The Microscopic Representation of Complex Macroscopic Phenomena

Critical Slowing Down - A Blessing in Disguise

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S. Solomon

e-mail: sorin@vms.huji.ac.il

Department of Physics, University College of Swansea

and

Racah Institute of Physics, Hebrew University of Jerusalem

(permanent address)

Abstract

Many complex systems are representable as macroscopic sets of elements which interact by simple rules.

The complex macroscopically relevant phenomena are then the result of the generic emergence of a space-time multi-scale dynamics.

Critical Slowing Down labels the emerging global features and describes their complex collective evolution.

This paradigm is quite universal and extends to a very wide range of systems and disciplines.
1 The Microscopic Representation Paradigm

If one throws a stone and wishes to predict its trajectory one needs only to consider it as a single rigid body.

If one wishes to study its light spectrum, one has to consider only its atomic and molecular structure.

If one wishes to study how the stone deforms under pressure, one may treat it as an elastic continuum.

However, if one increases the pressure and one wishes to know how it breaks, one has to consider it as a conglomerate of smaller stones (and stones within stones) with cracks and faults developing over many space and time scales down to the microscopic one.

The present paper concerns complex systems for which the origin of complexity can be traced to the very attempt by our perception to describe a macroscopic number of microscopic objects and events in terms of a limited number of macroscopic features.

We will discuss the techniques through which one can systematically follow the birth of the complex macroscopic phenomenology out of the simple fundamental microscopic laws.

In the field of fundamental physics, such understanding was obtained for a wide range of phenomena using theories based on a Microscopic Representation (in short, in the following text MicRep) paradigm.

The MicRep paradigm consists in deducing the macroscopic objects (Macros from now on) and their phenomenological complex ad-hoc laws in terms of a multitude of elementary microscopic objects (Micros from now on) interacting by simple fundamental laws. The Macros and their laws emerge then naturally from the collective dynamics of the Micros as its effective global large scale features.

However, the mere microscopic representation of a system cannot lead to a satisfactory and complete understanding of the macroscopic phenomena. Indeed, the mere copying on the computer of a real-life system with all its problems does not by itself constitute a solution to those problems.

It is clear that a satisfactory MicRep procedure of such complex systems has to be "multi-scale" i.e.:

- one has to recognize the relevant objects which describe effectively the system at each scale and
• one has to establish the relations between the objects and the laws prevailing at different scales.

Therefore, the MicRep approach is not trying to substitute the study of one scale for the study of another scale; one is trying to unify into a coherent picture the complementary descriptions of a one and the same reality.

In fact one can have a multitude of scales such that the Macros of one scale become the Micros of the next level. As such the "elementary" Micros of one MicRep do not need to be elementary in the fundamental sense: it is enough that the details of their internal structure and dynamics are irrelevant for the effective dynamics of the Macros.

More precise expressions of some of these ideas were encapsulated in the renormalization group (\textit{RG}) \cite{1} and in the multigrid (\textit{MG}) method \cite{2}.

\textit{MG} was offered for the last 20 years as a computational philosophy to accelerate the computations arising in various scientific fields. The idea was to accelerate algorithms by operating on various scales such that the computations related with a certain length scale are performed directly on the objects relevant at that scale.

In the present review, the multi-scale phenomena and the relevant Macros hierarchies are considered for their own interest even (sometimes) in the absence of a multi-scale algorithm which accelerates the computations.

The multi-scale concept is proposed as a tool to reformulate and reorganize the way of approaching the problematics of various fields. Thus its usefulness transcends by far the mere application of a computational technique and may induce in certain fields a shift in the concepts, the language, the techniques and even in their objectives (see chap. 4).

In the last decades, the MicRep method was used in the RG formulation of the quantum field gauge theories to explain the observed behavior of the electromagnetic, nuclear, radioactive and gravitational phenomena \cite{5}. One started by representing the continuous space and time as a discrete mesh (\textit{lattice}) and representing the field by elementary simple objects residing on the nodes and links of the mesh. Each elementary object interacts by simple microscopic laws with its immediate neighbors on the mesh (see chap. 3).

In spite of the fact that the fundamental interactions act only between immediate neighbors, the resulting dynamics of the macroscopic system is such that the effects of acting on the system in a particular point are significantly felt over a large distance \(\xi\) around that point.
This phenomenon by which a system defined by microscopic laws presents macroscopic phenomena is called "criticality" and $\xi$ is the "critical length".

The macroscopic dynamics of the critical systems, at scales larger than $\xi$ is "universal" i.e. largely independent on the details of the microscopic dynamics used in the microscopic definition of the system [4]. In particular, universality allows one to choose a highly simplified version of the microscopic definition of the system and still obtain an accurate description of its macroscopic properties.

Moreover, it was shown that the resulting macroscopic properties of the critical system can be appropriately described in terms of the effective dynamics of a limited number of macroscopic objects (e.g. "relevant operators" [3]).

Understanding the critical behavior of a microscopic system of Micro’s reduces then to the identification of the relevant Macro’s and the description of their long time-scales evolution. Conversely, finding an appropriate MicReRep explanation for a macroscopic complex phenomenon is to find a system of Micro’s whose effective macroscopic critical dynamics leads to Macro’s modeling well the macroscopic complex phenomenon.

In practice the iterative algorithms used to simulate the critical systems in terms of their simple microscopic laws required exceedingly long computer runs. This phenomenon was called Critical Slowing Down (CSD) (see chap. 2). A physical way to understand CSD is to realize that the relevant effective macroscopic processes governing the Macro’s involve typically a macroscopic number of elementary interactions between each typical pair of neighboring Micro’s. In fact CSD is a well known, experimentally measurable phenomenon in the physics of real critical systems [6].

Nonetheless its presence in the computer simulations is a major inconvenient and a lot of research was invested in the last decade to devise algorithms which give the properties of the critical systems without investing the enormous computer time implied by the presence of CSD [7].

The construction and study of such accelerated no-CSD or reduced-CSD (RCSD from now on) algorithms, lead to a finer understanding of the relation between CSD and the mechanisms which generally relate the simple microscopic definition of a system to its complex macroscopic dynamics.

For a system defined by given Micro’s, the RCSD algorithms were based on identifying and acting upon the relevant Macro’s. As such, these algorithms were at one and the same time
• the result,
• the expression and
• the source

of understanding the critical dynamics of the systems to which they applied. In chapter 3 we suggest a quantitative measure for the "knowledge" or "understanding" contained in an algorithm.

The emergence of this sort of algorithmic operational way of acquiring and expressing knowledge has a very far reaching methodological potential. This is even more evident when coupled with the modern computers simulation, visualization and animation capabilities [23, 24]. In fact, we performed many such experiments [23] (see following chapters) in which certain systems were successfully studied by performing computer experiments which involved the animation of their evolution under the action of various algorithms. The following research routine was common for most of these experiments:

• modeling the system as composite of many Micros.
• computer simulation and visualization of the resulting model.
• identification and computer experimental study of the CSD modes within the microscopic system.
• identification of the Macros.
• predictions based on the Macros behavior of the model.
• comparison with the experimental macroscopic behavior
• confirming or correcting the microscopic model in view of the comparison.
• starting new experiments based on these results.

The use of this dialogue with an artificial system created in the computer in order to understand its critical properties extends to systems away from equilibrium and to complex systems which are not characterized by an energy functional or by an asymptotic probability distribution.
For such general systems, some of the notions usually associated to criticality might become inapplicable. Yet, one needs criticality in order to insure the emergence of the universality properties which make the MicRep method reliable at macroscopic scales.

We propose to use the very emergence of CSD in the dynamics as a criterion for the legitimate use of universality. We turn the tables and transform the CSD from a curse into a blessing in disguise. We use CSD as the label which isolates the relevant Macro’s. We use the RCSD algorithms as operational proofs that the relevant Macro’s were efficiently identified and expressed algorithmically. We use the Macro’s in order to visualize and understand the emergence of the collective dynamics, in order to relate the salient complex phenomenology to the simple underlying microscopic causes.

We propose to use this understanding in the task of formulating and studying MicRep models for basic problems in a wide range of fields (as detailed in chap. 4). We are claiming that CSD is the key to characterize, build, identify and study such models.

2 CSD Appearance and Measurement

Let us give a short qualitative description of the appearance of CSD (and other critical features) in various systems. We explain below the fact that in many cases no fine tuning is necessary and that CSD is a widely encountered (in fact typical) phenomenon in nature.

This will become important later in chap. 4 where we advocate the universal application of the MicRep technique to a very wide range of complex systems (For CSD in the graph coloring problem see Hogg’s review in this volume).

One of the typical mechanisms leading to CSD is the following. At low enough energies, the systems get ordered and many of their degrees of freedom become effectively frozen. The freezing usually results into a breaking of one of the symmetries of the system (the spontaneous symmetry breaking phenomenon).

I.e., while the fundamental laws describing the system’s dynamics are symmetric under certain transformations (e.g. global rotation in the ferromagnetic case, translation in the crystal case), the low energy states tend
to pick up particular directions (or positionality) as preferential (e.g. the
direction of magnetization, or the particular position in which the molecules
become localized upon freezing).

One can show that in this situations, the long time dynamics of the system
is dominated by the dynamics of large homogenous sub-sets changing their
shape and position collectively (e.g. spin waves, crystal deformations, diffu-
sion, Weiss domains, fluid droplets, membrane deformations, inverse lapla-
cian etc.).

For continuous symmetries and local interactions, one can explicitly iden-
tify in these situations long(-macroscopic)-range correlations in space and
time (Goldstone modes- e.g. long-wavelength spin waves in isotropic mag-
nets) [8]).

As opposed to usual critical behavior which takes place only for very
special (fine-tuned) values of the parameters of the model (e.g. critical tem-
perature $T_c$), the long range excitations related to the Goldstone mechanism
take place naturally for ranges of non-zero measure in the parameters space
defining the systems (e.g. $T < T_c$).

More generally, if a system presents states which differ slightly in their
energy from the ground state, one can have a dynamics which inter-
polates between the various (quasi-)vacua and which takes macroscopic times (e.g
spin glasses [11]).

In general, this is not necessarily related to the breaking of a global sym-
metry. All it is necessary is that there exist collective changes over a large
subset of the system which do not result into a significant change of the
energy per degree of freedom [9]. Even though large numbers of elemen-
tary degrees of freedom are involved, it is therefore not possible in general
to expose long range correlation lengths within the system$^1$. It is therefore
important that even in the absence of a divergent $\xi$, the CSD phenomenon is
still there to signal that the system is still critical and universality may still
apply.

Moreover, criticality and universality labeled by CSD apply to systems
far from the equilibrium. For such systems, CSD may just be the result of

\footnote{Another such instance where criticality is not related to long range space-correlations
is the completely connected Ising model where all the sites are at distance 1 one from the
other and still criticality and CSD is present for "local" (or rather "single-spin") dynamics.
The proper terminology would be therefore "large sub-sets correlations" and the RCSD
dynamics may be called "large sub-sets dynamics".}
"branching": i.e. the existence of alternative courses of the far-from equilibrium dynamics which are equally likely to take place. When followed in time (or other parameter) the different courses of evolution might be separated by very small energy barriers at the beginning and by very large barriers in the end. This makes the branching region analogous to a phase transitions in equilibrium systems and the situation after branching analogous to the "magnetized phase" where the system has degenerate "free energy" minima and the dynamics breaks the symmetry which relates these minima by effectively choosing one of them.

Note that as one lowers the available thermal energy and the tunneling between vacuums becomes rarer and rarer, the corresponding tunneling time scale $\tau$ does not shrink. In fact $\tau$ diverges to higher and higher values until the time scales involved become longer than the typical runs can afford. For a finite system and non-zero energy, there is always a time above which the tunneling still can take place.

As a consequence, the Critical Slowing Down regime is not limited to a certain value of the parameters (e.g. for temperature $T_c$), but to all values of parameters for which the depth of the (local) minima in the "energy landscape" are larger than the typical thermal energy available within the system at typical times and locations.

In this sense, the CSD once installed never disappears as the available energy gets smaller and smaller: the usual critical point is not any more the singular point at which time correlations diverge, but in fact the point at which the CSD related to (quasi-)ground states degeneracy DISAPPEARS(!) as the temperature rises above the energy landscape.

Another effect which favorites the presence in nature of critical and CSD systems and gives them a non-zero probability measure to appear is the fact that once a system is getting into a critical state, the very presence of CSD is preventing it to leave this state within a microscopic time.

For instance if one lets water to cool from 200$^0C$ to −100$^0C$, the water will spend a non-zero fraction of time at exactly 100$^0C$ and respectively 0$^0C$ (in this particular case the transition is first order, so it is not clear how much of the macroscopic properties are independent on the microscopic details).

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2 In our examples in the next sections, such dynamical branching phenomena happen when one freezes a system with instantons and gets trapped in various topological sectors, or when the traffic flow chooses between organizing into an optimal flow or into a completely jammed configuration.
Also, if one is holding different boundaries of a water container at respectively 90°C and 110°C, a macroscopic fraction of the volume of the container will present the bubbling and large fluctuations characteristic to the critical temperature 100°C.

We will insist below on this relation between the emergence of objects with macroscopic space dimensions and the emergence of processes extending over macroscopic times.

This property holds for non-equilibrium systems too (actually especially for non-equilibrium systems). In both equilibrium and non-equilibrium systems, the system spends more time and/or more space in the critical region, in a natural way, without the necessity of fine tuning of the boundary or of the initial conditions.

It is worth mentioning that from the simulation point of view all the past runs aimed at bringing statistical mechanics and lattice field theory models towards the thermal equilibrium state, become particular instances in which the CSD-labeled non-equilibrium universal dynamics was studied and characterized (2D XY, roughening, Weiss domains, Dirac equation in gauge background etc).

Another general CSD generating mechanism further blurring the difference between static and dynamic simulations is the "decoupling" mechanism of certain (metastable) global degrees of freedom: if the freezing of the system takes place much faster than the typical times necessary for certain global structures to develop or disappear, the system gets "quenched" in configurations which are not typical for the thermal equilibrium at the frozen temperature. Indeed, in as far as the short range interactions are concerned, the system is at the freezing temperature but the large distance features are still typical for the higher temperature. By further lowering the temperature, the relaxation of the long-distance features towards the low energy configurations becomes slower rather than being accelerated. In particular the relaxation dynamics of the global features is not directly related to the various time scales of the external factors which pump or extract energy from the system. The various space scales of the global quenched objects are the ones which govern the corresponding multiscale hierarchy of relaxation times. The CSD becomes consequently the repository of the information related with the relevant Macros of the system.

In order to clarify quantitatively the relation between space scales and time scales, we devised a general tool: the Multi-Scale Dynamical Exper-
While for MSDE calibration, we used initially an equilibrium experiment, its main destination is in the questions of nonequilibrium dynamics mentioned above.

Following [12], we defined the elements of the space-time-correlation matrix to be:

$$M_{l,j}(t) = \langle S_l(0)S_j(t) \rangle - \langle S_l \rangle \langle S_j \rangle.$$  \hspace{1cm} (1)

where \(l\) and \(j\) are points on the lattice (e.g. in the square lattice \(l = (n_x, n_y), 1 \leq n_x \leq L_x, 1 \leq n_y \leq L_y\)), and \(S_l\) is any local observable (e.g. the spin at the point \(l\)). \(t\) is the (running) time between the measurement of \(S_l\) and the measurement of \(S_j\).

In the case of a translation invariant dynamics (e.g. as defined by the eq. (9)) one has:

$$M_{l+d,j+d}(t) = M_{l,j}(t)$$  \hspace{1cm} (2)

where \(d\) is any lattice vector. Thus, one can identify the eigenvectors of the space-time-correlation matrix as Fourier modes. Each eigenvalue \(\lambda_k(t)\) corresponding to the eigenvector \(e^{i\cdot k}\) satisfies:

$$\sum_j M_{l,j}(t)e^{i\cdot k} = \lambda_k(t)e^{i\cdot k}.$$  \hspace{1cm} (3)

For each wave-vector \(k\) one can assign a decorrelation time \(\tau_k\) via:

$$\lambda_k(t) \sim e^{-t/\tau_k}, \hspace{0.5cm} t \to \infty$$  \hspace{1cm} (4)

Note that for instance in the spin case, \(\tau_0\) (\(\tau\) for \(k = (0,0)\)) is identical to the usually defined exponential correlation time for the magnetization.

Measuring the decay of the eigenvalue, \(\lambda_k(t)\), we estimate \(z\) assuming the dynamical RG [13] scaling relation:

$$\tau_k \propto k^{-z}. \hspace{1cm} (5)$$

where \(k\) denotes the norm of \(k\). The reason for this assumption is that the corresponding length scale of a Fourier mode is: \(\Lambda \propto k^{-1}\). Therefore assuming dynamical universality one expects that \(\tau_k \propto \Lambda^z\).

The Simulation. — The model in which we calibrated our method is the 2D Ising on a square lattice [14]. We measured the correlation eigenvalues for both a global algorithm (Wolff [15]) and a local one (Parallel Checkerboard
Metropolis (PCM) dynamics). We use 2 lattice sizes \( L = 64 \) and \( L = 128 \) to show that the result is independent in \( L \).

We worked exactly at the critical temperature \( T = T_c = 2/\log(\sqrt{2} + 1) \) (we normalize \( J \) and \( k \) to 1). In the PCM case we employed the Wolff dynamics to decorrelate the system between the ”local” PCM episodes when we performed the measurements. In both cases we had periodic boundary conditions. In both cases we discard the first 10,000 Wolff sweeps and then took \( 3 \times 10^5 - 2 \times 10^6 \) different measurements of the space-time-correlation matrix, with ten Wolff sweeps between each measurement. Due to the translation invariance it is enough to compute only one line of the space-time-correlation matrix in each dimension, and the reflection symmetry reduces further the required measurements to almost half of a line in each dimension, altogether only \( (L/4 + 1)^2 \) elements. We repeated for 10 independent systems and estimated the error bars by the standard deviation. The fitting to an exponential, in order to estimate \( \tau_k \), and to power law to deduce \( z \), were done using nonlinear least square fitting \([16]\) which gives the error bars as well.

In figures 1-3 one can see a log-log plot of \( \tau_k(k) \), and the evaluated \( z \) for the Metropolis and Wolff dynamics, respectively (we ignored the smallest \( k \) values where the linear fit fails because the saturation of the time correlations due to finite size effects. In particular, in MSDE, the usual measurement of the total magnetization is a datum which we discard as a mere finite size artifact.

The final results of our fitting are \( z_{\text{met}} = 2.1 \pm 0.05 \), \( z_{\text{wolff}} = 0.1 \pm 0.2 \), for \( L=64 \), and \( z_{\text{met}} = 2.1 \pm 0.1 \), \( z_{\text{wolff}} = 0 \pm 0.15 \), for \( L=128 \).

MSDE display explicitly the simultaneous existence of all the time scales below the maximal scale in the system. For increasingly larger systems, this scale diverges and the critical system presents explicitly all the time scales as active scales in its dynamics. This produces in the infinite volume limit a power law. The MSDE method investigates the internal relations between the spatial and the time scales in statistical models and translates into practical numerical experiments the ideas of the dynamical RG.

The results in the 2D Ising model confirmed that the Dynamical RG assumption for the scaling relation \( \tau_\Lambda \propto \Lambda^z \) holds both for the Metropolis and the Wolff dynamics, suggesting that it does not depend on the particular kind of dynamics.

One can go further and use our method to analyze the internal structure of the Fourier subspaces as well as seeing the correspondence between the
equilibrium and non-equilibrium measurements.

The experiments relating the spatial scales and features with the time scales of CSD of the type reported above can help us identify the relevant macros in other models. In particular the ferromagnetic Ising model on certain lattices has at very low temperatures, in spite of the unique ground state some dynamical spin-glass-like properties.

For instance, for Ising spins defined on the triangles of a Random Triangularized Surface (RTS) [20], the presence of baby universes of arbitrary large size $L_{baby}$ leads to islands of uniform spin whose life-time diverge like $\tau_{baby} \sim e^{L_{baby}/T}$ (i.e. they are rigorously stable at $T = 0$) [3].

The natural branched polymer structure of both closed [41] and open [39, 40] RTS gives rise to an ultrametric structure of these states landscape which is very similar with the one typical for spin glasses [9].

It is encouraging that even in such conditions algorithms acting on large spin islands are effective in eliminating CSD [20]. The Ising model on an infinite coordination number lattice is also tractable by the cluster algorithms [21]. For low average coordination number, the global cluster algorithms may apply even in the frustrated case [22].

In both equilibrium and non-equilibrium systems we have used successfully interactive computer animation in order to identify the spatial structures and the Macros responsible for the CSD. In particular we ran in parallel local (CSD) and global (RCSD) algorithms and documented the evolution of the systems at various spatial scales (see last section).

3 Elimination of CSD as Understanding of Macroscopic Dynamics

The perception of Critical Slowing Down as an unwanted feature of the simulation, lead into oblivion in the past studies the fundamental importance of CSD as a tool in identifying the relevant degrees of freedom of the particular critical system under study.

However, in the context of RCSD algorithms the fact that the acceleration

$3^{\text{Even on a finite regular hexagonal lattice the life times of the convex islands of linear size } L \text{ are of the order } \tau_{\text{convex}} \sim L^3 e^{1/T} \text{ and therefore they become rigorously stable at } T = 0.}$
of the dynamics of a certain mode eliminates/reduces CSD is a clear sign that the critical degrees of freedom were correctly identified and their dynamics correctly understood.

RCSD algorithms express and validate in an objective way (by reducing the dynamical critical exponent \( z \)) our previous intuitive understanding of the collective macroscopic dynamics of large ensembles of microscopic elementary objects. In certain systems, which resist the conceptualization of their understanding in closed analytic formulae, this kind of "algorithmic operational understanding" might be the only alternative.

With this in mind, one may attempt to use CSD and its elimination (reduction) as a quantitative measure of the "understanding" of the model’s critical properties. For instance an algorithm leading to \( z = 2 \) would contain a very low level of understanding while the "ultimate" level of understanding is when one needs not simulate the model at all in order to get the static properties of every "lattice size" (analytic understanding).

At the other extreme are the simulations of spin glasses where the the correlation times increase exponentially with the size of the system. In simplicial quantum gravity the increase is faster than any function \(^{[26]}\). This would be the "ultimate" CSD: incomputability; that is, \textit{a priori} impossibility to compute within a systematic approximation procedure the numbers "predicted" by the theory \(^{[26]}\). The rise of incomputability in the context of the MicRep approach allows a new perspective on the issues of predictability and reductionism: the possibility arises that the future state the universe is physically totally \textit{determined} by its present state, yet the future state cannot be \textit{predicted} because its physical parameters as determined by the theory are mathematically uncomputable numbers. (Unfortunately this fascinating point falls outside the scope of present article.)

Until now, one could think that there are only two possibilities: to know or not to know. However, as explained above, during the last years research it became evident that intermediate levels of understanding exist. They are all categorized by the ability to eliminate or reduce the CSD. We have learned that in many models embedding a knowledge that we have about the model can result in a better (faster) dynamics.

Often, the way to a "more efficient algorithm" passed through the understanding of the "relevant degrees of freedom" and their dynamics. In order to maintain all the information stored by the elementary degrees of freedom, it is always necessary to define precisely the relevant Macros in terms of the
Micros. Once the Macros are defined in term of the Micros it is possible to find their dynamics in term of the translated Micro dynamics.

There are few "tests" to establish that for a given critical system the set of Macro’s was correctly identified:

- One would need to make sure that there is a "large" number of Macro’s. This requirement makes sure that a large fraction of the relevant degrees of freedom is indeed represented by the Macro’s that were discovered.

- Then one has to check that they are relevant in the sense that they are not just symmetries of the theory. In other words, a change in an Macro should have an influence on the important measurables of the system.

- One of the more stringent tests is to verify that the resulting Macro dynamics is "free". That is, in a typical configuration of Micro’s the resulting dynamics of the Macro seems to be free. This is a signal that the correct Macro have been identified.

An analogy for the relation between Macro’s and Micro’s can be found in language. The letters would be the Micro’s and the words will be the Macro’s. Of course, manipulating words amounts to manipulating letters. However when one "works" in the words-level one need not bother with the letter-level, even though these two levels co-exist.

In order to illustrate the way in which one can eliminate CSD by identifying and acting upon the appropriate Macros, let us consider the case of a free field on a Random Triangulated Surface (RTS).

In particular this example illustrates simply the relation between the rules constructing the Macros and the critical properties of the physical model.

In the example below the macros overlap rather than being separated by sharp boundaries. In fact, the same Micro may belong (to various degrees) to more than one Macro. This "fuzziness" rends the boundaries defining a Macro a "subjective" choice, a matter of degree/opinion, which, while not affecting the validity of the numerical algorithm, sets the scene for further conceptual elasticity. It suggests continuous interpolation and extrapolation procedures closer in their form and essence to the working of the natural human intelligence.

In fact, through substituting the binary logic of the Micros with the continuous one of the Macros, one may avoid the no-go theorems, the paradoxes
and the puzzles related to (un)computability, (i) reversibility and (creative) reasoning.

The precise yet "smeared" formulation of the Macros within the multiscale MicRep approach bypasses these classical conceptual puzzles arising in the reductionist method. In particular, while the Macros acquire a certain degree of reality, individuality and causal behavior at the macroscopic level, their conceptual boundaries are fuzzy enough to avoid the paradoxes which would arise should one try to apply Macro categories in the microscopic domain (their boundaries "dissolve" gracefully as one tries to resolve them with a too sharp "microscope"). In the MicRep Multiscale framework, there is no contradiction between considering the ocean as a set of molecules or a mass of waves. These are just complementary pictures relevant at different scales.

In the theory of random triangulations [19] one has usually to calculate expectation values, \( < O > \), of observables, \( O \), according to the formula:

\[
< O > = \sum_{\{X\}} O(\{X\}) P(\{X\})
\]  \hspace{1cm} (6)

where the sum is over all field configurations, \( \{X\} \), and

\[
P(\{X\}) = \frac{1}{Z} \exp(-\beta E(\{X\}))
\]  \hspace{1cm} (7)

\[
Z = \sum_{\{X\}} \exp(-\beta E(\{X\}))
\]

The expression for the energy, in presence of an external field, \( H_i \) is:

\[
E(\{X\}) = \sum_{<i,j>} J_{ij} x_i x_j + \sum_i H_i x_i + E_0,
\]  \hspace{1cm} (8)

where the first summation is over all the links, and \( x_i \) is the field on the site \( i \). In the free field case \( J_{ij} = -1 \), and \( J_{ii} = 3 + m^2 \) where \( m \) is the bare mass.

Note that on a lattice with arbitrary topology (which will become our case after the first coarseening step), for an arbitrary number of neighbors \( N_i \) of the point \( i \), the relation between the mass \( m \) and the coefficients is \( J_{ii} = N_i + m^2 \) and \( J_{ij} = -1 \). Therefore the signal that a particular link is susceptible of being part of the mechanism of creating long range correlations \( (m \approx 0) \) and CSD is \( J_{ij} \approx -J_{ii}/N_i \) while links for which \( -J_{ij} << J_{ii}/N_i \) are allowing each of their end nodes to equilibrate quite independently one
from the other \( m >> 0 \). Therefore they are not significant for long range effects and can be effectively treated by local algorithms. This will become important later in deciding the blocking procedure for the global algorithm.

The common local method to estimate (6) is the heat-bath (HB) [27] algorithm. It creates iteratively configurations according to the following procedure. Given a configuration \( \{ X \} \) one constructs a new configuration \( \{ X' \} \) by updating the field \( x_{i_0} \) at an arbitrary point \( i_0 \). More precisely, \( x_{i_0} \) will be changed to \( x'_{i_0} \) with a probability,

\[
W(\{ X \} \rightarrow \{ X' \}) = N \exp( - \sum_j J_{i_0 j} x'_{i_0} x_j + H_{i_0} x'_{i_0} ),
\]

where \( N \) is a normalization factor. That probability fulfills the detailed balance condition [27],

\[
W(\{ X \} \rightarrow \{ X' \}) P(\{ X \}) = W(\{ X' \} \rightarrow \{ X \}) P(\{ X' \}).
\]

The condition (10) is sufficient for taking care of the factor \( P(\{ X \}) \) in estimating (6). I.e., after creating \( n \) decorrelated configurations according to (10), one obtains (3) by simply computing the value of the observable \( O \) for each of those configurations and taking their average:

\[
< O > \approx \frac{1}{n} \sum_{\{ X \}} O(\{ X \}).
\]

Unfortunately, HB generates strongly time-correlated configurations as the number of the grid points increases. This critical slowing down (CSD), related with fields correlations, is well known [6] and analog to the CSD of laplacian-like Partial Differential Equations (PDE) solvers (e.g. Gauss-Seidel). Therefore, in constructing our algorithm, we borrow intuition from the non-dynamical PDE solvers terminology.

In particular, the process of eliminating the errors during the iterations of an Algebraic Multigrid (AMG) PDE solver is analogous to the elimination of correlations in the present Dynamical Algebraic Multigrid (DAMG) procedure (and to reaching the thermal equilibrium at the beginning of the runs).
The fact that this conceptual analogy works is of course a non-trivial statement on the critical dynamics of the system. Consequently, the success of our procedure expresses a nontrivial understanding (which turns out to be of degree $z = 0.2 \pm 0.2$).

As in the non-dynamical AMG solvers case, we define first an $\alpha$–strong connection. A point $i_0$ is $\alpha$–strongly connected to $j_0$, if

$$J_{i_0,j_0} > \alpha \cdot \max \{J_{i_0,k}\} \quad 0 \leq \alpha \leq 1,$$  \hspace{1cm}(12)

In the PDE solver case, the CSD originates in the fact that, after some relaxation sweeps, the error becomes smooth over strongly connected sets of points. While the smooth components of the error are not reduced efficiently by the usual relaxation, they are easily transported to the coarse grid where they can be efficiently reduced. Thus, coarsening, in the AMG scheme is done by blocking strongly connected points.

The strong connection link criterion is blocking together variables which interact via links corresponding to low mass contribution to the local energy (according to the comments following equation 8). These are the links which are most difficult to equilibrate by local updatings and which are responsible for long range correlations and CSD. One sees again how the building of a RCSD algorithm is in fact the expression of a more intimate understanding of the relation between the microscopically defined elementary interactions and their macroscopic consequences.

The connections $J_{ij}$ appear also as weights in expressing the values of functions (and operators) on the coarse grid points as weighted averages of the values on the fine grid points. It is interesting therefore that even though the model is definitely not a gauge model, its multiscale structure endows the $J_{ij}$ coefficients with parallel transporters properties.

We have constructed the coarse grid according to the prescription developed for PDE in [28].

However, in dynamical relaxations, utmost care has to be taken to fulfill the detailed balance condition, (10). This puts additional constraints on the way the transport of the variables (and operators) between the grids is performed. To see it, consider the change in the coarse grid energy caused by the updating of a coarse grid variable. When translated in terms of the fine grid variables, this coarse grid updating leads to a change in the fine grid energy. We will show now that the detailed balance condition is fulfilled if
this change in the fine grid energy is equal to the change in the coarse grid energy.

We will show it for a two-levels cycle as the similar property for multilevel cycles follows from iterating the two level argument.

The starting point is a fine grid configuration, \( \{ X \} \), obtained after some fine grid HB relaxation sweeps. The variables are \( x_i \) and their current values, are \( x_i^0 \). The aim is to update these values using a two levels cycle.

In the case of (non-dynamical) MG solvers what is transported on the coarser level is not the fine grid equation but the so called “error equation” [2]. By analogy, in the dynamical case (DAMG) we consider the following change of variables:

\[
x_i \rightarrow x_i^0 + q_i.
\]

(13)

(the new fine grid variables, \( q_i \), are analog to the fine grid error components in the non-dynamical algorithms). Every change in the new variables \( q_i \) expresses a corresponding updating of the variables \( x_i \). The variables \( q_i \) are defining a configuration, \( \{ Q \} \). By substituting (13) to (8) the energy is expressed in terms of \( q \)’s as:

\[
E(\{Q\}) = \sum_{i,j} J_{ij} q_i q_j + \sum_i H_i' q_i + E_0',
\]

\[
H_i' = H_i + 2 \sum_j J_{ij} x_i^0,
\]

\[
E_0' = E_0 + \sum_i H_i x_i^0 + \sum_{i,j} x_i^0 x_j^0.
\]

(14)

Note that even when starting with \( H_i = 0 \) in (8), non vanishing external fields \( H_i' \) are arising in (14).

The next step in the usual, (non-dynamical) AMG solver is to transport the configuration \( \{ Q \} \) to the coarse grid in order to accelerate the updating. In the dynamical case the coarsening corresponds to considering during a coarse sweep only a selected type of updatings, parameterized by a smaller number of parameters \( q^c \) than the number of degrees of freedom \( q \).

More precisely in DAMG we are considering only updatings which affect the following nonlocal combinations of fine grid variables:

\[
q_i = I_{ij} q_j^c,
\]

(15)

where in eq. (15) \( I_{ij} \) is the (rectangular) interpolation matrix, and the \( q_j^c \) can be called "coarse grid variables".
Since updating the $q^c$'s does not lead in general from every fine grid configuration to any other arbitrary fine grid configuration, the updatings parameterized by changes in the $q^c$'s forsake ergodicity. In exchange, these nonlocal steps are more efficient in traveling throughout the configurations space. The ergodicity is restored by the performance of usual sweeps on the fine grid.

The correct Boltzmann distribution is insured by taking care that the configurations as expressed by various values of $q^c$ are distributed according to their energy when expressed in terms of the fine grid variables $q$.

This means that the distribution of the variables $q^c$ is governed by the energy (14) when expressed in terms of (15):

$$E(\{Q_c\}) = \sum_{i,j} J^c_{ij} q^c_i q^c_j + \sum_i H^c_i q^c_i + E'_0$$

$$J^c = I^t J I,$$
$$H^c = I^t H'.$$

One can see now that when restricting the updates of the system to the coarse moves parameterized by the coarse variables $q^c$, one is lead to simulate a system of the same type as the initial one (14) with new coefficients $J^c_{ij}$. This allows one to proceed iteratively and apply the coarsening procedure to the system $q'$.

This completes the definition of the DAMG cycle. The coarsest system, which contains a very small number of variables and quite short range correlations can be simulated readily by HB or Metropolis methods without any worry of CSD.

**Correlations - The CSD in the dynamical process.**

The objective of the Dynamical Algebraic Multigrid (DAMG) algorithm is to eliminate/reduce the CSD.

In order to characterize CSD, one defines first the autocorrelation function, $C(n)$. Let $\{X\}_i$, $i = 1,..,N$, be a chain of configurations obtained by a Markov process such as the one produced by the HB procedure. Let $O(\{X_i\})$ be the results of measuring a certain observable $O$ on each of the configurations. The auto-correlation function is defined by:

$$C(n) = \langle O(\{X\}_i)O(\{X\}_{i+n}) \rangle - \langle O \rangle^2.$$  

At large $n$, $C(n)$ behaves like,

$$C(n) \sim C(0) \exp(-n/\tau),$$

$$19$$
where $\tau$ is the so called auto-correlation ‘time’.

$\tau$ measures the number of Markov steps one has to perform after obtaining a configuration in order to obtain a new configuration uncorrelated with the first one.

In critical systems, $\tau$ diverges with the dimension of the system. In the case of a regular $d$-dimensional grids of volume $L^d$, one has:

$$\tau \sim L^z. \quad (19)$$

which defines the dynamical critical exponent $z$. Typically $z \geq 2$, for local algorithms. In our case, since the lattice is not regular, we checked the dependence of $\tau$ on the number of triangles $N$:

$$\tau \sim N^\zeta, \quad (20)$$

which assuming an effective (Hausdorff) dimension $d_H$ of the lattice would be related to $z$ by:

$$z = \zeta d_H. \quad (21)$$

For a typical 2-dimensional critical system which displays CSD, $\zeta$ would be of order 1.

In the next section we show the comparison between the $\zeta$ value in DAMG runs vs. usual one level HB.

Results

The critical properties of the system are best represented by the surface "width":

$$O = \langle (x - \langle x \rangle)^2 \rangle \quad (22)$$

which in the string formalism is also called the mean square extent of the surface.

This quantity diverges with the size of the system and consequently displays CSD during local runs. Thus, we have measured $O$-related quantities, both using DAMG and using the usual one level HB for comparison.

In Figure 4 we demonstrate CSD by showing that the local HB run has problems in reaching the correct value within a reasonable time and has extremely long range correlations. As opposed to it, the DAMG has correlation times of the order 1 and covers very effectively the configuration space.

In order to make a fair comparison one has to take into account that a DAMG sweep involves more operations than a HB one. Consequently, we
compared the computations only after normalizing the units such that one unit represents the same CPU time in both cases. In practice, this reduces to counting the number of HB 'hits' performed during the run (a 'hit' is a local updating of a field upon a site) . In figure 4, the ordinate $n$ is in units of the number of hits performed upon the fine grid, that is

$$n = \frac{\text{number of hits performed}}{N}. \quad (23)$$

The lattice size of the run presented in Fig. 4 was $N = 8596$ triangles and one level measurements were made after each 'sweep', i.e. $N$ 'hits'. DAMG measurements were made after each 'cycle'. A 'cycle' contains a sweep upon the fine grid and all the coarser grids sweeps needed to update it.

Figure 5 is a comparison of the autocorrelation times, $\tau$, for DAMG vs. one level HB. The grid size was $N = 10000$, and using the formula (17) we have computed the logarithm of correlation function, $C(n)/C(0)$, for $O$. Here we have also normalized the running time in the way mentioned above, in order to have a common basis of comparison between the two methods.

We measured the slope of $\log C$ and got (see (18): $\tau = 1661 (143)$ for the one level HB, and $\tau = 2.13 (0.24)$, in the DAMG scheme where the numbers in parenthesis are the errors.

Figure 6 presents the autocorrelation time, $\tau$, as a function of the number of grid points, $N$. From (20) one can see that plotting it on a log−log scale, and measuring its slope would give the critical exponent $\zeta$. The results for usual HB is $\zeta = 1.5(0.2)$ while for DAMG $\zeta = 0.2(0.2)$.

Let us emphasize that beyond the computational gain of eliminating CSD completely, we obtained here nontrivial physical information on the character and the origins of the criticality in this model. In particular we learned about the relation between the microscopic "strong links" and the macroscopic long range and long time correlations. In this simple system, this can be traced to the relation between the microscopic bare mass $m$ appearing in the definition of link energy and the macroscopic correlation length corresponding to the physical mass: $m_{phys} \approx \frac{1}{\xi}$ [33].

The Dynamical Algebraic Multi-Grid for free fields on RTS is just an example of the general computational philosophy we applied in a full range of models. The very high precision physical measurements obtained with this new generation of efficient algorithms were reported elsewhere [29].

We summarize below some of those applications by providing for each of them a relevant reference, the Micros, the elementary INTERactions and the
relevant Macros. According to the description above, for real fields on RTS this summary has the form:

- Elimination of CSD for free fields simulations on arbitrary lattices by acting on the relevant collective variables [32].
  - Micro - value of field at a point
  - INTER - continuity on strong links, noise on weak links
  - Macro - large neighborhoods with strongly correlated fields.

For other systems, we have:

- Elimination of CSD by performing the relevant critical operations. The example of Valleys-to-Mountains-Reflection (VMR) algorithm in interface roughening dynamics [31].
  - Micro - the height of surface at certain point
  - INTER - height difference
  - Macro - valleys and mountains

- Elimination of CSD by acting on the relevant subsystems. The example of Ising spins and of discrete gaussian model on regular and RT Surfaces [20].
  - Micro - sign at a point
  - INTER - preference for the same sign neighbors
  - Macro - blocks of same sign

- Eliminating or reduction of CSD by acting on macroscopic Gauge invariant degrees of freedom:
  - Flux Tubes Dynamics [36].
    * Micro - gauge fields associated to links
    * INTER - plaquette frustration
    * Macro - flux tubes
  - Instanton Dynamics [37].

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Micro-gauge fields on links
* INTER - plaquette frustration
* Macro - topologically stable instantons

Polyakov Loop Dynamics \[38\].
* Micro - gauge fields on links
* INTER - plaquette frustration
* Macro - polyakov loop aligned blocks

Fermion Dynamics in Gauge background and Parallel Transported Multigrid \[25\].
* Micro - fermion fields
* INTER - local Dirac equation
* Macro - extended (topological) solutions to Dirac equation

Elimination of CSD in the geometry dynamics of 2D SQG \[11\].

Micro - points, triangles
* INTER - connectivity
* Macro - global geometry, topology.

Another phenomenon which appeared in the Valley-to-Mountain algorithm applications in the application of Instanton global moves and in the finite temperature polyakov loop application is the "hetergodicity". This term, denotes a situation in which the juxtaposition of steps belonging to two slow algorithms leads to an algorithm which does not present critical slowing down \[30, 37, 38\]. This turned out to give precious information on the structure of the critical modes subspace and its expression at the Markov matrix level \[37\].

4 MicRep use in Multiscale Phenomena

As explained in the previous sections, we propose to use the experience gathered in isolating the relevant macroscopic dynamics in order to study various complex models not necessarily at equilibrium.

A context in which the MicRep is unavoidable, is when the phenomena under study do involve a multitude of scales.
Moreover we have argued in chap. 2 that one may expect a reasonable degree of universality even in systems which did not undergo a fine tuning in their parameters and even in systems which do not admit a rigorous equilibrium or near-equilibrium Statistical Mechanics definition.

We sketch here a few examples and then concentrate in full detail on one of the applications.

**Coarse grained Molecular Dynamics**

Using a very simple minded “Coarse Grained Molecular Dynamics” CGMD method of simulation [42], we were able to describe and predict the nonequilibrium distributions of reactants in certain Chemical Reactions. The microscopic ”elementary” objects did not necessarily represent the molecules dynamics faithfully: the ”elementary” Micros moved on straight trajectories except for certain cross sections for elastic and inelastic collisions as well as a cross section for the chemical reaction to take place if the colliding Micros were in the appropriate excited internal state. In spite of its coarseness, this approach allowed the realistic treatment of reaction rates and chemical distributions arbitrarily far away from equilibrium.

In analogy to the scheme used in previous chapter, we may describe this application schematically as:

- **coarse grained molecular dynamics** in far from equilibrium systems [42].
  - Micro - ”molecules”
  - INTER - collisions, excitation, activation, bonding and dissociation.
  - Macro - reaction rate, density and energy distributions.

The use of this type of CGMD might be absolutely unavoidable in describing systems in which various reaction chains are mutually exclusive and a de facto segregation takes place in the system between regions in which various mutually-exclusive reactions take place. We have in mind the situation in which the same set of enzymes are capable in principle to sustain 2 or more different reaction chains but due to certain inhibitory cross-interactions taking place at certain intermediate steps between the chains, the chains cannot take place simultaneously in the same spatial region. There is preliminary evidence also for the opposite effect: reaction chains mutually sustaining one
another into active states which would not be viable in the absence of their coexistence and cross interaction \cite{43}.

The implicit assumption of continuity imposed by a continuum hydrodynamic - like or Landau - Ginsburg - like model is then invalid for such self-segregating reactive systems while the CGMD remains a valid MicRep of the system.

This situation is also representative for the typical biological systems where the same genes and proteins lead spontaneously to different reaction chains in different spatial regions of the cell/organism.

Another example is in linguistics in which the carriers of languages (speakers) are usually spatially segregated and the transitions between various languages are quite sharp rather than through a continuum of interpolating dialects.

Similar effects appear in the ecology of various species populations.

The coarse grained MicRep rather than the continuum approach is also necessary if one is interesting to go beyond continuous deformations of solids and study the appearance and spreading of cracks and fractures.

**psychophysics**

In psychology and psychophysics the identification of the appropriate microscopic elementary objects is still problematic but it was shown that it is not beyond the scope our quantitative tools.

As an example we considered the well defined problem of recognizing 3 dimensional global structure and motion from the 2 dimensional information available at different times and locations (on the retina).

It was argued by a combination of rigorous analytical results supported by a series of experiments \cite{46} that contrary to the previous data, the 3D structure is re-constructed by our visual system from point-like signals rather than contour portions passing through the receptors area.

Moreover, predicting and verifying macroscopic features such as visual illusions in certain controlled reproducible conditions, one was able to validate a theoretical model about the way the microscopic signals are integrated by the human visual system into an emerging 3D collective motion percept.

In particular, it appears that, in order to lower the computation time necessary for producing in real time the 3D percept, our visual system adopts quite "sloppy algorithms" which lead to coarse approximations rely on plain guessing and are often wrong (e.g. systematically mistake certain rigid mo-
tions as non-rigid) [4].

At the present stage, many details are still missing and the detailed microscopic representation model for the visual system is far from completed. However, as mentioned above, the prediction-validation dialogue has already started.

Some further clues might be provided by the study of the emergence of contours from the early children scribbles – a formal and visual analogy was observed between this process and the protein folding dynamics [45].

The schematic MicRep profile of this application is:

- **Microscopic seers and macroscopic sight** [46].
  - Micro - line elements, points.
  - INTER - curvature, continuity, integration, (mind) changes
  - Macro - shapes (coils, helixes, hashes)

We are listing below the MicRep scheme of a few other systems proposed in the literature after which we describe one of them (the stock market simulation) in full detail:

**Examples of MicRep’s, their Micro’s, their elementary INTER-ACTIONS and their Macro’s**

- **-microscopic drivers and macroscopic jams** [49].
  - Micro - cars
  - INTER - go ahead/give way at intersections.
  - Macro - traffic flow, jamming.

- **-microscopic Einsteins and the macroscopic universe geometry**
  - Open Boundary Random Triangulated Surfaces and the Wheeler DeWitt Equation for the wave function of the 2D universe [39, 40].
  - Micro - points, links, triangles

---

4 In view of the rigorous limitations for Turing machines computations on what can be computed (uncomputability theorems) and at which price (NP theorems), this general human brain tendency for sloppines, resorting to irrational beliefs and gambling on luck might turn out to be computationally sensible (and morally commendable).
– INTER - connectivity of points
– Macro - surface geometry, boundary length distribution (wave function of the universe).

• microscopical Alexander moves and macroscopic Reasoning/Theorem Proving systems
  – is Simplicial Quantum Gravity Turing Computable [26]?
    * Micro - links, letters.
    * INTER - simplexes, words.
    * Macro - global topology, statements.
  – inventive solution search [17],
    * Micro - postulates, statements
    * INTER - syllogisms, heuristic rules
    * Macro - contradictions, solutions, theorems.

• dramas - mathematical categories endowed with dynamics [44],
  – Micro - categories
  – INTER - relations, composition laws
  – Macro - (stories) dramas

• microscopic investors and macroscopic crashes [18],
  – Micro - investors, shares
  – INTER - sell/buy orders
  – Macro - market price (cycles, crushes, booms, stabilization by noise)

Let us describe this last MicRep in full detail.

economic simulations
In the study of economic systems, the simulation of the money market is usually based on the integration of a differential stochastic equation in which the average interest rate and the momentary dispersion (volatility) are parameters. In more involved estimations of futures values, various
versions of the resulting stochastic dynamics are considered and an average is performed over the possible histories.

Another treatment is to construct a network of simple elements whose interactions do not attempt to constitute an approximation of the actual microscopic dynamics of the model. Still, one attempts to obtain through iterative optimization an ”output” of the system which fits the real macroscopic data.

Yet another technique is to assume that the macroscopic dynamics can be modeled by a set of a few coupled stochastic differential equations, to guess their form and to isolate the universal features in the chaotic behavior of their solutions.

The MicRep formulation allows a more first-principles treatment \cite{48} by considering explicitly a set of investors which sell or buy equities according to simple rules of stochastically maximizing a utility function which is constructed on the basis of their previous history and prejudice.

The market price of the equity is then computed based on the bidding prices from all the participating investors. This in turn influences the wealth of the investors and their prejudices.

The collective emerging dynamics is by-and-large independent on the details of each individual buy-sell particularities and displays general features.

In particular we found the natural emergence of cycles of booms and crashes and the dependence of their timing and intensity on the dispersion of the individual buy-sell criteria.

By comparing the standard economic models simulations with the MicRep one, one sees that the relation is similar to the one between thermodynamics and statistical mechanics: while the thermodynamics can provide a general macroscopic framework which relates the possible values of certain parameters which parameterize the macroscopic dynamics, the statistical mechanics can address more detailed questions and eventually deduce the macroscopic parameters from the microscopic properties.

\textit{The model}

The simple system introduced in \cite{48} contains as its microscopic elements \( I \) investors interacting via buying and selling as detailed below.

The macroscopic dynamics are the trends in the stock market.

Each investor decides at each time how to divide its wealth \( W \) between two investment options:
It can invest in stock a proportion $0 < X(i) < 1$ of its wealth and

- Invest the rest of it $(1 - X(i))$ in bonds.

The objective of each investor (concretized in the strategy detailed below) is to maximize the expected utility $EU(X(i))$ which in most of our runs we take to be $\log W$.

The bond is a riskless investment yielding a constant return $RR$ after each period. Thus an investment of $(1 - X(i)) \times W$ dollars will yield $(1 - X(i)) \times W \times (1 + RR)$ in the next period.

We actually treat the bonds as depositing money in a bank at fixed return rate. Therefore the statements from now on that the bonds are riskless might be interpreted as dynamically equivalent with requiring for simplicity the existence of an ideal riskless (at least very safe compared with the equities) way of keeping ones money and still getting a return.

This simplifying assumption as well as most of the ones below can be easily relaxed and their effect can be studied in detail. In fact the strength of our approach is that it is very robust to the further introduction of any degree of realism one deems necessary.

The non-trivial fact is that already in the absence of the effects of real goods production, prime-matter price fluctuations, public policies interference, taxation, global politics, etc. the system presents realistic behavior of its nontrivial cycles dynamics.

This supports the view that the stock market has a structured (not pure noise) dynamics by itself, autonomous to a large extent from the details of its environment. This is an important prerequisite for any attempt to predict and characterize stock market future tendencies.

The stock is the risky asset. The return yielded by the stock is composed of two elements:

- $P_n$ - the price per share of the stock during the time period $n$. $P_n$ is determined collectively by all investors by the law of supply and demand as detailed below.

- The company gives dividends according to its profits. For simplicity, we assume here that the firm pays a constant dividend $D$ per share every period (the effects of variations in $D$ are presently under study).
The overall return per share per period is then:

\[ H_n = \frac{P_n - P_{n-1} + D}{P_{n-1}} \]

The investors prejudices are expressed by keeping track of the last \( K \) returns by the stock. This we call the stock’s history:

\[ H_n = \{H_n, H_{n-1}...H_{n-K+1}\} \]

For \( H_0 \) the \( H_j \)’s with \( j = 0, -1, ..., -9 \), are given as part of the initial conditions. In the initial model we assumed that the investor believes each one of the last \( K \) history elements \( H_j \) has an equal probability of \( 1/K \) to occur in the next period. Later we altered the assumption of equal probability without observing a qualitative change.

Let us explain in detail the rules which decide the bid of each investor and the establishing of the new stock price.

Suppose investor \( i \) holds at time \( n \) a number of shares \( N_n(i) \) at the price of \( P_n \) a share, and has a total wealth (including its bonds) of \( W_n(i) \). In particular the values \( N_0(i), P_0 \) and respectively \( W_0(i) \) are provided as initial conditions.

Before each trade period, each investor \( i \) decides according the procedure described below how many shares \( N_{n+1}(i) \) he wishes to hold as a function of the hypothetical new stock price \( P_h \) which the offer and demand may fix during that trade period (according to the procedure detailed later). Its bid is formalized in a function \( N_h(i, P_h) \) which expresses how many shares he orders for each hypothetical value of the market price \( P_h \). It has to specify a function rather than a firm shares number, because according to the bidding rules, the actual value of the price \( P_{n+1} \) resulting from the trade round is not yet known at the time of its bid.

The procedure followed by each investor to compute \( N_h(i, P_h) \) as a function of \( P_h \) is based on its attempt to maximize the utility function at time \( n + 2 \). Since \( N_n \) is already fixed, it is too late at this stage to try to affect the wealth at time \( n + 1 \). It is going to be

\[ W_h(i) = W_n(i) + N_n(i)(P_h - P_0) \]

independently on the \( N_{n+1} \) value. Therefore, in general, before the \( (n+1)^{-th} \) trade period, one tries to optimize the utility function:

\[ U_{n+2}(X_{n+1}(i)) = \log \left( (1-X_{n+1}(i))W_h(i)(1+RR) + X_{n+1}(i)W_h(i)\left(1+H_{n+1}\right) \right) \]
The first term in the $U_{n+2}$ formula is the contribution of the bond while the second term is the stock’s contribution. Of course the catch is that the investor does not know the value of $H_{n+1}$ ($H_1$ in the first round case) in this term. To supply for it, one maximizes in place of $U_{n+2}(X(i))$ a mean value of $U(X(i))$’s obtained by substituting respectively $H_{n+1}$ with each of the $H_j$’s belonging to $H_n$ into the $U_{n+2}(X(i))$ formula:

$$\mathcal{EU}_{n+2}(X(i)) = \frac{1}{K} \sum_{j=0}^{-K+1} \log \left[ (1-X(i))W_h(i)(1+RR) + X(i)W_h(i)(1+H_{n+j}) \right]$$

The optimum proportion of investment $X(i) = X_{n+1}(i)$ depends therefore of the knowledge and use which the investor makes of the history $H_n$ and of how it compares with $RR$.

The optimum proportion of investment $X_{n+1}(i)$ determines the number of shares: since $X_{n+1}(i)W_h(i)$ is the amount of money investor $i$ wishes to holds in shares at price $P_h$, the number of shares is:

$$N_h(i, P_h) = \frac{X_{n+1}(i)W_h(i)}{P_h}$$

The number of shares investor $i$ wishes to hold as function of the price of the share is its personal demand curve. Summing the demands of all investors gives us the collective demand curve. Since the number of shares in the market, denoted by $N$, is fixed, the collective demand curve sets the new price of the share $P_{n+1}$ by solving

$$N = \sum_{i=1}^{I} N_h(i, P_h)$$

with respect to $P_h$.

After the trading the wealth of each investor has changed:

$$W_{n+1}(i) = W_n(i) + N_n(i)(P_{n+1} - P_n)$$

Its new wealth equals its old wealth plus the change in price multiplied by the number of shares he had before the trade. The investors buys and sells shares at market value and does not gain or lose from this. It does gain or
lose money on the shares he holds. Therefore the decision on \( X_n(i) \) affects only the wealth at \( W_{n+1} \).

The number of shares held by investor \( i \) after the trade is:

\[
N_{n+1}(i) = \frac{X_{n+1}(i)W_{n+1}}{P_{n+1}} = \frac{X_{n+1}(i)\left(W_n(i) + N_n(i)\left(P_{n+1} - P_n\right)\right)}{P_{n+1}}
\]

Now follows a period of no trade at the end of which dividends and interest are received. During this period investor \( i \) holds \( N_{n+1}(i) \) shares, and has

\[
\left(\left(W_n(i) + N_n(i)(P_{n+1} - P_n)\right)\right)\left(1 - X_{n+1}(i)\right)
\]

invested in bonds.

After this period its wealth is:

\[
W_{n+1}(i) = W_n(i) + N_n(i)(P_{n+1} - P_n) + N_{n+1}(i)D
\]

\[
+ \left(\left(W_n(i) + N_n(i)(P_{n+1} - P_n)\right)\right)\left(1 - X_{n+1}(i)\right)RR
\]

where the third term is due to the dividends, and the fourth due the interest on the bonds.

The new price \( P_{n+1} \) also adds a new element to the history of the stock. The most recent return will be:

\[
H_{n+1} = \frac{P_{n+1} - P_n + D}{P_n}
\]

We can now update the history \( H_n \) to \( H_{n+1} \) by adding \( H_{n+1} \) to the list and discarding \( H_{n-K+1} \).

The model described so far is deterministic. The 'decision making' is done by maximizing the expected utility. We take into account the variability in the behavior of the various investors by adding a random variable to the optimal proportion of investment. For instance, we replace \( X(i) \) with

\[
X'(i) = X(i) + \epsilon(i)
\]

where \( \epsilon(i) \) is drawn at random from a normal distribution with standard deviation \( T \). \( \epsilon(i) \) is drawn separately for each investor. This randomness plays
the part of the ‘noise’ or ‘temperature’ in the system. ‘High temperature’ means a large T which means a wider distribution and a larger deviation from the deterministic case.

Computer Experiments and Results

In the simulations described here we chose the period of time to be one year. Accordingly we chose the rest of the parameters as follows.

We took the yearly riskless return $RR = 0.1$.

We chose the elements of $H_0$ randomly with mean $0.1001$ and variance $0.024$, so that the proportion of investment in the first round will be around 50%. We took the history (or rather- memory) length $K = 10$. The number of investors was $I = 100$ (small for computational reasons, but large enough to exhibit macroscopic phenomena). The total number of shares was $N = 10,000$.

The initial wealth of each investor was $W_0(i) = $1,000 (this is arbitrary as long as the initial price of the share is chosen accordingly). Initial price of share was $P_0 = $4.40 (chosen so that the first return by the stock will be close to the mean of the returns in $H_0$). We choose the dividend $D = $0.3 so that $D/P_0 = $0.068 i.e. a little less than $RR$.

As mentioned before, there was no fine tuning of the parameters.

Our results are shown in graphs 7 - 11. To understand these results let us first examine the deterministic case ($T = 0$) in graph 7. We see that the price of the stock rises sharply and then grows at a steady exponential rate. Let us first concentrate at the sharp rise. The first rate of return from the stock is relatively high. This produces a distribution $H_1$ which is higher than $H_0$. This means that investors are willing to increase their proportion of investment in the equity. This increase in the proportions of investment (demand), especially near the maximum proportion of investment, cause an immediate increase in the stock price.

After this sharp rise, the stock’s history of returns $H_n$ is very promising, and the proportions of investment in equities are fixed at their maximum. It can be shown, that under the condition of fixed investment proportions and zero temperature (no noise):

$$H_n = \frac{P_n - P_{n-1} + D}{P_{n-1}} = RR + \frac{D}{P_n} \left( \frac{1}{1 - X} \right)$$

We see that as $P_n \to \infty$, $H \to RR$. This explains the constant slope of the
exponential climb because:

\[ \frac{P_n}{P_{n-1}} \approx \frac{P_n - P_{n-1} + D}{P_{n-1}} \rightarrow RR \]

when \( P_{n-1} \) is large.

We also learn from this result that the differences between the \( H \)'s within the history \( H \) decrease as the average value of the \( H \)'s approaches \( RR \) - the fixed interest rate. Because \( H \) reaches \( RR \) from above in the limit \( P \rightarrow \infty \), the stock remains slightly preferable to the bond all the way. This situation, however, is highly unstable against the slightest noise. The instability is easy to understand: because the history of returns is only very slightly preferable to the bond, the smallest fluctuation in price can turn the tables and make the bond preferable. Because the variance in histories is very small (all the returns are close to \( RR \)) a small change in one of the histories causes a dramatic change in the proportions of investment. As the proportions of investment drop the price drops sharply producing a catastrophic crash.

In the total absence of noise \((T = 0)\) the small fluctuation which could trigger the crash never happens. However, for even very small \( T \) such a fluctuation will eventually happen and will be immediately amplified into a very spectacular crash. It is enough that one investor should deviate from the deterministic behavior, and decide to lower its proportion of investment in stocks a little, to cause the small fluctuation that will trigger the crash. This is exactly what happens when we 'turn on the heat', see graph 8. One can see that the exponential climb of the stock preceding the crash is not as smooth as before. The small temperature gives rise to small fluctuations in price. One of these fluctuations is large enough and causes the crash. The typical time for the crash to take place depends on the temperature and diverges when \( T \rightarrow 0 \).

After the first crash the price of the stock reaches rock bottom. When the price becomes very low, the fixed dividend ensures that the returns on the stock will be relatively high. After a few periods there are enough high returns in the history to make the stock an attractive investment again, the proportions of investment grow and the cycle begins all over again.

Note here a phenomenon reminding us of the recent history of certain authoritarian economic regimes: by enforcing artificially a narrow volatility (by strict control of the market) one could postpone indefinitely the crash (the time scale being related as we saw above to the narrowness of the dispersion...
function). However, the longer the crash is postponed, the deeper and more catastrophic it will be.

In contrast, a noisy "free market" displays a "stabilization by noise" effect i.e. it is quite jumpy on short range but it has rather mild long range instabilities. Moreover we have found the stability is helped if the periods of exchange and the quantization of the minimal transaction (share) are denser.

Indeed, in graph 9 one can see what happens when we turn on the heat. The temperature adds the random element to the deterministic run and has the effect of "smearing" the cycles, but underneath the noise one can still see the cyclic behavior. As the temperature grows the sharp rises and crashes become smoother. At first glance the prices in Fig. 10 ($T = 1.2$) seem random, but a closer look reveals the traces of cycles, in fact they resemble patterns from the real stock market (Fig. 11).

Our MicRep confirms the phenomenological observation that the ratio of the dividend yield to price is a reliable indicator for state of the market. In particular we confirm that when the dividend yield is relatively low it is a sign of a bear market and a crash is to be expected. The opposite holds if the dividend yield is high, exactly as we have found in our simulations.

Another common belief which finds a numerical echo in our experiments is that program trading is to be blamed for the 1987 crash. When many investors follow the same investment strategy a strong crash is more probable. Indeed we obtained strong crashes when we assumed homogenous decisions. However, once we 'turned on the heat' and created heterogeneous decision making the cycles became milder and the crashes are much smaller. The higher the temperature the lower the probability for a dramatic crash.

The implication for the optimal market structure is that the larger the number of brokers and investment consultants in the market and the the diversity of their operation, the smaller the chance of a crash. If there is one a guru in the stock market sharp fluctuations in the market are expected.

It is encouraging to discover such a wealth of phenomena rising from such a simple model. It is even more encouraging that these phenomena seem to fit reality, and other, completely different, macroscopic models so nicely. The model can be enriched by introducing different stocks, investors with different utility functions, borrowing, bankruptcy, fluctuating dividends and so on [48].

One can study also the breaking of symmetry which takes place when one starts from a state with all investors equally wealthy and with identical
criteria. Still, any initial small random fluctuations may get amplified. Does this increased polarization in wealth lead to increasing market instability?

As in most of the MicRep applications we mentioned in this review, the ability to model and simulate the system based on its most basic elements brings about deeper yet more direct and intuitive understanding of the macroscopic behavior we observe.

5 Visualization and Animation of MicRep as tools to identify the CSD modes and the Relevant Macros

As mentioned in the introduction, we regard the elimination of CSD as the numerical analog to an exact solution in the analytical treatment.

According to this analogy, the construction of an RCSD algorithm is an operational proof of the understanding of the relevant multi-scale mechanisms.

Usually a mathematical proof is only a consecration of an understanding which precedes and conditions it. In the same way, before the construction of a RCSD algorithm, and quite independent on its very existence one can accumulate a quite detailed understanding of the relevant Macros and their dynamics.

Often, analytical proofs might be intuitively less illuminating than an heuristic explanation of the considerations which lead to the proof. In the same way, a series of interactive animation experiments may prove as valuable as the construction of an RCSD algorithm.

The understanding might be expressed, validated and improved through an active dialogue between the researcher and the system. In such a dialogue the researcher makes predictions about the effects of various changes which he implements in the system and the simulation answers with confirmations of these predictions or suggests corrections.

This interactive computer simulation-visualization-animation paradigm was applied lately to the multi-scale study of various systems in Quantum Gravity, Random Triangulated Surfaces, Solid-on-Solid systems, Fermions dynamics, Gauge Theories, Topological Objects, Molecular Dynamics, Non-linear Dynamics, Fluid Dynamics, Psychophysics, Neural Network Learning,
Chaotic systems, Drawing Dynamics, Image Dynamics and a multitude of Quantum Field Theories and Statistical Mechanics systems [23, 24].

Such scientific experiments, documented in a series of computer animation movies [23] and follow-up papers, proved the usefulness of the interactive animation method in solving multi-scale MicRep problems by:

- suggesting new (including analytical) formulations,
- identifying visually the relevant Macros and their dynamics,
- generating more powerful mental representations of the Macros including their multi-scale hierarchy.

We give below 4 examples.

1) *Dynamics of Fermions in Lattice Gauge Background*

The Parallel transported Multigrid (PTMG) method is one of the promising numerical methods to eliminate CSD in the inversion of the Dirac operator in Lattice Gauge background [37].

The PTMG iterative solver represents the Dirac equation on a hierarchy of grids with various lattice spacing scales. During a PTMG iteration, the error, residual and the solution of the equation are transported between various grids using the gauge background field as a parallel transporter. This is analogous with the DAMG procedure in chap. 3 where the fields were transported between the grids with the help of the $J_{ij}$’s.

However, in PTMG, the gauge fields are also used as parallel transporters for the gauge background itself in the process of representing the Dirac equation on the coarse lattice.

The identification of the CSD modes as the Atiyah-Singer zero-modes and their relation to large scale spatial features (corresponding to instantons) was established in the Parallel Transported Multigrid method with the initial help of a detailed visual search by computer animation described below [23, 25].

The visual study was prompted initially by the puzzling explosion (rather than convergence) of the error during the PTMG iterations in certain gauge backgrounds.

We suspected that the problem might be with the well known difficulties of representing faithfully instantons and the associated fermionic Atiyah-Singer zero-modes on the lattice.
The visualization studies proved the contrary: both the gauge and the fermion fields of the continuum are quite faithfully represented by the lattice (modulo the doubling problem) and the representations on lattices with various lattice spacings scales are quite consistent one with another.

We identified visually the zero modes related to the continuum Atiyah-Singer theorem and discovered that, in the PTMG runs which diverge, the diverging "error" is always proportional to those zero-modes. Moreover we verified that PTMG transports faithfully the geometrical aspect of the error, residual and solution between fine and coarse grids and vice-versa.

It turned out that the divergence which alarmed us in the first place, rather than indicating a deficiency of the PTMG method, was the result of the presence of some zero-modes which are exactly represented on certain lattices. These modes lead to a degenerate family of solutions of the Dirac equation on those lattices. This family is then spanned by the "diverging" runs.

Based on these facts, and on a theoretical analysis of the chiral symmetry on the lattice, we predicted that at \(m = 0\), for instantons of dimensions \((4n) \times (4n)\) the PTMG process is strongly divergent, for instantons of dimensions \((2n+1) \times (2n+1)\) the process is strongly convergent, while for \((4n+2) \times (4n+2)\) the error approaches very quickly a constant and gets totally stuck there.

These predictions were dramatically confirmed by our subsequent runs.\[23\]

As a by-product of this visualization study, we were left with a significantly increased understanding of the geometrical properties of the lattice Dirac equation and of the Dirac equation on compact spaces in general.

2) The Wave-Function of the 2D Quantum Universe

This visualization was prompted by the study of the open-boundary RTS model which is a model for the 2 dimensional quantum gravity \[19\].

In order to simulate the RTS model, one has to generate stochastically triangulated surfaces composed of \(N\) triangles such that each triangulation with local 2-dimensional topology has equal chance to be generated. This distribution is realized usually by generating the triangulations recursively using a local algorithm (Alexander move) which updates the position of one link at a time.

We wished to measure the probability distribution \(P(L)\) of the length \(L\) of the boundary. \(P(L)\) corresponds in the continuum limit to the wave function of the 2D universe. The objective was to compare it with certain
analytical predictions from the continuum \[34, 35\].

We realized that the measured \[39\] mean value of the length

\[
\langle L \rangle \equiv \int L P(L) dL \approx 0.765N
\]

and the width of its distribution

\[
\sigma(L) \equiv \int (L - \langle L \rangle)^2 P(L) dL \approx 0.65 \sqrt{N}
\]

suggest a geometry in which most of the triangles touch the boundary and there is a random density of "branching" triangles (triangles where 3 otherwise disjoint surface regions are joint together).

By visualizing the triangulated "surfaces", we observed that this \(P(L)\) distribution is the result of the fact that the typical open "surface" looks rather like a tree of beads of width 1 which are branching at every scale. The branches are CSD Macros under the usual Alexander moves dynamics. This is due to the fact that the large scale branching features are topologically metastable under local geometry changes.

Using the intuitions emerging from this picture, we were able to write a rigorous recursion relation \[40\] which allowed us to estimate certain amplitudes (i.e. the relative number of RTS’s with a given topology) which were not available by any other method.

In particular we were able to estimate the amplitude corresponding to the "figure eight" diagram \[35\]: the amplitude for creating from the vacuum a pair of baby universes which touch in 1 point (the ancestor of a foam).

It turned out that this diagram represents the (long searched-for) solution to the Wheeler-DeWitt equation for the wave function of a universe starting from nothing \[40\].

3) **Visual study of the Solid-On-Solid (SOS) roughening simulations**

The SOS are models for the roughening transition of the interface between 2 phases (e.g. cristal vs. gas, or spin up vs. spin down regions in the 3D Ising model). The SOS models are defined on (usually) regular 2 dimensional lattices. The variables are integers defined on the lattice sites and represent the "height" (or "depth") of the interface with respect to a certain reference "horizontal" plane.

The energy of each configuration is minimal when the heights of neighboring sites are equal and increases with their difference.
The Valleys-to-Mountains-Reflection (VMR) algorithm views each configuration of the critical SOS system as a landscape with valleys and mountains as its Macros. The VMR algorithm creates stochastically new configurations by imagining that a mountain is sectioned by a horizontal plane along one of its equal-height contours. The new configuration is obtained by reflecting about the horizontal plane the mountain region situated above the plane. In this way, this region becomes a valley.

A similar transformation is devised for reflecting "valley" bottoms into "mountains". Care is taken that the actual procedure respects scrupulously the detailed balance (10).

The VMR algorithms allowed an improvement by an order of magnitude in the precision of the simulations of Solid-On-Solid (SOS) models and their Kosterlitz-Thouless transition (29).

In order to understand and generalize the successes of VMR in other contexts, we produced an animation movie which compares the dynamics of various VMR versions. We compared the VMR version which reflects surface regions with respect to horizontal planes situated at half integers heights (H-algorithm) with the VMR version which reflects with respect to integers (combined with local Metropolis or HB) (I-algorithm).

This allowed us to observe and study the metastable Macros which are responsible for the remaining CSD in each case.

In the case of the I algorithm, the CSD Macros turned out to be large terraces of height 1. The reflection about an integer, can transform such a terrace into a shallow "pond" of depth 1 with the same shape and position. However the reflection about an integer cannot generate, annihilate or deform such a height 1 / depth 1 feature.

The incapability of the I reflections to change significantly the shape and/or the position of these Macros is responsible for the CSD and leads to a dynamical critical exponent $z \approx 1$. for the I algorithm.

In the case of the H algorithm, the CSD ($z \approx 1.5$) is due to the incapability to perform efficiently the above "terrace-pond" switch. However, the H reflection is capable of generating and annihilating step 1 features.

The combined algorithm (alternating I and H steps) includes all the relevant Macro moves and consequently it eliminates completely CSD ($z \approx 0$).

4) Visualization of the Instanton tunneling dynamics

One of the simplest lattice gauge theories presenting topological sectors
indexed by an instanton charge $Q$ and the related Atyiah-Singer fermionic zero-modes is the 2-dimensional U(1) gauge model.

It can be shown that the typical local $Q$ density is about $1/(\xi \times \xi)$ where $\xi$ is the correlation length. Consequently, for a realistic simulation ($\xi \approx L/8$), the typical values for the total topological charge are between $-10$ and $10$.

In the process of studying the fermion propagators in gauge background, we realized that the usual local Metropolis or Heat Bath (HB)[27] algorithms (defined according to the rules in chap 3) never tunnel between $Q$-sectors for realistic $\xi$ ($> 5$) values. In fact, this is consistent with the theoretical estimation of the lattice energy barrier which separates the topological sectors [50].

Therefore, we were witnessing an extreme case of CSD where the Macros were topological instantons.

In order to prepare a representative ensemble of gauge configurations we had to devise a global algorithm which explicitly transformed a configuration into another configuration belonging to a different topological sector [50] while scrupulously fulfilling the detailed balance condition (10).

The resulting "instanton-offering" algorithm produced this time the correct $Q$ distribution consistent with the analytical estimates [37].

In order to study the way the new algorithm works, we visualized the evolution of the gauge configurations, and compared the dynamics of the local (HB) algorithm with the "instanton-offering" algorithm (plus local).

However, the differences between the 2 procedures are not apparent when the elementary plaquettes values are displayed. At short distances, the configurations obtained by the 2 algorithms are undistinguishable (all one sees is the thermal noise of non topological excitations).

In order to see the effects of the instanton dynamics one has to display rather, the contributions to $Q$ comming from coarser lattice subsets. As one displays plaquette averages over various scale sub-lattices, the dynamical effects of the global steps become apparent. For subsets of the order of $\xi$, the effects of the tunneling take the form of sudden jumps. Such jumps, or any other changes, are never observed in the local algorithm runs even for sublattices much larger than $\xi$. In particular the total topological charge never changes after the initial thermalization.

We also discovered visually the disturbing fact that, even though none of the measurements of the usual quantities shows CSD after introducing the global instanton-generating steps, certain metastabilities still persist in the
long range geometrical aspect of the configurations. These effects might be related to CSD in n-point correlation functions \((n > 2)\) and deserve further study. This is an interesting new situation in which the Macros are discovered before the CSD they produce.

The 4 examples above are only a sample of the multi-scale computer visualization experiments we performed for physical and other systems.

The interactive multiscale animation techniques proved so powerful that one had sometimes the feeling that they tap into a short-cut leading directly to the inner mental representation forms manipulated by the human mind in the process of understanding. A possible explanation for it is that the natural structures inherent to the human thought are intrinsically multi-scale. As such, they might be considered as a legitimate target for further MicRep study.

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Figure Captions

Figure 1:
The decorrelation time $\tau_k$ versus $k$, in Metropolis dynamics. The squares denote the data points with their corresponding error bars, and the solid line denotes the best fitting. Note that different modes ($k$'s) can have the same $k$. The linear lattice size is $L=64$, $z = 2.1 \pm 0.05$.

Figure 2:
Same as Fig. 1 for $L=128$, $z = 2.1 \pm 0.1$.

Figure 3:
The decorrelation time $\tau_k$ versus $k$, in Wolff dynamics.
(a) $L=64$, $z = 0.1 \pm 0.2$. (b) $L=128$, $z = 0 \pm 0.15$.

Figure 4:
The evolution of $O = < (x - <x>)^2 >$ during a DAMG run (crosses) vs. an usual HB run (continuous line). One can see that during the DAMG run the $O$ measurements effectively cover the support of its relevant distribution while the HB run does not even reproduce the correct mean value. The horizontal axis of each run is normalized such that the same point on the axis corresponds to equal CPU times. The lattice size was 8596 triangles.
Figure 5:
The logarithm of the time auto-correlation function $C$, for the observable $< (x-<x>)^2 >$ is plotted as a function of the number of sweeps $n$. The results from DAMG runs (crosses) and HB runs (diamonds) appear on the same graph for comparison. The ordinates $n$ of the two graphs are respectively normalized such that each point on the horizontal axis represents equal CPU times for DAMG and HB.

Figure 6:
The logarithm of the time auto-correlation function $C$, for the observable $< (x-<x>)^2 >$ is plotted as a function of the number of sweeps $n$. The results from DAMG runs (crosses) and HB runs (diamonds) appear on the same graph for comparison. The ordinates $n$ of the two graphs are respectively normalized such that each point on the horizontal axis represents equal CPU times.

Figure 7:
The the evolution of the stock value in the "deterministic" case ($T = 0$).

Figure 8:
The stock cycles in the presence of a small random dispersion of the market.
Figure 9:
The stock cycles in the presence of a larger random dispersion of the market.

Figure 10:
The stock cycles for a realistic random dispersion ($T = 1.2$) of the market.

Figure 11:
The real stock market evolution during the last decades.