Discounting the Past*

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Abstract

Stochastic games with discounted payoff, introduced by Shapley, model adversarial interactions in stochastic environments where two players try to optimize a discounted sum of rewards. In this model, long-term weights are geometrically attenuated based on the delay in their occurrence. We propose a temporally dual notion—called past-discounting—where agents have geometrically decaying memory of the rewards encountered during a play of the game. We study objective functions based on past-discounted weight sequences and examine the corresponding stochastic games with liminf, discounted, and mean payoffs. For objectives specified as the limit inferior of past-discounted reward sequences, we show that positional determinacy fails and that optimal strategies may require unbounded memory. To overcome this obstacle, we study an approximate windowed objective based on the idea of using sliding windows of finite length to examine infinite plays. On the other hand, for objectives specified as the discounted and average limits of past-discounted reward sequences we establish determinacy in mixed stationary strategies in the setting of concurrent stochastic games and show how the values of these games may be computed via reductions to standard discounted and mean-payoff games.

1 Introduction

Time preference refers to the tendency of rational agents to place a higher value on the desirable outcomes received at an earlier time as compared to a later time. This phenomenon is often codified as discounted payoff in mathematical models from diverse disciplines such as economics and game theory, control theory, and reinforcement learning. It stands to reason, then, that for undesirable outcomes the time preference of rational agents may be dual: a tendency to regret less the undesirable outcomes experienced in the distant past compared to those encountered recently. We study a temporally dual notion of “future” discounting as past-discounting and optimization and games over such preferences.

Two-player zero-sum stochastic games provide a natural model for adversarial interactions between rational agents with competing objectives in uncertain settings. Beginning with an initial configuration of an arena, these games proceed in discrete time, and at each step the players—named Min and Max—concurrently choose (stochastically) from a set of state-dependent actions. Based on their choices, a scalar weight is determined with the interpretation that this quantity represents a “payment” from Min to Max. Given the current state and the players’ action pair, a probabilistic transition function determines the next state and the process repeats infinitely. The goal of Max is to maximize a given objective—typically specified as some functional into the real numbers—defined on infinite sequences of payments \(x = (x_n)_{n \geq 0}\), while the goal of Min is the opposite. The most popular objectives are the limit payoff, the discounted payoffs, which

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are well-defined for discount factors $\lambda \in [0, 1)$, and the mean payoff:

\[
L(x) \overset{\text{def}}{=} \lim_{n \to \infty} \inf x_n \quad \text{(Limit Payoff)}
\]

\[
D_\lambda(x) \overset{\text{def}}{=} \lim_{n \to \infty} \sum_{k=0}^{n} \lambda^k x_k, \quad \text{(Discounted Payoff)}
\]

\[
M(x) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} x_k. \quad \text{(Mean Payoff)}
\]

The discount factor $\lambda$ can be interpreted as the complement of a probability $(1 - \lambda)$ that the game will halt at any given point in time. At time $k$, the quantity transferred to Max from Min is scaled by the probability that the game continues: $\lambda^k$. The players are interested in optimizing, in expectation, the total accumulated reward before termination. For an infinite sequence $x = \langle x_n \rangle_{n \geq 0}$, the discounted sum $(1 - \lambda)D_\lambda(x)$ may be viewed as a weighted average—a convex combination, in fact—in which elements occurring earlier in the sequence are weighted more heavily. On the other hand, when agents prefer long-run rewards over initial rewards, a more traditional average—the mean payoff objective—is employed. In this case, each element contributes equally to the total, and transient elements become less significant as time progresses.

Shapley, in his seminal work [36], showed that stochastic games played on finite arenas with discounted objectives are determined in stationary strategies. In other words, there are optimal strategies for both players that need only consider the current state of play. Bewley and Kohlb erg showed that stochastic games with mean payoff objective are also determined in stationary strategies [6] and related the discounted and mean payoff objectives asymptotically [5].

**Discounting the Past.** Using Shapley’s discounting as a conceptual basis, we study *past-discounting* as a temporally dual notion. For a finite sequence of weights $\langle x_0, x_1, \ldots, x_n \rangle$, the past-discounted sum with discount factor $\gamma \in [0, 1)$, is defined by the formula $\sum_{k=0}^{n} \gamma^{n-k}x_k$, and was introduced by Alur, et al. [3].

To motivate the idea of past-discounting, consider a setting where the weights in a stochastic game arena characterize the performance of an agent. Under such an interpretation, rewards obtained by the agent at each stage represent the immediate response to their actions from the environment. In this sense, the total sum of a finite sequence of rewards encountered by the agent provides a quantification of their reputation over that period of time. This sum may be thought of as a performance evaluation of the agent done by an evaluator considering each past action with equal significance. In contrast, the past-discounted sum of the finite sequence represents an evaluation of the agent in which actions taken more recently are given greater significance. Here, the discount factor $\gamma$ can be seen as a parameter encoding the evaluator’s preference for recent performance compared to performances occurring long ago. If we suppose the cause of this preference to be that the evaluator has imperfect memory that worsens over time, then $\gamma$ may represent the rate at which the evaluator forgets or forgives.

If, for instance, the agent is a politician, the reward incurred at each step could correspond to their popularity rating, and the past-discounted sum of these ratings would then be their (weighted) average popularity, placing more significance on recent ratings. In this scenario, the past-discounted evaluation is more relevant than the total sum evaluation, since favorability of a politician depends on the public’s recollection of their past actions, which tends to decay with time. More generally, past-discounting makes sense whenever the agent in question can learn and adapt and the evaluator either has a time-decaying memory or a time-decaying perception of significance.

We propose the extension of this notion to infinite weight sequences. Perhaps the most straightforward way to define the past-discounted objective for an infinite sequence $\langle x_n \rangle_{n \geq 0}$ of payoffs is to consider the limit $\lim_{n \to \infty} \sum_{k=0}^{n} \gamma^{n-k}x_k$ of the past-discounted sums of finite prefixes of increasing length. Even for bounded sequences, however, this summation may fail to converge. As an example consider the bounded infinite sequence $(1, 2, 1, 2, 1, 2, \ldots)$. The past-discounted sum for every even step $k = 2n$ is given as

\[
2 \cdot \frac{1 - (\gamma^2)^n}{1 - \gamma^2} + \gamma \cdot \frac{1 - (\gamma^2)^n}{1 - \gamma^2},
\]
while, for every odd step \( k = 2n + 1 \), we have

\[
\frac{1 - (\gamma^2)^n}{1 - \gamma^2} + 2\gamma \cdot \frac{1 - (\gamma^2)^n}{1 - \gamma^2}.
\]

Hence, every even subsequence converges to \( \frac{2\gamma}{1 - \gamma^2} \), while the odd subsequences converge to \( \frac{1 + 2\gamma}{1 - \gamma^2} \), and the limit of the sequence as whole does not exist.

In light this observation, \( \lim_{n \to \infty} \sum_{k=0}^{n} \gamma^{n-k} x_k \) is ill-posed and does not define a function from bounded sequences to scalars. Instead, we study the following variants which are well defined over any bounded weight sequence \( x = \langle x_n \rangle_{n \ge 0} \):

\[
P_\gamma(x) \overset{\text{def}}{=} \liminf_{n \to \infty} \sum_{k=0}^{n} \gamma^{n-k} x_k \quad \text{(Past-Discounted Payoff)}
\]

\[
D_\lambda P_\gamma(x) \overset{\text{def}}{=} \lim_{n \to \infty} \sum_{k=0}^{n} \lambda^k \sum_{i=0}^{k} \gamma^{k-i} x_i \quad \text{(Discounted Past-Discounted Payoff)}
\]

\[
MP_\gamma(x) \overset{\text{def}}{=} \liminf_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \sum_{i=0}^{k} \gamma^{k-i} x_i \quad \text{(Mean Past-Discounted Payoff)}
\]

The latter two objectives correspond to the Abelian and Cesàro summations of the past-discounted weights.

**Past-Discounted Payoff (Example).** Consider the bounded infinite sequence \( x = (3, 4, 5)^\omega \) and the corresponding sequence of past-discounted sums:

\[
P_\gamma(x) = \langle 3, (3\gamma + 4), (3\gamma^2 + 4\gamma + 5), (3\gamma^3 + 4\gamma^2 + 5\gamma + 3), (3\gamma^4 + 4\gamma^3 + 5\gamma^2 + 3\gamma + 4), \ldots \rangle.
\]

It is easy to see that the sequence \( P_\gamma(x) \) does not converge. However, the subsequence containing every third element \( (k = 3n) \) converges to \( \frac{3 + 4\gamma + 5\gamma^2}{1 - \gamma^2} \). Likewise, the subsequence with \( k = 3n + 1 \) converges to \( \frac{1 + 2\gamma + 3\gamma^2}{1 - \gamma^2} \) and the subsequence with \( k = 3n + 2 \) converges to \( \frac{1 + 2\gamma + 3\gamma^2}{1 - \gamma^2} \).

**Discounted Past-Discounted Payoff (Example).** Consider the \( \lambda \)-discounted sum of sequence \( P_\gamma(x) \):

\[
D_\lambda P_\gamma(x) = 3 + \lambda (3\gamma + 4) + \lambda^2 (3\gamma^2 + 4\gamma + 5) + \lambda^3 (3\gamma^3 + 4\gamma^2 + 5\gamma + 3) + \lambda^4 (3\gamma^4 + 4\gamma^3 + 5\gamma^2 + 3\gamma + 4) + \ldots
\]

\[
= (3 + 3\gamma \lambda + 3\gamma^2 \lambda^2 + \ldots) + (4\lambda + 4\lambda^2 \gamma + 4\lambda^3 \gamma^2 + \ldots) + (5\lambda^2 + 5\lambda^3 \gamma + 5\lambda^4 \gamma^2 + \ldots)
\]

\[
+ (3\lambda^3 + 3\gamma \lambda + 3\gamma^2 \lambda^2 + \ldots) + \ldots
\]

\[
= 3(1 + \gamma \lambda + \gamma^2 \lambda^2 + \ldots) + 4\lambda(1 + \gamma \lambda + \gamma^2 \lambda^2 + \ldots) + 5\lambda^2(1 + \gamma \lambda + \gamma^2 \lambda^2 + \ldots)
\]

\[
+ 3\lambda^3(1 + \gamma \lambda + \gamma^2 \lambda^2 + \ldots) + \ldots
\]

\[
= \frac{3 + 4\lambda + 5\lambda^2 + 3\lambda^3 + \ldots}{(1 - \gamma \lambda)}
\]

\[
= \frac{3 + 4\lambda + 5\lambda^2}{(1 - \gamma \lambda)(1 - \lambda^2)}
\]

Notice that the discounted past-discounted sum is equal to to the discounted sum but with weights scaled by a factor \( \frac{1}{1 - \gamma \lambda} \). We show that this is not a mere coincidence, and prove that this equality holds also for Markov chains and stochastic games. In doing so, we reduce the problem of solving games with discounted past-discounted payoff to standard discounted games.

**Mean Past-Discounted Payoff (Example).** Finally, consider the mean payoff of the past-discounted sequence defined as the limit inf of the following sequence:

\[
\langle 3, \frac{3\gamma + 4}{2}, \frac{3\gamma^2 + 4\gamma + 5}{3}, \frac{3\gamma^3 + 4\gamma^2 + 5\gamma + 3}{4}, \frac{3\gamma^4 + 4\gamma^3 + 5\gamma^2 + 3\gamma + 4}{5}, \ldots \rangle.
\]
Based on the classical Tauberian results, we may be tempted to conjecture that that mean past-discounted payoff $MP_\gamma(x)$ equals $\lim_{\lambda \uparrow 1}(1 - \lambda)D_\lambda P_\gamma(x)$, i.e.

$$MP_\gamma(x) = \lim_{\lambda \uparrow 1}(1 - \lambda) \frac{3 + 4\lambda + 5\lambda^2}{(1 - \gamma\lambda)(1 - \lambda^2)} = \lim_{\lambda \uparrow 1}\frac{3 + 4\lambda + 5\lambda^2}{(1 - \gamma\lambda)(1 + \lambda + \lambda^2)} = \frac{3 + 4 + 5}{3(1 - \gamma)}$$

Indeed, we show that this conjecture holds. Moreover, we show that to compute the value of mean past-discounted payoff in a stochastic game, it suffices to scale the weights by a factor of $\frac{1}{1-\gamma}$ to compute the optimal value for the mean past-discounted games.

**Contributions.** In summary, the contributions of this paper are itemized below.

**Past-Discounted Payoff.** We study the properties of past-discounted payoffs and show that while they are prefix-independent, they are not submixing (in the sense of \[20\]). Hence, their positionality does not follow from well known results. We prove that past-discounted games are indeed not positionally determined and establish that optimal strategies for past-discounted games may require unbounded memory. We consider an approximation method, formulated in terms of sliding windows of finite length over the infinite sequence of rewards, following the work of \[18, 19, 7\], called window past-discounted payoffs and show how to compute their values.

**Discounted Past-Discounted Payoff.** We provide a reduction from the values for concurrent stochastic discounted past-discounted games to those for standard concurrent stochastic discounted games. As a result, the computational and strategic complexity are equivalent for these games.

**Mean Past-Discounted Payoff.** We provide a reduction from the values of concurrent stochastic mean past-discounted games to those of concurrent stochastic mean payoff games. Computational and strategic complexity are therefore equivalent for these games. We prove a Tauberian theorem which asymptotically relates the values of discounted past-discounted games and mean past-discounted games on concurrent stochastic arenas.

## 2 Preliminaries

Let $X^*$ be the set of all finite words and $X^\omega$ be the set of all countably infinite words over a finite set $X$. A discrete probability distribution over a (possibly infinite) set $X$ is a function $d : X \to [0,1]$ such that $\sum_{x \in X}d(x) = 1$ and the support $\text{supp}(d) \triangleq \{x \in X : d(x) > 0\}$ of $d$ is countable. Let $\text{dist}(X)$ denote the set of all discrete distributions over $X$. A distribution $d$ is a point distribution if $d(x) = 1$ for some $x \in X$.

We consider two-player zero-sum games on finite stochastic game arenas (SGAs) between two players—player Min and player Max—who concurrently choose their actions to move a token along the edges of a graph. The next state is determined by a probabilistic transition function based on the current state and the players’ selected actions. A stochastic game arena (SGA) $G$ is a tuple $(S, A_{\text{Min}}, A_{\text{Max}}, w, p)$ where:

- $S$ is a finite set of states,
- $A_{\text{Min}}$ and $A_{\text{Max}}$ are finite sets of actions for players Min and Max;
- $w : S \times A_{\text{Min}} \times A_{\text{Max}} \to \mathbb{R}$ is a weight function, and
- $p : S \times A_{\text{Min}} \times A_{\text{Max}} \to \text{dist}(S)$ is a probabilistic transition function.

We write $A_{\text{Min}}(s) \subseteq A_{\text{Min}}$ and $A_{\text{Max}}(s) \subseteq A_{\text{Max}}$ for the set of actions available to players Min and Max, respectively, at the state $s \in S$. For states $s, s' \in S$ and actions $(a, b) \in A_{\text{Min}}(s) \times A_{\text{Max}}(s)$, we write $p(s', s, a, b)$ for $p(s, a, b)(s')$. An SGA is said to be deterministic when $p(\cdot | s, a, b)$ is a point distribution, for all possible states and action pairs. An SGA is a perfect-information or turn-based arena if, for all $s \in S$, at least one of the sets $A_{\text{Min}}(s)$ and $A_{\text{Max}}(s)$ is a singleton. If an SGA is not turn-based it is called concurrent. An SGA where one of the players has only a single choice of action from every state is called a one-player arena, otherwise it is considered to be two-player. One-player stochastic arenas are equivalent to Markov decision processes, and one-player deterministic arenas are equivalent to weighted directed graphs.
A play on $G$ is an infinite sequence $\pi = \langle (s_n, a_n, b_n) \rangle_{n \geq 0} \in (S \times A_{\text{Min}} \times A_{\text{Max}})^{\omega}$ such that $p(s_{k+1} | s_k, a_k, b_k) > 0$ for all $k \geq 0$. A finite play is a sequence in $S \times ((A_{\text{Min}} \times A_{\text{Max}}) \times S)^{\ast}$. Let $\pi_k^n$ represent the length $n - k$ contiguously of $\pi$, starting at $(s_k, a_k, b_k)$ and ending at $(s_n, a_n, b_n)$. Denote by $\text{last}(\pi)$ the final tuple comprising the finite play $\pi$. We write $\text{Plays}^G$ and $f\text{Plays}^G$, respectively, for the set of all plays and finite plays on the SGA $G$ and $\text{Plays}^G(s)$ and $f\text{Plays}^G(s)$ for the respective subsets of these for which $s$ is in the initial state.

Starting from an initial state $s \in S$ of an SGA $G$, players Min and Max produce an infinite run by concurrently choosing state-dependent actions, and then moving to a successor state determined by the transition function. A strategy of player Min in $G$ is a function $\nu : f\text{Plays}^G \times S \to \text{dist}(A_{\text{Min}})$ such that, for all plays $\pi \in f\text{Plays}^G$, we have that $\text{supp}(\nu(\pi, s)) \subseteq A_{\text{Min}}(s)$. A strategy $\chi$ of player Max is defined analogously. A strategy is pure if its image is a point distribution wherever it is defined; otherwise, it is mixed. We say that a strategy $\sigma$ is stationary if $\sigma(\pi, s) = \sigma(\pi', s)$, for any plays $\pi$ and $\pi'$. Strategies that are not stationary are called history dependent. A strategy is positional if it is pure and stationary. Let $\Sigma_{\text{Min}}$ and $\Sigma_{\text{Max}}$ be the sets of all strategies for the players. If the arena is not clear from context, we write $\Sigma_{\text{Min}}^G$ and $\Sigma_{\text{Max}}^G$.

For an SGA $G$, a state $s \in S$, and strategy pair $(\nu, \chi) \in \Sigma_{\text{Min}} \times \Sigma_{\text{Max}}$, let $\text{Plays}^{\nu, \chi}(s)$ (resp. $f\text{Plays}^{\nu, \chi}(s)$) denote the set of infinite (resp. finite) plays in which player Min and Max play according to $\nu$ and $\chi$, respectively. Given a finite play $\pi \in f\text{Plays}^{\nu, \chi}(s)$, a basic cylinder set $\text{cyl}(\pi)$ is the set of infinite plays in $\text{Plays}^{\nu, \chi}(s)$ for which $\pi$ is a prefix. Using standard results from probability theory one can construct a probability space $(\text{Plays}^{\nu, \chi}(s), \mathcal{F}^{\nu, \chi}(s), \text{Pr}^{\nu, \chi}(s))$ where $\mathcal{F}^{\nu, \chi}(s)$ is the smallest $\sigma$-algebra generated by the basic cylinder sets and $\text{Pr}^{\nu, \chi}(s) : \mathcal{F}^{\nu, \chi}(s) \to [0, 1]$ is a unique probability measure such that a finite play $\pi = \langle (s_k, a_k, b_k) \rangle_{k=0}^n$ has probability $\text{Pr}^{\nu, \chi}(\pi) = \prod_{k=0}^n p(s_{k+1} | s_k, a_k, b_k) \cdot \nu(\pi_k^{k-1}, s_k) \cdot \chi(\pi_k^{k-1}, s_k)$.

A real-valued random variable $f : \text{Plays} \to \mathbb{R}$, otherwise known as an objective function or synonymously a payoff function determines the optimization objective of a particular game. The expression $\text{E}^{\nu, \chi}_s(f)$ denotes the expectation of $f$ with respect to $\text{Pr}^{\nu, \chi}$. Given a payoff function $f : \text{Plays} \to \mathbb{R}$ and an SGA $G$, the pair $(G, f)$ fully specifies a game in which the goal of Max is to maximize the expected value of $f$ over $G$, while the goal of Min is the opposite. For every state $s \in S$, define the upper value $\text{Val}^u(f, s)$ as the minimum payoff player Min can ensure irrespective of player Max’s strategy. Symmetrically, the lower value $\text{Val}^l(f, s)$ of a state $s \in S$ is the maximum payoff player Max can guarantee irrespective of player Min’s strategy. Symbolically,

$$\text{Val}^u(f, s) \overset{\text{def}}{=} \inf_{\nu \in \Sigma_{\text{Min}}} \sup_{\chi \in \Sigma_{\text{Max}}} \text{E}^{\nu, \chi}_s(f) \quad \text{and} \quad \text{Val}^l(f, s) \overset{\text{def}}{=} \sup_{\chi \in \Sigma_{\text{Max}}} \inf_{\nu \in \Sigma_{\text{Min}}} \text{E}^{\nu, \chi}_s(f).$$

The inequality $\text{Val}^l(f, s) \leq \text{Val}^u(f, s)$ holds for all two-player zero-sum stochastic games. A game is determined when, for every state $s \in S$, the lower value and upper value are equal. In this case, we say that the value of the game exists with $\text{Val}(f, s) = \text{Val}^l(f, s) = \text{Val}^u(f, s)$ for every $s \in S$. If it is not clear from context, we write $\text{Val}_G$ to specify the arena over which the value is taken. For strategies $\nu \in \Sigma_{\text{Min}}$ and $\chi \in \Sigma_{\text{Max}}$ of players Min and Max, we define their values as

$$\text{Val}^u(\nu, s) \overset{\text{def}}{=} \sup_{\chi \in \Sigma_{\text{Max}}} \text{E}^{\nu, \chi}_s(f) \quad \text{and} \quad \text{Val}^l(\chi, s) \overset{\text{def}}{=} \inf_{\nu \in \Sigma_{\text{Min}}} \text{E}^{\nu, \chi}_s(f).$$

A strategy $\nu_1$ of player Min is called optimal if $\text{Val}^u(\nu_1, s) = \text{Val}(f, s)$. Likewise, a strategy $\chi_1$ of player Max is optimal if $\text{Val}^l(\chi_1, s) = \text{Val}(f, s)$. We say that a game is positionally determined if both players have positional optimal strategies. Likewise, a game is determined in stationary strategies when both players have optimal stationary strategies.

**Definition 1.** For an SGA $G$, the following payoffs of player Min to player Max have been considered extensively in literature (See, for example [23]).

**Limit Payoff.** The limit payoff $L : \text{Plays} \to \mathbb{R}$ of a play is defined as follows:

$$L \overset{\text{def}}{=} \langle (s_n, a_n, b_n) \rangle_{n \geq 0} \Rightarrow \lim_{n \to \infty} \inf_{n \to \infty} w(s_n, a_n, b_n).$$

**Discounted Payoff.** The $\lambda$-discounted payoff $D_{\lambda} : \text{Plays} \to \mathbb{R}$ of a play, for a given discount factor $\lambda \in (0, 1)$, is defined as follows:

$$D_{\lambda} \overset{\text{def}}{=} \langle (s_n, a_n, b_n) \rangle_{n \geq 0} \Rightarrow \lim_{n \to \infty} \sum_{k=0}^n \lambda^k w(s_k, a_k, b_k).$$
Mean Payoff. The mean payoff $M : \text{Plays} \to \mathbb{R}$ of a play is given by the long-run average-sum of the weight sequence of the play:

$$M \overset{\text{def}}{=} \langle (s_n, a_n, b_n) \rangle_{n \geq 0} \mapsto \liminf_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} w(s_k, a_k, b_k).$$

We refer to SGAs with the limit payoff objective as stochastic liminf games, to SGAs with a discounted payoff objective as stochastic discounted games, and to SGAs with the mean payoff objective as stochastic mean payoff games.

To each game $(G, f)$, there is an associated decision problem called the threshold problem or, alternatively, the value problem, which is defined relative to a rational number $t \in \mathbb{Q}$ and an initial state $s$ in the arena. The threshold problem asks whether the value of the game from state $s$ is bounded below by $t$; that is, to decide if $\text{Val}(f, s) \geq t$. When referring to the computational complexity of various classes of games or of solving various classes of games, we mean the complexity of deciding the value problem.

The next theorem recalls state of the art results on strategic and computational complexity of stochastic games with liminf and limsup payoff.

**Theorem 1** (Limit Games). All liminf games and limsup games are determined [24, 34]. Liminf games and limsup games over one-player deterministic arenas, one-player stochastic arenas, and two-player deterministic arenas are in PTIME [14, 16]. Liminf games and limsup games over turn-based two-player stochastic arenas are in UP $\cap \text{coUP}$ [24]. Over any type of non-concurrent arena, liminf games and limsup games are positionally determined [14, 16].

The following theorem summarizes state of the art results on strategic and computational complexity of stochastic games with discounted and mean payoff.

**Theorem 2** (Discounted and Mean Payoff Games). Stochastic discounted games [26] and stochastic mean payoff games [6] are determined in mixed stationary strategies. Both are positionally determined for turn-based game arenas [28, 23]. Stochastic discounted games are in FIXP and are SQRT-SUM-hard [24]. Stochastic mean payoff games are in EXPTIME and are PTIME-hard [13]. For turn-based SGAs, both types of games are in UP $\cap \text{coUP}$ [14].

Stochastic mean payoff games are intimately connected to stochastic discounted games, as evidenced by the subsequent Tauberian theorem.

**Theorem 3** (Blackwell Optimality [3]). As $\lambda$ tends to 1 from below, the following equation holds for every state $s$:

$$\text{Val}(M, s) = \lim_{\lambda \nrightarrow 1} (1 - \lambda) \text{Val}(D_{\lambda}, s).$$

Moreover, when $\lambda$ is close to 1, strategies optimal for $D_{\lambda}$ are optimal for $M$.

### 3 Past-Discounted Games

**Definition 2** (Past-Discounted Payoffs). Let $G = (S, A_{\text{Min}}, A_{\text{Max}}, w, p)$ be an SGA, $\pi = \langle (s_n, a_n, b_n) \rangle_{n \geq 0}$ be an infinite play with weights $\langle w_n \rangle_{n \geq 0} = \langle w(s_n, a_n, b_n) \rangle_{n \geq 0}$, and $\gamma \in [0, 1)$ be a discount factor. For every $n \in \mathbb{N}$ the $n$-step past-discounted objective is given by the following equation.

$$P^n_{\gamma}(\pi) \overset{\text{def}}{=} \sum_{k=0}^{n} \gamma^{n-k} w_k$$

(Finite Past-Discounted Payoff)

The lower and upper past-discounted objectives are characterized, respectively, by the following equations.

$$P_{\gamma}(\pi) \overset{\text{def}}{=} \liminf_{n \to \infty} P^n_{\gamma}(\pi) = \liminf_{n \to \infty} \sum_{k=0}^{n} \gamma^{n-k} w_k$$

(Lower Past-Discounted Payoff)

$$P_{\gamma}(\pi) \overset{\text{def}}{=} \limsup_{n \to \infty} P^n_{\gamma}(\pi) = \limsup_{n \to \infty} \sum_{k=0}^{n} \gamma^{n-k} w_k$$

(Upper Past-Discounted Payoff)
Since the lower and upper past-discounted payoffs can mutually simulate each other (by negating the sequences of finite-step past-discounted payoffs), we present our results in terms of the lower past-discounted payoff and simply refer to it as the past-discounted payoff written as \( P \). This is without loss of generality and our results apply to the upper past-discounted payoff as well.

### 3.1 Properties of the Past-Discounted Payoff

In [26] Gimbert provides a convenient sufficient condition on objectives that ensures positional determinacy in some situations. If an objective function is both prefix-independent and submixing, as defined below, then there are optimal positional strategies for the objective over stochastic one-player arenas.

**Definition 3.** Suppose that \( f \) is an arbitrary objective function and that \( x = \langle x_n \rangle_{n \in \mathbb{N}} \) and \( y = \langle y_n \rangle_{n \in \mathbb{N}} \) are infinite sequences of finite words over the finite set of possible weights \( W \), ie. \( x_n, y_n \in W^* \) for all \( n \). Define the shuffle of these sequences as \( x \uplus y = \langle x_1, y_1, x_2, y_2, \ldots \rangle \). The objective \( f \) is

- **prefix-independent** if \( f(uv) = f(v) \), for any sequences \( u \in W^* \) and \( v \in W^\omega \),
- **submixing** if \( f(x \uplus y) \leq \max \{ f(x), f(y) \} \).

**Proposition 1.** The past-discounted payoffs are prefix-independent.

**Proof.** Suppose that \( x = \langle x_0, \ldots, x_m \rangle \) a finite sequence of weights, \( y = \langle y_n \rangle_{n \geq 0} \) is an infinite sequence of weights, and let \( xy = \langle x_0, \ldots, x_m, y_1, y_2, \ldots \rangle \) be the infinite sequence created by prepending \( x \) to \( y \). For any \( n \) greater than \( m \), the \( n \)-step finite past-discounted payoff on \( xy \) may be written as

\[
P^n_\gamma(xy) = \sum_{k=m}^{n} \gamma^{n-k} y_{k-m} + \sum_{k=0}^{m} \gamma^{n-k} x_k,
\]

by separating the portions of the summation concerning elements from \( x \) and \( y \). Manipulating the summation indices, we may rewrite this equation as

\[
P^n_\gamma(xy) = \sum_{k=0}^{n-m} \gamma^{n-m-k} y_k + \sum_{k=0}^{m} \gamma^{n-k} x_k.
\]

Factoring \( \gamma^{n-m} \) from the summation of \( x \) elements yields the equality \( \sum_{k=0}^{m} \gamma^{n-k} x_k = \gamma^{n-m} \sum_{k=0}^{m} \gamma^{m-k} x_k \), and this implies the equation

\[
P^n_\gamma(xy) = P^n_\gamma(y) + \gamma^{n-m} P^n_\gamma(x).
\]

Since \( m \) is fixed and finite, \( \liminf_{n \to \infty} \gamma^{n-m} P^n_\gamma(x) = 0 \). Taking the limit inferior of both sides of this equation yields that \( P_\gamma(xy) = P_\gamma(y) \), and thus we conclude that \( P_\gamma \) is prefix-independent. \( \square \)

**Proposition 2.** The past-discounted payoffs are not submixing.

**Proof.** Consider the sequence \( z = \langle 200, 2, 100, 1 \rangle^\omega \) as the shuffle of the sequences \( x = \langle 2, 1, 200, 100 \rangle^\omega \) and \( y = \langle 200, 100, 2, 1 \rangle^\omega \), which is valid since \( z \) can be produced as an interleaving of the elements of \( x \) and \( y \) that preserves the relative ordering of the terms from each, as illustrated below.

\[
\begin{align*}
2 & \quad 1 \quad 200 \quad 100 \\
200 & \quad 2 \quad 100 \quad 1 \quad 200 \quad 2 \quad 100 \quad 1
\end{align*}
\]

Because these sequences are periodic, we get that \( P_\gamma(z) \) is equal to

\[
\min \left\{ \frac{200\gamma^3 + 2\gamma^2 + 100\gamma + 1}{1 - \gamma^4}, \frac{\gamma^3 + 200\gamma^2 + 2\gamma + 100}{1 - \gamma^4}, \frac{100\gamma^3 + \gamma^2 + 200\gamma + 2}{1 - \gamma^4}, \frac{2\gamma^3 + 100\gamma^2 + \gamma + 200}{1 - \gamma^4} \right\}.
\]
and $P_\gamma(x) = P_\gamma(y)$ are equal to

$$
\min \left\{ \frac{2\gamma^3 + \gamma^2 + 200\gamma + 100}{1 - \gamma^4}, \frac{100\gamma^3 + 2\gamma^2 + \gamma + 200}{1 - \gamma^4}, \frac{200\gamma^3 + 100\gamma^2 + 2\gamma + 1}{1 - \gamma^4}, \frac{\gamma^3 + 200\gamma^2 + 100\gamma + 2}{1 - \gamma^4} \right\}.
$$

Setting $\gamma = \frac{1}{10}$, these expressions evaluate approximately to $P_\gamma(z) \approx 11.2211$ and $P_\gamma(x) = P_\gamma(y) \approx 2.4002$. Therefore $P_\gamma(z) > \max \{P_\gamma(x), P_\gamma(y)\}$, establishing that $P_\gamma(z)$ is not submixing for every discount factor. \qed

In fact, the sequences $x, y, z$ from the proof of Proposition 2 provide a counterexample to submixing of $P_\gamma$, for all discount factors in the unit interval, other than those very close to 1. This is illustrated by the figure to the right, which displays, using the sequences from the proof of Proposition 2, $\max \{P_\gamma(x), P_\gamma(y)\}$ in red and $P_\gamma(z)$ in blue, for each possible discount factor $\gamma \in [0, 1)$.

Our next result settles the matter of positional determinacy for past-discounted games by establishing that optimal strategies may require unbounded memory in even the simplest of contexts: games on one-player deterministic arenas.

**Theorem 4.** Past-discounted games are not determined in stationary strategies and optimal strategies may require unbounded memory.

**Proof.** We proceed by exhibiting a one-player deterministic arena on which the optimal strategy, at any point in time, depends upon the entire history of play. Without loss of generality, we fix the single player in this context as Min whose goal is to minimize the past-discounted payoff. Consider the one-player deterministic arena shown in Figure 1 on which there are exactly two positional strategies. One strategy chooses action $a$ from state $s_1$ while the other selects action $b$. These strategies generate plays $\pi_a$ and $\pi_b$, respectively, with values

$$
P_\gamma(\pi_a) = \min \left\{ \frac{4 - 2\gamma}{1 - \gamma^2}, \frac{-2 + 4\gamma}{1 - \gamma^2} \right\} \quad \text{and} \quad P_\gamma(\pi_b) = \frac{-1}{1 - \gamma}
$$

If $\gamma = \frac{1}{3}$, then $P_\gamma(\pi_a) = 0$ and $P_\gamma(\pi_b) = -2$. A play $\pi_\sharp$ can achieve a better value, namely $P_\gamma(\pi_\sharp) = -2 - \frac{\gamma}{1 - \gamma} = -3$, if it is consistent with a strategy that chooses action $a$ from state $s_1$ infinitely often and also eventually selects $n$ consecutive $b$ actions, for every $n \geq 0$. Any strategy satisfying these conditions generates a play $\pi_\sharp$ with weight sequence $\langle w_n \rangle_{n \geq 0}$ where, for every $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ with $n \leq m$, such that $w_m = -2$ and $w_{m-k} = -1$, for $1 \leq k \leq n$. As a result, it holds that

$$
P_\gamma^m(\pi_\sharp) = -2 - \frac{\gamma(1 - \gamma^n)}{1 - \gamma} + \gamma^{n+1}P_\gamma^{m-n}(\pi_\sharp),
$$

and we get that $P_\gamma(\pi_\sharp) = -2 - \frac{\gamma}{1 - \gamma} = -3$ in the limit when $\gamma = \frac{1}{3}$.

Since a strategy generating $\pi_\sharp$ must choose distinct actions from the same state at various points in the game, it cannot be positional. Because mixed stationary strategies are weighted combinations of positional strategies, their expected values fall between those of the positional strategies, and thus do not achieve better values. Therefore past-discounted games are not determined in stationary strategies, even on one-player deterministic arenas. Since all other types of arenas under consideration generalize this type, it follows that past-discounted games are not determined in stationary strategies over any type of arena. Moreover,
an optimal strategy must be able to generate all optimal $n$-step subsequences have been achieved thus far in order to determine the number of times action $b$ should be pumped the next time the player is in state $s_1$. Therefore, we conclude that there is no optimal strategy with finitely bounded memory.

\section{Window Past-Discounted Games}

We now introduce window past-discounted games as a means to approximate the values of past-discounted games. This payoff is based on the sequence of finite past-discounted sums of fixed length and considers only the last $\ell$ elements of the play at any given point. The value of $\ell$ is considered as the \textit{window length}.

\begin{definition}
Let $G = (S, A_{\text{Min}}, A_{\text{Max}}, w, p)$ be an SGA, $\pi = \{(s_n, a_n, b_n)\}_{n \geq 0}$ be an infinite play with weight sequence $\langle w_n \rangle_{n \geq 0} = \langle w(s_n, a_n, b_n) \rangle_{n \geq 0}$, and $\ell \in \mathbb{N}$ be a window length. The window-past discounted payoff is characterized by the following equation.

$$W_\ell P_\gamma(\pi) \overset{\text{def}}{=} \liminf_{n \to \infty} P_\gamma^{\ell}(\pi|_{n-\ell}) = \liminf_{n \to \infty} \sum_{k = \max(0, n - \ell)}^{n} \gamma^{n-k} w_k$$ (Window Past-Discounted Payoff)

The following proposition formalizes the relationship between past-discounted payoffs, window past-discounted payoffs, and the liminf payoff.

\begin{proposition}
We have the following equalities.
1. $\lim_{\ell \to \infty} W_\ell P_\gamma = P_\gamma$.
2. If $\ell = 0$, then $W_\ell P_\gamma = L$.

Our next result establishes determinacy of window past-discounted games by characterizing their values in terms of the values of certain liminf games.

\begin{theorem}
Window past-discounted games are determined. For any window past-discounted game $(G, W_\ell P_\gamma)$, there is a liminf game $(H, L)$ such that, for every state $s$ in $G$, there exists a state $s'$ in $H$ where

$$\text{Val}_G(W_\ell P_\gamma, s) = \text{Val}_H(L, s').$$

\end{theorem}

\begin{proof}
We construct an arena $H$ for a liminf game $(H, L)$ as the product $H = G \times \mathcal{M}^\ell$, where $\mathcal{M}^\ell = (\Sigma, Q, q_0, \delta)$ is a deterministic finite automaton such that:

- $\Sigma = S \times A_{\text{Min}} \times A_{\text{Max}},$
- $Q = \{q_0\} \cup \{\pi \in \text{Plays}^G : |\pi| \leq \ell\},$
- $\delta(q_0, sab) = sab,$
- $\delta(s_0a_0b_0 \ldots s_{k-1}a_{k-1}b_{k-1}, sab) = \begin{cases} s_0a_0b_0 \ldots s_{k-1}a_{k-1}b_{k-1}sab & \text{if } k < \ell, \\ s_1a_1b_1 \ldots s_{k-1}a_{k-1}b_{k-1}sab & \text{if } k = \ell. \end{cases}$

More precisely, $H = G \times \mathcal{M}^\ell = (S^\ell, A_{\text{Min}}^\ell, A_{\text{Max}}^\ell, w^\ell, p^\ell)$ where

- $S^\ell = \{(s, q) \in S \times Q : q = q_0 \text{ or } \text{last}(q) = s_k a_k b_k \text{ and } p(s | s_k, a_k, b_k) > 0\},$
- $A_{\text{Min}}^\ell(s, q) = A_{\text{Min}}(s)$ and $A_{\text{Max}}^\ell(s, q) = A_{\text{Max}}(s),$
- $w^\ell((s, s_0a_0b_0 \ldots s_{k-1}a_{k-1}b_{k-1}), a, b) = w(s, a, b) + \sum_{i=0}^{k-1} \gamma^{k-i} w(s_i, a_i, b_i),$
- $p^\ell((s', q') | (s, q), a, b) = \begin{cases} p(s' | s, a, b) & \text{if } \delta(q, sab) = q', \\ 0 & \text{otherwise.} \end{cases}$
The number of states in $H$ is on the order of $|S^f| = O(|S|2^f)$.

Claim: for every strategy pair $\nu, \chi$ over $G$ there is a strategy pair $\iota, \alpha$ over $H$ such that $E^\nu\chi_s(W_\iota P_\gamma) = E^\iota\alpha_{(s,q_0)}(L)$, for all states $s$ in $G$.

Consider $\pi \in \text{Plays}^G$ and the finite play
\[
\pi^{[n]}_0 = ((s_0, a_0, b_0), \ldots, (s_\ell, a_\ell, b_\ell), \ldots, (s_{n-\ell}, a_{n-\ell}, b_{n-\ell}), \ldots, (s_n, a_n, b_n)).
\]
We know from the construction above that, for every $1 \leq k \leq \ell$, there exists a state $(s_k, \pi^{[k-1]}_0) = (s_k, s_0a_0b_0 \ldots s_{k-1}a_{k-1}b_{k-1})$ in $H$ and for $k = 0$ we can associate the state $(s_0, q_0)$ in $H$. Likewise, for every $\ell < k \leq n$, there exists a state $(s_k, \pi^{[k-\ell]}_0)$ in $H$. From the definition of the transition function $\delta$ of $M^f$ and the state set $S^f$ of $H$, it follows that $\pi^{[n]}_0 \in \text{Plays}^G$ implies the existence of $w^{[n]}_0 \in \text{Plays}^H$, where
\[
w^{[n]}_0 = (((s_0, q_0), a_0, b_0), ((s_1, \pi^{[n]}_0), a_1, b_1), \ldots, ((s_\ell, \pi^{[\ell-1]}_0), a_\ell, b_\ell), \ldots, ((s_{n-\ell}, \pi^{[n-\ell-1]}_0), a_{n-\ell}, b_{n-\ell}), \ldots, ((s_n, \pi^{[n-1]}_0), a_n, b_n)).
\]
If $\pi^{[n]}_0 \in \text{Plays}^\nu\chi$ for arbitrary strategies $\nu, \chi$, then it follows from the definition of $\text{Pr}^\nu\chi$ that
\[
\text{Pr}^\nu\chi_0(\pi^{[n]}_0) = \prod_{k=1}^n p(s_k | s_{k-1}, a_{k-1}, b_{k-1}) \cdot \nu(\pi^{[k-1]}_0, s_k) \cdot \chi(\pi^{[k-1]}_0, s_k)(b_k).
\]
Likewise, if $w^{[n]}_0 \in \text{Plays}^{\iota, \alpha}$ for arbitrary strategies $\iota, \alpha$, then we have that
\[
\text{Pr}^{\iota, \alpha}_{(s_0, q_0)}(w^{[n]}_0) = \prod_{k=1}^n p^f((s_k, q_k) | (s_{k-1}, q_{k-1}), a_{k-1}, b_{k-1}) \cdot \iota(w^{[k-1]}_0, (s_k, q_k))(a_k) \cdot \alpha(w^{[k-1]}_0, (s_k, q_k))(b_k).
\]
By definition of the probabilistic transition function $p^f$ of $H$, we have the equality
\[
p(s_k | s_{k-1}, a_{k-1}, b_{k-1}) = p^f((s_k, q_k) | (s_{k-1}, q_{k-1}), a_{k-1}, b_{k-1}).
\]
Therefore, $\text{Pr}^{\nu, \chi}_{s_0}(\pi^{[n]}_0) = \text{Pr}^{\iota, \alpha}_{(s_0, q_0)}(w^{[n]}_0)$, for all $n \in \mathbb{N}$, precisely when
\[
\nu(\pi^{[n]}_0, s) = \iota(w^{[n]}_0, (s, \pi^{[n-\ell]}_{\max(0,n-1-\ell)})), \quad \chi(\pi^{[n]}_0, s) = \alpha(w^{[n]}_0, (s, \pi^{[n-\ell]}_{\max(0,n-1-\ell)})).
\]
(1)

hold for all $n \in \mathbb{N}$. These equations facilitate translation between arbitrary strategy pairs over arenas $G$ and $H$, and so there indeed exist strategies in $H$ for any strategies in $G$, and vice versa, that induce equivalent probability measures. In combination with the definition of $w^f$, this implies the equality of expectations for the strategies $\nu, \chi$ on $G$ with that for the strategies $\iota, \alpha$ on $H$. To see why, expand out the definition of $E^{\nu, \chi}_s(W_\iota P_\gamma)$ to get the following.

\[
E^{\nu, \chi}_s(W_\iota P_\gamma) = E^{\nu, \chi}_s\left(\liminf_{n \to \infty} \sum_{k=\max(0,n-\ell)}^n \gamma^{n-k}w(s_k, a_k, b_k)\right)
\]

Expanding the definition of $E^{\iota, \alpha}_{(s,q_0)}(L)$ yields an almost identical result.

\[
E^{\iota, \alpha}_{(s,q_0)}(L) = E^{\iota, \alpha}_{(s,q_0)}\left(\liminf_{n \to \infty} w^f(s_k, a_k, b_k)\right) = E^{\iota, \alpha}_{(s,q_0)}\left(\liminf_{n \to \infty} \sum_{k=\max(0,n-\ell)}^n \gamma^{n-k}w(s_k, a_k, b_k)\right)
\]

Because of the equivalence we have established between the probability measures $\text{Pr}^{\nu, \chi}$ and $\text{Pr}^{\iota, \alpha}$, assuming that $\nu, \iota$ and $\chi, \alpha$ satisfy eq. (1), it follows that these expectations are equivalent as well:

\[
E^{\nu, \chi}_s(W_\iota P_\gamma) = E^{\iota, \alpha}_{(s,q_0)}(L).
\]
Since we have proven the above claim and established, in eq. 1, a bijection between strategy pairs that preserves expectation, we know that

\[
\inf_{\nu} \sup_{\chi} \mathbb{E}^{\nu,\chi}_s (W_t P_s) = \inf_{\alpha} \sup_{(s, q_0)} \mathbb{E}^{\nu,\alpha}_s (L),
\]

\[
\sup_{\chi} \inf_{\nu} \mathbb{E}^{\nu,\chi}_s (W_t P_s) = \sup_{\alpha} \inf_{(s, q_0)} \mathbb{E}^{\nu,\alpha}_s (L).
\]

Furthermore, we know liminf games are determined, allowing us to infer that

\[
\inf_{\nu} \sup_{\chi} \mathbb{E}^{\nu,\chi}_s (W_t P_s) = \sup_{\chi} \inf_{\nu} \mathbb{E}^{\nu,\chi}_s (W_t P_s) = \text{Val}_H (L, (s, q_0)),
\]

from which it follows that window past-discounted games are determined with

\[
\text{Val}_C (W_t P_s, s) = \text{Val}_H (L, (s, q_0)).
\]

\[\square\]

From Theorems 1 and 5 we obtain the following corollary on the strategic and computational complexity of window past-discounted games.

**Corollary 1** (Complexity of Window Past-Discounted Games).

1. For any window past-discounted game over a turn-based arena, both players have pure optimal strategies requiring $O(\ell)$ memory.

2. Window past-discounted games are in EXPTIME over stochastic one-player arenas and deterministic turn-based two-player arenas. For fixed window length, these games are in PTIME.

3. Turn-based stochastic window past-discounted games are in NEXPTIME $\cap$ coNEXPTIME, and are in UP $\cap$ coUP for fixed window length.

**Remark 1.** Algorithmic aspects of liminf games and limsup games over concurrent two-player arenas have not been extensively studied as far as we know (cf. Section 4.7 of [12]). As a result, strategic complexity and computational complexity of the these games on concurrent arenas is unknown. Resolving these questions would also provide complexity bounds for window past-discounted games over concurrent arenas.

### 4 Discounted Past-Discounted Games

**Definition 5.** Let $G = (S, A_{\text{Min}}, A_{\text{Max}}, w, p)$ be an SGA, $\pi = \{ (s_n, a_n, b_n) \}_{n \geq 0}$ be an infinite play with weight sequence $\langle w_n \rangle_{n \geq 0} = \langle w(s_n, a_n, b_n) \rangle_{n \geq 0}$, and $\lambda, \gamma \in [0, 1)$ be discount factors. The discounted past-discounted payoff is characterized by the following equation.

\[
D_{\lambda} P_\gamma (\pi) \overset{\text{def}}{=} \lim_{n \to \infty} \sum_{k=0}^{n} \lambda^k P_{\gamma, k} (\pi) = \lim_{n \to \infty} \sum_{k=0}^{n} \lambda^k \sum_{i=0}^{k} \gamma^{k-i} w_i \quad \text{(Discounted Past-Discounted Payoff)}
\]

We now show that discounted past-discounted games are determined in stationary strategies by reducing the problem to computing optimal strategies for a standard discounted game on the same arena. Our primary tool in showing the reduction is a classical theorem from real analysis (see [32], for instance).

**Theorem 6** (Mertens’ Theorem). Let $a = \langle a_n \rangle_{n \geq 0}$ and $b = \langle b_n \rangle_{n \geq 0}$ be two sequences, and let $c = \langle c_n \rangle_{n \geq 0}$ be the Cauchy product $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ of $a$ and $b$. If $\lim_{n \to \infty} \sum_{k=0}^{n} a_k = A$ and $\lim_{n \to \infty} \sum_{k=0}^{n} b_k = B$ and at least one of the sequences $(\sum_{k=0}^{n} a_k)_{n \geq 0}$ and $(\sum_{k=0}^{n} b_k)_{n \geq 0}$ converges absolutely, then the limit of the Cauchy product exists such that $\lim_{n \to \infty} \sum_{n=0}^{N} c_n = AB$.

\[\text{See Appendix A for further details.}\]
Theorem 7. Given a state $s$ of some SGA and discount factors $\gamma, \lambda \in [0,1)$, it holds that

$$\text{Val} (D_\lambda P_\gamma, s) = \frac{1}{1 - \gamma \lambda} \text{Val} (D_\lambda, s).$$

Proof. To prove this theorem, it suffices to show that $\mathbb{E}_\nu^\chi (D_\lambda P_\gamma) = \frac{1}{1 - \gamma \lambda} \mathbb{E}_\nu^\chi (D_\lambda)$ holds for any pair of strategies $(\nu, \chi) \in \Sigma_{\text{Min}} \times \Sigma_{\text{Max}}$ and consistent play $\pi = \langle (s_n, a_n, b_n) \rangle_{n \geq 0}$. In turn, this reduces to proving the equation $D_\lambda P_\gamma (\pi) = \frac{1}{1 - \gamma \lambda} D_\lambda (\pi)$. This can be done by observing the following sequence of equalities, in which $w_k = w(s_k, a_k, b_k)$:

$$D_\lambda P_\gamma (\pi) = \lim_{n \to \infty} \sum_{k=0}^{n} \lambda^k \sum_{i=0}^{k} \gamma^{k-i} w_i = \lim_{n \to \infty} \sum_{k=0}^{n} \sum_{i=0}^{k} \lambda^k \gamma^{k-i} w_i = \lim_{n \to \infty} \sum_{k=0}^{n} \sum_{i=0}^{k} (\gamma \lambda)^{k-i} \lambda^i w_i = \left( \lim_{n \to \infty} \sum_{k=0}^{n} (\gamma \lambda)^k \right) \left( \lim_{n \to \infty} \sum_{k=0}^{n} \lambda^k w_k \right) = \frac{D_\lambda (\pi)}{1 - \gamma \lambda}. \quad (6)$$

The equality (2) is from the definition of $D_\lambda P_\gamma$. Equalities (3) and (4) follow from basic algebra. The equality (5) follows from Mertens’ theorem, since $\sum_{i=0}^{k} (\gamma \lambda)^{k-i} \lambda^i w_i$ is the Cauchy product of $(\gamma \lambda)^k_{k \geq 0}$ and $\lambda^k_{k \geq 0}$. Step (6) is a consequence of the definition of geometric series and $D_\lambda$. \qed

Corollary 2. For discounted past-discounted stochastic games:

1. the value is determined in mixed stationary strategies over concurrent arenas;
2. the value is determined in positional strategies over turn-based arenas;
3. the complexity of the value problem is equivalent to that of discounted games.

5 Mean Past-Discounted Games

Definition 6. Let $G = (S, A_{\text{Min}}, A_{\text{Max}}, w, p)$ be an SGA, $\pi = \langle (s_n, a_n, b_n) \rangle_{n \geq 0}$ be an infinite play with weight sequence $\langle w_n \rangle_{n \geq 0} = \langle w(s_n, a_n, b_n) \rangle_{n \geq 0}$, and $\gamma \in [0,1)$ be a discount factor. The mean past-discounted payoff is characterized by the following equation.

$$MP_\gamma (\pi) \overset{\text{def}}{=} \liminf_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} P_\gamma (\pi) = \liminf_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \sum_{i=0}^{k} \gamma^{k-i} w_i \quad \text{(Mean Past-Discounted Payoff)}$$

Taking a similar approach to Section 4, we show that mean past-discounted games are determined in stationary strategies. The proof, however, uses alternate techniques, as Mertens’ theorem fails to apply.

Theorem 8. Given a state $s$ of some SGA and discount factor $\gamma \in [0,1)$ it holds that

$$\text{Val} (MP_\gamma, s) = \frac{1}{1 - \gamma} \text{Val} (M, s).$$
Proof. To prove this theorem, it suffices to show that \( E^\nu,\chi_{\gamma}(MP\gamma) = \frac{1}{1 - \gamma} E^\nu,\chi(M) \) for every pair of strategies \((\nu, \chi) \in \Sigma_{\text{Min}} \times \Sigma_{\text{Max}}\) and consistent play \(\pi = \langle (s_n, a_n, b_n) \rangle_{n \geq 0}\). This amounts to establishing the equation \(MP\gamma(\pi) = \frac{1}{1 - \gamma} M(\pi)\), which is shown in the following sequence of equalities, where \(w_k = w(s_k, a_k, b_k)\):

\[
MP\gamma(\pi) = \lim \inf_{n \to \infty} \frac{1}{n + 1} \sum_{k=0}^{n} \sum_{i=0}^{k} \gamma^{k-i} w_i \\
= \lim \inf_{n \to \infty} \sum_{k=0}^{n} \frac{w_k(1 - \gamma^{n+1-k})}{(n + 1)(1 - \gamma)} \\
= \lim \inf_{n \to \infty} \left( \sum_{k=0}^{n} \frac{w_k}{(n + 1)(1 - \gamma)} - \sum_{k=0}^{n} \frac{\gamma^{n+1-k} w_k}{(n + 1)(1 - \gamma)} \right) \\
= \lim \inf_{n \to \infty} \sum_{k=0}^{n} \frac{w_k}{(n + 1)(1 - \gamma)} \\
= \frac{M(\pi)}{1 - \gamma}. 
\]

The first and the last equalities, (7) and (11), follow from definition of the payoff functions \(MP\gamma\) and \(M\). The equality (8) is justified by the proof of Lemma 1. Going from (8) to (9) is done via basic algebra. The equality (10) follows from Lemma 2 and the super-additivity of \(\lim \inf\): for infinite sequences \(\langle a_n \rangle_{n \geq 0}\) and \(\langle b_n \rangle_{n \geq 0}\) the inequality \(\lim \inf_{n \to \infty} (a_n + b_n) \geq \lim \inf_{n \to \infty} a_n + \lim \inf_{n \to \infty} b_n\) holds, and if either sequence converges, then \(\lim \inf_{n \to \infty} (a_n + b_n) = \lim \inf_{n \to \infty} a_n + \lim \inf_{n \to \infty} b_n\).

Lemma 1. For every play \(\pi = \langle (s_n, a_n, b_n) \rangle_{n \geq 0}\) and prefix of length \(n \geq 0\)

\[
\frac{1}{n + 1} \sum_{k=0}^{n} \sum_{i=0}^{k} \gamma^{k-i} w(s_i, a_i, b_i) = \sum_{k=0}^{n} \frac{w(s_k, a_k, b_k)(1 - \gamma^{n+1-k})}{(n + 1)(1 - \gamma)}.
\]

Proof. We proceed by induction on \(n\). (see Appendix B)

Lemma 2. For every play \(\pi = \langle (s_n, a_n, b_n) \rangle_{n \geq 0}\) the following equation holds:

\[
\lim \inf_{n \to \infty} \sum_{k=0}^{n} \frac{\gamma^{n+1-k} w(s_k, a_k, b_k)}{(n + 1)(1 - \gamma)} = 0.
\]

Proof. See Appendix C

Corollary 3. For mean past-discounted stochastic games:

1. the value is determined in mixed stationary strategies over concurrent arenas;
2. the value is determined in positional strategies over turn-based arenas;
3. the complexity of the value problem is equivalent to games with mean payoff.

The results from section B along with classical results for discounted and mean-payoff games allow us to extend the Tauberian theorem of [5] and relate asymptotically the values of discounted past-discounted games with the values of mean past-discounted games.

Theorem 9 (Tauberian Theorem).

\[
\text{Val}(MP\gamma, s) = \lim_{\lambda \uparrow 1} (1 - \gamma) \text{Val}(D\lambda P\gamma, s)
\]
Proof. By Theorem 7 we obtain the equation
\[ \lim_{\lambda \uparrow 1} (1 \gamma) \text{Val}(D\lambda P\gamma, s) = \lim_{\lambda \uparrow 1} \frac{\text{Val}(\lambda, s)}{1 \gamma}. \]
Applying Theorem 3 to the right-hand side of this equation leads to the following:
\[ \lim_{\lambda \uparrow 1} (1 \gamma) \text{Val}(D\lambda P\gamma, s) = \text{Val}(M, s) \frac{1}{1 \gamma}. \]
Finally, applying Theorem 8 yields the desired equivalence:
\[ \lim_{\lambda \uparrow 1} (1 \gamma) \text{Val}(D\lambda P\gamma, s) = \text{Val}(MP\gamma, s). \]

6 Conclusion

The discounted and mean-payoff objectives have played central roles in the theory of stochastic games. A multitude of deep results exist connecting these objectives 5, 6, 32, 4, 17, 14, 38, 39, 40 in addition to an extensive body of work on algorithms for solving these games and their complexity 24, 33, 34, 17, 22, 13.

Past discounted sums for finite sequences were studied in the context of optimization 3 and are closely related to exponential recency weighted average, a technique used in nonstationary multi-armed bandit problems 37 to estimate the average reward of different actions by giving more weight to recent outcomes. However, to the best of our knowledge, past-discounting has not been formally studied as a payoff function for stochastic games until the present work.

Discounted objectives have found significant applications in areas of program verification and synthesis 20, 8. Relatively recently—although the idea of past operators is quite old 27—a number of classical formalisms including temporal logics such as LTL and CTL and the modal \( \mu \)-calculus have been extended with past-tense operators and with discounted quantitative semantics 21, 1, 2. A particularly significant result 31 around LTL with classical boolean semantics is that, while LTL with past operators is no more expressive than standard LTL, it is exponentially more succinct. It remains open whether this type of relationship holds for other logics and their extensions by past operators when interpreted with discounted quantitative semantics 2.

Regret minimization 9 is a popular criterion in the setting of online learning where a decision-maker chooses her actions so as to minimize the average regret—the difference between the realized reward and the reward that could have been achieved. We posit that imperfect decision makers may view their regret in a past-discounted sense, since a suboptimal action in the recent past tends to cause more regret than an equally suboptimal action in the distant past. We hope that the results of this work spur further interest in developing foundations of past-discounted characterizations of regret in online learning and optimization.

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A Additional Details from the Proof of Theorem 5

Define $h : \Sigma_{\text{Min}}^G \cup \Sigma_{\text{Max}}^G \rightarrow \Sigma_{\text{Min}}^H \cup \Sigma_{\text{Max}}^H$ and $g : \Sigma_{\text{Min}}^H \cup \Sigma_{\text{Max}}^H \rightarrow \Sigma_{\text{Min}}^G \cup \Sigma_{\text{Max}}^G$ as the functions satisfying the following equations, for $\sigma \in \Sigma_{\text{Min}}^G \cup \Sigma_{\text{Max}}^G$ and $\varsigma \in \Sigma_{\text{Min}}^H \cup \Sigma_{\text{Max}}^H$, corresponding to eq. [1]:

$$\sigma(\pi^n_0, s) = h(\sigma) \left( \varpi^n_0 \left( s, \pi^{n-1}_{\max(0,n-1)} \right) \right)$$

$$g(\varsigma(\pi^n_0, s), \varsigma) \left( \varpi^n_0 \left( s, \pi^{n-1}_{\max(0,n-1)} \right) \right)$$

Let $h(\Sigma_{\text{Min}}^G), h(\Sigma_{\text{Max}}^G)$ be the image of $h$ on all strategies for the players over $G$, and let $g(\Sigma_{\text{Min}}^H), g(\Sigma_{\text{Max}}^H)$ be the image of $g$ on all strategies of the players over $H$.

For every $\nu \in \Sigma_{\text{Min}}^G$ there exists $h(\nu) \in \Sigma_{\text{Min}}^H$ and for every $\chi \in \Sigma_{\text{Max}}^G$ there exists $h(\chi) \in \Sigma_{\text{Max}}^H$ such that:

$$E^{\nu}_{s} \left( W_{\nu} P_{\nu} \right) = E^{h(\nu), h(\chi)}_{(s, q_0)} \left( L \right)$$

$$\sup_{\chi \in \Sigma_{\text{Max}}^G} E^{\nu}_{s} \left( W_{\nu} P_{\nu} \right) = \sup_{\chi \in h(\Sigma_{\text{Max}}^G)} E^{h(\nu), h(\chi)}_{(s, q_0)} \left( L \right)$$

$$\inf_{\nu \in \Sigma_{\text{Min}}^G} \sup_{\chi \in \Sigma_{\text{Max}}^G} E^{\nu}_{s} \left( W_{\nu} P_{\nu} \right) = \inf_{\nu \in h(\Sigma_{\text{Min}}^G)} \sup_{\chi \in h(\Sigma_{\text{Max}}^G)} E^{h(\nu), h(\chi)}_{(s, q_0)} \left( L \right)$$

Symmetrically, we have

$$E^{\nu}_{s} \left( W_{\nu} P_{\nu} \right) = E^{h(\nu), h(\chi)}_{(s, q_0)} \left( L \right)$$

$$\inf_{\nu \in \Sigma_{\text{Min}}^G} \sup_{\chi \in \Sigma_{\text{Max}}^G} E^{\nu}_{s} \left( W_{\nu} P_{\nu} \right) = \inf_{\nu \in h(\Sigma_{\text{Min}}^G)} \sup_{\chi \in h(\Sigma_{\text{Max}}^G)} E^{h(\nu), h(\chi)}_{(s, q_0)} \left( L \right)$$

$$\sup_{\chi \in \Sigma_{\text{Max}}^G} \inf_{\nu \in \Sigma_{\text{Min}}^G} E^{\nu}_{s} \left( W_{\nu} P_{\nu} \right) = \sup_{\chi \in h(\Sigma_{\text{Max}}^G)} \inf_{\nu \in h(\Sigma_{\text{Min}}^G)} E^{h(\nu), h(\chi)}_{(s, q_0)} \left( L \right)$$

$$\inf_{\nu \in \Sigma_{\text{Min}}^G} \inf_{\chi \in \Sigma_{\text{Max}}^G} E^{\nu}_{s} \left( W_{\nu} P_{\nu} \right) = \inf_{\nu \in h(\Sigma_{\text{Min}}^G)} \inf_{\chi \in h(\Sigma_{\text{Max}}^G)} E^{h(\nu), h(\chi)}_{(s, q_0)} \left( L \right)$$

Lines (12) and (15) follow from the arguments given in the main body of the proof of Theorem 5. Because of the fact that $h$ is bijective, we have that $h(\Sigma_{\text{Min}}^G) = \Sigma_{\text{Min}}^H$, for Min, and likewise for Max, mutatis mutandis. This provides justification for obtaining (14) from (13) and for obtaining (17) from (16) in the above derivations.

For every $\nu \in \Sigma_{\text{Min}}^H$ there exist $g(\nu) \in \Sigma_{\text{Min}}^G$ and for every $\chi \in \Sigma_{\text{Max}}^H$ there exists an $g(\chi) \in \Sigma_{\text{Max}}^G$ such that:

$$E^{g(\nu), g(\chi)}_{s} \left( W_{g(\nu)} P_{g(\nu)} \right) = E^{g(\nu), g(\chi)}_{(s, q_0)} \left( L \right)$$

$$\sup_{\chi \in g(\Sigma_{\text{Max}}^G)} E^{g(\nu), g(\chi)}_{s} \left( W_{g(\nu)} P_{g(\nu)} \right) = \sup_{\chi \in h(\Sigma_{\text{Max}}^G)} E^{g(\nu), g(\chi)}_{(s, q_0)} \left( L \right)$$

$$\inf_{\nu \in g(\Sigma_{\text{Min}}^G)} \sup_{\chi \in g(\Sigma_{\text{Max}}^G)} E^{g(\nu), g(\chi)}_{s} \left( W_{g(\nu)} P_{g(\nu)} \right) = \inf_{\nu \in h(\Sigma_{\text{Min}}^G)} \sup_{\chi \in h(\Sigma_{\text{Max}}^G)} E^{g(\nu), g(\chi)}_{(s, q_0)} \left( L \right)$$

$$\inf_{\nu \in g(\Sigma_{\text{Min}}^G)} \inf_{\chi \in g(\Sigma_{\text{Max}}^G)} E^{g(\nu), g(\chi)}_{s} \left( W_{g(\nu)} P_{g(\nu)} \right) = \inf_{\nu \in h(\Sigma_{\text{Min}}^G)} \inf_{\chi \in h(\Sigma_{\text{Max}}^G)} E^{g(\nu), g(\chi)}_{(s, q_0)} \left( L \right).$$
Symmetrically, we have
\[
E^\nu_{\pi(x)}(W_t P_x) = E^\nu_{\pi(x)}(L) = 0.
\]

\[
\inf_{\nu \in g(\Sigma^H_{\text{min}})} E^\nu_{\pi(x)}(W_t P_x) = \inf_{\nu \in \Sigma^H_{\text{min}}} E^\nu_{\pi(x)}(L)
\]
\[
\sup_{\chi \in g(\Sigma^H_{\text{max}})} \inf_{\nu \in \Sigma^H_{\text{max}}} E^\nu_{\pi(x)}(W_t P_x) = \sup_{\chi \in \Sigma^H_{\text{max}}} \inf_{\nu \in \Sigma^H_{\text{max}}} E^\nu_{\pi(x)}(L)
\]
\[
\sum_{\nu \in \Sigma^H_{\text{min}}} \nu g(\Sigma^H_{\text{min}}) = 0.
\]

Lines (18) and (21) follow from the arguments given in the main body of the proof of Theorem 5. Because of the fact that \( g \) is bijective, we have that \( \sum_{\nu \in \Sigma^H_{\text{min}}} \nu g(\Sigma^H_{\text{min}}) = 0 \), and likewise for Max, mutatis mutandis. This provides justification for obtaining (22) from (23) and for obtaining (27) from (26) in the above derivations.

**B Proof of Lemma 1**

For ease of notation, let \( w_k = w(s_k, a_k, b_k) \).

**Base case:** Lemma holds for \( n = 0 \).

Let \( n = 0 \), then
\[
\frac{1}{n+1} \sum_{k=0}^{n} \left( \sum_{i=0}^{k} \gamma^{k-i} w_i \right) = w_0 = \sum_{k=0}^{n} w_k (1 - \gamma^{n+1-k}) (n+1)(1 - \gamma).
\]

**Inductive case:** If Lemma holds for \( n - 1 \), then it also holds for \( n \).

Firstly, observe the following derivation:
\[
\frac{1}{n+1} \sum_{k=0}^{n} \left( \sum_{i=0}^{k} \gamma^{k-i} w_i \right) = \sum_{k=0}^{n} \frac{1}{n+1} \sum_{i=0}^{k} \gamma^{k-i} w_i
\]
\[
= \sum_{k=0}^{\nu - 1} \sum_{i=0}^{k} \gamma^{k-i} w_k + \sum_{k=0}^{n} \gamma^{n-k} w_k
\]
\[
= \frac{1}{n+1} \sum_{k=0}^{\nu - 1} \sum_{i=0}^{k} \gamma^{k-i} w_k + \frac{1}{n+1} \sum_{k=0}^{n} \gamma^{n-k} w_k.
\]

Notice that the expression within the parentheses matches exactly the left-hand side of Lemma for \( n - 1 \). By the inductive hypothesis, we may rewrite this as follows:
\[
\frac{1}{n+1} \sum_{k=0}^{n} \left( \sum_{i=0}^{k} \gamma^{k-i} w_i \right) = \left( \frac{n}{n+1} \sum_{k=0}^{n-1} w_k (1 - \gamma^{n-k}) \right) + \left( \frac{1}{n+1} \sum_{k=0}^{n} w_k \gamma^{n-k} \right).
\]

On the right-hand side, the \( n \) terms in the left summand cancel, and factoring \( (1 - \gamma) \) out of the denominator leaves
\[
\left( \frac{1}{1-\gamma} \sum_{k=0}^{n-1} w_k (1 - \gamma^{n-k}) \right) + \left( \frac{1}{n+1} \sum_{k=0}^{n} w_k \gamma^{n-k} \right).
\]

Now, factoring again by \( \frac{1}{1-\gamma} \) and by \( \frac{1}{n+1} \) yields
\[
\frac{\sum_{k=0}^{n-1} w_k (1 - \gamma^{n-k}) + \sum_{k=0}^{n} w_k \gamma^{n-k} (1 - \gamma)}{(n+1)(1 - \gamma)}.
\]
Multiplying the terms within the summations in the numerator, we get the following expression:

\[
\frac{\left(\sum_{k=0}^{n-1} w_k - w_k \gamma^{n-k}\right) + \left(\sum_{k=0}^{n} w_k \gamma^{n-k} - w_k \gamma^{n+1-k}\right)}{(n + 1)(1 - \gamma)}.
\]

Taking advantage of additive cancelation in the numerator, we obtain

\[
\frac{\left(\sum_{k=0}^{n-1} w_k\right) + w_n - \left(\sum_{k=0}^{n} w_k \gamma^{n-k}\right)}{(n + 1)(1 - \gamma)}.
\]

With some simple regrouping of terms, the above may be rewritten as

\[
\frac{\sum_{k=0}^{n} w_k - \sum_{k=0}^{n} w_k \gamma^{n-k}}{(n + 1)(1 - \gamma)},
\]

and subsequently

\[
\frac{\sum_{k=0}^{n} w_k (1 - \gamma^{n+1-k})}{(n + 1)(1 - \gamma)}.
\]

Finally, by the linearity of summation, we obtain the desired equality:

\[
\frac{1}{n + 1} \sum_{k=0}^{n} \left( \sum_{i=0}^{k} \gamma^{k-i} w_i \right) = \sum_{k=0}^{n} \frac{w_k (1 - \gamma^{n+1-k})}{(n + 1)(1 - \gamma)}.
\]

\(\square\)

C Proof of Lemma 2

By assumption the sequence \(w(\pi) = (w(s_n, a_n, b_n))_{n \geq 0} = (w_n)_{n \geq 0}\) is bounded, and so the values \(\overline{w} = \sup w(\pi)\) and \(\underline{w} = \inf w(\pi)\) are well-defined as greatest and least scalar values occurring anywhere in \(w(\pi)\).

The following derivation shows that \(\sum_{k=0}^{n} \frac{w_k \gamma^{n+1-k}}{(n + 1)(1 - \gamma)}\) is bounded above by zero.

\[
\lim_{n \to \infty} \sum_{k=0}^{n} \frac{\overline{w} \gamma^{n+1-k}}{(n + 1)(1 - \gamma)} = \frac{1}{1 - \gamma} \lim_{n \to \infty} \frac{1}{(n + 1)(1 - \gamma)} \sum_{k=0}^{n} \frac{\overline{w} \gamma^{n+1-k}}{1 - \gamma} = \frac{1}{1 - \gamma} \lim_{n \to \infty} \frac{1}{n + 1} \left( \frac{\overline{w}(1 - \gamma^{n+1})}{1 - \gamma} \right) = \frac{1}{1 - \gamma} \cdot \frac{\overline{w}}{1 - \gamma} = 0
\]

Similarly, we may obtain an identical equation using \(\inf w(\pi)\):

\[
\lim_{n \to \infty} \sum_{k=0}^{n} \frac{\underline{w} \gamma^{n+1-k}}{(n + 1)(1 - \gamma)} = 0
\]

showing that \(\sum_{k=0}^{n} \frac{w_k \gamma^{n+1-k}}{(n + 1)(1 - \gamma)}\) is bounded below by zero as well. Since this sequence is bounded on both sides by zero, it must converge to zero.

\(\square\)