Inflation with Gauss-Bonnet coupling

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Abstract

We consider inflationary models with the inflaton coupled to Gauss-Bonnet term assuming a special relation \( \delta_1 = 2 \lambda \epsilon_1 \) between the two slow-roll parameters \( \delta_1 \) and \( \epsilon_1 \). For the slow-roll inflation, the assumed relation leads to the reciprocal relation between the Gauss-Bonnet coupling function \( \xi(\phi) \) and the potential \( V(\phi) \), and it leads to the consistency relation \( r = 16(1 - \lambda) \epsilon_1 \) that reduces the tensor to scalar ratio \( r \) by a factor of \( 1 - \lambda \). For the constant-roll inflation, we derive the analytical expressions for the scalar and tensor power spectra, the scalar and tensor spectral tilts and the tensor to scalar ratio to the first order of \( \epsilon_1 \) by using the method of Bessel function approximation. Comparing the derived \( n_s-r \) with the observations, we obtain the constraints on the model parameters \( \eta \) and \( \lambda \).

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I. INTRODUCTION

The flatness and horizon problems in standard cosmology can by solved by cosmic inflation [1–5], and the seeds of the large scale structure of our Universe are sowed by the quantum fluctuations of the inflaton during inflation that leave imprints on the cosmic microwave background radiation [6–11]. The simplest inflation model is a canonical scalar field with a flat potential minimally coupled to gravity. Since the current observations cannot tell the nature of the scalar field, there exist many other kinds of inflation models, one of them is the Gauss-Bonnet inflation. The Gauss-Bonnet term that is induced from the superstring theory, provides the possibility of avoiding the singularity problem of the Universe [12–16]. Gauss-Bonnet coupling is also a subclass of the Horndeski theory in which equations of motion are at most of the second order in the derivative of both the metric $g_{\mu\nu}$ and the scalar field $\phi$ in four-dimensions [17, 18]. The inflation models with the Gauss-Bonnet coupling have been studied in [19–38]. Among them, in Refs. [19–22] the authors calculated the scalar tilt $n_s$ and the tensor to scalar ratio $r$ under the slow-roll condition $\epsilon_i \ll 1$ and $\delta_i \ll 1$. The authors also find that if the coupling function $\xi(\phi)$ to the Gauss-Bonnet term and the potential $V(\phi)$ satisfy the relation $V(\phi)\xi(\phi) = \text{const}$, then the tensor to scalar ratio $r$ is reduced so that the result can be consistent with the observations [39, 40]. In this paper, we show that under the slow-roll approximation, the reciprocal relation between the coupling function $\xi(\phi)$ and the potential $V(\phi)$ can be derived from the relation

$$\delta_1 = 2\lambda\epsilon_1,$$

(1)

where $\lambda$ is an order one constant.

Besides the slow-roll inflationary scenario there exists a constant-roll inflationary scenario [41–59] in which one of the slow-roll parameter is regarded as constant instead of small and the slow-roll condition may be violated. The constant-roll inflation has a richer physics than the slow-roll inflation does. For example, it can generate large local non-Gaussianity and the curvature perturbation may grow on the super-horizon scales [43, 44, 60]. Furthermore, it can be used to generate the primordial black holes [61, 62]. In this paper, we study the constant-roll inflation with the Gauss-Bonnet coupling, with the assumed relation (1). To discuss more general cases, we introduce the slow-roll parameter $\tilde{\eta}_A = \epsilon_2 - A\epsilon_1$ with a constat $A$, and assume $\tilde{\eta}_A$ to be a constant. For $A = 0$, we have $\tilde{\eta}_0 = \epsilon_2$; and for $A = 2$, we have
\[ \tilde{\eta}_2 = -2\eta_H, \] so different constant-roll model corresponds to different choice of the value of \( A \).

This paper is organized as follows. In sections II A and II B, we briefly review the slow-roll Gauss-Bonnet inflation. In section II C, we show that under the slow-roll condition, the relation \( \xi(\phi)V(\phi) = \text{const} \) can be derived from the condition (1). We also discuss the effects of the Gauss-Bonnet coupling on the natural inflation and the \( \alpha \)-attractor with the condition (1). In section III, we study the constant-roll inflation models with the Gauss-Bonnet coupling under the condition (1). The paper is concluded in section IV.

II. THE SLOW-ROLL GAUSS-BONNET INFLATION

A. The background

In this section, we review the slow-roll inflation with the Gauss-Bonnet coupling. The action for the Gauss-Bonnet inflation is

\[ S = \frac{1}{2} \int \sqrt{-g} d^4x \left[ R - g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2V(\phi) - \xi(\phi)R^2_{\text{GB}} \right], \]

where \( R^2_{\text{GB}} = R^\mu_\nu R^\nu_\rho R^\rho_\sigma - 4R_{\mu\nu}R^{\mu\nu} + R^2 \) is the Gauss-Bonnet term which is a pure topological term in four dimensions, and \( \xi(\phi) \) is the Gauss-Bonnet coupling function. With the Friedmann-Robertson-Walker metric, the field equations are

\[ 6H^2 = \dot{\phi}^2 + 2V + 24\dot{\xi}H^3, \]

\[ 2\dot{H} = -\dot{\phi}^2 + 4\ddot{\xi}H^2 + 4\dot{\xi}H \left( 2\dot{H} - H^2 \right), \]

\[ \left( \ddot{\phi} + 3H\dot{\phi} \right) + V,\phi + 12\xi,\phi H^2 \left( \dot{H} + H^2 \right) = 0. \]

where a dot denotes the derivative with respect to time \( t \), and \( V,\phi = dV/d\phi \).

For the slow-roll inflation, we introduce the following slow-roll conditions

\[ \dot{\phi}^2 \ll V(\phi), \quad |\dot{\phi}| \ll 3H|\dot{\phi}|, \quad 4H|\ddot{\xi}| \ll 1, \quad |\dddot{\xi}| \ll H|\ddot{\xi}|. \]

Under these slow-roll conditions, Eqs. (3), (4) and (5) become

\[ H^2 \approx \frac{1}{3}V, \]

\[ \dot{H} \approx -\frac{1}{2}\dot{\phi}^2 - 2\ddot{\xi}H^3, \]

\[ \dot{\phi} \approx -\frac{1}{3H}(V,\phi + 12\xi,\phi H^4). \]
To quantify the slow-roll conditions, we introduce the hierarchy of Hubble flow parameters

\[ \epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_{i+1} = \frac{d\ln|\epsilon_i|}{d\ln a}, \quad i \geq 1, \tag{10} \]

and the hierarchy of the flow parameters for the coupling function \[ \delta_1 = 4\dot{\xi}H, \quad \delta_{i+1} = \frac{d\ln|\delta_i|}{d\ln a}, \quad i \geq 1. \tag{11} \]

In terms of these slow-roll parameters, the slow-roll conditions become

\[ \epsilon_1 \ll 1, \quad |\epsilon_2| \ll 1, \quad |\delta_1| \ll 1, \quad |\delta_2| \ll 1. \tag{12} \]

With the help of Eqs. (7), (8) and (9), the slow-roll parameters can be expressed by the potential \( V(\phi) \) and the coupling function \( \xi(\phi) \) as

\[ \epsilon_1 \approx \frac{Q V_{,\phi}}{2 V}, \tag{13} \]
\[ \epsilon_2 \approx -Q \left( \frac{V_{,\phi\phi}}{V_{,\phi}} - \frac{V_{,\phi}}{V} + \frac{Q_{,\phi}}{Q} \right), \tag{14} \]
\[ \delta_1 \approx -\frac{4}{3} \xi_{,\phi} Q V, \tag{15} \]
\[ \delta_2 \approx -Q \left( \frac{\xi_{,\phi\phi}}{\xi_{,\phi}} + \frac{V_{,\phi}}{V} + \frac{Q_{,\phi}}{Q} \right), \tag{16} \]

where \( Q = V_{,\phi}/V + 4\xi_{,\phi} V/3. \) The e-folding number \( N \) at the horizon exit before the end of the inflation can also be expressed by the potential and the coupling function

\[ N(\phi) \approx \int_{\phi_e}^{\phi} \frac{3V}{3V_{,\phi} + 4\xi_{,\phi} V^2} d\phi = \int_{\phi_e}^{\phi} \frac{d\phi}{Q}. \tag{17} \]

B. The power spectrum

1. The scalar perturbation

In the flat gauge \( \delta \phi = 0 \), the gauge invariant scalar perturbation becomes the curvature perturbation which is related to the metric perturbation by \( \delta g_{ij} = a^2(1 + 2\zeta)\delta_{ij} \). The Fourier component of the mode function \( v_k = z_s\zeta_k \) for the curvature perturbation \( \zeta \) satisfies the Mukhanov-Sasaki equation

\[ v''_k + \left( k^2 + \frac{z_s''}{z_s} \right) v_k = 0, \tag{18} \]
where a prime represents the derivative with respect to the conformal time \( \tau = \int a^{-1} dt \). The sound speed \( c_s \) and \( z_s \) are

\[
c_s^2 = 1 - \Delta^2 \frac{2\epsilon_1 + \frac{1}{2} \delta_1 (1 - 5\epsilon_1 - \delta_2)}{F},
\]

\[
z_s^2 = a^2 \frac{F}{(1 - \frac{1}{2}\Delta)^2},
\]

where \( \Delta = \delta_1/(1 - \delta_1) \), \( F = 2\epsilon_1 - \delta_1 (1 + \epsilon_1 - \delta_2) + 3\Delta\delta_1/2 \), and

\[
\frac{z_s''}{z_s} = a^2 H^2 \left[ 2 - \epsilon_1 + 3 \frac{\dot{F}}{2HF} + 3 \frac{\dot{\Delta}}{2H(1 - \frac{1}{2}\Delta)} + \frac{1}{2} \frac{\ddot{\Delta}}{H^2(1 - \frac{1}{2}\Delta)} - \frac{1}{4} \frac{\dot{F}^2}{H^2F^2}
\right.

\[
+ \frac{1}{2} \frac{\ddot{\Delta}}{H^2(1 - \frac{1}{2}\Delta)^2} + \frac{1}{2} \frac{\dot{\Delta}}{H(1 - \frac{1}{2}\Delta)} \dot{F}
\left. \right] = \frac{1}{\tau^2} \left( \nu^2 - \frac{1}{4} \right).
\]

With the slow-roll conditions (6), we get

\[
aH \approx -\frac{1}{(1 - \epsilon_1)\tau}.
\]

Substituting Eq. (22) into Eq. (21), we obtain

\[
\nu = \frac{3}{2} + \epsilon_1 + \frac{2\epsilon_1\epsilon_2 - \delta_1\delta_2}{2(2\epsilon_1 - \delta_1)}.
\]

Assuming that \( \nu \) is almost a constant, we get the power spectrum for the scalar perturbation expressed by the Hankel function

\[
\mathcal{P}_R = \frac{k^3}{2\pi^2} |\zeta_k|^2 = \frac{H^2}{8\pi} \frac{(1 - \Delta/2)^2}{(1 - \epsilon_1) F c_s^3} \left[ H^{(1)}_{\nu} \left( \frac{1}{1 - \epsilon_1 \frac{c_s k}{aH}} \right) \right]^2 \left( \frac{c_s k}{aH} \right)^3.
\]

On super-horizon scales, \( c_s k \ll aH \), using the asymptotic behaviour of the Hankel function, the power spectrum for the scalar perturbation becomes

\[
\mathcal{P}_R = 2^{2\nu - 3} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{(1 - \Delta/2)^2}{F c_s^3} \left( \frac{H}{2\pi} \right)^2 (1 - \epsilon_1)^{2\nu - 1} \left( \frac{c_s k}{aH} \right)^{3 - 2\nu} \bigg|_{c_s k = aH}.
\]

Therefore, the scalar spectral tilt is

\[
n_s - 1 = \frac{d\ln \mathcal{P}_R}{d\ln k} = 3 - 2\nu
\]

\[
= -2\epsilon_1 - \frac{2\epsilon_1\epsilon_2 - \delta_1\delta_2}{2\epsilon_1 - \delta_1}.
\]
2. The tensor perturbation

For the tensor perturbation $\delta g_{ij} = a^2 h_{ij}$, the mode function $u^\lambda_k(\tau) = z_T h^\lambda_k/2$ satisfies the equation [64-67]

$$\frac{d^2 u^\lambda_k}{d\tau^2} + \left( c^2 T k^2 - \frac{z''_T}{z_T} \right) u^\lambda_k = 0,$$

(27)

where “$\lambda$” stands for the “$+$” or “$\times$” polarizations and

$$z^2_T = a^2 (1 - \delta_1),$$

(28)

$$c^2_T = 1 + \Delta (1 - \epsilon_1 - \delta_2).$$

(29)

In terms of the slow-roll parameters, we have

$$\frac{z''_T}{z_T} = a^2 H^2 \left[ 2 - \epsilon_1 - \frac{3}{2} \Delta \delta_2 - \frac{1}{2} \Delta \delta_2 (-\epsilon_1 + \delta_2 + \delta_3) - \frac{1}{4} \Delta^2 \delta_2^2 \right]$$

(30)

By using the slow-roll conditions (6) and with the help of Eq. (22), we get

$$\mu \approx \frac{3}{2} + \epsilon_1.$$  

(31)

Assuming that $\mu$ is almost a constant, following the same procedure as that in scalar perturbation, we obtain the power spectrum for the tensor perturbation

$$P_T = \frac{k^3}{2\pi^2} \sum_{\lambda = +, \times} \left| \frac{2u^\lambda_k}{z_T} \right|^2 = \frac{H^2}{\pi(1 - \epsilon_1)(1 - \delta_1)c^2_T} \left[ H^{(1)}_\mu \left( \frac{1}{1 - \epsilon_1} \frac{c_T k}{aH} \right) \right]^2 \left( \frac{c_T k}{aH} \right)^3.$$  

(32)

On super-horizon scales, $c_T k \ll aH$, we have

$$P_T = \frac{2^{2\mu}}{(1 - \delta_1)c^2_T} \left[ \frac{\Gamma(\mu)}{\Gamma(3/2)} \right]^2 \left( \frac{H}{2\pi} \right)^2 (1 - \epsilon_1)^{2\mu-1} \left( \frac{c_T k}{aH} \right)^{3-2\mu} \bigg|_{c_T k = aH}.$$  

(33)

The tensor spectral tilt is [20]

$$n_T = \frac{d\ln P_T}{d\ln k} = 3 - 2\mu = -2\epsilon_1,$$  

(34)

and the tensor to scalar ratio is [20]

$$r = \frac{P_T}{P_R} = 16\epsilon_1 - 8\delta_1.$$  

(35)

The time at the horizon crossing, $aH = c_s k$, for the scalar perturbation is not exactly the same as that for the tensor perturbation, $aH = c_T k$, however to the lowest order of the slow-roll approximation, this difference is unimportant [20].
C. The models

In [21], the authors studied two special models with \( V(\phi) = V_0 \exp(-p\phi), \) \( \xi(\phi) = \xi_0 \exp(p\phi) \) and \( V(\phi) = V_0 \phi^p, \) \( \xi(\phi) = \xi_0 \phi^{-p}, \) respectively. The potentials and coupling functions in the two models satisfy the relation \( V(\phi)\xi(\phi) = \text{const}. \) Now we show that the reciprocal relation between the coupling function \( \xi(\phi) \) and the potential \( V(\phi) \) can be obtained from the condition (1) under the slow-roll conditions (6). Substituting Eqs. (13) and (15) into Eq. (1), and choosing the integration constant to be zero, we get

\[
\xi(\phi) = \frac{3\lambda}{4V(\phi)}. \tag{36}
\]

By using the condition (1), and the definitions (10) and (11), we obtain the relations for the other slow-roll parameters

\[
\delta_{i+1} = \epsilon_{i+1}, \quad i \geq 1. \tag{37}
\]

Substituting the relations (1) and (37) into Eqs. (26) and (35), we obtain

\[
n_s - 1 = -2\epsilon_1 - \epsilon_2, \tag{38}
\]

\[
r = 16(1 - \lambda)\epsilon_1. \tag{39}
\]

The result for the scalar spectral tilt \( n_s \) is the same as that in the canonical case without the Gauss-Bonnet coupling, but the result for the tensor to scalar ratio \( r \) is reduced by a factor of \( 1 - \lambda \) comparing with the canonical case without the Gauss-Bonnet coupling.

In the canonical case without the Gauss-Bonnet coupling, we have the Hubble flow and horizon flow slow-roll parameters, and different definitions give the same result. In the case with the Gauss-Bonnet coupling, different definitions may give different results. For comparison, we introduce the following slow-roll parameters

\[
\eta_H = -\frac{\ddot{H}}{2HH}, \quad \eta_{H\phi} = \frac{2H\phi\dot{\phi}}{H}, \quad \eta_\phi = -\frac{\ddot{\phi}}{H\phi}, \quad \eta_\xi = \frac{\ddot{\xi}}{H\xi}. \tag{40}
\]

Furthermore, we also introduce the potential slow-roll parameters

\[
\epsilon_V = \frac{1}{2} \left( \frac{V_{,\phi}}{V} \right)^2, \quad \eta_V = \frac{V_{,\phi\phi}}{V}. \tag{41}
\]

By using the condition (1), and the slow-roll conditions (6), to the first order of approxima-
tion, we obtain the relations

\begin{align}
\epsilon_V & \approx \frac{\epsilon_1}{1 - \lambda}, \\
\eta_H & = \frac{1}{2}(\epsilon_2 - 2\epsilon_1) \approx \eta_\phi, \tag{42} \\
\eta_{H\phi} & \approx -\frac{1}{2(1 - \lambda)}(\epsilon_2 - 2\epsilon_1), \\
\eta_\xi & = \epsilon_2 + \epsilon_1, \\
\eta_V & \approx -\frac{1}{2(1 - \lambda)}(\epsilon_2 - 4\epsilon_1). \tag{46}
\end{align}

These relations can be parameterized as

\[ \tilde{\eta}_A = \epsilon_2 - A\epsilon_1, \tag{47} \]

where \( A \) is a constant. For \( A = 0 \), we get \( \tilde{\eta}_0 = \epsilon_2 \). For \( A = 2 \), we get \( \tilde{\eta}_2 = -2\eta_H \). For \( A = 4 \), we get \( \tilde{\eta}_4 = -2(1 - \lambda)\eta_V \) under the slow-roll conditions.

1. **natural inflation**

For the natural inflation

\[ V = V_0 \left[ 1 + \cos \left( \frac{\phi}{f} \right) \right], \tag{48} \]

the potential slow-roll parameters are

\begin{align}
\epsilon_V & = \frac{\sin^2(\phi/f)}{2f^2[(\cos(\phi/f) + 1)^2],} \\
\eta_V & = -\frac{\cos(\phi/f)}{f^2[(\cos(\phi/f) + 1)].} \tag{50}
\end{align}

The value of the inflaton at the end of inflation is

\[ \phi_e = f \arccos \left( \frac{1 - 2\tilde{f}^2}{1 + 2\tilde{f}^2} \right), \tag{51} \]

where \( \tilde{f} = f/\sqrt{1 - \lambda} \). The e-folding number \( N \) at the horizon exit for a pivotal scale \( k_* \) is

\[ N = 2\tilde{f}^2 \ln \left[ \frac{\sin(\phi_e/2f)}{\sin(\phi_*/2f)} \right]. \tag{52} \]

The scalar spectral tilt and the tensor to scalar ratio \( r \) are,

\begin{align}
\eta_s - 1 & = -\frac{1}{f^2} \frac{1 + \exp[-\tilde{N}/\tilde{f}^2]}{1 - \exp[-\tilde{N}/\tilde{f}^2]} \tag{53}, \\
r & = \frac{8(1 - \lambda)}{\tilde{f}^2} \frac{\exp[-\tilde{N}/\tilde{f}^2]}{1 - \exp[-\tilde{N}/\tilde{f}^2]} \tag{54}.
\end{align}
FIG. 1. The marginalized 1σ, 2σ and 3σ confidence level contours for $n_s$ and $r$ from Planck 2015 and BICEP2/Keck data [39, 40] along with the observational constraints on $n_s$-$r$ for the natural inflation with different values of $\lambda$ and $f$. The solid black curve, the dashed red curve and the dashdotted blue curve represent the results for $\lambda = 0$, $\lambda = 0.3$ and $\lambda = 0.6$, respectively.

where

$$\tilde{N} = N - f^2 \ln \frac{2f^2}{1 + 2f^2}. \quad (55)$$

We compare the predictions from Eqs. (53) and (54) for different values of $f$ and $\lambda$ with the observations [39, 40] and the results are displayed in Fig. 1. For the natural inflation without the Gauss-Bonnet coupling, $\lambda = 0$, the predictions for $n_s$ and $r$ are only consistent within the 2σ confidence level of the observations because of the large tensor to scalar ratio $r$. With the help of the Gauss-Bonnet coupling, the natural inflation can be consistent with the observations at the 1σ confidence level if $\lambda$ is large enough.

2. $\alpha$-attractors

For the E-model, the potential is

$$V = V_0 \left[ 1 - \exp \left( -\sqrt{\frac{2}{3\alpha}} \phi \right) \right]^{2n}. \quad (56)$$
As an example, in the paper we consider the case \( n = 1/4 \) only. The scalar spectral tilt \( n_s \) and the tensor to scalar ratio \( r \) are

\[
n_s = 1 + 2 \frac{3\hat{\alpha} [g(N, \hat{\alpha}) + 1]}{g(N, \hat{\alpha}) + 1} - \frac{5}{6\hat{\alpha} [g(N, \hat{\alpha}) + 1]^2}, \tag{57}
\]

\[
r = \frac{4(1 - \lambda)}{3\hat{\alpha} [g(N, \hat{\alpha}) + 1]^2}, \tag{58}
\]

where \( \hat{\alpha} = \alpha/(1 - \lambda) \),

\[
g(N, \hat{\alpha}) = W_{-1} \left[ -\left( \frac{1}{6\hat{\alpha}} + \frac{v}{6\hat{\alpha}} + 1 \right) \exp \left( -1 - \frac{v + 2N + 1}{6\hat{\alpha}} \right) \right], \tag{59}
\]

with \( v = \sqrt{6\hat{\alpha} + 1} \), and \( W_{-1} \) is the lower branch of the Lambert \( W \) function. We compare the results \((57)\) and \((58)\) with the observations \([39, 40]\) and the comparisons are displayed in the left panel of Fig. 2.

For the T-model, the potential is

\[
V(\phi) = V_0 \tanh^{2n} \left( \frac{\phi}{\sqrt{6\hat{\alpha}}} \right), \tag{60}
\]

Similar to the E-model inflation, we consider the special case \( n = 1/4 \) only. The scalar spectral tilt \( n_s \) and the tensor to scalar ratio \( r \) are

\[
n_s = 1 - 2 \frac{N}{N + 1} \frac{(N + 1)\sqrt{36\hat{\alpha}^2 + (1 - 6\hat{\alpha})} - 3\hat{\alpha} - (3\hat{\alpha} - 2)N/2 + 1}{\left( \sqrt{9\hat{\alpha}^2 + (1 - 6\hat{\alpha})} + N + 1/2 \right)^2 - 9\hat{\alpha}^2}, \tag{61}
\]

\[
r = \frac{12(1 - \lambda)\hat{\alpha}}{\left( \sqrt{9\hat{\alpha}^2 + (1 - 6\hat{\alpha})} + N + 1/2 \right)^2 - 9\hat{\alpha}^2}, \tag{62}
\]

where \( \hat{\alpha} = \alpha/(1 - \lambda) \). We show the results along with the observational constraints in the right panel of Fig. 2.

III. THE CONSTANT-ROLL INFLATION

In this section, we study the constant-roll inflation by taking \( \tilde{\eta}_A \) defined in Eq. \((47)\) as a constant with the condition \((1)\). Take the derivative of Eq. \((47)\) with respect to time \( t \), and using the relation \((37)\), we obtain

\[
4\ddot{\xi} = 2\lambda \epsilon_1 [(A + 1)\epsilon_1 + \tilde{\eta}_A]. \tag{63}
\]
FIG. 2. Similar as Fig. 1. The left panel shows the results for the E-model $V = V_0 \left[1 - \exp\left(-\sqrt{2/3\alpha}\phi\right)\right]^{1/2}$, and the right panel shows the results for the T-model $V(\phi) = V_0 \tanh^{1/2} \left(\phi/\sqrt{6\alpha}\right)$.

Substituting the result into the background Eqs. (3) and (4), we obtain

$$\epsilon_1 = 2 \left(\frac{H_{,\phi}}{H}\right)^2 \frac{1 + \lambda(\tilde{\eta}_A - 1)}{1 - 2\lambda(A - 1)H_{,\phi}^2/H^2}, \quad (64)$$

and

$$\eta = \frac{2H_{,\phi}[1 + \lambda(\tilde{\eta}_A - 1) + \lambda(A - 1)\epsilon_1]}{H} - \frac{\lambda(A - 1)\tilde{\eta}_A\epsilon_1 + \lambda A(A - 1)\epsilon_1^2}{2[1 + \lambda(\tilde{\eta}_A - 1) + \lambda(A - 1)\epsilon_1]}, \quad (65)$$

Combining Eqs. (64) and (65), we get

$$\tilde{\eta}_A = (2 - A)\epsilon_1 - \frac{4H_{,\phi}[1 + \lambda(\tilde{\eta}_A - 1) + \lambda(A - 1)\epsilon_1]}{H} + \frac{\lambda(A - 1)\tilde{\eta}_A\epsilon_1 + \lambda A(A - 1)\epsilon_1^2}{1 + \lambda(\tilde{\eta}_A - 1) + \lambda(A - 1)\epsilon_1}. \quad (66)$$

where we used the relation $\epsilon_2 = 2\epsilon_1 - 2\eta_H$. Combining Eqs. (64) and (66), we may get the analytical form of $H(\phi)$ for constant $\tilde{\eta}_A$, and the potential can be obtained from the Hamilton-Jacobi equation

$$V(\phi) = (3 - 6\lambda\epsilon_1)H^2 - 2[1 + \lambda(\tilde{\eta}_A - 1) + \lambda(A - 1)\epsilon_1]^2H_{,\phi}^2. \quad (67)$$

Using the definition (10), we obtain the solution to $\epsilon_1$ in terms of the e-folding number $N$,

$$\epsilon_1(N) = \frac{\tilde{\eta}_A}{(A + \tilde{\eta}_A) \exp(\tilde{\eta}_A N) - A}, \quad (68)$$

where we used the relations $H dt = -dN$ and $\epsilon_1(0) = 1$ at the end of inflation. To find the relation between $aH$ and $\tau$, we use the relation

$$\frac{d}{d\tau} \left(\frac{1}{aH}\right) = -1 + \epsilon_1. \quad (69)$$
To the first order approximation of $\epsilon_1$, we get the following relation

$$\frac{1}{aH} \approx \left( -1 + \frac{\epsilon_1}{1 - \tilde{\eta}_A} \right) \tau. \quad (70)$$

Substitute Eq. (70) into Eq. (21), we obtain

$$\nu = \frac{1}{2} |3 + \tilde{\eta}_A| + \nu_A \epsilon_1, \quad (71)$$

where

$$\nu_A = \frac{a_0 + a_1 \tilde{\eta}_A + a_2 \tilde{\eta}_A^2 + a_3 \tilde{\eta}_A^3 + a_4 \tilde{\eta}_A^4}{2|\tilde{\eta}_A| + 3|1 - \tilde{\eta}_A|(1 - \lambda + \lambda \tilde{\eta}_A)} \quad (72)$$

and

$$a_0 = 3(2 + A)(1 - \lambda), \quad a_1 = (7 - A) + (2 + 7A)\lambda + 3\lambda^2, \quad (73)$$
$$a_2 = (2 - 2A) + 4\lambda + 5\lambda^2, \quad a_3 = -4(A\lambda + \lambda^2), \quad a_4 = -4\lambda^2.$$

Assuming that $\nu$ is almost a constant, we derive the power spectrum for the scalar perturbation

$$P_R = k^3 \frac{v_k}{2\pi} \left| z_s \right|^2 = \frac{H^2}{8\pi} \frac{1 - \tilde{\eta}_A}{(1 - \tilde{\eta}_A - \epsilon_1)} \left( 1 - \Delta/2 \right)^2 \left[ H^{(1)} \left( \frac{1 - \tilde{\eta}_A - c_s k}{1 - \tilde{\eta}_A - \epsilon_1 aH} \right) \right]^2 \left( \frac{c_s k}{aH} \right)^3. \quad (74)$$

On super-horizon scales, $c_s k \ll aH$, using the asymptotic behaviour of the Hankel function, the power spectrum for the scalar perturbation becomes

$$P_R = 2^{2\nu-3} \left[ \frac{\Gamma(\nu)}{\Gamma(3/2)} \right]^2 \frac{(1 - \Delta/2)^2}{c_s^2 F} \left( \frac{H}{2\pi} \right)^2 \left( \frac{1 - \tilde{\eta}_A - \epsilon_1}{1 - \tilde{\eta}_A} \right)^{2\nu-1} \left( \frac{c_s k}{aH} \right)^{3-2\nu} \bigg|_{aH = c_s k}. \quad (75)$$

The expression is almost the same as that for the slow-roll inflation except that the value of $\nu$ is different. Similar to the slow-roll inflation, we get the scalar spectral tilt

$$n_s - 1 = 3 - |3 + \tilde{\eta}_A| - \frac{2\nu_A \tilde{\eta}_A}{(A + \tilde{\eta}_A) \exp(\tilde{\eta}_A N) - A}. \quad (76)$$

For the case with $\lambda = 0$, the Gauss-Bonnet coupling is absent, the scalar spectral tilt becomes

$$n_s - 1 = 3 - |3 + \tilde{\eta}_A| - \frac{3(2 + A)\tilde{\eta}_A + (7 - A)\tilde{\eta}_A^2 + (2 - 2A)\tilde{\eta}_A^3}{|\tilde{\eta}_A + 3|(1 - \tilde{\eta}_A) [ (A + \tilde{\eta}_A) \exp(\tilde{\eta}_A N) - A ]}. \quad (77)$$

The results are consistent with those obtained in \[57, 58\].

Similarly, for the tensor perturbation, we obtain

$$\mu = \frac{3}{2} + \mu_1 \epsilon_1, \quad (78)$$

where

$$\mu_1 = \frac{3 + (1 - 3\lambda)\tilde{\eta}_A + 2\lambda\tilde{\eta}_A^2 + \lambda\tilde{\eta}_A^3}{3(1 - \tilde{\eta}_A)}. \quad (79)$$
which is independent of $A$. Assuming that $\mu$ is almost a constant, we obtain the power spectrum for the tensor perturbation

\[
P_T = \frac{2k^3}{2\pi^2} \left| \frac{2u_k}{z_T} \right|^2 = \frac{H^2}{\pi} \frac{1 - \bar{\eta}_A}{(1 - \bar{\eta}_A - \epsilon_1)(1 - \delta_1)c_T^2} \left[ H^{(1)}_\mu \left( \frac{1 - \bar{\eta}_A - c_T k}{1 - \bar{\eta}_A - \epsilon_1 aH} \right)^2 \left( \frac{c_T k}{aH} \right)^3 \right]. \tag{80}
\]

On super-horizon scales, $c_T k \ll aH$, using the asymptotic behaviour of the Hankel function, the power spectrum for the tensor perturbation becomes

\[
P_T = \frac{2^{2\mu}}{(1 - \delta_1)c_T^2} \left[ \frac{\Gamma(\mu)}{\Gamma(3/2)} \right]^2 \left( \frac{H}{2\pi} \right)^2 \left( \frac{1 - \bar{\eta}_A - \epsilon_1}{1 - \bar{\eta}_A} \right)^{2\mu - 1} \left( \frac{c_T k}{aH} \right)^{3 - 2\mu} \left| \frac{\bar{\eta}_A}{aH} \right]_{aH = c_T k}^2. \tag{81}
\]

The tensor spectral tilt is

\[
n_t = -\frac{2\mu_1 \bar{\eta}_A}{(A + \bar{\eta}_A) \exp(\bar{\eta}_A N) - A}. \tag{82}
\]

Combining Eqs. (75) and (81), we obtain the tensor to scalar ratio

\[
r = \frac{P_T}{P_R} = 16 \left[ 2^{3 - |3 + \bar{\eta}_A|}(1 - \lambda + \lambda \bar{\eta}_A) \times \frac{\Gamma(3/2)}{\Gamma(3/2 + \bar{\eta}_A/2)} \right] \epsilon_1 \times \left( \frac{\Gamma(3/2)}{\Gamma(3/2 + \bar{\eta}_A/2)} \right)^2 \times 2^{7 - |3 + \bar{\eta}_A|} \bar{\eta}_A [1 + \lambda \bar{\eta}_A/(1 - \lambda)] \left( A + \bar{\eta}_A \right) \exp(\bar{\eta}_A N) - A. \tag{83}
\]

If $\lambda = 0$, we have

\[
r = \left[ \frac{\Gamma(3/2)}{\Gamma(3/2 + \bar{\eta}_A/2)} \right]^2 \times \frac{2^{7 - |3 + \bar{\eta}_A|} \bar{\eta}_A}{(A + \bar{\eta}_A) \exp(\bar{\eta}_A N) - A}. \tag{84}
\]

The tensor to scalar ratio (83) for the model with the Gauss-Bonnet coupling is also reduced by the factor $1 - \lambda$ comparing to the case (84) without the Gauss-Bonnet coupling, in the constant-roll inflation.

**A. The case with $A = 0$**

Constant $\bar{\eta}_0$ is the constant-roll inflation with $\epsilon_2$ being a constant. In this case the scalar spectral tilt is

\[
n_s - 1 = 3 - |3 + \bar{\eta}_0| - \frac{2\nu_0}{\exp(\bar{\eta}_0 N)}, \tag{85}
\]

where

\[
\nu_0 = \frac{6(1 - \lambda) + (7 + 2\lambda + 3\lambda^2)\bar{\eta}_0 + (2 + 4\lambda + 5\lambda^2)\bar{\eta}_0^2 - 4\lambda^2\bar{\eta}_0^3 - 4\lambda^2\bar{\eta}_0^4}{2|\bar{\eta}_0 + 3|(1 - \bar{\eta}_0)(1 - \lambda + \lambda \bar{\eta}_0)}, \tag{86}
\]

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the tensor spectral tilt is
\[ n_t = -\frac{2\mu_1}{\exp(\tilde{\eta}_0 N)}, \tag{87} \]
and the tensor to scalar ratio is
\[ r = (1 - \lambda) \left[ \frac{\Gamma(3/2)}{\Gamma((3/2 + \tilde{\eta}_0/2))} \right]^2 \times \frac{2^{7 - |3 + \tilde{\eta}_0|} [1 + \lambda \tilde{\eta}_0 / (1 - \lambda)]}{\exp(\tilde{\eta}_0 N)}. \tag{88} \]
For the canonical case with \( \lambda = 0 \), we have \[ n_s - 1 = 3 - |3 + \eta_0| - \frac{6 + 7\tilde{\eta}_0 + 2\tilde{\eta}_0^2}{|\tilde{\eta}_0 + 3|(1 - \tilde{\eta}_0) \exp(\tilde{\eta}_0 N)}, \tag{89} \]
\[ r = 2^{7 - |3 + \tilde{\eta}_0|} \left[ \frac{\Gamma(3/2)}{\Gamma((3/2 + \tilde{\eta}_0/2))} \right]^2 \times \exp(-\tilde{\eta}_0 N). \tag{90} \]
We compare the predictions from Eqs. (85) and (88) with the observations \[ 39, 40 \] and find that the model is ruled out by the observations.

B. The case with \( A = 1 \)

In this case, we get \( \tilde{\eta}_1 = \epsilon_2 - \epsilon_1 \). For the model with constant \( \tilde{\eta}_1 \), Eqs. (66) and (67) become
\[ \tilde{\eta}_1 = 2\beta_1 \left( \frac{H,\phi}{H} \right)^2 - \frac{4\beta_1 H,\phi^2}{H}, \tag{91} \]
\[ V(\phi) = 3H^2 - (2\beta_1^2 + 12\lambda\beta_1)H^2, \tag{92} \]
where \( \beta_1 = 1 + \lambda(\tilde{\eta}_1 - 1) \). By using Eqs. (91) and (92), we get the potential
\[ V(\phi) = \begin{cases} V_0 \cosh^4 \left[ \sqrt{\gamma_1}(\phi - \phi_0) \right] \left( 1 + V_1 \tanh^2 \left[ \sqrt{\gamma_1}(\phi - \phi_0) \right] \right), & \gamma_1 > 0, \\ V_0 \cos^4 \left[ \sqrt{-\gamma_1}(\phi - \phi_0) \right] \left( 1 - V_1 \tan^2 \left[ \sqrt{-\gamma_1}(\phi - \phi_0) \right] \right), & \gamma_1 < 0, \end{cases} \tag{93} \]
where
\[ \gamma_1 = -\frac{\tilde{\eta}_1}{8\beta_1}, \quad V_1 = \frac{(1 + 5\lambda + \tilde{\eta}_1 \lambda)\tilde{\eta}_1}{3}. \tag{94} \]

The scalar spectral tilt is
\[ n_s - 1 = 3 - |3 + \tilde{\eta}_1| - \frac{2\nu_1 \tilde{\eta}_1}{(1 + \tilde{\eta}_1) \exp(\tilde{\eta}_1 N) - 1}, \tag{95} \]
where
\[ \nu_1 = \frac{9(1 - \lambda) + (6 + 9\lambda + 3\lambda^2)\tilde{\eta}_1 + (4\lambda + 5\lambda^2)\tilde{\eta}_1^2 - 4(\lambda + \lambda^2)\tilde{\eta}_1^3 - 4\lambda^2 \eta_1^4}{2|3 + \tilde{\eta}_1|(1 - \tilde{\eta}_1)(1 - \lambda + \lambda\tilde{\eta}_1)}. \tag{96} \]
The tensor spectral tilt is
\[ n_t = -\frac{2\mu_1\tilde{\eta}_1}{(1 + \tilde{\eta}_1) \exp(\tilde{\eta}_1 N) - 1}. \tag{97} \]

The tensor to scalar ratio is
\[ r = (1 - \lambda) \left[ \frac{\Gamma(3/2)}{\Gamma(|3/2 + \tilde{\eta}_1/2|)} \right]^2 \times \frac{2^{7 - |3 + \tilde{\eta}_1|} [1 + \lambda \tilde{\eta}_1/(1 - \lambda)]}{(1 + \tilde{\eta}_1) \exp(\tilde{\eta}_1 N) - 1}. \tag{98} \]

Comparing the predictions from Eqs. (95) and (98) with the observations [39, 40], we obtain the constraints on the parameters \(\tilde{\eta}_1\) and \(\lambda\) as shown in Fig. 3. The observations rule out the model without the Gauss-Bonnet coupling. With the Gauss-Bonnet coupling, the model is consistent with the observations if \(\lambda\) is large enough.

![Figure 3](image)

**FIG. 3.** The 1\(\sigma\), 2\(\sigma\) and 3\(\sigma\) constraints on \(\lambda\) and \(\tilde{\eta}_1\). The red, green and blue regions correspond to the 1\(\sigma\), 2\(\sigma\) and 3\(\sigma\) confidence levels, respectively.

### C. The case with \(A = 2\)

In this case, \(\tilde{\eta}_2 = -2\eta_H\), so constant \(\tilde{\eta}_2\) is the constant-roll inflation with \(\eta_H\) being a constant. The scalar spectral tilt is
\[ n_s - 1 = 3 - |3 + \tilde{\eta}_2| - \frac{2\nu_2\tilde{\eta}_2}{(2 + \tilde{\eta}_2) \exp(\tilde{\eta}_2 N) - 2}, \tag{99} \]
where
\[ \nu_2 = \frac{12(1 - \lambda) + (5 + 16\lambda + 3\lambda^2)\tilde{\eta}_2 + (-2 + 4\lambda + 5\lambda^2)\tilde{\eta}_2^2 - 4(2\lambda + \lambda^2)\tilde{\eta}_2^3 - 4\lambda^2\tilde{\eta}_2^4}{2|\tilde{\eta}_2 + 3|(1 - \tilde{\eta}_2)(1 - \lambda + \lambda\tilde{\eta}_2)}. \tag{100} \]

The tensor spectral tilt is
\[ n_t = -\frac{2\mu_1\tilde{\eta}_2}{(2 + \tilde{\eta}_2) \exp(\tilde{\eta}_2 N) - 2}. \tag{101} \]
The tensor to scalar ratio is

\[
    r = (1 - \lambda) \left[ \frac{\Gamma(3/2)}{\Gamma(|3/2 + \tilde{\eta}_2/2|)} \right]^2 \times \frac{2^{7-|3+\tilde{\eta}_2|}\tilde{\eta}_2[1 + \lambda\tilde{\eta}_2/(1 - \lambda)]}{(2 + \tilde{\eta}_2)\exp(\tilde{\eta}_2 N) - 2}.
\]  

(102)

For the canonical case \( \lambda = 0 \), we recover the results [58]

\[
    n_s - 1 = 3 - |3 + \tilde{\eta}_2| - \frac{12\tilde{\eta}_2 + 5\tilde{\eta}_2^2 - 2\tilde{\eta}_2^3}{|\tilde{\eta}_2 + 3|(1 - \tilde{\eta}_2)[(2 + \tilde{\eta}_2)\exp(\tilde{\eta}_2 N) - 2]},
\]  

(103)

\[
    r = 2^{7-|3+\tilde{\eta}_2|} \left[ \frac{\Gamma(3/2)}{\Gamma(|3/2 + \tilde{\eta}_2/2|)} \right]^2 \times \frac{\tilde{\eta}_2}{(2 + \tilde{\eta}_2)\exp(\tilde{\eta}_2 N) - 2}.
\]  

(104)

Comparing the predictions from Eqs. (99) and (102) with the observations [39, 40], we obtain the constraints on the parameters \( \tilde{\eta}_2 = -2\eta_H \) and \( \lambda \) as shown in Fig. 4. For the constant \( \eta_H \) inflation without the Gauss-Bonnet coupling, the predictions are consistent with the observations at the 2\( \sigma \) confidence level [58]. With the Gauss-Bonnet coupling, the predictions are consistent with the observations at the 1\( \sigma \) confidence level. If we take \( \lambda = 0.927, \tilde{\eta}_2 = -0.002 (\eta_H = 0.001) \), we get \( n_s = 0.968 \) and \( r = 0.01 \). Since the observations

![FIG. 4. The 1\( \sigma \), 2\( \sigma \) and 3\( \sigma \) constraints on \( \lambda \) and \( \tilde{\eta}_2 \). The red, green and blue regions correspond to the 1\( \sigma \), 2\( \sigma \) and 3\( \sigma \) confidence levels, respectively.](image)

require that \( \epsilon_1 \) and \( \tilde{\eta}_2 \) are both small, so the slow-roll conditions are satisfied and the constant-roll inflation with constant \( \eta_H \) is also a slow-roll inflation.

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D. The case with $A = 4$

Finally, we consider the model with constant $\tilde{\eta}_4$ which includes the slow-roll inflation with $\eta_V$ being a constant. The scalar spectral tilt $n_s$ and the tensor to scalar ratio $r$ are

$$n_s - 1 = 3 - |3 + \tilde{\eta}_4| - \frac{2\nu_4\tilde{\eta}_4}{(4 + \tilde{\eta}_4)\exp(\tilde{\eta}_4N) - 4},$$  
$$r = (1 - \lambda) \left[ \frac{\Gamma(3/2)}{\Gamma(|3/2 + \tilde{\eta}_4/2|)} \right]^2 \times \frac{27^{-|3 + \tilde{\eta}_4|} [1 + \lambda \tilde{\eta}_4/(1 - \lambda)]}{(4 + \tilde{\eta}_4)\exp(\tilde{\eta}_4N) - 4}.$$  

where

$$\nu_4 = \frac{18(1 - \lambda) + (3 + 30\lambda + 3\lambda^2)\tilde{\eta}_4 + (-6 + 4\lambda + 5\lambda^2)\tilde{\eta}_4^2 - 4(4\lambda + \lambda^2)\tilde{\eta}_4^3 - 4\lambda^2\tilde{\eta}_4^4}{2|\tilde{\eta}_4 + 3| (1 - \tilde{\eta}_4)(1 - \lambda + \lambda\tilde{\eta}_4)}.$$  

For the canonical case with $\lambda = 0$, we have 57

$$n_s - 1 = 3 - |3 + \tilde{\eta}_4| - \frac{18\tilde{\eta}_4 + 3\tilde{\eta}_4^2 - 6\tilde{\eta}_4^3}{|\tilde{\eta}_4 + 3| (1 - \tilde{\eta}_4) [(4 + \tilde{\eta}_4)\exp(\tilde{\eta}_4N) - 4]},$$  
$$r = 27^{-|3 + \tilde{\eta}_4|} \left[ \frac{\Gamma(3/2)}{\Gamma(|3/2 + \tilde{\eta}_4/2|)} \right]^2 \times \frac{\tilde{\eta}_4}{(4 + \tilde{\eta}_4)\exp(\tilde{\eta}_4N) - 4}.$$  

Comparing the predictions from Eqs. (105) and (106) with the observations 39 40, we obtain the constraints on the parameters $\tilde{\eta}_4$ and $\lambda$ as shown in Fig. 5. This model is consistent with the observations at the 1σ confidence level.

![FIG. 5. The 1σ, 2σ and 3σ constraints on $\lambda$ and $\tilde{\eta}_4$. The red, green and blue regions correspond to the 1σ, 2σ and 3σ confidence levels, respectively.](image-url)
E. The potentials for small $\tilde{\eta}_A$

If the constant $\tilde{\eta}_A$ is small, then the constant-roll inflation is also a slow-roll inflation. To the first order of approximation, Eqs. (66) and (67) become

$$\tilde{\eta}_A \approx 2(2 - A)\tilde{\lambda} \left(\frac{H,\phi}{H}\right)^2 - 4\tilde{\lambda}H,\phi, \quad (110)$$

$$V(\phi) \approx 3H^2 - (12\lambda\tilde{\lambda} + 2\tilde{\lambda}^2)H^2, \quad (111)$$

where $\tilde{\lambda} = 1 - \lambda$. For the slow-roll case, we can derive the potential for the model with constant $\tilde{\eta}_A$ by using Eqs. (110) and (111). For convenience, we introduce the function $X(\phi)$,

$$X(\phi) = H(\phi)\frac{\tilde{\eta}_A}{2}. \quad (112)$$

Substituting the function $X(\phi)$ into Eq. (110), we get

$$\frac{X,\phi\phi}{X} = \gamma_A, \quad (113)$$

where $\gamma_A = -A\tilde{\eta}_A/(8\tilde{\lambda})$. The solution to Eq. (113) is

$$X(\phi) = c_1 \exp[\sqrt{\gamma_A}(\phi - \phi_0)] + c_2 \exp[-\sqrt{\gamma_A}(\phi - \phi_0)], \quad (114)$$

if $\gamma_A > 0$, where $c_1$ and $c_2$ are integration constants. For any values of $c_1$ and $c_2$, we can choose the value of $\phi_0$ so that the solution falls into one of the following three classes

1. $X(\phi) = M \exp(\pm \sqrt{\gamma_A}\phi), \quad c_1c_2 = 0, \quad (115)$
2. $X(\phi) = M \sinh(\sqrt{\gamma_A}\phi), \quad c_1c_2 < 0, \quad (116)$
3. $X(\phi) = M \cosh(\sqrt{\gamma_A}\phi), \quad c_1c_2 > 0, \quad (117)$

where $M > 0$. The potential for the case (1) is

$$V(\phi) = V_0 \exp\left(\pm \frac{4\sqrt{\gamma_A}}{A}\phi\right), \quad (118)$$

the potential for the case (2) is

$$V(\phi) = V_0 \sinh^{4/A}(\sqrt{\gamma_A}\phi) \left[1 + V_A \coth^2(\sqrt{\gamma_A}\phi)\right], \quad (119)$$

and the potential for the case (3) is

$$V(\phi) = V_0 \cosh^{4/A}(\sqrt{\gamma_A}\phi) \left[1 + V_A \tanh^2(\sqrt{\gamma_A}\phi)\right], \quad (120)$$
where \( V_A = (5\lambda + 1)\tilde{\eta}_A/(3A) \).

For \( \gamma_A < 0 \), the solution to Eq. (113) is

\[
X(\phi) = M \cos[\sqrt{-\gamma_A}(\phi - \phi_0)],
\]

(121)

and the potential is

\[
V(\phi) = V_0 \cos^{4/A} \left[ \sqrt{-\gamma_A}(\phi - \phi_0) \right] \left( 1 - V_A \tan^2 \left[ \sqrt{-\gamma_A}(\phi - \phi_0) \right] \right).
\]

(122)

Note that if \( A = 1 \), Eqs. (120) and (122) gives the potential (93) for small \( \tilde{\eta}_1 \). The potentials (120) for \( A = 1, A = 2 \) and \( A = 4 \) are shown in Fig. 6.

![FIG. 6. The potentials (120) normalized by \( V_0 \) for \( A = 1, A = 2 \) and \( A = 4 \).](image)

IV. CONCLUSION

For the slow-roll inflation, the reciprocal relation \( \xi = 3\lambda/(4V) \) can be derived from the condition \( \delta_1 = 2\lambda\epsilon_1 \). With the help of the Gauss-Bonnet coupling and the condition \( \delta_1 = 2\lambda\epsilon_1 \), the tensor to scalar ratio \( r \) is reduced by a factor of \( 1 - \lambda \) so that the results become more favorable by the observations. Therefore, inflation models with large \( r \) can be saved by the Gauss-Bonnet coupling. For the model with large \( r \) such as the natural inflation ruled out by the observations at the 1\( \sigma \) confidence level, we find that if \( \lambda > 0.55 \), it will be consistent with the observations at the 1\( \sigma \) confidence level.

We use a general parametrization \( \tilde{\eta}_A = \epsilon_2 - A\epsilon_1 \) to discuss different constant-roll inflations with the condition (1). For the model with constant \( \tilde{\eta}_A \), we derive the formulae for the
power spectra of both the scalar and tensor perturbations. The formulae are applied for four specific models and the observational data are used to constrain the model parameters. For the case $A = 0$, we have $\tilde{\eta}_0 = \epsilon_2$ and this corresponds to the constant-roll inflation with constant $\epsilon_2$. Unfortunately, the constant-roll inflation with $\epsilon_2$ being a constant is ruled out by the observations at the $3\sigma$ confidence level. If $A = 1$, we have $\tilde{\eta}_1 = \epsilon_2 - \epsilon_1$, and the model with constant $\tilde{\eta}_1$ is consistent with the observations if $\lambda > 0.84$. The potential for the model is also obtained. For the case $A = 2$, we have $\tilde{\eta}_2 = -2\eta_H$ and this corresponds to the constant-roll inflation with constant $\eta_H$. The constraints on the model parameters $\tilde{\eta}_2$ and $\lambda$ are obtained. For the case $A = 3$, in the slow-roll approximation, constant $\tilde{\eta}_3$ corresponds to the constant-roll inflation with constant $\eta_N$. The model is consistent with the observations even when the Gauss-Bonnet coupling is absent. For the models with constant $\tilde{\eta}_4$, $\tilde{\eta}_3$ and $\tilde{\eta}_4$, the observations constrain the model parameter $\tilde{\eta}_A$ to be small, so these constant-roll inflations are also slow-roll inflations. Using the slow-roll approximation, the potentials for these models are obtained. In conclusion, the Gauss-Bonnet coupling and the condition $\delta_1 = 2\lambda \epsilon_1$ help inflation models to be consistent with the observations.

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