On the new universality class in structurally disordered $n$-vector model with long-range interactions

Dmytro Shapoval, 1,2 Maxym Dudka, 1,2 Yurij Holovatch 1,2,3

1 Institute of Condensed Matter Physics, National Academy of Sciences of Ukraine, UA – 79011 Lviv, Ukraine

2 L4 Collaboration & Doctoral College for the Statistical Physics of Complex Systems, Leipzig-Lorraine-Lviv-Coventry, Europe

3 Centre for Fluid and Complex Systems, Coventry University, Coventry, CV1 5FB, United Kingdom

Abstract

We study a stability border of a region where nontrivial critical behaviour of an $n$-vector model with long-range power-law decaying interactions is induced by the presence of a structural disorder (e.g. weak quenched dilution). This border is given by the marginal dimension of the order parameter $n_c$ dependent on space dimension, $d$, and a control parameter of the interaction decay, $\sigma$, below which the model belongs to the new dilution-induced universality class. Exploiting the Harris criterion and recent field-theoretical renormalization group results for the pure model with long-range interactions we get $n_c$ as a three loop $\epsilon = 2\sigma - d$-expansion. We provide numerical values for $n_c$ applying series resummation methods. Our results show that not only the Ising systems ($n = 1$) can belong to the new disorder-induced long-range universality class at $d = 2$ and $d = 3$.

Keywords: Long-range interaction, quenched disorder, renormalization group, marginal dimension
1 Introduction

Year 2022 marks the 110th birth anniversary of Oleksandr (a.k.a. A.S.) Davydov, an outstanding physicist, known for his seminal contributions in the fields of solid state theory, nuclear physics and biophysics, for many years he served as a director of the Bogolyubov Institute for Theoretical Physics in Kyiv. The authors of this paper who studied physics also from O. Davydov’s books [1, 2, 3] consider as a great honour to contribute to the Festschrift prepared on this occasion. In our paper we use the perturbative field theoretical renormalization approach refined by the resummation of asymptotic series expansions to study universal features of criticality. Although such problems were beyond the focus of attention of O. Davydov, the concepts called for their analysis: Symmetry, Space dimension, Range of interaction belong to the central ones in physics. In the paper we show how their interplay defines universal features of one of the key models currently used to understand quantitatively and to describe qualitatively the critical behaviour in condensed matter and beyond. Therefore, conceptually the results presented in this paper are related to those discussed in O. Davydov’s seminal works. For this reason we have chosen to present these results here.

Since inter-particle forces in various physical, chemical, and biological systems are often of a long-range nature, models with long-range interaction attract much attention. They have found their applications in studies of gravitational, dipolar, cold Coulomb systems, problems in plasma, atomic and nuclear physics, hydrodynamics and geophysical fluid mechanics (see [4, 5, 6] and reference therein). Systems with long-range interactions possess properties that differ from those with short-range interactions. To give an example, even weak long-range interactions effectively modify the critical properties and may induce the long-range order in one-dimensional systems [5, 6].

In this paper we will discuss possible changes in the critical behaviour of a many-particle system caused by mutual effects of long-range interactions and structural disorder. To this end, we will consider the – now standard – \( n \)-vector spin model with the Hamiltonian:

\[
\mathcal{H} = -\frac{1}{2} \sum_{\mathbf{x}, \mathbf{x}'} J(|\mathbf{x} - \mathbf{x}'|) \vec{S}_\mathbf{x} \cdot \vec{S}_{\mathbf{x}'}
\]

that describes a system of classical \( n \)-component vectors (‘spins’) \( \vec{S}_\mathbf{x} = (S^1_\mathbf{x}, S^2_\mathbf{x}, \ldots, S^n_\mathbf{x}) \) located at sites \( \mathbf{x} \) of a \( d \)-dimensional lattice and interacting via the distance-dependent potential \( J(x) \). The r.h.s of Eq. (1) contains a scalar product of spins and the sums over \( \mathbf{x}, \mathbf{x}' \) span all lattice sites. Influence
of long-range interactions on the critical behavior is usually exemplified by the power-law decaying interaction:

$$J(x) \sim x^{-d-\sigma},$$

where $\sigma > 0$ is a control parameter of the interaction decay.

As we discuss in more details below, in the case of a regular (non-disordered) lattices, the critical behaviour of the model (1) is governed by the triple of parameters $(d, n, \sigma)$: depending on their values, the model may manifest a low-temperature long-range order that emerges as a second-order phase transition. As long as the model Hamiltonian (1) is formulated in terms of elementary magnets – ‘spins’, the long-range ordered phase is usually associated with an emergence of spontaneous magnetization. We will use such magnetic terminology too, however let us note that the model itself as well as our discussion without the loss of generality concern much more wide range of types of ordering [7, 8] in physics and beyond. The transition to the ordered ‘magnetic’ phase is characterised by certain universal (i.e. independent on specific system details) features. It is said, that it belongs to certain universality class. Systems, that belong to the same universality class share the values of critical exponents, amplitude ratios, scaling functions. Our goal in this paper is to show how these universal features are changed if instead of a regular lattice structure, one considers the disordered one. Disorder in the lattice structure may be imposed e.g. by dilution, when a part of the lattice sites in (1) are not occupied by spins. Such situation mimics randomness and non-regularities that are so often met in nature and attract much interest in modern theory of critical phenomena (see e.g. [8] and references therein.) To quantify our analysis, we will calculate the marginal dimension $n_c(\sigma)$: for given space dimension $d$ it discriminates between different universality classes. The rest of the paper is organized as follows. In the next Section we give a short review of the results present so far, in Section 3 we describe the field-theoretical renormalization group picture of the critical behaviour for the long-range interacting $n$-vector model with disorder. The results for $n_c$ are given in Section 4 and we summarize our study in the last Section 5. Some lengthy expressions are given in the Appendix.

2 Review

In this section we briefly review some results relevant for our analysis. To proceed further, we explain terminology, used throughout the paper. The $n$-vector model (1) with short-range interactions – since
the corresponding free energy is invariant with respect to rotations in the $n$-dimensional magnetization space it is also called the $O(n)$-symmetric model – manifests the second order phase transition for the lattice space dimension $d > d_{lc}$. The lower critical dimension $d_{lc} = 1$ for the discrete (Ising) case $n = 1$ whereas $d_{lc} = 2$ for $n > 1$. Critical exponents and other universal properties of the short-range $n$-vector model depend on $n$ and $d$ in a non-trivial way in the region $d_{lc} < d \leq d_{uc}$. It is said that they belong to the short-range universality class. For $d$ larger than the upper critical dimension $d_{uc} = 4$ the model is governed by the mean-field exponents, see [9] for more details. Introducing the long-range interaction (2) drastically changes the picture of the critical behaviour of the $n$-vector model (1). Calculations performed for the three-dimensional spherical model [10] (it corresponds to the $n$-vector model at $d = 3, n = \infty$) show that for $\sigma > 2$ the critical properties are governed by the short-range critical exponents, while for $\sigma < 2$ one has two regimes depending on the value of $\sigma$: with mean-field critical exponents and with the $\sigma$-dependent ones. One-dimensional Ising model ($d = 1, n = 1$) with interaction (2) was proven to have phase transition to the long-range-ordered phase at non-zero temperature [11]. Field-theoretical renormalization group (RG) analysis of the long-range interacting $n$-vector model gives three universality classes in dependence on $\sigma$ [12]: (i) the mean-field critical behavior for $\sigma \leq d/2$, (ii) the short-range universality class for $\sigma \geq 2$ with critical exponents coinciding with those of the model with short-range interactions, (iii) the long-range universality class for $d/2 < \sigma < 2$, where critical exponents depend on $\sigma$. Later it was established that the actual border between the short-range and the long-range universality classes lies at $\sigma = 2 - \eta_{SR}$ [13, 14] rather than at $\sigma = 2$, $\eta_{SR}$ is the pair correlation function critical exponent of the short-range model. Such picture was corroborated by other approaches including non-perturbative variant of RG (NPRG) [15], Monte Carlo simulations [16] and conformal bootstrap [17] (for other references and discussion see the review [18]).

Another factor discussed in this paper is the structural disorder and its impact on the critical behaviour. The influence of structural inhomogeneity on the universal properties of physical systems continues to be a hot research topic both for academic and practical reasons, since almost all materials are characterized by a certain degree of disorder in their structure. Structural inhomogeneities in magnetic systems are of different nature, which in turn may lead to different changes in critical behavior. In the case of strong structural disorder, randomness is accompanied by frustration and percolation effects and often leads to absence of the long-range magnetic order. The case of weak structural disorder is not that obvious. Here we focus specifically on presence of the weak quenched
disorder in lattice structure, which may be implemented into model (1) via dilution by point-like uncorrelated (or short-range correlated) quenched non-magnetic inhomogeneities. Its relevance for the critical behaviour is given by the Harris criterion \[19\]. The criterion states that the structural disorder (quenched dilution) leads to a new universality class of the magnetic phase transition only if the heat capacity critical exponent of the undiluted (pure) system is positive, \(\alpha_p > 0\), i.e. if the heat capacity of the pure system diverges at the critical point. Correspondingly, the disorder is irrelevant if \(\alpha_p < 0\). For the short-range \(n\)-vector model at \(d = 3\) \(\alpha_p > 0\) for \(n = 1\) (Ising model), whereas \(\alpha_p < 0\) for \(n \geq 2\), therefore due to the Harris criterion the diluted \(n\)-vector model at \(n \geq 2\) shares the universal properties of its undiluted counterpart \[20, 21\]. Unlike the short-range \(n\)-vector model, the long-range one manifests the new universality class induced by dilution also in the region \(n \geq 2\) as it was shown in the RG study of Ref. \[22\] within two-loop approximation. This result was also corroborated by the low-temperature RG \[23\]. However, the estimates of the regions of values \((d, n, \sigma)\), where the new disordered long-range universality class governs the critical behaviour were not satisfactory \[22\]. The perturbative RG expansions having zero radius of convergence, additional resummation procedures have to be applied in order to get reliable numerical data on their basis \[7\]. Especially it concerns the Ising model \((n = 1)\), where degeneracy of the RG equations (similarly as in the short-range case \[20, 21\]) makes the expansion parameter to be \(\sqrt{\epsilon}\) \[24\]. Reliable results for the last case were obtained within a massive renormalization scheme with resummation of the RG functions \[25\].

A remarkable feature of the Harris criterion is that it allows to forecast structural-disorder-induced changes in the universality class of a pure system without explicit calculation of the RG functions for the diluted one. Indeed, if the structural disorder changes the universality class only when the heat capacity of the pure system diverges (i.e. when \(\alpha_p > 0\)), one can use the condition \(\alpha_p = 0\) as an equation to define parameters \((d, \sigma, n)\) that discriminate between different universality classes. For given space dimension \(d\) such equation defines the so-called marginal order-parameter dimension \(n_c(\sigma)\). Similar to critical exponents and critical amplitude ratios, the marginal dimensions are universal quantities, reachable in experiments and numerical simulations and are the subject of intensive studies \[26, 27, 28, 29, 30, 31\]. In the next sections we will calculate the marginal dimension \(n_c(\sigma)\) for the diluted long-range \(n\)-vector model at space dimensions \(d = 2\) and \(d = 3\). This marginal dimension line in the \((n, \sigma)\) parametric plane defines a border of stability between the pure long-range and the disordered long-range universality classes. To this end we will make use of the recent three-loop RG results for the critical exponents of the pure \(n\)-vector model with long-range power-law decaying
interactions [32].

3 Field-theoretical renormalization group description

The progress achieved in quantitative understanding and qualitative description of critical phenomena to a large extent is due to application of RG methods [7]. In the field-theoretical RG approach, the critical properties of the $n$-vector model (1) with the power-law decaying long-range interactions (2) are described by analysing the effective Hamiltonian [33]:

$$\mathcal{H} = \int d^d x \left\{ \frac{1}{2} \left( \nabla^{\sigma/2} \vec{\varphi} \right)^2 + r_0 \vec{\varphi}^2 \right\} + \frac{u_0}{4!} (\vec{\varphi}^2)^2, \tag{3}$$

where $\vec{\varphi} = \vec{\varphi}(x) = \{\varphi_1(x), \ldots, \varphi_n(x)\}$ is an $n$-component field, $u_0$ is unrenormalized coupling, $r_0$ defines the temperature distance to the critical point, and $\nabla^{\sigma/2}$ is a symbolic notation for the fractional derivative. The last is defined via its action in the momentum space and leads to the propagator term $q^{\sigma}$ rather than $q^2$ as in the case of short-range interactions. Power counting gives the upper critical dimension $d_{uc} = 2\sigma$, which at $\sigma = 2$ coincides with the traditional $d_{uc} = 4$. The effective Hamiltonian (3) is relevant for the case $0 < \sigma < 2 - \eta_{SR}$. To study crossover to the short-range case $\sigma \to 2$ one should include the traditional term $(\nabla \vec{\varphi})^2$ into (3).

In the field-theoretical RG approach, a critical point corresponds to a reachable and stable fixed point (FP) of the RG transformation. It has been found [12] that the nontrivial FP determining new long-range universality class is stable for $d/2 < \sigma$. Critical exponents within this universality class were calculated in $\epsilon = 2\sigma - d$-expansion up to order $\epsilon^2$ [12, 33, 34] and up to order $1/n$ [35] in $1/n$-expansion. Estimates for the critical exponents of three-dimensional systems were obtained within the massive renormalization approach completed by resummation in two-loop approximation [36]. The RG results in three-loop approximation were obtained only recently [32]. Monte Carlo estimates for the critical exponents in this universality class have been obtained only for the $n = 1$ Ising case mainly in one and two space dimensions (for collection of references see [32]). Only a few Monte Carlo results are available for the three-dimensional case [37].

The presence of uncorrelated non–magnetic impurities (weak quenched structural disorder) is usually modeled by fluctuations of the local phase transition temperature [38]. Introducing $\phi = \phi(x)$ as the field of local critical temperature fluctuations, one obtains the following effective Hamiltonian
for the structurally disordered system:

\[
\mathcal{H} = \int d^d x \left\{ \frac{1}{2} \left( (\nabla^\sigma \bar{\phi})^2 + (r_0 + \phi) \bar{\phi}^2 \right) + \frac{u_0}{4!} (\bar{\phi}^2)^2 \right\},
\]  

(4)

where the random variable \( \phi \) has a Gaussian distribution with zero mean and a correlator containing the second coupling \( v_0 \):

\[
\langle \phi(x) \rangle = 0, \quad \langle \phi(x)\phi(x') \rangle = v_0 \delta(x-x').
\]  

(5)

The angular brackets \( \langle \ldots \rangle \) indicate an average over the random variable \( \phi \) distribution.

The RG picture for the disordered long-range model (4) is similar to that for its short-range analog [20, 21]. In the parametric space of couplings \( (u, v) \) the critical properties of the model (4) are governed by four FPs \( (u^*, v^*) \) in dependence on values \( \epsilon = 2\sigma - d \) and \( n \) [22, 24]: Gaussian FP \( (u^* = 0, v^* = 0) \), unphysical FP \( (u^* = 0, v^* \neq 0) \), Heisenberg long-range or pure long-range FP \( (u^* = 0, v^* \neq 0) \) and disordered long-range FP \( (u^* \neq 0, v^* \neq 0) \). The Gaussian FP is always unstable below critical \( d_u = 2\sigma (\epsilon > 0) \) while the unphysical FP is always stable in this case, however, it is not accessible from initial conditions appropriate for the model described by (4) and (5). For \( \epsilon > 0 \) and \( n > n_c \), the long-range Heisenberg FP is stable and the disordered one is unstable, while for \( n < n_c \) the FPs swap their stability. Therefore for \( n > n_c \) the universal critical exponents of the diluted model (4) coincide with those of the model (3). For \( n < n_c \) model (4) belongs to the new disorder-induced long-range universality class. The border between these two regimes is determined by the marginal dimension \( n_c(d, \sigma) \).

So far, the value of \( n_c \) for the long-range \( n \)-vector model was known within the two-loop approximation [39], with the result

\[
n_c = 4 - 4[\psi(1) - 2\psi(\sigma/2) + \psi(\sigma)] \epsilon + O(\epsilon^2),
\]  

(6)

where \( \psi(x) \) is the digamma function. The asymptotic nature of this series together with its shortness made it difficult to get reliable numerical estimates on its basis. In the next section we will proceed getting the next order of the \( \epsilon \)-expansion and delivering numerical estimates for \( n_c(\sigma) \) at certain space dimensions with the help of resummation procedures.
4 Calculation of the marginal dimension

As it has been mentioned above, the marginal dimension \( n_c \) of a weakly diluted \( n \)-vector model with power-law decaying interactions can be obtained on the base of the critical exponents for the undiluted model. As a consequence of the Harris criterion, the master equation for determining \( n_c \) reads:

\[
\alpha_p(n_c, d, \sigma) = 0. \tag{7}
\]

The treatment of Eq. (7) by means of the field-theoretical RG approach can be performed in various schemes. Here we will exploit the results of dimensional regularisation with the minimal subtraction [40] allowing to obtain quantities of interest by familiar \( \epsilon \)-expansion with \( \epsilon = 2\sigma - d \) in our case. To get \( n_c(d, \sigma) \) we use the hyperscaling relation \( \alpha_p = 2 - d\nu_p \) and \( \epsilon \)-expansion for the critical exponent \( \nu_p \) of the \( n \)-vector model with long-range interactions, which is known in the three-loop approximation [32] in the following form:

\[
\nu_p^{-1} = \sigma - \frac{(n+2)}{n+8} \epsilon + \frac{(n+2)(7n+20)\alpha_{S,0}}{(n+8)^3} \epsilon^2 + \frac{(n+2)\epsilon^3}{(n+8)^5} - 4(5n+22)(7n+20)\alpha_{S,0}^2 \\
+ (n+8)^2(7n+20)(\alpha_{S,1} - 2\alpha_{D,1}\alpha_{S,0}) + (n+8)\left(-8(n-1)\alpha_T + 2(n^2 + 20n + 60)\alpha_U \right) \\
+ 2(n^2 + 24n + 56)\alpha_{I_1} + (5n^2 + 28n + 48)\alpha_{I_2} + (5n + 22)\alpha_{I_4}\right] + O(\epsilon^4), \tag{8}
\]

where \( \alpha_K \) with \( K = \{S, D, I_1, I_2, I_4, T, U\} \) are expressed in terms of the loop integrals (for details see [32]). \( \alpha_{S,i} \) and \( \alpha_{D,i} \) are the coefficients at \( \epsilon^i \) in the \( \epsilon \)-expansion series of \( \alpha_S \) and \( \alpha_D \). Explicit expressions for \( \alpha_K \) are given in the Appendix. We get the following expression for \( n_c \):

\[
n_c(d, \sigma) = 4 + 4\alpha_{S,0}\epsilon + (56\alpha_{I_1} + 40\alpha_{I_2} + 7\alpha_{I_4} - 192\alpha_{D,1}\alpha_{S,0} - 56\alpha_{S,0}^2 + 96\alpha_{S,1} - 4\alpha_T + 52\alpha_U)\frac{\epsilon^2}{24} + O(\epsilon^3). \tag{9}
\]

Taking into account the explicit expression for \( \alpha_{S,0} \) one can check that up to the first order in \( \epsilon \) Eq. (9) coincides with the two-loop result (6).

Formally, the numerical value of \( n_c \) at fixed \( d \) and \( \sigma \) can be calculated from the expansion (9) by using expressions for \( \alpha_K \) from the Appendix, recalling that \( \epsilon = 2\sigma - d \) and substituting the values of \( d \) and \( \sigma \). However the \( \epsilon \)-expansions are known to be asymptotic at best [7]. Therefore one has to apply special resummation procedures to restore their convergence in order to get reliable numerical
estimates on their basis. Doing so, we start our analysis by representing series (9) by means of the diagonal Padé approximant \([1/1](\epsilon)\) [41]. The result for \(n_c\) as a function of \(\sigma\) is shown by dashed lines in Fig. 1 a,b for fixed \(d = 2\) and \(d = 3\), correspondingly. As it was noted in the previous Section, in the region of \(n\) and \(\sigma\) above the lines the critical behaviour of the diluted model is the same as for an undiluted model with long-range interactions. For the values of \(n\) and \(\sigma\) below the lines, the new disorder long-range universality class is induced. It is known that in the short-range case the Padé approximants of the three-loop expansion give values of \(n_c\) that exceed the most accurate estimates [27]. It seems to be also in the long-range case, since the data of the NPRG approach (shown by black dots in Fig. 1) are located below the lines calculated via Padé approximant.\(^1\) To enhance our estimates we use also a more elaborated Padé-Borel resummation technique [42]. First, to weaken the factorial growth of the expansion coefficients we write the Borel transform for (9) as:

\[
\sum_{k=0}^{2} n_k \epsilon^k \rightarrow \sum_{k=0}^{2} \frac{n_k \epsilon^k}{k!}.
\]

An analytical continuation of the Borel transform is achieved by representing it in the form of the diagonal Padé approximant \([1/1]_B(\epsilon)\), where subscript \(B\) is used to distinguish from the Padé approximant of the original series (9). Finally, the resummed function is obtained via an inverse Borel

\(^1\)These data were obtained from interpolation of the NPRG results for critical exponents of the long-range \(n\)-vector model [15].

Figure 1: Resummed values of \(n_c(\sigma)\) obtained by the Padé-approximant [1/1] (dashed lines) and Padé-Borel resummation (solid lines) for \(d = 2\) (a) and \(d = 3\) (b). For the region of values \(n\) and \(\sigma\) above the lines, the pure long-range universality class holds, while the new disorder long-range universality class is induced for the values below the lines. Dots show results that follow from the interpolation of the NPRG data of Ref. [15]. The mean-field behaviour holds for \(\sigma < d/2\) (regions separated by vertical dashed lines, coloured online).
transform:

\[ n_c^{\text{res}}(\varepsilon) = \int_0^\infty d\tau e^{-\tau} [1/1]_B(\varepsilon \tau) . \] (11)

Results that follow from the Padé-Borel resummation are presented in Fig. 1 by solid lines for \( d = 2 \) and \( d = 3 \). For \( d = 3 \) the Padé-Borel resummation leads to lower values of \( n_c(\sigma) \) as compared to those obtained from the \([1/1]\) Padé approximant. This is a right tendency, as is seen also from comparing our results to the NPRG data, shown by dots in Fig. 1. As usual with the perturbative expansions, the accuracy of the results decreases with an increase of the expansion parameter, in our case it is \( \varepsilon = 2\sigma - d \). Therefore, our results are less accurate for \( d = 2 \), where the expansion parameter changes within \( 0 \leq \varepsilon \leq 2 \) for \( 1 \leq \sigma \leq 2 \). But even then the results of Padé-Borel resummation may serve as reliable estimates up to the moderate values of \( \sigma \approx 1.5 \). A remarkable feature of the plots presented in Figs. 1 a,b is that for a certain range of parameters \( \sigma \) there are regions in the \( n,\sigma \) plane that correspond to integer values of \( n = 1, 2, 3 \) and lie below the \( n_c(\sigma) \) curve. This means that the new disorder long-range universality class is induced in the \( n \)-vector model not only for the Ising (\( n = 1 \)) but also for the XY (\( n = 2 \)) and classical Heisenberg (\( n = 3 \)) cases.

5 Conclusions

In this study we were interested in the question how the critical behaviour of a many-particle system is changed under the competing influence of two factors: type of interaction and structural disorder? To this end, we have considered the archetype model to describe criticality, an \( n \)-vector spin model (1), and analysed changes in its critical behaviour provided the interaction between spins is of a long-range nature (2) and an underlying lattice structure is disordered. To be more specific, we considered the case when the weak quenched structural disorder leads to fluctuations in the local transition temperature (5). The literature available so far [22, 23] reported that second order phase transition in such a model can belong to the new, disorder induced long-range universality class. However, such a qualitative conclusion has to be supported by quantitative estimates of the region of model parameters where the new universality class can manifest.

To do so, we have calculated the marginal dimension \( n_c(\sigma) \) of the structurally-disordered long-range interacting \( n \)-vector model. For given space dimension \( d \) and interaction decay \( \sigma \), the model with the order parameter component number \( n < n_c \) belongs to the new disorder-induced long-range
universality class. Based on the recent results for the critical exponents of the pure long-range $n$-vector model \cite{32}, we used the Harris criterion to calculate $n_c(\sigma)$ with the record three-loop accuracy, Eq. (9). This enabled us to apply familiar resummation techniques to evaluate numerical values of the marginal dimension, as shown in Figs. 1 a,b for space dimensions $d = 2$ and $d = 3$. Obtained results serve as a solid argument that not only the Ising-like ($n = 1$) systems, but also systems that are described by the XY ($n = 2$) and Heisenberg ($n = 3$) models belong to the new universality class for the moderate values of $\sigma$ at space dimensions $d = 2$ and $d = 3$.

**Acknowledgements**

We thank the Editors, Yurij Naidyuk, Larissa Brizhik, and Oleksandr Kovalev, for their invitation to contribute to the Festschrift in memory of Oleksandr Davydov. This work was supported in part by the grant of the National Academy of Sciences of Ukraine for research laboratories/groups of young scientists No 07/01-2022(4)) (D.S.). M.D. thanks members of LPTMC for their hospitality during his academic visit in the Sorbonne University, where the part of this work was done.

**Appendix**

In this appendix, we list expressions for $\alpha_K$ as they were given in Ref. \cite{32}

\begin{align}
\alpha_D &= 1 + \frac{\epsilon}{2} \left[ \psi(1) - \psi(\frac{d}{2}) \right] + \frac{\epsilon^2}{8} \left[ \left( \psi(1) - \psi(\frac{d}{2}) \right)^2 + \psi_1(1) - \psi_1(\frac{d}{2}) \right], \\
\alpha_S &= 2\psi(\frac{d}{4}) - \psi(\frac{d}{2}) - \psi(1) + \frac{\epsilon}{4} \left[ 2\psi(\frac{d}{4}) - \psi(\frac{d}{2}) - \psi(1) \right] \left[ 3\psi(1) - 5\psi(\frac{d}{2}) + 2\psi(\frac{d}{4}) \right] \\
&\quad + 3\psi_1(1) + 4\psi_1(\frac{d}{4}) - 7\psi_1(\frac{d}{2}) - 4J_0(\frac{d}{4}), \\
\alpha_U &= \alpha_{I_2} = -\psi_1(1) - \psi_1(\frac{d}{2}) + 2\psi_1(\frac{d}{4}) + J_0(\frac{d}{4}), \\
\alpha_T &= \frac{1}{2} \left[ 2\psi(\frac{d}{4}) - \psi(\frac{d}{2}) - \psi(1) \right]^2 + \frac{1}{2} \psi_1(1) + \psi_1(\frac{d}{4}) - \frac{3}{2} \psi_1(\frac{d}{2}) - J_0(\frac{d}{4}), \\
\alpha_{I_1} &= \frac{3}{2} \left[ 2\psi(\frac{d}{4}) - \psi(\frac{d}{2}) - \psi(1) \right]^2 + \frac{1}{2} \psi_1(1) - \frac{1}{2} \psi_1(\frac{d}{4}), \\
\alpha_{I_4} &= 6 \frac{\Gamma(1+\frac{d}{4})^3 \Gamma(-\frac{d}{4})}{\Gamma(\frac{d}{2})} \left[ \psi_1(1) - \psi_1(\frac{d}{4}) \right].
\end{align} (12)
In the above expressions $\psi_i$ are the polygamma functions of order $i$, while $J_0$ is the following sum:

$$J_0(\frac{d}{4}) = \frac{1}{\Gamma(\frac{d}{4})^2} \sum_{n \geq 1} \frac{\Gamma(n + \frac{d}{4}) \Gamma(n + \frac{d}{2})^2}{n(n!) \Gamma(\frac{d}{4} + 2n)} \left[ 2\psi(n + 1) - \psi(n) - 2\psi(n + \frac{d}{4}) - \psi(n + \frac{d}{2}) + 2\psi(\frac{d}{2} + 2n) \right]. \quad (13)$$

References

[1] A. S. Davydov. *Quantum Mechanics*, Pergamon Press, Oxford (2013)

[2] A. S. Davydov. *Theory of Molecular Excitons*, Plenum Press, New York (1971)

[3] A. S. Davydov. *Solitons in Molecular Systems*, Springer, Dordrecht (1985)

[4] *Dynamics and Thermodynamics of Systems with Long-Range Interactions*, edited by T. Dauxois, S. Ruffo, E. Arimondo, M. Wilkens, Lect. Not. in Phys. 602, Springer-Verlag, New York (2002); *Dynamics and Thermodynamics of Systems with Long Range Interactions: Theory and Experiments*, edited by A. Campa, A. Giansanti, G. Morigi, F. Sylos Labini, AIP Conf. Proc. 970 (2008); A. Campa, T. Dauxois and S. Ruo, *Phys. Rep.*, 480, 57 (2009)

[5] A. Campa, T. Dauxois, D. Fanelli, S. Ruffo, *Physics of long-range interacting systems*, Oxford University Press, Oxford (2014)

[6] D. Mukamel, *Notes on the statistical mechanics of systems with long-range interactions* (2009) [arXiv: 0905.1457]

[7] D. J. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena*, World Scientific, Singapore (1989); J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Oxford University Press, Oxford (1996); H. Kleinert, V. Schulte-Frohlinde, *Critical Properties of $\phi^4$-Theories*, World Scientific, Singapore (2001)

[8] *Order, Disorder and Criticality. Advanced Problems of Phase Transition Theory*. Yu. Holovatch (editor). vols. 1–7, World Scientific, Singapore, 2004–2022

[9] B. Berche, T. Ellis, Yu. Holovatch, R. Kenna, *SciPost Phys. Lect. Notes* 60 (2022)

[10] G. S. Joyce, *Phys.Rev*. 146, 349 (1966)

[11] F. J. Dyson, *Comm. Math. Phys.* 12, 91 (1969)
[12] M. E. Fisher, S. K. Ma, and B. G. Nickel, *Phys. Rev. Lett.* **29**, 917 (1972)

[13] J. Sak, *Phys. Rev. B* **8**, 281 (1973)

[14] J. Sak, *Phys. Rev. B* **15**, 4344 (1977)

[15] N. Defenu, A. Trombettoni, and A. Codello, *Phys. Rev. E* **92**, 052113 (2015)

[16] E. Luijten and H. W. J. Blöte, *Phys. Rev. Lett.* **89**, 025703 (2002)

[17] C. Behan, *J. Phys. A: Math. Theor.* **52**, 075401 (2019)

[18] N. Defenu, A. Codello, S. Ruffo, A. Trombettoni, *J. Phys. A: Math. Theor.* **53**, 143001 (2020)

[19] A. B. Harris, *J. Phys. C: Solid State Phys.* **7**, 1671 (1974)

[20] A. Pelissetto, E. Vicari, *Phys. Rep.* **368**, 549 (2002)

[21] Yu. Holovatch, V. Blavats’ka, M. Dudka, C. von Ferber, R. Folk, and T. Yavors’kii, *Int. J. Mod. Phys. B* **16**, 4027 (2002)

[22] Y. Yamazaki, *Physica A* **90**, 547 (1978)

[23] M. X. Li, N. M. Duc, *Physica Status Solidi (b)*, **107**, 403 (1981)

[24] A. Theumann, *J. Phys. A: Math. Gen.* **14**, 2759 (1981)

[25] S. V. Belim, *JETP Lett.* **77**, 434 (2003)

[26] C. Bervillier, *Phys. Rev. B* **34**, 8141 (1986)

[27] M. Dudka, Yu. Holovatch, T. Yavors’kii, *J. Phys. Stud.* **5** (3/4), 233 (2001)

[28] H. Kleinert, V. Schulte-Frohlinde, *Phys. Lett. B* **342**, 284 (1995); J. M. Carmona, A. Pelissetto, E. Vicari, *Phys. Rev. B* **61**, 15136 (2000); R. Folk, Yu. Holovatch, T. Yavors’kii, *Phys. Rev. B* **61**, 15114 (2000); K. V. Varnashev, *Phys. Rev. B* **61**, 14660 (2000); R. Folk, Yu. Holovatch, T. Yavors’kii, *Phys. Rev. B* **62**, 12195 (2000); L. Ts. Adzhemyan, E. V. Ivanova, M. V. Kompaniets, A. Kudlis, A. I. Sokolov, *Nucl. Phys. B* **940**, 332 (2019)

[29] M. Dudka, Yu. Holovatch, T. Yavors’kii, *Acta Physica Slovaca* **52**, 323 (2002); M. Dudka, Yu. Holovatch, T. Yavors’kii, *J. Phys. A: Math. Gen.* **37**, 10727 (2004)
[30] H. Kawamura, *Phys. Rev. B* **38**, 4916 (1988); S. A. Antonenko, A. I. Sokolov, and K. B. Varnashev, *Phys. Lett. A* **208**, 161 (1995); P. Calabrese and P. Parruccini, *Nucl. Phys. B* **679**, 568 (2004); Yu. Holovatch, D. Ivaneyko, and B. Delamotte, *J. Phys. A: Math. Gen.* **37**, 3569 (2004); M.V. Kompaniets, A. Kudlis, and A.I. Sokolov, *Nucl. Phys. B* **950**, 114874 (2020)

[31] M. Dudka, R. Folk, Yu. Holovatch, G. Moser, *Condens. Matter Phys.* **15**, 43001 (2012)

[32] D. Benedetti, R. Gurau, S. Harribey, K. Suzuki, *J. Phys. A: Math. Theor.* **53**, 445008 (2020)

[33] M. Suzuki, Y. Yamazaki and G. Igarashi, *Phys. Lett. A* **42**, 313 (1972)

[34] Y. Yamazaki and M. Suzuki, *Prog. Theor. Phys.* **57**, 1886 (1977)

[35] M. Suzuki, *Phys. Lett. A* **42**, 5 (1972); *Id., Prog. Theor. Phys.* **49**, 424 (1973); *Id., Prog. Theor. Phys.* **49**, 1106 (1973); *Id., Prog. Theor. Phys.* **49**, 1440 (1973)

[36] S. V. Belim, *JETP Lett.* **77**, 112 (2003)

[37] S. V. Belim, I. B. Larionov, R. V. Soloneckiy, *Phys. Metals Metallogr.* **117**, 1079 (2016); S. V. Belim, I. B. Larionov, *Moscow Univ. Phys. Bull.* **73**, 394 (2018)

[38] G. Grinstein, A. Luther, *Phys. Rev. B* **13**, 1329 (1976)

[39] Y. Chen, Z.B. Li, *Mod. Phys. Lett. B* **17**, 1227 (2003)

[40] G.’t Hooft, M. Veltman, *Nucl. Phys. B* **44**, 189 (1972); G.’t Hooft, *Nucl. Phys. B* **61**, 455 (1973)

[41] G. A. Baker, Jr., P. Graves-Morris, *Padé Approximants*, Addison-Wesley, Reading, Mass (1981)

[42] G. A. Baker, Jr., B. G. Nickel, M. S. Green, D. I. Meiron, *Phys. Rev. Lett.* **36**, 1351 (1976); G. A. Baker, Jr., B. G. Nickel, D. I. Meiron, *Phys. Rev. B* **17**, 1365 (1978)