Weak magnetic field effects on chiral critical temperature in a nonlocal Nambu–Jona-Lasinio model.

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In this article we study the nonlocal Nambu–Jona-Lasinio model with a Gaussian regulator in the chiral limit. Finite temperature effects and the presence of a homogeneous magnetic field are considered. The magnetic evolution of the critical temperature for chiral symmetry restoration is then obtained. Here we restrict ourselves to the case of low magnetic field values, being this a complementary discussion to the existing analysis in nonlocal models in the strong magnetic field regime.

I. INTRODUCTION

In recent years there has been an increasing interest in studying the QCD phase diagram in the presence of a magnetic field. Particularly, the effect of the magnetic field on the critical temperature for chiral phase transition, has been studied in lattice QCD [1–5] as well as in different effective models [6–11]. Most results from model and lattice calculations have found that magnetic catalysis takes place, i.e., that the critical temperature for chiral phase transition becomes higher in the presence of a magnetic field. However, recent improved lattice calculations have found the opposite behavior [3–5].

The Nambu–Jona-Lasinio (NJL) model and extensions including the Polyakov Loop (PNJL) have been considered for the study of chiral and deconfinement phase transitions in the presence of strong magnetic field [6, 12, 13]. In [14], the authors use an extended version of the PNJL for weak magnetic field reproducing anticatalysis when considering a modified magnetic-dependent Polyakov-loop potential.

The nonlocal Nambu–Jona-Lasinio (nNJL) models are an attempt to improve NJL model in a more realistic way, inspired in low energy approaches as Dyson-Schwinger resummation, lattice results, instantons liquid model and one gluon exchange models [15–19]. In this context, the external magnetic field effects on the critical temperature for chiral restoration and deconfinement were studied in [20] for the case of a strong magnetic field, $eB > 10m_c^2$, where the approximation used was to cut the Landau series. For the weak magnetic field case it is necessary to sum over too many Landau levels in order to obtain an accurate result. Another approach is to expand the fermion propagator in powers of $eB$, which is possible if the magnetic field is smaller than the square of the lowest particle energy, in this case, the lowest Matsubara frequency, i.e., $eB < (\pi T)^2$ [21, 22]. In this article we consider magnetic effects in the nNJL model with a Gaussian regulator, in the regime of low magnetic field, in order to compare with previous discussions in the strong field case [20]. This will give also a better understanding on the validity of the expansion of the fermion propagator in powers of $eB$. To study this we restrict ourselves to the chiral limit, in which case the chiral phase transition at vanishing chemical potential is a second order one [23], and the result will be compared with NJL with and without the weak magnetic field expansion.

This article is organized as follows: In Sec. II the NJL and nNJL models are introduced and the appropriate gap equations are computed. In Sec. III the effect of the magnetic field on the critical temperature for the chiral phase transition is shown. In Sec. IV our conclusions are presented.

II. THE MODEL

The NJL model and its nonlocal variant (nNJL) have been vastly used to study the thermodynamics of the low energy limit of QCD (see e.g. [24–27] and references therein). Both models have an approximate chiral symmetry and present a chiral phase transition. The Lagrangian for the NJL model is

$$\mathcal{L} = \bar{\psi}(x)(i\partial - m)\psi(x) + \frac{G}{2}\left((\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\tau\psi)^2\right),$$  (1)

with $\tau$ the Pauli matrices in isospin space and $\psi(x)$ a quark field. In the mean field approximation, the quarks acquire an effective mass $M = m + G\langle\bar{\psi}\psi\rangle$. The dressed propagator can be written in Euclidean space as

$$S_E(q) = \frac{-g + M}{q^2 + M^2},$$  (2)

The value of the constituent mass $M$ can be determined through the gap equation, obtained by minimizing the effective potential with respect to the mean field [24, 25]

$$M - m = G \int \frac{d^4q}{(2\pi)^4} \text{tr}(S_E(q)),$$  (3)
where the trace goes over color, flavor and Lorentz indices. The NJL model is nonrenormalizable and the integral in the previous equation needs to be regularized. This can be done in different ways. Since we are interested in studying the model coupled to a magnetic field, the proper time regularization method turns out to be appropriate [28]. Inserting

$$S_E(q) = \int_\eta^\infty ds e^{-s(M^2 + q^2)},$$

(4)

where \( \eta = 1/\Lambda^2 \) is an UV cutoff, in Eq. (3) and performing the momentum integrals, the gap equation now reads

$$M - m = M G \frac{N_c}{2\pi^2} \int_\eta^\infty \frac{ds}{s^2} e^{-sM^2}.$$  

(5)

A finite temperature gap equation can be obtained by considering the Matsubara frequencies \( \omega_n \), such that

$$q_4 \rightarrow -\omega_n = -(2n+1)\pi T$$

(6)

$$\int \frac{dq_4}{2\pi} \rightarrow -T \sum_n$$

(7)

in Eq. (3). This yields

$$M - m = M G T \frac{N_c}{\pi^{3/2}} \sum_n \int_\eta^\infty \frac{ds}{s^{3/2}} e^{-s(\omega_n^2 + M^2)}.$$  

(8)

We are interested in studying the model coupled to a homogeneous magnetic field. The derivative in the Lagrangian (1) is replaced by a covariant derivative

$$D_\mu = \partial_\mu + iq_f A_\mu,$$  

(9)

where \( A^\mu \) is the vector potential corresponding to a homogeneous external magnetic field \( B = |B|\hat{z} \) and \( q_f \) is the electric charge of the quark fields (i.e. \( q_u = 2e/3 \) and \( q_d = -e/3 \)). In the symmetric gauge, \( A^\mu = B \frac{q_f}{2}(0,-y,x,0) \),

(10)

The Schwinger proper time representation for the propagator is given by

$$S_E(p) = \int_0^\infty ds e^{-s(p_0^2 + p_\perp^3 \tanh(qBs) + M^2)}$$

$$\times \left( (\cosh(qBs) - i\gamma_1 \gamma_2 \sinh(qBs)) + (M - p_0) - \frac{p_\perp}{\cosh(qBs)} \right),$$  

(11)

with \( p_0^2 = p_0^2 + p_\perp^3 \), \( p_\perp^3 = p_\perp^3 + p_\perp^3 \) and where \( q \) is the charge of the particle being \( B \) the magnetic field. Using this propagator, we can obtain a zero temperature gap equation in the presence of a magnetic field

$$M - m = GM \frac{N_c}{4\pi^2} \sum_{f=u,d} |q_f B|$$

$$\times \int_\eta^\infty \frac{ds}{s \tanh(|q_f B| s)} e^{-sM^2}.$$  

(12)

with \( q_u = 2e/3 \) and \( q_d = -e/3 \). Similarly, the finite temperature gap equation reads

$$M - m = GM \frac{N_c}{\pi^{3/2}} T \sum_{f=u,d} |q_f B|$$

$$\times \sum_n \int_\eta^\infty ds e^{-s(\omega_n^2 + M^2)}$$

$$\times \frac{1}{\sqrt{s} \tanh(|q_f B| s)}.$$  

(13)

A usual manipulation when working with the NJL model is to introduce bosonic fields \( \sigma(x) = G\bar{\psi}(x)\psi(x) \) and \( \pi(x) = G(\bar{\psi}(x)\gamma^5 \tau \psi(x)) \). The mean field approximation is then taken, i.e. \( \sigma \approx \bar{\sigma} = G(\bar{\psi}\psi) \) and \( \pi \approx \bar{\pi} = 0 \). The mean field value of the \( \pi \) fields is taken as vanishing due to isospin symmetry. In this way \( \bar{\sigma} = M - m \) and, since the \( \bar{\sigma} \) field is related to the chiral condensate, the temperature at which \( \bar{\sigma}(T) = 0 \) corresponds to the critical temperature for chiral phase transition. Equations (5), (8), (12) and (13) will allow us to get \( \bar{\sigma}(T,qB) \) and from it we can get \( T_c(qB) \) with \( T_c \) the critical temperature for chiral phase transition.

The Lagrangian for the nNJL model is

$$\mathcal{L} = \bar{\psi}(x)(i\gamma - m)\psi(x) + \frac{G}{2} j_a(x) j_a(x),$$  

(14)

with the nonlocal currents

$$j_a(x) = \int d^4 y d^4 z r(y-x)r(x-z)\bar{\psi}(y)\Gamma_a \psi(z)$$  

(15)

and where \( \Gamma_a = (1, i\gamma^5 \tau) \). The function \( r \) in the previous equation is called the regulator of the model. The regulator may take different forms [29, 30], e.g. a Gaussian regulator can be outlined from inspiration on the liquid instants model [17]. The usual bosonization procedure can be performed similarly to what we did in the NJL model. Taking the mean field approximation and a vanishing value for the vacuum expectation value of the pionic fields, the dressed propagator of the model can be written as

$$S_E(q) = \frac{-\bar{\sigma} + \Sigma(q)}{q^2 + \Sigma^2(q)}.$$  

(16)

As seen in Eq. (16), the constituent mass of the NJL model has now been replaced by \( \Sigma(q) = m + \bar{\sigma} r^2(q) \). As in the NJL model, the temperature for which \( \bar{\sigma}(T) = 0 \) is the critical temperature for the chiral phase transition. In order to get the thermal evolution of \( \bar{\sigma} \) one needs to solve the gap equation. The zero temperature gap equation reads

$$\bar{\sigma} = G \frac{N_c}{\pi^2} \int_0^\infty dp p^3 r^2(p)\Sigma(p^2)$$

(17)

The momentum integrals can now be computed since the nNJL model does not need a UV cutoff. However, an energy scale \( \Lambda \) is hidden within the regulator \( r(p) \). We can
obtain the corresponding finite temperature gap equation following the prescriptions given in Eqs. (6) and (7)

\[
\bar{\sigma} = GT \frac{4N_c}{\pi^2} \sum_n \int_0^\infty d\omega_n p^2 \frac{r^2(\omega_n, p)\Sigma(\omega_n, p)}{\omega_n^2 + p^2 + \Sigma^2(\omega_n, p)}. \tag{18}
\]

Once again, we will resort to the Schwinger representation of the propagator in order to obtain the gap equations in the presence of a homogeneous magnetic field. The zero temperature gap equation in the presence of a magnetic field reads

\[
\bar{\sigma} = G \frac{N_c}{4\pi^4} \sum_{f=u,d} \int_0^\infty ds \int_0^\infty d^4p r^2(p)\Sigma(p) \times e^{-s\left(\Sigma^2(p) + p^2_{\parallel} + \frac{\tan(bq_B)}{K^2} + p^2_{\perp}\right)}, \tag{19}
\]

The finite temperature gap equation is obtained from Eq. (18) by using the prescriptions in Eqs. (6) and (7).

### III. MAGNETIC CATALYSIS

The gap equations introduced in the previous section will allow us to compute \(\bar{\sigma}(T, qB)\) for both the NJL and nNJL models. We can then look for the critical temperature for the chiral phase transition obtaining its dependence on the magnetic field, i.e. \(T_c(qB)\). Since we are interested in studying how the magnetic field affects the critical temperature for the chiral phase transition, we will work in the critical limit \(m = 0\). If we do not do this, then the transition between the broken and restored phases is rather a crossover, which makes the definition of a critical temperature a bit ambiguous. Having fixed the value of the current mass \(m\) to zero, the NJL model still has two parameters that need to be fixed, namely \(\Lambda\) and the coupling constant \(G\). We take \(\Lambda = 1086\) MeV and \(GA^2 = 7.56\) from [25], being then our energy scale \(\Lambda\) much bigger than the temperatures and magnetic fields involved in our analysis. Using Eq. (5) we can also determine \(M(T = qB = 0) = \bar{\sigma}(T = qB = 0) = 200\) MeV. We can then use the gap equation in the presence of a magnetic field to get \(T_c(qB)\).

As seen in Fig. 1 and 2 the critical temperature for the chiral phase transition rises with \(qB\). For the case \(B = 0\), the critical temperature was found around \(T \approx 152\) MeV. This is known as magnetic catalysis. In order to test the validity of the expansion in powers of \(eB\) in the weak field case, we solved the NJL model in this limit finding similar results to those of the exact calculation.

For the nNJL model we will consider a Gaussian regulator inspired in the liquid instantons model [15, 17]

\[
r^2(q_E^2) = e^{-q_E^2/\Lambda^2}. \tag{20}
\]

We fix the free parameters of the model using as input the pion decay constant \(f_\pi\), and the chiral condensate \(\langle \bar{q}q \rangle\) [31]. This yields \(\Lambda = 914\) MeV and \(G = 21 \cdot 10^{-6}\) MeV\(^{-2}\). From the gap equation (17) we also get \(\bar{\sigma}(T = qB = 0) = 235\) MeV. For simplicity, we will consider weak magnetic fields and expand in power of \(eB\) up to order \((eB)^2\). The fermionic propagator in this limit is given by [32]

\[
S_E(p) = \frac{(\Sigma - \not{p})}{K^2 + m_f^2} + i \frac{\gamma_1 \gamma_2 (qB)(\Sigma - \not{p}_I)}{(p^2 + \Sigma^2)^2} \\
+ \frac{2(qB)^2 p_I^2}{(p^2 + \Sigma^2)^4} \left( (\Sigma - \not{p}_I) + \frac{\not{p}_I (\Sigma^2 + p_{\perp}^2)}{p_{\perp}^2} \right). \tag{21}
\]

In this case, the finite temperature gap equation in the presence of an homogeneous magnetic field is

\[
\bar{\sigma} = GT \frac{2N_c}{\pi^2} \sum_{f=u,d} \sum_n \int dp^2 r^2(\omega_n, p) \times \left[ \frac{\Sigma(\omega_n, p)}{p^2 + \omega_n^2 + \Sigma^2(\omega_n, p)} + \frac{4}{3} \frac{|qB|^2 (p^2 + \omega_n^2) \Sigma^2(\omega_n, p)}{(p^2 + \omega_n^2 + \Sigma^2(\omega_n, p))^2} \right]. \tag{22}
\]

From Eq. (18) we can determine the critical temperature for the chiral phase transition in the absence of a magnetic field to be \(T_c \approx 127\) MeV. Then, the weak field limit, \(qB < (\pi T)^2\), will be met for the magnetic fields we are considering.

![FIG. 1. Critical temperature for the chiral phase transition as a function of \(eB\) for the NJL and nNJL models.](image-url)
Fig. 2 shows the change in the critical temperature, normalized by its value at vanishing magnetic field in each model, as a function of the magnetic field. This allows us to appreciate that the slope is greater in the nNJL model. In this sense, if we consider the normalized critical temperature, the catalysis effect is stronger in the nNJL model.

Fig. 2. Critical temperature, normalized by its value at $B = 0$ for each model, for the chiral phase transition as a function of $eB$ for the NJL and nNJL models.

**IV. CONCLUSIONS**

We have studied the NJL and nNJL model in the presence of a homogeneous magnetic field. In both cases we found that the critical temperature for the chiral phase transition increases as the magnetic field grows. This is consistent with the catalysis found in most model calculations. We find that this effect is stronger in the nonlocal model if one considers the normalized critical temperature. We have also shown how to include magnetic field in the nNJL model in the weak field limit. We verified that the approximation taken in this limit is valid and gives similar results to the exact calculation in the NJL model.

The inverse magnetic catalysis recently found from lattice results is not observed. However, it has been suggested [33, 34] that the inclusion of thermal and magnetic corrections to the coupling, going beyond the mean field approximation, provide a temperature and magnetic field dependent mass. This effect produces a decreasing behavior for $T_c(qB)$, i.e. inverse magnetic catalysis. In principle, this type of behavior, could be also found in nNJL models when going beyond the mean field approximation.

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