EXAMPLES OF SMOOTH NON-GENERAL TYPE SURFACES IN $\mathbb{P}^4$

Sorin Popescu

0. Introduction

Smooth projective varieties with small invariants have got renewed interest in recent years, primarily due to the fine study of the adjunction mapping by Reider, Sommese, Van de Ven and others. For the special case of smooth surfaces in $\mathbb{P}^4$ the method goes back to the Italian geometers, who at the turn of the century used it for the study of the surfaces of degree less than 7, or sectional genus $\pi \leq 3$. Later on, for larger values of the invariants, there are contributions by Combesattini and especially Roth. For example, in [38], Roth tried to establish a classification of smooth surfaces with $\pi \leq 6$, but his lists are incomplete since he misses the non-special rational surfaces of degree 9 and the minimal bielliptic surfaces of degree 10. Nowadays, through the effort of several mathematicians (some references are given below), a complete classification of smooth surfaces in $\mathbb{P}^4$ has been worked out up to degree 10, and a partial one is available in degree 11.

But, apart from the general framework of classification problems concerning codimension two varieties, there is another strong motivation for the interest in these surfaces. Namely, in a recent paper Ellingsrud and Peskine [19] proved Hartshorne’s conjecture that there are only finitely many families of special surfaces in $\mathbb{P}^4$. More specifically, given an integer $a < 6$, they show that the degree of smooth surfaces with $K^2 \leq a \chi$ is bounded. In particular, there are only finitely many families of smooth surfaces in $\mathbb{P}^4$, not of general type. However, the question of an exact degree bound is still open. A recent work of Braun and Fløystad [10] improves the initial bound ($\sim 10000$) of Ellingsrud and Peskine to $d \leq 105$, but it is believed that the degree of the smooth, non-general type surfaces in $\mathbb{P}^4$ should be less than or equal to 15. A similar finiteness result for 3-folds in $\mathbb{P}^5$ was proved in [11], but the real degree bound is believed to be much higher in this case. Nevertheless, examples of smooth 3-folds in $\mathbb{P}^5$ not of general type are known only up to degree 18 (see [18] for more details and a complete list of known examples).

Another reason for the interest in studying surfaces in $\mathbb{P}^4$ is the small number of known liaison classes of such surfaces. Each new specimen of liaison classes is of

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real interest. In this direction, the work of Decker, Ein and Schreyer [17] provides a powerful and effective method of construction of surfaces in \( \mathbb{P}^4 \).

The aim of this paper is to provide a series of examples of smooth surfaces in \( \mathbb{P}^4 \), not of general type, in degrees varying from 12 up to 14, and to describe part of their geometry. In degree 15, two families of abelian surfaces [23], [3], [34], and a family of bielliptic surfaces [4] are currently known. We have tried to work out examples of degrees higher than 15 but failed in this attempt. The methods of construction we used are mainly the syzygy approach of [17] and liaison techniques. The families we construct are:

- minimal proper elliptic surfaces of degree 12 and sectional genus \( \pi = 13 \),
- two types of non-minimal proper elliptic surfaces of degree 12 and sectional genus \( \pi = 14 \),
- non-minimal \( K3 \) surfaces of degree 13 and sectional genus 16, and
- non-minimal \( K3 \) surfaces of degree 14 and sectional genus 19.

At this point it may be appropriate to recall some references for the list of the smooth surfaces in \( \mathbb{P}^4 \) of degrees less or equal to 11. The classification and construction of surfaces of degree \( \leq 7 \) was initiated in [38] and completed up to degree 8 in [24], [25], [30], [31], [32], supplemented for the case of rational surfaces of degree 8, sectional genus 5 by [1]. In degree 9, the rational surfaces are described in [1] and [2], the Enriques surfaces with \( \pi = 6 \) in [14] and [15], while the classification and description of the liaison classes is completed in [5]. In degree 10, the classification in terms of numerical invariants and the description of a large number of surfaces is achieved in the beautiful thesis of K. Ranestad [36]. The existence, the uniqueness and the geometry of bielliptic surfaces of degree 10, \( \pi = 6 \) were taken care by [42] and [4], respectively, the Enriques surfaces of degree 10, \( \pi = 8 \) were first constructed in [17] and further studied in [12], while the minimal abelian surfaces were first described about 20 years ago in [23]. Finally, a non-minimal \( K3 \) surface of degree 10, \( \pi = 9 \), lying on only one quartic hypersurface was constructed [34], thus giving a positive answer to the last open existence case in [36]. The remaining uniqueness problems, the syzygies and the description of the liaison classes in degree 10 were completed in [35]. Finally, 23 different families of smooth surfaces are known in degree 11, and [34] is an attempt at construction and classification in this degree.

1. Preliminaries

Double point formula. For smooth surfaces \( S \subset \mathbb{P}^4 \) the relation [22]:

\[
d^2 - c_2(N_S) = d^2 - 10d - 5H \cdot K - 2K^2 + 12\chi(S) = 0,
\]

expresses the fact that \( S \) has no double points.

Linkage [33]. Two surfaces \( S \) and \( S' \) in \( \mathbb{P}^4 \) are said to be linked \( (m,n) \) if there exist hypersurfaces \( V \) and \( V' \) of degree \( n \) and \( m \) respectively, without common components and such that \( V \cap V' = S \cup S' \). The standard sequences of linkage, namely

\[
0 \rightarrow \mathcal{O}_S(K) \rightarrow \mathcal{O}_S(m+n-5) \rightarrow \mathcal{O}_{S'}(m+n-5) \rightarrow 0
\]
yield then $\chi(S') = \chi(V \cap V') - \chi(\mathcal{O}_S(m+n-5))$ and a relation between the sectional genera: $\pi(S) - \pi(S') = \frac{1}{2}(m+n-4)(d(S) - d(S'))$.

The Eagon-Northcott complex method \[17\]. The aim of this method is to realize a surface $S \subset \mathbb{P}^4$ as the determinantal locus $S = D(\varphi) = \{ x \in \mathbb{P}^4 \mid \text{rk } \varphi(x) < e \}$ of a map $\varphi$ between two vector bundles $\mathcal{E}$ of rank $e$, and $\mathcal{F}$ of rank $e + 1$ on $\mathbb{P}^4$. In case $\varphi$ degenerates in (expected) codimension two, $D(\varphi)$ is locally Cohen-Macaulay and the Eagon-Northcott complex is exact and identifies coker $\varphi$ with the twisted ideal sheaf of $S$:

$$0 \rightarrow \mathcal{E} \xrightarrow{\varphi} \mathcal{F} \rightarrow \mathcal{I}_S(c_1(\mathcal{F}) - c_1(\mathcal{E})) \rightarrow 0.$$ 

In order to construct a surface $S$ with the desired invariants one has to find appropriate vector bundles $\mathcal{E}$ and $\mathcal{F}$, and a general method for this is to determine first the differentials of the Beilinson spectral sequence applied to the twisted ideal sheaf of $S$:

$$E_1^{pq} = H^q(\mathbb{P}^n, \mathcal{I}(p)) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)$$

converging to $\mathcal{I}$; i.e. $E_1^{pq} = 0$ for $p + q \neq 0$ and $\oplus E_1^{p-q} \mathcal{I}$ is the associated graded sheaf of a suitable filtration of $\mathcal{I}$.

All the $E_1$-terms are in the second quadrant and only finitely many of them are non-zero. Via canonical isomorphisms induced by contraction, $\text{Hom}(\Omega^i_{\mathbb{P}^n}(i), \Omega^j_{\mathbb{P}^n}(j)) \cong \Lambda^{i-j}V$, for $i \geq j$, cf. \[9\], the $d_1$-differentials

$$d_1^{pq} \in \text{Hom} \left( H^q(\mathbb{P}^n, \mathcal{I}(p)) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p), H^q(\mathbb{P}^n, \mathcal{I}(p+1)) \otimes \Omega_{\mathbb{P}^n}^{-p-1}(-p-1) \right)$$

can be identified with the natural multiplication maps in

$$\text{Hom} \left( H^0(\mathbb{P}^n, \mathcal{I}(1)) \otimes H^q(\mathbb{P}^n, \mathcal{I}(p)), H^q(\mathbb{P}^n, \mathcal{I}(p+1)) \right).$$

In our specific case, this means that to determine the $d_1$-differentials is equivalent to fixing the module structure of the Hartshorne-Rao modules $\oplus_p H^q(\mathbb{P}^4, \mathcal{I}(p))$, $q \in \{1, 2\}$.

Multisecants \[28\]. Some classical numerical formulas for multisecant lines to a smooth surface $S$ in $\mathbb{P}^4$ have been recently given a modern treatment by Le Barz. Consider the double curve $\Gamma$ of a general projection of such a surface $S$ to $\mathbb{P}^3$ and denote by

$$\delta = \left( \binom{d-1}{2} \right) - \pi$$

the degree of $\Gamma$, by

$$t = \left( \binom{d-1}{2} \right) - \pi(d-3) + 2\chi - 2,$$

where $\chi$ is the uniformity of $S$. Theorem 1.1. \[9\]. Let $\mathcal{G}$ be a coherent sheaf on $\mathbb{P}^n = \mathbb{P}(V)$. There exists a spectral sequence with $E_1$ terms

$$E_1^{pq} = H^q(\mathbb{P}^n, \mathcal{G}(p)) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)$$

converging to $\mathcal{G}$; i.e. $E_1^{pq} = 0$ for $p + q \neq 0$ and $\oplus E_1^{p-q} \mathcal{G}$ is the associated graded sheaf of a suitable filtration of $\mathcal{G}$.
the number of apparent triple points, i.e., the number of trisecants to \( S \) which meet a general point, and by

\[
h = \frac{1}{2}(\delta(\delta - d + 2) - 3t)
\]

the number of apparent double points on \( \Gamma \). Suppose there are no lines on \( S \) with positive self-intersection. Then the number of 6-secants (if finite) plus the number of exceptional lines on \( S \) is:

\[
N_6(d, \pi, \chi) = -\frac{1}{144}d(d - 4)(d - 5)(d^3 + 30d^2 - 577d + 786) + \delta(2\left(\frac{d}{4}\right) + 2\left(\frac{d}{3}\right) - 45\left(\frac{d}{2}\right) + 148d - 317) - \frac{1}{2}\left(\frac{\delta}{2}\right)(d^2 - 27d + 120) - 2\left(\frac{\delta}{3}\right) + h(\delta - 8d + 56) + t(9d - 3\delta - 28) + \binom{t}{2}.
\]

2. THREE FAMILIES OF SMOOTH ELLIPTIC SURFACES OF DEGREE 12 AND A SMOOTH K3 SURFACE OF DEGREE 13

We construct in the sequel several examples of smooth, regular, proper elliptic surfaces of degree 12 in \( \mathbb{P}^4 \). Previously known examples of non-general type surfaces with \( d = 12 \) were only the blown-up K3 surfaces with \( \pi = 14 \), with one exceptional quartic and ten exceptional lines constructed in [17].

Recall first that for a smooth surface \( S \) of degree 12 in \( \mathbb{P}^4 \) the double point formula reads

\[
K^2 = 47 - 5\pi + 6\chi,
\]

while Severi’s theorem [43] and Riemann-Roch give

\[
\pi = \chi + 8 + h^1(\mathcal{O}_S(H)) - h^0(\mathcal{O}_S(K - H)).
\]

**Proposition 2.1.** There exist smooth, regular, minimal proper elliptic surfaces \( S \subset \mathbb{P}^4 \), with \( d = 12 \), \( \pi = 13 \), \( \chi = 3 \), and 10 skew 6-secant lines.

*Proof.* For construction we use the syzygy approach in [17]. A promising Beilinson cohomology table is

\[
\begin{array}{c|c|c|c|c}
\hline
i & 2 & 2 & 4 & 5 \\
\hline
\end{array}
\]

\[
h^i(\mathcal{J}_S(p))
\]
where $h^i(J_S) = h^i(J_S(1)) = 0$ ($S$ being reduced and linearly normal, cf [43]),
$h^2(J_S) = h^1(\mathcal{O}_S) = q = 0$, $h^3(J_S) = h^2(\mathcal{O}_S) = p_g = 2$, $h^3(J_S(k)) = 0$, for $k \geq 1$ (since $\kappa(S) < 2$), while $h^0(J_S(m)) = 0$ for $m \leq 3$. We may set $E := 2\Omega(1) \oplus 2\Omega^3(3)$ and $\mathcal{F} := (\ker \psi)$, where $4\mathcal{R}(1) \hookrightarrow 15\mathcal{R}$ is the minimal free presentation of the graded $R = \mathbb{C}[x_0, x_1, x_2, x_3, x_4]$-module $H^1(J_S(\ast + 4)) = H^1(\mathcal{F}(\ast))$, and try to construct the surface as the degeneracy locus of a general morphism $\varphi \in \text{Hom}(E, \mathcal{F})$.

However, for a general choice of the matrix $\psi$, the module $M := \text{coker } \psi$ has a minimal free resolution of type

$$
M \hookrightarrow 4\mathcal{R}(1) \hookrightarrow 15\mathcal{R}(1) \oplus 10\mathcal{R}(2) \leftarrow 30\mathcal{R}(3) \leftarrow 21\mathcal{R}(4) \leftarrow 5\mathcal{R}(5) \leftarrow 0
$$

and thus $\text{Hom}(\Omega^3(3), \mathcal{F}) = 0$ in this case. What is needed for the construction to work is that $\psi$ has at least two linear syzygies of second order. We will choose a $\psi$ featuring such syzygies.

Let $F$ be the Horrocks-Mumford bundle (see [23]). It is a stable rank 2 vector bundle on $\mathbb{P}^4$ with Chern classes $c_1 = -1$, $c_2 = 4$, and its $H^1$-cohomology module has a minimal free resolution of type (cf. [16], see also [34])

$$
0 \hookrightarrow H^1(F(\ast)) \hookrightarrow 5\mathcal{R} \xrightarrow{\gamma} 15\mathcal{R}(1) \oplus 4\mathcal{R}(3) \leftarrow 2\mathcal{R}(3) \leftarrow 15\mathcal{R}(4) \oplus 35\mathcal{R}(5) \leftarrow 20\mathcal{R}(6) \leftarrow 2\mathcal{R}(8) \leftarrow 0,
$$

with $\gamma = (M_{z_0}(x) | M_{z_1}(x) | M_{z_2}(x))$, where $M_{z}(x) = (x_{i+j}z_{i-j})_{i,j \in \mathbb{Z}_5}$ are Moore blocks, and the parameters are say $z_0 = (1 : 0 : 0 : 0 : 0)$, $z_1 = (0 : 1 : 0 : 0 : 1)$ and $z_2 = (0 : 0 : 1 : 1 : 0)$.

From [23], or even just by looking at the above resolution since $F(-1)$ is the cokernel of the unique morphism $0 \to 2\Omega^3(2) \xrightarrow{\theta} Syz_1(H^1(F(\ast))),$ we have $h^0(F) = h^0(F(1)) = 0$ and $h^0(F(2)) = 4$.

Consider now a rank 3 vector bundle $E$ on $\mathbb{P}^4$ constructed as the extension

$$(2.2) \quad 0 \to F \to E \to \mathcal{O} \to 0$$

corresponding to a non-trivial element $0 \neq \xi \in H^1(F) = \text{Ext}^1(\mathcal{O}, F)$. Then $E$ has Chern classes $c_1(E) = -1$, $c_2(E) = 4$ and $c_3(E) = 0$, and is stable because $F$ is.

Also $h^1(E) = h^1(F) - 1 = 4$ by construction. We will assume in the sequel that the extension $E$ comes from a generic element $\xi \in H^1(F)$, meaning by this that $\xi$ satisfies the following two conditions:

- the natural map $\mathbb{C}\xi \otimes H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \to H^1(F(1))$ induced by multiplication with linear forms is injective, while
- the similar natural map $\mathbb{C}\xi \otimes H^0(\mathcal{O}_{\mathbb{P}^4}(2)) \to H^1(F(2))$ is surjective.

To see that such a choice is possible one either checks it for a random $\xi$ via [7], or one uses the invariance of $H^1(F(\ast))$ under the group $G = \mathbb{H}_5 \rtimes \mathbb{Z}_2$, where $\mathbb{H}_5$ is the Heisenberg group of level 5. Namely, using notations and facts from
[23], or [29], if \( \mathbb{P}^4 = \mathbb{P}(V) \) then we can find a basis \( e_0, \ldots, e_4 \) of \( V \) such that, under the Schrödinger representation of \( \mathbb{H}_5 \) on \( \mathbb{P}^4 \), \( H^1(F) = V_3 \), \( H^1(F(1)) = 2V_1^2 \), \( H^1(F(2)) = 2V_0^2 \), while the multiplication map \( V^* \otimes H^1(F) \to H^1(F(1)) \) is given by the projection on the second factor \( V^* \otimes V_3 \cong 3V_1 \oplus 2V_1^2 \to 2V_1^2 \). One checks easily that \( \xi := \sum_{i=0}^{4} (-1)^i e_i \in H^1(F) \) has the desired properties, and thus deduces that there is a Zariski open subset of elements of \( H^1(F) \) satisfying the two conditions.

With this choice of \( \xi \), the exact sequence (2.2) yields \( h^0(E(1)) = 0 \), \( h^0(E(2)) = 5 \), \( h^1(E(1)) = 5 \) and \( h^1(E(m)) = 0 \) for all \( m \geq 2 \), or \( m \leq -1 \). Summarizing, we deduce that \( M := H^1(E(\ast)) \) is an artinian module with Hilbert function \((4, 5)\) and with the desired syzygies.

As we can check in an example via [7], the dependency locus of two general sections in \( H^3(E(2)) \) is a smooth surface \( S \subset \mathbb{P}^4 \) with \( d = 12 \), \( \pi = 13 \) and \( \chi = 3 \). The ideal sheaf \( I_S \) has a resolution of type

\[
\begin{array}{ccccccccc}
0 & \to & 2\mathcal{O} & \to & E(2) & \to & I_S(5) & \to & 0
\end{array}
\]

This description of \( S \) is equivalent to that as the degeneracy locus of a general morphism \( \varphi \in \text{Hom}(E, \text{Syz}_1(H^1(E(\ast + 1)))) \). In particular, \( S \) has a minimal free resolution of type

\[
\begin{array}{ccccccccc}
0 & \leftarrow & 3\mathcal{O}(-5) & \oplus & 12\mathcal{O}(-6) & \leftarrow & 30\mathcal{O}(-7) & \leftarrow & 21\mathcal{O}(-8) & \leftarrow & 5\mathcal{O}(-9) & \leftarrow & 0
\end{array}
\]

Dualizing (2.3) we obtain

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{O}(-5) & \to & E^\vee(-2) & \to & 2\mathcal{O} & \to & \omega_S & \to & 0
\end{array}
\]

thus \( \omega_S \) is globally generated, \( p_g = 2 \), and since the double point formula yields \( K^2 = 0 \) we deduce that \( S \) is a minimal proper elliptic surface. Now Le Barz’s formula (1.2) gives \( N_3 = 10 \), hence there are ten 6-secant lines to \( S \) because \( E(2) \) is globally generated outside a 3-codimensional set.

**Corollary 2.4.** There exist smooth non-minimal \( K3 \) surfaces \( X \subset \mathbb{P}^4 \) with \( d = 13 \), \( \pi = 16 \) which are embedded by a linear system

\[
|H| = |H_{\text{min}} - 7E_0 - \sum_{i=1}^{10} E_i|
\]
Proof. A minimal proper elliptic surface $S \subset \mathbb{P}^4$ as constructed in Proposition 2.1 can be linked in the complete intersection of two quintic hypersurfaces to a surface $X$ with invariants $d = 13$, $\pi = 16$, $\chi = 2$ and a resolution of type

$$(2.5) \quad 0 \to \mathcal{E}^\vee(-2) \to 4\mathcal{O} \to \mathcal{I}_X(5) \to 0.$$ 

Smoothness can be checked again in an example. $X$ is cut out by quintic hypersurfaces, hence there are no 6-secant lines this time. On the other hand, Le Barz’s formula gives $N_6 = 10$, so there exist 10 exceptional lines on $X$, namely the 6-secant lines to $S$. To describe $X$ we will use adjunction theory. Let $X_1$ denote the image of $X$ under the adjunction map, defined by $|H + K|$, and $X_2$ denote the image of $X_1$ under the adjunction map defined by $|H_1 + K_1|$. We compute the following invariants:

$$
\begin{align*}
X_1 &\subset \mathbb{P}^{16} \\
H_1^2 &= 36 & H_1K_1 &= 6 & K_1^2 &= -1 & \pi_1 &= 22 \\
X_2 &\subset \mathbb{P}^{22} \\
H_2^2 &= 47 & H_2K_2 &= 5 & K_2^2 &= -1 + b & \pi_2 &= 27,
\end{align*}
$$

where $b$ is the number of $(-1)$-conics on $X$. The Hodge index theorem gives $K_2^2 = -1 + b \leq 0$, thus $X$ is either a $K3$, or a proper elliptic surface. Moreover, in case it is elliptic, $X$ has a $(-1)$ conic or a $(-1)$ cubic and the proper transform of the canonical divisor on the minimal model is an elliptic curve of degree 4 or 5, while in case $X$ is a $K3$ surface there is an exceptional rational septic curve on it. To prove the claim of the corollary it is enough to check that $p_2 = 1$, since in the elliptic case Kodaira’s formula for the canonical divisor gives $p_2 = h^0(\omega_X \otimes^2) \geq 2$. Dualizing $(2.5)$ we obtain a resolution of $\omega_X$

$$
0 \to \mathcal{O}(-5) \to 4\mathcal{O} \to E(2) \to \omega_X \to 0,
$$

and thus one for $\omega_X \otimes^2$:

$$
0 \to 6\mathcal{O} \to 4E(2) \to S^2(E)(4) \to \omega_X \otimes^2 \to 0.
$$

Splitting it up in short exact sequences and using the facts that $h^1(E(2)) = 0$ and $h^0(E(2)) = 5$ we deduce $p_2 = h^0(S^2(E)(4)) - 14$. Now, from $(2.2)$, $S^2(E)$ can be realized as an extension

$$
0 \to S^2(F) \to S^2(E) \to E \to 0,
$$

while $h^0(S^2(F)(4)) = 0$, $h^1(E(4)) = 0$, and the kernel of the coboundary morphism $H^0(E(4)) \to H^1(S^2(F)(4))$ has dimension 15, as one can easily check with $\text{SL}(2,\mathbb{Z}_5)$-representation theory. Hence $p_2 = 1$ and $X$ is a $K3$-surface of the claimed type with a minimal free resolution

$$
0 \leftarrow \mathcal{J}_X \leftarrow 4\mathcal{O}(-5) \oplus 5\mathcal{O}(-6) \leftarrow 10\mathcal{O}(-7) \leftarrow 2\mathcal{O}(-9) \leftarrow 0.
$$
Proposition 2.6. There exist smooth, regular, proper elliptic surfaces $S \subset \mathbb{P}^4$, with invariants $d = 12$, $\pi = 14$, $\chi = 3$ and embedded by one of the following linear systems

\[ a) \quad |H| = |H_{\text{min}}| - 2E_0 - \sum_{i=1}^{4} E_i, \]

\[ b) \quad |H| = |H_{\text{min}}| - \sum_{i=1}^{5} E_i. \]

Proof. Argueing as in the proof of Proposition 2.1 a possible Beilinson cohomology table is

Thus we may take this time $\mathcal{E} := 2\mathcal{O}(-1) \oplus 3\mathcal{O}(3)$ and $\mathcal{F} := \ker \psi$, for some epimorphism $\psi : 2\Omega^2(2) \oplus \Omega^1(1) \to \mathcal{O}$, and check for the degeneracy locus of a general morphism $\varphi \in \text{Hom}(\mathcal{E}, \mathcal{F})$. Identifying $\mathbb{P}^4 = \mathbb{P}(V)$, with $V = \text{span}_\mathbb{C}(e_0, \ldots, e_4)$, the morphism $\psi$ is induced by a triple $(\psi_{11}, \psi_{12}, \psi_2)$, where $\psi_{11}, \psi_{12} \in \Lambda^2 V$ and $\psi_2 \in V$. We check the various choices for $\psi$. Namely, if

$\alpha) \psi$ is generic, then the associated vector bundle $\mathcal{F}$ has a minimal free resolution of type

\[
0 \leftarrow \mathcal{F} \leftarrow 25\mathcal{O}(-1) \leftarrow 10\mathcal{O}(-2) \leftarrow \mathcal{O}(-3) \leftarrow \mathcal{O}(-5) \leftarrow 0,
\]

and a general morphism $\varphi \in \text{Hom}(\mathcal{E}, \mathcal{F})$ gives a smooth surface $S_\alpha \subset \mathbb{P}^4$ with minimal free resolution

\[
0 \leftarrow \mathcal{I}_{S_\alpha} \leftarrow 8\mathcal{O}(-5) \leftarrow 7\mathcal{O}(-6) \leftarrow \mathcal{O}(-7) \leftarrow \mathcal{O}(-9) \leftarrow 0.
\]

while, if

$\beta) \psi$ distinguishes a plane, e.g., say $\psi_{11} = 0, \psi_{12} = e_0 \wedge e_1$ and $\psi_2 = e_2$, then $\mathcal{F}$ has a resolution

\[
0 \leftarrow \mathcal{F} \leftarrow 25\mathcal{O}(-1) \oplus 12\mathcal{O}(-2) \oplus 2\mathcal{O}(-3) \leftarrow \mathcal{O}(-5) \leftarrow 0.
\]
and the generic \( \varphi \in \Hom(\mathcal{E}, \mathcal{F}) \) degenerates on a smooth surface \( S_\beta \subset \mathbb{P}^4 \) with syzygies
\[
0 \leftarrow J_{S_\beta} \leftarrow \bigoplus \bigoplus \bigoplus \bigoplus \bigoplus 2\mathcal{O}(-6) 5\mathcal{O}(-7) 4\mathcal{O}(-8) \mathcal{O}(-9) \leftarrow 0.
\]

Smoothness can be checked in examples on a computer via [7]. Also it is easily seen that all other choices of \( \psi \) lead to singular surfaces, or to determinantal loci which are not of the expected codimension. We determine next what type of surfaces we have constructed.

Let \( S_1 \) denote the image of \( S_\alpha \) (\( S_\beta \) resp.) under the adjunction map, and let \( S_2 \) be the image of \( S_1 \) under the adjunction map defined by \( |H_1 + K_1| \). Then
\[
\begin{align*}
S_1 & \subset \mathbb{P}^{16} \quad H_1^2 = 35 \quad H_1K_1 = 9 \quad K_1^2 = -5 + a \quad \pi_1 = 23 \\
S_2 & \subset \mathbb{P}^{24} \quad H_2^2 = 48 + a \quad H_2K_2 = 4 + a \quad K_2^2 = -5 + a + b \quad \pi_2 = 27 + a,
\end{align*}
\]

where \( a \) is the number of \((-1)\)-lines and \( b \) is the number of \((-1)\)-conics on \( S_\alpha \) (\( S_\beta \) resp.).

In case \( \alpha \), the ideal \( I_{S_\alpha} \) is generated by quintic hypersurfaces so \( S_\alpha \) has no 6-secant lines. Le Barz’s formula (1.2) gives \( N_6 = 4 \), hence there are 4 exceptional lines, say \( E_1, E_2, \ldots, E_4 \), on \( S_\alpha \). Let \( S_{\min} \) denote the minimal model of \( S_\alpha \) and assume first that it is a surface of general type. Then \( S_\alpha \) has at least two other exceptional curves \( F_1 \) and \( F_2 \) of degree \( \geq 2 \), and thus there would exist a curve in \( |K_{\min} - F_1| \) of degree \( HK_{\min} - 2 \leq 4 \) and arithmetic genus \( p_a(K_{\min}) \geq 2 \), which is a contradiction.

It follows that \( S_\alpha \) is elliptic, and thus \( K \sim K_{\min} + \sum_{i=1}^4 E_i + E_0 \), where \( E_0 \) is a \((-1)\) curve of degree \( \geq 2 \). On the other side a curve in \( |K_{\min} - E_0| \) has arithmetic genus one, so \( HE_0 \leq 3 \). We investigate first the case when \( HE_0 = 3 \). Then a curve \( D \in |K_{\min} - E_0| \) has degree 4 and arithmetic genus one, so it spans only a hyperplane in \( \mathbb{P}^4 \). The residual curve \( G \sim H - D \) has then degree 8 and genus 9. We check now that \( G \) lies on a quadric surface in \( \mathbb{P}^3 \). Namely, since Riemann-Roch gives \( \chi(\mathcal{O}_G(2H)) = 8 \) it is enough to show that \( h^1(\mathcal{O}_G(2H)) \leq 1 \). This follows from the cohomology of the exact sequence
\[
0 \longrightarrow \mathcal{O}_S(H + D) \longrightarrow \mathcal{O}_S(2H) \longrightarrow \mathcal{O}_G(2H) \longrightarrow 0
\]
since, in the examples above, \( h^1(\mathcal{O}_S(1)) = 3 \), \( h^1(\mathcal{O}_S(2)) = 2 \), \( h^2(\mathcal{O}_S(H + D)) = h^0(\mathcal{O}_S(K - D - H)) = 0 \), while the composite multiplication map
\[
H^1(\mathcal{O}_S(H)) \xrightarrow{D} H^1(\mathcal{O}_S(H + D)) \xrightarrow{H - D} H^1(\mathcal{O}_S(2H))
\]
drops rank at most one on \( \mathbb{P}^4 \), for a general choice of the morphism \( \varphi \in \Hom(\mathcal{E}, \mathcal{F}) \).

Now the curve \( D \) lies on two quadrics, thus \( h^0(J_H(4)) \geq 2 \), where \( H \) denotes the hyperplane section of \( S_\alpha \) cut out by the \( \mathbb{P}^3 \) of \( D \). This is a contradiction, since under the above assumptions of (minimal) cohomology we have \( h^0(J_H(4)) \leq h^1(J_S(3)) = 1 \), for all hyperplane sections \( H \). Therefore \( E_0 \) must be an exceptional conic and \( S_\alpha \) a non-minimal elliptic surface embedded by a linear system of type \( a \), as claimed in the statement of the proposition.

In case \( \beta \), it is easily seen that the distinguished plane \( \Pi = \mathbb{P}(\text{span}_C(e_0, e_1, e_2)) \) meets \( S_\beta \) along a plane quintic curve \( C \) and the point \( P = \mathbb{P}(\text{span}_C(e_0)) \) outside
C. Therefore \( S_\beta \) has infinitely many 6-secant lines, namely the pencil of lines in \( \Pi \) through \( P \), and Le Barz’s formula doesn’t apply in this case. Using the explicit form of the syzygies of \( J_{S_\beta} \), it is easily seen that these are in fact all the 6-secant lines to \( S_\beta \). Taking cohomology of the exact sequences

\[
0 \rightarrow J_{S_\beta}(k-1) \rightarrow J_{S_\beta}(k) \rightarrow J_H(k) \rightarrow 0 \quad k = 3, 4
\]

we observe that \( h^1(J_H(3)) = 3 \) for all hyperplane sections \( H \) of \( S_\beta \), and that \( h^1(J_H(4)) = 1 \) if and only if \( P \in H \). Therefore each hyperplane through \( P \) contains a plane \( \pi \) such that \( h^1(J_{\pi \cap S_\beta}(4)) = 1 \). In particular, the pencil of hyperplanes through \( \Pi \) determines a quadric cone

\[
Q = \{ \det \begin{pmatrix} l & m \\ x_3 & x_4 \end{pmatrix} = 0 \},
\]

where \( l \) and \( m \) are suitable linear forms, such that \( h^1(J_{\pi(\lambda; \mu) \cap S_\beta}(4)) = 1 \) holds for all planes \( \pi(\lambda; \mu) = \{ \mu l + \lambda m = \mu x_3 + \lambda x_4 = 0 \} \) in one of the rulings of \( Q \). The plane \( \Pi \) is obviously a member of the opposite ruling of \( Q \), and thus residual to \( C \) in the complete intersection \( S_\beta \cap Q \) there is a curve \( G \sim 2H - C \) of degree 19 and arithmetic genus 23. On the other side, the hyperplane sections through \( \Pi \) cut, residual to \( C \), a pencil \(|D|\) with base point \( P \), of curves of degree 7 and genus 3. It follows that the curve \( G \) splits as \( G = G_1 + G_2 \), where \( G_1 \) is a union of plane curves contained in planes of the ruling \( \pi(\lambda; \mu) \), while \( G_2 \) is a curve of degree 14 which maps down via projection from the vertex of \( Q \) to a complete intersection of type \((7, 7)\) on the quadric surface, which is the base of the cone. It is easily checked that \( G_1 \) splits as the union of 5 exceptional lines, say \( E_1, E_2, \ldots, E_5 \), on \( S_\beta \). Therefore, on the first adjoint surface \( S_1 \) we obtain \( K_1^2 \geq 0 \) and in fact, by the Hodge index theorem, the equality \( K_1^2 = 0 \) holds. We argue further as in case \( a \).

If \( S_\beta \) were a surface of general type, then it would contain further an exceptional curve \( E \) of degree \( \geq 2 \) and thus a curve \( N \in |K_{\min} - E| \) would have degree at most 5 and arithmetic genus at least 2. Thus the only case to exclude is \( K_{\min}^2 = 1 \), \( HE = 2 \) and \( HN = 5 \), \( p_a(N) = 2 \). If \( N \) spanned only a \( \mathbb{P}^3 \), then the residual curve \( H - N \) would have degree 7 and arithmetic genus 8, which is impossible. Therefore \( N \) spans all of \( \mathbb{P}^4 \) and necessarily splits as \( N = A + B \), with \( A \) a plane quartic curve and \( B \) a line disjoint from it. But then \( A^2 + B^2 = N^2 = 0 \), \( B^2 \leq -2 \) since \( B \) cannot be exceptional, while the Hodge index theorem yields \( A^2 \leq 16/12 \), which is a contradiction. As claimed, it follows this time that \( S_\beta \) is a non-minimal elliptic surface embedded by a linear system of type \( b \).

3. A FAMILY OF SMOOTH K3 SURFACES OF DEGREE 14

We construct in this chapter an example of a smooth non-minimal K3 surface of degree 14 in \( \mathbb{P}^4 \). Currently this is the only known family of smooth non-general type surfaces of this degree in \( \mathbb{P}^4 \). More precisely, we show

**Proposition 3.1.** There exist smooth non-minimal K3 surfaces \( S \subset \mathbb{P}^4 \), with \( d = 14, \pi = 19, K^2 = -15 \), and embedded via

\[
|H| = |H_{\min} - 4E_0 - \sum_{i=1}^4 2E_i - \sum_{j=1}^{14} E_j|,
\]
where $|H_{\text{min}}|$ is very ample on $S_{\text{min}}$ of degree 56 and dimension 29.

Proof. We will discuss two different approaches. First

**A syzygy construction.** A plausible Beilinson cohomology table for a surface with these invariants is

$$
\begin{array}{c|ccc}
  i & 1 & 7 & 7 & 3 \\
  \hline
  p & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

$h^i(J_S(p))$

In this case everything is determined by the structure of $M := \bigoplus_{m \in \mathbb{Z}} H^2(J_S(m + 4))$. We will consider the dual module $M^*$ and assume that it is generated, as $R = \mathbb{C}[x_0, \ldots, x_4]$-module, by $\text{Hom}_\mathbb{C}(H^2(J_S(3)), \mathbb{C})$. Thus $M^*$ has a minimal free resolution of type

$$
0 \leftarrow M^* \leftarrow 3R(-1) \xleftarrow{\psi} 8R(-2) \oplus mR(-3)
$$

with $m \geq 0$. The morphism $\psi = (\psi_1, \psi_2)$ is given by a $3 \times (8 + m)$-matrix with linear entries in $\psi_1$ and quadratic entries in $\psi_2$. Also, $m > 0$ if and only if $\psi_1$ has at least three non-trivial linear syzygies. However, this doesn’t occur for a general choice of $\psi_1$ and the cokernel of $\psi_1$ is an artinian graded module with Hilbert function $(3, 7, 7)$ in this case. In order to obtain a module with the desired Hilbert function, it is necessary that the number of linear syzygies of $\psi_1$ equals $m + 2$. The idea is to start with four planes $P_i = \{l_{i1} = l_{i2} = 0\}$, $i = 1, \ldots, 4$, and to consider the direct sum of the four Koszul complexes built on $\{l_{i1}, l_{i2}\}$

$$
4R \xleftarrow{\alpha} 8R(-1) \xleftarrow{\beta} 4R(-2).
$$

We take now as $\psi_1$ a morphism given by three general lines of $\alpha$, i.e., $\psi_1 = \gamma \alpha(-1)$, with $\gamma \in M_{3,4}(\mathbb{C})$ a random matrix, and as $\psi_2$ a general $3 \times 2$-matrix with quadratic entries (since $\psi_1$ has exactly 4 linear syzygies). $M^* := \text{coker } \psi$ is artinian, with Hilbert function $(3, 7, 7)$ and with a minimal free resolution of type

$$
0 \leftarrow M^* \leftarrow 3R(-1) \xleftarrow{\psi} 8R(-2) \oplus 4R(-3) \oplus 2R(-3) \oplus 5R(-4) \oplus 15R(-5) \oplus 38R(-6) \leftarrow 28R(-7) \leftarrow 7R(-8) \leftarrow 0
$$

We dualize, and set $\mathcal{F} := Syz_2(M)$. To get a hint for the second bundle, we compare the syzygies of $\mathcal{F}$ with Beilinson’s spectral sequence for $J_S$. Namely, the $E_\infty$-filtration yields an exact sequence

$$
0 \rightarrow \mathcal{F}(-1) \oplus \mathcal{H}^0(\mathcal{F}) \oplus \mathcal{F} \rightarrow \mathcal{J} \rightarrow J_S(4) \rightarrow 0.
$$
Furthermore $h^0(F) = 15$, so we may take $\mathcal{E} := \mathcal{O}(-1) \oplus 15\mathcal{O}$. One checks in examples, via [7], that the degeneration locus of a $\varphi \in \text{Hom}(\mathcal{E}, F) = \text{Hom}(\mathcal{O}(-1) \oplus (H^0(F) \otimes \mathcal{O}), F) = H^0(\mathcal{F}(1)) \oplus \text{Hom}(H^0(F) \otimes \mathcal{O}, \mathcal{F})$, given by a general section and the natural evaluation map, is a smooth surface $S$ with the desired numerical invariants and the desired cohomology. The minimal free resolution of the ideal sheaf of the surface is of type

$$
0 \leftarrow \mathcal{J}_S \leftarrow 4\mathcal{O}(-5) \oplus 2\mathcal{O}(-6) \\
4\mathcal{O}(-6) \oplus 8\mathcal{O}(-7) \leftarrow 3\mathcal{O}(-8) \leftarrow 0,
$$

and thus the homogeneous ideal is generated by 4 quintics and 4 sextics. Moreover, it follows from the construction of the module $M^*$ that the four quintics containing $S$ intersect in

$$
V((I_S)_{\leq 5}) = S \cup \bigcup_{i=1}^4 P_i,
$$

and a closer look at the syzygies of $M^*$ shows that $S$ cuts each plane $P_i$ along a sextic curve. Hence each of the planes $P_i$ contains an $\infty^2$ of 6-secant lines, and in particular Le Barz’s formula doesn’t apply to this example.

To determine the type of surface we have constructed, one can argue as follows. All the geometric facts needed in the sequel will follow from our second construction. One checks first that $S \cup \bigcup_{i=1}^4 P_i$ is an arithmetically Cohen-Macaulay scheme of degree 18 and sectional genus 39, with syzygies of type

$$
0 \leftarrow \mathcal{J}_{S \cup \bigcup_{i=1}^4 P_i} \leftarrow 4\mathcal{O}(-5) \oplus 2\mathcal{O}(-6) \leftarrow 0
$$

The minors of the above $4 \times 2$-submatrix $4\mathcal{O}(-5) \leftarrow 2\mathcal{O}(-6)$ vanish precisely along an exceptional quartic curve $E_0$ on $S$. Furthermore, one may check that there are exactly ten exceptional lines on the surface. Let now $S_1$ denote the image of $S$ under the adjunction map, and $S_2$ denote the image of $S_1$ under the map defined by $|H_1 + K_1|$. Then

| $S_1 \subset \mathbb{P}^{19}$ | $H_1^2 = 43$ | $H_1K_1 = 7$ | $K_1^2 = -5$ | $\pi_1 = 26$ |
|---|---|---|---|---|
| $S_2 \subset \mathbb{P}^{26}$ | $H_2^2 = 52$ | $H_2K_2 = 2$ | $K_2^2 = -5 + b$ | $\pi_2 = 28$, |

where $b$ is the number of $(-1)$-conics on $S$. But $K_2^2 = -5 + b \geq -H_2K_2 = -2$, and there exists already an exceptional quartic curve $E_0$ on $S$, so the only possibility is that $b = 4$ and $K_2$ is a $(-1)$ conic on $S_2$. As it turns out, $S = S_{\text{min}}(p_0, \ldots, p_{14})$ is a minimal $K3$ surface blown up in 15 points and

$$
|H| = |H_{\text{min}} - 4E_0 - \sum_{i=1}^4 2E_i - \sum_{j=5}^{14} E_j|,
$$

where $|H_{\text{min}}|$ is a very ample linear system on $S_{\text{min}}$ (the fourth adjoint surface), defining an embedding $S_{\text{min}} \subset \mathbb{P}^{29}$, with $\deg S_{\text{min}} = 56$.

We want in the sequel to recover an alternative linkage construction for the above $K3$ surface. General facts about linkage [33, Prop. 4.1] and [35, Rem. 0.13] ensure
that one can link (5, 5) the configuration $S \cup \bigcup_{i=1}^{4} P_i$ to a smooth surface $Y$ of degree 7 and sectional genus 6. The cohomology of the liaison exact sequence

$$0 \rightarrow \mathcal{O}_Y(K_Y) \rightarrow \mathcal{O}_{\Sigma_{5,5}}(5) \rightarrow \mathcal{O}_{S \cup \bigcup_{i=1}^{4} P_i}(5) \rightarrow 0,$$

where $\Sigma_{5,5}$ denotes the complete intersection of the two quintic hypersurfaces used in the linkage, gives $p_g(Y) = 2$ and $q(Y) = 0$ while the double point formula yields $K_Y^2 = 0$. Surfaces with these invariants are classified in [31] and are known to be arithmetically Cohen-Macaulay, minimal proper elliptic surfaces. More precisely, $|K_Y|$ is a pencil without base points of plane cubic curves; the planes spanned by its members being those in one ruling of the determinantal quadric defined by the linear syzygies in

$$0 \rightarrow 2\mathcal{O}(-5) \rightarrow 2\mathcal{O}(-4) \oplus \mathcal{O}(-2) \rightarrow \mathcal{I}_Y \rightarrow 0.$$

Once again linkage shows that $Y$ cuts each plane $P_i$ along a conic $C_i$, which is necessarily a section of the elliptic fibration, since there are no singular fibers and the fibration is by plane curves. In particular, this means that the rank of the Picard group of $Y$ is at least 6, while the Picard number of a generic elliptic surface of degree 7 in $\mathbb{P}^4$ is only 2 by [20]. Therefore $Y$ has to be chosen carefully in order to recover $S$ via liaison from the scheme $Y \cup \bigcup_{i=1}^{4} P_i$.

**A linkage construction.** The above facts suggest to us the following linkage construction for this family of $K3$ surfaces. Let $P, P_1, P_2, P_3, P_4$ be five planes in general position in $\mathbb{P}^4$ and denote by $\{p_{ij}\} = P_i \cap P_j$, for $1 \leq i < j \leq 4$, the mutual intersection points of the last four of them.

**Lemma 3.3.2.**

1. **The homogeneous ideal** $I_{P \cup \{p_{ij}, 1 \leq i < j \leq 4\}}$ **is generated by 3 quadrics and 4 cubics.**
2. **The three quadrics intersect along the plane $P$ and a rational normal quartic curve $Q$, which is trisecant to $P$ and goes through the points $p_{ij}$.**

**Proof.** The first part follows from the cohomology of the residual intersection sequences

$$0 \rightarrow \mathcal{I}_{\{p_{ij}, i < j\}}(m - 1) \rightarrow \mathcal{I}_{P \cup \{p_{ij}, i < j\}}(m) \rightarrow \mathcal{O}_{P^2}(m - 1) \rightarrow 0$$

where $m \in \mathbb{Z}$. For the second part observe that the plane $P$ is linked in the complete intersection of two of the hyperquadrics to a rational cubic scroll $T$. If $H_P \sim C_0^2 + 2f$, with $C_0^2 = -1$, $C_0f = 1$ and $f^2 = 0$, is the embedding of the scroll in $\mathbb{P}^4$, then $P \cap T \sim C_0 + f$ is a conic and the third hyperquadric cuts on $T$ the rational normal quartic curve $Q \sim C_0 + 3f$. Now $2P \cap Q = Q(C_0 + f) = 3$ and the lemma follows.

We consider now a general quadric $V \in H^0(I_{P \cup \{p_{ij}\}}(2))$ and denote with $C_i$ the conics $V \cap P_i$, for $i = 1, \ldots, 4$. They intersect pairwise in the points $\{p_{ij}\} := C_i \cap C_j$, $1 \leq i < j \leq 4$.

**Lemma 3.3.3.** There exists a unique rational normal quartic curve $E_0$ which is contained in $V$, passes through the points $p_{ij}$, $1 \leq i < j \leq 4$, and intersects the plane $P$ in exactly one point $p$.

**Proof.** The claim is closely related to a theorem of James [26], [40]. Consider the rational map $\sigma : \mathbb{P}^4 \dashrightarrow \mathbb{P}^5$ given by the quadrics through the rational normal.
quartic curve \( Q \) in lemma 3.2. It is one to one onto a smooth hyperquadric \( \Omega \subset \mathbb{P}^5 \), which we identify in the sequel with the image of the grassmannian of lines in \( \mathbb{P}^3 \) under the Plücker embedding. Let \( \mathbb{P}^4 \) be the blowing up of \( \mathbb{P}^4 \) along \( Q \) and denote by \( E \) the exceptional divisor and by \( \gamma : \mathbb{P}^4 \to \Omega \subset \mathbb{P}^5 \) the induced morphism. Then the trisecant planes of \( Q \) are mapped through \( \gamma \) to the planes of one generating system, say \( \alpha \)-planes, of the grassmannian \( \Omega \), while \( E \) is mapped by \( \gamma \) onto a sextic threefold ruled by \( \beta \)-planes. Each of the \( \beta \)-planes corresponds to the normal directions in \( \mathbb{P}^4 \) at points of \( Q \). We remark also that quadric cones through \( Q \) are mapped via \( \gamma \) to special linear complexes, i.e., to tangent hyperplane sections of \( \Omega \). To fix notations, let \( H \subset \mathbb{P}^5 \) be the hyperplane corresponding to \( V \) and let \( \beta_{ij}, \ 1 \leq i < j \leq 4 \), be the \( \beta \)-planes corresponding to the points \( p_{ij} \). Rational normal quartic curves which meet \( Q \) in six points are represented via \( \gamma \) by conics in which \( \Omega \) is met by planes. Thus, in order to prove the lemma, all we need to check is that there exists exactly one plane in \( H \) meeting all six lines \( H \cap \beta_{ij} \) and not contained in the quadric cone \( H \cap \Omega \). But this is clear since the Plücker embedding of the grassmannian of planes in \( \mathbb{P}^4 \) has degree 5, while the planes of the cone \( H \cap \Omega \) describe via the same Plücker embedding the union of two conics. The rational quartic curve \( E_0 \) represented by this unique plane meets \( P \) in one point because \( \gamma \) maps \( P \) to an \( \alpha \)-plane contained in \( H \).

**Lemma 3.4.** If \( T = P \cup \bigcup_{i=1}^{4} C_i \), then its homogeneous ideal \( I_T \) is generated by 1 quadric, 2 cubic and 4 quartic hypersurfaces.

**Proof.** One uses again the residual exact sequences

\[
0 \to J_{\bigcup_{i=1}^{4} C_i}(m-1) \to J_T(m) \to J_{P \cup \bigcup_{i=1}^{4} C_i \cap H}(m) \to 0
\]

where \( H \) is a general hyperplane through \( P \) and \( m \in \mathbb{Z} \), together with the fact that \( I_{\bigcup_{i=1}^{4} C_i} \) is generated by 1 quadric and 8 cubic hypersurfaces.

From the above lemma it follows that \( P \) can be linked in the complete intersection of \( V \) and a general quartic hypersurface \( W \in H^0(J_T(4)) \) to a smooth, minimal proper elliptic surface \( Y \subset \mathbb{P}^4 \) with \( \deg \ Y = 7, \ \pi(Y) = 6 \). By construction, the conics \( C_i \) all on \( Y \).

**Lemma 3.5.**

a) \( C_i^2 = -3 \) and \( K_Y C_i = 1 \) on \( Y \), so each conic \( C_i \) is a section of the elliptic fibration.

b) The planes \( P_i \) intersect \( Y \) exactly along the conics \( C_i \).

**Proof.** In any case \( K_Y C_i \geq 1 \) since there are no multiple fibers. On the other hand, we recall that the elliptic fibration is cut out on \( Y \) by the planes in one of the rulings of the cone \( V \). Thus if \( K_Y C_i \geq 2 \), then \( P_i \) would lie on \( V \) and this would contradict our choices. It follows that \( C_i^2 = -3 \) and \( K_Y C_i = 1 \). Part b) is set theoretically clear by construction. It is enough to remark that residual to each conic \( C_i \) there is a pencil \( |H_Y - C_i| \) of curves of degree 5 and genus 2, without base points since \( (H_Y - C_i)^2 = 0 \).

We need in the sequel some classical facts of projective geometry.

**Proposition 3.6 (Segre).** With any four general planes \( P_i, \ i = 1, \ldots, 4 \), there is associated a uniquely determined fifth plane \( P_5 \), such that all lines which meet the first four planes meet also the fifth.
Proof. As mentioned above, the Plücker embedding of the Grassmannian of lines in \( \mathbb{P}^4 \) has degree 5, thus the claim follows because the special linear complexes consist of lines meeting a given plane. See also [39] or [41].

Corollary 3.7 (Segre)[39]. The lines in \( \mathbb{P}^4 \), which meet the four initial planes, generate a cubic hypersurface \( X \) containing the five planes \( P_i, i = 1, \ldots, 5 \), and having singularities (nodes) exactly at the ten points at which the planes meet in pairs.

Proof. We briefly recall the arguments in [39]. The first part of the claim follows from Bezout’s theorem and from Schubert calculus in \( G(l) \) and since if \( a \) is a hyperplane \( H \) and \( \varphi : \tilde{X} \to X \subset \mathbb{P}^4 \) is given by the quadrics through the five points, while the nodes are the images of the lines joining any two of the points \( a_i \). We mention in the sequel some of the properties of this threefold (cf. [39], [41], [21]).

The Segre cubic primal \( X \) has a symmetrical system of 15 planes, of which 5 correspond to the exceptional divisors over the points \( a_i \); and 10 to the planes \( P_{ijk} = \varphi(\text{span}_C(a_i, a_j, a_k)) \), for \( \{i, j, k\} \subset \{1, 2, 3, 4, 5\} \). The symmetry of the the planes resides in the following properties:
- each plane contains four of the nodes,
- each plane is met in lines by 6 others, namely the plane corresponding to \( a_i \) by the planes \( P_{ijk} \), for all \( \{k, j\} \subset \{1, 2, 3, 4, 5\} \setminus \{i\} \), and the plane \( P_{ijk} \) by those corresponding to \( a_i, a_j, a_k \) and \( P_{\alpha, \beta, \gamma} \), with \( \alpha \in \{i, j, k\} \) and \( \{\beta, \gamma\} = \{1, 2, 3, 4, 5\} \setminus \{i, j, k\} \).

We will assume in the sequel that we have chosen the desingularization morphism \( \varphi \) such that the planes \( P_i, i = 1, \ldots, 4 \), correspond to the exceptional divisors over \( a_i \). Let now as above \( Z = Y \cup \bigcup_{i=1}^4 P_i \). It is a local complete intersection scheme, outside the points \( P_{ij} \) which are Cohen-Macaulay and where the tangent cone is linked to a plane in a complete intersection, and has invariants \( \deg Z = 11, \pi(Z) = 10, \chi = 3, q = 0 \). We remark here that Hodge index implies that there is no smooth surface in \( \mathbb{P}^4 \) with these invariants. By computing syzygies one shows that \( Z \) has a resolution of type

\[
0 \rightarrow 2 \mathcal{O}(-1) \oplus (\mathcal{H}^0(\mathcal{S}) \otimes \mathcal{O}) \rightarrow \mathcal{S} \rightarrow \mathcal{I}_Z(4) \rightarrow 0
\]

with \( \mathcal{S} := Syz_1(M^*)^*(3) \), where \( M^* \) is the graded artinian module in our first construction. In particular the homogeneous ideal \( \mathcal{I}_Z \) is generated by 3 quintic and 15 sextic hypersurfaces, and thus we can link \( Z \) in the complete intersection of two
quintics to a surface $S$ with $d = 14$, $\pi = 19$, $\chi = 2$, $q = 0$. One checks in examples via [7] that $S$ is smooth.

**Remark 3.8.** By liaison, each plane $P_i$, $i = 1, \ldots, 4$, intersects $S$ along a sextic curve $D_i$, thus each of them contains an $\infty^2$ of 6-secant lines to $S$.

**Lemma 3.9.**

a) $E_0$ is an exceptional quartic on $S$.

b) Each of the four planes $P_{ijk}$, with $\{i, j, k\} \subset \{1, 2, 3, 4\}$, cuts the surface $S$ along an exceptional conic.

**Proof.** The rational normal curve $E_0$ is contained in $V$ and intersects $W$ in a scheme of length 16, of which one point is on $P$. Thus, for general choices, $E_0$ cuts $Z = Y \cup \bigcup_{i=1}^{4} P_i$ along a scheme of length $15 + 6 = 21$ and, by Bezout’s theorem, lies on all quintic hypersurfaces containing $Z$, whence on $S$. We show now that, say $P_{123}$ cuts $S$ along a conic; the other cases being similar. Observe first that $P_{123}$ cuts $P_1$, $P_2$ and $P_3$ along the lines pairwise joining the points $p_{12}$, $p_{13}$, $p_{23}$, while $P_4$ and $P_5$ both meet this plane at the node $v_{45}$ corresponding to the line through $a_4$ and $a_5$. For general choices $P_{123}$ meets $Y$ in a scheme of length 7: $p_{12}$, $p_{13}$, $p_{23}$ and four extra points. Let $E_4$ denote the unique conic through these four points and the node $v_{45}$. It is easily seen that $E_4$ is a 11-secant conic to the configuration $Z$, so by Bezout’s theorem it necessarily lies on $S$.

By linkage $K_S + ((Y \cup \bigcup_{i=1}^{4} P_i) \cap S) \sim 5H_S$. On the other hand, the curve of degree 24 and arithmetic genus 37 represented by $(Y \cap S) \sim 5H_Y - K_Y - \sum_{i=1}^{4} C_i$, and the quartic $E_0$ lie both on the quadric cone $V$. It follows that $K_S + ((\bigcup_{i=1}^{4} P_i) \cap S) \sim 3H_S + E_0$, thus $K_S + \sum_{i=1}^{4} D_i \sim (X \cap S) + E_0$. Since $H(K - E_0 - \sum_{i=1}^{4} E_i) = 22 - 12 = 10$ and $K^2 = -15$ we deduce easily that $E_i$, $i = 0, \ldots, 4$, are exceptional curves on $S$.

Adjunction and the above lemma show also that $S$ must have 10 exceptional lines, thus it is a non-minimal $K3$ surface embedded by

$$|H| = |H_{\text{min}} - 4E_0 - \sum_{i=1}^{4} 2E_i - \sum_{j=5}^{14} E_j|.$$

**Corollary 3.10.** The Segre cubic primal $X$ intersects $S$ along the union of the 10 exceptional lines, the 4 exceptional conics and the 4 plane sextic curves $D_i$.

A similar linkage construction gives also the following

**Proposition 3.11.** There exist smooth, non-minimal general type surfaces $S \subset \mathbb{P}^4$ with invariants $d = 15$, $\pi = 22$, $p_g = 3$, $q = 0$, $K^2 = -6$, and with 9 exceptional lines.

**Proof.** This time one starts with a Castelnuovo surface $Y \subset \mathbb{P}^4$, i.e., with a smooth, arithmetically Cohen-Macaulay, rational surface with $d = 5$, $\pi = 2$ and $K^2 = 1$ (see [8] or [30]). $Y$ is linked to a plane in the complete intersection of a hyperquadric and a cubic hypersurface, and can be represented via the adjunction map as $\mathbb{F}_1$ blown up in 7 general points, thus it is embedded in $\mathbb{P}^4$ by

$$|H_Y| = |4l - 2E_0 - \sum_{i}^{7} E_i|.$$
Consider now the following conics on $Y$:

\[ C_0 = 3l - 2E_0 - \sum_{i=1}^{6} E_i \]
\[ C_1 = 2l - E_0 - E_7 - E_1 - E_3 - E_5 \]
\[ C_2 = 2l - E_0 - E_7 - E_1 - E_4 - E_6 \]
\[ C_3 = 2l - E_0 - E_7 - E_2 - E_4 - E_5 \]
\[ C_4 = 2l - E_0 - E_7 - E_2 - E_3 - E_6. \]

They intersect pairwise in one point and the planes they span, denoted in the sequel by $P_i$, for $i \in \{0, 4\}$, intersect the Castelnuovo surface $Y$ only along the conics $C_i$. The scheme $Z = Y \cup \bigcup_{i=0}^{4} P_i$ is regular, of degree 10 and sectional genus 7, and has a minimal free resolution of type

\[
0 \leftarrow I_Z \leftarrow \bigoplus_{i=10}^{34} \mathcal{O}(-6) \leftarrow 34 \mathcal{O}(-7) \leftarrow 27 \mathcal{O}(-8) \leftarrow 7 \mathcal{O}(-9) \leftarrow 0.
\]

The five quintics in the ideal intersect along $Z$ and the union of 9 skew lines. In fact, if, according to lemma 3.6, $Q_i$ denotes the unique Segre cubic hypersurface containing the planes $P_{i+1}, P_{i+2}, P_{i+3}$ and $P_{i+4}$, for $i \in \mathbb{Z}_5$, then one checks that $Q_1, Q_2, Q_3$ and $Q_4$ each contain 6 skew lines which are 6-secant to the configuration $Z$, while $Q_0$ contains only 5 such lines. On the other hand, the five Segre cubics $Q_i$ cut out an elliptic quintic scroll $T \subset \mathbb{P}^4$ (see [39], [41, Th.XXXIII, p.278]); each of the planes $P_i$ intersecting it along a cubic curve, section of the ruling. It follows that there are exactly 5 rulings of the scroll which are 6-secant to $Z$, and thus altogether 9 skew lines with this property. The scheme $Z$ can be linked in the complete intersection of two quintic hypersurfaces to a surface $S$ with the desired invariants, having the above 9 lines as exceptional curves. One computes the following cohomology table

\[
\begin{array}{cccc}
& & & \\
& & & h^i(J_S(p)) \\
& 3 & 7 & 8 & 4 \\
& & & & \\
\end{array}
\]

and a minimal free resolution of type

\[
0 \leftarrow J_S \leftarrow \bigoplus_{i=12}^{12} \mathcal{O}(-6) \leftarrow 12 \mathcal{O}(-7) \leftarrow 4 \mathcal{O}(-8) \leftarrow 0.
\]

Finally, we remark that each of the planes $P_i$ intersects $S$ along a sextic curve, and thus contains an $\infty^2$ of 6-secant lines to $S$.

Remark 3.12. $S$ and $Z$ are minimal elements in their even liaison classes.
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COLUMBIA UNIVERSITY, DEPARTMENT OF MATHEMATICS, MAIL CODE 4417, 2990 BROADWAY, NEW YORK, NY 10027, USA

E-mail address: psorin@math.columbia.edu