Level statistics for one-dimensional Schrödinger operators and Gaussian beta ensemble

Fumihiko Nakano *

December 30, 2013

Abstract

We study the level statistics for two classes of 1-dimensional random Schrödinger operators: (1) for operators whose coupling constants decay as the system size becomes large, and (2) for operators with critically decaying random potential. As a byproduct of (2) with our previous result [2] imply the coincidence of the limits of circular and Gaussian beta ensembles.

Mathematics Subject Classification (2000): 60H25, 34L20

1 Introduction

As one of the recent developments of the theory of random matrices, the continuum limit of the beta ensembles are recently revealed: Killip-Stoiciu [1] identified the limit of the circular beta ensemble($C_{\beta}$-ensemble, in short) by using the solution to a SDE. Valkó-Virág [5] identified the limit of Gaussian beta ensemble($G_{\beta}$-ensemble, in short) by using Brownian carousel. At the same time, it also gave a new insight to the level statistics problem of 1-dimensional random Schrödinger operators: In [1], they also studied the level statistics problem of the CMV matrices, that is, they studied the scaling

---

*Department of Mathematics, Gakushuin University, 1-5-1, Mejiro, Toshima-ku, Tokyo, 171-8588, Japan. e-mail : fumihiko@math.gakushuin.ac.jp
limit of the point process \( \xi_L \) whose atoms are composed of the scaled eigenvalues of the truncated matrices. When the diagonal components decay in the order of \( n^{-\alpha} \) they showed that, \( \xi_L \) converges to (i) \( \alpha > \frac{1}{2} \): the clock process, (ii) \( \alpha < \frac{1}{2} \): the Poisson process, (iii) \( \alpha = \frac{1}{2} \): the limit of \( G_\beta \)-ensemble. In [4], they studied the same problem for 1-dimensional discrete Schrödinger operators whose random potential decays in the order of \( n^{-\frac{1}{2}} \) and showed that \( \xi_L \) converges to the limit of \( G_\beta \)-ensemble. Moreover, they also studied the random Hamiltonians with system size \( L \) in which the coupling constant decays in the order of \( L^{-\frac{1}{2}} \). They identified the limit ("Sch".) of \( \xi_L \) and studied its various properties. In [2], they studied the 1-dimensional Schrödinger operators in the continuum with random decaying potential, for the case of \( \alpha > \frac{1}{2} \) and \( \alpha = \frac{1}{2} \), and the results obtained are parallel to that in [4]. This paper is basically a continuum analogue of [4]: (1) we consider the operator on \([0,L]\) where the coupling constant is equal to \( L^{-\alpha} \). We study the limit of \( \xi_L \) for the case of \( \alpha > \frac{1}{2} \) and \( \alpha = \frac{1}{2} \). (2) we consider the same operator to that in [2] for the critical decay \( \alpha = \frac{1}{2} \) and show that \( \xi_L \) converges to the limit of \( G_\beta \)-ensemble, which, together with the results in [2], implies that the limit of these two beta ensembles are equal. In the next subsection, we shall explain the motivation of the problem [1].

1.1 Motivation and Set ups

The localization length \( l_{loc} \) of the 1-dimensional Schrödinger operator \( H = -\Delta + \lambda V \) is typically in the order of \( \lambda^{-2} \). Thus, setting \( H_L := H|_{[0,L]} \), \( \lambda = L^{-\alpha} \), we expect:

(1) (extended case) \( \alpha > \frac{1}{2} \) : we have \( L \ll l_{loc} \) so that the particle would be extended.

(2) (localized case) \( \alpha < \frac{1}{2} \) : we have \( l_{loc} \ll L \) so that the particle would be localized.

(3) (critical case) \( \alpha = \frac{1}{2} \) : \( l_{loc} \simeq L \) so that it would correspond to the critical case.

Therefore if we consider the level statistics problem, \( \xi_L \) would converge to (1) \( \alpha > \frac{1}{2} \) : the clock process, (2) \( \alpha < \frac{1}{2} \) : the Poisson process, (3) \( \alpha = \frac{1}{2} \) : something which is intermediate between the clock and Poisson.

\(^1\)The author would like to thank F. Klopp for introducing this problem.
In this paper we consider this problem in the continuum setting: The Hamiltonian is defined by

\[ H_L := H_{\lambda L}|_{[0,L]} \]

with Dirichlet boundary condition, where \( H_{\lambda L} \) is the Schrödinger operator with the coupling constant \( \lambda_L \) which decays at certain rate as the system size \( L \) is large:

\[ H_{\lambda L} := -\frac{d^2}{dt^2} + \lambda_L F(X_t), \quad \lambda_L := L^{-\alpha}, \quad \alpha > 0. \]

\( (X_t)_{t \geq 0} \) is a Brownian motion on a compact Riemannian manifold \( M \) and \( F \in C^\infty(M) \) with \( \langle F \rangle := \int_M F(x)dx = 0. \)

Let \( \{E_n(L)\}_{n \geq 1} \) be the eigenvalues of \( H_L \) in the increasing order. Since we only consider the positive eigenvalues, we set

\[ n(L) := \min\{n|E_n(L) > 0\}. \]

Fix the reference energy \( E_0 > 0 \) arbitrary. To study the local distribution of \( E_n(L) \)'s near \( E_0 \), we set

\[ \xi_L := \sum_{n \geq n(L)} \delta_{L(\sqrt{E_n(L)} - \sqrt{E_0})}. \quad (1.1) \]

Here we take \( \sqrt{E_n(L)} \) instead of \( E_n(L) \) to unfold the eigenvalues with respect to the density of states. Our purpose is to study the behavior of \( \xi_L \) as \( L \) tends to infinity.

### 1.2 Results for Extended Case

If we consider the free Laplacian, we must take a subsequence in order that \( \xi_L \) converges to a point process. We need the same condition described below.

(A) The subsequence \( \{L_j\}_{j=1}^\infty \) satisfies \( L_j \xrightarrow{j \to \infty} \infty \) and

\[ \sqrt{E_0}L_j = m_j\pi + \beta + o(1), \quad j \to \infty \]

\( m_j \in \mathbb{N}, \beta \in [0,\pi). \)
Theorem 1.1 Assume (A) and $\alpha > \frac{1}{2}$. Then we have

$$\lim_{j \to \infty} E[e^{-\xi_{L_j}(f)}] = \exp \left( - \sum_{n \in \mathbb{Z}} f(n\pi - \beta) \right).$$

In other words, $\xi_{L_j}$ converges to a (deterministic) clock process with spacing $\pi$, in probability.

When the random potential is spatially decaying in the order of $\alpha > \frac{1}{2}$, $\xi_L$ also converges to a clock process but $\beta$ is random [2]. Here the effect of the random potential is rather weak compared to that in [2]. In fact, the solution to the eigenvalue equation $Hx_t = Ex_t$ approaches to the free solution in probability (Theorem 3.3). However, the randomness appear in the second order (Theorem 1.2). To see the spacing between eigenvalues, we renumber the eigenvalues near $E_0$ such that \( E_{-2}' < E_{-1}' < E_0' < E_1' < \ldots \).

Then by the argument of the proof of Theorem 1.1 for any $l \in \mathbb{Z}$,

$$\left( \sqrt{E_{l+1}'(L)} - \sqrt{E_l'(L)} \right) L - \pi \to 0$$

in probability. Hence it is reasonable to consider

$$X_j(n) := \left\{ \left( \sqrt{E_{m_j+n+1}(L_j)} - \sqrt{E_{m_j+n+j}(L_j)} \right) L_j - \pi \right\} L_j^{\alpha - \frac{1}{2}}, \quad n \in \mathbb{Z}$$

to study the second order asymptotics of eigenvalues near $E_0$, for $E_{m_j}(L_j)$ may be regarded as the closest eigenvalue to $E_0$.

Theorem 1.2 \{X_j(n)\}_{n \in \mathbb{Z}} converges in distribution to the Gaussian system with covariance

$$C(n, n') := \frac{C(E_0)}{8E_0} \begin{cases} 2 & (n = n') \\ -1 & (|n - n'| = 1) \\ 0 & (\text{otherwise}) \end{cases}$$

where

$$C(E) := \int_M |\nabla(L + 2i\sqrt{E})^{-1}F|^2 dx$$

and $L$ is the generator of $(X_t)$.
Here the covariance is short range, while it is not the case if the potential is spatially decaying [2].

**Remark 1.1** By definition of $X_j(n)$, we have
\[
\sqrt{E_{m_j + n}(L_j)} = \sqrt{E_{m_j}(L_j)} + \frac{n\pi}{L_j} + \frac{1}{L_j^{\alpha + \frac{1}{2}}} \sum_{l=0}^{n-1} X_j(l)
\]
so that the difference between $\sqrt{E_{m_j + n}(L_j)}$ and $\sqrt{E_{m_j}(L_j)}$ converges to the Gaussian in the second order.

**Remark 1.2** Suppose that we consider two reference energies $E_0, E'_0$, $E_0 \neq E'_0$ both satisfying (A) with the same subsequence. Then the corresponding $\{X_j(n)\}, \{X'_j(n')\}$ converge jointly to the two independent Gaussian systems each other. The same property also holds for Theorem 1.3, 1.4 stated below.

### 1.3 Results for Critical Case

In this subsection we set $\alpha = \frac{1}{2}$. By Theorem 4.4, the solution to the eigenvalue equation $Hx_t = Ex_t$ is bounded from above and below so that it is different from that in the critically decaying potential case studied in [2].

**Theorem 1.3** Assume $\alpha = \frac{1}{2}$ and (A). Then we have
\[
\lim_{j \to \infty} E[e^{-\xi L_j(f)}] = E \left[ \exp \left( - \sum_{n \in \mathbb{Z}} f(\Psi_1^{-1}(2n\pi - 2\beta)) \right) \right].
\]
where $\Psi_t(c)$ is a strictly-increasing function valued process such that for any $c_1, c_2, \cdots, c_m$, $\Psi_t(c_1), \cdots, \Psi_t(c_m)$ jointly satisfy the following SDE.
\[
d\Psi_t(c_j) = \left( 2c_j - Re \frac{i}{2E_0} \langle Fg_{E_0} \rangle \right) dt \\
+ \frac{1}{\sqrt{E_0}} \left\{ \sqrt{\frac{C(E_0)}{2}} \left( e^{i\Psi_t(c_j)} dZ_t + \sqrt{C(0)} dB_t \right) \right\} (1.2)
\]
j = 1, 2, \cdots, m, where $Z_t$ is a complex Brownian motion independent of a Brownian motion $B_t$ and
\[
g_{\sqrt{E_0}} := (L + 2i\sqrt{E_0})^{-1}F, \quad g := L^{-1}(F - \langle F \rangle), \\
C(E_0) := \int_M |\nabla g_{\sqrt{E_0}}|^2 dx, \quad C(0) := \int_M |\nabla g|^2 dx.
\]
This SDE is the same as that satisfied by the phase function of “Schτ” [4] up to constant. Hence the properties of “Schτ” derived in [4] such as strong repulsion, large gap asymptotics, explicit form of intensity and CLT, also hold for our case.

1.4 Results for decaying potential model with critical decay

In this subsection we consider
\[ H = -\frac{d^2}{dt^2} + V(t), \quad V(t) := a(t)F(X_t). \]
where \( a \in C^\infty \), \( a(-t) = a(t) \), \( a \) is decreasing on \([0, \infty)\), and
\[ a(t) = t^{\frac{1}{2}}(1 + o(1)), \quad t \to \infty. \]

\((X_t)\) and \( F \) satisfy the same conditions stated in subsection 1.1. Let \( H_L := H|_{[0,L]} \) be the finite box Hamiltonian with Dirichlet boundary condition and let \( \{E_n(L)\}_{n \geq n(L)} \) be the set of positive eigenvalues of \( H_L \). Let \( \xi_L \) defined as in (1.1). In [2], we proved that \( \xi_L \) converges to the limit of \( C_\beta \)-ensemble. That is, let \( \zeta_C = \sum_k \delta_{\lambda_k} \) be the continuum limit of the \( C_\beta \)-ensemble. Then
\[ \xi_L \xrightarrow{d} \tilde{\zeta}_C^C \quad (1.3) \]
where \( \tilde{\zeta}_C^C = \sum_k \delta_{\lambda_k}/2 \) and \( \beta = \frac{8E_0}{C(E_0)} \). Here we give a different description of the limit.

**Theorem 1.4** The limit \( \xi_\infty = \lim_{L \to \infty} \xi_L \) has the following property. Let \( N(\lambda) := \sharp \{ \text{atoms of } \xi_\infty \text{ in } [0, \lambda] \} \) be the counting function of \( \xi_\infty \). Then \( N(\lambda) \overset{d}{=} \frac{1}{2\pi} \Psi_{1,-}(\lambda) \) where \( \Psi_t(\lambda), t \in [0, 1) \) is the strictly-increasing function valued process which is the solution to
\[ d\Psi_t(\lambda) = 2\lambda dt + \frac{D(E_0)}{\sqrt{1-t}} Re\left( e^{i\Psi_t(\lambda)} - 1 \right) dZ_t, \quad \Psi_0(\lambda) = 0 \quad (1.4) \]
where \( D(E_0) = \sqrt{\frac{C(E_0)}{2E_0}} \).
This theorem is the continuum analogue of that in [4]. To see the significance of Theorem 1.4, let us recall the Gaussian beta ensemble whose joint density of ordered eigenvalues \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) is proportional to

\[
\exp \left( -\frac{\beta}{4} \sum_{k=1}^{n} \lambda_k^2 \right) \prod_{j<k} |\lambda_j - \lambda_k|^{\beta}.
\]

Then Valkó-Virág found the following representation of the continuum limit of the \( \mathbb{G}_\beta \)-ensemble.

**Theorem 1.5 (Valkó-Virág [5])**

Let \( \mu_n \) be the sequence such that \( n^{\frac{1}{6}} (2\sqrt{n} - |\mu_n|) \to \infty \). Then

\[
\sum_k \delta_{\Lambda_k^{(n)}} \Rightarrow \zeta_G^\beta, \quad \Lambda_k^{(n)} := \sqrt{4n - \mu_n^2} (\lambda_k - \mu_n)
\]

where \( \sharp \{ \text{atoms of } \zeta_G^\beta \text{ in } [0, \lambda] \} = \frac{1}{2\pi} \alpha_\infty(\lambda) \) where \( \alpha_\infty(\lambda) := \lim_{t \to \infty} \alpha_t(\lambda) \) and \( \alpha_t(\lambda) \) is the solution to

\[
d\alpha_t(\lambda) = \lambda \frac{\beta}{4} e^{-\frac{\beta t}{4}} dt + \text{Re}[(e^{i\alpha_t(\lambda)} - 1) dZ_t], \quad \alpha_0(\lambda) = 0. \tag{1.5}
\]

Note that \( \alpha_\infty(\lambda) \in 2\pi \mathbb{Z} \). By the time change \( t = 1 - e^{-ct} \), \( c = \frac{\beta}{4} \), \( \beta = \frac{8E_0}{C(E_0)} \), SDE (1.4) is transformed to (1.5) [4]. Therefore

**Corollary 1.6** For \( \zeta_G^\beta = \sum_k \delta_{\lambda_k} \), let \( \tilde{\zeta}_G^\beta = \sum_k \delta_{\lambda_k/2} \). Then \( \xi_L \xrightarrow{d} \tilde{\zeta}_G^\beta \) where \( \beta = \frac{8E_0}{C(E_0)} \).

By varying \( E_0 \in (0, \infty) \) or \( F \), any \( \beta > 0 \) can be realized. Hence by combining (1.3) and Corollary 1.6, we have the coincidence of the limit of two beta ensembles.

**Corollary 1.7** For any \( \beta > 0 \), \( \zeta_G^\beta = \zeta_G^\beta \).

This is known for \( \beta = 1, 2, 4 \). Valkó-Virág have a direct proof of this fact by showing that these two descriptions are equivalent [6].

The method of proof of Theorem 1.1 - 1.3 is essentially the same as that in [2]: we write the Laplace transform of \( \xi_L \) in terms of the Prüfer variables, and study the behavior of the relative Prüfer phase. The idea of proof of Theorem 1.4 is the same as that in [5, 4] with different techniques.
identify the scaling limit of the relative Prüfer phase as the solution to a SDE, and show that \( t \uparrow 1 \) limit of this solution gives the counting function of \( \xi_\infty \). The outline of this paper is as follows: Section 2 is the preparation of the basic notations and tools. In Section 3 - 5, we prove Theorem 1.1, Theorem 1.2, Theorem 1.3, and Theorem 1.4 respectively. In Appendix, we recall the techniques used in [3, 2]. In what follows, \( C \) denotes positive constants which is subject to change from line to line.

2 Preparation

For general 1-dim Schrödinger operator \( H = -\frac{d^2}{dx^2} + q \), let \( x_t \) be the solution to the equation \( Hx_t = \kappa^2 x_t, x_0 = 0, (\kappa > 0) \) which we write in the (modified) Prüfer variables:

\[
\begin{pmatrix}
x_t \\
x_t'/\kappa
\end{pmatrix} = \begin{pmatrix}
\sin \theta_t \\
\cos \theta_t
\end{pmatrix}, \quad \theta_0 = 0. 
\tag{2.1}
\]

We define \( \tilde{\theta}_t(\kappa) \) by

\[ \theta_t(\kappa) = \kappa t + \tilde{\theta}_t(\kappa). \]

Then it follows that

\[
\begin{align*}
r_t(\kappa) &= \exp \left( \frac{1}{2\kappa} \text{Im} \int_0^t q(s)e^{2i\theta_s(\kappa)}ds \right) \\
\tilde{\theta}_t(\kappa) &= \frac{1}{2\kappa} \int_0^t \text{Re}(e^{2i\theta_s(\kappa)} - 1)q(s)ds, \quad \tilde{\theta}_0(\kappa) = 0 \tag{2.2}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \theta_t(\kappa)}{\partial \kappa} &= \int_0^t \frac{r_s^2}{r_t^2} ds + \frac{1}{2\kappa^2} \int_0^t \frac{r_s^2}{r_t^2} q(s)(1 - \text{Re } e^{2i\theta_s(\kappa)})ds. \tag{2.3}
\end{align*}
\]

Since \( r_t \) is bounded from below (Theorem 3.3, 4.4 and Proposition 2.1), for any closed interval \( I \subset (0, \infty) \) we have \( \inf_{\kappa \in I} \frac{\partial \theta_t(\kappa)}{\partial \kappa} > 0 \) for sufficiently large \( t > 0 \) so that \( \theta_t(\kappa) \) is strictly-increasing as a function of \( \kappa \in I \). Set \( q(t) := \lambda_L F(X_t) \) and let \( \theta_{t,L}(\kappa), r_{t,L}(\kappa) \) be the corresponding Prüfer variables. Let

\[
\begin{align*}
\kappa_0 &= \sqrt{E_0} \\
\Psi_{t,L}(x) &= \theta_{t,L}(\kappa_0 + \frac{x}{L}) - \theta_{t,L}(\kappa_0), \quad 0 \leq t \leq L
\end{align*}
\]

8
be the relative Prüfer phase. Further write $\theta_{L,L}(\kappa_0)$ as

$$\theta_{L,L}(\kappa_0) = m(\kappa_0, L)\pi + \phi(\kappa_0, L), \quad m(\kappa_0, L) \in \mathbb{N}, \quad \phi(\kappa_0, L) \in [0, \pi).$$

Then we have the following representation of the Laplace transform of $\xi_L$ in terms of Prüfer variables [1].

**Lemma 2.1**

$$E[e^{-\xi_L(f)}] = E\left[\exp\left(-\sum_{n \geq n(L)-m(\kappa_0,L)} f(\Psi_{L,L}^{-1}(n\pi - \phi(\kappa_0, L)))\right)\right]$$

for $f \in C_c^+(\mathbb{R})$, where $\xi_L(f) := \int_{\mathbb{R}} f d\xi_L$.

**Proof.** Let

$$x_n = L\left(\sqrt{E_n(L)} - \kappa_0\right)$$

be the $n$-th atom of $\xi_L$. By Sturm oscillation theorem, $x = x_n$ if and only if $\theta_{L,L}(\kappa_0 + \frac{x}{L}) = n\pi$ so that we have

$$\{x_n\}_{n \geq n(L)} = \{x \mid \Psi_{L,L}(x) = n\pi - \phi(\kappa_0, L), \quad n \geq n(L) - m(\kappa_0, L)\}.$$ 

$\square$

### 3 Extended case

Throughout this section we set $\alpha > \frac{1}{2}$ to study the extended case.

#### 3.1 Proof of Theorem 1.1

First of all, we study the behavior of the following quantity.

$$J_{t,L}(\kappa) := \lambda_L \int_0^t e^{2i\theta_{t,L}(\kappa)} F(X_s)ds, \quad \kappa \geq 0 \quad (3.1)$$

where we set $\theta_{t,L}(0) := 0$ for convenience.
**Lemma 3.1** Let $\kappa \geq 0$.

(1) \[ \sup_{0 \leq t \leq L} |J_{t,\kappa}(\kappa)| \xrightarrow{L \to \infty} 0 \] in probability, compact uniformly for $\kappa \in (0, \infty)$.

(2) \[ L^{\alpha - \frac{1}{2}} J_{t,\kappa}(\kappa) = L^{\alpha - \frac{1}{2}} Y_{t,\kappa}(\kappa) + o(1) \] as $L \to \infty$, where

\[ Y_{t,\kappa}(\kappa) := \lambda_L \int_0^t e^{2i\theta_s,\kappa} dM_s(\kappa) \] (3.2)

and $M_s(\kappa)$ is a complex martingale defined in Lemma 6.1.

**Proof.** (1) We first assume that $\kappa > 0$. By Lemma 6.1

\[ J_{t,\kappa}(\kappa) = \lambda_L \left[ e^{2i\theta_s,\kappa} g_\kappa(X_s) \right]_0^t - \frac{2i}{2\kappa} \int_0^t \text{Re} \left( e^{2i\theta_s,\kappa} - 1 \right) \lambda_L F(X_s) e^{2i\theta_s,\kappa} g_\kappa(X_s) ds + \int_0^t e^{2i\theta_s,\kappa} dM_s(\kappa) \]

\[ =: I_{t,\kappa} + II_{t,\kappa} + Y_{t,\kappa}. \]

Then $I_{t,\kappa} = O(\lambda_L) = O(L^{-\alpha})$, $II_{t,\kappa} \leq C\lambda_L^2 t = O(L^{-2\alpha + 1})$ and $Y_{t,\kappa}(\kappa)$ is a martingale satisfying

\[ \langle Y_{t,\kappa}(\kappa), Y_{t,\kappa}(\kappa) \rangle_t, \quad \langle Y_{t,\kappa}(\kappa), \overline{Y_{t,\kappa}(\kappa)} \rangle_t \leq C\lambda_L^2 t. \]

Therefore by martingale inequality we have

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq L} |Y_{t,\kappa}(\kappa)|^2 \right] \leq (\text{Const.}) \mathbb{E}[|Y_{L,\kappa}(\kappa)|^2] = O(L^{-2\alpha + 1}). \]

A standard argument using Chebishev’s inequality yields the conclusion. The proof for $\kappa = 0$ is similar except that $\langle F \rangle = 0$ and $II_{t,\kappa} = 0$.

(2) It easily follows from the argument above. □
Lemma 3.2 For $\kappa \geq 0$, 

$$\sup_{0 \leq t \leq L} |\tilde{\theta}_{t,L}(\kappa)| \xrightarrow{L \to \infty} 0$$

in probability.

Proof. By (2.3) $\tilde{\theta}_{t,L}(\kappa) = \frac{1}{2\kappa} \text{Re} \left( J_{t,L}(\kappa) - J_{t,L}(0) \right)$. Then the conclusion follows from Lemma 3.1(1).

Proof of Theorem 1.1

By Lemma 3.2 and (A), 

$$\Psi_{L,L}(x) = x + \tilde{\theta}_{L,L}(\kappa_0 + \frac{x}{L}) - \tilde{\theta}_{L,L}(\kappa_0) = x + o(1)$$

$$\theta_{L_j,L}(\kappa_0) = \kappa_0 L_j + \tilde{\theta}_{L_j,L}(\kappa_0) = m_j \pi + \beta + o(1)$$

in probability. Hence for any subsequence of $\{L_j\}$, we can further find a subsequence $\{L_{j_k}\}$ of that such that

$$\lim_{k \to \infty} \phi(\kappa_0, L_{j_k}) = \beta, \quad a.s.$$ 

By using the fact that $\lim_{k \to \infty} (n(L_{j_k}) - m_{L_{j_k}}) = -\infty$ and Lemma 6.2 we have

$$\lim_{k \to \infty} \mathbb{E}[e^{-\xi_{L_{j_k}}(f)}] = \exp \left( - \sum_{n \in \mathbb{Z}} f(n\pi - \beta) \right).$$

Since this holds for any subsequence of $\{L_j\}$, we arrive at the conclusion.

3.2 Behavior of solutions

We study the behavior of the solution $x_t$ to the Schrödinger equation $H_L x_t = \kappa^2 x_t$.

Theorem 3.3 For a solution $x_{t,L}$ to the equation $H_L x_{t,L} = \kappa^2 x_{t,L}$ ($\kappa > 0, x_{0,L} = 0$), let $r_{t,L}, \theta_{t,L}$ be the corresponding Prüfer variables defined in (2.1). Then we have

$$\sup_{0 \leq t \leq L} |r_{t,L}(\kappa) - 1| \to_{p} 0$$

$$\sup_{0 \leq t \leq L} |\theta_{t,L}(\kappa) - \kappa t| \to_{p} 0$$

in probability so that $x_{t,L}$ approaches to the free solution.
Proof. It easily follows from (2.2), (2.3) and Lemma 3.1(1).

### 3.3 Second Limit Theorem

**Lemma 3.4**

\[ \sqrt{\bar{E}_{m_j+n}(L_j)} = \kappa_0 + \frac{n\pi - \beta + o(1)}{L_j} \]

in probability.

**Proof.** This lemma follows from Lemma 3.2 and the following computation.

\[
\left( \sqrt{\bar{E}_{m_j+n}(L_j)} - \kappa_0 \right) L_j \\
= \theta_{L_j,L_j} \left( \sqrt{\bar{E}_{m_j+n}(L_j)} - \kappa_0 L_j - \tilde{\theta}_{L_j,L_j} \left( \sqrt{\bar{E}_{m_j+n}(L_j)} \right) \right) \\
= (m_j + n)\pi - (m_j \pi + \beta + o(1)).
\]

For \( c_1, c_2, d_1, d_2 \in \mathbb{R} \), set

\[ \kappa_c := \kappa_0 + \frac{c}{n}, \quad c \in \mathbb{R} \]

\[ \Theta_t^{(n)}(c_1, c_2) := \tilde{\theta}_{nt,n}(\kappa_{c_1}) - \tilde{\theta}_{nt,n}(\kappa_{c_2}) \]

\[ C_t(c_1, c_2; d_1, d_2) := \frac{C(E_0)}{8E_0} \text{Re} \int_0^t (e^{2ic_1 u} - e^{2ic_2 u})(e^{-2id_1 u} - e^{-2id_2 u}) du. \]

We study the behavior of \( \Theta_t^{(n)}(c_1, c_2) \) as \( n \) tends to infinity, for fixed \( c_1, c_2 \).

**Lemma 3.5** As \( n \to \infty \), \( \{n^{\alpha-\frac{1}{2}}\Theta_t^{(n)}(c_1, c_2)\}_{t,c_1,c_2} \) converges to the Gaussian system \( \{G(t, c_1, c_2)\}_{t,c_1,c_2} \) with covariance \( \{C_t^{(n)}(c_1, c_2; d_1, d_2)\}_{t,c_1,c_2,d_1,d_2} \).

**Proof.** Set \( \kappa_j := \kappa_0 + \frac{c_j}{n}, \quad \kappa_j' := \kappa_0 + \frac{d_j}{n}, \quad j = 1, 2 \). By Lemma 3.1(2)

\[ n^{\alpha-\frac{1}{2}}\Theta_t^{(n)}(c_1, c_2) = n^{\alpha-\frac{1}{2}} \frac{1}{2\kappa} \text{Re} \left( J_{nt,n}(\kappa_1) - J_{nt,n}(\kappa_2) \right) + O(n^{-\frac{1}{2}}) \]

\[ = n^{\alpha-\frac{1}{2}} \frac{1}{2\kappa} \text{Re} \left( Y_{nt,n}(\kappa_1) - Y_{nt,n}(\kappa_2) \right) + o(1) \]
where $Y_{t,n}(\kappa)$ is defined in (3.2). Set

$$Z^{(n)}_t(\kappa_1, \kappa_2) := n^{\alpha - \frac{1}{2}} (Y_{nt,n}(\kappa_1) - Y_{nt,n}(\kappa_2)).$$

By Lemma 6.1 we have

$$\langle Z^{(n)}_t(\kappa_1, \kappa_2), Z^{(n)}_t(\kappa_1', \kappa_2') \rangle = C(E_0) \int_0^t (e^{2i\epsilon_1 u} - e^{2i\epsilon_2 u}) (e^{-2id_1 u} - e^{-2id_2 u}) \, du + o(1).$$

It then suffices to use the martingale central limit theorem. \[\square\]

**Proof of Theorem 1.2**

Let $b_n$ such that $\sqrt{E_{m_j+n}(L_j)} = \kappa_0 + \frac{b_n}{L_j}$. Then by Lemma 3.4, $b_n = n\pi - \beta + o(1)$ in probability. Taking difference between

$$(m_j + n + 1)\pi = \sqrt{E_{m_j+n+1}(L_j)} L_j + \tilde{\theta}_{L_j,L_j} \left( \sqrt{E_{m_j+n+1}(L_j)} \right)$$

yields

$$\{ \left( \sqrt{E_{m_j+n+1}(L_j)} - \sqrt{E_{m_j+n}(L_j)} \right) L_j - \pi \} L_j^{\alpha - \frac{1}{2}} = -L_j^{\alpha - \frac{1}{2}} \Theta^{(L_j)}(b_{n+1}, b_n).$$

By using Lemma 3.5 and Skorohard’s theorem, we obtain the conclusion. The statement of covariance follows from the following computation.

$$\text{Re} \int_0^1 \left( e^{2i((n+1)\pi\beta)}s - e^{2i(n\pi\beta)s} \right) \left( e^{-2i((n'+1)\pi\beta)s} - e^{-2i(n'\pi\beta)s} \right) ds$$

$$= \frac{1}{\pi} \int_0^{2\pi} \cos((n - n')\theta)(1 - \cos \theta) d\theta.$$

\[\square\]

### 4 Critical Case

In this section we set $\alpha = \frac{1}{2}$.  

13
4.1 Preliminaries

Lemma 4.1 Let $J_{t,L}(\kappa)$, $\kappa \geq 0$ be the one defined in (3.1).

(1) For $\kappa > 0$:

$$J_{t,L}(\kappa) = \frac{i}{2\kappa} \langle F g_\kappa \rangle \cdot \lambda_L^2 \cdot t + Y_{t,L}(\kappa) + Z_{t,L}(\kappa) + O(\lambda_L) + O(\lambda_L^3 \cdot t)$$

where $Y_{t,L}(\kappa)$, $Z_{t,L}(\kappa)$ are martingales such that

$$Y_{t,L}(\kappa) := -\frac{2i}{2\kappa} \cdot \lambda_L^2 \left( \frac{1}{2} K_{4,3}(\kappa) + \frac{1}{2} K_{0,3}(\kappa) - K_{2,3}(\kappa) \right)$$

$$K_{\beta,3}(\kappa) := \lambda_L^2 \int_0^t e^{i\beta s,L(\kappa)} d\tilde{M}^{(3)}_s(\kappa), \quad \beta = 0, 2, 4$$

$$Z_{t,L}(\kappa) := \lambda_L \int_0^t e^{2i\theta s,L(\kappa)} dM_s(\kappa).$$

(2) For $\kappa = 0$:

$$J_{t,L}(0) := Z_{t,L} + O(\lambda_L), \quad Z_{t,L}(0) := \lambda_L M_t$$

where $g_\kappa$, $M_s(\kappa)$, $M_s$, $\tilde{M}^{(3)}_s(\kappa)$ are defined in Lemma 6.1.

Proof. (1) By Lemma 6.1

$$J_{t,L}(\kappa) = \lambda_L \left\{ \left[ e^{2i\theta s,L(\kappa)} g_\kappa(X_s) \right]_0^t \right.$$  

$$- \frac{2i}{2\kappa} \int_0^t \text{Re} \left( e^{2i\theta s,L(\kappa)} - 1 \right) e^{2i\theta s,L(\kappa)} \lambda_L F(X_s) g_\kappa(X_s) ds$$

$$+ \int_0^t e^{2i\theta s,L(\kappa)} dM_s(\kappa) \right\}$$

$$=: I + II + III.$$

Clearly, $I = O(\lambda_L)$ and $III = Z_{t,L}(\kappa)$. For the second term $II$,

$$II = -\frac{2i}{2\kappa} \lambda_L^2 \int_0^t \left( e^{4i\theta s,L(\kappa)} + 1 - e^{2i\theta s,L(\kappa)} \right) F(X_s) g_\kappa(X_s) ds$$

$$=: -\frac{2i}{2\kappa} \left( \frac{1}{2} K_4(\kappa) + \frac{1}{2} K_0(\kappa) - K_2(\kappa) \right)$$
where

\[ K_\beta(\kappa) := \lambda_2^2 \int_0^t e^{i\beta \theta_s(L)} F(X_s) g_\kappa(X_s) ds, \quad \beta = 0, 2, 4. \]

For \( \beta = 2, 4 \) we use Lemma 6.1 again

\[
K_\beta(\kappa) = \left[ \lambda_2^2 e^{i\beta \theta_s(L)} h_{\kappa, \beta}(X_s) \right]^t_0 \nonumber
\]

\[
- \frac{i\beta}{2\kappa} \lambda_2^2 \int_0^t Re \left( e^{2i\theta_s(L)} - 1 \right) e^{i\beta \theta_s(L)} \lambda_L F(X_s) g_\kappa(X_s) ds
\]

\[
+ \lambda_2^2 \int_0^t e^{i\beta \theta_s(L)} d\tilde{M}_s^{(\beta)}(\kappa)
\]

\( =: K_{\beta,1} + K_{\beta,2} + K_{\beta,3}. \)

We have \( K_{\beta,1} = O(\lambda_2^2) = O(L^{-2\alpha}), K_{\beta,2} = O(\lambda_3^2 \cdot t) = O(L^{-3\alpha} \cdot t). \) Similarly for \( \beta = 0, \)

\[
K_0(\kappa) = \lambda_2^2 \langle F g_\kappa \rangle \cdot t + \lambda_2^2 [h_{0, \kappa}(X_s)]^t_0 + \lambda_2^2 \tilde{M}_t
\]

\( =: K_{0,1} + K_{0,2} + K_{0,3}. \)

We then have \( K_{0,2} = O(\lambda_2^2) = O(L^{-2\alpha}). \) Putting together

\[
II = - \frac{2i}{2\kappa} \lambda_2^2 \langle F g_\kappa \rangle \cdot t - \frac{2i}{2\kappa} \lambda_2^2 \left( \frac{1}{2} K_{4,3} + \frac{1}{2} K_{0,3} - K_{2,3} \right)
\]

\[ + O(\lambda_2^2) + O(\lambda_3^2 \cdot t) \]

proving (1).

(2) It immediately follows from Lemma 6.1 and the fact that \( \langle F \rangle = 0. \]

**Lemma 4.2** For any \( \gamma > 0, \)

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq L} e^{\gamma (Y_{t,L}(\kappa) + Z_{t,L}(\kappa))} \right] \leq C_\gamma.
\]

**Proof.** By Ito’s formula

\[
e^{\gamma (Y_{t,L}(\kappa) + Z_{t,L}(\kappa))}
\]

\[ = 1 + \gamma \int_0^t e^{\gamma (Y_{s,L}(\kappa) + Z_{s,L}(\kappa))} d(Y_{t,L} + Z_{t,L})_s
\]

\[
+ \frac{\gamma^2}{2} \int_0^t e^{\gamma (Y_{s,L}(\kappa) + Z_{s,L}(\kappa))} d(Y_{t,L} + Z_{t,L}, Y_{t,L} + Z_{t,L})_s
\]

\[ + \frac{\gamma^2}{2} \int_0^t e^{\gamma (Y_{s,L}(\kappa) + Z_{s,L}(\kappa))} d(Y_{t,L} + Z_{t,L}, Y_{t,L} + Z_{t,L})_s \] (4.1)
we note that
\[|d(Y_{L} + Z_{L}, Y_{L} + Z_{L})| \leq (\text{Const.})(\lambda_{L}^2 + \lambda_{L}^3 + \lambda_{L}^4)ds\]
\[|d(Y_{L} + Z_{L}, Y_{L} + Z_{L})| \leq (\text{Const.})(\lambda_{L}^2 + \lambda_{L}^3 + \lambda_{L}^4)ds.\]

Setting
\[f_{\gamma}(t) := E\left[e^{\gamma(Y_{L}(\kappa) + Z_{L}(\kappa))}\right]\]
we have by (4.1),
\[f_{\gamma}(t) \leq 1 + C_{L} \int_{0}^{t} f_{\gamma}(s)ds, \quad C_{L} := C \left(\lambda_{L}^2 + \lambda_{L}^3 + \lambda_{L}^4\right).\]

By Grownwall's inequality \(f_{\gamma}(t) \leq e^{C_{L} \cdot t}\) yielding
\[\sup_{0 \leq t \leq L} f_{\gamma}(t) \leq e^{C}.\]

By martingale inequality,
\[E\left[\sup_{0 \leq t \leq L} \left|\int_{0}^{t} e^{\gamma(Y_{L}(\kappa) + Z_{L}(\kappa))} d(Y_{L} + Z_{L})\right|^{2}\right] \leq (\text{Const.}) E\left[\left|\int_{0}^{L} e^{\gamma(Y_{L}(\kappa) + Z_{L}(\kappa))} d(Y_{L} + Z_{L})\right|^{2}\right] \leq (\text{Const.}) E\left[\int_{0}^{L} e^{2\gamma(Y_{L}(\kappa) + Z_{L}(\kappa))} d(Y_{L} + Z_{L}, Y_{L} + Z_{L})\right] + (\text{Const.}) E\left[\int_{0}^{L} e^{2\gamma(Y_{L}(\kappa) + Z_{L}(\kappa))} d(Y_{L} + Z_{L}, Y_{L} + Z_{L})\right] \leq (\text{Const.}) \int_{0}^{L} f_{2\gamma}(s)\lambda_{L}^2ds \leq (\text{Const.}).\]
Substituting to (4.1), we arrive at the conclusion.

**4.2 Proof of Theorem 1.3**

We set
\[\Psi_{L}(c) := 2c + 2\tilde{\theta}_{L,L} \left(\kappa_{0} + \frac{c}{L}\right) = 2\tilde{\theta}_{L,L} \left(\kappa_{0} + \frac{c}{L}\right) - 2\kappa_{0}L.\]
By definition, \(\Psi_{L}(c)\) is increasing with respect to \(c\) for large \(L\). As Lemma 2.1, we have
Lemma 4.3 Writing \( \kappa_0 L_j \) as
\[
\kappa_0 L_j = m_j \pi + \beta_j, \quad m_j \in \mathbb{Z}, \quad \beta_j \in [0, \pi),
\]
we have
\[
E[e^{-\xi L_j(f)}] = E \left[ \exp \left( - \sum_{n \geq n(L_j) - m_j} f \left( \Psi^{-1}_{L_j}(2n\pi - 2\beta_j) \right) \right) \right].
\]
It then suffices to study the behavior of \( \Psi_{L_j}(c) \) as \( j \to \infty \). Replacing \( L_j \) by \( n \), we set
\[
\kappa_c := \kappa_0 + \frac{c}{n}, \quad c \in \mathbb{R}
\]
\[
\Psi^{(n)}_t(c) := 2ct + 2\tilde{\theta}_{nt,n}(\kappa_c), \quad 0 \leq t \leq 1.
\]
For simplicity we set \( J^{(n)}_t(\kappa) := J_{nt,n}(\kappa) \). By (2.3), we have
\[
\tilde{\theta}_{nt,n}(\kappa_c) = \frac{1}{2\kappa} \Re \left( J^{(n)}_t(\kappa_c) - J^{(n)}_t(0) \right) + O(\lambda_n).
\]
By Lemma 4.1 and the fact that \( \langle Y_{-n}, Y_{-n} \rangle_t = O(\lambda_n^4 \cdot nt) = O(n^{-1}) \), we have
\[
J^{(n)}_t(\kappa_c) - J^{(n)}_t(0) = W^{(n)}_t(c) - \frac{i}{2\kappa_0} \langle F \kappa_0 \rangle t + o(1)
\]
in probability, where
\[
W^{(n)}_t(c) := Z_{nt,n}(\kappa_c) - Z_{nt,n}(0).
\]
Set \( \varphi := [g, g], \varphi_{\kappa_0} := [g_{\kappa_0}, g_{-\kappa_0}] \). Then by Lemma 6.1
\[
\langle W^{(n)}(c), W^{(n)}(d) \rangle_t = \langle \varphi \rangle t + o(1) \quad (4.2)
\]
\[
\langle W^{(n)}(c), \overline{W^{(n)}(d)} \rangle_t = \langle \varphi_{\kappa_0} \rangle \int_0^t e^{i(\Psi^{(n)}_u(c) - \Psi^{(n)}_u(d))} du + \langle \varphi \rangle t + o(1) \quad (4.3)
\]
in probability. The following estimate
\[
E \left[ |W^{(n)}_t(c) - W^{(n)}_s(c)|^2 \right] = E \left[ |\langle W^{(n)}_t(c), \overline{W^{(n)}_t(c)} \rangle_t - \langle W^{(n)}_s(c), \overline{W^{(n)}_s(c)} \rangle_s|^2 \right]
\leq \lambda^2_n \langle \varphi_{\kappa_0} \rangle \int_{ns}^{nt} du + \langle \varphi \rangle (t - s) + o(1)
\leq C(t - s) + o(1),
\]

17
together with martingale inequality and Kolmogorov’s theorem implies that the sequence of processes \((W_t^{(n)}(c))_{0 \leq t \leq 1}\) is tight. Therefore by taking subsequences further we may assume

\[ W_t^{(n)}(c) \to W_t(c) \quad \Psi_t^{(n)}(c) \to \Psi_t(c), \quad a.s. \]

Letting \(n \to \infty\) in (4.2), (4.3), martingale \(W(c)\) satisfies

\[
\langle W(c), W(d) \rangle_t = \langle \varphi \rangle t \\
\langle W(c), \omega(d) \rangle_t = \langle \varphi \rangle \int_0^t e^{i(\Psi_u(c) - \Psi_u(d))} du + \langle \varphi \rangle t
\]

so that we have

\[
W_t(c) = \sqrt{\langle \varphi \rangle} \int_0^t e^{i\Psi_s(c)} dZ_s + \sqrt{\langle \varphi \rangle} B_t
\]

where \(Z_t, B_t\) are mutually independent, complex and standard Brownian motions respectively. Since

\[
\Psi_t(c) = 2ct + \frac{1}{\kappa_0} \Re \left( W_t(c) - \frac{i}{2\kappa_0} \cdot t \cdot \langle F g_{\kappa_0} \rangle \right)
\]

we have

\[
d\Psi_t(c) = 2cdt + \frac{1}{\kappa_0} \left\{ \sqrt{\langle \varphi \rangle} \Re \left( e^{i\Psi_s(c)} dZ_s \right) + \sqrt{\langle \varphi \rangle} dB_t - \Re \frac{i}{2\kappa_0} \langle F g_{\kappa_0} \rangle dt \right\}.
\]

Similar arguments show that \(\Psi_t(c_1), \ldots, \Psi_t(c_m)\) jointly satisfy the SDE (1.2). This determines the process \(\Psi_t(c)\) uniquely, which is strictly-increasing by SDE comparison theorem. That \(\Psi_t^{(n)}(c) \to \Psi_t(c)\) also in the sense of strictly-increasing function valued process follows from [2], Proposition 9.2. By Lemma 6.2 we finish the proof of Theorem 1.3.

\subsection*{4.3 Behavior of Solutions for Critical Case}

\textbf{Theorem 4.4} Suppose that \(x_t\) is the solution to the Schrödinger equation:

\[ H x_t = \kappa^2 x_t, \quad \kappa > 0, \]

and let \(r_t(\kappa), \theta_t(\kappa)\) be the corresponding Prüfer variables. Then we can find \(C_\kappa < \infty\) such that for any \(a > 0\),

\[ P \left( \frac{1}{a} \leq r_t \leq a \text{ for any } t \in [0, L] \right) \geq 1 - \frac{C_\kappa}{a}. \]
Proof. By (2.2), Lemma 4.1, 4.2,
\[ \log r_t = B(t) + Y(t) + Z(t), \]
where
\[ M := \sup_{0 \leq t \leq L} |B(t)| < \infty, \quad E[\sup_{0 \leq t \leq L} e^{Y_t + Z_t}] \leq C_1, \quad E[\sup_{0 \leq t \leq L} e^{-(Y_t + Z_t)}] \leq C_2. \]
Then the conclusion follows from Chebyshev’s inequality. □

5 Decaying potentials with critical rate

For the proof of Theorem 1.4, we solve the eigenvalue equation
\[ Hx_t = k^2 x_t \]
with \( x_L = 0, \; x'_L / \kappa = 1 \). Here we suppose that the distribution of \( X_0 \) is
uniform on \( M \). Then \( (X_t)_{t \in \mathbb{R}} \) is stationary, so that if \( (Y_t)_{t \in \mathbb{R}} \) is an independent
 copy of \( (X_t) \), we may replace \( H_n \) by the following operator.
\[ \hat{H}_n := -\frac{d^2}{dt^2} + \hat{V}(t), \quad \hat{V}(t) := a(n - t)F(Y_t) \text{ on } L^2(0, n). \]
Moreover by [2], \( \xi_\infty = \lim_{n \to \infty} \xi_n \) is uniquely determined as far as \( a(t) = t^{-\frac{1}{2}}(1 + o(1)) \). Therefore, without loss of generality, we may suppose that
\[ a(s) = \frac{1}{\sqrt{s}}, \quad s \geq 1. \]
Furthermore we set
\[ \kappa_0 := \sqrt{E_0}, \quad \kappa_c := \kappa_0 + \frac{c}{n}, \quad c \in \mathbb{R} \]
\[ \Psi^{(n)}_t(c) := 2\theta_{nt}(\kappa_c) - 2\theta_{nt}(\kappa_0). \]

5.1 A priori estimate

In this subsection we show the following theorem.

Theorem 5.1 For any fixed \( T < 1 \) we have
\[ \Psi^{(n)}_t(c) = 2ct + \frac{1}{2\kappa_0} Re V^{(n)}_t(c) + \delta_{nt}(\kappa_c) - \delta_{nt}(\kappa_0) + O(n^{-\frac{1}{2}}), \quad 0 \leq t \leq T \]
where
\[
V_t^{(n)}(c) := Y_t^{(n)}(\kappa_c) - V_t^{(n)}(\kappa_0)
\]
\[
Y_t^{(n)}(\kappa) := \int_0^{nt} a(n - s)e^{2i\theta_s(\kappa)}dM_s(\kappa).
\]

The concrete form of \( \delta_{nt}(\kappa) \) is given in Lemma 5.2 below. Moreover
\[
E \left[ \sup_{0 \leq t \leq T} |V_t^{(n)}(c)| \right] \leq C \frac{1}{\sqrt{1 - T}}
\]
\[
E \left[ \sup_{0 \leq t \leq T} |\delta_{nt}(\kappa_c) - \delta_{nt}(\kappa_0)|^2 \right] \xrightarrow{n \to \infty} 0.
\]

First of all, by (2.3) it is easy to see
\[
\Psi_t^{(n)}(c) = 2ct + \frac{1}{\kappa_0} Re \left( J_t^{(n)}(\kappa_c) - J_t^{(n)}(\kappa_0) \right) + O(n^{-\frac{1}{2}}) \tag{5.1}
\]
where
\[
J_t^{(n)}(\kappa) := \int_0^{nt} e^{2i\theta_s(\kappa)}a(n - s)F(Y_s)ds.
\]

We decompose this integral by using Lemma 6.1. The result is:

**Lemma 5.2**

\[
J_t^{(n)}(\kappa) = C_t^{(n)}(\kappa) + \delta_{nt}(\kappa) + Y_t^{(n)}(\kappa), \quad \kappa > 0
\]

where
\[
C_t^{(n)}(\kappa) := -\frac{i}{2\kappa} \int_0^{nt} a(n - s)^2F(Y_s)g_{\kappa}(Y_s)ds
\]
\[
\delta_{nt}(\kappa) := \delta_{nt}^{(1)}(\kappa) + \delta_{nt}^{(2)}(\kappa)
\]
\[
\delta_{nt}^{(1)}(\kappa) := [a(n - s)e^{2i\theta_s(\kappa)}g_{\kappa}(Y_s)]_0^{nt} - \int_0^{nt} (a(n - s))'e^{2i\theta_s(\kappa)}g_{\kappa}(Y_s)ds
\]
\[
\delta_{nt}^{(2)}(\kappa) := \frac{i}{\kappa} \int_0^{nt} \left( \frac{e^{2i\theta_s(\kappa)}}{2} - 1 \right) e^{2i\theta_s(\kappa)}a(n - s)^2F(Y_s)g_{\kappa}(Y_s)ds
\]
\[
Y_t^{(n)}(\kappa) := \int_0^{nt} a(n - s)e^{2i\theta_s(\kappa)}dM_s(\kappa).
\]
To compute $J_t(n)(\kappa_c) - J_t(n)(\kappa_0)$ we estimate the difference of them:

**Lemma 5.3** For $0 \leq t < 1$

1. $\left| C_t(n)(\kappa_c) - C_t(n)(\kappa_0) \right| \leq C \frac{\log n}{n}$
2. $\delta_{nt}^{(1)}(\kappa_c) - \delta_{nt}^{(1)}(\kappa_0) = O(n^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}})$
3. $\delta_{nt}^{(2)}(\kappa) = -\frac{i}{2\kappa} Z_4(n)(\kappa) + \frac{i}{\kappa} Z_2(n)(\kappa) + O(n^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}})$

where

$$Z_\beta(n)(\kappa) := \int_0^{nt} a(n-s)^2 e^{i\beta s(\kappa)} d\tilde{M}_s(\kappa).$$

**Proof.** It is sufficient to show (3).

$$\delta_{nt}^{(2)}(\kappa) = -\frac{i}{2\kappa} D_4(n)(\kappa) + \frac{i}{\kappa} D_2(n)(\kappa)$$

where we set

$$D_\beta(n)(\kappa) := \int_0^{nt} a(n-s)^2 e^{i\beta s(\kappa)} F(Y_s) g_\kappa(Y_s) ds, \quad \beta = 2, 4.$$ 

Thus it suffices to estimate $D_\beta(n)(\kappa)$. By Lemma 6.1

$$D_\beta(n)(\kappa) = \left[ a(n-s)^2 e^{i\beta s(\kappa)} h_{\kappa,\beta}(Y_s) \right]_0^{nt}$$

$$- \int_0^{nt} (a(n-s)^2 e^{i\beta s(\kappa)} h_{\kappa,\beta}(Y_s) ds$$

$$- \frac{i\beta}{2\kappa} \int_0^{nt} Re \left(e^{2\beta s(\kappa)} - 1\right) e^{i\beta s(\kappa)} a(n-s)^3 F(Y_s) h_{\kappa,\beta}(Y_s) ds$$

$$+ \int_0^{nt} a(n-s)^2 e^{i\beta s(\kappa)} d\tilde{M}_s(\beta)(\kappa).$$

Since $\int_0^{nt} a(n-s)^2 ds = O(n^{-1}(1-t)^{-1})$ and $\int_0^{nt} a(n-s)^3 ds = O(n^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}})$, we have

$$D_\beta(n)(\kappa) = Z_\beta(n)(\kappa) + O(n^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}).$$

□

We next estimate $V_t(n)(c)$. 

21
Lemma 5.4  (1) If $0 < t < 1$, we have

$$E \left[ \langle V^{(n)}(c), \overline{V^{(n)}(c)} \rangle_t \right] \leq \frac{C}{1-t}$$

(2) For fixed $0 < T < 1$, we have

$$E \left[ \sup_{0 \leq t \leq T} |V_t^{(n)}(c)| \right] \leq (\text{const.}) \frac{C}{\sqrt{1-T}}$$

Proof.

\[
\begin{align*}
\langle V^{(n)}(c), \overline{V^{(n)}(c)} \rangle_t \\
= \int_0^t a(n-s)^2 \left| e^{2i(\theta_s(\kappa_c) - \theta_s(\kappa_0))} - 1 \right|^2 [g_{\kappa_0}, \overline{g_{\kappa_0}}] ds + O(n^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}})
\end{align*}
\]

Set $\varphi_{\kappa_0} := [g_{\kappa_0}, \overline{g_{\kappa_0}}]$. We compute by using Lemma 6.1

\[
\begin{align*}
\langle V^{(n)}(c), \overline{V^{(n)}(c)} \rangle_t \\
= \langle \varphi_{\kappa_0} \rangle \int_0^t a(n-s)^2 \left| e^{2i(\theta_s(\kappa_c) - \theta_s(\kappa_0))} - 1 \right|^2 ds + W_t^{(n)}(\kappa_0) + O(n^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}})
\end{align*}
\]

where

\[
\begin{align*}
W_t^{(n)}(\kappa_0) := \int_0^t a(n-s)^2 \left| e^{2i(\theta_s(\kappa_c) - \theta_s(\kappa_0))} - 1 \right|^2 dM_s(\varphi_{\kappa_0}, 0) \\
\langle W^{(n)}(\kappa_0), W^{(n)}(\kappa_0) \rangle_t = O(n^{-1}(1-t)^{-1}).
\end{align*}
\]

Taking expectation, martingale term vanishes and we have

\[
\begin{align*}
E \left[ \langle V^{(n)}(c), \overline{V^{(n)}(c)} \rangle_t \right] \\
= \langle \varphi_{\kappa_0} \rangle \int_0^t a(n-s)^2 E \left[ \left| e^{2i(\theta_s(\kappa_c) - \theta_s(\kappa_0))} - 1 \right|^2 \right] ds + O(n^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}) \\
\leq C \int_0^t \frac{n}{n(1-u)} du + O(n^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}) \\
\leq \frac{C}{1-t} + O(n^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}).
\end{align*}
\]

(2) follows (1) and the martingale inequality. []
Lemma 5.5 If $0 < T < 1$,

$$E \left[ \sup_{0 \leq t \leq T} |\delta_{nt}(\kappa_c) - \delta_{nt}(\kappa_0)|^2 \right]^{n \to \infty} \to 0.$$

Proof. By Lemma 5.3, it suffices to show that the martingale part converges to 0, that is,

$$E \left[ \sup_{0 \leq t \leq T} \left| Z^{(n)}_{t}(\kappa_c) - Z^{(n)}_{t}(\kappa_0) \right|^2 \right] \leq C \int_0^n a(n-s)^4 E \left[ \left| e^{i\delta_s(\kappa_c)} - e^{i\delta_s(\kappa_0)} \right|^2 \right] ds + O(n^{-2}(1 - T)^{-2}) \to 0.$$

By combining (5.1), Lemma 5.2, 5.3, 5.4 and 5.5, we obtain Theorem 5.1

5.2 Tightness

Lemma 5.6 For $0 \leq s < t \leq T < 1$ we have

$$E \left[ \left| V^{(n)}_t(c) - V^{(n)}_s(c) \right|^4 \right] \leq (\text{const.})(t-s)^2.$$

Proof.

$$E \left[ \left| V^{(n)}_t(c) - V^{(n)}_s(c) \right|^4 \right] \leq C E \left[ \left| V^{(n)}_t(c) - V^{(n)}_s(c) \right|^2 \right]^2 \leq C \left( \int_{ns}^{nt} a(n-u)^2 \left| e^{2i\theta_u(\kappa_c)} g_{\kappa_c} - e^{2i\theta_u(\kappa_0)} g_{\kappa_0}, e^{2i\theta_u(\kappa_c)} g_{\kappa_c} - e^{2i\theta_u(\kappa_0)} g_{\kappa_0} \right| du \right)^2 \leq C \left( \int_{ns}^{nt} a(n-u)^2 du \right)^2 \leq C(t-s)^2.$$

By using these lemmas, we can show the tightness.
Theorem 5.7 For any $c \in \mathbb{R}$, $\{\Psi_t^{(n)}(c)\}_{0 \leq t < 1}$ is tight. In fact, for any $0 < T < 1$, we have

1. $\lim_{A \to \infty} \mathbb{P} \left( \sup_{0 \leq t \leq T} |\Psi_t^{(n)}(c)| \geq A \right) = 0,$
2. $\lim_{\delta \downarrow 0} \lim_{n \to \infty} \mathbb{P} \left( \sup_{|t-s| < \delta, 0 \leq s, t < T} |\Psi_t^{(n)}(c) - \Psi_s^{(n)}(c)| > \rho \right) = 0, \quad \forall \rho > 0.$

Proof. (1) follows from Theorem 5.1 and (2) follows from Theorem 5.1 and Lemma 5.6.

5.3 Derivation of SDE

By Theorem 5.7 $\Psi_t^{(n)}(c)$ have a limit point $\Psi_t(c)$. Then by Skorohard’s theorem, we may assume

$$\Psi_t^{(n)}(c) \to \Psi_t(c), \quad a.s.$$ for some subsequence.

Theorem 5.8 $\Psi_t(c)$ satisfies the following SDE.

$$d\Psi_t(c) = 2cdt + \frac{D}{\sqrt{1-t}}\text{Re} \left( (e^{i\Psi_t(c)} - 1) \, dZ_t \right), \quad 0 \leq t < 1$$

where $D := \sqrt{\frac{\langle \Phi_{\kappa_0} \rangle}{\sqrt{2\kappa_0}}}.$

Proof. By Theorem 5.1

$$\Psi_t^{(n)}(c) = 2ct + \frac{1}{\kappa_0} \text{Re} \, V_t^{(n)}(c) + o(1)$$

in probability. Let $c, d \in \mathbb{R}$. By Lemma 6.1 we have

$$\langle V_t^{(n)}(c), V_t^{(n)}(d) \rangle_t = \int_0^t a(n-s)^2 \left( e^{2i\theta_s(\kappa_c)} - e^{2i\theta_s(\kappa_d)} \right) \left( e^{2i\theta_s(\kappa_d)} - e^{2i\theta_s(\kappa_0)} \right) \varphi_{\kappa_0}(Y_s) ds + O\left( n^{-1}(1-t)^{-1} \right)$$

$$= o(1)$$
\[ \langle V^{(n)(c)}, V^{(n)(d)} \rangle_t \]
\[ = \langle \varphi_{\kappa_0} \rangle \int_0^t a(n-s)^2 \left( e^{2i(\theta_s(\kappa_c) - \theta_s(\kappa_0))} - 1 \right) \left( e^{-2i(\theta_s(\kappa_d) - \theta_s(\kappa_0))} - 1 \right) ds + o(1) \]
\[ = \langle \varphi_{\kappa_0} \rangle \int_0^t n a(n-ns)^2 \left( e^{i\Psi_s(n)(c)} - 1 \right) \left( e^{-i\Psi_s(n)(d)} - 1 \right) ds + o(1) \]
\[ = \langle \varphi_{\kappa_0} \rangle \int_0^t \frac{1}{1-s} \left( e^{i\Psi_s(c)} - 1 \right) \left( e^{-i\Psi_s(d)} - 1 \right) ds + o(1) \]
in probability. Therefore
\[ V_t(c) := \lim_{n \to \infty} V^{(n)}_t(c) \]
is a $L^2$-continuous martingale such that
\[ \langle V(c), V(d) \rangle_t = 0 \]
\[ \langle V(c), V(d) \rangle_t = \langle \varphi_{\kappa_0} \rangle \int_0^t \frac{1}{1-s} \left( e^{i\Psi_s(c)} - 1 \right) \left( e^{-i\Psi_s(d)} - 1 \right) ds \]
Hence $V_t(c)$ satisfies
\[ dV_t = \sqrt{\langle \varphi_{\kappa_0} \rangle} \frac{1}{2} \sqrt{1-t} \left( e^{i\Psi_t(c)} - 1 \right) dZ_t \]
Since $\Psi_t(c) = 2ct + \frac{1}{\kappa_0} \text{Re } V_t(c)$, we are done. \(\square\)

### 5.4 Behavior of $\theta_{n-n^\beta}$

Let
\[ \{x\}_{2\pi Z} := x - \max\{2\pi k \mid k \in Z, 2\pi k \leq x\}. \]

**Theorem 5.9** For $0 < \beta < 1$ and $\kappa > 0$, $\{2\theta_{n-n^\beta}(\kappa)\}_{2\pi Z}$ converges to the uniform distribution on $[0, 2\pi)$.

**Proof.** It suffices to show
\[ \lim_{n \to \infty} E[e^{2mi\tilde{\theta}_{n-n^\beta}(\kappa)}] \to 0, \quad m \neq 0. \]
In what follows, we omit the $\kappa$-dependence. Set
\[ t = t_n = 1 - n^{\beta-1}. \]
We then have
\[ e^{2mi\tilde{\theta}_{n-\eta}(\kappa)} = 1 + \int_0^{nt} 2mi \frac{1}{2\kappa} Re \left( e^{2i\tilde{\theta}_s} - 1 \right) e^{2mi\tilde{\theta}_s} a(n - s) F(Y_s) ds \]
\[ = 1 + \frac{mi}{\kappa} \int_0^{nt} \left( \frac{e^{2iks + (2m+2)i\tilde{\theta}_s} + e^{-2iks + (2m-2)i\tilde{\theta}_s}}{2} - e^{2mi\tilde{\theta}_s} \right) a(n - s) F(Y_s) ds \]
\[ =: 1 + I + II + III. \]

By Lemma 6.1,
\[ I = \frac{mi}{2\kappa} \left\{ [a(n - s)e^{(2m+2)i\tilde{\theta}_s} + 2i\kappa g_n(Y_s)]_0^{nt} \right. \]
\[ - \int_0^{nt} (a(n - s))'e^{(2m+2)i\tilde{\theta}_s} + 2i\kappa g_n(Y_s) ds \]
\[ - \frac{(2m + 2)i}{2\kappa} \int_0^{nt} a(n - s)^2 Re \left( e^{(2m+2)i\tilde{\theta}_s} - 1 \right) e^{(2m+2)i\tilde{\theta}_s} + 2i\kappa g_n(Y_s) F(Y_s) ds \]
\[ + \int_0^{nt} a(n - s)e^{(2m+2)i\tilde{\theta}_s} + 2i\kappa dM_s(\kappa) \}
\[ =: I_1 + I_2 + I_3 + I_4. \]

Since \( n(1 - t) = n^\beta \), we have \( I_1, I_2 = O(n^{-\frac{\beta}{2}}) \). We further compute \( I_3 \) by using Lemma 6.1:
\[ I_3 = \frac{mi}{2\kappa} \cdot \frac{-2m + 2i}{2\kappa} \cdot \frac{1}{2} \int_0^{nt} a(n - s)^2 e^{2mi\tilde{\theta}_s} F(Y_s) g_n(Y_s) ds + O(n^{-\frac{\beta}{2}}) \]
\[ = \frac{mi}{2\kappa} \cdot \frac{-2m + 2i}{2\kappa} \cdot \frac{1}{2} (Fg_n) \int_0^{nt} a(n - s)^2 e^{im\tilde{\theta}_s} ds + O(n^{-\frac{\beta}{2}}) + (\text{martingale}) \]

Putting all together, we have
\[ I = \frac{mi}{2\kappa} \cdot \frac{-2m + 2i}{2\kappa} \cdot \frac{1}{2} (Fg_n) \int_0^{nt} a(n - s)^2 e^{2mi\tilde{\theta}_s} ds + O(n^{-\frac{\beta}{2}}) + (\text{martingale}) \]

By computing \( II, III \) in a similar manner we obtain
\[ e^{2mi\tilde{\theta}_n} \]
\[ = 1 + C_m \int_0^{nt} a(n - s)^2 e^{2mi\tilde{\theta}_s} ds + O(n^{-\frac{\beta}{2}}) + (\text{martingale}) \]
\[ = 1 + C_m \int_0^t na(n - nu)^2 e^{2mi\tilde{\theta}_n} du + O(n^{-\frac{\beta}{2}}) + (\text{martingale}) \] (5.2)
where
\[ C_m := \left( \frac{m(2m+2)}{2(2\kappa)^2} \langle Fg_\kappa \rangle + \frac{m(2m-2)}{2(2\kappa)^2} \langle Fg_{-\kappa} \rangle + \frac{m^2}{\kappa^2} \langle Fg \rangle \right). \]

Let \( \sigma_F \) be the spectral measure of \( L \) associated to \( F \). Because
\[ \langle Fg_\kappa \rangle = \int_M F(L+2i\kappa)^{-1}F dx = \int_{-\infty}^{0} \frac{1}{\lambda + 2i\kappa} d\sigma_F(\lambda), \]
\[ Re \langle Fg_\kappa \rangle = Re \langle Fg_{-\kappa} \rangle < 0 \text{ and } Re \langle Fg \rangle < 0 \text{ so that} \]
\[ Re C_m < 0, \quad m \neq 0. \]

Take expectation on (5.2) and set
\[ \rho_t^{(n)} := E[e^{2mi\tilde{\theta}nt}] \]
\[ f_n(t) := na(n-nt)^2. \]

Then
\[ \rho_t^{(n)} = 1 + C_m \int_0^t f_n(u)\rho_u^{(n)} du + g_n(u). \]

where
\[ g_n(u) = O(n^{-\frac{2}{3}}), \quad 0 \leq u \leq 1 - n^{\beta-1}. \]

It follows that
\[ \rho_t^{(n)} = 1 + C_m \int_0^t (f_n(s) + f_n(s)g_n(s)) \exp \left( C_m \int_s^t f_n(u) du \right) ds + g_n(t) \]
\[ =: A + B + C + D. \]

The first two terms are equal to
\[ A + B = 1 + C_m \int_0^t f_n(s) \exp \left( C_m \int_s^t f_n(u) du \right) ds \]
\[ = 1 + \left[ - \exp \left( C_m \int_s^t f_n(u) du \right) \right]_0^t \]
\[ = \exp \left( C_m \int_0^t f_n(u) du \right). \]

Because \( Re C_m < 0 \), we have
\[ \exp \left( C_m \int_0^1 n^{\beta-1} f_n(u) du \right) = \exp \left( C_m \int_0^1 \frac{1}{1-s} ds \right) \xrightarrow{n \to \infty} 0. \]

Similarly, \( C, D = O(n^{-\frac{2}{3}}) \). Therefore \( \lim_{n \to \infty} \rho_t^{(n)} = 0. \)
5.5 Proof of Theorem 1.4

Take $0 < \epsilon < 1$, $0 < \beta < 1$ arbitrary, and let

$$l_1 = n(1-\epsilon), \quad l_2 = n - n^\beta$$

so that $0 < l_1 < l_2$. Moreover let $a_*, a^* = a_* + 1 \in 2\pi\mathbb{Z}$ satisfying

$$a_* \leq \Psi^{(n)}_{1-\epsilon}(c) < a^*.$$

Lemma 5.10 For $n \gg 1$ and for $l_1 \leq l \leq l_2$ we have

$$\Psi^{(n)}_{l/n}(c) - \Psi^{(n)}_{1-\epsilon}(c) = 2 \cdot \frac{c}{n} \left( 1 + O(n^{-\frac{\beta}{2}}) \right) (l - l_1) + \frac{1}{\kappa_0} \int_{l_1}^{l} \text{Re} \left( e^{2i\theta_s(\kappa_c)} - e^{2i\theta_s(\kappa_0)} \right) a(n-s)F(Y_s)ds.$$

Thus by the comparison theorem,

$$\Psi^{(n)}_{l/n}(c) \geq a_*, \quad l_1 \leq l \leq l_2$$

for sufficiently large $n$.

Proof. By (2.3) we have

$$\Psi^{(n)}_{l/n}(c) - \Psi^{(n)}_{1-\epsilon}(c) = 2 \cdot \frac{c}{n} \left( 1 + O(n^{-\frac{\beta}{2}}) \right) (l - l_1) + \left( \frac{2}{2\kappa_c} - \frac{2}{2\kappa_0} \right) \int_{l_1}^{l} \text{Re} \left( e^{2i\theta_s(\kappa_c)} - 1 \right) a(n-s)F(Y_s)ds + \frac{2}{2\kappa_0} \int_{l_1}^{l} \text{Re} \left( e^{2i\theta_s(\kappa_0)} - e^{2i\theta_s(\kappa_0)} \right) a(n-s)F(Y_s)ds.$$

Then the following estimate yields the conclusion.

$$\left( \frac{2}{2\kappa_c} - \frac{2}{2\kappa_0} \right) \int_{l_1}^{l} \text{Re} \left( e^{2i\theta_s(\kappa_c)} - 1 \right) a(n-s)F(Y_s)ds \leq \frac{C}{n} \int_{l_1}^{l} \frac{1}{\sqrt{n-s}}ds \leq \frac{C}{n} \cdot \frac{1}{n^{\beta/2}} \cdot (l - l_1).$$

The following lemma is an straightforward consequence of (5.1), Lemma 5.2 and 5.3.
Lemma 5.11  For \( l_1 \leq l \leq l_2 \),
\[
\Psi_{l/n}(c) - \Psi_{1-\epsilon}(c) = 2 \cdot \frac{c}{n} \cdot (l - l_1) + \frac{1}{\kappa_0} \Re \left\{ \int_{K_1} \left( e^{2i\theta_s(\kappa_c)} - e^{2i\theta_s(\kappa_0)} \right) a(n - s) dM_s(\kappa_0) \right\} + O(n^{\frac{\beta}{2} - 1}) + O(n^{-\frac{\beta}{2} + \epsilon'}), \quad 0 < \epsilon' \ll 1.
\]
Let \( F_i := \sigma(Y_s; 0 \leq s \leq t) \).

Proposition 5.12
\[
E \left[ |\Psi_{1-n^{\beta-1}}(c) - \Psi_{1-\epsilon}(c)| \mid F_{l_1} \right] \leq C \left( d(\Psi_{1-\epsilon}(c), 2\pi Z) + \epsilon \right).
\]
This proposition follows from Lemma 5.13 below.

Lemma 5.13
\[
\begin{align*}
(1) \quad & E[|\Psi_{1-n^{\beta-1}}(c) - a_\ast||F_{l_1}] \leq C \left( \epsilon + (\Psi_{1-\epsilon}(c) - a_\ast) \right) \\
(2) \quad & E[|\Psi_{1-n^{\beta-1}}(c) - a_\ast||F_{l_1}] \leq C \left( \epsilon + (a_\ast - \Psi_{1-\epsilon}(c)) \right).
\end{align*}
\]

Proof. (1) By Lemma 5.11
\[
\left| E[\Psi_{1-n^{\beta-1}}(c) - \Psi_{1-\epsilon}(c)|F_{l_1}] \right| \leq 2 \cdot \frac{c}{n} (l_2 - l_1) + o(1). \quad (5.3)
\]
By Lemma 5.10
\[
E[|\Psi_{1-n^{\beta-1}}(c) - a_\ast||F_{l_1}] = E[|\Psi_{1-n^{\beta-1}}(c) - a_\ast||F_{l_1}] \leq E[|\Psi_{1-n^{\beta-1}}(c) - \Psi_{1-\epsilon}(c)||F_{l_1}] + (\Psi_{1-\epsilon}(c) - a_\ast).
\]
Substituting (5.3) and using \( \frac{\epsilon}{n}(l_2 - l_1) \leq C\epsilon \), we have the conclusion.

(2) Letting \( T := \inf \left\{ t \geq l_1 \mid \Psi_{t/n}(c) - a_\ast \geq 0 \right\} \),
we have
\[
E[|\Psi_{1-n^{\beta-1}}(c) - a_\ast|^+ |F_{l_1}] = E \left[ 1(T \leq l_2) E[|\Psi_{1-n^{\beta-1}}(c) - a_\ast| F_T] |F_{l_1} \right].
\]
If \( T \leq l_2 \), then \( \Psi_{T/n}(c) = a_\ast \) so that by Lemma 5.10 5.11
\[
0 \leq E \left[ 1(T \leq l_2) E \left[ \Psi_{1-n^{\beta-1}}(c) - \Psi_{T/n}(c) |F_T \right] |F_{l_1} \right] \leq \frac{c}{n} (l_2 - l_1) + o(1).
\]
Therefore
\[ E[(\Psi_{1-n^\beta-1}(c) - a^*)^+ | \mathcal{F}_{t_1}] \leq \frac{c}{n} \cdot (l_2 - l_1) + o(1) \leq C\epsilon. \]

On the other hand
\[ |E[\Psi_{1-n^\beta-1}(c) - a^* | \mathcal{F}_{t_1}]| \leq |E[\Psi_{1-n^\beta-1}(c) - \Psi_{1-\epsilon}(c) | \mathcal{F}_{t_1}]| + (a^* - \Psi_{1-\epsilon}(c)) \leq 2 \cdot \frac{c}{n} \cdot (l_2 - l_1) + (a^* - \Psi_{1-\epsilon}(c)). \]

Hence by using \(|a| = -a + 2a^+\) we have
\[
E[|\Psi_{1-n^\beta-1}(c) - a^*||\mathcal{F}_{t_1}|] = \left| \frac{c}{n} \cdot (l_2 - l_1) + (a^* - \Psi_{1-\epsilon}(c)) \right| \leq C\epsilon + (a^* - \Psi_{1-\epsilon}(c)).
\]

Let \(\kappa_\lambda := \kappa_0 + \frac{\lambda}{n}\) and let \(\theta^*_{n^\beta}(\kappa_\lambda)\) be the solution to the following equation.
\[
\theta^*_{n^\beta}(\kappa_\lambda) := \kappa_\lambda (n^\beta - n) + \frac{1}{2\kappa_\lambda} \int_{n^\beta}^{n} \Re \left( e^{2i\theta_s(\kappa_\lambda)} - 1 \right) a(n - s) F(Y_s) ds
\]
That is, \(\theta^*_{n^\beta}(\kappa_\lambda)\) is the Prüfer phase function solved from the right endpoint. By Sturm-Liouville theory
\[
\#\{\text{atoms of } \xi_n \text{ in } [\lambda_1, \lambda_2]\} = \#(\{2\theta_{n-n^\beta}(\kappa_\lambda_1) - 2\theta^*_{n^\beta}(\kappa_\lambda_1), 2\theta_{n-n^\beta}(\kappa_\lambda_2) - 2\theta^*_{n^\beta}(\kappa_\lambda_2)\} \cap 2\pi \mathbb{Z}) . \quad (5.4)
\]

**Lemma 5.14**
\[
\theta^*_{n^\beta}(\kappa_\lambda) - \theta^*_{n^\beta}(\kappa_0) \overset{P}{\rightarrow} 0
\]

**Proof.** By [2], Lemma 6.4 we have
\[
E[|\theta^*_{n^\beta}(\kappa_\lambda) - \theta^*_{n^\beta}(\kappa_0)|] \leq C \cdot \frac{t}{n} + \frac{1}{\sqrt{n}}.
\]
Setting $t = n^\beta$ yields the conclusion. \(\square\)

**Proof of Theorem 1.4**

Our goal is to show

\[ \sharp (\text{atoms of } \xi_n \text{ in } [0, \lambda], \cdots, [0, \lambda_d]) \rightarrow \frac{1}{2\pi} (\Psi_1(-\lambda_1), \cdots, \Psi_1(-\lambda_d)). \]

We show this convergence for $d = 1$, for the general case follow similarly. By Theorem 5.8 and Proposition 5.12,

\[ \Psi^{(n)}_{1-n^\beta-1}(\kappa_\lambda) \rightarrow \Psi_{1-}(\lambda). \] (5.5)

for some subsequence. Letting $\lambda_1 = 0$, $\lambda_2 = \lambda$ in (5.4) we have

\[ \sharp (\text{atoms of } \xi_n \text{ in } [0, \lambda]) \]
\[ = \sharp (\{2\theta_{n-n^\beta}(\kappa_0) - 2\theta^*_{n^\beta}(\kappa_0), 2\theta_{n-n^\beta}(\kappa_\lambda) - 2\theta^*_{n^\beta}(\kappa_\lambda)\} \cap 2\pi \mathbb{Z}). \]

The length of this interval is equal to, by Lemma 5.14 (5.5),

\[ 2 (\theta_{n-n^\beta}(\kappa_\lambda) - \theta^*_{n^\beta}(\kappa_\lambda)) - 2 (\theta_{n-n^\beta}(\kappa_0) - \theta^*_{n^\beta}(\kappa_0)) \]
\[ = \Psi_{1-n^\beta-1}(\lambda) + o(1) \, \mathbb{P} \rightarrow \Psi_{1-}(\lambda). \]

By conditioning on $Y_{n-n^\beta}$, we see that $\theta_{n-n^\beta}(\kappa_0)$ and $\theta^*_{n^\beta}(\kappa_0)$ are independent. Thus by Theorem 5.9 the left endpoint of this interval satisfies that its projection \(\{2\theta_{n-n^\beta}(\kappa_0) - 2\theta^*_{n^\beta}(\kappa_0)\} \cap 2\pi \mathbb{Z}\) to $[0, 2\pi)$ converges to the uniform distribution on $[0, 2\pi)$. Therefore

\[ \sharp (\text{atoms of } \xi_n \text{ in } [0, \lambda]) \rightarrow \Psi_{1-}(\lambda) \]

proving Theorem 1.4. \(\square\)

### 6 Appendix

In this section we recall basic tools used in this paper. The content below are borrowed from [2]. For $f \in C^\infty(M)$ let $R_{\beta} f := (L + i\beta)^{-1} f$ ($\beta > 0$), $R f := L^{-1}(f - \langle f \rangle)$. Then by Ito's formula,

\[ \int_0^t e^{i\beta s} f(X_s) ds = \left[ e^{i\beta s}(R_{\beta} f)(X_s) \right]_0^t + \int_0^t e^{i\beta s} dM_s(f, \beta) \]
\[ \int_0^t f(X_s) ds = \langle f \rangle t + [(R f)(X_s)]_0^t + M_t(f, 0). \]
$M_s(f, \beta), M_s(f, 0)$ are the complex martingales whose variational process satisfy

$$
\langle M(f, \beta), M(f, \beta) \rangle_t = \int_0^t [R\beta f, R\beta f](X_s) ds,
$$

$$
\langle M(f, \beta), \overline{M(f, \beta)} \rangle_t = \int_0^t [R\beta f, \overline{R\beta f}](X_s) ds
$$

$$
\langle M(f, 0), M(f, 0) \rangle_t = \int_0^t [Rf, Rf](X_s) ds,
$$

$$
\langle M(f, 0), \overline{M(f, 0)} \rangle_t = \int_0^t [Rf, \overline{Rf}](X_s) ds
$$

where

$$[f_1, f_2](x) := L(f_1 f_2)(x) - (L f_1)(x) f_2(x) - f_1(x)(L f_2)(x)
= (\nabla f_1, \nabla f_2)(x).$$

Then the integration by parts gives us the following formulas to be used frequently.

**Lemma 6.1**

(1) \[ \int_0^t b(s)e^{i\gamma s}e^{i\gamma \theta s} f(X_s) ds \]

\[ = \left[ b(s)e^{i\gamma \theta s}e^{i\beta s}(R\beta f)(X_s) \right]_0^t - \int_0^t b'(s)e^{i\gamma \theta s}e^{i\beta s}(R\beta f)(X_s) ds \]

\[ - \frac{i\gamma}{2\kappa} \int_0^t b(s)a(s) Re(e^{2i\theta s} - 1)e^{i\gamma \theta s}e^{i\beta s} F(X_s)(R\beta f)(X_s) ds \]

\[ + \int_0^t b(s)e^{i\beta s}e^{i\gamma \theta s} dM_s(f, \beta). \]

(2) \[ \int_0^t b(s)e^{i\gamma \theta s} f(X_s) ds \]

\[ = \langle f \rangle \int_0^t b(s)e^{i\gamma \theta s} ds \]

\[ + \left[ b(s)e^{i\gamma \theta s}(Rf)(X_s) \right]_0^t - \int_0^t b'(s)e^{i\gamma \theta s}(Rf)(X_s) ds \]

\[ - \frac{i\gamma}{2\kappa} \int_0^t a(s)b(s) Re(e^{2i\theta s} - 1)e^{i\gamma \theta s} F(X_s)(Rf)(X_s) ds \]

\[ + \int_0^t b(s)e^{i\gamma \theta s} dM_s(f, 0). \]
We will also use following notation for simplicity.

\[ g_\kappa := (L + 2i\kappa)^{-1}F, \quad g := L^{-1}(F - \langle F \rangle), \]

\[ h_{\kappa, \beta} := (L + 2i\beta\kappa)^{-1}Fg_\kappa \]

\[ M_s(\kappa) := M_s(F, 2\kappa), \quad M_s := M_s(F, 0), \]

\[ \tilde{M}^{(\beta)}_s(\kappa) := M_s(Fg_\kappa, \beta\kappa), \quad \tilde{M}_s := M_s(Fg_\kappa, 0). \]

**Lemma 6.2** Let \( \Psi_n, n = 1, 2, \cdots, \) and \( \Psi \) are continuous and increasing functions defined on a open set \( K \subset \mathbb{R} \) such that \( \lim_{n \to \infty} \Psi_n(x) = \Psi(x) \) pointwise. If \( y_n \in \text{Ran} \Psi_n, \ y \in \text{Ran} \Psi \) and \( y_n \to y \), then it holds that

\[ \Psi_n^{-1}(y_n) \xrightarrow{n \to \infty} \Psi^{-1}(y). \]

**Acknowledgement** This work is partially supported by JSPS grant Kiban-C no.22540140.

**References**

[1] Killip, R., Stoiciu, M., : Eigenvalue statistics for CMV matrices : from Poisson to clock via random matrix ensembles, Duke Math. 146, no. 3(2009),

[2] Kotani, S. and Nakano, F : Level statistics for one-dimensional random Schrödinger operator with random decaying potential, preprint, arXiv 1210.4224

[3] Kotani, S., Ushiroya, N. : One-dimensional Schrödinger operators with random decaying potentials, Comm. Math. Phys. 115(1988), 247-266.

[4] Kritchevski, E., Valkó, B., Virág, B., : The scaling limit of the critical one-dimensional random Sdhrödinger operators, Comm. Math. Phys. 314(2012), 775-806.

[5] Valkó, B. and Virág, V. : Continuum limits of random matrices and the Brownian carousel, Invent. Math. 177(2009), 463-508.

[6] Valkó, B. : private communication.