On the law of the iterated logarithm and strong invariance principles in stochastic geometry *

Johannes Krebs† ‡

March 10, 2020

Abstract

We study the law of the iterated logarithm (Khinchin (1933), Kolmogorov (1929)) and related strong invariance principles for functionals in stochastic geometry. As potential applications, we think of well-known functionals defined on the k-nearest neighbors graph and important functionals in topological data analysis such as the Euler characteristic and persistent Betti numbers.

Keywords: Binomial process; Euler characteristic; Strong invariance principles; Law of the iterated logarithm; Persistent Betti numbers; Poisson processes; Stochastic geometry; Strong stabilization; Topological data analysis.

MSC 2010: Primary: 60F15; Secondary: 60D05; 60G55.

In the present manuscript, we study the law of the iterated logarithm (LIL) and strong invariance principles (SIP) in stochastic geometry (Penrose (2003)) with a particular focus on its applications in topological data analysis (TDA), for the latter we refer to Zomorodian and Carlsson (2005), Carlsson (2009), Edelsbrunner and Harer (2010).

The original statement of the LIL is due to Khinchin (1933); another version is due to Kolmogorov (1929). The nowadays more general statement was proved by Hartman and Wintner (1941). Let \( (X_i)_{i \in \mathbb{N}} \) be independent and identically distributed with mean zero and unit variance, then

\[
\limsup_{n \to \infty} \sqrt{2n \log \log n} \sum_{i=1}^{n} X_i = 1 \quad \text{a.s.}
\]

Strassen (1964) introduced the striking functional form of the LIL which has then been recovered in various functional settings, in particular, martingales, see Heyde and Scott (1973), Hall and Heyde (1980), Wichura (1973), Philipp (1977).

Since then the LIL has been extensively studied in different models, such as martingales, see Stout (1970). In the context of dependence the LIL has been investigated for mixing processes with a fast decaying correlation structure, see Philipp (1969), Oodaira and Yoshihara (1971), Rio (1995), Schmuland and Sun (2004). For stationary processes the LIL was studied by Wu (2007), Zhao and Woodroofe (2008). The LIL has been studied for random fields by Li et al. (1992) and Jiang (1999). The applications of the LIL in statistical tests and U-statistics were studied by Arcones and Giné (1995), Dehling and Wendler (2009). For empirical processes the LIL was studied by Arcones (1997).

Strong invariance principles consider the asymptotic approximability of the functional of interest by a Brownian motion. In this context, many deep results have been obtained for time series under various dependence structures, see Philipp et al. (1975), Eberlein (1986), Wu (2007).

*This research was partially supported by the German Research Foundation (DFG), Grant Number KR-4977/1-1.
†Department of Mathematics, TU Braunschweig, 38106 Braunschweig, Germany, email: johannes.krebs@tu-braunschweig.de
‡Corresponding author
The present contribution of the LIL and the SIP for functionals in stochastic geometry and TDA relies on the idea of strongly stabilizing functionals, which dates back to Lindeberg (1997, 1999) and which has been popularized in the central limit theorem of Penrose and Yukich (2001). For an introduction to TDA and persistent homology, we refer to Chazal and Michel (2017).

We consider a certain class of translation invariant functionals \( H \) defined on point processes such as the homogeneous Poisson process of unit intensity on \( \mathbb{R}^d \). Let \( (X_n)_n \) be a sequence of finite subsamples of such a point process, specified in detail below. We show

\[
\limsup_{n \to \infty} \frac{1}{\sqrt{2n \log \log n}} (H(X_n) - \mathbb{E}[H(X_n)]) = \sigma \quad \text{a.s.,}
\]

where \( \sigma^2 = \lim_{n \to \infty} n^{-1} \text{Var}(H(X_n)) > 0 \). Technically our proofs rely on the observation of W. Philipp, who conjectured, the LIL holds for "any process for which the Borel-Cantelli Lemma, the central limit theorem with a reasonably good remainder and a certain maximal inequality are valid" (Philipp (1969)). Using an approach, which relies on martingale differences, we show that the functional \( H(X_n) - \mathbb{E}[H(X_n)] \) can indeed be decomposed in a main term consisting of a partial sum process w.r.t. to a stationary random field and a remainder which behaves "reasonably" nice given the stabilizing properties of the functional \( H \). We consider two point processes as input, namely, the Poisson process and the binomial process in the critical (thermodynamic) regime, see also Yogeshwaran et al. (2017). This setting means, that we observe the Poisson process on an increasing sequence of windows \( (W_n)_n \subset \mathbb{R}^d \).

Moreover, we consider the special case of Poisson and binomial processes defined on a domain \( \mathcal{D} \times [-n/2, n/2] \), which stretches in only one dimension. Apart from recovering the LIL also in this case, we establish a strong invariance principle. More precisely, we show that there is a standard Brownian motion on an enlarged probability space such that

\[
H(X_n) - \mathbb{E}[H(X_n)] = B(n \sigma^2) + r_n,
\]

where up to logarithmic factors and under suitable moment assumptions the remainder \( r_n \) is of order \( O_{a.s.}(n^{1/4}) \) in the Poisson sampling scheme and of order \( O_{a.s.}(n^{1/4+1/p}) \) in the binomial sampling scheme, for a certain \( p \in \mathbb{R}_+ \). The technique relies on the classical Skorokhod embedding for martingales, see Strassen (1964) and Strassen (1967). So in the Poisson case and given the technique of the Skorokhod embedding, the rate is maximal (up to a logarithmic factor).

In the binomial case, the rate suffers somewhat from the fact that all points are generated anew in each step. Improving the rate in the binomial case, can be considered as an independent problem.

In practice, the quantitative stabilizing properties of the functional \( H \) need to be checked of course individually. For many functionals these are however quite immediate if we think of \( k \)-nearest neighbor problems or the Euler characteristic. The latter being the oldest and simplest descriptor of the topological properties of (point cloud) data. For persistent Betti numbers an exact quantification of the stabilizing properties is still open, even though there are first results (Chatterjee and Sen (2017), who quantify the stabilization in the study of the asymptotic normality of the minimal spanning tree.

The methods of TDA have been applied successfully in the past to various fields such as finance (Gidea and Katz (2018), material science (Lee et al. (2017)) or biology (Yao et al. (2009)). For recent applications involving the Euler characteristic, see Adler (2008), Decreusefond et al. (2014), Crawford et al. (2016). Multivariate or functional central limit theorems for the Euler characteristic where proved in Hug et al. (2016), Thomas and Owada (2019). Ergodic theorems for the Euler characteristic were established in Schneider and Weil (2008).

So our present contribution is not limited to establishing limit theorems in TDA but also lays the groundwork of statistically sound testing procedures which rely on the LIL and the presented SIP, see also Robbins (1970), Lerche (1986), Wu (2005) for potential applications. The rest of the paper is organized as follows. We introduce the basic notation and the stochastic model in Section 1. The main results are stated in Section 2. The technical details are given
in Section 3, some of them are deferred to the Appendix A.

1 Preliminaries

Let $H$ be a real-valued functional defined for all finite subsets of $\mathbb{R}^d$. We assume that $H$ is translation invariant, i.e.,

$$H(P + x) = H(P)$$

for all finite $P \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$. Classical results on limit theorems in this context rely on the add-one cost function of $H$ which is defined for a finite set of points $P \subset \mathbb{R}^d$ by

$$\mathcal{D}_0(P) := H(P \cup \{0\}) - H(P).$$

Let $\mathcal{P}$ be a homogeneous Poisson process of unit intensity on $\mathbb{R}^d$. The functional $H$ is strongly stabilizing on $\mathcal{P}$ if there exists $a.s.$ finite random variables $S$ (the so-called radius of stabilization) and $\mathcal{D}_0(\mathcal{P}, \infty)$ such that

$$\mathcal{D}_0((\mathcal{P} \cap B(0, S)) \cup A) = \mathcal{D}_0(\mathcal{P}, \infty)$$

(1.1)

for all finite $A \subset \mathbb{R}^d \setminus B(0, S)$, where $B(z, w)$ is the closed $w$-neighborhood of $z$ w.r.t. the Euclidean distance $d$ on $\mathbb{R}^d$ for $z \in \mathbb{R}^d$ and $w \in \mathbb{R}^d$.

Before we introduce the models, we begin with some general notation. For $f \in L^q(S, \mathcal{G}, \mu)$, where $(S, \mathcal{G}, \mu)$ is some generic measure space and $q \geq 1$, we denote its $q$-norm by $\|f\|_q$. Given a $d$-dimensional point cloud $P$ and $z \in \mathbb{R}^d$, we write $P - z$ for the translated point cloud $\{p - z : p \in P\}$. For a set $A \subset \mathbb{R}^d$ and $\delta \in \mathbb{R}_+$, we write $A(\delta)$ for the collection of all points with a distance to $A$ at most equal to $\delta$. For a finite set $A$ we write $\#A$ for its cardinality. If $A \subset \mathbb{R}^d$ is infinite, we write $|A|$ for its Lebesgue measure. The diameter of $A$ is abbreviated by diam$(A)$.

Let $A, B \subset \mathbb{R}^d$, then we write $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

We will use a sequence of observation windows, which are given by $W_n = [-n^{1/d}/2, n^{1/d}/2]^d$, $n \in \mathbb{N}$. (A generalization to other observation windows is a routine, however, in the present contribution, we do not treat this any further.) The topological boundary of $W_n$ is $\partial W_n$. Let $Q_0 = [-1/2, 1/2]^d$ and $Q_z = Q_0 + z, z \in \mathbb{R}^d$.

Moreover, let $\mathcal{P}'$ be another homogeneous Poisson process of unit intensity on $\mathbb{R}^d$. $\mathcal{P}$ and $\mathcal{P}'$ are independent. Write $\mathcal{P}'_z$ for $(\mathcal{P} \setminus Q_z) \cup (\mathcal{P}' \setminus Q_z)$, $z \in \mathbb{R}^d$. We write $\mathcal{P}_n(\mathcal{P}'_n, \mathcal{P}'_n)$ for the restriction of $\mathcal{P}(\mathcal{P}'_n, \mathcal{P}'_n)$ to $W_n$.

Furthermore, for each $n, m \in \mathbb{N}$, we introduce a binomial process $\mathcal{U}_{n,m}$ of length $m$ on $W_n$, whose elements have a uniform distribution on $W_n$. We use the following coupling. Set $N_n = \mathcal{P}(W_n)$. $N_n$ is Poisson distributed with parameter $n$ and $N_0 \leq N_{n+1}$ for each $n \in \mathbb{N}$. Let the Poisson process $\mathcal{P}_n$ be given by $X_1, \ldots, X_{N_n}$. Let $V_{n,1}, V_{n,2}, \ldots$ be i.i.d. on $W_n$. Define the $n$th binomial process on $W_n$ by

$$\mathcal{U}_{n,m} = \{X_i : i \in \{1, \ldots, m \wedge N_n\}\} \cup \{V_{n,k} : k \in \{1, \ldots, (m - N_n) \wedge 0\}\}, \quad m \in \mathbb{N}. \quad (1.2)$$

Write $\preceq$ for the lexicographic ordering on $\mathbb{Z}^d$. For $z \in \mathbb{Z}^d$, let $\mathcal{F}_z$ be the smallest $\sigma$-field such that the Poisson points of $\mathcal{P}$ in $\bigcup_{y \preceq z} Q(y)$ are measurable, that is $\mathcal{F}_z = \sigma(\mathcal{P}|_{\bigcup_{y \preceq z} Q(y)} : y \in \mathbb{Z}^d, y \preceq z)$.

We study two models which have different stabilizing and growth properties. The functional $H$ satisfies a hard-thresholded stabilizing condition in model $\textbf{[M1]}$, e.g., $H$ is the Euler characteristic. In model $\textbf{[M2]}$ the functional $H$ is exponentially stabilizing but satisfies a polynomial growth condition, for functionals with these properties see Penrose and Yukich [2001].

\textbf{(M1) Hard-thresholded stabilization.} Let $P$ be a finite point cloud. Then $H$ satisfies (1.1), where $S \preceq S^*$ $a.s.$ for some $S^* \in \mathbb{R}_+$. Furthermore, for all locally finite point clouds $P$

$$H(P \cap W_n \cap B(z, S^*)) \leq C^* \left| \sum_{k=0}^{N} \binom{N}{k} \right|^q,
\quad (1.3)$$

where $C^*$ and $q$ are positive constants.
Exponentially stabilizing and polynomially bounded functionals. The uniform bounded moments condition is given in terms of the binomial process. Given a set \( A \) from the class of sets

\[
\mathcal{B} = \{W_n + z, B(z, w), (W_n + z) \cap B(z, w) : n \in \mathbb{N}, w \in \mathbb{R}_+, z \in \mathbb{R}^d\},
\]

let \( \mathcal{U}_{A,m} \) be a binomial process of length \( m \) on \( A \). We require a uniform bounded moments condition to be satisfied. The order of the moment is characterized by two parameters and differs depending on the sampling scheme as follows. For Poisson sampling let \( p \in \mathbb{N} \), \( p \) even, such that \( p > 2d \lor 4 \). Let \( r \equiv \infty \). For Binomial sampling let \( p \in \mathbb{N} \), \( p \) even, such that \( p > 2d \lor 6 \). Let \( r \in (1, (2p - 4)/(p + 2)) \).

Then

\[
\sup_{A \in \mathcal{B}} \sup_{\delta \in A \in [1/2, 1/3, 1/2] |A|} \mathbb{E} \left[ H(\mathcal{U}_{A,m} \cup \{0\}) - H(\mathcal{U}_{A,m}) \right]^{pr/(r-1)} < \infty \tag{1.4}
\]

Furthermore, the functional \( H \) is polynomially bounded such that \( |H(P)| \leq \nu (\text{diam}(P) + \#P)^{\nu} \) for a constant \( \nu \in \mathbb{R}_+ \).

Denote \( C_{y,n} \) the cube with center \( y \) and edge length \( n^{1/d} \), \( y \in \mathbb{R}^d \) and \( n \in \mathbb{N} \). Let \( V_1, V_2, \ldots \) be a sequence of i.i.d. random variables with uniform distribution on \( Q_0 \). Let \( x \in Q_0, k, n \in \mathbb{N} \) and \( y \in \mathbb{R}^d \). Consider the point cloud \( P(y, n, k, x) = ((\mathcal{P}|_{C_{y,n}} \cap Q_0) \cup \{V_1, \ldots, V_k\}) \setminus x \). There is an a.s. finite random variable \( S = S(y, n, k, x) \) such that

\[
\mathcal{D}_0(P(y, n, k, x) \cap B(0, S)) = \mathcal{D}_0([P(y, n, k, x) \cap B(0, S)] \cup A) \tag{1.5}
\]

for all finite \( A \subset \mathbb{R}^d \setminus B(0, S) \). The radius \( S \) satisfies: There are constants \( c_1, c_2 > 0 \) with

\[
\sup P(S > r) \leq c_1 \exp(-c_2 r), \quad r \in (0, n^{1/d}/2), \tag{1.6}
\]

where the supremum is taken over all possible parameters \( x \in Q_0, k, n \in \mathbb{N}, y \in \mathbb{R}^d \).

The stabilizing condition in model \( \text{(M1)} \) features some kind of \( m \)-dependence, we will see this later in the proofs. Similarly, the exponential stabilizing condition in \( \text{(M2)} \) features a certain analogy to exponential mixing conditions.

One finds that the growth condition from \( \text{(1.3)} \) and the hard-thresholded stabilization in model \( \text{(M1)} \) imply the uniform bounded moments condition \( \text{(1.4)} \) in model \( \text{(M2)} \) for each \( p \in \mathbb{N} \), see Lemma \( \text{A.1} \). However, in contrast to model \( \text{(M2)} \) we do not assume an overall polynomial bound in model \( \text{(M1)} \).

The order of the uniform bounded moments condition varies significantly, depending on the model. In the Poisson sampling scheme, the order is at least 6, whereas in the binomial sampling scheme, we require more than 48. It is very likely that this latter bound is suboptimal. However, improving the bounds without additional assumptions is probably very difficult, this can be rather be thought of as an independent problem. Technically, the binomial process suffers from the fact that the entire process is generated anew in each step.

Penrose and Yukich (2001) derived a CLT for strongly stabilizing and polynomially bounded functionals \( H \). They require a similar uniform bounded moments condition to be satisfied with \( p = 4 \). In order to establish the LIL and strong invariance principles, we need to quantify the impact of the moment condition on the order of the approximation.

In the context of normal approximation, many functionals have been extensively studied under various stabilizing conditions, see Lachieze-Rey et al. (2019). Popular examples are functionals defined the \( k \)-nearest neighbor graph (Bickel and Breiman (1983)) or functionals defined on Voronoi tessellations (Schulte (2012), Thäle and Yukich (2016)). In this contribution, we study the Euler characteristic of simplicial complexes and total edge length of the \( k \)-nearest
neighbor graph.

**Example 1.1** (Euler characteristic). Let \( P \subset \mathbb{R}^d \) be a finite point cloud. Let \( \mathcal{K} \) be a simplicial complex obtained from \( P \) by a rule which puts an upper bound on the diameter of a simplex, e.g., the Čech or the Vietoris-Rips complex for a certain filtration parameter \( r \in \mathbb{R}_+ \). Then the Euler characteristic of \( \mathcal{K} \) is \( \chi = \sum_{k=0}^{\infty} (-1)^k S_k(\mathcal{K}) \), where \( S_k \) is the number of \( k \)-dimensional simplices in \( \mathcal{K} \). As (persistent) Betti numbers, the Euler characteristic is a topological invariant, see Hatcher (2002). Since the diameter of each simplex in \( \mathcal{K} \) is bounded above by some \( r \in \mathbb{R}_+ \), when adding a point \( z \) to the point cloud \( P \), only those points in the \( r \)-neighborhood of \( z \) can form new simplices involving \( z \). Hence, the Euler characteristic has the hard-thresholding stabilizing property of model \([\text{M1}]\). Moreover, using the rule that \( m \) points can at most form \( \binom{m}{k+1} \) many \( k \)-simplices for \( 0 \leq k \leq m-1 \), one sees that also the growth conditions of model \([\text{M1}]\) are satisfied.

**Example 1.2** (Total edge length in the \( k \)-nearest neighbor graph). Let \( H \) be the total edge length obtained from the \( k \)-nearest neighbor graph of a set of points. Penrose and Yukich (2001) show the asymptotic normality of this functional, they also give details on the uniform bounded moments condition. Rates of convergence in the Kolmogorov distance are given in Lachieze-Rey et al. (2019), here one also needs to employ an exponential stabilizing condition similar as in (1.6). We show in the Appendix A that the total edge length \( H \) satisfies the uniform stabilization property from Model \([\text{M2}]\) on the homogeneous Poisson process \( \mathcal{P} \).

## 2 Main results

In this section we study the rate of convergence of \( n^{-1}(H_0(\mathcal{Q}_n) - \mathbb{E}[H_0(\mathcal{Q}_n)]) \). In general for stationary and ergodic processes \((X_n)_n\), their empirical mean \( n^{-1} \sum_{i=1}^{n} X_i \) can converge to \( \mathbb{E}[X_0] \) arbitrarily slowly (see Krengel (1985)). However, given appropriate moment conditions and decay rates for the dependence structure of the stochastic model as in \([\text{M1}]\) and \([\text{M2}]\) one obtains the optimal rates known from i.i.d. data, which are given in terms of the law of the iterated logarithm. In the following, we will need the notion of the random variable \( \Delta(0, \infty) \), which satisfies together with the radius of stabilization \( S_0 \), say, the requirement

\[
H([\mathcal{P} \cap B(0, S_0)] \cup A) - H([\mathcal{P}''_0 \cap B(0, S_0)] \cup A)
= H(\mathcal{P} \cap B(0, S_0)) - H(\mathcal{P}''_0 \cap B(0, S_0)) =: \Delta(0, \infty) \quad \text{a.s.}
\]

(2.1)

for all \( A \subset \mathbb{R}^d \setminus B(0, S') \). The existence of \( \Delta(0, \infty) \) follows from the assumptions in both models, see Penrose and Yukich (2001) for a rigorous proof for \([\text{M1}]\). We state the LIL for both models \([\text{M1}]\) and \([\text{M2}]\) for an underlying sequence of Poisson processes \((\mathcal{P}_n)_n\).

**Theorem 2.1** (LIL for the Poisson process). Assume that \( \mathcal{D}_0(\mathcal{P}, \infty) \) is nondegenerate. Then

\[
\limsup_{n \to \infty} \frac{1}{\sqrt{2n \log \log n}} (H(\mathcal{P}_n) - \mathbb{E}[H(\mathcal{P}_n)]) = \sigma \quad \text{a.s.,}
\]

where \( \sigma = \sqrt{\mathbb{E}[\mathbb{E}[\Delta(0, \infty) | \mathcal{F}_0]^2]} \).

In the same spirit, we obtain the LIL for an underlying sequence of binomial processes \((\mathcal{U}_{n,n})_n\).

**Theorem 2.2** (LIL for the binomial process). Assume that \( \mathcal{D}_0(\mathcal{P}, \infty) \) is nondegenerate. Then

\[
\limsup_{n \to \infty} \frac{1}{\sqrt{2n \log \log n}} (H(\mathcal{U}_{n,n}) - \mathbb{E}[H(\mathcal{U}_{n,n})]) = \tau \quad \text{a.s.,}
\]

where \( \tau = \sqrt{\sigma^2 - \alpha^2} \) for \( \sigma = \sqrt{\mathbb{E}[\mathbb{E}[(0, \infty) | \mathcal{F}_0]^2]} \) and \( \alpha = \mathbb{E}[\mathcal{D}_0(\mathcal{P}, \infty)] \).
The key idea of Theorem 2.1 is to approximate \((H(\mathcal{P}_n) - \mathbb{E}[H(\mathcal{P}_n)])_n\) by a stationary process \((F_z)_{z \in W_n}\) and to show that the difference is of order \(o(n^{1/2} \log \log n)\) with probability 1. Then we apply standard techniques for the LIL of stationary random fields also considered in Schmuland and Sun (2004). In order to obtain Theorem 2.2 we rely on the classical Poissonization trick and couple \(H(\mathcal{U}_{n,n})\) with \(H(\mathcal{P}_n)\) at an accuracy of \(o(n^{1/2} \log \log n)\).

2.1 The special case of a one-dimensional domain

In this section, we consider a slight modification of the above model. Instead of using observation windows \((W_n)_n\), we observe the Poisson or the binomial process on a cylinder-like domain \(\tilde{W}_n = D \times [-n/2, n/2]\), which stretches only in one dimension. Here \(D \subset \mathbb{R}^d\) is a bounded and convex set. In this case, we can easily recover the LIL for both models. Moreover, using a Skorokhod embedding, we obtain a strong invariance principle.

We need to adjust slightly the definitions to the new domain, that is all quantities describing the models [M1] and [M2] are defined in terms of the domain \(D \times \mathbb{R}\), in particular, the add-one cost function \(\Omega_0 = \mathcal{D}_0(\mathcal{P}||D\times\mathbb{R}, \infty)\), the stabilization radius \(S\) and the random variable \(\Delta(0, \infty) = \Delta_{\mathcal{D}_0(\mathcal{P}||D\times\mathbb{R})}(0, \infty)\). Also all point processes \(P = P_{\mathcal{D}_0(\mathcal{P}||D\times\mathbb{R})}\) are restricted to this domain. The \(\sigma\)-fields are adjusted, we use \(\tilde{\mathcal{F}}_z = \sigma(P_{\mathcal{D}_0(\mathcal{P}||D\times(-n/2, n/2))} : y \in \mathbb{Z}, y \leq z)\) instead.

A particular interesting functional \(H\) in this framework is the persistent Betti number obtained from the Vietoris-Rips or the Čech complex of a point cloud in \(D \times \mathbb{R}\). It is well-known that the persistent Betti number is polynomially bounded and satisfies the uniform bounded moments condition for each \(p \in \mathbb{N}\), this follows comparably directly from calculations similar as in Yogeshwaran et al. (2017) or Krebs and Polonik (2019). Moreover, since the domain \(D\) is bounded in this 1-dimensional set-up, we do not observe percolation effects and the radius of stabilization decays at an exponential rate as required for model [M2] see also Krebs and Hirsch (2020) for more details.

**Theorem 2.3** (1-dimensional LIL). Let either \(\Omega\) be a Poisson process of unit intensity on \(D \times \mathbb{R}\) and \(\Omega_n = \Omega|_{\tilde{W}_n}\), or let \(\Omega_n\) be a binomial process of length \(n\) and uniform distribution on \(\tilde{W}_n\). Assume that \(\mathcal{D}_0(\Omega|_{\mathcal{D}\times\mathbb{R}}, \infty)\) is nondegenerate. Then

\[
\limsup_{n \to \infty} \frac{1}{\sqrt{2n \log \log n}} (H(\Omega_n) - \mathbb{E}[H(\Omega_n)]) = \nu \quad \text{a.s.,}
\]

(2.2)

where \(\nu^2 \) equals either \(\mathbb{E}[\mathbb{E}[\Delta_{\mathcal{D}_0(\mathcal{P}||D\times\mathbb{R})}(0, \infty)||\mathcal{F}_0||^2] \text{ in the Poisson model or } \mathbb{E}[\mathbb{E}[\Delta_{\mathcal{D}_0(\mathcal{P}||D\times\mathbb{R})}(0, \infty)||\mathcal{F}_0||^2] - \mathbb{E}[\mathcal{D}_0(\Omega|_{\mathcal{D}\times\mathbb{R}}, \infty)|^2] \text{ in the binomial model.}\)

The strong invariance principle studied for the case of the stretched 1-dimensional domain means that the (discrete) path \(n \mapsto H(\Omega_n) - \mathbb{E}[H(\Omega_n)]\) can be approximated by Brownian motion. Consequently, asymptotic properties for the process of interest can be obtained from those of Brownian motion (up to a certain approximation error).

In order to state strong invariance principles, one usually needs to work on an enlarged probability space, which is rich enough to contain all those random variables necessary for the approximation. Also one needs to redefine the process on this probability space without changing its distribution. So, for brevity, we say, there is a richer probability space and a standard Brownian motion \(B\), such that \(n \mapsto H(\Omega_n) - \mathbb{E}[H(\Omega_n)]\) can be approximated by \(B\), see also Wu (2007) for this convention.

**Theorem 2.4** (1-dimensional SIP). Let the assumptions of Theorem 2.3 be satisfied. In model [M2] let \(p \geq 12\) additionally to the conditions which lead to (1.4). Then on a richer probability space, there is a standard Brownian motion \(B\) such that

\[
H(\Omega_n) - \mathbb{E}[H(\Omega_n)] = B(n \nu^2) + \mathcal{O}_{a.s.}(n^{1/4} (\log n)^{1/2} (\log \log n)^{1/4})
\]

if \((\Omega_n)_n\) is the sequence of Poisson processes and

\[
H(\Omega_n) - \mathbb{E}[H(\Omega_n)] = B(n \nu^2) + \mathcal{O}_{a.s.}(n^{1/4+1/p} (\log n)^{2})
\]
if $\{Q_n\}_m$ is the sequence of binomial processes.

Strong invariance principles are of considerable importance in probability theory and have received attention in the statistical inference of dependent processes. Motivated by Strassen’s work [Strassen (1964), Strassen (1967)] approximating partial sums by Brownian motions have been considered for i.i.d. sequences, martingale differences and stationary ergodic processes. Komlós et al. [Komlós et al. (1975), Komlós et al. (1976)] proved that for an i.i.d. sequence of random variables with finite $p$th moment, $p > 2$, the optimal approximation rate is $o(n^{1/p})$. Strong invariance principles for partial sums under dependence have been studied by Berkes and Philipp (1979), Philipp et al. (1975), Eberlein (1986), Shao (1993), Wu (2007) Regarding statistical applications, the SIP has been considered in change-point and trend analysis (Csorgo and Horváth (1997)).

In the present models (M1) and (M2) it is unclear whether a bound of order $\mathcal{O}(n^{1/p}(n))$, where $h$ is a slowly growing function, can be obtained for general $p$ and for the above sequence of observation windows $(W_n)_n$. The application of Strassen’s method based on the Skorokhod embedding appears to be limited to the case of the stretched domain $D \times [-n/2,n/2]$ and limits the approximation property to $n^{1/4}$ up to logarithmic factors. In the binomial setting the rate suffers somewhat from the fact that a Poissonization argument is applied and that the entire point process changes in each step.

3 Technical results

We use the following notation and abbreviations throughout the rest of this manuscript.

Convention about constants. To ease notation, most constants in this paper will be denoted by $C, c, c'$, etc. and their values may change from line to line. These constants may depend on parameters like the dimension and often we will not point out this dependence explicitly; however, none of these constants will depend on the index $n$, used to index infinite sequences, or on the index, which is used to index martingale differences.

The Poisson process. $\mathcal{P}$ and $\mathcal{P}'$ are independent homogeneous Poisson processes of unit intensity on $\mathbb{R}^d$. Let $z \in \mathbb{Z}^d$. $S_z$ is the random variable which satisfies

$$\Delta(z,n) := H(\mathcal{P}_n) - H(\mathcal{P}'_{n,z}) \to \Delta(z,\infty) := H(\mathcal{P} \cap B(z,S_z)) - H(\mathcal{P}'_{n,z} \cap B(z,S_z)) \text{ a.s. } (n \to \infty).$$

Moreover, we define for $z \in \mathbb{Z}^d$

$$\Delta'(z,n) := \#\mathcal{P}_n - \#\mathcal{P}'_{n,z} \to \Delta'(z,\infty) := \mathcal{P}(Q_z) - \#(\mathcal{P}'(Q_z)) \text{ a.s. } (n \to \infty).$$

Finally, put $F_z = \mathbb{E}[\Delta(z,\infty)|F_z]$ and $F'_z = \mathbb{E}[\Delta(z,\infty) - \Delta'(z,\infty)|F_z]$, $z \in \mathbb{Z}^d$. Note that $(F_z)_z$ and $(F'_z)_z$ are both stationary processes.

The coupled binomial processes. The binomial and the Poisson process are coupled as in (1.2). We write $I_n$ for the set which contains the integers between $n$ and $N_n$, i.e.,

$$I_n = \{n, \ldots, N_n - 1\} \cup \{N_n, \ldots, n - 1\}, \quad (3.1)$$

where the first set or the second set on the right-hand side is empty.

Moreover, we define the index set $B_n = \{z \in \mathbb{Z}^d : Q_z \cap W_n \neq \emptyset\}$. Then $#B_n/n \to 1$ as $n \to \infty$. Also set $D_n = \{z \in \mathbb{Z}^d : ||z|| \leq n\}$, where $|| \cdot ||$ is the $\infty$-norm on $\mathbb{Z}^d$. 

7
3.1 A general LIL

We begin with three general statements regarding the stationary process \((F_z : z \in \mathbb{Z}^d)\). The proofs are very similar to those given in [Schmuland and Sun (2004)], who study the LIL for random fields under modified \(\varphi\)-mixing conditions (which do not apply to our setting), and are deferred to the Appendix A.2.

For Lemma 3.1 Propositions 3.2 3.3 and Theorem 3.4 let \(h : \mathbb{R} \to \mathbb{R}\) be either \(h = \text{id}\) or \(h = (\cdot)^2 - E[F_0^2]\). Then in both cases \(E[h(F_0)] = 0\). Moreover we require in this section, \(E[h(F_0)^6] < \infty\). When deriving the LIL, \(h\) is the identity, and the latter condition is satisfied in both models [M1] and [M2]. When deriving the SIP, \(h\) also equals \((\cdot)^2 - E[F_0^2]\), in this case, we will then need \(E[F_0^2] < \infty\), see also the requirements in Theorem 2.4.

First, we define the quantity \(\tilde{\sigma}^2 := \sum_{z \in \mathbb{Z}^d} E[h(F_0)h(F_z)]\). We have

**Lemma 3.1** (Covariance). The definition of \(\tilde{\sigma}^2\) is meaningful for \(h = \text{id}\) and \(h = (\cdot)^2 - E[F_0^2]\).

Moreover, let \(X = G(\sum_{z \in I} h(F_z))\) and \(Y = \tilde{G}(\sum_{z \in I} h(F_z))\) for two bounded Borel functions \(G, \tilde{G}: \mathbb{R} \to \mathbb{R}\). Write \(d(I, J) = \min\{||z - z'|| : z \in I, z' \in J\}\). Then there are constants \(c_1, c_2 \in \mathbb{R}_+\), which do neither depend on \(I\) nor on \(J\) such that

\[
\text{Cov}(X, Y) \leq c_1(\#I + \#J) \exp(-c_2d(I, J)).
\]

**Proposition 3.2** (Normal approximation). Let \(\Phi\) be the distribution function of the standard normal distribution. Let \(\varepsilon \in (0, 1/2)\), then

\[
\sup_{z \in \mathbb{R}^n} \left| \mathbb{P}\left( \sum_{z \in D_n} h(F_z) \leq \tilde{\sigma}(\#D_n)^{1/2}z \right) - \Phi(z) \right| \leq Cn^{-\varepsilon/4}.
\]

**Proposition 3.3** (Maximal inequality). Let \(\beta > 1\). There is a \(\rho > 0\) such that

\[
\mathbb{P}\left( \max_{1 \leq j \leq n} \sum_{z : ||z|| \leq n} h(F_z) \right) \geq \beta \sqrt{2\tilde{\sigma}^2(2n + 1)^d \log \log n} \leq C(\log n)^{-(1+\rho)}.
\]

**Theorem 3.4** (The LIL). Given the present assumptions \(\limsup_{n \to \infty} \pm (2n \log \log n)^{-1/2} \sum_{z \in D_n} h(F_z) = \tilde{\sigma}, \) viz.,

\[
\limsup_{n \to \infty} \pm \frac{1}{\sqrt{2n \log \log n}} \sum_{z : ||z|| \leq n} \mathbb{E}[\Delta(z, \infty)|F_x] = \mathbb{E}[\mathbb{E}[\Delta(z, \infty)|F_x]^2]^{1/2},
\]

\[
\limsup_{n \to \infty} \pm \frac{1}{\sqrt{2n \log \log n}} \sum_{z : ||z|| \leq n} \left( \mathbb{E}[\Delta(z, \infty)|F_x]^2 - \sigma^2 \right) = \left( \sum_{z \in \mathbb{Z}^d} \mathbb{E}[F_z^2F_0^2 - \sigma^2] \right)^{1/2}.
\]

**Remark 3.5.** In the same spirit, the three statements above are valid for the process \((F'_z : z \in \mathbb{Z}^d)\), if \(E[h(F'_z)^6] < \infty\). We simply replace the variances; if \(h\) is the identity, we replace \(\sigma^2\) with \(\tau^2 = E[\mathbb{E}[\Delta(0, \infty) - \alpha\Delta'(0, \infty)]F_0]^2\), this last equality will be verified below in Lemma 3.10. Note that by definition \(E[|\Delta'(0, \infty)|^q] < \infty\) for all \(q \in \mathbb{N}\).

**Proof of Theorem 3.4** Set \(\varphi(n) = \sqrt{2\tilde{\sigma}^2(2n + 1)^d \log \log n}, n \in \mathbb{N}, n > e\). Using \(S_n = \sum_{z \in D_n} h(F_z)\),

\[
\limsup_{n \to \infty} \pm \frac{\sum_{z \in B_n} h(F_z)}{\sqrt{2\tilde{\sigma}^2#B_n \log \log #B_n}} = \limsup_{n \to \infty} \pm \frac{\sum_{z \in D_n} h(F_z)}{\sqrt{2\tilde{\sigma}^2#D_n \log \log #D_n}} = \limsup_{n \to \infty} \pm \frac{S_n}{\varphi(n)}.
\]

Let \(\varepsilon > 0\). First, we show \(\limsup_{n \to \infty} |S_n|/\varphi(n) < 1 + \varepsilon\) with probability 1. Set \(n_k = [(1 + \tau^2/\tilde{\sigma}^2)] + 1\) for \(\tau > 0\). Using Proposition 3.3 for each \(\gamma > 0\) there is a \(\rho > 0\) such that

\[
\sum_{k \in \mathbb{N}} P(\max_{1 \leq n \leq n_k} |S_n| > (1 + \gamma)\varphi(n)) \leq C \sum_{k \in \mathbb{N}} (\log n_k)^{-(1+\rho)} < \infty.
\]
Moreover, there is a $k_0 \in \mathbb{N}$ such that $\varphi(n_k) \leq (1 + 2\tau)^{d/2}\varphi(n_{k-1})$ for all $k \geq k_0$. Also, there are $\gamma \in (0, \varepsilon)$ and $\tau > 0$ such that $(1 + \varepsilon) > (1 + \gamma)(1 + 2\tau)^{d/2}$. Consequently,

$$\mathbb{P}(\limsup_{n \to \infty} |S_n|/\varphi(n) > (1 + \varepsilon)) \leq \mathbb{P}(\limsup_{k \to \infty} \max_{n_{k-1} \leq n \leq n_k} |S_n|/\varphi(n_{k-1}) > (1 + \varepsilon))$$

$$\leq \mathbb{P}(\limsup_{k \to \infty} \max_{n \leq n_k} |S_n|/\varphi(n_k) > (1 + \varepsilon)/(1 + 2\tau)^{d/2}) = 0.$$  

Second, we show $\limsup_{n \to \infty} |S_n|/\varphi(n) > 1 - \varepsilon$ with probability 1. We use two sequences defined by $n_k = k^{4k}$ and $m_k = n_k/k^2$. Consider the event $E_k = E_k(\lambda) = \{S_{n_k} - S_{m_k} \geq (1 - 2\lambda)\varphi(n_k)\}$. Then

$$\sum_{k \in \mathbb{N}} \mathbb{P}(E_k(\lambda)) = \infty$$  

(3.2)

for all $\lambda > 0$. Indeed, consider the inequality

$$\mathbb{P}(S_{n_k} \geq (1 - \lambda)\varphi(n_k)) \leq \mathbb{P}(E_k) + \mathbb{P}(S_{m_k} \geq \lambda\varphi(n_k)).$$  

(3.3)

One finds with the Markov inequality that the second term is negligible in the sense that

$$\sum_{k=1}^{\infty} \mathbb{P}(S_{m_k} \geq \lambda\varphi(n_k)) < \infty.$$  

(3.4)

Hence, using (3.3) and (3.4), it is enough to show $\sum_{k=1}^{\infty} \mathbb{P}(S_{n_k} \geq (1 - \lambda)\varphi(n_k)) = \infty$ in order to verify (3.2). An application of the normal approximation in Proposition 3.2 yields

$$\sum_{k=1}^{\infty} |\mathbb{P}(S_{n_k} \geq (1 - \lambda)\varphi(n_k)) - \Phi((1 - \lambda)\varphi(n_k)/(\sigma(D_{n_k})^{1/2}))| \leq C \sum_{k=1}^{\infty} (#D_{n_k})^{-\delta} < \infty$$

where $\delta > 0$ and where $\Phi$ is the distribution function of the standard normal distribution. Using the lower bound $1 - \Phi(x) \leq (2\pi)^{-1/2}x^{-1} \exp(-x^2/2)$, it is a routine to verify for the subsequence $(S_{n_k})_k$

$$\sum_{k=1}^{n} \Phi((1 - \lambda)\varphi(n_k)/(\sigma(D_{n_k})^{1/2})) = \sum_{k=1}^{n} \Phi((1 - \lambda)(2 \log \log(#D_{n_k}))^{1/2}) = \infty.$$ 

This shows (3.2). Next, we claim that this leads to $\mathbb{P}(E_k(\lambda) \text{ occurs i.o.}) = \mathbb{P}(\sum_{k=1}^{\infty} \mathbb{1}\{E_k(\lambda)\} = \infty) = 1$ for all $\lambda > 0$. Indeed, we show $\mathbb{P}(\sum_{k=1}^{\infty} \mathbb{1}\{E_k(\lambda)\} < \infty) = 0$ with the following considerations

$$\mathbb{P}\left(\sum_{k=1}^{\infty} \mathbb{1}\{E_k\} \leq \frac{1}{2} \sum_{k=1}^{n} \mathbb{P}(E_k)\right) \leq \mathbb{P}\left(\sum_{k=1}^{n} \mathbb{1}\{E_k\} \leq \frac{1}{2} \sum_{k=1}^{n} \mathbb{P}(E_k)\right)$$

$$\leq \mathbb{P}\left(\left|\sum_{k=1}^{n} \mathbb{1}\{E_k\} - \mathbb{P}(E_k)\right| \geq \frac{1}{2} \sum_{k=1}^{n} \mathbb{P}(E_k)\right)$$

$$\leq \frac{4\text{Var}(\sum_{k=1}^{n} \mathbb{1}\{E_k\})}{(\sum_{k=1}^{n} \mathbb{P}(E_k))^2} \leq \frac{4(\sum_{k=1}^{n} \mathbb{P}(E_k)) + c}{(\sum_{k=1}^{n} \mathbb{P}(B_{k}))^2} \to 0 \ (n \to \infty);$$

the last inequality is valid because

$$\sum_{i,j: i < j} |\text{Cov}(\mathbb{1}\{E_i\}, \mathbb{1}\{E_j\})| \leq \sum_{i,j: i < j} c_1 \exp\left(-c_2((i + j)^4 + j^4)\right) \leq \sum_{i,j: i < j} c_1 \exp\left(-c_2((i + j)^2\right) < \infty$$
with an application of Lemma 3.1 and where the second to last inequality follows from a short application of the mean-value theorem. Now
\[ E_k(\varepsilon/4) \subset \{ S_{n_k} \geq (1 - \varepsilon) \varphi(n_k) \} \cup \{ -S_{m_k} \geq \varepsilon/2 \varphi(n_k) \}. \]
As in (3.7) one finds \( \sum_{k \in \mathbb{N}} \mathbb{P}(|S_{m_k}| \geq \varepsilon/2 \varphi(n_k)) < \infty. \) Thus, \( -S_{m_k}/\varphi(n_k) \geq \varepsilon/2 \) only finitely many times. Moreover,
\[ 1 = \mathbb{P}(E_k(\varepsilon/4) \text{ occurs i.o.}) \leq \mathbb{P}(S_{n_k} \geq (1 - \varepsilon) \varphi(n_k) \text{ occurs i.o.}) + \mathbb{P}(-S_{m_k} \geq \varepsilon/2 \varphi(n_k) \text{ occurs i.o.}). \]
This means \( \lim\sup_{k \to \infty} S_{n_k}/\varphi(n_k) \geq 1 - \varepsilon \) with probability 1. One can show \( \lim\inf_{k \to \infty} S_{n_k}/\varphi(n_k) \leq -(1 - \varepsilon) \) with probability 1 in a similar fashion. This completes the proof.

### 3.2 The LIL for the Poisson process

**Proposition 3.6.** Assume model [M1] or [M2] and assume that \( p \in \mathbb{N}, p > 2d \) is even. If model [M2] is holds, let \( \varepsilon \in (0, 1/(2d) - 1/p) \), otherwise \( \varepsilon = 0 \). Then for all \( \delta > 0 \)
\[ \sum_{z \in B_n} \mathbb{E} [\Delta(z, n) - \Delta(z, \infty)|\mathcal{F}_z] = o_{n,p}(\varepsilon^{d+1/p+(d-1)/(2d)}). \]

**Proof.** Model [M1] We derive a maximal inequality, this will then enable us to proof the claim. We define \( Y_{n,z} = \mathbb{E} [\Delta(z, n) - \Delta(z, \infty)|\mathcal{F}_z] \) for \( n \in \mathbb{N} \) and \( z \in B_n \). Furthermore, we define \( B''_n \) \( = \{ z \in B_n : d(Q_z, \partial W_n) > S^* \} \). Then the cardinality of \( B''_n \) is at most \( CS^*n^{(d-1)/d} \) for a constant \( C > 0 \) and \( Y_{n,z} = 0 \) for all \( z \not\in B''_n \). Let \( p \in \mathbb{N} \) be even. Set \( p = p/2 \), then
\[ \mathbb{E} \left[ \left| \sum_{z \in B_n} \mathbb{E} [\Delta(z, n) - \Delta(z, \infty)|\mathcal{F}_z] \right|^p \right] = \mathbb{E} \left[ \left| \sum_{z \in B''_n} Y_{n,z} \right|^p \right]. \] (3.5)

Note that \( (Y_{n,z} : z \in I) \) and \( (Y_{n,z'} : z' \in J) \) are independent whenever \( d(Q_z, Q_{z'}) > 2S^* \) for all \( z \in I, z' \in J \) \( I \subset B''_n \). Using this observation, we can partition the set \( B''_n \) in \( w = [2S^*]^{d} \) disjoint sets \( B''_{n,j}, j \in \{1, \ldots, w \} \) such that \( (Y_{n,z}, z \in B''_{n,j}) \) is independent for each \( j \in \{1, \ldots, w \} \). Indeed, let \( z \) be the element in \( B''_n \) which satisfies \( z \leq y \) for all \( y \in B''_n \). Set
\[ B''_{n,1} = \{ z + \lceil 2S^* \rceil (k_1, \ldots, k_d) : k_1, \ldots, k_d \in \mathbb{N}_0 \}. \]
There are \( \lceil 2S^* \rceil^{d} - 1 \) translations \( x \) such that \( x + B''_{n,1} \) is a subsets of \( B''_n \). Denote these translated sets \( B''_{n,2}, \ldots, B''_{n, \lceil 2S^* \rceil^{d}} \). Then \( B''_{n,1}, \ldots, B''_{n, \lceil 2S^* \rceil^{d}} \) partition \( B''_n \), the independence requirement is satisfied and
\[ w \cdot \max_j \#B''_{n,j} \leq (n^{1/d} + \lceil 2S^* \rceil^{d}) \leq C \#B''_n. \]

Consequently, the right-hand-side of (3.5) is at most (up to a multiplicative constant)
\[ \sum_{j=1}^{w} \mathbb{E} \left[ \sum_{z \in B''_{n,j}} Y_{n,z} \right]^p = \sum_{j=1}^{w} \sum_{z_1, \ldots, z_w \in B''_{n,j}} \mathbb{E} [Y_{n,z_1}^{a_1} \cdots Y_{n,z_w}^{a_w}]. \]
where the equality follows from the fact that the expectation is zero whenever a certain \( Y_{n,z} \) only occurs once in the product because of the independence within \( (Y_{n,z} : z \in B''_{n,j}) \). Plainly, some of the non-negative integers \( a_1, \ldots, a_w \)
can also be zero and \( a_1 + \ldots + a_p = p \). Using the H"older inequality and Lemma \( \text{[A.2]} \), we obtain
\[
\mathbb{E} \left[ Y_{n,z_1}^{a_1} \cdots Y_{n,z_p}^{a_p} \right] \leq \sup_{n \in \mathbb{N}} \max_{z \in B_n^p} \mathbb{E} \left[ |\Delta(z, n) - \Delta(z, \infty)|^2 \right] \leq C_p.
\]

Consequently, there is a constant which only depends on \( p \in \mathbb{N} \) such that the right-hand side of (3.5) is of order \( C(\#B''_n)^{p/2} \) uniformly in \( n \in \mathbb{N} \). This puts us in position to derive the maximal inequality. We have
\[
\mathbb{E} \left[ \max_{k \leq n} \left| \sum_{z \in B_k} \mathbb{E} [\Delta(z, k) - \Delta(z, \infty)|F_z] \right|^p \right]^{1/p} \leq n^{1/p} \max_{k \leq n} \mathbb{E} \left[ \sum_{z \in B_k} \mathbb{E} |\Delta(z, k) - \Delta(z, \infty)|F_z] \right]^{1/p} \leq Cn^{1/p} \sum_{j=1}^{\#B''_n} (\#B''_n)^{1/2} \leq Cn^{1/p} w^{1/2} (\#B''_n)^{1/2} \leq C(S')^{d/2} n^{1/p+(d-1)/(2d)}. \tag{3.6}
\]
Hence, for \( \delta > 0 \),
\[
(\log n)^{-(1+\delta)} n^{-(1/p+(d-1)/(2d))} \left\| \max_{k \leq n} \sum_{z \in B_k} \mathbb{E} [\Delta(z, k) - \Delta(z, \infty)|F_z] \right\|_p \leq C(\log n)^{-(1+\delta)}, \tag{3.7}
\]
and an application of Lemma \( \text{[A.3]} \) yields
\[
\left| \sum_{z \in B_n} \mathbb{E} [\Delta(z, n) - \Delta(z, \infty)|F_z] \right| = o_{a.s.}(\log n)^{1+\delta} n^{1/p+(d-1)/(2d)}
\]
and proves the claim under the assumption \( \text{[M1]} \).

**Model \( \text{[M2]} \)** The proof follows very much the same ideas as for the model \( \text{[M1]} \). However, we first need to deal with the stabilizing property of the functional. Set \( r_n = n^{\gamma} \) for some \( \gamma \in (0, 1/d) \) sufficiently small, see below. Consider
\[
\Delta(z, n) - \Delta(z, \infty)
\]
\[
= H(\mathbb{P}_n) - H(\mathbb{P}''_{n,z}) - H(\mathbb{P}''_{z} - \mathbb{P}_n) + H(\mathbb{P}'_{z} - \mathbb{P}_n)
\]
\[
= H(\mathbb{P}_n) - H(\mathbb{P}''_{n,z}) - H(\mathbb{P}_n \cap B(z, r_n)) + H(\mathbb{P}''_{n,z} \cap B(z, r_n)) + H(\mathbb{P}'_{z} \cap B(z, r_n)) \tag{3.8}
\]
\[
+ H(\mathbb{P} \cap B(z, r_n)) - H(\mathbb{P}''_{z} \cap B(z, r_n)) - H(\mathbb{P} \cap B(z, r_n)) + H(\mathbb{P}'_{z} \cap B(z, r_n)) \tag{3.9}
\]
\[
+ H(\mathbb{P} \cap B(z, r_n)) - H(\mathbb{P}''_{z} \cap B(z, r_n)) - H(\mathbb{P} \cap B(z, r_n)) + H(\mathbb{P}'_{z} \cap B(z, r_n)). \tag{3.10}
\]
First consider the remainder terms in (3.8) and (3.10). Using the assumption on the radius of stabilization from (1.6) and the translation invariance of \( H \), we find that both
\[
\mathbb{P}(H(\mathbb{P}_n) - H(\mathbb{P}''_{n,z}) \neq H(\mathbb{P}_n \cap B(z, r_n)) - H(\mathbb{P}''_{n,z} \cap B(z, r_n)) \leq c_1 \exp(-c_2 r_n),
\]
\[
\mathbb{P}(H(\mathbb{P} \cap B(z, r_n)) - H(\mathbb{P}''_{z} \cap B(z, r_n)) \neq H(\mathbb{P} \cap B(z, r_n)) - H(\mathbb{P}'_{z} \cap B(z, r_n)) \leq c_1 \exp(-c_2 r_n).
\]
Moreover, using the uniform bounded moments condition from (1.3), it is straightforward to derive that the \( p \)th moment of the term in (3.8) and in (3.10) is uniformly bounded in \( n \) and \( z \), see Lemma \( \text{[A.2]} \).

Hence, in order to derive a maximal inequality of the same type as in (3.7) (this time under the assumptions of model \( \text{[M2]} \)), it is enough to consider the sum over the terms in (3.9). For this define
\[
\tilde{Y}_{n,z} := H(\mathbb{P}_n \cap B(z, r_n)) - H(\mathbb{P}''_{n,z} \cap B(z, r_n)) - H(\mathbb{P} \cap B(z, r_n)) + H(\mathbb{P}'_{z} \cap B(z, r_n)).
\]
Again, we can use the independence argument as before and partition \( B_n' = \{ z \in B_n : d(Q_z, \partial W_n) > r_n \} \) in \( w_n \) families of random variables, such that the elements within each single family are independent and \( w_n = \lceil 2r_n \rceil^d \).

Once more, an application of Lemma A.2 yields

\[
\sup_{n \in \mathbb{N}} \sup_{z \in B_n} \mathbb{E} \left[ \left| \hat{Y}_{n,z} \right|^p \right] \leq 2^{2p} \sup_{n \in \mathbb{N}} \sup_{z \in B_n} \left\{ \mathbb{E} \left[ \left| H(P_n \cap B(z, r_n)) - H(P_{n,z}'' \cap B(z, r_n)) \right|^p \right] + \mathbb{E} \left[ \left| H(P \cap B(z, r_n)) - H(P'' \cap B(z, r_n)) \right|^p \right] \right\} < \infty.
\]

Consequently, using the same reasoning as in the first part and that \( w_n = \mathcal{O}(n^{\gamma d}) \) as well as \( p > 2d \), we arrive at

\[
\mathbb{E} \left[ \max_{k \leq n} \left| \sum_{z \in B_k} \mathbb{E} \left[ \Delta(z, k) - \Delta(z, \infty) \right] \right|^p \right]^{1/p} \leq C(w_n)^{1/2 + 1/p + (d-1)/(2d)} \leq Cn^{d+1/p + (d-1)/(2d)},
\]

which generalizes the result in (3.6). By assumption \( p > 2d \). Hence, choosing \( \gamma > 0 \) sufficiently small, we obtain once again the analogous result in (3.7), which proves the statement in model \( (M2) \) and completes the proof.

We can now prove LIL for the Poisson process.

**Proof of Theorem 2.7** We use the following decomposition using martingale differences

\[
H(P_n) - \mathbb{E} [H(P_n)] = \sum_{z \in B_n} \mathbb{E} [H(P_n) - H(P''_{n,z})] = \sum_{z \in B_n} \mathbb{E} [\Delta(z, n)|F_z] + \sum_{z \in B_n} \mathbb{E} [\Delta(z, \infty)|F_z] = \sum_{z \in B_n} \mathbb{E} [\Delta(z, n) - \Delta(z, \infty)|F_z].
\]

The remainder in (3.11) is of order \( o_{a.s.}(\sqrt{n}) \) in both models, see the subsequent Proposition 3.6.

The leading term involving the \( F_z = \mathbb{E} [\Delta(z, \infty)|F_z], z \in B_n \) satisfies Theorem 3.4. We explain in Lemma 3.10 that \( \tau^2 \) is positive, hence, also \( \sigma^2 \) is positive. This completes the proof.

### 3.3 The LIL for the binomial process

**Proposition 3.7.** Assume model \( (M1) \) or \( (M2) \). Then for all \( \delta > 0 \)

\[
H(U_{n,n}) - H(U_{n,N_n}) - \alpha(n - N_n) = o_{a.s.}(n^{1/4+1/p}(\log n)^{1+\delta}).
\]

**Proof.** We distinguish the cases \( |N_n - n| \geq n^{1/2+1/p} \) and \( |N_n - n| < n^{1/2+1/p} \). An application of the Borel-Cantelli Lemma reveals that \( \{ |N_n - n| \geq n^{1/2+1/p} \} \) is empty for almost all \( n \in \mathbb{N} \), for each \( p > 0 \). In particular,

\[
\limsup_{n \to \infty} h(n) \left| H(U_{n,n}) - H(U_{n,N_n}) - \alpha(n - N_n) \right| \mathbb{1} \left\{ |N_n - n| \geq n^{1/2+1/p} \right\} = 0 \quad a.s.
\]

for any sequence \( (h(n))_n \subset \mathbb{R}_+ \).

Now we show the remainder of the claim on the set \( \{ |N_n - n| < n^{1/2+1/p} \} \) in two steps. In the first step, we compute the \( p \)th moment of (3.12), when conditioned on \( N_n \). In the second step, we conclude.

**Step 1.** Using the requirement that \( p \in \mathbb{N}_+ \) is even, we show

\[
\mathbb{E} \left[ \left| H(U_{n,n}) - H(U_{n,N_n}) - \alpha(n - N_n) \right|^p \middle| N_n \right] \mathbb{1} \left\{ |N_n - n| \leq n^{1/2+1/p} \right\} \leq C |N_n - n|^{p/2} \quad a.s.
\]
for a constant \( C \in \mathbb{R}_+ \), which does neither depend on \( N_n \) nor on \( n \). We use the decomposition

\[
|H(U_{n,n}) - H(U_{n,N_n}) - \alpha(n - N_n)| = \left| \sum_{m \in I_n} (H(U_{n,m+1}) - H(U_{n,m}) - \alpha) \right|. \tag{3.14}
\]

where \( I_n \) is given in **3.1**. Let \( n \in \mathbb{N} \) be arbitrary but fixed. Define \( Y_m := \{ U_{n,m+1} \setminus U_{n,m} \} \) as the unique point in \( U_{n,m+1} \) that is not contained in \( U_{n,m} \).

Computing the left-hand side of (3.13) with the equality in (3.14), amounts to calculate

\[
\sum_{(m_1, \ldots, m_p) \in I_n^p} E \left[ \prod_{i=1}^p (H(U_{n,m_i+1}) - H(U_{n,m_i}) - \alpha) \right]. \tag{3.15}
\]

(plainly some of the indices \( m_i \) can occur multiple times). Let \( (m_1, \ldots, m_p) \) be a generic tuple and consider a factor \( (H(U_{n,m_k+1}) - H(U_{n,m_k}) - \alpha) \) which occurs with a multiplicity of \( a_k \in \{1, \ldots, p\} \) in the product. The details depend now on the selected model.

**Model [MI]** If \( A_k = \{ d(Y_{m_k}, \cup_{j \neq k} Y_{m_j}) > 2S^* \} \) occurs, then this factor is independent of all other factors conditionally on the points \( Y_{m_1}, \ldots, Y_{m_p} \). And if additionally, \( a_k = 1 \), then the expectation of this factor is close to zero. Otherwise, if \( A_k \) does not occur, there is a certain dependence between this and other factors, however, \( 1 - \mathbb{P}(A_k) \leq cn^{-1} \).

Given the generic tuple \( (m_1, \ldots, m_p) \), we write

\( L = \{ k \in \{1, \ldots, p\} | m_i \neq m_k : \forall i \leq k, i \in \{1, \ldots, p\} \} \)

for all indices of the generic tuple which occur and \( L^* = \{ k \in L : a_k = 1 \} \) for the indices which occur exactly once. (So \( L \) and \( L^* \) both depend on the tuple \( (m_1, \ldots, m_p) \).) Moreover, denote \( Z_m = H(U_{n,m+1}) - H(U_{n,m}) - \alpha \).

Consider the expectation in (3.15) for a generic \( (m_1, \ldots, m_k) \) now in terms of the multiples \( a_k \). We have

\[
E \left[ \prod_{k \in L} Z_{m_k}^{a_k} \bigg| Y_{m_k} : k \in L \right]
\]

\[
= \prod_{k \in L^*} \left( \mathbb{1}\{A_k\} + \mathbb{1}\{A_k^c\} \right) \prod_{k \in L \setminus L^*} E \left[ Z_{m_k} \prod_{k \in L \setminus L^*} Z_{m_k}^{a_k} \bigg| Y_{m_k} : k \in L \right]
\]

\[
= \sum_{w \in \{0,1\}^{\#L^*}} \prod_{k \in L^*} \mathbb{1}\{A_k\}^{w_k} \mathbb{1}\{A_k^c\}^{1-w_k} \prod_{k \in L \setminus L^*} E \left[ Z_{m_k} \bigg| Y_{m_k} \right]
\]

\[
= \sum_{w \in \{0,1\}^{\#L^*}} \prod_{k \in L^*} \mathbb{1}\{A_k\}^{w_k} E \left[ Z_{m_k} \bigg| Y_{m_k} \right]
\]

\[
= \prod_{k \in L^*} \mathbb{1}\{A_k\}^{w_k} \prod_{k \in L \setminus L^*} Z_{m_k}^{a_k} \bigg| Y_{m_k} : k \in L \right]. \tag{3.16}
\]

Consider the conditional expectation of \( Z_{m_k} \) (in the situation where it occurs exactly once). First, note that

\[
E \left[ H\left( |\mathcal{P}_{m_k} \cap B(Y_{m_k}, S^*)| \cup \{ Y_{m_k} \} \right) - H\left( \mathcal{P}_{m_k} \cap B(Y_{m_k}, S^*) \right) \bigg| Y_{m_k} \right] = \alpha \text{ a.s.}
\]

where \( \mathcal{P}_{m_k} \) is the homogeneous Poisson process on \( \mathbb{R}^d \), which consists of the Poissonized binomial process \( U_{n,m_k} \) on
$W_n$ and an independent homogeneous Poisson process $\mathbb{P}^*$ on $\mathbb{R}^d \setminus W_n$. Note that $\mathbb{P}_{m_k}$ is independent of $Y_{m_k}$. Consider the probability

$$\mathbb{P}(\mathcal{U}_{n,m_k} \cap B(Y_{m_k}, S^*) \neq \mathbb{P}_{m_k} \cap B(Y_{m_k}, S^*) | Y_{m_k})$$

Then we apply with the Hölder inequality with $r \in (1, (2p - 4)/(p + 2))$

$$|\mathbb{E}[Z_{m_k} | Y_{m_k}]| = |\mathbb{E}[H(\mathcal{U}_{n,m_k+1} \cap B(Y_{m_k}, S^*)) - H(\mathcal{U}_{n,m_k} \cap B(Y_{m_k}, S^*)) | Y_{m_k}] - \mathbb{E}[H([\mathbb{P}_{m_k} \cap B(Y_{m_k}, S^*)] \cup \{Y_{m_k}\}) - H(\mathbb{P}_{m_k} \cap B(Y_{m_k}, S^*) ) | Y_{m_k}] |$$

$$\leq \left\{ \begin{array}{l} \mathbb{E}[H(\mathcal{U}_{n,m_k+1} \cap B(Y_{m_k}, S^*)) - H(\mathcal{U}_{n,m_k} \cap B(Y_{m_k}, S^*)) | (r/(r-1)) | Y_{m_k}]^{(r-1)/r} \\
+ \mathbb{E}[H([\mathbb{P}_{m_k} \cap B(Y_{m_k}, S^*)] \cup \{Y_{m_k}\}) - H(\mathbb{P}_{m_k} \cap B(Y_{m_k}, S^*) ) | (r/(r-1)) | Y_{m_k}]^{(r-1)/r} \\
\cdot \mathbb{P}(\mathcal{U}_{n,m_k} \cap B(Y_{m_k}, S^*) \neq \mathbb{P}_{m_k} \cap B(Y_{m_k}, S^*) | Y_{m_k}) \end{array} \right\}^{1/r}$$

$$\leq Cn^{-(1/2 - 1/p)/r},$$

where the last inequality follows from (3.17), Lemma A.2 and the constant $C$ is independent of $n$.

Furthermore, using the condition that $|N_n - n| \leq n^{1/2 + 1/p}$ and Lemma A.2 we find

$$\mathbb{E}[Z_{m_k} | Y_{m_k}] \vee \mathbb{E}[Z_{m_k} | Y_{m_k} : k \in L] \leq C,$$

uniformly in $m_k \in I_n$ and $n \in \mathbb{N}$. In addition, set $b = \#L^* - \|w\|_1$, where $\| \cdot \|_1$ is the $\ell^1$-norm. Let $\{k \in L^* : w_k = 0\}$ be given by $\{k_1, \ldots, k_b\}$. Then

$$\mathbb{P}(\cap_{k \in L^*: w_k=0} A_k) = \mathbb{P}(d(Y_{m_k}, \cup_{j \neq k} Y_{m_j}) \leq 2S^* \text{ for all } k \in L^* \text{ with } w_k = 0)$$

$$\leq \sum_{j_1, \ldots, j_b=1}^{p} \mathbb{P}(d(Y_{m_{k_1}}, Y_{m_{j_1}}) \leq 2S^*, \ldots, d(Y_{m_{k_b}}, Y_{m_{j_b}}) \leq 2S^*)$$

$$\leq C n^{-((\#L^* - \|w\|_1)/2)}.$$
tions for which \( \#L^* = j \) for each \( j \in \{1, \ldots, p\} \). This can be done with an elementary combinatorial argument: The number of combinations which determine the indices that occur exactly once is at most \( (\#I_n)^j \). The number of free positions in \( I_n \) is \( p - j \), however, each position has to be occupied by two indices, hence, it remain at most \( (\#I_n)^{(p-j)/2} \) combinations for the indices with a multiplicity of at least 2. This means the number of combinations for which \( \#L^* = j \) is at most \( (\#I_n)^{j+(p-j)/2} = (\#I_n)^{j/2 + p/2} \) because \( p \) is even. Consequently,

\[
\#\{(m_1, \ldots, m_p) \in I_p^n : L^*(m_1, \ldots, m_p) = j\} \leq (\#I_n)^{p/2 + j/2},
\]

where we write \( \lfloor x \rfloor \) for the integer \( z \leq x \) for \( x \in \mathbb{R} \). Consequently, since \( r \in (1, (2p - 4)/(p + 2)) \) and \( (\#I_n) \leq n^{1/2 + 1/p} \), (3.15) is at most (times a suitable constant which is independent of \( n \))

\[
\sum_{(m_1, \ldots, m_p) \in I^n} \frac{1}{n^{\#L^*(1/2 - 1/p)/r}} \leq \frac{(\#I_n)^{p/2 + j/2}}{n^{j/(1 - 1/p)}} \leq \frac{(\#I_n)^{p/2}}{n^{j/(1 - 1/p)}} \leq p(\#I_n)^{p/2}. \tag{3.21}
\]

This shows (3.13) for the model [M1].

**Model [M2]** Let \( n \in \mathbb{N} \) be arbitrary but fixed and let \( r \in (1, (2p - 4)/(p + 2)) \) and \( r_n := n^r \) for \( \gamma \in (0, ((2p - 4) - (p + 2)r)/(4dp)) \) (see also (3.26) below). We keep the notation introduced in the calculations for model [M1] and decompose the right-hand side of (3.14) in two terms as follows

\[
\sum_{m \in I_n} \left| H(\mathcal{U}_{n,m+1} - H(\mathcal{U}_{n,m}) - \left[ H(\mathcal{U}_{n,m+1} \cap B(\gamma_{Y,m}, r_n) - H(\mathcal{U}_{n,m} \cap B(\gamma_{Y,m}, r_n)) \right] \right| =: \sum_{m \in I_n} \tilde{Z}_m, \tag{3.22}
\]

\[
\sum_{m \in I_n} \left| H(\mathcal{U}_{n,m+1} \cap B(\gamma_{Y,m}, r_n) - H(\mathcal{U}_{n,m} \cap B(\gamma_{Y,m}, r_n)) \right| \leq \alpha. \tag{3.23}
\]

One can follow the calculations for model [M1] to show that conditional on the event \( \{|N_n - n| \leq n^{1/2 + 1/p}\} \) the \( p \)-th-moment of the term in (3.23) is of order

\[
\sum_{j = 0}^{p} \left( \frac{|I_n|^{p/2 + j/2}}{n^{j/(1 - 1/p)}} \right)^{j/r} \leq p|I_n|^{p/2},
\]

for the above choices of \( \gamma \) and \( r \) (compare with (3.17), (3.20), (3.21), where \( r_n \) corresponds to \( S^* \) and is constant). Note that in this case we need the uniform bounded moments condition to be satisfied for \( r/(r - 1) \).

We consider the approximation error in (3.22) on \( \{|N_n - n| \leq n^{1/2 + 1/p}\} \). If \( \tilde{Z}_m \neq 0 \), then the event

\[
E_m = \{ \mathcal{U}_{n,m} \cap B(\gamma_{Y,m}, r_n) \neq \mathcal{P}_n \cap B(\gamma_{Y,m}, r_n) \} \cup \{ \bar{S} > r_n \}
\]

occurs; here \( \bar{S} = \bar{S}(\gamma_{Y,m}, n, N, -\gamma_{Y,m}) \) is the radius of stabilization of the point clouds \( (\mathcal{P}_n - \gamma_{Y,m}) \) and \( \{0\} \) under the functional \( H \) from (1.5) and (1.6), with \( M := (\mathcal{P}_n - \gamma_{Y,m})(Q_0) \). Consequently,

\[
E \left[ \sum_{m \in I_n} \tilde{Z}_m \right] \leq \sum_{(m_1, \ldots, m_p) \in I^n} E \left[ \tilde{Z}_{m_1} \mathbb{1}\{ E_{m_1} \} \cdot \ldots \cdot \tilde{Z}_{m_p} \mathbb{1}\{ E_{m_p} \} \mid N_n \right] 
\leq \sum_{(m_1, \ldots, m_p) \in I^n} E \left[ |\tilde{Z}_{m_1} \cdot \ldots \cdot \tilde{Z}_{m_p}|^{r/(r-1)} \right]^{(r-1)/r} \mathbb{P}(E_{m_1} \cap \ldots \cap E_{m_p})^{1/r}. \tag{3.24}
\]

Using the uniform bounded moments condition from (1.4)

\[
E \left[ |H(\mathcal{U}_{n,m+1} - H(\mathcal{U}_{n,m})|^{pr/(r-1)} \right] \leq C
\]
If we choose for the derivation of the second but last inequality see (3.20). Once more, consider a generic tuple \((m_1, \ldots, m_p)\). Since \(P(S(y, n, k, x) > r_n)\) decays exponentially in \(n\) (uniformly in the parameters \(y, n, k, x\)), the relevant part of the probability in (3.20) is

\[
P(U_{n, m_i} \cap B(Y_{m_i}, r_n) \neq P_n \cap B(Y_{m_i}, r_n) \text{ for } i \in \{1, \ldots, p\})
\]

(3.25)

Again, we use the notation \(A_k = \{d(Y_{m_k}, \cup_{j \neq k} Y_{m_j}) > 2r_n\} \text{ for } k \in \{1, \ldots, p\}\). Let \(w \in \{0, 1\}^p\). Then using the properties of the Poisson process, we obtain for this \(w\)

\[
P\left(U_{n, m_i} \cap B(Y_{m_i}, r_n) \neq P_n \cap B(Y_{m_i}, r_n) \text{ for } i \in \{1, \ldots, p\} \text{ with } w_i = 1 \left| Y_{m_i}, I\{A_i\} \text{ for } i \in \{1, \ldots, p\} \text{ with } w_i = 1, U_{n, m_k} \cap B(Y_{m_k}, r_n) \text{ for } k \in \{1, \ldots, p\}\right)
\]

\[
= \prod_{i: w_i = 1} P\left(U_{n, m_i} \cap B(Y_{m_i}, r_n) \neq P_n \cap B(Y_{m_i}, r_n) \left| Y_{m_i}, I\{A_i\}, U_{n, m_i} \cap B(Y_{m_i}, r_n) \right) \right)
\]

\[
\leq \prod_{i: w_i = 1} C_{n}^{r_d} \frac{n}{n^{1/2-1/p}} \mathbb{E}\left[\left|N_n - n\right| + |n - m|\right] = O\left(\left(\frac{r_d}{n^{1/2-1/p}}\right)^{\|w\|_1}\right).
\]

Moreover, \(\Omega = \bigcup_{w \in \{0, 1\}^p}(\cap_{i: w_i = 1} A_i \cap \bigcap_{i: w_i = 0} A_i)\). Thus, (3.25) equals

\[
\sum_{w \in \{0, 1\}^p} P\left(U_{n, m_i} \cap B(Y_{m_i}, r_n) \neq P_n \cap B(Y_{m_i}, r_n) \text{ for } i \in \{1, \ldots, p\} \right) \cap \bigcap_{i: w_i = 1} A_i \cap \bigcap_{i: w_i = 0} A_i^c
\]

\[
\leq \sum_{w \in \{0, 1\}^p} C_{n}^{r_d} \frac{n}{n^{1/2-1/p}} \mathbb{E}\left[\left|N_n - n\right| + |n - m|\right] \left(\prod_{i: w_i = 1} A_i^c\right)
\]

\[
\leq \sum_{w \in \{0, 1\}^p} C_{n}^{r_d} \frac{n}{n^{1/2-1/p}} \mathbb{E}\left[\left|N_n - n\right| + |n - m|\right] \left(\prod_{i: w_i = 1} A_i\right)
\]

\[
\leq C\left(\frac{r_d}{n^{1/2-1/p}}\right)^p
\]

for the derivation of the second but last inequality see (3.20).

So the \(p\)th moment of (3.22) on \(\{\left|N_n - n\right| \leq n^{1/2+1/p}\}\) is bounded above by (up to a multiplicative constant)

\[
\left(\frac{r_d}{n^{1/2-1/p}}\right)^{p/r} \mathbb{P}\left[\left|N_n - n\right| \leq n^{1/2+1/p}\right] \leq \left(\frac{r_d}{n^{1/2-1/p}}\right)^{p/r} n^{(1/2+1/p)p/2} \mathbb{P}\left[\left|N_n - n\right| \leq n^{p/2}\right].
\]

(3.26)

If we choose \(r \in (1, (2p - 4)/(p + 2))\) and \(\gamma \in (0, ((2p - 4) - (p + 2)r)/(4dp))\), then the left-hand side of (3.26) is \(o\left(\left|N_n - n\right|^{p/2}\right)\). In particular, (3.13) is also satisfied in model [M2].

**Step 2.** Using Jensen’s inequality and properties of the maximum, we find with the result of the first step

\[
\mathbb{E}\left[\max_{k \in \{1, \ldots, n\}} \left|H(U_{k, k}) - H(U_{k, N_k}) - \alpha(k - N_k)\right] \mathbb{I}\left\{|N_k - k| \leq k^{1/2+1/p}\right\}\right]
\]

\[
\leq n^{1/p} \max_{k \in \{1, \ldots, n\}} \mathbb{E}\left[\left|H(U_{k, k}) - H(U_{k, N_k}) - \alpha(k - N_k)\right|^{p} \mathbb{I}\left\{|N_k - k| \leq k^{1/2+1/p}\right\}\right]^{1/p}
\]

\[
= n^{1/p} \max_{k \in \{1, \ldots, n\}} \mathbb{E}\left[\left|H(U_{k, k}) - H(U_{k, N_k}) - \alpha(k - N_k)\right|^{p} \mathbb{I}\left\{|N_k - k| \leq k^{1/2+1/p}\right\}\right]^{1/p}
\]
Proposition 3.8. Assume model (M1) or model (M2). Then

\[C \ell \] where the last inequality follows because the \( \ell \)th moment of a Poisson random variable with parameter \( \lambda \) is bounded above by \( C \ell \lambda^{\ell/2} \) for some constant \( C \ell \in \mathbb{R}_+ \) which only depends on \( \lambda \).

The claim follows now from Lemma A.3 using the additional factor \((\log n)^{1+\delta}\) for some \( \delta > 0 \). \( \square \)

Proposition 3.8. Assume model (M1) or model (M2). Then \( \mathbb{E} [H(\mu_{n,N_n}) - H(\mu_{n,n})] = O(n^{1/4}) \).

Proof. We will use the decomposition

\[ H(\mu_{n,N_n}) - H(\mu_{n,n}) = \sum_{m=n}^{N_n-1} H(\mu_{n,m+1}) - H(\mu_{n,m}) - \sum_{m=N_n}^{n-1} H(\mu_{n,m+1}) - H(\mu_{n,m}), \]

with the convention that the first or the second sum is zero depending on whether \( N_n < n \) or \( N_n > n \). Moreover, we have \( 0 = \mathbb{E}[\alpha(N_n - n)] = \mathbb{E}[\alpha(N_n - n) \mathbb{I}\{N_n > n\} - \alpha(n - N_n) \mathbb{I}\{N_n < n\}] \). Consequently, using another time the definition of the set \( I_n \) from (3.3), we obtain

\[ \mathbb{E} [H(\mu_{n,N_n}) - H(\mu_{n,n})] \leq \mathbb{E} \left[ \sum_{m \in I_n} \left( H(\mu_{n,m+1}) - H(\mu_{n,m}) - \alpha \right) \right] \]

\[ = \mathbb{E} \left[ \sum_{m \in I_n} \left( H(\mu_{n,m+1}) - H(\mu_{n,m}) - \alpha \right) \mathbb{I}\{|N_n - n| \leq n\} \right] \]

\[ + \mathbb{E} \left[ \sum_{m \in I_n} \left( H(\mu_{n,m+1}) - H(\mu_{n,m}) - \alpha \right) \mathbb{I}\{|N_n - n| > n\} \right]. \tag{3.27} \]

First consider (3.27). We deduce from (3.13) that

\[ \mathbb{E} \left[ \sum_{m \in I_n} \left( H(\mu_{n,m+1}) - H(\mu_{n,m}) - \alpha \right)^2 \right] \mathbb{I}\{|N_n - n| \leq n\} \leq C|N_n - n| \quad \text{a.s.} \]

Hence, the expectation in (3.27) is of order \( O(n^{1/4}) \) in both models (M1) and (M2).

Second, we consider (3.28). We begin with model (M1). Let \( n \in \mathbb{N} \) be arbitrary but fixed. Once more, write \( Y_m = \{U_{n,m+1} \setminus U_{n,m}\} \). Denote \( M_m \) the number of points of \( U_{n,m} \) in \( B(Y_m, S^*) \). Then, conditional on \( N_n \), \( M_{N_n} \) follows a binomial distribution with length \( N_n \) and success probability proportional to \( (S^*)^d/n \). Moreover, \( M_n \) follows a binomial distribution with length \( n \) and probability proportional to \( (S^*)^d/n \). So \( M_n \) tends to a Poisson distributed variable as \( n \to \infty \). Consequently, by the assumptions of (M1)

\[ |H(\mu_{n,m+1}) - H(\mu_{n,m})| \leq 2C^* \left| \sum_{k=0}^{M_{N_n}+1} \left( \begin{array}{c} M_{N_n}+1 \\ k \end{array} \right) \right|^q \leq 2C^* \left| \sum_{k=0}^{M_{N_n}+1} \left( \begin{array}{c} M_{N_n}+1 \\ k \end{array} \right) \right|^q. \]

This means, the asymptotic growth of (3.28) is determined by

\[ \mathbb{E} \left[ |N_n - n|^2 \left( \mathbb{E} \left[ \sum_{k=0}^{M_{N_n}+1} \left( \begin{array}{c} M_{N_n}+1 \\ k \end{array} \right)^q \right] \right) \mathbb{I}\{|N_n - n| \leq n\} + \mathbb{E} \left[ \sum_{k=0}^{M_{N_n}+1} \left( \begin{array}{c} M_{N_n}+1 \\ k \end{array} \right)^q \right] \right]^{1/2} \tag{3.29} \]

\[ \cdot \mathbb{P}(|N_n - n| > n)^{1/2}. \]
Clearly, the probability in (3.29) decays exponentially in \( n \). So it remains to show that the expectation grows polynomially in \( n \). Consider the first inner conditional expectation. Using the result from Lemma A.1, it is enough to compute \( \mathbb{E}[2^\delta M_n | N_n] \) for general \( \delta > 0 \), where \( M_n \) follows a binomial distribution \( \text{Bin}(N_n, c/n) \). We obtain
\[
\mathbb{E}[2^\delta M_n | N_n] \leq \sum_{k=0}^{N_n} \frac{(c2^\delta)^k}{k!} \left( \frac{N_n}{n} \right)^k \leq \exp \left( c2^\delta \frac{N_n}{n} \right).
\]
Moreover, \( \mathbb{E}[\exp(c2^\delta N_n/n)] = \exp(c2^\delta/n n) \to \exp(c2^\delta) \) as \( n \to \infty \). Furthermore, \( \mathbb{E}[2^\delta M_n] \leq \exp(c2^\delta) \).

Consequently, an application of the \( \text{H"older's inequality} \) to (3.29) yields an upper bound of polynomial order in \( n \). So, (3.28) is of order \( o(n^{1/4}) \) in model (M1).

Finally, consider (M2). The difference \( |H(\mathcal{U}_{n,m+1}) - H(\mathcal{U}_{n,m})| \) is of order \( O(n^{1/2 - 1/(2d)}) \). Define \( W_{n,z} = \mathcal{P}(Q_z \setminus B_n) \). The probability \( \mathbb{P}(|N_n - n| > n) \) decays exponentially. So, an application of the \( \text{H"older's inequality} \), shows that (3.28) is of order \( o(n^{1/4}) \) in model (M2) too.

**Lemma 3.9.** For all \( \delta > 0 \), \( \sum_{z \in B_n} \mathbb{E}[\Delta'(z, n) - \Delta'(z, \infty)|\mathcal{F}_z] = o_n. (n^{1/2 - 1/(2d)}) \log(n)^{2 + \delta} \).

**Proof.** We have that \( \mathbb{E}[\Delta'(z, n) - \Delta'(z, \infty)|\mathcal{F}_z] = -\mathbb{P}(Q_z \setminus B_n) + \mathbb{E}[\mathbb{P}(Q_z \setminus B_n)] \) which is zero if \( Q_z \subset B_n \). Write \( B_n'' \) for those \( z \in B_n \) such that \( Q_z \) is not a subset of \( B_n \). \#\( B_n'' \) is of order \( O(n^{(d-1)/d}) \). Define \( W_{n,z} = \mathcal{P}(Q_z \setminus B_n) \).

Consider the Laplace transform
\[
\mathbb{E}\left[ \exp(\gamma \sum_{z \in B_n'} W_{n,z} - \mathbb{E}[W_{n,z}]) \right] = \prod_{z \in B_n''} \mathbb{E}[\exp(\gamma W_{n,z})] \exp(-\gamma \mathbb{E}[W_{n,z}]).
\]
(3.30)

Since \( W_{n,z} \) is Poisson distributed with parameter \( |Q_z \setminus B_n| = \mathbb{E}[W_{n,z}] \), we have for all \( \gamma \in \mathbb{R}_+ \)
\[
\mathbb{E}[\exp(\gamma W_{n,z})] = \exp \left( \mathbb{E}[W_{n,z}] (e^\gamma - 1) \right).
\]
Hence, the right-hand side of (3.30) equals \( \exp(\sum_{z \in B_n''} \mathbb{E}[W_{n,z}] (e^\gamma - 1 - \gamma) ) \leq \exp(e^n \mathbb{E}[W_{n,z}] ) \). Consider
\[
\mathbb{E}\left[ \max_{1 \leq k \leq n} \left| \sum_{z \in B_k} W_{k,z} - \mathbb{E}[W_{k,z}] \right| \right]
\leq n^a \log \mathbb{E}\left[ \max_{1 \leq k \leq n} \left| \frac{1}{n^a} \sum_{z \in B_k} W_{k,z} - \mathbb{E}[W_{k,z}] \right| \right]
\leq n^a \log n + n^a \log \left\{ \max_{1 \leq k \leq n} \mathbb{E}\left[ \exp\left( \frac{1}{n^a} \sum_{z \in B_k} W_{k,z} - \mathbb{E}[W_{k,z}] \right) \right] \right\}
\leq n^a \log n + n^a \log \left\{ \max_{1 \leq k \leq n} \mathbb{E}\left[ \exp\left( -\frac{1}{n^a} \sum_{z \in B_k} W_{k,z} - \mathbb{E}[W_{k,z}] \right) \right] \right\}
\leq n^a \log n + cn^a - 2n^{(d-1)/d}.
\]
(3.32)

Hence, if \( d > 1 \) or \( d = 1 \), the choice \( a = (d - 1)/(2d) \) equalizes the rate in both terms, which is then \( n^{(d-1)/(2d)} \log n \). An application of Lemma A.3 yields the claim. \( \square \)

**Proof of Theorem 2.2** We apply the following fundamental decomposition.
\[
H(\mathcal{U}_{n,n}) - \mathbb{E}[H(\mathcal{U}_{n,n})] = H(\mathcal{U}_{n,N_n}) - \mathbb{E}[H(\mathcal{U}_{n,N_n})] - \alpha(N_n - n)
\]
\[
+ \left\{ H(\mathcal{U}_{n,n}) - H(\mathcal{U}_{n,N_n}) - \alpha(n - N_n) \right\}
\]
(3.33)
(3.34)
Consequently, \( \lim \) above introduced martingale difference sequence (note that \( X \) of (3.33) equals \( o \)) in (3.33) equals \( \sigma \) in (3.33) equals \( o \). Hence, \( o \). Regarding the second term, we infer from Lemma 3.9 that it equals \( o \). Assume model (M1) or model (M2). Then \( o \). Furthermore, \( o \). It remains to verify the equality. We use the representation from (3.33) to (3.35) and set \( o \). The first term in (3.37) is of order \( o \). We show that the term in (3.36) satisfies the LIL, whereas the two terms in (3.37) are negligible. We begin with the remainder terms in (3.37). The first term in (3.37) is of order \( o \), this is demonstrated in Proposition 3.6. Regarding the second term, we infer from Lemma 3.9 that it equals \( o \). Finally, consider (3.36). The LIL follows along the same lines as in the proof of Theorem 2.1; see also Remark 3.5. We omit the details. Hence,

\[
\limsup_{n \to \infty} \pm \frac{1}{\sqrt{2n \log \log n}} \sum_{y \in B_n} \mathbb{E} [\Delta(z, \infty) - \alpha \Delta'(z, \infty)|\mathcal{F}_z] = \mathbb{E} \mathbb{E} [\Delta(0, \infty) - \alpha \Delta'(0, \infty)|\mathcal{F}_0]^2]^{1/2} \quad \text{a.s.}
\]

We infer from Lemma 3.10 that the a.s.-limit on the right-hand side equals \( \tau^2 - \sigma^2 - \alpha^2 \) and is positive.

**Lemma 3.10.** Assume model (M1) or model (M2). Then \( \tau^2 = \mathbb{E} \mathbb{E} [\Delta(0, \infty) - \alpha \Delta'(0, \infty)|\mathcal{F}_0]^2] \geq 0 \).

**Proof.** If \( \mathcal{Q}_0(\mathcal{P}, \infty) \) is nondegenerate, \( \tau^2 \) is positive, see Penrose and Yukich (2001) (the proof applies to both models). It remains to verify the equality. We use the representation from (3.33) to (3.35) and set

\[
\begin{align*}
X_n &:= H(\mathcal{U}_{n,n}) - \mathbb{E} [H(\mathcal{U}_{n,n})] \\
Y_n &:= H(\mathcal{U}_{n,n}) - \mathbb{E} [H(\mathcal{U}_{n,n})] \\
Z_n &:= \alpha(N_n - n).
\end{align*}
\]

Then \( (n^{-1/2}Y_n)_n \) converges to \( N(0, \sigma^2) \) in distribution under the assumptions of model (M1) and (M2) by Proposition 3.2 and Proposition 3.6. Furthermore, \( (n^{-1/2}Z_n)_n \) converges in distribution to \( N(0, \alpha^2) \). Using the independence of \( X_n \) and \( Z_n \), we can infer with characteristic functions and the above approximation results that \( n^{-1/2}X_n \) converges to \( N(0, \sigma^2 - \alpha^2) \), for a sketch see the proof of Theorem 2.1 in Penrose and Yukich (2001). It remains to show the equality \( \tau^2 = \mathbb{E} \mathbb{E} [\Delta(0, \infty) - \alpha \Delta'(0, \infty)|\mathcal{F}_0]^2] \). We have

\[
\tau^2 \overset{\text{def}}{=} \sigma^2 - \alpha^2 = \lim_{n \to \infty} n^{-1} \text{Var} [H(\mathcal{U}_{n,n})] = \lim_{n \to \infty} n^{-1} \text{Var} [H(\mathcal{U}_{n,n})] + \alpha^2 - 2\alpha \lim_{n \to \infty} n^{-1} \text{Cov} (H(\mathcal{U}_{n,n}), N_n - n).
\]

Consequently, \( \lim_{n \to \infty} \text{Cov} (H(\mathcal{U}_{n,n}), N_n - n) = \alpha \). Moreover, the covariance can be expressed in terms of the above introduced martingale difference sequence (note that \( \mathbb{E} \mathbb{E} [\Delta'(z, n)|\mathcal{F}_z] = \mathcal{P}_n(Q_z) - \mathbb{E} [\mathcal{P}_n(Q_z)] \)).

\[
n^{-1} \text{Cov} (H(\mathcal{U}_{n,n}), N_n - n) = n^{-1} \mathbb{E} \left( \sum_{z \in B_n} \mathbb{E} [\Delta(z, n)|\mathcal{F}_z] \right) \left( \sum_{y \in B_n} \mathcal{P}_n(Q_y) - \mathbb{E} [\mathcal{P}_n(Q_y)] \right).
\]
as well as 
\[ \eta \] generated by 
\[ \tilde{\eta} \] (Proof of Theorem 2.4. We prove the invariance principle with a standard technique which relies on the Skorokhod embedding for martingales, (see, e.g., Hall and Heyde (1980)) . We begin with the Poisson process and introduce the definitions. 

In this section, we consider the special case for the 1-dimensional stretched domain \( D \times [-n/2, n/2] \). Given the technical details for the full domain (\( W_n \)), which stretches in each dimension, it is a routine to verify the LIL for the sequence \( (H(Q_n) - E[H(Q_n)])_n \). So, we skip the details of the proof of Theorem 2.3 here and focus on the SIP.

Proof of Theorem 2.4. We prove the invariance principle with a standard technique which relies on the Skorokhod embedding for martingales, (see, e.g., Hall and Heyde (1980)). We begin with the Poisson process and introduce the following definitions.

\[
\tilde{S}_{c,n} = \sum_{z=-[n/2]}^{[n/2]} n^{-1} \sum_{z \in B_n} E[\Delta(z, n)|F_z] = H(\mathcal{P}_n) - E[H(\mathcal{P}_n)], \\
\tilde{S}_n = \sum_{z=-[n/2]}^{[n/2]} \sum_{z \leq n} E[\Delta(z, \infty)|F_z], \\
S_n = \sum_{z=1}^{n} F_z.
\]

Then \( \tilde{S}_{c,n} = \tilde{S}_n + o_{a.s.}(n^{1/p+\varepsilon}) \) by Proposition 3.6 for any \( \varepsilon > 0 \) (in model (M1) a choice \( \varepsilon = 0 \) is also allowed), which is also valid in the case of the stretched domain with \( d = 1 \). Moreover, \( (\tilde{S}_n)_n \) equals \( (S_{n+1+1(n \text{ odd})})_n \) in law.

Using the Skorokhod martingale embedding, there is a standard Brownian motion \( B \) and non-negative random variables \( \tau_i, i \in \mathbb{N}_+ \), such that \( (S_n)_n \) equals \( (B(\sum_{i=1}^{n} \tau_i))_n \) a.s. and the \( \tau_i \) satisfy

\[ E[\tau_i | \mathcal{G}_{i-1}] = E[F_{i-1} | F_1, \ldots, F_{i-1}] \quad a.s. \]

as well as \( E[\tau_i^r | \mathcal{G}_{i-1}] \leq C_r E[F_{i-1}^r] \) for all \( r > 1 \) for a certain \( C_r \in \mathbb{R}_+ \) and where the \( \sigma \)-field \( \mathcal{G}_i \) is generated by \( (B(\sum_{k=1}^{j} \tau_k) : j \leq i) \) and by the \( B(t), t \in [0, \sum_{k=1}^{i} \tau_k] \). Also, \( \sum_{k=1}^{i} \tau_k \) is \( \mathcal{G}_i \)-measurable. We show

\[
B(\sum_{i=1}^{n} \tau_i) = B(n \sigma^2) + o_{a.s.} \left( n^{1/4} \log n \right)^{1/2} (\log \log n)^{1/4}. \quad (3.38)
\]
This shows then \( S_{\epsilon,n} \) equals \( B((n + 1 + \{ n \text{ odd} \}) \sigma^2) + O_{a.s.}(n^{1/4}(\log n)^{1/2}(\log \log n)^{1/4}) \) because \( \epsilon \) can be selected such that \( 1/p + \epsilon < 1/4 \). The desired approximation follows then from Lemma 3.7, as well as Proposition 3.8. So that we need in this step, the uniform bounded martingale LIL (Stout (1970)). This yields \( \sum_{n=1}^{\infty} \tau_i - n \sigma^2 \leq C n^{1/2}(\log \log n)^{1/2} =: t_n \ a.s. \), for a certain \( C \in \mathbb{R}_+ \).

Since \( t_n = o(n) \), we are in position to apply Lemma 3.4. More precisely,

\[
\left| B\left( \sum_{i=1}^{n} \tau_i \right) - B(n \sigma^2) \right| \leq \max_{k \leq n} \sup_{x: |x - k \sigma^2| \leq t_n} |B(x) - B(k \sigma^2)|
\leq C \sqrt{t_n \log(nt_n^{-1})} = O_{a.s.}(n^{1/4}(\log n)^{1/2}(\log \log n)^{1/4}).
\]

This shows (3.38).

Regarding the binomial process, we can use the decomposition from (3.33) to (3.35) and the results from Proposition 3.7 as well as Proposition 3.8. So that

\[
H(\tilde{U}_{n,n}) - \mathbb{E}[H(\tilde{U}_{n,n})] = H(\tilde{U}_{n,N_n}) - \mathbb{E}[H(\tilde{U}_{n,N_n})] - \alpha(N_n - n) + O_{a.s.}(n^{1/4+1/p}(\log n)^2) + \mathbb{E}[H(\tilde{U}_{n,N_n}) - H(\tilde{U}_{n,n})],
\]

where \( \mathbb{E}[H(\tilde{U}_{n,N_n}) - H(\tilde{U}_{n,n})] = O(n^{1/4}) \) and where the term \( H(\tilde{U}_{n,N_n}) - \mathbb{E}[H(\tilde{U}_{n,N_n})] - \alpha(N_n - n) \) equals \( \sum_{z \in B_n} \mathbb{E}[\Delta(z, \infty) - \alpha \Delta'(z, \infty)|F_z] \) plus an error of order \( o_{a.s.}((\log n)^2 n^{\varepsilon+1/p}) + o_{a.s.}((\log n)^3) \), where \( \varepsilon > 0 \) is arbitrarily small. This completes the proof in the binomial case.

3.5 Verification of the example

Proof of Example 1.2. For simplicity, we only consider the case for 2 dimensions, see also Penrose and Yukich (2001) for generalizations to higher dimensions or for the 1-dimensional case. We show that \( \mathbb{P}(\tilde{S} > r) \leq c_1 \exp(-c_2 r) \) for two constants \( c_1, c_2 \in \mathbb{R}_+ \) whenever \( r \in (0, n^{1/d}/2) \). It suffices to consider the case where \( x = 0 \). The (number of) points inside \( Q_0, V_1, \ldots, V_k \) is also not the main difficulty, as we will see. Depending on the parameters \( y, n, \) we need to distinguish two cases.

**Case 1.** \( B(0, r) \subset \mathcal{C}_{y,n} \). We follow the ideas of Penrose and Yukich (2001) and construct a neighborhood of 0 from 6 disjoint equilateral triangles \( T_j, j \in \{1, \ldots, 6\} \), with edge length \( r/4 \) as in the left panel of Figure 1. Then each \( T_j \) has Lebesgue measure \( \sqrt{3} r^2/16 = \lambda(r) \).

\[
P(T_j \text{ contains at most } k \text{ points from } \mathcal{P} \text{ for one } j \in \{1, \ldots, 6\}) \leq 6P(T_1 \text{ contains at most } k \text{ points from } \mathcal{P}) \leq 6(k + 1)(1 + \lambda(r)) \exp(-\lambda(r)) \leq c_1 \exp(-c_2 r),
\]

for certain constants \( c_1, c_2 \in \mathbb{R}_+ \). One finds that given each \( T_j \) contains at least \( k + 1 \) points, the radius of stabilization is at most \( 4(r/4) = r \). We omit the details here because they will be explained in detail in the next case.

**Case 2.** \( B(0, r) \not\subset \mathcal{C}_{y,n} \). In this case, \( B(0, r) \) either intersects with one or with two edges because \( r < 2^{1/4} n^{1/d}, \)
see the middle and the right panel of Figure 1. We use again the 6 disjoint equilateral triangles with edge length \( r/4 \). This time, we divide each equilateral triangle \( T_j \) in two disjoint isosceles triangles. Then the Lebesgue measure of each isosceles triangle is \( \lambda(r)/2 \).

We begin with the situation in the middle panel and consider the 6 isosceles triangles with edge length \( r/4 \) in the lower half (given in brown in the figure), \( T'_1, \ldots, T'_6 \), say. Note that in general the blue and green triangles in the upper half can intersect with \( \mathbb{R}^d \setminus \mathcal{E}_{y,n} \) depending on the distance 0 to the boundary of \( \mathcal{E}_{y,n} \).

Define the event

\[
A = \{ T'_i \setminus Q_0 \text{ contains at least } k + 1 \text{ points for } i \in \{1, \ldots, 6 \} \}
\]

Then, \( \mathbb{P}(A^c) \leq c_1 \exp(-c_2 r) \) for certain constants \( c_1, c_2 \in \mathbb{R}_+ \). We show that given the event \( A \), the radius of stabilization is at most \( 9r \).

Plainly, given \( A \), 0 has \((k + 1)\) points within distance \( r/4 \) from \( \mathbb{P}|\mathcal{E}_{y,n} \) because all 6 isosceles triangles are entirely contained within \( \mathcal{E}_{y,n} \) and each of these triangles contains at least \((k + 1)\) points.

Conversely, given \( A \), let 0 be a kNN of a point \( z \in (\mathbb{P}|\mathcal{E}_{y,n} \setminus Q_0) \cup \{V_1, \ldots, V_k\} \). If the shortest path of \( z \) to 0 intersects with one of the two equilateral triangles at the top (in green), then \( z \) lies in the induced convex cone within a distance of \( 2r \) (otherwise \( z \) lies outside of \( \mathcal{E}_{y,n} \), one can show this with elementary calculations).

Otherwise (if the path does not intersect with the green triangles), the path must pass through one of the six lower isosceles triangles (in brown) or through one of the two neighboring isosceles triangles (in blue). By construction each of the six lower isosceles triangles contains \((k + 1)\) points. Write \( C \) for the two blue equilateral triangles.

Then it holds: If \( z \) is not contained in \( \cup_i T'_i \cup C \) and if the shortest path of \( z \) does not pass through the two top green triangles, then 0 is no kNN of \( z \). Indeed, if the path of \( z \) passes through \( C \) but \( z \notin C \), then \( z \) lies closer to \((k + 1)\) points in a brown adjacent triangle than to 0. If the path of \( z \) passes through one of the \( T'_i \) but \( z \notin T'_i \), then the \((k + 1)\) points from this \( T'_i \) lie closer to \( z \) than 0.

Hence, if 0 is a kNN of \( z \), then either \( z \in \cup_i T'_i \cup C \) or the shortest path of \( z \) to 0 passes through the two top triangles and \( z \) lies in \( B(0, 2r) \). So in both cases \( z \in B(0, 2r) \) and has \((k + 1)\) points in distance \( 2r + r/4 \).

Next, let \{\( z, y \)\} be an edge which is removed from the graph due to the additional point 0, w.l.o.g., let 0 be a kNN of \( z \), so that \( y \) is a \((k + 1)\)-nearest neighbor of \( z \) when adding 0. Then \( y \in B(0, 2(2r + r/4)). \) If additionally \( z \) is not a kNN of \( y \), then the edge \{\( z, y \)\} is removed. Otherwise, \( z \) is a kNN of \( y \) and \( y \) has \( k \) points within distance \( 2(2r + r/4) \). Consequently, the configuration of the points in \( \mathcal{E}_{n,y} \) but outside of \( B(0, 4(2r + r/4)) \) is irrelevant for the choice of whether an edge \{\( z, y \)\} is removed from the graph. This completes the case for the situation in the middle panel of Figure 1 and the radius of stabilization is at most \( 9r \).
The arguments are very similar for the right panel. Consider the three brown isosceles triangles with edge length $r/4$, $T''_i$, say. Once more, set $A = \{T''_i \setminus Q_0 \text{ contains at least } k+1 \text{ points for } i \in \{1, \ldots, 6\}\}$. Then $P(A)$ is at least $1 - c_1 \exp(-c_2 r)$, so that we can restrict our considerations conditional on the event $A$. Clearly, $0$ contains $(k+1)$ points within distance $r/4$. Moreover, arguing as before if $0$ is a kNN of a $z$, then $z$ lies within a distance of $mr$, for some multiple $m \in \mathbb{R}_+$ and there are $(k+1)$ points within distance of $mr + r/4$.

Furthermore, let $\{z, y\}$ be an edge which is removed, and let $0$ be a kNN of $z$. Then $y \in B(0, 2(mr + r/4))$. If additionally $z$ is not a kNN of $y$, then $\{z, y\}$ is removed; otherwise $y$ has $k$ points within distance $2(mr + r/4)$. So, the configuration of the points in $C_{y,n}$ but not in $B(0, 2(mr + r/4))$ is irrelevant for the choice of whether an edge $\{z, y\}$ is removed from the graph. Consequently, in the situation in the right panel of Figure 1 the radius of stabilization is at most $(m+1)r$.

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### A Appendix

#### A.1 General results

**Lemma A.1.** Let $m \in \mathbb{N}$. There is a constant $C \in \mathbb{R}_+$, which does not depend on $m$, such that

$$\binom{m}{k} = \frac{m!}{k!(m-k)!} \leq C \frac{2^m}{m^{1/2}}, \quad \forall k \in \{0, \ldots, m\}.$$  

In particular, if $\lambda \in \mathbb{R}_+$ and $X \sim \text{Poi}(\lambda)$, then $E\left[|\sum_{k=0}^{X} \binom{X}{k}|^q\right] < \infty$ for all $q \in \mathbb{R}_+$.

Moreover, the uniform bounded moments condition (1.4) is also satisfied for model (M1) for any positive exponent.

**Proof.** The result relies on Stirling’s formula

$$\sqrt{2\pi n^{n+1/2}e^{-n}} \leq n! < e n^{n+1/2} e^{-n} \text{ for } n \in \mathbb{N}_+.$$  

It is well known that the binomial coefficient is maximal at $m/2$ if $m$ is even and at $(m+1)/2$ if $m$ is odd. Thus, if $m$ is even,

$$\frac{m!}{k!(m-k)!} \leq \frac{m!}{((m/2)!)^2} \leq \frac{m!}{2\pi m^{m+1} e^{-m}} \leq \frac{e 2^m}{\pi m^{1/2}}.$$  

A similar result is valid if $m$ is odd,

$$\frac{m!}{k!(m-k)!} \leq \frac{m!}{((m+1)/2)!((m-1)/2)!} \leq \frac{e^{3/2} 2^m}{\pi m^{1/2} + o(1)}.$$  

The claim regarding the moment of the Poisson random variable follows immediately because $E[e^{\delta X}] = \exp(\lambda(e^\delta - 1))$ is finite for all $\delta < \infty$. 

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Consider the uniform bounded moments condition from \((\ref{uniform-bounded-moments})\). Using the hard-thresholded stabilization, it is enough to compute \(\mathbb{E}[H(|A_m|)]\) for an \(A \in \mathbb{B}\) and \(m \in \{1/2, 3/2\}\) as well as \(p^2 \in \mathbb{R}_+\). This amounts to compute \(\mathbb{E}[2^{B_N}]\), where \(\delta > 0\) and \(N\) follows a binomial distribution of length \(m\) and a success probability proportional to \(|A|^{-1}\). Hence,

\[
E[2^{B_N}] = \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \left( \frac{e}{|A|} \right)^k \left( 1 - \frac{e}{|A|} \right)^{n-k} 2^\delta k \leq \exp(e2^\delta),
\]

where we use that \(|A|^{-1} < 3/2 m^{-1}\) and the right-hand side only depends on \(\delta\). This completes the proof. \(\square\)

**Lemma A.2** (Uniform bounded moments). Let \(\rho \in (0, 1/2)\).

(a) There is a constant \(C\), which only depends on the dimension \(d\), such that for all \(y \in \mathbb{R}^d\) and \(r \in \mathbb{R}_+\)

\[
\sup_{m \in \mathbb{N}} \mathbb{P}(\mathbb{N}_{m,n} \cap B(y, r) \neq \mathbb{N} \cap B(y, r) | \mathbb{N}_{m,n} \cap B(y, r)) \leq C \left( \frac{d}{n} \right)^{1/2-\rho} (A.1)
\]

with probability one.

(b) Assume model \((M1)\). Let \(p \in \mathbb{N}\), then

\[
\sup_{n \in \mathbb{N}} \sup_{y \in \mathbb{B}_n} \sup_{m \in \mathbb{N}} \mathbb{E} \left[ H((\mathbb{N}_{m,n} \cap B(y, r)) \cup \{y\}) - H(\mathbb{N}_{m,n} \cap B(y, r)) \right] < \infty,
\]

\[
\mathbb{E} \left[ H((\mathbb{N} \cap B(0, r)) \cup \{0\}) - H(\mathbb{N} \cap B(0, r)) \right] < \infty, (A.2)
\]

(c) Assume model \((M2)\). Let \(p \in \mathbb{N}\) be as in Condition \((\ref{uniform-bounded-moments})\) and let \(S_0\) be given by \((\ref{uniform-bounded-moments})\). Then

\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ H(\mathbb{N}) - H(\mathbb{N}_{n,z}) \right] < \infty,
\]

\[
\mathbb{E} \left[ H((\mathbb{N} \cap B(z, r)) - H(\mathbb{N}_{n,z} \cap B(z, r)) \right] < \infty,
\]

\[
\mathbb{E} \left[ H((\mathbb{N} \cap B(z, r)) - H(\mathbb{N}_{n,z} \cap B(z, r)) \right] < \infty, (A.3)
\]

\[
\sup_{n \in \mathbb{N}, z \in \mathbb{B}_n, r \in \mathbb{R}_+} \mathbb{E} \left[ H((\mathbb{N} - z) \cap B(0, S_0)) - H(\mathbb{N}_{n,z} - z) \cap B(0, S_0)) \right] < \infty.
\]

**Proof.** We begin with (a). Let \(N_n\) be the number of points of \(\mathbb{N}\) in \(B_n\). Then on \(B(y, r)\) the point processes can differ in at most \(|N_n - m|\) i.i.d. points, each of these points falls in \(B(y, r)\) with a probability proportional to \(r^d n^{-1}\). There is a constant \(C\), which only depends on \(d\), such that the left-hand side of \((A.1)\) is less than

\[
\frac{Cr^d}{n} \mathbb{E} \left[ |N_n - m| \right] \leq \frac{Cr^d}{n} \left( \mathbb{E} \left[ |N_n - m| \right] + |n - m| \right) \leq Cr^d \left( n^{-1/2} + n^{-1/2+\rho} \right). (A.4)
\]

This shows \((A.1)\).

We continue with (b) and the inequalities in \((A.2)\). Let \(m, n\) be arbitrary but fixed, \(m \in [n - n^{1/2+\rho}, n + n^{1/2+\rho}]\). Let \(y \in B_n\). Let \(M\) be the number of points of \(\mathbb{N}_{m,n}\) in \(Q_y^{(\delta)}\). By assumption,

\[
|H((\mathbb{N}_{m,n} \cap B(y, S^*))| \cup |H((\mathbb{N}_{m,n} \cap B(y, S^*))| \cup \{y\}) | \leq \sum_{k=0}^{M} \binom{M}{k} q^k,
\]
for some $q \in \mathbb{N}$.

By construction, $M$ is stochastically dominated by $1$ plus a binomial random variable of length $n + n^{1/2 + \rho}$ and a success probability proportional to $n^{-1}$. The claim follows now from a Poissonization argument and Lemma A.1. The second and the third inequality in (A.3) with Condition (1.4); the proof works similar as in Penrose and Yukich (2001) Lemma 3.1. The fourth statement follows from the polynomial growth of $H((P - z) \cap B(0, n))$ along with the exponential decay of $\mathbb{P}(S_0 \in (n - 1, n]) \leq \mathbb{P}(S_0 > n - 1)$.

**Lemma A.3.** Let $(X_n)_n$ be a sequence of random variables satisfying $(\log n)^{-\alpha} \| n^{-\beta} \max_{k \leq n} |X_k|\|_q \leq C(\log n)^{-(1+\delta)}$ for constants $C, \alpha, \beta \in \mathbb{R}_+$ (which are independent of $n$). Then $(\log n)^{-\alpha} n^{-\beta} \max_{k \leq n} |X_k| \leq \tilde{C}$ a.s. for all $n \geq 4$ for some $\tilde{C} \in \mathbb{R}_+$ (independent of $n$). In particular, $\limsup_{n \to \infty} (\log n)^{-\alpha} n^{-\beta} |X_n| \leq \tilde{C}$ a.s.

**Proof.** Clearly, $\sum_{j \in \mathbb{N}} (\log(2^j))^{-\alpha} 2^{-j\beta} \max_{k \leq 2^j} |X_k| \|_q < \infty$. Hence, $\max_{k \leq 2^j} |X_k| = \mathcal{O}_{a.c.}(\log(2^j))^{\alpha} (2^j)^{\beta}$. In particular, there is a constant $C^* \in \mathbb{R}_+$ such that $\max_{k \leq 2^j} |X_k| \leq C^* (\log(2^j))^{\alpha} 2^{j\beta}$ a.s. for all $j \in \mathbb{N}$. Let $n \in \mathbb{N}$, $n \geq 4$, and write $J$ for the largest integer such that $2^{J-1} < n < 2^J$. Then $J \geq 2$ and

$$(\log n)^{-\alpha} n^{-\beta} \max_{k \leq n} |X_k| \leq \frac{1}{(\log n)^{\alpha}} \frac{\rho}{2^{\beta}} \max_{k \leq 2^J} |X_k| \leq \frac{\tilde{J}}{1 - 1} 2^{\beta} C^* \leq 2^{\alpha + \beta} C^*.$$ 

Consequently, $\limsup_{n \to \infty} (\log n)^{-\alpha} n^{-\beta} |X_n| \leq \limsup_{n \to \infty} (\log n)^{-\alpha} n^{-\beta} \max_{k \leq n} |X_k| \leq 2^{\alpha + \beta} C^* < \infty$. $

**A.2 Deferred results for the law of the iterated logarithm**

**Proof of Lemma 3.1.** First, we show the amendment. Define $r = d(I, J)/2$. Put

$$\tilde{F}_z = \mathbb{E} \left[ H(P \cap B(z, r)) - H(P^\prime \cap B(z, r)) \bigg| F_z \right]$$

and $X' = G(\sum_{z \in I} h(\tilde{F}_z))$ and $Y' = \tilde{G}(\sum_{z \in J} h(\tilde{F}_z))$. Then

$$\text{Cov}(X, Y) = \text{Cov}(X - X', Y - Y') + \text{Cov}(X - X', Y') + \text{Cov}(X', Y - Y')$$

because $\text{Cov}(X', Y') = 0$. Moreover,

$$|\text{Cov}(X - X', Y - Y')| \leq \mathbb{E} \left[ |X - X'||Y - Y'| \mathbb{1} \left\{ F_z \neq \tilde{F}_z \text{ for one } z \in I \cup J \right\} \right] + \mathbb{E} \left[ |X - X'| \mathbb{1} \left\{ F_z \neq \tilde{F}_z \text{ for one } z \in I \right\} \right] \mathbb{E} \left[ |Y - Y'| \mathbb{1} \left\{ F_z \neq \tilde{F}_z \text{ for one } z \in J \right\} \right]$$

and similarly for $\text{Cov}(X - X', Y')$ and $\text{Cov}(X', Y - Y')$. In both models, there are constants such that $\mathbb{P}(F_z \neq \tilde{F}_z) \leq c_1 \exp(-c_2 r)$. This shows the amendment because both $G$ and $\tilde{G}$ are bounded functions.

Consider now the special case where $G = \text{id}$ and the index sets are singletons (the fact that $G$ is not bounded is irrelevant). If $h$ is the identity, then $\tilde{\sigma}^2$ is simply $\mathbb{E}[F_0^2]$. In the other case, use the definition in (A.5) with $r = d(z, 0)/2$.

$$\sum_{z \in \mathbb{Z}^d} |\text{Cov}(h(F_0), h(F_z))| \leq \mathbb{E}[h(F_0)^2] + \sum_{z \neq 0 \in \mathbb{Z}^d} \mathbb{E} \left[ h(F_0)h(F_z) \mathbb{1} \left\{ F_0 \neq \tilde{F}_0, F_z \neq \tilde{F}_z \right\} \right]$$

$$\leq \mathbb{E}[h(F_0)^2] + \sum_{z \neq 0 \in \mathbb{Z}^d} \mathbb{E} \left[ h(F_0)^3 \right]^{2/3} c_1^{1/3} \exp(-c_2 / 3 d(z, 0)/2) < \infty.$$ 

Thus, the definition of $\tilde{\sigma}^2$ is meaningful in both cases. This completes the proof. $

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Proof of Proposition 3.2. The proof is based on approximation techniques between the characteristic functions and a suitable partitioning of the index sets into large and small blocks (which is a quite common technique when working with random fields).

Let $\varepsilon \in (0, 1/2)$ and $n \geq 2^{1/\varepsilon}$. Set $p(p(n)) := \lfloor n^{1/2} \rfloor$, $q(n) := \lfloor n^{1/2-\varepsilon} \rfloor$ and $k(n) := \lceil (2n+1)/(2p+1+q) \rceil$. The set $[-n, n] \cap \mathbb{N}$ contains $k$ sets $I_1, \ldots, I_k$ of cardinality $2p+1$ with distance $q$ between them. For each $i \in \{1, \ldots, k\}^d$, define a cube $J_i = I_{i_1} \times \ldots \times I_{i_d}$ with cardinality $\#J_i = (2p+1)^d$. Then one sees

$$1 \geq \frac{\#(\bigcup J_i)}{\#D_n} = \frac{k^d (2p+1)^d}{(2n+1)^d} \geq \left(1 - \frac{4n^{1/2}}{2n+1}\right)^d \left(1 - \frac{n^{1/2-\varepsilon}}{n^{1/2}}\right)^d \geq 1 - 2d(2n^{-\varepsilon}). \quad (A.6)$$

Moreover, let

$$\xi_i := \sum_{z \in J_i} h(F_z), \quad i \in \{1, \ldots, k\}^d,$$

and let the family $\{\xi_i : i \in \{1, \ldots, k\}^d\}$ be i.i.d. with distribution $\xi_1$. (Note that the $\xi_i$ also depend on $n$.) We introduce the following random variables

$$Z_n := \frac{\sum_{z \in D_n} h(F_z)}{\sigma(\#D_n)^{1/2}}, \quad Z_n' := \sum_{i} \xi_i / \sigma(\#D_n)^{1/2},$$

$$\tilde{Z}_n := \frac{\sum_{i} \xi'_i}{\sigma(\#D_n)^{1/2}}, \quad Z_n^* := \frac{\sum_{i} \xi'_i}{k^d \text{Var}(\xi_1)}.$$

In particular, the variances of $Z_n, Z'_n, \tilde{Z}_n, Z_n^*$ are uniformly bounded in $n$. First, consider

$$|E[\exp(itZ_n)] - E[\exp(itZ'_n)]| \leq C |E[Z_n - Z'_n]|^{1/2}$$

$$= \frac{\text{Var}(\sum_{z \in D_n \setminus J_i} h(F_z))^{1/2}}{\sigma(\#D_n)^{1/2}} \leq Cn^{-\varepsilon/2} \quad (A.7)$$

because $\sum_{y} \text{Cov}(h(F_z), h(F_y)) \leq C \in \mathbb{R}_+$ for all $z \in \mathbb{Z}^d$ as in the proof of Lemma 3.1. Second, one finds

$$|E[\exp(it\tilde{Z}_n)] - E[\exp(itZ_n^*)]| \leq C \left( \sum_i \xi'_i \right)^2 \left( \frac{\sigma(\#D_n)^{1/2}}{\sigma(\#D_n)^{1/2} (k^d \text{Var}(\xi_1))^{1/2}} \right)^2^{1/2}$$

$$= C |1 - \frac{k^d \text{Var}(\xi_1)^{1/2}}{\sigma^2(\#D_n)}|$$

$$\leq C |1 - \frac{k^d (2p+1)^d}{(2n+1)^d} + C |1 - \frac{\text{Var}(\xi_1)}{\sigma^2(2p+1)^d}| \right) \quad (A.8)$$

where the last inequality follows from (A.6) and the observation: Let $J = [-m, m]^d$, then

$$|\sigma^2 \#J - \text{Var}(\sum_{z \in J} h(F_z))| = \left| \sum_{z \in J} \sum_{y \not\in J} \mathbb{E}[h(F_y)h(F_z)] \right|$$

$$\leq \sum_{r=0}^{m} \sum_{z \in J, y \not\in J, ||y-z||=r} \mathbb{E}[h(F_y)h(F_z)] + \sum_{r=m+1}^{\infty} \sum_{z \in J, y \not\in J, ||y-z||=r} \mathbb{E}[h(F_y)h(F_z)] \bigg|.$$

If $z \in J$ and $y \not\in J$, then $m-r < ||y|| \leq m$. The number of such $z$ is bounded above by $(2m+1)^d - (2(m-r)+1)^d$ which is at most $2dr(2m+1)^{d-1}$. So, for a given $r$ the sum in the first term is $0|2dr(2m+1)^{d-1}(2r+1)^d \exp(-cr)|$. Moreover, for a given $r$, the inner sum in the second term is of order $(2m+1)^d(2r+1)^d \exp(-cr)$. This shows
\[ |\bar{\sigma}^2 - \text{Var}(\sum_{z \in \mathcal{J}})/\#J| \leq \#J^{1/d}. \] Using the stationarity, we can apply this estimate to the second term in (A.8). We can apply Lemma 3.1 to see

\[
\mathbb{E} \left[ \exp \left( i t \sum \xi_i \right) - \exp \left( i t \sum \xi_i^* \right) \right] \leq \text{Cov} \left( \exp \left( i t \sum_{i: i \neq j} \xi_i \right), \exp \left( i t \xi_j \right) \right) + \mathbb{E} \left[ \exp \left( i t \sum_{i: i \neq j} \xi_i \right) - \exp \left( i t \sum \xi_i^* \right) \right] \mathbb{E} \left[ \exp \left( i t \xi_j \right) \right]
\]

(A.9)

Finally, by the Esseen’s lemma (Petrov (2012), p. 109, Lemma 1),

\[
\left| e^{-t^2/2} - \mathbb{E}[\exp \left( i t \sum \xi_i/(kd\mathbb{E} \left[ \xi_i^2 \right])^{1/2} \right)] \right| \leq 16 \frac{\mathbb{E}[\left| \xi_i \right|^3]}{\mathbb{E}[\xi_i^2]^{3/2}} k^{-d/2} |t|^{3} e^{-t^2/2} \leq C k^{-d/2} |t|^{3} e^{-t^2/2},
\]

(A.10)

whenever, \(|t| \leq k^{d/2}/4 \cdot \mathbb{E}[\xi_i^2]^{3/2} / \mathbb{E}[|\xi_i|^3] \). The last inequality in (A.10) follows because \(\mathbb{E}[|\sum_{z \in \mathcal{J}} h(F_z)|^4] \leq C(\#J)^2 \) for a constant \(C\) which does not depend on \(J\); this can be seen with similar stabilization techniques as in the proof of Lemma 3.1 we skip the details.

Using the estimates from (A.7) to (A.10) in combination with Esseen’s theorem (Esseen (1945)), yields two constants \(K_1, K_2\) independent of \(T\) such that

\[
\left| \mathbb{P}(Z_n \leq z) - \Phi(z) \right| \leq K_1 \int_{-T}^T \frac{tn^{-\varepsilon/2}}{t} dt + K_1 \int_{-T}^T k^d |D_n| e^{-c_2n^{1/2-\varepsilon}} \left/ t \right. \quad \text{and} \quad 1 \ dt
\]

\[
+ K_1 \int_{-T}^T \frac{tn^{-\varepsilon}}{t} dt + K_1 \int_{-T}^T k^d |t|^2 e^{-t^2/2} dt + \frac{K_2}{T} \leq Cn^{-\varepsilon/4},
\]

for a choice of \(T\) proportionally to \(n^{\varepsilon/4}\). This completes the proof. \(\square\)

**Proof of Proposition 3.3** Set \(\varphi(n) := \sqrt{2\bar{\sigma}^2(2n^2 + 1)d \log \log n}, S_n = \sum_{z: \|z\| \leq n} h(F_z)\) and \(r := \lfloor n^{1/6} \rfloor, k := \lfloor n/k \rfloor\). Also set

\[ E_j := \{ |S_i| < \beta \varphi(n), \forall i < j \} \cap \{ |S_j| \geq \beta \varphi(n) \}. \]

Then we obtain with the abbreviation \(\xi_i = \sum_{z: \|z\| = i} h(F_z)\)

\[
P(\max_{1 \leq j \leq n} |S_j| \geq \beta \varphi(n)) \leq \sum_{i=0}^{k-2} P \left( \bigcup_{j=1}^{r} \left( E_{ir+j} \cap |S_n - S_{ir+j}| \geq \varepsilon \varphi(n) \right) \right) + P \left( \bigcup_{\ell=(k-1)r+1}^{n} \left( E_{\ell} \cap |S_n - S_{\ell}| \geq \varepsilon \varphi(n) \right) \right) + P(|S_n| \geq \beta(1-\varepsilon) \varphi(n))
\]

\[
\leq \sum_{i=0}^{k-2} \sum_{j=1}^{r} P \left( \left| E_{ir+j} \right| \geq \varepsilon/2 \varphi(n) \right) + \sum_{i=0}^{k-2} \sum_{j=1}^{r} P \left( \left| E_{ir+j} \right| \geq \varepsilon/2 \varphi(n) \right) \quad \text{(A.11)}
\]

\[
+ \sum_{i=0}^{k-2} P \left( \left| \xi_{ir+1} \right| + \ldots + \left| \xi_{ir+2r} \right| \geq \varepsilon/2 \varphi(n) \right) \quad \text{(A.12)}
\]
Collecting the estimates for the terms in (A.11) to (A.13) yields for \( n \) sufficiently large
\[
\begin{align*}
&\quad + \mathbb{P}\left( |\xi_{(k-1)r+1}| + \ldots + |\xi_n| \geq \varepsilon \varphi(n) \right) \\
&\quad + \mathbb{P}(|S_n| \geq \beta(1 - \varepsilon)\varphi(n)).
\end{align*}
\] (A.13)

Consider the term in (A.11). Each single summand is
\[
\mathbb{P}\left( \bigcup_{j=1}^{r} \left\{ |S_{ir+j}| \geq \beta \varphi(n), |S_u| < \beta \varphi(n), \quad \forall u < ir + j \right\} \right) \\
\leq \mathbb{P}\left( \bigcup_{j=1}^{r} \left\{ |S_{ir+j}| \geq \beta \varphi(n), |S_u| < \beta \varphi(n), \quad \forall u < ir + j \right\} \right)
\cdot \mathbb{P}(|S_n - S_{(i+2)r}| \geq \varphi(n) \varepsilon / 2) + (2(2n + 1)^d)c_1e^{-c_2r},
\]
(A.14)
where we have used for the last inequality that the \( \alpha \)-mixing coefficient decays exponentially. Indeed, let \( I, J \subset \mathbb{Z}^d \) be two disjoint sets with \( r = d(I, J)/2 \). Then, using the stabilizing property of the functional \( H \), we find with some elementary calculations
\[
\mathbb{P}((F_2 : z \in I) \in A, (F_2 : z \in J) \in B) \leq \mathbb{P}((F_2 : z \in I) \in A)\mathbb{P}((F_2 : z \in J) \in B) + 4(|#I + #J)c_1e^{-c_2r},
\]
where the constants \( c_1, c_2 \in \mathbb{R}_+ \) depend on \( H \). This shows the \( \alpha \)-mixing property.

Moreover, the second factor in the first term in (A.14) is ultimately smaller than 1/2. Indeed,
\[
\mathbb{P}(|S_n - S_{(i+2)r}| \geq \varphi(n) \varepsilon / 2) \leq \left( \frac{\varepsilon}{2} \varphi(n) \right)^{-2} \mathbb{E} \left[ |S_n - S_{(i+2)r}|^2 \right] \\
= \left( \frac{\varepsilon}{2} \varphi(n) \right)^{-2} \sum_{y,z: \|y\| \leq d, \|z\| \leq n} \mathbb{E} \left[ h(F_y)h(F_z) \right] \\
\leq C \log \log((2n + 1)^d).
\]
We continue with (A.12). An application of the Markov inequality yields that this term is of order
\[
\sum_{i=0}^{k-2} (2r + 1)^{4} \varepsilon^{-4}(2n + 1)^{-2}(\log \log((2n + 1)^d))^{-2} \\
= (k - 1)(2r + 1)^{4} \varepsilon^{-4}(2n + 1)^{-2}(\log \log((2n + 1)^d))^{-2} \leq C \frac{r^4(k - 1)}{\varepsilon^4n^2(\log \log n)^2}.
\]
The same reasoning applies to (A.13), which is of order
\[
\frac{|n - (k - 1)r|^4}{\varepsilon^4n^2(\log \log n)^2} \leq C \frac{r^4}{\varepsilon^4n^2(\log \log n)^2}.
\]
Collecting the estimates for the terms in (A.11) to (A.13) yields for \( n \) sufficiently large
\[
\begin{align*}
&\quad \mathbb{P}(\max_{1 \leq j \leq n} |S_j| \geq \beta \varphi(n)) \leq C(2n + 1)^d c_1 e^{-c_2r} + C \frac{r^4k}{\varepsilon^4n^2(\log \log n)^2} \\
&\quad + \frac{1}{2} \mathbb{P}(\max_{1 \leq j \leq n} |S_j| \geq \beta \varphi(n)) + \mathbb{P}(|S_n| \geq \beta(1 - \varepsilon)\varphi(n)).
\end{align*}
\] (A.15)

Rearranging the terms in (A.15) and using the normal-approximation estimate from Proposition 3.2, we obtain
\[
\mathbb{P}(\max_{1 \leq j \leq n} |S_j| \geq \beta \varphi(n)) \leq C \frac{r^3}{\varepsilon^3n} + 2\mathbb{P}(|S_n| \geq \beta(1 - \varepsilon)\varphi(n)) \leq C(\log n)^{-(1 + \rho)}
\]
for some $\rho \in \mathbb{R}_+$. This completes the proof.

### A.3 Results for the invariance principle

**Lemma A.4.** Let $B$ be a standard Brownian motion. Let $(t_n)_n$ be of order $o(n)$. Then

$$\max_{k:k \leq n} \sup_{x:|x-k\sigma^2| \leq t_n} |B(x) - B(k\sigma^2)| \leq C \sqrt{t_n \log(nt_n^{-1})} \text{ a.s.}$$

**Proof.** We use the fact that $t \mapsto a^{-1/2}B(at)$ is a time-changed Brownian motion for $a > 0$ and make the definition $W(x) = B(x(n\sigma^2))(n\sigma^2)^{-1/2}$, $x \geq 0$. Then the left-hand side equals

$$\max_{k:k \leq n} \sup_{x:|x(\sigma^2)^{-1}-kn^{-1}| \leq t_n(\sigma^2)^{-1}} |W(x(n\sigma^2)^{-1}) - W(kn^{-1})(\sigma^2)^{1/2}|.$$

Plainly, the right-hand side is at most

$$\sup_{x,y:|x-y| \leq t_n(n\sigma^2)^{-1}} |W(x) - W(y)|(n\sigma^2)^{1/2}. \quad \text{(A.16)}$$

Lévy’s modulus of continuity theorem (in the global version) yields

$$\limsup_{\epsilon \downarrow 0} \sup_{x,y \in [0,1]} \frac{W(x) - W(y)}{\sqrt{2 \epsilon \log \epsilon^{-1}}} = 1.$$

Consequently, (A.16) is bounded above by

$$C \sqrt{2 \frac{t_n}{n\sigma^2} \log \left( \frac{n\sigma^2}{t_n} \right)} (n\sigma^2)^{1/2} \leq C \sqrt{t_n \log \left( \frac{n}{t_n} \right)}.$$

$\square$