Search for flow invariants in even and odd dimensions

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Abstract. A flow invariant in quantum field theory is a quantity that does not depend on the flow connecting the UV and IR conformal fixed points. We study the flow invariance of the most general sum rule with correlators of the trace $\Theta$ of the stress tensor. In even (four and six) dimensions we recover the results known from the gravitational embedding. We derive the sum rules for the trace anomalies $a$ and $a'$ in six dimensions. In three dimensions, where the gravitational embedding is more difficult to use, we find a non-trivial vanishing relation for the flow integrals of the three- and four-point functions of $\Theta$. Within a class of sum rules containing finitely many terms, we do not find a non-vanishing flow invariant of type $a$ in odd dimensions. We comment on the implications of our results.

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1. Introduction

The investigation of quantum field theory beyond the weakly coupled regime is one of the difficult open problems of theoretical physics. The renormalization group and anomalies provide powerful tools to study this issue in several situations. Anomalies are quantities with special features, which make them calculable to high orders in perturbation theory and sometimes even exactly. The Adler–Bardeen theorem [1] is an example of an exact result (in the sense of the resummed perturbative expansion) in quantum field theory. It has a variety of applications. For example, it ensures that certain anomalies are one-loop exact. It also justifies the ’t Hooft anomaly matching conditions [2], which put constraints on the strongly coupled limit of the theory.

The RG flow is intimately related to anomalies, the breaking of scale invariance being encoded in the anomaly associated with the trace of the stress tensor. In even dimensions, the trace anomaly in external gravity defines the so-called central charges, quantities which are particularly useful to characterize the ultraviolet and infrared fixed points of the RG flow. In supersymmetric theories, the two classes of anomalies, axial and trace, are related to each other, and in several models the Adler–Bardeen theorem can be used to compute the exact values of the central charges in the interacting fixed points of the flows [3].

A useful notion to understand the structure of the RG flow at a general level is the notion of flow invariant. We can actually define two different types of flow invariant, associated with the axial and trace anomalies, respectively:

(i) a flow invariant of the first type is a quantity \( b \) that does not depend on the energy scale, namely it is constant throughout the RG flow; in particular it satisfies

\[ b_{\text{UV}} = b_{\text{IR}}; \]

(ii) a flow invariant of the second type is a quantity \( \Delta A \) that does not depend on the particular flow connecting the same pair of UV and IR fixed points.

Examples of flow invariants of the first type are the ’t Hooft anomalies themselves. Consider a current \( J_a \) associated with a classical symmetry. The Adler–Bardeen theorem implies that if the current is conserved at the quantum level (i.e. the internal anomaly vanishes), then all external anomalies are one-loop exact. More explicitly, if the triangle \( \langle \partial J(x) J_{\text{int}}(y) J_{\text{int}}(z) \rangle \) vanishes for all gauge currents \( J_{\text{int}} \), then the correlation function \( \langle \partial J(x) J_{\text{ext}}(y) J_{\text{ext}}(z) \rangle \) is one-loop exact for all currents \( J_{\text{ext}} \) coupled to external fields. This correlation function is uniquely determined by a constant \( b \), which therefore satisfies \( b_{\text{UV}} = b_{\text{IR}} \).

Examples of flow invariants of the second type are provided by the central charges. In particular, a flow invariant \( \Delta A \) is in general the difference between the values of a central charge \( A \) at the fixed points:

\[ \Delta A = A_{\text{UV}} - A_{\text{IR}}. \]

In even dimensions well known central charges are the quantities \( c \) and \( a \) defined by the trace anomaly in external gravity (details are given below). The central charge \( a \) satisfies the inequality \( \Delta a \geq 0 \) in higher even-dimensional renormalizable unitary flows [4]. This property is known as irreversibility of the RG flow. Instead, the sign of \( \Delta c \) can be negative. In two dimensions there is only one central charge (\( c \) and \( a \) coincide, in some sense) and this does satisfy the irreversibility theorem [5, 6]. Finally, in odd dimensions the trace anomaly in external gravity is identically
zero at the critical points. A central charge $c$ can be defined using the stress-tensor two-point function, but there is no clear indication of the existence of a quantity like $a$.

In this paper, we report on the search for a central charge of type $a$ in odd dimensions, and the associated flow invariant. This research is meant to start a programme of systematic investigation of odd-dimensional quantum field theory as an RG interpolation between conformal fixed points.

By central charge of type $a$ we mean a function $A$ of the coupling constant or, more generally, of the energy scale, with the following properties.

1. **Flow invariance**, i.e. $\Delta A = A_{UV} - A_{IR}$ does not depend on the particular flow connecting the fixed points.

2. **Stationarity** at the fixed points, i.e. $\left. \frac{\partial A(\alpha)}{\partial \ln \alpha} \right|_{\alpha \sim \alpha^*} = \beta_\alpha$, or $\left. \frac{\partial A(\lambda)}{\partial \lambda} \right|_{\lambda \sim \lambda^*} = \beta_\lambda$.

   Here $\alpha$ denotes a gauge coupling, while $\lambda$ denotes a coupling of a Yukawa vertex, or $\phi^4$ in four dimensions, $\phi^6$ in three dimensions etc. Formally, the formulae for gauge and non-gauge couplings agree under the replacement $\ln \alpha \leftrightarrow \lambda$. In particular, $\beta_\alpha = d\ln \alpha/d\ln \mu$ and $\beta_\lambda = d\lambda/d\ln \mu$. The star refers to the critical values.

3. **Marginality**, i.e. $A$ is constant on families of continuously connected fixed points.

4. **Irreversibility**, i.e. $\Delta A \geq 0$.

   For the time being, we postpone the study of property (4). The first three properties are sufficient to discriminate between central charges of type $c$ and central charges of type $a$, at least in four dimensions [7, 3]. We look for a quantity satisfying (1), (2) and (3). More precisely, we study a class of sum rules for $\Delta A$ satisfying (2) and check whether they also satisfy (1) and (3). The result is that in even dimensions we do not find more flow invariants than the known one, $\Delta a$, while in odd dimensions we do not find a genuine flow invariant of type $a$, but we do find a nontrivial vanishing relation, that is to say a flow invariant of type (ii) identically equal to zero.

   In odd dimensions, the embedding in external gravity is not helpful to define a quantity $a$, because the trace anomaly of the theory embedded in external gravity identically vanishes at the fixed points. The reason is that the dimension of the stress tensor in units of mass equals the spacetime dimension, but no scalar of odd dimension can be constructed with the Riemann tensor and its covariant derivatives. On the other hand, there is evidence, both in even and odd dimensions, that the properties of the central charges $c$ and $a$ and the flow invariants $\Delta c$ and $\Delta a$ associated with them need more than the embedding in external gravity to be explained. For example, in even dimensions the subclass of conformal field theories having $c = a$ is singled out by a special structure $G_d$ in the trace anomaly [4, 8]. The peculiarity of the combination $G_d$ does not appear to follow from the properties of the embedding in external gravity. Second, the flows having $\Delta a = \Delta a'$ are associated with a 'pondered' extension of the Euler density [9, 4], whose existence, again, does not appear to follow from the mere existence of a gravitational embedding. Third, in Gaussian massive models there appears to be a nontrivial relation between $a'$ and $c$ [10, 4], both in even and in odd dimensions, but from the theoretical point of view this relation remains to be fully explained. The embedding in external gravity does not seem to be able to provide the missing understandings.

   These observations suggest that the explanation of the properties of central charges and flow invariants of type (ii) must lie beyond the gravitational embedding. Presumably, every property can be derived without using the gravitational embedding. In this paper, we show that
the quantity $a$ can be singled out, in even dimensions, using only flow invariance. We exploit
this fact to seek a similar quantity in odd dimensions.

Our empirical approach, based on the search for flow invariants in specific toy models
without using the embedding in external gravity, is not sufficient to rigorously prove the existence
of flow invariants. Nevertheless, this is the most economical method we have at the moment to
guide the search for flow invariants in odd dimensions. Note that the vanishing relations and
sum rules of this paper and [11] are universal, that is to say they hold in every quantum field
theory (renormalizable and non-renormalizable, unitary and non-unitary) interpolating between
well defined UV and IR fixed points.

It is impossible to mention here all of the works that can be found in the literature about
correlerators of composite operators, in particular the stress tensor and its trace, with or without the
embedding in external fields. We just point out that the idea of working without the gravitational
embedding is not new in itself. What is new is the context in which we use it, in odd dimensions,
and the purpose. Curiously, however, in the end we discover the relation (3.4) in three dimensions,
whose features suggest that it does follow from the gravitational embedding. It would be
interesting to rederive (3.4) from the gravitational embedding. This is more difficult than in
even dimensions and we expect that the techniques of [11] need deep improvements.

We now illustrate the plan of the research and the method.

We assume that the stress tensor is multiplicatively renormalizable, i.e. there is no
improvement term. Then the stress tensor is finite. In the presence of improvements, the
treatment can be generalized using the minimum principle of section 7 of [10]. Moreover, at
some point in the argument below we specialize, for pedagogical reasons, to classically conformal
theories, where the trace of the stress tensor is an ‘evanescent’ operator (because it is classically
zero) and its renormalized expression has the form

$$\Theta = \beta_j O_j$$  \hspace{1cm} (1.1)

for certain operators $O_j$. For example, $O = -F^2/4$ in Yang–Mills theory and $O = \varphi^4/4!$ in
the $\varphi^4$-theory in four dimensions. The formulae below are written for a gauge coupling, for
simplicity. The conclusions that we derive here also hold in the more general cases.

To have (2) guaranteed, we can assume that the function $A(\alpha(\mu))$, or $A(t)$, where $t = \ln \mu$,
is such that, roughly,

$$\frac{dA}{dt} \sim \int \langle \Theta \Theta \rangle + \int \langle \Theta \Theta \Theta \rangle + \cdots, \hspace{1cm} \frac{dA}{dt} \sim \beta^2, \hspace{1cm} \frac{\partial A}{\partial \ln \alpha} \sim \beta_a.$$  

This assures stationariness at the critical points. Below we make this argument rigorous, with the
help of the embedding in external gravity, eventually specializing to a conformally flat external
metric. First we give some details on the notation and the scheme conventions.

We consider the correlators

$$\langle \Theta(x_1) \cdots \Theta(x_i) \Theta(0) \rangle.$$  \hspace{1cm} (1.2)

In general, these correlators need subtractions at the coincident points (see [12] for a detailed
treatment of this issue in perturbation theory). In our case, we can deal with these subtractions as
follows. We distinguish the subtractions of ‘overall’ contact terms (i.e. the local terms where all
of the points coincide) from the subtractions of semilocal terms. The subtractions of semilocal
terms cancel out if we replace (1.2) with the functional derivatives $-\Gamma_{\kappa_1 \ldots \kappa_6}$ of the induced action
$\Gamma[\phi]$ with respect to the conformal factor $\phi$ at the points $x_1, \ldots, x_i, 0$. The reason is that $\Gamma[\phi]$ is
finite (after the subtraction of appropriate counterterms constructed with the metric) and the
functional derivatives of $\Gamma[\phi]$ are therefore finite [12]. The overall local terms are associated with the counterterms constructed with the metric. It was shown in [11] that if we calculate (1.2) in the framework of the dimensional-regularization technique and the minimal subtraction scheme, with a conformally flat external metric, the overall local terms vanish. Then the finite (anomalous) terms that have the appropriate structure and dimensionality to mix with (1.2) are collected in the ultraviolet limit $\Gamma_{\text{UV}}[\phi]$. In even dimensions, $\Gamma_{\text{UV}}[\phi]$ depends only on $a$ and $a'$. Expressions of $\Gamma_{\text{UV}}[\phi]$ in four and six dimensions are (2.7), (2.8), (2.13) and (2.16). In odd dimensions, instead, $\Gamma_{\text{UV}}[\phi]$ is zero, because the trace anomaly vanishes at the fixed points.

In conclusion, with our conventions it is appropriate to replace (1.2) with the functional derivatives $-\Gamma'_{\text{UV}}$ of the difference $\Gamma'$. We postulate sum rules of the form

$$\Delta A = -\int d^d x P_1(x) \Gamma'_{x_0} - \int d^d x d^d y P_2(x, y) \Gamma'_{xy0} - \int d^d x d^d y d^d z P_3(x, y, z) \Gamma'_{xyz0} + \cdots$$

where the $P_i(x_1, \ldots, x_i)$s are Lorentz-invariant ($SO(d)$-invariant, in the Euclidean framework) homogeneous functions of degree $d$ in the coordinates. In even dimensions, the $P_i$ are polynomials and, due to certain vanishing relations [11], the sum (1.4) is finite. The $P_i$ are model independent, and such that the integrals of (1.4) are convergent.

We insert identities

$$1 = \int_0^\infty d\zeta^2 \delta(\zeta^2 - F_i(x_1, \ldots, x_i)),$$

where the $F_i$ are positive homogeneous functions of degree two in the $x_1, \ldots, x_i$. For example, we can take

$$F_i(x_1, \ldots, x_i) = \sum_{k=1}^i |x_k|^2.$$

We then perform the rescaling $x_k \rightarrow \zeta x_k$, and use the Callan–Symanzik equations and the finiteness of $\Theta$. We obtain

$$\Delta A = -2 \int_0^\infty d\zeta \sum_{i=1}^\infty \int d^d x_k P_i(x_1, \ldots, x_i) \delta(1 - F_i(x_1, \ldots, x_i)) \Gamma'_{x_1 \ldots x_i 0} \alpha(\zeta \mu).$$

With $t = \ln \zeta$ and changing notation from $\alpha(\zeta \mu)$ to $\alpha(t)$, we can define the function $A(t)$
such that
\[ \dot{A}(t) = 2 \sum_{i=1}^{\infty} \int P_i(x_1, \ldots, x_i) \delta(1 - F_i(x_1, \ldots, x_i)) \Gamma'_{x_1 \ldots x_i 0}(\alpha(t)) \prod_{k=1}^{i} d^d x_k, \]
\[ \Delta A = - \int_{-\infty}^{+\infty} dt \dot{A}(t) = A(-\infty) - A(\infty), \quad A(t) = A(-\infty) + \int_{-\infty}^{t} dt' \dot{A}(t'). \]  

If the theory is classically conformal, \( \mu \) is the unique scale at the quantum level and \( A(t) \) depends only on \( t \). Otherwise, the function \( A(t) \) also depends on the ratios between \( \mu \) and the other dimensioned parameters, typically the masses. In even dimensions, \( A(\pm \infty) \) are both zero (vanishing relations) or the UV and IR values of the central charge \( a \). In odd dimensions only the difference \( A(t) - A(-\infty) \) is defined.

Let us focus on classically conformal theories. Here, the terms of (1.3) containing the derivatives of \( \Theta \) are evanescent and can be neglected in the sum rules [11], if the integrals (1.4) converge, which we have assumed. In this particular case, \( \Gamma'_{x_1 \ldots x_i 0} \) can be replaced with minus (1.2). Then, it is simple to prove that the function \( A(t) \) is stationary at criticality. Indeed, from the considerations made so far, equation (1.1) tells us that
\[ \langle \Theta(x_1) \ldots \Theta(x_i) \Theta(0) \rangle = \beta^{i+1}(\alpha) \langle O(x_1) \ldots O(x_i) O(0) \rangle. \]  

We recall that in general this correlator should contain additional local terms from the conformal anomaly, but these terms have been shifted to the definition of the \( \Theta \)-correlators, according to our notation and conventions. The \( O \)-correlators are regular throughout the RG flow. We conclude from (1.6) that
\[ \frac{\partial A(\alpha)}{\partial \ln \alpha} = 2 \sum_{i=1}^{\infty} \beta^i(\alpha) \int \prod_{k=1}^{i} d^d x_k P_i(x_1, \ldots, x_i) \delta(1 - F_i(x_1, \ldots, x_i)) \langle O(x_1) \ldots O(x_i) O(0) \rangle. \]  

This formula contains one factor of \( \beta \) less than (1.6), since \( \dot{A}(t) = -\beta \frac{\partial A(\alpha)}{\partial \ln \alpha} \). The surviving factors of \( \beta \) are however sufficient to prove that (1.8) vanishes at both critical points.

In odd dimensions the functions \( P_i \), which have dimension \( d \), are not polynomial. A priori, the sum rule (1.4) could contain infinitely many terms. Some nontrivial restrictions are provided by the requirement that the integrals be convergent. We will see in a concrete one-dimensional example that our approach does not allow us to study sum rules with infinitely many terms. We are forced to put some restrictions on the \( P_i \), for example that they do not contain negative powers of the coordinates and that they are sufficiently simple. Then, the surviving set of \( P_i \) is finite. The set we choose does not single out a flow invariant in odd dimensions, but exhibits the existence of a non-trivial vanishing relation among flow integrals of the three- and four-point correlators.

The paper is organized as follows. In section 2 we describe the procedure in even dimensions, four and six. We show that it is possible to ignore the existence of a coupling to external gravity and fix (1.4) so that it is flow invariant. We show that the unique non-trivial flow invariant is the one associated with the central charge \( a \). In section 3 we study the problem in odd dimensions, three and one. We show that the simplest class of \( P_i \) does not produce any flow invariant, but they give the non-trivial vanishing relation (3.4), which might be implied by the coupling to external gravity. In the conclusions we comment on the possibilities that remain open after our analysis.
2. Search for flow invariants in even dimensions

Before describing our calculations, let us recall how the embedding in external gravity defines the central charges \( c, a \) and \( a' \) in even dimensions \( d = 2n \).

The trace anomaly at criticality in even dimensions contains three types of terms constructed with the curvature tensors and their covariant derivatives:

(i) terms \( \mathcal{W}_i, i = 0, 1, \ldots, I \), such that \( \sqrt{g} \mathcal{W}_i \) are conformally invariant;

(ii) the Euler density

\[
G_d = (-1)^n \varepsilon_{\mu_1 \nu_1 \ldots \mu_n \nu_n} \varepsilon^{\alpha_1 \beta_1 \ldots \alpha_n \beta_n} \prod_{i=1}^{n} R_{\alpha_i \beta_i},
\]

(iii) covariant total derivatives \( \mathcal{D}_j, j = 0, 1, \ldots, J \), having the form \( \nabla_\alpha J_\alpha \), \( J_\alpha \) denoting a covariant current.

The coefficients multiplying these terms in the trace anomaly are denoted with \( c_d, a_d \) and \( a'_d \), respectively. We write \( c_d = c_d^0 \) and \( a'_d = a'_d^0 \).

We have

\[
\Theta^*_d = \frac{n!}{(4\pi)^n (d+1)!} \left[ c_d (d-2) \mathcal{W}_0 + \sum_{i=1}^{I} c'_i \mathcal{W}_i - \frac{2^{1-n}}{d} \left( a_d G_d + \sum_{j=0}^{J} a'_d \mathcal{D}_j \right) \right].
\]

(2.1)

Here \( c_2 \) is \( a_2 \). \( \mathcal{W}_0 \) is the unique term of the form \( W \Box^{n-2} W + \cdots \) such that \( \sqrt{g} \mathcal{W}_0 \) is conformally invariant, where the dots denote cubic terms in the curvature tensors. I have separated \( \mathcal{W}_0 \) from the other terms of the type \( \mathcal{W}_i \), because its coefficient \( c_d \) is also the coefficient of the stress-tensor two-point function:

\[
\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle = c_d \frac{\Gamma(n+1)\Gamma(n)}{8\pi^d (d+1)d(d-1)(d-2)\prod_{\mu\nu,\rho\sigma}^{(2)} \left( \frac{1}{|x|^{2d-4}} \right)},
\]

\[
\prod_{\mu\nu,\rho\sigma}^{(2)} = \frac{1}{2} (\pi_{\mu\rho} \pi_{\nu\sigma} + \pi_{\mu\sigma} \pi_{\nu\rho}) - \frac{1}{d-1} \pi_{\mu\nu} \pi_{\rho\sigma}, \quad \pi_{\mu\nu} = \partial_\mu \partial_\nu - \Box \delta_{\mu\nu},
\]

and is normalized so that for free fields \( (n_s \) real scalars, \( n_f \) Dirac fermions and, in even dimensions, \( n_v \) \( (n-1) \)-forms) it reads

\[
c_d = n_s + 2^{[n]-1}(d-1)n_f + \frac{d!}{2([n-1]!)^2} n_v.
\]

(2.3)

Definition (2.2) and formula (2.3) also hold if the spacetime dimension \( d \) is odd (but greater than one), if \( n_v \) is set to zero and \([n]\) denotes the integral part of \( n \). The covariant total-derivative term

\[
\mathcal{D}_0 = -\frac{2^n d}{2(d-1)} \Box^{n-1} R
\]

is also peculiar among the \( \mathcal{D}_j \), since it is the unique \( tD \) linear in the curvature tensors. We choose a basis such that all the \( \mathcal{D}_j, j > 0 \), are at least quadratic in the curvature tensors. Then, on conformally flat metrics, \( \mathcal{D}_0 \) contains the unique term of formula (2.1) that is linear in the conformal factor \( \phi \).
The quantities $\Delta a_d^{ij'}$ are not flow invariants of type (ii), because the $a_d^{ij'}$ are ill defined at criticality. Arbitrary, RG-invariant constants $\delta a_d^{ij'}$ can be added to these quantities. These ambiguities are due to the arbitrary additions of finite local terms $D_j$ to the action $\Gamma$:

$$\Gamma \to \Gamma - \frac{2^{1-n}n!}{(4\pi)^n(d+1)!} \delta a_d^{ij'} \int d^{2n}x \sqrt{g} D_j.$$  (2.4)

The $D_j$ are just the $\phi$-derivatives of the $D_j$:

$$\sqrt{g} D_j(x) = -\frac{\delta}{\delta \phi(x)} \int d^{2n}x \sqrt{g} D_j.$$  

The ambiguities disappear in the differences $\Delta a_d^{ij'}$, which are, however, no longer flow invariant.

2.1. Four dimensions

In four dimensions we have

$$\Theta_{d=4}^* = \frac{1}{120} \frac{1}{(4\pi)^2} \left[ c \, W^2 - \frac{a}{4} G_4 + \frac{2}{3} a' \Box R \right]$$

and the sum rules [11]

$$\Delta a = -\frac{5\pi^2}{2} \int d^4x |x|^4 \Gamma'_{x0} - \frac{5\pi^2}{2} \int d^4x d^4y x^2 y^2 \Gamma'_{xy0}$$

$$= -\frac{5\pi^2}{2} \int d^4x |x|^4 \Gamma'_{x0} - \frac{5\pi^2}{2} \int d^4x d^4y d^4z (x \cdot y)(x \cdot z) \Gamma'_{xyz0}.  \quad (2.5)$$

These sum rules have been derived using the coupling to external gravity. To prepare the search for flow invariants in odd dimensions, where the coupling to external gravity is more difficult to use, we derive the sum rules in an alternative way. We postulate sum rules of the form (1.4) and determine the polynomials $P_i$ so that $\Delta A$ is a flow invariant.

Let us fix restrictions on the polynomials $P_i(x_1, \ldots, x_i)$. First, we know on dimensional grounds that the polynomials have degree four. They have of course to be Lorentz invariant. It was shown in [11] that there exist equivalence relations among polynomials. In particular, there exists one irreducible monomial for $i = 1$, which is $|x_1|^4$, one for $i = 2$, which is for example $x_1^2 x_2^2$, one for $i = 3$, for example $(x_1 \cdot x_2)(x_1 \cdot x_3)$, and none for $i > 3$. The most general sum rule we have to inspect therefore reads

$$\Delta A = -\frac{5\pi^2}{2} \left[ \lambda_1 \int d^4x |x|^4 \Gamma'_{x0} + \lambda_2 \int d^4x d^4y x^2 y^2 \Gamma'_{xy0} + \lambda_3 \int d^4x d^4y d^4z (x \cdot y)(x \cdot z) \Gamma'_{xyz0} \right].  \quad (2.6)$$

Of the three unknown constants $\lambda_1, \lambda_2$ and $\lambda_3$, one is an overall normalization.
The three terms of (2.6) are in one-to-one correspondence with the terms of the most general local expression for the effective action $\Gamma[\phi]$ at criticality. We restrict ourselves to conformally flat metrics $g_{\mu\nu} = \delta_{\mu\nu} e^{2\phi}$ and only study the dependence of $\Gamma$ on the conformal factor $\phi$. Neglecting total derivatives, power counting gives

$$\Gamma^*[\phi] = \int d^4x \left( b_1 (\Box \phi)^2 + b_2 (\Box \phi) (\partial_\mu \phi)^2 + b_3 (\partial_\mu \phi)^4 \right).$$

If we knew that the action $\Gamma^*[\phi]$ was inherited from the coupling to external gravity, we would have only two independent terms, corresponding to $a$ and $a'$, instead of three [9]:

$$\Gamma^*[\phi] = \frac{1}{60} \frac{1}{(4\pi)^2} \int d^4x \left( a^2 (\Box \phi)^2 - (a - a') [\Box \phi + (\partial_\mu \phi)^2]^2 \right).$$

The results are

$$\int x^4 \Gamma^*_{x_0} d^4x = -384 \Delta b_1,$$

$$\int (x^2 y^2) \Gamma^*_{x_0} d^4x d^4y = 192 \Delta b_2,$$

$$\int (x \cdot y) (x \cdot z) \Gamma^*_{x_0} d^4x d^4y d^4z = 384 \Delta b_3.$$

We inspect the sum rules in the case of the higher-derivative scalar field, with action

$$S = \frac{1}{2} \int d^{2n}x \sqrt{g} [\varphi \Delta_4 \varphi + \beta m^2 (\partial_\mu \varphi) (\partial_\nu \varphi) g^\mu\nu + m^4 \varphi^2 + \eta R m^2 \varphi^2],$$

here written in generic spacetime dimension $d = 2n$ for later convenience. The field $\varphi$ has dimension $(d - 4)/2$ in units of mass. The fourth-order differential operator $\Delta_4$ (see for example [16])

$$\Delta_4 = \nabla^2 \nabla^2 + \nabla^\mu \left[ \frac{4}{d - 2} R^\mu\nu - \frac{d^2 - 4d + 8}{2(d - 1)(d - 2)} R^\mu\nu R \right] \Delta_v - \frac{d - 4}{4(d - 1)} \nabla^2 R$$

transforms as

$$\Delta_4 \rightarrow e^{-(n+2)\phi} \Delta_4 e^{(n-2)\phi}$$

under a Weyl rescaling $g_{\mu\nu} \rightarrow e^{2\phi} g_{\mu\nu}$.

We write $\beta = r + 1/r$, where $r^2$ is the ratio between the two poles of the propagator. Indeed, (2.9) reads in flat space

$$S = \frac{1}{2} \int d^{2n}x \varphi (-\Box + rm^2) (-\Box + m^2/r) \varphi,$$

which means that the theory propagates two fields (one of which is a ghost) with masses $rm^2$ and $m^2/r$. The results can be taken from [14]:

$$-\frac{5\pi^2}{2} \int d^4 |x|^4 \Gamma^*_{x_0} = u(r) = 960\pi^2 \Delta b_1.$$
We have therefore
\[ \Delta A = (\lambda_1 + \lambda_2 + \lambda_3) u(r) - \frac{28}{3} (\lambda_2 + \lambda_3). \]
Demanding flow invariance, that is to say independence of \( r \), we find
\[ \lambda_1 + \lambda_2 + \lambda_3 = 0, \]
and therefore
\[ \Delta A = -\frac{28}{3} \lambda_1. \] (2.11)
Now if \( \lambda_1 \neq 0 \), this is precisely the first formula of (2.5) for \( \Delta a \) (for \( \lambda_1 = 1 \)). If, instead, \( \lambda_1 = 0 \), then \( \lambda_3 = -\lambda_2 \) and we have a null flow invariant, that is to say a non-trivial vanishing relation \( (\Delta b_2 = 2\Delta b_3) \) which does not follow from the kinematic sum rules (see discussion about this point in [11]). This relation is the difference between the two expressions (2.5) for \( \Delta a \) and explains the restriction from (2.7) to (2.8).

We conclude that there are two flow invariants. Having explored only one explicit model, we can always arrange the invariants in one non-vanishing invariant, plus a remainder of vanishing invariants. Only exploring a larger class of models, for example the higher-derivative fermion, can we have true evidence that the non-vanishing invariant is unique. We will not do this here and proceed in other directions.

2.2. Six dimensions

We now repeat the procedure in six dimensions and confirm the conclusions derived in the previous section. In six dimensions the form of the anomaly at criticality \( \Theta^* \) is [15]
\[ \Theta_6^* = \frac{1}{840(4\pi)^3} \left[ \frac{c}{3} \mathcal{W}_0 + c_1 \mathcal{W}_1 + c_2 \mathcal{W}_2 - \frac{1}{24} \left( a G_6 + a' D_0 + \sum_{j=1}^{5} a'_j D_j \right) \right]. \] (2.12)

The conformally invariant terms are
\[ \mathcal{W}_0 = \mathcal{W}_{\mu\nu\rho\sigma} \left( \Big( \Box^* \delta^\mu_\alpha + 4 R^\mu_\sigma - \frac{6}{5} R \delta^\mu_\alpha \right) W^{\alpha\nu\rho\sigma} + \nabla_\mu J^\mu, \]
\[ \mathcal{W}_1 = \mathcal{W}_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} W^{\rho\sigma} , \quad \mathcal{W}_2 = \mathcal{W}_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} W^{\rho\sigma} , \]
where the precise expression of \( J^\mu \) can be found in [16, 17]. The total-derivative terms \( D_j \) have been classified by Bastianelli et al in [17], where it was shown that there are six such terms. For our purposes, we are only interested in the three \( D_j \) that do not vanish on a conformally flat metric. We choose a basis made by \( D_0 \) and
\[ D_1 = \frac{24\Box}{25} R^2 , \quad D_2 = -12 \Box R^2 - \frac{24}{5} R^2 - \frac{48}{5} \nabla_\mu (R^{\mu\nu} \nabla_\nu R). \]
We recall that for a conformally flat metric $W = 0$ and the only unambiguous term is $aG_6$. It is possible to integrate $\Theta$ to obtain a local action $\Gamma^*$ for the conformal factor $\phi$.

\[
\Gamma^*[\phi] = \frac{1}{840(4\pi)^3} \int d^6x \left\{ a' \phi \Box^3 \phi - (6a' - 13a^*_5 + 7a^*_4)(\Box \phi)^3 + (8a' - 8a^*_5) \Box \phi (\partial_\mu \partial_\nu \phi)^2 - (8a + 12a' - 78a^*_5 + 58a^*_4)(\Box \phi)^2 (\partial_\mu \phi)^2 + (8a + 8a' - 16a^*_5)(\partial_\mu \phi)^2 (\partial_\mu \partial_\nu \phi)^2 + (8a' - 8a^*_5) (\partial_\mu \phi)(\partial^\mu \phi)(\partial_\nu \phi)^2 - (12a + 24a' - 156a^*_5 + 120a^*_4)(\partial_\mu \phi)^4 \Box \phi - (8a + 16a' - 104a^*_5 + 80a^*_4)(\partial_\mu \phi)^6 \right\}.
\] (2.13)

If $a'_5 = a$ the action $\Gamma^*$ reduces to

\[
\Gamma^*[\phi] = \frac{a}{840 (4\pi)^3} \int d^6x \left( \phi \Box^3 \phi \right).
\]

This nice simplification of $\Gamma^*$, occurring for a specific choice of the $a'$, is due to the existence of the pondered Euler density [15]

\[
\sqrt{g} \tilde{G}_6 \equiv \sqrt{g} (G_6 + \nabla_\alpha J^\alpha) = 48 \Box^3 \phi
\] (2.14)

where

\[
J^\alpha_a = -\left(\frac{408}{5} - 20\xi\right) R^\nu_\mu R^{\alpha\mu\nu} - \left(\frac{36}{5} + 2\xi\right) R^{\alpha\mu\nu} \nabla_\mu R + \xi \nabla^\nu R^2 + \left(-\frac{144}{5} - 10\xi\right) \nabla^\nu (R^{\mu\nu} R^{\alpha\beta}) - \frac{24}{5} \nabla^\alpha \Box R
\]

and $\xi$ is arbitrary.

Using the procedure of [11] we can derive from (2.13) a set of equivalent sum rules for $\Delta a$ and the $\Delta a'_5$ and a set of vanishing relations. The ‘minimal’ sum rule for $\Delta a$ is

\[
\Delta a = \frac{7\pi^3}{36} \left(-6 \int x^5 \Gamma'_{x,0} d^6x + \int [8(x \cdot y)^3 - 9x^4 y^2] \Gamma'_{x,y,0} d^6x d^6y + 6 \int x^2 (y \cdot z)^2 \Gamma'_{x,y,z,0} d^6x d^6y d^6z \right).
\] (2.15)

In arbitrary even dimensions $2n$ the analogue minimal sum rule will contain terms up to the $(n+1)^{th}$ derivative of $\Gamma'$. The increasing complexity of the formulae with the spacetime dimension reflects the increasing complexity of the structure of the action in external gravity.

If we ignore the existence of an embedding in external gravity, the most general form for $\Gamma^*$, subject only to the constraint that it is polynomial in $\phi$ and its derivatives, and free of dimensionful parameters, is

\[
\Gamma^*[\phi] = \int d^6x \left( b_1 \phi \Box^3 \phi + b_2 (\Box \phi)^3 + b_3 \Box \phi (\partial_\mu \partial_\nu \phi)^2 + b_4 (\Box \phi)^2 (\partial_\mu \phi)^2 + b_5 (\partial_\mu \phi)^2 (\partial_\mu \partial_\nu \phi)^2 + b_6 (\partial_\mu \phi)(\partial^\mu \phi)(\partial_\nu \phi)^2 + b_7 (\partial_\mu \phi)^4 \Box \phi + b_8 (\partial_\mu \phi)^6 \right).
\] (2.16)

We can write a set of sum rules for $\Gamma'$ expressing the difference between the values of the $b_i$ at the critical points. They are collected in the appendix. We know, from the embedding in external gravity, that only four $b_i$ are independent, so there must be at least four vanishing relations among the $\Delta b_i$. These can be viewed as trivial flow invariants. Moreover, we know that another linear combination of the $\Delta b_i$ is flow invariant, namely $\Delta a$. This makes at least five flow invariants in total, predicted by the embedding in external gravity. This counting is confirmed by our method based only on flow invariance.
We study the sum rules in the model (2.9). Expressing the action (2.9) by means of \( \tilde{\phi} \equiv \phi e^{\phi} \), choosing a conformally flat metric and functionally differentiating with respect to \( \phi \) at fixed \( \tilde{\phi} \), we arrive at the stress-tensor trace up to terms proportional to the field equations:

\[
\Theta = -\beta m^2 e^{2\phi} (\partial_{\mu} \tilde{\phi}^2 + \frac{1}{2} \Box \tilde{\phi}^2 - \Box (\phi) \tilde{\phi}^2 - (\partial_{\mu} \phi)^2 \tilde{\phi}^2 - (\partial_{\mu} \phi)(\partial^\mu \tilde{\phi}^2))
- 2m^4 e^{4\phi} \tilde{\phi}^2 + 5\eta m^2 e^{2\phi} \Box (\tilde{\phi}^2).
\]

The terms proportional to the field equations cancel out in the combinations (1.3). We can choose the improvement parameter \( \eta \) as we wish, since the improvement term vanishes at both critical points and flow invariants are \( \eta \) independent, as shown in [14]. The most convenient choice for practical calculations is \( \eta = \frac{1}{16} \beta \), in which case the trace operator reduces to

\[
\Theta \equiv -\beta m^2 e^{2\phi} ((\partial_{\mu} \tilde{\phi})^2 - \Box (\phi) \tilde{\phi}^2 - (\partial_{\mu} \phi)^2 \tilde{\phi}^2 - (\partial_{\mu} \phi)(\partial^\mu \tilde{\phi}^2)) - 2m^4 e^{4\phi} \tilde{\phi}^2.
\]

We can evaluate the sum rules explicitly for the \( \Delta b_1 \). For our purposes it is sufficient to compute \( \Delta b_{1,6} \). The results are given in the appendix. Three combinations \( \sum_{i=1}^{6} \lambda_i \Delta b_i (r) \) are independent of \( r \):

\[
\Delta b_3 - \Delta b_6 = 0,
6\Delta b_2 + 4\Delta b_3 - \Delta b_4 - \Delta b_5 = 0,
\]

and

\[
\Delta a = 6720\pi^3 (8\Delta b_1 + 6\Delta b_2 + 2\Delta b_3 - \Delta b_4) = -\frac{16}{9}.
\]  

(2.17)

In the next section this value is verified computing the trace anomaly explicitly. In summary, we have found three flow invariants evaluating six sum rules. This implies that the remaining two \( \Delta b_i \) cannot produce more than five flow invariants in total, precisely the five flow invariants predicted by the embedding in external gravity. The remaining two vanishing relations among the \( \Delta b_i \) are

\[
2\Delta b_7 - 3\Delta b_8 = 0,
2\Delta b_7 + \Delta b_3 - 4\Delta b_4 - \Delta b_5 = 0.
\]

Finally, the quantities \( \Delta a' \) and \( a_{1,2}' \) are not flow invariant, as expected by the ambiguities (2.4) associated with them. They are also independent from one another. Their expressions are

\[
\Delta a' = 53,760\pi^3 \Delta b_1,
\Delta a'_1 = \frac{6720}{13}\pi^3 (104\Delta b_1 + 8\Delta b_2 - 7\Delta b_3),
\Delta a'_2 = 6720\pi^3 (8\Delta b_1 - \Delta b_3).
\]

2.3. Calculation of \( a \) for higher-derivative scalar fields

We use techniques similar to those of [18]. The induced generating functional \( \Gamma[g_{\mu\nu}] \) for the external metric is defined by

\[
\exp(-\Gamma[g_{\mu\nu}]) = \int [d\phi] \exp(-S[\phi, g_{\mu\nu}]),
\]

where the action is

\[
S[\phi, g_{\mu\nu}] = \frac{1}{2} \int d^n x \sqrt{g} \Delta_4 \phi.
\]
and \(2n = d\) is the spacetime dimension.

We have
\[
\Gamma[g_{\mu\nu}] = \frac{1}{2} \ln \det \Delta_4, \quad \langle \Theta \rangle = -\frac{\delta \Gamma[e^{2\phi}g_{\mu\nu}]}{\delta \phi} \bigg|_{\phi=0}.
\]

If \(\omega_k\) denote the eigenvalues of \(\Delta_4\) with degeneracies \(\delta_k\), we can write, using the zeta-function regularization,
\[
\Gamma[g_{\mu\nu}] = \frac{1}{2} \sum_{k=0}^{\infty} \delta_k \ln \omega_k = -\frac{1}{2} \frac{d}{ds} \ln \left[ \Delta_4^{-s} \right] \bigg|_{s=0} = -\frac{1}{2} \frac{d}{ds} \sum_{k=0}^{\infty} \frac{\delta_k}{\omega_k^s} \bigg|_{s=0} = -\frac{1}{2} \frac{d}{ds} \left[ \frac{\omega(s)}{2} \right] \bigg|_{s=0}.
\]

The first sum is only formal, because it diverges. Since \(\Delta_4\) transforms as \(\Delta_4 \to e^{-\phi} \Delta_4\) under a conformal rescaling \(g_{\mu\nu} \to e^{2\phi}g_{\mu\nu}\) of the metric, we have
\[
\int d^2x \langle \Theta(x) \rangle = -\int d^2x \frac{d}{ds} \left[ \frac{\omega(s)}{2} \right] \bigg|_{s=0} = 2\Gamma(0). \quad (2.18)
\]

We now work on a sphere, where the formulae simplify considerably. Using (2.18) and (2.1) and keeping in mind that the Euler characteristic of the sphere is equal to two, we obtain
\[
a_d = (-1)^{n+1} \frac{(2n+1)!}{n!(n-1)!} \Gamma(0).
\]

Now we compute \(\Gamma(0)\) on the sphere. We first observe that on Einstein spaces, which satisfy \(R_{\mu\nu} = \Lambda g_{\mu\nu}\), the operator \(\Delta_4\) factorizes into
\[
\Delta_4 = \left( \Box - \Lambda \frac{d(d-2)}{4(d-1)} \right) \left( \Box - \Lambda \frac{(d-4)(d+2)}{4(d-1)} \right) = \Box_s \left( \Box_s + \frac{2\Lambda}{d-1} \right),
\]
where
\[
-\Box_s = -\Box + \Lambda \frac{d(d-2)}{4(d-1)}
\]
is the kinetic operator of a unitary scalar field. Since \(-\Box_s\) has a complete spectrum of eigenfunctions with positive eigenvalues \(\lambda_k = \Lambda (k + d/2)(k + d/2 - 1)/(d - 1)\), \(\Delta_4\) has the same spectrum of eigenfunctions and its eigenvalues are \(\omega_k = \lambda_k \delta_k (d-2)/(d-1)\), with the same degeneracy \(\delta_k\) as the \(\lambda_k\). We have
\[
\omega_k = \frac{\Lambda^2}{(d-1)^2} \left( k + \frac{d}{2} - 1 \right) \left( k + \frac{d}{2} + 1 \right),
\]
\[
\delta_k = \frac{(2k + d - 1)(k + d - 2)!}{k!(d-1)!}.
\]

We can take \(\Lambda = d - 1\) to simplify some formulae. We obtain the sum
\[
\Psi(s) = \frac{1}{\Gamma(1-s)} \sum_{k=0}^{\infty} \frac{\omega_k^s}{\sigma_k^s} = \frac{1}{(d-1)!} \sum_{k=0}^{\infty} \frac{(2k + d - 1)(k + d - 2)!}{k!(k+n-2)!(k+n-1)!(k+n)!(k+n+1)!}.
\]

The \(s \to 0\) limit of this expression can be evaluated as follows. We write
\[
1 + \frac{1}{(d-1)!} \sum_{k=1}^{\infty} \frac{(2k + d - 1)(k + d - 2)!}{k! k^{k+1}} \left( 1 - s \sum_{i=1}^{4} \ln \left( 1 + \frac{n + 2 - i}{k} \right) \right).
\]
Higher powers in $s$ do not contribute in the limit, since all terms can be expressed by means of the Riemann zeta function, which has at worst a simple pole for $s \to 0$. We now expand the logarithms in series. Since the logarithms are multiplied by $s$, we just need to capture the poles of the zeta functions, which come from sums of the form $\sum_{k=1}^{\infty} 1/k$. For this reason, it is sufficient to truncate the series to $2n$:

$$1 + \frac{1}{(d-1)!} \sum_{k=1}^{\infty} \frac{(2k + d - 1)(k + d - 2)!}{k! k^{4s}} \left( 1 - s \sum_{i=1}^{2n} \sum_{j=1}^{i} (-1)^{j+1} \left( \frac{n + 2 - i}{k} \right)^j \right).$$

The $s \to 0$ limit of this sum can be evaluated using the analytic continuation of the zeta function in $s = 0$. A few results are

$$a_4 = -\frac{28}{3}, \quad a_6 = -\frac{16}{9}, \quad a_8 = -\frac{52}{35}, \quad a_{10} = -\frac{124}{135}, \quad a_{12} = -\frac{56302}{70875}, \quad a_{14} = -\frac{53044}{42525}, \quad a_{16} = -\frac{2977778}{4465125}.$$

Note that the values are all negative. The value $a_6 = -16/9$ agrees with the result (2.17) coming from the evaluation of the sum rule (2.15), while $a_4 = -28/3$ agrees with (2.11).

3. Search for flow invariants in odd dimensions

In this section we study the sum rules (1.4) in three- and one-dimensional flows. In three dimensions we again study the model (2.9) and find a non-trivial vanishing relation. In one dimension we study some simple Calogero-type models. In neither case do we find nonvanishing flow invariants.

3.1. Three dimensions

The stress-tensor trace is $\Theta = -\delta S/\delta \phi$ with the derivative taken at constant $\bar{\phi} = \phi e^{-\phi/2}$ and reads

$$\Theta = -\beta m^2 ((\partial_\mu \bar{\phi})^2 - \frac{1}{4} \Box \bar{\phi}^2 - \frac{1}{4} (\partial_\mu \phi)^2 \phi^2 - \frac{1}{4} \Box \phi \bar{\phi}^2 - \frac{1}{4} \bar{\phi} \partial^\mu \phi \partial^\nu \bar{\phi} \phi^2) e^{2\phi}$$

$$- \eta m^2 (\Box \bar{\phi}^2 + 3 (\partial_\mu \phi)^2 \bar{\phi}^2 + 3 \Box \phi \bar{\phi}^2 + 3 \partial_\mu \phi \partial^\nu \bar{\phi} \phi^2) e^{2\phi} - 2 m^4 \bar{\phi}^2 e^{4\phi}. \quad (3.1)$$

The field $\bar{\phi}$ has dimension $-1/2$ in units of mass and $\phi$ is dimensionless. The most convenient choice for the improvement parameter is $\eta = 1/12$, in which case all derivatives of the conformal factor $\phi$ disappear:

$$\Theta = -\beta m^2 ((\partial_\mu \bar{\phi})^2 - \frac{1}{6} \Box \bar{\phi}^2) e^{2\phi} - 2 m^4 \bar{\phi}^2 e^{4\phi}. \quad (3.2)$$

We have, in flat space,

$$\frac{\delta \Theta(x)}{\delta \phi(y)} = \delta(x - y)[2 \Theta(x) - 4 m^4 \bar{\phi}^2(x)],$$

$$\frac{\delta^2 \Theta(x)}{\delta \phi(y) \delta \phi(z)} = \delta(x - y) \delta(x - z)[4 \Theta(x) - 24 m^4 \bar{\phi}^2(x)].$$

The functions $P_i(x_1, \ldots, x_n)$ of the ansatz (1.4) cannot be polynomials in odd dimensions. The most general sum rule contains infinitely many terms. The best we can do is to explore a subclass of sum rules in which the $P_i$ are particularly simple, such that (1.4) contains finitely many terms. For example, we can take the $P_i$ which are the product of a polynomial of degree two
and the modulus of a distance between the points \(x_1, \ldots, x_i\) and 0. We recall that the equivalence relations [11]

\[
P_i(x_1, \ldots, x_i) \sim \begin{cases} P_i(x_1 - x_k, \ldots, x_{k-1} - x_k), \\ -x_k, x_{k+1} - x_k, \ldots, x_i - x_k) & \forall k = 1, \ldots, i, \\ 0 & \text{if } P_i \text{ is independent of any of the } x_k \end{cases}
\]

reduce the set of independent functions. Our subclass contains only three terms: for \(i = 1\) we have only \(P_1(x) = |x|^3\); for \(i = 2\) we have only \(P_2(x, y) = |x| |y|^2\), since \(P_2(x, y) = |x|^3 \sim 0, |x| x \cdot y \sim 0, |x - y| x^2 \sim |x - y| x \cdot y \sim |x| |y|^2\); for \(i = 3\) we have only \(P_3(x, y, z) = |x| y \cdot z\), since \(|x - y| z^2 \sim 2|x - y| x \cdot z \sim -2|x| y \cdot z\); for \(i > 3\) there is no nontrivial function \(P_i\) in our subclass. We have to study the expression

\[
\Delta A = -\lambda_1 \int |x|^3 \Gamma'_{x,0} d^3x - \lambda_2 \int |x| y^2 \Gamma'_{x,y,0} d^3x d^3y - \lambda_3 \int |x| y \cdot z \Gamma'_{x,y,z,0} d^3x d^3y d^3z.
\]

(3.3)

With the notation

\[
\pi T_2 = \int |x|^3 \langle \Theta(x) \Theta(0) \rangle d^3x
\]

\[
\pi T_2 F = m^4 \int |x|^3 \langle \Theta(x) \Phi^2(0) \rangle d^3x
\]

\[
\pi T_3 = \int |x| y^2 \langle \Theta(x) \Theta(y) \Theta(0) \rangle d^3x d^3y
\]

\[
\pi T_4 = \int |x| (y \cdot z) \langle \Theta(x) \Theta(y) \Theta(z) \Theta(0) \rangle d^3x d^3y d^3z
\]

\[
\pi T_2 F_1 = m^4 \int |x| y^2 \langle \Theta(x) \Theta(y) \Phi^2(0) \rangle d^3x d^3y
\]

\[
\pi T_2 F_2 = m^4 \int |x| (x \cdot y) \langle \Theta(x) \Theta(y) \Phi^2(0) \rangle d^3x d^3y
\]

(in every correlator \(\Theta\) is meant in flat space, i.e. at \(\phi = 0\), we find

\[
I_1 = -\int |x|^3 \Gamma'_{x,0} d^3x = \pi T_2
\]

\[
I_2 = -\int |x| y^2 \Gamma'_{x,y,0} d^3x d^3y = \pi (T_3 + 2T_2 - 4TF),
\]

\[
I_3 = -\int |x| y \cdot z \Gamma'_{x,y,z,0} d^3x d^3y d^3z = \pi (T_4 + 2T_3 - 4T_2 F_1 - 8T_2 F_2 - 4T_2 - 8TF).
\]

To write the last formula, we have used a kinematic vanishing relation of the type [11], namely

\[
\int d^{2n}x \prod_{i=1}^{k-1} d^{2n}x_i F(x_1, \ldots, x_{k-1}) \Gamma'_{x_1, \ldots, x_{k-1}, 0} = 0.
\]

This identity holds for every homogeneous function \(F(x_1, \ldots, x_{k-1})\) of degree \(d = 2n\). In particular, we have

\[
\int |x| (x \cdot y) \langle \Theta(x) \Theta(y) \Theta(0) \rangle d^3x d^3y = -2\pi T_2 + 4\pi TF.
\]
To study the $r$-dependence of $\Delta A(r)$, we have computed the values of $I_{1,2,3}$ and their second, fourth and sixth derivatives at $r = 1$ (the odd derivatives are related to the even derivatives in a simple way). We find

| $r = 1$ | $T_2$ | $T_4$ | $T_3$ | $T_2T_4$ | $T_3T_2$ | $T_4/T_2$ | $I_1/\pi$ | $I_2/\pi$ | $I_3/\pi$ |
|---------|-------|-------|-------|---------|---------|---------|---------|---------|---------|
| $d^2/dr^2$ at $r = 1$ | 16 | 1 | 2 | 9 | 45 | 2408 | 9 | 1858 | 12 |
| $d^4/dr^4$ at $r = 1$ | 43 | 9 | 16 | 108 | 30240 | 4 | 100399 | 5 | 945 |
| $d^6/dr^6$ at $r = 1$ | 48 | 0 | 953 | 610121 | 0 | 8284253 | 0 | 179 | 144 |

Analysing the data of this table, we do not find non-trivial flow invariants, but we do find a non-trivial vanishing relation, which reads

$$\int |x| y^2 \Gamma_{x,y,0} d^3 x d^3 y - 2 \int |x| y \cdot z \Gamma_{x,y,z,0} d^3 x d^3 y d^3 z = 0. \quad (3.4)$$

Presumably, it is possible to prove this relation in full generality studying the embedding in external gravity. This does not seem to be straightforward, however, since the $\phi$-dependence is trivial at the critical points and non-local at intermediate energies.

### 3.1.1. Improvement.

To perform the calculations, we have chosen a convenient value $\eta = 1/12$ of the improvement parameter in (3.1). Due to the non-locality of the functions $P_i(x_1, \ldots, x_i)$, the other values of $\eta$ produce undesirable divergent terms of the form

$$\int d^3 x \frac{\delta(x)}{|x|} G(|x|), \quad (3.5)$$

in the integrals $I_2$ and $I_3$. Here $G(|x|)$ denotes some regular function. It was shown in [14] that a flow invariant is insensitive to the value of the improvement parameter $\eta$. So, we expect that the divergences (3.5) cancel out in the combination (3.4). This happens if $G(0) = 0$. We are going to check that this is true inspecting the terms of the form (3.5) in (3.4) to the first order in $1 - 12\eta$.

Observe that because of the $\delta$-function in (3.5), the divergences (3.5) appear only in the integrals of correlators containing insertions of $\delta \tilde{\Theta}(x) / \delta \phi(y)|_{\phi=0}$ and $\delta^2 \tilde{\Theta}(x) / \delta \phi(y) \delta \phi(z)|_{\phi=0}$. They do not appear in the integrals of correlators containing only insertions of $\tilde{\Theta}$. In particular, no divergence (3.5) appears in $I_1$. Performing the differentiations, we find that, to the first order in $1 - 12\eta$, $G(0)$ is equal to $1 - 12\eta$ times

$$2 \int d^3 y d^3 z \cdot z \langle \tilde{\Theta}(y) \tilde{\Theta}(z) \hat{\varphi}^2(0) \rangle + m^2 \int d^3 y y^2 \langle 3 \tilde{\Theta}(y) - 8m^2 \hat{\varphi}^2(y) \rangle \hat{\varphi}^2(0). \quad (3.6)$$

Now, these integrals contain only local polynomials and can be computed easily for arbitrary $r$. We find

$$\int d^3 y d^3 z \cdot z \langle \tilde{\Theta}(y) \tilde{\Theta}(z) \hat{\varphi}^2(0) \rangle = \sqrt{r} (23r^4 + 100r^3 + 186r^2 + 100r + 23) / 16(r + 1)^5,$$

$$m^2 \int d^3 y y^2 \langle \tilde{\Theta}(y) \hat{\varphi}^2(0) \rangle = -\frac{5\sqrt{r}}{8(r + 1)},$$

$$m^4 \int d^3 y y^2 \langle \hat{\varphi}^2(0) \hat{\varphi}^2(0) \rangle = \frac{\sqrt{r} (r^4 + 5r^3 + 12r^2 + 5r + 1)}{8(r + 1)^5}.$$
The combination (3.6) is the unique vanishing linear combination of the three integrals just computed.

This result is further evidence that the vanishing of (3.4) is not a coincidence, but must be a consequence of the embedding in external gravity. Moreover, the convergence criterion $G(0) = 0$ can be used to fix the relation (3.4) with much less effort than the inspection of flow invariance. We expect that, with the methods developed in this section, many more non-trivial relations can be found.

3.2. One dimension

In one dimension the stress tensor coincides with its trace. The stress tensor two-point function does not define a central charge $c$. We therefore look for a central charge of type $a$. The simplest candidate sum rule for the difference $\Delta A$ between the critical values of a central charge of type $a$ reads

$$\Delta A = \int_0^\infty dt \left| \langle 0 | T(\Theta(t)) : \Theta(0) : | 0 \rangle \right| = 2 \int_0^\infty dt \langle 0 | : \Theta(t) : \Theta(0) : | 0 \rangle.$$

Here $\Theta(t) := \Theta(t) - \langle 0 | \Theta(0) | 0 \rangle$ is the normal product. Inserting a complete set of orthonormalized states $|n\rangle$, we obtain

$$\Delta A = 2 \int_0^\infty dt \sum_n \langle 0 | : \Theta(t) : | n \rangle \langle n | : \Theta(0) : | 0 \rangle = 2 \sum_{n>0} \frac{|\langle 0 | \Theta | n \rangle|^2}{(E_n - E_0)^2}. \quad (3.7)$$

We want to test the flow invariance of this expression in a family of flows interpolating between the same fixed points, or between two families of continuously connected fixed points. In one dimension, the classically conformal potential is

$$V(q) = \frac{1}{q^2}.$$

We consider the model

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^2 \dot{q}_i^2 + m^2 q_i^2 + \frac{g^2}{2q^2}, \quad (3.8)$$

where $q^2 = q_1^2 + q_2^2$ and $g$ is positive. We prefer to study this model, instead of the ‘two-body Calogero model’

$$\mathcal{L} = \frac{1}{2} \dot{q}^2 + m^2 q^2 + \frac{g^2}{2q^2}, \quad (3.9)$$

since (3.8) reduces to a harmonic oscillator for $g = 0$, while (3.9) does not. We use the Euclidean notation. We have a potential

$$V = \frac{1}{2} \left( m^2 q^2 + \frac{g^2}{q^2} \right),$$

with minima at

$$\bar{q} = \pm \sqrt{\frac{g}{m}}, \quad V(\bar{q}) = mg.$$

The stress tensor is equal to its trace:

$$\Theta = -m^2 q^2.$$
The eigenfunctions and eigenvalues are
\[
\psi_{n,l}(\rho, \theta) = \frac{m^\nu n!}{\pi (n+\nu)!} \rho^\nu L_n^\nu(m\rho^2), \quad E_{n,l} = m(2n + 1 + \nu),
\]
(3.10)
with \( \nu = \sqrt{g^2 + l^2} \). Here \( L_n^\nu \) is the associated Laguerre polynomial. When \( g \to 0 \) we recover the two-dimensional harmonic oscillator.

The UV fixed points of the model (3.8) are the one-parameter family of conformal field theories
\[
\mathcal{L} = \frac{1}{2} \sum_{i=1}^2 \dot{q}_i^2 + \frac{g^2}{2q^2}.
\]
If the central charge \( A_{UV} \) satisfied the marginality property (3) of the introduction, then it would be \( g \) independent.

The IR fixed point of (3.8) is the empty theory, since the potential is everywhere infinite for \( m \to \infty \). We therefore expect \( A_{IR} = 0 \) for every candidate central charge \( A \). We conclude that marginality and flow invariance require that \( \Delta A \) does not depend on \( g \). We want to check if our candidate \( \Delta A \) of (3.7) is \( g \) independent.

This does not turn out to be true. Indeed, using some standard properties of the associated Laguerre polynomials, we find
\[
\langle 0 | q^2 | n, l \rangle = \frac{(g + 1)!}{m} \sqrt{\frac{n!}{g!(n+g)!}} \delta_{l0} (\delta_{n0} - \delta_{n1})
\]
(3.11)
and
\[
\Delta A = \frac{1}{2} (1 + g).
\]
(3.12)
We can try with more general sum rules. Straightforward manipulations allow us to derive the generalization of formula (3.11), which reads
\[
\langle n, l | \Theta | n', l' \rangle = -m \delta_{l'll'} \delta_{nn'} (2n + \nu + 1) - \sqrt{\frac{(n+1)(n+\nu+1)}{n(n+\nu)}} \delta_{n,n'1}.
\]
Using this result, we can compute
\[
\sum_{n,n',n''>0 \atop n \neq n' \neq n''} \frac{\langle 0 | \Theta | n, l \rangle \langle n, l | \Theta | n', l' \rangle \langle n', l' | \Theta | n'', l'' \rangle \langle n'', l'' | \Theta | 0 \rangle}{E_0 - E_n - E_n - E_n - E_n - E_0} = -\frac{1}{16} (5 + 8g + 3g^2).
\]
A natural tentative sum rule containing infinitely many terms reads
\[
\Delta A = 2 \sum_{n>0} \frac{|\langle 0 | \Theta | n \rangle|^2}{(E_n - E_0)^2}
\]
\[
+ d_1 \sum_{n,n',n''>0 \atop n \neq n' \neq n''} \frac{\langle 0 | \Theta | n, l \rangle \langle n, l | \Theta | n', l' \rangle \langle n', l' | \Theta | n'', l'' \rangle \langle n'', l'' | \Theta | 0 \rangle}{E_0 - E_n - E_n - E_n - E_n - E_0} + \cdots,
\]
where the constants \( d_i \) should be fixed imposing that \( \Delta A \) be \( g \) independent. However, the infinite system of equations cannot be solved recursively. This is evident from the expressions (2.18) and (2.10) of the first two contributions. Our procedure does not allow us to fix a sum rule with infinitely many terms using only flow invariance.
4. Conclusions

Several properties of the trace anomalies in external gravity appear to lie beyond the gravitational embedding and need a more general treatment to be fully understood. In this paper we have made some steps in this direction. In even dimensions we have been able to recover the known results without relying on the gravitational embedding. In odd dimensions the situation is complicated by the fact that if a non-trivial flow invariant of type $a$ exists, the sum rule (1.4) probably contains infinitely many terms. The procedure of this paper does not allow us to explore sum rules containing infinitely many terms. Within a class of sum rules containing finitely many terms we find the nontrivial vanishing relation (3.4) in three dimensions, but we do not find a non-vanishing flow invariant of type $a$. Our results are an indication that such a quantity might not exist at all in odd dimensions.

We mention some directions for future investigations. The study of the irreversibility of the RG flow in even dimensions [4, 9, 15, 8] points out the peculiarity of classically conformal theories. This class of theories might be peculiar in odd dimensions also. The simplest odd-dimensional domain for this study is three-dimensional quantum field theory, in particular the flows constructed in [19]. Indeed, in one dimension classically conformal theories have trivial flows, because they are also conformal at the quantum level. Finally, the properties of the central charge $c$, defined using the stress-tensor two-point function, need to be studied separately. Progress in these directions will be reported soon.

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Appendix. Sum rules in six dimensions

Here we collect some lengthy formulae used in section 2.2. The sum rules for the $b_i$ of formula (2.16) are

\[
\int x^6 \Gamma_{x,0}^\prime \, d^6x = -46\,080\Delta b_1, \\
\int (x \cdot y)^3 \Gamma_{x,y,0}^\prime \, d^6x \, d^6y = -6912\Delta b_2 - 12\,672\Delta b_3, \\
\int x^4 y^2 \Gamma_{x,0}^\prime \, d^6x \, d^6y = -27\,648\Delta b_2 - 12\,288\Delta b_3, \\
\int x^2 (x \cdot y) (x \cdot z) \Gamma_{x,y,z,0}^\prime \, d^6x \, d^6y \, d^6z = -1536\Delta b_4 - 1536\Delta b_5 + 8448\Delta b_6, \\
\int x^4 (y \cdot z) \Gamma_{x,y,z,0}^\prime \, d^6x \, d^6y \, d^6z = -9216\Delta b_4 - 9216\Delta b_5 + 12\,288\Delta b_6, \\
\int x^2 (y \cdot z)^2 \Gamma_{x,y,z,0}^\prime \, d^6x \, d^6y \, d^6z = -5376\Delta b_4 + 384\Delta b_5 + 8448\Delta b_6, \\
\int x^2 y^2 (z \cdot k) \Gamma_{x,y,z,k,0}^\prime \, d^6x \, d^6y \, d^6z \, d^6k = 15 \,360\Delta b_7, \\
\int x^2 (y \cdot t) (z \cdot k) \Gamma_{x,y,z,k,t,0}^\prime \, d^6x \, d^6y \, d^6z \, d^6k \, d^6t = -46\,080\Delta b_8.
\]
Their evaluation in the model (2.9) gives

\[
(4\pi)^3 \Delta b_1 = -\frac{1}{5040(r^2 - 1)^5} (149r^{10} + 25r^8 - 1660r^6 + 1660r^4 - 25r^2 - 149 \\
- 420(r^{10} - r^8 - 2r^6 - 2r^4 - r^2 + 1) \log(r)),
\]

\[
(4\pi)^3 \Delta b_2 = \frac{1}{18900(r^2 - 1)^5} (3649r^{10} + 9475r^8 - 31550r^6 + 31550r^4 - 9475r^2 - 3649 \\
- 1890(5r^{10} - r^8 + 4r^6 + 4r^4 - r^2 + 5) \log(r)),
\]

\[
(4\pi)^3 \Delta b_3 = -\frac{1}{1575(r^2 - 1)^3} (167r^{10} + 775r^8 - 4000r^6 + 4000r^4 - 775r^2 - 167 \\
- 210(5r^{10} - 7r^8 - 2r^6 - 2r^4 - 7r^2 + 5) \log(r)),
\]

\[
(4\pi)^3 \Delta b_4 = \frac{1}{4725(r^2 - 1)^5} (3434r^{10} + 8975r^8 - 10075r^6 + 10075r^4 - 8975r^2 - 3434 \\
- 315(15r^{10} + 29r^8 + 64r^6 + 64r^4 + 29r^2 + 15) \log(r)),
\]

\[
(4\pi)^3 \Delta b_5 = \frac{1}{9450(r^2 - 1)^3} (71r^{10} - 8125r^8 + 21500r^6 - 21500r^4 + 8125r^2 - 71 \\
+ 210(5r^{10} - 9r^8 + 6r^6 + 6r^4 - 9r^2 + 5) \log(r)),
\]

\[
(4\pi)^3 \Delta b_6 = -\frac{1}{1575(r^2 - 1)^3} (167r^{10} + 775r^8 - 4000r^6 + 4000r^4 - 775r^2 - 167 \\
- 210(5r^{10} - 7r^8 - 2r^6 - 2r^4 - 7r^2 + 5) \log(r)).
\]

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