ON THE TOPOLOGY OF SOME QUASI-PROJECTIVE SURFACES

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ABSTRACT. Let $X$ be a surface with isolated singularities in the complex projective space $\mathbb{P}^3$ and let denote $Y$ the smooth part of $X$. In this note we discuss some aspects of the topology of such quasi-projective surfaces $Y$.

1. Introduction and statements of results

Surfaces theory is a classical subject, with a very rich history and a number of excellent textbooks, as for instance [2], [3]. It is quite surprising that new open questions arise even in such a classical subject, and one of the purposes of this note is to state such an open question in Example 2.2 related to the cubic surfaces in $\mathbb{P}^3$. These surfaces have been classified already by Cayley, see for a modern presentation [4]. Another open question in relation to the Zariski sextic with 6 cusps appears in Example 2.3.

Let $X$ be surface with isolated singularities in the complex projective space $\mathbb{P}^3$. Then it is well known that $X$ is simply-connected, see for instance [7]. Let $Y$ denote the smooth part of $X$. So $Y$ is obtained from $X$ by removing a finite number of points. However, unlike the case when $X$ is smooth, this operation alters sometimes the fundamental groups and we may get quasi-projective surfaces $Y$ with $\pi_1(Y) \neq 0$. We omit the base points in this note, since our spaces $Y$ are path-connected, hence the isomorphism class of $\pi_1(Y, y)$ is independent of $y \in Y$. For related results on fundamental groups of surfaces we refer to [1], [12] and [13].

The first result describes the first integral homology group $H_1(Y)$ of the surface $Y$, which is exactly the abelianization of the fundamental group $\pi_1(Y)$. In the sequel $\text{Tors}$ denotes the torsion part of a finitely generated abelian group.

**Theorem 1.1.** Let $X$ be a surface with isolated singularities in $\mathbb{P}^3$, let $Z$ be the singular set of $X$ and set $Y = X \setminus Z$. Then one has the following.

(i) $H_1(Y) = H^3(X)$. In particular, if $X$ is a $\mathbb{Q}$-manifold, e.g. when $X$ has only simple singularities of type $A_n$, $D_n$, $E_6$, $E_7$ and $E_8$, then $H_1(Y) = \text{Tors} H_2(X)$ is a finite group.

(ii) $H_3(Y)$ is a free abelian group of rank given by $|Z| - 1$. In particular, the surface $Y$ is not affine if $X$ has at least two singular points.

In fact, the surface $Y$ is never affine, since a regular function $\phi$ defined on $Y$ extends to $X$, as $X$ is normal, and hence $\phi$ has to be constant.

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The next result shows that there is a geometrically induced epimorphism

\[ \Gamma_g \to \pi_1(Y), \]

where \( \Gamma_g \) is the fundamental group of a smooth plane curve of genus

\[ g = \frac{(d - 1)(d - 2)}{2}. \]

**Proposition 1.2.** (i) For a generic plane \( H \) in \( \mathbb{P}^2 \), the intersection \( C = X \cap H \) is a smooth curve contained in \( Y \) and the inclusion \( i : C \to Y \) induces an epimorphism

\[ i_* : \pi_1(C) \to \pi_1(Y). \]

(ii) For any plane \( H \) in \( \mathbb{P}^2 \) such that the intersection \( C = X \cap H \) is a (possibly singular) curve contained in \( Y \), the inclusion \( i : C \to Y \) induces an epimorphism

\[ i_* : \pi_1(C) \to \pi_1(Y). \]

In particular, if \( Y \) contains a rational cuspidal plane curve \( C \), then \( \pi_1(Y) = 0 \).

**Corollary 1.3.** Let \( X \) be a surface with isolated singularities in \( \mathbb{P}^3 \). If \( X \) is a surface of degree 3, then the fundamental group \( \pi_1(Y) \) of its smooth part \( Y \) is abelian. More precisely, denote by \( X(4A_1) \) the cubic surface in \( \mathbb{P}^3 \) having as singularities 4 nodes \( A_1 \), and similarly for \( X(3A_2), X(A_1A_5) \) and \( X(2A_1A_3) \). Then the corresponding smooth quasi-projective surfaces \( Y(4A_1), Y(3A_2), Y(A_1A_5) \) and \( Y(2A_1A_3) \) have the following fundamental groups:

\[ \pi_1(Y(4A_1)) = \pi_1(Y(A_1A_5)) = \pi_1(Y(2A_1A_3)) = \mathbb{Z}/2\mathbb{Z} \text{ and } \pi_1(Y(3A_2)) = \mathbb{Z}/3\mathbb{Z}. \]

We have also the following.

**Proposition 1.4.** Let \( X \) be a surface with isolated singularities in \( \mathbb{P}^3 \), let \( Z \) be the singular set of \( X \) and set \( Y = X \setminus Z \). For each singular point \( z \in Z \), let \( L_z \) denote the link of the singularity \( (X, z) \). Then there is morphism

\[ \Pi_{z \in Z} \pi_1(L_z) \to \pi_1(Y), \]

where \( \Pi_{z \in Z} \pi_1(L_z) \) denotes the free product of the family of groups \( (\pi_1(L_z))_{z \in Z} \), whose image is not contained in any normal subgroup of \( \pi_1(Y) \).

One can ask about the mixed Hodge structures on the surfaces \( Y \). Here is the answer.

**Theorem 1.5.** Let \( X \) be a surface with isolated singularities in \( \mathbb{P}^3 \), let \( Z \) be the singular set of \( X \) and set \( Y = X \setminus Z \). Then \( H^3_c(Y) = H^3(X) \) is a pure Hodge structure of weight 3 and by duality, \( H^1(Y) \) is a pure Hodge structure of weight 1. In particular, there is no surjective morphism with a generic connected fiber from \( Y \) to \( \mathbb{P}^1 \setminus A \), where \( A \) is a finite set of cardinal \( |A| \geq 2 \).

The cohomology groups \( H^2_c(Y) = H^2(X) \) have a more complicated mixed Hodge structure, in general with several possible weights, see for more on this subject \[8\] and \[10\].

The Hodge theoretic result in Theorem \[15\] has the following pure topological consequence. Let \( C : g(x, y, z) = 0 \) be a reduced curve in \( \mathbb{P}^2 \) of degree \( d \). Consider
the surfaces \(X_C : g(x, y, z) + t^d = 0\), whose singularities are the degree \(d\) suspension of the singularities of the curve \(C\). Let \(Y_C\) be the smooth part of \(X_C\) as above and denote by \(F_C\) the Milnor fiber of \(g\), namely the affine smooth surface \(F_C : g(x, y, z) = 1\) in \(\mathbb{C}^3\). Then clearly \(F_C \subset Y_C\) is a Zariski open subset and hence the inclusion \(j : F_C \to Y_C\) induces an epimorphism \(\pi_1(F_C) \to \pi_1(Y_C)\). By duality, we get a monomorphism \(j^* : H^1(Y_C, \mathbb{Q}) \to H^1(F_C, \mathbb{Q})\).

**Proposition 1.6.** The image of the monomorphism \(j^* : H^1(Y_C, \mathbb{Q}) \to H^1(F_C, \mathbb{Q})\) is exactly \(H^1(F_C, \mathbb{Q}) \neq 1\), the non-fixed part of \(H^1(F_C, \mathbb{Q})\) under the monodromy action.

This result allows us to construct many examples of surfaces \(X_C\) such that \(H^1(Y_C, \mathbb{Q})\) (and presumably \(\pi_1(Y)\)) is quite large, e.g. using as \(C\) various line arrangements in \(\mathbb{P}^2\), see [18], [19], [14] for various monodromy computations in this case. To increase these groups, one may also use non-linear arrangements as well, as described for instance in Example 5.14 in [9]. One example using Zariski sextic curve with 6 cusps is given in Example 2.3 below.

Some open questions appear in Example 2.2 and Example 2.3. A major open question is to develop a general strategy for the computation of the fundamental groups for this class of surfaces.

This note gives a number of (very limited) answers to questions that Ciro Ciliberto asked me some time ago, and for which I would like to thank him.

2. The proofs

We consider first Theorem 1.1. We apply Lefschetz duality theorem, see for instance [16], p. 297 to the compact relative 4-manifold \((X, Z)\), where \(Z\) is the finite set of singular points of the surface \(X\). Since any pair of algebraic sets is triangulable, it follows that the pair \((X, Z)\) is taut, see [16], p. 291, and hence we have an isomorphism \(H_k(Y) = H^{4-k}(X, Z)\). To prove the first claim (i), we take \(k = 1\) and the long exact sequence of the pair \((X, Z)\) yields an isomorphism \(H^3(X, Z) = H^3(X)\). If \(X\) is a \(\mathbb{Q}\)-manifold, it follows that \(b_3(X) = b_1(X) = 0\) and hence \(H^3(X)\) is a finite group. It remains to use the standard fact that \(\text{Tors} H^3(X) = \text{Tors} H^3(X),\) see [16], p. 244. For the classification of simple singularities \(A, D, E\) and their properties we refer to [4], [7]. To prove the claim (ii), we take \(k = 3\) in the above isomorphism and get \(H_3(Y) = H^1(X, Z)\). Then the long exact sequence of the pair \((X, Z)\) yields

\[b_3(Y) = |Z| - 1.\]

We conclude by using the fact that an affine surface \(W\) has the homotopy type of a 2-dimensional CW-complex and hence \(H^3(W) = 0\).

The proof of Proposition 1.2 follows from the Zariski theorem of Lefschetz type stated for instance in [7], p. 25 and the fact that \(X\) admits the obvious Whitney regular stratification given by \(Y\) and the finite set \(Z\), see for instance in [7], p. 5. For the part (ii), one has to use the careful description of ”good” hyperplanes in this case given in loc.cit. Moreover, an irreducible projective curve is simply-connected.
if and only if it is a rational cuspidal curve, i.e. $C$ is rational and any singular point of $C$ is unibranch.

To prove Corollary 1.3 for the first claim we just apply Proposition 1.2 and the fact that $\Gamma_1 = \mathbb{Z}^2$ to get that $\pi_1(Y)$ is abelian.

For the computation of the fundamental groups of the surfaces, $Y(4A_1)$, $Y(3A_2)$, $Y(A_1A_5)$ and $Y(2A_1A_3)$, we use the corresponding results for the second homology groups described in [7], p. 165.

The proof of Proposition 1.4 follows by applying the van Kampen theorem to the open covering $Y, Z'$ of $X$, where $Z'$ is obtained as follows. Take a minimal tree $T$ formed of simple non-intersecting arcs connecting the points in $Z$. Add for each $z \in Z$ a small contractible neighborhood $B_z$ of $z$ in $X$ such that $B_z^* = B_z \setminus \{z\}$ is homotopically equivalent to the link $L_z$. Then $Z'$ is a tubular open neighborhood of $T \cup \cup_{z \in Z} B_z$. It follows that $Y \cap Z'$ has the homotopy type of the join of the links $L_z$, and hence

$$\pi_1(Y \cap Z') = \Pi_{z \in Z} \pi_1(L_z).$$

On the other hand, $Z'$ is contractible and $X$ is simply-connected, so van Kampen theorem implies that the inclusion $Y \cap Z' \to Y$ induces a morphism $\pi_1(Y \cap Z') \to \pi_1(Y)$ whose image is not contained in any normal subgroup of $\pi_1(Y)$.

We consider now Theorem 1.5. Note that the first part of this proof gives an alternative proof for the claim (i) in Theorem 1.1. The exact sequence with compact supports for the pair $(X, Z)$ yields the isomorphism $H^3_c(Y) = H^3(X)$ of MHS (short for mixed Hodge structures). The result follows using the fact that $H^3(X)$ is a pure Hodge structure, see [17]. For duality between $H^3_c(Y)$ and $H^1(Y)$ we refer to [15]. To show that there is no morphism $\phi : Y \to \mathbb{P}^1 \setminus A$ with the claimed properties, it is enough to notice that such a morphism induces a monomorphism $\phi^* : H^1(\mathbb{P}^1 \setminus A, \mathbb{Q}) \to H^1(Y, \mathbb{Q})$, and $H^1(\mathbb{P}^1 \setminus A, \mathbb{Q})$ is a nonzero Hodge structure of pure weight 2, see also [12] and [13] for details and the relation between such morphisms $\phi$ and the characteristic and the resonance varieties of $Y$.

Finally, to prove Proposition 1.6 we recall that we have a splitting

$$H^1(F_C, \mathbb{Q}) = H^1(F_C, \mathbb{Q})_1 \oplus H^1(F_C, \mathbb{Q})_{\neq 1},$$

where $H^1(F_C, \mathbb{Q})_1$ has pure weight 2 and $H^1(F_C, \mathbb{Q})_{\neq 1}$ has pure weight 1, see [10]. Moreover, it is shown in [10] and in [11] that

$$\dim H^1(Y_C, \mathbb{Q}) = \dim H^3(X_C, \mathbb{Q}) = \dim H^1(F_C, \mathbb{Q})_{\neq 1}.$$

This clearly completes the proof.

Remark 2.1. It follows from Theorem 1.1 (ii) that the three surfaces $X(4A_1)$, $X(A_1A_5)$ and $X(2A_1A_3)$ have distinct third Betti numbers, namely 3, 1 and 2, hence they are not homotopically equivalent to each other.

Example 2.2. It follows from Corollary 1.3 that each of the surfaces, $Y(4A_1)$, $Y(3A_2)$, $Y(A_1A_5)$ and $Y(2A_1A_3)$ has a finite unramified cover which is a simply-connected surface, i.e. the corresponding universal covering. In the case of the surface $Y(3A_2)$, we can choose the equation for $X(3A_2)$ to be

$$f = xyz - t^3 = 0.$$
and hence the map \( p : \mathbb{P}^2 \to X(3A_2) \) given by

\[
p(u : v : w) = (u^3 : v^3 : w^3 : uvw)
\]

is ramified precisely over the singular set \( Z \), see also [7], p. 166. Hence the universal cover of \( Y(3A_2) \) is obtained from \( \mathbb{P}^2 \) by deleting 3 points. It would be nice to have similar descriptions for the other three universal covering surfaces.

**Example 2.3.** Here is an example of a surface \( X \) such that the fundamental group \( \pi_1(Y) \) is infinite. Let \( X \) be the surface given by

\[
(x^2 + y^2)^3 + (y^3 + z^3)^2 + t^6 = 0,
\]

the degree 6 cover of \( \mathbb{P}^2 \) ramified over the Zariski sextic with 6 cusps on a conic. It is well known that \( b_3(X) = 2 \), see for instance in [7], p. 210. It would be interesting to check whether \( H^3(X) = \mathbb{Z}^2 \) and also to determine the fundamental group \( \pi_1(Y) \) in this classical example.

**Remark 2.4.** When \( b_1(Y) = b_3(X) > 0 \), then one can use Theorem 2 in [13] to compute the germs at the origin of the characteristic varieties \( V^1_r(Y) \), in terms of the resonance varieties germs \( R^1_r(\tilde{X})_0 \), where \( \tilde{X} \) is a resolution of singularities for \( X \).

**Remark 2.5.** Some of the considerations in this paper can be applied to a complete intersection surface \( X \) with isolated singularities in \( \mathbb{P}^n \) with \( n \geq 4 \). The case when \( X \) is the intersection of two quadrics in \( \mathbb{P}^4 \) is considered in Proposition 4.3 in [5], where the corresponding group \( H_1(Y) \) is computed and found to be \( \mathbb{Z}/2\mathbb{Z} \) in two cases, namely for \( Y(4A_1) \) and for \( Y(2A_1A_3) \), with an obvious notation. A generic hyperplane section of \( X \) in this case is again a curve of genus one, hence we get as in Corollary 1.3 that one has

\[
\pi_1(Y) = H_1(Y) = \mathbb{Z}/2\mathbb{Z},
\]

in these cases as well. It would be interesting to find an analog of Proposition 1.6 in the case of complete intersections of codimension > 1. One can also ask if the corresponding degree 2 universal covering spaces associated with \( Y(4A_1) \) and \( Y(2A_1A_3) \) have a simple geometrical description.

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