Coefficient Bounds for a New Subclasses of Bi-Univalent Functions Associated with Horadam Polynomials

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Abstract:
In this work we present and investigate three new subclasses of the function class of bi-univalent functions in the open unit disk defined by means of the Horadam polynomials. Furthermore, for functions in each of the subclasses introduced here, we obtain upper bounds for the initial coefficients $|a_2|$ and $|a_3|$. Also, we debate Fekete-Szegö inequality for functions belongs to these subclasses.

Keywords: Bi-univalent functions, Coefficient bounds, Fekete-Szegö inequality, Holomorphic function, Horadam polynomials.

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Introduction
Symbolized by $\mathcal{A}$ the function class of the shape:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are holomorphic in the open unit disk $\Delta = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$ and normalized under the conditions indicated by $f(0) = f'(0) - 1 = 0$. Furthermore, symbolized by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.

The Koebe One-Quarter Theorem [4] shows that the image of $\Delta$ includes a disk of radius $\frac{1}{4}$ under each function $f$ from $\mathcal{S}$. Thereby each univalent function of this kind has an inverse $f^{-1}$ which fulfills

$$f^{-1}(f(z)) = z \quad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f); \ r_0(f) \geq \frac{1}{4}),$$

where

$$f^{-1}(w) = g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.$$  

(2)

The function $f \in \mathcal{A}$ is considered bi-univalent in $\Delta$ if together $f^{-1}$ and $f$ are univalent in $\Delta$. Indicated by the Taylor-Maclaurin series expansion (1), the class of all bi-univalent functions in $\Delta$ can be symbolized by $\Sigma$. In the year 2010, Srivastava et al. [10] refreshed the study of various classes of bi-univalent functions. Moreover, many penmans explored bounds for different subclasses of bi-univalent functions (see, for example [3,5,6,11]). The coefficient estimate problem involving the bound of $|a_n|$ ($n \in \mathbb{N}\backslash\{1,2\}, \mathbb{N} = \{1,2,3,\ldots\}$) is still an open problem.

For two functions $\mathcal{D}$ and $\mathcal{Y}$, holomorphic in the open unit disk $\Delta$, we say that the function $\mathcal{D}(w)$ is subordinate to $\mathcal{Y}(w)$ in $\Delta$, and write

$$\mathcal{D}(w) < \mathcal{Y}(w) \quad (w \in \Delta),$$

if there exists a Schwarz function $\mathcal{T}(w)$, holomorphic in $\Delta$, with

$$\mathcal{T}(0) = 0 \text{ and } |\mathcal{T}(w)| < 1 \quad (w \in \Delta),$$

such that
In special, if the function $\mathcal{Y}$ is univalent in $\Delta$, the above subordination is equivalent to
\[
\mathcal{D}(0) = \mathcal{Y}(0) \text{ and } \mathcal{D}(\Delta) \subset \mathcal{Y}(\Delta).
\]

The following recurrence relation gives the Horadam polynomials $h_n(x)$ (see (8))
\[
h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x), \quad (x \in \mathbb{R}, \ n \in \mathbb{N}\backslash\{1,2\}, \mathbb{N} = \{1,2,3,\ldots\}),
\]
with $h_1(x) = k$, $h_2(x) = hx$ and $h_3(x) = px^2 + kq$ where $k, b, p$ and $q$ are some real constants. The characteristic equation of repetition relationship (3) is $t^2 - pxt - q = 0$. There are two real roots of this equation
\[
a_1 = \frac{px + \sqrt{p^2x^2 + 4q}}{2} \quad \text{and} \quad a_2 = \frac{px - \sqrt{p^2x^2 + 4q}}{2}.
\]

The generating function of the Horadam polynomials $h_n(x)$ is indicated by
\[
\Omega(x, z) = \sum_{n=1}^{\infty} h_n(x)z^{n-1} = \frac{k + (b - kp)xz}{1 - pxz - qz^2}.
\]

It should be noted that for specific values of $k, b, p$ and $q$, the Horadam polynomial $h_n(x)$ leads to different polynomials, among those, we list a few cases here (see, [7, 8], for more details):

a) If $k = b = p = q = 1$, then we get the Fibonacci polynomials $F_n(x)$.

b) If $k = 2$ and $b = p = q = 1$, then we have the Lucas polynomials $L_n(x)$.

c) If $k = q = 1$ and $b = p = 2$, then we attain the Pell polynomials $P_n(x)$.

d) If $k = b = p = 2$ and $q = 1$, then we have the Pell-Lucas polynomials $Q_n(x)$.

e) If $k = b = 1, p = 2$ and $q = -1$, then we obtain the Chebyshev polynomials $T_n(x)$ of the first kind.

f) If $k = 1, b = p = 2$ and $q = -1$, then we attain the Chebyshev polynomials $U_n(x)$ of the second kind.

### Coefficient bounds and Fekete–Szegő inequality for the class $\mathcal{K}_x(\beta, x)$

**Definition 1** A function $f \in \Sigma$ is said to be in the class $\mathcal{K}_x(\beta, x)$ for $0 \leq \beta \leq 1$ and $x \in \mathbb{R}$, if the following conditions of subordination are satisfied:
\[
(1 - \beta)f'(z) + \beta \left(1 + \frac{zf''(z)}{f'(z)}\right) < \Omega(x, z) + 1 - k
\]
and
\[
(1 - \beta)g'(w) + \beta \left(1 + \frac{wg''(w)}{g'(w)}\right) < \Omega(x, w) + 1 - k,
\]
where the function $g = f^{-1}$ is indicated by (2) and $k$ is real constant.

**Remark 1**
For $\beta = 0$, the class $\mathcal{K}_x(\beta, x)$ shortens to the class $\Sigma'$ presented and investigated by Alamoush [2].

For $\beta = 1$, the class $\mathcal{K}_x(\beta, x)$ shortens to the class $\mathcal{K}_x(x)$ presented and investigated by Abirami et al. [1].

**Theorem 1** Let the function $f \in \Sigma$ indicated by (1) be in the class $\mathcal{K}_x(\beta, x)$. Then
\[
|a_2| \leq \frac{|bx|}{\sqrt{|(3 - \beta)b - 4p|bx^2 - 4kq|}}
\]
and
\[ |a_3| \leq \frac{b^2 x^2}{4} + \frac{|bx|}{3(\beta + 1)}, \quad (8) \]

and for some \( \mu \in \mathbb{R} \),
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|bx|}{3(\beta + 1)} & \text{if } |\mu - 1| \leq \frac{|(3 - \beta)b - 4p|bx^2 - 4kq|}{3(\beta + 1)b^2 x^2} \\
|bx|^3|\mu - 1| & \text{if } |(3 - \beta)b - 4p|bx^2 - 4kq| \\
\frac{|(3 - \beta)b - 4p|bx^2 - 4kq|}{3(\beta + 1)b^2 x^2} & \text{if } |\mu - 1| \geq \frac{|(3 - \beta)b - 4p|bx^2 - 4kq|}{3(\beta + 1)b^2 x^2}.
\end{cases} \quad (9)
\]

**Proof.** Let \( f \in \mathcal{K}(\beta, x), 0 \leq \beta \leq 1 \) and \( x \in \mathbb{R} \). Then there are two holomorphic function \( v, u : \Delta \to \Delta \) indicated by
\[
v(z) = t_1 z + t_2 z^2 + t_3 z^3 + \cdots \quad (z \in \Delta)
\]

and
\[
u(w) = s_1 w + s_2 w^2 + s_3 w^3 + \cdots \quad (w \in \Delta),
\]

with \( v(0) = u(0) = 0, |v(z)| < 1 \) and \( |u(w)| < 1 \), \( z, w \in \Delta \), such that
\[
(1 - \beta)f'(z) + \beta \left( 1 + \frac{zf''(z)}{f'(z)} \right) < \Omega(x, v(z)) + 1 - k
\]

and
\[
(1 - \beta)g'(w) + \beta \left( 1 + \frac{wg''(w)}{g'(w)} \right) < \Omega(x, u(w)) + 1 - k.
\]

Or, in equivalent way,
\[
(1 - \beta)f'(z) + \beta \left( 1 + \frac{zf''(z)}{f'(z)} \right) = 1 + h_1(z) - k + h_2(z)v(z) + h_3(z)[v(z)]^2 + \cdots \quad (10)
\]

and
\[
(1 - \beta)g'(w) + \beta \left( 1 + \frac{wg''(w)}{g'(w)} \right) = 1 + h_1(x) - k + h_2(x)u(w) + h_3(x)[u(w)]^2 + \cdots. \quad (11)
\]

From (10) and (11), we attain
\[
(1 - \beta)f'(z) + \beta \left( 1 + \frac{zf''(z)}{f'(z)} \right) = 1 + h_2(z) t_1 z + [h_2(z)t_2 + h_3(z)t_3^2]z^2 + \cdots \quad (12)
\]

and
\[
(1 - \beta)g'(w) + \beta \left( 1 + \frac{wg''(w)}{g'(w)} \right) = 1 + h_2(x)s_1 w + [h_2(x)s_2 + h_3(x)s_3^2]w^2 + \cdots. \quad (13)
\]

Notice that if
\[
|v(z)| = |t_1 z + t_2 z^2 + t_3 z^3 + \cdots| < 1 \quad (z \in \Delta)
\]

and
\[
|u(w)| = |s_1 w + s_2 w^2 + s_3 w^3 + \cdots| < 1 \quad (w \in \Delta),
\]
then

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It follows from (12) and (13) that

\[ 2a_2 = h_2(x)t_1, \]
\[ 3(1 + \beta)a_3 - 4\beta a_2^2 = h_2(x)t_2 + h_3(x)t_1^2, \]
\[ -2a_2 = h_2(x)s_1 \]
and
\[ -3(1 + \beta)a_3 + 2(\beta + 3)a_2^2 = h_2(x)s_2 + h_3(x)s_1^2. \]

From (14) and (16), we find that
\[ t_1 = -s_1 \]
and
\[ 8a_2^2 = [h_2(x)]^2(t_1^2 + s_1^2). \]

If we add (15) to (17), we get
\[ (6 - 2\beta)a_2^2 = h_2(x)(t_2 + s_2) + h_3(x)(t_1^2 + s_1^2). \]

By using (19) in equation (20), we have
\[ \left(6 - 2\beta\right) - \frac{8h_3(x)}{[h_2(x)]^2} a_2^2 = h_2(x)(t_2 + s_2), \]
which yields
\[ |a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|[(3 - \beta)b - 4p]bx^2 - 4kq|}}. \]

Next, if we deduct (17) from (15), we get
\[ 6(\beta + 1)(a_3 - a_2^2) = h_2(x)(t_2 - s_2) + h_3(x)(t_1^2 - s_1^2). \]

In view of (18) and (19), equation (22) becomes
\[ a_3 = \frac{[h_2(x)]^2(t_1^2 + s_1^2)}{8} + \frac{h_2(x)(t_2 - s_2)}{6(\beta + 1)}. \]

Now, with the help of equation (3), we deduce that
\[ |a_3| \leq \frac{b^2x^2}{4} + \frac{|bx|}{3(\beta + 1)}. \]

Finally, by using (21) and (22) for some \( \mu \in \mathbb{R} \), we get
\[ a_3 - \mu a_2^2 = \frac{h_2(x)(t_2 - s_2)}{6(\beta + 1)} + \frac{[h_2(x)]^2(1 - \mu)(t_2 + s_2)}{(6 - 2\beta)[h_2(x)]^2 - 8h_3(x)} \]
\[ = \frac{h_2(x)}{2} \left[ \Psi(\mu, x) + \frac{1}{3(\beta + 1)} \right] t_2 + \left[ \Psi(\mu, x) - \frac{1}{3(\beta + 1)} \right] s_2, \]
where
\[ \Psi(\mu, x) = \frac{[h_2(x)]^2(1 - \mu)}{(3 - \beta)[h_2(x)]^2 - 4h_3(x)}. \]

Thus, we conclude that
and with respect to (3), it evidently completes the proof of theorem (1).

Remark 2 If we put \( \beta = 0 \) in Theorem (1), we get the outcomes which were indicated by Alamoush [2]. In addition, if we put \( \beta = 1 \) in Theorem (1), we get the outcomes which were indicated by Abirami et al. [1].

Coefficient bounds and Fekete–Szegő inequality for the class \( \mathcal{W}_2(\alpha, x) \)

Definition 2 A function \( f \in \Sigma \) is said to be in the class \( \mathcal{W}_2(\alpha, x) \) for \( 0 \leq \alpha \leq 1 \) and \( x \in \mathbb{R} \), if the following conditions of subordination are satisfied:

\[
zf'(z) + (2\alpha^2 - \alpha)zf''(z) < \Omega(x, z) + 1 - k \tag{23}
\]

and

\[
wg'(w) + (2\alpha^2 - \alpha)wg''(w) < \Omega(x, w) + 1 - k, \tag{24}
\]

where the function \( g = f^{-1} \) is indicated by (2) and \( k \) is real constant.

Remark 3 For \( \alpha = 0 \), the class \( \mathcal{W}_2(\alpha, x) \) shortens to the class \( \mathcal{W}_2(x) \) introduced and investigated by Srivastava et al. [9].

Theorem 2 Let the function \( f \in \Sigma \) indicated by (1) be in the class \( \mathcal{W}_2(\alpha, x) \). Then

\[
|a_3| \leq \frac{|bx|}{\sqrt{|(12\alpha^4 - 28\alpha^3 + 15\alpha^2 + 2\alpha + 1)b - (1 + 3\alpha - 2\alpha^2)c^2p|bx^2 - (1 + 3\alpha - 2\alpha^2)c^2k^2|}} \tag{25}
\]

and

\[
|a_3| \leq \frac{b^2x^2}{(1 + 3\alpha - 2\alpha^2)c^2} + \frac{|bx|}{2(2\alpha^2 + 1)}, \tag{26}
\]

and for some \( \mu \in \mathbb{R} \),

\[
|a_3 - \mu a_2^2| \leq \begin{cases} |\mu - 1| \leq \frac{|bx|}{2(2\alpha^2 + 1)} & \text{if} \\
\frac{|bx^3|}{2(2\alpha^2 + 1)} & \text{if} \end{cases} \tag{27}
\]

Proof. Let \( f \in \mathcal{W}_2(\alpha, x) \), \( 0 \leq \alpha \leq 1 \) and \( x \in \mathbb{R} \). Then there are two holomorphic function \( v, u : \Delta \rightarrow \Delta \) indicated by

\[
v(z) = t_1z + t_2z^2 + t_3z^3 + \ldots \quad (z \in \Delta)
\]

and

\[
u(0) = u(0) = 0, |v(z)| < 1 \text{ and } |u(w)| < 1, \ z, w \in \Delta, \text{ such that}
\]

\[
zf'(z) + (2\alpha^2 - \alpha)zf''(z) < \Omega(x, v(z)) + 1 - k
\]

and
Or, in equivalent way,

\[
\frac{z f'(z) + (2a^2 - \alpha) z^2 f''(z)}{4(a - a^2)z + (2a^2 - \alpha)zf'(z) + (2a^2 - 3\alpha + 1)f(z)} = 1 + h_1(x) - k + h_2(x) \nu(z) + h_3(x)[\nu(z)]^2 + \ldots
\]  \quad (28)

and

\[
\frac{wg'(w) + (2a^2 - \alpha)w^2g''(w)}{4(a - a^2)w + (2a^2 - \alpha)wg'(w) + (2a^2 - 3\alpha + 1)g(w)} = 1 + h_1(x) - k + h_2(x) \nu(w) + h_3(x)[\nu(w)]^2 + \ldots. \quad (29)
\]

From the equations (28) and (29), we attain

\[
\frac{zf'(z) + (2a^2 - \alpha)z^2 f''(z)}{4(a - a^2)z + (2a^2 - \alpha)zf'(z) + (2a^2 - 3\alpha + 1)f(z)} = 1 + h_2(x) t_1 z + [h_2(x) t_2 + h_3(x) t_1^2] z^2 + \ldots
\]  \quad (30)

and

\[
\frac{wg'(w) + (2a^2 - \alpha)w^2g''(w)}{4(a - a^2)w + (2a^2 - \alpha)wg'(w) + (2a^2 - 3\alpha + 1)g(w)} = 1 + h_2(x) s_1 w + [h_2(x) s_2 + h_3(x) s_1^2] w^2 + \ldots. \quad (31)
\]

Notice that if

\[
|\nu(z)| = |t_1 z + t_2 z^2 + t_3 z^3 + \ldots| < 1 \quad (z \in \Delta)
\]

and

\[
|\nu(w)| = |s_1 w + s_2 w^2 + s_3 w^3 + \ldots| < 1 \quad (w \in \Delta),
\]

then

\[
|t_i| \leq 1 \text{ and } |s_i| \leq 1 \quad (i \in \mathbb{N}).
\]

It follows from (30) and (31) that

\[
(1 + 3\alpha - 2\alpha^2) a_2 = h_2(x) t_1,
\]  \quad (32)

\[
(12\alpha^4 - 28\alpha^3 + 11\alpha^2 + 2\alpha - 1)a_2^2 + (4\alpha^2 + 2)a_3 = h_2(x) t_2 + h_3(x) t_1^2,
\]  \quad (33)

\[
-(1 + 3\alpha - 2\alpha^2) a_2 = h_2(x) s_1
\]  \quad (34)

and

\[
(12\alpha^4 - 28\alpha^3 + 19\alpha^2 + 2\alpha + 3)a_2^2 - (4\alpha^2 + 2)a_3 = h_2(x) s_2 + h_3(x) s_1^2.
\]  \quad (35)

From (32) and (34), we find that

\[
t_1 = -s_1
\]  \quad (36)

and

\[
2(1 + 3\alpha - 2\alpha^2) a_2^2 = [h_2(x)]^2 (t_1^2 + s_1^2).
\]  \quad (37)

If we add (33) to (35), we get

\[
(24\alpha^4 - 56\alpha^3 + 30\alpha^2 + 4\alpha + 2)a_2^2 = h_2(x)(t_2 + s_2) + h_3(x)(t_1^2 + s_1^2).
\]  \quad (38)

By using (37) in equation (38), we have

\[
\left[(24\alpha^4 - 56\alpha^3 + 30\alpha^2 + 4\alpha + 2) - \frac{2(1 + 3\alpha - 2\alpha^2) h_3(x)}{[h_2(x)]^2}\right] a_2^2 = h_2(x)(t_2 + s_2),
\]  \quad (39)
which yields

$$|a_2| \leq \frac{|b| \sqrt{|b|}}{\sqrt{((12a^4-28a^3+15a^2+2a+1)b - (1+3a-2a^2)^2p)bx^2 - (1+3a-2a^2)^2kq}}.$$  

Next, if we deduct (35) from (33), we obtain

$$4(2a^2 + 1)(a_1 - a_2^2) = h_2(x)(s_2 - t_2) + h_3(x)(t_1^2 - s_1^2).$$  

In view of (36) and (37), equation (40) becomes

$$a_3 = \frac{[h_2(x)]^2(t_1^2 + s_1^2)}{2(1 + 3a - 2a^2)^2} + \frac{h_2(x)(t_2 - s_2)}{4(2a^2 + 1)}.$$  

Now, with the help of equation (3), we deduce that

$$|a_3| \leq \frac{b^2x^2}{(1 + 3a - 2a^2)^2} + \frac{|b|}{2(2a^2 + 1)}.$$  

Finally, by using (39) and (40) for some $\mu \in \mathbb{R}$, we get

$$a_3 - \mu a_2^2 = \frac{h_2(x)(t_2 - s_2)}{4(2a^2 + 1)} + \frac{[h_2(x)]^2(1 - \mu)(t_2 + s_2)}{(24a^4 - 56a^3 + 30a^2 + 4a + 2)[h_2(x)]^2 - (1 + 3a - 2a^2)^2h_3(x)}$$

$$= \frac{h_2(x)}{2} \left( \Psi(\mu, x) + \frac{1}{2(2a^2 + 1)} \right) t_2 + \left( \Psi(\mu, x) - \frac{1}{2(2a^2 + 1)} \right) s_2,$$

where

$$\Psi(\mu, x) = \frac{[h_2(x)]^2(1 - \mu)}{(12a^4 - 28a^3 + 15a^2 + 2a + 1)[h_2(x)]^2 - (1 + 3a - 2a^2)^2h_3(x)}.$$  

Thus, we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|h_2(x)|}{2(2a^2 + 1)} & \text{if } 0 \leq |\Psi(\mu, x)| \leq \frac{1}{2(2a^2 + 1)} \\
|h_2(x)||\Psi(\mu, x)| & \text{if } |\Psi(\mu, x)| \geq \frac{1}{2(2a^2 + 1)} \end{cases}$$

and with respect to (3), it evidently completes the proof of the theorem (2).

**Remark 4** If we put $\alpha = 0$ in Theorem (2), we get the outcomes which were indicated by Srivastava et al. [9].

**Coefficient bounds and Fekete–Szegö inequality for the class $N_2(\alpha, \gamma, x)$**

**Definition 3** A function $f \in \Sigma$ is said to be in the class $N_2(\alpha, \gamma, x)$ for $0 \leq \alpha \leq 1$, $\gamma \in \mathbb{C}\setminus\{0\}$ and $x \in \mathbb{R}$, if the following conditions of subordination are satisfied:

$$1 + \frac{1}{\gamma} \left[ \frac{ax^3f'''(z) + (1 + 2\alpha)x^2f''(z) + zf'(z)}{ax^2f''(z) + zf'(z)} - 1 \right] < \Omega(x, z) + 1 - k$$

and

$$1 + \frac{1}{\gamma} \left[ \frac{aw^3g'''(w) + (1 + 2\alpha)w^2g''(w) + wg'(w)}{aw^2g''(w) + wg'(w)} - 1 \right] < \Omega(x, w) + 1 - k,$$

where the function $g = f^{-1}$ is indicated by (2) and $k$ is real constant.

**Theorem 3** Let the function $f \in \Sigma$ indicated by (1) be in the class $N_2(\alpha, \gamma, x)$. Then

$$|a_2| \leq \frac{|\gamma|b|\sqrt{|b|}}{\sqrt{|\gamma(2 + 4\alpha - 4\alpha^2)b - 4(1 + \alpha)^2p|bx^2 - 4(1 + \alpha)^2kq|}}$$

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and

\[ |a_3| \leq \frac{|y|^2 b^2 x^2}{4(1 + a)^2} + \frac{|y||bx|}{6(1 + 2a)}, \tag{44} \]

and for some \( \mu \in \mathbb{R}, \)

\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{|y||bx|}{6(1 + 2a)} & \text{if} \quad \frac{|y|(1+2\alpha-2a^2)b-2(1+\alpha)^2p|b|z^2-2(1+\alpha)^2kq|}{3\sqrt[3]{(1+2\alpha)b^2x^2}} \leq |\mu - 1| \leq \frac{|y|^2|bx|^3|\mu - 1|}{|y|(2+4\alpha-4a^2)b-4(1+\alpha)^2p|b|z^2-4(1+\alpha)^2kq|} & \text{if} \end{cases} \tag{45} \]

**Proof.** Let \( f \in \mathcal{S}_D(\alpha, \gamma, x), \) \( 0 \leq \alpha \leq 1, \gamma \in \mathbb{C}\setminus\{0\} \) and \( x \in \mathbb{R}. \) Then there are two holomorphic function \( v, w : \Delta \to \Delta \) indicated by

\[ v(z) = t_1z + t_2z^2 + t_3z^3 + \ldots \quad (z \in \Delta) \]

and

\[ u(w) = s_1w + s_2w^2 + s_3w^3 + \ldots \quad (w \in \Delta), \]

with \( v(0) = u(0) = 0, |v(z)| < 1 \) and \( |u(w)| < 1, \) \( z, w \in \Delta, \) such that

\[ 1 + \frac{1}{\gamma} \left[ \frac{|az^3f'''(z) + (1 + 2\alpha)z^2f''(z) + zf'(z)}{az^2f''(z) + zf'(z)} - 1 \right] < \Omega(x, v(z)) + 1 - k \]

and

\[ 1 + \frac{1}{\gamma} \left[ \frac{|aw^3g'''(w) + (1 + 2\alpha)w^2g''(w) + wg'(w)}{aw^2g''(w) + wg'(w)} - 1 \right] < \Omega(x, u(w)) + 1 - k. \]

Or, in equivalent way,

\[ 1 + \frac{1}{\gamma} \left[ \frac{|az^3f'''(z) + (1 + 2\alpha)z^2f''(z) + zf'(z)}{az^2f''(z) + zf'(z)} - 1 \right] = 1 + h_1(x) - k + h_2(x)u(z) + h_3(x)[v(z)]^2 + \ldots \tag{46} \]

and

\[ 1 + \frac{1}{\gamma} \left[ \frac{|aw^3g'''(w) + (1 + 2\alpha)w^2g''(w) + wg'(w)}{aw^2g''(w) + wg'(w)} - 1 \right] = 1 + h_1(x) - k + h_2(x)u(w) + h_3(x)[u(w)]^2 + \ldots. \tag{47} \]

From (46) and (47), we get

\[ 1 + \frac{1}{\gamma} \left[ \frac{|az^3f'''(z) + (1 + 2\alpha)z^2f''(z) + zf'(z)}{az^2f''(z) + zf'(z)} - 1 \right] = 1 + h_2(x)t_2z + [h_2(x)t_2 + h_3(x)t_2^2]z^2 + \ldots \tag{48} \]

and

\[ 1 + \frac{1}{\gamma} \left[ \frac{|aw^3g'''(w) + (1 + 2\alpha)w^2g''(w) + wg'(w)}{aw^2g''(w) + wg'(w)} - 1 \right] = 1 + h_2(x)s_1w + [h_2(x)s_2 + h_3(x)s_2^2]w^2 + \ldots. \tag{49} \]

Notice that if

\[ |v(z)| = |t_1z + t_2z^2 + t_3z^3 + \ldots| < 1 \quad (z \in \Delta) \]
and

\[ |u(w)| = |s_1 w + s_2 w^2 + s_3 w^3 + \cdots| < 1 \quad (w \in \Delta), \]

then

\[ |t_i| \leq 1 \text{ and } |s_i| \leq 1 \quad (i \in \mathbb{N}). \]

It follows from (48) and (49) that

\[ \frac{2(1 + \alpha)}{\gamma} a_2 = h_2(x) t_1, \tag{50} \]

\[ \frac{6(1 + 2\alpha)}{\gamma} a_3 - \frac{4(1 + \alpha)^2}{\gamma} a^2_2 = h_2(x) t_2 + h_3(x) t_2^2, \tag{51} \]

\[ - \frac{2(1 + \alpha)}{\gamma} a_2 = h_2(x) s_1 \tag{52} \]

and

\[ \frac{6(1 + 2\alpha)}{\gamma} (2a^2_2 - a_3) - \frac{4(1 + \alpha)^2}{\gamma} a^2_2 = h_2(x) s_2 + h_3(x) s_2^2. \tag{53} \]

From (50) and (52), we find that

\[ t_1 = -s_1 \tag{54} \]

and

\[ \frac{8(1 + \alpha)^2}{\gamma^2} a^2_2 = [h_2(x)]^2 (t_1^2 + s_1^2). \tag{55} \]

If we add (51) to (53), we get

\[ \frac{(4 + 8\alpha - 8\alpha^2)}{\gamma} a^2_2 = h_2(x) (t_2 + s_2) + h_3(x) (t_2^2 + s_2^2). \tag{56} \]

By using (55) in equation (56), we have

\[ \left[ \frac{(4 + 8\alpha - 8\alpha^2)}{\gamma} - \frac{8(1 + \alpha)^2 h_3(x)}{\gamma^2 [h_2(x)]^2} \right] a^2_2 = h_2(x) (t_2 + s_2), \tag{57} \]

which yields

\[ |a_2| \leq \frac{|\gamma| b x |\sqrt{b x}|}{\sqrt{|(2 + 4\alpha - 4\alpha^2) b - 4(1 + \alpha)^2 p b x^2 - 4(1 + \alpha)^2 q |}}. \]

Next, if we deduct (53) from (51), we get

\[ \frac{12(1 + 2\alpha)}{\gamma} (a_3 - a^2_2) = h_2(x) (t_2 - s_2) + h_3(x) (t_2^2 - s_2^2). \tag{58} \]

In view of (54) and (55), equation (58) becomes

\[ a_3 = \frac{\gamma^2 [h_2(x)]^2 (t_1^2 + s_1^2)}{8(1 + \alpha)^2} + \frac{\gamma h_2(x) (t_2 - s_2)}{12(1 + 2\alpha)} \]

Now, with the help of equation (3), we conclude that

\[ |a_3| \leq \frac{|\gamma|^2 b^2 x^2}{4(1 + \alpha)^2} + \frac{|\gamma| b x}{6(1 + 2\alpha)}. \]

Finally, by using (57) and (58) for some \( \mu \in \mathbb{R} \), we get

\[ a_3 - \mu a^2_2 = \frac{\gamma h_2(x) (t_2 - s_2)}{12(1 + 2\alpha)} + \frac{\gamma^2 [h_2(x)]^3 (1 - \mu)(t_2 + s_2)}{8(1 + \alpha)^2 h_3(x)} - \frac{\gamma^2 [h_2(x)]^3 (1 - \mu)(t_2 + s_2)}{8(1 + \alpha)^2 h_3(x)} \]

\[ \frac{\gamma^2 [h_2(x)]^3 (1 - \mu)(t_2 + s_2)}{8(1 + \alpha)^2 h_3(x)} \]

\[ \frac{\gamma}{4(1 + \alpha)^2} \left( \frac{\gamma}{6(1 + 2\alpha)} \right). \]
\[ yh_2(x) = \frac{1}{2} \left( \Psi(\mu, x) + \frac{1}{6(1+2\alpha)} t_2 + \left( \Psi(\mu, x) - \frac{1}{6(1+2\alpha)} \right) s_2 \right), \]

where

\[ \Psi(\mu, x) = \frac{\gamma [h_2(x)]^2(1-\mu)}{\gamma (2 + 4\alpha - 4\alpha^2)[h_2(x)]^2 - 4(1 + \alpha)^2 h_3(x)}. \]

Thus, we conclude that

\[ |a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|\gamma||h_2(x)|}{6(1 + 2\alpha)} & \text{if } 0 \leq |\Psi(\mu, x)| \leq \frac{1}{6(1 + 2\alpha)} \\
\frac{|\gamma||h_2(x)||\Psi(\mu, x)|}{6(1 + 2\alpha)} & \text{if } |\Psi(\mu, x)| \geq \frac{1}{6(1 + 2\alpha)}
\end{cases} \]

and with respect to (3), it evidently completes the proof of the theorem (3).

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