An example of non-uniqueness for the weighted Radon transforms along hyperplanes in multidimensions

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Abstract

We consider the weighted Radon transforms $R_W$ along hyperplanes in $\mathbb{R}^d$, $d \geq 3$, with strictly positive weights $W = W(x, \theta)$, $x \in \mathbb{R}^d$, $\theta \in S^{d-1}$. We construct an example of such a transform with non-trivial kernel in the space of infinitely smooth compactly supported functions. In addition, the related weight $W$ is infinitely smooth almost everywhere and is bounded. Our construction is based on the famous example of non-uniqueness of Boman (1993 J. d’Anal. Math. 61 395–401) for the weighted Radon transforms in $\mathbb{R}^2$ and on a recent result of Goncharov and Novikov (2016 Eurasian J. Math. Comput. Appl. 4 23–32).

Keywords: weighted Radon transforms, injectivity, non-injectivity

1. Introduction

We consider the weighted Radon transforms $R_W$, defined by the formulas:

$$R_W f(x, \theta) = \int_{s \theta = x} W(x, \theta) f(x) \sigma(dx), \quad (s, \theta) \in \mathbb{R} \times S^{d-1}, \quad x \in \mathbb{R}^d, \quad d \geq 2,$$

(1.1)

where $W = W(x, \theta)$ is the weight, $f = f(x)$ is a test function on $\mathbb{R}^d$, and $\sigma$ denotes the measure on the hyperplane $\{x \in \mathbb{R}^d : x \theta = s\}$ induced from Lebesgue measure on $\mathbb{R}^d$.

We assume that $W$ is real valued, bounded and strictly positive, i.e.:

$$W = \overline{W} \geq c > 0, \quad W \in L^\infty(\mathbb{R}^d \times S^{d-1}),$$

(1.2)

where $\overline{W}$ denotes the complex conjugate of $W$, $c$ is a constant.
At present, the transforms $R_W$ arise in different domains of pure and applied mathematics; see, e.g. [Bey84, Bom93], [BQ87, GN16], [Gon17, Kun92], [LB73, MQ85], [Natt01, Nov14], [Qui83, Rad17] and references therein.

If $W \equiv 1$, then $R_W$ is reduced to the classical Radon transform $R$ along hyperplanes in $\mathbb{R}^d$. This transform is invertible by the classical Radon inversion formulas; see [Rad17]. Note that, initially, in the Radon’s work [Rad17] the transform $R$ was introduced and studied from a pure mathematical point of view.

If $W$ is strictly positive, $W \in C^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})$, and $f \in C_0^\infty(\mathbb{R}^d)$, then in [Bey84] the inversion of $R_W$ is reduced to solving a Fredholm type linear integral equation. Besides, in [BQ87] it was proved that $R_W$ is injective (for example, in $L_2^0(\mathbb{R}^d)$) if $W$ is real-analytic and strictly positive. In addition, an example of $R_W$ in $\mathbb{R}^2$ with infinitely smooth strictly positive $W$ and with non-trivial kernel $\text{Ker} R_W$ in $C_0^\infty(\mathbb{R}^2)$ was constructed in [Bom93]. Here $C_0^\infty$, $L_2^0$ denote the spaces of functions from $C^\infty$, $L_2$ with compact support, respectively.

In connection with the most recent progress in inversion methods for weighted Radon transforms $R_W$, see [GN16].

We recall also that inversion methods for $R_W$ in $\mathbb{R}^3$ admit applications in the framework of emission tomographies (see [GN16]).

In the present work we construct an example of $R_W$ in $\mathbb{R}^d$, $d \geq 3$, with non-trivial kernel $\text{Ker} R_W$ in $C_0^\infty(\mathbb{R}^d)$. The related $W$ satisfies (1.2). In addition, our weight $W$ is infinitely smooth almost everywhere on $\mathbb{R}^d \times \mathbb{S}^{d-1}$. To our knowledge, this result is the first example of non-uniqueness for the transforms $R_W$ under the assumptions (1.2) for $d \geq 3$ (for example, in $L_2^0(\mathbb{R}^d)$). In particular, this example shows limitations for inversion methods for $R_W$ in dimension $d \geq 3$ developed in the literature; see, e.g. [Bey84, GN16], [Gon17].

In our construction we proceed from results of [Bom93] and [GN16]. In particular, this construction is motivated by relations between $R_W$ and weighted ray transforms arising in emission tomographies for $d = 3$; see [GN16].

In section 2, in particular, we recall the result of [GN16].

In section 3 we recall the result of [Bom93].

In section 4 we obtain the main result of the present work.

2. Relations between the Radon and the ray transforms

We consider also the weighted ray transforms $P_w$ in $\mathbb{R}^d$, defined by the formulas:

$$P_wf(x, \theta) = \int_\mathbb{R} w(x + t\theta, \theta) f(x + t\theta) \, dt, \quad (x, \theta) \in TS^{d-1},$$

$$TS^{d-1} = \{(x, \theta) \in \mathbb{R}^d \times \mathbb{S}^{d-1} : x\theta = 0\}, \quad d \geq 2,$$

where $w = w(x, \theta)$ is the weight and $f = f(x)$ is a test-function on $\mathbb{R}^d$.

We assume that $w$ is real valued, bounded and strictly positive, i.e.:

$$w = w \geq c > 0, \quad w \in L^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1}).$$

We recall that $TS^{d-1}$ can be interpreted as the set of all oriented rays in $\mathbb{R}^d$. In particular, if $\gamma = (x, \theta) \in TS^{d-1}$, then

$$\gamma = \{y \in \mathbb{R}^d : y = x + t\theta, \quad t \in \mathbb{R}\},$$

where $\theta$ gives the orientation of $\gamma$.
We recall that for \( d = 2 \), transforms \( P_w \) and \( R_w \) are equivalent up to the following change of variables:

\[
R_w f(s, \theta) = P_w f(s \theta, \theta^\perp), \ s \in \mathbb{R}, \ \theta \in \mathbb{S}^1, \tag{2.5}
\]

\[
W(x, \theta) = w(x, \theta^\perp), \ x \in \mathbb{R}^2, \ \theta \in \mathbb{S}^1, \tag{2.6}
\]

where \( f \) is a test-function on \( \mathbb{R}^2 \).

For \( d = 3 \), the transforms \( R_w \) and \( P_w \) are related by the following formulas (see [GN16]):

\[
R_w f(s, \theta) = \int_\mathbb{R} P_w f(s \theta + \tau [\theta, \alpha(\theta)], \alpha(\theta)) \, d\tau, \ (s, \theta) \in \mathbb{R} \times \mathbb{S}^2, \tag{2.7}
\]

\[
W(x, \theta) = w(x, \alpha(\theta)), \ x \in \mathbb{R}^3, \ \theta \in \mathbb{S}^2, \tag{2.8}
\]

\[
\alpha(\theta) = \begin{cases} 
\lvert \eta, \theta \rvert, & \text{if } \theta \neq \pm \eta, \\
\text{any vector } e \in \mathbb{S}^2, \text{ such that } e \perp \theta, & \text{if } \theta = \pm \eta,
\end{cases} \tag{2.9}
\]

where \( \eta \) is some fixed vector from \( \mathbb{S}^2 \), \([\cdot, \cdot]\) denotes the standard vector product in \( \mathbb{R}^3 \) and \( \perp \) denotes the orthogonality of vectors. Actually, formula (2.7) gives an expression for \( R_w \) on \( \mathbb{R} \times \mathbb{S}^2 \) in terms of \( P_w f \) restricted to the rays \( \gamma = \gamma(x, \theta) \), such that \( \theta \perp \eta \), where \( W \) and \( w \) are related by (2.8).

Below we present analogs of (2.7) and (2.8) for \( d > 3 \).

Let

\[
\Sigma(s, \theta) = \{ x \in \mathbb{R}^d : s \theta = s \}, \ s \in \mathbb{R}, \ \theta \in \mathbb{S}^{d-1}, \tag{2.10}
\]

\[
\Xi(v_1, \ldots, v_k) = \text{Span}\{v_1, \ldots, v_k\}, v_i \in \mathbb{R}^d, i = 1, \ldots, k, 1 \leq k \leq d, \tag{2.11}
\]

\[
\Theta(v_1, v_2) = \{ \theta \in \mathbb{S}^{d-1} : \theta \perp v_1, \theta \perp v_2 \} \cong \mathbb{S}^{d-3}, v_1, v_2 \in \mathbb{R}^d, v_1 \perp v_2, \tag{2.12}
\]

\((e_1, e_2, e_3, \ldots, e_d)\)-be some fixed orthonormal, positively oriented basis in \( \mathbb{R}^d \). \( \tag{2.13} \)

If \((e_1, \ldots, e_d)\) is not specified otherwise, it is assumed that \((e_1, \ldots, e_d)\) is the standard basis in \( \mathbb{R}^d \).

For \( d \geq 3 \), the transforms \( R_w \) and \( P_w \) are related by the following formulas:

\[
R_w f(s, \theta) = \int_{\mathbb{R}^{d-2}} P_w f(s \theta + \sum_{i=1}^{d-2} \tau_i \beta_i(\theta), \alpha(\theta)) \, d\tau_1 \ldots d\tau_{d-2}, \ (s, \theta) \in \mathbb{R} \times \mathbb{S}^{d-1}, \tag{2.14}
\]

\[
W(x, \theta) = w(x, \alpha(\theta)), \ x \in \mathbb{R}^d, \ \theta \in \mathbb{S}^{d-1}, \tag{2.15}
\]

where \( \alpha(\theta), \beta_i(\theta), i = 1, \ldots, d-2, \) are defined as follows:

\[
\alpha(\theta) = \begin{cases} 
\text{direction of one-dimensional intersection } \Sigma(s, \theta) \cap \Xi(e_1, e_2), \text{ where} \\
\text{the orientation of } \alpha(\theta) \text{ is chosen such that } \det(\alpha(\theta), \theta, e_3, \ldots, e_d) > 0 \text{ if } \theta \notin \Theta(e_1, e_2), \\
\text{any vector } e \in \mathbb{S}^{d-1} \cap \Xi(e_1, e_2) \text{ if } \theta \in \Theta(e_1, e_2),
\end{cases} \tag{2.16}
\]
\[(\alpha(\theta), \beta_1(\theta), \ldots, \beta_{d-2}(\theta))\] is an orthonormal basis on \(\Sigma(s, \theta)\) (or, more precisely, on \(\Sigma(0, \theta)\) considered as a linear space),

\[(2.17)\]

and \(\Sigma(s, \theta), \Theta(e_1, e_2)\) are given by \((2.10)\) and \((2.12)\), respectively. Here, in particular:

\[
\dim(\Sigma(s, \theta) \cap \Xi(e_1, e_2)) = 1 \text{ if } \theta \not\in \Theta(e_1, e_2); \\
\alpha(\theta) \in \mathbb{S}^{d-1} \cap \Xi(e_1, e_2) \simeq \mathbb{S}^1. \\
\]

Formula \((2.18)\) follows from the formulas:

\[
\dim \Sigma(s, \theta) = d - 1, \dim \Xi(e_1, e_2) = 2, \dim \Sigma(s, \theta) + \dim \Xi(e_1, e_2) = d + 1 > d, \\
\Xi(e_1, e_2) \not\subset \Sigma(s, \theta), \Sigma(s, \theta) \cap \Xi(e_1, e_2) \neq \emptyset. \\
\]

Formula \((2.19)\) follows from the definition.

Note that formulas \((2.14)-(2.18)\) are also valid for \(d = 3\). In this case these formulas are reduced to \((2.7)-(2.9)\), where \(e_3 = -\eta\).

Note that, formula \((2.14)\) gives an expression for \(R_{uf}\) on \(\mathbb{R} \times \mathbb{S}^{d-1}\) in terms of \(P_{uf}\) restricted to the rays \(\gamma = (x, \alpha)\), such that \(\alpha \in \mathbb{S}^{d-1} \cap \Xi(e_1, e_2)\).

**Remark 1.** In \((2.16)\) one can also write:

\[
\alpha(\theta) = (-1)^{d-1} * (\theta \wedge e_3 \wedge \cdots \wedge e_d) \text{ if } \theta \not\in \Theta(e_1, e_2), \\
\]

where * denotes the Hodge star and \(\wedge\) is the exterior product in \(\Lambda^*\mathbb{R}^d\) (exterior algebra on \(\mathbb{R}^d\)); see, for example, Chapters 2.1.c, 4.1.c of \cite{Mor01}.

Note that the value of the integral in the right hand-side of \((2.14)\) does not depend on the particular choice of \((\beta_1(\theta), \ldots, \beta_{d-2}(\theta))\) of \((2.17)\).

Note also that, due to \((2.8), (2.9), (2.15)\) and \((2.16)\), the weight \(W\) is defined everywhere on \(\mathbb{R}^d \times \mathbb{S}^{d-1}, d \geq 3\). In addition, this \(W\) has the same smoothness as \(w\) on \(\mathbb{R}^d\) and in \(\theta\) on \(\mathbb{S}^{d-1} \setminus \Theta(e_1, e_2)\), where \(\Theta(e_1, e_2)\) is defined in \((2.12)\) and has zero Lebesgue measure on \(\mathbb{S}^{d-1}\).

### 3. Boman’s example

For \(d = 2\), in \cite{Bom93} there were constructed a weight \(W\) and a function \(f\), such that:

\[
R_{uf} \equiv 0 \text{ on } \mathbb{R} \times \mathbb{S}^1, \\
\text{(3.1)}
\]

\[
W \equiv W \geq c > 0, \quad W \in C^\infty(\mathbb{R}^2 \times \mathbb{S}^1), \\
\text{(3.2)}
\]

\[
f \in C^\infty_0(\mathbb{R}^2), \quad f \neq 0, \quad \text{supp } f \subset \overline{B^1} = \{x \in \mathbb{R}^2 : |x| \leq 1\}, \\
\text{(3.3)}
\]

where \(c\) is some positive constant. In addition, as a corollary of \((2.5), (2.6), (3.1)-(3.3)\), we have that:

\[
P_{w_0f_0} \equiv 0 \text{ on } T\mathbb{S}^1, \\
\text{(3.4)}
\]

\[
w_0 = w_0 \geq c > 0, \quad w_0 \in C^\infty(\mathbb{R}^2 \times \mathbb{S}^1), \\
\text{(3.5)}
\]

\[
f_0 \in C^\infty_0(\mathbb{R}^2), \quad f_0 \neq 0, \quad \text{supp } f \subset \overline{B^2} = \{x \in \mathbb{R}^2 : |x| \leq 1\}, \\
\text{(3.6)}
\]

where
(3.7) \[ w_0(x, \theta) = W(x, -\theta^\perp), \ x \in \mathbb{R}^2, \ \theta \in S^1, \]

(3.8) \[ f_0 \equiv f. \]

The construction of [Bom93] is very non-trivial. But for the present work only formulas (3.1)–(3.3) are important.

4. Main results

Let

\[ B^d = \{ x \in \mathbb{R}^d : |x| < 1 \}, \]

(4.1)

\[ \overline{B}^d = \{ x \in \mathbb{R}^d : |x| \leq 1 \}, \]

(4.2)

\[(e_1, \ldots, e_d) \] be the canonical basis in \( \mathbb{R}^d \).

(4.3)

**Theorem 1.** There are \( W \) and \( f \), such that

\[ RWf \equiv 0 \] on \( \mathbb{R} \times S^{d-1} \),

(4.4)

\( W \) satisfies (1.2), \( \psi \in C_0^\infty(\mathbb{R}^d), f \neq 0, \)

(4.5)

where \( RW \) is defined by (1.1), \( d \geq 3 \). In addition,

\( W \) is \( C^\infty \)–smooth on \( \mathbb{R}^d \times (S^{d-1} \setminus \Theta(e_1, e_2)) \).

(4.6)

where \( \Theta(e_1, e_2) \) is defined by (2.12). Moreover, weight \( W \) and function \( f \) are given by formulas (2.15), (2.16), (4.7) (taking into account (2.19)) and (4.8), (4.9) in terms of the Boman’s weight \( w_0 \) and function \( f_0 \) of (3.7) and (3.8).

**Proof of theorem 1.** We define

\[ w(x, \alpha) = w(x_1, \ldots, x_d, \alpha) \overset{\text{def}}{=} w_0(x_1, x_2, \alpha_1, \alpha_2), \]

(4.7)

\[ f(x) = f(x_1, \ldots, x_d) \overset{\text{def}}{=} \psi(x_3, \ldots, x_d)f_0(x_1, x_2), \]

for \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \ \alpha = (\alpha_1, \alpha_2, 0, \ldots, 0) \in S^{d-1} \cap \Xi(e_1, e_2) \simeq S^1, \)

(4.8)

where

\[ \psi \in C_0^\infty(\mathbb{R}^{d-2}), \ \text{supp} \ \psi = \overline{B}^{d-2} \text{ and } \psi \not\equiv 0. \]

(4.9)

From (2.1), (3.4), (4.7)–(4.9) it follows that:

\[ P_wf(x, \alpha) = \int_{\mathbb{R}} w(x_1 + t\alpha_1, x_2 + t\alpha_2, x_3, \ldots, x_d, \alpha) f(x_1 + t\alpha_1, x_2 + t\alpha_2, x_3, \ldots, x_d) \ dt \]

\[ = \psi(x_3, \ldots, x_d) \int_{\mathbb{R}} w_0(x_1 + t\alpha_1, x_2 + t\alpha_2, \alpha_1, \alpha_2) f_0(x_1 + t\alpha_1, x_2 + t\alpha_2) \ dt \]

\[ = \psi(x_3, \ldots, x_d) P_wf_0(x_1, x_2, \alpha_1, \alpha_2) = 0, \]

\[ x \in \mathbb{R}^d, \ \alpha = (\alpha_1, \alpha_2, 0, \ldots, 0) \in S^{d-1} \cap \Xi(e_1, e_2) \simeq S^1. \]

(4.10)
Properties (4.4)–(4.6) for $W$ defined by (2.15), (2.16), (4.7) and $f$ defined by (4.8) and (4.9) follow from (2.14)–(2.17), (2.19), (2.22), (3.2), (3.3) and (4.10). In particular, property (4.4) follows from (2.14), (2.19) and (4.10).

Theorem 1 is proved. □

Note also that, according to (2.15) and (2.16), weight $W(x, \theta)$ for $\theta \in \Theta(e_1, e_2)$ can be specified as follows:

$$W(x, \theta) = W(x_1, \ldots, x_d, \theta) \overset{\text{def}}{=} w_0(x_1, x_2, e_1) \text{ for } \theta \in \Theta(e_1, e_2), x \in \mathbb{R}^d. \quad (4.11)$$

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References

[Bey84] Beylkin G 1984 The inversion problem and applications of the generalized Radon transform Commun. Pure Appl. Math. 37 579–99
[BQ87] Boman J and Quinto E 1987 Support theorems for real-analytic Radon transforms Duke Math. J. 55 943–8
[Bom93] Boman J 1993 An example of non-uniqueness for a generalized Radon transform J. d’Anal. Math. 61 395–401
[GN16] Goncharov F O and Novikov R G 2016 An analog of Chang inversion formula for weighted Radon transforms in multidimensions Eurasian J. Math. Comput. Appl. 4 23–32
[Gon17] Goncharov F O 2017 An iterative inversion of weighted Radon transforms along hyperplanes Inverse Problems 33 124005
[Kun92] Kunyansky L 1992 Generalized and attenuated Radon transforms: restorative approach to the numerical inversion Inverse Problems 8 809
[LB73] Lavrent’ev M M and Bukhgeim A L 1973 A class of operator equations of the first kind Funkt. Anal. Appl. 7 290–8
[MQ85] Markoe A and Quinto E 1985 An elementary proof of local invertibility for generalized and attenuated Radon transforms SIAM J. Math. Anal. 16 1114–9
[Mor01] Morita S 2001 Geometry of differential forms Am. Math. Soc.
[Natt01] Natterer F 2001 The Mathematics of Computerized Tomography (Philadelphia: SIAM)
[Nov14] Novikov R G 2014 Weighted Radon transforms and first order differential systems on the plane Mosc. Math. J. 14 807–23
[Qui83] Quinto E T 1983 The invertibility of rotation invariant Radon transforms J. Math. Anal. Appl. 91 510–22
[Rad17] Radon J 1917 Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten Ber. Saechs Akad. Wiss. Leipzig, Math-Phys 69 262–7