An extra structure of spacetime: a space of points, areas and volumes

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Abstract. A theory in which points, lines, areas and volumes are on on the same footing is investigated. All those geometric objects form a 16-dimensional manifold, called $C$-space, which generalizes spacetime. In such higher dimensional space fundamental interactions can be unified à la Kaluza-Klein. The ordinary, 4-dimensional, gravity and gauge fields are incorporated in the metric and spin connection, whilst the conserved gauge charges are related to the isometries of curved $C$-space. It is shown that a conserved generator of an isometry in $C$-space contains a part with derivatives, which generalizes orbital angular momentum, and a part with the generators of Clifford algebra, which generalizes spin.

1. Introduction
In current approaches to quantum gravity the starting point is often in assuming that at short distances there exists an underlying structure, based, e.g., on strings and branes, or spin networks and spin foams (see, e.g., [1]). It is then expected that the smooth spacetime manifold of classical general relativity will emerge as a sufficiently good approximation at large distances. However, it is feasible to assume that what we have, even at large distances, is in fact not just spacetime, but spacetime with certain additional structure. The approach discussed in this contribution suggests that the long distance approximation to a more fundamental structure is the space of extended events, i.e., points, lines, areas, 3-volumes, and 4-volumes [2]–[13]. All those objects can be elegantly represented [14] by Clifford numbers $x^M \gamma_M \equiv x^{\mu_1 \ldots \mu_r} \gamma_{\mu_1 \ldots \mu_r}$, $r = 0, 1, 2, 3, 4$, and therefore the corresponding space is called Clifford space ($C$-space).

It turns out that, since $C$-space is a higher dimensional space, it provides a consistent description of quantized string theory [9]. The underlying spacetime can remain 4-dimensional, there is no need for a 26-dimensional or a 10-dimensional spacetime. The extra degrees of freedom required for consistency of string theory, described in terms of variables $X^M (\tau, \sigma) \equiv X^{\mu_1 \ldots \mu_r} (\tau, \sigma)$, are due to extra dimensions of $C$-space, and they need not be compactified; they are due to the volume (area) evolution, and are thus physical. But since a generic component $X^{\mu_1 \ldots \mu_r}$ denotes an oriented $r$-volume, associated with an $(r - 1)$-brane (i.e., a $p$-brane for $p = r - 1$), we have that string itself (i.e., 1-brane) is not enough for consistency. Higher branes are automatically present in the description with functions $X^{\mu_1 \ldots \mu_r} (\tau, \sigma)$, although they are not described in full detail, but only up to the knowledge of oriented $r$-volume. Because of the presence of two parameters $\tau, \sigma$, we keep on talking about evolving strings, not in 26
or 10-dimensional spacetime, but in 16-dimensional Clifford space. In general, the number of parameters can be arbitrary, but less than 16, and so we have a brane in C-space, i.e., a generalized brane [4]. Since with the points of a flat C-space one can associate Clifford numbers (polyvectors), this automatically brings spinors (as members of left or right ideals of Clifford algebra) into our description. A polyvector $X^{\mu_1 \ldots \mu_r} \gamma_{\mu_1} \ldots \gamma_{\mu_r}$ can be rewritten in terms of a basis spanning four independent left ideals, and thus contains spinorial degrees of freedom. This means that by describing our branes in terms of the C-space embedding functions $X^M \equiv X^{\mu_1 \ldots \mu_r}$ we have already included spinorial degrees of freedom. We do not need to postulate them separately, as in ordinary string and brane theories, where besides Grassmann even (“bosonic”) variables $X^\mu$, there occur also Grassmann odd (“fermionic”) or spinorial variables. In this formulation we have a possible clue to the resolution of a big open problem, namely, what exactly is string theory.

If we release the constraint of flat C-space, then we encounter a fascinating possibility, namely that curved C-space provides a realization of Kaluza-Klein theory [10]–[13], and since all of its 16 dimensions are physically observable to us, there is no need for compactification of the “extra” dimensions. The extra components of the C-space metric tensor are related to the gauge fields that describe the fundamental interactions. When considering the generalized Dirac equation in curved C-space it turns out that the extra components of the C-space spin connection also manifest themselves as gauge fields [12, 13]. The conserved charges consist of two contributions: one due to the “orbital angular momentum”, and another one due to the “spin” in the “internal” space. This is analogous to the situation that we know from the ordinary Dirac theory: instead of spacetime we have C-space, and instead of gamma “matrices” we have now the full basis of Clifford algebra. So instead of the ordinary angular momentum and spin, we have the corresponding momenta in C-space, and in particular their components in the “internal” part of C-space, i.e., the part that goes beyond spacetime.

2. Extending spacetime to Clifford space

In a manifold we do not have points only. We also have higher objects, such as open and closed lines, surfaces, etc. In a flat manifold description of such objects can be done straightforwardly by employing vectors of various grades: ordinary vectors, bivectors, trivectors, etc., in general $r$-vectors or multivectors, which are Clifford numbers. Basis vectors (1-vectors) are generators of Clifford algebra satisfying

$$\gamma_\mu \cdot \gamma_\nu \equiv \frac{1}{2} (\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = g_{\mu\nu}$$  \hspace{1cm} (1)

Higher grade vectors are given by the wedge product of vectors:

$$\gamma_\mu \wedge \gamma_\nu \equiv \frac{1}{2!} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \equiv \frac{1}{2} [\gamma_\mu, \gamma_\nu]$$  \hspace{1cm} (2)

$$\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\alpha \equiv \frac{1}{3!} [\gamma_\mu, \gamma_\nu, \gamma_\alpha]$$  \hspace{1cm} (3)

$$\vdots$$

$$\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \ldots \wedge \gamma_{\mu_n} \equiv \frac{1}{n!} [\gamma_{\mu_1}, \gamma_{\mu_2}, \ldots, \gamma_{\mu_n}]$$  \hspace{1cm} (4)

An oriented line element is

$$dx = dx^\mu \gamma_\mu$$  \hspace{1cm} (5)

where $dx^\mu, \mu = 0, 1, 2, 3, \ldots$, are just the ordinary differentials of coordinates (that should not be confused with the notation in the calculus of differential forms). The $dx$ is a vector, expanded
in terms of basis vectors $\gamma_\mu$, its components being $dx^\mu = dx \cdot \gamma^\mu$. Here $\gamma^\mu$ are reciprocal basis vectors satisfying $\gamma^\mu \cdot \gamma_\mu = \delta^\mu_\nu$.

Oriented area element is the wedge product of two line elements $dx_1$ and $dx_2$:

$$dx_1 \wedge dx_2 = dx_1^\mu dx_2^\nu \gamma_\mu \wedge \gamma_\nu$$

An arbitrary multivector element of degree $r$ is

$$dx_1 \wedge dx_2 \wedge ... \wedge dx_r = dx_1^{\mu_1} ... dx_r^{\mu_r} \gamma_{\mu_1} \wedge ... \wedge \gamma_{\mu_r}$$

At this stage we do not yet specify the dimensionality $n$ of the underlying space(time); it can be arbitrary. The grade of a non vanishing $r$-vector can be at most $r = n$.

If we are in flat spacetime manifold, we can straightforwardly integrate the above infinitesimal $r$-vector line elements, and so we obtain finite $r$-vectors $x^{\mu_1...\mu_r}\gamma_{\mu_1...\mu_r}$ describing oriented $r$-dimensional areas ($r$-areas) associated with closed $(r-1)$-dimensional surfaces [6], or open $r$-dimensional surfaces. We have thus introduced a manifold of $r$-areas, with $x^{\mu_1...\mu_r}$ being their coordinates. The latter manifold is called Clifford space, or C-space.

This was just a manifold of abstract $r$-areas that generalizes the ordinary manifold of abstract points. We can make contact with physics by noting that in a spacetime manifold we have “matter” consisting of all sorts of physical objects that can be point particles or extended objects, such as strings and branes of various dimensionalities. A spacetime manifold can be considered as a space of all possible positions that a test particle can have.

Analogously, an extended object can have many positions in a multidimensional space of all possible object’s configurations, the so called configuration space. If extended objects are strings or branes, then their configuration space is infinite dimensional. In refs. [4] it was called $\mathcal{M}$-space. The latter space is thus a space of all possible configurations that a (generalized) brane can have. $\mathcal{M}$-space can have metric. Different choices of $\mathcal{M}$-space metric lead to different brane theories, with different actions in the underlying spacetime. From the point of view of spacetime we have different brane theories, but from the point of view of $\mathcal{M}$-space there is one theory which allows for different background $\mathcal{M}$-space metrics. The latter metric can become dynamical, if we include into the description a corresponding kinetic term [4]. Thus we arrive at a background independent theory that generalizes general relativity to $\mathcal{M}$-space.

In order to avoid infinite dimensional description of branes, one can introduce a quenched description [15, 16] working in a finite dimensional subspace of $\mathcal{M}$-space. The finite dimensional space is the space of oriented $r$-areas that we associate with closed $(r-1)$-branes, or open $r$-branes. We have thus a many-to-one mapping from the infinite dimensional objects, such as branes, to finite dimensional $r$-areas. In other words, we have a many-to-one mapping from infinite dimensional $\mathcal{M}$-space to finite dimensional C-space [2]–[13].

Clifford space is thus just a particular case of configuration space, associated with an extended object. It takes into account not only an object’s center of mass position, but also its extension, which is sampled by coordinates $x^{\mu_1...\mu_r}$.

In general, a “point” in C-space can be described by coordinates $x^{\mu_1...\mu_r}$. Two infinitesimally separated points, with coordinates $x^{\mu_1...\mu_r}$ and $x^{\mu_1...\mu_r} + dx^{\mu_1...\mu_r}$, define a polyvector

$$dX = \frac{1}{r!} \sum_{r=0}^{n} dx^{\mu_1...\mu_r} \gamma_{\mu_1...\mu_r} \equiv dx^M \gamma_M$$

which is a superposition of polyvectors of different grades, including the scalars.

The line element in C-space is given by the quadratic form [2]–[13]

$$dS^2 = dX^* dX = dx^M dx^N G_{MN} \equiv dx^M dx_M$$
Here ‘∗’ denotes the scalar product of two polyvectors $A$ and $B$:

$$A ∗ B = ⟨AB⟩_0 \quad (10)$$

where $⟨ ⟩_0$ denotes the scalar part of the Clifford product $AB$. Symbol ‘‡’ denotes reversion, i.e., the operation that reverses the order of vectors, e.g.,

$$⟨γ_1γ_2γ_3⟩^‡ = γ_3γ_2γ_1 \quad (11)$$

The metric is given by the scalar product of two basis Clifford numbers:

$$G_{MN} = γ^M_M ∗ γ^N_N \quad (12)$$

Reversion in the above definition is necessary for consistency reasons [9].

If the signature of $n$-dimensional spacetime we start from is pseudoeuclidean of the form $+−−−...$, then the signature of C-space is $(p, p)$, with $p = 2^n/2$. This has important consequences for string theory in particular, and for field theory in general [9, 17].

We can now envisage that physical objects are described in terms of $x^M = (ω, x^μ, x^{μν}, ...)$.

The first straightforward possibility is to introduce a single parameter $τ$ and consider a mapping

$$τ \rightarrow x^M = X^M(τ) \quad (13)$$

where $X^M(τ)$ are 16 embedding functions that describe a worldline in C-space. From the point of view of C-space, $X^M(τ)$ describe a wordline of a “point particle”: at every value of $τ$ we have a point in C-space. But from the perspective of the underlying 4-dimensional spacetime, $X^M(τ)$ describe an extended object, sampled by the center of mass coordinates $X^μ(τ)$ and the coordinates $X^{μ1μ2}(τ), X^{μ1μ2μ3}, X^{μ1μ2μ3μ4}(τ)$. They are a generalization of the center of mass coordinates in the sense that they provide information about the object 2-vector, 3-vector, and 4-vector extension and orientation.

The dynamics of such an object is determined by the action

$$I[X] = M \int dτ (\dot{X}^† ∗ \dot{X})^{1/2} = M \int dτ(\dot{X}^M ∗ \dot{X}_M)^{1/2} \quad (14)$$

The dynamical variables are given by the polyvector

$$X = X^Mγ_M = Ω^† + X^μγ_μ + X^{μ1μ2}γ_{μ1μ2} + ... + X^{μ1...μn}γ_{μ1...μn} \quad (15)$$

whilst

$$\dot{X} = \dot{X}^Mγ_M = 0^† + \dot{X}^μγ_μ + \dot{X}^{μ1μ2}γ_{μ1μ2} + ... + \dot{X}^{μ1...μn}γ_{μ1...μn} \quad (16)$$

is the velocity polyvector, where $\dot{X}^M ≡ dX^M/dτ$.

In the action (14) we have a straightforward generalization of the relativistic point particle in $M_4$:

$$I[X^μ] = m \int dτ(\dot{X}^μ \dot{X}_μ)^{1/2}, \quad μ = 0, 1, 2, 3 \quad (17)$$

If a particle is extended, then (17) provides only a very incomplete description. A more complete description is given by the action (14), in which the C-space embedding functions $X^M(τ)$ sample the objects extension.

1 A systematic and detailed treatment is in refs. [6].
3. Strings and Clifford space

Usual strings are described by the mapping \((\tau, \sigma) \rightarrow x^\mu = X^\mu(\tau, \sigma)\), where the embedding functions \(X^\mu(\tau, \sigma)\) describe a 2-dimensional worldsheet swept by a string. The action is given by the requirement that the area of the worldsheet be “minimal” (extremal). Such action is invariant under reparametrizations of \((\tau, \sigma)\). There are several equivalent forms of the action including the “\(\sigma\)-model action” which, in the conformal gauge, can be written as

\[
I[X^\mu] = \frac{\kappa}{2} \int d\tau d\sigma (\dot{X}^\mu \dot{X}_\mu - X'^\mu X'^\mu) \tag{18}
\]

where \(\dot{X}^\mu \equiv dX^\mu/d\tau\) and \(X'^\mu \equiv dX^\mu/d\sigma\). Here \(\kappa\) is the string tension, usually written as \(\kappa = 1/(2\pi\alpha')\).

If we generalize the action (18) to \(C\)-space, we obtain

\[
I[X] = \frac{\kappa}{2} \int d\tau d\sigma (\dot{X}^M \dot{X}^N - X'^M X'^N)G_{MN} \tag{19}
\]

Taking 4-dimensional spacetime, there are \(D = 2^4 = 16\) dimensions of \(C\)-space. Its signature \((+++\ldots−−−\ldots)\) has 8 plus and 8 minus signs. This particular form of metric suggest to define vacuum according to Jackiw et al. [18, 19] (see also [9, 17]). Then one finds that such generalized string theory is consistent. There are no negative norm states, and the Virasoro algebra has no central charges [10, 9]. My proposal is that, instead of adding extra dimensions to spacetime, we can start from 4-dimensional spacetime \(M_4\) with signature \((++−−\ldots)\) and consider the Clifford space \(C_{M_4}\) (\(C\)-space) whose dimensionality is 16 and signature \((8+, 8−)\). The necessary extra dimensions for consistency of string theory are in \(C\)-space. This also automatically brings spinors into the game. It is an old observation [20, 21] that spinors are the elements of left or right ideals of Clifford algebras. In other words, spinors are particular sort of polyvectors. Therefore, the string coordinate polyvectors contain spinors. This is an alternative way of introducing spinors into the string theory.

4. Curved Clifford Space

As we can pass in the ordinary theory of relativity from flat to curved spacetime manifold, so we can pass from flat to curved Clifford space \(C\). This is a manifold such that at every point \(X \in C\) its tangent space \(T_X C\) is Clifford algebra, a vector space, whose elements are polyvectors [13]. Among them one can chose those independent polyvectors which form a basis or frame. For variable \(X\) we have a frame field.

It is convenient to distinguish two kinds of frame field:

(i) Coordinate frame field

\[
\gamma^M = \gamma_{\mu_1 \ldots \mu_r} \tag{20}
\]

(ii) Orthonormal frame field

\[
\gamma^A = \gamma_{a_1 \ldots a_r} = \gamma_{a_1} \wedge \gamma_{a_2} \wedge \ldots \wedge \gamma_{a_r} \tag{21}
\]

While the basis elements \(\gamma^A\) are defined at every point point of \(C\)-space as the wedge product of vectors, this is not the case for \(\gamma^M\). The relation between \(\gamma^M\) and \(\gamma^A\) is given by

\[
\gamma^M = e_M^A \gamma_A \tag{22}
\]

where \(e_M^A\) is the \(C\)-space vielbein. Eqs. (2)–(4) may hold at one point \(X\), but not at different points of curved \(C\) space.
At every point \( X \) of \( C \) a polyvector, an element of the tangent space \( T_X C \), can be expanded, e.g., in terms of the basis (20) or (21). Since \( X \) may run over the manifold \( C \), we thus have a polyvector field.

At this point we encounter an important concept, namely, the derivative of a polyvector field. If acting on a scalar field, the derivative is just the ordinary partial derivative

\[
\partial_M \phi = \frac{\partial \phi}{\partial x^M}
\]  

(23)

If acting on a frame field, it defines connection, e.g., the connection for the coordinate frame field

\[
\partial_M \gamma_N = \Gamma^J_{MN} \gamma_J
\]

(24)

or the connection for the orthonormal frame field

\[
\partial_M \gamma_A = -\Omega^B_A \gamma_M \gamma_B
\]

(25)

If acting on a generic polyvector field, we have

\[
\partial_M (A^N \gamma_N) = \partial_M A^N \gamma_N + A^N \partial_M \gamma_N = (\partial_M A^N + \Gamma^N_{MK} \gamma_K) \gamma_N \equiv D_M A^N \gamma_N
\]

(26)

The components \( D_M A^N \) are covariant derivative of the tensor analysis.

Contrary to the usual practice, we use in eqs. (23)–(26) the same symbol \( \partial_M \). In ref. [13] we argue in detail why usage of different symbols for derivatives of different geometric objects is unnecessary.

Curvature of \( C \)-space is defined, as usually, by the commutator of derivatives acting on basis polyvectors:

\[
[\partial_M, \partial_N] \gamma_J = R^M_{NJK} \gamma_K
\]

(27)

or

\[
[\partial_M, \partial_N] \gamma_A = R^M_{NAB} \gamma_B
\]

(28)

Introducing the reciprocal basis polyvectors \( \gamma^M, \gamma^A \) satisfying

\[
(\gamma^M)^\dagger \gamma_N = \delta^M_N, \quad (\gamma^A)^\dagger \gamma_B = \delta^A_B
\]

(29)

we find from (27) or (28) the explicit expressions for the components of curvature in the corresponding basis:

\[
R^M_{NJK} = \partial_M \Gamma^K_{NJ} - \partial_N \Gamma^K_{MJ} + \Gamma^L_{NJ} \Gamma^K_{ML} - \Gamma^L_{MJ} \Gamma^K_{NL}
\]

(30)

or

\[
R^M_{NAB} = -(\partial_M \Omega_A^B \gamma_N - \partial_N \Omega_A^B \gamma_M + \Omega_A^C \Omega_B^D \gamma_M \gamma_C - \Omega_A^C \Omega_B^D \gamma_M \gamma_C)
\]

(31)

A consequence of non vanishing curvature is that after the parallel transport of a polyvector along a closed path we obtain a polyvector with different orientation. In particular this means that, if initially we have, e.g., a vector at a given point of \( C \), then after a round trip parallel transport to the same point we can obtain, e.g., a bivector, or in general any superposition of vectors, bivectors, 3-vectors, etc.

Instead of 4-dimensional spacetime we have thus 16-dimensional \( C \)-space. Since the latter space is higher dimensional, it can provide a realization of Kaluza-Klein theory. Good features of \( C \)-space are:

(i) There is no need for extra dimensions of spacetime. Extra dimensions are in \( C \)-space.

(ii) There is no need to compactify the “extra dimensions”. The extra dimensions of \( C \)-space, namely \( \omega, x^{\mu\nu}, x^{\mu\nu\rho}, x^{\mu\nu\rho\sigma} \) sample the extended objects, therefore they are physical dimensions.

(iii) The number of components \( G_{\mu\bar{M}}, \mu = 0, 1, 2, 3, \bar{M} \neq \mu \), is 12, which is the same as the number of the gauge fields in the standard model.
5. Spinors as members of left ideals of Clifford algebra and the generalized Dirac equation

Let us consider a polyvector valued field $\Phi(X)$ on a curved $C$-space manifold. At every point $X \in C$ a field $\Phi$ can be expanded, e.g., in terms of the orthonormal basis according to

$$\Phi = \phi^A \gamma_A, \quad A = 1, 2, ..., 16$$  (32)

where $\phi^A$ are complex valued scalar components.

Alternatively, we can use another basis with elements $\xi_i \equiv \xi_{\alpha i} \in I^L_i$, $\alpha = 1, 2, 3, 4; i = 1, 2, 3, 4$, where $I^L_i$ is the $i$-th left minimal ideal of Clifford algebra $\mathcal{C}$. Expansion of a polyvector field then reads $[12, 13]$:

$$\Phi \equiv \Psi = \psi^{\tilde{A}} \xi^{\tilde{A}}$$  (33)

Such object is the sum of four independent 4-component spinors, each in a different ideal $I^L_i$.

By assumption a field $\Psi$ has to satisfy

$$\partial \Psi \equiv \gamma^M \partial_M \Psi = 0$$  (34)

which generalizes the Dirac equation to $C$-space. The above equation is covariant, because the derivative $\partial_M$, if acting on a generalized spinorial basis elements $\xi^{\tilde{A}}$, gives the generalized spin connection:

$$\partial_M \xi^{\tilde{A}} = \Gamma^M_{\tilde{B} \tilde{A}} \xi^{\tilde{B}}$$  (35)

Using the latter relation, we can write eq. (34) in the form

$$\gamma^M \partial_M (\psi^{\tilde{A}} \xi^{\tilde{A}}) = \gamma^M (\partial_M \psi^{\tilde{A}} + \Gamma^M_{\tilde{B} \tilde{A}} \psi^{\tilde{B}}) \xi^{\tilde{A}} \equiv \gamma^M (D_M \psi^{\tilde{A}}) \xi^{\tilde{A}} = 0$$  (36)

An action which leads to eq. (34) is $[13]$:

$$I[\Psi, \Psi^\dagger] = \int d^n x \sqrt{|G|} i \psi^\dagger \partial \Psi = \int d^n x \sqrt{|G|} i \psi^\dagger \gamma^M \xi^{\tilde{A}} \psi^{\tilde{A}}$$  (37)

where $d^n x \sqrt{|G|}$ is the invariant volume element of the $2^n$-dimensional $C$-space, $G \equiv \text{det} G_{MN}$. We take $n = 4$.

A generic transformation in the tangent $C$-space $T_X C$ which maps a polyvector $\Psi$ (i.e., a generalized spinor) into another polyvector $\Psi'$ is given by

$$\Psi' = R \Psi S$$  (38)

where $R = \exp \Sigma_{AB} \alpha^{AB}$ and $S = \exp \Sigma_{AB} \beta^{AB}$, with $\alpha^{AB}$ and $\beta^{AB}$ being parameters of the left and right transformations, respectively. The generators $\Sigma_{AB}$ are defined as

$$\Sigma_{AB} = -\Sigma_{BA} = \begin{cases} \gamma_A \gamma_B, & \text{if } A < B \\ -\gamma_A \gamma_B, & \text{if } A > B \\ 0, & \text{if } A = B \end{cases}$$  (39)

Transformation (38) can be written in the form of a $16 \times 16$ matrix:

$$\psi'^{\tilde{A}} = U^{\tilde{A}}_{\tilde{B}} \psi^{\tilde{B}} \quad \text{or} \quad \psi' = U \psi, \quad U = R \otimes S^T$$  (40)

where $\mathbf{R}$ and $\mathbf{S}$ are $4 \times 4$ matrices representing the Clifford numbers $R$ and $S$. We see that $U$ is the direct product of $\mathbf{R}$ and the transpose $\mathbf{S}^T$ of $\mathbf{S}$.

$^2$ For a more detailed description see [13].
Under the transformation (40) the generalized spin connection transforms as
\[ \Gamma_{M\hat{A}} \hat{B} = U_D \hat{B} U_{\hat{C}} \Gamma^r_{M\hat{C}} \hat{D} + \partial_M U_{\hat{D}} U_D \hat{B}, \quad \text{i.e.,} \quad \Gamma_M = U \Gamma^r_M U^{-1} + U \partial_M U^{-1} \] (41)
while the covariant derivative transforms as
\[ D'_M \psi^{\hat{A}} = U \hat{A} B D_M \psi^{\hat{B}}, \quad \text{i.e.,} \quad D'_M \psi^\alpha = U D_M \psi^\alpha \] (42)
where \( D'_M \psi^{\hat{A}} = \partial_M \psi^{\hat{A}} + \Gamma'_M \hat{A} \psi^{\hat{B}} \) and \( D_M \psi^{\hat{A}} = \partial_M \psi^{\hat{A}} + \Gamma_M \hat{A} \psi^{\hat{B}} \). We see that \( \Gamma_M \) transforms as a non abelian gauge field.

The generally covariant equation in 16-dimensional \( C \)-space contains the coupling of spinor fields \( \psi^{\hat{A}} \) to non abelian gauge fields \( \Gamma_M \hat{A} \), which altogether form components of connection in the generalized spinor basis.

6. Conserved charges and isometries

In curved space in general there are no conserved quantities, unless there exist isometries which are described by Killing vector fields. Suppose we have a curved Clifford space which admits \( K \) Clifford numbers \( k^\alpha = k_M^\alpha \gamma^M \), \( \alpha = 1, 2, ..., K; M = 1, 2, ..., 16 \), where the components satisfy the condition for isometry, namely
\[ D_N k_N^\alpha + D_M k_M^\alpha = 0 \] (43)
the covariant derivative being defined in eq. (26). We assume that such \( C \)-space with isometries is not given ad hoc, but is a solution to the generalized Einstein equations that arise from the action which contains a “matter” term, such as (37), and a field term defined by means of the \( C \)-space curvature (see [13]).

Taking a coordinate system in which \( k^{\alpha\mu} = 0 \), \( k^{\alpha\hat{M}} \neq 0 \), \( \mu = 0, 1, 2, 3, \hat{M} \neq \mu \), the metric and the vielbein can be written as
\[ G_{MN} = \begin{pmatrix} G_{\mu\nu} & G_{\mu\hat{M}} \\ G_{\hat{M}\nu} & G_{\hat{M}\hat{N}} \end{pmatrix}, \quad e^A_M = \begin{pmatrix} e^\alpha_\mu \\ e^\alpha_{\hat{M}} \\ e^{\hat{A}}_\mu \\ e^{\hat{A}}_{\hat{N}} \end{pmatrix} \] (44)
Here
\[ e^{\hat{A}}_{\hat{M}} = 0 \] (45)
whilst the components \( e^{\hat{A}}_\mu \) can be written in terms of Killing vectors and gauge fields \( W^\alpha_\mu (x^\mu) \):
\[ e^{\hat{A}}_\mu = e^{\hat{A}}_{\hat{M}} k^{\alpha\hat{M}} W^\alpha_\mu, \quad \partial_{\hat{N}} W^\alpha_\mu = 0 \] (46)

If we set the \( C \)-space torsion to zero and calculate the connection \( \Omega_{ABM} \) by using eqs. (44)–(46), we obtain an analogous result as given, e.g., in ref. [22]:
\[ \Omega_{\hat{M}\hat{N}M} = \frac{1}{2} k^{\alpha}_{[\hat{M}, \hat{N}]} W^\alpha_\mu \] (47)
where \( k^{\alpha}_{[\hat{M}, \hat{N}]} \equiv \Omega_{\hat{N}\hat{M}M} k^{\alpha}_\mu - \Omega_{\hat{M}\hat{N}M} k^{\alpha}_\mu \)

Let us now rewrite the \( C \)-space Dirac equation by using eqs. (43)–(47). Omitting the terms due to the \( C \)-space torsion, we obtain
\[ \left[ \gamma^{(4)\mu} \left( \partial_\mu - \Omega_{ab\mu} \frac{1}{2} [\gamma^a, \gamma^b] - q^a W^a_\mu + ... \right) + \gamma^\hat{M} \partial_{\hat{M}} + ... \right] \psi = 0 \] (48)

\(^{3}\) This is a \( C \)-space analog of the Kaluza-Klein splitting usually performed in the literature. See, e.g., [22, 23].
where $\gamma^{(A)\mu} e_\mu^\alpha \gamma^a$ are 4-dimensional coordinate basis vectors, and

$$ q^\alpha = k^\alpha \bar{M} \partial_{\bar{M}} + \frac{1}{8} k^\alpha_{[\bar{M}, \bar{N}]} e_\mu^{\bar{M}} \bar{e}_\mu^{\bar{N}} \Sigma^{AB} $$

(49)

are the charges, conserved due to the presence of isometries $k^\alpha \bar{M}$. They are the sum of the coordinate part and the contribution of the spin angular momentum in the “internal” space, spanned by $\gamma^a$. The coordinate part is the projection of the linear momentum onto the Killing (poly)vectors, and can in particular be just the orbital angular momentum of the “internal” part of C-space. The first term that contributes to the charge $q^\alpha$ comes from the vielbein according to eq. (46), whilst the second term comes from the connection according to eq. (47). Both terms couple to the same 4-dimensional gauge fields $W^\alpha_\mu$, where the index $\alpha$ denotes which gauge field (which Killing polyvector), and should not be confused with the spinorial index, used in Dirac matrices.

In eq. (48) we explicitly displayed only the the most relevant terms which contain the the ordinary vierbein $e_\mu^\alpha$ and spin connection $\Omega_{ab\mu}$ (describing gravity and torsion), and also Yang-Mills gauge fields $W^\alpha_\mu$ which, as shown in eqs. (46),(47), occur in the C-space vielbein and in the C-space “spin” connection. We omitted the terms arising form the C-space torsion.

7. Discussion

Clifford space provides a promising approach to the unification of fundamental interactions. At first sight one might think that the signature $(p, p)$ brings ghosts into the description. This is not the case, if we adopt the Jackiw et al. definition of vacuum [18, 19, 17, 9]. Then such pseudoeuclidean signature turns out to be welcome for string theory [9] and for the resolution of the cosmological constant problem [17].

Another possible obstacle could be seen in the Coleman-Mandula theorem [24] which forbids nontrivial mixing of spacetime and internal symmetries. But all such theorems are based on certain, often tacit, assumptions. One such tacit assumption (technically expressed in terms of certain properties of S-matrix) in the derivation of the Coleman-Mandula theorem is that probability conservation refers to spacetime, and not to the “internal” space. In other words, the internal space was not taken on equal footing as spacetime at the very beginning. However, it was shown by Pelc and Horwitz [25] that Coleman-Mandula theorem can be extended to a higher dimensional space. In general, in Kaluza-Klein approach conserved generators associated with gauge charges come from the extra dimensions. Extra dimensions have in principle the same role as the ordinary four dimensions, and there are the transformations that transform one into the other. Thus internal symmetries, associated with extra dimensions, can in general nontrivially mix with four spacetime dimensions. Analogously holds for Clifford space. In particular, however, a curved higher dimensional space can admit isometries, and can then be written as the direct product of 4-dimensional spacetime and the internal space. We have shown that the conserved charges, which are due to the isometries of curved Clifford space, have two contributions: one from the “orbital” angular momentum in the “internal” part of C-space, and the other from the “internal” spin. This could have some important physical consequences that need to be explored further.

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