Short distance behaviour of correlators in the 2D Ising model in a magnetic field

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Abstract

We study the \(\langle \sigma \sigma \rangle\), \(\langle \sigma \epsilon \rangle\), \(\langle \epsilon \epsilon \rangle\) correlators in the 2d Ising model perturbed by a magnetic field. We compare the results of a set of high precision Montecarlo simulations with the predictions of two different approximations: the Form Factor approach, based on the exact S-matrix description of the model, and a short distance perturbative expansion around the conformal point. Both methods give very good results, the first one performs better for distances larger than the correlation length, while the second one is more precise for distances smaller than the correlation length. In order to improve this agreement we extend the perturbative analysis to the second order in the derivatives of the OPE constants.

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1 Introduction

In these last years much progress has been done in the study of two dimensional statistical systems in the neighbourhood of critical points. In the framework of quantum field theory these systems can be seen as Conformal Field Theories (CFTs) perturbed by some relevant operator. Since the seminal work of Belavin, Polyakov and Zamolodchikov [1] we have an almost complete understanding of CFT’s (at least for the minimal models): we have complete lists of all the operators of the theories and explicit expressions for the correlators. However much less is known on their relevant perturbations. In some cases it has been possible to show that these perturbations give rise to integrable models [3, 4]. In these cases again we have a rather precise description of the theory. In particular it is possible to obtain the exact asymptotic expression for the large distance behaviour of the correlators [4]. From this information several important results (and in particular all the universal amplitude ratios) can be obtained.

However in comparing with numerical simulations or with experiments one is often interested in the short distance behaviour of the correlators (short here means for distances smaller or equal than the correlation length) and this is not easily accessible in integrable systems. Moreover integrable perturbations represent only a small subset of the possible theories. For instance in the case of the Ising model both the purely thermal and the purely magnetic perturbations are integrable, but for any combination of them the exact integrability is lost.

For these reasons, besides the S-matrix results, it is important to develop a perturbative approach well defined in the short distance regime of the theory and such that it does not rely on the exact integrability of the model. This is however a rather difficult task. In fact any naive perturbative expansion of the (massless) CFT along a relevant direction, is affected by infrared divergences (IR) and some non-trivial strategy is needed.

Recently, in [5, 6], a new approach has been proposed to overcome this difficulty (see [7]-[15] for relevant related works and preexisting ideas). The method is based on Wilson’s operator product expansion (OPE). Roughly speaking the main idea of this new approach is that the Wilson’s coefficients of the OPE, being well defined at short distance, can be assumed to have a regular, IR safe, perturbative expansion with respect to the coupling. For this reason we shall refer to it in the following as the IRS (infrared safe)
perturbative approach.

The main requirement of this IRS approach is the knowledge of the Wilson coefficients (and their derivatives with respect to the perturbing coupling). For this reason it is particularly efficient if applied to perturbations of exactly solved theories like 2d critical CFT’s (but the framework is quite general and in principle could be extended also to higher dimensions).

The price one has to pay to control in this way the IR divergences is that one needs, as an external input information, the expectation values of the operators involved in the expansion. There are at this point two possibilities. The first one is to concentrate only on observables in which these expectation values exactly cancel. This is a small but very interesting subset of the informations that we can obtain with the IRS perturbation.

The second possibility is to obtain the desired expectation values with some other method or extract them from numerical simulations (an interesting numerical approach to obtain these VEV is based on the Truncated Conformal Space technique, see [16, 17] and references therein).

From this point of view, the IRS approach becomes particularly powerful if applied to integrable perturbations, since in this case some of the expectation values can be deduced from the S-matrix of the model.

The last step one has to face in comparing the results of the IRS method with simulations or experiments is the presence of a nonuniversal normalization factor between the operators in the continuum quantum field theory and their lattice discretizations. These normalizations (and the related normalization of the coupling of the perturbation) can be fixed if an exact solution of the lattice model exists at the critical point. Actually much less is needed. One only needs the exact expression (or even only its large distance asymptotic form) of a correlator involving the operators in which we are interested. This makes the Ising model perturbed by a magnetic field a perfect candidate for testing the IRS method. In fact it is well known that this model is exactly integrable [2, 3] and all the amplitude ratios and expectation values of the primary fields are known. Moreover the Ising model is exactly solvable at the critical point and the exact expression is known for several correlators [18, 19].

In fact the IRS approach was successfully tested with the magnetic perturbation of the Ising model in [6]. The aim of this paper is to make a further step in this direction. In particular we have three goals:
a] Compare the results of the method with new high precision Montecarlo simulations so as to test the range of applicability of the method.

b] Compare the IRS method with the results obtained in the S-matrix framework with the so called “form factor” (FF) approach.

c] Show that it is possible to extend the analysis of [5, 6] to higher orders in the perturbative coupling and discuss the technical problems that one has to afford following this route.

In particular we shall study in this paper, as an example, the second order term in the perturbative expansion of the \( \langle \epsilon \epsilon \rangle \) correlator. The reason for this choice is that this correlator has a very peculiar behaviour since its first order correction turns out to be exactly zero, thus our second order calculation is mandatory if one is interested to study the influence of the magnetic field on the simple critical behaviour of the correlator.

This paper is organized as follows. Sect. 2 is devoted to a general description of the Ising model in a magnetic field both on the lattice and in the continuum. The aim of this section is to fix conventions and normalizations which will be useful in the following. In sect. 3 we shall briefly describe the IRS method, while in sect. 4 we shall extend it to second order derivatives of the magnetic field. In sect. 5 we shall briefly describe our Montecarlo simulation while in sect. 6 we shall compare the results of our simulations with IRS and FF predictions. Finally sect. 7 will be devoted to some concluding remarks. The details of the calculation of the second order derivative of the Wilson coefficient are collected in the Appendix. We have reported in three tables at the end of the paper a sample of the results of our simulations.

2 Ising model in a magnetic field.

The continuum theory, which is the starting point of the IRS expansion is given by the action:

\[
\mathcal{A} = \mathcal{A}_0 + h \int d^2 x \sigma(x)
\]

(1)

where \( \mathcal{A}_0 \) is the action of the conformal field theory which describes the Ising model at the critical point. Let us start our analysis by looking in detail at this CFT.
2.1 The Ising model at the critical point

The Ising model at the critical point is described by the unitary minimal CFT with central charge \( c = 1/2 \). It contains three conformal families whose primary fields \( 1, \sigma, \epsilon \) have scaling dimensions \( x = 0, 1/8, 1 \) respectively. The fusion rule algebra is

\[
\begin{align*}
[\epsilon][\epsilon] &= [1] \\
[\sigma][\epsilon] &= [\sigma] \\
[\sigma][\sigma] &= [1] + [\epsilon].
\end{align*}
\]

Once the operator content is known, the only remaining information which is needed to completely identify the theory are the OPE constants. The OPE algebra is defined as

\[
\Phi_i(r)\Phi_j(0) = \sum_{\{k\}} C_{ij}^{\{k\}}(r)\Phi_{\{k\}}(0)
\]

where with the notation \( \{k\} \) we mean that the sum runs over all the fields of the conformal family \( [k] \). The structure functions \( C_{ij}^k(r) \) are \( c \)-number functions of \( r \) which must be single valued in order to take into account locality. In the large \( r \) limit they decay with a power like behaviour

\[
C_{ij}^k(r) \sim |r|^{-\text{dim}(C_{ij}^k)}
\]

whose amplitude is given by

\[
\hat{C}_{ij}^k \equiv \lim_{r \to \infty} C_{ij}^k(r) |r|^\text{dim}(C_{ij}^k). \tag{5}
\]

The actual value of these constants depends on the normalization of the fields, which can be chosen by fixing the long distance behaviour of, for instance, the \( \sigma \sigma \) and \( \epsilon \epsilon \) correlators. In this paper we follow the commonly adopted convention which is\[\text{1}]

\[
\langle \sigma(x)\sigma(0) \rangle = \frac{1}{|x|^{1/4}}, \quad |x| \to \infty \tag{6}
\]

\[
\langle \epsilon(x)\epsilon(0) \rangle = \frac{1}{|x|^2}, \quad |x| \to \infty. \tag{7}
\]

\[\text{1} \text{Notice the change of normalization with respect to } [\text{3}] \text{. The } \epsilon \text{ operator of the present paper corresponds to } 2\pi \text{ times that of ref. [3].}\]
With these conventions we have, for the structure constants among primary fields

\[ \hat{C}_{\sigma,\sigma} = \hat{C}_{\epsilon,\epsilon} = \hat{C}^\sigma_{\epsilon,\sigma} = 0 \]  
\[ \hat{C}^1_{\sigma,\sigma} = \hat{C}^\sigma_{\sigma,1} = \hat{C}^1_{\epsilon,\epsilon} = \hat{C}^\epsilon_{\epsilon,1} = 1 \]  

and

\[ \hat{C}^\sigma_{\sigma,\epsilon} = \hat{C}^\epsilon_{\sigma,\sigma} = \frac{1}{2}. \]  

2.2 The Ising model in a magnetic field

If we switch on the magnetic field \( h \), the structure functions acquire a \( h \) dependence so that we have in general

\[ \Phi_i(r)\Phi_j(0) = \sum_{\{k\}} C_{ij}^{(k)}(h, r)\Phi_{\{k\}}(0). \]  

Also the mean values of the \( \sigma \) and \( \epsilon \) operators acquire a dependence on \( h \). Standard renormalization group arguments allow one to relate this \( h \) dependence to the scaling dimensions of the operators of the theory and lead to the following expressions:

\[ \langle \sigma \rangle_h = A_\sigma h^{\frac{\Delta}{15}} + ... \]  
\[ \langle \epsilon \rangle_h = A_\epsilon h^{\frac{\Delta}{15}} + ... \]  

The exact value of the two constants \( A_\sigma \) and \( A_\epsilon \) can be found in [20] and [21] respectively

\[ A_\sigma = \frac{2C^2}{15(\sin \frac{\pi}{3} + \sin \frac{\pi}{5} + \sin \frac{\pi}{15})} = 1.27758227..., \]  

with

\[ C = \frac{4 \sin \frac{\pi}{5} \Gamma \left( \frac{1}{5} \right) \left( \frac{4\pi^2 \Gamma \left( \frac{3}{5} \right) \Gamma^2 \left( \frac{13}{16} \right) \Gamma \left( \frac{3}{5} \right) \Gamma^2 \left( \frac{13}{16} \right) \right)^{\frac{1}{5}}}{\Gamma \left( \frac{2}{5} \right) \Gamma \left( \frac{8}{15} \right) \Gamma \left( \frac{8}{15} \right) \Gamma \left( \frac{2}{5} \right) \Gamma^2 \left( \frac{16}{15} \right) \Gamma \left( \frac{1}{5} \right) \Gamma^2 \left( \frac{16}{15} \right)} \]  

and

\[ A_\epsilon = 2.00314... \]  

Notice however that these amplitudes are not universal. They depend on the details of the regularization scheme. Thus some further work is needed to obtain their value on the lattice.
2.3 The lattice model

The lattice version of the above model is defined by the following partition function:

\[ Z = \sum_{\sigma_i = \pm 1} e^{\beta(\sum_{(i,j)} \sigma_i \sigma_j + H \sum_i \sigma_i)} \]  \hspace{1cm} (17)

where the notation \( \langle i, j \rangle \) denotes nearest neighbour sites in the lattice which we assume to be a two dimensional square lattice of size \( L \). In order to select only the magnetic perturbation, \( \beta \) must be fixed to its critical value:

\[ \beta = \beta_c = \frac{1}{2} \log (\sqrt{2} + 1) = 0.4406868... \]

Finally, by defining \( h_l = \beta_c H \) we find

\[ Z = \sum_{\sigma_i = \pm 1} e^{\beta_c \sum_{(i,j)} \sigma_i \sigma_j + h_l \sum_i \sigma_i} \hspace{1cm} (18) \]

In the following we shall denote the lattice discretization of the operators \( \sigma \), \( \varepsilon \) with the index \( l \). The magnetization \( M(h) \) is defined as usual:

\[ M(h) \equiv \frac{1}{N} \frac{\partial}{\partial h_l} (\log Z)|_{\beta = \beta_c} = \langle \frac{1}{N} \sum_i \sigma_i \rangle. \hspace{1cm} (19) \]

where \( N \equiv L^2 \) denotes the number of sites of the lattice. This result suggests the following definition for the lattice discretization of \( \sigma \)

\[ \sigma_i \equiv \frac{1}{N} \sum_i \sigma_i, \hspace{1cm} (20) \]

so that the mean value of \( \sigma_i \) coincides with \( M(h) \):

\[ \langle \sigma_i \rangle \equiv M(h) \hspace{1cm} (21) \]

Similarly, we define the internal energy as:

\[ E(h) \equiv \frac{1}{2N} \frac{\partial}{\partial h_l} (\log Z)|_{\beta = \beta_c} = \langle \frac{1}{2N} \sum_{(i,j)} \sigma_i \sigma_j \rangle \hspace{1cm} (22) \]
For the energy operator one must also take into account the presence of an additional bulk contribution at the critical point. This constant can be easily evaluated (for instance by using Kramer-Wannier duality) to be \( \epsilon_0 = \frac{1}{\sqrt{2}} \).

This result suggests, for the lattice discretization of \( \epsilon \), the following definition

\[
\epsilon_l \equiv \frac{1}{2N} \sum_{(i,j)} \sigma_i \sigma_j - \frac{1}{\sqrt{2}} \quad (23)
\]

so that the mean value of \( \epsilon_l \) coincides with the singular part of \( E(h) \):

\[
\langle \epsilon_l \rangle \equiv E(h) - \frac{1}{\sqrt{2}} \quad (24)
\]

According to the above discussion we expect:

\[
\langle \sigma_l \rangle_h = A_l^\sigma h^{15}_l + \ldots \quad (25)
\]

\[
\langle \epsilon_l \rangle_h = A_l^\epsilon h^{8}_l + \ldots \quad (26)
\]

where the lattice amplitudes \( A_l^\sigma, A_l^\epsilon \) are different from the corresponding amplitudes evaluated in the continuum.

In order to relate the lattice results with the continuum ones, we must fix the relative normalizations of \( \sigma \) versus \( \sigma_l \), \( \epsilon \) versus \( \epsilon_l \) and \( h \) versus \( h_l \).

The simplest way to do this is to look at the analogous of eq.(6,7) at the critical point (namely for \( h_l = 0 \)) [22]. From the exact solution of the Ising model [18] we know that

\[
\langle \sigma_i \sigma_j \rangle_{h=0} = \frac{R_{\sigma}^2}{|r_{ij}|^{1/4}} \quad (27)
\]

where \( r_{ij} \) denotes the distance on the lattice between the sites \( i \) and \( j \) and we know from [19] that:

\[
R_{\sigma}^2 = e^{3\xi'(-1)}2^{5/24} = 0.70338\ldots \quad (28)
\]

By comparing this result with eq.(6) we find

\[
\sigma_l = R_{\sigma} \sigma = 0.83868\ldots \sigma \quad (29)
\]

\footnote{This essentially amounts to measure all the quantities in units of the lattice spacing. For this reason we can fix in the following the lattice spacing to 1 and neglect it.}
From this we can also obtain the normalization of the lattice magnetic field which must exactly compensate that of the spin operator in the perturbation term $h\sigma$. We find:

$$h_l = (R_\sigma)^{-1}h = 1.1923...h$$  \hspace{1cm} (30)

Combining these two results we obtain the value in lattice units of the constant $A_\sigma$

$$A^l_\sigma = (R_\sigma)^{16/15}A_\sigma = 1.058...$$  \hspace{1cm} (31)

In the case of the energy operator the connected correlator on the lattice, at $h_l = 0$ and for any value of $\beta$, has the following expression [23]:

$$\langle \epsilon_l(0)\epsilon_l(r) \rangle_c = \left(\frac{\delta}{\pi}\right)^2 \left[K^2_1(\delta r) - K^2_0(\delta r)\right]$$  \hspace{1cm} (32)

where $K_0$ and $K_1$ are modified Bessel functions, $\delta$ is a parameter related to the reduced temperature, defined as

$$\delta = 4|\beta - \beta_c|$$  \hspace{1cm} (33)

and with the index $c$ we denote the connected correlator (notice that thanks to the definition [23] no disconnected part must be subtracted at the critical point and the index $c$ becomes redundant). This expression has a finite value in the $\delta \to 0$ limit (namely at the critical point). In fact the Bessel functions difference can be expanded in the small argument limit as

$$\left[K^2_1(\delta r) - K^2_0(\delta r)\right] = \frac{1}{(\delta r)^2} + ...$$  \hspace{1cm} (34)

thus giving, exactly at the critical point:

$$\langle \epsilon_l(0)\epsilon_l(r) \rangle = \frac{1}{(\pi r)^2} \equiv \frac{R^2_\epsilon}{|r|^2}.$$  \hspace{1cm} (35)

Hence $R_\epsilon = 1/\pi$. By comparing this result with eq.\ref{eq:7} we find

$$\epsilon_l = R_\epsilon \epsilon = \frac{\epsilon}{\pi}$$  \hspace{1cm} (36)

and from this we obtain the expression in lattice units of $A_\epsilon$

$$A^l_\epsilon = (R_\sigma)^{8/15}(R_\epsilon)A_\epsilon = 0.58051...$$  \hspace{1cm} (37)

Hence $R_\epsilon = 1/\pi$. By comparing this result with eq.\ref{eq:7} we find

$$\epsilon_l = R_\epsilon \epsilon = \frac{\epsilon}{\pi}$$  \hspace{1cm} (36)

and from this we obtain the expression in lattice units of $A_\epsilon$

$$A^l_\epsilon = (R_\sigma)^{8/15}(R_\epsilon)A_\epsilon = 0.58051...$$  \hspace{1cm} (37)
2.4 Correlators

In the remaining part of this paper we shall be mainly interested in the dependence on the external magnetic field of the following correlators:

\[ G_{\sigma,\sigma} \equiv \langle \sigma(0)\sigma(r) \rangle \] (38)

\[ G_{\epsilon,\epsilon} \equiv \langle \epsilon(0)\epsilon(r) \rangle \] (39)

\[ G_{\sigma,\epsilon} \equiv k\langle \sigma(0)\epsilon(r) \rangle \] (40)

where \( k \equiv \text{sign}(h) \). We already know the behaviour at the critical point of the first two of them, which is given by eq. (6), (7) in the continuum (or equivalently eq. (27), (35) on the lattice), while the OPE constants reported in eq. (8) immediately tell us that \( \langle \epsilon(0)\sigma(r) \rangle = 0 \).

For small values of \( h \) we may expect to add to these results correction terms functions of \( h \) and \( r \). However standard renormalization group arguments show that these two variables are actually related and that there is a natural scaling variable which describes the short distance expansion of these correlators in a magnetic field which is \( t \equiv |h| |r|^{15/8} \). In order to obtain an explicit expansion in powers of \( t \) we must absorb the scaling dimensions of the various operators in the expansion \( \Phi \). To this end let us define

\[ F_{\sigma,\sigma} \equiv \langle \sigma(0)\sigma(r) \rangle |r|^{1/4} \] (41)

\[ F_{\epsilon,\epsilon} \equiv \langle \epsilon(0)\epsilon(r) \rangle |r|^2 \] (42)

\[ F_{\sigma,\epsilon} \equiv k\langle \sigma(0)\epsilon(r) \rangle |r|^{9/8} \] (43)

where \( k \equiv \text{sign}(h) \).

The powers which appear in the \( t \) expansion of the functions \( F \) can be immediately deduced from the analysis of the OPE via the IRS method, that will be described in the following section.

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Footnote: On the contrary, in the large distance regime where one may use the predictions obtained with the form factor approach, the natural normalization is that of the \( G_{\Phi,\Phi} \) functions defined above.
3 The infrared safe approach

The goal of the method presented in Ref. [5] is to obtain informations about the short distance behavior of a conformal field theory perturbed by relevant operators.

The general idea behind this approach, (for preexisting ideas see [9, 10, 11, 12, 13]) is the fact that Wilson Coefficients, being short distance objects, can be taken to have a regular, IR safe, perturbative expansion with respect to the coupling. This OPE approach leaves unfixed some constants that parameterize the vacuum expectation values of operators that appear in conformal field theory.

In [5] it was found that the correlators of the perturbed CFT are given in terms of the derivatives of the Wilson coefficients (calculated at $h = 0$ point). To be precise, they appear in the following way

$$\langle \Phi_i(r)\Phi_j(0) \rangle_h = \sum_{\{k\}} C_{ij}^{\{k\}}(h, r)\langle \Phi_{\{k\}}(0) \rangle_h =$$

$$\sum_{\{k\}} \left[ C_{ij}^{\{k\}}(0, r) + \partial_h C_{ij}^{\{k\}}(0, r) h + \frac{1}{2} \partial_h^2 C_{ij}^{\{k\}}(0, r) h^2 + \cdots \right] \langle \Phi_{\{k\}}(0) \rangle_h.$$

It was also shown that a general formula could be written for the $n$–th derivative of the Wilson Coefficients with respect to $h$. Here we will write only the first and the second order derivatives for the Wilson Coefficients,

$$\sum_b \partial_h^1 C_{a_1a_2}^{b} \langle \phi_b X_R \rangle =$$

$$= \int' d^2 z \langle [\sigma_z (\phi_a \phi_b - \sum_b C_{a_1a_2}^{b} \phi_b) X_R] \rangle$$

and

$$\sum_b \partial_h^2 C_{a_1a_2}^{b} \langle \phi_b X_R \rangle =$$

$$= \int' d^2 z \int' d^2 z' \langle [\sigma_z \sigma_{z'} (\phi_a \phi_b - \sum_b C_{a_1a_2}^{b} \phi_b) X_R] \rangle +$$

$$+ \sum_b \partial_h C_{a_1a_2}^{b} \int' d^2 z \langle [\sigma_z \phi_b X_R] \rangle.$$
The general structure of this formula is a sum of the “naive” perturbative term plus \( n \) (for the \( n - \text{th} \) order coefficient) infrared counterterms. The asterisk reminds that the sum on the counterterms is truncated and is performed up to a given infrared dimension (see again [5]).

The construction of the IRS expansion requires two steps.

- First, one must select by using the OPE rules which operators can appear in the various expansions, identify their scaling dimensions, select the dominant ones and give their expression in terms of the structure constants and of their derivatives.

- Second, one must evaluate the derivatives of the structure constants by using eq.(44) or (45) to reduce them to suitable integrals over correlators evaluated at the critical point. This allows in principle to complete the analysis, since the explicit form of all possible critical correlators is known. However in general these integrals are highly non trivial and their evaluation represents the major problem of the whole approach.

These two steps where performed in [6] for all the terms in the expansion involving at most first order derivatives of the structure constants. This allows to obtain the first three terms in the expansion of the \( \langle \sigma \epsilon \rangle \) and \( \langle \sigma \sigma \rangle \) correlators (which are reported for completeness at the end of this section). On the contrary for the \( \langle \epsilon \epsilon \rangle \) in this way one can only obtain the first two terms of the expansion. Moreover one can verify by an explicit calculation that the second of them is identically zero ([6]). Thus in the \( \langle \epsilon \epsilon \rangle \) correlator, in order to reach the first non trivial correction to scaling, it is mandatory to extend the analysis of [6] and to deal with second order derivatives of the Wilson coefficients. We shall address this problem in the next section. In particular, in sect. 4.1 we shall discuss the first step of the IRS analysis, and select among the possible candidates the one with the lowest power of \( t \) which, as anticipated, turns out to involve a second order derivative of a structure constant. Then in sect. 4.2 (and in the Appendix) we shall explicitly evaluate this contribution. Let us conclude this section by listing for all the three correlators the first three terms of the IRS expansion

\[
F_{\sigma,\sigma} = B_{\sigma\sigma}^1 + B_{\sigma\sigma}^2 t^{8/15} + B_{\sigma\sigma}^3 t^{16/15} + O(t^2) \tag{46}
\]

\[
F_{\epsilon,\epsilon} = B_{\epsilon\epsilon}^1 + B_{\epsilon\epsilon}^2 t^{16/15} + B_{\epsilon\epsilon}^3 t^2 + O(t^{32/15}) \tag{47}
\]
\[ \kappa F_{\sigma, \epsilon} = B_1^{\sigma} t^{1/15} + B_2^{\sigma} t + B_3^{\sigma} t^{23/15} + O(t^{31/15}) \] (48)

where the coefficients \( B_{i}^{\Phi, \Phi} \) are

\[
\begin{align*}
B_1^{\sigma\sigma} & = \bar{C}_1^{\sigma\sigma} \\
B_2^{\sigma\sigma} & = A_{\epsilon} \bar{C}_1^{\epsilon\sigma} \\
B_3^{\sigma\sigma} & = A_{\epsilon} \bar{\partial}_h \bar{C}_1^{\epsilon\sigma} \\
B_1^{\epsilon\epsilon} & = \bar{C}_1^{\epsilon\epsilon} \\
B_2^{\epsilon\epsilon} & = A_{\epsilon} \bar{\partial}_h \bar{C}_1^{\epsilon\epsilon} \\
B_3^{\epsilon\epsilon} & = \frac{1}{2} \bar{\partial}_h^2 \bar{C}_1^{\epsilon\epsilon} \\
B_1^{\sigma\epsilon} & = A_{\epsilon} \bar{C}_1^{\epsilon\sigma} \\
B_2^{\sigma\epsilon} & = \bar{\partial}_h \bar{C}_1^{\epsilon\sigma} \\
B_3^{\sigma\epsilon} & = A_{\epsilon} \bar{\partial}_h \bar{C}_1^{\epsilon\sigma}
\end{align*}
\]

the derivatives of \( C_{\Phi_i, \Phi_j}^{\Phi_k} \) are given by (44) and the notation \( \bar{\partial}_h \bar{C}_{\Phi_i, \Phi_j}^{\Phi_k} \) is the extension of the definition given in eq.(5) to the derivatives of the Wilson coefficients. The first order derivatives have been calculated in [6] we report here for completeness their numerical value. Notice a slight change with respect to [6] due to the different choice of normalizations for the \( \epsilon \) operator (the \( \epsilon \) of the present paper corresponds to 2\( \pi \) times that of ref. [6])

\[
\begin{align*}
\bar{\partial}_h \bar{C}_1^{\sigma\sigma} & = -0.40374 \\
\bar{\partial}_h \bar{C}_1^{\epsilon\epsilon} & = 0 \\
\bar{\partial}_h \bar{C}_1^{\epsilon\sigma} & = 3.29627 \\
\bar{\partial}_h \bar{C}_1^{\epsilon\sigma} & = -0.90900
\end{align*}
\]

The second order derivative which appears in last one in the \( \langle \epsilon\epsilon \rangle \) correlator requires a more involved calculation which we shall discuss in the next section and in the Appendix.

4 Second order corrections
4.1 Dimensional analysis

To estimate the higher order corrections to $\langle \epsilon \epsilon \rangle$ we must analyse two kinds of possible contributions.

- The expectation values of secondary operators multiplied by the Wilson coefficients and their first derivatives.
- The second derivatives of Wilson coefficients.

We would like to understand which is the most important of these terms. In the Ising model there are two secondary operators at first level, obtained by acting on $\epsilon$ and $\sigma$ with the Virasoro generator $L_{-1}$ and its hermitian conjugate (the action of $L_{-1}$ on $1$ gives 0). We start by considering

$$\epsilon^1 \equiv L_{-1}\bar{L}_{-1}\epsilon \quad (49)$$

and

$$\sigma^1 \equiv L_{-1}\bar{L}_{-1}\sigma \quad (50)$$

where $L_k$, $\bar{L}_k$ are Virasoro generators. It is clear that the expectation value of this kind of operators is zero being total derivatives. So let us go to second level of the algebra. There are two possible terms:

$$L_{-1}^2\bar{L}_{-1}^2\phi, \quad L_{-2}\bar{L}_{-2}\phi \quad (51)$$

where $\phi$ is a generic primary field. In this situation we have to consider also the identity operator.

In the identity sector the only contribution is given by

$$\bar{T}T = L_{-2}\bar{L}_{-2} \, 1 \quad (52)$$

i.e. the energy-momentum tensor. Again a simple dimensional analysis shows that

$$\dim \, TT = 4, \quad \langle \bar{T}T \rangle_h = A_{TT}|h|^{32/15} \quad (53)$$

giving

$$A_{TT} \, \hat{C}_{\epsilon\epsilon}^{TT} \, t^{32/15}. \quad (54)$$

It is clear that the terms containing secondary operators (of second level) of $\sigma$ and $\epsilon$ are of higher order in $t$ and will not be considered here.
A second possible contribution is given by the higher order derivative

\[ \sum_b \partial^2_h C^b_{a_1 a_2} \langle \phi_b X_R \rangle = \int' d^2 z \int' d^2 z' \langle [\sigma_z \sigma_{z'} (\phi_a \phi_b - \sum_b C^b_{a_1 a_2} \phi_b) X_R] \rangle + \]

\[ + \sum_b \partial_h C^b_{a_1 a_2} \int' d^2 z \langle [\sigma_z \phi_b X_R] \rangle. \]  

(55)

Let us fix \( X_R = 1 \). An elementary computation shows that the series is truncated and only those operators having \( x_b \leq \frac{15}{8} \) appear in it.

It follows that

\[ \partial^2_h C^1_{\epsilon \epsilon} = \]

\[ = \int' d^2 z \int' d^2 z' \langle \sigma_z \sigma_{z'} (\epsilon, \epsilon_0 - C^1_{\epsilon \epsilon}) \rangle + \]

\[ + \partial_h C^\sigma_{\epsilon \epsilon} \int' d^2 z \langle \sigma_z \sigma_0 \rangle \]  

(56)

but \( \partial_h C^\sigma_{\epsilon \epsilon} = 0 \), and we can say that

\[ \partial^2_h C^1_{\epsilon \epsilon} = \int' d^2 z \int' d^2 z' \langle \sigma_z \sigma_{z'} (\epsilon, \epsilon_0 - C^1_{\epsilon \epsilon}) \rangle. \]  

(57)

Again from dimensional analysis we get

\[ \text{dim} \partial^2_h C^1_{\epsilon \epsilon} = \frac{14}{8} \]  

(58)

and the contribution to \( \langle \epsilon \epsilon \rangle \) of the second order derivative is given by

\[ \frac{1}{2} \partial^2_h C^1_{\epsilon \epsilon} t^2. \]  

(59)

It is also clear that, being \( X_R = 1 \) the lowest dimension operator, derivatives of Wilson coefficients relative to \( \sigma \) and \( \epsilon \) will give terms with a higher power in \( t \).

Let us write finally the perturbative expansion of this correlator

\[ F_{\epsilon \epsilon} = C^1_{\epsilon \epsilon} + \frac{1}{2} \partial^2_h C^1_{\epsilon \epsilon} t^2 + O(t^{32/15}). \]  

(60)
4.2 The Wilson derivative

Let us remember that

\[
\partial^2_h C^1_{\epsilon\epsilon} = \int d^2 z \int d^2 z' \langle \sigma_z \sigma_{z'} (\epsilon, \epsilon - C^1_{\epsilon\epsilon}) \rangle
\]  

(61)

where \( \langle \sigma(z_1) \sigma(z_2) \epsilon(z_3) \epsilon(z_4) \rangle \) denotes the correlator at the critical point which can be written as

\[
\langle \sigma(z_1) \sigma(z_2) \epsilon(z_3) \epsilon(z_4) \rangle = \left| \frac{z_{12}(z_{32} + z_{42}) - 2z_{32}z_{42}}{4|z_{42}z_{32}z_{41}z_{31}| |z_{43}|^2 |z_{12}|^{1/4}} \right|^2.
\]  

(62)

By fixing the values of \( z_1 = z, z_2 = w, z_4 = 0 \) and by rescaling \( r \) we can choose \( z_3 = 1 \), we get

\[
\partial^2_h C^1_{\epsilon\epsilon} = \int d^2 w \int d^2 z' \frac{|z(1 - w) + w(1 - z)|^2}{4|w(1 - w)z(1 - z)||z - w|^{1/4}} + \cdots
\]

(63)

where the dots indicate the counterterms.

By fixing the values of \( z_1 = z, z_2 = w, z_4 = 0 \) and by rescaling \( r \) we can choose \( z_3 = 1 \), we get

\[
\partial^2_h C^1_{\epsilon\epsilon} = \int d^2 w \int d^2 z \frac{|z(1 - w) + w(1 - z)|^2}{4|w(1 - w)z(1 - z)||z - w|^{1/4}} + \cdots
\]

where the dots indicate the counterterms.

The explicit calculation of this integral can be done using a technique developed by Mathur, [24]. The general idea behind this approach is to factorize the integral in a holomorphic and antiholomorphic part using Stokes theorem. The calculation is reported in the Appendix. After this calculation, in order to get rid of the infrared cutoff we perform a Mellin transform of the integral and the infrared counterterm (see [3] for more details). In this we we end up with a finite result when the infrared cutoff goes to infinity. The final result is

\[
\partial^2_h C^1_{\epsilon\epsilon} = 97.5936 \ldots
\]

(64)

Let us stress that the techniques that we have discussed in this section can be extended to any order in the derivatives of the Wilson coefficients. This is a rather important observation, since it allows, in principle, to study the IRS corrections, in a consistent way, to any given order in \( t \).

5 The Montecarlo simulation

It has been recently shown [25] that in the case of the 2d Ising model in a magnetic field, algorithms based on the exact (or approximate) diagonalization of the transfer matrix are much more effective than standard Montecarlo
simulations. In particular this is true for all possible observables involved in the large distance behaviour of the model. The only exception is represented by the short distance behaviour of the point-point correlators which is the subject of the present paper. In fact in order to reach lattices as large as possible in the transfer-matrix programs discussed in [25] only zero momentum projected observables could be studied, while we are instead interested in point-point correlators. Moreover, we need to have a window as large as possible between the region (few lattice spacings) dominated by the lattice artifacts and the correlation length. This window shrinks to zero in the transfer matrix approach where only small values of the correlation length can be studied.

For these reasons we decided to perform our tests with standard Monte-carlo simulations. We used a Swendsen-Wang type algorithm, modified so as to take into account the presence of an external magnetic field. For a detailed description of the algorithm see for instance [26, 27].

5.1 Finite size effects.

As a preliminary test we performed a simulation at $h_l = 4.4069 \times 10^{-4}$ (which corresponds to $H = 0.001$) with lattice size $L = 128$ which exactly coincides with one of the simulations reported in [26] and found results in complete agreement with those quoted in [26]. Then for the same value of $h_l$ we performed a set of high precision simulations varying the lattice size so as to check the presence of possible finite size effects. In particular we compared our estimates of the mean magnetization, susceptibility and internal energy with the known exact results, extrapolated at the value of $h_l$ at which we performed the simulations.

The comparison is reported in tab.1. It turns out that lattice sizes at

---

4If one is interested in a high precision comparison, also the contribution of secondary fields should be taken into account in extracting these exact estimates. The amplitude of some of these secondary fields have been evaluated numerically in [25]. In the case of $M$ and $\chi$, for the values of $h_l$ in which we are interested, the contributions of the secondary fields are strictly smaller than the statistical errors of the results of our simulations and hence can be neglected in the comparison. On the contrary for the internal energy it turns out that the amplitude of the first correction is rather large (see [25] for details) and must be taken into account. In fact, if we would neglect it, instead of the value reported in tab.1 we would find $E = 0.71652$ in clear disagreement with the Montecarlo results. This represents a non trivial test of the results of [25].
Table 1: Finite size effects at $h_l = 4.4069 \times 10^{-4}$. In the first column we report the lattice sizes used in the simulations, in the remaining three columns the mean values of the magnetization, susceptibility and internal energy. In the last row we report the exact results obtained by using the known values of the amplitudes for these quantities.

| $L$ | $M$       | $\chi$  | $E$      |
|-----|-----------|---------|----------|
| 120 | 0.63110(16) | 113(1)  | 0.71638(3) |
| 140 | 0.63247(16) | 98.4(6) | 0.71645(3) |
| 160 | 0.63245(14)  | 96.4(4) | 0.71639(3) |
| 200 | 0.63255(11)  | 95.4(3) | 0.71643(3) |
| exact | 0.63260     | 95.7    | 0.71642   |

least larger than 12 times the correlation length are needed to be sure that finite size effects are under control (with this we mean that the systematic errors induced by the finite size of the lattice are smaller than the statistical errors of the simulations and can be neglected). A side consequence of this observation is that the simulations reported in [26] are indeed affected by rather large finite size effects.

It is interesting to notice that the magnetic observables are more affected by finite size effects than the thermal one. As one can easily expect the largest corrections appear in the case of the susceptibility.

5.2 The simulation

Once we were sure to have finite size effects under control we performed a set of high precision simulations of the model for three different values of the magnetic field.

An important quantity to understand the range of validity of the IRS approximation is the correlation length. Roughly speaking we expect that the IRS results should give a reasonable approximation for distances equal or smaller than the correlation length, while above it the form factor approach should give results of better quality. For this reason it is important to have a good estimate of $\xi$. This can be easily obtained from the knowledge of the spectrum of the theory. We find, in lattice units:
(see \cite{25} for details on the continuum to lattice conversion of $\xi$). In tab. 2 we have reported the expected values of $\xi$ in our cases.

Table 2: Values of the correlations lengths for the three choices of $h_l$.

| $h_l$       | $\xi$   |
|------------|---------|
| $4.4069 \times 10^{-4}$ | 15.4    |
| $2.2034 \times 10^{-4}$ | 22.4    |
| $1.1017 \times 10^{-4}$ | 32.2    |

For all the values of $h_l$ that we simulated, we studied the three correlators: $\langle \sigma(0) \sigma(r) \rangle$, $\langle \sigma(0) \epsilon(r) \rangle$ and $\langle \epsilon(0) \epsilon(r) \rangle$, for $r = 1, \ldots, L_{\text{max}}$, where the maximum distance $L_{\text{max}}$ was chosen to be roughly twice the correlation length. In this way we can test our results also in the large distance regime, where predictions from the form factor approach are expected to give very precise estimates for the correlators. Notice that when studying the large distance behaviour of correlators one is usually interested in the zero momentum projection of the connected part of the correlator. On the contrary in the present case we are interested in the point–point correlators without mean value subtraction or zero momentum projections. This must be taken into account when comparing the data with those obtained with the form factor approach. Some informations on the simulations are reported in tab. 3.

Table 3: Some informations on the simulations. $L_{\text{max}}$ denotes the maximum distance at which the correlators have been evaluated, its value almost coincides with twice the correlation length. $L$ denotes the lattice size, $h_l$ the magnetic field. In the third column we have reported the number of measures while in the fourth column we have reported the number of SW sweeps which separates two measures.

| $h_l$       | $L$    | measures | sweep/measures | $L_{\text{max}}$ |
|------------|--------|----------|----------------|-----------------|
| $4.4069 \times 10^{-4}$ | 200    | $4 \times 10^5$ | 5              | 30              |
| $2.2034 \times 10^{-4}$ | 300    | $2 \times 10^5$  | 5              | 45              |
| $1.1017 \times 10^{-4}$ | 400    | $1 \times 10^5$  | 10             | 65              |
We report an example of our results (for the value $h_l = 4.4069 \times 10^{-4}$) in the first columns of tabs. 7, 8 and 9. The quoted errors have been obtained with a standard jackknife method.

6 Discussion of the results

In figs 1-8 and tab.7, 8 and 9 we compare our estimates for the correlators with the IRS and form factor predictions. For completeness we briefly recall here the form factor results (see [28, 29] for details) and give the numerical values (once all the conversion factors are taken into account) of the constants in the IRS approach.

6.1 Form factor results

The scattering theory which describes the scaling limit of the Ising Model in a magnetic field \[3\] contains eight different species of self-conjugated particles $A_a$, $a = 1, \ldots, 8$ with masses

\[
m_2 = 2m_1 \cos \frac{\pi}{5} = (1.6180339887\ldots) m_1,
m_3 = 2m_1 \cos \frac{\pi}{30} = (1.9890437907\ldots) m_1,
m_4 = 2m_2 \cos \frac{7\pi}{30} = (2.4048671724\ldots) m_1,
m_5 = 2m_2 \cos \frac{2\pi}{15} = (2.9562952015\ldots) m_1,
m_6 = 2m_2 \cos \frac{\pi}{30} = (3.2183404585\ldots) m_1,
m_7 = 4m_2 \cos \frac{\pi}{5} \cos \frac{7\pi}{30} = (3.8911568233\ldots) m_1,
m_8 = 4m_2 \cos \frac{\pi}{5} \cos \frac{2\pi}{15} = (4.7833861168\ldots) m_1
\]

$m_1(h_l)$ denotes the overall mass scale and coincides with $1/\xi$, hence its value in lattice units is

\[
m_1(h_l) = 4.0104... h_l^{s/\xi}.
\]

From the knowledge of the masses and of the form factors it is possible to obtain a large distance approximation for the correlators by constructing
a spectral sum over a complete set of intermediate states. We thus find for any pair of local operators $\Phi_1$ and $\Phi_2$:

$$G_{\Phi_1,\Phi_2}(x) \equiv \langle \Phi_1(x)\Phi_2(0) \rangle$$

$$= \sum_{n=0}^{\infty} \int_{\theta_1>\theta_2>\cdots>\theta_n} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi} F^{\Phi_2*}_{a_1 \cdots a_n}(\theta_1, \ldots, \theta_n) F^{\Phi_1}_{a_1 \cdots a_n}(\theta_1, \ldots, \theta_n) \times e^{-|x|\sum_{k=1}^{n} m_k \cosh \theta_k} ,$$

where the form factors $F^{\Phi}_{a_1 \cdots a_n}(\theta_1, \ldots, \theta_n)$ are the matrix elements of the local operator $\Phi(x)$ on the asymptotic states $A_a$, i.e. they are defined as

$$F^{\Phi}_{a_1 \cdots a_n}(\theta_1, \ldots, \theta_n) = \langle 0 | \Phi(0) | A_{a_1}(\theta_1) \cdots A_{a_n}(\theta_n) \rangle.$$  \hfill (69)

The important point is that these form factors can be exactly computed in the integrable models once the $S$–matrix is known.

It is natural to organize the above expansion setting a reference value $m_r$ and keeping in the spectral sum only the states with a mass smaller than $m_r$. Looking at eq.(68) we see that we must expect, as a consequence of this truncation, a systematic error in the approximation of the order $O(e^{-m_rx})$. Up to $m_r = 2m_1$ (where $m_1$ is the fundamental mass of the model) only single particle states survive in the sum and eq.(68) greatly simplifies. In particular looking at eq.(66) we see that only the first three states (which are the only ones below the pair production threshold) survive. Eq.(68) becomes in this case, (choosing for instance the spin-spin correlator)

$$G_{\sigma\sigma}(r) \sim M^2(h_l)(1 + \sum_{i=1}^{3} \frac{|F_i^\sigma|^2}{\pi} K_0(m_{i}r))$$  \hfill (70)

where $K_0(x)$ is the modified Bessel function and $F_i^\sigma$ denotes the overlap (measured in units of the magnetization) of the $i^{th}$ state with the $\sigma$ operator. $M(h_l)$ denotes the magnetization and its $h_l$ dependence is given in (12, 31).

Similarly the spin-energy and energy-energy correlators are given by

$$kG_{\sigma\epsilon}(r) \sim M(h_l)E(h_l)(1 + \sum_{i=1}^{3} \frac{F_i^\sigma F_i^\epsilon}{\pi} K_0(m_{i}r))$$  \hfill (71)

$$G_{\epsilon\epsilon}(r) \sim E^2(h_l)(1 + \sum_{i=1}^{3} \frac{|F_i^\epsilon|^2}{\pi} K_0(m_{i}r))$$  \hfill (72)
The constants $F_σ^i$ and $F_ε^i$ have been evaluated in [28, 29]. Their value is reported in tab.4 and 5.

One can systematically improve the approximation by setting higher values of $m_r$. For instance, for $m_r = 3$ one must keep into account the first five single particle form factors together with the two-particle ones involving the (1,1), (1,2) and (1,3) pairs, and so on. By using the results of [28, 29] also these multiparticle form factors can be evaluated exactly.

Table 4: Overlap amplitudes for the spin operator.

\[
\begin{array}{c}
F_1^σ = -0.64090211 \\
F_2^σ = 0.33867436 \\
F_3^σ = -0.18662854 \\
F_4^σ = 0.14277176 \\
F_5^σ = 0.06032607 \\
F_6^σ = -0.04338937 \\
F_7^σ = 0.01642569 \\
F_8^σ = -0.00303607 \\
\end{array}
\]

Table 5: Overlap amplitudes for the energy operator.

\[
\begin{array}{c}
F_1^ε = -3.70658437 \\
F_2^ε = 3.42228876 \\
F_3^ε = -2.38433446 \\
F_4^ε = 2.26840624 \\
F_5^ε = 1.21338371 \\
F_6^ε = -0.96176431 \\
F_7^ε = 0.45230320 \\
F_8^ε = -0.10584899 \\
\end{array}
\]

As discussed above, in eq.s (70), (71) and (72) we expect systematic errors of the order $\mathcal{O}(e^{-2m_1r})$. For distances larger than the correlation length these deviations are very small but become increasingly relevant as the correlation length is approached. It would be important to have an estimate of the magnitude of these corrections. From this point of view the present case is a perfect laboratory since we know that all other possible sources of systematic
errors are under control. We report an example of the results obtained using eq.s (70), (71) and (72) (for the value $h_l = 4.4069 \times 10^{-4}$) in the last column of tabs. 7, 8 and 9. In order to gain further insight on the convergence of the approximation we have also evaluated the contribution of the first eight terms of the spectral series (i.e. with $m_r = 3m_1$). We report in fig.10 the result of this analysis, together with those with $m_r = 2m_1$ and $m_r = m_1$ for comparison, in the particular case of the $\sigma\sigma$ correlator.

6.2 IRS approach

By using the results of sect. 3 and 4 and the exact knowledge of the constants $R_\sigma$ and $R_\epsilon$ one can easily write the constants $B^i_{\Phi\Phi}$ in lattice units. They are reported in tab.6.

We report an example of our results (obtained by plugging the values of tab.6 in eqs.(46), (47) and (48) in the particular case $h_l = 4.4069 \times 10^{-4}$), in the second column of tabs. 7, 8 and 9.

Table 6: Coefficients of the IRS expansion in lattice units.

| $B^1_{\sigma\sigma}$ | 0.7034... |
| $B^2_{\sigma\sigma}$ | 0.6414... |
| $B^3_{\sigma\sigma}$ | -0.3007... |
| $B^1_{\sigma\epsilon}$ | 0.1013... |
| $B^2_{\sigma\epsilon}$ | 0 |
| $B^3_{\sigma\epsilon}$ | 3.4776... |
| $B^1_{\epsilon\sigma}$ | 0.1685... |
| $B^2_{\epsilon\sigma}$ | 0.7380... |
| $B^3_{\epsilon\sigma}$ | -0.3712... |

6.3 Comparison with MC results

To stress this fact we introduce in analogy to $h_l$ a new scaling variable $t_l = |h_l| |r|^{15/8}$. In the coefficients reported in tab.6 the conversion from $t$ to $t_l$ has already been taken into account, hence they refer to the IRS expansion in powers of $t_l$ of the correlators.

\footnote{In doing this conversion one must also take into account the $h$ factor contained in $t$. To stress this fact we introduce in analogy to $h_l$ a new scaling variable $t_l = |h_l| |r|^{15/8}$. In the coefficients reported in tab.6 the conversion from $t$ to $t_l$ has already been taken into account, hence they refer to the IRS expansion in powers of $t_l$ of the correlators.}
6.3.1 Lattice artifacts

It is interesting to see that lattice artifacts are confined to a remarkably small region of few lattice spacings, which shows a negligible dependence on the magnetic field or on the type of correlator. Since the lattice artifacts decrease so quickly it is rather easy to find where does the region of applicability of the IRS results starts, by simply looking at the distance $L_{\text{min}}$ where for the first time the IRS prediction becomes compatible (within the errors) with the MC data or, if this never happens, looking at the location of the minimum difference between IRS predictions and the MC simulations. It turns out that for all correlators and $h_l$ values $L_{\text{min}}$ ranges between 7 and 9 lattice spacings. This tells us that, at least in the case of the Ising model perturbed by a magnetic field, the IRS method has a large window of applicability, which becomes larger and larger as the critical point is approached. This is well exemplified by fig. 6 and 7 where the difference between MC data and IRS predictions is plotted, for the $\sigma\sigma$ correlator in the short range region, first in units of the lattice spacing and then in units of the correlation length. It would be interesting to test if this behaviour also holds for other models or for different realizations of this one.

6.3.2 $h_l$ dependence

In the range $L_{\text{min}} < r < \xi$ the agreement between IRS predictions and MC results is always very good. In particular, it seems that the method reaches its better results in the case of $\epsilon\sigma$ correlator. As expected, the IRS approximation becomes better and better as we approach the critical point (see fig.s 3, 4 and 5). First because the range of validity becomes larger and second because the systematic deviations due to the terms neglected in the expansion, which are proportional to higher powers of $h_l$ become less important. In particular for the $\epsilon\sigma$ correlator at $h_l = 1.1017 \times 10^{-4}$ there is a wide range (more than 40 lattice spacings) in which the IRS prediction coincides with the MC results within the errors (see fig.8).

---

\footnote{We may expect that higher orders in the $t$ expansion improve the large distance behaviour of the IRS results, but they give a negligible contribution around $L_{\text{min}}$ where the discrepancy between Montecarlo data and IRS predictions is completely dominated by lattice artifacts.}
6.3.3 IRS versus FF.

Looking at tables 7, 8, 9 and at the fig.s 1-5 we see that, as expected, the FF approach performs better than the IRS one for distances larger than the correlation length and that the opposite is true for distances smaller than $\xi$. It is interesting to see that for distances of the order of the correlation length the IRS and FF methods give comparable performances. Some interesting informations on the systematic errors involved in the two approximations can be extracted from the data.

1) While the systematic errors in the IRS approach have a polynomial behaviour in the distance $r$ (which is contained in $t_l$), those of the FF approach have an exponential behaviour. This is clearly visible in fig.2 where the deviations are plotted as a function of $r$ for the correlator $G_{\sigma\sigma}$ at $h_l = 4.4069 \times 10^{-4}$ and in fig.3-5 where they are plotted for all the correlators and all the values of $h_l$ as functions of $r/\xi$. This makes the IRS method still reasonably reliable even for distances twice the correlation length.

2) We may obtain a rough estimate of the magnitude of the systematic errors involved in the IRS approach with the following argument. Since $t_l \equiv |h_l| |r|^{15/8}$, looking at eq.(33) we see that for distances of the order of the correlation length we have $t_l \sim 0.06$. Depending on the correlator chosen, we would expect deviations of the order $O(t_l^2)$, $O(t_l^{31/15})$, $O(t_l^{32/15})$. Since in general the $B_{\phi\phi}$ constants are of order unity, this amounts to an expected deviation for the $F_{\phi\phi}$ functions of order $\delta F \sim 0.004$. This expectation is in good agreement with the values of $\delta F$ obtained by comparing the MC and IRS estimates at the distance $r = \xi$. (see for instance the data reported in tab.s 7,8 and 9. In using these data one must take into account the normalization between $F_{\phi\phi}$ and $G_{\phi\phi}$ functions).

3) A similar analysis can be performed in the case of the FF method. In this case we know that the systematic errors are of order $O(e^{-m_r})$. We shall further discuss them in sect.6.3.5 below. In the large distance regime $r \geq 1.5 \xi$ the performances of the FF approach are very good. For instance, in this region, for the lowest value of $h$ that we studied:
\[ h_l = 1.1017 \times 10^{-4} \] the FF predictions for the \( \epsilon \epsilon \) correlator coincide with the MC results within the errors (see fig.5).

### 6.3.4 Convergence of the IRS expansion

It is interesting to study the convergence properties of the IRS method, i.e. to see if the agreement with the Montecarlo data improves as higher terms are added in the expansion.

- \( \langle \epsilon \sigma \rangle \) and \( \langle \sigma \sigma \rangle \).

  In these two cases the agreement improves as new terms are added in the expansion. This is clearly visible in fig.8 where we have plotted the difference between the MC data for the correlator \( \langle \epsilon \sigma \rangle \) at \( h_l = 1.1017 \times 10^{-4} \) and the IRS results with one (pluses), two (crosses) and three (diamonds) terms in the expansion. The analogous plot for the \( \langle \sigma \sigma \rangle \) correlator shows exactly the same behaviour.

- \( \langle \epsilon \epsilon \rangle \).

  In this case we find exactly the opposite behaviour. As it is shown in fig.9 the new coefficient that we evaluated does not improve the agreement with the Montecarlo data which is, by the way, impressively good already with the simple zero order contribution to the correlator. We see two possible reasons for this, rather unexpected, behaviour.

1] As noticed in sect.4, the next term in the perturbative expansion of the correlator has an exponent \( 32/15 \), which is very near to the one that we evaluated. We cannot evaluate this further contribution since it would require the knowledge of the expectation value of the \( T \bar{T} \) operator. In principle this term could well compensate the deviation that we observe.

2] This behaviour could be an indication of the bad convergence properties of the IRS method. If this is the case it would be very interesting to test (in view of the good behaviour of the other two correlators) if this is a peculiar feature of the \( \langle \epsilon \epsilon \rangle \) correlator or a feature of the expansion itself. This issue could be settled in principle by looking to the higher perturbative terms in the expansions of the two other correlators. We plan to address this point in a forthcoming paper.
6.3.5 Convergence of the FF expansion

In order to study the convergence of the FF approximation we have compared the Montecarlo data in the case of the $\langle \sigma \sigma \rangle$ correlator at $h_l = 1.1017 \times 10^{-4}$ with the result of the FF approximation truncated at $m_r = m_1$, $m_r = 2m_1$, $m_r = 3m_1$ respectively. This corresponds to take into account one, three and eight states respectively in the spectral sum. The results of the comparison are reported in fig.10. Looking at this figure one may see that as higher orders are added the approximation smoothly converges to the MC data. By comparing the three approximations one may get a perception of the convergence rate of the method.

7 Concluding remarks

In this paper we have compared the predictions of the IRS and FF approximations for the $\sigma \sigma$, $\epsilon \sigma$ and $\epsilon \epsilon$ correlators with the results of a set of high precision MC simulations of the 2d Ising model perturbed by a magnetic field. To this end we have extended the IRS approach to second order derivatives of the structure constants. Our main results are:

- Lattice artifacts are confined in a small region of few lattice spacings.
- There is a wide region ranging from $\sim 7 - 9$ lattice spacings to the correlation length in which the MC data are in good agreement with the IRS results.
- The agreement improves as the critical point is approached.
- For distances smaller than $\xi$ the IRS gives a better approximation than the FF method, while the opposite is true for distances larger than $\xi$.

The IRS method can be extended in principle to any order in the derivatives of the Wilson coefficients, by using the integration method of [24] and the technique of the Mellin transform. However in the case that we studied, i.e. the $\langle \epsilon \epsilon \rangle$ correlator, the IRS method turns out to show rather bad convergence properties. It remains an open problem to understand if this is a limit of the method itself or if it is a peculiar feature of the correlator that we have chosen.
It would be very interesting to extend this analysis to other models in this same universality class. In particular one could study the model recently introduced in [30, 31] for which an exact bethe ansatz solution, out of the critical point exists. Another interesting application of the method would be the study of the correlators in the case of the most general perturbation of the Ising critical point (i.e. a mixed situation with both magnetic and thermal perturbations). In this case the exact integrability is lost but the IRS method is still valid and could give important informations on the behaviour of the correlators. In particular it would allow us to compare our approximation with the interesting results, directly obtained on the lattice, in [32].

**Appendix**

The evaluation of the coefficient \( \partial^2_2 C_1 \) involves the calculation of the integral

\[
Z = \int \, d^2 w \, |w|^\epsilon |1 - w|^r w^s (1 - w)^t \int \, d^2 z \, |z|^{\alpha} |1 - z|^\beta z^m (1 - z)^n |z - w|^\gamma \tag{73}
\]

where \( n, m, r, s \in \mathbb{N} \) and \( \alpha, \beta, \gamma, e, f \in \mathbb{R} \).

It is useful to introduce the following theorem (see [24], [3]). Let us consider an integral of the form

\[
I = \int \, d^2 w \, \sum_{\alpha,\beta=1}^N f_\alpha(w) Q_{\alpha\beta} \bar{f}_\beta(w^*) \tag{74}
\]

where \( \{ f_\alpha(w) \}_{\alpha=1,N} \), \( \{ \bar{f}_\beta(w^*) \}_{\beta=1,N} \) are two sets of independent functions and \( Q_{\alpha\beta} \) is a constant matrix. Let us assume that \( \bar{f}_\beta(w^*) \) e \( (f_\beta(w))^* \) have the same monodromies, in particular the two sets of functions \( f_\alpha(w) \) and \( g \equiv (f_\beta(w^*))^* \) must have the same branch points \( \{ w_k \}_{k=0}^{m+1} \) such that

\[
0 = |w_0| < |w_1| < \cdots < |w_m| < |w_{m+1}| = \infty \tag{75}
\]

and they have to be analytic elsewhere.

If we assume now that the matrix \( Q \) is invariant under the monodromy group action

\[
Q = M_k^* Q M_k^t, \quad \forall k \tag{76}
\]
where $M_k$ are the monodromy matrices of $f$ and $g$ related to the branch points $w_k$, it follows that we are able to express $I$ in terms of one-dimensional integrals (see [24], [6] for more details)

$$I = \frac{i}{2} \sum_{k=1}^{m} \mathcal{I}(k) \left[\left((1 - M_{k+1})^{-1} - (1 - M_k)^{-1}\right)^{t} Q\right]_{\alpha\beta} \tilde{\mathcal{I}}(k)$$

where $^t$ is the transposition and

$$\mathcal{I}(k) \equiv \int_{C_k} df(w)$$

$$\tilde{\mathcal{I}}(k) \equiv \int_{C_k} d\bar{w} \bar{f}(w)$$

where $C_k$ ($\tilde{C}_k$) are counter-clockwise (clockwise) circumferences enclosing all the branch points of modulus lower than $w_k$, starting at $w_{k+}$ (infinitesimally over the cut at $w_k$) and ending at $w_{k-}$ (infinitesimally under the cut at $w_k$).

Now we are able to evaluate both the $z$-plane and $w$-plane integrations of (73) using the previous lemma. First, to perform the $z$-plane integration, we pose

$$I_z(w, w^*) = \int d^2z |z|^\alpha |1 - \bar{z}|^\beta z^n(1 - z)^m |z - w|^\gamma$$

so the $z$-plane integration involves the following branch points

$$z_0 = 0, \quad z_1 = w, \quad z_2 = 1, \quad z_\infty = \infty.$$

Thus, the application of (77) gives

$$I_z(w, w^*) = \frac{i}{2} \left[ I_1(1) T_{12}^{(1)} \tilde{I}_2(1) \right] + \frac{i}{2} \left[ I_1(2) T_{12}^{(2)} \tilde{I}_2(2) \right]$$

where

$$T_{12}^{(1)} = \frac{e^{i\pi\alpha}(e^{i\pi\gamma} - 1)}{(e^{i\pi(\alpha+\gamma)} - 1)(e^{i\pi\alpha} - 1)}$$

and

$$T_{12}^{(2)} = \frac{e^{i\pi(\alpha+\gamma)}(e^{i\pi\beta} - 1)}{(e^{i\pi(\alpha+\beta+\gamma)} - 1)(e^{i\pi(\alpha+\gamma)} - 1)}$$
are the only non vanishing entries of the matrices

\[ T^{(1)} = \left( (1 - M_2)^{-1} - (1 - M_1)^{-1} \right) Q \]
\[ T^{(2)} = \left( (1 - M_\infty)^{-1} - (1 - M_2)^{-1} \right)^t Q. \]  (84)

This imply that we have to take in account only the integrals

\[ I_{(1)}^{(1)} = (e^{-i\pi\alpha} - 1) w^{1+\alpha/2+\gamma/2+n} \frac{\Gamma(\alpha/2 + n + 1)\Gamma(\gamma/2 + 1)}{\Gamma(\alpha/2 + \gamma/2 + 2 + n)} \cdot F(-\beta/2, \alpha/2 + n + 1; \alpha/2 + \gamma/2 + 2 + n; w); \]
\[ \bar{I}_{(1)}^{(1)} = (e^{i\pi\alpha} - 1) w^{1+\alpha/2+\gamma/2} \frac{\Gamma(\alpha/2 + 1)\Gamma(\gamma/2 + 1)}{\Gamma(\alpha/2 + \gamma/2 + 2)} \cdot F(-m - \beta/2, \alpha/2 + 1; \alpha/2 + \gamma/2 + 2; w^*) \]  (85)

and

\[ I_{(2)}^{(2)} = (e^{-i\pi(\alpha+\beta+\gamma)} - 1)(-\gamma)^m \frac{\Gamma(-\alpha/2 - \beta/2 - \gamma/2 - n - 1)\Gamma(\beta/2 + 1)}{\Gamma(-\alpha/2 - \gamma/2 - n)} \cdot F(-\gamma/2, -\alpha/2 - \beta - \gamma/2 - n - 1; -\alpha/2 - \gamma/2 - n; w); \]
\[ \bar{I}_{(2)}^{(2)} = (e^{i\pi(\alpha+\beta+\gamma)} - 1) \frac{\Gamma(-\alpha/2 - \beta/2 - \gamma/2 - m - 1)\Gamma(\beta/2 + 1 + m)}{\Gamma(-\alpha/2 - \gamma/2)} \cdot F(-\gamma/2, -\alpha/2 - \beta - \gamma/2 - m - 1; -\alpha/2 - \gamma/2; w^*). \]  (86)

Finally, putting all these relations in (81), we can recover the wanted result for \( I_z(w, w^*) \).

The \( w \)-plane integration is very similar to the previous one. Now we have to evaluate

\[ Z = \int d^2 w |w|^e |1 - w|^f w^{*r}(1 - w)^s I_z(w, w^*) \]  (87)

which involves \( w_0 = 0, w_1 = 1, w_\infty = \infty \) as branch points. Hence the solution is given by (77), i.e.

\[ Z = \frac{i}{2} \left[ \mathcal{I}^{(1)} T^{(1)} \bar{\mathcal{I}}^{(1)} \right] \]  (88)

where

\[ T^{(1)} = \left( (1 - M_\infty)^{-1} - (1 - M_0)^{-1} \right)^t Q. \]  (89)
The contribution coming from $I^{(1)}$ is

\[
I_1^{(1)} = (e^{i\pi e} - 1) \int_0^1 dz f_1 = (e^{i\pi e} - 1) J_1
\]

\[
I_2^{(1)} = (e^{-i\pi(e+\alpha+\gamma)} - 1) \int_0^1 dz f_2 = (e^{-i\pi(e+\alpha+\gamma)} - 1) J_2
\]

\[
\bar{I}_1^{(1)} = (e^{-i\pi e} - 1) \int_0^1 dz^* \bar{f}_1 = (e^{-i\pi e} - 1) \bar{J}_1
\]

\[
\bar{I}_2^{(1)} = (e^{i\pi(e+\alpha+\gamma)} - 1) \int_0^1 dz^* \bar{f}_2 = (e^{i\pi(e+\alpha+\gamma)} - 1) \bar{J}_2
\]

(90)

that, in terms of generalized hypergeometric functions, becomes

\[
J_1 = B(-\alpha/2 - \beta/2 - \gamma/2 - n - 1, -\alpha/2 + \beta/2 + 1) B(1 + e/2, 1 + f/2 + s) \\
\quad 3F_2\left(-\alpha - \beta - \gamma, -n - 1, -\gamma/2, 1 + \frac{e}{2} - \frac{\alpha - \gamma}{2} - n, 2 + \frac{e + f}{2} + s\right); 1)
\]

\[
J_2 = B(1 + \gamma/2, 1 + \alpha/2 + n) B(2 + e/2 + \alpha/2 + \gamma/2 + n, 1 + f/2 + s) \\
\quad 3F_2\left(-\beta/2, 2 + \frac{\alpha + \gamma}{2} + n + e/2, 1 + \frac{\alpha}{2} + n; 2 + \frac{\alpha + \gamma}{2} + n, 3 + + \frac{\alpha + \gamma + e + f}{2} + n + s\right); 1)
\]

\[
\bar{J}_1 = B(-\alpha/2 - \beta/2 - \gamma/2 - m - 1, -\alpha/2 + \beta/2 + 1 + m) B(1 + e/2 + k, 1 + f/2) \\
\quad 3F_2\left(-\alpha - \beta - \gamma, -m - 1, -\gamma, 1 + e/2 + k; -\frac{\alpha - \gamma}{2}, 2 + \frac{e + f}{2} + k\right); 1)
\]

\[
\bar{J}_2 = B(1 + \gamma/2, 1 + \alpha/2) B(2 + e/2 + \alpha/2 + \gamma/2 + k, 1 + f/2) \\
\quad 3F_2\left(-\beta/2, -m, 2 + \frac{\alpha + \gamma + e}{2} + k, 1 + \frac{\alpha}{2} + 2 + \frac{\alpha + \gamma}{2} + 3 + + \frac{\alpha + \gamma + e + f}{2} + k\right); 1).
\]

(91)

Thus the solution has the form

\[
Z = t_{11} J_1 \bar{J}_1 + t_{12} J_1 \bar{J}_2 + t_{21} J_2 \bar{J}_1 + t_{22} J_2 \bar{J}_2
\]

(92)

where the matrix elements $t_{ij}$ are the following

\[
t_{11} = \Delta^{-1} S(e/2) S(\beta/2) S((\alpha + \beta + \gamma)/2).
\]
\[
\cdot \left( S(\alpha/2)S(\beta/2)S(f/2)S((\alpha + \beta + e + f + 2\gamma)/2) + S(\gamma/2)S((\alpha + \beta + \gamma)/2)S((\alpha + e + f + \gamma)/2)S((f + \gamma + \beta)/2) \right)
\]

\[
t_{22} = \Delta^{-1}S(\alpha/2)S(\gamma/2)S((\alpha + e + \gamma)/2) \cdot \\
\cdot \left( S(\alpha/2)S(\beta/2)S((e + f)/2)S(1/2(\beta + f + \gamma)) + S(\gamma/2)S((\alpha + \beta + \gamma)/2)S(f/2)S((e + f + \gamma + \beta)/2) \right)
\]

\[
t_{12} = t_{12} = \Delta^{-1}S(\alpha/2)S(\beta/2)S(e/2)S(\gamma/2) \cdot \\
\cdot S((\alpha + e + \gamma)/2)S((\alpha + \beta + \gamma)/2)S((\beta + \gamma)/2)
\]

with
\[
\Delta = S((\alpha + \gamma)/2) \\
\cdot \left( S(\alpha/2)S(\beta/2)S((e + f)/2)S((\alpha + \beta + e + f + 2\gamma)/2) + S(\gamma/2)S((\alpha + \beta + \gamma)/2)S((\alpha + e + f + \gamma)/2)S((e + f + \gamma + \beta)/2) \right)
\]

and \( S(x) = \sin(\pi x) \).

For all details on the calculation we refer to [33].

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Fig. 1: Comparison of the Montecarlo estimates (crosses), the IRS results (asterisks) and the form factor results (pluses) for the correlator $G_{\sigma\sigma}$ at $h_t = 4.4069 \times 10^{-4}$. 
Fig. 2: Differences between MC estimates and IRS results (crosses) and between MC and form factor results (pluses) for the correlator $G_{\epsilon\sigma}$ at $h_{\text{f}} = 1.1017 \times 10^{-4}$. In this figure, and in all the following ones the errors in the MC estimates are not reported since they are smaller than the symbol size.
Fig. 3: Differences between MC estimates and IRS results (crosses for $h_l = 4.4069 \times 10^{-4}$, dotted squares for $2.2034 \times 10^{-4}$ and circles for $1.1017 \times 10^{-4}$) and between MC and form factor results (pluses for $h_l = 4.4069 \times 10^{-4}$, diamonds for $2.2034 \times 10^{-4}$ and filled squares for $1.1017 \times 10^{-4}$) for the correlator $G_{\sigma\sigma}$. Distances are measured in units of the correlation length.
**Fig. 4:** Same as fig.3, but for the $G_{\sigma\epsilon}$ correlator. Notice the different scale on the $y$ axis.
Fig. 5: Same as fig.3, but for the $G_{ee}$ correlator.
Fig. 6: Differences between MC estimates and IRS results for $h_l = 4.4069 \times 10^{-4}$ (pluses), $h_l = 2.2034 \times 10^{-4}$ (crosses) and $h_l = 1.1017 \times 10^{-4}$ (diamonds) for the $\epsilon\sigma$ correlator, in the short distance region dominated by the lattice artifacts.
**Fig. 7:** Same as in fig.6, but with distances measured in units of the correlation length.
Fig. 8: Difference between the MC data for the $\langle \epsilon \sigma \rangle$ correlator at $h_l = 1.1017 \times 10^{-4}$ and the IRS results with one (pluses), two (crosses) and three (diamonds) terms in the expansion.
Fig. 9: Difference between the MC data for the $\langle \epsilon \epsilon \rangle$ correlator at $h_l = 1.1017 \times 10^{-4}$ and the IRS results with one (pluses) and two (crosses) terms in the expansion.
Fig. 10: Difference between the MC data for the $\langle \sigma\sigma \rangle$ correlator at $h_l = 1.1017 \times 10^{-4}$ and the FF results with one (pluses), three (crosses) and eight (diamonds) terms in the spectral series.
Table 7: Comparison of the Montecarlo estimates (second column), the IRS results (third column) and the form factor (FF) results (fourth column) for the correlator $G_{\sigma\sigma}$ at $h_l = 4.4069 \times 10^{-4}$. In the first column is reported the distance in lattice units.

| $r$ | MC        | IRS        | FF         |
|-----|-----------|------------|------------|
| 1   | 0.71643( 2) | 0.71371  | 0.59385   |
| 2   | 0.61138( 3) | 0.60871  | 0.54528   |
| 3   | 0.55862( 4) | 0.55764  | 0.51757   |
| 4   | 0.52628( 5) | 0.52590  | 0.49852   |
| 5   | 0.50401( 6) | 0.50385  | 0.48425   |
| 6   | 0.48757( 6) | 0.48750  | 0.47303   |
| 7   | 0.47487( 7) | 0.47483  | 0.46393   |
| 8   | 0.46475( 7) | 0.46472  | 0.45638   |
| 9   | 0.45651( 7) | 0.45647  | 0.45002   |
| 10  | 0.44968( 8) | 0.44961  | 0.44459   |
| 11  | 0.44393( 8) | 0.44383  | 0.43991   |
| 12  | 0.43904( 8) | 0.43889  | 0.43584   |
| 13  | 0.43484( 9) | 0.43463  | 0.43229   |
| 14  | 0.43120( 9) | 0.43093  | 0.42916   |
| 15  | 0.42803( 9) | 0.42768  | 0.42639   |
| 16  | 0.42526(10) | 0.42481  | 0.42394   |
| 17  | 0.42281(10) | 0.42226  | 0.42175   |
| 18  | 0.42065(10) | 0.41998  | 0.41980   |
| 19  | 0.41874(10) | 0.41792  | 0.41805   |
| 20  | 0.41702(10) | 0.41606  | 0.41648   |
| 21  | 0.41549(11) | 0.41436  | 0.41506   |
| 22  | 0.41412(11) | 0.41280  | 0.41378   |
| 23  | 0.41290(11) | 0.41137  | 0.41263   |
| 24  | 0.41179(11) | 0.41003  | 0.41158   |
| 25  | 0.41079(11) | 0.40879  | 0.41064   |
| 26  | 0.40988(12) | 0.40762  | 0.40978   |
| 27  | 0.40907(12) | 0.40652  | 0.40899   |
| 28  | 0.40832(12) | 0.40547  | 0.40828   |
| 29  | 0.40765(12) | 0.40447  | 0.40763   |
| 30  | 0.40703(12) | 0.40351  | 0.40704   |
Table 8: Comparison of the Montecarlo estimates (second column), the IRS results (third column) and the form factor (FF) results (fourth column) for the correlator $G_{\epsilon \epsilon}$ at $h_l = 4.4069 \times 10^{-4}$. In the first column is reported the distance in lattice units.

| $r$ | MC       | IRS      | FF       |
|-----|----------|----------|----------|
| 1   | 0.104067(2) | 0.101321 | 0.002330 |
| 2   | 0.029348(2) | 0.025332 | 0.001735 |
| 3   | 0.012327(3) | 0.011262 | 0.001399 |
| 4   | 0.006674(2) | 0.006340 | 0.001169 |
| 5   | 0.004190(2) | 0.004064 | 0.000999 |
| 6   | 0.002879(2) | 0.002829 | 0.000867 |
| 7   | 0.002103(2) | 0.002087 | 0.000761 |
| 8   | 0.001606(2) | 0.001608 | 0.000674 |
| 9   | 0.001268(2) | 0.001281 | 0.000601 |
| 10  | 0.001029(2) | 0.001049 | 0.000540 |
| 11  | 0.000854(2) | 0.000880 | 0.000488 |
| 12  | 0.000720(2) | 0.000753 | 0.000443 |
| 13  | 0.000615(2) | 0.000657 | 0.000404 |
| 14  | 0.000534(2) | 0.000582 | 0.000371 |
| 15  | 0.000469(2) | 0.000524 | 0.000341 |
| 16  | 0.000415(2) | 0.000478 | 0.000315 |
| 17  | 0.000370(2) | 0.000442 | 0.000292 |
| 18  | 0.000334(2) | 0.000414 | 0.000272 |
| 19  | 0.000305(2) | 0.000392 | 0.000254 |
| 20  | 0.000280(2) | 0.000375 | 0.000238 |
| 21  | 0.000258(2) | 0.000362 | 0.000224 |
| 22  | 0.000239(2) | 0.000353 | 0.000212 |
| 23  | 0.000221(2) | 0.000347 | 0.000200 |
| 24  | 0.000205(2) | 0.000344 | 0.000190 |
| 25  | 0.000194(2) | 0.000342 | 0.000181 |
| 26  | 0.000184(2) | 0.000343 | 0.000173 |
| 27  | 0.000174(2) | 0.000345 | 0.000165 |
| 28  | 0.000166(2) | 0.000349 | 0.000159 |
| 29  | 0.000158(2) | 0.000354 | 0.000153 |
| 30  | 0.000152(2) | 0.000360 | 0.000147 |
Table 9: Comparison of the Monte Carlo estimates (second column), the IRS results (third column) and the form factor (FF) results (fourth column) for the correlator $G_{\sigma \epsilon}$ at $h_l = 4.4069 \times 10^{-4}$. In the first column is reported the distance in lattice units.

| $r$ | MC           | IRS          | FF            |
|-----|--------------|--------------|---------------|
| 1   | 0.104077(19) | 0.101011     | 0.025867      |
| 2   | 0.053137(12) | 0.050882     | 0.020745      |
| 3   | 0.034975(11) | 0.034286     | 0.017834      |
| 4   | 0.026275(11) | 0.026062     | 0.015840      |
| 5   | 0.021240(11) | 0.021181     | 0.014353      |
| 6   | 0.017966(11) | 0.017967     | 0.013190      |
| 7   | 0.015676(11) | 0.015704     | 0.012251      |
| 8   | 0.013990(10) | 0.014032     | 0.011476      |
| 9   | 0.012705(10) | 0.012753     | 0.010826      |
| 10  | 0.011699(10) | 0.011748     | 0.010274      |
| 11  | 0.010891(10) | 0.010942     | 0.009800      |
| 12  | 0.010231(11) | 0.010282     | 0.009391      |
| 13  | 0.009685(11) | 0.009736     | 0.009035      |
| 14  | 0.009228(11) | 0.009277     | 0.008723      |
| 15  | 0.008841(11) | 0.008888     | 0.008448      |
| 16  | 0.008510(11) | 0.008555     | 0.008206      |
| 17  | 0.008224(11) | 0.008268     | 0.007991      |
| 18  | 0.007978(11) | 0.008018     | 0.007800      |
| 19  | 0.007765(10) | 0.007800     | 0.007629      |
| 20  | 0.007577(10) | 0.007608     | 0.007476      |
| 21  | 0.007411(11) | 0.007438     | 0.007340      |
| 22  | 0.007265(11) | 0.007287     | 0.007217      |
| 23  | 0.007134(11) | 0.007152     | 0.007106      |
| 24  | 0.007018(11) | 0.007031     | 0.007006      |
| 25  | 0.006914(11) | 0.006922     | 0.006916      |
| 26  | 0.006822(11) | 0.006823     | 0.006834      |
| 27  | 0.006740(11) | 0.006733     | 0.006760      |
| 28  | 0.006667(11) | 0.006650     | 0.006693      |
| 29  | 0.006600(12) | 0.006575     | 0.006632      |
| 30  | 0.006539(12) | 0.006505     | 0.006577      |