Existence of solutions for a mathematical model related to solid-solid phase transitions in shape memory alloys*

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Abstract

We consider a strongly nonlinear PDE system describing solid-solid phase transitions in shape memory alloys. The system accounts for the evolution of an order parameter $\chi$ (related to different symmetries of the crystal lattice in the phase configurations), of the stress (and the displacement $u$), and of the absolute temperature $\vartheta$. The resulting equations present several technical difficulties to be tackled: in particular, we emphasize the presence of nonlinear coupling terms, higher order dissipative contributions, possibly multivalued operators. As for the evolution of temperature, a highly nonlinear parabolic equation has to be solved for a right hand side that is controlled only in $L^1$. We prove the existence of a solution for a regularized version, by use of a time discretization technique. Then, we perform suitable a priori estimates which allow us pass to the limit and find a weak global-in-time solution to the system.

Key words: nonstandard phase field system, nonlinear partial differential equations, initial-boundary value problem, existence of solutions

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1
1 Introduction

This paper deals with a strongly nonlinear differential system, which may be related to austenite-martensite phase transitions in shape memory alloys. These materials are characterized by the fact that they can be permanently deformed by mechanical loads and then recover their original shape just by heating. This phenomenon is justified by a change of symmetry of the mesoscopic structure, as the transition involves a deformation of the crystalline cells. In particular, the austenite phase (which is present at high temperatures) is more symmetric with respect to the martensite variants. The model we are considering (see [6] and [22] for a detailed derivation) couples a Ginzburg-Landau type equation, which describes the evolution of a phase (order) parameter \( \chi \), with the momentum balance (accounting for accelerations) written in the displacement \( u \), and the energy balance governing the evolution of the absolute temperature \( \vartheta \). Note that here, just for the sakes of simplicity and better readability of the paper, we let the displacement be a scalar variable. As a consequence, deformations are accounted for by \( \nabla u \) and the stress is a vector. In the more general situation (but in the small strain regime) deformations should be described by the linearized symmetric strain tensor.

Here is the resulting PDE system:

\[
\begin{align*}
(\epsilon_0 - \vartheta \alpha''(\vartheta)G'(\chi))\partial_t \vartheta - \vartheta \alpha'(\vartheta)G'(\chi)\partial_t \chi - \Delta \vartheta &= R_\Omega + |\partial_t \chi|^2 \quad (1.1) \\
\partial_t^2 u - \text{div} \, \sigma &= B_\Omega \quad \text{where} \quad \sigma = \nabla u - \gamma(\vartheta)e \quad (1.2) \\
\partial_t \chi - \Delta \chi + \partial_r F'(\chi) + \alpha(\vartheta)G'(\chi) - \sigma \cdot e \, \gamma'(\chi) &= 0 \quad (1.3)
\end{align*}
\]

in the unknown fields \( \vartheta, u, \chi \), with \( \sigma \) denoting the stress. As usual, the partial differential equations are meant to hold in a bounded domain \( \Omega \subset \mathbb{R}^3 \) and in some time interval \((0, T)\). In the above equations, \( \epsilon_0, \kappa \) and \( \vartheta_0 \) are positive constants, \( e \) is a fixed unit vector, and \( \alpha, F, G, \gamma \) are given nonlinear functions satisfying suitable properties (which in particular ensure a parabolic character for (1.1)). One may think to \( F \) as a potential with two wells located for instance in \([-1, 1] \); \( G(0) = G'(0) = 0 \); moreover, the function \( \gamma \) in (1.3) is related to \( G \) by

\[
\gamma(r) = G(r) \text{sign}(r), \quad r \in \mathbb{R}
\]

with \( \text{sign}(r) \) taking the values: \( +1 \) if \( r > 0 \), \( 0 \) if \( r = 0 \), \( -1 \) if \( r < 0 \). We point out that both \( G \) and \( \gamma \) are sufficiently smooth: for their precise regularity we refer to the subsequent assumptions (2.8), (2.11), and (2.13). Actually, with respect to the model introduced in [6] and [22], we are taking a smoother function \( \alpha \) in the energy functional. Indeed, in [6] and [22] it is postulated that \( \alpha \) is simply of the type \( \alpha(r) = (r - \vartheta_M)^+ \) for \( r \in \mathbb{R} \) (\( \vartheta_M > 0 \) being a critical transition temperature and \( \cdot^+ \) denoting the positive part function), which entails the embarrassing presence of a Dirac measure in the equation corresponding to (1.1). Here, instead, we consider \( \alpha \) smooth (to give a meaning to \( \alpha'' \) entering the definition of the specific heat), and bounded (for technical reasons). We point out that the boundedness assumption is, at the end, not restricting from a modeling point of view, as it preserves the required behavior between different phases and corresponds to a change in the free energy (preserving minima) just for very high temperatures, when only the austenite phase may be present. On the contrary, while in some classical models for (solid-solid) phase transitions (and in [6]), \( F \) is just a quartic double-well potential,
here we include the possibility that \( F \) accounts for internal (non-smooth) constraints on the phase variable. In particular, equation (1.3) has to be read as a differential inclusion if the monotone part of \( F' \) is replaced by a subdifferential (e.g., of the indicator function of the interval \([-1, 1]\), so that \( \chi \) is compelled to take values in a physically consistent range). Finally, \( R_\Omega \) and \( B_\Omega \) are given forcing terms.

The system (1.1)-(1.3) is then complemented with the proper initial conditions
\[
\vartheta(0) = \vartheta_0, \quad u(0) = u_0, \quad \partial_t u(0) = u'_0, \quad \chi(0) = \chi_0
\]
as well as the boundary conditions for the fluxes, namely
\[
\partial_\nu \vartheta = 0, \quad \sigma \cdot \nu = b_\Gamma, \quad \partial_\nu \chi = 0 \quad \text{on } \Gamma \times (0, T)
\]
where \( \Gamma := \partial \Omega \), \( \nu \) denotes the outward normal unit vector on \( \Gamma \), \( \partial_\nu := \nu \cdot \nabla \) stands for the normal derivative and \( b_\Gamma \) is a given datum on the boundary.

In this paper, we are mainly interested in the analytical study of the initial-boundary value problem, which represents an interesting mathematical issue in itself. Thus, before proceeding, we briefly comment on the main difficulties we are going to deal with.

First, let us point out the presence of the nonlinear coefficient of \( \partial_t \vartheta \) in (1.1), as well as of other nonlinear terms. In particular, the quadratic dissipative term on the right hand side of (1.1) has to be handled and is, a priori, estimated just in some \( L^1 \) space (once the time derivative of \( \chi \) is estimated, as expected, in \( L^2 \) from (1.3)). Hence, some \textit{ad hoc} techniques for equations with \( L^1 \) data have to be applied.

We also notice the presence of a non-smooth and possibly multivalued operator in (1.3). As for (1.3), our approach is very general and gives us the possibility to set some internal constraint on the phase without using any a posteriori maximum principle type technique. On the other hand, it is clear that the treatment of a possibly singular and multivalued operator leads to additional mathematical difficulties.

Next, we point out the presence of the inertial term \( \partial^2_t u \) in equation (1.2) which is evolutionary and hyperbolic. The coupling of (1.2) with other equations (1.1) and (1.3) and with conditions (1.5)-(1.6) provides an absolutely non-trivial problem.

Furthermore, even though some formal a priori estimates could be shown with rather standard techniques, the necessity of dealing with approximating problems makes the whole argument difficult. In particular, the precise choice of the regularization we make is crucial and its construction is necessarily involved. Eventually, such an approximating problem still couples a hyperbolic equation with two strongly nonlinear equations of parabolic type, whence its solvability is not obvious. This forces us to additionally use a time discretization technique, with turns out to be rather heavy.

Concerning the physical meaning of the system under investigation, at first we recall that several models describing austenite-martensite phase transitions have been introduced in the last years (see, among the others, [26, 27, 32] and references therein). In this paper we mainly refer to the Ginzburg-Landau theory describing changes in the internal order structure of the material. One of the main advantage of this approach consists in viewing the phase transition as a change of the order in the symmetry of the alloys, so that just one phase parameter is used instead of vectorial or tensorial parameters (see
The fact that the equation for the phase is scalar represents also a good point for numerical implementation. More precisely, we let $\chi$ describe the order structure, i.e., $\chi = 0$ stands for the presence of austenite, while different (oriented) variants of martensite are associated to $\chi = +1$ and $\chi = -1$. Some recent papers deal with this kind of problem. Let us mention [25] and [7, 24]: the former is concerned with a model for shape memory alloys characterized by an intermediate pattern between first and second order phase transitions; the other two papers focus on histeretic effects in the solid-solid phase transition both for the 1D and 3D cases.

Let us point out that in the set $\{\chi = 0\}$ equation (1.2) postulates an elastic behaviour of the material (as it is $\gamma(0) = 0$), while if $\chi = \pm 1$ a transformation stress appears, whose direction depends on the orientation of the martensitic variant.

Equations are recovered by balance laws and thermodynamic principles by virtue of the following Gibbs free energy functional, depending on the state variables $\chi$, $\nabla \chi$, $\vartheta$, and the stress vector $\sigma$:

$$
G(\vartheta, \chi, \nabla \chi, \sigma) = -c_0 \vartheta \log \vartheta - \frac{1}{\kappa} \sigma \cdot \left( \frac{1}{2} \sigma + \gamma(\chi)e \right) + \frac{1}{2} \vartheta c F(\chi) + \alpha(\vartheta) G(\chi) + \frac{1}{2} |\nabla \chi|^2.
$$

Here, $\vartheta_c$ represents a critical phase transition temperature. Let us point out that the constitutive relation in (1.2) comes from the relation between $\sigma$ and deformation (here $\nabla u$)

$$
\nabla u = -\frac{\partial G}{\partial \sigma} = \frac{1}{\kappa} (\sigma + \gamma(\chi)e).
$$

Let us refer to [6] and [22] for any further detail on the model.

Now, let us briefly review some contributions related to shape memory alloys models. Indeed, the mathematical analysis of such models produced a rather impressive literature and received a great deal of attention in recent years. Some of the authors of this article contributed to study the Frémond and other models for shape memory alloys (see, e.g., [10, 11, 13, 16, 17, 19, 21]). Concerning related phase transition models, we underline that a model for hydrogen storage in metal hydrides has been recently investigated in [12], by encountering the difficulty due to the term $|\partial \chi|^2$ in the energy balance equation, but for a simpler analytical form of the other equations. In the system studied in [12] the presence of the quadratic dissipative contribution $|\partial \chi|^2$ comes from a generalized form of the principle of virtual powers, accounting for micro-forces and micro-motions responsible for the phase transition. Concerning phase change models with microscopic motions, there is a comprehensive literature originating from the Frémond theory [26, 27]. We quote [14], in which the resulting system of phase field type is characterized by the occurrence of $|\partial \chi|^2$ and other nonlinearities which were not present in the classical formulation of phase field systems (not accounting for microscopic stresses). Several authors have dealt with this kind of problems and various situations have been analyzed. However, mainly for analytical difficulties, to our knowledge there is no global in time well-posedness result for the complete related system in the 3D (or 2D) case. A global existence result is proved in the 1D setting [30, 31] or for a non-diffusive phase evolution [20]. Other results have been obtained for some regularized versions of the problem [9].

In this paper, we mainly focus on the three-dimensional situation. However, our results cover the lower-dimensional cases $\Omega \subset \mathbb{R}^d$ with $d = 1, 2$ and with minor changes we hope to be able to improve a little the results if $d = 1, 2$. 
The paper is organized as follows. In the next section, we list the assumptions on the data of the problem and state the main existence result. In the same section, we also sketch the strategy of our existence proof, which is based on a double approximation, namely, first a regularization in terms of a parameter \( \varepsilon > 0 \) that also introduces the viscous contribution \(-\varepsilon \Delta \partial_t \chi \) in (1.3), and then a time discretization of the regularized problem. In Section 3, we keep \( \varepsilon \) fixed and solve such approximating problems. Section 4 is devoted to the proof of some uniform estimates, independent of the parameter \( \varepsilon \), on the approximating solutions; then, the passage to the limit procedure as \( \varepsilon \downarrow 0 \) is carefully detailed.

## 2 Assumptions and results

The aim of this section is to introduce precise assumptions on the functions and the data that enter the mathematical problem under investigation, and state our results. We assume the domain \( \Omega \subset \mathbb{R}^3 \) to be a bounded open set with a smooth boundary \( \Gamma \) and fix a final time \( T \in (0, +\infty) \). We set

\[
Q := \Omega \times (0, T).
\]

We introduce the notation

\[
V := H^1(\Omega), \quad H := L^2(\Omega), \quad V := V^3, \quad H := H^3
\]

\[
W := H^2(\Omega) \quad \text{and} \quad W_0 := \{ v \in W : \partial_{\nu}v|_{\Gamma} = 0 \}
\]

and endow the above spaces with their standard norms, for which we use a notation like \( \| \cdot \|_V \). However, we use the same symbol for the norm in a space and in any power of it and simply write \( \| \cdot \|_p \) for the usual norm in \( L^p(\Omega) \) for \( p \in [1, +\infty] \). Moreover, for such values of \( p \), the conjugate exponent of \( p \) is denoted by \( p' \). We identify \( H \) to a subspace of \( V^* \) in the usual way, i.e., in order that \( \langle v_1, v_2 \rangle = \int_{\Omega} v_1 v_2 \) for every \( v_1 \in H \) and \( v_2 \in V \). Finally, as no confusion can arise, if \( Z \) is any Sobolev space, we use the same symbol \( \langle \cdot, \cdot \rangle \) for the duality product between the dual space \( Z^* \) and \( Z \) itself.

For the structure of our system, we are given constants and functions in order that the conditions listed below hold true:

\[
c_0, \kappa, \vartheta_c > 0, \quad e \in \mathbb{R}^3 \quad \text{with} \quad |e| = 1
\]

\[
\alpha : [0, +\infty) \to \mathbb{R} \quad \text{is a} \ C^2 \ \text{nonnegative function such that} \ \alpha(0) = 0
\]

\[
F = F_1 + F_2 \quad \text{with} \quad F_1 : \mathbb{R} \to [0, +\infty] \quad \text{and} \quad F_2 : \mathbb{R} \to \mathbb{R}
\]

\[
F_1 \text{ is convex, proper, lower semicontinuous (l.s.c.) and} \ F_2 \text{ is of class} \ C^2
\]

\[
G : \mathbb{R} \to \mathbb{R} \text{ is a} \ C^1 \ \text{nonnegative function with} \ G(0) = 0.
\]

Moreover, we assume the following parabolicity, boundedness, and growth conditions (where the positive constant \( C \) can be the same, without loss of generality):

\[
c_0 - r \alpha''(r)G(s) \geq \lambda_0 \quad \text{for some} \ \lambda_0 \in (0, c_0) \quad \text{and every} \ r \geq 0, \ s \in \mathbb{R}
\]

\[
|\alpha(r)| + |r \alpha'(r)| + r |\alpha''(r)| \leq C \quad \text{for some} \ C > 0 \quad \text{and every} \ r \geq 1
\]

\[
F_2'', G, G' \text{ are bounded and} \ G' \text{ is Lipschitz continuous.}
\]
Note that \(2.5\) and \(2.10\) imply that \(\alpha, \alpha'\) and \(\alpha''\) are bounded in \([0, +\infty)\). On the other hand, we are not going to use the non-negativity property of \(\alpha\) in our proofs. We set for convenience
\[
\beta := \partial F_1, \quad \pi := F'_2.
\] (2.12)
In \((2.12)\), the symbol \(\partial F_1\) denotes the subdifferential of \(F_1\) (defined in the sense of convex analysis, see, e.g., \([15, \text{Ex. 2.1.4, p. 21}]\)). Let \(\gamma\) be defined by \((1.4)\). In view of \((2.8)\), \(G\) takes a minimum at \(0\), whence \(G'(0) = 0\). Consequently, \((2.8)\) and \((2.11)\) yield
\[
\gamma \text{ is a } C^1 \text{ function and } \gamma, \gamma' \text{ are bounded and Lipschitz continuous.} \tag{2.13}
\]
For the forcing terms and the initial data, we require that
\[
R := R_\Omega \in L^2(Q) \text{ and } R \geq 0, \quad B_\Omega \in L^2(Q) \text{ and } b_r \in H^1(0, T; \Gamma) \tag{2.14}
\]
\[
\vartheta_0 \in H, \quad \ln \vartheta_0 \in L^1(\Omega), \quad u_0 \in V, \quad u'_0 \in H, \quad \chi_0 \in V, \quad F_1(\chi_0) \in L^1(\Omega) \tag{2.15}
\]
and define \(B_r \in H^1(0, T; V^*)\) and \(B \in L^2(0, T; H) + H^1(0, T; V^*)\) (where the sum is meaningful in the sense of the embedding \(H \subset V^*\) mentioned at the beginning of the section) as follows
\[
\langle B_r(t), v \rangle := \int_\Gamma b_r(t)v \quad \text{for every } v \in V \text{ and } t \in [0, T], \quad B := B_\Omega + B_r. \tag{2.16}
\]
In particular, let us point out that the prescribed sign of \(R\) in \((2.14)\) helps in keeping \(\vartheta > 0\), which complies with thermodynamical laws. Let us come to the equations of our systems. The presence of the quadratic term \(|\partial_\chi|^2\) in \((1.1)\) forces the function \(\vartheta\) to be rather irregular. For that reason, it is convenient to introduce a related auxiliary function \(w\) and present the equation for \(\vartheta\) in a different form, namely
\[
\partial_t w - \alpha(\vartheta)\partial_\chi G(\chi) - \Delta \vartheta = R + |\partial_\chi|^2, \quad \text{where} \quad w := c_0 \vartheta + (\alpha(\vartheta) - \vartheta \alpha'(\vartheta)) G(\chi) \tag{2.17}
\]
complemented with the homogeneous Neumann boundary condition for \(\vartheta\) and the proper initial condition for \(w\) (derived from \((1.5)\)). More precisely, we can deal with a weak formulation of the resulting initial-boundary value problem: so, we state the problem under investigation in the precise form given below.

**Definition 2.1.** A sextuplet \((w, \vartheta, u, \sigma, \chi, \xi)\) of functions is a solution to our initial and boundary value problem if for some \(q \in (1, 3/2)\) the conditions
\[
w, \vartheta \in L^q(0, T; W^{1, q}(\Omega)) \quad \text{and} \quad \vartheta > 0 \quad \text{a.e. in } Q \tag{2.18}
\]
\[
\partial_t w \in L^1(0, T; (W^{1, q'}(\Omega))^*), \quad \text{with} \quad q' = \frac{q}{q - 1} > 3 \tag{2.19}
\]
\[
u \in H^2(0, T; V^*) \cap C^1([0, T]; H) \cap C^0([0, T]; V) \tag{2.20}
\]
\[
\sigma \in C^0([0, T]; H) \tag{2.21}
\]
\[
\chi \in H^1(0, T; H) \cap L^2(0, T; W_0) \tag{2.22}
\]
\[
\xi \in L^2(0, T; H) \quad \text{and} \quad \xi \in \beta(\chi) \quad \text{a.e. in } Q \tag{2.23}
\]
are fulfilled along with the following equalities
\[
w = c_0 \vartheta + (\alpha(\vartheta) - \vartheta \alpha'(\vartheta)) G(\chi) \quad \text{a.e. in } Q \tag{2.24}
\]
\[
\langle \partial_t w, v \rangle - \langle \alpha(\vartheta)\partial_\chi G(\chi), v \rangle + \int_\Omega \nabla \vartheta \cdot \nabla v = \langle R + |\partial_\chi|^2, v \rangle \quad \text{a.e. in } (0, T) \quad \text{and for every } v \in W^{1, q'}(\Omega) \tag{2.25}
\]
\[ \sigma = \kappa \nabla u - \gamma(\chi) e \quad \text{a.e. in } Q \]  
\[ \langle \partial_t^2 u, v \rangle + \int_\Omega \sigma \cdot \nabla v = \langle B, v \rangle \quad \text{a.e. in } (0, T) \text{ and for every } v \in V \]  
\[ \partial_t \chi - \Delta \chi + \partial_c (\xi + \pi(\chi)) + \alpha(\vartheta) G'(\chi) - \sigma \cdot e \gamma'(\chi) = 0 \quad \text{a.e. in } Q \]  
\[ w(0) = w_0, \quad u(0) = u_0, \quad \partial_t u(0) = u'_0, \quad \text{and} \quad \chi(0) = \chi_0 \]  

where
\[ w_0 := c_0 \vartheta_0 + (\alpha(\vartheta_0) - \vartheta_0 \alpha'(\vartheta_0)) G(\chi_0). \]  

**Remark 2.2.** We observe that the boundary condition for \( \chi \) given in (1.6) is contained in (2.22) (see (2.3)). On the other hand, the analogous boundary conditions for \( \vartheta \) and \( \sigma \) are included in the variational equations (2.25) and (2.27). Next, we note that the right hand side of the equality (2.25) makes sense although \( R + |\partial_\chi|^2 \) is just in \( L^1(0, T; L^1(\Omega)) \) (cf. (2.22)): indeed, as \( q' > 3 \), it turns out that \( W^{1,q}(\Omega) \) is compactly embedded in \( L^\infty(\Omega) \) and therefore \( L^1(\Omega) \subset (W^{1,q'}(\Omega))^* \). Moreover, the first initial condition in (2.29) is meaningful in \( (W^{1,q'}(\Omega))^* \) as well.

Our existence result reads as follows.

**Theorem 2.3.** Assume that (2.4) - (2.16) hold. Then, there exists at least a sextuplet \((w, \vartheta, u, \sigma, \chi, \xi)\) which is a solution to our problem in the sense of Definition 2.1. In particular, we have that
\[ \ln \vartheta \in L^\infty(0, T; L^1(\Omega)) \cap L^2(0, T; V) \]  
\[ w, \vartheta \in L^\infty(0, T; L^1(\Omega)) \cap L^{4q/3}(Q) \]  
and (2.18) hold for every \( q \in [1, 5/4) \).

Due to the highly nonlinear character of our problem, in particular, to the presence of the quadratic term \( |\partial_\chi|^2 \) on the right hand side of (2.25), our study passes through an approximating system, depending on a parameter \( \varepsilon \in (0, 1) \), whose solution is much smoother. Namely, we perturb equation (2.28) by adding a higher order term with \( \varepsilon \) in front of it; then, \( \varepsilon \) is intended to go to 0 in the limit. On the other side, we regularize the data and all the nonlinearities (in particular the subdifferential \( \beta \), as in the forthcoming (2.50)). Let us denote by \( \alpha_\varepsilon, F_\varepsilon, G_\varepsilon, \gamma_\varepsilon, B_\varepsilon \), the approximating functions, whose regularity will be specified later on.

This leads to the approximating problem of finding a quintuplet \((w_\varepsilon, \vartheta_\varepsilon, u_\varepsilon, \sigma_\varepsilon, \chi_\varepsilon)\) satisfying
\[ w_\varepsilon = c_0 \vartheta_\varepsilon + \left( \alpha_\varepsilon(\vartheta_\varepsilon) - \vartheta_\varepsilon + \varepsilon \alpha'_\varepsilon(\vartheta_\varepsilon) \right) G_\varepsilon(\chi_\varepsilon) \quad \text{a.e. in } Q \]  
\[ \int_\Omega \partial_t w_\varepsilon \vartheta - \int_\Omega \alpha_\varepsilon(\vartheta_\varepsilon) \partial_t G_\varepsilon(\chi_\varepsilon) \vartheta + \int_\Omega \nabla \vartheta_\varepsilon \cdot \nabla \vartheta = \int_Q \left( R + |\partial_t \chi_\varepsilon|^2 \right) \vartheta \]  
\[ \text{a.e. in } (0, T) \text{ and for every } \vartheta \in V \]  
\[ \sigma_\varepsilon = \kappa \nabla u_\varepsilon - \gamma_\varepsilon(\chi_\varepsilon) e \quad \text{a.e. in } Q \]
\[
\int_{\Omega} \partial_t^2 u_\varepsilon v + \int_{\Omega} \sigma_\varepsilon \cdot \nabla v = \int_{\Omega} B_\varepsilon v \quad \text{a.e. in } (0, T) \text{ and for every } v \in V
\]  
(2.36)

\[
\partial_t \chi_\varepsilon - \Delta \chi_\varepsilon - \varepsilon \partial_t \Delta \chi_\varepsilon + \partial_t F'_\varepsilon(\chi_\varepsilon) + \alpha_\varepsilon(\partial_t \chi_\varepsilon) G'_\varepsilon(\chi_\varepsilon) - \sigma_\varepsilon \cdot \varepsilon \gamma'_\varepsilon(\chi_\varepsilon) = 0 \quad \text{a.e. in } Q
\]  
(2.37)

\[
w_\varepsilon(0) = w_{0,\varepsilon}, \quad u_\varepsilon(0) = u_{0,\varepsilon}, \quad \partial_t u_\varepsilon(0) = u'_{0,\varepsilon}, \quad \text{and } \chi_\varepsilon(0) = \chi_{0,\varepsilon}
\]  
(2.38)

where the unknown functions have to fulfill rather strong regularity conditions, namely

\[
w_\varepsilon \in H^1(0, T; H) \cap L^\infty(0, T; V)
\]  
(2.39)

\[
\vartheta_\varepsilon \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W_0) \quad \text{and} \quad \vartheta_\varepsilon \geq 0 \quad \text{a.e. in } Q
\]  
(2.40)

\[
u_\varepsilon \in W^{2,\infty}(0, T; H) \cap W^{1,\infty}(0, T; V)
\]  
(2.41)

\[
\sigma_\varepsilon \in W^{1,\infty}(0, T; H)
\]  
(2.42)

\[
\chi_\varepsilon \in H^2(0, T; V) \cap W^{1,\infty}(0, T; W_0)
\]  
(2.43)

In (2.37), we simply wrote \(F'_\varepsilon(\chi_\varepsilon)\) since the function \(F_\varepsilon\) is constructed in the sequel in order that its derivative \(F'_\varepsilon\) is an approximation of the whole sum \(\beta + \pi\).

**Remark 2.4.** Equation (2.34) is the approximating version of (2.25): note that (with respect to (2.25)) here we are writing integrals in place of duality pairings. This is possible thanks to the further regularity expressed by (2.39) and (2.43), which ensures that \(\partial_t \chi_\varepsilon \in L^\infty(Q)\). In this setting, (2.34) can be replaced by the variational equation corresponding to (1.1), that is

\[
\int_{\Omega} \left( c_0 - (\vartheta_\varepsilon + \varepsilon)\alpha''_\varepsilon(\vartheta_\varepsilon) G'_\varepsilon(\chi_\varepsilon) \right) \partial_t \vartheta_\varepsilon v
\]

\[
- \int_{\Omega} (\vartheta_\varepsilon + \varepsilon) \alpha'_\varepsilon(\vartheta_\varepsilon) \partial_t G'_\varepsilon(\chi_\varepsilon) v + \int_{\Omega} \nabla \vartheta_\varepsilon \cdot \nabla v
\]

\[
= \int_{\Omega} (R + |\partial_t \chi_\varepsilon|^2) v \quad \text{a.e. in } (0, T) \text{ and for every } v \in V.
\]  
(2.44)

It is worth writing all such equations for a future convenience. Moreover, we point out that (2.34) and (2.44) can also be expressed in the strong form of boundary value problems, namely

\[
\partial_t w_\varepsilon - \alpha_\varepsilon(\vartheta_\varepsilon) \partial_t G(\chi_\varepsilon) - \Delta \vartheta_\varepsilon = R + |\partial_t \chi_\varepsilon|^2
\]  
(2.45)

and

\[
\left( c_0 - (\vartheta_\varepsilon + \varepsilon)\alpha''_\varepsilon(\vartheta_\varepsilon) G(\chi_\varepsilon) \right) \partial_t \vartheta_\varepsilon
\]

\[
- (\vartheta_\varepsilon + \varepsilon) \alpha'_\varepsilon(\vartheta_\varepsilon) G'_\varepsilon(\chi_\varepsilon) \partial_t \chi_\varepsilon - \Delta \vartheta_\varepsilon = R + |\partial_t \chi_\varepsilon|^2
\]  
(2.46)

with the boundary condition \(\partial_\nu \vartheta_\varepsilon = 0\) on \(\Gamma \times (0, T)\).

For the approximating functions \(\alpha_\varepsilon : [0, +\infty) \to \mathbb{R}\) we require that

\[
\alpha_\varepsilon(0) = \alpha'_\varepsilon(0) = \alpha''_\varepsilon(0) = 0 \quad \text{for every } \varepsilon \in (0, 1)
\]  
(2.47)

\[
|\alpha_\varepsilon(r)| + |\alpha'_\varepsilon(r)| + |r + \varepsilon|\alpha'_\varepsilon(r) + |r + \varepsilon|\alpha''_\varepsilon(r)| \leq C
\]

for some \(C > 0\) and all \(r \geq 0\) and \(\varepsilon \in (0, 1)\) \(\varepsilon \leq 0\) \(\varepsilon \leq 0\)

\[
\alpha_\varepsilon(r) \to \alpha(r), \quad (r + \varepsilon)\alpha'_\varepsilon(r) \to r\alpha'(r) \quad \text{and} \quad (r + \varepsilon)\alpha''_\varepsilon(r) \to r\alpha''(r)
\]  
(2.48)

uniformaly in \([0, +\infty)\) as \(\varepsilon \searrow 0\)
and the forthcoming Proposition 2.7 ensures that such an approximation actually exists. As far as the approximating function $F_\varepsilon$ is concerned, we define it along with the related function $F_{1,\varepsilon}, \beta_\varepsilon : \mathbb{R} \to \mathbb{R}$ as follows. We set

$$F_{1,\varepsilon}(s) := \min_{s' \in \mathbb{R}} \left( \frac{(s' - s)^2}{2\varepsilon} + F_1(s') \right), \quad \beta_\varepsilon := F'_1 \quad \text{and} \quad F_\varepsilon := F_{1,\varepsilon} + F_2. \quad (2.50)$$

Thus, $\beta_\varepsilon$ is the Yosida regularization of $\beta$ (see, e.g., [15, p. 28] and [15, Prop. 2.11, p. 39] for its basic properties). Here, we mention that $F_{1,\varepsilon}$ is convex and that $\beta_\varepsilon$ is monotone and Lipschitz continuous. Moreover, for every $s \in \mathbb{R}$,

$$0 \leq F_{1,\varepsilon}(s) \leq F_1(s), \quad F_{1,\varepsilon}(s) \nearrow F_1(s) \quad \text{monotonically as } \varepsilon \searrow 0 \quad (2.51)$$
$$|\beta_\varepsilon(s)| \leq |\beta^0(s)|, \quad \beta_\varepsilon(s) \text{ tends to } \beta^0(s) \text{ monotonically as } \varepsilon \searrow 0 \quad (2.52)$$

where $\beta^0(s)$ denotes the element of $\beta(s)$ having minimum modulus provided $\beta(s) \neq \emptyset$.

**Remark 2.5.** In order to make the sequel completely rigorous, we should have taken an approximation of $F_1$ that is more regular than the one given by (2.50). For instance, in the next section, we use the pointwise values of $F'_{1,\varepsilon}$, while $F''_{1,\varepsilon}$ is just Lipschitz continuous. However, we can quote [28, Sect. 3], where the reader can find how to smooth the Yosida regularization without loosing its basic properties. So, we behave later on as $F_\varepsilon$ were as smooth as needed.

For the approximating functions $G_\varepsilon$ and $\gamma_\varepsilon$, we still define the latter through the former as we did for $\gamma$ (cf. (1.4)), namely

$$\gamma_\varepsilon(s) := G_\varepsilon(s) \text{sign}(s) \quad \text{for } s \in \mathbb{R}. \quad (2.53)$$

Here, $G_\varepsilon : \mathbb{R} \to \mathbb{R}$ is defined in order that

$$G_\varepsilon \text{ and } \gamma_\varepsilon \text{ are of class } C^3 \text{ and } G_\varepsilon^{(k)} \text{ is bounded for } k = 0, \ldots, 3 \quad (2.54)$$
$$0 \leq G_\varepsilon(s) \leq \sup G \quad \text{and} \quad |G'_\varepsilon(s)| \leq \sup |G'| \quad \text{for every } s \in \mathbb{R} \quad (2.55)$$
$$G_\varepsilon^{(k)} \to G^{(k)} \quad \text{uniformly on every bounded interval for } k = 0, 1 \quad (2.56)$$

whence an analogous convergence follows for $\gamma_\varepsilon$. Such a $G_\varepsilon$ can be obtained this way. We introduce the function $G_\varepsilon$ defined by $G_\varepsilon(r) = G(r - \varepsilon)$ if $r \geq \varepsilon$, $G_\varepsilon(r) = G(r + \varepsilon)$ if $r \leq -\varepsilon$, and $G_\varepsilon(r) = 0$ if $|r| < \varepsilon$; we note that $G_\varepsilon$ is of class $C^1$ by (2.8), and construct $G_\varepsilon$ by convolution with a kernel supported in $(-\varepsilon, \varepsilon)$.

**Remark 2.6.** We observe once and for all that (2.9), (2.48)–(2.49) and (2.55) imply

$$\lambda_* \leq c_0 - (r + \varepsilon)\alpha''_\varepsilon(r)G_\varepsilon(s) \leq C_* \quad \text{for all } r \geq 0 \text{ and } s \in \mathbb{R} \quad (2.57)$$

where, e.g., $\lambda_* = \lambda_0/2$ and $C_*$ is some positive constant, provided that $\varepsilon$ is small enough. This has an important consequence, as we show at once. We have indeed

$$c_0 - (r + \varepsilon)\alpha''_\varepsilon(r)G_\varepsilon(s) = \partial_r \phi_\varepsilon(r, s) \quad (2.58)$$

where

$$\phi_\varepsilon(r, s) := c_0 r + (\alpha_\varepsilon(r) - (r + \varepsilon)\alpha'_\varepsilon(r))G(s) \quad \text{for } r \geq 0 \text{ and } s \in \mathbb{R}. \quad (2.59)$$
As $\phi_\varepsilon(0, s) = 0$ for every $s$ and (2.57) means $\lambda_s \leq \vartheta_\varepsilon \varphi_\varepsilon \leq C_s$, it follows that
\[
\lambda_s r \leq c_0 r + (\alpha_\varepsilon(r) - (r + \varepsilon)\alpha_\varepsilon'(r)) G_\varepsilon(s) \leq C_s r \quad \text{for all } r \geq 0 \text{ and } s \in \mathbb{R}.
\] (2.60)
Furthermore, we notice that (2.60) and the positivity of $\vartheta_\varepsilon$ given in (2.40) yield
\[
w_\varepsilon \geq \lambda_s \vartheta_\varepsilon \geq 0 \quad \text{and} \quad w_\varepsilon \leq C_s \vartheta_\varepsilon \quad \text{a.e. in } Q.
\] (2.61)
For the same reason, the similar inequalities
\[
w_{0, \varepsilon} \geq \lambda_s \vartheta_{0, \varepsilon} \geq 0 \quad \text{and} \quad w_{0, \varepsilon} \leq C_s \vartheta_{0, \varepsilon} \quad \text{a.e. in } \Omega
\] (2.62)
hold for the initial data we introduce below.

For the approximating data of the $\varepsilon$-problem we assume that
\[
B_{\Omega, \varepsilon}, B_{\Gamma, \varepsilon} \in H^1(0, T; H), \quad B_\varepsilon := B_{\Omega, \varepsilon} + B_{\Gamma, \varepsilon}
\] (2.63)
\[
\vartheta_{0, \varepsilon} \in V, \quad \vartheta_{0, \varepsilon} \geq 0 \quad \text{a.e. in } \Omega, \quad u_{0, \varepsilon} \in W, \quad u'_{0, \varepsilon} \in V, \quad \chi_{0, \varepsilon} \in W_0
\] (2.64)
\[
w_{0, \varepsilon} := c_0 \vartheta_{0, \varepsilon} + (\alpha_\varepsilon(\vartheta_{0, \varepsilon}) - (\vartheta_{0, \varepsilon} + \varepsilon)\alpha_\varepsilon'(\vartheta_{0, \varepsilon})) G_\varepsilon(\chi_{0, \varepsilon})
\] (2.65)
\[
\sigma_{0, \varepsilon} := \kappa \nabla u_{0, \varepsilon} - \gamma_\varepsilon(\chi_{0, \varepsilon}) \cdot e, \quad \text{div } \sigma_{0, \varepsilon} \in H, \quad \sigma_{0, \varepsilon} \cdot \nu = 0 \quad \text{on } \Gamma
\] (2.66)
and that the following boundedness and convergence properties are satisfied
\[
B_{\Omega, \varepsilon} \to B_\Omega \quad \text{strongly in } L^2(0, T; H)
\] (2.67)
\[
B_{\Gamma, \varepsilon} \to B_\Gamma \quad \text{strongly in } H^1(0, T; V^*)
\] (2.68)
\[
\vartheta_{0, \varepsilon} \to \vartheta_0 \quad \text{strongly in } H \quad \text{and} \quad -\int_\Omega \ln(\vartheta_{0, \varepsilon} + \varepsilon) \leq C
\] (2.69)
\[
u_{0, \varepsilon} \to u_0 \quad \text{strongly in } V \quad \text{and} \quad u'_{0, \varepsilon} \to u'_0 \quad \text{strongly in } H
\] (2.70)
\[
\chi_{0, \varepsilon} \to \chi_0 \quad \text{strongly in } V, \quad \varepsilon^{1/2} \|\Delta \chi_{0, \varepsilon}\|_H \leq C \quad \text{and}
\]
\[
\limsup_{\varepsilon \to 0} \int_\Omega F_{1, \varepsilon}(\chi_{0, \varepsilon}) \leq \int_\Omega F_1(\chi_0)
\] (2.71)
\[
w_{0, \varepsilon} \to w_0 \quad \text{strongly in } H
\] (2.72)
as $\varepsilon \searrow 0$, where $C$ denotes a constant independent of $\varepsilon$. Note that $B_{\Omega, \varepsilon}, B_{\Gamma, \varepsilon}$ and $u'_{0, \varepsilon}$ actually exist, just by density. Moreover, it is not difficult to check that (2.72) follows from (2.65), (2.69) and (2.71), thanks to the uniform boundedness and Lipschitz continuity properties expressed in (2.48) – (2.49) and (2.54) – (2.56).

On the other hand, it is not obvious that the remainning requirements can actually be fulfilled. So, we prove the existence of such approximating data and construct the approximating functions $\alpha_\varepsilon$ as well. We start from the latter.

**Proposition 2.7.** There exists a family $\{\alpha_\varepsilon\}_{\varepsilon \in (0, 1)}$ of $C^2$ functions $\alpha_\varepsilon : [0, +\infty) \to \mathbb{R}$ satisfying conditions (2.47) – (2.49).

**Proof.** We first introduce suitable continuous functions $D_\varepsilon, \zeta_\varepsilon : [0, +\infty) \to \mathbb{R}$ that approximate the Dirac mass at the origin and the null function, respectively. For $\varepsilon \in (0, 1)$
we set
\[ D_\varepsilon(r) := \frac{\lambda_\varepsilon}{2\varepsilon^2} r \quad \text{for } r \in [0, \varepsilon], \quad D_\varepsilon(r) := \frac{\lambda_\varepsilon}{r + \varepsilon} \quad \text{for } r \in (\varepsilon, \varepsilon^{1/2}) \]
\[ D_\varepsilon(r) := \frac{\lambda_\varepsilon(2\varepsilon^{1/2} - r)}{\varepsilon + \varepsilon^{3/2}} \quad \text{for } r \in [\varepsilon^{1/2}, 2\varepsilon^{1/2}], \quad D_\varepsilon(r) = 0 \quad \text{for } r > 2\varepsilon^{1/2} \]
where \( \lambda_\varepsilon > 0 \) is chosen in order that the conditions
\[ \int_0^{+\infty} D_\varepsilon(s) \, ds = 1, \quad 0 \leq D_\varepsilon(r) \leq \frac{2\lambda_\varepsilon}{r + \varepsilon} \quad \text{for every } r \geq 0 \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} \lambda_\varepsilon = 0 \quad (2.73) \]
are satisfied, as we show at once. Indeed, the second property in (2.73) is obvious for \( 0 \leq r \leq \varepsilon^{1/2} \) and for \( \varepsilon \geq 2\varepsilon^{1/2} \); on the other hand, we have for \( \varepsilon^{1/2} \leq r \leq 2\varepsilon^{1/2} \)
\[ 0 \leq (r + \varepsilon)D_\varepsilon(r) \leq (2\varepsilon^{1/2} + \varepsilon)D_\varepsilon(r) \leq (2\varepsilon^{1/2} + \varepsilon) \frac{\lambda_\varepsilon \varepsilon^{1/2}}{\varepsilon + \varepsilon^{3/2}} \leq 2\lambda_\varepsilon. \]
Moreover, \( D_\varepsilon \) is continuous. Furthermore, an elementary computation yields
\[ \int_0^{+\infty} D_\varepsilon(r) \, dr = \lambda_\varepsilon \left\{ \frac{1}{4} + \ln(1 + \varepsilon^{-1/2}) - \ln 2 + \frac{1}{2(1 + \varepsilon^{1/2})} \right\} \]
so that the proper choice of \( \lambda_\varepsilon \) that guarantees the first condition in (2.73) is obvious. The same choice clearly ensures the third condition as well. As far as \( \zeta_\varepsilon \) is concerned, we need that
\[ \zeta_\varepsilon(0) = 1, \quad |\zeta_\varepsilon(r)| \leq 1 \quad \text{for } r \geq 0, \quad \zeta_\varepsilon(r) = 0 \quad \text{for } r \geq \varepsilon \]
and \( \int_0^\infty \zeta_\varepsilon(r) \, dr = 0. \quad (2.74) \]
This is achieved by setting \( \zeta_\varepsilon(r) := \zeta(r/\varepsilon) \), where \( \zeta : [0, +\infty) \to \mathbb{R} \) is a continuous function satisfying the above properties written with \( \varepsilon = 1 \). At this point, by setting just for brevity \( a_1 := \alpha'(0) \) and \( a_2 := \alpha''(0) \), we define \( \alpha_\varepsilon : [0, +\infty) \to \mathbb{R} \) by
\[ \alpha_\varepsilon(r) := \alpha(r) - a_1 r + a_1 \int_0^r \left( \int_0^s D_\varepsilon(\tau) \, d\tau \right) ds - a_2 \int_0^r \zeta_\varepsilon(\tau) \, d\tau \, ds. \quad (2.75) \]
Then, \( \alpha_\varepsilon \) is a \( C^2 \) function. Moreover, it turns out that \( \alpha_\varepsilon(0) = 0 \) and, for \( r \geq 0 \),
\[ \alpha_\varepsilon'(r) = \alpha'(r) - a_1 + a_1 \int_0^r D_\varepsilon(s) \, ds - a_2 \int_0^r \zeta_\varepsilon(s) \, ds \quad (2.76) \]
\[ \alpha_\varepsilon''(r) = \alpha''(r) + a_1 D_\varepsilon(r) - a_2 \zeta_\varepsilon(r) \quad (2.77) \]
so that \( \alpha_\varepsilon'(0) = \alpha''(0) = 0 \) and
\[ \alpha_\varepsilon'(r) = \alpha'(r) \quad \text{for } r \geq 2\varepsilon^{1/2}. \quad (2.78) \]
The representation (2.76) and properties (2.73) and (2.74) ensure that
\[ |\alpha_\varepsilon'(r) - \alpha'(r)| \leq |a_1| \left| 1 - \int_0^r D_\varepsilon(s) \, ds \right| + |a_2| \left| \int_0^r \zeta_\varepsilon(s) \, ds \right| \leq |a_1| + \varepsilon|a_2| \quad \text{for } r \geq 0. \]
This, (2.78) and our assumption (2.10) on $\alpha'$ imply the boundedness and convergence conditions (2.48)--(2.49) involving $\alpha''_\varepsilon$. Furthermore, the inequality

$$|\alpha_\varepsilon(r) - \alpha(r)| \leq (|a_1| + |a_2|)2\varepsilon^{1/2} \quad \text{for } r \geq 0$$

follows as well. Thus, also uniform boundedness and convergence for $\alpha_\varepsilon$ are proved. Finally, we notice that (2.78) implies an analogous identity for $\alpha''_\varepsilon$. Moreover, (2.77) and (2.73) yield

$$|(r + \varepsilon)\alpha''_\varepsilon(r) - r\alpha''(r)| \leq (r + \varepsilon)|\alpha''_\varepsilon(r) - \alpha''(r)| + \varepsilon|\alpha''(r)|$$

$$\leq (r + \varepsilon)|a_1D_\varepsilon(r) - a_2\zeta_\varepsilon(r)| + \varepsilon|\alpha''(r)| \leq 2|a_1|\lambda_\varepsilon + |a_2|(2\varepsilon^{1/2} + \varepsilon) + \varepsilon|\alpha''(r)|$$

for $0 \leq r \leq 2\varepsilon^{1/2}$, whence (2.48)--(2.49) are completely shown, on account of (2.10). □

**Proposition 2.8.** There exist families $\{\vartheta_0,\varepsilon\}$, $\{u_0,\varepsilon\}$ and $\{\chi_0,\varepsilon\}$ satisfying the conditions contained in (2.64), (2.66) and the convergence and boundedness properties (2.69)--(2.71).

**Proof.** Concerning $\{\vartheta_0,\varepsilon\}$, we can take it as the family of solutions $\vartheta_0,\varepsilon \in W_0$ to the elliptic problem

$$\int_\Omega \vartheta_{0,\varepsilon}v + \varepsilon \int_\Omega \nabla \vartheta_{0,\varepsilon} \cdot \nabla v = \int_\Omega \vartheta_0 v \quad \text{for all } v \in V. \quad (2.79)$$

Then, it is not difficult to check that $\vartheta_{0,\varepsilon} \rightarrow \vartheta_0$ strongly in $H$ (weak convergence plus convergence of norms) as well as $\vartheta_{0,\varepsilon} \geq 0$ in $\Omega$ (positiveness of $\vartheta_0$ and maximum principle). In order to show the bound contained in (2.69), it suffices to take $v = -1/(\vartheta_{0,\varepsilon} + \varepsilon)$ in (2.79) and observe that $r \mapsto -1/(r + \varepsilon)$ is the derivative of the convex function $r \mapsto -\ln(r + \varepsilon)$, $r \geq 0$; then, we find out that

$$-\int_\Omega \ln(\vartheta_{0,\varepsilon} + \varepsilon) + \int_\Omega \ln(\vartheta_0 + \varepsilon) \leq -\int_\Omega \vartheta_{0,\varepsilon} - \vartheta_0$$

$$\leq -\int_\Omega \frac{\vartheta_{0,\varepsilon} - \vartheta_0}{\vartheta_{0,\varepsilon} + \varepsilon} + \int_\Omega \frac{1}{(\vartheta_{0,\varepsilon} + \varepsilon)^2}|\nabla \vartheta_{0,\varepsilon}|^2 = 0$$

whence

$$-\int_\Omega \ln(\vartheta_{0,\varepsilon} + \varepsilon) \leq -\int_\Omega \ln(\vartheta_0 + \varepsilon) \leq -\int_\Omega \ln \vartheta_0$$

and (2.69) follows from (2.15). As far as the families $\{u_0,\varepsilon\}$ and $\{\chi_{0,\varepsilon}\}$ are concerned, it is more convenient to construct first the latter and then the former. Thus, we proceed as follows. Let $\chi_{0,\varepsilon} \in W_0$ solve the elliptic equation

$$\chi_{0,\varepsilon} - \varepsilon \Delta \chi_{0,\varepsilon} + \varepsilon \beta_\varepsilon(\chi_{0,\varepsilon}) = \chi_0 \quad \text{a.e. in } \Omega. \quad (2.80)$$

Hence, let us test (2.80) by $(\chi_{0,\varepsilon} - \chi_0)$ and integrate by parts, take advantage of the convexity property

$$\int_\Omega (F_{1,\varepsilon}(\chi_{0,\varepsilon}) - F_{1,\varepsilon}(\chi_0)) \leq \int_\Omega \beta_\varepsilon(\chi_{0,\varepsilon})(\chi_{0,\varepsilon} - \chi_0)$$

and use the elementary Young inequality to obtain

$$\|\chi_{0,\varepsilon} - \chi_0\|^2_H + \varepsilon \left(\frac{1}{2}\|\nabla \chi_{0,\varepsilon}\|^2_H + \int_\Omega F_{1,\varepsilon}(\chi_{0,\varepsilon})\right) \leq \varepsilon \left(\frac{1}{2}\|\nabla \chi_0\|^2_H + \int_\Omega F_{1,\varepsilon}(\chi_0)\right). \quad (2.81)$$
Now, in view of (2.15) and (2.51), it follows that
\[ \chi_{0,\varepsilon} \rightarrow \chi_0 \text{ strongly in } H \] (2.82)
and dividing by \( \varepsilon \) in (2.81) leads to
\[ \frac{1}{2} \| \nabla \chi_{0,\varepsilon} \|_H^2 + \int_{\Omega} F_{1,\varepsilon}(\chi_{0,\varepsilon}) \leq \frac{1}{2} \| \nabla \chi_0 \|_H^2 + \int_{\Omega} F_1(\chi_0). \] (2.83)
Next, we can test (2.80) by \(-\Delta \chi_{0,\varepsilon}\) and integrate by parts. Using Young’s inequality once more and exploiting the monotonicity of \( F_{1,\varepsilon} \), we are led to
\[ \frac{1}{2} \| \nabla \chi_{0,\varepsilon} \|_H^2 + \varepsilon \| \Delta \chi_{0,\varepsilon} \|_H^2 \leq \frac{1}{2} \| \nabla \chi_0 \|_H^2. \] (2.84)
Thus, we can deduce
\[ \varepsilon^{1/2} \| \Delta \chi_{0,\varepsilon} \|_H \text{ bounded independently of } \varepsilon \]
and
\[ \limsup_{\varepsilon \searrow 0} \| \nabla \chi_{0,\varepsilon} \|_H^2 \leq \| \nabla \chi_0 \|_H^2. \] (2.85)
Then, as \( \chi_{0,\varepsilon} \rightarrow \chi_0 \) weakly in \( V \) by weak compactness and (2.82), \( \chi_{0,\varepsilon} \) strongly converges to \( \chi_0 \) in \( V \) thanks to the convergence of norms, which is ensured by (2.85). At this point, in view of (2.83) we easily recover the property
\[ \limsup_{\varepsilon \searrow 0} \int_{\Omega} F_{1,\varepsilon}(\chi_{0,\varepsilon}) \leq \int_{\Omega} F_1(\chi_0) \]
and (2.71) completely follows. Finally, let us arrive at the construction of \( \{ u_{0,\varepsilon} \} \). Recalling the definition of \( \sigma_{0,\varepsilon} \) in (2.66) and invoking the Lax-Milgram lemma, we can take \( u_{0,\varepsilon} \in V \) as the unique solution of the variational equality
\[ \int_{\Omega} u_{0,\varepsilon} v + \varepsilon \int_{\Omega} \sigma_{0,\varepsilon} \cdot \nabla v = \int_{\Omega} u_0 v \text{ for every } v \in V. \] (2.86)
Moreover, it is not difficult to check that \(-\varepsilon \text{ div } \sigma_{0,\varepsilon} + u_{0,\varepsilon} = u_0 \) or equivalently
\[ -\varepsilon \kappa \Delta u_{0,\varepsilon} + u_{0,\varepsilon} = u_0 - \varepsilon \gamma_\varepsilon'(\chi_{0,\varepsilon}) \nabla \chi_{0,\varepsilon} \cdot e \]
in the sense of distributions over \( \Omega \), whence by comparison \( \Delta u_{0,\varepsilon} \in H \) and consequently
\[ \sigma_{0,\varepsilon} \cdot \nu = 0 \text{ or } \kappa \partial_{\nu} u_{0,\varepsilon} = \gamma_\varepsilon(\chi_{0,\varepsilon})|_{\Gamma} e \cdot \nu \]
in the sense of traces on \( \Gamma \). Then, (2.66) holds and, as \( \gamma_\varepsilon(\chi_{0,\varepsilon})|_{\Gamma} e \cdot \nu \in H^{1/2}(\Gamma) \), standard elliptic regularity properties ensure that \( u_{0,\varepsilon} \in W \). Taking now \( v = \kappa(u_{0,\varepsilon} - u_0) \) in (2.86) and setting \( \sigma_0 := \kappa \nabla u_0 - \gamma(\chi_0) e \), we easily deduce that
\[ \frac{\kappa}{\varepsilon} \int_{\Omega} |u_{0,\varepsilon} - u_0|^2 + \int_{\Omega} \sigma_{0,\varepsilon} \cdot (\sigma_{0,\varepsilon} - \sigma_0 + (\gamma_\varepsilon(\chi_{0,\varepsilon}) - \gamma(\chi_0)) e) = 0 \]
and, with the help of the elementary Young inequality,\begin{equation}
\frac{\kappa}{\varepsilon} \int_{\Omega} |u_{0,\varepsilon} - u_0|^2 + \frac{1}{2} \int_{\Omega} |\sigma_{0,\varepsilon}|^2 \leq \frac{1}{2} \int_{\Omega} |\sigma_0 - (\gamma_{\varepsilon}(\chi_{0,\varepsilon}) - \gamma(\chi_0)) \varepsilon|^2.
\end{equation}

Then, in view of (2.53)–(2.56) and (2.71), we infer that $u_{0,\varepsilon} \to u_0$ strongly in $H$ and $\sigma_{0,\varepsilon}$ weakly converges in $\mathbf{H}$ to some limit which must coincide with $\sigma_0$ as $\varepsilon \to 0$. At this point, we can conclude the strong convergence of $\sigma_{0,\varepsilon}$ to $\sigma_0$, and consequently of $\nabla u_{0,\varepsilon}$ to $\nabla u_0$, in $\mathbf{H}$, and thus complete the proof of (2.70).

Now, we resume at the approximating problem in (2.33)–(2.38) and observe that even though it looks much smoother than the original problem (2.24)–(2.30), it is not obvious that it has at least a solution. The method we use to prove existence relies on a time discretization. For that reason, we introduce a notation.

**Notation 2.9.** Let $N$ be a positive integer, $\tau$ a positive parameter and $Z$ a vector space. Then, we define $\delta_{\tau} : Z^{N+1} \to Z^N$ as follows:

\begin{equation}
\text{for } z = (z_0, z_1, \ldots, z_N) \in Z^{N+1} \text{ and } w = (w_0, \ldots, w_{N-1}) \in Z^N \quad \delta_{\tau} z = w \quad \text{means that} \quad w_n := \frac{z_{n+1} - z_n}{\tau} \quad \text{for } n = 0, \ldots, N - 1.
\end{equation}

Then, for simplicity, we write $\delta_{\tau} z_n$ instead of $(\delta_{\tau} z)_n$ and use the same notation $\delta_{\tau}$ for different choices of the space $Z$. We also can iterate such a procedure and define, e.g., $\delta_{\tau}^2 z = (\delta_{\tau}^2 z_n)_{n=0}^{N-2} \in Z^{N-1}$. We have

\begin{equation}
\delta_{\tau}^2 z_n := \frac{\delta_{\tau} z_{n+1} - \delta_{\tau} z_n}{\tau} = \frac{z_{n+2} - 2z_{n+1} + z_n}{\tau^2} \quad \text{for } n = 0, \ldots, N - 2.
\end{equation}

The time discretization scheme we are introducing mainly corresponds to replace the time derivative $\partial_t$ by the different quotient operator $\delta_{\tau}$, the meaning of $\tau$ being $\tau := T/N$ from now on. However, we cannot ensure positivity for the discrete temperature. For that reason

we extend $\alpha_\varepsilon$ to the whole of $\mathbb{R}$ by setting $\alpha_\varepsilon(r) = 0$ for $r < 0$.

By (2.47), such an extension is a $C^2$ function. At this point, we are ready to go on. We define the vectors $(R_n)_{n=0}^N, (B_n)_{n=0}^N \in H^{N+1}$ by setting

\begin{equation}
R_n := \frac{1}{\tau} \int_{n\tau}^{(n+1)\tau} R(s) \, ds \quad \text{and} \quad B_n := B_\varepsilon(n\tau) \quad \text{for } n = 0, \ldots, N - 1
\end{equation}
and look for vectors \((\vartheta_n)_{n=0}^N, (u_n)_{n=-1}^N, (\chi_n)_{n=0}^N,\) and \((\sigma_n)_{n=0}^N\) satisfying the conditions listed below

\[
\vartheta_0, u_0 \text{ and } \chi_0 \text{ are the initial data } \vartheta_{0,\varepsilon}, u_{0,\varepsilon} \text{ and } \chi_{0,\varepsilon}, \text{ respectively} \tag{2.92}
\]
\[
u_{-1} := u_0 - \tau u_{0,\varepsilon} \tag{2.93}
\]
\[
\vartheta_n, \chi_n \in W_0 \text{ and } u_n \in V \text{ for } n = 1, \ldots, N \tag{2.94}
\]
\[
\sigma_n \in H \text{ for } n = 0, \ldots, N \tag{2.95}
\]
\[
\sigma_n = \kappa \nabla u_n - \gamma_\varepsilon(\chi_n)e \text{ for } n = 0, \ldots, N \tag{2.96}
\]
\[
(c_0 - (\vartheta_n + \varepsilon)\alpha'_\varepsilon(\vartheta_n)G_\varepsilon(\chi_n))\delta_\tau \vartheta_n - (\vartheta_n + \varepsilon)\alpha'_\varepsilon(\vartheta_n)G'_\varepsilon(\chi_n)\delta_\tau \chi_n - \Delta \vartheta_{n+1}
= R_n + |\delta_\tau \chi_n|^2 \text{ a.e. in } \Omega, \text{ for } n = 0, \ldots, N - 1 \tag{2.97}
\]
\[
\int_\Omega \delta_\tau u_{n-1} v + \int_\Omega \sigma_{n+1} \nabla v = \int_\Omega B_nv \quad \forall v \in V \text{ for } n = 0, \ldots, N - 1 \tag{2.98}
\]
\[
\delta_\tau \chi_n - \Delta \chi_{n+1} - \varepsilon \delta_\tau \Delta \chi_n + \vartheta_nF'_\varepsilon(\chi_{n+1}) \\
+ \alpha_\varepsilon(\vartheta_n)G'_\varepsilon(\chi_n) - \sigma_n \cdot e \cdot \gamma_\varepsilon(\chi_n) = 0 \quad \text{a.e. in } \Omega, \text{ for } n = 0, \ldots, N - 1. \tag{2.99}
\]

It is clear that all the vectors we are dealing with depend on both \(\tau\) and \(\varepsilon\), even though such a dependence is not stressed in the notation. We also remark that the definitions of the 0th components of the unknown vectors might not render the Cauchy data of the original problem. For instance, \(\chi_0\) is now given by \((2.92)\) and thus means \(\chi_{0,\varepsilon}\). Despite of the ambiguous notation, no confusion can arise in the sequel. Indeed, we deal with the discrete problem and the original problem in the next two sections, separately. Namely, in the former we solve problem \((2.92) - (2.93)\) and show that suitable interpolants of the discrete solutions converge to a solution of the approximating problem as \(\tau\) tends to zero (for a subsequence). In the latter, we let \(\varepsilon\) tend to zero and obtain a solution to the original problem \((2.24) - (2.30)\).

Now, we list a number of notations and well-known results we owe to throughout the paper. First of all, we use the Hölder inequality. Moreover, we account for the continuous embedding along with the corresponding Sobolev type inequality (holding in the three-dimensional case)

\[
W^{1,q}(\Omega) \subset L^p(\Omega) \text{ and } \|v\|_p \leq C_{p,q}\|v\|_{W^{1,q}(\Omega)} \text{ for every } v \in W^{1,q}(\Omega) \tag{2.100}
\]

provided that

\[
1 \leq p \leq q^* := \frac{3q}{3-q}, \quad 1 \leq p < +\infty, \quad 1 \leq p \leq +\infty
\]

according to \(q < 3, \ q = 3, \ q > 3\),

\[
\tag{2.101}
\]

respectively. In \((2.100)\), the constant \(C_{p,q}\) depends only on \(\Omega, p \) and \(q\). Moreover, \(L^\infty(\Omega)\) can be replaced by \(C^0(\Omega)\) in \((2.100)\) if \(q > 3\). The embedding \((2.100)\) is compact for every allowed \(p\) if \(q \geq 3\), while compactness is true only if \(1 \leq p < q^*\) if \(q < 3\). In particular

\[
V \subset L^p(\Omega) \text{ and } \|v\|_p \leq C_{p,2}\|v\|_V \text{ for } p \in [1, 6] \text{ and every } v \in V, \tag{2.102}
\]

the embedding being compact if \(p < 6\). We also take advantage of the compact embedding

\[
W \subset C^0(\Omega) \text{ and } \|v\|_\infty \leq C\|v\|_W \text{ for every } v \in W \tag{2.103}
\]
where $C$ depends only on $\Omega$. Besides, we account for the Poincaré type inequality
\[ \|v\|_V \leq C(\|\nabla v\|_H + \|v\|_1) \tag{2.104} \]
for every $v \in V$.

Again, $C$ depends only on $\Omega$. Furthermore, we repeatedly make use of the elementary identity and inequalities
\[ a(a-b) = \frac{1}{2} a^2 - \frac{1}{2} b^2 + \frac{1}{2} (a-b)^2 \geq \frac{1}{2} a^2 - \frac{1}{2} b^2 \tag{2.105} \]
for every $a, b \in \mathbb{R}$
\[ ab \leq \lambda a^2 + \frac{1}{4\lambda} b^2 \tag{2.106} \]
for every $a, b \in \mathbb{R}$ and $\lambda > 0$ (2.106)
(quote (2.106) as the elementary Young inequality), as well as of the discrete Gronwall lemma in the following form (see, e.g., [29, Prop. 2.2.1]): for nonnegative real numbers $M$ and $a_n, b_n, n = 0, \ldots, N$,
\[ a_m \leq M + \sum_{n=0}^{m-1} b_n a_n \quad \text{for } m = 0, \ldots, N \quad \text{implies} \]
\[ a_m \leq M \exp\left(\sum_{n=0}^{m-1} b_n\right) \quad \text{for } m = 0, \ldots, N. \tag{2.107} \]
Finally, we set
\[ Q_t := \Omega \times (0, t) \quad \text{for } t \in [0, T] \tag{2.108} \]
and, again throughout the paper, we use a small-case italic $c$ without subscripts $0, 1, \ldots$ (thus, in contrast with, e.g., $c_0$ in (2.4) and $C$ in (2.103), where a capital letter is used) for different constants, that may only depend on $\Omega$, the final time $T$, the shape of the nonlinearities $\alpha, F, G$, and the properties of the data involved in the statements at hand. Thus, the values of such constants might change from line to line and even in the same formula or chain of inequalities. A notation like $c_\lambda$ signals a constant that depends also on the parameter $\lambda$. Finally, we write capital letters (with or without subscripts) for precise values of constants we want to refer to.

3 The approximating problem

In this section, we prove an existence result for the approximating problem (2.33)–(2.38). It is understood that assumptions (2.4)–(2.16), (2.47)–(2.50), (2.53)–(2.56) and (2.63)–(2.72) on the structure, the approximation and the data are in force; moreover, by accounting for Remark 2.5, we assume $F'_{\varepsilon}$ to be Lipschitz continuous. Here is our existence result.

**Theorem 3.1.** Problem (2.33)–(2.38) has at least a solution $(w_{\varepsilon}, \vartheta_{\varepsilon}, u_{\varepsilon}, \sigma_{\varepsilon}, \chi_{\varepsilon})$ satisfying (2.39)–(2.43).

The first step consists in proving the existence of a solution to the discrete problem.

**Proposition 3.2.** Assume Notation 2.9. Then, there exists $\tau_* > 0$, depending only on $\vartheta_c$, $\pi$, and $\Omega$, such that the discrete problem (2.92)–(2.99) has a unique solution
\[ (\vartheta_n)_{n=0}^N, \quad (u_n)_{n=-1}^N, \quad (\chi_n)_{n=0}^N, \quad \text{and} \quad (\sigma_n)_{n=0}^N \quad \text{if } \tau < \tau_. \]
Proof. We point out that for the existence proof it is sufficient to construct an iterative method for the first three vectors, since the fourth one is simply given by (2.96) in terms of

\((u_n)_{n=0}^{N}\) and \((\chi_n)_{n=0}^{N}\). First, note that \(\vartheta_0, u_0\) and \(\lambda_0\) are given by (2.92) and \(u_{-1}\) is defined by (2.33). We compute the other components by the following steps (also accounting for the proper boundary conditions contained in (2.94)): inductively for \(n = 0, \ldots, N - 1\)

\[
i) \text{ solve (2.99) for } \chi_{n+1} \\
ii) \text{ solve (2.97) for } \vartheta_{n+1} \\
iii) \text{ solve (2.98) for } u_{n+1}.
\]

We have to prove that each of the above steps yields a well-posed problem.

\(i)\) As \(\delta_\tau \chi_n = (\chi_{n+1} - \chi_n)/\tau\), equation (2.99) has the form

\[
\frac{1}{\tau} \chi_{n+1} - \Delta \chi_{n+1} - \frac{\varepsilon}{\tau} \Delta \chi_{n+1} + \vartheta_c F'_\varepsilon(\chi_{n+1}) = f_{1,n}
\]

where \(f_{1,n} \in H\) is known by virtue of the previous step. Hence, the solutions to the corresponding homogeneous Neumann boundary value problem are the stationary points of the functional \(j_n : V \rightarrow \mathbb{R}\) defined by

\[
j_n(v) := \int_{\Omega} \left( \frac{1}{2} v^2 + \vartheta_c F'_\varepsilon(v) - f_{1,n} v \right) + \left( \frac{1}{2} + \frac{\varepsilon}{2\tau} \right) \int_{\Omega} |\nabla v|^2.
\]

We recall notations (2.12), (2.50), the regularity assumptions (2.11), and that \(F_\varepsilon\) is convex. Thus, \(F'_\varepsilon(s) \geq -\sup |\pi'|\) for every \(s \in \mathbb{R}\), so that \(j_n\) is strictly convex and coercive whenever \(1/\tau > 1/\tau_* := \vartheta_c \sup |\pi'|\). Therefore, for \(\tau < \tau_*\), the functional \(j_n\) has a unique stationary point (namely, a minimum point) and the problem to be solved has a unique weak solution \(\chi_{n+1} \in V\). By accounting for elliptic regularity, we then see that \(\chi_{n+1} \in W_0\).

\(ii)\) We set \(a_\varepsilon(r, s) := c_0 - (r + \varepsilon)\alpha''_\varepsilon(s)G(s)\) for \(r, s \in \mathbb{R}\) and \(a_n := a_\varepsilon(\vartheta_n, \chi_n)\), and observe that equation (2.97) has the form

\[
\frac{1}{\tau} a_n \vartheta_{n+1} - \Delta \vartheta_{n+1} = f_{2,n}
\]

where \(f_{2,n}\) is known as well as \(a_n\), since \(\chi_{n+1}\) has already been computed. Note that \(f_{2,n} \in H\) because, in particular, \(\delta_\tau \chi_n \in L^\infty(\Omega)\) by (2.103). Moreover, \(a_n\) is bounded and satisfies \(a_n \geq \lambda_s\) a.e. in \(\Omega\) thanks to (2.57). It follows that the corresponding homogeneous Neumann boundary value problem has a unique weak solution \(\vartheta_{n+1} \in V\) with \(\Delta \vartheta_{n+1} \in H\) by comparison. Elliptic regularity then gives \(\vartheta_{n+1} \in W_0\).

\(iii)\) As \(\delta^2 u_{n-1} = (u_{n+1} - 2u_n + u_{n-1})/\tau^2\) for \(1 \leq n < N\), equation (2.98) has the form

\[
\frac{1}{\tau^2} \int_{\Omega} u_{n+1} v + \kappa \int_{\Omega} \nabla u_{n+1} \cdot v = \langle f_{3,n}, v \rangle \quad \text{for every } v \in V
\]

where \(f_{3,n} \in V^*\) is known since \(\chi_{n+1}\) has already been computed in step \(i)\). Hence, the existence of a unique solution \(u_{n+1} \in V\) is ensured by the Lax-Milgram lemma. \(\square\)
As announced in the previous section, the strategy we use to solve the approximating problem (2.33)–(2.38) is the following. By using the solution to the discrete problem, we construct some interpolants and prove that they converge to the desired solution as \( \tau \) tends to zero by using compactness methods. Hence, by keeping \( \varepsilon \) fixed, we prove a number of estimates in terms of constants that might depend on \( \varepsilon \) but are independent of the time discretization parameter \( \tau \), at least for \( \tau \) small enough (i.e., for \( \tau > 0 \) smaller than some \( \tau_0 > 0 \) that can depend on \( \varepsilon \)). To start, we assume \( \tau \leq 1 \) and \( N \geq 2 \). Even though \( \varepsilon \) is kept fixed in the whole section, sometimes we distinguish between \( c_\varepsilon \) and \( c \), according to the general rule explained at the end of Section 2. Moreover, in order to unify some cases, we write sums that might have an empty set of indices. It is understood that such sums have to be ignored, or that they vanish by definition. Thus, we first introduce the interpolants. Then, we present some useful preliminary material. Finally, we start with the true proof of Theorem 3.1.

**Notation 3.3.** We use Notation 2.9 and recall that \( \tau := T/N \) with \( N \geq 2 \) (without stressing the dependence on \( \tau \) on \( N \)). We set \( I_n := ((n-1)\tau, n\tau) \) for \( n = 1, \ldots, N \) and define the interpolation maps from \( Z^{N+1} \) into spaces \( W^{k,\infty}(0, T; Z) \) as follows: for \( z = (z_0, z_1, \ldots, z_N) \in Z^{N+1} \), we associate a further coordinate \( z_{N+1} \) defined by

\[
z_{N+1} := 2z_N - z_{N-1}
\]

so that \( \delta_\tau z_N = \delta_\tau z_{N-1} \) and \( \delta_\tau^2 z_{N-1} = 0 \), and set

\begin{align*}
\bar{z}_\tau, \tilde{z}_\tau &\in L^\infty(0, T; Z), \quad \tilde{z}_\tau \in W^{1,\infty}(0, T; Z) \quad \text{and} \quad \bar{z}_\tau \in W^{2,\infty}(0, T; Z) \quad (3.3) \\
\bar{z}_\tau(t) &= z_n \quad \text{and} \quad \tilde{z}_\tau(t) = z_{n-1} \quad \text{for a.a.} \ t \in I_n, \ n = 1, \ldots, N \quad (3.4) \\
\tilde{z}_\tau(0) &= z_0 \quad \text{and} \quad \partial_t \tilde{z}_\tau(t) = \delta_\tau z_{n-1} \quad \text{for a.a.} \ t \in I_n, \ n = 1, \ldots, N \quad (3.5) \\
\bar{z}_\tau(0) &= z_0, \quad \partial_t \bar{z}_\tau(0) = \delta_\tau z_0 \quad \text{and} \quad \partial_t^2 \bar{z}_\tau(t) = \delta_\tau^2 z_{n-1} \quad \text{for a.a.} \ t \in I_n, \ n = 1, \ldots, N. \quad (3.6)
\end{align*}

**Remark 3.4.** The notation we have used recalls its meaning. Indeed, the maps defined by (3.4)–(3.5) provide the back/forth piece-wise constant and piece-wise linear interpolants of the discrete vectors, respectively, since we also have \( \tilde{z}_\tau(n\tau) = z_n \) for every \( n \), and the function (3.6) is \( C^1 \) and piece-wise quadratic. However, the relation between the latter and the original vector only passes through the vector \( (\delta_\tau z_n) \) of the difference quotients, for we have \( \partial_t \tilde{z}_\tau(n\tau) = \delta_\tau z_n \) for every \( n \), while no equality entering the values of \( \bar{z}_\tau \) and \( z_n \) with \( n > 0 \) is true.

In order to help the reader, we collect a number of properties involving the interpolants just introduced.

**Proposition 3.5.** With Notation 3.3, we have

\[
\partial_t \tilde{z}_\tau = v_\tau \quad \text{and} \quad \partial_t \bar{z}_\tau = \tilde{v}_\tau \quad \text{if} \quad v_n = \delta_\tau z_n, \quad \text{for} \ n = 0, \ldots, N. \quad (3.7)
\]

Moreover, if \( Z \) is a normed space, we also have

\begin{align*}
\|\bar{z}_\tau\|_{L^\infty(0,T;Z)} &= \max_{n=1,\ldots,N} \|z_n\|_Z, \quad \|\tilde{z}_\tau\|_{L^\infty(0,T;Z)} = \max_{n=0,\ldots,N-1} \|z_n\|_Z \quad (3.8) \\
\|\partial_t \tilde{z}_\tau\|_{L^\infty(0,T;Z)} &= \max_{0 \leq n \leq N-1} \|\delta_\tau z_n\|_Z, \quad \|\partial_t^2 \bar{z}_\tau\|_{L^\infty(0,T;Z)} = \max_{0 \leq n \leq N-2} \|\delta_\tau^2 z_n\|_Z \quad (3.9)
\end{align*}
and the same identities for the difference $\bar{z}_\tau - \tilde{z}_\tau$. Finally

\begin{align}
\|\partial_t \bar{z}_\tau - \partial_t \tilde{z}_\tau\|^2_{L^2(0,T;\mathbb{Z})} &= \frac{T^2}{3} \|\partial^2_t \bar{z}_\tau\|^2_{L^2(0,T;\mathbb{Z})} \\
\|\bar{z}_\tau - \tilde{z}_\tau\|^2_{L^\infty(0,T;\mathbb{Z})} &\leq \frac{T^2}{3} \|\partial^2_t \tilde{z}_\tau\|^2_{L^2(0,T;\mathbb{Z})}
\end{align}

\begin{proof}
Properties (3.7)–(3.11) and (3.14)–(3.15) are straightforward to verify by a direct computation. Relations (3.12)–(3.13) are a consequence of (3.10) since $\tilde{z}_\tau(t)$ is a convex combination of $z_{n-1}$ and $z_n$ for $t \in I_n$. Finally, (3.16) follows from the analogue of (3.15) for $\bar{z}_\tau - \hat{v}_\tau$ (see (3.7)), and (3.17) is immediately deduced by representing $\bar{z}_\tau - \tilde{z}_\tau$ as the integral of its derivative and applying Hölder’s inequality.
\end{proof}

We also collect a set of inequalities involving difference quotients that are useful in the sequel and prepare an easy lemma. Consider a vector $(v_n)_{n=0}^N$, where $v_n : \Omega \to \mathbb{R}$ are measurable functions and $f : \mathbb{R} \to \mathbb{R}$ is, say, continuous. Then, the definition of Lipschitz continuity, the first and second order Taylor expansions (around either $v_n(x)$ or $v_{n+1}(x)$ for a.a. $x \in \Omega$), and a standard convex inequality yield

\begin{equation}
|\delta_r f(v_n)| \leq \text{sup ess} |f'| |\delta_r v_n| \quad \text{a.e. in } \Omega, \quad \text{for } n = 0, \ldots, N - 1
\end{equation}

if $f$ is Lipschitz continuous,

\begin{align}
|\delta_r f(v_n) - f'(v_{n+1})\delta_r v_n| &\leq \text{sup ess} |f''| \tau |\delta_r v_n|^2 \quad \text{a.e. in } \Omega, \quad \text{for } n = 0, \ldots, N - 1 \\
|\delta_r f(v_n) - f'(v_n)\delta_r v_n| &\leq \text{sup ess} |f''| \tau |\delta_r v_n|^2 \quad \text{a.e. in } \Omega, \quad \text{for } n = 0, \ldots, N - 1 \\
|\delta^2_r f(v_n) - f'(v_{n+1})\delta^2_r v_n| &\leq \text{sup ess} |f''| (|\delta_r v_n|^2 + |\delta_r v_{n+1}|^2) \quad \text{a.e. in } \Omega, \quad \text{for } n = 0, \ldots, N - 2
\end{align}

if $f$ is $C^1$ and $f'$ is Lipschitz continuous, and

\begin{equation}
f'(v_{n+1})\delta_r v_n \geq \delta_r f(v_n) \quad \text{a.e. in } \Omega, \quad \text{for } n = 0, \ldots, N - 1
\end{equation}

if $f$ is $C^1$ and convex. Even though the notation we have used is self-explaining, we make it precise, e.g., for $\delta_r f(v_n)$: the vector we apply $\delta_r$ to is $(f(v_n))_{n=0}^N \in \mathbb{Z}^{N+1}$, where $\mathbb{Z}$ is
the vector space of all measurable functions. Similarly we behave throughout the section with the solution to the discrete problem, \( f \) being one of the nonlinearities involved in our system.

**Lemma 3.6.** Let \( p \geq 1 \) be an integer and assume that \((z_n)_{n=0}^{p} \in H^{p+1}\) and \((f_n)_{n=0}^{p-1} \in H^p\) satisfy
\[
\frac{1}{\varepsilon} z_{n+1} + \delta \tau z_n = f_n \quad \text{a.e. in } \Omega \quad \text{for } n = 0, \ldots, p-1. \tag{3.23}
\]
Then, we have for \( m = 0, \ldots, p-1 \)
\[
\frac{\tau}{\varepsilon} \sum_{n=0}^{m} \|z_{n+1}\|_H^2 + \|z_{m+1}\|_H^2 \leq \|z_0\|_H^2 + \varepsilon \tau \sum_{n=0}^{m} \|f_n\|_H^2 \quad \tag{3.24}
\]
\[
\|\delta \tau z_n\|_H^2 \leq 2 \|f_n\|_H^2 + c \|z_0\|_H^2 + c \varepsilon \tau \sum_{n=0}^{m} \|f_n\|_H^2. \tag{3.25}
\]

**Proof.** By multiplying \((3.23)\) by \(2 \tau z_{n+1}\), integrating over \(\Omega\) and owing to the elementary inequalities \((2.105)–(2.106)\), we easily obtain
\[
\frac{2 \tau}{\varepsilon} \int_{\Omega} |z_{n+1}|^2 + \int_{\Omega} |z_{n+1}|^2 - \int_{\Omega} |z_n|^2 \leq \frac{\tau}{\varepsilon} \int_{\Omega} |z_{n+1}|^2 + \varepsilon \tau \int_{\Omega} |f_n|^2.
\]
By rearranging and summing over \( n = 0, \ldots, m \leq p-1 \), we trivially deduce \((3.24)\). Now, by comparison in \((3.23)\) written with \( n = m \), we have
\[
\|\delta \tau z_m\|_H^2 \leq 2 \|f_m\|_H^2 + 4 \varepsilon \|z_{m+1}\|_H^2
\]
so that \((3.25)\) follows from \((3.24)\).

Next, it is convenient to collect a number of estimates involving the forcing terms and the initial data of the discrete problem. We recall definitions \((2.63)–(2.64)\) and \((2.91)–(2.93)\).

**Lemma 3.7.** If \((R_n)_{n=0}^{N}\) and \((B_n)_{n=0}^{N}\) are specified by \((2.91)\) and if \((2.92)–(2.99)\) are in force, we have that
\[
\tau \sum_{n=0}^{N-1} \|R_n\|_H^2 \leq \|R\|_{L^2(0,T;H)}^2 \leq c \tag{3.26}
\]
\[
\max_{0 \leq n \leq N} \|B_n\|_H \leq \|B_\varepsilon\|_{L^\infty(0,T;H)} \leq c_\varepsilon, \quad \tau \sum_{n=0}^{N-1} \|\delta \tau B_n\|_H^2 \leq \|\partial_t B_\varepsilon\|_{L^2(0,T;H)}^2 \leq c_\varepsilon \tag{3.27}
\]
\[
\|\partial_0\|_V \leq c_\varepsilon, \quad \|\delta \tau u_{-1}\|_V = \|u_0\|_V \leq c_\varepsilon, \quad \|\sigma_0\|_H \leq c_\varepsilon, \quad \|\chi_0\|_W \leq c_\varepsilon. \tag{3.28}
\]
Moreover, for \( \tau \) small enough, we have
\[
\|\delta \tau \chi_0\|_W \leq c_\varepsilon, \quad \|\delta \tau^2 u_{-1}\|_H \leq c_\varepsilon, \quad \|\delta \tau \sigma_0\|_H \leq c_\varepsilon. \tag{3.29}
\]
Proof. Estimates (3.26)–(3.28) follow from (2.63)–(2.64), (2.91)–(2.93) and (2.96) with \( n = 0 \), due to the boundedness of \( \gamma_\varepsilon \). In order to prove (3.29), we first estimate \( \| x_1 \|_V \) by testing (2.99) written for \( n = 0 \) by \( \tau \chi_1 \). As \( \alpha_\varepsilon(\vartheta_0) \in H \) (cf. (2.48)), owing to the Lipschitz continuity of the nonlinearities and the elementary inequalities (2.105)–(2.106), we easily obtain

\[
\frac{1}{2} \int_\Omega |x_1|^2 + \frac{\tau}{2} \int_\Omega |\nabla x_1|^2 + \varepsilon \int_\Omega |\nabla x_1|^2 \\
\leq \frac{1}{2} \int_\Omega |x_0|^2 + \frac{\varepsilon}{2} \int_\Omega |\nabla x_0|^2 + c_\varepsilon \tau \int_\Omega (1 + |\sigma_0|^2 + |x_1|^2)
\]

and conclude that

\[
\| x_1 \|_V \leq c_\varepsilon \tag{3.30}
\]

for \( \tau \) small enough. At this point, we rewrite (2.99) with \( n = 0 \) in the form

\[
\frac{1}{\varepsilon} (x_1 - \varepsilon \Delta x_1) + \delta_\tau (x_0 - \varepsilon \Delta x_0) \\
= f_0 := \frac{1}{\varepsilon} x_1 - \vartheta_\varepsilon F'_\varepsilon(x_1) - \alpha_\varepsilon(\vartheta_0) G'_\varepsilon(x_0) + \sigma_0 \cdot e_\gamma' \varepsilon(x_0) \tag{3.31}
\]

and notice that \( \| f_0 \|_H \leq c_\varepsilon \) due to the Lipschitz continuity of \( F'_\varepsilon \), the boundedness of the other nonlinearities, and estimate (3.30) just obtained. Now, we can apply Lemma 3.6 with \( p = 1 \) and \( z_n = \chi_n - \varepsilon \Delta \chi_n \in H \) for \( n = 0, 1 \); thus, we deduce that \( \| z_1 \|_H \leq c_\varepsilon \) and \( \| \delta_\tau z_0 \|_H \leq c_\varepsilon \). Then, the desired estimates follow by elliptic regularity because \( z_0 \in H \). As a by-product, we have an improvement of (3.30), namely, \( \| x_1 \|_W \leq c_\varepsilon \). Let us come to the second and third properties in (3.29). We take (2.98) written for \( n = 0 \) and subtract to both sides the term \( \int_\Omega \sigma_0 \cdot \nabla v \), then choose \( v = \kappa \delta_\tau^2 u_{-1} \) finding

\[
\kappa \int_\Omega |\delta_\tau^2 u_{-1}|^2 + \int_\Omega \delta_\tau \sigma_0 \cdot \nabla \kappa (\delta_\tau u_0 - \delta_\tau u_{-1}) = \kappa \int_\Omega B_0 \delta_\tau^2 u_{-1} - \int_\Omega \sigma_0 \cdot \nabla (\kappa \delta_\tau^2 u_{-1}). \tag{3.32}
\]

Next, we recall that \( \nabla (\kappa \delta_\tau u_0) = \delta_\tau \sigma_0 + \delta_\tau \gamma_\varepsilon(x_0) e \) (cf. (2.96)) and that \( \sigma_0 \) is nothing but the vector \( \sigma_{0,\varepsilon} \) defined in (2.66), so that we can integrate by parts in the last integral of (3.32). Moreover, we have \( \delta_\tau u_{-1} = u'_{0,\varepsilon} \) by (2.93). Hence, from (3.32) it follows that

\[
\kappa \int_\Omega |\delta_\tau^2 u_{-1}|^2 + \int_\Omega |\delta_\tau \sigma_0|^2 \\
= - \int_\Omega \delta_\tau \sigma_0 \cdot (\delta_\tau \gamma_\varepsilon(x_0) e - \nabla (\kappa u'_{0,\varepsilon})) + \kappa \int_\Omega (B_0 + \text{div } \sigma_{0,\varepsilon}) \delta_\tau^2 u_{-1}. \tag{3.33}
\]

Now, we want to apply the Young inequality (2.106) in the two integrals on the right hand side of (3.33). For the treatment of \( \delta_\tau \gamma_\varepsilon(x_0) \) we invoke (3.18) and the boundedness of \( \gamma'_\varepsilon \) along with the control \( \| \delta_\tau x_0 \|_H \leq c_\varepsilon \). Then, in view of (2.64), (2.71), (2.66) as well, we can proceed and deduce that

\[
\frac{\kappa}{2} \int_\Omega |\delta_\tau^2 u_{-1}|^2 + \frac{1}{2} \int_\Omega |\delta_\tau \sigma_0|^2 \leq c_\varepsilon.
\]

Consequently, (3.29) is completely proved. \( \square \)
At this point, we can start the true proof of Theorem 3.1.

**First a priori estimate.** We choose \( v = \kappa \delta_r u_n \) in (2.98), and observe that (2.96) yields \( \nabla v = \delta_r \sigma_n + \delta_r \gamma_\varepsilon (\chi_n) e \). Hence, for \( 0 \leq n < N \) we have

\[
\frac{\kappa}{\tau} \int_\Omega (\partial_r u_n - \partial_r u_{n-1}) \partial_r u_n + \int_\Omega \sigma_{n+1} \cdot \delta_r \sigma_n = \kappa \int_\Omega B_n \delta_r u_n - \int_\Omega \sigma_{n+1} \cdot e \delta_r \gamma_\varepsilon (\chi_n).
\]

By accounting for the Hölder and elementary inequalities (2.105)–(2.106), we obtain

\[
\frac{\kappa}{2\tau} \int_\Omega (|\partial_r u_n|^2 - |\partial_r u_{n-1}|^2) + \frac{1}{2\tau} \int_\Omega (|\sigma_{n+1}|^2 - |\sigma_n|^2) \\
\leq \int_\Omega (\kappa |B_n|^2 + \kappa |\partial_r u_n|^2 + c_\lambda |\sigma_{n+1}|^2) + \lambda \int_\Omega |\partial_r \gamma_\varepsilon (\chi_n)|^2 \\
\leq \int_\Omega (\kappa |B_n|^2 + \kappa |\partial_r u_n|^2 + c_\lambda |\sigma_{n+1}|^2) + \lambda \sup |\gamma'| \int_\Omega |\partial_r \chi_n|^2
\]

for every \( \lambda > 0 \), the last inequality by (3.18). Then, we choose \( \lambda \) such that \( \lambda \sup |\gamma'| \leq 1/4 \), multiply the inequality we get by \( \tau \) and sum over \( n = 0, \ldots, m \), where \( 0 \leq m < N \). Hence, by accounting for (3.27) and with some vanishing empty sum if \( m = 0 \), we have

\[
\frac{\kappa}{2} \int_\Omega |\partial_r u_m|^2 + \frac{1}{2} \int_\Omega |\sigma_{m+1}|^2 \\
\leq \frac{\kappa}{2} \int_\Omega |\partial_r u_{m-1}|^2 + \frac{1}{2} \int_\Omega |\sigma_0|^2 + c_\varepsilon + \tau \int_\Omega (\kappa |\partial_r u_m|^2 + c |\sigma_{m+1}|^2) \\
+ \tau \sum_{n=0}^{m-1} \int_\Omega (\kappa |\partial_r u_n|^2 + c |\sigma_{n+1}|^2) + \frac{\tau}{4} \sum_{n=0}^{m} |\partial_r \chi_n|^2
\]

whence also

\[
\frac{\kappa}{3} \int_\Omega |\partial_r u_m|^2 + \frac{1}{3} \int_\Omega |\sigma_{m+1}|^2 \\
\leq \frac{\kappa}{2} \int_\Omega |\partial_r u_{m-1}|^2 + \frac{1}{2} \int_\Omega |\sigma_0|^2 + c_\varepsilon \\
+ \tau \sum_{n=0}^{m-1} \int_\Omega (\kappa |\partial_r u_n|^2 + c |\sigma_{n+1}|^2) + \frac{\tau}{4} \sum_{n=0}^{m} |\partial_r \chi_n|^2
\]

for \( \tau \) small enough and \( 1 \leq m < N \). By Lemma 3.7 (see (3.28)), we can upgrade such an inequality as follows

\[
\frac{\kappa}{3} \int_\Omega |\partial_r u_m|^2 + \frac{1}{3} \int_\Omega |\sigma_{m+1}|^2 \leq c_\varepsilon + \tau \sum_{n=0}^{m-1} \int_\Omega (|\partial_r u_n|^2 + c |\sigma_{n+1}|^2) + \frac{\tau}{4} \sum_{n=0}^{m} |\partial_r \chi_n|^2 \quad (3.34)
\]

for \( 0 \leq m < N \). Next, we add \( \chi_{n+1} \) to both sides of (2.99), multiply the resulting equality by \( \partial_r \chi_n \), integrate over \( \Omega \) by accounting for (2.96) and (2.50), and rearrange. Owing to the boundedness of the involved nonlinear functions and to the elementary Young
inequality (2.106), we infer that
\[
\int_\Omega |\delta_\tau \chi_n|^2 + \int_\Omega \chi_{n+1} \delta_\tau \chi_n + \int_\Omega \nabla \chi_{n+1} \cdot \nabla \delta_\tau \chi_n + \varepsilon \int_\Omega |\delta_\tau \nabla \chi_n|^2 + \vartheta_\varepsilon \int_\Omega F'_{1,\varepsilon}(\chi_{n+1}) \delta_\tau \chi_n
\]
\[
= - \int_\Omega \alpha_\varepsilon (\vartheta_n) G'_n(\chi_n) \delta_\tau \chi_n + \int_\Omega \sigma_n \cdot e \gamma'_n(\chi_n) \delta_\tau \chi_n + \int_\Omega (\chi_{n+1} - \vartheta_\varepsilon F'_2(\chi_{n+1})) \delta_\tau \chi_n
\]
\[
\leq c_\varepsilon \int_\Omega (1 + |\sigma_n| + |\chi_{n+1}|) |\delta_\tau \chi_n|
\]
\[
\leq \frac{1}{2} \int_\Omega |\delta_\tau \chi_n|^2 + c_\varepsilon \int_\Omega |\sigma_n|^2 + c_\varepsilon \int_\Omega |\chi_{n+1}|^2 + c_\varepsilon.
\]
On the other hand, by applying (3.22) to $F_{1,\varepsilon}$, we obtain
\[
\int_\Omega F'_{1,\varepsilon}(\chi_{n+1}) \delta_\tau \chi_n \geq \int_\Omega \delta_\tau F_{1,\varepsilon}(\chi_n).
\]
Hence, by combining and applying the elementary inequality (2.105), we derive that
\[
\frac{1}{2} \int_\Omega |\delta_\tau \chi_n|^2 + \frac{1}{2} \int_\Omega |\chi_{n+1}|^2 - \frac{1}{2} \int_\Omega |\chi_n|^2 + \frac{1}{2} \int_\Omega |\nabla \chi_{n+1}|^2 - \frac{1}{2} \int_\Omega |\nabla \chi_n|^2
\]
\[
+ \varepsilon \int_\Omega |\delta_\tau \nabla \chi_n|^2 + \frac{\vartheta_\varepsilon}{\tau} \int_\Omega (F_{1,\varepsilon}(\chi_{n+1}) - F_{1,\varepsilon}(\chi_n)) \leq c_\varepsilon \int_\Omega |\sigma_n|^2 + c_\varepsilon \int_\Omega |\chi_{n+1}|^2 + c_\varepsilon
\]
for $\tau$ small enough. Now, we multiply by $\tau$ and sum over $n = 0, \ldots, m$, where $0 \leq m < N$, obtaining
\[
\frac{\tau}{2} \sum_{n=0}^m \int_\Omega |\delta_\tau \chi_n|^2 + \frac{1}{2} \int_\Omega |\chi_{m+1}|^2 + \frac{1}{2} \int_\Omega |\nabla \chi_{m+1}|^2
\]
\[
+ \varepsilon \tau \sum_{n=0}^m \int_\Omega |\delta_\tau \nabla \chi_n|^2 + \vartheta_\varepsilon \int_\Omega F_{1,\varepsilon}(\chi_{m+1})
\]
\[
\leq \frac{1}{2} \int_\Omega |\chi_0|^2 + \frac{1}{2} \int_\Omega |\nabla \chi_0|^2 + \vartheta_\varepsilon \int_\Omega F_{1,\varepsilon}(\chi_0)
\]
\[
+ c_\varepsilon \tau \sum_{n=0}^m \int_\Omega |\sigma_n|^2 + c_\varepsilon \tau \sum_{n=0}^m \int_\Omega |\chi_{n+1}|^2 + c_\varepsilon.
\]
We note that $\|\sigma_0\|_H \leq c_\varepsilon$ and $\|\chi_0\|_V \leq c_\varepsilon$ by (3.28). Moreover, $F_{1,\varepsilon}(\chi_{m+1})$ is nonnegative (see (2.51)) and $\chi_0 = \chi_{0,\varepsilon}$ (by (2.92)), whence $F_{1,\varepsilon}(\chi_0)$ is independent of $\tau$. Finally, we can absorb the term $c_\varepsilon \tau |\chi_{m+1}|^2$ that appears in the last sum by the corresponding one on the left hand side just by assuming that $\tau$ is small enough. So, we improve the above inequality and sum it to (3.33). We obtain
\[
\frac{\kappa}{3} \int_\Omega |\delta_\tau u_m|^2 + \frac{1}{3} \int_\Omega |\sigma_{m+1}|^2
\]
\[
+ \frac{\tau}{4} \sum_{n=0}^m \int_\Omega |\delta_\tau \chi_n|^2 + \frac{1}{4} \int_\Omega |\chi_{m+1}|^2 + \frac{1}{2} \int_\Omega |\nabla \chi_{m+1}|^2 + \varepsilon \tau \sum_{n=0}^m \int_\Omega |\delta_\tau \nabla \chi_n|^2
\]
\[
\leq c_\varepsilon + c_\varepsilon \tau \sum_{n=0}^{m-1} \int_\Omega (|\delta_\tau u_n|^2 + |\sigma_{n+1}|^2 + |\chi_{n+1}|^2).
\]
At this point, we can apply the discrete Gronwall lemma (2.107) and deduce that
\[ \| \delta_{\tau} u_m \|_H^2 + \| \sigma_{m+1} \|_H^2 + \| \chi_{m+1} \|_V^2 \\
+ \tau \sum_{n=0}^m \| \delta_{\tau} \chi_n \|_H^2 + \tau \sum_{n=0}^m \| \delta_{\tau} \nabla \chi_n \|_H^2 \leq c_\varepsilon \quad \text{for } 0 \leq m < N. \tag{3.35} \]

On the other hand, \( \kappa \nabla u_{m+1} = \sigma_{m+1} + \gamma_\varepsilon (\chi_{m+1}) e \), whence also a bound for \( \| \nabla u_{m+1} \|_H \) follows. In terms of the interpolants (see Notation 3.3 and Remark 3.4), this means that
\[ \| u_{m+1} \|_V + \| \nabla u_{m+1} \|_H \leq c_\varepsilon. \]

We infer that \( \hat{u}_\tau \) is bounded in \( L^\infty(0,T; H) \), thanks to (3.35) and the properties of the nonlinearities. By Lemma 3.6, we can conclude that
\[ \| \hat{u}_\tau \|_{L^\infty(0,T; H)} + \| \nabla \hat{u}_\tau \|_{L^\infty(0,T; H)} \leq c_\varepsilon. \tag{3.36} \]
\[ \| \hat{u}_\tau - \bar{u}_\tau \|_{L^\infty(0,T; H)} + \| \nabla \hat{u}_\tau - \bar{\nabla} \hat{u}_\tau \|_{L^\infty(0,T; H)} \leq c_\varepsilon \tau. \tag{3.37} \]

**Consequence.** We set for convenience \( z_n := \chi_n - \varepsilon \Delta \chi_n \) for \( 0 \leq n \leq N \). Then \( z_n \in H \) for every \( n \) and (2.99) can be rewritten in the form (3.23) with \( p = N \) and \( \| f_n \|_H \leq c_\varepsilon \) for every \( n \), thanks to (3.35) and the properties of the nonlinearities. By Lemma 3.6, we deduce that
\[ \| z_{m+1} \|_H + \| \delta_{\tau} z_m \|_H \leq c_\varepsilon \quad \text{for } m = 0, \ldots, N-1. \tag{3.38} \]

As \( z_m = \chi_m - \varepsilon \Delta \chi_m \) and \( \chi_m \in W_0 \), standard elliptic regularity results yield
\[ \| \chi_{m+1} \|_V + \| \delta_{\tau} \chi_m \|_V \leq c_\varepsilon \quad \text{for } m = 0, \ldots, N-1 \tag{3.39} \]
whence also (by the continuous embedding 2.103)
\[ \| \chi_{m+1} \|_\infty + \| \delta_{\tau} \chi_m \|_\infty \leq c_\varepsilon \quad \text{for } m = 0, \ldots, N-1. \tag{3.40} \]

In terms of interpolants, the above estimates read
\[ \| \nabla \chi \|_{L^\infty(Q)} + \| \delta_{\tau} \nabla \chi \|_{L^\infty(Q)} \leq c(\| \nabla \chi \|_{L^\infty(0,T; W)} + \| \delta_{\tau} \nabla \chi \|_{L^\infty(0,T; W)}) \leq c_\varepsilon \tag{3.41} \]
and the similar ones obtained by replacing \( \nabla \chi \) by \( \chi \) hold true as well.

**Second a priori estimate.** We add \( \partial_{n+1} \) to both sides of (2.97) for convenience. Then, we multiply by \( \delta_{\tau} \partial_n \) and integrate over \( \Omega \). Thanks to the parabolicity and elementary inequalities (2.57) and (2.105), we obtain for \( 0 \leq n < N \)
\[ \lambda_n \int_\Omega \| \delta_{\tau} \partial_n \|^2 + \frac{1}{2\tau} \int_\Omega \left( \| \partial_{n+1} \|^2 + \| \nabla \partial_{n+1} \|^2 \right) - \frac{1}{2\tau} \int_\Omega \left( \| \partial_n \|^2 + \| \nabla \partial_n \|^2 \right) \leq \int_\Omega \left( \partial_{n+1} + (\partial_n + \varepsilon) \alpha_\varepsilon G'(\chi_n) \delta_{\tau} \chi_n + R_n + \| \delta_{\tau} \chi_n \|^2 \right) \delta_{\tau} \partial_n. \]
Due to the boundedness of all the nonlinear functions involved and to (3.40), the right hand side of the above inequality is bounded by

$$c_\varepsilon \int_\Omega (|\vartheta_{n+1}| + |\vartheta_n| + |R_n| + 1) |\delta \vartheta_n|$$

$$\leq \frac{\lambda_*}{2} \int_\Omega |\delta \vartheta_n|^2 + c_\varepsilon \int_\Omega (|\vartheta_{n+1}|^2 + |\vartheta_n|^2 + |R_n|^2 + 1).$$

By combining, multiplying by \(\tau\), summing over \(n = 0, \ldots, m\) with \(0 \leq m < N\), and owing to (3.26), we deduce that

$$\lambda_* \tau \sum_{n=0}^{m} \int_\Omega |\delta \vartheta_n|^2 + \frac{1}{2} \|\vartheta_{m+1}\|_V^2 \leq \frac{1}{2} \|\vartheta_0\|_V^2 + c_\varepsilon \tau \int_\Omega |\vartheta_{m+1}|^2 + c_\varepsilon \tau \sum_{n=0}^{m} \int_\Omega |\vartheta_m|^2 + c_\varepsilon.$$

Now, we absorb the term on the right hand side that involves \(\vartheta_{m+1}\) by the left hand side provided \(\tau\) is small enough, and then apply the discrete Gronwall lemma (2.107). Thus, we conclude that

$$\tau \sum_{n=0}^{m} \int_\Omega |\delta \vartheta_n|^2 + \|\vartheta_{m+1}\|_V^2 \leq c_\varepsilon \quad \text{for} \quad m = 0, \ldots, N - 1. \quad (3.42)$$

For the interpolants, this implies that

$$\|\partial_t \hat{\vartheta}\|_{L^2(0,T;H)} + \|\hat{\vartheta}\|_{L^\infty(0,T;V)} + \|\hat{\vartheta}\|_{L^\infty(0,T;V)} \leq c_\varepsilon.$$

By representing \(\hat{\vartheta}\) by means of its initial value \(\vartheta_0\) and its derivative \(\partial_t \hat{\vartheta}\), observing that \(\|\vartheta_0\|_V \leq c_\varepsilon\) by (3.28), and owing to Proposition 3.5, we conclude that

$$\|\hat{\vartheta}\|_{H^1(0,T;H)} + \|\hat{\vartheta}\|_{L^\infty(0,T;V)} + \|\hat{\vartheta}\|_{L^\infty(0,T;V)} \leq c_\varepsilon \quad (3.43)$$

$$\|\hat{\vartheta} - \hat{\vartheta}\|_{L^2(0,T;H)} + \|\hat{\vartheta} - \hat{\vartheta}\|_{L^2(0,T;H)} \leq c_\varepsilon \tau. \quad (3.44)$$

**Consequence.** From (3.42) and the previous estimates, we derive a bound for a higher norm of \((\vartheta_n)\) by comparing terms in (2.97). Indeed, as the terms in front of \(\delta \vartheta_n\) and \(\delta \vartheta_n\) are bounded by the properties of the approximating nonlinearities (cf. (2.48) and (2.55)), estimates (3.40), (3.42), (3.26) and elliptic regularity immediately imply that

$$\tau \sum_{n=0}^{N-1} \|\Delta \vartheta_{n+1}\|_H^2 \leq c_\varepsilon \quad \text{and} \quad \tau \sum_{n=0}^{N-1} \|\vartheta_{n+1}\|_W^2 \leq c_\varepsilon.$$

In terms of the interpolant \(\overline{\vartheta}_n\), this reads

$$\|\overline{\vartheta}_n\|_{L^2(0,T;W)} \leq c_\varepsilon. \quad (3.45)$$

**Third a priori estimate.** By setting for convenience

$$\eta_n := \delta \tau u_n \quad \text{for} \quad n = -1, \ldots, N - 1 \quad (3.46)$$
and recalling (2.98), we see that
\[
\int_\Omega \delta \eta_{n-1} v + \int_\Omega \sigma_{n+1} \cdot \nabla v = \int_\Omega B_n v \quad \text{for every } v \in V \text{ and } n = 0, \ldots, N - 1. \quad (3.47)
\]
We perform a discrete differentiation with respect to time, i.e., we take the difference between (3.47) written with \( n + 1 \) in place of \( n \) and (3.47) itself and divide by \( \tau \). Then, we choose \( v = \kappa \delta \eta_n \) as a test function and obtain for \( n = 0, \ldots, N - 2 \)
\[
\frac{\kappa}{\tau} \int_\Omega (\delta \eta_n - \delta \eta_{n-1}) \delta \eta_n + \int_\Omega \delta \sigma_{n+1} \cdot \nabla (\kappa \delta \eta_n) = \int_\Omega \delta \eta_{n} B_n \delta \eta_n.
\]
On the other hand, (2.96) yields
\[
\nabla (\kappa \delta \eta_n) = \delta \sigma_n = \frac{\delta \sigma_n - \delta \sigma_{n-1}}{\tau} + \frac{\delta \gamma_n'(\chi_{n+1}) - \delta \gamma_n'(\chi_n)}{\tau} e = \frac{\delta \sigma_n - \delta \sigma_{n-1}}{\tau} + \delta \gamma_n'(\chi_n) e
\]
so that the above equality becomes
\[
\frac{\kappa}{\tau} \int_\Omega (\delta \eta_n - \delta \eta_{n-1}) \delta \eta_n + \frac{1}{\tau} \int_\Omega \delta \sigma_{n+1} \cdot (\delta \sigma_n - \delta \sigma_{n-1})
\]
\[
= \kappa \int_\Omega \delta \eta_n B_n \delta \eta_n - \int_\Omega \delta \sigma_{n+1} \cdot e \delta \gamma_n'(\chi_n).
\]
At this point, in view of the elementary inequality (2.105), we infer that
\[
\frac{\kappa}{2\tau} \int_\Omega |\delta \eta_n|^2 - \frac{\kappa}{2\tau} \int_\Omega |\delta \eta_{n-1}|^2 + \frac{1}{2\tau} \int_\Omega |\delta \sigma_{n+1}|^2 - \frac{1}{2\tau} \int_\Omega |\delta \sigma_n|^2
\]
\[
\leq c \int_\Omega |\delta \eta_{n} B_n|^2 + c \int_\Omega |\delta \eta_n|^2 + \int_\Omega |\delta \sigma_{n+1}|^2 + \int_\Omega |\delta \gamma_n'(\chi_n)|^2.
\]
Now, we multiply by \( \tau \) and sum over \( n = 0, \ldots, m \) with \( 0 \leq m \leq N - 2 \). By accounting for (3.27), we obtain
\[
\frac{\kappa}{2} \int_\Omega |\delta \eta_m|^2 + \frac{1}{2} \int_\Omega |\delta \sigma_{m+1}|^2
\]
\[
\leq \frac{\kappa}{2} \int_\Omega |\delta \eta_{m-1}|^2 + \frac{1}{2} \int_\Omega |\delta \sigma_{0}|^2 + c e
\]
\[
+ c \tau \sum_{n=0}^{m} \int_\Omega |\delta \eta_n|^2 + \tau \sum_{n=0}^{m} \int_\Omega |\delta \sigma_{n+1}|^2 + \tau \sum_{n=0}^{m} \int_\Omega |\delta \gamma_n'(\chi_n)|^2.
\]
Now, we compensate the terms on the right hand side that involve \( \eta_m \) and \( \sigma_{m+1} \) with the left hand side by assuming \( \tau \) small enough. Hence, we conclude that for \( m = 0, \ldots, N - 2 \) (with some vanishing empty sums if \( m = 0 \))
\[
\int_\Omega |\delta \eta_m|^2 + \int_\Omega |\delta \sigma_{m+1}|^2
\]
\[
\leq c \int_\Omega |\delta \eta_{m-1}|^2 + c \int_\Omega |\delta \sigma_{0}|^2 + c e
\]
\[
+ c \tau \sum_{n=0}^{m-1} \int_\Omega |\delta \eta_n|^2 + c \tau \sum_{n=0}^{m-1} \int_\Omega |\delta \sigma_{n+1}|^2 + c \tau \sum_{n=0}^{m-1} \int_\Omega |\delta \gamma_n'(\chi_n)|^2. \quad (3.48)
\]
The first two terms on the right hand side are estimated by \((3.29)\). On the other hand, we can apply \((3.31)\) with \(f = \gamma'_e\) and \(v_n = x_n\) and take advantage of \((3.40)\) this way

\[
|\delta^2 e_n' (x_n)| \leq c_\varepsilon (|\delta^2_e x_n| + |\delta \tau x_n|^2 + |\delta \tau x_{n+1}|^2) \leq c_\varepsilon (|\delta^2_e x_n| + 1) \quad \text{a.e. in } \Omega.
\]

Hence, inequality \((3.48)\) becomes

\[
\int_\Omega |\delta \tau \eta_n|^2 + \int_\Omega |\delta \tau \sigma_{m+1}|^2 \leq c_\varepsilon + c \sum_{n=0}^{m-1} \int_\Omega |\delta \tau \eta_n|^2 + c \sum_{n=0}^{m-1} \int_\Omega |\delta \tau \sigma_{n+1}|^2 + C \sum_{n=0}^{m-1} \int_\Omega |\delta \tau \zeta_n|^2 \quad (3.49)
\]

for \(m = 0, \ldots, N - 2\), where we have set for convenience

\[
\zeta_n := \delta \tau x_n
\]

and marked the constant in front of the last sum by using the capital letter \(C\) for a future reference. Now, we stop for a while and suitably test the equation obtained by differentiating \((2.99)\) in the discrete sense. Namely, we write \((2.99)\) with \(n + 1\) in place of \(n\), take the difference of the equality we obtain and \((2.99)\) itself and divide by \(\tau\). By keeping the notation \((3.50)\), we multiply by \(\delta \tau \zeta_n\) and have for \(n = 0, \ldots, N - 2\)

\[
\int_\Omega (|\delta \tau \zeta_n|^2 + \varepsilon |\nabla \delta \tau \zeta_n|^2) + \frac{1}{\tau} \int_\Omega \nabla \zeta_{n+1} \cdot (\nabla \zeta_{n+1} - \nabla \zeta_n) = -\tau \int_\Omega \delta \tau F'(x_{n+1}) \delta \tau \zeta_n - \int_\Omega \delta \tau (\alpha e (\delta \tau) C' (x_n)) \delta \tau \zeta_n + \int_\Omega \delta \tau \delta \tau (\sigma_n \cdot e \gamma'_e (x_n)) \delta \tau \zeta_n.
\]

Using the elementary inequality \((2.105)\) on the left hand side and exploiting the boundedness of the nonlinearities, inequality \((3.18)\) and estimates \((3.35), (3.40)\) on the right hand side, we easily deduce that

\[
\int_\Omega (|\delta \tau \zeta_n|^2 + \varepsilon |\nabla \delta \tau \zeta_n|^2) + \frac{1}{2 \tau} \int_\Omega |\nabla \zeta_{n+1}|^2 - \frac{1}{2 \tau} \int_\Omega |\nabla \zeta_n|^2 \leq c_\varepsilon \int_\Omega (|\delta \tau x_{n+1}| + |\delta \tau \eta_n| + |\delta \tau x_n| + |\delta \tau \sigma_n| + |\sigma_n| |\delta \tau x_n|) |\delta \tau \zeta_n|
\]

\[
\leq \frac{1}{2} \int_\Omega |\delta \tau \zeta_n|^2 + c_\varepsilon \int_\Omega (1 + |\delta \tau \eta_n|^2 + |\delta \tau \sigma_n|^2).
\]

Now, we rearrange and multiply by \((2C + 1)\tau\), where \(C\) is the marked constant in \((3.49)\). Then, we sum over \(n = 0, \ldots, m\) with \(0 \leq m \leq N - 2\), use \((3.42)\), and observe that \(\zeta_0 = \delta \tau x_0\) is bounded in \(V\) by the first of \((3.29)\). We deduce that

\[
(C + 1) \tau \left\{ \sum_{n=0}^{m} \int_\Omega (|\delta \tau \zeta_n|^2 + 2 \varepsilon |\nabla \delta \tau \zeta_n|^2) + \int_\Omega |\nabla \zeta_{m+1}|^2 \right\} \leq c_\varepsilon + c_\varepsilon \tau \sum_{n=0}^{m} \int_\Omega |\delta \tau \sigma_n|^2.
\]

Now, we add this inequality to \((3.49)\) and obtain for \(0 \leq m \leq N - 2\)

\[
\int_\Omega |\delta \tau \eta_n|^2 + \int_\Omega |\delta \tau \sigma_{m+1}|^2 + (C + 1) \tau \left\{ \sum_{n=0}^{m} \int_\Omega (|\delta \tau \zeta_n|^2 + 2 \varepsilon |\nabla \delta \tau \zeta_n|^2) + \int_\Omega |\nabla \zeta_{m+1}|^2 \right\}
\]

\[
\leq c_\varepsilon + c_\varepsilon \tau \sum_{n=0}^{m-1} \int_\Omega |\delta \tau \eta_n|^2 + c_\tau \sum_{n=0}^{m-1} \int_\Omega |\delta \tau \sigma_{n+1}|^2 + C_\tau \sum_{n=0}^{m} \int_\Omega |\delta \tau \zeta_n|^2 + c_\varepsilon \tau \sum_{n=0}^{m} \int_\Omega |\delta \tau \sigma_n|^2.
\]
At this point, we rearrange, apply the discrete Gronwall lemma (2.107), replace \( \eta_n \) and \( \zeta_n \) by their values (see (3.40) and (3.50)) and conclude that

\[
\| \delta^2 u_m \|_H^2 + \| \delta_r \sigma_{m+1} \|_H^2 + \tau \sum_{n=0}^{m} \| \delta^2 \chi_n \|_V^2 + \| \nabla \delta_r \chi_{m+1} \|_H^2 \leq c_\varepsilon \tag{3.51}
\]

for \( 0 \leq m \leq N - 2 \). As \( \kappa \delta_r \nabla u_{m+1} = \delta_r \sigma_{m+1} + \delta_r \gamma'_\varepsilon(\chi_{m+1}) \) and (3.40) holds, we easily deduce that \( \| \delta_r \nabla u_{m+1} \|_H \leq c_\varepsilon \). By also accounting for (3.28), we conclude that

\[
\| \delta_r u_m \|_V \leq c_\varepsilon \quad \text{for } m = 0, \ldots, N - 1. \tag{3.52}
\]

In other words, all this reads

\[
\| \partial^2 u \|_{L^\infty(0,T;H)} + \| \partial_r \sigma \|_{L^\infty(0,T;H)} + \| \partial_r \hat{u} \|_{L^\infty(0,T;V)} + \| \nabla \partial_r \hat{\chi} \|_{L^\infty(0,T;H)} \leq c_\varepsilon. \tag{3.53}
\]

Moreover, as both \( \hat{u}_\varepsilon(0) = u_0 \) and \( \partial_t \hat{u}_\varepsilon(0) = \delta_r u_0 \) are bounded in \( W \), thus in \( H \), by (3.28), we derive a bound for \( \hat{u}_\varepsilon \) itself in \( W^2,\infty(0, T; H) \). A similar argument yields an estimate for \( \hat{\chi}_\varepsilon \) in \( H^2(0, T; V) \) and Proposition 3.5 and 3.36 imply bounds for the different interpolants of the vector \( (\sigma_n) \). We collect here some of the consequences we can derive this way. They are useful in the sequel:

\[
\| \hat{u}_\varepsilon \|_{H^2(0, T; H)} + \| \hat{\chi}_\varepsilon \|_{H^2(0, T; V)} + \| \hat{\chi}_\varepsilon \|_{H^1(0, T; V)} \leq c_\varepsilon \tag{3.54}
\]

\[
\| \hat{u}_\varepsilon - \hat{u}_\varepsilon \|_{L^\infty(0,T;V)} + \| \sigma - \sigma \|_{L^\infty(0,T;H)}
\]

\[
+ \| \partial_t \hat{\chi} - \partial_t \hat{\chi} \|_{L^2(0,T;V)} + \| \hat{\chi} - \hat{\chi} \|_{L^\infty(0,T;V)} + \| \hat{\chi} - \hat{\chi} \|_{L^\infty(0,T;V)} \leq c_\varepsilon \tau. \tag{3.55}
\]

**Conclusion.** First of all, we rewrite the equations of the discrete problem as follows

\[
(c_0 - (\partial_r + \varepsilon)\alpha'' \delta(u_\varepsilon) G'_\varepsilon(\chi_\varepsilon)) \partial_t \hat{\chi}_\varepsilon - (\partial_r + \varepsilon)\alpha'_\varepsilon \delta(u_\varepsilon) G'_\varepsilon(\chi_\varepsilon) \partial_t \hat{\chi}_\varepsilon - \Delta \hat{\sigma}_\varepsilon = B_r \varepsilon + |\partial_t \hat{\chi}_\varepsilon|^2 \quad \text{a.e. in } Q \tag{3.56}
\]

\[
\sigma = \kappa \nabla u_\varepsilon - \gamma'_\varepsilon(\chi_\varepsilon) e, \quad \sigma = \kappa \nabla u_\varepsilon - \gamma'_\varepsilon(\chi_\varepsilon) e \quad \text{a.e. in } Q \tag{3.57}
\]

\[
\int \Omega \partial^2 \hat{u}_\varepsilon v + \int \Omega \sigma \cdot \nabla v = \int \Omega B_r \varepsilon v \quad \text{a.e. in } (0,T) \text{ and for every } v \in V \tag{3.58}
\]

\[
\partial_t \hat{\chi}_\varepsilon - \Delta \hat{\chi}_\varepsilon - \varepsilon \Delta \partial_t \hat{\chi}_\varepsilon + \partial_t \hat{\chi}_\varepsilon \left. \gamma'_\varepsilon(\chi_\varepsilon) \right| + \alpha'_\varepsilon \alpha' \delta(u_\varepsilon) G'' \varepsilon(\chi_\varepsilon) - \sigma \cdot e \gamma'_\varepsilon(\chi_\varepsilon) = 0 \quad \text{a.e. in } Q \tag{3.59}
\]

and observe that the proper boundary conditions for \( \sigma \) is contained in (3.58) in a weak sense, while the homogeneous Neumann boundary conditions for \( \partial_r \) and \( \hat{\chi}_\varepsilon \) follow from \( \partial_t, \chi_\varepsilon \in W_0 \) for every \( n \). Our aim is to let \( \tau \) tend to zero in such a problem by compactness methods. In the sequel, it is understood that the convergence we derive always holds for a subsequence, even though we never mention this fact. So, by the a priori estimates (3.36), (3.41) as well as its analogue involving \( \chi_\varepsilon \), (3.43), (3.45) and (3.53)–(3.54), we deduce that all the interpolants we are interested in converge weakly or weakly star to
some limits in the proper topologies. Moreover, the estimates of the differences given by (3.37), (3.44) and (3.55) imply that some of the weak limits coincide. Hence, we have

\[
\widehat{\vartheta}_r \to \vartheta \quad \text{and} \quad \widehat{\vartheta}_r, \check{\vartheta}_r \to \vartheta \quad \text{weakly star in} \\
H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W) \text{ and } L^\infty(0, T; V), \text{ resp.} \tag{3.60}
\]

\[
\check{u}_r \to u, \quad \widehat{u}_r \to u \quad \text{and} \quad \overline{u}_r, \underline{u}_r \to u \quad \text{weakly star in} \\
W^{2, \infty}(0, T; H), W^{1, \infty}(0, T; V) \text{ and } L^\infty(0, T; V), \text{ resp.} \tag{3.61}
\]

\[
\overline{\sigma}_r \to \sigma \quad \text{and} \quad \overline{\sigma}_r, \underline{\sigma}_r \to \sigma \quad \text{weakly star in} \\
W^{1, \infty}(0, T; H) \text{ and } L^\infty(0, T; H), \text{ resp.} \tag{3.62}
\]

\[
\check{\chi}_r \to \chi, \quad \widehat{\chi}_r \to \chi \quad \text{and} \quad \overline{\chi}_r, \underline{\chi}_r \to \chi \quad \text{weakly star in} \\
H^2(0, T; V), W^{1, \infty}(0, T; V) \cap L^\infty(0, T; W) \text{ and } L^\infty(0, T; W), \text{ resp.} \tag{3.63}
\]

and \(\widehat{\chi}_r, \check{\chi}_r, \underline{\chi}_r\) converge to \(\chi\) weakly star in \(L^\infty(Q)\) as well because of the continuous embedding \(W \subset L^\infty(\Omega)\) (see also (3.41)). The quadruplet \((\vartheta, u, \sigma, \chi)\) (we avoid writing the subscript \(\varepsilon\) for simplicity) is a candidate to satisfy (2.39)–(2.43) and be a solution to problem (2.33)–(2.38), where we forget about \(\varepsilon\) and consider the initial-boundary value problem for (2.46) in place of the variational equation (2.34). We prove that this actually is the case. The regularity requirements and the Cauchy conditions (2.38) and \(\vartheta_r(0) = \vartheta_{0,r}\) are clearly verified. The homogeneous Neumann boundary conditions for \(\vartheta\) and \(\chi\) are satisfied as well, since the trace operator \(\partial_r\) is continuous from \(L^2(0, T; W)\) into \(L^2(0, T; L^2(\Gamma))\). Thus, it remains to identify the limits of the nonlinear terms. To this end, some strong convergence is useful and we can derive what we need first by accounting for [33, Sect. 8, Cor. 4] and the compact embeddings (2.102) and (2.103), then by recalling the estimates on the differences between the interpolants. The following is sufficient for the sequel

\[
\check{\vartheta}_r \to \vartheta \quad \text{and} \quad \partial_r \widehat{\chi}_r \to \partial_r \chi \\
\quad \text{strongly in } C^0([0, T]; L^q(\Omega)) \text{ for } 1 \leq q < 6 \tag{3.64}
\]

\[
\widehat{\chi}_r \to \chi \quad \text{strongly in } C^0([0, T]; C^0(\overline{\Omega})) = C^0(Q). \tag{3.65}
\]

Indeed, the second convergence in (3.64), (3.41) and (3.16) imply that

\[
\partial_t \widehat{\chi}_r \to \partial_t \chi \quad \text{strongly in } L^2(0, T; H)
\]

whence

\[
|\partial_t \widehat{\chi}_r|^2 \to |\partial_t \chi|^2 \quad \text{strongly in } L^p(Q) \text{ for every } p < +\infty
\]

due to uniform boundedness in \(L^\infty(Q)\). Moreover, we infer that

\[
\overline{\vartheta}_r \to \vartheta \quad \text{and} \quad \underline{\chi}_r, \overline{\chi}_r \to \chi \quad \text{strongly in } L^2(0, T; H) \text{ and in } L^\infty(0, T; L^6(\Omega)), \text{ resp.}
\]

the former by (3.64) and (3.41), the latter by (3.65), (3.55) and the continuous embedding \(V \subset L^6(\Omega)\). This and the induced convergence almost everywhere imply a proper convergence for the nonlinear terms. For instance, we have \((\overline{\vartheta}_r + \varepsilon)\alpha''(\overline{\vartheta}_r) \to (\vartheta + \varepsilon)\alpha''(\vartheta)\) and \(G_\varepsilon(\overline{\chi}_r) \to G_\varepsilon(\chi)\) strongly in \(L^p(Q)\) for every \(p \in [1, +\infty)\). Indeed, \((\overline{\vartheta}_r + \varepsilon)\alpha''(\overline{\vartheta}_r)\) and \(G_\varepsilon(\overline{\chi}_r)\) are bounded in \(L^\infty(Q)\) since \(r \mapsto (r + \varepsilon)\alpha''(r)\) and \(G_\varepsilon\) are bounded functions (see
Thus, by also accounting for the H"older inequality, we immediately see that (a better convergence holds true indeed)

\[(\vartheta_\varepsilon + \varepsilon)M_\varepsilon(\vartheta_\varepsilon)G_\varepsilon(\chi_\varepsilon)\partial_\tau \vartheta_\varepsilon \to (\vartheta + \varepsilon)M_\varepsilon(\vartheta)G(\chi)\partial_\tau \vartheta \quad \text{weakly in } L^1(Q).
\]

As the other nonlinear terms and products in system (3.56)–(3.59) can be dealt with in a similar and even simpler way, we conclude that the quadruplet \((\vartheta, u, \chi, \sigma)\) we have constructed satisfies (2.46), (2.35) and (2.37), as well as an integrated form of (2.36), namely

\[
\int_Q \vartheta_\varepsilon^2 t u + \int_Q \sigma \cdot \nabla v = \int_Q B_\varepsilon v \quad \text{for every } v \in L^2(0, T; V)
\]

which is equivalent to (2.36) itself. It remains to show that the function \(\vartheta = \vartheta_\varepsilon\) we have constructed is nonnegative. More generally, we can show that the same properties holds for every solution to the approximating system, provided that the function \(\alpha_\varepsilon\) is extended by 0 on the negative half-line (cf. (2.90)) in order that the approximating problem is meaningful without assumptions on the sign of temperature. We write equation (2.44) at the time \(s\) with \(v = -\vartheta_\varepsilon^-(s)\), where \((-)\) denotes the negative part. Notice that such a choice of \(v\) yields

\[
\alpha_\varepsilon''(\vartheta_\varepsilon(s))v = \alpha_\varepsilon'(\vartheta_\varepsilon(s))v = 0 \quad \text{and} \quad (R(s) + |\partial_t \chi(s)|^2)v \leq 0
\]

since \(\alpha_\varepsilon\) vanishes on \((-\infty, 0]\) and \(R\) is nonnegative. Hence, after integrating over \((0, t)\) with respect to \(s\), where \(t \in (0, T)\) is arbitrary, and recalling that \(\vartheta_0, \vartheta \geq 0\), we obtain

\[
\frac{c_0}{2} \int_0^t |\vartheta_\varepsilon^-(t)|^2 + \int_{Q_t} |\nabla \vartheta_\varepsilon^-|^2 \leq \frac{c_0}{2} \int_\Omega |\vartheta_\varepsilon^-|^2 = 0.
\]

Therefore, \(\vartheta_\varepsilon^- = 0\), whence \(\vartheta_\varepsilon \geq 0\), and the proof is complete. \(\square\)

4 The existence result

In this section, we prove Theorem 2.3 by using compactness techniques as before and monotonicity arguments in addition. We prepare a useful energy equality for equations (2.26)–(2.27) and (2.29).

Lemma 4.1. Assume that

\[
u \in W^{2,1}(0, T; V^*) \cap W^{1,\infty}(0, T; H) \cap L^\infty(0, T; V) \tag{4.1}
\]

\[
\sigma \in L^\infty(0, T; H) \quad \text{and} \quad \chi \in H^1(0, T; H) \tag{4.2}
\]

satisfy (2.26)–(2.27) and (2.29). Then, \(u\) and \(\sigma\) satisfy (2.20)–(2.21) and the identity

\[
\frac{\kappa}{2} \int_\Omega |\partial_t u(t)|^2 - \frac{\kappa}{2} \int_\Omega |u_0'|^2 + \frac{1}{2} \int_\Omega |\sigma(t)|^2 - \frac{1}{2} \int_\Omega |\sigma(0)|^2 + \int_{Q_t} \sigma \cdot e \partial_t \gamma(\chi)
\]

\[
= \kappa \int_{Q_t} B_\Omega \partial_t u + \kappa \langle B_\Gamma(t), u(t) \rangle - \kappa \langle B_\Gamma(0), u_0 \rangle - \kappa \int_0^t \langle \partial_t B_\Gamma(s), u(s) \rangle \, ds \tag{4.3}
\]

holds true for every \(t \in [0, T]\).
Proof. On account of (2.26), we write (2.27) in the form
\[ \langle \partial_t^2 u, v \rangle + \kappa \int_\Omega \nabla u \cdot \nabla v = \langle B + \gamma_s, v \rangle \quad \text{a.e. in } (0, T) \] and for every \( v \in V \) \ (4.4)
where \( \gamma_s \in H^1(0, T; V^*) \) is defined by \( \langle \gamma_s(t), v \rangle := \int_\Omega \gamma(\lambda(t)) e \cdot \nabla v \) for a.a. \( t \in (0, T) \)
and every \( v \in V \). Let us read (4.4) as an abstract second order equation with a given
right hand side. Then, the Cauchy problem obtained by complementing (4.4) with the
first two initial conditions (2.29) has a unique solution \( u \) satisfying (2.20) and a unique
generalized solution in a class of functions satisfying regularity requirements that are
weaker than (4.1) (see, e.g., [3] Thms. 3.3 and 4.4 or [23]). Hence, (2.20) follows and
(2.21) is a trivial consequence, on account of (2.20) and (4.2). In particular, (4.3) actually
is meaningful for every \( t \). We also observe that (4.3) can be formally obtained by choosing
\( v = \partial_t u(s) \) in (4.4) written at the time \( s \) and then integrating over \( (0, t) \) with respect to \( s \).
However, such a choice of \( v \) is not allowed due to a lack of regularity. Therefore, for \( \lambda > 0 \),
we introduce the solution \( v_\lambda \) of the time dependent elliptic problem
\[ v_\lambda(t) \in W_0 \quad \text{and} \quad v_\lambda(t) - \lambda \Delta v_\lambda(t) = u(t) \quad \text{a.e. in } \Omega, \quad \text{for every } t \in [0, T] \]
and perform the above formal argument by replacing \( u \) by \( v_\lambda \) and observing that \( v_\lambda \) is much
smoother that \( u \). For our purpose, it is sufficient to notice that \( v_\lambda, \partial_t v_\lambda \in L^\infty(0, T; W_0) \)
and that the following convergence holds true as \( \lambda \downarrow 0 \) (see, e.g., [18] Appendix)
\[ v_\lambda(t) \to u(t) \quad \text{strongly in } V \text{ for every } t \in [0, T] \]
\[ v_\lambda \to u \quad \text{and} \quad \partial_t v_\lambda \to \partial_t u \quad \text{strongly in } L^2(0, T; V) \text{ and } L^2(0, T; H), \text{ respectively} \]
\[ \int_0^t \langle \partial_t^2 u(s), \partial_t v_\lambda(s) \rangle \, ds \to \frac{1}{2} \int_\Omega |\partial_t u(t)|^2 - \frac{1}{2} \int_\Omega |\partial_t u(0)|^2 \quad \text{for every } t \in [0, T] \]
\[ \int_{Q_t} \nabla u \cdot \nabla \partial_t v_\lambda \to \frac{1}{2} \int_\Omega |\nabla u(t)|^2 - \frac{1}{2} \int_\Omega |\nabla u(0)|^2 \quad \text{for every } t \in [0, T]. \]
So, we test (4.3) by \( \partial_t v_\lambda \) and take the limit as \( \lambda \downarrow 0 \). By the above formulas, the limit of
the left hand side of the equality we obtain is
\[ \frac{1}{2} \int_\Omega |\partial_t u(t)|^2 - \frac{1}{2} \int_\Omega |u_0|^2 + \kappa \frac{1}{2} \int_\Omega |\nabla u(t)|^2 - \kappa \frac{1}{2} \int_\Omega |\nabla u_0|^2. \] \ (4.5)
On the other hand, for \( \lambda > 0 \), the right hand side of the same equality can be written as
\[ \int_0^t \langle B(s) + \gamma_s(s), \partial_t v_\lambda(s) \rangle \, ds = \int_{Q_t} B_\Omega \partial_t v_\lambda + \int_0^t \langle B_T(s) + \gamma_s(s), \partial_t v_\lambda(s) \rangle \, ds \]
\[ = \int_{Q_t} B_\Omega \partial_t v_\lambda + \langle B_T(t) + \gamma_s(t), v_\lambda(t) \rangle - \langle B_T(0) + \gamma_s(0), v_\lambda(0) \rangle \]
\[ - \int_0^t \langle \partial_t B_T(s) + \partial_t \gamma_s(s), v_\lambda(s) \rangle \, ds \]
\[ = \int_{Q_t} B_\Omega \partial_t v_\lambda + \langle B_T(t), v_\lambda(t) \rangle - \langle B_T(0), v_\lambda(0) \rangle - \int_0^t \langle \partial_t B_T(s), v_\lambda(s) \rangle \, ds \]
\[ + \int_\Omega \gamma(\lambda(t)) e \cdot \nabla v_\lambda(t) - \int_\Omega \gamma(\lambda_0) e \cdot \nabla v_\lambda(0) - \int_{Q_t} \partial_t \gamma(\lambda) e \cdot \nabla v_\lambda. \]
Hence, its limit as \( \lambda \searrow 0 \) has to coincide with (4.3). Multiplying by \( \kappa \), we thus obtain
\[
\frac{\kappa}{2} \int_{\Omega} |\partial_t u(t)|^2 - \frac{\kappa}{2} \int_{\Omega} |u_0|^2 + \frac{\kappa^2}{2} \int_{\Omega} |\nabla u(t)|^2 - \frac{\kappa^2}{2} \int_{\Omega} |\nabla u_0|^2
= \kappa \int_{Q_t} B_\Omega \partial_t u + \kappa \langle B_\Gamma(t), u(t) \rangle - \kappa \langle B_\Gamma(0), u_0 \rangle - \kappa \int_0^t \langle \partial_t B_\Gamma(s), u(s) \rangle ds
+ \kappa \int_{\Omega} \gamma(\chi(t)) e \cdot \nabla u(t) - \kappa \int_{\Omega} \gamma(\chi_0) e \cdot \nabla u_0 - \kappa \int_{Q_t} \partial_t \gamma(\chi) e \cdot \nabla u.
\] (4.6)

On the other hand, by recalling the definition (2.26) of \( \sigma \) and that \( |e| = 1 \), we have
\[
|\sigma|^2 = \kappa^2 |\nabla u|^2 + |\gamma(\chi)|^2 - 2\kappa \gamma(\chi) e \cdot \nabla u
\]
and
\[
\sigma \cdot e \partial_t \gamma(\chi) = \kappa \partial_t \gamma(\chi) e \cdot \nabla u - \gamma(\chi) \partial_t \gamma(\chi).
\]

Therefore, we deduce that
\[
\frac{1}{2} \int_{\Omega} |\sigma(t)|^2 - \frac{1}{2} \int_{\Omega} |\sigma(0)|^2 + \int_{Q_t} \sigma \cdot e \partial_t \gamma(\chi)
= \frac{\kappa^2}{2} \int_{\Omega} |\nabla u(t)|^2 - \frac{\kappa^2}{2} \int_{\Omega} |\nabla u_0|^2
- \frac{\kappa^2}{2} \int_{\Omega} |\nabla u_0|^2 - \frac{1}{2} \int_{\Omega} |\gamma(\chi_0)|^2 + \kappa \int_{\Omega} \gamma(\chi_0) e \cdot \nabla u_0
+ \kappa \int_{Q_t} \partial_t \gamma(\chi) e \cdot \nabla u - \int_{Q_t} \gamma(\chi) \partial_t \gamma(\chi)
= \frac{\kappa^2}{2} \int_{\Omega} |\nabla u(t)|^2 - \frac{\kappa^2}{2} \int_{\Omega} |\nabla u_0|^2
- \kappa \int_{Q_t} \gamma(\chi(t)) e \cdot \nabla u(t) + \kappa \int_{\Omega} \gamma(\chi_0) e \cdot \nabla u_0 + \kappa \int_{Q_t} \partial_t \gamma(\chi) e \cdot \nabla u.
\]

By adding this to (4.6), we obtain (4.3). \( \square \)

**Remark 4.2.** An analogous identity holds for the approximating problem, namely
\[
\frac{\kappa}{2} \int_{\Omega} |\partial_t u_\varepsilon(t)|^2 - \frac{\kappa}{2} \int_{\Omega} |u_0^{\varepsilon}|^2 + \frac{1}{2} \int_{\Omega} |\sigma_\varepsilon(t)|^2 - \frac{1}{2} \int_{\Omega} |\sigma_\varepsilon(0)|^2 + \int_{Q_t} \sigma_\varepsilon \cdot e \partial_t \gamma_\varepsilon(x_\varepsilon)
= \kappa \int_{Q_t} B_{\Omega,\varepsilon} \partial_t u_\varepsilon + \kappa \langle B_{\Gamma,\varepsilon}(t), u_\varepsilon(t) \rangle - \kappa \langle B_{\Gamma,\varepsilon}(0), u_{0,\varepsilon} \rangle - \kappa \int_0^t \langle \partial_t B_{\Gamma,\varepsilon}(s), u_\varepsilon(s) \rangle ds
\] (4.7)
for every \( t \in [0, T] \), and the corresponding proof is much simpler. Indeed, one can test equation (2.36) directly by \( \partial_t u_\varepsilon \), since the solution and the data are smoother.

At this point, we recall (2.40) and Remark 2.6, in particular that both \( \partial_\varepsilon \) and \( w_\varepsilon \) are nonnegative, and start estimating.

**First a priori estimate.** Our strategy consists in suitably testing all the equations of the system and then summing up. We first take \( v = 1 \) in (2.31) and integrate over \((0, t)\)
with \( t \in (0, T) \). As \( R \in L^2(Q) \), and (2.48), (2.55) and (2.72) hold, using the Hölder and Young inequalities, we obtain
\[
\int_Ω w_ε(t) = \int_Ω w_0,ε + \int_{Q_t} α_ε(\vartheta_ε)∂_t G_ε(\chi_ε) + \int_{Q_t} (R + |∂_t \chi_ε|^2) \leq \frac{3}{2} \int_{Q_t} |∂_t \chi_ε|^2 + c. \tag{4.8}
\]

Next, we note that \(-1/(\vartheta_ε + ε)\) is meaningful and belongs to \( L^2(0, T; V) \). Hence, its values at \( s \in (0, T) \) can be chosen as a test function in (2.44) written at the time \( s \). By integrating over \((0, t)\) with respect to \( s \) and rearranging, we have
\[
-c_0 \int_Ω \ln(\vartheta_ε(t) + ε) + \int_{Q_t} |\nabla \ln(\vartheta_ε + ε)|^2 + \int_{Q_t} |∂_t \chi_ε|^2
= -c_0 \int_Ω \ln(\vartheta_0,ε + ε) - \int_{Q_t} \left\{ α''_ε(\vartheta_ε)G_ε(\chi_ε)∂_t \vartheta_ε + α'_ε(\vartheta_ε)\partial_t G_ε(\chi_ε)\right\} - \int_{Q_t} \frac{R}{\vartheta_ε + ε}
\]
and observe that the last integral on the left hand side is nonnegative. On the other hand, we have that \(-c_0 \int_Ω \ln(\vartheta_0,ε + ε) \leq c\) by (2.69) and \( R \geq 0 \) by (2.14)–(2.15), and the second integrand on the right hand side can be written as \( ∂_t\{α'_ε(\vartheta_ε)G_ε(\chi_ε)\} \). Moreover, (2.48) and (2.55) hold, so that both \( α'_ε \) and \( G_ε \) are uniformly bounded. Hence, the above equality implies
\[
-c_0 \int_Ω \ln(\vartheta_ε(t) + ε) + \int_{Q_t} |\nabla \ln(\vartheta_ε + ε)|^2 \leq c. \tag{4.9}
\]

Now, we write (2.36) at the time \( s \), choose \( v = 2\kappa ∂_t u_ε(s) \in V \) as a test function and observe that \( \nabla v = 2∂_t σ_ε(s) + 2γ_ε'(χ_ε(s))∂_t χ_ε(s) \) by (2.35). Then, we integrate over \((0, t)\) with respect to \( s \) and add the same term \( 2\int_{Q_t} u_ε \partial_t u_ε \) to both sides for convenience. As the norms \( ||u_0,ε||_{H} \), \( ||u_ε||_{V} \) and \( ||σ_ε(0)||_{H} \) of the initial values are bounded (see (2.70)–(2.71) and (2.66)), we obtain
\[
\kappa \int_Ω |∂_t u_ε(t)|^2 + \int_Ω |u_ε(t)|^2 + \int_Ω |σ_ε(t)|^2 + 2 \int_{Q_t} σ_ε \cdot ε \gamma_ε'(χ_ε)∂_t χ_ε
\]
\[
= \kappa \int_Ω |u_0,ε|^2 + \int_Ω |u_0,ε|^2 + \int_Ω |σ_ε(0)|^2 + \int_{Q_t} (2\kappa B_ε + 2u_ε)∂_t u_ε
\]
\[
\leq c + \int_{Q_t} |∂_t u_ε|^2 + \int_{Q_t} |u_ε|^2 + 2\kappa \int_{Q_t} B_ε ∂_t u_ε.
\]

We recall that \( B_ε = B_{Ω,ε} + B_{Γ,ε} \) (see (2.63) and that \( B_{Ω,ε} \) and \( B_{Γ,ε} \) are bounded in \( L^2(0, T; H) \) and in \( H^1(0, T; V^*) \), respectively (cf. (2.67)–(2.68)). Hence, for every \( λ > 0 \) we have that
\[
\int_{Q_t} B_ε ∂_t u_ε = \int_{Q_t} B_{Ω,ε} ∂_t u_ε
\]
\[
+ \int_{Q_t} B_{Γ,ε}(t) u_ε(t) - \int_{Ω} B_{Γ,ε}(0) u_0,ε - \int_0^t \langle ∂_t B_{Γ,ε}(s), u_ε(s) \rangle \, ds
\]
\[
\leq \int_{Q_t} |∂_t u_ε|^2 + λ||u_ε(t)||_{V}^2 + c \int_0^t ||u_ε(s)||_{V}^2 \, ds + c_λ.
\]
Therefore, by combining the last two inequalities, we deduce that

\[ \kappa \int_{Q_t} |\partial_t u_\varepsilon(t)|^2 + \int_{Q_t} |u_\varepsilon(t)|^2 + \int_{\Omega} |\sigma_\varepsilon(t)|^2 + 2 \int_{Q_t} \sigma_\varepsilon \cdot e \gamma'_\varepsilon(\chi_\varepsilon) \partial_t \chi_\varepsilon \]

\[ \leq c \int_{Q_t} (|\partial_t u_\varepsilon|^2 + |u_\varepsilon|^2 + |\nabla u_\varepsilon|^2) + \lambda \int_{Q_t} |u_\varepsilon(t)|^2 + \lambda \int_{\Omega} |\nabla u_\varepsilon(t)|^2 + c_\lambda. \quad (4.10) \]

Next, we add \( \chi_\varepsilon \) to both sides of (2.37), multiply the equality we get by \( 2 \partial_t \chi_\varepsilon \), rearrange, and integrate over \( Q_t \). Using the uniform boundedness of \( \alpha_\varepsilon \) given by (2.48), the Lipschitz continuity of \( F'_2 \), (2.55) and (2.71), we infer

\[ 2 \int_{Q_t} |\partial_t \chi_\varepsilon|^2 + 2 \varepsilon \int_{Q_t} |\nabla \chi_\varepsilon|^2 + \int_{\Omega} |\nabla \chi_\varepsilon(t)|^2 + \int_{Q_t} |\chi_\varepsilon(t)|^2 + 2 \varepsilon \int_{Q_t} F_{1,\varepsilon}(\chi_\varepsilon(t)) \]

\[ = \int_{\Omega} (|\nabla \chi_{0,\varepsilon}|^2 + |\chi_{0,\varepsilon}|^2 + 2 \varepsilon F_{1,\varepsilon}(\chi_{0,\varepsilon})) + \]

\[ + 2 \int_{Q_t} (\chi_\varepsilon - \partial_t F'_2(\chi_\varepsilon) - \alpha_\varepsilon(\varepsilon)(\chi_\varepsilon) D' \chi_\varepsilon) \partial_t \chi_\varepsilon + 2 \int_{Q_t} \sigma_\varepsilon \cdot e \gamma'_\varepsilon(\chi_\varepsilon) \partial_t \chi_\varepsilon \]

\[ \leq c + \frac{1}{4} \int_{Q_t} |\partial_t \chi_\varepsilon|^2 + c \int_{Q_t} |\chi_\varepsilon|^2 + 2 \int_{Q_t} \sigma_\varepsilon \cdot e \gamma'_\varepsilon(\chi_\varepsilon) \partial_t \chi_\varepsilon. \quad (4.11) \]

Finally, by rearranging (2.36) and squaring, applying the elementary Young inequality and recalling that \( \gamma_\varepsilon \) is uniformly bounded, we have

\[ \frac{\kappa^2}{4} \int_{\Omega} |\nabla u_\varepsilon(t)|^2 \leq \frac{1}{2} \int_{\Omega} |\sigma_\varepsilon(t)|^2 + c. \quad (4.12) \]

At this point, we sum (4.9)–(4.12) to each other. Then, two terms cancel and we eventually obtain

\[ \int_{\Omega} w_\varepsilon(t) - c_0 \int_{\Omega} \ln(\partial_\varepsilon(t) + \varepsilon) + \int_{Q_t} |\nabla \ln(\partial_\varepsilon + \varepsilon)|^2 \]

\[ + \kappa \int_{\Omega} |\partial_t u_\varepsilon(t)|^2 + (1 - \lambda) \int_{\Omega} |u_\varepsilon(t)|^2 + ((\kappa^2/4) - \lambda) \int_{\Omega} |\nabla u_\varepsilon(t)|^2 \]

\[ + \frac{1}{2} \int_{Q_t} |\sigma_\varepsilon|^2 + \frac{1}{4} \int_{Q_t} |\partial_t \chi_\varepsilon|^2 + \varepsilon \int_{Q_t} |\nabla \partial_t \chi_\varepsilon|^2 + \|\chi_\varepsilon(t)\|_{L^2(Q)}^2 \]

\[ \leq c \int_{Q_t} (|\partial_t u_\varepsilon|^2 + |u_\varepsilon|^2 + |\nabla u_\varepsilon|^2 + |\chi_\varepsilon|^2) + c_\lambda. \quad (4.13) \]

Now, we recall that \( w_\varepsilon \geq \lambda_\varepsilon \partial_\varepsilon \geq 0 \) (see (2.61)), whence \( \lambda_\varepsilon(\partial_\varepsilon + \varepsilon) \leq w_\varepsilon + \lambda_\varepsilon \varepsilon \), and observe that \( \lambda_\varepsilon r - c_0 \ln r \geq (\lambda_\varepsilon/2)(r + |\ln r|) - c \) for some constant \( c \) and every \( r > 0 \). Hence, if we choose \( \lambda \) small enough and apply the Gronwall lemma, we obtain

\[ \|w_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} + \|\partial_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} + \|\ln(\partial_\varepsilon + \varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \]

\[ + \|u_\varepsilon\|_{W^{1,\infty}(0,T;H)} + \|\sigma_\varepsilon\|_{L^\infty(0,T;H)} + \|\partial_t \chi_\varepsilon\|_{L^2(Q)} \]

\[ + \|\chi_\varepsilon\|_{L^\infty(0,T;V)} + \varepsilon^{1/2} \|\nabla \partial_t \chi_\varepsilon\|_{L^\infty(0,T;H)} \leq c. \quad (4.14) \]

**Consequence.** A comparison in (2.36) easily shows that

\[ \|\partial_\varepsilon^2 u_\varepsilon\|_{L^2(0,T;V^*)} \leq c. \quad (4.15) \]
Second a priori estimate. We rewrite (2.36) as
\[ -\Delta \chi_\varepsilon - \varepsilon \partial_t \Delta \chi_\varepsilon + \partial_c \beta_\varepsilon (\chi_\varepsilon) = f_\varepsilon \]  
where we have set \( f_\varepsilon := -\partial_t \chi_\varepsilon - \partial_c \pi (\chi_\varepsilon) - \alpha_\varepsilon (\partial_\varepsilon \pi (\chi_\varepsilon) + \sigma_\varepsilon \cdot e \gamma' (\chi_\varepsilon) . \) Observe that \( f_\varepsilon \) is bounded in \( L^2 (Q) \) by (4.14). By multiplying (4.16) by \(-\Delta \chi_\varepsilon \) and integrating over \( Q_t \), we thus obtain
\[
\int_{Q_t} |\Delta \chi_\varepsilon|^2 + \frac{\varepsilon}{2} \int_{\Omega} |\Delta \chi_\varepsilon(t)|^2 + \varepsilon \int_{Q_t} \beta'_\varepsilon (\chi_\varepsilon) |\nabla \chi_\varepsilon|^2 \\
\leq \frac{\varepsilon}{2} \int_{\Omega} |\Delta \chi_{0,\varepsilon}|^2 + \frac{1}{2} \int_{Q_t} |\Delta \chi_\varepsilon|^2 + c .
\]
As \( \beta'_\varepsilon \) is nonnegative and \( \varepsilon^{1/2} \| \Delta \chi_{0,\varepsilon} \|_H \) is bounded independently of \( \varepsilon \) by (2.71), we conclude that
\[ \| \Delta \chi_\varepsilon \|_{L^2 (Q)} \leq c \quad \text{whence also} \quad \| \chi_\varepsilon \|_{L^2 (0,T;W_0)} \leq c \]  
by (4.14) and elliptic regularity.

Consequences. We introduce \( \Delta^* : V \to V^* \) by setting
\[ \langle -\Delta^* v, z \rangle := \int_{\Omega} \nabla v \cdot \nabla z \quad \text{for every} \ v, z \in V . \]  
Then, for \( v \in L^2 (0,T;V) \), we have
\[
\int_0^T \langle -\partial_t \Delta^* \chi_\varepsilon (t), v (t) \rangle \ dt = \int_Q \nabla \partial_t \chi_\varepsilon \cdot \nabla v \leq \| \nabla \partial_t \chi_\varepsilon \|_{L^2 (0,T;H)} \| v \|_{L^2 (0,T;V)}
\]
and the estimate for \( \nabla \partial_t \chi_\varepsilon \) given by (4.14) implies that
\[ \varepsilon^{1/2} \| \partial_t \Delta^* \chi_\varepsilon \|_{L^2 (0,T;V^*)} \leq c . \]  
We deduce an estimate for \( \beta_\varepsilon (\chi_\varepsilon) \) as follows. We observe that equation (2.36) for \( \chi_\varepsilon \) complemented with \( \chi_\varepsilon \in L^2 (0,T;W_0) \) can be written as the abstract equation in \( V^* \)
\[
\partial_t \chi_\varepsilon - \Delta^* \chi_\varepsilon - \varepsilon \partial_t \Delta^* \chi_\varepsilon + \partial_c \beta_\varepsilon (\chi_\varepsilon) + \partial_c \pi (\chi_\varepsilon) + \alpha_\varepsilon (\partial_\varepsilon \pi (\chi_\varepsilon) + \sigma_\varepsilon \cdot e \gamma' (\chi_\varepsilon) = 0
\]
and that \( \Delta^* \chi_\varepsilon = \Delta \chi_\varepsilon \) since \( \Delta^* v = \Delta v \) whenever \( v \in W_0 \). Then, (4.14), (4.17) and (4.19) yield by comparison
\[ \| \beta_\varepsilon (\chi_\varepsilon) \|_{L^2 (0,T;V^*)} \leq c . \]  

Third a priori estimate. We adapt the technique of [8] to the present situation and give the details, for the reader’s convenience, since some modifications of the argument of [8] are spread in the calculation. Here, Remark 2.6 plays a role. For every nonnegative integer \( k \), we introduce the truncation function \( \mathcal{I}_k : [0, +\infty) \to \mathbb{R} \) and the set \( Q^k \) defined by
\[
\mathcal{I}_k (r) := \int_0^r \min \{(s - k)^+, 1\} \ ds \quad \text{for} \ r \geq 0
\]
\[ Q^k := \{(x,t) \in Q : k \leq w_\varepsilon (x,t) < k + 1\} . \]
and test (2.34) written at the time $t$ by $v = T'_k(w_\varepsilon(t))$. Then, we integrate over $(0, T)$ with respect to $t$ and rearrange. Once $k$ is fixed, we easily obtain

$$
\int_{\Omega} T_k(w_\varepsilon(T)) + \int_{Q^k} \nabla \vartheta \cdot \nabla w_\varepsilon
\leq \int_{\Omega} T_k(w_\varepsilon(0)) + \int_{Q^k} \left| (\alpha_\varepsilon(\vartheta) \partial_t G_\varepsilon(x_\varepsilon) + R + |\partial_t x_\varepsilon|^2) \right| |T'_k(w_\varepsilon)|.
$$

Now, we notice that the first integral on the left hand side is nonnegative. Moreover, the whole right hand side is bounded since $|T'_k| \leq 1$ and $\int_{\Omega} T_k(w_\varepsilon(0)) \leq \int_{\Omega} |w_\varepsilon(0)| \leq c$ by (2.38) and (2.72). Hence, we infer

$$
\int_{Q^k} \nabla \vartheta \cdot \nabla w_\varepsilon \leq c. \tag{4.21}
$$

On the other hand, in view of (2.33) we have $\nabla w_\varepsilon = a_\varepsilon \nabla \vartheta + b_\varepsilon \nabla x_\varepsilon$, where

$$
a_\varepsilon := c_0 - (\vartheta_\varepsilon + \varepsilon)\alpha''_\varepsilon(\vartheta_\varepsilon)G'_\varepsilon(x_\varepsilon) \quad \text{and} \quad b_\varepsilon := (\alpha'_\varepsilon(\vartheta_\varepsilon) - (\vartheta_\varepsilon + \varepsilon)\alpha'_\varepsilon(\vartheta_\varepsilon))G'_\varepsilon(x_\varepsilon)
$$

whence immediately

$$
\nabla \vartheta = \frac{\nabla w_\varepsilon - b_\varepsilon \nabla x_\varepsilon}{a_\varepsilon} \quad \text{and} \quad \nabla \vartheta \cdot \nabla w_\varepsilon = \frac{|\nabla w_\varepsilon|^2}{a_\varepsilon} - \frac{b_\varepsilon}{a_\varepsilon} \nabla x_\varepsilon \cdot \nabla w_\varepsilon. \tag{4.22}
$$

By accounting for (2.48), (2.55), (2.57) and estimate (4.14), we thus obtain

$$
\nabla \vartheta \cdot \nabla w_\varepsilon \geq \frac{|\nabla w_\varepsilon|^2}{C_*} - \frac{c}{\lambda_*} |\nabla x_\varepsilon| |\nabla w_\varepsilon| \geq \frac{|\nabla w_\varepsilon|^2}{2C_*} - c |\nabla x_\varepsilon|^2 \geq \frac{|\nabla w_\varepsilon|^2}{2C_*} - c
$$

and combining with (4.21), we conclude that

$$
\int_{Q^k} |\nabla w_\varepsilon|^2 \leq c. \tag{4.23}
$$

Assume now $q \in [1, 5/4)$ and let $|Q^k|$ be the Lebesgue measure of $Q^k$. As $Q = \bigcup_{k=0}^\infty Q^k$ by (2.61), we have

$$
\int_Q |\nabla w_\varepsilon|^q = \sum_{k=0}^\infty \int_{Q^k} |\nabla w_\varepsilon|^q \leq \sum_{k=0}^\infty \left( \int_{Q^k} |\nabla w_\varepsilon|^2 \right)^{q/2} |Q^k|^{(2-q)/2} \leq c_q \sum_{k=0}^\infty |Q^k|^{(2-q)/2}. \tag{4.24}
$$

On the other hand, it is clear that for every $k$

$$
\int_{Q^k} w_\varepsilon^{4q/3} \geq k^{4q/3} |Q^k| \quad \text{whence} \quad |Q^k| \leq k^{-4q/3} \int_{Q^k} w_\varepsilon^{4q/3}
$$

so that (4.24) and the Hölder inequality for infinite sums yield

$$
\int_Q |\nabla w_\varepsilon|^q \leq \sum_{k=0}^\infty k^{-2q(2-q)/3} \left( \int_{Q^k} w_\varepsilon^{4q/3} \right)^{(2-q)/2}
\leq \left( \sum_{k=0}^\infty k^{-4(2-q)/3} \right)^{q/2} \left( \sum_{k=0}^\infty \int_{Q^k} w_\varepsilon^{4q/3} \right)^{(2-q)/2}
= c_q \left( \int_Q w_\varepsilon^{4q/3} \right)^{(2-q)/2} = c_q \|w_\varepsilon\|_{L^{4q/3}(Q)}^{(2-q)/2} \tag{4.25}
$$
where \( c_q \) may denote the sum of the above numeric series. Notice that such a series actually converges since \( q > 5/4 \) implies \( 4(2-q)/3 > 1 \). Now, we choose \( v = w_\varepsilon(t) \) in the following interpolation and Sobolev-Poincaré inequalities
\[
\|v\|_{L^q/3} \leq \|v\|_1^{1/4} \|v\|_{3q/(3-q)}^{3/4}, \quad \|v\|_{3q/(3-q)} \leq c_q \left( \|\nabla v\|_q + \|v\|_1 \right) \quad \text{for every } v \in W^{1,q}(\Omega)
\]
and integrate over \((0,T)\). Recalling the estimate for \( w_\varepsilon \) given by (4.14) and combining with (4.25), we obtain
\[
\|w_\varepsilon\|_{L^{4q/3}(Q)} = \int_0^T \|w_\varepsilon(t)\|_{4q/3} \, dt \leq c_q \int_0^T \|w_\varepsilon(t)\|_{3q/(3-q)}^3 \, dt
\]
\[
\leq c_q \int_0^T \left( \|\nabla w_\varepsilon(t)\|_q^q + 1 \right) \, dt = c_q \|\nabla w_\varepsilon\|_{L^2(Q)}^q + c_q \leq c_q \|w_\varepsilon\|_{L^{4q/3}(Q)}^{(2-q)/2} + c_q.
\]
As \((2-q)/2 < 4q/3\), we infer that \( \|w_\varepsilon\|_{L^{4q/3}(Q)} \) is bounded. By using (4.25) again, we conclude that
\[
\|w_\varepsilon\|_{L^{4q/3}(Q)} \leq c_q \quad \text{for every } q \in [1,5/4).
\]
Due to (4.22), (2.57) and estimate (4.14), we derive that
\[
\|\partial_\varepsilon\|_{L^{4q/3}(Q)} \cap L^q(0,T;W^{1,q}(\Omega)) \leq c_q \quad \text{for every } q \in [1,5/4).
\]

**Consequence.** We write (2.31) for a.a. \( t \in (0,T) \) and take any \( v \in W^{1,q}(\Omega) \) as a test function, by noting that \( W^{1,q}(\Omega) \subset L^\infty(\Omega) \) since \( q > 5/4 \) implies \( q' > 3 \). As \( \alpha_\varepsilon \) and \( G_\varepsilon \) are uniformly bounded, we obtain for a.a. \( t \in (0,T) \)
\[
\int_0^T \partial_t w_\varepsilon(t) \, v = \int_\Omega \alpha_\varepsilon(\vartheta_\varepsilon(t)) \partial_t G_\varepsilon(\chi_\varepsilon(t)) \, v - \int_\Omega \nabla \vartheta_\varepsilon(t) \cdot \nabla v + \int_\Omega (R(t) + |\partial_t \chi_\varepsilon(t)|^2) \, v
\]
\[
\leq c \|\partial_t \chi_\varepsilon(t)\|_H \|v\|_\infty + \|\nabla \vartheta_\varepsilon(t)\|_q \|\nabla v\|_q + \|R(t)\|_H \|v\|_\infty + \|\partial_t \chi_\varepsilon(t)\|_H^2 \|v\|_\infty
\]
\[
\leq c \left( 1 + \|\partial_t \chi_\varepsilon(t)\|_H^2 + \|\vartheta_\varepsilon(t)\|_{W^{1,q}(\Omega)} + \|\partial_t \chi_\varepsilon(t)\|_H^2 \right) \|v\|_{W^{1,q}(\Omega)}.
\]
This means that
\[
\|\partial_t w_\varepsilon(t)\|_{(W^{1,q}(\Omega))^*} \leq c \left( 1 + \|\partial_t \chi_\varepsilon(t)\|_H^2 + \|\vartheta_\varepsilon(t)\|_{W^{1,q}(\Omega)} + \|R(t)\|_H \right).
\]
As \( R \in L^2(Q) \) and (4.14), (4.27) hold, we conclude that
\[
\|\partial_t w_\varepsilon\|_{L^1(0,T;W^{1,q}(\Omega))^*} \leq c_q.
\]

**Convergence and first consequences.** From (4.14)–(4.15), (4.17), (2.48), (4.20) and (4.26)–(4.27), we deduce that a sextuplet \((w, \vartheta, u, \sigma, \chi, \xi)\) and a pair \((a, \ell)\) exist such that, for a subsequence of \( \varepsilon \searrow 0 \) and for every \( q \in (1,5/4) \), the following convergence holds true
\[
w_\varepsilon \rightharpoonup w \quad \text{weakly in } L^{4q/3}(Q) \cap L^q(0,T;W^{1,q}(\Omega))
\]
\[
\vartheta_\varepsilon \rightharpoonup \vartheta \quad \text{weakly in } L^{4q/3}(Q) \cap L^3(0,T;W^{1,q}(\Omega))
\]
\[
\alpha_\varepsilon(\vartheta_\varepsilon) \rightharpoonup a \quad \text{weakly star in } L^\infty(Q)
\]
\[
\ln(\partial_\varepsilon + \varepsilon) \rightharpoonup \ell \quad \text{weakly in } L^2(0,T;V)
\]
\[
u_\varepsilon \rightharpoonup u \quad \text{weakly in } H^2(0,T;V^*) \cap W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V)
\]
\[
\sigma_\varepsilon \rightharpoonup \sigma \quad \text{weakly star in } L^\infty(0,T;H)
\]
\[
\chi_\varepsilon \rightharpoonup \chi \quad \text{weakly star in } H^1(0,T;H) \cap L^2(0,T;W_0)
\]
\[
\varepsilon \partial_\varepsilon \nabla \chi_\varepsilon \rightharpoonup 0 \quad \text{strongly in } L^2(0,T;H)
\]
\[
\beta_\varepsilon(\chi_\varepsilon) \rightharpoonup \xi \quad \text{weakly in } L^2(0,T;V^*).
Owing to Lemma 4.1 we see that the regularity requirements (2.18), (2.20)–(2.22) stated in Definition 2.1 and regarding \( w, \vartheta, u, \sigma, \lambda \) are fulfilled except for the positivity for \( \vartheta \). Moreover, the above convergence implies weak convergence at least in \( C^0([0, T]; V^\ast) \) for \( u_\varepsilon, \partial_t u_\varepsilon \) and \( \chi_\varepsilon \) and (2.38), (2.70)–(2.71) hold, so that the Cauchy conditions (2.23) for \( u, \partial_t u \) and \( \chi \) are satisfied (while more work is needed for \( w \)). Furthermore, (4.35) implies

\[ \chi_\varepsilon \to \chi \quad \text{weakly in } C^0([0, T]; V) \]  

(4.38)
due to the continuous embeddings \( H^1(0, T; H) \cap L^2(0, T; W) \subset C^0([0, T]; V) \). Now, we recall the compact embeddings \( V \subset H \) and \( W \subset V \), as well as the continuous embeddings \( W^{1,q}(\Omega) \subset L^q(\Omega) \subset L^1(\Omega) \subset (W^{1,q}(\Omega))^\ast \), the first one being compact. Then, thanks to estimate (4.28) and accounting for strong compactness results (see, e.g., [33, Sect. 8, Cor. 4]), we derive some strong and a.e. convergence (for a subsequence). Namely, we deduce that

\[ w_\varepsilon \to w \quad \text{strongly in } L^q(Q) \text{ and a.e. in } Q \]  

(4.39)
\[ u_\varepsilon \to u \quad \text{strongly in } C^1([0, T]; V^\ast) \cap C^0([0, T]; H) \text{ and a.e. in } Q \]  

(4.40)
\[ \chi_\varepsilon \to \chi \quad \text{strongly in } C^0([0, T]; H) \cap L^2(0, T; V) \text{ and a.e. in } Q \]  

(4.41)
for every \( q \in [1, 5/4) \). As a consequence of (4.41), the limits of all the nonlinear terms involving \( \chi_\varepsilon \), but \( \beta_\varepsilon(\chi_\varepsilon) \), can be correctly identified. Namely, we have (cf. (2.53)–(2.56))

\[ \phi_\varepsilon(\chi_\varepsilon) \to \phi(\chi) \quad \text{strongly in } L^p(Q) \text{ for } p < +\infty \text{ and a.e. in } Q \]  

(4.42)

where \( \phi = G, G', \gamma, \gamma' \).

Let us comment, e.g., on the limit of \( G_\varepsilon(\chi_\varepsilon) \). Due to (4.41) and assumption (2.56), we deduce that \( G_\varepsilon(\chi_\varepsilon) \) a.e. converges to \( G(\chi) \). Then, (4.42) with \( \phi = G \) follows for (2.55) implies that \( G_\varepsilon(\chi_\varepsilon) \) is bounded in \( L^\infty(Q) \). In addition, as (4.41) yields \( \pi(\chi_\varepsilon) \to \pi(\chi) \), e.g., strongly in \( C^0([0, T]; H) \) since \( \pi \) is Lipschitz continuous, we infer that (2.26) is satisfied and that

\[ \partial_t \chi - \Delta \chi + \partial_t (\xi + \pi(\chi)) + a G'(\chi) - \sigma \cdot e \gamma'(\chi) = 0 \quad \text{in } V^\ast, \text{ a.e. in } (0, T). \]  

(4.43)

Now, just by comparison in (4.43), we deduce that

\[ \xi \in L^2(0, T; H) \]  

(4.44)
i.e., the first condition in (2.23). Hence, equation (2.28) is satisfied as well once we prove that \( a = \alpha(\vartheta) \) and \( \xi = \beta(\chi) \). This will be done in the following. As far as (2.27) is concerned, we easily recover an integrated version of it (in fact equivalent to (2.27) itself), namely

\[ \int_0^T \left\langle \partial_t^2 u(t), v(t) \right\rangle dt + \kappa \int_Q \sigma \cdot \nabla v = \int_Q B_\Omega v + \int_0^T \left\langle B_\Gamma(t), v(t) \right\rangle dt \]  

(4.45)
for every \( v \in L^2(0, T; V) \). Indeed, the analogue of (4.45) for the approximating problem

\[ \int_0^T \left\langle \partial_t^2 u_\varepsilon(t), v(t) \right\rangle dt + \kappa \int_Q \sigma_\varepsilon \cdot \nabla v = \int_Q B_{\Omega, \varepsilon} v + \int_0^T \left\langle B_{\Gamma, \varepsilon}(t), v(t) \right\rangle dt \]  

holds true as well for every \( v \in L^2(0, T; V) \), so that it suffices to recall (2.67)–(2.68) and (1.33)–(1.34).
More identifications and properties. We can derive both (2.24) and positivity for \( \dot{v} \) (we just have \( \dot{v} \geq 0 \) for the moment, as a consequence of (4.30) and of \( \dot{v}_\varepsilon \geq 0 \) for \( \varepsilon > 0 \)), as well as we identify the weak limits \( a \) and \( \ell \) given by (4.31)–(4.32) as \( \alpha(\dot{v}) \) and \( \ln \dot{v} \), respectively. Regarding the first claim, we prove that

\[
\dot{v}_\varepsilon \to \dot{v} \quad \text{a.e. in } Q.
\] (4.46)

To this aim, we use the analogous convergence for \( w_\varepsilon \) (cf. (4.39)), the convergence a.e. for \( G_\varepsilon(x_\varepsilon) \) just remarked, and the uniform bounds and convergence properties of the approximating nonlinearities: see (2.49), (2.55), (2.57)–(2.59) and also (2.10)–(2.11). For two different indices \( \varepsilon, \varepsilon' \) of the subsequence we have

\[
w_\varepsilon - w_{\varepsilon'} = c_0 \dot{v}_\varepsilon + (\alpha_\varepsilon(\dot{v}_\varepsilon) - (\dot{v}_\varepsilon + \varepsilon)\alpha'_\varepsilon(\dot{v}_\varepsilon))G_\varepsilon(x_\varepsilon)
- c_0 \dot{v}_{\varepsilon'} + (\alpha_\varepsilon(\dot{v}_{\varepsilon'}) - (\dot{v}_{\varepsilon'} + \varepsilon)\alpha'_{\varepsilon}(\dot{v}_{\varepsilon'}))G_\varepsilon(x_{\varepsilon'})
+ (\alpha_\varepsilon(\dot{v}_\varepsilon) - (\dot{v}_\varepsilon + \varepsilon)\alpha'_\varepsilon(\dot{v}_\varepsilon) - \alpha_{\varepsilon'}(\dot{v}_{\varepsilon'}) + (\dot{v}_\varepsilon + \varepsilon')\alpha'_{\varepsilon'}(\dot{v}_{\varepsilon'}))G_\varepsilon(x_\varepsilon)
+ (\alpha_{\varepsilon'}(\dot{v}_{\varepsilon'}) - (\dot{v}_{\varepsilon'} + \varepsilon')\alpha'_{\varepsilon'}(\dot{v}_{\varepsilon'}))(G_\varepsilon(x_\varepsilon) - G_{\varepsilon'}(x_{\varepsilon'})).
\]

Thus, we deduce that

\[
|w_\varepsilon - w_{\varepsilon'}|
\geq \lambda_d|\dot{v}_\varepsilon - \dot{v}_{\varepsilon'}| - \sup_{r \geq 0} |\alpha_\varepsilon(r) - (r + \varepsilon)\alpha'_\varepsilon(r) - \alpha_{\varepsilon'}(r) + (r + \varepsilon')\alpha'_{\varepsilon'}(r)| \sup_{s \in \mathbb{R}} G_\varepsilon(s)
- \sup_{r \geq 0} |\alpha_{\varepsilon'}(r) - (r + \varepsilon')\alpha'_{\varepsilon'}(r)| \|G_\varepsilon(x_\varepsilon) - G_{\varepsilon'}(x_{\varepsilon'})\|
\]

which implies that \( \{\dot{v}_\varepsilon\} \) is a Cauchy sequence and consequently converges almost everywhere in \( Q \) to some measurable function \( \Theta \). Then, using (4.30) and the Egorov theorem, it is not difficult to find out that \( \Theta = \dot{v} \) and

\[
\dot{v}_\varepsilon \to \dot{v} \quad \text{strongly in } L^q(Q) \text{ for every } q \in [1,5/4).
\] (4.47)

In particular, (4.46) follows. Moreover, owing to (2.48), for every \( p < +\infty \) a strong convergence in \( L^p(Q) \) to the correct limits holds true for all the nonlinear terms involving \( \alpha_\varepsilon \), like \( \alpha_\varepsilon(\dot{v}_\varepsilon) \) and \( (\dot{v}_\varepsilon + \varepsilon)\alpha'_\varepsilon(\dot{v}_\varepsilon) \). Therefore, (2.24) comes out as a consequence. Next, we prove that \( \dot{v} > 0 \) a.e. in \( Q \) and that the weak limit \( \ell \) given by (4.32) coincides with \( \ln \dot{v} \). To this aim, we recall the bound for \( \ln(\dot{v}_\varepsilon + \varepsilon) \) given by (4.14). Thanks to (4.46) and to the Fatou lemma, we deduce that \( \ln \dot{v} \in L^1(Q) \), whence \( \dot{v} > 0 \) a.e. in \( Q \). More precisely, we have \( \int_{\Omega} |\ln(\dot{v}_\varepsilon(t) + \varepsilon)| \leq c \) for a.a. \( t \in (0,T) \), whence also \( \int_{\Omega} |\ln \dot{v}(t)| \leq c \), i.e., \( \ln \dot{v} \in L^\infty(0,T;L^1(\Omega)) \) and \( \ell = \ln \dot{v} \) as well.

Now, we aim to identify \( \xi \) in (4.43) as a selection from \( \beta(\chi) \) (see (2.23)). We introduce three nonnegative functionals on \( H, V \) and \( L^2(0,T;V) \), respectively, by setting (being understood that the integrals are possibly infinite)

\[
j_H(v) := \int_{\Omega} F_1(v) \quad \text{for } v \in H \quad \text{and} \quad j_V(v) := \int_{\Omega} F_1(v) \quad \text{for } v \in V
\]

\[
J_V(v) := \int_0^T j_V(v(t)) \, dt = \int_Q F_1(v) \quad \text{for } v \in L^2(0,T;V)
\]
(thus, \(j_{V}\) is the restriction of \(j_{H}\) to \(V\)). Their subdifferentials are (possibly multi-valued) maps from the above spaces to the corresponding dual spaces (see, e.g., [4, p. 52]). The characterization of \(j_{H}\) and \(J_{V}\) we use can be obtained by applying, e.g., [4, Ex. 3 and Prop. 2.8] and adapting the argument, respectively, while the property of \(\partial j_{V}\) we are going to mention can be found, e.g., in [5, Prop. 2.5]. So, by identifying \(H^{*}\) with \(H\) and \((L^{2}(0, T; V))^{*}\) with \(L^{2}(0, T; V^{*})\) as usual, we have

\[
\text{for } v \in H \text{ and } v^{*} \in H, \quad v^{*} \in \partial j_{H}(v) \text{ if and only if } v^{*}(x) \in \partial F_{1}(v(x)) = \beta(v(x)) \text{ for a.a. } x \in \Omega
\]

\[
\text{for } v \in L^{2}(0, T; V) \text{ and } v^{*} \in L^{2}(0, T; V^{*}) \quad v^{*} \in \partial J_{V}(v) \text{ if and only if } v^{*}(t) \in \partial j_{V}(v(t)) \text{ for a.a. } t \in (0, T)
\]

\[
\text{for } v \in V \text{ and } v^{*} \in H \quad v^{*} \in \partial j_{V}(v) \quad \text{if and only if } v^{*} \in \partial j_{H}(v).
\]

By recalling that \(\chi \in L^{2}(0, T; V)\) and \(\xi \in L^{2}(0, T; H)\) (cf. (4.44)), observing that \(\xi \in \beta(\chi)\) a.e. in \(Q\) if and only if \(\xi(t) \in \beta(\chi(t))\) a.e. in \(\Omega\), for a.a. \(t \in (0, T)\), and combining the above statements, we deduce that

\[
\xi \in \beta(\chi) \text{ a.e. in } Q \quad \text{if and only if } \xi \in \partial J_{V}(\chi).
\]

Thus, we prove that \(\xi \in \partial J_{V}(\chi)\), i.e.

\[
\int_{Q} F_{1}(\chi) + \int_{0}^{T} \langle \xi(t), z(t) - \chi(t) \rangle dt \leq \int_{Q} F_{1}(z) \text{ for every } z \in L^{2}(0, T; V) \tag{4.48}
\]

(in fact, the above duality is an integral since \(\xi \in L^{2}(0, T; H)\)). So, we fix \(z \in L^{2}(0, T; V)\) and assume that \(F_{1}(z) \in L^{1}(Q)\), without loss of generality. By convexity and \(\beta_{\varepsilon} = F_{1, \varepsilon}'\) (see (2.50)), we have for \(\varepsilon > 0\)

\[
\int_{Q} F_{1, \varepsilon}(\chi_{\varepsilon}) + \int_{0}^{T} \langle \beta_{\varepsilon}(\chi_{\varepsilon}), z - \chi_{\varepsilon}(t) \rangle dt \leq \int_{Q} F_{1, \varepsilon}(z).
\]

Moreover, the weak convergence (4.37) is coupled with the strong convergence (4.41) in the duality pairing on the left hand side of the above inequality, and \(F_{1, \varepsilon}(s) \leq F_{1}(s)\) for every \(s \in \mathbb{R}\) by (2.50). Therefore, (4.48) immediately follows once we prove that

\[
\int_{Q} F_{1}(\chi) \leq \liminf_{\varepsilon \searrow 0} \int_{Q} F_{1, \varepsilon}(\chi_{\varepsilon}). \tag{4.49}
\]

To this end, we fix \(\varepsilon' > 0\) for a while. By accounting for (4.41), the lower semicontinuity of \(F_{1, \varepsilon'}\) and the inequality \(F_{1, \varepsilon'}(s) \leq F_{1, \varepsilon}(s)\) for every \(s \in \mathbb{R}\) and \(\varepsilon \in (0, \varepsilon')\) (trivially from (2.50)), we obtain

\[
\int_{Q} F_{1, \varepsilon'}(\chi) \leq \liminf_{\varepsilon \searrow 0} \int_{Q} F_{1, \varepsilon'}(\chi_{\varepsilon}) \leq \liminf_{\varepsilon \searrow 0} \int_{Q} F_{1, \varepsilon}(\chi_{\varepsilon}). \tag{4.50}
\]

Now, we let \(\varepsilon'\) vary and recall that \(F_{1, \varepsilon'}(s) \searrow F_{1}(s)\) monotonically for every \(s \in \mathbb{R}\) as \(\varepsilon' \searrow 0\). Thus, the Beppo Levi monotone convergence theorem yields

\[
\int_{Q} F_{1}(\chi) = \lim_{\varepsilon' \searrow 0} \int_{Q} F_{1, \varepsilon'}(\chi). \tag{4.51}
\]
By combining (4.51) and (4.50), we obtain (4.49). Therefore, even (4.48) is established and the proof is complete.

**Further strong convergence.** In order to pass to the limit in (2.34) we need to prove that \( \partial_\varepsilon \chi \) strongly converges to \( \partial t \chi \) in \( L^2(Q) \). To this end, it suffices to show that

\[
\limsup_{\varepsilon \to 0} \int_Q |\partial_\varepsilon \chi_\varepsilon|^2 \leq \int_Q |\partial_t \chi|^2
\]  

(4.52)

where it is understood that \( \varepsilon \) tends to zero along the subsequence satisfying all the convergence properties just proved, in particular (4.35). To achieve (4.52), we compute the integral on the left hand side by testing (2.37) by \( \partial_\varepsilon \chi_\varepsilon \). We have

\[
\int_Q |\partial_\varepsilon \chi_\varepsilon|^2 = -\frac{1}{2} \int_\Omega |\nabla \chi(T)|^2 + \frac{1}{2} \int_\Omega |\nabla \chi_0|^2 - \varepsilon \int_Q |\partial_\varepsilon \nabla \chi_\varepsilon|^2 \\
- \varrho_\varepsilon \int_\Omega F_1(\chi(T)) + \varrho_\varepsilon \int_\Omega F_1(\chi_\varepsilon) \\
- \psi_\varepsilon \int_\Omega \pi(\chi_\varepsilon) \partial_\varepsilon \chi_\varepsilon - \int_Q \alpha(\varepsilon) G_\varepsilon'(\chi_\varepsilon) \partial_\varepsilon \chi_\varepsilon + \int_Q \sigma_\varepsilon \cdot e \gamma_\varepsilon' \partial_\varepsilon \chi_\varepsilon.
\]  

(4.53)

As \( \chi_\varepsilon(T) \to \chi(T) \) weakly in \( V \) and strongly in \( H \) (cf. (4.38) and (4.41)), we have

\[
\frac{1}{2} \int_\Omega |\nabla \chi(T)|^2 \leq \liminf_{\varepsilon \to 0} \frac{1}{2} \int_\Omega |\nabla \chi_\varepsilon(T)|^2
\]

and, arguing as in the proof of (4.49), we can show that

\[
\varrho_\varepsilon \int_\Omega F_1(\chi(T)) \leq \liminf_{\varepsilon \to 0} \varrho_\varepsilon \int_\Omega F_1(\chi_\varepsilon(T)).
\]

Moreover, in view of (2.71) we also infer that

\[
\lim_{\varepsilon \to 0} \left( \frac{1}{2} \int_\Omega |\nabla \chi_0|^2 + \varrho_\varepsilon \int_\Omega F_1(\chi_\varepsilon) \right) = \frac{1}{2} \int_\Omega |\nabla \chi_0|^2 + \varrho \int_\Omega F_1(\chi_0).
\]

Hence, using the weak convergence of \( \partial_\varepsilon \chi_\varepsilon \) in \( L^2(Q) \) and the strong convergences of \( \pi(\chi_\varepsilon) \) and \( \alpha(\varepsilon) G_\varepsilon'(\chi_\varepsilon) \) in \( L^2(Q) \), from (4.53) it is straightforward to deduce that

\[
\limsup_{\varepsilon \to 0} \int_Q |\partial_\varepsilon \chi_\varepsilon|^2 \leq -\frac{1}{2} \int_\Omega |\nabla \chi(T)|^2 - \varrho \int_\Omega F_1(\chi(T)) \\
+ \frac{1}{2} \int_\Omega |\nabla \chi_0|^2 + \varrho_\varepsilon \int_\Omega F_1(\chi_0) - \psi_\varepsilon \int_\Omega \pi(\chi) \partial_\varepsilon \chi \\
- \int_Q \alpha(\varepsilon) G_\varepsilon'(\chi_\varepsilon) \partial_\varepsilon \chi + \limsup_{\varepsilon \to 0} \int_Q \sigma_\varepsilon \cdot e \gamma_\varepsilon' \partial_\varepsilon \chi_\varepsilon.
\]  

(4.54)

Unfortunately, the last term of (4.54) cannot be immediately identified since it couples two weakly convergent factors. In order to estimate it, we compute the integral with the help of (4.47) written with \( t = T \), combine weak convergence for the terms involving the solution and strong convergence for the data (see (2.67)–(2.71)), and use weak semicontinuity as before. In particular, due to (4.33) and (4.40) we note that \( \partial_t u_\varepsilon(T) \to \partial_t u(T) \) weakly
in $H$ and $u_\varepsilon(T) \to u(T)$ weakly in $V$ (whence easily $\sigma_\varepsilon(T) \to \sigma(T)$ weakly in $H$ as well). Finally, we account for identity (4.3). We obtain

$$
\limsup_{\varepsilon \searrow 0} \int_Q \sigma_\varepsilon \cdot e \gamma'(\chi_\varepsilon) \partial_t \chi_\varepsilon
\leq -\frac{\kappa}{2} \int_\Omega |\partial_t u(T)|^2 + \frac{\kappa}{2} \int_\Omega |u'|^2 - \frac{1}{2} \int_\Omega |\sigma(T)|^2 + \frac{1}{2} \int_\Omega |\sigma(0)|^2
+ \kappa \int_Q B_\Omega \partial_t u + \kappa \langle B_T(T), u(T) \rangle - \kappa \langle B_T(0), u_0 \rangle - \kappa \int_0^T \langle \partial_t B_T(s), u(s) \rangle \, ds
= \int_Q \sigma \cdot e \gamma'(\chi) \partial_t \chi.
$$

Hence, observing also that

$$
\text{thanks to integration by parts, (2.23), and the chain rule shown e.g. in [15, Lemme 3.3, p. 73], the inequality (4.54) entails}
$$

$$
\limsup_{\varepsilon \searrow 0} \int_Q |\partial_t \chi_\varepsilon|^2 \leq -\int_Q (-\Delta \chi + \vartheta_c \xi) \partial_t \chi
- \vartheta_c \int_Q \pi(\chi) \partial_t \chi - \int_Q \alpha(\vartheta) G'(\chi) \partial_t \chi + \int_Q \sigma \cdot e \gamma'(\chi) \partial_t \chi. \tag{4.55}
$$

On the other hand, by testing (2.28) by $\partial_t \chi$ and integrating over $Q$, one immediately sees that the right hand side of (4.55) is precisely $\int_Q |\partial_t \chi|^2$. Therefore, (4.52) is proved.

**End of the proof.** Now, we can take the limit in (2.34). Taking $v \in W^{1,q'}(\Omega)$, with $q'$ as in (2.19), and integrating from 0 to $t \in (0, T]$, thanks to (2.38) we have that

$$
\langle w_\varepsilon(t), v \rangle = \langle w_{0,\varepsilon}, v \rangle + \int_{Q_t} \alpha_\varepsilon(\vartheta_\varepsilon) G'(\chi_\varepsilon) \partial_t \chi_\varepsilon \, v - \int_{Q_t} \nabla \vartheta_\varepsilon \cdot \nabla v + \int_{Q_t} (R + |\partial_t \chi_\varepsilon|^2) \, v.
$$

We observe that (4.39) yields $w_\varepsilon(t) \to w(t)$ strongly in $L^q(\Omega)$ for a.a. $t \in (0, T)$. On the other hand, owing to our convergence properties and (2.72), the above right hand side converges to the expected limit for every $t \in [0, T]$. Therefore, it turns out that

$$
\langle w(t), v \rangle = \langle w_0, v \rangle + \int_{Q_t} \alpha(\vartheta) G'(\chi) \partial_t \chi \, v - \int_{Q_t} \nabla \vartheta \cdot \nabla v + \int_0^t \langle R + |\partial_t \chi|^2, v \rangle \tag{4.56}
$$

for a.a. $t \in (0, T)$. In particular, $w$ belongs to $C^0([0, T]; (W^{1,q'}(\Omega))^*)$ and the initial condition for $w$ in (2.29) is satisfied. Furthermore, by differentiating (4.56) with respect to $t$, we finally recover (2.25) and the regularity (2.19) for $\partial_t w$.

About the $L^\infty(0, T; L^1(\Omega))$-regularity of $w$ (cf. (2.32)), (4.39) implies that $w_\varepsilon \to w$ in $L^1(0, T; L^1(\Omega))$, whence

$$
\|w_\varepsilon(t)\|_{L^1(\Omega)} \to \|w(t)\|_{L^1(\Omega)} \quad \text{for a.e. } t \in (0, T), \text{ at least for a subsequence.}
$$
Then, recalling (4.14) we infer that
\[ \|w(t)\|_{L^1(\Omega)} \leq \sup_{\varepsilon \in (0,1)} \|w_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq c, \]
and consequently \( w \in L^\infty(0,T;L^1(\Omega)) \). The same property can be deduced for \( \vartheta \), so that (2.32) holds. The proof of Theorem 2.3 is then complete.

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