Research Article

Partially Observed Nonzero-Sum Differential Game of BSDEs with Delay and Applications

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A class of partially observed nonzero-sum differential games for backward stochastic differential equations with time delays is studied, in which both game system and cost functional involve the time delays of state variables and control variables under each participant with different observation equations. A necessary condition (maximum principle) for the Nash equilibrium point to this kind of partially observed game is established, and a sufficient condition (verification theorem) for the Nash equilibrium point is given. A partially observed linear quadratic game is taken as an example to illustrate the application of the maximum principle.

1. Introduction

Game theory has penetrated into many fields of economics and attracted more and more attention. A series of studies on game theory was given by [1–9]. There have been many papers about the differential games driven by backward stochastic differential equations (BSDEs), such as [10, 11]. The prospective progress of many systems relies as much on their past history as on their current state. The optimal control problem of stochastic systems with time delays is studied by [12–22]. The game problem of stochastic systems with the time-delayed generator is discussed by [23–25].

Nevertheless, in the aforementioned control and game problems, it is assumed that the information is fully observed. This does not make sense in real life. In general, only partial information is available for controllers in most cases. The latest research studies on the partially observed optimal control issues of stochastic differential systems were given by [26–32]. The partially observed game issues of stochastic systems were studied by [33–36].

As far as we know, the results in regard to partially observed differential games corresponding to backward stochastic systems with time delays (BSDDE) are few. This problem will be investigated in this paper. Comparing the above results, our work differs in several aspects. Firstly, we research such a kind of partially observed differential game problem corresponding to the BSDDE, which enriches the game theory of backward stochastic systems. Secondly, under the circumstance of different observation equations for each participant, our controlled systems and utility functions include the delays of state variables and control variables. Thirdly, we study a class of linear quadratic (LQ) game corresponding to backward stochastic systems with the time-delayed generator and give the specific expression of the Nash equilibrium point.

The outline of this article is as follows. We present the main hypotheses and the partially observed differential game problem of BSDDE in Section 2. In Section 3, we obtain the necessary optimality conditions of the partially observed game of BSDDE. Section 4 is devoted to the sufficient maximum principle. In Section 5, we take a partially observed LQ game as an example to illustrate the application of our maximum principle. Section 6 is the conclusion of this paper.
2. Statement of the Problems

Throughout our article, \((\Omega, \mathcal{F}, [\mathcal{F}_t]_{t \geq 0}, P)\) is a complete filtered probability space, on which three mutually independent one-dimensional standard Brownian motions \(W(t), Y_1(t), \text{ and } Y_2(t)\) are defined. Let \(\mathcal{F}_{t}^0, \mathcal{F}_{t}^1, \text{ and } \mathcal{F}_{t}^2\) be the natural filtrations generated by \(W(t), Y_1(t), \text{ and } Y_2(t)\), respectively. We set \(\mathcal{F}_t = \mathcal{F}_{t}^0 \otimes \mathcal{F}_{t}^1 \otimes \mathcal{F}_{t}^2\). For all \(t \in [-\delta, 0]\), \(\mathcal{F}_t \equiv \mathcal{F}_0^0\), which is the trivial \(\sigma\)-field, and \(\mathcal{F}_t = \mathcal{F}^1_t \otimes \mathcal{F}^2_t\). Set \(\mathcal{F}^i_t = \sigma(Y_i(s); 0 \leq s \leq t), (i = 1, 2)\) and \(\mathcal{F}^0_t = \mathcal{F}^1_t \otimes \mathcal{F}^2_t \neq \phi, \forall t \in [-\delta, 0]\). The finite time duration is defined by \(T > 0\), and the constant time delays are defined by \(0 < \delta, \delta_1, \delta_2 < T\), respectively. The expectation on \((\Omega, \mathcal{F}, P)\) is denoted by \(\mathbb{E}\), and the conditional expectation under \(\mathcal{F}_t\) is denoted by \(\mathbb{E}^t = \mathbb{E}[.]|\mathcal{F}_t\). In \(\mathbb{R}\) and \(\mathbb{R}^{m \times n}, \langle \cdot, \cdot \rangle\) is the usual inner product and \(|\cdot|\) is the Euclidean norm. The symbol “\(\tau\)” that appears in the superscript represents the transpose of the matrix. In this article, all of the equalities and inequalities are in the sense of \(dt \times d\mathbb{P}\) almost surely on \([0, T] \times \Omega\).

We introduce the following notations:

\[
L^2(\mathcal{F}_T; \mathbb{R}) = \left\{ \xi: \xi \text{ is an } \mathbb{R} - \text{valued}, \mathcal{F}_T \text{- measurable random variable satisfying } \mathbb{E}[|\xi|^2 < \infty] \right\},
\]

\[
L^2_2(s, r; \mathbb{R}) = \left\{ v(t), s \leq t \leq r: v(t) \text{ is an } \mathbb{R} - \text{valued, } \mathcal{F}_t \text{- adapted process satisfying } \mathbb{E} \int_s^T |v(t)|^2 \, dt < \infty \right\}. 
\]

Let the nonempty set \(U_i \in \mathbb{R} (i = 1, 2)\) be convex and the admissible control set be defined as the following:

\[
U_i[0, T] = \left\{ v_i: [0, T] \times \Omega \rightarrow U_i \left| v_i \text{ is } \mathcal{F}_i \text{- adapted, } \mathbb{E} \int_0^T |v_i(t)|^4 \, dt < \infty \right\}, \quad (i = 1, 2).
\]

Every element in \(U_i\) is known as an admissible control to Player \(i (i = 1, 2)\). And \(U_1 \times U_2\) is known as the admissible control set to the players.

This work pays attention to a kind of partially observed games of BSDE, which stems from some attractive financial scenarios. Now let us elaborate on the problem. Take into account the following BSDE:

\[
\begin{align*}
-\text{d}y^{\nu_1, \nu_2}(t) &= f(t, \Theta^{\nu_1, \nu_2}(t), y^{\nu_1, \nu_2}(t)) \, dt - z^{\nu_1, \nu_2}(t) \, dW(t), \quad t \in [0, T], \\
y^{\nu_1, \nu_2}(T) &= \xi, \quad y^{\nu_1, \nu_2}(t) = \psi_0(t), \quad t \in [-\delta, 0], \\
v_1(t) &= \psi_1(t), \quad t \in [-\delta, 0], \\
v_2(t) &= \psi_2(t), \quad t \in [-\delta, 0],
\end{align*}
\]

where

\[
\Theta^{\nu_1, \nu_2}(t) = (t, y^{\nu_1, \nu_2}(t), y^{\nu_1, \nu_2}(t - \delta), z^{\nu_1, \nu_2}(t), v_1(t), v_1(t - \delta), v_2(t), v_2(t - \delta)),
\]

\[
\begin{align*}
\text{d}Y_i(t) &= h_i(t, y^{\nu_1, \nu_2}(t), y^{\nu_1, \nu_2}(t - \delta), z^{\nu_1, \nu_2}(t), v_1(t), v_2(t)) \, dt + dW_i(t), \\
Y_i(0) &= 0, \quad (i = 1, 2),
\end{align*}
\]

where \(W_1(\cdot)\) and \(W_2(\cdot)\) are \(\mathbb{R}\)-valued stochastic processes depending on \(v_1(\cdot)\) and \(v_2(\cdot)\) and \(h_i: [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, i = 1, 2\), are continuous functions.

We assume

\(\text{(H1)}\) (i) \(f\) and \(h_i, i = 1, 2\), are continuously differentiable with respect to \((y, y_0, z, v_1, v_{1\delta}, v_2, v_{2\delta})\)

(ii) \(f_y, f_{y_0}, f_z, f_{v_1}, f_{v_1\delta}, f_{v_2}, f_{v_{2\delta}}, h_{1y}, h_{1z}, h_{1v_1}, h_{1v_{1\delta}}, h_{2y}, h_{2z}, h_{2v_2}, h_{2v_{2\delta}}\), \(h_{iv_i}, i = 1, 2\), are bounded by \(c > 0\).
Now, if (H1) is true and both \( v_1(\cdot) \) and \( v_2(\cdot) \) are admissible controls, then BSDDE (3) has a unique solution \((y^{v_1;v_2}(\cdot), z^{v_1;v_2}(\cdot)) \in L_2^\delta(\delta, T; \mathbb{R}) \times L_2^\delta(\delta, T; \mathbb{R}) \) (see [14]).

Define \(d\mathbb{P}^{v_1;v_2} = Z^{v_1;v_2}(t)d\mathbb{P} \), where

\[
Z^{v_1;v_2}(t) = \exp\left\{\sum_{j=1}^{2} \int_0^t h_j(s, y^{v_1;v_2}(s), y^{v_1;v_2}(s - \delta), z^{v_1;v_2}(s), v_1(s), v_2(s))dY_j(s) \right. \\
\left. - \frac{1}{2} \sum_{j=1}^{2} \int_0^t \left| h_j(s, y^{v_1;v_2}(s), y^{v_1;v_2}(s - \delta), z^{v_1;v_2}(s), v_1(s), v_2(s)) \right|^2 ds \right\}.
\]

Obviously, \(Z^{v_1;v_2}(t) \) satisfies the subsequent SDE:

\[
\begin{aligned}
dZ^{v_1;v_2}(t) &= \sum_{j=1}^{2} h_j(t, y^{v_1;v_2}(t), y^{v_1;v_2}(t - \delta), z^{v_1;v_2}(t), v_1(t), v_2(t))Z^{v_1;v_2}(t)dY_j(t), \\
Z^{v_1;v_2}(0) &= 1.
\end{aligned}
\]

Hence, by (H1) and Girsanov’s theorem, we obtain a three-dimensional Brownian motion \((W(\cdot), W_1(\cdot), W_2(\cdot)) \) built on the probability space \((\Omega, \mathcal{F}, \mathbb{P}^{v_1;v_2}) \), in which \(\mathbb{P}^{v_1;v_2} \) is a probability measure.

Making sure to accomplish the target \(\xi\), each player owns his individual interest, which is the cost functional as follows:

\[
J_i(v_1(\cdot), v_2(\cdot)) = \mathbb{E}^{v_1;v_2}\left\{\int_0^T l_i(\Theta^{v_1;v_2}(t))dt + \Phi_i(y^{v_1;v_2}(0))\right\}, \quad (i = 1, 2),
\]

where \(\mathbb{E}^{v_1;v_2} \) is the expectation on \((\Omega, \mathcal{F}, \mathbb{P}^{v_1;v_2}) \) and

\[
l_i: f: [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \\
\Phi_i: \mathbb{R} \rightarrow \mathbb{R}, \quad (i = 1, 2).
\]

We also assume for \(i = 1, 2\),

(H2) \(l_i \) are continuously differentiable with respect to \((y, y_\theta, z, v_1, v_1, v_2, v_2)\), and their partial derivatives

\[
J_i(v_1(\cdot), v_2(\cdot)) = \mathbb{E}\left\{\int_0^T Z^{v_1;v_2}(t)l_i(\Theta^{v_1;v_2}(t))dt + \Phi_i(y^{v_1;v_2}(0))\right\}, \quad (i = 1, 2).
\]

So, the original problem (10) is the same thing as minimizing (11) over \((v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2 \) subject to (3) and (7). For the sake of convenience, we refer to the above game problem as Problem (PONBZ). If an admissible control \(u(\cdot) = (u_1(\cdot), u_2(\cdot)) \) which satisfied (10) can be found, then it is called as an equilibrium point of Problem (PONBZ), and the corresponding state trajectory is denoted by \((y(\cdot), z(\cdot)) = (y^{u}(\cdot), z^{u}(\cdot)) \).

3. A Partially Observed Necessary Maximum Principle

In the case of a convex admissible control set, the convex perturbation method is the classical method to obtain the necessary optimality condition. Let the equilibrium point of Problem (PONBZ) be \(u(\cdot) = (u_1(\cdot), u_2(\cdot)) \), and the corresponding optimal trajectory is \((y(\cdot), z(\cdot)) \). Let \((\overline{v}_1(\cdot), \overline{v}_2(\cdot)) \) are continuous in \((y, y_\theta, z, v_1, v_1, v_2, v_2)\) and bounded by \(c(1 + |y| + |y_\theta| + |z| + |v_1| + |v_1| + |v_2| + |v_2|)\)

(ii) \(\Phi_i \) are continuously differentiable, and \(\Phi_i y \) are bounded by \(c(1 + |y|)\)

Assume that every player wants to minimize the cost functional \(J_i(v_1(\cdot), v_2(\cdot)) \) by picking the appropriate admissible control \(v_i(\cdot) \) \((i = 1, 2)\). Then, our partially observed nonzero-sum stochastic differential game problem is to find out a pair of admissible controls \((u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2 \) such that

\[
\begin{aligned}
J_1(u_1(\cdot), u_2(\cdot)) &= \min_{v_1(\cdot)\in\mathcal{U}_1} J_1(v_1(\cdot), u_2(\cdot)), \\
J_2(u_1(\cdot), u_2(\cdot)) &= \min_{v_2(\cdot)\in\mathcal{U}_2} J_2(u_1(\cdot), v_2(\cdot)).
\end{aligned}
\]

Obviously, cost functional (8) can be converted to
be such that \((u_t (\cdot), \overline{v}_1 (\cdot), u_2 (\cdot) + \overline{v}_2 (\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2\). Since \(\mathcal{U}_1\) and \(\mathcal{U}_2\) are convex, for any \(0 \leq p \leq 1\), \((u_t^p (\cdot), u_2^p (\cdot)) = (u_t (\cdot) + p \overline{v}_1 (\cdot), u_2 (\cdot) + p \overline{v}_2 (\cdot))\) is also in \(\mathcal{U}_1 \times \mathcal{U}_2\). For the controls \((u_1^p (\cdot), u_2 (\cdot))\) and \((u_1 (\cdot), u_2^p (\cdot))\), the corresponding state trajectories of game system (3) are denoted by \((y_t^u (\cdot), z_t^u (\cdot))\) and \((y_t^v (\cdot), z_t^v (\cdot))\).

We introduce the subsequent symbols:

\[
\begin{align*}
\varphi (t) &= \varphi (t, y(t), y(t - \delta), z(t), u_1(t), u_1(t - \delta), u_2(t), u_2(t - \delta)), \\
\varphi_t^v (t) &= \varphi (t, y(t), y(t - \delta), z(t), v_1(t), v_1(t - \delta), v_2(t), v_2(t - \delta)), \\
\varphi_t^{u_1, u_2} (t) &= \varphi (t, y(t), y(t - \delta), z(t), u_1^l (t), u_1^l (t - \delta), u_2(t), u_2(t - \delta)), \\
\varphi_t^{u_1, u_2^l} (t) &= \varphi (t, y(t), y(t - \delta), z(t), u_1(t), u_1(t - \delta), u_2^l (t), u_2^l (t - \delta)),
\end{align*}
\]

where \(\varphi\) denotes one of \(f, l, i, \) and

\[
\begin{align*}
h_t (t) &= h_t (t, y(t), y(t - \delta), z(t), u_1(t), u_2(t)), \\
h_t^v (t) &= h_t (t, y(t), y(t - \delta), z(t), v_1(t), v_2(t)), \\
h_t^{u_1, u_2} (t) &= h_t (t, y(t), y(t - \delta), z(t), u_1^l (t), u_2(t)), \\
h_t^{u_1, u_2^l} (t) &= h_t (t, y(t), y(t - \delta), z(t), u_1(t), u_2^l (t)),
\end{align*}
\]

\((i = 1, 2)\).

The variational equations are as follows:

\[
\begin{align*}
-& \frac{dy_t^1 (t)}{dt} = \left[ f_y (t) y_t^1 (t) + f_{y_1} (t) y_t^1 (t - \delta) + f_y (t) z_t^1 (t) + f_{y_1} (t) \overline{v}_1 (t) + f_{y_2} (t) \overline{v}_1 (t - \delta) \right] dt - z_t^1 (t) dW (t), \quad t \in [0, T], \\
y_t^1 (T) &= 0, \quad y_t^1 (t) = 0, \quad t \in [- \delta, 0], \\
\overline{v}_1 (t) &= 0, \quad t \in [- \delta_1, 0], \\
\overline{v}_2 (t) &= 0, \quad t \in [- \delta_2, 0], \quad (i = 1, 2),
\end{align*}
\]

\[
\begin{align*}
\frac{dZ_t^i (t)}{dt} &= \sum_{j=1}^{2} \left[ Z_t^1 (t) h_{j} (t) + Z(t) \left( h_{j y_1} (t) y_t^1 (t) + h_{y_2} (t) y_t^1 (t - \delta) + h_{j z} (t) z_t^1 (t) + h_{z_1} (t) \overline{v}_1 (t) \right) \right] dY_j (t), \\
Z_t^i (0) &= 0, \quad (i = 1, 2).
\end{align*}
\]

From (H1), it is easy to see that (14) and (15) admit unique solutions \((y_t^{u_1, u_2^l} (\cdot), z_t^{u_1, u_2^l} (\cdot)) \in L^2_T (-\delta, T; \mathbb{R}) \times L^2_T (-\delta, T; \mathbb{R})\) and \(Z^1 (t) \in L^2_T (0, T; \mathbb{R})\), respectively.

For \(t \in [0, T]\) and \(p > 0\), we set

\[
\begin{align*}
\overline{y}_t^p (t) &= \frac{y_t^p (t) - y (t)}{p} - y_t^1 (t), \\
\overline{z}_t^p (t) &= \frac{z_t^p (t) - z (t)}{p} - z_t^1 (t), \\
\overline{Z}_t^p (t) &= \frac{Z_t^p (t) - Z (t)}{p} - Z_t^1 (t), \quad (i = 1, 2),
\end{align*}
\]

Similar to the arguments in Lemmas 3.1 and 3.2 in [34], it is easy to obtain subsequent Lemmas 1 and 2. Thus, we omit the details for simplicity.
\[
\begin{align*}
\text{d}x_i(t) &= \sum_{j=1}^2 \left[ h_{ij}(t)x_j^i(t) + h_{ij}(t)y_j^i(t) + h_{ij}(t)\overline{y}_i(t) \right] \text{d}W_j(t), \\
\Gamma_i(0) &= 0, \quad (i = 1, 2).
\end{align*}
\] (19)

From this and Lemma 1, we have the following.

**Lemma 2.** Assume (H1) and (H2) are true. Then, we get the following variational inequality:

\[
\begin{align*}
\mathbb{E}^{\mu_i, u_i} \left[ \int_0^T \left[ \frac{\partial h_{ij}(t)}{\partial y_j^i(t)} (t) + l_{ij}(t) y_j^i(t) - \frac{\partial^2 h_{ij}(t)}{\partial t^2} (t) \left( t - \delta \right) + l_{ij}(t) z_j^i(t) + l_{ij}(t) \overline{y}_i(t) \right] \text{d}t \right] + \mathbb{E}^{\mu_i, u_i} \left[ \Phi_{ij}(y(0))y_j^i(0) \right] &\geq 0, \quad (i = 1, 2).
\end{align*}
\] (20)

Our Hamiltonian function \( H_i \): \([0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), \( i = 1, 2 \), is defined as follows:

\[
\begin{align*}
H_i(t, y, y_\delta, z, v_1, v_{1\delta}, v_2, v_{2\delta}, p_1, Q_{11}, Q_{22}) &= -\langle p_1(t), f(t, y, y_\delta, z, v_1, v_{1\delta}, v_2, v_{2\delta}) \rangle \\
&+ \sum_{j=1}^2 \langle Q_{ij}(t), h_j(t, y, y_\delta, z, v_1, v_{1\delta}, v_2, v_{2\delta}) \rangle + l_i(t, y, y_\delta, z, v_1, v_{1\delta}, v_2, v_{2\delta}), \quad (i = 1, 2).
\end{align*}
\] (21)

Denote \( H_i(t) \equiv H_i(t, y, y_\delta, z, v_1, v_{1\delta}, v_2, v_{2\delta}, p_1, Q_{11}, Q_{22}) \) and its derivatives.

We note that the adjoint equation to (19) is a BSDE, whose solution is \( (P_i(\cdot), Q_{ii}(\cdot), Q_{12}(\cdot)) \):

\[
\begin{align*}
-\text{d}P_i(t) &= l_i(t) \text{d}t - \sum_{j=1}^2 Q_{ji}(t) \text{d}W_j(t), \\
P_i(T) &= 0, \quad (i = 1, 2),
\end{align*}
\] (22)

and the adjoint equation to (14) is an SDE, whose solution is \( p_i(\cdot) \):

\[
\begin{align*}
\text{d}p_i(t) &= -\left[ H_{ij}(t) + \mathbb{E}^{\nu_i, \xi_i} \left[ H_{ij}(t + \delta) \right] \right] \text{d}t - H_{iz}(t) \text{d}W_i(t), \\
p_i(0) &= -\Phi_{ij}(y(0)), \quad (i = 1, 2).
\end{align*}
\] (23)

**Remark 1.** It is easy to see that equation (23) is a linear anticipated SDE. Under (H1) and (H2), the unique solvability of equation (23) is assured by Theorem 2.2 in [14].

Based on variational inequality (20), we set out the main result of this section.

**Theorem 1** (partially observed necessary maximum principle). Assume (H1) and (H2) are true, an equilibrium point of Problem (POBNZ) is \( (u_1(\cdot), u_2(\cdot)) \), the optimal trajectory is \( (y(\cdot), z(\cdot)) \), and the solution of (7) is \( Z(\cdot) \). Let \( (P_i(\cdot), Q_{ii}(\cdot), Q_{12}(\cdot)), i = 1, 2 \), be the solution of (22) and \( p_i(\cdot) \) be the solution of adjoint equation (23). Then, the following maximum principle

\[
\begin{align*}
\mathbb{E}^{\mu_i, u_i} \left[ \left( H_{1v_1}(t) + \mathbb{E}^{\nu_i, \xi_i} \left[ H_{1v_1}(t + \delta) \right] \right) (v_1 - u_1(t)) \right] \geq 0, \\
\mathbb{E}^{\mu_i, u_i} \left[ \left( H_{2v_2}(t) + \mathbb{E}^{\nu_i, \xi_i} \left[ H_{2v_2}(t + \delta) \right] \right) (v_2 - u_2(t)) \right] \geq 0,
\end{align*}
\] (24) (25)

for any \( (v_1, v_2) \in U_1 \times U_2, \ a.e., \ t \in [0, T] \), in which the Hamiltonian function \( H \) is defined as (21).

**Proof.** For \( i = 1 \), using Itô’s formula to \( \langle y_1^i(t), p_1(t) \rangle \) + \( \langle \Gamma(t), p_1(t) \rangle \), from variational equations (14) and (15), variational inequality (20), and adjoint equations (22) and (23), we obtain
\[ E^{u_i, u_j} \int_0^T \left[ l_1(t) \Gamma_1(t) + l_{1y_i}(t) y_i^1(t) + l_{1y_j}(t) y_j^1(t) + l_{1z}(t) z_1(t) + l_{1v_i}(t) v_i(t) + l_{1v_j}(t) v_j(t) - (t - \delta) \right] dt \\
+ E^{u_i, u_j} \left[ \Phi_{1y}(y(0)) y_i^1(0) \right] \\
= E^{u_i, u_j} \int_0^T \left\langle f_{y_i}(t) p_i(t) + \sum_{j=1}^2 h_{y_{y_j}}(t) Q_{y_j}(t) + I_{1v_i}(t) \right. \\
+ E^{u_i, u_j} \left[ f_{y_j}^\top(t + \delta) p_j(t + \delta) + \sum_{j=1}^2 h_{y_{y_j}}^\top(t + \delta) Q_{y_j}(t + \delta) + I_{1v_i}(t + \delta) \right| F_1^i, v_i(t) \rangle dt \\
\geq 0. \] (26)

Because \( v_i(t) \) satisfies \( u_i(t) + v_i(t) \in U_1 \), we have
\[ E^{u_i, u_j} \int_0^T \left\langle H_{1y_i}(t) + E^{u_i, u_j} \left[ H_{1y_i}(t + \delta) \right] F_1^i, v_i \right. \\
- u_i(t) \rangle dt \geq 0, \quad \forall v_i \in U_1. \] (27)

This implies that
\[ E^{u_i, u_j} \left[ H_{1y_i}(t) + E^{u_i, u_j} \left[ H_{1y_i}(t + \delta) \right] F_1^i, v_i - u_i(t) \right] \geq 0, \quad \forall v_i \in U_1. \] (28)

Now, assume that \( F \) is an arbitrary element of \( \sigma \)-algebra \( F_1^i \) and \( v_i(t) \in U_1 \) is a deterministic element. Let

\[ E^{u_i, u_j} \left\langle H_{1y_i}(t) + E^{u_i, u_j} \left[ H_{1y_i}(t + \delta) \right] F_1^i, v_i - u_i(t) \right\rangle \left| F_1^i \right| \geq 0, \quad \forall v_i \in U_1, \text{ a.e.t.} \in [0, T], \text{ a.s.} \] (31)

Similar to the aforementioned method, we can get the other inequality for any \( v_2 \in U_2 \). The proof of Theorem 1 is completed. \( \square \)

4. A Partially Observed Sufficient Maximum Principle

In this section, we explore a sufficient maximum principle to Problem (POBNZ). Let \( (y(t), z(t), u_1(t), u_2(t)) \) be a quintuple that satisfies (3), and assume that there is a solution \( p_i(t) \) corresponding to adjoint SDE (23). We assume the following:

(\text{H3}) For \( i = 1, 2 \), for all \( t \in [0, T] \), \( H_i(t, \cdot, y, \cdot, y, \cdot, y, \cdot, v_i) \) is convex in \( (y, y_3, z, v_i, v_{1\delta}, v_2, v_{2\delta}) \), and \( \Phi_i(y) \) is convex in \( y \)

For \( i = 1, 2 \), let

\[ H_i(t) = H_i(t, y(t), y(t) - \delta, z(t), u_1(t), u_1(t) - \delta, u_2(t), u_2(t) - \delta, p_i(t), Q_{1i}(t), Q_{2i}(t)), \]
\[ H_{iY_i}(t) = H_i(t, y(t), y(t) - \delta, z(t), v_i(t), v_i(t) - \delta_1, u_2(t), u_2(t) - \delta_2, p_i(t), Q_{1i}(t), Q_{2i}(t)), \]
\[ H_{iY_j}(t) = H_i(t, y(t), y(t) - \delta, z(t), u_1(t), u_1(t) - \delta_1, v_2(t), v_2(t) - \delta_2, p_i(t), Q_{1i}(t), Q_{2i}(t)), \]
\[ h_i(t) = h_i(t, y(t), y(t) - \delta, z(t), u_1(t), u_2(t)), \]
\[ h_{iY_i}(t) = h_i(t, y(t), y(t) - \delta, z(t), v_i(t), u_2(t)), \]
\[ h_{iY_j}(t) = h_i(t, y(t), y(t) - \delta, z(t), u_1(t), v_2(t)), \]
\[ \varphi(t) = \varphi(t, y(t), y_3(t), z(t), v_i(t), u_1(t), u_1(t) - \delta_1, u_2(t), u_2(t) - \delta_2)), \]
\[ \varphi_{v_i}(t, \cdot) = \varphi(t, y(t), y_3(t), z(t), v_i(t), v_i(t) - \delta_1, u_2(t), u_2(t) - \delta_2)), \]
\[ \varphi_{v_2}(t, \cdot) = \varphi(t, y(t), y_3(t), z(t), u_1(t), u_1(t) - \delta_1, v_2(t), v_2(t) - \delta_2)), \] (32)
where \( \varphi = f, i = 1, 2 \).

**Theorem 2** (partially observed sufficient maximum principle). Assume (H1)–(H3) are true. Moreover, the maximum conditions (24) and (25) of the partial observation are true; then, \((u_1 (\cdot), u_2 (\cdot))\) is the equilibrium point to Problem (POBNZ).

**Proof.** For any \( v_1 (\cdot) \in U_1 \), we consider

\[
I_1 (v_1 (\cdot), u_2 (\cdot)) - J_1 (u_1 (\cdot), u_2 (\cdot))
= \mathbb{E} \int_0^T l_1 (t) [Z_i^n (t) - Z (t)] dt + \mathbb{E} [\Phi_1 (y^n_i (0)) - \Phi_1 (y (0))] + \mathbb{E}^{\nu_2} \int_0^T [l_i^n (t) - l_1 (t)] dt
= I_1 + I_2 + I_3.
\]

Using Itô's formula to \( \langle P_1 (t), Z_i^n (t) - Z (t) \rangle \) on \([0, T] \), we deduce

\[
I_1 = \mathbb{E}^{\nu_2} \int_0^T \sum_{j=1}^2 Q_{ij} (t) [\hat{h}_j^n (t) - \hat{h}_j (t)] dt.
\] (34)

Then, we have

\[
I_2 \geq \mathbb{E}^{\nu_2} \langle H_{1y} (y (0)), y^n_i (0) - y (0) \rangle
= -\mathbb{E}^{\nu_2} \int_0^T \langle y^n_i (t) - y (t), H_{1y} (t) + \mathbb{E}^{\nu_2} [H_{1y} (t + \delta)] \rangle dt
- \mathbb{E}^{\nu_2} \int_0^T \langle z^n_i (t) - z (t), H_{1z} (t) \rangle dt + \mathbb{E}^{\nu_2} \int_0^T \langle P_1 (t), f^n_i (t) - f (t) \rangle dt.
\] (35)

By the virtue of convexity of \( H_1 \) to \((y, y_\delta, z, v_1, v_{1\delta}, v_2, v_{2\delta})\), we deduce

\[
H_{i}^n (t) - H_1 (t) \geq \langle y^n_i (t) - y (t), H_{1y} (t) \rangle + \langle y^n_\delta (t) - y_\delta (t), H_{1y} (t) \rangle + \langle z^n_i (t) - z (t), H_{1z} (t) \rangle + \langle v_1 (t) - u_1 (t), H_{1v_1} (t) \rangle + \langle v_{1\delta} (t) - u_{1\delta} (t), H_{1v_{1\delta}} (t) \rangle.
\] (37)

Notice the truth that

\[
\mathbb{E} \int_0^T \langle y^n_\delta (t) - y_\delta (t), H_{1y} (t) \rangle dt - \mathbb{E} \int_0^T \langle y^n_i (t) - y (t), H_{1y} (t + \delta) \rangle dt
= \mathbb{E} \int_0^T \langle y^n_i (t) - y_\delta (t), H_{1y} (t) \rangle dt - \mathbb{E} \int_0^{T+\delta} \langle y^n_\delta (t) - y_\delta (t), H_{1y} (t) \rangle dt
= \mathbb{E} \int_0^{T+\delta} \langle y^n_\delta (t) - y_\delta (t), H_{1y} (t) \rangle dt - \mathbb{E} \int_0^T \langle y^n_i (t) - y_\delta (t), H_{1y} (t) \rangle dt
= 0.
\] (38)
Then, we get
\[
J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \geq \mathbb{E} \int_0^T \langle H_{1v_1}(t) + \mathbb{E}F_i[H_{1u_2}(t + \delta)] \rangle, v_1(t) - u_1(t) \rangle \, dt.
\]
(39)

Finally, by necessary optimality conditions (24), we obtain
\[
J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \geq 0.
\]
Then, it implies
\[
J_1(u_1(\cdot), u_2(\cdot)) = \min_{v_1(\cdot) \in \mathcal{W}_1} J_1(v_1(\cdot), u_2(\cdot)).
\]
(41)

In the same way,
\[
J_2(u_1(\cdot), u_2(\cdot)) = \min_{v_2(\cdot) \in \mathcal{W}_2} J_2(u_1(\cdot), v_2(\cdot)).
\]
(42)

So, we come to the expected conclusion. The proof is completed.

5. Application

In this section, we construct a partially observed LQ differential game with regard to backward stochastic systems with time delays. Using the classical filtering theory and the aforementioned theoretical results, we attempt to give a specific expression of the Nash equilibrium point. Let us think about the subsequent linear BSDDE:

\[
\begin{align*}
\dot{y}^{v_1, v_2}(t) &= (A(t)y^{v_1, v_2}(t) + \bar{A}(t)y^{v_1, v_2}(t - \delta) + B(t)z^{v_1, v_2}(t) + C_1(t)v_1(t) + C_2(t)v_2(t))dt - z^{v_1, v_2}(t)dW(t), \quad t \in [0, T], \\
y^{v_1, v_2}(T) &= \xi, \quad z^{v_1, v_2}(t) = \psi(t), \quad t \in [-\delta, 0],
\end{align*}
\]

and the observation
\[
\begin{align*}
\dot{Y}_i(t) &= D_i(t)dt + dW_i(t), \\
Y_i^{v_1, v_2}(0) &= 0, \quad (i = 1, 2).
\end{align*}
\]
(44)

We introduce the following cost functional:
\[
J_i(v_1(\cdot), v_2(\cdot)) = \frac{1}{2} \mathbb{E} \int_0^T M_i(t)(y_i(t))^2 \, dt + N_i(y_i(0))^2, \quad (i = 1, 2),
\]
(45)

where constant \( N_i \geq 0 \), functions \( A(\cdot), \bar{A}(\cdot), B(\cdot), C_i(\cdot), D_i(\cdot), M_i(\cdot), i = 1, 2 \), are deterministic and bounded, and \( M_i(\cdot) \) is bounded. Our partially observed nonzero-sum LQ differential game is to find out a pair of admissible controls \((u_1(\cdot), u_2(\cdot)) \in \mathcal{W}_1 \times \mathcal{W}_2\) satisfying

\[
\begin{align*}
-\dot{y}^{v_1, v_2}(t) &= (A(t)y^{v_1, v_2}(t) + \bar{A}(t)y^{v_1, v_2}(t - \delta) + B(t)z^{v_1, v_2}(t) + C_1(t)M_1(t)\xi)\mathbb{E}[F_i[p_1(t)\mathcal{F}_t]] - z^{v_1, v_2}(t)dW(t), \quad t \in [0, T], \\
\dot{p}_i(t) &= [A^T(t)p_i(t) + \mathbb{E}F_i[A^T(t)p_{i\delta}(t)]]\, dt + B^T(t)p_i(t)dW(t), \quad t \in [0, T], \quad (i = 1, 2), \\
y^{v_1, v_2}(T) &= \bar{\xi}, \quad z^{v_1, v_2}(t) = \psi(t), \quad t \in [-\delta, 0], \\
p_i(0) &= N_iy(0), \quad p_i(t) = 0, \quad t \in [T, T + \delta], \quad (i = 1, 2).
\end{align*}
\]
(46)

6. Conclusion

In this research, we have explored a class of partially observed game problem of the backward stochastic system with delay. More specially, based on the convex variational method, we establish the necessary and sufficient conditions with regard to Nash equilibrium in our game issue. The theoretical results of this paper are applied to an LQ game, for which the unique equilibrium point is expressed explicitly. On account of that the LQ model is usually used to depict many financial and economic phenomena, we expect that our LQ game result of BSDDEs can be widely used in these fields. As far as we know, the partially observed nonzero-sum backward game problem with the time-delay generator is firstly investigated in our paper.
Notwithstanding that we are committed to the above game problem, we are likewise able to progress some consequences of optimal control for BSDDEs, for example, [14, 34].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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