Exact solution for two unequal counter-rotating black holes

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(Dated: August 30, 2012)

The complete solution for two unequal counter-rotating black holes separated by a massless strut, is developed in terms of four arbitrary parameters involving two quantities $\sigma_1$ and $\sigma_2$ as the half length of the two rods representing the black hole horizons, the total mass $M$ and the relative distance $R$ between the centers of the horizons. A further attempt for describing the explicit form of this solution in terms of the physical parameters: The two Komar masses $M_1$ and $M_2$, Komar angular momenta per unit mass $a_1$ and $a_2$ ($a_1$ and $a_2$ have opposite sign), and the coordinate distance $R$, guided us to a 4-parameter subclass in which the five physical parameters satisfy a simple algebraic relation and the interaction force in this scheme looks like Schwarzschild type.

PACS numbers: 04.20.Jb, 04.70.Bw, 97.60.Lf

I. INTRODUCTION

One of the first attempts for describing the physical properties for two rotating black holes, was made by Varzugin [1], by solving the corresponding Riemann-Hilbert problem. In such problem was defined a quantity $\sigma_1$ as irreducible mass, described as the half length of the rod representing the event horizon of the $i$th black hole located on the symmetry axis. Where the regularity conditions for the symmetry axis and event horizon were formulated first, and the black holes are separated by a massless strut (conical singularity) $2$ in order to prevent the falling onto each other. Nonetheless, due that this quantity is related with the respective surface gravity and area of the black hole horizon by means the Smarr mass formula $3$, one could express it in terms of physical parameters. However, due the cumbersome expressions derived, he was able to calculate explicitly the analytical solution for the problem describing two identical counter-rotating black holes, where the interaction force has the same aspect as one Schwarzschild type. The corresponding unique quantity $\sigma$ was described by only three parameters, which are: The Komar mass $m$ and Komar angular momentum per unit mass $a$ for one black hole $4$ (the other one has the parameters $m$ and $-a$), and coordinate distance $R$ between both constituents.

Later, Manko et al [5] using the previous result about the explicit form of $\sigma$, were capable to construct the 3-parametric solution. By writing explicit expressions for the Ernst potential and the entire metric outside the symmetry axis, describing the simplest configuration of two counter-rotating black holes. Which has the property as one equatorially antisymmetric solution $6$, where the regularity conditions on the symmetry axis are satisfied naturally, avoiding pathological regions involving closed timelike curves due the presence of the NUT sources. In this particular problem the total angular momentum of the system $J = 0$. Nevertheless, the most general solution describing a system of two unequal counter-rotating black holes separated by a massless strut, must be depicted by five physical parameters: The two Komar masses $M_1$, $M_2$ of each constituent, their respective Komar angular momenta per unit mass $a_1$, $a_2$ ($a_1$ and $a_2$ having opposite sign) and the relative coordinate distance $R$ between the centers of the black hole horizons. However, in such case the difficult task for describing the general solution of this problem, can be done only if we are capable of making null the presence of the NUT sources and we could provide the explicit form of the two quantities $\sigma_1$ and $\sigma_2$ in terms of physical quantities. Nowadays, the 5-parameter subclass of the well-known double-Kerr solution [6] and its analytical representation in terms of physical parameters has not yet been obtained and this can be due in part to the complexity of the problem.

In the present paper, we will derive first the entire metric describing the system composed by two rotating black holes, by using the facilities that offer us the Sibgatullin’s method $7, 8$. Then, applying the conditions in order to make regular the symmetry axis and solving the corresponding equations for the case of two unequal counter-rotating black holes separated by a massless strut, will give us later the possibility of rewriting the entire solution involving the two quantities $\sigma_1$ and $\sigma_2$, as a 4-parameter subclass of the double-Kerr problem $9$. Later, we will try to calculate these quantities by means the Komar integrals for the mass $M_i$ and the angular momentum $J_i$, and we want to show that the interaction force can be observed as the same form as one Schwarzschild type, if the 5 parameters satisfy an algebraic relation which contains in a more general form the two assumptions made by Bonnor [10], in order to remove the additional contribution made by the massless spinning rods outside the sources, where the total angular momentum of the system is exactly $J = M_1a_1 + M_2a_2$, containing only the contribution of the two sources.
II. THE GENERAL SOLUTION FOR A BINARY SYSTEM OF KERR SOURCES

The Papapetrou line element describing stationary axisymmetric spacetimes is
\[ ds^2 = f^{-1} \left[ e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2 \right] - f(dt - \omega d\varphi)^2, \]  

in which \( f, \omega, \gamma \) are unknown functions depending only of cylindrical coordinates \((\rho, z)\). According to the Ernst formalism [11] the Einstein vacuum equations for this particular case can be written in the following form
\[ (\text{Re}\mathcal{E}) \Delta \mathcal{E} = \nabla \mathcal{E} \cdot \nabla \mathcal{E}, \]  

where \( \nabla \) and \( \Delta \) denote respectively the gradient and Laplace operators defined in cylindrical coordinates and acting over the complex potential \( \mathcal{E} = f + i \Psi \). If one can know any solution of the equation (2), then the metric functions \( \omega \) and \( \gamma \) of the line element (1) can be obtained from the following system of differential equations
\[ \omega_{\rho} = -\rho f^{-2} \Psi_{z}, \quad \omega_{z} = \rho f^{-2} \Psi_{\rho}, \]  
\[ 4\gamma_{\rho} = \rho f^{-2} \left( |\mathcal{E}_{\rho}|^2 - |\mathcal{E}_{z}|^2 \right), \quad 2\gamma_{z} = \rho f^{-2} \text{Re}(\mathcal{E}_{\rho} \bar{\mathcal{E}}_{z}). \tag{3} \]

Here \(|x|^2 = x \bar{x}\), the bar over a symbol represents the complex conjugate and the subscript \( \rho \) or \( z \) denotes partial differentiation. In order to solve the non-linear equation (2), we will use the powerful mathematical technique based in the soliton theory named Sibgatullin’s method [1], where the extended double-Kerr problem can be constructed easily by applying this method in the same way like in the reference [3] for the case in which \( N = 2 \), but avoiding the electromagnetic field \((\Phi = 0)\). Let us start by defining the Ernst potential on the upper part of the symmetry axis as follows
\[ \mathcal{E}(\rho = 0, z) = e(z) = 1 + \frac{e_{1}}{z - \beta_{1}} + \frac{e_{2}}{z - \beta_{2}}, \tag{4} \]

being the set of parameters \( \{e_{j}, \beta_{j}\} \) complex constants, having a total of eight real parameters related with the multipolar terms. The complex potential in whole space can be obtained from the integral
\[ \mathcal{E}(\rho, z) = \frac{1}{\pi} \int_{-1}^{1} \frac{\mu(\zeta) e(\xi) d\zeta}{\sqrt{1 - \zeta^2}}, \tag{5} \]

whose unknown function \( \mu(\zeta) \) satisfies an integral equation
\[ \int_{-1}^{1} \frac{\mu(\zeta)[e(\xi) + \bar{e}(\eta)] d\zeta}{(\zeta - \eta)\sqrt{1 - \zeta^2}} = 0, \tag{6} \]

and a condition of normalization
\[ \frac{1}{\pi} \int_{-1}^{1} \frac{\mu(\zeta) d\zeta}{\sqrt{1 - \zeta^2}} = 1, \tag{7} \]

where \( \bar{e}(\eta) = \bar{e}(\bar{\eta}) \) and \( f \) is representing a principal value integral. Besides \( e(\xi) \) is the local holomorphic continuation of \( e(z) \) on the complex plane \( z + i \rho \), with \( \xi = z + i\rho \zeta \), \( \eta = z + i\rho \tau \), \( \forall \zeta, \tau \in [-1, 1] \). Since \( e(z) \) is a rational function, the corresponding \( \mu(\zeta) \) can be proposed in the form
\[ \mu(\zeta) = A_{0} + \sum_{n=1}^{4} A_{n}(\xi - \alpha_{n})^{-1}, \tag{8} \]

being \( A_{0} \) and \( A_{n} \) coefficients to be found from the equations (6)-(7), and the constants \( \alpha_{n} \) are indicating the location of the sources on the symmetry axis as the roots of the characteristic equation (see Fig. 1)
\[ e(z) + \bar{e}(z) = 0, \tag{9} \]

and doing the substitution of (4) into (9), it can be possible to relate the old parameters \( \{e_{j}, \beta_{j}\} \) and the new ones \( \{\alpha_{n}, \beta_{j}\} \), through the following relations
\[ e_{1} = \frac{2(\beta_{1} - \alpha_{1})(\beta_{1} - \alpha_{2})(\beta_{1} - \alpha_{3})(\beta_{1} - \alpha_{4})}{(\beta_{1} - \beta_{2})(\beta_{1} - \beta_{3})(\beta_{1} - \beta_{4})}, \]
\[ e_{2} = \frac{2(\beta_{2} - \alpha_{1})(\beta_{2} - \alpha_{2})(\beta_{2} - \alpha_{3})(\beta_{2} - \alpha_{4})}{(\beta_{2} - \beta_{1})(\beta_{2} - \beta_{3})(\beta_{2} - \beta_{4})}. \tag{10} \]

After several cumbersome calculations one can obtain the general solution describing the extended double-Kerr problem, writing the Ernst complex potential \( \mathcal{E} \) and the corresponding metric functions \( f, \omega, \gamma \) in the following explicit form
This latest is an eight parametric solution represented by the values of the quantities \( \alpha_n \), \( n = 1, 4 \) and \( \beta_j \), \( j = 1, 2 \). However, when is fixed the values of the \( \alpha_n \) as reals, we are representing the general solution for a binary system composed by two Kerr black holes. Finalizing this section, it is very important to comment that the above solution was constructed assuming asymptotic flatness at spatial infinity, where \( f \rightarrow 1 \), \( \gamma \rightarrow 0 \) and \( \omega \rightarrow 0 \) (in the absence of NUT sources), but besides the metric functions \( \gamma \) and \( \omega \) are obeying automatically the following conditions on the symmetry axis: \( \gamma(\alpha_1 < z < \infty) = \gamma(-\infty < z < \alpha_4) = 0 \) and \( \omega(\alpha_1 < z < \infty) = 0 \), establishing an elementary flatness on the upper part of the symmetry axis.

### III. THE FOUR-PARAMETRIC SOLUTION

If the binary system is located on the symmetry axis in such way that the roots \( \alpha_n \) satisfy \( \sum \alpha_n = 0 \), it can be reduced one parameter, describing the solution by only seven parameters. Following with the idea of vanishing the presence of the NUT sources between the objects and lower part: regions \( (\alpha_3 < z < \alpha_2) \) and \( (-\infty < z < \alpha_4) \), for making regular the symmetry axis outside the sources, and determine the solution for two counter-rotating black holes separated by a massless strut representing the well-known conical line singularity [2]. We must impose the following two conditions on the metric function \( \omega \):

\[
\omega(\rho = 0, \alpha_2 < z < \alpha_3) = 0, \quad \omega(\rho = 0, -\infty < z < \alpha_4) = 0.
\]

By noting that the second condition in (12) is similar to vanishing the gravitomagnetic monopole (NUT parameter), which also can be determined asymptotically in the Ernst potential on the symmetry axis [4] as follows

\[
\text{Im}[c_1 + e_2] = 0,
\]

being \( e_1 \) and \( e_2 \) defined in (10). A straightforward simplification over these two conditions in (12) lead us to the following compact algebraic equations

\[
\begin{align*}
\text{Im} & \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & \gamma_{11} & \gamma_{12} \\
1 & \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & 1 & 0 \\
1 & \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & 0 & \kappa_{11} \\
1 & -\gamma_{11} & -\gamma_{12} & \gamma_{13} & \gamma_{14} & 0 & \kappa_{21} \\
0 & \kappa_{12} & \kappa_{13} & \kappa_{14} & 0 & \kappa_{22} \\
0 & \kappa_{22} & \kappa_{23} & \kappa_{24} & 0 & \kappa_{23} \\
0 & \kappa_{23} & \kappa_{24} & 0 & \kappa_{24}
\end{pmatrix} = 0, \\
\text{Im} & \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & -\gamma_{11} & -\gamma_{12} \\
1 & -\gamma_{11} & -\gamma_{12} & \gamma_{13} & \gamma_{14} & 1 & 0 \\
1 & \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & 0 & \kappa_{11} \\
1 & -\gamma_{21} & -\gamma_{22} & \gamma_{23} & \gamma_{24} & 0 & \kappa_{21} \\
0 & \kappa_{12} & \kappa_{13} & \kappa_{14} & 0 & \kappa_{22} \\
0 & \kappa_{22} & \kappa_{23} & \kappa_{24} & 0 & \kappa_{23} \\
0 & \kappa_{23} & \kappa_{24} & 0 & \kappa_{24}
\end{pmatrix} = 0.
\end{align*}
\]

If we are capable to solve these equations without constraints, it can be possible to reduce the seven parametric solution into a five parametric solution and represent the entire metric in similar way like in the double Reissner-Nordström problem [12, 13]. Unfortunately, this difficult task is far away of our purpose and we will restrict our solution in order to describe only the four parametric subclass.

![FIG. 1. Location of two unequal Kerr black holes on the symmetry axis, where \( \sum \alpha_n = 0 \), and the rods are separated if the condition \( R > \sigma_1 + \sigma_2 \) is satisfied.](image-url)
real constants \( \alpha_n \) which determine the location of the two Kerr black hole sources on the symmetry axis, in our particular case the constants \( \alpha_n \) can be reparametrized in the following form

\[
\alpha_1 = \frac{R}{2} + \sigma_1, \quad \alpha_2 = \frac{R}{2} - \sigma_1, \\
\alpha_3 = -\frac{R}{2} + \sigma_2, \quad \alpha_4 = -\frac{R}{2} - \sigma_2, \tag{15}
\]

where \( \sigma_1 \) and \( \sigma_2 \) describe the half length of the two rods representing the black hole horizons and \( R \) is the relative distance between the two centers as we show in the Figure 1. One important point to say, is that \( \sigma_1 \) and \( \sigma_2 \) can be expressed in terms of physical parameters like the Komar individual masses \( M_1 \) and \( M_2 \), Komar angular momenta per unit mass \( \alpha_1 \) and \( \alpha_2 \), and coordinate distance \( R \) \cite{12, 13}, and satisfy a relationship if we impose an extra condition in \cite{14} for solving these complicated equations. Where the Komar integrals for the individual mass \( M_i \) and angular momentum \( J_i \) can be calculated with the aid of Tomimatsu’s formulae \cite{14}

\[
M_i = -\frac{1}{8\pi} \int_{H_i} \omega \Psi_z d\phi dz, \\
J_i = -\frac{1}{8\pi} \int_{H_i} \omega \left( 1 + \frac{1}{2} \omega \Psi_z \right) d\phi dz, \tag{16}
\]

whose integrals are taken over the black hole horizon \( H_i = \{ \alpha_{2i} \leq z \leq \alpha_{2i-1}, \ 0 \leq \phi \leq 2\pi, \rho \rightarrow 0 \}, \ i = 1, 2. \) Moreover, the total mass \( M \) can be considered as the sum of the individual masses \( M_1 \) and \( M_2 \) if the conditions established in \cite{12} for making regular the symmetry axis outside the sources are true. In order to solve the system of equations \cite{14}, we look again the expression \cite{4}, where the total mass \( M \), can be calculated asymptotically by making a first order expansion in the Ernst potential on the symmetry axis, having the next relationship

\[
\text{Re}[\epsilon_1 + \epsilon_2] = -2M, \tag{17}
\]

and the substitution of \cite{10} into \cite{17} yields the equation

\[
\beta_1 + \beta_2 + \tilde{\beta}_1 + \tilde{\beta}_2 = -2M, \tag{18}
\]

implying several possibilities on the relations between the betas and the total mass \( M \). The simplest choice who is helping us to obtain an analytical solution for the algebraic equations \cite{14} is the relation \( \beta_1 + \beta_2 = -M \), where it can be possible to describe the unequal counter-rotating case and derive an explicit form for the betas.

A simple calculation lead us to the following result

\[
\beta_{1,2} = -\frac{M \pm \sqrt{p + q}}{2}, \\
p = R^2 - M^2 + 2(\epsilon_1 - \epsilon_2 R), \\
q = \frac{2\sqrt{(R^2 - M^2)(M^2 R^2 - \epsilon_2^2)(M^2 - 2\epsilon_1 M^2 + \epsilon_2^2)}}{M(MR + \epsilon_2)}, \tag{19}
\]

\[
\epsilon_{1,2} = \sigma_1^2 \pm \sigma_2^2,
\]

where the subindex 1 and 2 are associated with the sign + and − respectively. Next we present the four parametric solution for two unequal counter-rotating black holes, expressing the Ernst potential and metric functions in terms of the parameters \( M, \ R, \sigma_1 \) and \( \sigma_2 \).

\[
\mathcal{E} = \frac{\Lambda + 2\Gamma}{\Lambda - 2\Gamma}, \quad f = \frac{|\Lambda|^2 - 4|\Gamma|^2}{|\Lambda|^2}, \quad \omega = -2\text{Im}[\left( \Lambda - 2\Gamma \right)G], \\
\epsilon^{2\gamma} = \frac{|\Lambda|^2 - 4|\Gamma|^2}{256\sigma_1^2 \sigma_2^2 (M^2 R^2 - \epsilon_2^2)^2 r_1 r_2 r_3 r_4}, \\
\Lambda = 4\sigma_1 \sigma_2 (M^4 - \epsilon_2^2) (r_1 r_2 + r_3 r_4) + \left( M^4 R^2 + \epsilon_2^2 (R^2 - 2M^2) \right) (r_1 - r_2) (r_3 - r_4) + \left( M^4 - 2M^2 R^2 + \epsilon_2^2 \right), \\
\Gamma = \sigma_1 (M + \epsilon_2 / M) [2\sigma_2 M^2 (\epsilon_2 - R^2) (r_3 + r_4) - R (M^2 - \epsilon_2) (r_3 - r_4)] - \sigma_2 (M - \epsilon_2 / M), \\
G = 2z \Gamma + 4\sigma_1 \sigma_2 R (M^4 - \epsilon_2^2) (r_1 r_2 - r_3 r_4) + \sigma_1 (R^2 + \epsilon_2) \left( M^2 - \epsilon_2 \right) (r_1 r_2 + r_3 r_4), \\
\sigma_2 (R^2 - \epsilon_2) (M^2 + \epsilon_2) (r_1 r_2 - r_3 r_4) + 2i \epsilon_2 \delta (r_1 r_2 - r_3 r_4) - \sigma_1 (M + \epsilon_2 / M), \\
\times \{ 2\sigma_2 R \left[ \epsilon_2^2 + M^2 (M^2 - R^2 - \epsilon_2^2) (r_3 + r_4) + 2M^2 \epsilon_1 (R^2 - \epsilon_2) + (2M^2 - R^2) \epsilon_2^2 - M^2 R^2 \right] (r_3 - r_4) \}
\]

\[+ \delta \left[ R (r_3 - r_4) - 2\sigma_2 (r_3 + r_4) \right] + \sigma_2 (M - \epsilon_2 / M) \{ 2\sigma_1 R \left[ \epsilon_2^2 + M^2 (M^2 - R^2 - \epsilon_2^2) \right] (r_1 + r_2) \}
\]

\[+ \left[ 2M \epsilon_1 (R^2 + \epsilon_2) + (2M^2 - R^2) \epsilon_2^2 - M^2 R^2 \right] (r_1 - r_2) - \delta \left[ R (r_1 - r_2) + 2\sigma_1 (r_1 + r_2) \right] \}, \tag{20}
\]

having \( r_n \) the following parametrized form

\[
r_{1,2} = \sqrt{\rho^2 + \left( \frac{1}{2} R \mp \sigma_1 \right)^2}, \\
r_{3,4} = \sqrt{\rho^2 + \left( \frac{1}{2} R \mp \sigma_2 \right)^2}, \tag{21}
\]

being 1, 3 and 2, 4 associated with − and + respectively. Obviously the solution \cite{20} has not the equatorial anti-symmetry property in the sense of \cite{6}, but only in the
case where both constituents are equal. Nevertheless, interchanging the physical properties between the constituents (i.e. \(1 \leftrightarrow 2\)) and applying the transformation \(z \rightarrow -z\), only the metric function \(\omega\) has a change in the global sign. Furthermore, rewriting the Ernst potential on the upper part of the symmetry axis we have that
\[
e(z) = \frac{e_+}{e_-} \rightarrow e(z) = e_+ + e_- \to e(z) = \frac{2M^3 - MR^2 - 2M\epsilon_1 + 2\epsilon_2 R}{4M} - i \frac{(R^2 - M^2)(M^2R^2 - \epsilon_2^2)(M^4 - 2\epsilon_1 M^2 + \epsilon_2^2)}{2M(MR - \epsilon_2)} \tag{22}
\]
such that the total angular momentum of the system can be calculated asymptotically by making a second order expansion, having
\[
J = \frac{\epsilon_2}{2M} \sqrt{\frac{(R^2 - M^2)(M^4 - 2\epsilon_1 M^2 + \epsilon_2^2)}{M^2R^2 - \epsilon_2^2}} \tag{23}
\]
and there exist a change in the global sign of the total angular momentum, at the moment of interchange the properties of both constituents: \(J = -J_{(1 \leftrightarrow 2)}\), which means that \(\text{[20]}\) is representing the case of two unequal counter-rotating black holes. On the other hand, the analysis of the energy-momentum tensor associated with the strut, give us the expression for the interaction force between the constituents via the formula \([2, 15]\)
\[
\mathcal{F} = \frac{1}{4} (e^{-\gamma_0} - 1) = \frac{M^4 - (\sigma_1^2 - \sigma_2^2)^2}{4M^2(R^2 - M^2)^2} \tag{24}
\]
being \(\gamma_0\) the constant value of the metric function \(\gamma\) evaluated on the corresponding region of the strut. It is worthwhile to mention, that the interaction force for two equal counter-rotating black holes \((M_1 = M_2 = m, a_1 = -a_2 = a)\) in the non-extreme case: \(M = 2m, \sigma_1 = \sigma_2 = \sigma\), and the extreme case: \(M = 2m, \sigma_1 = \sigma_2 = 0\), has the same following expression \([1, 14, 18]\)
\[
\mathcal{F} = \frac{m^2}{R^2 - 4m^2} \tag{25}
\]
In the absence of rotation: \(a_1 = a_2 = 0, \sigma_1 = M_1, \sigma_2 = M_2\), and \(M = M_1 + M_2\), we recover the well-known expression for the interaction force between two Schwarzschild black holes \([13, 17]\)
\[
\mathcal{F} = \frac{M_1 M_2}{R^2 - (M_1 + M_2)^2} \tag{26}
\]

IV. THE PHYSICAL PARAMETRIZATION

The calculation of the individual Komar mass and Komar angular momentum, can help us to understand the relation between the quantities \(\sigma_1, \sigma_2\) and the physical parameters of the system. In order to observe this, one can use the Tomimatsu’s formulae \([16]\) in the following simplified form \([14, 18]\)
\[
M_i = \frac{\omega_i}{4} |\Psi_{\rho=0, z=\alpha_1} - \Psi_{\rho=0, z=\alpha_2}|, \\
J_i = \frac{\omega_i}{2}(M_i - \sigma_i), \quad i = 1, 2, \tag{27}
\]
being \(\omega_i\) the constant value of the corresponding metric function \(\omega\) evaluated over the horizon of each constituent. In this case the horizons are defined as two null hypersurfaces: \(\rho = 0, -\sigma_1 < z - R/2 < \sigma_1\) and \(\rho = 0, -\sigma_2 < z + R/2 < \sigma_2\), separated by a massless strut. A straightforward calculation lead us to the following system of equations for solving
\[
M_1 = \frac{M^2 + \epsilon_2}{2M}, \quad M_2 = \frac{M^2 - \epsilon_2}{2M}, \tag{28}
\]
\[
J_1 = \frac{M_1}{2M} \sqrt{\frac{(R + M)(MR - \epsilon_2)(M^4 - 2\epsilon_1 M^2 + \epsilon_2^2)}{(R - M)(MR + \epsilon_2)}}, \\
J_2 = -\frac{M_2}{2M} \sqrt{\frac{(R + M)(MR + \epsilon_2)(M^4 - 2\epsilon_1 M^2 + \epsilon_2^2)}{(R - M)(MR - \epsilon_2)}}. \tag{29}
\]

From the equation \([25]\) it can be possible to show that the total mass \(M = M_1 + M_2\), but besides we obtain the relation
\[
\sigma_1^2 - \sigma_2^2 = M_1^2 - M_2^2, \tag{30}
\]
and after the substitution of \([30]\) into \([29]\), one can obtain the following expressions
\[
\sigma_1 = \sqrt{M_1^2 - a_1^2 \frac{(R - M_2)^2 - M_1^2}{(R + M_2)^2 - M_1^2}}, \\
\sigma_2 = \sqrt{M_2^2 - a_2^2 \frac{(R - M_1)^2 - M_2^2}{(R + M_1)^2 - M_2^2}}. \tag{31}
\]
whereas the relation \([30]\) implies that is necessary satisfy a relation between the five parameters
\[
M_1 a_1 + M_2 a_2 + R(a_1 + a_2) - M_2 a_1 - M_1 a_2 = 0, \tag{32}
\]
where this relation contains in a more general form the two assumptions made by Bonnor in order to remove the contribution from the massless spinning rods outside the sources \([10]\). In this particular case, the expression \([23]\) is representing exactly the total angular momentum \(J\) as the sum of the individual angular momenta of both constituents, defined as
\[
J = M_1 a_1 + M_2 a_2, \tag{33}
\]
and the interaction force depicted by the expression \([24]\), can be reduced to the simple formula for the interaction force between two non-rotating black holes (two
Schwarzschild black holes) described by \(26\), with the simple difference that the two angular momenta per unit mass are related by the equation \(32\). Nonetheless, as we mention before, this condition can be interpreted that on the symmetry axis are being removed the contributions in the interaction force made by the two semi-infinite massless spinning rods located on the upper and lower part, just like the finite massless spinning rod between the two constituents \(10\), and hence the force seems to be as one Schwarzschild type. Moreover, if the condition \(32\) relating the five independent parameters is not satisfied, it could be possible to observe the proper contribution of the spin-spin interaction to this force \(18\). On the other hand, we notice that the above expressions \(31\) for the quantities \(\sigma_1\) and \(\sigma_2\), can recover the case for one isolate black hole, by imposing \(R \to \infty\) or just making zero the physical properties of the other body (in this case the respective mass). In the table we show some numerical values described with 3 decimals.

### Table I. Particular values for the 4-parameter subclass of the DK problem.

| \(\sigma_1\) | \(\sigma_2\) | \(M_1\) | \(M_2\) | \(a_1\) | \(a_2\) | \(R\) | \(J\) |
|-------------|-------------|--------|--------|-------|-------|-------|-------|
| 4.973       | 1.931       | 5      | 2      | 1.304 | -3    | 8     | 0.819 |
| 1.609       | 5.881       | 2      | 6      | 5.444 | -2.333 | 10    | -3.111 |
| 0.681       | 1.861       | 1      | 2      | 2.5   | -1.5  | 4     | -0.5  |
| 1.972       | 1.972       | 2      | 2      | 1     | -1    | 5     | 0     |
| 3           | 1           | 3      | 1      | 0.667 | -2    | 4     | 0     |

In the table the first three sets present different values for the masses and angular momenta per unit mass, where the angular momentum of each component have different sign. The fourth set is describing the case of two equal counter-rotating black holes. However, the fifth set of values is interesting in the sense that we are describing the case in which the two angular velocities are null, just like the total angular momentum of the system \((J = 0)\), representing a static case in which the horizons of the two constituents are touching each other and the system is stopped, where the system of two bodies becomes into one Schwarzschild black hole. Below are displayed the final expressions for \(E\), \(f\), \(\gamma\) and \(\omega\), describing the four parametric solution for two unequal counter-rotating black holes.

\[
E = \frac{\Lambda + 2\Gamma}{\Lambda - 2\Gamma}, \quad f = \frac{|\Lambda|^2 - 4|\Gamma|^2}{|\Lambda - 2\Gamma|^2}, \quad \omega = -\frac{2\text{Im}[(\Lambda + 2\Gamma)G]}{|\Lambda - 2\Gamma|^2}, \quad c^{2\gamma} = \frac{|\Lambda|^2 - 4|\Gamma|^2}{16\sigma_1^2\sigma_2^2[R^2 - (M_1 - M_2)^2]^2r_1r_2r_3r_4},
\]

\[
\Lambda = 4\sigma_1\sigma_2M_1M_2(r_1r_2 + r_3r_4) - \rho(r_1 - r_2)(r_3 - r_4) + \sigma_1\sigma_2(R^2 - M_1^2 - M_2^2)(r_1 + r_2)(r_3 + r_4) - iv[\sigma_1(r_1 - r_2)(r_3 + r_4) - \sigma_2(r_1 + r_2)(r_3 - r_4)],
\]

\[
\Gamma = -\sigma_1M_1[\sigma_2(R^2 - M_1^2 - M_2^2)(r_1 + r_2) + 2M_2^2R(r_3 - r_4)] - \sigma_2M_2[\sigma_1(R^2 + M_1^2 - M_2^2)(r_1 + r_2) - 2M_2^2R(r_1 - r_2)] - iv[\sigma_1M_1(r_3 - r_4) - \sigma_2M_2(r_1 - r_2)],
\]

\[
G = 2\Gamma + 4\sigma_1\sigma_2M_1M_2R(r_1r_2 - r_3r_4) + \sigma_1M_1^2(R^2 + M_1^2 - M_2^2)(r_1 + r_2)(r_3 - r_4) + \sigma_2M_2^2(R^2 - M_1^2 + M_2^2)(r_1 - r_2)(r_3 + r_4) + iv[\sigma_1M_1(R(r_3 - r_4) - 2\sigma_2(r_3 + r_4)) + \sigma_2M_2[R(r_1 - r_2) + 2\sigma_1(r_1 + r_2)]],
\]

with

\[
\mu = (1/2) \left[ (\sigma_1^2 + \sigma_2^2)(R^2 - M_1^2 - M_2^2) - (M_1^2 + M_2^2)R^2 + (M_1^2 - M_2^2)^2 \right],
\]

\[
\nu = (1/\sqrt{2})(R - M_1 - M_2) \sqrt{\alpha_1^2(R + M_1 - M_2)^2 + a_2^2(R - M_1 + M_2)^2},
\]

and the Ernst potential on the symmetry axis is defined as follows

\[
e(z) = \frac{e_+}{e_-},
\]

\[
e_\pm = z^2 \mp Mz + M_1M_2 - \frac{R^2}{4} + \left(\frac{R^2 - M^2}{4}\right) F^{\pm1} - \frac{R - M}{R + M} \left[ \frac{R + M}{2} \right] F^{\pm1/2} - \frac{i}{\sqrt{2}} \sqrt{a_1^2F + a_2^2F^{-1}},
\]

with \(a_1\) and \(r_n\) were defined by the formulas \(31\) and \(21\) respectively. But besides \(a_1\) and \(a_2\) are related by the expression \(32\).

Reviewing now the thermodynamical properties of black holes, it can be mentioned that for each component
of the binary system, the Smarr formula for the mass [3] holds
\[ M_i = \frac{\kappa_i S_i}{4\pi} + 2\Omega_i a_i M_i = \sigma_i + 2\Omega_i a_i M_i, \quad i = 1, 2, \quad (37) \]
where \( \kappa_i \) is the surface gravity, \( S_i \) is the area of the horizon, \( \Omega_i \) the angular velocity and \( a_i \) the angular momentum per unit mass for each constituent. For calculating \( \kappa_i \) and \( \Omega_i \), one can use the following formulas [18, 19]
\[ \kappa_i = \sqrt{-\omega_i^{-2} e^{2\gamma}}, \quad \Omega_i = \omega_i^{-1}, \quad (38) \]
being \( \omega_i \) and \( \gamma_i \) the constant values of the corresponding metric functions \( \omega \) and \( \gamma \) evaluated on the horizon of each constituent, where \( e^{2\gamma} \) is negative at the horizon [18]. Using the solution defined by (34), it can be possible to obtain straightforwardly the following expressions for the angular velocity \( \Omega_i \), surface gravity \( \kappa_i \) and area of the horizon \( S_i \)
\[ \Omega_i = \frac{a_i ((R - M_2)^2 - M_i^2)}{2M_i (M_i + \sigma_i) [(R + M_2)^2 - M_i^2]}, \]
\[ \kappa_i = \frac{\sigma_i (R + M_1 - M_2)}{2M_i (M_i + \sigma_i) (R + M)}, \]
\[ S_i = \frac{8\pi M_i (M_i + \sigma_i) (R + M)}{(R + M_1 - M_2)}, \]
\[ \Omega_2 = \Omega_{i(1+2)}, \quad \kappa_2 = \kappa_{i(1+2)}, \quad S_2 = S_{i(1+2)}. \]
When the sources are far away from each other, one can arrive to
\[ \Omega_i = \frac{a_i F_i}{2M_i (M_i + \sigma_i)}, \]
\[ F_1 = 1 - \frac{4M_2}{R} + 8\frac{M_2^2}{R^2} + O\left(\frac{1}{R^3}\right), \quad (40) \]
\[ F_2 = 1 - \frac{4M_1}{R} + 8\frac{M_1^2}{R^2} + O\left(\frac{1}{R^3}\right), \]
having the possibility to see, that the proper contribution to the angular velocity \( \Omega_i \) from the angular momentum \( J_i = M_i a_i \), begins at the third order expansion (i.e \( \Omega_i = O(1/R^3) \)) [3]. Finally, as one special case of our solution it can be possible to recover the simplest case related with two equal counter-rotating black holes by applying the assumptions: \( M_1 = M_2 = m, \quad \sigma_1 = \sigma_2 = \sigma \) and \( a_1 = -a_2 = a \), having the unique \( \sigma \) as
\[ \sigma = \sqrt{m^2 - a^2 \left(\frac{R - 2m}{R + 2m}\right)}, \quad (41) \]
turning this particular case into a 3-parameter subclass of the general double-Kerr solution [9], where the total angular momentum of the system \( J = 0 \).

V. CONCLUDING REMARKS

In the present paper we worked out an exact solution describing two unequal counter-rotating black holes separated by a massless strut. This problem can be more interesting with respect the case when both constituents are equal [5], in the sense that one can see how the physical properties of one body are being affected by the other one. Nonetheless, it is worthwhile to mention that the technical details for removing the presence of the NUT sources outside the two rotating black holes is not a trivial problem, and for that reason, we had success to study only the counter-rotating case.

In a near future, we hope to find an alternative way for calculating the more general problem related with the counter/co-rotating case.

Due the complexity for expressing the quantities \( \sigma_1 \) and \( \sigma_2 \) in terms of physical parameters, we have been capable to derive a 4-parameter subclass involving a simple algebraic relation between the five physical parameters. This relation is containing in a more general form the two assumptions made by Bonnor [10], in order to avoid the contribution of the spinning massless rods located on the symmetry axis, outside the regions of the rods describing the black holes and therefore the interaction force seems to be like Schwarzschild type. On the other hand, in the extreme case: \( \sigma_1 = 0 \) and \( \sigma_2 = 0 \), it seems like the unequal and opposite angular momentums per unit mass, are exceeding in absolute value to the respective (positive) value of the masses: \( |a_1| > M_1 > 0 \), according to the expressions
\[ a_1 = \epsilon M_1 \sqrt{\frac{(R + M_1)^2 - M_1^2}{(R - M_1)^2 - M_1^2}}, \quad (42) \]
\[ a_2 = -\epsilon M_2 \sqrt{\frac{(R + M_2)^2 - M_2^2}{(R - M_2)^2 - M_2^2}}, \quad \epsilon = \pm 1. \]

However, in this particular case, the condition established between the five parameters is satisfied only if both constituents are equal: \( M_1 = M_2 = m \) and \( a_1 = -a_2 = a \), where the total angular momentum of the system is zero, establishing a distance \( R \) in which the extremality condition occurs [3]
\[ R = \frac{2m(a^2 + m^2)}{a^2 - m^2}, \quad |a| > m > 0, \quad (43) \]
Finally it is worth mentioning that the expressions defined by (42), relating the masses and angular momenta, have the same form as the relations presented in [20] referring to a binary system of two extreme Reissner-Nordström black holes separated by a strut.

ACKNOWLEDGEMENTS

We would like to thank N. Gürlebeck, V. Perlick and E. Ruiz for helpful and interesting discussions about this topic. This work was supported by the Deutscher Akademischer Austausch Dienst (DAAD) Germany, Grant No. A/10/77743. CL would like to acknowledge support through the DFG Research Training Group
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