LAX KLEISLI-VALUED PRESHEAVES AND COALGEBRAIC WEAK BISIMULATION

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ABSTRACT. We generalize the work by P. Sobociński on relational presheaves and their connection with weak (bi)simulation for labelled transition systems to a coalgebraic setting. We show that the coalgebraic notion of saturation studied in our previous work can be expressed in the language of lax Kleisli-valued presheaves in terms of existence of a certain adjoint situation between presheaf categories. This observation allows us to generalize the notion of the coalgebraic (weak) bisimulation to lax Kleisli-valued presheaves. At this level of generality interesting properties of strong and weak bisimilarity emerge: in the family of \( p \)-bisimilarities, which arises naturally in this setting, the former is the finest and the latter is the coarsest relation.

1. INTRODUCTION

We have witnessed a rapid development of the theory of coalgebras as a unifying theory for state-based systems [9, 24]. A coalgebra can be thought of as an abstract representation of a single step of computation of a given process. The theory of coalgebras provides a good setting for the study of bisimulation [13, 24, 31]. The notion of a strong bisimulation for different transition systems plays an important role in theoretical computer science. A weak bisimulation is a relaxation of this notion by allowing silent, unobservable transitions. Here, we focus on the weak bisimulation and weak bisimilarity proposed by R. Milner [20, 21] (see also [25]). One of several (equivalent) ways to define Milner’s weak bisimulation on a labelled transition system \( \alpha \) is to consider it as a strong bisimulation on its saturation \( \alpha^* \) [25]. Labelled transition systems saturation boils down to finding the smallest LTS containing all transitions of the original structure and satisfying the rules [20]:

\[
\begin{align*}
&x \xrightarrow{x} x \\
&x \xrightarrow{\alpha} x' \quad x' \xrightarrow{\alpha'} x'' \\
&x \xrightarrow{\alpha} x' \quad x' \xrightarrow{\alpha'} x'' \\
&x \xrightarrow{\alpha} x' \quad x' \xrightarrow{\alpha'} x''
\end{align*}
\]

In our recent papers [1, 5] we show that from the point of view of the theory of coalgebra the systems with silent moves should be considered as coalgebras over a monadic type. This allows to abstract away from a specific structure on labels and consider systems of the type \( X \rightarrow TX \) for a monad \( T \). The rules presented above that describe a saturated LTS structure can be restated in the abstract setting of coalgebras over order enriched monads by the following two axioms [1, 5]:

\[
1 \leq \alpha \quad \text{and} \quad \alpha \cdot \alpha \leq \alpha.
\]

Intuitively, these two rules say that a coalgebra satisfying them is reflexive and transitive. Hence, for several types of coalgebras over order enriched monads, the

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saturation can be thought of as a reflexive and transitive closure of a given structure. In particular, this abstract treatment encompasses classical notions of weak bisimulation for labelled transition systems and Segala systems [4, 27]. Coalgebraic saturation from our work can also be described as the left adjoint to the inclusion functor from the category of saturated (i.e. reflexive and transitive) coalgebras to the category of all coalgebras (however, in both cases one has to consider lax homomorphisms as the categorical morphisms). See [4] for details.

The purpose of this paper is to offer a more general perspective on saturation and state it in the setting of lax Kleisli-valued presheaves. Here, a lax Kleisli-valued presheaf is a lax functor \( \pi : \mathcal{D}^{op} \to \mathcal{Kl}(T) \) from an arbitrary (small) category \( \mathcal{D} \) to the order enriched Kleisli category for the monad \( T \).

**Content and organization of the paper.** The main contributions of this paper are the following:

- we introduce the notion of a lax Kleisli-valued presheaf and generalize the results by P. Sobociński on existence of a left adjoint to the change of base functor between categories of lax Kleisli-valued presheaves,
- we show that the coalgebraic saturation from our previous work described as the left adjoint to the inclusion functor is in fact a consequence of an adjoint situation between certain presheaf categories,
- we define the notion of weak bisimulation in the setting of lax presheaves on monoid categories in two ways and show their equivalence,
- the setting of lax presheaves allows us to observe a special characteristic of weak bisimulation. Namely, we show that in the family of the so-called \( p \)-bisimilarities, which arises naturally in this setting, weak bisimilarity is the coarsest relation. On the other hand, strong bisimilarity turns out to be the finest.

In Section 2 we recall basic definitions and properties required in the remainder of the paper. In Section 3 we define the notion of a lax Kleisli-valued presheaf and present some examples of presheaf categories and relate them to certain categories of coalgebras. Section 4 is devoted to the notion of weak bisimulation and \( p \)-bisimulation presented for lax presheaves and study of their properties. Section 5 is a summary.

**Related work.** Our approach generalizes the work by P. Sobociński on relational presheaves and weak (bi)simulation for labelled transition systems [29]. There, the author notices that a labelled transition system with an internal label \( \tau \) can be expressed as a relational presheaf \((A_\tau)^* \to \mathcal{R} = \mathcal{Kl}((\mathcal{P}))\), where \((A_\tau)^*\) is the free monoid category over the set of labels \( A_\tau = A + \{\tau\} \). It is shown that the LTS saturation can be described as the left adjoint to the so-called change of base functor between suitable categories of relational presheaves: \( \mathcal{R}(A_\tau^*) \xrightarrow{\sim} \mathcal{R}(A^*) \).

In our paper we go one step further as we hide labels inside a monad, which allows us to formulate more general and more canonical results about existence of a left adjoint to the change of base functor and characterisation of saturation. The change of base functor and its left adjoint was also considered in [22] in the context of relational presheaves with functional morphisms (in P. Sobociński’s paper [29] it was studied in the setting of relational presheaves with relational morphisms). Our work subsumes both treatments.
Our results are also similar to [7]. There, the authors study saturation in the context of Set-valued presheaves. The main ingredient in their construction is an adjoint situation between slice categories \( \text{Cat}/D \xrightarrow{\pi} \text{Cat}/E \), where the right adjoint is the change of base functor. Our setting is more coalgebraic oriented, but as we will see in sections to come it is very similar in its formulation.

2. Basic notions and properties

We assume the reader is familiar with standard notions in category theory [19]. We will now briefly recall basics needed in this paper.

**Coalgebras.** Let \( C \) be a category and let \( F : C \to C \) be a functor. An \( F \)-coalgebra is a morphism \( \alpha : X \to FX \) in \( C \). The domain \( X \) of \( \alpha \) is called carrier and the morphism \( \alpha \) is sometimes also called structure. A homomorphism from an \( F \)-coalgebra \( \alpha : X \to FX \) to an \( F \)-coalgebra \( \beta : Y \to FY \) is a morphism \( f : X \to Y \) in \( C \) such that \( F(f) \circ \alpha = \beta \circ f \). The category of all \( F \)-coalgebras and homomorphisms between them is denoted by \( C_F \). Many transition systems can be captured by the notion of coalgebra. The most important from our perspective are listed below.

Let \( \Sigma \) be a fixed set and put \( \Sigma^* = \Sigma + \{\varepsilon\} \). The label \( \tau \) (sometimes replaced with the label \( \varepsilon \)) is considered a special label called silent or invisible label.

**Labelled transition systems.** \( P(\Sigma^* \times \mathbb{I}) \)-coalgebras are labelled transition systems over the alphabet \( \Sigma \) \([20, 21, 24, 25]\). Here, \( P \) denotes the powerset functor. In this paper we will also consider labelled transition systems with a monoid structure on labels, i.e. coalgebras of the type \( P(M \times \mathbb{I}) \) for a monoid \((M, \cdot, \varepsilon)\).

**Nondeterministic automata.** \( P(\Sigma^* \times \mathbb{I} + 1) \)-coalgebras are nondeterministic automata with \( \varepsilon \)-transitions \([12, 28]\). It is useful sometimes to extend the type of \( \varepsilon \)-NA’s and work with \( P(\Sigma^* \times \mathbb{I} + \Sigma^*) \)-coalgebras instead (see \([3]) \) for a discussion.

**Fully probabilistic systems.** \( D(\Sigma^* \times \mathbb{I}) \)-coalgebras are fully probabilistic systems \([2, 30]\). Here, \( D \) denotes the substochastic distribution functor assigning to any set \( X \) the set \( \{\mu : X \to [0, 1] \mid \sum_{x \in X} \mu(x) \leq 1\} \) of substochastic distributions and to any map \( f : X \to Y \) the map \( Df : DX \to DY ; \mu \mapsto Df(\mu) \) with \( Df(\mu)(y) = \sum_{x \in X} \mu(x) f(x, y) \).

**Segala systems.** \( CM(\Sigma^* \times \mathbb{I}) \)-coalgebras are Segala systems \([27]\). The functor \( CM : \text{Set} \to \text{Set} \) of convex distributions \([10]\) is defined as follows. Let \( \mathbb{R}_0^+ \) denote the semiring \((\mathbb{R}_+ \cup \{0\}, +, \cdot)\) of non-negative real numbers with standard addition and multiplication. For any set \( X \) define \( MX \) to be the carrier of the free module over \( X \) and put \( CMX = \{U \subseteq MX \mid U = \overline{U} \text{ and } U \neq \emptyset\} \), where for any subset \( U \subseteq MX \) we define \( \overline{U} = \{\sum_{i=1, \ldots, n} r_i \cdot u_i \mid u_i \in U, r_i \in \mathbb{R}_0^+ \& \sum_i r_i = 1\} \). For any map \( f : X \to Y \) put \( CM(f) : CMX \to CMY; U \mapsto \overline{f(U)} \). It is important to note that the Segala systems were originally modelled as \( PD(\Sigma^* \times \mathbb{I}) \)-coalgebras \([30]\). We choose \( CM \) over \( PD \) as the latter fails to carry a monadic structure due to the lack of a distributive law between monads \( P \) and \( D \) \([33]\). See \([4, 10]\) for a discussion of this treatment.

**Filter coalgebras.** \( F \)-coalgebras, where \( F : \text{Set} \to \text{Set} \) denotes the filter functor (see e.g. \([10]\)) which assigns to any set \( X \) the set \( FX \) consisting of all filters on \( X \) and to any mapping \( f : X \to Y \) the mapping \( Ff : FX \to FY ; \mathcal{G} \mapsto \{Y' \subseteq Y \mid f^{-1}(Y') \in \mathcal{G}\} \).
Monads and their Kleisli categories. A monad on $C$ is a triple $(T, \mu, \eta)$, where $T : C \to C$ is an endofunctor and $\mu : T^2 \Rightarrow T$, $\eta : I_d \Rightarrow T$ are two natural transformations for which $\mu \circ T \mu = \mu \circ T \mu$ and $\mu \circ \eta_T = id_T = \mu \circ T \eta$. The transformation $\mu$ is called multiplication and $\eta$ unit. Each monad gives rise to a canonical category - Kleisli category for $T$. This category, denoted by $\mathcal{Kl}(T)$, has the class of objects equal to the class of objects of $C$ and for two objects $X, Y$ in $\mathcal{Kl}(T)$ we have $\text{Hom}_{\mathcal{Kl}(T)}(X, Y) = \text{Hom}_C(X, TY)$ with the composition $\cdot$ in $\mathcal{Kl}(T)$ defined between two morphisms $f : X \to TY$ and $g : Y \to TZ$ by $g \cdot f := \mu_Z \circ T(g) \circ f$ (here, $\circ$ denotes the composition in $C$). Since in many cases we will work with two categories at once: $C$ and $\mathcal{Kl}(T)$, morphisms in $C$ will be denoted using the standard arrow $\to$, whereas for morphisms in $\mathcal{Kl}(T)$ we will use the symbol $\Rightarrow$. For any object $X$ in $C$ (or equivalently in $\mathcal{Kl}(T)$) the identity map from $X$ to itself in $C$ will be denoted by $id_X$ and in $\mathcal{Kl}(T)$ by $1_X$ or simply 1 if the domain can be deduced from the context. The category $C$ is a subcategory of $\mathcal{Kl}(T)$ where the inclusion functor $\mathcal{I}$ sends each object $X \in C$ to itself and each morphism $f : X \to Y$ in $C$ to the morphism $f^\mathcal{I} : X \Rightarrow Y$ given by $f^\mathcal{I} : X \to TY; f^\mathcal{I} = \eta_Y \circ f$.

Example 2.1. Although the setting presented in this paper works for arbitrary category the main examples of our monads are Set based monads. In the table below we summarize the most important examples we use and give the explicit formulae.

| Name | Functor | $g \cdot f(x) = \{ \cdot \}$ | Remarks |
|------|---------|----------------|----------|
| Powerset monad | $\mathcal{P}$ | $\{ \cdot \}$ | $\{ \cdot \}$ |
| LTS monad | $\mathcal{P}(\Sigma_r \times I_d)$ | $\{(a, z) \mid x \dnf f y \dnf g z \}$ | $\{ \cdot \}$ |
| LTS monad with a monoid structure on labels $\circ$ | $\mathcal{P}(M \times I_d)$ | $\{(a \cdot b, z) \mid x \dnf f y \dnf g z \}$ | $\{ \cdot \}$ |
| $\varepsilon$-NA monad | $\mathcal{P}((\Sigma \times I_d) + (\Sigma \times I_d)$ | $\{(s_1, s_2, z) \mid x \dnf f y \dnf g z \}$ | $\{ \cdot \}$ |
| Convex distribution monad $\mathcal{CM}$ | $\bigcup_{x \in f(x)} \sum_{y \in \text{supp}(x)} \{ \phi(y) \psi | \psi \in g(y) \}$ | $\{ \cdot \}$ |
| Filter monad $\mathcal{F}$ | $\{ \cdot \}$ | $\{ \cdot \}$ | $\{ \cdot \}$ |
| Subdistribution monad $\mathcal{D}$ | $\sum_{y \in \text{supp}(x)} f(x)(y) \cdot g(y)$ | $\{ \cdot \}$ | $\{ \cdot \}$ |
Order enriched categories. In this paper we work with basic 2-categorical notions. However, since the only type of a 2-category considered in this paper is the order enriched category, we recall some basic definitions only from the point of view of the structures we are interested in. The reader is referred to e.g. [17] [32] for a more general perspective. A category is order enriched if each hom-set is a poset with order preserved by the composition. Any ordinary category can be turned into an order enriched category by introducing the equality relation as the partial order on hom-sets. A functor between two order enriched categories is locally monotonic if it preserves order. A functor-like assignment $F$ from a category $C$ to an order enriched category $D$ is called lax functor if:

- $id_{FX} \leq F(id_X)$ for any object $X \in C$,
- $F(f) \circ F(g) \leq F(f \circ g)$ for any two composable morphisms $f, g \in C$.

Let $F, F' : C \to D$ be two lax functors. A family $\phi = \{\phi_X : FX \to F'X\}_{X \in C}$ of morphisms in $D$ is called lax natural transformation if for any $f : X \to Y$ in $C$ we have $\phi_Y \circ Ff \leq F'f \circ \phi_X$. Note that in the more general 2-categorical setting a lax functor and a lax natural transformation are assumed to additionally satisfy extra coherence conditions [32]. In our setting of order enriched categories these conditions are vacuously true, hence we do not list them here.

The Klesli category for any monad from Example 2.1 is order enriched with the order on hom-sets imposed by the natural pointwise order summarized in the table below. For $f, g : X \to TY$ in $\text{ Kl}(T)$ for a suitable monad we have:

| Monads | $f \leq g$ if and only if | References |
|--------|---------------------------|------------|
| $P, P(\Sigma \times I) \text{d}, P(M \times I)\text{d}, P(\Sigma^{+} \times I)\text{d} + \Sigma^{+} \text{d}, CM$ | $f(x) \subseteq g(x)$ for any $x \in X$ | [5] [15] [16] |
| $F$ | $f(x) \supseteq g(x)$ for any $x \in X$ | [8] [26] |
| $D$ | $f(x)(y) \leq g(x)(y)$ for any $x \in X, y \in Y$ | e.g. [13] |

Coalgebras with internal moves. Originally [12] [28], coalgebras with internal moves have been introduced in the context of trace semantics as coalgebras of the type $T(F + I)$ for a monad $T$ and an endofunctor $F$ on $C$. If we take $T \in \{P, CM, D\}$ and $F = \Sigma \times I d$ then we have $T(F + I d) = T(\Sigma \times I d + I d) \cong T(\Sigma^{+} \times I d)$, i.e. we get the LTS functor, the Segala systems functor and the fully probabilistic systems functor respectively. In [4] we showed that given some mild assumptions on $T$ and $F$ we may either introduce a monadic structure on $T(F + I d)$ or embed it into the monad $TF^{*}$, where $F^{*}$ is the free monad over $F$. In particular the LTS functor$^1$ the Segala systems functor and the fully probabilistic systems functor carry monadic structures that turn the invisible moves into a part of the unit of the given monads [4]. The trick of modelling the invisible steps via a monadic structure allows us not to specify the internal moves explicitly. Instead of considering $T(F + I d)$-coalgebras we consider $T'$-coalgebras for a monad $T'$ on an arbitrary category $C$. From now on all coalgebras considered in this paper are coalgebras over a monad.

$^1$The monadic structure on the LTS functor that arises this way is exactly the one mentioned two paragraphs ago
Coalgebras and functional simulations. From now on we assume that the monad \((T, \mu, \eta)\) on \(C\) gives rise to an order-enriched category \(Kl(T)\). By \(C_{T, \leq}\) we denote the category whose objects are exactly the objects from \(C_T\) and whose morphisms are lax homomorphisms. A morphism \(f : X \to Y\) in \(C\) is a lax homomorphism between \(T\)-coalgebras \(\alpha : X \to TX\) and \(\beta : Y \to TY\) if \(f^\sharp \cdot \alpha \leq \beta \cdot f^\sharp\). We can restate this inequality in terms of the composition in \(C\) as in the diagram on the right above.

Coalgebras with morphisms satisfying a similar condition were studied in e.g. \[11\] in the context of forward simulations. However, in \[11\] these morphisms are taken from the Kleisli category not the base category. In order to emphasize the fact that our morphisms come from \(C\) we will often refer to lax homomorphisms as functional simulations. They can be intuitively understood as functional morphisms preserving (and not necessarily reflecting) transitions. In the case of labelled transition systems, the category \(\text{Set}(\Sigma \times I)^\leq\) has all LTS as objects and as morphisms maps between the carriers satisfying the following implication:

\[ x \xrightarrow{\alpha} x' \implies f(x) \xrightarrow{\alpha} f(x'). \]

Note that, obviously, the category \(C_T\) is a subcategory of \(C_{T, \leq}\).

Order saturation monads and coalgebraic weak bisimulation. A monad \(T\) whose Kleisli category is order enriched is called ordered saturation monad \[4\] provided that in \(Kl(T)\) for any morphism \(\alpha : X \rightrightarrows X\) there is a morphism \(\alpha^* : X \rightrightarrows X\) which is the least morphism satisfying

\[ 1 \leq \alpha^* \text{ and } \alpha \leq \alpha^* \text{ and } \alpha^* \cdot \alpha^* \leq \alpha^*. \]

In the definition of order saturation monad there is one extra condition we omit here. A curious reader is referred to \[4\] for details. All monads from Example \[2.1\] except for \(D\) are order saturation monads. In \[4\] one can find a proof of this statement for the LTS monad and the \(CM\) monad. In the same paper the reader can also find an argument under which \(D\) fails to satisfy the desired conditions. In \[5\] we implicitly state that the \(\varepsilon\)-NA monad is order saturation monad. Indeed, if the Kleisli category for a monad \(T\) is order enriched in which hom-sets admit all non-empty suprema which are preserved by the composition then \(T\) is an order saturation monad with \(\alpha^* = \bigvee_n \alpha^n\). The \(\varepsilon\)-NA monad satisfies this condition. Moreover, so do the LTS monad \(\text{Set}(\Sigma \times I)^\leq\) with a monoid structure on labels (this is straightforward to verify and is left to the reader) and the filter monad (see e.g. \[8, 26\]).

If a monad \(T\) is an order saturation monad then there are several alternatives for defining weak bisimulation on a \(T\)-coalgebra \(\alpha\) \[4, 5\] and one of them is to define it as a strong bisimulation on \(\alpha^*\).

Let \(C_{T, \leq}\) be the full subcategory of \(C_{T, \leq}\) consisting only of saturated coalgebras, i.e. coalgebras satisfying \(1 \leq \alpha\) and \(\alpha \cdot \alpha \leq \alpha\).\(\text{C}_{T, \leq} \subseteq \text{C}_{T, \leq}\)

As it will be recalled in sections to come (see Theorem \[3.10\]) the process of saturation can be described in terms of a left adjoint to the inclusion functor from \(C_{T, \leq}\) to \(C_{T, \leq}\). Moreover, the category \(C_{T, \leq}\) will turn out to be an instance of a special lax \(Kl(T)\)-valued presheaf category. We will also show in sections to come that the aforementioned adjunction can be obtained by composing two adjoint situations with one being an adjunction between presheaf categories (see Theorem \[3.20\]).
3. T-coalgebras and lax \( KL(T) \)-valued presheaves

In this section we will define lax \( KL(T) \)-valued presheaves and study their properties. Assume that \( D \) is an arbitrary category.

**Definition 3.1.** A lax \( KL(T) \)-valued presheaf, or simply a presheaf, is a lax functor from \( D^{op} \) to \( KL(T) \).

We define the category \( T^{D^{op}} \) whose objects are lax \( KL(T) \)-valued presheaves and morphisms lax natural transformations of the form \( \phi = \{ \phi^D : \pi D \rightarrow \pi' D \}_{D \in D} \) for a family \( \{ \phi_D : \pi D \rightarrow \pi' D \}_{D \in D} \) of morphisms in \( C \). In other words, for any morphism \( d : D' \rightarrow D \) in \( D \) we have the following diagrams in \( KL(T) \) and \( C \) resp.:

\[
\begin{array}{c}
\pi D \xrightarrow{\pi D} \pi D' \\
\phi_D \downarrow \quad \downarrow \phi_D^d
\end{array} \quad \begin{array}{c}
\pi D \xrightarrow{\phi D} \pi D' \\
\downarrow \pi D \quad \downarrow \pi D
\end{array}
\]

The category \( T^{D^{op}} \) is in fact an order enriched category with the order between lax transformations \( \psi = \{ \psi^D \}, \phi = \{ \phi^D \} \) given by:

\[ \psi \leq \phi \iff \psi^D \leq \phi^D \text{ for any } D \in D. \]

**Remark 3.2.** We denote our category by \( T^{D^{op}} \) and not by, say, \( KL(T)^{D^{op}} \) to emphasize the fact that the transformations between presheaves in \( T^{D^{op}} \) are functional, i.e. their components come from the base category \( C \). In order to consider the transformations whose components are from \( KL(T) \) we take \( C = KL(T) \) and \( T = Id \). In this case the class of objects of \( T^{D^{op}} \) is the same as the class of objects of \( T^{D^{op}} \). However, in terms of morphisms, the category \( T^{D^{op}} \) is richer than \( T^{D^{op}} \). In particular this means that our setting encompasses the setting of relational presheaves and functional transformations presented in e.g. [22, 23] (if we put \( C = \text{Set} \) and \( T = \mathcal{P} \)) and relational presheaves with relational transformations from [29] (in this case \( C = KL(\mathcal{P}) \) and \( T = Id \)).

### 3.1. Examples of lax Kleisli-valued presheaves

In this paper we focus mainly on presheaf categories for monoid categories. The most important examples are

- \( \bullet \cdot \): the point category with object \(*\) and the identity morphism \( id : * \rightarrow * \),
- \( \omega \): the monoid category on object \(*\) obtained from the monoid \((\omega, +, 0)\) of natural numbers with ordinary addition.

If \( D \) is one of the above then \( D = D^{op} \) and hence we will often drop the superscript \( ^{op} \) and abbreviate the notation.

**Presheaves in \( T^\omega \).** We will now describe presheaves in \( T^\omega \) and show the relation between this category and the category \( C_{T, \leq} \). For a presheaf \( \pi \in T^\omega \) define

\[ \pi_n = \pi(n) : \pi(*) \rightarrow \pi(*) \text{ (i.e. } \pi_n : \pi(*) \rightarrow T \pi(*) \text{ in } C). \]

Note that any presheaf \( \pi \) in \( T^\omega \) is determined by its sequence \( (\pi_n)_{n \in \omega} \) and any transformation \( \phi : \pi \Rightarrow \pi' \) between two presheaves \( \pi \) and \( \pi' \) in \( T^\omega \) is determined by the morphism

\[ \phi_\pi : \pi(*) \rightarrow \pi'(*) \]

in \( C \). Therefore, for the sake of simplicity of notation, presheaves in \( T^\omega \) will be considered as sequences of coalgebras with a common carrier, and morphisms between presheaves as morphisms between the given carriers in \( C \).
The following three propositions are straightforward to verify and hence are left without proofs.

**Proposition 3.3.** A sequence $\pi = (\pi_n)_{n \in \omega}$ of $T$-coalgebras with a common carrier is a presheaf in $T^\omega$ if and only if the following conditions are satisfied:

1. $1 \leq \pi_0$.
2. $\pi_n \cdot \pi_m \leq \pi_{n+m}$ for any $n, m = 0, 1, 2, \ldots$.

**Example 3.4.** Consider the presheaf category $P((\Sigma \times I)^\omega)$. A sequence $(\pi_n : X \to P((\Sigma \times X)))_{n \in \omega}$ of LTS coalgebras is an object of this category if and only if it satisfies

$$x \tau \to \pi_0 x \xrightarrow{\alpha} \pi_n x \xrightarrow{\tau} \pi_n x \xrightarrow{x} \pi_n x \xrightarrow{\tau} \pi_n x \xrightarrow{x} \pi_n x \xrightarrow{\tau} \pi_n x \xrightarrow{x} \pi_n x$$

**Proposition 3.5.** Given two presheaves $\pi, \pi'$, a morphism $\phi : \pi(*) \to \pi'(*)$ in $\mathcal{C}$ is a morphism between presheaves $\pi$ and $\pi'$ in $T^\omega$ if and only if the following condition holds for any $n \in \omega$:

$$\pi(*) \xrightarrow{\phi} \pi'(*)$$

$$\pi_n \leq \pi'_n$$

$$T \phi(*) \xrightarrow{T} T \pi'(*)$$

For any coalgebra $\alpha : X \to TX$ define a presheaf $\alpha \in T^\omega$ whose sequence $(\alpha_n)$ is given by $\alpha_n = \alpha^n$. For any lax homomorphism $f : X \to Y$ between coalgebras $\alpha : X \to TX$ and $\beta : Y \to TY$ in $C_{T, \leq}$ put $\underline{f} = f$. Clearly, the assignment $(\underline{-}) : C_{T, \leq} \to T^\omega$ is functorial. Moreover, we have the following.

**Proposition 3.6.** The functor $(\underline{-})$ is a full and faithful embedding of the category $C_{T, \leq}$ into $T^\omega$.

Now, we introduce a functor $(\underline{-})_1 : T^\omega \to C_{T, \leq}$ which assigns to any presheaf $\pi$ the coalgebra $\pi_1$ and any presheaf morphism $\phi$ is assigned to itself. Note that the composition of $(\underline{-})$ and $(\underline{-})_1$ is the identity functor on $C_{T, \leq}$.

**Proposition 3.7.** We have the following adjunction:

$$C_{T, \leq} \xrightarrow{(\underline{-})} T^\omega \xleftarrow{(\underline{-})_1}$$

**Proof.** The statement follows directly by the fact that for any coalgebra $\alpha$ and any presheaf $\pi$ we have $Hom_{C_{T, \leq}}(\alpha, \pi_1) = Hom_{T^\omega}(\alpha, \pi)$. Indeed, it is clear that $Hom_{C_{T, \leq}}(\alpha, \pi_1) \supseteq Hom_{T^\omega}(\alpha, \pi)$. To see the opposite inclusion takes a lax homomorphism $f : X \to Y$ between $\alpha : X \to TX$ and $\pi_1 : Y \to TY$. This means that $f^\sharp \cdot \alpha \leq \pi_1 \cdot f^\sharp$. Inductively, we prove $f^\sharp \cdot \alpha^n \leq \pi_n \cdot f^\sharp$. Since for the presheaf $\pi \in T^\omega$ we have $\pi_1 \leq \pi_n$ we directly get that $f^\sharp \cdot \alpha^n \leq \pi_n \cdot f^\sharp$. This proves the assertion. \qed
Presheaves in $T^\bullet$. Any presheaf $\pi$ in $T^\bullet$ is determined by the underlying coalgebra $\pi(id) : \pi(*) \rightarrow \pi(*)$ and any presheaf morphism is a morphism in $C$ between the carriers of the underlying coalgebras. Hence, we will identify presheaves and presheaf morphisms in $T^\bullet$ with their coalgebras and carrier morphisms.

We have the following proposition which is a direct consequence of the definition of a presheaf.

**Proposition 3.8.** A $T$-coalgebra $\alpha$ is a presheaf in $T^\bullet$ if and only if it satisfies the following: $1 \leq \pi$ and $\pi \cdot \pi \leq \pi$. A morphism $\phi$ in $C$ is a transformation between presheaves $\pi$ and $\pi'$ in $T^\bullet$ if and only if it is a lax homomorphism between the underlying coalgebras $\pi, \pi'$. Hence, $C_{T,\leq} = T^\bullet$.

**Example 3.9.** We will now describe objects in the presheaf category $T^\bullet$ for several examples of $T$.

- $P^\bullet$: note that $\mathcal{Kl}(P)$ can be thought of as the category of sets as objects and relations as morphisms with relation composition as morphism composition. A relation on a given set is a member of $P^\bullet$ if and only if it is reflexive and transitive, i.e. it is a preorder.
- $P(S, \times \text{Id})^\bullet$: an LTS coalgebra $\alpha : X \rightarrow P(S, \times \text{Id})$ is an object of this category iff:
  $$x \xrightarrow{\alpha} x \quad \xrightarrow{\tau} x \quad \xrightarrow{\alpha} x$$
  $$x \xrightarrow{\alpha} x \quad \xrightarrow{\alpha} x$$

- $P(M \times \text{Id})^\bullet$ for a monoid $(M, \cdot, \varepsilon)$: an LTS coalgebra $\alpha : X \rightarrow P(M \times \text{Id})$ with a monoid structure on labels is an object of this category iff:
  $$x \xrightarrow{\alpha} x \quad \xrightarrow{\alpha} x \quad \xrightarrow{\alpha} x$$

- $P(S, \times \text{Id} + S)^\bullet$: a coalgebra $\alpha$ is a member of this category iff:
  $$x \xrightarrow{\alpha} x \quad \xrightarrow{\alpha} x$$
  $$x \xrightarrow{\alpha} x$$

- $CM^\bullet$: a structure $\alpha : X \rightarrow CMX$ is an object of this category provided that it satisfies the following rules:
  $$x \rightarrow \sum_i r_i \cdot x_i$$
  $$x \rightarrow \sum_i r_i s_{i j} \cdot x_{i j}$$

- $F^\bullet$: is isomorphic to the category $\text{Top}$ of topological spaces and continuous maps $[8][14]$.
- $D^\bullet$: a coalgebra $\alpha : X \rightarrow DX$ is a member of this category iff $\alpha = 1_X$, i.e. if $\alpha$ is the $X$-component of the unit of $D$. Any map $f : X \rightarrow Y$ between $1_X, 1_Y \in D^\bullet$ is a morphism in $D^\bullet$. Hence, $D^\bullet \cong \text{Set}$.

Given the examples above we see that the members in $T^\bullet$ are very important from our perspective. In our previous work we have proved the following.

**Theorem 3.10.** [14] If $T$ is an order saturation monad then the functor assigning to each coalgebra $\alpha$ the structure $\alpha^*$ and which is the identity on morphism is the left adjoint to the inclusion functor from $T^\bullet$ to the category $C_{T,\leq}$: $C_{T,\leq} \xrightarrow{\bot} T^\bullet$. 
We will use this result to develop the notion of weak (bi)simulation for lax Kleisli-valued presheaves. We will show that $T^\bullet$ is a subcategory of an arbitrary presheaf category $TD^{op}$ and in the setting of presheaves the left adjoint to the inclusion functor will account for saturation.

**Remark 3.11.** Another way of looking at the category $T^\bullet$ is via the following coincidence. There is a one-to-one correspondence between lax functors $\bullet \to D$ and monads in $D$ for an arbitrary $2$-category $D$ [3, 17]. Here, whenever $\mathcal{Kl}(T)$ is order enriched, the monad unit and the monad multiplication are $2$-cells $1 \leq \pi$ and $\pi \cdot \pi \leq \pi$ respectively. Theorem above states that, in categorical terms, the coalgebraic saturation assigns to a $T$-coalgebra $\alpha : X \rightarrow X$ the free monad $\alpha^* : X \rightarrow X$ over $\alpha$ in the order enriched category $\mathcal{Kl}(T)$.

**Other examples of presheaf categories.** The category $T^\omega$ can be thought of as a category whose presheaves represent a single process and its (approximations of) iteration. We will now focus on the category $T^{A^\ast}$, where $A^\ast$ is the free monoid category over the set $A$. Clearly, the generalized versions of Proposition 3.3 and 3.5 can be easily derived for $T^{A^\ast}$. Hence, we will not do it explicitly. Instead, we will give an interpretation to members of $T^{A^\ast}$. Just like a presheaf in $T^\omega$ can be understood as a single process (coalgebra) and its iteration, a presheaf $\pi$ in $T^{A^\ast}$ can be thought of as several transition systems $\pi(a_1), \ldots, \pi(a_n)$ for $A = \{a_1, \ldots, a_n\}$ on the same state space that interact with one another in terms of composition in $\mathcal{Kl}(T)$.

**Example 3.12.** Let $A = \{a_1, \ldots, a_n\}$ be arbitrary coalgebras on $X$. Define a presheaf $\pi \in T^{\{1, 2, \ldots, n\}^\ast}$ by putting:
\[
\pi(x) = 1_X, \quad \pi(i) = a_i,
\]
\[
\pi(w) = \pi(a_1) \cdot \pi(a_2) \cdots \pi(a_n) \text{ for } w = a_1 \ldots a_n \text{ and } a_i \in \{1, \ldots, n\}.
\]

To see a more concrete example consider $\Sigma = \{a, b\}$, $T = \mathcal{P}(\Sigma^\ast \times Id)$ for the free monoid $\Sigma^\ast$ over $\Sigma$ and $X = \{x, y, z\}$. Define coalgebras $\alpha_1, \alpha_2 : X \rightarrow \mathcal{P}(\Sigma^\ast \times Id)$ as in the first two diagrams below. Define $\pi \in T^{\{1, 2\}^\ast}$ such that $\pi(1) = \alpha_1, \pi(2) = \alpha_2$ and $\pi$ on arbitrary words from $\{1, 2\}^\ast$ is defined like above. In particular, this means that e.g. $\pi(11)$ and $\pi(21)$ are respectively represented by the last two diagrams:

Coalgebras provide a unifying abstract theory of a single process. We think that Kleisli-valued presheaves can extend this theory in the study of several processes and their interactions. In the sequel we will focus on developing a tool to compare the *cumulative* behaviour of presheaves: weak bisimilarity of presheaves.

**3.2. Change of base functor and its left adjoint.** Before we state the definition of weak bisimulation between presheaves we need one technical result regarding the so-called change of base functor and its properties. Any functor $p : D \rightarrow E$ yields a locally monotonic functor $T^p : TE^{op} \rightarrow TD^{op}$ defined as follows. For any presheaf $\pi \in TE^{op}$ put $T^p(\pi) = \pi \circ p$ and for any transformation $\phi$ between presheaves $\pi$ and $\pi'$ in $TE^{op}$ the $D$-component of $T^p(\phi) : \pi \circ p \Rightarrow \pi' \circ p$ is given by $\phi_{pD}^D$ for $\phi_{pD} = \pi(pD) \rightarrow \pi'(pD)$, in other words:
\[
T^p(\phi)^D_{pD} = \phi_{pD}^D.
\]
It is easy to check that the functor $T^p$ is locally monotonic. In \cite{23,29} in the context of relational presheaves the functor $T^p$ was denoted by $p^*$. Since we want to avoid an extra use of $\ast$ which would not be directly related to coalgebraic saturation, we change the notation.

The general case. In this paragraph we assume the following:

- $p : D \rightarrow E$ is a functor between small categories,
- $C$ has all small coproducts (which implies that $Kl(T)$ has them),
- partially ordered hom-sets in $Kl(T)$ are complete lattices with arbitrary suprema preserved by the composition (i.e. $Kl(T)$ is a quantaloid \cite{23}) and also by arbitrary cotupling in $Kl(T)$, i.e.

$$[(\bigvee_{i,j} f_{i,j})] = \bigvee_i [\{f_{i,j}\}] \quad (\bigvee_i f_i) \cdot g = \bigvee_i f_i \cdot g \quad f \cdot \bigvee_{j} g_j = \bigvee_{j} f \cdot g_j.$$ 

**Theorem 3.13.** The functor $T^p : T^{E^{op}} \rightarrow T^{D^{op}}$ admits a left adjoint $\Sigma_p$.

**Proof.** The proof of this theorem is divided into three parts. In the first part, we establish basic notation and facts about Kleisli categories used in the remainder of the proof. In the second part, we present an assignment $\Sigma_p$ and show it is a well-defined functor between suitable categories. In the third part we show that $\Sigma_p$ is the left adjoint to $T^p$.

**Preliminaries** For a family $\{X_i\}_{i \in I}$ of objects in $C$ and let $\sum_i X_i$ together with the coprojections $i_{X_i} : X_i \rightarrow \sum_i X_i$ be its coproduct in $C$. Then the object $\sum_i X_i$ together with the morphisms $i_{X_i} := \text{id}_{X_i} : X_i \rightarrow \sum X_i$ is the coproduct in the Kleisli category $Kl(T)$. For a family $\{f_i : X_i \rightarrow Y_i\}$ of morphisms by $\sum_i f_i : \sum_i X_i \rightarrow \sum Y_i$ we denote the coproduct of morphisms in $C$, i.e. the cotupling $\{i_{Y_i} \circ f_i\}$. Note that for a family of morphisms $\alpha_i : X_i \rightarrow Y_i$ in $Kl(T)$ we have $\sum \alpha_i = \{T(\sum_{i} i_{Y_i} \circ \alpha_i)\}$, where $\sum$ denotes the coproduct in $Kl(T)$ and the cotupling on the right hand side is taken in $C$. If $\{f_i : X_i \rightarrow Y_i\}$ is a family of morphisms in $C$ then we have

$$\sum_i f_i = (\sum_i X_i) \ast Y_i \circ \sum_i f_i = \{\{i_{Y_i} \circ \sum_{i} i_{X_i}, \circ f_i\}\} = i_{Y_i} \circ \sum_{i} i_{X_i}, \circ f_i = (\sum f_i)_i.$$

In the above the first coproduct is in $Kl(T)$. The remaining coproduct and cotupling are computed in $C$.

**Part 1.** For any two objects $X, Y$ in $Kl(T)$ let $\perp$ denote the least element in the poset $Hom_{Kl(T)}(X, Y)$. For any presheaf $\pi \in T^{D^{op}}$ define an assignment $\Sigma_p(\pi)$ from the category $E^{op}$ to $Kl(T)$ on an object $E \in E$ and a morphism $\epsilon : E_1 \rightarrow E_2 \in E$ by:

$$\Sigma_p(\pi)(E) = \sum_{D : pD = E} \pi D \text{ and } \Sigma_p(\pi)(\epsilon) = \bigvee_{D : pD = E \perp} \pi D,$$

where $\pi D$ is given as follows. Let $d : D_1 \rightarrow D_2$ and $pD_1 = E_1$, $pD_2 = E_2$. Define the morphism $\pi d$ via cotupling in $Kl(T)$ by:

$$\pi d = \delta_D \ast \{\sum_{D : pD = E_2}\}$$

where $\delta_D$ is given as follows. Let $d : D \rightarrow D'$ and $pD' = E_2$. Define the morphism $\pi d$ via cotupling in $Kl(T)$ by:

$$\delta_D = \{\begin{array}{ll}
\text{in}_{D_1} : \pi d & : D_2 \rightarrow \sum_{D' : pD' = E_2} \pi D' \text{ if } D = D_2 \\
\perp & : \pi D \rightarrow \sum_{D' : pD' = E_2} \pi D' \text{ otherwise.}
\end{array}$$

We will now show that for any presheaf $\pi \in T^{D^{op}}$ the assignment

$$\Sigma_p(\pi) : E^{op} \rightarrow Kl(T)$$
is a presheaf in $T^{op}$. Indeed, let $\pi \in T^{D^{op}}$ and take $id_E : E \to E$ in $E$. We have: $\Sigma_p(\pi)(id_E) = \bigvee_{d: pD = E} \pi_d$. There can be two cases. If there is no $D$ mapped onto $E$ by the functor $p$ then $\Sigma_p(\pi)(E)$ is the initial object in $KL(T)$. In this case the identity morphism on $\Sigma_p(\pi)(E)$ and the morphism $\Sigma_p(\pi)(id_E)$ are both equal to the least morphism $\bot$. Now, for any object $D$ such that $pD = E$ we have $p(id_D) = id_E$, $\pi(id_D) \geq 1_{\pi D}$ and by the fact that cotupling preserves suprema we get

$$\Sigma(\pi)(id_E) = \bigvee_{d: pD = E} \pi_d \geq \bigvee_{D: pD = E} \pi(id_D) \geq \bigvee_{D: pD = E} \pi_D = 1_{\Sigma_p(\pi)(E)}.$$

Now take $E_1 \to E_2 \to E_3$ in $E$. We have:

$$\Sigma_p(\pi)(e' \circ e) = \bigvee_{d: pD = E'} \pi_d \geq \bigvee_{pd_1 = e', pd_2 = e} \pi(d_2) \circ \pi(d_1) \geq \bigvee_{pd_1 = e', pd_2 = e} \pi(d_2) \cdot \pi(d_1) = \Sigma_p(\pi)(e') \cdot \Sigma_p(\pi)(e').$$

The equation marked with $(\circ)$ requires some explanation. If $d_1$ and $d_2$ are composable then $\pi(d_2) \circ \pi(d_1) = \pi(d_2) \cdot \pi(d_1)$. If they are not composable then $\pi(d_2) \cdot \pi(d_1) = \bot$, so clearly this equation holds.

For any transformation $\phi : \pi \Rightarrow \pi'$ between presheaves $\pi, \pi'$ in $T^{D^{op}}$ put $\Sigma_p(\phi) : \Sigma_p(\pi) \Rightarrow \Sigma_p(\pi')$ whose $E$-component is given by:

$$\Sigma_p(\phi)_E = \sum_{pD = E} \pi_D \cdot \phi_D \cdot \sum_{pD' = E} \pi'(D).$$

Note that the $E$-component of $\Sigma_p(\phi)$ comes from the base category $C$ since

$$\Sigma_p(\phi)_E = \sum \phi_D = (\sum \phi_D)^E.$$

In the above the last coproduct is evaluated in $C$. It is clear that $\Sigma_p$ is functorial. This part of the proof is now completed.

**Part 2.** We will now prove that $\Sigma_p$ is the left adjoint to $T^p$. Here we should note that the remaining part of the proof is almost the same as the proof of a similar statement concerning relational persheaves [29]. For any presheaf $\pi \in T^{D^{op}}$ define a transformation $\eta_\pi : \pi \Rightarrow T^p(\Sigma_p(\pi)) = \Sigma_p(\pi) \circ p$ whose $D$-component is given by the coprojection into the component of the coproduct indexed with $D$:

$$(\eta_\pi)_D : \pi D \to \Sigma_p(\pi)(pD) = \sum_{D' : pD' = pD} \pi D'; \quad (\eta_\pi)_D = \ln_{pD}.$$
(c) \( \eta \) is a natural transformation from the functor \( \mathcal{I}d : T^{D^{op}} \to T^{D^{op}} \) to the functor \( T^p \circ \Sigma_p \).

We will check that \( \eta \) satisfies the universal property of units. Consider any transformation \( \phi : \pi \Rightarrow T^p(\pi') = \pi' \circ p \) in \( T^{D^{op}} \) for \( \pi', \pi \in T^{E^{op}} \). By the universal properties of the coproduct there is a unique morphism \( \psi_E : \sum_{D : pD = E} \pi D \to \pi' \) in \( \mathcal{Kl}(T) \) for which the following diagram commutes.

\[
\begin{array}{ccc}
\pi D & \xrightarrow{\eta_E(\pi)} & \sum_{D : pD = E} \pi D' \\
\phi_D & \downarrow \psi_D & \downarrow \pi'(pD)
\end{array}
\]

By (3) and properties of the coproduct in Kleisli category we directly see that \( \psi_E = (\psi_E')^\sharp \) for a morphism \( \psi_E' \) in \( C \). In order to complete the proof we need to show that the family \( \psi = (\psi_E')_{E \in E} \) is a transformation from \( \Sigma_p(\pi) \) and \( \pi' \) in \( T^{E^{op}} \).

We need to show that for any \( e : E \to E' \) in \( E \) we have:

\[
\begin{array}{l}
\Sigma_p(\pi)(E) \xrightarrow{\psi_E^\sharp} \pi'(E) \\
\Sigma_p(\pi)(E) = \bigvee_{d : pD = e} \pi D \\
\Sigma_p(\pi)(E') \xrightarrow{\psi_E'^\sharp} \pi'(E')
\end{array}
\]

Clearly, it is enough if we focus on morphisms from \( E \) which are images of morphisms from \( D \) under \( p \). Indeed, if \( e \) is not of this form then the diagram above lax commutes as \( \Sigma_p(\pi)(e) = \bot \). By our assumptions about \( \phi \) and by (1) the front square and the parallelogram on the back in the diagram below lax commute for arbitrary \( d : D' \to D \) in \( D \). By the fact that cotupling preserves all suprema the parallelogram on the right also lax commutes.

\[
\begin{array}{ccc}
\pi D & \xrightarrow{\eta_E(\pi)} & \sum_{D : pD = E} \pi D' \\
\phi_D & \downarrow \psi_D & \downarrow \pi'(pD')
\end{array}
\]

This completes the proof. \( \square \)

As mentioned before, the theorem above encompasses results presented in [22] (for \( C = \text{Set} \) and \( T = \mathcal{P} \)) and [29] (for \( C = \mathcal{Kl}(\mathcal{P}) \) and \( T = \mathcal{I}d \)).

As witnessed in the previous subsection the most important presheaf category from our perspective is \( T^\bullet \). Let \( ! \) denote the unique functor from a category \( D \) to \( \bullet \). The theorem below follows directly by the definition of \( T^p \).

**Proposition 3.14.** For any category \( D \) the functor \( T^l : T^\bullet \to T^{D^{op}} \) is faithful.

**Example 3.15.** For \( D = \omega \) this gives rise to the functor \( T^l : T^\bullet \to T^\omega \). Consider a presheaf \( \pi \in T^\bullet \). The presheaf \( T^l(\pi) = \pi! \in T^\omega \) is given by the following sequence of coalgebras: \( T^l(\pi)_n = \pi \) for \( n = 0, 1, \ldots \). In other words, \( T^l(\pi) \) is the constant sequence of coalgebras with all terms equal to \( \pi \). For a lax homomorphism \( f \) between coalgebras \( \pi, \pi' \in T^\bullet \) we have \( T^l(\pi) = f \). Note that the composition of \( T^l \) and \( (-)_1 : T^\omega \to \mathcal{C}_{T, \le} \) is the inclusion functor \( T^\bullet \to \mathcal{C}_{T, \le} \).
Adjunction between presheaf categories on monoids. It is worth noting that the restrictive assumptions from the previous paragraph can be relaxed a little in the case when we deal with monoid categories $D$ and $E$ and a full functor $p : D \to E$. Here, in order to get an adjoint situation, we only assume the following:

- in $\mathit{Kl}(T)$ the partial order on hom-sets admits all non-empty joins which are preserved by the Kleisli composition.

The relaxation of the assumptions allows us to put more monads into the setting. Except for the monads whose Kleisli category is a quantaloid, i.e. $P(\Sigma \times Id)$, $P(M \times Id)$ and $P^2$ the $\varepsilon$-NA monad also satisfies these assumptions [5].

Let $p : D \to E$ be a monoid epimorphism (i.e. a full functor between the monoid categories $D$ and $E$). We define the functor $\Sigma_p : T^D \to T^E$ as follows. For any presheaf $\pi \in T^D$ and a transformation $\phi : \pi \Rightarrow \pi'$ between presheaves in $T^D$ we put:

$$\Sigma_p(\pi)(e) = \bigvee_{d : pd = e} \pi(d) \quad \text{and} \quad \Sigma_p(\phi) = \phi.$$

**Theorem 3.16.** The functor $\Sigma_p$ is a left adjoint to $T^p$.

**Proof.** The proof of this theorem is a simplified version of the proof of Theorem 3.13. Since $D$ and $E$ are one-object categories and since $p$ is full we do not require the least morphism $\bot$ to exist, nor the cotupling is used. □

**Example 3.17.** Consider $\pi \in T^{(1,2)}$ from Example 3.12. Given the definition of $\Sigma_1$ above, we see that in this case the presheaf $\Sigma_1 \pi = \bigvee_w \pi(w) \in T^*$ is given as follows:

![Diagram]

We see that if we think about a presheaf $\pi$ in $T^{(1,2)}$ as two independent processes on the same state space interacting with one another in terms of composition, then $\Sigma_1 \pi$ describes the process containing cumulative information about the processes described by $\pi$ and all possible interactions between them.

Presheaf adjunction $T^\omega \rightleftarrows T^*$ and coalgebraic saturation. A careful reader might have noticed that the convex distribution monad was not mentioned in the previous paragraph as one of the monads that fit the framework. The reason for that is it simply does not satisfy the assumptions. Although the hom-sets in $\mathit{Kl}(\mathcal{CM})$ admit all non-empty suprema, the composition does not preserve them in general. The preservation of arbitrary non-empty suprema holds only in the 2nd component, i.e. $g \cdot \bigvee_i f_i = \bigvee_i g \cdot f_i$. In the first component directed suprema are preserved [10]. However, the most important adjunction from coalgebraic saturation perspective, i.e. $T^\omega \rightleftarrows T^*$, still exists. In order to state the theorem below we need recall one classical notion. We call an order enriched category $\omega$-$\text{cpo}$ if each countable ascending family $f_1 \leq f_2 \leq \ldots$ of morphisms admits the supremum which is preserved by the composition in both components.

---

[2] It is straightforward to show that LTS monads give rise to the Kleisli categories which are quantaloids. The category $\mathit{Kl}(\mathcal{F})$ is also a quantaloid (for a proof see e.g. [5, 20]).
Theorem 3.18. Let the monad $T$ satisfy the following:

- $\mathcal{K}(T)$ admits all non-empty suprema,
- for arbitrary non-empty countable families $\{\alpha_i : X \to X\}_{i \in \omega}$ and $\{\beta_i : Y \to Y\}_{i \in \omega}$ in $\mathcal{K}(T)$ and a morphism $f : X \to Y$ in $\mathcal{C}$ we have:

$$\forall i \in I \quad \frac{X \xrightarrow{f} Y}{TX \xrightarrow{f} TY} \quad \alpha_i \leq \beta_i \implies \forall i \in I \quad \frac{X \xrightarrow{f} Y}{TX \xrightarrow{f} TY}$$

- $\mathcal{K}(T)$ is an $\omega$-cpo.

Then the change of base functor $T^\uparrow : T^\bullet \to T^\omega$ admits a left adjoint.

Proof. For any presheaf $\pi \in T^\omega$ put $\Sigma(\pi)$ to be the assignment $\bullet \mapsto \mathcal{K}(T)$ defined as follows.

$$\Sigma_\uparrow(\pi)(\star) = \pi(\star) \text{ and } \Sigma_\uparrow(\pi)(id_\uparrow) = \bigvee_{m \in \omega} \bigvee_{n \in \omega} m_n \uparrow n.$$  

For any presheaf transformation $\phi : \pi \Rightarrow \pi'$ define $\Sigma_\uparrow(\phi) = \phi$.

**Part 1.** We will show that $\Sigma_\uparrow(\pi)$ is a presheaf in $T^\bullet$. For the sake of simplicity of notation we will identify the assigment $\Sigma_\uparrow(\pi)$ with its coalgebra $\Sigma_\uparrow(\pi)(id_\uparrow)$. We have:

$$1 \leq \Sigma_\uparrow(\pi),$$

$$\Sigma_\uparrow(\pi) \cdot \Sigma_\uparrow(\pi) = \bigvee_{m \in \omega} \bigvee_{n \in \omega} (\bigvee_{m \in \omega} \bigvee_{n \in \omega} m_n \uparrow n)^m \leq \bigvee_{m_1, m_2 \in \omega} \bigvee_{n \in \omega} (\bigvee_{m \in \omega} \bigvee_{n \in \omega} m_n \uparrow n)^{m_1 + m_2} = \Sigma_\uparrow(\pi).$$

The first inequality follows by one of the properties of presheaves in $T^\omega$, namely $1 \leq \pi_0 \leq \bigvee_{n} \pi_n$. The equality marked with $(\star)$ follows by the fact that the family $\{(\bigvee_{m} \pi_n)^m\}_{m \in \omega}$ is an ascending chain.

In order to complete the first part of the proof we will show that $\Sigma_\uparrow$ is functorial. It is enough to prove that $\Sigma_\uparrow(\phi) = \phi$ is a transformation between $\Sigma_\uparrow(\pi)$ and $\Sigma_\uparrow(\pi')$. This follows directly by our second and third assumption.

**Part 2.** We will now prove that the left adjoint to $T^\uparrow$, by proving the following identity for any $\pi \in T^\omega$ and $\pi' \in T^\bullet$: $\text{Hom}_T(\pi, T^\uparrow(\pi')) = \text{Hom}_T(\Sigma_\uparrow(\pi), \pi')$.

Indeed, we have the following:

$$\forall n \quad \pi(\star) \xrightarrow{\phi} \pi'(\star) \quad \iff \quad \pi(\star) \xrightarrow{\phi} \pi'(\star) \quad \iff \quad \pi(\star) \xrightarrow{\phi} \pi'(\star)$$

This completes the proof. \( \square \)

**Corollary 3.19.** For the monad $T = \mathcal{C}M$ the functor $T^\uparrow : T^\bullet \to T^\omega$ admits a left adjoint.

Proof. We need to prove that the three assumptions from Theorem 3.18 are satisfied for $T = \mathcal{C}M$. The proof of the first and last property can be found in [10]. The 2nd property follows easily by the fact that the composition in $\mathcal{K}(\mathcal{C}M)$ preserves arbitrary non-empty suprema in the 2nd component [10]. Indeed, we have:

$$f^\uparrow \cdot \alpha_i \leq \beta_i \cdot f^\uparrow \leq (\bigvee_i \beta_i) \cdot f^\uparrow$$

Hence,

$$f^\uparrow \cdot \bigvee_i \alpha_i = \bigvee_i f^\uparrow \cdot \alpha_i \leq (\bigvee_i \beta_i) \cdot f^\uparrow$$

\( \square \)
We can now relate the theory of lax presheaves to coalgebra saturation studied by us in [4]. As we stated in Theorem 3.10 the coalgebra saturation can be described via the left adjoint to the inclusion functor $J : T^\bullet \to C_{T,\leq}$. We have the following.

**Theorem 3.20.** If $T^l : T^\bullet \to T^\omega$ admits a left adjoint then so does the inclusion functor $J : T^\bullet \to C_{T,\leq}$.

**Proof.** The statement follows directly by composing two adjoint situations.

$$
\begin{array}{ccc}
C_{T,\leq} & \xrightarrow{\sim} & T^\omega \\
\downarrow & \circlearrowright & \downarrow \\
\downarrow & \circlearrowleft & \downarrow \\
J & \xrightarrow{\sim} & T^\bullet
\end{array}
$$

**Remark 3.21.** In particular, the theorem above states that for $T$-coalgebras over the type $T \in \{P(\Sigma \times Id), P(M \times Id), P(\Sigma^* \times Id + \Sigma^*), CM, F\}$ the coalgebraic saturation is a consequence of the adjoint situation $T^\omega \iff T^\bullet$ between presheaf categories.

**Kleisli category for the saturation monad.** Assume the change of base functor $T^l : T^\bullet \to T^\omega$ admits a left adjoint. Two adjunctions $C_{T,\leq} \iff T^\omega \iff T^\bullet$ yield the saturation monad $*: C_{T,\leq} \to C_{T,\leq}$. Assume the following:

- $*: C_{T,\leq} \to C_{T,\leq}$ is the identity on morphisms.

Consider the Kleisli category $Kl(*)$. Objects in $Kl(*)$ are $T$-coalgebras. For two coalgebras $\alpha$ and $\beta$ a morphism $\phi \in Hom_{Kl(*)}(\alpha, \beta)$ if and only if

$$
\begin{array}{c}
X \xrightarrow{\phi} Y \\
\circlearrowleft \leq \circlearrowleft \beta^* \\
TX \xrightarrow{\tau_\phi} TY
\end{array}
$$

The category $Kl(*)$ is called the category of coalgebras and weak functional simulations between them.

**Example 3.22.** For $T = P(\Sigma \times Id)$ and two LTS $\alpha : X \to P(\Sigma \times X)$ and $\beta : Y \to P(\Sigma \times Y)$ a map $f : X \to Y$ is a morphism between coalgebras $\alpha$ and $\beta$ in $Kl(*)$ if and only if the following condition is satisfied:

$$
x \overset{a}{\sim}_\alpha x' \implies f(x) \overset{a}{\sim}_\beta f(x') \text{ for any } a \in \Sigma_t,
$$

where $\sim_\beta$ denotes the saturated version of $\to_\beta$, i.e.

$$
\sim_\beta = \left\{ \begin{array}{ll}
(\overset{\tau}{\sim}_\beta)^* \circ \overset{\tau}{\sim}_\beta \circ (\overset{\tau}{\sim}_\beta)^* & \text{for } a \neq \tau, \\
(\overset{\tau}{\sim}_\beta)^* & \text{otherwise.}
\end{array} \right.
$$

4. Weak bisimulation for presheaves

The purpose of this section is to introduce the notion of (weak) bisimulation between presheaves on one-object categories. Although the presheaf (weak) bisimulation can serve as an extension of coalgebra (weak) bisimulation in future applications when dealing with more processes and their iteration modelled by coalgebras, the main reason we introduce these notions is to show their special characteristics that are visible only on this level of generality (see Theorem 4.45). In this section we assume the following:
• D is a one-object category. In this case we have the forgetful functor $U : T^{D^{op}} \to C$ which sends any presheaf $\pi \in T^{D^{op}}$ and any transformation $\phi : \pi \Rightarrow \pi'$ to $U(\pi) = \pi(*) \quad U(\phi) = \phi_*$.

This allows us to use the notion of a relation in $C$ in the presheaf setting. To be more precise, a relation between objects $X, Y$ in $C$ is a jointly monic span $X \xleftarrow{\text{proj}_1} R \xrightarrow{\text{proj}_2} Y$ in $C$.

• $T^i : T^* \to T^{D^{op}}$ admits a left adjoint $\Sigma_i : T^{D^{op}} \to T^*$.

• $\Sigma_i$ is the identity on morphisms. Together with the fact that so is $T^i$ we get that $* = T^i \circ \Sigma_i : T^{D^{op}} \to T^{D^{op}}$ is also the identity on morphisms.

• $\Sigma_i$ and $T^i$ preserve strict transformations.

The last two assumptions are technical conditions used in the proof of Theorem 4.4.

For any monad $T \in \{ P(\Sigma \times Id), P(M \times Id), P(\Sigma^* \times Id + \Sigma^*), F \}$ these assumptions are satisfied, which follows directly by the definition of $\Sigma_i$ from the previous section. In order to be able to speak about weak bisimulation for presheaves we need to be a bit picky as far as the choice of morphisms for our underlying category is concerned. Just like in the case of $C_T, \leq$ and $C_T$ where bisimulation is defined in terms of strict homomorphisms between suitable coalgebras we will define (weak) bisimulation for presheaves in $T^{D^{op}}$ in terms of strict natural transformations.

**Definition 4.1.** Let $\pi, \pi' \in T^{D^{op}}$. A relation $U(\pi) \xleftarrow{\text{proj}_1} R \xrightarrow{\text{proj}_2} U(\pi')$ in $C$ is called a (presheaf) bisimulation between $\pi$ and $\pi'$ if there is a presheaf $\rho \in T^{D^{op}}$ such that $U(\rho) = R$ for which $\text{proj}_1$ and $\text{proj}_2$ are components of strict transformations from $\rho$ to $\pi$ and $\pi'$ respectively.

The definition above is an analogue of the coalgebraic bisimulation via span, i.e. a bisimulation in Aczel-Mendler style [1] [31]. Let us recall this notion now. A relation $X \xleftarrow{\alpha} R \xrightarrow{\beta} Y$ is Aczel-Mendler bisimulation, or bisimulation in short, between coalgebras $\alpha : X \to TX$ and $\beta : Y \to TY$ if there is a structure $\gamma : R \to TR$ making the following diagram commute:

$$
\begin{array}{ccc}
X & \xleftarrow{\alpha} & R \\
\downarrow & \gamma & \downarrow \\
TX & \xrightarrow{\beta} & TY
\end{array}
$$

It is important to note that bisimulation via span is one of several options available for the definition of bisimulation we could introduce in the presheaf setting. We choose this one as we want to be able to relate it to an equivalence in $Kl(*) = Kl(T^i \circ \Sigma_i)$ via Theorem 4.4. In general, however, there is no need to restrict ourselves to bisimulation via span. We can use other options summerized in [31].

We would also like to emphasize that the above definition can be used to introduce the Aczel-Mendler bisimulation for coalgebras. The following result is a direct consequence of the adjoint situation $C_T, \leq \Rightarrow T^\omega$ from Proposition 3.7.

**Theorem 4.2.** For two coalgebras $\alpha : X \to TX$ and $\beta : Y \to TY$, a relation between $X$ and $Y$ is an Aczel-Mendler bisimulation between $\alpha$ and $\beta$ if and only if it is a presheaf bisimulation in $T^\omega$ between $\alpha$ and $\beta$.

**Proof.** Assume $R$ is a bisimulation between coalgebras $\alpha$ and $\beta$. This yields a coalgebraic structure $\gamma : R \to TR$ satisfying the above properties. The presheaf
\( \rho := \gamma \) turns \( R \) into a presheaf bisimulation between \( \alpha \) and \( \beta \). For the converse implication let \( \rho \) be a presheaf which makes a relation \( R = U(\rho) \) a bisimulation between \( \alpha \) and \( \beta \). Then the desired coalgebraic structure on \( R \) which turns it into a bisimulation between coalgebras \( \alpha \) and \( \beta \) is given by \( \gamma := \rho_1 \). \( \Box \)

**Definition 4.3.** Let \( \pi, \pi' \in T^{D^{op}} \). A relation \( U(\pi) \xleftarrow{\text{proj}_1} R \xrightarrow{\text{proj}_2} U(\pi') \) in \( C \) is weak bisimulation between \( \pi \) and \( \pi' \) if it is a strong bisimulation between \( \pi^* \) and \( \pi'^* \).

Note that the above definition can be restated using the Kleisli category \( K_l(\ast) \) for the saturation monad \( \ast = T ! \circ \Sigma ! : T D^{op} \rightarrow T D^{op} \). To be more precise, a relation \( U(\pi) \xleftarrow{\text{proj}_1} R \xrightarrow{\text{proj}_2} U(\pi') \) is a weak bisimulation between \( \pi \) and \( \pi' \) if and only if there is a presheaf \( \rho \) with \( U(\rho) = R \) for which \( \text{proj}_1 \) and \( \text{proj}_2 \) are strict transformations between \( \rho \) and \( \pi, \pi' \) in \( K_l(\ast) \).

Given the assumptions at the beginning of this section the following result is straightforward to verify and hence is left without a proof.

**Theorem 4.4.** A relation is a weak bisimulation between presheaves \( \pi \) and \( \pi' \) in \( T^{D^{op}} \) iff it is a bisimulation between \( \Sigma ! \pi \) and \( \Sigma ! \pi' \) in \( T^* \).

The framework presented above can be extended in the following way. We can relax the notion of weak bisimulation as a bisimulation in \( T^* \). Indeed, for a one-object category \( E \) if \( p : D \rightarrow E \) is a functor and \( \Sigma_p : T^{D^{op}} \rightarrow T^{E^{op}} \) is the left adjoint to the change of base functor \( T^p : T^{E^{op}} \rightarrow T^{D^{op}} \) then we may define the notion of \( p \)-bisimulation of presheaves \( \pi, \pi' \in T^{D^{op}} \) as a bisimulation between \( \Sigma_p(\pi) \) and \( \Sigma_p(\pi') \) in \( T^{E^{op}} \). We can easily generalize all results presented in this section to \( p \)-bisimulation (given the assumptions from the beginning are satisfied in the general case). In this terminology weak bisimulation would be \(!\)-bisimulation and strong bisimulation the \( \text{Id}-\)bisimulation.

**Theorem 4.5.** Assume \( p : D \rightarrow E \) is a functor such that \( T^p \) admits a left adjoint \( \Sigma_p \) and that the adjoint pairs \( \Sigma !, T^! \) and \( \Sigma_p, T^p \) satisfy assumptions from the beginning of this section. We have:

\[
\text{strong bisimulation} \implies p\text{-bisimulation} \implies \text{weak bisimulation}.
\]

**Proof.** This follows by the fact that \(! \circ p = !\) and that composition of two adjoint situations yields an adjoint situation.

5. **Summary and future work**

We presented a framework of lax Kleisli-valued presheaves as a setting that generalizes the setting of coalgebras and lax homomorphisms in which we can introduce the notion of bisimulation. Just like a single coalgebra is a process, a presheaf can be understood as a system of processes. We showed that in many cases, the change of base functor between lax presheaf categories admits a left adjoint and that the
adjunction $T^\omega \rightleftharpoons T^\bullet$ plays an important role in coalgebraic saturation and coalgebraic weak bisimulation. Using the lax presheaf adjunction we introduced the notion of weak bisimulation for presheaves over monoid categories. This relation takes into account the cumulative behaviour of a presheaf. Finally, we showed that in the family of $p$-bisimulations, which arises naturally in the presheaf setting, two bisimulations are special, namely the strong bisimilarity and the weak bisimilarity. The first one turns out to be the finest, the second the coarsest.

Although the setting presented in this paper is, in our opinion, very elegant, the biggest disappointment is that the subdistribution monad fails to fit it. It would be interesting to see if it is possible to generalize the setting to arbitrary 2-categories as Kleisli categories and if there are any examples of transition systems with weak bisimulation that would fit the generalized framework and do not fit the one presented here.

Moreover, it would be interesting to get to know other basic categorical properties of lax presheaf category $T^D_\omega$. It is worth noting that in [22] in the case $C = \mathbb{Set}$ and $T = P$ the author studies the existence of the right adjoint to $T^p$. It would be nice to see if this functor is of any practical value from our perspective.

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