Boundary Problems for Three-Dimensional Dirac Operators and Generalized MIT Bag Models for Unbounded Domains

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Abstract. We consider the operators of the following boundary problems

\[ \mathcal{D}_{A,\Phi} u = \begin{cases} \mathcal{D}_{A,\Phi} u \text{ on } \Omega & \mathfrak{B} u|_{\partial \Omega} = 0 \text{ on } \partial \Omega \end{cases} \tag{1} \]

in unbounded domains \( \Omega \subset \mathbb{R}^3 \), where \( \mathcal{D}_{A,\Phi} \) is the \( 3-D \) Dirac operator

\[ \mathcal{D}_{A,\Phi} = [\alpha \cdot (i\nabla + A) + \alpha_0 m + \Phi I_4] = \sum_{j=1}^{3} \alpha_j (i\partial_{x_j} + A_j)\alpha_0 m + \Phi I_4 \]

defined on the distributions \( u = (u_1, u_2, u_3, u_4) \in H^1(\Omega, \mathbb{C}^4) \), where \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 \) are Dirac matrices, \( A \in L^\infty(\Omega, \mathbb{R}^4) \) and \( \Phi \in L^\infty(\mathbb{R}^3) \) are the magnetic and electrostatic potentials, \( m \in \mathbb{R} \) is the mass of a particle. Let \( C^1 \ni u = (u^1, u^2) \in C^2 \oplus C^2 \). We assume that the operator \( \mathfrak{B} \) of the boundary condition is

\[ \mathfrak{B} u|_{\partial \Omega} = b_1 u^1|_{\partial \Omega} + b_2 u^2|_{\partial \Omega}. \tag{2} \]

where \( b_1, b_2 \) are \( 2 \times 2 \) matrices, \( u^1, u^2 \in H^{1/2}(\partial \Omega, \mathbb{C}^2) \), \( j = 1, 2 \), are restrictions of distributions \( u_j \in H^1(\Omega, \mathbb{C}^2) \) on \( \partial \Omega \). The class of the boundary condition (2) in a particular case contains the boundary conditions of the MIT bag model and its generalizations which describe the confinement of the quarks to the domain \( \Omega \).

We give conditions of self-adjointness of unbounded operators \( \mathcal{D}_{A,\Phi,\mathfrak{B}} \) associated with the boundary problem (2) and give a description of the essential spectrum of \( \mathcal{D}_{A,\Phi,\mathfrak{B}} \) for certain unbounded domains by applying the limit operators method.

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1.1. INTRODUCTION AND NOTATION

Massive relativistic particles of spin-\( \frac{1}{2} \) such as electrons and quarks are described by the Dirac operator, on which relativistic quantum mechanics is based (see, for instance, [3, 4]). The boundary problems for \( 3-D \) Dirac operators arise in relativistic quantum mechanics for the descriptions of the particles confined in domains in \( \mathbb{R}^3 \). One important model of the confinement of particles is the three-dimensional MIT bag model suggested in the 1970s by physicists (see, for instance, [5, 6, 7]). In dimension two, the Dirac operators with special boundary conditions similar to the MIT bag model are used in the description of graphene [8, 9], see also [10].

The rigorous mathematical investigation of the MIT bag model as a boundary value problem for the \( 3D \) Dirac operator in a domain \( \Omega \subset \mathbb{R}^3 \) with the special boundary conditions was started in the papers [8, 9, 10], where the authors apply modern techniques of extension theory of symmetric operators to the study of self-adjointness of the MIT bag model. We note also the very recent paper [11] devoted to the spectral theory of boundary value problems for the \( 3D \) Dirac operator with the MIT bag model boundary conditions and some of its generalizations. The approach of that paper is based on abstract boundary triple techniques from extension theory of symmetric operators and a thorough study of certain classes of (boundary) integral operators that appear in a Krein type resolvent formula. In the paper, the authors also study the relationship between the \( 3D \) Dirac operators with singular \( \delta \)-shell interactions and the MIT bag model.

It should be noted that there is an extensive literature devoted to Fredholm theory and to the index of boundary problems for multi-dimensional Dirac type operators in bounded domains (see, for instance, [12, 13, 14, 15, 16] and references cited there).
In the present paper we consider operators of boundary value problems for 3D Dirac operators with variable magnetic and electrostatic potentials, containing as a particular case the operators of MIT bag model (see, for instance, [?]), where \( D_{A,\Phi} \) is a 3D Dirac operator defined on the distributions \( u = (u_1, u_2, u_3, u_4) \in H^1(\Omega, \mathbb{C}^4) = H^1(\Omega) \otimes \mathbb{C}^4 \), where \( H^1(\Omega) \) is the Sobolev space on \( \Omega \) of order 1. We denote by \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 \) the Dirac matrices satisfying the relations

\[
\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}I_4, \quad i, j = 0, 1, 2, 3, 4,
\]

\( I_n \) is the \( n \times n \) unit matrix, \( A = (A_1, A_2, A_3) \in L^\infty(\mathbb{R}^3, \mathbb{C}^3) \) is the magnetic potential, \( \Phi \in L^\infty(\mathbb{R}^3) \) is the electrostatic potential, \( m \in \mathbb{R} \) is the mass of the particle. We use the system of coordinates for which the Planck constant \( h = 1 \), the speed of light is \( c = 1 \), and the charge of the particle \( e = 1 \).

We introduce boundary operators of the form

\[
((\mathfrak{B}u)(x') = b_1(x')u^1_{\partial\Omega}(x') + b_2(x')u^2_{\partial\Omega}(x'), \quad x' \in \partial\Omega,
\]

where

\[
u = (\nu_1, \nu_2, \nu_3)
\]

is the unit outward normal vector to \( \partial\Omega \). One can see that the rank of the matrix \( I_4 + i\alpha_0(\alpha \cdot \nu) \) is 2 and the boundary condition (??) is equivalent to the condition

\[
u^1 + (i\sigma \cdot \nu)\nu^2 = 0, \quad \sigma \cdot \nu = \sigma_1\nu_1 + \sigma_2\nu_2 + \sigma_3\nu_3
\]

of the form (??), where \( \sigma_1, \sigma_2, \sigma_3 \) are the \( 2 \times 2 \) Pauli matrices. Moreover, the boundary condition

\[
\theta(I_4 + i\alpha_0(\alpha \cdot \nu))u_{\partial\Omega} = (I_4 + i\alpha_0(\alpha \cdot \nu))\alpha_0u_{\partial\Omega}
\]

introduced in the paper [?] is equivalent to the condition

\[
(\theta - 1)\nu^1 + (\theta + 1)(\sigma \cdot \nu)\nu^2 = 0
\]

of the form (??).

We associate to the boundary value problem (??) the bounded operator

\[
\mathfrak{D}_{A,\Phi,\mathfrak{B}}u(x) = \begin{cases} \mathfrak{D}_{A,\Phi}u(x), & x \in \Omega, \\ \mathfrak{B}u_{\partial\Omega}(x') = b_1(x')u^1_{\partial\Omega}(x') + b_2(x')u^2_{\partial\Omega}(x') = 0, & x' \in \partial\Omega, \end{cases}
\]

acting from \( H^1(\Omega, \mathbb{C}^4) \) to \( L^2(\Omega, \mathbb{C}^4) \). We also associate with the operator \( \mathfrak{D}_{A,\Phi,\mathfrak{B}} \) the operator \( \mathfrak{D}_{A,\Phi,\mathfrak{B}} \), which is unbounded in \( L^2(\Omega, \mathbb{C}^4) \) and generated by the 3D Dirac operator \( \mathfrak{D}_{A,\Phi} \) with domain

\[
H_{\mathfrak{B}}^2(\Omega, \mathbb{C}^4) = \{ u \in H^1(\Omega, \mathbb{C}^4) : \mathfrak{B}u_{\partial\Omega} = 0 \text{ on } \partial\Omega \}.
\]
The paper has two principal aims: (1) to obtain conditions for the operator $\mathcal{D}_{A,\Phi,3}$ to be self-adjoint in $L^2(\Omega, C^1)$ for bounded and unbounded domains $\Omega$ and (2) to study the essential spectrum of $\mathcal{D}_{A,\Phi,3}$ for unbounded domains $\Omega$.

(1) Our approach to self-adjointness of the operators $\mathcal{D}_{A,\Phi,3}$ is based on the study of invertibility of the parameter-dependent boundary problems

$$\mathbb{D}_{A,\Phi,3}(i\mu) = \mathbb{D}_{A,\Phi,3} - i\mu I_d, \quad \mu \in \mathbb{R},$$

for large values $|\mu|$ of the parameter $\mu \in \mathbb{R}$. We introduce the local parameter-dependent Lopatinsky–Shapiro conditions for the operator $\mathbb{D}_{A,\Phi,3}(i\mu)$ and establish the following result.

- Let: (i) $\partial\Omega$ be a $C^2$-uniformly regular surface, (ii) let the parameter-dependent Lopatinsky–Shapiro conditions for $\mathbb{D}_{A,\Phi,3}(i\mu)$ be satisfied uniformly on $\partial\Omega$, (iii) let the unbounded operator $\mathcal{D}_{A,\Phi,3}$ with the domain $H^2_3(\Omega, C^1)$ be symmetric in $L^2(\Omega, C^1)$. Then $\mathcal{D}_{A,\Phi,3}$ is a self-adjoint operator in $L^2(\Omega, C^1)$.

Unlike the above-cited papers [?, ?, ?, ?], where the boundary problems were considered for domains with compact $C^2$-boundary, our approach enables us to prove the self-adjointness for a wide class of boundary value problems for 3D Dirac operators in domains with bounded and unbounded boundary; in particular, for the operators of the MIT bag models and their generalizations.

(2) We also study the Fredholm property of the operators $\mathbb{D}_{A,\Phi,3} : H^1(\Omega, C^1) \to L^2(\Omega, C^1)$ and the essential spectrum of the associated unbounded operators $\mathcal{D}_{A,\Phi,3}$ in unbounded domains in $\mathbb{R}^3$ with $C^2$-uniformly regular boundary applying the limit operator method [?, ?, ?, ?]. We consider the following cases:

- (a) $\Omega$ is the exterior of a bounded domain, that is, $\Omega = \mathbb{R}^3 \setminus \Omega'$, where $\Omega'$ is a bounded domain with a $C^2$-boundary $\partial \Omega$.

- (b) $\Omega$ is an unbounded domain with $C^2$-boundary which has a conic exit at infinity, i.e., $\Omega$ coincides with a conic set outside a ball $B_R = \{ x \in \mathbb{R}^3 : |x| < R \}$, $R > 0$.

Following the papers [?, ?], we define a family $\text{Lim}(\mathbb{D}_{A,\Phi,3})$ of limit operators $\mathbb{D}_{A,\Phi,3}^h$ for $\mathbb{D}_{A,\Phi,3}$ and obtain the following result.

- Let the standard Lopatinsky–Shapiro conditions for $\mathbb{D}_{A,\Phi,3}$ be satisfied uniformly on $\partial\Omega$. Then the essential spectrum $\text{sp}_{\text{ess}} \mathcal{D}_{A,\Phi,3}$ is defined by the formula

$$\text{sp}_{\text{ess}} \mathcal{D}_{A,\Phi,3} = \bigcup_{\mathbb{D}_{A,\Phi,3} \subset \text{Lim}(\mathbb{D}_{A,\Phi,3})} \text{sp} \mathbb{D}_{A,\Phi,3}^h,$$

where $\mathbb{D}_{A,\Phi,3}^h$ are the unbounded operators associated with $\mathbb{D}_{A,\Phi,3}$.

Note that, in case (a), the limit operators $\mathbb{D}_{A,\Phi,3}^h$ are Dirac operators on $\mathbb{R}^3$ and, in the case (b), the limit operators are operators of boundary value problems for half-spaces. Under some additional assumptions, the spectra of limit operators can be obtained in an explicit form, i.e., formula (??) gives a complete description of $\text{sp}_{\text{ess}} \mathcal{D}_{A,\Phi,3}$.

Let $\Omega = \mathbb{R}^3 \setminus \overline{\Omega}$ be the exterior of a bounded domain $\Omega'$ with $C^2$-boundary $\partial\Omega$. We assume that the potentials $A$ and $\Phi$ are real-valued and slowly oscillating at infinity (see Definition ??); then the essential spectrum of the operator $\mathcal{M}_{A,\Phi,3}$ of the MIT bag model defined by the Dirac operator $\mathcal{D}_{A,\Phi}$ with domain

$$\text{dom} \mathcal{M}_{A,\Phi,3} = \{ u \in H^1(\Omega, C^1) : \exists R \mathcal{M}u = u^1_{|\partial\Omega} + (i\sigma \cdot \nu) u^2_{|\partial\Omega} = 0 \text{ on } \partial\Omega \}$$

is given by

$$\text{sp}_{\text{ess}} \mathcal{M}_{A,\Phi,3} = (-\infty, M_{\Phi}^{\sup} - |m|) \bigcup \{|M_{\Phi}^{\inf} + |m|, +\infty\},$$

where

$$M_{\Phi}^{\sup} = \limsup_{x \to \infty} \Phi(x), \quad M_{\Phi}^{\inf} = \liminf_{x \to \infty} \Phi(x).$$

Thus, $\text{sp}_{\text{ess}} \mathcal{M}_{A,\Phi,3}$ is independent of the slowly oscillating magnetic potential $A$, and if $M_{\Phi}^{\sup} - M_{\Phi}^{\inf} < 2 |m|$, then

$$\text{sp}_{\text{dis}} \mathcal{M}_{A,\Phi,3} \subset (M_{\Phi}^{\sup} - |m|, M_{\Phi}^{\inf} + |m|)$$
and, if $M_{\Phi}^{\text{sup}} - M_{\Phi}^{\text{inf}} \geq 2|\mu|$, then

$$\text{sp} \cdot \mathcal{M}_{A, \Phi, \Omega} = \text{sp} \cdot \mathcal{M}_{A, \Phi, \Omega} = (-\infty, +\infty).$$

Let $\Omega$ be a domain with a conic exit to infinity and the potentials $A$ and $\Phi$ be real-valued and slowly oscillating at infinity; in this case, if $m > 0$, then the essential spectrum of $\mathcal{M}_{A, \Phi, \Omega}$ is given by formulas (14.6), (14.7) and, if $m \leq 0$, then the sp $\mathcal{M}_{A, \Phi, \Omega} = \text{sp} \cdot \mathcal{M}_{A, \Phi, \Omega} = (-\infty, +\infty)$.

Note that the approach based on the investigation of parameter-dependent operators and on limit operators was applied earlier for the investigation of the $\delta$–interaction problems for Schrödinger operators in [?, ?].

### 1.1. Notations

- If $X, Y$ are Banach spaces, then we denote by $\mathcal{B}(X, Y)$ the Banach space of bounded linear operators acting from $X$ into $Y$, and by $\mathcal{K}(X, Y)$ its subspace of compact operators. In the case of $X = Y$, we briefly write $\mathcal{B}(X)$.

- An operator $A \in \mathcal{B}(X, Y)$ is said to be a Fredholm operator if

$$\ker A = \{x \in X : A x = 0\} \quad \text{and} \quad \text{coker} A = Y/\text{Im} A$$

are finite-dimensional spaces.

- Let $\mathcal{A}$ be a closed unbounded operator in a Hilbert space $\mathcal{H}$ with domain $\text{dom} \mathcal{A}$ dense in $\mathcal{H}$. Then $\mathcal{A}$ is said to be a Fredholm operator if $\mathcal{A} : \text{dom} \mathcal{A} \to \mathcal{H}$ is a Fredholm operator as a bounded operator, where $\text{dom} \mathcal{A}$ is equipped by the graph norm

$$\|u\|_{\text{dom} \mathcal{A}} = \left(\|u\|_{\mathcal{H}}^2 + \|\mathcal{A} u\|_{\mathcal{H}}^2\right)^{1/2}$$

(see, for instance, [?]).

- The essential spectrum $\text{sp} \cdot \mathcal{A}$ of an unbounded operator $\mathcal{A}$ is the set of $\lambda \in \mathbb{C}$ such that $\mathcal{A} - \lambda I$ is not a Fredholm operator as an unbounded operator, and the discrete spectrum $\text{sp} \cdot \text{dis} \mathcal{A}$ of $\mathcal{A}$ is the set of isolated eigenvalues of finite multiplicity. As is well known, if $\mathcal{A}$ is a self-adjoint operator, then

$$\text{sp} \cdot \text{dis} \mathcal{A} = \text{sp} \mathcal{A} \setminus \text{sp} \cdot \mathcal{A}.$$

- We denote by $L^2(\Omega, C^d), L^2(\partial \Omega, C^d)$ the Hilbert spaces of $d$–dimensional vector-functions $u(x) = (u^1(x), \ldots, u^d(x))$, $x \in \Omega$ or $x \in \partial \Omega$, with the inner products

$$\langle u, v \rangle_{L^2(\Omega, C^d)} = \int_{\Omega} u(x) \cdot v(x) dx, \quad \langle \varphi, \psi \rangle_{L^2(\partial \Omega, C^d)} = \int_{\partial \Omega} \varphi(x') \cdot \psi(x') dx',$$

where $h \cdot g = \sum_{j=1}^d h_j g_j$.

- We denote by $H^s(\mathbb{R}^n, C^d)$ the Sobolev space of distributions $u \in \mathcal{D}'(\mathbb{R}^n, C^d) = \mathcal{D}'(\mathbb{R}^n) \otimes C^d$ such that

$$\|u\|_{H^s(\mathbb{R}^n, C^d)} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s \|\hat{u}(\xi)\|_{C^d}^2 d\xi\right)^{1/2} < \infty, \quad s \in \mathbb{R},$$

where $\hat{u}$ is the Fourier transform of $u$ in the sense of distributions. If $\Omega$ is a domain in $\mathbb{R}^d$, then $H^s(\Omega, C^d)$ is the space of restrictions of the distributions $u \in H^s(\mathbb{R}^n, C^d)$ on $\Omega$ with the norm

$$\|u\|_{H^s(\Omega, C^d)} = \inf_{l u \in H^s(\mathbb{R}^n, C^d)} \|l u\|_{H^s(\mathbb{R}^n, C^d)},$$

where $l u$ is an extension of $u$ to $\mathbb{R}^d$. If $s > 1/2$, then the distributions in $H^s(\Omega, C^d)$ have traces on $\partial \Omega$, and we denote by $H^{s-1/2}(\partial \Omega, C^d)$ the Sobolev space on $\partial \Omega$ consisting of these traces.

- We denote by $C_b(\mathbb{R}^n)$ the $C^*$–algebra of bounded continuous functions on $\mathbb{R}^n$, by $C_b^k(\mathbb{R}^n), k \in \mathbb{N}$, the sub-algebra of $C_b(\mathbb{R}^n)$ consisting of $k$–times differentiable functions $u$ such that $\partial^\alpha u \in C_b(\mathbb{R}^n)$ for every multi-index $\alpha : |\alpha| \leq k$. If $\Omega$ is an unbounded domain in $\mathbb{R}^n$, then $C_b(\Omega), C_b^k(\Omega), C_b(\partial \Omega), C_b^k(\partial \Omega)$ are algebras consisting of the restrictions of functions from $C_b(\mathbb{R}^n), C_b^k(\mathbb{R}^n)$ to $\Omega, \partial \Omega$, respectively.
We say that a domain $\Omega \subset \mathbb{R}^n$ has a $C^2$—uniformly regular boundary $\partial \Omega$ ([25, 28]) if $\partial \Omega$ is a $C^2$—hypersurface, and, (i) for fixed $r > 0$ and for every point $x_0 \in \partial \Omega$, there exist a ball $B_r(x_0) = \{ x \in \mathbb{R}^3 : |x - x_0| < r \}$ and a homeomorphism $\varphi_{x_0} : B_r(x_0) \rightarrow B_1(0)$ such that
\[
\varphi_{x_0}(B_r(x_0) \cap \Omega) = B_1(0) \cap \mathbb{R}^n, \quad \mathbb{R}^n_+ = \{ y \in \mathbb{R}^n : y_n > 0 \},
\]
\[
\varphi_{x_0}(B_r(x_0) \cap \partial \Omega) = B_1(0) \cap \mathbb{R}^{n-1}, \quad \mathbb{R}^{n-1} = \{ y \in \mathbb{R}^n : y_n = 0 \};
\]
(ii) if $\varphi_{x_0}^i, i = 1, \ldots, n$, are the coordinate functions of the mappings $\varphi_{x_0}$, then
\[
\sup_{x_0 \in \partial \Omega} \sup_{|x| \leq 2, x \in B_r(x_0)} \left| \partial_{x}^i \varphi_{x_0}(x) \right| < \infty, \quad i = 1, \ldots, n.
\]

Note that, if $\Omega$ is a bounded domain with $C^2$—boundary $\partial \Omega$, then $\partial \Omega$ is a uniformly regular surface.

1.2. Free Dirac Operators

We denote by
\[
\mathcal{D} = \alpha \cdot i\mathbf{\nabla} + \alpha_0 m = \sum_{j=1}^{3} \alpha_j D_{x_j} + \alpha_0 m, D_{x_j} = i\partial_{x_j}
\]
the free Dirac operator (see, for instance, [?]), where $\alpha_j, j = 0, 1, 2, 3$, are the $4 \times 4$ Dirac matrices
\[
\alpha_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3, \tag{15}
\]
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{16}
\]
are the $2 \times 2$ Pauli matrices, which satisfy the relations
\[
\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk}I_2, \quad j, k = 1, 2, 3. \tag{17}
\]
Relations (??) imply
\[
\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk}I_4; \quad j, k = 0, 1, 2, 3, \tag{18}
\]
where $I_n$ is the $n \times n$ identity matrix. From equation (??) it follows that
\[
(\alpha \cdot i\mathbf{\nabla})^2 = -\Delta I_4,
\]
where $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$ is the 3D Laplacian. Moreover,
\[
\mathcal{D}^2 = (-\Delta + m^2) I_4.
\]
As is well-known, the unbounded operator $\mathcal{D}$ in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ generated by the free Dirac operator $\mathcal{D}$ with domain $H^1(\mathbb{R}^3, \mathbb{C}^4)$ is self-adjoint and
\[
\text{sp } \mathcal{D} = \text{sp}_{\text{ess}} \mathcal{D} = (-\infty, -|m|] \bigcup [ |m|, +\infty)
\]
(see, for instance, [?]).

2. PARAMETER-DEPENDENT BOUNDARY PROBLEMS FOR 3D DIRAC OPERATORS

2.1. Lopatinsky-Shapiro Conditions

We consider the parameter-dependent operator of the boundary value problem
\[
\mathcal{D}_{A, \Phi, \mathbf{b}}(i\mu) \mathbf{u} = \begin{cases} \mathcal{D}_{A, \Phi} - i\mu I_4 \mathbf{u} = \mathcal{D}_{A, \Phi}(i\mu) \mathbf{u} = \mathbf{f} & \text{on } \Omega, \quad \mu \in \mathbb{R}, \\
\mathbb{B} \mathbf{u} = b_1 u_{\partial \Omega} + b_2 u_{\partial \Omega}^2 = 0 & \text{on } \partial \Omega,
\end{cases} \tag{19}
\]
where $\Omega \subset \mathbb{R}^3$ is a domain with $C^2$—uniformly regular boundary and
\[
A \in L^\infty(\Omega, \mathbb{C}^4), \quad \Phi \in L^\infty(\Omega), \quad b_k = \left( \frac{b_{ij}}{\lambda} \right)_{i,j=1}^2, \quad b_{ij} \in C_b(\partial \Omega), \quad k = 1, 2. \tag{20}
\]
and \( \mathbf{u} = (u^1, u^2) \in H^1(\Omega, C^4) = H^1(\Omega, C^2) \oplus H^1(\Omega, C^2), u^j_\partial \in H^{1/2}(\partial \Omega, C^2) \) are the boundary values of the vector-functions \( u^j \) on \( \partial \Omega, j = 1, 2 \). We study the invertibility of the operator \( D_{A, \Phi, \mathcal{B}}(i\mu) : H^1(\Omega, C^4) \to L^2(\Omega, C^4), \mu \in \mathbb{R} \), for large values of \( |\mu| \). We follow the well-known paper \([7]\), where the authors consider the invertibility of general parameter-dependent boundary value problems in smooth bounded domains.

We construct the locally inverse operators at every chosen point \( x \in \Omega \), and then obtain the globally inverse operator for large values of \( |\mu| \) by gluing together the locally inverse operators by means of a countable partition of unity of finite multiplicity.

We need the Sobolev spaces \( H^s_\mu(\mathbb{R}^3, C^4) \) of distributions \( \mathbf{u} \in H^s(\mathbb{R}^3, C^4) \) with the norm depending on the parameter \( \mu \in \mathbb{R} \),

\[
\| \mathbf{u} \|_{H^s_\mu(\mathbb{R}^3, C^4)} = \left( \int_{\mathbb{R}^3} (1 + |\xi|^2 + \mu^2)^s |\hat{\mathbf{u}}(\xi)|^2 d\xi \right)^{1/2} < \infty, \quad s \geq 0, \quad \mu \in \mathbb{R},
\]

where

\[
\hat{\mathbf{u}}(\xi) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} \mathbf{u}(x) dx
\]

is the Fourier transform. We denote by \( H^s(\Omega, C^4) \) the space of the restrictions to the domain \( \Omega \) of the distributions \( \mathbf{u} \in H^s_\mu(\mathbb{R}^3, C^4) \) with the norm

\[
\| \mathbf{u} \|_{H^s(\Omega, C^4)} = \inf_{l \in H^s_\mu(\mathbb{R}^3, C^4)} \| l \mathbf{u} \|_{H^s_\mu(\mathbb{R}^3, C^4)},
\]

where \( l \mathbf{u} \) is the extension of \( \mathbf{u} \in H^s(\Omega, C^4) \) on \( \mathbb{R}^3 \). Note that

\[
\| \mathbf{u} \|_{H^s_{\mu}(\Omega, C^4)} \leq (1 + \mu^2)^{\frac{s}{2}} \| \mathbf{u} \|_{H^s_{\mu}(\mathbb{R}^3, C^4)}, \quad r \geq s, \quad \mu \in \mathbb{R}, \tag{21}
\]

and the trace operator \( \gamma_\partial \) is bounded as a mapping from \( H^r_{\mu}(\Omega, C^4) \) to \( H^{r/2}_{\mu}(\partial \Omega, C^4) \) if \( \partial \Omega \) is a \( C^4 \) surface, \( l > s \).

We regard \( D_{A, \Phi, \mathcal{B}} \) as an operator acting from \( H^1_{\mu}(\Omega, C^4) \) to \( L^2(\Omega, C^4) \).

Let \( D(\mu) = \alpha \cdot \nabla - i\mu I_4 \) be the main part of the parameter-dependent operator \( D_{A, \Phi}(i\mu) = D_{A, \Phi} - i\mu I_4 \). Since

\[
(\alpha \cdot \xi + i\mu I_4)(\alpha \cdot \xi - i\mu I_4) = \left( |\xi|^2 + \mu^2 \right) I_4,
\]

it follows that \( D_{A, \Phi} - i\mu I_4 \) is a uniformly elliptic operator with parameter \( \mu \in \mathbb{R} \) (see, for instance, \([7, 7]\)).

For a point \( x_0 \in \partial \Omega \), we choose a local system of orthogonal coordinates \( y = (y_1, y_2, y_3) \), where \( y' = (y_1, y_2) \) belongs to the tangent plane to \( \partial \Omega \) at the point \( x_0 \) and the axis \( y_3 = z \) is directed along the inward pointing normal vector \( \nu_{x_0} \) to \( \partial \Omega \) at the point \( x_0 \), and assign to the point \( x_0 \in \partial \Omega \) the following family of 1D boundary value problems on the half-line \( \mathbb{R}^+ = \{ z \in \mathbb{R} : z > 0 \} \),

\[
\hat{\mathcal{D}}_{\mathcal{B}_{x_0}}(\xi', \mu) \psi(z) = \left\{ \begin{array}{ll}
(\alpha' \cdot \xi' + i\alpha_3 \frac{d}{dz} + i\mu I_4) \psi(z), & z \in \mathbb{R}^+, \\
(\alpha' \cdot \xi' - i\mu I_4) \psi(0) = b^1(x_0) \psi^1(+0) + b^2(x_0) \psi^2(+0) = 0,
\end{array} \right. \tag{22}
\]

acting from \( H^1(\mathbb{R}^+, C^4) \) to \( L^2(\mathbb{R}^+, C^4) \), where \( \psi^j(+0) = \lim_{z \to +0} \psi^j(z), j = 1, 2 \).

We are looking for exponentially decreasing solutions of the equation

\[
\hat{\mathcal{D}}_{\mathcal{B}_{x_0}}(\xi', \mu) \psi = 0 \quad \text{on} \quad \mathbb{R}^+. \tag{23}
\]

Note that

\[
(\alpha' \cdot \xi' + i\alpha_3 \frac{d}{dz} + i\mu I_4) (\alpha' \cdot \xi' + i\alpha_3 \frac{d}{dz} - i\mu I_4) = \left( |\xi'|^2 + \mu^2 - \frac{d^2}{dz^2} \right) I_4. \tag{24}
\]

Hence, the equation

\[
(\alpha' \cdot \xi' + i\alpha_3 \frac{d}{dz} - i\mu I_4) \psi(z) = 0 \tag{25}
\]

has exponentially decreasing solutions on \( \mathbb{R}^+ \) of the form

\[
\psi(z) = h(\xi', \mu) e^{-\rho z}, \quad \rho = \sqrt{|\xi'|^2 + \mu^2} > 0, \quad z > 0, \quad (\xi', \mu) \in \mathbb{R}^3,
\]

where the vector \( h(\xi', \mu) \in C^4 \) satisfies the equation

\[
(\alpha' \cdot \xi' - i\alpha_3 \rho - i\mu I_4) h = 0. \tag{27}
\]
The general solution of equation (2.50) has the form
\[ h(\xi', \mu) = \Theta(\xi', \mu) f, \] (28)
where \( f \in \mathbb{C}^4 \) is an arbitrary vector and
\[
\Theta(\xi', \mu) = \alpha' \cdot \xi' + i\rho \sigma_3 + i\mu I_4 = \begin{pmatrix} i\mu I_2 & \Lambda(\xi', \mu) \\ \Lambda(\xi', \mu) & i\mu I_2 \end{pmatrix},
\]
\[
\Lambda = \Lambda(\xi', \mu) = \sigma' \cdot \xi' - i\rho \sigma_3 = \begin{pmatrix} -i\rho & \zeta \\ \zeta & i\rho \end{pmatrix}, \quad \zeta = \xi_1 + i\xi_2,
\]
(29)
\[
\sigma' \cdot \xi' = \sigma_1 \xi_1 + \sigma_2 \xi_2.
\]
(30)

We set
\[
e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad h_1 = h_1(\xi', \mu) = \Theta(\xi', \mu) \begin{pmatrix} e_1 \\ 0 \end{pmatrix} = \begin{pmatrix} i\mu e_1 \\ \Lambda(\xi', \mu) e_1 \end{pmatrix},
\]
\[
h_2 = h_2(\xi', \mu) = \Theta(\xi', \mu) \begin{pmatrix} 0 \\ e_2 \end{pmatrix} = \begin{pmatrix} \Lambda(\xi', \mu) e_2 \\ i\mu e_2 \end{pmatrix},
\]
(32)
where
\[
\Lambda(\xi', \mu) e_1 = \begin{pmatrix} -i\rho \\ \zeta \end{pmatrix}, \quad \Lambda(\xi', \mu) e_2 = \begin{pmatrix} \zeta \\ i\rho \end{pmatrix}.
\]

The vectors \( h_1(\xi', \mu), h_2(\xi', \mu) \) are orthogonal and satisfy equation (2.50). Hence,
\[
\{ h_1(\xi', \mu) e^{-\rho z}, h_2(\xi', \mu) e^{-\rho z} \}
\]
is a fundamental system of solutions of equation (2.50) in \( L^2(\mathbb{R}_+, \mathbb{C}^4) \). Then every solution \( u \in L^2(\mathbb{R}_+, \mathbb{C}^4) \) of equation (2.50) on \( \mathbb{R}_+ \) is of the form
\[ \psi(z) = C_1 h_1 e^{-\rho z} + C_2 h_2 e^{-\rho z}, \quad \rho > 0, \]
(33)
where \( C_1, C_2 \in \mathbb{C} \). Substituting \( \psi \) into the boundary condition
\[ b_1(x_0) \psi^1(+0) + b(x_0) \psi^2(+0) = 0, \]
we obtain a system of linear equations
\[
(b_1(x_0) h_1^1 + b_2(x_0) h_1^2) C_1 + (b_1(x_0) h_2^1 + b_2(x_0) h_2^2) C_2 = 0
\]
(34)
with respect to \( C_1, C_2 \). Let \( \mathcal{L}(x_0, \xi', \mu) = (b_1(x_0, \xi', \mu), b_2(x_0, \xi', \mu)) \) be the matrix with columns
\[
b_1(x_0, \xi', \mu) = b_1(x_0) h_1^1 (\xi', \mu) + b_2(x_0) h_1^2 (\xi', \mu),
\]
\[
b_2(x_0, \xi', \mu) = b_1(x_0) h_2^1 (\xi', \mu) + b_2(x_0) h_2^2 (\xi', \mu).
\]
(35)
System (3.5) has the trivial solution if and only if
\[
\det \mathcal{L}(x_0, \xi', \mu) \neq 0.
\]

**Definition 1.** (i) We say that the operator \( \mathbb{D}_{A, \Phi, \Xi}(i\mu) \) satisfies the local parameter-dependent Lopatinsky–Shapiro condition at \( x_0 \in \partial \Omega \) if
\[
\det \mathcal{L}(x_0, \xi', \mu) \neq 0 \quad \text{for every} \ (\xi', \mu) : |\xi'|^2 + \mu^2 = 1.
\]
(36)

(ii) We say that the operator \( \mathbb{D}_{A, \Phi, \Xi}(i\mu) \) satisfies the uniform parameter-dependent Lopatinsky–Shapiro condition if
\[
\inf_{x \in \partial \Omega, \mu^2 + |\xi'|^2 = 1} |\det \mathcal{L}(x, \xi', \mu)| > 0.
\]
(37)

Note that if the boundary \( \partial \Omega \) is a compact set and the local parameter-dependent Lopatinsky–Shapiro condition (3.5) is satisfied at every point \( x \in \partial \Omega \), then the uniform Lopatinsky–Shapiro condition (3.5) holds.
2.2. Standard Lopatinsky–Shapiro Condition

We introduce the standard Lopatinsky–Shapiro condition for the operator $D_{A,\Phi,B}$ for domains $\Omega \subset \mathbb{R}^3$ with a $C^2$-boundary. As above, for a chosen point $x_0 \in \partial \Omega$, we introduce the local system of coordinates $y = (y_1, y_2, y_3)$, where $y_1, y_2 \in \mathbb{T}_{x_0}$ (the tangent plane to $\partial \Omega$ at the point $x_0$) and the axis $y_3 = z$ is directed along the outward pointing normal vector $\nu$ to $\partial \Omega$ at the point $x_0$.

Let

\[
\psi_1(\xi', z) = h_1(\xi')e^{-|\xi'|z}, \quad \psi_2(\xi', z) = h_2(\xi')e^{-|\xi'|z}, \quad z > 0,
\]

where

\[
h_1(\xi') = \begin{pmatrix} \Lambda(\xi')e_1 \\ 0 \end{pmatrix}, \quad h_2(\xi') = \begin{pmatrix} 0 \\ \Lambda(\xi')e_2 \end{pmatrix}.
\]

We introduce the $2 \times 2$ matrix

\[
\mathcal{L}(x_0, \xi') = (b_1(x_0)\Lambda(\xi')e_1, b_2(x_0)\Lambda(\xi')e_2),
\]

which coincides with the matrix $\mathcal{L}(x_0, \xi', 0)$.

**Definition 2.** We say that the operator $D_{A,\Phi,B}$ satisfies the local standard Lopatinsky–Shapiro condition at the point $x_0 \in \partial \Omega$ if

\[
\det \mathcal{L}(x_0, \xi') \neq 0 \quad \text{for all} \quad \xi' \in S^1
\]

and the operator $D_{A,\Phi,B}$ satisfies the standard Lopatinsky–Shapiro conditions uniformly if

\[
\inf_{x_0 \in \partial \Omega, \xi' \in S^1} |\det \mathcal{L}(x_0, \xi')| > 0.
\]

2.3. Invertibility of $D_{A,\Phi,B}(i\mu)$ for Large Values of $|\mu|$

**Theorem 3.** Let $\Omega \subset \mathbb{R}^3$ be a domain with a $C^2$–uniformly regular boundary $\partial \Omega$, let (the magnetic potential) $A = (A_1, A_2, A_3) \in L^\infty(\Omega, \mathbb{C}^3)$, let (the electrostatic potential) $\Phi \in L^\infty(\Omega)$, $b_j \in C_0(\partial \Omega) \otimes \mathcal{B}(\mathbb{C}^3), j = 1, 2$, and let the uniform parameter-dependent Lopatinsky–Shapiro condition (5) for $D_{A,\Phi,B}(i\mu), \mu \in \mathbb{R}$, be satisfied. Then there exists $\mu_0 > 0$ such that the operator $D_{A,\Phi,B}(i\mu) : H^1(\Omega, \mathbb{C}^4) \to L^2(\Omega, \mathbb{C}^4)$ is invertible for every $\mu \in \mathbb{R}, |\mu| > \mu_0$.

**Proof.** The idea of the proof is similar to that of the paper [2] (see also [3, Sec. 3]). However, we consider the parameter-dependent boundary problems for unbounded domains, and therefore, we need an infinite partition of unity and estimates associated with it.

Since the Dirac operator $D_{A,\Phi}(i\mu)$ is a uniformly elliptic parameter-depending operator on $\Omega$, the uniform Lopatinsky–Shapiro condition (5) holds, and $\partial \Omega$ is a $C^2$–uniformly regular surface, it follows that there exist $r > 0$ and $\mu_0 > 0$ such that, for every point $x_0 \in \Omega$, there exist operators

\[
L_{x_0}(\mu), R_{x_0}(\mu) \in \mathcal{B}(L^2(\Omega, \mathbb{C}^4), H^1(\Omega, \mathbb{C}^4))
\]

such that

\[
\sup_{x_0 \in \Omega, |\mu| \geq \mu_0} \|L_{x_0}(\mu)\|_{\mathcal{B}(L^2(\Omega, \mathbb{C}^4), H^1(\Omega, \mathbb{C}^4))} = d_L < \infty, \quad \sup_{x_0 \in \Omega, |\mu| \geq \mu_0} \|R_{x_0}(\mu)\|_{\mathcal{B}(L^2(\Omega, \mathbb{C}^4), H^1(\Omega, \mathbb{C}^4))} = d_R < \infty,
\]

and for every function $\varphi \in C_0^\infty(B_r(x_0))$,

\[
L_{x_0}(\mu)D_{A,\Phi,B}(i\mu)\varphi I = \varphi I, \quad \varphi D_{A,\Phi,B}(i\mu)R_{x_0}(\mu) = \varphi I.
\]

From the covering $\bigcup_{j \in \mathbb{N}} B_r(x_0) \supset \Omega$, we choose a countable subcovering $\bigcup_{j \in \mathbb{N}} B_r(x_j) \supset \Omega$ of a finite multiplicity $N \in \mathbb{N}$ and construct a partition of unity

\[
\sum_{j \in \mathbb{N}} \theta_j(x) = 1, \quad x \in \Omega,
\]

subordinated to the covering $\bigcup_{j \in \mathbb{N}} B_r(x_j)$, with $\theta_j \in C_0^\infty(B_r(x_j))$, $0 \leq \theta_j(x) \leq 1$, such that, for every $x \in \Omega$, the sum $\sum_{j \in \mathbb{N}} \theta_j(x)$ contains at most $N$ nonzero terms. Let $\varphi_j \in C_0^\infty(B_r(x_j)), 0 \leq \varphi_j(x) \leq 1$, and $\theta_j \varphi_j = \theta_j$. We set

\[
L(\mu)f = \sum_{j \in \mathbb{N}} \theta_j L_{x_j}(\mu)\varphi_j f, \quad f \in C_0^\infty(\Omega, \mathbb{C}^4),
\]

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where $C_0^\infty(\bar{\Omega}, \mathbb{C}^4)$ is the space of restrictions of functions in $C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ to $\bar{\Omega}$. Taking into account the fact that the covering \{B_r(x_j)\}$_{j \in \mathbb{N}}$ has finite multiplicity $N$, we obtain the estimates

$$
\|L(\mu) f\|_{L_2^1(\Omega, \mathbb{C}^4)} \leq C \sup_{j \in \mathbb{N}} \|L_j(\mu)\| \|f\|_{L_2(\bar{\Omega}, \mathbb{C}^4)} \leq C d_L \|f\|_{L_2(\bar{\Omega}, \mathbb{C}^4)},
$$
\(\text{for every } |\mu| \geq \mu_0 > 0 \text{ and } f \in C_0^\infty(\bar{\Omega}, \mathbb{C}^4) \) with a constant $C > 0$ independent of $f$. Estimates (47), (48) imply that the operators $L(\mu), R(\mu)$ can be continued to bounded operators acting from $L^2(\Omega, \mathbb{C}^4)$ to $H^1_\mu(\Omega, \mathbb{C}^4)$. Let $\psi_j \in C_0^\infty(B_r(x_j)), 0 \leq |\psi_j(x)| \leq 1 \varphi_j, \psi_j \in C_0^\infty(B_r(x_j)), 0 \leq |\psi_j(x)| \leq 1$, satisfy $\varphi_j \psi_j = \varphi_j$. Then

$$
L(\mu)D_{A, \phi, \mathbb{B}}(i\mu) = \sum_{j \in \mathbb{N}} \varphi_j D_{x_j}(i\mu)\varphi_j D_{A, \phi, \mathbb{B}}(i\mu)\psi_j I = I + T_1(\mu),
$$
where

$$
T_1(\mu) = \sum_{j \in \mathbb{N}} \varphi_j D_{x_j}(i\mu)\varphi_j D_{A, \phi, \mathbb{B}}(i\mu)\psi_j I, \quad \text{and} \quad D_{A, \phi, \mathbb{B}}(i\mu)\varphi_j I = D_{A, \phi, \mathbb{B}}(i\mu)\varphi_j I - \varphi_j D_{A, \phi, \mathbb{B}}(i\mu).
$$

Applying estimate (47), we obtain

$$
\|[D_{A, \phi, \mathbb{B}}(i\mu)\varphi_j I]\|_{\mathcal{B}(H^1_\mu(\Omega, \mathbb{C}^4), L_2^1(\bar{\Omega}, \mathbb{C}^4))} \leq C \frac{C}{|\mu|^{N/2}}, \quad |\mu| \geq \mu_0,
$$
with a constant $C > 0$ independent of $j \in \mathbb{N}$. Applying the finite multiplicity of the covering \{B_r(x_j)\}$_{j \in \mathbb{N}}$ again, we obtain from (47) and (48) the estimate

$$
\|T_1(\mu)\|_{\mathcal{B}(H^1_\mu(\Omega, \mathbb{C}^4), H^1_\mu(\Omega, \mathbb{C}^4))} \leq C \frac{C}{|\mu|^{N/2}}, \quad |\mu| \geq \mu_0.
$$

Hence, there exists $\mu_1 \geq \mu_0$ such that

$$
\sup_{|\mu| \geq \mu_1} \|T_1(\mu)\|_{\mathcal{B}(H^1_\mu(\Omega, \mathbb{C}^4), H^1_\mu(\Omega, \mathbb{C}^4))} < 1.
$$

Thus, the operator $D_{A, \phi, \mathbb{B}}(i\mu)$ has the left inverse operator $\mathbb{L}(\mu) = (I + T_1(\mu))^{-1} L(\mu)$ for every $\mu \in \mathbb{R}$, $\mu \geq \mu_1$. In the same way, we can prove that there exists a right inverse operator $\mathbb{R}(\mu)$ of $D_{A, \phi, \mathbb{B}}(i\mu)$ for $|\mu| \geq \mu_2 > \mu_1$. Hence, the operator $\mathbb{D}_{A, \phi, \mathbb{B}}(i\mu) : H^1_\mu(\Omega, \mathbb{C}^4) \to L^2(\Omega, \mathbb{C}^4)$ is invertible for every $|\mu| \geq \mu_2$. \qed

**Corollary 4.** Let the assumptions of Theorem 27 be satisfied. Then there exists a $\tilde{\mu} > 0$ such that the operator

$$
\mathbb{D}_{A, \phi, \mathbb{B}}(i\mu) : H^1(\Omega, \mathbb{C}^4) \to L^2(\Omega, \mathbb{C}^4)
$$

is invertible for every $\mu \in \mathbb{R}, |\mu| \geq \tilde{\mu}$.

**Proof.** For every chosen $\mu \in \mathbb{R}$, the norm in the space $H^1_\mu(\Omega, \mathbb{C}^4)$ is equivalent to the norm in the usual Sobolev spaces $H^1(\Omega, \mathbb{C}^4)$ without the parameter $\mu$. This implies the invertibility of the operator $\mathbb{D}_{A, \phi, \mathbb{B}}(i\mu) : H^1(\Omega, \mathbb{C}^4) \to L^2(\Omega, \mathbb{C}^4)$ for every $\mu : |\mu| \geq \tilde{\mu}$. \qed

### 3. Self-Adjointness of the Unbounded Operator $\mathbb{D}_{A, \phi, \mathbb{B}}$

Now we consider the self-adjointness in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ of the unbounded operators $\mathbb{D}_{A, \phi, \mathbb{B}}$ associated with the operator $D_{A, \phi, \mathbb{B}}$ defined by the Dirac operator

$$
\mathbb{D}_{A, \phi} = \alpha \cdot (i\nabla + A) + \alpha_0 m + \Phi I_4,
$$
where $A \in L^\infty(\Omega, \mathbb{C}^4)$ and $\Phi \in L^\infty(\Omega)$, with domain

$$
H^1_{B_1}(\Omega, \mathbb{C}^4) = \left\{ u \in H^1(\Omega, \mathbb{C}^4) : \mathbb{B}u(x') = b_1(x')u'_{\partial \Omega}(x') + b_2(x')u^2_{\partial \Omega}(x') = 0, \quad x' \in \partial \Omega, b_j \in C_0(\partial \Omega, \mathbb{R}^2) \right\}.
$$

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\textbf{Theorem 5.} Let (i) $\Omega \subset \mathbb{R}^3$ be a domain with $C^2$-uniformly regular boundary, (ii) the vector-valued potential $\mathbf{A} \in L^{\infty}(\Omega, \mathbb{R}^4)$ and the electrostatic potential $\Phi \in L^{\infty}(\Omega)$ be real-valued, (iii) the uniformly parameter-dependent Lopatinsky-Shapiro condition

$$\inf_{x' \in \partial \Omega, |\xi'|^2 + \mu^2 = 1} |\det \mathcal{L}(x', \xi', \mu)| > 0$$

hold; (iv) the coefficients $b_j \in C_b(\partial \Omega) \otimes \mathcal{B}(\mathbb{C}^2), j = 1, 2$ be such that the operator $\mathcal{D}_{\mathbf{A}, \Phi, \mathbb{B}}$ is symmetric in $L^2(\Omega, \mathbb{C}^4)$.

Then the operator $\mathcal{D}_{\mathbf{A}, \Phi, \mathbb{B}}$ is self-adjoint in $L^2(\Omega, \mathbb{C}^4)$.

\textbf{Proof.} Corollary ?? implies that there exists a $|\mu|$ large enough and a constant $C > 0$ such that, for every $u \in H^1_b(\Omega, \mathbb{C}^4)$,

$$\|u\|_{H^1(\Omega, \mathbb{C}^4)} \leq C \left( \|\mathcal{D}_{\mathbf{A}, \Phi} u\|_{L^2(\Omega, \mathbb{C}^4)} + |\mu| \|u\|_{L^2(\Omega, \mathbb{C}^4)} \right),$$

(54)

It follows from the a priori estimate (??) that the operator $\mathcal{D}_{\mathbf{A}, \Phi, \mathbb{B}}$ is closed. Moreover, it follows from Corollary ?? that the deficiency indices of $\mathcal{D}_{\mathbf{A}, \Phi, \mathbb{B}}$ are equal 0. Hence (see, for instance, [7], p. 100), the operator $\mathcal{D}_{\mathbf{A}, \Phi, \mathbb{B}}$ is self-adjoint.

\textbf{Corollary 6.} Let

$$H^1_b(\Omega, \mathbb{C}^4) = \{u \in H^1(\Omega, \mathbb{C}^4) : \mathcal{B}u(x') = u_{\partial \Omega}(x') + b(x')u_{\partial \Omega}^2(x') = 0, x' \in \partial \Omega\},$$

where $b_j \in C_b(\partial \Omega) \otimes \mathcal{B}(\mathbb{C}^2)$ and

$$b^*(\sigma \cdot \nu) + (\sigma \cdot \nu) b = 0 \quad \text{on} \quad \partial \Omega,$$

(55)

where $\nu$ is the outward pointing unit normal vector to $\partial \Omega$. Then the operator $\mathcal{D}_{\mathbf{A}, \Phi, \mathbb{B}}$ is symmetric. Hence, if conditions (i), (ii), (iii) of Theorem ?? hold, then the operator $\mathcal{D}_{\mathbf{A}, \Phi, \mathbb{B}}$ is self-adjoint in $L^2(\Omega, \mathbb{C}^4)$.

\textbf{Proof.} Integrating by parts, we obtain

$$\langle \mathcal{D}_{\mathbf{A}, \Phi} u, v \rangle_{L^2(\Omega, \mathbb{C}^4)} - \langle u, \mathcal{D}_{\mathbf{A}, \Phi} v \rangle_{L^2(\Omega, \mathbb{C}^4)} = \langle (-i \alpha \cdot \nu) u_{\partial \Omega}, v_{\partial \Omega} \rangle_{L^2(\partial \Omega, \mathbb{C}^4)},$$

$$u, v \in H^1_b(\Omega, \mathbb{C}^4).$$

Taking into account (??), we see that

$$(\alpha \cdot \nu) u_{\partial \Omega} \cdot v_{\partial \Omega} = -i (\sigma \cdot \nu) u_{\partial \Omega}^2 \cdot v_{\partial \Omega} - i (\sigma \cdot \nu) u_{\partial \Omega}^1 \cdot v_{\partial \Omega}^2 = b^* i (\sigma \cdot \nu) + i (\sigma \cdot \nu) b) u_{\partial \Omega}^1 \cdot v_{\partial \Omega}^2 = 0.$$ (56)

Hence, $\mathcal{D}_{\mathbf{A}, \Phi, \mathbb{B}}$ is a symmetric operator and, by Theorem ??, $\mathcal{D}_{\mathbf{A}, \Phi, \mathbb{B}}$ is self-adjoint.

\textbf{3.0.1. Self-adjointness of the generalized MIT bag model.} We consider the operator of generalized MIT bag model

$$\mathcal{M}_{\mathbf{A}, \Phi, \mathbb{B}} u(x) = \begin{cases} \mathcal{D}_{\mathbf{A}, \Phi} u(x), & x \in \Omega, \\ \mathfrak{M}_a(x') u_{\partial \Omega}(x') = u_{\partial \Omega}^1(x') + i a(x') (\sigma \cdot \nu) u_{\partial \Omega}^2(x'), & x' \in \partial \Omega, \end{cases}$$

(57)

where $a \in C_b(\partial \Omega)$ is a real-value function. Note that, if $a = 1$, then we obtain the boundary condition of the MIT bag model (see [7, ? , ? , ?]). Note that the boundary condition given in the paper [7],

$$\theta (I_4 + ia_0(\alpha \cdot \nu)) u_{\partial \Omega} = (I_4 + ia_0(\alpha \cdot \nu)) a_0 u_{\partial \Omega}$$

(58)

with $\theta \in C_b(\partial \Omega)$, can be written as

$$\begin{cases} (\theta - 1) u^1 + (\theta + 1) i (\sigma \cdot \nu) u^2 = 0, \\ - (\theta - 1) i (\sigma \cdot \nu) u^1 + (\theta + 1) u^2 = 0. \end{cases}$$

(59)

The boundary condition (??) is equivalent to the condition $\mathfrak{M}_a(x') u_{\partial \Omega}(x') = 0, x' \in \partial \Omega$, where

$$a = \frac{\theta + 1}{\theta - 1} \in C_b(\partial \Omega) \quad \text{if} \quad \inf_{x \in \partial \Omega} |\theta(x) - 1| > 0.$$

(60)
Note that the matrix \( b = ia (\sigma \cdot \nu) \) satisfies condition (??). Hence, the unbounded operator \( \mathcal{M}_{A,\Phi,\mathfrak{M}} \) associated with \( M_{A,\Phi,\mathfrak{M}} \) is symmetric.

We now consider the uniform parameter-dependent Lopatinsky–Shapiro condition for the operator \( \mathcal{D}_{A,\Phi,\mathfrak{M}}(i\mu) \). Applying formulas (??), we obtain

\[
\mathcal{L}(x, \xi', \mu) = (h_1^1 + ia\sigma_3 h_2^1, h_2^1 + ia\sigma_3 h_2^2) = (i\mu e_1 + ia\sigma_3 \Lambda e_1, \Lambda e_2 + ia\sigma_3 \mu e_2).
\]

Formulas (??) and (??) yield

\[
\det \mathcal{L}(x, \xi', \mu) = \det \begin{pmatrix} i\mu + a(x)\rho & \xi \\ -ia(x)\zeta & i\rho + a(x)\mu \end{pmatrix} = \mu \rho (a^2(x) - 1) + 2ia(x)\rho^2, \quad x \in \partial \Omega.
\]

This implies that the parameter-dependent Lopatinsky–Shapiro condition is satisfied on \( \partial \Omega \) uniformly if

\[
\inf_{x \in \partial \Omega} |a(x)| > 0.
\]

Thus, Theorem 5 yields the following result.

**Theorem 7.** Let \( \Omega \subset \mathbb{R}^3 \) be a domain with \( C^2 \)-uniformly regular boundary, let \( A_j \in \mathbb{L}^\infty(\Omega), j = 1, 2, 3 \), \( \Phi \in \mathbb{L}^\infty(\Omega), a \in C_b(\partial \Omega) \) be real-valued functions, and let condition (??) hold. Then the unbounded operator \( \mathcal{D}_{A,\Phi,\mathfrak{M},\mathfrak{M}} \) is self-adjoint in \( \mathcal{L}^2(\Omega, \mathbb{C}^3) \).

**Corollary 8.** Theorem 7 implies that the operator \( \mathcal{D}_{A,\Phi,\mathfrak{M},\mathfrak{M}} \), where \( a = \frac{1 + \delta(x)}{1 - \delta(x)} \) with \( A_j \in \mathbb{L}^\infty(\Omega), j = 1, 2, 3 \), \( \Phi \in \mathbb{L}^\infty(\Omega) \), and

\[
0 < \inf_{x \in \partial \Omega} |a(x)| \leq \sup_{x \in \partial \Omega} |a(x)| < \infty,
\]

is self-adjoint in \( \mathcal{L}^2(\Omega, \mathbb{C}^3) \). In the particular case \( a = 1 \), we see that the operator of MIT bag model is self-adjoint.

**Remark 9.** The self-adjointness of operators of the MIT bag models for domains with bounded \( C^2 \)-boundaries \( \partial \Omega \subset \mathbb{R}^3 \) was studied in the recent papers [??, ??, ??, ??] by means of different approaches.

### 4. Fredholm Theory and the Essential Spectrum

#### 4.1. Fredholm Property of the Operator \( \mathcal{D}_{A,\Phi,\mathfrak{B}} \) for Bounded Domains

**Theorem 10.** Let \( \Omega \) be a bounded domain with \( C^2 \)-boundary, let

\[
A \in \mathbb{C}^{1}(\bar{\Omega}, \mathbb{C}^3), \quad \Phi \in \mathbb{C}^{1}(\bar{\Omega}), \quad b_j \in \mathbb{C}(\partial \Omega, \mathfrak{B}(\mathbb{C}^2)), \quad j = 1, 2,
\]

and let the local standard Lopatinsky–Shapiro condition hold at every point \( x \in \partial \Omega \). Then

\[
\mathcal{D}_{A,\Phi,\mathfrak{B}} : H^1(\Omega, \mathbb{C}^3) \to \mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^4)
\]

is a Fredholm operator.

**Proof.** Since \( \mathcal{D}_{A,\Phi} \) is an elliptic operator, this theorem follows from standard elliptic theory (see, for instance, [??, ??, ??, ??]). \( \square \)

**Corollary 11.** Let the assumptions of Theorem 5 hold and let the domain \( \Omega \) be bounded with \( C^2 \)-boundary. Then the operator \( \mathcal{D}_{A,\Phi,\mathfrak{B}} \) is self-adjoint and has the discrete real spectrum.

#### 4.2. Fredholm Property of the Operator \( \mathcal{D}_{A,\Phi,\mathfrak{B}} \) for Unbounded Domains

Let \( \chi \in \mathbb{C}^{\infty}_c(\mathbb{R}^n) \) satisfy \( 0 \leq \chi(x) \leq 1, \chi(x) = 1 \) for \( |x| \leq 1, \chi(x) = 0 \) for \( |x| \geq 2 \), and \( \chi_R(x) = \chi\left(\frac{x}{R}\right), \psi_R(x) = 1 - \chi_R(x) \).

**Definition 12.** Let \( \Omega \subset \mathbb{R}^3 \) be an unbounded domain. (i) We say that the operator

\[
\mathcal{D}_{A,\Phi,\mathfrak{B}} : H^1(\Omega, \mathbb{C}^4) \to \mathcal{L}^2(\Omega, \mathbb{C}^4)
\]
is a locally Fredholm operator if, for every $R > 0$, there exist operators $\mathcal{L}_R, \mathcal{R}_R \in \mathcal{B}(L^2(\Omega, C^4), H^1(\Omega, C^4))$ such that
\[
\mathcal{L}_R \mathcal{B}_{A, \Phi, B} \chi_R I = \chi_R I + T'_R, \quad \mathcal{R}_R \mathcal{B}_{A, \Phi, B} = \chi_R I + T''_R,
\]
where $T'_R \in \mathcal{X}(H^1(\Omega, C^4))$ and $T''_R \in \mathcal{X}(L^2(\Omega, C^4))$.

(ii) We say that the operator
\[
\mathcal{D}_{A, \Phi, B} : H^1(\Omega, C^4) \to L^2(\Omega, C^4)
\]
is locally invertible at infinity if there exist an $R > 0$ and operators $\mathcal{L}'_R, \mathcal{R}'_R \in \mathcal{B}(L^2(\Omega, C^4), H^1(\Omega, C^4))$ such that
\[
\mathcal{L}'_R \mathcal{D}_{A, \Phi, B} \psi_R I = \psi_R I, \quad \psi_R \mathcal{D}_{A, \Phi, B} \mathcal{R}'_R = \psi_R I.
\]

**Proposition 1.** [?] The operator
\[
\mathcal{D}_{A, \Phi, B} : H^1(\Omega, C^4) \to L^2(\Omega, C^4)
\]
is Fredholm if and only if $\mathcal{D}_{A, \Phi, B}$ is a locally Fredholm and locally invertible operator at infinity.

We denote by $\tilde{\mathbb{R}}^3$ the compactification of $\mathbb{R}^3$ obtained by adding to the every ray
\[
l_\omega = \{x \in \mathbb{R}^3 : x = t\omega, t > 0, \omega \in S^2\}
\]
the infinitely distant point $\vartheta_\omega$. Introduce a topology in $\tilde{\mathbb{R}}^3$ so that $\tilde{\mathbb{R}}^3$ will become homeomorphic to the unit closed ball $B_1(0)$. The fundamental system of neighborhoods of a point $\vartheta_{\omega_0}$ is formed by the conical sets $U_{\omega_0, R} = F_{\omega_0} \times (R, +\infty)$, where $R > 0$ and $F_{\omega_0}$ is a neighborhood of the point $\omega_0$ on the unit sphere $S^2$. We define the cut-off function $\varphi_{\vartheta_{\omega_0}}$ of the infinitely distant point $\vartheta_{\omega_0}$ as
\[
\varphi_{\vartheta_{\omega_0}} = \varphi_{\omega_0}(x) = \varphi_{\omega_0}(x) I, \quad \varphi_{\omega_0}(\omega) \in C_0^\infty(F_{\omega_0}) \quad \text{and} \quad \varphi_{\omega_0}(\omega) = 1
\]
in a neighborhood $F'_{\omega_0}$ such that $F'_{\omega_0} \subset F_{\omega_0}$.

If $\Omega \subset \mathbb{R}^3$ is an unbounded domain, then we denote by $\tilde{\Omega}, \partial \tilde{\Omega}$ the closure of $\Omega, \partial \Omega$ in $\tilde{\mathbb{R}}^3$ and by $\Omega_\infty, \partial \Omega_\infty$ the associated sets of infinitely distant points.

**Definition 14.** We say that the operator $\mathcal{D}_{A, \Phi, B} : H^1(\Omega, C^4) \to L^2(\Omega, C^4)$ is locally invertible at the infinitely distant point $\vartheta_\omega$ if there exist a neighborhood $U_{\vartheta_\omega}$ of the point $\vartheta_\omega$ and operators
\[
\mathcal{L}_{\vartheta_\omega}, \mathcal{R}_{\vartheta_\omega} \in \mathcal{B}(L^2(\Omega, C^4), H^1(\Omega, C^4))
\]
such that
\[
\mathcal{L}_{\vartheta_\omega} \mathcal{D}_{A, \Phi, B} \varphi_{\vartheta_\omega} I = \varphi_{\vartheta_\omega} I, \quad \varphi_{\vartheta_\omega} \mathcal{D}_{A, \Phi, B} \mathcal{R}_{\vartheta_\omega} = \varphi_{\vartheta_\omega} I,
\]
where $\varphi_{\vartheta_\omega}$ is the cut-off function of the infinitely distant point $\vartheta_\omega$.

**Proposition 15.** [?] The operator $\mathcal{D}_{A, \Phi, B} : H^1(\Omega, C^4) \to L^2(\Omega, C^4)$ is a Fredholm operator if and only if $\mathcal{D}_{A, \Phi, B}$ is a locally Fredholm operator and $\mathcal{D}_{A, \Phi, B}$ is locally invertible at every infinitely distant point $\vartheta_\omega \in \Omega_\infty \cup \partial \Omega_\infty$.

4.2.1. Fredholm property and essential spectra in the exterior of a bounded domain. Let $\Omega \subset \mathbb{R}^3$ be an unbounded domain with $C^2$--boundary such that $\Omega' = \mathbb{R}^3 \setminus \Omega$ is a bounded domain. We consider the Fredholm property of the operator
\[
\mathcal{D}_{A, \Phi, B} : H^1(\Omega, C^4) \to L^2(\Omega, C^4).
\]
We assume as above that $A_j, \Phi \in C^1_b(\partial \Omega)$ and $b_j \in C^1(\partial \Omega, \mathcal{B}(C^2)), j = 1, 2$.

Following the book [?] and the paper [?], we describe the Fredholm property of $\mathcal{D}_{A, \Phi, B}$ in terms of limit operators.

We give a definition of the limit operators. Let $f \in C^1_b(\bar{\Omega})$ and let (a sequence) $\mathbb{R}^3 \ni g_m \to \vartheta_\omega \in \mathbb{R}^3$. The family of functions $\{f(\cdot + g_m)\}$ is uniformly bounded and equicontinuous on $\Omega$. Then the Arzelà--Ascoli Theorem implies that there exists a subsequence $h_m, g_m, h_m$ and a limit function $f^h \in C_b(\bar{\Omega})$ such that
\[
\lim_{m \to \infty} \sup_{x \in K} |f(x + h_m) - f^h(x)| = 0
\]
Hence, formula (??) for every compact set \( K \subset \Omega \).

Let \( h_m \to \partial_\omega \) be a sequence such that

\[
A(x + h_m) \to A^h(x), \quad \Phi(x + h_m) \to \Phi^h(x)
\]

in the sense of the convergence defined by formula (??). The operator \( \mathcal{D}_{A,\Phi}^h = \mathcal{D}_{A^h,\Phi^h} \) is called the limit operator defined by the sequence \( h_m \), and we denote by \( \lim h_{A,\Phi} \mathcal{D}_{A,\Phi} \) the set of all limit operators defined by the sequences \( h_m \to \partial_\omega \). We set

\[
\lim \mathcal{D}_{A,\Phi} = \bigcup_{h \in \mathbb{R}} \lim h_{A,\Phi} \mathcal{D}_{A,\Phi}.
\]

**Theorem 16.** Let \( A \in C^1(\overline{\Omega}, \mathbb{C}^3), \Phi \in C^1(\overline{\Omega}), b_j \in C^1(\partial \Omega), j = 1, 2; i, j = 1, 2 \), and let the local standard Lopatinsky–Shapiro condition hold at every point \( x \in \partial \Omega \). Then

\[
\mathcal{D}_{A,\Phi} : H^1(\Omega, \mathbb{C}^4) \to L^2(\Omega, \mathbb{C}^4)
\]

is a Fredholm operator if and only if all limit operators \( \mathcal{D}_{A,\Phi}^h \in \lim \mathcal{D}_{A,\Phi} \) taking \( H^1(\Omega, \mathbb{C}^4) \) to \( L^2(\Omega, \mathbb{C}^4) \) are invertible.

**Proof.** Since the boundary \( \partial \Omega \subset \mathbb{R}^3 \) is a compact surface, the operator \( \mathcal{D}_{A,\Phi} \) is elliptic on \( \overline{\Omega} \), and the Lopatinsky–Shapiro condition holds at every point \( x \in \partial \Omega \), it follows that the operator \( \mathcal{D}_{A,\Phi,\bar{B}} \) is locally Fredholm. Hence, by Proposition ??, \( \mathcal{D}_{A,\Phi,\bar{B}} \) is a Fredholm operator if and only if \( \mathcal{D}_{A,\Phi,\bar{B}} \) is locally invertible at infinity. The operator \( \mathcal{D}_{A,\Phi,\bar{B}} \) coincides with \( \mathcal{D}_{A,\Phi} \) outside the set \( \overline{\Omega} \). Applying the results of the paper [??], we see that \( \mathcal{D}_{A,\Phi} \) is locally invertible at infinity if and only if, for every \( \sigma_\omega \in \mathbb{R}_+ \), all limit operators \( \mathcal{D}_{A,\Phi}^h \in \lim \mathcal{D}_{A,\Phi} \) taking \( H^1(\mathbb{R}^3, \mathbb{C}^4) \) to \( L^2(\mathbb{R}^3, \mathbb{C}^4) \) are invertible.

**Corollary 17.** Let conditions of Theorem ?? hold. Then

\[
\operatorname{sp ess} \mathcal{D}_{A,\Phi,\bar{B}} = \bigcup_{\mathcal{D}_{A,\Phi}^h \in \lim \mathcal{D}_{A,\Phi}} \operatorname{sp} \mathcal{D}_{A^h,\Phi^h}.
\]

**Definition 18.** We say that a function \( a \in C^1(\mathbb{R}^3) \) is slowly oscillating at infinity and belongs to the class \( SO^1(\mathbb{R}^3) \) if

\[
\lim_{x \to -\infty} \partial_\omega a(x) = 0, \quad j = 1, 2, 3.
\]

If \( \Omega \subset \mathbb{R}^3 \) is an unbounded domain, we denote by \( SO^1(\Omega) \) the class of functions which are the restrictions to \( \Omega \) of functions in \( SO^1(\mathbb{R}^3) \).

Note that if \( f \in SO^1(\mathbb{R}^3) \) and there exists a limit function \( f^h \) in the sense of formula (??), then \( f^h \in \mathbb{C} \) (see, for instance, [??, p. 228]).

Let \( A_j, \Phi \in SO^1(\Omega) \). Then the limit operators \( \mathcal{D}_{A,\Phi}^h \) are of the form

\[
\mathcal{D}_{A,\Phi}^h = \mathcal{D}_{A^h,\Phi^h} + \alpha \cdot (i \nabla + A^h) + \alpha_0 m + \Phi^h I_4,
\]

where \( A^h \in \mathbb{C}^3, \Phi^h \in \mathbb{C} \). If \( A_j, \Phi \) are real-valued functions, then the operator \( \mathcal{D}_{A,\Phi}^h \) is self-adjoint and

\[
\operatorname{sp} \mathcal{D}_{A,\Phi}^h = \{ -\infty, \Phi^h - \| m \| \} \bigcup \{ \Phi^h + \| m \|, +\infty \}.
\]

Hence, formula (??) yields the following result.

**Theorem 19.** Let \( \Omega \) be the exterior of a bounded domain with \( C^2 \) boundary \( \partial \Omega \), let

\[
A \in SO^1(\Omega, \mathbb{R}^3), \quad \Phi \in SO^1(\Omega, \mathbb{R}),
\]

and let the local standard Lopatinsky–Shapiro condition be satisfied at every point \( x \in \partial \Omega \). Then

\[
\operatorname{sp ess} \mathcal{D}_{A,\Phi,\bar{B}} = (-\infty, \mathcal{M}_{\Phi}^{\sup} - \| m \|] \bigcup [\mathcal{M}_{\Phi}^{\inf} + \| m \|, +\infty),
\]

where

\[
\mathcal{M}_{\Phi}^{\sup} = \limsup_{x \to -\infty} \Phi(x), \mathcal{M}_{\Phi}^{\inf} = \liminf_{x \to -\infty} \Phi(x).
\]

**Corollary 20.** Under the assumptions of Theorem 19,

\[
\operatorname{sp dis} \mathcal{D}_{A,\Phi,\bar{B}} \subset \mathcal{M}_{\Phi}^{\sup} - \| m \|, \mathcal{M}_{\Phi}^{\inf} + \| m \|
\]

if \( \mathcal{M}_{\Phi}^{\sup} - \mathcal{M}_{\Phi}^{\inf} < 2 \| m \| \) and, if \( \mathcal{M}_{\Phi}^{\sup} - \mathcal{M}_{\Phi}^{\inf} > 2 \| m \| \), then

\[
\operatorname{sp} \mathcal{D}_{A,\Phi,\bar{B}} = \operatorname{sp ess} \mathcal{D}_{A,\Phi,\bar{B}} = (-\infty, +\infty).
\]
4.3. Fredholm Property and Essential Spectrum in Domains with Conical Structure at Infinity

Let $\Omega \subset \mathbb{R}^3$ be a connected open domain with $C^2$ boundary. We say that $\Omega$ has a conic exit at infinity if

$$\Omega \cap B'_R = \{ x \in \mathbb{R}^3 : x = t\omega, t > R, \omega \in \Sigma \},$$

where $B'_R = \{ x \in \mathbb{R}^3 : |x| > R \}$ and $\Sigma \subset S^2$ is an open set with $C^2$ boundary $\partial \Sigma$. We denote by $\hat{\Omega}, \hat{\partial} \Omega$ the compactifications of $\Omega, \partial \Omega$ in the topology of $\hat{\mathbb{R}}^3$.

We consider the operator $\mathbb{D}_{A,\Phi,\mathbb{B}}$ in domains with conic exit at infinity with potentials $A_j$, where $\Phi \in SO^1(\Omega)$ and $b_j \in SO^1(\partial \Omega, \mathcal{B}(C^2)) = SO^1(\partial \Omega) \otimes \mathcal{B}(C^2), j = 1, 2$. We define the limit operators of the operator $\mathbb{D}_{A,\Phi,\mathbb{B}}$ similarly to their definition in [?, ?].

- If $\partial_\omega \notin \partial \Omega_\infty$, then the limit operators defined by the sequence $h_m \to \partial_\omega$ are the Dirac operators $\mathbb{D}^{A_h,\Phi_h}$ with spectrum given by formula (76).

- Let $\partial_\omega \in \partial \Omega_\infty$ and $l^R_\omega = \{ x \in \mathbb{R}^3 : x = t\omega, t > R \}$, let $\mathcal{T}_{\partial_\omega}$ be the tangent plane to $\partial \Omega$ at the ray $l^R_\omega$, and let $\nu(\omega)$ be the outward pointing normal vector to $\partial \Omega$ at the points of the ray $l^R_\omega$. We denote by

$$\mathbb{R}^3_{\partial_\omega} = \{ y = (y', y_3) \in \mathbb{R}^3 : y' \in \mathcal{T}_{\partial_\omega}, y_3 = t\nu(\omega), t > 0 \}$$

the half-space in $\mathbb{R}^3$ with the boundary $\mathcal{T}_{\partial_\omega}$. Following the paper [?], we obtain the limit operators of $\mathbb{D}_{A,\Phi,\mathbb{B}}$ defined by the sequences $h_m \to \partial_\omega$ as

$$\mathbb{D}^{A_h,\Phi_h}_h \Phi (x) = \begin{cases} \mathcal{D}^{A_h,\Phi_h}_h \Phi (x), & x \in \mathbb{R}^3_{\partial_\omega}, \\ \mathcal{B}^{h} \Phi (s) = b_1^h \mathcal{D}^{A_h,\Phi_h}_h \Phi (s) + b_2^h \mathcal{D}^{A_h,\Phi_h}_h \Phi (s) = 0, & s \in \mathcal{T}_{\partial_\omega}, \end{cases}$$

where $A_h \in \mathbb{C}^3$, $\Phi_h \in \mathbb{C}$, $b_j^h \in \mathcal{B}(\mathcal{T}_{\partial_\omega}, \mathcal{B}(C^2))$, $u_h \in H^1(\mathbb{R}^3_{\partial_\omega}, \mathbb{C}^4)$, $u_{\partial_\omega}^2 = \gamma_{\partial_\omega} u^1 \in H^{1/2}(\mathcal{T}_{\partial_\omega}, \mathbb{C}^2)$.

**Theorem 21.** Let (i) $\partial \Omega$ be a $C^2$-surface with a conic exit at infinity, (ii) $A_j, \Phi \in SO^1(\Omega)$ and $b_j \in SO^1(\partial \Omega, \mathcal{B}(C^2)) = SO^1(\partial \Omega) \otimes \mathcal{B}(C^2)$,

(iii) the Lopatinsky-Shapiro condition be satisfied at every point $x \in \partial \Omega$. Then the operator $\mathbb{D}_{A,\Phi,\mathbb{B}} : H^1(\mathbb{R}^3, \mathbb{C}^4) \to L^2(\Omega, \mathbb{C}^4)$ is a Fredholm operator if and only if, for every $\partial_\omega \in \partial \Omega_\infty$, all limit operators $\mathbb{D}^{A_h,\Phi_h}_{A,\Phi,\mathbb{B}}$ defined by formulas (77) are invertible.

**Proof.** The operator $\mathbb{D}_{A,\Phi,\mathbb{B}}$ is locally Fredholm, since the Dirac operator $\mathbb{D}_{A,\Phi}$ is elliptic and the Lopatinsky condition holds at every point $x \in \partial \Omega$. According to Proposition 15, $\mathbb{D}_{A,\Phi,\mathbb{B}}$ is a Fredholm operator if and only if $\mathbb{D}_{A,\Phi,\mathbb{B}}$ is locally invertible at every infinitely distant point $\partial_\omega \in \partial \Omega_\infty$. Following the monograph [?] and the paper [ ?, ?], we obtain the statement of Theorem 21. \qed

**Corollary 22.** Let the assumptions of Theorem 21 hold. Then

$$\text{sp}_{\text{ess}} \mathbb{D}_{A,\Phi,\mathbb{B}} = \bigcup_{\mathbb{D}^{A_h,\Phi_h}_{A,\Phi,\mathbb{B}}} \text{sp}_{\text{ess}} \mathbb{D}^{A_h,\Phi_h}_{A,\Phi,\mathbb{B}},$$

where $\mathbb{D}^{A_h,\Phi_h}_{A,\Phi,\mathbb{B}}$ are unbounded operators associated with the limit operators $\mathbb{D}^{A_h,\Phi_h}_{A,\Phi,\mathbb{B}}$.

4.4. Essential Spectrum of the Operator of the MIT Bag Model in Domains with Conic Exit to Infinity

We consider operators of the MIT bag model in the domain $\Omega$ with $C^2$ boundary conical at infinity

$$\mathbb{M}_{A,\Phi,\mathbb{B}} u(x) = \begin{cases} \mathcal{D}_{A,\Phi} u(x), & x \in \Omega, \\ \mathcal{B} u = u_{\partial_\Omega}^1 + i (\sigma \cdot \nu) u_{\partial_\Omega}^2 = 0 & \text{on } \partial \Omega. \end{cases}$$

We assume that the potentials $A \in SO^1(\Omega, \mathbb{R}), \Phi \in SO^1(\Omega, \mathbb{R})$ are real-valued. Since the conical at infinity $C^2$ surface is uniformly regular, it follows that the unbounded operator $\mathbb{M}_{A,\Phi,\mathbb{B}}$ associated with $\mathbb{M}_{A,\Phi,\mathbb{B}} : H^1(\Omega, \mathbb{C}^4) \to L^2(\Omega, \mathbb{C}^4)$ is self-adjoint in $L^2(\Omega, \mathbb{C}^4)$ and $\text{sp}_{\text{ess}} \mathbb{M}_{A,\Phi,\mathbb{B}}$ is defined by formula (78). That is,

$$\text{sp}_{\text{ess}} \mathbb{M}_{A,\Phi,\mathbb{B}} = \bigcup_{\mathbb{M}^{A_h,\Phi_h}_{A,\Phi,\mathbb{B}}} \text{sp}_{\text{ess}} \mathbb{M}^{A_h,\Phi_h}_{A,\Phi,\mathbb{B}}.$$
where $\mathcal{M}_h^{A, \Phi, M}$ are limit operators of the operator $\mathcal{M}_h^{A, \Phi, M}$.

Let a sequence $h_m$ converge to $\partial_{\omega} \in \Omega_{\infty} \setminus \partial_\Omega_{\infty}$. Then the limit operators are of the form $\mathcal{M}_h^{A, \Phi, M} = \mathcal{D}_h^{A, \Phi, h}$ and

$$sp_{\mathcal{M}_h^{A, \Phi, M}} = sp \mathcal{D}_h^{A, \Phi, h} = (-\infty, \Phi^h - |m|) \cup [\Phi^h + |m|, +\infty).$$

(77)

Let the sequence $h_m$ converge to $\partial_{\omega} \in \partial_\Omega_{\infty}$. Then, without loss of generality, we assume that $T_{\partial_{\omega}} = \mathbb{R}^2_{x'} = \{x = (x', x_3): x_3 = 0\}$. Hence, the limit operators of $\mathcal{M}_h^{A, \Phi}$ are of the form

$$M_h^{A, \Phi, M} u(x) = \begin{cases} \mathcal{D}_h^{A, \Phi, h} u(x), & x \in \mathbb{R}^3, \\ M_{h_0} u(x') = u(x',+0) + i\sigma_3 u^2(x',+0) = 0 & on \mathbb{R}^2_{x'} \end{cases}$$

(78)

The gauge transformation $x \rightarrow e^{-iA^h \cdot x} x$ reduces the study of the spectrum of $M_h^{A, \Phi, M}$ to that of the spectrum of the operator $M_{h_0, \Phi, \Phi} = M_{h_0, \Phi} + \Phi^h I_4$, where

$$M_{h_0, \Phi} u = \begin{cases} (i\alpha \cdot \nabla + a_0 m) u(x), & x \in \mathbb{R}^3, \\ \Theta u(x') = u(x',+0) + i\sigma_3 u^2(x',+0) = 0, & x' \in \mathbb{R}^2_{x'} \end{cases}$$

(79)

After the Fourier transform in (78) with respect to $x' \in \mathbb{R}^2$, we obtain a family of one-dimensional Dirac operators depending on the parameter $\xi' \in \mathbb{R}^2$,

$$\mathcal{L}(\xi') v(z) = \begin{cases} (\alpha' \cdot \xi' + i\sigma_3 \frac{d}{dz} + a_0 m) v(z), & z \in \mathbb{R}_+, \xi' \in \mathbb{R}^2, \\ v^i(+0) + i\sigma_3 v^2(+0) = 0 \end{cases}$$

(80)

The operator $\mathcal{L}(\xi')$ has the essential spectrum

$$sp_{ess} \mathcal{L}(\xi') = (-\infty, -\sqrt{\xi'^2 + |m|^2}) \cup \left[\sqrt{\xi'^2 + |m|^2}, +\infty\right)$$

and a possible discrete spectrum

$$sp_{dis} \mathcal{L}(\xi') \subset \left(-\sqrt{\xi'^2 + m^2}, \sqrt{\xi'^2 + m^2}\right).$$

(81)

We are looking for $sp_{dis} \mathcal{L}(\xi')$ as follows. The equation

$$\left(\alpha' \cdot \xi' + i\alpha_3 \frac{d}{dz} + a_0 m - \lambda I_4\right) v(z) = 0, \quad z > 0, \quad \lambda \in \mathbb{R},$$

(83)

has exponentially decreasing solutions of the form $v(z) = h e^{-\rho z}$, $\rho = \frac{\sqrt{\xi'^2 + m^2 - \lambda^2}}{\sqrt{\xi'^2 + m^2}} > 0$ with $|\lambda| < \sqrt{\xi'^2 + m^2}$, where the vector $h$ satisfies the equation

$$\Theta^-(\xi', \lambda) h = (\alpha' \cdot \xi' - i\rho \alpha_3 + a_0 m - \lambda I_4) h = 0.$$  

(84)

Equation (84) has the general solution

$$h = h(\xi', \lambda) = \Theta^+(\xi', \lambda) f,$$

(85)

where $f \in \mathbb{C}^4$ is an arbitrary vector,

$$\Theta^+(\xi', \lambda) = \alpha' \cdot \xi' - i\rho \alpha_3 + a_0 m + \lambda I_4 = \begin{pmatrix} (\lambda + m) I_2 & \Lambda(\xi', \lambda) \\ \Lambda(\xi', \lambda) & (\lambda - m) I_2 \end{pmatrix},$$

and

$$\Lambda = \Lambda(\xi', \lambda) = \sigma' \cdot \xi' - i\rho \sigma_3 = \begin{pmatrix} -i\rho & \xi \\ \xi & i\rho \end{pmatrix}, \quad \varsigma = \xi_1 + i\xi_2.$$  

(86)

Let $h_1, h_2$ be two linear independent solutions of equation (84) of the form

$$h_1 = h_1(\xi', \lambda) = \Theta(\xi', \lambda) \begin{pmatrix} e_1 \\ 0 \end{pmatrix} = \begin{pmatrix} (\lambda + m) e_1 \\ \Lambda(\xi', \lambda) e_1 \end{pmatrix},$$

(87)
\[
    h_2 = h_2(\xi', \lambda) = \Theta(\xi', \lambda) \begin{pmatrix} 0 \\ e_2 \end{pmatrix} = \begin{pmatrix} \Lambda(\xi', \lambda)e_2 \\ (\lambda - m)e_2 \end{pmatrix},
\]

where
\[
    \Lambda(\xi', \lambda)e_1 = \begin{pmatrix} -i\rho \\ \zeta \end{pmatrix}, \quad \Lambda(\xi', \lambda)e_2 = \begin{pmatrix} \bar{\zeta} \\ i\rho \end{pmatrix}.
\]

Then the general solution of equation (\ref{eq:2.1}) is
\[
    v(z) = C_1 h_1 e^{-\rho z} + C_2 h_2 e^{-\rho z}, \quad z > 0.
\]

Substituting (\ref{eq:2.1}) into the boundary condition \( v^1(+0) + i\sigma_3 v^2(+0) = 0 \), we obtain a system of linear equations with respect to \( C_1, C_2 \),
\[
    C_1(h_1^1 + i\sigma_3 h_1^2) + C_2(h_2^1 + i\sigma_3 h_2^2) = 0.
\]

System (\ref{eq:2.1}) has a nontrivial solution if and only if
\[
    \det(h_1^1 + i\sigma_3 h_1^2, h_2^1 + i\sigma_3 h_2^2) = \det \begin{pmatrix} \lambda + m + \rho & -i\zeta \\ -i\zeta & \rho - i(\lambda - m) \end{pmatrix} = 2i\rho(\rho + m) = 0. \tag{92}
\]

We consider two cases:

1. The mass of the particle is \( m \geq 0 \). In this case, equation (\ref{eq:2.1}) has no positive solutions, and \( \text{sp} \mathcal{M}^h_{A, \Phi, \Theta} \) is given by formula (\ref{eq:s3}). This implies that \( \text{sp} \mathcal{M}^h_{A, \Phi, \Theta} \) is given by formulas (\ref{eq:2.1}), (\ref{eq:2.2}).

2. Let the mass be \( m < 0 \). Then the equation (\ref{eq:2.1}) has the positive solution \( \rho = |m| \). It follows from the equation
\[
    \rho = \sqrt{|\xi'|^2 + m^2 - \lambda^2} = |m|
\]

that the operator \( \mathbb{L}^0(\xi') \) has eigenvalues \( \lambda_{\pm}(\xi') = \pm|\xi'| \), for every \( \xi' \in \mathbb{R}^2 \setminus 0 \). This gives
\[
    \text{sp} \mathcal{M}^h_{A, \Phi, \Theta} = (-\infty, +\infty). \tag{93}
\]

Taking into account (\ref{eq:2.1}) and (\ref{eq:2.2}), we obtain
\[
    \text{sp} \mathcal{D}_{A, \Phi, \Theta} = \text{sp} \mathcal{D}_{A, \Phi, \Theta} = (-\infty, +\infty).
\]

Thus, we have established the following result.

**Theorem 23.** Let \( A \in SO^1(\Omega, \mathbb{R}^3) \) and \( \Phi \in SO^1(\Omega) \) be real valued functions and let \( \Omega \) be a domain with \( C^2 \) boundary having a conic exit at infinity. Then the essential spectrum of the MIT bag model is defined by the following formulas:

(a) If \( m \geq 0 \), then
\[
    \text{sp} \mathcal{M}^h_{A, \Phi, \Theta} = (-\infty, M^\text{sup}_\Phi - |m|] \cup [M^\text{inf}_\Phi + |m|, +\infty), \tag{94}
\]

where
\[
    M^\text{sup}_\Phi = \limsup_{x \to -\infty} \Phi(x), \quad M^\text{inf}_\Phi = \liminf_{x \to -\infty} \Phi(x). \tag{95}
\]

(b) If \( m < 0 \), then
\[
    \text{sp} \mathcal{M}^h_{A, \Phi, \Theta} = (-\infty, +\infty).
\]

It follows from Theorem 23 that the operator \( \mathcal{M}^h_{A, \Phi, \Theta} \) may have a discrete spectrum on the interval \((M^\text{sup}_\Phi - |m|, M^\text{inf}_\Phi + |m|)\) if \( m > 0 \) and \( M^\text{sup}_\Phi - M^\text{inf}_\Phi < 2m \), and \( \mathcal{M}^h_{A, \Phi, \Theta} \) does not have any discrete spectrum if \( m < 0 \).

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