Consistency Conditions On The S-Matrix Of Massless Particles

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Abstract

We introduce a set of consistency conditions on the S-matrix of theories of massless particles of arbitrary spin in four-dimensional Minkowski space-time. We find that in most cases the constraints, derived from the conditions, can only be satisfied if the S-matrix is trivial. Our conditions apply to theories where four-particle scattering amplitudes can be obtained from three-particle ones via a recent technique called BCFW construction. We call theories in this class constructible.

We propose a program for performing a systematic search of constructible theories that can have non-trivial S-matrices. As illustrations, we provide simple proofs of already known facts like the impossibility of spin $s > 2$ non-trivial S-matrices, the impossibility of several spin 2 interacting particles and the uniqueness of a theory with spin 2 and spin 3/2 particles.

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I. INTRODUCTION

The power of the constraints that Lorentz invariance imposes on the S-matrix of four dimensional theories has been well known at least since the work of Weinberg \[1, 2\]. Impressive results like the impossibility of long-range forces mediated by massless particles with spin \(> 2\), charge conservation in interactions mediated by a massless spin 1 particle, or the universality of the coupling to a massless spin 2 particle are examples beautifully obtained by simply using the pole structure of the S-matrix governing soft limits in combination with Lorentz invariance \[2, 3\].

Weinberg’s argument does not rule out the possibility of non-trivial Lagrangians describing self-interacting massless particles of higher spins. It rules out the possibility of those fields producing macroscopic effects. Actually, the theory of massless particles of higher spins has been an active research area for many years (see reviews \[4, 5\] and reference therein, also see \[6, 7\] for alternative approaches). Lagrangians for free theories have been well understood while interactions have been a stumbling block. Recent progress shows that in spaces with negative cosmological constant it is possible to construct consistent Lagrangian theories but no similar result exists for flat space-time \[8, 9\]. Despite the difficulties of constructing an interactive Lagrangian, several attempts have been made in studying the consistency of specific couplings among higher spin particles. For example, cubic interactions have been studied in \[10, 11, 12, 13\]. Also, very powerful techniques for constructing interaction vertices systematically have been developed using BRST-BV cohomological methods \[14, 15, 16\] and references therein.

In this paper we introduce a technique for finding theories of massless particles that can have non-trivial S-matrices within a special set of theories we call constructible. The starting point is always assuming a Poincaré covariant theory where the S-matrix transformation is derived from that of one-particle states which are irreducible representations of the Poincaré group. There will also be implicit assumptions of locality and parity invariance.

The next step is to show that for complex momenta, on-shell three-particle S-matrices of massless particles of any spin can be uniquely determined. As is well known, on-shell three-particle amplitudes vanish in Minkowski space. That this need not be the case for amplitudes in signatures different from Minkowski or for complex momenta was explained by Witten in \[17\].
We consider theories for which four-particle tree-level S-matrix elements can be completely determined by three-particle ones. These theories are called constructible. This is done by introducing a one parameter family of complex deformations of the amplitudes and using its pole structure to reconstruct it. The physical singularities are on-shell intermediate particles connecting physical on-shell three-particle amplitudes. This procedure is known as the BCFW construction [18, 19]. One can also introduce the terminology fully constructible if this procedure can be extended to all \( n \)-particle amplitudes. Examples of fully constructible theories are Yang-Mills [19] and General Relativity [20, 21, 22] (the fact that cubic couplings could play a key role in Yang-Mills theory and General Relativity was already understood in [23, 24]).

The main observation is that by using the BCFW deformation, the four-particle amplitude is obtained by summing over only a certain set of channels, say the \( s \)- and the \( u \)-channels. However, if the theory under consideration exists, then the answer should also contain the information about the \( t \)-channel. In particular, one could construct the four-particle amplitude using a different BCFW deformation that sums only over the \( t \)- and the \( u \)-channel.

Choosing different deformations for constructing the same four-particle amplitude and requiring the two answers to agree is what we call the four-particle test. This simple consistency condition turns out to be a powerful constraint that is very difficult to satisfy.

It is important to mention that the constraints are only valid for constructible theories. Luckily, the set of constructible theories is large and we find many interesting results. We also discuss some strategies for circumventing this limitation.

As illustrations of the simplicity and power of the four-particle test we present several examples. The first is a general analysis of theories of a single spin \( s \) particle. We find that if \( s > 0 \) all theories must have a trivial S-matrix except for \( s = 2 \) which passes the test. As a second example we allow for several particles of the same spin. We find that, again in the range \( s > 0 \), the only theories that can have a nontrivial S-matrix are those of spin 1 with completely antisymmetric three-particle coupling constants which satisfy the Jacobi identity and spin 2 particles with completely symmetric three-particle coupling constants which define a commutative and associative algebra. We also study the possible theories of particles of spin \( s \), without self-couplings and with \( s > 1 \), that can couple non-trivially to a spin 2 particle. In this case, we find that only \( s = 3/2 \) passes the test. Moreover, all
couplings in the theory must be related to that of the three-spin-2 particle amplitude. Such a theory is linearized $\mathcal{N} = 1$ supergravity.

The paper is organized as follows. In section II, we review the construction of the S-matrix and of scattering amplitudes for massless particles. In section III, we discuss how three-particle amplitudes are non-zero and uniquely determined up the choice of the values of the coupling constant. In section IV, we apply the BCFW construction to show how, for certain theories, four-particle amplitudes can be computed from three-particle ones. A theory for which this is possible is called constructible. We then introduce the four-particle test. In section V, we discuss sufficient conditions for a theory to be constructible. In section VI, we give examples of the use of the four-particle test. In section VII, we conclude with a discussion of possible future directions including how to relax the constructibility constraint. Finally, in the appendix we illustrate one of the methods to relax the constructibility condition.

II. PRELIMINARIES

A. S-Matrix

In this section we define the S-matrix and scattering amplitudes. We do this in order to set up the notation. Properties of the S-matrix, which we exploit in this paper like factorization, have been well understood since at least the time of the S-matrix program \cite{25, 26, 27}.

Recall that physically, one is interested in the probability for, say, two asymptotic states to scatter and to produce $n - 2$ asymptotic states. Any such probability can be computed from the matrix elements of momentum eigenstates

\begin{equation}
\langle p_1 \ldots p_{n-2} | p_a p_b \rangle_{\text{in}} = \langle p_1 \ldots p_{n-2} | S | p_a p_b \rangle
\end{equation}

where $S$ is a unitary operator. As usual, it is convenient to write $S = \mathbb{1} + iT$ with

\begin{equation}
\langle p_1 \ldots p_{n-2} | iT | p_a p_b \rangle = \delta^{(4)} \left( p_a + p_b - \sum_{i=1}^{n-2} p_i \right) M(p_a, p_b \rightarrow \{p_1, p_2, \ldots, p_{n-2}\}).
\end{equation}

$M(p_a, p_b \rightarrow \{p_1, p_2, \ldots, p_{n-2}\})$ is called the scattering amplitude (see for example chapter 4 in \cite{28}).
Assuming crossing symmetry one can write \( p_a = -p_{n-1} \) and \( p_b = -p_n \) and introduce a scattering amplitude where all particles are outgoing. Different processes are then obtained by analytic continuation of

\[
M_n = M_n(p_1, p_2, \ldots, p_{n-1}, p_n).
\]

(3)

\( M_n \) is our main object of study. Our goal is to determine when \( M_n \) can be non-zero. Up to now we have exhibited only the dependence on momenta of external particles. However, if they have spin \( s > 0 \) one also has to specify their free wave functions or polarization tensors. We postpone the discussion of the explicit form of polarization tensors until section V.

\section*{B. Massless Particles Of Spin \( s \)}

It turns out that all the information needed to describe the physical information of an on-shell massless spin \( s \) particle is contained in a pair of spinors \( \{\lambda_a, \tilde{\lambda}_a\} \), left- and right-handed respectively, and the helicity of the particle \([17, 29, 30, 31]\). Recall that in a Poincaré invariant theory, irreducible massless representations are classified by their helicity which can be \( h = \pm s \) with \( s \) any integer or half-integer known as the spin of the particle.

The spinors \( \{\lambda_a, \tilde{\lambda}_a\} \) transform in the representations \((1/2, 0)\) and \((0, 1/2)\) of the universal cover of the Lorentz group, \( SL(2, \mathbb{C}) \), respectively. Invariant tensors are \( \epsilon^{ab}, \epsilon^\dot{a}\dot{b} \) and \((\sigma^\mu)_{a\dot{a}}\) where \( \sigma^\mu = (\bar{1}, \bar{\sigma}) \). The most basic Lorentz invariants, from which any other is made of, can be constructed as follows:

\[
\lambda_a \lambda'_b \epsilon^{ab} \equiv \langle \lambda, \lambda' \rangle, \quad \tilde{\lambda}_a \tilde{\lambda}'_b \epsilon^{\dot{a}\dot{b}} \equiv [\lambda, \lambda'].
\]

(4)

Finally, using the third invariant tensor we can define the momentum of the particle by \( p^\mu = \lambda^a (\sigma^\mu)_{a\dot{a}} \tilde{\lambda}^\dot{a} \), where indices are raised using the first two tensors. A simple consequence of this is that the scalar product of two vectors, \( p^\mu \) and \( q^\mu \) is given by \( 2p \cdot q = \langle \lambda^\mu, \lambda'^\mu \rangle [\tilde{\lambda}^\mu, \tilde{\lambda}'^\mu] \).

\section*{III. THREE PARTICLE AMPLITUDES: A UNIQUENESS RESULT}

In this section we prove that three-particle amplitudes of massless particles of any spin can be uniquely determined.
The statement that on-shell scattering amplitudes of three massless particles can be non-zero might be somewhat surprising. However, as shown by Witten [17], three-particle amplitudes are naturally non-zero if we choose to work with the complexified Lorentz group $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$, where $(1/2, 0)$ and $(0, 1/2)$ are completely independent representations and hence momenta are not longer real. In other words, if $\tilde{\lambda}_\alpha \neq \pm \bar{\lambda}_\alpha$ then $p^\mu$ is complex.

Let us then consider a three-particle amplitude $M_3(\{\lambda^{(i)}, \tilde{\lambda}^{(i)}, h_i\})$ where the spinors of each particle, $\lambda^{(i)}$ and $\tilde{\lambda}^{(i)}$, are independent vectors in $\mathbb{C}^2$.

Momentum conservation $(p_1 + p_2 + p_3)_{\dot{a}\dot{a}} = 0$ and the on-shell conditions, $p_i^2 = 0$, imply that $p_i \cdot p_j = 0$ for any $i$ and $j$. Therefore we have the following set of equations

$$\langle 1, 2 \rangle [1, 2] = 0, \quad \langle 2, 3 \rangle [2, 3] = 0, \quad \langle 3, 1 \rangle [3, 1] = 0.$$  \hspace{1cm} (5)

Clearly, if $[1, 2] = 0$ and $[2, 3] = 0$ then $[3, 1]$ must be zero. The reason is that the spinors live in a two dimensional vector space and if $\tilde{\lambda}^{(1)}$ and $\tilde{\lambda}^{(3)}$ are proportional to $\tilde{\lambda}^{(2)}$ then they must also be proportional.

This means that the non-trivial solutions to (5) are either $\langle 1, 2 \rangle = \langle 2, 3 \rangle = \langle 3, 1 \rangle = 0$ or $[1, 2] = [2, 3] = [3, 1] = 0$.

Take for example $[1, 2] = [2, 3] = [3, 1] = 0$ and set $\tilde{\lambda}_a^{(2)} = \alpha_2 \tilde{\lambda}_a^{(1)}$ and $\tilde{\lambda}_a^{(3)} = \alpha_3 \tilde{\lambda}_a^{(1)}$. Then momentum conservation implies that $\lambda_a^{(1)} + \alpha_2 \lambda_a^{(2)} + \alpha_3 \lambda_a^{(3)} = 0$ which is easily seen to be satisfied if $\alpha_2 = -\langle 1, 3 \rangle / \langle 2, 3 \rangle$ and $\alpha_3 = -\langle 1, 2 \rangle / \langle 3, 2 \rangle$.

The conclusion of this discussion is that three-particle amplitudes, $M_3(\{\lambda^{(i)}, \tilde{\lambda}^{(i)}, h_i\})$, which by Lorentz invariance are only restricted to be a generic function of $\langle i, j \rangle$ and $[i, j]$ turn out to split into a “holomorphic” and an “anti-holomorphic” part\(^1\). More explicitly

$$M_3 = M_3^H(\langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 3, 1 \rangle) + M_3^A([1, 2], [2, 3], [3, 1]).$$ \hspace{1cm} (6)

It is important to mention that we are considering the full three-particle amplitude and not just the tree-level one. Therefore $M_3^H$ and $M_3^A$ are not restricted to be rational functions\(^2\). In other words, we have purposefully avoided to talk about perturbation theory. We will be forced to do so later in section V but we believe that this discussion can be part of a more general analysis.

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\(^1\) Using “holomorphic” and “anti-holomorphic” is an abuse of terminology since $\tilde{\lambda}_\alpha \neq \pm \bar{\lambda}_\alpha$. We hope this will not cause any confusion.

\(^2\) We thank L. Freidel for discussions on this point.
A. Helicity Constraint and Uniqueness

One of our basic assumptions about the S-matrix is that the Poincaré group acts on the scattering amplitudes as it acts on individual one-particle states. This in particular means that the helicity operator must act as

\[(\lambda_i \frac{\partial}{\partial \lambda_i} - \tilde{\lambda}_i \frac{\partial}{\partial \tilde{\lambda}_i}) M_3(h_1, h_2, h_3) = -2h_i M_3(h_1, h_2, h_3).\]  

(7)

Equivalently,

\[(\lambda_i \frac{\partial}{\partial \lambda_i} + 2h_i) M_3^H(\{1, 2\}, \{2, 3\}, \{3, 1\}) = 0\]  

(8)

on the holomorphic one and as

\[(\tilde{\lambda}_i \frac{\partial}{\partial \tilde{\lambda}_i} - 2h_i) M_3^A([1, 2], [2, 3], [3, 1]) = 0\]  

(9)

on the anti-holomorphic one.

It is not difficult to show that if

\[d_1 = h_1 - h_2 - h_3, \quad d_2 = h_2 - h_3 - h_1, \quad d_3 = h_3 - h_1 - h_2,\]

then

\[F = \langle 1, 2 \rangle^{d_3} \langle 2, 3 \rangle^{d_1} \langle 3, 1 \rangle^{d_2}, \quad G = [1, 2]^{-d_3} [2, 3]^{-d_1} [3, 1]^{-d_2}\]  

(10)

are particular solutions of the equations (8) and (9) respectively.

Therefore, \(M_3^H/F\) and \(M_3^A/G\) must be “scalar” functions, i.e., they have zero helicity.

Let \(x_1\) be either \(\langle 2, 3 \rangle\) or \([2, 3]\) depending on whether we are working with the holomorphic or the antiholomorphic pieces. Also let \(x_2\) be either \(\langle 3, 1 \rangle\) or \([3, 1]\) and \(x_3\) be either \(\langle 1, 2 \rangle\) or \([1, 2]\). Finally, let \(\mathcal{M}\) be either \(M_3^H/F\) or \(M_3^A/G\). Then we find that

\[x_i \frac{\partial \mathcal{M}(x_1, x_2, x_3)}{\partial x_i} = 0\]  

(11)

for \(i = 1, 2, 3\). Therefore, up to solutions with delta function support which we discard based on analyticity, the only solution for \(\mathcal{M}\) is a constant. Let such a constant be denoted by \(\kappa_H\) or \(\kappa_A\) respectively.

We then find that the exact three-particle amplitude must be

\[M_3(\{\lambda^{(i)}, \tilde{\lambda}^{(i)}, h_i\}) = \kappa_H \langle 1, 2 \rangle^{d_3} \langle 2, 3 \rangle^{d_1} \langle 3, 1 \rangle^{d_2} + \kappa_A [1, 2]^{-d_3} [2, 3]^{-d_1} [3, 1]^{-d_2}.\]  

(12)

Finally, we have to impose that \(M_3\) has the correct physical behavior in the limit of real momenta. In other words, we must require that \(M_3\) goes to zero when both \(i, j\) and
\([i, j]\) are taken to zero\(^3\). Simple inspection shows that if \(d_1 + d_2 + d_3\), which is equal to 
\(-h_1 - h_2 - h_3\), is positive then we must set \(\kappa_A = 0\) in order to avoid an infinity while if 
\(-h_1 - h_2 - h_3\) is negative then \(\kappa_H\) must be zero. The case when \(h_1 + h_2 + h_3 = 0\) is more 
subtle since both pieces are allowed. In this paper we restrict our study to \(h_1 + h_2 + h_3 \neq 0\) and leave the case \(h_1 + h_2 + h_3 = 0\) for future work.

**B. Examples**

Let us consider few examples, which will appear in the next sections, as illustrations of 
the uniqueness of three-particle amplitudes.

Consider a theory of several particles of a given integer spin \(s\). Since all particles have the 
same spin we can replace \(h = \pm s\) by the corresponding sign. Let us use the middle letters 
of the alphabet to denote the particle type.

There are only four helicity configurations:

\[
M_3(1^-_m, 2^-_r, 3^+_s) = \kappa_{mrs} \left( \frac{\langle 1, 2 \rangle^3}{\langle 2, 3 \rangle \langle 3, 1 \rangle} \right)^s, \quad M_3(1^+_m, 2^+_r, 3^-_s) = \kappa_{mrs} \left( \frac{[1, 2]^3}{[2, 3][3, 1]} \right)^s \tag{13}
\]

and

\[
M_3(1^-_m, 2^-_r, 3^-_s) = \kappa'_{mrs} (\langle 1, 2 \rangle \langle 2, 3 \rangle \langle 3, 1 \rangle)^s, \quad M_3(1^+_m, 2^+_r, 3^+_s) = \kappa'_{mrs} ([1, 2][2, 3][3, 1])^s \tag{14}
\]

The subscripts on the coupling constants \(\kappa\) and \(\kappa'\) mean that they can depend on the particle 
type\(^4\). We will use the amplitudes in (13) in section VI.

A simple but important observation is that if the spin is odd then the coupling constant 
must be completely antisymmetric in its indices. This is because due to crossing symmetry 
the amplitude must be invariant under the exchange of labels.

This leads to our first result, a theory of less than three massless particles of odd spin 
must have a trivial three-particle S-matrix. Under the conditions of constructibility, this 
can be extended to higher-particle sectors of the S-matrix and even to the full S-matrix.

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\(^3\) Taking to zero \(\langle i, j \rangle\) means that \(\lambda^{(i)}\) and \(\lambda^{(j)}\) are proportional vectors. Therefore, all factors \(\langle i, j \rangle\) can be 
taken to be proportional to the same small number \(\epsilon\) which is then taken to zero.

\(^4\) Note that here we have implicitly assumed parity invariance by equating the couplings of conjugate 
amplitudes.
IV. THE FOUR-PARTICLE TEST AND CONSTRUCTIBLE THEORIES

In this section we introduce what we call the four-particle test. Consider a four-particle amplitude $M_4$. Under the assumption that one-particle states are stable in the theory, $M_4$ must have poles and multiple branch cuts emanating from them at locations where either $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$ or $u = (p_3 + p_1)^2$ vanish.\(^5\)

We choose to consider only the pole structure. Branch cuts will certainly lead to very interesting constraints but we leave this for future work. Restricting to the pole structure corresponds to working at tree-level in field theory.

As we will see, under certain conditions, one can construct physical on-shell tree-level four-particle amplitudes as the product of two on-shell three-particle amplitudes (evaluated at complex momenta constructed out of the real momenta of the four external particles) times a Feynman propagator. In general this can be done in at least two ways. Roughly speaking, these correspond to summing over the $s$-channel and $u$-channel or summing over the $t$-channel and $u$-channel. A necessary condition for the theory to exists is that the two four-particle amplitudes constructed this way give the same answer. This is what we call the four-particle test. It might be surprising at first that a sum over the $s$- and $u$-channels contains information about the $t$-channel but as we will see this is a natural consequence of the BCFW construction which we now review.

A. Review Of The BCFW Construction And Constructible Theories

The key ingredient for the four-particle test is the BCFW construction [19]. The construction can be applied to $n$-particle amplitudes, but for the purpose of this paper we only need four-particle amplitudes.

We want to study $M_4(\{\lambda^{(i)}_a, \tilde{\lambda}^{(i)}_\dot{a}, \lambda^{(j)}_a, \tilde{\lambda}^{(j)}_\dot{a}, h_i\})$. Recall that momenta constructed from the spinors of each particle are required to satisfy momentum conservation, i.e., $(p_1 + p_2 + p_3 + p_4)^\mu = 0$.

Choose two particles, one of positive and one of negative helicity\(^6\), say $i^{+s_i}$ and $j^{-s_j}$,

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\(^5\) We have introduced the notation $s$ for the center of mass energy in order to avoid confusion with the spin $s$ of the particles.

\(^6\) Here we do not consider amplitudes with all equal helicities.
where \( s_i \) and \( s_j \) are the corresponding spins, and perform the following deformation

\[
\lambda^{(i)}(z) = \lambda^{(i)} + z\lambda^{(j)}, \quad \bar{\lambda}^{(j)}(z) = \bar{\lambda}^{(j)} - z\bar{\lambda}^{(i)}. \tag{15}
\]

All other spinors remain the same.

The deformation parameter \( z \) is a complex variable. It is easy to check that this deformation preserves the on-shell conditions, i.e., \( p_k(z)^2 = 0 \) for any \( k \) and momentum conservation since \( p_i(z) + p_j(z) = p_i + p_j \).

The main observation is that the scattering amplitude is a rational function of \( z \) which we denote by \( M_4(z) \). This fact follows from \( M_4(1^{h_1}, \ldots, 4^{h_4}) \) being, at tree-level, a rational function of spinor products. Being a rational function of \( z \), \( M_4(z) \) can be determined if complete knowledge of its poles, residues and behavior at infinity is found.

**Definition:** We call a theory *constructible* if \( M_4(z) \) vanishes at \( z = \infty \). As we will see this means that \( M_4(z) \) can only be computed from \( M_3 \) and hence the name.

In the next section we study sufficient conditions for a theory to be constructible. The proof of constructibility relies very strongly on the fact that on-shell amplitudes should only produce the two physical helicity states of a massless particle\(^7\). In this section we assume that the theory under consideration is constructible.

Any rational function that vanishes at infinity can be written as a sum over its poles with the appropriate residues. In the case at hand, \( M_4(z) \) can only have poles of the form

\[
\frac{1}{(p_i(z) + p_k)^2} = \frac{1}{\langle \lambda^{(i)}(z), \lambda^{(k)} \rangle[i,k]} = \frac{1}{\langle \langle i,k \rangle + z\langle j,k \rangle \rangle[i,k]} \tag{16}
\]

where \( k \) has to be different from \( i \) and \( j \).

As mentioned at the beginning of this section, \( M_4(z) \) can be constructed as a sum over only two of the three channels. The reason is the following. For definiteness let us set \( i = 1 \) and \( j = 2 \), then the only propagators that can be \( z \)-dependent are \( 1/(p_1(z) + p_4)^2 \) and \( 1/(p_1(z) + p_3)^2 \). By construction \( 1/(p_1 + p_2)^2 \) is \( z \)-independent.

The rational function \( M_4(z) \) can thus be written as

\[
M_4^{(1,2)}(z) = \frac{c_t}{z - z_t} + \frac{c_u}{z - z_u} \tag{17}
\]

\(^7\) This in turn is simply a consequence of imposing Lorentz invariance.\[2\].
where \( z_t \) is such that \( t = (p_1(z) + p_4)^2 \) vanishes, i.e., \( z_t = -\langle 1, 4 \rangle /\langle 2, 4 \rangle \) while \( u = (p_1(z) + p_3)^2 \) vanishes, i.e., \( z_u = -\langle 1, 3 \rangle /\langle 2, 3 \rangle \). Note that we have added the superscript \((1, 2)\) to \( M_4(z) \) to indicate that it was obtained by deforming particles 1 and 2.

Finally, we need to compute the residues. Close to the location of one of the poles, \( M_4(z) \) factorizes as the product of two on-shell three-particle amplitudes. Note that each of the three-particle amplitudes is on-shell since the intermediate particle is also on-shell. See figure 1 for a schematic representation. Therefore, we find that

\[
M_4^{(1,2)}(z) = \sum_h M_3(p_1^{h_1}(z_t), p_4^{h_4}, -P_{1,4}^h(z_t)) \frac{1}{P_{1,4}^2(z)} M_3(p_2^{h_2}(z_t), p_3^{h_3}, P_1^{h_4}(z_t)) + \\
\sum_h M_3(p_1^{h_1}(z_u), p_3^{h_3}, -P_{1,3}^h(z_u)) \frac{1}{P_{1,3}^2(z)} M_3(p_2^{h_2}(z_u), p_4^{h_4}, P_1^{h_3}(z_u)),
\]

(18)

where the sum over \( h \) runs over all possible helicities in the theory under consideration and also over particle types if there is more than one.

The scattering amplitude we are after is simply obtained by setting \( z = 0 \), i.e, \( M_4(\{\lambda^{(i)}, \bar{\lambda}^{(i)}, h_i\}) = M_4^{(1,2)}(0) \).

Recall that we assumed \( h_1 = s_1 \) and \( h_2 = -s_2 \). Let us further assume that \( h_4 = -s_4 \). Therefore we could repeat the whole procedure but this time deforming particles 1 and 4. In this way we should find that \( M_4(\{\lambda^{(i)}, \bar{\lambda}^{(i)}, h_i\}) = M_4^{(1,4)}(0) \).

We have finally arrived at the consistency condition we call the four-particle test. One has to require that

\[
M_4^{(1,2)}(0) = M_4^{(1,4)}(0).
\]

(19)

As we will see in examples, this is a very strong condition that very few constructible theories satisfy non-trivially. In other words, most constructible theories satisfy (19) only if all three-particle couplings are set to zero and hence four-particle amplitudes vanish. If the theory is fully constructible, this implies that the whole S-matrix is trivial.

**B. Simple Examples**

We illustrate the use of the four-particle test by first working out the general form of \( M_4^{(1,2)}(0) \) and \( M_4^{(1,4)}(0) \) for a theory containing only integer spin particles\(^8\). We then specialize to the case of a theory containing a single particle of integer spin \( s \). It turns out that

\(^8\) Including half-integer spins is straightforward and we give an example in section VI.
\[ M_{4}^{(1,2)} = \sum_{h} \frac{1}{P_{14}^{2}} \left( \kappa_{(1+h_{1}+h_{4}+h)}^{H} \langle 1, 4 \rangle^{h_{1}+h_{4}} \langle 4, \hat{P}_{1,4}, 1 \rangle^{h_{4}} \right) + \sum_{h} \frac{1}{P_{13}^{2}} \left( \kappa_{(1+h_{2}+h_{3}+h)}^{H} \langle 3, 2 \rangle^{h_{2}+h_{3}+h} \langle 2, \hat{P}_{1,4}, 3 \rangle^{h_{2}+h_{3}+h} \right) + \sum_{h} \langle 4 \leftrightarrow 3 \rangle. \]

Here the subscripts on the three-particle couplings denote the dimension of the coupling. The range of values of the helicity of the internal particle depends on the details of the specific theory under consideration. Even though (20) is completely general we choose to
exclude theories where $h$ can take values such that $h + h_1 + h_2 = 0$ or $-h + h_2 + h_3 = 0$. The main reason is that formulas will simplify under this assumption.

Note also that we have kept the two pieces of all three-particle amplitudes entering in (20). However, recall that we should set either the holomorphic or the anti-holomorphic coupling to zero. As we will now see this condition is very important for the consistency of (20).

Let us solve the condition $P_{1,4}(z)^2 = 0$. As mentioned above this leads to $z_t = -\langle 1, 4 \rangle / \langle 2, 4 \rangle$. Since $P_{1,4}(z_t)$, which we denoted by $\hat{P}_{1,4}$, is a null vector, it must be possible to find spinors $\lambda^{(P)}$ and $\bar{\lambda}^{(P)}$ such that $\hat{P}_{1,4}^a = \lambda^{(P)a} (\sigma^\mu)_{ab} \bar{\lambda}^{(P)b}$. Clearly, given $\hat{P}_{1,4}$ it is not possible to uniquely determine the spinors since any pair of spinors $\{t\lambda^{(P)}, t^{-1}\bar{\lambda}^{(P)}\}$ gives rise to the same $\hat{P}_{1,4}$. This ambiguity drops out of (20) as we will see.

After some algebra we find that

$$P_{1,4}(z_t) = \hat{P}_{1,4} = \left[ \begin{array}{c} 1, 4 \\ 1, 3 \end{array} \right] \lambda^{(4)} \bar{\lambda}^{(3)}.$$ (21)

Therefore we can choose

$$\lambda^{(P)} = \alpha \lambda_4, \quad \bar{\lambda}^{(P)} = \beta \bar{\lambda}_3, \quad \text{with} \quad \alpha \beta = \left[ \begin{array}{c} 1, 4 \\ 1, 3 \end{array} \right].$$ (22)

Moreover, it is also easy to get

$$\hat{\lambda}_1 = \left[ \begin{array}{c} 2, 1 \\ 2, 4 \end{array} \right] \lambda_4, \quad \hat{\lambda}_2 = \left[ \begin{array}{c} 1, 2 \\ 1, 3 \end{array} \right] \bar{\lambda}_3.$$ (23)

Using the explicit form of all the spinors one can check that the three-particle amplitude with coupling constant $\kappa_{(1+h_1+h_4+h)}^H$ in (20) possesses a factor of the form $\langle 4, 4 \rangle = 0$ to the power $-h_1 - h_4 - h$. From our discussion in section III, if $-h_1 - h_4 - h$ is less than zero then the coupling $\kappa_{(1+h_1+h_4+h)}^H = 0$. In this way a possible infinity is avoided. Therefore we get a contribution from the term with coupling $\kappa_{(1+h_1-h_4-h)}^A$ whenever $h > -(h_1 + h_4)$.

Now, if $-h_1 - h_4 - h$ is positive then $\kappa_{(1+h_1+h_4+h)}^H$ need not vanish but the factor multiplying it vanishes. In this case $\kappa_{(1+h_1-h_4-h)}^A$ must be zero and we find no contributions. This means that the only non-zero contributions to the sum over $h$ can only come from the region where $h > -(h_1 + h_4)$.

Turning to the other three-particle amplitude, we find that the piece with coupling $\kappa_{(1-h_2+h_3+h)}^A$ has a factor $[3, 3] = 0$ to the power $-h_2 + h_3$. A similar analysis shows that the only non-zero contributions come from regions where $h > (h_2 + h_3)$. 

13
Putting the two conditions together we find that the first term gives a non-zero contribution only when \( h > \max(-(h_1 + h_4), (h_2 + h_3)) \).

Simplifying we find

\[
M^{(1,2)} (0) = \sum_{h > \max(-(h_1 + h_4), (h_2 + h_3))} \left( \kappa^A_{1-h_1-h_4} \kappa^H_{1+h_2+h_3-h} \frac{(-P_{3,4}^2)^h}{P_{1,4}^2} \right) \left( \frac{[1,4][3,4]}{[1,3]} \right) ^{h_1} \left( \frac{\langle 3,4 \rangle}{\langle 2,3 \rangle \langle 2,4 \rangle} \right) ^{h_2} \left( \frac{\langle 2,4 \rangle}{\langle 2,3 \rangle \langle 3,4 \rangle} \right) ^{h_3} + \sum_{h > \max(-(h_1 + h_3), (h_2 + h_4))} (4 \leftrightarrow 3).
\]

Finally, it is easy to obtain \( M^{(1,4)} (0) \) from (24) by simply exchanging the labels 2 and 4.

Next we will write down all formulas explicitly for the case when \( |h_i| = s \) for all \( i \).

2. Theories Of A Single Spin \( s \) Particle

Consider now the case \( h_1 = s, h_2 = -s, h_3 = s \) and \( h_4 = -s \). We also assume that the theory under consideration has a single particle of spin \( s \). This restriction is again for simplicity. If one decided to allow for more internal particles then the different terms would have to satisfy the four-particle test independently since the dimensions of the coupling constants would be different.

Using (24) we find that the first sum contributing to \( M^{(1,2)} (0) \) allows only for \( h = s \) while the second one allows for \( h = -s \) and \( h = s \). Using momentum conservation to simplify the expressions we find

\[
M^{(1,2)} (0) = \kappa^A_{1-s} \kappa^H_{1-s} \left( \frac{\langle 2,4 \rangle^{3}[1,3]}{\langle 1,2 \rangle \langle 3,4 \rangle} \right) ^s \frac{1}{\langle 1,4 \rangle \langle 1,3 \rangle} + \kappa^A_{1-s} \kappa^H_{1-s} \left( \frac{[1,3]^3 \langle 4,2 \rangle}{[4,3][1,2]} \right) ^s \frac{1}{\langle 1,3 \rangle \langle 1,3 \rangle} + \kappa^A_{1-s} \kappa^H_{1-s} \left( \frac{[1,3][3,4]}{[1,3][1,3]} \right) ^{2s} \frac{(-P_{3,4}^2)^{4s}}{P_{1,3}^2}.
\]

We would like to set all couplings with the same dimension to the same value. In other words, we define \( \kappa = \kappa^A_{1-s} = \kappa^H_{1-s} \). We also choose to study the case \( \kappa' = \kappa^A_{1-3s} = \kappa^H_{1-3s} = 0 \).

---

9 There might be cases where the dimensions might agree by accident. Such cases might actually lead to new interesting theories. We briefly elaborate in section VII but we leave the general analysis for future work.

10 One can easily show that momentum conservation for four particles implies that \( \langle a, b \rangle / \langle a, c \rangle = -[d, c] / [d, b] \) for any choice of \( \{a, b, c, d\} \).
It turns out that if we had chosen $\kappa = 0$ and $\kappa'$ non-zero the resulting theories would not have been constructible. In section VII we explore strategies for relaxing this condition.

As mentioned above we can write $M^{(1,4)}(0)$ by simply exchanging the labels 2 and 4. We then find

$$
M^{(1,2)}(0) = \kappa^2 \left( \frac{\langle 2,4 \rangle^3 [1,3]}{\langle 1,2 \rangle \langle 3,4 \rangle} \right)^s \frac{1}{\langle 1,4 \rangle [1,4]} + \kappa^2 \left( \frac{[1,3]^3 [4,2]}{[4,3] [1,2]} \right)^s \frac{1}{[1,3] [1,3]},
$$

$$
M^{(1,4)}(0) = \kappa^2 \left( \frac{\langle 4,2 \rangle^3 [1,3]}{\langle 1,4 \rangle \langle 3,2 \rangle} \right)^s \frac{1}{\langle 1,2 \rangle [1,2]} + \kappa^2 \left( \frac{[1,3]^3 [2,4]}{[2,3] [1,4]} \right)^s \frac{1}{[1,3] [1,3]}.
$$

Both amplitudes can be further simplified to

$$
M^{(1,2)}(0) = -(-1)^s \kappa^2 \frac{[1,3] [2,4]^{2s}}{stu} \times s^{2s}, \quad M^{(1,4)}(0) = -(-1)^s \kappa^2 \frac{[1,3] [2,4]^{2s}}{stu} \times t^{2s}.
$$

Finally, the four-particle test requires $M^{(1,2)}(0) = M^{(1,4)}(0)$ or equivalently $M^{(1,2)}(0)/M^{(1,4)}(0) = 1$. The latter gives the condition $(s/t)^{2s} = 1$ which can only be satisfied for generic choices of kinematical invariants if $s = 2$. If $s \neq 2$ the four-particle test $M^{(1,2)}(0) = M^{(1,4)}(0)$ then requires $\kappa = 0$ and hence a trivial S-matrix.

V. CONDITIONS FOR CONSTRUCTIBILITY

The example in the previous section showed that the only theory of a single massless spin $s$ particle that passes the four-particle test is that with $s = 2$. This theory turns out to be linearized General Relativity. For $s = 1$, the result is also familiar: a single photon should be free. However, if $s = 0$ one knows that a single scalar can have a non-trivial S-matrix. The reason we did not find $s = 0$ as a possible solution in the previous example is that precisely for $s = 0$ the four-particle amplitude is not constructible. Therefore our calculation was valid only for $s > 0$.

In this section we study the criteria for constructibility in more detail. Unfortunately, we do not know a way of carrying out this discussion without first assuming the existence of a Lagrangian. The conditions for constructibility will therefore be given in terms of conditions on the interaction vertices of a Lagrangian. We will also assume that it is possible to perform a perturbative expansion using Feynman diagrams. The starting point of all theories we consider is a canonical kinetic term (free Lagrangian) which for $s = 0, 1, 2$ is very well known and for $s > 2$ can be found for example in [4, 32, 34].
The first ingredient is the polarization tensors of massless particles of spin $s$. Polarization tensors of particles of integer spin $s$ can be expressed in terms of polarization vectors of spin 1 particles as follows:

$$
\epsilon_{a_1 \tilde{a}_1, \ldots, a_s \tilde{a}_s}^+ = \prod_{i=1}^{s} \epsilon_{a_i \tilde{a}_i}^+, \quad \epsilon_{a_1 \tilde{a}_1, \ldots, a_s \tilde{a}_s}^- = \prod_{i=1}^{s} \epsilon_{a_i \tilde{a}_i}^-.
$$

(28)

For half-integer spin $s + 1/2$ they are

$$
\epsilon_{a_1 \tilde{a}_1, \ldots, a_s \tilde{a}_s, b}^+ = \tilde{\lambda}_b \prod_{i=1}^{s} \epsilon_{a_i \tilde{a}_i}^+, \quad \epsilon_{a_1 \tilde{a}_1, \ldots, a_s \tilde{a}_s, b}^- = \lambda_b \prod_{i=1}^{s} \epsilon_{a_i \tilde{a}_i}^-.
$$

(29)

and where polarization vectors of spin 1 particles are given by

$$
\epsilon^+_{a \tilde{a}} = \frac{\mu_a \tilde{\lambda}_{\tilde{a}}}{\langle \mu, \lambda \rangle}, \quad \epsilon^-_{a \tilde{a}} = \frac{\lambda_a \tilde{\mu}_{\tilde{a}}}{[\lambda, \mu]}.
$$

(30)

with $\mu_a$ and $\tilde{\mu}_{\tilde{a}}$ arbitrary reference spinors.

This explains how all the physical data of a massless particle can be recovered from $\lambda, \tilde{\lambda}$ and $h$. A comment is in order here. The presence of arbitrary reference spinors means that polarization tensors cannot be uniquely fixed once $\{\lambda, \tilde{\lambda}, h\}$ is given. If a different reference spinor is chosen, say, $\mu'$ for $\epsilon^+_{a \tilde{a}}$ then

$$
\epsilon^+_{a \tilde{a}}(\mu') = \epsilon^+_{a \tilde{a}}(\mu) + \omega \lambda_a \tilde{\lambda}_{\tilde{a}}
$$

(31)

where

$$
\omega = \frac{\langle \mu', \mu \rangle}{\langle \mu', \lambda \rangle \langle \lambda, \mu \rangle}.
$$

If the particle has helicity $h = 1$ then it is easy to recognize (31) as a gauge transformation and the amplitude must be invariant.

However, one does not have to invoke gauge invariance or assume any new principle. As shown by Weinberg in 
for any spin $s$, the only way to guarantee the correct Poincaré transformations of the S-matrix of massless particles is by imposing invariance under (31). In that sense, there is no assumption in this section that has not already been made in section II. In other words, Poincaré symmetry requires that $M_n$ gives the same answer independently of the choice of reference spinor $\mu$.

A. Behavior at Infinity

If a theory comes from a Lagrangian then the three-particle amplitudes derived in section III can be computed as the product of three polarization tensors times a three-particle vertex
that contains some power of momenta which we denote by $L_3$. Simple dimensional arguments indicate that if all particles have integer spin then $L_3 = |h_1 + h_2 + h_3|$. Let us denote the power of momenta in the four-particle vertex by $L_4$.

We are interested in the behavior of $M_4$, constructed using Feynman diagrams, under the deformation of $\lambda^{(1)}$ and $\tilde{\lambda}^{(2)}$ defined in (15) as $z$ is taken to infinity.

Feynman diagrams fall into three different categories corresponding to different behaviors at infinity. Representatives of each type are shown in figure 2. The first kind corresponds to the $(1,2)$-channel ($s$-channel). The second corresponds to either the $(1,3)$-channel ($u$-channel) or the $(1,4)$-channel ($t$-channel). Finally, the third kind is the four-particle coupling.

Under the deformation $\lambda^{(1)}(z) = \lambda^{(1)} + z\lambda^{(2)}$ and $\tilde{\lambda}^{(2)}(z) = \tilde{\lambda}^{(2)} - z\tilde{\lambda}^{(1)}$, polarization tensors give contributions that go as $z^{-s_1}$ and $z^{-s_2}$ respectively in the case of integer spin and like $z^{-s_1+1/2}$ and $z^{-s_2+1/2}$ in the case of half-integer spin. Recall that we chose particle 1 to have positive helicity while particle 2 to have negative helicity. Had we chosen the opposite helicities, polarization tensors would have given positive powers of $z$ at infinity. For simplicity, let us restrict the rest of the discussion in this section to integer spin particles.

For the first kind of diagrams, only a single three-particle vertex is $z$ dependent and gives $z^{L_3}$. Combining the contributions we find $z^{L_3-s_1-s_2}$. Therefore, we need $s_1 + s_2 > L_3$.

For the second kind of diagrams, two three-particle vertices contribute giving $z^{L_3+L_3'}$. This time a propagator also contributes with $z^{-1}$. Combining the contributions we get $z^{L_3+L_3'-s_1-s_2-1}$. Therefore we need $s_1 + s_2 > L_3 + L_3' - 1$.

Finally, for the third kind of diagrams, only the four-particle vertex contributes giving $z^{L_4}$. Combining the contributions we find $z^{L_4-s_1-s_2}$. Therefore we need $s_1 + s_2 > L_4$.

Summarizing, a four-particle amplitude is constructible, i.e., $M_4^{(1,2)}(z)$ vanishes as $z \to \infty$ if $s_1 + s_2 > L_3$, $s_1 + s_2 > L_3 + L_3' - 1$ and $s_1 + s_2 > L_4$. It is important to mention that these are sufficient conditions but not necessary. Recall that we are interested in the behavior of the whole amplitude and not on that of individual diagrams. Sometimes it is possible that the sum of Feynman diagrams vanishes at infinity even though individual diagrams do not. Also possible is that since our analysis does not take into account the precise structure of interaction vertices, there might be cancellations within the same diagram. In other words, our Feynman diagram analysis only provides an upper bound on the behavior at infinity.

Let us go back to the example in the previous section. There $s_1 = s_2 = s$, $L_3 = L_3' = s$. 17
FIG. 2: The three different kinds of Feynman diagrams which exhibit different behavior as \( z \to \infty \). They correspond to the s-channel, t (u)-channel and the four-particle coupling respectively.

Note that \( s_1 + s_2 > L_3 \) implies \( s > 0 \), as mentioned at the beginning of this section. The second condition is empty and the third implies that \( L_4 < 2s \). Thus, our conclusions in the example are valid only if \( s > 0 \) and four-particle interactions have at most \( 2s - 1 \) derivatives. Note that for \( s = 1 \) this excludes \( (F^2)^2 \) terms and for \( s = 2 \) this excludes \( R^2 \) terms. We will comment on possible ways to make these theories constructible in section VII.

B. Physical vs. Spurious Poles

There is an apparent contradiction when in section IV we used that the only poles of \( M_4(z) \) come from propagators and when earlier in this section we used that polarization tensors behave as \( z^{-s} \).

The resolution to this puzzle is very simple yet amusing. Recall that polarization tensors are defined only up to the choice of a reference spinor \( \mu \) or \( \tilde{\mu} \) of positive or negative chirality depending on the helicity of the particle. The \( z \)-dependence in polarization tensors comes from the factors in the denominator of the form \( \langle \lambda(z), \mu \rangle^s \) or \( [\tilde{\lambda}(z), \tilde{\mu}]^s \). The deformed spinors are given by \( \lambda(z) = \lambda + z\lambda' \) (or \( \tilde{\lambda}(z) = \tilde{\lambda} + z\tilde{\lambda}' \)) where \( \lambda' \) (or \( \tilde{\lambda}' \)) are the spinors of a different particle. Now we see that if \( \mu \) is not proportional to \( \lambda' \) then individual Feynman diagrams go to zero as \( z \) becomes large due to the \( z \) dependence in the polarization tensors. In the same way, individual Feynman diagrams possess more poles than just those coming from propagators. Now let us choose \( \mu \) proportional to \( \lambda' \). Then the \( z \) dependence in polarization tensors disappears. We then find that individual Feynman diagrams do not vanish as \( z \) becomes large but they show only poles at the propagators. Recall that we are
not interested in individual Feynman diagrams, but rather in the full amplitude, which is independent of the choice of reference spinor. Therefore, since $M_4(z)$ vanishes for large $z$ for some choice of reference spinors it must also do so for any other choice. This means that the pole at infinity is spurious. Similarly, poles coming from polarization tensors are spurious as well.

VI. MORE EXAMPLES

In this section we give more examples of how the four-particle test can be used to constrain many theories. In previous sections we studied theories of a single particle of integer spin $s$ and found that only $s = 2$ admits self-interactions. Here we allow for several particles of the same spin. In this section we consider the coupling of a particle of spin $s$ and one of spin 2. The spin $s$ can be integer or half-integer.

A. Several Particles Of Same Integer Spin

Consider theories of several particles of the same integer spin $s$. The idea is to see whether allowing for several particles relaxes the constraint found in section IV.B.2 that sets $s = 2$.

We are interested in four-particle amplitudes where each particle carries an extra quantum number. We can call it a color label. The data for each particle is thus $\{\lambda(i), \bar{\lambda}(i), h_i, a_i\}$. As discussed in section III.B, the most general three-particle amplitudes possess coupling constants that can depend on the color of the particles. Here we drop the superscripts $H$ and $A$ in order to avoid cluttering the equations and define $\kappa_{a_1a_2a_3} = \kappa_{1-s}f_{a_1a_2a_3}$ where $f_{a_1a_2a_3}$ are dimensionless factors. The subscript $(1-s)$ is the dimension of the coupling constant.

Repeating the calculation that led to (26) but this time keeping in mind that we have to sum not only over the helicity of the internal particle but also over all possible colors, we find

\[
M_4^{(1,2)}(0) = \kappa_{1-s}^2 \sum_{a_i} f_{a_1a_3a_1}f_{a_1a_3a_2}A + \kappa_{1-s}^2 \sum_{a_i} f_{a_1a_3a_1}f_{a_1a_3a_2}B, \tag{32}
\]

while

\[
M_4^{(1,4)}(0) = \kappa_{1-s}^2 \sum_{a_i} f_{a_1a_3a_1}f_{a_1a_3a_4}C + \kappa_{1-s}^2 \sum_{a_i} f_{a_1a_3a_1}f_{a_1a_3a_4}D \tag{33}
\]
with

\[ A = \frac{(2, 4)^4}{\langle 1, 2 \rangle \langle 2, 3 \rangle \langle 3, 4 \rangle \langle 4, 1 \rangle} \left( \frac{(2, 4)^3[1, 3]}{\langle 1, 2 \rangle \langle 3, 4 \rangle} \right)^{s-1}, \]
\[ B = \frac{(2, 4)^3}{\langle 1, 2 \rangle \langle 4, 3 \rangle \langle 3, 1 \rangle} \left( \frac{(2, 4)^3[1, 3]}{\langle 1, 2 \rangle \langle 3, 4 \rangle} \right)^{s-1}, \]
\[ C = \frac{(2, 4)^4}{\langle 1, 2 \rangle \langle 2, 3 \rangle \langle 3, 4 \rangle \langle 4, 1 \rangle} \left( \frac{(2, 4)^3[1, 3]}{\langle 1, 4 \rangle \langle 2, 3 \rangle} \right)^{s-1}, \]
\[ D = \frac{(2, 4)^3}{\langle 1, 3 \rangle \langle 3, 2 \rangle \langle 4, 1 \rangle} \left( \frac{(2, 4)^3[1, 3]}{\langle 1, 4 \rangle \langle 2, 3 \rangle} \right)^{s-1}. \]

(34)

In order to understand why we have chosen to factor out the pieces that survive when \( s = 1 \) let us study this case in detail.

1. Spin 1

Before setting \( s = 1 \) it is important to recall that three-particle amplitudes for any odd integer spin did not have the correct symmetry structure under the exchange of particle labels. At the end of section III, we concluded that if no other labels were introduced then the three-particle couplings had to vanish. Now we have theories with a color label. In this case, it is easy to check that in order to ensure the correct symmetry properties we must require \( f_{a_1 a_2 a_3} \) to be completely antisymmetric in its indices.

Let us now set \( s = 1 \). The four-particle test requires \( M_4^{(1,2)}(0) - M_4^{(1,4)}(0) = 0 \). First note that the factor in front of \( B \) and \( D \) are equal up to a sign (due to the antisymmetric property of \( f \)). Therefore they can be combined and simplified to give

\[ \sum_{a_1} f_{a_1 a_3 a_1} f_{a_1 a_4 a_2} (B + D) = - \sum_{a_1} f_{a_1 a_3 a_1} f_{a_1 a_4 a_2} \left( \frac{(2, 4)^4}{\langle 1, 2 \rangle \langle 2, 3 \rangle \langle 3, 4 \rangle \langle 4, 1 \rangle} \right) \]

(35)

where the right hand side was obtained by a simple application of the identity \( \langle 1, 2 \rangle \langle 3, 4 \rangle + \langle 1, 4 \rangle \langle 2, 3 \rangle = \langle 1, 3 \rangle \langle 2, 4 \rangle \) which follows from the fact that spinors are elements of a two-dimensional vector space\(^{11}\).

Note that the right hand side of (35) can nicely be combined with the other terms to give rise to the following condition

\[ \sum_{a_1} f_{a_1 a_3 a_1} f_{a_1 a_3 a_2} + \sum_{a_1} f_{a_1 a_3 a_1} f_{a_1 a_4 a_2} + \sum_{a_1} f_{a_1 a_2 a_1} f_{a_1 a_3 a_4} = 0. \]

(36)

\(^{11}\) Readers familiar with color-ordered amplitudes possibly have recognized (35) as the \( U(1) \) decoupling identity, i.e., \( A(1, 2, 3, 4) + A(2, 1, 3, 4) + A(2, 3, 1, 4) = 0 \).
This condition is nothing but the Jacobi identity! Therefore, we have found that the four-particle test implies that a theory of several spin 1 particles can be non-trivial only if the dimensionless coupling constants \( f_{a_1a_2a_3} \) are the structure constants of a Lie algebra.

2. Spin 2

After the success with spin 1 particles, the natural question is to ask whether a similar structure is possible for spin 2. Once again, before setting \( s = 2 \) let us mention that like in the case of odd integer spin particles, the requirement of having the correct symmetry properties under the exchanges of labels implies that the dimensionless structure constants, \( f_{a_1a_2a_3} \), must be completely symmetric for even integer spin particles.

Imposing the four-particle test using (32) and (33) we find that the most general solution requires

\[
\sum_{a_1} f_{a_1a_4a_1} f_{a_2a_3a_2} = \sum_{a_1} f_{a_1a_3a_1} f_{a_2a_4a_2}
\]

which due to the symmetry properties of \( f_{abc} \) implies that all the other products of structure constants are equal and they factor out of (32) and (33) leaving behind the amplitudes for a single spin 2 particle which we know satisfy the four-particle test.

Note that (37) implies that the algebra defined by

\[
\mathcal{E}_a \star \mathcal{E}_b = f_{abc} \mathcal{E}_c
\]

must be commutative and associative. It turns out that those algebras are reducible and the theory reduces to that of several non-interacting massless spin 2 particles. This proves that it is not possible to define a non-abelian generalization of a theory of spin 2 particles that is constructible\(^{12} \). The same conclusion was proven by using BRST methods in [35].

Finally, let us mention that for \( s > 2 \) there is no non-trivial way of satisfying the four-particle test.

B. Coupling Of A Spin s Particle To A Spin 2 Particle

Our final example of the use of the four-particle test is to theories of a single spin \( s \) particle (\( \Psi \)) and a spin 2 particle (\( G \)). Here we assume that the spin 2 particle only has

\(^{12} \text{We thank L. Freidel for useful discussions about this point.} \)
cubic couplings of the form $++-$ and $--+$. This means that we are dealing with a graviton. Let the coupling constant of three gravitons be $\kappa$ while that of a graviton to two $\Psi$’s be $\kappa'$. Assume that the graviton coupling preserves the helicity of the $\Psi$ particle. This implies that $\kappa$ and $\kappa'$ have the same dimensions. Also assume that there is no any cubic coupling of $\Psi$’s.

We need to analyze two different 4 particle amplitudes: $M_4(G_1, G_2, \Psi_3, \Psi_4)$ and $M_4(\Psi_1, \Psi_2, \Psi_3, \Psi_4)$.

Consider first $M_4(\Psi^-_1, \Psi^+_2, \Psi^-_3, \Psi^+_4)$ under a BCFW deformation. A Feynman diagram analysis shows that the theory is constructible, i.e., the deformed amplitude vanishes at infinity, for $s > 1$. This implies that the following discussion applies only to particles $\Psi$’s with spin higher than 1.

Let us consider the four-particle test. We choose to deform $(1^-, 2^+)$ and $(1^-, 4^+)$. Taking the ratio of the quantities in (39) leads to:

$$\frac{M_4^{(1,2)}}{M_4^{(1,4)}} = \frac{\left(\frac{s}{t}\right)^{2s-3}}{s},$$

where $s = P^2_{12}$ and $t = P^2_{14}$. This ratio is equal to one only if $s = 3/2$. Thus, the only particle with spin higher than 1 which can couple to a graviton, giving a constructible theory, has the same spin as a gravitino in $\mathcal{N} = 1$ supergravity.

At this point the couplings $\kappa$ and $\kappa'$ are independent and it is not possible to conclude that the theory is linearized supergravity. Quite nicely, the next amplitude constrains the couplings.

Consider the four-particle test on the amplitude $M_4(G_1, G_2, \Psi_3, \Psi_4)$. Again we choose to

---

13 This last condition is not essential since such a coupling would have dimension different from that of $\kappa$ and $\kappa'$ and hence it would have to satisfy the four-particle test independently.
deform (1, 2) and (1, 4):

\[
M_4^{(1,2)} = - (\kappa')^2 \frac{\langle 1, 3 \rangle^2 [2, 4]^{2s+2}}{[1, 2]^2 [3, 4]^{2[2, 3]}^{2s-4}} \frac{s}{tu} \\
M_4^{(1,4)} = \kappa' \frac{\langle 1, 3 \rangle^2 [2, 4]^{2s+2}}{[1, 4]^2 [2, 3]^{2s-2}} \left( \frac{\kappa}{s} + \frac{\kappa'}{u} \right)
\]

(41)

where \( u = P_{13}^2 \).

Taking their ratio and setting \( s = 3/2 \), we get

\[
\frac{M_4^{(1,4)}}{M_4^{(1,2)}} = 1 - \frac{u}{t} \left( \frac{\kappa}{\kappa'} - 1 \right).
\]

(42)

Requiring the right hand side to be equal to one implies that \( \kappa' = \kappa \). This means that this theory is unique and turns out to agree with linearized \( \mathcal{N} = 1 \) supergravity.

An interesting observation is that the local supersymmetry of this theory arises as an accidental symmetry. The only symmetry we used in our derivation was under the Poincaré group; not even global supersymmetry was assumed. It has been known for a long time \[36\] that if one imposes global supersymmetry, then \( \mathcal{N} = 1 \) supergravity is the unique theory of spin 2 and spin 3/2 massless particles. The uniqueness of \( \mathcal{N} = 1 \) supergravity was successively \[37\] derived from the non-interactive form by using gauge invariances. More recently and by using cohomological BRST methods, the assumption of global supersymmetry was dropped \[38\].

Finally, let us stress that this analysis does not apply to the coupling of particles with spin \( s \leq 1 \) since the deformed amplitude under the BFCW deformation does not vanish at infinity. This simply means that we need to implement our procedure in a different way. We discuss this briefly in the next section as well as in the appendix.

VII. CONCLUSIONS AND FUTURE DIRECTIONS

Starting from the very basic assumptions of Poincaré invariance and factorization of the S-matrix, we have derived powerful consistency requirements that constructible theories must satisfy. We also found that many constructible theories satisfy the conditions only if the S-matrix is trivial. Non-trivial S-matrices seem to be rare.

The consistency conditions we found came from studying theories where four-particle scattering amplitudes can be constructed out of three-particle ones via the BCFW construction. While failing to satisfy the four-particle constraint non-trivially means that the
theory should have a trivial S-matrix, passing the test does not necessarily imply that the interacting theory exists. Once the four-particle test is satisfied one should check the five- and higher-particle amplitudes. A theory where all \( n \)-particle amplitudes can be determined from the three-particle ones is called fully constructible.

It is interesting to note that Yang-Mills \([19]\) and General Relativity \([22]\) are fully constructible. This means that the theories are unique in that once the three-particle amplitudes are chosen (where the only ambiguity is in the value of the coupling constants) then the whole tree-level S-matrix is determined. In the case of General Relativity it turns out that general covariance emerges from Poincaré symmetry. In the case of Yang-Mills, the structure of Lie algebras, i.e., antisymmetric structure constants that satisfy the Jacobi identity, also emerges from Poincaré symmetry. In both cases, the only non-zero coupling constants of three-particle amplitudes were chosen to be those of \( M_3(\pm \pm \pm) \) and \( M_3(\pm \pm \pm) \). It is important to mention that our analysis does not discard the possibility of theories with three-particle amplitudes of the form \( M_3(\pm \pm \pm) \) and \( M_3(\pm \pm \pm) \). Dimensional analysis shows that these theories are non-constructible due to the high power of momenta in the cubic vertex. For example, if \( s = 2 \) one finds six derivatives. Indeed, for spin 2, Wald \([40]\) found consistent classical field theories that propagate only massless spin 2 fields and which are not linearized General Relativity. Those theories do not possess general covariance and the simplest of them possesses cubic couplings with six derivative interactions. In this class of theories might be the spin 3 self-interaction, which seems to be possible from \([41]\), as well as the recent proposal for spin 2 and spin 3 interaction of \([42]\).

There are some natural questions for the future. One of them is to ask what the corresponding statements are if one replaces Poincaré symmetry by some other group. In particular, it is known that interactions of higher spins are possible in anti-de Sitter space (see \([39]\) and references therein). It would be interesting to reproduce such results from an S-matrix viewpoint.

The constraints we obtained in this paper only concern the pole structure of the S-matrix. It is natural to expect that branch cuts might lead to more constraints. In field theory one is very familiar with this phenomenon; some theories that are classically well defined become anomalous at loop level. It would be very interesting to find out whether the approach presented in this paper can lead to constraints analogous to anomalies. Speculating even more, one could imagine that since three-particle amplitudes are determined exactly, even
non-perturbatively, then it might be possible to find constraints that are only visible outside perturbation theory.

A well known way to handle quantum corrections is supersymmetry. A natural generalization of the results of this paper is to replace Poincaré symmetry by super Poincaré and then explore consistency conditions for theories involving different supermultiplets.

All of these generalizations, if possible, will only be valid for the set of constructible theories. In order to increase the power of these constraints one has to find ways of relaxing the condition of constructibility. Two possibilities are worth mentioning.

The first approach is to compose several BCFW deformations [43] so that more polarization tensors vanish at infinity and make the amplitude constructible. This procedure works in many cases but it is not very useful for four particles since deforming three particles means that one has to sum over all channels at once and the four-particle constraint is guaranteed to be satisfied. One can however go to five and more particles and then there will be non-trivial constraints.

Some peculiar cases can arise because, as it was stressed in section V, the behavior at infinity obtained by a Feynman diagram analysis is only an upper bound. It turns out in many examples that a Feynman diagram analysis shows a non-zero behavior at infinity under a single BCFW deformation and a vanishing behavior under a composition of BCFW deformations. Using the composition, one computes the amplitude which naturally comes out in a very compact form. When one takes this new compact, but equivalent, form of the amplitude and looks again at the behavior under a single BCFW deformation, one finds that it does go to zero at infinity! This shows that there are cancellations that are not manifest from Feynman diagrams. It would be very interesting if there was a simple and systematic way of improving the Feynman diagram analysis so that it will produce tighter upper bounds. It would be even more interesting to find a way of carrying out the analysis only in terms of the S-matrix.

The second possibility is to introduce auxiliary massive fields such that quartic vertices with too many derivatives arise as effective couplings once the auxiliary field is integrated out. Propagators of the auxiliary field create poles in $z$ whose location is proportional to the mass of the auxiliary field. The theory is then constructible, in the sense that no poles are located at infinity. Once the amplitudes are obtained one can take the mass of the auxiliary field to infinity and then recover the original theory. This gives a nice interpretation to the
physics at infinity of some non-constructible theories: the presence of poles at infinity implies that the theory is an effective theory where some massive particles have been integrated out. The simplest example is a theory of a massless scalar $s = 0$. Recall that one condition for a theory to be constructible is that the quartic interaction has to have $l < 2s$ derivatives. In the case at hand, with $s = 0$, this means that the quartic interaction must be absent. Therefore, a scalar theory with a $\lambda \phi^4$ interaction is not constructible. In the appendix, we show that this theory can be made constructible by introducing an auxiliary field (and deforming three particles).

A necessary ingredient to carry out the program of auxiliary fields is to find three-particle amplitudes where one or more particles are massive. More generally, it will be interesting to extend our methods for general massive representations of the Poincaré group. A good reason to believe that this might be possible is the analysis of \cite{44} where amplitudes of massive scalars and gluons were constructed using a suitable modification of BCFW deformations. In the case of massive particles of higher spins one might try to generate a mass term using the Higgs mechanism.

Finally, there are two more directions that, in our view, deserve further study. The first is the extension to theories in higher or less number of dimensions, including theories in ten dimensions. The second is to carry out a systematic search for theories where several three-particle amplitudes might have coupling constants with different dimensions but that when multiplied to produce four-particle amplitudes produce accidental degeneracies. Such degeneracies might lead to new consistent non-trivial theories which we might call exceptional theories.

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APPENDIX A: RELAXING CONSTRUCTIBILITY: AUXILIARY FIELDS

Our proposal for studying arbitrary spin theories is very general, but it suffers from the fact that some interesting theories are not constructible. In section VII, we mentioned several ways of trying to extend the range of applicability of our technique. One of them was the introduction of auxiliary fields. In this appendix we illustrate the idea by showing how the $\lambda\phi^4$ theory, which is not constructible (even under compositions of BCFW deformations), can be thought of as the effective theory of a constructible theory which contains a massive field. The constructibility here is under a composition of two BCFW deformations.

The failure to be constructible of the four-particle amplitude in the $\lambda\phi^4$ theory is understood as a consequence of sending the mass of the heavy auxiliary field to infinity.

Let us start with a massless scalar with a $\lambda\phi^4$ interaction:

$$L(\phi) = \frac{1}{2} \left( \partial_{\mu} \phi \right) \left( \partial^{\mu} \phi \right) - \frac{\lambda}{4!} \phi^4. \quad (A1)$$

We can remove the quartic coupling by introducing a massive auxiliary field $\chi$:

$$L(\phi, \chi) = \frac{1}{2} \left( \partial_{\mu} \phi \right) \left( \partial^{\mu} \phi \right) + \frac{1}{2} \left( \partial_{\mu} \chi \right) \left( \partial^{\mu} \chi \right) - \frac{1}{2} m^2 \chi^2 - g \chi \phi^2. \quad (A2)$$

It is straightforward to check that (A1) can be obtained from (A2) by integrating out the field $\chi$ taking the limit of large $g$ and large $m_\chi$, and by keeping $g^2/2m^2_\chi \equiv \lambda/4!$ finite.

The theory (A2) now has only cubic interactions. Since massless scalar fields do not possess polarization tensors that can be made to vanish at infinity, the theory with only cubic interactions is still not constructible under a BCFW deformation of two particles. This problem is resolved by applying a composition and deforming three particles.

Another problem one has to deal with is that the new vertex in (A2) involves a massive scalar. This implies that the analysis of section III is not readily applicable. However, in this specific case, the three particle amplitude is simply given by the coupling constant $g$.

Since we are interested in the scattering of the massless scalars represented by the field $\phi$, we consider only amplitudes where $\chi$ appears as an internal particle. This means that an internal propagator takes the form

$$\frac{1}{P^2 - m_\chi^2}. \quad (A3)$$

Let $M_4(\phi_1, \phi_2, \phi_3, \phi_4)$ be the four particle amplitude of interest. From Feynman diagrams,
it is easy to see that it is given by
\[ M_4(\phi_1, \phi_2, \phi_3, \phi_4) = \sum_{j=2}^{4} \frac{g^2}{P_{ij}^2 - m_\chi^2}, \]  
(A4)
where \( P_{ij} = p^{(1)} + p^{(j)} \). Already from (A4), one can see that the correct limit leads to the four point vertex of the original theory:
\[ \frac{g^2}{P_{ij}^2 - m_\chi^2} \to -\frac{g^2}{m_\chi^2} \sim \lambda. \]  
(A5)

Let us apply a three-particle deformation:
\begin{align*}
\tilde{\lambda}^{(1)}(z) &= \lambda^{(1)} - z \left( \frac{[1, 3]}{[2, 3]} \lambda^{(2)} + \frac{[1, 3]}{[3, 4]} \lambda^{(4)} \right) \\
\lambda^{(2)}(z) &= \lambda^{(2)} + z \frac{[1, 3]}{[2, 3]} \lambda^{(1)} \\
\lambda^{(4)}(z) &= \lambda^{(4)} + z \frac{[1, 3]}{[3, 4]} \lambda^{(1)}. 
\end{align*}  
(A6)

A Feynman diagram analysis shows that the deformed amplitude vanishes at infinity as \( z^{-1} \). Taking the t-channel as an example, the deformed propagator in this channel is:
\[ \frac{1}{P_{14}^2(z) - m_\chi^2}, \quad P_{14}(z) = P_{14} - z \frac{[1, 3]}{[2, 3]} \lambda^{(1)} \tilde{\lambda}^{(2)}, \]  
(A7)
and its pole is given by
\[ z_u = \frac{[2, 3]}{[1, 3]} \left( \frac{P_{14}^2 - m_\chi^2}{\langle 1, 4 \rangle [2, 4]} \right). \]  
(A8)

The momentum \( P_{14} \) on-shell becomes:
\[ P_{14}(z_u) = P_{14} - \frac{(P_{14}^2 - m_\chi^2)}{\langle 1, 4 \rangle [2, 4]} \lambda^{(1)} \tilde{\lambda}^{(2)}. \]  
(A9)

As stated at the beginning of the appendix, the three-particle amplitude is just the coupling constant \( g \), so it is easy to reconstruct the result (A4) and, as a consequence, (A5).

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