VOLUME FLUCTUATIONS OF RANDOM ANALYTIC VARIETIES IN THE UNIT BALL

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ABSTRACT. Given a Gaussian analytic function $f_L$ of intensity $L$ in the unit ball of $\mathbb{C}^n$, $n \geq 2$, consider its (random) zero variety $Z(f_L)$. We study the variance of the $(n-1)$-dimensional volume of $Z(f_L)$ inside a pseudo-hyperbolic ball of radius $r$. We first express this variance as an integral of a positive function in the unit disk. Then we study its asymptotic behaviour as $L \to \infty$ and as $r \to 1^-$. Both the results and the proofs generalise to the ball those given by Jeremiah Buckley for the unit disk.

1.Definitions and statements

Let $B_n$ denote the unit ball in $\mathbb{C}^n$ and let $\nu$ denote the Lebesgue measure in $\mathbb{C}^n$ normalised so that $\nu(B_n) = 1$. Explicitly $\nu = \frac{\beta}{\pi} d\nu = \frac{\beta}{\pi} |z|^{2n+1}$, where $\beta = \frac{i}{2\pi} \partial \bar{\partial} |z|^2$ is the fundamental form of the Euclidean metric.

For $L > n$ consider the weighted Bergman space

$$B_L(B_n) = \{ f \in H(B_n) : \| f \|_{n,L} := c_{n,L} \int_{B_n} |f(z)|^2(1-|z|^2)^L d\mu(z) < +\infty \},$$

where

$$d\mu(z) = \frac{d\nu(z)}{(1-|z|^2)^{n+1}},$$

and $c_{n,L} = \frac{\Gamma(L)}{n!\Gamma(L-n)}$ is chosen so that $\|1\|_{n,L} = 1$.

Let

$$e_\alpha(z) = \left( \frac{\Gamma(L+|\alpha|)}{\alpha!\Gamma(L)} \right)^{1/2} z^\alpha$$

denote the normalisation of the monomial $z^\alpha$ in the norm $\| \cdot \|_{n,L}$, so that $\{ e_\alpha \}_\alpha$ is an orthonormal basis of $B_L(B_n)$. As usual, here we denote $z = (z_1, \ldots, z_n)$ and use the multi-index notation $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$ and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

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The hyperbolic Gaussian analytic function (GAF) of intensity $L$ is defined as
\[
f_L(z) = \sum_{\alpha} a_{\alpha} \left( \frac{\Gamma(L + |\alpha|)}{\alpha! \Gamma(L)} \right)^{1/2} z^{\alpha} \quad z \in \mathbb{B}_n,
\]
where $a_{\alpha}$ are i.i.d. complex Gaussians of mean 0 and variance 1 ($a_{\alpha} \sim N_{\mathbb{C}}(0, 1)$).

The sum defining $f_L$ can be analytically continued to $L > 0$, which we assume henceforth.

The characteristics of the hyperbolic GAF are determined by its covariance kernel, which is given by (see [ST04, Section 1], [Sto94, p.17-18])
\[
K_L(z, w) = \mathbb{E}[f_L(z)f_L(w)] = \sum_{\alpha} \frac{\Gamma(L + |\alpha|)}{\alpha! \Gamma(L)} z^{\alpha} \bar{w}^{\alpha} = \sum_{m=0}^{\infty} \frac{\Gamma(L + m)}{m! \Gamma(L)} \sum_{\alpha : |\alpha| = m} \frac{1}{\alpha!} z^{\alpha} \bar{w}^{\alpha}
\]
\[
= \sum_{m=0}^{\infty} \frac{\Gamma(L + m)}{m! \Gamma(L)} (z \cdot \bar{w})^m = \frac{1}{(1 - z \cdot \bar{w})^L}.
\]

A main feature of the hyperbolic GAF is that the distribution of the (random) integration current of its zero variety $Z_{f_L}$,
\[
[Z_{f_L}] = \frac{i}{2\pi} \partial \bar{\partial} \log |f_L|^2,
\]
is invariant under automorphisms of the unit ball (see [ST04 Section 1] or [BMP14]). Given $w \in \mathbb{B}_n$ there exists $\phi_w \in \text{Aut}(\mathbb{B}_n)$ such that $\phi_w(w) = 0$ and $\phi_w(0) = w$, and all automorphisms are essentially of this form: for all $\psi \in \text{Aut}(\mathbb{B}_n)$ there exist $w \in \mathbb{B}_n$ and $U$ in the unitary group such that $\psi = U \phi_w$ (see [Rud08 2.2.5]). Then the pseudo-hyperbolic distance $\varrho$ in $\mathbb{B}_n$ is defined as
\[
\varrho(z, w) = |\phi_w(z)|, \quad z, w \in \mathbb{B}_n,
\]
and the corresponding pseudo-hyperbolic balls as
\[
E(w, r) = \{ z \in \mathbb{B}_n : \varrho(z, w) < r \}, \quad r < 1.
\]

The Edelman-Kostlan formula (see [HKPV09 Section 2.4] and [Sod00 Theorem 1]) gives the so-called first intensity of the GAF:
\[
\mathbb{E}[Z_{f_L}] = \frac{i}{2\pi} \partial \bar{\partial} \log K_L(z, z) = L \omega(z),
\]
where $\omega$ is the invariant form
\[
\omega(z) = \frac{i}{2\pi} \partial \bar{\partial} \log \left( \frac{1}{1 - |z|^2} \right) = \frac{1}{(1 - |z|^2)^2} \sum_{j,k=1}^{n} [(1 - |z|^2)\delta_{j,k} + z_k \bar{z}_j] \frac{i}{2\pi} dz_j \wedge d\bar{z}_k.
\]

In this paper we study the fluctuations of the $(n-1)$-dimensional volume of the random variety $Z_{f_L}$ inside a pseudo-hyperbolic ball $E(z, r)$. By the invariance under automorphisms,
this is equivalent to measuring the \((n-1)\)-th volume of \(Z_{fL}\) inside \(B(0, r)\). This volume is given by the integral

\[
I_{fL}(r) = \int_{B(0, r) \cap Z_{fL}} \omega_{n-1} = \int_{B(0, r)} \frac{i}{2\pi} \partial \bar{\partial} \log |f_L|^2 \wedge \omega_{n-1},
\]

where for \(p = 1, \ldots, n\), \(\omega_p = \omega^p/p!\).

Notice that now \(\omega^n = d\mu\), so the Edelman-Kostlan formula yields trivially

\[
\mathbb{E}[I_{fL}(r)] = L \int_{B(0, r)} \frac{\omega^n}{(n-1)!} = \frac{L}{(n-1)!} \mu(B(0, r)) = \frac{L}{(n-1)!} (1-r^2)^n.
\]

Our main goal is to study the variance

\[
\text{Var} I_{fL}(r) = \mathbb{E}[(I_{fL}(r) - \mathbb{E}(I_{fL}(r)))^2],
\]

and, particularly, to describe its asymptotic behaviour as \(L \to \infty\) and as \(r \to 1^-\).

The computations are much simpler if we consider the Euclidean volume instead of the invariant volume defined above. Let

\[
E_{fL}(r) = \int_{B(0, r) \cap Z_{fL}} \beta_{n-1} = \int_{B(0, r)} \frac{i}{2\pi} \partial \bar{\partial} \log |f_L|^2 \wedge \beta_{n-1},
\]

where, for \(p = 1, \ldots, n\), \(\beta_p = \beta^p/p!\).

The key result is the following reduction of \(\text{Var} E_{fL}(r)\) to an integral of a positive function in the unit disk, together with the relation between \(\text{Var} I_{fL}(r)\) and \(\text{Var} E_{fL}(r)\).

**Theorem 1.** Let \(n \geq 2\). For \(L > 0\) and \(r \in (0, 1)\),

(a) \(\text{Var} E_{fL}(r) = \frac{r^{4n} L^2 (1-r^2)^{2L-2}}{(n-1)! (n-2)!} \int_{\mathbb{D}} \frac{(1-|w|^2)^{n-2}}{|1-r^2 w|^2 L - (1-r^2)^{2L} |1-r^2 w|^2} \frac{|1-w|^2}{\pi} dm(w);\)

(b) \(\text{Var} I_{fL}(r) = \frac{\text{Var} E_{fL}(r)}{(1-r^2)^{2n-2}}.\)

From this we obtain the leading term in the asymptotics of \(\text{Var} I_{fL}(r)\) as \(L \to \infty\).

**Theorem 2.** Let \(n \geq 2\) and fix \(r \in (0, 1)\). Then, as \(L \to \infty\),

\[
\text{Var} I_{fL}(r) = \frac{1}{4\sqrt{\pi}} \frac{\zeta(n+1/2)}{(n-1)!} \frac{r^{2n-1}}{(1-r^2)^n} \frac{1}{L^{n-3/2}} [1+o(1)].
\]

**Remarks.** 1. Even though the proof of Theorem 2 is only valid for \(n \geq 2\), the statement matches with the corresponding result for \(n = 1\) (see [Bucl13, Theorem 2(a)] and the references therein). Notice also that \(\text{Var} E_{fL}(r) = \text{Var} I_{fL}(r)\) when \(n = 1\).

2. As explained in [SZ06, Section 2.2], Theorem 2 can also be obtained with the methods used in the proof of the analogous result in the context of compact manifolds. Let \(p_N\) be a Gaussian holomorphic polynomial in \(\mathbb{C}\mathbb{P}^n\) or, more generally, a section of a power \(L^N\) of a
positive Hermitian line bundle \( L \) over an \( n \)-dimensional Kähler manifold \( M \). Given a domain \( \mathcal{U} \subset M \) with sufficiently regular boundary, define

\[
A_{pN}(\mathcal{U}) = \int_{Z_{pN} \cap \mathcal{U}} \omega^{n-1} (n-1)! ,
\]

where \( \omega \) denotes the Kähler form of \( M \). According to B. Shiffman and S. Zelditch [SZ08, Theorem 1.4.] (with \( k = 1 \))

\[
\text{Var} A_{pN}(\mathcal{U}) = \frac{1}{N^{n-3/2}} \left[ \frac{\pi^{n-5/2}}{8} \zeta(n + 1/2) \sigma(\partial \mathcal{U}) + O(N^{\epsilon-1/2}) \right] ,
\]

where \( \sigma(\partial \mathcal{U}) \) denotes the \((2n-1)\)-volume of the boundary \( \partial \mathcal{U} \). Notice that \( r^{2n-1} \) is the \((2n-1)\)-volume (with respect to the invariant form) of \( \partial B(0, r) \).

The proof of this result is based on a (pluri)bipotential expression of \( \text{Var} I_{f_L}(r) \) (see the beginning os Section 2) and on good estimates of the covariance kernel, something we certainly have for the hyperbolic GAF in the ball.

3. Theorem 2 shows a strong form of “self-averaging” of the volume \( I_{f_L}(r) \), in the sense that the fluctuations are of smaller order than the expected value (see remarks also in [SZ08]). More precisely, as \( L \to \infty \)

\[
\frac{\text{Var} I_{f_L}(r)}{\left( \mathbb{E}[I_{f_L}(r)] \right)^2} = O \left( \frac{1}{L^{n+1/2}} \right) .
\]

Notice also that the rate of self-averaging increases with the dimension.

We also study the behaviour of \( \text{Var} I_{f_L}(r) \) as \( r \to 1^- \).

**Theorem 3.** Let \( n \geq 2 \) and fix \( L > 0 \). Then, as \( r \to 1^- \):

(a) If \( L < n/2 \) then,

\[
\text{Var} I_{f_L}(r) = C(L, n) \frac{1}{(1-r^2)^2(n-L)[1 + o(1)]}
\]

where

\[
C(L, n) = \frac{L^2}{\sqrt{\pi}} \frac{2^{n-1}}{4^L(n-1)!} \frac{\Gamma(n/2 - L)\Gamma(n + 1/2 - L)}{(\Gamma(n-L))^2} .
\]

(b) If \( L = n/2 \) then,

\[
\text{Var} I_{f_L}(r) = C(n/2, n) \frac{1}{(1-r^2)^n} \log \left( \frac{1}{1-r^2} \right) [1 + o(1)]
\]

where

\[
C(n/2, n) = \frac{(n/2)^2}{(n-1)!(\Gamma(n/2))^2} .
\]
(c) If \( L > n/2 \) then,

\[
\text{Var} I_{f_L}(r) = C(L, n) \frac{1}{(1 - r^2)^n}[1 + o(1)]
\]

where

\[
C(L, n) = \frac{L^2}{4\sqrt{\pi(n-1)!}} \sum_{k=1}^{\infty} \frac{\Gamma(Lk - \frac{n}{2})\Gamma(Lk - \frac{n-1}{2})}{(\Gamma(Lk + 1))^2} (Lk + \frac{n(n-1)}{2}).
\]

Remarks. (a) Notice that for \( L \) fixed and \( r \to 1^- \) there is also a strong self-averaging of the volume \( I_{f_L}(r) \), which also increases with the dimension:

\[
\frac{\text{Var} I_{f_L}(r)}{\mathbb{E}[I_{f_L}(r)]^2} = \begin{cases} 
O \left( (1 - r^2)^{2L} \right) & L < n/2 \\
O \left( (1 - r^2)^n \log \left( \frac{1}{1-r^2} \right) \right) & L = n/2 \\
O \left( (1 - r^2)^n \right) & L > n/2.
\end{cases}
\]

(b) As before, the proofs we give are valid only for \( n \geq 2 \) but the values \( C(n, L) \) above match those obtained by J. Buckley in the disk \([\text{Buc13, Theorem 1}]\) (since \( 1 - r^2 = 2(1 - r) + o(1 - r) \)).

(c) As in dimension 1, there is a change of regime at \( L = n/2 \), which we don’t know how to explain.

(d) Using the asymptotics \( \lim_{k \to \infty} \frac{\Gamma(k+a)}{\Gamma(k)k^a} = 1 \) we see that as \( L \to \infty \)

\[
C(L, n) = \frac{L^2}{4\sqrt{\pi\Gamma(n)}} \left( \sum_{k=1}^{\infty} \frac{1}{(Lk)^{n+1/2}} \right) (1 + o(1)) = \frac{1}{4\sqrt{\pi\Gamma(n)}} \frac{\zeta(n+1/2)}{L^{n-3/2}} (1 + o(1))
\]

and in particular, by Theorem 2 the limits in \( r \) and \( L \) can be interchanged:

\[
\lim_{L \to \infty} \lim_{r \to 1^-} L^{n-3/2}(1 - r^2)^n \text{Var} I_{f_L}(r) = \lim_{r \to 1^-} \lim_{L \to \infty} L^{n-3/2}(1 - r^2)^n \text{Var} I_{f_L}(r)
\]

\[
= \frac{\zeta(n+1/2)}{4\sqrt{\pi\Gamma(n)}}.
\]

The scheme of the proofs of Theorems 1, 2, and 3 is the same as in the one dimensional case (see [Buc13]), although some of the computations are considerably more involved.

The paper is structured as follows. Section 2 contains the proof of Theorem 1, which is a long and sometimes tricky computation. In Section 3 we prove Theorem 2, while Section 4 is devoted to the proof of Theorem 3.

### 2. Proof of the Theorem 1

(a) As in dimension \( n = 1 \), or as in the context of compact manifolds, the variance we want to study can be expressed through the bipotential. Denote \( S(0, r) = \{ \zeta \in \mathbb{C}^n : |\zeta| = r \} \) and
Here $\rho_L(z, w) = \log(L((\theta(z, w)) L))$, where $\log(x) = \sum_{m=1}^\infty \frac{x^m}{m^2}$ and

$$\theta(z, w) = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z \cdot \bar{w}|^2}.$$

Notice that $\theta L^{1/2}(z, w)$ coincides with the modulus of the normalised kernel of the GAF.

For the sake of completeness we sketch the proof of this identity given in [SZ06, Proposition 3.5]. Let $U = B(0, r)$. By the Edelman-Kostlan formula

$$E_{f_L}(r) - \mathbb{E}[E_{f_L}(r)] = \int_U \frac{i}{2\pi} \partial \bar{\partial} \log |f_L|^2 \wedge \beta_{n-1} = \int_{\partial U} \frac{i}{2\pi} \partial \bar{\partial} \log |f_L|^2 \wedge \beta_{n-1},$$

where

$$f_L(z) = \frac{f_L(z)}{\sqrt{K_L(z, z)}}$$

is the normalised GAF. Then

$$\text{Var}(E_{f_L}(r)) = \mathbb{E}[(E_{f_L}(r) - \mathbb{E}[E_{f_L}(r)])^2]$$

$$= \mathbb{E} \left[ \int_{\partial U} \frac{i}{2\pi} \partial \bar{\partial} \log |f_L(z)|^2 \wedge \beta_{n-1}(z) \right] \int_{\partial U} \frac{i}{2\pi} \partial \bar{\partial} \log |f_L(w)|^2 \wedge \beta_{n-1}(w)$$

$$= \int_{\partial U} \mathbb{E} \left[ \frac{i}{2\pi} \partial \bar{\partial} \log |f_L(z)|^2 \wedge \frac{i}{2\pi} \partial \bar{\partial} \log |f_L(w)|^2 \wedge \beta_{n-1}(z) \wedge \beta_{n-1}(w) \right]$$

$$= \int_{\partial U} \int_{\partial U} \left( \frac{i}{2\pi} \right)^2 \partial \bar{\partial} \mathbb{E} \left[ \log |f_L(z)|^2 \log |f_L(w)|^2 \wedge \beta_{n-1}(z) \wedge \beta_{n-1}(w) \right].$$

The result follows from the fact that (see for instance [HKPV09, Lemma 3.5.2])

$$\mathbb{E}[\log |f_L(z)|^2 \log |f_L(w)|^2] = \rho_L(z, w) \boxtimes.$$

To compute the integrals above we use that $\beta_n = \bigwedge_{j=1}^n i \frac{i}{2\pi} dz_j \wedge d\bar{z}_j$. For $j = 1, \ldots, n$ define the $(n - 1, n)$-forms

$$\gamma_j(z) = \frac{i}{2\pi} d\bar{z}_j \wedge \bigwedge_{k \neq j} \frac{i}{2\pi} dz_k \wedge d\bar{z}_k = \frac{i}{2\pi} d\bar{z}_j \wedge \beta_{n-1}(z).$$

Denoting $S = S(0, 1)$ and letting $z = r\xi$, $w = r\eta$, where $\xi, \eta \in S$ we have, from the expression above,

$$\text{Var}E_{f_L}(r) = \int_S \int_S \sum_{j,k=1}^n \frac{\partial^2 \rho_L}{\partial z_j \partial w_k}(r\xi, r\eta) \gamma_j(r\xi) \gamma_k(r\eta).$$
Lemma 4. Let \( r \in (0, 1) \) and \( \xi, \eta \in S \). Then

\[
\frac{\partial^2 \rho_L}{\partial z_j \partial \bar{w}_k}(r \xi, r \eta) = \frac{L^2(1 - r^2)^{2L - 2} \rho^2}{|1 - r^2 \xi \cdot \eta|^{2L} - (1 - r^2)^{2L}} \times \\
\times \left[ (1 - r^2) \eta_j (1 - r^2 \xi \cdot \eta) \frac{(1 - r^2) \xi_k - \eta_k (1 - \xi \cdot \eta)}{|1 - r^2 \xi \cdot \eta|^2} \right].
\]

Proof. Since \( \text{Li}'(x) = \frac{1}{x} \log \left( \frac{1}{1-x} \right) \), we have

\[
\frac{\partial \rho_L}{\partial z_j} = \text{Li}'(\theta^L) \frac{\partial \theta^L}{\partial z_j} \frac{\partial \theta}{\partial z_j} = \frac{1}{\theta^L} \log \left( \frac{1}{1 - \theta^L} \right) \frac{\partial \theta}{\partial z_j}
\]

and

\[
\frac{\partial^2 \rho_L}{\partial z_j \partial \bar{w}_k} = \frac{L}{\theta^2} \frac{\partial \theta}{\partial \bar{w}_k} \log \left( \frac{1}{1 - \theta^L} \right) \frac{\partial \theta}{\partial z_j} + \frac{L}{\theta^L} \frac{\partial \theta^L}{\partial \theta} \frac{\partial \theta}{\partial z_j} \frac{\partial \theta}{\partial \bar{w}_k} + \frac{L}{\theta} \log \left( \frac{1}{1 - \theta^L} \right) \frac{\partial^2 \theta}{\partial z_j \partial \bar{w}_k}
\]

Using the definition of \( \theta \) given above,

\[
\frac{\partial \theta}{\partial z_j} = \frac{-z_j (1 - |w|^2)}{|1 - z \cdot w|^2} + \frac{(1 - |z|^2)(1 - |w|^2)}{(1 - z \cdot w)(1 - z \cdot w)^2} w_j = \frac{1 - |w|^2}{|1 - z \cdot w|^2} \left( \frac{1 - |z|^2}{1 - z \cdot w} w_j - z_j \right).
\]

We deduce that

\[
\frac{\partial^2 \theta}{\partial z_j \partial \bar{w}_k} - \frac{1}{\theta} \frac{\partial \theta}{\partial z_j} \frac{\partial \theta}{\partial \bar{w}_k} = 0
\]

and therefore,

\[
\frac{\partial^2 \rho_L}{\partial z_j \partial \bar{w}_k} = \frac{L^2}{\theta^2} \frac{\theta^L}{1 - \theta^L} \frac{\partial \theta}{\partial z_j} \frac{\partial \theta}{\partial \bar{w}_k}
\]

\[
= \frac{L^2}{\theta^2} \frac{\theta^L}{1 - \theta^L} \left( \frac{1 - |z|^2}{1 - z \cdot w} w_j - z_j \right) \left( \frac{1 - |w|^2}{1 - z \cdot w} z_k - w_k \right)
\]

Substituting \( z = r \xi, \ w = r \eta \), and using the identity

\[
\frac{\partial^L(r \xi, r \eta)}{1 - \theta^L(r \xi, r \eta)} = \frac{(1 - r^2)^{2L}}{|1 - r^2 \xi \cdot \eta|^{2L}} \frac{1}{1 - \frac{(1 - r^2)^{2L}}{|1 - r^2 \xi \cdot \eta|^{2L}}} = \frac{(1 - r^2)^{2L}}{|1 - r^2 \xi \cdot \eta|^{2L} - (1 - r^2)^{2L}}
\]

we get the result.

\[\]

Plugging this into (5) we finally have

\[
\text{Var} I_{f,L}(r) = L^2(1 - r^2)^{2(L-1)} r^2 \mathcal{I}(L, r),
\]

(4)
where
\[ I(L, r) = \int_S \int_S \frac{1}{|1 - r^2 \eta \cdot \xi |^{2L} - (1 - r^2)^{2L}} \frac{\Omega(r \xi, r \eta)}{|1 - r^2 \xi \cdot \eta|^2} \]
and \( \Omega(r \xi, r \eta) \) is the \((n - 1, n - 1)\)-form (in \( \xi \) and \( \eta \)) given by
\[
\Omega(r \xi, r \eta) = \sum_{j,k=1}^{n} [(1 - r^2) \eta_j - \xi_j (1 - r^2 \xi \cdot \eta)] [(1 - r^2) \xi_k - \eta_k (1 - r^2 \xi \cdot \eta)] \gamma_j (r \xi) \gamma_k (r \eta).
\]

We operate
\[
[(1 - r^2) \eta_j - \xi_j (1 - r^2 \xi \cdot \eta)] [(1 - r^2) \xi_k - \eta_k (1 - r^2 \xi \cdot \eta)] =
(1 - r^2)^2 \xi_k \eta_j - (1 - r^2) (1 - r^2 \xi \cdot \eta) \xi_j \xi_k - (1 - r^2) (1 - r^2 \xi \cdot \eta) \eta_j \eta_k + |1 - r^2 \xi \cdot \eta|^2 \xi_j \eta_k
\]
and split \( I(L, r) = \sum_{m=1}^{4} I_m(L, r) \), where
\[
I_1(L, r) = \int_S \int_S \frac{(1 - r^2)^2}{|1 - r^2 \eta \cdot \xi |^{2L} - (1 - r^2)^{2L}} \frac{1}{|1 - r^2 \xi \cdot \eta|^2} \left( \sum_{j=1}^{n} \eta_j \gamma_j (r \xi) \right) \left( \sum_{k=1}^{n} \xi_k \gamma_k (r \eta) \right)
\]
\[
I_2(L, r) = -\int_S \int_S \frac{1 - r^2}{|1 - r^2 \eta \cdot \xi |^{2L} - (1 - r^2)^{2L}} \frac{1}{1 - r^2 \xi \cdot \eta} \left( \sum_{j=1}^{n} \xi_j \gamma_j (r \xi) \right) \left( \sum_{k=1}^{n} \xi_k \gamma_k (r \eta) \right)
\]
\[
I_3(L, r) = -\int_S \int_S \frac{1 - r^2}{|1 - r^2 \eta \cdot \xi |^{2L} - (1 - r^2)^{2L}} \frac{1}{1 - r^2 \xi \cdot \eta} \left( \sum_{j=1}^{n} \eta_j \gamma_j (r \xi) \right) \left( \sum_{k=1}^{n} \eta_k \gamma_k (r \eta) \right)
\]
\[
I_4(L, r) = \int_S \int_S \frac{1}{|1 - r^2 \eta \cdot \xi |^{2L} - (1 - r^2)^{2L}} \left( \sum_{j=1}^{n} \xi_j \gamma_j (r \xi) \right) \left( \sum_{k=1}^{n} \eta_k \gamma_k (r \eta) \right).
\]

In order to compute these integrals we first fix \( \eta \) and consider a unitary transformation \( \mathcal{U} \) taking \( e_1 = (1, 0, \ldots, 0) \) to \( \eta \). Denote \( v^1 = \eta \) and \( v^j = \mathcal{U}(e_j), j > 1 \), where \( e_j = (0, \ldots, 1, \ldots, 0) \), so that \( \eta, v^2, \ldots, v^n \) is an orthonormal system.

Consider the change of variables \( \xi = \mathcal{U}(\alpha) = \sum_{j=1}^{n} \alpha_j v^j \). Then
\[
\xi \cdot \bar{\eta} = \mathcal{U}(\alpha) \cdot \overline{\mathcal{U}(e_1)} = \alpha \cdot \bar{e}_1 = \alpha_1.
\]

Also
\[
\xi_k = \sum_{m=1}^{n} \alpha_m v_k^m, \quad \bar{\xi}_k = \sum_{j=1}^{n} \bar{\alpha}_m \bar{v}_k^m,
\]
and therefore
\[
d\xi_k = \sum_{m=1}^{n} v_k^m d\alpha_m, \quad d\bar{\xi}_k = \sum_{m=1}^{n} \bar{v}_k^m d\bar{\alpha}_m.
\]
Since $\beta(\xi) = \frac{i}{2\pi} \partial \bar{\partial} |\xi|^2$ is invariant by unitary transformations

(5) \hspace{1cm} \gamma_j(r\xi) = \frac{i}{2\pi} r d\xi_j \wedge \beta_{n-1}(r\xi) = \frac{i}{2\pi} \left( r \sum_{m=1}^n \bar{v}_j^m \, d\bar{\alpha}_j \right) \wedge \beta_{n-1}(r\alpha) = \sum_{m=1}^n \bar{v}_j^m \gamma_m(r\alpha).

Now parametrise $\alpha = U^{-1}(\xi) \in S$ with coordinates $w = (w_1, \ldots, w_{n-1}) \in \mathbb{B}_{n-1}, \psi \in [0, 2\pi)$ in the following way

\[
\begin{align*}
\alpha_j &= w_j \\
\alpha_n &= \sqrt{1 - |w|^2} e^{i\psi}.
\end{align*}
\]

Let us write the forms $\gamma_j(r\alpha)$ in this parametrisation.

**Lemma 5.** Let $d\beta_{n-1}(w) = \bigwedge_{k=1}^{n-1} \frac{i}{2\pi} dw_k \wedge d\bar{w}_k$. Then

\[
\begin{align*}
\gamma_j(\alpha) &= \frac{i}{2\pi} \bar{w}_j \frac{d\psi}{\sqrt{1 - |w|^2}} \wedge d\beta_{n-1}(w) & j = 1, \ldots, n-1; \\
\gamma_n(\alpha) &= \sqrt{1 - |w|^2} e^{-i\psi} \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w)
\end{align*}
\]

**Proof.** Directly from the definition we have

\[
\begin{align*}
d\alpha_n &= \sum_{l=1}^{n-1} \left( \frac{-\bar{w}_l e^{i\psi}}{2\sqrt{1 - |w|^2}} dw_l + \frac{-w_l e^{i\psi}}{2\sqrt{1 - |w|^2}} d\bar{\alpha}_l \right) + \sqrt{1 - |w|^2} ie^{i\psi} d\psi \\
d\bar{\alpha}_n &= \sum_{l=1}^{n-1} \left( \frac{-\bar{w}_l e^{-i\psi}}{2\sqrt{1 - |w|^2}} dw_l + \frac{-w_l e^{-i\psi}}{2\sqrt{1 - |w|^2}} d\bar{\alpha}_l \right) - \sqrt{1 - |w|^2} ie^{-i\psi} d\psi.
\end{align*}
\]

Assume first that $j < n$. Then $\bigwedge_{k\neq j} \frac{i}{2\pi} d\alpha_k \wedge d\bar{\alpha}_k$ contains the factor $\frac{i}{2\pi} d\alpha_n \wedge d\bar{\alpha}_n$, so

\[
\begin{align*}
\bigwedge_{k\neq j} \frac{i}{2\pi} d\alpha_k \wedge d\bar{\alpha}_k &= \bigwedge_{k\neq j, n} \frac{i}{2\pi} d\alpha_k \wedge d\bar{\alpha}_k \wedge \frac{i}{2\pi} d\alpha_n \wedge d\bar{\alpha}_n \\
&= \bigwedge_{k\neq j} \frac{i}{2\pi} dw_k \wedge d\bar{w}_k \wedge \frac{i}{2\pi} \left( \frac{-\bar{w}_j e^{i\psi}}{2\sqrt{1 - |w|^2}} dw_j + \frac{-w_j e^{i\psi}}{2\sqrt{1 - |w|^2}} d\bar{\alpha}_j + \sqrt{1 - |w|^2} ie^{i\psi} d\psi \right) \wedge \\
&\wedge \left( \frac{-\bar{w}_l e^{-i\psi}}{2\sqrt{1 - |w|^2}} dw_l + \frac{-w_l e^{-i\psi}}{2\sqrt{1 - |w|^2}} d\bar{\alpha}_l - \sqrt{1 - |w|^2} ie^{-i\psi} d\psi \right) \\
&= \frac{1}{2\pi} d\psi \wedge (\bar{w}_j dw_j + w_j d\bar{w}_j) \wedge \bigwedge_{k\neq j} \frac{i}{2\pi} dw_k \wedge d\bar{w}_k.
\end{align*}
\]
Therefore
\[
\gamma_j(\alpha) = \frac{i}{2\pi} d\bar{\alpha}_j \wedge \bigwedge_{k \neq j} \frac{i}{2\pi} d\alpha_k \wedge d\bar{\alpha}_k
\]
\[
= \frac{i}{2\pi} d\bar{w}_j \wedge \frac{1}{2\pi} d\psi \wedge (\bar{w}_j dw_j + w_j d\bar{w}_j) \wedge \bigwedge_{k \neq j} \frac{i}{2\pi} dw_k \wedge d\bar{w}_k
\]
\[
= \bar{w}_j \frac{d\psi}{2\pi} \wedge \bigwedge_{k=1}^{n-1} \frac{i}{2\pi} dw_k \wedge d\bar{w}_k
\]

Assume now that \(j = n\). Then
\[
\gamma_n(\alpha) = \frac{i}{2\pi} d\bar{\alpha}_n \wedge \bigwedge_{k < n} \frac{i}{2\pi} d\alpha_k \wedge d\bar{\alpha}_k = \sqrt{1 - |w|^2} e^{-i\psi} \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w)
\]

We finally use the parametrisation of Lemma 5 to compute the integrals \(I_j(L, r)\). We begin with \(I_4\).

\(I_4(L, r)\). First make the change of variables \(\xi = U(\alpha)\). By invariance by unitary transformations,
\[
\sum_{j=1}^{n} \xi_j \gamma_j(r\xi) = \frac{i}{2\pi} \bar{\partial}[\xi]^2 \wedge \beta_{n-1}(r\xi) = \sum_{j=1}^{n} \alpha_j \gamma_j(r\alpha),
\]

and therefore
\[
I_4(L, r) = \int_S \int_S \frac{1}{|1 - r^2 \alpha_1|^2 - (1 - r^2)^2L} \left( \sum_{j=1}^{n} \alpha_j \gamma_j(r\alpha) \right) \left( \sum_{k=1}^{n} \eta_k \gamma_k(r\eta) \right)
\]
\[
= A_n \int_S \int_S \frac{1}{|1 - r^2 \alpha_1|^2 - (1 - r^2)^2L} \left( \sum_{j=1}^{n} \alpha_j \gamma_j(r\alpha) \right),
\]

where, by Stokes and the identity \(\beta^n = d\nu\),
\[
A_n := \int_S \sum_{k=1}^{n} \eta_k \gamma_k(r\eta) = r^{2n-1} \int_S \frac{i}{2\pi} \bar{\partial}[\eta]^2 \wedge \beta_{n-1}(\eta) = r^{2n-1} \int_{\beta_n} \frac{\beta^n(z)}{(n-1)!} = \frac{r^{2n-1}}{(n-1)!}.
\]

We compute the integral in \(\alpha\) using the parametrisation of Lemma 5. Since \(\gamma_j(r\alpha) = r^{2n-1} \gamma_j(\alpha)\) and
\[
\sum_{j=1}^{n} \alpha_j \gamma_j(\alpha) = \sum_{j=1}^{n-1} w_j \bar{w}_j \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) + \sqrt{1 - |w|^2} e^{i\psi} \sqrt{1 - |w|^2} e^{-i\psi} \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w)
\]
\[
= (|w|^2 + 1 - |w|^2) \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) = \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w),
\]

\(\Box\)
we have, after integrating \( w_2, \ldots, w_{n-1}, \psi \),

\[
\mathcal{I}_4(L, r) = A_n \int_{S_{n-1}} \int_0^{2\pi} \frac{r^{2n-1}}{|1 - r^2 w_1|^{2L} - (1 - r^2)^{2L}} \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w)
\]

\[
= A_n \int_{S_{n-1}} \frac{r^{2n-1}}{|1 - r^2 w_1|^{2L} - (1 - r^2)^{2L}} \frac{1}{(n-2)!} (1 - |w_1|^2)^{n-2} dm(w_1) \pi
\]

\[
= \frac{A_n r^{2n-1}}{(n-2)!} \int_{S_{n-1}} \frac{1}{|1 - r^2 w_1|^{2L} - (1 - r^2)^{2L}} dm(w_1) \pi.
\]

\( \mathcal{I}_1(L, r) \). As in the previous case, first we change \( \xi = \mathcal{U}(\alpha) \). By (5), and since \( \eta = v^1 \) and the system \( \{ \bar{v}^l \}_{l=1}^n \) is orthonormal, the form to integrate is

\[
\Gamma_1 := \sum_{j,k=1}^n \eta_j \xi_k \gamma_j(r\xi) \gamma_k(r\eta) = \sum_{j,k=1}^n \eta_j \left( \sum_{l=1}^n \alpha_l v^l_k \right) \left( \sum_{m=1}^n \bar{v}_j^m \gamma_m(r\alpha) \right) \gamma_k(r\eta)
\]

\[
= \sum_{k,l,m=1}^n \left( \sum_{j=1}^n v^l_j \bar{v}_j^m \right) \alpha_l v^l_k \gamma_m(r\alpha) \gamma_k(r\eta) = \sum_{k=1}^n \left( \sum_{l=1}^n \alpha_l v^l_k \right) \gamma_1(r\alpha) \gamma_k(r\eta).
\]

Now we use the parametrisation given in Lemma[5] Then

\[
\sum_{l=1}^n \alpha_l v^l_k = w_1 \eta_k + \sum_{l=2}^{n-1} w_l v^l_k + \sqrt{1 - |w|^2} e^{i\psi} v^n_k,
\]

and therefore

\[
\left( \sum_{l=1}^n \alpha_l v^l_k \right) \gamma_1(r\alpha) = r^{2n-1} \left( w_1 \eta_k + \sum_{l=2}^{n-1} w_l v^l_k + \sqrt{1 - |w|^2} e^{i\psi} v^n_k \right) \bar{w}_1 \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w)
\]

\[
= r^{2n-1} \left( |w|^2 \eta_k + \sum_{l=2}^{n-1} \bar{w}_l w_l v^l_k + \bar{w}_1 \sqrt{1 - |w|^2} e^{i\psi} v^n_k \right) \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w).
\]

This form will be integrated against a function which does not depend on \( w_2, \ldots, w_{n-1}, \psi \).

The last term in the sum will vanish when integrating in \( \psi \), while the second term will vanish when integrating in \( w_2, \ldots, w_n \). Thus, in terms of the integration we want to perform,

\[
\Gamma_1 = r^{2n-1} \sum_{k=1}^n |w|^2 \eta_k \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) \gamma_k(\eta) + \text{vanishing terms}
\]

\[
= r^{2n-1} \left( \sum_{k=1}^n \eta_k \gamma_k(\eta) \right) |w|^2 \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) + \text{vanishing terms}.
\]
Letting \( A_n \) be the constant \((5)\), and integrating \( w_2, \ldots, w_{n-1}, \psi \), we obtain

\[
\mathcal{I}_1(L, r) = A_n \int_{S_{n-1}} \int_0^{2\pi} \frac{r^{2n-1}}{1 - r^2 w_1^2 L - (1 - r^2)^2 L} \frac{(1 - r^2)^2}{|w_1|^2} \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w)
\]

\[
= \frac{r^{2n-1} A_n}{(n-2)!} \int_{\mathbb{D}} \frac{(1 - |w_1|^2)^{n-2}}{1 - r^2 w_1^2 L - (1 - r^2)^2 L} \frac{(1 - r^2)^2}{|w_1|^2} \frac{dm(w_1)}{\pi}
\]

\( \mathcal{I}_2(L, r) \). Once more, changing \( \xi = \mathcal{U}(\alpha) \) and parametrising as in Lemma \(5\),

\[
\Gamma_2 := \sum_{j,k=1}^n \xi_j \xi_k \gamma_j(r\xi) \gamma_k(r\eta) = \sum_{j,k=1}^n \left( \sum_{l=1}^n (\alpha_l v_j^l) \right) \left( \sum_{m=1}^n (\alpha_m v_k^m) \right) \gamma_j(r\alpha) \gamma_k(r\eta)
\]

\[
= \sum_{k,l,m,t=1}^n \alpha_l \alpha_m v_k^m v_l^l \gamma_t(r\alpha) \gamma_k(r\eta) = \left( \sum_{l=1}^n (\alpha_l \gamma_l(r\alpha)) \right) \left( \sum_{k,m=1}^n (\alpha_m v_k^m \gamma_k(r\eta)) \right)
\]

Hence

\[
\sum_{k,m=1}^n \alpha_m v_k^m \gamma_k(r\eta) = \sum_{m=1}^{n-1} w_m \left( \sum_{k=1}^n v_k^m \gamma_k(r\eta) \right) + \sqrt{1 - |w|^2 e^{i\psi}} \left( \sum_{k=1}^n v_k \gamma_k(r\eta) \right)
\]

\[
= w_1 \sum_{k=1}^n \eta_k \gamma_k(r\eta) + \sum_{m=2}^{n-1} w_m \left( \sum_{k=1}^n v_k^m \gamma_k(r\eta) \right) + \sqrt{1 - |w|^2 e^{i\psi}} \left( \sum_{k=1}^n v_k \gamma_k(r\eta) \right)
\]

and therefore, using \((7)\),

\[
\Gamma_2 = r^{2n-1} \left[ w_1 \sum_{k=1}^n \eta_k \gamma_k(r\eta) + \sum_{m=2}^{n-1} w_m \left( \sum_{k=1}^n v_k^m \gamma_k(r\eta) \right) + \sqrt{1 - |w|^2 e^{i\psi}} \left( \sum_{k=1}^n v_k \gamma_k(r\eta) \right) \right] \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w)
\]

As before, the third term in the bracket will vanish when integrating in \( \psi \), while the second term will vanish when integrating in \( w_2, \ldots, w_{n-1} \). Thus, finally,

\[
\mathcal{I}_2(L, r) = -A_n \int_{S_{n-1}} \int_0^{2\pi} \frac{r^{2n-1}}{1 - r^2 w_1^2 L - (1 - r^2)^2 L} \frac{1 - r^2}{|w_1|} \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w)
\]

\[
= -\frac{r^{2n-1} A_n}{(n-2)!} \int_{\mathbb{D}} \frac{(1 - |w|^2)^{n-2}}{1 - r^2 w_1^2 L - (1 - r^2)^2 L} \frac{1 - r^2}{|w_1|} \frac{dm(w_1)}{\pi}
\]
Here the form to integrate is, in the terms of the parametrisation,

\[ I_3(L, r) = \sum_{j, k=1}^{n} \eta_j \eta_k \gamma_j(r \xi) \gamma_k(r \eta) = \sum_{k=1}^{n} \eta_k \sum_{j=1}^{n} v_j^1 \left( \sum_{m=1}^{n} \bar{v}_j^m \gamma_m(r \alpha) \right) \gamma_k(r \eta) \]

\[ = \sum_{k=1}^{n} \eta_k \sum_{m=1}^{n} \left( \sum_{j=1}^{n} v_j^1 \bar{v}_j^m \right) \gamma_m(r \alpha) \gamma_k(r \eta) = r^{2n-1} \left( \sum_{k=1}^{n} \eta_k \gamma_k(r \eta) \right) \bar{w}_1 \frac{d \psi}{2 \pi} \wedge d \beta_{n-1}(w). \]

Hence

\[ I_3(L, r) = -A_n \int_{\beta_{n-1}}^{2\pi} \int_0^{2\pi} \frac{r^{2n-1}}{1 - r^2 w_1^{2L} - (1 - r^2)2L} \frac{1 - r^2}{1 - r^2 \bar{w}_1} \frac{d \psi}{2 \pi} \wedge d \beta_{n-1}(w) \]

\[ = -\frac{r^{2n-1}A_n}{(n-2)!} \int_{D} \frac{(1 - |w|^2)^{n-2}}{1 - r^2 w_1^{2L} - (1 - r^2)2L} \frac{1 - r^2}{1 - r^2 \bar{w}_1} dm(w_1). \]

Finally, adding up the expressions of \( I_j(L, r) \) and using that

\[ 1 + \frac{(1 - r^2)^2}{1 - r^2 w_1^2} - \frac{1 - r^2}{1 - r^2 w_1} - \frac{1 - r^2}{1 - r^2 \bar{w}_1} = \left| 1 - \frac{(1 - r^2)w_1}{1 - r^2 w_1} \right|^2 = \frac{|1 - w_1|^2}{1 - r^2 w_1^2}, \]

we obtain

\[ I(L, r) = \sum_{m=1}^{4} I_m(L, r) = \frac{r^{2n-1}A_n}{(n-2)!} \int_{D} \frac{(1 - |w|^2)^{n-2}}{1 - r^2 w_1^{2L} - (1 - r^2)2L} \frac{|1 - w_1|^2}{1 - r^2 w_1^2} \ dm(w_1). \]

Plugging this into (4) and writing down the value \( A_n \) (see (6)) we get Theorem 1(a).

(b) We just need to repeat the proof of (a) replacing \( \beta \) by \( \omega \). The corresponding bipotential formula for the variance is now

\[ \text{Var } E_{fL}(r) = \int_{S(0, r)} \int_{S(0, r)} \sum_{j,k=1}^{n} \frac{\partial^2 \rho_t}{\partial \bar{z}_j \partial \bar{w}_k}(z, w) \frac{i}{2 \pi} d \bar{z}_j \wedge \omega_{n-1}(z) \wedge \frac{i}{2 \pi} d \bar{w}_k \wedge \omega_{n-1}(w). \]

Replacing \( \gamma_j(z) \) by

\[ \Gamma_j(z) = \frac{i}{2 \pi} d \bar{z}_j \wedge \omega_{n-1}(z) \]

in the calculations above we get the “invariant” version of (4):

\[ \text{Var } I_{fL}(r) = L^2 (1 - r^2)^{2(L-1)} r^2 \sum_{m=1}^{4} I_m^L(L, r), \]

where the integrals \( I_m^L(L, r) \) are obtained from \( I_m(L, r) \) replacing the \( \gamma_j \) by \( \Gamma_j \).

As before, change now \( \alpha = U^{-1}(\xi) \in S \) and parametrise \( \alpha \) with the same coordinates coordinates \( w = (w_1, \ldots, w_{n-1}) \in \mathbb{B}_{n-1}, \psi \in [0, 2\pi) \). We will be done as soon as we prove the following lemma.
Lemma 6. For $r < 1$ and $j = 1, \ldots, n$

$$\Gamma_j(r\alpha) = \frac{\gamma_j(r\alpha)}{(1 - r^2)^{n-1}}.$$

Proof. From the definition we have

$$\omega_{n-1}(z) = \frac{1}{(1 - |z|^2)^n} \left\{ \sum_{k=1}^{n} (1 - |z_k|^2) \int_{j \neq k} \frac{i}{2\pi} dz_j \land d\bar{z}_j + \sum_{j,k=1 \atop j \neq k} \bar{z}_j z_k \frac{i}{2\pi} dz_j \land d\bar{z}_k \right\}.$$

Therefore

$$\Gamma_m(z) = \frac{i}{2\pi} d\bar{z}_m \land \omega_{n-1}(z) = \frac{1}{(1 - |z|^2)^n} \left\{ (1 - |z_m|^2) \gamma_m(z) - \sum_{j \neq m} \bar{z}_m z_j \gamma_j(z) \right\},$$

and in particular

$$\Gamma_m(r\alpha) = \frac{1}{(1 - r^2)^n} \left\{ (1 - r^2 |\alpha_m|^2) \gamma_m(r\alpha) - r^2 \sum_{j \neq m} \bar{\alpha}_m \alpha_j \gamma_j(r\alpha) \right\}.$$

For $m < n$ we have then

$$\Gamma_m(r\alpha) = \frac{1}{(1 - r^2)^n} \left\{ (1 - r^2 |w_m|^2) \gamma_m(r\alpha) - r^2 \sum_{j \neq m} \bar{w}_m w_j \gamma_j(r\alpha) - r^2 \sqrt{1 - |w|^2} e^{i\psi} \bar{w}_m \gamma_n(r\alpha) \right\}$$

$$= \frac{r^{2n-1}}{(1 - r^2)^n} \left\{ (1 - r^2 |w_m|^2) \bar{w}_m \frac{d\psi}{2\pi} \land d\beta_{n-1}(w) - r^2 \sum_{j \neq m} \bar{w}_m |w_j|^2 \frac{d\psi}{2\pi} \land d\beta_{n-1}(w) - r^2 (1 - |w|^2) \bar{w}_m \frac{d\psi}{2\pi} \land d\beta_{n-1}(w) \right\}$$

$$= \frac{r^{2n-1}}{(1 - r^2)^n} \left\{ (1 - r^2 |w_m|^2 - r^2 \sum_{j \neq m} |w_j|^2 - r^2 (1 - |w|^2) \bar{w}_m \frac{d\psi}{2\pi} \land d\beta_{n-1}(w) \right\}$$

By Lemma 5 we have thus $\Gamma_m(r\alpha) = (1 - r^2)^{1-n} \Gamma_m(r\alpha), m < n.$
If \( m = n \) we have similarly

\[
\Gamma_m(r\alpha) = \frac{r^{2n-1}}{(1 - r^2)^n} \left[ (1 - r^2(1 - |w|^2))\sqrt{1 - |w|^2} e^{-i\psi} \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) - \right.
\]

\[
\left. -r^2 \sum_{j=1}^{n-1} w_j \sqrt{1 - |w|^2} e^{-i\psi} \bar{w}_j \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w) \right]
\]

\[
= \frac{r^{2n-1}}{(1 - r^2)^{n-1}} \sqrt{1 - |w|^2} e^{-i\psi} \frac{d\psi}{2\pi} \wedge d\beta_{n-1}(w).
\]

Again by Lemma 5 we have \( \Gamma_n(r\alpha) = (1 - r^2)^{1-n}\Gamma_n(r\alpha) \). ■

### 3. Proof of the Theorem 2

We start with the following consequence of Theorem 1(a).

**Lemma 7.** Let \( n \geq 2 \). For \( L > 0 \) and \( r \in (0, 1) \),

\[
\text{Var } E_{fL}(r) = \frac{2L^2(1 - r^2)^{n-2}}{\pi(n-1)!(n-2)!} J(L, r),
\]

where

\[
J(L, r) = \int \frac{s^{2L-n}}{1 + s^{2L}} \int_0^{\alpha(s,r)} (2 \cos \theta - 2 \cos \alpha(s,r))^{n-2} \left( s + \frac{1}{s} - 2 \cos \theta \right) d\theta ds
\]

and \( \alpha(s,r) = \arccos \left[ \frac{1}{2} \left( (1 + r^2)s + \frac{1 - r^2}{s} \right) \right] \).

**Proof.** Starting from the expression given in Theorem 1(a), write

\[
\text{Var } E_{fL}(r) = \frac{r^{4n}L^2(1 - r^2)^{-4}}{\pi(n-1)!(n-2)!} J(L, r),
\]

where

\[
J(L, r) = \int_{\mathbb{D}} \frac{(1 - |w|^2)^{n-2}}{1 - \left( \frac{1 - r^2}{|1 - r^2 w|} \right)^2} \left( \frac{1 - r^2}{|1 - r^2 w|} \right)^{2L+2} |1 - w|^2 dm(w)
\]

We write this integral in (a sort of) polar coordinates \( s, \theta \): let

\[
w = \frac{1}{r^2} + \frac{1 - r^2}{r^2 s} e^{i(\pi - \theta)},
\]

so that

\[
\frac{1 - r^2}{|1 - r^2 w|} = s.
\]
Let $p(s, r)$ denote the intersection point of the unit circle and the circle $|w - 1/r^2| = 1/r_s^2$. In these coordinates, $w \in \mathbb{D}$ if and only if $s \in (1/r^2, 1)$ and $\theta \in (-\alpha(s, r), \alpha(s, r))$, where $\alpha(s, r)$ is the angle between the complex numbers $-1/r^2$ and $p(s, r) - 1/r^2$.

In these coordinates

\begin{equation}
1 - |w|^2 = 1 - \left| \frac{1}{r^2} + \frac{1 - r^2}{r^2 s} e^{i(\pi - \theta)} \right|^2 = \frac{1 - r^2}{r^4} \left[ 2 \cos \theta - 2 \cos \alpha(s, r) \right],
\end{equation}

since

\begin{equation}
2 \cos \alpha(s, r) = (1 + r^2) s + \frac{1 - r^2}{s}.
\end{equation}

Also

\[ |1 - w|^2 = \left| 1 - \frac{1}{r^2} + \frac{1 - r^2}{r^2 s} e^{i(\pi - \theta)} \right|^2 = \frac{(1 - r^2)^2}{r^4 s} \left( s + \frac{1}{s} - 2 \cos \theta \right). \]

Since $dm(w) = \frac{(1 - r^2)^2}{r^4 s^3} d\theta ds$, and letting $\mathbb{D}_+ = \{ z \in \mathbb{D} : \text{Im} \ z > 0 \}$,

\[ J(L, r) = 2 \int_{\mathbb{D}_+} \frac{(1 - |w|^2)^{n-2}}{1 - \left| \frac{1 - r^2}{|1 - r^2 w|} \right|^{2L}} \left( \frac{1 - r^2}{|1 - r^2 w|} \right)^{2L+2} (1 - w)^2 dm(w) = \frac{2(1 - r^2)^{n+2}}{r^{4n}} K(L, r), \]

where

\[ K(L, r) = \int_{1/r^2}^{1} \int_{0}^{\alpha(s, r)} \frac{s^{2L-n}}{1 - s^{2L}} \left( 2 \cos \theta - 2 \cos \alpha(s, r) \right)^{n-2} \left( s + \frac{1}{s} - 2 \cos \theta \right) d\theta ds. \]

Going back to (8) we obtain Lemma 7.

Our starting point in the proof of Theorem 2 is the expression of $\text{Var} E_{f_L}(r)$ given by Lemma 7.

Fix $r < 1$ and, in order to simplify the notation, let $\alpha(s) = \alpha(s, r)$.

1. In the first place we observe that for the asymptotics of $\text{Var} E_{f_L}(r)$ as $L \to \infty$, only the part of the integral $K(L, r)$ corresponding to $s$ close to 1 is relevant. Fix $\epsilon > 0$ and let us see that the part of the integral up to $s = 1 - \epsilon$ decays exponentially in $L$; we will see later that the part corresponding to $s \in (1 - \epsilon, 1)$ decays polynomially. From (9) we have

\[ 2 \cos \theta - 2 \cos \alpha(s, r) \leq \frac{r^4 s}{1 - r^2}. \]
and therefore, for $L$ big enough,
\[
\int_{\frac{1-v^2}{1+v^2}}^{1-\epsilon} \frac{s^{2L-n}}{1-s^{2L}} \int_0^{\alpha(s)} (2 \cos \theta - 2 \cos (s, r))^{n-2} \left( s + \frac{1}{s} - 2 \cos \theta \right) d\theta \, ds \\
\leq \frac{(1-\epsilon)^{2L-n}}{1-(1-\epsilon)^{2L}} \int_{\frac{1-v^2}{1+v^2}}^{1-\epsilon} \frac{r^4 s^{n-2}}{1-r^2} \left[ \left( s + \frac{1}{s} \right) \alpha(s) - 2 \sin \alpha(s) \right] d\theta \, ds \\
\leq \frac{(1-\epsilon)^{2L-2} r^{4n-2}}{(1-r^2)^{n-2}} \int_{\frac{1-v^2}{1+v^2}}^1 \left[ \left( s + \frac{1}{s} - 2 \right) \alpha(s) + 2 \alpha(s) \right] ds \\
\leq \frac{(1-\epsilon)^{2L-2} r^{4n-2}}{(1-r^2)^{n-2}} \int_{\frac{1-v^2}{1+v^2}}^1 \left[ \left( s + \frac{1}{s} - 2 \right) \frac{\pi}{2} + \pi \right] ds = C_{n,r}(1-\epsilon)^{2L-2}.
\]

2. For $s$ near 1 we can assume that $\alpha(s)$ is small. More precisely, given $\delta > 0$, there exists $\epsilon = \epsilon(\delta) > 0$ such that $\lim_{\delta \to 0} \epsilon(\delta) = 0$ and
\[
s \in (1-\epsilon, 1) \implies \alpha(s) \leq \arccos(1-\delta) = O(\sqrt{\delta}).
\]
To see this notice that, by (10), $\alpha(s) \leq \arccos(1-\delta)$ precisely when $(1+r^2)s + \frac{1-s^2}{1+s^2} \geq 2(1-\delta)$, which is equivalent to $s^2 - \frac{2(1-\delta)}{1+r^2} s + \frac{1-s^2}{1+s^2} \geq 0$, and to
\[
s \notin \left( \frac{1}{1+r^2}(1-\delta - \sqrt{(1-\delta)^2 - (1-r^4)}), \frac{1}{1+r^2}(1-\delta + \sqrt{(1-\delta)^2 - (1-r^4)}) \right).
\]
Hence it is enough to take
\[
\epsilon = 1 - \frac{1}{(1+r^2)}(1-\delta + \sqrt{(1-\delta)^2 - (1-r^4)}).
\]

3. For $\alpha(s)$ close to 0,
\[
\alpha^2(s) = (1+r^2) \frac{1-s}{s} \left( s - \frac{1-r^2}{1+r^2} \right) + \cdots
\]
(here and in the remaining of this section, + · · · will indicate terms of lower order as $s \to 1^-$.)

To see this notice that, from (10),
\[
2 - 2 \cos \alpha(s) = -\frac{(1-s)^2}{s} + \frac{r^2}{s} - sr^2 = (1+r^2) \frac{1-s}{s} \left( s - \frac{1-r^2}{1+r^2} \right),
\]
which together with the approximation $2 - 2 \cos \alpha(s) = \alpha^2(s) + o(\alpha^2(s))$ gives the statement.

4. We will see next that the whole integral, with $s$ from $\frac{1-s^2}{1+s^2}$ up to 1, has polynomial decay. As seen in the previous point, for $s$ close to 1 we have $\alpha^2(s) = 2r^2(1-s) + \cdots$. On the other
hand, by \((10)\)

\[
2 \cos \theta - 2 \cos \alpha(s) = -2(1 - \cos \theta) + 2 - s - \frac{1}{s} + r^2 \left( \frac{1}{s} - s \right)
\]

\[
= 2r^2(1 - s) - 2(1 - \cos \theta) - (1 - r^2) \sum_{j=2}^{\infty} (1 - s)^j,
\]

and

\[
s + \frac{1}{s} - \cos \theta = s + \frac{1}{s} - 2 + 2(1 - \cos \theta) = 2(1 - \cos \theta) + \sum_{j=2}^{\infty} (1 - s)^j.
\]

Together with the approximation \(2(1 - \cos \theta) = \theta^2 + \cdots\), and writing only the leading term around \(s = 1\), this yields

\[
A(s) := \int_0^{\alpha(s)} (2 \cos \theta - 2 \cos \alpha(s))^{n-2} \left( s + \frac{1}{s} - 2 \cos \theta \right) d\theta
\]

\[
= \int_0^{\sqrt{2r^2(1-t)}} (2r^2(1 - s) - \theta^2)^{n-2} \theta^2 d\theta + \cdots
\]

\[
= \sum_{j=0}^{n-2} \binom{n-2}{j} (2r^2(1 - s))^{n-2-j} (-1)^j \int_0^{\sqrt{2r^2(1-t)}} \theta^{2j+2} d\theta + \cdots
\]

\[
= (2r^2(1 - s))^{n-1/2} \sum_{j=0}^{n-2} \binom{n-2}{j} \frac{(-1)^j}{2j + 3} + \cdots
\]

A straightforward computation shows that for \(m \in \mathbb{N}\) and \(z \in \mathbb{C} \setminus \{0, 1, \ldots, m\}\),

\[
\sum_{j=0}^{m} \binom{m}{j} \frac{(-1)^j}{z + j} = \frac{m!}{z(z+1) \cdots (z+m)} = \frac{m! \Gamma(z)}{\Gamma(z+m+1)}.
\]

(12)

Applying this to \(z = 3/2, m = n - 2\), and using that \(\Gamma(1/2) = \sqrt{\pi}\) we have

\[
A(s) = \frac{(n-2)! \sqrt{\pi}}{4\Gamma(n+1/2)} (2r^2(1 - s))^{n-1/2}.
\]
Since for $L$ big the integral for $s \in (0, \frac{1-r^2}{1+r^2})$ tends to 0, this implies that

$$K(L, r) = \frac{(n-2)!\sqrt{\pi}}{4\Gamma(n+1/2)} \int_{\frac{1-r^2}{1+r^2}}^1 \frac{s^{2L-n}}{1-s^{2L}} (2r^2(1-s))^{n-1/2} ds + \ldots$$

$$= \sqrt{\frac{\pi}{2}} \frac{(n-2)!2^{n-2}}{\Gamma(n+1/2)} r^{2n-1} \sum_{k=0}^\infty \int_0^1 s^{2L+2LK-n} (1-s)^{n-1/2} ds + \ldots$$

$$= \sqrt{\frac{\pi}{8}} \frac{(n-2)!2^{n-2}}{\Gamma(n+1/2)} r^{2n-1} \sum_{k=0}^\infty \frac{\Gamma(2L+2kL-n)\Gamma(n+1/2)}{\Gamma(2L+2kL+3/2)} + \ldots$$

Using the asymptotics $\lim_{k \to \infty} \frac{\Gamma(k+a)}{\Gamma(k)k^a} = 1$ we have then

$$K(L, r) = \sqrt{\frac{\pi}{8}} (n-2)!\zeta(n+1/2) r^{2n-1} \frac{1}{L^{n+1/2}} + \ldots$$

Finally, by Lemma 7 we get

$$\text{Var } E_{fl}(r) = \frac{2L^2(1-r^2)^{n-2}}{\pi(n-1)!(n-2)!} \frac{\sqrt{\pi}}{8} (n-2)!\zeta(n+1/2) r^{2n-1} \frac{1}{L^{n+1/2}} + \ldots$$

$$= \frac{1}{4\sqrt{\pi}} \frac{\zeta(n+1/2)}{(n-1)!} r^{2n-1}(1-r^2)^{n-2} L^{3/2-n} + \ldots$$

This and Theorem 1(b) finish the proof.

4. PROOF OF THE THEOREM 3

Let us see first that the order of growth as $r \to 1^-$ is as stated. Later on we will see how the constants $C(L, n)$ can be determined. The notation $\simeq$ indicates that there exists a constant $C$ independent of $r$ such that $C^{-1}A \leq B \leq CA$.

According to Lemma 7 it is enough to study the asymptotics of $K(L, r)$ as $r \to 1^-$. Since $2 \cos x - 2 \cos a \simeq (\sin a)(a-x)$ for $0 \leq x \leq a \leq \pi/2$, we see that in the range of integration of $\theta$ in $K(L, r)$

$$2 \cos \theta - 2 \cos \alpha(s, r) \simeq (\sin \alpha(s, r))(\alpha(s, r) - \theta) \simeq \alpha(s, r)(\alpha(s, r) - \theta).$$

Therefore

$$K(L, r) \simeq \int_{1-r^2}^1 \frac{s^{2L-n}}{1-s^{2L}} (\alpha(s, r))^{n-2} \int_0^{\alpha(s, r)} (\alpha(s, r) - \theta)^{n-2} \left[ \frac{(1-s)^2}{s} + 2(1-\cos \theta) \right] d\theta ds$$
Denote temporarily $\alpha = \alpha(s, r)$. Using now that $1 - \cos \theta \simeq \theta^2$ for $\theta \in [0, \pi/2]$ we can estimate the integral in $\theta$:

\[
\int_0^\alpha (\alpha - \theta)^{n-2} \left[ \frac{(1-s)^2}{s} + 2(1-\cos \theta) \right] d\theta \simeq \frac{(1-s)^2}{s} \int_0^\alpha (\alpha - \theta)^{n-2} d\theta + \int_0^\alpha (\alpha - \theta)^{n-2} \theta^2 d\theta \\
\simeq \frac{(1-s)^2}{s} \int_0^\alpha (\alpha - \theta)^{n-2} d\theta + \int_0^\alpha (\alpha - \theta)^{n-2} d\theta + 2\alpha \int_0^\alpha (\alpha - \theta)^{n-1} d\theta + \alpha^2 \int_0^\alpha (\alpha - \theta)^{n-2} d\theta \\
\simeq \frac{(1-s)^2}{s} \alpha^{n-1} + \alpha^{n+1} = \alpha^{n-1} \left[ \frac{(1-s)^2}{s} + \alpha^2 \right].
\]

Hence

\[
K(L, r) \simeq \int_{1-r^2}^1 \frac{s^{2L-n}}{1-s^{2L}} (\alpha(s, r))^{2n-3} \left[ \frac{(1-s)^2}{s} + \alpha^2(s, r) \right] ds.
\]

Using (II) we have also,

\[
\alpha(s, r)^2 \simeq 2 - 2 \cos \alpha(s, r) = (1 + r^2) \frac{1-s}{s} \left( s - \frac{1-r^2}{1+r^2} \right),
\]

and therefore

\[
K(L, r) \simeq \int_{1-r^2}^1 \frac{s^{2L-n}}{1-s^{2L}} \left( \frac{1-s}{s} \left( s - \frac{1-r^2}{1+r^2} \right) \right)^{n-3/2} \left[ \frac{(1-s)^2}{s} + \frac{1-s}{s} \left( s - \frac{1-r^2}{1+r^2} \right) \right] ds \\
\simeq \int_{1-r^2}^1 \frac{s^{2L-2n+1/2}}{1-s^{2L}} (1-s)^{n-1/2} \left( s - \frac{1-r^2}{1+r^2} \right)^{n-3/2} \left[ (1-s) + (s - \frac{1-r^2}{1+r^2}) \right] ds \\
\simeq \int_{1-r^2}^1 \frac{s^{2L-2n+1/2}}{1-s^{2L}} (1-s)^{n-1/2} \left( s - \frac{1-r^2}{1+r^2} \right)^{n-3/2} ds.
\]

Let us see now that for the asymptotics as $r \to 1^-$ it is enough to take care of $s$ small. Notice, for instance, that the portion of the integral where $s \in (1/2, 1)$ tends to the constant

\[
\int_{1/2}^1 \frac{s^{2L-2n+1/2}}{1-s^{2L}} (1-s)^{n-1/2} s^{-3/2} ds = \int_{1/2}^1 \frac{s^{2L-n-1}}{1-s^{2L}} (1-s)^{n-1/2} ds.
\]

On the other hand, for $s \leq 1/2$ we have $1-s \simeq 1 - s^{2L} \simeq 1$ and therefore,

\[
\int_{1-r^2}^{1/2} \frac{s^{2L-2n+1/2}}{1-s^{2L}} (1-s)^{-1/2} \left( s - \frac{1-r^2}{1+r^2} \right) s^{-3/2} ds \simeq \int_{1-r^2}^{1/2} \frac{s^{2L-2n+1/2}}{1-s^{2L}} \left( s - \frac{1-r^2}{1+r^2} \right) s^{-3/2} ds \\
\simeq \begin{cases} (1-r^2)^{2L-n} + \cdots & \text{if } 2L - n < 0 \\ \log \frac{1}{1-r^2} + \cdots & \text{if } 2L - n = 0 \\ 1 & \text{if } 2L - n > 0. \end{cases}
\]
This gives the order of growth of $\Var E_{f_L}(r)$ as stated in Theorem \ref{thm:order_of_growth}. Once we know this we can determine the values $C(L, n), L > 0$.

Case $L \leq n/2$. Here $K(L, r)$ tends to $\infty$ as $r \to 1^-$, at speed $(1 - r^2)^{2L-n}$, so it is enough to consider the terms giving this maximal order of growth. Since

$$s + \frac{1}{s} - 2 \cos \theta = \left(\frac{1 - s}{s}\right)^2 + 2(1 - \cos \theta) = \frac{1}{s} + \cdots$$

and

$$\frac{1}{1 - s^{2L}} = 1 + \cdots$$

(where the dots indicate lower order terms) we have

$$K(L, r) = \int_{\frac{1}{1 + r^2}}^{1} \int_{0}^{\alpha(s, r)} s^{2L-n-1} (2 \cos \theta - 2 \cos \alpha(s, r))^{n-2} d\theta \, ds + \cdots$$

$$= \int_{\frac{1}{1 + r^2}}^{1} \int_{0}^{\alpha(s, r)} s^{2L-n-1} \sum_{j=0}^{n-2} \left(\frac{n - 2}{j}\right)(-1)^j (2 \cos \alpha(s, r))^j (2 \cos \theta)^{n-2-j} d\theta \, ds + \cdots$$

In order to apply Fubini’s theorem –and to simplify the notation– denote $\epsilon(r) = \frac{1 - r^2}{1 + r^2}$. The domain of the double integral above is thus given by the conditions

$$\begin{cases}
\epsilon(r) \leq s \leq 1 \\
0 \leq \theta \leq \alpha(s, r).
\end{cases}$$

To determine the global range of $\theta$, notice that the function

$$h(s) := 2 \cos \alpha(s, r) = (1 + r^2)s + \frac{1 - r^2}{s}$$

has a minimum at $s = \sqrt{\epsilon(r)}$ and that $h(\sqrt{\epsilon(r)}) = 2\sqrt{1 - r^4}$. Therefore

$$0 \leq \theta \leq \alpha_0(r) := \arccos(\sqrt{1 - r^2}).$$

Once $\theta$ is fixed, the inequalities above determine the range of $s$:

$$s \in A(\theta) := \{s : \epsilon(r) \leq s \leq 1 , \ 2 \cos \theta \geq (1 + r^2)s + \frac{1 - r^2}{s}\}.$$

Therefore

$$K(L, r) = \sum_{j=0}^{n-2} \left(\frac{n - 2}{j}\right)(-1)^j \int_{0}^{\alpha_0(r)} (2 \cos \theta)^{n-2-j} \left[\int_{A(\theta)} (2 \cos \alpha(s, r))^j ds\right] d\theta + \cdots$$

Here we wish to determine the leading term of

$$J := \int_{A(\theta)} (2 \cos \alpha(s, r))^j ds = \int_{A(\theta)} ((1 + r^2)s + \frac{1 - r^2}{s})^j ds.$$
After the change of variable \( s = \sqrt{e(r)} x \) the restrictions on \( s \) imposed by \( A(\theta) \) are equivalent to the conditions
\[
\sqrt{e(r)} \leq x \leq 1/\sqrt{e(r)}, \quad (x + 1/x)\sqrt{1 - r^4} \leq 2 \cos \theta,
\]
hence
\[
J = (e(r))^{L-\frac{r}{4}} (1 - r^4)^{j/2} \int_{\sqrt{e(r)} \leq x \leq 1/\sqrt{e(r)}} x^{2L-n-1}(x + 1/x)^j dx + \cdots
\]
\[
= \frac{(1 - r^2)^{L-\frac{r}{4} + j}}{2^{L - \frac{r}{2} - j}} \int_{\sqrt{e(r)} \leq x \leq 1/\sqrt{e(r)}} x^{2L-n-1}(x + 1/x)^j dx + \cdots
\]
We split the integral above into two parts, depending on whether or not \( x \leq 1 \). For both parts we perform the same change of variables \( y = x + 1/x \) and extract the terms of maximal order.

(i) Assume \( x \leq 1 \). Here \( y = 1/x + \cdots \) and therefore \( x^{2L-n-1} = y^{n+1-2L} + \cdots \). Similarly
\[
dy = (1 - \frac{1}{x^2})dx = -\frac{dx}{x^2} + \cdots = -y^2 dx + \cdots
\]
As for the limits of integration, notice that if \( x = \sqrt{e(r)} \) then \( y = \sqrt{e(r)} + 1/\sqrt{e(r)} = \frac{2}{\sqrt{1-r^4}} \).
This yields
\[
A_1 := \int_{\sqrt{e(r)} \leq x \leq 1/\sqrt{e(r)}} x^{2L-n-1}(x + 1/x)^j dx = \int_2^{2 \cos \theta} y^{n+1-2L} y^j dy + \cdots
\]
\[
= \frac{1}{n - 2L + j} \left( \frac{2 \cos \theta}{\sqrt{1-r^4}} \right)^{n-2L+j} + \cdots = \frac{2^{\frac{r}{4}-L+rac{j}{2}}(\cos \theta)^{n-2L+j}}{(n-2L+j)(1-r^2)^{\frac{r}{2}-L+rac{j}{2}}} + \cdots
\]
(ii) Let us see now that the integral corresponding to \( x \geq 1 \) is of smaller order. Here \( y = x + \cdots \) and \( dy = dx + \cdots \), thus
\[
A_2 := \int_{1 \leq x \leq 1/\sqrt{e(r)}} x^{2L-n-1}(x + 1/x)^j dx = \int_2^{2 \cos \theta} y^{2L-n-1+j} dy + \cdots
\]
If \( 2L-n+j < 0 \) these integrals tend to a constant. If \( 2L-n+j = 0 \) then this is \( O(\log(\frac{1}{1-r^2})) \), which is obviously of smaller order than \( (1-r^2)^{L-\frac{r}{2}-\frac{j}{2}} = (1-r^2)^{-j} \). Finally, if \( 2L-n+j > 0 \) then
\[
A_2 = \frac{1}{2L-n+j} \left( \frac{2 \cos \theta}{\sqrt{1-r^4}} \right)^{2L-n+j} = o\left( \frac{1}{(1-r^2)^{\frac{r}{2}-L+rac{j}{2}}} \right).
\]
Therefore the leading term of the integral in (14) is $A_1$, and by (15) we get
\[
J = \frac{(1 - r^2)^{L - \frac{n}{2} + \frac{j}{2}}}{2^{L - \frac{n}{2} + \frac{j}{2}} (n - 2L + j)(1 - r^2)^{\frac{1}{2} - L - \frac{j}{2}}} + \ldots
\]
\[
= \frac{2^{n-2L+j}}{n-2L+j} (\cos \theta)^{n-2L+j} (1 - r^2)^{2L-n} + \ldots
\]
Plugging this into (13) we obtain
\[
K(L, r) = 2^{n-2L-2} \sum_{j=0}^{n-2} \binom{n-2}{j} \frac{(-1)^j}{n-2L+j} \left[ \int_0^{\pi/2} (\cos \theta)^{2n-2L-2} d\theta \right] (1 - r^2)^{2L-n} + \ldots
\]
The sum in $j$ is computed using (12) with $z = n - 2L$ and $m = n - 2$. The integrals in $\theta$ are taken care of by the following identity, which is a simple computation: for $m \in \mathbb{N}$
\[
\int_0^{\pi/2} (\cos \theta)^m d\theta = \frac{\sqrt{\pi} \Gamma\left(\frac{m+1}{2}\right)}{2 \Gamma\left(\frac{m}{2} + 1\right)}.
\]
We have thus
\[
K(L, r) = 2^{n-2L-2} \frac{(n-2)\Gamma(n-2L) \sqrt{\pi} \Gamma(n - L - 1/2)}{\Gamma(2n-2L-1) \Gamma(n-L)} (1 - r^2)^{2L-n} + \ldots
\]
and therefore
\[
\text{Var} \ E_{f_L}(r) = \frac{L^2}{\sqrt{\pi} (n-1)!} \Gamma(n-2L) \Gamma(n-L) (1 - r^2)^{2L-n} + \ldots
\]
This expression can be simplified by means of the duplication formula for the $\Gamma$-function:
\[
\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma(z+1).
\]
Taking $z = n - L - 1/2$ we see that
\[
\frac{2^{n-2L-2} \Gamma(n - L - 1/2)}{\sqrt{\pi} \Gamma(2n-2L-1)} = \frac{1}{\Gamma(n-L)}
\]
and therefore
\[
\text{Var} \ E_{f_L}(r) = \frac{L^2}{(n-1)!} \frac{\Gamma(n-2L)}{\Gamma(n-L)^2} (1 - r^2)^{2L-n} + \ldots
\]
Applying once more the duplication formula, now with $z = n/2 - L$, we finally get
\[
\text{Var} \ E_{f_L}(r) = \frac{L^2}{(n-1)!} \frac{2^{n-2L-1} \Gamma\left(\frac{n}{2} - L\right) \Gamma\left(\frac{n+1}{2} - L\right)}{\sqrt{\pi} \Gamma(n-L)^2} (1 - r^2)^{2L-n} + \ldots
\]
Case $L = n/2$. Proceeding as in the previous case we arrive at
\[
K(L, r) = \sum_{j=0}^{n-2} \binom{n-2}{j} (-1)^j \int_0^{\alpha_{o(r)}} (2 \cos \theta)^{n-2-j} \left[ \int_{A(\theta)} \frac{1}{s} (2 \cos \alpha(s, r))^j ds \right] d\theta + \ldots
\]
where
\[ J := \int_{A(\theta)} \frac{1}{s} (2 \cos \alpha(s, r))^j ds = 2^{j/2}(1 - r^2)^{j/2} \int_{\sqrt{1/r}}^{\sqrt{1/\sqrt{r}}} \frac{1}{x} (x + 1/x)^j dx + \cdots \]

As before we split the integral into two parts, depending on whether \( x \leq 1 \) or not. Unlike the previous case here both parts have the same order.

(i) Assume \( x \leq 1 \). Then
\[ A_1 := \int_{\sqrt{1/r}}^{1/r} \frac{1}{x} (x + 1/x)^j dx = \int_{2}^{\sqrt{1-r^4}} y^{j-1} dy + \cdots \]
\[ = \begin{cases} \log\left(\frac{2 \cos \theta}{\sqrt{1-r^4}}\right) + \cdots = \frac{1}{2} \log\left(\frac{1}{1-r^2}\right) + \log(\cos \theta) \cdots & \text{if } j = 0 \\ \frac{1}{j} \left(\frac{2 \cos \theta}{\sqrt{1-r^4}}\right)^j + \cdots = \frac{2^{j/2}(\cos \theta)^j}{j(1-r^2)^{j/2}} + \cdots & \text{if } j \geq 1 \end{cases} \]

(ii) For \( x > 1 \) we have the same values as in the previous case:
\[ A_2 := \int_{1/r}^{1/\sqrt{r}} \frac{1}{x} (x + 1/x)^j dx = \int_{2}^{\sqrt{1-r^4}} y^{j-1} dy + \cdots \]

Then
\[ J = \begin{cases} \log\left(\frac{1}{1-r^2}\right) + 2 \log(\cos \theta) + \cdots & \text{if } j = 0 \\ \frac{2^{j+1}}{j} (\cos \theta)^j + \cdots & \text{if } j \geq 1 \end{cases} \]

With this we get
\[ K(L, r) = \int_{0}^{\alpha_0(r)} (2 \cos \theta)^{n-2} \left(\log\left(\frac{1}{1-r^2}\right) + 2 \log(\cos \theta)\right) d\theta + \]
\[ + \sum_{j=1}^{n-2} \binom{n-2}{j} (-1)^j \int_{0}^{\alpha_0(r)} (2 \cos \theta)^{n-2-j} \frac{2^{j+1}}{j} (\cos \theta)^j d\theta. \]

Since \( (\cos \theta)^{n-2} \log(\cos \theta) \) is integrable in \([0, \pi/2]\) and the sum in \( j \geq 1 \) is bounded independently of \( r \), the leading term of \( K(L, r) \) is given by the factor \( \log(1/(1-r^2)) \) above. Then (16)
yields
\[
K(L, r) = 2^{n-2} \int_0^{\pi \alpha(r)} (\cos \theta)^{n-2} \log \left( \frac{1}{1 - r^2} \right) + \cdots = 2^{n-2} \int_0^{\pi/2} (\cos \theta)^{n-2} \log \left( \frac{1}{1 - r^2} \right) + \cdots
\]
\[
= 2^{n-3} \sqrt{\pi} \frac{\Gamma \left( \frac{n-1}{2} \right)}{\Gamma \left( \frac{n}{2} \right)} \log \left( \frac{1}{1 - r^2} \right) + \cdots
\]

By Lemma 7 we get then
\[
\text{Var} E_{\alpha L}(r) = \frac{\left( \frac{n}{2} \right)^2}{\sqrt{\pi(n-1)!}(n-2)!} \frac{\Gamma \left( \frac{n-1}{2} \right)}{\Gamma \left( \frac{n}{2} \right)} \left( 1 - r^2 \right)^{n-2} \log \left( \frac{1}{1 - r^2} \right) + \cdots
\]

The duplication formula (17) with \( z = \frac{n-1}{2} \) yields \( \frac{2^{n-2} \Gamma(n-1)}{\sqrt{\pi(n-2)!}} = \frac{1}{\Gamma(n/2)} \) and thus the stated result.

Case \( L > n/2 \). Here \( K(L, r) \) tends, as \( r \to 1^- \), to the constant
\[
K(L, 1) = \int_0^1 \int_0^{\arccos s} \left( 2 \cos \theta - 2s \right)^{n-2} \left( s + \frac{1}{s} - 2 \cos \theta \right) d\theta \, ds.
\]

As before, we expand the power and apply Fubini’s theorem:
\[
K(L, 1) = 2^{n-2} \int_0^1 \int_0^{\arccos s} \sum_{k=0}^{\infty} \sum_{j=0}^{n-2} s^{2L+2Lk-n-2j} \binom{n-2}{j} (-1)^j \frac{1}{s^j} \left( \cos \theta \right)^{n-2-j} \left( s + \frac{1}{s} - 2 \cos \theta \right) d\theta \, ds
\]
\[
= 2^{n-2} \sum_{k=0}^{\infty} \sum_{j=0}^{n-2} \binom{n-2}{j} (-1)^j \int_0^{\pi/2} \left( \cos \theta \right)^{n-2-j} \int_0^{\arccos s} s^{2L(1+k)-n+j} \left( s + \frac{1}{s} - 2 \cos \theta \right) ds d\theta.
\]

Re-indexing the sum in \( k \) and performing the integral in \( s \) we get
\[
K(L, 1) = 2^{n-2} \sum_{j=0}^{n-2} \sum_{k=1}^{\infty} \binom{n-2}{j} (-1)^j \left[ \int_0^{\pi/2} \frac{\left( \cos \theta \right)^{2Lk}}{2Lk-n+j+2} d\theta + \int_0^{\pi/2} \frac{\left( \cos \theta \right)^{2Lk-2}}{2Lk-n+j} d\theta \right]
\]
\[
- 2 \int_0^{\pi/2} \frac{\left( \cos \theta \right)^{2Lk}}{2Lk-n+j+1} d\theta.
\]

Using (16) to compute the integrals in \( \theta \), and (12) to compute the sum in \( j \), we obtain
\[
K(L, 1) = \sqrt{\pi(n-2)!} 2^{n-3} \sum_{k=1}^{\infty} \left[ \frac{\Gamma(Lk+1/2) \Gamma(2Lk-n+2)}{\Gamma(Lk+1) \Gamma(2Lk+1)} + \frac{\Gamma(Lk-1/2) \Gamma(2Lk-n)}{\Gamma(Lk) \Gamma(2Lk-1)} \right]
\]
\[
- 2 \frac{\Gamma(Lk+1/2) \Gamma(2Lk-n+2)}{\Gamma(Lk+1) \Gamma(2Lk)}.
\]
Lemma 8. For $M > 0$

$$\frac{\Gamma(M + 1/2) \Gamma(2M - n + 2)}{\Gamma(M + 1) \Gamma(2M + 1)} + \frac{\Gamma(M - 1/2) \Gamma(2M - n)}{\Gamma(M) \Gamma(2M - 1)} - 2 \frac{\Gamma(M + 1/2) \Gamma(2M - n + 1)}{\Gamma(M + 1) \Gamma(2M)} = 2^{-n} \frac{\Gamma(M - n/2) \Gamma(M - n/2 - 1)}{(\Gamma(M + 1))^2} (M + \frac{n(n - 1)}{2})$$

This with $M = Lk$ yields

$$K(L, 1) = \frac{\sqrt{\pi}(n - 2)!}{8} \sum_{k=1}^{\infty} \frac{\Gamma(Lk - n/2) \Gamma(Lk - (n - 1)/2)}{(\Gamma(Lk + 1))^2} (LK + \frac{n(n - 1)}{2})$$

which together with the identity of Lemma 7 gives the stated result.

Proof of Lemma 8. Denote by $S$ the left hand side of identity in the lemma. Then

$$S = \frac{\Gamma(M + 1/2) \Gamma(2M - n + 2)}{\Gamma(M + 1) \Gamma(2M + 1)} - \frac{\Gamma(M + 1/2) \Gamma(2M - n + 1)}{\Gamma(M + 1) \Gamma(2M)} +$$

$$+ \frac{\Gamma(M - 1/2) \Gamma(2M - n)}{\Gamma(M) \Gamma(2M - 1)} - \frac{\Gamma(M + 1/2) \Gamma(2M - n + 1)}{\Gamma(M + 1) \Gamma(2M)}$$

$$= \frac{\Gamma(M + 1/2) \Gamma(2M - n + 2)}{\Gamma(M + 1) \Gamma(2M + 1)} \left( \frac{1}{2M} - \frac{1}{2M - n + 1} \right) +$$

$$+ \frac{\Gamma(M + 1/2) \Gamma(2M - n + 1)}{\Gamma(M + 1) \Gamma(2M)} \left( \frac{2M}{2M - n - 1} \right)$$

Using the duplication formula (17) for $z = M - n$ and $z = M + 1$ we get finally

$$S = \frac{\Gamma(M + 1/2) 2^{2M-n-1} \Gamma(M - n/2) \Gamma(M - n/2 - 1)}{2^{2M} \Gamma(M + 1/2) \Gamma(M + 1)} (2M + n(n - 1))$$

$$= \frac{1}{2^n} \frac{\Gamma(M - n/2) \Gamma(M - n/2 - 1)}{(\Gamma(M + 1))^2} (M + \frac{n(n - 1)}{2})$$.

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