INTERPOLATIONS OF JENSEN’S INEQUALITY

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Abstract. Weighted and unweighted interpolations of general order are given for Jensen’s integral inequality. Various upper-bound estimates are made for the differences between the interpolates and some convergence results derived. The results generalise and subsume a body of earlier work and employ streamlined proofs.

1. Introduction

A central tool in the applied literature is Jensen’s weighted integral inequality, the basic form of which is as follows.

**Theorem 1.** Let \( f, g : [a, b] \to \mathbb{R} \) be measurable and denote by \( I \) the convex hull of the image of \([a, b]\) under \( f \). Let \( \phi : I \to \mathbb{R} \) be convex and suppose that \( g, f \cdot g \) and \((\phi \circ f) \cdot g\) are all integrable on \([a, b]\). If \( g(t) \geq 0 \) on \([a, b]\) and \( \int_a^b g(t) \, dt > 0 \), then

\[
\phi \left( \frac{\int_a^b f(t) g(t) \, dt}{\int_a^b g(t) \, dt} \right) \leq \frac{\int_a^b (\phi \circ f)(t) g(t) \, dt}{\int_a^b g(t) \, dt}.
\]

(1.1)

A convenient standardisation is suggested by the ubiquitous applications of Jensen’s inequality in probability. If we define

\[
p(t) := \frac{g(t)}{\int_a^b g(t) \, dt},
\]

then \( p \) is nonnegative and satisfies \( \int_a^b p(t) \, dt = 1 \) and so may be regarded as a probability density function on \([a, b]\). With this notation, (1.1) takes the simple form

\[
\phi \left( \int_a^b f(t)p(t) \, dt \right) \leq \int_a^b (\phi \circ f)(t)p(t) \, dt.
\]

(1.2)

Without loss of generality we may work with this simpler canonical form.

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175
Recently Dragomir and Goh [10] derived an estimate for the difference between the two sides of a multivariate version of (1.1). In our present notation, the univariate case of their estimate is

\[
0 \leq \int_a^b (\phi \circ f)(t) p(t) \, dt - \phi \left( \int_a^b f(t) p(t) \, dt \right)
\]

\[
\leq \int_a^b (\phi' \circ f)(t) \cdot f(t) p(t) \, dt
- \int_a^b (\phi' \circ f)(t) p(t) \, dt \cdot \int_a^b f(t) p(t) \, dt,
\]

(1.3)

provided that all the integrals exist and \( \phi \) is differentiable convex on \( \mathbb{R} \).

In this paper we give some refinements of these results. For notational convenience, we introduce the \( k \)-variate linear integral operator

\[
I_k \{ \cdot \} := \int_a^b \ldots \int_a^b (\cdot) p(t_1) \ldots p(t_k) dt_1 \ldots dt_k.
\]

In this notation, (1.2) now becomes

\[
\phi[I_1 \{ f(t) \}] \leq I_1 \{ (\phi \circ f)(t) \}
\]

(1.4)

and (1.3) reads

\[
0 \leq I_1 \{ (\phi \circ f)(t) \} - \phi[I_1 \{ f(t) \}]
\]

\[
\leq I_1 \{ (\phi' \circ f)(t) \cdot f(t) \} - I_1 \{ (\phi' \circ f)(t) \} \cdot I_1 \{ f(t) \}.
\]

(1.5)

In Section 2 we interpolate (1.4), using both weighted and unweighted (that is, uniformly weighted) forms. The \( k \)-th order weighted and unweighted interpolates are respectively

\[
\varphi_k(u) := I_k \left\{ \phi \left( \sum_{i=1}^k u_i f(t_i) \right) \right\}
\]

and

\[
\varphi_k := I_k \left\{ \phi \left( \frac{1}{k} \sum_{i=1}^k f(t_i) \right) \right\}.
\]

Here \( u = (u_1, \ldots, u_k) \) is a set of probability weights, that is, each \( u_i \geq 0 \) and \( \sum_{i=1}^k u_i = 1 \), and it is envisaged that \( k \) is a fixed positive integer. When we wish to vary the order \( k \) the extended notation \( u^{(k)} = (u_{1,k}, \ldots, u_{k,k}) \) will be used.

The basic result is Theorem 2, which generalises a number of known results. We shall see that the \( k \)-th order weighted interpolate \( \varphi_k(u) \) is minimised by the unweighted interpolate \( \varphi_k \), that is, when each \( u_i = 1/k \). In Section 3 we give upper bounds for the difference between the first and third terms in (2.1) below. By virtue of the noted minimisation result, our estimates include as a special case an upper bound for the difference.
between the first and second terms in (2.1). A convergence theorem is established for the difference with \( k \to \infty \).

In Section 4 we treat the sequence \((\varphi_k(u^{(k)}) - \varphi_k)_{k \geq 1}\). Some results for the sequence \((\varphi_k - \varphi_{k+1})_{k \geq 1}\) are deduced in Section 5. We conclude in Section 6 with some remarks on applications to Hadamard’s inequalities.

Our arguments exploit the standardisation of \( p \) being a probability density. Suppose \( Y_1, \ldots, Y_k \) are independent random variables with common density function \( p \) and define \( X_1, \ldots, X_k \) by \( X_i = f(Y_i) \) \((i = 1, \ldots, k)\). We shall also write \( X, Y \) for a generic pair \( X_i, Y_i \). Then \( I_k \) is simply the expectation operator with respect to the minimal completed sigma field \( F_k \) generated by \( Y_1, \ldots, Y_k \). Denoting the mean of \( X_1 \) by \( E(X_1) \), as is customary, we then have \( E(X_1) = I_1 \{ f(t_1) \} \). Since \( F_1 \) is a sub sigma field of \( F_k \), we have also \( E(X_1) = I_k \{ f(t_1) \} \). We may now express (1.2), (1.5) slightly more succinctly and considerably more evocatively as respectively

\[
\phi(E(X)) \leq E(\phi(X))
\]

and

\[
0 \leq E(\phi(X)) - \phi(E(X)) \leq E(X\phi'(X)) - E(\phi'(X))E(X).
\]

We shall lean heavily on this probabilistic formulation both for compact notation within our proofs and for streamlining the algebra involved in them. The assumptions of Theorem 1 are presumed throughout without further comment and with the standardisation that \( g \) is replaced by a probability density function \( p \). A number of useful bounds arise \( via \) the Cauchy–Schwarz inequality. In each such connection we shall assume in addition without further comment that \( f^2 \) is integrable, and introduce

\[
\sigma := \left[ I_1 \{ f^2(t) \} - (I_1 \{ f(t) \})^2 \right]^{1/2}.
\]

Probabilistically this states that

\[
\sigma^2 = E(X^2) - [E(X)]^2 =: \text{var}(X),
\]

the variance of \( X \). The basic probabilistic results we shall invoke are that \( E(U^2) = \text{var}(U) \) when \( E(U) = 0 \) and that for independent random variables \( X_1, \ldots, X_k \) and constants \( u_1, \ldots, u_k \), we have

\[
\text{var} \left( \sum_{i=1}^{k} u_i X_i \right) = \sum_{i=1}^{k} u_i^2 \text{var}(X_i).
\]

For notation convenience, it will be convenient to introduce into our discussion the auxiliary random variables

\[
Z_1 = Z_{1,k} := \sum_{i=1}^{k} u_i X_i
\]

and

\[
W_k := \frac{1}{k} \sum_{i=1}^{k} X_i.
\]
It is immediate that $E(Z_1) = E(W_k) = E(X)$ and that $\varphi_k(u) = E(\phi(Z_1))$ and $\varphi_k = E(\phi(W_k))$.

2. Basic Results

Our first result relates expectations involving weighted and unweighted interpolates and refines (1.1).

**Theorem 2.** For each $k \geq 1$ and set of probability weights $u^{(k)}$, we have

$$\phi(I_1 \{f(t)\}) \leq \varphi_k \leq \varphi_k(u^{(k)}) \leq I_1 \{(\phi \circ f)(t)\}. \quad (2.1)$$

**Proof.** In probabilistic terms, the result to be proved is that

$$\phi(E(X)) \leq E(\phi(W_k)) \leq E(\phi(Z_1)) \leq E(\phi(X)). \quad (2.2)$$

By Jensen’s integral inequality we have

$$E\{\phi(W_k)\} \geq \phi(E\{(W_k)\}) = \phi(E(X)),$$

the first inequality in the enunciation.

For fixed $k$ put $X_{i+k} := X_i$ and for $1 \leq j \leq k$ define

$$Z_j := \sum_{i=1}^{k} u_i X_{i+j-1},$$

which is consistent with the definition of $Z_1$. Then $E(Z_j) = E(X)$ and $E(\phi(Z_j)) = E(\phi(Z_1))$.

By Jensen’s discrete inequality we have

$$\phi\left(\frac{1}{k} \sum_{i=1}^{k} Z_i\right) \leq \frac{1}{k} \sum_{i=1}^{k} \phi(Z_i),$$

and since $\sum_{i=1}^{k} Z_i = kW_k$, we derive

$$\phi(W_k) \leq \frac{1}{k} \sum_{j=1}^{k} \phi(Z_j).$$

Taking expectations provides

$$E\{\phi(W_k)\} \leq \frac{1}{k} E\left(\sum_{j=1}^{k} \phi(Z_j)\right) = E\{\phi(Z_1)\}. \quad (2.3)$$

This gives the second inequality in the enunciation.
Finally, by Jensen’s discrete inequality again, we have

$$\phi(Z_1) \leq \sum_{i=1}^{k} u_i \phi(X_i).$$

Taking expectations provides the final desired inequality.

If we choose $u_{i,k+1} = 1/k$ for $1 \leq i \leq k$ and $u_{k+1,k+1} = 0$, then $\varphi_{k+1}(u)$ becomes $\varphi_k$. Thus we have $\varphi_{k+1} \leq \varphi_k$ and so $(\varphi_k)_{k \geq 1}$ is a nonincreasing sequence. We have also $\varphi_1 = E(\phi(X))$, of course.

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The choices $f(t) := t$ and $p(t) := 1/(b-a)$ provide the following interpolation of the Hadamard integral inequalities, which we exhibit in extenso.

**Corollary 1.** Suppose $\phi$ is convex on $[a, b]$ and that $u_i$ $(1 \leq i \leq k)$ is a set of probability weights. Then

$$\phi\left(\frac{a + b}{2}\right) \leq \frac{1}{(b-a)^k} \int_a^b \ldots \int_a^b \phi\left(\frac{1}{k} \sum_{i=1}^{k} t_i\right) dt_1 \ldots dt_k \leq \frac{1}{(b-a)^k} \int_a^b \ldots \int_a^b \phi\left(\sum_{i=1}^{k} u_i t_i\right) dt_1 \ldots dt_k \leq \frac{1}{b-a} \int_a^b \phi(t) dt. \quad (2.4)$$

This subsumes several known results: the first inequality was proved in [11], the second in [8] and the last in [4]. We pick up these threads again in Section 6.

### 3. Bounds for the Difference $\varphi_k(u) - \phi(I_1\{f(t)\})$

**Theorem 3.** Denote by $\phi'_{+}$ the right derivative of $\phi$ on the interior $\tilde{I}$ of $I$. Then

$$0 \leq \varphi_k(u) - \phi(I_1\{f(t)\}) \leq I_k\left\{\phi'_{+}\left(\sum_{i=1}^{k} u_i f(t_i)\right)\sum_{j=1}^{k} u_j f(t_j)\right\} - I_k\left\{f(t)\right\} - I_k\left\{\phi'_{+}\left(\sum_{i=1}^{k} u_i f(t_i)\right)\right\}. \quad (3.1)$$

**Proof.** We already have the first inequality and wish to prove the second. We may express (3.1) probabilistically as

$$0 \leq E\{\phi Z_1\} - E(X) \leq E\{Z_1\phi'_{+}\} - E(X) \cdot E\{\phi'_{+}\}.$$

Since $\phi$ is convex on $I$,

$$\phi(x) - \phi(y) \geq \phi'_{+}(y)(x - y) \text{ for all } x, y \in \tilde{I}$$

$$E\{\phi Z_1\} - E(X) \leq E\{Z_1\phi'_{+}\} - E(X) \cdot E\{\phi'_{+}\}.$$
and \( \phi'_+ (\cdot) \) is nonnegative on \( \hat{I} \). Taking \( x = E(X) \) and \( y = Z_1 \), we deduce that
\[
\phi(E(X)) - \phi(Z_1) \geq \phi'_+ (Z_1) [E(X) - Z_1].
\]

Taking expectations yields
\[
E \{ \phi(Z_1) \} - \phi(E(Z_1)) \leq E \{ Z_1 \phi'_+(Z_1) \} - E \{ Z_1 \} \cdot E \{ \phi'_+(Z_1) \},
\]
whence we have the desired result.

When each \( u_i = 1/k \), we may exploit symmetry in \( j \) of the summand in (3.1) to simplify the conclusion of the last theorem to
\[
0 \leq \phi_k(u) - \phi \left( I_1 \{ f(t) \} \right)
\]
\[
\leq I_k \left\{ f(t_1) \phi'_+ \left( \frac{1}{k} \sum_{i=1}^{k} f(t_i) \right) \right\} - I_1 \left\{ f(t) \right\} \cdot I_k \left\{ \phi'_+ \left( \frac{1}{k} \sum_{i=1}^{k} f(t_i) \right) \right\}.
\]

The previous theorem may be extended as follows.

**Theorem 4.** For \( k \geq 1 \) we have
\[
0 \leq I_k \left\{ \phi'_+ \left( \sum_{i=1}^{k} u_i f(t_i) \right) \sum_{j=1}^{k} u_j f(t_j) \right\} - I_1 \left\{ f(t) \right\} \cdot I_k \left\{ \phi'_+ \left( \sum_{i=1}^{k} u_i f(t_i) \right) \right\}
\]
\[
\leq \sigma \sqrt{\sum_{j=1}^{k} u_j^2 \left( I_k \left\{ \left[ \phi'_+ \left( \sum_{i=1}^{k} u_i f(t_i) \right) \right]^2 \right\} \right)^{1/2}}.
\]

**Proof.** Since \( E(X) = E(Z_1) \), the middle term in (3.3) can be cast probabilistically as
\[
E \left\{ \phi'_+(Z_1) \times [Z_1 - E(Z_1)] \right\},
\]
which by the Cauchy–Schwarz inequality is less than or equal to
\[
\left\{ E \left[ \left[ \phi'_+(Z_1) \right]^2 \right] \right\}^{1/2} \left\{ E \left[ [Z_1 - E(Z_1)]^2 \right] \right\}^{1/2}.
\]

Further \( E[Z_1 - E(Z_1)] = 0 \), so we have
\[
E([Z_1 - E(Z_1)]^2) = \text{var}(Z_1) = \sum_{i=1}^{k} u_i^2 \text{var}(X_i) = \sigma^2 \sum_{i=1}^{k} u_i^2,
\]
and the desired result follows.
As with the previous theorem, (3.3) simplifies when \( u_i = 1/k \) for each \( i \), becoming

\[
0 \leq I_k \left\{ \phi'_+ \left( \frac{1}{k} \sum_{i=1}^{k} f(t_i) \right) f(t_1) \right\} - I_1 \left\{ f(t) \cdot I_k \left\{ \phi'_+ \left( \frac{1}{k} \sum_{i=1}^{k} f(t_i) \right) \right\} \right.
\]

\[
\leq \sigma k^{-1/2} \left[ I_k \left\{ \left[ \phi'_+ \left( \frac{1}{k} \sum_{i=1}^{k} f(t_i) \right) \right]^2 \right\} \right]^{1/2}.
\]

for all \( k \geq 1 \).

**Corollary 2.** Suppose that

\[
M := \sup_{x \in I} \left| \phi'_+ (x) \right| < \infty
\]

and

\[
\sum_{j=1}^{k} u_{j,k}^2 \to 0 \text{ as } k \to \infty.
\]

Then

\[
\phi_k \left( u^{(k)} \right) \to \phi \left( E_1 \left\{ f(t) \right\} \right) \text{ as } k \to \infty.
\]

We note that the second assumption is automatically satisfied in the particular case \( u_{j,k} = 1/k \) for \( 1 \leq j \leq k \).

The conclusion of the corollary may be expressed probabilistically as

\[
E \left\{ Z_{1,k} \right\} \to \phi(E(X)) \text{ as } k \to \infty.
\]

### 4. Bounds for \( \phi_k (u) - \phi_k \)

The difference between the outermost terms in (2.1) can be used to provide a crude upper bound for \( \phi_k (u) - \phi_k \). Here we provide tighter bounds.

**Theorem 5.** For \( k \geq 1 \) we have

\[
0 \leq \phi_k (u) - \phi_k \leq I_k \left\{ \phi'_+ \left( \sum_{i=1}^{k} u_i f(t_i) \right) \sum_{j=1}^{k} u_j f(t_j) \right\} - I_k \left\{ \phi'_+ \left( \sum_{i=1}^{k} u_i f(t_i) \right) \frac{1}{k} \sum_{j=1}^{k} f(t_j) \right\}.
\]

**Proof.** By the convexity of \( \phi \)

\[
\phi(W_k) - \phi(Z_1) \geq \phi'_+ (Z_1) (W_k - Z_1).
\]

Taking expectations provides

\[
E(\phi(Z_1)) - E(\phi(W_k)) \leq E \left\{ Z_1 \phi'_+(Z_1) \right\} - E \left\{ W_k \phi'_+(Z_1) \right\},
\]
which is the desired result in probabilistic form.

The estimate is continued by the next theorem.

**Theorem 6.** For each \( k \geq 1 \),

\[
I_k \left\{ \phi'_+ \left( \sum_{i=1}^{k} u_i f(t_i) \right) \sum_{j=1}^{k} u_j f(t_j) \right\} - I_k \left\{ \phi'_+ \left( \sum_{i=1}^{k} u_i f(t_i) \right) \right\} \leq \sigma \left[ \sum_{i=1}^{k} (u_i - 1/k)^2 \times \left( I_k \left\{ \phi'_+ \left( \sum_{j=1}^{k} u_j f(t_j) \right) \right\} \right)^2 \right]^{1/2}.
\]

**Proof.** The left–hand side of this inequality is

\[
E \left\{ \phi'_+ (Z_1) \times (Z_1 - W_k) \right\},
\]

which by the Cauchy–Schwarz inequality is less than or equal to

\[
\left( E \left\{ [\phi'_+ (Z_1)]^2 \right\} \right)^{1/2} \times \left( E \left\{ [Z_1 - W_k]^2 \right\} \right)^{1/2}.
\]

Since \( E(Z_1 - W_k) = 0 \), we may compute the second term in parentheses as

\[
\text{var}\left\{ \sum_{j=1}^{k} \left( u_j - \frac{1}{k} \right) X_j \right\} = \sum_{j=1}^{k} \left( u_j - \frac{1}{k} \right)^2 \text{var}(X_j) = \sigma^2 \sum_{j=1}^{k} \left( u_j - \frac{1}{k} \right)^2,
\]

from which we deduce the desired estimate.

**Corollary 3.** If (3.4) applies, then

\[
0 \leq \varphi_k (u) - \varphi_k \leq M \sigma \left[ \sum_{i=1}^{k} \left( u_i - \frac{1}{k} \right)^2 \right] .
\]

It follows that subject to (3.4), a sufficient condition for

\[
\lim_{k \to \infty} \varphi_k (u) = \phi (I_1 \{ f(t) \})
\]

is that

\[
\lim_{k \to \infty} \sum_{i=1}^{k} (u_{i,k} - 1/k)^2 = 0.
\]

By virtue of the relation \( \sum_{i=1}^{k} u_{i,k} = 1 \), this is the same condition as (3.5).
5. Upper Bounds for $\varphi_k - \varphi_{k+1}$

From (2.2), we have

$$\phi(E(X)) \leq \varphi_{k+1} \leq \varphi_k \leq \ldots \leq E(\phi(X))$$  \hspace{1cm} (5.1)

for $k \geq 1$, so that the difference $\varphi_k - \varphi_{k+1}$ is nonnegative and can be ascribed a uniform upper bound $E(\phi(X)) - \phi(E(X))$ which is independent of $k$. The next theorem refines this to a tighter and $k$–dependent bound.

**Theorem 7.** For each $k \geq 1$,

$$0 \leq \varphi_k - \varphi_{k+1} \leq \frac{1}{k+1}\left[ I_k \left\{ \phi' \left( \frac{1}{k} \sum_{i=1}^{k} f(t_i) \right) \right\} - I_k \left\{ \phi' \left( \frac{1}{k} \sum_{i=1}^{k} f(t_i) \right) \right\} \right].$$  \hspace{1cm} (5.2)

**Proof.** As $\phi$ is convex,

$$\phi\left( \frac{1}{k+1} \sum_{i=1}^{k+1} X_i \right) - \phi\left( \frac{1}{k} \sum_{i=1}^{k} X_i \right) \geq \phi' \left( \frac{1}{k} \sum_{i=1}^{k} X_i \right) \left( \frac{1}{k+1} \sum_{i=1}^{k+1} X_i - \frac{1}{k} \sum_{j=1}^{k} X_j \right)$$

$$= \phi' \left( \frac{1}{k} \sum_{i=1}^{k} X_i \right) \left[ \frac{1}{k+1} \sum_{i=1}^{k+1} X_i - \frac{1}{k} \sum_{j=1}^{k} X_j \right]$$

for all $k \geq 1$.

Taking expectations provides

$$\varphi_{k+1} - \varphi_k \geq \frac{1}{k+1}\left[ E\left\{ \phi' \left( \frac{1}{k} \sum_{i=1}^{k} X_i \right) \right\} E(X) - E\left\{ \phi' \left( \frac{1}{k} \sum_{i=1}^{k} X_i \right) \right\} \right]$$

$$= \frac{1}{k+1}\left[ E\left\{ \phi' \left( \frac{1}{k} \sum_{i=1}^{k} X_i \right) \right\} E(X) - E\left\{ X_1 \phi' \left( \frac{1}{k} \sum_{i=1}^{k} X_i \right) \right\} \right].$$

where symmetry has been coupled with a change of variables to provide the last step.

The above result is continued by the following one.

**Theorem 8.** For each $k \geq 1$ we have

$$\frac{1}{k+1}\left[ I_k \left\{ \phi' \left( \frac{1}{k} \sum_{i=1}^{k} f(t_i) \right) \right\} f(t) \right] - I_k \left\{ \phi' \left( \frac{1}{k} \sum_{i=1}^{k} f(t_i) \right) \right\} \right].$$

$$\leq \frac{\sigma}{\sqrt{k(k+1)}} \left[ I_k \left\{ \left[ \phi' \left( \frac{1}{k} \sum_{i=1}^{k} f(t_i) \right) \right]^2 \right\} \right]^{1/2}. $$
Proof. The left-hand side can be expressed as
\[
\frac{1}{k + 1} E \left\{ \phi' \left( \frac{1}{k} \sum_{i=1}^{k} X_i \right) \left[ X_{k+1} - \frac{1}{k} \sum_{j=1}^{k} X_j \right] \right\},
\]
which by the Cauchy–Schwarz inequality is less than or equal to
\[
\left( E \left\{ \left[ \phi' \left( \frac{1}{k} \sum_{i=1}^{k} X_i \right) \right]^2 \right\} \right)^{1/2} \times \left( E \left\{ \left[ X_{k+1} - \frac{1}{k} \sum_{i=1}^{k} X_i \right]^2 \right\} \right)^{1/2}.
\]

Since \( E(X_{k+1} - (1/k) \sum_{i=1}^{k} X_i) = 0 \), the expression within the second pair of parentheses is
\[
\text{var} \left( X_{k+1} - \frac{1}{k} \sum_{i=1}^{k} X_i \right) = \text{var} \left( X_{k+1} \right) + \frac{1}{k^2} \sum_{i=1}^{k} \text{var}(X_i) = \frac{(k + 1)\sigma^2}{k},
\]
from which we deduce the desired result.

Finally we have the following corollary.

Corollary 4. If (3.4) holds, then for all \( \alpha \in [0, 1) \) we have
\[
\lim_{n \to \infty} (\varphi_n - \varphi_{n+1}) n^\alpha = 0.
\]

Proof. By the two preceding theorems,
\[
0 \leq \varphi_n - \varphi_{n+1} \leq \frac{M \sigma}{\sqrt{n(n+1)}}, \tag{5.3}
\]
for all \( n \geq 1 \), whence the result.

6. Applications to Hadamard’s Inequalities

We conclude by resuming from Corollary 1 and the observations made there. Hadamard’s inequality states that if \( \phi : I \to \mathbb{R} \) is convex on the interval \( I = [a, b] \) of real numbers, then
\[
\phi \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b \phi(x)dx \leq \frac{\phi(a) + \phi(b)}{2}. \tag{6.1}
\]

Denote by
\[
J_k \{ \cdot \} := \frac{1}{(b - a)^k} \int_a^b \ldots \int_a^b (\cdot) dx_1 \ldots dx_k
\]
the special case of $I_k$ when $p(x) := 1/(b-a)$ on $[a, b]$. Dragomir, Pečarić and Sándor [11] have interpolated the first inequality in (6.1) as
\[
\phi\left(\frac{a+b}{2}\right) \leq J_{k+1}\left\{\phi\left(\frac{1}{k+1}\sum_{i=1}^{k+1} x_i\right)\right\} \leq J_k\left\{\phi\left(\frac{1}{k}\sum_{i=1}^{k} x_i\right)\right\} \leq \ldots \leq J_1\{\phi(x)\} \tag{6.2}
\]
for all $k \geq 1$. This is a particular case of (5.1).

Dragomir [4] has also established a weighted interpolation, in our notation
\[
\phi\left(\frac{a+b}{2}\right) \leq J_k\left\{\phi\left(\sum_{i=1}^{k} v_i x_i\right)\right\} \leq J_1\{\phi(x)\}, \tag{6.3}
\]
of Hadamard’s first inequality. This was subsequently improved by Dragomir and Bușe [8] who proved inter alia that
\[
J_k\left\{\phi\left(\frac{1}{k}\sum_{i=1}^{k} x_i\right)\right\} \leq J_k\left\{\phi\left(\sum_{i=1}^{k} u_i x_i\right)\right\}. \tag{6.4}
\]
This is Theorem 2 with $f(x) := x$ (and so $X_i = Y_i$).

From Corollary 2 we can obtain the following result which was derived by a different argument in [9].

Suppose $\phi : I \to \mathbb{R}$ is convex, (3.4) holds and that
\[
\lim_{n \to \infty} \frac{\sum_{i=1}^{n} v_i^2}{\left(\sum_{i=1}^{n} v_i\right)^2} = 0.
\]
Then if $V_n := \sum_{i=1}^{n} v_i > 0$, we have
\[
\lim_{n \to \infty} J_n\left\{\phi\left(\sum_{i=1}^{n} v_i x_i / V_n\right)\right\} = \phi\left(\frac{a+b}{2}\right).
\]

Write $h_n$, $h_n(u)$ respectively for $\varphi_n$, $\varphi_n(u)$ in the case $p(x) = 1/(b-a)$ on $[a, b]$. We have the following.

**Proposition 1.** Let $\phi : I \to \mathbb{R}$ be convex and suppose (3.4) holds. Then for all $a, b \in I$ with $a < b$, we have
\[
0 \leq h_n - h_{n+1} \leq \frac{M(b-a)}{2\sqrt{3}\sqrt{n(n+1)}}
\]
for all positive integers $n$.

**Proof.** The result is (5.3) with
\[
\sigma^2 = \frac{\int_{a}^{b} t^2 \, dt}{b-a} - \left(\frac{\int_{a}^{b} t \, dt}{b-a}\right)^2 = \frac{(b-a)^2}{12}.
\]
The consequence

$$\lim_{n \to \infty} [n^\alpha (h_n - h_{n+1})] = 0 \text{ for } \alpha \in [0, 1)$$

is an improvement on the results of [7].

The weighted case is embodied in the following proposition.

**Proposition 2.** With the assumptions of Proposition 1,

$$0 \leq h_n (u) - h_n \leq \frac{M (b-a)}{2\sqrt{3}} \left[ \sum_{i=1}^{n} (u_i - 1/n^2) \right]^{1/2}$$

for all $n \geq 1$.

For other results connected with Hadamard’s inequality see [1]–[9], where further references are given.

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