THE FUNDAMENTAL LEPAGE FORM IN TWO INDEPENDENT VARIABLES: A GENERALIZATION USING ORDER-REDUCIBILITY

ZBYNĚK URBAN AND JANA VOLNÁ

Abstract. A second-order generalization of the fundamental Lepage form of geometric calculus of variations over fibered manifolds with 2-dimensional base is described by means of insisting on (i) equivalence relation “Lepage differential 2-form is closed if and only if the associated Lagrangian is trivial” and, (ii) the principal component of Lepage form, extending the well-known Poincaré–Cartan form, preserves order prescribed by a given Lagrangian. This approach completes several attempts of finding a Lepage equivalent of a second-order Lagrangian possessing condition (i), which is well-known for first-order Lagrangians in field theory due to Krupka and Betounes.

1. Introduction

Lepage forms play a basic role in the calculus of variations of both simple- and multiple-integral problems over fibered manifolds and Grassmann fibrations. Among the well-known examples of Lepage forms we mention namely: the Cartan form of classical mechanics and its generalization in higher-order mechanics (Krupka [11]); in first-order field theory, the Poincaré–Cartan form (García [5]), the Carathéodory form (Carathéodory [2]), and the fundamental Lepage form (also known as the Krupka–Betounes form) (Krupka [12], Betounes [1]); in second-order field theory, the generalized Poincaré–Cartan form (Krupka [13]), the generalized Carathéodory form (Crampin and Saunders [3], Urban and Volná [26]), the fundamental Lepage form for second-order, homogeneous Lagrangians (Saunders and Crampin [22]). See also Gotay [8], Goldschmidt and Sternberg [4], Rund [21], Dedecker [4], Horák and Kolár [9], Krupka [13], Krupka and Stěpánková [14], Saunders [22], Sniatycki [24]. Further recent attempts of generalization and study of the fundamental Lepage equivalent for first- and second-order Lagrangians include also [19, 20, 25]. For a review of basic properties and results, see Krupka, Krupková, and Saunders [17].

Replacing the initial Lagrangian by its Lepage equivalent, the corresponding variational functional is preserved, and in addition the basic variational properties as variations, extremals, or conservation laws can be formulated and studied using geometric operations (such as the exterior derivative, the Lie derivative) acting on the corresponding Lepage equivalent of a Lagrangian.
Our aim in this note is to study a generalization of the fundamental Lepage equivalent \( Z_\lambda \) of a second-order Lagrangian \( \lambda \). This Lepage equivalent was introduced for first-order Lagrangians in variational theory over fibered manifolds with an \( n \)-dimensional base (see [12] [1]), and it obeys the following crucial property:

\[
\omega = \text{dim} \quad \lambda \quad \text{if and only if} \quad E_{\lambda} = 0,
\]

that is, the Lepage equivalent of a Lagrangian is closed if and only if the Lagrangian is trivial (i.e., the corresponding Euler–Lagrange expressions vanish identically).

For 2-dimensional base (i.e., two independent variables), we show that a Lepage equivalent \( Z_\lambda \) of a second-order Lagrangian \( \lambda \), obeying the aforementioned equivalence property, does not exist in general when \( Z_\lambda = \Theta_\lambda \) plus a 2-contact part, where \( \Theta_\lambda \) is the principal component (Lepage form) of \( Z_\lambda \). Nevertheless, we describe here the fundamental Lepage form, associated with a second-order Lagrangian which assures that the principal component of a Lepage form (the generalized Poincaré–Cartan form) has the same order as the initial Lagrangian. This order reducibility assumption is also motivated by the first-order theory and include, for instance, important class of Lagrangians linear in second derivatives.

Recent studies on the fundamental Lepage form include namely Saunders and Cramerp [28] (for two independent variables, generalization of the fundamental form is given for higher-order, homogeneous Lagrangians on tangent bundles), and Pales, Rossi, and Zanello [19] (on the basis of integration by parts, possible generalization of the fundamental form for a second-order Lagrangian is discussed while, however, differs from our result for \( n = 2 \)).

Basic underlying geometric structures, well adapted to the present paper, can be found in book chapters Volda and Urbana [28], and Krupka [16]. Throughout, we use the standard geometric concepts: the exterior derivative \( d \), the contraction \( \iota_{\xi}\rho \) of a differential form \( \rho \) with respect to a vector field \( \xi \), and the pull-back operation \( \star \) acting on differential forms.

If \( (U, \varphi) \), \( \varphi = (x^r) \), is a chart on smooth manifold \( X \), the local volume element is denoted by \( \omega_0 = dx^1 \wedge \ldots \wedge dx^n \), and we put

\[
\omega_j = \iota_{\partial_\gamma / \partial x^0} \omega_0 = \frac{1}{(n-1)!} \varepsilon_{i_1 i_2 \ldots i_n} dx^{i_1} \wedge \ldots \wedge dx^{i_n},
\]

where \( \varepsilon_{i_1 i_2 \ldots i_n} \) is the Levi-Civita permutation symbol. We denote by \( Y \) a fibered manifold of dimension \( n + m \) over an \( n \)-dimensional base manifold \( X \) with projection \( \pi : Y \to X \) the surjective submersion. \( J^r \) denotes the \( r \)-th order jet prolongation of \( Y \) whose elements are \( r \)-jets \( J^r \gamma \) of sections \( \gamma \) of \( \pi \) with source at \( x \in X \) and target at \( \gamma(x) \in Y \). The canonical jet bundle projection is denoted by \( \pi^{r,0} : J^r \to J^{r,0} \). Every fibered chart \( (V, \psi) \), \( \psi = (x^i, y^\sigma) \), \( 1 \leq i \leq n, 1 \leq \sigma \leq m \) on \( Y \) induces the associated chart \( (U, \varphi) = (x^i) \) on \( X \), and the associated fibered chart \( (V', \psi') \) on \( J^r \), where \( U = \pi(V) \), \( V'^r = (\pi^{r,0})^{-1}(V) \), and \( \psi'^r = (x^i, y^\sigma, \ldots, y^\sigma_{j_1 \ldots j_r}) \), where

\[
y^\sigma_{j_1 \ldots j_r}(J^r_x) = D_{j_1} \ldots D_{j_r} (\rho^{x^r x^r})(\varphi(x)), \quad 0 \leq k \leq r.
\]

A tangent vector \( \xi \in T_y Y \) is called \( \pi \)-vertical, if \( T \pi : \xi = 0 \), and a differential form \( \rho \) on \( Y \) is called \( \pi \)-horizontal, if for every point \( y \in Y \) the contraction \( \iota_{\xi} \rho(y) \) vanishes whenever \( \xi \in T_y Y \) is \( \pi \)-vertical.

We denote by \( \Omega^*_q \) the \( \Omega^*_q Y \)-module of smooth differential \( q \)-forms defined on \( J^r \). \( \pi^r \)-horizontal \( q \)-forms on \( J^r \) constitute a submodule of \( \Omega^*_q Y \), denoted by \( \Omega^r_{q, \pi} \). For a fibered manifold \( \pi : Y \to X \) there exists a unique morphism \( h : \)
\(\Omega^r Y \rightarrow \Omega^{r+1} Y\) of exterior algebras of differential forms such that for any fibered chart \((V, \psi), \psi = (x^1, y^k)\), on \(Y\), and any differentiable function \(f : J^r Y \rightarrow \mathbb{R}\),

\[hf = f \circ \pi^{r+1, r}, \quad hdf = (d_i f) dx^i,\]

where \(d_i\) (resp. \(d'_i\)) is the \(i\)-th formal derivative (resp. the cut \(i\)-th formal derivative) operator associated with \((V, \psi), \psi\),

\[d_i = d'_i + \sum_{j_1 \leq \ldots \leq j_r} \frac{\partial}{\partial y^i_{j_1 \ldots j_r}} y^i_{j_1 \ldots j_r},\]

and

\[(1.2) \quad d'_i = \frac{\partial}{\partial x^i} + \sum_{k=0}^{r-1} \sum_{j_1 \leq \ldots \leq j_k} \frac{\partial}{\partial y^i_{j_1 \ldots j_k}} y^i_{j_1 \ldots j_k}.\]

A differential form \(q\)-form \(\rho \in \Omega^r Y\) satisfying \(h \rho = 0\) is called contact, and every contact form \(\rho\) is generated by contact 1-forms

\[\omega^\sigma_{j_1 \ldots j_s} = dy^i_{j_1 \ldots j_s} - y^i_{j_1 \ldots j_s} dx^i, \quad 0 \leq s \leq r - 1.\]

Any differential \(q\)-form \(\rho \in \Omega^r Y\) has a unique invariant decomposition,

\[(\pi^{r+1, r})^* \rho = h \rho + \sum_{k=1}^q p_k \rho,\]

where \(p_k \rho\) is the \(k\)-contact component of \(\rho\), containing exactly \(k\) exterior product factors \(\omega^\sigma_{j_1 \ldots j_s}\) with respect to any fibered chart \((V, \psi)\).

2. Lepage Equivalents in Field Theory

We summarize basic facts about Lepage differential forms on finite-order jet prolongations of fibered manifolds and, in particular, we discuss distinguished examples of Lepage equivalents of first- and second-order Lagrangians; for more details see [13, 16, 18, 28, 25, 26].

By a Lagrangian \(\lambda\) for a fibered manifold \(\pi : Y \rightarrow X\) of order \(r\) we mean an element of the submodule \(\Omega^r_{n, X} Y\) of \(\pi^r\)-horizontal \(n\)-forms in the module of \(n\)-forms \(\Omega^r_{\ast, X} Y\), defined on the \(r\)-th jet prolongation \(J^r Y\). In a fibered chart \((V, \psi), \psi = (x^i, y^k)\), Lagrangian \(\lambda \in \Omega^r_{n, X} Y\) has an expression

\[\lambda = \mathcal{L} \omega_0,\]

where \(\omega_0 = dx^1 \wedge \ldots \wedge dx^n\) is the (local) volume element, and \(\mathcal{L} : V^r \rightarrow \mathbb{R}\) is said to be the Lagrange function associated to \(\lambda\) and \((V, \psi)\).

An \(n\)-form \(\rho \in \Omega^r_{n} Y\) on \(J^r Y\) is called a Lepage form, if one of the following equivalent conditions is satisfied:

(i) \(p_1 d \rho\) is a \(\pi^{r+1, 0}\)-horizontal \((n + 1)\)-form,

(ii) \(h \xi d \rho = 0\) for arbitrary \(\pi^{r, 0}\)-vertical vector field \(\xi\) on \(J^r Y\),

(iii) For every fibered chart \((V, \psi)\) on \(Y\), \(\rho\) satisfies

\[(\pi^{r+1, r})^* \rho = f_0 \omega_0 + \sum_{k=0}^s f^i_{\sigma j_1 \ldots j_k} \omega^\sigma_{j_1 \ldots j_k} \wedge \omega_i + \eta,\]
Let $\lambda \in \Omega^r_{n,X} Y$ be a Lagrangian for $\pi : Y \to X$. A Lepage form $\rho \in \Omega^r_n Y$ is called a Lepage equivalent of $\lambda$, if $h \rho = \lambda$ (up to a canonical jet projection). The following theorem describes the structure of Lepage equivalents of a Lagrangian.

**Theorem 1.** Let $\lambda \in \Omega^r_{n,X} Y$ be a Lagrangian of order $r$ for $\pi : Y \to X$, locally expressed by (2.1) with respect to a fibered chart $(V, \psi)$. An $n$-form $\rho \in \Omega^r_n Y$ is a Lepage equivalent of $\lambda$ if and only if it obeys the following decomposition,

$$\rho = \Theta_\lambda + d\mu + \eta,$$

where $n$-form $\Theta_\lambda$ is defined on $V^{2r-1}$ by

$$\Theta_\lambda = \mathcal{L} \omega_0 + \sum_{k=0}^{r-1} \left( \sum_{l=0}^{r-1-k} (-1)^l d_{p_1} \cdots d_{p_l} \frac{\partial \mathcal{L}}{\partial y_{j_1} \cdots y_{j_k} p_1 \cdots p_l} \right) \omega_{j_1} \cdots \omega_{j_k} \wedge \omega_i,$$

$\mu$ is a contact $(n-1)$-form, and an $n$-form $\eta$ has the order of contactness $\geq 2$.

**Proof.** See [28].

$\Theta_\lambda$, given by (2.3) on $V^{2r-1}$, is called the principal Lepage equivalent of $\lambda$ with respect to fibered chart $(V, \psi)$. This Lepage form is uniquely determined by imposing that a Lepage form is $\pi^{2r-1,r-1}$-horizontal and it has the order of contactness $\leq 1$. We note that, in general, decomposition (2.2) is not uniquely determined with respect to contact forms $\mu$ and $\eta$, although the Lepage equivalent $\rho$ satisfying (2.2) is a globally defined differential form on $J^sY$. For a first-order Lagrangian $\lambda$, $\Theta_\lambda$ (2.3) is the well-known Poincaré–Cartan form defined on $J^1Y$ (cf. [5]),

$$\Theta_\lambda = \mathcal{L} \omega_0 + \frac{\partial \mathcal{L}}{\partial y_j} \omega_j \wedge \omega_j.$$

For a second-order Lagrangian $\lambda$, $\Theta_\lambda$ (2.3) is the generalised Poincaré–Cartan form defined on $J^2Y$ (cf. [13, 19]),

$$\Theta_\lambda = \mathcal{L} \omega_0 + \frac{\partial \mathcal{L}}{\partial y_j} \omega_j \wedge \omega_j + \frac{\partial \mathcal{L}}{\partial y_{ij}} \omega_i \wedge \omega_j.$$

We point out that for Lagrangians of order $r \geq 3$, local expressions (2.3) need not define differential forms on $J^{2r-1}Y$ globally (cf. [9, 13]).

The well-known Euler–Lagrange mapping of the calculus of variations assigns to a Lagrangian $\lambda \in \Omega^r_{n,X} Y$ the Euler–Lagrange form

$$E_\lambda = E_\sigma(\mathcal{L}) \omega^\sigma \wedge \omega_0,$$

with coefficients

$$E_\sigma(\mathcal{L}) = \sum_{k=0}^{r} (-1)^k d_{i_1} \cdots d_{i_k} \frac{\partial \mathcal{L}}{\partial y_{i_1} \cdots i_k}$$

the Euler–Lagrange expressions associated to $\mathcal{L} : V^r \to \mathbb{R}$. Note that the 1-contact and $\pi^{r,0}$-horizontal $(n+1)$-form $E_\lambda$ is defined by means of (2.6), (2.7) on
THE FUNDAMENTAL LEPAGE FORM IN TWO INDEPENDENT VARIABLES

Let \( \lambda \in \Omega^r_{n,X} Y \) be a Lagrangian of order \( r \) for \( \pi : Y \to X \) and let \( \rho \in \Omega^s_{n,Y} \) be a Lepage equivalent of \( \lambda \). Then

\[
(\pi^{s+1,r})^\ast \rho = \lambda + E, \tag{2.6}
\]

where \( E \) is the Euler–Lagrange form associated to \( \lambda \), and \( F \) is an \((n + 1)\)-form with order of contactness \( \geq 2 \). In particular, \( E \) coincides with the 1-contact component of the exterior derivative of a Lepage equivalent of Lagrangian \( \lambda \), i.e., on \( J^{2r} Y \),

\[
E = p_1 \rho. \tag{2.8}
\]

Proof. See [16, 18, 28]. \( \square \)

Beside the principal Lepage equivalent \( \Theta_\lambda \), given by (2.4) and (2.5) for a first- and second-order Lagrangian \( \lambda \), respectively, we recall the other known examples of Lepage equivalents, determined by means of additional requirements.

Lemma 3. (a) Let \( \lambda \in \Omega^1_{n,X} Y \) be a non-vanishing first-order Lagrangian for \( \pi : Y \to X \), locally expressed by (2.1). Then the local expression

\[
\Lambda_\lambda = \frac{1}{\mathcal{L}^{n-1}} \sum_{j=1}^{n} \left( \mathcal{L} dx^j + \frac{\partial \mathcal{L}}{\partial y^j} \omega^\sigma \right) \tag{2.9}
\]

defines a \( \pi^{1,0} \)-horizontal differential \( n \)-form \( \Lambda_\lambda \in \Omega^1_n Y \), which is a Lepage equivalent of \( \lambda \).

(b) Let \( \lambda \in \Omega^2_{n,X} Y \) be a non-vanishing second-order Lagrangian for \( \pi : Y \to X \), locally expressed by (2.1). Then the local expression

\[
\Lambda_\lambda = \frac{1}{\mathcal{L}^{n-1}} \sum_{j=1}^{n} \left( \mathcal{L} dx^j + \left( \frac{\partial \mathcal{L}}{\partial y^j} - d_i \frac{\partial \mathcal{L}}{\partial y^j_i} \right) \omega^\sigma + \frac{\partial \mathcal{L}}{\partial y^j_i} \omega^i \right) \tag{2.10}
\]

defines a \( \pi^{3,1} \)-horizontal differential \( n \)-form \( \Lambda_\lambda \in \Omega^1_n Y \), which is a Lepage equivalent of \( \lambda \).

Proof. See [2, 26]. \( \square \)

\( \Lambda_\lambda \) (2.9) is the well-known Carathéodory form, associated to a non-vanishing, first-order Lagrangian \( \lambda \) (cf. [2]), whereas \( \Lambda_\lambda \) (2.10) is its generalization for second-order Lagrangians, recently studied in [26].

Remark 4. Note that the Carathéodory form \( \Lambda_\lambda \) (2.10) is decomposed as a sum of the generalized Poincaré–Cartan form \( \Theta_\lambda \) (2.5) and a \( \pi^{3,1} \)-horizontal, 2-contact differential \( n \)-form. For further purpose, we give this decomposition explicitly for
We have the dimension of base \( n = 2 \):

\[
\Lambda_\lambda = \Theta_\lambda + \frac{1}{Z} \frac{\partial L}{\partial y_1^i} \left( \frac{\partial L}{\partial y_1^j} - d_i \frac{\partial L}{\partial y_1^i} \right) \omega^j \wedge \omega^\nu
\]

(2.11)

\[
+ \frac{1}{Z} \frac{\partial L}{\partial y_1^j} \left( \frac{\partial L}{\partial y_1^i} - d_i \frac{\partial L}{\partial y_1^j} \right) \omega^i \wedge \omega^\nu
\]

Lemma 5. Let \( \lambda \in \Omega_{n,X}^1 Y \) be a first-order Lagrangian for \( \pi : Y \to X \), locally expressed by (2.1). There exists a unique Lepage equivalent \( Z_\lambda \in \Omega_{n,Y}^1 \) of \( \lambda \), which satisfies \( Z_\lambda = (\pi^{1,0})^* \rho \) for any n-form \( \rho \in \Omega_n^0 W \) on \( W \) such that \( h\rho = \lambda \). With respect to a fibered chart \((V, \psi)\), \( Z_\lambda \) has an expression

\[
(2.12) \quad Z_\lambda = L \omega_0 + \sum_{k=1}^n \frac{1}{(n-k)!} \frac{\partial^k L}{\partial y_1^{j_1} \ldots \partial y_1^{j_k}} \omega^{j_1 \ldots j_k} \wedge \omega^{\nu_1 \ldots \nu_m} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_m}.
\]

Proof. See [12, 1]. \( \square \)

\( Z_\lambda \) (2.12) is known as the fundamental Lepage form, or Krupka–Betounes form, associated to first-order Lagrangian \( \lambda \); original sources are [12] and [1], further recent contributions include [19, 20, 25].

Remark 6. One can directly verify that the Lepage equivalent \( Z_\lambda \) (2.12) satisfies the equivalence relation “\( Z_\lambda \) is closed if and only if the associated Lagrangian \( \lambda = h Z_\lambda \) is trivial”, that is “\( dZ_\lambda = 0 \) if and only if \( E_\lambda = 0 \)” Since \( E_\lambda \equiv p_1 dZ_\lambda \), it is immediate that \( \lambda \in \Omega_{n,X}^1 Y \) is trivial, provided \( Z_\lambda \) is closed. However, the converse implication if a non-trivial one. A construction of (local, 1-contact) Lepage equivalents of higher-order Lagrangians satisfying this equivalence relation has been recently described in [27].

Another remarkable property which \( Z_\lambda \) (2.12) satisfies is the following: \( Z_\lambda \) is \( \pi^{1,0} \)-projectable if and only if \( E_\lambda \) is \( \pi^{2,1} \)-projectable.

3. Trivial Lagrangians

A Lagrangian \( \lambda \) is said to be variationally trivial (or null), if the associated Euler–Lagrange form \( E_\lambda \) vanishes identically. The following theorem describes variationally trivial Lagrangians.

Theorem 7. Let \( \lambda \in \Omega_{n,X}^r Y \) be a Lagrangian of order \( r \) for \( \pi : Y \to X \). The following conditions are equivalent:

(i) \( \lambda \) is trivial.

(ii) For every fibered chart \((V, \psi)\) on \( Y \) there exists an \((n-1)\)-form \( \mu \in \Omega_{n-1}^r Y \) such that on \( V^r \),

\[
\lambda = h \mu.
\]

(iii) For every fibered chart \((V, \psi)\) on \( Y \) there exist functions \( g^i : V^r \to \mathbb{R} \) such that \( \lambda = L \omega_0 \) on \( V^r \), where

\[
L = d_i g^i,
\]

(3.1)
Lemma 8. \[\text{(a) \(\lambda\) is trivial if and only if it satisfies the following conditions}\]

\[\frac{\partial \mathcal{L}}{\partial y^i} - d_i' \frac{\partial \mathcal{L}}{\partial y^i} + d_i d_j \frac{\partial \mathcal{L}}{\partial y^i_{ij}} = 0,\]

\[\frac{\partial^2 \mathcal{L}}{\partial y^i_{pq} \partial y^i_{ij}} + 2d_i' \left( \frac{\partial^2 \mathcal{L}}{\partial y^i_{pq} \partial y^i_{ij}} \right) = 0 \quad \text{Sym(pqi)},\]

\[\frac{\partial^3 \mathcal{L}}{\partial y^i_{pq} \partial y^i_{ij}} = 0 \quad \text{Sym(stj), Sym(pqi)},\]

\[\frac{\partial^2 \mathcal{L}}{\partial y^i_{pq} \partial y^i_{ij}} = 0 \quad \text{Sym(pqi)},\]

where \(d_i'\) is the cut formal derivative operator [12].

(b) For every fibered chart \((V, \psi)\) on \(Y\) there exist functions \(g^i : V^2 \rightarrow \mathbb{R}\) such that \(\lambda = \mathcal{L} \omega_0\) on \(V^2\), where

\[\mathcal{L} = d_ig^i,\]

and

\[\frac{\partial g^i}{\partial y^s_{jk}} + \frac{\partial g^j}{\partial y^s_{ki}} + \frac{\partial g^k}{\partial y^s_{ij}} = 0.\]

Proof. Consider the Euler–Lagrange expressions \(E_\sigma(\mathcal{L})\), associated to a second-order Lagrangian \(\lambda = \mathcal{L} \omega_0\). We have

\[d_i \frac{\partial \mathcal{L}}{\partial y^i} = \frac{\partial^2 \mathcal{L}}{\partial x^i \partial y^i} + \frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^i_{ij}} y^i_{ij} + \frac{\partial^2 \mathcal{L}}{\partial y^i_{pq} \partial y^i_{ij}} y^i_{pq} + \frac{\partial^2 \mathcal{L}}{\partial y^i_{pq} \partial y^i_{ij}} y^i_{pq},\]

and

\[d_i d_j \frac{\partial \mathcal{L}}{\partial y^i_{ij}} = d_i \left( d_j' \frac{\partial \mathcal{L}}{\partial y^i_{ij}} + \frac{\partial^2 \mathcal{L}}{\partial y^i_{pq} \partial y^i_{ij}} y^i_{pq} \right)\]

\[= d'_i d_j' \frac{\partial \mathcal{L}}{\partial y^i_{ij}} + d_i' \left( \frac{\partial^2 \mathcal{L}}{\partial y^i_{pq} \partial y^i_{ij}} y^i_{pq} + \frac{\partial^2 \mathcal{L}}{\partial y^i_{pq} \partial y^i_{ij}} y^i_{pq} \right),\]
Thus (3.2) now reads,
\[
E_\sigma(L) = \frac{\partial L}{\partial y^\nu} - d'_i d'_j \frac{\partial L}{\partial y^\sigma_{ij}} + \left( \frac{\partial^2 L}{\partial y^\nu_{pq} \partial y^\sigma_{ij}} - \frac{\partial L}{\partial y^\nu_{pq} y^\sigma_{ij}} + 2d'_j \left( \frac{\partial^2 L}{\partial y^\nu_{pq} \partial y^\sigma_{ij}} \right) \right) y^\nu_{pqij} + \frac{\partial^3 L}{\partial y^\nu_{pq} \partial y^\sigma_{ij} \partial y^\tau_{kl}} y^\nu_{pqij},
\]
(3.8)

hence \(E_\sigma(L)\) vanish if and only if conditions (3.4) hold, proving (a).

Condition (b) follows from Theorem 7, (iii), where (3.7) is implied by equation (3.1) satisfied identically on \(V^2 \subset J^2Y\).

4. The fundamental Lepage equivalent of a second-order Lagrangian: order reduction

First, we present an order reduction condition, which allows us to construct a generalization of the fundamental Lepage equivalent \(Z_\lambda\) (2.12) for a second-order Lagrangian \(\lambda\). From the Theorem 1 on the structure of every Lepage equivalent of a Lagrangian, it is immediate that a differential \(n\)-form to be found must be decomposable as \(Z_\lambda = \Theta_\lambda + d\mu + \eta\) (up to a canonical jet projection), where \(\Theta_\lambda\) is the generalized Poincaré–Cartan form (2.5), \(\mu\) is a contact \((n-1)\)-form, and an \(n\)-form \(\eta\) has the order of contactness \(\geq 2\).

Let us assume that \(\Theta_\lambda\) (2.5) is of the same order as the Lagrangian \(\lambda\). Note that this condition is automatically satisfied for first-order Lagrangians but, nevertheless, for the second-order it restricts the class of Lagrangians under consideration.

Lemma 9. Let \(\lambda \in \Omega^2_{n,X}Y\) be a second-order Lagrangian for \(\pi : Y \rightarrow X\). Then the generalized Poincaré–Cartan form \(\Theta_\lambda\) (2.5) is of second-order, if and only if for every chart \((V, \psi)\), \(\psi = (x^i, y^\sigma)\), on \(Y\),

\[
\frac{\partial^2 L}{\partial y^\nu_{pq} \partial y^\sigma_{ij}} + \frac{\partial^2 L}{\partial y^\nu_{ip} \partial y^\sigma_{qj}} + \frac{\partial^2 L}{\partial y^\nu_{qi} \partial y^\sigma_{pj}} = 0,
\]
(4.1)

where \(L\) is the Lagrange function, associated to \(\lambda\) (2.1).

For \(n = 2\), (4.1) read

\[
\frac{\partial^2 L}{\partial y^\nu_{11} \partial y^\sigma_{11}} = 0, \quad \frac{\partial^2 L}{\partial y^\nu_{11} \partial y^\sigma_{12}} = 0, \quad \frac{\partial^2 L}{\partial y^\nu_{22} \partial y^\sigma_{21}} = 0, \quad \frac{\partial^2 L}{\partial y^\nu_{22} \partial y^\sigma_{22}} = 0,
\]
(4.2)

Proof. The necessary and sufficient condition (4.1) follows immediately from the chart expression (2.5) of \(\Theta_\lambda\), by means of annihilating terms linear in coordinates \(y^\nu_{pqij}\).

Lemma 10. Let \(\lambda \in \Omega^2_{n,X}Y\) be a second-order Lagrangian for \(\pi : Y \rightarrow X\) such that the generalized Poincaré–Cartan form \(\Theta_\lambda\) (2.5) is of second-order. Then \(\lambda\) is
variationally trivial, if and only if for every chart \((V, \psi), \psi = (x^1, y^\sigma), \) on \(Y,\)

\[
\frac{\partial L}{\partial y^\sigma} - d_i \frac{\partial L}{\partial y_i} + d_j \frac{\partial L}{\partial y_j} = 0, \quad \frac{\partial^2 L}{\partial y_i \partial y_j} - \frac{\partial^2 L}{\partial y_j \partial y_i} = 0 \quad \text{Sym}(pqi).
\]

For \(n = 2,\)

\[
\frac{\partial L}{\partial y^i} - d_i \frac{\partial L}{\partial y_i} + d_j \frac{\partial L}{\partial y_j} = 0, \quad i, j = 1, 2,
\]

\[
\frac{\partial^2 L}{\partial y_j \partial y_i} = 0, \quad \frac{\partial^2 L}{\partial y_i \partial y_j} = 0,
\]

\[
2 \frac{\partial^2 L}{\partial y_i \partial y_{12}} - 2 \frac{\partial^2 L}{\partial y_i \partial y_{12}} + \frac{\partial^2 L}{\partial y_1 \partial y_{12}} - \frac{\partial^2 L}{\partial y_2 \partial y_{12}} = 0,
\]

\[
2 \frac{\partial^2 L}{\partial y_j \partial y_{12}} - 2 \frac{\partial^2 L}{\partial y_j \partial y_{12}} + \frac{\partial^2 L}{\partial y_1 \partial y_{12}} - \frac{\partial^2 L}{\partial y_2 \partial y_{12}} = 0.
\]

Proof. Necessary and sufficient conditions for a variationally trivial Lagrangian are nothing but reduction of \(6.8\)–\(6.9\), Lemma 8, with the help of \(4.11.\) \(\square\)

In the next Theorem we construct the fundamental Lepage equivalent \(Z_\lambda\) of a second-order Lagrangian \(\lambda\) over fibered manifolds \(\pi: Y \to X,\) where \(\dim X = 2.\)

Analogously to the Carathéodory form \(\Lambda_\lambda\) \(2.1\), associated to second-order Lagrangian \(\lambda \in \Omega^2_{2,Y} Y,\) suppose that \(Z_\lambda\) is decomposed as a sum of the generalized Poincaré–Cartan form \(\Theta_\lambda\) \(2.5\) and contact terms, generated by the wedge products \(\omega^\sigma \wedge \omega^\nu,\ \omega^\sigma \wedge \omega^\nu,\) and \(\omega^\sigma \wedge \omega^\nu,\) i.e.

\[
Z_\lambda = \Theta_\lambda + \frac{1}{2} P_{\sigma \nu} \omega^\sigma \wedge \omega^\nu + Q^1_{\sigma \nu} \omega^\sigma \wedge \omega^\nu + \frac{1}{2} R^{i,j}_{\sigma \nu} \omega^\sigma \wedge \omega^\nu,
\]

where \(P_{\sigma \nu}, Q^1_{\sigma \nu},\) and \(R^{i,j}_{\sigma \nu}\) are real-valued functions on \(V^3 \subset J^2 Y\) such that \(P_{\sigma \nu}\) is skew-symmetric in \((\sigma, \nu),\) and \(R^{i,j}_{\sigma \nu}\) is skew-symmetric in pairs \((i, \sigma), (j, \nu).\)

**Theorem 11.** Let \(\lambda \in \Omega^2_{2,Y} Y\) be a second-order Lagrangian for \(\pi: Y \to X\) such that \(1.1\) holds. The following two conditions are equivalent:

(i) If \(\lambda\) is variationally trivial, then \(Z_\lambda\) \(1.5\) is closed.

(ii) For every chart \((V, \psi), \psi = (x^1, y^\sigma),\) on \(Y, \ Z_\lambda\) \(1.5,\) is uniquely determined by means of real-valued functions \(P_{\sigma \nu}, Q^1_{\sigma \nu},\) and \(R^{i,j}_{\sigma \nu},\) defined on \(V^2 \subset J^2 Y\) as

\[
P_{\sigma \nu} = \frac{1}{2} \left( \frac{\partial^2 L}{\partial y_i \partial y_j} - \frac{\partial^2 L}{\partial y_j \partial y_i} \right) + d_i \left( \frac{\partial^2 L}{\partial y_j \partial y_{12}} - \frac{\partial^2 L}{\partial y_i \partial y_{12}} \right) + d_j \left( \frac{\partial^2 L}{\partial y_i \partial y_{12}} - \frac{\partial^2 L}{\partial y_j \partial y_{12}} \right),
\]

\[
Q^1_{\sigma \nu} = 2 \left( \frac{\partial^2 L}{\partial y_j \partial y_{12}} - \frac{\partial^2 L}{\partial y_i \partial y_{12}} \right) - 2 d_i \left( \frac{\partial^2 L}{\partial y_j \partial y_{12}} - \frac{\partial^2 L}{\partial y_i \partial y_{12}} \right) - 2 d_j \left( \frac{\partial^2 L}{\partial y_i \partial y_{12}} - \frac{\partial^2 L}{\partial y_j \partial y_{12}} \right),
\]

\[
Q^2_{\sigma \nu} = -2 \left( \frac{\partial^2 L}{\partial y_j \partial y_{12}} + \frac{\partial^2 L}{\partial y_i \partial y_{12}} \right) + d_i \left( \frac{\partial^2 L}{\partial y_j \partial y_{12}} + \frac{\partial^2 L}{\partial y_i \partial y_{12}} \right) + d_j \left( \frac{\partial^2 L}{\partial y_i \partial y_{12}} + \frac{\partial^2 L}{\partial y_j \partial y_{12}} \right),
\]

\[
R^{1,2}_{\sigma \nu} = -2 \left( \frac{\partial^2 L}{\partial y_j \partial y_{12}} + \frac{\partial^2 L}{\partial y_i \partial y_{12}} \right),
\]

\[
R^{2,1}_{\sigma \nu} = 0, \quad R^{3,2}_{\sigma \nu} = 0,
\]
Proof. Suppose that Lagrangian \( \lambda \in \Omega^2_X Y \) is variationally trivial and the generalized Poincaré–Cartan form \( \Theta_\lambda \) is defined on \( J^2 Y \). Thus, in arbitrary fibered chart \( (V, \psi) \), \( \psi = (x^i, y^\sigma) \), on \( Y \), the associated Lagrange function \( \mathcal{L} : V^2 \to \mathbb{R} \) satisfies conditions (3.3)–(3.5) of Lemma 8, and (4.1) of Lemma 9. Note first that condition (4.1) already implies (3.6), and using (4.1),

\[
d_i \frac{\partial \mathcal{L}}{\partial y^\sigma_{ij}} = d_i \frac{\partial \mathcal{L}}{\partial y^\sigma_{ij}}.
\]

Hence, the exterior derivative of \( \Theta_\lambda \) reads,

\[
d\Theta_\lambda = d\mathcal{L} \wedge \omega_0 + \left. d \left( \frac{\partial \mathcal{L}}{\partial y^\sigma_{ij}} - d_i \frac{\partial \mathcal{L}}{\partial y^\sigma_{ij}} \right) \right| \wedge \omega^\sigma \wedge \omega_j + \left. d \left( \frac{\partial \mathcal{L}}{\partial y^\sigma_{ij}} - d_i \frac{\partial \mathcal{L}}{\partial y^\sigma_{ij}} \right) \right| d\omega^\sigma \wedge \omega_j
\]

\[+ d \left( \frac{\partial \mathcal{L}}{\partial y^\sigma_{ij}} \right) \wedge \omega^\sigma \wedge \omega_j + \left. \frac{\partial \mathcal{L}}{\partial y^\sigma_{ij}} \right| d\omega^\sigma \wedge \omega_j\]

(4.7)

\[
= \left( \frac{\partial \mathcal{L}}{\partial y^\sigma} - d_j \frac{\partial \mathcal{L}}{\partial y^\sigma_j} + d_i \frac{\partial \mathcal{L}}{\partial y^\sigma_i} \right) \omega^\sigma \wedge \omega_0 - \frac{\partial \mathcal{L}}{\partial y^\sigma} \left( \frac{\partial \mathcal{L}}{\partial y^\sigma_j} - d_i \frac{\partial \mathcal{L}}{\partial y^\sigma_i} \right) \omega^\sigma \wedge \omega_j
\]

\[+ \left( \frac{\partial^2 \mathcal{L}}{\partial y^\sigma \partial y^\kappa_l} - \frac{\partial}{\partial y^\kappa_l} \left( \frac{\partial \mathcal{L}}{\partial y^\sigma_j} - d_i \frac{\partial \mathcal{L}}{\partial y^\sigma_i} \right) \right) \omega^\sigma \wedge \omega^\kappa_l \wedge \omega_j
\]

\[- \frac{\partial}{\partial y^\kappa_l} \left( \frac{\partial \mathcal{L}}{\partial y^\sigma_j} - d_i \frac{\partial \mathcal{L}}{\partial y^\sigma_i} \right) \right| \omega^\sigma \wedge \omega^\kappa_l \wedge \omega_j - \frac{\partial^2 \mathcal{L}}{\partial y^\kappa_l \partial y^\sigma_{ij}} \omega^\kappa_l \wedge \omega^\sigma \wedge \omega_j
\]

\[- \frac{\partial^2 \mathcal{L}}{\partial y^\kappa_l \partial y^\sigma_{ij}} \omega^\kappa_l \wedge \omega^\sigma \wedge \omega_j.
\]

From (4.5), we have

\[
dZ_\lambda = d\Theta_\lambda
\]

\[+ \frac{1}{2} dP_{\sigma\tau} \wedge \omega^\sigma \wedge \omega^\tau + \frac{1}{2} P_{\sigma\tau} d \left( \omega^\sigma \wedge \omega^\tau \right) + dQ_{\sigma\tau}^i \wedge \omega^\sigma \wedge \omega^\tau + Q_{\sigma\tau}^i d \left( \omega^\sigma \wedge \omega^\tau \right)
\]

\[+ \frac{1}{2} dR_{\sigma\tau}^{i,j} \wedge \omega^\sigma \wedge \omega^\tau + \frac{1}{2} R_{\sigma\tau}^{i,j} d \left( \omega^\sigma \wedge \omega^\tau \right),
\]

and combining with (4.7) we get a decomposition of \( dZ_\lambda \) containing independent base terms,

\[
dZ_\lambda = E_\lambda + F_0 + F_1 + F_2,
\]

where \( E_\lambda \) is the Euler–Lagrange form of \( \lambda \), \( F_0 \) and \( F_1 \) are the 2-contact parts and \( F_2 \) is the 3-contact part of \( dZ_\lambda \). For indices \( i, j \) running through \( \{1, 2\} \), and \( \kappa(1) = 2 \),
\( \kappa(2) = 1 \), we have

\[
F_0 = (-1)^{j-1} \left( P_{\sigma \nu} + d_{\kappa(j)} Q_{\sigma, \nu}^{(j)} + \frac{\partial^2 \mathcal{L}}{\partial y^\sigma \partial y_{12}^{\nu}} - \frac{\partial}{\partial y_{12}^{\nu}} \left( \frac{\partial \mathcal{L}}{\partial y_j^\sigma} - d_j^i \frac{\partial \mathcal{L}}{\partial y_{ij}^\nu} \right) \right) \omega^\sigma \wedge \omega_{12}^\nu \wedge \omega_j
\]

\[\quad + (-1)^{j-1} \left( Q_{\sigma, \nu}^{(j)} + 2(-1)^{j-1} \frac{\partial \mathcal{L}}{\partial y_{12}^{\nu}} \left( \frac{\partial \mathcal{L}}{\partial y_j^\sigma} - d_j^i \frac{\partial \mathcal{L}}{\partial y_{ij}^\nu} \right) \right) \omega^\sigma \wedge \omega_{12}^\nu \wedge \omega_j
\]

\[\quad + (-1)^{j-1} \left( R_{\sigma, \nu}^{(j), j} + 2(-1)^{j-1} \frac{\partial \mathcal{L}}{\partial y_{12}^{\nu}} \right) \omega_{12}^\nu \wedge \omega_{12}^\nu \wedge \omega_j
\]

\[\quad + (-1)^{j-1} R_{\sigma, \nu}^{(j), j} \omega_{12}^\nu \wedge \omega_{12}^\nu \wedge \omega_{12}^\nu,
\]

and

\[
F_1 = \left( \frac{\partial}{\partial y^\sigma} \left( \frac{\partial \mathcal{L}}{\partial y_j^\nu} - d_j^i \frac{\partial \mathcal{L}}{\partial y_{ij}^\nu} \right) + (-1)^{j-1} \frac{1}{2} d_{\kappa(j)} P_{\sigma \tau} \right) \omega^\sigma \wedge \omega^\tau \wedge \omega_j \quad \text{Alt}(\sigma \tau)
\]

\[\quad + (-1)^{j-1} \left( d_{\kappa(j)} Q_{\sigma, \nu}^{(j)} + \frac{\partial^2 \mathcal{L}}{\partial y^\sigma \partial y_j^\nu} - \frac{\partial}{\partial y_j^\nu} \left( \frac{\partial \mathcal{L}}{\partial y_j^\sigma} - d_j^i \frac{\partial \mathcal{L}}{\partial y_{ij}^\nu} \right) \right) \omega^\sigma \wedge \omega_j^\nu \wedge \omega_j
\]

\[\quad - \frac{\partial}{\partial y_j^\nu} \left( \frac{\partial \mathcal{L}}{\partial y_j^\sigma} - d_j^i \frac{\partial \mathcal{L}}{\partial y_{ij}^\sigma} \right) \omega^\sigma \wedge \omega_j^\nu \wedge \omega_j
\]

\[\quad + \frac{1}{2} (-1)^{j-1} \left( Q_{\sigma, \nu}^{(j)} - Q_{\nu, \sigma}^{(j)} + (-1)^j \left( \frac{\partial^2 \mathcal{L}}{\partial y_{12}^{\nu} \partial y_{12}^{\sigma}} - \frac{\partial^2 \mathcal{L}}{\partial y_{12}^{\sigma} \partial y_{12}^{\nu}} \right) \right) \omega_{12}^\sigma \wedge \omega_{12}^\nu \wedge \omega_j
\]

\[\quad + \frac{1}{2} (-1)^{j-1} \left( d_{\kappa(j)} R_{\sigma, \nu}^{(j), j} + \frac{\partial^2 \mathcal{L}}{\partial y_j^\sigma \partial y_j^\nu} - \frac{\partial^2 \mathcal{L}}{\partial y_j^\nu \partial y_j^\sigma} \right) \omega_j^\sigma \wedge \omega_j^\nu \wedge \omega_j
\]

\[\quad + (-1)^{j-1} \left( \frac{1}{2} d_{\kappa(j)} \right) \left( R_{\sigma, \nu}^{(1), 2} - R_{\nu, \sigma}^{(2), 1} \right) + (-1)^j Q_{\nu, \sigma}^{(1)} + \frac{\partial^2 \mathcal{L}}{\partial y_j^\sigma \partial y_j^\nu} - \frac{\partial^2 \mathcal{L}}{\partial y_j^\nu \partial y_j^\sigma} \right) \omega_{12}^\sigma \wedge \omega_{12}^\nu \wedge \omega_j
\]

and

\[
F_2 = \frac{1}{6} \left( \frac{\partial P_{\sigma \nu}}{\partial y^\sigma} + \frac{\partial P_{\sigma \nu}}{\partial y^\nu} + \frac{\partial P_{\sigma \nu}}{\partial y^\sigma} \right) \omega^\sigma \wedge \omega^\nu \wedge \omega^\nu
\]

\[\quad + \frac{1}{2} \left( \frac{\partial P_{\sigma \nu}}{\partial y_j^\sigma} - \frac{\partial Q_{\sigma, \nu}^{(1)}}{\partial y_j^\nu} + \frac{\partial Q_{\nu, \sigma}^{(1)}}{\partial y_j^\nu} \right) \omega^\sigma \wedge \omega^\nu \wedge \omega_j^\nu
\]

\[\quad + \frac{1}{2} \frac{\partial P_{\sigma \nu}}{\partial y_j^\sigma} \omega_j^\tau \wedge \omega^\nu \wedge \omega^\nu + \frac{1}{2} \frac{\partial P_{\sigma \nu}}{\partial y_j^\nu} \omega_j^\tau \wedge \omega^\sigma \wedge \omega^\nu
\]

\[\quad + \frac{1}{2} \frac{\partial P_{\sigma \nu}}{\partial y_j^\sigma} \omega_j^\tau \wedge \omega^\nu \wedge \omega^\nu + \frac{1}{2} \frac{\partial P_{\sigma \nu}}{\partial y_j^\nu} \omega_j^\tau \wedge \omega^\sigma \wedge \omega^\nu
\]
The equivalence of (i) and (ii) is therefore proved.

Remark is zero. (cf. Theorem 2). As we require that \( \lambda \) identically. To this can be, however, directly verified by means of the assumption on derivative operators.

In particular, without the assumption (4.1) on order-reducibility of the generalized P satisfies (4.2) and the symmetric component of \( \mathcal{L} \) annihilating (4.8) we obtain the functions (4.6), which are correctly defined since \( \mathcal{L} \) satisfies (4.2) and the symmetric component of

\[
\frac{\partial^2 \mathcal{L}}{\partial y^\sigma \partial y^\nu_{12}} - \frac{\partial}{\partial y^\nu_{12}} \left( \frac{\partial \mathcal{L}}{\partial y^\sigma} - d_i \frac{\partial \mathcal{L}}{\partial y^\sigma_{ij}} \right) = 0,
\]

is zero.

It remains to show that all coefficients of forms \( F_1 \) and \( F_2 \) vanish identically. To this can be, however, directly verified by means of the assumption on variational triviality of \( \lambda \), namely conditions (4.4) of Lemma 10 as well as partial derivative operators \( \partial / \partial y^\nu_{12}, \partial / \partial y^\nu_{12} \) applied to condition (4.5). In particular, the following identities follow from (4.5) and are employed,

\[
\begin{align*}
\frac{\partial^3 \mathcal{L}}{\partial y^\sigma \partial y^\gamma \partial y^\nu_{12}} + \frac{\partial^3 \mathcal{L}}{\partial y^\nu \partial y^\gamma \partial y^\nu_{12}} + \frac{\partial^3 \mathcal{L}}{\partial y^\nu \partial y^\sigma \partial y^\nu_{12}} & = 0, \\
\frac{\partial^3 \mathcal{L}}{\partial y^\nu \partial y^\nu_{12} \partial y^\nu_{12}} - \frac{\partial^3 \mathcal{L}}{\partial y^\nu \partial y^\nu_{12} \partial y^\nu_{12}} & = 0, \\
\frac{\partial^3 \mathcal{L}}{\partial y^\nu \partial y^\nu_{12} \partial y^\nu_{12}} - \frac{\partial^3 \mathcal{L}}{\partial y^\nu \partial y^\nu_{12} \partial y^\nu_{12}} & = 0,
\end{align*}
\]

The equivalence of (i) and (ii) is therefore proved.

Remark 12. The proof of Theorem 11 shows that \( Z_3 \) satisfying the equivalence relation "\( Z_3 \) is closed if and only if \( \lambda \) is variationally trivial" does not exist in general. In particular, without the assumption (4.1) on order-reducibility of the generalized
Poincaré–Cartan form $\Theta$, the coefficients $P_{\sigma \nu}, Q_{\sigma \nu}^{\lambda}$, and $R_{\sigma \tau \nu}^{i}$ of $Z$ are generally not well-defined to ensure the closure condition of $Z$ for a trivial Lagrangian $\lambda$.

As an example illustrating this property, we refer to the Camassa–Holm equation and its Lagrangian on the second jet prolongation of fibered manifold $S^{1} \times \mathbb{R} \times \mathbb{R}$ over $S^{1} \times \mathbb{R}$ (see [10]),

$$L_{CH} = \frac{1}{2} \left( y_1(y_2)^2 + \frac{(y_{12})^2}{y_1} \right),$$

which is quadratic in second derivatives and does not satisfy the order-reducibility condition (4.1). Following the proof of Theorem 11 it can be directly shown that $Z_{\lambda}$ (4.5), associated to Lagrangian (4.11) and satisfying the equivalence relation (1.1), does not exist.

Note that the result of Theorem 11 can be also directly verified as follows: consider $Z_{\lambda}$ (4.5) with coefficients (4.6), where the associated Lagrange function $L$ is of the form $L = d_{d}g^{i}$, where (3.7) of Lemma 8 holds. One can then verify that $dZ_{\lambda} = 0$ in a straightforward way.

A possible generalization of the fundamental form, associated to a second-order Lagrangian in field theory, has been recently studied in [19]; the obtained result is, however, not the case described by Theorem 4 since the corresponding Lepage form does not obey the closure condition (1.1).

Remark 13. Clearly, the order-reducibility condition (4.1) imposed beforehand is satisfied for Lagrangians $\lambda \in \Omega_{n,\lambda}^{2} Y$ linear in second derivatives $y_{ij}$. In this case, if

$$L = A + B_{ij} y_{ij} = A + B_{11} y_{11} + 2B_{12} y_{12} + B_{22} y_{22}$$

is a Lagrange function satisfying (1.2), then

$$P_{\sigma \nu} = \frac{1}{2} \left( \frac{\partial^2 A}{\partial y_{1}^{\sigma} \partial y_{2}^{\nu}} - \frac{\partial^2 A}{\partial y_{1}^{\nu} \partial y_{2}^{\sigma}} + \left( \frac{\partial^2 B_{ij}^{\nu}}{\partial y_{1}^{\sigma} \partial y_{2}^{\nu}} - \frac{\partial^2 B_{ij}^{\sigma}}{\partial y_{1}^{\nu} \partial y_{2}^{\sigma}} \right) y_{ij} \right)$$

$$+ (-1)^{\sigma + \nu} 2B_{ij} \left( \frac{\partial B_{ij}^{\lambda}}{\partial y_{1}^{\sigma}} - \frac{\partial B_{ij}^{\lambda}}{\partial y_{2}^{\nu}} \right),$$

$$Q_{\sigma \nu}^{\lambda} = \frac{\partial B_{ij}^{\lambda}}{\partial y_{1}^{\sigma}} - \frac{\partial B_{ij}^{\lambda}}{\partial y_{1}^{\nu}} - \frac{\partial B_{ij}^{\lambda}}{\partial y_{2}^{\sigma}} - \frac{\partial B_{ij}^{\lambda}}{\partial y_{2}^{\nu}} + 2B_{ij} \frac{\partial B_{ij}^{\lambda}}{\partial y_{1}^{\sigma}} - \frac{\partial B_{ij}^{\lambda}}{\partial y_{1}^{\nu}}$$

$$+ 2B_{ij} \frac{\partial B_{ij}^{\lambda}}{\partial y_{2}^{\sigma}} - \frac{\partial B_{ij}^{\lambda}}{\partial y_{2}^{\nu}}.$$

Moreover, from (3.8) it is easy to see that the order-reducibility (4.1) is a weaker condition than the requirement on a second-order Lagrangian leading to second-order Euler–Lagrange equations, cf. [4], which is the case of the Einstein–Hilbert gravitation Lagrangian of General Relativity. For the example of interaction of gravitational and electromagnetic fields Lagrangian, see [28].

Let $\lambda \in \Omega_{n,\lambda}^{2} Y$ be a Lagrangian for $\pi : Y \to X$ such that the generalized Poincaré–Cartan form $\Theta$ (2.5) is of second-order. We call $Z_{\lambda}$ (4.5), satisfying one of the equivalent conditions of Theorem 11 the fundamental Lepage equivalent, associated to a second-order Lagrangian $\lambda$.

Our conjecture is that this kind of generalization of the fundamental form using order reducibility can be provided for a general dimension of a base manifold, which will be studied in a future work.
References

[1] Betounes, D.E. Extension of the classical Cartan form. *Phys. Rev. D* 1984, 29, 599–606.

[2] Carathéodory, C. Über die Variationsrechnung bei mehrfachen Integralen. *Acta Szeged Sect. Sci. Math.* 1929, 4, 193–216.

[3] Crampin, M.; Saunders, D.J. The Hilbert–Carathéodory form and Poincaré–Cartan forms for higher-order multiple integral variational problems. *Houston J. Math.* 2004, 30, 657–689.

[4] Dedecker, P. On the generalization of symplectic geometry to multiple integrals in the calculus of variations. In *Differential Geometrical Methods in Mathematical Physics*, Lecture Notes in Mathematics 570, Bleuler, K., Reetz, A., Eds.; Springer: Berlin, 1977; pp. 395–456.

[5] García, P.L. The Poincaré–Cartan invariant in the calculus of variations. In *Symposia Mathematica XIV* (Convegno di Geometria Simplicetta e Fisica Matematica, INDAM, Rome, 1973); Academic Press: London, 1974; pp. 219–246.

[6] Giachetta, G.; Mangiarotti, L; Sardanashvily, G. *Advanced classical field theory*. World Scientific: Singapore, 2009.

[7] Goldschmidt, H.; Sternberg, S. The Hamilton–Cartan formalism in the calculus of variations. *Ann. Inst. Henri Poincaré* 1973, 23, 203–267.

[8] Gotay, M. J. An exterior differential systems approach to the Cartan form. In: Geométrie Symplectique et Physique Mathematique, P. Donato et al. (Eds.), Proc. Colloq., Aix-en-Provence, 1990; *Prog. Math.* 1991, 99, 160–188.

[9] Horák, M.; Kolář, I. On the higher-order Poincaré–Cartan forms. *Czech. Math. J.* 1983, 33, 467–475.

[10] Kouranbaeva, S.; Shkoller, S. A variational approach to second-order multisymplectic field theory. *J. Geom. Phys.* 2000, 35(4), 333–366.

[11] Krupka, D. Some Geometric Aspects of Variational Problems in Fibered Manifolds, Facultas Scientiarum Naturalium Universitatis Purkynianae Brunensis, Vol. 14; J. E. Purkyně University: Brno, 1973, 65 pp., arXiv: [math-ph/0110005](https://arxiv.org/abs/math-ph/0110005).

[12] Krupka, D. A map associated to the Lepagean forms of the calculus of variations in fibered manifolds. *Czech. Math. J.* 1977, 27, 114–118.

[13] Krupka, D. Lepagean forms in higher-order variational theory. In *Modern Developments in Analytical Mechanics*, Academy of Sciences of Turin, Elsevier: 1983, pp. 197–238.

[14] Krupka D.; Štěpánková, O. On the Hamilton form in second order calculus of variations. In: *Proc. of the Meeting “Geometry and Physics”*, Florence, October 1982; Pitagora Editrice Bologna: 1983, 85–101.

[15] Krupka, D.; Musilová, J. Trivial lagrangians in field theory, *Diff. Geom. Appl.* 1998, 9, 293–305.

[16] Krupka, D. Global variational theory in fibred spaces. In *Handbook of Global Analysis*; Krupka D., Saunders, D., Eds.; Elsevier: Amsterdam, 2007; pp. 727–791.

[17] Krupka D.; Krupková, O.; Saunders, D. The Cartan form and its generalizations in the calculus of variations. *Int. J. Geom. Meth. Mod. Phys.* 2010, 7(4), 631–654.

[18] Krupka, D. *Introduction to Global Variational Geometry*. Atlantis Studies in Variational Geometry I; Atlantis Press: Amsterdam, 2015.

[19] Palese, M.; Rossi, O.; Zanello, F. Geometric integration by parts and Lepage equivalents. *Diff. Geom. Appl.* 2022, 81, 101866.

[20] Pérez Álvarez, J. On the Cartan–Betounes form. *Math. Nachrichten* 2019, 292(8), 1791–1799.

[21] Rund, H. A Cartan form for the field theory of Carathéodory in the calculus of variations of multiple integrals, *Lecture Notes in Pure and Appl. Math.* 100 (1985), 455–469.

[22] Saunders, D.J. *The Geometry of Jet Bundles*. Cambridge University Press: Cambridge, 1989.

[23] Saunders D.J.; Crampin, M. The fundamental form of a homogeneous Lagrangian in two independent variables. *J. Geom. Phys.* 2010, 60(11), 1681–1697.

[24] Sniatycki, J. On the geometric structure of classical field theory in Lagrangian formulation, *Proc. Camb. Phil. Soc.* 1970, 68, 475–484.

[25] Urban, Z.; Brajerčík, J. The fundamental Lepage form in variational theory for submanifolds. *Int. J. Geom. Meth. Mod. Phys.* 2018, 15(6), 1850103.

[26] Urban, Z.; Volná, J. On the Carathéodory form in higher-order variational field theory. *Symmetry* 2021, 13, 800.
[27] Voicu, N.; Garoiu, S.; Vaisan, B. On the closure property of Lepage equivalents of Lagrangians, *Diff. Geom. Appl.* 81 (2022), 101852.

[28] Volná, J.; Urban, Z. First-Order Variational Sequences in Field Theory. In *The Inverse Problem of the Calculus of Variations, Local and Global Theory*; D.V. Zenkov, Ed.; Atlantis Press: Amsterdam, 2015, pp. 215–284.

Z. Urban and J. Volná, Department of Mathematics, Faculty of Civil Engineering, VŠB-Technical University of Ostrava, Ludvíka Poděště 1875/17, 708 33 Ostrava-Poruba, Czech Republic

*Email address: zbynek.urban@vsb.cz; jana.volna@vsb.cz*