ADAPTATION AND NONLINEAR PARAMETRIZATION: NONLINEAR DYNAMICS PROSPECTIVE

Ivan Tyukin * Cees van Leeuwen *

* Lab. for Perceptual Dynamics, RIKEN Brain Science Institute, 2-1, Hirosawa, Wako-shi, Saitama, Japan

Abstract: We consider adaptive control problem in presence of nonlinear parametrization of uncertainties in the model. It is shown that despite traditional approaches require for domination in the control loop during adaptation, it is not often necessary to use such energy inefficient compensators it in wide range of applications. In particular, we show that recently introduced adaptive control algorithms in finite form which are applicable to monotonic parameterized systems can be extended to general smooth non-monotonic parametrization. These schemes do not require any damping or domination in control inputs.

Keywords: adaptive systems, nonlinear parameterization, finite form algorithms, convergence, nonlinear persistent excitation

1. INTRODUCTION

The standard techniques to address nonlinear parameterization of the uncertainty in adaptive control literature involve applying domination functions (see, for instance, (Marino and Tomei, 1993; Lin and Qian, 2002a; Lin and Qian, 2002b)) or damping of the unknown nonlinearity (Loh et al., 1999). Both operate on the principle of invoking auxiliary crutches ensuring Lyapunov stability of the extended system. These methods yield effective analytical solutions to the problem of adaptation in presence of nonlinear parameterized uncertainties.

Yet, in a wide range of applications in control and reverse bio-engineering, physics and biology, it is desirable to refrain from domination or damping. The issue of non-dominating adaptation is critical in control engineering applications where systematic overshooting in control results in fast wearing off of the actuators, waste of energy and undesired chattering. A typical example is traction/braking control under unknown tire-road conditions. Tire-road conditions enter into the equations describing the slip dynamics as uncertainties that are highly nonlinear in their parameters (Pacejka and Bakker, 1993; Canudas de Wit and Tsiotras, 1999). The issue is how under these conditions the slip could be effectively controlled during braking without applying unnecessary large braking/trackgin torques.

In bio-engineering, the motivation to use non-conventional adaptation mechanisms is even more stronger. In these systems physiological considerations motivate local adaptation, at the level of the single node (Brenner et al., 2000),(Webster et al., 2002) (visual system, cognitive processing e.t.c.). Even simplest mathematical models of these nodes usually are nonlinearly parameterized (Dayan and Abbott, 2001). Due to the necessity to respond sensitively to different stimulation, use of external domination is not desirable. On the other hand, because the number of interacting nodes is large, the adaptation, should be non-dominating in order to avoid inefficient consumption of energy.
From control-theoretic prospective, perhaps, the most challenging task is to find the principles, which will allow to control behavior of nonlinear systems by gentle and non-dominating parametric adaptation. Available non-dominating solutions, however, are either local (Karsenti et al., 1996), or assume monotonic parametrization of uncertainties (Tyukin, 2003; Tyukin et al., 2003b). Therefore, new approaches are needed in order to design adaptive algorithms capable of ensuring asymptotic reaching of the control goal for large class of nonlinear parameterized systems.

Finding gentle solutions to control problems with nonlinear uncertainties will involve a change in design methodology. Conventional design methods in adaptive control theory often favor Lyapunov-based methodology. For many decades Lyapunov methods were successfully applied in design and analysis of nonlinear and adaptive systems (see, for example, (Isidori, 1989; Sastry, 1999; Narendra and Annaswamy, 1989; Miroshnik et al., 1999)). Furthermore, as a rule of thumb, Lyapunov stability serves as a measure of acceptable performance in mechanics and engineering simply for its formulation guaranteeing small deviations from the equilibrium in case of small perturbations in initial conditions.

When adaptive control is required, however, deviations of the unknown parameters are likely to be large in order for the adaptation to make practical sense. Therefore, requirement of Lyapunov stability for systems with general parametrization does not seem to be necessary unless it provides certain advantages in specific applications in addition to mere asymptotic reaching of the control goal.

On the other hand, adaptation processes in many physical and biological phenomena are far from being stable in conventional sense. Examples of such non-stable adaptation include (but are not limited to) high-frequency oscillation of the eye (tremor) that allows us to see static pictures better (Ditchburn and L., 1952; Martinez-Conde et al., 2004), perception of the ambiguous figures (perceptual switches) (Ito et al., 2003), evolution in social and ecological systems (Sole et al., 1999; Bak and Sneppen, 1993). This motivates us to seek for replacement of conventional Lyapunov stability-based methodology for dealing with problems of adaptation in nonlinear systems.

Doing so would enable, in principle, a specter of entirely new applications and associated problems is emerging. Theoretical findings in physics and biology such as intermittent synchronization (Gauthier and C., 1996; Kaneko and Tsuda, 2000; Kaneko, 1994; Kaneko and Tsuda, 2003), homeostasis in the living cells stability (see (Moreau and Sontag, 2003) and references therein) and, last but not least, self-organized criticality - phenomena often observed in the earthquakes (Bak et al., 2002), in the neuronal activity of the human brain (Beggs and Plenz, 2003) which can be modeled by the arrays of phase oscillators with nonlinear driving (Corral et al., 1995) show the importance of systems on the boundary of stability. Understanding the principles of learning and adaptation in such systems could eventually benefit greatly from the principles of adaptive control.

We start with brief analysis of design strategies for adaptive systems with nonlinear parametrization. We show that that certainty-equivalence adaptive control inevitably requires damping or domination functions to guarantee global stability in presence of general nonlinear parametrization. In order to avoid these difficulties we replace requirement of Lyapunov stability of the system with mere reaching a neighborhood of the goal manifold. After the new control goal is defined we analyze a class of nonlinear systems where uncertainties are given by one-dimensional nonlinear parameterized function. For this class of systems we provide adaptive control algorithms capable of steering the system state to a small neighborhood of the target manifold. The control algorithm should not involve neither domination function, nor should it require for additional damping unknown nonlinearity. After this step is accomplished we extend our approach to multidimensional parameterizations.

Throughout the paper we will use the following notations. Symbol $x(t, x_0, t_0)$ denotes solution of a system of differential equations starting at the point $x_0$ at time instant $t_0$; symbol $C^r$ denotes the space of $r$ times differentiable functions; symbol $\mathbb{R}$ stand for the space of reals; $\mathbb{R}_+$ defines non-negative real numbers, symbol $\mathbb{I}m$ denotes image of the map. We say that $\nu : \mathbb{R}_+ \to \mathbb{R}$ belongs to $L_2$ iff $L_2(\nu) = \int_0^\infty \nu^2(\tau) d\tau < \infty$. The value $\|\nu\|_2 = \sqrt{L_2(\nu)}$ stands for the $L_2$ norm of $\nu(t)$. Function $\nu : \mathbb{R}_+ \to \mathbb{R}$ belongs to and $L_\infty$ iff $L_\infty(\nu) = \sup_{t \geq 0} ||\nu(t)|| < \infty$, where $|| \cdot ||$ is the Euclidean norm. The value of $\|\nu\|_\infty = L_\infty(\nu)$ stands for the $L_\infty$ norm of $\nu(t)$. Symbol $U_t(x)$ denotes the set of all $x' : \|x - x'\| \leq \epsilon$. Let $X \subset \mathbb{R}^n$, distance $\text{dist}(x, X) = \inf_{x' \in X} \|x - x'\|$. Symbol $U_t(X)$ denotes the set $\{x' \in \mathbb{R}^n | x' : \text{dist}(x', X) \leq \epsilon\}$. Symbol $L_t \psi$ stands for the Lie-derivative of function $\psi(x)$ w.r.t. the vector field $f(x)$.

The paper is organized as follows. Section 2 contain preliminary notations and statement of the problem. In Section 3 we provide main results of the paper. Section 4 concludes the paper.
2. PROBLEM FORMULATION AND PRELIMINARIES

Let us consider standard adaptive control problem where the uncertain system be given as follows

\[ \dot{x} = f(x, \theta) + g(x)u, \ x_0 \in \Omega_x \]  

(1)

where \( x \in \mathbb{R}^n \) is state vector, \( f : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n, \ g : \mathbb{R}^n \to \mathbb{R}^n \) are \( C^1 \)-smooth vector-fields, \( \theta \in \Omega_\theta \subset \mathbb{R}^d \) is vector of parameters, \( u \) is control input, and \( \Omega_x \subset \mathbb{R}^n \) is the set of initial conditions \( x_0 \). Functions \( f, g \) are known, vector of parameters \( \theta \) is assumed to be unknown a-priori. We also assume that \( g(x) \) is bounded w.r.t. \( x \).

Let the control goal be to reach asymptotically a neighborhood of the following manifold given implicitly by

\[ \psi(x) = 0, \ \psi \in C^2 \]  

(2)

In addition to (2) we will require that

\[ \psi(x(t)) \in L_\infty \Rightarrow x(t) \in L_\infty \]  

(3)

This requirement ensures that any bounded deviation from the goal manifold does not result in unbounded growth of the norm \( \|x\| \). In addition we assume that \( |L_g(x)\psi(x)| \geq \delta_0 > 0 \).

Let us select the class of admissible control functions which, allowing for the goal to be reached, can compromise between performance and domination issues. The most natural way would be to define this class on the ground of certainty-equivalence principle. In particular, consider control function

\[ u(x, \theta) = (L_g(x)\psi(x))^{-1}(-L_f(x, \theta)\psi(x) - \varphi(\psi) + v(t)) \]  

(4)

which transforms the system (1) into the following error model

\[ \dot{\psi} = f(x, \theta) - f(x, \hat{\theta}) - \varphi(\psi) + v(t), \]  

(5)

where \( f(x, \theta) = L_f(x, \theta)\psi(x), \ \varphi \in C_\varphi \subset C^0 : \mathbb{R} \to \mathbb{R} \) and \( \varphi(\psi) > 0 \ \forall \ \psi \neq 0, \ v, d : \mathbb{R}_+ \to \mathbb{R} \). The desired dynamics (performance) of the system is given by equation \( \dot{\psi} = -\varphi(\psi) \), and \( v(t) \) stands for auxiliary control or disturbances depending on the context.

**Definition 1.** Adaptive control law (4) is called non-dominating in class \( C_\varphi \) if for any \( \epsilon > 0 \) there exists such function \( \hat{\theta}(x, t, \delta(\epsilon)) \in \Omega_\theta, \ \delta(\epsilon) : \mathbb{R}_+ \to \mathbb{R}_+ \) and, time \( t^* \) that \( |\psi(x(t, x_0, t_0))| < \epsilon \) for any \( t \geq t^*, \ x_0 \in \Omega_x, \ \theta \in \Omega_\theta, \ \varphi \in C_\varphi \) and \( v(t) \equiv 0 \).

In the present study we will restrict class \( C_\varphi \) to the following class of functions:

\[ C_\varphi(k) = \{ \varphi : \mathbb{R} \to \mathbb{R} | \varphi \in C^1, \ \varphi(\varphi(\psi)) \geq \psi^2k, \ k \in \mathbb{R}_+ \} \]  

(6)

The fact that adaptation is non-dominating in this class of functions means that for any arbitrary small gain \( k > 0 \) in feedback \( \varphi(\psi) = k\psi \) there exists function \( \hat{\theta}(x, t, \delta) \) such that control goal is reached in finite time. In order to specify desired performance of the adaptive algorithm itself we require that \( \hat{\theta}(x, t, \delta) \) does not change along the manifold \( f(x, \theta) - f(x, \hat{\theta}) = 0 \) and norm \( \|\theta - \hat{\theta}\| \) does not grow with time.

In conventional certainty-equivalence adaptive control the problem of adaptation is usually viewed as the problem of design of operator \( A(\psi, x, \hat{\theta}, t) \) such that solutions of

\[ \dot{\hat{\theta}} = A(\psi, x, \hat{\theta}, t) \]  

(7)

together with (4) ensure the goal relation (2).

Function \( A(\psi, x, \hat{\theta}, t) \), in addition, neither should depend on unknown \( \theta \), nor it should require derivative \( \dot{x} \). The standard way to solve this problem for \( v(t) \equiv 0 \) is to design function \( A(\psi, x, \hat{\theta}, t) \) such that \( \psi(x) = 0 \), \( \hat{\theta} = \theta \) is stable manifold in Lyapunov sense.

In general nonlinear setup this condition, however, is hardly ever met for every \( \theta \in \Omega_\theta \) and \( x(t_0) \in \mathbb{R}^n \). To show this it is enough to decompose system (1)–(7) into the following form

\[ \begin{pmatrix} \dot{\psi} \\ \dot{\hat{\theta}} \end{pmatrix} = \begin{pmatrix} -\varphi(\psi) \\ A(\psi, x, \theta, \hat{\theta}) \end{pmatrix} \begin{pmatrix} \psi \\ \hat{\theta} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v(t), \ \hat{\theta} = \theta - \hat{\theta} \]  

(8)

where functions \( \varphi, F(x, \theta, \hat{\theta}), A(\psi, x, \theta, \hat{\theta}) \) follow from Hadamard lemma. Unlike in linear parametrization case, explicit dependance of function \( F(x, \theta, \hat{\theta}) \) on unknown \( \theta \) does not allow to compensate for uncertainty by choosing appropriate function \( A(\psi, x, \theta, \hat{\theta}) \) in (8). Therefore, in general, it is necessary to use additional control input \( v(t) \) in order to ensure that system (8) is Lyapunov stable and that \( \psi(x) \to 0 \) as \( t \to \infty \). Success of this strategy is reported in (Loh et al., 1999), (Lin and Qian, 2002a), (Lin and Qian, 2002b).

One step forward towards obtaining non-dominating adaptive control would be to reformulate the problem as follows: design function \( \hat{\theta}(x, t) \) such that either

\[ \lim_{t \to \infty} f(x, \theta) - f(x, \hat{\theta}(x, t)) = 0, \]  

(9)

or

\[ f(x, \theta) - f(x, \hat{\theta}(x, t)) \in L_2 \]  

(10)

hold. One possible way to achieve goal (9), (10) is to use information about the difference

\[ \begin{pmatrix} f(x, \theta) \\ f(x, \hat{\theta}(x, t)) \end{pmatrix} = \int_0^t \frac{\partial F(x, \theta, \hat{\theta}(x, t), t)}{\partial x} \psi \, dt + \varphi, \ \varphi = \int_0^t \frac{\partial A(x, \theta, \hat{\theta}(x, t), t)}{\partial x} \psi \, dt \]  

\[ A(\psi, x, \theta, \hat{\theta}) = \int_0^t \frac{DA(\psi, x, \theta, \hat{\theta})}{\partial x} \psi \, dt \]  

(11)

(12)

1. In particular, these functions can be calculated as follows

2. Similar idea was proposed in (Ortega et al., 2002) as adaptive “root-searching” strategy
\[ f(x, \theta) - f(x, \hat{\theta}(x, t)) \] explicitly in the adaptation algorithm rather than using external control \( v(t) \).

For a class of nonlinear parameterized systems this additional information can be introduced into adjustment schemes by mere structural changes in the adjustment law. In particular it is suggested in (Tyukin et al., 2003) to use adaptive algorithms in differential-integral or finite form instead of using adaptive algorithms in differential form. Extended system in this case can be described as follows:

\[
\begin{pmatrix}
\dot{\psi} \\
\dot{\theta}
\end{pmatrix} = \left( -\phi \psi \quad F(x, \theta, \hat{\theta}) \quad 0 - F(x, \theta, \hat{\theta}) \alpha(x, t) \right) \begin{pmatrix}
\psi \\
\theta
\end{pmatrix},
\]

where \( F(x, \theta, \hat{\theta}) \alpha(x, t) \) is positive semi-definite time-varying matrix. Sufficient conditions for existence of such algorithms are given in (Tyukin et al., 2003a), (Tyukin et al., 2004), where the problem is reduced to solution of specific partial differential equation (or explicit realizability condition) by embedding the system into the one of the higher order. While solution to this problem was shown to exist for wide range of functions \( \alpha(x, t) \) and \( \psi(x) \), it is not always possible to guarantee that \( F(x, \theta, \hat{\theta}) \alpha(x, t) \) is positive semi-definite for arbitrary \( f(x, \theta) \) and any \( \theta \in \Omega_0, x(t_0) \in \mathbb{R}^n \). As a result of that Lyapunov stability of the whole system becomes problematic, if not at all impossible, for general nonlinear parameterizations.

These observations suggest that in both conventional problem statement (5), (7) and nonconventional ones (8), (11) ensuring Lyapunov stability for adaptive control system in case of nonlinear parametrization inevitably leads either to domination of the nonlinearity or to additional restrictions on the class of nonlinear parameterizations. In the other words, the problem of Lyapunov stable and non-dominating adaptive control is ill-posed in general. As a candidate for replacement of Lyapunov stability one could think of the set attractivity (Guckenheimer and Holmes, 2002; Milnor, 1985) of the goal manifold. This concept allows us to design systems which being unstable in Lyapunov sense have bounded solutions and also are capable of reaching the goal asymptotically. The main problem with this concept, however, is that there is a number of conditions to check which critically depend on precise knowledge of the vector fields of adaptive system. This knowledge includes in particular properties of yet unknown function \( \theta(x, t, \delta) \).

Therefore perhaps the most reasonable and the least demanding concept of the control goals would be the notion of \( \omega \)-limit set

**Definition 2.** A point \( p \in \mathbb{R}^n \) is called an \( \omega \)-limit point \( \omega(x(t, x_0, t_0)) \) of \( x_0 \in \mathbb{R}^n \) if there exists sequence \( \{ t_i \} \), \( t_i \to \infty \), such that \( x(t, x_0, t_0) \to p \).

The set of all limit points \( \omega(x(t, x_0, t_0)) \) is the \( \omega \)-limit set of \( x_0 \).

Let, therefore, the control goal be to ensure that for some positive \( \varepsilon \) the set

\[
\Omega_\varepsilon(\varepsilon) = \{ x \in \mathbb{R}^n | |\psi(x)| \leq \varepsilon \}
\]

contains the \( \omega \)-limit set of \( \Omega_\varepsilon \) for non-autonomous system (1), (4) with \( v(t) \equiv 0 \). Hence, the main question of our current study is the following: is there exists a non-dominating adaptive scheme and reasonably large class of parameterizations of the uncertainty such that all \( \Omega_\varepsilon(\varepsilon) \) contains the \( \omega \)-limit set of the adaptive system for any arbitrary small \( \varepsilon > 0 \), all trajectories of the system are bounded, and volume of the domain of uncertainty is decreasing with time? The answer to this question is provided in the next section.

3. MAIN RESULTS

To begin with, let us consider the case where function \( f(\cdot, \cdot) \) is parameterized by scalar \( \theta \in \Omega_0 = [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}, \underline{\theta} < \bar{\theta} \). For each \( \theta \in \Omega_\theta \) and nonnegative \( \Delta \in \mathbb{R}_{\geq 0} \) we introduce the following equivalence relation

\[
\theta \sim \theta' \Leftrightarrow |f(x, \theta) - f(x, \theta')| \leq \Delta \forall x \in \mathbb{R}^n
\]

and corresponding equivalence classes \( [\theta]_\Delta = \{ \theta' \in \Omega_\theta | \theta \sim \theta' \} \).

For the given functions \( \varphi(\psi) \) and \( \alpha(t) : \mathbb{R}_+ \to \mathbb{R} \), \( \alpha(t) \in C^1 \) let us define the following function

\[
S_\delta(\varphi(\psi), \alpha(t)) = \begin{cases} 
1, & |\varphi(\psi) + \alpha(t)| > \delta \\
0, & |\varphi(\psi) + \alpha(t)| \leq \delta 
\end{cases}
\]

With the function \( S_\delta(\varphi(\psi(x(t))), \alpha(t)) \) we associate the time sequence \( T = \{ \underline{t}_0 \leq \overline{t}_0 < \underline{t}_1 < \overline{t}_1 < \ldots < \underline{t}_i < \overline{t}_{i+1} < \overline{t}_{i+1} < \ldots \} \), where

\[
\underline{t}_i = \inf_{t \geq \underline{t}_{i-1}} \{ t : |\varphi(\psi(x(t))) + \alpha(t_i)| < \delta \}
\]

\[
\overline{t}_i = \inf_{t \geq \underline{t}_{i-1}} \{ t : |\varphi(\psi(x(t))) + \alpha(t)| > \delta \}
\]

The elements of this sequence are time instances \( \underline{t}_i \) (or \( \overline{t}_i \)) at which the sum \( \varphi(\psi(x(t))) + \alpha(t) \) leaves (or enters) domain \( |\varphi(\psi(x(t))) + \alpha(t)| \leq \delta \). We define that \( \underline{t}_0 = \overline{t}_0 \) if \( |\varphi(\psi(x(t_0))) + \alpha(t_0)| < \delta \).

Let us, in addition, introduce function \( \lambda \) with the following properties:

\[
\lambda : \mathbb{R} \to [\underline{\theta}, \bar{\theta}], \lambda \in C^1, \quad \mathcal{I}_m(\lambda) \supset [\underline{\theta}, \bar{\theta}]
\]

\[
\forall s \in \mathbb{R}, \theta \in \Omega_\theta \quad \exists T, \tau(s) > 0 : \quad \theta = \lambda(s + \tau(s)), \quad 0 < \tau(s) < T
\]

An example of such function is

\[
\lambda(s) = \underline{\theta} + \bar{\theta} \frac{1}{2} \sin(s) + 1
\]
As a candidate for $\hat{\theta}(x, t, \delta)$ we choose the following adaptation algorithm:

$$\dot{\theta}(x, t, \delta) = \lambda(\hat{\theta}_0(x, t, \delta))$$

$$\dot{\theta}_0(x, t, \delta) = \gamma \left( \dot{\theta}_P(x, t) + \theta_I(t) + C_\theta(t) \right)$$

$$\dot{\theta}_P(x, t) = \psi(x) \left( \alpha(t) \frac{1}{2} \dot{\psi}(x) \right)$$

$$\dot{\theta}_I = S_\theta(\varphi, \alpha(t)) \left[ \psi(x) - \varphi(\psi(x)) \right]$$

$$\alpha(t) = (1, 0)(\xi_1, \xi_2)^T r$$

$$\left( \xi_1, \xi_2 \right) = \left( \begin{array}{c} a_1 \xi_2 + b_1 \psi(x) \\ a_2 \psi(x) \end{array} \right)$$

$$b_1 \neq 0, a_1, a_2 < 0,$$

$$C_\theta(t) = \left( \begin{array}{c} 1 \dot{\theta}(\xi_{t-1}) - \dot{\theta}_I(\xi_{t-1}) \end{array} \right)$$

Properties of algorithm (15) are summarized in the following theorem:

**Theorem 3.** Let system (1) with control function (4) and corresponding error model (5) be given. Let function $\psi(t), \dot{\psi}(t) \in L^2$ and $||\psi(t)||_\infty \leq \Delta$.

Let in addition function $f(x, \theta)$ be bounded. Then 1) for any $\varepsilon > 0$ and $\varphi \in C_\psi(k), k > 0$ there exist functions $\delta_0 : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \delta_0 \in C^0, \delta(0) = 0,$ $\delta(\varepsilon, \Delta) = \delta_0(\varepsilon) + \Delta,$ and function $\hat{\theta}(x, t, \delta)$, given by (15) with arbitrary $\gamma \in \mathbb{R}, \gamma > 0$ and initial conditions such that $\Omega_\psi(\varepsilon + \frac{\Delta}{k})$ contains the $\omega$-limit set of system (1); 2) all trajectories of the system are bounded and solutions $x(t, x_0, t_0)$ converge into the domain $\Omega_\psi(\varepsilon + \frac{\Delta}{k})$ in finite time; 3) if for any $\theta \in \Omega_\theta$ there exist constants $T_1 > 0, M > 2\delta_0(\varepsilon) + \Delta > 0$ and function $\tau(t) : \mathbb{R} \rightarrow (0, T_1)$ such that

$$\left| f(x(t + \tau(t), \theta)) - f(x(t + \tau(t), \hat{\theta})) \right| > M$$

$$\forall \hat{\theta} \in \Omega_\theta \setminus \mathcal{U}_\varepsilon(\hat{\theta})$$

then $\hat{\theta}$ converges to $\mathcal{U}_\varepsilon(\theta)$ in finite time.

**Proof of the Theorems are given in Appendix.**

Theorem 3 states that for arbitrary $C^1$-smooth and bounded function $f(x, \theta)$ there exists non-dominating adaptation algorithm in class $C_\psi(k)$. To see this it is enough to let $\Delta = 0$. In presence of the unknown perturbation $\psi(t)$ we guarantee convergence of the trajectories $x(t, x_0, t_0)$ to arbitrary small neighborhood of $\Omega_\psi(\varepsilon + \frac{\Delta}{k})$ subject to the choice of $\delta_0 > 0$.

Adaptation algorithm (15) ensures boundedness of the solutions in the extended system and furthermore, under assumption (16), it guarantees convergence of the estimates $\hat{\theta}$ to arbitrary small neighborhood $\mathcal{U}_\varepsilon(\theta)$. In case $\mathcal{U}_\varepsilon(\hat{\theta}) = \theta$ it guarantees convergence to the small neighborhood of the actual value of $\theta$.

Condition (16), which we require for convergence of the parameter $\hat{\theta}$ into $\mathcal{U}_\varepsilon$, can be regarded as a new version of *nonlinear persistent excitation* (Cao et al., 2003). Our condition, however, is more easy to verify. In addition, this condition is consistent with *linear persistent excitation* condition (Narendra and Annaswamy, 1989): $\exists T > 0, \rho > 0 : \int_0^T x(t) \dot{x}(t) dt > \rho I_n$. Indeed $\int_0^T x(t) \dot{x}(t) dt > \rho I_n \Rightarrow \forall \hat{\theta} \neq \theta' : \dot{x}(t) \dot{x}(t) + \rho \| \hat{\theta} - \theta' \|^2 > \| \hat{\theta} - \theta' \|^2 > M, \forall \theta \in \Omega_\theta, M = \frac{\rho}{2}$

Let us generalize the statements of Theorem 3 to the case where $\theta \in \Omega_\theta \subset \mathbb{R}^d$. For that reason we introduce the following assumption:

**Assumption 1.** Let $\Omega_\theta$ be bounded and there exist $C^1$-smooth function $\eta : [\theta, \theta] \rightarrow \mathbb{R}^d$ such that that for any $\theta \in \Omega_\theta$ there exists $\lambda^*(\theta) \in [\theta, \theta]$: $|f(x, \theta) - f(x, \eta(\lambda^*))| \leq \Delta, \forall x \in \mathbb{R}^n$

Applicability of algorithms (15) to multi-dimensional $\theta$ then follows explicitly from Theorem 3.

**Theorem 4.** Let system (1) with control function (4) and corresponding error model (5) be given, function $f(x, \theta)$ be bounded, and Assumption 1 hold. Then statements 1) – 3) of Theorem 3 follow.

Adaptation algorithm (15) can be considered as a nonlinear dynamical system which, however, is not internally globally stable in Lyapunov sense. The algorithm, nonetheless, does not lead to unbounded growth of its internal state, neither it fails to ensure reaching of the control goal for arbitrary initial conditions $x(t_0), \hat{\theta}_I(t_0), \xi_{t_0}$. The properties of this algorithm essentially rely on two ideas: *monotonic evolution* of $\theta_0(t)$, and *multiple equilibriums* in the corresponding differential equation for $\theta_0$ given by (17) in Appendix.

Multiple equilibriums of are guaranteed by function $\lambda(\cdot)$ defined by (14) and invariance of $\theta_0(t)$ on the following set $\{ x : \theta_0(t_0) = f(x, \theta) - f(x, \lambda(\theta_0)) = 0 \}$. Monotonic character of $\theta_0(t)$ and multiple equilibriums ensure existence of the limit $\lim_{t \rightarrow \infty} \theta_0(t) = \bar{\theta}_{0, \infty}$ for any initial conditions $x(t_0), \hat{\theta}_I(t_0), \xi_{t_0}$. This fact is used in the proof of Theorem 3 to show convergence of the trajectories $x(t, x_0, t_0)$ to the set specified by (12). Notice also that existence of these multiple equilibriums in case of persistent excitation for
Conceptual difference between our method and conventional Lyapunov-based design is illustrated with Fig. 1. In Fig. 1 the upper plot depicts the solution curve \( \theta_0(t, \theta_0(t_0), t_0) \) of differential equation (17), where the arrows point towards increase of the independent variable along the curve. For the given value of \( \theta \) and initial condition \( \theta_0(t_0) \) function \( \lambda(\theta_0) \) generates infinitely many equilibria \( \hat{\theta}_0, i \in \mathbb{N} \). If the perturbation is applied to the system the system will eventually escape its current equilibrium (for instance, \( \hat{\theta}_0 = \hat{\theta}_{0,1} \)) equilibrium and move along the axis \( \hat{\theta}_0 \). Due to the monotonicity of \( \theta_0(t, \theta_0(t_0), t) \) with respect to \( t \) it eventually reaches a neighborhood of the point \( \theta_0 = \hat{\theta}_{0,2} \) and stops there if the perturbation is released. In order to prevent unbounded growth of \( \theta_0 \) under persistent perturbations it may be necessary to change the sign of \( \gamma \) in (17) upon solution \( \hat{\theta}_0(t) \) reaches certain bounds.

In the lower plot we have shown the typical would be solution curves of Lyapunov asymptotically stable estimator. Disturbances will move the solution away from the equilibrium and the system will restore its original state after the disturbing terms vanish. Despite obvious advantages of this type of behavior, i.e. small perturbations induce small deviations, in adaptive control problems the parametric perturbations are usually large. Furthermore, as it has been already pointed out, gentle control ensuring Lyapunov stability for arbitrary nonlinear parameterized system is problematic.

One of the distinctive features of algorithms (15) is that they can in principle take into account information about (available a-priori) distribution of the unknown quasi-stationary \( \theta \) as a function of time. This information is then to be accounted for by the choice of functions \( \eta(\cdot) \) and \( \lambda(\cdot) \). Illustration to this choice is given in Figure 2. In Fig. 2 the curve \( \eta(\lambda) \) is designed to visit neighborhood of the center more frequently (16 times per period) than other points (only 2 times per period) of the domain. Despite the problem of choosing these curves \( \eta(\lambda) \) (which fit given distributions of \( \theta \)) is not trivial, such optimal choice, if successful, can provide room for further enhancements of performance in the adaptive systems. The second step would be to adjust or tune functions \( \eta(\lambda) \) adaptively thus enabling self-tuning of the adaptive algorithm itself. These topics, however, are beyond the goals of our current study.

4. CONCLUSION

In this paper we have proposed new technique for adaptive control of nonlinear dynamical systems with nonlinear parametrization. In contrast to conventional concept of Lyapunov stable adaptive control, and as a result domination of the nonlinearity by high-gain feedbacks, we use adaptation schemes which are not stable in Lyapunov sense. Yet, these algorithms guarantee reaching of arbitrary small neighborhood of the desired target set. Moreover the resulting control function is non-dominating.
The ideology beyond our method is somewhat similar to the one introduced in (Ilchman, 1997; Pomet, 1992). The results, however, are substantially different. First, we do not require neither exponential stability nor asymptotic stability of the target dynamics. Second, the speed of adaptation in our case is not to be slowed down with time. Last but not least is that the method can be used to identify nonlinear systems of rather general class without requesting for linearization of the nonlinearities.

Technique for adaptive control introduced in our paper rather well coincides with the thesis of "nonlinear philosophy for nonlinear systems" declared in (Fradkov, 2000). Furthermore, it fits well empirical observations that complex natural systems poses clearly detectable patterns of instability in the empirical data.

Despite we proved only that the target set will be reached in the system with our algorithms, we hope that some type of robustness w.r.t. unknown external perturbations can be also shown. In fact, robustness of the system to unmodeled dynamics with known $L_{\infty}$ norm can be easily ensured by enlarging the value of $\delta$ in our algorithms. Whether or not robust behavior can be achieved by choice of another parameters like functions $\eta(\cdot)$ or $\lambda(\cdot)$ will be the topics of our future study.

REFERENCES

Bak, P. and K. Sneppen (1993). Punctuated equilibirum and criticality in a simple model of evolution. Physical Review Letters.

Bak, P., K. Christensen, L. Danon and T. Scanlon (2002). Unified scaling law for earthquakes. Physical Review Letters 88, 178501.

Beggs, J. M. and D. Plenz (2003). Neuronal avalanches in neocortical circuits. J. Neurosci. 22(35), 11167–11177.

Brenner, N., W Bialek and R. de Ruyter van Steveninck (2000). Adaptive rescaling maximizes information transmission. Neuron 26(3), 695–702.

Canudas de Wit, C and P. Tsiotras (1999). Dynamic tire models for vehicle traction control. In: Proceedings of the 38th IEEE Control and Decision Conference.

Cao, C., A.M. Annaswamy and A. Kojic (2003). Parameter convergence in nonlinearly parametrized systems. IEEE Trans. on Automatic Control 48(3), 397–411.

Corral, A., C. J. Perez, A. Daz-Guilera and A. Arenas (1995). Self-organized criticality and synchronization in a lattice model of integrate-and-fire oscillators. Physical Review Letters 74, 118–121.

Dayan, P. and L.F. Abbott (2001). Theoretical Neuroscience: Computational and Mathematical Modeling of Neural Systems. MIT Press.

Ditchburn, R. W. and Ginzburg B. L. (1952). Vision with a stabilized retinal image. Nature 170(4314), 36–37.

Fradkov, A. L. (2000). A nonlinear philosophy for nonlinear systems. In: 39-th IEEE Conf. on Decisions and Control, pp. 4397–4402.

Gauthier, D. J. and Bienfang J. C. (1996). Intermittent loss of synchronization in coupled chaotic oscillators: Toward a new criterion for high-quality synchronization. Physical Review Letters 77(9), 1751–1754.

Guckenheimer, J. and P. Holmes (2002). Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields. Springer.

Ilchman, A. (1997). Universal adaptive stabilization of nonlinear systems. Dynamics and Control (7), 199–213.

Isidori, A. (1989). Nonlinear control systems: An Introduction, second ed.. Springer–Verlag.

Ito, J., A. Nikolaev, M. Luman, M. F. Aukes, C. Nakatani and C. van Leeuwen (2003). Perceptual switching, eye-movements, and the bus-paradox. Perception 32, 681–698.

Kaneko, K. (1994). Relevance of dynamic clustering to biological networks. Physica D 75, 137–172.

KaneKO, K. and I. Tsuda (2000). Complex Systems: Chaos and Beyond. Springer.

KaneKO, K. and I. Tsuda (2003). Chaotic itinerancy. Chaos 13(3), 926–936.

Karsenti, L., F. Lamnabhi-Lagarrigue and G. Bastin (1996). Adaptive control of nonlinear systems with nonlinear parameterization. System and Control Letters 27, 87–97.

Lin, W. and C. Qian (2002a). Adaptive control of nonlinearly parameterized systems: A nonsmooth feedback framework. IEEE Trans. Automatic Control 47(5), 757–773.

Lin, W. and C. Qian (2002b). Adaptive control of nonlinearly parameterized systems: The smooth feedback case. IEEE Trans. Automatic Control 47(8), 1249–1266.

Loh, Ai-Poh, A.M. Annaswamy and F.P. Skantze (1999). Adaptation in the presence of general nonlinear parameterization: An error model approach. IEEE Trans. on Automatic Control 44(9), 1634–1652.

Marino, R. and P. Tomei (1993). Global adaptive output-feedback control of nonlinear systems, part II: Nonlinear parameterization. IEEE Trans. Automatic Control 38(1), 33–48.

Martinez-Conde, S., S. L. Macknik and D. H. Hubel (2004). The role of fixational eye movements in visual perception. Nature Reviews. Neuroscience 5(3), 229–240.

Milnor, J. (1985). On the concept of attractor. Commun. Math. Phys. 99, 177–195.
Proof of Theorem 3. Let us consider system (15) and calculate derivative $\dot{\theta}_0$ with respect to independent variable $t$:

$$
\dot{\theta}_0 = \gamma S_\delta(\psi, \alpha(t))(\varphi(\psi) + \dot{\psi})(\varphi(\psi) + \alpha(t)),
$$

(17)

To proceed further we will need auxiliary Lemma 5 and Lemma 6.

Lemma 5. Consider system (1), (5). Let function $f(x, \theta)$ in (5) be bounded w.r.t. $x, \theta$ and $\psi(t), \dot{\psi}(t) \in L_\infty$. Then $\dot{\theta}(t), \theta(t) \in L_\infty$ imply that $\dot{\psi} \in L_\infty$.

Proof of Lemma 5. First, we observe that $\psi(x) \in L_\infty$. This follows immediately from boundedness of $f(x, \theta)$ and the fact that function $\varphi$ belongs to the class $C_\varphi$ specified by (6). Therefore, according to assumption (3) on function $\psi(x)$, state $x \in L_\infty$. Moreover, given that $\psi \in L_\infty$ and $\varphi \in C_\varphi$, we can derive from (5) that $\dot{\psi} \in L_\infty$. Let us consider $\dot{\psi}$.

$$
\dot{\psi} = \left( \frac{\partial f(x, \theta)}{\partial x} - \frac{\partial f(x, \theta)}{\partial \theta} \right) (f(x, \theta) + g(x)u) + \frac{\partial f(x, \theta)}{\partial \psi}\dot{\theta} + \frac{\partial g(x)}{\partial \psi}\dot{\psi} + \dot{\psi}(t).
$$

Derivatives $\frac{\partial f(x, \theta)}{\partial x}, \frac{\partial f(x, \theta)}{\partial \theta}$ are continuous functions with respect to $x$ as $\dot{\psi} \in C_\varphi$, $f(x, \theta) \in C_\varphi$. Therefore they are bounded as $x, \dot{\theta}, \theta \in L_\infty$. In addition, continuity of $\frac{\partial f(x, \theta)}{\partial \psi}$ and $\frac{\partial g(x)}{\partial \psi}$ results in their boundedness. To conclude the proof it is enough to notice that function $u$ is bounded for bounded $x, \dot{\theta}, \theta$. The lemma is proven.

Lemma 6. Let function $\psi(t)$ be given and its second time-derivative is bounded: $|\ddot{\psi}| < \beta_1, \beta_1 > 0$. Then there exists differential filter

$$
\begin{pmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-a_1 & a_2
\end{pmatrix}
\begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix} +
\begin{pmatrix}
b_1 \\
b_2
\end{pmatrix} \psi
$$

(18)

$$
y = (c_1, 0) (\xi_1, \xi_2)^T,
$$

where $b_1 \neq 0, a_1, a_2 < 0$, and time $t_1 > 0$ such that for any positive constant $\epsilon > 0$ the following estimate holds: $|y(t) - \dot{\psi}(t)| \leq \epsilon \forall t > t_1$. In particular, if $c_1b_1 = -a_1, b_2 = a_2b_1$ then filter output satisfies the following inequality:

$$
|y(t) - \dot{\psi}(t)| \leq \frac{a_2b_1}{a_1} + \delta(t),
$$

(19)

where $\delta(t)$ decays exponentially fast to the origin.

Proof of Lemma 6. Let $b_2 = a_2b_1$. Then system (18) has the following equivalent representation:

$$
\begin{pmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
-a_1 & a_2
\end{pmatrix}
\begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix} +
\begin{pmatrix}
b_1 \\
b_2
\end{pmatrix} \psi
$$

(20)

$$
y = (c_1, 0) (\xi_1, \xi_2)^T.
$$

Denoting

$$
A = \begin{pmatrix}
0 & 1 \\
-a_1 & a_2
\end{pmatrix}, \quad b_1 = \begin{pmatrix} 0 \\ b_1 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ 0 \end{pmatrix},
$$

we may write the filter equations in a vector-matrix form:
\[ \dot{\xi} = A\xi + b_1\dot{\psi} \]
\[ y = c^T\xi \]  
\[ (21) \]
Consider output $y$ of system (21):
\[ y = c^T\left(e^{A t}\xi_0 + e^{A t}\int_0^t e^{-A r}b_1\dot{\psi}(r)dr\right). \]  
\[ (22) \]
Matrix $A$ is Hurwitz and, therefore, invertible. Hence taking into account existence of $\hat{\psi}(t)$ and that matrices $A^{-1}$, $e^{A t}$ commute, we can rewrite equality (22) as follows:
\[ y = c^T(e^{A t}\xi_0 - e^{A t}A^{-1}e^{-A t}b_1\dot{\psi}(t)|_0^t + e^{A t}A^{-1}\int_0^t e^{-A r}b_1\dot{\psi}(r)dr), \]
Hence
\[ y = c^T(e^{A t}\xi_0 - A^{-1}b_1\dot{\psi}(t)|_0^t + A^{-1}\int_0^t e^{A(t-r)}b_1\dot{\psi}(r)dr). \]
Consider the following difference $|y(t) - \hat{\psi}(t)|$:
\[ |y(t) - \hat{\psi}(t)| = |c^T(e^{A t}\xi_0 - A^{-1}b_1\dot{\psi}(t) + A^{-1}\times e^{A t}b_1\dot{\psi}(t)|_0^t + A^{-1}\int_0^t e^{A(t-r)}b_1\dot{\psi}(r)dr) - \hat{\psi}(t)| \]
\[ \leq |c^T(e^{A t}\xi_0 - A^{-1}b_1\dot{\psi}(t) + A^{-1}\times e^{A t}b_1\dot{\psi}(t)|_0^t + A^{-1}\int_0^t e^{A(t-r)}b_1\dot{\psi}(r)dr) - \hat{\psi}(t)| + |c^T A^{-1}\int_0^t e^{A(t-r)}b_1\dot{\psi}(r)dr| \]
\[ = |c^T(e^{A t}\xi_0 - A^{-1}b_1\dot{\psi}(t) + A^{-1}\times e^{A t}b_1\dot{\psi}(t)|_0^t + A^{-1}\int_0^t e^{A(t-r)}b_1\dot{\psi}(r)dr) - \hat{\psi}(t)| + |c^T A^{-2}e^{A t}b_1 - c^T A^{-2}b_1|, \]
where $A^{-2} = A^{-1}A^{-1}$. Let us denote $\delta(t) = |c^T(e^{A t}\xi_0) + |c^T A^{-1}e^{A t}b_1\dot{\psi}(t)|_0^t + |c^T A^{-1}e^{A t}b_1|$. Term $\delta(t) \to 0$ at $t \to \infty$ as matrix $A$ is Hurwitz. Therefore for any $\delta(t) > 0$ there exists time $t_1 > 0$ such that for any $t > t_1$ the following inequality holds:
\[ |y(t) - \hat{\psi}(t)| \leq |c^T A^{-1}b_1| + |c^T A^{-2}b_1| + \delta_1, \]
\[ (23) \]
Let us consider term $c^T A^{-1}b_1$ in (23). Matrix
\[ A^{-1} = \begin{pmatrix} 0 & 1 \\ a_1 & a_2 \end{pmatrix}^{-1} = \frac{1}{a_1} \left( \begin{array}{cc} -a_2 & 1 \\ -1 & 0 \end{array} \right) \]
Then
\[ c^T A^{-1}b_1 = (c_1 0) A^{-1} \left( \begin{array}{c} 0 \\ b_1 \end{array} \right) = \frac{b_1 c_1}{a_1} \]
Notice that $\frac{b_1 c_1}{a_1} = -1$ due to the lemma assumptions. Therefore, using inequality (23) we can derive the following estimate:
\[ |y(t) - \hat{\psi}(t)| \leq |c^T A^{-2}b_1| + \delta_1, \]
where
\[ A^{-2} = \frac{1}{a_1^2} \left( \begin{array}{cc} a_2^2 + a_1 & -a_2 \\ -a_2 a_1 & 0 \end{array} \right). \]
Hence
\[ |y(t) - \hat{\psi}(t)| \leq \left| (c_1 0) A^{-2} \left( \begin{array}{c} 0 \\ b_1 \end{array} \right) \right| + \delta_1 \]
\[ = \left| a_2 b_1 \right| + \delta_1. \]
\[ (24) \]
Inequality (24) proves Lemma 6.

Lemmas 5, 6 will allow us to show that $\hat{\theta}(x, t, \delta)$, $\delta = \delta_0 + \Delta$, $\delta_0 \in \mathbb{R}_+$ has a limit as $t \to \infty$. First we observe that $\hat{\theta}(x, t, \delta)$ is bounded. Moreover $\hat{\theta}(x, t, \delta)$ is continuous and has bounded time-derivative. Therefore, according to Lemma 5 derivative $\hat{\psi}$ is bounded. Therefore, as it follows from Lemma 6, there exist parameters $a_3, a_4, b_1, b_2$ of filter (20) such that $|\alpha(t) - \hat{\psi}(t)| < \delta_0/4 + |\delta_1(t)|$, where $\delta_1(t) \to 0$ as $t \to \infty$. In particular, there exist time $t_1 > 0$ such that $|\delta_1(t)| < \delta_0/4$ for any $t > t_1$. Notice, that function $\hat{\theta}(x, t, \delta)$ is bounded for $t \leq t_1$ (as a sum and integral of bounded functions in time). Let us show that for any $t > t_1$ function $\hat{\theta}(x, t, \delta)$ is monotonically increasing with respect to time $t$.

Let us consider function $\alpha(t)$, $t > t_1$. Taking into account that $|\alpha(t) - \hat{\psi}(t)| < \delta_0/2$ for any $t > t_1$ we can estimate function $\alpha(t)$ in the following way:
\[ \alpha(t) = \hat{\psi}(t) + \mu(t), |\mu(t)| \leq \delta_0/2 \]
\[ (25) \]
Therefore, for any $\alpha(t), \psi$: $|\alpha(t) + \varphi(\psi)| > \delta$ we have
\[ (\hat{\psi} + \varphi(\psi))(\alpha(t) + \varphi(\psi)) = (\alpha + \varphi(\psi) - \mu(t))(\alpha(t) + \varphi(\psi)) = (\alpha(t) + \varphi(\psi))^2 - \mu(t)\alpha(t) + \varphi(\psi)) \geq |\alpha(t) + \varphi(\psi)| \delta - \frac{\delta_0}{2} \]
\[ (26) \]
Hence, according to equations (17), (26) we can conclude that function $\hat{\theta}_0(t, x)$ is monotonic, continuous and not decreasing with time.

Let us show that, function $\hat{\theta}_0(x, t, \delta)$ is also bounded from above. Assume that for any $D > 0$ exists $t_2 > 0$ such that
\[ |\hat{\theta}_0(x(t_1), t_1, \delta) - \hat{\theta}_0(x(t_2), t_2, \delta)| > D \]
\[ (27) \]
Let us select $D > T$, where the values of $T$ are specified in (14). Function $\hat{\theta}_0(x(t), \tau, \delta)$ is continuous w.r.t. $\tau$ and bounded for any $\tau \in [t_1, t_2]$ (boundedness of $\hat{\theta}_0(x(t), \tau, \delta)$ follows from boundedness of its derivative over finite time interval). Therefore, according to the intermediate value theorem, for any $0 < T_0 < D$ there exists $t_3 \in [t_1, t_2]$ such that $\hat{\theta}_0(x(t_3), t_3, \delta) = \hat{\theta}_0(x(t_1), t_1, \delta) + T_0$. Taking into account (14) we can conclude that for any $\theta \in [\theta_0, \theta_1]$ there exists $t_3 > 0$ such that $\theta = \lambda(\hat{\theta}_0(x(t_3), t_3, \delta)), t_3 \in [t_1, t_2]$. According to the properties of function $\hat{\theta}_0(x, t, \delta)$, however, derivative $\hat{\theta}_0(t) \equiv 0$ for any $t \geq t_3$ as
\[ |\alpha(t_3) + \varphi(\psi(x(t_3)))| = \hat{\psi} + \varphi(\psi(x(t_3))) + \mu(t_3) \]
\[ = |f(x, \lambda(\hat{\theta}_0(x(t_3), t_3, \delta))) - f(x, \theta)| = |\dot{\theta}(x, t) + \varphi(\psi)| \leq |\varphi(\psi)| + |\mu(t)| \leq \delta_0/2 + \Delta \quad \text{s.t.} \quad t \geq t_3 \]
Therefore $\hat{\theta}_0(x(t_2), t_2, \delta) = \hat{\theta}_0(x(t_3), t_3, \delta) \leq \hat{\theta}_0(x(t_1), t_1, \delta) + T$ which contradicts to (27).

So far we have showed that $\hat{\theta}_0(x, t, \delta)$ is continuous, monotonic and bounded function for any $t > t_1$. Therefore, applying Bolzano-Weierstrass,
theorem we can derive that there exists the following limit
\[
\lim_{t \to \infty} \hat{\theta}(x(t), \tau, \delta) = \hat{\theta}_{0,\infty}
\] (28)

Taking into account equality (17) we can rewrite function \(\hat{\theta}_0(x(t), t, \delta)\) for \(t > t_1\) as follows:
\[
\hat{\theta}_0(x(t), t, \delta) = \hat{\theta}_0(x(t_1), t_1, \delta) + \int_{t_1}^{t} S(\phi(\psi(\tau)), \alpha(\tau))(\psi(\tau) + \varphi(\psi(\tau))) d\tau
\] (29)

Equations (29), (28) and inequality (26) result in the following estimate \(0 < \int_{t_1}^{\infty} S(\varphi(\psi(\tau)))\) for \(t \in [t_1, t_2]\):
\[
|\varphi(\psi(t)) + \varphi(\psi)| \leq |\varphi(\psi(t_1)) + \varphi(\psi)| + (T_1 - L_1)D_{|\tau|},
\]
where \(D_{|\tau|} = \max_{t \in [t_1, T_1]} \frac{d}{d\tau} \varphi(\psi) + \alpha(\tau)\). Function \(D_{|\tau|}\) is the bounded function of time and therefore
\[
\lim_{t \to \infty} \varphi(\psi(t)) + \alpha(\tau) = \delta
\] (30)

Taking (30) into account we can derive that dynamics of function \(\psi(t)\) for \(t > t_1\) can be described as follows:
\[
\dot{\psi} = \mu(t) - \varphi(\psi), \quad \lim_{t \to \infty} \|\mu(t)\| \leq \frac{1}{2} \delta_0 + \Delta
\] (31)

According to the definition of \(\limsup\) there exists a time instant \(t_4\) such that \(\mu(t) < 2\delta_0 + \Delta\) for any \(t < t_4\). Let us consider the following function
\[
\psi(t) = \int_{0}^{\psi(t)} \phi(\xi, \nu) d\xi,
\]
where \(\psi(t)\) is the solution of system (31) for \(t > t_4\) with initial condition \(x(t_4)\). Function \(\varphi(\psi) \in C_\varphi(k)\), therefore \(\varphi(\psi) \leq k\psi^2\). Let us select \(\nu = (2\delta_0 + \Delta)/k\) and consider derivative \(\dot{\psi}\):
\[
\dot{\psi} = \phi(\psi, \nu) \leq \varphi(\psi)\mu(t) - \varphi(\psi) \leq \psi(\psi) \mu(t) - k\psi^2 \leq 0.
\]

Moreover, it follows from Barbalat’s lemma that \(\phi(\psi, \nu) \to 0\) as \(t \to \infty\). This automatically implies that there exist time instant \(t_5\) such that
\[
\|\psi(x(t))\|_\infty \leq (3\delta_0 + \Delta)/k \quad \text{for any } t \geq t_5.
\]

So far we have shown that for any \(\delta > \Delta + \delta_0\), \(\delta_0 > 0\), \(\varphi \in C_\varphi(k)\) and \(\mathbf{x}_0 \in \Omega_\varphi\) there exists algorithm (15) with arbitrary \(\gamma > 0\) and initial conditions such that \(\Omega_\varphi(2\delta_0/\Delta + (\Delta)/k)\) contains the \(\omega\)-limit set of system (1), (4) with \(v(t) : \|v(t)\|_\infty = \Delta\). Furthermore, we have shown that trajectories \(\mathbf{x}(t, \mathbf{x}_0, t_0)\) converge into the domain \(\Omega_\varphi(3\delta_0/k + \Delta/k)\) in finite time. Therefore in order to complete the proof, given arbitrary \(\varepsilon > 0\), it is enough to pick \(\delta_0 < 1/3k\varepsilon\). Hence, statements 1 and 2 of the theorem are proven.

Let us show that statement 3 of the theorem holds. Notice that, in order to complete the proof it is enough to show that \(\lambda(\hat{\theta}_{0,\infty}) \in \mathcal{U}_\varphi(\{\theta_{\Delta}\})\), where \(\hat{\theta}_{0,\infty}\) is defined as in (28). Let us assume that \(\lambda(\hat{\theta}_0(x, t_1, \delta))\) does not belong to \(\mathcal{U}_\varphi(\{\theta_{\Delta}\})\), and consider equality (25) for \(t > t_1\). According to (25) the following equality holds:
\[
\dot{\psi} + \varphi(\psi) + \mu(t) = \alpha(t) + \varphi(\psi) = f(x(t), \theta) - f(x, \hat{\theta})
\]

This automatically implies \(\lim_{t \to \infty} \varphi(\psi) + \mu(t) = \delta_{\psi} + \Delta\). Therefore there exists \(t' > t_1\) such that \(\varphi(\psi(t)) + \mu(t') > 2\delta_0 + \Delta\), and, consequently \(\varphi(\psi(t)) + \mu(t') \geq \delta_{\psi} + \Delta + \delta_0 + \Delta, t + \tau(t) > t_1\). Let us denote \(t_6 = t + \tau(t)\). Notice that \(\hat{\theta}(t) \in \mathcal{U}_\varphi(\{\theta_{\Delta}\})\). Hence, there exist constant \(\tau_1 > 0\) such that \(\varphi(\psi(t)) + \mu(t) > \delta_{\psi} + \Delta\) for any \(t \in [t_6, t_6 + \tau_1]\). Furthermore, function \(\hat{\theta}(x, t, \delta)\) is bounded and, according to (26), (16) increases by at least \(N\tau_1\delta_2/2\) over time interval \([t_1, t] > t_1 + N\tau_1\) until \(\lambda(\hat{\theta}(0)) \in \mathcal{U}_\varphi(\{\theta_{\Delta}\})\). On the other hand, we have shown that function \(\hat{\theta}_0(x, t, \delta)\) is bounded from above and, furthermore, converges to the limit \(\theta_{0,\infty} < \infty\). Therefore, there exists time instant \(t_7\) for any \(t > t_7\). Let us assume that \(\lambda(\hat{\theta}_0(x, t_1, \delta)) \in \mathcal{U}_\varphi(\{\theta_{\Delta}\})\). Then there are two possibilities. First, \(\lambda(\hat{\theta}_0(x, t, \delta)) \in \mathcal{U}_\varphi(\{\theta_{\Delta}\})\) for all \(t > t_1\), and second, \(\lambda(\hat{\theta}_0(x, t, \delta))\) eventually leaves the domain \(\mathcal{U}_\varphi(\{\theta_{\Delta}\})\). In this case, however, the previously provided arguments apply. The theorem is proven.

**Proof of Theorem 4.** Let us denote \(v(t) = f(x(t), \theta) - f(x, \eta(\hat{\theta}))\). According to Assumption 1 function \(v(t) \in \mathcal{L}_\infty\) and moreover \(\|v(t)\|_\infty = \Delta\). Then \(x\) is bounded. Hence, applying the same arguments as in the proof of Theorem 3 we conclude that \(\hat{\theta}, \hat{\theta}\) are bounded. Therefore, \(\dot{v}(t) \in \mathcal{L}_\infty\). Then Theorem 3 applies, which proves the theorem.