Geometric modeling of solutions of the direct and inverse tasks of geometric optics on a plane

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Abstract. The paper presents solutions of the direct and inverse tasks of geometric optics on a plane: given a carrier of a bundle of rays and a reflector (mirror curve), the carrier of a bundle of reflected rays is determined; given carriers of bundles of rays, their common reflector is determined. The method of determining the reflector for pairs consisting of carriers of central, diffused or parallel bundles of lines modeling various emitters or receivers of illumination is based on optical properties of cyclographic projection of curved line.

1. Introduction
The laws of geometric optics of emission are applicable in the areas, where emission wavelength approaches zero. The laws of geometric optics might appear the only way of optical device development in case it is possible to neglect physical properties of the designed object. One of the major problems studied by geometric optics is the behavior of “emitter-reflector-emitter” systems, elements of which perform optical transformation of one bundle of rays into another [1].

The solutions of optical transformation tasks are urgent in such areas as radar, illumination engineering, optics, acoustics, laser technique, etc. [1–3]. The papers [1-3] come up with various methods and calculations of reflectors of various shape: parabolic, elliptic, complex curvilinear, compound, etc. Generally, the mentioned methods require solution of differential equations in order to acquire equations of reflector curve, which complicates solution of a task. Normally, solutions to such tasks encounter in scientific literature only for simpler cases. The cases where emitter, receiver or reflector are represented by complex shapes are given much less consideration. Therefore, the development of geometric models of solution of similar tasks featuring accessible computational algorithms is considered relevant.

2. Problem definition
The geometric model of a multitude of rays on a plane can be represented as a central, diffused (non-linear) or parallel bundle of straight lines. Some combinations of optical transformation of one of the mentioned bundles into another are thoroughly studied in scientific literature. Such combinations include transformations of a central bundle into a parallel bundle and of one central bundle into another central bundle. The reflectors acquired in such cases are parabola and ellipse respectively. The other cases, e.g. focusing the emission into a curve of certain geometry, are still either not adequately investigated, or not investigated at all; therefore necessity of solution of such tasks demands the development of approach, which would allow us to solve each possible combination of pairs of optical transformation: central, diffused and parallel bundles on a plane.

3. Theory
In the present paper the solution method of optical transformation of pairs of bundles of lines on a plane is based on cyclographic representation of a spatial curve on a plane. In cyclographic projection
a point of space is placed in correspondence with a certain projecting cone of revolution. The cone, as it intersects with projection plane $\Pi_1(xy)$, generates a circle in the area of the intersection. The circle serves as a base of the cone; it is directed and known as a “cycle” in cyclography. The direction of the mentioned circle depends on the position of the vertex of the cone, which is located on the initial spatial curve: if the vertex is positioned above projection plane $\Pi_1(xy)$, i.e. its coordinate $z$ is positive, then the direction of the circle is counter-clockwise, otherwise, if the coordinate $z$ is negative, then the direction of the circle is clockwise [6-8]. In classic cyclography the angle at the vertex of the cone between its axis and generatrix equals 45°, therefore, the height of the cone, i.e. coordinate $z$ of the point of curve, is equal to the radius of the cone base. Such cone is called $\alpha$-cone [9].

In order to acquire a cyclographic image of a spatial curve, it is required to construct a one-parameter set of $\alpha$-cones with vertexes belonging to the given curve and bases lying in projection plane $\Pi_1(xy)$, thus generating a one-parameter multitude of cycles (figure 1). The envelope of the multitude of cycles is a cyclographic projection of the spatial curve, while the orthogonal projection of the initial curve, the center line of the cycles, represents the reflector [10]. Envelopes $\overline{P}_1(t)$ and $\overline{P}_2(t)$, and center line $\overline{P}_1(t)$ generate a triad of curves possessing an optical capability. This capability is known in scientific literature [6] considering cyclographic $\alpha$-projection of a curve. Let us consider curves $\overline{P}_1(t)$, $\overline{P}_2(t)$, and $\overline{P}_1(t)$ as profiles of cylindrical surfaces $\Phi_1$, $\Phi_2$, and $\Phi_1$ correspondingly, projecting in relation to plane $\Pi_1(xy)$. Then a ray of light emitted normally from emitter surface in direction of, for example, $\Phi_1$, will be reflected from cylindrical surface $\Phi$ normally with respect to surface $\Phi_1$. Obviously, one of the curves of cyclographic projection of a spatial curve, for example, curve $\overline{P}_1(t)$, can perform the role of emitter, the other – curve $\overline{P}_2(t)$ – the role of receiver and vice versa, while the role of reflector can be performed by the orthogonal projection of the initial spatial curve, i.e. curve $\overline{P}_1(t)$ [11].

Let us put central, parallel and diffused bundles of lines in correspondence with spatial $\alpha$-images, i.e. linear $\alpha$-surfaces providing cyclographic representation. Central bundle of lines corresponds to any $\alpha$-cone, the vertex of which can be projected orthogonally on plane of projection $\Pi_1(xy)$ and has the same coordinates as those of the center of the bundle. The parallel bundle of lines corresponds to an $\alpha$-plane inclined in relation to the plane of projection on angle 45°; the trace of the $\alpha$-plane is a basic geometric object, which constitutes the carrier of said bundle. The diffused bundle of lines defined on a plane by a certain curve $n_0$, which constitutes the carrier of said bundle, corresponds to an $\alpha$-surface with generatrixes inclined on angle 45° to plane $\Pi_1(xy)$ (figure 7). Each generatrix passes through a
pair of corresponding points, one of which belongs to the curve \( n_0 \), while the other belongs to a spatial curve \( m \) with its orthogonal projection \( m_1 \), at that, the ordinate \( z \) of each point of line \( m \) represents curvature radius of curve \( n_0 \) in the corresponding point, while curve \( m_1 \) represents evolute of \( n_0 \).

In the process of the inverse task solution it is required to put each bundle of lines modeling the emitter and the receiver into correspondence with a spatial \( \alpha \)-image and acquire the curve of their intersection i.e. the curve of intersection of \( \alpha \)-surfaces. Orthogonal projection of the acquired curve on plane \( \Pi_1(xy) \) represents the sought reflector. The process of solution of the inverse task is considered in detail in paper [11]. Visualization of the solution is depicted on figure 2.

The solution to the direct task is different in the way that only a single object of “emitter-receiver” pair is given, however, the reflector is given as well. The given emitter (or receiver), on the analogy with the inverse task solution, is put into correspondence with its spatial \( \alpha \)-image, i.e. \( \alpha \)-surface, while the given reflector is put into correspondence with projecting cylindrical surface. A certain curve is acquired upon intersection of the mentioned surfaces. In general case, the cyclographic projection of the acquired curve constitutes two branches of the envelope of one-parameter set of cycles on projection plane \( \Pi_1(xy) \), one of them representing the emitter, the other representing the receiver.

Let’s consider the examples of direct task solution. Consider a source given as a central bundle of lines \( K_1(x_0, y_0) \) and a certain planar curve \( s_1 \) that represents the reflector of rays and redirects them to a certain unknown receiver (figure 3). It is required to acquire the shape of the receiver.

![Figure 2. Visualization of solution of the inverse task on the example of optical transformation of a central bundle into a parallel bundle of rays and vice versa.](image)

![Figure 3. The source of central bundle of lines \( K_1 \) and mirror line \( s_1 \) defined on a plane.](image)
The central bundle of lines is put into correspondence with a spatial $\alpha$-image — a certain $\alpha$-cone of revolution $\Psi$ with base of arbitrary radius $R$:

$$Z = R - \sqrt{(X - x_0)^2 + (Y - y_0)^2}$$

The mirror curve is put into correspondence with a cylindrical surface $\Phi$. Simultaneous solution of the equations of the conical and the cylindrical surfaces allows us to acquire the equation of spatial curve of their intersection $s$ (figure 4). Let us define the cyclographic projection of the acquired curve $s$ through the use of the known formulas [10]:

$$x_{s(1,2)} = x + z \frac{-x' \cdot z' \pm y' \cdot \sqrt{(x')^2 + (y')^2 - (z')^2}}{(x')^2 + (y')^2},$$

$$y_{s(1,2)} = y + z \frac{-y' \cdot z' \pm x' \cdot \sqrt{(x')^2 + (y')^2 - (z')^2}}{(x')^2 + (y')^2},$$

where $x', y', z'$ represent derivative functions of coordinates $x(t), y(t), z(t)$ with respect to parameter $t$.

The acquired lines $s_{11}$ and $s_{12}$ are represented on figure 4. The line $s_{11}$ matches the base of the cone, which represents an $\alpha$-image of the given central bundle of lines ($K_1$), while the line $s_{12}$ constitutes the sought receiver for the given initial data. Figure 5 represents the result of the task solution on a plane.

![Figure 4. Visualization of receiver (emitter) curve determination given a central bundle and a reflector.](image-url)
Figure 5. The solution of receiver (emitter) curve determination task on a plane.

Usage of spatial $\alpha$-images of bundles of lines on a plane in order to determine the line of receiver has certain advantages: upon variation of coordinate $z$ of vertex $K$ of the cone modeling the central bundle $(K,\Omega)$ and, correspondingly, its base (cycle) radius $R$, it is possible to acquire a one-parameter multitude of receiver curves for constant source $K_1$ and reflector $s_1$. Therefore, the solution of similar tasks presents an opportunity to optimize the position of the receiver in relation to the given reflector and emitter from technical requirements standpoint. Possible variation of coordinate $z$ of the cone vertex and corresponding variation of receiver curve position in spatial $\alpha$-images and on projection plane $\Pi_1(xy)$ are depicted on figure 6.

Figure 6. Variation of position of receiver curve $s_{12}$ in relation to variation of coordinate $z$ of the vertex of the cone modeling central line bundle (in units of length):

a) $z = 2$; b) $z = 6$; c) $z = 10$. 
Let us consider a different case. Consider a certain diffused bundle of lines given as a source and a certain planar curve given as a reflector. As mentioned earlier, a diffused bundle is modeled on a plane as an arbitrary carrier \( n_0 \), while its corresponding \( \alpha \)-image is a certain linear \( \alpha \)-surface. In order to define the equation of \( \alpha \)-surface it is required to construct an evolute \( m_1 \) to the given initial curve \( n_0 : x_n(t), y_n(t) \). The evolute is defined by formulas known in differential geometry [12]:

\[
\begin{align*}
x_m(t) &= x_n(t) + y'_n(t) \frac{(x'_n(t))^2 + (y'_n(t))^2}{x'_n(t) \cdot y'_n(t) - x'_n(t) \cdot y'_n(t)}, \\
y_m(t) &= y_n(t) + x'_n(t) \frac{(x'_n(t))^2 + (y'_n(t))^2}{x'_n(t) \cdot y'_n(t) - x'_n(t) \cdot y'_n(t)}.
\end{align*}
\]

The radius of curvature in any point of evolute allows us to recreate the spatial curve \( m \). In order to acquire the coordinate \( z \) of any point of the curve \( m \), let us engage the formula [9]:

\[
z_m(t) = \pm \sqrt{(x_n(t) - x_m(t))^2 + (y_n(t) - y_m(t))^2}.
\]

The equation of the curve \( n_0 \) and the acquired spatial curve \( m \) are then put into the equation of \( \alpha \)-surface \( \Omega \):

\[
\begin{align*}
X(t, l) &= x_m(t) + l \cdot [x_n(t) - x_m(t)], \\
Y(t, l) &= y_m(t) + l \cdot [y_n(t) - y_m(t)], \\
Z(t, l) &= z_m(t) - l \cdot (1 - l), T_0 \leq t \leq T, L_0 \leq l \leq L.
\end{align*}
\]

As in the previous task, the spatial \( \alpha \)-image corresponding to the reflector \( s_1 \) is a cylindrical surface \( \Phi \). Simultaneous solution of the equations of the linear and the cylindrical surfaces results in acquiring the equations of the curve \( s \) of their intersection. Then, on the basis of mentioned parametric equations, the cyclographic projection of the curve \( s \) is acquired, where one of the branches of the envelope matches the given source, i.e. \( s_{11} = n_0 \) while the other – the curve \( s_{12} \) – constitutes the sought receiver. Visualization of the solution is depicted on figure 7.

![Figure 7. Spatial visualization of receiver (emitter) curve determination given a diffused bundle \( n_0 \) and reflector \( s_1 \).](image)
In contrast to the previous example, in the current case it is not possible – due to unambiguity – to vary the position of the receiver in relation to the given emitter and reflector by similar means.

4. Results of experiments
Visualization of calculation results of the considered examples has been performed by means of computer-aided algebra. In order to demonstrate the solution of the direct task depicted on figure 2, 3, 4, a central bundle of lines centered at coordinates \( x_0 = 0 \); \( y_0 = 0 \) was introduced. The value of coordinate \( z \) and, correspondingly, radius \( R \) of cone base, which constitutes the spatial image of central bundle \( K_1 \), was specified equal to 10 units of length. A reflector was specified as a curve of the third order: \( x_1 = 3t^2 - 3t - 5; y_1 = 2t^3 - 6t^2 + 9t + 2; z = 0 \), where \( 0 \leq t \leq 1 \).

In order to perform visualization of calculation results of the other direct task depicted on figure 6, the diffused bundle and the reflector were specified by means of two parabolas (correspondingly):

\[
\begin{align*}
x_1 &= 1.5 - \frac{1}{4}t_1^2; y_1 = 1 + t_1; z_1 = 0, \quad -2 \leq t_1 \leq 0, \\
x_2 &= -\frac{1}{8}t_2^2; y_2 = 1 + t_2; z_2 = 0, \quad -2 \leq t_2 \leq 1.
\end{align*}
\]

5. Consideration of results
The capability to acquire mirror curves given emitter and receiver or to acquire emitter or receiver given a mirror curve and the other element of the pair by means of cyclographic representation method allows us to solve every possible combination of optical transformations of pairs of bundles of lines on a plane. The validity of the results is readily verified by solving known tasks, i.e. transformation of a central bundle into a parallel bundle.

6. Conclusion
In the present paper the cyclographic representation method of solution of geometric optics on a plane is considered. The possibility of solution of the direct task of optical transformation – acquiring receiver curve given emitter and reflector – is thoroughly illustrated. The given examples are reinforced by analytic computations and computer visualization.

The method of cyclographic modeling applied in solution of tasks of geometric optics on a plane allows us to solve any of the tasks of optical transformation of bundles of lines on a plane. The results of the present paper can find application in the areas of science and technique, where the modeled processes of certain emission are subject to the laws of geometric optics.

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