Time-of-arrival probabilities and quantum measurements: II Application to tunneling times

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Abstract
We formulate quantum tunneling as a time-of-arrival problem: we determine the detection probability for particles passing through a barrier at a detector located a distance $L$ from the tunneling region. For this purpose, we use a Positive-Operator-Valued-Measure (POVM) for the time-of-arrival determined in [1]. This only depends on the initial state, the Hamiltonian and the location of the detector. The POVM above provides a well-defined probability density and an unambiguous interpretation of all quantities involved. We demonstrate that for a class of localized initial states, the detection probability allows for an identification of tunneling time with the classic phase time. We also establish limits to the definability of tunneling time.

We then generalize these results to a sequential measurement set-up: the phase space properties of the particles are determined by an unsharp sampling before their attempt to cross the barrier. For such measurements the tunneling time is defined as a genuine observable. This allows us to construct a probability distribution for its values that is definable for all initial states and potentials. We also identify a regime, in which these probabilities correspond to a tunneling-time operator.

1 Introduction
This paper is a continuation of Ref ([1]), in which a procedure was sketched for the construction of a Positive-Operator-Valued Measure (POVM) for the time-of-arrival for a particle described by a Hamiltonian $\hat{H}$. Here, we extend this POVM to cover the case of particles tunneling through a barrier. This procedure

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allows us to provide an unambiguous determination for the tunneling-time as it can be measured in time-of-arrival type of measurements.

Quantum tunneling refers to the escape of a particle from a region through a potential barrier, whose peak corresponds to an energy higher than that carried by the particles. There are two important questions (relevant to experiments) that can be asked in this regard. The first is, how long does it take a particle to cross the barrier (i.e. what is the tunneling time?). The second is, what is the law that determines the rate of the particle’s escape through the barrier? In this paper, we develop a formalism that provides an answer to these questions and we apply it to the first one. The issue of the decay probability will be taken up in [2].

The issue of tunneling time has received substantial attention in the literature, especially after the 1980’s—see the reviews [3, 4]. The reason is that there is an abundance of candidates and a diversity of viewpoints with no clear consensus. There are roughly three classes of approaches: (i) Wave packet methods: one follows the particle’s wave packet across the barrier and determines the tunneling time through a “delay in propagation” [5, 6], (ii) one defines suitable variables for the particle’s paths and one obtains a probability distribution (or an average) for the transversal time corresponding to each path. These paths can be constructed either through path-integral methods [4, 7, 8], through Bohmian mechanics [9], or through Wigner functions [10], and (iii) the use of an observable for time: this can take the form of an additional variable playing the role of a clock [11, 12], or of a formal time operator [13]. In general, these methods lead to inequivalent definitions and values for the tunneling time.

1.1 Our approach

The basic feature of our approach to this problem is its operational character. We identify the tunneling time by constructing probabilities for the outcome of specific measurements. We assume that the quantum system is prepared in an initial state \( \psi(0) \), which is localized in a region on one side of a potential barrier that extends in a microscopic region. At the other side of the barrier and a macroscopic distance \( L \) away from it, we place a particle detector, which records the arrival of particles. Using an external clock to keep track of the time \( t \) for the recorder’s clicks, we construct a probability distribution \( p(t) \) for the time of arrival. The fact that the detector is a classical macroscopic object and that it lies at a macroscopic distance away from the barrier allows one to state (using classical language) that the detected particles must have passed through the barrier (quantum effects like a particle crossing the barrier and then backtracking are negligible). Hence, at the observational level, the probability \( p(t) \) contains all information about the temporal behavior for the ensemble of particles: all probabilistic quantities referring to tunneling can be reconstructed from it.

With the considerations above, both problems of determining the tunneling
time and the escape probability as a function of time (see [2]) are mapped to the
single problem of determining the time-of-arrival at the detector’s location for
an ensemble of particles described by the wave function \( \psi_0 \) at \( t = 0 \) and evolving
under a Hamiltonian with a potential term. To solve this problem, we elaborate
on the result of [1], namely the construction of a Positive Operator Valued
Measure (POVM) for the time-of-arrival for particles for a generic Hamiltonian
\( \hat{H} \)–see [14] and [15] for definition, properties and interpretation of POVMs. This
POVM provides a unique determination of the probability distribution \( p(t) \) for
the time-of-arrival. It is important to emphasize that by construction \( p(t) \) is
linear with respect to the initial density matrix, positive-definite, normalized
(when the alternative of non-detection is also taken into account) and a genuine
density with respect to time.

Since our results depend on the POVM for the time-of-arrival constructed in
[1], we review here the basic physical considerations involved in its construction.
The technical aspect, namely the construction of this POVM for the problem
at hand is undertaken in Sec. 2.

The POVM of [1] involves no structures other than the ones of standard
quantum mechanics: the Hamiltonian, the initial state and the location of the
recording device. It also involves a smearing function with respect to time,
but we employ it in the regime in which the results are independent of such a
choice. The first step in the derivation arises from the remark that the notion
of arrival-time is well-defined when one considers histories for a physical system
(both in classical and in quantum probability). We assume that the detector
lies at \( x = L \) and that the initial state is localized in the region \( I = \{ x, x < L \} \).
Moreover, we assume a discretization \( t_0, t_1, t_2, \ldots, t_n \) of a time interval \([0, T]\).
One asks at any instant \( t_i \) of time, whether the particle lies in region \( I \) or in
region \( II = \{ x, x > L \} \). The set of all possible successive alternatives forms a
Boolean algebra. The key point is that one can construct a subalgebra of events
labeled by the time of first crossing (together with the event of no crossing),
namely by the first instant \( t_i \) that the particle is found in region \( II \). This
implies that propositions about the time-of-arrival have a well-defined algebraic
structure, which is compatible with the Hilbert space description of quantum
mechanics. The algebra of propositions for the time-of-arrival is a special case of
the so-called spacetime coarse-grainings [16, 17] that have been studied within
the consistent histories approach to quantum mechanics [18, 19, 20, 21].

The construction above takes place at the discrete-time level. One should
then implement the continuum limit within the quantum mechanical formalism.
The problem is that there is no proper continuous limit if one works at the level
of probabilities (for the same mathematical reason that leads to the quantum
Zeno effect). However, there is a proper continuum limit for this algebra if
one works at the level of amplitudes. More specifically, one can implement the
continuous limit at the level of the decoherence functional, an object introduced
in the consistent histories approach.

The decoherence functional is a hermitian, bilinear functional on the space
of histories that contains all probability and phase information for the histories
of the system. The restriction of the decoherence functional to the algebra of propositions about the time-of-arrival effectively yields a hermitian function $\rho(t,t')$ which is a density with respect to both of its arguments.

The decoherence functional contains sufficient information for the construction of POVMs for measurements that involve variables that refer to more than one instant of time. This has been established for sequential measurements \[24\] and for time-extended measurements \[25\]. In these cases one can compare the results to ones obtained from single-time quantum mechanics, but for the time-of-arrival, there is no analogous construction without the use of histories. Nonetheless, the method provides a definition of POVM for the time-of-arrival through a suitable smearing of the diagonal elements of the decoherence functional. For a free particle, this reproduces Kijowski’s POVM \[26\] in the appropriate regime.

The important point in the procedure above is that the POVM of \[1\] is valid for a generic Hamiltonian. The time parameter entering the POVM is the external Newtonian time and the identification of the time-of-arrival is done through purely kinematical arguments. Hence, this result can also be applied to the specific Hamiltonian operators that are relevant to tunneling. This is the content of Sec. 2.

Summarizing, there are three basic features in our approach: a) the reformulation of tunneling as a time-of-arrival problem, b) the use of POVMs for the determination of the probabilities for the tunneling particles, and c) the basic ideas of the histories approach that enable us to construct a suitable POVM.

1.2 Relation to other approaches

There are some common points and some points of divergence with previous work on the tunneling time issue. Yamada has employed the decoherence functional showing that different definitions of tunneling time correspond to different definitions of the alternatives for the ‘paths’ considered in the definition \[27\]. The construction of the decoherence functional is different from ours in one respect: the (coarse-grained) histories we consider refer to the paths’ first crossing of the surface $x=L$, which lies a macroscopic distance away from the barrier. In \[27\], the histories refer to the crossing of the barrier and the ambiguity in the definition of the tunneling time reflects the inability to decide which of all possible spacetime coarse-grainings provides the true measure of tunneling time. This is due to the fact that quantum ‘mechanical’ paths may cross and then reenter the barrier region. In our case, this is not an issue. The detector is far away from the barrier region (at a macroscopic distance $L$) and the probability that a particle crossing $x=L$ would ever backtrack to the barrier is practically zero. Another difference is that Yamada argues within the context of the decoherent histories programme that deals with closed systems \[28\]. While we employ the methods and (many) conceptual tools of consistent histories, our approach is

\footnote{Alternatively, it can be viewed as a generating functional for all possible temporal correlation functions of the system \[22\ \[23\].}
strictly operational within the Copenhagen interpretation. The probabilities we construct refer to measurement outcomes in a statistical ensemble. The decoherence functional is only used as a mathematical object that allows us to construct a POVM and the particle crossing of the surface \( x = L \) is viewed as corresponding to an (irreversible) act of measurement by a device located there.

The fact that the measurement of the particle takes place far away from the barrier region suggests that our results should be compatible with the asymptotic analysis of wave packets. Indeed, as we shall see, our expression for tunneling time (whenever this can be defined) corresponds to the classic Bohm-Wigner phase time \[5\]. However, the methodology is different: we do not identify time through the peak \( x(t) \) of the wave-packet (or through its center-of-mass), but the probability distribution for the detection time is obtained from a POVM that is defined for all possible initial states. Unlike time of detection, a sharp definition for the tunneling time is only possible for initial states characterized by a strong peak in their momentum distribution. However, the generality of our construction allows us to fully specify the limits in the definability of tunneling time.

From the technical point of view, our approach has more in common with the second class of proposals we mentioned in the beginning: time being identified at the kinematical level from the properties of ‘paths’. In particular, the formalism bears substantial resemblance to the Feynman path integral derivation of tunneling times of Sokolovski and Baskin \[7\]. However, our boundary conditions are different, and more importantly the probabilities we obtain arise from proper probability densities with respect to time. While the time-averaged quantities in \[7\] are linear with respect to a restricted propagator, such propagators appear in a quadratic form in our expression for the probability. It was argued extensively in \[1\] that this is necessary, in order to obtain a genuine probability density in a way that respects the convexity of the space of quantum states. The present construction also shares these properties and this implies that the issue of complex tunneling times does not arise.

### 1.3 Our results

In Sec.3, we apply the POVM we constructed in Sec. 2 to a simple case of a particle in one-dimension. We consider a potential barrier \( V(x) \), which takes non-zero values only in a bounded (microscopic) region of width \( d \) around \( x = 0 \). A wave packet approaches the barrier from the negative real axis, while the detection of the particle takes place at \( x = L >> a \).

We assume that the initial wave -function is well localized in position and in momentum (e.g. a coherent state). In addition, we require that \( \sigma/k_0 << 1 \), where \( k_0 \) the mean momentum and \( \sigma \) the momentum spread of the initial state. We then find (for a rather general regime for the values of the parameters characterizing the system) that the probability distribution for the time of arrival is sharply peaked around a value \( t_m \).

From this probability distribution, we identify the delay due to the presence of the barrier as the difference between the time \( t_m \) and the time it would
take a classical particle of momentum \( k_0 \) to travel from the center of the initial wavepacket to the location of the detector. This ‘delay time’ equals

\[
  t_d = \frac{M}{k_0} \left( \frac{\partial T_k}{\partial k} \right)_{k=k_0},
\]

where \( T_k \) is the transmission amplitude corresponding to the potential \( V(x) \). The delay time may be negative: the tunneling time is obtained as the sum of \( t_d \) with the time it would take the particle to cross the barrier: it coincides with the classic phase time.

Note that the delay and tunneling times defined this way are not observables of the system: they cannot be defined for a generic initial state, but only for states well localized in momentum and for values of the parameters that lead to a probability distribution \( p(t) \) characterized by a sharp peak. In this case, one can use classical arguments for their definition. However, if either the initial distribution has a substantial momentum spread, or if \( p(t) \) exhibits a more complex structure, there is no unambiguous way to define tunneling time. While the value for the time-of-arrival is a genuine observable (a random variable on the sample space of the POVM), the tunneling time as we define it here requires the knowledge of the corresponding time-of-arrival for a free particle: and this cannot be defined, unless the initial value of momentum is known.

Hence, in this approach the tunneling time is a parameter of the detection probability. It can only be identified for specific initial states, and not for any state, because its definition involves a correspondence argument to classical physics.

In Sec. 5, we propose a generalization of the results above that leads to a definition of the tunneling time as a genuine random variable. The idea is to consider a sequential measurement set-up: the phase space properties of the particle are determined through an unsharp phase space sampling before this attempts to cross the barrier, and then the time-of-arrival for the particles that crossed the barrier is measured. The sample space corresponding to such sequential measurements accommodates the definition of the tunneling time as a genuine quantum observable and it allows us to construct a marginal POVM that provides its probabilities for a generic initial state. In a specific regime, this POVM becomes independent of the details of the first measurement: as such it defines an ideal probability distribution for the delay and the tunneling times: this distribution suggests a definition for a delay-time and for a tunneling-time operator.

\[\text{Strictly speaking, the above definition of tunneling time involves counterfactual reasoning. However, in the operational setting we consider here this is not a problem, as long as we keep in mind that the tunneling time (whenever it can be defined) is a ‘property’ of the ensemble of detected particles and not of any individual one.}\]
2 The general probability measure

In this section, we review the construction of the POVM for the time-of-arrival in [1], and we extend it for the case relevant to tunneling.

2.1 The histories formalism

The POVM of [1] is constructed using some notions of quantum mechanical histories, as they appear in the consistent histories approach to quantum theory of Griffiths [18], Omnes [19], Gell-Mann and Hartle [20, 21]. We should note however that these objects are used in the present context differently, namely in an operational approach to quantum theory–see [23, 24]

A history intuitively corresponds to a sequence of properties of the physical system at successive instants of time. A discrete-time history $\alpha$ is then represented by a string $\hat{P}_{t_1}, \hat{P}_{t_2}, \ldots, \hat{P}_{t_n}$ of projectors, each labeled by an instant of time. From them, one can construct the class operator

$$\hat{C}_\alpha = \hat{U}^\dagger(t_1)\hat{P}_{t_1}\hat{U}(t_1)\ldots\hat{U}^\dagger(t_n)\hat{P}_{t_n}\hat{U}(t_n) \quad (2.1)$$

where $\hat{U}(s) = e^{-i\hat{H}s}$ is the time-evolution operator. A candidate probability for the realisation of this history is

$$p(\alpha) = Tr\left(\hat{C}_\alpha^\dagger\hat{\rho}_0\hat{C}_\alpha\right), \quad (2.2)$$

where $\hat{\rho}_0$ is the density matrix describing the system at time $t = 0$.

However, the expression above does not define a probability measure in the space of all histories, because the Kolmogorov additivity condition cannot be satisfied: if $\alpha$ and $\beta$ are exclusive histories, and $\alpha \lor \beta$ denotes their conjunction as propositions, then it is not true that

$$p(\alpha \lor \beta) = p(\alpha) + p(\beta). \quad (2.3)$$

The histories formulation of quantum mechanics does not, therefore, enjoy the status of a genuine probability theory on the space of all histories.

However, an additive probability measure is definable, when we restrict to particular sets of histories. These are called consistent sets. They are more conveniently defined through the introduction of a new object: the decoherence functional. This is a complex-valued function of a pair of histories given by

$$d(\alpha, \beta) = Tr\left(\hat{C}_\beta^\dagger\hat{\rho}_0\hat{C}_\alpha\right). \quad (2.4)$$

A set of exclusive and exhaustive alternatives is called consistent, if for all pairs of different histories $\alpha$ and $\beta$, we have $\text{Re} \ d(\alpha, \beta) = 0$. In this case, one can use equation (2.2) to assign a probability measure to this set.
2.2 Time-of-arrival histories

Using the histories formalism we construct a decoherence functional for time-of-arrival histories with $N$ time steps $t_1, t_2, \ldots, t_N$ (discrete-time). The reason for this construction is that the decoherence functional has a good continuous time limit (unlike the probabilities for histories).

We consider a particle in one dimension for concreteness, even though the results obtained here only use abstract Hilbert space operators and hold more generally. We split the line into the interval $(-\infty, L]$ and the interval $[L, \infty)$. Let $\hat{P}_-$ and $\hat{P}_+$ be the corresponding projectors. Our aim is to identify histories that correspond to the statement that the particle crossed from the $-$ region to the $+$ region during a particular time step. If we assume that at $t = 0$ the particle lies at the $-$ region then it is easy to verify that the history

$$\alpha_m := (\hat{P}_-, t_1; \hat{P}_-, t_2; \ldots, \hat{P}_-, t_m; \hat{P}_+, t_{m+1}; 1, t_{m+2}; \ldots, 1, t_N)$$

(2.5)

corresponds to the proposition that the particle crossed $x = L$ for the first time between the $m$-th and the $m + 1$-th time step. The sequence $\tilde{\alpha} = (\hat{P}_-, t_1; \hat{P}_-, t_2; \ldots, \hat{P}_-, t_m; \ldots, \hat{P}_-, t_N)$ corresponds to the proposition that the particle did not cross $x = L$ within the $n$-time steps.

The set of histories $\alpha_m$ together with $\tilde{\alpha}$ is exhaustive and exclusive (a sublattice of the lattice of history propositions) – see also [16, 29]. The decoherence functional is then defined on this set of histories: it is a hermitian bilinear functional on a sample space consisting of the points $(t_1, \ldots, t_n)$ together with the point $N$ corresponding to no crossing

$$d(t_n, t_m) = d(\alpha_n, \alpha_m)$$

(2.6)
$$d(N, t_n) = d(\tilde{\alpha}, \alpha_n)$$

(2.7)
$$d(N, N) = d(\tilde{\alpha}, \tilde{\alpha}).$$

(2.8)

We next consider two discretisations $\{t_0 = 0, t_1, t_2, \ldots, t_N = T\}$ and $\{t'_0 = 0, t'_1, t'_2, \ldots, t'_{N'} = T\}$ of the time interval $[0, T]$ with time-step $\delta t = T/N$, and $\delta t' = T/N'$. We construct the decoherence functional $d([t, t + \delta t], [t', t' + \delta t'])$, where $n = tN/T$ and $m = t'N'/T$. This reads

$$d([t, t + \delta t], [t', t' + \delta t']) = Tr \left( \rho_0 [e^{i\hat{H}\delta t'} \hat{P}_-]^n e^{i\hat{H}\delta t} \hat{P}_+ \right.$$  
$$\times e^{i\hat{H}(t' - t)} \hat{P}_+ e^{-i\hat{H}\delta t}[\hat{P}_- e^{-i\hat{H}\delta t}]^m \right).$$

(2.9)

We then take the limit $N, N' \to \infty$, while keeping $t$ and $t'$ fixed. Assuming that $\rho_0$ lies within the range of $\hat{P}_-$, i.e. $\hat{P}_- \rho_0 \hat{P}_- = \rho_0$ we obtain

$$d([t, t + \delta t], [t', t' + \delta t']) = \delta t \delta t' Tr \left( e^{i\hat{H}(t' - t)} \hat{P}_+ \hat{H} \hat{P}_- \hat{C}_t \rho_0 \hat{C}_t^\dagger \hat{P}_- \hat{P}_+ \right),$$

(2.10)

where $\hat{C}_t = \lim_{n \to \infty} (\hat{P}_- e^{-i\hat{H} t/n} \hat{P}_-)^n$. Writing

$$\rho(t, t') = Tr \left( e^{i\hat{H}(t' - t)} \hat{P}_+ \hat{H} \hat{P}_- \hat{C}_t \rho_0 \hat{C}_t^\dagger \hat{P}_- \hat{P}_+ \right)$$

(2.11)
we see that the decoherence functional defines a complex-valued density on $[0, T] \times [0, T]$. The additivity of the decoherence functional (which reflects the additivity of quantum mechanical amplitudes) allows us to obtain a continuum limit, something that could not be done if we worked at the level of probabilities.

2.3 The tunneling Hamiltonian

For the simple case of a particle at a line with Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2M} + V(\hat{x})$, where the potential $V(x)$ is bounded from below, we employ a result in [16, 30] that the restricted propagator $\hat{C}_t$ is obtained from the Hamiltonian $\hat{H}$ by imposing Dirichlet boundary conditions at $x = L$. If we also denote by $G_0(x, x'|t)$ the full propagator in the position basis (corresponding to $e^{-i\hat{H}t}$), we obtain

$$\rho(t, t') = \frac{1}{4M^2} \partial_x (\hat{C}_t \psi_0)^* (L) \partial_x (\hat{C}_t \psi_0) (L) G_0 (L, L|t-t') \quad (2.12)$$

where $\rho_0 = |\psi_0 \rangle \langle \psi_0 |$, with $\psi_0$ having support for $x < L$.

We now specialize to a case relevant for tunneling. We assume that the potential is short-range: it is significantly different from zero only in a region of width $d$ around $x = 0$. The distance $L$ is macroscopic, while $a$ is microscopic. This means that in the neighborhood of $x = L$, the propagator is effectively that of a free particle. Hence, we can substitute $G_0(L, L|t-t')$ in Eq. (2.12) with the corresponding expression for the free particle

$$G(L, L|t) = \left( \frac{M}{2\pi it} \right)^{1/2}. \quad (2.13)$$

The considerations above also specify the range of values for $L$ that are relevant to our problem. The first condition on $L$ is that the propagator may be substituted by that of the free particle, as in Eq. (2.13). The second is that $L$ is sufficiently far away from the tunneling region so that the probability amplitude of a particle backtracking to the barrier region from $L$ is practically zero. Physically one expects that this is the case for all initial states $\psi_0$ for which the position spread $\Delta q(t)$ remains at all times much smaller than $L$.

Clearly, with the considerations above it is not necessary that $L$ is a macroscopic distance in the literal sense of the word: the requirement that $L$ be macroscopic is a sufficient but not a necessary condition.

We next consider the Hamiltonian $\hat{H}_D$ that is obtained from the original Hamiltonian $\hat{H}$ by imposing Dirichlet boundary conditions at $x = L$. We distinguish two cases: (i) if $x$ takes value in the half-line, the spectrum of $\hat{H}_D$ is expected to be discrete; (ii) if $x$ takes values in the full real axis, at least the positive energy spectrum will be continuous. (We restrict to Hamiltonians having this property.) Either way, for $x >> a$, $V(x) = 0$ and the solution of the Schrödinger equation $\hat{H}_D \psi_E(x) = E \psi_E(x)$ with Dirichlet boundary conditions is proportional to $\sin k(L - x)$, where $k = (2ME)^{1/2}$. We choose to label the eigenstates of $\hat{H}_D$ by $k$, namely we write $|k\rangle_D$ as a solution to the equation

$$\hat{H}|k\rangle_D = \frac{k^2}{2M}|k\rangle_D. \quad (2.14)$$
with Dirichlet boundary conditions.

Normalizing $|k\rangle_D$ so that

$$D\langle k|k'\rangle_D = \delta(k,k'),$$  \hspace{1cm} (2.15)

(and similarly in the discrete-spectrum case) we write

$$\langle x|k\rangle_D = D_k \sin k(L-x),$$  \hspace{1cm} (2.16)

where the form of the normalization factor $D_k$ is specified the Hamiltonian’s (generalized) eigenstates.

For the study of tunneling, we assume that the initial state of the system has support only in the positive energy spectrum of $\hat{H}$. Hence,

$$\langle x|\hat{C}_t|\psi_0\rangle = \sum_k e^{-ik^2t/2M}D_k \sin k(L-x)c_k,$$  \hspace{1cm} (2.17)

where $c_k = D\langle k|\psi_0\rangle$ and $\sum_k$ denotes the integration with respect to the spectral measure of $\hat{H}_D$. Substituting into Eq. (2.12), we obtain

$$\rho(t,t') = \frac{1}{4M\sqrt{2\pi iM(t-t')}} \sum_{kk'} D_k D_{k'}^* c_k c_{k'}^* e^{-i\frac{\tau}{M}k^2t^2},$$  \hspace{1cm} (2.18)

2.4 Construction of the POVM

The decoherence functional contains sufficient information for the construction of POVMs for the probabilities of measurement outcomes for magnitudes that have an explicit time-dependence. In particular, the probabilities for the measurement outcomes for single-time, sequential and extended-in-time measurements (obtained through the standard formalism) can be identified with suitable diagonal elements of the decoherence functional—see [24, 25]. In other words, one can define POVMs by suitable smearing of the decoherence functional and in the cases above, these POVMs coincide with ones obtained from the standard methods in quantum measurement theory. In the case of the time-of-arrival there is no analogous expression obtained from standard methods. However, the smeared form of the decoherence functional still defines a POVM, and the main assumption in this paper is that this POVM yields the correct probabilities.

With this assumption, we obtain the following probability density for the time-interval $[0,T]$

$$p^\tau(t) = \int_0^T ds \int_0^T ds' \sqrt{f^\tau(t,s)} \sqrt{f^\tau(t,s')} \rho(s,s'),$$  \hspace{1cm} (2.19)

here $f_\tau(s,s')$ is a family of smeared delta functions $f_\tau(s,s')$ characterized by the parameter $\tau$. The functions $f_\tau$ satisfy the following property

$$\int_0^T ds f_\tau(s,s') = \chi_{[0,T]}(s'),$$  \hspace{1cm} (2.20)
where $\chi_{[0,T]}$ is the characteristic function of the interval $[0,T]$: $\chi_{[0,T]}(s) = 1$ if $s \in [0,T]$, and $\chi_{[0,T]}(s) = 0$ otherwise. The functions $f^\tau$ incorporate specific features of the instrument that records particles crossing the surface $x = L$.

Essentially, the key assumption in our approach (stated above) is that the functions $f^\tau$ appearing in the definition of (2.19) are analogous to the smearing functions that appear in the definition of POVMs for usual observables (i.e. ones other than the time of arrival). In [1], we showed that this assumption leads for the case of free particles to Kijowski’s POVM [26].

The decoherence functional satisfies an hermiticity condition
\[ \rho(s,s') = \rho^*(s',s), \]
which together with the positivity condition for its diagonal elements
\[ \int_a^b ds \int_a^b \rho(s,s') \geq 0 \] (2.21)
guarantees that $p^\tau(t)$ is positive-definite for all values of $t$.

The density (2.19) is linear with respect to the initial density matrix. Together with the probability of no-detection
\[ p^\tau(N) = 1 - \int_0^T ds p^\tau(s) \] (2.22)
they define a POVM $\hat{\Pi}$ on the space $[0,T] \cup \{N\}$. This POVM describes the time of detection of a particle by an instrument located at $x = L$.

In this paper, we will be interested in taking $T \to \infty$, i.e. taking $t \in [0,\infty)$. It is convenient to work with Gaussian smearing functions
\[ f^\tau(s,s') = \frac{1}{\sqrt{2\pi \tau}} e^{-\frac{(s-s')^2}{2\tau^2}}. \] (2.23)

However, these Gaussians are smeared delta-functions with respect to the whole real axis and not with respect to the time-interval $[0,\infty)$. To remedy this problem, we note that by Eq. (2.18), $\rho(-s,-s') = \rho^*(s,s') = \rho(s',s)$ and that the probability (2.19) is symmetric to an exchange of $s$ and $s'$. We also note that the contribution of terms that mix positive and negative $s$ are significant only at times $|t|$ of order $\tau$. Hence, the probability (2.19) with the Gaussian (2.23) is substituted in place of $f^\tau$ and the integration limits taken from $-\infty$ to $\infty$, is twice the probability density that is defined with an integral over the positive half-axis. Hence, the use of Gaussian smearing functions only involves dividing $p^\tau(t)$ in (2.19) by a factor of 2 (for times $t >> \tau$).

Inserting (2.23 into (2.19), we change variables to $u = \frac{1}{2}(s + s')$ and $v = s - s'$ noting that
\[ \sqrt{f^\tau(t,s)} \sqrt{f^\tau(t,s')} = f^\tau(u-t)e^{-\frac{u^2}{2\tau^2}}. \] (2.24)

We substitute in the integration $f^\tau(u-t)$ with a delta function $\delta(u-t)$. We then obtain
\[ p^\tau(t) = \frac{1}{8M \sqrt{2\pi M}} \sum_{kk'} D_k D_{k'} c_k c_{k'} e^{-\frac{i s^2}{2\tau^2}} R \left( \frac{k^2 + k'^2}{4M} \right), \] (2.25)
where

\[ R(\epsilon) = \int_{-\infty}^{\infty} dv \frac{e^{-v^2/2\tau^2 - iv}}{\sqrt{i\tau}} = 2\sqrt{\pi} \int_{0}^{\infty} dy \frac{e^{-y^2/2\tau^2} \left[ \cos(2\epsilon\tau y) + \sin(2\epsilon\tau y) \right]}{\sqrt{y}} \]  

(2.26)

At the limit of \( \epsilon \tau >> 1 \), i.e. if the detection time is much larger than \( \epsilon^{-1} \)

\[ \int_{0}^{\infty} dy \frac{e^{-y^2/2\tau^2} \left[ \cos(2\epsilon\tau y) + \sin(2\epsilon\tau y) \right]}{\sqrt{y}} \approx \sqrt{\frac{\pi}{\epsilon \tau}} \]  

(2.27)

Hence, \( R(\epsilon) = 2\sqrt{\pi/\epsilon} \). It follows that

\[ p(t) = \frac{1}{2\sqrt{2M}} \sum_{kk'} D_k^* c_k c_{k'} e^{-i(k^2 + k'^2) - i\frac{k'k}{2}} \]  

(2.28)

The probability for the time-of-arrival then becomes independent of the parameter \( \tau \), and it is expressed solely in terms of the system’s Hamiltonian, the initial state and the value of \( L \).

Eq. (2.28) is simplified if the spread \( \Delta k \) of the initial state \( |\psi_0\rangle \) (\( \hat{k} = \sqrt{2MH_D} \)) is much smaller than the corresponding mean value \( \bar{k} \): in this case, \( k^2 + k'^2 \approx 2kk' \), hence

\[ p(t) = \left| \sum_{k} D_k c_k \sqrt{\frac{k}{4M}} e^{-i\frac{k^2}{2M} t} \right|^2 \]  

(2.29)

It was shown in [1] that for the test case of a free particle (in which \( D_k = (2\pi)^{-1/2} \)) the probability distribution above reproduces the one of Kijowski [26].

3 The detection probability

In this section, we use the probability density (2.28) in a specific context that allows us to determine a magnitude that corresponds to the time the particle spends in the forbidden region. In effect, we identify tunneling-time by the delay caused by the presence of the barrier to the particles’ time-of-arrival (see Sec. 4). This turns out to be the same definition as the one employed in the methods involving the wave packet analysis. However, we do not identify any specific features of the wave-packet (these objects have no natural probabilistic or operational interpretation in quantum mechanics), but we work directly at the level of measurement outcomes, namely the probability distribution for the time-of-arrival.

We consider the simplest possible example of a particle tunneling through a potential barrier. We assume that the potential \( V(x) \geq 0 \) takes non-zero values in a region of width \( d \) around \( x = 0 \). Let \( V_0 \) be the maximum value of this potential, which is valid if the mean energy of the initial state is much larger than the energy uncertainty, and it is accurate for all times \( t \gg \tau \).
potential. In classical mechanics no particle with energy \( E < V_0 \) can cross the barrier, hence the probability of detection at \( x = L \) is zero at all times. We next consider this problem in quantum theory.

Eq. \( \text{(2.28)} \) involves the eigenstates of the Hamiltonian with Dirichlet boundary conditions at \( x = L \). Since \( x \in (-\infty, \infty) \), the spectrum of the Dirichlet Hamiltonian is continuous. The summation over \( k \) is then substituted by an integral \( \int_0^\infty dk \).

The first step is to construct the generalized eigenstates of the Hamiltonian with Dirichlet boundary conditions. To do so, we first study the solutions to the Schrödinger equation

\[
-\frac{1}{2M} \partial_x^2 u(x) + V(x)u(x) = \frac{k^2}{2M} u(x). \tag{3.1}
\]

There are two linearly independent solutions for each value of \( k \). It will necessary to construct an orthonormal basis of generalized eigenstates from these solutions. We pick one class of solutions \( u_k^\pm(x) \) that correspond to a particle propagating from \( -\infty \) and scattering on the potential

\[
u^+_k(x) = \begin{cases} A_k^+ \left( e^{ikx} + R_k^+ e^{-ikx} \right) & x < -d/2 \\ A_k^+ T_k^+ e^{ikx} & x > d/2 \end{cases}, \tag{3.2}
\]

where \( R_k^+ \) and \( T_k^+ \) is the reflection and transmission coefficient respectively, while \( A_k^+ \) is a normalization factor so that \( \int dx u_k^+(x) u_k^+(x) = \delta(k-k') \). Let \( u_k^- \) be a normalized linearly independent solution that satisfies \( \int dx u_k^-(x) u_k^-(x) = 0 \). Its form will be the following

\[
u^-_k(x) = \begin{cases} A_k^- \left( T_k^- e^{-ikx} + S_k e^{ikx} \right) & x < -d/2 \\ A_k^- \left( e^{-ikx} + R_k^- e^{ikx} \right) & x > d/2 \end{cases} \tag{3.3}
\]

Note that there is no reason for \( u_k^- \) to have a physical interpretation in terms of left-moving particles, and the labels \( T_k^-, R_k^- \) are chosen for convenience: they do not correspond to a transmission and reflection coefficient of any sort. We also note that the coefficients in \( u_k^+, u_k^- \) are not independent. For any two solutions \( \psi, \phi \) to the Schrödinger equation with the same energy, the Wronskian \( \psi' \phi - \phi' \psi \) must be \( x \)-independent. This yields the following conditions

\[
T_k^+ = T_k^- - S_k R_k^+ \tag{3.4}
\]

\[
S_k = T_k^- R_k^- + T_k^+ R_k^+ \tag{3.5}
\]

\[
|T_k^+|^2 + |R_k^-|^2 = 1 \tag{3.6}
\]

\[
|T_k^-|^2 + |R_k^-|^2 = 1 + |S_k|^2. \tag{3.7}
\]

To impose the Dirichlet boundary conditions on these solutions, we take a linear combination \( v_k(x) \) of \( u_k^+(x) \) and \( u_k^-(x) \) and require that \( v_k(L) = 0 \). This yields

\[
v_k(x) = C_k \left[ A_k^- (1 + R_k^- e^{2ikL}) u_k^+(x) - A_k^+ T_k^+ e^{2ikL} u_k^- (x) \right], \tag{3.8}
\]
where
\[
C_k = \frac{1}{\sqrt{|A_k^-|^2 + |R_k^- e^{2ikL}|^2 + |A_k^+|^2 |T_k^+|^2}} \tag{3.9}
\]
is a normalization constant chosen so that \( \int dx \bar{v}_k(x)v_k'(x)dx = \delta(k - k') \).

For \( x > d/2 \), we obtain
\[
v_k(x) = -2iC_k A_k^- A_k^+ T_k^+ e^{ikL} \sin k(L - x). \tag{3.10}
\]
Hence,
\[
D_k = -2iC_k A_k^- A_k^+ T_k^+ e^{ikL} \tag{3.11}
\]

We now consider a Gaussian initial state \( \psi_0 \) centered around \( x_0 < -d/2 \) and having mean momentum \( k_0 > 0 \)
\[
\psi_0(x) = \frac{1}{(2\pi \sigma^2)^{1/4}} e^{-\frac{(x-x_0)^2}{4\sigma^2}} + ik_0x, \tag{3.12}
\]
where \( \delta \) is the spread in position and we assume that \( \delta << |x_0 + d/2| \) so that the initial state does not overlap with the region where the potential is non-zero. In this region,
\[
v_k(x) = C_k A_k^- A_k^+ \left[ (1 + R_k^- e^{2ikL} - T_k^+ S_k e^{2ikL}) e^{ikx} + (R_k^+ R_k^- e^{-2ikL} - T_k^+ T_k^- e^{-2ikL}) e^{-ikx} \right]. \tag{3.13}
\]
The coefficients \( c_k = D \langle k|\psi_0 \rangle \) are then given by
\[
c_k = \bar{C}_k A_k^- A_k^+ \left[ 1 + (\bar{R}_k^- - T_k^+ S_k) e^{-2ikL} \right] \frac{1}{(2\pi \sigma^2)^{1/4}} e^{-\frac{(k-k_0)^2}{4\sigma^2}} - i\lambda_0(k+x_0) \tag{3.14}
\]
where we set \( \sigma = (2\delta)^{-1} \) the momentum spread. The assumption that \( \sigma/k_0 << 1 \) allowed us to drop a term of order \( e^{-\frac{k^2}{4\sigma^2}} \).

The probability for the time-of-arrival at \( x = L \) is then given by \( p(t) = |z(t)|^2 \), where
\[
\begin{align*}
z(t) &= \int_0^\infty dk B_k e^{ikL} \frac{1}{(2\pi \sigma^2)^{1/4}} e^{-\frac{(k-k_0)^2}{4\sigma^2}} - i\lambda_0(k+x_0) \sqrt{\frac{k}{4M}} e^{-ik^2t/2M}. \tag{3.15}
\end{align*}
\]
In [3.15] we defined
\[
B_k = -2i\sqrt{2\pi |C_k|^2 |A_k^-|^2 |A_k^+|^2} \left[ 1 + (\bar{R}_k^- - T_k^+ S_k) e^{-2ikL} \right] T_k^+. \tag{3.16}
\]
Since \( \sigma/k_0 << 1 \), we can expand \( B_k \) around its value at \( k = k_0 \). As a term \( \sqrt{k} \) also appears in the integral outside the exponential, we expand together
\[
\sqrt{k}B_k \simeq \sqrt{k_0}B_{k_0} e^{(\xi_{k_0} + i\lambda_{k_0})(k-k_0)}, \tag{3.17}
\]
where
\[ \xi_{k_0} = \frac{1}{2k_0} + \left( \frac{\partial \log |B_k|}{\partial k} \right)_{k=k_0} \]  
(3.18)
\[ \lambda_{k_0} = \left( \frac{\partial \arg[B_k]}{\partial k} \right)_{k=k_0}. \]  
(3.19)

Within the same approximation, we take the limits of integration in Eq. (3.15) from \(-\infty\) to \(\infty\). We then obtain
\[ z(t) = B_{k_0} e^{ik_0 L} \sqrt{\frac{k_0}{4M}} \frac{1}{(2\pi \sigma^2)^{1/4}} \times \int_{-\infty}^{\infty} dk e^{-\frac{(k-k_0)^2}{4\sigma^2} + i(|x_0|+L+\lambda_{k_0}+i\xi_{k_0})(k-k_0)e^{-ik^2 t/2M}}. \]  
(3.20)

The expression above involves a standard Gaussian integral. Its evaluation gives
\[ z(t) = B_{k_0} e^{-ik_0^2 t/2M} e^{ik_0 L} \sqrt{\frac{k_0}{4M}} \frac{(8\pi \sigma^2)^{1/4}}{\sqrt{1 + 2\sigma^2 t/M}} \times \exp \left[ -\sigma^2 \left( |x_0| + L + \lambda_{k_0} - \frac{2\sigma^2 t M}{k_0} - i\xi_{k_0} \right)^2 \right]. \]  
(3.21)

Hence,
\[ p(t) = |z(t)|^2 = |B_{k_0}|^2 e^{2\sigma^2 \xi_{k_0}^2} \frac{k_0}{4M} \sqrt{\frac{8\pi \sigma^2}{1 + 4\sigma^2 t^2/M^2}} \times \exp \left\{ -\frac{2k_0^2 \sigma^2 / M^2}{1 + 4\sigma^2 t^2 / M^2} \left[ (1 + 2\xi_{k_0} \sigma^2 / k_0) t - \frac{M(|x_0| + L + \lambda_{k_0})}{k_0} \right]^2 \right\} \]  
(3.22)

This expression is the probability distribution for the time-of-arrival, as it would be measured by a device located at distance \(L\) from the barrier. In the following section, we analyze its properties: in particular, we identify the delay caused by the presence of the barrier.

### 4 Delay-time and tunneling time

#### 4.1 The identification of delay time

For a sufficiently monochromatic wave packet \((\sigma / k_0 \rightarrow 0)\), we assume that \(\xi_{k_0} \sigma^2 / k_0 \ll 1\), hence Eq. (3.22) yields
\[ p(t) = |z(t)|^2 = |B_{k_0}|^2 e^{2\sigma^2 \xi_{k_0}^2} \frac{k_0}{4M} \sqrt{\frac{8\pi \sigma^2}{1 + 4\sigma^2 t^2 / M^2}} \times \exp \left\{ -\frac{2k_0^2 \sigma^2 / M^2}{1 + 4\sigma^2 t^2 / M^2} \left[ (1 + 2\xi_{k_0} \sigma^2 / k_0) t - \frac{M(|x_0| + L + \lambda_{k_0})}{k_0} \right]^2 \right\} \]  
(4.1)
The term $1 + 4t^2\sigma^4/M^2$ corresponds to the spread in the particle’s wave function due to time evolution. Since we want a configuration in which the determination of time is as sharp as possible, we assume that the value of $\sigma$ is so small that this spread is negligible at the time $t_m = \frac{M(|x_0| + L + \lambda_{k_0})}{k_0}$, namely that $t_m^2\sigma^4/M << 1$. Then we obtain

$$p(t) = |B_{k_0}|^2e^{2\sigma^2\xi_{k_0}}\frac{k_0}{4M}\sqrt{8\pi\sigma^2}\exp\left\{-\frac{2k_0^2\sigma^2}{M^2}\left[t - \frac{M(|x_0| + L + \lambda_{k_0})}{k_0}\right]^2\right\} (4.2)$$

Then $t = t_m$ is a sharp peak for the mean value of the time-of-detection. A classical particle (or in quantum theory a narrow wavepacket) that starts from $x_0$ with momentum $k_0$ in absence of the potential barrier will arrive at $x = L$ at (average) time $t_0 = M\frac{|x_0| + L}{k_0}$. Hence, the barrier causes a ‘delay’ $t_d = t_m - t_0$ to the time-of-arrival (of the particles that are not reflected)

$$t_d = M\lambda_{k_0}/k_0. \quad (4.3)$$

The presence of the barrier has increased the effective length that has to be traversed by the particle by a factor of $\lambda_{k_0}$. In fact, $\lambda_{k_0}$ may be negative: the time it takes the particle to cross the forbidden region of the barrier is $t_{tun} = M(\lambda_{k_0} + d_{k_0})/k_0$, where $d_{k_0} = x_2(k_0) - x_1(k_0) \geq 0$, where $x_{1,2}(k_0)$ are the points that determine the forbidden region: they are respectively the lowest- and highest-valued solutions of the equation $\frac{k^2}{2M} = V(x)$. The total tunneling time has to be positive, but it is not necessary that it is larger than the time $M\lambda_{k_0}/k_0$ that the forbidden region is traversed by a classical free particle.

We next calculate $\lambda_{k_0}$ in terms of the absorption and reflection coefficients corresponding to the potential $V(x)$. From Eq. (3.16) we see that the only term contributing to a phase in $B_k$ is the product $[1 + f_{k_0} e^{-2ikL}]T_k^+$, where $f_k = (R_k - T_k^+ S_k)$. We then obtain

$$\lambda_{k_0} = Im\left(\frac{\partial \log T_k^+}{\partial k}\right)_{k = k_0} + Im\frac{f_{k_0} - 2iL f_{k_0}}{1 + f_{k_0} e^{-2ikL}} e^{-2ik_0 L}. \quad (4.4)$$

The second term in the right-hand-side of (4.4) oscillates very fast with $L$, because $L$ is much larger than the de-Broglie wavelength $2\pi/k_0$ of the particle. These oscillations are an artifact of our modeling the detection process by a crossing of the sharply defined surface $x = L$. In a realistic detection scheme the particle detection cannot take place with an accuracy greater than their de Broglie wavelength. For this reason, we can formally average $L$ within a region of size $l << L$. Indeed, using a Gaussian smearing function $\rho(L) = (\pi l^2)^{-1/2} e^{-(L - L_0)^2}/l^2$ we obtain a suppression factor of order $e^{-k_0^2 l^2} << 1$ for the oscillating terms.

Hence, the effective tunneling time is

$$t_{tun} = \frac{Md_{k_0}}{k_0} + \frac{M}{k_0} Im\left(\frac{\partial \log T_k^+}{\partial k}\right)_{k = k_0}, \quad (4.5)$$
i.e. we recover the expression for the Bohm-Wigner phase time \([3]\). It is important to emphasize that this derivation did not employ any characteristics of the wave-packets (e.g. the trajectory followed by their peak, or their ‘center-of-mass’). It is a natural operational definition at the level of the probability density that corresponds to the measurement outcomes.

Note that a precise treatment involves smearing the probability function \(p(t)\) of \([3, 22]\). The only \(L\)-dependent objects that appear in this equation are the term \(B_{k_0}\) and the Gaussian exponential. If \(\frac{1}{\sigma} \gg l\), the effect of smearing is to substitute \(L\) by the mean value \(L_0\): the expression is not affected. The effect of smearing on \(B_{k_0}\) is to suppress the oscillations; it leads to an effective expression \(\tilde{B}_{k_0}\)

\[
\tilde{B}_{k_0} = -2i\sqrt{2\pi} \frac{|A_{k_0}^\pm|^2|A_{k_0}^-|^2}{|A_{k_0}^-|^2(1 + |R_{k_0}^-|^2) + |A_{k_0}^+|^2|T_{k_0}^+|^2} T_{k_0}^+. \tag{4. 6}
\]

Note that to a first (very rough) approximation, \(|A_{k_0}^\pm|\) can be taken equal to \((2\pi)^{-1/2}\), i.e. the value taken if the contribution of the region with no zero potential is considered to be negligible. Then

\[
\tilde{B}_{k_0} \simeq -\frac{i}{\sqrt{2\pi}} T_{k_0}^+. \tag{4. 7}
\]

Before continuing, we summarize the approximations involved in the results we obtained in this section. Eq. \([3. 15]\) only involves the assumption that \(\sigma/k_0 \ll 1\). Eq. \([3. 22]\) involves the additional assumption that the function \(\log B_k\) varies slowly around \(k = k_0\) so that it is sufficient to keep the first order in its Taylor expansion. This approximation amounts to the condition \(\frac{B''}{B_{k_0}} - \frac{B'}{B_{k_0}} |\sigma| \ll 1\). Eq. \([4. 1]\) involves the additional assumption that \(\xi_{k_0}\sigma^2/k_0 \ll 1\). Finally, Eq. \([4. 2]\) involves the assumption that \(t_m^2 \sigma^2/M \ll 1\). This implies that \(L\) cannot be too large, because the spread of the wave function due to the free propagation will induce a large uncertainty in the determination of tunneling time.

### 4.2 Special cases

**Parity-invariant potentials.** The expression for the mode functions and for \(B_k\) simplifies greatly if the potential is invariant under parity, namely if \(V(x) = V(-x)\). This implies that the eigenstate \(u_k^+(x)\) can be identified with the parity transform of \(u_k^-(x)\). Hence \(S_k = 0, T_k^+ = T_k^- := T_k, R_k^+ = R_k^- := R_k\) and \(A_k^+ = A_k^- := A_k\). We then obtain,

\[
B_k = -2i\sqrt{2\pi} \frac{|A_k|^2}{|1 + R_k e^{2ikL}|^2 + |T_k|^2} |1 + R_k e^{-2ikL}|T_k \tag{4. 8}
\]
The square potential barrier We apply our results to the simplest example of a square potential barrier: \( V(x) = V_0 \) for \( x \in [-d/2, d/2] \). Defining \( \gamma_k = \sqrt{2MV_0 - k^2} \), we obtain the following values for the coefficients \( T_k, R_k \)

\[
T_k = \frac{2k}{\gamma_k} e^{-ikd} \frac{2k\gamma_k \cosh \gamma_k d - i(\gamma_k^2 - k^2) \sinh \gamma_k d}{4k^2\gamma_k^2 + (\gamma_k^2 + k^2) \sinh^2 \gamma_k d} \quad (4.9)
\]

\[
R_k = -ie^{-ikd} \frac{(\gamma_k^2 + k^2)[2k\gamma_k \cosh \gamma_k d - i(\gamma_k^2 - k^2) \sinh \gamma_k d]}{4k^2\gamma_k^2 + (\gamma_k^2 + k^2) \sinh^2 \gamma_k d} \quad (4.10)
\]

There are two limits, in which the results are particularly simple. The limit of a long barrier \( \gamma_k d >> 1 \), for which

\[
T_k \simeq e^{-ikd} e^{-\gamma_k d} \frac{4k\gamma_k}{(\gamma_k^2 + k^2)^2} [2k\gamma_k - i(\gamma_k^2 - k^2)] \quad (4.11)
\]

\[
R_k \simeq e^{-ikd} - (\gamma_k^2 - k^2) + ik\gamma_k \frac{1}{4\gamma_k^2} \quad (4.12)
\]

In this limit, the parameter \( \lambda_{k_0} \) is

\[
\lambda_{k_0} = -d + \frac{2}{\gamma_{k_0}}, \quad (4.13)
\]

i.e. it takes negative values (since \( \gamma_{k_0} d >> 1 \)). The tunneling time is therefore \( t_{tun} = \frac{2M}{\gamma_{k_0}} \).

The other limit is that of the delta function (very short) barrier. It is obtained by letting \( V_0 \to \infty \) and \( d \to 0 \) such that \( V_0d \) is a constant (we denote this constant as \( \kappa/M \)). At this limit, \( \gamma_k d \simeq \sqrt{\kappa d} \) and

\[
T_k = \frac{1}{1 + ik/\kappa}, \quad (4.14)
\]

\[
R_k = \frac{1}{1 + ik/\kappa}. \quad (4.15)
\]

Hence,

\[
\lambda_{k_0} = \frac{\kappa}{k_0^2 + \kappa^2} \quad (4.16)
\]

Since \( d = 0 \) the tunneling time is \( t_{tun} = \frac{M\kappa}{k_0^2 + \kappa^2} \).

4.3 Comments

4.3.1 Domain of validity

It is important to emphasize that our identification of a tunneling time \( t_{tun} \) relies on the fact that the probability of detection has a unique sharp maximum at a specific moment of time. This is only possible for specific initial states. For
example, it is easy to demonstrate that a superposition of Gaussians centered at different values of momentum will lead to a probability distribution with an oscillating behavior. While there is still a mean detection time, we cannot read from it a time delay for the particle, because the momentum uncertainty does not allow one to specify uniquely a corresponding time for free particle evolution. Hence, the tunneling time is not a proper observable (i.e. a random variable on the sample space upon which the POVM is defined) in our description: it is only a parameter that appears in the detection probability for a class of initial states, which has an intuitive interpretation in terms of classical concepts.

The fact that the concept of tunneling time has a restricted domain of validity is highlighted by another point. We saw that for a long square potential the tunneling time equals \( t_{\text{tun}} = \frac{2M\gamma_{k_0}}{\lambda_{k_0}} \). If \( d \) is very large, the condition \( \gamma_{k_0}d \gg 1 \) can be satisfied even if \( \gamma_{k_0} \) takes very small values, i.e. if the particle’s mean energy \( \frac{k^2_0}{2M} \) is very close to \( V_0 \). Hence, it is in principle possible to construct configurations, in which \( t_{\text{tun}} \) is arbitrarily small: the effective ‘velocity’ \( d/t_{\text{tun}} \) in the crossing of the barrier is then super-luminal. This is a well known effect in tunneling (the Hartman effect [31]). A full treatment in the present context involves the consideration of relativistic systems–this we will undertake in future work. Here, we only note that the regime of very large values for \( d/t_{\text{tun}} \), (very small values for \( \gamma_{k_0} \)) is one for which the approximation involved in Eq. (3.17) fails. The tunneling probability increases rapidly in this regime and one would have to include further terms in the expansion of \( \log B_k \), which would lead to a substantially deformed probability distribution \( p(t) \) with no clear peak. The definition of \( t_{\text{tun}} \) would then be highly problematic, and so would be the notion of a mean velocity in the tunneling region.

### 4.3.2 Uncertainty in the specification of tunneling time

The uncertainty in the determination of the peak in the probability distribution (4.2) is \( \frac{M}{k_0\sigma} \). In order for the delay time \( \frac{M\lambda_{k_0}}{k_0} \) to be distinguishable (if we ignore all other sources of uncertainty) it is necessary that \( \sigma|\lambda_{k_0}| \gg 1 \). In order for the tunneling time to be distinguishable, it is also necessary to take into account the uncertainty in the quantity \( \frac{Md_{k_0}}{k_0} \). To leading order in \( \sigma \) this equals \( a_{k_0}\sigma \), where

\[
a_{k_0} = \frac{k_0}{M} \left( \frac{1}{V'[x_2(k_0)]} - \frac{1}{V'[x_1(k_0)]} \right) - \frac{Md_{k_0}}{k_0^2}.
\] (4.17)

The overall uncertainty in the determination of the tunneling time \( t_{\text{tun}} = M(\lambda_{k_0} + d_{k_0})/k_0 \) is of the order

\[
\frac{M}{k_0\sigma} + |a_{k_0}|\sigma.
\] (4.18)

This expression is bounded from below by \( 2\sqrt{Mk_0a_{k_0}} \). Hence, a necessary condition for tunneling time \( t_{\text{tun}} \) to be distinguishable is

\[
t_{\text{tun}} \gg \sqrt{Ma_{k_0}}/k_0.
\] (4.19)
We note that for a parity symmetric potential \( a_{k_0} = -Md_{k_0}/k_0^2 \), hence the condition becomes \( t_{\text{run}} > M\sqrt{d_{k_0}/k_0} \). For the long square barrier, this implies that

\[ \frac{\gamma_{k_0}^2 d_{k_0}}{k_0} << 1. \] (4. 20)

This condition can only be satisfied if \( \gamma_{k_0}/k_0 << 1 \). This is inadmissible, because the expansion (3. 17) is not adequate in this regime. Hence, for the long square barrier the operational definition of the tunneling time is not meaningful. On the other hand, there is no problem in the short barrier limit \( (d \to 0) \).

4.3.3 The dependence on \( L \)

Finally, we comment on the assumption that \( L >> d \). The consideration of a detector at a macroscopic distance away from the barrier region greatly simplifies our results: it leads to an expression for the tunneling time, which essentially coincides with the results of the asymptotic analysis of the wave packets. This assumption enters at two steps. First, in the construction of the POVM, we assume that \( L \) is sufficiently removed from the barrier region, so that the value for the particle’s propagator at \( x = L \) can be substituted by the corresponding value for the free particle. This condition is satisfied exactly if the corresponding Hamiltonian has no (generalized) eigenstates with an asymptotic behavior that does not correspond to that of a free particle (e.g. negative energy states). This is the case we considered in this section. Hence, the only place where the assumption of large \( L \) enters in a non-trivial way in the construction, is when we smear the probability distribution in order to remove the contribution of the terms oscillating as \( e^{ik_0 L} \). This implies that (at least formally), the expression (4. 4) for the parameter \( \lambda_{k_0} \) is valid for all values of \( L \) such that the first condition stated above holds. We therefore obtain an expression for the tunneling time, even if the detector is located near the tunneling region. Clearly, this will have a very sensitive dependence on \( L \), because the presence of the detector close to the barrier affects the configuration of the system. Note however that this result is rather formal, since it involves the idealization of the detection process by the crossing of the sharply defined surface \( x = L \). In a realistic treatment the detailed physics of the detector are expected to influence the tunnelling time.

For example, for a parity symmetric potential \( (S_k = 0) \), we obtain the following expression for the parameter \( \lambda_{k_0} \),

\[ \lambda_{k_0} = \theta'_{k_0} \frac{r_{k_0}(2L + \theta'_{k_0})(1 + \cos(2k_0L + \theta_{k_0})) + r'_{k_0} \sin(2k_0L + \theta_{k_0})}{1 + r_{k_0}^2 + 2r_{k_0} \cos(2k_0L + \theta_{k_0})}, \] (4. 21)

where we wrote \( R_k = r_k e^{i\theta_k} \) and the prime denotes differentiation with respect to \( k \).
5 A POVM for the tunneling time through sequential measurements

We saw in the last section that the determination of tunneling time through the time-of-arrival probability is only meaningful for a specific class of initial states, because the delay time is not a proper random variable on the sample space of the POVM. It depends on the particle’s initial momentum (and position) and as such it cannot be inferred unless both the initial state and the detection probability have very sharp maxima.

However, this problem can be alleviated if we make a change in the experimental set-up, namely if we consider that a measurement of momentum takes place before any recording of the time-of-arrival. In effect, if one considers sequential measurements, it is possible to construct a POVM for which the tunneling time is a genuine random variable and no mixed classical-quantum arguments are needed for its identification.

The procedure is the following. Let $\hat{Q}(x, k)$ be a POVM for unsharp phase space measurements. Let us also assume that the corresponding device is placed at the left-hand-side of the barrier; we perform an unsharp phase space measurement to any particle that moves towards the barrier that allows us to determine unsharp values for its position $q$ and momentum $p$. The measurement is assumed to be non-destructive, hence the particles continue their motion, some of them cross the barrier and they are detected at distance $L$ away. In other words, we have a sequential measurement: first an unsharp phase space measurement and then a time-of-arrival measurement. For each particle, the outcomes of this sequential measurement is encoded in the three numbers $(x, k, t)$ that span a sample space $\Omega$.

The key point is that from the knowledge of $\hat{Q}$ and $\hat{\Pi}$ (the time-of-arrival POVM), it is possible to construct a POVM $\hat{E}$ on $\Omega$. The procedure is standard, see [24] for a detailed analysis. The POVM $\hat{E}$ consists of the positive operators

$$\hat{E}(t, x, k) = \sqrt{\hat{Q}(x, k)\hat{\Pi}(t)}\sqrt{\hat{Q}(x, k)}, \quad (5.1)$$

and of the positive operator

$$\hat{E}(N, x, k) = \sqrt{\hat{Q}(x, k)\hat{\Pi}(N)}\sqrt{\hat{Q}(x, k)}, \quad (5.2)$$

that corresponds to a phase space measurement and then no detection. By construction it satisfies

$$\int dx dk 2\pi \left( \int_0^\infty dt \hat{E}(t, x, k) + \hat{E}(N, x, k) \right) = 1. \quad (5.3)$$

For an initial state $\hat{\rho}_0$, the joint probability density on the sample space $\Omega$ is given by

$$P(t, x, k) = Tr \left( \hat{\rho}_0 \hat{E}(t, x, k) \right). \quad (5.4)$$
The key benefit in the consideration of such a POVM is that the delay-time
\[ t_d = t - \frac{M(L - x)}{k}, \quad (5.5) \]
and the tunneling time
\[ t_{\text{tun}} = t_d + \frac{Md_k}{k}, \quad (5.6) \]
are both random variables on the sample space \( \Omega \). Hence, it is possible to
define a POVM on the space in which they take values. We will do so after we
construct explicitly the POVM \( \hat{E} \).

We consider POVMs for the unsharp phase-space measurements of the form
\[ \hat{Q}(x, k) = \int \frac{dk_0 dx_0}{2\pi} f(x - x_0, k - k_0)|x_0, k_0\rangle\langle x_0, k_0|, \quad (5.7) \]
where \( |x_0, k_0\rangle \) is the coherent state \( (3.12) \), and \( f \) is a positive-valued function
that determines the phase space resolution of the apparatus. Since
\[ \int \frac{dx dk}{2\pi} \hat{\Pi}(x, k) = 1, \]
\( (5.8) \) it is necessary that the function \( f \) satisfies
\[ \int \frac{dx dk}{2\pi} f(x, k) = 1. \quad (5.8) \]
The minimum resolution measurements correspond to \( f(x, k) = 2\pi\delta(x)\delta(k) \),
in which case \( \hat{Q}(x, k) = |xk\rangle\langle xk| \). For simplicity, we will consider minimum
resolution measurements in what follows.

We obtain the following probability density on \( \Omega \)
\[ P(t, x, k) = \langle xk|\hat{\rho}|xk\rangle \langle xk|\hat{\Pi}(t)|xk\rangle. \quad (5.9) \]
We note that \( \langle xk|\hat{\Pi}(t)|xk\rangle \) equals the probability density \( p(t) \) of Eq. \( (3.22) \).
We write this as \( p_{x,k}(t) \), in order to express its dependence on the initial state
\( k = k_0 \) and \( x = x_0 \) in Eq. \( (5.22) \). We then obtain
\[ P(t, x, k) = \langle xk|\hat{\rho}|xk\rangle p_{x,k}(t). \quad (5.10) \]

We next change variables in \( (5.10) \) from \( t \) to the delay time \( t_d \). We note that
on the full sample space, the relation between \( t_d \) and \( t \) is not one-to-one. First, the random variable \( t_d \) takes values in the whole real axis, while \( t \) only on the
positive real axis. It is therefore convenient to define the probability \( P(t, x, k) \)
for \( t \) running to all reals. This involves defining \( p_{x,k}(t) \) for all \( t \in \mathbb{R} \); we saw in
Sec. 2 that this is obtained by doubling the values of \( p_{x,k}(t) \) for \( t \in [0, \infty) \). With
\( t \) defined over all reals, we note that for each value of \( t \), one obtains the same
value for \( t_d \) twice, since \( t_d \) is the same at points \( (t, x, p) \) and \( (t, 2L - x, -p) \). We
perform the change of variables taking the facts above into account, and then
we integrate over \( x \) and \( k \), in order to obtain a marginal probability distribution
over \( t_d \)
\[ P_d(t_d) = 4 \int \frac{dx dk}{2\pi} \langle xk|\hat{\rho}|xk\rangle p_{x,k}(t_d + \frac{L - x}{k}), \quad (5.11) \]
The same procedure leads to a marginal probability distribution for the tunneling time

\[ P_{tun}(t_{tun}) = 4 \int \frac{dx dk}{2\pi} \langle xk|\hat{\rho}|xk\rangle p_{x,k}(t_{tun} + \frac{L - x + d_k}{k}). \] (5.12)

The two equations above are completely general, and they hold without any approximations. They simplify significantly if we assume that for all values of \(k\) in the support of the initial state, the following two conditions hold: (i) \(p_{x,k}(t)\) is appreciably different from zero only for times \(t\) such that \(t^2\sigma^2/M << 1\), and (ii) \(\sigma\xi_k << 1\). The dependence on \(x\) of \(p_{x,k}\) is then absorbed in the definition of the variable \(t_d\), and we obtain

\[ P_d(t_d) = \sqrt{8\pi\sigma^2} \int dk \langle k|\hat{\rho}_0|k\rangle |\tilde{B}_k|^2 \frac{|k|}{M} \exp \left\{ -\frac{2k^2\sigma^2}{M^2} \left[ t_d - \frac{M\lambda_k}{k} \right]^2 \right\}. \] (5.13)

Similarly,

\[ P_{tun}(t_{tun}) = \sqrt{8\pi\sigma^2} \int dk \langle k|\hat{\rho}_0|k\rangle |\tilde{B}_k|^2 \frac{|k|}{M} \exp \left\{ -\frac{2k^2\sigma^2}{M^2} \left[ t_{tun} - \frac{M(\lambda_k + d_k)}{k} \right]^2 \right\}. \] (5.14)

Note that neither \(P_d\) nor \(P_{tun}\) are normalized to unity. The delay and tunneling times are only defined for the fraction of the ensemble that corresponds to particles that have crossed the barrier. To normalize it, we have to divide by the probability corresponding to the detected particles \(1 - Tr(\hat{\rho}_0\hat{E}(N))\).

Hence, we have constructed a positive definite probability density for the delay and the tunneling times, which is valid for an arbitrary initial state (with the restriction that its position support lies on the left side of the barrier). This probability is definable in the context of a sequential measurement: there is no other way to define these times as quantum observables otherwise: the definition in Sec.4 involved a mixture of quantum mechanics and classical argumentation and was only meaningful for a specific class of initial states. We have to keep in mind though that the experimental set-up for which these probabilities are valid involves keeping track of the phase space properties of individual particles and then comparing them with the registered arrival time. It requires relatively precise measurements at a microscopic scale, and it cannot be implemented when working with particle beams.

We should also note that both probabilities \(P_d\) and \(P_{tun}\) are contextual, i.e. they depend strongly on specific features of the apparatus that performs the phase space sampling. They both have a strong dependence on the parameter \(\sigma\), which defines the family of coherent states: in the present context \(\sigma\) is the inherent uncertainty in the specification of momentum.\(^5\) At the limit \(\sigma \to 0\), both \((5.13)\) and \((5.14)\) vanish. There is, however, a limit in which the results become \(\sigma\)-independent. If the initial state has support on values of \(k\), such that

\(^5\)For the contextuality of sequential measurements, see the extended discussion in [23].
the mean of the Gaussian in either probability density is much larger than its spread, then we can approximate it by a delta function. This condition implies

$$\sigma|\lambda_k| >> 1, \quad (5.15)$$

for (5.13) and

$$\sigma(\lambda_k + d_k) >> 1 \quad (5.16)$$

for (5.14).

At these regimes, we obtain

$$P_d(t_d) = 2\pi \int dk \langle k|\hat{\rho}_0|k\rangle|\tilde{B}_k|^2\delta(t_d - \frac{M\lambda_k}{k}), \quad (5.17)$$

$$P_{tun}(t_{tun}) = 2\pi \int dk \langle k|\hat{\rho}_0|k\rangle|\tilde{B}_k|^2\delta(t_{tun} - \frac{M(\lambda_k + d_k)}{k}). \quad (5.18)$$

In other words, the values of $P_d(t_d)$ and of $P_{tun}(t_{tun})$ are determined by the value of the probability distribution of the initial’s state momentum at values of $k$ that are solutions of the algebraic equations $t_d = \frac{M\lambda_k}{k}$ and $t_{tun} = \frac{M(\lambda_k + d_k)}{k}$ respectively. These expressions for the probability distribution are independent of the detailed characteristics of the phase space POVM: they only depend on the initial state and on the characteristics of the potential.\footnote{Recall that by virtue of smearing the value of $L$, there is no $L$-dependence in $\tilde{B}_k$; hence the marginal probability distributions are also $L$-independent.}

They can therefore be considered as ideal distributions of delay and tunneling times respectively that exhibit little sensitivity to the measurement scheme employed for their determination.

We can further simplify the expressions for $P_d$ and $P_{tun}$ using the estimation (4.7) for $\tilde{B}_0$:

$$P_d(t_d) = \int dk \langle k|\hat{\rho}_0|k\rangle|T_k|^2\delta(t_d - \frac{M\lambda_k}{k}), \quad (5.19)$$

$$P_{tun}(t_{tun}) = \int dk \langle k|\hat{\rho}_0|k\rangle|T_k|^2\delta(t_{tun} - \frac{M(\lambda_k + d_k)}{k}). \quad (5.20)$$

In effect, the probability for $t_d$ and $t_{tun}$ are defined from the corresponding values of the momentum distribution weighted by the transmission probability. Defining the functions $F_d(k) := \frac{M\lambda_k}{k}$ and $F_{tun}(k) := \frac{M(\lambda_k + d_k)}{k}$, we see that the probabilities (5.19) and (5.20) are obtainable from the operators $\hat{T}_d = F_d(\hat{p})$ and $\hat{T}_{tun} = F_{tun}(\hat{p})$ ($\hat{p}$ is the momentum operator) when these act on the state

$$\hat{\rho}_{cross} = \int dk |T_k|^2 \hat{P}_k \hat{\rho}_0 \hat{P}_k, \quad \hat{P}_k = |k\rangle\langle k| \quad (5.21)$$

that describes the sub-ensemble of particles that have crossed the barrier. One could therefore call $\hat{T}_d$ and $\hat{T}_{tun}$ time-delay and tunneling-time operators respectively.\footnote{There is an ambiguity in their definition at $k = 0$. However, this does not affect the probabilities (5.19) and (5.20), because $|T_{k=0}| = 0$.}
We end this section, by examining the domain of validity of conditions (5.16) and (5.15) for the square potential barrier. At the large barrier limit, they read

\[
\sigma | - d + \frac{2}{\gamma_k} | \gg 1 \quad (5.22)
\]

\[
\sigma / \gamma_k \gg 1. \quad (5.23)
\]

They are satisfied if the position \(\sigma^{-1}\) spread of the coherent states is much smaller than the effective lengths corresponding to delay and tunneling time respectively. For the delta function barrier, these conditions imply

\[
\frac{\sigma}{\kappa^2 + \kappa^2} \gg 1, \quad (5.24)
\]

which is only possible if \(\kappa\) is extremely small (a rather unphysical case).

We see therefore that the ideal probability distributions (5.17) and (5.18) can only be obtained if the initial phase space measurement has a resolution for position substantially smaller than the dimensions of the barrier. This is a type of measurement that is not explicitly forbidden by quantum mechanics, but clearly it would be extremely difficult to achieve in practice.

6 Conclusions

We reformulated tunneling as a problem in the determination of probability for the time-of-arrival. This allowed us to identify the classic Bohm-Wigner time as the most suitable measure for the tunneling time. However, this identification only holds for a specific class of initial states and potentials; in other regimes, there is no operational definition of the concept. There is one way to go around this problem by considering a sequential measurement set-up: we first measure the phase space properties of the particles (before they attempt to cross the barrier) and then we determine their times-of-arrival. In this context, it is possible to construct a probability measure for the tunneling time that is valid for all initial states.

The key feature of our construction is that there is neither interpretational nor probabilistic ambiguity. The probabilities we derive are obtained through a POVM, hence they are always positive and they respect the convexity of the space of quantum states. The interpretation of these objects is concretely operational, in the sense that it is tied to the statistics for the measurement of particles' arrival times. Tunneling time is solely defined in terms of the statistics of measurement outcomes.

In another paper [2], the POVM we constructed here will be employed for the study of the decay probability of unstable states through tunneling.

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