A Reexamination of the Canonical Structure of the Einstein-Hilbert Action in First-Order Form

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Abstract

A canonical analysis of the Einstein-Hilbert action $S_d = \int d^d x \sqrt{-g} R$ ($d > 2$) is considered, using the first order form with the metric and affine connection as independent fields. We adopt a conservative approach to using the Dirac constraint formalism; we do not use equations of motion which are independent of time derivatives and correspond to first class constraints to eliminate fields. Applying the Dirac procedure, we find that the primary constraints lead to secondary constraints which are equations of motion not involving time derivatives, and that those secondary constraints which are first class imply novel tertiary constraints which are also first class. Once the constraints and their associated gauge conditions are used to eliminate the non-dynamical degrees of freedom in $S_d$, there are $d(d - 3)$ degrees of freedom left in phase space. We also consider the simpler limiting case of the non-interacting graviton in the first order formalism as well as the effect of adding the action for a massless scalar field to the Einstein-Hilbert action.
I. INTRODUCTION

Any analysis of the canonical structure of $d$-dimensional Einstein-Hilbert action

$$S_d = \int d^d x \sqrt{-g} R$$  \hspace{1cm} (1)

is greatly complicated by symmetries which appear because of the presence of first class constraints. Disentangling the physical degrees of freedom from those that serve only to maintain manifest invariance under symmetry transformations is a principal goal of any examination of the canonical structure of $S_d$. Having a clear understanding of this structure would be crucial in any quantization procedure for the gravitational field.

Einstein's first formulation of general relativity (GR) was solely in terms of the metric $g_{\mu\nu}(x)$, but he later \[1\] showed that if $d > 2$, then $S_d$ can be considered with the metric and the affine connection $\Gamma^\lambda_{\mu\nu}$ being taken as independent. Such a “first order” (in derivatives) form of $S_d$ yields the same equations of motion as the original “second order” form in which $S_d$ depends solely on the metric with the affine connection being identified with the Christoffel symbol $\{^\lambda_{\mu\nu}\}$. (Palatini is often credited with this result \[2\].) This is because the equation of motion for $\Gamma^\lambda_{\mu\nu}$ when $S_d$ is written in first order form is $\Gamma^\lambda_{\mu\nu} = \{^\lambda_{\mu\nu}\}$ when $d > 2$; if $d = 2$ then $\Gamma^\lambda_{\mu\nu}$ is not uniquely determined by $g_{\mu\nu}$ \[3\].

Geometrical variables other than $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ are often used to characterize $S_d$. A second-order form can employ the vierbein $e^a_\mu$ while a first order form could use the vierbein and spin connection $\omega^\mu_{ab}$. Indeed, if spinors occur in curved space, these geometric quantities must be used \[4\]. It is not even apparent that the formulation of $S_d$ in terms of $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ is fully equivalent to that in terms of $e^a_\mu$ and $\omega^\mu_{ab}$ \[5\].

The various choices of geometrical quantities to characterize $S_d$ have all been used when analyzing its canonical structure. The first order form of $S_d$ in which both $e^a_\mu$ and $\omega^\mu_{ab}$ appear as basic fields has been treated \[6\] using the constraint formalism of Dirac \[7, 8, 9, 10, 11, 12\]. If the one basic quantity is the spin connection, then the program of “loop quantum gravity” can be developed \[13, 14, 15\].

Early treatments of the canonical structure of $S_d$ involve taking the metric or the metric and affine connection to be the fundamental fields \[16, 17, 18\]. In his analysis of the action in second order form when $d = 4$ \[17, 18\], Dirac considers the metric to be fundamental and discards those portions of $\sqrt{-g} R$ that are the divergence of a vector, keeping only the
“\(g \Gamma\)” part, thereby breaking covariance of the Lagrangian. Also, he characterizes each space-like surface in the theory by a distinct value of the time parameter \(t\). We adopt the same assumption here, and do not discuss the question of whether in Einstein’s theory selecting such a time coordinate is feasible.

The canonical structure of \(S_4\) in first order form was first discussed by Arnowitt, Deser and Misner (ADM) [19, 20, 21, 22]. (See also the texts of refs. [23, 24].) In this treatment, all of the equations of motion that do not involve time derivatives (the “algebraic constraints”) are solved for a number of the fundamental fields at the level of the Lagrangian. These solutions are then used to eliminate these fields from the action, by which one obtains a so called “reduced” action; eq. (3.3) of ref. [25] for example. The canonical analysis of the action starts at this point \(^1\). Therefore one expects that the four ADM first class constraints \(\mathcal{H}_i\) and \(\mathcal{H}\) that are obtained by working with this form of the Lagrangian lead to generators of a transformation which is the invariance of the “reduced” action, and possibly the gauge invariance of the original EH action.\(^2\)

An essential difference between the canonical analysis of the first order form of the EH action presented in this paper and that of previous treatments is that the Dirac constraint formalism is applied only using equations of motion corresponding to second class constraints to eliminate fundamental fields at the Lagrangian level. As it will be seen, this leads to a constraint structure sharply distinct from that of ADM. As a matter of fact, applying the Dirac constraint analysis to the first order form of \(S_2\) has been shown [28, 29, 30, 31, 32] to lead to a gauge transformation that is distinct from a coordinate transformation, even though the Lagrangian is manifestly invariant under a coordinate transformation. It might be interesting to make connections between this unexpected result and those of ref. [33], where the class of all symmetries of the second order Einstein equations of motion in \(d = 4\) are studied. It might very well be that having a new symmetry is a feature particular to \(d = 2\).

In the next section the canonical analysis of \(S_d\) in the first order form is given in detail.

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\(^1\) The first order form of \(S_4\), where \(g_{\mu \nu}\) and \(\Gamma^\lambda_{\mu \nu}\) are the fundamental fields, is treated explicitly using this procedure in refs. [20, 24]. The approach of ref. [26] to constrained systems with first order Lagrangians is much the same as that of refs. [20, 24].

\(^2\) An account of the derivation of the diffeomorphism invariance of the EH action in second order form can be found in ref. [27], however, the authors of this paper are unaware of such an account for the first order ADM analysis.
This program has been outlined in ref. [30] although here we use a different set of canonical variables. The linearized version of $S_d$ (i.e. the first order form of the spin-two field [19]) is treated using this formalism in appendix A. The effect on the PB algebra of a free massless scalar field is considered in appendix B. The inclusion of a cosmological constant, massive scalar fields, Maxwell gauge fields and Yang-Mills fields is considered in [34]. A summary of our results for the constraint structure of the first order EH action appears in ref. [35].

II. THE EH ACTION IN D DIMENSIONS

In this section we will use the Dirac constraint formalism to analyze the first order form of the EH action in $d$ dimensions. Since this is a rather lengthy procedure, subheadings will be used to itemize each of the steps.

A. Choice of Variables

The EH action of eq. (1) when written in terms of the metric $g_{\mu\nu}$ and the affine connection $\Gamma^\lambda_{\mu\nu}$ is

$$S_d = \int d^d x \sqrt{-g} g^{\mu\nu} \left( \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\lambda\mu,\nu} + \Gamma^\lambda_{\mu\nu} \Gamma^\sigma_{\sigma\lambda} - \Gamma^\lambda_{\sigma\mu} \Gamma^\sigma_{\lambda\nu} \right).$$

It is convenient to re-express this in terms of the variables

$$h^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$$

$$G^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \frac{1}{2} (\delta^\lambda_{\mu} \Gamma^\sigma_{\nu\sigma} + \delta^\lambda_{\nu} \Gamma^\sigma_{\mu\sigma})$$

so that

$$S_d = \int d^d x h^{\mu\nu} \left( G^\lambda_{\mu\nu,\lambda} + \frac{1}{d-1} G^\lambda_{\mu\nu} G^\sigma_{\sigma\nu} - G^\lambda_{\sigma\mu} G^\sigma_{\lambda\nu} \right).$$

If $d \neq 2$, then $g^{\mu\nu}$ can be expressed in terms of $h^{\mu\nu}$ since

$$\det h^{\mu\nu} = -(\sqrt{-g})^{d-2}.$$

For convenience, we integrate the first term in eq. (5) by parts and drop the surface term. If $h = h^{00}, h^i = h^{0i}, \pi = -G^0_{00}, \pi_i = -2G^0_{0i}, \pi_{ij} = -G^0_{ij}, \xi^i = -G^i_{00}, \xi^i_j = -2G^i_{j0}$ and
\( \xi_{jk} = -G_{jk}^i \), then eq. (5) can be written as

\[
S_d = \int d^d x \left[ (\pi h_{,0} + \pi_i h^i_{,0} + \pi_{ij} h^{ij}_{,0}) + \frac{2-d}{d-1} \left( h\pi^2 + h^i \pi_i + \frac{1}{4} h^{ij} \pi_i \pi_j \right) \right. \\
+ \xi^i (h^i_{,i} - h \pi_i - 2h^j \pi_{ij}) \\
+ \xi^j (h^j_{,i} + \frac{1}{d-1} h \pi \delta^j_i + \frac{1}{2(d-1)} h^k \pi_k \delta^j_i - \frac{1}{2} h^j \pi_i - h^{jk} \pi_{ik}) \\
+ \xi^j k (h^{jk}_{,i} + \frac{1}{d-1} \pi (\delta^j_i h^k + \delta^j_k h^i) + \frac{1}{2(d-1)} (\delta^j_i h^{kl} + \delta^j_k h^{il}) \pi_l) \\
+ \frac{1}{4} \left( \frac{1}{d-1} \xi^k \xi^l_i - \xi^k \xi^l_k \right) h + \left( \frac{1}{d-1} \xi^k \xi^l_i - \xi^k \xi^l_k \right) h + \left( \frac{1}{d-1} \xi^k \xi^l_i - \xi^k \xi^l_k \right) h \\
\left. \right]
\]

At this stage we do not use equations of motion that are independent of time derivatives in order to eliminate any of the fields in eq. (7), unlike refs. [19, 20, 21, 22, 23, 24, 25].

We can further simplify the form of eq. (7) by first separating the trace of \( \xi^i \)

\[
\xi^i_j = \bar{\xi}_j^i + \frac{1}{d-1} \delta^i_j t
\]

where \( \bar{\xi}_i^i = 0 \), and then shifting \( \bar{\xi}_j^i \) to decouple \( \bar{\xi}_j^i \) from \( \xi^i_{jk} \) in the action,

\[
\bar{\xi}_j^k = \bar{\xi}_j^k - \frac{2}{h} \left( \xi^k_{lm} - \frac{1}{d-1} \delta^k_l \xi^l_{jm} \right) h^m,
\]

so that eq. (7) becomes

\[
S_d = \int d^d x \left[ (\pi h_{,0} + \pi_i h^i_{,0} + \pi_{ij} h^{ij}_{,0}) + \frac{2-d}{d-1} \left( h\pi^2 + h^i \pi_i + \frac{1}{4} h^{ij} \pi_i \pi_j \right) \right. \\
+ \xi^i (h^i_{,i} - h \pi_i - 2h^j \pi_{ij}) + \frac{t}{d-1} (h^j_{,j} + h \pi - h^{jk} \pi_{jk}) \\
+ \bar{\xi}_j^i \left( h^j_{,i} - \frac{1}{2} h^j \pi_i - h^{jk} \pi_{ik} \right) - \frac{h}{4} \bar{\xi}_k^i \bar{\xi}_j^k \\
+ \xi^j k (h^{jk}_{,i} + \frac{1}{h} (h^j h^k),_i + \frac{1}{(d-1)h} (\delta^j_i h^k + \delta^j_k h^i)(h^i_l - \frac{1}{2} h^l \pi_l - h^{im} \pi_{lm} + h \pi) \\
+ \frac{1}{h} h^j h^k \pi_i + \frac{1}{h} (h^j h^k + h^k h^l) \pi^i_d + \frac{1}{2(d-1)} (\delta^j_i h^{kl} + \delta^j_k h^{il}) \pi_l) \\
+ \left. H^{ij} \left( \xi^k_i \xi^l_{kj} - \frac{1}{d-1} \xi^k_i \xi^l_{ki} \right) \right]
\]

where

\[
H^{ij} = \frac{1}{h} h^i h^j - h^{ij}.
\]

At this point, it is convenient to replace \( h^{ij} \) by \( H^{ij} \). If we define

\[
\omega = \pi - \frac{h^i h^j}{h^2} \pi_{ij} , \quad \omega_i = \pi_i + 2 \frac{h^j}{h} \pi_{ij} , \quad \omega_{ij} = -\pi_{ij}
\]
it follows that
\[ \pi h^i_0 + \pi_i h^i_0 + \pi_{ij} h^{ij}_0 = \omega h^i_0 + \omega_i h^i_0 + \omega_{ij} H^{ij}_0. \] (13)

The action of eq. (10) now becomes
\[
S_d = \int d^d x \left[ \omega h^i_0 + \omega_i h^i_0 + \omega_{ij} H^{ij}_0 
+ \frac{2 - d}{d-1} \left( h (\omega + \frac{1}{2} h^{i}_0) \right)^2 - \frac{1}{4} H^{ij} (\omega_i + \frac{2 \omega_{im} h^m}{h}) (\omega_j + \frac{2 \omega_{jn} h^n}{h}) \right] + \bar{t} \left( \chi^i - \frac{h^{ij} h^k}{h^2} \xi^i_{jk} \right),
\]
where
\[
\chi = h^j + h \omega - H^{jk} \omega_{jk}, \quad \chi^i = h_{ij} - h \omega_i, 
\]
\[
\bar{\xi}^i = \xi^i - \frac{h^{ij} h^k}{h^2} \xi^i_{jk}, \quad \bar{t}^i = t + \frac{1}{h} \left( \delta^i_j h^k + \delta^i_k h^j \right) \xi^j_{jk},
\]

and
\[
\lambda^j_i = h^j_i - \frac{1}{2} h^j \omega_i - H^{jk} \omega_{ik}, \quad \sigma^j_i = -H^{jk}_i + \frac{1}{h} (h^{ij} H^{kl} + h^k H^{ji}) \omega_{il} - \frac{1}{d-1} (\delta^j_i H^{kl} + \delta^j_k H^{il}) \left( \frac{1}{2} \omega_l + \omega_{lm} \frac{h^m}{h} \right). \] (20)

At this stage one might decompose \( \xi^i_{jk} \) into \( \bar{\eta}^i_{jk} \), \( t_i \) and \( s^i \) where \( \bar{\eta}^i_{jk} = 0 = H^{jk} \bar{\eta}^i_{jk} \) by the equations
\[
\xi^i_{jk} = \bar{\xi}^i_{jk} + \frac{1}{d} (\delta^j_i t_k + \delta^j_k t_l), \quad (\bar{\xi}^i_{jk} = 0) \] (21)
\[
\bar{\xi}^i_{jk} = \bar{\eta}^i_{jk} + \frac{1}{d(d - 2)(d + 1)} \left( d H_{jk} s^i - (\delta^i_j H_{km} + \delta^i_k H_{jm}) s^m \right) \] (22)
with \( H^{ip} H_{pj} = \delta^i_j \). This however does not simplify the canonical analysis.

The canonical analysis of the EH action written in the form of eq. (14) can now proceed.

**B. Primary and Secondary Constraints**

Since eq. (14) is first order in the time derivatives, we see immediately that the momenta associated with the fields \( \omega, \omega_i \) and \( \omega_{ij} \) are all zero while the momenta associated with \( h, \)
\( h^i \) and \( H^{ij} \) are \( \omega, \omega_i \) and \( \omega_{ij} \) respectively. These constitute a set of \( d(d+1) \) primary second class constraints [7, 8, 9, 10, 11, 12].

The momenta associated with the fields \( \bar{t} \) and \( \xi^i \) also vanish. As \( \bar{t} \) and \( \xi^i \) only enter eq. (14) linearly, the vanishing of their momenta form a set of \( d \) primary first class constraints.

From eq. (14) the canonical Hamiltonian is

\[
H = \frac{d - 2}{d - 1} \left( h (\omega + \frac{1}{2} h^i \omega_i)^2 - \frac{1}{4} H^{ij} (\omega_i + \frac{2 \omega_m m^m}{h}) (\omega_j + \frac{2 \omega_n n^n}{h}) \right) - \bar{\xi} \chi - \bar{\xi} \chi_i - \frac{\bar{t}}{d - 1} \chi - \lambda^j \bar{\xi}^j - \sigma^j \bar{\xi}^j - \frac{h}{4} \bar{\xi} \bar{\xi} - H^{ij} \left( \xi^k l^l - \frac{1}{d - 1} \xi^k l^l \right).
\]

In order to describe the dynamics of the gravitational field, instead of forming the total Hamiltonian by supplementing the canonical Hamiltonian of eq. (23) with primary constraints by means of Lagrange multipliers, we adopt a different approach. In this approach, it is not necessary to fix Lagrange multipliers by the emergence of second class constraints that may arise because of the consistency conditions, but Dirac brackets are introduced instead of Poisson brackets and second class constraints are set strongly equal to zero.

Having the momenta associated with \( \bar{t} \) and \( \xi^i \) vanish means that these momenta must have a vanishing PB with \( H \) in eq. (23); we thus obtain the secondary constraints

\[
\chi = 0 \quad (24)
\]

\[
\chi_i = 0. \quad (25)
\]

By using test functions to evaluate the PB of \( \chi \) and \( \chi_i \) we find that

\[
\{\chi_i, \chi\} = \chi_i \quad (26)
\]

while

\[
\{\chi_i, \chi_j\} = 0 = \{\chi, \chi\}. \quad (27)
\]

As has been noted above after eq. (7), we do not use equations of motion that have no time derivatives to eliminate fields from the action. In particular, two of these equations of motion are the trace of eq. (A3) and eq. (A4) of ref. [25], and these are identical to our constraints \( \chi = \chi_i = 0 \) of eqs. (24, 25).

Since by eq. (27) it is possible at this stage that the constraints \( \chi \) and \( \chi_i \) are first class, it is necessary to find the PB of these constraints with \( H \) to see if there are any tertiary constraints. We then must determine if \( \chi \) and \( \chi_i \) continue to be first class once these tertiary
constraints are included, and to find what class the tertiary constraints belong to. If the tertiary constraints are not seen to be immediately second class, the possibility of “fourth generation” constraints must be considered and the procedure continues until all constraints are found and classified.

The presence in eq. (24) of terms quadratic in $\bar{\zeta}^i_j$ and $\xi^i_{jk}$ implies that there are also second class secondary constraints to be considered. Such constraints do not arise if $d = 2$, considerably simplifying the canonical structure of the two dimensional EH action [28, 29, 30, 31, 32].

C. Tertiary constraints

The momenta associated with the traceless quantities $\bar{\zeta}^i_j$ and the quantity $\xi^i_{jk}$ all vanish; this leads to $[(d - 1)^2 - 1] + \left[\frac{1}{2}d(d - 1)^2\right] = \frac{1}{2}d(d^2 - 3)$ primary constraints. Taking the PB of these constraints with $H$ given in eq. (23) results in $\frac{1}{2}d(d^2 - 3)$ additional secondary constraints, each of which is linear in either $\bar{\zeta}^i_j$ or $\xi^i_{jk}$. Consequently, all of these constraints must be second class; in total there are $d(d^2 - 3)$ second class constraints. The equations of motion that are secondary second class constraints correspond to eq. (A2) and the traceless part of eq. (A3) of ref. [25].

We can in fact solve these equations of motion and eliminate the variables $\bar{\zeta}^i_j$ and $\xi^i_{jk}$ in the Hamiltonian provided we use the appropriate DB [7, 8, 9, 10, 11, 12]. Being able to solve these second class constraints in order to eliminate $\bar{\zeta}^i_j$ and $\xi^i_{jk}$ is quite unlike the situation for the first class constraints $\chi$ and $\chi_i$ of eqs. (24,25) which cannot be used to eliminate fields in the Dirac constraint formalism.

We first write the portion of the Hamiltonian of eq. (23) that generates the secondary second class constraints as

$$A = -\bar{\zeta}^i_j \lambda^j_i + \frac{h}{4} \bar{\zeta}^i_j \bar{\zeta}^j_i$$  (28)

$$B = -\xi^i_{jk} \sigma^j_k - H^{ij} \left( \xi^k_i \xi^l_j - \frac{1}{d-1} \xi^k_i \xi^l_j \right)$$

$$\equiv -\xi^i_{jk} \sigma^j_k - \xi^k_{lm} \left( M_{k}^{lm}_{de} \right) \xi^c_{de}$$  (29)
where

\[ M^{lm}_{k c} = \frac{1}{4} \left[ H^{me} \left( \delta^l_k \delta^d_c - \frac{1}{d-1} \delta^l_k \delta^d_c \right) + H^{md} \left( \delta^l_k \delta^e_c - \frac{1}{d-1} \delta^l_k \delta^e_c \right) \right. \]

\[ + H^{le} \left( \delta^m_k \delta^d_c - \frac{1}{d-1} \delta^m_k \delta^d_c \right) + H^{ld} \left( \delta^m_k \delta^e_c - \frac{1}{d-1} \delta^m_k \delta^e_c \right) \right]. \]  

(30)

If

\[ M^{-1 x y z l m}_{k} = \frac{1}{2} \left[ H^{+} \delta^{x}_{k} \delta^{y}_{m} + H^{t} \delta^{x}_{m} \delta^{y}_{k} + H_{m} \delta^{k}_{y} \delta^{m}_{x} + H_{m z} \delta^{k}_{y} \delta^{z}_{x} \right] \]

\[ + \frac{2}{d-2} \left( H^{k x} H_{l m} H^{y z} - H^{k y} (H^{l z} H_{m y} + H^{l y} H_{m z}) \right) \]

then it follows that

\[ (M^{-1 x y z l m}_{k}) (M^{lm}_{k c}) = \frac{1}{2} \delta^{s}_{k} (\delta^{d}_{s} \delta^{c}_{d} + \delta^{d}_{s} \delta^{c}_{d}) \].

The equations of motion for \( \tilde{\xi}_{j} \) and \( \xi_{jk} \) that follow from \( A \) and \( B \) in eqs. (28,29) imply that

\[ \tilde{\xi}_{j} = \frac{2}{h} \left( \delta^{m}_{j} \delta^{n}_{j} - \frac{1}{d-1} \delta^{j}_{m} \delta^{n}_{j} \right) \lambda^{m}_{n} \]  

(32)

\[ \xi_{jk} = \frac{-1}{2} (M^{-1 i l}_{j k m n}) \sigma^{mn}_{l} \].

(33)

Substitution of eqs. (32,33) into eqs. (28,29) respectively results in

\[ A = -\frac{1}{h} \left( \lambda^{x}_{j} \lambda^{j}_{i} - \frac{1}{d-1} \lambda^{x}_{i} \lambda^{j}_{j} \right) \]  

(34)

\[ B = \frac{1}{4} \sigma^{j k}_{i} (M^{-1 i l}_{j k m n}) \sigma^{mn}_{l} \].

(35)

Replacing \( A \) and \( B \) as given in eqs. (28,29) with \( A \) and \( B \) as given in eqs. (34,35) leads to the Hamiltonian of eq. (23) being expressed as a function that depends exclusively on \( (h, \omega), (h^{i}, \omega_{i}), (H^{ij}, \omega_{ij}), i \) and \( \tilde{\xi} \). We then drop explicit dependence on \( \chi \) and \( \chi_{i} \) occurring in the Hamiltonian of eq. (23), leading to the following weak Hamiltonian,

\[ H_{w} = h \omega^{2} + h^{i} \omega_{i} - \frac{d-3}{4(d-2)} H^{ij} \omega_{i} \omega_{j} - \frac{h^{m}}{h} H^{ij} \omega_{i} \omega_{j} - \frac{1}{h} H_{ik} H_{ik} \omega_{i} \omega_{i} \]

\[ + \frac{1}{h} h^{i} h^{j} \omega_{i} + \frac{2}{h} h^{i} H^{jk} \omega_{i} - \frac{h^{i}}{h} H^{ij} \omega_{j} + \frac{1}{2(d-2)} H_{jk} H_{ik} H^{im} \omega_{m} \]

\[ - \frac{1}{h} h^{i} h^{j} + \frac{1}{2} H^{jk} H_{jq} H^{i q} + \frac{1}{4} H^{ip} H_{kr,i} H^{k r} + \frac{1}{4(d-2)} H^{ip} H_{jk} H_{ik} H_{qr} H^{q r} \).

(36)

Evaluation of the PB of \( \chi \) and \( \chi_{i} \) with the Hamiltonian provides the time change of these constraints \(^{3}\). However, since we are only interested in what constraints arise from \( \chi \) and \( \chi_{i} \)

\(^{3}\) At this stage, since a set of second class constraints have been set to zero and solved for a number of fundamental fields in the action, Poisson Brackets should be replaced by Dirac Brackets. However, as it is shown in eq. (52) below, for the purpose of our calculations we may safely use PBs instead of DBs.
at this stage, we may by eqs. (26,27) use $H_w$ instead of the full Hamiltonian. From $\chi_i$, the following quantity $\bar{\tau}_i$ is obtained,

$$\bar{\tau}_i = \{\chi_i, \int dy H_w(y)\}$$

$$= (H^{pq})_i \omega_{pq} - 2(H^{pq}\omega_{qj})_p - h^p \omega_{i,p} - \omega_i H^{pq}\omega_{pq} + 2H^{pq}\omega_{pq,i} + h^p \omega_{p,i}.$$ 

Using the form of $\chi_i$ given in eq. (16), we find that this is equivalent to taking

$$\tau_i = h(\frac{1}{h}H^{pq}\omega_{pq})_i + H^{pq}\omega_{pq,i} - 2(H^{pq}\omega_{qj})_p$$

$$= \bar{\tau}_i + h^p \left[ \left( \frac{\chi_p}{h} \right)_i - \left( \frac{\chi_i}{h} \right)_p \right] - \frac{\chi_i}{h} H^{pq}\omega_{pq}$$

to be the tertiary constraint following from $\chi_i$. Similarly, if $\{\chi(x), \int dy H_w(y)\} \approx 0$ we find that

$$\bar{\tau} = H_w + \partial_i \delta^i \approx H + \partial_i \delta^i$$

must weakly vanish. Remarkably, $\bar{\tau}$ equals the weak Hamiltonian of eq. (36) plus the divergence of a vector

$$\delta^i = -H^{ij} + \frac{1}{h} (h^i h^j)_j + 2h^i \omega - H^{ij}(\omega_j + \frac{2\omega_{jm}h^m}{h}) - \frac{d}{d-1} \frac{h^i}{h} \chi.$$ 

Carefully combining terms in the Hamiltonian $H_w$ of eq. (36) and $\partial_i \delta^i$, it follows that

$$\tau = \tau + \frac{h^i}{h} \tau_i + \frac{h^i}{h} \chi, - \frac{h^j h^i}{h^2} \chi_i + \frac{H^{jk}\omega_{jk}}{h} \chi - \frac{h^i \omega}{h} \chi$$

$$+ \frac{2}{h^2} h^k H^{ij} \omega_{ik} \chi_j + \omega \chi - \frac{d}{d-1} \left( \frac{h^i}{h} \chi \right)_i,$$

where

$$\tau = -H^{ij}_{ij} - (H^{ij}_i \omega_j)_i - \frac{d-3}{4(d-2)} H^{ij}_i \omega_j + \frac{1}{2(d-2)} H_{kl} H^{kl}_{i} H^{ij}_i \omega_j$$

$$- \frac{1}{h} H^{jk} H^j_i (\omega_{jk} \omega_{it} - \omega_{ik} \omega_{jt}) + \frac{1}{2} H^{jk}_i H_{jl} H^i_{kl} + \frac{1}{4} H^{ij} H_{kl,i} H^{kl}_j$$

$$+ \frac{1}{4(d-2)} H^{ij} H_{kl} H^{kl}_{i} H_{mn} H^{mn}_j.$$ 

Once again, we are forced to impose a tertiary constraint in order to ensure that $d\chi/dt \approx 0$; we take this tertiary constraint to be $\tau$ in eq. (42).

An alternate way of obtaining the tertiary constraints is to work with the Hamiltonian in the form of eq. (23) without eliminating $\tilde{\zeta}_j^i$ and $\xi^i_{jk}$. This means using a DB in place of a PB if $\tilde{\zeta}_j^i$ or $\xi^i_{jk}$ are involved.
To illustrate how this works, it is convenient to consider a simplified model in which we have the action

\[ S = \int dt \{ p_i \dot{q}_i - [ H_0(q_i, p_i) + \lambda_A \chi_A(q_i, p_i) + f_I^a(q_i, p_i) Q_I^a - \frac{1}{2} Q_I^a g_{ij}^a(q_i) Q_J^b ] \}, \]

where

\[ \{ \chi_A, \chi_B \} = C_{ABC} \chi_C. \quad (44) \]

Eqs. (43,44) are analogues of eqs. (14,26-27) respectively, with \( q_i \) representing \((h, h^i, H^i)\), \( p_i \) representing \((\omega, \omega_i, \omega_{ij})\), \( Q_I^a \) representing \((\tilde{c}_j^i, \xi_{jk})\) and \( \lambda_A \) representing \((\tilde{t}, \tilde{\xi}_i^j)\). The momenta conjugate to \( Q_I^a \) and \( \lambda_A \) (\( P_I^a \) and \( \pi_A \)) are zero; these primary constraints immediately give rise to the secondary constraints

\[ \bar{\theta}_I^a \equiv f_I^a(q_i, p_i) - g_{ij}^a(q_i) Q_J^b = 0 \quad (45) \]

and

\[ \gamma_A = \chi_A(q_i, p_i) = 0. \quad (46) \]

The constraints \( \theta_I^a = P_I^a = 0 \) and \( \bar{\theta}_I^a \) of eq. (45) are obviously second class while \( \gamma_A \) of eq. (46) may be first class on account of eq. (44). (Subsequent tertiary constraints may change these constraints to second class.)

In order to eliminate the second class constraints from the action, we need to form the appropriate DBs. Since

\[ \{ \theta_I^a, \bar{\theta}_J^b \} = g_{IJ} \delta_{IJ} \quad (47) \]

and

\[ \{ \bar{\theta}_I^a, \bar{\theta}_J^b \} = \{ f_I^a(q_i, p_i) - g_{IK}^a(\tilde{c}_j^i) Q_K^m, f_J^b(q_i, p_i) - g_{JK}^b(\xi_{jk}) Q_L^n \} \equiv M_{IJ}^{ab} \quad (48) \]

then the matrix \( d_{\alpha\beta} = \{ \chi_\alpha, \chi_\beta \} \), where \( \chi_\alpha \) and \( \chi_\beta \) are second class constraints to be eliminated \([2, 8, 9, 10]\), takes the form

\[ d = \begin{pmatrix}
0 & g_1 & 0 & 0 \\
-g_1 & M_{11} & 0 & M_{12} \\
0 & 0 & 0 & g_2 \\
0 & -M_{12} & -g_2 & M_{22}
\end{pmatrix} \quad (49) \]
with the indices $I$ and $J$ in eq. (45) taking on two values, corresponding to $\tilde{\zeta}_i^j$ and $\tilde{\xi}_i^{jk}$. Using the relation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I & B \\ 0 & D \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & I \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} + D^{-1} \end{pmatrix}$$ (50)

we find that

$$d^{-1} = \begin{pmatrix} g^{-1}_1 M_{11} g^{-1}_1 & -g^{-1}_1 g^{-1}_1 M_{12} g^{-1}_2 & 0 \\ g^{-1}_1 & 0 & 0 & 0 \\ -g^{-1}_2 M_{12} g^{-1}_1 & 0 & g^{-1}_2 M_{22} g^{-1}_2 & -g^{-1}_2 \\ 0 & 0 & g^{-1}_2 & 0 \end{pmatrix}.$$ (51)

From eq. (51), the definition of the DB [7, 8, 9, 10],

$$\{A, B\}^* = \{A, B\} - \{A, \chi_\alpha\} (d^{-1})^{\alpha\beta} \{\chi_\beta, B\},$$

shows that in this system

$$\{q_i, p_j\}^* = \delta_{ij},$$ (52)

$$\{q_i, Q_{Ia}\}^* = \{q_i, \tilde{\theta}^c_i\} (g^{-1})^{ca}_{IJ} \delta_{IJ},$$ (53)

$$\{p_i, Q_{Ia}\}^* = \{p_i, \tilde{\theta}^c_i\} (g^{-1})^{ca}_{IJ} \delta_{IJ},$$ (54)

$$\{Q^a_I, Q^b_J\}^* = (g^{-1})^{ab}_{IK} M_{KL}^{mn} (g^{-1})^{nb}_{LJ}.$$ (55)

An explicit calculation shows that the matrices $M_{KL}^{mn}$ in eq. (55) are non local. This makes the use of eq. (55) somewhat ambiguous, but we will see that in the process of evaluating the tertiary constraints corresponding to the secondary constraints $\chi$ and $\chi_i$ we luckily don’t need them. In fact, using the constraint $\tilde{\theta}^a_i$ to express the Hamiltonian that follows from eq. (43) in the form

$$H = H_0 + \lambda_A \chi_A + \frac{1}{2} Q^a_I g^{ab}_{IJ} Q^b_J,$$ (56)

it follows from eqs. (44,53,54) that

$$\frac{d\chi_A}{dt} \approx \{\chi_A, H\}^* \approx \{\chi_A, H_0\} + \{\chi_A, \tilde{\theta}^a_i\} Q^a_i + \frac{1}{2} Q^a_I \{\chi_A, g^{ab}_{IJ} \} Q^b_J.$$ (57)

Eq. (57) can be used to find the tertiary constraints $\tau_i$ and $\tau$ that follow from the secondary constraints of eqs. (24,25).
It is now necessary to see how the constraints \( \chi, \chi_i, \tau \) and \( \tau_i \) are to be classified, and if any further “fourth generation” constraints are required in order to ensure that \( \tau \) and \( \tau_i \) have weakly vanishing time derivatives.

D. Algebra of Constraints

In addition to the PB of eqs. (26,27), one can show easily that

\[
\{ \chi, \tau \} = 0 .
\] (58)

Another direct calculation (one that is somewhat more difficult) leads to

\[
\{ \chi, \tau \} = \tau .
\] (59)

It is also possible to show that

\[
\{ \chi_i, \tau \} = 0
\] (60)

and

\[
\{ \chi_i, \tau_j \} = 0 .
\] (61)

A rather involved calculation leads to

\[
f \{ \tau_i, \tau_j \} g = g(\partial_j f)\tau_i - f(\partial_i g)\tau_j ,
\] (62)

where \( f \) and \( g \) are test functions. More explicitly, eq. (62) can be written as

\[
\int dx \, dy \, f(x) \{ \tau_i(x), \tau_j(y) \} \, g(y)
\] (63)

\[
= \int dx \, [g(x) (\partial_j f(x)) \tau_i(x) - f(x) (\partial_i g(x)) \tau_j(x)]
\]

\[
= \int dx \, dy \, \left[ f(x) \left( -\partial_j^x \delta(x-y)\tau_i(y) + \tau_j(x) \partial_i^y \delta(x-y) \right) \right] \, g(y) ,
\]

so that we have the non-local PB

\[
\{ \tau_i(x), \tau_j(y) \} = -\partial_j^x \delta(x-y)\tau_i(y) + \tau_j(x) \partial_i^y \delta(x-y) .
\] (64)

This is identical to the PB of the constraints \( \mathcal{H}_i \) appearing in refs. [19, 20, 21, 22, 25], even though \( \tau_i \) and \( \mathcal{H}_i \) are distinct.

As mentioned, a disadvantage of the Dirac Brackets introduced in Section C is that the matrices \( M_{KL}^{mn} \) occurring in eq. (55) are non local. Therefore, at the stage developed in this
paper, it is not straightforward how the PBs of the tertiary constraints $\tau_i$ and $\tau$, and of $\tau$ and $\tau$, and their time derivatives must be computed using them. As a result, in order to find these PBs of first class constraints and their time derivatives, we use the alternative method where we solved $\xi_{ijk}$ and $\bar{\zeta}_i$ in terms of $h$, $h^i$, $H^{ij}$, $\omega$, $\omega_i$ and $\omega_{ij}$ by means of the second class constraints occurring in the theory.

When computing the PBs $\{\tau, \tau\}$ and $\{\tau_i, \tau\}$ we are confronted with huge expressions which are rather difficult to arrange into combinations of first class constraints. However, it is indeed necessary to show that these PBs are weakly zero if $\tau$ and $\tau_i$ are to be identified as first class constraints. It must also be shown that the time derivatives of these tertiary constraints do not lead to fourth generation constraints. We now explain how these two problems are intimately connected, and how this connection helps to resolve the algebraic difficulty of computing the PBs $\{\tau, \tau\}$ and $\{\tau_i, \tau\}$.

The observation that the first class constraint $\tau$ of eq. (42) weakly differs from the Hamiltonian by a total divergence $\partial_i \delta^i$ is useful. Based on the number of degrees of freedom in the non interacting graviton field, one expects that all tertiary constraints are first class and therefore no higher generation of constraints should arise. One then concludes that the time change of $\tau$ and $\tau_i$, and therefore, $f\{\tau_i, \int H_w dy\}$ and $f\{\tau, \int H_w dy\}$, where $H_w$ is given by eq. (36) should be written as a linear combination of first class constraints. But since $\tau \approx H_w + \partial_i \delta^i$, one concludes that $f\{\tau_i, \int \tau dy\}$ $\approx$ $f\{\tau_i, \int H_w dy\}$ and also that $f\{\tau, \int \tau dy\}$ $\approx$ $f\{\tau, \int H_w dy\}$. In other words, $f\{\tau_i, \int \tau dy\}$ and $f\{\tau, \int \tau dy\}$ should be expressible in terms of first class constraints. These expressions, though still enormous, have turned out to be manageable. They not only lead us to first class expressions for the time change of $\tau$ and $\tau_i$, but also infer how some of the terms appearing in $f\{\tau_i, \tau\}g$ and $f\{\tau, \tau\}g$ can be written in terms of linear combinations of constraints.

Having these considerations in mind, we first compute the time change of the constraint $\tau$ and find that it is given by a linear combination of constraints
\[
 f\{\tau, \int dy H_w\} = \partial_t f\frac{H^{ij}}{h^2} (h\tau_j - H^{mn}\omega_{mn}\chi_j + 2H^{mn}\omega_{mj}\chi_n). \tag{65}
\]

The structure of the last two expressions on the right hand side of this equation resembles that of the last two terms in the constraint $\tau_i$ of eq. (35). This suggests a redefinition of the constraint $\tau_i$ in order to obtain a simpler algebra that might be closer to that of the ADM
algebra\(^4\), but so far this effort has not been successful. Using eq. (65) for the time change of \(\tau\), we are aided in finding that the PB \(\{\tau, \tau\}\) is

\[
\{\tau, \tau\} g = (g \partial_{i} f - f \partial_{i} g) \frac{H_{ij}^{\tau}}{h^2} \left( h \partial_{j} \tau - H_{mn}^{\tau \omega} \chi_{j} + 2H_{mn}^{\tau \omega m j} \chi_{n} \right).
\] (66)

In much the same way, the time change of \(\tau_{i}\) is expressible as a linear combination of constraints,

\[
\{\tau_{i}, \int dy \, H_{w}\} = \frac{(f h)_{i}}{h} \tau + \frac{d - 3}{2(d - 2)} f \left( \frac{1}{h} H_{i j}^{\omega} \chi_{j} \right)_{,j} - \frac{d - 3}{2(d - 2)} f H_{i l}^{\omega} \left( \frac{\chi_{k}}{h} \right)_{,k} \] (67)

and this helps us show that

\[
\{\tau_{i}, \tau\} g = g \left( \frac{f h)_{i}}{h} \tau - f g_{i} \tau \right) - \frac{d - 3}{2(d - 2)} f g H_{i l}^{\omega} \left( \frac{\chi_{k}}{h} \right)_{,k} \] (68)

Eqs. (66), (67), (68) all show that amongst themselves, \(\chi, \chi_{i}, \tau\) and \(\tau_{i}\) are first class and their PB algebra is highly unusual.

The Jacobi identities for the PBs of the first class constraint triplets \((\tau, \tau, \chi), (\tau, \tau_{i}, \chi)\) and \((\tau, \tau, \tau)\) have been verified by explicit computation, providing a non-trivial consistency test for the PBs of eqs. (66) and (68).

III. DISCUSSION

We have found the complete constraint structure for the action \(S_{d}\) of eq. (5) if \(d > 2\). In particular, we have the \(d(d + 1)\) primary second class constraints resulting from the

\[
\{H(x), H(y)\} = (H_{i}(x) + H_{i}(y)) \partial_{i} \delta(x - y),
\]

\[
\{H_{i}(x), H(y)\} = H(x) \partial_{i} \delta(x - y),
\]

\[
\{H_{i}(x), H_{j}(y)\} = (H_{i}(y) \partial_{j} + H_{j}(x) \partial_{i}) \delta(x - y).
\]

\(^4\) This is
identification of $-G^0_{00}, -2G^0_{0i}$ and $-G^0_{ij}$ with the canonical momenta conjugate to $h, h^i$ and $h^{ij}$. We have already noted that there are $d(d^2 - 3)/2$ primary second class constraints associated with the vanishing of the canonical momenta for $\bar{\zeta}^i_j$ and $\xi^i_{jk}$ and that these in turn lead to a further $d(d^2 - 3)/2$ secondary second class constraints associated with the equations of motion for $\bar{\zeta}^i_j$ and $\xi^i_{jk}$. In total there then are $d(d + 1) + d(d^2 - 3) = d^3 + d^2 - 2d$ second class constraints. We also have $d$ primary first class constraints (the momenta associated with $\bar{t}$ and $\bar{\xi}^i$) as well as $d$ secondary first class constraints ($\chi$ and $\chi^i$) and $d$ tertiary first class constraints ($\tau$ and $\tau_i$). When we include the gauge conditions associated with each of these $3d$ first class constraints, there are $3d + 3d + d^3 + d^2 - 2d = d(d^2 + d + 4)$ restrictions on the $d(d + 1)^2$ variables in phase space (the $h^{\mu\nu}, G^\lambda_{\mu\nu}$ and their conjugate momenta). There are thus $d(d + 1)^2 - d(d^2 + d + 4) = d(d - 3)$ independent degrees of freedom in phase space.

If $d = 3$, there are no degrees of freedom while if $d = 4$, there are the two polarizations of the graviton as well as their conjugate momenta. This is in agreement with the expectations of ref. [30].

In the ADM approach to the first order action of eq. (2) (refs. [20, 25]) in $d = 4$ dimensions, six of the ten components of the metric fields are dynamical and the remaining four become Lagrange multipliers, related to the “lapse” and “shift” functions. Thirty equations of motion that correspond to the secondary constraints of eqs. (24, 25, 32, 33) do not contain time derivatives and are used to eliminate components of the affine connections. (The first four of these equations, $\chi = \chi^i = 0$, which are first class constraints in our treatment, if used to eliminate $\omega$ and $\omega^i$, would reduce the Hamiltonian of eq. (36) to the ADM Hamiltonian, eq. (3.3) of ref. [25].) Furthermore, once the elimination has taken place, all of the affine connections $\Gamma^\mu_{00}$ disappear from the action and are not considered to be dynamical in the ADM approach. (In the analysis presented in this paper, $\Gamma^i_{00}$ and $\Gamma^0_{00}$ are associated with the Lagrange multipliers $\xi^i$ and $t$ respectively.)

There are then six remaining components of the affine connection that form the momenta conjugate to those components of the metric which are dynamical. When these constraints are combined with their associated gauge conditions, only the two transverse degrees of freedom associated with the metric plus their conjugate momenta remain in phase space. We thus see how the analysis presented in this paper, which uses exclusively the Dirac constraints formalism [7, 8, 9, 10, 11, 12], is related to the more conventional ADM approach to the canonical structure of $S_d$ of eq. (1) [19, 20, 21, 22, 23, 24, 25].
The relationship between the Dirac approach and that of ref. [26] is discussed in ref. [37]. There it is shown how the Dirac procedure can be cast into a form that is the same as that of ref. [26]. However, there does not make it clear how to classify the constraints that arise at each step of ref. [26], or if the PB algebra of the resulting constraints is identical to that of the constraints obtained by applying the Dirac procedure exclusively. Consequently, it is important to know the connection between the ADM constraints and the constraints found in this paper. In attempting to do this, we might try to find linear combinations of constraints that simplify our algebra. As a matter of fact, by replacing $\bar{\tau}$ in eq. (41) by $\tau$ in eq. (42), the algebra of PB of constraints has already been simplified, as the PB $\{\chi_i, \bar{\tau}\}$ is non local and $d$-dependent,

$$f\{\chi_i, \bar{\tau}\}g = fg \tau_i - \frac{2}{d-1} f \partial_i g \chi - \frac{h_i}{h} f \partial_i g \chi_j - \frac{h_j}{h} f \partial_j g \chi_i,$$

in contrast to eq. (60). It is quite possible that even more simplification occurs if the first class constraints were combined in a judicious manner. For example, if $\tilde{\chi}_i = \chi_i/h$ then $\{\tilde{\chi}_i, \chi\} = 0$ in place of eq. (26). It remains to be seen if the PBs of eqs. (64-68) could be similarly simplified. This could also possibly provide some insight into the geometrical significance of the first class constraints $\chi, \chi_i, \tau$ and $\tau_i$ which is not immediately apparent. We note though that no matter what the most convenient form of the first class constraints may be, there will always be tertiary constraints which will necessarily lead to transformations involving second derivatives of the gauge functions. This is to be expected as the coordinate transformation of the affine connection lead to such second derivatives. If the second order form of the EH action were considered, then only secondary first class constraints would arise as in ref. [40]. In the first order formalism in which the vierbein and $e^a_i$ and the spin connection $\omega^a_{\mu} \dot{h}$ are the independent fields, only secondary constraints should arise, as both the vierbein and affine connection are covariant under a coordinate transformation, and hence only first derivatives of the gauge functions occur, consistent with the results of ref. [6].

The most obvious problem that follows from our analysis that should be addressed is the question of finding the gauge transformation associated with the first class constraints. Having the gauge invariance for the fields $h^{\mu\nu}$ and $G_{\mu\nu}^A$ makes it possible to apply the

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5 A way of simplifying the ADM PB algebra is given in refs. [38, 39].
quantization procedure outlined in refs. [41, 42, 43]. When this was done in two dimensions [44], the transformations to be considered were other than diffeomorphism and the resulting radiative effects appear to cancel. It would be quite interesting to see what radiative effects follow from eq. (5), especially since it is only a cubic polynomial in the fields.

Extending our analysis to systems which include Bosonic matter fields such as massive scalar fields, Maxwell gauge fields and Yang-Mills fields has been done in [34]. Having a coupling between the gravitational field and spinors would mean [4] that the canonical analysis would have to be done using the vierbein and spin connection as geometrical fields as in ref. [6]. This analysis would be quite distinct from the one done here in terms of the metric and affine connection.

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[1] A. Einstein, *Sitz. Preuss Akad. Wiss. Phys.-Math. K1*, 414 (1925).
[2] M. Ferraris, M. Francaviglia and C. Reine, *Gen. Relativ. Gravit. 14*, 243 (1982).
[3] U. Lindstrom and M. Rocek, *Class. Quant. Grav. 4*, L79, (1987), J. Gegenberg, P.F. Kelly, R.B. Mann and D. Vincent, *Phys. Rev. D 37*, 3463 (1988).
[4] H. Weyl, *Z. Phys. 56*, 330 (1929), T.W.B. Kibble, *J. Math. Phys. 2*, 212, (1961).
[5] E. Witten, hep-th 0706.3359.
[6] L. Castellani, P. van Nieuwenhuizen and M. Pilati, *Phys. Rev. D 26*, 352 (1982), W. Kummer and H. Schütz, *Eur. Phys. J. C42*, 227 (2005), S. Y. Alexandrov and D. V. Vassilevich, *Phys. Rev. D 58*, 124029 (1998), I.A. Nikolic, *Class. Quantum Grav. 12*, 3103 (1995), R.D. Stefano and R.T. Rauch, *Phys. Rev. D 26*, 1214 (1982).
[7] P.A.M. Dirac, *Can. J. Math. 2*, 129 (1950).
[8] P.A.M. Dirac, *Lectures on Quantum Mechanics* (Dover, Mineola, 2001).
[9] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton U. Press, Princeton, 1992).

[10] K. Sundermeyer, *Constrained Dynamics* (Springer-Verlag, Berlin, 1982).

[11] L. Castellani, *Ann. Phys.(NY)* **143**, 357 (1982).

[12] M. Henneaux, C. Teitelboim and J. Zanelli, *Nucl.Phys.* **B332**, 169 (1990); R. Banerjee, H. J. Rothe and K. D. Rothe, *Phys. Lett.* **B462**, 248 (1999).

[13] A. Ashtekar, *Lectures on Non-Perturbative Quantum Gravity* (World Scientific, Singapore, 1991).

[14] T. Thiemann, hep-th 0608210

[15] H. Nicolai, K. Peters and M. Zamaklar, *Class. Quant. Grav.* **22** R193 (2005).

[16] F.A.E. Pirani, A. Schild and S. Skinner, *Phys. Rev.* **87** 87, 452 (1952).

[17] P.A.M. Dirac, *Proc. Roy. Soc. (London)* **A246**, 333 (1958).

[18] P.A.M. Dirac, *Phys. Rev.* **114**, 924 (1959).

[19] R. Arnowitt and S. Deser, *Phys. Rev.* **113**, 745 (1959).

[20] R. Arnowitt, S. Deser and C.W. Misner, *Phys. Rev.* **116**, 1322 (1959).

[21] R. Arnowitt, S. Deser and C.W. Misner, *Phys. Rev.* **117**, 1595 (1960).

[22] R. Arnowitt, S. Deser and C.W. Misner, in *Gravitation: An Introduction to Modern Research* (L. Witten, ed., Wiley, NY, 1962) also gr-qc 0405109.

[23] R.M. Wald, *General Relativity* (U. of Chicago Press, Chicago, 1971).

[24] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation* (Freeman Press, San Francisco, 1971).

[25] L.D. Faddeev, *Sov. Phys. Usp.* **25**, 130 (1982).

[26] L.D. Faddeev and R. Jackiw, *Phys. Rev. Lett.* **60**, 1692 (1988).

[27] P. Mukherjee and A. Saha, hep-th 0705.4358

[28] N. Kiriushcheva, S.V. Kuzmin and D.G.C. McKeon, *Mod. Phys. Lett.* **A20**, 1895 (2005).

[29] N. Kiriushcheva, S.V. Kuzmin and D.G.C. McKeon, *Mod. Phys. Lett.* **A20**, 1961 (2005).

[30] N. Kiriushcheva, S.V. Kuzmin and D.G.C. McKeon, *Int. J. Mod. Phys.* **A21**, 3401 (2006).

[31] N. Kiriushcheva and S.V. Kuzmin, *Ann. Phys.(NY)* **321**, 958 (2006).

[32] R.N. Ghalati, D.G.C. McKeon and T.N. Sherry, *Int. J. Mod. Phys.* **A22** 4833 (2007).

[33] C.G. Torre and I.M. Anderson, *Phys. Rev. Lett.* **70**, 3525 (1993).

[34] R.N. Ghalati gr-qc 0803.3651
[35] R.N. Ghalati and D. G. C. McKeon, gr-qc 07112543.
[36] S.V. Kuzmin and D.G.C. McKeon, Ann. Phys. (NY) 318, 495 (2005).
[37] J.Antonio-Garcia and J. M. Pons, Int. J. Mod. Phys. A12, 451 (1997).
[38] J. D. Brown and K. V. Kuchar, Phys. Rev. D 51, 5600 (1995).
[39] F. G. Markopoulou, Class. Quantum Grav. 13, 2577 (1996).
[40] N. Kiriushcheva, S.V. Kuzmin, C. Racknor and S.R. Valluri, to be published in Phys. Lett. A.
[41] L.D. Faddeev and V.N. Popov, Sov. Phys. Usp. 16, 777 (1975).
[42] B.S. DeWitt, Phys. Rev. 162, 1195 (1967).
[43] M. Henneaux, Phys. Rep. 126, 1 (1985).
[44] D.G.C. McKeon, Class. Quant. Grav. 23, 3037 (2006).
[45] N.S. Baaklini and M. Tuite, J. Phys. A1, L13 (1979).
[46] D.G.C. McKeon, Can. J. Phys. 57, 2096 (1979).
[47] A.F. Ferrari et al., Phys. Lett. B652, 174 (2007).
[48] T. Padmanabhan, gr-qc 0409089.
[49] M. Leclerc, gr-qc 0612125.
[50] M. Leclerc, Class. Quant. Grav. 24 4337 (2007).
[51] M. Leclerc, gr-qc 0703048.
[52] R.N. Ghalati, hep-th 0703268.
[53] K. Green, N. Kiriushcheva and S.V. Kuzmin, gr-qc 0710.1430.
[54] M. Fierz and W. Pauli, Proc. R. Soc. A73, 211 (1939).
[55] J. Schwinger, Particle, Sources and Fields (Addison-Wesley, Boston, 1970).
APPENDIX A: CANONICAL ANALYSIS OF THE SPIN-TWO FIELD IN FIRST ORDER FORMALISM

In this appendix we examine the canonical structure of linearized gravity in first order form using the Dirac constraint formalism. It differs in interesting ways from the structure of the full theory outlined in the body of this paper. Various aspects of this problem are considered in refs. [45, 46, 47, 48, 49, 50, 51, 52, 53].

In order to linearize the action of eq. (5), we merely replace it by \(\tilde{S}_d = \int d^d x \left[ h^{\mu\nu} G_{\mu\nu,\lambda} + \eta^{\mu\nu} \left( \frac{1}{d-1} G^\lambda_{\lambda\mu} G_{\sigma\nu}^\sigma - G^\lambda_{\sigma\mu} G_{\sigma\nu}^\lambda \right) \right] \) (A1)

where \(\eta^{\mu\nu} = \text{diag}(-, +, +, \ldots, +)\) is the flat space metric.

Eqs. (30, 31) can be used to solve the equations of motion of \(G_{\mu\nu}^\lambda\), expressing \(G_{\mu\nu}^\lambda\) in terms of \(h^{\mu\nu}\). Using this in order to eliminate \(G_{\mu\nu}^\lambda\) in eq. (A1), we find that \(\tilde{S}_d = \frac{1}{2} \int d^d x \left[ h^{\mu\nu} G_{\mu\nu,\lambda} + \frac{1}{2} h^{\mu\nu} h_{\mu\nu,\lambda} + \frac{1}{2(d-2)} h^{\mu\nu} h_{\mu\nu,\nu} \right] \) (A2) provided \(d \neq 2\). (This case will be dealt with presently.) If \(d = 4\), eq. (A2) is seen to be the action for a spin-two field appearing in refs. [54, 55].

The momentum conjugate to \(h^{00} = h_0\), \(h^{0i} = h_i\) and \(h^{ij}\) (upon integration by parts in the first term of eq. (A1)) are respectively

\[
\pi = -G_{00}^0, \quad \pi_i = -2G_{0i}^0, \quad \pi_{ij} = -G_{ij}^0. \quad (B3 - B5)
\]

If now we define

\[
\xi^k = G_{00}^k, \quad \xi^i_j = 2G_{ji}^0 = \tilde{\xi}^i_j + \frac{1}{d-1} \delta^i_j t, \quad \xi^i_{jk} = G_{jk}^i \quad (B6 - B8)
\]

where \(t = \xi^i_i\), then the canonical Hamiltonian is

\[
H = \pi h_{0} + \pi_i h^i_{0} + \pi_{ij} h^i_{j} - L \quad (A9)
\]

\[
= \frac{2 - d}{d - 1} \left( \pi^2 - \frac{1}{4} \pi_i \pi^i \right) + \xi^k (\pi_k + h_{k}) + \frac{t}{d - 1} (-\pi_{ii} - \pi + h^i_i) + \xi^i_{ji} h^j_{i} + \frac{1}{d - 1} \delta^i_j \pi_k + \xi^i_{jk} \xi^j_{ik} - \frac{1}{d - 1} \xi^i_{ik} \xi^j_{jk} \]

Many features of the Hamiltonian of eq. (A9) resemble those of eq. (23). In particular, the momenta associated with \(t\) and \(\xi^i\) vanish; these primary first class constraints result in the
secondary constraints

\[ \chi_k = h_k + \pi_k \quad (A10) \]

\[ \chi = h^k_k - \pi - \pi_{kk} . \quad (A11) \]

They have the PB

\[ \{ \chi, \chi \} = \{ \chi_i, \chi_j \} = \{ \chi, \chi_j \} = 0, \quad (A12) \]

in contrast to those of eqs. (26,27). Furthermore, the momenta conjugate to \( \bar{\zeta}_j^i \) and \( \xi_{jk}^i \) also vanish. These primary constraints are second class as they lead to second class secondary constraints, which are the equations of motion for \( \bar{\zeta}_j^i \) and \( \xi_{jk}^i \) and these variables enter the equations of motion linearly. Eliminating \( \bar{\zeta}_j^i \) and \( \xi_{jk}^i \) from the Hamiltonian of Eq. (B.9) using their equations of motion results in

\[
H = 2 - \frac{d}{d-1} \pi_i \pi_j + \frac{d-3}{4(d-2)} \pi_i \pi_j + \xi^k (\pi_k + h_k) - \frac{t}{d-1} (\pi_{ii} + \pi - h_{ii}) \quad (A13)
\]

\[
+ \left( \pi_{ij} \pi_{ij} - \frac{1}{d-1} \pi_{ii} \pi_{jj} - 2 \pi_{ij} h_{ij} + \frac{2}{d-1} \pi_{kk} h_{ij} + \frac{d-2}{d-1} h_{ij} h_{ij} \right)
\]

\[
- \left( \frac{1}{2(d-2)} h_{ij} \pi_j + \frac{1}{2} h_{ij} h_{ij} + \frac{1}{2} h_{ij} h_{ij} + \frac{1}{4(d-2)} h_{mn} h_{mn} - \frac{1}{4} h_{ij} h_{mn} \right) .
\]

One must now see if the secondary constraints of eqs. (A10-A11) imply any further constraints. As

\[
\{ H, \chi \} = \tau \quad (A14)
\]

\[
\{ H, \chi_k \} = 2 \left( \frac{d-2}{d-1} \chi_k - \tau_k \right) \quad (A15)
\]

there are \( d \) tertiary constraints

\[ \tau = h_{ij,ij} + \pi_{ii} \quad (A16) \]

\[ \tau_k = \pi_{ii,k} - \pi_{ik,i} \quad (A17) \]

Any pair of the constraints of eqs. (A10-A11, A16-A17) have vanishing PB and consequently all are first class. There are no fourth generation constraints as

\[ \{ \tau, H \} = 0 \quad (A18) \]

\[ \{ \tau_k, H \} = -\frac{1}{2} \tau_{,k} . \quad (A19) \]

It is now possible to find the gauge transformations implied by the constraints \( \chi, \chi_k, \tau, \) and \( \tau_k \) as well as the first class constraints \( \Pi \) and \( \Pi_k \), the momenta associated with \( t \) and
The algebra of constraints for this spin-two theory is quite simple in comparison to that of the full theory of general relativity, making application of refs. [9, 12] relatively easy. For this, we need eqs. (A12, A14, A15, A18, A19) as well as

\[ \{\Pi, H\} = -\chi, \quad \{\Pi_k, H\} = \frac{1}{d-1} \chi_k. \]  

\[ (B20 - B21) \]

This constraint structure is unusual in that derivatives of constraints appear in the PB algebra. A general analysis of the gauge transformations implied by the first class constraints in such cases appears in ref. [52].

The form of the generator of gauge transformations is given by

\[ G = \tilde{\mu}\Pi + \tilde{\mu}^k\Pi_k + \mu\chi + \mu^k\chi_k + \mu\tau + \mu^k\tau_k. \]  

\[ (A22) \]

Upon using the formulation of refs. [9, 12, 52] we find that this generator leaves the action of eq. (A1) invariant provided

\[ \dot{\mu} + \frac{1}{d-1} \tilde{\mu} + 2 \left( \frac{d-2}{d-1} \right) \mu_{k,k} = 0 \]  

\[ (A23) \]

\[ \dot{\mu}^k - \tilde{\mu}^k = 0 \]  

\[ (A24) \]

\[ \dot{\mu} - \mu + \frac{1}{2} \mu_{k,k} = 0 \]  

\[ (A25) \]

\[ \dot{\mu}_k + 2\mu_k = 0, \]  

\[ (A26) \]

so that \( G \) in eq. (A22) becomes

\[ G = \left[ -(d-1)\tilde{\mu} + \frac{1}{2} (d-3) \dot{\mu}_{k,k} \right] \Pi + \left[ -\frac{1}{2} \tilde{\mu}^k \right] \Pi_k + \left[ \dot{\mu} + \frac{1}{2} \mu_{k,k} \right] \chi \]  

\[ (A27) \]

\[ + \left[ -\frac{1}{2} \dot{\mu}_k \right] \chi_k + \mu\tau + \mu^k\tau_k. \]

If now \( \epsilon = \mu \) and \( \epsilon_k = \frac{1}{2} \mu_k \), then we find that

\[ \tilde{\delta}h = \{h, G\} = \dot{\epsilon} + \epsilon_k \]  

\[ (A28) \]

\[ \tilde{\delta}h_k = \{h_k, G\} = -\dot{\epsilon}_k - \epsilon_k \]  

\[ (A29) \]

\[ \tilde{\delta}h_{ij} = \{h_{ij}, G\} = (\dot{\epsilon} - \epsilon_{k,k})\delta_{ij} + \epsilon_{i,j} + \epsilon_{j,i}. \]  

\[ (A30) \]

This is consistent with

\[ \delta h^{\mu\nu} = \partial^\mu f^\nu + \partial^\nu f^\mu - \eta^{\mu\nu}\partial\cdot f \]  

\[ (A31) \]

which is the form of the gauge transformation for eq. (A2) discussed in ref. [46]. Eq. (A31) is in fact the linearized form of the diffeomorphism transformation. It remains to be seen if
the linearized form of the gauge transformation of the full action of eq. (5) implied by its first class constraints is given by eq. (A31).

In the case $d = 2$, the equation of motion for $G^\lambda_{\mu\nu}$ that follows from eq. (A1) cannot be solved to express $G^\lambda_{\mu\nu}$ in terms of $h_{\mu\nu}$. However, if we were to set

$$G^\lambda_{\mu\nu} = V^\lambda_{\mu\nu} + \bar{G}^\lambda_{\mu\nu} \quad (\bar{G}^\lambda_{\mu\nu} \eta^{\mu\nu} = 0)$$  \hspace{1cm} (A32)$$

then eq. (A1) when $d = 2$ becomes

$$\tilde{S}_2 = \int d^2 x \left[ - h^{\mu\nu}_{\lambda,\mu} V^\lambda_{\nu\mu} - h^{\mu\nu}_{\rho,\mu} \bar{G}^\lambda_{\rho,\nu} + (\bar{G}^\lambda_{\lambda\mu} \bar{G}^\sigma_{\sigma\nu} - \bar{G}^\lambda_{\sigma\mu} \bar{G}^\sigma_{\lambda\nu}) \eta^{\mu\nu} \right]$$  \hspace{1cm} (A33)$$

and it is possible to express $\bar{G}^\lambda_{\mu\nu}$, the traceless part of $G^\lambda_{\mu\nu}$, in terms of $h_{\mu\nu}$. If we take

$$\frac{\delta \bar{G}^\lambda_{\mu\nu}}{\delta G^\gamma_{\gamma\delta}} = \delta^\lambda_{\sigma} \left[ \frac{1}{2} (\delta^\gamma_{\mu} \delta^\lambda_{\nu} + \delta^\gamma_{\nu} \delta^\lambda_{\mu}) - \frac{1}{2} \eta^{\gamma\delta} \eta^{\mu\nu} \right]$$  \hspace{1cm} (A34)$$

then the equation of motion for $\bar{G}^\lambda_{\mu\nu}$ results in

$$\bar{G}^\lambda_{\mu\nu} = -\frac{1}{2} (h^\lambda_{\mu,\nu} + h^\lambda_{\nu,\mu}) + \frac{1}{4} (h^\rho_{\rho,\mu} \delta^\lambda_{\nu} + h^\rho_{\rho,\nu} \delta^\lambda_{\mu}) + \frac{1}{2} h^\lambda_{\mu,\nu} + \frac{1}{2} \eta_{\mu\nu} (h^\lambda_{\rho,\rho} - h^\rho_{\rho,\lambda})$$  \hspace{1cm} (A35)$$

If eq. (A35) is substituted back into eq. (A33), then the action $\tilde{S}_2$ collapses down to

$$S_2 = - \int d^2 x h^{\mu\nu}_{\lambda,\mu} V^\lambda_{\nu\mu} ,$$  \hspace{1cm} (A36)$$

showing the triviality of the spin-two field in two dimensions.

If we were to define

$$h = h^{00} , \quad h^1 = h^{01} , \quad \pi = -G^0_{00} , \quad \pi_1 = -G^0_{01} \quad (A37 - A40)$$

$$\pi_{11} = -G^0_{11} , \quad \xi = G^1_{00} , \quad \xi_1 = 2G^1_{01} , \quad \xi_{11} = G^1_{11} \quad (A41 - A44)$$

then eq. (A1) when $d = 2$ becomes

$$\tilde{S}_2 = \int d^2 x \left[ h_{00} \pi + h^0_{10} \pi_1 + h^0_{01} \pi_{11} - \xi (h_{11} + \pi_1) - \xi_1 (h^1_{11} - \pi - \pi_{11}) - \xi_{11} (h^1_{11} + \pi_1) \right] .$$  \hspace{1cm} (A45)$$

These secondary constraints are

$$\phi_1 = h_{11} + \pi_1 ,$$  \hspace{1cm} (A46)$$

$$\phi = h^1_{11} - \pi - \pi_{11} ,$$  \hspace{1cm} (A47)$$

$$\phi^1 = h^1_{11} + \pi_1 .$$  \hspace{1cm} (A48)$$

These are analogous to the secondary constraints that arise from the first order EH action in two dimensions. The PB of any two of these constraints vanishes.
APPENDIX B: INCLUSION OF SCALARS

We can supplement $S_d$ of eq. (1) with

$$S_\phi = \frac{1}{2} \int d^4x \sqrt{-g} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi. \quad (B1)$$

The primary and secondary constraints of sections (IIB) and (IIC) are not altered by the inclusion of $S_\phi$. However, as a result of this extra contribution to the action, the field $\phi$ has an associated momentum

$$p = h (\partial_0 \phi) + h^i (\partial_i \phi) \quad (B2)$$

which leads to a Hamiltonian density

$$\tilde{H}_\phi = \left[ \frac{p^2}{2h} + \frac{H^{ij} \partial_{i \phi} \partial_{j \phi}}{2} \right] - \frac{p h^i \partial_i \phi}{h} \quad (B3)$$

$$\equiv H_\phi - \frac{p h^i \partial_i \phi}{h}. \quad (B4)$$

Since

$$\{ \chi, \tilde{H}_\phi \} = \tilde{H}_\phi \quad (B5)$$

$$\{ \chi_i, \tilde{H}_\phi \} = -p \partial_i \phi \quad (B6)$$

where $\chi$ and $\chi_i$ are the secondary constraints of eqs. (24,25), the tertiary constraints of eqs. (38,39) become

$$T_i = \tau_i - p \partial_i \phi \quad (B7)$$

$$\tilde{T} = \tilde{\tau} + \tilde{H}_\phi. \quad (B8)$$

If we now set

$$T = \tau + H_\phi \quad (B9)$$

then

$$\tilde{T} - T - \frac{h^i}{h} T_i = \tilde{\tau} - \tau - \frac{h^i}{h} \tau_i. \quad (B10)$$
We now find that

\[ f \{ -p \partial_i \phi, -p \partial_j \phi \} g = g \partial_j f(-p \partial_i \phi) - f \partial_i g(-p \partial_j \phi) \quad (B11) \]

\[ f \{ H, H \} g = (g \partial_i f - f \partial_i g) \frac{H^i}{h}(-p \partial_j \phi) \quad (B12) \]

\[ \{ \chi, H \} = H \phi \quad (B13) \]

\[ \{ \chi, H \} = 0 \quad (B14) \]

\[ \{ \tau, H \} = \frac{1}{h}(H^m g H^n \omega_{kl} - H^m H^n \omega_{kl})(\partial_m \phi \partial_n \phi) \quad (B15) \]

\[ \{ \tau, -p \partial_j \phi \} = 0 \quad (B16) \]

\[ \{ \tau, -p \partial_j \phi \} = 0 \quad (B17) \]

and

\[ f \{ \tau_i - p \partial_i \phi, H \} g \quad (B18) \]

\[ = (fh)_i g + \frac{H^m}{2h} \phi_{,m \phi, n} g + \frac{1}{2} (f H^m)_{,i} \phi_{,m \phi, n} g + (pf)_{,i} p \frac{H^m \phi_{,m}}{h} g - \phi_{,i} (f H^m \phi_{,m} g)_{,n} \]

\[ = \left[ \frac{1}{h} (fh)_i g - fg_{,i} \right] \phi + \left( \frac{fg \phi^2}{2h} \right)_{,i} - (fg H^m \phi_{,m} \phi_{,n})_{,i} + \frac{1}{2} (fg H^m \phi_{,m} \phi_{,n})_{,i} \]

The total divergences appearing in eq. (B18) can be neglected. It now follows from eqs. (B11-B18) that the PBs of eqs. (58-64,66,68) (which arise when dealing with pure gravity defined by eq. (1)) can be modified to accommodate the scalar field by simply replacing \( \tau \) and \( \tau_i \) by \( T \) and \( T_i \) respectively. This result shows that the gauge transformation implied by the first class constraints in pure gravity and in pure gravity supplemented by a free scalar field are clearly related.