A cellular automaton model of gravitational clustering

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Gravitational clustering of a random distribution of point masses is dominated by the effective short-range interactions due to large-scale isotropy. We introduce a one-dimensional cellular automaton to reproduce this effect in the most schematic way: at each time particles move towards their nearest neighbours with whom they coalesce on collision. This model shows an extremely rich phenomenology with features of scale-invariant dynamics leading to a tree-like structure in space-time whose topological self-similarity are characterised with universal exponents. Our model suggests a simple interpretation of the non-analytic hierarchical clustering and can reproduce some of the self-similar features of gravitational N-body simulations.

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The basic mechanism of large-scale pattern formation by gravity remains mainly elusive, in spite of elaborate studies in the past few decades [1]. Models of pattern formation, such as diffusion-limited aggregation (DLA) and its many variants [2], which have been remarkably successful in wide ranges of disciplines, do not appear to represent, even at a simplified level, gravitational clustering. A different class of irreversible aggregation models, described by Smoluchowski coagulation equation and its various associated kernels, gives rise to a power-law distribution of the masses of the aggregates which are, however, randomly distributed in space [3–8].

Gravitational clustering, in general, leads to a power law distribution of the masses, known as the Press-Schechter mass function [9], but with a space distribution which, at least up to some scale, is not homogeneous and can be described by fractal geometry [10–12]. The key difference between the aggregations one encounters in statistical physics and gravitational clustering is that in the former, forces are short-ranged, and the nonlinear dynamics, driven by collisions, erase the memory of the initial conditions rather fast. Gravity, on the other hand, is long-ranged with a deterministic evolution which strongly depends on the initial conditions even after the dynamics become nonlinear.

Despite these notorious features of gravity, it has long been shown by Chandrasekhar [13] that for the Poisson distribution of particle positions, gravity is effectively short-ranged: it can be well-described by an effective short-range interaction may at first appear puzzling. However, the reason for this simplification is that the isotropy of Poisson distribution is only broken at small scales where the granularity becomes important.

As the system evolves, particle positions become correlated, but the self-similar nature of gravitational clustering (see for example [9,15]) can ensure that at progressively larger scales the functional form of the Holtsmark distribution remains intact. Hence, under an appropriate renormalisation of the masses and the distances, determined by the scale of granularity at a given time, gravitational interactions can be taken to be effectively short-ranged, even at times comparable to the dynamical time of the system.

Inspired by these facts, we introduce an extremely simple one-dimensional automaton model which focuses precisely on this granular and self-similar nature of the gravitational clustering phenomenon. Unlike most previous models such as DLA, Smoluchowski or 1-dimensional Burgers, our aggregation rule is position-dependent and independent of the masses and initial velocities of the aggregates. Unlike the conventional boolean cellular automata, we distribute particles randomly on a line rather than on discrete lattice points. We focus mainly on the universal features of the distribution of the masses of the aggregates and on the space-time topological properties of the full aggregation history, i.e., on the universal characteristics of the “merger tree” of our model.

The basic idea of this model, that the particles move towards their nearest neighbours, is implemented by the following algorithm. We distribute $10^5$ particles randomly on a periodic line with a length of $10^5$ units, so that the average density is equal to one. Particles move towards their nearest neighbours either by one unit at each time step or by half their separations, whichever
that is shorter. If two particles are closer than a lower threshold, which we set equal to 2 units, and, in addition, are mutual nearest-neighbours they coalesce at the mid-point of their separation, conserving mass. This aggregation rule for a particle \( i \) at the position \( x_i(t) \) at time \( t \) can be written as,

\[
\begin{align*}
    x_i(t + \Delta t) &= x_i(t) + \min \left( \frac{x_r(t)}{2}, 1 \right) \quad \text{if } x_r(t) < |x_i(t)| \\
    x_i(t + \Delta t) &= x_i(t) + \max \left( \frac{x_r(t)}{2}, -1 \right) \quad \text{if } x_r(t) > |x_i(t)|
\end{align*}
\]

(2)

where \( x_r(t) = x_{i+1}(t) - x_i(t) \) and \( x_l(t) = x_{i-1}(t) - x_i(t) \). This process is illustrated in Fig. 1.

FIG. 1. A schematic illustration of the aggregation rule (2). The essence of our model is that particles move towards their nearest neighbours and coalesce on collision.

As time progresses, masses and distances of the nearest neighbours rescale and the same rule, given by (2), is followed by more massive and farther apart aggregates. The continuation of this process will eventually lead to the total coagulation of the colloidal particles into one single mass. This aggregation mechanism traces a self-similar tree-like structure in space-time as shown in Fig. 2.

The tree structure of the aggregation process in space-time is a manifestation of topological self-similarity [16], which is a property of many branched structures such as river-networks [17] and bronchial trees [18] and can be quantified by various scaling exponents. A most common of these exponents is the Strahler index, \( s \), given by

\[
P(\eta) \sim \eta^{-s},
\]

(3)

where \( P(\eta) \) is the probability that a point in the network is connected to \( \eta \) other points uphill, also known as the drainage area. The Strahler index, \( s \), is a measure of the bushiness of a branched structure and has an upper limit of 1.5 for river networks [17].

In our model, the function \( P(\eta) \), given by (3), interprets as the probability that a newly-formed aggregate has a mass equal to \( \eta \). We observe a larger value for Strahler index \( s = 2 \), given by the slope of the plot in the upper inset of Fig. 3) than is expected for river networks and which seems to be a characteristic of Cayley tree structures. A notable example of such a structure, also with exponent 2, has recently been observed for the property of internet connections [19]. The difference between our model and the conventional river-networks could be the stringent requirement of mass conservation here.

Topological scaling, is believed to emerge from a self-similar growth process. Self-similar growth, or dynamical scaling, is a dominant feature of hierarchical gravitational clustering [15], and is the fundamental reason for the emergence of distribution functions such as Press-Schechter mass function in cosmology [9]. An appropriate way to analyse dynamical scaling is, indeed, to study the mass distribution function \( n(m, t) \). Note that our function differs from the usual number density by a constant factor which is given by the size of our system.

It can be easily inferred from Fig. 2, and we have verified numerically that the average mass, \( \langle m \rangle \), and the mass variance, \( \langle (m^2) - \langle m \rangle^2 \rangle \), grow linearly with time. It is worth comparing this with the growth of average mass in one-dimensional Burgers, for example with uniform initial velocities and positions, where an exponent of 2/3 has been obtained [20,21]. The linear growth of average mass also holds for Smoluchowski equation with a constant kernel [8]. The universality between our model and Smoluchowski arises inspite of the fact that particle trajectories are not Brownian here. In addition to this common feature, our model and Smoluchowski equation have similar asymptotic states. In both of these models, clusters collide until all the mass falls into one final clump, whereas in 1-dimensional Burgers, the asymptotic state can contain many clumps with zero momentum.

The distribution of the masses deviates slightly from a simple Gaussian for small masses where it develops a power-law as is shown in Fig. 4. The inset clearly demonstrates that a self-similar condensation process, rather
similar to the one observed in gravitational N-body simulations (compare with Fig. 1 of [9]), has set in. In the process of coagulation, the shape of the mass spectrum seems to become fixed, and the curves move in parallel to the right (increasing aggregate mass).

FIG. 3. Topological scaling: the scaling, over many decades, of the probability distribution of the masses of the newly-borned aggregates, \( P(\eta) \), (upper right inset) and of the corresponding cumulative probability, \( P_{>\eta} \), i.e. the probability that the mass of a newly-formed aggregate is larger than \( \eta \), (main plot). The sketch in lower left inset illustrates the standard procedure of Horton-Strahler ordering and link amplitude to find the Strahler index: \( P(\eta) \) is the frequency of occurrence of a number shown in that sketch.

The results presented so far provide strong evidence that the mass distribution function has the general scaling form:

\[
n(m,t) \sim m^\alpha t^\beta \exp\left(-m^\gamma/t^\lambda\right) ,
\]

where the value of the exponents \( \alpha, \beta, \gamma \) and \( \lambda \) will be found in what follows.

The conservation of the total mass in our model leads to the exponent identity

\[
\beta + \frac{\lambda}{\gamma} (2 + \alpha) = 0 .
\]  

In addition, the linear rate of growth of the average mass, \( \langle m \rangle \sim t \), leads to the further scaling identity,

\[
\frac{\lambda}{\gamma} = 1 .
\]  

At this point one can already see the emergence of a topological scaling, namely a power-law behaviour in the time-integrated mass distribution function, at small masses. We comment that this is different from Strahler index we found in Fig. 3, since the latter refers only to newly-formed aggregates. This topological scaling, namely the power-law behaviour at small masses of the time-integrated mass distribution function, \( \int n(m,t)dt \sim 1/m^l \), is implied by the scaling relation (5), which fixes the value of the exponent \( l \) to \( l = 1 \). This value is also confirmed by our numerical simulations.

FIG. 4. Dynamical scaling: self-similar growth of mass distribution function \( n(m,t) \). Fits on the main plot are obtained with a simple Gaussian. The inset shows the self-similar growth of the fraction, \( f(m,t) \), of mass in objects of mass smaller than \( m \), spanning over the significant part of the dynamical time.

A further test of our scaling identities (5) and (6), is provided by the evolution of the maxima, \( n_{\text{max}} \), of the mass distribution function, i.e., by the rate of decay of the peaks of the main plot in Fig. 4. Using the fact that the average mass grows linearly with time, in the scaling expression (4), we obtain \( n_{\text{max}}(m,t) \sim 1/t^2 \), which is again confirmed by our simulations.

Thus, the scaling identities (5) and (6), reduce our scaling ansatz (4) to the self-similar form

\[
t^2 n(m,t) \sim \left(\frac{m}{t}\right)^\alpha e^{-m^\gamma/t^\lambda} .
\]

The factor of \( 1/t^2 \) in (7) is a consequence of mass conservation and the linear growth rate of the average mass, and has also been observed for the constant-kernel solution to Smoluchowski equation [8]: \( n(m,t) = 4t^{-2}\exp(-2m/t) \). This solution of Smoluchowski equation, is obtained from our general solution (7) by the transformation \( t \rightarrow 2t \) and by using the following values of the exponents: \( \alpha = 0, \gamma = 1 \). We shall soon show that these exponents take different values in our model.

To put our notation in accordance with that used in cosmology, we replace the time variable by the cut-off mass \( m^*(t) \). As we have mentioned previously, our simulations show that a typical mass grows linearly with time, i.e. \( m^* \sim t \). The exponents \( \alpha \) and \( \gamma \) in expression (7) can be found by plotting \( n(m,t) m^{2\alpha} \) against the ratio \( m/m^* \), which we have done in Fig. 5.
and we finally arrive at the approximate expression for fractal structures in space. The present model does not generate asymptotic preserved and not destroyed as is the case here. In this large-scale structures are built while substructures are usual statistical models which generate fractals, where small to large scales. Thus, our model differs from the innated by the granular properties which are shifted from early with time and the growth of the structures is dominated by the Poisson distribution. The crossover length increases linearly at large scales where it reminiscs the initial characteristic cutoff mass, $m^*$, grows linearly with time.

We show in Fig. 5 that for over two decades in time scale the functional form of the mass function, given by the RHS of (7), is preserved. The fit in Fig. 5 sets the value of the unknown exponents in (7) to $\alpha \sim 3$ and $\gamma \sim 2$ and we finally arrive at the approximate expression for the mass distribution function. We emphasis that, our solution is different from the constant-kernel solution to Smoluchowski equation which is an special case of our general scaling solution (7), as previously noted. We comment that unlike some of the previous aggregation models used in cosmology [6,7], our distribution function cannot be formed from a white noise initial spectrum. It remains to be seen if, as for Smoluchowski with additive kernel, the introduction of a mass-dependent factor in our aggregation rule (2), would give rise to a Press-Schechter type mass function.

We have also analysed the density-density correlation function which has a power-law behaviour with exponent $-1$ at small scales, indicating that the mass is distributed on zero-dimensional objects, and a crossover to a constant value at large scales where it reminisces the initial Poisson distribution. The crossover length increases linearly with time and the growth of the structures is dominated by the granular properties which are shifted from small to large scales. Thus, our model differs from the usual statistical models which generate fractals, where large-scale structures are built while substructures are preserved and not destroyed as is the case here. In this sense the present model does not generate asymptotic fractal structures in space.

In conclusion, the seminal result of Chandrasekhar, that the long-range gravitational interactions between randomly distributed particles can be almost exactly replaced by nearest-neighbour interactions, stimulated us to present a simple aggregation model which captures this profound feature of gravitating systems. We have shown that our model exhibits topological self-similarity over many decades of mass scale and dynamical scaling over many decades of temporal scale. These properties make it a simple and intuitive model for the study of gravitational hierarchical clustering.

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