The Vibration of a Simply Supported Visco-Thermoelastic Nano-beam of Silicon Nitride Induced by Thermal Shock

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Abstract

In this article, the vibration of visco-thermoelastic isotropic homogeneous nano-beam has been studied. A general model of visco-thermoelasticity theory under simply supported conditions of one relaxation time has been used. The Laplace transformation has been applied for the governing equations. The inverse Laplace transformation has been considered by using Tzou procedure. When subjected to thermal shock loading and simply supported conditions, the numerical results have been validated for a visco-thermoelastic rectangular nano-beam of silicon nitride. Figures of this paper represent numerical results to describe the effects of the visco-thermoelastic parameters and the thickness of the nano-beam. The beam’s thickness and the visco-thermoelastic parameters have been significant effects on all the studied state-functions.

Keyword: Vibration; Visco-thermoelastic; Nano-beam; Silicon Nitride; Thermal shock; Fourier law

Introduction

The concept of heat conduction has been considered using mathematical models for example dual-phase lag (DPL), which was suggested by Tzou [1,2]. The heat flux and temperature gradient have been recognized by DPL model. Several researchers used DPL model in heat transfer problems [3], physical systems [4-8]. One of the types of heat conduction is the theory of coupled thermoelasticity that consists of two differential partial equations: the equation of energy conservation and the equation of motion, based on Fourier’s law of heat conduction [9-12]. Lord and Shulman introduced the relaxation time in case of an isotropic body by modifying Fourier’s law of heat conduction, such that including the heat flux and its time derivative. In other word, non-Fourier’s law of heat conduction replaces to Fourier’s law [13]. In this case, the heat equation in the theory of coupled thermoelasticity is a hyperbolic equation that detects and removes the infinite speeds of propagation [14].

Mechanical signal processing, ultrasensitive mass detection, scanning probe microscopes, actuators, signal processing components and ultrafast sensors etc. are applications based on micro and nano-electromechanical beam resonators [15-18]. Nano-beam’s vibration is the most important of the micro/nano beam resonator.

Alghamdi [9] used the DPL thermoelasticity theory to studied the damping thermoelastic vibration of beam resonator with voids. Sharma and Grover [19] investigated the transverse thermoelastic vibration of isotropic and homogenous micro/nano thin beam resonators with voids. But the study of the damping thermoelastic vibration was done by Sun and Saka [20] for microplate circular resonators. They added factor K = (1 + υ)/(1 − 2υ) to formula of thermoelastic damping, so their formula becomes different from that of Lifshitz and Roukes [21], where υ is Poisson’s ratio. Several researchers have investigated the heat transfer process and the vibration of nano-beams [22-26]. The study of the vibration of nano-beam gold subjected to thermal shock was done by Eman and Hamdi [23]. Kidawa [25] used the properties of the Green functions to study the effects of internal and external damping on beam’s vibrations caused a moving heat source. Boley [24] investigated the affect a thermal shock on vibrations of a rectangular nano-beam that was simply supported. Manolis and Beskos [26] studied vibration of thermoelastic nano-beam’s dynamic response under the effect of thermal loading by using a numerical method of analysis. Al-Huniti et al. [22] used a high-power moving laser beam to investigated the thermally induced displacements and stresses of heated rod and it’s dynamical behaviour by using the Laplace transforms technique.

The study of visco-thermoelasticity has become important in mechanics. Biot [27,28] discussed the theory of visco-thermoelasticity and the principles in thermodynamics vibration. Drozdov [29] derived a thermo-visco-elasticity model at finite strains. Ezzat and El-

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Karmany [30] used a model of thermo-viscoelasticity for isotropic material to studying the effects of volume properties of viscoelastic media in thermoelasticity. Carcione et al. [31] used a numerical algorithm in an elastic material using the Kelvin–Voigt model. Grover [32–34] studied vibrations in visco-thermoelastic micro-beam resonators. Grover and Seth [35] used a dual phase lagging model to study visco-thermoelastic microbeam resonators.

**Basic Equations**

We will consider a thermally isotropic homogenous conducting, Kelvin–Voigt type thermo-viscoelastic solid in systems of Cartesian coordinate initially unformed and at a uniform temperature $T_0$. The governing equations of motion and heat conduction in the context of generalized (non-Fourier) thermoelasticity for displacement vector $\mathbf{u}(x, y, z, t) = (u, v, w)$ and temperature change $T(x, y, z, t)$, in the absence of heat sources and body forces, are given by [34]:

$$\sigma_{ij,j} = \rho \ddot{u}_i, i, j = x, y, z \quad (1)$$

$$\sigma_{ij} = \lambda_0 \delta_{ij} e_{kk} + 2\mu e_{ij} - \beta \delta_{ij} (T - T_0), i, j = x, y, z \quad (2)$$

$$KT_{ii} = \left( \frac{\partial}{\partial t} + \frac{\alpha_T}{\alpha_0} \frac{\partial^2}{\partial z^2} \right) \left( \rho C_v T + \beta T \frac{\partial e_{ij}}{\partial z} \right), i, j = x, y, z \quad (3)$$

$$e_{ij} = \frac{1}{2} (u_{ij} + u_{ji}), i, j = x, y, z \quad (4)$$

$$\lambda = \lambda_0 \left( 1 + \frac{\alpha}{\alpha_0} \frac{\partial}{\partial t} \right), \mu = \mu_0 \left( 1 + \mu_1 \frac{\partial}{\partial t} \right), \beta = (3\lambda + 2\mu) \alpha_T \quad (5)$$

Where $\rho$ is the density, $\alpha_T$ is the coefficient of linear thermal expansion, $\lambda_0, \mu_1$ are the viscoelastic relaxation times, $\tau_0$ is the thermal relaxation time, $\lambda_0, \mu_0$, Lamé’s parameter in usual case, $C_v$ is the specific heat and $K$ is the thermal conductivity.

**Problem Formulation**

We will consider small flexural deflections of an elastic thin beam of length $\ell (0 \leq x \leq \ell)$, width $b \left( -\frac{b}{2} \leq y \leq \frac{b}{2} \right)$ and thickness $h \left( -\frac{h}{2} \leq z \leq \frac{h}{2} \right)$, for which the $x$, $y$, and $z$ axes are defined along the longitudinal, width, and thickness directions of the beam, respectively. In equilibrium, the beam is unstrained, unstressed, without a damping mechanism, and the temperature is $T_0$ everywhere [6].

In the present work, the Euler–Bernoulli assumption [32] is adopted, so, any plane cross-section, initially perpendicular to the axis of the beam remains plane and perpendicular to the neutral surface during bending. Thus, the displacements are given by

$$u = -z \frac{\partial w(x, t)}{\partial x}, v = 0, w(x, y, z, t) = w(x, t) \quad (6)$$

The flexural moment of cross-section is given by

$$M(x, t) = (\lambda + 2\mu) I \frac{\partial^2 w}{\partial x^2} + \beta M_T \quad (7)$$

and $M_T$ is the thermal moment of the beam which is given by:

$$M_T = b \int \frac{h}{2} \theta dz \quad (8)$$

$I = \frac{bh^3}{12}$ is the moment of inertia of the cross-section about $x$-axis.

Hence, the thermally differential equation induced the beam lateral vibration expressed in the form [32]:

$$(\lambda + 2\mu) I \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} + (3\lambda + 2\mu) \alpha_T \frac{\partial^2 w}{\partial x^2} = 0 \quad (9)$$

where $w(x, t)$ is the lateral deflection, $A = bh$ is the cross-section area, and $\theta = (T - T_0)$ is the temperature increment of the resonator.

The non-Fourier heat conduction equation has the following form [32]:

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \left( \frac{\partial}{\partial t} + \frac{\alpha_T}{\alpha_0} \frac{\partial^2}{\partial z^2} \right) \left( \frac{\rho C_v}{k} \frac{\partial \theta}{\partial z} + \frac{(3\lambda + 2\mu) \alpha_T}{k} \right) \quad (10)$$

where $e = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}$ is the volumetric strain which gives from (6) that:

$$e = -\frac{\partial^2 w}{\partial x^2} \quad (11)$$

From the relation in (5), we have the following:

$$\lambda + 2\mu = (\lambda_0 + 2\mu_0) \left( 1 + \beta_1 \frac{\partial}{\partial t} \right), (3\lambda + 2\mu) = (3\lambda_0 + 2\mu_0) \left( 1 + \beta_2 \frac{\partial}{\partial t} \right) \quad (12)$$

where $\beta_1 = \frac{(3\lambda + 2\mu) \alpha_T}{\alpha_0}$, $\beta_2 = \frac{(3\lambda + 2\mu) \alpha_T}{(3\lambda_0 + 2\mu_0) \alpha_T}$ are the combination of the viscoelastic relaxation times parameters.

Because there is no heat flow across the lower and upper beam surfaces, so that $\frac{\partial \theta}{\partial z} = 0$ at $z = \pm h/2$. For a very thin beam and assuming the temperature varies in terms of $a \sin(pz)$ function along the thickness direction, where $p = \pi/h$, gives [36]:

![Fig. 1: Rectangular Nanobeam](image-url)
\( \theta(x, z, t) = \vartheta(x, t) \sin(pz) \)  

(13)

Hence, equations (8), (9), and (13) gives:

\[
(\lambda_0 + 2\mu_0) \left( 1 + \beta_1 \frac{\partial}{\partial t} \right) \theta + \frac{12\rho}{h^2} \frac{\partial^3 w}{\partial t^3} + \frac{24(3\lambda_0 + 2\mu_0)}{h^2} \frac{\partial^2 w}{\partial t^2} \bigg( 1 + \frac{12\beta_2}{\partial t} \frac{\partial^2 w}{\partial x^2} = 0
\]

(14)

and equation (10) gives:

\[
\left( \frac{\partial^2 \vartheta}{\partial z^2} - p^2 \vartheta \right) \sin(pz) = \left( \frac{\partial^2 \vartheta}{\partial t^2} \right) \frac{\rho \varepsilon_0}{\kappa} \vartheta \sin(pz) - \frac{(3\lambda_0 + 2\mu_0)}{k} \frac{\partial}{\partial z} \bigg( 1 + \frac{\beta_2}{\partial t} \frac{\partial^2 w}{\partial x^2} \bigg) z \sin(pz) dz = 0
\]

(15)

After doing the integrations, equation (14) takes the form

\[
(\lambda_0 + 2\mu_0) \left( 1 + \beta_1 \frac{\partial}{\partial t} \right) \theta + \frac{12\rho}{h^2} \frac{\partial^3 w}{\partial t^3} + \frac{24(3\lambda_0 + 2\mu_0)}{h^2} \frac{\partial^2 w}{\partial t^2} \bigg( 1 + \frac{12\beta_2}{\partial t} \frac{\partial^2 w}{\partial x^2} = 0
\]

(16)

In equation (15), we multiply the both sides by \( z \) and integrating with respect to \( z \) from \( -\frac{h}{2} \) to \( \frac{h}{2} \), then we obtain

\[
\frac{\partial^2 \vartheta}{\partial z^2} - p^2 \vartheta = \left( \frac{\partial^2 \vartheta}{\partial t^2} \right) \left( \varepsilon_0 - \frac{\tau_0 h^2 (3\lambda_0 + 2\mu_0)}{24k} \bigg( 1 + \frac{12\beta_2}{\partial t} \frac{\partial^2 w}{\partial x^2} \right)
\]

(17)

where \( \varepsilon = \frac{\rho C_0}{\kappa} \)

Now, we will use the non-dimensional variables [19]:

\[
(x', w', h', \varepsilon) = \varepsilon C_0 (x, w, h, \varepsilon), (t', \tau_0, \beta_1, \beta_2) = \varepsilon C_0 (t, \tau_0, \beta_1, \beta_2), \sigma' = \frac{\lambda_0 + 2\mu_0}{\rho}.
\]

(18)

Then, we have

\[
(1 + \beta_1 \frac{\partial}{\partial t}) \frac{\partial^2 w}{\partial x^2} + \varepsilon_1 \frac{\partial^2 w}{\partial t^2} + \varepsilon_2 \left( 1 + \beta_2 \frac{\partial}{\partial t} \right) \frac{\partial^2 w}{\partial x^2} = 0
\]

(19)

And

\[
\frac{\partial^2 \vartheta}{\partial x^2} - \varepsilon_3 \vartheta = \left( \frac{\partial^2 \vartheta}{\partial t^2} \right) \bigg( \varepsilon_4 - \frac{1 + \beta_2}{\partial t} \frac{\partial^2 w}{\partial x^2} \bigg)
\]

(20)

\[
\sigma_{xx} = \left( 1 + \beta_1 \frac{\partial}{\partial t} \right) \left( 1 + \beta_2 \frac{\partial}{\partial t} \right) \vartheta \sin(pz)
\]

where \( \varepsilon_1 = \frac{12}{h^2}, \varepsilon_2 = \frac{24\gamma}{24k \varepsilon_0^2}, \varepsilon_3 = p^2, \varepsilon_4 = \frac{\pi^2 h (3\lambda_0 + 2\mu_0) T \tau_0}{24k \varepsilon_0^2}, \gamma = \frac{(3\lambda_0 + 2\mu_0) T \tau_0 \beta_0}{(\lambda_0 + 2\mu_0)}.
\]

We have dropped the prime for convenience

**Formulation the Problem in the Laplace Transform Domain**

The Laplace transform for equations (19) and (20), which is defined by the following formula will be applied:

\[
f(s) = \int_0^\infty f(t) e^{-st} \; dt
\]

(22)

Hence, we obtain the following system of differential equations:

\[
(1 + \beta_1 s) \frac{\partial^2 \vartheta}{\partial x^2} + \varepsilon_1 s^2 \vartheta + \varepsilon_2 \left( 1 + \beta_2 s \right) \frac{\partial^2 \vartheta}{\partial x^2} = 0
\]

(23)

And

\[
\frac{\partial^2 \vartheta}{\partial x^2} - \varepsilon_3 \vartheta = (s + \tau_0 s^2) \left( \varepsilon_4 - (1 + \beta_2 s) \frac{\partial^2 \vartheta}{\partial x^2} \right)
\]

(24)

\[
\sigma_{xx} = (1 + \beta_2 s) \vartheta - \gamma (1 + \beta_2 s) \vartheta \sin(pz)
\]

(25)

\[
\vartheta(x, 0) = w(x, 0) = \frac{\partial w(x, 0)}{\partial t} = 0
\]

(27)

We can re-write the above system to be in the forms:

\[
(\mathcal{D}^4 + \varepsilon_3) \vartheta = -\varepsilon_4 \mathcal{D}^2 \vartheta
\]

(28)

And

\[
(\mathcal{D}^2 - \varepsilon_3) \vartheta = -\varepsilon_4 \mathcal{D} \vartheta
\]

(29)

where \( \mathcal{D}^2 = \frac{\partial^2}{\partial x^2}, \mathcal{D} = \frac{\partial}{\partial x} \)

Eliminating \( \vartheta \) between the equations of the above system, then, we get

\[
[D^6 - LD^4 + MD^2 - N] \hat{\vartheta} = 0
\]

(30)

In similar, eliminating \( \vartheta \) gives:

\[
[D^6 - LD^4 + MD^2 - N] \hat{\vartheta} = 0
\]

(31)

where \( L = \varepsilon_3 + \varepsilon_6 \varepsilon_3 \), \( M = \varepsilon_3 N = \varepsilon_6 \varepsilon_7 \)

The solutions of the equations (30) and (31) take the forms:

\[
\hat{\vartheta}(x, s) = -\varepsilon_6 \sum_{i=1}^{3} c_i k_i^2 \sinh(k_i(\ell - x))
\]

(32)

And

\[
\hat{\vartheta}(x, s) = \sum_{i=1}^{3} c_i (k_i^2 - \varepsilon_7) \sinh(k_i(\ell - x))
\]

(33)

where \( \pm k_1, \pm k_2, \pm k_3 \) are the roots of the characteristic equation.
To calculate the constants $c_i$, $i = 1, 2, 3$, we must apply any set of boundary conditions, so we consider that the beam is thermally shocked and simply supported as following:

$$w(0, t) = \frac{\partial^2 w(0, t)}{\partial x^2} = 0, \ \vartheta(0, t) = \vartheta_0 H(t)$$

(35)

And

$$w(\ell, t) = \frac{\partial^2 w(\ell, t)}{\partial x^2} = \vartheta(\ell, t) = 0$$

(36)

where $H(t)$ is the unit step function and $\vartheta_0$ is constant which gives the strength of the thermal shock.

Apply the Laplace transform, we have

$$\tilde{w}(0, s) = \frac{\partial^2 \tilde{w}(0, s)}{\partial x^2} = 0, \ \tilde{\vartheta}(0, s) = \frac{\vartheta_0}{s}$$

(37)

and

$$\tilde{w}(\ell, s) = \frac{\partial^2 \tilde{w}(\ell, s)}{\partial x^2} = \tilde{\vartheta}(\ell, s) = 0$$

(38)

Then, we obtain the following system of linear equations:

$$\sum_{i=1}^{3} c_i k_i^2 \sinh(k_i \ell) = -\frac{\vartheta_0}{s \epsilon_0}$$

(39)

$$\sum_{i=1}^{3} c_i (k_i^2 - \epsilon_i) \sinh(k_i \ell) = 0$$

(40)

And

$$\sum_{i=1}^{3} c_i (k_i^2 - \epsilon_i) k_i^2 \sinh(k_i \ell) = 0$$

(41)

After solving the above system equations, then, we get the solutions in the Laplace transform domain as:

$$\tilde{\vartheta}(x, s) = \frac{\vartheta_0}{s \epsilon_0 \sinh(k_1 \ell)} \left[ \frac{(x - k_1 \ell)(x - k_2 \ell)}{k_1^2 - k_2^2}\sinh(k_1 \ell) + \frac{(k_1^2 - k_2^2)(k_1^2 - k_3^2)}{k_2^2 - k_3^2}\sinh(k_2 \ell) + \frac{(k_1^2 - k_3^2)(k_2^2 - k_3^2)}{k_3^2 - k_1^2}\sinh(k_3 \ell) \right]$$

(42)

$$\tilde{w}(x, s) = \frac{\vartheta_0}{s \epsilon_0 \sinh(k_1 \ell)} \left[ \frac{1}{k_1 - k_2}(x - k_2 \ell)\sinh(k_1 \ell) + \frac{1}{k_2 - k_3}(x - k_3 \ell)\sinh(k_2 \ell) + \frac{1}{k_3 - k_1}(x - k_1 \ell)\sinh(k_3 \ell) \right]$$

(43)

And

$$\tilde{\epsilon}(x, s) = \frac{\vartheta_0}{s \epsilon_0 \sinh(k_1 \ell)} \left[ \frac{k_1^2}{k_1 - k_2}(x - k_2 \ell)\sinh(k_1 \ell) + \frac{k_2^2}{k_2 - k_3}(x - k_3 \ell)\sinh(k_2 \ell) + \frac{k_3^2}{k_3 - k_1}(x - k_1 \ell)\sinh(k_3 \ell) \right]$$

(44)

The Stress and the Strain-Energy

The stress-strain energy which is generated on the beam is given by:

$$W(x, z, t) = \sum_{i,j=1}^{3} \sigma_{ij}(x, z, t)e_{ij}(x, z, t)$$

(45)

Hence, we have:

$$W(x, z, t) = \frac{1}{2} [L^{-1}(\sigma_{xx}(x, z, s))] [L^{-1}(\epsilon_{xx}(x, z, s))]$$

(46)

where $L^{-1}[\cdot]$ is the inversion of the Laplace transform.

Numerical Inversion of the Laplace Transform

We use the Riemann-sum approximation method to determine the solutions in the time domain and obtain numerical results. In this method, we can invert any function in the Laplace domain to the time domain as:

$$f(t) = \frac{e^{\alpha t}}{\pi} \left[ \sum_{n=1}^{N} (-1)^{n} \tilde{f}(\kappa + \frac{in\pi}{t}) \right]$$

(47)

where $Re$ is the real part and $i$ is an imaginary number unit. For faster convergence, many numerical experiments have shown that $\kappa t \approx 4.7$ Tzou [2].

Numerical Results and Discussion

We will discuss a numerical result. The physical constants of silicon nitride used as the thermoelastic material are set to the following values [34]:

$$k = 43.5W/(mK), \alpha_r = 2.71 (10^{-6})K^{-1}, \rho = 3200 kg/m^3, T_0 = 293K, C_v = 630J/[(kgK), \lambda_0 = 217 \times 10^9 N/m^2, \mu_0 = 108 \times 10^{-13} N/m^2, \tau_0 = 4.32 \times 10^{-12}s, \lambda_1 = \mu_1 = 6.89 \times 10^{-13}s]$$

We will assume that the aspect ratios of the beam as $\ell/h = 5$, $b = h/2$ and the range of the beam length is $\ell(1 - 100) \times 10^{-12} m$ for the nanoscale beam. The original time $t$ and the relaxation time $\tau_0$ of order $10^{-12}$ sec and $10^{-14}$sec, respectively.

The figures were set by using the non-dimensional variables for nano-beam length $\ell = 1.0, \theta_0 = 1.0, z = h/4$ and $d = 1.0$.

Figures 2-6 represent the temperature increment, the lateral vibration, the deformation, the stress, and the stress-strain energy distributions, respectively, for the Biot model (the model based on Fourier law of thermoelectricity heat conduction) for thermoelastic case and visco-thermal case. It has been noted that the

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temperature increment is nearly the same for the two cases, while the visco-thermoelastic parameters are significant effect on the lateral vibration, the deformation, the stress, and the stress-strain energy distributions. The peak points of the lateral vibration and the deformation decrease in the case of visco-thermoelasticity, while the absolute value of the stress increases in the same case. The peak point of the stress-strain energy distribution increases in the context of visco-thermoelastic model.

Figures 7-11 represent the temperature increment, the lateral vibration, the deformation, the stress, and the stress-strain energy distributions, respectively, for the Lord-Shulman model (L-S) (the model based on non-Fourier law of thermoelasticity heat conduction) for the thermoelastic case and the visco-thermoelastic case. It has been noted that the visco-thermoelastic parameters are significant effect on temperature increment, the lateral vibration, the deformation, the stress, and the stress-strain energy distributions. The peak points of the temperature increment, the lateral vibration and the deformation decrease in the case of visco-thermoelasticity, while the absolute value of the stress increases in the same case. The peak point of the stress-strain energy distribution increases in the context of visco-thermoelastic model.

Figures 12-16 represent the temperature increment, the lateral vibration, the deformation, the stress, and the stress-strain energy distributions, respectively, for the Lord-Shulman model (L-S) for the visco-thermoelastic case with different values of the beam’s thickness $z = (h/4, h/6)$ to stand on the thickness effect on all the studied functions. It has been noted that the visco-thermoelastic parameters are significant effect on temperature increment, the deformation, the stress, and the stress-strain energy distributions while its effect on the lateral vibration is null. The peak points of the temperature increment, the deformation, the stress, and the stress-strain energy increase in the case of $z = h/4$, while it is almost the same for the lateral vibration.

**Conclusion**

When the visco-thermoelastic beam has been thermally shocked and simply supported, the visco-thermoelastic parameters have significant effects on the lateral vibration, the deformation, the stress, and the stress-strain energy distributions and null effect on the temperature increment distribution in the context of Fourier law of heat conduction. In the context of the non-Fourier law of heat conduction, the visco-thermoelastic parameters and the thickness of the nano-beam have important effects on all the studied state-functions.
Fig. 5: The stress distribution for Biot model

Fig. 6: The stress-strain energy distribution for Biot model

Fig. 7: The temperature increments distribution for L-S model

Fig. 8: The lateral deflection distribution for L-S model

Fig. 9: The deformation distribution for L-S model

Fig. 10: The stress distribution for L-S model
Fig. 11: The stress-strain energy distribution for L-S model

Fig. 12: The temperature increments distribution with variance thickness h

Fig. 13: The lateral deflection distribution with variance thickness h

Fig. 14: The deformation distribution with variance thickness h

Fig. 15: The stress distribution with variance thickness h

Fig. 16: The stress-strain energy distribution with variance thickness h
References

[1] Tzou D (1989) On the thermal shock wave induced by a moving heat source. Journal of Heat Transfer 111 (2):232-238
[2] Tzou D, Transfer M-tMH (1997) The Lagging Behavior. Taylor Francis, Washington
[3] Xu M, Guo J, Wang L, Cheng L (2011) Thermal wave interference as the origin of the overshooting phenomenon in dual-phase-lagging heat conduction. International Journal of Thermal Sciences 50 (5):825-830
[4] Al-Hunti NS, Al-Nimr M (2004) Thermoelastic behavior of a composite slab under a rapid dual-phase-lag heating. Journal of Thermal Stresses 27 (7):607-62
[5] Ho J-R, Kuo C-P, Jiaung W-S (2003) Study of heat transfer in multilayered structure within the framework of dual-phase-lag heat conduction model using lattice Boltzmann method. International Journal of Heat and Mass Transfer 46 (1):55-69
[6] Lee Y-M, Tsai T-W (2007) Ultra-fast pulse-laser heating on a two-layered semi-infinite material with interfacial contact conductance. International communications in heat and mass transfer 34 (1):45-51
[7] Liu K-C (2007) Numerical analysis of dual-phase-lag heat transfer in a layered cylinder with nonlinear interface boundary conditions. Computer physics communications 177 (3):307-314
[8] Ramadan K (2009) Semi-analytical solutions for the dual phase lag heat conduction in multilayered media. International Journal of Thermal Sciences 48 (14):14-25
[9] Alghamdi N (2017) Dual-Phase-Lagging Thermoelastic Damping Vibration in Micro-Nano Scale Beam Resonators with Voids. Int J of Multidisciplinary and Current research 5
[10] Alghamdi NA, Youssef HM (2017) Dual-phase-lagging thermoelastic damping in extensional vibration of rotating nano-rod. Microsystem Technologies 23 (10):4333-4343
[11] Biot MA (1956) Thermoelasticity and irreversible thermodynamics. Journal of applied physics 27 (3):240-253
[12] Youssef HM, Alghamdi N (2015) Thermoelastic damping in nanomechanical resonators based on two-temperature generalized thermoelectricity theory. Journal of Thermal Stresses 38 (12):1345-1359
[13] Lord HW, Shulman Y (1967) A generalized dynamical theory of thermoelasticity. Journal of the Mechanics and Physics of Solids 15 (5):299-309
[14] Dhaliwal RS, Sherief HH (1980) Generalized thermoelasticity for anisotropic media. Quarterly of Applied Mathematics 38 (1):1-8
[15] Hoang CM (2015) Thermoelastic damping depending on vibration modes of nano beam resonator. Communications in Physics 25 (6):317
[16] Naik AK, Hanay M, Hiebert W, Feng X, Roukes ML (2009) Towards single-molecule nanomechanical mass spectrometry. Nature nanotechnology 4 (7):445
[17] O’Connell AD, Hofheinz M, Ansmann M, Bialczak RC, Lenander M, Lucero E, Neeley M, Sank D, Wang H, Weides M (2010) Quantum ground state and single-phonon control of a mechanical resonator. Nature 464 (7289):697
[18] Van Beek J, Puers R (2011) A review of MEMS oscillators for frequency reference and timing applications. Journal of Micromechanics and Microengineering 22 (1):013001
[19] Sharma J, Grover D (2011) Thermoelastic vibrations in micro-/nano-scale beam resonators with voids. Journal of Sound and Vibration 330 (12):2964-2977
[20] Sun Y, Saka M (2010) Thermoelastic damping in micro-scale circular plate resonators. Journal of Sound and Vibration 329 (3):328-337
[21] Lifshitz R, Roukes ML (2000) Thermoelastic damping in micro- and nanomechanical systems. Physical review B 61 (8):5600
[22] Al-Hunti NS, Al-Nimr M, Naji M (2001) Dynamic response of a rod due to a moving heat source under the hyperbolic heat conduction model. Journal of Sound and Vibration 242 (4):629-640
[23] Al-Lehaibi EA, Youssef HM (2015) Vibration of gold nano-beam with variable young’s modulus due to thermal shock. World Journal of Nano Science and Engineering 5 (04):194
[24] Boyle BA (1972) Approximate analyses of thermally induced vibrations of beams and plates. Journal of Applied Mechanics 39 (1):212-216
[25] Kidawa-Kukla J (2003) Application of the Green functions to the problem of the thermally induced vibration of a beam. Journal of sound and vibration 262 (4):865-876
[26] Manolis G, Beskos D (1980) Thermally induced vibrations of beam structures. Computer Methods in Applied Mechanics and Engineering 21 (3):337-355
[27] Biot MA (1954) Theory of stress-strain relations in anisotropic viscoelasticity and relaxation phenomena. Journal of Applied Physics 25 (11):1391-1391
[28] Biot MA (1955) Variational principles in irreversible thermodynamics with application to viscoelasticity. Physical Review 97 (6):1463
[29] Drozdov A (1999) A constitutive model in finite thermoviscoelasticity based on the concept of transient networks. Acta mechanica 133 (1-4):13-37
[30] Ezzat MA, El-Karamany AS (2003) The relaxation effects of the volume properties of viscoelastic material in generalized thermoelasticity. International journal of engineering science 41 (19):2281-2298
[31] Carcione JM, Poletto F, Gell D (2004) 3-D wave simulation in anelastic media using the Kelvin–Voigt constitutive equation. Journal of Computational Physics 196 (1):282-297
[32] Grover D (2012) Viscothermoelastic vibrations in micro-scale beam resonators with linearly varying thickness. Canadian Journal of Physics 90 (5):487-496
[33] Grover D (2015) Damping in thin circular viscothermoelastic plate resonators. Canadian Journal of Physics 93 (12):1597-1605
[34] Grover D (2013) Transverse vibrations in micro-scale viscothermoelastic beam resonators. Archive of Applied Mechanics 83 (2):303-314
[35] Grover D, Seth R (2018) Viscothermoelastic micro-scale beam resonators based on dual-phase lagging model. Microsystem Technologies 24 (3):1667-1672
[36] Youssef H, Al-Ghamdi NA Vibration of Gold Nano Beam in Context of Two-Temperature Generalized Thermoelasticity without energy dissipation. In: ICTEA: International Conference on Thermal Engineering, 2017.