On QCD predictions for the chiral Lagrangian coefficients

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Based on a previous study of deriving the chiral Lagrangian (CL) from QCD, we illustrate the main feature of how QCD predicts the CL coefficients (CLC) in certain approximations. We first show that, in the large-$N_c$ limit, the anomaly part contributions to the CLC are exactly cancelled by certain terms in the normal part contributions (NPC), so that the final results only concern the remaining NPC depending on QCD interactions. We then do the calculation in a simple approach with further approximations. The obtained CLC and quark condensate are consistent with the experiments.

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\section{I. INTRODUCTION}

Studying low energy hadron physics in QCD is a long standing difficult problem due to its nonperturbative nature. A widely used approach is the theory of chiral Lagrangian (CL) based on the global symmetry of the system and the momentum expansion without dealing with the nonperturbative QCD dynamics. In such an approach, the CL coefficients (CLC) are all unknown parameters being determined by experimental inputs. Studying the relation between the CL and the underlying theory of QCD will not only be theoretically interesting for a deeper understanding of the CL, but will also be helpful for increasing the predictive power of the CL.

The CLC are contributed both by the anomaly part (from the quark functional measure) and the normal part (from the QCD Lagrangian). In the literature, the anomaly part contributions (APC) are carefully calculated using the heat kernel technique, while the normal part contributions (NPC) have not been calculated as carefully. In a previous paper, Ref. [3], the CL was formally derived from the first principles of QCD without taking approximations, and all the NPC to the CLC are expressed in terms of certain Green's functions in QCD. To compare the APC and NPC carefully, a unified regularization scheme and new technique feasible for the calculations of both APC and NPC is needed since the heat kernel technique is hard to be applied to the NPC which contains complicated functions of the momentum from nonperturbative QCD dynamics.

In this letter, we take the Schwinger proper time regularization scheme for the calculations of both APC and NPC with the technique developed in [3]. Thus APC and NPC are treated on equal footing. We then take a simple approach with a series of approximations to illustrate the main feature of how QCD predicts the CLC. The first approximation is to take the large-$N_c$ limit in which the effective actions can be evaluated in the saddle-point approximation, and all the CLC should be free from ultraviolet divergence since the only ultraviolet divergence in the CLC comes from the meson expansion without taking approximations, and all the NPC to the CLC are expressed in terms of certain Green's functions in QCD. To compare the APC and NPC carefully, a unified regularization scheme and new technique feasible for the calculations of both APC and NPC is needed since the heat kernel technique is hard to be applied to the NPC which contains complicated functions of the momentum from nonperturbative QCD dynamics.

We first calculate the APC. The anomaly term in the path integral can be expressed by the following effective action

\begin{equation}
S_{\text{eff}}^{\text{(anom)}} = -i \times \text{anomaly terms} = -i N_c [\text{Tr} \ln (i \not{\partial} + J) - \text{Tr} \ln (i \not{\partial} + \not{J})] = i N_c [\text{Tr} \ln (i \not{\partial} + \not{J}) + \cdots],
\end{equation}

where $\Omega$ is related to the nonlinearly realized meson field $U$ by $U = \Omega^2$, $J$ is the external source containing scalar, pseudoscalar, vector, and axial vector components, $\not{J}$ is $J$ chirally rotated by $\Omega$, and the ellipsis denotes $\Omega$-independent ($U$-independent) terms which is irrelevant to the CLC. The APC to the CLC concerns only the real part of the $\text{Tr} \ln (i \not{\partial} + \not{J})$ term in (1), which is positively definite in the Euclidean space-time. Now we evaluate it by using the Schwinger proper time regularization with parameters $\Lambda$ and $\kappa$ regularizing the UV and IR divergences, respectively. In [3], we see that the APC to the $O(p^2)$ CLC is exactly cancelled by a term in the NPC, so that we can concentrate on the $O(p^4)$ CLC. After lengthy but elementary calculations, we obtain

\begin{align}
L_1^{\text{(anom)}} &= \frac{N_c}{384 \pi^2}, \quad L_2^{\text{(anom)}} = \frac{N_c}{192 \pi^2}, \quad L_3^{\text{(anom)}} = \frac{N_c}{96 \pi^2}, \\
L_4^{\text{(anom)}} &= \frac{N_c}{192 \pi^2}, \quad L_5^{\text{(anom)}} = 0, \quad L_7^{\text{(anom)}} = \frac{N_c}{1152 \pi^2}, \\
L_8^{\text{(anom)}} &= \frac{N_c}{384 \pi^2}, \quad L_9^{\text{(anom)}} = \frac{N_c}{48 \pi^2}, \quad L_10^{\text{(anom)}} = \frac{N_c}{96 \pi^2}, \\
H_1^{\text{(anom)}} &= \frac{N_c}{96 \pi^2} \lim_{\kappa \to 0} \lim_{\Lambda \to \infty} (\ln \frac{\kappa^2}{\Lambda^2} + \gamma + \frac{1}{2}), \\
H_2^{\text{(anom)}} &= \frac{N_c}{192 \pi^2} + \lim_{\Lambda \to \infty} \frac{N_c \Lambda^2}{32 \pi^2 B_0},
\end{align}

(2)
These are exactly the results in [3] for $M_Q = 0$. When taking $N_c = 3$, the results given in (2) are close to the experimental results [3] except that $L_7$ and $L_8$ are of wrong signs. This gives people an impression that the APC might play the major role in the CLC, and the NPC might only contribute small corrections [3,4]. We argue that the results in (2) do not really appear in the final expressions for the CLC because: (i) they are independent of $\Pi$ in the present approximation, (ii) the final CLC should be finite as $\Lambda \rightarrow \infty$ in the large-$N_c$ limit, while $H_1$ and $H_2$ in (2) are divergent as $\Lambda \rightarrow \infty$. We shall see later that the terms in (2) are indeed exactly cancelled by certain terms in the NPC, and they do not really appear in the final expressions for the CLC.

Now, we calculate the NPC with the same regularization scheme. In the large-$N_c$ limit, the saddle-point approximation reduces the normal part effective action $S_{\text{eff}}^{(\text{norm})}$ to

$$S_{\text{eff}}^{(\text{norm})} = -iN_c\text{Tr} \ln[i\partial + J_\Omega - \Pi_{\Omega c}] + N_c \int d^4x d^4x' \times \Phi(x, x') \Pi_{\Omega c}^{\sigma}(x, x') + N_c \sum_{n=2}^\infty \int d^4x_1 \cdots d^4x_n \times \left( -\frac{i^n (N_c g_s^2)^{n-1}}{n!} \right) \times \Phi_{\Omega c}^{\sigma \rho \nu}(x_1, x_1') \cdots \Phi_{\Omega c}^{\sigma \rho \nu}(x_n, x_n') \right)_{\Pi_{\Omega c} = 0} = 0 \right) (3)$$

satisfying the useful relation [3]

$$\left. \frac{dS_{\text{eff}}^{(\text{norm})}}{dJ^\rho_{\Omega c}(x)} \right|_{U, \text{fix, anomaly ignored}} = N_c \Phi^{\sigma \rho}_{\Omega c}(x, x), \quad (4)$$

where $\Phi$ and $\Pi$ are auxiliary fields, $\Phi_{\Omega c}$ and $\Pi_{\Omega c}$ are classical fields of the chirally rotated (by $\Omega$) $\Phi$ and $\Pi$ satisfying the saddle-point equations, and $\hat{G}$ is the general gluon Green’s function defined in [2]. In the present approximation, (3) reduces to

$$-i[(i\partial + J_\Omega - \Pi_{\Omega c})^{-1}]^{\sigma \rho}(x, x) = \Phi^{\sigma \rho}_{\Omega c}(x, x). \quad (5)$$

We see that $\Pi_{\Omega c}$ and $\Phi_{\Omega c}$ play the roles of the quark self-energy and the quark propagator, respectively, in the case with $J_3 \neq 0$.

Next, we decompose $S_{\text{eff}}^{(\text{norm})}$ into a part $S_{\text{eff}}^{(\text{norm}, \Pi_{\Omega c} = 0)}$ independent of $\Pi_{\Omega c}$ and a part $S_{\text{eff}}^{(\text{norm}, \Pi_{\Omega c} \neq 0)}$ depending on $\Pi_{\Omega c}$. $S_{\text{eff}}^{(\text{norm}, \Pi_{\Omega c} = 0)}$ can be extracted from (3) by setting $\Pi_{\Omega c} = 0$, i.e.

$$S_{\text{eff}}^{(\text{norm}, \Pi_{\Omega c} = 0)} = -iN_c\text{Tr} \ln[i\partial + J_\Omega] \times \hat{G}_{\rho_1 \cdots \rho_n}(x_1, x_1') \cdots \times \Phi^{\sigma \rho \nu}_{\Omega c}(x_n, x_n') \right]_{\Pi_{\Omega c} = 0}. \quad (6)$$

It can be shown that the last term in (6) is actually $\Omega$-independent [8]. Therefore, (6) can be written as

$$S_{\text{eff}}^{(\text{norm}, \Pi_{\Omega c} = 0)} = -iN_c[\text{Tr} \ln(i\partial + J_\Omega) + \cdots]. \quad (7)$$

where the ellipsis denotes $\Omega$-independent terms irrelevant to the CLC. Comparing the $\Omega$-dependent terms in (3) and (7), we see that they are of the same form but with an opposite sign. Thus their contributions to the CLC exactly cancel each other to all orders in the momentum expansion. Hence the results in (3) do not really appear in the final expressions for the CLC. The CLC are actually contributed by the remaining normal part effective action $S_{\text{eff}}^{(\text{norm}, \Pi_{\Omega c} \neq 0)}$. This is our first new conclusion in this study. Then the final formulae for the CLC given in [3] (cf. eqs.(62) and (63) in [5]) becomes

$$L_1 = \frac{1}{32} \hat{K}_4 + \frac{1}{16} \hat{K}_5 + \frac{1}{16} \hat{K}_{13} - \frac{1}{32} \hat{K}_{14},$$

$$L_2 = \frac{1}{16} (\hat{K}_4 + \hat{K}_6) + \frac{1}{8} \hat{K}_{13} - \frac{1}{16} \hat{K}_{14},$$

$$L_3 = \frac{1}{16} (\hat{K}_3 - 2 \hat{K}_4 - 6 \hat{K}_{13} + 3 \hat{K}_{14}),$$

$$L_4 = \frac{1}{16} \hat{K}_{12} - \frac{1}{16} \hat{K}_{11} + \frac{1}{16} \hat{K}_5,$$

$$L_7 = -\frac{1}{16} \hat{K}_1 - \frac{1}{16} \hat{K}_{10} - \frac{1}{16} \hat{K}_{15},$$

$$L_8 = \frac{1}{16} (\hat{K}_1 + \frac{1}{B_0} \hat{K}_7 - \frac{1}{B_0} \hat{K}_9 + \frac{1}{B_0} \hat{K}_{15}),$$

$$L_9 = \frac{1}{16} (4 \hat{K}_{13} - \hat{K}_{14}),$$

$$H_1 = -\frac{1}{4} (\hat{K}_2 + \hat{K}_{13}),$$

$$H_2 = \frac{1}{4} (\hat{K}_3 + \frac{1}{B_0} \hat{K}_7 + \frac{1}{B_0} \hat{K}_9 - \frac{1}{B_0} \hat{K}_{15}) \quad (8)$$

in which $\hat{K}_i \equiv {K}_i^{(\text{norm}, \Pi_{\Omega c} \neq 0)} - {K}_i^{(\text{norm}, \Pi_{\Omega c} = 0)}$, and the $K$s are components with various Lorentz structures of the related QCD Green’s functions defined in [3]. These $O(p^4)$ CLC depend on QCD interactions through $\Pi_{\Omega c}$ as it should be.

The effective action $S_{\text{eff}}^{(\text{norm})}$ in [3] has never been carefully evaluated in the literature. As the first time of doing the calculation, we take further approximations to

*Different from the approach in [3], our present approach does not put in a constituent quark mass $M_Q$ by hand. Instead, it is naturally included in the quark self-energy reflecting chiral symmetry breaking in the normal part. So that our results should be compared with the $M_Q = 0$ results in [3].
simplify the evaluation. Now we take the assumption of keeping only the leading order in dynamical perturbation [1], which means taking account of only the nonperturbative QCD dynamics through the quark self-energy reflecting chiral symmetry breaking, and neglecting all QCD corrections in positive powers of $g_s$. Thus the complicated last term in (3) is neglected. Moreover, from the equation of motion of $\Phi_{Qc}$, we can see that the second term is of the same order as the last term so that it should be neglected as well. Then only the first term in (3) is kept. As we have mentioned, $\Pi_{Qc}$ plays the role of the quark self-energy. From the local gauge transformation property of $\Pi_{Qc}$, we know that $\Pi_{Qc}$ is related to the conventional quark self-energy $\Sigma(-p^2)$ by

$$\Pi_{Qc}(x, y) = (\Sigma(x, y))^{\text{norm}} \delta^4(x - y),$$  \hspace{1cm} (9)

where \(\Sigma(x, y) = \partial^\mu_x \partial^\mu_y(x)\). Then the simplified $S_{\text{eff}}^{\text{norm}}$ can be written as

$$S_{\text{eff}}^{\text{norm}} = -i N_c \text{Tr} \ln[i\partial + J_\Omega - \Sigma(\nabla^2)].$$  \hspace{1cm} (10)

This can be evaluated in the Schwinger proper time regularization scheme with the technique developed in [1]. The calculation is first performed in the Euclidean space-time and then converted into the Minkowskian space-time. The calculation is lengthy and the final results in the Minkowskian space-time are

$$F_0^2 B_0 = 4 \int dp \Sigma_p X_p,$$

$$F_0^2 = 2 \int dp \left[ -2 \Sigma_p^2 + 2 \Sigma_p X_p \right],$$

$$K_1(n_{\text{norm}}) = 2 \int dp \left[ -2 \Sigma_p^2 + 2 \Sigma_p X_p \right],$$

$$K_2(n_{\text{norm}}) = 2 \int dp \left[ -2 \Sigma_p^2 + 2 \Sigma_p X_p \right],$$

$$K_3(n_{\text{norm}}) = 2 \int dp \left[ -2 \Sigma_p^2 + 2 \Sigma_p X_p \right],$$

$$K_4(n_{\text{norm}}) = 2 \int dp \left[ -2 \Sigma_p^2 + 2 \Sigma_p X_p \right],$$

$$K_5(n_{\text{norm}}) = K_6(n_{\text{norm}}) = K_8(n_{\text{norm}}) = K_{10}(n_{\text{norm}}) = K_{12}(n_{\text{norm}}) = 0,$$

$$K_7^{(n_{\text{norm}})} = 2 \int dp \left[ 3 \Sigma_p^2 + 2 \Sigma_p X_p \right].$$

in which the short notations are $\Sigma_p \equiv \Sigma(-p^2)$, \(\int dp \equiv i N_c \int \frac{d^4p}{(2\pi)^4} \exp \{ (p^2 - \Sigma_p^2)/\Lambda^2 \}, \) $X_p \equiv 1/(p^2 - \Sigma_p^2)$, and $A_p$, $B_p$, $C_p$, $D_p$, $E_p$, and $F_p$ are functions of $\Sigma_p$ with lengthy expressions given in [3].

Taking $\Lambda \to \infty$ the expression for $F_0^2$ in (11) is just the well-known Pagels-Stokar formula [1]. It is easy to check that the $K_i(n_{\text{norm}})$ $(i = 1, \ldots, 15)$ in (11) do contain the $\Sigma_p$-independent ($\Pi_{Qc}$-independent) piece mentioned before which exactly cancel the APC in (3). It can also be checked that the results in (11) are all finite when $\Lambda \to \infty$ as it should be.

Subtracting the $\Sigma_p$-independent piece from (11), we get the desired $K_i(\Sigma_p)$ $(i = 1, \ldots, 15)$ needed in obtaining the $O(p^4)$ CLC in [3].

The final step of the calculation is to calculate $\Sigma(p^2)$ from the Schwinger-Dyson equation. We further take the improved ladder approximation as in the literature,

$$\Sigma(p^2) = \frac{3N_c}{2} \int d^4q \alpha_s(p - q) \frac{\Sigma(q^2)}{4\pi^3 (p - q)^2 q^2 + \Sigma(q^2)} = 0.$$  \hspace{1cm} (12)

So far we only know the large momentum behavior of the running coupling constant $\alpha_s(p - q)$. The low momentum behavior of it is not know yet due to the ignorance of nonperturbative QCD dynamics. Inevitably, we have to take certain QCD motivated model for it as in the literature. We shall take the following Model A from [1], and Model B and Model C from [11] as examples to do the calculation. They are
A: $\alpha_s(p) = \frac{12\pi}{(33 - 2N_f)}$, for $\ln(p^2/\Lambda_{QCD}^2) \leq -2$;

$$\{7 - \frac{4}{9}[2 + \ln(p^2/\Lambda_{QCD}^2)]^2\} \frac{12\pi}{(33 - 2N_f)},$$

for $-2 \leq \ln(p^2/\Lambda_{QCD}^2) \leq 0.5$;

$$= \frac{1}{\ln(p^2/\Lambda_{QCD}^2)} \frac{12\pi}{(33 - 2N_f)},$$

for $0.5 \leq \ln(p^2/\Lambda_{QCD}^2)$.

B: $\alpha_s(p) = 4\pi^3 \mu^2 p^2 \delta^4(p) + \frac{12\pi}{(33 - 2N_f) \ln(2 + p^2/\Lambda_{QCD}^2)}$;

C: $\alpha_s(p) = \frac{1}{\mu^2 p^2 e^{-p^2/\Lambda^2}} + \frac{12\pi}{(33 - 2N_f) \ln(2 + p^2/\Lambda_{QCD}^2)}$.

We take the original values $\Lambda_{QCD} = 484$ MeV (Model A), $\Lambda_{QCD} = 230$ MeV (Models B and C), and $p_0 = 380$ MeV (Model C), and determine other parameters by taking $F_0 = 93$ MeV as input. We further take the usual approximated $\alpha_s(p - q) \approx \theta(p^2 - q^2)\alpha_s(p^2) + \theta(q^2 - p^2)\alpha_s(q^2)$ with which the integral equation (12) can be converted into differential equation which is easy to solve numerically. We have found the numerical solutions with the desired asymptotic behavior $\Sigma(p^2) \rightarrow -\infty$ $\lim[\ln(p^2/\Lambda_{QCD}^2)]^{\gamma/2}/p^2$ $(\gamma = (9N_c)/(2(33 - 2N_f)))$ characterizing chiral symmetry breaking for the three models. Then we obtain the values of the $O(p^2)$ CLC from (8) and (11) which are listed in TABLE I together with the experimental values [3] for comparison. We see from TABLE I that: (a) the obtained CLC are not so sensitive to the forms of $\alpha_s(p)$, (b) all $L_1, \ldots, L_{10}$ are of the right orders of magnitude and the right signs, (c) the consistency with the experiments of $L_1$, $L_2$, $L_4$, $L_6$, and $L_{10}$ is at $1\sigma$ level, and that of $L_3$, $L_5$, $L_7$ and $L_8$ is at $2\sigma$ level, (d) only $L_9$ deviates from the experimental value by $(3 - 4)\sigma$. Considering the large theoretical uncertainty in this simple approach, the obtained $L_1, \ldots, L_{10}$ are all consistent with the experiments.

In addition to $L_1, \ldots, L_{10}$, we can also calculate the quark condensate from the $O(p^2)$ CLC $F_0^2 B_0$ with the relation in the present approximation $\langle \bar{\psi}\psi \rangle = -N_f F_0^2 B_0$ [6]. This is divergent when taking $\Lambda \rightarrow \infty$, so that it needs to be renormalized. We take the renormalization counter term such that $\Lambda$ is replaced by a scale parameter $\mu$. The values of the renormalized quark condensate at $\mu = 1$ GeV for the three models are

A: $\langle \bar{\psi}\psi \rangle_r = -(296 \text{ MeV})^3$,

B: $\langle \bar{\psi}\psi \rangle_r = -(296 \text{ MeV})^3$,

C: $\langle \bar{\psi}\psi \rangle_r = -(301 \text{ MeV})^3$.

Considering the large theoretical uncertainty in this calculation, the obtained $\langle \bar{\psi}\psi \rangle_r$ is also consistent with the experimentally determined value $\langle \bar{\psi}\psi \rangle_{\text{expt}} = -(250 \text{ MeV})^3$ from the QCD sum rule at the scale of the typical harmonic mass [12].

In conclusion, we have calculated the CLC from QCD in certain approximations. We have first shown the exact cancellation between the APC and the $\Pi_{\Omega_c}$-independent part of the NPC, so the final results of the CLC concern only the $\Pi_{\Omega_c}$-dependent part of the NPC. Our obtained CLC and quark condensates are all consistent with the experiments. Although the present approximations are rather crude, it does reveal the main feature of how QCD predicts the CLC. We see that the quark self-energy reflecting chiral symmetry breaking plays an important role in the predictions for the CLC. Study on improved approximations is in progress.

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**TABLE I.** The obtained values of the $O(p^4)$ CLC (in units of $10^{-3}$) for Model A, B, and C with $\Lambda \rightarrow \infty$ together with the experimental values for comparison.

|     | $L_1$ | $L_2$ | $L_3$ | $L_4$ | $L_5$ | $L_6$ | $L_7$ | $L_8$ | $L_9$ | $L_{10}$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|---------|
| A   | 1.10  | 2.20  | -7.82 | 0     | 1.62  | 0     | -0.70 | 1.75  | 5.07  | -7.06   |
| B   | 0.921 | 1.84  | -6.73 | 0     | 1.43  | 0     | -0.673| 1.64  | 3.80  | -6.22   |
| C   | 0.948 | 1.90  | -6.90 | 0     | 1.29  | 0     | -0.632| 1.56  | 3.95  | -6.21   |
| Expt| 0.9 ± 0.3 | 1.7 ± 0.7 | -4.4 ± 2.5 | 0 ± 0.5 | 2.2 ± 0.5 | 0 ± 0.3 | -0.4 ± 0.15 | 1.1 ± 0.3 | 7.4 ± 0.7 | -6.0 ± 0.7 |