Spinors in non-relativistic
Chern-Simons electrodynamics

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Abstract

It is shown that the non-relativistic ‘Dirac’ equation of Lévy-Leblond, we used recently to describe a spin \( \frac{1}{2} \) field interacting non-relativistically with a Chern-Simons gauge field, can be obtained by lightlike reduction from \( 3 + 1 \) dimensions. This allows us to prove that the system is Schrödinger symmetric. A spinor representation of the Schrödinger group is presented. Static, self-dual solutions, describing spinor vortices are given and shown to be the non-relativistic limits of the fermionic vortices found by Cho et al. The construction is extended to external harmonic and uniform magnetic fields.

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1. Introduction

The ‘Chern-Simons’ version of electromagnetism in $2+1$ dimensions, where the electromagnetic field, $(B, \vec{E})$, and the current, $(\varrho, \vec{J})$, satisfy the identities

$$\kappa B \equiv \kappa \epsilon^{ij} \partial_i A^j = -e \varrho, \quad \kappa E^i \equiv \kappa \left( -\partial_i A^0 - \partial_t A^i \right) = e \epsilon^{ij} J^j$$

(1.1)

rather than the Maxwell equations, provides a promising framework for understanding the quantized Hall effect [1] as well as high $T_c$ superconductivity [2].

The theory supports, just like the conventional Abelian Higgs model [3], vortices [4]. Exact solutions were found, in particular, in the non-relativistic model of Jackiw and Pi [5] who chose $\varrho = \bar{\Phi} \Phi$ and $\vec{J} = 1/(2im) \left[ \bar{\Phi} \vec{D} \Phi - \bar{\Phi} (\vec{D} \Phi)^* \right]$, where $D_\alpha \equiv \partial_\alpha - ie A_\alpha$ (with $\alpha = 0, 1, 2$), the massive scalar field $\bar{\Phi}$ satisfying the planar, gauged, non-linear Schrödinger (NLS) equation,

$$i D_t \bar{\Phi} = \left[ -\frac{\vec{D}^2}{2m} - \Lambda \bar{\Phi} \Phi \right] \bar{\Phi}. \quad (1.2)$$

The model is non-relativistic: it admits the Schrödinger group as symmetry [5], [6]. Let us stress the importance of such a development for the physical applications: condensed matter physics is non-relativistic.

For the special value $\Lambda = e^2/(mk)$ of the non-linearity, the static second-order equation (1.2) can be reduced to the first-order ‘self-duality’ (SD) equations,

$$(D_1 \pm iD_2) \bar{\Phi} = 0. \quad (1.3)$$

In a suitable gauge, this leads to Liouville’s equation, and can therefore be solved [5].

Motivated by the above mentioned potential applications, recently [7] we generalized the Jackiw-Pi theory to spinors: we described a spin $\frac{1}{2}$ non-relativistic field by the $(2+1)$-dimensional version of the non-relativistic ‘Dirac’ equation proposed some time ago by Lévy-Leblond [8],

$$\begin{cases} 
(\vec{\sigma} \cdot \vec{D}) \Phi + 2m \chi = 0, \\
D_t \Phi + i(\vec{\sigma} \cdot \vec{D}) \chi = 0, 
\end{cases} \quad (1.4)$$

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D_t \Phi + i(\vec{\sigma} \cdot \vec{D}) \chi = 0, 
\end{cases} \quad (1.4)$$
coupled to the Chern-Simons field (1.1) through

\[ \varrho = |\Phi|^2 \quad \text{and} \quad \vec{J} = i(\Phi^\dagger \vec{\sigma} \chi - \chi^\dagger \vec{\sigma} \Phi). \] 

(1.5)

Here \( \Phi \) and \( \chi \) are two component ‘Pauli’ spinors and \( \vec{\sigma} \cdot \vec{D} \equiv \sum_{j=1}^{2} \sigma^j D_j \), with \( \sigma^j \) denoting the Pauli matrices. Then, Eq. (1.4) is readily seen to split into two ‘chiral’ components.

In a suitable basis, the chiral components combine according to

\[ \Phi = \left( \begin{array}{c} \Phi_+ \\ \Phi_- \end{array} \right), \quad \chi = \left( \begin{array}{c} \chi_- \\ \chi_+ \end{array} \right), \] 

(1.6)

so that \( \Phi \) and \( \chi \) themselves are not chiral. Eliminating the \( \chi \)-components in Eq. (1.4), we could finally reduce our equations to

\[ iD_t \Phi_\pm = \left[ -\frac{\vec{D}^2}{2m} \pm \lambda \Phi_\dagger \Phi_\pm \right] \Phi_\pm \] 

(1.7)

with

\[ \lambda \equiv \frac{e^2}{2m\kappa}, \] 

(1.8)

which is Eq. (1.2) but with non-linearities half of the special value used by Jackiw and Pi in Ref. [5]. Generalizing the SD equations (1.3) allowed us to construct new, purely magnetic \((A_t = 0)\), spinorial vortex solutions [7] of the coupled system (1.1,4,5). The relation of the LHS of Eq. (1.7) to the square of a first-order planar system has already been noted [5]. Our result here provide a physical interpretation to this observation.

In another paper [6], we investigated the Jackiw-Pi model in a ‘non-relativistic Kaluza-Klein’ framework: we started with a \((3 + 1)\)-dimensional Lorentz manifold \((M, g)\) endowed with a complete covariantly-constant null vector \( \xi \), called a ‘Bargmann space’ [9]. Non-relativistic spacetime, \( Q \), is the quotient of \( M \) by the integral curves of \( \xi \). On \( M \), we posited the relations

\[ \kappa f_{\mu \nu} = e\sqrt{-g} \epsilon_{\mu \nu \rho \sigma} \xi^\rho j^\sigma \quad \text{and} \quad \partial_{[\mu} f_{\nu \rho]} = 0. \]

(1.9)

The first of these equations — we will call the FCI — lifts the field/current identity, Eq. (1.1), while the second guarantees that \( f_{\mu \nu} \) derives (locally) from a 4-potential \( a_\mu \), i.e.
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\[ f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu. \]

The non-linear Schrödinger equation, Eq. (1.2), is in turn obtained from a non-linear wave equation for a massless scalar field, \( \tilde{\phi} \), on \( M \), when \( \tilde{\phi} \) is required to be equivariant (with respect to the group generated by \( \xi \)), viz \( D_\xi \tilde{\phi} = im \tilde{\phi} \). As emphasized in Ref. [6], this framework is particularly convenient to describe the symmetries and the associated conserved quantities of the system.

In this paper, we extend our ‘Kaluza-Klein-type’ framework to spinors. In particular, we show that the Lévy-Leblond (LL) equation (1.4) is in fact the lightlike reduction of the massless Dirac equation on a simple example of a Bargmann space [10,11], namely flat Minkowski space. However, the lightlike dimensional reduction is more general and works for any Bargmann space. In the general case, the conformal invariance of the reduced system can be shown along the same lines as in [6]. We demonstrate the usefulness of this ‘Kaluza-Klein’ framework by explicitly computing all the conserved quantities that belong to the various residual conformal symmetries in the spinor Chern-Simons (CS) theory. In addition to this, we also show how it enables one to derive the spinor representation of the Schrödinger group.

It is important to know some exact solutions of the theory. Here we rederive the static, purely magnetic, self-dual spinor vortices, already found in Ref. [7] in a slightly novel way.

Finally, again motivated by the potential physical applications, we extend our theory to background harmonic oscillator and uniform magnetic fields. We achieve this by adapting the geometric framework described in [12] to the spinor theory and using it we present explicit exact solutions — again in the form of spinor vortices — in these background fields.

The simplest case of our theory (i.e. when the Bargmann space is just flat Minkowski space) is closely related to the model considered by Cho et al. in Ref. [13] where the CS gauge fields are coupled to relativistic spinor fields in \( 2+1 \) dimensions. Here we show that, within the Kaluza Klein framework, this model is obtained by using a spacelike rather than a lightlike covariantly constant \( \xi \) vector. Furthermore, we show that our static vortices
can be viewed in fact as the non-relativistic limits of the fermionic vortices of Cho et al.

Our paper is organized as follows: in Sec. 2 we present the coupled gauged Lévy-Leblond and Chern-Simons equations. Sec. 3 contains the discussion of the conformal symmetry in the case of a general Bargmann space. In Sec. 4 we derive the spinor representation of the Schrödinger group, while in Sec. 5 we present the conserved quantities belonging to the residual conformal symmetries. In Sec. 6 the static solutions are presented. Sec. 7 is devoted to the external field problem, and in Sec. 8 we discuss the relation with the model considered in Ref. [13]. In Sec. 9 we briefly describe the problem of the Lévy-Leblond equations in higher dimensions. In Sec. 10 we collected several remarks and comments on the various possible extensions of our theory. Finally, some technical computations needed to establish the conformal invariance are collected in Appendix A.

2. The coupled, gauged, Lévy-Leblond and Chern-Simons equations

Let \((M, g, \xi)\) be a four-dimensional Bargmann manifold of signature \((-+, +, +, +)\), which is assumed to be endowed with a spin structure, and choose Dirac matrices \(\gamma^\mu\) (with \(\mu = 0, 1, 2, 3\)) such that \(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2g^\mu\nu\). Let \(a_\mu\) be a U(1)-gauge potential on \(M\) and denote by \(D_\mu \equiv \nabla_\mu - ie a_\mu\) the gauge and metric-covariant derivative of a spinor fields. The latter reads explicitly as

\[
\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{8} [\gamma^\rho, \partial_\mu \gamma_\rho - \Gamma^\sigma_{\mu \rho} \gamma^\sigma] \psi,
\]

with \(\Gamma^\sigma_{\mu \rho}\) denoting the Christoffel symbols of the metric. Let us now consider the gauged massless Dirac equation on \(M\),

\[
\mathcal{D} \psi \equiv \gamma^\mu D_\mu \psi = 0.
\]

This equation is chiral-symmetric: the chirality operator, which is defined using an orthonormal basis by

\[
\Gamma \equiv \gamma^5 = -\sqrt{-g} \varepsilon_{\mu \nu \rho \sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma,
\]
anticommutes with \( \bar{\mathcal{D}} \). The 4-component equation (2.2) splits therefore into two uncoupled 2-component equations for the two spinor fields \( \psi_+ \) and \( \psi_- \), which are in fact the chiral components, defined by

\[
\Gamma \psi_\pm = \mp i \psi_\pm.
\]

Our system of equations has to be supplemented by the chirality constraint (2.4).

The space of spinors carries a hermitian structure of signature \((+, +, -, -)\), defined by \( \bar{\psi} \equiv \psi^\dagger G \), where \( G \) is determined by the requirements \( \gamma_\mu = G^{-1} \gamma_\mu^\dagger G = \gamma_\mu \) and \( G^\dagger = G \).

For the metric-covariant derivative of the Dirac adjoint, \( \bar{\psi} \), the relation \( \bar{\nabla}_\mu \psi = \nabla_\mu \bar{\psi} \) holds. Therefore, \( \bar{\mathcal{D}} \bar{\psi} = 0 \) whenever \( \psi \) solves the Dirac equation (2.2). It follows that the current

\[
j^\mu = \bar{\psi} \gamma^\mu \psi
\]

is conserved, \( \nabla_\mu j^\mu = 0 \). Then the coupled system (1.9) and (2.2,4,5) becomes self-consistent.

We now reduce the system from 4 to 3 dimensions. We can choose on our four-dimensional Bargmann manifold \((M, g, \xi)\) a trivializing coordinate system \((t, x, y, s)\) such that \( \xi = \partial_s \). Then the quotient, \( Q \), of \( M \) by the flow of \( \xi \) is a Newton-Cartan spacetime [9] which is, hence, parametrized by non-relativistic time and position, \( t \) and \( \vec{x} = (x, y) \), respectively.

Owing to the FCI in (1.9), the field strength \( f_{\mu\nu} \) is the lift of a closed two-form \( F_{\alpha\beta} \) on \( Q \). The 4-potential \( a_\mu \) can be chosen therefore to be the lift from \( Q \) of a 3-potential \( A_\alpha \) (with \( \alpha = 0, 1, 2 \)) such that \( F_{\alpha\beta} = 2 \partial_{[\alpha} A_{\beta]} \). As we demonstrated in Ref. [6], the lightlike reduction of the FCI in (1.9) yields precisely the Jackiw-Pi field/current identity, Eq. (1.1). Similarly, the 4-current \( j^\mu \) projects onto the 3-current \( J^\alpha = (\varrho, \vec{J}) \).

To reduce the spinor field we further require it to be equivariant,

\[
D_\xi \psi = im\psi,
\]

(2.6)
for some constant $m$, interpreted as the mass. Therefore, putting
\[ \psi = e^{ims} \begin{pmatrix} \Phi \\ \chi \end{pmatrix}, \] (2.7)
the two-component spinors $\Phi$ and $\chi$ do not depend on $s$ and Eq. (2.2) descends to one on $Q$.

Now it follows from the general theory of spinors that, in three dimensions, ‘Dirac’ spinors only have 2 components. This fact manifests itself in our theory in that the reduction commutes with chirality. Each of the two chiral components of Eq. (2.2) project therefore to two uncoupled systems in $2 + 1$ dimensions. Identifying the chiral components with two-spinors, we end up with two uncoupled, 2-component equations.

In summary, the equations that describe the interacting spinor and Chern-Simons gauge fields and the ones we intend to study are the following:

\[ \bar{D} \psi = 0, \quad \Gamma \psi = -i\varepsilon \psi, \quad D \xi \psi = im\psi, \]
\[ \kappa f_{\mu\nu} = e \sqrt{-g} \epsilon_{\mu\nu\rho\sigma} \xi^\rho j^\sigma, \quad \partial_{[\mu} f_{\nu\rho]} = 0, \] (2.8)
\[ j^\mu = \bar{\psi} \gamma^\mu \psi, \]

with ‘helicity’ $\varepsilon = \pm 1$, mass $m$ and coupling constants $\kappa$ and $e$.

**Minkowski-space**

Let us exemplify these general statements in the simplest case where $M$ is Minkowski space, $\mathbb{R}^{3,1}$, with its metric written in light-cone coordinates as $d\bar{x}^2 + 2dtds$. The Dirac matrices
\[ \gamma^t = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\gamma} = \begin{pmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \quad \gamma^s = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, \] (2.9)
as well as
\[ \Gamma = \begin{pmatrix} -i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}, \] (2.10)
are hermitian, $\bar{\gamma}^\mu = \gamma^\mu$, with respect to the hermitian structure defined by $G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. 
The massless Dirac equation \( \bar{\psi} \psi = 0 \), Eq. (2.2), on \( M \) reduces, when equivariance, Eq. (2.6), is imposed, to the first-order system

\[
\begin{align*}
(\bar{\sigma} \cdot \bar{D})\Phi + 2m\chi &= 0, \\
D_t \Phi + i(\bar{\sigma} \cdot \bar{D})\chi &= 0.
\end{align*}
\]  

(2.11)

which is formally the same as the non-relativistic ‘Dirac’ equation of Lévy-Leblond [8] (albeit in \( 2 + 1 \) rather than \( 3 + 1 \) dimensions). Note that \( \Phi = \begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix} \) and \( \chi = \begin{pmatrix} \chi_- \\ \chi_+ \end{pmatrix} \) are not the chiral components. For \( \Gamma \) as in (2.10), these latter read rather

\[
\Psi_+ = \begin{pmatrix} \Phi_+ \\ 0 \\ 0 \\ \chi_+ \end{pmatrix} \quad \text{and} \quad \Psi_- = \begin{pmatrix} 0 \\ \Phi_- \\ \chi_- \\ 0 \end{pmatrix},
\]

(2.12)

respectively. Then, Eq. (2.11) splits into the two uncoupled systems of prescribed chirality

\[
\begin{align*}
(D_1 + iD_2)\Phi_+ + 2m\chi_+ &= 0, \\
D_t \Phi_+ + i(D_1 - iD_2)\chi_+ &= 0;
\end{align*}
\]

\[
\begin{align*}
(D_1 - iD_2)\Phi_- + 2m\chi_- &= 0, \\
D_t \Phi_- + i(D_1 + iD_2)\chi_- &= 0.
\end{align*}
\]  

(2.13)

As will be shown in Sec. 4, the \( \psi_+ \) and \( \psi_- \) Bargmann spinor fields (see (2.7,12)) span the spin \( \frac{1}{2} \) and \(-\frac{1}{2}\) ‘spinor representations’ of the Schrödinger group in \( 2 + 1 \) dimensions, thus they are indeed ‘spinor’ fields. (1) Each of them separately would be sufficient to describe (in general different) physical phenomena in \( 2 + 1 \) dimensions. Nevertheless, for the ease of presentation, we keep in the sequel all four components of \( \psi \), since in this way we can describe the spin \( \frac{1}{2} \) and spin \(-\frac{1}{2}\) fields simultaneously.

(1) Since Euclidian rotations form an Abelian subgroup of the Galilei group in \( 2 + 1 \) dimensions, spin is a subtle object. In fact, we introduce spin by dimensional reduction so that the Poincaré helicity is actually a Galilei invariant, whence the justification of the notion of spin \( \pm \frac{1}{2} \).
It follows from the singular form of $\gamma^t$ in (2.9) that, as found already by [8], the mass (or particle) density, $\varrho \equiv J_\mu \xi^\mu$, viz $\varrho = J_s = J^t$, only depends on the ‘upper’ component, $\Phi$:

$$
\varrho = |\Phi|^2 = |\Phi_+|^2 + |\Phi_-|^2
$$

(2.14)

and is positive definite allowing for a probabilistic interpretation. For the spatial components of the current we find

$$
\vec{J} = i(\Phi^\dagger \vec{\sigma} \chi - \chi^\dagger \vec{\sigma} \Phi).
$$

(2.15)

Combining Eqs (2.14,15) with the FCI in (1.9) shows that the two different chirality spinors, if present simultaneously, couple to each other only through the CS gauge fields. Now $\chi$ can be eliminated using (2.11), to yield the expression in terms of $\Phi$ only:

$$
\vec{J} = \frac{1}{2im} \left( \Phi^\dagger \vec{D} \Phi - (\vec{D} \Phi)^\dagger \Phi \right) + \vec{\nabla} \times \left( \frac{1}{2m} \Phi^\dagger \sigma_3 \Phi \right).
$$

(2.16)

Note here the new term, due to the spin.

Using $(\vec{\sigma} \cdot \vec{D})^2 = \vec{D}^2 + eB\sigma_3$, we find that the component-spinors satisfy

$$
\begin{aligned}
    iD_t \Phi &= -\frac{1}{2m} \left[ \vec{D}^2 + eB\sigma_3 \right] \Phi, \\
    iD_t \chi &= -\frac{1}{2m} \left[ \vec{D}^2 + eB\sigma_3 \right] \chi - \frac{e}{2m} (\vec{\sigma} \cdot \vec{E}) \Phi.
\end{aligned}
$$

(2.17)

Thus, $\Phi$ solves a ‘Pauli equation’, while $\chi$ also couples to the electric field through the Darwin term, $\vec{\sigma} \cdot \vec{E}$.

Expressing $\vec{E}$ and $B$ through the FCI in (1.9) or (1.1) and inserting them into our equations, we get finally the remarkable system

$$
\begin{aligned}
    iD_t \Phi &= \left[ -\frac{1}{2m} \vec{D}^2 + \frac{e^2}{2m\kappa} |\Phi|^2 \sigma_3 \right] \Phi, \\
    iD_t \chi &= \left[ -\frac{1}{2m} \vec{D}^2 + \frac{e^2}{2m\kappa} |\Phi|^2 \sigma_3 \right] \chi - \frac{e^2}{2m\kappa} (\vec{\sigma} \times \vec{J}) \Phi.
\end{aligned}
$$

(2.18)

Note that the non-linearity comes from the Pauli term upon the use of the FCI.

If we restrict the chirality of $\psi$ to $+1$ (respectively $-1$), this system describes non-relativistic spin $\frac{1}{2}$ (spin $-\frac{1}{2}$) fields interacting with a Chern-Simons gauge field. Leaving
the chirality of $\psi$ unspecified, the system describes two spinor fields of spin $±\frac{1}{2}$, interacting with each other and the Chern-Simons gauge field.

Since the lower component of the spinor field is simply

$$\chi = -\frac{1}{2m} (\vec{\sigma} \cdot \vec{D}) \Phi,$$

(2.19)it is enough to solve the $\Phi$-equation. In the general case this latter is still a coupled, 2-component equation, because $\varrho = |\Phi|^2$ involves both components of $\Phi$. For the $±$ chiral components in Eq. (1.6), the ‘Pauli’ equation for $\Phi$ reduces to the equations (1.7) with the value (1.8) of the non-linearity coupling constant.

### 3. Conformal symmetry

In studying the symmetries of the NLS-CS equation, Ref. [6], we considered ‘non-relativistic conformal’ transformation [9,11], i.e. those mappings $h : M \rightarrow M$ such that

$$h^{\ast} g_{\mu\nu} = \Omega^2 g_{\mu\nu} \quad \text{and} \quad h^{\ast} \xi^\mu = \xi^\mu,$$

(3.1)for some smooth function $\Omega > 0$. Likewise, the non-relativistic conformal vector fields, $X$, are such that, for some smooth function $k$, there holds

$$L_X g = k g \quad \text{and} \quad L_X \xi = 0.$$

(3.2)

In this Section, we show that on any Bargmann space, the $\xi$-preserving conformal transformations (3.1) still act as symmetries on the coupled spinor-CS system described by Eqs (1.9) and (2.2,4,5,6) — or Eqs (2.8). The proof is given in three steps: first we deal with the conformal invariance of the purely metric massless Dirac equation, then we prove the Bargmann conformal invariance of the ungauged LL equations and, in the final step, we consider the complete system consisting of the gauged LL equations and the FCI.
Massless Dirac equation

Let us consider the purely metric Dirac operator, $\nabla \equiv \gamma^\mu \nabla_\mu$, on a $(3+1)$-dimensional Bargmann manifold $(M, g, \xi)$ equipped with Dirac matrices satisfying the Clifford relations $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2g^{\mu \nu}$, where $\nabla_\mu$ is the metric covariant derivative (2.1) of spinor fields. To show the conformal invariance of $\nabla \psi = 0$, we recall that the Lie derivative of a spinor field with respect to a vector field $X$ is defined as [14]

$$L_X \psi = \nabla_X \psi - \frac{1}{4} \gamma^\mu \gamma^\nu \partial_{[\mu} X_{\nu]} \psi. \quad (3.3)$$

Let $X$ be an infinitesimal conformal transformation, i.e. a vector field on $M$ such that

$$L_X g_{\mu \nu} = 2 \nabla_{(\mu} X_{\nu)} = k g_{\mu \nu} \quad (3.4)$$

for some function $k$ on $M$ (notice that $k = \frac{1}{2} \nabla_\mu X^\mu$).

Evaluating the commutator $[L_X, \nabla]$ (see the details in Appendix A) one finds

$$[L_X, \nabla]\psi = -\frac{1}{2} k \nabla \psi + \frac{3}{4} \gamma^\mu \partial_\mu k \psi.$$

Therefore, defining the infinitesimal action of conformal transformations on Dirac spinors to be rather

$$\delta_X \psi \equiv L_X \psi + \frac{3}{8} (\nabla_\mu X^\mu) \psi, \quad (3.5)$$

one has $[\delta_X, \nabla] \psi = -\frac{1}{4} (\nabla_\mu X^\mu) \nabla \psi$. Thus, whenever $\psi$ is a solution of the massless Dirac equation, the same is true for

$$\psi_\epsilon \equiv \psi + \epsilon \delta_X \psi + \cdots \quad (3.6)$$

for any conformal-Killing vector field $X$ of $(M, g)$:

$$\nabla \psi = 0 \implies \nabla \delta_X \psi = 0. \quad (3.7)$$
The ungauged Lévy-Leblond system

Next, we prove the invariance of the ungauged Lévy-Leblond system

\[ \nabla / \psi = 0 \quad \text{and} \quad \nabla \xi \psi = im \psi \]  \hspace{1cm} (3.8)

under the Lie algebra of the conformal-Killing vector fields which also preserve the ‘vertical’ vector field \( \xi \), cf. (3.2).

Since the question of conformal invariance of the massless Dirac equation has already been settled, one only has to prove that the equivariance condition (or mass constraint) is preserved. Clearly, \([L_X, L_\xi] = L_{[X, \xi]} = 0\) entails \([\delta X, L_\xi] \psi = 0\), as a result of the definitions and of \(\xi^\mu \partial_\mu k = 0\). Thus, if \(\psi\) is a solution of the Lévy-Leblond equation, the same is true for \(\psi_\epsilon\), Eq. (3.6), for any Bargmann-conformal vector field \(X\), since

\[ L_\xi \psi = im \psi \quad \implies \quad L_\xi \delta X \psi = im \delta X \psi. \]  \hspace{1cm} (3.9)

Using the expression of the Lie derivative of gamma matrices, Eq. (3.10) below, it is straightforward to prove that the chirality condition (2.4) is also Bargmann-conformally invariant,

\[ [\delta X, \Gamma] \psi = 0. \]

The gauged Lévy-Leblond/Chern-Simons system

Let us now turn to our final goal, i.e., to prove the Bargmann-conformal invariance of the coupled, gauged, Lévy-Leblond/Chern-Simons system, expressed on our extended ‘Bargmann’ spacetime as Eqs (2.8). It follows from the definition (3.3) that

\[ L_X \gamma_\mu = \frac{1}{2} (L_X g_{\mu\nu}) \gamma^\nu. \]  \hspace{1cm} (3.10)

Hence, for any conformal-Killing vector field \(X\),

\[ \delta X \gamma_\mu \equiv L_X \gamma_\mu - \frac{1}{2} k \gamma_\mu = 0. \]  \hspace{1cm} (3.11)
Putting $\hat{D} \equiv \gamma^\mu D_\mu$ with $D_\mu \equiv \nabla_\mu - ie a_\mu$, we find
\begin{equation}
[\delta_X, \hat{D}]\psi = -ie\gamma^\mu (L_X a)_\mu \psi - \frac{1}{2} k \hat{D} \psi,
\end{equation}
so that whenever $\psi$ and $a$ solve the gauged, massless, Dirac equation (2.2), the same is true for $\psi_\epsilon$, Eq. (3.6), and
\begin{equation}
a_\epsilon \equiv a + \epsilon \delta_X a + \cdots
\end{equation}
i.e.
\begin{equation}
\hat{D} \psi = 0 \implies \hat{D} \delta_X \psi - ie\gamma^\mu (\delta_X a)_\mu \psi = 0,
\end{equation}
provided the CS-field transforms as
\begin{equation}
\delta_X a = L_X a, \quad \text{hence} \quad \delta_X f \equiv L_X f.
\end{equation}
The mass constraint, Eq. (2.6), is also Bargmann-conformally invariant: in fact,
\begin{equation}
[\delta_X, D_\xi]\psi = -ie(L_X a)(\xi)\psi, \quad \text{proving} \quad L_\xi \delta_X \psi - ie(\delta_X a)(\xi)\psi = im \delta_X \psi, \quad \text{i.e. that, up to higher order terms},
\end{equation}
\begin{equation}
(\hat{D}_\epsilon) \psi_\epsilon = im \psi_\epsilon.
\end{equation}
The CS-field equations (1.9) are readily seen to be Bargmann-conformally invariant since, under the transformations (3.15), they change as
\begin{equation}
\partial_{[\mu} \delta_X f_{\nu\rho]} = 0 \quad \text{and} \quad \kappa \delta_X f_{\mu\nu} = e\sqrt{-g} \epsilon_{\mu\nu\rho\sigma} \xi^\rho \delta_X j^\sigma,
\end{equation}
where $\delta_X j \equiv L_X j + 2k j$. The proof is completed by establishing the consistency of these equations, using Eqs (3.5) and (2.5) identifying the CS and Dirac currents:
\begin{equation}
\delta_X (\bar{\psi}\gamma^\mu \psi) = \delta_X \bar{\psi}\gamma^\mu \psi + \bar{\psi}\gamma^\mu \delta_X \psi = L_X (\bar{\psi}\gamma^\mu \psi) + 2k \bar{\psi}\gamma^\mu \psi.
\end{equation}
In conclusion, the Bargmann-conformal vector fields $X$ on $(M, g, \xi)$,
\begin{equation}
\delta_X g \equiv L_X g - \frac{1}{2} (\nabla_\mu X^\mu) g = 0 \quad \text{and} \quad \delta_X \xi \equiv L_X \xi = 0,
\end{equation}
provide symmetries of the coupled LL-CS Eqs, associated with the representation (3.5) and (3.15) on the spinor and gauge fields, respectively. These vector fields form a finite dimensional Lie algebra that can be integrated (see below) to a finite dimensional Lie group of $\xi$-preserving conformal transformations.
4. Spinor representation of the Schrödinger group

In the case of Minkowski space $M = \mathbb{R}^{3,1}$, those conformal vector fields which preserve $\xi = \partial_s$ (see (3.18)) form a 9-parameter Lie algebra, called the (extended) ‘Schrödinger’ algebra [15]. It consists of the vector fields $X = X^t \partial_t + X^j \partial_j + X^s \partial_s$ (with $j = 1, 2$) of the form [9,11]

$$
(X^\mu) = \begin{pmatrix} 
\kappa t^2 + 2\chi t + \epsilon \\
\omega \vec{x} + t\vec{\beta} + \vec{\gamma} + (\chi + \kappa t)\vec{x} \\
-\frac{1}{2}\kappa r^2 - \vec{\beta} \cdot \vec{x} + \eta
\end{pmatrix} \tag{4.1}
$$

with $r = ||\vec{x}||$ and $\omega_{ij} = \omega \epsilon_{ij}$ where $\omega \in \mathbb{R}$; $\vec{\beta}, \vec{\gamma} \in \mathbb{R}^2$; $\epsilon, \kappa, \chi, \eta \in \mathbb{R}$.

Integrating the infinitesimal action we get the (extended) Schrödinger group [16,9] as a 9-dimensional subgroup of $O(4,2)$ which acts on $M$ according to $(t, \vec{x}, s) \rightarrow (t^*, \vec{x}^*, s^*)$:}

$$
\begin{align*}
t^* &= \frac{dt + e}{ft + g} \\
\vec{x}^* &= \frac{R\vec{x} + \vec{b}t + \vec{c}}{ft + g} \\
s^* &= s + \frac{f}{2} \left( \frac{R\vec{x} + \vec{b}t + \vec{c}}{ft + g} \right)^2 - \vec{b} \cdot R\vec{x} - \frac{t}{2} \vec{b}^2 + h
\end{align*} \tag{4.2}
$$

while its induced action on spacetime, $Q = \mathbb{R} \times \mathbb{R}^2$, is given by $(t, \vec{x}) \rightarrow (t^*, \vec{x}^*)$, i.e. by ‘forgetting’ about the $s$ coordinate in (4.2). Here, $R \in SO(2)$ is a rotation in the plane, and $\vec{b}, \vec{c} \in \mathbb{R}^2$ are the boosts and the space translations, respectively; at last $d, e, f, g \in \mathbb{R}$ with $dg - ef = 1$ parametrize the SL$(2, \mathbb{R})$ subgroup formed by the time translations ($d = 1, f = 0$), dilations ($e = f = 0$), and expansions ($d = 1, e = 0$). Denote by

$$
u = (a, \vec{b}, \vec{c}, d, e, f, g, h) \tag{4.3}
$$

a typical element of the so-called ‘Spin-Schrödinger’ group where $a = \exp(\frac{i}{2} \theta \sigma_3)$ is such that $a(\vec{\sigma} \cdot \vec{x})a^{-1} = \vec{\sigma} \cdot (R\vec{x})$.

We record, for further use, that dilations and expansions form, indeed, a 2-dimensional Lie subgroup consisting of elements of the form $(1, \vec{0}, 0, g^{-1}, 0, f, g, 0)$ with $g > 0$ and $f \in \mathbb{R}$.
It will also be convenient to think of any element (sitting in an open dense subset where \( d = g^{-1}(1 + ef) \)) of the Schrödinger group as the product of elements of the 7-dimensional Bargmann group (a central extension of the Galilei group in \( 2 + 1 \) dimensions) and of that 2-dimensional group, a Borel subgroup of \( SL(2, \mathbb{R}) \), namely

\[
u = \left( a, g\tilde{b} - f\tilde{c}, g^{-1}\tilde{c}, 1, g^{-1}e, 0, 1, h + \frac{1}{2}fg^{-1}\tilde{c}^2 \right) \cdot \left( 1, \tilde{o}, \tilde{o}, g^{-1}e, 0, f, g, 0 \right).
\]

(4.4)

The infinitesimal transformations

As previously shown, infinitesimal conformal transformations, \( X \), act on the space of solutions of the massless Dirac equation, namely through the operators \( \frac{1}{i} \delta_X \) where

\[
\delta_X \psi \equiv \left( X^\mu \nabla_\mu - \frac{1}{8} [\gamma^\mu, \gamma^\nu] \nabla_\mu X_\nu + \frac{3}{8} \nabla_\mu X^\mu \right) \psi.
\]

(4.5)

In the Minkowski case, the infinitesimal conformal transformations form the \( o(4, 2) \) Lie algebra,

\[
\begin{align*}
P_\mu &= -i \partial_\mu \\
M_{\mu\nu} &= -i (x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{i}{4} [\gamma_\mu, \gamma_\nu] \\
d &= -ix_\mu \partial_\mu - \frac{3i}{2} \\
K_\mu &= -i (x_\nu x_\nu \partial_\mu - x_\mu (3 + 2x_\nu \partial_\nu)) - \frac{i}{2} [\gamma_\mu, \gamma_\nu] x_\nu
\end{align*}
\]

(4.6)

The Schrödinger algebra is precisely the subalgebra which leaves the covariantly constant vector \( \xi \) invariant. The most convenient way of finding it is to view \( \xi \) as the generator of vertical translations, \( P_s = -i \partial_s \), and find those generators in \( o(4, 2) \), Eq. (4.6), which commute with it. These operators preserve also the equivariance condition, Eq. (2.6), as well as the chirality condition, Eq. (2.4). We find that Eq. (4.5) reads in flat Minkowski spacetime

\[
\delta_X \psi = X^\mu \partial_\mu \psi + \begin{pmatrix}
-i\frac{\sigma_3}{2} - \frac{1}{2}(\chi + \kappa t) & 0 \\
\frac{i}{2} \vec{\sigma} \cdot (\vec{\beta} + \kappa \vec{x}) & -i\frac{\sigma_3}{2} + \frac{1}{2}(\chi + \kappa t)
\end{pmatrix} \psi + \frac{3}{2} (\chi + \kappa t) \psi
\]

(4.7)

where the Bargmann conformal vector fields \( X \) are given by Eq. (4.1).
The finite transformations

Now we derive the formulæ for finite transformations by integrating these expressions. Firstly, we only consider dilations and expansions.

For expansions only, one gets from Eq. (4.2) $t^* = t/(ft + 1)$, also $\vec{x}^* = \vec{x}/(ft + 1)$ and $s^* = s + \frac{1}{2} f \vec{x}^2/(ft + 1)$. Some tedious calculation then leads to the corresponding integration at the spinor level: the general solution $f \rightarrow \psi_f$ of the differential equation $d\psi/df = \delta_X \psi$ (with all coefficients set zero in Eq. (4.7) with the exception of $\kappa = -1$) reads

$$
\psi_f(t, \vec{x}, s) = \begin{pmatrix}
(f + 1)^{-1} & 0 \\
\frac{f}{2i} (\vec{\sigma} \cdot \vec{x}) (ft + 1)^{-2} & (ft + 1)^{-2}
\end{pmatrix} \psi(t^*, \vec{x}^*, s^*). \tag{4.8}
$$

Putting $\Psi(t, \vec{x}) \equiv \psi(t, \vec{x}, s)e^{-ims}$, one readily gets

$$
\Psi_f(t, \vec{x}) = \frac{1}{(ft + 1)^2} \begin{pmatrix}
ft + 0 \\
\frac{f}{2i} (\vec{\sigma} \cdot \vec{x}) & ft + 1
\end{pmatrix} \Psi\left(\frac{t}{ft + 1}, \frac{\vec{x}}{ft + 1}\right) \exp\left(-\frac{imf \vec{x}^2}{2(ft + 1)}\right), \tag{4.9}
$$

and one verifies that we have, indeed, obtained a representation $(\Psi_f)_{f'} = \Psi_{f+f'}$ of the (additive) group of pure expansions on the solutions of the LL equation. As for pure dilations, a straightforward calculation yields

$$
\Psi_d(t, \vec{x}) = \begin{pmatrix}
d & 0 \\
0 & d^2
\end{pmatrix} \Psi(d^2 t, d \vec{x}), \tag{4.10}
$$

which, again, turns out to be a genuine representation. Combining these two results by invoking the product of a dilation and an expansion $(d, 0) \cdot (1, f) = (d, f/d)$, one then finds the following (anti-)representation, $\pi$, of this 2-dimensional Borel subgroup

$$
\pi(u)\psi(t, \vec{x}, s) = \frac{1}{(ft + g)^2} \begin{pmatrix}
ft + g & 0 \\
\frac{f}{2i} (\vec{\sigma} \cdot \vec{x}) & 1
\end{pmatrix} \psi(t^*, \vec{x}^*, s^*), \tag{4.11}
$$

where $(t^*, \vec{x}^*, s^*) = u \cdot (t, \vec{x}, s)$ is the image, (4.2), by $u = (1, \vec{0}, \vec{0}, g^{-1}, 0, f, g, 0)$. One directly checks that we have indeed an anti-representation of the Borel subgroup, viz $\pi(uv) = \pi(v)\pi(u)$. 
On the other hand, a simple calculation gives the following (anti-)representation, we still denote $\pi$, of the Bargmann group (a mere group of isometries), originally due to Lévy-Leblond [8]

$$
\pi(u)\psi(t, \vec{x}, s) = \left( \begin{array}{cc}
a^{-1} & 0 \\
\frac{i}{2} a^{-1} \vec{\sigma} \cdot \vec{b} & a^{-1}
\end{array} \right) \psi(t^*, \vec{x}^*, s^*),
$$

(4.12)

where $(t^*, \vec{x}^*, s^*) = u \cdot (t, \vec{x}, s)$ with $u = (a, \vec{b}, \vec{c}, 1, e, 0, 1, h)$ in the Spin-Bargmann group, see (4.3).

We then use the previous factorization of the Schrödinger group to define the sought representation, again called $\pi$, of the Spin-Schrödinger group $\pi(u) = \pi(u'')\pi(u')$ associated with the decomposition (4.4) of the group element $u = u'u''$. With the help of Eqs (4.11,12), one ends up with the following (anti-)representation of the full Spin-Schrödinger group

$$
\pi(u)\psi(t, \vec{x}, s) = \frac{1}{(ft + g)^2} \left( \begin{array}{cc}
a^{-1}(ft + g) & 0 \\
\frac{1}{2i} a^{-1} \vec{\sigma} \cdot (fR\vec{x} - g\vec{b} + f\vec{c}) & a^{-1}
\end{array} \right) \psi(t^*, \vec{x}^*, s^*),
$$

(4.13)

where $(t^*, \vec{x}^*, s^*) = u \cdot (t, \vec{x}, s)$ with $u = (a, \vec{b}, \vec{c}, d, e, f, g, h)$.

By equivariance, one finally obtains the ‘natural’ (anti-)representation of the full Spin-Schrödinger group on the space of solutions of the free LL equation, that is

$$
\pi(u)\Psi(t, \vec{x}) =
$$

$$
\frac{1}{(ft + g)^2} \left( \begin{array}{cc}
a^{-1}(ft + g) & 0 \\
\frac{1}{2i} a^{-1} \vec{\sigma} \cdot (fR\vec{x} - g\vec{b} + f\vec{c}) & a^{-1}
\end{array} \right) \Psi \left( \frac{dt + e}{ft + g}, \frac{R\vec{x} + \vec{b}t + \vec{c}}{ft + g} \right)
$$

$$
\times \exp \left( -im \left\{ \frac{(fR\vec{x} + \vec{b}t + \vec{c})^2}{2(ft + g)} - \vec{b} \cdot R\vec{x} - \frac{t}{2} \vec{b}^2 + h \right\} \right)
$$

(4.14)

where the group element $u$ is as in (4.3).

Observe that the action of the Schrödinger group commutes with that of $\Gamma$, and descends therefore to the chiral components $\psi_\pm$. This chiral representation is also of the ‘lower triangular’ form: though the $\Phi_\pm$ components are mapped into themselves, the $\chi_+$
(resp. \( \chi_- \)) components transform into combinations of \( \Phi_+ \) and \( \chi_+ \) (resp. \( \Phi_- \) and \( \chi_- \)) under boosts and expansion. It is, therefore, not possible to further reduce the representation provided by the \( \psi_\pm \).

5. Conserved quantities

In Ref. [6] we constructed, following Souriau [17], a symmetric, traceless, conserved energy-momentum tensor, \( \vartheta_{\mu\nu} \), for the NLS-CS system and used it to associate the conserved charge

\[
Q_X = \int \vartheta_{\mu\nu} X^\mu \xi^\nu \sqrt{\gamma} d^2x, \tag{5.1}
\]

where \( \gamma = \det(g_{ij}) \), to each \( \xi \)-preserving conformal vector field \( X \) of Bargmann space (here, the integral is taken over ‘2-space at time \( t \)’). Now we extend and apply these results to spinors (see, e.g. [18] and references therein). Remember that the massless Dirac equation can be obtained by varying the matter action

\[
S = \int_M \Im \{ \bar{\psi} \partial^\mu \psi \} \sqrt{-g} d^4x. \tag{5.2}
\]

We note that the variational derivative with respect to the vector potential, \( \delta S/\delta a_\mu \), yields the current \( j^\mu \) in Eq. (2.5), whose conservation, \( \nabla_\mu j^\mu = 0 \), follows simply from the invariance of \( S \) with respect to gauge transformations.

Denote \( \mathcal{L} \) the Lagrangian density in definition (5.2), namely \( S = \int_M \mathcal{L} \). Then a short calculation resorting to the action of a conformal vector field, \( X \), on Dirac spinors, Eq. (3.5), and on the CS field, Eq. (3.15), together with the technical results (3.11,12), shows that

\[
\delta_X \mathcal{L} = -\frac{1}{2} k \mathcal{L} + L_X \mathcal{L} \tag{5.3}
\]

where \( k = \frac{1}{2} \nabla_\mu X^\mu \) (see (3.18)). Now \( \mathcal{L} = 0 \) in view of the Euler-Lagrange equations (2.2), it thus follows, since \( \mathcal{L} \) is represented by a closed 4-form, that \( \delta_X \mathcal{L} = 0 \) modulo a surface term that stems from the Lie derivative in the RHS of Eq. (5.3). Thus, for any conformal
vector field $X$, we have
\begin{equation}
\delta_X S = 0 \tag{5.4}
\end{equation}

at the critical points of the matter action, $S$, which again proves the conformal symmetry of our system. The energy-momentum tensor $\partial_{\mu\nu} = -2 \frac{\delta S}{\delta g^{\mu\nu}}$, viz
\begin{equation}
\partial_{\mu\nu} = -2 \frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{2} \Im \{ \overline{\psi} (\gamma_{\mu} D_{\nu} + \gamma_{\nu} D_{\mu}) \psi \}, \tag{5.5}
\end{equation}
is therefore traceless as well as conserved and automatically symmetric. Hence, a conserved quantity, Eq. (5.1), is associated to each $\xi$-preserving conformal vector field. Upon using Eqs. (5.1, 5) and (4.1) as well as the reduced Dirac equation, Eq. (2.11), and assuming all surface terms vanish, the associated conserved quantities read
\begin{equation*}
\begin{cases}
\vec{P} = \int \left\{ \frac{1}{2i} \left( \Phi^\dagger \vec{D} \Phi - (\vec{D} \Phi)^\dagger \Phi \right) \right\} d^2\vec{x} & \text{linear momentum}, \\
M = m \int |\Phi|^2 \, d^2\vec{x} & \text{mass \times particle number}, \\
J = \int \vec{r} \times \vec{P} \, d^2\vec{x} + \frac{1}{2} \int \Phi^\dagger \sigma_3 \Phi \, d^2\vec{x} & \text{angular momentum}, \\
\vec{G} = t\vec{P} - m \int |\Phi|^2 \, \vec{x} \, d^2\vec{x} & \text{boost}, \\
H = \int \left\{ \frac{1}{2m} |\vec{D} \Phi|^2 + \lambda |\Phi|^2 \Phi^\dagger \Phi \right\} d^2\vec{x} & \text{energy}, \\
D = 2tH - \int \vec{P} \cdot \vec{x} \, d^2\vec{x} & \text{dilation}, \\
K = -t^2H + tD + \frac{m}{2} \int |\Phi|^2 \, \vec{x}^2 \, d^2\vec{x} & \text{expansion},
\end{cases}
\end{equation*}

where the constant $\lambda = e^2/(2m\kappa)$ is as in (1.8).

**A (2 + 1)-dimensional formulation**

As pointed out in Sec. 2, when the massless Dirac equation is reduced to 2 + 1 dimensions, the resulting equations split, consistently with chiral symmetry, into two independent
sets of equations. This can be seen also by looking at the action. Identifying (with some abuse of notation) $\Psi_\pm$ with two-component spinors,

$$\Psi_+ = \begin{pmatrix} \Phi_+ \\ \chi_+ \end{pmatrix} \quad \text{and} \quad \Psi_- = \begin{pmatrix} \Phi_- \\ \chi_- \end{pmatrix},$$

and introducing the two sets of $2 \times 2$ matrices

$$\Sigma^t_\pm = \frac{1}{2}(1 + \sigma_3), \quad \Sigma^1_\pm = -\sigma_2, \quad \Sigma^2_\pm = \pm \sigma_1, \quad \Sigma^s_\pm = \frac{1}{2}(1 - \sigma_3)$$

(5.8)

taking into account the mass constraint, the Lagrangian density in Eq. (5.2) is seen in fact to split into two parts,

$$\Im \left\{ \psi^\dagger_+ (\Sigma^t_+ D_t + \Sigma^i_+ D_i - 2im\Sigma^s_+) \psi_+ \right\} + \Im \left\{ \psi^\dagger_- (\Sigma^t_- D_t + \Sigma^i_- D_i - 2im\Sigma^s_-) \psi_- \right\},$$

(5.9)

whose variational equations are

$$\left[ \Sigma^t_\pm D_t + \Sigma^i_\pm \cdot \vec{D} - 2im\Sigma^s_\pm \right] \Psi_\pm = 0,$$

(5.10)

i.e. the Lévy-Leblond equations (2.13), in another form. Augmenting (5.9) by the Lagrangian density of the CS gauge fields,

$$\mathcal{L}_{CS} = \frac{\kappa}{4} \epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma},$$

(5.11)

we get a local $(2+1)$-dimensional Lagrangian for the complete spinor-CS system. Thus, just like in the scalar case, while there is no action for the coupled system in $3 + 1$ dimensions, in $2 + 1$ dimensions there is one.

The existence of the ‘reduced’ action is important for at least two reasons: on the one hand, it makes it possible to apply the standard canonical or Hamiltonian description to the spinor-CS system, and on the other, it opens up an alternative way to discuss the symmetries and the conserved quantities. These investigations go beyond the scope of the present paper and here we merely give the canonical Hamiltonian density:

$$\mathcal{H} = A_t \left( e \sum_{i=\pm} \psi^\dagger_1 \Sigma^t_i \psi_i + \kappa F_{12} \right) + 2m \sum_{i=\pm} \psi^\dagger_1 \Sigma^s_i \psi_i - \Im \left\{ \sum_{i=\pm} \psi^\dagger_1 \Sigma^\dagger \cdot (\vec{D} \psi_i) \right\}.$$

(5.12)

The static equations of motion can also be obtained by variation of the Hamiltonian. It is easy to see — using the equations of motions — that on solutions $\int \mathcal{H} d^2\vec{x}$ takes (up to an irrelevant sign) the same value as $H$, Eq. (5.6).
6. Static, self-dual solutions

Reduction to the Liouville equation

In Ref. [7], we constructed static, purely magnetic vortices of definite chirality by solving the static versions of Eqs (2.17) and (1.1) using the first-order ‘self-dual’ Ansatz

\[(D_1 \pm iD_2)\Phi = 0.\] (6.1)

Here we arrive at this same equation in a more direct way. Let us consider in fact the static version of the first-order equations (2.13), written in chiral components:

\[
\begin{align*}
(D_1 + iD_2)\Phi_+ &+ 2m\chi_+ = 0, \\
-eA_t\Phi_+ &+ (D_1 - iD_2)\chi_+ = 0; \\
(D_1 - iD_2)\Phi_- &+ 2m\chi_- = 0, \\
-eA_t\Phi_- &+ (D_1 + iD_2)\chi_- = 0.
\end{align*}
\] (6.2)

Then, requiring the solution to be purely magnetic, \(A_t = 0\), and setting \(\chi_+ = 0\) for the positive and \(\chi_- = 0\) for the negative chirality, these equations reduce directly to the self-duality conditions (6.1). In this sense, the first-order LL equations (2.13) already contain self-duality. It is readily seen that the current, Eq. (2.15), vanishes identically for positive chirality spinors with \(\chi_+ = 0\) (negative chirality ones with \(\chi_- = 0\)), thus with \(A_t \equiv 0\) the static version of the Chern-Simons equations, \(\kappa E^i = e\epsilon^{ij}J^j\), are also satisfied.

The SD conditions (6.1), as well as the remaining Chern-Simons equation, \(B = -e\varrho/\kappa\), where \(\varrho = |\Phi_+|^2\) (or \(\Phi_-|^2\)) were solved in [7] along the lines of Ref. [5]: in the gauge \(\Phi_\pm = \varrho^{1/2}\), (2) Eq. (6.1) leads to

\[\vec{A} = \pm \frac{1}{2e} \vec{\nabla} \times \ln \varrho.\] (6.3)

(2) The gauge transformation needed to bring \(\Phi_\pm\) to this form may be singular at the zeros of \(\varrho\).
Thus \( B = -e\rho/\kappa \) reduces, in both cases, to the Liouville equation

\[
\triangle \ln \rho = \pm \frac{2e^2}{\kappa} \rho. \tag{6.4}
\]

A normalizable solution is obtained for \( \Phi_+ \) when \( \kappa < 0 \), and for \( \Phi_- \) when \( \kappa > 0 \). These correspond precisely to having an attractive non-linearity in Eq. (2.19).

**The properties of the solutions**

As these static solutions have only the \( \Phi_+ \) or \( \Phi_- \) components of \( \Phi \) not identically zero, they involve only one of the spinor fields \( \Psi_\pm \) in 2 + 1 dimensions (see (2.12)), depending upon the sign of \( \kappa \).

Furthermore, as the total magnetic flux of the solutions,

\[
\int B \, d^2\vec{x} = -\frac{e}{\kappa} \int \rho \, d^2\vec{x} \equiv -\frac{e}{\kappa} N, \tag{6.5}
\]

is nonvanishing if \( N \neq 0 \), we call them spinor vortices. Also, as \( A_t \equiv 0 \) (and hence \( \vec{E} = 0 \)), they are purely magnetic ones.

We can now confirm our results on self-duality. The same argument as in the scalar case [5] shows that static solutions must have vanishing energy: it is the only possibility to cancel the apparent t-dependence in the conserved quantities \( D \) (dilation) and \( K \) (expansion). Using the algebraic identity: \( |\vec{D}\Phi|^2 = |(D_1 \pm iD_2)\Phi|^2 \pm \rho \) (up to total derivatives integrating to surface terms), which is established along the same lines as in the scalar case [5], the energy, Eq. (5.7), is re-written as

\[
H = \frac{1}{2m} \int |(D_1 \pm iD_2)\Phi|^2 \, d^2\vec{x} \mp 2\lambda \int |\Phi_\pm|^2 \Phi^* \Phi \, d^2\vec{x}, \tag{6.6}
\]

where \( \Phi = \left( \begin{array}{c} \Phi_+ \\ \Phi_- \end{array} \right) \). Now it is easy to see that this expression indeed vanishes for the (anti) self dual configurations described above: taking the upper (resp. lower) sign for \( \kappa < 0 \) (resp. \( \kappa > 0 \)) we get a positive semidefinite expression, and \( \Phi_- \equiv 0 \) (resp. \( \Phi_+ \equiv 0 \)) together with the first (resp. second) equation in (6.1) guarantee that it is indeed zero.
It is straightforward to verify that the static field equations (6.1) can also be obtained by variation of $\int H d^2 \vec{x}$. However, as, in contrast to the scalar case [5], $H$ is not a (semi) definite expression, we can not rule out the existence of non self-dual static solutions.

The general solution of the Liouville equation is

$$\varrho = \mp \frac{4\kappa}{e^2} \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2}$$

(6.7)

with $z = x^1 + i x^2$ and $f(z)$ complex analytic, but for our purposes $f(z)$ must be chosen such that $\varrho$ is well behaved. To describe explicit self-dual solitons we use the radially symmetric solution of Eq. (6.7), given by [5]:

$$\varrho_n(r) = \mp \frac{4n^2 \kappa}{e^2 r^2} \left[ \left( \frac{r_0}{r} \right)^n + \left( \frac{r}{r_0} \right)^n \right]^{-2}$$

(6.8)

where $r = |z|$, with $r_0$ and $n$ two free parameters. However, as explained in [5], the aforementioned (singular) gauge transformation, which not only removes the phases of $\Phi_{\pm}$, but also cancels the singularities in $B$, must be single valued. This condition requires $n$ to be an integer, which, without loss of generality, can be chosen to be positive. Integrating $\varrho_n$ over 2-space yields

$$N = \frac{\kappa}{e^2} 4\pi n.$$  

(6.9)

This quantity then determines the actual values of all the conserved charges for our solitons: both the magnetic flux, $-eN/\kappa = -(\text{sign } \kappa/e)4\pi n$, and the mass, $M = mN$, are the same as for the scalar soliton of [5], while the total angular momentum, $J = \mp N/2$ is merely half of the corresponding value taken by the scalar soliton (the conserved quantities $\vec{P}, \vec{G}$ and $D$ vanish for our spinor solitons much in the same way as they do for the scalar one).

7. Spinor vortices in external fields

Physical applications as the Fractional Quantum Hall Effect [1] would require to extend the spinor theory to background fields. In the scalar case, the ‘empty’ space solution has been ‘exported’ to provide solutions in a uniform magnetic or harmonic force field [19].
In Ref. [12] we gave a geometric framework to this procedure and showed that the only
cases where it works correspond to the time-dependent uniform magnetic and electric or
harmonic forces. Now we generalize this approach for the case of spinors.

To describe the external fields one can use the most general ‘Bargmann’ metric found
long ago by Brinkmann [20]:

\[
g_{ij}(t, \vec{x})dx^i dx^j + 2dt \left[ ds + A_i(t, \vec{x}) dx^i \right] - 2U(t, \vec{x}) dt^2,
\]

(7.1)

where the ‘transverse’ metric \( g_{ij} \) (with \( i, j = 1, 2 \)) as well as the ‘vector potential’ \( \vec{A} \) and the
‘scalar potential’ \( U \) are functions of \( t \) and \( \vec{x} \) only. Clearly, \( \xi = \partial_s \) is a covariantly constant
null vector. The null geodesics of this metric describe particle motion in curved transverse
space in external electromagnetic fields \( \vec{E} \sim -\partial_t \vec{A} - \vec{\nabla} U \) and \( B \sim \vec{\nabla} \times \vec{A} \) [9]. In the sequel,
we make the simplifying assumption that the transverse 2-space is flat, \( g_{ij} = \delta_{ij} \).

Consider now the Dirac-Chern-Simons system, Eqs (2.8), in the background (7.1).
Using that the non-vanishing components of the inverse metric are
\( g^{ij} = \delta^{ij}, g^{ts} = -A^t, \)
\( g^{ss} = 2U + A_i A^i, \)
\( g^{ts} = g^{st} = 1, \)
we find that an appropriate set of ‘curved space’ Dirac
matrices can be written as

\[
\gamma^t = \gamma^T, \quad \gamma^i = \gamma^I, \quad \gamma^s = -A^i \gamma^I + U \gamma^T + \gamma^S,
\]

(7.2)

where \( \gamma^T, \gamma^I \) and \( \gamma^S \) denote the constant, flat space Dirac matrices (2.9):

\[
\gamma^T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \gamma^I = \begin{pmatrix} -i\sigma^I & 0 \\ 0 & i\sigma^I \end{pmatrix}, \quad \gamma^S = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}.
\]

(7.3)

Since the determinant of the metric, hence \( \sqrt{-g} \) as well as \( \gamma^i \) and \( \gamma^t \) coincide with their
flat Minkowski space counterparts, both the (3 + 1)-dimensional and the reduced versions
of the FCI, Eq. (1.9) and Eq. (1.1) respectively, retain their previous, flat space form.
Nevertheless the external potentials, \( A_i \) and \( U \), do make their presence felt through the
Dirac equation. The first term in the massless Dirac equation (2.2):

\[
\gamma^\mu D_\mu \psi \equiv \gamma^\mu D_\mu \psi - \frac{1}{8} [\gamma^\rho, \partial_\mu \gamma^\rho - \Gamma^\sigma_{\mu \rho} \gamma^\sigma] \psi = 0,
\]

(7.4)
(where \( D_\mu \equiv \partial_\mu - ie a_\mu \)) reads, if we put \( \psi \equiv e^{i m s} \Psi \) and use the Dirac matrices (7.2),

\[
\left( \gamma^T (D_t + i m U) + \gamma^J (D_j - i m A_j) + i m \gamma^S \right) \Psi.
\]

Thus, the effect of these terms is simply to modify the flat space CS-covariant derivatives \( D_\mu \) into ‘external field’ ones, \( D_\mu \), viz

\[
D_t = D_t - i m A_t, \quad D_j = D_j - i m A_j,
\]

where \( A_t = -U \). Using the Christoffel symbols of the metric (7.1),

\[
\Gamma^i_{tt} = \partial_t A_i + \partial_i U,
\]

\[
\Gamma^i_{jt} = \frac{1}{2} (\partial_j A_i - \partial_i A_j),
\]

\[
\Gamma^s_{ij} = \frac{1}{2} (\partial_i A_j + \partial_j A_i),
\]

\[
\Gamma^s_{it} = -\partial_i \left( U + \frac{1}{4} \vec{A}^2 \right) + \frac{1}{2} A^j \partial_j A_i,
\]

\[
\Gamma^s_{tt} = -A^i \partial_i U - \partial_t \left( U + \frac{1}{2} \vec{A}^2 \right),
\]

one finds that the net contribution of the second term to the massless Dirac equation (7.4) has the form:

\[-\frac{1}{16} \left( \partial_i A_j - \partial_j A_i \right) [\gamma^I, \gamma^J] \gamma^T \Psi.
\]

Collecting these terms, we get (without imposing the chirality constraint) that the reduced massless Dirac equation takes the form:

\[
\begin{aligned}
(\vec{\sigma} \cdot \vec{D}) \Phi + 2m \chi
&= 0, \\
D_t \Phi + \frac{i}{4} B \sigma_3 \Phi + i (\vec{\sigma} \cdot \vec{D}) \chi
&= 0,
\end{aligned}
\]

(7.5) where \( B = e^{ij} \partial_i A_j \) denotes the background ‘magnetic’ field.

Eliminating the lower component we see that \( \Phi \) satisfies the ‘Pauli equation’ with anomalous magnetic moment (compare (2.18):

\[
i D_t \Phi = -\frac{1}{2m} \left[ \vec{D}^2 + e B \sigma_3 \right] \Phi - \frac{B}{4} \sigma_3 \Phi.
\]

(7.6)
In order to identify the ‘magnetic’ component of our metric ($A_j$) with the potential of a genuine magnetic field, one has to make the replacement $A_j \rightarrow e A_j/m$. Then the $B$-dependent term in Eqs (7.5,6), $eB/(4m)\sigma_3 \Phi$, shows that our reduced spinor fields — since they have a non vanishing magnetic moment — interact also directly with the background magnetic field, in addition to the minimal coupling appearing in $D_\mu$. For chiral spinors the sign of the magnetic moment depends on the sign of the 4-dimensional chirality.

The harmonic oscillator field

Physically, the most interesting external fields describe either a uniform magnetic field or a harmonic oscillator force field. The latter case is obtained by choosing $A_i = 0$ and $U = \frac{1}{2} \omega^2 r^2$ in (7.1) [12], so that, in the oscillator background, the Dirac matrices are:

$$\gamma^t = \gamma^T, \quad \gamma^i = \gamma^I, \quad \gamma^s = \frac{1}{2} \omega^2 r^2 \gamma^T + \gamma^S.$$  \hspace{1cm} (7.7)

The clue that makes possible to obtain the solutions of the oscillator problem from that of the Minkowski space, is that the mapping $(t, \vec{x}, s) \rightarrow (T, \vec{X}, S) \equiv (\vec{x}^\mu)$:

$$T = \tan \frac{\omega t}{\omega}, \quad X^i = \frac{x^i}{\cos \omega t}, \quad S = s - \frac{\omega r^2}{2} \tan \omega t,$$  \hspace{1cm} (7.8)

carries the oscillator metric

$$d\vec{x}^2 + 2 dt ds - \omega^2 r^2 dt^2,$$  \hspace{1cm} (7.9)

conformally into the flat metric $d\vec{X}^2 + 2 dT dS$ [12]. The sought connection between the two sets of solutions can be determined by implementing this conformal mapping on the spinors. This can be done in three steps: invoking the equivariance condition, one writes the solution of the oscillator problem as

$$\psi = \exp \left( i m \left( s - \frac{1}{2} \omega r^2 \tan \omega t \right) \right) \tilde{\Psi}(T, \vec{X})$$

and substitutes it into Eq. (7.4) with Dirac matrices as in Eq. (7.7). In this way, one gets

$$\exp \left( i m \left( s - \frac{1}{2} \omega r^2 \tan \omega t \right) \right) \left[ \vec{\gamma}^t D_T + \vec{\gamma}^j D_J + i m \vec{\gamma}^s \right] \tilde{\Psi}(T, \vec{X}) = 0,$$  \hspace{1cm} (7.10)
where

\[
\tilde{\gamma}^t = \gamma^T \frac{1}{\cos^2 \omega t},
\]

\[
\tilde{\gamma}^i = \gamma^T \frac{x^i \omega \sin \omega t}{\cos^2 \omega t} + \gamma^l \frac{1}{\cos \omega t},
\]

\[
\tilde{\gamma}^s = \gamma^S - \gamma^l x_i \omega \tan \omega t - \gamma^T \frac{\omega^2 \gamma^2}{2} \tan^2 \omega t.
\]

It is straightforward to verify that \(\tilde{\gamma}^\mu\) satisfy the anticommutation relations

\[
\tilde{\gamma}^\mu \tilde{\gamma}^\nu + \tilde{\gamma}^\nu \tilde{\gamma}^\mu = -2 \tilde{\eta}^{\mu \nu} = -\frac{2}{\cos^2 \omega t} \eta^{\mu \nu}
\]

where \(\eta^{\mu \nu}\) denotes the flat Minkowski metric. This equation implies that \(\tilde{g}_{\mu \nu} = \eta_{\mu \nu} \cos^2 \omega t\), therefore as a second step we make a conformal rescaling by \(\Omega = \cos \omega t\). Now using that \(\hat{D} \hat{\Psi} = \Omega^{-5/2} \hat{D} \hat{\Psi}\) if \(\hat{\Psi} = \Omega^{-3/2} \hat{\Psi}\) and adding to this the observation that \(\hat{\gamma}^\mu = \tilde{\gamma}^\mu \cos \omega t\) satisfy the anticommutation rules of flat Minkowski space, one can rewrite Eq. (7.10) as

\[
\exp \left( i m \left( s - \frac{1}{2} \omega r^2 \tan \omega t \right) \right) \frac{(\cos \omega t)^{-5/2}}{\cos \omega t} \left[ \hat{\gamma}^t \hat{D}_T + \hat{\gamma}^i \hat{D}_J + im \hat{\gamma}^s \right] \hat{\Psi}(T, \vec{X}) = 0.
\]

The next step is to find operators \(U\) and \(U^{-1}\) that ‘rotate’ \(\hat{\gamma}^\mu\) into the flat space Dirac matrices:

\[
\hat{\gamma}^t = U \gamma^T U^{-1}, \quad \hat{\gamma}^i = U \gamma^l U^{-1}, \quad \hat{\gamma}^s = U \gamma^S U^{-1}.
\]

Making a suitable lower triangular Ansatz for \(U\) and exploiting the explicit form of the flat space Dirac matrices, one readily finds that

\[
U^{-1} = f \begin{pmatrix} 1 & 0 \\ -i \frac{\omega}{2} (\vec{\sigma} \cdot \vec{x}) \sin \omega t & \cos \omega t \end{pmatrix}, \quad U = f^{-1} \begin{pmatrix} 1 & 0 \\ i \frac{\omega}{2} (\vec{\sigma} \cdot \vec{x}) \tan \omega t & (\cos \omega t)^{-1} \end{pmatrix}
\]

do this for any \(f\). The most natural choice is to require \(\det U = 1\), this yields then \(f = (\cos \omega t)^{-1/2}\). Since \(\hat{D}_\mu\) is a ‘covariant derivative’ we can write

\[
\hat{\gamma}^\mu \hat{D}_\mu \hat{\Psi} = U \gamma^\mu U^{-1} \hat{D}_\mu \hat{\Psi} = U \gamma^\mu \hat{D}_\mu (U^{-1} \hat{\Psi})
\]

in Eq. (7.13). Therefore, denoting the solution of the original Lévy-Leblond equation by \(\hat{\Psi}^0(T, \vec{X})\), the solution of the oscillator problem can be written as

\[
\hat{\Psi}(t, \vec{x}) = \exp \left( -im \frac{\omega r^2}{2} \tan \omega t \right) (\cos \omega t)^{-3/2} U \hat{\Psi}^0(T, \vec{X}),
\]
Spinors in Chern-Simons

\[ \Psi(t, \vec{x}) = \exp \left( -im \frac{\omega r^2}{2} \tan \omega t \right) (\cos \omega t)^{-1} \begin{pmatrix} 1 & 0 \\ i\frac{\omega}{2}(\vec{\sigma} \cdot \vec{x}) \tan \omega t & (\cos \omega t)^{-1} \end{pmatrix} \Psi^0(T, \vec{X}). \quad (7.15) \]

Note the two interesting properties of this final expression:

(a) The various powers of \( \cos \omega t \) combine nicely to yield a pure \(-1\) for the upper component, \( \Phi \), of \( \Psi^0 \); i.e. \( \Phi \) transforms with the same ‘time-dependent dilation’ factor as a scalar field. In the same time the ‘lower’ component, \( \chi \), has a conformal factor \((\cos \omega t)^{-2}\) as well as an inhomogeneous part, which is linear in \( \Phi \).

(b) Applying the transformation (7.15) to the static spinor vortices in Sec. 6 yields in particular time-dependent background-field solutions with non-vanishing ‘lower’ component as well as a (Chern-Simons) electric field.

Constant magnetic field

The constant magnetic field background can be described by setting

\[ A_i = -\frac{e}{2m} \epsilon_{ij} B x^j, \quad U = 0 \quad (7.16) \]

in (7.1); here we already implemented the \( A_j \rightarrow e A_j/m \) scaling. The key point is to realize that the mapping (which amounts to switching to a rotating frame with angular velocity \( \omega = eB/(2m) \)): \((t, \vec{x}, s) \rightarrow (t_{\text{osc}}, \vec{x}_{\text{osc}}, s_{\text{osc}})\) given by

\[ t_{\text{osc}} = t, \quad \vec{x}_{\text{osc}} = R(-\omega t) \vec{x}, \quad s_{\text{osc}} = s, \quad (7.17) \]

takes the ‘constant \( B \)-metric’ (7.16) into the oscillator metric (7.9) [12]. Just as in Sec. 4. the matrix \( R(\theta) \) is a rotation by angle \( \theta \) in the plane and \( \vec{\sigma} \cdot (R \vec{x}) = a(\vec{\sigma} \cdot \vec{x}) a^{-1} \) with \( a(\theta) \equiv \exp(i\theta \sigma_3/2) \). Repeating the same procedure as in the case of the oscillator problem one finds that the transition between the uniform \( B \) field and the oscillator can be implemented by

\[ \Psi^B(t, \vec{x}) = \begin{pmatrix} e^{i\omega t \sigma_3/2} & 0 \\ -i\frac{\omega}{2}(\vec{\sigma} \times \vec{x}) e^{i\omega t \sigma_3/2} & e^{i\omega t \sigma_3/2} \end{pmatrix} \Psi^\text{osc}(t, \vec{x}_{\text{osc}}). \quad (7.18) \]
Combining the two conformal mappings in Eqs (7.8) and (7.17) gives finally a map from the uniform $B$ metric to the flat one [7]:

$$T = \frac{\tan \omega t}{\omega}, \quad \vec{X} = \frac{1}{\cos \omega t} R^{-1}(\omega t) \vec{x},$$

and using the explicit expressions to implement these transformations on the spinors, Eqs (7.15) and (7.18), finally we get [7]:

$$\Psi^B(t, \vec{x}) = e^{-(im\omega r^2 \tan \omega t)/2} \begin{pmatrix} e^{i\omega t\sigma_3/2} \\ i\frac{\omega}{2} \left[ \tan \omega t (\vec{\sigma} \cdot \vec{x}) - (\vec{\sigma} \times \vec{x}) \right] e^{i\omega t\sigma_3/2} \\ 0 \end{pmatrix} \Psi^0(T, \vec{X}),$$

$$A^B_\alpha = \partial_\alpha X^\beta A^0_\beta.$$ (7.20)

8. Relativistic fermions and the non-relativistic limit

In Ref. [13], Cho et al. consider the massive gauged Dirac equation $\slashed{D} \psi = iM\psi$ on four-dimensional Minkowski space with metric given by

$$dt^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

and require that the chiral components, denoted by $\psi_\pm$, satisfy

$$\psi_\pm(t, x, y, z) = e^{ip_\pm z} \Psi_\pm(t, x, y).$$ (8.2)

Then their massive Dirac equation becomes, with $c = 1$,

$$(i\gamma_\pm D_\alpha - p_\pm) \Psi_\pm = \pm M\sigma_3 \Psi_\pm,$$ (8.3)

where $\alpha = 0, 1, 2$ and $(\gamma^\alpha_\pm) = (\pm \sigma^3, i\sigma^2, -i\sigma^1)$. Requiring the gauge field dynamics to be governed by the Chern-Simons rule,

$$\frac{1}{2} \kappa e^{\alpha\beta\gamma} F_{\beta\gamma} = e(\overline{\Psi}_+ \gamma^\alpha_+ \Psi_+ + \overline{\Psi}_- \gamma^\alpha_- \Psi_-),$$ (8.4)
allows them to construct purely magnetic solutions. They note also that for \( p_+ = p_- \) and for chiral spinors with one vanishing component, their equations reduce formally to the same system as the one studied by Jackiw and Pi [5]. This is the point we will be discussing in this Section, together with the relation between our work and that of Ref. [13].

Setting \( M = 0 \) in Eq. (8.3) results in two relativistic Dirac equations on \( \mathbb{R}^{2,1} \) if one interprets \( p_\pm \) as the masses of the chiral components \( \Psi_\pm \). Apart from the ‘mass’ term \( M \) (that destroys conformal invariance), the main difference of Eq. (8.3) with our approach is that the reduction here is \emph{spacelike}, while we used so far a lightlike reduction. To proceed, we thus outline the relativistic version of our theory, namely the coupled Dirac-Chern-Simons system (2.8) where, this time,

\[
 g_{\mu\nu} \xi^\mu \xi^\nu = c^{-2}. 
\] (8.5)

Relativistic spinors in Kaluza-Klein framework

Let us start with \( \mathbb{R}^{3,1} \), endowed with the metric [10]

\[
d\ell^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = 2 dt ds + dx^2 + dy^2 + c^{-2} ds^2 
\] (8.6)

where \( c = \text{const} < +\infty \). The (covariantly) constant vector field

\[
 \xi = \frac{\partial}{\partial s} 
\] (8.7)

is chosen as the generator of vertical translations. Note that \( \xi \) is now spacelike, Eq. (8.5).

The contravariant metric associated with (8.6) descends as a contravariant metric on the quotient \( \mathbb{R}^{2,1} = \mathbb{R}^{3,1}/\mathbb{R}\xi \) which is plainly Minkowski spacetime, with the metric

\[-c^2 dt^2 + dx^2 + dy^2.\]

Consider, as before but with \( c < +\infty \), the gauged, massless, Dirac equation on \( \mathbb{R}^{3,1} \)

\[
 \gamma^\mu D_\mu \psi = 0, 
\] (8.8)
with Dirac matrices
\[
\begin{align*}
\gamma^t &= \begin{pmatrix} 0 & c^{-2} \\ 1 & 0 \end{pmatrix}, & \overline{\gamma} &= \begin{pmatrix} -i\sigma & 0 \\ 0 & i\sigma \end{pmatrix}, & \gamma^s &= \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix},
\end{align*}
\]
(8.9)
which indeed satisfy \(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2g^{\mu\nu}\); these are again hermitian with respect to the metric \(G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) on the spinor space, \(\mathbb{C}^2,2\), as is the chirality operator
\[
\Gamma = \begin{pmatrix} -i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}
\]
(8.10)
which retains the same form as in (2.10). The latter still anticommutes with the Dirac operator, and also commutes with reduction with respect to the mass constraint
\[
\xi^\mu D_\mu \psi = im\psi.
\]
(8.11)

The massless Dirac equation (8.8), as well as its reduction, splits therefore into two sets of prescribed chirality,
\[
\Gamma\psi_\pm = \mp i\psi_\pm.
\]
(8.12)

With the same assumption as in Sec. 2 and with the notation (2.7), the projection of our system onto spacetime, \(\mathbb{R}^{2,1}\), reads now
\[
\begin{align*}
\left\{ \begin{array}{l}
c^{-2} i D_t \chi + (\bar{\sigma} \cdot \vec{D}) \Phi + 2m\chi = 0, \\
D_t \Phi + i(\bar{\sigma} \cdot \vec{D})\chi = 0,
\end{array} \right. & (8.13)
\end{align*}
\]
while the chiral components are still given by (2.12). Note here the new term with \(c^{-2}\).

The gauge field dynamics is again governed by the CS field equations (1.9) which project as [6]
\[
\frac{1}{2} \kappa \epsilon^{\alpha\beta\gamma} F_{\beta\gamma} = e J^\alpha \quad \text{and} \quad \partial_\alpha F_{\beta\gamma} = 0
\]
(8.14)
(with \(\alpha, \beta, \gamma = 0, 1, 2\)). The current in Eq. (2.5), descends as
\[
J^\alpha = e \overline{\Psi} \gamma^\alpha \Psi
\]
(8.15)
so that \(\varrho \equiv J^t = |\Phi|^2 + c^{-2}|\chi|^2\) is still positive definite and \(\vec{J}\) is as in (2.15).
Symmetry

Equations (8.13–15) stem from the relativistic system (2.8) in 3 + 1 dimensions, whose invariance with respect to the ξ-preserving conformal transformations had been established in Sec. 3. Now we show that these transformations span merely the trivial extension

\[ e(2, 1) \times \mathbb{R} \tag{8.16} \]

of the Poincaré Lie algebra \( e(2, 1) \) by the vertical translations generated by \( ξ \).

To prove this, remember that the algebra \( o(4, 2) \) is spanned by those vector fields

\[ Z^μ = Λ^μ_ν x^ν + Γ^μ + K^μ x_ν x^ν - 2 x^μ K_ν x^ν + α x^μ \tag{8.17} \]

where \( Λ \in o(3, 1) \) (Lorentz transformation), \( Γ, K \in \mathbb{R}^{3,1} \) (translation, special conformal transformation) and \( α \in \mathbb{R} \) (dilation). Demanding \( L_ξ Z = 0 \) results, since \( ξ \) is (covariantly) constant, in the extra condition \( \nabla_ξ Z = 0 \). Since \( Z^μ \) is a quadratic expression of the coordinates, this condition yields, for any \( x^μ \),

\[ K^μ (x_ν ξ^ν) - x^μ (K_ν x^ν) - (K_ν x^ν) ξ^μ = 0 \quad \text{and} \quad Λ^μ_ν ξ^ν + α ξ^μ = 0. \]

It follows from the first relation that \( K_ν ξ^ν = 0 \) and \( K^μ = a ξ^μ \) for some \( a \in \mathbb{R} \). Using that \( ξ \) is non-null, we have \( a = 0 \), hence \( K = 0 \): the special conformal transformations are broken by the reduction. Similarly, the second relation gives \( Λ^μ_ν ξ_μ ξ^ν = α ξ_μ ξ^μ = 0 \) since \( Λ_μ^μ \) is skew-symmetric, therefore \( α = 0 \): the dilations are broken and the Lorentz transformations preserve the ‘vertical’ vector \( ξ \). Thus, since \( ξ \) is spacelike, \( Λ \) sits in \( o(2, 1) \).

In conclusion, the residual symmetry is spanned by

\[ Z^μ = Λ^μ_ν x^ν + Γ^μ \quad \text{with} \quad Λ^μ_ν ξ^ν = 0. \tag{8.18} \]

These vector fields form the 7-dimensional Lie algebra \( o(2, 1) \circledast \mathbb{R}^{3,1} \) which is \( e(2, 1) \times \mathbb{R} \).

We thus contend that the system (8.13–15) admits the relativistic symmetry (8.16).
Relation to the work of Cho et al.

In order to exhibit the relation of our relativistic system to that of Ref. [13], introduce the new coordinate system

$$\tilde{t} = t, \quad \tilde{x} = y, \quad \tilde{y} = -x, \quad \tilde{z} = s/c + ct$$

(8.19)

in terms of which the metric (8.6) and the vertical vector $\xi$, Eq. (8.7), respectively read

$$d\ell^2 = -c^2 d\tilde{t}^2 + d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2$$

and

$$\xi = \frac{1}{c} \frac{\partial}{\partial \tilde{z}},$$

(8.20)

and correspond precisely to those used in Ref. [13] (see Eqs (8.1,2)). The associated metric on $\mathbb{R}^{2,1} = \mathbb{R}^{3,1}/\mathbb{R}\xi$, which is again (2+1)-dimensional Minkowski spacetime, is

$$-c^2 d\tilde{t}^2 + d\tilde{x}^2 + d\tilde{y}^2.$$

The Dirac matrices in this new coordinate system are given by

$$\tilde{\gamma}^t = \begin{pmatrix} 0 & c^{-2} \\ 1 & 0 \end{pmatrix}, \quad \tilde{\gamma}^x = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \quad \tilde{\gamma}^y = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}, \quad \tilde{\gamma}^z = \begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix},$$

(8.21)

and the chirality operator remains unchanged (compare Eq. (8.10)),

$$\tilde{\Gamma} = \begin{pmatrix} -i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}. \quad (8.22)$$

Now, the key point is that our Dirac-CS system (2.8) — with (8.5) — is intrinsic; as such, it can be projected upon (2+1)-dimensional spacetime using the equivariance relation (8.11) and the fact that the CS field strength is basic, independently of any coordinate system. The former reads $\tilde{\xi}^\mu \tilde{D}_\mu \tilde{\psi} = im\tilde{\psi}$ (where $\tilde{D}_\mu \equiv \partial/\partial \tilde{x}^\mu - ie \tilde{a}_\mu$), whence

$$\tilde{\psi}(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}) = e^{imc\tilde{z}} \tilde{\Psi}(\tilde{t}, \tilde{x}, \tilde{y})$$

(8.23)

in a gauge where $\tilde{a}_\mu d\tilde{x}^\mu$ is basic and, in view of (8.12,22),

$$\tilde{\Psi} = \begin{pmatrix} \tilde{\Phi}_+ \\ \tilde{\Phi}_- \\ \tilde{\lambda}_- \\ \tilde{\lambda}_+ \end{pmatrix}. \quad (8.24)$$
The gauged, massless, Dirac equation $\tilde{\gamma}^\mu \tilde{D}_\mu \tilde{\psi} = 0$ then yields the couple of equations

$$\begin{pmatrix} \tilde{D}_t + imc & \tilde{D}_1 - i\tilde{D}_2 \\ -(\tilde{D}_1 + i\tilde{D}_2) & -c^{-2} \tilde{D}_t + imc^{-1} \end{pmatrix} \begin{pmatrix} \tilde{\Phi}_+ \\ \tilde{\chi}_+ \end{pmatrix} = 0 \tag{8.25}$$

and

$$\begin{pmatrix} -c^{-2} \tilde{D}_t + imc^{-1} & \tilde{D}_1 - i\tilde{D}_2 \\ -(\tilde{D}_1 + i\tilde{D}_2) & \tilde{D}_t + imc \end{pmatrix} \begin{pmatrix} \tilde{\chi}_- \\ \tilde{\Phi}_- \end{pmatrix} = 0 \tag{8.26}$$

Upon putting $c = 1$ and defining

$$\tilde{\Psi}_+ = \begin{pmatrix} \tilde{\Phi}_+ \\ \tilde{\chi}_+ \end{pmatrix} \quad \text{and} \quad \tilde{\Psi}_- = \begin{pmatrix} \tilde{\chi}_- \\ \tilde{\Phi}_- \end{pmatrix}, \tag{8.27}$$

this system, Eqs (8.25,26), is interestingly rewritten as

$$(i\tilde{\gamma}_\pm^\alpha \tilde{D}_\alpha - m) \tilde{\Psi}_\pm = 0 \tag{8.28}$$

where $\alpha = 0, 1, 2$ and $(\tilde{\gamma}_\pm^\alpha) = (\pm \sigma^3, i\sigma^2, -i\sigma^1)$, i.e. precisely the equations of Cho et al. with the same chiral masses, $p_\pm = mc$ (no parity violation) and with $M = 0$ (see (8.3)).

At last, tedious but straightforward calculations confirm that our projected current

$$\tilde{J}^\alpha = \overline{\tilde{\Psi}} \tilde{\gamma}_\pm^\alpha \tilde{\Psi} \tag{8.29}$$

(see (8.15)) actually reproduces the current which appears in (8.4),

$$\tilde{J}^\alpha = \overline{\tilde{\Psi}}_+ \tilde{\gamma}_+^\alpha \tilde{\Psi}_+ + \overline{\tilde{\Psi}}_- \tilde{\gamma}^-_\alpha \tilde{\Psi}_-, \tag{8.30}$$

where is is understood that the hermitian structures on the chiral spinors (8.27) in this last expression are respectively $G_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $G_- = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

The non-relativistic limit

The coordinate system (8.6) is particularly suited for studying the non-relativistic limit $c \to +\infty$ of the previous system. In these coordinates, the constant vector field $\xi$ remains unchanged when $c \to +\infty$, but tends to a null vector as the metric tends to the
canonical flat Bargmann metric $2dt ds + dx^2 + dy^2$. The Dirac matrices also tend smoothly to their lightlike form (2.9) used in the non-relativistic theory. Thus, both the massless Dirac equation and the equivariance constraint reduce to the ones we used in Sec. 2.

Our system is hence the non-relativistic limit of that of Cho et al. To explain this in slightly different way, observe that, in terms of the projected quantities, our transformation formula (8.23) relates Cho et al.’s spinor fields to ours, via the familiar high frequency oscillating factor

$$\tilde{\Psi}(\tilde{t}, \tilde{x}, \tilde{y}) = e^{-imc^2\tilde{t}} \Psi(t, x, y).$$

This results in transforming Eqs (8.25,26) into ours, Eq. (8.13). Then taking the non-relativistic limit amounts simply to dropping the term $c^{-2} D_t \chi$, which leaves us with the Lévy-Leblond equation (1.4).

Also, the 7-dimensional relativistic symmetry algebra $e(2,1) \times \mathbb{R}$ (see (8.16)) gets enlarged into the 9-dimensional ‘Schrödinger’ algebra (4.1). In fact, for $\xi$ lightlike, one special conformal transformation remains unbroken: $K^\mu = a\xi^\mu$ yields precisely the expansion $a \in \mathbb{R}$. Similarly, in the relation $\Lambda^\mu_\nu \xi^\nu + \alpha \xi^\mu = 0$ the dilation $\alpha \in \mathbb{R}$ remains arbitrary. The relativistic dilation combines, hence, with a Lorentz transformation to yield the non-relativistic dilation in Eq. (4.1).

Let us stress that, in the limit $c \to +\infty$, all geometric structures and equations remain well behaved on $\mathbb{R}^{3,1}$ — unlike on the (2+1)-dimensional quotient where singularities occur.

**Self-duality**

Relativistic, purely-magnetic fermionic vortex solutions were constructed by Cho et al. in Ref. [13]. Remarkably, their solutions are again associated with the Liouville equation, just as in (6.5). Let us finish this Section by explaining how this comes about. Observe that, for static and purely magnetic fields

$$D_t \Psi = (\partial_t - ieA_t) \Psi = 0.$$
Then the gauged, massless, Dirac equation (8.13) and the associated SD equations retain the same form in either the relativistic or non-relativistic regime, because the term involving $c^{-2}$ vanishes along with $D_t \chi$.

9. The Lévy-Leblond equation in higher dimensions

Our non relativistic Kaluza-Klein framework can also be applied to derive the field equations of the non relativistic spinor fields in $3 + 1$ dimensions. One starts with a 5-dimensional Bargmann space, $M$. In 5 dimensions the spinor fields, $\psi$, have $2^{[5/2]} = 4$ components, and in the absence of a non trivial analogue of the chirality operator, $\Gamma$, they form — even in the massless case — a single irreducible representation of $\text{o}(4, 1)$. \(^{(3)}\) Since the FCI, Eq. (1.9), have no obvious analogue in 5 dimensions, $\psi$ can only couple to the ‘standard’ non-relativistic Maxwell electrodynamics, described, e.g., in [9]. If $M$ is flat Minkowski space with metric $\sum_{j=1}^{3} (dx^j)^2 + 2 dt ds$, then we can choose the Dirac matrices

$$
\gamma^t = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} -i\sigma^j & 0 \\ 0 & i\sigma^j \end{pmatrix}, \quad \gamma^s = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}
$$

\hspace{1cm} (9.1)

with $j = 1, \ldots, 3$. Using these $\gamma^\mu$ in the massless Dirac equation $\not{\partial} \psi = 0$, together with the mass constraint, Eq. (2.6), leads to a system having the same form as Eq. (2.11), the only difference being is that now $\hat{\mathbf{\sigma}} \cdot \hat{\mathbf{D}} = \sum_{j=1}^{3} \sigma^j D_j$. This system is identical to that proposed by Lévy-Leblond [8].

The absence of the analogue of $\gamma^5$ implies that this reduced system does not now split into two independent spinor equations, although the $\Phi$ part can again be solved separately.

The fact that $M$ has now 5 dimensions changes slightly the discussion of conformal invariance and the description of the spinor representation of the Schrödinger group. In particular

$$
\hat{\not{\partial}} \hat{\psi} = \Omega^{-3} \hat{\not{\partial}} \psi \quad \text{with} \quad \hat{\psi} = \Omega^{-2} \psi,
$$

\hspace{1cm} (9.2)

\(^{(3)}\) This $\text{o}(4, 1)$ is the Lie algebra of the symmetry group of the tangent space of $M$. It is a subalgebra of the $\text{o}(5, 2)$ Lie algebra formed by the infinitesimal conformal transformations of $M$. 

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so that the current scales as \( j^\mu \to \tilde{j}^\mu = \Omega^{-5} j^\mu \). Also, the coefficient of the last term in Eq. (4.5) changes from 3/8 to 7/16. As a consequence, some numerical factors in the expressions of \( d \) and \( K_\mu \) in (4.6) change (e.g. the 3/2 in \( d \) changes to 35/16), but basically this equation (with \( \mu \) running over \( t, s \) and 1, 2, 3) determines the action of the generators of \( o(5,2) \) on the solutions of \( \hat{D}\psi = 0 \). Naively, this seems to be a contradiction, since \( o(5,2) \) has no four-dimensional representation. Indeed, since its fundamental representation is seven-dimensional, the only candidate is its eight-dimensional spinor representation, but it is irreducible even in the massless case, thus it cannot split into two four dimensional ones. However, we note that (4.6) implies that these \( o(5,2) \) generators act as matrix differential operators on \( \psi \), i.e. the representation given is not a matrix representation. \(^{(4)}\) Put in a slightly different way, the solution space of the massless Dirac operator in 5 dimensions is \( 4 \times \) infinite dimensional.

10. Miscellany and discussion

As explained in Ref. [6], a Maxwell term can be consistently included into the FCI and Eq. (1.9) becomes thus modified, as

\[
\sqrt{-g} \epsilon_{\mu\nu\rho\sigma} \xi^\rho \nabla_\tau f^{\tau\sigma} + \kappa f_{\mu\nu} = e \sqrt{-g} \epsilon_{\mu\nu\rho\sigma} \xi^\rho j^\sigma \quad \text{and} \quad \partial_{[\mu} f_{\nu\rho]} = 0 \quad (10.1)
\]

(all indices running from 0 to 3). This equation descends, as before, to the quotient, \( Q \).

In the Minkowski case, for example, it reads

\[
\kappa B = -e g, \quad \partial_i B + \kappa E_i = e \epsilon_{ij} J^j \quad \text{and} \quad \vec{\nabla} \times \vec{E} + \partial_t B = 0, \quad (10.2)
\]

generalizing the FCI (1.1) in the non-relativistic framework. Clearly, the new system, Eqs (10.2) and (1.4, 5), is still invariant with respect to the \( \xi \)-preserving conformal transformations: the RHS of Eq. (10.1) is plainly invariant under the rescalings \( g_{\mu\nu} \to \Omega^2 g_{\mu\nu} \).

\(^{(4)}\) The same remark applies to the representation obtained in Sec. 4 in the case of four-dimensional Minkowski space: imposing the chiral projection on \( \psi \), we get a 2-dimensional representation, though \( o(4,2) \) has no 2-dimensional matrix representation.
and $\xi^\mu \to \xi^\mu$ as a short calculation shows that the current scales as $j^\mu \to \Omega^{-4} j^\mu$, which is precisely compensated for by $\sqrt{-g} \to \Omega^4 \sqrt{-g}$. Now, in $n$ dimensions, the Maxwell term scales as

$$\nabla_\tau f^{\tau\sigma} \to \Omega^{-4} \nabla_\tau f^{\tau\sigma} + (n - 4) \Omega^{-3} f^{\tau\sigma} \partial_\tau \Omega,$$

(10.3)

with the sought conformal weight (the last term vanishes here, as $n = 3 + 1$). Thus the LHS of Eq. (10.1) is also invariant. So, the combined LL-Maxwell-CS system is, indeed, Schrödinger symmetric.

It is worth mentioning that the non-relativistic conformal invariance is maintained when modifying the massless Dirac equation (2.2) by an ‘anomalous’ term,

$$\mathcal{D}\psi = ik \frac{\sqrt{-g}}{2} \epsilon_{\mu\nu\rho\sigma} \gamma^{\mu} \xi^{\nu} f^{\rho\sigma} \psi,$$

(10.4)

where $k = \text{const}$, because both sides scale with the weight $\Omega^{-5/2}$. This amounts to modifying, in the previous developments, the covariant derivative according to

$$D_\mu \to D_\mu - ik \frac{\sqrt{-g}}{2} \epsilon_{\mu\nu\rho\sigma} \xi^{\nu} f^{\rho\sigma},$$

(10.5)

yielding a non-relativistic version of the model studied in Ref. [21].

A peculiar feature of our non-relativistic ‘Kaluza-Klein’ approach to Chern-Simons theory is that it lacks an action principle for deriving the four-dimensional form of the FCI, Eq. (1.9). This difficulty is a consequence of the null character of the fibration over Galilei spacetime; it forced us to deal essentially with the field equations themselves in the study of the coupled LL-CS system. It is worth noticing that, on the other hand, Carroll, Field and Jackiw [22] have discovered a four-dimensional version of Chern-Simons theory admitting an action principle in the relativistic case, which is governed by a space-like fibration discussed in Sec. 8 in connection with relativistic spinor vortices.

Let us finally mention that our non-relativistic spinor vortices studied in Secs 6 and 7 turn out to be different from those found before by Leblanc et al. [23], which are rather embedded scalar solutions.
Appendix A

Consider a 4-dimensional spin manifold \((M, g)\) with Clifford relations \(\gamma^{(\mu} \gamma^{\nu)} + g^{\mu\nu} = 0\) and denote \(\nabla \psi\) the covariant derivative of a spinor field \(\psi\), Eq. (2.1). The spin-curvature, defined by

\[
\nabla_\mu \nabla_\nu \psi - \nabla_\nu \nabla_\mu \psi = \gamma(R)_{\mu\nu} \psi,
\]

reads

\[
\gamma(R)_{\mu\nu} \equiv \frac{1}{4} \gamma^\alpha \gamma^\beta R_{\alpha\beta\mu\nu}.
\]  

(We use the convention \(R_{\alpha\beta\gamma\delta} = \partial_\alpha \Gamma^\lambda_{\beta\mu} + \cdots\).) A tedious calculation (using the Bianchi identities and the symmetries of the Riemann tensor) leads to \(R_{\mu\nu\alpha\beta} \gamma^\nu \gamma^\alpha \gamma^\beta = -2 R_{\mu\nu} \gamma^\nu\) where the Ricci tensor is defined by \(R_{\beta\mu} = R_{\alpha\beta\mu\nu} g^{\alpha\nu}\), hence, in view of Eq. (A.2) to the useful formula

\[
\gamma^\mu \gamma(R)_{\mu\nu} = \frac{1}{2} \gamma^\mu R_{\mu\nu}.
\]

Recall the definition of the Lie derivative of a spinor field, \(\psi\), with respect to a vector field, \(X\), Eq. (3.2),

\[
L_X \psi \equiv \nabla_X \psi - \frac{1}{2} \gamma(dX)\psi
\]

where we have put \(\overline{X} \equiv g(X, \cdot)\) as a shorthand notation and used the convention (A.2). From now on assume that \(X\) be an infinitesimal conformal transformation, i.e.

\[
L_X g_{\mu\nu} = 2 \nabla_{(\mu} X_{\nu)} = k g_{\mu\nu}
\]

(with \(k = \frac{1}{2} \nabla_\mu X^\mu\)). Calling \(\nabla \equiv \gamma^\mu \nabla_\mu\) the Dirac operator, we find, with the help of Eqs (A.4) and (A.1), that

\[
[L_X, \nabla] \psi = X^\nu \nabla_\nu (\gamma^\mu \nabla_\mu \psi) - \frac{1}{2} \gamma(d\overline{X}) \gamma^\mu \nabla_\mu \psi - \gamma^\mu \nabla_\mu (X^\nu \nabla_\nu \psi) + \frac{1}{2} \gamma^\mu \nabla_\mu (\gamma(d\overline{X}) \psi) =
\]

\[
X^\nu \gamma^\mu \gamma(R)_{\nu\mu} \psi + \frac{1}{8} [\gamma^\mu, [\gamma^\alpha, \gamma^\beta]] (\nabla_\alpha X_\beta) \nabla_\mu \psi - \gamma^\mu (\nabla_\mu X^\nu) \nabla_\nu \psi + \frac{1}{8} \gamma^\mu [\gamma^\alpha, \gamma^\beta] (\nabla_\mu \nabla_\alpha X_\beta) \psi.
\]

We then compute separately each term in this last expression. By using

\[
[\gamma^\mu, [\gamma^\alpha, \gamma^\beta]] = 8 \gamma^\mu [\alpha]\cdot g^{\beta\mu}
\]
together with the partial result
\begin{equation}
\gamma^\mu [\gamma^\alpha, \gamma^\beta] \nabla_\mu \nabla_\alpha X_\beta = -2 \nabla^\mu \nabla_\mu X_\nu \gamma^\nu - 4 \gamma (R)_{\mu\nu} \gamma^\mu X^\nu + 4 \gamma^\mu \partial_\mu k
\end{equation}

and (taking the divergence of both sides of Eq. (A.5))
\begin{equation}
\nabla^\nu \nabla_\nu X_\mu + R_{\mu\nu} X^\nu = -\partial_\mu k
\end{equation}

one finds, using Eq. (A.3), that
\begin{equation}
[L_X, \nabla] \psi = -\frac{1}{2} R_{\mu\nu} \gamma^\mu X^\nu \psi - \frac{1}{2} \gamma^\nu (\nabla_\mu X_\nu + \nabla_\nu X_\mu) \nabla^\mu \psi + \frac{1}{4} (\partial_\mu k + R_{\mu\nu} X^\nu) \gamma^\mu \psi + \frac{1}{4} R_{\mu\nu} \gamma^\mu X^\nu \psi + \frac{1}{2} \partial_\mu k \gamma^\mu \psi
\end{equation}

and, finally,
\begin{equation}
[L_X, \nabla] \psi = -\frac{1}{2} k \nabla \psi + \frac{3}{4} \gamma (dk) \psi \tag{A.7}
\end{equation}

where \( \gamma (dk) \equiv \gamma^\mu \partial_\mu k \). Notice the disappearance of the terms involving the curvature in this last expression.
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