The Age of the Universe from Joint Analysis of Cosmological Probes

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Abstract. Analyses of various cosmological probes, including the latest Cosmic Microwave Background anisotropies, the 2dF Galaxy Redshift Survey and Cepheid-calibrated distance indicators suggest that the age of expansion is $13 \pm 3$ Gyr. We discuss some statistical aspects of this estimation, and we also present results for joint analysis of the latest CMB and Cepheid data by utilizing ‘Hyper-Parameters’. The deduced age of expansion might be uncomfortably close to the age of the oldest Globular Clusters, in particular if they formed relatively recently.

1. Introduction

Estimating the age of the Universe in the framework of the Big Bang model is an old problem. The rapid progress in observational Cosmology in recent years has led to more accurate values of the fundamental cosmological parameters, including the age of the Universe. We summarize the basics in Section 2, we point out possible problems in joint analysis of cosmological probes in section 3, and we summarize some recent results in Section 4. In Section 5 we introduce a new method of ‘Hyper-Parameters’ for combining different data sets, and we apply it in sections 6 and 7 to the latest Cepheid and CMB data. In section 8 we contrast the age of expansion with the ages of Globular Clusters.

2. The Age of Expansion

In the standard Big Bang model the age of the Universe is found by integrating $dt = H^{-1} da/a$, where $a$ is the scale factor and $H \equiv \dot{a}/a$ is the Hubble parameter as given by Einstein’s equations. This gives the present age in terms of three present-epoch parameters, $H_0 = 100h$ km/sec/Mpc = (9.78 Gyr)$^{-1} h$, the mass density parameter $\Omega_m$ and the scaled cosmological constant $\Omega_\Lambda \equiv \Lambda/(3H_0^2)$:

$$t_0 = H_0^{-1} \int_0^1 da \ a^{1/2} [\Omega_\Lambda a^3 + (1 - \Omega_m - \Omega_\Lambda)a + \Omega_m]^{-1/2}.$$  

(1)

The difficulty is that in practice the parameters $(H_0, \Omega_m, \Omega_\Lambda)$, when estimated from various cosmic probes, are commonly correlated with each other. For a flat universe $(\Omega_m + \Omega_\Lambda = 1)$, supported by the recent Cosmic Microwave Background (CMB) experiments, this integral has an analytic solution in terms of only two
free parameters:

\[ t_0 = \frac{2}{3} H_0^{-1} \Omega_A^{-1/2} \ln[(1 + \sqrt{\Omega_A})(1 - \Omega_A)^{-1/2}] \]  

(2)

which is well approximated (e.g. Peacock 1999) by:

\[ t_0 \approx \frac{2}{3} H_0^{-1} \Omega_m^{-0.3} . \]  

(3)

This gives an insight to the way the errors propagate in the determination of the cosmic age:

\[ \frac{\Delta t_0}{t_0} \approx \frac{\Delta H_0}{H_0} + 0.3 \frac{\Delta \Omega_m}{\Omega_m} . \]  

(4)

This shows that the fractional error in \( H_0 \) is about three times more important than the fractional error in \( \Omega_m \). Typically the quoted error on \( H_0 \) is 10% (e.g. Freedman et al 2000). The range of recent quoted values for the density parameter suggests \( \Omega_m \sim 0.3 \pm 50\% \), so the expected fractional error in age is about 25% (e.g. for \( t_0 \approx 13 \) Gyr, \( \Delta t_0 \approx 3 \) Gyr). Again, \( \Omega_m \) and \( H_0 \) are not always measured independently. For example redshift survey constrains the shape of the Cold Dark Matter (CDM) power-spectrum via the product \( \Gamma \equiv \Omega_m h \), while the CMB angular power-spectrum constrains \( \omega_m = \Omega_m h^2 \).

3. Cosmological Parameters from a Joint Analysis: a Cosmic Harmony?

A simultaneous analysis of the constraints placed on cosmological parameters by different kinds of data is essential because each probe (e.g. CMB, SNe Ia, redshift surveys, cluster abundance and peculiar velocities) typically constrains a different combination of parameters. By performing joint likelihood analyses, one can overcome intrinsic degeneracies inherent in any single analysis and so estimate fundamental parameters much more accurately. The comparison of constraints can also provide a test for the validity of the assumed cosmological model or, alternatively, a revised evaluation of the systematic errors in one or all of the data sets. Recent papers that combine information from several data sets simultaneously include Webster et al. (1998); Lineweaver (1998); Gawiser & Silk (1998), Bridle et al. (1999, 2001), Eisenstein, Hu & Tegmark 1999; Efstathiou et al. 1999; and Bahcall et al. (1999).

While joint Likelihood analyses employing both CMB and LSS data allow more accurate estimates of cosmological parameters, they involve various subtle statistical issues:

- There is the uncertainty that a sample does not represent a typical patch of the FRW Universe to yield reliable global cosmological parameters.
- The choice of the model parameter space is somewhat arbitrary.
- One commonly solves for the probability for the data given a model (e.g. using a Likelihood function), while in the Bayesian framework this should be modified by the prior for the model and its parameters.
If one is interested in a small set of parameters, should one marginalize over all the remaining parameters, rather than fix them at certain (somewhat ad-hoc) values?

The ‘topology’ of the Likelihood contours may not be simple. It is helpful when the Likelihood contours of different probes ‘cross’ each other to yield a global maximum (e.g. in the case of CMB and SNe), but in other cases they may yield distinct separate ‘mountains’, and the joint maximum Likelihood may lie in a ‘valley’.

Different probes might be spatially correlated, i.e. not necessarily independent.

What weight should one give to each data set?

In a long term collaboration in Cambridge (Bridle et al. 1999, 2001; Efstathiou et al. 1999; Lahav et al. 2000) we have compared and combined in a self-consistent way the most powerful cosmic probes: CMB, galaxy redshift surveys, galaxy cluster number counts, type Ia Supernovae and galaxy peculiar velocities. Our analysis suggests, in agreement with studies by other groups, that we live in a flat accelerating Universe, with comparable amounts of dark matter and ‘vacuum energy’ (the cosmological constant Λ).

4. Some Recent ‘Best Fit’ Cosmological Parameters

To give the flavor of favoured parameters we quote below two recent studies. These and numerous other studies support a Λ-CDM model with $\Omega_m = 1 - \Omega_\Lambda \sim 0.3$ and $h \sim 0.75$, which corresponds to an expansion age of $t_0 \sim 12.6 \text{ Gyr}$ (and $H_0 t_0 = 0.96$).

4.1. Combining CMB, Supernovae Ia and Peculiar Velocities

A recent study (Bridle et al. 2001) is an example of combining 3 different data sets. We compared and combined likelihood functions for the matter density parameter $\Omega_m$, the Hubble constant $h$, and the normalization $\sigma_8$ (in terms of the variance in the mass density field measured in an $8h^{-1}\text{ Mpc}$ radius sphere) from peculiar velocities, CMB (including the earlier Boomerang and Maxima data) and type Ia Supernovae. These three data sets directly probe the mass in the Universe, without the need to relate the galaxy distribution to the underlying mass via a “biasing” relation.

Our analysis assumes a flat Λ-CDM cosmology with a scale-invariant adiabatic initial power spectrum and baryonic fraction as inferred from Big Bang Nucleosynthesis. We find that all three data sets agree well, overlapping significantly at the 2σ level. This therefore justifies a joint analysis, in which we find a best fit model and 95% confidence limits of $\Omega_m = 0.28 (0.17, 0.39)$, $h = 0.74 (0.64, 0.86)$, and $\sigma_8 = 1.17 (0.98, 1.37)$. In terms of the natural parameter combinations for these data $\sigma_8 \Omega_m^{0.6} = 0.54 (0.40, 0.73)$, $\Omega_m h = 0.21 (0.16, 0.27)$. Also for the best fit point, $Q_{\text{rms}} = 19.7\mu\text{K}$ and the age of the Universe is 13.0 Gyr.
4.2. The 2dF Galaxy Redshift Survey

The 2dF Galaxy Redshift Survey (2dFGRS) has now measured in excess of 160,000 galaxy redshifts and is the largest existing galaxy redshift survey. A sample of this size allows large-scale structure statistics to be measured with very small random errors. An initial analysis of the power-spectrum of the 2dFGRS (Percival et al. 2001) yields 68% confidence limits on the total matter density times the Hubble parameter $\Omega_m h = 0.20 \pm 0.03$, and the baryon fraction $\Omega_b/\Omega_m = 0.15 \pm 0.07$, assuming scale-invariant primordial fluctuations and a prior on the Hubble constant ($h = 0.7 \pm 10\%$). Although the $\Lambda$-CDM model with comparable amounts of dark matter and dark energy is not so elegant, it is remarkable that various measurements show such good consistency.

5. Hyper-Parameters

We have addressed recently (Lahav et al. 2000; Lahav 2001) the issue of combining different data sets, which may suffer different systematic and random errors. We generalized the standard procedure of combining likelihood functions by allowing freedom in the relative weights of various probes. This is done by including in the joint likelihood function a set of ‘Hyper-Parameters’, which are dealt with using Bayesian considerations. The resulting algorithm, which assumes uniform priors on the logarithm of the Hyper-Parameters, is simple to implement. Here we show some examples of and results from the joint analysis of the latest CMB and Cepheid data sets.

Assume that we have two independent data sets, $D_A$ and $D_B$ (with $N_A$ and $N_B$ data points respectively) and that we wish to determine a vector of free parameters $w$ (such as the density parameter $\Omega_m$, the Hubble constant $H_0$ etc.). This is commonly done by minimizing

$$\chi^2_{\text{joint}} = \chi^2_A + \chi^2_B,$$

(or, more generally, maximizing the product of Likelihood functions).

Such procedures assume that the quoted observational random errors can be trusted, and that the two (or more) $\chi^2$s have equal weights. However, when combining ‘apples and oranges’ one may wish to allow freedom in the relative weights. One possible approach is to generalize Eq. 5 to be

$$\chi^2_{\text{joint}} = \alpha \chi^2_A + \beta \chi^2_B,$$

where $\alpha$ and $\beta$ are ‘Hyper-Parameters’, which are to be dealt with the following Bayesian way. There are a number of ways to interpret the meaning of the HPs. One way is to understand $\alpha$ and $\beta$ as controlling the relative weight of the two data sets. It is not uncommon that astronomers accept and discard measurements (e.g. by assigning $\alpha = 1$ and $\beta = 0$) in an ad-hoc way. The HPs procedure gives an objective diagnostic as to which measurements are problematic and deserve further understanding of systematic or random errors.

How do we eliminate the unknown HPs $\alpha$ and $\beta$? This is done by marginalization over $\alpha$ and $\beta$ with Jeffreys’ uniform priors in the log, $P(\ln \alpha) = P(\ln \beta) = 1$. We can then get the probability for the parameters $w$ given the data sets:

$$-2 \ln P(w|D_A, D_B) = N_A \ln(\chi^2_A) + N_B \ln(\chi^2_B).$$
To find the best fit parameters $w$ requires us to minimize the above probability in the $w$ space. It is as easy to calculate this statistic as the standard $\chi^2$, and it can be generalized for any number of data sets.

Since $\alpha$ and $\beta$ have been eliminated from the analysis by marginalization they do not have particular values that can be quoted. Rather, each value of $\alpha$ and $\beta$ has been considered and weighted according to the probability of the data given the model. It can be shown that the ‘weights’ are $\alpha_{\text{eff}} = \frac{N_A}{\chi^2_A}$ and $\beta_{\text{eff}} = \frac{N_B}{\chi^2_B}$, both evaluated at the joint peak.

6. $H_0$ from Cepheids

One of the most important results for the Hubble constant comes from the Hubble Space Telescope Key Project (Freedman et al. 2000). The method is based on luminosity-period Cepheid calibration of several secondary distance indicators measured over distances of 400 to 600 Mpc. Freedman et al. 2000 (see also in this volume) combined the different measurements by several statistical methods and derived as the ‘final result’ $H_0 = 72 \pm (3)_{\text{r}} \pm (7)_{\text{s}}$ km/sec/Mpc (1-sigma random and systematic errors).

Given the importance of this work, we have attempted to combine the data by a different method, using the HPs. We used the raw data given by Freedman et al. (2000) for Surface Brightness Fluctuations (SBF), Supernovae Ia (SNIa), Tully Fisher (TF) and Fundamental Plane (FP). To the random errors given in the tables we added (in quadrature) the quoted systematic errors.

The results are shown on the right top and bottom panels in Figure 1. We see that the four methods give a range of values for $H_0$, with the most discrepant result being the FP ($H_0 = 88$ km/sec/Mpc). However, using the HPs, the most probable result, $H_0 = 73$ km/sec/Mpc, agrees well with the result of Freedman et al. We also see that in this case the standard joint $\chi^2$ and the HPs give a very similar answer. The resulting HPs (‘weights’) for SBF, TF, SN and FP are 3.4, 2.7, 1.9 and 0.5, respectively.

7. Combining Cepheids and CMB Data

The latest Boomerang (Netterfield et al. 2001, de Bernardis et al. 2001), Maxima (Stomper et al. 2001) and Dasi (Pryke et al. 2001) CMB anisotropy measurements indicate 3 acoustic peaks. Parameter fitting to a $\Lambda$-CDM model suggests consistency between the different experiments, and a ‘best fit Universe’ with zero curvature, and an initial spectrum with spectral index $n = 1$ (e.g. Wang et al. 2001 and references therein). Unlike the earlier Boomerang and Maxima results, the new data also show that the baryon contribution is consistent with the Big-Bang Nucleosynthesis value $\Omega_b h^2 \sim 0.02$ (O’Meara et al. 2001).

Several authors (e.g. Wang et al. 2001) have pointed out that the Hubble constant $h$ itself cannot be determined accurately from CMB data alone. As the CMB constrains well the combination $\omega_m \equiv \Omega_m h^2$, the curvature $\Omega_k$ and $\Omega_\Lambda$,
Figure 1. Probabilities for the Hubble constant. Top right: $\chi^2$ statistic for four Cepheid-calibrated distance indicators from Freedman et al. (2000): SBF(dashed-dotted line) SN Ia(long-dashed), TF(dotted) and FP(dashed). The joint $\chi^2$ for the four Cepheid data is shown by the solid line, centred on $h = 0.74$. Bottom right: The probability based on the Hyper-Parameters approach for the Hubble constant given the four Cepheid-calibrated data set. The maximum probability is at $h = 0.73$. Top left: $\chi^2$ statistic derived for a compilation of the latest CMB data including Boomerang, Maxima and Dasi from Wang et al. 2001) and a grid of CMB models for an assumed $\Lambda$-CDM model with with $n = 1, \Omega_m = 1 - \lambda = 0.3, \Omega_bh^2 = 0.02$ (BBN value), and $Q_{\text{rms}} = 18\mu K$ (COBE-normalization). The maximum probability is at $h = 0.75$ (see also Figure 2). The distribution will be wider if some of the above parameters are kept free. Bottom left: The probability based on the Hyper-Parameters for the four Cepheid-calibrated data sets and the CMB compilation. The maximum probability is at $h = 0.73$. 

Figure 2. A compilation of the latest CMB $\Delta T$ data points against spherical harmonic $l$ (from Wang et al. 2001). The line shows the predicted angular power-spectrum for a $\Lambda$-CDM model with $n = 1, \Omega_m = 1 - \lambda = 0.3, \Omega_b h^2 = 0.02$ (BBN value), $Q_{\text{rms}} = 18\mu K$ (COBE normalization) and $h = 0.75$. A similar model (with $h = 0.7$) is also the best fit to the the 2dF galaxy power-spectrum (Percival et al. 2001). Hence we see good agreement of two entirely different data sets.

the Hubble constant can be derived from

$$h = \sqrt{\omega_m/(1 - \Omega_k - \Omega_\Lambda)}.$$ 

It is not surprising therefore that estimates of $h$ from the latest CMB data strongly depend on the assumed set of free parameters, and on the assumed ‘priors’ from other probes. For example, from the latest Boomerang data, Netterfield et al. (2001) derive for ‘weak priors’ $h = 0.56 \pm 0.11$ and age $t_0 = 15.4 \pm 2.1$ Gyr, while with ‘strong priors’ ($h = 0.66 \pm 0.05; t_0 = 14.0 \pm 0.6$ Gyr). Wang et al. (2001) find e.g. ($h = 0.42 \pm 0.23; t_0 = 20.5 \pm 9.0$ Gyr) from CMB alone, and ($h = 0.57 \pm 0.30; t_0 = 14.2 \pm 4.3$ Gyr) by combining the CMB with the IRAS PSCz data.

Here, for simplicity, we take the approach of fixing all the other parameters, apart from $h$. We assume that CMB fluctuations arise from adiabatic initial conditions with Cold Dark Matter and negligible tensor component, in a flat Universe with $\Omega_m = 0.3, \Omega_\Lambda = 1 - \Omega_m = 0.7, n = 1, Q_{\text{rms}} = 18\mu K$ (COBE-normalization) and $\Omega_b h^2 = 0.02$. This choice is motivated by numerous other studies which combined CMB data with other cosmological probes (e.g. Bridle et al. 2000; Hu et al. 2000; Wang et al. 2001; section 4 above). Of course, one may keep more free parameters, and marginalize over some of them, as done in numerous other studies. We obtain theoretical CMB power-spectra using the CMBFAST and CAMB codes (Sleijak & Zaldarriaga 1996; Lewis, Challinor & Lasenby 2000). Increasing $h$ decreases the height of the first acoustic peak, and makes few other significant changes to the angular power spectrum (e.g. Hu et al. 2000). The range in $h$ investigated here is $(0.5 < h < 1.1)$. 
Different CMB data sets can be combined in different ways (e.g. Jaffe et al. 2000; Lahav et al. 2000; Lahav 2001). For simplicity we use here a compilation of 24 $\Delta T/T$ data points from Wang et al. (2001), which is based on 105 band-power measurements (including the latest Boomerang, Maxima and Dasi). The left top panel of Figure 1 shows the CMB likelihood function (with the correlation matrix not yet taken into account) when the only free parameter is the Hubble constant. It favours a value of $h \sim 0.75$. Figure 2 shows the CMB data points, and we also projected our 'best-fit' model (for the above set of assumptions). A similar model (with $h = 0.7$) also fits well other cosmological measurements, e.g. the 2dF galaxy power-spectrum (Percival et al. 2001). The left bottom panel of Figure 1 shows the Hyper-parameters joint probability for the CMB and the four Cepheid-calibrated data sets. The maximum probability is at $h \sim 0.73$. We note that if more cosmological parameters are left free and then marginalized over, the error in $h$ would typically be much larger. We also note that in the context of CDM models we can estimate the Hubble constant from the ratio

$$h = \omega_m/\Gamma$$

(see section 2). The recent CMB data suggest $\omega_m \sim 0.15$ (e.g. Netterfield et al. 2001) and from various redshift surveys $\Gamma \sim 0.2$, so $h \sim 0.75$, in good agreement with our derived value. Of course $h$ and other parameters can be derived more quantitatively by joint likelihood analysis of CMB and redshift surveys (e.g. Webster et al. 1998).

8. Discussion

Joint analyses of cosmic probes suggests a flat Universe with $\Omega_m = 1 - \Omega_\Lambda \approx 0.3$. The measurement of the Hubble constant from Cepheids and from the CMB suggests $H_0 \approx 75 {\rm km/sec/Mpc}$. This set of 'best-fit' parameters yields an expansion age $t_0 = 13 \pm 3$ Gyr. The error estimate is due to errors in both $H_0$ and $\Omega_m$ (based on the range of recently quoted values). While this is currently the most popular model there are potential problems with this set of parameters:

(i) There is no simple theoretical explanation why the present epoch contributions to matter $\Omega_m$ and 'dark energy' ($\Omega_\Lambda$) are nearly equal.

(ii) The age of expansion is commonly compared with the age $t_{GC}$ of Globular Clusters (GC) and other old objects. More precisely, one requires

$$t_0 = t_f + t_{GC},$$

(10)

where $t_f$ is the epoch of formation.

The age of the oldest GC was estimated to be $t_{GC} = 11.5 \pm 1.3$ Gyr by Chaboyer (1998), but he revised it upwards to $t_{GC} = 13.2 \pm 1.5$ Gyr (see this volume). A recent radioactive dating (using $^{238}$U, with half-life of 4.5 Gyr) of a very metal-poor star in the Galaxy, gives an age of $12.5 \pm 3$ Gyr (Cayrel et al. 2001).

We see that the ages of old objects might be uncomfortably close to the age of expansion (despite having a non-vanishing cosmological constant, which tends to stretch the age of the Universe). Furthermore, it is commonly assumed that $t_f \sim 0.5 - 2$ Gyr (e.g. Chaboyer 1998), and hence $t_f$ is neglected in eq.
However, we point out that $t_f$ is model dependent, and it can span a wide range of values. For example Peebles & Dicke (1968) suggested (before dark matter was recognized as a major component) that GC can be identified with the Jeans’ mass after recombination, i.e. $z_f \sim 100 - 1000$ and indeed in this case $t_f$ is negligible.

On the other hand, the model of Fall & Rees (1985) for the formation of GC in the Galactic halo as a result of thermal instability suggests $z_f \sim 1 - 3$, i.e. $t_f \sim 2 - 5$ Gyr for the above world model. These and other models will be discussed elsewhere (Gnedin, Lahav & Rees, in preparation). Obviously, late formation of the oldest GC means that the derived age of expansion $t_0$ is too short compared with $t_f + t_{GC}$. This possible ‘age crisis’ might indicate a potential problem for either the cosmological model or the age estimation of GC.

The age of the Universe and other cosmological parameters will be revisited soon with larger and more accurate data sets such as the big redshift surveys (2dF, SDSS) and CMB (MAP, Planck) data. Other relevant probes for the Hubble constant are the Sunyaev-Zeldovich effect (e.g. Mason et al. 2001), the gravitational lensing time delay (e.g. Williams & Saha 2000) and the baryon fraction in clusters (e.g. Ettori et al. 2001; Douspis et al. 2001). New high-quality data sets will allow us to study a wider range of models and parameters.

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