Stability of Iterative Decoding of Multi-Edge Type Doubly-Generalized LDPC Codes Over the BEC

Enrico Paolini, Mark F. Flanagan, Marco Chiani and Marc P. C. Fossorier

Abstract—Using the EXIT chart approach, a necessary and sufficient condition is developed for the local stability of iterative decoding of multi-edge type (MET) doubly-generalized low-density parity-check (D-GLDPC) code ensembles. In such code ensembles, the use of arbitrary linear block codes as component codes is combined with the further design of local Tanner graph connectivity through the use of multiple edge types. The stability condition for these code ensembles is shown to be succinctly described in terms of the value of the spectral radius of an appropriately defined polynomial matrix.

I. INTRODUCTION

Multi-edge type (MET) low-density parity-check (LDPC) codes were originally proposed in [1] as a framework to capture both degree-1 variable nodes (VNs) and punctured bits in the analysis of LDPC code ensembles, and to achieve a finer control of the connectivity between VNs and check nodes (CNs) in the ensemble definition. The new framework allowed the design of powerful finite-length LDPC codes over the additive white Gaussian noise (AWGN) channel, with a very good compromise between waterfall performance and error-floor. Since then, several aspects of MET LDPC codes have been investigated such as, for instance, their average weight distribution [2]. Traditional unstructured irregular LDPC code ensembles parametrized through their degree distribution pair [3] may be seen as MET ensembles where all edges in the Tanner graph are of the same type. On the other hand, LDPC codes based on protographs [4] may be seen as MET LDPC codes such that no two edges connected to the same VN or to the same CN are of the same type.

Another way to extend the original framework of unstructured LDPC code ensembles consists of replacing the VNs and the CNs with linear block codes other than repetition codes and single parity-check (SPC) codes, respectively. The resulting LDPC-like codes are called doubly-generalized LDPC (D-GLDPC) codes [5], and extend the original idea of generalized LDPC (GLDPC) codes [6], where only the CNs were replaced with generic linear block codes. Several theoretical aspects of unstructured D-GLDPC codes have been recently clarified, such as their stability condition over the binary erasure channel (BEC) [7], and the analysis of the exponent of their weight distribution [8].

The two different extensions can be considered together, leading to the concept of MET D-GLDPC code ensemble. This represents a very general framework for design and analysis of LDPC-like codes, enabling to handle different variable and check component codes along with puncturing; VNs and CNs with local minimum distance 1, including degree-1 VNs (state variables), can be also considered. The asymptotic weight enumerators for MET D-GLDPC codes were investigated in [9], while EXIT analysis to calculate the threshold of MET D-GLDPC codes over the BEC was developed in [10].

In this paper, we analyze the convergence properties of the belief-propagation (BP) decoder for MET D-GLDPC code ensembles over the BEC by developing its stability condition, i.e., the condition under which the erasure-free state attracts the decoder, in the asymptotic setting where the codeword length tends to infinity. If and only if the condition is satisfied, BP decoding can in principle succeed, provided the BEC erasure probability is below the threshold which can be calculated using the technique in [10]. The stability condition is obtained in the case where the are no punctured bits and where the local minimum distance of each VN and CN is at least 2. It can be shown that the obtained condition coincides with that developed in [1], [11, Ch. 7] in the special case of MET LDPC codes.

II. PRELIMINARY DEFINITIONS

A. Concept of D-GLDPC Codes

A D-GLDPC code consists of a set of CNs and a set of VNs. Each CN corresponds to some arbitrary linear ‘local’ code. On the other hand, each VN corresponds to some arbitrary linear ‘local’ code, together with its encoder (i.e., generator matrix). Graphically, each CN and each VN may be viewed as having a set of sockets corresponding to the bits in the local codeword. The sockets of the CNs are connected by edges to the sockets of the VNs in a one-to-one fashion; the resulting graph is called the Tanner graph of the D-GLDPC code. A codeword of the D-GLDPC code is an assignment of an information word to each VN such that the local encoding of this word at each VN assigns an encoded bit to each edge of the Tanner graph in such a way that the resulting configuration forms a valid local codeword from the perspective of every CN. It is easily seen that if the local code at each CN is a single parity-check code and if the local code at each VN is a repetition code, the resulting D-GLDPC code reduces to an ordinary LDPC code.

B. MET D-GLDPC Code Ensemble Definition

In MET D-GLDPC codes, we distinguish between $n_e$ different edge types. Each edge type is identified by an index...
in the set \( \delta = \{ 1, 2, \ldots, n_e \} \). Furthermore, we distinguish between different VN types and different CN types. Each VN type is identified by a triplet \( \gamma = (v_\gamma, b_\gamma, d_\gamma) \), where:

- \( v_\gamma \) identifies a \((q_\gamma, k_\gamma)\) variable component code, where \( q_\gamma \) is the code length and \( k_\gamma \) the code dimension, and a specific encoder (i.e., generator matrix \( G_\gamma \)) of it;
- \( b_\gamma \) is a binary vector of length \( k_\gamma \) which specifies the local puncturing pattern for the VN. Specifically, for \( i \in \{ 1, 2, \ldots, k_\gamma \} \), if \( b_{\gamma,i} = 0 \) then the corresponding encoded bit of the D-GLDPC code is punctured, and it is not punctured otherwise;
- \( d_\gamma \) is a vector of length \( q_\gamma \) whose \( i \)-th element \( d_{\gamma,i} \in \delta \) specifies the edge type of the \( i \)-th VN socket.

Each CN type is identified by a pair \( \delta = (c_\delta, d_\delta) \), where:

- \( c_\delta \) identifies an \((s_\delta, h_\delta)\) check component code (regardless of its representation), where \( s_\delta \) is the code length and \( h_\delta \) the code dimension;
- \( d_\delta \) is a vector of length \( s_\delta \) whose \( i \)-th element \( d_{\delta,i} \in \delta \) specifies the edge type of the \( i \)-th CN socket.

The set of all VN types \( \gamma \) is denoted by \( \mathcal{F}_V \), and set of all CN types \( \delta \) by \( \mathcal{F}_C \). Moreover, the fraction of edges of type \( l \in \mathcal{E} \) connected to VNs of type \( \gamma \in \mathcal{F}_V \) is denoted by \( \lambda_{\gamma,l} \), while the fraction of edges of type \( l \in \mathcal{E} \) connected to CNs of type \( \delta \in \mathcal{F}_C \) by \( \rho_{\delta,l} \). We have \( \lambda_{\gamma,l} > 0 \) if and only if the generic VN of type \( \gamma \) has at least one soccket of type \( l \), and \( \lambda_{\gamma,l} = 0 \) otherwise. An analogous statement can be made regarding \( \rho_{\delta,l} \). Also, \( q_{\gamma,l} \) and \( s_{\delta,l} \) denote the number of sockets of type \( l \) for a VN of type \( \gamma \) and for a CN of type \( \delta \), respectively. The constraints \( \sum_{l \in \mathcal{E}} q_{\gamma,l} = q_\gamma \forall \gamma \in \mathcal{F}_V \) and \( \sum_{l \in \mathcal{E}} s_{\delta,l} = s_\delta \forall \delta \in \mathcal{F}_C \) hold.

An example of D-GLDPC code ensemble is depicted in Fig. 1. As opposed to single-edge type codes, where a unique encoder (i.e., generator matrix \( G_\gamma \)) is the case, we have two parity-check equations and each CN of type \( \delta \) one parity-check equation, then the overall code dimension is \( K = 8 \). Since four bits are punctured, the overall codeword length is \( N = 28 \), and the code rate is \( R = 2/7 \).

### C. Further Definitions

Throughout the paper, vectors are intended as column vectors. We define \( \mathbf{0} \) and \( \mathbf{1} \) as the vectors of length \( n_e \) whose elements are all equal to 0 and all equal to 1, respectively. Moreover, we define \( \mathbf{1}_e \) as the vector of length \( n_e \) whose elements are all equal to 0 except the element in position \( e \) which is equal to 1. The subset of VN types \( \gamma \) with local minimum distance 2 is denoted by \( \mathcal{F}_{V2} \subseteq \mathcal{F}_V \), and the subset of CN types \( \delta \) with local minimum distance 2 by \( \mathcal{F}_{C2} \subseteq \mathcal{F}_C \).

For \( \delta \in \mathcal{F}_{C2} \) and \( l, m \in \mathcal{E} \), we denote by \( \xi_{\delta}^{(l)}(l,m) \) the number of ordered pairs of sockets of a CN of type \( \delta \), such that the first socket is of type \( l \) and the second of type \( m \), and such that the assignment of a ‘1’ to these sockets and a ‘0’ to all other CN sockets results in a (weight-2) local codeword. Note that \( \xi_{\delta}^{(l)}(l,m) = \xi_{\delta}^{(l)}(m,l) \). For \( l, m \in \mathcal{E} \), we define the nonnegative real parameter \( C_l^{m} \) as

\[
C_l^{m} := \sum_{\delta \in \mathcal{F}_{C2}} \left( \rho_{\delta,l} \right) \xi_{\delta}^{(l)}(l,m). \tag{1}
\]

If CNs of type \( \delta \in \mathcal{F}_{C2} \) have no sockets of type \( l \) (in which case \( \rho_{\delta,l} = 0 \) and \( s_{\delta,l} = 0 \)), then we set \( C_l^{m} = 0 \) by definition. Denoting by \( A_2^{(l)}(l,m) \) the number of weight-2 local codewords of a CN of type \( \delta \in \mathcal{F}_{C2} \) such that one of the two ‘1’ local encoded bits corresponds to a socket of type \( l \) and the other to a socket of edge type \( m \), we have \( \xi_{\delta}^{(l)}(l,m) = A_2^{(l)}(l,m) \) for \( l \neq m \) and \( \xi_{\delta}^{(l)}(l,l) = 2A_2^{(l)}(l,l) \).

If \( \gamma \in \mathcal{F}_{V2} \) and \( l, m, \in \mathcal{E} \), we denote by \( \chi_{\delta,u}^{(l)}(l,m) \) the number of ordered pairs of sockets of a VN of type \( \gamma \) such that the first socket is of type \( l \) and the second of type \( m \), and such that the assignment of a ‘1’ to these sockets and a ‘0’ to all other VN sockets results in a (weight-2) local codeword generated by a local input word of weight \( u \). Similarly to the CN case, we have \( \chi_{\delta,u}^{(l)}(l,m) = \chi_{\delta,u}^{(m)}(m,l) \). The nonnegative polynomial \( P_l^{m}(x) \) (with real coefficients) is defined as

\[
P_l^{m}(x) := \sum_{\gamma \in \mathcal{F}_{V2}} \left( \lambda_{\gamma,l} \right) \sum_{u=1}^{k_\gamma} \chi_{\delta,u}^{(l)}(l,m) x^u. \tag{2}
\]

If VNs of type \( \gamma \in \mathcal{F}_{V2} \) have no sockets of type \( l \) (in which case \( \lambda_{\gamma,l} = 0 \) and \( q_{\gamma,l} = 0 \)), then we set \( P_l^{m}(x) = 0 \) by definition. Moreover, denoting by \( B_{\delta,u}^{(l)}(l,m) \) the number of weight-2 local codewords of a VN of type \( \gamma \in \mathcal{F}_{V2} \) generated
by local weight-$u$ input words, and such that one of the two ‘1’
local encoded bits corresponds to a socket of edge type $l$ and
the other to a socket of edge type $m$, we have $\chi^{(\gamma)}(l, m) = B^{(\gamma)}_2(l, m)$ for $l \neq m$ and $\chi^{(\gamma)}(l, l) = 2B^{(\gamma)}_2(l, l)$.

Remark 2.1: For $l, m \in \mathcal{E}$ and $l \neq m$, in general we have $C^{l,m} = C^{m,l}$ and $P^{l,m} = P^{m,l}$.

Remark 2.2: For single-edge type codes, $\mathcal{E}$ is a singleton
$\mathcal{E} = \{l\}$, $P^{l,m}$ reduces to the polynomial $P^l(x)$ and $C^{l,m}$ to
the parameter $C$ characterizing ordinary D-GLDPC codes [7], [8]. For codes constructed from protographs, no two sockets
of a VN or CN are of the same type. Hence, in this case we have $P^{l,m}(x) = 0$ and $C^{l,m} = 0$ for all $l \in \mathcal{E}$.

D. Multi-Type Information Functions

Although in this paper we make some assumptions on the
VNIs and CNs (see the last paragraph of Sec. [II-B]), the
definitions provided in this subsection are more general and
do not rely on such assumptions.

Consider a CN of type $\delta = (c_\delta, d_\delta) \in \mathcal{F}_C$, and let $G_\delta$
be any generator matrix for the associated component code.
From $G_\delta$, form $n_e$ matrices $G_{\delta,l}$, where $G_{\delta,l}$ is the $(s_\delta \times s_{\delta,l})$ matrix composed of the columns of $G_\delta$ associated with the
bit positions of type $l \in \mathcal{E}$ (irrespective of the order of these columns).
Then, for any integer $n_e$-tuple $g = (g_1, g_2, \ldots, g_{n_e})$
satisfying $0 \leq g_l \leq s_{\delta,l}$ for all $l \in \mathcal{E}$, the $g$-th multi-type
information function of the CN is defined as

$$
\hat{e}^{(\delta)}_{g}(\delta) := \sum_{S^{(\delta)}_g} \text{rank} \left( S^{(\delta)}_g \right)
$$

where $S^{(\delta)}_g$ is a matrix formed by selecting $g_l$ columns in $G_{\delta,l}$
(irrespective of the order of these columns) and where $\sum_{S^{(\delta)}_g}$
denotes the summation over all $\prod_{l=1}^{n_e} (s_{\delta,l})$ matrices $S^{(\delta)}_g$.

For a VN of type $\gamma = (v_\gamma, b_\gamma, d_\gamma)$, let $G_\gamma$
be the specific generator matrix identified by $v_\gamma$. Moreover, let $b_\gamma$ be the
Hamming weight of $b_\gamma$. From $G_\gamma$, form $n_e$ matrices $G_{\gamma,l}$,
where $G_{\gamma,l}$ is the $(k_\gamma \times q_{\gamma,l})$ matrix composed of the columns of $G_\gamma$
associated with the bit positions of type $l \in \mathcal{E}$ (irrespective of the order of these columns). Then, for any integer $n_e$-tuple $g = (g_1, g_2, \ldots, g_{n_e})$
satisfying $0 \leq g_l \leq q_{\gamma,l}$ for all $l \in \mathcal{E}$, and for any integer $0 \leq u \leq k_\gamma$, the $g$-th $u$-th multi-type
split information function of the VN is defined as

$$
\hat{e}^{(\gamma)}_{g,u}(u) := \sum_{S^{(\gamma)}_{g,u}} \text{rank} \left( S^{(\gamma)}_{g,u} \right)
$$

where $S^{(\gamma)}_{g,u}$ is a matrix formed by selecting $g_l$ columns in $G_{\gamma,l}$
(irrespective of the order of these columns) and $u$ columns among the $b_\gamma$ columns of $I_\gamma$ (order-$k_\gamma$ identity matrix)
corresponding to the support of $b_\gamma$.

1If for some $l \in \mathcal{E}$ the CN has no sockets of type $l$, then $g_l$ is conventionally set to 0. This convention shall be adopted also for the multi-type split information functions defined for the VNs.

III. EXIT ANALYSIS AND BP DECODER STABILITY

EXIT analysis of a MET D-GLDPC code ensemble with $n_e$
edge types consists of modeling the average behavior of the
iterative decoder, in the asymptotic case where the codeword
length tends to infinity, through an $n_e$-dimensional discrete
dynamical system tracking $n_e$ average extrinsic information
values, one for each edge type. For $e \in \mathcal{E}$, the $e$-th value we
track is the average extrinsic information over the edges of
type $e$, outgoing from the VN set towards the CN set.

Let $\ell \geq 1$ denote the decoding iteration index. Let $I_{E_{V},e}^\ell$
and $I_{E_{C},e}^\ell$ be the average extrinsic information over the edges of
type $e$ outgoing from the VN set and from the CN set, at the
$\ell$-th decoding iteration, respectively. Moreover, let $I_{A_{V},e}^\ell$
and $I_{A_{C},e}^\ell$ be the average a priori information over the edges of
type $e$ incoming towards the VN set and towards the CN
set, at the $\ell$-th decoding iteration, respectively. EXIT analysis equations of a MET D-GLDPC code ensemble over a BEC
with erasure probability $\epsilon$ may be expressed as

$$
\begin{align}
I_{E_{V,1}}^\ell &= I_{E_{V,1}}^{EV}(I_{A_{V,1}}^{\ell-1}, \ldots, I_{A_{V,n_e}}^{\ell-1}, \epsilon) \\
& \quad \vdots \\
I_{E_{V,n_e}}^\ell &= I_{E_{V,n_e}}^{EV}(I_{A_{V,1}}^{\ell-1}, \ldots, I_{A_{V,n_e}}^{\ell-1}, \epsilon)
\end{align}
$$

and

$$
\begin{align}
I_{E_{C,1}}^\ell &= I_{E_{C,1}}^{EC}(I_{A_{C,1}}^{\ell-1}, \ldots, I_{A_{C,n_e}}^{\ell-1}) \\
& \quad \vdots \\
I_{E_{C,n_e}}^\ell &= I_{E_{C,n_e}}^{EC}(I_{A_{C,1}}^{\ell-1}, \ldots, I_{A_{C,n_e}}^{\ell-1})
\end{align}
$$

The $2n_e$ equations (5) and (6), together with $I_{A_{C,1}}^\ell = I_{E_{V,i}}^\ell \forall e \in \mathcal{E}$, $I_{A_{V,i}}^\ell = I_{E_{C,i}}^\ell \forall i \in \mathcal{E}$, and $I_{A_{V},i}^\ell = 0 \forall i \in \mathcal{E}$, define a recursion that can be expressed in the compact form

$$
I_{E_{V}}^{\ell+1} = f(I_{E_{V}}^{\ell}, \epsilon)
$$

for $\ell \geq 0$ and where $I_{E_{V}} = [I_{E_{V,1}}, I_{E_{V,2}}, \ldots, I_{E_{V,n_e}}]^T$
is a column vector whose elements are the $n_e$ values to be
tracked. The $n_e$-dimensional discrete dynamical system (7)
models the asymptotic (in terms of codeword length) evolution of
the BP decoder over a BEC with erasure probability $\epsilon$.

The function $f(\cdot)$ can be evaluated exploiting results developed in [10]. In more detail, neglecting the iteration index
$\ell$, for $e \in \mathcal{E}$ we have $I_{E_{V,e}}(I_{A_{V,1}}, I_{A_{V,2}}, \ldots, I_{A_{V,n_e}}, \epsilon) = \sum_{\gamma \in \mathcal{F}_V} \lambda_{\gamma,e} E_{V,e}^{(\gamma)}(I_{A_{V,1}}, I_{A_{V,2}}, \ldots, I_{A_{V,n_e}}, \epsilon)$
where

$$
\begin{align}
E_{V,e}^{(\gamma)}(I_{A_{V,1}}, I_{A_{V,2}}, \ldots, I_{A_{V,n_e}}, \epsilon) &= 1 - \frac{1}{q_{\gamma,e}} \sum_{z=0}^{q_{\gamma,e}-1} (1 - \epsilon)^{q_{\gamma,e}-1-t_e} \\
& \times \sum_{t_1=0}^{q_{\gamma,e}-1} (1 - I_{A_{V,1}})^{t_1} (I_{A_{V,1}}^{\gamma})^{q_{\gamma,e}-1-t_1} \\
& \times \sum_{t_2=0}^{q_{\gamma,e}-1} (1 - I_{A_{V,e}})^{t_2} (I_{A_{V,e}}^{\gamma})^{q_{\gamma,e}-1-t_2} \\
& \times \sum_{t_{n_e}=0}^{q_{\gamma,e}-1} (1 - I_{A_{V,n_e}})^{t_{n_e}} (I_{A_{V,n_e}}^{\gamma})^{q_{\gamma,e}-1-t_{n_e}} q_{\gamma,e}^{(\gamma)}(e)
\end{align}
$$
and
\[ a_{t,h}^{(\gamma,e)} = (q_{\gamma,e} - t_e) e_{q_{\gamma,e}}^{(t)} - (t_e + 1) e_{q_{\gamma,e}}^{(-1)} - t_{\gamma,e} - t_{\gamma,h} - h. \] (9)

Moreover, we have
\[ \delta_{EC,\ell}(I_{AC,1}, I_{AC,2}, \ldots, I_{AC,n_e}) = \sum_{\delta \in F_\ell} P_{\delta,e}(I_{EC,\ell}(I_{AC,1}, I_{AC,2}, \ldots, I_{AC,n_e})) \]
where
\[ \delta_{EC,\ell}(I_{AC,1}, I_{AC,2}, \ldots, I_{AC,n_e}) = 1 - \frac{1}{s_{\delta,e}} \sum_{t=0}^{s_{\delta,e}-1} (1 - I_{AC,e})^t (I_{AC,e})^{s_{\delta,e}-1-t} \]
\[ \times \sum_{t=0}^{s_{\delta,n_e}} (1 - I_{AC,n_e})^t (I_{AC,n_e})^{s_{\delta,n_e}-t} \]
\[ \times \sum_{t=0}^{s_{\delta,D-GLDPC}} (1 - I_{AC,D-GLDPC})^t (I_{AC,D-GLDPC})^{s_{\delta,D-GLDPC}-t} \]
and
\[ a_{t,e}^{(\delta,e)} = (s_{\delta,e} - t_e) e_{s_{\delta,e}-1}^{(t)} \]
\[ - (t_e + 1) e_{s_{\delta,e}-1}^{(-1)} - t_{\gamma,e} - t_{\gamma,h} - h. \] (10)

Lemma 3.1: For a MET D-GLDPC code ensemble such that all VNs and CNs have local minimum distance at least 2 and such that no encoded bit is punctured (\( b_e \)) is the all-1 vector for all \( \gamma \in F_\ell \), \( I_{EV,1}, I_{EV,2}, \ldots, I_{EV,n} \) is a fixed point of (7) regardless of \( e \) if \( f(1, e) = 1 \) for all \( e \in (0, 1) \).

The fixed point \( I_{EV} = 1 \) corresponds to a state of the system in which no erasure messages are exchanged between the VN set and the CN set, i.e., in which all encoded bits of the D-GLDPC code are known. A transmission over the BEC may then be modeled as a perturbation of the system state from \( I_{EV} = 0 \) to \( I_{EV} = [I_{EV,1}, 0, \ldots, 0, 0, \ldots, 0, 0] \) calculated in the fixed point. Specifically, the fixed point is a local attractor when the magnitude of all eigenvalues of the Jacobian matrix is less than 1 or, equivalently, if and only if the spectral radius of the Jacobian matrix is less than 1. Hence, we need to prove that \( J(1, e) = P(\epsilon) C \), where \( J(1, e) \) is the Jacobian matrix of \( f(I_{EV}, e) \), calculated in \( I_{EV} = 1 \).

For \( l, m \in \{1, 2\} \), the \( (l, m) \)-th entry of \( J(1, e) \), is given by
\[ J^{l,m}(1, e) = 2 \sum_{\epsilon = 1} \frac{\partial I_{EV,\ell}}{\partial A_{l,m}} (1, e) \cdot \Delta I_{AC,\ell} \cdot \Delta I_{AC,m}. \] (13)

Consider now a generic \((\gamma_1, k_1)\) VN of type \( \gamma \in F_\ell \) having at least one codeword edge of type \( e \). Using (8) and (9), it is easy to show that
\[ \lim_{k \to \infty} \frac{J^{l,m}(\gamma_1, k)}{I_{AC,\ell}} = \frac{1}{q_{\gamma_1}} \sum_{z=0}^{k_1} z^l - a_{0,z}^{(\gamma_1, l)} - a_{1,z}^{(\gamma_1, l)} \]
where the last equality is due to \( a_{0,z}^{(\gamma_1, l)} = 0 \). In fact, for \( l = 1 \) we have \( a_{0,z}^{(\gamma_1, 1)} = q_{\gamma_1} - q_{\gamma_1} q_{\gamma_2} k_z - q_{\gamma_2} k_z - q_{\gamma_2} k_z = q_{\gamma_1} q_{\gamma_2} k_z - q_{\gamma_2} k_z - q_{\gamma_2} k_z = 0 \), and in an analogous way we can show that \( a_{0,z}^{(\gamma_1, l)} = 0 \). Next, we develop (14), assuming \( l = 1 \), in the two cases \( e = [1, 0]^T \) and \( e = [0, 1]^T \). From being \( \sigma(A) \) the spectral radius of a square matrix \( A \), i.e., the magnitude of the eigenvalue of \( A \) with the largest magnitude.

Interestingly, inequality (13) represents the “natural” extension to the MET framework of the condition \( P(\epsilon) C < 1 \) proved in [7] for the single-edge type case. A sketch of proof of Theorem 3.1 is provided in Section V. Theorem 3.1 allows us to develop a simple sufficient condition for local stability of fixed point \( I_{EV} = 1 \), as follows.

Corollary 3.1: Consider a MET D-GLDPC code ensemble with \( n_e \) edge types. Assume that there are no VNs and CNs with local minimum distance 1 and that no D-GLDPC encoded bit is punctured. Moreover, assume that the following condition is satisfied: If a socket of VN of type \( \gamma \in F_\ell \), associated with the support of a weight-2 local codeword, is of type \( e \) \( \in \delta \), then for all \( \delta \in F_\ell \) a CN of type \( \delta \) has no sockets of type \( e \) associated with the support of a weight-2 local codeword. Then, the fixed point \( I_{EV} = 1 \) of (7) is locally stable for any BEC erasure probability \( \epsilon \).

Proof: Simply observe that in this case \( P(\epsilon) C \) is the all-zero matrix and, consequently, \( \sigma(P(\epsilon) C) = 0 \) for all \( \epsilon \).
and from the definition of multi-type information function in Section II-13 we have

\[ k_{\gamma} \sum_{z=0}^{k_{\gamma}} \epsilon^z (1 - \epsilon)^{k_{\gamma}-z} a^{(\gamma,1)}_{\gamma}[1,0],z = \sum_{z=0}^{k_{\gamma}} \epsilon^z (1 - \epsilon)^{k_{\gamma}-z} \]

\[ \times \left[ (q_{\gamma},1 - 1)e^{(\gamma)}_{q_{\gamma},1-1,q_{\gamma},2,k_{\gamma}-z} - 2e^{(\gamma)}_{q_{\gamma},1-2,q_{\gamma},2,k_{\gamma}-z} \right] \]

\[ = 2 \sum_{z=0}^{k_{\gamma}} \epsilon^z (1 - \epsilon)^{k_{\gamma}-z} \]

\[ \times \frac{k_{\gamma}}{2} - e^{(\gamma)}_{q_{\gamma},1-2,q_{\gamma},2,k_{\gamma}-z} \]

\[ = 2 \sum_{z=0}^{k_{\gamma}} \epsilon^z (1 - \epsilon)^{k_{\gamma}-z} \]

\[ \sum_{z=0}^{k_{\gamma}} (k_{\gamma} - \text{rank}(S_{q_{\gamma},1-2,q_{\gamma},2,k_{\gamma}-z})) \]  \hspace{1cm} (15)

\[ \sum_{z=0}^{k_{\gamma}} \epsilon^z (1 - \epsilon)^{k_{\gamma}-z} a^{(\gamma,1)}_{\gamma}[1,0],z = \sum_{z=0}^{k_{\gamma}} \epsilon^z (1 - \epsilon)^{k_{\gamma}-z} \]

\[ \times \left[ q_{\gamma},1 \epsilon^{(\gamma)}_{q_{\gamma},1-2,q_{\gamma},2-1,k_{\gamma}-z} - \epsilon^{(\gamma)}_{q_{\gamma},1-1,q_{\gamma},2-1,k_{\gamma}-z} \right] \]

\[ = \sum_{z=0}^{k_{\gamma}} \epsilon^z (1 - \epsilon)^{k_{\gamma}-z} \]

\[ \times \sum_{z=0}^{k_{\gamma}} \sum_{z=0}^{k_{\gamma}} (k_{\gamma} - \text{rank}(S_{q_{\gamma},1-1,q_{\gamma},2-1,k_{\gamma}-z})) \cdot \]  \hspace{1cm} (16)

It is possible to show that (15) and (16) are equivalent to 2 \( B_{2,u}^{(\gamma)} (1,1) \) and \( \sum_{u=1}^{k_{\gamma}} \chi_{2,u}^{(\gamma)} (1,1) e^{u} = \sum_{u=1}^{k_{\gamma}} \chi_{2,u}^{(\gamma)} (1,2) e^{u} \), respectively. Both expressions are obtained through an argument along the same line as that used, in the one-edge type case, to prove Lemma 4 in [7]. Incorporating these expressions into (14), recalling that \( I_{AV} (I_{AV}, \epsilon) = \sum_{\gamma} \gamma \chi_{\gamma} (1) k_{\gamma} \), and recalling (2), we finally obtain

\[ \frac{\partial I_{AV} (I_{AV}, \epsilon)}{\partial I_{AV} (I_{AV}, \epsilon)} = \sum_{\gamma} \chi_{\gamma} (1) k_{\gamma} \sum_{u=1}^{k_{\gamma}} \chi_{2,u}^{(\gamma)} (1,1) e^{u} = P_{1,1}^{(\gamma)} (\epsilon) \]  \hspace{1cm} (17)

\[ \frac{\partial I_{AV} (I_{AV}, \epsilon)}{\partial I_{AV} (I_{AV}, \epsilon)} = \sum_{\gamma} \chi_{\gamma} (1) k_{\gamma} \sum_{u=1}^{k_{\gamma}} \chi_{2,u}^{(\gamma)} (1,2) e^{u} P_{1,2}^{(\gamma)} (\epsilon) \].  \hspace{1cm} (18)

Note that in both (17) and (18) the summation is over \( \gamma \), since, for any \( \gamma \in \mathcal{F}_{V} \setminus \mathcal{F}_{V} \), we have \( \chi_{2,u}^{(\gamma)} (1,1) = \chi_{2,u}^{(\gamma)} (1,2) = 0 \). The same proof technique leading to (17) and (18) yields \( \frac{\partial I_{AV} (I_{AV}, \epsilon)}{\partial I_{AV} (I_{AV}, \epsilon)} = P_{2,2}^{(\gamma)} (\epsilon) \) and \( \frac{\partial I_{AV} (I_{AV}, \epsilon)}{\partial I_{AV} (I_{AV}, \epsilon)} = P_{2,1}^{(\gamma)} (\epsilon) \).

We now need to develop \( \frac{\partial I_{EC}}{\partial I_{AC,m}} (1) \) in the right-hand side of (13). To this purpose, simply observe that a CN of type \( \delta \in \mathcal{F}_{C} \) may be regarded as a VN whose \( k_{\delta} \) local information bits are all punctured (\( b_{\delta} = 0 \)). Note that this is equivalent to assuming a channel erasure probability \( \epsilon = 1 \) for the VN. In this case, the right-hand sides of (17) and (18) become \( \sum_{\gamma} \frac{\lambda_{\gamma}}{q_{\gamma}^2} k_{\gamma} \sum_{u=1}^{k_{\gamma}} \chi_{2,u}^{(\gamma)} (1,1) = \sum_{\gamma} \chi_{2,u}^{(\gamma)} (1,1) \) and \( \sum_{\gamma} \frac{\lambda_{\gamma}}{q_{\gamma}^2} k_{\gamma} \sum_{u=1}^{k_{\gamma}} \chi_{2,u}^{(\gamma)} (1,2) = \sum_{\gamma} \chi_{2,u}^{(\gamma)} (1,2) \), respectively. Thus, we have

\[ \frac{\partial I_{EC,1}}{\partial I_{AC,1}} (1) = \sum_{\delta} \epsilon_{\delta} (1,1) = C_{1,1}^{1,1} \]  \hspace{1cm} (19)

\[ \frac{\partial I_{EC,1}}{\partial I_{AC,2}} (1) = \sum_{\delta} \epsilon_{\delta} (1,2) = C_{1,2}^{1,2} \]  \hspace{1cm} (20)

and also \( \frac{\partial I_{EC,2}}{\partial I_{AC,2}} (1) = C_{2,2}^{2,2} \) and \( \frac{\partial I_{EC,2}}{\partial I_{AC,2}} (1) = C_{2,1}^{2,1} \). Hence, for \( l, m \in \{1,2\} \), the \( (l,m) \)-th entry of \( J_{1} (1, \epsilon) \) is given by \( J_{1,1}^{1,2} (1, \epsilon) = \sum_{l=1}^{2} \frac{P_{l} (\epsilon)}{2} C_{l,m} \), i.e., \( J_{1} (1, \epsilon) = P (\epsilon) C \).

V. Examples

In this section, the stability of the iterative decoder is analyzed for two MET ensembles.

Example 5.1: Consider the two-edge-type ensemble (\( \mathcal{E} = \{1, 2\} \)) whose Tanner graph is depicted in Fig.2 where edges of type \( 1 \in \mathcal{E} \) are depicted in red and edges of type \( 2 \in \mathcal{E} \) in blue. There are \( N \) VNIs, all of the same type \( \gamma \).

Each VN is a length-2 repetition code with \( G_{\gamma} = [1, 1] \), with one socket of type \( 1 \in \mathcal{E} \) and the other of type \( 2 \in \mathcal{E} \). Thus, we have \( \mathcal{F}_{V} = \mathcal{F}_{V,2} = \{ \gamma \} \). There are two CN types \( \mathcal{F}_{C} = \{ \delta_{1}, \delta_{2} \} \), where CNs of type \( \delta_{1} \) are \( (s_{1}, h_{1}) \) codes, depicted in yellow, and CNs of type \( \delta_{2} \) are \( (s_{2}, h_{2}) \) codes, depicted in green. All \( s_{1} \) sockets of a type-\( \delta_{1} \) CN are of type \( 1 \in \mathcal{E} \), while all \( s_{2} \) sockets of a type-\( \delta_{2} \) CN are of type \( 2 \in \mathcal{E} \).

The number of CNs of types \( \delta_{1} \) and \( \delta_{2} \) are \( N/s_{1} \) and \( N/s_{2} \) respectively, so each edge interleaver is for \( N \) edges.

Assuming that CNs of both types have minimum distance \( 2 \) (\( \mathcal{F}_{C,2} = \{ \delta_{1}, \delta_{2} \} \)), we obtain

\[ P (\epsilon) = \begin{bmatrix} 0 & \epsilon \\ \epsilon & 0 \end{bmatrix} \] and \( C = \begin{bmatrix} 2 A_{2}^{(\delta_{1})}/s_{1} & 0 \\ 0 & 2 A_{2}^{(\delta_{2})}/s_{2} \end{bmatrix} \),

where \( A_{2}^{(\delta_{1})} \) and \( A_{2}^{(\delta_{2})} \) are the multiplicities of weight-2 local codewords of CNs of types \( \delta_{1} \) and \( \delta_{2} \) respectively. From Theorem 3.1 the condition for local stability of the erasure-free state is

\[ \epsilon < \frac{1}{2} \sqrt{\frac{s_{1}s_{2}}{A_{2}^{(\delta_{1})} A_{2}^{(\delta_{2})}}} \]. \hspace{1cm} (21)

where the right-hand side is an upper bound on the iterative decoding threshold called the stability bound.

From (21), we see how the multiplicities \( A_{2}^{(\delta_{1})} \) and \( A_{2}^{(\delta_{2})} \) may jeopardize the decoder stability, and how increasing \( s_{1} \) or \( s_{2} \) is beneficial in terms of stability. We may also observe that the erasure-free fixed point for this ensemble is a stable attractor if the CNs of at least one type are characterized.
by minimum distance larger than 2, irrespective of the local weight spectrum of CNs of the other type (all diagonal entries as well as at least one off-diagonal entry of $C$ are zero in this case). In practice, for large $N$ this model gives a good indication of stability for the ensemble of product codes which are obtained by taking $N = s_1 s_2$ and choosing appropriately the two edge interleavers $\Pi_1$ and $\Pi_2$ [12].

Example 5.2: Consider the two-edge-type ensemble $(\mathcal{E} = \{1, 2\})$ whose Tanner graph is depicted in Fig. 5, where edges of type 1 $\in \mathcal{E}$ are depicted in red and edges of type 2 $\in \mathcal{E}$ in blue. There are two VN types $\mathcal{F}_V = \{\gamma_1, \gamma_2\}$, where the $N_{\gamma_2}$ VNs of type $\gamma_1$, depicted in cyan, are $(q, k)$ codes generated by some generator matrix $G_{\gamma_1}$, and the $N_{\gamma_2} = N_{\gamma_1} q$ VNs of type $\gamma_2$, depicted in pink, are length-$2$ repetition codes with $G_{\gamma_2} = [1, 1]$. All $q$ sockets of a type-$\gamma_1$ VN are of type 1 $\in \mathcal{E}$, while both sockets of a type-$\gamma_2$ VN are of type 2 $\in \mathcal{E}$. Moreover, there are $N_{\gamma_1} q$ CNs, all of the same type $\delta$. Each CN is a $(3, 2)$ SPC code having one socket of type 1 $\in \mathcal{E}$ and two sockets of type 2 $\in \mathcal{E}$. Hence, we have $\mathcal{F}_C = \mathcal{F}_C = \{\delta\}$.

Assuming that VNs of type $\gamma_1$ have minimum distance 2 ($\mathcal{F}_V = \{\gamma_1, \gamma_2\}$), we obtain

$$P(\epsilon) = \begin{bmatrix} 2^{-k} \sum_{u=1}^{k} B_{2,u}^{(q_1)} e^u & 0 \\ 0 & \epsilon \end{bmatrix}$$

and $C = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$,

where $B_{2,u}^{(\gamma_1)}$ denotes the number of weight-$2$ local codewords of VNs of type $\gamma_1$ generated by local input words of length $k$ through $G_{\gamma_1}$. Again applying Theorem 5.1, we obtain the following condition for stability of the erasure-free fixed point:

$$\frac{4}{q} \sum_{u=1}^{k} B_{2,u}^{(\gamma_1)} e^{u+1} < 1 - \epsilon . \tag{22}$$

Again, an increase in the multiplicity of weight-$2$ local codewords of VNs of type $\gamma_1$ has a negative effect on the stability of the fixed point $I_{EV} = 1$, as it reduces the range of channel erasure probabilities over which such a fixed point is locally stable (just note that all coefficients of the polynomial on the left-hand side of (22) are positive). Moreover, increasing $q$ has a positive effect on the stability. We also point out that the fixed point $I_{EV} = 1$ must be locally stable if the minimum distance of the type-$\gamma_1$ VNs is larger than 2 (in fact, in this case we obtain $\epsilon < 1$). Finally, we observe that, upon a proper choice of the edge interleaver $\Pi_2$, the Tanner graph depicted in Fig. 5 corresponds to the serial concatenation of a $(q, k)$ linear block encoder $G_{\gamma_1}$ with an accumulator. Hence, this class of codes may be seen as a generalization of repeat-accumulate (RA) codes [13]. An RA code is obtained when type-$\gamma_1$ VNs are length-$q$ repetition codes.

VI. CONCLUSION

In this paper, the stability condition for iterative BP decoding of MET D-GLDPC codes over the BEC has been developed. The obtained inequality is compact, and naturally extends to the MET ensemble parametrization the previously obtained condition for unstructured irregular (single-edge type) D-GLDPC codes. Although this point is not addressed in the present work, we mention that for LDPC-like codes, the stability condition has a further practical impact on code design through its relationship with the average weight distribution of the ensemble.

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