SKLYANIN INVARIANT INTEGRATION

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Abstract. The Sklyanin algebra admits realizations by difference operators acting on theta functions. Sklyanin found an invariant metric for the action and conjectured an explicit formula for the corresponding reproducing kernel. We prove this conjecture, and also give natural biorthogonal and orthogonal bases for the representation space. Moreover, we discuss connections with elliptic hypergeometric series and integrals and with elliptic 6j-symbols.

1. Introduction

The study of solvable models in statistical mechanics and related areas of physics led to the introduction of quantum groups in the 1980’s. In one of the earliest papers on the subject [S1], Sklyanin introduced what has become known as the Sklyanin algebra. Its commutation relations were obtained from the Boltzmann weights of the eight-vertex model. Since these involve elliptic functions, the Sklyanin algebra is an example of an elliptic quantum group.

The development of elliptic quantum groups has been slow, especially regarding analytic aspects. We think here of concrete problems in harmonic analysis, typically resulting in explicit identities involving special functions. In a recent paper [Ro2], we made some progress by explaining how analytically continued elliptic 6j-symbols appear in connection with the Sklyanin algebra. These symbols are Boltzmann weights for a generalization of the eight-vertex model [DJMO], see also [DJKMO], [FT]. In [Ro2], they appeared as matrix elements for the change between natural bases of finite-dimensional representations. The difference from the case of Lie groups, or even simpler quantum groups, is that in that situation “natural” would mean the eigenbasis of a Lie algebra element, whereas for the Sklyanin algebra one must consider a generalized eigenvalue problem \( Y_1 v = \lambda Y_2 v \) involving two different algebra elements.

In the present paper we use the results of [Ro2] to further develop harmonic analysis on the Sklyanin algebra. In particular, we are interested in questions connected with invariant integration on a fixed representation (as opposed to questions connected with the Haar measure). We work with representations found by Sklyanin [S2], where the algebra acts by difference operators on spaces of higher order theta functions, that is, on sections of certain line bundles on a torus. Sklyanin introduced a measure on the torus which is invariant in the sense that his algebra generators are self-adjoint on the corresponding \( L^2 \)-space.

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In spite of its elegance, so far there seems to have been no applications of Sklyanin's result. At least part of the reason must be that until now there has been no example of two functions whose scalar product can be computed explicitly. However, for a metric to be useful it seems necessary to have a rich class of such examples. In particular, one would like to know an explicit orthogonal basis and an explicit expression for the reproducing kernel.

Sklyanin conjectured an explicit formula for the reproducing kernel, a problem that has remained open. Our first main result, Theorem 3.3, settles this conjecture. In our second main result, Theorem 3.4, we give explicit biorthogonal bases, which are “natural” in the sense alluded to above. By specialization, one may obtain orthogonal bases, which allows us to prove another conjecture of Sklyanin, Proposition 3.5, concerning the continuation of his scalar product in the parameter \( q \). Our main tools are the alternative description of the Sklyanin algebra due to Rains [R2] (see also the Appendix), together with the generalized eigenvalue and tridiagonal equations from [Ro2], expressing how Rains’ operators act on our bases.

The plan of the paper is as follows. Section 2 contains preliminaries and Section 3 statements of the main results. The proofs of these are given in Sections 4–8. In the remaining three sections we comment briefly on relations to other topics. In Section 9 we explain how the most fundamental identity for elliptic hypergeometric series, the Frenkel–Turaev summation, arises naturally as the bridge between our two main results. In Section 10 we point out another by-product, the evaluation of an elliptic hypergeometric double integral, which we have not been able to reduce to known results. In the final Section 11 we explain the relevance of the present work to elliptic 6j-symbols: it explains their self-duality or, in the language of harmonic analysis, the duality between the spectral and geometric variables. In the Appendix we provide some details about the relation between Rains’ and Sklyanin’s difference operators.

2. Preliminaries

2.1. Theta functions. Throughout, \( \tau \) and \( \eta \) will be fixed parameters, and we let

\[
p = e^{2\pi i \tau}, \quad q = e^{4\pi i \eta}.
\]

We assume that \( \tau \in i \mathbb{R}_{>0} \), or equivalently \( 0 < p < 1 \). The parameter \( \eta \) will be either real or purely imaginary, so that \( q \) is unimodular or positive, and will later become subject to further restrictions.

Elliptic functions are built from theta functions similarly as rational functions are built from first degree polynomials. We take as our building block Jacobi’s
function $\theta_1(x|\tau)$, which we denote for short by $\theta(x)$. Thus,

$$\theta(x) = \theta_1(x|\tau) = i \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i (n^2/2) + \pi i (2n-1) x}$$

(2.1)

$$= ip^{1/8} e^{-\pi ix} (p; e^{2\pi i x}, pe^{-2\pi i x}; p)_\infty,$$

where, in general,

$$(a_1, \ldots, a_n; p)_\infty = \prod_{j=0}^{\infty} (1 - a_1 p^j) \cdots (1 - a_n p^j).$$

The function $\theta$ is entire with zeroes $\mathbb{Z} + \tau \mathbb{Z}$. We will often use short-hand notation such as

(2.2a) \quad \begin{aligned} \theta(a_1, \ldots, a_n) &= \theta(a_1) \cdots \theta(a_n), \\
\theta(a \pm b) &= \theta(a + b) \theta(a - b), \end{aligned}

(2.2b)

and, for $x \in \mathbb{C}$, $y \in \mathbb{C}^n$,

$$\theta(x + \vec{y}) = \theta(x + y_1, \ldots, x + y_n).$$

The function $\theta$ satisfies the elementary identities

(2.3) \quad \begin{aligned} \theta(-x) &= -\theta(x), \\
\overline{\theta(x)} &= \theta(\bar{x}), \\
\theta(x + 1) &= -\theta(x), \\
\theta(x + \tau) &= -e^{-\pi i (2x + \tau)} \theta(x), \\
\theta(2x) &= \frac{ip^{1/8}}{(p; p)_\infty^3} \theta \left( x, x + \frac{1}{2}, x + \tau, \frac{x}{2}, x - \frac{1}{2} - \frac{\tau}{2} \right), \\
\theta(x \pm y, u \pm v) &= \theta(u \pm x, v \pm y) - \theta(u \pm y, v \pm x) \end{aligned}

and the more general identity [WW, p. 451], see also [Ro1, Section 4],

(2.6) \quad \begin{aligned} \sum_{k=1}^{n} \prod_{j=1}^{n} \theta(x_k - y_j) \prod_{j=1, j \neq k}^{n} \theta(x_k - x_j) = 0, \\
\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j. \\
\end{aligned}

By (2.1),

(2.7) \quad \theta'(0) = \lim_{x \to 0} \frac{\theta(x)}{x} = 2\pi p^{1/8}(p; p)_\infty^3.

We also mention the modular transformation

(2.8) \quad \theta_1(x/\tau | -1/\tau) = -i(\tau/i)^{1/2} e^{\pi ix^2/\tau} \theta_1(x|\tau).
Another useful result is Jacobi’s identity [WW, p. 468], which we write as
\begin{equation}
\theta \left( \frac{\vec{b} - B}{2} \right) = \frac{1}{2} \left\{ \theta(\vec{b}) + \theta \left( \frac{\vec{b} + 1}{2} \right) + e^{\pi i(\tau + B)} \theta \left( \frac{\vec{b} + \tau}{2} \right) - e^{\pi i(\tau + B)} \theta \left( \frac{\vec{b} + 1 + \tau}{2} \right) \right\},
\end{equation}
where \( b = (b_1, b_2, b_3, b_4) \) and \( B = b_1 + b_2 + b_3 + b_4 \). In algebraic geometry, \((2.9)\) is often called Riemann’s relation [M]. A direct proof is easy; it can also be obtained from the case \( n = 5 \) of \((2.6)\) after substituting
\begin{align*}
x &= \left( 0, \frac{1}{2}, \frac{\tau}{2}, -\frac{1}{2}, \frac{B}{2} \right), \\
y &= \left( -b_1, -b_2, -b_3, -b_4, \frac{B}{2} \right)
\end{align*}
and using \((2.4)\) together with the identity
\begin{equation}
\theta \left( \left( \frac{1}{2}, \frac{\tau}{2}, -\frac{1}{2}, \frac{\tau}{2} \right) \right) = \frac{2}{ip^{1/8}} \left( \frac{p}{p} \right)^3 \infty,
\end{equation}
which follows from \((2.4)\) after dividing with \( \theta(x) \) and letting \( x \to 0 \).

Occasionally, we will write \([x] = \theta(2\eta)x\) and denote elliptic shifted factorials by
\begin{align*}
[x]_k &= [x][x + 1] \cdots [x + k - 1], \\
[x_1, \ldots, x_n]_k &= [x_1]_k \cdots [x_n]_k.
\end{align*}
Note that
\begin{equation}
\lim_{\eta \to 1} \lim_{p \to 0} \frac{[x]_k}{[y]_k} = \frac{x(x + 1) \cdots (x + k - 1)}{y(y + 1) \cdots (y + k - 1)},
\end{equation}
which exhibits \([x]_k\) as a two-parameter deformation of the classical Pochhammer symbol.

2.2. Higher order theta functions. Let \( \Theta_N \) denote the space of even theta functions of order \( 2N \) with quasi-period \((1, \tau)\) and zero characteristics. That is, \( \Theta_N \) consists of entire functions satisfying
\begin{align*}
f(x + 1) &= f(x), \\
f(x + \tau) &= e^{-2\pi i N(2x + \tau)} f(x), \\
f(-x) &= f(x).
\end{align*}
This space has dimension \( N + 1 \) and is spanned by functions of the form
\begin{equation}
\prod_{j=1}^{N} \theta(a_j \pm x).
\end{equation}
In [Ro2] we were led to choose \( \{a_j\} \) as the union of two arithmetic progressions. More precisely, let us write
\begin{equation}
e_k(x) = e_k^N(x; a, b) = \prod_{j=0}^{k-1} \theta(a \pm x + 2j\eta) \prod_{j=0}^{N-k-1} \theta(b \pm x + 2j\eta).
\end{equation}
Then \[Ro2\] Remark 5.2 \((e_k)_k^{N=0}\) form a basis for \(\Theta_N\) if and only if

\[
\begin{align*}
(2.11a) & \quad a - b + 2j\eta \notin \mathbb{Z} + \tau \mathbb{Z}, \quad j = 1 - N, 2 - N, \ldots, N - 1, \\
(2.11b) & \quad a + b + 2j\eta \notin \mathbb{Z} + \tau \mathbb{Z}, \quad j = 0, 1, \ldots, N - 1.
\end{align*}
\]

We think of \(\Theta_N\) as a deformation of the space of polynomials of degree \(\leq N\) and of \(e_k(x)\) as an analogue of \((a + x)^k(b + x)^{N-k}\).

We note that

\[
(2.12) \quad \overline{e_k^N(x; a, b)} = e_k^N(\bar{x}; \pm \bar{a}, \pm \bar{b}),
\]

where the plus sign is chosen for \(\eta \in \mathbb{R}\) and the minus sign for \(\eta \in i\mathbb{R}\). Moreover, one has

\[
\begin{align*}
(2.13) & \quad e_k^N(x; a + 1, b) = e_k^N(x; a, b), \\
& \quad e_k^N(x; a + \tau, b) = e^{-2\pi i k(\tau + 2a + 2(k-1)\eta)} e_k^N(x; a, b),
\end{align*}
\]

and similarly for \(b\) since \(e_k^N(x; a, b) = e_{N-k}^N(x; b, a)\).

2.3. Sklyanin algebra. For \(a = (a_1, a_2, a_3, a_4)\) such that \(\sum a_i = 0\), we let \(\Delta(a)\) denote the difference operator

\[
(2.14) \quad \Delta(a)f(x) = \frac{\theta(x - a + \frac{1}{2} N\eta) f(x + \eta) - \theta(x - a - \frac{1}{2} N\eta) f(x - \eta)}{\theta(2x)}.
\]

It is easy to check that \(\Delta(a)\) preserves the space \(\Theta_N\). We note the quasi-periodicity

\[
\begin{align*}
\Delta(a_1 + 1, a_2 - 1, a_3, a_4) &= \Delta(a_1, a_2, a_3, a_4), \\
\Delta(a_1 + \tau, a_2 - \tau, a_3, a_4) &= e^{2\pi i (a_2-a_1-\tau)} \Delta(a_1, a_2, a_3, a_4),
\end{align*}
\]

and similarly for the other parameters by symmetry.

The operators \(\Delta(a)\) were introduced by Rains \[R1\], who also observed \[R2\] that they form the degree one subspace of representations of the Sklyanin algebra discovered by Sklyanin \[S2\]. Namely, Sklyanin introduced four operators \(S_0, S_1, S_2\) and \(S_3\), corresponding to his four algebra generators. One can check that each \(S_i\) is proportional to an operator of the form \(\Delta(a)\) and, conversely, that any \(\Delta(a)\) is a linear combination of the \(S_i\). Although we will not need these facts, for the benefit of the interested reader we provide the details in the Appendix.

In \[Ro2\] Proposition 6.2], we considered the action of the operators \(2.14\) on the basis vectors \(2.10\). In the present notation, we proved that

\[
(2.15) \quad \Delta(a) e_k^N(x; a_1 - \frac{1}{2} N\eta + \eta, a_2 - \frac{1}{2} N\eta + \eta) \\
= -\theta(a_1 + a_2 + N\eta, a_1 + a_3 + (2k - N)\eta, a_2 + a_3 + (N - 2k)\eta) \\
\times e_k^N(x; a_1 - \frac{1}{2} N\eta, a_2 - \frac{1}{2} N\eta),
\]

where the plus sign is chosen for \(\eta \in \mathbb{R}\) and the minus sign for \(\eta \in i\mathbb{R}\). Moreover, one has

\[
\begin{align*}
(2.13) & \quad e_k^N(x; a + 1, b) = e_k^N(x; a, b), \\
& \quad e_k^N(x; a + \tau, b) = e^{-2\pi i k(\tau + 2a + 2(k-1)\eta)} e_k^N(x; a, b),
\end{align*}
\]

and similarly for \(b\) since \(e_k^N(x; a, b) = e_{N-k}^N(x; b, a)\).
and that

\[ \Delta(a) e^N_k(x; \lambda, \mu) = \sum_{j=k-1}^{k+1} C_j e^N_j(x; \lambda + \eta, \mu + \eta) \]

for some coefficients \( C_j \). We need to know \( C_{k\pm 1} \) explicitly. This can be achieved by choosing \( x = \lambda + \eta(2k - 1) \) and \( x = \mu + \eta(2N - 2k - 1) \) in (2.16), giving

\[
C_{k-1} = \frac{\theta(\lambda + \bar{a} + \eta(2k - 1 - \frac{1}{2}N), 2k\eta, \lambda - \mu + 2k\eta)}{\theta(\lambda + \mu + 2N\eta, \lambda - \mu + 2(2k - N - 1)\eta, \lambda - \mu + 2(2k - N)\eta)}.
\]

\[
C_{k+1} = \frac{\theta(\mu + \bar{a} + \eta(\frac{3}{2}N - 2k - 1), 2(k - N)\eta, \lambda - \mu + 2(k - N)\eta)}{\theta(\lambda + \mu + 2N\eta, \lambda - \mu + 2(2k - N)\eta, \lambda - \mu + 2(2k - N + 1)\eta)}.
\]

2.4. Involution. Several of our results are most conveniently stated in terms of the involution \( \sigma \) on \( \Theta_N \) defined by

\[
(\sigma f)(x) = e^{2\pi i N(\frac{1}{4} + \frac{\tau}{2} + x)} f \left( x + \frac{1}{2} + \frac{\tau}{2} \right) = e^{2\pi i N(\frac{1}{4} + \frac{\tau}{2} - x)} f \left( x - \frac{1}{2} - \frac{\tau}{2} \right).
\]

It is easy to check that \( \sigma \) preserves \( \Theta_N \) and that \( \sigma \circ \sigma = \text{id} \).

We mention the easily verified identity

\[
\sigma e^N_k(x; a, b) = e^{2\pi i (ak - b(N - k) + (N - 1)(2k - N)\eta + \frac{1}{2}N(\tau - 1))} \times e^N_k(x; a + \frac{1}{2} + \frac{\tau}{2}, b - \frac{1}{2} - \frac{\tau}{2}).
\]

We also need to know how \( \Delta(a) \) behaves under conjugation by \( \sigma \). One first computes

\[
(\sigma \Delta(a) \sigma f)(x) = \frac{e^{\pi i (\tau + 4\bar{a})}}{\theta(2x)} \left\{ e^{-2\pi i N\eta} \theta(x + \bar{a} + \frac{1}{2} + \frac{\tau}{2} - \frac{1}{2}N\eta) f(x + \eta) - e^{2\pi i N\eta} \theta(x - \bar{a} + \frac{1}{2} + \frac{\tau}{2} + \frac{1}{2}N\eta) f(x - \eta) \right\},
\]

which may be rewritten in the form

\[
\sigma \circ \Delta(a) \circ \sigma = e^{2\pi i (a_1 + a_4 + \frac{\tau}{2})} \Delta \left( a_1 + \frac{1}{2} + \frac{\tau}{2}, a_2 + \frac{1}{2} - \frac{\tau}{2}, a_3 - \frac{1}{2} + \frac{\tau}{2}, a_4 - \frac{1}{2} - \frac{\tau}{2} \right).
\]

The apparent loss of symmetry is needed to preserve the condition \( \sum a_i = 0 \).

2.5. Invariant integration. The following metric on \( \Theta_N \) was introduced by Sklyanin [S2]:

\[
\langle f, g \rangle = \int \int_{C/(Z \times Z)} f(u) \overline{g(u)} M(u, \bar{u}) \, dx dy,
\]
where \( u = x + iy \) and

\[
M(u, v) = \frac{\theta(2u)\theta(2v)}{e^{2\pi i u(N+2)} \prod_{k=0}^{N+1} \theta(u \pm v + (2k - N - 1)\eta + \frac{1}{2} + \frac{\tau}{2})}.
\]

It can be viewed as a deformation of the standard SU(2)-invariant metric on polynomials of degree \( \leq N \):

\[
\text{Const} \int_C \frac{f(u)g(u)}{(1 + |u|^2)^{N+2}} dxdy.
\]

It is easy to check that, for \( f, g \in \Theta_N \), the integrand in (2.21) has indeed double-period \((1, \tau)\). The weight \( M(u, \bar{u}) \) is free from poles if

\[
(2.22) \quad (2k - N - 1)\eta \notin \left( \mathbb{Z} + \frac{1}{2} + i\mathbb{R} \right) \cup \left( \tau(\mathbb{Z} + \frac{1}{2}) + \mathbb{R} \right), \quad k = 0, 1, \ldots, N+1,
\]

and it is non-negative if \( \eta \in \mathbb{R} \cup i\mathbb{R} \), since then

\[
M(u, \bar{u}) = \frac{p^{1(N+2)}(p; p)^{2N+2}}{|\theta(2u)|^2 \prod_{k=0}^{N+1} \left| \frac{1}{(-\sqrt{p} e^{2\pi i (u \pm \bar{u} + (2k - N - 1)\eta)}/p)_{\infty}} \right|^2}.
\]

When both these conditions are satisfied, we have a genuine scalar product. In particular, this happens for \( \eta \in \mathbb{R}_N \cup i\mathbb{I}_N \), where

\[
\mathbb{R}_N = \{ \eta \in \mathbb{R}; |\eta| < 1/(N + 1) \}, \quad \mathbb{I}_N = \{ \eta \in i\mathbb{R}; |\eta| < \tau/(2(N + 1)i) \}.
\]

3. Statement of results

Our main tool is the invariance of the metric (2.21) with respect to the operators (2.14). Sklyanin proved that for \( \eta \in \mathbb{R}_N \) his operators \( S_i \) are self-adjoint. Working with the more general operators (2.14) allows one to simplify the proof, since one need not then consider the \( S_i \) one by one. Moreover, we want to extend Sklyanin’s result to the case \( \eta \in \mathbb{I}_N \). Although the two cases are related by the modular transformation (2.8), we prefer to treat them in parallel. For these two reasons, and since Sklyanin’s presentation is rather sketchy, we give a detailed proof in Section 4.

**Proposition 3.1.** Assume that \( \eta \in \mathbb{R}_N \), and let \( \Delta(a) \), \( \sum a_i = 0 \), be an operator of the form (2.14). Then its adjoint with respect to the metric (2.21) is given by \( \Delta(a)^* = -\sigma \circ \Delta(-\bar{a}) \circ \sigma \). Similarly, if \( \eta \in \mathbb{I}_N \), then \( \Delta(a)^* = \sigma \circ \Delta(\bar{a}) \circ \sigma \).

It is easy to check that \( \sigma^* = \sigma \), which shows that Proposition 3.1 is consistent with \( \Delta(a)^{**} = \Delta(a) \). Note also that, by (2.20), there exist in both cases a constant \( C \) and parameters \( b_i \) such that \( \Delta(a)^* = C \Delta(b) \). However, the formulation involving \( \sigma \) is more convenient for our purposes.

Using (A.1) and (2.20), one may check that Proposition 3.1 agrees with the result of Sklyanin mentioned above.
Corollary 3.2. For $\eta \in \mathbb{R}_N$, one has $S_i^* = S_i$, $i = 0, 1, 2, 3$, whereas for $\eta \in \mathbb{I}_N$, $S_i^* = -S_i$, $i = 0, 1, 2, 3$.

Our first main result concerns the reproducing kernel for $\Theta_N$ with respect to the metric (2.21). Sklyanin conjectured that it is given, up to a multiplicative constant, by

$$K_v(u) = K(u, \bar{v}) = e^{2\pi i u N} \prod_{k=0}^{N-1} \theta(u \pm \bar{v} + (2k - N + 1)\eta + \frac{1}{2} + \frac{\tau}{2}).$$

Note that

(3.1) $K(u, \bar{v}) = K(v, \bar{u}) = K(\bar{v}, u)$.

In Section 5 we will prove Sklyanin’s conjecture, and in Section 6 the multiplicative constant will be computed. We summarize the result as follows.

Theorem 3.3. For $\eta \in \mathbb{R}_N \cup \mathbb{I}_N$, the reproducing kernel of $\Theta_N$ with the metric (2.21) is given by $C^{-1}K_v$, where

(3.2) $C = \frac{2\eta p^{3/8}}{\theta(2(N + 1)\eta)(p; p)_\infty^3}$.

When $\eta = 0$, the expression for $C$ has a removable singularity and should be interpreted as, cf. (2.7),

$$\lim_{\eta \to 0} C = \frac{p^{1/4}}{2\pi(N + 1)(p; p)_\infty^6}.$$

We have also obtained biorthogonal bases for $\Theta_N$. Indeed, we can find the dual of any basis of the form (2.10).

Theorem 3.4. Suppose that

$$e_k(x) = e_k^N(x; a_1 - \frac{1}{2}N\eta, a_2 - \frac{1}{2}N\eta), \quad k = 0, \ldots, N,$$

form a basis for the space $\Theta_N$. Let

$$f_k(x) = \sigma e_k^N(x; \mp \bar{a}_2 - \frac{1}{2}N\eta + \eta, \mp \bar{a}_1 - \frac{1}{2}N\eta + \eta),$$

where the minus sign is chosen if $\eta \in \mathbb{R}_N$ and the plus sign if $\eta \in \mathbb{I}_N$. Then

$$\langle e_k, f_l \rangle = C \Gamma_k \delta_{kl},$$

where $C$ is given by (3.2) and

$$\Gamma_k = e^{\pi i N(\tau-1)/2} \frac{[\lambda]}{[\lambda + 2k]} \frac{[1, \lambda + N + 1]}{[-N, \lambda]} \frac{[\lambda + 1, (a_1 + a_2 - N\eta)/2\eta]_N}{[\lambda + 1, (a_1 + a_2 - N\eta)/2\eta]_N},$$

with $\lambda = (a_1 - a_2 - 2N\eta)/2\eta$. 
In the case $\eta = 0$, the expression for $\Gamma_k$ should be interpreted as the limit
\[
\lim_{\eta \to 0} \Gamma_k = e^{\pi i N(\tau - 1)/2} \frac{(-1)^k}{N^k} \theta(a_1 + a_2) \theta(a_1 - a_2)^N.
\]
We also observe that, by (2.12), for $\eta \in \mathbb{R}_N$ and $\eta \in \mathbb{I}_N$ alike we have
\[
(3.3) \quad \bar{f}_k(x) = e^{2\pi i N(x - \frac{1}{2} + \frac{\tau}{2})} e_N^N(x + \frac{1}{2} + \frac{\tau}{2}; -a_2 - \frac{1}{2} N \eta + \eta, -a_1 - \frac{1}{2} N \eta + \eta).
\]

The proof of Theorem 3.4 is divided into two parts. For the biorthogonality we use the generalized eigenvalue equation (2.15) and for the norm computation the generalized tridiagonal equation (2.16). The details are given in Sections 7 and 8, respectively.

Sklyanin conjectured [S2, p. 277] that his metric extends from $\eta \in \mathbb{R}_N$ to the larger parameter range $\eta \in \mathbb{R}_{N-1}$. This follows quite easily from Theorem 3.4.

**Proposition 3.5.** The analytic continuation in $\eta$ of $C^{-1}(f, g)$ is positive definite for $\eta \in \mathbb{R}_{N-1} \cup \mathbb{I}_{N-1}$.

**Proof.** We first specialize the parameters in Theorem 3.4 to obtain an orthogonal basis. By (2.18), $f_k(x)$ is proportional to
\[
e_k^N(x; \mp a_2 - \frac{1}{2} N \eta + \eta + \frac{1}{2} + \frac{\tau}{2}, \mp a_1 - \frac{1}{2} N \eta + \eta - \frac{1}{2} - \frac{\tau}{2}).
\]
In particular, if $a_2 = \mp a_1 + \eta - \frac{1}{2} - \frac{\tau}{2}$ then, using also (2.13), $e_k$ and $f_k$ are proportional, so that $(e_k)^N_{k=0}$ is orthogonal. We know that $C^{-1}||e_k||^2$ is given by an exponential factor times $\Gamma_k$, and that it is positive for $\eta \in \mathbb{R}_N \cup \mathbb{I}_N$. By continuity, for generic $a_1$ it must remain positive as long as $[1]_k/[-N]_k$ is well-defined and non-zero. This is indeed true for $\eta \in \mathbb{R}_{N-1} \cup \mathbb{I}_{N-1}$. \qed

## 4. Proof of Proposition 3.1

Consider first the case $\eta \in \mathbb{R}_N$. We write
\[
\langle \Delta(a)f, g \rangle = \int \frac{\theta(u + a - \frac{1}{2} N \eta)f(u + \eta) - \theta(u - a + \frac{1}{2} N \eta)f(u - \eta)}{\theta(2u)} g(\bar{v}) M(u, v) \, dxdy
\]
\[
= \int \frac{\theta(u + a - \frac{1}{2} (N + 2) \eta)f(u)g(\bar{v}) M(u, v)}{\theta(2u - 2 \eta)} \, dxdy
\]
\[
- \int \frac{\theta(u - a + \frac{1}{2} (N + 2) \eta)f(u)g(\bar{v}) M(u + \eta, v)}{\theta(2u + 2 \eta)} \, dxdy.
\]

We wish to replace the contours of integration with $v = \bar{u}$. In general, we have that
\[
\int \int \frac{\theta(u + a - \frac{1}{2} (N + 2) \eta)f(u)g(\bar{v}) M(u - \eta, v)}{\theta(2u - 2 \eta)} \, dxdy
\]
is independent of $\gamma$ as long as we avoid the poles of $M(u - \eta, \bar{u} - \gamma)$, that is, for

$$\gamma \notin \mathbb{R} + \tau \left( \mathbb{Z} + \frac{1}{2} \right),$$

$$\gamma - \eta + (2k - N - 1)\eta \notin \mathbb{Z} + \frac{1}{2} + i\mathbb{R}, \quad k = 0, 1, \ldots, N + 1.$$

In particular, the region of analyticity containing $\gamma = \eta$ is given by

$$|\text{Im}(\gamma)| < \frac{\tau}{2i}, \quad |\text{Re}(\gamma) - \eta \pm (N + 1)\eta| < 1/2.$$ 

If we make the temporary assumption $\eta \in \mathbb{R}_{N+1}$, then this region contains $\gamma = 0$, so that the contour may indeed be replaced with $v = \bar{u}$. This also holds for the integral on $v = \bar{u} + \eta$, giving

$$\langle \Delta(a)f, g \rangle = \int_{v=\bar{u}} f(u)g(\bar{v}) \left\{ \frac{\theta(u + \bar{a} - \frac{1}{2}(N+2)\eta)M(u - \eta, v)}{\theta(2u - 2\eta)} - \frac{\theta(u - \bar{a} + \frac{1}{2}(N+2)\eta)M(u + \eta, v)}{\theta(2u + 2\eta)} \right\} \, dx dy. \tag{4.1}$$

Using (2.19), the same argument, still assuming $\eta \in \mathbb{R}_{N+1}$, gives

$$\langle f, \sigma \Delta(\bar{b})\sigma g \rangle = \int_{v=\bar{u}} f(u)g(\bar{v}) e^{\pi i(\tau - 4v)} \left\{ e^{2\pi i(N+2)\eta} \frac{\theta(v + \bar{b} + \frac{1}{2} - \frac{\tau}{2} - \frac{1}{2}(N+2)\eta)M(u, v - \eta)}{\theta(2v - 2\eta)} - e^{-2\pi i(N+2)\eta} \frac{\theta(v - \bar{b} + \frac{1}{2} - \frac{\tau}{2} + \frac{1}{2}(N+2)\eta)M(u, v + \eta)}{\theta(2v + 2\eta)} \right\} \, dx dy. \tag{4.2}$$

We are thus reduced to proving

$$\langle \Delta(a)f, g \rangle = e^{i\pi(\tau - 4v)} \left\{ e^{-2\pi i(N+2)\eta} \frac{\theta(v + \bar{a} + \frac{1}{2} - \frac{\tau}{2} + \frac{1}{2}(N+2)\eta)M(u, v + \eta)}{\theta(2v + 2\eta)} - e^{2\pi i(N+2)\eta} \frac{\theta(v - \bar{a} + \frac{1}{2} - \frac{\tau}{2} + \frac{1}{2}(N+2)\eta)M(u, v - \eta)}{\theta(2v - 2\eta)} \right\}. \tag{4.3}$$
Multiplying (4.3) with $e^{2\pi i(N+2)\eta} \prod_{k=0}^{N+2} \theta(u \pm \nu + (2k - N - 2)\eta + \frac{1}{2} + \frac{\tau}{\theta})$ and simplifying gives

(4.4) \[ \theta(2\nu) \left\{ e^{2\pi i(N+2)\eta} \theta(u \pm \nu + (N+2)\eta + \frac{1}{2} + \frac{\tau}{\theta}, u + \bar{a} - \frac{1}{2}(N+2)\eta) \\
- e^{-2\pi i(N+2)\eta} \theta(u \pm \nu - (N+2)\eta + \frac{1}{2} + \frac{\tau}{\theta}, u - \bar{a} + \frac{1}{2}(N+2)\eta) \right\} \]

\[= e^{i\tau(\nu-\nu)} \theta(2\nu) \times \left\{ e^{-2\pi i(N+2)\eta} \theta(u \pm \nu - (N+2)\eta + \frac{1}{2} + \frac{\tau}{\theta}, v + \bar{a} + \frac{1}{2} - \frac{\tau}{\theta} + \frac{1}{2}(N+2)\eta) \\
- e^{2\pi i(N+2)\eta} \theta(u \pm \nu + (N+2)\eta + \frac{1}{2} + \frac{\tau}{\theta}, v - \bar{a} + \frac{1}{2} - \frac{\tau}{\theta} - \frac{1}{2}(N+2)\eta) \right\}. \]

This is equivalent to the case $n = 4$ of (2.6), which we rewrite as

\[\theta(x_4 - x_3) \left\{ \theta(x_2 - x_3, x_2 - x_4, x_1 - \bar{y}) - \theta(x_3 - x_1, x_4 - x_2, x_2 - \bar{y}) \right\} = \theta(x_2 - x_1) \left\{ \theta(x_4 - x_1, x_2 - x_4, x_3 - \bar{y}) - \theta(x_3 - x_1, x_2 - x_3, x_4 - \bar{y}) \right\},\]

valid for $y = (y_1, y_2, y_3, y_4)$ with $\sum_i x_i = \sum_i y_i$. Indeed, substituting

\[(x_1, x_2, x_3, x_4) = \left( -u + \frac{1}{2}(N+2)\eta, u + \frac{1}{2}(N+2)\eta, \\
-\nu - \frac{1}{2} + \frac{\tau}{2} - \frac{1}{2}(N+2)\eta, \nu + \frac{1}{2} - \frac{\tau}{2} - \frac{1}{2}(N+2)\eta \right),\]

letting $y_i = a_i$ and repeatedly using (2.3), one obtains (4.1).

We have assumed $\eta \in \mathbb{R}_{N+1}$, but the result extends to $\eta \in \mathbb{R}_N$. Namely, \( \langle \Delta(a)f, g \rangle + \langle f, \sigma \Delta(-\bar{a})\sigma g \rangle \) is analytic in $\eta$ as long as $f$ is satisfied, thus, it is zero for $\eta \in \mathbb{R}_N$.

Repeating the calculation for imaginary $\eta$, one sees that (4.1) holds also for $\eta \in \mathbb{I}_{N+1}$, while in (4.2), $\eta$ should be replaced by $\bar{\eta} = -\eta$ everywhere on the right-hand side. However, this has the same effect as replacing $b$ by $-b$ and multiplying the whole expression with $-1$. Thus, we are again reduced to the identity (4.3).

The extension from $\eta \in \mathbb{I}_{N+1}$ to $\eta \in \mathbb{I}_N$ follows as before.

5. Proof of Sklyanin’s conjecture

In this section we prove Sklyanin’s conjecture, that is, that

\[(5.1) \quad f(u) = C^{-1}(f, K_u), \quad f \in \Theta_N, \quad u \in \mathbb{C} \]

for some constant $C$.

We need to know that the kernels $(K_u)_{u \in \mathbb{C}}$ span $\Theta_N$. This can be seen, for instance, by considering the elements

\[e_k^N(x; a, 2\eta - a) = \prod_{j=k-N}^{k-1} \theta(a \pm x + 2j\eta), \quad k = 0, 1, \ldots, N.\]
These are all proportional to a kernel $K_u(x)$. Moreover, by (2.11), for generic $a$ they form a basis for $\Theta_N$ as long as $2j\eta \notin \mathbb{Z} + \tau\mathbb{Z}$, $1 \leq j \leq N$; in particular, this happens for $\eta \in \mathbb{R}_N \cup \mathbb{I}_N$ (and even for $\eta \in \mathbb{R}_{N-1} \cup \mathbb{I}_{N-1}$), $\eta \neq 0$.

It is thus enough to prove (5.1) for $f = K_u$, that is, introducing the kernel

$$\Phi_v(u) = \Phi(u, \bar{v}) = \langle K_u, K_u \rangle,$$

that

$$\Phi_v = CK_v.$$  

Our main tool for proving (5.2) is the existence of Sklyanin algebra elements that act nicely on the kernel $K_v$. Note first that, by (3.1), we may write

$$K_v(x) = e^{2\pi i N} \tau_N(x; a_1 + \eta - \frac{1}{2} N\eta, a_2 + \eta - \frac{1}{2} N\eta),$$

where $a_1 = \bar{v} - \frac{1}{2} N\eta + \frac{1}{2} + \frac{\tau}{2}$ and $a_2$ is arbitrary. Thus, (2.15) gives

$$\Delta(a)K_v = -e^{2\pi i N\eta} \theta(a_1 + a_2 + N\eta, a_1 + a_3 + N\eta, a_2 + a_3 - N\eta) K_v - \eta,$$

where $a_3$ is arbitrary and $a_4 = -a_1 - a_2 - a_3$.

Assume first that $\eta \in \mathbb{R}_N$. We consider the equality

$$\langle \Delta(a)K_v, K_u \rangle = \langle K_v, \Delta(a)K_u \rangle,$$

where we know that the left-hand side is

$$-e^{2\pi i N\eta} \theta(a_1 + a_2 + N\eta, a_1 + \eta + a_3 + N\eta, a_2 + a_3 - N\eta) \Phi(u, \bar{v} - \eta).$$

As for the right-hand side, we use Proposition 3.1 together with the identity

$$\sigma K_u(x) = e^{\pi i N(1-\tau)/2} \tau_N(x; w, \bar{u} + (1 - N)\eta),$$

with $w$ arbitrary. Choosing $a_2 = \frac{1}{2} N\eta - u$, we find that the right-hand side of (5.3) equals

$$-\theta(a_1 + a_2 - N\eta, a_1 + \eta + a_3 + N\eta, a_2 + a_3 - N\eta) \Phi(u - \eta, \bar{v}).$$

Identifying (5.4) and (5.5) we obtain the difference equation

$$\Phi(u, \bar{v} - \eta) = e^{2\pi i N} \frac{\theta(u - \bar{v} + N\eta + \frac{1}{2} + \frac{\tau}{2})}{\theta(u - \bar{v} - N\eta + \frac{1}{2} + \frac{\tau}{2})} \Phi(u - \eta, \bar{v}),$$

which, after replacing $\bar{v}$ with $\bar{v} + \eta$ and iterating yields

$$\Phi(u, \bar{v}) = e^{2\pi i N \eta k} \prod_{j=1}^{k} \frac{\theta(u - \bar{v} + (2j - 1 - N)\eta + \frac{1}{2} + \frac{\tau}{2})}{\theta(u - \bar{v} - (2j - 1 + N)\eta + \frac{1}{2} + \frac{\tau}{2})} \Phi(u - k\eta, \bar{v} + k\eta).$$

We now plug $u = u_k = \bar{v} + (2k - 1 - N)\eta - \frac{1}{2} - \frac{\tau}{2}$ into (5.7), which makes the numerator zero. The denominator is then $\prod_{j=1}^{k} \theta(2(k - j - N)\eta)$, which is non-zero if $1 \leq k \leq N$ and $\eta \in \mathbb{R}_N$, $\eta \neq 0$. Thus, $\{u_k\}_{k=1}^{N}$ are zeroes of the function $\Phi_v$. Since $\Phi_v \in \Theta_N$, the points $\pm u_k + \mathbb{Z} + \tau\mathbb{Z}$ are also zeroes. These are precisely the
zeros of \( K_v \), so \( \Phi_v/K_v \) is a \((1, \tau)\)-periodic entire function, and thus constant by Liouville’s theorem.

In the case \( \eta \in \mathbb{I}_N, \eta \neq 0 \), the same proof goes through, although one then finds instead of (5.6) the equation
\[
\Phi(u, \bar{v} - \eta) = e^{-2\pi i N\eta} \frac{\theta(u + \bar{v} - N\eta + \frac{1}{2} + \frac{\tau}{2})}{\theta(u + \bar{v} + N\eta + \frac{1}{2} + \frac{\tau}{2})} \Phi(u + \eta, \bar{v}).
\]

For the above proof to work it is essential that \( \eta \neq 0 \), though the case \( \eta = 0 \) is included by continuity, cf. the remark following Theorem 3.3.

6. Computation of the constant

Knowing that the constant \( C \) of Theorem 3.3 exists, we shall now compute it. To this end, let \( (e_k)_{k=0}^N \) be an orthonormal basis of the space \( \Theta_N \), so that
\[
\frac{1}{C} K(u, \bar{v}) = \sum_{k=0}^N e_k(u) e_k(v).
\]
If we put \( u = v \) in this identity and integrate we obtain
\[
\frac{1}{C} \int\int K(u, \bar{u}) M(u, \bar{u}) \, dxdy = \sum_{k=0}^N \|e_k\|^2 = N + 1,
\]
that is,
\[
C = \frac{1}{N + 1} \int\int K(u, \bar{u}) M(u, \bar{u}) \, dxdy = \frac{e^{-4\pi i (N+1)\eta}}{N + 1} \int\int \frac{\theta(2u)\theta(2\bar{u})}{\theta(u \pm \bar{u} \pm \gamma)} \, dxdy,
\]
where \( \gamma = (N + 1)\eta + \frac{1}{2} + \frac{\tau}{2} \). Note that \( \eta \in \mathbb{R}_N \cup \mathbb{I}_N \) means that \( 0 < \text{Re}(\gamma) < 1 \), \( 0 < \text{Im}(\gamma) < \tau/i \).

Applying the following lemma now completes the proof of Theorem 3.3.

**Lemma 6.1.** Suppose \( 0 < \text{Re}(\gamma) < 1 \) and \( 0 < \text{Im}(\gamma) < \tau/i \). Then
\[
\int\int \frac{\theta(2u)\theta(2\bar{u})}{\theta(u \pm \bar{u} \pm \gamma)} \, dxdy = \frac{p^{-1/8}(2\gamma - 1 - \tau)}{\theta(2\gamma)(p; p)_\infty^3}.
\]

Our proof will be similar to the proof of [R1, Lemma 3.3]. We first note that applying \( \frac{\partial}{\partial x} \big|_{x=y} \) to both sides of (2.5) gives
\[
\frac{\theta(u \pm v)}{\theta(u \pm y)\theta(v \pm y)} = \frac{1}{\theta'(0)\theta(2y)} \left( \frac{\theta'(u + y)}{\theta(u + y)} - \frac{\theta'(u - y)}{\theta(u - y)} + \frac{\theta'(v - y)}{\theta(v - y)} - \frac{\theta'(v + y)}{\theta(v + y)} \right).
\]
Lemma 6.2. For $I, J$ after simplification.

Since the last term is 1-periodic, we obtain indeed

where $\theta(2x + \gamma) = \frac{\theta'(2x + \gamma)}{\theta(2x + \gamma)}$, $\theta(2iy + \gamma) = \frac{\theta'(2iy + \gamma)}{\theta(2iy + \gamma)}$.

We are thus reduced to computing integrals of the form

$I(\gamma) = \int_0^1 \frac{\theta'(2x + \gamma)}{\theta(2x + \gamma)} \, dx, \quad \gamma \notin \mathbb{R} + \tau\mathbb{Z},$

$J(\gamma) = \int_0^{\tau/i} \frac{\theta'(2iy + \gamma)}{\theta(2iy + \gamma)} \, dy, \quad \gamma \notin \mathbb{Z} + i\mathbb{R}.$

**Lemma 6.2.** For $0 < \text{Im}(\gamma) < \tau/i$ one has $I(\gamma) = -i\pi$. For other values of $\gamma$, $I(\gamma)$ is determined by $I(\gamma + \tau) = -2\pi i + I(\gamma)$. Similarly, for $0 < \text{Re}(\gamma) < 1$ one has $J(\gamma) = 2\pi(\frac{1}{2} - \gamma - \tau)$; for other values, $J(\gamma)$ is determined by $J(\gamma) = J(\gamma + 1)$.

Although Lemma 6.2 is easily deduced from known results, we include a proof for completeness. The functional equations for $I$ and $J$ follow from (2.3). By analyticity, it is then enough to assume $\text{Im}(\gamma) = \tau/2i$ and $\text{Re}(\gamma) = 1/2$, respectively.

In the case $\text{Im}(\gamma) = \tau/2i$, we may write

$$\theta(2x + \gamma) = i\tau^{1/8}(p; p)_\infty e^{-\pi i(2x+\gamma)} \left|(e^{2\pi i(2x+\gamma)}; p)_\infty\right|^2,$$

and thus

$$\frac{\theta'(2x + \gamma)}{\theta(2x + \gamma)} = -i\pi + \frac{d}{dx} \log \left|(e^{2\pi i(2x+\gamma)}; p)_\infty\right|.$$

Since the last term is 1-periodic, we obtain indeed $I(\gamma) = -i\pi$.

Similarly, if $\text{Re}(\gamma) = 1/2$ we may write

$$\theta(2iy + \gamma) = i\tau^{1/8} e^{-\pi i(2iy+\gamma)} (p, e^{-2\pi(2y+\delta)}, -pe^{2\pi(2y+\delta)}; p)_\infty,$$

where $\delta = (\gamma - 1/2)/i \in \mathbb{R}$. This gives

$$\frac{\theta'(2iy + \gamma)}{\theta(2iy + \gamma)} = -i\pi + \frac{1}{2i} \frac{d}{dy} \log(e^{-2\pi(2y+\delta)}, -pe^{2\pi(2y+\delta)}; p)_\infty,$$

which is integrated to

$$J(\gamma) = \int_0^{\tau/i} \frac{\theta'(2iy + \gamma)}{\theta(2iy + \gamma)} \, dy = -\pi \tau + \frac{1}{2i} \log \left(e^{2\pi i(2\tau\delta)}, -pe^{2\pi i(\delta+\tau)}; p\right)_\infty$$

$$= 2\pi \left(\frac{1}{2} - \gamma - \tau\right).$$

Using Lemma 6.2, we can now compute the integral in Lemma 6.1 as

$$\frac{p^{-1/8}}{2\pi \theta(2\gamma)(p; p)_\infty^3} \left(\frac{\tau}{\pi} (I(\gamma) - I(-\gamma)) + J(-\gamma) - J(\gamma)\right),$$

where $I(\pm\gamma) = \mp \pi i$, $J(\pm\gamma) = 2\pi(\pm(\frac{1}{2} - \gamma) - \tau)$, which gives the desired result after simplification.
7. Biorthogonality

In this section we begin the proof of Theorem 3.1 by showing that \( \langle e_k, f_l \rangle = 0 \) for \( k \neq l \). Recall that

\[
e_k(x) = e^N_k(x; a_1 - \frac{1}{2}N\eta, a_2 - \frac{1}{2}N\eta),
\]

\[
f_k(x) = \sigma e^N_k(x; \mp\bar{a}_2 - \frac{1}{2}N\eta + \eta, \mp\bar{a}_1 - \frac{1}{2}N\eta + \eta),
\]

where the minus sign is chosen for \( \eta \in \mathbb{R}_N \) and the plus sign for \( \eta \in \mathbb{I}_N \). We will also write

\[
e^+_k(x) = e^N_k(x; a_1 - \frac{1}{2}N\eta + \eta, a_2 - \frac{1}{2}N\eta + \eta),
\]

\[
f^-_k(x) = \sigma e^N_k(x; \mp\bar{a}_2 - \frac{1}{2}N\eta, \mp\bar{a}_1 - \frac{1}{2}N\eta).
\]

Consider the identity

\[
\langle \Delta(a)e^+_k, f_l \rangle = \mp\langle e^+_k, \sigma \Delta(\mp\bar{a})\sigma f_l \rangle,
\]

where \( a_3 \) is arbitrary and \( a_4 \) is fixed by \( \sum a_i = 0 \). By (2.15), the left-hand side is given by

\[-\theta(a_1 + a_2 + N\eta, a_1 + a_3 + (2k - N)\eta, a_2 + a_3 + (N - 2k)\eta) \langle e_k, f_l \rangle\]

and the right-hand side by

\[-\theta(a_1 + a_2 - N\eta, a_1 + a_3 + (2l - N)\eta, a_2 + a_3 + (N - 2l)\eta) \langle e^+_k, f^-_l \rangle.\]

Assuming that the denominator is non-zero, this gives

\[
\langle e_k, f_l \rangle = \frac{\theta(a_1 + a_2 - N\eta, a_1 + a_3 + (2l - N)\eta, a_2 + a_3 + (N - 2l)\eta)}{\theta(a_1 + a_2 + N\eta, a_1 + a_3 + (2k - N)\eta, a_2 + a_3 + (N - 2k)\eta)} \langle e^+_k, f^-_l \rangle.
\]

We now choose \( a_3 = (2l - N)\eta - a_2 \). This gives \( \langle e_k, f_l \rangle = 0 \), as long as the denominator in (7.1) is non-zero, that is, for

\[
a_1 + a_2 + N\eta, a_1 - a_2 + 2(k + l - N)\eta, 2(l - k)\eta \notin \mathbb{Z} + \tau\mathbb{Z}.
\]

If \( k \neq l \), the last condition holds for \( \eta \in \mathbb{R}_N \cup \mathbb{I}_N, \eta \neq 0 \). The other two conditions follow from the assumption that \( \langle e^+_k, f^-_l \rangle \) form a basis, that is, that (2.11) holds with \( (a, b) \) replaced by \( (a_1 - \frac{1}{2}N\eta, a_2 - \frac{1}{2}N\eta) \).

This proves that \( \langle e_k, f_l \rangle = 0 \) when \( k \neq l \). As a by-product, choosing \( k = l \) in (7.1) gives the identity

\[
\langle e_k, f_k \rangle = \frac{\theta(a_1 + a_2 - N\eta)}{\theta(a_1 + a_2 + N\eta)} \langle e^+_k, f^-_k \rangle,
\]

which will be used in the next section.
In this section we complete the proof of Theorem 3.4 by computing the constants $\Gamma_k = C^{-1}\langle e_k, f_k \rangle$. To this end, we consider the equality

\begin{equation}
\langle \Delta(A)e_k, f_{k+1}^- \rangle = \langle e_k, \Delta(A)^* f_{k+1}^- \rangle, \tag{8.1}
\end{equation}

where $A = (A_1, A_2, A_3, A_4)$ is arbitrary. By (2.16) and Proposition 3.1,

$$\Delta(A)e_k = c_{k-1}e_{k-1}^- + c_k e_k^+ + c_{k+1}e_{k+1}^+,$$

$$\Delta(A)^* f_{k+1}^- = d_k f_k + d_{k+1} f_{k+1} + d_{k+2} f_{k+2},$$

for some constants $c_i$ and $d_i$. By the results of the previous section, (8.1) then reduces to

$$c_{k+1} \langle e_{k+1}^+, f_{k+1}^- \rangle = \bar{d}_k \langle e_k, f_k \rangle,$$

which, by (172), yields the recursion

$$\Gamma_{k+1} = \frac{\theta(a_1 + a_2 - N\eta) \bar{d}_k}{\theta(a_1 + a_2 + N\eta) e_{k+1}^-} \Gamma_k.$$

Using (2.17), we compute

$$c_{k+1} = \frac{\theta(a_2 + \bar{A} + \eta(N - 2k - 1), 2(k - N)\eta, a_1 - a_2 + 2(k - N)\eta)}{\theta(a_1 + a_2 + N\eta, a_1 - a_2 + 2(2k - N)\eta, a_1 - a_2 + 2(2k - N + 1)\eta)},$$

$$\bar{d}_k = \frac{\theta(a_2 + \bar{A} + \eta(N - 2k - 1), 2(k + 1)\eta, a_1 - a_2 + 2(k + 1)\eta)}{\theta(a_1 + a_2 - N\eta, a_1 - a_2 + 2(2k - N + 1)\eta, a_1 - a_2 + 2(2k - N + 2)\eta)}.$$

giving

$$\Gamma_{k+1} = \frac{\theta(2(k + 1)\eta, a_1 - a_2 + 2(k + 1)\eta, a_1 - a_2 + 2(2k - N)\eta)}{\theta(2(k - N)\eta, a_1 - a_2 + 2(k - N)\eta, a_1 - a_2 + 2(2k + 2 - N)\eta)} \Gamma_k.$$

Upon iteration, this yields

$$\Gamma_k = \frac{[\lambda]_{k}}{[\lambda + 2k]_{k}} \frac{[1, \lambda + N + 1]_k}{[-N, \lambda]_k} \Gamma_0,$$

where $\lambda = (a_1 - a_2 - 2N\eta)/2\eta$.

We are thus reduced to computing $\Gamma_0 = C^{-1}\langle e_0, f_0 \rangle$, for which we observe that

$$f_0 = e^{\pi i N(\tau - 1)/2} K_{a_1 - \frac{1}{2} N\eta},$$

and thus, by Theorem 3.3,

$$\Gamma_0 = e^{\pi i N(\tau + 1)/2} e_0(a_1 - \frac{1}{2} N\eta) = e^{\pi i N(\tau - 1)/2} [\lambda + 1, (a_1 + a_2 - N\eta)/2\eta]_N.$$

This completes the proof of Theorem 3.4.
9. Elliptic hypergeometric series

By the general theory of reproducing kernel Hilbert spaces (which reduces to
linear algebra in the present, finite-dimensional, case) we have, in the notation of
Theorems 3.3 and 3.4

\[(9.1) \quad K_v(u) = \sum_{k=0}^{N} \frac{e_k(u)f_k(v)}{\Gamma_k}. \]

Writing this out explicitly, using (3.3), one obtains after simplification

\[(9.2) \quad \sum_{k=0}^{N} \frac{[a + 2k]}{[a]} \frac{[a, b, c, d, e, -N]_k}{[1, a + 1 - b, a + 1 - c, a + 1 - d, a + 1 - e, a + 1 + N]_k} = \frac{[a + 1, a + 1 - b - c, a + 1 - b - d, a + 1 - c - d]_N}{[a + 1 - b, a + 1 - c, a + 1 - d, a + 1 - b - c - d]_N}, \]

where

\[
(a, b, c, d, e) = \frac{1}{2\eta} \left( a_1 - a_2 - 2N\eta, -a_2 - \frac{1}{2}N\eta + \bar{v} + \frac{1}{2} + \tau \right),
\]

\[
- a_2 - \frac{1}{2}N\eta + \eta - \bar{v} - \frac{1}{2} - \tau, a_1 - \frac{1}{2}N\eta + u, a_1 - \frac{1}{2}N\eta - u \right). \]

These parameters satisfy \( b + c + d + e = 2a + N + 1 \) but are otherwise generic.
The summation formula (9.2) was first obtained by Frenkel and Turaev [FT].
When \( p = 0 \) it reduces to the Jackson sum [GR, (II.22)]. Its appearance here is
another example of the connection between elliptic quantum groups and elliptic
hypergeometric series, see further [KNR, Ro2]. Note also that, conversely, taking
the scalar product of both sides of (9.1) with \( f_l \) gives

\[
f_l(v) = \sum_{k=0}^{N} \frac{\langle e_k, f_l \rangle f_k(v)}{\Gamma_k},
\]

and we recover \( \langle e_k, f_l \rangle = \delta_{kl} \Gamma_k \). Thus, if we are willing to assume (9.2), we
obtain an alternative proof of Theorem 3.4 as a consequence of Theorem 3.3.

10. Elliptic hypergeometric integrals

Theorem 3.3 is equivalent to the integral formula

\[
\int_{\mathbb{C}/(2\pi i\mathbb{Z})} K(v, \bar{z}) K(z, \bar{w}) M(z, \bar{z}) \, dx dy = CK(v, \bar{w}), \quad z = x + iy.
\]

We write this out explicitly, using the notation

\[
\Gamma(x; p, q) = \prod_{j, k=0}^{\infty} \frac{1 - p^{j+1}q^{k+1}/x}{1 - p^j q^k x}
\]
for Ruijsenaars’ elliptic gamma function [Ru], which satisfies
\[ \frac{\Gamma(qx; p, q)}{\Gamma(x; p, q)} = (x, p/x; p, q)_\infty, \quad \Gamma(x; p, q)\Gamma(pq/x; p, q) = 1. \]

After making the substitution \( e^{2\pi iz} \mapsto z \) and similarly for \( v \) and \( \bar{w} \), we obtain after simplification the identity
\[
\int \int_{\{|z| < 1\}} \frac{\Gamma(-p^{1/2}q^{N+1/2}v^\pm z^\pm, -p^{1/2}q^{N+1/2}w^\pm \bar{z}^\pm; p, q)}{\Gamma(z^2, \bar{z}^2, p\bar{z}^{-2}, p\bar{z}^{-2}, -p^{1/2}q^{N+1/2}z^\pm \bar{z}^\pm; p, q)} \ dx dy \mid z \mid^4
\]
\[ = \frac{2\pi \log(q)p^{-1/2}q^{N+1/2}}{(p, p, q^{N+1}, pq^{-N-1}, p)_{\infty}} \Gamma(-p^{1/2}q^{N+1/2}v^\pm \bar{w}^\pm; p, q), \quad z = x + iy, \]
in standard short-hand notation analogous to (2.2). We have proved this for \( \eta \in \mathbb{R}_N \cup \mathbb{I}_N \), but it extends immediately to the region
\[ |\text{Re}(\eta)| < 1/2(N + 1), \quad |\text{Im}(\eta)| < \tau/2(N + 1)i \]
or, equivalently,
\[ p^{-1/(N+1)} < |q| < p^{1/(N+1)}, \quad |\text{arg}(q)| < 2\pi/(N + 1). \]

The identity (10.1) is quite similar in structure to Spiridonov’s elliptic beta integral [Sp], and to its double and multiple extensions conjectured by van Diejen and Spiridonov [DS] and proved by Rains [R1]. Initially, we tried to prove Theorem 3.3 by deducing (10.1) from such known results, but we did not succeed with this approach.

11. Elliptic 6j-symbols

In [Ro2], we studied the change of base coefficients \( R_k^l = R_k^l(a, b, c, d; N; q, p) \) occurring in
\[ e_k^N(x; a, b) = \sum_{l=0}^N R_k^l e_l^N(x; c, d). \]

It turned out that they can be identified with analytically continued elliptic 6j-symbols. Note that, because of the different normalizations used for theta functions, the quantity denoted \( R_k^l(a, b, c, d; N; q, p) \) here equals
\[ q^{(l-k)(k+l-N)} e^{2\pi i(-ak-b(N-k)+c+d(N-l))} R_k^l(e^{2\pi ia}, e^{2\pi ib}, e^{2\pi ic}, e^{2\pi id}; N; q, p) \]
in the notation of [Ro2].

Using Theorem 3.4 we may interpret \( R_k^l \) as a scalar product of two basis vectors. Namely, taking the scalar product of both sides of (11.1) with
\[ \sigma e_l^N(x; \eta(1 - N) \mp \bar{a}, \eta(1 - N) \mp \bar{c}) \]
gives

\[
R^l_k(a, b, c, d; N; q, p) = \frac{C^{-1} e^{\pi i N(1-\tau)/2}}{[\lambda + 1, (c + d)/2\eta]_N [\lambda + 2l]} \frac{[-N, \lambda]_l}{[1, \lambda + N + 1]_l} \times \langle e^N_k(x; a, b), \sigma e^N_l(x; \eta(1 - N) \mp \bar{d}, \eta(1 - N) \mp \bar{c}) \rangle,
\]

where \( \lambda = (c - d - 2N\eta)/2\eta \). As before, the minus sign is taken for \( \eta \in \mathbb{R}_N \) and the plus sign for \( \eta \in \mathbb{I}_N \).

The main interest in (11.2) is that it explains the self-duality of elliptic 6j-symbols. To this end, we apply the symmetry \( \langle f, g \rangle = \langle g, f \rangle \) to (11.2), using that \( \sigma^* = \sigma \) and that, by (2.12),

\[
R^l_k(a, b, c, d; N; q, p) = R^l_k(\pm \bar{a}, \pm \bar{b}, \pm \bar{d}, \eta(1 - N) \mp \bar{d}, \eta(1 - N) \mp \bar{c}); N; q, p).
\]

Combining these facts we obtain the end result

\[
R^l_k(a, b, c, d; N; q, p) = \frac{[\mu + 1, (a + b)/2\eta]_N [\lambda + 2l]}{[\lambda + 1, (c + d)/2\eta]_N [\lambda + 2l]} \frac{[-N, \lambda]_l}{[1, \lambda + N + 1]_l} \frac{[\mu + 2k]}{[\mu + 2k]} \frac{[-N, \mu]_k}{[1, \mu + 1]_k} \times R^k_l(\eta(1 - N) - d, \eta(1 - N) - c, \eta(1 - N) - b, \eta(1 - N) - a; N; q, p),
\]

where \( \lambda = (c - d - 2N\eta)/2\eta \) and \( \mu = (a - b - 2N\eta)/2\eta \). This symmetry is not obvious from (11.1), although it is clear from the explicit expression for \( R^l_k \) as an elliptic hypergeometric series given in [Ro2, Theorem 3.3].

**Appendix. Sklyanin’s generators**

In this appendix we provide the details of the relation between the operators (2.14) and Sklyanin’s generators. This is mostly based on personal communication from Eric Rains.

Sklyanin operators [S2, Theorem 2] have the form

\[
S_i f(x) = \frac{s_i(x - \frac{1}{2}N\eta)f(x + \eta) - s_i(-x - \frac{1}{2}N\eta)f(x - \eta)}{\theta(2x)},
\]

where, in our notation,

\[
s_0(x) = \theta(\eta, 2x),
\]

\[
s_1(x) = \theta \left( \eta + \frac{1}{2}, 2x + \frac{1}{2} \right),
\]

\[
s_2(x) = e^{\pi i(\frac{1}{2} + \frac{1}{2} + \eta + 2x)} \theta \left( \eta + \frac{1}{2} + \frac{\tau}{2}, 2x + \frac{1}{2} + \frac{\tau}{2} \right),
\]

\[
s_3(x) = -e^{\pi i(\frac{1}{2} + \eta + 2x)} \theta \left( \eta + \frac{\tau}{2}, 2x + \frac{\tau}{2} \right),
\]
Using (2.4), it is easy to check that

\[
S_0 = \frac{ip^{1/8}\theta(\eta)}{(p; p)_\infty^3} \Delta \left( 0, \frac{1}{2}, -\frac{1}{2} - \frac{\tau}{2} \right),
\]

\[
S_1 = -\frac{ip^{1/8}\theta(\eta + \frac{1}{2})}{(p; p)_\infty^3} \Delta \left( \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4} - \frac{\tau}{2} \right),
\]

(A.1)

\[
S_2 = \frac{ip^{1/8}e^{\pi i \theta(\eta + \frac{1}{2})}}{(p; p)_\infty^3} \Delta \left( \frac{1}{4}, \tau, \frac{1}{4}, -\frac{1}{4} - \frac{\tau}{4}, -\frac{3}{4} - \frac{\tau}{4} \right),
\]

\[
S_3 = \frac{ip^{1/8}e^{\pi i \theta(\eta + \frac{\tau}{2})}}{(p; p)_\infty^3} \Delta \left( \frac{\tau}{4}, \frac{\tau}{4}, -\frac{3}{4} - \frac{\tau}{4}, -\frac{1}{2} - \frac{\tau}{4} \right).
\]

Conversely, every \( \Delta(a) \) is a linear combination of the \( S_i \). Namely,

\[
\Delta(a) = \frac{1}{2} \left\{ \frac{\theta(a_1 + a_4, a_2 + a_4, a_3 + a_4)}{\theta(\eta)} S_0 
- \frac{\theta(a_1 + a_4 + \frac{1}{2}, a_2 + a_4 + \frac{1}{2}, a_3 + a_4 + \frac{1}{2})}{\theta(\eta + \frac{1}{2})} S_1 
- e^{\pi i (\frac{\tau}{2} + 2a_4 - \eta)} \frac{\theta(a_1 + a_4 + \frac{1}{2} + \frac{\tau}{2}, a_2 + a_4 + \frac{1}{2} + \frac{\tau}{2}, a_3 + a_4 + \frac{1}{2} + \frac{\tau}{2})}{\theta(\eta + \frac{\tau}{2})} S_2 
+ e^{\pi i (\frac{\tau}{2} + 2a_4 - \eta)} \frac{\theta(a_1 + a_4 + \frac{\tau}{2}, a_2 + a_4 + \frac{\tau}{2}, a_3 + a_4 + \frac{\tau}{2})}{\theta(\eta + \frac{\tau}{2})} S_3 \right\}.
\]

Writing this out explicitly, one is reduced to the theta function identity

\[
\theta(x + a_1, x + a_2, x + a_3, x + a_4) = \frac{1}{2} \left\{ \theta(a_1 + a_4, a_2 + a_4, a_3 + a_4, 2x) 
- \theta(a_1 + a_4 + \frac{1}{2}, a_2 + a_4 + \frac{1}{2}, a_3 + a_4 + \frac{1}{2}, 2x + \frac{1}{2}) 
+ e^{\pi i (\tau + 2a_4 + 2x)} \theta(a_1 + a_4 + \frac{1 + \tau}{2}, a_2 + a_4 + \frac{1 + \tau}{2}, a_3 + a_4 + \frac{1 + \tau}{2}, 2x + \frac{1 + \tau}{2}) 
- e^{\pi i (\tau + 2a_4 + 2x)} \theta(a_1 + a_4 + \frac{\tau}{2}, a_2 + a_4 + \frac{\tau}{2}, a_3 + a_4 + \frac{\tau}{2}, 2x + \frac{\tau}{2}) \right\},
\]

which is obtained from (2.3) after substituting \( b = (a_1 + a_4, a_2 + a_4, a_3 + a_4, -2x) \).

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