Loop States in Lattice Gauge Theories

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We solve the Gauss law as well as the corresponding Mandelstam constraints of (d+1) dimensional SU(2) lattice gauge theory in terms of harmonic oscillator prepotentials. This enables us to explicitly construct a complete orthonormal and manifestly gauge invariant basis in the physical Hilbert space. Further, we show that this gauge invariant description represents networks of unoriented loops carrying certain non-negative abelian fluxes created by the harmonic oscillator prepotentials. The loop network is characterized by 3(d−1) gauge invariant integers at every lattice site which is the number of physical degrees of freedom. Time evolution involves local fluctuations of these loops. The loop Hamiltonian is derived. The generalization to SU(N) gauge group is discussed.

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I. INTRODUCTION

The idea that gauge theories should be formulated completely in terms of loops in space carrying electric fluxes is quite old, appealing and has long history [1]. It is widely believed that QCD written in terms of such gauge invariant colorless loops, instead of colored quarks and gluons, is better suited to study non-perturbative long distance physics like color confinement. Further, since the introduction of SL(2,C) Yang Mills connections as the basic variables for gravity [2], the loop formalism goes beyond the color invariant description of gauge theories and has a much wider reach. To this end, the Wilson loop approach, though geometrical and manifestly gauge invariant, suffers from the serious problem of over completeness due to Mandelstam constraints [3]. Therefore, a most economical as well as complete description of gauge theories in d dimension in terms of gauge invariant loop states is an important issue and the subject of the present work. One would like to solve the Mandelstam constraints in the loop basis and study the loop dynamics without making any approximations or taking any particular limits [3]. We use the recently proposed prepotential formulation of lattice gauge theories [4] to explicitly construct an orthonormal and manifestly gauge invariant basis in the physical Hilbert space and thus solve the SU(2) Gauss law as well as the associated Mandelstam constraints. The ideas can be generalized to SU(N) group and discussed in the last section. In the SU(2) case, we show that the basis vectors describe networks of loops which carry positive integer abelian fluxes created by the prepotential operators. Further, the action of the Hamiltonian on the loop basis too has simple interpretation of counting, creating and destroying the above abelian flux lines on the links. The loop network is characterized by 3(d−1) angular momentum quantum numbers at every lattice site which is the number of the physical degrees of freedom of the SU(2) theory. Therefore, this loop state description is also a duality transformation [5, 6] where the effect of compactness of the gauge group is contained in the discrete angular momentum quantum numbers labeling the loop states. In the simpler context of compact (2+1) and (3+1) U(1) gauge theories such duality transformations are known to isolate the topological magnetic monopole degrees of freedom leading to confinement [7].

The plan of the paper is as follows. After a brief introduction, we first construct the loop states in d=2 and study their dynamics. This keeps the discussion simple and also illustrates all the ideas involved. The corresponding analysis and results in arbitrary d dimension is then obvious and done next. The generalization to SU(N) case is discussed at the end.

We start with SU(2) lattice gauge theory in (d+1) dimension. The Hamiltonian is [8]:

$$H = \sum_{n,i} \text{tr} E(n,i)^2 + K \sum_{\text{plaquettes}} \text{tr} \left( U_{\text{plaquette}} + h.c \right). \tag{1}$$

where K is the coupling constant. The index n labels the site of a d-dimensional spatial lattice and i, j (=1, 2, ...d) denote the unit vectors along the links. Each link (n,i) is associated with a symmetric top whose configuration, i.e. the rotation matrix from space fixed to body fixed frame, is given by the operator valued SU(2) matrix $U(n,i)$. The angular momenta with respect to space fixed and the body fixed frames are denoted by $E_L^n(n,i)$ and $E_R^n(n+i,i)$. More explicitly, $E_L(n,i)$ and $E_R(n+i,i)$ generate the gauge transformations at the lattice sites n and n+i respectively. They commute with each other and satisfy: $E_L(n,i)E_L(n,i) = E_R(n+i,i)E_R(n+i,i) \equiv E(n,i)$. $E(n,i)$ as the total angular momentum is same in both the frames. Therefore, a complete basis at every link (n,i) is given by $|j(n,i), m(n,i), \bar{m}(n,i) >$
where $j(n,i), m(n,i), \tilde{m}(n,i)$ are the eigenvalues of $E(n,i), E(n,i), \bar{E}(n,i)$ and $\bar{E}(n + i,i)$ respectively. We now exploit the Schwinger boson representation of the angular momentum algebra to define harmonic oscillator potentials on the links:

$$E_L(n,i) = a^\dagger(n,i) \frac{a(n,i)}{2};$$

$$E_R(n + i,i) = b^\dagger(n + i,i) \frac{a(n + i,i)}{2}.$$ 

The gauge transformation properties of the angular momenta, $E_L(n,i) \rightarrow \Lambda(n)E_L(n,i)\Lambda^\dagger(n)$ and $E_R(n + i,i) \rightarrow \Lambda(n + i)E_R(n + i,i)\Lambda^\dagger(n+i)$, imply that the Schwinger bosons belong to the fundamental representations of the gauge group, i.e:

$$a^\dagger(n,i) \rightarrow \Lambda(n)_{\alpha\beta}a^\dagger(n,i); \quad b^\dagger(n,i) \rightarrow \Lambda(n)_{\alpha\beta}b^\dagger(n,i).$$

Therefore, $a^\dagger(n,i)$ (left oscillator) and $b^\dagger(n+i,i)$ (right oscillator) create spin half charges at left and right ends of the link (n,i) respectively. The total angular momentum being same in both the frames implies:

$$a^\dagger(n,i)a(n,i) = b^\dagger(n+i,i)b(n+i,i) \equiv N(n,i).$$

Thus, besides SU(2) gauge invariance at every lattice site, we get an addition abelian gauge invariance on every lattice link:

$$a^\dagger(n,i) \rightarrow (exp i\theta(n,i)) a^\dagger(n,i);$$

$$b^\dagger(n+i,i) \rightarrow (exp -i\theta(n,i)) b^\dagger(n+i,i).$$

So we have gone from the electric field (or angular momentum), link operator description of Kogut-Susskind Hamiltonian to an equivalent description of SU(2) lattice gauge theory which is in terms of harmonic oscillator potentials with SU(2)⊗U(1) gauge invariance. With the simple SU(2)⊗U(1) gauge transformations and 4, we are well equipped to construct explicitly a manifestly gauge invariant and orthonormal loop basis.

## II. THE LOOP STATES IN D=2

The U(1) gauge invariance and its Gauss law simply state that the number of the left ($a^\dagger(nii)$) and the right ($b^\dagger(n+i,ii)$) oscillators is the same on any link. We denote this integer number by $2j(n,i)$. The abelian Gauss law constraints are solved easily by drawing $2j(n,i)$ lines on every link (n,i). Each of these $2j(n,i)$ lines represents the U(1) charge ($\pm 1$) of $a^\dagger(n,i)$ (see 1) and henceforth will be called abelian charge or abelian flux line. To illustrate, a simple example with $j(n,1) = j(n,2) = j(n-1,1) = j(n-2,2) = 1$ is shown in Figure 1. With U(1) Gauss law satisfied, we now deal with SU(2) gauge invariance. Under SU(2) gauge transformations, we note that $a^\dagger(n,i)$ located at the starting point of the link (n,i)] and $b^\dagger(n,i)$ [located at the end point of the link (n-i,i)] transform together by $\Lambda(n)$.

Therefore, it is convenient to group them together and define $a^\dagger[n,i] = a^\dagger(n,1), a^\dagger(n,2) = a^\dagger(n,2), a^\dagger[n,3] = b^\dagger(n,1), a^\dagger[n,4] = b^\dagger(n,2)$.

Thus, each of the four $a^\dagger[n,i]$ can be represented by a Young tableau (YT) box belonging to the SU(2) group which acts at the site n. Thus, to get SU(2) gauge invariance, we have to construct all possible spin singlets out of \((\sum_{i=1}^{2d=4} 2j[n,i])\) YT boxes. This is a simple problem: all possible spin zero operators are of the form $a^\dagger[n,i]a^\dagger[n,j] \equiv \epsilon_{i\beta}a^\dagger[n,i]a^\dagger[n,j]$, where $\epsilon_{i\beta}$ is the completely antisymmetric tensor and corresponds to putting two boxes of type i and j in a vertical column. In Figure 1, we represent this by linking a line of type i with a line of type j (i $\neq$ j) (see Figure 2). Thus for SU(2) gauge invariance all abelian flux lines must be mutually linked and no self linking is allowed. Therefore, the necessary condition on the number of abelian flux lines on the links [n,i] to give SU(2) gauge invariant state(s) at site n is:

$$2j(n,i) = \sum_{j \neq i} l_{ij}, \quad l_{ij} = l_{ji}, \quad l_{ij} \in Z_+,$$ 

where $Z_+$ denotes the set of all non-negative integers and $l_{ij}$ are the linking numbers amongst i and j types of abelian flux lines. The partition represents the manifestly SU(2) gauge invariant state:

$$|\bar{\ell}\rangle = \prod_{i,j=1}^{2d=4} (a^\dagger[i]a^\dagger[j])^{l_{ij}} |0\rangle.$$

The pattern \(|l_{12}(n), l_{13}(n), l_{14}(n), l_{23}(n), l_{24}(n), l_{34}(n)\rangle\) will be used to characterize the states in 1. Thus given any network of loops on the lattice we can construct a manifestly $SU(2)\otimes U(1)$ gauge invariant basis characterized by $d(2d-1)$ integer quantum numbers at every
The disadvantage of the basis (7) is that (like Wilson Loop basis) it is not orthonormal and it is over complete. To show this, we consider three distinct basis vectors contained in the set (7): $|\vec{l}_1\rangle = |100001\rangle, |\vec{l}_2\rangle = |010010\rangle,$ and $|\vec{l}_3\rangle = |001100\rangle.$ We find that they are linearly related: $|\vec{l}_1\rangle = |\vec{l}_2\rangle - |\vec{l}_3\rangle$ due to the identity:
\[
(a^\dagger[1],\tilde{a}^\dagger[2]) \equiv (a^\dagger[1],\tilde{a}^\dagger[3]) (a^\dagger[2],\tilde{a}^\dagger[4]) - (a^\dagger[1],\tilde{a}^\dagger[4]) (a^\dagger[2],\tilde{a}^\dagger[3]).
\] (8)

Infact, the identity (8) is the basic SU(2) Mandelstam identity written in terms of prepotentials. To solve the problem of over-completeness, we notice that the states (7) obtained by different possible contractions of the abelian flux lines are all characterized by $|J[n, i],J[n, i] = j[n, i] (j[n, i] + 1), J = J_{total} = 0\rangle.$ However, the intermediate angular momentum quantum number labels are missing (5). Therefore, we choose the following angular momentum addition scheme: $|J[1] + J[2] \rightarrow J[12] + J[3] \rightarrow J[(12)3] + J[4] = J = 0\rangle,$ and label the the common eigenvectors by the corresponding eigenvalues: $|j_1,j_2,j_12,j_{123},J_{total} = 0\rangle \equiv |j_1,j_2,j_12,j_{123},J\rangle.$ Thus, we get the (missing) operator $\langle J[1, n] + J[2, n]\rangle^2$ in this scheme which is yet to be diagonalized in the basis (7) with eigenvalue $j_{12}.$ This diagonalization problem is again simple: after linking $l_{12}$ boxes from $2j_2$ YT boxes on the link (n,1) with $l_{12}$ boxes from $2j_2$ YT boxes on the link (n,2) we should be left with 2$j_{12}$ boxes which are not linked (and symmetrized).

Therefore, $l_{12} = j_1 + j_2 - j_{12}.$ As total angular momentum is zero this also fixes $l_{134} = j_3 + j_4 - j_{12}.$ This, in the example of Figure 1, is illustrated in Figure 2. The final orthonormal and manifestly SU(2) gauge invariant states are:
\[
|j_1,j_2,j_{12},(j_3),j_{123}\rangle = (j_4) \equiv |j_1,j_2,j_{12}\rangle
\] (9)

In (9), the prime over the summation means that the linking numbers $l_{13},l_{14},l_{23},l_{24}$ are summed over all possible values which are consistent with (14). More simply, in the loop network language, the summations are over all possible contractions of the abelian flux lines keeping $l_{12}$ contractions fixed (14). The normalization constant $N(j) = N(j_1,j_2,j_{12})N(j_{12},j_3,j_{123})N(j_{123} = (j_4), j_4, 0)$, where $N(j_1,j_2,j_3) = ((j_1 - j_2 + j_3)!(-j_1 + j_2 + j_3)!j_1 + j_2 - j_3)^2(j_3)(2j_3 + j_1 + j_2)^2.$ In (4), on the left hand site $j_3$ and $j_4$ are within brackets as they are associated with the previous sites due to the U(1) Gauss law (4). Thus in d=2 the SU(2)$\otimes$U(1) gauge invariant loop network basis is characterized by three physical (gauge invariant) quantum numbers per lattice site. We summarize the results obtained so far: In d=2, the complete set of orthonormal gauge invariant states is isomorphic to the set of all possible loops which are labeled by the number of forward loop lines $j(n,1), j(n,2)$ and the linking number $l_{12}(n)$ at every lattice site. The explicit construction is given by (8).

### A. The Loop Space Dynamics

The Hamiltonian (11) has a very simple interpretation in the dual loop basis (9). The electric field term is now like potential energy term which simply counts the number of abelian flux lines. It’s contribution to the energy is: $\sum_{\text{links}(l)}j(l)(j(l) + 1).$ The plaquette term in (11) too has a simple meaning: it creates or annihilates the abelian flux lines on the plaquette. This can be seen by writing the link operator in terms of prepotentials:
\[
U(l)_{\alpha\beta} = F(l)(\tilde{a}^\dagger_{\alpha}(l)b^\dagger_\beta(l) + \tilde{a}_{\alpha}(l)b_\beta(l))F(l) = U_{\alpha\beta}(l) + U_{\alpha\beta}^{-}(l). \tag{10}
\]

Above, $F(l) = \frac{1}{\sqrt{\langle a_\dagger(l),a(l)\rangle + 1}}.$ The transformation to prepotentials (10) is obvious from the $SU(2)\otimes U(1)$ gauge transformations (5) and (6) respectively. Looking at (10) we realize that the operator $U^+(l) (U^{-}(l)$ creates (destroys) an abelian flux line on the link (l) like in the case of compact QED (see also (8)). We now compute the matrix elements of the Hamiltonian in the loop basis (9) for d=2. For convenience, we denote the four corners: n,n+1,n+1+2,n+2 of the plaquette located at n by a,b,c,d respectively and the associated loop basis vector as $|j_{abcd}\rangle = \prod_{l=a,b,c,d} |j_{12}^l, j_{12}^l, j_{12}^l, j_{12}^l\rangle.$ We consider the following $SU(2)\otimes U(1)$ gauge invariant part of the plaquette term in (11). $U_{\text{plaq}} = \langle a^\dagger[1],\tilde{a}^\dagger[2] \rangle_a \langle a^\dagger[2],\tilde{a}^\dagger[3] \rangle_b \langle a^\dagger[3],\tilde{a}^\dagger[4] \rangle_c \langle a^\dagger[1],\tilde{a}^\dagger[4] \rangle_d + \text{h.c.}$ The matrix elements are (10):
\[
<\hat{j}_{abcd;U_{\text{plaq}};j_{abcd}^*} = \left( N_+ \delta_{j_{12}^a,j_{12}^a} \delta_{j_{12}^b,j_{12}^b} \delta_{j_{12}^c,j_{12}^c} + N_- \delta_{j_{12}^a,j_{12}^a} \delta_{j_{12}^b,j_{12}^b} \delta_{j_{12}^c,j_{12}^c} \right) \left( \begin{array}{c|c|c|c} j_{12}^a & j_{12}^a & j_{12}^a & j_{12}^a \\ \hline j_{12}^b & j_{12}^b & j_{12}^b & j_{12}^b \\ \hline j_{12}^c & j_{12}^c & j_{12}^c & j_{12}^c \\ \hline j_{12}^d & j_{12}^d & j_{12}^d & j_{12}^d \\ \end{array} \right)
\]

(11)
Above \( N_q \) are the constants depending on the angular momentum quantum numbers on the plaquette \((abcd)\). The trivial \( \delta \) functions over the quantum numbers which do not change are not shown. The \( 6 \)-j symbols simply reflect the spin half nature of the prepotentials. The details will be given elsewhere \[11\] (also see \[3\]).

III. THE LOOP STATES IN D DIMENSION

It is easy to generalize \( d=2 \) construction of the previous section. We extend the angular momentum ladder and choose: 
\[
J[1] + J[2] \rightarrow J[12] + J[3] \rightarrow J[123] + \ldots \rightarrow J[12 \cdots 2d - 1] + J[2d] = J = 0.
\]
The states are now characterized as: 
\[
(\ldots j_{d-1}, (j_{d-1}), \ldots ) \quad (\ldots l_{d-1}, (l_{d-1}), \ldots ) = (j_{2d}) \equiv (j_{1}, j_{2}, \ldots, j_{d-1}, j_{d-1}, \ldots j_{2d-1}) = (j_{2d}) \equiv (j_{1}, j_{2}, \ldots, j_{d-1}, j_{d-1}, \ldots j_{2d-1}).
\]
Thus the loop network is labeled by \( 3(d-1) \) gauge invariant angular momentum quantum numbers at every lattice site which is the number of transverse physical degrees of the freedom of the gluons \[3\]. The states are again given by \[12\] with the constraints on the linking numbers which have to be generalized. The Young tableau arguments in \( d=2 \) case lead to: 
\[
l_{12} = j_{1} + j_{2} - j_{d-1} + l_{d-1} = j_{12} + j_{3} - j_{123} + \ldots , l_{12d} + l_{2d} - \ldots l_{2d-1,d} = 2j_{2d-1} \equiv 2j_{2d}.
\]
Note that the last equation is an identity. As in the \( d=2 \) case, all possible contractions consistent with the above constraints and with the number of flux lines on the links \[12\] are required to get the manifestly gauge invariant orthogonal loop basis. The dynamical matter fields are easy to incorporate, they will provide SU(2) charge sources and sinks to the abelian flux lines at their end points (see Figure 1) leading to additional color singlets.

IV. SU(N) LATTICE GAUGE THEORY

The SU(N) group has \((N-1)\) fundamental representations. Therefore, the defining equations for the SU(N) harmonic oscillator prepotentials \[12\] on the links are:
\[
E^a_L \equiv \sum_{r=1}^{(N-1)} a^\dagger [r] \frac{\lambda^a[r]}{2} a[r], E^b_R \equiv \sum_{r=1}^{(N-1)} b^\dagger [r] \frac{\lambda^a[r]}{2} b[r]
\]
where the prepotential oscillators \(a\) and \(b\) are defined at the initial and the end point of the link \( l \) respectively, the index \( r \) varies over the rank of the SU(N) group. Thus, the SU(N) lattice gauge theories in terms of prepotentials will have \( SU(N) \otimes U(1)^{(N-1)} \) gauge invariance. This will lead to \((N-1)\) types of abelian flux lines in the SU(N) loop space. The role of \((N-1)\) abelian gauge groups in the confinement mechanism of SU(N) gauge theories has been emphasized by ‘t Hooft through the idea of abelian projection \[13\]. It is interesting to imagine the background SU(2) quark anti-quark pair located at two different lattice sites in the prepotential formulation. The SU(2) fluxes will be neutralized locally but the abelian gauge invariance will demand the formation of an abelian flux line (string) between quark anti-quark pair leading to color confinement in the strong coupling limit. The construction of SU(N) loop basis, the issue of color confinement and especially \( N \rightarrow \infty \) limit are under investigation.

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[14] Our derivation of \[11\] in arbitrary \( d \) dimension (see section III) is by exploiting the properties of SU(2) coherent states and will be published elsewhere \[10\].