APPENDIX A. CONVEX CONE STRUCTURES

To keep this work self-consistent we present in this appendix basic definitions and properties about convex cone structure theory.

Suppose $X$ is a finite dimensional real vector space and $C \subset X$ is a closed convex cone. We assume that $C$ is pointed, that means $C \cap -C = \{0\}$. A closed pointed convex cone is in one-to-one correspondence with partial order in $X$, by $x \geq y \iff x - y \in C$ for each $x, y \in X$. If we additionally assume that the cone is generating, that is for each $x \in X$ there exists $u, w \in C$ such that $x = u - w$, then a nonempty set $C \subseteq X$ satisfying all above properties will be called a proper cone in space $X$. Let $X^*$ be a dual space with duality $\langle \cdot | \cdot \rangle$. Then, we introduce a partial order in $X^*$ as well with dual cone $C^* = \{ f \in X^* : \langle f | z \rangle \geq 0, \forall z \in C \}$. Observe that the cone $C^*$ is also closed and convex.

Moreover, if $C$ is generating in space $X$, then $C^*$ is pointed, so we can introduce the partial order in $X^*$ given by

$$f \geq g \iff f - g \in C^*$$

for all $f, g \in X^*$.

Next, consider a linear space with fixed inner product. If $X$ is an inner product space, then the Riesz representation theorem \[1\] holds that the inner product determines an isomorphism between $X$ and $X^*$. Therefore, the cone $C$ is equal to $C^*$.

An interior point $e \in \text{int}(C)$ of a cone $C$ is called an order unit if for each $x \in X$, there exists $\lambda > 0$ such that $\lambda e - x \in C$. Whereas, a base of $C$ is defined as compact and convex subset $B \subset C$ such that for every $z \in C \setminus \{0\}$, there exists unique $t > 0$ and an element $b \in B$ such that $z = tb$. It can be shown that the set

$$B = \{ z \in C : \langle e | z \rangle = 1 \}$$

is the base of $C$ (determined by element $e$) if and only if an element $e$ is an order unit and $e \in \text{int}(C^*)$. Finally, we define the base norm as

$$\|x\|_B = \{ \alpha + \beta, x = \alpha b_1 - \beta b_2, \alpha, \beta \geq 0, b_1, b_2 \in B \}.$$
It can be shown [2] that the base norm is expressed as

\[ ||x||_B = \sup_{\tilde{\mathbf{b}} \in \tilde{B}} ||\tilde{\mathbf{b}}^{1/2}x\tilde{\mathbf{b}}^{1/2}||_1, \]

where \( \tilde{B} = \{ \tilde{\mathbf{b}} \in \mathcal{C} : \text{tr}(\tilde{\mathbf{b}}) = 1, \forall \mathbf{b} \in B \} \).

References

[1] W. Rudin, “Functional analysis 2nd ed.” International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991.
[2] A. Jenčová, “Base norms and discrimination of generalized quantum channels,” Journal of Mathematical Physics, vol. 55, no. 2, p. 022201, 2014.