Modeling Convective Heat Transfer in Subduction Zone

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Abstract. The results of mathematical modeling convection of lithospheric plate in a subduction zone are given. Driven lithospheric plate collides with a continental plate and is immersed in mantle. The gravitational acceleration changed under the linear law. As a result of the numerical decision of a task the fields of temperature, velocity, stream function, vortex and local Nusselt number at the upper and lower border of calculating area were received.

1. Introduction
The paper considers a 2-D model of the continuous convection of the lithospheric plate close to oceanic trench with regard for the heat of the phase transition. The subduction zone is considered, where lithospheric plate collides with continental, and then on a trench, which axis is located under an angle \( \varphi \) to the land surface, is immersed in mantle. The depth of immersing of lithospheric plate in mantle is determined as a result of the task decision. It is accepted, that the gravitational acceleration on mantle depth varies under the linear law.

By development of mathematical model the following assumptions are accepted:
1. The substance, of which lithosphere and spreading it mantle consists, is considered as very high viscous Newtonian liquid.
2. Border between lithosphere and mantle is isotherm of solid phase temperature \( T_s \).
3. Thermal conductivity, density, viscosity and the heat flux of substance are determined depending on its temperature.

Relative quantity of a solid phase \( \gamma \) depending on temperature approximates by cubic spline. Boundary value for relative quantity of a solid phase is \( \gamma = 0.95 \). The zone of partial melting settles down, where \( \gamma < 0.95 \) and at \( \gamma > 0.95 \) substances are in a crystal condition.

2. Mathematical model
The subduction zone is considered, where lithospheric plate collides with continental, and then on a trough, which axis is located under an angle \( \varphi \) to the land surface, is immersed in mantle. The depth of immersing of lithospheric plate in mantle is determined as a result of the task decision. It is accepted, that the gravitational acceleration on mantle depth varies under the linear law \( g = Ay + B \).

Constructing the mathematical model, the following assumptions have been adopted: the lithospheric plate and the underlying mantle are considered as the non-compressible Newtonian liquid with a very high viscosity. The temperature at a boundary between the mantle and the plate is constant and equals to the temperature of solid state \( T_s \). The thermal conductivity \( \lambda \), the viscosity of the substance \( \mu \) and the heat flux \( q_v \) are determined with account of their temperature dependence:
1. \( \{\mu, \lambda, q_s\} = \begin{cases} \{\mu_1, \lambda_1, q_{s1}\}, & T \leq T_s, \\ \{\mu_2, \lambda_2, q_{s2}\}, & T > T_s. \end{cases} \)

Index 1 denotes the lithosphere parameters, 2 - the mantle parameters. The dependence of the density \( \rho \) on the temperature of the medium is assumed to be

\[
\rho = \begin{cases} \rho_1(1 - \beta T), & T \leq T_s, \\ \rho_2(1 - \beta T), & T > T_s \end{cases}
\]

2. Border between lithosphere and mantle is isothermal of solid phase with value of temperature \( T_s \).

3. The relative quantity of the solid phase \( \gamma \) (characterizing melting and crystallization of the substance) contained in the mantle or lithosphere is determined depending on the state of the substance (a solid state with a temperature \( T_s \) or a liquid state with a temperature \( T_L \)). The relative quantity of the solid phase \( \gamma \), depending on temperature, approximates by cubic spline

\[
\gamma = \begin{cases} 1, & T < T_L, \\ 2 \left( \frac{T - T_s}{T - T_L} \right)^3 - 3 \left( \frac{T - T_s}{T_L - T_s} \right)^2 + 1, & T_s \leq T \leq T_L, \\ 0, & T > T_L. \end{cases}
\]

A set of parameters for the lithosphere and the mantle used for the calculations are listed in the table.

Following the hypothesis propounded in 1960 by Hess, the lithospheric plate motions are going on in a rectilinear manner starting from the areas of formation of the plates up to the zones of under thrusting, which allows us to solve the problem in a plane vertical layer. It results from systematic investigations of the island arc and continental margin zones that the depth of subsidence of the under thrusting plate does not exceed 700 km. Therefore, the extent of the area calculated along the Y-axis is assumed to be 1000 km. According to the results [1], the extent of the calculated area along the X-axis is assumed to be 3000 km.

**Table 1. Thermophysical properties.**

| Physical properties | Lithosphere | Mantle |
|---------------------|-------------|--------|
| Viscosity \( \mu \) (Pa s) | \( 10^{22} \) | \( 10^{20} \) |
| Thermal conductivity \( \lambda \) (W/mK) | 3 | 5 |
| Density \( \rho \) (kg/m\(^3\)) | 3000 | 3300 |
| Heat flux \( q_v \) (W/m\(^3\)) | \( 5 \times 10^{-6} \) | \( 10^{-9} \) |
| Specific heat capacity \( C_p \) (J/kgK) | 1200 | 1200 |
| Volumetric expansion coefficient \( \beta \) (1/K) | \( 3 \times 10^{-7} \) | \( 3 \times 10^{-7} \) |
| Heat of melting \( L_f \) (J/kg) | \( 4 \times 10^5 \) | \( 4 \times 10^5 \) |

Figure 1 shows physical setting of a problem. The horizontal oceanic plate moves towards the continental plate with a constant velocity \( U_o \) and subsides in the asthenosphere in the trench zone at an angle \( \varphi \) to the land surface with the same velocity. In turn, the continental plate moves towards the oceanic plate with a velocity \( U_k \). Most often the lithospheric plates are subsiding in the subduction zones at an angle of 45°, though in some sectors of the island arcs the angles of subsidence from 30° to 90° have been marked. Here \( U_k \) - velocity of a continental plate, \( U_o \) - velocity of oceanic (lithospheric) plate; \( \varphi \) - angle of immersing of oceanic plate; \( \Gamma 1, \Gamma 2, \Gamma 3 \) - right, lower and left border of a trench, on which the plate is immersed, accordingly.
Location of trough, on which the plate is immersed, is considered equal \( Lx / 2 \). The lower border \( \Gamma 2 \), up to which the velocity of immersing of a plate is known, is determined on the interval of temperature melting. At achievement of melting temperature (liquid) \( T_L \) the lithospheric plate melts and depth of its immersing in mantle is determined.

The mathematical statement of a task describing convection in subduction zone includes the equations of movement of an incompressible liquid in the Boussinesq approximation, continuity and energy with account internal heat sources. The governing dimensionless equations can be written as:

\[
\begin{align*}
\frac{\partial \Psi}{\partial Y} \frac{\partial W}{\partial X} - \frac{\partial \Psi}{\partial X} \frac{\partial W}{\partial Y} & = \frac{\eta}{\phi} \frac{1}{\text{Re}} \frac{\partial \Delta W}{\partial X} + \gamma \frac{A V L}{\text{Re}^2} \frac{\partial \Theta}{\partial X}, \\
W & = -2 \frac{\gamma}{\eta} \left[ \frac{\partial^2 \Psi}{\partial Y^2} + \frac{\partial^2 \Psi}{\partial X^2} \right], \quad \frac{\partial \Psi}{\partial Y} \frac{\partial \Theta}{\partial X} - \frac{\partial \Psi}{\partial X} \frac{\partial \Theta}{\partial Y} = \frac{a}{\text{Pe}} \frac{\partial \Theta}{\partial Y} + \frac{Q}{\phi \text{Pe}}.
\end{align*}
\]

Where
\[
X = \frac{x}{L_y}, \quad Y = \frac{y}{L_y}, \quad U = \frac{u}{U_0}, \quad V = \frac{v}{U_0}, \quad \Theta = \frac{T - T_c}{T_h - T_c}, \quad \lambda = \frac{\lambda}{\lambda_2}, \quad \phi = \frac{\rho}{\rho_2}, \quad \eta = \frac{\mu}{\mu_2}, \\
Q = \frac{q_s L_y^2}{\lambda_2 (T_h - T_c)} , \quad W = \frac{W_L L_y}{U_0}, \quad \Psi = \frac{\Psi}{U_0 L_y}, \quad C = \frac{C_{ef}}{C_p}, \quad a = \frac{\Lambda}{\phi \cdot C_p}, \quad \Theta_s = \frac{T_s - T_c}{T_h - T_c}, \quad \Theta_L = \frac{T_L - T_c}{T_h - T_c}.
\]

Stefan, Reynolds, Grashof and Peclet numbers.

\[
\begin{align*}
C_{ef} & = \begin{cases} 
C_p, & T < T_s, T > T_L \\
C_p - L_F \frac{\partial \gamma}{\partial T}, & T_s \leq T \leq T_L 
\end{cases} \\
C & = \begin{cases} 
1, & \Theta < \Theta_s, \quad \Theta > \Theta_L, \\
1 - \frac{1}{\text{Ste}} \frac{\partial \gamma}{\partial \Theta}, & \Theta_s \leq \Theta \leq \Theta_L. 
\end{cases}
\end{align*}
\]

\[
\frac{\partial \gamma}{\partial \Theta} = \begin{cases} 
\frac{6(\Theta - \Theta_s)(\Theta - \Theta_L)}{(\Theta_L - \Theta_s)^3}, & \Theta_s \leq \Theta \leq \Theta_L, \\
0, & \Theta < \Theta_s, \quad \Theta > \Theta_L.
\end{cases}
\]

\[
\eta = \frac{\mu_i}{\mu_2} \left( 1 + (H - 1) \right) \left( 0.5 + \frac{1}{\pi} \tan \left( \frac{T_s - (T_h - T_c) \Theta - T_c}{(1 - Y) \cdot L_y} \right) \right),
\]

where \( H = [0;1] \) – the dimensionless mantle depth.
Next, we consider the boundary conditions, in dimensionless coordinates, the conditions are transformed to the form (figure 1)

\[
\Theta|_{Y=0} = 1, \quad \Theta|_{Y=1} = 0, \quad \frac{\partial \Theta}{\partial X}|_{X=0} = \frac{\partial \Theta}{\partial X}|_{X=Lx/Ly} = 0, \quad \frac{\partial \Theta_1}{\partial n}|_{\Gamma_{1,2,3}} = \frac{\lambda_1}{\lambda_2} \frac{\partial \Theta_1}{\partial n}|_{\Gamma_{1,2,3}};
\]

where \( \frac{\partial \Theta_k}{\partial n} = \frac{\partial \Theta_k}{\partial X} \cos \varphi + \frac{\partial \Theta_k}{\partial Y} \sin \varphi, \) \( (k = 1, 2, 3), \)

\[
\Theta_1|_{\Gamma_{1,3}} = \Theta_2|_{\Gamma_{1,3}}; \quad \Theta_1|_{\Gamma_{2,1}} = \Theta_2|_{\Gamma_{2,1}} = \Theta_L. \quad \frac{\partial \Psi}{\partial Y} = \frac{U_k}{U_0} \quad \text{at} \quad X \in \left[0, \frac{Lx - Ln}{2Ly}\right]; \quad Y = 1.
\]

\[
\frac{\partial \Psi}{\partial Y} = -1 \quad \text{at} \quad X \in \left[\frac{Lx - Ln}{2Ly}, \frac{Lx}{Ly}\right]; \quad Y = 1.
\]

\[
\frac{\partial \Psi}{\partial X} = 0, \quad \frac{\partial^2 \Psi}{\partial X^2} = 0 \quad \text{at} \quad X \in \left[0, \frac{Lx}{Ly}\right]; \quad \frac{\partial^2 \Psi}{\partial X \partial Y}|_{Y=1} = 0; \quad \Psi|_{Y=0} = 0.
\]

\[
\frac{\partial \Psi}{\partial X}|_{\Gamma_{1,1,3}} = -\cos(\varphi), \quad \frac{\partial \Psi}{\partial Y}|_{\Gamma_{1,1,3}} = \sin(\varphi) \quad \text{and inside a trench.}
\]

For intensity of a vortex \( W \) the boundary conditions were calculated with the use of boundary conditions for stream function. In accounts the non-uniform grid with a condensation in trench region and at the upper boundary of the area \( (Y = 1) \) was applied. Dimensionless local Nusselt numbers on top \( (Y = 1) \) and bottom \( (Y = 0) \) to border of calculated area were determined by the expression \( \frac{Nu}{Y=1,0} = -\frac{\partial \Theta}{\partial Y}|_{Y=1,0} \) (the derivative was calculated on three points with the second order of approximation). The problem was solved numerically using the Patankar method [2].

3. Results and conclusions
The values of temperature at upper and lower border are accepted equal \( T_c = 1400 \) K, \( T_h = 2400 \) K. The extent of area on an axis \( x \) is accepted equal 3000 km, and on an axis \( y \) - 1000 km. Lithosphere thickness \( Ln = 100 \) km. The Temperatures of solid state \( T_s \) and liquid state \( T_L \) are 1800 K and 1900 K accordingly. Velocity of movement of a continental plate \( U_k = 1 \) cm/year, and velocity of oceanic plate \( U_0 \) changed in an interval 1 - 9 cm/year. Angle of a feat under continental plate 45° is submitted.

In a figure 2 the results of calculation for velocity of movement oceanic plate 1 cm/year are submitted. \( Re = 1.05 \times 10^{19}; \quad Gr = 3.2 \times 10^{14}; \quad Pe = 251; \quad Ste = 3. \) Gravitational acceleration value \( g = 9.8 \) is constant. In trench region isothermals "bend" to lower boundary of the area (figure 2). Under oceanic and continental plates two large-scale convective cells are formed, the liquid in which is gone in opposite directions. And besides convective cell and vortex under continental plate flow round immersed lithosphere and penetrate into area of oceanic convective cell and vortex, located to the right of subduction zone, moving them to the right. The depth of immersing lithosphere \( h \) achieving the value of about 290 km. The distributions of Nusselt numbers on the upper and lower boundaries of the area are given. At value \( x = 1700 \) km (figure 2) in the collision region of plates takes place minimum heat flow on the upper border of calculated area that will be agreed with known experiments [1].
In Figure 2, calculated fields for $U_o = 1$. Temperature (a); stream function (b); vortex intensity (c); velocity vector field (d); Nusselt number distribution on upper and lower boundaries of the area (e, f).

In Figure 3 the results for velocity of movement oceanic plate $U_o = 5$ cm/year are given. The depth of immersing achieves value of about 450 km. The temperature field differs from result in figure 2. Oceanic convective cell has practically completely superseded continental in area to the left of an axis of a trench (figure 3). $Re = 5.25 \cdot 10^{-19}$; $Gr = 3.2 \cdot 10^{14}$; $Pe = 1255$; $Ste = 3$; $g = 9.8$.

In a figure 4 the results for velocity of movement of oceanic plate $U_o = 9$ cm/year are submitted. The depth of immersing is increased, achieving value of about 630 km. The temperature field has changed significantly in comparison with the previous result. Oceanic convective cell occupies practically all area (figure 4). $Re = 9.45 \cdot 10^{-19}$; $Pe = 2259$; $Ste = 3$; $g = 9.8$. 
Analyzing the results of modeling we can make the following conclusion:
- not taking into account of buoyancy forces conducts to significant changes in the structure of current;
- the account of gravitational acceleration from mantle depth practically does not influence at thermal and hydro dynamical processes in subduction zone in comparison with a case when gravitational acceleration was to constants.

Thus, the mathematical model suggested and the results obtained contribute to the available information on investigation of convection in the Earth’s interior and may be of use for better understanding and explanation of such phenomena as sea-floor spreading and subduction, lithospheric plate motion and heat flow change at Earth’s surface.

4. References
[1] Zharkov V N, Trubitsyn V P and Samsonenko L V 1971 Physics of the Earth and planets Figures and inner structure (Moscow: Nauka, Chief editorial board of phys.-math. lit.) p 384
[2] Patankar C 1980 Numerical heat transfer and fluid flow (Hemisphere Publishing Corporation, New York) p 152