Abstract—Insufficiency of linear coding for the network coding problem was first proved by providing an instance which is solvable only by nonlinear network coding (Dougherty et al., 2005). Based on the work of Effros, et al., 2015, this specific network coding instance can be modeled as a groupcast index coding (GIC) instance with 74 messages and 80 users (where a message can be requested by multiple users). This proves the insufficiency of linear coding for the GIC problem. Using the systematic approach proposed by Maleki et al., 2014, the aforementioned GIC instance can be cast into a unicast index coding (UIC) instance with more than 200 users, each wanting a unique message. This confirms the necessity of nonlinear coding for the UIC problem, but only for achieving the entire capacity region. Nevertheless, the question of whether nonlinear coding is required to achieve the symmetric capacity (broadcast rate) of the UIC problem remained open. In this paper, we settle this question and prove the insufficiency of linear coding, by directly building a UIC instance with only 36 users for which there exists a nonlinear index code outperforming the optimal linear code in terms of the broadcast rate.

I. INTRODUCTION

Index coding problem was first introduced by Birk and Kol in the context of satellite communication in which there is a single server, broadcasting $m$ messages to a number of users via a noiseless shared channel. Each user requests one specific message from the server and may already know some other messages as its side information. Exploiting the side information of the users, the server might be able to reduce the overall number of coded messages in order to communicate the $m$ messages. The main objective can be summarized as finding the minimum number of transmissions so that all users will be able to decode their requested message. The simple model established in index coding problem can be employed to study several important communication settings, including network coding, distributed storage, coded caching, and topological interference management.

The connection between network coding and index coding problem was established in [4], in which a reduction method was provided to map any instance of network coding to a corresponding instance of index coding. This connection between network coding and index coding was extended in [5] to include the general encoding and decoding functions so that the solution of an index coding instance will be suitably converted as a solution for the equivalent network coding instance.

While in the unicast index coding (UIC), each message can be requested by only one user, this scenario can be extended to allow multiple users to request the same message, which is referred to as groupcast index coding (GIC). In fact, the aforementioned equivalence, turns any instance of network coding to an instance of index coding such that some messages are requested by more than one user. In [10], a systematic approach was proposed to construct an equivalent UIC instance for an arbitrarily GIC instance such that each message requested by multiple users is mapped to multiple distinct auxiliary messages requested by only one user. However, this construction method requires that the rate of auxiliary messages to be different from the rate of other messages, resulting in an asymmetric rate UIC problem.

Index coding schemes are broadly categorized into linear and nonlinear codes. Although linear index coding has been the center of attention due to its straightforward encoding and decoding processes, for the general index coding problem, they can be outperformed by nonlinear codes [4], [5], [11]–[13]. Insufficiency of linear codes was first proved in [11] in the context of network coding by providing a network instance which is not solvable by any linear codes, while it can be solved by a nonlinear network code. Having established an efficient method of reducing a network coding instance to a corresponding instance of index coding [4], [5], the necessity of nonlinear coding was proved for the GIC problem. Furthermore, the construction technique of mapping an arbitrary GIC instance to an equivalent UIC instance means that linear coding is also insufficient for the UIC problem, but only for achieving the entire capacity region.

Open Problem [10] Remark 1: For the UIC problems, the question of whether linear coding is necessary to achieve the broadcast rate remained as an open problem, despite the fact that broadcast rate is most commonly used and studied in the index coding literature.

In this paper, we settle this open question by constructing a UIC instance for which there exists a nonlinear index code that can outperform the optimal linear coding in terms of the broadcast rate. This UIC instance consists of two separate UIC subinstances, which are connected to each other in some specific ways. In fact, the characterization of these subinstances is inspired by the two network coding subinstances in [11], where for the first one, linear coding cannot be optimal over any finite field with odd characteristic (i.e., a field with odd cardinality), and for the second one, linear coding is not able to achieve the broadcast rate over any finite field with characteristic two (i.e., field with even cardinality). In this work, the side information set of each user is designed, aiming at requiring the optimal index coding solution to satisfy the same specific constraints which must be met by any network code solving the network coding counterexample in [11].

One main merit of the UIC instance built in this paper, is its considerable simplicity with regard to the number of users (or messages) compared to the instances obtained based on mapping methods in [5] and [10]. In fact, mapping the network coding counterexample in [11] by the reduction method in [5] results in a GIC instance, including 74 messages and 80 users (indeed, 148 users if each user requests only one specific message). This GIC instance using the mapping in [10] is turned into its equivalent asymmetric rate UIC instance, consisting of more than 200 users. However, the UIC instance provided in this paper comprises only 36 users.

Broadcast Rate Requires Nonlinear Coding in a Unicast Index Coding Instance of Size 36

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A. Summary of Contributions

1) By directly designing a UIC instance, we prove that there exists a nonlinear index code which outperforms the optimal linear coding to achieve the symmetric capacity rate. This implies that linear coding is insufficient for the general UIC problem for achieving the broadcast rate. This UIC instance consists of two distinct subinstances which are connected in two different ways (Section III).

2) We prove that for the first subinstance, its broadcast rate is not achievable by any linear coding over finite field with odd characteristic, while it can be achieved by a scalar binary linear code (Section IV).

3) For the second subinstance, we prove that linear coding over any finite field with characteristic two will not be able to achieve its broadcast rate. However, we show that there exists a scalar binary nonlinear index code which is optimal (Section IV).

II. SYSTEM MODEL AND BACKGROUND

This section provides an overview of the system model and relevant background and definitions in index coding problem.

A. Notation

Scalar small letters such as $n$ denote an integer number where $[n] := \{1, ..., n\}$. Scalar capital letters such as $L$ denote a set, whose cardinality is denoted by $|L|$ and $n_L := \{n_l, l \in \mathbb{L}\}$. Symbols in bold face such as $I$ and $L$ denote a vector and a matrix, respectively, with $L^T$ denoting the transpose of matrix $L$. A calligraphic symbol such as $L$ is used to denote a set whose elements are sets.

We write $X$ to denote a finite alphabet. We use $F_q$ to denote a finite field of size $q$ and write $F_q^{n \times m}$ to denote the vector space of all $n \times m$ matrices over the field $F_q$. Throughout the paper, $I_n$ denotes the $n \times n$ identity matrix, and $0_{n \times m}$ represents a full-zero matrix of size $n \times m$.

B. System Model

Consider a broadcast communication system in which a server transmits a set of $mt$ messages $X = \{x_{i,j}, i \in [m], j \in [l]\}$, $x_{i,j} \in X$, to a number of users $U = \{u_i, i \in [m]\}$ through a noiseless broadcast channel. Each user $u_i$ wishes to receive a message of length $t$, $X_i = \{x_{i,j}, j \in [l]\}$ and may have a prior knowledge of a subset of the messages $S_i := \{x_{i,j}, l \in A_i, j \in [l]\}, A_i \subseteq [m]\{i\}$, which is referred to as its side information set. The main objective is to minimize the number of coded messages which is required to be broadcast so as to enable each user to decode its requested message. An instance of index coding problem $\mathcal{I}$ can be characterized by either the side information set of all users as $\mathcal{I} = \{A_i, i \in [m]\}$, or by their interfering message set $B_i = \{m\}\{A_i \cup \{i\}\}$ as $\mathcal{I} = \{B_i, i \in [m]\}$.

C. General Index Code

Definition 1 (Index Code for $\mathcal{I}$). Given an instance of index coding problem $\mathcal{I} = \{A_i, i \in [m]\}$, a $(t, r)$ index code is defined as $C_2 = (\phi_2, \psi_2)$, where

- $\phi_2 : \mathcal{X}^{mt} \rightarrow \mathcal{X}^r$ is the encoder function which maps the $mt$ message symbol $x_{i,j}$ in $X$ to the $r$ coded messages as $Y = \{y_1, ..., y_r\}$, where $y_k \in \mathcal{X}, \forall k \in [r]$.
- $\psi_2$ represents the decoder function, where for each user $u_i, i \in [m]$, the decoder $\psi_2 : \mathcal{X}^r \times \mathcal{X}^{A_i} \rightarrow \mathcal{X}^{l}$ maps the received $r$ coded messages $y_k \in Y, k \in [r]$ and the $|A_i|t$ messages $x_{i,j} \in S_i$ in the side information to the $t$ messages $\psi_2(Y, S_i) = \{\tilde{x}_{i,j}, j \in [l]\}$, where $\tilde{x}_{i,j}$ is an estimate of $x_{i,j}$.

Definition 2 ($\beta(C_2)$: Broadcast Rate of $C_2$). Given an instance of index coding problem $\mathcal{I}$, the broadcast rate of a $(t, r)$ index code $C_2$ is defined as $\beta(C_2) = \frac{t}{r}$.

Definition 3 ($\beta_2$: Broadcast Rate of $\mathcal{I}$). Given an instance of index coding problem $\mathcal{I}$, the broadcast rate $\beta_2$ is defined as

$$\beta_2 = \inf_{t} \inf_{r} \beta(C_2).$$

Definition 4 (Scalar and Vector Index Code). The index code $C$ is considered to be scalar if $t = 1$. Otherwise, it is called a vector code. For scalar codes, we use $x_i = x_{i,1}, \forall i \in [m]$, for simplicity.

D. Linear Index Code

We can assume that the finite alphabet $X$ is selected as a finite field $F_q$, so linear operations are well-defined. Let $x = [x_1, ..., x_m]^T \in F_q^{mt \times 1}$ denote the message vector, where $x_i = [x_{i,1}, ..., x_{i,l}] \in F_q^{l \times 1}$ is the requested message vector by user $u_i, \forall i \in [m]$. We denote the side information vector of $u_i$ as $x_{A_i} = [x_{1,i}, ..., x_{m,i}]^T$, where $x_{j,i} = x_j, \text{ if } j \in A_i$ and $x_{j,i} = 0_{l \times 1}, \text{ otherwise}$.

Definition 5 ($L_2$-Linear Index Code for $\mathcal{I}$). Given an instance of index coding problem $\mathcal{I} = \{A_i, i \in [m]\}$, a $(t, r)$ linear index code is defined as $L_2 = (H, \{\psi_i\})$, where

- $H : F_q^{mt \times t} \rightarrow F_q^{r \times 1}$ is the encoder matrix which maps the message vector $x = [x_1, ..., x_m]^T \in F_q^{mt \times 1}$ to a coded message vector $y = [y_1, ..., y_r]^T \in F_q^{r \times 1}$ as follows

$$y = Hx = \sum_{i \in [m]} H_i x_i^T.$$ Here $H_i \in F_q^{r \times t}$ is the encoder matrix of the $i$-th message vector $x_i$, such that $H = [H_1, ..., H_m] \in F_q^{r \times mt}$.

- $\psi_i$ represents the linear decoder function for user $u_i, i \in [m]$, where $\psi_i(y, x_{A_i})$ maps the received coded message $y$ and its side information vector $x_{A_i}$ to $\tilde{x}_i$, which is an estimate of the requested message vector $x_i$.

Proposition 1. It can be shown [12] that the necessary and sufficient condition for linear decoder $\psi_i, \forall i \in [m]$ to be able to correctly decode the requested message vector $x_i$ is

$$\text{rank } H_{(i) \cup B_i} = \text{rank } H_{B_i} + t,$$

where $H_L$ denotes the matrix $[H_1 \mid \cdots \mid H_m] \in F_q^{r \times mt}$.

Definition 6 ($\lambda_1(L_2)$: Linear Broadcast Rate of $L_2$ over $F_q$). Given an instance of index coding problem $\mathcal{I}$, the linear broadcast rate of a $(t, r)$ index code $L_2$ over field $F_q$ is defined as $\lambda_1(L_2) = \frac{t}{r}$.

Definition 7 ($\lambda_{L_2}$: Linear Broadcast Rate of $\mathcal{I}$ over $F_q$). Given an instance of index coding problem $\mathcal{I}$, the linear broadcast rate $\lambda_{L_2}$ over field $F_q$ is defined as

$$\lambda_{L_2} = \inf_{t} \inf_{r} \lambda_1(L_2).$$

Definition 8 ($H_{\ast q}$: Optimal Encoder Matrix of $\mathcal{I}$ over $F_q$). Given an instance of index coding problem $\mathcal{I}$, an encoder matrix, satisfying the decoding condition in (2) for all users, is said to be optimal over $F_q$ if the number of its rows is equal to $\lambda_{L_2}$. Such an optimal encoder matrix is denoted by $H_{\ast q}$.

Definition 9 ($\lambda_2(L_2)$: Linear Broadcast Rate for $\mathcal{I}$). Given an instance of index coding problem $\mathcal{I}$, the linear broadcast rate is defined as

$$\lambda_2 = \min_{q} \lambda_{L_2,q}.$$
which means that the new instance $\mathcal{I}$ is a concatenation of the two substANCES $\mathcal{I}_1$ and $\mathcal{I}_2$ such that each user in $\mathcal{I}_1$ has all the messages requested by the users in $\mathcal{I}_2$ in its side information set and vice versa.

**Proposition 3** (Blasiak et al. [19]). Let $\lambda_{\mathcal{I}_1,q}$ and $\lambda_{\mathcal{I}_2,q}$, respectively, denote the linear broadcast rate of $\mathcal{I}_1$ and $\mathcal{I}_2$ over $\mathbb{F}_q$. Then, for the linear broadcast rate of $\mathcal{I}_3 = \mathcal{I}_1 \leftrightarrow \mathcal{I}_2$ and $\mathcal{I}_4 = \mathcal{I}_1 \leftrightarrow \mathcal{I}_2$ over $\mathbb{F}_q$, we have

$$\lambda_{\mathcal{I}_3,q} = \lambda_{\mathcal{I}_1,q} + \lambda_{\mathcal{I}_2,q},$$

$$\lambda_{\mathcal{I}_4,q} = \max\{\lambda_{\mathcal{I}_1,q}, \lambda_{\mathcal{I}_2,q}\}.$$  

**Theorem 1.** Linear coding is insufficient for achieving the broadcast rate of the UIC problems.

**Proof.** In the next two sections, two UIC instances $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively, with $m_1 = 10$ and $m_2 = 26$ messages will be characterized with the following properties:

- In Theorem 2 first, we show that $\lambda_{\mathcal{I}_1,2} = \beta_{\mathcal{I}_1} = 6$ by designing a scalar binary linear code $C_{\mathcal{I}_1} = (H_{\mathcal{I}_1,2}, \{x^{(\mathcal{I}_1)}\})$. This implies the optimality of the binary linear code. Then, we prove that $\lambda_{\mathcal{I}_2,q} > 26, \forall q = 2k, k \geq 1$, which means that linear coding over any finite field with odd characteristic is not optimal.

- In Theorem 3 first, we prove that $\lambda_{\mathcal{I}_2,q} > \beta_{\mathcal{I}_2} = 6, \forall q = 2k, k \geq 1$, which means that linear coding over any finite field with characteristic two is not optimal. Then, we show that there exists a scalar binary nonlinear code $C_{\mathcal{I}_2} = (\phi_{\mathcal{I}_2}, \psi_{\mathcal{I}_2}^{(\mathcal{I}_2)})$, which is optimal, i.e., $\beta(C_{\mathcal{I}_2}) = 6$.

This implies that, if the finite field $\mathbb{F}_q$ has either characteristic two or odd characteristic, then one of $\lambda_{\mathcal{I}_1,q}$ and $\lambda_{\mathcal{I}_2,q}$ will always be greater than $6$. Hence, according to (3) and Proposition 3, we have

$$\lambda_{\mathcal{I}_1,q} \geq \lambda_{\mathcal{I}_2,q} \geq \beta_{\mathcal{I}_2} \geq \beta_{\text{MAIS}(\mathcal{I}_1)},$$

$$\lambda_{\mathcal{I}_2,q} > 6.$$  

**III. INSUFFICIENCY OF LINEAR CODING FOR THE UIC PROBLEM IN TERMS OF BROADCAST RATE**

This section gives a description of the two specific ways of connecting the two instances $\mathcal{I}_1$ and $\mathcal{I}_2$, which will be used in this paper for proving the insufficiency of linear coding for achieving the broadcast rate of the UIC problem.

**Definition 14** ($\mathcal{I}_1 \leftrightarrow \mathcal{I}_2$: No-way Connection of $\mathcal{I}_1$ and $\mathcal{I}_2$). Given two index coding instances $\mathcal{I}_1 = \{I_1, i \in [m_1]\}$ and $\mathcal{I}_2 = \{I_2, i \in [m_2]\}$, a no-way connection of $\mathcal{I}_1$ and $\mathcal{I}_2$, denoted by $\mathcal{I}_1 \leftrightarrow \mathcal{I}_2$, is defined as a new index coding instance $\mathcal{I} = \{I, i \in [m]\}$, where $m = m_1 + m_2$

$$\lambda_{\mathcal{I}_2,q} \geq \lambda_{\mathcal{I}_2,2} \geq \beta_{\mathcal{I}_2} \geq \beta_{\text{MAIS}(\mathcal{I}_1)}.$$  

**Proposition 2** (Var-Yossif et al. [18]). Given the index coding instance $\mathcal{I}$ and a finite field $\mathbb{F}_q$, we have

$$\lambda_{\mathcal{I},q} \geq \lambda_{\mathcal{I},q} \geq \beta_{\mathcal{I}} \geq \beta_{\text{MAIS}(\mathcal{I})},$$

$$\lambda_{\mathcal{I},q} \geq \lambda_{\mathcal{I},q} \geq \beta_{\mathcal{I}} \geq \beta_{\text{MAIS}(\mathcal{I})}.$$  

**Remark 2.** It can be easily verified that the decoding condition in (3) in the properties of the rank function give the following results.

rank $H_{i} \cup B_i = \text{rank} H_{B_i} + t, \forall B_i \subseteq B_i, \forall i \in [m]$.

rank $H_{i} = t, \forall i \in [m]$.

rank $H_{L_1} \leq \text{rank} H_{L_2}, \forall L_1 \subseteq L_2 \subseteq [m]$.

**Lemma 1.** Assume $L \subseteq [m]$ is an acyclic set of $\mathcal{I}$. Then, the condition in (7) for all $i \in L$ requires rank $H_{L_i} = |L|t$.

**Proof.** If $L$ is an acyclic set, then we can find a sequence of its elements $l_1, \ldots, l_L \subseteq L$ such that $L_i \subseteq B_{l_i}, \forall i \in [L]$, where $L_i = \{l_{i+1}, \ldots, l_L\} \subseteq \{l_{i+1}, \ldots, l_L\} = \emptyset$. Note $L = \{l_1\} \cup L_1$ and $L_2 = \{l_{i+1} \cup L_1 \cup L_{i+1}\} \cup \{l_{i+1} \cup L_1 \cup L_{i+1}\} = \{l_{i+1} \cup L_1 \cup L_{i+1}\}$. By applying the condition in (7) for each $i = l_1, \ldots, l_L$, we have

rank $H_{L = \{l_1\} \cup L_1} = \text{rank} H_{L_1 = \{l_2\} \cup L_2} + t$.
First, note that for any set \( L \subseteq [m] \) we consider the encoder matrix \( H \). In order to have rank \( H_L = ([L] - 1)t \), we must have \( \mathcal{H} = \{ H_j \mathcal{M}_{i,j}, \forall l \in L \} \) such that each \( \mathcal{M}_{i,j} \in \mathbb{F}^{q \times t} \) is invertible.

**Lemma 2.** Let \( L \subseteq [m] \) be a minimal cyclic set of \( \mathcal{I} \) with encoder matrix \( H \). In order to have rank \( H_L = ([L] - 1)t \), we must have \( \mathcal{H} = \{ H_j \mathcal{M}_{i,j}, \forall l \in L \} \) such that each \( \mathcal{M}_{i,j} \in \mathbb{F}^{q \times t} \) is invertible.

**Proof.** First, note that for any \( l \in L \), set \( L \setminus \{l\} \) is an acyclic set. Then, according to Lemma 1,

\[
\text{rank } H_L \setminus \{l\} = ([L] - 1)t, \quad \forall l \in L. \tag{10}
\]

So, having rank \( H_L = ([L] - 1)t \) requires \( H_L = \{ H_j \mathcal{M}_{i,j}, \forall l \in L \} \) to be invertible, then rank \( H_L \setminus \{l\} < ([L] - 1)t \), which contradicts (10). Thus, all \( \mathcal{M}_{i,j} \) must be invertible.

**Lemma 3.** Assume \( L \subseteq [m] \) is an independent set of \( \mathcal{I} \) and let \( j \in [m] \setminus L \). Now, if \( j \in B_i \), for some \( i \in L \), then in order to have rank \( H_{(j)\cup L} = [L]t \), one must satisfy \( H_j = \mathcal{M}_{i,j} \) for some invertible matrix \( \mathcal{M}_{i,j} \).

**Proof.** First, because \( L \) is an independent set, then \( L \setminus \{j\} \subseteq B_i, \forall i \in L \). Moreover since \( L \) is an acyclic set, then rank \( H_L = [L]t \). So, in order to have rank \( H_{(j)\cup L} = [L]t \), we must have \( \mathcal{H} = \{ H_j \mathcal{M}_{i,j}, \forall l \in L \} \). Let \( j \in [m] \) and \( j \in B_i \), for some \( i \in L \) which leads to \( \{j\} \cup L \setminus \{j\} \subseteq B_i \). Now, assume \( \mathcal{M}_{i,j} \) is a nonzero matrix (so, rank \( \mathcal{M}_{i,j} \) is 1). Then,

\[
\text{rank } H_{(j)\cup L} = \text{rank } H_j \mathcal{M}_{i,j} = [L]t. \tag{11}
\]

where (11) is due to (7), (12) is because of the property of the rank function by removing the term \( \sum_{l \in L \setminus \{j\}} H_j \mathcal{M}_{i,j} \) from \( \sum_{l \in L} H_j \mathcal{M}_{i,j} \) as it is a linear combination of the columns of \( H_{L \setminus \{j\}} \). (12) is based on Lemma 1 and the fact that \( L \) is an acyclic set. Thus, the column space of \( H_L \) is linearly independent of column space of \( H_{L \setminus \{j\}} \). Finally, (13) is due to the fact that \( H_j \) is full-rank and rank \( \mathcal{M}_{i,j} \geq 1 \). The result in (14) contradicts the assumption that rank \( H_{(j)\cup L} = [L]t \), and hence, we must have \( \mathcal{M}_{i,j} = 0 \). The argument is the same for \( i \in L \setminus \{j\} \) gives \( M_{i,j} = 0, \forall i \in L \setminus \{j\} \).

Therefore, the binary linear code is optimal.

**Theorem 2.** For the index coding instance \( \mathcal{I}_1 \), \( \lambda_{\mathcal{I}_1,2} = 2t + 1 \).

However, \( \lambda_{\mathcal{I}_1,2} = \beta_{\mathcal{I}_1,2} = 6 \). Each claim is proved separately in Propositions 4 and 5 respectively.

**Proposition 4.** \( \lambda_{\mathcal{I}_1,2} = 6 \). This means Binary linear coding achieves the broadcast rate of \( \mathcal{I}_1 \).

**Proof.** First, it can be observed that \( V' = \{1, 2, 3, 8, 9, 10\} \) is a MAIS set of \( \mathcal{I}_1 \). So, \( \beta_{\mathcal{MAIS}(\mathcal{I}_1)} = 6 \), which verififies that the following scalar binary linear code achieves the MAIS bound:

\[
\begin{align*}
\begin{bmatrix}
1 & y_1 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
1 & y_2 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
1 & y_3 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\
1 & y_4 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10}
\end{bmatrix}
\end{align*}
\]

Therefore, the binary linear code is optimal.

**Proposition 5.** \( \lambda_{\mathcal{I}_2,2} > 6 \) for all \( q = \text{odd} \). This means linear coding over any field with odd characteristic cannot achieve the broadcast rate of \( \mathcal{I}_1 \).

**Proof.** First, note that \( \mathcal{I}_2 = \mathcal{I}_2' \cup \mathcal{I}_2'' \), where \( \mathcal{I}_2' = \{ A_i, i \in [7] \} \) and \( \mathcal{I}_2'' = \{ A_8, A_9, A_{10} \} \). For the subinstance \( \mathcal{I}_2'' \), \( L_1 = \{8, 9, 10\} \) forms a MAIS set, so \( \beta_{\mathcal{MAIS}(\mathcal{I}_2'')} = 3 \), which means uncoded transmission \( y_8 = x_8, y_9 = x_9, y_{10} = x_{10} \) is optimal, now, we prove that for the subinstance \( \mathcal{I}_2 = \{ B_i', i \in [7] \} \), where \( B_i' = B_i \cap [7] \) (local interfering message set), any linear coding over a field with odd characteristic cannot be optimal. It can be seen that set \( L_2 = \{1, 2, 3\} \) is a MAIS set of \( \mathcal{I}_2' \), so \( \beta_{\mathcal{MAIS}(\mathcal{I}_2')} = 3 \). To achieve \( \lambda_{\mathcal{I}_2,2} = 3 \), the decoding condition in (2) and Lemma 1 respectively, give

\[
\text{rank } H_{B_i' \cup L} = 2t, \quad \forall i \in [7], \tag{17}
\]

\[
\text{rank } (H_{(1,2,3)} = [H_1 \ H_2 \ H_3]) = 3t. \tag{18}
\]

So, to have rank \( H = 3t \), other \( H_i, i \in \{4, 5, 6, 7\} \) must be expressed as a linear combination of \( H_1, H_2, H_3 \). It can also be observed that each set \( B_i', i \in [7] \) is a minimal cyclic set. Thus, because we want rank \( H_{B_i'} = 2t \), then based on Lemma 2 we will have seven constraints as follows (for simplicity, in this proof, each matrix \( M_j \) is numbered by only one index)

\[
\begin{align*}
B_1' & \rightarrow H_5 = H_2M_1 + H_3M_2, \tag{19} \\
B_2' & \rightarrow H_7 = H_3M_3 + H_4M_4, \tag{20} \\
B_3' & \rightarrow H_6 = H_4M_5 + H_5M_6, \tag{21} \\
B_4' & \rightarrow H_7 = H_1M_7 + H_5M_8, \tag{22} \\
B_5' & \rightarrow H_7 = H_2M_9 + H_6M_{10}, \tag{23} \\
B_6' & \rightarrow H_4 = H_1M_{11} + H_2M_{12}, \tag{24} \\
B_7' & \rightarrow H_6 = H_1M_{13} + H_3M_{14}, \tag{25}
\end{align*}
\]

where the matrices \( M_j \in \mathbb{F}^{q \times t}, j \in [14] \) must be all invertible. We show that meeting these seven constraints will lead to a contradiction over any field with odd characteristic. Now, in (20), (22) and (23), we replace \( H_4, H_5, H_6 \) with their equal term, respectively, in (24), (19) and (25), which leads to

\[
\begin{align*}
& (20), (24) \rightarrow H_7 = H_1M_{11} + H_2M_{12} + H_3M_{14}, \tag{26} \ \\
& (22), (19) \rightarrow H_7 = H_1M_7 + H_2M_9 + H_3M_{12}, \tag{27} \ \\
& (23), (25) \rightarrow H_7 = H_1M_{13} + H_2M_{10} + H_3M_{14}, \tag{28}
\end{align*}
\]
Due to (18), it can be seen that (26), (27) and (28) are equal if $f$ the binary field which can achieve the broadcast rate.

Now , in (35), we substitute \[ \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \] which results in

\[ M_{12}M_{11}^{-1}M_{13} = M_1M_2^{-1}M_{14}. \tag{32} \]

On the other hand, in (21), we replace $H_4, H_5, H_6$ with their equal term, respectively, in (29) and (30), which gives

\[ M_{13} = M_1M_5, \tag{33} \]
\[ M_{14} = M_2M_6, \tag{34} \]
\[ M_{12}M_5 + M_1M_6 = 0. \tag{35} \]

Now, in (33), (34), respectively, which gives

\[ M_{12}M_{11}^{-1}M_{13} + M_1M_2^{-1}M_{14} = 0. \tag{36} \]

Finally, since all the $M_i$'s are invertible, from (32) and (36), we must have $I_1 = -I_1$, which is not possible over any field with odd characteristic. This completes the proof.

\[ \Box \]

V. THE INDEX CODING INSTANCE $I_2$

This section provides the index coding instance $I_2$ where its broadcast rate is $\beta_{I_2} = 6$. We prove that this rate is not achievable by linear coding over any finite field with characteristic two. However, we show that, there exists a nonlinear code over the binary field which can achieve the broadcast rate.

The index coding instance $I_2 = \{B_i, i \in [26]\}$ is characterized as follows:

\[
\begin{align*}
B_1 & = \{2,3,5,11,12,13,15,22,23,24,25,26\}, \\
B_2 & = \{1,3,6,11,12,13,16,21,22,24,25,26\}, \\
B_3 & = \{1,2,4,11,12,13,14,21,22,23,24,26\}, \\
B_4 & = \{5,6,14,15,16\}, \\
B_5 & = \{4,6,14,15,16\}, \\
B_6 & = \{4,5,14,15,16\}, \\
B_7 & = \{17\}, \\
B_8 & = \{1,5,7,11,15,17,18\}, \\
B_9 & = \{2,6,7,12,16,17,19\}, \\
B_{10} & = \{3,4,7,13,14,17,20\}, \\
B_{11} & = \{1,2,3,5,12,13,15,21,23,24,25,26\}, \\
B_{12} & = \{1,2,3,6,11,13,16,21,22,23,25,26\}, \\
B_{13} & = \{1,2,3,4,11,12,14,21,22,23,24,25\}, \\
B_{14} & = \{4\}, \\
B_{15} & = \{5\}, \\
B_{16} & = \{6\}, \\
B_{17} & = \{7\}, \\
B_{18} & = \{1,5,7,8,11,15,17\}, \\
B_{19} & = \{2,6,7,9,12,16,17\}, \\
B_{20} & = \{3,4,7,10,13,14,17\}, \\
B_{21} & = \{4,6,14,16,22\}, \\
B_{22} & = \{4,6,14,16,21\}, \\
B_{23} & = \{4,5,14,15,24\}, \\
B_{24} & = \{4,5,14,15,23\}, \\
B_{25} & = \{5,6,15,16,26\}, \\
B_{26} & = \{5,6,15,16,25\}.
\end{align*}
\] \tag{37}

**Theorem 3.** $\lambda_{I_2,q} > 6, \forall q = 2k, k \geq 1$. This means that linear coding over any finite field with characteristic two cannot achieve the broadcast rate of $I_2$.

\[ \begin{align*}
\text{Proof.} \text{ First, it can be observed that } L = \{1,2,3,11,12,13\} \text{ is an independent set of } I_2. \text{ Thus, based on Lemma 3,} \\
\text{rank } H_L = 6t. \tag{38} \\
\end{align*} \]

Now, if $\lambda_{I_2,q} = 6$, then $\text{rank } H = 6t$, and thus, each $H_j, j \in [26] \setminus L$ must be expressed as a linear combination of $H_j, j \in L$ as $H_j = \sum_{l \in L} H_l M_{j,l}$. In the following, through **Steps 1 to 5**, we determine the constraints which must be held on the linear space of each $H_i, i \in [27] \setminus L$, which will finally lead to a contradiction if the field has characteristic two.

**Step 1:** In this step, we determine the constraints on the linear space of $H_i, i \in [21, ..., 26]$.

Lemma 3 gives the following results.

\[ \begin{align*}
21 & \in B_i, \forall i \in L \setminus \{1\} \rightarrow H_{21} = H_1M_{21,1}, \tag{39} \\
22 & \in B_i, \forall i \in L \setminus \{11\} \rightarrow H_{22} = H_{11}M_{22,11}, \tag{40} \\
23 & \in B_i, \forall i \in L \setminus \{2\} \rightarrow H_{23} = H_2M_{23,2}, \tag{41} \\
24 & \in B_i, \forall i \in L \setminus \{12\} \rightarrow H_{24} = H_{12}M_{24,12}, \tag{42} \\
25 & \in B_i, \forall i \in L \setminus \{3\} \rightarrow H_{25} = H_3M_{25,3}, \tag{43} \\
26 & \in B_i, \forall i \in L \setminus \{13\} \rightarrow H_{26} = H_{13}M_{26,13}. \tag{44}
\end{align*} \]

for some invertible matrices $M_{21,1}, M_{22,11}, M_{23,2}, M_{24,12}, M_{25,3}$ and $M_{26,13}$, respectively.

**Step 2:** In this step, we determine the constraints on the linear space of $H_{(i,t)+10}, i \in \{4,5,6\}$, which we write as

\[ H_{(i,t)+10} = \begin{bmatrix} H_{(1,11)}N_{i,1} & H_{(2,12)}N_{i,2} & H_{(3,13)}N_{i,3} \end{bmatrix}, \tag{45} \]

where,

\[ N_{i,j} = \begin{bmatrix} M_{i,j} & M_{i,j+10} \\ M_{i,j+10}^{-1} & M_{i,j+10}^{-1} \end{bmatrix}. \tag{46} \]

Let $C'_1 = \{2,3,12,13\} \subset C_1 = \{2,3,5,12,13,15\}, C'_2 = \{1,3,11,13\} \subset C_2 = \{1,3,6,11,13,16\}$ and $C'_3 = \{1,2,11,12\} \subset C_3 = \{1,2,4,11,12,14\}$. Then, for all $i = 1,2,3$, we have

\[ 6t = \text{rank } H \tag{47} \]
\[ \geq \text{rank } H_{(i,t)+10}\cup C_i \tag{48} \]
\[ = \text{rank } H_{(i,t)+10}\cup C_i + t \tag{49} \]
\[ = \text{rank } H_{C_i} + 2t \tag{50} \]
\[ \geq \text{rank } H_{C_i} + 2t \tag{51} \]
\[ = 6t, \tag{52} \]

where (47) is because we desire $\lambda_{I_2,q} = 6$. (48) is due to (49) and (50) are, respectively, because of (47) and the fact that \( (i,t)+10 \cup C_i \subset B_i \) and \( C_i \subset B_{i+10} \). (51) is due to \( C'_i \subset C_i \), and finally, (52) follows from the fact that each \( C'_i \) is an independent set. Now, based on (47), ..., (52), we have

\[ \text{rank } H_{C_i} = \text{rank } H_{C'_i} = 4t, \quad \forall i = 1,2,3, \tag{53} \]

which implies that each $H_{C'_i \setminus C_i}$ must be expressed as a linear combination of $H_{C'_i}$ for $i = 1,2,3$. This, respectively, results in

\[ \begin{align*}
H_{(5,15)} &= \begin{bmatrix} H_{(2,12)}N_{2,2} & H_{(3,13)}N_{3,3} \end{bmatrix}, \tag{54} \\
H_{(6,16)} &= \begin{bmatrix} H_{(1,11)}N_{1,1} & H_{(3,13)}N_{3,3} \end{bmatrix}, \tag{55} \\
H_{(4,14)} &= \begin{bmatrix} H_{(1,11)}N_{1,1} & H_{(2,12)}N_{2,4} \end{bmatrix}. \tag{56}
\end{align*} \]
Step 3: In this step, we show that each $N_{5,2}$, $N_{5,3}$, $N_{6,1}$, $N_{6,3}$, $N_{4,1}$ and $N_{4,2}$ is invertible. First, let $D_1 = \{4, 14, 21, 22\}$, $D_2 = \{4, 14, 23, 24\}$, $D_3 = \{5, 15, 23, 24\}$, $D_4 = \{5, 15, 25, 26\}$, $D_5 = \{6, 16, 21, 22\}$ and $D_6 = \{6, 16, 25, 26\}$. Then, for $D_1$, we have

$$4t = \text{rank } H_{D_1} = \text{rank } [H_{(4,14)} | H_{(21,22)}] = \text{rank } [H_{(4,14)} | H_{(11,1)} [M_{21,1} M_{22,11}]^T]$$

where (57) follows from the fact that each $H_{(4,14)}$ and $H_{(11,1)}$ must be met on the space of $H$. Step 4: In this step, we determine the three constraints which must be met on the space of $H_{(7,17)}$. First, let $L_3 = \{2, 3, 12, 13\}$. Then,

$$4t = \text{rank } H_{L_3} = \text{rank } H_{(2,12)} + \text{rank } H_{(3,13)}$$

where (70) and (71) follows from the fact that $L_3$ is an independent set. (72) is due to the invertibility of $N_{4,2}$, which implies that $H_{(2,12)}$ must be a subspace of $H_{(11,1)}$, and (73) is also due to the (59), which indicates that $H_{(4,14)}$ is linearly independent of $H_{(3,13)}$. Now, based on (70), (71), (72), (73) and (74), we have

$$\text{rank } H_{(3,4,13,14)} = 4t$$

Moreover, using the same argument in (70), (71), (72), (73) and (74) for $L_4 = \{1, 2, 11, 12\}$ and $L_5 = \{1, 3, 11, 13\}$ by considering the facts that (i) each $L_4$ and $L_5$ is an independent set, (ii) $N_{5,1}$ and $N_{5,3}$ are invertible, respectively, due to (69) and (66), (iii) $H_{(1,11)} N_{6,1}$ and $H_{(2,12)} N_{5,3}$, respectively, are subspace of $H_{(6,16)}$ and $H_{(5,15)}$ due to (55) and (53), and finally (iv) $H_{(6,16)}$ and $H_{(5,15)}$, respectively, are linear independent of $H_{(2,12)}$ and $H_{(1,11)}$, respectively, due to (55) and (53), we will have

$$\text{rank } H_{(2,6,12,16)} = 4t, \quad \text{rank } H_{(1,5,11,15)} = 4t.$$
which means that each $H_{(4,14)}$, $H_{(5,15)}$ and $H_{(6,16)}$ can be expressed as a linear combination of the other two, resulting in

$$\text{rank } H_{(i),iB_i} = \text{rank } H_{B_i}, \quad \forall i = 4, 5, 6, \tag{97}$$

which contradicts the decoding condition in (2) for $i = 4, 5, 6$. Thus, users $u_4$, $u_5$ and $u_6$ are not able to decode their requested messages over any field with characteristic two. This completes the proof.

Proposition 6. There exists a scalar nonlinear code over the binary field which can achieve the broadcast rate of $I_2$.

Proof. First, note that the set $L = \{1, 2, 3, 11, 12, 13\}$ is a MAIS set of $I_2$. So, $\beta_{\text{MAIS}}(x_2) = 6$. Now, we have that $\beta(C_{x_2}) = 6$ for a scalar nonlinear index code $C_{x_2} = (\phi_{x_2}, \{\psi_x\}_x)$, where the encoder and decoder functions are as follows. First, encoder $\phi_{x_2}$ maps the messages $x_i, i \in [26]$ to the coded messages $z_j, j \in [6]$ as below:

$$\begin{align*}
z_1 &= x_1 \oplus x_4 \oplus x_6 \oplus x_7 \oplus x_{10} \oplus x_{21}, \\
z_2 &= x_{11} \oplus x_{14} \oplus x_{16} \oplus x_{17} \oplus x_{20} \oplus x_{22} \oplus (x_{24} \oplus x_{27} \oplus x_{29}), \\
z_3 &= x_2 \oplus x_4 \oplus x_5 \oplus x_7 \oplus x_{8} \oplus x_{23}, \\
z_4 &= x_{12} \oplus x_{14} \oplus x_{15} \oplus x_{17} \oplus x_{18} \oplus x_{24} \oplus (x_{24} \oplus x_{27} \oplus x_{29}), \\
z_5 &= x_2 \oplus x_4 \oplus x_6 \oplus x_7 \oplus x_{8} \oplus x_{25}, \\
z_6 &= x_{13} \oplus x_{15} \oplus x_{16} \oplus x_{17} \oplus x_{19} \oplus x_{26} \oplus (x_{24} \oplus x_{27} \oplus x_{29}).
\end{align*}$$

Now each decoder $\psi_{x_2}, i \in [26]$ decodes the requested message $x_i$ using the received coded messages $z_j, j \in [6]$ and the side information $S_i$ as follows:

- Users $u_1, u_{11}, u_2, u_{12}, u_4$ and $u_{13}$ can decode their requested message, respectively, from $z_1, z_2, z_3, z_4, z_5$ and $z_6$.
- Users $u_7$ and $u_{17}$ both can decode $x_7$ from either $z_1, z_3$ or $z_5$. Then, $u_17$ can decode $x_{17}$ from either $z_2, z_4$ or $z_6$.
- Users $u_{14}, u_{15}$ and $u_{16}$, respectively, first decode $x_4$ from $z_1, x_5$ from $z_3$ and $x_6$ from $z_5$. Then, they can decode, respectively, $x_{14}$ from $z_2, x_{15}$ from $z_4$ and $x_{16}$ from $z_6$.
- Users $u_{21}$ and $u_{22}$ both first decode both $x_4$ from $z_3$ and $x_6$ from $z_5$. Then, $u_{21}$ can decode $x_{21}$ from $z_1$. User $u_{22}$ decodes $x_{14}$ from $z_4$ and $x_{16}$ from $z_6$. Now, $u_{22}$ can decode $x_{22}$ from $z_2$.
- Users $u_{23}$ and $u_{24}$ both first decode $x_4$ from $z_3$ and $x_5$ from $z_5$. Then, $u_{23}$ can decode $x_{23}$ from $z_3$. User $u_{24}$ decodes $x_{14}$ from $z_2$ and $x_{15}$ from $z_6$. Now, $u_{24}$ can decode $x_{24}$ from $z_4$.
- Users $u_{25}$ and $u_{26}$ both first decode $x_5$ from $z_3$ and $x_6$ from $z_5$. Then, $u_{25}$ can decode $x_{25}$ from $z_3$. User $u_{26}$ decodes $x_{15}$ from $z_4$ and $x_{16}$ from $z_2$. Now, $u_{26}$ can decode $x_{26}$ from $z_5$.
- Users $u_{8}, u_{9}$ and $u_{10}$, respectively, decode $(x_5 \oplus x_7)$ from $z_3, (x_6 \oplus x_7)$ from $z_1$ and $(x_4 \oplus x_5)$ from $z_2$. Then, they are able to decode their requested messages $x_8$ from $z_3$, $x_9$ from $z_5$, $x_{10}$ from $z_1$.
- User $u_{18}$, first decodes $x_5 \oplus x_7$ from $z_3$. Then, it adds $x_2 \oplus z_6$ to achieve (after removing the messages in its side information) $x_{18} \oplus (x_4 \oplus x_6)(x_7 \oplus x_5)$ (note that the term $x_6x_7$ is canceled out). Now, because it has $x_4$ and $x_6$ in its side information and has already decoded $x_6 \oplus x_7$, then it will be able to decode its desired message $x_{19}$.
- User $u_{20}$, first decodes $x_4 \oplus x_7$ from $z_3$. Then, it adds $x_2 \oplus z_4$ to achieve (after removing the messages in its side information) $x_{20} \oplus (x_5 \oplus x_6)(x_4 \oplus x_7)$ (note that the term $x_6x_7$ is canceled out). Now, because it has $x_5$ and $x_6$ in its side information and has already decoded $x_1 \oplus x_7$, then it will be able to decode its desired message $x_{20}$.
- Users $u_4, u_5$ and $u_6$ do as follows. Note, in binary field, $x_i^2 = x_i$. User $u_4$ first decodes $(x_4 \oplus x_5)$ and $(x_4 \oplus x_5)$, respectively, from $z_1$ and $z_3$. Then, it multiplies them to achieve $x_4 \oplus x_4 \oplus x_5 \oplus x_5 \oplus x_5 \oplus x_5$. User $u_5$ first decodes $(x_4 \oplus x_5)$ and $(x_5 \oplus x_6)$, respectively, from $z_3$ and $z_5$. Then, it multiplies them to achieve $x_5 \oplus x_4 \oplus x_5 \oplus x_5 \oplus x_5 \oplus x_5$. User $u_6$ first decodes $(x_4 \oplus x_5)$ and $(x_6 \oplus x_6)$, respectively, from $z_1$ and $z_5$. Then, it multiplies them to achieve $x_5 \oplus x_4 \oplus x_5 \oplus x_5 \oplus x_5 \oplus x_5$. On the other hand, we add $z_2 \oplus z_4 \oplus z_6$ to cancel the terms $x_{14}, x_{15}, x_{16}$ and $x_{4}x_7, x_5x_7, x_6x_7$. Now, it can be observed that users $u_4, u_5$ and $u_6$ will obtain $x_{14}x_6 \oplus x_{15}x_5 \oplus x_{16}x_5$, and so, they are able to decode their desired message $x_{4}x_5$ and $x_5$, respectively.

Remark 3. The specific constraints on the optimal solution of each index coding subinstances $I_1$ and $I_2$ were inspired by the constraints on the solution of each network coding subinstances in [11] (which were denoted there by $N_1$ and $N_2$). Using similar techniques as in [11], one can show that linear coding is insufficient over non-commutative rings and modules, where linear operations are well-defined [20]. [21]. We leave the technical details for a future version of this work.

VI. CONCLUDING REMARKS

In this paper, we addressed the open problem of proving the necessity of nonlinear coding for achieving the symmetric rate of the unicast index coding problem. This proof was made by providing a unicast index coding instance, consisting of two separate subinstances which are connected in a two-way method or no-way (disjoint) method in terms of their side information. We proved that for the first instance linear coding is optimal only over a finite field with characteristic two. However, for the second instance, we proved that linear coding with characteristic two cannot be optimal while an optimal nonlinear code was provided over the binary field. This, in turn, settles the insufficiency of linear codes for unicast setting with symmetric message rate. One main advantage of the structure of our instance is its significant simplicity, having only 36 users while the example in [5] has 74 messages and 80 users, and also the instance in [10] contains more than 200 users.

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