Strong disorder real-space renormalization for the many-body-localized phase of random Majorana models

Cécile Monthus

Institut de Physique Théorique, Université Paris Saclay, CNRS, CEA, 91191 Gif-sur-Yvette, France

E-mail: cecile.monthus@cea.fr

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Abstract
For the many-body-localized phase of random Majorana models, a general strong disorder real-space renormalization procedure known as RSRG-X (Pekker et al 2014 Phys. Rev. X 4 011052) is described to produce the whole set of excited states, via the iterative construction of the local integrals of motion (LIOMs). The RG rules are then explicitly derived for arbitrary quadratic Hamiltonians (free-fermions models) and for the Kitaev chain with local interactions involving even numbers of consecutive Majorana fermions. The emphasis is put on the advantages of the Majorana language over the usual quantum spin language to formulate unified RSRG-X rules.

Keywords: random Majorana models, many-body-localization, strong disorder renormalization

1. Introduction

Strong disorder renormalization procedures introduced for ground-states of random quantum models by Ma–Dasgupta–Hu [1, 2] and Daniel Fisher [3–5] (see the review [6] and references therein) are usually formulated in terms of quantum spins. Although one can indeed argue that the language of quantum spins $S = 1/2$ or q-bits is the most natural framework for quantum models or quantum information, another appealing point of view is that it is much more advantageous to use instead the language of Majorana fermions in order to reveal the true underlying structure of the model, that could be otherwise somewhat hidden in the spin formulation (see for instance the two recent works [7, 8] where the Majorana language is instrumental to classify possible phases).

In the present paper, the goal is thus to formulate strong disorder renormalization rules for generic random Majorana models. Besides the construction of the ground-state mentioned
above, the strong disorder renormalization approach has been recently extended to construct
the whole set of excited eigenstates via the RSRG-X procedure [9–14], or to obtain the effective
dynamics via the RSRG-t procedure [15, 16]. These two closely related procedures [17]
actually identify iteratively the local integrals of motion called LIOMs [18–37] that are known
to characterize the many-body-localized phase existing in some isolated random quantum
interacting models (see the many recent reviews [38–46] and references therein).

The paper is organized as follows. In section 2, the notations for general random Majorana
models with parity-interactions are introduced. In section 3, the general RSRG-X procedure
is described with the simplest example of the random Kitaev chain. In section 4, the RSRG-X
rules are given for arbitrary quadratic Hamiltonians (free-fermions). In section 5, the RSRG-X
rules are derived for the random Majorana chain with local interactions involving only con-
nsecutive Majorana operators. The conclusions are summarized in section 6. The appendix
contains a short reminder of the dictionary between Majorana fermions, Dirac fermions and
quantum spin chains.

2. Notations for random Majorana models with parity-interactions

2.1. Majorana operators

In the present paper, we wish to study models defined in terms of 2N Majorana operators \( \gamma_j \)
with \( j = 1, \ldots, 2N \) (see appendix for the dictionary between Majorana fermions, Dirac fer-
mions and quantum spin chains.). These Majorana operators are Hermitian

\[ \gamma_j^\dagger = \gamma_j \]  

square to unity

\[ \gamma_j^2 = 1 \]  

and anti-commute with each other

\[ \{ \gamma_j, \gamma_l \} \equiv \gamma_j \gamma_l + \gamma_l \gamma_j = 0 \quad \text{for } j \neq l. \]

So the first advantage of the Majorana formulation over Dirac fermions or quantum spins is
clearly the symmetric role played by the 2N Majorana operators instead of the creation and
annihilation operators for the Dirac fermions, or the three Pauli matrices for quantum spins
(see appendix). One thus expects that the Majorana language is more appropriate to formulate
unified renormalization rules.

2.2. Parity operators

It is convenient to associate to any even number \((2k)\) with \( k = 1, 2, \ldots, N \) of Majorana operators
labelled by \( 1 \leq j_1 < j_2 < \ldots < j_{2k} \leq 2N \) the parity operator

\[ p^{(2k)}_{j_1, j_2, \ldots, j_{2k}} \equiv i^k \gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \cdots \gamma_{j_{2k-1}} \gamma_{j_{2k}}. \]  

For \( k = 1 \) and \( k = 2 \), they represent the usual interactions between two and four Majorana
operators, respectively

\[ p^{(2)}_{j_1, j_2} = i \gamma_{j_1} \gamma_{j_2} \]

\[ p^{(4)}_{j_1, j_2, j_3, j_4} = -\gamma_{j_1} \gamma_{j_2} \gamma_{j_3} \gamma_{j_4} \]  

\[ \text{appendix} \]
while for \( k = N \), the only possibility is \( j_q = q \) leads to the standard total parity of the whole system
\[
P_{\text{tot}} \equiv P_{1,2,\ldots,2N \rightarrow 1,2N} = i^N \gamma_1 \gamma_2 \gamma_3 \gamma_4 \ldots \gamma_{2N-1} \gamma_{2N}.
\] (6)
The parity operators of equation (4) are Hermitian
\[
(P_{j_1, j_2, \ldots, j_2k})_j^\dagger = P_{j_1, j_2, \ldots, j_2k}
\] (7)
square to unity
\[
(P_{j_1, j_2, \ldots, j_2k})^2 = 1
\] (8)
and they commute or anti-commute
\[
P_{j_1, j_2, \ldots, j_2k} P_{l_1, l_2, \ldots, l_2q} = (-1)^{p_c} P_{l_1, l_2, \ldots, l_2q} P_{j_1, j_2, \ldots, j_2k}
\] (9)
depending on the parity \((-1)^{p_c}\) of the number \( p_c \) of common Majorana operators between the two sets \( \{j_1, \ldots, j_{2k}\} \) and \( \{l_1, \ldots, l_{2q}\} \).

2.3. General Hamiltonian commuting with the total parity
The most general Hermitian Hamiltonian commuting with the total parity \( P_{\text{tot}} \) of equation (6) can be expanded into all the parity operators of equation (4)
\[
\mathcal{H} = \sum_{k=1}^{N} \mathcal{H}^{(2k)}
\]
\[
\mathcal{H}^{(2k)} = \sum_{1 \leq j_1 < j_2 < \ldots < j_{2k} \leq 2N} K^{(2k)}_{j_1, j_2, \ldots, j_{2k}} P^{(2k)}_{j_1, j_2, \ldots, j_{2k}}
\] (10)
where \( K^{(2k)}_{j_1, j_2, \ldots, j_{2k}} \) are the real couplings defining the model.

For instance, \( \mathcal{H}^{(2)} \) corresponds to the most general quadratic Hamiltonian
\[
\mathcal{H}^{(2)} = \sum_{1 \leq j_1 < j_2 \leq 2N} K^{(2)}_{j_1, j_2} P^{(2)}_{j_1, j_2} = \sum_{1 \leq j_1 < j_2 \leq 2N} K^{(2)}_{j_1, j_2} \gamma_j \gamma_j
\] (11)
while \( \mathcal{H}^{(4)} \) contains all the possible four-Majorana-interactions
\[
\mathcal{H}^{(4)} = \sum_{1 \leq j_1 < j_2 < j_3 \leq 2N} K^{(4)}_{j_1, j_2, j_3, j_4} P^{(4)}_{j_1, j_2, j_3, j_4} = \sum_{1 \leq j_1 < j_2 < j_3 \leq 2N} K^{(4)}_{j_1, j_2, j_3, j_4} \gamma_j \gamma_j \gamma_j \gamma_j.
\] (12)

Before specializing to specific models, it is useful to define first a general RSRG-X procedure for the generic Hamiltonian of equation (10), as described in the following section.

3. General RSRG-X procedure for random Majorana models
In this section, we consider the generic Majorana Hamiltonian of equation (10) with random coupling constants \( K^{(2k)}_{j_1, j_2, \ldots, j_{2k}} \), and we describe the RSRG-X procedure based on the decimation of the strongest two-Majorana-coupling.
3.1. Strongest two-Majorana-coupling

Let us choose the biggest two-Majorana-coupling in absolute value $|K_{nm}^{(2)}|$ with $1 \leq n < n \leq 2N$

$$|K_{nm}^{(2)}| = \max_{1 \leq j_1 < j_2 \leq 2N} |K_{j_1 j_2}^{(2)}|^2. \quad (13)$$

The corresponding elementary two-Majorana Hamiltonian involves only the parity $P_{nm}^{(2)}$

$$h_{nm} = K_{nm}^{(2)} p_{nm}^{(2)} = iK_{nm}^{(2)} \gamma_n \gamma_m \quad \text{so the two eigenvalues labelled by } \epsilon = \pm 1$$

are associated to the two orthogonal projectors

$$\pi_{nm}^\epsilon = \frac{1 + \epsilon p_{nm}^{(2)}}{2} = \frac{1 + i \epsilon \gamma_n \gamma_m}{2}. \quad (16)$$

3.2. Perturbation theory in the other couplings

The projection of the full Hamiltonian on the two energy branches labelled by $\epsilon = \pm 1$ (equation (15)) reads at second-order perturbation theory in all the other couplings

$$H_{nm}^{\text{eff}} = H_{nm} + \mathcal{O}(K_{nm}^{(2)})$$

$$H_{nm}^{\epsilon} = \pi_{nm}^\epsilon H \pi_{nm}^\epsilon + \frac{(\pi_{nm}^\epsilon H \pi_{nm}^-)(\pi_{nm}^- H \pi_{nm}^\epsilon)}{2\epsilon K_{nm}^{(2)}}. \quad (17)$$

To evaluate how the Hamiltonian $H$ behaves between two equal $\epsilon = \epsilon'$ or opposite $\epsilon = -\epsilon'$ projectors of equation (16), it is useful to decompose $H$ into the four terms

$$H = H_{nm}^{10} + H_{nm}^{01} + i\gamma_n H_{nm}^{10} + i\gamma_m H_{nm}^{01} \quad (18)$$

where the $H_{nm}^{\alpha\beta}$ involve only the other Majorana fermions ($\gamma_j$) with $j \neq (n, m)$. In particular, $H_{nm}^{10}$ and $H_{nm}^{01}$ contain an even number of these other Majorana operators, while $H_{nm}^{10}$ and $H_{nm}^{01}$ contain an odd number of these other Majorana operators. As a consequence, the part $H_{nm}^{\text{comm}}$ of $H$ that commutes with the parity $P_{nm}^{(2)} = i\gamma_n \gamma_m$ reads

$$H_{nm}^{\text{comm}} = H_{nm}^{10} + i\gamma_n \gamma_m H_{nm}^{11} \quad (19)$$

while the contribution $H_{nm}^{\text{anti}}$ of $H$ that anticommutes with the parity $P_{nm}^{(2)} = i\gamma_n \gamma_m$ is

$$H_{nm}^{\text{anti}} = i\gamma_n H_{nm}^{10} + i\gamma_m H_{nm}^{01}. \quad (20)$$

Between two identical projectors $\epsilon = \epsilon'$, only the commuting part survives and gives the contribution

$$\pi_{nm}^\epsilon H \pi_{nm}^\epsilon = \pi_{nm}^\epsilon H_{nm}^{00} + \epsilon H_{nm}^{11} \quad (21)$$

Between two orthogonal projectors, only the anticommuting part survives and yields

$$\pi_{nm}^\epsilon H \pi_{nm}^- = \pi_{nm}^\epsilon \gamma_n \pi_{nm}^- \pi_{nm} = \pi_{nm}^\epsilon H_{nm}^{\text{anti}} \pi_{nm}^- \quad (22)$$

so that the the numerator of equation (17) becomes
(\pi^e_{nm}H^\text{anti}_{nm})(\pi^e_{nm}H^\text{anti}_{nm}) = \pi^{e^2}_{nm}(H^\text{anti}_{nm})^2\pi^e_{nm}.  
(23)

Since $H^{01}_{nm}$ and $H^{10}_{nm}$ contain an odd number of the other Majorana operators ($\gamma$) with $j \neq (n,m)$, one obtains that the square of equation (20) reads

$$ (H^{\text{anti}}_{nm})^2 = -\left(\gamma_n H^{10}_{nm} + \gamma_m H^{01}_{nm}\right)^2 = (H^{10}_{nm})^2 + (H^{01}_{nm})^2 - \gamma_n \gamma_m [H^{01}_{nm}, H^{10}_{nm}]  
(24)$$

so that its projection reads

$$\pi^{e^2}_{nm}(H^{\text{anti}}_{nm})^2\pi^e_{nm} = (H^{10}_{nm})^2 + (H^{01}_{nm})^2 + i[H^{01}_{nm}, H^{10}_{nm}].  
(25)$$

Putting everything together, one obtains that the effective Hamiltonian of equation (17) for the remaining Majorana operators reads

$$H^e_{nm} = H^{00}_{nm} + \epsilon H^{11}_{nm} + \epsilon \frac{(H^{10}_{nm})^2 + (H^{01}_{nm})^2}{2K^{(2)}_{nm}} + \frac{i[H^{01}_{nm}, H^{10}_{nm}]}{2K^{(2)}_{nm}}  
(26)$$

in terms of the decomposition of equation (18). To see how this procedure works in practice, let us now describe the simplest possible case.

### 3.3. Simplest application: the random Kitaev chain

As recalled in appendix, the Kitaev chain [47] with random nearest-neighbor-two-Majorana couplings $K^{(2)}_{j,j+1}$

$$H^{\text{Kitaev}} = i \sum_{j=1}^{2N-1} K^{(2)}_{j,j+1} \gamma_j \gamma_{j+1}  
(27)$$

corresponds to the random transverse field Ising chain (RTFIC) of equation (A.7). Since the RTFIC is one of the basic model where the strong disorder RG approach has been developed [4], it is useful to mention how the RSRG-X procedure described above works for the random Kitaev chain of equation (27).

One chooses the biggest coupling in absolute value (equation (13))

$$|K^{(2)}_{n,n+1}| = \max_{1 \leq j \leq 2N-1} |K^{(2)}_{j,j+1}|  
(28)$$

and one computes the corresponding decomposition of equation (18)

$$\mathcal{H} = H^{00}_{n,n+1} + i \gamma_n H^{10}_{n,n+1} + i \gamma_{n+1} H^{01}_{n,n+1} + \gamma_n \gamma_{n+1} H^{11}_{n,n+1}$$

$$H^{00}_{n,n+1} = i \sum_{j=1}^{n-2} K^{(2)}_{j,j+1} \gamma_j \gamma_{j+1} + i \sum_{j=n+2}^{2N-1} K^{(2)}_{j,j+1} \gamma_j \gamma_{j+1}$$

$$H^{10}_{n,n+1} = -K^{(2)}_{n-1,n} \gamma_{n-1}$$

$$H^{01}_{n,n+1} = K^{(2)}_{n+1,n+2} \gamma_{n+2}$$

$$H^{11}_{n,n+1} = K^{(2)}_{n,n+1}  
(29)$$

in order to obtain the effective Hamiltonian via equation (26)

$$H^{e}_{n,n+1} = H^{00}_{nm} + \epsilon \left(K^{(2)}_{n,n+1} + \frac{(K^{(2)}_{n-1,n})^2 + (K^{(2)}_{n+1,n+2})^2}{2K^{(2)}_{n,n+1}}\right) + \frac{i\gamma_{n-1} K^{(2)}_{n-1,n+2} K^{(2)}_{n,n+1}}{K^{(2)}_{n,n+1} \gamma_{n-1} \gamma_{n+2}}  
(30)$$
So besides the first term $H_{nm}^{(0)}$ representing the part of the chain that is left unchanged by the decimation of the pair $(\gamma_n, \gamma_{n+1})$ and the second term proportional to $\epsilon$ representing the direct energy contribution of the decimation, the third term means that the Majorana operators $\gamma_{n-1}$ and $\gamma_{n+2}$ that become nearest-neighbor after the decimation are now coupled by the renormalized coupling

$$K_{n-1,n+2}^{(2)} = \frac{K_{n-1,n+1}^{(2)} + K_{n+1,n+2}^{(2)}}{K_{n+1}^{(2)}}$$

that is independent of the energy branch $\epsilon = \pm 1$ chosen for the decimation. This independence is of course not surprising, since it is a direct consequence of the notion of ‘free fermions’, but it is nevertheless important to stress here the difference with the RSRG-X rules formulated in the spin language, where the choice $\epsilon = \pm 1$ of the energy branch explicitly appear in the renormalization of the couplings [9].

In the remainder of the paper, we analyze two different generalizations of this random Kitaev chain. We first describe how the RSRG-X procedure works for arbitrary quadratic Hamiltonians in section 4. We then consider the random Kitaev chain in the presence of local interactions involving even numbers of consecutive Majorana fermions in section 5.

### 4. Application to arbitrary quadratic Hamiltonians

In this section, the RSRG-X procedure described in the previous section is applied to any random quadratic Hamiltonians (free-fermions).

#### 4.1. Decomposition of equation (18)

When the Hamiltonian contains only pair-interaction between Majorana operators (only $k = 1$ in equation (10))

$$\mathcal{H} = \mathcal{H}^{(2)} = \sum_{1 \leq j_1 < j_2 \leq 2N} K_{j_1,j_2}^{(2)} \gamma_{j_1} \gamma_{j_2} = i \sum_{1 \leq j_1 < j_2 \leq 2N} K_{j_1,j_2} \gamma_{j_1} \gamma_{j_2}$$

the decomposition of equation (18) with respect to the pair $(\gamma_n, \gamma_m)$ reads

$$H_{nm}^{(0)} = \sum_{1 \leq j_1 < j_2 \leq 2N, j_1 \neq (n,m), j_2 \neq (n,m)} iK_{j_1,j_2} \gamma_{j_1} \gamma_{j_2}$$

$$H_{nm}^{(10)} = \sum_{n+1 \leq j_1 \leq 2N} (-K_{mj}^{(2)}) \gamma_j + \sum_{n+1 \leq j_1 \leq 2N, j \neq m} K_{nj} \gamma_j = \sum_{j \neq (n,m)} K_{nj}^{(2)} \gamma_j$$

$$H_{nm}^{(11)} = \sum_{1 \leq j_1 \leq m-1, j \neq n} (-K_{mj}^{(2)}) \gamma_j + \sum_{m+1 \leq j_1 \leq 2N} K_{mj} \gamma_j = \sum_{j \neq (n,m)} K_{mj}^{(2)} \gamma_j$$

$$H_{nm}^{11} = K_{nm}$$

(33)

where we have introduced the notation for $j_2 > j_1$

$$K_{j_1,j_2}^{(2)} = -K_{j_2,j_1}^{(2)}$$

Since $H_{nm}^{(10)}$ and $H_{nm}^{(11)}$ are linear in the other Majorana operators, their squares are constants

$$\langle H_{nm}^{(10)} \rangle^2 = \sum_{j \neq (n,m)} (K_{nj}^{(2)})^2$$

$$\langle H_{nm}^{(11)} \rangle^2 = \sum_{j \neq (n,m)} (K_{mj}^{(2)})^2$$

(35)
while their commutator is quadratic

\[
[H^{(1)}_{nm}, H^{(0)}_{nm}] = \sum_{j_1 \neq (n,m), \ j_2 \neq (n,m)} K_{nj_1}^{(2)} \sum_{j_2 \neq (n,m)} K_{mj_2}^{(2)} [\gamma_{j_1}, \gamma_{j_2}] \\
= \sum_{j_1 \neq j_2, \ j_1 \neq (n,m), \ j_2 \neq (n,m)} \left( K_{nj_1}^{(2)} K_{mj_2}^{(2)} - K_{nj_2}^{(2)} K_{mj_1}^{(2)} \right) 2 \gamma_{j_1} \gamma_{j_2}.
\] (36)

4.2. RSRG-X rules

Putting everything together, equation (26) becomes

\[
H^{(c)}_{nm} = H^{(0)}_{nm} + \epsilon \left( K_{nm}^{(2)} + \sum_{j \neq (n,m)} \frac{(K_{nj})^{(2)} + (K_{mj})^{(2)}}{2K_{nm}^{(2)}} \right) + i \sum_{j_1 < j_2, j_1 \neq (n,m), j_2 \neq (n,m)} K_{j_1 j_2}^{R(2)} \gamma_{j_1} \gamma_{j_2}
\] (37)

with the renormalized couplings between the remaining Majorana operators

\[
K^{R(2)}_{j_1 j_2} = K^{(2)}_{j_1 j_2} + \frac{K_{nj_1}^{(2)} K_{mj_2}^{(2)} - K_{nj_2}^{(2)} K_{mj_1}^{(2)}}{2K_{nm}^{(2)}}.
\] (38)

These RSRG-X rules are thus closed for any quadratic Hamiltonian, and represent a direct generalization of the rule discussed above for the Kitaev chain in equation (31). Again, the choice of the energy branch \( \epsilon = \pm 1 \) appears only in the constant energy contribution of the decimation (second term of equation (37)) but not in the renormalized couplings of equation (38) as a consequence of the notion of ‘free-fermions’.

5. Application to the Majorana chain with consecutive-parity-interactions

After the free-fermion models considered in the previous section, let us now focus on the random Majorana chain with local interactions.

5.1. Majorana chain with consecutive-parity-interactions

In this section, we focus on the case where the parity operators appearing in the Hamiltonian (equation (10)) are only those involving strings of \( (2k) \) consecutive operators (instead of the general case of equation (4)), so that it is convenient to introduce the simplified notation

\[
p^{(2k)}_{j_1j_2} \equiv p^{(2k)}_{j_1j_2j_1+2j_2-2} = \gamma_{j_1} \gamma_{j_2} \gamma_{j_1+2} \gamma_{j_2-2} \gamma_{j_1+2k-1}.
\] (39)

The Hamiltonian of equation (10) is thus replaced by

\[
H^{(2k)} = \sum_{k=1}^{N} H^{(2k)}
\]

\[
H^{(2k)} = \sum_{j_1}^{2N-2k+1} K^{(2k)}_{j_1j_1+2k-1} p^{(2k)}_{j_1j_1+2k-1}.
\] (40)

In particular, \( H^{(2)} \) corresponds to the random Kitaev chain of equation (27)
\[ \mathcal{H}^{(2)} = \sum_{j=1}^{2N-1} K_{[jj+1]}^{(2)} p_{[jj+1]}^{(2)} = \frac{1}{2} \sum_{j=1}^{2N-1} K_{[jj+1]}^{(2)} \gamma_j \gamma_{j+1} \]  

(41)

while \( \mathcal{H}^{(4)} \) contains only four-Majorana-interactions between four consecutive operators

\[ \mathcal{H}^{(4)} = \sum_{j=1}^{2N-3} K_{[jj+3]}^{(4)} p_{[jj+3]}^{(2k)} = - \sum_{j=1}^{2N-3} K_{[jj+3]}^{(4)} \gamma_{j+1} \gamma_{j+2} \gamma_{j+3}. \]  

(42)

The translation of this model in the quantum spin language is given in equations (A.7), (A.9) and (A.10) of appendix.

5.2. Renormalized consecutive parities

After the elimination of the two Majorana operators \( (\gamma_n, \gamma_{n+1}) \) corresponding to the biggest coupling in absolute value (equation (13))

\[ |K_{[nn+1]}^{(2)}| = \max_{1 \leq j \leq 2N-1} |K_{[jj+1]}^{(2)}| \]  

(43)

the operators \( \gamma_{n-1} \) and \( \gamma_{n+2} \) have become neighbors. One then needs to introduce the renormalized consecutive-parity-operators across the decimated pair like the one already encountered in equation (30) for the Kitaev chain

\[ p_{[n-1,n+2]}^{R(2)} \equiv i \gamma_{n-1} \gamma_{n+2}. \]  

(44)

Here we will need more generally the other renormalized consecutive parities

\[ p_{[j,j+2k-1]}^{R(2k-2)} \equiv i^{k-1} (\gamma_{j+2} \cdots \gamma_{j+2k-1}) (\gamma_{j-2} \cdots \gamma_{j-2k+1}) \]  

(45)

for \( j \leq n-1 \) and \( j + 2k - 1 \geq n + 2 \).

5.3. Decomposition of equation (18)

In the decomposition of equation (18)

\[ \mathcal{H} = H_{00}^{00} + i \gamma_n H_{10}^{00} + i \gamma_{n+1} H_{01}^{00} + i \gamma_n \gamma_{n+1} H_{11}^{00} \]  

(46)

\( H_{00}^{00} \) contains all the terms of the Hamiltonian included in \([1, \ldots, n - 1]\) or included in \([n + 2, \ldots, 2N]\)

\[ H_{00}^{00} = \sum_{k=1}^{N} \left( \sum_{j=1}^{n-2k} K_{[jj+2k-1]}^{(2k)} p_{[jj+2k-1]}^{(2k)} + \sum_{j=n+2}^{2N-2k+1} K_{[jj+2k-1]}^{(2k)} p_{[jj+2k-1]}^{(2k)} \right) \]  

(47)

while \( H_{11}^{00} \) reads in terms of the renormalized consecutive-parity-operators of equation (45)

\[ H_{11}^{00} = \sum_{k \geq 2} K_{[nn+1]}^{(2k)} + \sum_{k \geq 2} \left( K_{[n+2-2k,n+1]}^{(2k-2)} p_{[n+2-2k,n+1]}^{(2k-2)} + K_{[n+2-2k,n+1]}^{(2k-2)} p_{[n+2-2k,n+1]}^{(2k-2)} \right) \]  

\[ + \sum_{k \geq 2} \sum_{j=n+1-2k}^{n-1} K_{[jj+2k-1]}^{(2k)} p_{[jj+2k-1]}^{R(2k-2)} \]  

(48)
\( H_{n,n+1}^{01} \) can be obtained from all the parity operators beginning exactly at \( j = n + 1 \), and it is thus convenient to factor out the common operator \( \gamma_{n+2} \) to rewrite

\[
H_{n,n+1}^{01} = \gamma_{n+2} \left( K_{[n+1,n+2]}^{(2)} + \sum_{k \geq 2} K_{[n+1,n+2k]}^{(2k)} P_{[n+3,n+2k]}^{(2k-2)} \right). \tag{49}
\]

Similarly, \( H_{n,n+1}^{10} \) can be obtained from all the parity operators ending exactly at \( j + 2k - 1 = n \), and one can factor out the common operator \( \gamma_{n-1} \) to rewrite

\[
H_{n,n+1}^{10} = - \left( K_{[n-1,n]}^{(2)} + \sum_{k \geq 2} K_{[n+1-2k,n]}^{(2k)} P_{[n+1-2k,n-2]}^{(2k-2)} \right) \gamma_{n-1}. \tag{50}
\]

Then their squares simplify into

\[
(H_{n,n+1}^{01})^2 = \left( K_{[n+1,n+2]}^{(2)} + \sum_{k \geq 2} K_{[n+1,n+2k]}^{(2k)} P_{[n+3,n+2k]}^{(2k-2)} \right)^2
= \sum_{k \geq 1} \left( K_{[n+1,k+2]}^{(2k)} \right)^2 + 2 \sum_{1 \leq k_1 < k_2} K_{[n+1,2k_1]}^{(2k_1)} K_{[n+1,2k_2]}^{(2k_2)} P_{[n+1+k_1,n+2k_2]}^{(2k_2-2k_1)} \tag{51}
\]

and

\[
(H_{n,n+1}^{10})^2 = \left( K_{[n-1,n]}^{(2)} + \sum_{k \geq 2} K_{[n+1-2k,n]}^{(2k)} P_{[n+1-2k,n-2]}^{(2k-2)} \right)^2
= \sum_{k \geq 1} \left( K_{[n+1,2k]}^{(2k)} \right)^2 + 2 \sum_{1 \leq k_1 < k_2} K_{[n+1,2k_1]}^{(2k_1)} K_{[n+1,2k_2]}^{(2k_2)} P_{[n+1+k_1,n-2k_2]}^{(2k_2-2k_1)} \tag{52}
\]

while their commutator reads in terms of the renormalized consecutive-parity-operators of equation (45)

\[
\frac{i}{2} [H_{n,n+1}^{01}, H_{n,n+1}^{10}]
= \left( K_{[n+1,n+2]}^{(2)} + \sum_{k_1 \geq 2} K_{[n+1+2k_1,n]}^{(2k_1)} P_{[n+1+2k_1,n+2k_2]}^{(2k_2-2k_1)} \right) \left( K_{[n-1,n]}^{(2)} + \sum_{k \geq 2} K_{[n+1-2k,n]}^{(2k)} P_{[n+1-2k,n+2k_2]}^{(2k_2-2k)} \right)
= \left( K_{[n+1,n+2]}^{(2)} + \sum_{k_1 \geq 2} K_{[n+1+2k_1,n]}^{(2k_1)} P_{[n+1+2k_1,n+2k_2]}^{(2k_2-2k_1)} \right) + \sum_{k \geq 2} K_{[n+1-2k,n]}^{(2k)} P_{[n+1-2k,n+2k_2]}^{(2k_2-2k)}
+ \sum_{k_1 \geq 2} K_{[n+1+2k_1,n]}^{(2k_1)} \sum_{k_2 \geq 2} K_{[n+1+2k_1,n+2k_2]}^{(2k_2)} P_{[n+1+2k_1,n+2k_2]}^{(2k_2-2k_1)}
= \sum_{k_1 \geq 2} \sum_{k_2 \geq 2} K_{[n+1+2k_1,n]}^{(2k_1)} K_{[n+1+2k_1,n+2k_2]}^{(2k_2)} P_{[n+1+2k_1,n+2k_2]}^{(2k_2-2k_1)} \tag{53}
\]
5.4. Renormalized Hamiltonian

Putting everything together, equation (26) yields

\[ H_{n,n+1}^\epsilon = H_{n,n+1}^{00} + \epsilon H_{n,n+1}^{11} + \epsilon \left( \frac{H_{n,n+1}^{10}}{2K_{n,n+1}^{(2)}} \right)^2 + \frac{i[H_{n,n+1}^{01}, H_{n,n+1}^{10}]}{2K_{n,n+1}^{(2)}} \]

\[ = \sum_{k=1}^{N} \sum_{j=1}^{n-k} K_{n-j+2k-1}^{(2k)} P_{n-j+2k-1}^{(2k)} + \sum_{j=0}^{n-2k} K_{n-j+2k-1}^{(2k)} P_{n-j+2k-1}^{(2k)} \]

\[ + \epsilon K_{n,n+1}^{(2)} + \epsilon \sum_{k \geq 1} \left( [K_{[n+1-2k,n]}^{(2k)}]^2 + [K_{[n+1,n+2k]}^{(2k)}]^2 \right) \]

\[ + \sum_{k \geq 1} \sum_{k \geq 1} \left( K_{[n+1-2k,n]}^{(2k)} P_{[n+1-2k,n]}^{(2k)} + K_{[n+1,n+2k]}^{(2k)} P_{[n+1,n+2k]}^{(2k)} \right) \]

\[ + \sum_{k \geq 1} \sum_{k \geq 1} \left( K_{[n+1-2k+n]}^{(2k)} P_{[n+1-2k+n]}^{(2k)} + K_{[n+1,n+2k+2]}^{(2k)} P_{[n+1,n+2k+2]}^{(2k)} \right) \]

\[ + \sum_{k \geq 1} \sum_{k \geq 1} \left( K_{[n+1-2k+n]}^{(2k+1)} P_{[n+1-2k+n]}^{(2k+1)} + K_{[n+1,n+2k+2]}^{(2k+1)} P_{[n+1,n+2k+2]}^{(2k+1)} \right) \]

To clarify the meaning of the various terms, it is useful to distinguish four types of contributions

\[ H_{n,n+1}^\epsilon = E_{n,n+1}^\epsilon + H_{n,n+1}^{\text{Left}} + H_{n,n+1}^{\text{Right}} + H_{n,n+1}^{\text{Middle}} \]  

(55)

The first term is simply the constant contribution produced directly by the decimation that depends on the energy branch \( \epsilon = \pm 1 \)

\[ E_{n,n+1}^\epsilon = \epsilon \left( K_{[n,n+1]}^{(2)} + \sum_{k \geq 1} \left( [K_{[n+1-2k,n]}^{(2k)}]^2 + [K_{[n+1,n+2k]}^{(2k)}]^2 \right) \right) \]

(56)

The second term contains the parity-operators localized on the left \([1,\ldots,n-1]\) of the decimated pair

\[ H_{n,n+1}^{\text{Left}} = \sum_{k \geq 1} \sum_{l \leq n-1} \left( K_{[n-l+2k+2]}^{(2k)} \delta_{n-l-1} + \frac{K_{[n-l+2k+1]}^{(2k+1)}}{K_{n,n+1}^{(2)}} \right) \]

(57)

The third term contains the parity-operators localized on the right \([n+2,\ldots,2N]\) of the decimated pair

\[ H_{n,n+1}^{\text{Right}} = \sum_{k \geq 1} \sum_{j \geq n+2} \left( K_{[n-j+2k-1]}^{(2k+1)} \delta_{n-j+2} + \frac{K_{[n-j+2k]}^{(2k+2)}}{K_{n,n+1}^{(2)}} \right) \]

(58)

Finally the fourth term contains the renormalized parity-operators of equation (45) that begin before the decimated pair and that end after the decimated pair.
\( H_{\text{Middle}}^{n,n+1} = \sum_{k \geq 1} \sum_{j=n+1-2k}^{n-1} \left( \epsilon K_{j,j+2k+1}^{(2k+2)} + \frac{K_{j,n}^{(n+1-j)} K_{n+1,j+2k+1}^{(2k+j+1-n)}}{K_{n,n+1}^{(2)}} \right) P_{j,j+2k+1}^{(2k)} \). \tag{59}

5.5. RSRG-X rules

The RSRG-X rules for the couplings between the surviving Majorana operators can be thus summarized as follows.

(i) The coupling associated to the parity operator \( P_{l,l+1-2k,l}^{(2k)} \) living on the left of the decimated pair \( l \leq n-1 \) (equation (57)) follows the RG rule

\[
K_{l,l+1-2k,l}^{R(2k)} = K_{l,l+1-2k,l}^{(2k)} + \epsilon \left( K_{l+1-2k,l}^{(2k+2)} \delta_{l,l-1} + \frac{K_{l+1-2k,n}^{(n-l)} K_{n+1,l}^{(2k+l-n)}}{K_{n,n+1}^{(2)}} \right). \tag{60}
\]

Besides its initial value \( K_{l,l+1-2k,l}^{(2k)} \), the new contributions come from the ‘degradation’ of the higher-order couplings \( K_{n,n+1}^{(2k+2)} \) and \( K_{n+1,l}^{(2k+l-n)} \) of order \( 2k + n - l \geq 2k + 2 \) and depend on the choice \( \epsilon = \pm \) of the energy branch.

(ii) The coupling associated to the parity operator \( P_{j,j+2k+1-1}^{(2k)} \) living on the right of the decimated pair \( j \geq n + 2 \) (equation (58)) follows the RG rule

\[
K_{j,j+2k+1-1}^{R(2k)} = K_{j,j+2k+1-1}^{(2k)} + \epsilon \left( K_{n,n+2k+1}^{(2k+2)} \delta_{j,j+1} + \frac{K_{n+1,j}^{(n-j-1)} K_{n+1,j+2k+1}^{(2k+j-n-1)}}{K_{n,n+1}^{(2)}} \right). \tag{61}
\]

Here again, besides its initial value \( K_{j,j+2k+1-1}^{(2k)} \), the new contributions come from the ‘degradation’ of the higher-order couplings \( K_{n,n+2k+1}^{(2k+2)} \) and \( K_{n+1,j+2k+1}^{(2k+j-n-1)} \) of order \( 2k + j - n - 1 \geq 2k + 2 \) and depend on the choice \( \epsilon = \pm \) of the energy branch.

(iii) The renormalized parity operator \( P_{j,j+2k+1+1}^{(2k)} \) that begins before the decimated pair \( j \leq n - 1 \) and that ends after the decimated pair \( n + 2 \leq j + 1 + 2k \) (equation (59)) is associated to the new renormalized couplings

\[
K_{j,j+2k+1+1}^{R(2k)} = \epsilon K_{j,j+2k+1}^{(2k+2)} + \frac{K_{j,n}^{(n+1-j)} K_{n+1,j+2k+1}^{(2k+j+1-n)}}{K_{n,n+1}^{(2)}}. \tag{62}
\]

The first terms corresponds again to the ‘degradation’ of the higher-order coupling \( K_{j,j+2k+1}^{(2k+2)} \) and depends on the choice \( \epsilon = \pm \) of the energy branch. The second term is the generalization of the basic rule of equation (31) concerning the Kitaev chain and does not depend on the choice \( \epsilon = \pm \) of the energy branch. In the present procedure, this second term is the only mechanism where new higher order couplings can be generated from two couplings of smaller orders \( 2k_1 = n + 1 - j \) and \( 2k_2 = 2k + j + 1 - n = 2k + 2 - 2k \).

In conclusion, the Majorana chain with consecutive parity-interactions of equation (40) remains closed for the RSRG-X procedure with the renormalized rules described above.
To see more clearly how it works in practice, it is now useful to consider the following simplest example.

5.6. First RG step for the initial chain involving only two and four Majorana interactions

Let us consider the case where the initial Hamiltonian of equation (40) contains only interactions between two and four consecutive Majorana operators (equations (41) and (42))

\[ H_{ini} = \frac{2N-1}{i} \sum_{j=1}^{K^{(2)}_{|j|+1}} \gamma_j \gamma_{j+1} - \frac{2N-3}{i} \sum_{j=1}^{K^{(4)}_{|j|+3}} \gamma_j \gamma_{j+1} \gamma_{j+2} \gamma_{j+3}. \]  

(63)

The RSRG-X rules for the first decimation of the biggest coupling \( K_{n,n+1}^{(2)} \) in absolute value are the following.

(i) The RG rule of equation (60) for the left of the decimated pair gives new contributions only for

\[ K_{|n-2,n-1|}^{R(2)} = K_{|n-2,n-1|}^{(2)} + \epsilon K_{|n-2,n+1|}^{(4)} \]  

(64)

and

\[ K_{|n-3,n-2|}^{R(2)} = K_{|n-3,n-2|}^{(2)} + \frac{\epsilon K_{|n-3,n|}^{(4)} K_{|n-1,n|}^{(2)}}{K_{n,n+1}^{(2)}} \]  

(65)

representing the ‘degradation’ of the four-Majorana-couplings \( K_{|n-2,n+1|}^{(4)} \) and \( K_{|n-3,n|}^{(4)} \) into contributions of couplings of order \( 2k = 2 \) that were already existing.

(ii) The RG rule of equation (61) for the right of the decimated pair gives new contributions only for

\[ K_{|n+2,n+3|}^{R(2)} = K_{|n+2,n+3|}^{(2)} + \epsilon K_{|n,n+3|}^{(4)} \]  

(66)

and

\[ K_{|n+3,n+4|}^{R(2)} = K_{|n+3,n+4|}^{(2)} + \frac{\epsilon K_{|n+1,n+2|}^{(2)} K_{|n+1,n+4|}^{(4)}}{K_{n,n+1}^{(2)}} \]  

(67)

representing also the ‘degradation’ of the four-Majorana-couplings \( K_{|n+3,n+3|}^{(4)} \) and \( K_{|n+1,n+4|}^{(4)} \) into contributions of couplings of order \( 2k = 2 \) that were already existing.

(iii) The RG rule of equation (62) for the renormalized parities across the decimated pair gives new couplings of various orders. The only renormalized coupling of order \( 2k = 2 \) is

\[ K_{|n-1,n+2|}^{R(2)} = \epsilon K_{|n-1,n+2|}^{(4)} + \frac{K_{|n-1,n|}^{(2)} K_{|n+1,n+2|}^{(2)}}{K_{n,n+1}^{(2)}} \]  

(68)

containing the ‘degradation’ of the four-Majorana coupling \( K_{|n-1,n+2|}^{(4)} \) and the renormalized contribution already seen for the Kitaev chain (equation (31)). The only renormalized couplings of order \( 2k = 4 \) are...
KR(4)_{[n-3,n+2]} = \frac{K^{(4)}_{[n-3,n]} K^{(2)}_{[n+1,n+2]}}{K^{(2)}_{n,n+1}} \quad (69)

and

KR(4)_{[n-1,n+4]} = \frac{K^{(2)}_{[n-1,n]} K^{(4)}_{[n+4,n+4]}}{K^{(2)}_{n,n+1}}. \quad (70)

Finally, there is one new renormalized coupling of order 2k = 6

KR(6)_{[n-3,n+4]} = \frac{K^{(4)}_{[n-3,n]} K^{(4)}_{[n+1,n+4]}}{K^{(2)}_{n,n+1}}. \quad (71)

This example shows that the generation of higher-order couplings remains rather limited, while there are many mechanisms of ‘degradation’ into smaller-order couplings. So we hope that these Majorana RSRG-X procedure can be applied numerically on large sizes without the proliferation of too many new renormalized couplings. This numerical implementation clearly goes beyond the scope of the present work and is left for other authors with more numerical possibilities (see [9] for the specific numerical problems related to the choice of different energy branches at each step of the RSRG-X).

6. Conclusion

In this work, we have formulated a general RSRG-X procedure for random Majorana models in their many-body-localized (MBL) phase. We have then derived the explicit RG rules for arbitrary quadratic Hamiltonians (free-fermions models) and for the random Kitaev chain with local interactions involving even numbers of consecutive Majorana fermions. However, these two examples of application are not restrictive, and one can apply the general rule of equation (26) to any other Majorana model of interest.

Along the paper, we have stressed the advantages of the Majorana language over the usual quantum spin language to formulate unified RG rules:

(a) the symmetric role played by the 2N majorana operators allows to classify the various terms of the Hamiltonian by the even number (2k) and the locations 1 < j_1 < ... < j_{2k} < 2N of the Majorana operators (while the spin language requires the distinction between different types of couplings in terms of Pauli matrices as recalled in appendix).

(b) in the strong disorder renormalization perspective, the unique elementary decimation then corresponds to the pairing between the two Majorana operators that are the most strongly coupled in absolute value and thus leads to unified RSRG-X rules (while the spin language requires the distinction between the decimations of different types of couplings in terms of Pauli matrices). In addition in free fermions models, the renormalized rule for the renormalized couplings is independent of the energy branch (equations (31) and (38)).

(c) this ‘deconstruction’ into Majorana fermions suggests that the simplest MBL model is actually the random Kitaev chain with interactions involving four consecutive Majorana fermions (equation (63)), while the standard model of MBL, namely the random-field XXZ chain actually corresponds to a Majorana ladder with some degeneracy in the couplings J_x = J_y = J_z (see appendix) so that the RSRG-X rules are more complicated.
as described in Ref [17]. It would be thus interesting in the future to apply numerically the RSRG-X rules to the simplest MBL model of equation (63) as discussed after equation (71).

Appendix. Dictionary between Majorana fermions, Dirac fermions and quantum spin chains

A.1. Majorana formulation of Dirac fermions models

In any dimension, a model involving $N$ Dirac Fermions described by annihilation and creation operators $(c_j, c_j^\dagger)$ for $j = 1, \ldots, N$ satisfying the canonical anti-commutation relations

$$\{c_j, c_k\} = 0 = \{c_j^\dagger, c_k\}$$
$$\{c_j, c_k^\dagger\} = \delta_{jk} \quad (A.1)$$

can be rewritten in terms of the real and imaginary parts

$$\gamma_{2j-1} \equiv c_j^\dagger + c_j$$
$$\gamma_{2j} \equiv i(c_j^\dagger - c_j) \quad (A.2)$$

that correspond to the $(2N)$ Majorana operators of equations (1)–(3) by the simple substitution

$$c_j = \frac{\gamma_{2j-1} + i\gamma_{2j}}{2}$$
$$c_j^\dagger = \frac{\gamma_{2j-1} - i\gamma_{2j}}{2} \quad (A.3)$$

A.2. Majorana formulation of quantum spin chains

For a chain of $N$ quantum spins described by Pauli matrices, if the Hamiltonian commutes with the total parity

$$P_{\text{tot}} = \prod_{k=1}^N (-\sigma_z^j) \quad (A.4)$$

it can be rewritten via the standard Jordan–Wigner transformation in terms of the $(2N)$ string operators

$$\gamma_{2j-1} \equiv \left(\prod_{k=1}^{j-1} \sigma_z^k\right) \sigma_x^j$$
$$\gamma_{2j} \equiv \left(\prod_{k=1}^{j-1} \sigma_z^k\right) \sigma_y^j \quad (A.5)$$

that correspond to the $(2N)$ Majorana operators of equations (1)–(3).

For instance, the simplest local terms commuting with the total parity have for translation
\[ \sigma_j^x = -i \gamma_{j-1} \gamma_j \]
\[ \sigma_j^y \sigma_{j+1}^y = -i \gamma_{j+1} \gamma_j \]
\[ \sigma_j^z \sigma_{j+1}^y = i \gamma_{j+2} \gamma_j \]
\[ \sigma_j^x \sigma_{j+1}^z = -i \gamma_{j-2} \gamma_{j+1} \gamma_j \gamma_{j+2} \]
\[ \sigma_j^y \sigma_{j+2}^z = -i \gamma_{j+2} \gamma_{j+1} \gamma_{j+2} \gamma_{j+3} \]  \quad (A.6)

In particular, the random transverse field Ising chain (RTFIC) translates into the random Kitaev chain of equation (27)

\[ H^{\text{RTFIC}} = - \sum_{j=1}^{N} h_j \sigma_j^z - \sum_{j=1}^{N-1} J_j^x \sigma_j^x \sigma_{j+1}^x \]
\[ = i \sum_{j=1}^{2N-1} K_j^{(2)} \gamma_j \gamma_{j+1} = H^{\text{Kitaev}} \]  \quad (A.7)

with the correspondence

\[ h_j = K_j^{(2)} \gamma_j \gamma_{j+1}, \]
\[ J_j^x = K_j^{(2)} \gamma_j \gamma_{j+1}. \]  \quad (A.8)

The well-known duality between fields \( h_j \) and couplings \( J_j^x \) thus becomes obvious in the Majorana language where they correspond to odd and even two-Majorana-couplings respectively.

The additional interactions between four consecutive Majorana operators of the Hamiltonian \( \mathcal{H}^{(4)} \) of equation (42) translates into

\[ \mathcal{H}^{(4)} = - \sum_{j=1}^{2N-3} K_j^{(4)} \gamma_j \gamma_{j+1} \gamma_{j+2} \gamma_{j+3} \]
\[ = \sum_{j=1}^{N-1} K_j^{(4)} \gamma_j \gamma_{j+1} \gamma_{j+2} \gamma_{j+3} + \sum_{j=1}^{N-2} K_j^{(4)} \gamma_j \gamma_{j+1} + \sum_{j=1}^{N-2} K_j^{(4)} \gamma_j \gamma_{j+1} \gamma_{j+2}. \]  \quad (A.9)

The first term in \( \sigma_j^x \sigma_{j+1}^y \) is the standard nearest-neighbor interaction term in the field of quantum spin chains, while the second term \( \sigma_j^z \sigma_{j+2}^z \) between next-nearest-neighbor is less usual but nevertheless interesting to consider, as discussed in [48, 49] for the case of pure Majorana models.

More generally, the Hamiltonian \( \mathcal{H}^{(2k)} \) of equation (40) involving the consecutive parity operators of equation (39) reads in the spin language

\[ \mathcal{H}^{(2k)} = \sum_{j=1}^{2N-2k+1} K_j^{(2k)} \gamma_j \gamma_{j+1} \gamma_{j+2} \gamma_{j+2k-1} \]
\[ = (-1)^k \sum_{j=1}^{N+1-k} K_j^{(2k)} \gamma_j \gamma_{j+1} \gamma_{j+2} \gamma_{j+2k-1} + (-1)^k \sum_{j=1}^{N-k} K_j^{(2k)} \gamma_j \gamma_{j+1} \gamma_{j+2} \gamma_{j+2k-1} \]  \quad (A.10)

where the first term involves \( k \) consecutive Pauli matrices \( \sigma^x \), while the second term involves only two Pauli matrices \( \sigma^x \) separated by the distance \( k \).
As a final remark, let us mention that the Jordan–Wigner transformation of equation (A.5) is of course specific to one dimension, but for certain bidimensional quantum spin models, other relations have been introduced between quantum spins and Majorana fermions [50–52].

ORCID iDs

Cécile Monthus https://orcid.org/0000-0002-4356-4493

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