional condition. In Section 4 we show the following theorem. Denote \( G^0 \) the connected component of the neutral element of \( G \).

Theorem 5. Let \((X, G, \omega, \mu)\) be a Hamiltonian manifold with \( H^1_{dR}(G^0) = 0 \). Let the orbit \( G \cdot x \) be totally real and \( \omega \) restricted to this orbit be exact. Then there exists an invariant local potential on some \( G \)-stable neighborhood of \( x \).

In the last section we give examples where there is no local invariant potential away from the momentum zero level (Section 7).

1 Complexified group and orbits

Let \( G \) be a Lie group and \( X \) a complex \( G \)-manifold. By this we mean that \( G \) acts smoothly by holomorphic transformations.

In many cases of interest the \( G \)-action on \( X \) is given by restricting a holomorphic action of the complexified group \( G^C \) on a complex manifold \( Z \) which contains \( X \) as an open \( G \)-stable subset. Provided \( G \) is contained in \( G^C \) this is not always the case. If otherwise \( G \) is not contained in \( G^C \) this is even never the case for an effective \( G \)-action, e.g. consider the group \( H = SL_2(\mathbb{R}) \) acting by left multiplication on \( H^C = SL_2(\mathbb{C}) \). This action is proper and free and therefore there is an open \( H \)-stable neighborhood \( X \) of \( H \) in \( H^C \) which is \( H \)-equivariantly diffeomorphic to \( H \times \Sigma \) where \( \Sigma \subset \mathbb{R}^3 \) denotes a ball in \( \mathbb{R}^3 \). The universal covering \( \tilde{G} = \tilde{H} \) of \( H \) acts holomorphically on the universal covering \( \tilde{X} \) of \( X \) and this \( G \)-action cannot be realized as a restriction of a holomorphic action of \( G^C = H^C = SL_2(\mathbb{C}) \).

1.1 The \( G \)-tube

As a first step we complexify the group \( G \). As these complexifications are desired to be shrinkable we need them to fulfill the Runge property relative to each
other.

**Definition 6.** An open subset \( Y \) of a complex manifold \( X \) is said to be *Runge* if the restriction map \( \mathcal{O}(X) \to \mathcal{O}(Y) \) has dense image in \( \mathcal{O}(Y) \).

**Definition 7.** Let \( G \) act on \( X \). A continuous function \( \rho : X \to \mathbb{R} \) is called *\( G \)-exhaustive* or a *\( G \)-exhaustion* if it is \( G \)-invariant and the sets \( \{ x \in X \mid \rho(x) < c \} / G \) are relatively compact in \( X/G \) for all \( c < \sup \rho \). We just write *exhaustion/exhaustive* if the trivial group acts.

Recall that classical results from Grauert show that given a strictly plurisubharmonic exhaustion \( \rho : X \to \mathbb{R} \geq 0 \) the relatively compact sets \( X_c = \{ x \in X \mid \rho(x) < c \} \) are Runge (cf. e.g. [HL98, Thm. 1.3 (v)]). The existence of a strictly plurisubharmonic exhaustion on \( X \) is equivalent for \( X \) to be a Stein manifold. If we drop the assumption of \( \rho \) being an exhaustion but keep that \( X \) is a Stein manifold the Runge property is still satisfied as the following shows.

**Lemma 8.** Let \( X \) be a Stein manifold and \( \rho_Y : X \to \mathbb{R} \geq 0 \) a strictly plurisubharmonic function. Then \( Y = \{ y \in Y \mid \rho_Y(y) < 1 \} \) is Runge in \( X \).

**Proof.** We fix some strictly plurisubharmonic exhaustion function \( \rho : X \to \mathbb{R} \geq 0 \) and for \( n \in \mathbb{N} \) we set \( \rho_n = \max \{ \frac{1}{n} \rho, \rho_Y \} \). The function \( \rho_n \) a strictly plurisubharmonic in the sense of perturbation and an exhaustion. It follows that

\[
Y_n = \{ x \in X \mid \rho_n(x) < 1 \}
\]
is an open Stein submanifold of \( X \). Moreover \( Y_n \) is Runge in \( X \). For \( n > 1 \) we set

\[
K_n = \{ x \in X \mid \rho(x) \leq n - \frac{1}{2}, \rho_Y(x) \leq 1 - \frac{1}{n} \}.
\]

These are compact subsets of \( Y_n \) with \( K_n \subset K_{n+1} \) and \( \bigcup_{n \geq 2} K_n = Y \). For every holomorphic function \( f \in \mathcal{O}(Y) \) there is a function \( f_n \in \mathcal{O}(X) \) such that \( \| f - f_n \|_{K_n} < \frac{1}{n} \) holds in the sup-norm on \( K_n \). This shows that \( f_n|_Y \) converges to \( f \), i.e. \( \mathcal{O}(X) \) has dense image in \( \mathcal{O}(Y) \).

**Definition 9.** A Stein \( G \)-manifold \( G^* \) is said to be a *\( G \)-tube* if

1. there is a \( G \)-equivariant embedding \( \kappa : G \to G^* \) where \( G \) acts on \( G \) by left multiplication,
2. every \( G \)-orbit in \( G^* \) is a totally real submanifold of \( G^* \),
3. there is a real submanifold \( \Sigma \subset G^* \) with \( \kappa(e) \in \Sigma \) such that
   a. the map \( G \times \Sigma \to G^*, (g, \gamma) \mapsto g \cdot \gamma \) is a \( G \)-equivariant diffeomorphism,
   b. \( \Sigma \) is diffeomorphic to a convex bounded open neighborhood of 0 in \( \mathbb{R}^k \) with \( k = \dim G \),
4. there is a strictly plurisubharmonic \( G \)-exhaustion \( \rho : G^* \to \mathbb{R} \) which becomes minimal on \( \kappa(G) \).
5. the manifold $G^*$ is “shrinkable”, i.e. every $G$-stable neighborhood $U$ of $\kappa(G)$ contains a Stein $G$-submanifold $T \supset \kappa(G)$ such that $\Sigma_T = \Sigma \cap T$ is convex, $\rho|_{\Sigma_T}$ is an exhaustion and $T$ is Runge in $G^*$.

Condition (2) implies that the $G$-action on $G^*$ is proper and free. We will refer to $\Sigma$ as a “slice” for the $G$-action on $G^*$ and will identify $\kappa(G)$ with $G$.

A smooth function $\rho$ on a complex manifold $X$ is strictly plurisubharmonic if and only if $\omega = i\partial\bar{\partial}\rho = dd^c\rho$ is a Kähler form. Note that $d^c = \frac{i}{2\pi}(\partial - \bar{\partial})$ is a real operator.

Later we will use $G$-tubes to construct local slice models of the $G$-action on a complex manifold $X$ around totally real $G$-orbits. For this we have to fix a compact subgroup $K$ of $G$ which will be given as an isotropy group of the $G$-action later.

**Theorem 10. (Existence of a $G$-tube)** For every Lie group $G$ with fixed compact subgroup $K$ there exists a $G$-tube $G^*$ such that the $(G \times K)$-action on $G$ given by $(g,k) \cdot x = g x k^{-1}$ extends to $G^*$ as an action by holomorphic transformations.

**Proof.** We view $G$ as a $(G \times K)$-manifold. Then there is a complex $(G \times K)$-manifold $T$ which is $(G \times K)$-equivariantly diffeomorphic to a $(G \times K)$-stable neighborhood of the zero section of the tangent bundle $TG$ of $G$. In order to see this one has to choose a real analytic $(G \times K)$-invariant Riemannian metric on $G$ and apply the main results in [LS91, GS92]. By a result of Winkelmann ([Win93]) the manifold $T$ which we identify with a subset of $TG$ can be chosen to be a Stein manifold (see [HHK96]). On $TG$ we have the function $\rho(v) = \|v\|^2$, $v \in T_g G$, where $\|\cdot\|$ is defined by the $(G \times K)$-invariant metric on $G$. The function is strictly convex on $T_g G$ and constant on the zero section, a totally real submanifold of maximal dimension. Thus $\rho$ is strictly plurisubharmonic on a neighborhood of the zero section in $T$. As $\rho$ is $(G \times K)$-invariant this neighborhood can be chosen $(G \times K)$-stable. This shows that for some $c_0 > 0$ and all $0 < c \leq c_0$ the $(G \times K)$-stable sets $T_c = \{v \in T \mid \rho(v) < c\}$ intersect $T_g G$ in a convex bounded set and with $\Sigma = T_{c_0} \cap T_g G$ the function $\rho|_{\Sigma}$ is strictly convex. The restriction $\rho|_{\Sigma}$ is a $G$-exhaustion by construction becoming minimal on $\kappa(G)$. Since $T$ is a Stein manifold and the boundary of $T_c \subset T$ is pseudoconvex, the $(G \times K)$-stable neighborhoods of the zero section $G \times \{0\} \subset T$ are open Stein submanifolds of $T$ ([DG60]). Finally, for every $G$-stable neighborhood $U$ of $\kappa(G)$ there is a $0 < c_1 < c_0$ such that $T_{c_1} \subset U$. Lemma [N] shows that $T_{c_1}$ is Runge in $T_{c_0}$. Thus $T_{c_1}$ fulfills all the desired properties.

The next proposition shows that we can consider a $G$-tube the prototype set to which the real analytic orbit map extends as a holomorphic map.

**Proposition 11.** Given a complex manifold $X$ and a Lie group $G$ acting by holomorphic transformations, let $\varphi : G \to X, g \mapsto g \cdot x_0$ be the orbit map for some point $x_0 \in X$. Then $\varphi$ extends to some $G$-tube $G^*$ as a holomorphic map $\varphi^* : G^* \to X$. 

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**Proof.** The map $\varphi$ being real analytic extends as a holomorphic map $\psi$ to some neighborhood $\Omega \subset G^*$ of $e \in G$. We may assume that $\Omega_\Sigma = \Omega \cap \Sigma$ is connected. Define $T = G \cdot \Omega_\Sigma$ which is $G$-equivariantly diffeomorphic to $G \times \Omega_\Sigma$. So, we can define the $G$-equivariant map $\varphi^c : T \to X, g \cdot x \mapsto g \cdot \psi(x)$. For each point $x \in \Omega_\Sigma$ there is a neighborhood $U(x)$ such that the map $\psi$ is locally $G$-equivariant on $U(x)$. So the maps $\varphi^c$ and $\psi$ are locally $G$-equivariant and identical on $\Omega_\Sigma$, thus they are locally identical. Hence $\varphi^c$ is holomorphic in a neighborhood of $\Omega_\Sigma$ and globally by $G$-equivariance. Restricted to $G \subset G^*$ both maps $\varphi$ and $\varphi^c$ coincide by construction. Hence $\varphi^c$ extends $\varphi$ as a holomorphic function on a $G$-stable neighborhood of $G$. By definition the neighborhood $T$ can be shrunk to a $G$-tube.

1.2 Complexified orbits

We will now establish the notion of a complexified $G$-orbit for an action of a real group by holomorphic transformations. It is meant to be a substitute for a $G^C$-orbit in the case where $G^C$ does not act.

**Definition 12.** For each point $x \in X$, the orbit map $\varphi : G \to X, g \mapsto g \cdot x$ is real analytic and extends to a holomorphic map $\varphi^c : G^* \to X$ on some $G$-tube. We denote by $\tau_G$ the smallest topology containing the sets $\varphi^c(G^*) \subset X$ as open sets for all orbit maps $\varphi$ and all $G$-tubes to which $\varphi$ extends.

Observe that Proposition 11 guarantees the existence of the extensions $\varphi^c$ to some $G$-tube.

**Lemma 13.** The images $\varphi^c(G^*) \subset X$ used in the definition of the topology form a basis of the topology $\tau_G$.

The proof of this lemma is given after Lemma 16.

**Definition 14.** The complexified $G$-orbit of $x \in X$ is the connected component of $x$ in the topology $\tau_G$ denoted by $B_X(x)$ or just by $B(x)$.

In the following we collect some properties of the topology $\tau_G$ and provide the postponed proof of Lemma 13.

In a first step we endow the $G$-tube $G^*$ itself with the topology $\tau_G$ and use this later as a model.

**Lemma 15.** Given a $G$-stable open subset $U \subset G^*$ and $\gamma \in U$ there is a $G$-tube $T \subset G^*$ and a holomorphic extension $\varphi^c : T \to G^*$ of the orbit map $\varphi : G \to G^*, g \mapsto g \cdot \gamma$ such that $\varphi^c(T) \subset U$.

**Proof.** The $G$-orbit $G \cdot \gamma$ is totally real by definition of the $G$-tube. Proposition 11 shows that $\varphi$ extends holomorphically to some $G$-tube $T_0 \subset G^*$ as $\varphi^c : T_0 \to G^*$. Since $\varphi$ is injective and immersive, the extension $\varphi^c$ is an injective immersion on some $G$-stable neighborhood of $G$. This set can be shrunk to a $G$-tube $T$ by definition. Since $\varphi^c$ is open for dimension reasons, $\varphi^c$ maps $T$ biholomorphically to its image in $U$. \qed
Recall that the $G$-tube $G^*$ is assumed to be $G$-equivariantly diffeomorphic to $G \times \Sigma$. We consider $\Sigma$ as a subset of $G^*$ via its identification with $\{e\} \times \Sigma$.

**Lemma 16.** A subset $U \subset G^*$ is $\tau_G$-open if and only if it is open and $G$-stable. Such a set is $\tau_G$-connected if and only if its intersection with $\Sigma$ is connected, in particular $(G^*, \tau_G)$ is locally connected and the $G$-tube itself is $\tau_G$-connected.

**Proof.** Lemma 15 shows that any open, $G$-stable subset of $G^*$ is $\tau_G$-open and vice versa, that each point in a $\tau_G$-open set in $G^*$ contains a $G$-stable, open subset, i.e. each $\tau_G$-open set in $G^*$ is $G$-stable and open.

Let $U \subset G^*$ be a $\tau_G$-open set and cover it by two disjoint $\tau_G$-open sets. This induces a cover of $U \cap \Sigma$. Since $U$ is $G$-stable, $U$ is $\tau_G$-connected if and only if $U \cap \Sigma$ is connected. The result follows from the fact, that $\Sigma$ is locally connected and connected.

**Proof of Lemma 15.** We want to show that the sets $\varphi^c(T)$ form a basis of the topology. Given two points $x_1, x_2 \in X$ and for $i = 1, 2$ some holomorphic extensions $\varphi^c_i : T_i \to X$ of the corresponding orbit maps $\varphi_i : G \to X, g \mapsto g \cdot x_i$ to $G$-tubes $T_1, T_2$ respectively. Suppose there is a point $y \in \varphi^c_1(T_1) \cap \varphi^c_2(T_2)$. Lemma 15 provides extensions $\tilde{\varphi}^c_i : \tilde{T}_i \to X$ of the orbit map of $y$ with $\tilde{\varphi}^c_i(\tilde{T}_i) \subset \varphi^c_i(U_i)$ for $i = 1, 2$. Since both holomorphic maps $\tilde{\varphi}^c_i$ extend the same map, they coincide on some possibly smaller $G$-tube $T_0 \subset G^*$ which can even be chosen to be contained in $\tilde{T}_1 \cap \tilde{T}_2$. The latter implies that $\tilde{\varphi}^c_1(T_0)$ is contained in $\varphi^c_1(T_1) \cap \varphi^c_2(T_2)$.

The following lemma shows that the equivariant holomorphic maps behave in a natural way compatible with the $\tau_G$-topology.

**Lemma 17.** Let $G$ act on $X$ and $Y$ by holomorphic transformations and let $\psi : X \to Y$ be a $G$-equivariant holomorphic map. Then $\psi : (X, \tau_G) \to (Y, \tau_G)$ is continuous and open.

**Proof.** Given a $\tau_G$-open set $U_X \subset X$, set $V_Y = \psi(U_X)$. For a given point $y \in V_Y$ choose $x \in U_X$ in the $\psi$-fiber of $y$. By definition of $\tau_G$, there is a $G$-stable, open set $U \subset G^*$ such that $\varphi^c : U \to U_X$ is a holomorphic extension of the orbit map $g \mapsto g \cdot x$. Due to $G$-equivariance of $\psi$, the map $\psi \circ \varphi^c$ is in fact a holomorphic extension of the orbit map $g \mapsto g \cdot y$ with $\psi \circ \varphi^c(U) \subset V_Y$. Thus $V_Y$ is $\tau_G$-open and therefore $\psi$ is $\tau_G$-$\tau_G$-open.

For a given $\tau_G$-open set $V_Y \subset Y$ set $U_X = \psi^{-1}(V_Y)$ and fix $x \in U_X$. We choose a holomorphic extension $\varphi^c : U \to X$ of the orbit map $g \mapsto g \cdot x$. If $(\psi \circ \varphi^c)^{-1}(V_Y)$ admits an open neighborhood of $G \subset G^*$, we may restrict $\varphi^c$ to this set and are done, since $U_X$ contains a $\tau_G$-open neighborhood of $x$. So suppose the contrary. There are sequences $t_n \in G^*$ and $g_n \in G$ such that $t_n^c = g_n \cdot t_n$ converges to $e$ and $\psi \circ \varphi^c(t_n) \notin V_Y$. The set $V_Y$ being $G$-stable and $\varphi^c$ and $\psi$ being $G$-equivariant, we conclude $\psi \circ \varphi^c(t_n) \notin V_Y$. But $\psi \circ \varphi^c$ is a holomorphic extension of the orbit map $g \mapsto g \cdot y$ and hence extends to some neighborhood of $G \subset U$ with image in $V_Y$, since $V_Y$ is $\tau_G$-open. So, $\psi$ is $\tau_G$-$\tau_G$-continuous.

**Lemma 18.** The space $(X, \tau_G)$ is locally connected. In particular, the connected components, namely the complexified $G$-orbits, are $\tau_G$-closed and $\tau_G$-open.
Proof. For topological spaces in general connected components are closed and for locally connected topological spaces they are also open. So, we are left to show that \((X, \tau_G)\) is locally connected. Given a point \(x \in X\) and a holomorphic extension \(\varphi^c : T \to X\) of the orbit map of \(x\) to a \(G\)-tube \(T\). Then \(T\) is \(\tau_G\)-connected as shown in Lemma 16. Since \(\varphi^c\) is \(\tau_G\)-continuous and \(\tau_G\)-\(\tau_G\)-open (Lemma 17), the image \(\varphi^c(T)\) is \(\tau_G\)-connected and \(\tau_G\)-open as well, hence a \(\tau_G\)-neighborhood of \(x\). Therefore \((X, \tau_G)\) is locally connected.

**Lemma 19.** Assume that the action of \(G\) is proper. Then the quotient topology of \(\tau_G\) on \(X/G\) is Hausdorff.

Proof. The quotient topology of the classical topology is finer than that of \(\tau_G\). But for a proper action the quotient \(X/G\) is Hausdorff for the first one already.

We will now show that complexified \(G\)-orbits coincide with \(G^C\)-orbits in the case where the \(G\)-action is given as a restriction of a holomorphic \(G^C\)-action.

**Lemma 20.** In the case where \(G^C\) acts holomorphically on \(X\) the complexified \(G\)-orbits are exactly the \(G^C\)-orbits.

Proof. Let \(B\) be the \(G^C\)-orbit of \(x \in X\). We have to show that \(B\) is \(\tau_G\)-open, \(\tau_G\)-closed and \(\tau_G\)-connected. The holomorphic action map \(\alpha : G^C \times X \to X\) induces a holomorphic orbit map \(\alpha_x : G^C \to X, g \mapsto g \cdot x\) and the \(G\)-equivariant orbit map \(\varphi : G \to B, g \mapsto g \cdot x\) factorizes via \(i : G \to G^C\) over \(\alpha_x\) i.e. \(\varphi = \alpha_x \circ i\). The map \(i\) extends as \(G\)-equivariant a holomorphic map \(i^c\) to some \(G\)-tube \(G^*\) as \(i^c : G^* \to G^C\). So there is an extension \(\varphi^c = \alpha_x \circ i^c : G^* \to B\). The set \(\varphi^c(G^*)\) is by definition a \(\tau_G\)-open neighborhood of \(x \in B\). Thus \(B\) is \(\tau_G\)-open. This implies that every \(G^C\)-orbit in \(X\) is \(\tau_G\)-open and since the complement of a \(G^C\)-orbit is a union of \(G^C\)-orbits every \(G^C\)-orbit is \(\tau_G\)-open and \(\tau_G\)-closed. We are left to show that \(B\) is \(\tau_G\)-connected. Endow \(G^C\) with the topology \(\tau_G\). By the same argument as in Lemma 16 it can be seen that the \(\tau_G\)-open subsets of \(G^C\) are the \(G\)-stable open subsets and that a \(\tau_G\)-open subset \(U \subset G^C\) is \(\tau_G\)-connected if and only if \(U/G \subset G^C/G\) is connected. Therefore \(G^C\) is \(\tau_G\)-connected. The map \(\varphi^c : G^C \to B\) being \(\tau_G\)-\(\tau_G\)-continuous (see Lemma 17) the image \(B = \varphi^c(G^C)\) is \(\tau_G\)-connected, as well.

## 2 The compact case

In this section we will reformulate the results of [Hei91] in the terminology of \(K\)-tubes and complexified \(K\)-orbits of Sections 1.1 and 1.2 in the case of a compact group \(K\) acting, which is well-understood. This shall illustrate the geometry, furthermore we will make decisive use of these known results in the sequel.
2.1 Key results on quotients of Stein manifolds

The results of this section are shown in [Hei91].

**Definition 21.** For a $G$-manifold $Y$ we define the equivalence relation $y_1 \sim y_2$ if $f(y_1) = f(y_2)$ for all $f \in O^G(Y)$, the algebra of $G$-invariant holomorphic functions. We call the quotient formal Hilbert quotient denoted $\pi : Y \to Y//G$. We define the sheaf $O_{Y//G} = \pi_* O^G_Y$ by associating to every open subset $V \subset Y//G$ the algebra $O^G_Y(\pi^{-1}(V))$. We call $(Y//G, O_{Y//G})$ the analytic Hilbert quotient if it is a (reduced) complex space.

**Theorem 22.** Let the compact Lie group $K$ act on the Stein manifold $X$ by holomorphic transformations. Then

1. the quotient $X//K$ endowed with the sheaf $O_{X//K}$ of $K$-invariant holomorphic functions on $X$ is an analytic Hilbert quotient,
2. $(X//K, O_{X//K})$ is a Stein space, and
3. any $K$-invariant holomorphic map $\psi : X \to Y$ from $X$ into a complex space $Y$ factorizes over $X//K$.

**Theorem 23.** Let the compact Lie group $K$ act on a Stein manifold $X$ by holomorphic transformations. Then $X$ can be realized as an open $K$-invariant subset of a Stein manifold $X^C$ such that

1. the complexified Lie group $K^C$ acts holomorphically on $X^C$
2. $X^C = K^C \cdot X$
3. $X$ is Runge in $X^C$.

**Remark.** This result fails in general for a non-compact group $G$ acting.

2.2 Complexified orbits and orbits of the complexified group

In a Stein manifold with a holomorphic $K^C$-action the complexified $K$-orbits are exactly the $K^C$-orbits (Lemma[22]). In the context in which a complexification of the space exists as in Theorem[23] there is another support that the complexified $K$-orbit is the suitable generalization of $K^C$-orbits.

**Proposition 24.** Let the compact Lie group $K$ act on a Stein manifold $X$ by holomorphic transformations and $X^C$ be its complexification in the sense of Theorem[23]. We regard $X$ as a $K$-stable subset of $X^C$. Then for any point $x \in X$

$$K^C \cdot x \cap X = B_X(x)$$

and in particular

$$K^C \cdot x \cap X = B_X(x)$$

The closures are meant in the ambient spaces respectively, i.e. $K^C \cdot x \subset X^C$ and $B_X(x) \subset X$. 

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Proof. Observe first that $X$ is a $K$-stable subset of $X^C$ such that the topology $\tau_K$ on $X$ is the relative topology of the topology $\tau_K$ on $X^C$. Denote $L = K^C \cdot x \cap X$. In $X^C$, each point in the $K^C$-orbit $K^C \cdot x$ admits a $\tau_K$-open neighborhood in the orbit as well as each point $y \notin K^C \cdot x$ admits a $\tau_K$-open neighborhood in the orbit $K^C \cdot y$ hence not intersecting with $K^C \cdot x$. For points $x, y \in X$ the same holds via intersecting the obtained neighborhoods with $X$. Therefore $L$ is the union of $\tau_K$-connected components. In order to see that $L$ is in fact one single $\tau_K$-connected component, we need a result from [Hei91]. The subset $X \subset X^C$ is shown to be "orbit-convex" [Hei91, Theorem, sect. 3.4]. This implies that $L/K$ is connected, hence a single complexified $K$-orbit. This shows $K^C \cdot x \cap X = B_X(x)$. Consequently $B_X(x) = K^C \cdot x \cap X \subset K^C \cdot x \cap X$ which implies $B_X(x) \subset K^C \cdot x \cap X$. Finally it is shown in [Hei91] that $Y = K^C \cdot x \cap B_X(x)$ is a $K$-stable analytic set such that $\dim K^C \cdot y \cap X < \dim B_X(x)$ for all $y \in Y$. From this $K^C \cdot x \cap X \subset B_X(x)$ follows.

2.3 The fibers of the analytic Hilbert quotient

In this section we give a geometrical description of the fibers of the analytic Hilbert quotient in terms of complexified $K$-orbits.

**Theorem 25.** For every $p \in X//K$ and every $x \in \pi^{-1}(p)$

1. $B_X(x)$ is analytic and the union of complexified $K$-orbits $B_X(y)$ satisfying $\dim B_X(y) < \dim B_X(x)$ for each $y \notin B_X(x)$.

2. there is exactly one complexified $K$-orbit of lowest dimension, denoted by $E_X(p)$. This complexified $K$-orbit is closed.

3. $E_X(p)$ lies in the closure of each complexified $K$-orbit in the fiber, formally $E_X(p) \subset B_X(x)$.

4. $\pi^{-1}(p)$ consists of the points whose complexified $K$-orbit close up in $E_X(p)$.

For a detailed proof we refer to [Hei91]. If $K^C$ acts on $X$, the set $\overline{K^C \cdot x}$ consists of $K^C$-orbits with

$$\dim K^C \cdot y < \dim K^C \cdot x \quad \text{for all } y \in \overline{K^C \cdot x} \setminus K^C \cdot x$$

The complexified $K^C$-orbits of lowest dimension are closed, otherwise their closure would contain a complexified $K^C$-orbit of lower dimension. The closure of a $K^C$-orbit contains only one single $K^C$-orbit of lowest dimension. The dimension observation also provides that this closed $K^C$-orbit of lowest dimension lies in the closure of each $K^C$-orbit in $\overline{K^C \cdot x}$. Thus the theorem holds for $X^C$.

When we combine these observations with Theorem 23 and in particular Proposition 24 we obtain the statements of the above Theorem 25.
2.4 The free action case

Later we will show that every totally real $G$-orbit admits a neighborhood which is $G$-equivariantly biholomorphic to a “local slice model” (Section 3). These models arise as analytic Hilbert quotients $G^* \times //K S = (G^* \times S)///K$ of $G^* \times S$ with respect to a free action of a compact group $K$. We collect the main results of free actions of a compact group.

**Proposition 26.** Let $K$ act freely on a Stein manifold $X$ and let $Y$ be an open $K$-stable Runge Stein subset of $X$.

1. $X//K$ is non-singular.
2. Each fiber of $\pi : X \to X//K$ consists of one single complexified $K$-orbit.
3. The projection maps $\pi : X \to X//K$ and $\pi_Y : Y \to Y//K$ are submersions.
4. $Y//K$ is a complex space and the inclusion $i : Y \to X$ induces an open embedding $i//K : Y//K \to X//K$.
5. $Y//K$ is Runge in $X//K$.
6. For every $y \in Y$ we have $B_Y(y) = Y \cap B_X(y)$, in particular $\pi_Y^{-1}(p) = Y \cap \pi^{-1}(p)$ holds for all $p \in Y//K \subset X//K$.

The Runge property will be helpful later to pass from a model to a smaller submodel.

**Proof.** Theorem 23 implies that $Y$ and $X$ are open subsets of $X^C$. The Linearization Lemma in [Hei91, sect. 5.1] provides for a free action that for each $p \in X//K \cong X^C//K^C$ there is a point $x \in X$ with $\pi(x) = p$ and a local complex manifold $S \subset X$ containing $x$ such that $K^C \times S \to X, (k, s) \mapsto k \cdot s$ is an open embedding. Hence $X^C$ is a $K^C$-principal bundle over the smooth quotient $X//K \cong X^C//K^C$. Each fiber of $\pi : X \to X//K$ is in fact the intersection of $X$ with the fiber of $\pi^C : X^C \to X^C//K^C$ which is a single closed $K^C$-orbit. Hence Proposition 24 implies that this intersection is a single complexified $K$-orbit.

Theorem 22 ensures that the quotient $Y//K$ is a Stein space. All $K^C$-orbits are closed, since $K$ and also $K^C$ act freely such that all complex orbits have the same maximal dimension.

Next we show that each fiber of $\pi_Y : Y \to Y//K$ is the intersection of a $K^C$-orbit with $Y$. Let $K^C \cdot x$ be a (closed) $K^C$-orbit for some $x \in Y$ and $F_Y = K^C \cdot x \cap Y$ which is closed in $Y$. We will show that $F_Y//K$ is connected. Suppose the contrary, then there is a function $f$ on $F_Y$ that is constant on each connected component taking different values on each component. Of course, $f$ is holomorphic as it is locally constant. Since $Y$ is a Stein manifold there is a holomorphic function $\tilde{f}$ on $Y$ such that $\tilde{f}|_{F_Y} = f$. The set $Y$ is assumed to be a Runge subset, so for each relatively compact subset $V \subset Y$ a holomorphic function $g$ on $X$ can be chosen which is arbitrarily close to $\tilde{f}$. Choosing $V$ stable by $K$ we
can average over $K$, such that $g$ can be assumed to be $K$-invariant. But since $F_Y \subset K^c \cdot x \cap X$ and every $K$-invariant holomorphic function on $X$ is constant on $K^c \cdot x \cap X$ the function $g$ is constant on that set and locally arbitrarily close to a locally constant function with different values providing a contradiction.

So, we proved that each intersection of $Y$ with a single $K^c$-orbit is contained in a single $\pi_Y$-fiber. Since two different $K^c$-orbits are separated by $K$-invariant holomorphic functions on $X$, their intersection with $Y$ are separated as well. This proves that each $\pi_Y$-fiber is the intersection of a $K^c$-orbit with $Y$ and hence $i//_K : Y//K \to X//K$ is injective.

All fibers of $\pi$ and $\pi_Y$ have the same dimension and $X, Y, X//K$ and $Y//K$ are non-singular, so both projections are submersions.

In order to see the Runge property, choose some exhausting sequence $A_n \subset Y$ of compact subsets with $A_n \subset A_{n+1}$ and $\bigcup A_n = Y$. Given a holomorphic function $f$ on $Y//K$ and a compact subset $D \subset Y//K$. For some $N \in \mathbb{N}$ the set $D \subset \pi(A_N)$. The holomorphic function $\pi^* f$ can be approximated by a global function on $X$ which can be averaged over $K$. This averaging process does not destroy the approximation property. The obtained function can be considered as a function on $X//K$ approximating $f$ on $D$. This shows the Runge property. 

3 Local slice model

After introducing the relevant objects and recalling the situation of a compact group acting, we will now consider proper actions of Lie groups by holomorphic transformations more closely. Our goal in this section is to construct local models for proper actions around totally real $G$-orbits.

3.1 Definition of the local slice model

Let $K$ be a compact subgroup of a Lie group $G$ and $G^*$ a $G$-tube of $G$. Theorem 10 shows that we can choose $G^*$ to be $K$-stable, i.e. the $(G \times K)$-action

$$\alpha_G : \quad (G \times K) \times G \rightarrow G$$

$$(g, k, x) \mapsto (g \cdot x \cdot k^{-1})$$

extends to a $(G \times K)$-action $\alpha_{G^*}$ by holomorphic transformations on $G^*$. Now let $V$ be a complex $K$-representation with (analytic) Hilbert quotient $\pi_V : V \to V//K$ and $S \subset V$ a $K$-stable neighborhood of the origin. With the given $K$-action on $S$ we can define a $(G \times K)$-action on the product $G^* \times S$ via

$$\alpha_{G^* \times S} : \quad (G \times K) \times (G^* \times S) \rightarrow G^* \times S$$

$$(g, k, \gamma, s) \mapsto (g \cdot \gamma \cdot k^{-1}, k \cdot s)$$

This action induces a $G$-action on the formal Hilbert quotient $G^* \times//K S = (G^* \times S)//K$ in the following way. Since the $G$- and the $K$-action commute, the induced $(G \times K)$- and hence the $G$-action on $O(G^* \times S)$ stabilizes the subalgebra of
K-invariant function $\mathcal{O}^K(G^* \times S)$. Thus the $(G \times K)$-action pushes down to the formal Hilbert quotient, with $K$ contained in the ineffectivity. Denoting by $[\gamma, s] = \pi(\gamma, s)$ a point in the image of the quotient map $\pi : G^* \times S \to G^* \times //KS$ we may write this as

$$\alpha : G \times (G^* \times //KS) \to G^* \times //KS \quad (g, [\gamma, s]) \mapsto [g \cdot \gamma, s]$$

Let us further assume $S$ to be a Stein manifold. On a Stein manifold an action of a compact group is known to provide an analytic Hilbert quotient which is a complex space, even more, Theorem 22 shows that the quotient is a Stein space. Since in our case the $K$-action is free, the quotient $G^* \times //KS$ is non-singular (Proposition 26), hence a Stein manifold. Finally we will ask $S$ to be Runge in $V$. This implies that the inclusion $S \to V$ induces an inclusion $S//K \to V//K$ (Proposition 26.1) and will provide a shrinking property later (Theorem 32 and Corollary 33). Finally we may suppose $S$ to be contractible in order to avoid cohomological obstructions. Choosing a $K$-invariant scalar product on $V$ and set $\rho = \|\cdot\|^2$ with respect to that scalar product, all sets $S = \{v \in V \mid \rho(v) < c\}$ for $c > 0$ are contractible Stein Runge neighborhoods of the origin, forming a neighborhood basis.

We will use quotients of this type as local models, precisely:

**Definition 27.** Let $G^*$, $K$ and $S$ be as above. An open subset $Y \subset X$ is called a local slice model of $x$ if there is a $G$-equivariant biholomorphic map $\psi : G^* \times //KS \to Y$ mapping $[c, 0]$ to $x \in X$.

For simplicity we sometimes identify $S$ with its image in $Y$. Note that $Y$ is automatically $G$-stable.

### 3.2 Complexified orbits in slice, product and quotient

In the subsequent section 3.3 we will show that for a local slice model $Y = G^* \times //KS$ the ringed spaces $(Y//G, \mathcal{O}_{Y//G})$ and $(S//K, \mathcal{O}_{S//K})$ are isomorphic. Since $S//K$ is a complex space, this will be the way to endow $Y//G$ with a natural complex structure.

In this section, we will prepare this result in showing a one-to-one correspondence between the complexified $K$-orbits in $S$ with the complexified $(G \times K)$-orbits in the product $G^* \times S$ and the complexified $G$-orbits in the local slice model $G^* \times //KS$, denoted $B^G_K$, $B^G_{G \times KS}$ and $B^G_{G^* \times KS}$ respectively.

We consider the projection $\kappa : G^* \times S \to S$ onto the second component, the quotient map $\pi : G^* \times S \to G^* \times //KS$ and their induced maps $\kappa^{-1} : \mathcal{P}(S) \to \mathcal{P}(G^* \times S)$ and $\pi^{-1} : \mathcal{P}(G^* \times //KS) \to \mathcal{P}(G^* \times S)$ on the set of subsets.

**Proposition 28.** The maps $\kappa^{-1}$ and $\pi^{-1}$ induce bijections $\kappa^{-1} : B^G_K \to B^G_{G \times KS}$ and $\pi^{-1} : B^G_{G \times KS} \to B^G_{G^* \times KS}$ between the corresponding complexified orbits. Furthermore both bijections define bijections on the level of closed complexified orbits.
For the proof we start with a technical lemma.

**Lemma 29.** Let \( p : X \to Y \) be a continuous, open, surjective map between locally connected spaces and let each fiber be connected. Then \( p^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X) \) induces a bijection between the connected components.

**Proof.** A subset of a locally connected space is a connected component if and only if it is closed, open and connected. Writing \( Y \) as the disjoint union of its connected components, \( Y = \bigcup C_\alpha \), then \( X = \bigcup D_\alpha \) with \( D_\alpha = p^{-1}(C_\alpha) \). The set \( D_\alpha \) is open and due to \( D_\alpha = X \setminus \bigcup_{\beta \neq \alpha} D_\beta \) it is also closed. Now let \( E \) be a connected component of \( X \) hence contained in some set \( D_\alpha \). For each point \( d \in E \) the fiber \( p^{-1}(p(d)) \) is connected by assumption hence entirely contained in \( E \), thus \( E \) is \( p \)-saturated, i.e. \( E = p^{-1}(p(E)) \). Both set \( p(E) \) and \( p(D_\alpha \setminus E) \) are open by assumption of \( p \) being open and since they are complementary in \( C_\alpha \) also closed. Thus one of them must be empty, hence \( D_\alpha = E \). We showed that the map \( p^{-1} \) maps connected components to connected components. It is surjective, since each point is contained in some set \( D_\alpha \). Finally, \( p^{-1} \) is injective, since \( p \) is assumed to be surjective. \( \Box \)

**Proof of Proposition 28.** We will apply Lemma 29 to the maps \( \kappa : G^* \times S \to S \) and \( \pi : G^* \times S \to G^* \times /\!\!/ K \times S \). In order to ensure continuity and openness of \( \pi \) and \( \kappa \) with respect to the \( \tau \)-topologies, we will use Lemma 17 which shows that a \( G \)-equivariant holomorphic map is \( \tau_G \)-continuous and \( \tau_G \)-open. For this, we define a \((G \times K)\)-action on \( S \) by extending the \( K \)-action ineffectively to \( G \). The induced topologies \( \tau_K \) and \( \tau_{G \times K} \) are identical. Thus Lemma 17 shows that \( \kappa \) is \( \tau_{G \times K} \)-continuous and \( \tau_G \)-open. For the projection \( \pi \) in turn, observe that the \( G \)-action on the local slice model \( G^* \times /\!\!/ K \times S \) is defined as induced by the \((G \times K)\)-action on \( G^* \times S \) pushed down to the quotient with \( \Delta = \{ (k^{-1}, k) | k \in K \} \subseteq G \times K \) contained in the inefficiency. Thus the topologies \( \tau_{G \times K} \) and \( \tau_G \) are identical. Analogously Lemma 17 provides that \( \pi \) is \( \tau_{G \times K} \)-continuous and \( \tau_G \)-open. In order to be able to apply Lemma 29 we are left to show that the fibers of \( \kappa \) and \( \pi \) are \( \tau_{G \times K} \)-connected. The \( G \)-action and the \( K \)-action on the product \( G^* \times S \) induce finer topologies \( \tau_G \) and \( \tau_K \), i.e. \( \tau_K \subseteq \tau_{G \times K} \) and \( \tau_K \subseteq \tau_{G \times K} \). A \( \kappa \)-fiber \( G^* \times \{ s \} \) is a complexified \( G \)-orbit (Lemma 16), hence \( \tau_G \)-connected and therefore connected with respect to the stronger topology \( \tau_{G \times K} \). Proposition 26(2) shows that each \( \pi \)-fiber is a complexified \( K \)-orbit, hence \( \tau_K \)-connected, and therefore connected with respect to the stronger topology \( \tau_{G \times K} \). Thus we are allowed to apply Lemma 29 to both maps \( \kappa \) and \( \pi \) and obtain the desired bijections \( \kappa^{-1} : B_S^G \to B_{G \times K} S \) and \( \pi^{-1} : B_{G^* \times /\!\!/ K} S \to B_{G^* \times S} \).

Now we turn back to the classical topologies. Both projections \( \pi \) and \( \kappa \) are continuous and open (Proposition 26(3)). Thus both induce a bijection between the saturated closed sets and closed set in the target and in particular inverse images of closed complexified orbits are closed complexified orbits.. \( \Box \)
3.3 Quotients of the local slice model

In this section we can give $Y//G$ the structure of a complex space. The sheaf $\mathcal{O}_{Y//G}$ is defined as $\pi_*\mathcal{O}_Y^G$, i.e. given an open subset $U \subset Y//G$ then $f \in \mathcal{O}_{Y//G}(U)$ if $\pi^*f$ is a ($G$-invariant) holomorphic function on $\pi^{-1}(U)$.

**Theorem 30.** Let $Y$ be a local slice model with slice $S \subset Y$ and denote $\pi_S : S \to S//K$ the analytic Hilbert quotient and $\pi : Y \to Y//G$ the formal Hilbert quotient. Then the embedding $i_S : S \to Y$ induces an isomorphism $(S//K, \mathcal{O}_{S//K}) \to (Y//G, \mathcal{O}_{Y//G})$.

Besides the result on the correspondence of complexified $K$, $(G \times K)$- and $G$-orbits in $S$, $G^* \times S$ and $Y$ from Section 3.2 above we need some results stating similar geometrical properties for the fibers of $\pi : Y \to Y//G$ analogous as described in Theorem 25.

**Proposition 31.** Let $Y$ be a local slice model with slice $S \subset Y$ and denote $\pi_S : S \to S//K$ the analytic Hilbert quotient and $\pi : Y \to Y//G$ the formal Hilbert quotient. Then each complexified $G$-orbit in $Y$ corresponds to a complexified $K$-orbit in $S$. Furthermore, for every $p \in Y//G$ and every $x \in \pi^{-1}(p)$

1. the closure of $\overline{B_Y(x)}$ is the union of complexified $G$-orbits $B_Y(y)$ with $\dim B_Y(y) < \dim B_Y(x)$ for each $y \notin B_Y(x)$.
2. there is exactly one complexified $G$-orbit of lowest dimension, denoted by $E_Y(p)$. This complexified $G$-orbit is closed.
3. $E_Y(p)$ lies in the closure of each complexified $G$-orbit in the fiber, formally $E_Y(p) \subset \overline{B_Y(x)}$.

The embedding $i_S : S \to Y$ induces a homeomorphism $\varphi_S : S//K \to Y//G$ in the sense that the following diagram commutes.

$$
\begin{array}{ccc}
S & \xrightarrow{i_S} & Y \\
\downarrow \pi_S & & \downarrow \pi \\
S//K & \xrightarrow{\varphi_S} & Y//G
\end{array}
$$

**Proof.** For $x_0 = [\gamma_0, s_0] \in Y$ the complexified $G$-orbit $B_Y(x_0)$ is the image of $i_{s_0} : G^* \to Y, \gamma \mapsto [\gamma, s_0]$. For each $s_0 \in S$ the complexified $K$-orbit $B_S(s_0)$ is the intersection $B_Y([e, s_0]) \cap S$. This realizes the one-to-one correspondence between complexified $K$-orbits in $S$ and complexified $G$-orbits in $Y$. $B_S(s_0)$ is closed if and only if $B_Y([e, s_0])$ is closed. Since each $\pi_S$-fiber contains a unique complexified $K$-orbit (Theorem 25), the same is true for $\pi$-fibers. This implies that the results of Theorem 25 applied to $S$ transform to the corresponding results on $Y$. Consequently, the map $S//K \to Y//G$ represents the identification of these closed complexified orbits, imposing a bijection which is a continuous map by construction. In order to establish that this identification is a homeomorphism...
we will show that it is a proper map. For this, let \( s_n \in S \) be a sequence. The induced sequence \( \pi_S(s_n) \in S//K \) is mapped via the considered identification to \( \pi([e, s_n]) \in Y//G \). Assume that this sequence converges and we aim to show that then \( \pi_S(s_n) \) converges. Every \( G \)-invariant holomorphic function \( f \) on \( Y \) induces a continuous function \( h \) on \( Y//G \). As \( h(\pi([e, s_n])) \) converges for every such function \( h \), the sequence \( f([e, s_n]) = h(\pi([e, s_n])) \) converges for every \( G \)-invariant holomorphic function \( f \) on \( Y \). As any \( K \)-invariant holomorphic map \( f_S \) on \( S \) can be interpreted as a \( G \)-invariant holomorphic map \( f \) on \( Y \), we deduce that \( f_S(s_n) \) converges for every \( K \)-invariant holomorphic function. As any holomorphic function on the quotient \( S//K \) can be seen as a \( K \)-invariant holomorphic function on \( S \), the image of the sequence \( \pi_S(s_n) \) under any holomorphic function on \( S//K \) converges. Finally, since \( S//K \) is a Stein space, this imposes that \( \pi_S(s_n) \) itself converges. This shows that \( S//K \to Y//G \) is a homeomorphism.

**Proof of Theorem 32.** Induced from the inclusion \( \iota_S : S \to Y \) there is a homeomorphism \( \varphi_S : S//K \to Y//G \) (Proposition 31). Thus for an open subset \( U \subset Y//G \) the intersection \( W_S \) of \( W = \pi^{-1}(U) \) with \( S \) equals \( \pi_S^{-1}(\varphi_S^{-1}(U)) \). Hence \( V_S \) is \( \pi_S \)-saturated. Given a holomorphic function \( f : U \to \mathbb{C} \) the pull-back \( \pi^*f \) is \( G \)-invariant and as \( V_S \) is \( K \)-stable the function’s restriction \( \pi^*f|_{V_S} \) is a \( K \)-invariant holomorphic function \( g : \varphi_S^{-1}(U) \to \mathbb{C} \). This defines a map \( \mathcal{O}_{Y//G}(U) \to \mathcal{O}_{S//K}(\varphi_S^{-1}(U)) \).

Now we want to define a map \( \mathcal{O}_{S//K}(V) \to \mathcal{O}_{Y//G}(\varphi_S(V)) \) for any open subset \( V \subset S//K \). On the level of (open) sets the identifications in Proposition 28 show that \( G^* \times//K \pi_S^{-1}(V) = \pi^{-1}(\varphi_S(V)) \). A holomorphic function \( g : V \to \mathbb{C} \) arises by definition from a \( K \)-invariant holomorphic function \( \pi_Sg : \pi_S^{-1}(V) \to \mathbb{C} \). We define \( h : G^* \times \varphi_S^{-1}(V) \to \mathbb{C} \) by \( h(e, s) = \pi_Sg(s) \). It is a \((G \times K)\)-invariant holomorphic function which pushes down to the \( G \)-invariant holomorphic function \( h : G^* \times//K \pi_S^{-1}(V) \to \mathbb{C} \). This identification defines the desired inverse.

We showed that for every local slice model \( Y \) the quotient \( \pi : Y \to Y//G \) is an analytic Hilbert quotient.

### 3.4 Shrinking of the local slice model

In our proof of the existence of local slice models \( G^* \times//K S \) around special points (Section 5.4) and later for the existence of invariant local potentials around these points (Section 4) we will need to shrink \( G^* \times//K S \) to an arbitrarily small local slice model \( T \times//K S_0 \). Herein \( T \subset G^* \) is a smaller \( G \)-tube as provided by definition of \( G^* \) and \( S_0 \subset S \) is a smaller slice with \( S_0 \) Runge in \( S \) and \( S_0 \) Stein and \( K \)-stable.

**Theorem 32.** (Shrinking Theorem of Local Slice Model) Let the set \( Y = G^* \times//K S \subset X \) be a local slice model around \( x \) and \( U \subset Y \) a \( G \)-stable neighborhood of \( x \). Then there is a \( G \)-subtube \( T \subset G^* \) and an open \( K \)-stable Stein neighborhood \( S_0 \) of the origin, Runge in \( S \), such that \( Y_0 = T \times//K S_0 \) is a local slice model around \( x \).
Corollary 33. The open embedding \( \iota_0 : Y_0 \rightarrow Y \) of the shrunk local slice model in Theorem 32 induces an open embedding \( \iota_0 : Y_0 / G \rightarrow Y / G \).

Proof of Theorem 32 and Corollary 33. Let \( p : G^* \times S \rightarrow G^* \times /K S \) denote the projection of the analytic Hilbert quotient and set \( \hat{U} = p^{-1}(U) \). There are open subsets \( A \subset G^* \) and \( B \subset S \) such that \( (e, 0) \in A \times B \subset \hat{U} \). The definition of \( G \)-tubes provides a \( G \)-tube \( T \subset A \) that is Runge in \( G^* \). The manifold \( S \) can be realized as a \( K \)-stable open neighborhood of the origin in a unitary \( K \)-representation \( V \). Let \( \rho_V \) be the squared norm function of the \( K \)-invariant norm on \( V \). For a sufficiently small \( \delta > 0 \), the set \( S_0 = \{ x \in V \mid \rho_V(x) < \delta \} \subset S \) is a Stein Runge subset of \( S \). Since the diagonal \( K \)-action on \( G^* \times S \) is free and \( T \times S_0 \) is an open Runge Stein subset of \( G^* \times S \), Proposition 26 implies that \( Y_0 = T \times /K S_0 \) can be seen as an open Runge Stein subset of \( Y = G^* \times /K S \). Part 31 of this proposition shows that the inclusion \( S_0 \hookrightarrow S / K \) induces an open embedding \( S_0 /K \hookrightarrow S / K \). From Theorem 32 we obtain canonical biholomorphisms \( S_0 /K \cong Y_0 /G \) and \( S /K \cong Y /G \). Combining these facts we see that the inclusion \( Y_0 \hookrightarrow Y \) induces a holomorphic open embedding \( Y_0 /G \rightarrow Y /G \).

3.5 Existence of the local slice model

As a next step we construct a local slice at a complexified \( G \)-orbit through \( x \in X \) if the orbit \( G \cdot x \) is a totally real submanifold of \( X \). The construction will give a local slice model of \( x \) for the \( G \)-action on \( X \).

Theorem 34. (Local Slice Theorem) Let \( X \) be a Stein manifold and \( G \) a Lie group acting properly by holomorphic transformations. If \( G \cdot x \) is a totally real submanifold of \( X \), there is a local slice model \( Y \) at \( x \).

Let \( K \) denote the compact isotropy group at \( x \). The slice \( S \subset Y \) can be chosen \( K \)-stable such that \( S \) is \( K \)-equivariantly biholomorphic to an open neighborhood of the origin of a \( K \)-stable complex linear subspace in \( T_x X \). This subspace is in fact complementary to the complex subspace generated by \( T_x \mathbb{G}(G \cdot x) \subset T_x X \). Note that the complex subspace generated by \( T_x \mathbb{G}(G \cdot x) \) is the right candidate for the tangent space of the complexified \( G \)-orbit through \( x \).

We first show the following technical lemma.

Lemma 35. Let \( G \) act properly on \( X_2 \) and let \( \varphi : X_1 \rightarrow X_2 \) be a \( G \)-equivariant local diffeomorphism with \( \varphi(x_1) = x_2 \) and suppose that \( \varphi|_{G \cdot x_1} \) is injective. Then there are \( G \)-stable neighborhoods \( U_i \) around \( x_i \) such that \( \varphi|_{U_i} : U_1 \rightarrow U_2 \) is a diffeomorphism.

Proof. First we recall that the action on \( X_1 \) is proper as well. Suppose that in each neighborhood of the orbit \( G \cdot x_1 \) there is a pair of points \( u \neq v \) mapping to the same point. So we get sequences \( u_n, v_n \) with \( \varphi(u_n) = v_n \) and \( u_n \neq v_n \). Using the action, we may assume that \( u_n \) converges to \( u \in G \cdot x_1 \). There is a sequence \( g_n \in G \) such that \( \tilde{v}_n = g_n \cdot v_n \) converges to a point in \( G \cdot x_1 \). This can even be arranged such that \( \tilde{v}_n \) converges to \( u \) likewise. Thus we get two sequences both
converging to \( f(u) \), namely \( f(\tilde{v}_n) \) and \( g_n^{-1} f(\tilde{v}_n) = f(v_n) = f(u_n) \). Properness of the action implies the existence of a convergent subsequence of \( g_n \), which for simplicity we also denote \( g_n \). The sequence \( f(\tilde{v}_n) \) converges therefore to \( f(u) \) as well as to \( g f(u) = f(gu) \), hence \( f(u) = f(gu) \). Injectivity on the orbit \( G \cdot x_1 \) provides \( u = gu \). So, \( u \) is the limit of both \( u_n \) and \( v_n \). Around \( u \), the map \( f \) is a local diffeomorphism, so \( u_n = v_n \) for large \( n \), contradicting the assumption. □

**Proof of Theorem 34.** We have to prove the existence of a local slice model at \( x \). Let \( K = G_x \) denote the isotropy group at \( x \) which is compact since \( G \) acts properly on \( X \). On a complex manifold an action of a compact group is linearizable in a neighborhood of a fixed point. This implies that \( x \) has an open \( K \)-stable neighborhood \( U \) of \( x \) which is \( K \)-equivariantly biholomorphic to an open neighborhood \( W \) of 0 in \( T_x X \) where \( K \) acts on \( T_x X \) by the isotropy representation of \( K \) at \( x \). The complex span of \( T_x(G \cdot x) \subset T_x X \) is a \( K \)-invariant subrepresentation of \( T_x X \). We choose a \( K \)-invariant complex complement \( V \) and set \( S_W = V \cap W \).

We use the linearization map to identify \( S_W \) with a \( K \)-stable local complex submanifold \( S \) of \( X \) with \( x \in S \). The restriction \( \alpha : G \times S \to X, (g,y) \mapsto g \cdot y \) of the action map is real analytic and for fixed \( g \in G \) the map \( \alpha|_{\{g\}\times S} : \{g\} \times S \to X \) is holomorphic. It follows that \( \alpha \) extends to a holomorphic map \( \alpha^c : \Omega \to X \) where \( \Omega \subset G^* \times S \) is an open neighborhood of \( G \times \{x\} \) in \( G^* \times S \). The map \( \alpha^c \) is locally \( G \)-equivariant and \( G^* \) is \( G \)-equivariantly diffeomorphic to \( G \times \Sigma \) by definition. This implies that \( \alpha^c \) extends \( G \)-equivariantly and holomorphically to a \( G \)-stable open neighborhood, i.e. after shrinking of \( G^* \) and \( S \) we may assume \( \Omega = G^* \times S \). The map \( \alpha^c : G^* \times S \to X \) is \( K \)-invariant where \( K \) acts on \( G^* \times S \) by \( k \cdot (\gamma,y) = (\gamma \cdot k^{-1}, k \cdot y) \) and after shrinking we may assume that it is a submersion onto its open image.

Since \( \alpha^c \) is \( K \)-invariant, it pushes down to a holomorphic map

\[
\varphi : G^* \times//K S \to X
\]

submersive to its image. Denote \( B = \alpha^c(G^* \times \{0\}) = \varphi(G^* \times//K \{0\}) \). Since the \( G \)-orbit is totally real,

\[
\dim G^* // K = \dim \mathbb{R} G/K = \dim B .
\]

Thus a calculation of the dimension

\[
\dim \left( G^* \times//K S \right) = \dim (G^* // K) + \dim S = \dim B + \dim S = \dim X
\]

shows that \( \varphi \) is in fact a \( G \)-equivariant local diffeomorphism. Lemma 35 in turn shows that this becomes a diffeomorphism to the image after restricting \( \varphi \) to some \( G \)-stable neighborhood of \( [e,0] \). In a last step, using the Shrinking Theorem of Local Slice Models (Theorem 32), this can be restricted to a neighborhood of the form \( G^* \times//K S \) by sufficient shrinking of \( G^* \) and \( S \). □
4 Local potentials

The construction of both the complex structure (Section 5) and in particular the Kähler structure (Section 6) make use of local potentials. The main purpose of this section is to prove Theorem 3 stating the existence of a local potential around each point in the momentum zero level \( \mathcal{M} \).

Additionally, the construction of a local potential gives rise to the question whether away from the momentum zero level local potentials do always exist as well. For this, we have to impose further natural conditions, namely the \( G \)-orbit of the considered point has to be totally real and the Kähler form restricted to the orbit has to be exact.

The \( G \)-orbits in the momentum zero level are isotropic and therefore the orbits are perforce totally real and \( \omega \) restricted to the orbit is exact. So, on \( \mathcal{M} = \mu^{-1}(0) \) these conditions are fulfilled automatically.

4.1 Definition of a local potential

Definition 36.

1. Given a Hamiltonian Kähler manifold \( (X, G, \omega, \mu) \) a potential is a smooth \( G \)-invariant strictly plurisubharmonic function \( \rho \) such that \( \omega = dd^c \rho \) and for each \( v \in \text{Lie}(G) \) the component of the momentum map satisfies \( \mu^v = -i_v d^c \rho \). Here \( \hat{v} \) is the vector field of \( v \) induced by the \( G \)-action.

2. A function is called a local potential around \( x \in X \) if it is a potential on some \( G \)-stable neighborhood of \( x \).

Observe that this makes sense as we obtain the momentum map condition

\[
    d\mu^v = -di_v d^c \rho = i_v dd^c \rho = i_v \omega
\]

from \( G \)-invariance of \( \rho \) and \( d^c \rho \) since invariance reads in formula

\[
    0 = L_v d^c \rho = i_v dd^c \rho + di_v d^c \rho .
\]

Vice versa, given a smooth \( G \)-invariant strictly plurisubharmonic function \( \rho \), it defines a Hamiltonian structure, i.e. an invariant Kähler form \( \omega = dd^c \rho \) and an equivariant momentum map defined by \( \mu^v = -i_v d^c \rho \).

We obtain strong relations between the momentum zero level \( \mathcal{M} \) and the analytic Hilbert quotient if we impose our exhaustion property of the potential (Definition 7).

Remark. On Stein manifolds an exhaustive potential \( \rho \) always exists [Gra58]. By averaging over a compact group \( K \) it can be made \( K \)-invariant.

The moment zero level \( \mathcal{M} \) is \( G \)-stable by construction. Therefore the inclusion \( i: \mathcal{M} \to X \) induces a map \( i_G: \mathcal{M}/G \to X//G \) to the formal Hilbert quotient. In the situation of a compact group we have the following theorem.
Theorem 37. Let the compact Lie group $K$ act on the Stein manifold $X$ by holomorphic transformations and let a $K$-exhaustive potential $\rho$ induce a Hamiltonian structure on $X$. Then the map $i_K : M/K \to X//K$ is a homeomorphism. In particular, for all $p \in X//K$ the set $\mathcal{M}_p = M \cap \pi^{-1}(p)$ is not empty and $\mathcal{M}_p \subset E(p)$.

For the proof see [HK95]. There it applies only for proper potentials unbounded from above. An exhaustive potential $\rho$ in the sense of this paper can be made unbounded from above keeping properness by choosing a suitable strictly convex diffeomorphism $\chi : (-\infty, \sup \rho) \to \mathbb{R}$ and using $\tilde{\rho} = \chi \circ \rho$ as exhaustive potential. This does not change the momentum zero level $\mathcal{M}$ (compare Lemma 44).

Corollary 38. Let $X$ be a Stein $(G \times K)$-manifold, $K$ acts freely and $\rho$ is a strictly plurisubharmonic $(G \times K)$-exhaustive potential inducing a Kähler form and a momentum map $\mu_K : X \to \text{Lie}(K)^*$. Then $M = \mu_K^{-1}(0)$, $M/K$ and $X//K$ are manifolds with induced $G$-action, the $G$-equivariant inclusion $M \to X$ induces a $G$-equivariant homeomorphism and $\rho|_M$ pushes down to a strictly plurisubharmonic $G$-exhaustion on $X//K$.

Proof. Proposition 26 shows that $X//K$ is a manifold. As the $K$-action is free the momentum map $\mu_K$ is submersive thus $M$ is a manifold. Since $K$ acts freely on $M$ the quotient $M/K$ is a manifold as well and $M \to M/K$ is a $K$-principle bundle. Since the $G$- and the $K$-action commute, $M$ is $G$-stable and both quotients $M/K$ and $X//K$ inherit a proper $G$-action such that the quotient maps $p : M \to M/K$ and $\pi : X \to X//K$ are $G$-equivariant. For $y \in X//K$ the $\pi$-fiber $F = \pi^{-1}(y)$ is a Stein manifold. Since the isotropy of the $G$-action at $y \in X//K$ is compact, the restriction $p|_F$ is a $K$-exhaustion. Thus [HK95] shows that $M$ intersects $F$ exactly in a single $K$-orbit. Hence the $G$-equivariant inclusion $i : M \to X$ induces a $G$-equivariant bijection $\varphi : M/K \to X//K$. By the same arguments as in [HK95] the map $\varphi|_G : (M/K)/G \to (X//K)/G$ is a homeomorphism and since the $G$-action on $M/K$ and $X//K$ is proper the bijection $\varphi$ is a homeomorphism as well.

The restricted function $\rho|_M$ is a $K$-invariant $G$-exhaustion and induces therefore a $G$-exhaustion on the quotient $M/K$. For every point $y \in M/K$ the principle bundle $p : M \to M/K$ possesses a local section $\sigma$. Locally the pushed down function can be written as $\sigma^* \rho$ independently of the choice of $\sigma$. For the point $x = \sigma(y)$ the covector $d^\ast \rho|_{T_x M}$ has $T_x(G \cdot x) = \ker(p)_x$ contained in its kernel. Thus $\sigma^* d^\ast \rho = d^\ast \sigma^* \rho$. Further $T_x(G \cdot x)$ is the kernel of $\omega|_{T_x M}$. Hence $dd^\ast \sigma^* \rho = \sigma^* \omega$ is a Kähler form and $\sigma^* \rho$ is strictly plurisubharmonic.

4.2 Potential on complexified $G$-orbit

In this section we will take a look at the $G$-invariant potential on a complexified $G$-orbit. They turn out to be minimal and critical on “isolated” $G$-orbit.

Definition 39. Given a smooth $G$-invariant function $\rho$. We say that $\rho$ has an local isolated minimum locus at $p$ if there is a $G$-stable neighborhood $U$ of $p$ such that $\rho(x) > \rho(p)$ for all $x \in U \setminus (G \cdot p)$.
We get the following local version on a complexified $G$-orbit.

**Lemma 40.** Let $B$ be a complexified $G$-orbit, $\rho$ a $G$-invariant strictly plurisubharmonic function and $p_0 \in B$ a critical point of $\rho$. If $\Sigma_B$ is a local submanifold transversal to the $G$-orbit $G \cdot p_0$ at $p_0$ of maximal dimension, then the restriction $\rho|_{\Sigma_B}$ has positive Hessian at $p_0$ and the $G$-orbit is a local isolated minimum locus of $\rho$.

**Proof.** Denote $\tilde{v}$ the fundamental vector field of $v \in \text{Lie}(G)$. Fix a linear subspace $W \subset \text{Lie}(G)$ such that the linear map $W \rightarrow T_{p_0}(G \cdot p_0), v \mapsto \tilde{v}_{p_0}$ is a bijection. For each $v \in W$, $v \neq 0$, we calculate

$$J\tilde{v}(J\tilde{v}(\rho))_{p_0} = -(iJ\hat{\xi}d\hat{\xi}\hat{\rho})_{p_0} = (i\hat{\xi}d\hat{\rho})_{p_0} = \omega_{p_0}(\tilde{v}, J\tilde{v}) > 0$$

since $L_\xi d\hat{\rho} = d\hat{\xi}\hat{\rho} = 0$ and $\omega_{p_0}(\tilde{v}, J\tilde{v}) > 0$ for all $\tilde{v}_{p_0} \neq 0$. From this we conclude that $T_{p_0}(G \cdot p_0)$ and $JT_{p_0}(G \cdot p_0)$ intersect trivially, hence the orbit is totally real and it is of maximal dimension since $T_{p_0}(G \cdot p_0) + JT_{p_0}(G \cdot p_0)$ span the tangent space of the complexified orbit $B$ by definition. Denote $\tilde{v}$ the induced vector field of $v \in \text{Lie}(G)$ and for a vector field $\xi$ the flow for time $t$ by $\varphi^t(\xi)$ starting at point $x \in X$. Then there is an open neighborhood $U$ of the origin in $W$ such that $U \rightarrow B, v \mapsto \varphi^t(\tilde{v})$ is an immersion and its image $\Sigma$ is a local submanifold transversal to the $G$-orbit and of maximal dimension for which the above calculations shows that the Hessian of $\rho$ is positive with respect to local linear coordinates coming from the identification $U \rightarrow \Sigma$. As $\rho$ is supposed to be critical a $p_0$, $\rho|_{\Sigma}$ has a local isolated minimum at $p_0$.

From the exhaustion property of a function $\rho$ we get a result on the existence of critical points.

**Lemma 41.** Let $G$ act properly on the manifold $X$, assume $X/G$ to be connected and let $\rho : X \rightarrow \mathbb{R}$ be a smooth $G$-exhaustive potential. If $\rho$ possesses two different $G$-orbits $G \cdot p_1$ and $G \cdot p_2$ which are local isolated minimum loci, then there is a critical point $q \notin G \cdot p_1 \cup G \cdot p_2$ such that $q$ is not a minimum locus.

**Proof.** First let us pass to the quotient, i.e. introduce the quotient map $\beta : X \rightarrow Z = X/G$, set $\bar{p}_1 = \beta(p_1)$ and define $\tilde{\rho}$ by $\tilde{\rho} \circ \beta = \rho$. Denote for each $b \in \mathbb{R}$ the sets $X_b = \{ x \in X \mid \rho(x) < b \}$ and $Z_b = \{ z \in Z \mid \tilde{\rho}(z) < b \}$. The function $\rho$ is exhaustive and $Z$ is connected, so there is a $C \in \mathbb{R}$ such that $\bar{p}_1, \bar{p}_2$ lie in the same connected component of $Z_C$.

We may assume that $\tilde{\rho}(\bar{p}_1) \geq \tilde{\rho}(\bar{p}_2)$ and claim that there is a $c > \tilde{\rho}(\bar{p}_1)$ such that the points $\bar{p}_1$ and $\bar{p}_2$ lie in different connected components of $Z_c$. By assumption $\bar{p}_1$ is a local isolated minimum point of $\tilde{\rho}$, i.e. there is an open neighborhood $V$ of $\bar{p}_1$ such that

$$\{ v \in V \mid \tilde{\rho}(v) > \tilde{\rho}(\bar{p}_1) \} = V \setminus \{ \bar{p}_1 \}.$$
So there is a $c > \hat{\rho}(\tilde{p}_1)$ such that $U_c = \{ v \in V \mid \hat{\rho}(v) < c \} = Z_c \cap V$ is relatively compact in $V$. Thus $Z_c$ is the disjoint union of $U_c$ and $Z_c \setminus V$, the sets are disconnected and $\tilde{p}_1 \in U_c$ and $\tilde{p}_2 \in Z_c \setminus V$. Let $\kappa$ be the supremum of all $c > \hat{\rho}(\tilde{p}_1)$ such that the points $\tilde{p}_1$ and $\tilde{p}_2$ lie in different connected components of $Z_c$.

Assume $d\rho(q) \neq 0$ for all $q \in \partial X_\kappa$. As $\partial Z_\kappa$ is compact, there is a $\tilde{\kappa} > \kappa$ such that $X_{\kappa}$ is a deformation retract of $X_{\tilde{\kappa}}$, e.g. just by choosing the negative gradient flow of $\rho$. Additionally this deformation can be chosen to be $G$-equivariant, so that $Z_{\kappa}$ is a deformation retract of $Z_{\tilde{\kappa}}$. Hence for each $i = 1, 2$ the connected component of $\tilde{p}_i$ in $Z_{\tilde{\kappa}}$ is retracted to the connected component of $\tilde{p}_i$ in $Z_{\kappa}$. Thus the points $\tilde{p}_1$ and $\tilde{p}_2$ still lie in different connected components of $Z_{\kappa}$.

This contradicts the choice of $\kappa$ as a supremum with this property. So, we may conclude that there is a point $q \in \partial X_\kappa$ with $d\rho(q) = 0$. As each neighborhood of $q$ intersects $X_\kappa$ the function $\rho$ does not become locally minimal at $q$.

Combining the results of the previous two lemmata we obtain the following global statement on complexified $G$-orbits as a corollary.

**Corollary 42.** Let $B$ be a complexified $G$-orbit, $\rho : B \to \mathbb{R}$ a $G$-exhaustive strictly plurisubharmonic function and $p \in B$ a critical point of $\rho$. Then the $G$-orbit through $p$ contains the only points on which $\rho$ becomes critical or minimal.

**Proof.** Since $\rho$ is $G$-exhaustive, there is at least one $G$-orbit on which $\rho$ becomes minimal and therefore critical. Assume there is a second $G$-orbit on which $\rho$ becomes critical, then by Lemma 40 both orbits are local isolated minimum loci and Lemma 41 shows that there is a third point $q$ which is critical but not a minimum. This in turn contradicts Lemma 40.

### 4.3 Existence of a local potential

Before proving Theorem 3, namely the existence of a local potential, we will treat first the simpler product case. For this purpose we have to consider a real, closed $(1, 1)$-form $\omega$ which might not be a Kähler form but admit a map $\mu : X \to \text{Lie}(G)^*$ satisfying $d\mu^\nu = i_\xi \omega$ which we still call a momentum map. In this context, we call an invariant plurisubharmonic function $\rho$ which might not be strictly plurisubharmonic but induces $\omega$ and $\mu$ still a potential.

**Lemma 43.** Let $G^*$ be a $G$-tube associated to a Lie group $G$. Furthermore let $S$ be a 1-connected Stein manifold and let $G$ act on $X = G^* \times S$ on the first factor. Let a $G$-invariant real closed $(1, 1)$-form $\omega$ be given on $X$ with a momentum map $\mu$. Then there is a potential $\rho$ on $X$.

**Proof.** We will treat the two factors $G^*$ and $S$ separately, construct two functions $\rho_G$ and $\rho_S$ associated to the directions and finally add them to the desired function $\rho = \rho_G + \rho_S$.

For the factor $S$ define the embedding

$$i_S : S \to X$$

$$s \mapsto (e, s)$$
with $e$ denoting the neutral element in $G \subset G^*$. Since $S$ is 1-connected, there is a 1-form $\alpha$ on $S$ such that $i_S^* \omega = d \alpha$ and since $S$ is a Stein manifold, it follows that there is a smooth function $\rho_S$ on $S$ fulfilling

$$dd^c \rho_S = i_S^* \omega$$

We extend $\rho_S$ constantly to the $G^*$-factor.

Now let us turn to the $G^*$-direction. We will show that there is a unique smooth function $\rho_G$ on some $G$-stable neighborhood of $S \subset X$ satisfying for all $v \in \text{Lie}(G)$

$$\tilde{v}(\rho_G) = 0 \quad (2)$$
$$J\tilde{v}(\rho_G) = \mu^v \quad (3)$$

and normalized on $S$ to

$$\rho_G|_S = 0 \quad (4)$$

In order to establish this we define a 1-form $\eta$ by

$$\eta(\tilde{v}) = 0 \quad \text{for all } v \in \text{Lie}(G)$$
$$\eta(J\tilde{v}) = \mu^v \quad \text{for all } v \in \text{Lie}(G)$$
$$\eta(\xi) = 0 \quad \text{for all } \xi \in T_{(\gamma,s)}S$$

where $T_{(\gamma,s)}S$ denotes the tangent vectors on $G^* \times S$ tangential to the $S$-direction. We claim that pulled back to each $G^*$-direction, i.e. pulled back via $\varphi_s : G^* \rightarrow G^* \times S, \gamma \mapsto (\gamma, s)$, the form is closed, i.e. $d\varphi_s^* \eta = 0$ for all $s \in S$. For this we use $[L_\zeta, t_\xi] = 0$. Applied to the 1-form $\eta$ gives

$$d\eta(\zeta, \xi) = \omega(\tilde{v}, \tilde{w}) - \omega(\tilde{v}, J\tilde{w}) - \eta([\tilde{v}, \tilde{w}]) \quad (5)$$

for arbitrary vector fields $\zeta, \xi$. For $v, w \in \text{Lie}(G)$ we obtain

$$d\eta(\tilde{v}, \tilde{w}) = 0 \quad (6)$$

since $\eta(\tilde{v}) = \eta(\tilde{w}) = \eta([\tilde{v}, \tilde{w}]) = 0$. For the next step we observe that $J\tilde{v}(\mu^v) = \omega(\tilde{w}, J\tilde{v}), \omega(\zeta, \xi) = \omega(J\zeta, J\xi)$ and $[J\tilde{v}, J\tilde{w}] = -[\tilde{v}, \tilde{w}]$. Thus

$$d\eta(J\tilde{v}, J\tilde{w}) = \omega(\tilde{w}, J\tilde{v}) - \omega(\tilde{v}, J\tilde{w}) - \eta([J\tilde{v}, J\tilde{w}])$$
$$= \omega(\tilde{w}, J\tilde{v}) + \omega(J\tilde{v}, \tilde{w}) = 0 \quad (7)$$

And finally using $[\tilde{v}, J\tilde{w}] = J[\tilde{v}, \tilde{w}]$ and $[\tilde{v}, \tilde{w}] = -[\tilde{v}, \tilde{w}]$ we obtain

$$d\eta(\tilde{v}, J\tilde{w}) = \omega(\tilde{w}, J\tilde{v}) - J\tilde{w}(\eta(\tilde{v}) - \eta(\tilde{J}[\tilde{v}, \tilde{w}]))$$
$$= \omega(\tilde{v}, \tilde{w}) - \mu^{-[v,w]} = 0 \quad (8)$$

where the last equality is the infinitesimal version of the $G$-equivariance of $\mu$. Since $\tilde{v}$ and $J\tilde{v}$ combined for all $v \in \text{Lie}(G)$ span each tangent space $T_\gamma G^*$ the
equations (6), (7) and (8) show that the form $\varphi^*_s\eta$ is closed for all $s$. We choose some 1-connected open set $\Omega_G \subset G$ containing $e$. The set

$$\Omega = \Omega_G \times \Sigma \subset G \times \Sigma \cong G^*$$

is 1-connected as well and for all $\gamma \in \Omega$ we can define the smooth function $\rho_G$ by

$$\rho_G(\gamma, s) = \int_{\gamma e} \varphi^*_s\eta$$

which satisfies equations (2), (3) and (4). By construction $\rho_G$ is locally $G$-invariant and can therefore be extended $G$-invariantly to $G^* \times S$.

The desired function will be $\rho = \rho_G + \rho_S$. The last step is to see that $dd^c \rho = \omega$ holds. We will establish this by applying both sides to all possible vector fields. From local $G$-invariance of $\rho$, namely

$$0 = L_{\tilde{\xi}} d\omega = d\iota_{\tilde{\xi}} d\rho + \iota_{\tilde{\xi}} dd^c \rho$$

we deduce for all $v \in \mathrm{Lie}(G)$

$$\iota_{\tilde{\xi}} dd^c \rho = -d\iota_{\tilde{\xi}} d\rho = d\mu^v = \iota_{\tilde{\xi}} \omega$$

Thus for any vector field $\xi$

$$\iota_{\xi} dd^c \rho = \iota_{\xi} \omega$$

Similarly we obtain for any vector field $\xi$

$$\iota_{\xi} J\iota_{\tilde{\xi}} dd^c \rho = -\iota_{J\xi} \iota_{\tilde{\xi}} dd^c \rho$$

$$= -\iota_{J\xi} \iota_{\tilde{\xi}} \omega = \iota_{\xi} J\iota_{\tilde{\xi}} \omega$$

So we are left to the case of two vector fields $\zeta, \xi \in \Gamma(TS)$ seen as vector fields on $X$ independent of the $G^*$-direction. On $S \subset X$

$$\iota_{\xi} \iota_{\zeta} \omega = \iota_{\zeta} \iota_{\xi} \omega$$

holds by construction. $G$-invariance of $\rho$ and $\omega$ extends this equation to $G^* \times S$.

Finally, the equation holds on $X$ entirely, if for all $v \in \mathrm{Lie}(G)$

$$L_{J\iota_{\xi}} (\iota_{\zeta} \iota_{\xi} dd^c \rho) = L_{J\iota_{\xi}} (\iota_{\zeta} \iota_{\xi} \omega)$$

holds. It is sufficient to establish equation (9) for commuting vector fields $\zeta$ and $\xi$, so for simplicity we may assume $[\zeta, \xi] = 0$ in the sequel. We will use twice the following formula for general vector fields $X$ and $Y$ and a general 1-form $\eta$

$$\iota_X \iota_Y d\eta = \iota_X d\iota_Y \eta - \iota_Y d\iota_X \eta - \iota_{[X,Y]} \eta$$
as already mentioned above as equation (5) as well as $\omega(JX, JY) = \omega(X, Y)$ for the real $(1, 1)$-forms $dd^c \rho$ and $\omega$ as well as the fact that $\zeta$ and $\xi$ commute with $J$. 

$$\mathcal{L}_{\tilde{v}}(\iota \xi \iota \zeta dd^c \rho) = \iota \xi \iota \zeta (\mathcal{L}_{\tilde{v}}dd^c \rho)$$

$$= \iota \xi \iota \zeta dt_{J\tilde{v}}dd^c \rho$$

$$= \iota \xi dt_{J\xi J\tilde{v}}dd^c \rho - \iota \xi dt_{\iota \zeta J\tilde{v}}dd^c \rho$$

$$= -\iota \xi dt_{J\xi J\tilde{v}}dd^c \rho + \iota \xi dt_{J\xi J\tilde{v}}dd^c \rho$$

$$= -\iota \xi dt_{J\xi J\tilde{v}}dd^c \rho + \iota \xi dt_{J\xi J\tilde{v}}dd^c \rho$$

$$= \iota \xi \iota \xi dt_{J\tilde{v}}\omega$$

$$= \iota \xi \iota \zeta \mathcal{L}_{\tilde{v}}\omega$$

$$= \mathcal{L}_{\tilde{v}}(\iota \xi \iota \zeta \omega)$$

Thus $\rho$ is a $G$-invariant function fulfilling both conditions

$$\omega = dd^c \rho$$

$$\mu^v = -\iota \xi dd^c \rho.$$

Now we consider the general situation. We aim to prove the existence of a $G$-invariant potential on some $G$-stable neighborhood of each point $x \in \mathcal{M}$.

**Proof of Theorem 3** Theorem 34 shows that each point in $\mathcal{M}$ admits a neighborhood isomorphic to a local slice model, therefore we are left to the slice situation. For simplicity we may assume

$$X = G^* \times /K S$$

Hereby $S$ as well as $G^*$ can be shrunk if necessary provided the latter set stays $G$-stable. The point $x$ corresponds to $[e, s] \in G^* \times /K S$. Define the projection

$$\pi : G^* \times S \to G^* \times /K S$$

and lift the objects to the product $G^* \times S$, namely $\tilde{\omega} = \pi^* \omega$ and $\tilde{\mu} = \pi^* \mu$ to some $G$-stable neighborhood of $(e, s)$. Lemma [43] provides a $G$-invariant plurisubharmonic potential $\tilde{\rho}$ on some $G$-stable neighborhood of $(e, s)$ which can be chosen $K$-stable.

Our aim is to push down $\tilde{\rho}$ to $X$. Therefore we have to show that $\tilde{\rho}$ is $K$-invariant and locally $K^c$-invariant with respect to the diagonal $K$-action.

In a first step, $\tilde{\rho}$ can be made $K$-invariant by averaging. So for any induced vector field $v \in \text{Lie}(K)$ of the diagonal action, $\mathcal{L}_{\tilde{v}}\tilde{\rho} = 0$ and consequently $\mathcal{L}_{\tilde{v}}dd^c \tilde{\rho} = 0$. We conclude that

$$\iota \xi dd^c \tilde{\rho} \text{ is constant} \quad (10)$$
\[ \text{since} \quad d\hat{v}(\hat{\rho}) = L^\hat{v}d^c\hat{\rho} - \iota_{\hat{v}}dd^c\hat{\rho} = -\iota_{\hat{v}}\hat{\omega} = 0 \]

By assumption \(\hat{\mu}(e, s) = 0\), so we get
\[ \iota_{\hat{v}}d^c\hat{\rho}(e, s) = -\hat{\mu}(e, s) = 0 \]

and with observation \((10)\) as desired
\[ \iota_{\hat{v}}d^c\hat{\rho} = 0 \]

This shows
\[ J\hat{v}(\hat{\rho}) = 0 \quad \text{and} \quad \hat{\nu}(\hat{\rho}) = 0 \]

the latter due to invariance. Thus \(\hat{\rho}\) vanishes on the fibers of \(\pi\) since each fiber is a single complexified \(K\)-orbit according to Proposition 26(2), and therefore pushes down to the quotient \(X = G^* \times /K S\) fulfilling the desired properties.

Theorem 5 states the existence of local potentials away from \(M\), if the orbit \(G \cdot x\) is totally real, the form \(\omega\) pulled back to \(G \cdot x\) is exact and the first de Rham cohomology group \(H^1_{dR}(G^0)\) of the identity connected component of \(G\) is zero.

**Proof of Theorem 5.** The \(G\)-orbit \(G \cdot x\) being totally real, we can apply again the local slice theorem, lift \(\omega\) and \(\mu\) to the product \(G^* \times S\) and construct a plurisubharmonic function \(\hat{\rho}\) by Lemma 43 as above, again \(K\)-invariant with respect to the diagonal action. The same argument provides that

\[ \iota_{\hat{v}}d^c\hat{\rho} \text{ is constant.} \]

The exactness of \(\omega\) on the considered \(G\)-orbit implies exactness on some neighborhood, so after further shrinking of \(X\) we have \(\omega = d\beta\). For \(\hat{\beta} = \pi^* \beta\) and any induced vector field \(v \in \text{Lie}(K)\) of the diagonal action we have
\[ \iota_{\hat{v}}\hat{\beta} = 0 \]

Thus the form \(\eta = d^c\hat{\rho} - \hat{\beta}\) is closed and therefore exact since \(H^1_{dR}(X_0) = H^1_{dR}(G^0) = 0\) for each connected component \(X_0\) of \(G^* \times S\). This shows that \(\eta = df\) for some \(K\)-invariant function \(f\), this implies
\[ \iota_{\hat{v}}d^c\hat{\rho} = \iota_{\hat{v}}d^c\hat{\rho} - \iota_{\hat{v}}\hat{\beta} = \iota_{\hat{v}}\eta = \hat{\nu}(f) = 0 \]

and we obtain
\[ J\hat{v}(\hat{\rho}) = 0 \quad \text{and} \quad \hat{\nu}(\hat{\rho}) = 0 \]

Therefore the potential \(\hat{\rho}\) pushes down to the quotient \(X = G^* \times /K S\) fulfilling the desired properties.
4.4 Modifying potentials

We start by showing that a $G$-exhaustive strictly plurisubharmonic function can be modified to be greater than any given $G$-invariant continuous function.

**Lemma 44.** Let $\nu$ be a $G$-exhaustive strictly plurisubharmonic function on $X$ with connected image $I = \nu(X) \subset \mathbb{R}$. Given a continuous $G$-invariant function $\lambda : X \to \mathbb{R}$, then there is a function $\chi : I \to \mathbb{R}$ such that $\nu_{\chi} = \chi \circ \nu$ is strictly plurisubharmonic, $G$-exhaustive with $\nu_{\chi} > \lambda$ and $\sup_{X} \nu_{\chi} = \infty$.

If furthermore there is a $G$-stable open subset $U \subset X$ with $\lambda(x) \leq \nu(x)$ for all $x \in U$, the function $\nu_{\chi}$ can be chosen such that $\nu_{\chi}(x) = \nu(x)$ for all $x \in U$.

**Proof.** We calculate

$$dd^c \nu_{\chi} = d(\chi' \circ \nu \cdot d^c \nu) = \chi'' \circ \nu \cdot d \nu \wedge d^c \nu + \chi' \circ \nu \cdot dd^c \nu$$  \hspace{1cm} (11)

Since $dd^c \nu$ is a Kähler form and $d \nu \wedge d^c \nu$ is a non-negative $(1,1)$-form $\nu_{\chi}$ is strictly plurisubharmonic whenever $\chi'$ and $\chi''$ are positive. We define the function

$$\alpha : I \to \mathbb{R}, \quad t \mapsto \sup\{\lambda(x) \mid \nu(x) = t\}$$  \hspace{1cm} (12)

which is continuous. Demanding $\chi > \alpha$ implies $\nu_{\chi} > \lambda$. But there is in fact a smooth function $\chi$ on $I$ with $\chi > \alpha$, $\chi' > 0$, $\chi'' > 0$ and $\sup \chi = \infty$. Furthermore for all $t \in \nu(U)$ we may choose $\chi(t) = t$, since $\alpha(t) \leq t$.

This next lemma will be used to construct $G$-exhaustive strictly plurisubharmonic functions on a local slice model which coincides on a sufficiently large set with a given strictly plurisubharmonic function.

**Lemma 45.** Let $Y = G^* \times /K S$ be a local slice model. Then

1. there is a $G$-exhaustive strictly plurisubharmonic function $\nu : Y \to \mathbb{R}$ which can be chosen to be greater than a given $G$-invariant continuous function $\lambda$ and

2. given a $G$-invariant strictly plurisubharmonic function $\rho : Y \to \mathbb{R}$, a $G$-stable open subset $U \subset Y$ with $U/G \subset \subset Y/G$ there is a $G$-exhaustive strictly plurisubharmonic function $\nu_{U,\rho} : Y \to \mathbb{R}$ such that $\nu_{U,\rho}|U = \rho|U$.

**Proof.** By construction there is a $G$-exhaustive strictly plurisubharmonic function $\nu_G$ on $G^*$ and an exhaustive strictly plurisubharmonic function $\nu_S$ on $S$. Lemma 44 shows that we may assume after modification the supremum of both functions to be infinity. We may assume $\nu_S$ to be $K$-invariant and $\nu_G$ invariant with respect to the $K$-action from the right, both can be obtained by averaging. On the product, we may define

$$\nu_P : G^* \times S \to \mathbb{R}, \quad (\gamma, s) \mapsto \nu_G(\gamma) + \nu_S(s)$$  \hspace{1cm} (13)
which is $K$-invariant with respect to the diagonal $K$-action. It induces a Kähler form $\omega_P$ and a momentum map $\mu_P$ with respect to the $K$-action. As the $K$-action is free the zero level $\mathcal{M}_P$ is a (real) submanifold which is both $K$- and $G$-stable as $\nu_P$ is $(G \times K)$-invariant. By Corollary 38 the restriction $\nu_P|_{\mathcal{M}_P}$ pushes down to $\mathcal{M}_P/K$ and by the identification $\mathcal{M}_P/K \cong G^* \times^{/K} S$ to a strictly plurisubharmonic $G$-exhaustion $\nu : G^* \times^{/K} S \to \mathbb{R}$. This proves the first assertion in part (1) while the second assertion follows from Lemma 44. Given a $G$-invariant strictly plurisubharmonic function $\rho$, a $G$-stable open subset $U \subset Y$ with $U/G \subset \subset Y/G$ and a $G$-exhaustive strictly plurisubharmonic function $\nu$ established from part (1). Choose a smooth $G$-invariant function $\chi : Y \to [0, 1]$ such that with $V = \{ y \in Y \mid \chi(y) \neq 0 \}$ the set $V/G$ is relatively compact in $Y/G$ and $\chi|_U = 1$. Since $\rho$ is strictly plurisubharmonic, there is a $t_0 > 0$ such that $\rho - t_0 \cdot \chi \cdot \nu$ is strictly plurisubharmonic. Lemma 44 provides a $G$-exhaustive strictly plurisubharmonic function $\nu_0$ such that $\nu_0|_U = \nu|_U$ and $\nu_0 > \nu + \frac{\rho_{t_0}}{t_0}$ on $Y \setminus \bar{V}$. The function $\nu_{U, \rho} = \rho - t_0 \cdot \chi \cdot \nu + t_0 \cdot \nu_0$ equals $\rho$ on $U$ and is greater than the $G$-exhaustion $\nu$ outside $\bar{V}$ and therefore a $G$-exhaustion. As a sum of two strictly plurisubharmonic functions it is strictly plurisubharmonic. This proves part (2).

5 Complex structure on symplectic reduction

The purpose of this section is to give the proofs for the main Theorem 2 and therewith Theorem 1. Recall that Theorem 2 states that for every point $x \in \mathcal{M}$ there is a $G$-stable neighborhood $Y \subset X$ such that $Y//G$ is a (reduced) normal complex space and a $G$-stable open neighborhood $U \subset Y$ with $U/G \subset \subset Y/G$ and $\pi(U) = U//G$ defining $\mathcal{M}_U = \mathcal{M} \cap U$ the diagram

$$
\begin{array}{ccc}
\mathcal{M}_U & \stackrel{i_U}{\hookrightarrow} & U \subset Y \\
\downarrow p & & \downarrow \pi \\
\mathcal{M}_U/G & \stackrel{\varphi_U}{\rightarrow} & U//G \subset Y//G
\end{array}
$$

commutes such that $\varphi_U : \mathcal{M}_U/G \to U//G$ is a homeomorphism and biholomorphic in the sense that it induces an isomorphism $(\varphi_U)^*$ between the sheaves $\mathcal{O}_{Y//G}$ and $\mathcal{O}_{\mathcal{M}_U/G}$ whenever the map is defined.

In the following Section 5.1 we establish first that the induced map $\varphi_U$ in diagram (14) is a homeomorphism. In section 5.2 we will show the main result that $\varphi_U$ defines a biholomorphism, i.e. local correspondences providing the complex structure.

5.1 Local homeomorphism to local analytic Hilbert quotient

A key ingredient for the proof of the topological part of Theorem 2 is the existence of a $G$-invariant strictly plurisubharmonic potential on a local slice model.
and its behaviour on closed and on non-closed complexified $G$-orbits. The existence of such a function has already been established, so in the sequel we will analyse its behaviour.

**Theorem 46.** Let $Y = G^* \times// K S$ be a local slice model and $\rho$ a $G$-exhaustive strictly plurisubharmonic function inducing a Kähler form $\omega$, a momentum map $\mu$ and the zero level $M = \mu^{-1}(0)$. Then for each fiber $F$ of $\pi : Y \to Y//G$ the function $\rho$ becomes critical where it attains a local minimum. This is exactly on a single $G$-orbit contained in the unique closed complexified $G$-orbit in $F$ and this $G$-orbit is the set $F \cap M$. The map $\varphi : M//G \to Y//G$ induced from the inclusion $M \hookrightarrow Y$ is a homeomorphism.

The first step will be to analyse the following special case of a local slice model. So in the sequel, the special local slice model $Z = G^* \times// L \Delta$ is considered where $L \subset G$ is isomorphic to $S^1$ and $\Delta$ is a bounded connected open subset of the origin in $\mathbb{C}$ on which $L$ acts non-trivially as the restriction of a complex linear representation. Let $z_0 = [e, 0] \in Z$ then $B(z_0) = G^* \times// L \{0\}$ is a closed complexified $G$-orbit in $Z$ and let $z_1 \notin B(z_0)$ then $B(z_1) = G^* \times// L (\Delta \setminus \{0\})$ is the complement and henceforth an open complexified $G$-orbit. Thus this model is special as on the one hand side $Z$ is a manifold and on the other hand side $Z$ splits into just two complexified $G$-orbits, the closed submanifold $B(z_0)$ and its open complement $B(z_1)$. We will show in the next two lemmata that a $G$-exhaustive strictly plurisubharmonic function has a unique critical $G$-orbit in $B(z_0)$ and no critical point in the open set $B(z_1)$.

**Lemma 47.** Let $\rho$ be a $G$-exhaustive strictly plurisubharmonic function on the local slice model $Z = G^* \times// L \Delta$ introduced above. Then there is a unique critical orbit $G \cdot p \subset B(z_0)$ for $\rho|_{B(z_0)}$ and for each local submanifold $\Sigma_z$ through $p$ transversal to the $G$-orbit $G \cdot p$ the Hessian of $\rho|_{\Sigma_z}$ is strictly positive and the orbit is a local isolated minimum locus of $\rho$ in $Z$.

**Proof.** The subset $B(z_0)$ is a closed submanifold which in turn is a complexified $G$-orbit as it is $G$-equivariantly biholomorphic to $G^* // L$. The function $\rho$ is assumed to be a $G$-exhaustion, so is the function $\rho|_{B(z_0)}$, and hence there is a point $p \in B(z_0)$ with respect to which $\rho|_{B(z_0)}$ is critical. By Lemma 46 the $G$-orbit $G \cdot p$ is a local isolated minimum locus of $\rho|_{B(z_0)}$. Corollary 42 shows that this orbit $G \cdot p$ is the only critical orbit of $\rho|_{B(z_0)}$ providing the uniqueness assertion claimed in the lemma. This local minimum locus is of maximal dimension, i.e.

$$\dim_{\mathbb{R}} G \cdot p = \dim_{\mathbb{C}} G^* // L = \dim_{\mathbb{C}} G^* - 1 = \dim_{\mathbb{R}} G - 1$$

So this action has a 1-dimensional isotropy group at $p$. Denote $H$ its identity component. The group $H$ is compact and therefore isomorphic to $S^1$.

We claim that $p$ is a critical point of $\rho : Z \to \mathbb{R}$. In order to see this we first note that $G \cdot p$ is $H$-stable. Thus the subspace $T = T_p(G \cdot p)$ is $H$-stable as well as $V = JT$. By construction $T \oplus V = T_p B(z_0)$. Further we may choose an $H$-stable complement $W \cong \mathbb{C}$ to $T \oplus V$ in $T_p Z$. Keep in mind that the $H$-representation on $W$ is non-trivial. On the one hand side $p$ is selected to be a critical point.
of $\rho_{|B(\infty)}$, so $d\rho$ vanishes on $T \oplus V$. On the other hand $d\rho$ is $H$-invariant on $W \cong \mathbb{C}$ hence vanishes on $W$ as well. This shows the claim.

In order to simplify the argumentation we will linearize the setting by pulling it up to a neighborhood of the origin in $T_pZ$ in the next step. There is an $H$-equivariant biholomorphism from an open neighborhood of the origin in $T_pZ$ to an open neighborhood of $p$ in $Z$ mapping the origin to $p$ such that the derivative $T_0T_pZ \rightarrow T_pZ$ becomes the identity after the natural identification $T_0T_pZ \cong T_pZ$. In the sequel we will argue in $T_pZ$ and denote $\hat{\rho}$ the pullback of $\rho$ to an open neighborhood of the origin there.

Our purpose is now to show that $\hat{\rho}$ restricted to $V \oplus W$ has strictly positive Hessian and the $G$-orbit is therefore a local isolated minimum locus of $\rho$. For this, we will analyse the second order terms of $\hat{\rho}$ at 0, namely we set

$$\Theta(v_1,v_2) = v_1(v_2(\hat{\rho}))_0 \quad \text{for} \ v_1,v_2 \in T_0T_pZ \cong T_pZ$$

and have to show that the symmetric bilinear form $\Theta$ has $T$ as its kernel and that the restriction $\Theta|_{V \oplus W}$ is positive.

We denote by $\text{Sym}^2(E)$ the space of (real) symmetric bilinear forms on a vector space $E$ and by $(E_1^* \otimes E_2^*)$ the space generated by elements of the form $e_1^* \otimes e_2^* = \frac{1}{2}(e_1^* \otimes e_2^* + e_2^* \otimes e_1^*) \in \text{Sym}^2(E_1 \oplus E_2)$ where $e_j \in E_j^*$. Note that the decomposition $E = \bigoplus_{j=1}^N E_j$ induces a decomposition

$$\text{Sym}^2(E) = \bigoplus_{j=1}^N \text{Sym}^2(E_j) \oplus \bigoplus_{j<k}^N (E_j^* \otimes E_k^*)$$

(15)

If additionally $E$ is an $H$-representation, we denote the set of invariant elements in the respective sets by the exponent $H$, i.e. $\text{Sym}^2(E)^H$. If $E = \bigoplus E_j$ is an $H$-invariant decomposition, then the space of invariant symmetric bilinear forms is the sum of invariant elements with respect to the decomposition (15).

We note that $\Theta$ is $H$-invariant. The aim in the sequel will be to find a decomposition $V = V_W \oplus V_N$ into $H$-stable subspaces and consequently $T_W = J \cdot V_W, T_N = J \cdot V_N$ such that with respect to the decomposition

$$T_pZ = T_W \oplus T_N \oplus V_W \oplus V_N \oplus W$$

the form $\Theta$ decomposes as $\Theta = \Theta_W + \Theta_N$ with

$$\Theta_W \in \text{Sym}^2(T_W \oplus V_W \oplus W)^H \quad \text{and} \quad \Theta_N \in \text{Sym}^2(T_N \oplus V_N)^H$$

and further all the components $\text{Sym}^2(T)^H, (T^* \otimes V^*)^H, (T^* \otimes W^*)^H$ of $\Theta$ and hence $\Theta_W$ vanish while for each of the remaining spaces

$$\text{Sym}^2(V_W)^H, \text{Sym}^2(W)^H \quad \text{and} \quad (V_W^* \otimes V^*)^H, (V_W^* \otimes W^*)^H$$

of the $H$-invariant elements we will be able to find a particular form of the corresponding components of $\Theta_W$ such that finally we are reduced to an ansatz

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for ΘW with 3 parameters for which we establish the positivity of the Hessian. Let us start to see that Θ does vanish on V, i.e. that all components of Θ in Sym²(T)H, (T*⊗V*)H and (T*⊗W*)H vanish. For a given vector v2 ∈ T there is a fundamental vector field ̂ν such that ̂νv = v2 ∈ T. But as local G-invariance reads ̂ν(ρ) = 0 the linear map Θ(·, v2) vanishes.

Now, define the following linear map Θv reads ̂ν ∈ M the squared modulus on W ∼= C which has positive Hessian. We introduce a complex linear coordinate x1, x2 on T W inducing complex linear coordinates zj = xj + i · yj on T W and ̂ν ∈ V the subspace V W ∼= V ⊕ W ∈ V. Restricted to V the bilinear form Θ|V is strictly positive (Lemma 40). Let ΘW = Θv be the orthogonal complement of VN with respect to Θ|V so that V = V W ⊕ VN. Thus there are ΘW ∈ Sym²(VW ∨ W)H and ΘN ∈ Sym²(VN)H such that Θ = ΘW + ΘN.

In order to analyse ΘW observe that there is an element ε ∈ H whose induced endomorphism on W coincides with multiplication by i. We choose a non-vanishing linear map w* ∈ W*. Since w*, εw* form a (real) basis of W*, there are functionals v*, ̂ν* ∈ V* such that the component ΘM of ΘW in (VW ∨ W) equals ΘM = v* ̂νw* + ̂ν*εw*. We think of this expression as the “mixed terms” of Θ. The element ε was chosen such that εεw* = −w* and the equation εεM = ΘM holds due to H-invariance of ΘM. We calculate

\[ \begin{aligned}
εεM &= ε̂νw* + ε̂νw* \\
&= \ThetaM = v* ̂νw* + ̂ν*εw*.
\end{aligned} \]

Again using the invariance εεM = ΘM assuming linear dependence over R and ΘM ≠ 0, i.e. εv* = λ · v*, leads to the contradiction λ² = −1, thus V W is 2-dimensional. In summary, we developed a splitting of T ∨ V ∨ W into subspaces such that the only term of Θ involving both V and W is a multiple of ΘM, a bilinear form on V W ∨ W.

We now go back to the function ̂ρ having induced the bilinear form Θ. The second order terms of ̂ρ with respect to the linear structure on TpZ around the origin define a function θ : TpZ → R which is H-invariant strictly plurisubharmonic and induces the same bilinear form. With T W = JW W the subspace T W ∨ V W ∨ W is a 3-dimensional complex vector space, T W ∨ V W ∼= C² and W ∨ C, and we define the H-invariant strictly plurisubharmonic function θW = θ|TW ∨ VW ∨ W.

We introduce a complex linear coordinate w = s + i · t and (real) linear coordinates x1, x2 on T W inducing complex linear coordinates zj = xj + i · yj on T W.
$T_W \oplus V_W$. With this the function $\theta_W$ splits into a sum of three functions, the first depending only on the variables $x_1, x_2, y_1, y_2$, the third depending only on $s, t$ and the second $\theta_M$ depending on the remaining “mixed terms”. In view of equation (16) the coordinates can be chosen such that there is an $\alpha_2 \in \mathbb{R}$ such that

$$\theta_M = 4\alpha_2 \cdot (y_1s + y_2t) .$$

Further we established that $\theta_W$ does not depend on $x_1$ and $x_2$. So $H$-invariance forces this term to be a multiple of $y_1^2 + y_2^2$. Finally the last term depending only on $s$ and $t$ has to be a multiple of $s^2 + t^2$ again due to $H$-invariance. In summary, there are $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\theta_W = 2\alpha_1(y_1^2 + y_2^2) + 4\alpha_2(y_1s + y_2t) + \alpha_3(s^2 + t^2)$$

As $\theta_W$ is strictly plurisubharmonic on $T_W \oplus V_W \oplus W$ so are the restriction to both $T_W \oplus V_W$ and $W$, so $\alpha_1$ and $\alpha_3$ are positive. We calculate the Levi matrix with respect to the coordinates $z_1, z_2, w$ as

$$L \theta_W = \begin{pmatrix}
\alpha_1 & 0 & -i \cdot \alpha_2 \\
0 & \alpha_1 & \alpha_2 \\
i \cdot \alpha_2 & \alpha_2 & \alpha_3
\end{pmatrix}$$

with determinant $\alpha_1(\alpha_1\alpha_3 - 2\alpha_2^2)$ which has to be positive by assumption that $\rho$ and hence $\theta_W$ are strictly plurisubharmonic. In summary, we know

$$\alpha_1 > 0, \quad \alpha_3 > 0 \quad \text{and} \quad \alpha_1\alpha_3 - 2\alpha_2^2 > 0 \quad (17)$$

Now, we calculate the Hessian restricted to the coordinates $y_1, s, y_2, t$

$$Hess \theta_W = \begin{pmatrix}
4\alpha_1 & 4\alpha_2 & 0 & 0 \\
4\alpha_2 & 2\alpha_3 & 0 & 0 \\
0 & 0 & 4\alpha_1 & 4\alpha_2 \\
0 & 0 & 4\alpha_2 & 2\alpha_3
\end{pmatrix}$$

This matrix is strictly positive if the positivity conditions (17) derived from the Levi matrix are fulfilled.

The Hessian of the other part $\theta_N = \theta|_{V_N}$ is strictly positive by Lemma 40 since the $G$-orbit is a local isolated minimum locus in $B(z_0)$. Finally, the constructed decomposition ensures $\theta = \theta_W + \theta_N$, so restricted to a local submanifold transversal to the $G$-orbit the function $\theta$ has in fact a strictly positive Hessian. Note that this holds for the easy case $V = V_N$ as well since in this case $\Theta_W = \alpha_3(s^2 + t^2)$ with $\alpha_3 > 0$. Therefore the $G$-orbit through $p$ is a local isolated minimum locus of $\rho$.

Now, we can use the particular structure of this special local slice model $Z = G^* \times \mathbb{C} - \Delta$, namely that it is a manifold and consists of just two complexified $G$-orbits, a closed one $B(z_0)$ and an open one $B(z_1)$. 32
Lemma 48. Let \( G \cdot p_0 \) be the critical orbit in the closed complexified \( G \)-orbit \( B(z_0) \) in the special local slice model \( Z = G^* \times //L \Delta \) established in the previous Lemma. This orbit is the only critical and minimal locus and hence a global minimum locus.

Proof. Let us assume there is a point \( p_1 \notin G \cdot p_0 \) at which \( \rho \) becomes critical. As \( G \cdot p_0 \) is the only critical locus of \( \rho|_{B(z_0)} \) according to Corollary \( \| \) we know \( p_1 \in B(z_1) \). Lemma \( \| \) shows that there is a further critical point \( q \) which is not a local minimum. Again \( q \notin B(z_0) \) by Corollary \( \| \). So the point \( q \) lies in the open complexified \( G \)-orbit \( B(z_1) \). Lemma \( \| \) shows that restricted to a complexified \( G \)-orbit \( \rho \) becomes critical only if it becomes locally minimal. This contradicts the existence of \( q \). \( \square \)

After analysing the behaviour of the function \( \rho \) on a special local slice model the following lemma provides a first decisive link of this special local slice model to the general local slice model.

Lemma 49. Let \( \rho : G^* \times ///K S \to \mathbb{R} \) be a \( G \)-exhaustive strictly plurisubharmonic smooth function and \( Z = G^* \times ///L \Delta \) as in Lemma \( \| \) with \( B(z_1) \subset Z \) the unique open complexified \( G \)-orbit. If \( \varphi : G^* \times ///L \Delta \to G^* \times ///K S \) is a \( G \)-equivariant holomorphic map and if \( q \in \varphi(Z) \) is a local minimum for \( \rho|_{B(q)} \), then \( q \notin \varphi(B(z_1)) \).

The proof is postponed. It consists in pulling back \( \rho \) by \( \varphi \) and modifying this pulled back function with another \( G \)-exhaustive strictly plurisubharmonic function \( \eta \) on \( Z \). Assuming that \( \rho \) admits a local minimum on a complexified \( G \)-orbit through a point in \( \varphi(B(z_1)) \) the modified function on \( Z \) will have a local isolated minimum on \( B(z_1) \). For this we need the following two lemmata.

Recall that a proper \( G \)-action on a manifold \( Z \) admits to a given point \( z \in Z \) a (local) slice \( \Sigma_z \) which can be characterized as follows. First the slice is a local submanifold of \( Z \) containing \( z \) and which is \( G_z \)-stable where \( G_z \) denotes the isotropy group of the \( G \)-action at \( z \). Second the tangent space \( T_z Z \) splits into \( T_z(G \cdot z) \) and \( T_z \Sigma_z \). When chosen \( \Sigma_z \) sufficiently small, this induces a \( G \)-equivariant open embedding

\[
G \times G_z \Sigma_z \to Z
\]

\[
[g, \sigma] \mapsto g \cdot \sigma
\]

Here \( G \times G_z \Sigma_z \) is the quotient of \( G \times \Sigma_z \) with respect to the following (diagonal) \( G_z \)-action.

\[
\alpha : G_z \times G \times \Sigma_z \to G \times \Sigma_z
\]

\[
(h, (g, \sigma)) \mapsto (g \cdot h^{-1}, h \cdot \sigma)
\]

Finally, we recall that \( \Sigma_z \) can be chosen \( G_z \)-equivariantly diffeomorphic to a \( G_z \)-stable open neighborhood of the origin of \( T_z \Sigma_z \) with \( z \) mapped to the origin.

Lemma 50. Let \( M \) and \( N \) be manifolds, \( G \) act properly on \( N \) and let \( \psi : M \to N \) be a \( G \)-equivariant submersion with \( \psi(x) = y \). Then there are respective slices \( \Sigma_x \) and \( \Sigma_y \) such that the restriction of \( \psi \) becomes a \( G_z \)-equivariant submersion \( \theta : \Sigma_x \to \Sigma_y \).
Proof. Observe that $G$ acts properly on $M$ as well. We have to show that there are linear subspaces $V \subset T_xM$ and $W \subset T_yN$ such that $V$ is complementray to $T_x(G \cdot x)$ and $W$ to $T_y(G \cdot y)$ and the restriction of $\psi_* : T_xM \to T_yN$ to $V$ becomes a $G_x$-equivariant submersion $\theta : V \to W$. In order to construct this, we observe that $\psi(G \cdot x) = G \cdot y$. Hence the $G_x$-equivariant map $\psi_*$ restricts to a $G_x$-equivariant submersion $\psi_*|_{T_x(G \cdot x)} : T_x(G \cdot x) \to T_y(G \cdot y)$. Denote by $E$ the kernel of $\psi_*|_{T_x(G \cdot x)}$ which is $G_x$-stable. Now choose a $G_y$-stable linear subspace $W \subset T_yN$ complementary to $T_y(G \cdot y)$. The $G_x$-stable preimage $(\psi_*)^{-1}(W)$ contains $E$ as a $G_x$-stable linear subspace. Denote by $V$ a $G_x$-stable complement to $E$ in $(\psi_*)^{-1}(W)$. The induced restricted map $\theta : V \to W$ is a $G_x$-equivariant submersion. Further $V$ is complementary to $T_x(G \cdot x)$ what finishes the proof.

The next lemma is needed for the proof of Lemma 49 deals with modifying a function “in fiber direction” when only having an isolated minimum transversal to the fibers.

**Lemma 51.** Let the Lie group $G$ act properly on the manifolds $M$ and $N$ and $\varphi : M \to N$ be a smooth, $G$-equivariant submersion with $\varphi(p) = q$. Furthermore let $\eta : M \to \mathbb{R}$ and $\rho : N \to \mathbb{R}$ be $G$-invariant smooth functions such that $G \cdot q$ is a local isolated minimum for $\rho$ and $G \cdot p$ is a local isolated minimum for $\eta$. Then for each $G$-stable neighborhood $U$ of $p$ there is a $G$-stable open neighborhood $V \subset U$ and a $\delta > 0$ such that for every $t \in (0, \delta)$ the function $\rho_t = \varphi^* \rho + t \cdot \eta$ attains a local isolated minimum on a $G$-orbit in $V$.

**Proof.** By the previous Lemma 49 there are slices $\Sigma_p$ and $\Sigma_q$ for the $G$-action at $p$ and $q$ respectively and a $G_p$-equivariant submersion $\theta : \Sigma_p \to \Sigma_q$. In this setting $\rho_0 = \rho|_{\Sigma_p}$ has a local isolated minimum at $q$. Let $\eta_0 = \eta|_{\Sigma_p}$. The restricted function $\eta_0|_{\varphi^{-1}(q)}$ has a local isolated minimum at $p$. For simplicity we assume the local minima to be global after further shrinking of $\Sigma_p$ and $\Sigma_q$ and $\rho_0(q) = \eta_0(p) = 0$. We choose an open neighborhood $\Omega \subset \Sigma_p$ which is relatively compact in $\Sigma_p$ and $\partial \Omega$ its boundary in $\Sigma_p$. The function $\theta^* \rho_0$ is positive on $\partial \Omega \setminus \varphi^{-1}(q)$ while $\eta_0$ is positive on $\partial \Omega \cap \varphi^{-1}(q)$. Due to continuity of $\theta^* \rho_0$ and $\eta_0$ the set $\partial \Omega$ can be covered by open sets $W_\alpha$ such that on each set $W_\alpha$ at least one of the functions $\theta^* \rho_0$ and $\eta_0$ is positive. Hence for each $W_\alpha$ there is a $\delta_\alpha > 0$ such that for each $t \in (0, \delta_\alpha)$ the function $\rho_t = \rho_0 \circ \theta + t \cdot \eta_0$ is positive. As $\partial \Omega$ is compact, we may pass to a finite subcover $W_{\alpha_i}$ of $\partial \Omega$ and set $\delta = \min_i \delta_{\alpha_i} > 0$. Then $\rho_t$ is positive for all $t \in (0, \delta)$ on $\partial \Omega$. But as $\rho_t(p) = 0$ and $\min_{\partial \Omega} \rho_t > 0$ the minimum is attained in $\Omega$. So for each $G$-stable open neighborhood of $G \cdot p$ there is a $\delta > 0$ such that $\rho_t = \varphi^* \rho + t \cdot \eta$ attains a local minimum on some $G$-orbit in this neighborhood.

**Proof of Lemma 49.** Recall that first there is a special local slice model $Z = G^* \times \mathbb{H} \Delta$ given (compare Lemma 47). There are points $z_0, z_1 \in Z$ such that $Z$ decomposes into a closed complexified $G$-orbit $B(z_0) = G^* \times \mathbb{H} \{0\}$ and a
complementary open complexified \(G\)-orbit \(B(z_1)\). Second there is a local slice model \(Y = G^* \times^{/K} S\) and a \(G\)-equivariant holomorphic map
\[
\varphi : G^* \times^{/L} \Delta \to G^* \times^{/K} S.
\]
Note that \(\varphi|_{B(z_1)}\) is automatically submersive to its image due to \(G\)-equivariance. Thus \(\varphi(B(z_1))\) is a local submanifold of \(Y\). By assumption there is a \(G\)-exhaustive strictly plurisubharmonic function \(\rho\) on \(Y\) attaining a local minimum at \(q \in Y\). We aim to show that the point \(q\) is not contained in \(\varphi(B(z_1))\) and therefore assume the contrary \(q \notin \varphi(B(z_1))\). Lemma [48] provides the existence of a \(G\)-exhaustive strictly plurisubharmonic function on \(Z\) which we denote \(\eta\). The \(G\)-stable set \(W = \varphi^{-1}(G \cdot q) \subset Y\) is closed, such that \(\eta\) attains a minimum at some point \(p\). Without restriction we may choose \(p \in \varphi^{-1}(q)\). Further without restriction we may add to \(\eta\) a \(G\)-invariant smooth function \(\chi\) having \(G \cdot p\) as local isolated minimum locus such that the modified \(G\)-invariant function stays strictly plurisubharmonic and \(G\)-exhaustive. For simplicity we denote this modified function again by \(\eta\). Lemma [51] shows that for a \(G\)-stable neighborhood of \(p\) in \(B(z_1)\) there is a \(\delta > 0\) such that for each \(t \in (0, \delta)\) the function \(\rho_t = \varphi^*\rho + t \cdot \eta\) has a \(G\)-orbit as a local minimum locus in that neighborhood. Additionally, this function is strictly plurisubharmonic as a sum of a plurisubharmonic and a strictly plurisubharmonic function and as \(\eta\) is \(G\)-exhaustive, so is \(\rho_t\). Choose \(t_0 \in (0, \delta)\) and let \(G \cdot p_0\) be this local minimum locus in \(B(z_1)\) for the function \(\rho_{t_0}\). As above we may modify \(\rho_{t_0}\) slightly such that the modified function is strictly plurisubharmonic and \(G\)-exhaustive and additionally has the orbit \(G \cdot p_0\) as a local isolated minimum locus. Since \(p_0 \in B(z_1)\), this contradicts Lemma [48] and shows that \(q \notin \varphi(B(z_1))\).

**Lemma 52.** Let \(Y = G^* \times^{/K} S\) be a local slice model and \(\rho\) a \(G\)-exhaustive strictly plurisubharmonic function with corresponding momentum map \(\mu\). Let \(\pi : Y \to Y//G\) be the analytic Hilbert quotient. Then for all \(c \in \mathbb{R}\) we have for \(U_c = \{y \in Y | \rho < c\}\)

1. \(\pi(U_c)\) is open
2. \(\pi(U_c) = \pi(U_c \cap \mathcal{M})\) where \(\mathcal{M} = \mu^{-1}(0)\) and
3. \(\pi^{-1}(\pi(U_c)) = \{y \in Y | B(y)\text{ intersects } U_c\}\) = \(\{y \in Y | B(y)\text{ intersects } U_c \cap \mathcal{M}\}\)

holds.

**Proof.** For a given set \(U_c\) let \(U_c^B\) be the set of all points whose complexified \(G\)-orbit intersects \(U_c\). First we will show that \(U_c^B\) is open. In fact, \(U_c^B\) is the smallest set containing \(U_c\) such that for all open subsets \(W \subset U_c^B\) such that given \(\xi \in \text{Lie}(G)\) such that \(\varphi^\xi(W) \subset Y\) implies \(\varphi^\xi(W) \subset U_c^B\). Hence \(U_c^B\) is open.

Now we will prove that \(U_c^B\) is \(\pi\)-saturated. Proposition [31] states that a \(\pi\)-fiber \(F\) is the disjoint union of a unique closed complexified \(G\)-orbit \(E\) and non-closed complexified \(G\)-orbits \(B_i\) and \(E \subset \partial B_i\) for all \(i\). As \(\rho\) restricted to \(F\) is
a $G$-exhaustion, the minimum is attained but not attained on each non-closed complexified $G$-orbit $B_i$. From this we conclude

$$\inf_B \rho = \min_E \rho$$  \hspace{1cm} (18)

Hence the set $B$ is contained in $U^B$ if and only if $E \subset U^B$, thus either the fiber $F$ is entirely contained in $U^B$ or the sets are disjoint. Hence $U^B$ is $\pi$-saturated. Therefore $\pi(U_c)$ is open, since $\pi^{-1}(\pi(U_c)) = U^B$ is open. With the above notation for $B_i$ and $E$, we obtain $\pi(U_c \cap B) = \pi(U_c \cap E) = \pi(U_c \cap \mathcal{M})$. □

**Proof of Theorem [46]**. In the first part of the proof we show that a $G$-exhaustive strictly plurisubharmonic function $\rho$ cannot become critical, and therefore also not minimal, on a non-closed complexified $G$-orbit in a local slice model $Y = G^* \times//K S$. Assume the contrary, i.e. let $B \subset Y$ be a non-closed complexified $G$-orbit in $Y$ with critical point $p \in B$ of $\rho$. Introduce the projection $\pi_K : G^* \times S \to Y$ and set $p = \pi_K(\gamma, s)$. Proposition [28] shows that there is a non-closed complexified $K$-orbit $B_S \subset S$ such that $\pi_K^{-1}(B) = G^* \times B_S$. Recall $S$ can be realized as an open Runge subset of a $K$-representation $V$ constructed as $S = \{ v \in V \mid \nu_V(v) < c \}$ for some $K$-invariant strictly plurisubharmonic exhaustion $\nu_V$ on $V$. Since $B_S$ is not closed the Hilbert Lemma (cf. for example [Kra84] III.2.4]) shows that there is an element $k \in K$ and subgroup $L \subset K$ isomorphic to $S^1$ with an induced group homomorphism $\lambda_C : \mathbb{C}^* \cong L^C \to K^C$ such that the following holds. The $\mathbb{C}^*$-orbit through $s_0 = k \cdot s$ closes up in a single fixed point of the $\mathbb{C}^*$-action and this closure is a 1-dimensional complex subspace of $V$, in particular $\lim_{t \to -\infty} \lambda_C(e^t)s_0$ exists. Define $\Omega = S \cap \mathbb{C}^* \cdot s_0$. As the map $t \mapsto \nu_V(\lambda_C(e^t)s_0)$ is strictly convex, $\Omega$ is a connected 1-dimensional (reduced) complex space, admitting an $L$-action with a single closed complexified $L$-orbit, namely a fixed point of $L$, and an open complexified $L$-orbit. We may normalize $\Omega$ to $\Delta$ which is $L$-equivariant and so $\Delta$ is $L$-equivariant biholomorphic to the unit disk in $C$ with $L$ acting non trivially and linearly on $\mathbb{C}$. The $L$-equivariant holomorphic map $\Delta \to S$ induces a $G$-equivariant holomorphic map $\varphi : G^* \times//L \Delta \to G^* \times//K S$ with the local minimum point $p$ of $\rho$ in the image, since $\pi_K(\gamma, s) = \pi_K(\gamma \cdot h^{-1}, h \cdot s)$. Let $z_0, z_1$ be two points in the special local slice model $Z = G^* \times//L \Delta$. We have $Z = B(z_0) \cup B(z_1)$ where $B(z_0) = G^* \times//L \{0\}$ is closed and $B(z_1)$ its open complement. Restricted to the open complexified $G$-orbit $B(z_1)$ the map $\varphi$ becomes a $G$-equivariant holomorphic submersion onto its image in the complexified $G$-orbit $G^* \times//K B_S$ in $G^* \times//K S$, since for each $y$ in the image of $\varphi$ the tangent space $T_y(G \cdot y)$ spans the complex tangent space to the complexified $G$-orbit. Lemma [49] gives a contradiction to $p$ being in the image of $\varphi(B(z_1))$. Thus we showed that $\rho$ does not become critical on a non-closed complexified $G$-orbit.

On the other hand, the existence of two distinct $G$-orbits in a closed $G$-orbit $B \subset Y$ on which $\rho$ becomes critical is excluded by Corollary [12]. In summary for every fiber $F$ of $\pi : Y \to Y//G$ the function $\rho|_F$ has to become minimal and critical on the unique closed complexified $G$-orbit $B$. On this complexified $G$-orbit $\rho|_B$ becomes critical exactly in a single $G$-orbit. Exactly on this
set \( \rho|_F \) becomes minimal. On the non-closed complexified \( G \)-orbits in \( F \) in turn the function \( \rho \) does not become critical at all. Thus the map \( \varphi : \mathcal{M}/G \to Y//G \) induced from the inclusion \( \mathcal{M} \hookrightarrow Y \) is a bijection.

We are left to show that \( \varphi \) is a homeomorphism. Observe that \( \varphi \) is continuous by construction. Given a convergent sequence \( z_n \in Y//=G \) Lemma 52 shows that there is a \( c < \sup_{\mathcal{Y}} \rho \) defining the \( G \)-stable open set \( U_c = \{ y \in Y \mid \rho(y) < c \} \) and the \( G \)-stable set \( \mathcal{M}_c = U_c \cap \mathcal{M} \) and there are points \( y_n \in \mathcal{M}_c \) such that \( \pi(y_n) = z_n \). As \( \rho \) is a \( G \)-exhaustion, \( U_c/G \) is relatively compact in \( Y//G \). Hence \( p(\mathcal{M}_c) \) is relatively compact in \( \mathcal{M}/G \). Therefore by possibly passing to a subsequence \( p(y_n) \) converges in \( \mathcal{M}/G \). By construction \( p(y_n) = \varphi^{-1}(z_n) \), hence \( \varphi \) is open and therefore a homeomorphism.

\[ \square \]

5.2 Local biholomorphism to local analytic Hilbert quotient

As mentioned in the introduction we endow the symplectic reduction \( \mathcal{M}/G \) with a structure sheaf \( \mathcal{O}_{\mathcal{M}/G} \) in the following way. Denote by \( p : \mathcal{M} \to \mathcal{M}/G \) the quotient map. Then for an open subset \( Q \subset \mathcal{M}/G \) the space of sections \( \mathcal{O}_{\mathcal{M}/G}(Q) \) consists of those maps \( f : Q \to \mathbb{C} \) such that \( p^*f : p^{-1}(Q) \to \mathbb{C} \) extends to a holomorphic function on some neighborhood of \( p^{-1}(Q) \subset Y \). So \( (\mathcal{M}/G, \mathcal{O}_{\mathcal{M}/G}) \) is a ringed space.

\[ \textbf{Lemma 53.} \text{ Let } Y = G^* \times^//K S \text{ be a local slice model, } \rho \text{ a } G \text{-exhaustive strictly plurisubharmonic function inducing a Kähler form } \omega, \text{ a momentum map } \mu \text{ and the zero level } \mathcal{M} = \mu^{-1}(0). \text{ Let } V \subset Y \text{ be an open subset such that } \mathcal{M}_V = \mathcal{M} \cap V \text{ is } G \text{-stable and let } f : V \to \mathbb{C} \text{ be a holomorphic function such that } f|_{\mathcal{M}_V} \text{ is } G \text{-invariant. Then there is a unique holomorphic function } h \text{ on } U = \pi^{-1}(\pi(\mathcal{M}_V)) \text{ which is constant on the } \pi \text{-fibers such that } f|_{\mathcal{M}_V} = h|_{\mathcal{M}_V} \text{ holds.} \]

\[ \textbf{Proof.} \text{ For } x \in \mathcal{M}_V \text{ choose a local slice model } Y_0 = G^* \times^//K_0 S_0 \subset V \text{ around } x. \text{ Let } A \text{ be the smallest closed analytic subset in } Y_0 \text{ containing } \mathcal{M}_0 = \mathcal{M} \cap Y_0 \subset V. \text{ This set } A \text{ is } G \text{-stable and in particular contains all closed complexified } G \text{-orbits in } Y_0 \text{ which intersect } \mathcal{M}_0. \text{ The restriction } f|_A \text{ is } G \text{-invariant and therefore the restriction } f|_{A_S} \text{ to the } K_0 \text{-stable intersection } A_S = A \cap S_0 \text{ is } K_0 \text{-invariant. Since } S_0 \text{ is a Stein manifold, } f|_{A_S} \text{ extends to } S_0 \text{ holomorphically. This extension can be made } K_0 \text{-invariant by averaging. We may consider this function as a holomorphic function on the quotient } S_0//K_0. \text{ Via the identification } Y_0//G \cong S_0//K_0 \text{ the function can be pulled back to a } G \text{-invariant holomorphic function } h_0 \text{ on } Y_0. \text{ On } A \text{ and hence on } \mathcal{M}_0 \text{ the functions } f \text{ and } h_0 \text{ coincide. This process can be made for every point in } \mathcal{M}_V. \text{ Since the extensions are unique, these extensions glue together to a } G \text{-invariant holomorphic function on a } G \text{-stable neighborhood } V_0 \subset U \text{ of } \mathcal{M}_V, \text{ in particular this function is constant on the } \pi|_{V_0} \text{-fibers. This function extends uniquely to a function } h : U \to \mathbb{C} \text{ which is constant on the } \pi \text{-fibers and coincides with } f \text{ on } \mathcal{M}_V. \text{ We claim that this function is holomorphic. In order to see this let } W \subset V \text{ consists of those points } w \text{ which are mapped by } \pi \text{ to a non-singular point such that the derivative } \pi_*
at \( w \) has full rank. This set \( W \) is the union of complexified \( G \)-orbits and \( V \setminus W \) is a proper analytic subset. Restricted to \( W \) the map \( \pi \) is submersive to its image. The set of points at which \( h \) is not holomorphic is a union of connected components of \( \pi \)-fibers. But for any point \( w \in W \) each connected component of the complexified orbit \( B(w) \) intersects \( V_0 \). Thus \( h|_W \) is holomorphic and continuous on \( V \) hence holomorphic on \( V \).

**Lemma 54.** Let \( Y = G^* \setminus K S \) be a local slice model, \( \rho \) a \( G \)-exhaustive strictly plurisubharmonic function inducing a Kähler form \( \omega \), a momentum map \( \mu \) and the zero level \( \mathcal{M} = \mu^{-1}(0) \). Then the homeomorphism \( \varphi : \mathcal{M}/G \rightarrow Y//G \) established in Theorem 46 induces an isomorphism of the sheaves \( \mathcal{O}_{\mathcal{M}/G} \) and \( \mathcal{O}_{Y//G} \).

**Proof.** Restriction induces a sheaf morphism \( \mathcal{O}_{Y//G} \rightarrow \varphi_* \mathcal{O}_{\mathcal{M}/G} \) in a trivial way. In order to show that this is an isomorphism we will define an inverse. Recall the notation of the quotient maps \( p : \mathcal{M} \rightarrow \mathcal{M}/G \) and \( \pi : Y \rightarrow Y//G \). For an open set \( R \subset Y//G \) we set \( Q = \varphi^{-1}(R) \). An element in \( \varphi_* \mathcal{O}_{\mathcal{M}/G}(R) \) is given by an open subset \( V \subset \mathcal{M} \) such that \( \mathcal{M}_V = \mathcal{M} \cap V \) is \( G \)-stable and \( \rho(\mathcal{M}_V) = Q \) together with a holomorphic function \( f : V \rightarrow \mathbb{C} \) which is \( G \)-invariant on \( \mathcal{M}_V \). Recall that \( R = \pi(\mathcal{M}_V) \) and denote \( U = \pi^{-1}(R) \). On this saturated open set there is a unique holomorphic function \( h : U \rightarrow \mathbb{C} \) constant on the \( \pi \)-fibers which coincides with \( f \) on \( \mathcal{M}_V \) (Lemma 53). This holomorphic function can be seen as an element of \( \mathcal{O}_{Y//G}(R) \). By this extension process the sheaf morphism \( \varphi_* \mathcal{O}_{\mathcal{M}/G} \rightarrow \mathcal{O}_{Y//G} \) is defined which is inverse to the sheaf morphism arising from restriction. Thus with these structure sheaves the homeomorphism \( \varphi \) becomes a biholomorphism.

Now we will summarize the partial results to prove the central Theorem 2. It states that each point \( x \in \mathcal{M} \) admits a \( G \)-stable neighborhood \( U \) such that with \( \mathcal{M}_U = \mathcal{M} \cap U \) there is an induced homeomorphism \( \varphi : \mathcal{M}_U/G \rightarrow U//G \) which induces an isomorphism between the sheaves \( \mathcal{O}_{\mathcal{M}_U/G} \) and \( \mathcal{O}_U//G \).

**Proof of Theorem 2.** For a given a point \( x \in \mathcal{M} \) Theorem 34 provides a neighborhood \( Y \) isomorphic to a local slice model with quotient map \( \pi : Y \rightarrow Y//G \). Theorem 3 implies the existence of a local potential \( \rho \) on \( Y \). For a given point there is a \( \pi \)-saturated open neighborhood \( U \subset Y \) of \( x \) such that \( \pi(U) \subset Y//G \) is relatively compact and Runge. Then there is a \( G \)-stable open neighborhood \( V \) of \( \mathcal{M} \cap U \) such that \( V/G \) is relatively compact in \( Y/G \).Lemma 54 shows that there is a strictly plurisubharmonic \( G \)-exhaustion \( \tilde{\rho} \) with \( \tilde{\rho}|_V = \rho|_V \) inducing a momentum zero level \( \tilde{\mathcal{M}} \) with \( \tilde{\mathcal{M}} \cap U = \mathcal{M} \cap U \). The induced map \( \tilde{\varphi} : \tilde{\mathcal{M}}/G \rightarrow Y//G \) is shown to be a homeomorphism (Theorem 34) inducing an isomorphism of sheaves (Lemma 54). Its restriction

\[
\varphi : \mathcal{M}_U/G \rightarrow \varphi(\mathcal{M}_U/G) = \pi(U) \subset Y//G
\] (19)

is a homeomorphism inducing an isomorphism of sheaves making \( \mathcal{M}_U/G \) a reduced normal complex space in a natural way. According to Lemma 54 a \( G \)-invariant holomorphic function on \( U \) is uniquely given by its restriction to \( \mathcal{M}_U \).
and in particular constant on the $\pi$-fibers. Therefore $\pi(U) = U//G$, hence

$$\varphi : \mathcal{M}_U/G \to U//G$$

is a biholomorphism.

Theorem 1 states that $(\mathcal{M}/G, \mathcal{O}_{\mathcal{M}/G})$ is a reduced normal complex space and now a consequence of Theorem 2.

Proof of Theorem 1. Theorem 2 shows that $(\mathcal{M}/G, \mathcal{O}_{\mathcal{M}/G})$ is locally biholomorphic to a reduced normal complex space. Since $\mathcal{M}/G$ is a Hausdorff space due to the properness of the action, the entire space arises from gluing reduced normal complex spaces.

6 Kähler structure on symplectic reduction

In this section we will show that the symplectic reduction $\mathcal{M}/G$ inherits a Kähler structure.

The space $X$ and hence the momentum zero level $\mathcal{M}$ can be given a stratification into smooth parts if we compound all points of $\mathcal{M}$ having the same isotropy group $H \subset G$ up to conjugation in a stratum $S_H \subset \mathcal{M}$. The sets $S_H/G$ induce a stratification of the symplectic reduction $\mathcal{M}/G$ into smooth parts. The restrictions $p|_{S_H} : S_H \to S_H/G$ are fiber bundles. We call this the stratification by $G$-orbit type.

The situation of the Kähler form on each stratum $S_H/G$ is rather simple. To explain this, choose a stratum $S = S_H$ and for some point $x \in S \subset \mathcal{M}$ restrict the bilinear form $\omega_x$ to $T_x S$. The vector space $T_x S$ splits into $T_x (G \cdot x)$ and $T_x S \cap J T_x S$, where $J$ denotes the complex structure. The form is non-degenerate on the latter (complex) linear space while $T_x (G \cdot x)$ is the kernel of the restricted form. Thus given any local section $\sigma$ to $p_S : S \to S/G$ the pullback $\sigma^* \omega$ is closed and non-degenerate, hence symplectic. Furthermore $\sigma^* \omega = \dd c \sigma^* \rho$, since $\sigma^* \dd c \rho = \dd c \sigma^* \rho$, thus $\sigma^* \omega$ is a Kähler form on $S/G$. In order to define a Kähler structure on the usually singular complex space $\mathcal{M}/G$ across the strata we have the following definition.

Definition 55. A Kähler structure on a complex space is given by an open covering $\{U_\alpha\}$ and strictly plurisubharmonic functions $\rho_\alpha$ on $U_\alpha$ such that $\rho_\alpha - \rho_\beta$ is plurisubharmonic on $U_\alpha \cap U_\beta$, i.e. locally the real part of some holomorphic function. A plurisubharmonic function $\rho$ is said to be strictly plurisubharmonic if the plurisubharmonicity is stable under perturbation, i.e. if for each smooth function $h$ and each relatively compact open set $U$ there is a $\delta > 0$ such that $\rho + t \cdot h$ is plurisubharmonic for all $t$ with $|t| < \delta$ on $U$.

Remark. The different notions of plurisubharmonicity across singularities are known to coincide ([FNS80]).
It cannot be hoped that the induced functions $\rho_\alpha$ on $\mathcal{M}/G$ are smooth even if $\omega$ is smooth on $X$ and $G$ is connected. Consider the simple example ([HHL94]) of the $S^1$-representation on $\mathbb{C}^2$, $(t,z_1,z_2) \mapsto (t z_1, t^{-1} z_2)$ with (analytic) Hilbert quotient $\pi : \mathbb{C}^2 \to \mathbb{C}^2//S^1$. The quotient can be realized by the biholomorphism $\psi : \mathbb{C}^2//S^1 \to \mathbb{C}, [z_1, z_2] \mapsto z_1 + z_2$. The function $\rho(z_1, z_2) = |z_1|^2 + |z_2|^2$ defines an $S^1$-invariant strictly plurisubharmonic function on $\mathbb{C}^2$ inducing $\omega$, $\mu$ and $\mathcal{M} = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| = |z_2|\}$. Hence there is a function $\theta : \mathbb{C} \to \mathbb{R}$ such that $\theta \circ \psi \circ \pi_{|\mathcal{M}} = \rho_{|\mathcal{M}}$. But $\theta(w) = 2|w|$ is not differentiable. Corollary 4 states the existence of a Kähler form in the sense of the above Definition 55.

Proof of Corollary 4. For every point in the momentum zero level $\mathcal{M}$ there is a local slice model with a $G$-invariant potential (Theorem 54). Let $\mathcal{M} = \bigcup \alpha Y_\alpha$ be a covering of $\mathcal{M}$ by local slice models $Y_\alpha$ with $G$-invariant potentials $\rho_\alpha$. This provides a cover of $\mathcal{M}/G$ by open sets $Z_\alpha = p(Y_\alpha \cap \mathcal{M})$. Restricted to $Y_\alpha \cap \mathcal{M}$ the function $\rho_\alpha$ is continuous and $G$-invariant, thus pushes down to continuous functions $\rho_\alpha^Z : Z_\alpha \to \mathbb{R}$, i.e. $\rho_\alpha^Z \circ p = \rho_\alpha$ on $Y_\alpha \cap \mathcal{M}$. We claim that these functions $\rho_\alpha^Z$ form the Kähler structure.

First we show that these functions glue together in the expected way. For a point $x \in \mathcal{M} \cap Y_\alpha \cap Y_\beta$ choose a $G$-stable neighborhood $\Omega \subset Y_\alpha \cap Y_\beta$ such that $\pi(\Omega) = \pi(\mathcal{M} \cap \Omega)$ and every complexified $G$-orbit closes up in $\mathcal{M}$. This is possible just by choosing a local slice model $\Omega_0 \subset Y_\alpha \cap Y_\beta$ around $x$ and setting $\Omega = \pi^{-1}(\pi(\mathcal{M} \cap \Omega_0))$. The difference $\delta = \rho_\alpha - \rho_\beta$ is plurisubharmonic and $G$-invariant thus $Jv(\delta)$ is constant on each $\pi$-fiber for all $v \in \text{Lie}(G)$. By construction $Jv(\delta)|_{\mathcal{M} \cap \Omega} = 0$ and each fiber intersects $\mathcal{M} \cap \Omega$, thus the derivatives $Jv(\delta)$ vanishes and hence $\delta$ is constant on each $\pi$-fiber. Locally $\delta$ is the real part of some holomorphic function $f$ which has to be constant on the $\pi$-fibers as well. This function is the pullback of a holomorphic function $g$ on an open subset around $\pi(x)$. Here $\rho_\alpha^Z - \rho_\beta^Z$ is the real part of $g$. Thus the functions $\rho_\alpha^Z$ glue together in the desired way.

It remains to show that $\rho_\alpha^Z$ defines a strictly plurisubharmonic function on $Z_\alpha$.

In the sequel we will drop the index. The stratification by $G$-orbit type induces a stratification on $\mathcal{M}/G$. As explained above in this section on each stratum the functions $\rho^Z$ are known to be strictly plurisubharmonic. We claim that their extensions across the strata are strictly plurisubharmonic. We may check plurisubharmonicity by pulling back the functions via a holomorphic map from the unit disk ([FNS90]). There, they extend to subharmonic functions. This result is shown in a much more general setting in [GR56].

Finally in order to see that $\rho$ is strictly plurisubharmonic choose for some point $y \in \mathcal{M} \cap Y$ a relatively compact open subset $U_Y \subset Y$. Its image $U_Z = \pi(U_Y)$ is open and relatively compact. Let $h$ be a smooth function on $Z$. There is some $\delta > 0$ such that $\rho_t = \rho + t \cdot \pi^* h$ is strictly plurisubharmonic on $U_Y$ for all $|t| < \delta$ since $\rho$ is strictly plurisubharmonic. Moreover, as these functions $\rho_t$ are $G$-invariant and strictly plurisubharmonic, they define momentum maps $\mu_t$ on $Y$ and $\mu_t^{-1}(0) = \mathcal{M} \cap \langle Y \rangle$ holds. Therefore the pushed down functions equal $\rho^Z + t \cdot h$. By the same arguments as above, they are plurisubharmonic. Hence
7 Examples without invariant potential

Finally we give examples where there is no G-invariant potential. We found cases for a solvable group and semisimple groups.

7.1 Examples with large isotropy group

Recall that given a (real) manifold $M$ with a proper action of $G$, a closed $G$-invariant 2-form $\tau$ with momentum map $\nu$, this setting can be complexified to a “Stein extension” in the following way. There is a Stein $G$-manifold $X$, a $G$-invariant Kähler form $\omega$ and a momentum map $\mu$ on $X$ and a totally real embedding $i : M \to X$ with $\dim \mathbb{R} M = \dim \mathbb{C} X$ such that $\tau = i^* \omega$ and $\nu = i^* \mu$ ([Str01]). The idea is that given a $G$-invariant potential $\rho$ the restriction of the 1-form $d\rho$ to $M$ is a $G$-invariant 1-form $\beta$ with $d\beta = \tau$. But if the isotropy group is “too large”, there is no non-vanishing invariant 1-form.

**Proposition 56.** Let $K$ be a compact subgroup of $G$, $\tau$ a non-vanishing $G$-invariant 2-form with momentum map $\nu$ on $M = G/K$. The group $K$ acts on the cotangent space $T^*_e M$ where $e$ is the neutral element of $G$ and $[e] \in G/K$ its image under the projection. If the only $K$-fixed point in $T^*_e M$ is 0, then $M$ admits no $G$-invariant 1-form $\beta$ with $d\beta = \tau$ and therefore the complexification of $M$ admits no $G$-invariant potential.

**Proof.** Given a $G$-invariant 1-form $\beta$ then $\beta_e = 0$ in $T^*_e M$ and by $G$-invariance $\beta = 0$. Since $\tau$ is not vanishing entirely, $d\beta \neq \tau$. \(\square\)

From this observation we obtain explicit examples.

7.2 Case of semisimple groups

Let $G = SU(n,1)$ act by holomorphic transformations transitively on the ball $B = \{ x \in \mathbb{C}^n \mid \|x\| < 1 \}$ with isotropy $K = SU(n)$. The Kähler form $\tau$ corresponding to the Bergman metric is kept invariant by $G$ and is exact. By construction of the metric one can see that $\tau$ admits a momentum map $\nu$. The only fixed point of the $K$-action on $T_0 B$ is 0. Now consider $B$ as a real manifold and let $X$ be a Stein extension. All conditions of Proposition 56 are satisfied, so there is no $G$-invariant potential on the complexified spaces.

7.3 Case of solvable groups

Let $G$ be the group of isometries of $\mathbb{R}^2$ with euclidean metric, $G = S^1 \rtimes \mathbb{R}^2$, and extend the action to $\mathbb{C}^2$ such that $G$ acts by holomorphic transformations. Within the standard coordinates on $\mathbb{C}^2$, $z_i = x_i + iy_i$, the Kähler form

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$$
is $G$-invariant. We fix a basis for the Lie algebra $\text{Lie}(G)$

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial \varphi} = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2}.$$  

The Lie algebra structure with respect to this basis is given by

$$\begin{bmatrix}
\frac{\partial}{\partial \varphi} \\
\frac{\partial}{\partial x_1} \\
\frac{\partial}{\partial x_2}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial \varphi} \\
\frac{\partial}{\partial x_1} \\
\frac{\partial}{\partial x_2}
\end{bmatrix}
= \frac{\partial}{\partial x_2},$$

$$\begin{bmatrix}
\frac{\partial}{\partial \varphi} \\
\frac{\partial}{\partial x_1} \\
\frac{\partial}{\partial x_2}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial \varphi} \\
\frac{\partial}{\partial x_1} \\
\frac{\partial}{\partial x_2}
\end{bmatrix}
= -\frac{\partial}{\partial x_1},$$

$$\begin{bmatrix}
\frac{\partial}{\partial x_1} \\
\frac{\partial}{\partial x_2}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial \varphi} \\
\frac{\partial}{\partial x_1} \\
\frac{\partial}{\partial x_2}
\end{bmatrix}
= 0$$

We have

$$\begin{array}{ll}
\frac{\iota_{\frac{\partial}{\partial \varphi}}}{\omega} &= dy_i \\
\frac{\iota_{\frac{\partial}{\partial x_1}}}{\omega} &= x_2dy_1 - y_2dx_1 - x_1dy_2 + y_1dx_2 \\
&= d(-x_1y_2 + x_2y_1)
\end{array}$$

If $\mu$ is a momentum map, then

$$\begin{array}{ll}
\mu_{\frac{\partial}{\partial \varphi}} &= y_i + c_i \\
\mu_{\frac{\partial}{\partial x_1}} &= x_2y_1 - x_1y_2 + c
\end{array}$$

for some constants $c_1, c_2, c \in \mathbb{R}$. Equivariance of $\mu$ gives $\iota_v d\mu^w = \mu_{[v,w]}$ for all $v, w \in \text{Lie}(G)$. This implies

$$\begin{array}{ll}
\frac{\iota_{\frac{\partial}{\partial \varphi}}}{d\mu_{\frac{\partial}{\partial \varphi}}} &= y_2 = y_2 + c_2
\end{array}$$

and therefore $c_2 = 0$ and analogously $c_1 = 0$. This shows that for each $c \in \mathbb{R}$ there is a momentum map $\mu_c$. The function

$$\rho_0 = \frac{1}{2}(y_1^2 + y_2^2)$$

is a $G$-invariant potential for $\mu_0$. Suppose there is a potential $\rho_c$ for $\mu_c$. Then $\delta = \rho_c - \rho_0$ is a $G$-invariant pluriharmonic function. Since $\delta$ is $\mathbb{R}^2$-invariant and pluriharmonic, there are constants $a, a_1, a_2$ such that $\delta = a_1y_1 + a_2y_2 + a$. The $S^1$-invariance applied to $\delta$ shows that $d\delta(0) = 0$, thus $\delta$ is constant.

This shows that for $\mu_c$, $c \neq 0$, there is no invariant potential.

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DEPARTMENT OF MATHEMATICS  
RUHR-UNIVERSITÄT BOCHUM  
44780 BOCHUM  
GERMANY

P. Heinzner  
*E-mail address* peter.heinzner@ruhr-universitaet-bochum.de

B. Stratmann  
*E-mail address* bernd.x.stratmann@ruhr-universitaet-bochum.de