Towards Gravity From a Color Symmetry

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Using tools from color-kinematics duality we propose a holographic construction of gravitational amplitudes, based on a 2d Kac-Moody theory on the celestial sphere. In the $N \to \infty$ limit the gauge group corresponds to $u_{1+\infty}$, due to the $U(N)$ generators enjoying a simple quantum group structure, which is in turn inherited from a twistor fiber over the celestial sphere.

We show how four-dimensional momentum-space is emergent in this picture, which connects directly to the so-called kinematic algebra of the tree-level S-Matrix. On the other hand, the framework can be embedded within a celestial CFT to make contact with holographic symmetry algebras previously observed in the soft expansion. Kac-Moody currents play the role of a graviton to all orders in such expansion, and also lead to a natural notion of Goldstone modes for $u_{1+\infty}$. Focusing on MHV amplitudes, main examples are a BCFW type recursion relation and holomorphic three-point amplitudes.

INTRODUCTION

Diverse aspect of gravitational theories suggest that their dynamics enjoy integrable properties. For one, the relation between (super)gravity and (super)YM theories via the double copy/color-kinematics duality [1] suggests certain that avatars of Yangian symmetry [2–4] can emerge in perturbative gravity amplitudes [5–7]. Classical solutions of interest, such as the Kerr black hole and its deformations, also enjoy hidden symmetries reflecting integrability in their dynamics [8, 9]. The relation between such classical and quantum notions of integrability is a topic of active research unearthing intriguing connections, see e.g. [10–13].

On a parallel venue, the celestial holography program [14–19] aims to reformulate scattering in asymptotically flat spacetimes as governed by a putative CFT in the celestial sphere. The defining recipe is to recast the four-dimensional S-Matrix in a boost eigenstate basis where it can be interpreted as a correlation function. From a pragmatical perspective it is expected that celestial correlation functions can be determined with help of the constraints from conformal symmetry and extensions thereof, alleviating the need for quantum gravity computations. A first motivation for this was the realization of universal soft theorems in gravity in the language of the celestial CFT (CCFT) [20–23]. By pushing the soft analysis to arbitrary orders within the conformal framework [22], recently a novel hierarchy based on the $u_{1+\infty}$ symmetry algebra has been uncovered in the gravitational CCFT [24, 25]. This algebra and its quantization have a long history with deep connections to integrability [26–29] and Penrose’s twistor construction [30, 31]. Because of this, it is pressing to understand its implications in the gravitational S-Matrix, directly in momentum space, investigating to what extent scattering is constrained.

In this letter we take a first step in this direction by introducing a 2d Kac-Moody theory that reproduces certain gravitational amplitudes through a celestial dictionary. Following the lore of color-kinematics duality, we show how four-dimensional momentum space emerges as the $u_{1+\infty} \cong U(N \to \infty)$ color group of the theory, thus explicitly manifests the hierarchy in the S-Matrix language.

FROM THE S-MATRIX TO $u_{1+\infty}$

To present the idea we first point out that the algebra has a universal imprint in the S-Matrix via collinear factorizations. In perturbative gravity amplitudes, such are given by splitting functions which also persist in a class of loop amplitudes [32, 33], nevertheless here we will focus on the tree-level instance.

Consider the scattering amplitude $A_n(p_i)$ dressed with momentum conservation. For massless particles collinear singularities correspond to the three-point on-shell vertices which are unique for given helicities [34]. We consider only helicity-preserving vertices whose coupling $16\pi G = 1$ is further fixed by the equivalence principle, see Appendix II for the generic case. The collinear limit is then universal and reads

$$A_{n+1}(k^{\pm 2}, p_1^{\pm h}, \ldots, p_n) \xrightarrow{k \cdot p_1} \langle \epsilon^+, p_1 \rangle^2 \frac{\epsilon^+ \cdot p_1}{k \cdot p_1} A_n((k+p_1)^{\pm h}, \ldots, p_n),$$

(1)

We have emphasized the helicities through superscripts $p_i^{\pm h}$. Hereafter we use $(2, 2)$ signature where the Lorentz group is $SL(2, \mathbb{R})_\pm \times SL(2, \mathbb{R})_\pm$ [35]: We write momentum vectors in terms of spinors as [36]

$$p_i = |\eta_i\rangle [\lambda_i], \quad k = |\eta\rangle [\lambda],$$

(2)

so that $|\eta_i\rangle = (1, z_i)$, and $|\lambda_i\rangle$ is an independent spinor carrying the energy scale. Similarly for the graviton momenta $k$, which has coordinate e.g. $z_p$ and $|\lambda\rangle$. Our collinear limit is then $z_p \to z_1$ where $\lambda, \lambda_1$ are kept...
generic. This yields \( k + p_1 = |\eta_1| (|\lambda| + |\lambda_1|) \). With a
concurring change of notation, eq. (1) can be written
\[
A_n^{\lambda_1 \cdots \lambda_n} (z_p, z_1, \ldots, z_n) \sim \left[ \frac{[\lambda_1]}{z_p^{1+}} \right] A_n^{1+ \cdots \lambda_n} (z_1, \ldots, z_n),
\]
with \([\lambda_1] = e^{a_1} \lambda_1 \xi_1\). In the following we will reinterpret
\( \lambda \) as \( \mathcal{G}(2, \mathbb{R}) \) carrying \( \mathcal{O}(\tau) \), where
\( \tau \) is a quantization parameter such that as \( \tau \to 0 \)
this reduces to a Poisson bracket. The generators possess an
obvious \( SL(2, \mathbb{R}) \) symmetry \( \mu'_{\alpha} = \Lambda_{\alpha}^{\beta} \mu_\beta \). In this language the \( w_{1+\infty} \) algebra emerges from the universal enveloping algebra of (5) (sometimes called Weyl algebra),
i.e. that given by totally symmetric polynomials of the operators
\[
W_{\alpha_1 \cdots \alpha_s} := \mu_{(\alpha_1 \cdots \alpha_s)}, \quad \alpha_i = \pm,
\]
with \( W^0 = 1 \) is included in \( w_{1+\infty} \) (as opposed to \( w_{\infty} \)). As
aforementioned, the \( W^s \) correspond to higher-spin fields under
\( SL(2, \mathbb{R}) \) [83]. Their algebra is also \( SL(2, \mathbb{R}) \) covariant OPE associated to a
graviton operator and a massless helicity-\( h \) field:
\[
G^\lambda (z_p) O^\lambda_h (z_1) \sim \left[ \frac{[\lambda_1]}{z_p^{1+}} \right] O^{1+\lambda_1}_h (z_1)
\]
where we emphasize that the antichiral components \( \lambda, \lambda_1 \)
carrying \( \mathcal{O}(\tau) \) weight are not collinear but generic.
Because of this the OPE holds not only in the soft sector
but for generic energies. In turn, it will become clear
that writing the OPE (4) allows us to identify \( G^\lambda (z_p) \)
as Kac-Moody currents for a \( w_{1+\infty} \) gauge group, where
the adjoint representation is parametrized by \( [\lambda] \), i.e. the
kinematic data! We will see this provides a precise realization of a 2d color-kinematics duality à la [37]. Furthermore
the OPE suggests that scattering amplitudes such as (3) are then computed as correlations of such currents, as
we will discuss.

\( w_{1+\infty} \) as a color group

Historically, the \( w_{1+\infty} \) algebra and its quantum version \( W_{1+\infty} \) arised as vertex algebras of higher-spin fields in a 2d CFT, see [28, 38] for a review. In turn, the novel
OPE (4) is written simply in terms of spin-1 currents and so obscures this interpretation. It turns out that
the connection can be made through the language of quantum
groups. To see this let us first provide a canonical definition of \( w_{1+\infty} \). Consider two abstract generators \( \mu_{\pm} \)
satisfying
\[
[\mu_{\alpha}, \mu_{\beta}] = i \tau \epsilon_{\alpha\beta}, \quad \epsilon_{++} = 1,
\]
where \( \tau \) is a quantization parameter such that as \( \tau \to 0 \)
this reduces to a Poisson bracket. The generators possess an
obvious \( SL(2, \mathbb{R}) \) symmetry \( \mu'_{\alpha} = \Lambda_{\alpha}^{\beta} \mu_\beta \). In this language the \( w_{1+\infty} \) algebra emerges from the universal enveloping algebra of (5) (sometimes called Weyl algebra),
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aforementioned, the \( W^s \) correspond to higher-spin fields under
\( SL(2, \mathbb{R}) \) [83]. Their algebra is also \( SL(2, \mathbb{R}) \) covariant:
\[
[W^m_{\alpha_1 \cdots \alpha_m}, W^n_{\beta_1 \cdots \beta_n}] = i \tau m n \epsilon_{(\beta_1} (W^{m+n-2}_{\alpha_2 \cdots \alpha_m})_{\beta_2 \cdots \beta_n}) + \mathcal{O}(\tau^2)
\]
This construction is closely related to classical twistor space [31, 39], locally \( \mathbb{CP}^1 \times \mathbb{C}^2 \), where \( \mu_+, \mu_- \) are simply
coordinates on \( \mathbb{C}^2 \). This motivates us to connect to momentum space through a Fourier transform \( \{ \mu_+ \} \to \{ \lambda_+ \} \). Thus we introduce the momentum wavefunctions
\[
G^\lambda := e^{[\lambda]} = \sum_{s=0}^{\infty} \frac{1}{s!} \lambda^1 \cdots \lambda^s W_{1+1}^{1+ \cdots s} \lambda_+ \in \mathbb{R}^2.
\]
Remarkably, in this basis the Weyl algebra (7) takes the
form
\[
[G^{\lambda_1}, G^{\lambda_2}] = i \tau [\lambda_1, \lambda_2] G^{\lambda_1+\lambda_2} + \mathcal{O}(\tau^2).
\]
This is precisely what appears in the numerator of (4) and
can be interpreted as structure constants of \( SU(N) \) when
\( N \to \infty \) as noted in the seminal work of [40]. As a warm
up for the next section let us see how this proceeds. First
note that (5) admits only infinite dimensional representations:
Indeed the enveloping algebra (7) is essentially the linear space of regular functions of \( \mu_+, \mu_- \) equipped with symplectic form (5). To connect to \( SU(N) \) or \( U(N) \)
we will instead obtain finite, \( N \)-dimensional, representations.
For this we “exponentiate” the algebra by introducing \( g_\pm = e^{\mu_\pm} \) so that (5) becomes
\[
g_+ g_- = q g_- g_+ , \quad q := e^{i \tau},
\]
where in the classical limit \( q \to 1 \) we see that \( g_\pm \) are
c-numbers in \( \mathbb{R}^2 \). Let now \( N \) be an odd integer. If we set
\[
\tau = 4 \pi / N,
\]
it essentially follows from (10) that \( g_\pm^N = 1 \) [84]. Thus \( g_\pm \)
now live in a lattice \( Z_N \times Z_N \) with modular parameter
\( i \tau = 4 \pi i / N \). This is where \( U(N) \) acts. More precisely,
there exist \( N \times N \) matrices \( g_{ij} \) satisfying (10), which is a
Heisenberg group. The generators are then

\[
\frac{G^\lambda}{4 \pi i / N} := e^{[\lambda]} = q^{1+ \lambda^- / 2} g_+ g_-^{1- \lambda^-}, \quad \lambda_+ \in Z_N \times Z_N.
\]
Thus in this language the \( U(N) \) generators \( G^\lambda \) carry an
adjoint color index given by a 2-spinor \( \lambda \) (To get \( SU(N) \),
which is associated to \( w_\infty \) instead of \( w_{1+\infty} \), we simply
discard \( G^{(0,0)} = W^0 = 1 \)). Moreover, the quantum group
relation (10) implies the Berezin formula
\[
G^{\lambda_1} G^{\lambda_2} = \frac{q^{||12||/2}}{4 \pi i / N} G^{\lambda_1+\lambda_2}.
\]
Indeed, by evaluating \( [G^{\lambda_1}, G^{\lambda_2}] \) from here one can precisely
match the structure constants for \( U(N) \) [40]. Since \( \lambda^c \)
corresponds to momenta on a lattice, this realizes a
two-dimensional color-kinematics duality in the sense of
[37]. On the other hand, for generic, non-discrete, values of \( \tau \), eq. (13) still applies [85], but the momenta is not
quantized. In any case, as \( N \to \infty, \tau \to 0 \) we obtain a semiclassical limit, where the momenta spinor \( \lambda \) again lives in \( \mathbb{R}^2 \) and

\[
G^{\lambda_1} G^{\lambda_2} = (N/4\pi i + [12]/2) G^{\lambda_1 + \lambda_2} + O(1/N) .
\] (14)

The divergence \( \sim N \) does not contribute to the commutator, which becomes precisely (9), using \( G^\lambda \to i \tau G^\lambda \).

As a final remark, for discrete \( N \) we have written the algebra in terms of \( N \times N \) matrices \( g_{\pm} \). The operators \( \mu_{\pm} \) are defined as \( \mu_{\pm} = \log g_{\pm} \) only in a formal sense and are unbounded. From the 4d theory perspective this is expected since they correspond to Goldstone modes as we now explain.

4D KINEMATIC ALGEBRA

Having constructed 2-dimensional color space as parametrized by \( \lambda \), it is easy to upgrade the construction to 4-dimensional kinematic space. The motivation comes again from twistor theory where, at least classically, the positions \( \mu_\alpha \in \mathbb{C}^2 \) are fibered over \( z = \eta_2/\eta_1 \in \mathbb{C}P^1 \) [31, 39]. This motivates us to construct the quantum operators from sections of this bundle, i.e. operators \( \mu_\alpha(z) \) satisfying

\[
\mu_\alpha(z_1) \mu_\beta(z_2) \sim i \tau \epsilon_{\alpha \beta} \frac{\delta(z_1 - z_2)}{z_{12}} .
\] (15)

This is the affine extension of the 2d algebra (5) [86]. More importantly, recalling that \( p = |\eta|/|\lambda| \) we can identify \( z = \eta_2/\eta_1 \) as living in the celestial sphere. As we explain in Appendix, this algebra generalizes the one previously introduced for celestial Goldstone modes in the soft sector [41]. Indeed here the operators \( \mu_\alpha \) themselves can be interpreted as graviton Goldstone modes valid for all energies. To see this we introduce the normal-ordered vertex operators

\[
i \tau G^\lambda(z) = : e^{[\lambda \mu(z)]} : ,
\] (16)

for which the analog of the Berezin formula (13) takes the form

\[
G^{\lambda_1}(z_1) G^{\lambda_2}(z_2) = \left( g^{[12]/z_{12} / i \tau} : e^{[\lambda_1 \mu(z_1)] + [\lambda_2 \mu(z_2)]} : \right) \sim \left( g_{[12]/z_{12} / i \tau} \right) G^{\lambda_1 + \lambda_2}(z_2) \sim \frac{[12]}{z_{12}} G^{\lambda_1 + \lambda_2}(z_2) + O(\tau) .
\] (17)

This mimics (13): In the second line we have taken the OPE limit \( z_{12} \to 0 \) which discards the \( \sim N \) terms, and in the third line the classical limit \( \tau = 4\pi N \to 0 \). An important comment is now in order. In the OPE (15) both operators have conformal weight 1/2 in \( z \), which requires \( \lambda_\alpha \) to have weight \(-1/2\). This is familiar from the construction of gravitational Goldstone operators [41]. However, after the \( \tau \to 0 \) limit is taken in (17) we can take \( z \) and \( \lambda \) to be independent, after all the latter only enter through the structure constants of \( U(\infty) \). We can make this even more explicit by writing, at strict \( \tau = 0 \),

\[
G^{\lambda_1}(z_1) G^{\lambda_2}(z_2) \sim \frac{\int d^2 \lambda_3 f^{\lambda_1, \lambda_2, \lambda_3}}{z_{12}} G^{\lambda_3}(z_2) ,
\] (18)

where the kinematic structure constants

\[
f^{\lambda_1, \lambda_2, \lambda_3} = f^{\lambda_1, \lambda_2, -\lambda_3} = [12] \delta^2(\lambda_1 + \lambda_2 - \lambda_3) ,
\] (19)

are completely antisymmetric and correspond to the kinematic algebra [42]. Seen in this way, the OPE pertains Kac-Moody currents with weight 1 under \( SL(2, \mathbb{R})_z \), where the other \( SL(2, \mathbb{R})_\bar{z} \) has been promoted to a color \( w_{1+\infty} \) gauge group. From a CFT perspective, this reproduces the chiral OPE block of [24] which resums all \( SL(2, \mathbb{R}) \) descendants, see also Appendix I.

Furthermore, we can use standard arguments to show that associativity of the OPE (18) immediately yields the kinematic Jacobi relation of [42]

\[
\int d^2 \lambda (f^{\lambda_1, \lambda_2, \lambda_4} + f^{\lambda_1, \lambda_3} f^{\lambda_3, \lambda_4} ) = 0 .
\] (20)

Note that this Lie algebra structure was completely invisible from the collinear factorization/S-Matrix perspective of (3), in striking contrast to its OPE realization. Furthermore, given this Lie algebra it is natural to further incorporate 2d primary fields of arbitrary helicity transforming in the adjoint representation (18), i.e.

\[
G^{\lambda_1}(z_1) O_h^{\lambda_2}(z_2) \sim \frac{\int d^2 \lambda_3 f^{\lambda_1, \lambda_2, \lambda_3}}{z_{12}} O_h^{\lambda_3}(z_2) ,
\] (21)

which is precisely (4) as promised. We elaborate on this result in the next section. Here let us point out that the algebraic analysis of refs. [43, 44], which studied the Jacobi identity in the celestial conformal basis is here synthesized in the Lie algebra relation (20), closer to the recent analysis of [45]. Such references also studied the different helicities case, which we discuss briefly in Appendix II.

TOWER OF SOFT CHARGES IN GRAVITY

Let us now provide example-based evidence that gravitational amplitudes can indeed be derived from the preceding Kac-Moody OPEs. We do so by computing correlation functions of soft modes. Indeed, as is well known, a systematic study of gravitational scattering can be carried out through the IR expansion of the S-Matrix [46], which is also easily realized as a conformal soft expansion in the celestial sphere [22, 43, 47]. Notably, as we further flesh out in the Appendix, this expansion of a graviton
state amounts precisely to the decomposition given in eq. (8) and thus allow us to identify the tower of soft modes. In the operator language, it follows from (16) that

$$G^\lambda(z) = \sum_n \frac{1}{n!} \lambda^{\alpha_1} \cdots \lambda^{\alpha_n} W_{\alpha_1 \cdots \alpha_n}(z),$$

where

$$W_{\alpha_1 \cdots \alpha_n}(z) = \frac{1}{iz^2} \mu_{\alpha_1} \cdots \mu_{\alpha_n} :z^2:,$$

is the 4d analog of (6). Very nicely, their affine algebra can be computed either directly from the Goldstone modes (15) or by expanding both sides of (17) in soft modes. At $\tau \to 0$ the result is

$$W_{\alpha_1 \cdots \alpha_n}(z_1) W_{\beta_1 \cdots \beta_n}(z_2) \sim \frac{mn}{z_{12}^n} \delta(\beta_1(a_1 W_{\alpha_2 \cdots \alpha_n})_2 \cdots \beta_n)(z_2).$$

This agrees with the algebra of [25, 33, 48] as we show in Appendix I. Note here that the $\lambda^a$ is precisely our starting eq. (15). Thus the operator $W^0$ is required to generate the tower. We will recognize it as a central extension of the 4d Poincare algebra.

To understand correlation functions involving arbitrary helicity fields we consider (21). It is convenient to rewrite it as an exponential soft theorems [22, 49–51], i.e.

$$e^{i \lambda^a \mu(z_1)} :O^{\lambda^a}(z_2) \sim i^2 \int_{z_2} \frac{12}{z_1^2} e^{i \lambda^a \frac{\partial}{\partial z_1}} O^{\lambda^a}(z_2),$$

from where, expanding both sides in $\lambda^a$, according to (22)-(23) immediately yields a tower of soft operator OPEs

$$W^0(z_1) O^{\lambda^a}(z_2) = \text{regular}$$

$$W^1(z_1) O^{\lambda^a}(z_2) \sim \frac{\lambda^a}{z_1^2} O^{\lambda^a}(z_2)$$

$$W^2(z_1) O^{\lambda^a}(z_2) \sim \frac{2 \lambda^a \frac{\partial}{\partial z_1}}{z_1^2} O^{\lambda^a}(z_2)$$

$$W^3(z_1) O^{\lambda^a}(z_2) \sim \frac{3 \lambda^a \frac{\partial^2}{\partial z_1^2}}{z_1^2} O^{\lambda^a}(z_2),$$

etc... From here we confirm that $W^0$ is trivial, consistent with the expectation that it has no scattering amplitudes. Moving on, we recognize $W^1$ as a spacetime momentum: Introducing $\eta_a = (1, z)$ the translation generator is [87]

$$P_{aa} = \frac{(i \tau)^{-1}}{2 \pi i} \oint dz \eta_a(z) \mu_a(z) \quad a, a = \pm.$$

Contour-integrating in (27) this gives $[P_{aa}, O^{\lambda^a}(z)] = \eta_a \lambda_a O^{\lambda^a}(z)$ as expected. Analogously, we recognize $W^2 = i \mu^2 :\mu^2:$ as the angular momentum generator $\lambda_{\alpha \beta \gamma} \frac{\partial}{\partial \alpha \beta \gamma}$ [52]. Similarly to the Sugawara construction, it can be reinterpreted as the stress-energy tensor in celestial CFT, see Appendix. The generator $W^3$ closely resembles a conformal transformation [52].

**SOFT THEOREMS AND RECURRENCE**

From the S-Matrix perspective, these generators are responsible for Weinberg soft theorems and their subleading extensions [46, 53], when inserted into generic correlation functions. Omitting $SL(2, \mathbb{R})$ indices for simplicity we have

$$\langle W^n(z_1) \cdots O^{\lambda^a}(z_n) \rangle \sim \frac{1}{2 \pi i} \oint \frac{dw}{w-z} \langle W^n(w) O^{\lambda^1}(z_1) \cdots O^{\lambda_n}(z_n) \rangle$$

$$= \frac{1}{2 \pi i} \int \frac{dw}{w-z} \langle W^n(w) O^{\lambda^1}(z_1) \cdots O^{\lambda_n}(z_n) \rangle + \mathcal{P}^r(z)$$

where the first term controls the colinear factorizations that are present in the scattering amplitudes. The extra term $\mathcal{P}^r(z)$ denotes additional contributions: Multiparticle singularities and polynomial terms at $z \to \infty$. The former are likely associated to Virasoro descendants and we leave their study for future work.

To gain further insight into this picture, let us write (31) as the soft theorems in [22, 54]. Resorting to the celestial dictionary [17], the correlation functions in the LHS of (31) correspond to the S-Matrix with a graviton insertion,

$$A_{n+1}(k^2, p_1, \ldots, p_n) \leftrightarrow \langle G^\lambda(z) O^{\lambda^1}(z_1) \cdots O^{\lambda_n}(z_n) \rangle.$$

Using (22) we can then translate the relation (31) into

$$A_{n+1}(k^2, p_1, \ldots, p_n) = \sum_i \langle \epsilon^+ \cdot p_i \rangle J_i A_n(p_1, \ldots, p_n) + \cdots,$$

where $J_i = [\lambda^a, \partial/\partial \alpha^a]$ amounts to a Lorentz generator on particle $i$. We refer to [22] for further details, see also [55]. The $\cdots$ denote the contribution from $\mathcal{P}^r$. Their contribution can be understood as follows. Let $A_n$ and $A_n$ correspond to the dressed and stripped amplitude, respectively

$$A_n(p_i) = \delta^2 \left( \sum_i \lambda_i \right)^2 \frac{\delta^2 \left( \sum_i z_i \lambda_i \right)}{\delta^2 \left( \sum_i \lambda_i \right) A_n(p_i)}.$$

Then, it is easy to see from (33) that $A_{n+1}$ develops the expected support on $\delta^2(\lambda + \sum_i \lambda_i)$, but not necessarily in $\delta^2(z \lambda + \sum_i z_i \lambda_i)$. The former is a consequence of our color $U(\infty)$ Lie algebra while the latter can be interpreted as a two-dimensional holomorphic contour, see discussion below. The presence of the polynomial $\mathcal{P}^r$ is precisely to enforce that contour. Moreover, in the case of pure graviton MHV amplitudes (33) defines a BCFW-type recursion relation $A_n \to A_{n+1}$, for we can show explicitly
that \( \mathcal{P}'(z_0) \) is given by the operator
\[
\sum_{i=1}^{n-1} \frac{[\lambda_i]}{z_{ni}} \sum_{j=0}^{r-4} \left( \frac{z_{m}}{z_{ni}} \right)^j \sum_{m=0}^{r-4-j} \left( \frac{r-1}{m} \right) (J_i - J_n)^{r-1-m} J_n m
\]
acting on the \( n \)-point amplitude \( A_n \). This illustrates that \( \mathcal{P}'(z_0) \) is a polynomial of degree \( r - 4 \) in \( z_0 \). In particular \( \mathcal{P}' = 0 \) for \( r \leq 3 \), and thus \( W^1, W^2, W^3 \) in (31) generate the standard (sub)leading soft theorems [53] as expected.

**DISCUSSION AND FUTURE DIRECTIONS**

In this work we have provided evidence that recent developments in **celestial holography, color-kinematics duality, and twistor theory** are intrinsically connected. A natural three-way bridge between these topics is given by the OPE language, which exposes the properties of a Kac-Moody algebra, such as associativity, completely concealed by the S-Matrix picture. On the other hand, a Kac-Moody algebra, such as associativity, completely exposed by the OPE language, which exposes the properties of developments in celestial holography, color-kinematics duality.

In a strict sense this is a level-0 theory which is therefore non unitary (this is expected for tree-level scattering). Moreover, the connection to self-dual gravity also suggests that there should not be a multiparticle S-Matrix due to integrability. Recent work has shown, however, that integrable theories can be reduced to non-trivial scattering in presence of defects or other auxiliary fields [61–63]. This resonates with an old conjecture of Ward [64, 65] stating that self-dual systems yield rich dynamics under compactification.

The properties unveiled through celestial holography, such as IR modes and their soft theorems, also point towards a construction more general than the self-dual sector. As a step towards such, in Appendix II we introduce negative helicity graviton states, corresponding to a 2d scalar field \( \Phi^\lambda \). We find
\[
\langle G^{\lambda_1}(z_1)G^{\lambda_2}(z_2)\Phi^{\lambda_3}(z_3) \rangle = \frac{[12]^6}{[13]^2[23]^2} \delta^4 \left( \sum_i p_i \right).
\]

In this formula four-momentum conservation emerges naturally from OPEs, contrasting with the usual CCFT picture [19, 66]. Here we interpret two delta functions as poles defining contours in \( z \). In particular we find
\[
\int \frac{dz_1dz_2}{2\pi i} \langle G^{\lambda_1}(z_1)G^{\lambda_2}(z_2)\Phi^{\lambda_3}(z_3) \rangle = f^{\lambda_1,\lambda_2,\lambda_3}.
\]

Moving on to the four-point function \( \langle GGG\Phi \rangle \), we find via the recursion that it does not vanish identically, but instead we have
\[
\int \frac{dz_1dz_2dz_3}{2\pi} \langle G^{\lambda_1}(z_1)G^{\lambda_2}(z_2)G^{\lambda_3}(z_3)\Phi^{\lambda_4}(z_4) \rangle = (20)
\]

Thus its residue vanishes identically only if the Jacobi identity holds! In any case, the correlation itself is non-zero but has support only when all points \( z_i \) coincide. This is precisely the situation in twistor space, where \( A_4(1^+2^+3^+4^-) \) is supported on a degree zero curve [52]! Further exploration of this fascinating direction is left for future work.

Many questions regarding the nature of the proposed holographic theory remain open. The most pressing one pertains multiparticle singularities as well as the loop amplitudes, perhaps paralleling the developments of [61, 62]. Relatedly, it is tempting to explore the finite \( N \) regime as a fully quantum theory. In this case the naive OPE (17) necessarily yields higher-spin operators as in string theory.

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**Note added.** While finalizing the writing stage of this paper we were informed of the parallel works [67]–[68], which have also discussed the color-kinematics duality in this context.

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[83] To compare with the generators of [24] we take $W^a = \omega^{1+s/2}$. See Appendix I.

[84] We assume that the center of the group is trivial.

[85] It corresponds to the Moyal product familiar in the context of non-commutative geometry, see e.g. [69, 70]. We thank K. Costello and R. Monteiro for pointing this out.

[86] It has also appeared as a worksheet OPE in the related context of the twistor sigma model [48, 71].

[87] In Appendix we interpret $W^1$ as a weight 3/2 operator, in which case these are its global modes.

## Appendices

### I. Relation to the conformal primary basis

In celestial holography, the S-Matrix is cast into a boost eigenstate basis that exhibits its 2d conformal symmetry [18]. In the present work we have in contrast attempted to formulate fundamental properties of correlation functions strictly in momentum space. The latter idea fits into a more general framework we shall elaborate further in [72]. In the meantime we will outline here one of its key points: namely that both approaches are connected through the holomorphic soft expansion [53].

In scattering, incoming massless particles such as gravitons can be represented by insertion of creation operators depending on an energy scale $\omega$ and coordinates $z, \bar{z}$ [73]

$$a^\dagger_{z\bar{z}}(\omega, z, \bar{z}) \ , \ \omega > 0 , \quad (39)$$

where $+2$ indicates the 4-dimensional helicity of a graviton. Here the momenta $p = |\eta| |\lambda|$ is parametrized as $|\eta| = \sqrt{\omega} (1 \ z)$ and $|\lambda| = \sqrt{\omega} (1 \ \bar{z})$. They have an energetic Laurent expansion starting by the IR singularity $\omega^{-1}$ in tree level amplitudes, which we parametrize it as

$$a^\dagger_{z\bar{z}}(\omega, z, \bar{z}) = \sum_{k=0}^{\infty} \omega^{k-1} H_{1-k}(z, \bar{z}) . \quad (40)$$

With some abuse of notation, and following the lore of [73], we interpret the operators $H_{1-k}(z, \bar{z})$ as inserting states into 2d correlators rather than in the 4d S-Matrix. Equivalently, they can be recast as conformal primaries by introducing $G^\Delta_{\Lambda} = \int_0^{\infty} d\omega \omega^{\Delta-1} a^\dagger_{z\bar{z}}(\omega, z, \bar{z})$ so that the soft modes in (40) are extracted as $H_k = \text{Res}_{\Lambda=k} G^\Delta_{\Lambda}$ [74]. These are precisely the operators constructed in [24, 25, 74] at integer boost weight $\Delta = h + \bar{h} = 1 - k$, and we refer to those references for more details. Now, since for a graviton $h - \bar{h} = 2$ these states have

$$H_{1-k}(z, \bar{z}) : (h, \bar{h}) = \left( \frac{3 - k}{2}, -\frac{k + 1}{2} \right) . \quad (41)$$

We will consider split signature $(+ + - -)$ where $z$ and $\bar{z}$ are independent and we do not distinguish between incoming/outgoing operators (this amounts to an overall sign in the momenta). Accordingly, the 4d Lorentz group is realized as $SL(2, \mathbb{R})_z \times SL(2, \mathbb{R})_{\bar{z}}$ acting independently on $z, \bar{z}$ with weights $(h, \bar{h})$. Since $\bar{h} < 0$ for $H_k$ (recall
group transformation such that (40) and (44), multiplet. The only caveat is that by comparing this with (44) we immediately identify the Penrose transform light-transform in coordinates |" under this transformation and becomes, combining (40) and (44),

\[
H_{4-2p}(z, \bar{z}) = \frac{1}{(2p-2)!} \sum_{n=1-p}^{p-1} \binom{2p-2}{n-1-p} w_n^p(z) \bar{z}^{p-1-n},
\]

where \(w_n^p(z)\) each have \(SL(2, \mathbb{R})_z\) weight \(3-p\), as dictated by (41). On the other hand, it was found in [25] that for fixed \(p\) the \(2p-1\) states can be nicely assembled into a \(SL(2, \mathbb{R})_z\) multiplet of spin \(p\). Now, a very natural spin-\(p\) representation of \(SL(2, \mathbb{R})\) is given by totally symmetric rank \(2p\) \(2\)-tensors (see e.g. [75]), so this suggests we can identify

\[
w_n^p = W^{2p-2}_{\alpha_1 \cdots \alpha_{2p-2}}(\omega, z, \bar{z})\lambda^{\alpha_1} \cdots \hat{\lambda}^{\alpha_{2p-2}}.
\]

i.e. as the independent components of the \(SL(2, \mathbb{R})\) tensor \(W^{2p-2}_{\alpha_1 \cdots \alpha_{2p-2}}\). Moreover, introducing the spinor \(|\hat{\lambda}| = (1, \bar{z})\) we see that the combinatorial factor of (18) gets reabsorbed and the sum is simply a contraction:

\[
H_{4-2p=1-k}(z, \bar{z}) = \frac{1}{(2p-2)!} W^{2p-2}_{\alpha_1 \cdots \alpha_{2p-2}}(\omega, z, \bar{z})\lambda^{\alpha_1} \cdots \hat{\lambda}^{\alpha_{2p-2}}.
\]

Since \(W^{2p-2}\) carry \(SL(2, \mathbb{R})_z\) indices, we have made “almost” manifest the transformation properties of the \(w_n^p\) multiplet. The only caveat is that \(|\hat{\lambda}| = (1, \bar{z})\) is an inhomogeneous spinor and picks up a Jacobian factor when transforming under \(SL(2, \mathbb{R})_z\). This is remedied easily by attaching the energy factor, i.e. we can perform a little group transformation such that

\[
|\eta\rangle = \sqrt{\omega(1, z)} |\lambda\rangle = \sqrt{\omega(1, \bar{z})} \rightarrow |\eta\rangle = (1, z), |\lambda\rangle = \omega(1, \bar{z}) = \omega|\hat{\lambda}|.
\]

Since the graviton \(a_{+2}^+(\omega, z, \bar{z})\) has helicity +2 it picks up a factor \(\omega^2\) under this transformation and becomes, combining (40) and (44),

\[
G^\lambda = \omega^2 a_{+2}^+(\omega, z, \bar{z}) = \sum_{k=0}^{\infty} \omega^k H_{1-k}(z, \bar{z}) = \sum_{k=0}^{\infty} \omega^k H_{1-k}(z, \bar{z}) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} W^{k+1}_{\alpha_1 \cdots \alpha_{k+1}}(z) \lambda^{\alpha_1} \cdots \hat{\lambda}^{\alpha_{k+1}} = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} W^{k+1}_{\alpha_1 \cdots \alpha_{k+1}}(z) \lambda^{\alpha_1} \cdots \lambda^{\alpha_{k+1}}.
\]

which is nothing but the expansion induced from vertex operators (22), as long as we include \(W^0 = 1\) as a central term. A number of comments are in order. First, we see from the third line that this is a soft expansion in the (anti)holomorphic coordinate, namely in \(\lambda = \omega \hat{\lambda}(\bar{z})\). But this is precisely the holomorphic soft expansion that was used in [53] to derive the subleading soft theorems! Here it emerges naturally to all orders in \(\omega\) and we shall pursue further applications in [72]. Second, considering \(SL(2, \mathbb{R})\) tensors, rather than modes \(w_n^p\), immediately bypasses the need to operate with the light-transform [43] of conformal primaries to obtain the \(w_{1, \infty}\) algebra.

Why did the light-transform play a predominant role in previous discussions of \(w_{1, \infty}\) [25, 43]? This can be easily understood by resorting to the homogeneous formalism used in [72] (see also [76]). In that reference the inverse light-transform in coordinates \(|\lambda\rangle = (1, \bar{z}), |\eta\rangle = (1, z)|\) is given by

\[
H_{4-2p}(z, \eta) = \frac{1}{\Gamma(2p-1)} \oint \frac{d\lambda d\bar{\lambda}}{2\pi i} |\lambda\rangle^{2p-2} |\bar{\lambda}\rangle^{2p-2} L[H_{4-2p}](z, \lambda), \ p > 0,
\]

By comparing this with (44) we immediately identify the Penrose transform

\[
W^{2p-2}_{\alpha_1 \cdots \alpha_{2p-2}} = \oint \frac{d\lambda d\bar{\lambda}}{2\pi i} \lambda^{\alpha_1} \cdots \bar{\lambda}^{\alpha_{2p-2}} L[H_{4-2p}](z, \lambda).
\]
Further using the identification (43), using $[\lambda d\lambda] = d\bar{z}$, this is

$$w_n^p(z) = \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{p-n-1} \bar{z}[H_{2p-2}](z, \bar{z}),$$

(49)

Hence $w_n^p(z)$ are nothing but the modes of the light-transform $L[H_{2p-2}]$. Indeed, our representation landed us straight into the so-called wedge of the $w_{1+\infty}$ algebra derived in [33, 43]

$$w_n^p(z_1)w_m^q(z_2) \sim 2 \frac{m(p-1) - n(q-1)}{\bar{z}_{12}} w_{m+n}^{p+q-2}(z_2), \quad |m| \leq p - 1,$$

which, by virtue of (44), can be shown to be equivalent to the $SL(2, \mathbb{R})$ covariant form (24). In particular we realize that $L_n = \frac{i}{2} w_n^0(0)$ satisfy the $SL(2, \mathbb{R})$ algebra and act as

$$[L_n, w_m^n] = (m - n(q - 1)) w_m^{n-1},$$

(50)

and so this defines a 2d energy-momentum tensor $\bar{T}(\bar{z})$ for the CFT, for after some work, it can be shown to be equivalent to

$$\bar{T}(\bar{z})G^{\lambda_2}(z_2) \sim \frac{\bar{h}}{\bar{z}_{12}} G^{\lambda_2}(z_2) + \frac{1}{\bar{z}_{12}} \partial_2 G^{\lambda_2}(z_2),$$

(51)

where $\bar{T}(\bar{z}) := \frac{1}{2} \sum_n \bar{w}_n^2(0)$ and $\bar{h} = -\frac{1}{2} \omega \partial_\omega$ is the helicity weight on $\lambda_2 = \omega \lambda$. As argued in [77], see also [78, 79], $\bar{T}$ is essentially the Shadow (or light) transform of $H_0(z, \bar{z}) = \frac{i}{2} \sum_n \left( \frac{z^2}{2} \right) w_n^2(z_2)$, for which (51) is equivalent to

$$H_0(z_1, \bar{z}_1)G^{\lambda_2}(z_2) \sim \frac{\bar{z}_{12}}{\bar{z}_{12}} \omega \partial_\omega + \bar{z}_{12} \partial_2 G^{\lambda_2}(z_2),$$

which in turn is nothing but (28) written in the coordinates $\lambda_2 = \omega \lambda(\bar{z})$.

**Relation to the PC system in the soft sector**

As a follow up of the preceding discussion it is worth commenting on the relation with the gravitational Goldstone operator $C(z, \bar{z})$ introduced in [41] for the leading soft mode. Denoting $P(z) = \partial H_1(z, \bar{z})$ that reference studied the system

$$P(z)C(w, \bar{w}) \sim \frac{i}{z - \bar{w}},$$

(52)

This then allowed to construct hard states in CCFT as the dressings

$$\mathcal{O}_h(\omega, z, \bar{z}) = e^{i\omega C(z, \bar{z})}\bar{\mathcal{O}}_h(\omega, z, \bar{z})$$

(53)

where $\bar{\mathcal{O}}_h$ does not interact with soft gravitons. By direct comparison, we can identify the PC algebra with our $\{\mu_+, \mu_-\}$ system (15). Indeed, from eqs. (44)-(23), we see that in our construction

$$H_1(z, \bar{z}) = \mu_+(z) + \bar{z} \mu_-(z) \Rightarrow P(z) = \mu_-(z).$$

(54)

On the other hand, we have also constructed hard gravitons (to all orders in energy) as (16), and thus we can identify

$$e^{i[\lambda \mu]} : \leftrightarrow e^{i\omega C(z, \bar{z})},$$

(55)

in the soft sector. Furthermore the leading soft-theorem derived there

$$P(z)e^{i\omega C(w, \bar{w})} \sim \frac{i\omega}{z - \bar{w}} e^{i\omega C(z, \bar{z})},$$

(56)

is nothing but (27) of the main text. We can see that identifying $C(z, \bar{z}) = \mu_+ + \bar{z} \mu_- = H_1$ is consistent with (52), (55) and (56), but also generalizes the Goldstone mode to all the orders in the soft expansion, i.e. not only to $H_1$ but for all $H_{1-k}, k \geq 1$. Dressed states of the form $O_h^\lambda(z) = e^{i[\lambda \mu]}\bar{O}_h(z)$ satisfy the tower of soft theorems (27)-(29), but may not be the most general states to do so. The difference with the framework of [41] relies mainly in the level terms $\langle CC\rangle$ which are fixed from the IR divergence. The identification $C = H_1$ done here precludes us from incorporating them and thus may be corrected at loop level. It would be fascinating to work out this connection.
II. Negative helicity states and one-minus amplitudes

Here we introduce negative helicity states that pair to $G^\lambda(z)$ through two-point functions. Indeed, to compute non-trivial correlations explicitly we need to specify the identity terms in the OPE. First, we can read off the color 2-point structure in momentum space from the $U(N)$ matrices in (13). We get

$$d^{\lambda_1, \lambda_2} := \frac{1}{N(c^2)} \text{Tr}(G^{\lambda_1} G^{\lambda_2}) = \delta^2(\lambda_1 + \lambda_2),$$

which is luckily diagonal as the usual CFT inner product. However, this does not appear as a singularity in the $G^\lambda(z) G^\lambda(z)$ OPE (17), in turn leading to their interpretation as zero-level Kac-Moody currents. This is consistent at tree-level, since a vanishing level leads to vanishing of correlation functions involving only $G^\lambda(z)$ operators, which should correspond to all-plus amplitudes, $\langle \lambda \cdots \lambda \rangle = 0$.

To obtain amplitudes with insertion of negative helicity gravitons, we can include a field $\Phi^\lambda$ with chiral OPE

$$G^{\lambda_1} \Phi^{\lambda_2} \sim \delta(z_{12}) d^{\lambda_1, \lambda_2} + \oint \frac{d\lambda_3 J_{\lambda_1, \lambda_2} \Phi^{\lambda_3}}{z_{12}},$$

where we recall the second term follows from universal colinear singularities and thus holds for generic helicities. The first term is a contact interaction which requires the theory to be defined in Lorentzian 2d signature (i.e. split 4d signature). Since in this picture $G^\lambda(z)$ has weight one, it also entails that $\Phi^\lambda(z)$ must transform as a scalar under $SL(2, \mathbb{R})$. From the Kac-Moody perspective the emergence of such a colored scalar is natural: One can consider a level term between the $(1,0)$ current $J^a_\lambda$ and the Shadow of the $(0,1)$ current $J^b_\lambda$, namely $J^a_\lambda S[J^b_\lambda] \sim 1/z_{12}^a d^{a,b}$ [80]. This is equivalent to $J^a_\lambda J^b_\lambda = \delta^2(z_{12}) d^{a,b}$, or to $J^a_\lambda \Phi^b = \delta(z_{12}) d^{a,b}$ in Lorentzian signature, if we identify $\Phi$ with the light-transform of $J$.

Let us further justify the contact term from the S-Matrix perspective. Negative helicity gravitons are canonically conjugate to positive ones. In the notation of the previous Appendix, the disconnected S-Matrix is given by the two-point function

$$\langle a_{-2}(w_2, z_2, \bar{z}_2) a_{+2}(w_1, z_1, \bar{z}_1) \rangle = \frac{\delta(\omega_1 + \omega_2)}{\omega_1} \delta(z_1 - z_2) \delta(\bar{z}_1 - \bar{z}_2),$$

up to an irrelevant normalization. Recall now we introduce homogeneous coordinates via $\lambda = \omega \lambda(\bar{z})$. Since $a_{-2}$ has 4d helicity $-2$ under the transformation (45), the appropriate operator becomes $\Phi^\lambda(z) = \omega^2 a_{-2}(\omega, z, \bar{z})$, c.f. (46). All together, the $\omega^2$ Jacobian cancels out in (59) and so we obtain

$$\langle G^{\lambda_1}(z_1) \Phi^{\lambda_2}(z_2) \rangle = \delta(\omega_1 + \omega_2) \delta(z_1 - z_2) \delta(\omega_1 \bar{z}_1 + \omega_2 \bar{z}_2)$$

$$= \delta(z_{12}) \delta^2(\lambda_1 + \lambda_2) = \delta(z_{12}) d^{\lambda_1, \lambda_2},$$

precisely as anticipated. This thus manifests the color structure at the level of two-point functions. The next task is to compute the anti-MHV amplitude $\langle GG \Phi \rangle$, which is slightly different from the usual computation due to the contact terms. Consider the limits $z_{12} \to 0$ and $z_{23} \to 0$. Using the GG OPE (17) together with (58) we find, respectively,

$$\langle G^{\lambda_1}(z_1) G^{\lambda_2}(z_2) \Phi^{\lambda_3}(z_3) \rangle \sim \frac{[12]}{z_{12}} \delta(z_{23}) \delta^2(\lambda_1 + \lambda_2 + \lambda_3)$$

$$\sim \frac{[23]}{z_{23}} \delta(z_{12}) \delta^2(\lambda_1 + \lambda_2 + \lambda_3).$$

The two limits are compatible if we identify $\frac{1}{2} \leftrightarrow 2\pi i \delta(z)$. This would render the function $\langle G^{\lambda_1} G^{\lambda_2} \Phi^{\lambda_3} \rangle$ to be genuinely holomorphic, while entailing that it is only defined on the support of a certain contour integral. Indeed, in the absence of other kinematic singularities, the two-dimensional delta function $\delta(z_{12})\delta(z_{23})$ is the natural contour prescription for $z_1, z_2$. After integration we get

$$\oint \frac{dz_1 dz_2}{2\pi i} \langle G^{\lambda_1}(z_1) G^{\lambda_2}(z_2) \Phi^{\lambda_3}(z_3) \rangle = \frac{[12]}{\pi} \delta^2(\lambda_1 + \lambda_2 + \lambda_3)$$

$$= f^{\lambda_1, \lambda_2, \lambda_3}.$$
(The antisymmetry of \( f^{\lambda_1,\lambda_2,\lambda_3} \) is reflected in the contour prescription, e.g. \( \oint \frac{dz_1 dz_2}{2\pi i} (\cdots) = -\oint \frac{dz_1 dz_2}{2\pi i} (\cdots) \), etc.). For real-valued \( z_i \), corresponding to the 2d Lorentzian slice, we have

\[
\frac{1}{2\pi i} \langle \mathbf{G}^{\lambda_1}(z_1) \mathbf{G}^{\lambda_2}(z_2) \Phi^{\lambda_3}(z_3) \rangle = f^{\lambda_1,\lambda_2,\lambda_3} \delta(z_{23}) \delta(z_{12})
\]

\[
= [12] \delta(z_{23}) \delta(z_{12}) \delta(\lambda_1 + \lambda_2 + \lambda_3)
\]

\[
= \left( \frac{[12][3]}{[23]} \right)^2 \delta^2(\lambda_1 + \lambda_2 + \lambda_3) \delta(z_1 \lambda_1 + z_2 \lambda_2 + z_3 \lambda_3)
\]

\[
= A_3(p_i) \delta^3(\sum_{i=1}^{3} p_i),
\]

which is the desired 3-point dressed S-Matrix. Note that momentum conservation has emerged from both the color and the holomorphic structure.

We now aim to study \( \langle \mathbf{G}^{\lambda_1} \mathbf{G}^{\lambda_2} \mathbf{G}^{\lambda_3} \Phi^{\lambda_4} \rangle \). Again, we consider first different OPE limits. Starting with \( z_{12} \to 0 \) we obtain

\[
\langle \mathbf{G}^{\lambda_1}(z_1) \mathbf{G}^{\lambda_2}(z_2) \mathbf{G}^{\lambda_3}(z_3) \Phi^{\lambda_4}(z_4) \rangle \to \frac{[12]}{z_{12}} \langle \mathbf{G}^{\lambda_1+\lambda_2}(z_2) \mathbf{G}^{\lambda_3}(z_3) \Phi^{\lambda_4}(z_4) \rangle.
\]

Consider first the Lorentzian case \( z_i \in \mathbb{R} \). From (63) the three-point function has support only on \( z_2 = z_3 = z_4 \). Thus the OPE limit \( z_1 \to z_2 \) implies that all four punctures coincide, which is a very singular configuration from the point of view of 4d kinematics. For generic kinematics (subjected to momentum conservation) this singularity vanishes. Repeating the argument for \( z_{13}, z_{14} \to 0 \) we see that the full correlation function has no singularities for generic kinematics and hence must vanish (a purely contact term is not allowed by conformal symmetry). In fact, this statement is already made in the original twistor construction of SYM amplitudes [52]! In that case the one-minus amplitude \( A_4(1^+2^+3^+4^-) \) only has support if the corresponding real twistor variables collapse to a single point, and hence vanishes generically.

On the other hand, we have proposed here to interpret the holomorphic momentum conservation as a two-dimensional contour prescription. Unlike the 3-point case however, the function \( \mathbf{G}\mathbf{G}\Phi \) carries the additional singularities of the type (64) and so it is natural to integrate it against a three-dimensional contour. In fact, since the overall \( SL(2,\mathbb{R}) \) weight is 3, the natural contour is

\[
\oint \frac{dz_1 dz_2 dz_3}{2\pi i} \langle \mathbf{G}^{\lambda_1}(z_1) \mathbf{G}^{\lambda_2}(z_2) \mathbf{G}^{\lambda_3}(z_3) \Phi^{\lambda_4}(z_4) \rangle
\]

\[
= f^{\lambda_1,\lambda_2} f^{\lambda_3,\lambda_4} + f^{\lambda_1,\lambda_3} f^{\lambda_4,\lambda_2} + f^{\lambda_2,\lambda_3} f^{\lambda_1,\lambda_4},
\]

which extends the 3-point statement (62). Here we have employed the form of the singularities (64) together with the double residue (62). This combination vanishes due to associativity (20) and can be interpreted as a residue theorem (see e.g. [81] and references within for evaluation of multidimensional residues). This can be thought as the momentum space version of the argument given in [82], where the same residues where analyzed in the moduli space. This is also precisely the condition found recently in momentum space in [45], which also extended the analysis to helicity-flipping terms. For completeness we present here the OPE for general helicity fields (the momentum-space version of the OPEs in [43])

\[
O^{\lambda_1}(z_1)O^{\lambda_2}(z_2) \sim \sum_h \kappa^{h_1 h_2}_h [12]^{h_1+h_2-h-1} z_{12}^{h_1+h_2} O^{\lambda_1+\lambda_2}_h,
\]

where imposing the vanishing of the analog of (65), or equivalently OPE associativity, leads to the constraints in \( \kappa^{h_1 h_2}_h \) presented in [44, 45]. It would be interesting to find a kinematic Lie algebra realization in the cases where \( h_2 \neq h \), which a priori do not correspond to adjoint \( U(\infty) \) representations.