STRONGLY 1-BOUNDED VON NEUMANN ALGEBRAS

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ABSTRACT. Suppose $F$ is a finite set of selfadjoint elements in a tracial von Neumann algebra $M$. For $\alpha > 0$, $F$ is $\alpha$-bounded if $\mathbb{P}^\alpha(F) < \infty$ where $\mathbb{P}^\alpha$ is the free packing $\alpha$-entropy of $F$ introduced in [9]. We say that $M$ is strongly 1-bounded if $M$ has a 1-bounded finite set of selfadjoint generators $F$ such that there exists an $x \in F$ with $\chi(x) > -\infty$. It is shown that if $M$ is strongly 1-bounded, then any finite set of selfadjoint generators $G$ for $M$ is 1-bounded and $\delta_0(G) \leq 1$; consequently, a strongly 1-bounded von Neumann algebra is not isomorphic to an interpolated free group factor and $\delta_0$ is an invariant for these algebras. Examples of strongly 1-bounded von Neumann algebras include (separable) II$_1$-factors which have property $\Gamma$, have Cartan subalgebras, are non-prime, or the group von Neumann algebras of $SL_n(\mathbb{Z}), n \geq 3$. If $M$ and $N$ are strongly 1-bounded and $M \cap N$ is diffuse, then the von Neumann algebra generated by $M$ and $N$ is strongly 1-bounded. In particular, a free product of two strongly 1-bounded von Neumann algebras with amalgamation over a common, diffuse von Neumann subalgebra is strongly 1-bounded. It is also shown that a II$_1$-factor generated by the normalizer of a strongly 1-bounded von Neumann subalgebra is strongly 1-bounded.

INTRODUCTION

Given a finite set of selfadjoint elements $F = \{x_1, \ldots, x_n\}$ in a tracial von Neumann algebra $(M, \varphi)$, the $(m, k, \gamma)$-microstate space for $F$, $\Gamma(F; m, k, \gamma)$, consists of all $n$-tuples of selfadjoint $k \times k$ complex matrices $(a_1, \ldots, a_n)$ such that for $1 \leq p \leq m$ and $1 \leq i_1, \ldots, i_p \leq n$

$$|tr_k(a_{i_1} \cdots a_{i_p}) - \varphi(x_{i_1} \cdots x_{i_p})| < \gamma$$

where $tr_k$ is the normalized tracial state on the $k \times k$ matrices. These microstate spaces are subsets of Euclidean spaces and hence, Lebesgue volume and packing dimension can be applied to analyze them. Voiculescu introduced these notions in [16] and used them to define the free entropy $\chi(F)$ of $F$ and the free entropy dimension $\delta_0(F)$ of $F$. $\chi(F)$ is an asymptotic logarithmic volume of the microstates of $F$ and $\delta_0(F)$ is an asymptotic packing/Minkowski dimension of the microstates of $F$.

The issue in microstates theory is the invariance problem for $\delta_0$: if $F$ and $G$ are finite sets of selfadjoint elements in $M$, and $F$ and $G$ generate the same von Neumann algebra, then is it the case that $\delta_0(F) = \delta_0(G)$? An affirmative answer to this would show the nonisomorphism of the free group factors.

[9] studied the microstate spaces with elementary techniques from fractal geometry. This attempt to strengthen the connections between microstate theory and geometric measure theory was driven in part by the following two facts, one from free probability, the other from geometric measure theory.

On the free probability side all applications in [5] and [17] show that von Neumann algebras with certain decomposition properties (Property $\Gamma$, Cartan subalgebras, tensor decomposition) satisfy the condition that for any finite generating set $F$ of the von Neumann algebra, $\delta_0(F) \leq 1$. Assuming the algebra embeds into the ultraprodut of the hyperfinite II$_1$-factor, [7] shows that $1 \leq \delta_0(F)$ so that $\delta_0(F) = 1$. Thus, $\delta_0$ is an invariant for such von Neumann algebras and their free entropy dimension is 1. The free group factor $L(F_\infty)$ has a finite set of selfadjoint generators $X$ such that $\delta_0(X) = n$, thus, [5] and [17] show that in fact a free group factors cannot have any of these decomposition

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properties. Significantly, these were the first known kind with separable predual which are prime or fail to have Cartan subalgebras (Popa shows in [13] that the von Neumann algebra on a free group with uncountable many generators is prime and has no Cartan subalgebras; but unfortunately, these von Neumann algebras are inseparable).

On the geometric measure theory side, Besicovitch classified metric spaces with finite Hausdorff 1-measure (these sets automatically have Hausdorff dimension 1). His study concluded with the following, fairly complete answer: any such space \( \Omega \) breaks up into a good and bad part. The good part consists of some Cantor dust and a set which has a tangent at almost all of its points; moreover, this latter set can be contained in a countable union of rectifiable curves. The bad part is a totally irregular set (all local densities have different upper and lower bounds) and no point of the set has a tangent. For the study of sets with nondegenerate Hausdorff \( r \)-measure with \( r > 1 \) the situation was much more complicated and it was some time (about 50 years after Besicovitch’s work initial work) before some of the basic problems were resolved (see [3] for an overview).

The analysis in [5] and [17] decomposes microstate spaces of von Neumann algebras with certain properties into microstates of hyperfinite algebras and sets of negligible packing entropy. Similarly Besicovitch’s work decomposes \( \Omega \) into rectifiable curves and sets of measure 0 (ignoring the irregular part). Both approaches express their respective problems in terms of well-understood spaces - the injective one in the microstate setting, and the real line in the fractal setting. Given some of the already existing connections between the microstates theory and geometric measure theory as well as Besicovitch’s success in classifying sets with finite Hausdorff measure 1, it seems plausible that the invariance problem for \( \delta_0 \) can be answered affirmatively for finite sets with dimension 1. More specifically, is it true that if \( \delta_0(F) = 1 \), then for any other finite set of selfadjoint generators \( G \) for \( F'' \), \( \delta_0(G) = 1 \)? Under some additional conditions the answer is “yes”, and moreover, one can use a decomposition argument akin to Besicovitch’s where amenability takes the place of \([0, 1]\).

Suppose \( F \subset M \) is a finite set of selfadjoint elements such that \( \mathbb{P}^1(F) < \infty \). This analytic condition on \( F \) is called 1-boundedness. Here \( \mathbb{P}^1(F) \) is a kind of packing 1-measure (this assumption can be likened to the assumption in Besicovitch’s classification that the set have bounded Hausdorff 1-measure, though strictly speaking, our assumption is stronger). Assume moreover that \( F \) contains an element \( x \) with finite free entropy. We show then that any other finite set of selfadjoint generators \( G \) for \( F'' \) is 1-bounded and that \( \delta_0(G) \leq 1 \). The argument uses a microstate decomposition relative to a hyperfinite algebra, an idea which appeared in qualitative form in [11]. It can be regarded as a Fubini-type theorem where, as in Besicovitch’s theorem the decomposition breaks up the good part of the space into a negligible set of ”Cantor dust” (relative microstates) and a ”rectifiable subset” (hyperfinite microstates). Now, when \( F'' \) embeds into the ultraproduct of the hyperfinite II\(_1\)-factor, then for any such generating \( G \) for \( F'' \), \( \delta_0(G) = 1 \) (this is a consequence of [7]). If this is not the case, then \( \delta_0(G) = -\infty \). From these facts, it follows that \( \delta_0 \) is indeed an invariant for all von Neumann algebras with such a generating set \( F \). We say that a von Neumann algebra \( M \) is strongly 1-bounded if it has such a generating set \( F \). It is a consequence of [5] and [17] that if \( M \) has property \( \Gamma \), a Cartan subalgebra, a nontrivial tensor product decomposition, or if \( M \) is a group von Neumann algebra of \( SL_n(\mathbb{Z}) \), \( n \geq 3 \), then \( M \) is strongly 1-bounded.

\( \delta_0 \) is an invariant for strongly 1-bounded von Neumann algebras and their amplifications. Moreover, if \( M \) and \( N \) are strongly 1-bounded and \( A \) and \( B \) are diffuse von Neumann subalgebras of \( M \) and \( N \), respectively, such that \( A \) and \( B \) generate a strongly 1-bounded von Neumann algebra, then the von Neumann algebra generated by \( M \) and \( N \) is strongly one-bounded and thus, not isomorphic to an interpolated free group factor. This implies, in particular, that if \( D = M \cap N \) is diffuse, then \( M \ast_D N \) is not isomorphic to a free group factor.

The outline of the paper is as follows. We start with a brief list of notation followed by the first section which defines \( \alpha \)-bounded sets, remarks on some equivalent formulations, and has a short list of examples. The second section collects some facts on the decomposition of microstate spaces relative
to a single selfadjoint; it is the decomposition of the Fubini/Besicovitch-type described above. The third section states and proves the main result. The fourth consists of nonisomorphism applications to amalgamated free products and other types of von Neumann algebras. The fifth and final section is a generalization of [5] and [17]. It will imply, in particular, that if a $II_1$-factor can be generated by the normalizer of a strongly 1-bounded von Neumann subalgebra, then $N$ is strongly 1-bounded. It follows that an interpolated free group factor $L(F_r)$ cannot be isomorphic to the crossed product of a strongly 1-bounded von Neumann algebra with a group action.

NOTATION

Throughout $M$ denotes a tracial von Neumann algebra with separable predual. For any $k, n \in \mathbb{N}, R > 0$, $M_k^{sa}(\mathbb{C})$ denotes the $k \times k$ complex selfadjoint matrices, $(M_k^{sa}(\mathbb{C}))^n$ denotes $n$-tuples of entries in $M_k^{sa}(\mathbb{C})$ and $(M_k^{sa}(\mathbb{C}))_R$ denotes the elements of $M_k^{sa}(\mathbb{C})$ with operator norms no greater than $R$. For $\xi = (\xi_1, \ldots, \xi_n) \in (M_k^{sa}(\mathbb{C}))^n$ we write $|\xi|_2 = (\sum_{i=1}^{n} |tr_k(\xi_i^2)|)^{1/2}$ where $tr_k$ is the tracial state on $M_k(\mathbb{C})$, the $k \times k$ matrices, and for a $k \times k$ unitary $u$, $u\xi u^* = (u\xi_1 u^*, \ldots, u\xi_n u^*)$.

1. $\alpha$-BOUNDED SETS

In this section assume $F$ is a finite set of selfadjoint elements of $M$. Recall the definitions of $\mathbb{K}_{\epsilon,\infty}(F)$ and $\mathbb{P}_{\epsilon,\infty}(F)$ introduced in [11]. We have the following definition:

**Definition 1.1.** For $\alpha > 0$, $F$ is said to be $\alpha$-bounded if for some $K, \epsilon_0 > 0$ and any $\epsilon_0 > \epsilon > 0$,

$$\mathbb{K}_{\epsilon,\infty}(F) \leq \alpha \cdot |\log \epsilon| + K.$$  

**Remark 1.2.** It is immediate from [8] and Lemma 2.2 of [11], that if $F$ is $\alpha$-bounded, then $\delta_0(F) \leq \alpha$.

Recall that cutoff constants for the operator norm were used in the definition of $\mathbb{K}_{\epsilon} (F)$ and $\mathbb{P}_{\epsilon}(F)$ and that $\epsilon$ quantities $\mathbb{K}_{\epsilon,R}(F)$ and $\mathbb{P}_{\epsilon,R}(F)$ were introduced in [8] where the microstates spaces have cutoff constants.

**Lemma 1.3.** For any $\epsilon > 0$ and $R \geq \max_{x \in F} \{\|x\|\}$ we have

$$\mathbb{P}_{4\epsilon,\infty}(F) \leq \mathbb{K}_{2\epsilon,\infty}(F) \leq \mathbb{K}_{\epsilon,R}(F) \leq \mathbb{K}_{\epsilon}(F) \leq \mathbb{K}_{\epsilon,\infty}(F) \leq \mathbb{P}_{4\epsilon,\infty}(F).$$

**Proof.** By Lemma 2.1 of [11] $\mathbb{K}_{2\epsilon,\infty}(F) \leq \mathbb{K}_{\epsilon,R}(F) \leq \mathbb{K}_{\epsilon}(F)$. The rest of the statement follows from the fact that for any metric space $\Omega$ and $\epsilon > 0$, $P_{\epsilon}(\Omega) \geq K_{2\epsilon}(\Omega) \geq P_{4\epsilon}(\Omega)$ where $P_{\epsilon}(\Omega)$ is the maximum number in a collection of disjoint open $\epsilon$-balls with centers in $\Omega$ and $K_{\epsilon}(\Omega)$ is the minimum number of open $\epsilon$-balls required to cover $\Omega$.

The free packing $\alpha$-entropy of $F$ was defined in [9] as $\mathbb{P}_{\alpha}(F) = \sup_{R > 0} \mathbb{P}_{R}^{\alpha}(F)$ where $\mathbb{P}_{R}^{\alpha}(F) = \limsup_{\epsilon \to 0} \mathbb{P}_{\epsilon,R}(F) + \alpha \log 2\epsilon$. The free packing $\alpha$-entropy has the same relationship to $\delta_0$ that Hausdorff measure (free Hausdorff entropy) has to Hausdorff dimension (free Hausdorff dimension). From Lemma 1.3 and Lemma 3.11 of [9] we have:

**Corollary 1.4.** Suppose $F = \{x_1, \ldots, x_n\}, \{s_1, \ldots, s_n\}$ is a semicircular family free with respect to $F$, and $R \geq \max_{x \in F} \{\|x\|\}$. The following conditions are equivalent:

- $F$ is $\alpha$-bounded.
- There exist $K, \epsilon_0 > 0$ such that for all $\epsilon_0 > \epsilon > 0$, $\mathbb{K}_{\epsilon,R}(F) \leq \alpha \cdot |\log \epsilon| + K$.
- There exist $K, \epsilon_0 > 0$ such that for all $\epsilon_0 > \epsilon > 0$, $\mathbb{P}_{\epsilon,\infty}(F) \leq \alpha \cdot |\log \epsilon| + K$.
- There exist $K, \epsilon_0 > 0$ such that for all $\epsilon_0 > \epsilon > 0$, $\mathbb{P}_{\epsilon,R}(F) \leq \alpha \cdot |\log \epsilon| + K$.
- $\mathbb{P}_{\alpha}(F) < \infty$.
- $\limsup_{\epsilon \to 0} \left(\chi(x_1 + \epsilon s_1, \ldots, x_n + \epsilon s_n : s_1, \ldots, s_n) + (n - \alpha)|\log \epsilon|\right) < \infty$.  


Example 1.5. Suppose $F''$ can be generated by a sequence of Haar unitaries $\langle u_j \rangle_{j=1}^{\infty}$ such that for each $j$, $u_{j+1}^* u_j u_{j+1} \in \{ u_1, \ldots, u_j \}''$. By [5] $F$ is 1-bounded. Indeed, it is a consequence of Lemma 5.1 of [5] that when $F$ consists of contractions and $\epsilon \in (0, 1)$, $\mathbb{K}_\epsilon(F) \leq |\log \epsilon| + 100 + \frac{1}{2} \log(\#F)$, and this condition clearly implies that $F$ is 1-bounded for general $F$.

This class of von Neumann algebras considered by Ge and Shen include the following cases: 1) $F$ generates a von Neumann algebra with a regular, diffuse, hyperfinite von Neumann subalgebra; 2) $F$ generates a von Neumann algebra of the form $A \otimes B$ where $A$ and $B$ are diffuse; 3) $F$ generates the group von Neumann algebras obtained from $\text{SL}_n(\mathbb{Z})$ for $n \geq 3$. In Section 5 we will generalize this example.

Example 1.6. Suppose $F'' = M$ and $M^\omega$ is the ultraproduct of $M$ for some nontrivial $\omega$. If $A \subset M^\omega$ is a diffuse, hyperfinite subalgebra such that the von Neumann algebra generated by the normalizer of $A$ contains $B$, then using the free entropy formulation in Corollary 1.4, it follows from Theorem 7.3 of [17] that $F$ is 1-bounded. In particular, if $M$ has property $\Gamma$, then $F$ is 1-bounded.

Example 1.7. Suppose $F$ is a finite generating set of the type considered in [10] for the interpolated free group factor $L(F_r)$, $r > 1$, of Dykema and Radulescu ([2] and [14]). It follow that $F$ is $r$-bounded. By Theorem 7.4 of [9] this also holds if $F$ can be broken up into a free collection of subsets, each of which generates a finite dimensional algebra.

2. Microstates Relative to a Single Selfadjoint

In this section we want to find packing entropy estimates with respect to finite sets of selfadjoints of the form $\{ x \} \cup F$ where $\chi(x) > -\infty$. These will come from the microstates of $F$ relative to a fixed sequence of microstates for $x$. Much of this will be a regurgitation of the material in [11] but we do so for completeness and because specific properties of such sets will be exploited to give more quantitative estimate. For the remainder of this section $F \subset M$ is a finite set of selfadjoint elements and $x \in M$ is selfadjoint. Fix $R > 0$ such that $R$ is greater than the operator norm of any element in $\{ x \} \cup F$. It is easy to see that there exists a sequence $\langle x_k \rangle_{k=1}^{\infty}$ such that for each $k$ $x_k \in M^a_k(\mathbb{C})$, $\|x_k\| < R$, and for any $m \in \mathbb{N}$ and $\gamma > 0$ $x_k \in \Gamma(\{ x \} \cup F; m, k, \gamma)$ for sufficiently large $k$. Fix this sequence $\langle x_k \rangle_{k=1}^{\infty}$. Recall from [11] the microstate spaces $\Xi(F; m, k, \gamma)$ for $F$ relative to $\langle x_k \rangle_{k=1}^{\infty}$.

$$\Xi(F; m, k, \gamma) = \{ \xi : (x_k, \xi) \in \Gamma(\{ x \} \cup F; m, k, \gamma) \}.$$

Define successively for $\epsilon > 0$,

$$\mathbb{K}_\epsilon(\Xi(F; m, \gamma)) = \limsup_{k \to \infty} k^{-2} \cdot \log K_\epsilon(\Xi(F; m, k, \gamma)),$$

$$\mathbb{K}_\epsilon(\Xi(F)) = \inf \{ \mathbb{K}_\epsilon(F; m, \gamma) : m \in \mathbb{N}, \gamma > 0 \},$$

where the packing quantities are taken with respect to $| \cdot |_2$. In a similar fashion, we define $\mathbb{P}_\epsilon(\Xi(F))$ by replacing the $K_\epsilon$ above with $P_\epsilon$. We will also use the notation $\Xi_R(F; m, k, \gamma)$ and $K_{\epsilon,R}(\Xi(F; m, \gamma))$, $P_{\epsilon,R}(\Xi(F; m, \gamma))$ for the quantities and sets where the cutoff constant $R$ is used (so these are the relative microstates and associated quantities where the operator norms of the entries are all less than $R$).

For a finite set of selfadjoint elements $X$ write $\chi(X)$ for the quantity obtained by replacing the $\limsup_{k \to \infty}$ in the definition of $\chi(X)$ with a $\liminf_{k \to \infty}$. $\chi(X) \geq \chi(X)$ (when equality occurs $X$ is said to be regular, see [18]). We also write $\mathbb{H}^\alpha(X)$ and $\mathbb{P}^\alpha(X)$ for the quantities obtained by replacing the $\limsup_{k \to \infty}$ in the definitions of $\mathbb{H}^\alpha(X)$ and $\mathbb{P}^\alpha(X)$ with a $\liminf_{k \to \infty}$.

Lemma 2.1. If $\{ x \} \cup F$ is an $\alpha$-bounded set and $\chi(x) > -\infty$, then there exist constants $C, \epsilon_1 > 0$ dependent on $\{ x \} \cup F$ such that for $\epsilon_1 > t > 0$
Proof. By Proposition of 4.5 of [16] and Lemma 3.7 of [9] we have \( H_1^1(x) = \chi(x) + \frac{1}{2} \log(\frac{2}{\pi e}) = \chi(x) + \frac{1}{2} \log(\frac{2}{\pi e}) > -\infty \). Thus, there exists an \( r > 0 \) such that for all \( r > t > 0, H_1^1(x) \geq c \) where \( c = \chi(x) + \frac{1}{2} \log(\frac{2}{\pi e}) - 1 \). It is easy to see that for such \( t \),

\[
c \leq H_1^1(x) \leq P_t(x) + \log 4t.
\]

Thus, for \( r > t > 0, c - \log 4 + |\log t| < P_t(x) \). Also, \( \{x\} \cup F \) is \( \alpha \)-bounded; let \( \epsilon_0 > 0 \) and \( K \) be as in the definition of \( \alpha \)-boundedness.

Suppose \( 0 < t < \min\{r, \epsilon_0\} \). There exist \( m_1 \in \mathbb{N} \) and \( \gamma_1 > 0 \) such that for all \( m > m_1 \) and \( 0 < \gamma < \gamma_1 \),

\[
\mathbb{P}_t(F; m, \gamma) \leq r \cdot |\log t| + K.
\]

Clearly for all \( m \in \mathbb{N} \) and \( \gamma > 0 \),

\[
c - \log 4 + |\log t| \leq P_t(x; m, \gamma).
\]

By [7] and [11], there exist \( m_2 \in \mathbb{N} \) and \( \gamma_2 > 0 \) such that if \( a, b \in \Gamma(x, m_2, k, \gamma_2) \), then there exists a \( k \times k \) unitary \( u \) such that \( |uau^* - b|_2 < \frac{1}{100} \). Combining this with (2) it follows that there exist \( m \geq m_1 + m_2, 0 < \gamma < \min\{\gamma_1, \gamma_2\} \), and \( k \) sufficiently large, unitaries \( \langle v_{\lambda k} \rangle_{\lambda \in \Lambda_k} \) such that the balls of \( \Gamma(x; m, k, \gamma) \) with centers \( \langle v_{\lambda k} \rangle_{\lambda \in \Lambda_k} \) and radii \( \frac{99t}{100} \) form a collection of disjoint subsets of \( \Gamma(x; m, k, \gamma) \) and \( \liminf_{k \rightarrow \infty} k^{-2} \cdot \log \#\Lambda_k \rightarrow P_t(x; m, \gamma) \). For each \( m \geq m_1 + m_2 \), \( \min\{\gamma_1, \gamma_2\} > \gamma > 0 \), and sufficiently large \( k \) pick a subset \( \langle \xi_{jk} \rangle_{j \in J_k} \) of \( \Xi(F; m, k, \gamma) \) of maximal cardinality with respect to the condition that the \( \epsilon \)-balls of \( \Xi(F; m, k, \gamma) \) with centers \( \xi_{jk} \) are disjoint. It is easy to see that the balls of \( \Gamma(\{x\} \cup F; m, k, \gamma) \) with centers

\[
\langle (v_{\lambda k} x_k v_{\lambda k}^*, v_{\lambda k} \xi_{jk} v_{\lambda k}^*) \rangle_{(\lambda,j) \in \Lambda_k \times J_k}
\]

and radii \( \frac{99t}{100} \) are a pairwise disjoint. So, using (1) and (2) we have for \( m \geq m_1 + m_2 \) and \( 0 < \gamma < \min\{\gamma_1, \gamma_2\} \)

\[
\alpha |\log t| + K \geq \mathbb{P}_{t} \cdot 1(\{x\} \cup F; m, \gamma)
\geq \limsup_{k \rightarrow \infty} k^{-2} \cdot \log (\#\Lambda_k \cdot \#J_k)
\geq \liminf_{k \rightarrow \infty} k^{-2} \cdot \log \#\Lambda_k + \limsup_{k \rightarrow \infty} k^{-2} \cdot \log P_t(\Xi(F; m, k, \gamma))
\geq \mathbb{P}_t(x; m, \gamma) + \limsup_{k \rightarrow \infty} k^{-2} \cdot \log P_t(\Xi(F; m, k, \gamma))
\geq c - \log 4 + |\log t| + \mathbb{P}_t(\Xi(F)).
\]

Grouping the constants together on one side we have for all \( \min\{r, \epsilon_0\} > t > 0, \)

\[
(\alpha - 1) |\log t| + K - c + \log 4 \geq \mathbb{P}_t(\Xi(F)) \geq K_{2t}(\Xi(F)).
\]

\[\square\]

Lemma 2.2. Suppose \( B = (\Sigma_{y \in \{x\} \cup F} \|x\|^2)^{\frac{1}{2}} \). There exists \( L, \epsilon_2 > 0 \) independent of \( \{x\} \cup F \) such that for \( 0 < \epsilon < \epsilon_2 \),

\[
K_{\epsilon, R}(\{x\} \cup F) \leq \log((4B + 6)L) + |\log \epsilon| + K_{\frac{2B + 6}{2B + 6} R}(\Xi(F)).
\]
Proof. By [15] or [19] there are $L, \epsilon_0 > 0$ such that for $\epsilon_0 > \epsilon > 0$ and for any $k \in \mathbb{N}$ there exists an $\epsilon$-net for $U_k$ with respect to the quotient metric induced by $| \cdot |_\infty$ with cardinality no greater than $\left(\frac{L}{\epsilon}\right)^{k^2}$. Suppose $m \in \mathbb{N}$ and $\gamma > 0$. Observe that there exists $\epsilon > r > 0$ so small that for any $k$, if $(\xi, \eta) \in \Gamma_R(\{x\} \cup F; m, k, \gamma/2)$, $|(\xi, \eta) - (a, b)|_2 < r$, and all the entries of $(a, b)$ have operator norms less than or equal to $\epsilon$, then $(x, a) \in \Gamma_R(\{x\} \cup F; m, k, \gamma)$. There also exist $m_1 \in \mathbb{N}$ and $\gamma_1 > 0$ such that if $y, z \in \Gamma(x; m_1, k, \gamma_1)$, then there exists a $k \times k$ unitary $u$ satisfying $|uu^* - z|_2 < r$. Finally, we can find $m_2$ and $\gamma_2$ such that for any $(\xi, \eta) \in \Gamma(\{x\} \cup F; m_2, k, \gamma_2)$, $|(\xi, \eta)|_2 < B + 1$. Set $m_3 = m + m_1 + m_2$ and $\gamma_3 = \min\{\gamma/2, \gamma_1, \gamma_2\}$.

For each $k$ find an $\epsilon$-net $(\eta_{jk})_{j \in J_k}$ for $\Xi_R(F; m, k, \gamma)$ with respect to $| \cdot |_2$ of minimum cardinality. Find for each $k$ a set of unitaries $(u_{gk})_{g \in G_k}$ such that they form $\epsilon$-net with respect to the operator norm and such that

$$\#G_k \leq \left(\frac{L}{\epsilon}\right)^{k^2}.$$ Consider

$$\langle(u_{gk}x_ku_{gk}^*, u_{gk}\eta_{jk}u_{gk}^*)\rangle_{(g, j) \in G_k \times J_k}.$$ I claim that this set is a $(4B + 6)\epsilon$-net for $\Gamma_R(\{x\} \cup F; m_3, k, \gamma_3)$.

To see this suppose $(\xi, \eta) \in \Gamma_R(\{x\} \cup F; m_3, k, \gamma_3)$. By the selection of $m_3$ and $\gamma_3$ there exists a $u \in U_k$ such that $|uu^* - \xi|_2 < r$. Taking into account the stipulation on $r$ this implies that $(u^*x_ku, \eta) \in \Gamma_R(\{x\} \cup F; m, k, \gamma) \iff (x_k, uu^*) \in \Gamma_R(\{x\} \cup F; m, k, \gamma)$, whence $uu^* \in \Xi_R(F; m, k, \gamma)$. There exists an $g \in G_k$ such that $\|u - u_{gk}\| < \epsilon$. Consequently,

$$|uu^* - \xi|_2 < 2(B + 1)\epsilon + |uu^* - \xi|_2 \leq (2B + 3)\epsilon.$$ $uu^* \in \Xi_R(F; m, k, \gamma)$ so there exists a $j \in J_k$ such that $|\eta_{jk} - uu^*|_2 < \epsilon$. Now,

$$|uu^* - \eta_j|_2 < 2(B + 1)\epsilon + |uu^* - \eta_j|_2 < (2B + 3)\epsilon.$$ $|(uu^* - \eta_j)(u^* - \eta_j)|_2 < (4B + 6)\epsilon$ as desired.

It follows that

$$\mathbb{K}_{(4B + 6)\epsilon, R}(\{x\} \cup F; m_3, \gamma_3) \leq \limsup_{k \to \infty} k^{-2} \cdot \log(\#G_k \cdot \#J_k) \leq \log L + |\log \epsilon| + \limsup_{k \to \infty} k^{-2} \cdot \log K_{\epsilon}(\Xi_R(F; m, k, \gamma)).$$

Given $0 < \epsilon < \epsilon_0$ and any $m \in \mathbb{N}$ and $\gamma > 0$ we produced $m_3 \in \mathbb{N}$ and $\gamma_3 > 0$ so that the above inequality holds. Thus

$$\mathbb{K}_{(4B + 6)\epsilon, R}(\{x\} \cup F) \leq \log L + |\log \epsilon| + \mathbb{K}_{\epsilon, R}(\Xi(F)).$$ This clearly implies that for $0 < \epsilon < (4B + 6)\epsilon_0$

$$\mathbb{K}_{\epsilon, R}(\{x\} \cup F) \leq \log((4B + 6)L) + |\log \epsilon| + \mathbb{K}_{\epsilon, R}(\Xi(F)).$$
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We now come to our main result which says that if \( \{x\} \cup F \) is a 1-bounded set of selfadjoint generators for \( M \) such that \( \chi(x) > -\infty \), then any other finite set of selfadjoint generators \( G \) for \( M \) is 1-bounded. First a lemma:

**Lemma 3.1.** If \( X \) and \( Y \) are finite sets of selfadjoint elements such that \( Y \subseteq X'' \), then for any \( R, \epsilon > 0 \), \( \mathbb{K}_{\epsilon,R}(X) \leq \mathbb{K}_{\epsilon,R}(X \cup Y) \). In particular, if \( X \cup Y \) is \( r \)-bounded, then \( X \) is \( r \)-bounded.

**Proof.** It suffices to prove the first statement. This is a repetition of Lemma 3.6 in [9]. Suppose \( R \) exceeds the operator norms of any of the elements in \( X \cup Y \). Given \( m \in \mathbb{N} \) and \( \epsilon, \gamma > 0 \) there exist \( m_1 \in \mathbb{N} \), \( R, \gamma_1 > 0 \) and a \( \#Y \)-tuple \( f \) of polynomials in \( n \) noncommutative variables such that if \( \xi \in \Gamma_R(X; m_1, k, \gamma_1) \) then

\[
(\xi, f(\xi)) \in \Gamma_R(X \cup Y; m, k, \gamma).
\]

For each \( k \) this map from \( \Gamma_1(X; m_1, k, \gamma_1) \) to \( \Gamma_R(X \cup Y; m, k, \gamma) \) defined by sending \( \xi \) to \( (\xi, f(\xi)) \) increases distances with respect to \( | \cdot |_2 \). Hence

\[
\mathbb{K}_{\epsilon,R}(X; m_1, \gamma_1) \leq \mathbb{K}_{\epsilon,R}(X \cup Y; m, \gamma).
\]

This being true for any \( m, \gamma, \epsilon \) the result follows from Corollary 1.4. \( \square \)

**Theorem 3.2.** If \( \{x\} \cup F \) is a 1-bounded finite set of selfadjoint generators for \( M \) such that \( \chi(x) > -\infty \), then for any other finite set of selfadjoint generators \( G \) for \( M \), \( G \) is a 1-bounded set. In particular, for such \( G \), \( \delta_0(G) \leq 1 \).

**Proof.** Suppose \( G \) is a finite set of selfadjoint generators for \( M \). By Lemma 3.1 in order to show that \( G \) is 1-bounded, it suffices to show that \( \{x\} \cup F \cup G \) is 1-bounded.

Fix \( R \) such that \( R \) exceeds the operator norms of any of the elements in \( \{x\} \cup F \cup G \). Find a sequence \( \{x_k\}_{k=1}^{\infty} \) with the associated \( C, \epsilon_1 > 0 \) as constructed in Lemma 2.1 satisfying the covering bound for the relative microstates \( \Xi(F; m, k, \gamma) \). Applying Lemma 2.2 to the set \( F \cup G \), if \( B = (\Sigma_{y \in \{x\} \cup F \cup G} \|y\|_2^2)^{1/2} \), then there exist constants \( K, \epsilon_2 > 0 \) with \( K \) dependent only on \( B \) such that for all \( \epsilon_2 > \epsilon > 0 \),

\[
\mathbb{K}_{\epsilon,R}(\{x\} \cup F \cup G) \leq K + | \log \epsilon| + \mathbb{K}_{4B+6,R}(\Xi(F \cup G)).
\]

Set \( D = 4B + 6 \) and suppose \( \epsilon_2 > \epsilon > 0 \). There exists an \( \#G \)-tuple of polynomials in \( \#F + 1 \) noncommuting variables, \( \Phi \), such that \( |\Phi(x, F) - G|_2 < \epsilon(10D)^{-1} \). Moreover, regarding \( \Phi \) as a map from \( (M_k^{sa}(C))_{R}^{\#F+1} \rightarrow (M_k^{sa}(C))^{\#G} \), \( \Phi \) has a bounded Lipschitz constant \( L_k \) with respect to the \( | \cdot |_2 \)-norm and \( \sup_{k \in \mathbb{N}} L_k < \infty \). Choose \( L \) greater than \( \sup_{k \in \mathbb{N}} L_k \) and greater than \( \max\{\epsilon^{-1}, 1\} \).

The selection of \( C, \epsilon_1 > 0 \) implies that for \( 2^{-1} \cdot \min\{\epsilon(10(DL))^{-1}, \epsilon_1\} > t > 0 \),

\[
C \geq \mathbb{K}_t(\Xi(F)).
\]

So, there exist \( m_1 \in \mathbb{N} \) and \( \gamma_1 > 0 \) such that for \( m \geq m_1 \) and \( \gamma > \gamma \),

\[
(4) \quad C + 1 \geq \lim_{k \to \infty} \sup k^{-2} \cdot \log K_t(\Xi_R(F; m, k, \gamma)).
\]

Now there clearly exist \( m_2 \in \mathbb{N} \) and \( \gamma_2 > 0 \) such that if \( m > m_2 \) and \( \gamma_2 > \gamma > 0 \), then for any \( k \), \( (\xi, \eta) \in \Xi(F \cup G; m, k, \gamma) \Rightarrow |\Phi(x_k, \xi) - \eta|_2 \leq \epsilon(10D)^{-1} \). Suppose \( m \in \mathbb{N} \) and \( \gamma > 0 \) with \( m > m_1 + m_2 \) and \( \min\{\gamma_1, \gamma_2\} > \gamma \). By (4) we can find for \( k \) sufficiently large, an \( t \)-net \( \{\xi_j\}_{j \in J_k} \) for \( \Xi_R(F; m, k, \gamma) \) which satisfies
Thus, \( \epsilon \) algebras.

\[ \epsilon D^{-1}\text{-cover for } \Xi_R(F \cup G; m, k, \gamma). \] Indeed, suppose \( \tilde{\epsilon}(\tilde{\xi}, \tilde{\eta}) \in \Xi_R(F \cup G; m, k, \gamma) \). Then by definition, \( \tilde{\xi} \in \Xi_R(F; m, k, \gamma) \) so that there exists some \( j_0 \in J_k \) with \(|\tilde{\xi} - \tilde{\xi}_{j_0}| \leq \epsilon(10DL)^{-1} \). Since both \( \tilde{\xi} \) and \( \tilde{\xi}_{j_0} \), are \#-tuples of operators with norms no greater than \( R \), \( |\Phi(x_k, \tilde{\xi}) - \Phi(x_k, \tilde{\xi}_{j_0})| \leq L t \leq \epsilon(10D)^{-1} \). On the other hand, \( |\tilde{\eta}| \leq \epsilon(10D)^{-1} \) so that \( |\Phi(x_k, \tilde{\xi}_{j_0}) - \tilde{\eta}| \leq \epsilon(5D)^{-1}. \) Thus, \( |\tilde{\epsilon}(\tilde{\xi}, \tilde{\eta}) - (\tilde{\epsilon}_{j_0k}, \Phi(x_k, \tilde{\xi}_{j_0k}))| \leq t + \epsilon(5D)^{-1} < \epsilon D^{-1}. \)

By the preceding paragraph and (5) we conclude that for \( m > m_1 + m_2 \), \( \min{\gamma_1, \gamma_2} > \gamma > 0 \) and \( k \) sufficiently large,

\[ k^{-2} \cdot \log \left[ K_{4k^2 + 6R}^{\Xi_R(\Xi(F \cup G; m, k, \gamma))} \right] \leq C + 1. \]

Taking a \( \limsup \) on both sides yields

\[ \limsup_{k \to \infty} K_{4k^2 + 6R}^{\Xi_R(\Xi(F \cup G))} \leq \limsup_{k \to \infty} K_{4k^2 + 6R}^{\Xi_R(\Xi(F \cup G; m, k, \gamma))} \leq C + 1 \]

Stuffing (6) into (3) yields for all \( 0 < \epsilon < \epsilon_2 \)

\[ \limsup_{k \to \infty} K_{\epsilon R}^{\Xi_R(\{x\} \cup F \cup G)} \leq K + C + 1 + |\log \epsilon|. \]

By Corollary 1.4, \( \{x\} \cup F \cup G \) is 1-bounded, and thus, by Lemma 3.1 so is \( G \).

1-boundedness is a condition on a finite set and its microstate spaces, and makes no direct reference to the generated algebra. The point of Theorem 3.2 is that 1-boundedness of an appropriate finite set imposes 1-boundedness on any other generating set of the von Neumann algebra of the initial set. In this way, 1-boundedness is a property of the set which propagates to a property of the generated von Neumann algebra.

In view of Theorem 3.2, we make the following definition:

**Definition 3.3.** \( M \) is strongly 1-bounded if \( M \) has a 1-bounded finite set of selfadjoint generators \( F \) such that \( F \) contains an element \( x \) with \( \epsilon(x) > -\infty. \)

**Remark 3.4.** Strong 1-boundedness of a tracial von Neumann algebra requires firstly, that the algebra have a finite set of selfadjoint generators with one element having finite free entropy and secondly, that the free packing 1-entropy of such a generating set be finite. The first condition is equivalent to having the von Neumann algebra possess a finite generating set. It is unknown whether an arbitrary von Neumann algebra (with separable predual) can be generated by a finite set of selfadjoint elements. However, it is well-known that the examples considered in Section 1 are finitely generated when factoriality is imposed. Suppose \( M \) is a \( \Pi_1 \)-factor (with separable predual, as is always tacitly assumed). By [4] if \( M \) has property \( \Gamma \), then \( M \) can be generated by a finite number of elements; thus, by Example 1.6 it follows that \( M \) is strongly 1-bounded. Similarly, by [6], if \( M \) has a regular, diffuse, hyperfinite von Neumann subalgebra or is non-prime, then \( M \) has a finite set of selfadjoint generators and by Example 1.5, it follows that \( M \) is strongly 1-bounded.

By Theorem 3.2 we have:

**Corollary 3.5.** If \( M \) is strongly 1-bounded, then for any finite set of selfadjoint generators \( X \) of \( M \), \( \delta_0(X) \leq 1 \). Moreover, if \( M \) is embeddable into the ultrapower of the hyperfinite \( \Pi_1 \)-factor, then for any finite set of selfadjoint generators \( X \) of \( M \), \( \delta_0(X) = 1 \). Thus, \( \delta_0 \) is an invariant for these algebras.

We have by Lemma 5.2 of [12]
Lemma 4.1. Suppose there exists a sequence \( \chi \) (the von Neumann algebra generated by \( \alpha \)) to the contrary that if \( \alpha > 0 \), then for any finite set of selfadjoint generators \( X \) for \( M_\alpha \), the amplification of \( M \) by \( \alpha \), \( \delta_0(X) \leq 1 \).

Recall that Dykema and Radulescu ([2], [14]) defined a family of von Neumann algebras, \( L(F_r) \), \( 1 < r \leq \infty \) such that for integer values \( r \in \mathbb{N} \cup \{ \infty \} \), \( L(F_r) \) coincides with the free group factor on \( r \) generators. These von Neumann algebras are called the interpolated free group factors. Voiculescu was the first to show ([16]) that there exists a finite set of generators \( X \) for \( L(F_r) \) such that \( \delta_0(X) = r \). Thus, by Corollary 3.5 if \( M \) is strongly 1-bounded, then \( M \) cannot be isomorphic to \( L(F_r) \) for \( 1 < r < \infty \). We include \( \epsilon \) more by shows that \( M \) cannot be isomorphic to \( L(F_\infty) \):

Lemma 3.7. If \( 1 < s \in \mathbb{N} \cup \{ \infty \} \) and \( \langle M_i \rangle_{i=1}^s \) is a sequence of finitely generated diffuse von Neumann subalgebras of \( M \) such that each \( M_i \) embeds into the ultraproduct of the hyperfinite II_1-factor, then \( \ast_{i=1}^s M_i \) cannot be strongly 1-bounded.

Proof. Suppose to the contrary that \( \ast_{i=1}^s M_i \) can be strongly 1-bounded. By definition there exists a finite set of selfadjoint generators \( X \) for \( \ast_{i=1}^s M_i \). There exist finite sets of selfadjoint elements, \( F_j \) such that \( F_j'' = M_j \ast_{i=1}^s M_i, j = 1, 2 \). By [7] and [17] it follows that \( 2 \leq \delta_0(F_1 \cup F_2) \). The embeddability assumption on the \( M_i \) and the asymptotic freeness results of [18] imply that \( \delta_0(F_1 \cup F_2) \leq \delta_0(X \cup F_1 \cup F_2) \). On the other hand, \( X \cup F_1 \cup F_2 \) is a finite set of generators for \( \ast_{i=1}^s M_i \) so Theorem 3.2 implies that \( X \cup F_1 \cup F_2 \) is 1-bounded; consequently by Remark 1.2, \( \delta_0(X \cup F_1 \cup F_2) \leq 1 \). Putting this together we have

\[
2 \leq \delta_0(F_1 \cup F_2) \leq \delta_0(X \cup F_1 \cup F_2) \leq 1.
\]

This is preposterous. \( \ast_{i=1}^s M_i \) cannot be strongly 1-bounded. \( \square \)

In [1], Nate Brown constructed finite sets \( X \) of selfadjoint elements such that \( \chi(X) > -\infty, \#X > 1 \), and \( X'' \neq L(F_r) \). In particular by [17] \( \delta_0(X) = \#X \neq 1 \) for such \( X \). Combining all this with Corollary 3.5 and Lemma 3.7 above we have

Corollary 3.8. If \( M \) is strongly 1-bounded, then \( M \) is neither isomorphic to an interpolated free group factor nor to the free perturbation algebras considered in [1].

4. AN APPLICATION TO AMALGAMATED FREE PRODUCTS

In this section we want to produce some other examples of 1-bounded sets. In particular we show (see Corollary 4.2 and Corollary 4.4) that if \( A \) and \( B \) are von Neumann algebras which are nonprime, have Cartan subalgebras, or have property \( \Gamma \) and \( D \) is a diffuse subalgebra of \( A \) and \( B \), then \( A \ast_D B \) is not isomorphic to an interpolated free group factor. The proof will again rest on the relative hyperfinite decomposition results in Section 2.

If \( M \) and \( N \) are von Neumann algebras acting on the same Hilbert space, then we denote by \( M \vee N \) the von Neumann algebra generated by \( M \) and \( N \).

Lemma 4.1. Suppose \( F_1 \) and \( F_2 \) are finite sets of selfadjoint elements in \( M \) and \( x^* = x \in M \) satisfies \( \chi(x) > -\infty \). If \( \{ x \} \cup F_1 \) and \( \{ x \} \cup F_2 \) are \( \alpha_1 \)-bounded and \( \alpha_2 \)-bounded, then \( \{ x \} \cup F_1 \cup F_2 \) is \( (\alpha_1 + \alpha_2 - 1) \)-bounded.

Proof. Fix \( R \) greater than the operator norms of any of the elements in \( F_1 \cup F_2 \). By Section 2 and Lemma 2.1 there exists a sequence \( \langle x_k \rangle_{k=1}^\infty \) such that there exist constants \( C_1, C_2, \epsilon_1, \epsilon_2 > 0 \) such that for any \( m \in \mathbb{N} \) and \( \gamma > 0 \), \( x_k \in \Gamma(x; m, k, \gamma) \) for \( k \) sufficiently large and such that the microstate spaces \( \Xi(F_i; m, k, \gamma), i = 1, 2 \) relative to the sequence \( \langle x_k \rangle_{k=1}^\infty \) satisfy for all \( \epsilon_i > t > 0 \)

\[
(\alpha_i - 1)|\log t| + C_i \geq \mathbb{K}_t(\Xi(F_i)).
\]
Notice that we have arranged the same sequence $x_k$ with respect to which we consider the conditioned microstates for $F_1$ and $F_2$ (this is exactly what was arranged in Lemma 2.1 and Lemma 2.2). Because $\Xi(F_1; m, k, \gamma) \times \Xi(F_2; m, k, \gamma) \supset \Xi(F_1 \cup F_2; m, k, \gamma)$, it follows that for all $\min\{\epsilon_1, \epsilon_2\} > \epsilon > 0$,

$$\left(\alpha_1 + \alpha_2 - 2\right)\log \epsilon + C_1 + C_2 \geq \mathbb{K}_\epsilon(\Xi(F_1)) + \mathbb{K}_\epsilon(\Xi(F_2)) \geq \mathbb{K}_{\sqrt{2}}(\Xi(F_1 \cup F_2)).$$

By Lemma 2.2 there exist $C, \epsilon_3 > 0$ such that for all $\epsilon_3 > \epsilon > 0$

$$\mathbb{K}_{\epsilon, R}(\{x\} \cup F_1 \cup F_2) \leq C + |\log \epsilon| + \mathbb{K}_{\epsilon, R}(\Xi(F_1 \cup F_2)) \leq C + C_1 + C_2 + 4(\alpha_1 + \alpha_2) + (\alpha_1 + \alpha_2 - 1)|\log \epsilon|.$$  

By Corollary 1.4 $\{x\} \cup F_1 \cup F_2$ is $(\alpha_1 + \alpha_2 - 1)$-bounded. \hfill \Box

**Corollary 4.2.** If $M$ and $N$ are strongly 1-bounded and $M \cap N$ is diffuse, then $M \lor N$ is strongly one-bounded.

**Proof.** By hypothesis, $M$ and $N$ can be generated by finite sets of selfadjoint elements $F$ and $G$, respectively, such that $F$ and $G$ each contain a selfadjoint with finite free entropy. Pick a semicircular element $z \in M \cap N$. By Theorem 3.2 $\{z\} \cup F$ is 1-bounded as is $\{z\} \cup G$. Thus, by Lemma 4.1, $\{z\} \cup F \cup G$ is a 1-bounded set. Clearly $\{z\} \cup F \cup G$ generates $M \lor N$ and $\chi(z) > -\infty$ by [14], so by definition $M \lor N$ is strongly one-bounded. \hfill \Box

**Corollary 4.3.** Suppose $M$ and $N$ are strongly 1-bounded von Neumann algebras with diffuse von Neumann subalgebras $A$ and $B$, respectively. If $A \lor B$ is strongly 1-bounded, then $M \lor N$ is strongly one-bounded. In particular, $M \lor N$ is not isomorphic to an interpolated free group factor or to the free perturbation algebras in [1].

**Proof.** By Corollary 4.2 the von Neumann algebras $M \lor B = M \lor (A \lor B)$ and $N \lor A = N \lor (B \lor A)$ are strongly 1-bounded. Since $A \lor B$ is a diffuse von Neumann algebra contained in both $M \lor B$ and $N \lor A$, Corollary 4.2 implies that $M \lor N$ is strongly 1-bounded. The rest is given by Corollary 3.8. \hfill \Box

**Corollary 4.4.** If $M$ and $N$ are strongly 1-bounded and $D \subset M \cap N$ is a diffuse subalgebra, then the amalgamated free product $M \ast_D N$ is not isomorphic to the interpolated free group factors or free perturbations algebras in [1].

5. **NORMALIZERS AND STRONGLY 1-BOUNDED VON NEUMANN ALGEBRAS**

We now turn to a generalization of the von Neumann algebras in [5] and [17]. In this last section $A \subset M$ is an inclusion of tracial von Neumann algebras and $\{x\} \cup F$ is finite set of selfadjoint generators for $A$. Assume $R > 1$ exceeds the norms of any of the elements of $\{x\} \cup F$. The relative microstates $\Xi()$ and associated quantities written below will all be taken with respect to a fixed sequence $\langle x_k \rangle_{k=1}^\infty$ of microstates for $x$ as discussed in Section 2.

**Lemma 5.1.** Suppose that $u \in M$ is a unitary such that for some diffuse selfadjoint $y \in A, uyu^* \in A$, and $z \in M$ is a selfadjoint such that $z'' = u''$. If $1 > \epsilon, r > 0$, then there exist $m_0 \in \mathbb{N}, \gamma_0 > 0$ and a constant $L(\epsilon) > 1$ dependent on $\epsilon$ such that for any $m > m_0, \gamma_0 > \gamma > 0$, and $\epsilon L(\epsilon)^{-1}$-net $\langle \xi_{sk} \rangle_{s \in \Sigma_k}$ for $\Xi_R(F; m, k, \gamma)$, there exist an index set $\Theta_k$ satisfying $\#\Theta_k < \epsilon^{-rk^2}$ and for each $s \in S_k$ a collection $\langle \eta_{bsk} \rangle_{b \in \Theta_k}$ such that $\langle (\xi_{sk}, \eta_{bsk}) \rangle_{(s,b) \in \Sigma_k \times \Theta_k}$ is an $\epsilon$-cover for $\Xi_R(F \cup \{z\}; m, k, \gamma)$. 

Proof. Suppose $1 > \epsilon, r > 0$. There exists a polynomial $h$ in one $*$-variable such that $|h(u) - z|_2 < \epsilon(40)^{-1}$. There also exists a constant $K > 1$ so that regarding $h$ as a function from $(M_k(\mathbb{C}))_2$ into $M_k(\mathbb{C})$, $h$ has a Lipschitz constant no greater than $K$, $K$ is independent of $k$. Find $n \in \mathbb{N}$ satisfying $nr|\log \epsilon| > \log(40K) + |\log \epsilon|$. Choose mutually orthogonal projections $e_1, \ldots, e_n \in \{y\}^n$, all with trace $n^{-1}$.

$$u = \sum_{i=1}^n u e_i = \sum_{i=1}^n (ue_i u^*) u e_i.$$  

For each $1 \leq i \leq n$ set $f_i = u e_i u^*$ and observe that $e_i, f_i \in A$. Thus, $\sum_{i=1}^n f_i u e_i = u$ and $|h[\sum_{i=1}^n f_i u e_i] - z|_2 < \epsilon(10)^{-1}$.

From the first paragraph it follows that there exist noncommutative, selfadjoint polynomials $\Phi_i, \Psi_i$ in $#F$-variables, $1 \leq i \leq n$, a polynomial $g$ in one variable, and $m_0 \in \mathbb{N}$, $\gamma_0 > 0$ such that the following conditions are satisfied for any $m > m_0, \gamma_0 > \gamma > 0$:

- If $(\xi, \eta) \in \Gamma_R(\mathbb{N}; m, k, \gamma)$, then $|h[\sum_{i=1}^n \Phi_i(\xi) g(\xi) \Phi_i(\xi)] - \eta|_2 < \frac{\epsilon}{6}.$
- For any $#F$-tuple $\xi$ of selfadjoint elements in a von Neumann algebra with entries with operator norm no greater than $R$ or any single selfadjoint element $\eta$ in the von Neumann algebra with operator norm no greater than $R$, $\Phi_i(\xi), \Psi_i(\xi)$, and $g(\eta)$ are all contractions.
- If $\xi \in \Gamma(F; m, k, \gamma)$, then there exists projections $p_i, q_i \in M_k(\mathbb{C})$ all with normalized trace $n^{-1}$ such that the $p_i$ are mutually orthogonal, the $q_i$ are mutually orthogonal, $|\Phi_i(\xi) - q_{ij}|_2 < \epsilon(40nK)^{-1}$, $|\Psi_i(\xi) - p_i|_2 < \epsilon(40nK)^{-1}$, and

$$|h[\sum_{i=1}^n \Psi_i(\xi) g(\xi) \Phi_i(\xi)] - h[\sum_{i=1}^n \Phi_i(\xi) g(\xi) \Psi_i(\xi)]|_2 < \epsilon(10K)^{-1}.$$  

Observe that each of the $\Phi_i$ and $\Psi_i$ are Lipschitz when considered as maps from $(M_k(\mathbb{C}))_2$ into $M_k(\mathbb{C})$ where the Lipschitz constants are independent of $k$ and the domains and ranges of these polynomial maps are endowed with the $\|\cdot\|_2$-norm. Thus, there exists a constant $C > 1$ which exceeds the Lipschitz constants of any of the $\Phi_i$ or $\Psi_i$, so regarded. Set $L(\epsilon) = 40CKn$.

Now suppose for each $k \in \mathbb{N}$ there is an $\epsilon L(\epsilon)^{-1}$-net for $\Xi_R(F; m, k, \gamma)$. For each $s \in \Sigma_k$ fix a family of projections $\langle p_{isk} \rangle_{i=1}^n \cup \langle q_{isk} \rangle_{i=1}^n$ all with normalized trace $n^{-1}$ such that the $p_{isk}$ are mutually orthogonal, the $q_{isk}$ are mutually orthogonal, $|\Phi_i(\xi_{sk}) - q_{isk}|_2 < \epsilon(40nK)^{-1}$ and $|\Psi_i(\xi_{sk}) - p_{isk}|_2 < \epsilon(40nK)^{-1}$ (this is possible by the third condition of the second paragraph). Consider $\sum_{i=1}^n q_{isk}(M_k(\mathbb{C}))_1 p_{isk} \subset (M_k(\mathbb{C}))_1$. Find an $\epsilon(40K)^{-1}$-net $\langle \eta_{bsk} \rangle_{b \in \Theta_k}$ for $\sum_{i=1}^n q_{isk}(M_k(\mathbb{C}))_1 p_{isk}$ such that

$$\# \Theta_k \leq \left(\frac{40K}{\epsilon}\right)^{\frac{k^2}{n}} \leq \left(\frac{1}{\epsilon}\right)^{rk^2}.$$  

Consider $\langle (\xi_{sk}, h(\eta_{bsk})) \rangle_{(s,b) \in \Sigma_k \times \Theta_k}$. I claim that this is an $\epsilon$-net for $\Xi_R(F; m, k, \gamma)$. Towards this end suppose $(\xi, \eta) \in \Gamma_R(F; m, k, \gamma)$. Denote by $p_i$ and $q_i$ the projections provided for in the third condition of the second paragraph. There exists an $s_0 \in \Sigma_k$ such that $|\xi - \xi_{sk}|_2 < \epsilon L(\epsilon)^{-1}$. For any $1 \leq i \leq n$

$$|p_i - p_{isk}|_2 \leq |p_i - \Phi_i(\xi)|_2 + |\Phi_i(\xi) - \Psi_i(\xi_{sk})|_2 + |\Psi_i(\xi_{sk}) - p_{isk}|_2 \leq \epsilon(20nK)^{-1} + \epsilon(40nK)^{-1} = \epsilon(10nK)^{-1}.$$  

Similarly, $|q_i - q_{isk}|_2 < \epsilon(10nK)^{-1}$. From this it follows that $|\sum_{i=1}^n p_i g(\eta) q_i - \sum_{i=1}^n p_{isk} g(\eta) q_{isk}|_2 < \epsilon(5K)^{-1}$ whence, $h(\sum_{i=1}^n p_i g(\eta) q_i) - h(\sum_{i=1}^n p_{isk} g(\eta) q_{isk})|_2 < \epsilon 5^{-1}$. Since $\|g(\eta)\| \leq 1$, it follows that there exists some $b_0 \in \Theta_k$ satisfying
Putting this all together we get

$$|\eta - h(\eta_{baok})|_2 \leq |\eta - h[\sum_{i=1}^{n} \Psi_i(\xi) g(\eta) \Phi_i(\xi)]|_2 + |h[\sum_{i=1}^{n} \Psi_i(\xi) g(\eta) \Phi_i(\xi)] - h[\sum_{i=1}^{n} p_i g(\eta) q_i]|_2 + |h[\sum_{i=1}^{n} p_i g(\eta) q_i] - h[\sum_{i=1}^{n} p_{baok} g(\eta) q_{baok}]|_2 + |h[\sum_{i=1}^{n} p_{baok} g(\eta) q_{baok}] - h(\eta_{baok})|_2 \leq \frac{\epsilon}{6} + \frac{\epsilon}{10} + \frac{\epsilon}{5} + \frac{\epsilon}{40} < \frac{3\epsilon}{5}. $$

Thus, $$|\langle \xi, \eta \rangle - \langle \xi_{baok}, \eta_{baok} \rangle|_2 < \epsilon L(\epsilon)^{-1} + \frac{3\epsilon}{5} < \epsilon.$$ This completes the proof. \(\square\)

Now suppose $$\langle u_i \rangle_{i=1}^{\infty}$$ is a sequence of unitaries in $$M$$. For each $$i$$ let $$z_i$$ be a selfadjoint contraction such that $$\{z_i\}'' = \{u_i\}''$$. By scaling the $$z_i$$, we can arrange it so that $$\sum_{i=1}^{\infty} \|z_i\|_2^2 < 1$$. Denote by $$A_i$$ the von Neumann algebra generated by $$A$$ and $$\{u_1, \ldots, u_i\}$$. Assume that $$u_1 \in A$$ and for each each $$i$$, $$u_{i+1} u_i u_{i+1}^* \in A_i$$.

**Lemma 5.2.** If $$A$$ is strongly 1-bounded, then there exist $$K, \epsilon_0 > 0$$ such that for any $$i \in \mathbb{N}$$ and $$\epsilon_0 > \epsilon > 0$$

$$\mathbb{K}_{\epsilon, R}(\Xi(F \cup \{z_1, \ldots, z_i\})) < K.$$

**Proof.** We produce $$C, \epsilon_0 > 0$$ such that for any $$i \in \mathbb{N}$$ and $$\epsilon_0 > \epsilon > 0$$,

$$\mathbb{K}_{\epsilon}(\Xi_R(F \cup \{z_1, \ldots, z_i\})) \leq C + \sum_{i=1}^{j} 2^{-i}.$$ 

We will demonstrate this by induction on $$i$$.

Because $$\{x\} \cup F$$ is a 1-bounded set and $$\chi(x) > -\infty$$ there exists by Lemma 2.1 constants $$C, \epsilon_0 > 0$$, $$1 > \epsilon_0$$, dependent on $$\{x\} \cup F$$ such that for any $$\epsilon_0 > \epsilon > 0$$,

(7) $$C \geq \mathbb{K}_{\epsilon}(\Xi(F)).$$

We can now start the induction. Suppose $$1 > \epsilon_0 > \epsilon > 0$$. Now $$1 > \epsilon, 4^{-1}\epsilon > 0$$ so applying Lemma 5.1 with $$r = 4^{-1}\epsilon$$ yields the corresponding $$m \in \mathbb{N}, \gamma > 0$$, and constant $$L(\epsilon) > 1$$ dependent on $$\epsilon$$ such that the conclusion of Lemma 5.1 holds. $$\epsilon L(\epsilon)^{-1} < \epsilon_0$$ so (7) provides $$m_1 > m$$ and $$\gamma > \gamma_1 > 0$$ so that $$\mathbb{K}_{\epsilon, R}(\Xi_R(F; m_1, \gamma_1)) \leq C + 4^{-1}$$. Thus, for $$k$$ sufficiently large there exists an $$\epsilon L(\epsilon)^{-1}$$-net $$\langle \xi_{sk} \rangle_{s \in \Sigma_k}$$ for $$\Xi_R(F; m_k, k, \gamma_1)$$ such that $$\#\Sigma_k < e^{\epsilon^2(2^{-1})}.$$ Applying the conclusion of Lemma 5.1 yields

$$\mathbb{K}_{\epsilon}(\Xi_R(F \cup \{z_1\})) \leq \mathbb{K}_{\epsilon}(\Xi_R(F \cup \{z_1\}; m_1, \gamma_1) \leq C + 4^{-1} \leq C + 2^{-1}.$$ 

Now suppose the statement is true at $$i \in \mathbb{N}$$. Suppose $$\epsilon_0 > \epsilon > 0$$. Again, applying Lemma 5.1 where $$F$$ is replaced by $$F \cup \{z_1, \ldots, z_i\}$$ and is replaced by $$z_{i+1}$$, and $$4 > \epsilon, 4^{-i}\epsilon = r > 0$$, there exists an $$m \in \mathbb{N}, \gamma > 0$$, and $$L(\epsilon) > 1$$ dependent on $$\epsilon$$ such that the conclusion of Lemma 5.1 holds. By the inductive hypothesis,
To finish the proof then, it suffices to bound the last term on the dominating sum above. This will
function from the
respect to the

\[ \mathbb{K}_{\varepsilon L(e)^{-1}}(\Xi_R(F \cup \{z_1, \ldots, z_i\})) \leq C + \Sigma_{j=1}^{i} 2^{-j}. \]

Consequently, there exists \( m_1 \in \mathbb{N} \) and \( \gamma_1 \) such that for \( k \) sufficiently large there exists an \( \varepsilon L(e)^{-1} \)-net \( \langle \zeta_{nk} \rangle_{n \in \Omega_k} \) for \( \Xi_R(F \cup \{x_1, \ldots, x_i\}; m_1, k, \gamma_1) \) such that \( \Omega_k < e^{k^2(C+4^{-(i+1)}+\Sigma_{j=1}^{i} 2^{-j})} \). Now applying
the conclusion of Lemma 5.1 to these nets, it follows that

\[ \mathbb{K}_{\varepsilon}(\Xi_R(F \cup \{z_1, \ldots, z_{i+1}\})) \leq \mathbb{K}_{\varepsilon}(\Xi_R(F \cup \{z_1, \ldots, z_{i+1}\}; m_1, \gamma_1) \leq C + \Sigma_{j=1}^{i} 2^{-j} + 4^{-(i+1)} + 4^{-(i+1)} \varepsilon \log \varepsilon | \leq C + \sum_{j=1}^{i+1} 2^{-j}. \]

\[ \square \]

We are now ready for the main result of the section. Its proof runs very much like that of Corollary
3.2, except that issues with normalizers complicate the argument (hence the preceding preparatory
lemmas). For efficiency’s sake, we could have subsumed Theorem 3.2 with the lemmas and the
following general theorem of this section. But the relation between the invariance issue and strong
1-boundedness is clearer in the less cluttered context of Theorem 3.2, and for clarity’s sake (clarity
and efficiency not being the same) we have opted to repeat (more or less) the argument of Theorem
3.2 below.

**Theorem 5.3.** If \( A_{\infty} \) is the von Neumann algebra generated by \( A \) and \( \langle u_i \rangle_{i=1}^{\infty} \), then any finite set of
selfadjoint generators \( G \) for \( A_{\infty} \) is 1-bounded.

**Proof.** Let \( K, \varepsilon_0 \) be as in the preceding lemma. Without loss of generality we can assume the operator norms of any of the elements in \( G \) is no greater than \( R \). Recall the universal constant \( L \) in Lemma 2.2 and find a \( D > 4|\Sigma_{y \in G}||y||_{2}^{2} + ||x||_{2}^{2} + \Sigma_{i=1}^{\infty}||z_{i}||_{2}^{2} + 6| + L \). By Lemma 3.1 and Lemma 2.2 it follows
that for any \( i \in \mathbb{N} \) and \( \varepsilon_0 > \varepsilon > 0 \),

\[ \mathbb{K}_{\varepsilon, R}(G) \leq \mathbb{K}_{\varepsilon, R}(\{x\} \cup F \cup \{z_1, \ldots, z_i\} \cup G) \leq \log D + |\log \varepsilon| + \mathbb{K}_{\varepsilon, D^{-1}, R}(\Xi(F \cup \{z_1, \ldots, z_i\} \cup G)) \]

To finish the proof then, it suffices to bound the last term on the dominating sum above. This will
follow from arguing just as in Theorem 3.2.

Fix \( \varepsilon \) with \( \varepsilon_0 > \varepsilon > 0 \). There exists an \( N \) so large and a \( \#G \)-tuple of noncommuting polynomials \( \Phi \) in \((\#F + N + 1)\)-variables satisfying \( |\Phi(x, F, z_1, \ldots, z_n) - G|_2 < \varepsilon (4D)^{-1} \). Regarding \( \Phi \) as a function from \((\mathbb{M}_k^{sa}(\mathbb{C})_R))^F + N + 1 \) into \((\mathbb{M}_k^{sa}(\mathbb{C}))^{\#G} \), \( \Phi \) has bounded Lipschitz constant \( L_k \) with respect to the \( |.|_2 \)-norm and moreover, \( \sup_{k \in \mathbb{N}} L_k < \infty \). Choose \( L > \sup_{k \in \mathbb{N}} L_k + 1 \). Suppose \( t \) satisfies \( \varepsilon (4DL)^{-1} > t > 0 \). By Lemma 5.2

\[ \mathbb{K}_{t, R}(\Xi(F \cup \{z_1, \ldots, z_{n}\})) < K. \]

Thus, there exist \( m_1 \in \mathbb{N} \) and \( \gamma_1 > \gamma > 0 \) such that for \( m \geq m_1 \), and \( \gamma_1 > \gamma > 0 \),

\[ (8) \quad K + 1 \geq \limsup_{k \to \infty} k^{-2} \cdot \log K_t(\Xi_R(F \cup \{z_1, \ldots, z_n\}; m, k, \gamma)). \]

There exist \( m_2 \in \mathbb{N} \) and \( \gamma_2 > 0 \) such that if \( m > m_2 \) and \( \gamma_2 > \gamma > 0 \), then for any \( k, (\xi, \eta, \zeta) \in \Xi(\{x\} \cup F \cup \{z_1, \ldots, z_n\} \cup G; m, k, \gamma) \Rightarrow |\Phi(x_k, \xi, \eta, \zeta)|_2 \leq \varepsilon (4D)^{-1} \). Suppose \( m > m_1 + m_2 \)
and \( \min \{ \gamma_1, \gamma_2 \} > \gamma > 0 \). By (8) for \( k \) sufficiently large we can find a \( t \)-net \( \{ (\xi_{sk}, \eta_{sk}) \}_{s \in \Sigma_k} \) for \( \Xi_R(F \cup \{ x_1, \ldots, x_n \}; m, k, \gamma) \) which satisfies

\[
\# \Sigma_k \leq e^{(K+1)k^2}.
\]

For each such \( k \) sufficiently large consider \( \{ (\xi_{sk}, \eta_{sk}, \Phi(x_k, \xi_{sk}, \eta_{sk})) \}_{s \in \Sigma_k} \). This set is an \( \epsilon D^{-1} \)-cover for \( \Xi_R(F \cup \{ z_1, \ldots, z_n \} \cup G; m, k, \gamma) \). To see this, suppose \( (\xi, \eta, \zeta) \in \Xi_R(F \cup \{ z_1, \ldots, z_n \} \cup G; m, k, \gamma) \). By definition \( (\xi, \eta) \in \Xi_R(F \cup \{ z_1, \ldots, z_n \}; m, k, \gamma) \) so there exists some \( s_0 \in \Sigma_k \) with \( |(\xi_{s_0k}, \eta_{s_0k}) - (\xi, \eta)|_2 \leq t \leq \epsilon (4D)^{-1} \). Both \( (\eta, \zeta) \) and \( (\eta_{s_0k}, \xi_{s_0k}) \) are \( (#F + n) \)-tuples of operators with norms no greater than \( R \) so \( |\Phi(x_k, \xi, \eta) - \Phi(x_k, \xi_{s_0k}, \eta_{s_0k})|_2 \leq L \cdot t \leq \epsilon (4D)^{-1} \). But \( |\Phi(x_k, \xi, \eta) - \zeta|_2 < \epsilon (4D)^{-1} \) so \( |\Phi(x_k, \xi_{s_0k}, \eta_{s_0k}) - \zeta|_2 < \epsilon (2D)^{-1} \). Combining this with \( |(\xi_{s_0k}, \eta_{s_0k}) - (\xi, \eta)|_2 < t \) yields

\[
|(\xi, \eta, \zeta) - (\xi_{s_0k}, \eta_{s_0k}, \Phi(x_k, \xi_{s_0k}, \eta_{s_0k}))|_2 < \epsilon D^{-1}.
\]

It now follows that for \( m > m_1 + m_2 \) and \( \min \{ \gamma_1, \gamma_2 \} > \gamma > 0 \) and \( k \) sufficiently large,

\[
k^{-2} \cdot \log \left[ K \epsilon_D^{-1}(\Xi_R(F \cup \{ z_1, \ldots, z_n \} \cup G; m, k, \gamma)) \right] \leq K + 1.
\]

This implies

\[
K \epsilon_D^{-1}(\Xi(F \cup \{ z_1, \ldots, z_n \} \cup G)) \leq K + 1.
\]

\( K \) is dependent only on \( \{ x \} \cup F \) and thus, by the first paragraph, we’re done. \( \square \)

In Theorem 5.3 if \( A_\infty \) is a \( \Pi_1 \)-factor, then a trivial modification of the techniques in [6] shows that \( A_\infty \) can be generated by a finite set of selfadjoint elements. By Theorem 5.3 then, \( A_\infty \) is strongly 1-bounded. We thus have,

**Corollary 5.4.** Suppose \( A \) is a strongly 1-bounded von Neumann algebra and \( \langle u_1^{-\infty} \rangle_{i=1}^\infty \) is a sequence of Haar unitaries such that \( u_1 \in A \) and for each \( i \in \mathbb{N} \) \( u_{i+1} u_i u_{i+1}^* \in (A \cup \{ u_1, \ldots, u_i \})'' \). If the von Neumann algebra \( A_\infty \) generated by \( A \) and \( \langle u_i \rangle_{i=1}^\infty \) is a \( \Pi_1 \)-factor, then \( A_\infty \) is not isomorphic to an interpolated free group factor, \( L(F_r) \), \( 1 < r \leq \infty \). In particular, a \( \Pi_1 \)-factor generated by the normalizer of a strongly 1-bounded von Neumann subalgebra is not an interpolated free group factor.

**Remark 5.5.** It is immediate from [7] that any diffuse, hyperfinite von Neumann algebra is strongly 1-bounded. Thus, Theorem 5.3 provides an alternate way of seeing that the von Neumann algebras considered in [5] or [17] have free entropy dimension no greater than 1.

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