LANDAU DISPERSION RELATIONSHIP IN SELF-CONSISTENT FIELD THEORY

S.A. Stepin

Abstract The spectral problem is studied associated with Maxwell-Boltzmann equations describing collisionless plasma. Formula for instability index is obtained and effective conditions of two-stream instability are given.

§ 1. Maxwell-Boltzmann equations of self-consistent field

The main object of our considerations will be the system of equations describing collisionless plasma consisting of two types of the particles — electrons and ions in the presence of electromagnetic field. Distribution of the particles in plasma is characterized by the corresponding densities (distribution functions) \( f_e \) and \( f_i \) dependent on time \( t \), space coordinate \( x \) and velocity \( v \) of the particles. Thus the system under consideration is composed of Maxwell equations for the components of electromagnetic field and kinetic (collisionless) Boltzmann equations for distribution functions. In spatially one-dimensional case magnetic induction is trivial \( B(t, x) \equiv 0 \) and the system of equations in question has the form

\[
\begin{align*}
\frac{\partial f_e}{\partial t} + v \frac{\partial f_e}{\partial x} - \frac{e}{m} E \frac{\partial f_e}{\partial v} &= 0, \\
\frac{\partial f_i}{\partial t} + v \frac{\partial f_i}{\partial x} + \frac{Ze}{M} E \frac{\partial f_i}{\partial v} &= 0, \\
\frac{\partial E}{\partial t} + 4\pi e \int_{-\infty}^{\infty} v (Zf_i - f_e) \, dv &= 0, \\
\frac{\partial E}{\partial x} - 4\pi e \int_{-\infty}^{\infty} (Zf_i - f_e) \, dv &= 0,
\end{align*}
\]

where \( e \) and \( Ze \) stand for the charges, \( m \) and \( M \) are the masses of electrons and ions respectively, while \( E = E(t, x) \) denotes the electric field strength. Each of the kinetic equations admits representation in Liouvillean form and expresses the fact that the total derivative of distribution function vanishes along the trajectory for the particle of the proper type.

Henceforth a solution to the stationary system such that

\[
f_e = f_{0e}(v), \quad f_i = f_{0i}(v), \quad E_0(x) \equiv 0
\]
is chosen as an unperturbed one. Following [1] we will assume that ions distribution function \( f_0(v) \) is fixed and linearize the system with respect to the stationary solution \( f_{0e}(v) \), \( E_0(x) \equiv 0 \). As a result we obtain the system

\[
\begin{align*}
\frac{\partial f}{\partial t} &= -v \frac{\partial f}{\partial x} + \frac{e}{m} f'_0(v) E, \\
\frac{\partial E}{\partial t} &= 4\pi e \int_{-\infty}^{\infty} v f \, dv
\end{align*}
\]

where \( f = f(t, x, v) \) is the perturbation of electrons stationary distribution function \( f_0(v) = f_{0e}(v) \) while \( E(t, x) \) stands for electric field strength.

Electromagnetic field induced by the electrons traffic and in turn affecting the evolution of their density is known to be called self-consistent. It is assumed that distribution functions of the particles and components of electromagnetic field decay at infinity sufficiently rapidly. According to charge conservation law one has

\[
\langle f \rangle = \int \int f(t, x, v) \, dx \, dv = 0.
\]

Note that equation

\[
\frac{\partial E}{\partial x} = -4\pi e \int_{-\infty}^{\infty} f \, dv
\]

is compatible with the system (1)-(2) under consideration in the sense that equation (1) integrated with respect to velocity variable

\[
\frac{\partial}{\partial t} \int_{-\infty}^{\infty} f \, dv + \frac{\partial}{\partial x} \int_{-\infty}^{\infty} v f \, dv = 0,
\]

by virtue of (2) and (3) becomes an identity \( E_{tx} = E_{xt} \).

Equation (1) treated as inhomogeneous one with respect to \( f(t) \) can be reduced to Duhamel type integral equation

\[
f(t) = U(t) f^{(0)} + \frac{e}{m} f'_0(v) \int_0^t U(t - s) E(s) \, ds
\]

where the group of translations \( U(t) : \varphi(x, v) \mapsto \varphi(x - vt, v) \) specifies evolution generated by homogeneous equation \( \partial_t f + v \partial_x f = 0 \) while initial perturbation of distribution function \( f\big|_{t=0} = f^{(0)} \) satisfies zero mean condition \( \langle f^{(0)} \rangle = 0 \).

Substitution of \( f(t) \) expressed from (1) into equation (2) implies integro-differential equation for the strength \( E(t, x) \) subject to condition \( E\big|_{t=-\infty} = 0 \) which agrees with the original setting of the problem about stability or instability of plasma oscillations. Provided that initial perturbation of the density \( f^{(0)} \) is symmetric with respect to velocity \( f^{(0)}(x, v) = f^{(0)}(x, -v) \) while stationary (unperturbed) distribution function \( f_0(v) \) satisfies condition \( f_0(0) = 0 \) the strength of electric field \( E = E(t, x) \) proves to be a solution to Fredholm type equation

\[
E = E^{(0)} + KE
\]
where \( E^{(0)}(t, x) = -4\pi e \int_{-\infty}^{x} dy \int_{-\infty}^{\infty} f^{(0)}(y - vt, v) \, dv \) with integral operator \( K \) given by the formula
\[
K : E(t, x) \mapsto \frac{4\pi e^2}{m} \int_{-\infty}^{\infty} f_0(v) \, dv \int_{0}^{t} s \, E(t - s, x - vs) \, ds .
\]

The question about existence and uniqueness of solution to equation (5) in different functional spaces is reduced in [1] to evaluation of corresponding norms for integral operator \( K \) while solution itself is given by perturbation theory series known as Neumann expansion
\[
E = E^{(0)} + K E^{(0)} + K^2 E^{(0)} + \ldots .
\]

Another approach to investigation of solvability problem for equation (5) takes advantage of Laplace-Fourier transformation in variables \( t \) and \( x \). After the passage to associated representation the solvability condition is formulated in terms of the corresponding Fredholm resolvent denominator
\[
\Delta(k, \lambda) = 1 + \frac{4\pi e^2}{ikm} \int_{-\infty}^{\infty} \frac{f_0(v)}{\lambda + ikv} \, dv
\]
where \( \lambda \in \mathbb{C} \) is the spectral parameter while \( k \in \mathbb{R} \) is the wave number in \( x \)-axis direction. Namely condition \( \Delta(k, \lambda) \neq 0 \) implies invertibility of operator \( I - K \).

In due turn the roots of the so called Landau dispersion relationship
\[
\Delta(k, \lambda) = 1 + \frac{4\pi e^2}{ikm} \int_{-\infty}^{\infty} \frac{f_0'(v)}{\lambda + ikv} \, dv = 0 \tag{6}
\]
give rise to solutions of harmonic wave type depending on \( t \) and \( x \) exponentially \( \exp(\lambda t + ikx) \) which induce unstable eigenmode regimes (undamped oscillations) in plasma provided that \( \text{Re} \lambda > 0 \). In what follows for notational convenience we will assume that \( 4\pi e^2/m = 1 \).

§ 2. Spectral problem and Schur complement

Linear dynamical system (1)-(2) under consideration is associated with infinitesimal operator \( T \) given by the formula
\[
T : \left( \begin{array}{c} f(x, v) \\ g(x) \end{array} \right) \mapsto \left( \begin{array}{c} -v \partial_x f + \varphi(v) g \\ \int_{\mathbb{R}} v \, f \, dv \end{array} \right) ,
\]
where $g(x) = \frac{1}{4\pi e^2} E(x)$ and $\varphi(v) = \frac{4\pi e^2}{m} f'(v)$. Within the framework of stability problem setting it is crucial to study the spectral properties of operator $T$ acting in the underlying space prescribed by the physical arguments

$$\{L_1(\mathbb{R}_x, dx) \otimes L_1(\mathbb{R}_v, (1 + |v|) \, dv)\} \oplus L_1(\mathbb{R}_x, dx).$$

General block operator matrix acting in the direct sum of Banach spaces $X \oplus Y$ admits (see [2]) the following factorization

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & J \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & J \end{pmatrix}$$

(7)

where $A : X \to X$ is assumed to be invertible, $B : Y \to X$ and $C : X \to Y$ are bounded operators, while $I$ and $J$ stand for identity operators in $X$ and $Y$ respectively.

**Definition** Expression $D - CA^{-1}B$ is called the Schur complement of the block $A$ of operator matrix (7).

Due to invertibility of upper and lower triangle factors in the above formula it readily implies

**Statement 1** Point $\lambda \not\in \sigma(A)$ belongs to the resolvent set of operator matrix (7) acting in $X \oplus Y$ if and only if Schur complement

$$S(\lambda) := D - \lambda J - C(A - \lambda I)^{-1}B$$

is boundedly invertible in $Y$.

For the problem under consideration we let

$$Y = L_1(\mathbb{R}_x, dx), \quad X = Y \otimes L_1(\mathbb{R}_v, (1 + |v|) \, dv).$$

In order to put our setting into the general operator theoretic context we define the entries of block operator matrix

$$T = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

as follows: operator $A = -v \frac{\partial}{\partial x}$ is acting in $X$ with the natural dense domain

$$\{f \in X : f(\cdot, v) \text{ absolutely continuous}, \ v \partial f/\partial x \in X\}, \quad C = \int_\mathbb{R} \cdot v \, dv$$

and $B$ is an operator of multiplication by function $\varphi(v)$ satisfying condition

$$\int_\mathbb{R} (1 + |v|) |\varphi(v)| \, dv < \infty.$$  

(8)
Theorem 1  The spectrum of operator $T$ consists of two components

\[ \sigma(T) = \sigma(A) \cup \{ \lambda \in \mathbb{C} \setminus i\mathbb{R} : -1 \in \sigma(Q(\lambda)) \} \]

where $\sigma(A) = i\mathbb{R}$, while operator

\[ Q(\lambda) = \lambda^{-1}C(A - \lambda I)^{-1}B \]

acting in $Y$ is in fact a superposition of multiplication operator $B : Y \to X$ and an integral operator $\lambda^{-1}C(A - \lambda I)^{-1} : X \to Y$.

Proof For $\lambda = \sigma + i\tau$, $\sigma \neq 0$, the resolvent $R_0(\lambda) := (A - \lambda I)^{-1}$ of operator $A$ proves to be (see e.g. [3]) an integral operator of the form

\[ R_0(\lambda)f(x, v) = \begin{cases} \frac{1}{\nu} \int_{x}^{\infty} \exp \left( \frac{\lambda}{\nu} (\xi - x) \right) f(\xi, v) \, d\xi, & \sigma v < 0, \\ \frac{-1}{\nu} \int_{-\infty}^{x} \exp \left( \frac{\lambda}{\nu} (\xi - x) \right) f(\xi, v) \, d\xi, & \sigma v > 0. \end{cases} \]

To be specific one can assume that $\sigma = \text{Re}\lambda > 0$ and evaluate the norm

\[
\| R_0(\lambda) f \|_X = \int_{-\infty}^{0} (1 - v) \, dv \int_{-\infty}^{\infty} dx \left| \frac{1}{v} \int_{x}^{\infty} \exp \left( \frac{\lambda}{v} (\xi - x) \right) f(\xi, v) \, d\xi \right| + \\
+ \int_{0}^{\infty} (1 + v) \, dv \int_{-\infty}^{\infty} dx \left| \frac{1}{v} \int_{-\infty}^{x} \exp \left( \frac{\lambda}{v} (\xi - x) \right) f(\xi, v) \, d\xi \right| \leq \\
\leq \int_{-\infty}^{0} (v - 1) \, dv \int_{-\infty}^{\infty} dx \left( \frac{1}{v} \int_{x}^{\infty} \exp \left( \frac{\sigma}{v} (\xi - x) \right) |f(\xi, v)| \, d\xi \right) + \\
+ \int_{0}^{\infty} (1 + v) \, dv \int_{-\infty}^{\infty} dx \left( \frac{1}{v} \int_{-\infty}^{x} \exp \left( \frac{\sigma}{v} (\xi - x) \right) |f(\xi, v)| \, d\xi \right) = \\
= \frac{1}{\sigma} \int_{-\infty}^{0} (1 + |v|) \, dv \int_{-\infty}^{\infty} |f(\xi, v)| \, d\xi = \frac{1}{\sigma} \| f \|_X.
\]

Thus we have obtained that $\mathbb{C} \setminus i\mathbb{R}$ belongs to the resolvent set of operator $A$ and moreover for arbitrary $\lambda \notin i\mathbb{R}$ the estimate

\[ \| R_0(\lambda) \|_{X \to X} \leq |\text{Re}\lambda|^{-1} \]

is valid. Let us show now that spectrum of operator $A$ is purely continuous and occupies the axis $\sigma(A) = i\mathbb{R}$. To this end given $\mu \in i\mathbb{R}$ it suffices to produce in $X$ a noncompact family of approximate eigenvectors

\[ f_\delta(x, v) = \frac{1}{\delta} \exp \left( -\mu \frac{x}{v} - x^2 \right) \left\{ \theta \left( \frac{v}{\delta} + 1 \right) - \theta \left( \frac{v}{\delta} - 1 \right) \right\}, \quad \delta > 0, \]

where $\theta$ is Heaviside step function, so that as $\delta \downarrow 0$ one has

\[ \| f_\delta \|_X = \frac{1}{\delta} \int_{-\infty}^{\infty} dx \int_{-\delta}^{\delta} \exp \left( -\mu \frac{x}{v} - x^2 \right) |1 + |v|| \, dv = (2 + \delta)\sqrt{\pi}. \]
\[ \|(A - \mu I)f_\delta\|_X = \frac{2}{\delta} \int_{-\infty}^{\infty} |x| \, dx \int_{-\delta}^{\delta} \exp \left( -\mu \frac{x}{v} - x^2 \right) \left( |v| + \nu^2 \right) \, dv = 2\delta(1 + 2\delta/3), \]

and moreover

\[ \|Cf_\delta\|_Y = \frac{1}{\delta} \int_{-\infty}^{\infty} \left| \int_{-\delta}^{\delta} \exp \left( -\mu \frac{x}{v} - x^2 \right) \, v \, dv \right| \leq 2\delta. \]

Thus vector functions \((f_\delta(x, v), 0)\) provides us with a non-compact in \(X \oplus Y\) family of approximate eigenvectors for operator \(T\) corresponding to the point \(\mu \in i\mathbb{R}\) and hence \(i\mathbb{R} \subset \sigma(T)\).

To complete the proof of Theorem 1 one should just take into account that \(\sigma(A) = i\mathbb{R}\) and make usage of Statement 1 according to which \(\lambda \in \sigma(T) \setminus i\mathbb{R}\) if and only if \(-1 \in \sigma(Q(\lambda))\) where \(Q(\lambda) = \lambda^{-1}CR_0(\lambda)B\) is an integral operator given in \(Y\) by a superposition formula

\[ Q(\lambda)g(x) = \frac{1}{\lambda} \int_{-\infty}^{\infty} R_0(\lambda)\varphi(v)g(x) \, dv. \]

Theorem 1 enables us to specify and effectively localize the zone at the complex plane \(\mathbb{C}\) disjoint with the spectrum of \(T\) in which operator function \(J + Q(\lambda)\) proves to be invertible due to an appropriate smallness of \(Q(\lambda) : Y \to Y\).

**Proposition 1**  
Resolvent set of operator \(T\) contains the domain \(\Omega \subset \mathbb{C}\) specified by condition

\[ |\text{Re} \lambda| > \frac{1}{|\lambda|} \int_{-\infty}^{\infty} |\varphi(v)| \, |v| \, dv . \quad (9) \]

**Proof**  
By virtue of Theorem 1 it suffices to establish the inequality

\[ \|Q(\lambda)\|_{Y \to Y} \leq \frac{1}{|\lambda| |\sigma|} \int_{-\infty}^{\infty} |\varphi(v)| \, |v| \, dv \]

provided that \(\sigma = \text{Re} \lambda \neq 0\). To be definite let us consider the case \(\sigma = \text{Re} \lambda > 0\) when operator \(Q(\lambda)\) is given by the expression

\[ Q(\lambda)g(x) = \frac{1}{\lambda} \left\{ \int_{-\infty}^{0} \varphi(v) \, dv \int_{x}^{\infty} \exp \left( \frac{\lambda}{v} (\xi - x) \right) g(\xi) \, d\xi - \right. \]

\[ \left. - \int_{0}^{\infty} \varphi(v) \, dv \int_{-\infty}^{x} \exp \left( \frac{\lambda}{v} (\xi - x) \right) g(\xi) \, d\xi \right\}. \]

6
For such \( \lambda \) the following estimate

\[
\|Q(\lambda)g\|_Y \leq \frac{1}{|\lambda|} \int_{-\infty}^{\infty} dx \left\{ \int_{-\infty}^{0} |\varphi(v)| dv \int_{x}^{\infty} \exp \left( \frac{\sigma}{v} (\xi - x) \right) |g(\xi)| d\xi + \right.
\]

\[
+ \int_{0}^{\infty} |\varphi(v)| dv \int_{-\infty}^{x} \exp \left( \frac{\sigma}{v} (\xi - x) \right) |g(\xi)| d\xi \right\} = \frac{1}{|\lambda|} \int_{-\infty}^{\infty} |\varphi(v)| |v| dv \int_{-\infty}^{\infty} |g(\xi)| d\xi
\]

holds true and similarly the case \( \sigma = \text{Re} \lambda < 0 \) is dealt with.

Condition (9) clearly implies the inequality \( \|Q(\lambda)\|_{Y \to Y} < 1 \) and thus guarantees invertibility of operator \( J + Q(\lambda) \) which is necessary and sufficient for \( \lambda \not\in i\mathbb{R} \) to belong to the resolvent set of operator \( T \). The boundary of the domain \( \Omega \) parametrized in variables \( \sigma = \text{Re} \lambda \) and \( \tau = \text{Im} \lambda \) proves to be two-component algebraic curve

\[
\sigma^4 + \sigma^2 \tau^2 - c^2 = 0, \quad c = \int_{-\infty}^{\infty} |\varphi(v)||v| dv,
\]

intersecting the real axis at points \( \lambda = \pm \sqrt{c} \). Connected component of the boundary \( \partial \Omega \) located in the right half-plane \( \mathbb{C}_+ \) approaches the imaginary axis at infinity with asymptotics \( \sigma \sim \pm \frac{c}{\tau} \), while in the vicinity of the real axis it can be approximated by a parabolic pattern \( \sigma - \sqrt{c} \sim -\frac{\tau^2}{4\sqrt{c}} \).

**Remark** The above information about the resolvent set of operator \( T \) associated with the problem in question specifies and supplements localization of spectrum free zone established in [1].

### § 3. Fourier representation and Landau dispersion relationship

Let us denote by \( \Phi \) the standard Fourier transform in the space variable

\[
\Phi : Y \ni f \longmapsto \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx.
\]

The image \( \hat{Y} = \Phi Y \) will be regarded as a Banach space equipped by the induced norm \( \|\hat{f}\|_{\hat{Y}} = \|f\|_Y \) with respect to which mapping \( \Phi \) clearly becomes an isometric isomorphism.
Proposition 2  After passing to Fourier representation the normalized Schur complement
\[ -\lambda^{-1}\Phi S(\lambda)\Phi^{-1} = \Phi \left( J + Q(\lambda) \right) \Phi^{-1} \]
associated with block matrix \( T \) becomes an operator of multiplication by the function
\[ \Delta(k, \lambda) = 1 + \frac{1}{ik} \int_{-\infty}^{\infty} \frac{\varphi(v)}{ikv + \lambda} \, dv \]
defined for \( k = 0 \) by continuity
\[ \Delta(0, \lambda) = 1 - \frac{1}{\lambda^2} \int_{-\infty}^{\infty} v \varphi(v) \, dv. \]

Proof  Provided that \( f \) is taken from the domain of operator \( A \) one has
\[ \Phi(A - \lambda I)f(k) = (-ik - \lambda)\Phi f(k) \]
and consequently the resolvent \( \Phi R_0(\lambda) \Phi^{-1} : \hat{Y} \to \hat{Y} \) is just multiplication by the function \( -(ik + \lambda)^{-1} \). Therefore operator \( J + Q(\lambda) = -\lambda^{-1}S(\lambda) \) written in Fourier representation takes the form
\[ \Phi \left( J + Q(\lambda) \right) \Phi^{-1} : \hat{f}(k) \mapsto \left\{ 1 - \frac{1}{\lambda} \int_{-\infty}^{\infty} \frac{v \varphi(v)}{ikv + \lambda} \, dv \right\} \hat{f}(k), \]
where
\[ \frac{1}{\lambda} \int_{-\infty}^{\infty} \frac{v \varphi(v)}{ikv + \lambda} \, dv = - \frac{1}{ik} \int_{-\infty}^{\infty} \frac{\varphi(v)}{ikv + \lambda} \, dv \]
since \( \int_{-\infty}^{\infty} \varphi(v) \, dv = 0 \). Finally let us evaluate the difference
\[ \int_{-\infty}^{\infty} \frac{v \varphi(v)}{ikv + \lambda} \, dv - \frac{1}{\lambda} \int_{-\infty}^{\infty} v \varphi(v) \, dv = - \frac{ik}{\lambda} \int_{-\infty}^{\infty} \frac{v^2 \varphi(v)}{ikv + \lambda} \, dv \]
as \( k \to 0 \). To be definite consider the case when \( \tau = \text{Im} \lambda > 0 \) so that for \( k > 0 \) small enough the following inequalities
\[ \left| \frac{ik}{\lambda} \int_{-\infty}^{-k^{-1/2}} \frac{v^2 \varphi(v)}{ikv + \lambda} \, dv \right| \leq \frac{2}{|\lambda|} \left( 1 + \frac{\tau}{|\sigma|} \right) \int_{-\infty}^{-k^{-1/2}} |v| |\varphi(v)| \, dv, \]
\[ \left| \frac{ik}{\lambda} \int_{-k^{-1/2}}^{0} \frac{v^2 \varphi(v)}{ikv + \lambda} \, dv \right| \leq \frac{\sqrt{k}}{|\lambda| |\sigma|} \int_{-\infty}^{\infty} |v| |\varphi(v)| \, dv, \]
\[ \left| \frac{ik}{\lambda} \int_{k^{-1/2}}^{\infty} \frac{v^2 \varphi(v)}{ikv + \lambda} \, dv \right| \leq \frac{1}{|\lambda|} \int_{k^{-1/2}}^{\infty} |v| |\varphi(v)| \, dv, \]
are valid where $\sigma = \Re \lambda \neq 0$. Due to condition (8) the right-hand sides of the above estimates vanish as $k \downarrow 0$ and therefore

$$
\lim_{k \downarrow 0} \int_{-\infty}^{\infty} \frac{v \varphi(v)}{ikv + \lambda} \, dv = \frac{1}{\lambda} \int_{-\infty}^{\infty} v \varphi(v) \, dv
$$

provided that $\tau = \Im \lambda > 0$. The remaining cases are dealt with similarly.

In physical literature (see e.g. [4]) equation

$$
\Delta(k, \lambda) = 0
$$

is known to be called Landau dispersion relationship. It specifies the values of spectral parameter $\lambda$ and wave number $k$ for which the problem (1)-(2) possesses unstable modes corresponding to undamped plasma oscillations. The roots $\lambda = \lambda(k)$ of equation (10) will be regarded as singular values of the problem in question associated to given $k \in \mathbb{R}$.

**Lemma 1** Suppose that function $\varphi(v)$ satisfies the following condition

$$
\int_{-\infty}^{\infty} |\varphi(v)| \, |v|^3 \, dv < \infty.
$$

Then for arbitrary $\lambda = \sigma + i\tau$, $\sigma \neq 0$, function $\phi(k, \lambda) = \Delta(k, \lambda) - 1$ is bounded uniformly together with its two first derivatives in $k$ and moreover the asymptotic estimate is valid

$$
|\phi(k, \lambda)| + |\phi'(k, \lambda)| + |\phi''(k, \lambda)| = O(\frac{1}{|k|^{3/2}}), \quad k \to \pm \infty.
$$

**Proof** Taking advantage of the formula $\phi(k, \lambda) = -\frac{1}{\lambda} \int_{-\infty}^{\infty} \frac{v \varphi(v)}{ikv + \lambda} \, dv$ we obtain the estimate

$$
|\phi(k, \lambda)| \leq \frac{1}{|\lambda| |\sigma|} \int_{-\infty}^{\infty} |\varphi(v)| \, |v| \, dv.
$$

To evaluate $\phi(k, \lambda)$ as $k \to \pm \infty$ it makes sense to use representation

$$
\phi(k, \lambda) = \frac{1}{ik} \left\{ \int_{-\infty}^{-|k|^{-1/2}} + \int_{-|k|^{-1/2}}^{|k|^{-1/2}} + \int_{|k|^{-1/2}}^{\infty} \right\} \frac{\varphi(v)}{ikv + \lambda} \, dv,
$$

where

$$
\left| \int_{-|k|^{-1/2}}^{|k|^{-1/2}} \frac{\varphi(v)}{ikv + \lambda} \, dv \right| \leq \frac{2}{|\sigma| \sqrt{|k|}} \max_{|v| \leq 1} |\varphi|
$$
and
\[ \left| \left\{ \int_{-\infty}^{-|k|^{-1/2}} + \int_{|k|^{-1/2}}^{\infty} \right\} \frac{\varphi(v)}{ikv + \lambda} \, dv \right| \leq \frac{2\sqrt{|k|}}{|k| - \tau^2} \int_{-\infty}^{\infty} |\varphi(v)| \, dv \]

for \(|k|\) sufficiently large. Along the same lines one can estimate the derivatives
\[
\phi'(k, \lambda) = \frac{i}{\lambda} \int_{-\infty}^{\infty} \frac{v^2 \varphi(v)}{(ikv + \lambda)^2} \, dv = -\frac{1}{ik^2} \int_{-\infty}^{\infty} \varphi(v) \, dv - \frac{1}{k} \int_{-\infty}^{\infty} \frac{v \varphi(v)}{(ikv + \lambda)^2} \, dv,
\]
\[
\phi''(k, \lambda) = \frac{2}{\lambda} \int_{-\infty}^{\infty} \frac{v^3 \varphi(v)}{(ikv + \lambda)^3} \, dv = \frac{2}{ik^3} \int_{-\infty}^{\infty} \frac{\varphi(v)}{ikv + \lambda} \, dv +
\]
\[
+ \frac{2}{k^2} \int_{-\infty}^{\infty} \frac{v \varphi(v)}{(ikv + \lambda)^2} \, dv + \frac{2i}{k} \int_{-\infty}^{\infty} \frac{v^2 \varphi(v)}{(ikv + \lambda)^3} \, dv.
\]

Specification of the resolvent set for operator \(T\) in terms of the corresponding Landau dispersion relationship is given by

**Theorem 2** If \(\lambda \not\in i\mathbb{R}\) proves not to be a root of equation [10] for any \(k \in \mathbb{R}\) then operator \(J + Q(\lambda)\) is boundedly invertible in \(Y\) and moreover point \(\lambda\) belongs to the resolvent set of operator \(T\).

**Proof** By Proposition 2 operator \((J + Q(\lambda))^{-1}\) after passing to Fourier representation reduces to multiplication by the function \(1/\Delta(k, \lambda)\). Due to this fact and with the assumption \(\min_{k \in \mathbb{R}} |\Delta(k, \lambda)| > 0\) taken into account in virtue of Lemma 1 the functions
\[
h(k, \lambda) := \frac{1}{\Delta(k, \lambda)} - 1 = -\frac{\phi(k, \lambda)}{\Delta(k, \lambda)},
\]
\[
h'(k, \lambda) = -\frac{\phi'(k, \lambda)}{\Delta(k, \lambda)^2}, \quad h''(k, \lambda) = -\frac{\phi''(k, \lambda)}{\Delta(k, \lambda)^2} + 2\frac{\phi'(k, \lambda)^2}{\Delta(k, \lambda)^3}
\]
are absolutely integrable in variable \(k\) on the whole axis. It follows readily that \(g(x) = \Phi^{-1}h(x) \in Y\). In fact \(h(\pm \infty, \lambda) = h'(\pm \infty, \lambda) = 0\) and hence
\[
\sqrt{2\pi} g(x) = \int_{-\infty}^{\infty} h(k, \lambda)e^{ikx} \, dk =
\]
\[
= -\frac{1}{ix} \int_{-\infty}^{\infty} h'(k, \lambda)e^{ikx} \, dk = -\frac{1}{x^2} \int_{-\infty}^{\infty} h''(k, \lambda)e^{ikx} \, dk
\]
is uniformly bounded and sufficiently rapidly decreasing at infinity.
Therefore for arbitrary \( f \in Y \) one has
\[
\Phi (J + Q(\lambda))^{-1} \hat{\Phi} f(k) = (1 + h(k, \lambda)) \hat{f}(k) = (1 + \Phi g(k)) \Phi f(k) = \Phi (f - f * g)(k),
\]
where the convolution \( (f * g)(x) = \int_{-\infty}^{\infty} f(x - y)g(y)\,dy \) is absolutely integrable on \( \mathbb{R} \) so that
\[
\| f * g \|_Y \leq \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |f(x - y)| \cdot |g(y)|\,dy = \| f \|_Y \cdot \| g \|_Y.
\]
Finally we obtain the following estimate
\[
\| (J + Q(\lambda))^{-1} f \|_Y = \| \Phi (J + Q(\lambda))^{-1} \hat{\Phi} f \|_{\hat{Y}} = \| \Phi (f - f * g) \|_{\hat{Y}} = \| f - f * g \|_Y \leq (1 + \| g \|_Y) \| f \|_Y,
\]
where
\[
\| g \|_Y = \frac{1}{\sqrt{2\pi}} \left( \int_{-1}^{1} \left| \int_{-\infty}^{\infty} h(k, \lambda) e^{ikx}\,dk \right| \,dx + \int_{-\infty}^{\infty} h''(k, \lambda) e^{ikx}\,dk \right| \frac{dx}{x^2} \right) \leq \sqrt{2 \pi} \int_{-\infty}^{\infty} \left( |h(k, \lambda)| + |h''(k, \lambda)| \right) \,dk.
\]
Thus under the hypothesis of Theorem 2 operator \( J + Q(\lambda) \) proves to be boundedly invertible in \( Y \) and moreover
\[
\| (J + Q(\lambda))^{-1} \|_{Y \rightarrow Y} \leq 1 + \sqrt{2 \pi} \int_{-\infty}^{\infty} \left( |h(k, \lambda)| + |h''(k, \lambda)| \right) \,dk.
\]
To complete the proof it suffices to apply Theorem 1 according to which \( \lambda \not\in i\mathbb{R} \) belongs to the resolvent set of operator \( T \) if and only if \( -1 \not\in \sigma(Q(\lambda)) \).

\section*{§ 4. Formula for instability index}

In what follows we will denote by \( \Lambda(k) \) the set of singular values of the problem in question corresponding to a given \( k \in \mathbb{R} \):
\[
\Lambda(k) := \{ \lambda \in \mathbb{C} \setminus i\mathbb{R} : \Delta(k, \lambda) = 0 \}.
\]
Lemma 2  The set $\Lambda(k) \subset \mathbb{C}$ is symmetric with respect to imaginary axis.

In fact provided that $\lambda = \sigma + i \tau \in \Lambda(k)$ one has $-\lambda = -\sigma + i \tau \in \Lambda(k)$ since

$$
\text{Re} \Delta(k, -\lambda) = 1 - \frac{1}{k} \int_{-\infty}^{\infty} \frac{\varphi(v)(kv + \tau)}{(kv + \tau)^2 + \sigma^2} dv = \text{Re} \Delta(k, \lambda),
$$

$$
\text{Im} \Delta(k, -\lambda) = \frac{\sigma}{k} \int_{-\infty}^{\infty} \frac{\varphi(v)}{(kv + \tau)^2 + \sigma^2} dv = -\text{Im} \Delta(k, \lambda).
$$

An effective estimate for the instability index $N(k)$, i.e. the total number of singular values $\lambda = \lambda(k)$ corresponding to fixed $k \in \mathbb{R}$ and such that $\text{Re} \lambda(k) > 0$, is given by

Theorem 3 Assume that $\varphi \in L_1(\mathbb{R})$, $\varphi(v) \to 0$ as $v \to \pm \infty$, and the derivative $\varphi'(v)$ is bounded. Every zero $s$ of function $\varphi$ is supposed to be non-degenerate and each of them satisfy the following condition

$$1 - \frac{1}{k^2} \int_{-\infty}^{\infty} \frac{\varphi(v)}{v-s} dv \neq 0 \quad (11)$$

for a certain $k \in \mathbb{R} \setminus \{0\}$. Then the total number $N(k)$ of the roots of equation (10) located in the right half-plane are evaluated by the formula

$$N(k) := \# \{ \lambda \in \Lambda(k) : \text{Re} \lambda > 0 \} = N_+(k) - N_-(k),$$

where

$$N_+(k) = \# \{ s \in \mathbb{R} : \varphi(s) = 0, \varphi'(s) > 0, 1 - \frac{1}{k^2} \int_{-\infty}^{\infty} \frac{\varphi(v)}{v-s} dv < 0 \},$$

$$N_-(k) = \# \{ s \in \mathbb{R} : \varphi(s) = 0, \varphi'(s) < 0, 1 - \frac{1}{k^2} \int_{-\infty}^{\infty} \frac{\varphi(v)}{v-s} dv < 0 \}.$$  

The right half-plane contains at most one root of equation $\Delta(0, \lambda) = 0$ and, moreover, $\Lambda(0) \cap \{ \text{Re} \lambda > 0 \} \neq \emptyset$ if and only if

$$\int_{-\infty}^{\infty} v \varphi(v) dv > 0.$$

Proof Let us fix $k \in \mathbb{R} \setminus \{0\}$, introduce notation $\psi(v) := \varphi(v)/k^2$ and consider the function

$$w(z) = 1 - \int_{-\infty}^{\infty} \frac{\psi(v)}{v-z} dv = \Delta(k, -ikz)$$

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analytic in $\mathbb{C}_\pm$ whose boundary values at the real axis are calculated by Sokhotski-Plemelj formulas

$$w(s \pm i0) = 1 - \text{v.p.} \int_{-\infty}^{\infty} \frac{\psi(v)}{v - s} \, dv + i\pi \psi(s).$$

In order to evaluate the total number of zeroes of the function $w(z)$ in the upper half-plane $\mathbb{C}_+$ which coincides with the required quantity $\# \{ \lambda \in \Lambda(k) : \Re \lambda > 0 \}$ we will take advantage of the argument principle (see e.g. [5]). Beforehand let us verify that $w(z) \to 1$ as $\mathbb{C}_+ \ni z \to \infty$. In fact for $z = s + it \in \mathbb{C}_+$ one has

$$|w(z) - 1| = \left| \int_{-\infty}^{\infty} \frac{\psi(v)}{v - s - it} \, dv \right| \leq \frac{1}{t} \int_{-\infty}^{\infty} |\psi(v)| \, dv \to 0$$

as $t = \Im z \to \infty$. To estimate absolute value $|w(z) - 1|$ in the case when $s = \Re z \to +\infty$ it makes sense to represent the corresponding integral in the form

$$1 - w(z) = \left\{ \int_{-\infty}^{s/2} + \int_{s/2}^{s-\varepsilon} + \int_{s-\varepsilon}^{s+\varepsilon} + \int_{s+\varepsilon}^{\infty} \right\} \frac{\psi(v)}{v - s - it} \, dv,$$

where $\varepsilon \in (0, s/2)$, and treat the above summands separately:

$$\left| \int_{s-\varepsilon}^{s+\varepsilon} \frac{\psi(v)}{v - s - it} \, dv - \psi(s) \left( \pi - 2 \arctan \left( \frac{t}{\varepsilon} \right) \right) \right| =$$

$$= \left| \int_{s-\varepsilon}^{s+\varepsilon} \frac{\psi(v) - \psi(s)}{v - s - it} \, dv \right| \leq 2\varepsilon \max |\psi'(v)|$$

and, besides,

$$\left| \int_{-\infty}^{s/2} \frac{\psi(v)}{v - s - it} \, dv \right| \leq \frac{2}{s} \int_{-\infty}^{\infty} |\psi(v)| \, dv \to 0,$$

$$\left| \left\{ \int_{s/2}^{s-\varepsilon} + \int_{s+\varepsilon}^{\infty} \right\} \frac{\psi(v)}{v - s - it} \, dv \right| \leq \frac{1}{\varepsilon} \int_{s/2}^{\infty} |\psi(v)| \, dv \to 0.$$

Thus the following inequality

$$|w(z) - 1| \leq 2\varepsilon \max |\psi'(v)| + \pi |\psi(s)| + \frac{2}{s} \int_{-\infty}^{\infty} |\psi(v)| \, dv + \frac{1}{\varepsilon} \int_{s/2}^{\infty} |\psi(v)| \, dv$$

holds true where the first summand on the right-hand side can be made arbitrarily small under an appropriate choice of $\varepsilon \in (0, s/2)$ while the second, the third and the fourth ones vanish for fixed $\varepsilon > 0$ as $s \to +\infty$. Similarly one can carry out estimation of the absolute value $|w(z) - 1|$ in the case when $s = \Re z \to -\infty$. 
According to Cauchy argument principle given arbitrary closed contour $\gamma \subset \mathbb{C}_+$ encircling all the zeroes of function $w(z)$ their total multiplicity $N = N(k)$ is equal to the sum of logarithmic residues of $w(z)$ associated with the interior of $\gamma$. It coincides with the index of the point $w = 0$ with respect to the curve $\Gamma = w(\gamma)$ also known as winding number, i.e. the total number of times that curve $\Gamma$ travels counterclockwise around the origin:

$$N = \text{Ind}_\Gamma(0) = \frac{1}{2\pi i} \oint_\gamma \frac{w'(z)}{w(z)} \, dz = \frac{1}{2\pi} \oint_\gamma \text{d} \arg w(z).$$

Such a method of counting the roots of dispersion relationship is related to the stability criterion due to H. Nyquist (see [4]) which has been rigorously justified in the context of plasma stability problem by O. Penrose in [6].

In the present setting we choose contour $\gamma$ to be composed of the segment $[-R,R]$ and the half-circle $z = R e^{i\theta}$, $\theta \in [0,\pi]$. Radius $R > 0$ is to be taken large enough so that according to the above calculations values $w(R e^{i\theta})$ would belong to sufficiently small neighborhood of the point $w = 1$ being separated away from the origin. At the same time the image $w([-R,R])$ proves to be a path with parametrization

$$\text{Re } w(s + i0) = 1 - \text{v.p.} \int_{-\infty}^{\infty} \frac{\psi(v)}{v-s} \, dv,$$

$$\text{Im } w(s + i0) = -\pi \psi(s), \quad s \in [-R,R],$$

and such that its endpoints corresponding to $s = \pm R$ belong to the prescribed neighborhood of $w = 1$. Moreover curve $\Gamma = w(\gamma)$ intersects the negative semi-axis $\mathbb{R}_-$ at points specified by the values of parameter $s_j \in [-R,R]$ enumerated in the ascending order and such that

$$\psi(s_j) = 0, \quad 1 - \int_{-\infty}^{\infty} \frac{\psi(v)}{v-s_j} \, dv < 0.$$

The gradient of transversal intersection corresponding to the zero $s = s_j$ of function $\psi$ is determined by the derivative $\psi'(s_j) \neq 0$ so that the arc $w([s_j, s_{j+1}])$ between two subsequent intersections of $\Gamma$ with $\mathbb{R}_-$ due to condition (9) produce the following increment of the argument

$$\int_{s_j}^{s_{j+1}} \text{d} \arg w(s + i0) = \pi \left( \text{sign } \psi'(s_j) + \text{sign } \psi'(s_{j+1}) \right).$$

In this way the winding number for the curve $\Gamma$ is given by the expression

$$N(k) = \text{Ind}_\Gamma(0) = \int_{-\infty}^{\infty} \text{d} \arg w(s + i0) =$$
\[
\frac{1}{2} \text{sign } \psi'(s_1) + \frac{1}{2} \sum_{j=1}^{n-1} \left( \text{sign } \psi'(s_j) + \text{sign } \psi'(s_{j+1}) \right) + \\
+ \frac{1}{2} \text{sign } \psi'(s_n) = N_+(k) - N_-(k).
\]

**Remark**  Singular values \( \lambda = \lambda(k) \) corresponding to a fixed \( k \neq 0 \) can be located at the imaginary axis \( i\mathbb{R} \) if and only if
\[
\psi(i\lambda/k) = 0, \quad 1 - \int_{-\infty}^{\infty} \frac{\psi(v)}{v - i\lambda/k} dv = 0.
\]
Thus condition (9) means that operator \( T \) does not have any singular values embedded into its continuous spectrum.

**Lemma 4**  Under hypotheses of Theorem 3 condition
\[
|k|^4 > 8 \max |\varphi'| \int_{-\infty}^{\infty} |\varphi(v)| dv
\]
guarantees absence of singular values \( \lambda = \lambda(k) \) in the right half-plane:
\[
\Lambda(k) \cap \{\Re \lambda > 0\} = \emptyset.
\]
In fact provided that \( \varphi(s) = 0 \) for arbitrary \( \varepsilon > 0 \) one has
\[
\left| \int_{-\infty}^{\infty} \frac{\varphi(v)}{v - s} dv \right| \leq \left| \int_{s-\varepsilon}^{s+\varepsilon} \frac{\varphi(v) - \varphi(s)}{v - s} dv \right| + \left| \left\{ \int_{-\infty}^{s-\varepsilon} + \int_{s+\varepsilon}^{\infty} \right\} \frac{\varphi(v)}{v - s} dv \right| \leq \\
\leq 2\varepsilon \max |\varphi'(v)| + \frac{1}{\varepsilon} \int_{-\infty}^{\infty} |\varphi(v)| dv.
\]
An optimal choice of parameter \( \varepsilon > 0 \) minimizing the right-hand side of the above estimate implies
\[
\left| \int_{-\infty}^{\infty} \frac{\varphi(v)}{v - s} dv \right| \leq 2\sqrt{2} \left( \max |\varphi'(v)| \right)^{1/2} \left( \int_{-\infty}^{\infty} |\varphi(v)| dv \right)^{1/2}.
\]
Hence for arbitrary \( k \in \mathbb{R} \) satisfying condition (9) and any zero \( v = s \) of function \( \varphi \) the following inequality
\[
1 - \frac{1}{k^2} \int_{-\infty}^{\infty} \frac{\varphi(v)}{v - s} dv > 0
\]
is valid and therefore \( N_+(k) = N_-(k) = 0 \) so that \( \# \{ \lambda \in \Lambda(k) : \Re \lambda > 0 \} = 0 \) by virtue of Theorem 3.
§ 5. The model of two-stream instability

A simple sufficient condition is known (see [1],[4]) to be formulated in terms of unperturbed distribution function $f_0(v)$ which forbids existence of unstable plasma oscillatory perturbations. Namely provided that $f_0(v)$ possesses a unique extremum (maximum) $v = a$ the problem under consideration has no singular values in the right half-plane: $\Lambda(k) \cap \{\text{Re} \lambda > 0\} = \emptyset$ for arbitrary $k \in \mathbb{R}$. In fact one has $\psi(v)/(v-a) < 0$ and hence $N_-(k) = N_+(k) = 0$ for $k \neq 0$ since

$$1 - \int_{-\infty}^{\infty} \frac{\psi(v)}{v-a} \, dv > 0.$$

Moreover $\int_{-\infty}^{\infty} (v-a) \varphi(v) \, dv \leq 0$ so that $\Lambda(0) \cap \{\text{Re} \lambda > 0\} = \emptyset$ by Theorem 3.

In the situation when $f_0(v)$ has two maxima a phenomenon may happen called two-stream instability. To this end critical points $v = a$ and $v = b$ corresponding to maximal values of the unperturbed distribution function are to be situated so that condition

$$\int_{a}^{b} \frac{\psi(v)}{v-c} \, dv > 1 + \left\{ \int_{-\infty}^{c} + \int_{b}^{\infty} \right\} \frac{\psi(v)}{c-v} \, dv$$

holds true where $v = c \in (a,b)$ is the critical point of $f_0(v)$ corresponding to its minimum and moreover the higher and the wider maximum peaks should be the farther they are located (cf. Lemmas 5 and 6 below).

To study the effect of two-stream instability we will consider the case when all the hypotheses of Theorem 3 are satisfied and moreover $f_0(a) = f_0(b) = M$ while $f_0(c) = 0$ so that

$$\int_{-\infty}^{\infty} \psi(v) \, dv = \int_{c}^{a} \psi(v) \, dv = \int_{c}^{b} \psi(v) \, dv = \int_{b}^{\infty} \psi(v) \, dv$$

where $\psi(v) > 0$ for $v \in (-\infty,a) \cup (c,b)$ and $\psi(v) < 0$ when $v \in (a,c) \cup (b,\infty)$ respectively. As a consequence one has

$$\int_{-\infty}^{\infty} \frac{\psi(v)}{v-a} \, dv = \int_{-\infty}^{a} \frac{\psi(v)}{v-a} \, dv + \left\{ \int_{a}^{c} + \int_{c}^{b} \right\} \frac{\psi(v)}{v-a} \, dv + \int_{b}^{\infty} \frac{\psi(v)}{v-a} \, dv < 0$$

and similarly

$$\int_{-\infty}^{\infty} \frac{\psi(v)}{v-b} \, dv = \int_{-\infty}^{a} \frac{\psi(v)}{v-b} \, dv + \left\{ \int_{a}^{c} + \int_{c}^{b} \right\} \frac{\psi(v)}{v-b} \, dv + \int_{b}^{\infty} \frac{\psi(v)}{v-b} \, dv < 0$$

so that $N_-(k) = 0$ and $N_+(k) \leq 1$. A criterion of two-stream instability within the present setting is as follows

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Statement 2  Given \( k \in \mathbb{R} \setminus \{0\} \) one has \( N_+(k) = 1 \) or equivalently \( \Lambda(k) \cap \{\text{Re} \lambda > 0\} \neq \emptyset \) if and only if
\[
\int_{-\infty}^{\infty} \frac{\varphi(v)}{v-c} \, dv > k^2.
\]

In order to formulate sufficient conditions which guarantee existence of unstable eigenmodes in different terms we introduce additional notation. For the critical point \( v = a \) given \( \mu \in (0, M) \) set
\[
a_<(\mu) = \sup \{ v < a : f_0(v) \leq \mu \}, \quad a_>(\mu) = \inf \{ v > a : f_0(v) \leq \mu \},
\]
so that quantity \( (a_>(\mu) - a_<(\mu)) \) is the width of corresponding maximum peak at level \( \mu \), i.e. a diameter of the connected component of preimage \( f_0^{-1}([\mu, \infty)) \) containing the point \( v = a \). Similarly the values \( b_<(\mu) \) and \( b_>(\mu) \) are defined for the critical point \( v = b \).

Lemma 5  Let the inequality
\[
\frac{(a_> - a_<)}{(c - a_>) (c - a_<)} \xi + \frac{(b_> - b_<)}{(b_< - c) (b_> - c)} \eta > k^2
\]
be valid for certain \( \xi, \eta \in (0, M) \), where \( a_> = a_>(\xi) \), \( a_< = a_<(\xi) \) and \( b_> = b_>(\eta) \), \( b_< = b_<(\eta) \). Then \( N_+(k) = 1 \) so that
\[
\# \{ \lambda \in \Lambda(k) : \text{Re} \lambda > 0 \} = 1.
\]

In fact one has \( \int_{a_<}^{a_>} \psi(v) \, dv = \int_{a_<}^{a_>} \psi(v) \, dv \), hence \( \int_{a_<}^{a_>} \frac{\psi(v)}{v-c} \, dv > 0 \) and therefore
\[
\int_{-\infty}^{c} \frac{\psi(v)}{v-c} \, dv > \left\{ \int_{-\infty}^{a_<} + \int_{a_>}^{c} \right\} \frac{\psi(v)}{v-c} \, dv >
\]
\[
> \frac{1}{a_< - c} \int_{-\infty}^{a_<} \psi(v) \, dv + \frac{1}{a_> - c} \int_{a_>}^{c} \psi(v) \, dv = \frac{(a_> - a_<) \xi / k^2}{(c - a_>) (c - a_<)}
\]
since \( \int_{-\infty}^{a_<} \psi(v) \, dv = \int_{a_>}^{a_>} \psi(v) \, dv = \xi / k^2 \). Similarly the inequality
\[
\int_{c}^{\infty} \frac{\psi(v)}{v-c} \, dv > \left\{ \int_{c}^{b_<} + \int_{b_>}^{\infty} \right\} \frac{\psi(v)}{v-c} \, dv >
\]
\[
> \frac{1}{b_< - c} \int_{c}^{b_<} \psi(v) \, dv + \frac{1}{b_> - c} \int_{b_>}^{\infty} \psi(v) \, dv = \frac{(b_> - b_<) \eta / k^2}{(b_< - c) (b_> - c)}
\]
is verified where \( \int_c^b \psi(v) \, dv = \int_c^b \psi(v) \, dv = \eta/k^2 \). To complete the proof of Lemma it suffices to apply Statement 2. A somewhat different type condition of instability is given by

**Lemma 6** If there exist \( \sigma \in (0, c - a) \) and \( \tau \in (0, b - c) \) such that the inequality

\[
\frac{\sigma f_0(a + \sigma)}{(c - a - \sigma)(c - a)} + \frac{\tau f_0(b - \tau)}{(b - c - \tau)(b - c)} > k^2
\]

holds true then the right half-plane \( \text{Re} \lambda > 0 \) contains just one singular value \( \lambda = \lambda(k) \) corresponding to given \( k \in \mathbb{R} \setminus \{0\} \).

Really taking into account that \( \int_{-\infty}^c \psi(v) \, dv = \int_c^a \psi(v) \, dv \) one has

\[
\int_{-\infty}^c \frac{\psi(v)}{v - c} \, dv > \frac{1}{a - c} \int_c^a \psi(v) \, dv + \int_c^c \frac{1}{v - c} \int_a^c \psi(v) \, dv = \int_c^c \left( \frac{1}{v - c} - \frac{1}{a - c} \right) \psi(v) \, dv
\]

where \( \psi(v) < 0 \) for \( v \in (a, c) \) and hence

\[
\int_{-\infty}^c \frac{\psi(v)}{v - c} \, dv > \int_{a+\sigma}^c \left( \frac{1}{v - c} - \frac{1}{a - c} \right) \psi(v) \, dv \geq \left( \frac{1}{c - a - \sigma} - \frac{1}{c - a} \right) \int_c^{a+\sigma} \psi(v) \, dv = \frac{\sigma f_0(a + \sigma)/k^2}{(c - a - \sigma)(c - a)}.
\]

Along the same lines we obtain the inequality

\[
\int_c^\infty \frac{\psi(v)}{v - c} \, dv > \int_c^{b-\tau} \left( \frac{1}{v - c} - \frac{1}{b - c} \right) \psi(v) \, dv \geq \left( \frac{1}{b - c - \tau} - \frac{1}{b - c} \right) \int_c^{b-\tau} \psi(v) \, dv = \frac{\tau f_0(b - \tau)/k^2}{(b - c - \tau)(b - c)}
\]

and thus the required sufficient condition proves to be a straightforward corollary of Statement 2.

**REFERENCES**

1. V. P. Maslov, M. V. Fedoryuk  The linear theory of Landau damping, Sbornik Mathematics, 1986, v. 55, № 2, p. 437-465.

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2. P. R. Halmos  A Hilbert space problem book, Springer-Verlag, 1982.

3. T. Kato  Perturbation theory for linear operators  Springer-Verlag, 1966.

4. T. H. Stix  Waves in plasmas, New York, American Institute of Physics, 1992.

5. M. A. Lavrentiev, B. V. Shabat  Methods of the theory of functions of the complex variable, Moscow, Nauka, 1973.

6. Penrose O.  Electrostatic instabilities of a uniform non-Maxwellian plasma, Physics of Fluids, 1960, v. 2, № 2, p. 258-264.