The Marshall-Olkin Gumbel-Lomax distribution: properties and applications

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ABSTRACTS

We introduce a new flexible five parameter lifetime distribution called Marshall-Olkin Gumbel-Lomax distribution. Some characterizations of the distribution such as the quantile function, moments, Trimmed L-moments, moment generating function, and order statistics are derived. The unknown parameters of the new distribution are estimated using the maximum likelihood approach. And the performance of the MLEs is examined through simulation studies. The potentials of the new distribution are illustrated using two real life data sets.

1. Introduction

The Lomax distribution which is also referred to as Pareto Type II distribution was initially proposed for modeling business failure (Lomax, 1954). Its application has been found in many other areas such as income, and size of city by Ahsanullah (1991), Receiver Operating Characteristics (ROC) by Campbell and Ratnapaikhi (1993), reliability and testing study (see Hassan and Al-Ghamdi, 2009; and Ahsanullah, 1991) and estimation of the Stress-Strength Reliability by Salman and Hamad (2019). A random variable X follows a Lomax distribution if the cumulative density function (cdf) and the probability density function (pdf) are respectively defined as

\[ F(x) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha} ; \quad x > 0, \]

and

\[ f(x) = \frac{\alpha}{\lambda} \left(1 + \frac{x}{\lambda}\right)^{-(\alpha + 1)} ; \quad x > 0, \quad \alpha > 0, \quad \lambda > 0, \]

where \( \alpha > 0 \) is the shape parameter and \( \lambda > 0 \) is the scale parameter.

The Lomax distribution has been extended and made more flexible to model data sets which ordinarily would not have provided adequate fit by means of addition of one or more parameters. Some of such extensions are Marshall-Olkin extended Lomax distribution by Ghitany et al. (2007), McDonald Lomax by Lemonte and Cordeiro (2011), Transmuted-Lomax by Asghar and Eitehiwy (2013), Beta-Lomax by Rajab et al. (2013), Gamma-Lomax by Cordeiro et al. (2013), Gumbel-Lomax distribution by Tahir et al. (2015), Logistic Lomax by Zubair et al. (2017), Half-Logistic Lomax by Anwar and Zahoor (2018) and generalized Lomax by Maurya et al. (2019).

Ugwuowo and Nwezza (2018) introduced a new class of distributions called the Marshall-Olkin Gumbel family of distribution with cdf given by

\[ G(x) = \begin{cases} \exp\left[-B\left(\frac{F(x)}{F(x)}\right)\right]^{\frac{1}{\sigma}} & \text{if } \frac{F(x)}{F(x)} \leq 1 \\ 1 - p\left[1 - \exp\left[-B\left(\frac{F(x)}{F(x)}\right)\right]^{\frac{1}{\sigma}}\right] & \text{if } \frac{F(x)}{F(x)} > 1 \end{cases} \]

here, \( F(x) = 1 - F(x) \) and \( F(x) \) is the cdf of baseline distribution.

By taking \( \frac{F(x)}{F(x)} = \left(1 + \frac{x}{\lambda}\right)^{-(\alpha + 1)} \), we propose a new extension of Lomax distribution called Marshall-Olkin Gumbel-Lomax (MOG-L) distribution. The cdf of the new distribution is given by

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where

\[ B = \exp \left( \frac{1}{p} \right), p \text { is an additional shape parameter, } \sigma \text { is additional scale parameter, } \mu \text { is a location parameter and } p = 1 - p. \]

The corresponding pdf to (1) is given by

\[
g(x) = \frac{pB \left( \frac{x}{\mu} \right)^{\alpha-1} \exp \left\{ -B \left( \left( \frac{x}{\mu} \right)^{\alpha} - 1 \right)^{\frac{1}{\beta}} \right\}}{\sigma \left( \left( \frac{x}{\mu} \right)^{\alpha} - 1 \right)^{\frac{1}{\beta}} \left\{ 1 - \exp \left( -B \left( \left( \frac{x}{\mu} \right)^{\alpha} - 1 \right)^{\frac{1}{\beta}} \right) \right\}^{\frac{1}{\beta}}}
\]

The new distribution generalizes Gumbel-Lomax by Tahir et al. (2015) and reduces to Gumbel-Lomax when \( p = 1. \)

The main motivation behind this paper is to develop a more flexible and heavy-tailed distribution that is capable of modeling and providing a better fit than already existing distribution. As shown in Figure 1, the new distribution possesses a heavier tail than the Lomax distribution.

The rest of the paper is structured as follows: some mathematical properties such as the quantile function, shape of the pdf, moments, Trimmed L-moments and moment generating function of the distribution are considered in section 2. In section 3, some reliability functions such as the hazard function, mean residual function are presented. The Lorenz curve function is presented in section 4. In section 5, we derived the entropy. The distribution of the \( p \)th ordered statistics is derived in section 6. In section 7, the estimation of the unknown parameters of the distribution is considered using the maximum likelihood approach. In section 8, the performance of the maximum likelihood estimates is studied by means of simulation. The importance of the new distribution is considered through two applications to real life data sets in section 9. In section 10, we provided the concluding remarks.

2. Mathematical properties

**Theorem 2.1.** The quantile function of a random variable \( X \) having MOG-L distribution is given by

\[
Q(u) = \lambda \left\{ 1 - B \left( \frac{\log \left( \frac{u - u_{fl}}{1 - u_{fl}} \right)}{1 - u_{fl}} \right)^{-\frac{1}{\beta}} \right\}
\]

Here, \( u \) is a random variable with uniform distribution in \((0, 1)\). Different quantile of interest can be obtained by substituting different values of \( u \in (0, 1) \).

**Corollary 2.1.1.** The Moors’s measure of kurtosis due to Moors (1988) which is based on the quantile function can be obtained by

\[
kurtosis = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(3/4) - Q(1/4)}
\]

2.1. Shape of MOG-L pdf

The shape of MOG-L pdf can be analyzed numerically. The critical points of the pdf are the roots of (3). By (3), there may be more than one root. Some graphically plots of the shapes of MOG-L pdf are shown in Figure 2.
\[ \frac{d^2 \log \tau(x)}{dx^2} = \frac{(1 - x)}{x(1 + \frac{x}{2})^2} + \frac{\alpha B}{\sigma \lambda^2}(1 + \frac{x}{2})^{2-\lambda} \left[ \left(1 + \frac{x}{2}\right)^\alpha - 1 \right] \left(\frac{x + 1}{2}\right)^{\lambda-1} \left[ (1 + \lambda)^\sigma \right] - \frac{1}{\sigma + 1} \alpha \]

+ \frac{\alpha (x + 1) (1 + \frac{x}{2})^{\lambda-1}}{\lambda\left[(1 + \lambda)^\sigma - 1\right]} + \left(\frac{\alpha - 1}{\lambda^2} \right) \left[\left(1 + \frac{x}{2}\right)^\alpha - 1\right] \left[ 1 - \exp\left(-B\left(1 + \frac{x}{2}\right)^\alpha - 1\right) \right] \left(\frac{\alpha - 1}{\lambda^2} \right)

\[ + 2 \left( \frac{\rho \sigma}{\alpha \lambda} \right) \frac{\alpha (x + 1) (1 + \frac{x}{2})^{\lambda-1}}{\lambda\left[(1 + \lambda)^\sigma - 1\right]} \left[\left(1 + \frac{x}{2}\right)^\alpha - 1\right] \left[ 1 - \exp\left(-B\left(1 + \frac{x}{2}\right)^\alpha - 1\right) \right] \left(\frac{\alpha - 1}{\lambda^2} \right) \]

\[ \frac{x}{(1 + x)^2} \left[\left(1 + \frac{x}{2}\right)^\alpha - 1\right] - \frac{1}{\sigma^2} \left[\left(1 + \frac{x}{2}\right)^\alpha - 1\right] \left[ 1 - \exp\left(-B\left(1 + \frac{x}{2}\right)^\alpha - 1\right) \right] \left(\frac{\alpha - 1}{\lambda^2} \right) \]

**Theorem 2.2.** Suppose a random variable \( Y \) follows Marshall-Olkin Gumbel distribution, and a random variable \( X = \lambda \left(1 + \exp(y)\right)^{\frac{1}{x}} - 1 \) follows Marshall-Olkin Gumbel-Lomax distribution.

**Proof**

The pdf of the random variable \( Y \) is given by

\[ r(y) = \frac{\rho B \exp\left(-\frac{y}{\sigma}\right) \exp\left(-B\exp\left(-\frac{y}{\sigma}\right)\right)}{\sigma \left(1 - \exp\left(-B\exp\left(-\frac{y}{\sigma}\right)\right)\right)^2} \]

By transformation, the pdf of the random variable \( X \) is defined as

\[ g(x) = r(y) \left| \frac{dy}{dx} \right| \]

where

\[ \frac{dy}{dx} = \frac{\alpha (x + 1)^{\alpha-1}}{\lambda\left[(1 + \frac{x}{2}) - 1\right]} \]

By substituting appropriately, we have that

\[ g(x) = \frac{\alpha B (x + 1)^{\alpha-1} - \exp\left(-B\left(1 + \frac{x}{2}\right)^\alpha - 1\right)}{\sigma \left(1 - \exp\left(-B\left(1 + \frac{x}{2}\right)^\alpha - 1\right)\right)^2} \]

\[ \left[\left(1 + \frac{x}{2}\right)^\alpha - 1\right] \left[ 1 - \exp\left(-B\left(1 + \frac{x}{2}\right)^\alpha - 1\right) \right] \left(\frac{\alpha - 1}{\lambda} \right) \]

**2.2. Moments**

The \( r \)th moment about the origin of a random variable having pdf \( f(x) \) is defined as

\[ E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx. \]

**Corollary 2.2.1.** The \( r \)th moment about the origin of a random variable \( X \sim \text{MOG-L} \) is given
2.3. Trimmed L-moments

Elamir and Scheult (2003) introduced the Trimmed L-moments which is more robust to outlier and has the sample variances and covariance in close form. The $n$th trimmed L-moment of a random variable $X$ for $t_1 = t_2 = t$ is given by

$$\lambda_{n-t} = \frac{1}{n} \sum_{k=0}^{n-t} (-1)^k \binom{n-1}{k} E(Y_{n,t-k+t})$$

where $n = 1, 2, ...$

Applying general binomial expansion, we have that

$$E(Y_{n,t-k+t}) = \frac{(n + 2t)!}{(n + t - k - 1)!(t + k)!} \int x^k G(x)^{n-t-k-1}(1 - G(x))^{t+k} dx.$$
\[ x_r' = \frac{\Psi p \sigma B}{\alpha \beta} \sum_{j=m}^{\infty} \frac{(-1)^{m+k}}{h!} \binom{n-t-k+i+m}{j} \binom{h+1}{m+q} x^{(n+1)} \frac{(n+1)}{(n+t-k-1)!} (t+k)! \]

where

\[ \Psi = \frac{1}{r} \sum_{k=0}^{r-1} \sum_{i=0}^{k} (-1)^{i+k} \binom{n-1}{i} \binom{t+k}{i} \frac{(n+2t)!}{(n+t-k-1)! (t+k)!} \]

For \( t = 1 \) and \( r = 1, 2, 3, 4 \), we can also obtain the population measures of location, scale, skewness and kurtosis.

### 2.4. Moment generating function

Theorem moment generating function of a random variable \( X \) with pdf of Eq. (2) is given by

\[ M_X(t) = \sum_{j=0}^{\infty} w_j \Gamma(n + 1), \]

where

![Figure 3. Plots of MOG-L hrf for selected parameter values: a) constant hrf, b) constant-monotone increasing, c) right skewed, d) monotone increasing-decreasing shape.](image-url)
\[ w_j = (-1)^j \sum_{k=0}^{j} \frac{(-1)^{j-k}}{k!} (1-p)^k \binom{j}{i} \left( \frac{k}{\sigma} + \frac{1}{\alpha} + m \right)^i \left( \frac{ak}{\sigma} + \frac{\alpha m + 1}{n} \right) (B(1+j))^i \]

**Proof**

By definition,

\[ M_0(t) = \int_0^t \exp(tx) \, dx = \int_0^\infty \exp(x) \, dx \]

\[ M_i(t) = \sum_{j=0}^\infty w_j \Gamma(n+1). \]

### 3. Reliability functions

Suppose random variable \( X \) denoting the lifetime of a product follows MOG-L distribution, the failure rate function (hrf) is given by

\[ hrf = \frac{\rho a \left( \frac{1}{\alpha} - 1 \right) \left( \frac{1}{\alpha} \right)^n - 1}{\sigma \lambda \left( \frac{1}{\alpha} - 1 \right) \left( \frac{1}{\alpha} \right)^n + 1} \left( 1 - \exp \left( -B \left( \frac{1}{\alpha} - 1 \right)^n \right) \right) \left( 1 - \exp \left( -B \left( \frac{1}{\alpha} - 1 \right)^n \right) \right) \]

**Figure 3** shows some shapes of the failure rate function for selected parameter values.

The survival function defined as \( p(X > x) \) denoted by \( S(x) \) of MOG-L distribution is given by

\[ S(x) = \frac{\rho - \rho \exp \left( -B \left( \frac{1}{\alpha} - 1 \right)^n \right)}{1 - \exp \left( -B \left( \frac{1}{\alpha} - 1 \right)^n \right)} \]

### 3.1. Mean residual life function

The random residual life of a production is of great interest in many areas of reliability studies such as engineering, biology, insurance. The expected value of this random residual life referred to as Mean residual life function is defined as

\[ m(t) = \frac{1}{1-F(t)} \int_t^\infty u f(u) \, du - t. \]

**Theorem 3.1.** If a random variable \( X \) having cdf as defined in Eq. (2), then the mean residual life function is given by

\[ m(t) = \sum_{j=0}^\infty h_{ok} \left( 1 + \frac{1}{\lambda} \right) - \left( \frac{1}{\alpha} - 1 \right)^n \left( \sum_{j=0}^\infty h_{ok} \left( 1 + \frac{1}{\lambda} \right) - \left( \frac{1}{\alpha} - 1 \right)^n \right) \]

where

\[ h_{ok} = \rho \sum_{j=0}^\infty \left( \frac{1}{\alpha} - 1 \right)^n \left( \frac{1}{\alpha} - 1 \right)^n \left( \frac{1}{\alpha} - 1 \right)^n \left( \frac{1}{\alpha} - 1 \right)^n \]

and

\[ h_{ok} = \sum_{j=0}^\infty \left( \frac{1}{\alpha} - 1 \right)^n \left( \frac{1}{\alpha} - 1 \right)^n \left( \frac{1}{\alpha} - 1 \right)^n \left( \frac{1}{\alpha} - 1 \right)^n \]

**Proof**

\[ m(t) = \frac{1}{1-F(t)} \int_t^\infty u f(u) \, du - t. \]

Applying the general binomial series expansion, the integral becomes
4. Lorenz curve function

The Lorenz curve is introduced by Lorenz (1905) to measure the distribution of income. Many densities having positive support range have been proposed for the modeling the distribution of income. The Lorenz curve function is defined by

\[
L(x) = \frac{1}{\mu} \int_0^1 q(t)\,dt.
\]

Theorem 4.1. If \( X \sim \text{MOG-L} \) distribution, the Lorenz curve function of is given by

\[
(1 + \frac{1}{\lambda})^{-\alpha} \left(\frac{1}{\lambda \sigma} \right) \left(1 + \frac{1}{\lambda}ight)\left(\frac{1}{\lambda \sigma} \right) - 1 - \lambda \sigma (1 + k + m \sigma)
\]

Proof

\[
L(x) = \frac{1}{\mu} \int_0^1 q(t)\,dt = \frac{1}{\mu} \int_0^1 p \exp \left(-B \left(1 + \frac{1}{B} - 1\right)^{\frac{1}{B}}\right)\,dt
\]

\[
= \frac{W}{\mu} \int_0^1 \left(\frac{1}{\lambda}ight)^{\alpha-1} dt
\]

\[
= \frac{W \lambda \sigma \left(1 + \frac{1}{\lambda}ight)\left(\frac{1}{\lambda \sigma} \right) - 1 - \lambda \sigma (1 + k + m \sigma)
\]

However, \( \mu = \frac{W (\alpha^2)}{\alpha (1 + k + m \sigma)|\alpha (1 + k + m \sigma) - \sigma} \)

Then

\[
L(x) = \frac{(1 + \frac{1}{\lambda})^{-\alpha} \left(\frac{1}{\lambda \sigma} \right) \left(1 + \frac{1}{\lambda}ight)\left(\frac{1}{\lambda \sigma} \right) - 1 - \lambda \sigma (1 + k + m \sigma)}{\lambda \sigma}
\]

5. Entropy

The measure of uncertainty of a random variable \( X \) having pdf \( f(x) \) is determined by its entropy. The Renyi entropy is defined as

\[
I_k(\gamma) = \frac{1}{1 - \gamma} \log \int f(x)\,dx
\]

Suppose a random variable \( X \) follows MOG-L distribution, the Renyi entropy of \( X \) is given by

\[
I_k(\gamma) = \frac{\gamma \log (\alpha \lambda B)}{1 - \gamma} - \frac{\gamma \log (\sigma B)}{1 - \gamma}
\]

\[
+ \frac{1}{1 - \gamma} \log \left(\sum_{n=0}^{\infty} \Phi_n \left(\alpha (k + \gamma + (m + \gamma) \sigma - \sigma)\right)\right)
\]

6. Order statistics

Let \( X_1, X_2, \ldots, X_n \) be random sample of size \( n \) drawn from MOG-L population and \( X_{(1)}, X_{(2)}, \ldots, X_{(n)} \) are ordered sample such that \( X_{(1)} < X_{(2)} < \ldots < X_{(n)} \). The pdf \( f^{(\ell)} \) ordered statistics defined as

\[
f(x_{(\ell)}) = \frac{n!}{(i-1)! (n-i)!} \Phi(x) \sum_{j=0}^{n-i} (-1)^j (n-i-j) \binom{n-i}{j} \binom{\alpha (k + \gamma + (m + \gamma) \sigma - \sigma)}{\alpha (1 + \gamma + (m + \gamma) \sigma - \sigma)}
\]

By applying some general binomial series expansion, we have that

\[
f(x_{(\ell)}) = \frac{n!}{(i-1)! (n-i)!} \Phi(x) \sum_{j=0}^{n-i} (-1)^j (n-i-j) \sum_{k=0}^{\infty} w_n (1 + \frac{1}{\lambda})\left(\frac{1}{\lambda \sigma} \right) - 1 - \lambda \sigma (1 + k + m \sigma)
\]

\[
= \frac{n!}{(i-1)! (n-i)!} \Phi(x) \sum_{j=0}^{n-i} (-1)^j (n-i-j) \sum_{k=0}^{\infty} (1 + \frac{1}{\lambda})\left(\frac{1}{\lambda \sigma} \right) - 1 - \lambda \sigma (1 + k + m \sigma)
\]

7. Parameter estimation

Suppose \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) drawn from MOG-L population, the estimators, say \( \hat{\theta} = (\hat{\beta}, \hat{\mu}, \hat{\sigma}, \hat{\lambda}, \hat{\alpha}) \), of the unknown parameters \( \theta = (\beta, \mu, \sigma, \lambda, \alpha) \) of MOG-L distribution can be obtained using the maximum likelihood method. The log-likelihood function, say \( \ell(\theta) \) for the parameters of the distribution is given by
### Table 1. Mean estimate values, biases, mean square errors of simulations with parameter values $p = 2, \mu = 0.8, \sigma = 3, \alpha = 0.7, \lambda = 1.4$ for sample sizes 50, 150, and 300.

| Sample size (n) | Parameter | Mean value | Bias | MSE       |
|----------------|-----------|------------|------|-----------|
| 50             | $p = 2$   | 5.4170     | 3.4170 | 31.9457   |
|                | $\mu = 0.8$ | 0.8115    | 0.0114 | 13.6678   |
|                | $\sigma = 3$ | 2.3599  | -0.6400 | 4.5358    |
|                | $\alpha = 0.7$ | 0.5001 | -0.1995 | 0.1851    |
|                | $\lambda = 1.4$ | 2.5200 | 1.2200  | 39.6032   |
| 150            | $p = 2$   | 5.0801     | 3.0800 | 26.1663   |
|                | $\mu = 0.8$ | 0.4895    | -0.3105 | 7.4179    |
|                | $\sigma = 3$ | 2.3149  | -0.6851 | 2.3752    |
|                | $\alpha = 0.7$ | 0.5144 | -0.1856 | 0.1053    |
|                | $\lambda = 1.4$ | 1.9792 | 0.6792  | 15.5207   |
| 300            | $p = 2$   | 4.2216     | 2.2217 | 17.0251   |
|                | $\mu = 0.8$ | 0.5629    | -0.2370 | 4.8827    |
|                | $\sigma = 3$ | 2.4143  | -0.5857 | 1.9266    |
|                | $\alpha = 0.7$ | 0.5553 | -0.1447 | 0.0664    |
|                | $\lambda = 1.4$ | 1.7195 | 0.4195  | 7.7226    |

### Table 2. Mean estimate values, biases, mean square errors of simulations with parameter values $p = 1.5, \mu = 0.5, \sigma = 3, \alpha = 0.4, \lambda = 0.9$ for sample sizes 50, 150, and 300.

| Sample size (n) | Parameter | Mean value | Bias | MSE       |
|----------------|-----------|------------|------|-----------|
| 50             | $p = 1.5$ | 3.7732     | 2.2732 | 14.7903   |
|                | $\mu = 0.5$ | -0.0666   | -0.5667 | 9.9212    |
|                | $\sigma = 3$ | 2.2567  | -0.7433 | 2.8955    |
|                | $\alpha = 0.4$ | 0.2758 | -0.1241 | 0.0374    |
|                | $\lambda = 0.9$ | 2.3119 | 1.4119  | 35.8294   |
| 150            | $p = 1.5$ | 3.2095     | 1.7096 | 8.4803    |
|                | $\mu = 0.5$ | -0.0776   | -0.5776 | 5.7277    |
|                | $\sigma = 3$ | 2.3342  | -0.6659 | 1.5944    |
|                | $\alpha = 0.4$ | 0.2979 | -0.1021 | 0.0247    |
|                | $\lambda = 0.9$ | 1.6580 | 0.7580  | 14.2065   |
| 300            | $p = 1.5$ | 2.7727     | 1.2727 | 6.7307    |
|                | $\mu = 0.5$ | 0.1013    | -0.3987 | 3.8151    |
|                | $\sigma = 3$ | 2.4884  | -0.5116 | 1.0632    |
|                | $\alpha = 0.4$ | 0.3237 | -0.0762 | 0.0169    |
|                | $\lambda = 0.9$ | 1.2807 | 0.3805  | 7.7442    |

### Table 3. MLEs, standard errors in parentheses of failure times of 63 aircraft windshield.

| Distributions | Estimates (Standard error in parentheses) |
|---------------|------------------------------------------|
| MOG - $L(p, \mu, \sigma, \alpha, \lambda)$ | 23.6608, (48.9757), -1.0716, (2.7859), 1.6599, (0.9729), 62.2514, (96.3189), 28.1483 (46.6959) |
| GL($\mu, \sigma, \alpha, \lambda$) | 4.9796, (2.2189), 3.8779, (1.3104), 56.964, (36.0295), 15.3481 (12.2518) |
| Lognormal($\alpha, \lambda$) | 40.1154, (39.8489), 83.0763 (84.0361) |
| Beta($\alpha, \beta, \alpha, \lambda$) | 1.9241, (0.3189), 27.0754, (473.0507), 4.7631, (83.4709), 140.7073 (258.5772) |
| ExponentialLognormal($\alpha, \beta, \alpha, \lambda$) | 1.9699, (0.3564), 24.1423, (20.0515), 32.7179 (28.1523) |
| KumarLognormal($\alpha, \beta, \alpha, \lambda$) | 1.6921, (0.2089), 21.3309, (26.9086), 0.0106, (0.0069), 7.3699 (7.8553) |
| GammaLognormal($\alpha, \beta, \alpha, \lambda$) | 1.9294, (0.3233), 0.0165, (0.0157), 50.7331 (41.1968) |
| ExpGL($\alpha, \beta, \alpha, \lambda$) | 7.7389, (10.7018), 1.9232, (0.3475), 0.0118, (0.0081), 7.7389 (10.7018) |
### Table 4. MLEs, standard errors in parentheses of failure times of 84 aircraft windshield.

| Distributions | Estimates (Standard error in parentheses) |
|---------------|------------------------------------------|
| MOG - \( L(p, \mu, \sigma, a, i) \) | 48.4816, (61.9641), 0.0715, (2.7644), 0.0715, (2.7644), 2.2266, (1.4595), 2.2266, (1.4595) |
| GL\((\mu, \sigma, a, i)\) | 13.7533, (6.4174), 7.1132, (3.1595), 43.3723, (34.8640), 15.7034 (21.4921) |
| Lamax(a,i) | 43.3723, (34.8640), 0.0715, (2.7644), 110.5311 (90.1214) |
| BL\((a, b, a, i)\) | 3.499, (0.5246), 33.2277, (46.0618), 3.6515, (0.6302), 59.2348, (65.7008), 7.6453, (7.2406) |
| ExpLamax(a, a, i) | 3.6515, (0.6302), 41.2290, (31.3498), 3.6515, (0.6302), 77.7915 (42.4022) |
| KumLamax(b, a, i) | 3.6515, (0.6302), 37.9354, (38.8801), 2.6493, (0.3158), 2.6493, (0.3158) |
| Gamma1L\((a, \alpha, \lambda)\) | 3.5413, (0.5315), 0.0182, (0.0103), 0.0182, (0.0103) |

### Table 5. Goodness-of-fit statistics of failure times of 63 aircraft windshield.

| Distributions | W* | A* | AIC | BIC | CAIC | – f |
|---------------|-----|----|-----|-----|------|-----|
| MOG - \( L(p, \mu, \sigma, a, i) \) | 0.0340 | 0.2151 | 206.1838 | 216.8995 | 207.236 | 98.0919 |
| GL\((\mu, \sigma, a, i)\) | 0.1206 | 0.7285 | 210.0851 | 218.6577 | 210.7748 | 101.0426 |
| Lamax(a,i) | 0.1718 | 1.0429 | 223.6148 | 227.9011 | 223.8148 | 109.8074 |
| BL\((a, b, a, i)\) | 0.1879 | 1.3805 | 213.9750 | 222.5475 | 214.6646 | 102.9875 |
| ExpLamax(a, a, i) | 0.2237 | 1.3542 | 214.5441 | 220.9735 | 214.9509 | 104.2721 |
| KumLamax(b, a, i) | 0.1286 | 0.7801 | 210.1840 | 218.7566 | 210.8731 | 101.0920 |
| Gamma1L\((a, \alpha, \lambda)\) | 0.1949 | 1.1799 | 212.4614 | 218.8908 | 212.8682 | 103.9024 |

### Table 6. Goodness-of-fit statistics of failure times of 84 aircraft windshield.

| Distributions | W* | A* | AIC | BIC | CAIC | – f |
|---------------|-----|----|-----|-----|------|-----|
| MOG - \( L(p, \mu, \sigma, a, i) \) | 0.0663 | 0.5133 | 266.3805 | 278.5445 | 267.1597 | 128.1952 |
| GL\((\mu, \sigma, a, i)\) | 0.0941 | 0.8805 | 274.7789 | 284.5021 | 275.2852 | 133.3894 |
| Lamax(a,i) | 0.1361 | 1.1993 | 331.3167 | 336.1783 | 331.4648 | 163.6584 |
| BL\((a, b, a, i)\) | 0.1683 | 1.4176 | 283.1507 | 292.8739 | 283.6570 | 137.5753 |
| ExpLamax(a, a, i) | 0.2279 | 1.8059 | 287.3927 | 294.6852 | 287.6927 | 140.6964 |
| KumLamax(b, a, i) | 0.0721 | 0.7212 | 271.3459 | 281.0692 | 271.8522 | 131.6729 |
| Gamma1L\((a, \alpha, \lambda)\) | 0.1686 | 1.4195 | 281.1808 | 288.4732 | 281.4808 | 137.5904 |
| ExpGL\((a, b, a, i)\) | 0.2226 | 1.7717 | 288.8673 | 298.5905 | 289.3736 | 140.4336 |

**Figure 4.** Estimated plots of 63 failure times of aircraft windshield: a) pdfs, b) cdfs with empirical cdf.
The log-likelihood function can be maximized directly using some software such as R statistical software, Maple, mathematica to obtain MLEs. More so, the maximum likelihood estimators \( \hat{\Theta} \) can be obtained by solving the system of non-linear equations given below. The equations below are also referred to as score functions.

\[
\ell'(\Theta) = n \log(\alpha) + \frac{p\mu}{\sigma} + n \log(p) + (\alpha - 1) \sum_{i=1}^{n} \log\left(1 + \frac{x_i}{\lambda}\right) - \exp\left(\frac{\mu}{\sigma}\right) \sum_{i=1}^{n} \left[\left(1 + \frac{x_i}{\lambda}\right)^{\alpha} - 1\right]^{\frac{1}{\alpha}} \\
- n \log(\sigma) - n \log(\lambda) - \left(\frac{1}{\sigma} + 1\right) \sum_{i=1}^{n} \log\left(\left(1 + \frac{x_i}{\lambda}\right)^{\alpha} - 1\right) \\
- 2 \sum_{i=1}^{n} \log\left\{1 - p \left[1 - \exp\left(-B\left[\left(1 + \frac{x_i}{\lambda}\right)^{\alpha} - 1\right]^{\frac{1}{\alpha}}\right]\right]\right\}
\]

The log-likelihood function can be maximized directly using some software such as R statistical software, Maple, mathematica to obtain MLEs. More so, the maximum likelihood estimators \( \hat{\Theta} \) can be obtained by solving the system of non-linear equations given below. The equations below are also referred to as score functions.

\[
\frac{d\ell'(\Theta)}{d\alpha} = \frac{n}{\alpha} + \frac{1}{\alpha} \sum_{i=1}^{n} \log\left(1 + \frac{x_i}{\lambda}\right) + \sum_{i=1}^{n} \frac{B}{\sigma} \left(1 + \frac{x_i}{\lambda}\right)^{\frac{1}{\alpha}} \left(1 + \frac{x_i}{\lambda}\right)^{\alpha} \log\left(1 + \frac{x_i}{\lambda}\right) - \frac{\left(1 + \frac{1}{\alpha}\right) \sum_{i=1}^{n} \left(1 + \frac{x_i}{\lambda}\right)^{\alpha} \log\left(1 + \frac{x_i}{\lambda}\right)}{\left(1 + \frac{x_i}{\lambda}\right) - 1} \\
- \frac{nB \exp\left(-B\left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha}} \left(1 + \frac{1}{\lambda}\right)^{\alpha} \log\left(1 + \frac{1}{\lambda}\right)\right)}{\sigma\left(\left(1 + \frac{x_i}{\lambda}\right)^{\alpha} - 1\right)^{\frac{1}{\alpha}} \left[1 - p \left[1 - \exp\left(-B\left[\left(1 + \frac{x_i}{\lambda}\right)^{\alpha} - 1\right]^{\frac{1}{\alpha}}\right]\right]\right]} = 0.
\]
\[
\frac{\partial^2 \ell(\theta)}{\partial \mu^2} = n \frac{B}{\sigma} \sum_{i=1}^{n} \left( (1 + \frac{\lambda}{\mu}) - 1 \right)^{\frac{1}{2}} - B \left( (1 + \frac{\lambda}{\mu}) - 1 \right) \left( \left( (1 + \frac{\lambda}{\mu}) - 1 \right)^{\frac{1}{2}} \right) = 0.
\]

\[
\frac{\partial^2 \ell(\theta)}{\partial \sigma^2} = -n \sigma + \mu B \sum_{i=1}^{n} \left( (1 + \frac{\lambda}{\mu}) - 1 \right)^{\frac{1}{2}} - B \left( (1 + \frac{\lambda}{\mu}) - 1 \right) \left( \left( (1 + \frac{\lambda}{\mu}) - 1 \right)^{\frac{1}{2}} \right) = -n \sigma.
\]

\[
-\frac{n}{\sigma^2} \sum_{i=1}^{n} \frac{\log \left( (1 + \frac{\lambda}{\mu}) - 1 \right)}{\left( (1 + \frac{\lambda}{\mu}) - 1 \right)^{\frac{1}{2}}} - 2 \sum_{i=1}^{n} \frac{\mu B \left( (1 + \frac{\lambda}{\mu}) - 1 \right)^{\frac{1}{2}} + 2 \left( (1 + \frac{\lambda}{\mu}) - 1 \right) \left( \left( (1 + \frac{\lambda}{\mu}) - 1 \right)^{\frac{1}{2}} \right)}{\sigma^2} - 1 - \exp \left\{ - B \left[ (1 + \frac{\lambda}{\mu}) - 1 \right] \right\} = 0.
\]

\[
\frac{\partial^2 \ell(\theta)}{\partial \lambda^2} = \frac{(1 - \alpha)}{\lambda} + \frac{n}{\sigma^2} \frac{\alpha B \left( (1 + \frac{\lambda}{\mu}) - 1 \right)^{\frac{1}{2}} \left( (1 + \frac{\lambda}{\mu}) - 1 \right)^{\frac{1}{2}}}{(1 + \frac{\lambda}{\mu}) - 1} - \frac{n}{\sigma^2} \frac{\lambda}{(1 + \lambda)} \sum_{i=1}^{n} \frac{\alpha \left( (1 + \frac{\lambda}{\mu}) - 1 \right)^{\frac{1}{2}}}{(1 + \frac{\lambda}{\mu}) - 1} = 0.
\]

\[
\frac{\partial^2 \ell(\theta)}{\partial \theta \partial \mu} = \frac{n}{\sigma^2} \sum_{i=1}^{n} \left( (1 + \frac{\lambda}{\mu}) - 1 \right)^{\frac{1}{2}} - B \left( (1 + \frac{\lambda}{\mu}) - 1 \right) \left( \left( (1 + \frac{\lambda}{\mu}) - 1 \right)^{\frac{1}{2}} \right) = 0.
\]

The fisher’s information matrix, say \( I(\theta)_{ij} = (\partial^2 \ell(\theta))/\partial \theta_i \partial \theta_j \), is obtained by taking the negative expectation of second derivatives of the score function with respect to every parameter, where \( a_{ij} = -E \left( \frac{\partial^2 \ell(\theta)}{\partial \theta_i \partial \theta_j} \right) \). However, under some regularity conditions it is known that \( \sqrt{n}(\hat{\theta} - \theta) \) asymptotically follows multivariate normal distribution \( N_n(0, \Sigma) \), where \( \Sigma \) the variance-covariance matrix is obtained by taking the inverse of the \( I(\theta) \) given by \( I(\theta)^{-1} \). The \( (1 - \alpha) \) 100% confidence interval for the parameters of the distribution can be obtained by \( \hat{\theta} \pm Z_{\alpha/2} \sqrt{\Sigma} \).

8. Simulation

In this section, we studied the performance of the MLEs of MOG-L distribution through simulations. The sample data used are obtained through inverse transform and are of sizes \( n = 50, 150, \) and 300. The process is repeated 1000 times using selected parameter values of the distribution \( \rho = 2, \mu = 0.8, \sigma = 3, \alpha = 0.7, \lambda = 1.4 \) and \( \rho = 1.5, \mu = 0.5, \sigma = 3, \alpha = 0.4, \lambda = 0.9 \). In each replication the maximum likelihood estimation is performed, and the average estimates, bias, Mean square error are obtained. The results of the simulations about the average estimates, biases, and mean square error are shown in Tables 1 and 2. The results indicate that the MSE decreases as the sample size \( n \) increases which satisfies the first order asymptotic theory.

9. Applications

We used two real-life data sets to illustrate the flexibility and potentials of the MOG-L model. The MOG-L model is compared with other models having Lomax (including Lomax distribution) as the baseline distribution such as Gumbel-Lomax (GL), Beta-Lomax (BL), Exponentiated Lomax (ExpLomax), Kumaraswamy-Lomax (KumL), Gamma-Lomax (GammaL), Exponentiated Generalized Lomax (ExpGL). The compar-
son is based on Cramer von Mises (W), Anderson Darling (A), Akaike Information criterion (AIC), Consistent Akaike Information criterion (CAIC), and Bayesian Information Criterion (BIC) goodness of fit statistics, and negative log-likelihood ($-\ell$). The model with least goodness of fit statistics provides the best fit for the data set (Chen and Balakrishnan, 1995). Tables 3 and 4 shows the MLEs and standard errors (in parentheses) of the first and second data set respectively. The estimated cramers von Mises ($W^*$) and Anderson Darling ($A^*$), AIC, CAIC, BIC and $-\ell$ of the first and second data set are shown in Tables 5 and 6 respectively. The values in Tables 5 and 6 indicate that the goodness of fit statistics of the MOG-L are the least among the competing models. Figures 4 and 5 show the t statistics of the estimated pdfs and cdfs with empirical cdf of the two data sets. These values also indicate that MOG-L model provides the best fits among the competing models. The pdfs of the competing models are given by

$$f_{\text{fi}}(x) = \frac{aB(1 + \frac{x}{2})^{-\alpha - 1}}{\sigma^\alpha [\left(1 + \frac{x}{2}\right)^\alpha - 1]}; a, \lambda, \sigma > 0, -\infty < \mu < \infty, B = \exp\left(\frac{\mu}{\sigma}\right).$$

$$f_{\text{glm}}(x) = \frac{aB(1 + x^2)^{-\alpha - 1}}{\sigma^\alpha [\left(1 + x^2\right)^\alpha - 1]}; a, \lambda, \sigma > 0.$$

$$f_{\text{expGL}}(x) = \frac{aB(1 + x^2)^{-\alpha - 1}}{\sigma^\alpha [\left(1 + x^2\right)^\alpha - 1]}; a, \lambda, \sigma > 0.$$

$$f_{\text{expL}}(x) = \frac{aB(1 + x^2)^{-\alpha - 1}}{\sigma^\alpha [\left(1 + x^2\right)^\alpha - 1]}; a, \lambda, \sigma > 0.$$

The two data sets used are on aircraft windshield failure and service times applied by Murthy et al. (2004) to compare various modified Weibull distributions and used by Ramos et al. (2013) to model exponentiated Logam Poisson distribution.

**First data set**: The first data set is on 63 service times (thousand hours) of aircraft windshield (unit in thousand hours). The data are as follow: 0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

**Second data set**: The second data set is on the 84 failure times (thousand hours) of aircraft windshield. The data are as follow: 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.823, 4.635, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

10. Conclusion

We have introduced a new flexible five parameter lifetime distribution called MOG-L distribution. The idea behind the introduction of this new distribution is primarily to develop more flexible distribution with heavy kurtosis to model more complex data sets. Some characterizations such as quantile function, moments, trimmed L-moments, moment generating function, entropy, order statistics, and reliability functions of the new distribution are presented. Furthermore, estimation of the unknown parameters of the new distribution through maximum likelihood approach is presented. We studied the performance of maximum likelihood estimates of the new distribution using simulation. Two real life data sets are used to illustrate the potentials and flexibility of the new distribution and comparison with other distributions having Lomax as baseline distribution is made. The bases of comparison are on the goodness of fit statistics such as Cramer von-Mises, Anderson-Darling, AIC, BIC, and CAIC. The results of the goodness of fit statistics indicate that the MOG-L distribution provides better fit than other competing models.

### Declarations

**Author contribution statement**

Nwegwe Elebe: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Fidelis Ugwuowo: Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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The authors declare no conflict of interest.
References

Ahsanullah, M., 1991. Record values of the Lomax distribution. Stat. Neerl. 45, 21–29.
Anwar, M., Zahoor, J., 2018. The half-logistic lomax distribution for lifetime modeling. J. Prob. Stat.
Ashour, S.K., Eltehiwy, M.A., 2013. Transmuted lomax distribution. Am. J. Appl. Math. Stat. 1 (6), 121–127.
Campbell, G., Ramakrishna, M.V., 1993. An application of Lomax distributions in receiver operating characteristic (ROC) curve analysis. Commun. Stat. Theor. Methods 22, 1681–1697.
Chen, G., Balakrishnan, N., 1995. A general purpose approximate goodness-of-fit test. J. Qual. Technol. 27, 154–161.
Cordeiro, G.M., Edwin, M.M., Ortega, E.M.M., Bozidar, V., Popović, B.V., 2013. The gamma-Lomax distribution. J. Stat. Comput. Simulat.
Elamir, E.A.H., Scheult, A.H., 2003. Trimmed L-moments. Comput. Stat. Data Anal. 43, 299–314.
Ghitany, M.E., Al-Awadhi, F.A., Alkhalfan, L.A., 2007. Marshall–Olkin extended lomax distribution and its application to censored data. Commun. Stat. Theor. Methods 36 (10), 1855–1866.
Hassan, A.S., Al-Ghamdi, A.S., 2009. Optimum step stress accelerated life testing for lomax distribution. J. Appl. Sci. Res. 5, 2153–2164.
Lemonte, A.J., Cordeiro, G.M., 2011. An extended Lomax distribution. Statistics. J. Theor. Appl. Stat.
Lomax, R.K., 1954. Business Failures: another example of the analysis of failure data. J. Am. Stat. Assoc. 49 (268), 847–852.
Lorenz, M.O., 1905. Methods of measuring the concentration of Wealth. Am. Stat. Assoc. 9 (70), 2009–2219.
Maurya, R.K., Tripathi, Y.M., Lodhi, C., Rastogi, M.K., 2019 July. On a generalized Lomax distribution. Int. J. Syst. Assur. Eng. Manag. 1–14.
Moors, J.J.A., 1988. A quantile alternative for kurtosis. Statistician 37, 25–32.
Murthy, D.N.P., Xie, M., Jiang, R., 2004. Weibull Models. John Wiley & Sons. Rajab, M., Aleem, M., Nawaz, T., Daniyal, M., 2013. On five parameter Beta lomax distribution. J. Stat. 20, 102–118.
Ramos, M.W.A., Marinho, P.R.D., da Silva, R.V., Cordeiro, G.M., 2013. The exponentiated Lomax Poisson distribution with an application to lifetime data. Adv. Appl. Stat. 34, 107–115.
Salman, A.N., Hamad, A.M., 2019. On estimation of the stress–strength reliability based on lomax distribution. JIP Conf. Ser. Mater. Sci. Eng.
Tahir, M.H., Hussain, A.M., Cordeiro, G.M., Hamedani, G.G., Mansoor, M., Zubair, M., 2015. The gumbel-lomax distribution: properties and applications. J. Stat. Theory. Appl. 15 (1), 61–79.
Ugwuowo, F.I., Nwezza, E.E., 2018. A new Marshall–Olkin Gumbel family of distributions: properties and applications. Far East J. Theor. Stat. 54 (5), 477–501.
Zubair, M., Cordeiro, G.M., Tahir, M.H., Mahmood, M., Mansoor, M., 2017. A study of logistic-lomax distribution and its applications. J. Prob. Stat. Sci. 15 (1), 29–46.