Abstract—In this paper, we address the distributed filtering and prediction of time-varying random fields. The field is observed by a sparsely connected network of agents/sensors collaborating among themselves. We develop a Kalman filter type consensus+innovations distributed linear estimator of the dynamic field termed as Consensus+Innovations Kalman Filter. We analyze the convergence properties of this distributed estimator. We prove that the mean-squared error of the estimator asymptotically converges if the degree of instability of the field dynamics is within a pre-specified threshold defined as tracking capacity of the estimator. The tracking capacity is a function of the local observation models and the agent communication network. We design the optimal consensus and innovation gain matrices yielding distributed estimates with minimized mean-squared error. Through numerical evaluations, we show that the distributed estimator with optimal gains converges faster and with approximately 3dB better mean-squared error performance than previous distributed estimators.

Index Terms—Kalman filter, distributed estimation, multi-agent networks, distributed algorithms, consensus.

I. INTRODUCTION

For decades, the Kalman-Bucy filter [1], [2] has played a key role in estimation, detection, or prediction of time-varying noisy signals. The Kalman filter is found in a wide variety of applications ranging from problems in navigation to environmental studies, computer vision to bioengineering, signal processing to econometrics. More recently, algorithms inspired by the Kalman filter have been applied to estimate random fields monitored by a network of sensors. In these problems, we distinguish two distinct layers: (a) the physical layer of the time-varying random field; and (b) the cyber layer of sensors observing the field.

A centralized approach to field estimation poses several challenges. It requires that all sensors communicate their measurements to a centralized fusion center. This is fragile to central node failure and severely taxes computationally the fusion center. Moreover, it also requires excessive communication bandwidth to and from the fusion center. Hence, the centralized approach is inelastic to estimation of large-scale time-varying random fields, like, when estimating temperature, rainfall, or wind-speed over large geographical areas [3], [4].

In [5], we proposed the Distributed Information Kalman Filter (DIKF) that is a distributed estimator of time-varying random fields consisting of two substructures. The first is the Dynamic Consensus on Pseudo-Observations (DCPO), a distributed estimator of the global average of the pseudo-observations (modified versions of the observations) of the agents. The second substructure uses these average estimates of pseudo-observations to estimate the time-varying random field. In this paper, we develop a distributed Kalman filter like estimator, the Consensus+Innovations Kalman Filter (CIKF), that instead of using the pseudo-observations uses distributed estimates of the pseudo-state (modified version of a state) to estimate the field. We show how to design optimally the gain matrices of the CIKF. We prove that the CIKF converges in the mean-squared error (MSE) sense when the degree of instability of the dynamics of the random field is within the network tracking capacity [6], a threshold determined by the cyber network connectivity and the local observation models.

We review related prior research on distributed estimation of time-varying random fields. We classify prior work into two categories based on the time-scales of operation: (a) two time-scale and (b) single time-scale. In two-time scale distributed estimators, see Fig [12], agents exchange their information multiple number of times between each dynamics/observations time-scale [7]–[15], so that average consensus occurs between observations. In contrast, in single time-scale approaches [5], [6], [16]–[24], the agents collaborate with their neighbors only once in between each dynamics/observation evolution. In other words, the dynamics, observation, and communication follow the same time scale as depicted in Fig [12].

The two time-scale approach demands fast communication between agents. In most practical applications, this is not true. Further, references [13], [14] developed distributed state estimators assuming local observability of the dynamic state in the physical layer. Such assumption is not feasible in large-scale systems. References [25], [26] assume a complete cyber network, which is not scalable.

We are interested in a single time-scale approach. We consider that the time-varying system is not locally observable at each agent, and we assume that the communication network in the cyber layer is sparsely connected. In the single time-scale category, references [19], [20] propose distributed Kalman filters where the agents communicate among themselves using the Gossip protocol [27]. Although Gossip filters require very low communication bandwidth, their MSE is higher and their convergence rate is lower than the consensus+innovations based distributed approaches. References [21], [22], [24] introduced consensus+innovations type distributed estimator for parameter (static state) estimation. This approach is extended to estimating time-varying random states in [6], [16]–[18]. Distributed consensus+innovations dynamic state estimators converge in MSE sense if their degree of instability of the state dynamics is below the network tracking capacity, see [6].

Consensus+Innovations Distributed Kalman Filter with Optimized Gains

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Abstract—In this paper, we address the distributed filtering and prediction of time-varying random fields. The field is observed by a sparsely connected network of agents/sensors collaborating among themselves. We develop a Kalman filter type consensus+innovations distributed linear estimator of the dynamic field termed as Consensus+Innovations Kalman Filter. We analyze the convergence properties of this distributed estimator. We prove that the mean-squared error of the estimator asymptotically converges if the degree of instability of the field dynamics is within a pre-specified threshold defined as tracking capacity of the estimator. The tracking capacity is a function of the local observation models and the agent communication network. We design the optimal consensus and innovation gain matrices yielding distributed estimates with minimized mean-squared error. Through numerical evaluations, we show that, the distributed estimator with optimal gains converges faster and with approximately 3dB better mean-squared error performance than previous distributed estimators.

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in this paper, we derive the tracking capacity for the CIKF. Its tracking capacity is a function of the local observation models and of the agents communication connectivity.

The single time-scale consensus+innovations distributed estimators that we introduced in [28]–[31] run a companion filter to estimate the global average of the pseudo-innovations, a modified version of the innovations. The DIKF, we proposed in [5], uses averaged pseudo-observations (linearly transformed observations) rather than pseudo-innovations. In contrast, this paper uses estimates of the pseudo-state, a linear transformation of the dynamic state. In centralized information filter, the pseudo-state is directly available from the observations. In the distributed setting, since not all the observations are available to the local sensors, the CIKF has to distributedly estimate the pseudo-state through a consensus step. Using pseudo-state rather than pseudo-innovations [28]–[31] or pseudo-observations [5] leads to significant better performance as we show here. The MSE of the CIKF is lower than that of the distributed estimators in [5], [19], [20], [28]–[31].

Developing distributed estimators for time-varying random fields (“distributed Kalman filter”) has gained considerable attention over the last few years. The goal has been to achieve MSE performance as close as possible to the optimal centralized Kalman filter. We show in this paper that the CIKF converges to a bounded MSE solution requiring minimal assumptions, namely global detectability and connected network, and not requiring the additional distributed observability assumption, as needed by the DIKF. The distributed optimal time-varying field estimation is, in general, NP-hard. The distributed parameter estimation [21], [32] have shown asymptotic optimality, in the sense that the distributed parameter estimator is asymptotically unbiased, consistent, and efficient converging at the same rate as the centralized optimal estimator. However, estimation of time-varying random fields adds another degree of complication since, while information diffuses through the network, the field itself evolves. So, this lag causes a gap in performance between distributed and centralized field estimators. Numerical simulations show that the proposed CIKF improves the performance by 3dB over the DIKF, reducing by half the gap to the centralized (optimal) Kalman filter, while showing a faster convergence rate than the DIKF. These improvements significantly distinguish the CIKF from the DIKF.

The rest of the paper is organized as follows. We describe the three aspects of the problem setup in Section II. Section III introduces the pseudo-state and presents the proposed optimal gain distributed Kalman filter (CIKF). In Section IV we analyze the dynamics of the error processes and their covariances. Section V includes the analysis of the tracking capacity of the proposed CIKF. We design the optimal gain matrices to obtain distributed estimates in Section VI. Numerical simulations are in Section VII. We present the concluding remarks in Section VIII. Proofs of the proposition, lemmas and theorems are in the Appendix A, Appendix B-E and Appendix F-I, respectively.

II. SYSTEM, OBSERVATION, AND COMMUNICATION

The distributed estimation framework consists of three components: dynamical system, local observation, and neighborhood communications. These three parts include two layers: the physical layer and the cyber layer. For the sake of simplicity, we motivate the model with the example of a time-varying temperature field over a large geographical area monitored by a sensor network.

A. Physical layer: dynamical system

Consider a time-varying temperature field distributed over a large geographical area, as shown in Fig 2a. A first-order approximation and discretization of the temperature field provide spatio-temporal discretized temperature variables $x_i^1, j = 1, \cdots, M$, of $M$ sites at discrete time indexes $i = 1, \cdots, T$. We stack the $M$ field variables in a temperature state vector $x_i = [x_i^1, \cdots, x_i^M]^T \in \mathbb{R}^M$. The evolution of the time-varying temperature field, $x_i$, can be represented by a discrete time linear dynamical system

$$x_{i+1} = Ax_i + v_i,$$  

where the first-order dynamics matrix $A \in \mathbb{R}^{M \times M}$ contains the coupling effects between the $M$ temperature variables, and the residual $v_i = [v_i^1, \cdots, v_i^M]^T \in \mathbb{R}^M$ is the system noise driving the dynamical temperature field. At each of the $M$ sites, the input noise $v_i^j, j = 1, \cdots, M$ accounts for the deviations in the temperature after the overall field dynamics. The field dynamics $A$ incorporates the sparsity pattern and connectivity of the physical layer consisting of the dynamical system (1).
B. Cyber layer: local observations

The physical layer consisting of the field dynamics is observed by a cyber layer consisting of a network of agents (sensors). In Fig 2b we see that there are N sensors, where each sensor observes the temperatures of only a few sites. Denote the number of sites observed by sensor n by $M_n$, $M_n \ll M$, and its measurements at time $i$ by $z^n_i \in \mathbb{R}^{M_n}$. The observations of the agents in the cyber layer can be represented by a linear model:

$$ z^n_i = H_n x_i + r^n_i, \quad n = 1, \ldots, N, $$

where, the observation matrix $H_n \in \mathbb{R}^{M_n \times M}$ contains the observation pattern and strength information, and the observation noise $r^n_i \in \mathbb{R}^{M_n}$ reflects the inaccuracy in measurements due to sensor precision, high frequency fluctuations in temperature, and other unavoidable constraints.

To illustrate, consider that sensor 1 is observing the temperature of 3 sites $\{x^1_i, x^2_i, x^3_i\}$, i.e., $M_1 = 3$. An example snapshot of the observation model of agent $n$ is:

$$
\begin{pmatrix}
z^1_i \\
z^2_i \\
z^3_i
\end{pmatrix} =
\begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
0 & 5 & 0 & \ldots & 0 \\
0 & 0 & 5 & \ldots & 0
\end{bmatrix}
\begin{pmatrix}
x^1_i \\
x^2_i \\
x^3_i
\end{pmatrix} +
\begin{bmatrix}
r^1_i \\
r^2_i \\
r^3_i
\end{bmatrix} =
\begin{bmatrix}
x^1_i + \epsilon^1_i \\
x^2_i + \epsilon^2_i \\
x^3_i + \epsilon^3_i
\end{bmatrix}.
$$

Now, for the ease of analysis, we aggregate the noisy local temperature measurements, $z^1_i, \ldots, z^N_i$ of all the N agents in a global observation vector, $z_i \in \mathbb{R}^{\sum_{n=1}^N M_n}$:

$$
\begin{pmatrix}
z^1_i \\
z^2_i \\
z^3_i
\end{pmatrix} =
\begin{pmatrix}
H_1 \\
H_2 \\
H_3
\end{pmatrix}
\begin{pmatrix}
x_i \\
r^1_i \\
r^2_i \\
r^3_i
\end{pmatrix},
$$

where, the global observation matrix is $H \in \mathbb{R}^{\sum_{n=1}^N M_n \times M}$ and the stacked measurement noise is $r_i \in \mathbb{R}^{\sum_{n=1}^N M_n}$. Note that, in general, the temperature measurement model is non-linear. For non-linear cases, refer to distributed particle filter in [33] and the references cited therein. Here we perform a first-order approximation to obtain a linear observation sequence.

C. Cyber layer: neighborhood communication

In the cyber layer, the agents exchange their temperature readings or current estimates with their neighbors. In many applications, to reduce communications costs, neighbors communicate only with their geographically nearest agents as shown in Fig 2c.

Formally, the agent communication network is defined by a simple (no self-loops nor multiple edges), undirected, connected graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of sensors (nodes or agents) and $\mathcal{E}$ is the set of local communication channels (edges or links) among the agents. The open $\Omega_n$ and closed $\overline{\Omega}_n$ neighborhoods of agent $n$ are:

$$
\begin{align*}
\Omega_n &= \{l | (n, l) \in \mathcal{E}\}, \\
\overline{\Omega}_n &= n \cup \{l | (n, l) \in \mathcal{E}\}.
\end{align*}
$$

In Fig 2c the open and closed neighborhoods of agent 1 are $\Omega_1 = \{2, 3\}$ and $\overline{\Omega}_1 = \{1, 2, 3\}$, respectively. The Laplacian matrix of $G$ is denoted by $L$. The eigenvalues of the positive semi-definite matrix $L$ are ordered as $0 = \lambda_1(L) \leq \lambda_2(L) \leq \ldots \leq \lambda_N(L)$. For details on graphs refer to [34]. The communication network is sparse and time-invariant.

D. Modeling assumptions

We make the following assumptions.

Assumption 1 (Gaussian processes). The system noise, $v_i$, the observation noise, $r_i$, and the initial condition of the system, $x_0$, are Gaussian sequences, with

$$
v_i \sim \mathcal{N}(0, V), \quad r^n_i \sim \mathcal{N}(0, R_n), \quad x_0 \sim \mathcal{N}(\bar{x}_0, \Sigma_0),$$

where, $V \in \mathbb{R}^{M \times M}$, $R_n \in \mathbb{R}^{M_n \times M_n}$ and $\Sigma_0 \in \mathbb{R}^{M \times M}$ are the corresponding covariance matrices. The noise covariance matrix $R$ of the global noise vector $v_i$ in (3) is block-diagonal, i.e., $R = \text{blockdiag}\{R_1, \ldots, R_N\}$, and positive-definite, i.e., $R > 0$.

Assumption 2 (Uncorrelated sequences). The system noise, the observation noise, and the initial condition: $\{v_i\}_i, \{r_i\}_i, x_0 \geq 0$ are uncorrelated random vector sequences.

Assumption 3 (Prior information). Each agent in the cyber layer knows the system dynamics model, $A$ and $V$, the
initial condition statistics, $\bar{x}_0$ and $\Sigma_0$, the parameters of the observation model, $H$ and $R$, and the communication network model, $G$.

In large-scale system applications, the dynamics, observation, and network Laplacian matrices, $A$, $H_n$, and $L$, are sparse, with $M_n \ll M$, and the agents communicate with only a few of their neighbors, $|\Omega_n| \ll N, \forall n$. In the dynamics (4) and observations (2), we assume that there is no deterministic input. The results are readily extended if there is a known deterministic input.

E. Centralized Information filter

Although not practical in the context of the problem we study, we use the centralized information filter to benchmark our results on distributed estimator. In a centralized scheme, all the agents in the cyber layer communicate their measurements to a central fusion center, as depicted in Fig 3. The fusion center performs all needed computation tasks. Refer to [5], [35] for the filter, gain, and error equations of the centralized information filter.

III. DISTRIBUTED FILTERING AND PREDICTION

This section considers our single time-scale distributed solution. We start with the introduction and derivation of the key components of our distributed estimator and then present our distributed field estimator.

A. Pseudo-state model

In a centralized information filter [35], all the observations are converted into pseudo-observations [5] to obtain the optimal estimates. Following (2), the pseudo-observation $\tilde{z}^n_i$ of agent $n$ is

$$\tilde{z}^n_i = H^T_n R_n^{-1} z^n_i = \overline{H}_n^T \bar{x}_i + H^T_n R_n^{-1} r^i_n,$$

where,

$$\overline{H}_n = H^T_n R_n^{-1} H_n.$$  

The centralized information filter computes the sum, $\overline{z}_i$, of all the pseudo-observations

$$\overline{z}_i = \sum_{n=1}^{N} \tilde{z}^n_i = G \bar{x}_i + H^T R^{-1} r_i$$

where,

$$G = \sum_{n=1}^{N} H^T_n R_n^{-1} H_n = \sum_{n=1}^{N} \overline{H}_n.$$  

The aggregated pseudo-observation, $\overline{z}_i$, is the key term in the centralized filter. It provides the innovations term in the filter updates enabling the filter to converge with minimum MSE estimates. However, in the distributed solution, each agent $n$ does not have access to all the pseudo-observations; instead it can only communicate with its neighbors. To address this issue, in [5] we introduced a dynamic consensus algorithm to compute the distributed estimates of the averaged pseudo-observations, $\overline{z}_i$, at each agent. In (6), we note that the crucial term is $G \bar{x}_i$ which carries the information of the dynamic state, $\bar{x}_i$, and the second term in $\overline{z}_i$ in (6) is noise. We refer to it as the pseudo-state, $y_i$,

$$y_i = G \bar{x}_i.$$  

The pseudo-state, $y_i$, is also a random field whose time dynamics can be represented by a discrete-time linear dynamical system. The pseudo-observations $\tilde{z}^n_i$ are its linear measurements. We summarize the state-space model for the pseudo-state in the following proposition. See Appendix A for the details of the proof.

Proposition 1. The dynamics and observations of the pseudo-state $y_i$ are:

$$y_{i+1} = \tilde{A}y_i + Gv_i + \tilde{A}x_i$$

$$\tilde{z}^n_i = \tilde{H}_n y_i + H^T_n R_n^{-1} r^i_n + \tilde{H}_n x_i.$$  

The pseudo-dynamics matrix $\tilde{A}$, pseudo-observations matrix $\tilde{H}_n$, and the matrices $\tilde{A}, \tilde{H}_n$, and $\tilde{H}$ at agent $n$ are:

$$\tilde{A} = G A^\dagger$$

$$\tilde{H}_n = H^T_n R_n^{-1} H_n G^\dagger = \overline{H}_n G^\dagger$$

$$\tilde{H} = G A I$$

$$\tilde{I} = I - G^\dagger G,$$

where, $G^\dagger$ denotes the Moore-Penrose pseudo-inverse of $G$.

In [5], the distributed information Kalman filter (DIKF) assumes distributed observability, i.e., it considers the case where $G$ is invertible. Under this assumption, $G^\dagger = G^{-1} \tilde{I} = 0$. In this paper we relax the requirement of invertibility of $G$, proposing a distributed estimator for general dynamics-observation models under the assumption of global detectability. In most cases $\tilde{I}$ is low-rank. In (9), the term $(Gv_i + \tilde{A}x_i = \xi_i, say)$ can be interpreted as the pseudo-state input noise, which follows Gauss dynamics

$$\xi_i \sim \mathcal{N}(\tilde{A}x_i, G V G + \tilde{A} \Sigma_i \tilde{A}^T)$$

where,

$$\Sigma_i = E[ (x_i - \bar{x}_i)(x_i - \bar{x}_i)^T].$$

Similarly, in (10), the term $(\delta^n_i = H^T_n R_n^{-1} r^i_n + \tilde{H}_n x_i)$ is the pseudo-state observation noise at agent $n$, which is Gaussian

$$\delta^n_i \sim \mathcal{N}(\tilde{H}_n \bar{x}_i, \overline{H}_n + \tilde{H}_n \Sigma \tilde{H}_n).$$
For the ease of analysis, we express the pseudo-state observation model in vector form by \( \bar{z}_i \in \mathbb{R}^{N_i \times M} \), the aggregate of the noisy local temperature measurements, \( \bar{z}_i, \cdots , \bar{z}_i^N \), of all the agents, the minimized MSE filter and prediction estimates are each agent \( y_i, \cdots , y_i^N \), of the state \( x_i \), and observations model of the pseudo-state, \( y_i \), of the state \( x_i \), in the cyber layer to implement the Consensus\+Innovations Kalman Filter (CIKF) and thereby obtain the unbiased minimized MSE distributed estimates of the dynamic state \( x_i \). Each agent \( n \) runs Algorithm 1 locally.

Later in Section VII, we analyze and compare the performance of CIKF with that of the distributed information Kalman filter (DKF) [5] and of the centralized Kalman filter (CKF). The centralized filter collects measurements from all

C. CIKF: Assumptions

The Consensus\+Innovations Kalman Filter (CIKF) achieves convergence given the following assumptions:

**Assumption 4 (Global detectability).** The dynamic state equation (1) and the observations model (3) are globally detectable, i.e., the pair \((A, H)\) is detectable.

**Assumption 5 (Connectedness).** The agent communication network is connected, i.e., the algebraic connectivity \( \lambda_2(L) \) of the Laplacian matrix \( L \) of the graph \( G \) is strictly positive.

By Assumption 4, the state-observation model (1)-(3) is globally detectable but not necessarily locally detectable, i.e., \((A, H_n), \forall n\), are not necessarily detectable. Note that these two are minimal assumptions. Assumption 4 is mandatory even for a centralized system, and Assumption 5 is required for consensus algorithms to converge. Further, note that in this paper we do not consider distributed observability (invertibility of \( G \)) of the model setup, which is the strong and restrictive assumption taken in [5], [21], [22] and [36] and similar to weak detectability presented in [19].

D. CIKF: Update algorithm

In this subsection, we present the step-by-step tasks executed by each agent \( n \) in the cyber layer to implement the Consensus\+Innovations Kalman Filter (CIKF) and thereby obtain the unbiased minimized MSE distributed estimates of the dynamic state \( x_i \). Each agent \( n \) runs Algorithm 1 locally.
The distributed filter and prediction estimates, $\hat{x}_{i|n}$ and $\hat{y}_{i|n}$, of the pseudo-state and state at agent $n$ by
\begin{align}
\hat{e}_{i|n} &= y_i - \hat{y}_{i|n}, \\
\hat{e}_{i+1|n} &= x_i - \hat{x}_{i+1|n}.
\end{align}

Similarly, represent the prediction error processes $e_{i+1|i}$ and $e_{i+1+i|1}$ of the pseudo-state and the state at agent $n$ by
\begin{align}
e_{i+1|i} &= y_{i+1} - \hat{y}_{i+1|i}, \\
e_{i+1+i|1} &= x_{i+1} - \hat{x}_{i+1+i|1}.
\end{align}

We establish that the CIKF provides unbiased estimates of the state and pseudo-state in the following lemma whose proof is sketched in Appendix E.

**Lemma 1.** The distributed filter and prediction estimates, $\hat{y}_{i|n}$, $\hat{x}_{i|n}$, $\hat{y}_{i+1|n}$, $\hat{x}_{i+1|n}$, $\hat{e}_{i|n}$, $\hat{e}_{i+1|n}$, $e_{i+1|i}$, and $e_{i+1+i|1}$ are zero-mean at all agents $n$:
\begin{align}
E[e_{i|n}] &= 0, \ E[e_{i+1|i}] = 0, \ E[e_{i+1+i|1}] = 0. \tag{29}
\end{align}

Each agent exchanges their estimates with their neighbors, hence their error processes are correlated. It is not feasible to analyze the error process of each agent separately as they depend on each other. To analyze all of them together, we stack the estimates and the errors of all the agents as we have done earlier for observations (3) and pseudo-observations (16):

We summarize the dynamics of the error processes in the following lemma whose proof is in Appendix C.

**Lemma 2.** The error processes, $e_{i|n}$, $e_{i+1|i}$, $e_{i+1+i|1}$, and $e_{i+1+i|1}$ are Gaussian and their dynamics are:
\begin{align}
e_{i|n} &= (I_{N} - B_{i}^{-1}B_{i}^{T} D_{H}) e_{i-1|n} - B_{i}^{-1} D_{H} \Sigma_{i-1|n}^{-1} r_{i}, \tag{30} \\
e_{i|n} &= (I_{N} - K_{i} (I_{N} \otimes G)) e_{i-1|n} + K_{i} e_{i|i}, \tag{31} \\
e_{i+1|i} &= (I_{N} \otimes A) e_{i|i} + (I_{N} \otimes \tilde{A}) e_{i|i} + I_{N} \otimes (G v_{i}), \tag{32} \\
e_{i+1+i|1} &= (I_{N} \otimes A) e_{i+1|i} + I_{N} \otimes r_{i}, \tag{33}
\end{align}

where, $B_{i}^{C}$ is the consensus gain matrix and $B_{i}^{T}$. $K_{i}$ are the innovations gain matrices for the pseudo-state and state estimation, respectively. The block diagonal matrices are $D_{H} = blockdiag\{H_{1}, \ldots, H_{N}\}$ and $D_{H} = blockdiag\{H_{1}, \ldots, H_{N}\}$.

The symbol $\otimes$ denotes the Kronecker matrix product. Lemma 1 established that the error processes (30)-(33) are unbiased. It then follows that the filter and prediction error covariances of the pseudo-state and state are simply:
\begin{align}
P_{i|n} &= \mathbb{E}[e_{i|n} e_{i|n}^{T}], \tag{34} \\
P_{i+1|i} &= \mathbb{E}[e_{i+1|i} e_{i+1|i}^{T}], \tag{35} \\
\Sigma_{i|n} &= \mathbb{E}[e_{i|i} e_{i|i}^{T}], \tag{36} \\
\Sigma_{i+1|i} &= \mathbb{E}[e_{i+1|i} e_{i+1|i}^{T}]. \tag{37}
\end{align}

Note that the state estimates $\hat{x}_{i|n}$, $\hat{x}_{i+1|n}$ depend on the pseudo-state estimates $\hat{y}_{i|n}$, $\hat{y}_{i+1|n}$. Hence the error process (30)-(33) are not uncorrelated. The filter and prediction cross-covariances are:
\begin{align}
\Pi_{i|n} &= \mathbb{E}[e_{i|n} e_{i|n}^{T}], \tag{38} \\
\Pi_{i+1|i} &= \mathbb{E}[e_{i+1|i} e_{i+1|i}^{T}], \tag{39} \\
\Gamma_{i} &= \mathbb{E}[e_{i|i} e_{i|i}^{T}]. \tag{40}
\end{align}

In the following theorem, we define and derive the evolution of the state, pseudo-state, and cross error covariances.

**Theorem 2.** The filter error covariances, $P_{i|n}$, $\Sigma_{i|n}$, $\Pi_{i|n}$, and the predictor error covariances, $P_{i+1|i}$, $\Sigma_{i+1|i}$, $\Pi_{i+1|i}$, follow Lyapunov-type iterations (34)-(41), where, $J = (I_{N} \otimes F_{N}) \otimes I_{M}$ and the initial conditions are $\Sigma_{i-1|n} = J \otimes \Sigma_{0}$, $P_{0|i} = J \otimes (G_{i} \Sigma_{0})$, $\Pi_{0|i} = J \otimes (\Sigma_{0})$.

The proof of the theorem is in Appendix C. The iterations (34) and (41) combined together constitute the distributed version of the discrete algebraic Riccati equation. The MSE of the proposed CIKF is the trace of the error covariance, $\Sigma_{i+1|i}$.
The optimal design of the gain matrices, $B_i$ and $K_i$, such that the CIKF minimizes the MISE, is discussed in Section VI. Before that in Section V we derive the conditions under which the CIKF converges, in other words, the dynamics of the filter error processes, bounded error covariances that in turn guarantee the convergence of the CIKF. Note that if the dynamics of the prediction error processes, $e_{i+1|1}$, $e_{i+1|1}$ are asymptotically stable, then the dynamics of the filter error processes, $e_{i|1}$, $e_{i|1}$ are also asymptotically stable. That is why we study the dynamics of only one of the error processes and in this paper we consider the prediction error processes.

### A. Asymptotic stability of error processes

To analyze the stability of the error processes, we first write the evolution of the prediction error processes, combining (30)-(33),

$$e_{i+1|1} = (I_N \otimes \bar{A}) (I_{MN} - B_i^c - B_i^T \bar{D}_H) e_{i|1} + \bar{\phi}_i,$$  \hspace{1cm} (48)

$$e_{i+1|1} = (I_N \otimes A) (I_{MN} - K_i (I_N \otimes G)) e_{i|1} + \bar{\phi}_i,$$  \hspace{1cm} (49)

where, the noise processes $\bar{\phi}_i$ and $\phi_i$ are

$$\bar{\phi}_i = (I_N \otimes \bar{A}) e_{i|1} - (I_N \otimes \bar{A}) B_i^T \bar{D}_H e_{i|1} + 1_N \otimes (G u_i),$$

$$\phi_i = (I_N \otimes A) K_i e_{i|1} + 1_N \otimes v_i.$$  \hspace{1cm} (50)

The statistical properties of the noises, $\bar{\phi}_i$ and $\phi_i$, of the error processes, $e_{i+1|1}$ and $e_{i+1|1}$, are stated in the following Lemma, and the proof is included in Appendix D.

**Lemma 3.** The noise sequences $\bar{\phi}_i$ and $\phi_i$ are zero-mean Gaussian that follow $\bar{\phi}_i \in \mathcal{N}(0, \Phi_i)$ and $\phi_i \in \mathcal{N}(0, \Phi_i)$.

The dynamics of the error processes are characterized by (48)-(49) and Lemma 3. Let $\rho(.)$ and $\|\cdot\|_2$ denote the spectral radius and the spectral norm of a matrix, respectively. The error processes are asymptotically stable if and only if the spectral radii of $\bar{F}$, $F$ are less than one, i.e.,

$$\rho(\bar{F}) < 1, \quad \rho(F) < 1$$  \hspace{1cm} (52)

and the noise covariances, $\bar{\Phi}_i$, $\Phi_i$ are bounded, i.e., $\|\bar{\Phi}_i\|_2 < \infty$, $\|\Phi_i\|_2 < \infty$, $\forall i$. Now if (52) holds, then the prediction error covariances $P_{i+1|1}$, $\Sigma_{i+1|1}$ are bounded; this ensures the error process convergences $P_{i|1}$, $\Sigma_{i|1}$ are also bounded. Further, the model noise covariances $V$ and $R$ are bounded. Then, by (67)-(68), the noise covariances $\bar{\Phi}_i$ and $\Phi_i$ are bounded if the spectral radii are less than one. Thus, (52) are the necessary and sufficient conditions for the convergence of the CIKF algorithm.

### B. Tracking capacity for unstable systems

The stability of the underlying dynamical system (1) in the physical layer is determined by the dynamics matrix $A$. If the system is asymptotically stable, i.e., $\rho(A) < 1$, then there always exist gain matrices $B_i$, $K_i$ such that (52) holds true. Hence for stable systems, the CIKF always converges with a bounded MSE solution. In contrast, for an unstable dynamical system (1), $\rho(A) > 1$, it may not always be possible to find gain matrices $B_i$, $K_i$ satisfying (52) conditions. There exists an upper threshold on the degree of instability of the system dynamics, $A$, that guarantees the convergence of the proposed CIKF. The threshold, similar to Network Tracking Capacity in [6], is the tracking capacity of the CIKF algorithm, and it depends on the agent communication network and observation models, as summarized in the following theorem.

**Theorem 3.** The tracking capacity of the CIKF is, $C$,

$$C = \max_{B_i^c, B_i^T} \lambda_1 \lambda_m \frac{\|I_{MN} - B_i^c - B_i^T \bar{D}_H\|_2}{\lambda_1 \lambda_m}$$  \hspace{1cm} (53)

where, $B_i^c$ has the same block sparsity pattern as the graph Laplacian $L$, $B_i^T$ is a block diagonal matrix, and, $\lambda_1$ and $\lambda_m$ are the minimum and maximum non-zero eigenvalues of $G$, $0 < \lambda_1 \leq \cdots \leq \lambda_m$. If $\|A\|_2 < C$, then there exists $B_i$, $K_i$ such that the CIKF (21)-(24) converges with bounded MSE.

The proof is in Appendix E. In the above theorem, the structural constraints on the gain matrices $B_i^c$, $B_i^T$ ensure that each agent combines its neighbors’ estimates for consensus
and its own pseudo-observations for the innovation part of the CIKF. The block sparsity pattern of $B_i^c$ being similar to that of $L$ implies that the tracking capacity is dependent on the connectivity of the communication network. Similarly, since $D_H$ is a block-diagonal matrix containing the observation matrices $H_n$, we conclude that the tracking capacity is also a function of the observation models.

The tracking capacity increases with the increase in communication graph connectivity and observation density. For instance, the tracking capacity is infinite if all agents are connected with everyone else (complete graph) or all the agents observe the entire dynamical system (local observability). Given the tracking capacity is satisfied for the system, observation, and communication models (1)-(4), the question remains how to design the gain matrices $B_i$ and $K_i$ to minimize the MSE of the CIKF, which we discuss in the following section.

VI. OPTIMAL GAIN DESIGN

The asymptotic stability of the error dynamics guarantees convergence of CIKF and bounded MSE, but here we discuss how to design the $B_i$ and $K_i$ such that the MSE is not only bounded but also minimum.

A. New uncorrelated information

In CIKF Algorithm 1 at any time $i$ each agent $n$ makes pseudo-observation $\tilde{z}_i^n$ of the state and receives prior estimates $\{\tilde{y}_{i|j-1}^n\}_{j \in \Omega_n}$ from its neighbors. The CIKF algorithm employs this new information to compute the distributed filter estimates of the pseudo-state and state. Denote the new information for the pseudo-state and state filtering by $\theta_i^n$ and $\theta_i^\nu$, respectively,

$$\tilde{\theta}_i^n = \begin{bmatrix} y_{i|j-1}^l \\ \vdots \\ \tilde{y}_{i|d_n}^l \\ \tilde{z}_i^n \end{bmatrix}, \quad \theta_i^n = \tilde{y}_{i|j-1}^n$$

where, $\{l_1, \ldots, l_{d_n}\} = \Omega_n$ and $d_n = |\Omega_n|$ is the degree of agent $n$. Note the new information, $\theta_i^n$ and $\theta_i^\nu$, are Gaussian since they are linear combinations of Gaussian sequences. However, $\tilde{\theta}_i^n$ and $\tilde{\theta}_i^\nu$ are correlated with the previous estimates $\tilde{y}_{i|j-1}^n$ and $\tilde{z}_{i|j-1}^n$. So, we transform them into uncorrelated new information and then combine the uncorrelated information with the previous estimates $\tilde{y}_{i|j-1}^n$ and $\tilde{z}_{i|j-1}^n$ to compute the current filtered estimates.

**Lemma 4.** The new uncorrelated information $\tilde{\nu}_i^n$ and $\nu_i^n$ for filtering update at agent $n$ are,

$$\tilde{\nu}_i^n = \tilde{\theta}_i^n - \tilde{\theta}^n_i, \quad \tilde{\theta}^m_i = \mathbb{E} \left[ \tilde{\theta}_i^n | \tilde{z}_{i|j-1}^n, \{\tilde{y}_{i|j-1}^l\}_{j \in \Omega_n} \right]$$

that expands to

$$\tilde{\nu}_i^n = \begin{bmatrix} y_{i|j-1}^l - \tilde{y}_{i|l-1}^n \\ \vdots \\ \tilde{y}_{i|d_n}^l - \tilde{y}_{i|d_n-1}^n \\ \tilde{z}_i^n - H_n \tilde{y}_{i|j-1}^n - H_n \tilde{z}_{i|j-1}^n \end{bmatrix}, \quad \nu_i^n = \tilde{y}_{i|j-1}^n - G \tilde{z}_{i|j-1}^n. \quad \text{(57)}$$

The uncorrelated sequences $\tilde{\nu}_i^n$ and $\nu_i^n$ are zero-mean Gaussian random vectors. Hence $\tilde{\nu}_i^n$ and $\nu_i^n$ are independent sequences.

The proof is in Appendix 3. We write the CIKF filter updates (21) and (22) in terms of the new uncorrelated information $\tilde{\nu}_i^n$ and $\nu_i^n$ from (57).

$$\tilde{y}_{i|j}^n = \tilde{y}_{i|j-1}^n + \tilde{B}_i^n \tilde{\nu}_i^n \quad \text{(58)}$$

$$\tilde{z}_{i|j}^n = \tilde{z}_{i|j-1} + K_i^n \nu_i^n \quad \text{(59)}$$

where, $\tilde{B}_i^n$ are the building blocks of the pseudo-state gain matrix $B_i$.

B. Consensus and innovation gains

Here, we present the methods to: (a) design the matrices $\tilde{B}_i^n$ and $K_i^n$, and (b) obtain the optimal gains $B_i$ and $K_i$ from them. These optimal gains provide the distributed minimized MSE estimates of the field. At agent $n$, we define the matrix $\hat{B}_i^n$ as

$$\hat{B}_i^n = \begin{bmatrix} B_{i|l_1}^{n, i_1}, \ldots, B_{i|l_{d_n}}^{n, i_{d_n}}, & B_{i|n}^{n} \end{bmatrix} \quad \text{(60)}$$

where, $\{l_1, \ldots, l_{d_n}\} = \Omega_n$. The gain matrix $B_i$ is a linear combination of $B_i^c$ and $B_i^T$, where $B_i^c$ has the same block structure as the graph Laplacian $L$ and $B_i^T$ is a block diagonal. The $(n,l)^{th}$ blocks of the $n^{th}$ row of $B_i^n$ are:

$$[\hat{B}_i^n]_{nl} = \begin{cases} -B_{i|l}^{n}, & \text{if } l \in \Omega_n \\ \sum_{j=1}^{d_n} B_{i|l}^{n}, & \text{if } l = n \\ 0, & \text{otherwise} \end{cases} \quad \text{(61)}$$

The $(n,n)^{th}$ blocks of the diagonal block matrices $B_i^T$ and $K_i$ are:

$$[B_i^T]_{nn} = B_{i|n}^{n}, \quad [K_i]_{nn} = K_{i|n}^{n} \quad \text{(62)}$$

Hence, once we design the matrices $\tilde{B}_i^n$ and $K_i^n$, it will provide the optimal gain matrices $B_i^c$, $B_i^T$, and $K_i$.

**Theorem 4.** The optimal gains for the CIKF algorithm are

$$\hat{B}_i^n = \Sigma_{y_i^n, \tilde{\nu}_i^n} \left( \Sigma_{\nu_i^n} \right)^{-1}, \quad \hat{K}_i^n = \Sigma_{x_i^n, \nu_i^n} \left( \Sigma_{\nu_i^n} \right)^{-1}$$

where, $\Sigma_{\tilde{\nu}_i^n}, \Sigma_{\nu_i^n}$ are the covariances of the new uncorrelated information $\tilde{\nu}_i^n$ and $\nu_i^n$; and, $\Sigma_y^n, \Sigma_{\tilde{\nu}_i^n}, \Sigma_{x_i^n, \nu_i^n}$ are cross-covariances between $y_i^n, \tilde{\nu}_i^n, \nu_i^n$ and $x_i^n, \nu_i^n$, respectively. These covariance and cross-covariance matrices are related to the error covariance matrices, $P_{i|l-1}^{n}, P_{i|l|n}, \Sigma_{i|l-1}^{n}, \Gamma_i^n, \hat{B}_i^n$, by the following functions:

$$\Sigma_{y_i^n, \tilde{\nu}_i^n} = \begin{bmatrix} P_{i|l-1}^{n-1} - P_{i|l|n-1}^{n-1} & \ldots & P_{i|l-1}^{n} - P_{i|l|n}^{n} & P_{i|l|n}^{n} \\ P_{i|l|n}^{n} & P_{i|l|n}^{n} & \ldots & P_{i|l|n}^{n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{i|l|n}^{n} & P_{i|l|n}^{n} & \ldots & P_{i|l|n}^{n} \end{bmatrix} G - \Gamma_i^n \quad \text{(63)}$$

that expands to

$$\tilde{\nu}_i^n = \begin{bmatrix} y_{i|j-1}^l - \tilde{y}_{i|j-1}^n \\ \vdots \\ y_{i|d_n}^l - \tilde{y}_{i|d_n}^n \\ \tilde{z}_i^n - H_n \tilde{y}_{i|j-1}^n - H_n \tilde{z}_{i|j-1}^n \end{bmatrix} \quad \text{(57)}$$
\[\begin{aligned}
&S_{q_s}^{\nu} = GS_{q_s}^{\nu n} (G - G^T G)^{-1} G + \nu^T
\end{aligned}\]
where, \([S_{q_s}^{\nu}]_{q_s}\) denotes the \((q, s)\)th block of the \((d_n + 1) \times (d_n + 1)\) block matrix \(S_{q_s}^{\nu n}\).

The proof is in Appendix I. By the Gauss-Markov theorem, the CKIF algorithm, along with this design of the consensus and innovation gain matrices, as stated in Theorem 4 results in the minimized MSE distributed estimates of the dynamic random field \(x_i\). The gain matrices are deterministic. Hence they can be precomputed offline and saved for online implementation. In Algorithm 2 we state the steps that each agent \(n\) runs to compute the optimal gain matrices.

Algorithm 2 Gain Design of CIKF

\begin{algorithm}
\begin{algorithmic}
\State **Input:** Model parameters \(A, V, H, R, G, L, \Sigma_0\).
\State **Initialize:** \(\Sigma_{0|0} = J \otimes \Sigma_0, P_{0|0} = J \otimes (G \Sigma_0 G), \Pi_{0|0} = J \otimes (\Sigma_0 G)\).
\While {\(i \geq 0\)}
\State **Optimal gains:**
\State \(\text{Compute } B_i^n, K_i^n \text{ using Theorem 4}\)
\State \(\text{Using } (61)-(65), \text{ obtain } B_i^n, B_i^T, K_i^n \text{ from } \hat{B}_i^n \text{ and } K_i^n\).
\State **Prediction error covariance updates:**
\State \(\text{Update } P_{i+1|i}, \Sigma_{i+1|i}, \Pi_{i+1|i} \text{ using } (42)-(47).\)
\EndWhile
\end{algorithmic}
\end{algorithm}

The offline Algorithm 2 along with the online Algorithm 1 completes our proposed distributed solution to obtain minimized MSE estimates of the dynamic field \(x_i\) at each agent in the cyber network.

VII. NUMERICAL EVALUATION

We numerically evaluate the MSE performance of the CIKF and compare it against the centralized Kalman filter (CKF) and the distributed information Kalman filter (DIKF) in [5]. To this objective, we build a time-varying random system, observation and network model that satisfies the Assumptions 1-3. The Algorithms 1, 2 run on these model parameters. First the Algorithm 2 computes and save the gain matrices and the error covariance matrices. The traces of the error covariance matrices provide the theoretical MSE trajectory of the CIKF with time. Then, we Monte-Carlo simulate Algorithm 1 to compute the numerical MSE of the distributed estimators, CIKF and DIKF, and the centralized estimator CKF.

A. Model specifications

Here, we consider a time-varying field, \(x_i\), with dimension \(M = 50\). The physical layer, consisting of \(M = 50\) sites, is monitored by a cyber layer consisting of \(N = 50\) agents. Each agent in the cyber layer observes \(M_n = 2\) sites of the physical layer. We build the field dynamics matrix \(A\) to be sparse and distributed. The dynamics \(A\) possess the structure of a Lattice graph, where the time evolution of a field variable depends on the neighboring field variables. For illustration, we consider an unstable field dynamics with \(\|A\|_2 = 1.05\) to test the resilience of the algorithms under unstable conditions.

The observation matrices, \(H_n \in \mathbb{R}^{2 \times 50}, n = 1, \ldots, 50\), are sparse \(0 - 1\) matrices with one non-zero element at each row corresponding to the site of \(x_i\) observed by the \(n\)th agent. The local observations \(z_i^n\) are \(2 \times 1\) random vectors. The mean \(\mu_0\) of the initial state vector is generated at random. The system noise covariance \(V\), the observation noise covariances \(R_n\), and the initial state covariance \(\Sigma_0\) are randomly generated symmetric positive definite matrices. The norms of the covariance matrices are: \(\|V\| = 4, \|R_n\| = 8\), and \(\|\Sigma_0\| = 16\). The agents in the cyber layer communicate among themselves following a randomly generated Erdős-Rényi graph \(G\) with 50 nodes and \(E = 138\) edges. The average degree of each node/agent is approximately 5.5. The communication network \(G\) is also sparse.

For Monte-Carlo simulations, we generate the noises, \(v_i, r_i^n\), and the initial condition, \(x_0\) as Gaussian sequences, with \(v_i \sim \mathcal{N}(0, V)\), \(r_i^n \sim \mathcal{N}(0, R_n)\), \(x_0 \sim \mathcal{N}(x_0, \Sigma_0)\). The sequences \(\{\{v_i\}_i, \{r_i\}_i, x_0\}_i\) are generated to be uncorrelated. Each agent \(n\) in the cyber layer has access to the system parameters \(A, V, H, R, x_0, \Sigma_0\), and \(G\). This numerical model satisfies Assumptions 1-3. The pair \((A, H)\) is detectable and the pairs, \((A, H_n)\) \(\forall n\), are not detectable. The agent communication graph \(G\) is connected with the algebraic connectivity of the Laplacian \(\lambda_2(L) = 0.7 > 0\). Hence the Assumptions 4, 5 hold true for this numerical system, observation, and network model.

B. Optimized gains and theoretical MSE

We run Algorithm 2 on the numerical model to obtain the gain matrices and the theoretical error covariances of the CIKF. We compute the gain matrices and the error covariances of the centralized Kalman filter (CKF) and of the distributed information Kalman filter (DIKF). In Fig. 4a we plot the MSE, trace of the predictor error covariance matrices \(\Sigma_{i+1|i}\), for each of these cases up to time \(i = 30\). The MSE of the optimal CKF is the smallest (recall the CKF, if feasible, would be optimal) and the objective of the distributed estimators is to achieve MSE performance as close as possible to that of the CKF.

From Fig. 4a we see that the MSE of the proposed CIKF is 3dB more than the CKF but is 3dB less than the DIKF. From the plot we see that the CIKF converges faster than the DIKF. Hence, the proposed CIKF provides faster convergence and 3dB MSE performance improvement over the DIKF. The performance of the CKF is 3dB better than the CIKF’s due to the fact that the CKF has access to the observations of all the sensors at every time steps. In contrast in CIKF, each agent has access to its own observations and the current estimates of its neighbors only; the impact of the observations from the other agents propagate through the network with delay. As the time-varying field \(x_i\) is evolving with input noise \(v_i\), lack of access to all the observations containing the driving input \(v_i\), combined with the network diffusion delay, causes a performance gap between the CKF and the CIKF.
C. Monte-Carlo Simulations

We empirically compute the MSE of the distributed estimates given by the CIKF Algorithm 1. We implement the algorithms using Matlab in the Microsoft Azure cloud. Given the computation load because of the large system (M = 50) and network (N = 50) models, we run our simulation on Azure DS13 (8 cores, 56 GB memory) virtual machine (VM). The MSE computation for the CIKF, CKF, and DIKF algorithms with 1000 Monte-Carlo runs require approximately 30 hours in the Azure DS13 VM. Once we obtain the field prediction estimates for the three algorithms, we compute the empirical prediction error covariance matrices \( \hat{\Sigma}_{i+1} \) and then obtain the Monte-Carlo MSE from their trace. From the Monte-Carlo simulations, we see that the empirical plots follow closely the theoretical plots in Fig 4a. From the Monte-Carlo MSE plot in Fig 4b, we see that the empirical plots closely match the theoretical plots in Fig 4a.

Both the theoretical Fig 4a and Monte-Carlo simulated Fig 4b MSE performance confirms our CIKF analysis in Sections IV VI. The novel Consensus+Innovations Kalman Filter (CIKF) proposed in Section III along with the optimized gain designs in Section VI provides unbiased distributed estimates with bounded and minimized MSE for the Consensus+Innovations distributed solution. The CIKF achieves nearly 3dB better performance than the DIKF [5].

VIII. CONCLUSIONS

Summary: In this paper, we propose a Consensus+Innovations Kalman Filter (CIKF) that obtains unbiased minimized MSE distributed estimates of the pseudo-states and real-time employs them to obtain the unbiased distributed filtering and prediction estimates of the time-varying random state at each agent. The filter update iterations are of the Consensus+Innovations type. Using the Gauss-Markov principle, we designed the optimal gain matrices that yield approximately 3dB improvement over previous available distributed estimators like the DIKF in [5].

Contributions: The three primary contributions of this paper are: (a) introduction of the concept of pseudo-state; (b) design of a filter and corresponding gain matrices to obtain minimized MSE distributed estimates at each agent under minimal assumptions; and (c) a theoretical characterization of the tracking capacity and distributed version of the algebraic Riccati equation.

APPENDIX

The Appendices prove the proposition, lemmas, and theorems stated in the paper.

Proof of Proposition

A. Proof of Proposition 7

First we derive the dynamics (9) of the pseudo-state \( y_i \). Using (8) and (1),
\[
y_i+1 = Gx_{i+1} = G(\hat{x}_i + v_i).
\]
By (26)
\[
y_i = G\left(G^\dagger G + \tilde{I}\right)x_i + Gv_i, \quad \text{[by (15), ]} I = G^\dagger G + \tilde{I}
\]
\[
= GAG^\dagger y_i + Gv_i + GA\hat{x}_i = \tilde{\bar{y}}_i + Gv_i + \hat{A}x_i, \quad \text{[by (8)]}.
\]

Now, we derive the observations (10) of the pseudo-observations \( \tilde{z}_i^n \). Using (4) and (2),
\[
\tilde{z}_i^n = H_n^TR_n^{-1}H_nx_i + H_n^TR_n^{-1}r_i^n
= H_n^TR_n^{-1}H_n\left(G^\dagger G + \tilde{I}\right)x_i + H_n^TR_n^{-1}r_i^n
= H_n^TR_n^{-1}H_nG^\dagger Gx_i + H_n^TR_n^{-1}r_i^n + H_n^TR_n^{-1}H_n\hat{A}x_i = \tilde{\bar{y}}_i + H_n^TR_n^{-1}r_i^n + \bar{H}_n\hat{x}_i.
\]

Proof of Lemmas

B. Proof of Lemma 7

Consider the filtering error definitions (25)-(26). We take expectations on both sides,
\[
E\left(e_{i|i}^n\right) = E\left[y_i - \tilde{y}_{i|i}^n\right]
= E\left[y_i - \tilde{y}_{i|i}^n | \tilde{z}_{i|i}^n, \{\tilde{y}_{i|l}^n | z_{i|l} \}_{l<i}\right]
= E\left[\tilde{y}_{i|i}^n - \tilde{y}_{i|i}^n\right] = 0 \quad \text{[by (17)]}
\]
\[
E\left(e_{i|i}^n\right) = E\left[x_i - \tilde{x}_{i|i}^n\right]
= E\left[E\left[x_i - \tilde{x}_{i|i}^n | \tilde{y}_{i|i}^n\right]\right]
= E\left[\tilde{x}_{i|i}^n - \tilde{x}_{i|i}^n\right] = 0 \quad \text{[by (26)].}
\]
Similarly, taking expectations on prediction errors (27)–(18),
\[
\mathbb{E}\left[ \epsilon_{i+1|i}^n \right] = \mathbb{E}\left[ y_{i+1} - \tilde{y}_{i+1|i}^n \right] = \mathbb{E}\left[ y_{i+1} - \tilde{y}_{i+1|i}^n \right] = \mathbb{E}\left[ \tilde{y}_{i+1|i}^n - \tilde{y}_{i+1|i}^n \right] = 0 \quad \text{[by (19)]}
\]
\[
\mathbb{E}\left[ e_{i+1|i}^n \right] = \mathbb{E}\left[ x_{i+1} - \tilde{x}_{i+1|i}^n \right] = \mathbb{E}\left[ x_{i+1} - \tilde{x}_{i+1|i}^n \right] = \mathbb{E}\left[ \tilde{x}_{i+1|i}^n - \tilde{x}_{i+1|i}^n \right] = 0 \quad \text{[by (28)].}
\]

C. Proof of Lemma 2

We write the pseudo-state filtering update (21) in vector form,
\[
\hat{y}_{i|i} = \hat{y}_{i|i-1} - B_i^e \hat{x}_{i|i-1} + B_i^e \left( \hat{\theta}_i - \left( \hat{D}_H \hat{y}_{i|i-1} + \hat{D}_H \hat{x}_{i|i-1} \right) \right),
\]
where,\[ B_i^e = -B_i^{nl} \quad \forall n \neq l, \quad B_i^{nl} = \begin{cases} B_i, & \text{if } n = l, \\ 0, & \text{otherwise}. \end{cases} \]
The block-diagonal matrices: \( D_H = \text{blockdiag} \{ H_1, \ldots, H_N \}, \hat{D}_H = \text{blockdiag} \{ \hat{H}_1, \ldots, \hat{H}_N \} \). Note that \( B_i^e \) is (11 \( \otimes \)) \( y_i \). Using this relation and the vector form of \( \hat{y}_{i|i} \), we expand the pseudo-state filter error process \( e_{i|i} \),
\[
e_{i|i} = 1_{N \otimes A} y_i - \hat{y}_{i|i} = 1_{N \otimes A} y_i - \left( \hat{D}_H \hat{y}_{i|i-1} + \hat{D}_H \hat{x}_{i|i-1} \right),
\]
where, \( K_i = \text{blockdiag} \{ K_1^e, \ldots, K_N^e \} \). Using the relation \( y_{i|i} = (1_{N \otimes A} \otimes y_i - e_{i|i}) = (1_{N \otimes A} \otimes A) (1_{N \otimes A} \otimes x_i - e_{i|i}) \), we expand the filter error process \( e_{i|i} \),
\[
e_{i|i} = 1_{N \otimes A} x_i - \tilde{x}_{i|i} = 1_{N \otimes A} x_i - \tilde{x}_{i|i} = \left( 1_{N \otimes A} \otimes A \right) (1_{N \otimes A} \otimes x_i - e_{i|i}).
\]

D. Proof of Lemma 3

Lemma 2 and Assumption 1 guarantee that \( e_{i|i}, e_{i+1|i} \), \( e_{i|i}, v_i, r_i \) are Gaussian. The error noises \( \Omega_i \) and \( \phi_i \) are therefore Gaussian as they are linear combinations of the error processes and the model noises \( e_{i|i}, e_{i+1|i}, e_{i|i}, v_i, r_i \).

By Lemma 1 we have \( \mathbb{E}\left[ e_{i|i} \right] = \mathbb{E}\left[ e_{i+1|i} \right] = \mathbb{E}\left[ e_{i+i} \right] = 0 \).

From Assumption 1 we know \( \mathbb{E}\left[ r_i \right] = \mathbb{E}\left[ v_i \right] = 0 \). We take expectation on both sides of (50), (51) and apply these relations
\[
\mathbb{E}\left[ \tilde{\phi}_i \right] = (I_{N \otimes A}) \mathbb{E}\left[ e_{i|i} \right] - (I_{N \otimes A}) B_i^e \hat{D}_H \mathbb{E}\left[ e_{i+1|i} \right] + \mathbb{E}\left[ (G \mathbb{E}[v_i]) \right] - (I_{N \otimes A}) B_i^e \hat{D}_H \mathbb{E}\left[ e_{i+i} \right] = 0.
\]
\[
\mathbb{E}\left[ \phi_i \right] = (I_{N \otimes A}) K_i \mathbb{E}\left[ e_{i|i} \right] + 1_{N \otimes (G \mathbb{E}[v_i])} = 0.
\]

Combining (31), (32) and (50), we have
\[
\hat{\phi}_i = F_1 e_{i|i-1} + F_2 e_{i|i-1} - F_3 \hat{D}_H R_{ii}^{-1} r_i + 1_{N \otimes (G \mathbb{E}[v_i])}
\]
where,
\[
F_1 = (I_{N \otimes A}) (I_{N \otimes A} - K_i (1_{N \otimes A} \otimes G) K_i^T) - (I_{N \otimes A}) B_i^e \hat{D}_H
\]
\[
F_2 = (I_{N \otimes A}) K_i (I_{N \otimes A} - B_i^e \hat{D}_H)
\]
\[
F_3 = (I_{N \otimes A}) B_i^e.
\]

Since \( \phi_i \) and \( \hat{\phi}_i \) are zero-mean, the noise covariances are,
\[
\Phi_i = \mathbb{E}\left[ \phi_i \phi_i^T \right] = F_1 \Sigma_i F_1^T + F_2 \Sigma_i F_2^T + F_3 \mathbb{E}[v_i] + J \mathbb{E}[v_i] + J \mathbb{E}[v_i] + J \mathbb{E}[v_i],
\]
\[
\Phi_i = \mathbb{E}\left[ \phi_i \phi_i^T \right] = (I_{N \otimes A}) K_i \Sigma_i K_i^T (I_{N \otimes A} \otimes A) + J \mathbb{E}[v_i]
\]
where, \( F_1, F_2, F_3 \) are defined in (64)–(66).

E. Proof of Lemma 4

We first compute the conditional means of \( \theta_i^n \) and \( \theta_i^n \) of the new information \( \tilde{\theta}_i^n \) and \( \phi_i^n \) from (54). The means \( \theta_i^n \) and \( \tilde{\theta}_i^n \) depend on the conditional means of \( \tilde{y}_{i|i-1}^n, \tilde{z}_{i|i-1}^n \) and \( y_{i|i}^n \).

\[
\mathbb{E}\left[ \tilde{y}_{i|i-1}^n \right] = n_{i|i-1} \in \Omega_i \]
\[
\mathbb{E}\left[ \tilde{z}_{i|i-1}^n \right] = n_{i|i-1} \in \Omega_i \]
\[
\mathbb{E}\left[ \tilde{y}_{i|i-1}^n \right] = n_{i|i-1} \in \Omega_i \]
\[
\mathbb{E}\left[ \tilde{z}_{i|i-1}^n \right] = n_{i|i-1} \in \Omega_i \]
\[
\mathbb{E}\left[ \tilde{y}_{i|i-1}^n \right] = n_{i|i-1} \in \Omega_i \]
\[
\mathbb{E}\left[ \tilde{z}_{i|i-1}^n \right] = n_{i|i-1} \in \Omega_i \]

Since the state \( x_i \), pseudo-state \( y_i \), their initial condition and all the noises are Gaussian, their estimates are also Gaussian making all the filtering and prediction errors Gaussian.
The conditional means of $\hat{y}_{i|j-1}^n$ and $\hat{y}_{i|j-1}^n$ show that the new uncorrelated information defined in (55)-(56) expands to the vectors $\tilde{v}_i^n$ and $\nu_i^n$ in (57). Note that $\nu_i^n$ and $\nu_i^n$ are zero mean, by definition, and, are Gaussian since they are linear combination of Gaussian vectors. Now to prove $\tilde{v}_i^n$ and $\nu_i^n$ are sequence of uncorrelated vectors, we have to show:

$$E \left[ \tilde{v}_i^n \tilde{v}_j^{nT} \right] = E \left[ \nu_i^n \nu_j^{nT} \right] = 0, \forall i \neq j, \forall n.$$ 

First, we write $\tilde{v}_i^n$ and $\nu_i^n$ in terms of the filtering and prediction error processes using (10), (25)-(28),

$$\tilde{v}_i^n = \begin{bmatrix} e_{i|1}^{n-1} - e_i^1 \\ \vdots \\ e_{i|1}^{n-1} - e_i^{l_{i|1}} \\ H_i e_{i|1}^{n-1} + H_i e_{i|1}^{n-1} + H_i R_i^n r_i^n \end{bmatrix}, \nu_i^n = G e_{i|1}^{n-1} - e_i^n.$$ 

Without loss of generality, we consider $i > j$. The rest of the proof is similar to the proof of Lemma 5 in [5]. Here, the only difference is that we should condition on $\{\tilde{z}_{i|1}^{n} \}_{i \in \Omega}$, $\{\hat{y}_{i|1-1}^{n} \}_{i \in \Omega}$. 

**Proof of Theorems**

**F. Proof of Theorem 7**

In Lemma 4 we showed that $\tilde{v}_i^n$ and $\nu_i^n$ are independent Gaussian sequences. By the Innovations Property [25], there are 1-1 correspondence between $\{\tilde{z}_{i|1}^{n} \}_{i \in \pi_n}$ and $\tilde{v}_i^n$, and between $\{\hat{y}_{i|1}^{n} \}_{i \in \pi_n}$ and $\nu_i^n$. The Innovations Property guarantees that there exists a unique way to get one from the other.

$$\tilde{y}_{i|1}^{n} = E \left[ y_i | \tilde{z}_{i|1}^{n}, \{\tilde{y}_{i|1-1}^{n} \}_{i \in \pi_n} \right] \iff \tilde{y}_{i|1}^{n} = E \left[ y_i | \tilde{v}_i^n \right]$$

$$\tilde{x}_{i|1}^{n} = E \left[ x_i | \hat{y}_{i|1}^{n} \right] \iff \tilde{x}_{i|1}^{n} = E \left[ x_i | \nu_i^n \right]$$

By the Gauss-Markov principle,

$$\tilde{y}_{i|1}^{n} = \hat{y}_{i|1-1}^{n} + \tilde{B}_i^n \tilde{v}_i^n$$

$$\tilde{x}_{i|1}^{n} = \hat{x}_{i|1-1}^{n} + K_i^n \nu_i^n$$

where $\tilde{B}_i^n$ are the non-zero blocks of the $n$th row of $B_i$. Now we expand the term $\tilde{B}_i^n \tilde{v}_i^n$ by multiplying the gain blocks $B_i^{ul}$ with the corresponding $(\hat{y}_{i|1-1}^{n} - \tilde{y}_{i|1-1}^{n})$ and the gain block $B_i^{ul}$ with $(\tilde{z}_{i|1}^{n} - \tilde{H}_i \hat{y}_{i|1}^{n-1} - \tilde{H}_i \tilde{x}_{i|1}^{n-1})$. This gives us the consensus+innovations filtering pseudo-state update (27).

The pseudo-state and state prediction updates are

$$\tilde{y}_{i|1}^{n} = E \left[ y_{i|1} | \tilde{z}_{i|1}^{n}, \{\hat{y}_{i|1-1}^{n} \}_{i \in \pi_n} \right]$$

$$= E \left[ A_{i|1} y_i + X_{i|1} + A \nu_i | \tilde{z}_{i|1}^{n}, \{\hat{y}_{i|1-1}^{n} \}_{i \in \pi_n} \right]$$

$$= A_{i|1} \hat{y}_{i|1}^{n} + A \tilde{x}_{i|1}^{n}$$

$$\tilde{x}_{i|1}^{n} = E \left[ x_{i|1} | \tilde{y}_{i|1}^{n} \right] = E \left[ A x_i + v_i | \hat{y}_{i|1}^{n} \right] = A \tilde{x}_{i|1}^{n}.$$ 

**G. Proof of Theorem 2**

By Lemma 1 and Lemma 2, the error processes, $e_{i|1}^{n}$, $e_{i|1}^{n}$, $e_{i|1}^{n}$, and $e_{i|1}^{n}$ are zero-mean Gaussian. The Lyapunov-type iterations (42)-(47) of the filter and predictor error covariances, $\Sigma_{i|1}$, $\Sigma_{i|1}$, $\Sigma_{i|1}$, $\Sigma_{i|1}$, and $\Sigma_{i|1}$, follow directly from the definitions (34)-(39) and error dynamics (30)-(33) by algebraic manipulations.

**H. Proof of Theorem 3**

For any square matrix, $\rho(\tilde{F}) \leq \| \tilde{F} \|$. Hence if $\| \tilde{F} \| < 1$, then it implies that $\rho(\tilde{F}) < 1$. We derive the tracking capacity with the sufficient condition. $\| \tilde{F} \| < 1$,

$$\| \tilde{F} \| = \left\| \left( I_{n} \otimes \hat{A} \right) \left( I_{n} - B_i^c - B_i^c \tilde{D}_i \right) \right\|$$

$$\leq \left\| I_{n} \otimes \hat{A} \left\| \left( I_{n} - B_i^c - B_i^c \tilde{D}_i \right) \right\|$$

$$\leq \| G \| \left\| \left( I_{n} - B_i^c - B_i^c \tilde{D}_i \right) \right\|$$

$$\leq \frac{\lambda_m}{\lambda_1} \| A \| \left\| \left( I_{n} - B_i^c - B_i^c \tilde{D}_i \right) \right\| < 1,$$ 

then $\| \tilde{F} \| < 1$ and also $\rho(\tilde{F}) < 1$. The bound on the spectral norm of $A$ is,

$$\| A \| < \frac{\lambda_1}{\lambda_m} \left\| \left( I_{n} - B_i^c - B_i^c \tilde{D}_i \right) \right\| = C.$$ 

Thus as long as $\| A \| < C$, there exists $B_i^c$ and $B_i^c$ such that $\rho(\tilde{F}) < 1$. By global detectability Assumption 4 there exists $K_i$ such that $\rho(\tilde{F}) < 1$. Refer to [35], for the convergence conditions of the centralized information filters. Further, by Lemma 3 the Gaussian noise processes $\phi_i$ and $\phi_i$ have bounded noise covariances. Thus, if $\| A \| < C$, then from (48)-(49) we conclude that the CIKF (21)-(24) converges with bounded MSE.
\[
\sum_{i \in \Omega} = \mathbb{E} \left[ (x_i - \bar{x}_i) \nu_i^T \right] = \mathbb{E} \left[ \left( e_{n+1|\nu} - e_{n+1|\nu}^T \right) \left( \tilde{y}_{n+1} - \tilde{y}_{n+1}^T \right) \right]
\]

...