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Weakly Compact Operators and the Dunford-Pettis Property on Uniform Spaces

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ABSTRACT. Let $(X, \mathcal{U})$ be a Hausdorff uniform space and $C_b(X)$ the space of all bounded continuous real-valued functions on $X$. The subspace of $C_b(X)$, consisting of the all uniformly continuous functions with respect to $\mathcal{U}$, is denote by $C_{ub}(X)$. In this paper we give a characterization of weakly compact operators and $\beta_u$ - continuous defined from $C_{ub}(X)$ into a Banach space $E$, where $\beta_u$ is the finest locally convex topology agreeing with the pointwise topology on each uniformly equicontinuous and bounded subsets of $C_{ub}(X)$. We also show that $(C_{ub}(X), \beta_u)$ has the Strict Dunford-Pettis Property and the Dunford-Pettis Property, both under special conditions.

1. INTRODUCTION AND NOTATIONS

All uniform spaces $(X, \mathcal{U})$ are assumed to be Hausdorff uniform spaces. Basic references for the measures theory on topological spaces are in Varadarajan [7]. We will denote by $C_b(X)$ the space of all real-valued bounded continuous functions defined on $X$, and $C_{ub}(X)$, the subspace of $C_b(X)$, consists of those functions which are uniformly continuous. $\mathcal{H}$ will denote the collection of all uniformly equicontinuous and bounded (U.E.B.) subsets of $C_{ub}(X)$. $\beta_u$ will denote the finest locally convex topology agreeing with the pointwise topology on each $H \in \mathcal{H}$. A uniform measure on $X$ is defined to be a bounded linear functional on $C_{ub}(X)$ which is pointwise continuous on each $H \in \mathcal{H}$ (see [1], [2], [4]) and the space of all uniform measures will be denoted by $\mu_u(X)$. It is well known that the dual of $(C_{ub}(X), \beta_u)$ is $\mu_u(X)$.

Let $M(X)$ be the dual of $(C_{ub}(X), \|\cdot\|)$, where the $\|\cdot\|$ denotes the supremum norm. We denote by $\mathcal{H}$-top the locally convex topology on $M(X)$ of uniform convergence on the U.E.B. sets. Let $M_d(X)$ be the subspace of $M(X)$ generated by the Dirac measures. It was proved in [4] that the $\mathcal{H}$-top closure of $M_d(X)$ is the space $M_u(X)$.

We denote by $\beta_k$ the locally convex topology on $C_b(X)$ agreeing with the compact-open topology on each norm-bounded subset of $C_b(X)$. Sentilles [6] proved that the

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dual of \((C_b(X), \beta_t)\) is the space \(M_t(X)\) of all tight measures on \(X\). One of the Sentilles's results that we will use here is the following: "a subset \(A\) of \(M_t(X)\) is uniformly tight if, and only if, \(A\) is \(\beta_t\)-equicontinuous. It is also known that \(C_{ub}(X)\) is \(\beta_t\)-dense on \(C_b(X)\) (see [1]).

The proof of the following technical lemma is a simple verification and it will be omitted.

**Lemma 1.** The canonical mapping \(\Phi : (C_{ub}(X), \beta_u) \to (C_{ub}(\hat{X}), \beta_u)\) defined by \(f \to \hat{f}\) is an isomorphism, where \(\hat{X}\) denotes the completion of \((X, \mathcal{U})\) and \(\hat{f}\) is the unique uniform extension of \(f\) to \(\hat{X}\).

2. **Weakly Compact Operators**

Let \(E\) be a Banach space. In this section we will study \(E\)-valued linear norm-continuous operators on \(C_{ub}(X)\), in particular, weakly compact operators.

**Definition 1.** Let \(T\) be a \(E\)-valued linear continuous operator on \(C_{ub}(X)\). We shall say that \(T\) is a tight additive operator if its restriction to the unit ball of \(C_{ub}(X)\) is continuous for the compact-open convergence topology.

Note that \(T\) is tight additive if, and only if, \(T\) is \(\beta_t\)-continuous and so, by the density of \(C_{ub}(X)\) in \(C_b(X)\) via the topology \(\beta_t\), \(T\) has a unique continuous extension to \(C_b(X)\).

From now on we will assume that \(T : C_{ub}(X) \to E\) is a weakly compact operator, that is, \(T\) transforms the unit ball of \(C_{ub}(X)\) into a relatively weakly compact subset of \(E\). By the denseness, \((C_{ub}(X), \beta_t)' = (C_b(X), \beta_t)' = M_t(X)\), the space of all tight measures (see [6]), and by the weakly compactness of \(T\), \(T''(C_{ub}(X)'' \subset E\). On the other hand, \(T\) has a unique \(\beta_t\)-continuous extension \(\widetilde{T}\) to \(C_b(X)\) and, by the latter,

\[
\widetilde{T}''(C_b(X)'' = T''(M_t(X)') = T''(C_{ub}(X)'') \subset E.
\]

Then, \(\widetilde{T}\) is also a weakly compact operator.

The following theorem will give a characterization of tight operators and the proof is based on the well known result which says that \(\beta_u\) and the norm topology have the same bounded sets (see [1]).

**Theorem 2.** Suppose that \(\mathcal{U}\) is metrizable and \((X, \mathcal{U})\) is complete. Then, the following statements are equivalent:

(a) \(T\) is \(\beta_u\)-continuous

(b) \(T\) is tight additive
Proof. Since \((X, \mathcal{U})\) is metrizable and complete, we have that the pointwise topology and the compact-open topology coincide on each U.E.B. subset of \(C_{ub}(X)\).

(a) \iff (b) Since \(T\) is \(\beta_u\)-continuous, we have that \(\{x' \circ T : \|x'\| \leq 1\} \subseteq M_u(X)\).

On the other hand, since \(T\) is also weakly compact operator,

\[ T'\left(\{x' : \|x'\| \leq 1\}\right) = \{x' \circ T : \|x'\| \leq 1\} \]

is relatively \(\sigma(M(X), M(X)')\)-compact. By this fact and since \(M(X)\) is an AL-space, the closure of the solid hull of \(\{x' \circ T : \|x'\| \leq 1\}\) is \(\sigma(M(X), M(X)')\)-compact (see Cor. 8.8, p. 119, \[5\]). Since \(\sigma(M(X), M(X)')\) is finer than \(\sigma(M(X), C_{ub}(X))\), we have that \(\{|x' \circ T| : \|x'\| \leq 1\}\) is relatively \(\sigma(M(X), C_{ub}(X))\)-compact. Then, by \[1\], p. 239, \(\{|x' \circ T| : \|x'\| \leq 1\}\) is a tight set.

Now, take a net \((f_\alpha)_{\alpha \in I}\) in the unit ball of \(C_{ub}(X)\) converging to 0 in the topology of compact-open convergence; hence by the above and the tightness of

\[ \{|x' \circ T| \circ (f_\alpha) : \|x'\| \leq 1\}, \]

we have \(|x' \circ T(f_\alpha)| \leq |x' \circ T| \circ (f_\alpha) \to 0\) uniformly for \(\|x'\| \leq 1\). This argument shows that \(\|Tf_\alpha\| \to 0\).

(b) \iff (a) By the tightness of \(T\), its restriction to each U.E.B. set is continuous for the topology of compact-open convergence. But, this topology coincides with the topology of pointwise convergence on each U.E.B. set; therefore \(T\) is \(\beta_u\) - continuous.

In the next theorem we shall use the following notations: If \((X, \mathcal{U})\) is a uniform space and \(d\) is uniformly continuous pseudometrics (u.c.p.) on \(X\), then \(\hat{X}_d\) denotes the completion of the metric space which comes from \(X, d\) and the corresponding projection, \(\pi_d\).

**Theorem 3.** Let \(T\) be a weakly compact \(E\) - valued operator defined on \(C_{ub}(X)\). Then, the following statements are equivalent:

1. \(T\) is \(\beta_u\) - continuous
2. \(T|_H\) is pointwise continuous for each U.E.B. set \(H\).
3. \(\{x' \circ T : x' \in E'; \|x'\| \leq 1\}\) is \(\beta_u\) - equicontinuous.
4. \(\{|x' \circ T| : x' \in E'; \|x'\| \leq 1\}\) is relatively \(\sigma(M_u(X), C_{ub}(X))\) - compact
5. For each u.c.p. \(d\) on \(X\), \(\pi_d \circ T\) (natural definition) is a tight additive operator.
Proof. The equivalences \((3) \Leftrightarrow (1) \Leftrightarrow (2)\) are clear. The equivalence \((3) \Leftrightarrow (4)\) follows from [1, p. 228 and 241].

(1) \Rightarrow (5) Let \(d\) be a u.c.p. on \(X\). Since \(\pi_d\) is uniformly continuous, we have that \(\pi_d \circ T : C_{wb}(X) \to E\), defined by \((\pi_d \circ T)(\hat{f}) = T \left(\hat{f} \circ \pi_d\right)\), is \(\beta_u\) – continuous and a weakly compact operator. Therefore, by Th. 2.2, \(\pi_d \circ T\) is a tight additive operator. (5) \Rightarrow (1) Let \(H \in \mathcal{H}\); hence \(d_H(x, y) = \sup \{|f(x) - f(y)| : f \in H\}\) is a u.c.p. on \(X\).

Denote by \(\pi_H\) the corresponding projection of \(d_H\) and defined by \(\hat{f}(\pi_H(x)) = f(x), \) for any \(f \in H\). It is not difficult to see that the function \(\hat{f}\) is well defined, it belongs to \(C_{wb}(X^*_d)\) and the \(\hat{H} = \left\{\hat{f} : f \in H\right\}\) is a U.E.B. subset of \(C_{wb}(X^*_d)\).

Take a net \((f_\alpha)_{\alpha \in I}\) in \(H\) such that \(f_\alpha \to 0\) pointwise. It easily follows that \(\hat{f}_\alpha \to 0\) pointwise and, from the hypothesis, \(\left\|(\pi_d \circ T)(\hat{f}_\alpha)\right\| \to 0\). Therefore, since \(\|Tf\| = \left\|(\pi_d \circ T)(\hat{f})\right\|\), for any \(f \in H\), we get that \(T\) is \(\beta_u\) – continuous.

3. DUNFORD-PETTIS AND STRICT DUNFORD-PETTIS PROPERTY

In this section we will analyze the Strict Dunford-Pettis and the Dunford-Pettis Property of the locally convex space \((C_{wb}(X), \beta_u)\). We begin with the definition of these properties which were given by Grothendieck in his well known paper “Sur les applications linéaires faiblement compact d’espace du type \(C(K)\)”, Canad. J. Math. 5(1974), 183-201.

Definition 2. We shall say that a Hausdorff locally convex space \(E\) has the Dunford-Pettis Property (resp. Strict Dunford-Pettis Property) if for any Banach space \(F\) and every linear continuous and weakly compact operator \(T : E \to F\), \(T(C)\) is relatively compact (resp. \(\{T(x_n)\}\) is Cauchy) in \(F\) for any absolutely convex weakly compact subset \(C\) (resp. weak Cauchy sequence \(\{x_n\}\)) in \(E\).

Theorem 4. If \(U\) is metrizable, then \((C_{wb}(X), \beta_u)\) has the Dunford-Pettis and Strict Dunford-Pettis Properties.

Proof. First we shall assume that \((X, U)\) is a complete metrizable uniform space. Let \(T\) be a weakly compact and \(\beta_u\) – continuous linear operator defined from \(C_{wb}(X)\) into an arbitrary Banach space \(F\). By Th. 2.2, \(T\) is a tight operator and then \(T\) admits a unique extension \(\tilde{T}\) to \(C_b(X)\) which is \(\beta_u\) – continuous.

We shall first prove that \((C_{wb}(X), \beta_u)\) has the Strict Dunford-Pettis Property. Let \(\{f_n\}_{n \in \mathbb{N}}\) be a \(\sigma(C_{wb}(X), M_u(X))\) – Cauchy sequence in \(C_{wb}(X)\). Since \(M_u(X) = \)
\(M_t(X)\) and \(\{f_n\}_{n \in \mathbb{N}}\) is in \(C_b(X)\), we have that this sequence is \(\sigma(C_b(X), M_t(X))\) - Cauchy. Now, by [3], we know that \((C_b(X), \beta_t)\) has the Strict Dunford-Pettis Property, therefore \(\{Tf_n\}_{n \in \mathbb{N}} = \{\tilde{T} f_n\}_{n \in \mathbb{N}}\) is convergent in \(F\).

If \(\mathcal{U}\) is metrizable, then the conclusion follows from Lemma 1.1, since

\[
T \circ \Phi : C_{ub}(\hat{X}) \to F
\]

is \(\beta_u\) - continuous and weakly compact operator.

Now, to prove that \((C_{ub}(X), \beta_u)\) has the Dunford-Pettis Property, we again assume that \((X, \mathcal{U})\) is complete metrizable uniform space and follows the similar argument given above.

One of the open problems that we still face, is whether or not \((C_{ub}(X), \beta_u)\) has the Strict Dunford-Pettis Property. We already proved that the answer is yes if \(\mathcal{U}\) is metrizable. In the next theorem we will assume that \((C_{ub}(X), \beta_u)\) has the Strict Dunford-Pettis Property and we will prove that it has the Dunford-Pettis Property under the condition that \(X\) is \(\sigma\)-compact.

**Theorem 5.** If \((C_{ub}(X), \beta_u)\) possesses the Strict Dunford-Pettis Property and \(X\) is \(\sigma\)-compact, then \((C_{ub}(X), \beta_u)\) has the Dunford-Pettis Property.

**Proof.** Let \(\{K_n\}_{n \in \mathbb{N}}\) be an increasing sequence of compact subsets of \(X\) such that \(\bigcup_{n=1}^{\infty} K_n\) is dense on \(X\). We will denote by \(L_n\) the closed absolutely convex hull of \(K_n\) in \(M_u(X)\) \((X\) is a uniform subspace of \(M_u(X))\). Since \(K_n\) is a compact subset of \(M_u(X)\) in the \(H\)-top and \((M_u(X), H\text{-top})\) is complete, we have that \(L_n\) is a compact subset of \(M_u(X)\). Moreover, \(\{L_n\}_{n \in \mathbb{N}}\) is an increasing sequence.

We claim that \(\bigcup_{n=1}^{\infty} L_n\) is \(H\)-top dense in \(M_u(X)\). In fact, take \(\mu \in M_u(X)\) and a balanced neighborhood \(V\) of \(\mu\). Since \(M_d(X)\) is \(H\)-top dense in \(M_u(X)\), \(V\) contains some element \(\nu = \sum_{i=1}^{p} \alpha_i \delta_{x_i}\) of \(M_d(X)\), with \(x_1, x_2, \ldots, x_p \in X\). Suppose that \(0 < \alpha = \sum_{i=1}^{p} |\alpha_i| \leq 1\) \((\alpha = 0, V \cap \bigcup_{n=1}^{\infty} L_n) \neq \emptyset\) and we are done) and take neighborhoods \(W_i\) of \(\delta_{x_i}\), \(i = 1, 2, \ldots, p\), such that \(\sum_{i=1}^{p} \alpha_i W_i \subset V\). Since \(W_i \cap X\) is a neighborhood of \(\delta_{x_i}\) in \(X\), we get \(\delta_{y_i} \in K_{n_i}\) such that \(\delta_{y_i} \in W_i \cap X\). Thus, \(\sum_{i=1}^{p} \alpha_i \delta_{y_i} \in \sum_{i=1}^{p} \alpha_i W_i \subset V\) and \(\sum_{i=1}^{p} \alpha_i \delta_{y_i} \in L_N\), where \(N = \max \{n_i : i = 1, 2, \ldots, p\}\). Therefore, \(V \cap \bigcup_{n=1}^{\infty} L_n\) \(\neq \emptyset\).

Suppose now that \(\alpha > 1\); hence \(\alpha \sum_{i=1}^{p} \frac{\alpha}{\alpha} \delta_{x_i} \in V\) and so \(\sum_{i=1}^{p} \frac{\alpha}{\alpha} \delta_{x_i} \in \frac{1}{\alpha} V \subset V\). Applying
the above argument to $\frac{1}{\alpha}V$, we get $V \cap \left( \bigcup_{n=1}^{\infty} L_n \right) \neq \emptyset$.

From this, we have that $(M_u(X), \mathcal{H} - \text{top})$ is a σ-compact space, which implies that $(M_u(X), \sigma(M_u(X), C_u(X)))$ is also a σ-compact space. The conclusion of the theorem follows from [3], Th. 1

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