Propagation of sound on line vortices in superfluids: role of ergoregions

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Abstract
We (re)consider the propagation of small disturbances (sound waves) in the presence of an irrotational vortex in a superfluid pinned/wound on a stiff central wire, with the help of the formalism of acoustic spacetimes. We give closed formulas for the scattering angle for sound rays, formulate the sound-propagation problem in the Hamiltonian form and discuss the form of boundary conditions at the core of the vortex, where the Hamiltonian has a singular point. The wave equation is simplified to a single ordinary differential equation of Mathieu type. We give an extensive discussion of perturbations localized close to the core, which are similar to what is known as the Kelvin waves for vortices that are bendable (not pinned). The spectra of modes depend strongly on the type of boundary condition employed close to the vortex core. The gapless mode with the angular number $-1$, the Kelvin mode usually discussed in the context of unpinned vortices in superfluid helium or rotating Bose–Einstein condensates, turns out either not to exist or to have a completely different dispersion relation if the vortex is pinned. The question of whether or not the acoustic spacetime admits an ergoregion turns out to have a decisive (qualitative) influence on many aspects of sound-propagation phenomena.

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1. Introduction

1.1. Background
Vortices provide quantum fluids with a way to carry angular momentum [1, 2]. In the simplest case of a vortex line, the stationary velocity field $v(x)$ is purely azimuthal and depends as $|v| = \text{const}/r$ on the distance $r$ from the line. This flow is locally irrotational but nonetheless possesses a finite global circulation $\Gamma = \int v \cdot dx$. For fluids with phase coherence, $\Gamma$ must be an integer multiple of $h/m_B$, where $m_B$ is the mass of the bosonic constituent of the fluid, e.g., the mass of a single helium-4 atom for the superfluid helium-4. A way to probe rotating
superfluids is provided by sound waves [3]. In investigations of sound, choices need to be made with regard to the type of description of the fluid and the structure of the vortex core (the innermost part of the vortex). The initial research on this topic was mainly concerned with the superfluid helium-4 described by equations of fluid dynamics [4, 3]. Due to the growing experimental accessibility of quantum vapors (BECs), recent works tend to deal with this type of fluid, employing various approaches stemming from the Bogoliubov theory [5–7]. The correspondence to earlier research is provided by (but also limited to the range of applicability of) the relation between these detailed descriptions and the fluid dynamics.

In the case of experiments with vortices in helium, the typical configurations consist of cylindrical cryostats filled with the superfluid, with the possibility of placing a wire (or rod) on the axis of symmetry of the cylinder. For not too large angular momenta of the fluid, the velocity field corresponding to the stationary flow is exactly the one of the vortex line, while for higher angular momenta a vortex lattice is formed [3]. The interesting problem of finding the most stable stationary state of the fluid for given radii of the cylinders at a given temperature and angular momentum of the fluid has been intensively investigated by many authors [8–10].

On the other hand, in the case of BECs, the typical configurations consist of trapped rotating clouds of supersaturated vapors. For repulsively interacting constituents, the vapor settles down to a metastable state, which is well described as a weakly interacting Bose gas [11, 12]. The most stable type of configuration of a trapped gas with a given angular momentum depends strongly on the form of the trapping potential (the potential observed by the constituents in the plane perpendicular to the axis of rotation): for harmonic trapping, an arrangement of single vortices (each carrying a single quantum of circulation) is the stable configuration for not too large angular momenta. Harmonic trapping has the special feature that above a certain angular momentum the BEC cloud begins to shed atoms radially. However, for trapping potentials ‘steeper’ at large distances from the axis, larger angular momenta can be accommodated by the cloud. When this is done, a configuration with a giant vortex in the middle of the trap is the expected final configuration at large angular momenta [9]. A single vortex (simple or giant) provides the velocity field of a line vortex.

The axis of rotation of an idealized vortex line is a singular place because $|v| = \text{const}/r$ cannot continue to $r = 0$. Dynamical models of vortex cores regularize this singularity. On the level of fluid dynamics, one may distinguish two typical types of models: for superfluids, if an equation of state (EOS) is given (and the symmetry of the flow is assumed), then we are lead to a vortex model with an interior surface at the radius where the pressure reaches the lowest allowed value (the saturated vapor pressure (SVP)). In this way, an irrotational everywhere regular velocity field is obtained. Alternatively, for normal fluids, one assumes (as in, e.g., the Rankine [13] or Lamb–Oseen [14] models) that there is a distribution of vorticity mostly localized in the innermost part of the vortex. In this way, the irrotational $\text{const}/r$ profile is not continued toward the core, but rather the fluid velocity remains finite everywhere. Such vortex-core models are apparently adequate for the description of real phenomena in aerodynamics [14]. On the other hand, in the case of rotating quantum vapors (BECs), there exist natural descriptions of the innermost region, alternative to fluid dynamics [7, 12, 11], perhaps the simplest of which is the Gross–Pitaevskii description. While in regions where the condensate density is large and slowly varying in space the BEC can be described as a (super)fluid, this is no longer the case in the core. Even though the equations of the fluid dynamics fail there, the mean-field Gross–Pitaevski (or other) description is still valid, and essential characteristics of excitations of the vortex can be computed reliably (numerically); see, e.g., [7, 5]. While these characteristics are very important for experimental verifications, they are nonetheless specific to the case of weakly interacting quantum vapors. In this paper, we chose to work within the universality class provided by equations of fluid dynamics. We show that a qualitative
Figure 1. Schematic picture of the setup considered in this work. Sound propagates on the rotating fluid configuration, which is a line vortex with a locally irrotational velocity profile \( v_\phi = \frac{\Gamma_1}{2\pi r} \). The vortex rotates in the positive direction and is not allowed to bend, because of "pinning" (i.e. being wound) on a central stiff wire.

The difference is expected for sound propagation phenomena once the region to the interior of \( r_c = \frac{\Gamma_1}{c_s} \) (with \( c_s \) denoting the local speed of sound) is accessible to the fluid.

In the variety of configurations outlined above, the problem of sound propagation has been considered by many authors. The simplest approach, also used in this work, is to employ the macroscopic, fluid-dynamic description. The problem is further simplified if the perturbation of density and velocity can be assumed small (linearization). Care must be exercised with this assumption if the position of the vortex is allowed to move as a result of the perturbation. If the velocity of the flow to be perturbed is large and close to the core (i.e. if the modification of const/r profile is localized at small \( r \)) then even a small shift of the axis of rotation may result in a large perturbation of the velocity field. In this work, we consider only pinned vortices (see figure 1), for which the above obstruction does not occur. The other way to address the above problem is to consider regularized core models [14–16] with sufficiently large cores so that the background velocity is not too large anywhere. The study of perturbations naturally splits into the consideration of sound scattering and of perturbations bound to the vortex.

For the scattering, the natural problem is to derive the phase shifts in the partial-wave expansion of perturbations. Approximate approaches to the problem start with the work of Fetter [4] and were continued with growing sophistication. Some renewed activity in the field appeared in the last decade due to the connection to the Iordanskii force problem [18, 17, 13] and the controversy associated with it. We remark that the general consensus on independence of scattering characteristics (for small frequencies) from the details of the vortex core can...
easily be shown to be unfounded in the case of a pinned vortex admitting an ergoregion. However, we do not focus on this problem in this work.

As to the study of disturbances localized close to the vortex core, it is important to note that generally—for each model of the vortex core—there exists a large family of such disturbances. The characteristics of the allowed modes, such as the dispersion relations for the (many) branches existing for each angular number \( m \), depend strongly on the type of regularization of the vortex core [14–16]. This family is usually referred to as the family of Kelvin waves. In many cases, there exists an important mode of vibrations with the angular number \( m = -1 \), positive frequency \( \omega \) and the gapless dispersion relation \( \omega(p) \approx p^2 \ln(1/p) \) for small wavenumbers \( p \) (wavenumber along the vortex). This mode is usually referred to as the Kelvin mode [3, 19]. Modes with \( |m| = 1 \) (there are many such modes for a given \( p \) [15, 16]) generally put the cores of unpinned vortices into motion. This has lead many authors to seek simple explanations of these modes on the phenomenological level (e.g. by considering forces acting on the core, as if it were a string with a specific tension [3, 8]). It should be remarked that a version of the Kelvin mode has also been apparently found in BEC systems [19, 5], although it is puzzling why none of the other modes of the family (which is usually very rich) has ever been reported to be found in this context.

Because for vortices pinned on sound-reflecting wires a qualitatively different dynamics in the innermost part is at work, and because such vortices do not allow for deformations of the core (a vortex line cannot be considered as a deformable string), we shall refer to the modes localized close to the core/wire discussed in this paper as the quasi-Kelvin waves (or whispering-gallery modes [20]). This is further justified by the fact that, as we shall show, the familiar Kelvin mode, with its characteristic gapless dispersion relation, is absent in the system for the two types of one-parameter boundary conditions that we consider.

The main purpose of this paper is to report on new results for sound propagation in the presence of a vortex line pinned (wound) on a central stiff wire. We use the fluid-mechanical description and employ the formalism of acoustic spacetimes, which has been under intensive development recently [21]. In this way, the mathematical structure of the problem receives additional attention. The sound fields are parameterized by a single scalar potential, and a simple Hamiltonian structure is available [22]. Quite remarkably, many technical aspects of the problem turn out to be related to other important areas of theoretical physics, including the stability of (classical and quantum) fields in the presence of ergoregions, behavior of fields in the presence of strongly binding potentials and the Aharonov–Bohm scattering. While the last relation is well known among authors [18, 17], we point to its limits once the acoustic spacetime contains an ergoregion.

To the best of our knowledge, the following results are new: an exact formula for the scattering angle in ray-acoustics approximation (section 2), discussion of the general form of boundary conditions at the core (‘core conditions’, section 5), (semi-numerical) determination of spectra of bound states (quasi-Kelvin/whispering-gallery waves) localized close to the vortex for Neumann and ‘core’ boundary conditions (section 6.1.2) and the discussion of various sections of the parameter space displaying the characteristics of these states (section 6.2).

The recurrent theme throughout the paper is the qualitative change of the character of the problem once the acoustic spacetime contains ergoregions. Some of the problems related to the stability of the flow are formulated, but left to a future investigation (see section 7).

This paper should be regarded as an attempt to strengthen the understanding of the mathematical structure of the sound-propagation problem for perturbations of the vortex flow. The additional features reported here in the fluid-dynamical approach (e.g. the specific characteristics of the full family of quasi-Kelvin waves) may or may not find their counterparts in the more sophisticated descriptions of concrete superfluids, such as the Bogoliubov
descriptions of weakly interacting rotating Bose–Einstein condensates [5, 6]. In section 8, we present a short discussion of the possible experimental regimes where these features might be detectable. The concrete case of superfluid helium-4 is considered in greater detail elsewhere [20]. In our opinion, problems of superfluid rotation are of such importance that progress achieved even in very simplified models may turn out very valuable for future research.

2. Acoustic ray description

In this paper, we investigate the propagation of sound in the presence of an irrotational flow of a vortex line. The vortex position is assumed to be fixed (i.e. it is a straight line irrespective of the action of perturbations), and the density perturbation is assumed to be ‘small’ (linear regime). The propagation of sound for such a flow can be derived from a single partial differential equation for a single scalar potential $\psi$. The velocity and density perturbations are expressed as [21, 22]

$$\delta v_i = \partial_i \psi,$$

$$\delta \rho = -\frac{\rho}{c_s^2} (\partial_i \psi + v^i \partial_i \psi),$$

where $c_s$ denotes the local speed of sound, $v$ denotes the local velocity field and $\rho$ denotes the local mass density of the background fluid configuration. The potential $\psi$ fulfills the d’Alembert equation (massless wave equation):

$$\Box_g \psi = 0$$

associated with the so-called acoustic metric $g_{ab}$, the form of which is

$$\ dx^2 = g_{ab} dx^a dx^b = \frac{\rho}{c_s^2} \left[ -c_s^2 dr^2 + \delta_{ij} (dx^i - v^i dr) (dx^j - v^j dr) \right].$$

In the stationary, cylindrically symmetric case of an irrotational vortex line, it reduces to

$$\ dx^2 = \frac{\rho}{c_s^2} \left[ -f dr^2 - 2\kappa dr d\phi + r^2 d\phi^2 + dr^2 + dz^2 \right],$$

where $f = c_s^2 - \kappa^2 / r^2$, and $\kappa = \nu \psi = \frac{\kappa}{2\pi} = \frac{1}{2\pi} \int \nu \ dx$ is a constant of dimension $[\kappa]=[\text{cm}^2 \text{s}^{-1}]$. Positive $\kappa$ corresponds to a fluid rotating in the positive direction. This constant, together with a reference value of $c_s$, e.g., at $\infty$, allows us to construct a length scale $r_c = \kappa / c_s$ and a time scale $t_c = \kappa / c_s^2$. In what follows, we will use the dimensionless quantities $t/t_c$ and $r/r_c$, denoted by the same letters $(t, r)$ for brevity.

From now on, we will make the simplifying assumption that $c_s$ and $\rho$ do not depend on the distance $r$ from the axis. Whether or not this is a restrictive assumption depends on the type of superfluid considered. We discuss the deviations from this assumption for superfluid helium-4 in appendix A. The case with $\rho$ and $c_s$ depending on $r$ is also considered in a separate work [20]. With the assumption, and the rescaling, we are lead to considering the physics of scalar fields in the spacetime with the metric

$$\ dx^2 = - \left(1 - \frac{1}{r^2} \right) dr^2 - 2 \ dr d\psi + r^2 d\psi^2 + dr^2 + dz^2.$$ 

The ranges of coordinates are standard. In cases where we restrict the ‘arena’ to the region exterior to ‘a central wire’ placed on the axis of rotation (around which the background flow rotates), $r$ will be restricted from below by $r_0$.

Ray approximation. The formalism provided by the sound potential $\psi$ is sufficient for acoustics, and one could proceed directly to solving the wave equation $\Box_g \psi = 0$ by convenient ansätze (see section 4). Before we do that, however, it is instructive to solve the ray-acoustics problem
in part because the methods known from general relativity can directly be applied here and provide an exact and insightful solution.

Because the equation is ‘massless’, ray-acoustics approximation requires us to determine the null geodesics of the acoustic spacetime. The equations ‘of motion’ fulfilled by such geodesics can be derived from the Lagrangian

\[ L = \frac{1}{2} \left( -f \dot{t}^2 - 2i \dot{\phi} + r^2 \dot{\phi}^2 + \dot{r}^2 + \dot{z}^2 \right) = \frac{1}{2} \dot{x}_a \dot{x}^a = 0, \]  

where \( f = 1 - \frac{1}{r^2} \), and where differentiations are performed with respect to the parameter \( \tau \) of the geodesics, as \( x^a = x^a(\tau) \). The spacetime contains an ergoregion if points with \( r < r_0 \) are accessible. This may, or may not, be the case, depending on the experimental configuration. If the vortex is pinned on a central wire, then the (rescaled) radius of the wire provides the lowest value of \( r \), i.e. \( r_0 \). The problem of integrating the geodesic equations reduces to quadratures, because there are four first integrals: the momentum along the vortex \( p = \dot{z} \), the ‘angular momentum’ \( J \), the ‘energy’ \( E \),

\[ J = r^2 \dot{\phi} - i, \]  

\[ E = f \dot{t} + \dot{\phi}, \]  

and the Lagrangian \( L \). Note that objects moving along the zero-angular-momentum curves \( J = 0 \) possess the (coordinate) angular velocity \( \frac{\dot{\phi}}{\dot{t}} = \frac{1}{r^2} \) and, thus, remain at rest with respect to the fluid particles (i.e. move together with fluid particles as they rotate around the axis of the vortex).

With the help of \( E, J \) and the Lagrangian, we obtain the identity

\[ \frac{1}{2} \left( -E \dot{t} + J \dot{\phi} + \dot{r}^2 + \dot{z}^2 \right) = 0, \]  

which is fulfilled along the geodesic. The reparametrization invariance of null geodesics allows us to set \( J \) or \( E \) to any value; we chose \( E = 1 \), leading to \( \dot{t} = 1 \) at \( r = \infty \) (provided geodesics reach that point). The velocities \( (\dot{t}, \dot{\phi}) \) are eliminated by

\[ \dot{t} = r^2 \dot{\phi} - J, \]  

\[ \dot{\phi} = \frac{1 + fJ}{r^2} \Rightarrow \dot{t} = 1 - \frac{J}{r^2}. \]  

(11)

(The value of \( t \) could, as it stands, decrease\(^1\) along a null geodesic with \( J > 0 \) if this geodesic, coming from \( r = \infty \), crossed the radius \( r_{\text{cusp}}^2 = J \). This does not happen, however, as we shall see below.)

The only equation to solve is the radial equation, which now assumes the form

\[ \dot{r}^2 = \left( 1 - \frac{J}{r^2} \right) (1 + fJ) - J - p^2. \]  

(12)

2.1. General properties of rays

Let us consider rays that reach \( r = \infty \) and call them scattering rays. The limit \( r \to \infty \) of the radial equation and equation (11) yields

\[ \dot{r}_\infty^2 + p^2 = 1, \quad \dot{\phi} = \frac{1 + fJ}{r^2}. \]  

(13)

Consequently, it is possible and advantageous to introduce the standard scattering parameters. All scattering rays are specified by the projection of the velocity on the radial direction

\[ v \overset{\text{def}}{=} \dot{r}_\infty = \cos(\alpha) \]  

(14)

\(^1\) The ‘\( t \)’ coordinate in the acoustic spacetime models has a special role, because it coincides with flow of time in the laboratory frame so that events in the acoustic spacetime can be directly interpreted. On this ground, it is evident that no ray propagating initially with \( t > 0 \) can switch to \( t < 0 \), at least as long as the ray acoustics is valid.
and by the ‘impact parameter’ \( b \) defined in the way familiar from scattering theory
\[
\dot{\varphi} = \frac{b v}{r^2}, \quad b v = 1 + J, \tag{15}
\]
(at \( r = \infty \)). We obtain the following form of the classical scattering problem:
\[
\dot{r}^2 = \left[ 1 - \frac{J}{r^2} \right] \left[ J + 1 - \frac{J}{r^2} \right] - (J + 1) + v^2. \tag{16}
\]
Let us mention that the second type of rays appearing is obtained by taking \( |p| > E = 1 \), i.e. \( v^2 < 0 \). These rays do not reach \( r = \infty \) and therefore are rays bound to the vortex. Setting \( E = v^2 \), and allowing for negative \( E \), we investigate the polynomial in \( 1/r \) on the right-hand side of
\[
\dot{r}^2 = J^2 \left( \frac{1}{r} \right)^4 - J(J + 2) \left( \frac{1}{r} \right)^2 + E, \tag{17}
\]
which is just a rearrangement of (16). Zeros of this polynomial correspond to peri-/apocenters of the rays, and motion can only take place in regions where the polynomial is positive. For \( E > 0 \), motion will always be possible at the two limits: \( r \to \infty \) and \( r \to 0 \). Setting \( \frac{1}{r} = w \), we note that the product of the zeros is equal to \( E/J^2 \). For the bound rays, with \( E < 0 \), exactly one zero of the polynomial is always positive, say \( (w_+) \), and it corresponds to exactly one positive root for \( \frac{1}{r} \). Therefore, the bound rays are coming from \( r = 0 \), reach the apocenter and fall toward \( r = 0 \). The case of scattering rays, with \( E > 0 \), requires a closer look.

### 2.2. Determination of orbits for scattering rays

In this section, we consider the scattering rays and determine the form of their orbits, i.e. the functional dependence of \( 1/r = u \) on the angle \( \varphi \). Quite generally, we shall see that geodesics starting from \( r = \infty \) either reach a pericenter (and scatter back to \( \infty \)) or propagate toward and end on the vortex core. The problem is integrable in terms of elliptic functions. The range of the angular momentum \( J \) leading to absorption is, as expected, asymmetric due to the rotation of the vortex.

We start with
\[
\dot{r} = \frac{du}{d\varphi} \frac{dr}{d\varphi} = -u'[J + 1 - Ju^2] \tag{18}
\]
and find from (17)
\[
(u')^2 [J + 1 - Ju^2] = [J + 1 - Ju^2][1 - Ju^2] - (J + 1) + v^2. \tag{19}
\]
Thus, our problem is the classical Jacobi inversion problem for \( u(\varphi) \) on a Riemann surface of genus one:
\[
\int_{u(\varphi)}^{u_0} \frac{du}{\sqrt{P_4(u)}} = \varphi, \tag{20}
\]
with
\[
P_4(u) = J^2u^4 - J(J + 2)u^2 + v^2. \tag{21}
\]
In order to compute the scattering angle \( \Delta \), we set \( u_0 \) to the lowest positive root of \( P_4 \) (pericenter), where we can assume \( \varphi = 0 \) and compute the integral up to \( u(\Delta) = 0 \) (corresponding to \( r \to \infty \)). The total scattering angle \( \theta \) will be \( \theta = \pi - 2\Delta \). Before establishing the dependence of \( \Delta(v, b) \), let us first consider the case when no scattering takes place, because there is no pericenter, and the ray propagates to the vortex core.
Fall onto the core. The polynomial $P_4(u)$ is bi-quadratic. Let $w = u^2$. Both roots of $P_2(w) = P_4(\sqrt{w})$ have the same sign due to $w_+w_- = v^2/J^2$. In two ranges of parameters $(J, v)$, none of the roots $w_{\pm}$ is real and positive (i.e. they do not lead to real values of $u$, and therefore, the ray continues $r = 0$); they are not real unless $(J + 2)^2 - 4v^2 \geq 0$, and they are real but negative for $J \in [2v-2, 0]$. These ranges of parameters have been depicted in figure 2. For other values of $(J, v)$, there are two real, positive roots and, thus, four real roots of $P_4(u)$, two of which are positive, say $u_{\pm}$. The rays starting at $r = \infty$ get reflected at $u_-$, while the ones starting at $r = 0$ get reflected at $u_+$.

By considering the limiting form of the radial and axial equations, we find the following asymptotic behavior of rays falling onto $r = 0$:

$$\dot{r}^2 = \frac{J^2}{r^2}, \quad \dot{\varphi} = -\frac{J}{r^2},$$

(22)

from which it turns out that the asymptotic trajectory is an (independent of $J$) hyperbolic spiral:

$$-\frac{dr}{r^2} = d\varphi \Rightarrow \varphi = \varphi_0 + \frac{1}{r}.$$  

(23)

Exact results for the scattering angle $\Delta$. Appearance of a quartic in the radical indicates a solution in terms of complete elliptic integrals. We recall the definitions [23]

$$K(m) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}}, \quad E(m) = \int_0^1 \frac{1-\sqrt{1-mt^2}}{\sqrt{1-t^2}} dt,$$  

(24)

and note the following identities:

$$\int_0^{x_{\pm}} \frac{dx}{\sqrt{x^4-Ax^2+1}} = \frac{1}{x_{\pm}} K(x_{\pm}^4), \quad \int_0^{x_{\pm}} \frac{x^2 dx}{\sqrt{x^4-Ax^2+1}} = x_{\pm} [K(x_{\pm}^4) - E(x_{\pm}^4)],$$  

(25)
Figure 3. Exact (numerical) solutions for sound rays. (Left) Acoustic rays coming from \( r = \infty \), which get (1) scattered by the vortex and (2) absorbed by the vortex. (Right) Rays emitted from the core which get reabsorbed. Paths have been plotted only up to the peri-/apocenter, where appropriate. Also visible is the boundary of the ergoregion, located at \( r = 1 \). The flow in the vortex is positive oriented. Therefore, a bound ray co-rotating with the vortex is presented in the right figure. It can be shown that the apocenter of such rays never exceeds \( r = 1 \), but it can exceed this value in the case of counter-rotating rays, in accordance with the results of section 6.2.

where \( A > 0 \), \( x_- \) is the smallest positive root of the radicand (which has, per assumption, four real roots), and \( x_+x_- = 1 \). Taking \( v \) out of the radicand and introducing a new variable \( z = u \sqrt{|J|/v} \), we obtain expressions of the above form with the result

\[
\Delta(J,v) = \frac{J + 1}{\sqrt{v|J|}} z_+ K(z_+) - \frac{\text{sgn}(J) \sqrt{v}}{\sqrt{|J|}} z_+ [K(z_+) - E(z_+)],
\]

(26)

where \( z_\pm \) are the (positive) roots of

\[
z^2 = \frac{1}{2} [A \pm \sqrt{A^2 - 4}], \quad \text{with} \quad A = \frac{J + 2}{v} \text{sgn}(J) > 0.
\]

(27)

The total scattering angle is just \( \theta = \pi - 2\Delta \). The scattering angles have an interesting expansion for large values of the impact parameter \( b = (J + 1)/v \):

\[
\Delta(b,v) \approx \frac{\pi}{2} + \frac{\pi(v^2 - 1)}{8vb^2} - \frac{\pi}{2vb^3} + O(b^{-4}), \quad b \to +\infty,
\]

\[
-\Delta(b,v) \approx \frac{\pi}{2} + \frac{\pi(v^2 + 1)}{8vb^2} + \frac{\pi}{2vb^3} + O(b^{-4}), \quad b \to -\infty,
\]

where the negative values of \( b \) correspond to rays coming from below the \( \hat{x} \) axis in the left part of figure 3. The symmetry of the first two terms, as well as an overall focusing character of the vortex scattering\(^2\), drew the attention of authors before [24]. On the ‘positive absorption edge’ \( J = 0 \), we have \( \Delta \to +\infty \), while \( \Delta \to -\infty \) on the ‘negative absorption edge’ \( J = -2v - 2 \). Note also that due to \( v^2 < 1 \) there holds

\[
w_- = \frac{J + 2 - \sqrt{(J + 2)^2 - 4v^2}}{2J} < \frac{1}{J} = w_{\text{caus}},
\]

(28)

and therefore, geodesics with \( J > 0 \) coming from \( \infty \), which potentially could enter the region with \( t < 0 \), get reflected before reaching that region.

\(^2\) An angle \( \Delta > \pi/2 \) for rays coming from above \( \hat{x} \)-axis corresponds to a focusing behavior of the scattering.
3. Mathematical structure of wave-propagation problem

In this section, the mathematical structure associated with the wave propagation is developed using tools available from classical field theory in curved spacetimes. The goal is to put subsequent considerations (determination of modes by partial-wave decomposition (section 4) and questions of stability (section 7)) on a more systematic mathematical footing. In particular, function spaces will be introduced (with respective ‘norms’ and ‘projection operators’), Hamiltonian structure will be constructed (equation 45), and the notions of energy and the angular momentum will be given a precise definition (section 3.3). Finally, a link to unsolved problems of quantum fields in strong binding potentials (and in rotating spacetimes) will be established (section 3.2.2).

3.1. Geometrical preliminaries: constant time (t) foliation

The acoustic spacetime has the line element

\[ ds^2 = -f \, dt^2 - 2 \, dt \, d\phi + r^2 \, d\phi^2 + dr^2 + dz^2 \]  

(29)

with \( f = 1 - r^{-2} \). We have \( g \overset{\text{def}}{=} \det (g_{ab}) = -r^2 \), and (displaying \((t, \phi)\) components only)

\[ g_{ab} = \begin{pmatrix} -f & -1 \\ -1 & r^2 \end{pmatrix}, \quad g^{ab} = \frac{1}{r^2} \begin{pmatrix} r^2 & 1 \\ 1 & -f \end{pmatrix}. \]

(30)

We may distinguish \( t = \text{const} \) surfaces \( C_t \), which foliate the acoustic spacetime. The unit normal vector \( N^a \) orthogonal to these surfaces \( N_a = -\partial_a (t) \) is always timelike. The Killing vector of \( t \)-translations \( T^a = (\partial_t)^a \) may be orthogonally decomposed as

\[ T^a = \alpha N^a + \beta^a, \]

(31)

with

\[ \alpha = \frac{1}{\sqrt{-g}} = 1, \quad \beta^a = \begin{pmatrix} 0, -\frac{1}{r^2}, 0, 0 \end{pmatrix}, \quad \beta_a = \begin{pmatrix} \frac{1}{r^2}, -1, 0, 0 \end{pmatrix}, \]

(32)

and finally

\[ N^a = \begin{pmatrix} 1, \frac{1}{r^2}, 0, 0 \end{pmatrix}. \]

(33)

The vector fields \( e^a_t = N^a \) and \( e^a_\phi = -r \beta_a = (0, \frac{1}{r^2}, 0, 0) \) form an orthonormal tetrad.

For every covariantly conserved current \( J^a \) by the Gauss theorem, we have that

\[ Q = \int_{C_t} dS^a J_a \]

(34)

is independent of the foliation parameter \( t \) (time), provided the fluxes through boundaries of \( C_t \) are either absent or integrate to zero. The volume element of \( C_t \) is \( dS^a = \sqrt{h} \, d^3 y \, N^a \), which for the special case of the vortex spacetime is just \( d^3 x \, \delta^a_t \), because the internal geometry of \( C_t \) is flat:

\[ ds^2 |_{C_t} = h_{ab} \, dx^a \, dx^b = r^2 \, d\phi^2 + dr^2 + dz^2. \]

Note that in our case, the determinants fulfill \(-g = h = r^2\).
3.2. Sound propagation as a linear dynamical system

3.2.1. Function spaces. In this subsection, we introduce appropriate function spaces, so that the propagation of sound can be considered as an evolution of a vector (in the relevant space), as is customary in classical and quantum mechanics. This type of consideration, however, is not usually presented in texts on vortex acoustics. The formalism of velocity potential \( \psi \) and the acoustic-spacetime formulation naturally allow for such a consideration, and therefore, we develop it here.

Let us consider the space of ‘initial data’ of the evolution of the scalar potential \( \psi(t, x) \) in time \( t \). Because the equation governing the evolution (the d’Alembert equation) is of second order in \( \partial_t \), the space of initial data on \( \mathcal{C} \) will necessarily contain the information about the potential \( \psi \) and about its ‘time derivative’ \( p = N^a \partial_a \psi \). The standard structure of a linear dynamical system associated with this problem will be given below.

First, we introduce the space of real solutions of the wave equation \( S_R \). Because of the uniqueness of time evolution, this space is equivalent to the space of initial data, i.e. to the space of elements of the form \( \Phi = (\psi, p) \) on any of the Cauchy surfaces \( \mathcal{C} \). In what follows, we will make no distinction in notation between both spaces. We equip the structure with the Poisson bracket

\[
\sigma[\Phi_1, \Phi_2] = \int_{\mathcal{C}} dS^a j_a[\Phi_1, \Phi_2] = \int d^3x (\psi_1 p_2 - p_1 \psi_2),
\]

(35)

where the rightmost form is specific to the vortex acoustic spacetime, and \( j_a \) stands for the covariantly conserved current

\[
j_a[\Phi_1, \Phi_2] = \Phi_1 \partial_a \Phi_2 - \partial_a \Phi_1 \Phi_2.
\]

(36)

We also consider the space of complex solutions, or complex initial data \( S_C \), equipped with the same Poisson bracket (but with the first argument complex conjugated). We mention that the dynamics of the system will be governed by a Hamiltonian of equation (39).

3.2.2. Relation to quantum fields in curved spacetimes. The above structures are sufficient from the point of view of classical field theory and therefore will also be sufficient for all subsequent purposes of this paper. However, in this subsection, we present a digression on the relation to the problem of quantized scalar fields in rotating spacetimes. As it turns out, many unsolved problems of this difficult theory present themselves already in the present setting of superfluid vortex acoustics, and it would be natural to attack them in this simpler (and more ‘applied’) context first. Here, we merely present how the QFT structure is constructed and what the connections to vortex acoustics are. We offer no solutions to outstanding problems of QFT in rotating spacetimes.

There are reasons to believe that quantized density fluctuations in fluids with phase coherence (such as BECs or superfluid helium-4) do exist and behave as a quantum field (see, however, [2]). Indeed, this type of reasoning provides the basis for sonic analogs of Hawking, Unruh and other phenomena traditionally restricted for quantum fields in flat and curved spacetimes [21].

Quantization of fields usually consists of two steps: the relatively easy one\(^3\) concerned with the formulation of the algebra of field observables, and the more difficult one concerned with a construction of a well-behaved representation of this algebra on a Hilbert space (such as the Fock space). The construction of a ground-state representation for fields in stationary axisymmetric spacetimes without ergoregions was given by Kay [25]. Technically what is needed is a Schrödinger-type description of the evolution of the ‘positive-frequency’ solutions

\(^3\) This step is relatively easy for free fields, which is the case that we consider here.
of the field equation. In order to achieve this, one forms a complex Hilbert space of (one-component) functions $\mathcal{H}$, usually known under the name of one-particle Hilbert space. The elements of $\mathcal{H}$ are meant to correspond to the ‘positive-frequency’ solutions in $\mathcal{S}_C$, with different choices of ‘positive frequencies’ leading to quantized-field representations related by a Bogoliubov transformation. The time evolution should be represented in $\mathcal{H}$ as a unitary operator $U : \mathcal{H} \rightarrow \mathcal{H}$.

In simple cases, such as the propagation of fields in Minkowski spacetime, the way to project elements of $\mathcal{S}_C$ into $\mathcal{H}$ is provided by an operator $j : \mathcal{S}_C \rightarrow \mathcal{H}$. For a solution ‘of frequency $\omega$’, the action of $j$ would have the form

$$j(\Phi) = \frac{1}{\sqrt{2\omega}} (\omega \psi - i p),$$

(37)

where $p = \partial_t \psi$ in that case. In our case, however, the frequency corresponding to an element of the initial data space $\mathcal{S}_R$ is not yet defined. The standard definition of the frequency is that it is the eigenvalue of the Hamilton operator $H$, which is $t$-independent for the stationary spacetimes. What is needed for the definition of the projection operator $j$, and thus also for the construction of the ground-state representation, is an operator $h$, such that

$$h \psi = \omega \psi,$$

$$h p = \omega p,$$

for an eigenvector of the Hamiltonian $H$ of the form $(\psi, p) = \Phi_\omega$. Then, the operator $j$ will fulfill

$$j[\Phi(t)] = e^{i H t} j[\Phi(0)], \quad \text{i.e.} \quad U(t) = e^{i H t}.$$

(38)

In the simplest case of a scalar field on the Minkowski space, the choice is $h = \sqrt{-\nabla^2}$, but we do not know if an appropriate choice can be made in the case of stationary spacetimes with ergoregions. We note that the investigation of quantum sound fields can be carried further using methods developed by Kay [25], although we stress that these methods will not be available if an ergoregion is present. In the latter case, no ground-state representation of the above type is known, and only innovative/alternative solutions of the problem are available [26, 27]. To the fascinating question of the physical interpretation of these constructions in the context of known superfluids, we can unfortunately offer no answer here.

### 3.2.3. Time evolution.

Returning to the classical fields and denoting by $T(t)\Phi_0$ the time evolution of a phase-space vector $\Phi_0 \in \mathcal{S}_C$, we derive the form of the Hamiltonian

$$i \frac{\partial}{\partial t} [T(t)\Phi]|_{t=0} = H \Phi_0 = -\sigma_2 A \Phi_0,$$

(39)

where for technical reasons we have distinguished the operator

$$A = \begin{pmatrix} -\nabla^2 & \beta \partial_t \\ \beta^\dagger \partial_t & 1 \end{pmatrix},$$

(40)

while $\sigma_2$ stands for the Pauli matrix. Anticipating the separation of the angular dependence via $e^{im\phi}$ (with $m \in \mathbb{Z}$), leading to the partial-wave expansion, we give the form of $A$ for partial waves:

$$A = \begin{pmatrix} -\nabla^2 & -i m \\ -i m & 1 \end{pmatrix},$$

(41)

where $\nabla^2$ is the Laplace operator (of a flat three-dimensional space).

We note the connection to a technically very similar problem of complex scalar fields in electrostatic potentials, as investigated in [26]. Setting the (external) vector potential to zero ($\vec{A} = 0$) there, and denoting the mass of the field by $\mu$, we obtain

$$A = \begin{pmatrix} -\nabla^2 + \mu^2 & iV \\ -iV & 1 \end{pmatrix},$$

(42)
where $V$ is the scalar potential. Therefore, the correspondence between [26] and our situation is $V \equiv i\beta \partial$, which for partial waves reduces to $V \equiv m/r^2$. Thus, from the Hamiltonian point of view, the behavior of (complex) sound potential in a vortex background is equivalent to the behavior of a charged scalar field in strongly attractive/repulsive (depending on $m$) potentials. Such problems have been investigated in the literature before, see, e.g., [28] and appendix B of [27].

For the linear space of Cauchy data, one may introduce a natural, positive-definite product

$$
\langle \Phi, \Phi \rangle = \int \mathcal{C} \, d^3y \, \sqrt{h} \, \Phi^\dagger \Phi = \int \mathcal{C} \, d^3x \, \Phi^\dagger \Phi,
$$

and also the product resulting from the Poisson bracket between $\Phi^\dagger$ and $\Phi$, which (conforming to the literature) we shall call ‘the norm of $\Phi$’,

$$
\| \Phi \|^2 = (\Phi, \Phi)_K \overset{\text{def}}{=} -\langle \Phi, \sigma_2 \Phi \rangle = i \int \mathcal{C} \, d^3y \, \sqrt{h} \, (\overline{\psi}p - \overline{p}\psi) = i\sigma [\overline{\Phi}, \Phi]
$$

(44)

(the last equality for $\psi$ of the form $e^{-i\omega t + i\sigma \varphi(r)}$). The space $\mathcal{S}_C$ equipped with the product $\langle \cdot, \cdot \rangle$ is a Hilbert space, while the space $\mathcal{K}$ equipped with $(\cdot, \cdot)_K$ is a Krein (indefinite-product) space. This product corresponds exactly to equation (34) of [26] (with $p = (\partial_t - iA_0)\psi$) and to (2.15) of [27].

The merit of introducing the additional function spaces lies in the special properties that the Hamiltonian $H$ enjoys when considered in these spaces. This leads us to structural results on the properties of $H$, which will shape the more traditional discussion of ‘sound modes’ in subsequent sections 4 and 6.

3.2.4. Properties of the Hamiltonian. The Hamilton operator, i.e. $H = -\sigma_2 A$, which is explicitly given by

$$
H = \begin{pmatrix}
 i\beta \partial & i \\
 i\nabla^2 & i\beta \partial
\end{pmatrix},
$$

for partial waves, $H = \begin{pmatrix}
 m/r^2 & i \\
 i\nabla^2 & m/r^2
\end{pmatrix},
$$

(45)

is a symmetric operator on the space $\mathcal{K}$, i.e. $(\Phi, H\Psi)_K = (H\Phi, \Psi)_K$. Additionally, in some cases, the operator $A$ is a positive operator w.r.t. the product $(\cdot, \cdot)$ (i.e. $(\Phi, A\Phi) \geq 0$ for all $\Phi \in \mathcal{S}_C$). In these cases, one may proceed as Kay and define the Hilbert (positive-product) space $\mathcal{A}$ with the product $(\cdot, \cdot)_\mathcal{A}$

$$
\langle \Psi, \Phi \rangle_\mathcal{A} = (\Psi, A\Phi),
$$

(46)

on which the Hamiltonian $H = -\sigma_2 A$ could be shown to be self-adjoint. In such cases, therefore, $H$ has the standard properties of self-adjoint operators in Hilbert spaces, such as the reality of eigenvalues $\omega$, the orthogonality of the eigenvectors and the completeness of the family of eigenvectors. All these properties, therefore, hold for sound modes propagating on a vortex background provided that the boundary conditions at the innermost part of the vortex lead to a self-adjoint $H$.

In the cases where $A$ is not positive definite, one is left with the Krein space $\mathcal{K}$ and with weaker results for $H$ (see appendix A), in particular, eigenvalues are not necessarily real (but do come in pairs of complex-conjugate numbers) and a broader class of vectors is necessary for completeness. As we shall see below, the distinction between cases—for stationary axisymmetric spacetimes—corresponds exactly to the distinction between spacetimes with/without ergoregions.
3.3. Energy and angular momentum

The notions of energy and momentum of a given configuration of perturbations are clearly important from the physical point of view, even though they actually are pseudo-energy, and pseudo-(angular)momentum in this context. These functionals of the sound potential $\psi$ are defined on each surface of constant time $t$ and are preserved during the course of evolution. The form of the functionals is unambiguously defined with the help of tools of classical field theory in curved spacetimes.

Let $\xi^\mu = (\partial_t)^\mu$ denote the Killing vector corresponding to time translation. The current associated with the notion of energy has the form

$$ J_t^\mu[\psi] = T^\mu_{\nu a b}[\psi] \xi^\nu, $$

where $T^ab[\psi]$ stands for the energy–momentum tensor of the solution $\psi(t, x)$:

$$ T_{ab}[\psi] = \partial_a \overline{\psi} \partial_b \psi - \frac{1}{2} g_{ab} \partial_c \overline{\psi} \partial^c \psi, $$

and the index-brackets denote symmetrization, $A_{(ab)} = \frac{1}{2} (A_{ab} + A_{ba})$. The energy is obtained by integrating $J_t^\mu$ over any Cauchy surface $C_t$:

$$ E[\psi] = \int_{C_t} d^3 x J_t^0[\psi]. $$

Note that formally (if relevant boundary terms vanish)

$$ E[\Phi_\omega] = \frac{1}{2} \langle \Phi_{\omega}, A \Phi_{\omega} \rangle = -\frac{1}{2} \langle \Phi_{\omega}, \sigma_2 H \Phi_{\omega} \rangle = \frac{1}{2} \langle \Phi_{\omega}, H \Phi_{\omega} \rangle_{\mathbb{K}} = \frac{1}{2} \omega^2 \| \Phi_{\omega} \|^2. $$

(50)

For scalar fields in Minkowski space, the norm $\| \Phi_{\omega} \|^2$ corresponding to positive-frequency ($\omega > 0$) solutions is always positive; here, this does not need to be the case (because the positivity of $E[\Phi_\omega]$ is not guaranteed, see section 6.1.5 for a discussion of the relation between the sign of energy and the sign of the norm in the context of bound states of vortex sound).

We may also regard energy as a functional of the solutions of the wave equation $\psi(t, x)$:

$$ E[\psi] = \int d^3 x T_{ab} N^a T^b = \frac{1}{2} \int d^3 x \left\{ |\partial_t \psi|^2 + \frac{1}{r^2} \left( 1 - \frac{1}{r^2} \right) |\partial_r \psi|^2 + |\partial_\phi \psi|^2 + |\partial_z \psi|^2 \right\} = \frac{1}{2} \langle \Phi, A \Phi \rangle, $$

(51)

where the integration by parts, $-\int d^3 x \overline{\psi} (\nabla^2 \psi) = \int d^3 x |\nabla \psi|^2$, was performed. (All boundary conditions employed in what follows are such that the—a priori present—boundary terms do vanish.) The energy functional is evidently positive when $r$ is bounded from below by $r_0 > 1$, i.e. in the absence of ergorégs. (See appendix B for a brief general discussion of the positivity of the energy functional.)

The angular momentum associated with a field configuration $\psi$, i.e. $J[\psi]$, will be found as the Cauchy-surface integral of the current associated with the Killing vector field corresponding to rotations, $\xi^\mu = (\partial_\phi)^\mu$:

$$ J[\psi] = \int d^3 x T_{ab} N^a \xi^b = \int d^3 x \text{Re}(\overline{\partial_\phi \psi}) = m \int d^3 x \left( \omega - \frac{m}{r^2} \right) |\psi|^2, $$

(52)

with the last equality holding for the solutions of the form $e^{-im r \cos \phi} \psi(r)$, for which we also find

$$ J[\psi] = \frac{1}{2} m \| \psi \|^2. $$

(53)

This completes our construction of mathematical tools providing structure to the concrete considerations of modes of sound in the presence of a background vortex flow, which now follows. From the mathematical point of view, the concrete considerations are just the spectral analysis of the Hamiltonian (45). The Hamiltonian still needs to be supplied with appropriate boundary conditions. This is done in section 5.
4. Partial-wave decomposition

In this section, the concrete task of solving the wave equation for the sound potential $\psi$ will be attacked by separation of variables. While this straightforward method leads directly to the main equation of this paper (equation (61)) we proceed with caution, and remark on delicate problems of completeness of the family of modes derived this way. Also, even though it might be tempting to consider the main equation as a Schrödinger-like problem, it is not quite correct due to the different nature of the mathematical structure behind it. In particular, the 'frequency' is not just the eigenvalue of the operator considered, but rather appears in an inseparable way in two terms in the main equation (61).

The scalar d'Alembert equation reads

$$\Box g\psi = \frac{1}{\sqrt{-g}} \partial_i (g^{ij} \sqrt{-g} \partial_j \psi) = \frac{1}{r} \partial_r (r \partial_r \psi) + \left[ -\partial_t^2 - \frac{2}{r^2} \partial_r \psi + \left( \frac{1}{r^2} - \frac{1}{r^4} \right) \partial^2_\theta + \partial^2_z \right] \psi = 0 \quad (54)$$

and is equivalent to a Schrödinger-type equation

$$i \partial_t \Phi = H \Phi, \quad \text{with} \quad \Phi_\omega = (\psi, p) \in \mathcal{K}. \quad (55)$$

The problem of finding 'modes' either simplifies solely to the eigenvalue problem for the Hamilton operator

$$H \Phi_\omega = \omega \Phi_\omega, \quad (56)$$

or (in case $H$ is not a normal operator) leads also to equations for Jordan associated eigenvectors (which are rarely considered in investigations of vortex acoustics)

$$H \Phi^a_\omega = \omega \Phi^a_\omega + \Phi^{a-1}_\omega, \quad (57)$$

with a (potentially infinite) chain numbered by the index $a \in \mathbb{N}$ and $\Phi^0_\omega = \Phi_\omega$. The Jordan eigenvectors evolve in time via

$$\Phi^a_\omega(t) = e^{-i\omega t} [\Phi^a_\omega(0) - i \Phi^{a-1}_\omega(0)], \quad (58)$$

fulfill equation (55) and therefore are associated with the solutions of the homogeneous wave equation $\Box \psi = 0$. If we had a finite-dimensional case at hand, the associated vectors would be necessary to obtain a complete set of vectors in the linear Hilbert space with the product $\langle \cdot, \cdot \rangle$. We presume this also to be the case in this context, should Jordan vectors exist for the given $H$ with chosen boundary conditions.

Proceeding further with our concrete problem of determining modes of sound, we separate the angular dependence and the $z$-dependence, as is usually done, by

$$\Phi(t, r) = e^{i(mp + r\omega)} \Psi(t, r), \quad m \in \mathbb{Z}, \quad p \in \mathbb{R}. \quad (59)$$

The domains of $m$ and $p$ are justified by single valuedness of $\Phi$ and self-adjointness of $-i\partial_z$, respectively. The problem is stationary, and for this reason, we separate the time dependence by

$$\Psi(t, r) = e^{-i\omega t} \Psi_\omega(r),$$

where, however, the $\omega$’s are not a priori restricted to be real numbers. Let us consider the eigenvalue problem for $H$ (equation (56)) in a sector of fixed $(m, p)$. The $\mathcal{K}$-product (equation (44)) of two eigenvectors of $H$ assumes the form

$$(\Phi_{\omega_1}, \Phi_{\omega_2})_{\mathcal{K}} = \int d^3x \left( \frac{\omega_1 + \omega_2 - 2m}{r^2} \right) \bar{\Psi}_{\omega_1} \Psi_{\omega_2}, \quad (60)$$
where $\Phi_1^\omega = (\psi_\omega, -i(\omega - \frac{m}{r^2})\psi_\omega)$.

The problem of determining sound modes supported by the vortex finally reduces to solving the radial equation for $\psi \equiv \psi_\omega$, which is the main equation of this paper:

$$\psi'' + \frac{1}{r} \psi' + \left[ k^2 - \frac{M^2}{r^2} + \frac{m^2}{r^4} \right] \psi = 0,$$

(61)

with $k^2 = \omega^2 - p^2$, $M^2 = m(m + 2\omega)$.

Neither $M^2$ nor $k^2$ need to be positive; negative values of these parameters will correspond to physically distinguished situations. From now on, we will use the notation

$$D^2_r \equiv \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}$$

for the (often appearing) radial part of the two-dimensional Laplace operator. The frequency $\omega$ as an eigenvalue of symmetric operator $H$ on a Krein space does not need to be real. The radial equation has the symmetry $(m, \omega) \rightarrow (-m, -\omega)$ so that finding all positive frequency solutions would be enough. We note the singularity at $r = 0$, which in the usual case (i.e. if we were dealing with the usual Hilbert space of square-integrable functions on $\mathbb{R}^3$) leads to well-known ambiguities associated with self-adjoint extensions of the Hamilton operator [29].

If not for the scalar product, the problem would be of the Schrödinger form with

$$- (D^2_r \psi) + \left[ \frac{M^2}{r^2} - \frac{m^2}{r^4} \right] \psi = k^2 \psi,$$

(62)

with the potential $V(r)$ diverging to $-\infty$ as $r \rightarrow 0$.

Momenta along the vortex, denoted here by $p$, are not necessarily forced to obey the relation $p^2 \leq \omega^2$. Such a restriction in usual scattering problems comes from the required behavior of the innermost parts of the ‘space’, for example from a boundary condition at a fixed $r$, or from a regularity requirement at $r = 0$. We will investigate this problem in section 5.

For comparison with ray acoustics, we note the relations

$$\dot{i} = E - \frac{J}{r^2}, \quad \dot{\psi} = \frac{E + J \left(1 - \frac{1}{r^2}\right)}{r^2},$$

(63)

and

$$\dot{r}^2 = (E^2 - p^2) - \frac{J(J + 2E)}{r^2} + \frac{J^2}{r^4},$$

so that for apt analogy with (62) we should make identifications $m \leftrightarrow J$, $\omega \leftrightarrow E$ and $k^2 = E^2 - p^2$. We also note the correspondence between $ip\psi = (\omega - \frac{m}{r^2})\psi$ and $i$ of the acoustic rays.

---

4 The equation(s) leading to the Jordan associated eigenvectors (elements of the Jordan chain) is an inhomogeneous one:

$$\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \psi_\alpha + \left[ k^2 - \frac{M^2}{r^2} + \frac{m^2}{r^4} \right] \psi_\alpha = 2 \left( \frac{m}{r^2} - \omega \right) \psi_\alpha,$$

where $\psi_\alpha$ is either the solution of the homogeneous equation, or the solution at the previous position in the Jordan chain, as the problem has the form $(H - \omega)\psi_{n+1} = \psi_n$. 

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5. Boundary condition at/near the vortex core

In this section, we investigate the appropriate form of the boundary condition at the innermost region of the vortex. The simplest solution, also pursued extensively in the following sections, consists of placing a simple boundary condition at some finite distance from the axis. Because it is the gradient of \( \psi \) that gives the velocity of the sound (density perturbation), it is natural to pose the Neumann condition \( \partial_r \psi(r) = 0 \) at a finite \( r = r_0 \). This condition corresponds exactly to a physical situation with a stiff wire of radius \( r_0 \) with its symmetry axis coinciding with the axis of rotation of the vortex. Such a boundary condition is sufficient, mathematically allowed and allows for the investigation of the dependence of the spectral properties of the Hamiltonian (45) on the ‘wire radius’ \( r_0 \). However, the mathematical nature of the problem at small \( r \) is very interesting, and below we attempt to develop an appropriate boundary condition to be placed at the vortex core, i.e., at \( r = 0 \), by analogy to the solution of the analogous problem in the Hilbert-space context. Besides mathematical curiosity, the ‘core boundary conditions’ developed below provide us with some basis tools to investigate the question: how do the spectral properties of Hamiltonian (45) depend on the details of the innermost parts of the vortex? In contrast to what is mostly reported by the authors, the dependence will turn out to be significant, but only if the regions below \( r = 1 \) are allowed for sound propagation. This behavior is similar to what happens in for the Dirac equation in the Coulomb field of a charge \( Z > 137 \).

The Hamiltonian is defined by its formal expression (45), together with appropriate boundary conditions. We recall that in the case of operators in Hilbert spaces choosing ‘wrong’ boundary conditions, a symmetric operator (fulfilling \( \langle \psi, H \chi \rangle = \langle H \psi, \chi \rangle \) on some product space) can be made non-self-adjoint and this therefore opens up the possibility of complex eigenvalues of \( H \) (frequencies \( \omega \)). For a one-dimensional differential operator, posing a boundary condition at a regular point of the corresponding ODE is relatively unproblematic. Furthermore, for posing boundary conditions at singular points of the ODE, there is a well-established theory due to Weyl [30]. In the vortex case, assuming that the whole space \( r \in [0, \infty) \) is available, we have the problem of posing the boundary condition at \( r = 0 \). This point is a singular point of the radial equation, but Weyl’s theory cannot be directly used because the product space is not a (positive-definite) Hilbert space. We are left, therefore, with an interesting mathematical problem for which no general theory exists. We shall proceed modestly by examining examples, and trying to replicate their results here.

Let us recall the important example of the Schrödinger problem (in the Hilbert space) with the potential \( V = -\frac{m^2}{2r^4} \) (the case proved in [29]) that for small \( r \) the eigenfunctions necessarily have the asymptotic form \( \psi \approx r \cos \left( \frac{m}{r} + B \right) \) with the same value of the phase \( B \) for all \( \omega \). Posing a boundary condition at \( r = 0 \) is equivalent to a choice of \( B \), which then becomes the essential property of the operator. This inhomogeneous condition replaces the more commonly employed homogeneous boundary conditions (such as the Dirichlet/Neumann conditions) at \( r = 0 \), and it is necessary for the mutual orthogonality w.r.t. \( \langle \cdot, \cdot \rangle \) of eigenfunctions to different eigenvalues (a ‘must’ for symmetric operators on a Hilbert and Krein spaces)\(^5\). A reasoning, similar to what has been done in [29], can be repeated in our case.

Close to \( r = 0 \), the radial equation assumes the form, which it always has for \( k = 0 \) (i.e., in the case of waves with the \( \hat{z} \) momentum equal to the frequency, \( p = \omega \in \mathbb{R} \)). Introducing \( u = 1/r \), we find

\[
u^2 \psi'' + u \psi' + (m^2 u^2 - M^2) \psi = 0,
\]

(64)
and thus,
\[ \psi = \text{Bessel}_M(|m|/r), \quad \text{for} \quad r \ll 1 \quad \text{or} \quad k = 0, \]
where \( \text{Bessel}_M(x) \) stands for an arbitrary regular Bessel function \( (J, Y \text{ or } H^{1,2}) \) of order \( M \). We recall that even for \( \omega \in \mathbb{R} \) the parameter \( M \) can be real or imaginary. (The latter case occurs for \( m \in (-2\omega, 0) \), which is the wave analog of ray capture\(^6\).

The asymptotic expansions of the radial function at \( r \ll 1 \) for the distinguished Bessel functions read
\[
J_M\left(\frac{|m|}{r}\right) \approx \cos\left(\frac{\pi}{4} + \frac{M\pi}{2} - \frac{|m|}{r}\right) \cdot \sqrt{\frac{2r}{\pi}} \quad (66)
\]
\[
Y_M\left(\frac{|m|}{r}\right) \approx -\sin\left(\frac{\pi}{4} + \frac{M\pi}{2} - \frac{|m|}{r}\right) \cdot \sqrt{\frac{2r}{\pi}} \quad (67)
\]
For any solution \( \psi \) of the radial equation (61), the asymptotic expansion at small \( r \) can be characterized by a phase \( B \):
\[
\psi \approx \cos B \cdot J_M\left(\frac{|m|}{r}\right) - \sin B \cdot Y_M\left(\frac{|m|}{r}\right) \approx \cos\left(\frac{\pi}{4} + \frac{M\pi}{2} - \frac{|m|}{r} - B\right) \cdot \sqrt{\frac{2r}{\pi}} \quad (68)
\]
or (shifting \( B \) by \( \frac{\pi}{4} + \frac{M\pi}{2} \))
\[
\psi \approx \cos\left(\frac{|m|}{r} + B\right) \cdot \sqrt{\frac{2r}{\pi}} \quad (69)
\]
We now repeat the case’s reasoning in order to reveal the type of boundary condition at \( r = 0 \) necessary in our case. Let \( \Phi_1 \) and \( \Phi_2 \) be two such solutions corresponding to the frequencies \( \omega_1 \) and \( \omega_2 \) and characterized by phases \( B_1 \) and \( B_2 \) close to the core. Assume that both correspond to the same values of the separation constants \( (m, p) \). In order to simplify matters at \( r = \infty \), we assume both of the functions to fulfill a Dirichlet boundary condition at some large \( r = R \) and investigate them on the interval \( r \in [0, R] \). Multiplying the radial equations (61) fulfilled by each function by the other function, subtracting them and integrating the result from 0 to \( R \), we obtain
\[
r[\overline{\psi_1} \partial_r \psi_2 - \overline{\psi_2} \partial_r \psi_1]_0^R \equiv (\omega_2 - \overline{\omega_1})(\Phi_1, \Phi_2)_{\mathcal{K}} \quad (70)
\]
(More precisely, there should stand only the radial part of the \( \mathcal{K} \)-product on the rhs.) Because eigenfunctions of \( \mathcal{K} \)-self-adjoint operator to different eigenvalues \( (\omega_1, \omega_2) \) must be \( \mathcal{K} \)-orthogonal (see appendix A), using the asymptotic expansion at \( r = 0 \) (equation (69)), we arrive at
\[
|m| \sin(B_1 - B_2) \equiv (\omega_2 - \overline{\omega_1})(\Phi_1, \Phi_2)_{\mathcal{K}} \equiv 0 \quad (71)
\]
where on the lhs the (regular) limit \( r \rightarrow 0 \) has been performed. Thus, for each partial-wave problem (fixed \( (m, p) \)), the phases \( B \) must be independent of the frequency \( \omega \). This is so unless we are dealing with a pair corresponding to mutually conjugate, complex frequencies, which however might appear in the problem. The boundary condition, as far as it can be fixed by the orthogonality-argument above, amounts to setting the phase \( B(m, p) \) (a function of \( m \) and \( p \)) to be independent of \( \omega \).

In real physical situations, should indeed regions far below \( r = 1 \) be accessible, the values of the phases \( B(m, p) \) will be related to the details of physics of the innermost parts of the vortex (e.g. they may be related to the surface tension on the interior ‘wall’ of the fluid, see the model in appendix E). In subsequent sections, we will illustrate the fact that the choice of these

\(^{6}\) Restoring the arbitrary value of \( E \), the acoustic rays with \( E^2 \approx p^2 \) would reach the core for \( J \in [-2E, 0] \).
phases does influence the essential physical characteristics of sound propagation on the vortex. We will do so by taking $B(m, p)$ to be independent of $(m, p)$, essentially choosing either the Bessel $J$ or $Y$ asymptotics for all $m$ and $p$. We do not expect this type of choice to be related to any particularly distinguished physical situation. This, apart from the ‘stiff-wire = Neumann’ boundary condition introduced in the beginning of this section, will provide two families of one-parameter boundary conditions for investigation of the spectrum of vortex-sound modes.

6. Solutions to the radial equation

This is the section containing the main new technical results of this paper. The only equation to be solved is the ‘radial’ equation (61), but the solutions must be examined for all values of relevant separation constants $(\omega, p, m)$, as well as for the values of the parameter specifying the boundary condition, $r_0$ for the ‘stiff-wire’ condition, or the phase $B$ for the ‘core’ boundary condition. Apart from the general considerations below we will solely focus on the ‘bound-state-like’ solutions of the radial equation, leaving the consideration of scattering states to a separate investigation.

6.1. Structural results

6.1.1. General considerations. A general-enough consideration of the radial equation (61) assumes only $m \in \mathbb{Z}$ and $p \in \mathbb{R}$. It is important to restrict the number of parameters to an essential set. Recalling that a priori $\omega \in \mathbb{C}$, and thus, $k^2 = \omega^2 - p^2$ is in general a complex number, we introduce a new variable $x_2^2 = \left| \frac{k}{m} \right|^2 r^2$, $x \in [0, \infty)$ and transform the radial equation (61) into the form

$$D^2_x \psi - \frac{M^2}{x^2} \psi + K^2 \left[ e^{i \arg(k^2)} + \frac{1}{x^4} \right] \psi = 0,$$

(72)

where $M^2 = m(m + 2\omega)$ and $|mk| = K^2$. We note that by a simple transformation of variables the equation can be shown to be equivalent to the Mathieu equation, taken along various contours in the complex plane (see appendix C). In the case of real frequencies, the equation assumes the form(s)

$$D^2_x \psi - \frac{M^2}{x^2} \psi + K^2 \left[ \pm 1 + \frac{1}{x^4} \right] \psi = 0,$$

(73)

where the $+$ sign corresponds to solutions having an interpretation of scattering states, occurring for $k^2 = \omega^2 - p^2 > 0$, while the $-$ sign to ‘bound states’, possibly occurring for $k^2 < 0$. Generally, unless $k^2$ is real positive, exactly one of two linearly independent solutions will be exponentially damped at $x \to \infty$.

For the scattering solutions, with $k^2 > 0$ on the two extremes of the domain of $x$ (0 and $\infty$), we drop the fastest decaying terms $x^{-4}$ or $x^4 = u^{-4}$ and obtain two Bessel equations with equal indices

$$D^2_x \psi - \frac{M^2}{x^2} \psi = 0, \quad x \gg 1,$$

$$D^2_u \phi - \frac{M^2}{u^2} \phi = 0, \quad u \gg 1,$$

where $u = x^{-1}$. Therefore, if regions of large $u$ are allowed, we are lead to the classical scattering problem on a real line. This problem will not be considered here further and must be relegated to a separate study.
6.1.2. Bound states in the \( (K^2, M^2) \) plane. Having established the general form of the radial equation, we now study its distinguished solutions more closely. As remarked before, in the case of negative \( k^2 = \omega^2 - p^2 \), the equation has the form

\[
D^2 \psi - \frac{M^2}{x^2} \psi + K^2 \left[ -1 + \frac{1}{x^2} \right] \psi = 0,
\]

(74)

with \( x \in (0, \infty) \). The solutions of this equation behave as modified Bessel functions \( (K_M(Kx) \text{ and } I_M(Kx)) \) at \( x \to \infty \). An important question arises that whether for a given boundary condition at a finite (or zero) \( x \) there exist values of \( M \) and \( K \), for which the solution would be ‘bound-state-like’ with the asymptotics containing only the exponentially damped Bessel function. One might expect to be able to answer this question with the help of the specific investigations of the Mathieu equation (e.g. [31, 32, 23]). So far, however, we have not been able to locate the answer in the literature. For that reason, we present here numerical results, from which important physical conclusions can be drawn. We shall use the following two types of boundary conditions.

(I) The family of ‘core’-boundary conditions provided by the orthogonality argument of section 5. The values of the constant \( B \) corresponding to asymptotic functions of the first \((J)\) and second \((Y)\) kind will be considered. We will only consider the (illustrative) cases, where the phase \( B \) is independent of \( m \) and \( p \); otherwise the boundary condition would depend on the position in the \( (K^2, M^2) \) plane.

(II) We consider the family of ‘stiff-wire’ (i.e. Neumann) boundary conditions posed at a finite \( r = r_0 \). This allows us to consider the effect of the presence/absence of an ergoregion, which is included in the spacetime if points with \( r < 1 \) belong to it.

6.1.3. Scaling transformations and positions of bound states. Both types of boundary conditions, the Neumann condition and the ‘core’ condition (69), can (for given values of regular parameters \((m, p, \omega, r_0 \text{ or } B)\)) be written purely in terms of the essential parameters/variables \((x, M, K)\). For the Neumann condition, we require

\[
\frac{\hat{\psi}}{\hat{x}} \bigg|_{x=x_0} = 0
\]

(75)

(with the \( x_0 \) computable from \( r_0 \), and from the other regular parameters), whereas for the ‘core’ condition, we require

\[
\psi \approx \cos \left( \frac{K}{x} + B \right) \sqrt{x}.
\]

(76)

With this reformulation performed, we consider the radial (Mathieu) problem in the space of essential parameters. The results (such as positions of bound states in the \( (K^2, M^2) \) plane) can always be recalculated to the space of regular parameters. An important ambiguity, however, occurs, because the number of essential parameters, \((x_0, K, M)\) for the Neumann condition, and \((B, K, M)\) for the ‘core’ condition, is smaller than the number of regular parameters, \((r_0, m, p, \omega)\) or \((B, m, p, \omega)\), there exists a one-parameter family of transformations for the regular parameters, leaving the essential parameters invariant. Recalling the relations

\[
x^2 = \left| \frac{k}{m} \right|^2, \quad K^2 = |mk|, \quad M^2 = m(m + 2\omega),
\]

(77)

we see immediately that essential parameters remain invariant if \((r_0, m_0, k_0, \omega_0)\) is transformed into

\[
r_\lambda = \lambda r_1, \quad m_\lambda = \lambda m_1, \quad k_\lambda = \frac{1}{\lambda} k_1, \quad \Omega_\lambda = \frac{1}{\lambda^2} \Omega_1.
\]

(78)
where

$$\Omega_\lambda = 1 + \frac{2m_\lambda}{m_\lambda}. \quad (79)$$

The only restriction on $\lambda$ is, that $m_\lambda$ should also be an integer. The frequency transforms according to

$$\omega_\lambda = \frac{m_1}{2} \left( 1 - \frac{1}{\lambda} \right) + \frac{\omega_1}{\lambda}, \quad (80)$$

6.1.4. Positive/negative frequency bound states. The $\lambda$-transformations allow to flip the sign of the frequency of a bound-state solution. Introducing an abbreviation $\mu_1 = \frac{m_1}{2} \omega_1$, we consider

$$\omega_\lambda = \frac{\omega_1}{\omega_1} = \mu_1 \left( \frac{1}{\lambda} - \lambda \right) + \frac{1}{\lambda}. \quad (81)$$

An inspection of the above function shows that unless $\mu_1 \in [-1, 0]$ (i.e. unless the $(\omega_1, m_1)$ parameters fulfill $m_1 \in [-2\omega_1, 0]$, which is equivalent to $M^2 \leq 0$) the range of the function is the whole real set $\mathbb{R}$, and therefore, the sign of the frequency will be flipped where $\omega_\lambda < 0$. The change of the sign of $\omega_\lambda$ occurs for large $\lambda$ if $\mu_1 > 0$ and for small $\lambda$ if $\mu_1 < -1$. Thus, a solution corresponding to $\omega_1 > 0, m_1 > 0$ can always be $\lambda$-transformed to produce a solution with $\omega_\lambda < 0, m_\lambda > 0$. This is true in the ‘existential sense’, i.e. it allows us to state that such solutions would exist in certain boundary-value problems. The $\lambda$-transformation for the Neumann boundary condition changes the radius $r_0$ (where the condition is posed) and therefore changes the problem considered. Solutions with the same sign of $\omega$ and $m$ rotate in the positive direction (co-rotate with the vortex).

6.1.5. Negative norm states for positive frequencies. For principal reasons, it is important to investigate whether a positive-frequency solution could in some cases correspond to negative ‘norms’:

$$||\Phi_\omega||^2 = 2 \int d^3x \left( \omega - \frac{m}{r^2} \right) |\psi_\omega|^2. \quad (82)$$

In this regard, let us stress that for positive $M^2$ the continuous nature of the positions of bound states in the $(K^2, M^2)$ plane allows us to investigate cases where $M^2 = m^2$, with $m \in \mathbb{Z}$. Because of $M^2 = m(m + 2\omega)$, we may regard $\omega$ to be a linear function of $M^2$ indexed by the branch-numbering, discrete parameter $m$:

$$\omega_m(M^2) = \frac{1}{2} \left( \frac{M^2}{m} - m \right). \quad (83)$$

Then, $\omega_m(M^2)$ crosses zero at $M^2 = m^2$, as we move along the bound-state curve in $(K^2, M^2)$ plane. It is clear from (82) that in regions of negligible $\omega$ and positive $m$ the ‘norm’ of the corresponding solution will be negative. Therefore, states with $M^2$ slightly greater than the branch-defining $m^2$ will have a positive frequency $\omega_m(M^2)$ and a negative norm $||\Phi_{\omega_m(M^2)}||^2$. (Whether or not the norm of such a solution will turn positive as we continue along the bound-state-line in the $(K^2, M^2)$ is not clear.)

On the other hand, the states with negative $M^2$ always correspond non-zero frequencies; the lowest possible frequency $\omega_m(M^2)$ for a given negative $M^2$ is $\omega = \sqrt{-M^2}$. (This happens only in the case that $m = -\omega$ is an integer.) The question whether or not negative norm solutions for positive frequencies exist for $M^2 < 0$ does not appear to be answerable by general arguments of the above type.
6.1.6. Bound states and scaling transformations. The $\lambda$-transformations allow us to tell whether bound states are necessarily related to the presence of an ergoregion in the acoustic spacetime. We will consider the ‘stiff-wire’ conditions posed at $r_0$, with ergoregions disappearing from the acoustic spacetime once the condition is posed at $r_\lambda > 1$.

Let us take one solution related to a bound state, with the Neumann condition at $r_1$, and consider the $\lambda$-transformations. It is clear that following the $\lambda$-trajectory, the new position of the boundary $r_\lambda = \lambda r_1$ will—for sufficiently large $\lambda$—be greater than 1. Focusing on a fixed ‘original problem’ with the parameters $(r_0, m_0, k_0, \omega_0)$, we note that for very large $\lambda$’s the following limits arise: $m_\lambda \gg 1$, $k_\lambda^2 \ll 1$ that is $p_\lambda^2 \approx \omega_\lambda^2$, with the frequency $\omega_\lambda = -\frac{m_\lambda^2}{2}$ (and therefore $|\omega_\lambda| \gg 1$). Such states, therefore, correspond to disturbances rotating in a negative direction (against the vortex rotation), with large angular quantum numbers, large frequencies and equally large momenta along the vortex.

Let us remark on how figures such as figure 4 are to be interpreted. First, the value of $x_0$ is the specific characteristic distinguishing between the spectra (lines in the $(K^2, M^2)$ plane corresponding to bound states). Second, $k = \sqrt{p^2 - \omega^2}$ will be known, once $x_0, r_0$ and $m$ are known:

$$|k| = |m| \left( \frac{x_0}{r_0} \right)^2,$$

which allows us to write

$$K^2 = m^2 \left( \frac{x_0}{r_0} \right)^2.$$

This equation allows for multiple interpretations of a given point $(K^2, M^2)$ on the ‘bound-state line’, depending on the chosen value of $m \in \mathbb{Z}$. Thus, from given $(K^2, M^2, m)$, we compute the radius $r_0$ and the frequency $\omega_m(M^2)$:

$$r_0 = \frac{|m|x_0}{\sqrt{K^2}},$$

$$\omega_m(M^2) = \frac{1}{2} \left( \frac{M^2}{m} - m \right).$$

Furthermore, once $r_0$ is known, $|k|$ and (together with the known $\omega_m$) $|p| = \sqrt{k^2 + \omega_m^2}$ can be computed.

Points lying near $M^2 = n^2$ ($n \in \mathbb{N}$) are distinguished, because they can correspond to zero-frequency modes. For instance, from figure 4, with $x_0 = 0.1$, we find a bound state at
Figure 5. Positions of the bound states in the \((K^2, M^2)\) plane for core boundary conditions, together with distinguished lines \(M \in \mathbb{N}\): (left) for the boundary condition \(\psi \approx J_M(K/x)\) and (right) for \(\psi \approx Y_M(K/x)\). The lines are zeros of the Wronskian of the exact (numerical) solution with the Bessel function \(K_M(Kx)\), which is the asymptotic form of the exponentially decaying solutions at \(x = \infty\). We note that none of the branches seem to enter/exist in the \(M^2 \leq 0\) quadrant, which suggests that there are no bound states there (for the ‘core’ boundary conditions).

\[
M^2 = 1, K^2 \approx 0.034, \text{ corresponding the zero-frequency, } \omega_1(1) = 0, \text{ solution with } |m| = 1 \text{ and }
\]
\[
r_0 \approx 0.54, \quad p = \pm \left(\frac{x_0}{r_0}\right)^2 \approx \pm 0.034. \quad (86)
\]

Thus, in this case, it is a problem with an ergoregion, and the (essentially static) perturbation has a form of a helical structure, with weak \(z\)-dependence, localized close to the vortex \((\psi \sim K_1(pr) \text{ for large } r)\). By considering states close to \(M^2\), with the same \(|m|\), we obtain problems (characterized by \(r_0\)‘s) with similar solutions, but either co-rotating (for \(M^2 > 1\)) or counter-rotating (for \(M^2 < 1\)) with the vortex.

On the other hand, we could have interpreted the above values of \((K^2, M^2)\) as coming from the second branch \(|m| = 2\), for which we would obtain
\[
r_0 \approx 1.09, \quad \omega_{-2}(1) = 0.75, \quad p = \pm \sqrt{2 \left(\frac{x_0}{r_0}\right)^2 + \omega^2} \approx \pm 0.75. \quad (87)
\]

This solution, therefore, corresponds to a problem without an ergoregion, and the perturbation (localized very close to the vortex) is counter-rotating \((\text{sgn}(\omega m) < 0)\) and moving along the vortex.

In figure 5, we have presented the dispersion relations for the case of two different ‘core’ boundary conditions. The apparent lack of bound states for \(M^2 < 0\) provides further evidence of the sensitivity of the spectrum to the conditions in the innermost part of the vortex.

6.2. Location of bound states in the space of physical parameters; dispersion relations

The scaling transformations in the previous section allowed us to formulate ‘existential’ statements to the effect that the existence of one bound state for certain values of physical parameters \(r_0, \omega, m, p\) was equivalent to the existence of bound states for other values of these parameters. From the physical point of view, it is, however, of interest to determine
also the full ‘spectrum’ of solutions for a given boundary problem. In other words, for a given type of boundary condition at \( r = 0 \), or at a finite \( r = r_0 \), we shall compute the lines corresponding to bound states in cuts of the \((p, \omega, m)\) parameter space. This is a rather large space of parameters, and below we will attempt to extract the essential characteristics of the solution of the problem, i.e. essential features of the three-dimensional surfaces embedded in the four-dimensional space \((r_0, p, \omega, m)\). We restrict this numerical consideration to the boundary conditions of Neumann type posed at a finite \( r = r_0 \). Only the case \( \omega > 0 \) will be presented, as the case \( \omega < 0 \) follows from the symmetry \((\omega, m) \leftrightarrow (-\omega, -m)\) of the problem.

The form of the dispersion relations is qualitatively different for co- and counter-rotating solutions, as defined by the sign of \( m \). (There are no bound states with \( m = 0 \), as the radial functions would need to be \( K_0(kr) \), which however is a monotonic function and cannot fulfill the Neumann boundary condition at any \( r_0 \) (we recall that \( k = \sqrt{p^2 - \omega^2} \)).) For the counter-rotating solutions \((m < 0)\), we obtain a number of branches concentrated just below the diagonal \( \omega = p \) (see figure 6). Solutions generally exist for all values of \( r_0 \), but small frequencies are only possible in the presence of an ergoregion, i.e. if \( r_0 < 1 \). The group velocities approach (but are always less than) 1, meaning that packets constructed out of the wavefunctions corresponding to bound states would move along the vortex with almost the speed of sound. Given the observation that each branch of solutions appears to begin at \( \omega = p \), we may ask: what is the lowest frequency \( \omega \) of a solution(s) corresponding to various \( m \) for a given \( r_0 \)? In figure 7, we have plotted the positions of the branches of bound states in the \((\omega, m)\) plane (with \( p = \omega + 0.002 \)). There is an apparent qualitative change between \( r_0 < 1 \) (ergoregion) and \( r_0 > 1 \) (no ergoregion): in the latter case no solutions with \( \omega \approx 0 \) are possible. In the cases barely admitting ergoregions \((r_0 \lesssim 1)\), the only branch approaching \( \omega = 0 \) is the leftmost one, and it does so for rather large \(|m|\) (see the central plot of figure 7). For \( r_0 > 1 \), the solutions with small \(|m|\) are the ones leading to lowest \( \omega \)'s, although not necessarily the \( m = -1 \) mode. These modes are all gapped in contrast to the gapless Kelvin mode in the unpinned situation [3, 16, 5].

The co-rotating solutions \((m > 0)\) exist only in the presence of an ergoregion. The branches are characterized by negative group velocities, all of which start at the diagonal \( \omega = p \) and end on the line \( \omega = 0 \) (see figure 8). As the ‘wire radius’ \( r_0 \) is increased (for a
Figure 7. Positions of counter-rotating bound states with $p = \omega + 0.002$ for $r_0 = \{0.5, 0.9, 1.2\}$. Zero-frequency states exist only for $r_0 < 1$ and are related to leftmost branches in the plots. From $r_0 > 1$ up to $r_0 \approx 8$, the state with $m = -1$ is the lowest \( \omega \) state, while for larger $r_0$, this role is played by states with ever lower $m$.

Figure 8. Typical ‘band-structure’ for bound states localized close to the core, propagating along the vortex with a momentum $p$, and co-rotating with it for (left) $(r_0, m) = (0.1, 5)$, (central) $(r_0, m) = (0.3, 5)$ and (right) $(r_0, m) = (0.1, 1)$. The plots correspond to a spacetime with an ergoregion; otherwise, co-rotating bound states do not seem to exist.

given $m$), the branches move toward small $p$, and some of them disappear. For each $m$, there exist a critical radius $r_0$, above which no co-rotating solutions exist.

In figure 9, we have depicted the positions of bound states in the $(r_0, m)$ plane, assuming fixed, small values of frequencies and momenta, with $\omega \approx p$. Given the way position of bound states moves as $r_0$ is increased (figure 8), an appearance of state at $p \approx \omega \approx 0$ signifies the entrance of a new branch into the $(p, \omega)$ plot (dispersion relation). The rightmost line in figure 9 is the line of occurrence of the first co-rotating bound state, i.e. at a fixed $m$ no bound states exist if $r_0$ is greater than the position of this line. For $r_0$ just smaller than 1, the co-rotating states exist only with very large angular numbers $m$.

7. Preliminary stability considerations: the absence of complex frequencies and loophole in Friedman’s theorem

In this section, we give preliminary answers to the question of linear stability of the vortex flow. We do that by considering the question of existence of complex-frequency solutions.
Additionally, we show the relation between the present problem and the issue of stability of compact (astrophysical) objects with ergoregions but without horizons.

In the fluid-dynamical context, by considering small (test) perturbations of a given flow, one can distinguish two types of instabilities. The first type develops, when the global energy of the fluid is lowered upon introduction of some perturbation. The second type comes about when modes of perturbation with complex eigenvalues are allowed, i.e. fulfill the appropriate wave equation and the boundary conditions. An example of the first, milder type of instability is provided by the creation of excitations in superfluid flows exceeding the Landau velocity \cite{33}. It is natural to expect this type of instability in the presence of ergoregions because of the non-positivity of the energy density of perturbations there. However, it is difficult to estimate how rapid the dynamics associated with the instability will be, as the rate of creation of (negative-energy) excitations depends on external factors such as the surface roughness for capillary superfluid flows. In the vortex case, one can easily think of situations where the fluid will not encounter any factors efficiently generating negative-energy excitations. Invoking the spacetime analogy, it is difficult to estimate \textit{a priori} the probabilities of Penrose-type processes \cite{34}.

Due to relation (50) negative energies are associated with positive-frequency, negative-norm states\footnote{Of course, by \((\omega, m) \rightarrow (\omega, -m)\) symmetry, with negative-frequency, positive-norm solutions.}. Naturally, this cannot happen when \(m < 0\) (for counter-rotating modes), because the norm is explicitly positive definite in this case. For co-rotating \((m > 0)\) modes, we can generally assess that at least the energies of the modes close to \(\omega = 0\) will be negative. This is always possible (see figure 8) whenever such modes exist, but as we have argued it requires the presence of an ergoregion (see e.g. figure 9).

In regard to this first type (negative-energy) of instability, let us note the connection to the work of Friedman, where it is argued that astrophysical spacetimes with ergoregions but without horizons are unstable \cite{35}. The theorem proved there states that unless a test, massless field can settle down to a non-radiative time-dependent state, the energy of the system in the bounded vicinity of the object will tend to \(-\infty\), radiating an infinite amount of energy to null
infinity, provided the system starts from a negative energy state. The vortex spacetime provides a model of a spacetime where an analogous statement should also hold.\(^8\) Note, however, that our co-rotating bound states provide exactly the type of solutions the ‘unless’ clause would exclude: they are time-dependent, possibly negative-energy, non-radiative solutions of the d’Alembert equation in a stationary axisymmetric spacetime with an ergoregion. We do not see how, in the astrophysical context, the existence of the bound-state-like states can be excluded, but such states decaying exponentially at large \(r\) (as in our vortex case) would violate the assumed asymptotic form of the solutions of the wave equation (equation (23)) in \([35]\). Moreover, as the existence of bound states with negative energy in our work is seen to be related to the presence of the ergoregion, we doubt that one can argue against the existence of such states (in any context) by examining fall-off conditions of the field at null infinity alone (as is done in \([35]\)).

If the completeness property for eigenmodes of the Hamiltonian turns out to hold in the ergoregion case, where it is a symmetric operator on a Krein space only, one will argue that every wavepacket can be expanded into the eigenmodes and the part associated with scattering states will be radiated away, carrying positive energy, while the part associated with bound states will remain close to the innermost part of the system forever (see figure 10).

Let us finally comment on the second type of instability, associated with complex frequencies, which is typical of unstable configurations of fluids. By the known properties of symmetric operators in Krein spaces (see appendix A), complex isolated eigenvalues of the Hamiltonian (39) always appear in pairs of mutually conjugate numbers. The radial part of the Hamiltonian and the boundary condition associated with it depend, however, on three free, real parameters: \(p\), \(r_0\) and \(m\). Let us call such an operator \(H_\xi\), where for the purpose of the argument we investigate the dependence of the spectrum on one of the parameters, say \(\xi\), with the other fixed.\(^9\) Relying on the analogy with similar (Hamiltonian, not necessarily self-adjoint) problems \([26, 27, 36]\), and on the experience with finite-dimensional operators \([37]\), we expect that one signature of an appearance of complex frequencies would be if a pair of real frequencies (two lines in the \((\xi, \omega)\) plane, say for \(\xi > \xi^*\)) were to ‘merge’ at a

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\(^8\) Friedman’s work is related to the astrophysical context and therefore assumes asymptotic flatness. Although the vortex (acoustic) spacetime due to its cylindrical symmetry does not conform to this assumption, we just use it to make the point that bound states would provide a way out of the conclusion of the theorem of Friedman.

\(^9\) We also extend the domain of \(m\) to real numbers.
Figure 11. Typical positions of bound states: (left) in the \((r_0, \omega)\) plane (Neumann problem) for fixed \((m, p) = (1, 6)\), (center) in the \((m, \omega)\) plane for fixed \((r_0, p) = (0.9, 6)\) and (right) in the \((p, \omega)\) plane for fixed \((r_0, m) = (0.2, -5)\). In none of the figures do the curves corresponding to bound states develop points with 'vertical tangent', symptomatic of the development of a pair of mutually complex frequencies.

point \((\xi^*, \omega^*)\), where the tangent line would be vertical \(\frac{d\omega}{d\xi} \bigg|_{\xi^*} = \infty\). Beyond this point (i.e. for \(\xi < \xi^*\)), only two mutually conjugate complex frequencies would remain. In figure 11, we present the dependence of \(\omega\) on various choices of \(\xi\) in what we observe to be the general behavior: no points of vertical tangent are to be found in the spectrum. We therefore conjecture that no complex frequency modes ever develop from a merger of real frequencies (in the vortex problem). However, the other way for complex frequencies to enter the spectrum is to enter from the continuous spectrum, i.e. from the region \(|\omega| > |p|\). Such an entry would leave no trace on plots of the positions of bound states in \((\xi, \omega)\) planes (because on these plots \(\omega \in \mathbb{R}\) per assumption) and requires further investigation.

In order to give an example of how complex frequencies can manifest themselves in problems involving sound propagating on rotating background flows, we recall the result of Sipp and Jacquin [16]. In this work, an inviscid fluid forming a Lamb–Oseen vortex, \(\Gamma(r) = \text{const} \left(1 - e^{-r^2/r_0^2}\right)\), is investigated and it is found that many of the Kelvin waves are damped, i.e. possess frequencies with negative imaginary parts. Moreover, as one moves along a single branch (mode) in the \((p, \omega)\) plane with an (initially) real frequency \(\omega_i\), a non-zero damping rate suddenly emerges at some critical value of \(p\). The relation of this result to our above argument is however not straightforward, because Lamb–Oseen vortex involves vorticity of the background flow and cannot be handled by the simple acoustic-spacetime description\(^{10}\). This description leads to the structure with \(H\) being a symmetric operator on a Krein space (see section 3.2).

8. Rotating superfluid flows accessible to experiments

In this section, we summarize the commonly available experimental setups in order to assess the relevant orders of magnitude of the scale parameters used in this work. As we currently see two main areas of application, i.e. supersaturated quantum vapors (BECs) and the superfluid helium-4, we shall focus on these two types of fluids.

\(^{10}\) An investigation, comparable to the acoustic-spacetime description, would need to use the extended formalism of [22]. While this, to the best of our knowledge, has not yet been attempted, it would provide an interesting extension of the work of Flaig and Fischer [13].
8.1. Supersaturated quantum vapors

8.1. Scattering length, mass, interaction strength, vortex circulation for $^{87}\text{Rb}$. Common experiments with rotating BECs (see e.g. [19, 6]) use doubly spin-polarized $^{87}\text{Rb}$ atoms. The relevant scattering length in this case is the triplet one $^{11}a_t = 105 \pm 4a_0$, i.e. $a_t = 5 \times 10^{-7}$ cm. Using the atomic mass of $^{12}^{87}\text{Rb}$, i.e. $m_B = 87u = 1.4 \times 10^{-22}$ g, one estimates the interaction strength

$$g = \frac{4\pi \hbar^2 a}{m_B} = 4.5 \times 10^{-38} \text{ g cm}^5 \text{ s}^{-2}. \quad (88)$$

The quantum of circulation is $\Gamma_0 = \int v_i \, dx' = 2\pi v_p = \frac{2\pi \hbar}{m_B} = 4.5 \times 10^{-5} \text{ cm}^2 \text{s}^{-1}$. In the acoustic metric (5), there appears

$$\kappa = v_p = 7 \times 10^{-6} \text{ cm}^2 \text{s}^{-1}. \quad (89)$$

(Some) experimental values of $n, c_s, \xi$ and $r_c$. Typical contemporary experiments involve condensate clouds with $N = 3 \times 10^5$ atoms in a cigar shape of characteristic dimensions $10^3 \times 100 \mu \text{m}^3 = 10^{-8} \text{ cm}^3$ corresponding to a typical number density of $n = 3 \times 10^{13} \text{ cm}^{-3}$. The typical speeds of sound are therefore of the order

$$c_s = \sqrt{\frac{\partial p}{\partial \rho}} = \sqrt{\frac{gn}{m_B}} = 10^{-1} \text{ cm s}^{-1}. \quad (90)$$

For vortices, we get the critical radius (radius of the ‘ergoregion’)

$$r_c = \frac{\kappa}{c_s} = 7 \times 10^{-5} \text{ cm} = 0.7 \mu \text{m}, \quad (90)$$

which is exactly the coherence length (below which the density and $c_s$ begin to vary with $r$) multiplied with $\sqrt{2}$:

$$\xi = \frac{\hbar}{\sqrt{2m_Bgn}} = r_c/\sqrt{2} = 0.5 \mu \text{m}. \quad (91)$$

The hydrodynamic description of BECs becomes inadequate on distances comparable with $\xi$, and therefore, typical BEC setups correspond to acoustic spacetimes without ergoregions. As a consequence of the relation $r_c = \sqrt{2}\xi$, modifications of the setups involving changes to the number density $n$, interaction strength $g$, mass/type of the boson $m_B$ would not change this state of affairs.

Qualitatively, new situation arises for vortices in condensates trapped by anharmonic external fields due to the modification of vortex structure in such setups [5, 9]. Namely, e.g., for condensates trapped by potential growing quicker than $r^2$ (or for condensates trapped by cylinders with solid walls), a new picture arises [9]. The regular Tkachenko lattice of vortices begins to get replaced at the center by an empty core (giant vortex), where condensate density is zero. The position of this core can be further stabilized by a central repulsive (external) potential. The velocity circulation around this core is finite, but the velocity of sound becomes negligible due to the vanishing of condensate density. Even though there are (usually) still vortices in the bulk, and the density is far from being constant everywhere $^{13}$, the system exhibits characteristics of a spacetime with an ergoregion.

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11 $a_0 = 5 \times 10^{-9}$ cm is the Bohr radius.
12 We denote the mass of the boson constituent of the condensate by $m_B$.
13 Density varies according to $n(r) \sim r^2 - R_0^2$ and the speed of sound as $c_s \sim \sqrt{n(r)}$. For highest angular momenta, as the giant vortices begins to emerge, there remains a fraction of the Tkachenko lattice for $r > R_0$ with a constant lattice spacing $\ell$ (see [9]); the velocity field corresponding to the missing lattice (for $r < R_0$) still circulates around the core of the giant vortex.
8.2. Superfluid helium-4

Single vortices. The typical speed of (first) sound in the superfluid helium-4 is \( c_s = 2.4 \times 10^4 \text{ cm s}^{-1} \). Together with the mass of \(^4\text{He} \) atom, \( m_{\text{He}} \approx 4u = 6.64 \times 10^{-24} \text{ g} \), this corresponds to \( \kappa = \frac{\Gamma_0}{2\pi} = 1.588 \times 10^{-4} \text{ cm}^2\text{s}^{-1} \), where \( \Gamma_0 = \frac{h}{m_{\text{He}}} \) is the quantum of circulation; with these values, we obtain

\[
\kappa = 1.588 \times 10^{-4} \text{ cm}^2\text{s}^{-1},
\]

which is less than the van der Waals radius of the helium atom, \( r_{vdW} \approx 1.4 \times 10^{-8} \text{ cm} \). Consequently, single vortices in superfluid helium-4, when pinned on a stiff wire on the axis of symmetry, provide an arena relevant to this work, but with the Neumann radius \( r_0 \) (divided by \( r_c \)) greater than 1 (no ergoregion). As shown in section 6.2, counter-rotating bound states do appear also in such situations, but their frequencies are rather large (the frequency scale for single vortices such that \( \omega = 1 \) in our paper corresponds to about 3600 GHz and is larger than the roton frequency for a single vortex).

There exists a possibility of trapping a larger number \( N \) of circulation quanta around a wire of fixed radius \( r_0 \). This has the effect of enlarging linearly \( r_c \) (and therefore reducing \( r_0/r_c \)) and reducing linearly the frequency scale (see appendix \( \text{E} \) for an assessment of such a possibility). It is not clear at present whether sufficiently large \( N \) can be (at least metastably) wound on the central wire to make the first bound states detectible by accessible acoustic devices. We expect these states to act as an efficient ‘acoustic transmission line’ in the direction parallel to the vortex, with most of the transmitted sound energy localized close to the wire (with characteristic spatial modulations, as described by the corresponding bound-state wavefunction \( \psi(r) \)).

9. Summary

In the paper, we have investigated propagation of sound in the presence of line vortices with irrotational velocity profiles, such as the ones appearing for rotating superfluids. The vortex was assumed not-bendable (e.g. wound around a stiff wire), in order to separate the acoustic effects from effects due to vortex bending (true Kelvin waves). This assumption allowed for the use of the formalism of acoustic spacetimes, where the density perturbation (sound) is parametrized by a single scalar function \( \psi(t, x) \). The solutions of the wave equation fulfilled by \( \psi \) were systematically investigated: first, in the acoustic-ray approximation, and later, in the full wave regime. It was determined that a large family of bound states, i.e. states exponentially localized in the vicinity of the vortex, is present in the problem. This family was extensively investigated for two one-parameter types of boundary conditions, and in the various ‘cuts’ through the space of parameters of the waves, including the frequency, the angular momentum and the momentum along the vortex. Furthermore, the mathematical structure of the problem was extensively explored, including formulation of the problem in terms of a Hamiltonian structure, discussing the relevant boundary condition at the singular point of the relevant potential, and discussing the relation of the vortex-sound problem to the problem of scalar fields in curved (rotating) spacetimes. Finally, a brief summary of the relevant experimental regimes was presented, in order to give the experimentally plausible ranges of dimensionless parameters appearing in the considerations.
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Appendix A. Spectral properties of self-adjoint operators in Krein spaces

Let $H$ be a self-adjoint operator on a Krein space with the indefinite product $(\cdot, \cdot)$ (we drop the index $H$ here). The following properties hold (see [39, 40, 26]):

(a) if $\omega \neq \omega'$, then the corresponding eigenvectors are orthogonal, $(\Psi_\omega, \Psi_{\omega'}) = 0$,
(b) for each eigenvalue $\omega$, there also exists an eigenvector to the eigenvalue $\omega$, $(\Psi_\omega, \Psi_\omega) = 0$,
(c) if an eigenvalue $\omega$ is not real, then the corresponding eigenvector is of zero norm, $(\Psi_\omega, \Psi_\omega) = 0$,
(d) if to a certain eigenvalue $\omega$ there also exists an associated eigenvector, $H \Psi_\omega = \omega \Psi_\omega$ (where $\Psi_\omega$ is the normal eigenvector to $\omega$), then the normal eigenvector is of zero norm, $(\Psi_\omega, \Psi_\omega) = 0$.

According to [26], in the case considered there (electrostatic potential $V(x)$), the eigenfunctions together with the associated eigenfunctions form a complete set so that any element of the phase space can be decomposed as (see equation (3.33) of [27])

$$\Phi(t) = \sum_i \left[ a_i \Psi_{\omega_i} e^{-i\omega_i t} + b_i \left[ \Psi_{\omega_i}^a - i\Psi_{\omega_i}^b \right] e^{-i\omega_i t} \right], \quad \text{with} \quad \Psi_{\omega_i} = \Psi_{\omega_i}(x). \quad (A.1)$$

Appendix B. Positivity of the energy functional

We introduce the decomposition $g_{ab} = -N_a N_b + e^b_a e^b_a + s_{ab}$ and $T_a = \alpha N_a + \beta e^b_a$ (for vortex spacetime $\beta = -\frac{1}{r}$, and $\alpha = 1$), and examine the integrand of the energy-functional $E[\psi]$. Using the notation $\partial_N \equiv N^a \partial_a$, $\partial_\phi \equiv e^b_a \partial_a$, $(\partial_\phi \psi)^2 \equiv \gamma^{ab} \partial_\phi \psi \partial_\phi \psi$, we find that

$$T_{ab} N^a T^b = \frac{\alpha}{2} \left[ (\partial_N \psi)^2 + 2 \frac{\beta}{\alpha} (\partial_N \psi)(\partial_\phi \psi) + (\partial_\phi \psi)^2 + (\partial_\phi \psi)^2 \right], \quad (B.1)$$

which is positive only as long as $|\beta/\alpha| < 1$. As this condition is not fulfilled in the ergoregion (in our case for $r < 1$), the density that is integrated over to obtain the energy $E[\psi]$ can be made negative for some classical field configurations.

Appendix C. Relation to general theory of Bloch/Floquet/Mathieu

The equation

$$\left[ D_y^2 - \frac{M^2}{x^2} + K^2 \left( 1 + \frac{1}{x^2} \right) \right] \psi = 0 \quad (C.1)$$

upon substitution $u = \ln x$, $u \in [-\infty, \infty]$, becomes the standard modified Mathieu equation

$$\psi_{uu} - [M^2 - K^2 \cosh(2u)] \psi = 0, \quad (C.2)$$

which in turn results from the Mathieu equation

$$\psi_{zz} + [M^2 - K^2 \cos(2z)] \psi = 0 \quad (C.3)$$
Figure C1. Relation between the Mathieu functions and the solutions of the radial equation for real frequencies $\omega$. The region $z \to i\infty$, corresponding to $x \to \infty$, is where the bound state functions should exponentially vanish.

taken along the pure imaginary axis $z = iu$. In the literature [23] the following definitions of constants in the Mathieu equations are employed:

$$M^2 \equiv a, \quad K^2 = q.$$ \hspace{1cm} (C.4)

This is especially convenient for the reason that the bound state problem (which appears in case there stands a $-1$ instead of $+1$ in the regular bracket of the original equation) is just the original problem taken along the line(s) $u = w \pm i\frac{\pi}{4}$, and with the parameter $q = \pm iK^2$ (the $x$-equation fulfilled by the bound state is recovered by substitution $x = e^w$). Our study of solutions of the radial equation in the space of essential parameters is thus seen equivalent to the study of the solutions of the Mathieu equation:

- for real and positive $a$’s and $q$’s (corresponding to positive $M^2$ and $K^2$, scattering states for non-absorbed rays), for $z = iu$,
- for real $a$’s and $q$’s with a negative $a$ (corresponding to $M^2 = -\mu^2 < 0$, scattering of ‘absorbed rays’) $z = iu$,
- for real $a$ and purely imaginary $q$ (bound states) with $z = iu \pm i\frac{\pi}{4}$. The wavefunctions need to vanish at $x \to +\infty$, which corresponds to $\text{Im}(z) \to +\infty$.

Unfortunately, at this point none of the standard solutions of the Mathieu equation appears to play a role for the vortex-bound-state problem. On the other hand, standard approximative methods associated with the theory of Mathieu equations [23, 41] do lead to efficient approximations for the positions of bound states and for the scattering characteristics, in particular, in the case of large $M^2$.

Appendix D. Note about the employed numerical method

The Mathieu nature of the sole non-trivial differential equation of this paper makes the use of numerical methods unnecessary. However, it proved more efficient to actually use them when dealing with the equation (due to the boundary conditions in a form not standard for Mathieu problems and due to the robustness of numerical methods for solving ODEs). The
Figure E1. The profile of density (solid curve) and speed of sound (dashed curve) for a simple model of a vortex in superfluid helium-4 constructed using the EOS (E.1) [38]. The chosen pressure at infinity is the maximal allowed pressure, and the vortex carries the single circulation quantum. Density is shown w.r.t. the SVP density $\rho_{\text{SVP}} = \rho_0$. Distances are given in [cm], and the innermost part should not be taken literally, because it is of the size of the van der Waals radius for $^4$He.

radial equation has therefore been implemented in Mathematica, and solutions for arbitrary $\omega$, $p$, $m$ were obtained starting from the appropriate boundary condition at small $r$. Positions of bound states were obtained by searching for zeros of the Wronskian of the numerical solution with the asymptotic solution that is damped for $r \to \infty$ (functions of Bessel-K type, $K_M(Kx)$, see section 6.1.1).

Appendix E. Simple model of a vortex in superfluid helium-4

In this appendix, we provide the reader with a very simple model of the vortex in superfluid helium-4. The goal is to assess the quality of the assumption that the superfluid density and the speed of sound are constant throughout the fluid. Moreover, the model allows for an assessment of the possibility of trapping large number of circulation quanta on the central wire so that the critical radius would exceed the radii available to the fluid. (In the model, based on an EOS, the fluid possesses an empty region close to the axis of rotation; the model itself is not very imaginative and should not be considered as an original contribution of the author.)

Some simple structure of vortex lines in superfluid helium-4 can be deduced from the empirical, $T = 0$, EOS published long time ago by Brooks and Donnelly [38]:

$$p = \sum_{n=1}^{3} A_n (\rho - \rho_0)^n, \quad (E.1)$$

with $A_0 = 560$, $A_1 = 1.097 \times 10^4$, $A_2 = 7.33 \times 10^4$ (with units giving pressure in atm), where $\rho_0 = 0.145 \text{ g cm}^{-3}$ is the density at the SVP. The equation gives approximately correct compressibilities, as well as the Grüneisen constant $U_G = \frac{\rho}{p} \frac{dp}{d\rho}|_T$, up to the pressure where helium solidifies $p_{\text{max}} \approx 25$ atm. Below, we illustrate the structure of the vortex based on this EOS, in order to give the reader a rough estimate of how realistic is the assumption $\rho = \text{const}$, which we have used in this work. Because the fluid at $T = 0$ can only exist in the superfluid phase for pressures between SVP and $p_{\text{max}}$ the density is only allowed to be in the range $[\rho_0, 1.188 \rho_0]$. From the EOS, one computes the enthalpy density $w(p) = \int_p^{p_{\text{max}}} \frac{dp}{\rho(p)}$, which for
The density (and thus enthalpy) decreases toward the center, and a boundary of the fluid develops at \( r = r_\ast \), with \( \rho(r_\ast) = \rho_0 \). In the case that one has neglected the nonlinear terms in the EOS (i.e., had one assumed a constant speed of sound), one would have obtained \( \rho(r) = \rho_\infty \exp \left[ -\frac{x}{2} \right] \), where we recognize the dimensionless combination \( \frac{1}{2}(\frac{x}{r_\ast})^2 \) in the exponent. Let us call such a case the simplified model. Generally, the speed of sound decreases toward the core of the vortex, and therefore, the ratio \( \frac{c(r)}{v(r)} \) grows monotonically. It is of interest to determine the highest possible value of this parameter; a value greater than 1 would signify the presence of an ergoregion. Let \( \rho_\infty = x\rho_0 \); in the simplified model, one finds

\[
\rho_\ast = \frac{r_c}{\sqrt{2\log x}}.
\]

While this is valid for any fluid with the simplified EOS \( p(\rho) = c_\infty^2 \rho \) (\( c_\infty \) a constant), it is curious that the maximal available \( x \) for superfluid helium is 1.188, making \( r_\ast \approx 1.7r_c \). Therefore, even though the size of the vortex \( r_c \) can be increased to avoid the difficulty with the van der Waals radius by increasing the circulation \( \kappa \) (i.e., by trapping more vortices on the central wire), the ratio \( v(r)/c(r) \) will never exceed 1/1.7 \( \approx 0.59 \). No higher value of the ratio can be produced with superfluid helium not contained by external walls, within the simplified model (including the dependence of \( c \) on \( \rho \) does not change this conclusion significantly).

If, on the other hand, the fluid were contained by an external cylinder of radius \( R \) (at pressure \( \leq 25 \text{ atm} \)), it would be possible to have almost arbitrarily large \( v/c \). The values \( r_\ast \leq r_c \) are achieved, however, only for relatively small diameters \( R \); for instance, in order to obtain \( r_\ast = r_c \), one would need to take 14 \( x = 1.65 \) leading to \( R = 2.359 \times 10^{-5} \text{ cm} \) and \( r_\ast = 2.175 \times 10^{-3} \text{ cm} \). Even though both of these radii can be appropriately scaled up by choosing a larger circulation \( \kappa \), \( R \) will always remain only by 8% larger than \( r_\ast \).

The SVP density \( \rho_0 \) and the maximal density at which the superfluid remains liquid \( x\rho_0 \) are clearly the thermodynamic properties of the substance under consideration. It is difficult to give a principal reason why their ratio should not lead to \( 1/\sqrt{2\log x} < 1 \), although this is not the case in superfluid helium-4.

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