Violation of Strong Energy Condition in Effective Loop Quantum Cosmology

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In loop quantum cosmology nonperturbative modification to a scalar matter field at short scales implies inflation which also means a violation of the strong energy condition. In the framework of effective Hamiltonian we discuss the issue of violation of the strong energy condition in the presence of quantum geometry potential. It shows that the appearance of quantum geometry potential strengthens the violation of the strong energy condition in small volume regions. In the small volume regions superinflation can easily happen. Furthermore, when the evolution of the universe approaches the bounce scale, this trend of violating the strong energy condition can be greatly amplified.

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I. INTRODUCTION

In classical general relativity the singularity theorems play an important role in the quest of the property of spacetime evolution. These theorems predict that if the matter stress tensor satisfies the so-called strong energy condition then the backward evolution of a globally hyperbolic spacetime is singular. It means that the spacetime is geodesically incomplete. Specialized to the context of isotropic and homogeneous cosmology, the singularity theorem tells us that if the matter satisfies the strong energy condition the scale factor (or the size of the universe) will vanish in a finite time when the universe evolves backward. This means that the universe begins from a singular point (corresponding to the vanishing scale factor), and at the singular point matter density and spacetime curvature diverge.

The finite expansion time of universe implies that there exists a particle horizon, which is defined by the proper distance a particle traveled in the past in the expanding universe. The particle horizon demarcates the causal contact region in which events can connect each other by light signal. So the presently observable universe is limited by the particle horizon in principle. The particle horizon now is far bigger than that in the early universe, but it does not correspond to the only one horizon in the early universe when evolved backward. In fact, according to the standard big bang model the horizon of the present universe is developed from many regions which had no causal contact between them in the early universe. While according to the observed data the cosmic microwave background (CMB) radiation highly abides the Planck black body radiation spectrum. This shows that the early universe in a large region is in a thermal equilibrium state. The contradiction between the present homogeneous CMB radiation in a large region and the apparently small horizon in the early universe is called the horizon puzzle in cosmology. In the 1980s Guth proposed an inflation mechanism to solve this problem. It was supposed that the early universe experienced a short time inflation such that the particle horizon increased quickly. In this inflation model the early universe is described by a scalar field with self-interaction potential, while it seems unnatural that in order to have sufficient amount of inflation with graceful exit the potential must be fine-tuned. However, the inflation mechanism is still a successful way to solve a series of problems in cosmology. Finally, it is clear that the inflation model violates the strong energy condition.

We have known that when the scale factor tends to zero there exists a singular spacetime point with diverging matter density. It is generally accepted that the appearance of singularity in classical general relativity means the failure of classical theory in the very small volume regions of spacetime. So it is expected that in the small volume regions (Planck scale) quantum gravity theory should replace the classical gravity and resolve the singularity appearing in classical general relativity. So far, there are two promising candidates for quantum gravity. One is string theory and the other is loop quantum gravity (LQG). LQG is background independent and nonpertubative. The underlying geometry is discrete at Planck scale. Loop quantum cosmology (LQC) inherits this feature by introducing symmetry reduction on the level of quantum state (known as spin network state). Applying techniques developed in the full theory (LQG) the classical Hamiltonian constraint which is reformulated in terms of a new set of variables is quantized and the obtained quantum Hamiltonian constraint can be interpreted as a quantum difference equation which evolves
non-singularly through the big bang point\textsuperscript{[6, 7]}. The quantum difference equation contains the information as to the evolution of the universe at the Planck scale. However the difference equation is difficult to solve. There are lacks of exact solutions as well as physical inner product. All these prohibit us to know about the semiclassical behavior of the quantum difference equation. Approximated way is developed to tackle this issue. An exact coherent state was constructed, but the explicit form of this coherent state depends on numerical result of the quantum behavior of the quantum difference equation. Approximated way is developed to tackle this issue. An exact coherent state taking an effective Hamiltonian to investigate these phenomena carefully. The effective continuum Hamiltonian is extracted from the difference equation and assumed to live on the pseudo-Riemann manifold. This depends on two steps. Namely, a continuum approximation was taken such that the fundamental difference is replaced by a second-order differential equation (Wheeler-DeWitt equation) and then a WKB route is followed to obtain an effective Hamiltonian\textsuperscript{[9]}. The valid region of the effective Hamiltonian is above a bounce scale which also sets the smallest scale for semiclassical region\textsuperscript{[16]}. Below the bounce scale the effective Hamiltonian breaks down and is carried over by the quantum difference equation. At large volume regions ($p \gg p_0$, $p_0$ demarcates the different scale for effect of gravity) the effective Hamiltonian recovers the classical form (classical Hamiltonian constraint).

In the framework of effective Hamiltonian the inflation and the bounce happen independently of the initial condition and quantization ambiguity parameter. These effects are natural results of the quantum geometry. It shows that the inflationary cosmology avoids singularity by experiencing a bounce. This implies that the strong energy condition (SEC) is violated in the effective theory. However the violation of SEC does not only come from the contribution of the modified matter field. In the effective theory there is the “quantum geometry potential” which is suppressed in large scale ($p \gg p_0$), moreover, although the quantum geometry potential appears from the WKB approximation it roots in the quantum difference equation and predicts the existence of bounce as well as the bounce scale\textsuperscript{[10]}. Therefore, it is necessary to investigate its role in the evolution of the early universe. At the region $p_{\text{bounce}} < p < p_j$ ($p_{\text{bounce}}$ denotes the bounce scale) it is necessary to consider about the effect of the quantum geometry potential, especially in the region approaching the bounce scale where the quantum geometry potential greatly changes the behavior of the effective Hamiltonian. In this paper we discuss the issue of violation of SEC in flat cosmology ($\eta = 0$) and mainly focus on the role of the quantum geometry potential. The appearance of the quantum geometry potential introduces a bounce which avoids the singular evolution of the universe. So it can be expected that the existence of the quantum geometry potential can help the violation of SEC. We shall show in this paper that the appearance of quantum geometry potential will strengthen the violation of SEC in small volume regions, furthermore, when the evolution of the universe approaches the bounce scale, this trend of violating SEC can be greatly amplified.

Based on the same slowly varying condition as in \textsuperscript{[9]}, there is a direct way to obtain the effective Hamiltonian by using WKB approximation which is referred as discrete correction to the effective Hamiltonian\textsuperscript{[17]}. In small volume as noted in \textsuperscript{[17]} the modified matter density and pressure go over to those defined in \textsuperscript{[9]}. In this paper our discussion of violation of SEC is in small volume, so we do not consider about the discrete correction to the effective Hamiltonian. Recently, different from the WKB way there is another approximated method based on constructing the semiclassical state which predicts the quadratic energy density modifications to the Friedmann equation\textsuperscript{[18]}. However, in \textsuperscript{[8]} the analysis shows that the WKB approximation matches well with the semiclassical state till approaching the Planck scale. So, it is not expected that our analysis based on the WKB way can be changed appreciably by the new approximated method.

This article is organized as follows. In Sec\textsuperscript{[11]} we simply introduce the process of obtaining the effective Hamiltonian and present the effective state parameter equation in the context of effective Hamiltonian. Then in Sec\textsuperscript{[11]} for a simple scalar field we analyze the violation of SEC in the presence of the quantum geometry potential. Finally, the Sec\textsuperscript{[11]} is the conclusion.
II. SEC IN THE FRAMEWORK OF EFFECTIVE HAMILTONIAN

In LQC the Hamiltonian constraint is described by a difference equation\(^7\)

\[
0 = A_{\mu+4\mu_0} \psi_{\mu+4\mu_0} - (2 + 4\mu_0^2\gamma^2\eta) A_{\mu} \psi_{\mu} + A_{\mu-4\mu_0} \psi_{\mu-4\mu_0} + 8\kappa\gamma^2 \mu_0 \left( \frac{1}{6\gamma^2} \right) l_p^2 H_m(\mu) \psi_{\mu}, \forall \mu \in R \tag{1}
\]

where \(A_{\mu} := |\mu + \mu_0|^{3/2} - |\mu - \mu_0|^{3/2}\), and \(H_m(\mu)\) is the eigenvalue of matter Hamiltonian which is assumed coupling with gravity via metric component and \(\mu_0\) is a dimensionless parameter whose value is fixed by the length of fiducial curve. Here, \(\eta = 0\) and \(\eta = 1\) correspond to flat and closed models, respectively.

Because of the nonseparable structure of the kinematical Hilbert space, there are infinite solutions for the Hamiltonian constraint. In the absence of a physical inner product it is helpful to stipulate the slowly varying property for a class of the solutions. So the solution \(\psi_{\mu}\) and coefficients in the difference equation can be expressed as the function of the continuous variable \(p(\mu)\), \(p(\mu) := \frac{1}{6} \gamma^2 l_p^2 \mu_0\). In terms of slowly varying \(\psi(p)\) the difference equation can be approximated as a second order differential equation

\[
0 = B_0(p, p_0) \psi(p) + 4p_0 B_-(p, p_0) \psi'(p) + 8p_0^2 B_+(p, p_0) \psi''(p), p_0 := \frac{1}{6} \gamma^2 l_p^2 \mu_0, \tag{2}
\]

where

\[
B_0(p, p_0) = A(p + 4p_0) - \left( 2 + 144\gamma^2 \eta \right) A(p) + A(p - 4p_0) + \left( 288\kappa \gamma^2 \right) H_m(\mu),
\]

\[
B_{\pm}(p, p_0) = A(p + 4p_0) \pm A(p - 4p_0),
\]

and

\[
A(p) = |p + p_0|^{3/2} - |p - p_0|^{3/2}.
\]

And then the WKB way is used to get an effective Hamiltonian which is given by

\[
H^{eff}(p, K, \phi, \phi) = -\frac{1}{\kappa} \left[ \frac{B_+(p, p_0)}{4p_0} K^2 + \eta \frac{A(p)}{2p_0} \right] + V_g + H_m(p, \phi, \phi), \tag{3}
\]

where the Poisson bracket between the extrinsic curvature \(K\) and the triad variable \(p\) is \(\frac{\mu_0}{288\kappa l_p^2}\). Here, \(V_g = \left( \frac{\mu_0}{288\kappa l_p^2} \right) \{ B_+(p) - 2A(p) \} \). The quantum geometry potential is denoted as \(V_g\) and is negative for \(p > 0\) (we only consider the positive value of \(p\) because the negative \(p\) corresponds to an inverse orientation universe). For \(p \gg p_0\), the classical Hamiltonian constraint is recovered. Next, we are limited in the flat cosmology, i.e., \(\eta = 0\).

In terms of the effective Hamiltonian and comparing with the usual FRW equation, the effective perfect fluid density and pressure are identified as

\[
\rho^{eff} = \frac{32}{3} \frac{\alpha}{a^4} (H_m + V_g),
\]

\[
P^{eff} = \frac{32}{9} \frac{\alpha}{a^4} \left\{ \left( 1 - \frac{a a'}{\alpha} \right) H_m - \frac{\partial}{\partial a} H_m \right\} + \frac{32}{9} \frac{\alpha}{a^4} \left\{ \left( 1 - \frac{a a'}{\alpha} \right) V_g - a V'_g \right\},
\]

where \(\alpha = \frac{B_+(p)}{4p_0}\), \(a\) is the FRW scale factor and the relation between \(p\) and \(a\) is \(p = \frac{a^3}{\alpha}\). The prime denotes \(\frac{\partial}{\partial a}\).

In the context of isotropic and homogeneous cosmology the SEC becomes \(4\pi G(\rho + 3P) \geq 0\), \(\rho \geq 0\), where \(\rho\) and \(P\) are the total energy density and pressure of the mater field. Usually SEC is described by a state parameter equation which is defined as

\[
\omega := \frac{P}{\rho}.
\]

In the framework of the effective Hamiltonian we still work in the continuous spacetime as in classical theory, so SEC can be employed in this effective theory. In terms of \(\omega^{eff}\) the effective state parameter is expressed as

\[
\omega^{eff} = \frac{P^{eff}}{\rho^{eff}} = \frac{1}{3} \left( 1 - \frac{a a'}{\alpha} \right) \left[ \frac{a}{\alpha} H_m + \frac{V'_g}{3} H_m + V_g \right]. \tag{5}
\]
In the large volume regions, the quantum geometry potential \( V_g \) is suppressed and tends to vanish, so the effective state parameter equation becomes

\[
\omega^{\text{eff}} \rightarrow \frac{1}{3} \left( 1 - \frac{a a'}{\alpha} \right) - \frac{a}{3} \frac{\varphi}{H_m}. \tag{7}
\]

The violation of SEC has been carefully discussed in the absence of the quantum geometry potential in [19] where in the small volume regions for a positive scalar field potential \( \omega^{\text{eff}} \rightarrow -1 \).

For the flat model (\( \eta = 0 \)), the effective Hamiltonian constraint behaves as

\[
H^{\text{eff}} = -\frac{1}{\kappa} \frac{B_+(p, p_0)}{4p_0} \kappa^2 + V_g + H_m = 0. \tag{8}
\]

The above equation implies that

\[
H_m + V_g = \frac{1}{\kappa} \frac{B_+(p, p_0)}{4p_0} \geq 0. \tag{9}
\]

In equation (9) the equality indicates an occurrence of bounce which also suggests that there exists a smallest scale (bounce scale \( p_{\text{bounce}} \)) and below this scale it is in a classically inaccessible region[16]. At the bounce scale, \( \rho^{\text{eff}} = 0 \).

So \( p_{\text{bounce}} \) is a singular point for the effective state parameter equation, in other words the effective state parameter equation defines illegally at \( p_{\text{bounce}} \). However the Hamiltonian equation behaves well at bounce scale. The region what we care about is above bounce scale. What is more, the effective Hamiltonian [8] needs that matter field \( H_m \) must be positive definite because the kinetic term and the quantum geometry potential are also negative value.

The occurrence of bounce (which also means \( \rho^{\text{eff}} \rightarrow 0 \) when \( p \) approaches \( p_{\text{bounce}} \)) essentially depends on the existence of the quantum geometry potential. Next, we will show the violation of SEC for a minimal coupled scalar field.

III. VIOLATION OF SEC FOR A SCALAR FIELD

In LQC in the semiclassical region (\( p_{\text{bounce}} < p < p_j \)) a classical scalar field gets modification. The modified scalar field is obtained by replacing the \( a^{-3} \) in the kinetic term by a function coming from the definition of the inverse triad operator [20, 21]. The modified scalar field with a self-interaction potential is given by

\[
H_m = \frac{1}{2} \left| \tilde{F}_{j,l}(a) \right|^{3/2} \varphi^2 + a^3 V(\phi), \tag{10}
\]

where \( \tilde{F}_{j,l}(a) = \left( \frac{3}{4} \gamma \mu_0 j!^2 p^{-1} \right) F_l \left( \frac{1}{4} \gamma \mu_0 j!^2 p^{-1} a^2 \right) \) is a smooth approximation (except at one point) of the inverse scale operator, and

\[
F_l(q) = \left( \frac{3}{2(l+2)(l+1)} \right) \left( (l+1) \left[ (q+1)^{l+2} - |q-1|^{l+2} \right] - (l+2)q \left( (q+1)^{l+1} - sgn(q-1) |q-1|^{l+1} \right) \right)^{1/7} \rightarrow \begin{cases} \frac{q^{-1}}{\left( \frac{3q}{l+1} \right)^{1/7}}, & (0 < q \ll 1). \end{cases} \tag{11}
\]

The \( j \) and \( l \) are two quantization ambiguity parameters with \( j \) being a half integer and \( l \in (0, 1) \). For large value of \( j \) it can lead to observable effects [20]. For the modified scalar field, the effective state parameter is

\[
\omega^{\text{eff}} = \frac{1}{3} \left( 1 - \frac{a a'}{\alpha} \right) - \frac{a}{3} \left( \frac{F_{j,l}(a)^{1/2} F_{j,l}(a)'}{p_\varphi^2 + 3a^2 V(\phi)} - \frac{\varphi}{H_m + V_g} \right) \tag{12}
\]

\[
= \frac{1}{3} \left( 1 - \frac{a a'}{\alpha} \right) - \frac{a}{3} \left( \frac{\varphi}{H_m + V_g} \right) \left( \frac{V_g'}{H_m + V_g} \right) \frac{1}{H_m + V_g} + a^3 V(\phi) - \frac{a}{3} \frac{V_g'}{H_m + V_g}, \tag{12}
\]
where \( q = (\frac{1}{3} \gamma \mu_0 j_p^2)^{-1} a^2 \).

In the small volume regions, i.e., \( 0 < q \ll 1 \), the effective state parameter equation is

\[
\omega^{eff} = \frac{1}{3} \left( 1 - \frac{a^{\prime \prime}}{\alpha} \right) - \frac{l}{1-l} + \frac{a^3 V(\phi)}{(1-l)(H_m + V_g)} - \frac{3V_g + (1-l) a V^\prime_g}{3(1-l)(H_m + V_g)}. \tag{13}
\]

The difference equation (11) evolves forward with the fixed step \( 4p_0 \). When scale is below the fixed step it is in deep quantum range where the continuous approximation becomes poor. So we can safely say that the region (semiclassical region) what we consider about should be above the fixed step. For convenience, we take quantum range where the continuous approximation becomes poor. So we can safely say that the region (semiclassical range) what we consider about should be above the fixed step. For convenience, we take quantum range where the continuous approximation becomes poor.

\[
-3V_g + (1-l) a V^\prime_g = \begin{cases} 4.71 + (1-l)12.73 \frac{p_0}{p}, & p = 5p_0 \\ \to 0, & p \gg p_0 \end{cases}
\]  \tag{15}

Because \( -\frac{3V_g + (1-l) a V^\prime_g}{3(1-l)(H_m + V_g)} \) < 0, the contribution of the quantum geometry potential always strengthens the violation of SEC. This shows that the appearance of the quantum geometry potential not only determines the occurrence of a bounce, but also leads to an accelerated expansion of the universe.

So the effect of the first term becomes negligible compared with the left three terms.

The second term \( \frac{1}{3} (1 - \frac{a^{\prime \prime}}{\alpha}) \) in (13) is independent of the particular form of the matter field and completely comes from the modification of the inverse scale factor to the matter field in the small volume regions. This depends on that in the small volume regions the inverse scale factor is an increasing function with power \( \frac{1}{1-l} \) which greatly differs from its classical behavior. The value of the second term is only determined by the quantization ambiguity parameter \( l \). It is clear that \( l \in (0, 1) \) makes \( \frac{1}{1-l} > 1 \). So this term causes \( \omega^{eff} \ll -1 \) and implies the possibility of a superinflation. The final result will rely on this term and the last two terms.

The last two terms in (13) correspond to the matter part and the quantum gravity potential, respectively. From (9) we know that the denominators of these two terms are all positive. Whether these two terms strengthen or weaken a violation of SEC will depend on the signs of their numerators. The third term indicates that for the violation of SEC the potential of the scalar field behaves to differ from the view in (13) where an appearance of a positive scalar potential always leads to a violation of SEC and makes \( \omega^{eff} \to -1 \) in small volume regions. However, here a negative scalar potential always strengthens the violation of SEC. Conversely, a positive potential weakens the violation.

The last two terms purely comes from the quantum geometry potential. In this term the numerator is a monotonically decreasing function whose value is positive and infinitely tends to zero for \( p \gg p_0 \), and

\[
-3V_g + (1-l) a V^\prime_g = \begin{cases} \frac{4.71}{p_0} + \frac{(1-l)12.73}{p}, & p = 5p_0 \\ \to 0, & p \gg p_0 \end{cases}
\]  \tag{15}

Because \( -\frac{3V_g + (1-l) a V^\prime_g}{3(1-l)(H_m + V_g)} \) < 0, the contribution of the quantum geometry potential always strengthens the violation of SEC. This shows that the appearance of the quantum geometry potential not only determines the occurrence of a bounce, but also leads to an accelerated expansion of the universe.

From the above analysis we know that in the small volume regions if \( V(\phi) < 0 \), then \( \omega^{eff} \ll -1 \). Since \( \omega^{eff} \ll -1 \) there exists a phase of superinflation. Compared to the result in (12) a superinflation is more easily attained in this context. In (13) a superinflation exists only for a massless scalar field. Here, a negative or vanishing scalar potential can lead to superinflation inevitably in the small regions. Because of the existence of the quantum geometry potential, even for a positive scalar potential

\[
V(\phi) \leq \frac{1}{3 l a^3} \left[ -3V_g + (1-l) a V^\prime_g \right]
\]  \tag{16}

the effective state parameter equation is still \( \omega^{eff} \ll -\frac{l}{1-l} \). Therefore, here a superinflation can happen in small volume regions only if the scalar potential satisfies the condition (16). So the happening of superinflation is independent of initial condition completely. Although the quantization ambiguity parameter \( l \) appears in the condition (16), \( l \) only limits the upper bound of a scalar potential for occurrence of a superinflation. The condition (16) shows that if \( l \to 0^+ \), there must be a superinflation which is independent of the scalar potential. If \( l \to 1^- \), a superinflation can happen for \( V(\phi) \leq \frac{1}{a^3} \) (the upper bound is a positive potential) and at the same time \( \omega^{eff} \ll -1 \).

For the effective equation (13) there is an effect that when \( p \) approaches the bounce scale the last two terms can be greatly amplified because of the small value of their numerators (i.e., \( H_m + V_g \sim 0 \)). For the last term when \( p \to p_{bounce} \),

\[
-\frac{3V_g + (1-l) a V^\prime_g}{3(1-l)(H_m + V_g)} \gg 1.
\]

So, if \( V(\phi) \leq 0 \), \( \omega^{eff} \ll -1 \) at the region \( p \to p_{bounce} \).
There is another issue of the graceful exit. Only in the presence of the modification to matter field from the inverse scale factor the inflationary phase automatically ends when the peak of $F_j(q)$ is reached \[12\]. However, from equation \[12\] it shows that the existence of the quantum geometry potential can prolong the ”exit time”, i.e., $p_{\text{exit}} \approx p_{\text{peak}}$. ($p_{\text{exit}}$ denotes the value of $p$ at the end of inflation in the presence of the quantum geometry and $p_{\text{peak}}$ corresponds to the value of $p$ at the peak of $F_j(q)$), because at the right side of the equation \[12\] the third term is independent of the modification from the inverse scale factor and it is always a negative value. But still a graceful exit can happen for some value of $p$ because when $p$ increases the quantum geometry potential tends to vanish.

Now let us concern the region discussed in this paper. As in \[8, 16, 19\], we work in the valid regions of the effective Hamiltonian which also corresponds to the semiclassical regions, i.e., $p_0 < p_{\text{bounce}} < p < p_j$. Here $p_0$ is the quantum geometry scale and $p_j$ is the inverse scale factor scale. When $p \gg p_j$ it is in the classical regions.

IV. CONCLUSION

In this paper we mainly discuss the issue of the violation of SEC in the presence of the quantum geometry potential. It shows that the appearance of the quantum geometry potential strengthens the trend of the violation in the small volume regions, especially in the region next to the bounce scale. A superinfaltion can happen if the scalar potential satisfies the given condition \[17\]. In conclusion, the inflation from quantum geometry is raised by two parts in which one is the modified matter and the other is the quantum geometry potential. In the small volume regions they are all important for the violation of SEC. As in usual cosmology described by the classical general relativity a negative pressure can lead to an accelerated expansion of the universe. Similarly, in effect the quantum geometry potential behaves like a negative pressure for the modified FRW equation by LQC.

One open issue is the range of the semiclassical region. In the semiclassical region $p_{\text{bounce}}$ and $p_j$ are two basic scales. In \[8\] the numerical result shows that the effective theory based on the WKB approximation can be trusted to approach the Planck scale $l_p$. So the bounce scale is comparable with $p_0$. As for $p_j$, it is an open problem to determine the value of $j$. But it is expected that $j$ is large enough to ensure that the semiclassical region is big enough in order that the phenomena can leave observable effects \[22\]. Therefore, our discussion in this paper essentially assumes that $p_j \gg p_0$.

In this paper we mainly care about the effect of the quantum geometry potential for the violation of SEC. However, because the quantum geometry potential appears in the effective Hamiltonian, basing on the effective Hamiltonian, many phenomena can be investigated again. And, it is expected that in the presence of the quantum geometry potential the results obtained in the before can be changed quantitatively or qualitatively in the small volume regions.

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