Limits to the scope of applicability of extended formulations for LP models of combinatorial optimization problems

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Abstract: We show that new definitions of the notion of “projection” on which some of the recent “extended formulations” works (such as Kaibel (2011); Fiorini et al. (2011; 2012); Kaibel and Walter (2013); Kaibel and Weltge (2013) for example) have been based can cause those works to over-reach in their conclusions in relating polytopes to one another when the sets of the descriptive variables for those polytopes are disjoint.

Keywords: Linear Programming; Combinatorial Optimization; Computational Complexity, Extended Formulations.

1 Background definitions

Definition 1 (“Standard EF Definition” (Yannakakis (1991); Conforti et al. (2010; 2013)) )
An “extended formulation” for a polytope $X \subseteq \mathbb{R}^p$ is a polyhedron $U = \{(x, w) \in \mathbb{R}^{p+q} : Gx + Hw \leq g\}$ the projection, $\varphi_x(U) := \{x \in \mathbb{R}^p : \exists w \in \mathbb{R}^q : (x, w) \in U\}$, of which onto $x$-space is equal to $X$ (where $G \in \mathbb{R}^{m \times p}, H \in \mathbb{R}^{m \times q}$, and $g \in \mathbb{R}^m$).

Definition 2 (“Alternate EF Definition #1” (Kaibel (2011); Fiorini et al. (2011; 2012)) )
A polyhedron $U = \{(x, w) \in \mathbb{R}^{p+q} : Gx + Hw \leq g\}$ is an “extended formulation” of a polytope $X \subseteq \mathbb{R}^p$ if there exists a linear map $\pi : \mathbb{R}^{p+q} \rightarrow \mathbb{R}^p$ such that $X$ is the image of $U$ under $\pi$ (i.e., $X = \pi(U)$; where $G \in \mathbb{R}^{m \times p}, H \in \mathbb{R}^{m \times q}$, and $g \in \mathbb{R}^m$). Kaibel (2011), Kaibel and Walter (2013), and Kaibel and Weltge (2013) refer to $\pi$ as a “projection.”

Definition 3 (“Alternate EF Definition #2” (Fiorini et al. (2012)) ) An “extended formulation” of a polytope $X \subseteq \mathbb{R}^p$ is a linear system $U = \{(x, w) \in \mathbb{R}^{p+q} : Gx + Hw \leq g\}$ such that $x \in X$ if and only if there exists $w \in \mathbb{R}^q$ such that $(x, w) \in U$. (In other words, $U$ is an EF of $X$ if $(x \in X \iff (\exists w \in \mathbb{R}^q : (x, w) \in U))$ (where $G \in \mathbb{R}^{m \times p}, H \in \mathbb{R}^{m \times q}$, and $g \in \mathbb{R}^m$).

Remark 4 The purpose of this note is to point out that the scope of applicability of EF work based on the above definitions is limited to cases where $U$ cannot be equivalent reformulated in terms of the $w$ variables only. For simplicity of exposition, without loss of generality, we will say that $G = 0$ in the above definitions iff there exists a description of $U$ which is in terms of the $w$-variables only and has the same or smaller complexity order of size. Or, equivalently, without loss of generality, we will say that $G \neq 0$ in the above definitions iff the $x$- and $w$-variables are
required in every valid inequality description of $U$ which has the same or smaller complexity order of size as the description at hand.

In particular, if every constraint of $U$ which involves the $x$-variables is redundant in the description of $U$, then clearly, every one of those constraints as well as the $x$-variables themselves can be dropped (without loss) from the description of $U$, with the result that $U$ would be stated in terms of the $w$-variables only. Also, in some cases (all of) the constraints involving the $x$-variables may become redundant only after other constraints in the description of $U$ are re-written and/or new constraints are added (as exemplified by the case of the minimum spanning tree problem (MSTP) in section 4 of this note). If either of these two cases is applicable, we will say that $G = 0$ in the above definitions. Otherwise, we will say that $G \neq 0$.

2 Summary of the general results

All the definitions above are equivalent when $G \neq 0$. However, this is not true when $G = 0$. Our main claim is below.

Claim 5 EF developments for relating the inequality descriptions of $U$ and $X$ in the above definitions are not applicable to cases where $G = 0$ in the above definitions.

We highlight the reasons for the claim in the following remarks. We only consider the case where $U \neq \emptyset$ and $X \neq \emptyset$, since the claim is trivial when $U = \emptyset$ or $X = \emptyset$.

Remark 6 When $G = 0$, $U$ cannot be an EF of $X$ according to Definition 1

The reason is that we would have: $\varphi_x(U) = \{x \in \mathbb{R}^p : (\exists w \in \mathbb{R}^q : (x, w) \in U)\} = \{x \in \mathbb{R}^p : (\exists w \in \mathbb{R}^q : Hw \leq g)\} = \mathbb{R}^p \neq X$ (since $X$ is a polytope and thus bounded, whereas $\mathbb{R}^p$ is unbounded).

Remark 7 When $G = 0$, $U$ cannot be an EF of $X$ according to Definition 3

The reason is that we would have that: $(\exists w \in \mathbb{R}^q : 0x + Hw \leq g) \iff (\exists w \in \mathbb{R}^q : Hw \leq g) \iff (\forall x \in \mathbb{R}^p, (x, w) \in U)$, so that $(\exists w \in \mathbb{R}^q : 0x + Hw \leq g)$ could not (necessarily) imply $x \in X$, so that the "if and only if" stipulation of the definition would not hold.

Now, focusing on Definition 2 we make the following observations.

Remark 8 The EF notion under Definition 2 becomes degenerate/loses meaningfulness when $G = 0$.

The reason is that a polytope can also be stated in terms of its extreme points and that a linear map (as stipulated in the definition) could be inferred from this statement without any reference to an inequality description of the polytope. This is illustrated in the following example.

Note that if $G = 0$ in Definition 2 then the linear inequality description of $U$ involves the $w$-variables only. Hence, for the sake of simplicity (but without loss of generality), let $\overline{U} \subset \mathbb{R}^5$ be described in the $w$-variables only as $\overline{U} = \{w_1 + w_2 = 5; w_1 - w_2 = 1; w_3 + w_4 + w_5 = 0; \text{ non-negativities}\}$. Then, the vertex-description of $\overline{U}$ is $\{w : w \in \text{Conv} \left\{ (3,2,0,0,0,0)^T \right\}\}$ (where $(\cdot)^T$ and $\text{Conv} (\cdot)$ denote the transpose of $(\cdot)$ and the convex hull of $(\cdot)$, respectively). Now, let $X \subset \mathbb{R}^3$ be specified by its vertex-description as $X = \{x : x \in \text{Conv} \left\{ (2,1,5)^T \right\}\}$. (In other words, $X$ consists of the point in $\mathbb{R}^3$, $(2,1,5)^T$.)
Then, the following would be true:

\[(x \in X) \text{ and } (w \in \overline{U}) \implies x = Aw,\]

where, among other possibilities, \(A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & -1 & 5 & 6 & 7 \\ 1 & 1 & 8 & 9 & 10 \end{bmatrix}\).

Since \(x = Aw\) in the above is a linear map between \(\overline{U}\) and \(X\), under Definition 2, \(\overline{U}\) would be an EF of every one of the infinitely-many possible inequality descriptions of \(X\). However, clearly, this EF relationship can only be a degenerate one, since it cannot be used to make any meaningful inferences when comparing the inequality descriptions of \(\overline{U}\) and \(X\).

An example of an inequality description of \(X\) above is: \(\overline{X} = \{x \in \mathbb{R}^3 : x_1 - x_2 + x_3 = 6; x_1 + x_2 \geq 3; x_1 + x_3 \leq 7; x_2 + x_3 \geq 6; x_1 \leq 2; \text{non-negativities}\}\). (It easy to verify that the feasible set of \(\overline{X}\) is indeed \{(2,1,5)\}.) Let \(U'\) denote \(\overline{U}\) augmented with the constraints of the linear map, \(x - Aw = 0\). We are arguing that no meaningful relationship exists between the inequality description \(\overline{U}\) of \(U\) and the inequality description \(\overline{X}\) of \(X\), even though \(U'\) does project to \(X\) under the standard definition (Definition 1). In other words, \(U\) would project to \(X\) under Definition 2 but this would be in a degenerate/meanless sense only, since it could not be used to make meaningful inferences in relating the inequality description of \(U\) to that of \(X\).

A fundamental notion in extended formulations theory is that the addition of redundant variables and constraints to the inequality description of a polytope does not change the EF relationships for that polytope. We use this fact to generalize the degeneracy/loss of meaningfulness which arises out of Definition 2 when \(G = 0\) to Definitions 1 and 3 as follows.

**Remark 9** Under Definitions 1, 2, and 3 provided redundant constraints and variables can be arbitrarily added to the descriptions of polytopes for the purpose of establishing EF relationships, the descriptions of any two non-empty polytopes expressed in non-overlapping (disjoint) variable spaces can be respectively augmented into being extended formulations of each other.

In other words, let \(x^1 \in \mathbb{R}^{n_1} (n_1 \in \mathbb{N}_+)\) and \(x^2 \in \mathbb{R}^{n_2} (n_2 \in \mathbb{N}_+)\) be vectors of variables with no components in common. Then, provided redundant constraints and variables can be arbitrarily added to the descriptions of polytopes for the purpose of establishing EF relationships, the inequality-description of every non-empty polytope in \(x^1\) can be augmented into an EF of the inequality-description of every other non-empty polytope in \(x^2\), and vice versa.

**Proof.** The proof is essentially by construction.

Let \(P_1\) and \(P_2\) be polytopes specified as:

\[P_1 = \{x^1 \in \mathbb{R}^{n_1} : A_1x^1 \leq a_1\} \neq \emptyset \text{ (where } A_1 \in \mathbb{R}^{p_1 \times n_1}, \text{ and } a_1 \in \mathbb{R}^{p_1}\); \]

\[P_2 = \{x^2 \in \mathbb{R}^{n_2} : A_2x^2 \leq a_2\} \neq \emptyset \text{ (where } A_2 \in \mathbb{R}^{p_2 \times n_2}, \text{ and } a_2 \in \mathbb{R}^{p_2}\).\]

Clearly, \(\forall (x^1, x^2) \in P_1 \times P_2; \forall q \in \mathbb{N}_+, \forall B_1 \in \mathbb{R}^{q \times n_1}, \forall B_2 \in \mathbb{R}^{q \times n_2},\) there exists \(u \in \mathbb{R}^q\) such that the constraints

\[B_1x^1 + B_2x^2 - u \leq 0\]  \hspace{1cm} (1)

are valid for \(P_1\) and \(P_2\), respectively (i.e., they are redundant for \(P_1\) and \(P_2\), respectively).
Now, consider:

\[ W := \{(x^1, x^2, u) \in \mathbb{R}^{n1} \times \mathbb{R}^{n2} \times \mathbb{R}_\succ^q : \]

\[ C_1 A_1 x^1 \leq C_1 a_1; \]

\[ B_1 x^2 + B_2 x^1 - u \leq 0; \]

\[ C_2 A_2 x^2 \leq C_2 a_2 \}

(2)

(3)

(4)

(where: \( C_1 \in \mathbb{R}^{p_1 \times p_1} \) and \( C_2 \in \mathbb{R}^{p_2 \times p_2} \) are diagonal matrices with non-zero diagonal entries).

Clearly, \( W \) augments \( P_1 \) and \( P_2 \) respectively. Hence:

\( W \) is equivalent to \( P_1 \), and \( W \) is equivalent to \( P_2 \). \( \) (5)

(6)

Also clearly, we have:

\( \varphi(x^1) = P_1 \) (since \( P_2 \neq \emptyset \), and (2) and (3) are redundant for \( P_1 \)), and \( \varphi(x^2) = P_2 \) (since \( P_1 \neq \emptyset \), and (2) and (3) are redundant for \( P_2 \)). \( \)

(7)

(8)

It follows from the combination of (5) and (6) that \( P_1 \) is an extended formulation of \( P_2 \).

It follows from the combination of (7) and (8) that \( P_2 \) is an extended formulation of \( P_1 \). \( \square \)

**Example 10**

Let

\[ P_1 = \{ x \in \mathbb{R}_\succ^2 : 2x_1 + x_2 \leq 6 \}; \]

\[ P_2 = \{ w \in \mathbb{R}_\succ^3 : 18w_1 - w_2 \leq 23; 59w_1 + w_3 \leq 84 \}. \]

For arbitrary matrices \( B_1, B_2, C_1, \) and \( C_2 \) (of appropriate dimensions, respectively); say \( B_1 = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \), \( B_2 = \begin{bmatrix} 5 & -6 & 7 \\ -10 & 9 & -8 \end{bmatrix} \), \( C_1 = [7] \), and \( C_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \); \( P_1 \) and \( P_2 \) can be augmented into extended formulations of each other using \( u \in \mathbb{R}_\succ^2 \) and \( W \):

\[ W = \left\{ (x, w, u) \in \mathbb{R}_\succ^{2+3+2} : \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 42; \]

\[ \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 & -6 & 7 \\ -10 & 9 & -8 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} - \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \]

\[ \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 18 & -1 & 0 \\ 59 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \leq \begin{bmatrix} 46 \\ 42 \end{bmatrix} \right\}. \]

\( \square \)
3 Application to the Fiorini et al. (2011; 2012) “barriers”

Fiorini et al. (2012) is a re-organized and extended version of Fiorini et al. (2011). The key extension is the addition of another alternate definition of extended formulations (page 96 of Fiorini et al. (2012)) which is recalled in this paper as Definition 3. This new alternate definition is then used to re-arrange “section 5” of Fiorini et al. (2011) into “section 2” and “section 3” of Fiorini et al. (2012). Hence, the developments in “section 5” of Fiorini et al. (2011) which depended on “Theorem 4” of that paper, are “stand-alones” (as “section 3”) in Fiorini et al. (2012), and “Theorem 4” in Fiorini et al. (2011) is relabeled as “Theorem 13” in Fiorini et al. (2012).

Claim 11 The developments in Fiorini et al. (2011) are not valid for relating the inequality descriptions of $U$ and $X$ in Definitions 1-3 when $G = 0$.

Proof. Using the terminology and notation of Fiorini et al. (2011), the main results of section 2 of Fiorini et al. (2011) are developed in terms of $Q := \{(x, y) \in \mathbb{R}^{d+k} \mid Ex + Fy = g, \ y \in C\}$ and $P := \{x \in \mathbb{R}^d \mid Ax \leq b\}$, with $Q$ (in Fiorini et al. (2011)) corresponding to $U$ in Definitions 1-3 and $P$ (in Fiorini et al. (2011)) corresponding to $X$ in Definitions 1-3. Hence, $G = 0$ in Definitions 1-3 corresponds to $E = 0$ in Fiorini et al. (2011). Hence, firstly, assume $E = 0$ in the expression of $Q$ (i.e., $Q := \{(x, y) \in \mathbb{R}^{d+k} \mid 0x + Fy = g, \ y \in C\}$). Then, secondly, consider Theorem 4 of Fiorini et al. (2011) (which is pivotal in that work). We have the following:

(i) If $A \neq 0$ in the expression of $P$, then the proof of the theorem is invalid since that proof requires setting “$E := A$” (see Fiorini et al. (2011, p. 7));

(ii) If $A = 0$, then $P := \{x \in \mathbb{R}^d \mid 0x \leq b\}$. This implies that either $P = \mathbb{R}^d$ (if $b \geq 0$) or $P = \emptyset$ (if $b < 0$). Hence, $P$ would be either unbounded or empty. Hence, there could not exist a non-empty polytope, Conv$(V)$, such that $P = Conv(V)$ (see Fiorini et al. (2011, 16-17), among others). Hence, the conditions in the statement of Theorem 4 of Fiorini et al. (2011) would be ill-defined/impossible.

Hence, the developments in Fiorini et al. (2011) are not valid for relating $U$ and $X$ in Definitions 1-3 when $G = 0$ in those definitions. ■

Claim 12 The developments in Fiorini et al. (2012) are not valid for relating the inequality descriptions of $U$ and $X$ in Definitions 1-3 when $G = 0$.

Proof. First, note that “Theorem 13” of Fiorini et al. (2012, p. 101) is the same as “Theorem 4” of Fiorini et al. (2011). Hence, the proof of Theorem 11 above is applicable to “Theorem 13” of Fiorini et al. (2012). Hence, the parts of the developments in Fiorini et al. (2012) that hinge on this result (namely, from “section 4” onward in Fiorini et al. (2012)) are not valid for relating $U$ and $X$ in Definitions 1-3 when $G = 0$.

Now consider “Theorem 3” of Fiorini et al. (2012) (section 3, page 99). The proof of that theorem hinges on the statement that (using the terminology and notation of Fiorini et al. (2012) which is similar to that in Fiorini et al. (2011)):

$$Ax \leq b \iff \exists y : E^\leq x + F^\leq y \leq g^\leq , \ E^=} x + F^=} y \leq g^= . \quad (9)$$
Note that \( G = 0 \) in Definitions 1-3 would correspond to \( E^\leq = E^\geq = 0 \) in (9). Hence, assume \( E^\leq = E^\geq = 0 \) in (9). Then, clearly, the “if and only if” stipulation of (9) cannot be satisfied in general, since

\[
(\exists y : 0 \cdot x + F^\leq y \leq g^\leq, \ 0 \cdot x + F^\geq y \leq g^\geq) \text{ cannot imply } (Ax \leq b) \text{ in general.}
\]

Hence, Theorem 3 of Fiorini et al. (2012) is not valid for relating \( U \) and \( X \) in Definitions 1-3 when \( G = 0 \).

Hence, the developments in Fiorini et al. (2012) are not valid for relating \( U \) and \( X \) in Definitions 1-3 when \( G = 0 \) in those definitions.

4 The case of the Minimum Spanning Tree Problem

Without the refinement brought by the distinction we make between the cases of \( G = 0 \) and \( G \neq 0 \) in Definitions 1-3 the case of the MSTP would mean that it is possible to extend an exponential-sized model into a polynomial-sized one by (simply) adding redundant variables and constraints to it (i.e., augmenting it), which is a clearly-unreasonable/out-of-the-question proposition. To see this, assume (as is normally done in EF work) that the addition of redundant constraints and variables does not matter as far EF relationships are concerned. Since the constraints of Edmonds’ model (Edmonds (1970)) are redundant for the model of Martin (1991), one could augment Martin’s formulation with these constraints. The resulting model would still be considered a polynomial-sized one. But note that this particular augmentation of Martin’s model would also be an augmentation of Edmonds’ model. Hence, the conclusion would be that Edmonds’ exponential-sized model has been augmented into a polynomial-sized one, which is an impossibility, since one cannot reduce the number of facets of a given polytope by simply adding redundant constraints to the description of that polytope. The distinction we are bringing to attention in this note explains the paradox, as further detailed below.

Example 13 We show that Martin’s polynomial-sized LP model of the MSTP is an EF of Edmonds’ exponential LP model of the MSTP in a degenerate/meaningless sense only, by showing that there exists a reformulation of Martin’s model which does not require the variables of Edmonds’ model (which is essentially the equivalent of having \( G = 0 \) in the description of \( U \) in Definitions 1-3).

- Using the notation in Martin(1991), i.e.:
  - \( N := \{1, \ldots, n\} \) (Set of vertices);
  - \( E : \) Set of edges;
  - \( \forall S \subseteq N, \gamma(S) : \) Set of edges with both ends in \( S \).

- Exponential-sized/“sub-tour elimination” LP formulation (Edmonds (1970))

\((P)\):
Minimize: \[ \sum_{e \in E} c_e x_e \]

Subject To: \[ \sum_{e \in E} x_e = n - 1; \]
\[ \sum_{e \in \gamma(S)} x_e \leq |S| - 1; \quad S \subset E; \]
\[ x_e \geq 0 \quad \text{for all } e \in E. \]

- **Polynomial-sized LP reformulation (Martin (1991))**

(Q):

Minimize: \[ \sum_{e \in E} c_e x_e \]

Subject To: \[ \sum_{e \in E} x_e = n - 1; \]
\[ z_{k,i,j} + z_{k,j,i} = x_e; \quad k = 1, \ldots, n; \quad e \in \gamma\{i, j\}; \]
\[ \sum_{s > i} z_{k,i,s} + \sum_{h < i} z_{k,i,h} \leq 1; \quad k = 1, \ldots, n; \quad i \neq k; \]
\[ \sum_{s > k} z_{k,k,s} + \sum_{h < k} z_{k,k,h} \leq 0; \quad k = 1, \ldots, n; \]
\[ x_e \geq 0 \quad \text{for all } e \in E; \quad z_{k,i,j} \geq 0 \quad \text{for all } k, i, j. \]

- **Re-statement of Martin’s LP model**

For each \( e \in E \):
  - Denote the ends of \( e \) as \( i_e \) and \( j_e \), respectively;
  - Fix an arbitrary node, \( r_e \), which is not incident on \( e \) (i.e., \( r_e \) is such that it is not an end of \( e \)).

Then, one can verify that \( Q \) is equivalent to:

(Q'):
Minimize: \[ \sum_{e \in E} c_e z_{r_e,i_e,j_e} + \sum_{e \in E} c_e z_{r_e,j_e,i_e} \]

Subject To: \[ \sum_{e \in E} z_{r_e,i_e,j_e} + \sum_{e \in E} z_{r_e,j_e,i_e} = n - 1; \]

\[ z_{k,i_e,j_e} + z_{k,j_e,i_e} = z_{r_e,i_e,j_e} + z_{r_e,j_e,i_e}; \quad k = 1, \ldots, n; \quad e \in E; \]

\[ \sum_{s > i} z_{k,i,s} + \sum_{h < i} z_{k,i,h} \leq 1; \quad i, k = 1, \ldots, n : i \neq k; \]

\[ \sum_{s > k} z_{k,k,s} + \sum_{h < k} z_{k,k,h} \leq 0; \quad k = 1, \ldots, n; \]

\[ z_{k,i,j} \geq 0 \quad \text{for all } k, i, j. \]

\[ \square \]

Claim 14 We claim that the reason EF work relating formulation sizes does not apply to the case of the MSTP is that although Martin’s model can be made to project to Edmond’s model, that projection is degenerate/non-meaningful in the sense we have described in this note.
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