Communication Systems With Amplitude Detection: An Asymptotic Approach

Amit Agarwal, Constantinos Psomas, Senior Member, IEEE, and Ioannis Krikidis, Fellow, IEEE

Abstract—Amplitude detection (AD) is a low-complexity nonlinear scheme for information retrieval, as it only considers the envelope of the received signal. This makes it ideal in applications with low-cost low-power devices such as the Internet of Things (IoT). We consider a basic Point-to-Point (P2P) setup and develop a unified analytical framework for the asymptotic symbol error rate (SER) performance of the maximum-likelihood (ML) decoder with M-ary amplitude-shift keying (ASK). The developed framework is general and can be applied for a large class of fading channels. The SER can be characterized by two parameters: the diversity order and the coding gain at a sufficiently high signal-to-noise regime. We derive closed-form expressions for the exact diversity order along with a lower and upper bound on the coding gain. We show that the diversity order is equal to m for Nakagami-m fading, whereas for the other considered models, it is equal to one. We also prove that among all possible binary modulations with equal power, the modulation with one symbol equal to zero [i.e., on-off keying (OOK)] achieves the minimum asymptotic SER. The mathematical framework developed for the P2P setup is extended for cooperative relay networks. Specifically, we consider a basic amplify-and-forward (AF) three-node relay topology that employs AD at both the relay and the destination. We analytically derive the ML decoder and study asymptotic bounds on the SER by showing that the proposed AF-AD relay setup can be transformed into an equivalent P2P-AD with respect to the ML decoder.

Index Terms—Amplitude detection (AD), amplitude-shift keying (ASK), asymptotic error performance, maximum likelihood (ML), relay.

I. INTRODUCTION

FUTURE wireless networks are expected to be fully automated and remotely managed with millions of devices and sensors embedded into cities, vehicles, homes, industries, and other environments. As a result, next-generation massive Internet of Things (IoT) networks envisage a rapid increase in wireless connectivity among a large number of small size devices and sensors [1]. Most of these devices are battery powered due to cost, convenience, and the need for untethered operation. Therefore, they are required to have long operational lifetimes from a few days to possibly several years [2]. Hence, designing low-complexity and energy-efficient receivers for IoT-enabled wireless networks is crucial. Conventional communication receivers employ linear detection, which requires expensive and complex circuitry, such as linear amplifiers and frequency synthesizers [3]–[5]. On the other hand, nonlinear detection techniques are of lower complexity since they process only the amplitude/phase of the received signal. Even though this approach affects the performance and decreases degrees of freedom, it is more energy and cost efficient and hence ideal for massive IoT applications [6].

A simple and well-known nonlinear detection technique is amplitude detection (AD), which only processes the envelope of the received signal and can be implemented mainly by a diode and a low-pass filter [7]. Since AD only differentiates between symbols with different magnitudes, a suitable modulation scheme is amplitude-shift keying (ASK). The simplest form of ASK is on–off keying (OOK) that has increasingly being used in energy-constrained wireless applications such as sensor networks, where efficient design is required to minimize the power consumption and extend the operation lifetime [2], [6]. In [8], a closed-form expression for the asymptotic error probability of OOK modulation with envelope detection over additive white Gaussian noise (AWGN) channels is derived. It is shown that the error probability associated with each OOK symbol contributes equally to the total error probability. In [9], the ratio of the asymptotic probability of error for the two symbols is derived over Rician channels in a tabular form. Recently, Psaltopoulos et al. [10]–[14] studied various fundamental aspects of AD, such as the achievable rate and diversity–multiplexing tradeoff, for a basic multiple-input–multiple-output (MIMO) setup. Contrary to the noncoherent OOK [9], they propose a coherent OOK system, where channel knowledge is available at the receiver. In addition to the proposed theoretical framework, a practical MIMO setup is implemented and the error performance for OOK modulation over Rayleigh channels is validated.

Even though OOK is energy efficient, it suffers from a low-spectral efficiency. Hence, multilevel ASK (M-ASK) can be used to overcome this limitation [15]–[20]. The error performance of AD with M-ASK over lognormal channels has been studied in [15]. The error performance of a low-complexity energy detector with M-ASK over Rayleigh channels with receive diversity and no channel knowledge at the
receiver is studied in [16]. Though the receiver structure is simple, it suffers from a high symbol error rate (SER) performance, especially at high signal-to-noise ratios (SNRs) when \( M > 2 \). To overcome this high SER effect, schemes with an additional knowledge of channel magnitude are proposed in [17] and [20]. Specifically, error performance of low-complexity amplitude detectors has been studied over Rayleigh [17]–[19] and Rician channels [20]. Moreover, [18] and [20] restrict their discussion only to equidistant ASK.

However, to the best of our knowledge, there is no unifying framework that studies the maximum-likelihood (ML) error performance of the nonlinear AD setup with \( M \)-ASK over various fading models.\(^1\) Thus, this study provides the optimum error performance that can be achieved by any decoder with input as the magnitude of the received signal for a large class of fading models.

Another promising technique for massive IoT networks is cooperation via signal relaying. In particular, since IoT networks consist of various distributed devices, the concept of relaying can provide various benefits, such as wider coverage, lower transmit power, diversity, as well as reduced interference [22]–[24]. Therefore, relay systems that employ nonlinear AD can be used as a potential low-cost and energy-efficient solution for applications such as massive IoT. This combination allows us to exploit the benefits associated with nonlinear detection schemes and those associated with relay systems. Cooperative relaying can be broadly categorized as decode-and-forward (DF) (regenerative) and amplify-and-forward (AF) (nonregenerative). Compared to an AF relay, the complexity of a DF one is significantly higher due to its full processing capability. An AF relay simply amplifies the received signal from the source node and then retransmits it toward the destination node without decoding. While a DF relay decodes, remodulates, and retransmit the received signal.

In this article, we first consider a Point-to-Point (P2P) setup that employs nonlinear AD at the destination, and study bounds on the SER performance of the ML decoder over various fading models with general \( M \)-ASK modulation. Next, the framework developed for the P2P setup is extended to a basic three-node relay setup that employs nonlinear AD. Thus, we present a nonlinear relaying scheme and provide an adequate theoretical basis for further investigation on this topic as a potential solution for the energy-efficient distributed networks. The main contributions of this article are threefold as follows.

1) We introduce a closed-form analytical framework that quantifies the SER in a low-complexity P2P communication setup where the destination employs nonlinear AD. First, we derive upper and lower bounds on the SER for the P2P-AD with \( M \)-ASK modulation. Then, closed-form asymptotic bounds for the average SER are derived to gain valuable insights into the impact of various channel models (and their parameters) on the diversity order and coding gain. This study is general and can be used to evaluate the error performance for a large class of fading models, e.g., Rayleigh, Nakagami-\( n \), Nakagami-\( q \), and Nakagami-\( m \). The theoretical results are validated via Monte Carlo simulations, which show that the derived asymptotic bounds accurately represent their corresponding simulation SER at high SNRs.

2) Our analysis reveals that the asymptotic lower and upper bounds on the SER have the same diversity order, which provides the exact diversity order of the considered system. We show that under Nakagami-\( m \) fading, the diversity order is equal to \( m \) whereas for the other considered models, it is equal to one. Furthermore, using these bounds with equal diversity order, we derive a closed interval for the coding gain. The simplicity of the derived results allows us to design the optimum binary modulation set in terms of the asymptotic error performance. We analytically show that among all possible binary ASK modulations (with the same average power), OOK modulation provides the minimum asymptotic error performance, regardless of the channel model used.

3) The developed framework for P2P-AD setups is extended for basic three-node AF relay setups, where AD is employed at both the relay and the destination. We analytically derive the ML decoder for the three-node relay setup. Our analysis reveals that the AF-AD topology can be transformed to an equivalent P2P-AD setup with respect to the ML decoder. Given this transformation, the mathematical framework used for the P2P-AD systems is applied for the SER performance of relay systems over various fading models.

To the best of our knowledge, no previous work has studied the framework discussed in this article to analyze diversity order and coding gain in the context of nonlinear AD receivers for P2P and relay setups. The application of the proposed framework may also be to understand the limiting SER performance of various applications that depend on the magnitude (or its function) of the received signal for information decoding, such as integrated simultaneous wireless power and information power transfer (SWIPT) and backscatter communications.

The remainder of this article is organized as follows. In Section II, the system model along with the ML decoder for a P2P-AD system is presented. Section III analyzes the SER performance for the considered P2P-AD system with \( M \)-ASK modulation over various channel models. In Section IV, we extend the analytical framework developed for the P2P-AD system to a basic three-node relay system and analyze the SER performance for \( M \)-ASK. Numerical results are shown and discussed in Section V, and are followed by our conclusions in Section VI.

II. SYSTEM MODEL

We consider a basic P2P communication setup consisting of a source and a destination that employs AD over a flat-fading channel; we refer to this setup as P2P-AD. If \( x_s \) is the transmitted symbol from the source with power \( \rho = \mathbb{E}[|x_s|^2] \),

\(^1\)Note that this work is applicable for general \( M \)-ASK modulation, that is, any value of \( M \geq 2 \) and any possible choice of the alphabet symbols (not necessarily the equidistant ASK).
The symbol detecting the received signal, and a full CSI is assumed at the receiver. The received signal can be written as

\[ y_D = |h_i x_s + n| \quad (1) \]

where \( h_i \in \mathbb{C} \) is the channel realization, and \( n \) is the complex AWGN at the receiver with mean zero and variance \( \sigma^2 \), i.e., \( n \sim \mathcal{CN}(0, \sigma^2) \). An ML symbol-by-symbol decoder is used for detecting the received signal, and a full CSI is assumed at the destination [14]. The symbol \( x_s \) is drawn from an equiprobable modulation alphabet set \( \mathcal{X} \); for a given channel realization, the ML decision rule is given by

\[ \hat{x}_s = \arg \max_{x_s \in \mathcal{X}} f_{yD}(yD|x_s, h) \quad (2) \]

where \( f_{yD}(yD|x_s, h) \) is the conditional probability density function (PDF) or the likelihood of the received symbol. For the received signal given in (1), the likelihood function is Rician and can be written as [7]

\[
 f_{yD}(yD|x_s, h) = \frac{2yD}{\sigma^2} \exp\left(-\frac{yD^2 + |x_s|^2|h|^2}{\sigma^2}\right) \times I_0\left(\frac{2yD|x_s||h|}{\sigma^2}\right). \quad (3)
\]

It is clear from (3) that the likelihood function depends on the amplitude of the transmit symbol \( x_s \) and not on the phase and therefore the ML can only differentiate between symbols with different amplitudes. Thus, we can deduce that an appropriate choice of modulation set is \( M \)-ASK, i.e.,

\[ \mathcal{X} = \{x_0, x_1, \ldots, x_{M-1}\} \quad (4) \]

such that \( x_i \in \mathbb{R}^+ \) and \( 0 \leq x_0 < x_1 < \cdots < x_{M-1} \). Hence, the transmit SNR can be expressed as

\[ \gamma \triangleq \frac{\rho}{\sigma^2} = \frac{1}{M\sigma^2} \sum_{i=0}^{M-1} x_i^2. \quad (5) \]

The probability of error, \( P_e \), for a given channel realization, depends on the transmit SNR \( \gamma \) and the random channel gain \( \beta = |h|^2 \). For the equiprobable \( M \)-ASK, we have [25]

\[ P_e = \frac{1}{M} \sum_{i=0}^{M-1} P_e|x_i \quad (6) \]

where \( P_e|x_i \) is the probability of error when the symbol \( x_i \) is transmitted and is given by

\[ P_e|x_i = \begin{cases} f_{yD}^{\infty}(yD|x_i, h)dy_D, & \text{for } i = 0 \\ f_{yD}^{\infty}(yD|x_i, h)dy_D + \int_{0}^{k_i} f_{yD}(yD|x_i, h)dy_D, & \text{for } 0 < i < M - 1 \\ \int_{0}^{k_i} f_{yD}(yD|x_i, h)dy_D, & \text{for } i = M - 1 \end{cases} \quad (7) \]

with

\[ \int_{0}^{k_i} f_{yD}(yD|x_i, h)dy_D = 1 - Q_1\left(\frac{\sqrt{2k_i^2\gamma \beta}}{\sigma}\right) \]

where \( k_i \) is the normalized \( i \)th transmit symbol given as

\[ k_i \triangleq \frac{x_i}{\sqrt{M\sigma^2\gamma}}, i = 0, 1, \ldots, M - 1 \quad (8) \]

and \( \lambda_i \) is the optimal detection threshold of the ML decoder that corresponds to the solution of

\[ f_{yD}(yD|x_i, h)\bigg|_{y_D=\lambda_i} = f_{yD}(yD|x_{i+1}, h)\bigg|_{y_D=\lambda_i}, \quad i = 0, 1, \ldots, M - 2. \quad (9) \]

Thus, we can rewrite (6) as

\[ P_e = \frac{1}{M} \left[ \sum_{i=0}^{M-2} \int_{\lambda_i}^{\infty} f_{yD}(yD|x_i, h)dy_D + \sum_{i=0}^{M-2} \int_{\lambda_i}^{\infty} f_{yD}(yD|x_{i+1}, h)dy_D \right] \quad (10a) \]
where by using (7) in (10a) along with some algebraic manipulations, we have the expression in (10b). Then, the average SER can be evaluated as [25]

\[
\bar{P}_e \triangleq \mathbb{E}[P_e] = \int_0^\infty P_e f_\beta(\beta) \, d\beta
\]

where \( f_\beta(\beta) \) is the PDF of the channel gain \( \beta \) (we consider well-known channel models, e.g., Nakagami-\( q \), Nakagami-\( m \), and Nakagami-\( n \)). Moreover, at high SNRs, the SER in (11) can be represented as \( P_{e,\infty} \approx (gy)^{-d} \), where \( g \) and \( d \) are the coding gain and the diversity order, respectively [25]. Then, we can evaluate the diversity order as

\[
d = \lim_{\gamma \to \infty} \frac{-\log P_{e,\infty}}{\log \gamma}.
\]

The diversity order determines the slope of the SER versus the SNR curve (in a log-log scale), at high SNRs. On the other hand, the coding gain (in decibels) determines the shift of the SER curve in SNR relative to a benchmark SER curve of \( \gamma^{-d} \).

### III. ERROR ANALYSIS FOR P2P-AD SYSTEMS

In this section, we study the asymptotic SER performance for the considered P2P-AD system over various channel models. We note that, first, due to the complexity and intractability of the Marcum Q-function and, second, due to the lack of closed-form expression for the detection thresholds, it is not straightforward to evaluate the average SER performance in (11). Therefore, in what follows, we first obtain tractable upper and lower bounds on the conditional error probability. Then, we derive the average SER performance bounds at high SNRs and study the diversity order and the coding gain of a P2P-AD system.

#### A. Bounds on the Error Performance

We first state a lower bound on the optimal detection threshold of the ML decoder in the following proposition.

**Proposition 1:** The optimal detection threshold, \( \lambda_i \), \( i = 0, 1, b, \ldots, M-2 \), can be lower bounded by

\[
\lambda_i \geq \frac{(x_i + x_{i+1})\sqrt{\beta}}{2}, \quad i = 0, 1, \ldots, M-2
\]

where the equality holds at high SNRs, i.e., \( \lim_{\gamma \to \infty} \lambda_i = \frac{((x_i + x_{i+1})\sqrt{\beta})}{2} \), \( i = 0, 1, \ldots, M-2 \).

**Proof:** See Appendix A.

The above result is used in the derivation of an upper bound on the conditional error probability in Proposition 2.

**Proposition 2:** The conditional (i.e., for a given channel model) probability of error for the M-ASK modulation is upper bounded by

\[
P_{e,\beta}^{ub} = \frac{1}{2M} \left[ \sum_{i=0}^{M-2} 3 \exp \left( -\frac{M\gamma\beta}{4} (k_{i+1} - k_i)^2 \right) \right. \\
\left. - \sum_{i=0}^{M-2} \exp \left( -\frac{M\gamma\beta}{4} (3k_{i+1} + k_i)^2 \right) \right]
\]

and lower bounded by

\[
P_{e,\beta}^{lb} = \frac{1}{M} \left[ Q \left( (k_1 - k_0)\sqrt{\frac{M\beta\gamma}{2}} \right) + Q \left( (k_{M-1} - k_{M-2})\sqrt{\frac{M\beta\gamma}{2}} \right) \right. \\
\left. + \sum_{i=1}^{M-2} Q \left( (k_{i+1} - k_i)\sqrt{\frac{M\beta\gamma}{2}} \right) + Q \left( (k_i - k_{i-1})\sqrt{\frac{M\beta\gamma}{2}} \right) \right].
\]

**Proof:** See Appendix B.

Note that the above probability expressions are conditioned on the channel gain \( \beta \). Therefore, by using (11), we can evaluate the bounds on the SER for any channel distribution. Moreover, we can see that the upper bound has an exponential form, and hence the corresponding SER can be evaluated in terms of the moment generating function (MGF) of the random variable \( \beta \). Specifically, for a given MGF \( \mathcal{M}_\beta(x) \triangleq \int_0^\infty \exp(x\beta) f_\beta(\beta) \, d\beta \), we can write the exact closed-form expression for the upper bound as

\[
\bar{P}_e^{ub} = \int_0^\infty P_{e,\beta}^{ub}(\beta) \, d\beta = \frac{1}{2M} \left[ \sum_{i=0}^{M-2} 3 \mathcal{M}_\beta \left( -\frac{M\gamma\beta}{4} (k_{i+1} - k_i)^2 \right) \right. \\
\left. - \sum_{i=0}^{M-2} \mathcal{M}_\beta \left( -\frac{M\gamma\beta}{4} (3k_{i+1} + k_i)^2 \right) \right].
\]

For instance, when the channel is Nakagami-\( m \), we have \( \mathcal{M}_\beta(x) = (1 - [x/m])^{-m} \), and we can write

\[
\bar{P}_e^{ub} = \frac{2^{2m-1}}{M} \left[ 3 \sum_{i=0}^{M-2} \left( 1 + \frac{M\gamma(k_{i+1} - k_i)^2}{4m} \right)^{-m} \right. \\
\left. - \sum_{i=0}^{M-2} \left( 1 + \frac{M\gamma(3k_{i+1} + k_i)^2}{4m} \right)^{-m} \right].
\]

Similarly, one can also derive an exact closed-form lower bound on the SER, i.e.,

\[
\bar{P}_e^{lb} = \int_0^\infty P_{e,\beta}^{lb}(\beta) \, d\beta
\]

since integration with the \( Q(\cdot) \) function is known for many channel models in terms of their MGFs and is given as [26]

\[
I(a, \gamma) \triangleq \int_0^\infty Q \left( a\sqrt{\beta\gamma} f_\beta(\beta) \right) \, d\beta = \frac{1}{\pi} \int_0^{\pi/2} M_\beta \left( \frac{a^2}{2\sin^2(\theta)} \right) \, d\beta.
\]
smoothness of SNRs is given by by the near-origin behavior of $f$ on the performance curve, which are more in closed form, they do not provide useful insights into the fading parameters on the performance. In what follows, we first derive SER bounds at sufficiently high SNRs and then analyze the diversity order and the coding gain.

Asymptotic Analysis

Although the derived novel expressions in (17) and (21) are in closed form, they do not provide useful insights into the system performance. This work mainly focuses on the effect of fading parameters on the performance curve, which are attributes of a high SNR regime, such as diversity order and coding gain. To this end, we use the fact that for a conditional error function $P_e$, which is nonzero around the origin and approaches zero as $\gamma$ increases, the SER is dominated by the near-origin behavior of $f_\beta(\beta)$ at high SNRs [27]. For most channel models, $f_\beta(\beta)$ can be approximated by a single polynomial term near the origin as $f_\beta(\beta) \approx a \beta^t$, where $a$ is a positive constant and the parameter $t$ quantifies the order of smoothness of $f_\beta(\beta)$ near the origin. Thus, the SER at high SNRs is given by

$$P_{e,\infty} \Delta \int_0^\infty P_e a \beta^t \, d\beta$$

where the values of the parameters $t$ and $a$ are provided in Table II for various channel models.

**Table II**

Near-Origin Parameters $(t, a)$ for Various Channel Models [27]

| Channel type | $f_\beta(\beta)$ | $t$ | $a$ |
|--------------|------------------|-----|-----|
| Rayleigh     | $\exp(-\beta)$   | 0   | 1   |
| Nakagami-$q$ | $\frac{1+\gamma^2}{2q} \exp\left(-\frac{(1+\gamma^2)^2}{4q^2}\right) I_0\left(\frac{(1+\gamma^2)/2}{q}\right)$ | 0   | $\frac{1+\gamma^2}{2q}$ |
| Nakagami-$m$ | $\frac{m^m}{\Gamma(m)} \exp(-m\beta)$ | $m-1$ | $\frac{m^m}{\Gamma(m)}$ |
| Nakagami-$n$ | $(1+n^2) \exp(-n^2) \exp\left(-(1+n^2)\beta I_0(2n\sqrt{(1+n^2)\beta})\right)$ | 0   | $(1+n^2) \exp(-n^2)$ |

**Theorem 1:** If the PDF of the channel gain $\beta$ is approximated as $f_\beta(\beta) \approx a \beta^t$ near the origin, then the upper bound on the SER performance at high SNRs is given by

$$P_{e,\infty}^{ub} = \left[\frac{2^{2t+1}a\Gamma(t+1)}{(t+1)M^t+2} \sum_{m=0}^{M-2} 3(k_{m+1}-k_m)^{-2(t+1)} \right] \gamma^{-(t+1)}$$

and the lower bound is given by

$$P_{e,\infty}^{lb} = \left[\frac{2^{2t+1}a\Gamma(t+\frac{3}{2})}{\sqrt{\pi(t+1)M^t+2}} \right] \times \left(\frac{(k_1-k_0)^{-2(t+1)} + (k_{M-1}-k_{M-2})^{-2(t+1)} + \sum_{i=1}^{M-2} (k_{i+1}-k_i)^{-2(t+1)} + \sum_{i=1}^{M-2} (k_{i}-k_{i-1})^{-2(t+1)}\right) \gamma^{-(t+1)}.$$  

**Proof:** See Appendix C.

Note that the term inside the braces $[\cdot]$ is independent of $\gamma$, this is because $(a, t)$ pair depends only on the fading type and $k_i$’s are normalized modulation symbols. Therefore, (24) can be written in the form $(g_2(\gamma))^{-(t+1)}$, where $g_2$ represents the coding gain and $t+1$ is the diversity gain. Similar observation holds for $P_{e,\infty}^{lb}$.

From Theorem 1, the asymptotic SER performance can be bounded as $P_{e,\infty}^{lb} \leq P_{e,\infty} \leq P_{e,\infty}^{ub}$. Let $g_1$ and $g_2$ be the coding gains associated with the upper bound and the lower bound, respectively. Thus, we can write

$$(g_2(\gamma))^{-(t+1)} \leq (g_1(\gamma))^{-d} \leq (g_2(\gamma))^{-(t+1)}.$$  

Then, it is simple to deduce that $t+1 \leq d \leq t+1$ and this implies $d = t+1$; further, it can be verified that $g_1 \leq g \leq g_2$. We state the above result in the following corollary.

**Corollary 1:** The diversity order and the coding gain are given, respectively, by

$$d = t+1, \quad g \in [g_1, g_2]$$

where $g_1$ and $g_2$ are the coding gains corresponding to the upper bound (23) and lower bound (24), respectively.
Therefore, the developed mathematical framework provides a simple and general methodology to derive the exact diversity order and a closed interval for the coding gain for all of the considered channel models, which are summarized in Table II. For instance, for the case of OOK modulation $X = \{0, A\}$ with a Rayleigh fading, by using $t = 0$ and $a = 1$ along with $k_0 = 0$ and $k_1 = 1$ in Corollary 1, we get the diversity order

$$d = 1$$

and the coding gain

$$g \in [0.6923, 2].$$

To further elaborate on Corollary 1, we consider an equidistant ASK modulation alphabet set $X = \{0, A, 2A, 3A\}$ and provide the diversity order and the coding gain over various channel models in Table III. In order to obtain the values in Table III, we substitute the value of the normalized symbol as $k_i = (i/\sqrt{4})$, $i = 0, 1, 2, 3$, along with the near-origin parameters $(t, a)$ associated with each channel type provided in Table II. We can see that the diversity order is one for Nakagami-$q$ and Nakagami-$n$ fading, while it is equal to $m$ for Nakagami-$m$ fading. One interesting remark is that for Rayleigh fading the coding gain lies in the interval $[0.0645, 0.1905]$, while for the binary OOK case, it lies in the interval $[0.6923, 2]$ [see (27)], and the diversity order is the same. Thus, the error performance for the quaternary modulation is inferior as compared to the binary case, which is also true for the linear detection schemes.

### C. Optimum Modulation for $M = 2$

From Theorem 1, it is clear that the diversity depends only on the channel (through the parameter $t$) and is independent of the choice of the modulation alphabet set, while the coding gain depends on the selected modulation set. Hence, it is interesting to design the optimum modulation alphabet set that maximizes the coding gain. Such a design for a general $M$ is not analytically tractable; however, in the following, we derive the optimum alphabet set for the case with $M = 2$.

**Corollary 2:** Among all the binary ASK modulations with an average transmit power $\rho$, the OOK modulation $X = \{0, \sqrt{2\rho}\}$ is optimum in terms of the coding gain.  

1. The ML decoder discussed in our work [see (3)] can only differentiate between symbols with different magnitudes. Symbols transmitted using the pulse position modulation (PPM) have the same magnitude and are therefore indistinguishable at the receiver. Thus, PPM modulation cannot be used for the information transfer for the considered ML decoder.

### Table III

| Channel type | $g$ | $d$ |
|--------------|-----|-----|
| Nakagami-$q$ | $[0.0645^{2q_{1+q^2}}, 0.1905^{2q_{1-q^2}}]$ | 1 |
| Nakagami-$n$ | $[0.0645/(1+n^2)e^{-n^2}, 0.1905/(1+n^2)e^{-n^2}]$ | 1 |
| Nakagami-$m$ | $\frac{8}{14m} (9 - 3 - 2m - 7- 2m - 11 - 2m)^{-1/4}$, $\frac{1}{14m} (3^{m+6} + 3^{m+4})^{-1/2}$ | $m$ |

### Proof: See Appendix D.

From the above corollary, one can deduce that the first symbol of the $M$-ASK modulation should be placed at the origin, i.e., $x_0 = 0$. In the next section, we discuss how the analytical framework developed for the error performance analysis of P2P-AD systems can be extended to cooperative relay systems.

### IV. Relay Systems With AD

Consider a basic three-node relay setup operating in a two-hop mode that employs AD at both the relay and the destination. Furthermore, we consider the AF cooperative protocol at the relay (see Fig. 1). In what follows, we first analytically characterize the ML decoder for the AF-AD system and then study its asymptotic error performance for the $M$-ASK modulation in terms of the diversity order and coding gain. In the first hop, the received signal at the relay node is given by

$$y_R = |h_1 x_s + n_R|$$

where $h_1 \in \mathbb{C}$ is the channel coefficient between the source and the relay, $n_R$ is the complex AWGN with mean zero and variance $\sigma_R^2$ at the relay, i.e., $n_R \sim \mathcal{C}\mathcal{N}(0, \sigma_R^2)$, and $x_s$ is the transmitted symbol from the source with power $\rho$. The AF relay amplifies the received signal and forwards it to the destination with power $\rho$ (same as the source power). Thus, the signal transmitted by the relay is expressed as

$$\alpha y_R = \alpha |h_1 x_s + n_R|$$

where $\alpha = \sqrt{\rho/(|h_1|^2 \rho + \sigma_R^2)}$ denotes the amplification factor [28]. Then, the signal received at the destination is given by

$$y_D = |h_2 \alpha y_R + n_D| = |h_2 \alpha |h_1 x_s + n_R| + n_D|$$

[2]
where \( h_2 \in \mathbb{C} \) is the channel coefficient between the relay and the destination, and \( n_D \) is the complex AWGN at the destination with zero mean and variance \( \sigma^2_D \), i.e., \( n_D \sim \mathcal{CN}(0, \sigma^2_D) \). Further, \( \gamma_D \triangleq (\rho/\sigma^2_D) \) is the SNR for the first hop and \( \gamma_D \triangleq (\rho/\sigma^2_D) \) is the SNR for the second hop.

A. ML Characterization and SER Performance

At the destination, we use an ML symbol-by-symbol decoder and we assume that the destination knows both \( h_1 \) and \( h_2 \). When \( x_i \) is drawn from the equiprobable alphabet set \( \mathcal{X} \), the output of the ML decoder is given in the following theorem.

**Theorem 2:** The ML decoder for an AF-AD relay system is given by

\[
\hat{x}_i = \arg \max_{x_i \in \mathcal{X}} \exp \left( -\frac{|x_i h_1 h_2 \alpha|^2}{\sigma^2_D (1 + \nu |h_2 \alpha|^2)} \right) \times I_0 \left( \frac{2 \gamma_D |x_i h_1 h_2 \alpha|}{\sigma^2_D (1 + \nu |h_2 \alpha|^2)} \right)
\]

(31)

where \( \nu \triangleq (\sigma^2_D/\sigma^2_D) \in \mathbb{R}^+ \) represents the ratio of the noise variances between the relay and the destination.

Furthermore, the error performance for the above ML decoder is identical to the ML decoder obtained for an equivalent P2P-AD system with a received signal equal to \( r = [\tilde{h} x_i + n] \), where

\[
\tilde{h} = \frac{h_1 h_2 \alpha}{\sqrt{1 + \nu |h_2 \alpha|^2}}
\]

(32)

denotes the effective channel and \( n \sim \mathcal{CN}(0, \sigma^2_N) \).

**Proof:** See Appendix E.

From the above discussion, it can be seen that an AF-AD relay system can be transformed into an equivalent P2P-AD with respect to the ML decoder (i.e., the error performance is the same) with an effective channel gain given as

\[
\tilde{\beta} = \frac{|\tilde{h}|^2}{\sqrt{1 + \nu |h_2 \alpha|^2}} = \frac{v |h_1|^2 |h_2|^2}{\nu |h_1|^2 + |h_2|^2}.
\]

(33)

Given this transformation, the derivation of the error performance for an AF-AD relay system follows directly from the expressions obtained for the P2P case. Specifically, from (10a), the probability of error for a given channel realization with M-ASK modulation can be expressed as

\[
P_e^{AF} = \frac{1}{M} \left[ M - 1 + \sum_{i=0}^{M-2} Q_1 \left( \frac{2k_i^2 M y_i \tilde{\beta}}{2 \sigma^2}, \frac{\sqrt{2} \lambda_i}{\sigma} \right) \right.
\]

(34)

where \( \lambda_i \) is the optimum detection threshold and can be evaluated numerically by solving the expression

\[
f_r(r|x_i, \tilde{h}) \big|_{r=\lambda_i} = f_r(r|x_{i+1}, \tilde{h}) \big|_{r=\lambda_i}, \quad i = 0, 1, \ldots, M - 2
\]

(35)

where \( f_r(r|x_i, \tilde{h}) \) can be obtained from (3). Subsequently, the exact average SER can be evaluated by averaging (34) with respect to the effective channel gain \( \tilde{\beta} \), i.e., \( P_e^{AF} = \mathbb{E}_{\tilde{\beta}}[P_e^{AF}] \). Further, the bounds follow directly from Proposition 2 after replacing \( \beta \) with \( \tilde{\beta} \) followed by the expectation, i.e.,

\[
\mathbb{E}_{\tilde{\beta}}]\left[ \rho_{\text{ub}} \right] \leq P_e^{AF} \leq \mathbb{E}_{\tilde{\beta}}]\left[ \rho_{\text{lb}} \right].
\]

(36)

It is worth noting that we explicitly use \( \tilde{\beta} \) in the subscript of the expectation operator, in order to emphasize the key difference between P2P-AD and AF-AD setups.

B. Asymptotic Analysis

Following Theorem 1, the exact diversity order and the closed interval for the coding gain can be obtained as a function of the near-origin parameters \((t, a)\) of the effective channel gain’s PDF. In the following, the near-origin parameters corresponding to \( f_{\tilde{\beta}}(\tilde{\beta}) \) are provided.

The effective channel gain can be expressed as \( \tilde{\beta} = \frac{(v|h_1|^2|h_2|^2)/(v|h_1|^2 + |h_2|^2)}{\nu} \) at high SNRs, since \((1/\nu) \rightarrow 0 \). Further, from [30, Proposition 1], if the PDFs of \(|h_1|^2\) and \(|h_2|^2\) are nonzero at the origin (it holds for Rayleigh, Nakagami-\(q\), and Nakagami-\(n\) distributions) with \( f_{|h_1|^2}(0) = (1/v)\int_{h_1|^2} f_{|h_1|^2}(0) = (x_0/v) \) and \( f_{|h_2|^2}(0) = \gamma_0 \), then we can approximate \( f_{\tilde{\beta}}(\tilde{\beta}) \) as \( f_{\tilde{\beta}}(\tilde{\beta}) \approx a \tilde{\beta}^2 \) around the origin, where

\[
t = 0 \quad \text{and} \quad a = \frac{x_0}{v} + \gamma_0.
\]

(37)

For instance, if the first and the second links are Nakagami-\(n\) with parameters \( n_1 \) and \( n_2 \), respectively, then we have \( t = 0 \) and \( a = (x_0/v) + \gamma_0 = ((1 + n_1^2) \exp(-n_1^2)/v) + (1 + n_2^2) \exp(-n_2^2) \). For the sake of completeness, we provide a brief discussion on how the nonlinear relay setup could be analyzed for the DF case.

**DF Relaying:** It is worth noting that the proposed AF-AD relay setup can easily be modified and analyzed for the DF relaying protocol (DF-AD). Since in DF, the relay first decodes the received symbol and then retransmits it, the DF-AD can be seen as two consecutive P2P-AD links. Thus, SER can be upper bounded as

\[
\text{SER}^{DF} \leq \text{SER}^{ub}_{SR} + (1 - \text{SER}^{lb}_{SR}) + \text{SER}^{ub}_{RD}
\]

(38)

and can be lower bounded as

\[
\text{SER}^{DF} \geq \text{SER}^{lb}_{SR} + (1 - \text{SER}^{ub}_{SR}) + \text{SER}^{lb}_{RD}
\]

(39)

where \( \text{SER}^{ub/lb}_{SR/RD} \) represents the upper/lower bound on the SER performance of the source-to-relay/relay-to-destination P2P-AD link. Note that in order to derive the upper bound in (38), we use both the upper and lower bounds on the SER performance of the source-to-relay P2P-AD link; a similar observation applies for the expression in (39).

V. NUMERICAL RESULTS

Computer simulations are carried out in order to validate the proposed analytical framework. In all simulation results, we
Fig. 2. SER versus SNR for binary modulations \( \{kx_1, x_1\}, k \in [0, 1] \) over Nakagami-\( m \) channels. (a) \( m = 0.5 \). (b) \( k = 0 \) (OOK).

Fig. 3. SER versus SNR for the OOK modulation over Nakagami-\( n \) channels.

lower and an upper bound on the coding gain and validates our diversity and coding gain analysis. Note that these observations hold for the following figures as well.

Fig. 2(a) demonstrates the effect of the modulation parameter \( k \) on the SER performance over Nakagami-\( m \) channels with \( m = 0.5 \). Recall that, an alternative representation for the binary modulation set is given by \( \mathcal{X} = \{kx_1, x_1\}, k \in [0, 1] \) (for details see Appendix A).

It is observed that the slope of the SER curves depends only on \( m \), regardless of the value of the variable \( k \). This implies that the diversity is independent of the binary modulation considered. It is also clear that, for a fixed SNR, both the upper and lower bounds decrease with \( k \) and the case where \( k \) is equal to zero (OOK modulation), provides the minimum asymptotic error performance (as shown in Corollary 2). Fig. 2(b) illustrates the effect of the parameter \( m \) on the SER performance for the binary ASK modulation with \( k = 0 \). It is observed that the slope of the SER curves (diversity order) is equal to \( m \). Moreover, the asymptotic behavior of the bounds shows up at relatively high SNRs for larger values of \( m \).

Fig. 3 shows the effect of parameter \( n \) of Nakagami-\( n \) channel on the SER performance for the OOK modulation. Note that the Nakagami-\( n \) channel is the same as the Rician-\( K \) channel with the scale factor \( K = n^2 \) [27]. All the key observations made for Nakagami-\( m \) channel in Fig. 2(b) also hold for the Nakagami-\( n \) channels, i.e., the asymptotic approximations for the bounds have the same slope, the exact SER lies in between the bounds, and the asymptotic behavior of the bounds shows up at relatively high SNRs for larger values of \( n \). An important remark is that the slope of the SER curves is independent of \( n \), whereas for Nakagami-\( m \) channels, the slope depends on the parameter \( m \). However, the SER curve shifts toward the left as \( n \) increases, and therefore the coding gain increases. Note that this observation is also in line with the analytical results derived, as both the upper and the lower bounds on the coding gain obtained by using (23) and (24) [with \( r = 0 \) and \( a = (1 + n^2 \exp(-n^2)) \)], respectively, are increasing functions.

\(^3\)Note that for the simulation of AF-AD relay setup, we consider only the case when \( v = 1 \) (i.e., \( \sigma_D^2 = \sigma^2_R \)), since the value of the parameter \( v \) does not change the behavior of the curves. Further, the effect of parameter \( v \) can be lumped into the channel gain \( |h_1|^2 \).
of $n$. Similarly, in Fig. 4, we show the SER performance for quaternary ($M = 4$) ASK modulation $\mathcal{X} = \{0, A, 2A, 3A\}$ with SNR over a Nakagami-$n$ fading channel with $n = \{0, 2\}$. The simulation results validate that our proposed framework and all the observations are in line with our previous observations for the binary case.

Fig. 5 deals with the SER performance as a function of SNR per-hop for an AF-AD system with OOK modulation. We consider two scenarios for the fading distributions of the source-to-relay and relay-to-destination links: 1) both links are subject to Rayleigh fading and 2) source-to-relay and relay-to-destination links: 1) both links are subject to Nakagami-$n$ channels.

![Fig. 4. SER performance for the ASK modulation with $M = 4$ over Nakagami-$n$ channels.](image)

![Fig. 5. SER performance for the AF-AD relay system versus SNR per hop with OOK modulation.](image)

approximations at high SNRs. This observation validates the transformation of the AF-AD setup into an equivalent P2P-AD setup. Furthermore, we observe that the diversity for both the scenarios is equal to one, whereas the coding gain for the Rayleigh scenario is lower. As a result, the asymptotic SER performance for the Rician scenario is better; this observation is expected, as Rician fading corresponds to better channel conditions.

**VI. Conclusion**

In this article, we developed a mathematical framework to study bounds on the SER performance for P2P-AD systems with ASK modulation and also derived asymptotic approximations of these bounds as a function of the channel parameters. The developed framework is general and can be used to study the asymptotic performance (i.e., diversity order and coding gain) for a large class of fading channels. Theoretical results are validated via Monte Carlo simulations. We have shown that the asymptotic approximation for each bound perfectly matches with their corresponding exact value at high SNRs. Moreover, they depend on the channel only through the near-origin parameters of the channel gain’s PDF. We also extended our mathematical framework to a basic three-node AF relay topology that employs AD at both the relay and the destination. The application of the proposed mathematical framework to other network topologies, e.g., multihop relaying, multiple-antenna systems, etc., will be considered for future work.

**APPENDIX A**

**Proof of Proposition 1**

The optimum detection threshold for the ML decoder, $\lambda_i$, $i = 0, 1, \ldots, M - 2$, corresponds to the solution of

$$f_{YD}(y_D|x_i, h)|_{y_D = \lambda_i} = f_{YD}(y_D|x_{i+1}, h)|_{y_D = \lambda_i}$$

$$\Rightarrow \frac{I_0\left(\frac{2\lambda_i x_i|h|}{\sigma^2}\right)}{I_0\left(\frac{2\lambda_i x_{i+1}|h|}{\sigma^2}\right)} = \exp\left(-\frac{(\lambda_i^2 - x_{i+1}^2)|h|^2}{\sigma^2}\right) \tag{40}$$

where $i = 0, 1, \ldots, M - 2$; we now obtain a lower bound on the threshold. To this end, we consider the two cases: 1) $x_i = 0$ and 2) $x_i \neq 0$, separately.

1) **Case With $x_i = 0$** For this case, (40) can be written as

$$I_0\left(\frac{2\lambda_i x_{i+1}|h|}{\sigma^2}\right) = \exp\left(-\frac{x_{i+1}^2|h|^2}{\sigma^2}\right).$$

A close approximate solution to the above equation is given by [6, eq. (4.15)]

$$\lambda_i \approx \sqrt{\sigma^2 + \frac{x_{i+1}^2|h|^2}{4}} \geq \frac{x_{i+1}|h|}{2}. \tag{41}$$

Further, the noise variance $\sigma^2 \to 0$ at high SNRs, and hence the threshold for the high SNR regime is given by

$$\lambda_i^\infty = \lim_{\gamma \to \infty} \lambda_i = \frac{x_{i+1}|h|}{2}. \tag{42}$$
From (41) and (42), we deduce that

\[ \lambda_i \geq \frac{x_{i+1} + |h_i|}{2} \geq 2, \quad \text{when } x_i = 0 \]  
(43)

where (a) follows from the fact that \( x_i = 0 \). Note that from the definition of the M-ASK modulation in (4), this case (\( x_i = 0 \)) is only possible for \( i = 0 \).

2) Case With \( x_i \neq 0 \): When \( x_i \neq 0 \), i.e., \( x_i > 0 \), we use the following inequality [31, eqs. (2.7) and (2.10)]:

\[ \exp(-y - x) < \frac{I_0(y)}{I_0(x)} < \frac{y}{x} \exp(-(y - x)), \quad y > x > 0. \]  
(44)

By substituting \( y = (2\lambda_i x_{i+1} |h|)/\sigma^2 \) and \( x = (2\lambda_i |h|)/\sigma^2 \) to the above inequality, we can write

\[ \exp\left(-\frac{2\lambda_i (x_{i+1} - x_i) |h|}{\sigma^2}\right) < \frac{I_0\left(\frac{2\lambda_i x_{i+1} |h|}{\sigma^2}\right)}{I_0\left(\frac{2\lambda_i x_i |h|}{\sigma^2}\right)} \]

\[ < \exp\left(-\frac{2\lambda_i (x_{i+1} - x_i) |h|}{\sigma^2}\right) \frac{x_{i+1}}{x_i}, \quad x_{i+1} > x_i > 0. \]  
(45)

From (45), by utilizing

\[ \exp\left(-\frac{2\lambda_i (x_{i+1} - x_i) |h|}{\sigma^2}\right) < \frac{I_0\left(\frac{2\lambda_i x_{i+1} |h|}{\sigma^2}\right)}{I_0\left(\frac{2\lambda_i x_i |h|}{\sigma^2}\right)} \]

in (40), we have

\[ \lambda_i > \frac{(x_{i+1} + x_i) |h|}{2}. \]  
(46)

Similarly, from (45), by using

\[ \frac{I_0\left(\frac{2\lambda_i x_{i+1} |h|}{\sigma^2}\right)}{I_0\left(\frac{2\lambda_i x_i |h|}{\sigma^2}\right)} < \exp\left(-\frac{2\lambda_i (x_{i+1} - x_i) |h|}{\sigma^2}\right) \frac{x_{i+1}}{x_i} \]

in (40), we have

\[ \lambda_i < \frac{(x_{i+1} + x_i) |h|}{2} + \frac{0.5 \sigma^2}{(x_{i+1} - x_i) |h|} \log \frac{x_{i+1}}{x_i}. \]  
(47)

At high SNRs, we have \( \sigma^2 \to 0 \), hence (47) becomes \( \lambda_i < \frac{(x_{i+1} + x_i) |h|}{2} \). Therefore, by using \( \lambda_i < \frac{(x_{i+1} + x_i) |h|}{2} \) (it holds for high SNRs) along with (46) (it holds for all SNRs), we can write

\[ \lambda_i^{\infty} = \frac{(x_{i+1} + x_i) |h|}{2} \leq \lambda_i, \quad \text{when } x_i \neq 0 \]  
(48)

with equality at high SNRs. This concludes the proof for the case with \( x_i \neq 0 \). Finally, from (43) and (48) along with \( \beta = |h|^2 \), we get

\[ \lambda_i \geq \frac{(x_{i+1} + x_i) \sqrt{\beta}}{2}, \quad i = 0, 1, \ldots, M - 2 \]  
(49)

and the equality holds at high SNRs; this completes the proof.

\[ P_e \leq \frac{1}{M} \left[ \sum_{i=0}^{M-2} Q_1\left( \sqrt{2k_i^2 M \gamma \beta}, \frac{\sqrt{2\lambda_i^{\infty}}}{\sigma} \right) \right. 
+ \sum_{i=0}^{M-2} \left. 1 - Q_1\left( \sqrt{2k_i^2 M \gamma \beta}, \frac{\sqrt{2\lambda_i^{\infty}}}{\sigma} \right) \right]. \]  
(52)
Next, by using $k_i = x_i / \sqrt{M \sigma^2 \gamma}$, we substitute $\lambda_i \rightarrow \infty = ((k_i + k_{i+1})/\sqrt{M \sigma^2 \gamma \beta^2})$ in (52) and therefore, we can rewrite (52) as

$$P_e \leq \frac{1}{M} \left[ \sum_{i=0}^{M-2} Q \left( \sqrt{2 M \gamma \beta} \left( \frac{k_i + k_{i+1}}{2} \right) \right) \right]$$

$$+ \left[ \sum_{i=0}^{M-2} \left( 1 - Q \left( \sqrt{2 M \gamma \beta} \left( \frac{k_{i+1} + k_{i+1}}{2} \right) \right) \right) \right]$$

(53)

In the following, we further upper bound (53) by deriving upper bounds on $T_1$ and $T_2$. To this end

$$T_1 \overset{(a)}{=} \exp \left( - \frac{M \gamma \beta}{4} (k_i - k_{i+1})^2 \right)$$

(54)

where (a) follows from the inequality $Q_1(c_1, c_2) \leq \exp(-((c_1 - c_2)^2)2)$, $c_2 > c_1 \geq 0$ [29, eq. (3)]; with $c_1 = k_i \sqrt{2 M \gamma \beta}$ and $c_2 = \sqrt{2 M \gamma \beta} ((k_{i+1} + k_{i+1})/2)$; $c_2 > c_1$ follows from the fact that $k_{i+1} = (k_i + k_{i+1})/2$ (since $k_{i+1} > k_i$). We now provide an upper bound on $T_2$

$$T_2 \overset{(a)}{=} \frac{1}{2} \left[ \exp \left( - \frac{M \gamma \beta}{4} (k_i - k_{i+1})^2 \right) - \exp \left( - \frac{M \gamma \beta}{4} (3k_{i+1} + k_{i+1})^2 \right) \right]$$

(55)

where (a) follows from the inequality $Q_1(c_1, c_2) = 1 - (1/2) \exp(-(c_1 - c_2)^2)/2 - \exp((-c_1 + c_2)^2)/2), c_1 > c_2 \geq 0$ [29, eq. (4)]; with $c_1 = k_{i+1} \sqrt{2 M \gamma \beta}$ and $c_2 = \sqrt{2 M \gamma \beta} ((k_{i+1} + k_{i+1})/2)$; $c_1 > c_2$ follows from the fact that $k_{i+1} > k_i$.

Therefore, by using (54) and (55) in (53), we have

$$P_e^{ub} = \frac{1}{2M} \sum_{i=0}^{M-2} \left[ \exp \left( - \frac{M \gamma \beta}{4} (k_i - k_{i+1})^2 \right) - \exp \left( - \frac{M \gamma \beta}{4} (3k_{i+1} + k_{i+1})^2 \right) \right]$$

(56)

which corresponds to an upper bound on $P_e$. Next, we present a lower bound based on the analysis in [10].

Lower Bound: The ML error performance of the system with an output $r = |h_x| + n$ provides a lower bound on the error performance of the P2P-AD system under study, i.e., $y_d = |h_x| + n$ [10]. The error performance of the system model $r = |h_x| + n$ is well studied (this is a conventional AWGN channel for a given channel realization) and the corresponding conditional error probability for the considered M-ASK modulation is given by [25]

$$P_e^{lb} = \frac{1}{M} \left[ Q \left( \frac{k_i - k_0}{\sqrt{M \gamma \beta}} \right) + \frac{M-2}{2} \right]$$

$$+ \left[ \sum_{i=0}^{M-2} Q \left( \frac{k_{i+1} - k_i}{\sqrt{M \gamma \beta}} \right) \right]$$

(57)

which corresponds to a lower bound on $P_e$.

Thus, from (56) and (57), we obtain an upper and lower bound on the conditional error probabilities. Therefore, bounds on SER can be obtained by averaging the corresponding conditional error probabilities over $\beta$. This completes the proof.

Appendix C

Proof of Theorem 1

We derive the asymptotic upper and lower bounds by averaging the corresponding conditional error functions over the random variable $\beta$. We utilize the fact that asymptotic SER performance can be evaluated by using the approximation $f_\beta(\beta) \approx a^\beta$.

Asymptotic Upper Bound: The asymptotic upper bound is given as

$$\bar{P}_{e, \infty} = \int_0^\infty P_{e, \beta} a^{\beta} d\beta$$

(58)

with

$$P_{e, \beta} = \frac{1}{2M} \sum_{i=0}^{M-2} \left[ \exp \left( - \frac{M \gamma \beta}{4} (k_i - k_{i+1})^2 \right) - \exp \left( - \frac{M \gamma \beta}{4} (3k_{i+1} + k_{i+1})^2 \right) \right]$$

(59)

Then, by using the Gamma function defined by $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$, (58) can be written as

$$\bar{P}_{e, \infty} = \frac{1}{2M} \sum_{i=0}^{M-2} \int_0^\infty \left[ \exp \left( - \frac{M \gamma \beta}{4} (k_i - k_{i+1})^2 \right) - \exp \left( - \frac{M \gamma \beta}{4} (3k_{i+1} + k_{i+1})^2 \right) \right] a^{\beta} d\beta$$

$$= 2^{a+1} a^{\gamma(t+1) - 1} \int_0^\infty \left[ \sum_{i=0}^{M-2} 3(k_{i+1} - k_i)^{2(t+1)} - \sum_{i=0}^{M-2} (3k_{i+1} + k_{i+1})^{2(t+1)} \right] a^{\beta} d\beta$$

(60)

which is an asymptotic upper bound on the SER performance for M-ASK. We now derive an asymptotic lower bound.

Asymptotic Lower Bound: If the condition error function $P_e = Q(\sqrt{b} y)$, then from [27, Proposition 1], we have

$$\lim_{y \to \infty} \mathbb{E}[Q(\sqrt{b} y)] = \int_0^\infty Q(\sqrt{b} y) a^{\beta} d\beta$$

$$= 2^{a+1} a^{\gamma(t+1)} (k_y)^{-(t+1)}.$$
By using the above expression, we can obtain the asymptotic lower bound as

$$
\hat{P}_{e,\infty}^{lb} = \int_{0}^{\infty} P_{e}^{lb} a^{b'} dt \\
= 2^{|a|+1} a \Gamma \left( t + \frac{3}{2} \right) M^{-\gamma} (t+2) \\
\times \left[ (k_1 - k_0)^{-2(t+1)} + (k_{M-1} - k_{M-2})^{-2(t+1)} \\
+ \sum_{i=1}^{M-2} (k_{i+1} - k_i)^{-2(t+1)} + (k_i - k_{i-1})^{-2(t+1)} \right] \\
\tag{61}
$$

and the proof is completed.

**APPENDIX D**

**PROOF OF COROLLARY 2**

Any binary ASK modulation set $\mathcal{X} = \{x_0, x_1\}$, $0 \leq x_0 < x_1$ can be alternatively represented by $\mathcal{X} = \{kx_1, x_1\}$ such that $k \triangleq (x_0/x_1)$, with $k \in [0, 1]$. Note that $k = 0$ corresponds to the OOK modulation. The average power of the modulation set is $(1 + k^2)x_1^2/2)$, which is constrained to be equal to $\rho$, i.e.,

$$
\frac{(1 + k^2)x_1^2}{2} = \rho \implies x_1 = \sqrt{\frac{2\rho}{1 + k^2}}.
$$

Thus, for a given transmit power $\rho$, the binary modulation set can be expressed as $\mathcal{X} = \{k\sqrt(2\rho/(1 + k^2)), \sqrt(2\rho/(1 + k^2))\}$, $k \in [0, 1]$. Next, the variable $k$ is optimized in order to achieve the maximum coding gain. Therefore, the optimization problem can be formulated as

$$
\arg \max_{k \in [0, 1]} g \in [g_1, g_2].
$$

We solve this problem by utilizing the derived upper and lower bounds, since the exact expression for coding gain $g$ is not known. For the binary modulation $\mathcal{X} = \{k\sqrt(2\rho/(1 + k^2)), \sqrt(2\rho/(1 + k^2))\}$, the normalized modulation symbols are given by $k_0 = (k\sqrt(1 + k^2))$ and $k_1 = (1/\sqrt(1 + k^2))$. Subsequently, the asymptotic upper bound in (23) can be written as

$$
\hat{P}_{e,\infty}^{ub} = \frac{2^{t-1} a \Gamma (t + 1)}{y^{t+1}} \left( 3 \left( \frac{1 - k}{\sqrt(1 + k^2)} \right)^{-2(t+1)} \\
- \left( \frac{3 + k}{\sqrt(1 + k^2)} \right)^{-2(t+1)} \right) \\
\tag{63}
$$

and the asymptotic lower bound in (24) is given by

$$
\hat{P}_{e,\infty}^{lb} = \frac{2^{t} a \Gamma (t + \frac{3}{2})}{\sqrt(\Gamma (t + 1))} \left( \frac{1 - k}{\sqrt(1 + k^2)} \right)^{-2(t+1)} \\
\tag{64}
$$

Thus, by comparing them with the expression $(g'\gamma)^{-t+1}$, the corresponding coding gains are given by

$$
g_1 = \frac{1}{1 + k^2} \left( a \Gamma (t + 1) 2^{t-1} \left( 3(1 - k)^{-2(t+1)} \\
- (3 + k)^{-2(t+1)} \right) \right)^{t+1} \\
\tag{65}
$$

and

$$
g_2 = \frac{(1 - k)^2}{1 + k^2} \left( \frac{2^t a \Gamma (t + \frac{3}{2})}{\sqrt(\Gamma (t + 1))} \right)^{t+1} \\
\tag{66}
$$

Next, we prove that both $g_1$ and $g_2$ are decreasing functions of parameter $k \in [0, 1]$.

**Proof of the Fact That $g_1$ Is a Decreasing Function of Parameter $k$:** By letting $h_1(k) \triangleq 1/(1 - k)$ and $h_2(k) \triangleq [1/(3 + k)]$, followed by some algebraic manipulations, (65) can be rewritten as

$$
g_1 = \left( a \Gamma (t + 1) 2^{-1} \left( 3(h_1(k))^{2(t+1)} \\
- (h_2(k))^{2(t+1)} \right) \right)^{t+1} \\
\tag{67}
$$

It is clear that $h_1(k)$ and $h_2(k)$ are monotonically increasing and decreasing function of $k \in [0, 1]$, respectively. Thus, we can write

$$
h_1(k_1) < h_1(k_2) \text{ and } h_2(k_1) > h_2(k_2), \ k_1 < k_2. \tag{68}
$$

Now, the following inequality follows directly from (68):

$$
3(h_1(k_1))^{2(t+1)} - (h_2(k_1))^{2(t+1)} < 3(h_1(k_2))^{2(t+1)} - (h_2(k_2))^{2(t+1)}, \ k_1 < k_2. \tag{69}
$$

By multiplying the L.H.S. of (69) by $(1 + k_1^2)_{t+1}$ and R.H.S. by $(1 + k_2^2)_{t+1}$, we get

$$
\frac{g_1|_{k=k_1}}{a \Gamma (t + 1) 2^{-1}} < \frac{g_1|_{k=k_2}}{a \Gamma (t + 1) 2^{-1}}, \ k_1 < k_2 \tag{70a}
$$

and

$$
\frac{g_1|_{k=k_1}}{a \Gamma (t + 1) 2^{-1}} < \frac{g_1|_{k=k_2}}{a \Gamma (t + 1) 2^{-1}}, \ k_1 < k_2. \tag{70b}
$$

Note that in (70a) the inequality sign remains unaffected, since $(1 + k_1^2)^t < (1 + k_2^2)^t$. By using (67) in (70a) yields (70b). And, (70c) follows directly from (70b), which proves the fact that $g_1$ is a decreasing function of $k$. We now prove that $g_2$ is a decreasing function of the parameter $k$.

**Proof of the Fact That $g_2$ Is a Decreasing Function of Parameter $k$:** It is easy to see that $[(1 - k)^2/(1 + k^2)]$ is a decreasing function of $k \in [0, 1]$ and therefore $g_2 = [(1 - k)^2/(1 + k^2)]^{-1/(t+1)}$ monotonically decreases with $k$.

Thus, the value $k = 0$ maximizes both the upper bound $g_1$ and the lower bound $g_2$. Therefore, we can deduce that, for a given power $\rho$ among all possible binary modulations, the OOK modulation $\mathcal{X} = [0, \sqrt(2\rho)]$ maximizes the coding gain or minimizes the asymptotic error performance. This completes the proof.
APPENDIX E
PROOF OF THEOREM 2

The output of the ML decoder on the received signal \( y_D = |h_2a|h_1x_i + n_D|_+ n_D | \) of an AF-AD relay system is given by

\[
\hat{x}_s = \arg \max_{x_i \in \mathcal{X}} f_{y_D}(y_D|x_s, h_1, h_2).
\]  

(71)

For the sake of analysis, we define \( u \triangleq |h_1x_i + n_D| \), then we can write

\[
f_{y_D}(y_D|x_s, h_1, h_2) = \int_{0}^{\infty} f_{y_D}(y_D|u, h_2) f_{u}(u|h_1, h_2) \, du.
\]

(72)

Since \( f_{y_D}(y_D|u, h_2) \) and \( f_{u}(u|h_1, h_2) \) are Rician PDFs, (72) can be written as

\[
f_{y_D}(y_D|x_s, h_1, h_2) = \int_{0}^{\infty} \frac{2y_D}{\sigma_D^2} \exp \left( -\frac{y_D^2 + |h_2au|^2}{\sigma_D^2} \right) \frac{2y_D h_2au}{\sigma_D^2} \, du \times \frac{2u^2}{\sigma_R^2} \exp \left( -\frac{u^2 + |x_i h_1|^2}{\sigma_R^2} \right) \int_{0}^{\infty} u \exp \left( -\frac{u^2}{\sigma_R^2} \right) \, du
\]

\[
= \frac{2y_D}{\sigma_D^2 (1 + \nu|h_2a|^2)} \exp \left( -\frac{y_D^2 + |x_i h_1 h_2a|^2}{\sigma_D^2 (1 + \nu|h_2a|^2)} \right) \frac{y_D h_1 h_2a}{\sigma_D^2 (1 + \nu|h_2a|^2)} \int_{0}^{\infty} \exp \left( -\frac{u^2}{\sigma_R^2} \right) \, du
\]

\[
= \frac{2y_D}{\sigma_D^2 (1 + \nu|h_2a|^2)} \exp \left( -\frac{y_D^2 + |x_i h_1 h_2a|^2}{\sigma_D^2 (1 + \nu|h_2a|^2)} \right) \frac{2y_D h_1 h_2a}{\sigma_D^2 (1 + \nu|h_2a|^2)} \int_{0}^{\infty} \exp \left( -\frac{u^2}{\sigma_R^2} \right) \, du
\]

\[
= \frac{2y_D}{\sigma_D^2 (1 + \nu|h_2a|^2)} \exp \left( -\frac{y_D^2 + |x_i h_1 h_2a|^2}{\sigma_D^2 (1 + \nu|h_2a|^2)} \right) \frac{2y_D h_1 h_2a}{\sigma_D^2 (1 + \nu|h_2a|^2)} \int_{0}^{\infty} \exp \left( -\frac{u^2}{\sigma_R^2} \right) \, du
\]

(73)

where the last step follows from the inequality [32, ET II 63(1)];

\[
\int_{0}^{\infty} x e^{-a_1 x^2} I_0(b_1 x) I_0(b_2 x) \, dx = \frac{1}{2a_1} \exp \left( b_1^2 + b_2^2 \right) \int_{0}^{\infty} \exp \left( \frac{b_1 x}{2a_1} \right) \, dx
\]

\[
= \frac{2y_D}{\sigma_D^2 (1 + \nu|h_2a|^2)} \exp \left( -\frac{y_D^2 + |x_i h_1 h_2a|^2}{\sigma_D^2 (1 + \nu|h_2a|^2)} \right) \frac{2y_D h_1 h_2a}{\sigma_D^2 (1 + \nu|h_2a|^2)} \int_{0}^{\infty} \exp \left( -\frac{u^2}{\sigma_R^2} \right) \, du
\]

(74)

which proves the first statement of the theorem.

We now prove the second statement of this theorem. To this end, from (2), the ML decoder for the P2P-AD system with received signal as \( y_D = |h_1h_2ax_i + n\), where \( \tilde{n} \sim \mathcal{CN}(0, \sigma_D^2(1 + \nu|h_2a|^2)) \) is given by

\[
\hat{x}_s = \arg \max_{x_i \in \mathcal{X}} \exp \left( -\frac{|x_i h_1 h_2a|^2}{\sigma_D^2 (1 + \nu|h_2a|^2)} \right) \times I_0 \left( \frac{2y_D |x_i h_1 h_2a|}{\sigma_D^2 (1 + \nu|h_2a|^2)} \right)
\]

\[
= \arg \max_{x_i \in \mathcal{X}} \exp \left( -\frac{|x_i h_1 h_2a|^2}{\sigma_D^2 (1 + \nu|h_2a|^2)} \right) \times I_0 \left( \frac{2y_D |x_i h_1 h_2a|}{\sigma_D^2 (1 + \nu|h_2a|^2)} \right)
\]

(75)

The last step follows by considering only those terms which depend on \( x_i \). Since the normalization does not affect the performance of the ML decoder, (75) also characterizes the ML decoder for the P2P-AD system with a received signal \( r = (y_D/\sqrt{1 + \nu|h_2a|^2}) = |h_1x_i + n| \) with \( \tilde{n} = [(h_1h_2a)/\sqrt{1 + \nu|h_2a|^2}] \) and \( n \sim \mathcal{CN}(0, \sigma_D^2) \). From (74) and (75), the two ML decoders are identical; consequently, an AF-AD system can be transformed to an equivalent P2P-AD setup with respect to the ML decoder. This completes the proof.

REFERENCES

[1] “Cisco visual networking index: Global mobile data traffic forecast update, 2017–2022,” Cisco, San Jose, CA, USA, Rep., Feb. 2019.
[2] D. C. Daly and A. P. Chandrakasan, “An energy-efficient OOK transceiver for wireless sensor networks,” IEEE J. Solid-State Circuits, vol. 42, no. 5, pp. 1003–1011, May 2007.
[3] A. Y. Wang, S. Cho, C. G. Sodini, and A. P. Chandrakasan, “Energy efficient modulation and MAC for asymmetric RF microsensor systems,” in Proc. Internat. Symp. Low Power Electron. Design, Huntington Beach, CA, USA, Aug. 2001, pp. 106–111.
[4] Z. Zhou, M. Dong, K. Ota, Z. Liu, J. Wu, and T. Sato, “Error probability analysis of joint signal detection with base station sleeping and cooperation;” in Proc. IEEE Int. Conf. Commun., Aug. 2014, pp. 2443–2448.
[5] I.-H. Lee and J.-B. Kim, “Average symbol error rate analysis for non-orthogonal multiple access with M-ary QAM signals in Rayleigh fading channels,” IEEE Commun. Lett., vol. 23, no. 8, pp. 1328–1331, Aug. 2019.
[6] G. K. Psaltopoulos and A. Wittneben, “Affordable nonlinear MIMO systems,” Ph.D. dissertation, Doctor Sci., ETH Zürich, Zürich, Switzerland, 2011.
[7] M. Schwartz, W. R. Bennett, and S. Stein, Communication Systems and Techniques. New York, NY, USA: IEEE Press, 1995.
[8] J. M. Geist, “Asymptotic error rate behavior for noncoherent on-off keying,” IEEE Trans. Commun., vol. 42, no. 234, p. 225, Feb.–Mar. 1994.
[9] A. Annamalai and V. K. Bhargava, “Asymptotic error-rate behavior for noncoherent on-off keying in the presence of fading,” IEEE Trans. Commun., vol. 47, no. 9, pp. 1293–1296, Sep. 1999.
[10] G. K. Psaltopoulos and A. Wittneben, “Diversity and spatial multiplexing of MIMO amplitude detection receivers,” in Proc. IEEE 26th Int. Symp. Pers, Indoor Mobile Radio Commun., Tokyo, Japan, Sep., 2009, pp. 202–206.
[11] G. K. Psaltopoulos, C. Sulser, and A. Wittneben, “sMILE: The first MIMO envelope detection testbed,” in Proc. IEEE Veh. Technol. Conf. Fall, San Francisco, CA, USA, Sep., 2011, pp. 1–5.
[12] G. K. Psaltopoulos and A. Wittneben, “Channel estimation for very low power MIMO envelope detectors,” in Proc. IEEE Int. Conf. Commun., May 2010, pp. 1–6.
[13] G. K. Psaltopoulos and A. Wittneben, “Nonlinear MIMO: Affordable MIMO technology for wireless sensor networks,” IEEE Trans. Wireless Commun., vol. 9, no. 2, pp. 824–832, Feb. 2010.
[14] G. K. Psaltopoulos, F. Troesch, and A. Wittneben, “On achievable rates of MIMO systems with nonlinear receivers,” in Proc. IEEE Int. Symp. Inf. Theory, Nice, France, 2007, pp. 1071–1075.
Amit Agarwal received the B.Tech. degree in electronics and communication engineering from the National Institute of Technology Kurukshetra, Kurukshetra, India, in 2011, the M.Tech. degree in telecommunication technology from the Indian Institute of Technology Delhi (IIT Delhi), New Delhi, India, in 2014, and the Ph.D. degree from the Electrical Engineering Department, IIT Delhi in 2019.

From 2021 to 2022, he worked as a Postdoctoral Researcher with the Computer Science and Electrical Engineering Department, University of Cyprus, Nicosia, Cyprus, where he was a part of Project PRIME. He is currently an Assistant Professor with the Department of Electronics and Communication Engineering, LNMIIT, Jaipur, India. His research interests include visible light communication, wireless communication theory, 5G/BSG communication systems, and cooperative networks.

Constantinos Psomas (Senior Member, IEEE) received the B.Sc. degree in computer science and mathematics from the Royal Holloway, University of London, Egham, U.K., the M.Sc. degree in applicable mathematics from the London School of Economics, London, U.K., and the Ph.D. degree in mathematics from The Open University, Milton Keynes, U.K.

He is currently a Research Fellow with the Department of Electrical and Computer Engineering, University of Cyprus, Nicosia, Cyprus. From 2011 to 2014, he was a Postdoctoral Research Fellow with the Department of Electrical Engineering, Computer Engineering and Informatics, Cyprus University of Technology, Limassol, Cyprus. His current research interests include wireless-powered communications, intelligent reflecting surfaces, cooperative networks, and full-duplex communications.

Dr. Psomas received an Exemplary Reviewer certificate by the IEEE TRANSACTIONS ON COMMUNICATIONS for 2020 and by the IEEE WIRELESS COMMUNICATIONS LETTERS for 2015 and 2018. He serves as an Associate Editor for the IEEE WIRELESS COMMUNICATIONS LETTERS and the Frontiers in Communications and Networking.

Ioannis Krikidis (Fellow, IEEE) received the Diploma degree in computer engineering from the Computer Engineering and Informatics Department, University of Patras, Patras, Greece, in 2000, and the M.Sc. and Ph.D. degrees in electrical engineering from the Ecole Nationale Supérieure des Télécommunications (ENST), Paris, France, in 2001 and 2005, respectively.

From 2006 to 2007, he worked as a Postdoctoral Researcher with ENST, and from 2007 to 2010, he was a Research Fellow with the School of Engineering and Electronics, University of Edinburgh, Edinburgh, U.K. He is currently an Associate Professor with the Department of Electrical and Computer Engineering, University of Cyprus, Nicosia, Cyprus. His current research interests include wireless communications, cooperative networks, 5G/BSG communication systems, wireless-powered communications, and intelligent reflected surfaces.

Dr. Krikidis was the recipient of the 2013 Young Researcher Award from the Research Promotion Foundation, Cyprus, the 2016 IEEEComSoc Best Young Professional Award in Academia, and the 2019 IEEE SIGNAL PROCESSING LETTERS Best Paper Award. He has been recognized by the Web of Science as a Highly Cited Researcher for 2017–2021. He has received the prestigious ERC Consolidator Grant. He serves as an Associate Editor for IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS, IEEE TRANSACTIONS ON GREEN COMMUNICATIONS AND NETWORKING, and IEEE WIRELESS COMMUNICATIONS LETTERS.