DEDEKIND $\eta$-FUNCTION AND QUANTUM GROUPS

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Abstract. In this paper we realize some powers of Dedekind $\eta$-function as the trace of an operator on quantum coordinate algebras.

1. Introduction

1.1. History. The partition function $p(n)$ of a positive integer $n$ and its numerous variants have a long history in combinatorics and number theory. A natural method to study these functions defined on the set of integers is considering their generating functions (for example: $\psi(x) = \sum_{n \geq 0} p(n) x^n$) to study their analytical properties, the algebraic equation they satisfy or the (quasi-)symmetries under group actions and so on.

In the case of partition function, although $p(n)$ augments rapidly and it is not possible to express $\psi(x)$ in a compact form, its inverse $\psi(x)^{-1}$, which seems to be more complicated, simplifies the story by the formula

$$\psi(x)^{-1} = \prod_{n \geq 1} (1-x^n).$$

This inverse of $\psi(x)$, denoted by $\varphi(x)$, is a fundamental object in mathematics as many modular forms can be constructed starting from $\varphi(x)$. For example, $\eta(x) = x^{\frac{1}{24}} \varphi(x)$ is the Dedekind $\eta$-function and $\Delta(x) = \eta(x)^{24}$ is a modular form of weight 12 whose expansion into power series in $x$ gives the famous Ramanujan’s $\tau$-function as coefficients.

Various powers of $\varphi(x)$ are studied by Euler, Jacobi and some other people in various domains of mathematics such as combinatorics, number theory, $\theta$-functions, index theorems and so on. As an example, Euler showed a relation between $\varphi(x)$ and the pentagon numbers

$$\varphi(x) = \sum_{n \in \mathbb{Z}} (-1)^n x^{\frac{3n^2-n}{2}}$$

and then Jacobi deduced the expression of $\varphi(x)^3$ by triangle numbers in his study of elliptic functions:

$$\varphi(x)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) x^\frac{n(n+1)}{2}.$$
some affine root systems. As an example, the Jacobi identity above can be interpreted through combinatorial informations arising from the affine root system of type $A_1$.

To be more precise, for any reduced root system on a finite dimensional real vector space $V$ with the standard bilinear form $(\cdot,\cdot)$, we can associate to it a complex Lie algebra $\mathfrak{g}$. The following formula (formula (0.5) in [13]) is obtained as the specialization of the Weyl denominator formula:

$$(1) \quad \eta(x)^d = \sum_{\mu \in M} d(\mu) x^{(\mu+\rho,\mu+\rho)/2g},$$

where $M$ is some set contained in the set of dominant integral weights, $d = \dim \mathfrak{g}$, $d(\mu)$ is the dimension of the irreducible representation of $\mathfrak{g}$ of highest weight $\mu$, $g = \frac{1}{2}((\phi + \rho, \phi + \rho) - (\rho, \rho))$, $\phi$ is the highest root of $\mathfrak{g}$ and $\rho$ is half of the sum of positive roots.

When the root system in the MacDonald’s identity [11] arises from a complex compact simple Lie group $G$ which is moreover simply connected, B. Kostant [8] made the set $M$ precise by connecting it with the trace of a Coxeter element $c$ in the Weyl group $W$ acting on the subspace of weight zero $V_1(\lambda)_0$ in the irreducible representation $V_1(\lambda)$ associated to a dominant integral weight $\lambda \in \mathcal{P}_+$. If the Lie group is simply laced (i.e., type A,D,E), the formula due to Kostant reads:

$$(2) \quad \eta(x)^{\dim G} = \sum_{\lambda \in \mathcal{P}_+} \Tr(c, V_1(\lambda)_0) \dim V_1(\lambda) x^{(\lambda+\rho,\lambda+\rho)}.$$

A similar result holding for general $G$ can be found in [8] (see Theorem 2).

These formulas have various explications by using different tools in Lie theory, a summary of corresponding results can be found in a Bourbaki seminar talk [4] by M. Demazure.

1.3. Quantum groups and representations. Quantum groups (quantized enveloping algebras) appear in the middle of eighties after the work of Drinfeld and Jimbo in the aim of finding solutions of the Yang-Baxter equation; it can be looked as deformations (of Hopf algebras) of classical enveloping algebras associated to symmetrizable Kac-Moody Lie algebras.

This quantization procedure deforms not only the enveloping algebras themselves but also structures related to them: integrable representations, Weyl groups and so on. Moreover, some new structures and tools appear only after this process: $R$-matrices, canonical (crystal) bases, integral forms, specialization to roots of unity and so on.

The appearance of the parameter $q$ in the quantum groups enriches the internal structure of the enveloping algebra as the latter can be recovered from the former by specializing $q$ to 1. For example, this one-dimensional freedom allows us to separate some kinds of knots or links in labeling different crossings by this parameter.

1.4. Quantum Weyl group. The Weyl group associated to a finite dimensional simple Lie algebra reveals its internal symmetries by permuting root spaces. Moreover, it acts simultaneously on the integrable representations which makes it possible to give a classification of them.
In the quantization procedure mentioned above, when algebras and their representations are deformed, it is natural to study the behaviors of automorphism groups under this procedure: for instance, the Weyl group.

The quantization of Weyl groups acting on integrable representations is archived after the work of Kirillov-Reshetikhin [7] and Levendorski-Soibelman [10] in aim of giving an explicit formula of the R-matrix. As this procedure arises from the deformation of the Poisson-Lie group structure [10], it is compatible with the whole quantization picture. To be more precise, the action of Weyl groups on the integrable representations are deformed in a way preserving the Coxeter commutation relations but increasing the order of generators which results a lift of the Weyl groups to Artin braid groups.

This tool is essential in the study of quantum groups: constructions of PBW basis and R-matrices are direct from the action of quantum Weyl groups. Moreover, the conjugation by the quantum Weyl group gives an action of the Artin braid group on the quantized enveloping algebras, which is shown in [7], [10] and [15] to agree with Lusztig's automorphisms $T_i$.

1.5. **Main results.** The main objective of this paper is to prove identities in the spirit of formulae (1) and (2) in the framework of quantum groups. It is surprising that powers of Dedekind $\eta$-function can be expressed as a trace of an explicit operator on quantum coordinate algebras, which gives compact forms of identities cited above.

Let $g$ be a finite dimensional simple Lie algebra and $U_q(g)$ the associated quantum group over $\mathbb{C}(q)$. The Artin braid group $B_g$ associated to the Weyl group $W$ of $g$ acts on the irreducible representation $V(\lambda)$ of $U_q(g)$ with dominant weight $\lambda \in P_+$. Let $\{\sigma_1, \cdots, \sigma_l\}$ be the set of generators of $B_g$, $\Pi = \sigma_1 \cdots \sigma_l$ be a Coxeter element and $h$ be the Coxeter number of the Weyl group. Then $\Pi \otimes \text{id}$ acts on the quantum coordinate algebra $\mathbb{C}_q[G] = \bigoplus_{\lambda \in P_+} V(\lambda) \otimes V(\lambda)^*$ componentwise and we obtain finally

**Theorem.** The following identity holds:

$$\text{Tr}(\Pi \otimes \text{id}, \mathbb{C}_q[G]) = \left( \prod_{i=1}^l \varphi(q^{(a_i, \alpha_i)}) \right)^{h+1}.$$ 

These formulas serve as new interpretations of results due to Macdonald and Kostant in the classical case.

1.6. **Constitution of this paper.** After giving recollections on quantum groups and quantum Weyl groups in Section 2 and 3, we explain the relation between quantum Weyl groups and $R$-matrices in Section 4. Section 5 is devoted to computing the action of the centre of the Artin braid group on the irreducible representations of Lie algebra $g$. This will lead to the main theorem in Section 6.

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2. Quantum Groups

This section is devoted to giving some recollections on different definitions of quantum groups.

2.1. Notations. We fix notations for Lie algebras and their representations.

1. \( g \) is a finite dimensional simple Lie algebra with a fixed Cartan subalgebra \( \mathfrak{h} \). We let \( l = \text{dim} \mathfrak{h} \) denote the rank of \( g \) and \( I \) be the index set \( \{1, \ldots, l\} \).

2. \( \Phi : g \times g \to \mathbb{C} \) is the Killing form given by \( \Phi(x, y) = \text{Tr}(ad x ad y) \) for \( x, y \in g \).

3. \( \Delta (\Delta_+) \subset \mathfrak{h}^* \) is the set of (positive) roots of \( g \).

4. \( \Pi = \{\alpha_1, \ldots, \alpha_l\} \) is the set of simple roots of \( g \).

5. \( C = (c_{ij})_{l \times l} \), where \( c_{ij} = 2\Phi(\alpha_i, \alpha_j)/\Phi(\alpha_i, \alpha_i) \), is the Cartan matrix of \( g \).

6. \( W \) is the Weyl group of \( g \) generated by simple reflections \( s_i : \mathfrak{h}^* \to \mathfrak{h}^* \) for \( i \in I \) where \( s_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i \).

7. A Coxeter element is a product of all simple reflections, Coxeter elements are conjugate in \( W \).

8. \( D = \text{diag}(d_1, \ldots, d_l) \) is the diagonal matrix with integers \( d_i \) relatively prime such that \( A = DC \) is a symmetric matrix.

9. \( (\cdot, \cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{Q} \) is the normalized bilinear form on \( \mathfrak{h}^* \) such that \( (\alpha_i, \alpha_j) = a_{ij} \).

10. \( \mathcal{Q} = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_l \) is the root lattice and \( \mathcal{Q}_+ = \mathbb{N}\alpha_1 + \cdots + \mathbb{N}\alpha_l \).

11. \( \{\varpi_1, \ldots, \varpi_l\} \) is the set of fundamental weights in \( \mathfrak{h}^* \) such that \( (\varpi_i, \alpha_j) = \delta_{ij} \) where \( \delta_{ij} \) is the Kronecker notation.

12. \( \mathcal{P} = \mathbb{Z}\varpi_1 + \cdots + \mathbb{Z}\varpi_l \) is the weight lattice and \( \mathcal{P}_+ = \mathbb{N}\varpi_1 + \cdots + \mathbb{N}\varpi_l \) is the set of dominant integral weights.

13. For \( \lambda \in \mathcal{P}_+ \), \( V_1(\lambda) \) is the finite dimensional irreducible representation of \( g \) of highest weight \( \lambda \).

2.2. Definition. From now on, we suppose that \( q \) is a variable and \( q_i = q^{d_i} \). The \( q \)-numbers are defined by

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_{q^1} = \prod_{i=1}^{n} [i]_q.
\]

**Definition 1.** The quantized enveloping algebra (quantum group) \( U_q(g) \) is the associative \( \mathbb{C}(q) \)-algebra with unit generated by \( E_i, F_i, K_i, K_i^{-1} \) for \( i \in I \) and relations

\[
K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad K_iE_jK_i^{-1} = q_i^{c_{ij}}E_j, \quad K_iF_jK_i^{-1} = q_i^{-c_{ij}}F_j, \quad [E_i, F_j] = \delta_{ij}K_i - K_i^{-1} \frac{K_j - K_j^{-1}}{q_i - q_i^{-1}},
\]

for \( i \neq j \in I \),

\[
\sum_{r=0}^{1-c_{ij}} \left[ 1 - \frac{c_{ij}}{r} \right]_q E_i^{1-c_{ij}-r} E_j E_i^r = 0, \quad \sum_{r=0}^{1-c_{ij}} \left[ 1 - \frac{c_{ij}}{r} \right]_q F_i^{1-c_{ij}-r} F_j F_i^r = 0.
\]
There exists a unique Hopf algebra structure on $U_q(\mathfrak{g})$: for $i \in I$,
\[
\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\mp 1}, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,
\]
\[
\varepsilon(K_i^{\pm 1}) = 1, \quad \varepsilon(E_i) = 0, \quad \varepsilon(F_i) = 0,
\]
\[
S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i, \quad S(K_i) = K_i^{-1}, \quad S(K_i^{-1}) = K_i.
\]

**Remark 1.** This definition works for any symmetrizable Kac-Moody Lie algebras, our restriction is determined by the nature of the problem.

For $\lambda \in \mathcal{P}_+$, we let $V(\lambda)$ denote the finite dimensional irreducible representation of $U_q(\mathfrak{g})$ of highest weight $\lambda$ and type 1.

The following normalized generators will also be used
\[
E_i^{(n)} = \frac{E_i^n}{[n]_q!}, \quad F_i^{(n)} = \frac{F_i^n}{[n]_q!}.
\]

**2.3. An $\hbar$-adic version.** There is an $\hbar$-adic version of the quantum group which is the original definition of Drinfeld’s. As we will switch between these two versions of a quantum group several times in the later discussion, the definition is recalled in this subsection.

Let $\hbar$ be a variable and $\mathbb{C}[[\hbar]]$ be the ring of formal series in the parameter $\hbar$.

**Definition 2.** The $\hbar$-adic version of a quantized enveloping algebra $U_\hbar(\mathfrak{g})$ is the associative algebra with unit over $\mathbb{C}[[\hbar]]$, generated by $X_i^+, X_i^-, H_i$ for $i \in I$ and relations
\[
[H_i, H_j] = 0, \quad [H_i, X_j^+] = a_{ij} X_j^+, \quad [H_i, X_j^-] = -a_{ij} X_j^-,
\]
\[
[X_i^+, X_j^-] = \delta_{ij} \frac{e^{\hbar d_i H_i} - e^{-\hbar d_i H_i}}{e^{\hbar d_i} - e^{-\hbar d_i}},
\]
for $i \neq j \in I$,
\[
\sum_{r=0}^{1-c_{ij}} \left[ 1 - \frac{c_{ij}}{r} \right]_{q_i} (X_i^+)_{1-c_{ij}-r} X_j^\pm (X_i^\mp)^r = 0.
\]

There exists a unique Hopf algebra structure on $U_\hbar(\mathfrak{g})$:
\[
\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta(X_i^+) = X_i^+ \otimes e^{\hbar d_i H_i} + 1 \otimes X_i^+,
\]
\[
\Delta(X_i^-) = X_i^- \otimes 1 + e^{-\hbar d_i H_i} \otimes X_i^-,
\]
\[
\varepsilon(X_i^+) = 0, \quad \varepsilon(X_i^-) = 0, \quad \varepsilon(H_i) = 0,
\]
\[
S(X_i^+) = -X_i^+ e^{-\hbar d_i H_i}, \quad S(X_i^-) = -e^{\hbar d_i H_i} X_i^-, \quad S(H_i) = -H_i.
\]

**Remark 2.**

(1) To obtain $U_q(\mathfrak{g})$ inside $U_\hbar(\mathfrak{g})$, it suffices to take $q = e^\hbar$ and $K_i = e^{\hbar d_i H_i}$.

(2) The advantage of working in the $\hbar$-adic framework is that the $R$-matrix can be well defined when a completion of the tensor product is properly chosen. But as a disadvantage, we could not specialize $U_\hbar(\mathfrak{g})$ to any complex number except 0.
2.4. **Specialization.** Let $U(g)$ be the enveloping algebra associated to $g$ with generators $e_i, f_i, h_i$ for $i \in I$.

It should be remarked that $U_q(g)$ has a $\mathbb{Z}[q, q^{-1}]$-form which is called an integral form (for example, see Chapter 9 in [3] for details). This integral form allows us to specialize $U_q(g)$ to any non-zero complex number. We let $\lim_{q \to 0} U_q(g)$ denote the specialized Hopf algebra. It is well known that $\lim_{q \to 0} U_q(g)$ is isomorphic to $U(g)$.

To be more precise, under the specialization $q \to 1$, $K_i$ is sent to $1$ and $[K_i; 0] = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ gives $h_i$ for $i \in I$.

Moreover, finite dimensional representations of $U_q(g)$ can be specialized: for example, when $q$ tends to $1$, the representation $V(\lambda)$ will be specialized to $V_1(\lambda)$.

3. **Quantum Weyl groups**

This section is devoted to giving a summary for the definition of quantum Weyl groups in the ordinary and $\hbar$-adic cases. Then we discuss their relations and specializations.

3.1. **Braid groups associated to Weyl groups.** We start from considering some specific elements in the Artin braid group associated to a Weyl group.

For $i, j \in I$, if the product $c_{ij}c_{ji} = 0, 1, 2, 3, 4$, we let $m_{ij} = 2, 3, 4, 6, \infty$ respectively.

**Definition 3.** The Artin braid group $B_g$ associated to the Weyl group $W$ of $g$ is a group generated by $\sigma_1, \ldots, \sigma_l$ and relations

$$\sigma_i \sigma_j \cdots \sigma_i \sigma_j = \sigma_j \sigma_i \cdots \sigma_j \sigma_i,$$

where lengths of words in both sides are $m_{ij}$.

For example, if the Lie algebra $g$ is of type $A_l$, then $m_{ij} = 3$ if $|i - j| = 1$, otherwise $m_{ij} = 2$. The Artin group $B_g$ is the usual braid group $B_{l+1}$.

We let $\Pi = \sigma_1 \cdots \sigma_l$ be a product of generators in $B_g$ and call it a Coxeter element. We let $h$ denote the Coxeter number of the Weyl group $W$. The following proposition explains some properties concerning the Coxeter element $\Pi$.

For an element $w$ in the Weyl group $W$ with reduced expression $w = s_{i_1} \cdots s_{i_r}$, we let $T(w) = \sigma_{i_1} \cdots \sigma_{i_r}$ be the element in $B_g$. It is well-known that $T(w)$ is independent of the reduced expression. Let $w_0$ be the longest element in $W$. We call $\Delta = T(w_0)$ the Garside element in $B_g$.

**Proposition 1 ([2], Lemma 5.8 and Satz 7.1).**

1. Let $\Delta$ be the Garside element in $B_g$. Then $\Pi^h = \Delta^2$.
2. If $g$ is not isomorphic to $\mathfrak{sl}_2$, the centre $Z(B_g)$ of $B_g$ is generated by $\Delta^2$.
3. If $g \cong \mathfrak{sl}_2$, the centre $Z(B_g)$ is generated by $\sigma_1 = \Delta$.

3.2. **Braid group acting on representations.** We define the $q$-exponential function by

$$\exp_q(x) = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} q^{\frac{k(k-1)}{2}} x^k.$$
The objective of this section is to recall an action of Artin braid group associated to the Weyl group of $\mathfrak{g}$ on the integrable modules of $U_q(\mathfrak{g})$, following [7], [10] and [15]. We start from the $sl_2$ case.

For $i \in I$, we let $U_q(\mathfrak{g})_i$ denote the subalgebra of $U_q(\mathfrak{g})$ generated by $E_i$, $F_i$ and $K_i^{\pm 1}$. It inherits a Hopf algebra structure from $U_q(\mathfrak{g})$. Moreover, as a Hopf algebra, $U_q(\mathfrak{g})_i$ is isomorphic to $U_q(sl_2)$.

Let $V(n)$ denote the $(n+1)$-dimensional irreducible representation of $U_q(\mathfrak{g})_i$ of type 1. For $i \in I$, we define an endomorphism $S_i \in \text{End}(V(n))$ by

$$S_i = \exp_{q_i}^{-1}(q_i^{-1}E_iK_i^{-1})\exp_{q_i}^{-1}(-F_i)\exp_{q_i}^{-1}(q_iE_iK_i)q_i^{H_i(H_i+1)/2},$$

where $q_i^{H_i(H_i+1)/2}$ sends $v \in V(n)$ to $q_i^{m(m+1)/2}$ if $K_i.v = q_i^m.v$. This operator $S_i \in \text{End}(V(n))$ is well-defined as both $E_i$ and $F_i$ act nilpotently on $V(n)$.

We want to obtain an explicit form for the action of $S_i$. If a basis $v_0, \ldots, v_n$ of $V(n)$ is chosen in such a way that

$$E_i.v_0 = 0, \quad F_i(k).v_0 = v_k, \quad K_i.v_0 = q_i^nv_0,$$

we have the following result.

**Lemma 1 ([15]).** The action of $S_i$ on $V(n)$ is given by

$$S_i.v_k = (-1)^{n-k}q_i^{(n-k)(k+1)}v_{n-k}.$$

As a direct corollary, $S_i \in \text{End}(V(n))$ is an automorphism.

We turn to the general case. Let $M$ be an integrable $U_q(\mathfrak{g})$-module. As $M$ is a direct sum of irreducible $U_q(\mathfrak{g})_i$-modules, $S_i \in \text{End}(M)$ is well-defined. As these $S_i$ are invertible, we could consider the group generated by $\{S_i \mid i \in I\}$ in $\text{End}(M)$. Using relations between $S_i$ and Lusztig’s automorphism $T_i$ which will be recalled later, Saito proved the following result:

**Proposition 2 ([15]).** Let $M$ be an integrable $U_q(\mathfrak{g})$-module. The assignment $\sigma_i : S_i \mapsto T_i$ extends to a group homomorphism between the the Artin group $\mathcal{B}_\mathfrak{g}$ and the subgroup of $\text{Aut}(M)$ generated by $\{S_i \mid i \in I\}$.

### 3.3. Lusztig’s automorphism.

There is another action of the Artin braid group on $U_q(\mathfrak{g})$ constructed by Lusztig, see [12] for details.

**Definition 4.** There exist algebra automorphisms $T_i : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined by

$$T_i(E_i) = -F_iK_i, \quad T_i(F_i) = -K_i^{-1}E_i, \quad T_i(K_j) = K_jK_i^{-a_{ij}}$$

and for any $i \neq j$,

$$T_i(E_j) = \sum_{k=0}^{\alpha_{ij}}(-1)^{k}q_i^{-k}E_i^{(-a_{ij}-k)}E_jE_i^{(k)},$$

$$T_i(F_j) = \sum_{k=0}^{\alpha_{ij}}(-1)^{k}q_i^{k}F_jF_i^{(-a_{ij}-k)}.$$

These $T_i$ are called Lusztig’s automorphisms.
It is proved by Lusztig that these $T_i$ satisfy the relations in the Artin braid group associated to the Weyl group of $g$, which gives the following proposition:

**Proposition 3 ([12]).** The assignment $\sigma_i \mapsto T_i$ extends to a group homomorphism between the Artin braid group $\mathfrak{B}_g$ and the subgroup of $\text{Aut}(U_q(g))$ generated by $\{T_i | i \in I\}$.

The quantum group $U_q(g)$ acts on the integrable module $M$ and the two braid group actions above are closely related in the following way:

**Proposition 4 ([15]).** For any $x \in U_q(g)$ and any integrable $U_q(g)$-module $M$, $T_i(x) = S_i x S_i^{-1}$ in $\text{End}(M)$.

After Proposition 2 and 3, we can define a Coxeter element $\Pi$ in both $\text{Aut}(M)$ and $\text{Aut}(U_q(g))$ by $S_1 \cdots S_t$ and $T_1 \cdots T_t$ respectively. Results in Proposition 4 hold for these elements in both of the automorphism groups.

### 3.4. Quantum Weyl group: $\mathfrak{h}$-adic version.

We start from the $sl_2$ case as above.

Let $U_h(sl_2)$ be the $\mathfrak{h}$-adic quantized enveloping algebra associated to $sl_2$. Let $V(n)$ be the irreducible representation of $U_h(sl_2)$ of rank $n + 1$. It is a free $\mathbb{C}[[\mathfrak{h}]]$-module with basis $u_0, \cdots, u_n$ such that

$$H.u_k = (n - 2k)u_k, \quad X^+.u_k = [n - k + 1]_q u_{k-1}, \quad X^-.u_k = [k + 1]_q u_{k+1}.$$ 

For $r, s = 0, \cdots, n$, we define linear functions $C_{r,s}^{(n)} : U_h(sl_2) \to \mathbb{C}[[\mathfrak{h}]]$ by: for any $x \in U_h(sl_2),$

$$x.u_s = \sum_{r=0}^{n} C_{r,s}^{(n)}(x) u_r.$$ 

Let $F_h(sl_2) \subset U_h(sl_2)^*$ denote its $\mathbb{C}[[\mathfrak{h}]]$-subalgebra generated by the set $\{C_{r,s}^{(n)} | n \in \mathbb{N}, \ 0 \leq r, s \leq n\}$. We define a linear form $w_h \in F_h(sl_2)^*$ by:

$$w_h(C_{r,s}^{(n)}) = \begin{cases} (-1)^r e^{4\mathfrak{h}n^2 + hr} & \text{if } r + s = n, \\ 0 & \text{otherwise}. \end{cases}$$

Notice that there is a natural embedding $U_h(sl_2) \to F_h(sl_2)^*$, we let $\tilde{U}_h(sl_2)$ denote the subalgebra of $F_h(sl_2)^*$ generated by $U_h(sl_2)$ and $w_h$ and call it the quantum Weyl group of $U_h(sl_2)$. The commutation relations between $w_h$ and elements in $U_h(sl_2)$ are given in Section 8.2 of [3].

There is another element $\tilde{w}_h$ defined by

$$\tilde{w}_h = w_h \exp\left(-\frac{\mathfrak{h}}{4}H^2\right) = \exp\left(-\frac{\mathfrak{h}}{4}H^2\right) w_h \in F_h(sl_2)^*.$$ 

We will see later that $\tilde{w}_h$ is closely related to the Artin braid group action defined above.

We turn to the general case where $g$ is a simple Lie algebra. We notice that $U_h(g)$ is generated by $U_h(g)_i$, where $U_h(g)_i$ is the sub-Hopf algebra of $U_h(g)$ generated by $X_i^+$, $X_i^-$ and $H_i$ which is moreover isomorphic to $U_{d,h}(sl_2)$ as a Hopf algebra.
The inclusion $U_h(g)_i \to U_h(g)$ induces a projection $\mathcal{F}_h(g) \to \mathcal{F}_h(g)_i$ given by the restriction where $\mathcal{F}_h(g)$ is the Hopf algebra generated by matrix elements of all finite dimensional irreducible representations of $U_h(g)$ of type 1, and then induces an inclusion $\mathcal{F}_h(g)_i^* \to \mathcal{F}_h(g)^*$.

As $\mathcal{F}_h(g)_i^*$ is isomorphic to $\mathcal{F}_{d,h}(sl_2)^*$, we can pull back $w_h \in \mathcal{F}_{d,h}(sl_2)^*$ to obtain an element $w_{h,i} \in \mathcal{F}_h(g)_i^*$. We define

$$\tilde{w}_{h,i} = w_{h,i} \exp \left( -\frac{hd_i H_i^2}{4} \right) = \exp \left( -\frac{hd_i H_i^2}{4} \right) w_{h,i} \in \mathcal{F}_h(g)_i^*.$$

**Definition 5.** The quantum Weyl group $\tilde{U}_h(g)$ associated to $U_h(g)$ is the subalgebra of $\mathcal{F}_h(g)^*$ generated by $U_h(g)$ and $w_{h,i}$ for $i \in I$.

In fact, the quantum Weyl group is a Hopf algebra, after the following proposition.

**Proposition 6.** After the identification $q = e^h$ and $K_i = e^{hd_i H_i}$, for any integrable $U_q(g)$-module $M$, $\tilde{w}_{h,i}$ and $S_i$ coincide as elements in $\text{End}(M)$.

**Proof.** It suffices to consider the $sl_2$ case and the representation $V(n)$, which is the case we can compute.

Choosing a basis of $V(n)$ as in Section 3.2, we want to compute the action of $w_h$ on $v_i$. As $V(m)$ is a finite dimensional left $U_h(sl_2)$-module, it inherits a right $\mathcal{F}_h(sl_2)$-comodule structure and then a left $\mathcal{F}_h(sl_2)^*$-module structure; after an easy computation, we have

$$w_h v_i = \sum_{j=0}^{n} w_h(C^{(n)}_{j,i}) v_j = (-1)^{n-i} e^{\frac{h}{4} (n^2 + n - i)} v_{n-i}.$$

As $\exp \left( -\frac{h}{4} H_i^2 \right)$ acts as a scalar $e^{-\frac{h}{4} (n-2i)^2}$ on $v_i$, the action of $\tilde{w}_h$ on $v_i$ is given by

$$\tilde{w}_h v_i = (-1)^{n-i} e^{h (n-i)(i+1)} v_{n-i},$$

which coincides with the action of $S_i$ in Lemma 11. \hfill \Box

As a direct consequence, for any $x \in U_q(g)$,

$$\tilde{w}_{h,i} x \tilde{w}_{h,i}^{-1} = T_i(x)$$
as endomorphisms in \( \text{End}(M) \), where in the left hand side, we consider \( x \) as an element in \( U_h(\mathfrak{g}) \).

3.5. **Specialization.** For \( \lambda \in \mathcal{P}_+ \), as \( V(\lambda) \) is an integrable \( U_q(\mathfrak{g}) \)-module, \( S_i \in \text{End}(V(\lambda)) \). Once the variable \( q \) is specialized to 1, \( V(\lambda) \) goes to \( V_1(\lambda) \) and \( S_i \) is sent to
\[
S_i = \exp(e_i)\exp(-f_i)\exp(e_i) \in \text{End}(V_1(\lambda)),
\]
which coincides with the action of the simple reflection \( s_i \) in the Weyl group \( W \) on the integrable representations of \( U(\mathfrak{g}) \).

4. **R-matrix**

One of the remarkable properties of quantum groups comes from the quasi-triangularity.

4.1. **Definition and construction.**

**Definition 6.** Let \( H \) be a Hopf algebra and \( R \in H \otimes H \) be an invertible element. The pair \((H, R)\) is called a quasi-triangular Hopf algebra (QTHA) if

(1) For any \( x \in H \), \( \Delta^{op}(x)R = R\Delta(x) \).

(2) \( (\Delta \otimes \text{id})(R) = R_{13}R_{23}, \) \( (\text{id} \otimes \Delta)(R) = R_{13}R_{12}, \) where for \( R = \sum s_i \otimes t_i \), \( R_{13} = \sum s_i \otimes 1 \otimes t_i, \) \( R_{12} = \sum s_i \otimes t_i \otimes 1, \) \( R_{23} = \sum 1 \otimes s_i \otimes t_i \).

If this is the case, we call \( R \) an R-matrix.

If \((H, R)\) is a quasi-triangular Hopf algebra, the R-matrix will satisfy the famous Yang-Baxter equation
\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},
\]
which endows a braid structure on the category of \( H \)-modules.

The following well-known theorem due to Drinfel’d gives one of the advantages of quantum groups.

**Theorem 1.** There exists \( R \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g}) \) such that \((U_h(\mathfrak{g}), R)\) is a QTHA.

**Remark 4.** In fact, the R-matrix of \( U_h(\mathfrak{g}) \) exists only in a completion of \( U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g}) \), see [6], Chapter XVII for a detailed discussion on this problem.

4.2. **Construction of R-matrix.** The aim of this subsection is to recall an explicit construction of the R-matrix, which will be used in the proof of the main theorem.

Let \( w \in W \) be an element in the Weyl group of \( \mathfrak{g} \) with a reduced expression \( w = s_{i_1} \cdots s_{i_t} \). Lusztig defined an automorphism \( T_w \in \text{Aut}(U_q(\mathfrak{g})) \) by \( T_w = T_{i_1} \cdots T_{i_t} \) and showed that it is independent of the reduced expression chosen in the very beginning.

Now let \( w_0 \in W \) be the longest element in the Weyl group. We fix a reduced expression \( w_0 = s_{i_1} \cdots s_{i_N} \) where \( N \) is the cardinal of the set of positive roots in \( \mathfrak{g} \). We denote
\[
\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \beta_3 = s_{i_1}s_{i_2}(\alpha_{i_3}), \cdots, \beta_N = s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N});
\]
the set \( \{\beta_1, \cdots, \beta_N\} \) coincides with the set of positive roots. The root vectors in \( U_q(\mathfrak{g}) \) are defined by: for \( r = 1, \cdots, N \),
\[
E_{\beta_r} = T_{i_1} \cdots T_{i_{r-1}}(E_{i_r}), \quad F_{\beta_r} = T_{i_1} \cdots T_{i_{r-1}}(F_{i_r}).
\]
Then the set
\[ \{ E_{\beta_1} \cdots E_{\beta_N} K_{s_1} \cdots K_{s_l} F_{\beta_1} \cdots F_{\beta_N} \mid r_1, \cdots, r_N, t_1, \cdots, t_N \in \mathbb{N}, s_1, \cdots, s_l \in \mathbb{Z} \} \]
forms a linear basis of \( U_q(\mathfrak{g}) \).

For the \( h \)-adic version of quantized enveloping algebras, the same construction can be applied to construct \( X^+_\beta \) and \( X^-_\beta \) through replacing \( E_{\beta} \) by \( X^+_\beta \) and \( F_{\beta} \) by \( X^-_\beta \) in the definition of Lusztig’s automorphisms \( T_i \) and the root vectors \( E_{\beta_i}, F_{\beta_i} \).

The \( R \)-matrix of \( U_h(\mathfrak{g}) \) is given by
\[
R_h = \exp \left( h \sum_{i,j} B_{ij} H_i \otimes H_j \right) \prod_{\beta \in \Delta_+} \exp_q \left( (1 - q^{-2}) X^+_\beta \otimes X^-_\beta \right)
\]
where \( B = (B_{ij}) \) is the inverse of the Cartan matrix \( C \) and \( q_\beta = e^{h d_\beta} \) (if the positive root \( \beta \) is conjugate to \( \alpha \), under the action of Weyl group). Moreover, the product is taken in the ordering on the set of positive roots induced by the fixed decomposition of \( w_0 \).

For example, in the case of \( U_h(\mathfrak{sl}_2) \), the \( R \)-matrix can be written explicitly as:
\[
R_h = \left( \sum_{m=0}^{\infty} \left( \frac{h}{2} \right)^m \frac{1}{m!} H^m \otimes H^m \right) \left( \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n]_q!} q^{n(n+1)/2} (X^+)^n \otimes (X^-)^n \right).
\]

### 4.3. Drinfel’d element \( u \)

Let \( R = \sum s_i \otimes t_i \) be the \( R \)-matrix of the \( h \)-adic quantized enveloping algebra \( U_h(\mathfrak{g}) \).

Drinfel’d defined an invertible element \( u = \sum S(t_i) s_i \in U_h(\mathfrak{g}) \) in [3] such that the square of the antipode \( S^2 \) coincides with the adjoint action of \( u \). As \( S^2 \) is given by the adjoint action of \( K_{2\rho} = e^{h H_{2\rho}} \), where \( H_{2\rho} = H_{\beta_1} + \cdots + H_{\beta_N} (H_{\beta_i} = H_{\alpha_{i+1}} + \cdots + H_{\alpha_{j+1}} \) if \( \beta_i = \alpha_{j_1} + \cdots + \alpha_{j_l} \) using notations in the last section, the element \( K_{-2\rho} u = u K_{-2\rho} \) is in the centre of \( U_h(\mathfrak{g}) \).

As \( K_{-2\rho} u \) is in the centre of \( U_h(\mathfrak{g}) \), it acts on \( V(\lambda) \) as a scalar after Schur lemma. The following lemma is well-known:

**Lemma 2.** The central element \( K_{-2\rho} u \) acts as multiplication by \( q^{-(\lambda, \lambda + 2\rho)} \) on \( V(\lambda) \).

We let \( V(\lambda)_0 \) denote the subspace of weight 0 in \( V(\lambda) \). As \( K_{-2\rho} \) acts as identity on \( V(\lambda)_0 \), \( u \) acts as a scalar \( q^{-(\lambda, \lambda + 2\rho)} \) on it.

For example, using the explicit expression of the \( R \)-matrix, we have the following formula of \( u \) in the case of \( sl_2 \): (we write \( E \) for \( X^+ \) and \( F \) for \( X^- \)):
\[
u = \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n]_q!} q^{n(n+1)/2} S(F^n) \left( \sum_{m=0}^{\infty} \left( -\frac{h}{2} \right)^m \frac{1}{m!} H^{2m} \right) E^n.
\]

As \( \mathfrak{g} \) is an Artin braid group, the square of \( w_{h,i} \) will not give identity in general. In fact, it is closely related to the Drinfel’d element, as will be explained in the following result. It should be remarked that it is slightly different from Proposition 8.2.4 in [3].

**Proposition 7 ([7]).** For any \( j \in I \), let \( u_{\alpha_j} = \sum S(t_i)s_i \), where \( R_{\alpha_j} = \sum s_i \otimes t_i \) is the \( R \)-matrix of \( U_h(\mathfrak{g})_j \). Then
\[
w_{h,j}^2 = u_{\alpha_j}^{-1} \exp (hd_j H_j) \varepsilon_j,
\]
where \( \varepsilon_j \in \mathcal{F}_h(sl_2) \) is defined by \( \varepsilon_j(C^{(n)}_{r,s}) = (-1)^n \).

**Proof.** It suffices to show this in the case of \( sl_2 \). We adopt notations in Section 3.2 then for \( 0 \leq k \leq n \), from the computation in Proposition 6

\[
 w_h^2 v_k = w_h \left( (-1)^{n-k} e^{\frac{h}{2} n^2 + n - k} v_{n-k} \right) = (-1)^n e^{\frac{h}{2} n^2 + n} v_k.
\]

On the other side, after Lemma 2

\[
 u^{-1} K \varepsilon v_k = (-1)^n q \frac{h^2 n^2 + n}{v_k}.
\]

\[\square\]

### 5. Action of Central Element

The main target of this section is to compute the action of \( Z(\mathfrak{B}_g) \) on \( V(\lambda) \).

**5.1. Action on extremal vectors.** The Artin braid group \( \mathfrak{B}_g \) acts on \( V(\lambda) \). From Proposition 1 let \( \theta = \Delta^2 = \Pi^h \) be the generator of the centre \( Z(\mathfrak{B}_g) \); we compute the action of \( \theta \) on the highest weight vector \( v_\lambda \) in this subsection.

Let \( w_0 = s_{i_1} \cdots s_{i_N} \) be the fixed reduced expression of \( w_0 \) as in Section 4.2 then we have

**Lemma 3.** The generator of \( Z(\mathfrak{B}_g) \) can be written as

\[
 \theta = \sigma_{i_1} \cdots \sigma_{i_N} \sigma_{i_N} \cdots \sigma_{i_1}.
\]

**Proof.** As both sides have the same length, it suffices to show that the right hand side is in the centre of \( \mathfrak{B}_g \). Since the Garside element \( \Delta \) has the property \( \Delta \sigma_i = \sigma_{t-i} \Delta \), for any \( 1 \leq t \leq l \),

\[
 \sigma_t \sigma_{i_1} \cdots \sigma_{i_N} \sigma_{i_N} \cdots \sigma_{i_1} = \sigma_{i_1} \cdots \sigma_{i_N} \sigma_{l-i} \sigma_{i_N} \cdots \sigma_{i_1} = \sigma_{i_1} \cdots \sigma_{i_N} \sigma_{i_N} \cdots \sigma_{i_1}. \]

\[\square\]

Return to our situation, when acting on \( V(\lambda) \), the central element \( \theta \) in \( \mathfrak{B}_g \) has the following expression after Proposition 1

\[
 \theta = S_{i_1} \cdots S_{i_{N-1}} S_{i_N}^2 S_{i_{N-1}} \cdots S_{i_1} = S_{i_1} \cdots S_{i_{N-2}} T_{i_{N-1}} (S_{i_N}^2) S_{i_{N-1}} S_{i_{N-2}} \cdots S_{i_1} = T_{i_1} \cdots T_{i_{N-1}} (S_{i_1}^2) S_{i_1} \cdots S_{i_{N-1}} S_{i_{N-2}} \cdots S_{i_1}.
\]

We start from computing \( S_{i_1}^2 v_\lambda \). Combining Proposition 6 and 7 for any \( k = 1, \ldots, l \),

\[
 S_k^2 = \tilde{w}_{h,k}^2 = w_{h,k}^2 \exp \left( -\frac{h}{2} d_k H_k^2 \right) = u_k^{-1} \exp (hd_k H_k) \exp \left( -\frac{h}{2} d_k H_k^2 \right) \varepsilon_k.
\]
When $S^2_i$ is acted on $v_\lambda$, $\varepsilon_i$ gives a constant $c_1 \in \{\pm 1\}$ and $\exp(hd_i H_{i_1})$ gives $q^{(\lambda, \alpha_{i_1})}$. We compute the action of
\[
u_{i_1} = \sum_{n=0}^{\infty} Q(n) S(F^m_{i_1}) \left( \sum_{m=0}^{\infty} \left( \frac{hd_{i_1}}{2} \right)^m \frac{1}{m!} S(H^m_{i_1}) H^m_{i_1} \right) E^m_{i_1}
\]
on the highest weight vector where $Q(n)$ is a rational function in $q$ such that $Q(0) = 1$: it is clear that only the middle part containing $H_{i_1}$ contributes, which is given by:
\[
\sum_{m=0}^{\infty} \left( -\frac{hd_{i_1}}{2} \right)^m \frac{1}{m!} H^{2m}_{i_1}.v_\lambda = e^{-\frac{h}{2}d_{i_1}(\lambda, \alpha_{i_1})^2}.
\]
As a consequence, $u_{i_1}^{-1}$ acts as a scalar $\exp\left(-\frac{h}{2}d_{i_1}(\lambda, \alpha_{i_1})^2\right)$ on $v_\lambda$. As
\[
\exp\left(\frac{h}{2}d_{i_1} H^2_{i_1}\right).v_\lambda = e^{-\frac{h}{2}d_{i_1}(\lambda, \alpha_{i_1})^2},
\]
we finally obtain that
\[
S^2_i.v_\lambda = c_1 q^{(\lambda, \alpha_{i_1})} v_\lambda.
\]
We turn to consider the action of a general term $T_{i_1} \cdots T_{i_{k-1}}(S^2_{i_k})$ on $v_\lambda$. The same argument as above can be applied here: we let
\[
\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}),
\]
then
\[
T_{i_1} \cdots T_{i_{k-1}}(S^2_{i_k}) = T_{i_1} \cdots T_{i_{k-1}} \left( u_{i_k}^{-1} \exp(hd_{i_k} H_{i_k}) \exp\left(-\frac{h}{2}d_{i_k} H^2_{i_k}\right) \varepsilon_{i_k} \right).
\]
Since $T_{i_1} \cdots T_{i_{k-1}}$ is an algebra morphism, we compute each part in the right hand side:
\[
T_{i_1} \cdots T_{i_{k-1}} \left( \exp\left(-\frac{h}{2}d_{i_k} H^2_{i_k}\right) \right) = \exp\left(-\frac{h}{2}d_{i_k} H^2_{i_k} \beta_k \right),
\]
and
\[
T_{i_1} \cdots T_{i_{k-1}}(u_{i_k})
\]
\[
= T_{i_1} \cdots T_{i_{k-1}} \left( \sum_{n=0}^{\infty} (\ast) S(F^m_{i_k}) \left( \sum_{m=0}^{\infty} \left( -\frac{hd_{i_k}}{2} \right)^m \frac{1}{m!} H^{2m}_{i_k} \right) \right) T_{i_1} \cdots T_{i_{k-1}}(E^m_{i_k})
\]
\[
= T_{i_1} \cdots T_{i_{k-1}} \left( \sum_{n=0}^{\infty} (\ast) S(F^m_{i_k}) \left( \sum_{m=0}^{\infty} \left( -\frac{hd_{i_k}}{2} \right)^m \frac{1}{m!} H^{2m}_{i_k} \right) \right) E^m_{i_k} \beta_k.
\]
When acting on the highest weight vector, the only part which contributes is the middle one consisting of $H_{i_k}$, so it suffices to compute
\[
T_{i_1} \cdots T_{i_{k-1}} \left( \sum_{m=0}^{\infty} \left( -\frac{hd_{i_k}}{2} \right)^m \frac{1}{m!} H^{2m}_{i_k} \right) = \exp\left(-\frac{h}{2}d_{i_k} H^2_{i_k} \beta_k \right),
\]
so $T_{i_1} \cdots T_{i_{k-1}}(u_{i_k}^{-1})$ acts as $\exp\left(\frac{h}{2}d_{i_k} H^2_{i_k} \beta_k \right)$ on $v_\lambda$. 
Combining the computation above, we have: there exists a constant \( c_k \in \mathbb{C} \setminus \{0\} \) such that
\[
T_i \cdots T_{i_{k-1}}(S_{ik}^2).v_\lambda = c_kq^{(\lambda, \beta_k)}v_\lambda.
\]
As \( \beta_1, \ldots, \beta_N \) run over the set of positive roots, we have proved the following result:

**Proposition 8.** There exists a constant \( c \in \{\pm 1\} \) such that
\[
\theta.v_\lambda = cq^{(\lambda, \rho)}v_\lambda.
\]

As \( \theta \in Z(\mathfrak{B}_g) \), it acts by the same constant on each \( \mathfrak{B}_g \)-orbit in \( V(\lambda) \).

5.2. **Central automorphism action.** Let \( U_q^{\geq 0}(\mathfrak{g}) \), \( U_q^{\leq 0}(\mathfrak{g}) \), \( U_q^{< 0}(\mathfrak{g}) \) denote the sub-Hopf algebra of \( U_q(\mathfrak{g}) \) generated by \( E_i, K_i^{\pm 1} \) (resp. \( F_i, K_i^{\pm 1} \)). We compute the action of \( T_{w_0}^2 \) on PBW basis of \( U_q^{< 0}(\mathfrak{g}) \) in this subsection. It is known that \( T_{w_0} \) permutes \( U_q^{\geq 0}(\mathfrak{g}) \) and \( U_q^{< 0}(\mathfrak{g}) \), so \( T_{w_0}^2 \) is an automorphism of \( U_q^{< 0}(\mathfrak{g}) \) and \( U_q^{\geq 0}(\mathfrak{g}) \).

For \( i \in I \), we let \( \hat{i} \) denote the index satisfying \( w_0(\alpha_i) = \alpha_\hat{i} \).

**Lemma 4.** For \( i \in I \), the following identities hold:
\[
T_{w_0}^2(E_i) = q^{(\alpha_i, \alpha_i)}K_i^{-2}E_i, \quad T_{w_0}^2(F_i) = q^{(\alpha_i, \alpha_i)}K_i^2F_i, \quad T_{w_0}^2(K_i) = K_i.
\]

**Proof.** A similar computation as in [11] Section 5.7 gives
\[
T_{w_0}(E_i) = -F_iK_i, \quad T_{w_0}(F_i) = -K_i^{-1}E_i, \quad T_{w_0}(K_i) = K_i^{-1}.
\]
Then the lemma is clear as \( w_0(\alpha_i) = \alpha_\hat{i} \). \( \square \)

We turn to consider the action of \( T_{w_0}^2 \) on a root vector \( F_{\beta_k} = T_i \cdots T_{i_{k-1}}(F_{i_k}) \). As \( T_{w_0}^2 \in Z(\mathfrak{B}_g) \) (here \( \mathfrak{B}_g \) is the Artin braid group generated by \( \{T_i \mid i \in I\} \)),
\[
T_{w_0}^2T_i \cdots T_{i_{k-1}}(F_{i_k}) = T_i \cdots T_{i_{k-1}}T_{w_0}^2(F_{i_k})
\]
\[
= T_i \cdots T_{i_{k-1}}(q^{(\alpha_i, \alpha_i)}K_i^2F_{i_k})
\]
\[
= q^{(\beta_k, \beta_k)}K_{\beta_k}^2F_{\beta_k},
\]
where \( q^{(\alpha_i, \alpha_i)} = q^{(\beta_k, \beta_k)} \) as the bilinear form is invariant under the action of Weyl group.

So in general, we have
\[
T_{w_0}^2(F_{\beta_j_1} \cdots F_{\beta_j_r}) = q^{\sum_{k=1}^r(\beta_j_k, \beta_k)}K_{\beta_j_1}^2F_{\beta_j_1} \cdots K_{\beta_j_r}^2F_{\beta_j_r}
\]
\[
= q^{\sum_{k=1}^r(\beta_j_k, \beta_k) + \sum_{k < l}2(\beta_j_k, \beta_j_l)}K_{\beta_j_1}^2 \cdots K_{\beta_j_r}^2F_{\beta_j_1} \cdots F_{\beta_j_r}
\]
\[
= q^{(\beta, \beta)}K^{2}F_{\beta_j_1} \cdots F_{\beta_j_r},
\]
where \( \beta = \beta_j_1 + \cdots + \beta_j_r \). These calculations give the following

**Proposition 9.** Let \( x_{\beta} \in U_q^{< 0}(\mathfrak{g})_{-\beta} \). Then
\[
T_{w_0}^2(x_{\beta}) = q^{(\beta, \beta)}K^{2}x_{\beta}.
\]
5.3. **Central element action on Weyl group orbits.**

**Proposition 10.** There exists a constant $c \in \{\pm 1\}$ such that for any non-zero vector $v \in V(\lambda)_0$ of weight $0$,

$$\theta.v = cq^{(\lambda,\lambda+2\rho)}v.$$ 

**Proof.** It should be remarked that if $\lambda$ is not in the root lattice $Q$, there will be no vector of weight $0$. So if $v \in V(\lambda)_0$ is a non-zero vector, $\lambda \in Q_+$ and there exists $x \in \mathcal{U}_{<0}(g)_{-\lambda}$ such that $v = x.v_\lambda$.

After Proposition 8 and 9, we have the following computation:

$$\theta.v = S_{i_1} \cdots S_{i_N} x.v_\lambda$$

$$= T_{i_1} \cdots T_{i_N} T_{i_1}(x) \theta.v_\lambda$$

$$= T_{\omega_0}(x) \theta.v_\lambda$$

$$= q^{(\lambda,\lambda)}K_\lambda x \theta.v_\lambda$$

$$= cq^{(\lambda,\lambda+2\rho)}v,$$

where $c \in \{\pm 1\}$ comes from Proposition 8 by evaluating on the highest weight vector, so it does not depend on the choice of $v \in V(\lambda)_0$. □

As a consequence, $\theta$ acts on $V(\lambda)_0$ as a constant $cq^{(\lambda,\lambda+2\rho)}$.

Moreover, this method can be applied to compute the action of the central element on each $B_g$-orbit. For example, if $g = sl_3$ and $\lambda = \alpha_1 + \alpha_2$, then $V(\lambda)$ is the adjoint representation of dimension 8. $\theta$ acts as $q^4$ on the outer cycle and $q^6$ on the inner one (it is the zero-weight space in this case).

5.4. **Trace of Coxeter element.** We compute the trace of the Coxeter element when it acts on $V(\lambda)$. The following observation simplifies the computation.

Let $wt(V(\lambda))$ denote the set of weights appearing in $V(\lambda)$. Then as $S_i(V(\lambda)_\mu) \subset V(\lambda)_{\mu-(\mu, \alpha_i)\alpha_i}$ after the definition of $S_i$, the action of Artin braid group $B_g$ on $wt(V(\lambda))$ is identical with that of the Weyl group $W$. The action of $\Pi$ is the same as the Coxeter element $c = s_1 \cdots s_l \in W$ which has no fixed point in $wt(V(\lambda)) \setminus \{0\}$.

A standard proof of the statement above can be found in [1], Chapitre V, n° 6.2.

As an immediate consequence of this observation, we have

$$\text{Tr}(\Pi, V(\lambda)) = \text{Tr}(\Pi, V(\lambda)_0).$$

Moreover, generators of $B_g$ preserve the zero-weight space $V(\lambda)_0$, so we can also look $B_g$ as a subgroup of $\text{Aut}(V(\lambda)_0)$.

Notice that $\Pi^h = \Delta^2 = \theta$, so after Proposition 10 $\Pi^h$ acts as a scalar $c q^{(\lambda,\lambda+2\rho)}$ on $V(\lambda)_0$ for some $c \in \{\pm 1\}$. If we let $\Lambda$ denote the set of roots of the equation $x^h = c$ in $\mathbb{C}$, then the eigenvalues of $\Pi$ belong to the set

$$\{y.q^{(\lambda,\lambda+2\rho)} | y \in \Lambda\}$$

and the trace $\text{Tr}(\Pi, V(\lambda)_0)$ is given by $\delta q^{(\lambda,\lambda+2\rho)}$ for some $\delta \in \mathbb{C}$. As a summary, we have proved that
Proposition 11. There exists a constant $\delta \in \mathbb{C}$ which does not depend on $q$ such that
\[ \text{Tr}(\Pi, V(\lambda)) = \delta q^{\frac{(\lambda,\lambda+2\rho)}{h}}. \]

We will determine this constant in the next section and see that it is in fact non-zero.

6. Main theorem

6.1. Dedekind $\eta$-function. We give a recollection on $\eta$-function in this subsection.

For a positive integer $n$, we let $p(n)$ denote the partition number of $n$ and $p(0) = 1$. Let $\psi(x) = \sum_{n \geq 0} p(n) x^n$ be their generating function and $\varphi(x) = \psi(x)^{-1}$. Then $\varphi(x)$ has a simple expression
\[ \varphi(x) = \prod_{n=1}^{\infty} (1 - x^n). \]

This $\varphi(x)$ is the simplest hypergeometric series and is closely related to the theory of modular forms.

Let $\eta(x) = x^{1/24} \varphi(x)$. Then $\eta^{24}(x)$ is a modular form of weight 12 and $\eta(x)$ is called the Dedekind $\eta$-function. The expansion of $\eta(x)^{24}$ into power series contains Ramanujan’s $\tau$-function as coefficients.

6.2. A theorem due to Kostant. We preserve notations for a simple Lie algebra $\mathfrak{g}$ given in Section 2.1.

Let $V_1(\lambda)$ be the irreducible representation of $\mathfrak{g}$ of highest weight $\lambda$ and $V_1(\lambda)_0$ be the subspace of $V_1(\lambda)$ of weight 0. Then the Weyl group $W$ acts on $V_1(\lambda)$ and therefore on $V_1(\lambda)_0$. We let
\[ \tau_\lambda : W \to \text{Aut}(V_1(\lambda)_0) \]
denote this representation. Let $c = s_1 \cdots s_l$ be a Coxeter element in $W$, $c(\lambda) = \Phi(\lambda + \rho, \lambda + \rho) - \Phi(\rho, \rho)$ where $\Phi$ is the Killing form on $\mathfrak{h}^*$. Let $h$ be the Coxeter number of $W$.

Theorem 2 (\cite{8}). The following identity holds:
\[ \left( \prod_{i=1}^{l} \varphi(x^{h\Phi(\alpha_i, \alpha_i)}) \right)^{h+1} = \sum_{\lambda \in \mathcal{P}_+} \text{Tr}(c, V_1(\lambda)_0) \dim V_1(\lambda) x^{c(\lambda)}. \]

In particular, if $\mathfrak{g}$ is simply laced (i.e. of type $A,D,E$), the identity above has the form
\[ \varphi(x)^{\dim \mathfrak{g}} = \sum_{\lambda \in \mathcal{P}_+} \text{Tr}(c, V_1(\lambda)_0) \dim V_1(\lambda) x^{c(\lambda)}. \]

Moreover, $\text{Tr}(c, V_1(\lambda)_0) \in \{-1, 0, 1\}$.

To simplify the notation, we let $\varepsilon(\lambda)$ denote $\text{Tr}(c, V_1(\lambda)_0)$.

Some discussions on particular cases of Theorem 2 can be found in \cite{4} and \cite{8}.

6.3. Main result. We give an explanation of the identity in Theorem 2 in the framework of quantum groups.
6.3.1. Coxeter numbers and Killing forms. The Killing form on $\mathfrak{h}^*$ is proportional to the inner product on the root system, i.e., there exists a constant $k \in \mathbb{C} \setminus \{0\}$ such that for any $x, y \in \mathfrak{h}^*$,

$$k \Phi(x, y) = (x, y).$$

For any simple Lie algebra $\mathfrak{g}$, we define a constant $r_\mathfrak{g} = \frac{k h}{k}$. The following table gives the explicit values of $r_\mathfrak{g}$ where the values of $h$ and $k$ are taken from [1].

| $\mathfrak{g}$ | $h$  | $k$  | $r_\mathfrak{g}$ |
|----------------|------|------|-----------------|
| $A_l$          | $l+1$| $2(l+1)$ | 2               |
| $B_l (l \geq 2)$ | $2l$ | $4l-2$ | $\frac{2l-1}{2}$ |
| $C_l (l \geq 2)$ | $2l$ | $4l+4$ | $\frac{2l+2}{2}$ |
| $D_l (l \geq 3)$ | $2l-2$ | $4l-4$ | 2               |
| $E_6$          | 12   | 24   | 2               |
| $E_7$          | 18   | 36   | 2               |
| $E_8$          | 30   | 60   | 2               |
| $F_4$          | 12   | 18   | $3/2$           |
| $G_2$          | 6    | 24   | 4               |

6.3.2. Main theorem. Let

$$\mathbb{C}_q[G] = \bigoplus_{\lambda \in \mathcal{P}_+} \text{End}(V(\lambda)) = \bigoplus_{\lambda \in \mathcal{P}_+} V(\lambda) \otimes V(\lambda)^*$$

be the quantum coordinate algebra which can be viewed as a deformation of the algebra of regular functions of a semi-simple algebraic group $G$.

It is clear that there is a canonical embedding

$$\bigoplus_{\lambda \in \mathcal{P}_+} \text{End}(V(\lambda)) \otimes \text{End}(V(\lambda)^*) \to \text{End}(\mathbb{C}_q[G]).$$

Keeping notations in previous sections, we state the main theorem of this paper.

Theorem 3. Let $\Pi$ be the Coxeter element in the Artin braid group $\mathcal{B}_\mathfrak{g}$ and $V(\lambda)_0$ be the zero-weight space in $V(\lambda)$ for $\lambda \in \mathcal{P}_+$. We denote $c(\lambda) = \Phi(\lambda + \rho, \lambda + \rho) - \Phi(\rho, \rho)$.

1. We have

$$\text{Tr}(\Pi, V(\lambda)) = \text{Tr}(\Pi, V(\lambda)_0) = \varepsilon(\lambda) q^{r_\mathfrak{g} c(\lambda)}.$$ 

2. The following identity holds

$$\text{Tr}(\Pi \otimes \text{id}, \mathbb{C}_q[G]) = \left( \prod_{i=1}^{l} \phi(q^{(\alpha_i, \alpha_i)}) \right)^{h+1},$$

where we look $\Pi \otimes \text{id}$ as in $\text{End}(\mathbb{C}_q[G])$ through the embedding above.

3. In particular, if $\mathfrak{g}$ is simply laced, i.e., of type $A, D, E$, then

$$\text{Tr}(\Pi \otimes \text{id}, \mathbb{C}_q[G]) = \phi(q^2)^{\text{dim } \mathfrak{g}}.$$
Proof. We start from proving (2) and (3) by supposing (1) holds.

The point (2) comes from the following computation using (1) and Theorem 2:

\[
\text{Tr}(\Pi \otimes \text{id}, C_q[G]) = \text{Tr} \left( \Pi \otimes \text{id}, \bigoplus_{\lambda \in \mathcal{P}_+} V(\lambda) \otimes V(\lambda)^* \right)
\]
\[
= \sum_{\lambda \in \mathcal{P}_+} \text{Tr}(\Pi \otimes \text{id}, V(\lambda) \otimes V(\lambda)^*)
\]
\[
= \sum_{\lambda \in \mathcal{P}_+} \dim_{C_q}(\lambda) \text{Tr}(\Pi, V(\lambda))
\]
\[
= \sum_{\lambda \in \mathcal{P}_+} \varepsilon(\lambda) \dim_{C_q}(\lambda) q^{r_c(\lambda)}
\]
\[
= \left( \prod_{i=1}^{l} \varphi(q^{r_c(\lambda_i)}) \right)^{h+1}
\]
\[
= \left( \prod_{i=1}^{l} \varphi(q^{r_c(\lambda_i)}) \right)^{h+1}.
\]

To show the point (3), it suffices to notice that in the simply laced case, \((\alpha_i, \alpha_i) = 2\) and \(l(h+1) = \dim g\).

Now we proceed to prove (1).

After Proposition 11, there exists some constant \(\delta \in \mathbb{C}\) such that

\[
\text{Tr}(\Pi, V(\lambda)) = \delta q^{\frac{\lambda(\lambda+2\rho)}{h}} = \delta q^{r_c(\lambda)}.
\]

To determine this constant, we consider the specialization of \(\Pi\) and \(V(\lambda)\). As remarked in Section 3.5, when \(q\) is specialized to 1, the automorphism \(S_i \in \text{End}(V(\lambda))\) goes to \(s_i \in \text{End}(V_1(\lambda))\) in the Weyl group \(W\) and therefore \(\Pi\) is specialized to the Coxeter element \(c \in W\). In the formula above, the left hand side has limit \(\text{Tr}(c, V_1(\lambda))\) when \(q\) tends to 1. On the other hand, as \(\delta \in \mathbb{C}\), the right hand side has limit \(\delta\), from which \(\delta = \varepsilon(\lambda)\) and the theorem is proved. \(\square\)

6.3.3. Variations. As a variant of the above identity, we can form the following series

\[
\varphi(q, t) = \sum_{\lambda \in \mathcal{P}_+} \text{Tr}(\Pi, V(\lambda)) \dim_{C_q}(\lambda) V(\lambda) t^{r_c(\lambda)}
\]

as in the Theorem 2. Then it is not difficult to show that

\[
\varphi(q, t) = \left( \prod_{i=1}^{l} \varphi(q^{r_c(\lambda_i)}) \right)^{h+1}.
\]

When \(g\) is simply laced, the identity above gives

\[
\varphi(q, t) = \varphi(q^2 t)^{\dim g} = \left( \prod_{n \geq 1} (1 - q^{2n} t^n) \right)^{\dim g}.
\]
6.3.4. Example: $U_q(sl_2)$ case. When $g = sl_2$, results in Theorem 3 can be directly verified.

In this case, there is only one generator $S$ in the Artin braid group. Let $V(n)$ be the irreducible representation of dimension $n + 1$ of $U_q(sl_2)$ of type 1 with a basis chosen as in Section 3.2. The Coxeter element in this case is given by $S$.

If $n$ is odd, there is no zero-weight space in $V(n)$, in this case, $\text{Tr}(S, V(n)) = 0$.

If $n$ is even, the zero-weight space in $V(n)$ is of dimension 1 which is generated by $v_m$, where $n = 2m$.

The action of $S$ on $v_m$ is given by

$$S.v_m = (-1)^m q^{(n-m)(m+1)} v_m = (-1)^m q^{m(m+1)} v_m.$$ 

From which the trace $\text{Tr}(S \otimes \text{id}, C_q[G])$:

$$\text{Tr}(S \otimes \text{id}, C_q[G]) = \sum_{m=0}^{\infty} (-1)^m (2m + 1) q^{m(m+1)},$$

which coincides with $\varphi(q^2)^{\frac{3}{2}}$, after Jacobi’s identity.

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