Large subgraphs without complete bipartite graphs

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Abstract
In this note, we answer the following question of Foucaud, Krivelevich and Perarnau. What is the size of the largest $K_{r,s}$-free subgraph one can guarantee in every graph $G$ with $m$ edges? We also discuss the analogous problem for hypergraphs.

1 Introduction
Motivated by the classical Turán problem, Foucaud, Krivelevich and Perarnau [3] proposed to study the size of the largest $H$-free subgraph one can always find in every graph $G$ with $m$ edges. Denote this function by $f(m, H)$. It is easy to determine $f(m, H)$ asymptotically if $H$ is not bipartite. In [3], the authors studied this problem when forbidding all even cycles in the subgraph up to length $2k$ and obtained estimates that are tight up to a logarithmic factor. They also asked to determine $f(m, H)$ when $H$ is a complete bipartite graph. The goal of this note is to resolve this question.

2 Complete bipartite graphs
Let $K_{r,s}$ be the complete bipartite graph with parts of order $r$ and $s$, where $2 \leq r \leq s$. The following theorem gives a lower bound on $f(m, K_{r,s}).$

Theorem 2.1. Every graph $G$ with $m$ edges contains a $K_{r,r}$-free subgraph of size at least $\frac{1}{4}m^{\frac{r+1}{2r}}$.

To prove this theorem we need an upper bound on the maximum number of copies of $K_{r,r}$ which one can find in a graph with $m$ edges. The problem of maximizing the number of copies of a fixed graph $H$ was solved by Alon [1] for all graphs and by Friedgut and Kahn [4] for all hypergraphs. For our purposes the following easy estimate will suffice.

Lemma 2.2. Every graph $G$ with $m$ edges contains at most $2m^r$ copies of $K_{r,r}$.

Proof. Note that every copy of $K_{r,r}$ in $G$ contains a matching of size $r$. Clearly the number of such matchings in $G$ is at most $\binom{m}{r}$. Also note that every matching in $G$ of size $r$ can appear in at most $2^r \binom{m}{r} \leq 2m^r$. □

Using this lemma, together with a simple probabilistic argument, one can prove a lower bound on $f(m, K_{r,s}).$

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Proof of Theorem 2.1. Let $G$ be a graph with $m$ edges. Consider a random subgraph $G'$ of $G$, obtained by choosing every edge randomly and independently with probability $p = \frac{1}{2} m^{-1/(r+1)}$. Then the expected number of edges in $G'$ is $mp$. Also, by Lemma 2.2, the expected number of copies of $K_{r,r}$ in $G'$ is at most $2p^2 m^r$. Delete one edge from every copy of $K_{r,r}$ contained in $G'$. This gives a $K_{r,r}$-free subgraph of $G$, which by linearity of expectation, has at least

$$pm - 2p^2 m^r \geq \frac{1}{2} m^\frac{r}{r+1} - \frac{1}{8} m^\frac{r}{r+1} \geq \frac{1}{4} m^\frac{r}{r+1}$$

edges on average. Hence, there exists a choice of $G'$ which produces a $K_{r,r}$-free subgraph of $G$ of size at least $\frac{1}{4} m^\frac{r}{r+1}$. □

Next we show that this gives an estimate on $f(m, K_{r,s})$ which is tight up to a constant factor depending on $s$ by taking $G$ to be an appropriately chosen complete bipartite graph with $m$ edges.

Theorem 2.3. Let $2 \leq r \leq s$ and let $G$ be a complete bipartite graph with parts $U$ and $W$, where $|U| = m^{1/(r+1)}$ and $|W| = m^r/(r+1)$. Then $G$ has $m$ edges and the largest $K_{r,s}$-free subgraph of $G$ has at most $sm^r/(r+1)$ edges.

Proof. The proof is a simple application of the counting argument of Kővári-Sós-Turán [5]. Let $G'$ be a $K_{r,s}$-free subgraph of $G$ and let $d = e(G')/|W|$ be the average degree of vertices of $G'$ in $W$. If $d \geq s$, then, by convexity,

$$\sum_{w \in W} \binom{d_{G'}(w)}{r} \geq |W| \binom{d}{r} \geq \binom{s}{r} m^r/(r+1) \geq sm^r/(r+1)/r!.$$ 

On the other hand, since $G'$ is $K_{r,s}$-free we have that

$$\sum_{w \in W} \binom{d_{G'}(w)}{r} < s \binom{|U|}{r} \leq s|U|^r/r! = sm^r/(r+1)/r!.$$ 

This contradiction completes the proof of the theorem. □

Remarks.

- Since $K_{2,2}$ is also a 4-cycle, our result improves by a logarithmic factor an estimate obtained by Foucaud, Krivelevich and Perarnau [3].

- Since the Turán number for $K_{r,s}$ is not known in general, it is somewhat surprising that one can prove a tight bound on the size of the largest $K_{r,s}$-free subgraph in graphs with $m$ edges.

3 Hypergraphs

The results presented in the previous section can be extended to $k$-uniform hypergraphs, which, for brevity, we call $k$-graphs. Given a fixed $k$-graph $H$, let $f(m,H)$ denote the size of the largest $H$-free subgraph one can always find in every $k$-graph $G$ with $m$ edges. Let $K_{r_1,...,r}$ denote the complete $k$-partite $k$-graph with parts of size $r$.

Theorem 3.1. Every $k$-graph $G$ with $m$ edges contains a $K_{r_1,...,r}$-free subgraph of size at least $\frac{1}{4} m^{q-1}$, where $q = \frac{k}{r_1-1}$.
Proof. Let \( G \) be a \( k \)-graph with \( m \) edges. Every copy of \( K_{r,...,r}^{(k)} \) in \( G \) contains a matching of size \( r \) and the number of such matchings is at most \( \binom{m}{r} \). On the other hand, every matching in \( G \) of size \( r \) can appear in at most \( (k!)^r \) copies of \( K_{r,...,r} \). This implies that the total number of such copies is at most \( (k!)^r \binom{m}{r} \).

Consider a random subgraph \( G' \) of \( G \), obtained by choosing every edge randomly and independently with probability \( p = \frac{1}{2}m^{-1/q} \). Then the expected number of edges in \( G' \) is \( mp \) and the expected number of copies of \( K_{r,...,r}^{(k)} \) in \( G' \) is at most \( (k!)^r p^k \binom{m}{r} \). Delete one edge from every copy of \( K_{r,...,r}^{(k)} \) contained in \( G' \). This gives a \( K_{r,...,r}^{(k)} \)-free subgraph of \( G \) with at least

\[
pm - (k!)^r p^k \binom{m}{r} \geq \frac{1}{4} m \frac{a - 1}{q}
\]

expected edges. Hence, there exists a choice of \( G' \) which produces a \( K_{r,...,r}^{(k)} \)-free subgraph of \( G \) of this size. \( \square \)

We can again see that this estimate is tight up to a constant factor depending on \( r \).

**Theorem 3.2.** Let \( 2 \leq r, k, q = \frac{r^k - 1}{r - 1} \) and let \( G \) be a complete \( k \)-partite \( k \)-graph with parts \( U_i, 1 \leq i \leq k \), such that \( |U_i| = m^{r^{i-1}/q} \). Then \( G \) has \( m \) edges and the largest \( K_{r,...,r}^{(k)} \)-free subgraph of \( G \) has at most \( rm^{(q-1)/q} \) edges.

The proof of this theorem uses a similar counting argument to the graph case but is more involved. It follows from the following statement, which one can prove by induction. This technique has its origins in a paper of Erdős [2].

**Proposition 3.3.** Let \( G \) be a \( k \)-partite \( k \)-graph with parts \( U_i, 1 \leq i \leq k \), such that \( |U_i| = n^{r^{i-1}} \) and with a sum of \( |U_i| \) edges and \( a \geq r \). Then \( G \) contains at least \( \binom{a}{r} \prod_{i \leq k-1} \binom{|U_i|}{r} \) copies of \( K_{r,...,r}^{(k)} \).

**Proof.** We prove this by induction on \( k \). The base case \( k = 1 \) is trivial, by properly interpreting empty products as one.

Now suppose we know the statement for \( k - 1 \). For every vertex \( x \in U_k \), denote by \( G_x \) the \((k-1)\)-partite \((k-1)\)-graph which is the link of vertex \( x \) (i.e., the collection of all subsets of size \( k - 1 \) which together with \( x \) form an edge of \( G \)). Let \( a_x \prod_{i \leq k-2} |U_i| \) be the number of edges in \( G_x \). By definition, \( \sum_x a_x = a |U_k| = an^{r^k} \). By the induction hypothesis, each \( G_x \) contains at least \( \binom{a_x}{r} \prod_{i \leq k-2} \binom{|U_i|}{r} \) copies of \( K_{r,...,r}^{(k-1)} \). By convexity, the total number of such copies added over all \( G_x \) is at least

\[
\binom{a}{r} n^{r^k} \prod_{i \leq k-2} \left( \frac{|U_i|}{r} \right) = \binom{a}{r} |U_{k-1}| \prod_{i \leq k-2} \left( \frac{|U_i|}{r} \right) \geq r! \binom{a}{r} \prod_{i \leq k-1} \left( \frac{|U_i|}{r} \right)
\]

For every subset \( S \) which intersects each \( U_i \) with \( i \leq k - 1 \) in exactly \( r \) vertices, denote by \( d(S) \) the number of vertices \( x \in U_k \) such that \( x \) forms an edge of \( G \) together with every subset of \( S \) of size \( k - 1 \) which contain one vertex from each \( U_i \). By the above discussion, we have that \( \sum_S d(S) \geq a \prod_{i \leq k-1} \left( \frac{|U_i|}{r} \right) \), that is, at least the number of all copies of \( K_{r,...,r}^{(k-1)} \) in all \( G_x \). On the other hand, by the definition of \( d(S) \), the number of copies of \( K_{r,...,r}^{(k)} \) in \( G \) equals \( \sum_S (d(S)) \). Since the total number of sets \( S \) is \( \prod_{i \leq k-1} \left( \frac{|U_i|}{r} \right) \), the result now follows by convexity. \( \square \)

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