Static Response of Functionally Graded Plates using Higher Order Theories

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Abstract. This work presents an assessment of higher order theories for the static response of functionally graded plates. The governing equations, variationally consistent boundary conditions, force and moment resultants of higher order shear deformable theories are derived using principle of virtual work. Material properties are assumed to vary smoothly in the thickness direction with power law variation of ceramic volume fraction. The effective material properties of FG material are obtained using rule of mixture as well as Mori-Tanaka homogenization scheme. Analytical solution using a double Fourier series expansion is obtained for simply supported plates. The computer program has been developed in MATLAB and the results are validated by comparing with the results available in the literature. The values of deflection, inplane and transverse stresses are presented for different theories, span to thickness ratio and inhomogeneity parameter.

Keywords: FGM plates, Higher order theory, Rule of Mixture (ROM), Mori-Tanaka Method (MTM), Analytical Fourier series solution

1. Introduction
Functionally graded materials (FGMs) were introduced as engineering materials by Cavanagh and his co-workers [1], for developing a crack-free thermal barrier coating of gas turbine blades. FGMs are essentially composite material where the composition of constituent materials gradually varying from one surface to another. The large jumps in the inplane and transverse shear stresses due to varied material properties in the adjacent layers, which have been found as a leading cause of delamination in bonded composite structures when they are used in thermal environment, have been eliminated using FGM concept. Theoretical modelling of FGM plates require a micromechanical model for computing effective material properties, a suitable and accurate 2D plate theory for the kinematic analysis and a solution technique. Detailed literature on the modelling and analysis of FGM plates can be found in [2]. Mostly, Voigt's rule of mixtures (ROM) has been employed as an effective and easy means for obtaining the effective properties [3 - 6].

For kinematic modelling, various 2D and 3D solutions with varied accuracy are available in the literature. The accuracy has been primarily depends on the assumptions employed in deriving the theory and the degree of the functions for higher order theories considered in defining the displacement field. The most simpler theory is the classical plate theory (CPT) where the shear
deformation effects are completely neglected, which has been employed by many researchers [7, 8]. This theory gives accurate results for thin plates but unable to predict accurate results for thick plates. The absence of the shear deformation effects in the CPT have been rectified by formulating first order shear deformation theory (FSDT) [4] and various higher order shear deformation theory (HOT) [9-12] by adding first and higher order shear deformation terms in the displacement field. These theories give accurate results for thick plates but their accuracy depends on the order of the functions considered in the displacement field. However, 3D exact [15-17] and FE solutions found to yield the most accurate results.

This work aims to present short comparison of various recently developed higher order theories for the static analysis of FGM plates. The system equations have been derived using principle of virtual work. The effective properties are obtained using rule of mixture (ROM) and Mori-Tanaka homogenization schemes. Analytical solution has been obtained for simply supported plates considering a double Fourier series expansion with \((m \times n)\) terms.

2. Problem Formulation
Consider an FGM plate having dimensions (length \(x\) width \(x\) height \(= ax b h\)) (Fig. 1) made with two distinct isotropic materials by evenly mixing them so that the volume fractions of the constituents are smoothly changing in the thickness direction. The effective values of Young’s modulus and Poisson’s ratio at any thickness point in the FGM plate are determined using the rule of mixtures (ROM)

\[
Y = Y_m V_m + Y_c V_c, \quad \nu = \nu_m V_m + \nu_c V_c
\]  

(1)

where \(V_c\) is the volume fraction of ceramic phase and \(V_m\) is the volume fraction of metal phase and \(V_c + V_m = 1\). In the Mori-Tanaka homogenization scheme, the effective values of shear modulus \((G)\) and bulk modulus \((K)\) are obtained using

\[
\frac{K - K_m}{K_c - K_m} = \frac{V_c}{1 + (1 - V_c) \frac{K_c - K_m}{K_m + 4G_m/3}}, \quad \frac{G - G_m}{G_c - G_m} = \frac{V_c}{1 + (1 - V_c) \frac{G_c - G_m}{G_m + \frac{f_1}{3}}}
\]  

(2)

where \(f_1 = G_m (9K_m + 8G_m)/6(K_m + 2G_m)\). The effective values of Young’s modulus \((Y)\) and Poisson’s ratio \((\nu)\) are computed from \(K\) and \(G\) as

\[
Y = \frac{9KG}{3K + G}; \quad \nu = \frac{3K - 2G}{2(3K + G)}
\]  

(3)

In this study, we consider ceramic volume fraction \((V_c)\) vary smoothly in the thickness direction according to the relation as

\[
V_c = (0.5 + z/h)^p
\]  

(4)

where \(p\) is known as inhomogeneity parameter or power law index. The kinematic relations of the plate are obtained using third order theory. Let \(u_x, u_y\) and \(w\) be the in-plane and transverse displacements, the strains are expressed in terms of displacements as

\[
\varepsilon_x = u_{xx}, \varepsilon_y = u_{yy}, \gamma_{xy} = u_{yx} + u_{xy},
\]

\[
\gamma_{xz} = w_x + u_{xz}, \quad \gamma_{yz} = w_y + u_{yz}
\]  

(5)

where \(\partial()/\partial x\). Mostly, 2D plate theories neglect transverse normal stress \(\sigma_z\) in comparison to other stress components, the constitutive equations of linear elasticity reduce to

\[
\sigma = Q(z)\varepsilon, \quad \tau = \tilde{Q}(z)\gamma
\]  

(6)

where;

\[
\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix}, \quad Q(z) = \begin{bmatrix} Q_{11}(z) & Q_{12}(z) & 0 \\ Q_{12}(z) & Q_{22}(z) & 0 \\ 0 & 0 & Q_{11}(z) \end{bmatrix}
\]  

(7)
\[
\tau = \begin{bmatrix} \tau_{xx} \\ \tau_{yy} \end{bmatrix}, \quad \gamma = \begin{bmatrix} \gamma_{xx} \\ \gamma_{yy} \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} Q_{55}(z) & 0 \\ 0 & Q_{44}(z) \end{bmatrix}
\]

\(Q_{ij}(z)\) are the reduced elastic stiffness coefficients defined for isotropic materials as

\[
Q_{11}(z) = Q_{22}(z) = Y(z)/(1 - \nu(z)^2); \quad Q_{12}(z) = v(z)Y(z)/(1 - \nu(z)^2)
\]

\(Q_{44}(z) = Q_{55}(z) = Q_{66}(z) = Y(z)/(1 + \nu(z))\)

(8)

\(Y(z)\) and \(v(z)\) are functions of \(z\), which are computed using the micromechanical model described in Eqs. 1 and 2 for a given profile of volume fraction \(V_c\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Geometry and coordinate systems used in FGM plate.}
\end{figure}

For the higher order theories (HOTs), transverse deflection \(w\) is taken independent of \(z\) and the in-plane displacements are assumed to follow higher order functions in \(z\).

\[
w(x, y, z, t) = w(x, y, t)
\]

(9)

\[
u = f_0(z)\tilde{u}_1
\]

(10)

where \(\tilde{u}_1 = [u_{0x}, u_{0y}, -w_{0x}, -w_{0y}, \psi_{0x}, \psi_{0y}]^T\), \(f_0(z) = [I_2 \quad zI_2 \quad R(z)I_2]\), \(u_0\) and \(w_0\) denote the mid-surface displacements and shear strains of the plate, \(I_2\) is 2\(^{nd}\) order identity matrix and \(R(z)\) is a higher order function which is defined as

\[
R(z) = \begin{cases} 
\frac{z - 4z^3/3h^2}{\sin(\pi z/h)} & \text{(TOT)} \\
\frac{(z - (h/n) \sin(\pi z/h))/\cosh(\pi/2) - 1}{\exp(-2(z/h)^2/2)} & \text{(Sinusoidal theory)} \\
\frac{\exp(-2(z/h)^2)}{\text{(Exponential theory)}} 
\end{cases}
\]

(11)

The strains are obtained for the HOT by substituting the displacement field of Eq. (10) in Eq (6) as

\[
\varepsilon = \varepsilon^0 + z\kappa^0 + R(z)\psi_{0d}, \quad \gamma = R'(z)\psi_0
\]

(12)

where \(R'(z)\) is the first derivative of the higher order function \(R(z)\) with respect to \(z\) and

\[
\varepsilon^0 = \begin{bmatrix} u_{0x,x} \\ u_{0y,y} \\ u_{0y,x} + u_{0x,y} \end{bmatrix}, \quad \kappa^0 = -\begin{bmatrix} w_{0,xx} \\ w_{0,yy} \\ 2w_{0,xy} \end{bmatrix}, \quad \psi_{0d} = \begin{bmatrix} \psi_{0x,x} \\ \psi_{0y,y} \\ \psi_{0x,y} + \psi_{0y,x} \end{bmatrix}, \quad \psi_0 = \begin{bmatrix} \psi_{0x} \\ \psi_{0y} \end{bmatrix}
\]

(13)

We use variational principle to derive the governing equations, force, moment and higher order moment resultants for 2D theories of FGM plates. The governing equations for the FGM plates are be expressed as

\[
N_{xx} + N_{yy,x} = 0, \quad N_{yy,x} + N_{y,y} = 0,
M_{xxx} + M_{yy,yy} + 2M_{yy,yy} + p_3 = 0,
P_{xx} + P_{yy} - Q_x = 0, \quad P_{yy} + P_{y,y} - Q_y = 0
\]

(14)

where \(N, M, Q\) and \(P\) consist of the in-plane stress resultants, bending moment resultants, shear resultants and the higher order bending moments, respectively, which are defined by
where

\[ A = \langle Q(z) \rangle; \quad E = \langle R(z) Q(z) \rangle; \quad \bar{A} = \langle R'(z) \bar{Q}(z) \rangle \]

\[ B = \langle zQ(z) \rangle; \quad F = \langle zR(z) Q(z) \rangle \]

\[ D = \langle z^2 Q(z) \rangle; \quad H = \langle R(z)^2 Q(z) \rangle \]

### 3. Analytical Solution for Simply Supported Plates

The solution of the resulting equations is expanded in the following Double Fourier series, which identically satisfies the simply supported boundary conditions at the four edges:

\[
\begin{bmatrix}
    w_0 \\
    u_{0x} \\
    u_{0y}
\end{bmatrix} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \begin{bmatrix}
    (w_0)_{mn} \sin(ax) \sin(by) \\
    (u_{0x})_{mn} \cos(ax) \sin(by) \\
    (u_{0y})_{mn} \sin(ax) \cos(by)
\end{bmatrix}
\]

where \( \alpha = m\pi/a \) and \( \beta = n\pi/b \). Substituting the expansions given by Eq. (17) in Eq. (14) yields for \((m, n)^{th}\) Fourier component, the following equation

\[
K \bar{U}^{mn} = \bar{F}^{mn}
\]

where

\[
\bar{U}^{mn} = \begin{bmatrix}
    u_{0x} \\
    u_{0y} \\
    w_0 \\
    \psi_{0x} \\
    \psi_{0y}
\end{bmatrix}, \quad \bar{F}^{mn} = [P_1, P_2, P_3, P_4, P_5]^T
\]

where \( K \) is symmetric stiffness matrix, \( \bar{U} \) is the displacement field and \( \bar{F} \) is the load vector.

### 4. Results and Discussions

To check the correctness and accuracy of the present formulation, we compare our results with those available in the literature. For this purpose, we consider FGM plate made of Al/ZrO₂, simply supported at all the four edges and bending due to applied uniform pressure of intensity \( p_0 \) on the top surface. The non-dimensionalized deflection and stresses are \( \hat{w} = 100Y_m h^3 w/12a^4(1-\nu_m^2) p_0^2 \) and \( \hat{\sigma}_x = \sigma_x h/a^2 p_0 \) respectively, where \( Y_m \) is the Young's modulus of the metal phase. The material properties of ceramic and metal phases considered in this study are (a) Aluminium: \( Y=70 \text{ GPa}, \nu = 0.3 \) (b) Zirconia(\( \text{ZrO}_2 \)): \( Y=200 \text{ GPa}, \nu = 0.3 \) (c) Silicon Carbide: \( Y=302 \text{ GPa}, \nu = 0.17 \). The non-dimensional values of central deflection \( \hat{w} \) and stress \( \hat{\sigma}_x \) for the FG plate with different values of thickness and inhomogeneity parameter are presented and compared in Table 1 with the published results. For ensuring the convergence, the present results are reported for 100 number of Fourier terms in our analytical solution. It can be seen from the Table 1 that the results obtained using TOT, sinusoidal and exponential theories are very close to the reported values presented in the literature for both \( a/h = 5 \) and \( 10 \). However, the hyperbolic shear deformation theory does not yield good results for thick plates but its accuracy increases as the plate becomes thinner.

The variation of centre point deflection \( \hat{w} \), stresses \( \hat{\sigma}_x \) and \( \bar{r} \) with thickness ratio \( a/h \) has been presented in Fig. 2 for \( p = 1 \) and 5 using TOT. It is seen that the values of \( \hat{w} \) and \( \hat{\sigma}_x \) decrease as \( a/h \) increases and attains a constant value after \( a/h = 40 \). However, the values of \( \bar{r}_{xz} \) increase with \( a/h \).

Next, we present the variation of deflection \( \hat{w}(x/a, 0.5b) \), stresses \( \hat{\sigma}_x(x/a, 0.5b) \) & \( \bar{r}(x/a, 0.5b) \) for Al/ZrO₂ and Al/SiC plates under uniform pressure loading in Fig. 3 and 4 respectively for different values of \( p \). The values of deflections as well as stresses are changing when the value of \( p \) changes. The effect of \( p \) is large when the difference in the properties of the constituent properties is large.
Table 1. Non-dimensional deflection $\tilde{w}$ and inplane stress $\tilde{\sigma}_x$ of a square $Al/ZrO_2$ FG plate under uniform pressure loading and simply supported at the ends.

| $p$ | Entity | $a/h$ | Homog. | Analytical Solution | FE solution |
|-----|--------|-------|--------|--------------------|-------------|
|     |        |       |        | Hyp    | Sine | Exp | TOT | Poly | Sine | TOT |
| 1   | $\tilde{w}$ | 5     | ROM   | 0.1963 | 0.2139 | 0.2317 | 0.2318 | 0.2330 | 0.2310 | 0.2313 |
|     |         |       | MTM   | 0.2158 | 0.2554 | 0.2552 | 0.2555 | 0.2548 | 0.2548 |
|     |         | 10    | ROM   | 0.1952 | 0.2041 | 0.2041 | 0.2040 | 0.2035 | 0.2035 |
|     |         |       | MTM   | 0.2146 | 0.2245 | 0.2244 | 0.2245 | 0.2239 | 0.2239 |
|     | $\tilde{\sigma}_x$ | 5     | ROM   | 0.3484 | 0.3603 | 0.3609 | 0.3593 | 0.3595 | 0.3595 |
|     |         |       | MTM   | 0.3718 | 0.3861 | 0.3868 | 0.3854 | 0.3839 | 0.3839 |
|     |         | 10    | ROM   | 0.3497 | 0.3527 | 0.3528 | 0.3524 | 0.3524 | 0.3524 |
|     |         |       | MTM   | 0.3733 | 0.3769 | 0.3771 | 0.3767 | 0.3760 | 0.3760 |
| 1   | $\tilde{w}$ | 5     | ROM   | 0.2300 | 0.2718 | 0.2716 | 0.2717 | 0.2730 | 0.2729 | 0.2714 |
|     |         |       | MTM   | 0.2480 | 0.2955 | 0.2953 | 0.2956 | 0.2950 | 0.2950 |
|     |         | 10    | ROM   | 0.2287 | 0.2392 | 0.2391 | 0.2390 | 0.2388 | 0.2388 |
|     |         |       | MTM   | 0.2465 | 0.2585 | 0.2584 | 0.2585 | 0.2580 | 0.2580 |
|     | $\tilde{\sigma}_x$ | 5     | ROM   | 0.3853 | 0.3990 | 0.3997 | 0.3979 | 0.3985 | 0.3985 |
|     |         |       | MTM   | 0.4097 | 0.4264 | 0.4271 | 0.4255 | 0.4244 | 0.4244 |
|     |         | 10    | ROM   | 0.3868 | 0.3902 | 0.3904 | 0.3900 | 0.3903 | 0.3903 |
|     |         |       | MTM   | 0.4114 | 0.4156 | 0.4158 | 0.4154 | 0.4152 | 0.4152 |
| 2   | $\tilde{w}$ | 5     | ROM   | 0.2616 | 0.3135 | 0.3133 | 0.3133 | 0.3147 | 0.3147 | 0.313 |
|     |         |       | MTM   | 0.2752 | 0.3335 | 0.3334 | 0.3335 | 0.3328 | 0.3328 |
|     |         | 10    | ROM   | 0.2601 | 0.2730 | 0.2730 | 0.2769 | 0.2726 | 0.2726 |
|     |         |       | MTM   | 0.2734 | 0.2881 | 0.2880 | 0.2880 | 0.2875 | 0.2875 |
|     | $\tilde{\sigma}_x$ | 5     | ROM   | 0.4220 | 0.4383 | 0.4391 | 0.4369 | 0.4376 | 0.4376 |
|     |         |       | MTM   | 0.4497 | 0.4691 | 0.4700 | 0.4681 | 0.4674 | 0.4674 |
|     |         |       |       | 0.4516 | 0.4565 | 0.4567 | 0.4562 | 0.4566 | 0.4566 |

Figure 2. Variation of non-dimensional centre deflection, stresses with thickness ratio $a/h$ for $Al/ZrO_2$ plate under uniform pressure loading for different inhomogeneity parameter $p$ using ROM for calculating effective properties.
5. Conclusion

Higher order theories presented above yield good accuracy. The results obtained using third order theory, exponential theory and sinusoidal theory are close to each other for very thick and thick plates. These values are very well compared with TOT FE and other FE solutions available in the literature. The hyperbolic shear deformation theory yields similar values of deflection and stresses for both very thick \((a/h = 5)\) and thick \((a/h = 10)\) FGM plate which should be different for different thickness. Therefore, hyperbolic shear deformation theory does not give accurate results for very thick plates.

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