A classification of $G$-charge Thouless pumps in 1D invertible states

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Abstract

Recently, a theory has been proposed that classifies cyclic processes of symmetry protected topological (SPT) quantum states. For the case of spin chains, i.e. one-dimensional bosonic SPT’s, this theory implies that cyclic processes are classified by zero-dimensional SPT’s. This is often described as a generalization of Thouless pumps, with the original Thouless pump corresponding to the case where the symmetry group is $U(1)$ and pumps are classified by an integer that corresponds to the charge pumped per cycle. In this paper, we review this one-dimensional theory in an explicit and rigorous setting and we provide a proof for the completeness of the proposed classification for compact symmetry groups $G$.

1 Introduction

Symmetry protected topological states are states of a spatially extended $d$-dimensional quantum many-body system that are symmetric with respect to a certain on-site symmetry group $G$ and that can be adiabatically connected to a product state, but only if the adiabatic path passes through states that are not $G$-symmetric. The idea is hence that the interesting topological characteristics of these states is only present due to the $G$-symmetry, which explains the nomenclature. Much progress has been made towards classifying such states: While the very notion of equivalence was introduced in [1], classifications were first proposed in $d = 1$ [2, 3, 4], with a first general proposal for all dimensions using group cohomology was given in [5]. Among the parallel classification schemes, we mention string order [6, 7] as well as the entanglement spectrum [8, 9]. More mathematical works are recent, with the cohomological classification given a fully rigorous treatment in [10, 11] in one dimension, and [12, 13] in two dimensions.

In the present paper, we restrict ourselves to $d = 1$ and to bosonic systems, i.e. spin chains. Given that restriction, we will consider a class of states that is a priori broader than SPT states, namely invertible states on a spin chains, to be defined in Section 2.3. Let us immediately say that it is believed that for spin chains, the two classes (SPT states and invertible states) actually coincide, and for finite symmetry groups $G$ this was proven in [12]. For us, the class of invertible states is the natural class for which the main result of the paper can be readily formulated.

Our main interest lies in classifying not the $G$-symmetric invertible states themselves, but loops of $G$-symmetric invertible states. Physically, these loops can be thought of as describing periodic pumps. The most well-known example of this is given by considering pumping protocols where a $U(1)$-charge is conserved, often referred to as Thouless pumps. The phenomenon of
topological quantization of charge transport, which is relevant to the integer quantum Hall effect, is then \([15, 16, 17, 18, 19, 20]\): *The average charge transported per cycle is an integer* when expressed in an appropriate, pump-independent, unit. Of course, this statement needs a restrictive assumption that excludes fractional charge transport in states with non-trivial topological order, cf. the fractional Quantum Hall effect. In our setting, it is precisely the property of invertibility that excludes such topologically ordered states.

In this paper, we construct an index associated with loops of \(G\)-symmetric invertible states. The index takes values in the first group cohomology group \(H^1(G, S^1)\) of the symmetry \(G\), and we shall refer to its value as a charge. In order to classify the loops, we introduce an equivalence relation on the set of loops which is given by a new and specific form of homotopy of paths of states, see Section 3. With this in hand, Theorem 1 states that the cohomological index yields a full classification of the loops: two loops with the same basepoint are equivalent if and only if they have the same index. We show that all possible values of the index can be realized. Finally, we prove that the group structure of \(H^1(G, S^1)\) reflects both the composition of cyclic processes and the stacking of physical systems. While this is reminiscent of the classification of SPT states discussed above, it is describing a different aspect of the manifold of states, namely its fundamental group. The discussion of Section 3 will however make the connection between the two, in relating the index of a loop in one dimension to an index of a particular SPT state in zero dimension.

![Figure 1: Bulk (a) and edge (b) characterization of charge transport.](image)

Before heading to the main text, we present the idea behind our classification in heuristic terms, and for the sake of clarity for \(G = U(1)\), see Figure 1. There are two ways to measure charge transport. In the ‘bulk’ characterization used e.g. in the original work of Thouless, transport is obtained as the difference of charge in reservoirs before and after the process has taken place; equivalently, one can compute the integrated current. For the purpose of this paper, we prefer the ‘edge’ characterization: Transport is determined by running the process truncated to only one half-line and by measuring the charge accumulated at the edge: specifically by measuring the difference between the amount of charge in the left part of the system before and after the truncated process took place. In the setting of spin chains, the idea goes back to [21] and it was developed more explicitly in [22, 23, 24, 25], often under the name of ‘Floquet phases’, see also [26] for a review. The problem of classifying loops of states (and more general families of states) has received much interest in the recent physics literature, see in particular [27, 28, 29, 30, 31].

Finally, we mention an auxiliary result, see Proposition 6, which may be of interest in its own, namely that small perturbations of a trivial on-site interaction have a unique gapped ground state. Results of that kind have been established in great generality with various techniques, see [32, 33, 34, 35, 36]. However, the specific claim that we need does not seem to appear in the literature, mainly because we consider the infinite-volume setup and because the class of interactions we consider is broader. The fact that the unperturbed ground state is a product state allows for a simpler argument — but still along the lines of [34].
Plan of the paper

We develop the setup in Sections 2 and 3. The results and relevant examples are stated in Sections 3 and 4. The rest of the paper is devoted to the proofs. In Sections 5 and 6 we state some general preliminaries, and the remaining Sections 7, 8, 9, 10 contain specific aspects of the proofs, see Section 4.3 for a more detailed overview.

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Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interests

The authors have no relevant financial or non-financial interests to disclose.

2 Setup

2.1 Algebras

We define a spin chain $C^*$-algebra $A$ in the standard way. To any site $j \in \mathbb{Z}$, we associate a finite-dimensional algebra $A_j$ isomorphic to $M_{n_j}(\mathbb{C})$, the algebra of $n_j \times n_j$ matrices with complex entries. The algebra $M_{n_j}(\mathbb{C})$ is equipped with its natural operator norm and $*$-operation (Hermitian adjoint of a matrix) making it into a $C^*$-algebra. The spin chain algebra $A$ is the inductive limit of algebra’s $A_S = \otimes_{j \in S} A_j$, with $S$ finite subsets of $\mathbb{Z}$. It comes naturally equipped with subalgebra’s $A_X, X \subset \mathbb{Z}$. We refer to standard references [37, 38, 39] for more background and details.

2.2 Almost local evolutions and their generators

We introduce the framework to discuss processes, i.e. time-evolutions. Because of the infinite-volume setup, and in contrast to quantum mechanics with a finite number of degrees of freedom, most evolutions cannot be generated by a Hamiltonian that is itself an observable, i.e. an element of the spin chain algebra $A$ and we need, instead, to consider so-called interactions.

2.2.1 Interactions

Let $\mathcal{F}$ be the class of non-increasing, strictly positive functions $f : \mathbb{N}^+ \to \mathbb{R}^+$, with $\mathbb{N}^+ = \{1, 2, \ldots\}$, satisfying the fast decay condition $\lim_{r \to \infty} r^p f(r) = 0$ for any $p > 0$. An interaction is a collection $H = (H_S)$ labelled by finite subsets $S$ of $\mathbb{Z}$ such that $H_S = H_S^* \in A_S$ and

$$||H||_f = \sup_{j \in \mathbb{Z}} \sum_{S \ni j} \frac{||H_S||}{f(1 + \text{diam}(S))}$$

is finite for some $f \in \mathcal{F}$. This excludes for example power law interactions. As in the above expression, sums over $S$ or $S_1, S_2, \ldots$ will always be understood to run over finite subsets of $\mathbb{Z}$. 
2.2.2 Almost local algebra $\mathcal{A}_{al}$

If $H$ is an interaction with finite $||\cdot||_f$ norm and it satisfies the additional property that $H_S = 0$ unless $0 \in S$, then $\sum_S H_S$ is convergent in the topology of $\mathcal{A}$ and we define

$$\iota(H) = \sum_S H_S \in \mathcal{A},$$

as a map from interactions to observables. The algebra generated by all such elements is a norm-dense subalgebra of $\mathcal{A}$ that is usually called the almost local algebra and that we denote by $\mathcal{A}_{al}$, see also [12]. The main interest of this algebra is that it is invariant under the adjoint action associated with an interaction. Indeed, if $A \in \mathcal{A}_{al}$ and $||H||_f < \infty$, then $\sum_S [H_S, A]$ is norm convergent and its sum, denoted $[H, A] = [\iota(H), A]$, is again an element of $\mathcal{A}_{al}$.

For later purposes, we also let $\iota(\Lambda) = \sum_{S \subseteq \Lambda} H_S$ whenever the sum is convergent. In particular $\iota_Z = \iota$, and if $Z$ is a finite set, then $\iota_Z(H) \in \mathcal{A}_Z$.

2.2.3 Almost local evolutions

We will consider families of interactions $H(s)$ parametrized by $s \in [0, 1]$ and call them ‘time-dependent interactions’ (TDI) provided that they satisfy some regularity conditions to be formulated below. Since the risk of confusion is small, we will often denote them $H$ as well. An interaction $H$ is a TDI if there is $f \in F$ such that $s \mapsto H(s)$ is $||\cdot||_f$ bounded and strongly measurable. On such functions, we use the supremum norms

$$||H||_f = \sup_{s \in [0, 1]} ||H(s)||_f.$$  

The role of a TDI $H$ is to generate almost local evolutions $\alpha_H = (\alpha_H(s))_{s \in [0, 1]}$, namely the one-parameter family of strongly continuous $^*$-automorphisms $\alpha_H(s)$ on $\mathcal{A}$ defined as a particular solution of the Heisenberg evolution equations on $\mathcal{A}_{al}$:

$$\alpha_H(s)[A] = A + i \int_0^s du \alpha_H(u)\{[H(u), A]\}. \quad (2.1)$$

The integral on the right hand side is to be understood in the sense of Bochner, and the strong measurability of the integrand follows from strong measurability of $s \mapsto H(s)$ and the strong continuity of $s \mapsto \alpha_H(s)$.

Specifically, $\alpha_H(s)[A]$ is defined as the limit in the topology of $\mathcal{A}$ of solutions of (2.1) where $H$ is replaced with $\iota_{\Lambda_n}(H)$ for an increasing and absorbing sequence $(\Lambda_n)_{n \in \mathbb{N}}$ of subsets of $\Gamma$. By a standard argument using the Lieb-Robinson bound [41, 42], this procedure is well-defined and the limit solves (2.1).

We conclude this section by noting that all definitions introduced so far can be applied to the tensor product algebra $\tilde{\mathcal{A}} = \mathcal{A} \otimes \mathcal{A}'$ of two spin chain algebras $\mathcal{A}$ and $\mathcal{A}'$. The site algebras $\tilde{\mathcal{A}}_j$ are defined as $\mathcal{A}_j \otimes \mathcal{A}'_j$ and $\tilde{\mathcal{A}}$ is then the inductive limit of (tensor products of) these site algebras.

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1 One can replace the site 0 by any other $j \in \mathbb{Z}$ without changing the subalgebra obtained from the procedure described here.

2 namely, it is the limit of a sequence of simple functions, pointwise almost everywhere, where the limit at any $s$ is taken in $||\cdot||_f$-norm, see [40]
2.3 States

States are normalized positive linear functionals on the spin chain algebra \( \mathcal{A} \). The set of states is denoted by \( \mathcal{P}(\mathcal{A}) \) and a natural metric on states is derived from the Banach space norm

\[
||\psi - \psi'|| = \sup_{A \in \mathcal{A}, ||A||=1} |\psi[A] - \psi'[A]|. \tag{2.2}
\]

The set \( \mathcal{P}(\mathcal{A}) \) is convex and its extremal points are called the pure states. In all what follows, we consider only pure states.

A distinguished class of states is that of spatial product states \( \phi \), i.e. states that satisfy \( \phi(A_j A_i) = \phi(A_j) \phi(A_i) \), \( A_j \in \mathcal{A}_j, A_i \in \mathcal{A}_i, i \neq j \).

**Definition 2.1** (Invertible state). A pure state \( \psi \) on a spin chain algebra \( \mathcal{A} \) is invertible if there is a pure state \( \overline{\psi} \) on a spin chain algebra \( \mathcal{A}' \), a TDI \( H \) on \( \mathcal{A} = \mathcal{A} \otimes \mathcal{A}' \) and a product state \( \phi \) on \( \mathcal{A} \) such that,

\[
\psi \otimes \overline{\psi} = \phi \circ \alpha_H(1). \tag{2.3}
\]

In this context, the state \( \overline{\psi} \) is usually referred to as an inverse of \( \psi \). Invertibility, first coined by \cite{21} is a way of expressing that a state has no intrinsic topological order. For spin chains, invertibility has recently been shown \cite{12} to be equivalent to a stronger property, namely being stably short-range entangled, which amounts to the additional requirement that \( \overline{\psi} \) itself is a product state. The notion of invertibility is however essential when adding symmetries.

2.4 Symmetries

Let \( G \) be a compact topological group. We equip the chain algebra with a strongly continuous on-site action of \( G \): For any \( j \in \mathbb{Z} \), there is an automorphism \( \gamma_j(g) \) of \( \mathcal{A}_j \) for each \( g \in G \) such that \( \gamma_j(g') \circ \gamma_j(g) = \gamma_j(g'g) \), and \( g \mapsto \gamma_j(g)[A] \) is continuous for any \( A \in \mathcal{A}_j \). For any \( A \in \mathcal{A}_S \) with finite \( S \), we let \( \gamma(g)[A] = \otimes_{j \in S} \gamma_j(g)[A] \) and extend this action to a strongly continuous action on \( \mathcal{A} \) by density. A state \( \psi \) is \( G \)-invariant if \( \psi \circ \gamma(g) = \psi \) for all \( g \in G \). A TDI \( H \) is called \( G \)-invariant if all its local terms are \( G \)-invariant:

\[
\gamma(g)[H_S(s)] = H_S(s), \quad \text{for all } s \in [0, 1], \text{ any finite } S \subset \mathbb{Z} \text{ and all } g \in G.
\]

Note these notions of invariance depend on the choice of action \( \gamma \), which is not well reflected in the nomenclature. In particular, any state is \( G \)-invariant if the action \( \gamma \) is chosen trivial.

**Definition 2.2** (\( G \)-invertible state). A \( G \)-invariant pure state \( \psi \) on a spin chain algebra \( \mathcal{A} \) is \( G \)-invertible if it is invertible, and its inverse \( \overline{\psi} \) and the interpolating family of states \( \phi \circ \alpha_H(s) \) (see \cite{23}) can be chosen to be \( G \)-invariant as well.

As we shall see shortly, an equivalent condition for a \( G \)-invariant, invertible pure state \( \psi \) to be \( G \)-invertible, is that the TDI \( H \) in \cite{23} can be chosen to be \( G \)-invariant.

3 Processes

Processes should intuitively be defined as continuous paths or loops of states. This presupposes in particular a suitable topology on the space of states. However, the standard topologies on states (e.g. norm topology or weak topology) do not fit physical intuition, in particular concerning locality properties, and it has become standard in this setting to view a curve of states as ‘continuous’ if it can be generated by a TDI, even if there is no known topology that substantiates this. This idea is sometimes referred to as *automorphic equivalence* and it was introduced in the works \cite{43, 44}. Concretely, we now define paths of pure states.
Definition 3.1 (Paths). Let $\mathcal{A}$ be a spin chain algebra and let $\gamma$ be a group action. A path is a function $[0,1] \ni s \mapsto \psi(s) \in \mathcal{P}(\mathcal{A})$ where each $\psi(s)$ is a pure state, and there is a TDI $H$ such that $\psi(s) = \psi(0) \circ \alpha_H(s)$. A path is called a loop if $\psi(0) = \psi(1)$. Finally, a path $\psi(\cdot)$ is $G$-invariant if all $\psi(s)$ are $G$-invariant.

Two remarks are in order. First of all, the group action (as well as the algebra) are fixed along the path. Secondly, while we did not require that a $G$-invariant path is generated by a $G$-invariant TDI, it is always possible to choose the TDI $G$-invariant. Indeed, if a $G$-invariant path is generated by a TDI $H$, then the TDIs given by $\gamma(g)|H(s)\rangle_S$ for $g \in G$ all generate the same path since $\psi$ and $\psi(s)$ are $G$-invariant, and so $\hat{H}_S(s) = \int_G d\mu(g) \gamma(g)|H_S(s)\rangle$, with $\mu(\cdot)$ the normalized Haar measure on $G$, generates the same path. Now $H$ is $G$-invariant and $||H||_f \leq ||H||_f$ for any $f \in \mathcal{F}$ since $\gamma(g)$ are automorphisms.

3.1 Charge and equivalence

We can now use this notion of path to define connected components of the set of states on a spin chain algebra. We present this in the form of an equivalence relation.

Definition 3.2 (Equivalence). A pair of pure states $\psi, \psi'$ on a spin chain algebra $\mathcal{A}$ are called equivalent if and only if there is a connecting path $s \mapsto \psi(s)$ such that $\psi(0) = \psi$ and $\psi(1) = \psi'$. If $\psi, \psi'$ are $G$-invariant, the pair is $G$-equivalent if the connecting path is $G$-invariant.

We first settle the classification of the $G$-equivalence classes of the set of pure $G$-invariant states for zero-dimensional systems that we define below. In this work, we will not consider the classification problem on spin chains, see however Section 3.3.1.

3.1.1 Zero-dimensional systems

One relevant case where the classification problem is straightforward is for 0-dimensional pure $G$-invariant states, called 0-dim $G$-states for brevity. By this we mean that $\mathcal{A} = \mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. The 0-dim $G$-states are $G$-invariant pure normal states on $\mathcal{B}(\mathcal{H})$ and we will assume that there is at least one such 0-dim $G$-state. It follows that the $G$-action on $\mathcal{B}(\mathcal{H})$ can be implemented by a family of unitaries $U(g)$, i.e. $\gamma(g)|A\rangle = U(g)|A\rangle U(g)^\dagger$, and $U(\cdot)$ is a strongly continuous unitary representation of $G$.

A pure normal states $\psi$ on $\mathcal{B}(\mathcal{H})$ is given by a ray in Hilbert space as $\psi(B) = \langle \Psi|B\Psi\rangle$ for some $\Psi \in \mathcal{H}$ with $||\Psi|| = 1$, which we call a representative of $\psi$. The equivalence question for 0-dim $G$-states is stated more easily than in the case of states on spin chain algebras, because now the natural topologies are physically relevant. We simply call a pair of 0-dim $G$-states $\psi, \psi'$ $G$-equivalent if they can be connected by a norm-continuous path of 0-dim $G$-states. Note that if the action $\gamma$ is trivial, then any pair of 0-dim $G$-states is $G$-equivalent, by the connectedness of the projectivization of $\mathcal{H}$. For general compact groups $G$, we will provide a full classification of $G$-equivalence classes of 0-dim $G$-states in Proposition 1.

3.1.2 $G$-charge

We consider the group $H^1(G) = \text{Hom}(G, S^1)$ of continuous group homomorphisms $G \to S^1$, equipped with the topology of uniform convergence. We use additive notation for the Abelian group $H^1(G)$. Because of the compactness of $G$, $H^1(G)$ is a discrete space in this topology, and this fact plays a central role in our reasoning.

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3See Lemma 6.1 and its proof
4Choosing a weaker topology would not change the classification
The next proposition shows that $H^1(G)$ completely classifies the 0-dim $G$-states. Recall that equivalence is defined between states of the same algebra equipped with the same group action.

**Proposition 1.** To every ordered pair of 0-dim $G$-states $\psi_1, \psi_2$ of $\mathcal{B}(\mathcal{H})$ equipped with $G$-action $\gamma$ we can associate an element $h_{\psi_2/\psi_1} \in H^1(G)$ such that

i. $\psi_1, \psi_2$ are $G$-equivalent if and only if $h_{\psi_2/\psi_1} = 0$

ii. $h_{\psi_3/\psi_2} + h_{\psi_2/\psi_1} = h_{\psi_3/\psi_1}$

iii. If $\psi'_1, \psi'_2$ are $G$-invariant states of $\mathcal{B}(\mathcal{H}')$ equipped with $G$-action $\gamma'$, then $h_{(\psi_2 \otimes \psi'_2)/(\psi_1 \otimes \psi'_1)} = h_{\psi_2/\psi_1} + h_{\psi'_2/\psi'_1}$

iv. If $\delta : \mathcal{B}(\mathcal{H}') \to \mathcal{B}(\mathcal{H})$ is a *-isomorphism such that $\delta \circ \gamma'(g) = \gamma(g) \circ \delta$, then $h_{\psi_2/\psi_1} = h_{(\psi_2 \circ \delta)/(\psi_1 \circ \delta)}$

We call henceforth $h_{\psi_2/\psi_1}$ the relative charge of $\psi_2$ and $\psi_1$. An alternative view on $G$-charge would be to restrict the algebra to $G$-invariant elements, in which case we could phrase the above proposition as describing the superselection sectors.

**Proof.** We choose a unitary representation $U(\cdot)$ of $G$ that implements $\gamma(\cdot)$ on $\mathcal{B}(\mathcal{H})$. If a state $\psi_j$ is $G$-invariant then any of its representatives $\Psi_j$ is invariant up to phase: $U(g)\Psi_j = z_j(g)\Psi_j$, and $G \to U(1) : g \mapsto z_j(g)$ is a group homomorphism. We then define $h_{\psi_2/\psi_1}(g) \in \mathbb{S}^1$ by

$$e^{ih_{\psi_2/\psi_1}(g)} = \frac{z_2(g)}{z_1(g)}.$$  

It is apparent that $z_j(g)$ does not depend on the choice of representatives $\Psi_j$. The unitary representation $U(\cdot)$ is uniquely determined up to multiplication with a $U(1)$ representation of $G$, which does affect $z_j(g)$ but not $h_{\psi_2/\psi_1}(g)$, and we conclude that $h_{\psi_2/\psi_1} \in H^1(G)$ is well-defined. To prove $i)$, we note that if two states are $G$-equivalent with interpolating path $\psi(s)$, then $s \mapsto h_{\psi(s)/\psi_1}$ is continuous and hence, by the discreteness of $H^1(G)$, constant. Reciprocally, if two states have zero relative charge, then there is a continuous path of $G$-invariant unitaries $V_s$ acting non-trivially only in the subspace spanned by their representatives $\Psi_1, \Psi_2$ and interpolating between the two. The vector states given by $V_s\Psi_1$ form the connecting path. $ii)$ is checked using the relation $\frac{z_2}{z_1} = \frac{z_4}{z_3} \frac{z_5}{z_4}$ and $iii)$ follows similarly. To prove $iv)$, we pick a unitary representation $U'$ implementing $\gamma'$ on $\mathcal{B}(\mathcal{H}')$. By the intertwining property we have then that $\gamma(g)\cdot [\cdot] = \delta(U'(g))[\cdot]\delta(U'(g))^*$ and the claim follows from $z_j(g) = \psi_j(U(g)) = \psi_j(\delta(U'(g)))$.  

3.2 Homotopy

We introduce a notion of a homotopy of two loops, say $\psi_0(\cdot), \psi_1(\cdot)$, generated by the TDIs $H_0, H_1$, respectively. If we had an appropriate topology on the set of states, we would say that a homotopy of these loops is a continuous function $(s, \lambda) \mapsto \psi_\lambda(s)$, such that $\psi_\lambda$ is a loop for each $\lambda$, reducing to the two given loops for $\lambda = 0$ and $\lambda = 1$. Though there is no topology, we have already specified in Definition 3.1 what it means for a state-valued function to be ‘continuous’ and we impose that definition here for the dependence on the parameter $\lambda$ as well. That is, we demand that $s \mapsto \psi_\lambda(s), \lambda \mapsto \psi_\lambda(s)$ are generated by TDIs that we note as $H_\lambda(\cdot), F_\lambda(\cdot)$:

$$\psi_\lambda(s) = \psi_\lambda(0) \circ \alpha_{H_\lambda}(s), \quad \psi_\lambda(s) = \psi_0(s) \circ \alpha_{F_\lambda}(\lambda), \quad \text{for any } s, \lambda \in [0, 1] \quad (3.1)$$

7
Additionally, we demand a uniform bound on the generating TDIs:

$$\sup_{\lambda} ||H_{\lambda}||_f < \infty, \quad \sup_s ||F_{s}||_f < \infty$$

(3.2)

A more intuitive way of phrasing our definition of homotopy is to say that motion on the two-dimensional sheet of states \((s, \lambda) \mapsto \psi_{\lambda}(s)\) is generated by an interaction-valued vector field \(H \frac{\partial}{\partial s} + F \frac{\partial}{\partial \lambda}\), see Figure 2, but we do not formalize this, as it would require technical conditions that are ultimately not relevant for our goals (but see [28]).

Figure 2: Intuitive notion of homotopy: the state \(\psi\) at the endpoint of the curve \(\Gamma : [0,1] \rightarrow [0,1]^2\) is given by \(\psi = \psi_0(0) \circ \alpha_{\Gamma}(1)\), with \(\alpha_{\Gamma}\) the almost local evolution generated by the TDI \(L(z), z \in [0,1]\) defined as \(L(z) = H \frac{\partial T(z)}{\partial s} + F \frac{\partial T(z)}{\partial \lambda}\).

**Definition 3.3** (Homotopy and G-homotopy). A pair of loops \(\psi(\cdot)\) and \(\psi'(\cdot)\) is homotopic if there exists a homotopy \((s, \lambda) \mapsto \psi_{\lambda}(s)\) in the sense of eqs. (3.1, 3.2) such that \(\psi_0(\cdot) = \psi(\cdot)\) and \(\psi_1(\cdot) = \psi'(\cdot)\). A pair of loops is G-homotopic if the homotopy can be chosen such that \(\psi_{\lambda}(s)\) are G-invariant for all \(s, \lambda\).

One could opt for a different definition of paths and their homotopies by requiring that the map \((s, \lambda) \mapsto H_{\lambda}(s)\) is sufficiently smooth (allowing e.g. for isolated singularities) in \(||\cdot||_f\)-norm, for some \(f \in F\). Such a condition would imply our definition of homotopy and all our results would still hold if we stuck to this definition but we find it unnatural as it puts the stress squarely on the generators rather then on the states.

On the other hand, it is not possible to drop the boundedness property (3.2). The necessity of this boundedness property is best understood on an example. Let the TDI \(H\) generate a loop \(\psi(\cdot)\). Then, for any \(\lambda\),

\[
H_{\lambda}(s) = \begin{cases} 
0 & s \leq \lambda \\
\frac{1}{1-\lambda} H\left(\frac{s}{1-\lambda}\right) & s > \lambda
\end{cases}
\]

generates a loop \(\psi_{\lambda}(\cdot)\). For any \(\lambda < 1\), this loop is simply a reparametrization of the original loop, namely

\[
\psi_{\lambda}(s) = \begin{cases} 
\psi(0) & s \leq \lambda \\
\psi\left(\frac{s}{1-\lambda}\right) & s > \lambda
\end{cases}
\]

and hence these loops are homotopic to each other by any reasonable definition. However, for \(\lambda = 1\), we get the constant loop \(\psi_1(s) = \psi(0)\). Hence, we arguably need some uniform boundedness property on the generators, akin to (3.2), to avoid the conclusion that any loop is homotopic to a constant loop.
3.2.1 Concatenation of loops

If \( \psi(\cdot) \) is a loop, we refer to the state \( \psi(0) \) as its basepoint. We define the concatenation \( \psi' \square \psi \) of two loops \( \psi', \psi \) with the same basepoint, in the obvious way:

\[
(\psi' \square \psi)(s) = \begin{cases} 
\psi(2s) & s \leq \frac{1}{2} \\
\psi'(2s - 1) & s \geq \frac{1}{2}
\end{cases}
\]  

(3.3)

and we note that \( \psi' \square \psi \) is also a loop with the same basepoint as \( \psi, \psi' \). It is quite intuitive that, if \( \psi' \) is a constant loop, then \( \psi' \square \psi \) and \( \psi \square \psi' \) are homotopic to \( \psi \). Along the same lines, for a loop \( \psi \), we can define the time-reversed loop \( \psi^\theta(\cdot) = \psi(1 - s) \) and it turns out that \( \psi^\theta \square \psi \) and \( \psi \square \psi^\theta \) are homotopic to a constant loop. Tools to prove such statements will be furnished in Section 5. Hence, just as for the conventional notion of homotopy, our notions of homotopy and concatenation lead to a group structure on homotopy equivalence classes of loops.

3.3 Stable equivalence and homotopy

3.3.1 Stable equivalence

We return to the question of equivalence of states. There are some works classifying certain \( G \)-invariant pure states on spin chain algebras, see in particular [11], but most authors [46, 2, 12] choose to relax the notion of equivalence to allow for tensoring with a product state on an auxiliary spin chain algebra, analogously to our definition of invertibility (2.3): One identifies \( \psi \) on a spin chain algebra \( \mathcal{A} \) with \( \psi \otimes \phi \) on \( \mathcal{A} \otimes \mathcal{A}' \) if \( \phi \) is a pure product state on \( \mathcal{A}' \). This gives states on spin chain algebras the structure of a monoid, with product states being the unit element. In that sense invertible states are indeed those that have an inverse in this monoid.

**Definition 3.4 (Stable \((G)\)-equivalence).** A pair of states \( \psi_1, \psi_2 \) on spin chain algebras \( \mathcal{A}_1, \mathcal{A}_2 \) are \((G)\)-stably equivalent if and only if there are \((G)\)-invariant product states \( \phi_1, \phi_2 \) on spin chain algebras \( \mathcal{A}'_1, \mathcal{A}'_2 \) such that the states \( \psi_1 \otimes \phi_1, \psi_2 \otimes \phi_2 \) are \((G)\)-equivalent.

The relaxation of the equivalence property by stabilization is natural, if only because it allows to compare states defined on spin chain algebras with distinct on-site dimensions \( n_j \). On the other hand, a case could be made that our notion of equivalence is too generous and that one should only allow for tensoring with product states of which all the factors have zero \( G \)-charge with respect to some fixed reference state, see [12] for a similar restriction. For example, consider a \( U(1) \)-invariant spin chain where, for each site \( i \), we specify a reference zero-dimensional \( G \)-invariant state \( \phi_i \) and we consider the \( G \)-invariant product state \( \psi = \otimes_i \psi_i \) such that \( h_{\psi_i/\phi_i} = i \), i.e., the charges grow linearly as one moves outwards. By our definition, \( \psi \) is trivially stably \( G \)-equivalent to \( \phi \), but it is not by the alternative definition, which seems in this case physically more relevant.

The classification of states up to stable \((G)\)-equivalence is usually referred to as the classification of (symmetry protected) topological phases. It has been a central theme in the recent literature.

3.3.2 Stable Homotopy

Just as for the notion of equivalence, it is natural to relax the notion of homotopy by allowing tensoring with product states.

**Definition 3.5 (Stable \((G)\)-homotopy).** A pair of loops \( \psi_1(\cdot), \psi_2(\cdot) \) on spin chain algebras \( \mathcal{A}_1, \mathcal{A}_2 \) are \((G)\)-stably homotopic if and only if there are \((G)\)-invariant product states \( \phi_1, \phi_2 \) on spin chain algebras \( \mathcal{A}'_1, \mathcal{A}'_2 \) such that the following loops are \((G)\)-homotopic \( \psi_1(\cdot) \otimes \phi_1, \psi_2(\cdot) \otimes \phi_2 \)
We note that the adjoined loops on $\mathcal{A}_1, \mathcal{A}_2$ are constant and that the role of the unit element in the monoidal structure of loops is played by constant loops with product basepoint.

### 3.4 Classification of loops

To streamline the discussion of our results, we introduce some terminology tailored to our results. This also serves to summarize the definitions made in the previous sections.

**Definition 3.6 (G-state).** A $G$-state is a triple $(\mathcal{A}, \gamma, \psi)$ where

i. $\mathcal{A}$ is a spin chain algebra.

ii. $\gamma$ is a strongly continuous $G$-action on $\mathcal{A}$ by on-site $*$-automorphisms.

iii. $\psi$ is a pure, $G$-invariant and $G$-invertible state on $\mathcal{A}$.

The extension to loops is then self-explanatory.

**Definition 3.7 (G-loop).** A $G$-loop is a triple $(\mathcal{A}, \gamma, \psi(\cdot))$ where

i. For any $s \in [0, 1]$, $(\mathcal{A}, \gamma, \psi(s))$ is a $G$-state.

We note that both $G$-states and $G$-loops come equipped with the monoidal structure of taking tensor products. If $\psi, \psi'$ are $G$-loops, then $s \mapsto \psi(s) \otimes \psi'(s)$ is a $G$-loop. Additionally, $G$-loops $\psi, \psi'$ that have a common basepoint, admit another natural operation, namely concatenation, see Section 3.2.1. To be precise, saying that two $G$-loops ‘have a common basepoint’ presupposes that they are defined on a common spin chain algebra $\mathcal{A}$, equipped with the same $G$-action $\gamma$.

We now come to our main result, the classification of loops, up to stable $G$-homotopy.

**Theorem 1. [Classification of loops]** There is a map $h$ assigning to any $G$-loop $\psi$ an element $h(\psi) \in H^1(G)$ such that

i. If $\psi$ is a constant $G$-loop, i.e. $\psi(s) = \psi(0)$ for $s \in [0, 1]$, then $h(\psi) = 0$.

ii. For any $h \in H^1(G)$, there is a $G$-loop $\psi$ such that $h(\psi) = h$.

iii. If a pair of $G$-loops $\psi$ and $\psi'$ is stably $G$-homotopic, then $h(\psi) = h(\psi')$.

iv. If the $G$-loops $\psi, \psi'$ have the same basepoint, then $h(\psi' \Box \psi) = h(\psi') + h(\psi)$.

v. For $G$-loops $\psi, \psi'$, we have $h(\psi' \otimes \psi) = h(\psi') + h(\psi)$.

vi. Let $\psi, \psi'$ be $G$-loops whose basepoints are stably $G$-equivalent and satisfying $h(\psi) = h(\psi')$, then $\psi, \psi'$ are stably $G$-homotopic.

We point out that the surjectivity $ii$ presupposes that $\mathcal{A}$ and the action $\gamma$ can be varied.

### 4 Pumping index

In this section, we provide an explicit construction of the map $h$ whose existence was postulated in Theorem 1. This echoes the informal discussion of the Thouless pump in the introduction.
4.1 Relative charge associated to cut loop

We consider a loop \( \psi(\cdot) \) generated by the \( G \)-invariant TDI \( H \). We restrict the generating TDI \( H \) at the origin to get the truncated TDI \( H' \) given by

\[
H'_S(s) = \begin{cases} 
H_S(s) & S \subset \{ \ldots, -2, -1, 0 \} \\
0 & \text{otherwise}
\end{cases}
\]  (4.1)

This TDI is again \( G \)-invariant, but the path it generates is in general no longer a loop because it can fail to be closed. That is, its endpoint, which shall henceforth call the pumped state,

\[
\psi' := \psi(0) \circ \alpha_H(1)
\]

need not be equal to the basepoint \( \psi(0) \). Nevertheless, far to the left of the origin, \( \psi' \) resembles \( \psi(0) \circ \alpha_H(1) = \psi(0) \) because far to the left \( H \) is equal to \( H' \). Far to the right, \( \psi' \) trivially resembles \( \psi(0) \). Making these observations precise, we will show that

\[
||(\psi(0) - \psi')_{<-r}|| \leq f(r), \quad ||(\psi(0) - \psi')_{>-r}|| \leq f(r)
\]  (4.2)

for some \( f \in \mathcal{F} \). Here we abbreviated \( \zeta_X = \zeta|_{A_X} \) for the restriction of functionals \( \zeta \) on \( A \) to the subalgebra \( A_X \), with \( X = \{ < -r \} \) or \( X = \{ \geq r \} \) in the above examples. Let \( \mathcal{H} \) be the Hilbert space carrying the GNS representation associated with \( \psi(0) \). Then the above bounds lead to

Proposition 2. If the pure state \( \psi(0) \) is invertible, then the pumped state \( \psi' \), is normal with respect to \( \psi(0) \). That is, it is represented by a density matrix on \( \mathcal{H} \).

We postpone the proof to Section [10]. By standard considerations (see Lemma [6.1]), we obtain a unitary action \( g \mapsto U(g) \) satisfying the requirements of Proposition [7] in Section [3.1.1] and therefore the pumped charge \( h_{\psi' / \psi(0)} \) is well-defined. We then define the index

\[
I(\psi(0), H) := h_{\psi' / \psi(0)}.
\]

We will show that it only depends on the loop \( \psi \), not on the TDI \( H \) generating the loop and so the map \( \psi \mapsto I(\psi(0), H) \) is well-defined as a map on \( G \)-loops.

Theorem 2. The map \( h : \psi \mapsto I(\psi(0), H) \) satisfies all the properties of Theorem [11].

We will therefore write the index as \( h(\psi) = I(\psi(0), H) \). We will use the notation \( I(\psi(0), H) \) whenever we locally want to emphasize the dependence on \( H \).

4.2 Example

We present a class of examples that realize a loop for every element \( h \in H^1(G) \). We take the on-site algebra \( A_j \simeq \mathcal{B}(\mathcal{H}) \) with \( \mathcal{H} \) a 3-dim Hilbert space with a distinguished orthonormal basis, denoted by \( \{| - h \rangle, |0\rangle, |h\rangle\} \). The group action is implemented by the unitary on-site operator \( U(g) \) defined on the basis by

\[
U(g)|h\rangle = e^{-ih\langle g|} |h\rangle,
\]  (4.3)

and hence the automorphism on the on-site algebra \( \mathcal{B}(\mathcal{H}) \) is given by \( \gamma(g)[A] = U(g)AU^*(g) \). Recalling the classification of zero-dimensional systems, we have on each site the pure states \( |h'\rangle \cdot |h\rangle \) with charge \( h' \) relative to the state \( |0\rangle \cdot |0\rangle \).

Given an operator \( O \in \mathcal{B}(\mathcal{H}) \), we write \( O_j \) for \( O \otimes 1_{A_{j+1}} \), viewing \( O \) as an element in \( A_j \). Similarly, given an operator \( O \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \), we write \( O_{j,j+1} \) for \( O \otimes 1_{A_{j,j+1}} \in A_{j,j+1} \).
This will be used now to construct the TDI. We define the charge pair creation and annihilation operators \( a(h), a^*(h) \) in \( \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \) by

\[
a(h') = |0,0\rangle \langle -h', h'|, \quad a^*(h') = |−h', h'\rangle \langle 0, 0|. \tag{4.4}
\]

Now we define two families of interactions \( E(h'), O(h') \) parameterized by \( h' \):

\[
E(h')_{(i,i+1)} = (a(h') + a^*(h'))_{i,i+1}, \quad \text{if } i \text{ is even} \tag{4.5}
\]

\[
O(h')_{(i,i+1)} = (a(h') + a^*(h'))_{i,i+1}, \quad \text{if } i \text{ is odd}. \tag{4.6}
\]

and \( E(h')_S, O(h')_S = 0 \) for all other \( S \). The TDI \( H = H(s) \) is then

\[
H(s) = \pi \begin{cases} 
E(h) & s \in [0,1/2] \\
O(-h) & s \in [1/2, 1]
\end{cases}. \tag{4.7}
\]

The basepoint of the loop is chosen as the product state \( \phi \) over zero-charge states defined by \( \phi[A] = \langle 0|A|0 \rangle \) for any \( A \in \mathcal{B}(\mathcal{H}) \) and \( \phi[A_iA'_i] = \phi[A_i]\phi[A'_i] \) for any \( A_i, A'_i \in \mathcal{B}(\mathcal{H}) \).

To understand the loop, it is helpful to graphically represent \( \phi \) as

\[
\ldots |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \ldots
\]

During the timespan \([0,1/2]\), the vector \(|0,0\rangle \rangle \rangle \) is rotated into \(|-h,h\rangle \) at pairs \((i, i+1)\) with \( i \) even, and vice versa. Vectors orthogonal to the 2-dimensional space spanned by \(|0,0\rangle \rangle \rangle \) and \(|-h,h\rangle \rangle \rangle \), are left invariant. Similarly, during the time \([1/2, 1]\], the vector \(|h,-h\rangle \rangle \rangle \) is rotated into \(|0,0\rangle \rangle \rangle \) at pairs \((i, i+1)\) with \( i \) odd, and vice versa. This gives hence

\[
\ldots |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \ldots \Rightarrow \ldots |-h\rangle \otimes |h\rangle \otimes |-h\rangle \otimes |h\rangle \ldots \Rightarrow \ldots |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \ldots
\]

In Figure 3 we give a graphical representation of \( \phi \), \( \phi \circ \alpha_H(1/2) \) and \( \phi \circ \alpha_H(1) \). This shows that indeed \( \phi = \phi \circ \alpha_H(1) \) and that therefore \( H \) does generate a loop. To determine the index of the loop, one goes through a similar calculation, to find

\[
\ldots |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \ldots \Rightarrow \ldots |-h\rangle \otimes |h\rangle \otimes |0\rangle \otimes |0\rangle \ldots \Rightarrow \ldots |0\rangle \otimes |h\rangle \otimes |0\rangle \otimes |0\rangle \ldots
\]

The apparent violation of charge conservation is because the opposite charge \(-h\) has been evacuated to \(-\infty\). It is not visible in the final state, and the index simply measures the charge \( h \) of the final state.

### 4.3 Overview of the proof

We are now ready to turn to proofs. We start in Section 5 with locality results on almost local evolutions; although they are known to some level from the literature, full proofs are not readily available in the precise setting we are considering here. In Section 6, we describe the Hilbert space setting obtained through the GNS representation. By comparing the initial state and the pumped state, this allows us to define an index following the lines sketched above and show that it is invariant under small perturbations of the TDI generating it. Section 7 starts the analysis of homotopies by showing that a loop with product basepoint can be deformed to what we call a short loop, namely a loop generated by a TDI with small norm. We show that this is doable through a \( G \)-homotopy provided the index vanishes. Section 8 finalizes this step by proving that a short loop can be further deformed to a constant loop; unlike the previous section which relies on splitting the chain, this section uses perturbative techniques on gapped ground states of quantum spin chains. Section 9 concludes the proof of our main result in the case of a fixed product basepoint, by proving the claimed additivities on the one hand, and showing that loops have the same index if and only if they are homotopic. Finally, Section 10 extends the results to loops with arbitrary \((G\)-invertible\) basepoints, concluding the proof of Theorem 1.
5 Technical preliminaries

In this section, we establish some technical facts and tools concerning interactions and TDIs.

5.1 Anchored interactions and TDIs

In Section 2, we defined interactions and we equipped them with a family of norms $|| \cdot ||_f$. We will often need to express that interactions are restricted to the vicinity of a region $X \subset \mathbb{Z}$. We say that an interaction $F$ is $X$-anchored if

\[ S \cap X = \emptyset \Rightarrow F_S = 0 \]

and it will be convenient to have an associated norm

\[ ||F||_{X,f} = \begin{cases} 
||F||_f & F \text{ is } X\text{-anchored} \\
\infty & \text{otherwise} 
\end{cases} \]  \hspace{1cm} (5.1)

If $F$ is $X$-anchored for a finite $X$, then $F$ determines in a natural way an element $\iota(F)$ of $\mathcal{A}$, see Section 2.2.2 where it was used primarily to define the almost local algebra $\mathcal{A}_{al}$.

Anchored TDIs are defined in a similar fashion, with norm $|||H|||_{X,f} = \sup_{s \in [0,1]} ||H(s)||_{X,f}$. We note that if $X$ is a finite set and $|||H|||_{X,f} < \infty$, the almost local evolution is generated by a time-dependent family of self-adjoint elements of $\mathcal{A}$, namely $\iota(H(s))$, see Lemma 5.5.

5.2 Manipulating interactions

We need some results about manipulating interactions and TDIs. We first introduce these manipulations algebraically and then we state most of the relevant bounds.
Commutators

Given interactions $H, H'$ we define the commutator interaction

\[
([H, H'])_S = \sum_{S_1, S_2: S_1 \cup S_2 = S, S_1 \cap S_2 \neq \emptyset} [H_{S_1}, H'_{S_2}]
\]  

(5.2)

The same symbol was already used in Section 2.2.2 to denote the action of an interaction on $A_{al}$. These two usages are consistent with each other: If $H'$ is anchored in a finite set, then $\iota(H') \in A_{al}$. In particular, $[H, \iota(H')]$ is well defined and $\iota([H, H']) = [H, \iota(H')]$. If also $H$ is anchored in a finite set, then the latter expression equals $[\iota(H), \iota(H')]$ with $[\cdot, \cdot]$ the standard commutator on $A$.

Derivations

The action of an interaction $H$ on $A_{al}$ by $[H, \cdot]$ is an (unbounded) derivation. It is the derivation, rather than the interaction itself, that determines the almost local evolution $\alpha_H$. One should notice that distinct $H \neq H'$ can give rise to the same derivation, which complicates some of our calculations. Therefore, we introduce the relation $\equiv$ to indicate equality of derivations:

$H \equiv H' \iff [H, A] = [H', A] \forall A \in A_{al}$

Weak sums

Given a family of interactions $H^{(n)}$ indexed by $n \in \mathbb{Z}$, we define the weak sum $\sum_n H^{(n)}$ by

\[
\langle \sum_n H^{(n)} \rangle_S = \sum_n H^{(n)}_S,
\]  

provided the right-hand side is absolutely norm-convergent for every $S$.

5.3 Automorphisms

Since we consider spin chain algebras $A$ where for every finite $S$, $A_S$ is finite-dimensional, there is a well-defined tracial state $\tau$ on $A$. For a region $X$, we write $\tau_{\cdot X}: A \to A$ for the corresponding conditional expectation $1_{A_X} \otimes \tau_{\cdot X}$. This linear map is contracting: $\|\tau_{\cdot X}[A]\| \leq \|A\|$. We define a canonical decomposition of observables $A \in A$ into finitely supported terms centered at a given site $j \in \mathbb{Z}$, $A = \sum_{k \in \mathbb{N}} A_{j,k}$, defined by

\[
A_{j,k} = \begin{cases} 
\tau_{B_k^c(j)}[A] - \tau_{B_{k-1}^c(j)}[A] & k > 0 \\
\tau_{B_0^c(j)}[A] & k = 0
\end{cases},
\]  

(5.4)

where we used the balls $B_k(j) = \{ \text{dist}(\cdot, j) \leq k \}$. Given an interaction $F$ that is $X$-anchored and a *-automorphism $\beta$, we define $\beta[F]$ by

\[
(\beta[F])_{B_k(j)} = \sum_{S \subseteq X, S \cap X^c} \frac{1}{|X \cap S|} (\beta[F_S])_{j,k}
\]  

(5.5)

for any $k \in \mathbb{N}$ and $j \in X$. For $S$ that are not of the form $B_k(j)$ with $j \in X$, we set $(\beta[F])_S = 0$. This definition satisfies the following constraint on the associated derivations.

\[
[\beta[F], \cdot] = \beta \circ [F, \cdot] \circ \beta^{-1}
\]  

(5.6)

and the integrated version for a TDI $H$ follows from Definition 2.1

\[
\alpha_{\beta[H]} = \beta \circ \alpha_H \circ \beta^{-1}
\]  

(5.7)
where $\beta[H](s) = \beta[H(s)]$. However, the above definition of $\beta[F]$ is of course not the unique definition satisfying these properties. For example, if $F$ is $X$-anchored, then it is also $X'$-anchored, whenever $X \subset X'$. The transformed interaction resulting from the definition (5.1) does in general change upon replacing $X$ by $X'$, whereas the associated derivations are equal. Let us illustrate a particular and relevant consequence of this. Let $\alpha_H$ be an almost local evolution and $F$ an interaction, so that the family $\alpha_H(s)[F]$ is defined by (5.5) for every $s \in [0, 1]$. Then in general the Heisenberg-like evolution equation $\alpha_H(s)[F] - F = i \int_0^s du \alpha_H(u)\{[H(u), F]\}$ is not correct as an equation between interactions, but the equality does hold on the level of derivations, i.e. we do have

$$\alpha_H(s)[F] - F \equiv i \int_0^s du \alpha_H(u)\{[H(u), F]\}$$

with $\equiv$ introduced in Section 5.2

### 5.4 Bounds on transformed interactions

To state bounds on transformed interactions, we will need to adjust the functions $f \in F$ as we go along. It would be too cumbersome to define each time explicitly the adjusted function, and therefore we introduce the following generic notion:

**Definition 5.1.** We say that $\hat{f}$ is affiliated to $f$ (and depends on some parameter $C$) if

1. Both $f, \hat{f}$ are elements of $F$ and $\hat{f}$ depends only on $f$ (and on the the parameter $C$).
2. If $f(r) \leq C\beta e^{-c_\beta r^\beta}$ for some $\beta < 1$ and $c_\beta, C_\beta > 0$, then $\hat{f}(r) \leq C'\beta e^{-c'_\beta r^\beta}$ for some $C'_\beta, c'_\beta > 0$.

We will also use repeatedly that $\max(f_1, f_2) \in F$ if $f_1, f_2 \in F$.

**Lemma 5.2.** Let $F, F_n, n \in \mathbb{Z}$ be interactions and $H$ a TDI. Let $X, X_n$ be (possibly infinite) regions and let $f, f_1, f_2$ be elements of $F$.

1. There is a function $\hat{f}$ affiliated to $\max\{f_1, f_2\}$ such that

$$||[F_1, F_2]||_{X, f} \leq \hat{f}(1 + \text{dist}(X_1, X_2))||F_1||_{X_1, f_1}||F_2||_{X_2, f_2}.$$  

2. If all $X_n$ are mutually disjoint, then

$$||\sum_n F^{(n)}||_{X, \hat{f}} \leq \sup_n ||F^{(n)}||_{X_n, f}, \quad X = \cup_n X_n$$

for $\hat{f}$ affiliated to $f$.

3. Assume that $||H||_{f_1} \leq C_H$ for some $C_H < \infty$. Then

$$||\alpha_H(s)[F]||_{X, \hat{f}} \leq ||F||_{X, f_2}, \quad s \in [0, 1]$$

for $\hat{f}$ affiliated to $f = \max(f_1, f_2)$ and depending on $C_H$.

**Proof.** (i) Let

$$f = \max(f_1, f_2), \quad g(r) = 2 \max_{r_1 + r_2 = r + 1} f(r_1)f(r_2), \quad \hat{f} = \sqrt{g}.$$
One checks that \( f, g \) and \( \hat{f} \) are in \( \mathcal{F} \). We also write \( f_i(S_i) = f_i(1 + \text{diam}(S_i)) \) for brevity. Definition \((5.2)\) implies immediately that the commutator is anchored in \( X_1 \). We then estimate

\[
|||F_1, F_2|||_{X_1, f} \leq \sum_{S_1 \cap X_1 \neq \emptyset, S_2 \cap X_2 \neq \emptyset} \frac{||(F_1)_1|| ||(F_2)_2||}{f_1(S_1) f_1(S_2)} \frac{2f_1(S_1)f_2(S_2)}{f(1 + \text{diam}(S_1 \cup S_2)).}
\]

We now bound

\[
2f_1(S_1)f_2(S_2) \leq g(1 + \text{diam}(S_1 \cup S_2)) \leq \hat{f}(1 + \text{dist}(X_1, X_2)) \hat{f}(1 + \text{diam}(S_1 \cup S_2)),
\]

where we used that \( \text{diam}(S_1 \cup S_2) \geq \text{dist}(X_1, X_2) \), and the claim follows.

(ii) Recalling Definition \((5.3)\), we pick \( S \subset \mathbb{Z} \) and we must estimate the norm of \( \sum_n F^{(n)}_S \). Since the sets \( X_n \) are mutually disjoint, there are at most \( \text{diam}(S) \) indices \( n \) in the sum such that \( F^{(n)}_S \neq 0 \). Hence

\[
\left\| \sum_n F^{(n)}_S \right\| \leq \text{diam}(S) \sup_n \|F^{(n)}_S\| \leq \text{diam}(S) \|f(1 + \text{diam}(S))\| \sup_n \|F^{(n)}\|_{X_n, f},
\]

which yields the claim since \( \text{diam}(S) \|f(1 + \text{diam}(S))\| \leq \hat{f}(1 + \text{diam}(S)) \) with \( \hat{f} \) affiliated to \( f \).

Item (iii) is a corollary of the Lieb-Robinson bound \([41]\). We refer to similar statements in \([48, 49]\) and their proofs, see also the proof of Lemma \(5.5\). \(\square\)

### 5.5 Inversion, time-reversal, and composition of almost local evolutions

We will often need to invert, time-reverse, and compose almost local evolutions. For TDIs \( H, H_1, H_2 \), we set

\[
(\alpha^{-1}_H(s)) = (\alpha_H(s))^{-1}, \quad (5.8)
\]

\[
(\alpha_H^\theta(s)) = \alpha^{-1}_H(1) \circ \alpha_H(1 - s), \quad (5.9)
\]

\[
(\alpha_{H_2} \circ \alpha_{H_1}(s)) = \alpha_{H_2}(s) \circ \alpha_{H_1}(s). \quad (5.10)
\]

**Lemma 5.3.** The families of automorphisms defined in \((5.3)\), (5.8) and (5.10), parametrized by \( s \in [0, 1] \) are almost local evolutions. More precisely:

i. The TDI \(-\alpha_H(s)[H(s)]\) generates the inverse \( \alpha^{-1}_H \).

ii. The TDI \(-H(1 - s)\) generates \( \alpha^\theta_H \), the time-reversal of \( \alpha_H \).

iii. The TDI \( H_1(1) + \alpha^{-1}_{H_1}(s)[H_2(s)] \) generates the composition \( \alpha_{H_2} \circ \alpha_{H_1} \).

**Proof.** The proposed families of interactions have a finite \(||| \cdot |||_{f}\)-norm for some \( f \in \mathcal{F} \) by Lemma \(5.2\). They are strongly measurable: If \( H(s) \) is strongly measurable, it is the pointwise limit of a piecewise constant interaction; The definition \((2.1)\) and the bounds of Lemma \(5.2\) further imply that \( \alpha_H(s) \) can be approximated by a piecewise constant automorphism. The fact that they generate the given families of automorphisms follows from a direct computation. \(\square\)

If \( \psi(\cdot) \) is a loop generated by a TDI \( H \), then the time-reversal \( \alpha^\theta_H \) generates the time-reversed loop \( \psi^\theta(\cdot) \) that was defined in Section \(3.2.1\). Indeed, using that \( \psi \) and \( \psi^\theta \) have the same basepoint \( \psi(0) \), we verify

\[
\psi(0) \circ \alpha^\theta_H(s) = \psi(0) \circ \alpha^{-1}_H(1) \circ \alpha_H(1 - s) = \psi(0) \circ \alpha_H(1 - s) = \psi(1 - s) = \psi^\theta(s).
\]
We now introduce the notion of a cocyle, associated to an almost local evolution. The cocycle \( \alpha_H(s, u) \), with \( u \leq s \), is defined as
\[
\alpha_H(s, u) = \alpha_H^{-1}(u) \circ \alpha_H(s)
\] (5.11)
so that in particular \( \alpha_H(s) = \alpha_H(s, 0) \). It corresponds to the evolution from time \( u \) to time \( s \).

The cocyle satisfies the cocycle equation
\[
\alpha_H(s, u) = \alpha_H(s', u) \circ \alpha_H(s, s').
\] (5.12)

and the evolution equation
\[
\alpha_H(s, u)[A] = A + i \int_u^s ds' \alpha_H(s', u)\{[H(s'), A]\}.
\]

In particular, \( \alpha_H(s, u) \) satisfies the same bound as \( \alpha_H(s) \) in Lemma 5.2(iii). Now we can then state the Duhamel principle:
\[
\alpha_H^2(s) - \alpha_H^1(s) = \int_0^s \alpha_H^2(u)\{[H^2(u) - H^1(u), \alpha_H^1(s, u)]\} du
\] (5.13)
which is established in the usual way from the Heisenberg equation (2.1).

Lemma 5.4. If \( H_1, H_2 \) are TDIs, then

i. The TDI \( \alpha_{H_1}(s)[H_2(s) - H_1(s)] \) generates the almost local evolution \( \alpha_{H_2} \circ \alpha_{H_1}^{-1} \).

ii. The TDI
\[
H_2(s) - H_1(s) + i\alpha_{H_2}^{-1}(s) \left( \int_0^s \alpha_{H_2}(u)\{[H_2(u) - H_1(u), \alpha_{H_1}(s, u)]\} du \right)
\]

generates the almost local evolution \( \alpha_{H_1}^{-1} \circ \alpha_{H_2} \).

Proof. Both claims follow by applying Lemma 5.3. For the second claim, one uses the Duhamel principle applied to interactions, more concretely,
\[
\alpha_{H_2}(s)[H(s)] - \alpha_{H}(s)[H(s)] = i \int_0^s \alpha_{H_2}(u)\{[H_2(u) - H(u), \alpha_{H}(s, u)]\} du.
\] (5.15)

\[\square\]

5.6 Applications of the Duhamel principle

We establish a few consequences of the bounds in the previous sections. Even though they are straightforward, they will be used so often that it is worthwhile to highlight them. First of all, we introduce the \( L^1 \) norm on TDIs, given by
\[
\|\| H \|\|_{X,f} := \int_0^1 ds \| H(s) \|_{X,f}
\]

Of, course, we always have \( \|\| H \|\|_{X,f} \leq ||| H |||_{X,f} \), but some bounds are stated more naturally in terms of the \( L^1 \)-norm.

Lemma 5.5. Let \( X, Y \) be (possibly infinite) intervals or complements thereof and let \( A \in \mathcal{A}_Y \).
i. If $F$ is an interaction anchored in $X$, then
\[ ||[F, A]|| \leq ||A|| ||F||_{X, f} \hat{f}(1 + \text{dist}(X, Y))(1 + |X \cap Y|) \quad (5.16) \]
with $\hat{f}$ affiliated to $f$.

ii. Let $H_1, H_2$ be TDIzs satisfying $||H_1||_f \leq C_H$, $||H_2||_f \leq C_H$ for some constant $C_H$. If $H_1 - H_2$ is anchored in $X$, then
\[ ||\alpha_{H_1}(s)[A] - \alpha_{H_2}(s)[A]|| \leq ||A|| ||H_1 - H_2||_{X, f} \hat{f}(1 + \text{dist}(X, Y))(1 + |X \cap Y|) \quad (5.17) \]
with $\hat{f}$ affiliated to depending on $C_H$.

Note that the right hand side of (5.16) is allowed to be infinite, but if it is finite then $[F, A]$ is a well-defined element of $\mathcal{A}$.

**Proof.** We obtain
\[ ||[F, A]|| \leq \sum_{y \in Y} \sum_{s \in Y \cap X \neq \emptyset} 2||F_s|| ||A|| \leq 2||A|| \sum_{y \in Y} f(1 + \text{dist}(y, X)) ||F||_{X, f}. \]
The bound $2\sum_{y \in Y} f(1 + \text{dist}(y, X)) \leq \hat{f}(1 + \text{dist}(X, Y))(1 + |X \cap Y|)$ now holds in the particular case of intervals. For ii) we use Duhamel’s formula (5.13) to get
\[ \alpha_{H_1}(s)[A] - \alpha_{H_2}(s)[A] = i \int_0^s du \alpha_{H_1}(u) \circ \alpha_{H_2}(s, u) \{ [\alpha^{-1}_{H_2}(s, u) \{ H_1(s) - H_2(s) \}, A] \} \quad (5.18) \]
By the results above, $\alpha^{-1}_{H_2}(s, u)$ is generated by a TDI whose norm is upper bounded by an expression depending on $C_H$. Therefore, we invoke Lemma 5.2 item iii) to get $||\alpha_{H_2}^{-1}(s, u) \{ H_1(s) - H_2(s) \}||_{X, f} \leq ||H_1(s) - H_2(s)||_{X, f}$ uniformly in $u, s \leq 1$. Using now item i) above, he argument of $\{ \}$ in (5.18) is in norm bounded by
\[ ||H_1(s) - H_2(s)||_{X, f} ||A|| \hat{f}(1 + \text{dist}(X, Y))(1 + |X \cap Y|), \]
which proves the claim since $\alpha_{H_1}(u)$ and $\alpha_{H_2}(s, u)$ preserve the norm.

The next lemma elaborates on the situation when a TDI $K$ is anchored in a set $X$. If $X$ is finite, then $\alpha_K(s)$ is an inner automorphism, i.e. we retrieve the formulas familiar from finite quantum systems:
\[ \alpha_K(s)[A] = \text{Ad}(V(s))[A] \quad (5.19) \]
with $V(s) \in \mathcal{A}$ a unitary family that solves
\[ V(s) = I + i \int_0^s du V(u) i(K(u)). \]

**Lemma 5.6.** Let $H$ be a TDI and $B$ a TDI anchored in $X$,

i. There are TDIs $K, K'$ anchored in $X$ such that
\[ \alpha_{H+B} = \alpha_H \circ \alpha_K = \alpha_{K'} \circ \alpha_H. \]

If $H, B$ are $G$-invariant, then so are $K, K'$. 

18
ii. Let $\psi$ be a pure state and let $X$ be finite. Then the states $\psi \circ \alpha_{H+B}$ and $\psi \circ \alpha_H$ are mutually normal. If $\psi$ and $H, B$ are $G$-invariant, then the relative charge of $\psi \circ \alpha_{H+B}$ and $\psi \circ \alpha_H$ is zero.

**Proof.** Item i). We use Lemma 5.3 and Lemma 5.4 to derive expressions for $K, K'$. Then we use Lemma 5.2 to pass the property of being anchored in $X$ from $B$ to $K, K'$. Item ii). By i) and the fact that $K$ is anchored on a finite set, the remark preceding the lemma ensures that there is a unitary $V \in A$ such that $\psi' = \psi \circ \text{Ad}(V)$. In particular they are normal with respect to each other. If $V$ is $G$-invariant, then they have zero relative charge.

### 5.7 Concatenation of almost local evolutions

First, we remark that one can often construct almost local evolutions by time-rescaling. This is used so often that we put it in a lemma.

**Lemma 5.7.** Let $H$ be a TDI and let $j : [0, 1] \rightarrow [0, 1]$ be a piecewise smooth function, with $|\frac{dj(s)}{ds}| \leq C_j$ except for a finite set of times $s$. Then $\alpha_H^{-1}(j(0)) \circ \alpha_H(j(\cdot))$ is an almost local evolution.

**Proof.** We set $K = \frac{dj(s)}{ds} H(s)$, except for the finite set of times where the bound $|\frac{dj(s)}{ds}| \leq C_j$ does not hold, where we set $K(s) = 0$. We see that $|||K|||_f \leq C_j|||H|||_f$ for any $f \in F$ so that the TDI satisfies the necessary bound. The fact that it generates the family $\alpha_H^{-1}(j(0)) \circ \alpha_H(j(\cdot))$ follows from direct computation.

For almost local evolutions $\alpha, \alpha'$, we define the concatenated evolution $\alpha \Box \alpha'$ by

$$(\alpha \Box \alpha')(s) = \begin{cases} 
\alpha(2s) & 0 \leq s < 1/2 \\
\alpha(1) \circ \alpha'(2s - 1) & 1/2 \leq s < 1
\end{cases} \quad (5.20)$$

The family of automorphisms $(\alpha \Box \alpha')(s)$ is again an almost local evolution by Lemma 5.3. We also note the relation to the concatenation of loops defined in (3.3): If $\psi, \psi'$ are loops with common basepoint and generated by $\alpha, \alpha'$, then $\psi \Box \psi'$ is a loop with the same basepoint and generated by $\alpha \Box \alpha'$. We will now state that composition of almost local evolutions is, in a certain sense, homotopic, to concatenation of almost local evolutions.

**Definition 5.8.** A family $(s, \lambda) \mapsto \alpha(\lambda)(s)$ is a $G$-homotopy of almost local evolutions if

$$\sigma^{(0)}(\lambda) \circ \alpha(\lambda)(s) = \alpha^{(0)}(s) \circ \sigma^{(s)}(\lambda), \quad \text{for any } s, \lambda \in [0, 1]$$

with $\alpha^{(\lambda)}, \sigma^{(s)}$, for each $\lambda, s$, generated by $G$-invariant TDI $H_\lambda, F_s$ satisfying the uniform boundedness property (3.2) and such that

$$\sigma^{(0)}(\lambda) = \sigma^{(1)}(\lambda) = \text{Id} \quad (5.21)$$

In Section 3.2 the homotopy was defined as a property of a function $(s, \lambda) \mapsto \psi_\lambda(s)$ of states. The connection with the above definition is the following: If $(s, \lambda) \mapsto \alpha(\lambda)(s)$ is a homotopy of almost local evolutions, then $(s, \lambda) \mapsto \nu \circ \alpha(\lambda)(s)$ is a homotopy of states, for any choice of state $\nu$. Moreover, if $s \mapsto \nu \circ \alpha(\lambda)(s)$ is a loop for some $\lambda$, then it is a loop (with the same basepoint $\nu$) for any $\lambda$. To complete the vocabulary, two almost local evolutions $\alpha^{(0)}, \alpha^{(1)}$ are called $G$-homotopic if there exists a $G$-homotopy $\alpha(\lambda)$ reducing to $\alpha^{(0)}$ for $\lambda = 0$ and to $\alpha^{(1)}$ for $\lambda = 1$. The main reason to consider homotopy of almost local evolutions, is given by the following lemma.
Lemma 5.9.
i. Let $H$ be a $G$-invariant TDI and let $j: [0, 1] \to [0, 1]$ be a piecewise smooth function, whose derivative is bounded (except for a finite set), and such that $j(0) = 0$ and $j(1) = 1$. Then $\alpha_H(\cdot)$ and $\alpha_H(j(\cdot))$ are $G$-homotopic.

ii. Let $H_1, H_2$ be two $G$-invariant TDIs. Then $\alpha_{H_2} \circ \alpha_{H_1}$ is $G$-homotopic to $\alpha_{H_2} \Box \alpha_{H_1}$.

Proof. Item i) We consider the family

$$\alpha^\lambda(s) = \alpha(k(s, \lambda)), \quad k(s, \lambda) = (1 - \lambda)s + \lambda j(s)$$

For any $s$, the function $k(s, \cdot)$ satisfies the conditions of Lemma 5.7 and hence it yields the required almost local evolution $\sigma^\lambda(\cdot) = \alpha^{-1}(s) \alpha(k(s, \cdot))$. The condition (5.21) follows from the fact that $j(0) = 0$ and $j(1) = 1$.

Item ii) For simplicity, let us denote $\alpha(s) = \alpha_{H_1}(s)$ and $\beta(s) = \alpha_{H_2}(s)$. We will construct a family $\alpha^\lambda(s)$ interpolating between $\beta \circ \alpha$ and $\beta \Box \alpha$ as $\lambda$ ranges between 0 and 1. Let

$$d(\lambda) = \frac{1}{2}(1 + \lambda, 1 - \lambda) \in [0, 1]^2, \quad \lambda \in [0, 1].$$

Let $j(\lambda): [0, 1] \to [0, 1]^2: s \mapsto j(\lambda)(s) = (j^1(\lambda)(s), j^2(\lambda)(s))$ be the continuous and piecewise smooth map defined by

$$j(\lambda)(s) = \begin{cases} 2s d(\lambda) & s \leq 1/2 \\ d(\lambda) + 2(s - \frac{1}{2})((1, 1) - d(\lambda)) & s > 1/2 \end{cases}$$

We note that

$$j^{(0)}(s) = (s, s), \quad j^{(1)}(s) = \begin{cases} (2s, 0) & s \leq 1/2 \\ (1, 2s - 1) & s > 1/2 \end{cases}$$

Then we define the family of automorphisms

$$\alpha^\lambda(s) = \beta(j^1(\lambda)(s)) \circ \alpha(j^2(\lambda)(s)), \quad \lambda \in [0, 1].$$

We verify

i. $\alpha^\lambda(0) = \text{Id}$ and $\alpha^\lambda(1) = \beta(1) \circ \alpha(1)$ because $j^{(0)}(0) = (0, 0), j^{(1)}(1) = (1, 1)$.

ii. $\alpha^{(0)}(s) = (\beta \circ \alpha)(s)$.

iii. $\alpha^{(1)}(s) = (\beta \Box \alpha)(s)$.

iv. For each $\lambda$, $\alpha^\lambda$ is an almost local evolution.

v. $\alpha^\lambda(s) = \alpha^{(0)}(s) \circ \sigma^\lambda(s)$, with

$$\sigma^\lambda(s) = \alpha(s)^{-1} \circ \beta(s)^{-1} \circ \beta(j^1(\lambda)(s)) \circ \alpha(j^2(\lambda)(s))$$

vi. For each $s$, $\sigma^\lambda(s)$ is an almost local evolution.

vii. $\sigma^{(0)}(\lambda) = \sigma^{(1)}(\lambda) = \text{Id}$, so item (v) means that $(s, \lambda) \mapsto \alpha^\lambda(s)$ is a homotopy. Item (i–iii) and (vii) are immediate consequences of the definition, using (5.22). To check (iv), we use Lemma 5.7 to conclude that both $\beta(j^1(\lambda)(\cdot))$ and $\alpha(j^2(\lambda)(\cdot))$ are almost local evolutions and so is their composition by Lemma 5.3. Item (v) follows from the definition of $\alpha^\lambda(s)$ and (ii). Item (vi) follows from Lemma 5.7 upon noting that $\lambda \mapsto j^{(\lambda)}(s)$ has uniformly bounded derivatives and $j^{(0)}(s) = (s, s)$.

\[\square\]
6 Hilbert space theory for states equivalent to a product state

Although most of our reasoning stays on the level of the spin chain algebra $A$, some steps are easier taken within the GNS representation. In this section, we establish the tools that we will need from Hilbert space theory.

6.1 GNS construction

We will make use of the GNS representation of state $\psi$ on $A$, see [50]. As before, we denote the GNS triple associated to $\psi$ by $(H, \Omega, \pi)$ (Hilbert space $H$, normalized vector $\Omega \in H$, $*$-representation $\pi : A \to B(H)$), such that

$$\psi(A) = \langle \Omega, \pi[A]\Omega \rangle,$$

for any $A \in A$.

From the purity of $\psi$, it follows that $\pi(A)' = \mathbb{C}1$ and hence the von Neumann algebra $\pi(A)''$ equals the full algebra $B(H)$. The following lemma, already used in Section 3.1.2, verifies that we are in the setting outlined in Section 3.1.1; We omit its standard proof which relies on the uniqueness of the GNS representation.

Lemma 6.1. If $\psi$ is $G$-invariant, then there is a unique strongly continuous unitary representation $G \to U(H) : g \mapsto U(g)$, such that

$$\pi \circ \gamma(g) = \text{Ad}(U(g)) \circ \pi$$

(6.1)

We further recall that two pure states $\psi, \psi'$ are said to be mutually normal iff $\psi'$ is represented as a pure density matrix $\rho_{\psi'}$ in the GNS representation of $\psi$: $\psi'(A) = \text{Tr}_H(\rho_{\psi'} \pi(A))$. It then follows that the same holds with the roles of $\psi, \psi'$ reversed.

6.2 GNS representation of a pure product state

We construct the GNS representation of a pure, $G$-invariant product state $\phi$ in a pedagogical way to render the objects that follow more tangible. Recall that $A_i$ is a matrix algebra of $n_i \times n_i$ complex matrices, that we write as $A_i = \mathbb{B}(\mathbb{C}^{n_i})$ For every $i$, we choose an orthonormal basis of $\mathbb{C}^{n_i}$ labelled by $\sigma = 0, \ldots, n_i - 1$ such that

$$\phi(A) = \langle 0|A|0 \rangle, \quad A \in A_i.$$

The state $\phi$ corresponds hence intuitively to the — a priori ill-defined — vector

$$\ldots \otimes |0\rangle_{i-1} \otimes |0\rangle_i \otimes |0\rangle_{i+1} \otimes \ldots$$

(6.2)

The representation Hilbert space $H$ is chosen as

$$\mathcal{H} := l^2(M_Z)$$

where $M_X$, for $X \subset \mathbb{Z}$, is the space of sequences $m : X \to \{0, 1, 2, \ldots\}$ such that $m(j) < n_j$ and the set $\{j : m(j) \neq 0\}$ is finite. We write henceforth $H_X = l^2(M_X)$ and we note the tensor product structure $H = H_X \otimes H_{X'}$. We now define an isometry $K_S : \otimes_{j \in S} \mathbb{C}^{n_j} \to H_S$ for finite $S$, by

$$K_S(\otimes_{j \in S}|m_j\rangle)(m') = \begin{cases} 1 & m' = m, \quad m, m' \in M_S \\ 0 & m' \neq m \end{cases}$$

The term ‘equivalent’ is used more often in the literature, but we avoid this term since it has already another natural meaning within this paper. However, in the zero-dimensional case, the two notions actually coincide.
and we lift it to an isomorphism of $C^*$-algebras by setting 
$$\pi_S : \mathcal{A}_S \to \mathcal{B}(\mathcal{H}_S) : A \mapsto KSAK_S^{-1}.$$  

Then the representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is fixed by the requirement that its restrictions 
$$\pi|_{\mathcal{A}_S} : \mathcal{A}_S \otimes I_{S^c} \to \mathcal{B}(\mathcal{H}_S) \otimes I_{H_{S^c}}$$  
coincide with $\pi_S$, for finite $S$. We note that for finite $S$, $\pi_S$ is an isomorphism between $\mathcal{A}_S$ and $\mathcal{B}(\mathcal{H}_S)$, whereas for infinite $X$, $\pi(\mathcal{A}_X)$ is a strict subset of $\mathcal{B}(\mathcal{H}_X)$. In the former case, the dual space of bounded linear functionals on $\mathcal{A}_S$ is isomorphic to the trace-class operators on $\mathcal{H}_S$, equipped with the trace norm $\|\cdot\|_1$. The state $\phi$ is now represented by the vector $\Omega = \delta_{m_0} \in \mathcal{H}$ with $m_0(j) = 0$ for all $j$;  
$$\phi(A) = \langle \Omega, \pi[A]\Omega \rangle$$  
and we see that $\Omega$ indeed corresponds to the heuristic (6.2), and it factors as $\Omega = \Omega_X \otimes \Omega_{X^c}$. We denote by $\Pi_X$ (and again $\Pi = \Pi_Z$) the orthogonal projection on the range of $\Omega_X$ and we abuse notation by also writing $\Pi_X$ to denote $\Pi_X \otimes 1_{X^c}$.

We now state a relevant condition for a pure state $\psi$ to be normal with respect to the product state $\phi$. We fix intervals in $\mathbb{Z}$ centered on the sites $\{0, 1\}$:  
$$I_1 = \emptyset, \quad I_r = [2 - r, r - 1], \quad r > 1.$$  

Then

**Lemma 6.2.** Let $\phi, \psi$ be pure states on $\mathcal{A}$ and let $\phi$ be a product. If $\|\langle (\psi - \phi)_{I_r}\rangle\| \to 0$ as $r \to \infty$, then $\psi$ is normal w.r.t. $\phi$.

**Proof.** We use the notation introduced above. Let $\rho_{I_r}$ be the density matrix on the finite-dimensional Hilbert space $\mathcal{H}_{I_r}$ representing the state $\psi_{I_r}$. We consider the sequence 
$$\rho_{I_r} \otimes \Pi_{I_r^c}$$  
(6.4)

of density matrices on $\mathcal{B}(\mathcal{H})$ indexed by $r$, and we prove now that this sequence is a Cauchy sequence in trace-norm $\|\cdot\|_1$. Let us denote $\delta(r) = \|\langle (\psi - \phi)_{I_r}\rangle\|$ and abbreviate $\rho_r = \rho_{I_r}$, $\Pi_r = \Pi_{I_r \setminus I_r}$, for $r < r'$, and let $\text{Tr}_{I_r}, \text{Tr}_{I_r^c}$ be the traces on $\mathcal{H}_{I_r}, \mathcal{H}_{I_r^c}$, respectively. The key observation is that, writing $P = 1 - \Pi_{I_r^c}$,  
$$\|\rho_r P\|_2^2 = \text{Tr}_{I_r} [\rho_r P] = \psi(P) = (\psi - \phi)(P)$$  
where we used the noncommutative H"older inequality $\|AB\|_1 \leq \|A\|_2 \|B\|_2$ with $A = \sqrt{\rho_r}, B = \sqrt{\rho_{r'}} P$ and $P = P_{I_r^c}$, to get the first inequality. Therefore,  
$$\|\rho_{r'}(1 - \Pi_{I_r^c})\|_1 \leq \sqrt{\delta}.$$  
(6.5)

Since $\rho_r = \text{Tr}_{I_r^c}[\rho_{r'}]$, we have that 
$$\rho_r \otimes \Pi_{r', r'} = \text{Tr}_{I_r^c}[\Pi_{r', r'} \rho_{r'} \Pi_{r', r'}] \otimes \Pi_{r', r'} + E_1$$  
(6.6)

$$= \Pi_{r', r'} \rho_{r'} \Pi_{r', r'} + E_1 = \rho_{r'} + E_1 + E_2$$  
(6.7)

where $\|E_1\|_1 \leq \sqrt{\delta}$ and $\|E_2\|_1 \leq 3\sqrt{\delta}$, by (6.5). This shows that (6.4) is indeed Cauchy, and hence it converges to a density matrix $\rho$. Therefore, the normal states on $\mathcal{A}$, induced by the density matrices (6.4) form a Cauchy sequence as well and they have a limit. The limit is necessarily equal to $\psi$ since $\psi$ is obviously a limit point (hence the unique limit point) in the weak* topology. We conclude that $\psi$ is represented by $\rho$. \qed
As remarked in Section 4.1, this lemma also follows from Corollary 2.6.11 in [50]. Unlike there, the proof given here yields quantitative bounds, which we collect now. We use the notations above, in particular \( \delta(r) = \| (\psi - \phi)_{I_f} \| \). Then, taking \( r' \to \infty \) in the proof of Lemma \ref{lemma:boundedness}, we have

\[
\| \rho - \rho_{I_f} \otimes \Pi_{I_f} \|_1 \leq 4\sqrt{\delta(r)}, \tag{6.8}
\]

\[
\| \rho - \Pi_{I_f} \rho \Pi_{I_f} \|_1 \leq 3\sqrt{\delta(r)}. \tag{6.9}
\]

From \( \ref{lemma:boundedness} \), we also deduce

\[
\| \rho - \frac{\Pi_{I_f} \rho \Pi_{I_f}}{\text{Tr}[\Pi_{I_f} \rho \Pi_{I_f}]} \|_1 \leq 12\sqrt{\delta(r)}, \tag{6.10}
\]

provided that \( \text{Tr}[\Pi_{I_f} \rho \Pi_{I_f}] \neq 0 \). The claim is trivial if \( 12\sqrt{\delta(r)} \geq 2 \). If \( 6\sqrt{\delta(r)} < 1 \), we denote \( \tilde{\rho} = \Pi_{I_f} \rho \Pi_{I_f} \) and have that \( \| \rho - (\text{Tr} \tilde{\rho}^{-1})\tilde{\rho} \|_1 \leq 3\sqrt{\delta(r)} + \| \text{Tr} \tilde{\rho}^{-1} \| \| 1 - \text{Tr} \tilde{\rho} \|_1 \), which yields the claim since \( \ref{lemma:boundedness} \) implies \( |1 - \text{Tr} \tilde{\rho}| \leq 3\sqrt{\delta(r)} \leq 1/2 \) and further \( \text{Tr} \tilde{\rho} \geq 1/2 \).

### 6.3 The pumped state and its index

In this section we analyze the pumped state that was introduced in Section 4.1 for the special case in which the basepoint of the loop is a pure \( G \)-invariant product \( \phi \), as in the previous subsections. We fix this state \( \phi \) in what follows, and we consider a \( G \)-invariant TDI \( H \) that generates a loop \( \phi \circ \alpha_{H}(\cdot) \), satisfying \( \| H \|_{\| f \|} \leq C_H \) for some \( f \in \mathcal{F} \). As in Section 4.1, we cut the TDI by defining \( H' \) to contain only the terms \( H_S \) supported on the left, \( S \leq 0 \), obtaining the pumped state

\[
\psi = \phi \circ \alpha_{H}(1). \tag{6.11}
\]

We first establish that \( \psi \) is normal w.r.t. \( \phi \).

**Lemma 6.3.** There is \( \hat{f} \) affiliated to \( f \) and depending on \( C_H \) such that

\[
\| (\psi - \phi)_{I_f} \| \leq \hat{f}(r). \tag{6.12}
\]

Therefore, as a consequence of Lemma 6.2, \( \psi \) is normal w.r.t. \( \phi \)

**Proof.** Clearly, \( \psi \) is a product \( \psi = \psi_{\leq 0} \otimes \psi_{\geq 1} \), and \( \psi_{\geq 1} = \phi_{\geq 1} \). It follows that

\[
\| (\psi - \phi)_{I_f} \| = \| (\psi - \phi)_{<2-r} \| = \sup_{A \in A_{<2-r}, \| A \| \leq 1} |\psi(A) - \phi(A)|
\]

and we now estimate the last quantity:

\[
|\psi(A) - \phi(A)| = |\phi(\alpha_{H}(1)[A] - \alpha_{H}(1)[A])| \leq \hat{f}(r)\| A \|
\]

where the equality follows because \( \phi \circ \alpha_{H}(\cdot) \) is a loop, and the inequality follows from Lemma 5.5 item ii), with the input that \( H - H' \) is anchored in \( \{ \geq 1 \} \) and that all constants that appear can be bounded by functions of \( C_H \).

Since \( \psi \) is normal w.r.t. \( \phi \), and the \( G \)-action is implemented by a unitary representation, see Lemma 6.1, we can indeed define the index \( I(\phi, H) \) introduced in Section 4.1. We will now derive some topological properties of the map \( H \mapsto I(\phi, H) \).
6.3.1 Properties of the index map $I(\phi, \cdot)$

We fix again the product state $\phi$ and its GNS triple. We also fix $f \in \mathcal{F}$ and a constant $C_H$. We let $S = \mathcal{S}(C_H, f, \phi)$ be the set of $G$-invariant TDIs such that

i. $\|\|H\|\|_f \leq C_H$

ii. $\phi \circ \alpha_H(\cdot)$ is a loop

To any $H \in S$, we associate the pumped state $\hat{\psi} = \phi \circ \alpha_H(1)$. Let $T_{1,*}(\mathcal{H})$ the set of pure $G$-invariant density matrices on $\mathcal{H}$. In particular, $\rho_\psi$, the density matrix representing the state $\psi$ on $\mathcal{H}$, is in $T_{1,*}(\mathcal{H})$. We define the function

$$\hat{\rho} : S \to T_{1,*}(\mathcal{H}) : H \mapsto \rho_\psi.$$ 

Next, let $g \mapsto U(g)$ be the unitary representation on $\mathcal{H}$ given in Lemma 6.1. Since $\rho$ is pure, it is a one-dimensional projection and the $G$-invariance implies that $U(g)$ reduces to a one-dimensional representation on the range of the projection. In particular, $\text{Tr}(\rho U(g))$ is a complex number on the unit circle. We set

$$\hat{h} : T_{1,*}(\mathcal{H}) \to H^1(G) : \rho \mapsto \hat{h}(\rho)$$

where $\hat{h}(\rho)$ is the homomorphism $G \to S^1$ defined by

$$e^{i\hat{h}(\rho)(g)} = \text{Tr}(\rho U(g)) \in U(1). \quad (6.14)$$

Because of the specific choice of $U(\cdot)$, we find that $\hat{h}(\rho_\psi) = h_{\psi/\phi}$. We consider $S$, $T_{1,*}(\mathcal{H})$, $H^1(G)$ with topologies induced by, respectively, the norm $\|\|_f^{(1)}$, the trace norm $\|\|_I^{(1)}$, and the metric $d_\infty(h, h') = \sup_{g \in G} |h(g) - h'(g)|_{S^1}$. With these choices, the map $\hat{h}$ is Lipschitz. Indeed, for any $\rho^{(1)}, \rho^{(2)} \in T_{1,*}(\mathcal{H})$,

$$d_\infty(\hat{h}(\rho^{(1)}), \hat{h}(\rho^{(2)})) \leq C_{S^1} \sup_{g \in G} \left| \text{Tr}\left[\rho^{(1)}(g) U(g)\right] - \text{Tr}\left[\rho^{(2)}(g) U(g)\right] \right| \leq C_{S^1} \|\rho^{(1)} - \rho^{(2)}\|_I.$$ 

with $C_{S^1} = \sup_{\theta \in [0, \pi]} \frac{\theta}{|e^{\theta} - 1|} < \infty$. We now show

**Lemma 6.4.** The maps $\hat{\rho} : S \to T_{1}(\mathcal{H})$ and

$$I(\phi, \cdot) = \hat{h} \circ \hat{\rho} : S \to H^1(G)$$

are uniformly continuous.

Note that the value of $\hat{h} \circ \hat{\rho}$ is the same as the value of the index in Theorem 2 but the map $\hat{h}$ there differs from $\hat{h} \circ \hat{\rho}$ here because it is a function of a loop of states there, while it is here a function of the Hamiltonians generating a particular loop with fixed product basepoint.

**Proof.** We have already remarked that $\hat{h}$ is Lipschitz, so both claims will be proven once we show that $\hat{\rho}$ is uniformly continuous. Consider $H^{(1)}, H^{(2)} \in S$ and let $\rho^{(j)} = \hat{\rho}(H^{(j)})$. We have

$$\|\rho^{(1)} - \rho^{(2)}\|_I \leq \|\rho^{(1)}_{L^c} \otimes \Pi_{L^c} - \rho^{(2)}_{L^c} \otimes \Pi_{L^c}\|_I + \|\rho^{(1)}_{L^c} \otimes \Pi_{L^c} - \rho^{(1)}\|_I + \|\rho^{(2)}_{L^c} \otimes \Pi_{L^c} - \rho^{(2)}\|_I$$

$$\leq \|\rho^{(1)}_{L^c} - \rho^{(2)}_{L^c}\|_I + 8\sqrt{f(r)}$$
where we used the bound (6.8) with \( \delta(r) = \hat{f}(r) \), by Lemma 6.6. To bound the difference \( \|\rho^{(1)}_{I_{r}} - \rho^{(2)}_{I_{r}}\|_1 \), we rewrite it as

\[
\sup_{A \in A_{r}, \|A\| \leq 1} |\phi \circ \alpha_{(H^{(1)})^{\gamma}}(1) [A] - \phi \circ \alpha_{(H^{(2)})^{\gamma}}(1) [A]| 
\]  

(6.15)

which is bounded by \( \hat{f}(1) \|I_{r}\| \|H^{(1)} - H^{(2)}\|_{f}^{(1)} \) by Lemma 6.5. We hence get

\[
\|\rho^{(1)} - \rho^{(2)}\|_1 \leq \hat{f}(1) \|I_{r}\| \|H^{(1)} - H^{(2)}\|_{f}^{(1)} + 8\sqrt{\hat{f}(r)}
\]

It is important to note that, since \( f \) is fixed, \( \hat{f} \) can be fixed as well. Since \( \hat{f}(r) \to 0 \) as \( r \to 0 \), we can now find a modulus of continuity for the function \( \hat{\rho} \).

Finally, we formulate a proposition that will be used crucially in the proof of homotopy invariance of the index map \( I(\phi, \cdot) \).

**Proposition 3.** There is an \( \epsilon > 0 \), such that, if \( \|H^{(1)} - H^{(2)}\|_{f}^{(1)} \leq \epsilon \) for \( H^{(1)}, H^{(2)} \in S \), then \( I(\phi, H^{(1)}) = I(\phi, H^{(2)}) \).

**Proof.** Let \( h \in H^1(G) \). The discreteness of \( H^1(G) \), see Section 3.1.2 implies that there is an \( \epsilon(h) > 0 \) such that \( d_{\infty}(h, h') \leq \epsilon(h) \) implies \( h = h' \). However, since the metric on \( H^1(G) \) is homogeneous, namely \( d_{\infty}(h, h') = d_{\infty}(h - h', 0) \), we find \( \epsilon(h) = \epsilon(0) \) and one can choose \( \epsilon \) uniformly on \( H^1(G) \). The claim of the proposition follows then from the uniform continuity of the index map \( I(\phi, \cdot) \) on \( S \), i.e. from Lemma 6.4. \( \square \)

### 6.4 Connecting states by adiabatic flow

States that are mutually normal, can be connected via an almost local evolution whose generating TDI is anchored in a finite set. In other words, the almost local evolution consists of inner automorphisms. In this section, we establish this statement in a quantitative way, but restricted to the specific setting in which we will need it. A result very similar to the crucial Lemma 6.6 below already appeared in [12].

#### 6.4.1 Parallel transport for 0-dimensional systems

We first exhibit how to connect states on 0-dimensional systems, i.e. states on \( B(H) \), with \( H \) a Hilbert space. We say that a bounded, measurable function \( [0,1] \ni s \mapsto E(s) \) with \( E(s) = E^{s}(s) \in B(H) \) is a zero-dimensional TDI and we set \( \|E\| = \sup_{s \in [0,1]} \|E(s)\| \). The evolution \( s \mapsto \alpha_{E}(s) \) is then defined as the family of automorphisms that is the unique solution to the equation

\[
\alpha_{E}(s) = \mathbb{I} + i \int_{0}^{s} du \alpha_{E}(s)\{[E(u), \cdot]\}.
\]

**Lemma 6.5.** Let \( \nu, \omega \) be pure normal states on \( B(H) \). Then there exist zero-dimensional TDI \( E \), such that

i. \( \|E\| < 8\|\nu - \omega\| \)

ii. \( \nu = \omega \circ \alpha_{E}(1) \)

iii. If both \( \nu, \omega \) are invariant under a \( G \)-action by automorphisms \( \gamma(g) \), and \( h_{\nu/\omega} = 0 \), then \( E \) can be chosen \( G \)-invariant as well.

The proof of this lemma uses only basic linear algebra and we postpone it to Appendix A.
6.4.2 Parallel transport for states normal to $\phi$

Let $\psi$ be a pure state normal w.r.t. the pure product state $\phi$. Recall the definition of the intervals $I_r$ in (6.3).

**Lemma 6.6.** Let $\psi$ be a pure state and $\phi$ a pure product state, such that

$$|||\psi - \phi|||_{F^r} \leq f(r), \quad f \in F. \tag{6.16}$$

Then there exist a TDI $K$ such that

i. $|||K|||_{(0,1), j} < 1$ for $f$ affiliated to $\phi$.

ii. $\psi = \phi \circ \alpha_K(1)$

iii. If both $\phi, \psi$ are G-invariant, with zero relative charge $h_{\psi/\phi} = 0$, then $K$ can be chosen $G$-invariant.

**Proof.** By Lemma 6.2 $\psi$ is normal w.r.t. $\phi$. Let $\rho$ be the pure density matrix on $\mathcal{H}$ representing $\psi$. We set

$$\tilde{\mu}_r = \frac{\Pi_{F^r} \rho \Pi_{F^r}}{\text{Tr}[\Pi_{F^r} \rho \Pi_{F^r}]} \quad \text{provided } \text{Tr}[\Pi_{F^r} \rho \Pi_{F^r}] \neq 0 \tag{6.17}$$

By the bounds (6.9) and (6.10) with $\delta(r) = f(r)$ and the fact that $f$ is non-increasing, there is a finite $r_0 > 0$ such that for $r \geq r_0$, the denominator (6.17) is not zero. For $1 \leq r \leq r_0$, we set simply $\tilde{\mu}_r = \Pi$ (which coincides with the definition (6.17) for $r = 1$). In any case, the density matrices $\tilde{\mu}_r$ are pure, by the purity of $\rho$, and of the form $\tilde{\mu}_r = \mu_r \otimes \Pi_{F^r}$ with $\mu_r$ pure as well. We have

$$||\mu_{r+1} - \mu_r \otimes \Pi_{I_{r+1} \setminus I_r}|| = ||\tilde{\mu}_{r+1} - \tilde{\mu}_r|| \leq ||\tilde{\mu}_r - \rho|| + ||\tilde{\mu}_{r+1} - \rho|| \leq 4\sqrt{f(r)} + 4\sqrt{f(r+1)} \leq 8\sqrt{f(r)}$$

by (6.10) with $\delta(r) = f(r)$.

We now apply Lemma 6.5 to obtain a zero-dimensional TDI $E_r(\cdot)$ of operators on $\mathcal{H}_{I_{r+1}}$ (see Section 6.2) transporting $\mu_r \otimes \Pi_{I_{r+1} \setminus I_r}$ into $\mu_{r+1}$, satisfying the bound

$$\sup_s ||E_r(s)|| \leq 8\sqrt{f(r)} \tag{6.18}$$

and we identify $E_r(\cdot)$ with $E_r(\cdot) \otimes \Pi_{F^{r+1}}$ so that $E_r(\cdot)$ are viewed as operators on $\mathcal{H}$ transporting $\rho_r$ into $\rho_{r+1}$. Since the restricted GNS representation $\pi|_{\mathcal{A}_X} : \mathcal{A}_X \to \mathcal{B}(\mathcal{H}_X)$ is bijective for $X$ finite, we can define

$$K_r(s) = \pi^{-1}(E_r(s)) \in \mathcal{A}_{I_{r+1}}$$

It remains to assemble this family of time-dependent operators into a single TDI. Let $T_r \subset [0, 1]$ be half-open intervals $[t, t')$ of length $(\pi^2/6)^{-1} \times 1/r^2$ such that $\min T_1 = 0$ and $\sup T_r = \min T_{r+1}$. Note that $\cup_r T_r = [0, 1)$. Then, for $s \in [0, 1]$,

$$K_S(s) = \begin{cases} \frac{1}{|T_r|} K_r\left(\frac{s}{|T_r|}\right) & S = I_{r+1}, s \in T_r \text{ for some } r \in \mathbb{N}^+ \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $\phi \circ \alpha_K(\sup T_r)$ is the state corresponding to the density matrix $\rho_{r+1}$ and so $\phi \circ \alpha_K(\sup T_r) \to \psi$ as $r \to \infty$. The bound $|||K|||_{(0,1), j} < 1$ now follows from (6.18). The other claims simply carry over from Lemma 6.5. \qed
7 A loop with trivial index and product basepoint is homotopic to a short loop

By a short loop, we mean a loop with a the generator of small norm.

**Proposition 4.** Let $\phi$ be a $G$-invariant pure product state on $A$ and let $H$ be a $G$-invariant TDI such that $\phi \circ \alpha_H(\cdot)$ is a loop with trivial index, $I(\phi, H) = 0$. Assume that $\|\|H\|\|_f \leq C_H$ for some $f \in \mathcal{F}$. Then there exists $G$-invariant TDIs $\tilde{H}^{(R)}$, with $R \in 2\mathbb{N}$ such that $\phi \circ \alpha_{\tilde{H}^{(R)}}(\cdot)$ is a loop that is $G$-homotopic to $\phi \circ \alpha_H(\cdot)$, and such that

$$\lim_{R \to \infty} \|\|\tilde{H}^{(R)}\|\|_f = 0.$$ 

for $\tilde{f}$ affiliated to $f$ and depending on $C_H$. The homotopy can be chosen to have fixed basepoint $\phi$.

We work throughout the section under the assumptions of the proposition. All constants $C$ that appear can depend on $C_H$ and $f$. In particular, we write $\tilde{f}$ for a generic function affiliated to $f$ and depending on $C_H$. Just like for constants $C$, it is understood that the function $\tilde{f}$ can change from line to line.

7.1 Splitting of the loop at a single edge

We say that a TDI $F$ is split at an edge $(j, j + 1)$ iff

$$\min S \leq j < \max S \Rightarrow \forall s : F_s(s) = 0.$$ 

In such a case, we can write $F = F_{\leq j} + F_{> j}$ where, obviously, $F_{\leq j}, F_{> j}$ consist of terms $F_s$ with $\max S \leq j$ and $\min S > j$, respectively. The aim of this section is to modify the TDI $H$ locally, $H = H + E(j)$, such that $H + E(j)$ is split at $(j, j + 1)$ while still generating a loop with basepoint $\phi$. We write

$$H = H_{\leq j} + H_{> j} + B_j, \quad \text{where} \quad (B_j)_S = \chi(\min S \leq j < \max S)H_S$$

such that $H - B_j$ is split at $(j, j + 1)$. Note that for $j = 0$, we previously used the notation $H'$ for $H_{\leq 0}$. We recall that the pumped state $\phi \circ \alpha_{H_{\leq 0}}(1)$ is assumed to have zero charge with respect to $\phi$. The following lemma uses and strengthens this statement.

**Lemma 7.1.** Let

$$\psi_- = \phi \circ \alpha_{H_{\leq j}}(1), \quad \psi_+ = \phi \circ \alpha_{H_{> j}}(1).$$

Then the states $\psi_-|_{\leq j}, \psi_+|_{> j}$ are normal and have zero relative charge with respect to $\phi|_{\leq j}, \phi|_{> j}$, respectively.

**Proof.** Since $\psi_- = \psi_-|_{\leq j} \otimes \phi|_{> j}$ and $\psi_+ = \phi|_{\leq j} \otimes \psi_+|_{> j}$, we have

$$h(\psi_-|_{\leq j} \phi|_{> j}) = h(\psi_-|_0), \quad h(\psi_+|_{> j} \phi|_{> j}) = h(\psi_+|_0)$$

(7.1)

from item iii) of Proposition 4. Therefore, in what follows, we abuse notation and write $\psi_-, \psi_+$ also for $\psi_-|_{\leq j}, \psi_+|_{> j}$, as the potential for confusion is small. Then

$$\psi_- \otimes \psi_+ = \phi \circ \alpha_{H - B_j}(1) = \phi \circ \alpha_H(1) \circ \alpha_L(1) = \phi \circ \alpha_L(1)$$

where, by Lemma 5.6, $L$ is a $G$-invariant TDI anchored in $\{j, j + 1\}$. The last equality follows because $H$ generates a loop. By invoking the last item of Lemma 5.6, we conclude that $\psi_- \otimes \psi_+$.
has zero charge wrt. \( \phi \). Therefore, using (7.1) and Proposition 1 we have \( h_{\psi_- / \phi} + h_{\psi_+ / \phi} = 0 \). If \( j = 0 \), then the state \( \psi_- \) is the pumped state \( \psi' \) defined in Section 4.1. For \( j \neq 0 \), TDIs \( H' = H_{\leq 0} \) and \( H_{\leq j} \) differ by an interaction anchored in a finite set, therefore the resulting states have equal charge, invoking again Lemma 5.6. Since we assumed that the loop generated by \( H \) has index zero, the claims follow.

\[ \text{Lemma 7.2. For any site } j \in \mathbb{Z}, \text{ there is a } G \text{-invariant TDI } E^{(j)} \text{ with} \]

\[ |||E^{(j)}|||_{(j,j+1),f} \leq C, \]

such that \( H + E^{(j)} \) is split at the edge \( (j,j+1) \) and it generates a loop with basepoint \( \phi \).

Proof. We fix \( j \) and we use again the notation \( \psi_- \), \( \psi_+ \) as above. By the same analysis as the one in the proof of Lemma 6.3, we get

\[ |||(\phi - \psi_-)|_{<j-r-2}| \leq \hat{f}(r), \quad |||(\phi - \psi_+)|_{>j+r}| \leq \hat{f}(r). \]

We will use Lemma 6.6 to connect these states. Even though Lemma 6.6 is formulated for states on a spin chain algebra, i.e. referring to the graph \( A = \mathbb{A}_\mathbb{Z} \) of course holds just as well for states on \( A_X \) with \( X \subset \mathbb{Z} \). We apply it once with \( X = \{ \leq j \} \) and once with \( X = \{ > j \} \), to obtain TDIs \( K_- \), \( K_+ \), supported on \( \{ \leq j \} \) and \( \{ > j \} \), and such that

\[ \psi_- \otimes \psi_+ = \phi \circ \alpha K(1), \quad K = K_- + K_+, \quad |||K|||_{(j,j+1),f} \leq 1, \]

with \( \hat{f} \) chosen independently of \( j \): Indeed, in both Lemmas 6.3 and 6.6, the function \( \hat{f} \) depends only on \( f \) and \( C_H \), in particular not on \( j \). Hence we have now obtained that the path of states

\[ \phi \circ \alpha H - B_j \circ \alpha^{-1}_K(\cdot) \]

is a loop with basepoint \( \phi \) and it is factorized between \( \{ \leq j \} \) and \( \{ > j \} \). The TDI generating this loop is, by Lemma 5.4,

\[ H + E^{(j)}, \quad E^{(j)}(s) = \alpha K(s)[-K(s) - B_j(s)] + (\alpha K(s)|H(s)| - H(s)). \]

Since \( H - B_j \) and \( K \) are split at \( (j,j+1) \), we deduce that \( H + E^{(j)} \) is split as well. To get the desired bounds on the TDI \( E^{(j)} \), we use Lemma 5.2 and Duhamel’s formula.

\[ \text{□} \]

7.2 Approximate splitting into factors

In the previous subsection, we constructed anchored interactions \( E^{(j)} \) that each split the loop at the edge \( (j,j+1) \). Now we will try to split a loop at all edges \( (Rn, Rn+1) \) at once, with \( n \in \mathbb{Z} \) and \( R \) a large even integer. We will however only succeed to do this approximatively, and we will obtain the representation

\[ \alpha_H = \alpha_F \circ \alpha_W \circ \alpha_Z \]

where \( F, W, Z \) are \( G \)-invariant TDIs depending on the parameter \( R \) such that

i. \( W \) is small in the sense that \( |||W|||_f \leq \hat{f}(R) \).

ii. \( F \) is factorized over the intervals \( I_n \) and \( Z \) is factorized over the intervals \( J_n \) (both \( I_n, J_n \) are defined below and have length \( R \), see Figure 4).
iii. $F$ is close to generating a loop: for each $n$,

$$
\|(\phi - \phi \circ \alpha_F(1))_{I_n}\| \leq \hat{f}(R),
$$

and analogously for $Z$ and the intervals $J_n$.

In the present section 7.2 and the next section 7.3 all TDIs and all states will be $G$-invariant, and we will not repeat this.

7.2.1 Splitting the loop into factorized quasiloops

Let $R \in \mathbb{N}$. We introduce the intervals

$$I_n = [Rn + 1, R(n + 1)], \quad J_n = [R(n - 1/2) - 1, R(n + 1/2)] \quad (n \in \mathbb{Z}).$$

We consider the TDI

$$\tilde{F} = H + \sum_{n \in \mathbb{Z}} E^{(Rn)}$$

where $E^{(Rn)}$ are defined as in Lemma 7.2. The infinite sum is understood in a weak sense, as in Section 5.2. Since $E^{(Rn)}$ is anchored at $\{Rn, Rn + 1\}$, Lemma 5.2 item ii) implies that $\|\|\| \tilde{F} \|\|_f \leq C$. Because of the tails of the anchored interactions $E^{(Rn)}$, $F$ is not exactly split over the intervals $I_n$. Therefore, we explicitly define the split interaction $F$ by

$$F_{S} = \begin{cases} 
\tilde{F}_{S} & S \subset I_n \text{ for some } n \\
0 & \text{otherwise}
\end{cases}$$

Let $\tilde{Z}$ be a TDI such that $\alpha_{\tilde{Z}} = \alpha_{F}^{-1} \circ \alpha_{H}$. We similarly define its split version, but unlike $F$, the terms are now supported on the intervals $J_n$:

$$Z_{S} = \begin{cases} 
\tilde{Z}_{S} & S \subset J_n \text{ for some } n \\
0 & \text{otherwise}
\end{cases}$$

We will now express that the differences $F - \tilde{F}, Z - \tilde{Z}$ are small. We note first that all four of these interactions depend on the parameter $R$ and this is the parameter that controls the smallness, namely

**Lemma 7.3.**

$$\|\|F - \tilde{F}\|\|_f \leq \hat{f}(R)$$

To avoid confusion, we remind that the two functions called $\hat{f}$ in the above expression need not be equal, as explained at the beginning of Section 7.
Proof. The difference $F - \tilde{F}$ contains only those terms that cross at least one of the cuts between two intervals $I_n$. Since $H + E(Rn)$ is split at site $Rn$, the only interaction terms in $\tilde{F} = H + \sum_{n'} E(n')$ that contain $\{Rn, Rn + 1\}$ arise from $E(Rn')$ with $n' \neq n$ and hence they correspond to a set $S$ with $\text{diam}(S) \geq R$ since $E(Rn')$ is anchored at $\{Rn', Rn' + 1\}$. It follows that every term in $F - \tilde{F}$ is of the form $\tilde{F}_S$ with $\text{diam}(S) \geq R$. Therefore
\[
\|F - \tilde{F}\| \leq \sqrt{\hat{f}(R)\|\tilde{F}\|}. 
\]
As we already remarked that $\|\tilde{F}\| \leq C$, the lemma is proven. \hfill \Box

Lemma 7.4.
\[
\|Z - \tilde{Z}\| \leq \hat{f}(R).
\]

Proof. Since $\tilde{Z}$ is such that $\alpha_{\tilde{Z}} = \alpha_{F}^{-1} \circ \alpha_H$, Lemma 5.4 yields
\[
\tilde{Z}(s) = H(s) - F(s) - i\alpha_H^{-1}(s) \left[ \int_0^s \alpha_H(u) \left\{ [H(u) - F(u), \alpha_F(s, u)\{F(s)\}] \right\} du \right]. \tag{7.3}
\]
We write
\[
H - F = D_1 + D_2, \quad D_2 = H - \tilde{F} = \sum_n E(Rn), \quad D_1 = \tilde{F} - F, \tag{7.4}
\]
and we note the following properties
\[
\|D_1\| \leq \hat{f}(R), \quad \|D_2\| \leq C.
\]

The bound on $D_1$ follows from Lemma 7.3. The bound on $D_2$ is by Lemma 5.2 item ii) and Lemma 7.2. We now plug the decomposition (7.4) of $H - F$ into (7.3). Lemma 5.2 items i,iii) then yields that $\tilde{Z}$ has a representation similar to $H - F$, namely
\[
\tilde{Z} = D_1' + D_2', \quad \|D_1'\| \leq \hat{f}(R), \quad \|D_2'\| \leq C. \tag{7.5}
\]

We recall now that $Z$ is the split version of $\tilde{Z}$, over the intervals $J_n$. Since $D_2'$ is anchored at the intersection of intervals $I_n$, $(D_2')_S \neq 0$ implies then $\text{diam}(S) \geq R/2$ if $S$ intersects more than a single interval $J_n$. We can therefore conclude as in the proof of the previous lemma that
\[
\|Z - \tilde{Z}\| \leq \|D_1'\| + \sqrt{\hat{f}(R)\|D_2'\|} \leq \sqrt{\hat{f}(R)\|D_2'\|},
\]
which, together with (7.5), yields the claim. \hfill \Box

7.2.2 $F$ and $Z$ generate quasiloops

We have established that $F$ and $Z$ are manifestly split; $F$ is decoupled over intervals $I_n$ and $Z$ is decoupled over intervals $J_n$. So in particular we have that both $\phi \circ \alpha_F$ and $\phi \circ \alpha_Z$ are factorized over intervals of length $R$. We now investigate how close those states are to $\phi$, namely how close $F, Z$ are to generating genuine loops.

Lemma 7.5. Let $\psi = \phi \circ \alpha_F(1)$. Then
\[
\| (\psi - \phi)_{I_n} \| \leq \hat{f}(R).
\]
Proof. First of all, Lemma 7.3 together with Lemma 5.5 imply that \( \| \alpha_{\mathcal{F}}(1)[A] - \alpha_{\mathcal{F}}(1)[A] \| \leq R \hat{f}(R) \| A \| \) for any \( A \in \mathcal{A}_{J_n} \). We absorb \( R \) into \( \hat{f} \) and we conclude that \( \| (\psi - \phi \circ \alpha_{\mathcal{F}}) \| _{I_n} \leq \hat{f}(R) \).

It remains therefore to compare \( \phi \) with \( \phi \circ \alpha_{\mathcal{F}} \). For this, we partition, see Figure 4

\[
I_n = I_- \cup I_+, \quad I_- = I_n \cap J_n, \quad I_+ = I_n \cap J_{n+1}.
\]

Let us first estimate \( \| (\psi - \phi) \| _{I_+} \). Let \( F_n = H + E^{(Rn)} \). Then \( F_n \) generates a loop with basepoint \( \phi \) by Lemma 7.2 and we have, for \( A \in \mathcal{A}_{I_+} \),

\[
\phi(A) - \phi \circ \alpha_{\mathcal{F}}(A) = \phi(\alpha_{\mathcal{F}}(1)[A] - \alpha_{\mathcal{F}}(1)[A]).
\]

(7.6)

We note now that the TDI \( \tilde{F} - F_n = \sum_{w \neq n} E^{(Rn)} \) is anchored at distance at least \( R/2 \) from \( I_+ \), see Figure 4 and Lemma 5.2. Therefore, by Lemma 5.5, the integrand is bounded by \( \hat{f}(R) \| A \| \).

We next estimate \( \| (\psi - \phi) \| _{I_-} \) by the same argument but this time using \( F_{n+1} = H + E^{(R(n+1))} \), so that, again \( \tilde{F} - F_{n+1} \) is anchored at distance \( R/2 \) from the relevant region \( I_- \). We have thus obtained

\[
\| (\psi - \phi) \| _{I_-} \leq \hat{f}(R).
\]

These estimates on each half \( I_\pm \) of \( I_n \) suffice to obtain the claim by applying Lemma A.1 where the projectors correspond to the (pure) restriction of the product state \( \phi \) to \( I_\pm \), and the density matrices are the reduced density matrices of \( \psi \) to \( I_\pm \). \( \square \)

Lemma 7.6. Let \( \psi = \phi \circ \alpha_{\mathcal{F}}^{-1}(1) \). Then

\[
\| (\psi - \phi) \| _{J_n} \leq \hat{f}(R).
\]

Proof. Recalling that \( \alpha_{\mathcal{F}} = \alpha_{\mathcal{F}}^{-1} \circ \alpha_{\mathcal{H}} \), we have

\[
\psi = \phi \circ \alpha_{\mathcal{F}}^{-1}(1) \circ \alpha_{\mathcal{F}}(1) \circ \alpha_{\mathcal{Z}}(1) \circ \alpha_{\mathcal{Z}}^{-1}(1) = \phi \circ \alpha_{\mathcal{F}}(1) \circ \alpha_{\mathcal{Z}}(1) \circ \alpha_{\mathcal{Z}}^{-1}(1),
\]

where the last equality follows because \( \mathcal{H} \) generates a loop with basepoint \( \phi \). Now we estimate

\[
|\psi(A) - \phi(A)| \leq |\phi \circ \alpha_{\mathcal{F}}(1)[A] - \phi[A]| + |A - \alpha_{\mathcal{Z}}(1) \circ \alpha_{\mathcal{Z}}^{-1}(1)[A]|.
\]

(7.7)

for \( A \in \mathcal{A}_{J_n} \), and we deal with both terms separately. For the first term, we use that \( J_n \) is contained in the union \( I_n \cup I_{n+1} \). Employing Lemma 7.5 for the intervals \( I_n, I_{n+1} \) and Lemma A.1 we can bound this term by \( \| A \| \hat{f}(R) \). For the second term in (7.7), we note that \( \alpha_{\mathcal{Z}} \circ \alpha_{\mathcal{Z}}^{-1} \) is generated by \( W = \alpha_{\mathcal{Z}}[\mathcal{Z} - \mathcal{Z}] \), see Lemma 5.3. The smallness of \( \mathcal{Z} - \mathcal{Z} \), see Lemma 7.4 leads via Lemma 5.2 to \( \| W \| _{\hat{f}} \leq \hat{f}(R) \) and hence to the estimate

\[
\| A - \alpha_{\mathcal{W}}(1)[A] \| \leq \int_0^1 ds || W(s), A || \leq \hat{f}(R) \| J_n \|| A ||
\]

Since \( |J_n| \) grows linearly in \( R \), it can be absorbed in \( \hat{f}(R) \). Therefore, (7.7) is bounded by \( \hat{f}(R) \| A \| \), as required. \( \square \)

Let us check that we achieved the result described at the beginning of Section 7.2. We write

\[
\alpha_{\mathcal{H}} = \alpha_{\mathcal{F}} \circ \alpha_{\mathcal{F}}^{-1} \circ \alpha_{\mathcal{H}} = \alpha_{\mathcal{F}} \circ (\alpha_{\mathcal{Z}} \circ \alpha_{\mathcal{Z}}^{-1}) \circ \alpha_{\mathcal{Z}} = \alpha_{\mathcal{F}} \circ \alpha_{\mathcal{W}} \circ \alpha_{\mathcal{Z}}
\]

(7.8)

and the desired properties of \( \mathcal{F}, \mathcal{Z}, \mathcal{W} \) are manifest from the results above.
7.3 Final stage of the proof of Proposition 4

7.3.1 Closing the product quasi-loops

Up to now, we have defined the TDIs $F, Z$ that are split on intervals of length $R$. They do not generate loops, but they fail to do so by an error which decays fast in $R$. We now show that, by modifying $F, Z$ by a small term that is split as well, the quasi-loops are made into genuine loops.

Lemma 7.7. There are TDIs $E = E^{(R)}$, $L = L^{(R)}$ such that

i. $|||E|||_{f} \to 0$ and $|||L|||_{f} \to 0$, as $R \to \infty$.

ii. $E$ is split over the intervals $I_{n}$ and $L$ is split over the intervals $J_{n}$.

iii. $\phi \circ \alpha_{F} \circ \alpha_{E}$ and $\phi \circ \alpha_{L} \circ \alpha_{Z}$ are loops with basepoint $\phi$.

Proof. Both $\phi \circ \alpha_{F}(1)$ and $\phi$ are factorized exactly over intervals $I_{n}$. We consider each interval separately and we note that we are exactly in the framework of Lemma 6.5. For each $n$, we obtain a time-dependent observable $E_{n}(\cdot)$ in $A_{I_{n}}$ that generates parallel transport from $(\phi \circ \alpha_{F}(1))_{I_{n}}$ to $\phi_{I_{n}}$, with norm bounded by

$$|||E_{n}||| \leq 8 |||\phi \circ \alpha_{F}(1) - \phi|||_{I_{n}} \leq \hat{f}(R),$$

(7.9)

see Lemma 7.5. We then assemble the different $E_{n}(\cdot)$ into a TDI: Let $E_{S}(s) = E_{n}(s)$ for $S = I_{n}$ for some $n$ and $E_{S}(s) = 0$ otherwise. This TDI $E$ is manifestly split and it satisfies

$$\phi \circ \alpha_{F}(1) \circ \alpha_{E}(1) = \phi.$$

Since the non-zero contributions to $E$ are given explicitly by $E_{n}$ that are strictly supported in the finite intervals $I_{n}$, the norm of $E$ is immediately bounded, for any $h \in F$, by

$$|||E|||_{h} \leq \sup_{n} \sup_{s} |||E_{n}(s)|||_{h}(R).$$

(7.10)

We choose $h = \sqrt{\hat{f}}$, and so (7.9) and (7.10) yield $|||E|||_{\sqrt{\hat{f}}} \leq \sqrt{\hat{f}(R)}$. This proves all claims concerning $E$. We proceed similarly with the generator of $\alpha_{Y}^{-1}$ given by Lemma 5.3, but replacing now $I_{n}$ by $J_{n}$ and calling the resulting (split and small) interaction $Y$. This achieves that $\phi = \phi \circ \alpha_{Y}^{-1}(1) \circ \alpha_{Y}(1)$ and hence also

$$\phi = \phi \circ \alpha_{Y}^{-1}(1) \circ \alpha_{Y}(1) = \phi \circ \alpha_{L}(1) \circ \alpha_{Z}(1).$$

where $L$ generates the inverse of $\alpha_{Y}$.

7.3.2 Contracting product loops

In the above sections, we have produced $G$-loops factorized over a collection of intervals $Y_{n}$. In our case $Y_{n} = I_{n}$ or $Y_{n} = J_{n}$. We will now show that such a loop is contractible, i.e. homotopic to a constant loop.

Lemma 7.8. Consider a $G$-loop $\psi$ that is factorized over intervals $Y_{n}$, with $Y_{n}$ forming a partition of $Z$. If $\text{diam}(Y_{n})$ is bounded uniformly in $n$, then $\psi$ is $G$-homotopic to a constant loop with the same basepoint.
It is intuitively clear that this lemma relies on the fact that zero-dimensional systems don’t allow for nontrivial loops because \( \mathcal{B}(\mathcal{H}) \) is simply connected. However, our notion of loops and their homotopy for states on spin chain algebras, does not simply reduce, in the case the chain is finite, to the standard notion of continuous loops and homotopy on topological spaces. The reason is that we require the loops to be generated by a TDI, which, even for loops on \( \mathcal{B}(\mathcal{H}) \), is a stronger notion than continuity. Actually, the requirement of being generated by a zero-dimensional TDI \( E(\cdot) \), as defined before Lemma 6.6, is in that case equivalent to absolute continuity of the loop. For this reason, we actually need an additional lemma on zero-dimensional loops.

**Lemma 7.9.** Let \( s \mapsto \nu(s) \) be a loop of pure normal states on \( \mathcal{B}(\mathcal{H}) \), with \( \mathcal{H} \) a finite-dimensional Hilbert space, and \( E(\cdot) \) a zero-dimensional TDI with \( \|E\| = \sup_s \|E(s)\| < \infty \) such that

\[
\nu(s) = \nu(0) \circ \alpha_{E}(s).
\]

Then, there is a two-dimensional family of pure normal states \( \nu_{\lambda}(s) \) with \( \lambda, s \in [0, 1] \), and zero-dimensional TDIs \( E_{\lambda}(\cdot), F_{s}(\cdot) \) such that

i. \( \sup_{\lambda} \|E_{\lambda}\| \leq 80 \|E\| \) and \( \sup_{s} \|F_{s}\| \leq 208 \)

ii. \( \nu_{1}(s) = \nu_{\lambda}(0) = \nu_{\lambda}(1) = \nu(0) = \nu(1) \) for all \( s, \lambda \).

iii. \( \nu_{\lambda}(s) = \nu(0) \circ \alpha_{E_{\lambda}}(s) \)

iv. \( \nu_{\lambda}(s) = \nu(s) \circ \alpha_{F_{s}}(\lambda) \)

v. If the loop \( \nu(\cdot) \) is \( G \)-invariant, then the family \( \nu_{\lambda}(s) \) and the families \( E_{\lambda}(s), F_{s}(\lambda) \) can also be chosen \( G \)-invariant.

We refer to the appendix for proof of this lemma, relying on standard Hilbert space theory.

**Proof of Lemma 7.8.** Let \( E \) be the TDI generating the factorized loop \( \psi \), with basepoint \( \phi \). Then \( \psi(s)_{Y_{n}} \) is a loop of pure, \( G \)-invariant states on \( \mathcal{A}_{Y_{n}} \). Since \( \mathcal{A}_{Y_{n}} \) is isomorphic to \( \mathcal{B}(\mathcal{H}_{Y_{n}}) \) through the map \( \pi^{-1} \), we can apply Lemma 7.9 for every \( n \) separately. We first obtain then the time-dependent operators \( E_{\lambda, n}(\cdot) \) and \( F_{s,n}(\cdot) \) in \( \mathcal{B}(\mathcal{H}_{Y_{n}}) \). Proceeding as in the proof of Lemma 7.7 we assemble these time-dependent operators in two TDIs \( E_{\lambda}, F_{s} \). Both have finite \( f \)-norm for any \( f \in \mathcal{F} \) since \( \sup_{n} \|Y_{n}\| < \infty \) and \( \sup_{n} \|E_{n,\lambda}\| < \infty \). □

### 7.3.3 Proof of Proposition 4

We start from the identity (7.8) and use Lemma 7.7 by bringing in the almost local evolutions \( E, L \) defined therein and bracketing terms judiciously to get:

\[
\alpha_{H} = (\alpha_{F} \circ \alpha_{E}) \circ (\alpha_{E}^{-1} \circ \alpha_{Z} \circ \alpha_{Z}^{-1} \circ \alpha_{L}^{-1}) \circ (\alpha_{L} \circ \alpha_{Z}).
\]

The first and third bracketed evolutions generate loops with basepoint \( \phi \). Since \( \alpha_{H} \) also generates a loop with basepoint \( \phi \), we conclude that also the second bracketed term generates a loop with basepoint \( \phi \). By Lemma 5.9 the above almost local evolution is \( G \)-homotopic to the concatenation of 3 loops:\footnote{Concatenation of loops is not an associative operation. However, Lemma 5.9 shows that \( (\alpha_{H_{1}} \circ \alpha_{H_{2}}) \circ \alpha_{H_{3}} \) is homotopic to \( \alpha_{H_{1}} \circ (\alpha_{H_{2}} \circ \alpha_{H_{3}}) \), since the composition \( \circ \) is associative. Therefore we allow ourselves here to write simply \( \ldots \circ \ldots \circ \ldots \).}

\[
(\alpha_{F} \circ \alpha_{E}) \circ (\alpha_{Z}^{-1} \circ \alpha_{E} \circ \alpha_{Z}^{-1}) \circ \alpha_{L}^{-1} \circ (\alpha_{L} \circ \alpha_{Z})
\]

\(33\)
The first and third loop are product loops, hence contractible by Lemma 7.8. It follows that the loop \( \phi \circ \alpha_H \) is \( G \)-homotopic to the loop \( \tilde{\phi} \circ \tilde{\alpha}_H \), where

\[
\alpha_H = \alpha_E^{-1} \circ \left( \alpha_Z \circ \alpha_Z^{-1} \right) \circ \alpha_L^{-1}.
\]

Just as all of \( E, \tilde{Z}, Z, L \) constructed above, the TDI \( \tilde{H} \) depends on the parameter \( R \), and we denote it \( \tilde{H}^{(R)} \). Since \( Z - \tilde{Z} \) is small (Lemma 7.4) and \( E, L \) are small in the same sense, we get from Lemma 5.4 and Lemma 5.2 that \( \tilde{H}^{(R)} \) is also small, namely \( ||\tilde{H}^{(R)}||_f \leq \hat{f}(R) \). □

8 Contractibility of short loops with product basepoint

In this section, we prove that short loops, i.e. where the TDI has small norm, with product basepoint, are \( G \)-homotopic to a constant loop. The more precise statement is below in Proposition 5. As before, we fix \( \phi \) to be a product \( G \)-state.

**Proposition 5.** For any \( f \in F \), there is an \( \epsilon_1(f) \), such that, if \( ||H||_f \leq \epsilon_1(f) \) and the \( G \)-invariant TDI \( H \) generates a loop \( \phi \circ \alpha_H \), then this loop is \( G \)-homotopic to the trivial loop with basepoint \( \phi \), and the homotopy can be chosen to have constant basepoint.

8.1 Construction of a loop of ground states

First we rescale the given \( G \)-loop \( \phi \circ \alpha_H \) to

\[
\psi(s) = \begin{cases} 
\phi \circ \alpha_H(2s) & s \leq 1/2, \\
\phi & s > 1/2,
\end{cases}
\]

so that now \( \psi(1/2) = \phi \). The resulting loop \( \psi \) is \( G \)-homotopic to the original one by Lemma 5.9. Next, we construct a specific \( G \)-invariant interaction \( F \), tailored to the product state \( \phi \). \( F \) has only on-site terms \( F_{\{i\}} = I - P_i \), which is the orthogonal projection on the kernel of the state \( \phi \). It follows that for any finite \( S \), \( \sum_{i \in S} F_{\{i\}} \) has a simple eigenvalue 0, corresponding to the state \( \phi \mid_S \), and it has no other spectrum below 1. We now define the \( G \)-invariant interaction

\[
Z(s) = \begin{cases} 
\alpha_H^{-1}(2s)[F] & s \leq 1/2, \\
(2s - 1)F + (2 - 2s)Z(1/2) & s > 1/2.
\end{cases}
\]

We have obtained a loop of interactions, since \( Z(1) = Z(0) = F \). We contract it to a point (namely \( F \)) by setting

\[
Z_\lambda(s) = Z(0) + (1 - \lambda)(Z(s) - Z(0)), \quad \lambda \in [0, 1].
\]

This two-parameter family of interactions has the property that it remains close to \( F \), as we remark now.

**Lemma 8.1.** If \( ||H||_f \leq \epsilon \leq 1 \), then, for \( \hat{f} \) affiliated to \( f \),

i. \( ||Z_\lambda(s) - F||_f \leq \epsilon \)

ii. There are \( G \)-invariant TDIs \( X_\lambda(\cdot), X'_\lambda(\cdot) \) such that

\[
Z_\lambda(s) - Z_\lambda(0) = \int_0^s duX_\lambda(u), \quad Z_\lambda(s) - Z_0(s) = \int_0^\lambda d\lambda'X'_\lambda(\lambda').
\]
iii. The TDIs from item ii) are uniformly bounded

\[ \sup_{s, \lambda} (|||X_\lambda|||_f + |||X'_\lambda|||_f) \leq \epsilon \]

**Proof.** Item i) follows directly from the definition of \( Z \), using Lemma 5.2, and \(|||F|||_f = f(1)\). For items ii) and iii), we pick

\[
X_\lambda(s) = \begin{cases} 
-2i \left[ H(2s), \alpha_H^{-1}(2s)[F] \right] & s \leq 1/2 \\
2F - 2Z(1/2) & s > 1/2 
\end{cases}, \quad X'_\lambda(\lambda) = -(Z(s) - Z(0))
\]

and we obtain the bounds similarly to item i).

To continue, we recall the notion of a ground state in the infinite-volume setting of quantum lattice systems. We use implicitly the discussion of Section 2.2.2.

**Definition 8.2.** A state \( \psi \) on a spin chain algebra \( \mathcal{A} \) is a ground state associated to an interaction \( K \) iff

\[ \psi(A^*[K, A]) \geq 0, \quad \forall A \in \mathcal{A}_{\text{al}} \]

We will need two particular properties of a ground state. The first one is its invariance under the dynamics generated by the constant TDI \( K(s) = K \):

\[ \psi = \psi \circ \alpha_K(s), \quad s \in [0, 1] \quad (8.2) \]

It follows by using that \([K, \cdot]\) is a derivation. Indeed, \([K, A^*A] = A^*[K, A] - (A^*[K, A])^* \) and hence \( \psi([K, A^*A]) = 0 \) for a ground state \( \psi \), which implies (8.2). The other property is the following variational statement, see Theorem 6.2.52 in [37].

**Lemma 8.3.** Let \( X \) be a finite subset of \( \mathbb{Z} \) and let \( K \) be an interaction. Consider the functional \( e_{X,K} : \mathcal{P}(A) \to \mathbb{C} \) (recall that \( \mathcal{P}(A) \) is the set of states) given by

\[ e_{X,K}(\psi') = \sum_{S: S \cap X \neq \emptyset} \psi'(K_S) \]

Then, any ground state \( \psi \) satisfies

\[ e_{X,K}(\psi) = \min_{\psi': \psi'|_{X^c} = \psi|_{X^c}} e_{X,K}(\psi') \]

This property makes explicit the fact that ground states minimize the energy locally. The relevance to our problem is that we have a loop of ground states of explicitly given interactions, namely

**Lemma 8.4.** For all \( s \in [0, 1] \), \( \psi(s) \) defined in (8.1) is a ground state of the interaction \( Z(s) \).

**Proof.** Definition 8.2 makes it explicit that the property of being a ground state depends only on the derivation association to the interaction \( Z(s) \). For \( s \in [0, 1/2] \), we have

\[ \psi(s)(A^*[Z(s), A]) = \phi(B(s)^*[F, B(s)]), \quad B(s) = \alpha_H(2s)(A), \]

and therefore the fact that \( \psi(s) \) is a ground state of \( Z(s) \) follows directly from \( \phi \) being a ground state of \( F \). For \( s > 1/2 \), we have \( Z(s) \equiv (2s - 1)F + (2 - 2s)\alpha_H^{-1}(1)[F] \) and we use that \( \phi \circ \alpha_H(1) = \phi \). Therefore \( \phi \) is a ground state of both \( F \) and \( \alpha_H^{-1}(1)[F] \) and hence also of their convex combinations. □
8.2 Construction of the homotopy via spectral flow

Our main idea to prove Proposition 5 is to consider the 2-parameter family of states given by groundstates of the two-parameter family $Z_\lambda(s)$. We first remark, in Proposition 6, that these groundstates are unique in the perturbative regime (small perturbations of product states). Results of that kind have been established in great generality (see [35] for results that are very close to ours, and also [32, 33, 34, 51, 36]) but the specific claim we need does not seem to appear in the literature. Therefore, we provide a proof in Appendix B.

Proposition 6. For any $h \in F$, there is an $\epsilon_2(h) > 0$ such that, if $||W||_h \leq \epsilon_2(h)$, then the interaction $F + W$ has a unique ground state.

If we choose $\epsilon_1(f)$ in Proposition 5 small enough, then, by Lemma 8.1, all of the interactions $Z_\lambda(s)$ satisfy the condition of the above proposition and hence they have a unique ground state, that we call $\psi_\lambda(s)$. This two-dimensional family $(s, \lambda) \mapsto \psi_\lambda(s)$ is the homotopy that will realise Proposition 5. Indeed, for any $\lambda$, we have $Z_\lambda(0) = Z_\lambda(1) = F$, which has $\phi$ as the unique ground state, and hence $\psi_\lambda(0) = \psi_\lambda(1) = \phi$. For $\lambda = 0$, the family $\psi_\lambda(s)$ reduces to the loop $\psi(\cdot)$, as we had already remarked that this was a loop of ground states of $Z(\cdot)$. For $\lambda = 1$, we have that $Z_1(s) = F$, and hence $\psi_1(\cdot)$ is the constant loop $\phi$. It remains to show that the family $(s, \lambda) \mapsto \psi_\lambda(s)$ is generated by a TDI in $\lambda$ and $s$ directions.

Such TDI will be obtained from the so-called spectral flow [43, 44, 52], combined with results on stability of the spectral gap. In particular, Proposition 7 below, follows from [35], except that the condition on spatial decay of potentials is relaxed slightly. The possibility of doing this is clear when inspecting the proofs of [35]. To avoid a clash of notation, we use $z \in [0, 1]$ (instead of $s, \lambda$) as time-parameter in a TDI.

Proposition 7. For any $h \in F$, there is $0 < \epsilon_3(h) < 1$ such that, for any pair of TDIs $W$ and $Y$ satisfying

$$|||W|||_h \leq \epsilon_3(h), \quad |||Y|||_h < \infty, \quad W(z) - W(0) = \int_0^z dz' Y(z'),$$

the following holds: There exists a TDI $\hat{Y}$ satisfying $|||\hat{Y}|||_{h'} \leq |||Y|||_{h'}$ for $h' \in F$ depending only on $h$, such that, if the pure state $\nu(0)$ is a ground state of $F + W(0)$, then

$$\nu(z) = \nu(0) \circ \alpha_{\hat{Y}}(z), \quad z \in [0, 1]$$

is a ground state of $F + W(z)$. If $Y$ is $G$-invariant, then also $\hat{Y}$ can be chosen $G$-invariant.

Based on this result we can finish in a straightforward way the

Proof of Proposition 5. By choosing $\epsilon$ in Lemma 8.1 small enough, we can apply the spectral flow technique, i.e. Proposition 7, to get generating TDIs for the paths $s \mapsto \psi_\lambda(s)$, and $\lambda \mapsto \psi_\lambda(s)$. They are uniformly bounded in $||\cdot||_{h'}$ by Proposition 7 and hence they provide the desired homotopy.

By combining Proposition 5 with Proposition 4 we conclude

Theorem 3. Let $\phi$ be a pure $G$-invariant product state and let $\psi(\cdot) = \phi \circ \alpha_H(\cdot)$ be a $G$-loop such that $I(\phi, H) = 0$, i.e. its index is zero, then the loop is homotopic to the constant loop $\text{Id}_\phi$, via a homotopy with constant basepoint.
9 Classification of loops with product basepoint

We are now ready to prove the full result for the case of loops with a fixed product basepoint. Throughout this section, \( A \) is a fixed chain algebra equipped a fixed group action and \( \phi \) is again a fixed pure, \( G \)-invariant product state.

9.1 Well-definedness of the index

Up to now, the index \( I(\phi, H) \) was defined as a function of the TDI \( H \) and the basepoint \( \phi \), see Section 6.3. We prove now that for product basepoint, it actually depends only on the loop itself and not on the choice of TDI that generates it.

**Lemma 9.1.** Let \( H_1, H_2 \) be \( G \)-invariant TDIs that generate loops with basepoint \( \phi \). If the loops are equal, namely \( \phi \circ \alpha_{H_1}(s) = \phi \circ \alpha_{H_2}(s) \) for all \( s \in [0, 1] \), then the relative charge of the pumped states \( \phi \circ \alpha_{H_1}(s) \), \( \phi \circ \alpha_{H_2}(s) \) is zero and hence

\[
I(\phi, H_1) = I(\phi, H_2).
\]

**Proof.** Let \( \xi(s) = \phi \circ \alpha_{H_1}(s) \circ \alpha_{H_2}^{-1}(s) \). Then

\[
\xi(s) - \phi = \phi \circ (\alpha_{H_1}(s) \circ \alpha_{H_2}^{-1}(s) - \alpha_{H_1}(s) \circ \alpha_{H_2}^{-1}(s))
\]

By similar reasoning as in the proof of Lemma 6.3 we find then some \( f \in \mathcal{F} \) such that

\[
\sup_{s \in [0,1]} \| (\xi(s) - \phi)_{\mathcal{F}} \| \leq f(r),
\]

(9.1)

In particular, by Lemma 6.2, \( \xi(s) \) are pure states that are normal with respect to \( \phi \); they can be represented as density matrices \( \rho(s) \) on \( \mathcal{B}(\mathcal{H}) \). Using (9.1), Lemma 6.2 and its proof imply that \( \sup_{s \in [0, 1]} \| \rho(s) - \rho_r(s) \otimes \Pi_{\mathcal{F}} \| \to 0 \), as \( r \to \infty \), with \( \rho_r(s) \otimes \Pi_{\mathcal{F}} \) as in the proof of Lemma 6.2. Moreover, \( s \mapsto \xi(s) \) is weakly-*continuous (by the strong continuity of \( s \mapsto \alpha_{H_i}(s) \) with \( i = 1, 2 \)) and therefore the function \( s \mapsto \rho_r(s) \otimes \Pi_{\mathcal{F}} \) is actually norm-continuous, since its range is contained in a finite dimensional vector space. Hence, \( s \mapsto \rho(s) \) is the uniform limit of a sequence of norm-continuous functions \( s \mapsto \rho_r(s) \otimes \Pi_{\mathcal{F}} \), and thereby itself norm-continuous. Therefore, the states \( \rho(s) \) are \( G \)-equivalent to \( \rho_\phi \) (i.e. to \( \Pi \)), and so item i) of Proposition 11 implies that the states \( \rho(s) \) have zero relative charge with respect to \( \rho_\phi \). Item iv) of the proposition and the definition of the index (Section 6.3) for mutually normal states then imply that \( \phi \circ \alpha_{H_1}(s) \) and \( \phi \circ \alpha_{H_2}(s) \) have zero relative charge. The claim of the lemma then follows by item ii) of Proposition 11.

Since the basepoint is fixed in this section, this result means that we can now consistently use the notation \( h(\psi) \) for a loop \( \psi \), instead of the more tedious object \( I(\phi, H) \).

9.2 Additivity of the index

**Lemma 9.2.** Let the TDIs \( H_1 \) and \( H_2 \) generate \( G \)-loops \( \psi_1, \psi_2 \) with common product basepoint \( \phi \). Then

\[
h(\psi_2 \boxdot \psi_1) = h(\psi_2) + h(\psi_1)
\]

**Proof.** The pumped state of the loop \( \psi_2 \boxdot \psi_1 \) is

\[
\phi \circ \alpha_{H_1}(1) \circ \alpha_{H_2}(1)
\]

(9.2)
where, as before, $H'_j$ are the truncated TDI. We let $H'_2(r)$ denoted the TDI truncated between sites $-r$ and $-r+1$ instead of the sites 0 and 1. By Lemma 5.6 we can replace $H'_2$ by $H'_2(r)$, such that the resulting pumped state

$$\psi'(r) = \phi \circ \alpha_{H'_1}(1) \circ \alpha_{H'_2(r)}(1)$$

has the same charge as the state $\psi(9.2)$, for any $r$. Let $C_H$ be a constant such that $|||H_1|||_f, |||H_2|||_f \leq C_H$ for some $f \in \mathcal{F}$. Let now $K_1, K_2(r)$ by the parallel transport TDIs constructed in Lemma 6.6 satisfying

$$\phi \circ \alpha_{H'_1}(1) = \phi \circ \alpha_{K_1}(1), \quad \phi \circ \alpha_{H'_2}(1) = \phi \circ \alpha_{K_2}(1)$$

and the bounds

$$||K_1||_{(0,1), f} \leq 1, \quad \sup_r ||K_2||_{(-r, -r+1), f} \leq 1$$

for $\hat{f}$ affiliated to $f$ and depending on $C_H$. Then we calculate

$$\psi'(r) = \phi \circ \alpha_{K_1}(1) \circ \alpha_{H'_2(r)}(1) = \phi \circ \alpha_{H'_2(r)}(1) \circ \alpha_{K_1+E(r)}(1)$$

$$= \phi \circ \alpha_{K_2(r)}(1) \circ \alpha_{K_1+E(r)}(1)$$

(9.4)

The first and third equality follow from (9.3), and the second equality follows by the commutation formula (5.1) with $E(r) = \alpha_{H'_2(r)}^{-1}(1) - K_1$. Locality estimates based on the fact that $H'_2(r)$ is anchored in a region at distance $r$ from the origin, yield $||E(r)||_{(0,1), f} \to 0$ as $r \to \infty$ and therefore also

$$||\psi'(r) - \phi \circ \alpha_{K_2(r)}(1) \circ \alpha_{K_1}(1)|| \to 0$$

(9.5)

As $K_1, K_2(r)$ are anchored in a finite set, we have $K_1 = \text{Ad}(V_1), K_2(r) = \text{Ad}(V_2(r))$ and by the locality properties we deduce that $|||V_1, \gamma(g)||V_2(r)||| \to 0$. We let the pure density matrix $\rho^{(r)}$ represent the (pure) pumped state $\psi'(r)$ and we write $W_j = \pi(V_j)$. Then (9.5) reads in the GNS representation

$$||\rho^{(r)} - \text{Ad}(W_2^*W_2^{(r)}(r))\Pi||_1 \to 0,$$

(9.6)

with $\Pi$, as before, the pure density matrix corresponding to the state $\phi$, and

$$|||W_1, U(g)W_2(r)U^*(g)||| \to 0$$

(9.7)

by the corresponding property for $V_1, V_2(r)$ above. We now bound

$$d(h(\psi_2 \square \psi_1), h(\psi_2) + h(\psi_1)) \leq C_{\Sigma_1} \sup_{g \in G} |\text{Tr} \left[ \rho^{(r)} U(g) \right] - e^{i(h_1(g) + h_2(g))}|,$$

(9.8)

for any $r \in \mathbb{N}^+$, with $C_{\Sigma_1} = \sup_{\theta \in [0,\pi]} \frac{\theta}{|e^{i\theta} - 1|}$. We use (9.6), (9.7) and the definitions of the separate charges

$$U(g) \text{Ad}(W_j^*)\Pi = e^{ik_j(g)} \text{Ad}(W_j^*)\Pi, \quad j = 1, 2,$$

to argue that the right hand side of (9.8) vanishes as $r \to \infty$, which concludes the proof.

9.3 Homotopy invariance

**Proposition 8.** Let $\psi_0$ and $\psi_1$ be $G$-invariant loops with common basepoint $\phi$. If $\psi_0$ and $\psi_1$ are $G$-homotopic, then $h(\psi_1) = h(\psi_0)$. 38
Proof. We consider a homotopy \( \psi_\lambda(s) \) relating \( \psi_0 \) and \( \psi_1 \), as defined in Section 3.2, with the families of TDIs \( H_\lambda \) and \( F_\lambda \), and we set
\[
C_0 = \sup_{s, \lambda} (|||H_\lambda|||_f + |||F_\lambda|||_f),
\]
with \( C_0 \) finite by the condition (3.2). Moreover, for every \( \lambda \), the TDI \( H_\lambda \) generates a loop. For later reference, let us also define the loop
\[
\lambda \mapsto \nu(\lambda) = \psi_\lambda(0) = \psi_\lambda(1).
\]
where the last equality follows from the fact that \( \psi_\lambda(\cdot) \) is a loop for every \( \lambda \). Moreover, \( \nu \) is a loop with basepoint \( \phi \) because the loops \( \psi_0 \) and \( \psi_1 \) both have basepoint \( \phi \).

We now take the set \( S = S(f, \phi, C_0) \) as defined in Section 6.3 and we take \( \epsilon \) as furnished by Proposition 3. Note that \( \epsilon \) depends on \( S \). We set \( N = 1/[C_0 \epsilon] \) and \( \mathcal{T}_N = \{k/N : k = 0, \ldots, N\} \). For any \( t \in \mathcal{T}_{2N} \setminus \{1\} \), we denote by \( \hat{t} \) its successor, i.e. \( t = t + \frac{1}{2N} \). We consider functions \( x : \mathcal{T}_{2N} \to \mathcal{T}_N^2 \) that satisfy

i. \( x(0) = (0,0) \)

ii. \( x(1) = (1,1) \)

iii. \( x(\hat{t}) - x(t) \) equals either \((1/N,0)\) or \((0,1/N)\).

and we call such functions admissible walks. By drawing a straight line between \( x(t) \) and \( x(\hat{t}) \), we obtain a piecewise linear path in the unit square, starting at the bottom-left corner \((0,0)\) and ending in the top-right corner \((1,1)\), and always moving either upwards or to the right, see Figure 5 and 7 for examples. To every admissible walk \( x \), we associate a TDI \( D_x(\cdot) \) as follows: for \( s \in (t, \hat{t}) \), we set
\[
D_x(s) = \begin{cases} 
2H_{x_1(t)} ((x_2(t) + 2(s-t)) & \text{if } x(\hat{t}) - x(t) = (0, \frac{1}{N}) \\
2F_{x_2(t)} ((x_1(t) + 2(s-t)) & \text{if } x(\hat{t}) - x(t) = (\frac{1}{N},0) 
\end{cases}
\]
This construction is illustrated in Figure 5. Every one of such TDIs \( D_x \) generates a loop with basepoint \( \phi \). This follows from the definition of homotopy given in Section 3.2, in particular from the fact that \( H_\lambda \) and \( F_\lambda \) generate motion on the two-dimensional sheet of states \((s, \lambda) \mapsto \psi_\lambda(s)\).

We say that two distinct admissible walks \( x, y \) are adjacent if \( x(t) = y(t) \) for all, but one value of \( t \in \mathcal{T}_{2N} \), that we call \( t_* \). It follows that \( x, y \) are obtained from each other by flipping the steps between preceding and following the time \( t_* \), i.e. and upwards step becomes right-moving, and vice versa, see Figure 6.

We then note that for \( x, y \) adjacent admissible walks, we have
\[
|||D_x - D_y|||^1_1 \leq C_0/(2N)
\]
By Proposition 3 and the choice of \( N \) made above, we have \( h(D_x) = h(D_y) \). Therefore, two admissible walks give rise to loops with the same index if those walks can be deformed into each other by switching a walk into an adjacent one. Consider then the two admissible walks
\[
x(t) = \begin{cases} 
(0, 2t) & t \leq \frac{1}{2} \\
(2(t - \frac{1}{2}), 1) & t > \frac{1}{2}
\end{cases} \quad y(t) = \begin{cases} 
(2t, 0) & t \leq \frac{1}{2} \\
(1, 2(t - \frac{1}{2})) & t > \frac{1}{2}
\end{cases}
\]
These walks can be transformed into each other, as we illustrate in Figure 7. The walk \( x \) correspond to the loop \( \psi_1 \cap \nu \) and the walk \( y \) corresponds to \( \nu \cap \psi_0 \). These two loops have hence the same index and by using the additivity Lemma 9.2, we conclude that \( h(\nu) + h(\psi_0) = h(\psi_1) + h(\nu) \) which implies \( h(\psi_0) = h(\psi_1) \). \( \square \)
9.4 Homeomorphism property of $h$

We have already established that $h$ maps the concatenation of loops into the sum of charges. We would like to state that $h$ is a homeomorphism, but as it stands, the concatenation of loops admits neither an identity nor an inverse. This can be remedied by using $(G)$-homotopy to define equivalence classes of loops. The following lemma shows that $h$ acts as a homeomorphism from homotopy classes to $H^1(G)$. Let $\text{Id}_\phi$ be the constant loop with basepoint $\phi$, i.e. $\phi(s) = \phi$, and recall the definition of the inverse $\psi^\theta(s) = \psi(1 - s)$ associated to a loop $\psi$.

**Lemma 9.3.** Let $\psi$ be a $G$-invariant loop with basepoint $\phi$. Then

i. $h(\text{Id}_\phi) = 0$

ii. $\psi^\theta \Box \psi$ is $G$-homotopic to $\text{Id}_\phi$.

iii. $h(\psi) = -h(\psi^\theta)$

**Proof.** Item i) follows because a possible TDI generating constant loops is $H = 0$, and for that TDI, the pumped state is equal to the basepoint. To show item ii), we consider the following homotopy:

$$
\zeta_\lambda(s) = \begin{cases} 
\psi((1 - \lambda)2s) & s \leq 1/2 \\
\psi^\theta(\lambda + (1 - \lambda)(2s - 1)) & s > 1/2
\end{cases}
$$

We note that

$$
\psi((1 - \lambda)2s) = \psi^\theta(\lambda + (1 - \lambda)(2s - 1)), \quad \text{for } s = 1/2
$$
by the definition of $\psi^\theta$, resolving the apparent incongruence between $s \leq 1/2$ and $s > 1/2$ in the definition of $\zeta_\lambda(s)$. The homotopy property of $(s, \lambda) \mapsto \zeta_\lambda(\cdot)$ now follows because $\psi, \psi^\theta$ are loops and using Lemma 5.9 item i). We then remark that $\zeta_0(\cdot) = \psi^\theta \Box \psi$ and $\zeta_1(\cdot) = \text{Id}_\phi \Box \text{Id}_\phi = \text{Id}_\phi$ which settles item ii). Finally, by additivity and item i), $h(\psi^\theta) + h(\psi) = h(\psi^\theta \Box \psi) = h(\text{Id}_\phi) = 0$ which implies item iii).

□

9.5 Completeness of classification

**Proposition 9.** If a pair of $G$-invariant loops $\psi_1(\cdot)$ and $\psi_2(\cdot)$ with common basepoint $\phi$ have equal index, then they are $G$-homotopic.

**Proof.** The loop $\psi_2^\theta \Box \psi_1$ has index zero, by Lemma 9.3 and Lemma 9.2. By Theorem 3, this loop is $G$-homotopic to the constant loop $\text{Id}_\phi$. Let $\zeta_\lambda(s)$ be this homotopy, satisfying also $\zeta_\lambda(0) = \zeta_\lambda(1) = \phi$. We now construct two derived homotopies $\nu$ and $\omega$ by

\[
\nu_\lambda(s) = \begin{cases} 
\zeta_\lambda(s) & s \leq 1/2 \\
\zeta_\lambda(2s-1)\lambda(1/2) & s > 1/2
\end{cases}, \quad \omega_\lambda(s) = \begin{cases} 
\zeta_\lambda(1-s) & s \leq 1/2 \\
\zeta_\lambda(2s-1)\lambda(1/2) & s > 1/2
\end{cases}.
\]

The apparent incongruence at $s = 1/2$ is resolved by noting that $\zeta_\lambda(s) = \zeta_\lambda(2s-1)\lambda(1/2)$ for $s = 1/2$. The fact that $s \mapsto \nu_\lambda(s)$ and $\lambda \mapsto \nu_\lambda(s)$ are generated by uniformly bounded TDIs, follows from the fact that $(s, \lambda) \mapsto \zeta_\lambda(s)$ is a homotopy and from Lemma 5.7. Then, by construction, $\zeta_0(1/2) = \phi$ and $\zeta_\lambda(0) = \phi$ so that $\nu_\lambda(\cdot)$ are loops with basepoint $\phi$. This confirms that $(s, \lambda) \mapsto \nu_\lambda(s)$ is indeed a homotopy. In an analogous way, one checks that $(s, \lambda) \mapsto \omega_\lambda(s)$ is a homotopy. In Figure 8, we illustrate how $\nu$ and $\omega$ are derived from $\zeta$. We define also the path $\kappa(z) = \zeta_{z}(1/2)$ and note that this path is in fact a loop. Indeed, $\kappa(1) = \phi$ because the original homotopy $\zeta$ reduces for $\lambda = 1$ to $\text{Id}_\phi$, and $\zeta_0(1/2) = \phi$ as already used above. We are now ready to analyze the homotopies $\nu$ and $\omega$, following the definitions. Since $\zeta_0 = \psi_2^\theta \Box \psi_1$, we have that $\zeta_0(s) = \psi_1(2s)$ for $s \leq 1/2$. Hence,

\[
\nu_0 = \text{Id}_\phi \Box \psi_1.
\]

For $\lambda = 1$, $\zeta_0 = \text{Id}_\phi$ while $\zeta_{2-2s}(1/2) = \kappa^\theta(2s)$ and so

\[
\nu_1 = \kappa^\theta \Box \text{Id}_\phi.
\] (9.12)
Similarly, \( \zeta_0(1-s) = \psi_2(2s) \) and so

\[
\omega_0 = \text{Id}_\phi \square \psi_2
\]

and \( \omega_1 = \nu_1 \). We conclude that \( \text{Id}_\phi \square \psi_1 \) and \( \text{Id}_\phi \square \psi_2 \) are homotopic to the same loop and hence to each other. It follows that \( \psi_1 \) is homotopic to \( \psi_2 \).

\[ \square \]

10 Proof of main theorems

In Section 9, we proved all results of this paper in the special case of loops with product basepoint. In the present section, we prove our results without this restriction by relying on the results in Section 9. Indeed, by the assumption of invertibility, we can reduce all our results to statements about loops with product basepoint. We use the terminology introduced at the beginning of Section 3.4.

10.1 Proof of Proposition 2

Let \( \psi \circ \alpha_H(\cdot) \) be loop. We let \( \tilde{\psi} \) be an inverse to the basepoint \( \psi \), such that

\[
\phi = \psi \otimes \tilde{\psi} \circ \beta^{-1}.
\]

(10.1)

where \( \phi \) is a product state and \( \beta = \alpha_E(1) \) for some TDI \( E \). As in the text preceding Proposition 2, we let \( \psi' = \psi \circ \alpha_{H'}(1) \) with \( H' \) the truncated TDI. Let also

\[
\phi' = \psi' \otimes \tilde{\psi} \circ \beta^{-1}
\]

(10.2)

By the uniqueness of the GNS-representation, normality of \( \psi' \) w.r.t. \( \psi \) is equivalent to normality of \( \psi' \otimes \tilde{\psi} \) w.r.t. \( \psi \otimes \tilde{\psi} \). By (10.1) and (10.2), it is therefore also equivalent to normality of \( \phi' \) w.r.t. \( \phi \). By equation (5.7), we have (here we write \( \alpha_{H'} \) for \( \alpha_{H'} \otimes \text{id} \))

\[
\phi' = \phi \circ \beta \circ \alpha_{H'}(1) \circ \beta^{-1} = \phi \circ \alpha_{H'}(1).
\]

Since \( \phi \circ \alpha_{H'}(\cdot) \) is a loop with product basepoint, the state \( \phi \circ \alpha_{H'}(1) \) is normal with respect to \( \phi \) by Lemma 6.3. Moreover, by Lemma 5.6, and the fact that \( \beta(H') - \beta(H')' \) is a TDI anchored in a finite set, we get that \( \phi \circ \alpha_{H'}(1) \) is normal with respect to \( \phi \circ \alpha_{H'}(1) \). Since normality of pure states is a transitive relation, the claim is proven.

10.2 Loops whose basepoint is \( G \)-equivalent to a product

As a first step in dealing with loops with non-product basepoint, we formulate a lemma for loops whose basepoint is \( G \)-equivalent (as opposed to stably \( G \)-equivalent) to a product state.

Lemma 10.1. Let \( \psi \) be a \( G \)-loop generated by the TDI \( H \), such that, for some \( G \)-invariant TDI \( K \) and a product state \( \phi \),

\[
\phi = \psi(0) \circ \alpha_K^{-1}(1).
\]

Then the loop \( \psi \) is homotopic to the loop

\[
\nu = \phi \circ \left( (\alpha_K \square \alpha_H) \square \alpha_K^{\theta} \right)
\]

(10.3)

which has product basepoint \( \phi \). Moreover, their indices are equal, i.e.

\[
I(\psi(0), H) = h(\nu)
\]

(10.4)
Recalling the definition of Section 5.7, we note that the loop \( \nu \) is such that \( \alpha_K \) is traversed between times \( s = 0 \) and \( s = 1/4 \), \( \alpha_H \) is traversed between \( s = 1/4 \) and \( s = 1/2 \), and \( \alpha_{K'}^\theta \) is traversed between \( s = 1/2 \) and \( s = 1 \). We define the homotopy, illustrated in Figure 9:

\[
\mu_\lambda(s) = \begin{cases} 
\nu(\lambda/4 + (1 - \lambda)s) & s \leq 1/4 \\
\nu(s) & 1/4 < s < 1/2 \\
\nu(1/2 + (s - 1/2)(1 - \lambda)) & 1/2 \leq s
\end{cases}
\]

We note first that there is no incongruence at \( s = 1/4 \) and \( s = 1/2 \) since, for any \( \lambda \),

\[
\mu_\lambda(1/4) = \nu(1/4) = \psi(0) = \psi(1) = \nu(1/2) = \mu_\lambda(1/2)
\]

From Lemma 5.7, we see that the family \( (s, \lambda) \mapsto \mu_\lambda(s) \) is generated by uniformly bounded TDIs in \( s \) and \( \lambda \) direction. To conclude that this family is homotopy, we check that it is a loop for all \( \lambda \), using the definition of time-reversed loops:

\[
\mu_\lambda(0) = \nu(\lambda/4) = \psi(\lambda) = \psi^\theta(1 - \lambda) = \nu(1 - \lambda/2) = \mu_\lambda(1).
\]

This homotopy connects the loop \( \nu \) with the loop \( (\text{Id}_{\psi(0)} \square \psi) \square \text{Id}_{\psi(0)} \) and the latter is clearly homotopic to \( \psi \), which settles the first claim of the lemma. Since we have not yet proven in general that homotopy preserves the index, we need another argument to get (10.4). The pumped state of the loop \( \nu(\cdot) \) is

\[
\phi \circ \alpha_{K'}(1) \circ \alpha_{H'}(1) \circ \alpha_{K''}(1) = \phi \circ \alpha_{K'}(1) \circ \alpha_{H'}(1) \circ \alpha_{K'}^{-1}(1) = \phi \circ \alpha_{\alpha_{K'}^{-1}(1)}[H'](1)
\]

where the first expression follows because \( (K')^\theta = (K')^\theta \) with \( K^\theta \) the TDI generating \( \alpha_{K'}^\theta \), i.e. \( K^\theta(s) = -K(1 - s) \). Then, we observe that

\[
\alpha_{K}(1)[H'] - \alpha_{K'}(1)[H']
\]

is a \( G \)-invariant TDI anchored in \( \{0, 1\} \). This follows from Duhamel’s formula (5.13) and Lemma 5.2 since \( [K - K', \alpha_{K'}(\cdot)[H']] \) is anchored in \( \{0, 1\} \). This implies then by Lemma 5.6 item ii) that the pumped state (10.5) has zero relative charge w.r.t. the state

\[
\phi \circ \alpha_{\beta}[H'](1) = \phi \circ \beta \circ \alpha_{H'}(1) \circ \beta^{-1}
\]

where we abbreviated \( \beta = \alpha_K(1) \). Hence to prove the claim of the lemma, we have to show that the relative charge of

\[
\phi \circ \beta \circ \alpha_{H'}(1) \circ \beta^{-1} \quad \text{w.r.t.} \quad \phi
\]

equals the relative charge of

\[
\phi \circ \beta \circ \alpha_{H'}(1) \quad \text{w.r.t.} \quad \phi \circ \beta
\]

since the latter are the pumped state and basepoint, respectively, of the loop \( \psi \). This equality of relative charges follows then from item v) of Proposition 1, taking \( \delta = \beta \).
10.3 Tensor products

Given two $G$-loops $\psi_1, \psi_2$ on spin chain algebra’s $A_1, A_2$, generated by TDIs $H_1, H_2$, we can consider the product loop $\psi_1 \otimes \psi_2$ on the spin chain algebra $A_1 \otimes A_2$, generated by the TDI

$$H_1 \otimes \text{Id} + \text{Id} \otimes H_2.$$ 

The truncation of this TDI to the left equals the sum of truncated TDIs $H'_1 \otimes \text{Id} + \text{Id} \otimes H'_2$ and it then follows that the pumped state is a tensor product of pumped states $\psi_1(0) \circ \alpha_{H'_1}(1) \otimes \psi_2(0) \circ \alpha_{H'_2}(1)$ and so

$$I(\psi_1(0) \otimes \psi_2(0), H_1 \otimes \text{Id} + H_2 \otimes \text{Id}) = I(\psi_1(0), H_1) + I(\psi_2(0), H_2).$$

Proposition 1 item iii).

We will establish in the next sections that the index depends only on the loop and not on the generating TDI, but we can already see that, if a loop is generated by the TDI $H = 0$, then its index is zero. Hence, any $G$-loop $\psi$ generated by $H$ can be extended to a $G$-loop $\psi \otimes \text{Id}_{\nu}$, with $\nu$ a $G$-state, generated by $H \otimes \text{Id}$, such that the index does not change, i.e.

$$I(\psi(0) \otimes \nu, H \otimes \text{Id}) = I(\psi(0), H).$$ (10.6)

10.4 The associated loop

The main tool that we will use in the remaining proofs is that every $G$-loop can be related to a loop with product basepoint, such that they have the same index. We consider hence a general $G$-loop $\psi$ and we let $\psi(0)$ be a $G$-inverse to $\psi(0)$. By the discussion in Section 10.3, the loop $\psi \otimes \text{Id}_{\psi(0)}$ has the same index as $\psi$. However, the former loop has a basepoint that is $G$-equivalent to a product state and hence we can invoke Lemma 10.1 to construct a loop $\psi^A$, which we call the associated loop, that has a product basepoint, and it is such that

Lemma 10.2. Let $\psi$ be a $G$-loop generated by the TDI $H$. There is a $G$-loop $\psi^A$ with product basepoint such that $h(\psi^A) = I(\psi(0), H)$.

10.5 Proof of Theorems 1 and 2

10.5.1 Well-definedness of the index

If we have two TDIs $H_1, H_2$ that both generate a loop $\psi$, then by definition, that the associated loops can be chosen equal. Hence, by the results in Section 9.1, their index is equal and hence also $I(\psi(0), H_1) = I(\psi(0), H_2)$. Therefore, also for loops with a non-product basepoint, the index $I(\cdot, \cdot)$ depends only on the loop itself. We can henceforth use the notation $h(\cdot)$. In particular, this means that Theorem 2 is proven once we establish all the properties of the index $h(\cdot)$ claimed in Theorem 1, which we do now.

10.5.2 Proof of Theorem 1

We now check the items of Theorem 1 one by one.

Item i) If a loop is constant, we can choose the generating TDI $H = 0$. In that case the pumped state equals the basepoint and the index is manifestly equal to zero.

Item ii) The examples described in Section 4.2 realize every $h \in H^1(G)$.

We prove item iii) by reducing it to the already treated case of a product basepoint using associated loops. Let the $G$-loops $\psi_1, \psi_2$ be stably $G$-homotopic. Then by definition (see Section 3.3.2) these loops are $G$-homotopic to each other upon adjoining constant loops with
product basepoint. From Section [10.3], we know that this extension does not change their respective index. Hence without loss of generality we assume henceforth that $ψ_1, ψ_2$ are $G$-homotopic. Consider an associated loop $ψ_i^A$ with product basepoint $ϕ$, obtained by adjoining an inverse $ψ_i(0)$. Since $ψ_1(0)$ and $ψ_2(0)$ are $G$-equivalent states, so are $ψ_1(0) ⊗ ψ_1(0)$ and $ψ_2(0) ⊗ ψ_1(0)$ and hence also $ψ_2$ has an associated loop $ψ_j^A$ with basepoint $ϕ$. By construction, the two loops $ψ_1^A, ψ_2^A$ are loops of states on the same algebra. Moreover, $ψ_j^A$ are stably homotopic to $ψ_j$ for $j = 1, 2$ and since the latter are $G$-homotopic to each other by assumption, we conclude that $ψ_1^A$ is $G$-homotopic to $ψ_2^A$. Since they share a product basepoint, these loops have equal index, by the results in Section [9]. By Lemma [10.2] it then follows that also $ψ_1, ψ_2$ have equal index.

Item iv) Here again, we can reduce the problem tho that of loops with product basepoints. For a pair $ψ_1, ψ_2$ of loops with common basepoint, we find a pair of associated loops $ψ_1^A, ψ_2^A$ with common basepoint $ϕ$. We note first that

$$(ψ_2 ⊗ ψ_1)^A \text{ is homotopic to } ψ_2^A ⊗ ψ_1^A$$

which follows from Lemma [9.3] item ii. Since these are loops with product basepoint, the homotopy implies that they have the same index. The claim then follows from Lemma [10.2] and the additivity of the index for loops with product basepoint.

Item v) This was already proven in Section [10.3]

Item vi) If the loops $ψ_1, ψ_2$ have stably $G$-equivalent basepoints, then by adjoining a product state, the basepoints are $G$-equivalent. Without loss of generality, we assume hence that the basepoints of $ψ_1, ψ_2$ are $G$-equivalent. Let $ψ_1(0)$ be a $G$-inverse to $ψ_1(0)$, then we consider the loops

$$ψ_j ⊗ Id_{ψ_i(0)} ⊗ Id_{ψ_1(0)}, \quad j = 1, 2$$

(10.7)

and we claim that they are $G$-homotopic. This yields the required stable $G$-homotopy of the original loops, since the adjoined product of constant loops is homotopic to a constant loop with product basepoint. As in item iii), the loops $ψ_j ⊗ Id_{ψ_i(0)}$ are $G$-homotopic to the associated loops $ψ_j^A$ which can be chosen to have the same product basepoint. The assumption $h(ψ_1) = h(ψ_2)$ implies that $h(ψ_1^A) = h(ψ_2^A)$ and hence $ψ_1^A, ψ_2^A$ are $G$-homotopic by Proposition [8]. This in turn implies that the loops in (10.7) are $G$-homotopic indeed.

A Appendix: basic tricks in Hilbert space

In this appendix we collect a few lemma’s dealing with zero-dimensional systems, i.e. we deal with Hilbert spaces without any local structure.

A.1 An estimate for bipartite systems

Lemma A.1. Consider a bipartite system $H = H_a ⊗ H_b$. Let $P = P_a ⊗ P_b$ where $P_a, P_b$ are rank-one projectors acting on Hilbert spaces $H_a, H_b$. Let $ρ$ be a density matrix on $H$ and let $ρ_a, ρ_b$ be the reduced density matrices on $H_a, H_b$. Then

$$||ρ - P||_1 \leq 6\sqrt{||ρ_a - P_a||_1} + 6\sqrt{||ρ_b - P_b||_1}$$

Proof. We abuse notation by writing $P_i$ to denote $P_i ⊗ 1$. First,

$$||ρ - P_iρP_i||_1 \leq ||P_iρP_i||_1 + ||P_iρP_i||_1 + ||P_iρP_i||_1 \leq ||P_i\sqrt{ρ}||_2||\sqrt{ρ}P_i||_2 + ||P_i\sqrt{ρ}||_2||\sqrt{ρ}P_i||_2 + Tr(P_iρ) \leq 3\sqrt{Tr(P_iρ)} \leq 3\sqrt{||ρ_i - P_i||_1}$$

45
The second inequality follows from the Cauchy-Schwarz inequality and the last one is because

$$\text{Tr}(\bar{P}_a \rho) = \text{Tr}((\bar{P}_a - P_1) \rho) \leq \|\rho - P_1\|_1,$$

where we used that $P_1$ is rank one. Then, we bound $\|P_b \rho P_b - P_b P_a \rho P_a P_b\|_1 \leq \|\rho - P_a \rho P_a\|_1$ and so

$$\|\rho - P_b P_a \rho P_a P_b\|_1 \leq \sum_{i=a,b} \|\rho - P_i \rho P_i\|_1 \leq \delta := 3 \sum_{i=a,b} \sqrt{\|\rho - P_i\|_1}.$$

Since $P = P_a P_b$ is rank one, we conclude that

$$|1 - \text{Tr}(\bar{P}\rho)| = |\text{Tr}(\rho - \text{Tr}(\bar{P}\rho)P)| \leq \|\rho - \text{Tr}(\bar{P}\rho)P\|_1 = \|\rho - P\rho P\|_1 \leq \delta.$$

Hence $\|\rho - P\|_1 \leq \|\rho - \text{Tr}(\bar{P}\rho)P\|_1 + \|\text{Tr}(\bar{P}\rho)P\|_1 \leq 2\delta$. \qed

**A.2 Parallel transport**

We now turn to the proof of Lemma \[\ref{lem:parallel_transport}\]. Let the normalized vectors $\Omega, \Psi \in \mathcal{H}$ be representatives of $\omega, \nu$. They can be chosen such that $a := \langle \Psi, \Omega \rangle$ is real, with $0 \leq a \leq 1$. Firstly, we note that $\|\nu - \omega\| = \sqrt{1 - a^2}$. We write then $\Psi = a\Omega + \sqrt{1 - a^2}\Psi_\perp$ where $\Psi_\perp$ is orthogonal to $\Omega$ and has unit norm. Let now

$$\Psi(s) := y(s)\Omega + \sqrt{1 - y(s)^2}\Psi_\perp, \quad y = a + (1 - a)(1 - (1 - s)^2)$$

Then,

- i. $||\Psi(s)|| = 1$,
- ii. $\Psi(0) = \Psi$ and $\Psi(1) = \Omega$,
- iii. $|\partial_s y|, |\partial_s\sqrt{1 - y(s)^2}| \leq 2\sqrt{1 - a}$.

Let $P(s)$ denote the orthogonal rank-one projector onto the span of $\Psi(s)$. We consider the adiabatic generator

$$K(s) = i(P(s)\dot{P}(s) - \dot{P}(s)P(s)),$$

which satisfies

$$\dot{P}(s) = i[K(s), P(s)].$$

We then have

$$||K(s)|| \leq 2||\dot{P}(s)|| \leq 4||\dot{\Psi}(s)|| \leq 8\sqrt{1 - a} \leq 8||\nu - \omega||.$$

This proves items (i,ii). We turn to item (iii). If $h_{\nu/\omega} = 0$, then $U(g)\Omega = z_\nu(g)\Omega$ and $U(g)\Psi = z_\omega(g)\Psi$ with $z_\nu(g) = z_\omega(g) = z(g)$. Therefore $U(g)\Psi_\perp = z(g)\Psi_\perp$ and so $U(g)\Psi(s) = z(g)\Psi(s)$ for all $s$. It follows that $P(s), P(s)$ are invariant and so is $K(s)$ by definition \[\ref{def:adiabatic_generator}\]. \qed

**A.3 Contracting loops**

In this section, we prove Lemma \[\ref{lem:contracting_loops}\] on the contractibility of loops in Hilbert space.

Throughout this work, we often need to appeal to the fundamental theorem of calculus for Banach-space valued functions defined on the interval $[0, 1]$. Let $X$ be an arbitrary Banach space. We first recall that a strongly measurable Banach-space valued function $J : [0, 1] \to X$:
$s \mapsto J(s)$ is Bochner integrable if and only if $\int_0^s du \| J(u) \| < \infty$, see [40]. We consider functions $F : [0, 1] \to X : s \mapsto F(s)$ that satisfy the following property:

$$F(s) - F(0) = \int_0^s du J(u), \quad \sup_{s \in [0,1]} \|J(s)\| < \infty,$$

where $s \mapsto J(s)$ strongly measurable. In particular the function

$$s \mapsto \alpha_H(s)[A],$$

for a TDI $H$ and $A \in \mathcal{A}_{al}$ satisfies this property, with $J = \alpha_H(u)\{[H(u), A]\}$.

For the application we have in mind here, it suffices to restrict our attention to the case of finite-dimensional Banach spaces. In that case (more generally in any Banach space with the Radon-Nikodym property with respect to the Lebesgue measure), a function $s \mapsto F(s)$ satisfies the property above if and only if it is Lipschitz continuous, in which case $J(s)$ is its derivative almost everywhere. We say that $C_F$ is a Lipschitz bound if $\sup_{s \in [0,1]} \|J(s)\| \leq C_F$. We now come to the

**Proof of Lemma 7.9.** As remarked in (5.19), the evolved state $\nu(s) = \nu \circ \alpha_E(s)$ is of the form

$$\nu(s) = \nu(0) \circ \text{Ad}(U^*(s)), \quad U(s) = \mathbb{I} + i \int_0^s du E(u) U(u).$$

We choose a unit vector representative $\Omega$ of $\nu(0)$, and we set

$$\tilde{\Psi}(s) = U(s) \Omega.$$

Then $s \mapsto \tilde{\Psi}(s)$ is Lipschitz, and its almost sure derivative is $iE(s)\tilde{\Psi}(s)$. Of course, changing the phase of $\tilde{\Psi}(s)$ does not affect the state $\nu(s)$ and we use this to introduce a crucial modification of $s \mapsto \tilde{\Psi}(s)$.

**Lemma A.2.** There exists a Lipschitz function $a : [0, 1] \mapsto U(1)$, with Lipschitz bound $4|||E|||$, and such that

$$\Psi(s) = a(s)\tilde{\Psi}(s)$$

satisfies

$$k(s) := \text{Re}(\Psi(s), \Omega) \geq -1/2.$$

**Proof.** Let $\tilde{k}(s) = \text{Re}(\tilde{\Psi}(s), \Omega)$. As $s \mapsto \tilde{\Psi}(s)$ is Lipschitz continuous with bound $|||E|||$, so is $\tilde{k}(s)$. Then the sets

$$\kappa_0 = \{ s \in [0,1], k(s) > 0 \}, \quad \kappa_1 = \{ s \in [0,1], k(s) < -1/2 \}$$

are open and

$$\text{dist}(\kappa_0, \kappa_1) \geq \frac{1}{2|||E|||}.$$

We can therefore construct a continuous, $U(1)$-valued function $a$ with Lipschitz bound $4|||E|||$ such that $a(\kappa_0) = 1, a(\kappa_1) = -1$ and the vector $a(s)\tilde{\Psi}(s)$ satisfies the required properties. \qed

We started from a loop $\nu(\cdot)$ and we now have a function $s \mapsto \Psi(s)$. The latter function is not necessarily a loop of unit vectors in the Hilbert space, but it is a family of representatives of $\nu(s)$, in particular

$$\Psi(0) = \Omega, \quad \Psi(1) \propto \Omega. \quad (A.2)$$
We now set for $\lambda \in [0, 1]$,
\[
\Psi_\lambda(s) := \frac{1}{\sqrt{N(s, \lambda)}} (\lambda \Omega + (1 - \lambda) \Psi(s)), \quad N(s, \lambda) = \lambda^2 + (1 - \lambda)^2 + 2\lambda(1 - \lambda)k(s).
\]

With this definition and from the above construction of $\Psi(s)$ and \[A.2\], it follows that

i. $N \geq 1/4$,

ii. $|||\Psi_\lambda(s)||| = 1$,

iii. $\Psi_\lambda(0) = \Omega$ and $\Psi_\lambda(1) \propto \Omega$,

iv. $\Psi_0(s) = \Psi(s)$ and $\Psi_1(s) = \Omega$.

Since $k(s)$ is Lipschitz, and $N \geq 1/4$, we infer that $\frac{1}{\sqrt{N}}$ is Lipschitz. Therefore, and since $s \mapsto a(s)$ and $\Psi$ are Lipschitz, also $s \mapsto \Psi_\lambda(s)$ is Lipschitz. A short calculation yields $|||\partial_s \Psi_\lambda(s)||| \leq 20 |||E|||$. We denote by $P = P(\lambda, s)$ the rank-one projector on the range of $\Psi_\lambda(s)$. Writing $P = |\Psi(\Psi)$, the Lipschitz continuity of $s \mapsto \Psi_\lambda(s)$ gives that $s \mapsto P(\lambda, s)$ is Lipschitz with $|||\partial_s P(\lambda, s)||| \leq 40 |||E|||$. We can now set
\[
E_\lambda(s) = -i [\partial_s P(\lambda, s), P(\lambda, s)]
\]
to satisfy item iii) of the lemma. Moreover, $|||E_\lambda||| \leq 80 |||E|||$. To construct the family $F_\alpha(\cdot)$, we proceed analogously, but the considerations are simpler because the functions $\lambda \mapsto \Psi_\lambda(s)$ are clearly Lipschitz in $\lambda$. Here we find $|||\partial_\lambda \Psi_\lambda(s)||| \leq 52$ and so $|||F_\alpha||| \leq 208$.

\[B\] \textbf{Proof of Proposition 6}

We will use finite-volume restrictions of interactions $H^{(L)}_\omega = \chi(S \subset [-L, L])$ that inherit bounds since $|||H^{(L)}|||_f \leq |||H|||_f$. We note that $\iota(H^{(L)})$ (recall the definition in Subsection 2.2.2) is a Hermitian element of the finite-dimensional $C^*$-algebra $A_L = A_{[-L, L]}$, which is isomorphic to a full matrix algebra. We write $\text{Tr}_L(\cdot)$ for the trace on $A_L$ and we say that $H^{(L)}$ has a spectral gap $\Delta$ if the minimum of the spectrum of $\iota(H^{(L)})$ is a simple eigenvalue $E_0$ and the next smallest eigenvalue is no smaller than $E_0 + \Delta$. We let $P_L$ denote the one-dimensional spectral projector corresponding to $E_0$ and we write $\bar{P}_L = 1 - P_L$. Further, $\omega_{P_L}$ denotes the state on $A_L$ given by
\[
\omega_{P_L}(A) = \text{Tr}(P_LA), \quad A \in A_L.
\]

\[B.1\] \textbf{Preliminaries}

For a function $v \in L^1(\mathbb{R})$, we define the map $K_v^H : A \to A$
\[
K_v^H[A] = \int_{-\infty}^{\infty} v(t) \alpha_H(t)[A]dt
\]

Note that we used the evolution $\alpha_H(t)$ here for $t \in \mathbb{R}$ instead of $[0, 1]$.

\textbf{Lemma B.1.} Let $H^{(L)}$ have a spectral gap $\Delta$, uniformly in $L$. There is a function $v \in L^1$ such that

i. $\int_{-\infty}^{\infty} v(t)dt = 1$ and $v \geq 0$;

ii. $K_v^H[A_{al}] \subset A_{al}$

48
iii. $K_v^{H(L)}(P_LAP_L) = K_v^{H(L)}(\tilde{P}_LAP_L) = 0$ for any $A \in A_L$.

iv. For any $A \in A_{al}$, \( \lim L K_v^{H(L)}(A) = K_v^H(A) \).

This Lemma is a special case of the, by now, standard constructions introduced in [43, 44]. The following lemma is a collection of facts appearing in the proof of Proposition 7, see again [35].

**Lemma B.2.** Assume the setup of Proposition 7 with $H = F + W$. Then $H^{(L)}$ has a spectral gap $\Delta = \frac{1}{2}$ uniformly in $L$. Moreover, for any $A \in A$ with finite support, the weak limit

$$\nu(A) = \lim L K_v^{H(L)}(A)$$

exists. By density, $\nu$ extends to a state on $A$ and it is a ground state for $H$.

**B.2 Proof of Proposition 6**

For any ground state $\psi$ associated to $H$, (8.2) and item iv Lemma B.1 imply that

$$\psi(A) = \psi(K_v^{H(A)}) = \lim L \psi(K_v^{H(L)}(A))$$

for $A$ with finite support and with $v$ as in Lemma B.1. Next,

$$\psi(A) = \lim L (\psi(K_v^{H(L)}(P_LAP_L)) + \psi(K_v^{H(L)}(\tilde{P}_LAP_L)))$$

(B.1)

$$= \lim L \psi(P_LAP_L) + \psi(K_v^{H(L)}(\tilde{P}_LAP_L))$$

(B.2)

with the first equality follows from item iii) of Lemma B.1 and where the second equality follows because $P_L$ is a one-dimensional spectral projection of $H^{(L)}$. For $L$ such that $[-L, L]$ contains the support of $A$, we have hence

$$\psi(P_LAP_L) = \psi(P_L) \text{Tr}_L(P_LA).$$

Since $\psi(P_L) \in [0, 1]$, there is a subsequence $L_n$ such that the limit $\lim L \psi(P_{L_n})$ exists. We call this limit $a \in [0, 1]$. We then see from Lemma B.2 that

$$\lim L \psi(P_{L_n}AP_{L_n}) = a\nu(A)$$

and from (B.2), it then follows that

$$\psi(A) = a\nu(A) + (1 - a)\mu(A)$$

(B.3)

where, in case $a < 1$,

$$\mu(A) = \lim L \frac{1}{\psi(P_{L_n})} \psi(\tilde{P}_{L_n}K_v^{H(L_n)}(A)\tilde{P}_{L_n})$$

and we used Lemma B.1 item iii) to commute the projectors. The existence of the limit follows because the first term in (B.2) has a limit. We now claim that $\mu$ extends to a state on $A$. Indeed, positivity follows from the nonnegativity of the function $v$. Normalization follows from $K_v^{H(L_n)}(I) = I$ since $\int v = 1$, and the extension is then by density of operators with local support. Therefore, also (B.3) extends to any $A \in A$.

Now, since $\psi$ is assumed to be pure, (B.3) means that either $a = 0$ or $a = 1$, since we can easily check that $\nu$ and $\mu$ are not equal. We will exclude the case $a = 0$, which will end the proof.
We define the boundary operator

\[ B_L = \sum_{S: S \cap [-L,L] \neq \emptyset, S \cap [-L,L]^c \neq \emptyset} W_S \]

satisfying \( \| B_L \| \leq 2f(1)\| W \|_f \). By the variational principle Lemma 8.3

\[ \psi(H^{(L)} + B_L) \leq (\omega_{P_L} \otimes \psi|_{[-L,L]^c})(H^{(L)} + B_L) \]

which implies

\[ \psi(H^{(L)}) \leq \omega_{P_L}(H^{(L)}) + 2\| B_L \| \]

By the gap assumption, we have also

\[ \psi(H^{(L)}) \geq \psi(P_L)E_{0,L} + (1 - \psi(P_L))(E_{0,L} + \Delta) \]

and hence

\[ \psi(P_L)E_{0,L} + (1 - \psi(P_L))(E_{0,L} + \Delta) \leq E_{0,L} + 2\| B_L \| \]

from which we conclude that

\[ (1 - \psi(P_L)) \leq \frac{2\| B_L \|}{\Delta} \leq \frac{4f(1)\| W \|_f}{\Delta}. \]

If \( \| W \|_f \) is small enough, we get that \( a = \lim \inf_L \psi(P_L) > 0 \) and so the alternative \( a = 0 \) is indeed excluded. \( \square \)

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