The Policy Iteration Algorithm for Average Continuous Control of Piecewise Deterministic Markov Processes

O.L.V. Costa ∗†
Departamento de Engenharia de Telecomunicações e Controle
Escola Politécnica da Universidade de São Paulo
CEP: 05508 900-São Paulo, Brazil.
phone: 55 11 30915771; fax: 55 11 30915718.
e-mail: oswaldo@lac.usp.br

F. Dufour
Universite Bordeaux I
IMB, Institut Mathématiques de Bordeaux
INRIA Bordeaux Sud Ouest, Team: CQFD
351 cours de la Libération
33405 Talence Cedex, France
e-mail : dufour@math.u-bordeaux1.fr

February 16, 2009

Abstract

The main goal of this paper is to apply the so-called policy iteration algorithm (PIA) for the long run average continuous control problem of piecewise deterministic Markov processes (PDMP’s) taking values in a general Borel space and with compact action space depending on the state variable. In order to do that we first derive some important properties for a pseudo-Poisson equation associated to the problem. In the sequence it is shown that the convergence of the PIA to a solution satisfying the optimality equation holds under some classical hypotheses and that this optimal solution yields to an optimal control strategy for the average control problem for the continuous-time PDMP in a feedback form.

Keywords: piecewise-deterministic Markov Processes, continuous-time, long-run average cost, optimal control, integro-differential optimality inequation, policy iteration algorithm.
AMS 2000 subject classification: 60J25, 90C40, 93E20

∗This author received financial support from CNPq (Brazilian National Research Council), grant 304866/03-2 and FAPESP (Research Council of the State of São Paulo), grant 03/06736-7.
†Author to whom correspondence should be sent to.
This paper studies the policy iteration algorithm (PIA) for the average cost control problem of a class of continuous-time Markov processes, namely piecewise-deterministic Markov processes (PDMP’s). These processes have been introduced in the literature by M.H.A. Davis [7] as a general class of stochastic models. They are a family of Markov processes involving deterministic motion punctuated by random jumps. The motion of the PDMP \( \{X(t)\} \) depends on three local characteristics, namely the flow \( \phi \), the jump rate \( \lambda \) and the transition measure \( Q \), which specifies the post-jump location. Starting from \( x \) the motion of the process follows the flow \( \phi(x,t) \) until the first jump time \( T_1 \) which occurs either spontaneously in a Poisson-like fashion with rate \( \lambda(\phi(x,t)) \) or when the flow \( \phi(x,t) \) hits the boundary of the state-space. In either case the location \( Z_1 \) of the process at the jump time \( T_1 \) is selected by the transition measure \( Q(\phi(x, T_1), .) \). Starting from \( Z_1 \), we now select the next interjump time \( T_2 - T_1 \) and postjump location \( X(T_2) = Z_2 \) in a similar way. This gives a piecewise deterministic trajectory for \( \{X(t)\} \) with jump times \( \{T_k\} \) and postjump locations \( \{Z_k\} \), and which follows the flow \( \phi \) between two jumps. A suitable choice of the state space and the local characteristics \( \phi, \lambda, \) and \( Q \) provide stochastic models covering a great number of problems of operations research [7].

The present work is a continuation of a series of papers: [4, 5]. It deals with the long run average cost control problem of PDMP’s taking values in a general Borel space. At each point \( x \) of the state space a control variable is chosen from a compact action set \( U(x) \) and is applied on the jump parameter \( \lambda \) and transition measure \( Q \). The long run average cost is composed of a running cost and a boundary cost (which is added each time the PDMP touches the boundary). In this context, we follow the idea developed in [4, 5] consisting of writing the optimality equation for the long run average cost control problem of the PDMP \( \{X(t)\} \) in terms of a discrete-time optimality equation related to the embedded Markov chain given by the post-jump location of the process \( \{X(t)\} \). As pointed out in [4], this discrete-time optimality equation is different from those classical ones encountered within the context of discrete-time Markov decision processes. The two main reasons for doing that is to use the powerful tools developed in the discrete-time framework (see for example the references [2, 8, 11, 13]) and to avoid working with the infinitesimal generator associated to a PDMP, which in most cases has its domain of definition difficult to be characterized.

The PIA has received considerable attention in the literature and consists of three steps: initialization, policy evaluation, which is related to the Poisson equation (PE) associated to the transition law defining the Markov decision process, and policy improvement. Without attempting to present here an exhaustive panorama of the literature for the PIA, we can mention the surveys [1, 3, 14, 15, 16] and the references therein and more specifically the references [12, 15] that analyze in details the PIA for general Markov decision processes and provide conditions which guarantee its converge.

The paper is organized as follows. We shall formulate in section 2 the control problem while in section 3 some of the main assumptions are presented. In our context, the policy evaluation step is connected to a kind of PE which we call a pseudo-Poisson equation. This equation is clearly different from a classical PE encountered in the literature of the discrete-time Markov control processes, see Remark 4.2. However, although different, we can show in section 4 that this pseudo-Poisson equation still has the good properties that we might expect to satisfy in order to guarantee the convergence of the policy iteration algorithm. These results are not straightforward to obtain due to the specific structure of this discrete-time optimality equation. Finally in section 5, the PIA is studied in details. It is first shown that the convergence of the PIA to a solution satisfying the optimality equation holds under some classical hypotheses. In the sequence it is shown that this optimal solution yields to an optimal control strategy for the average control problem for the continuous-time PDMP in a feedback form.
2 Definitions and problem formulation

2.1 Presentation of the control problem

In this section we present some standard notation and some basic definitions related to the motion of a PDMP \( \{x(t)\} \), and the control problems we will consider throughout the paper. For further details and properties the reader is referred to [7]. The following notation will be used in this paper: \( \mathbb{N} \) denotes the set of natural numbers, \( \mathbb{R} \) the set of real numbers, \( \mathbb{R}_+ \) the set of positive real numbers and \( \mathbb{R}^d \) the \( d \)-dimensional euclidian space. We write \( \eta \) as the Lebesgue measure on \( \mathbb{R} \). For \( X \) a metric space \( \mathcal{B}(X) \) represents the \( \sigma \)-algebra generated by the open sets of \( X \). \( \mathcal{M}(X) \) (respectively, \( \mathcal{P}(X) \)) denotes the set of all finite (respectively probability) measures on \((X, \mathcal{B}(X))\). Let \( X \) and \( Y \) be metric spaces. The set of all Borel measurable (respectively bounded) functions from \( X \) into \( Y \) is denoted by \( \mathcal{M}(X; Y) \) (respectively \( \mathcal{B}(X; Y) \)). Moreover, for notational simplicity \( \mathcal{M}(X) \) (respectively \( \mathcal{B}(X), \mathcal{M}(X; \mathbb{R}), \mathcal{M}(X; \mathbb{R}_+), \mathcal{B}(X; \mathbb{R}_+) \)). For \( g \in \mathcal{M}(X) \) with \( g(x) > 0 \) for all \( x \in X \), \( \mathcal{C}(X) \) is the set of functions \( v \in\mathcal{M}(X) \) such that \( \|v(x)\|_g = \sup_{x \in X} \frac{|v(x)|}{g(x)} < +\infty \).

\( \mathcal{C}(X) \) denotes the set of continuous functions from \( X \) to \( \mathbb{R} \). For \( h \in \mathcal{M}(E) \), \( h^+ \) (respectively \( h^- \)) denotes the positive (respectively, negative) part of \( h \).

Let \( E \) be an open subset of \( \mathbb{R}^n \), \( \partial E \) its boundary, and \( \overline{E} \) its closure. A controlled PDMP is determined by its local characteristics \((\phi, \lambda, Q)\), as presented in the sequel. The flow \( \phi(x, t) \) is a function \( \phi: \mathbb{R}^n \times \overline{\mathbb{R}_+} \to \mathbb{R}^n \) continuous in \((x, t)\) and such that \( \phi(x, t+s) = \phi(x, t), s \). For each \( x \in E \) the time the flow takes to reach the boundary starting from \( x \) is defined as \( t_*(x) = \inf\{t > 0 : \phi(x, t) \in \partial E\} \). For \( x \in E \) such that \( t_*(x) = \infty \) (that is, the flow starting from \( x \) never touches the boundary), we set \( \phi(x, t_*(x)) = \Delta \), where \( \Delta \) is a fixed point in \( \partial E \). We define the following space of functions absolutely continuous along the flow with limit towards the boundary:

\[
\mathcal{M}^{ac}(E) = \{ g \in \mathcal{M}(E) : (g(\phi(x, t)) : [0, t_*(x)) \to \mathbb{R} \text{ is absolutely continuous for each } x \in E \}
\text{ and whenever } t_*(x) < \infty \text{ the limit } \lim_{t \to t_*(x)} g(\phi(x, t)) \text{ exists} \}.
\]

For \( g \in \mathcal{M}^{ac}(E) \) and \( z \in \partial E \) for which there exists \( x \in E \) such that \( z = \phi(x, t_*(x)) \) we define \( g(z) = \lim_{t \to -t_*(x)} g(\phi(x, t)) \) (note that the limit exists by assumption). As shown in Lemma 2 in [4] for \( g \in \mathcal{M}^{ac}(E) \) there exists a function \( Xg \in \mathcal{M}(E) \) such that for all \( x \in E \) and \( t \in [0, t_*(x)) \)
\[
g(\phi(x, t)) = g(x) + \int_0^t Xg(\phi(x, s))ds.
\]

The local characteristics \( \lambda \) and \( Q \) depend on a control action \( u \in \mathcal{U} \) where \( \mathcal{U} \) is a compact metric space (there is no loss of generality in assuming this property for \( \mathcal{U} \), see Remark 2.8 in [4]), in the following way: \( \lambda \in \mathcal{M}(E \times \mathcal{U})^+ \) and \( Q \) is a stochastic kernel on \( E \) given \( E \times \mathcal{U} \). For each \( x \in E \) we define the subsets \( \mathcal{U}(x) \) of \( \mathcal{U} \) as the set of feasible control actions that can be taken when the state process is \( x \in E \), that is, the control action that will be applied to \( \lambda \) and \( Q \) must belong to \( \mathcal{U}(x) \). The following assumptions, based on the standard theory of Markov decision processes (see for example [11]), will be made throughout the paper:

**Assumption 2.1** For all \( x \in \overline{E}, \mathcal{U}(x) \) is a compact subspace of \( \mathcal{U} \).

**Assumption 2.2** The set \( K = \{(x, a) : x \in \overline{E}, a \in \mathcal{U}(x)\} \) is a Borel subset of \( \overline{E} \times \mathcal{U} \).

We present next the definition of an admissible control strategy and the associated motion of the controlled process. A control policy \( U \) is a pair of functions \((u, u_\theta) \in \mathcal{M}(\mathbb{N} \times E \times \mathbb{R}_+; \mathcal{U}) \times \mathcal{M}(\mathbb{N} \times E; \mathcal{U})\) satisfying \( u(n, x, t) \in \mathcal{U}(\phi(x, t)) \), and \( u_\theta(n, x) \in \mathcal{U}(\phi(x, t_*(x))) \) for all \( (n, x, t) \in \mathbb{N} \times E \times \mathbb{R}_+ \). The class
of admissible control strategies will be denoted by $U$. Consider the state space $\mathcal{E} = E \times E \times \mathbb{R}_+ \times \mathbb{N}$. For a control policy $U = (u, u_0)$ let us introduce the following parameters for $\dot{x} = (x, z, s, n) \in \mathcal{E}$: the flow $\dot{\varphi}(x, t) = (\varphi(x, t), z, s + t, n)$, the jump rate $\dot{\lambda}(x) = \lambda(x, z, n)$, and the transition measure $\dot{Q}^{\lambda}(\dot{x}, A \times B \times \{0\} \times \{n + 1\}) = \begin{cases} Q(x, (u(n, z, s)); A \cap B) & \text{if } x \in E, \\ Q(x, u_0(n, z); A \cap B) & \text{if } x \in \partial E, \end{cases}$ for $A$ and $B$ in $B(E)$. From [7, section 25], it can be shown that for any control strategy $U = (u, u_0) \in U$ there exists a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \{P_{\lambda}^{U}\}_{\lambda \in \mathcal{E}})$ such that the piecewise deterministic Markov process $\{\mathcal{X}^{\lambda}(t)\}$ with local characteristics $(\dot{\varphi}, \dot{\lambda}, Q^{\lambda})$ may be constructed as follows. For notational simplicity the probability $P_{\lambda}^{U}$ will be denoted by $P_{\lambda}(x, k)$ for $x_0 = (x, x, 0, k) \in \mathcal{E}$. Take a random variable $T_1$ such that

$$P_{\lambda}(x, k)(T_1 > t) = \begin{cases} e^{-\Lambda(x, k, t)} & \text{for } t < t(x) \\ 0 & \text{for } t \geq t(x) \end{cases}$$

where for $x \in E$ and $t \in [0, t(x)]$, $\Lambda(x, k, t) = \int_0^t \lambda(\varphi(x, s), u(k, x, s))ds$. If $T_1$ is equal to infinity, then for $t \in \mathbb{R}_+$, $\mathcal{X}^{\lambda}(t) = (\varphi(x, t), x, t, k)$. Otherwise select independently an $\mathcal{E}$-valued random variable (labelled $\mathcal{X}_1^{\lambda}$) having distribution

$$P_{\lambda}(x, k)(\mathcal{X}_1^{\lambda} \in A \times B \times \{0\} \times \{k + 1\}) = \begin{cases} Q(\varphi(x, T_1), u(k, x, T_1)); A \cap B) & \text{if } \varphi(x, T_1) \in E, \\ Q(\varphi(x, T_1), u_0(k, x); A \cap B) & \text{if } \varphi(x, T_1) \in \partial E. \end{cases}$$

The trajectory of $\{\mathcal{X}^{\lambda}(t)\}$ starting from $(x, x, 0, k)$, for $t \leq T_1$, is given by

$$\mathcal{X}^{\lambda}(t) = \begin{cases} (\varphi(x, t), x, t, k) & \text{for } t < T_1, \\ \mathcal{X}_1^{\lambda} & \text{for } t = T_1. \end{cases}$$

Starting from $\mathcal{X}^{\lambda}(T_1) = \mathcal{X}_1^{\lambda}$, we now select the next inter-jump time $T_2 - T_1$ and post-jump location $\mathcal{X}^{\lambda}(T_2) = \mathcal{X}_2^{\lambda}$ in a similar way. Let us define the components of the PDMP $\{\mathcal{X}^{\lambda}(t)\}$ by

$$\mathcal{X}^{\lambda}(t) = (X(t), Z(t), \tau(t), N(t)).$$

For notational convenience, we have omitted to write explicitly the dependence of $U$ on the components: $X(t)$, $Z(t)$, $\tau(t)$ and $N(t)$. From the previous construction, it is easy to see that $X(t)$ corresponds to the trajectory of the system, $Z(t)$ is the value of $X(t)$ at the last jump time before $t$, $\tau(t)$ is the time elapsed from the last jump up to time $t$, and $N(t)$ is the number of jumps of the process $\{X(t)\}$ up to time $t$. As in Davis [4], we consider the following assumption to avoid any accumulation point of the jump times:

**Assumption 2.3** For any $x \in E$, $U \in U$, and $t \geq 0$, we have $E_{(x, 0)}^{U} \left[ \sum_{i=1}^{\infty} I_{(T_i \leq t)} \right] < \infty.$

**Remark 2.4** In particular, a consequence of **Assumption 2.3** is that $T_m \to \infty$ as $m \to \infty$ $P_{(x, 0)}^{U}$ for all $x \in E$, $U \in U$.

The costs of our control problem will contain two terms, a running cost $f$ and a boundary cost $r$, satisfying the following properties:
Assumption 2.5 \( f \in \mathcal{M}(\mathbb{E} \times \mathbb{U})^+ \), and \( r \in \mathcal{M}(\partial \mathbb{E} \times \mathbb{U})^+ \).

Define for \( \alpha \geq 0, t \in \mathbb{R}_+ \), and \( U \in \mathcal{U} \),
\[
\mathbf{J}^\alpha(U, t) = \int_0^t e^{-\alpha s} f(X(s), u(N(s), Z(s), \tau(s))) ds + \int_0^t e^{-\alpha s} r(X(s), u(N(s), Z(s))) dp^\ast(s),
\]
where \( p^\ast(t) = \sum_{i=1}^\infty I\{T_i \leq t\} \tilde{I}\{X(T_i) \in \partial \mathbb{E}\} \) counts the number of times the process hits the boundary up to time \( t \) and, for notational simplicity, set \( \mathbf{J}(U, t) = \mathbf{J}^0(U, t) \). The long-run average cost we want to minimize over \( \mathcal{U} \) is given by: \( \mathcal{A}(U, x) = \lim_{t \to +\infty} \frac{1}{t} \mathbf{E}^U(x, 0)[\mathbf{J}(U, t)] \). We need the following assumption, to avoid infinite costs for the discounted case, see [4].

Assumption 2.6 For all \( \alpha > 0 \) and all \( x \in E \), \( \inf_{U \in \mathcal{U}} \mathbf{E}^U(x, 0)[\mathbf{J}^\alpha(U, \infty)] < \infty \).

2.2 Discrete-time relaxed and ordinary controls

We present in this sub-section the set of discrete-time relaxed and ordinary controls. Consider \( \mathbb{C}(\mathcal{U}) \), and equipped with the topology of uniform convergence and \( \mathcal{M}(\mathcal{U}) \) equipped with the weak* topology \( \sigma(\mathcal{M}(\mathcal{U}), \mathbb{C}(\mathcal{U})) \). For \( x \in E \), define \( \mathcal{P}_x(\mathcal{U}) \) as the set of measures \( \mu \in \mathcal{P}(\mathcal{U}) \) satisfying \( \mu(U(\phi(x, t_s(x)))) = 1 \), \( \mathcal{P}(\mathcal{U}) \) and \( \mathcal{P}_x(\mathcal{U}) \) for \( x \in E \) are subsets of \( \mathcal{M}(\mathcal{U}) \) and equipped with the relative topology.

Let \( \mathbb{V}^r \) (respectively \( \mathbb{V}^r(x) \) for \( x \in E \)) be the set of all \( \eta \)-measurable functions \( \mu \) defined on \( \mathbb{R}_+ \) with value in \( \mathcal{P}(\mathcal{U}) \) such that \( \mu(t, U) = 1 \) \( \eta \)-a.e. (respectively \( \mu(t, U(\phi(x, t))) = 1 \) \( \eta \)-a.e.). It can be shown (see sub-section 3.1 in [4]) that \( \mathbb{V}^r(x) \) is a compact set of the metric space \( \mathbb{V}^r \). A sequence \( (\mu_n)_{n \in \mathbb{N}} \) in \( \mathbb{V}^r(x) \) converges to \( \mu \) if and only if for all \( g \in L^1(\mathbb{R}_+; \mathbb{C}(\mathcal{U})) \)
\[
\lim_{n \to \infty} \int_{\mathbb{R}_+} \int_{\mathcal{U}(\phi(x,t))} g(t, u) \mu_n(t, du) dt = \int_{\mathbb{R}_+} \int_{\mathcal{U}(\phi(x,t))} g(t, u) \mu(t, du) dt.
\]

The set of relaxed controls can be defined as follows: \( \mathbb{V}^r(x) = \mathbb{V}^r(x) \times \mathcal{P}_x(\mathcal{U}) \), for \( x \in E \) and \( \mathbb{V}^r = \mathbb{V}^r \times \mathcal{P}(\mathcal{U}) \). The set of ordinary controls, denoted by \( \mathbb{V} \) (respectively \( \mathbb{V}(x) \) for \( x \in E \)), is defined as above except that it is composed of deterministic functions instead of probability measures. More specifically we have \( \mathbb{V}(x) = \{ \nu \in \mathcal{M}(\mathbb{R}_+, \mathcal{U}) : (\forall t \in \mathbb{R}_+), \nu(t) \in \mathcal{U}(\phi(x, t)) \} \), \( \mathbb{V}(x) = \mathbb{V}(x) \times \mathcal{U}(\phi(x, t_s(x))) \), \( \mathbb{V} = \mathcal{M}(\mathbb{R}_+, \mathcal{U}) \times \mathcal{U} \). Consequently, the set of ordinary controls is a subset of the set of relaxed controls \( \mathbb{V}^r \) (respectively \( \mathbb{V}^r(x) \) for \( x \in E \)) by identifying any control action \( u \in \mathcal{U} \) with the Dirac measure concentrated on \( u \). Thus we can write that \( \mathbb{V} \subset \mathbb{V}^r \) (respectively \( \mathbb{V}(x) \subset \mathbb{V}^r(x) \) for \( x \in E \)) and from now on we will consider that \( \mathbb{V} \) (respectively \( \mathbb{V}(x) \) for \( x \in E \)) will be endowed with the topology generated by \( \mathbb{V}^r \). The necessity to introduce the class of relaxed control \( \mathbb{V}^r \) is justified by the fact that in general there does not exist a topology for which \( \mathbb{V} \) and \( \mathbb{V}(x) \) are compact sets.

As in [11], page 14, we need that the set of feasible state/relaxed-control pairs is a measurable subset of \( \mathcal{B}(E) \times \mathcal{B}(\mathbb{V}^r) \), that is, we need the following assumption.

Assumption 2.7 \( \mathcal{K} = \{ (x, \Theta) : \Theta \in \mathbb{V}^r(x), x \in E \} \in \mathcal{B}(E) \times \mathcal{B}(\mathbb{V}^r) \).

A sufficient condition is presented in [4, Proposition 3.3] to ensure that Assumption 2.7 holds.

2.3 Discrete-time operators and measurability properties

In this sub-section we present some important operators associated to the optimality equation of the discrete-time problem. We consider the following notation \( w(x, \mu) \doteq \int_{\mathcal{U}} w(x, u) \mu(du) \) and \( Qh(x, \mu) \doteq \int_{\mathcal{U}} Qh(x, u) \mu(du) \)
\[ \int_U \int_E h(z)Q(x, u; dz)\mu(du), \text{ and } \lambda Q h(x, \mu) = \int_U \lambda(x, u) \int_E h(z)Q(x, u; dz)\mu(du) \text{ for } x \in \overline{E}, \mu \in \mathcal{P}(U), \]

\[ h \in \mathcal{M}(E)^+ \text{ and } w \in \mathcal{M}(E \times U)^+. \]

The following operators will be associated to the optimality equations of the discrete-time problems that will be presented in the next sections. For \( \Theta = (\mu, \mu_0) \in \mathcal{V}^r, (x, A) \in E \times \mathcal{B}(E), \alpha \in \mathbb{R}, \) according to Lemma 2 in [3, Appendix 5] define

\[ \Lambda^\mu(x, t) = \int_0^t \lambda(\phi(y, s), \mu(s))ds \]

\[ G_\alpha(x, \Theta; A) = \int_0^{t(x)} e^{-\alpha s - \Lambda^\mu(x, s)} \lambda Q I A(\phi(x, s), \mu(s))ds + e^{-\alpha t(x) - \Lambda^\mu(x, t(x))} Q(\phi(x, t(x)), \mu_0; A). \] (2)

For \( h \in \mathcal{M}(E)^+ \), we define \( G_\alpha h(x, \Theta) = \int_E h(y)G_\alpha(x, \Theta; dy). \) For \( x \in E, \Theta = (\mu, \mu_0) \in \mathcal{V}^r, v \in \mathcal{M}(E \times U)^+, \alpha \in \mathbb{R}, \) introduce

\[ L_\alpha v(x, \Theta) = \int_0^{t(x)} e^{-\alpha s - \Lambda^\mu(x, s)} v(\phi(x, s), \mu(s))ds, \]

\[ H_\alpha w(x, \Theta) = e^{-\alpha t(x) - \Lambda^\mu(x, t(x))} w(\phi(x, t(x)), \mu_0). \] (4)

For \( h \in \mathcal{M}(E) \) (respectively, \( v \in \mathcal{M}(E \times U) \)), \( G_\alpha h(x, \Theta) = G_\alpha h^+(x, \Theta) - G_\alpha h^-(x, \Theta) \) (respectively, \( L_\alpha v(x, \Theta) = L_\alpha v^+(x, \Theta) - L_\alpha v^-(x, \Theta) \)) provided the difference has a meaning. It will be useful in the sequel to define the function \( \mathcal{L}_\alpha(x, \Theta) \) as follows: \( \mathcal{L}_\alpha(x, \Theta) = L_\alpha I_{E \times U}(x, \Theta). \) In particular for \( \alpha = 0 \) we write for simplicity \( G_0 = G, L_0 = L, H_0 = H, L_0 = L. \) Measurability properties of the operators \( G_\alpha, L_\alpha, \) and \( H_\alpha \) are shown in [4, Proposition 3.4].

We present now the definitions of the one-stage optimization operators.

**Definition 2.8** Let \( \alpha \in \mathbb{R}_+, \rho \in \mathbb{R}, \) and \( h \in \mathcal{M}(E). \) Assume that for any \( x \in E \) and \( \Upsilon \in \mathcal{V}(x), \)

\[ -\rho \mathcal{L}_\alpha(x, \Upsilon) + L_\alpha f(x, \Upsilon) + H_\alpha r(x, \Upsilon) + G_\alpha h(x, \Upsilon) \]

is well defined. The (ordinary) one-stage optimization operator is defined by

\[ T_\alpha(\rho, h)(x) = \inf_{\Upsilon \in \mathcal{V}(x)} \left\{ -\rho \mathcal{L}_\alpha(x, \Upsilon) + L_\alpha f(x, \Upsilon) + H_\alpha r(x, \Upsilon) + G_\alpha h(x, \Upsilon) \right\}. \]

Assume that for any \( x \in E \) and \( \Theta \in \mathcal{V}^r(x), \) \( -\rho \mathcal{L}_\alpha(x, \Theta) + L_\alpha f(x, \Theta) + H_\alpha r(x, \Theta) + G_\alpha h(x, \Theta) \)

is well defined. The relaxed one-stage optimization operator is defined by

\[ R_\alpha(\rho, h)(x) = \inf_{\Theta \in \mathcal{V}^r(x)} \left\{ -\rho \mathcal{L}_\alpha(x, \Theta) + L_\alpha f(x, \Theta) + H_\alpha r(x, \Theta) + G_\alpha h(x, \Theta) \right\}. \]

In particular for \( \alpha = 0 \) we write for simplicity \( T_0 = T, \) and \( R_0 = R. \)

The sets of measurable selectors associated to \((\mathcal{U}(x))_{x \in E}, (\mathcal{V}(x))_{x \in E}, (\mathcal{V}^r(x))_{x \in E}\) are defined by \( \mathcal{S}_\mathcal{U} = \left\{ u \in \mathcal{M}(\overline{E}, \mathcal{U}) : (\forall x \in \overline{E}), u(x) \in \mathcal{U}(x) \right\}, \)

\( \mathcal{S}_\mathcal{V} = \left\{ (\nu, \nu_0) \in \mathcal{M}(E, \mathcal{V}) : (\forall x \in E), (\nu(x), \nu_0(x)) \in \mathcal{V}(x) \right\}, \)

\( \mathcal{S}_{\mathcal{V}^r} = \left\{ (\mu, \mu_0) \in \mathcal{M}(E, \mathcal{V}^r) : (\forall x \in E), (\mu(x), \mu_0(x)) \in \mathcal{V}^r(x) \right\}. \)

For \( \alpha \in \mathbb{R}_+, \rho \in \mathbb{R}, \) and \( v \in \mathcal{M}(E), \) the one-stage optimization problem associated to the operator \( T_\alpha(\rho, v), \) respectively \( R_\alpha(\rho, v), \) consists of finding a measurable selector \( \Upsilon \in \mathcal{S}_\mathcal{V}, \) respectively \( \Theta \in \mathcal{S}_{\mathcal{V}^r}, \) respectively.
such that for all \( x \in E, T_\alpha(\rho, v)(x) = -\rho L_\alpha(x, \Upsilon) + L_\alpha f(x, \Upsilon) + H_\alpha r(x, \Upsilon) + G_\alpha v(x, \Upsilon) \) and respectively \( R_\alpha(\rho, v)(x) = -\rho L_\alpha(x, \Theta) + L_\alpha f(x, \Theta) + H_\alpha r(x, \Theta) + G_\alpha v(x, \Theta) \).

Finally we conclude this section by recalling (see Propositions 3.8 and 3.10 in [4]) that there exist two natural mappings from \( S_V \) to \( S_Y \) and from \( S_U \) to \( U \).

**Definition 2.9** For \( u \in S_U \), define the measurable mapping \( u_\phi \) of the space \( E \) into \( V \) by \( u_\phi : x \to (u(\phi(x,)), u(\phi(x, t_s(x)))) \).

**Definition 2.10** For \( u \in S_U \), define the measurable mapping \( U_\phi \) of the space \( N \times E \times \mathbb{R}_+ \) into \( U \times U \) by \( U_\phi : (n, x, t) \to (u(\phi(x, t)), u(\phi(x, t_s(x)))) \) of the space \( N \times E \times \mathbb{R}_+ \) into \( U \times U \).

**Remark 2.11** The measurable selectors of the kind \( u_\phi \) as in Definition 2.9 are called ordinary feedback control measurable selectors in the class \( S_V \subset S_Y \) and the control strategies of the kind \( U_\phi \) as in definition 2.10 are called ordinary feedback control strategies in the class \( U \).

3 Assumptions

In order to prove our main results presented in section 5, we need to impose some conditions. Assumptions 3.1, 3.2 and 3.3 are needed to guarantee some convergence and continuity properties of the one-stage optimization operators, and the existence of a measurable selector. These properties are important to ensure the convergence of the policy iteration algorithm as shown in section 5.

**Assumption 3.1** For each \( x \in E \), the restriction of \( \lambda(x, \cdot) \) to \( U(x) \) is continuous, for \( t \in [0, t_s(x)) \),

\[
\int_0^t \sup_{a \in U(\phi(x, s))} \lambda(\phi(x, s), a) \, ds < \infty \quad \text{and if} \quad t_s(x) < \infty \quad \text{then} \quad \int_0^{t_s(x)} \sup_{a \in U(\phi(x, s))} \lambda(\phi(x, s), a) \, ds < \infty.
\]

**Assumption 3.2** For all \( y \in \overline{E} \), the restriction of \( f(y, \cdot) \) to \( U(y) \) is continuous and for all \( z \in \partial E \), the restriction of \( r(z, \cdot) \) to \( U(z) \) is continuous.

**Assumption 3.3** For all \( x \in \overline{E} \) and \( h \in \mathcal{B}(E) \), the restriction of \( Qh(x, \cdot) \) to \( U(x) \) is continuous.

The next assumption is mainly used to show that the policy iteration algorithm converges to the optimal cost and gives an optimal feedback control as shown in section 5.2. This condition is somehow related to the so-called expected growth condition (see, for instance, Assumption 3.1 in [10] for the discrete-time case, or Assumption A in [3] for the continuous-time case).

**Assumption 3.4** Suppose that there exist \( b \geq 0, c > 0, \delta > 0, M \geq 0 \) and \( g \in \mathcal{M}^{ac}(E), g \geq 1, \tau \in \mathcal{M}(\partial E), \tau(z) \geq 0 \), satisfying for all \( x \in E \)

\[
\sup_{a \in U(x)} \left\{ \mathcal{X}g(x) + cg(x) - \lambda(x, a) [g(x) - Qg(x, a)] \right\} \leq b, \tag{5}
\]

\[
\sup_{a \in U(x)} \left\{ f(x, a) \right\} \leq M g(x), \tag{6}
\]

and for all \( x \in E \) with \( t_s(x) < \infty \)

\[
\sup_{a \in U(\phi(x, t_s(x)))} \left\{ \tau(\phi(x, t_s(x))) + Qg(\phi(x, t_s(x)), a) \right\} \leq g(\phi(x, t_s(x))), \tag{7}
\]

\[
\sup_{a \in U(\phi(x, t_s(x)))} \left\{ r(\phi(x, t_s(x)), a) \right\} \leq \frac{M}{c + \delta} \tau(\phi(x, t_s(x))). \tag{8}
\]
In the next assumption notice that for any $u \in S_U$, $G(x, u_\Phi)$ can be seen as the stochastic kernel associated to the post-jump location of a PDMP. This assumption is related to some geometric ergodic properties of the operator $G$ (see for example the comments on page 122 in [13] or Lemma 3.3 in [10] for more details on this kind of assumption).

**Assumption 3.5** There exist $a > 0$, $0 < \kappa < 1$ and for any $u \in S_U$ there exists a probability measure $\nu_u$, such that $\nu_u(g) < +\infty$ and

$$|G^k h(x, u_\Phi) - \nu_a(h)| \leq a \|h\|_g \kappa^k g(x),$$

for all $h \in B_g(E)$ and $k \in \mathbb{N}$.

The following hypothesis is given by a Lyapunov-like inequality yielding an expected growth condition on the function $g$ with respect to $G$ (for further comments on this kind of assumption, see for example section 10.2 in [13, page 121]).

**Assumption 3.6** There exist $0 < k_g < 1$ and $K_g \geq 0$ such that for all $x \in E$, $\Gamma \in V(x)$,

$$Gg(x, \Gamma) \leq k_g g(x) + K_g.$$  (10)

The final assumption is:

**Assumption 3.7** There exist $\lambda \in \mathbb{M}(E)^+$, and $K_\lambda \in \mathbb{R}_+$ such that

1) $\lambda(y, a) \geq \lambda(y)$ for all $y \in E$ and $a \in U(y)$,

2) $\int_0^{t_+(x)} e^{ct - \int_0^t \Delta(\phi(x,s))ds} dt \leq K_\lambda$, for all $x \in E$,

3) $\lim_{t \to +\infty} \int_0^t \lambda(\phi(x,s))ds = 0$, for all $x \in E$ with $t_*(x) = +\infty$,

4) $\lim_{t \to +\infty} \int_0^t \lambda(\phi(x,s))ds g(\phi(x,t)) = 0$, for all $x \in E$ with $t_*(x) = \infty$,

5) $\int_0^{t_+(x)} e^{-\int_0^t \Delta(\phi(x,s))ds} \sup_{a \in U(\phi(x,t))} f(\phi(x,t), a) dt < \infty.$

**Remark 3.8** Notice the following consequences of Assumption [3.7]:

i) Assumption [3.7] c) implies that $G_\alpha(x, \Theta; A) = \int_0^{t_+(x)} e^{-\alpha s - \Lambda^x(s)} \lambda Q_{\Lambda}(\phi(x,s), \mu(s)) ds$, and $H_\alpha w(x, \Theta) = 0$, for any $x \in E$ with $t_*(x) = +\infty$, $A \in B(E)$, $\alpha \geq -c$, $\Theta = (\mu, \mu_\partial) \in V^r(x)$, $w \in \mathbb{M}(\partial E \times U)$.

ii) Assumptions [3.7] a) and b) imply that $L_\alpha(x, \Theta) \leq K_\lambda$ for any $\alpha \geq -c$, $x \in E$, $\Theta \in V^r(x)$.
4 A pseudo-Poisson equation

We introduce in Definition 4.1 a pseudo-Poisson equation associated to the stochastic kernel $G$. Proposition 4.3 shows that there exists a solution for such an equation. Moreover, it is proved in Proposition 4.4 that this equation has the important characteristic of ensuring the policy improvement property in the set $S_U$.

Definition 4.1 Consider $u \in S_U$. A pair $(\rho, h) \in \mathbb{R} \times \mathbb{B}_g(E)$ is said to satisfy the pseudo-Poisson equation associated to $u$ if

$$h(x) = -\rho \mathcal{L}(x, u_\phi(x)) + Lf(x, u_\phi(x)) + Hr(x, u_\phi(x)) + Gh(x, u_\phi(x)). \quad (11)$$

Remark 4.2 This equation is clearly different from a classical Poisson equation encountered in the literature of the discrete-time Markov control processes see for example equation (2.13) in [12]. In particular, the constant $\rho$, that will be shown to be the optimal cost, appears here as a multiplicative factor of the mapping $\mathcal{L}(x, u_\phi(x))$ and the costs $f$ and $r$ appear through the terms $Lf(x, u_\phi(x))$ and $Hr(x, u_\phi(x))$. However, it will be shown in the following propositions that this pseudo-Poisson equation has still the good properties that we might expect to satisfy in order to guarantee the convergence of the policy iteration algorithm.

Proposition 4.3 For arbitrary $u \in S_U$ the following assertions hold:

(a) Set $D_u = \int_E \mathcal{L}(y, u_\phi(y)) \nu_u(dy)$. Then $0 < D_u \leq K_\lambda$.

(b) If $v \in \mathbb{B}_g(E)$ and $b \in \mathbb{R}$ are such that for all $x \in E$,

$$v(x) = b \mathcal{L}(x, u_\phi(x)) + Gv(x, u_\phi(x)) \quad \text{(12)}$$

then $b = 0$ and for some $c_0 \in \mathbb{R}$, $v(x) = c_0$ for all $x \in E$.

(c) Let $w_u$ be the mapping in $\mathbb{M}(E)$ defined by $w_u(x) = Lf(x, u_\phi(x)) + Hr(x, u_\phi(x)) - \rho_u \mathcal{L}(x, u_\phi(x))$ for $x \in E$. Define $(\rho_u, h_u)$ by

$$\rho_u = \int_E \frac{[Lf(y, u_\phi(y)) + Hr(y, u_\phi(y))] \nu_u(dy)}{D_u} \geq 0, \quad (13)$$

$$h_u(x) = \sum_{k=0}^{\infty} G^k w_u(x, u_\phi(x)). \quad (14)$$

Then $(\rho_u, h_u) \in \mathbb{R} \times \mathbb{B}_g(E)$ and it is the unique solution to the Poisson equation (11) associated to $u$ that satisfies

$$\nu_u(h_u) = 0. \quad (15)$$

Moreover

$$\|h_u\|_g \leq \frac{aM_u}{1 - \kappa}, \text{ with } M_u := \max \left\{ \rho_u K_\lambda, \frac{M(1 + bK_\lambda)}{c} \right\}. \quad (16)$$
Proposition 4.4

Consider \( u \in \mathcal{S}_U \). Then there exists \( \tilde{u} \in \mathcal{S}_U \) such that

\[
\mathcal{R}(\rho_u, h_u)(x) = -\rho_u \mathcal{L}(x, u_\phi(x)) + Lf(x, u_\phi(x)) + Hr(x, u_\phi(x)) + G h_u(x, u_\phi(x)),
\]

and \( \rho_{\tilde{u}} \leq \rho_u \).
Step 2: Policy Evaluation - At the

Step 3: Policy Improvement - Determine

Step 1: Initialize with an arbitrary $u_0 \in \mathcal{S}_U$ and set $n = 0$.

Step 2: Policy Evaluation - At the $n^{th}$-iteration consider $u_n \in \mathcal{S}_U$ and evaluate $(\rho_n, h_n) \in \mathbb{R} \times \mathbb{B}_g(E)$ the

(21)

with $\nu_{u_n}(h_n) = 0$.

Step 3: Policy Improvement - Determine $u_{n+1} \in \mathcal{S}_U$ such that

(22)

Notice that from Propositions 4.3 and 4.4 the sequence $(\rho_n, h_n) \in \mathbb{R} \times \mathbb{B}_g(E)$ and $u_n \in \mathcal{S}_U$ is well-defined and moreover, $\rho_n \geq \rho_{n+1} \geq 0$. We set $\rho = \lim_{n \to \infty} \rho_n$. 

5 The Policy Iteration Algorithm

Having studied the pseudo-Poisson equation defined in section 4 we are now in position to analyze the policy iteration algorithm. In the first part, it is shown that the convergence of the policy iteration algorithm holds under a classical hypothesis (see for example assumption (H1) of Theorem 4.3 in [12]). Roughly speaking, it means that if the PIA computes a solution $(\rho_n, h_n)$ at the $n$th step then $(\rho_n, h_n) \to (\rho, h)$ and $(\rho, h)$ satisfies the optimality equation (24). However it is far from obvious to claim that $\rho$ is actually the optimal cost for the long run average cost problem of the PDMP \{X(t)\} and that there exists an optimal control. In the second part of this section, these two issues are studied. In particular, we show that $\rho = \inf_{U \in U} A(U, x)$ and the measurable selector $\hat{u}_\phi$ of the optimality equation (24) provides an optimal control of the feedback form $U_{\hat{u}_\phi}$ for the process \{X(t)\}: \inf_{U \in U} A(U, x) = A(U_{\hat{u}_\phi}, x). 

The policy iteration algorithm performs the following steps:

1. Initialize with an arbitrary $u_0 \in \mathcal{S}_U$, and set $n = 0$.

2. Policy Evaluation - At the $n^{th}$-iteration consider $u_n \in \mathcal{S}_U$ and evaluate $(\rho_n, h_n) \in \mathbb{R} \times \mathbb{B}_g(E)$ the

(24)

3. Policy Improvement - Determine $u_{n+1} \in \mathcal{S}_U$ such that

(25)

Notice that from Propositions 4.3 and 4.4 the sequence $(\rho_n, h_n) \in \mathbb{R} \times \mathbb{B}_g(E)$ and $u_n \in \mathcal{S}_U$ is well-defined and moreover, $\rho_n \geq \rho_{n+1} \geq 0$. We set $\rho = \lim_{n \to \infty} \rho_n$. 

11
5.1 Convergence of the PIA

First we present in the next result some convergence properties of \( G, H, L \) and \( \mathcal{L} \).

**Proposition 5.1** Consider \( h \in \mathcal{B}_g(E) \) and a sequence of functions \( \{h_k\}_{k \in \mathbb{N}} \in \mathcal{B}_g(E) \) such that for all \( x \in E \), \( \lim_{k \to \infty} h_k(x) = h(x) \) and there exists \( K_h \) satisfying \( |h_k(x)| \leq K_h g(x) \) for all \( k \) and all \( x \in E \). For \( x \in E \), consider \( \Theta_n = (\mu_n, \mu_{\partial_n}) \in \mathcal{V}^r(x) \) and \( \Theta = (\mu, \mu_{\partial}) \in \mathcal{V}^r(x) \) such that \( \Theta_n \to \Theta \). We have the following results:

\[
\begin{align*}
& a) \lim_{n \to \infty} \mathcal{L}(x, \Theta_n) = \mathcal{L}(x, \Theta), \\
& b) \lim_{n \to \infty} Lf(x, \Theta_n) = Lf(x, \Theta), \\
& c) \lim_{n \to \infty} Hr(x, \Theta_n) = Hr(x, \Theta), \\
& d) \lim_{n \to \infty} G_{hn}(x, \Theta_n) = G_{h}(x, \Theta).
\end{align*}
\]

**Proof:** The proof of item \( a) \) is the same as in Proposition 5.7 in [4] and it is essentially based on the fact that \( \lim_{n \to \infty} \Lambda^{\mu_n}(x, t) = \Lambda^\mu(x, t) \) by using assumption 3.1.

Item b) We have for \( x \in E \),

\[
L f(x, \Theta_n) = \int_0^{t_+(x)} \left[ e^{-\Lambda^{\mu_n}(x,t)} - e^{-\Lambda^\mu(x,t)} \right] f(\phi(x,t), \mu_n(t)) dt + \int_0^{t_+(x)} e^{-\Lambda^\mu(x,t)} f(\phi(x,t), \mu_n(s)) dt.
\]

By combining items \( a) \) and \( c) \) of assumption 3.7 and the dominated convergence theorem we obtain

\[
\lim_{n \to \infty} \int_0^{t_+(x)} \left| e^{-\Lambda^{\mu_n}(x,t)} - e^{-\Lambda^\mu(x,t)} \right| f(\phi(x,t), \mu_n(t)) dt = 0.
\]

Therefore, we obtain item \( b) \) by using assumption 3.2.

Item c) Let us consider first that \( t_+(x) = \infty \). From item \( i) \) of remark 3.3 it follows that \( Hr(x, \Theta_n) = Hr(x, \Theta) = 0 \). Suppose now that \( t_+(x) < \infty \) and set \( z = \phi(x, t_+(x)) \). From assumption 3.2 it follows that \( \lim_{n \to \infty} r(z, \mu_{\partial_n}) = r(z, \mu_{\partial}) \) showing item \( c) \).

Item d) Let \( \{\alpha_k\} \) a non increasing sequence of positive numbers with \( \alpha_k \downarrow 0 \). We have clearly \( \lim_{n \to \infty} G_{hn}(x, \Theta_n) \geq \lim_{n \to \infty} G_{\alpha_n} h_n(x, \Theta) \). It follows that \( \lim_{n \to \infty} G_{hn}(x, \Theta_n) \geq G_{h}(x, \Theta) \) by applying Proposition 3.18 in [3]. Replacing \( h_n \) by \( -h_n \) it gives that \( \lim_{n \to \infty} G_{hn}(x, \Theta_n) \leq G_{h}(x, \Theta) \), completing the proof of item \( d) \).

We shall consider now the following assumption.

**Assumption 5.2** There exists a subsequence \( \{h_k\} \) of \( \{h_n\} \) and \( h \in \mathcal{M}(E) \) such that for each \( x \in E \),

\[
\lim_{k \to \infty} h_k(x) = h(x).
\]

The following theorem is the main result of this subsection. It shows the convergence of the PIA and ensures the existence of a measurable selector for the optimality equation.
Theorem 5.3 We have that $(\rho, h) \in \mathbb{R} \times \mathbb{B}_g(E)$ satisfies the optimality equation:

$$h(x) = \mathcal{R}(\rho, h)(x).$$

Moreover there exists $\tilde{u} \in \mathcal{S}_\mathcal{V}$ such that

$$h(x) = -\rho \mathcal{L}(x, \Theta(x)) + Lf(x, \tilde{u}_\phi(x)) + Hr(x, \tilde{u}_\phi(x)) + Gh(x, \tilde{u}_\phi(x)).$$

Proof: From (16) and recalling that $\rho_n \geq \rho_{n+1}$ we get that for all $k$,

$$\|h_k\| \leq M := \frac{aM_u}{1 - \kappa}, \quad M_u := \max\{\rho_0 K_\lambda, \frac{M(1 + bK_\lambda)}{c}\}.$$  \hspace{1cm} (26)

From (26) we get that $h \in \mathbb{B}_g(E)$, where $h$ is as in (23). Consider $u_k \in \mathcal{S}_\mathcal{V}$ the measurable selector associated to $(\rho_k, h_k)$ as in (21). We have that for each $x \in E$, $\mathcal{V}(x)$ is compact and $\{(u_k)_\phi\}$ is a sequence in $\mathcal{S}_\mathcal{V}$. Then according to Proposition 8.3 in [12] (see also [17]) there exists $\Theta \in \mathcal{S}_\mathcal{V}$ such that $\Theta(x) \in \mathcal{V}(x)$ is an accumulation point of $\{(u_k)_\phi\}$ for each $x \in E$. Therefore for every $x \in E$, there exists a subsequence $k_i = k_i(x)$ such that $\lim_{i \to \infty}(u_{k_i})_\phi(x) = \Theta(x)$. We fix now $x \in E$ and we consider the sub-sequence $k_i = k_i(x)$ as above. From Proposition 5.1 and taking the limit in (21) for $n = k_i$ as $i \to \infty$ we have that

$$h(x) = -\rho \mathcal{L}(x, \Theta(x)) + Lf(x, \tilde{u}_\phi(x)) + Hr(x, \tilde{u}_\phi(x)) + Gh(x, \tilde{u}_\phi(x)),$$

and thus clearly $h(x) \geq \mathcal{R}(\rho, h)(x)$. On the other hand from (21) and (22) we have that

$$\mathcal{R}(\rho_{n-1}, h_{n-1})(x) + (\rho_{n-1} - \rho_n) \mathcal{L}(x, (u_n)_\phi(x)) + G(h_n - h_{n-1})(x, (u_n)_\phi(x))$$

$$= -\rho_n \mathcal{L}(x, (u_n)_\phi(x)) + Lf(x, (u_n)_\phi(x)) + Hr(x, (u_n)_\phi(x)) + Gh_n(x, (u_n)_\phi(x))$$

$$= h_n(x).$$ \hspace{1cm} (27)

From (28) it is immediate that for any $\tilde{\Theta} \in \mathcal{S}_\mathcal{V}$

$$h_n(x) \leq -\rho_{n-1} \mathcal{L}(x, \tilde{\Theta}(x)) + Lf(x, \tilde{\Theta}(x)) + Hr(x, \tilde{\Theta}(x)) + Gh_{n-1}(x, \tilde{\Theta}(x))$$

$$+ (\rho_{n-1} - \rho_n) \mathcal{L}(x, (u_n)_\phi(x)) + G(h_n - h_{n-1})(x, (u_n)_\phi(x)).$$ \hspace{1cm} (29)

Fix $x$ and $k_i = k_i(x)$ as before and notice that for any $y \in E$, $\lim_{i \to \infty}(h_{k_i}(y) - h_{k_i-1}(y)) = 0$ and from (26), $\|h_{k_i} - h_{k_i-1}\|_g \leq \tilde{M}$. Applying Proposition 5.1 into (29) replacing $n$ by $k_i$ and taking the limit as $i \to \infty$ yields that

$$h(x) \leq -\rho \mathcal{L}(x, \tilde{\Theta}(x)) + Lf(x, \tilde{\Theta}(x)) + Hr(x, \tilde{\Theta}(x)) + Gh(x, \tilde{\Theta}(x)),$$

and from (30) we get that $h(x) \leq \mathcal{R}(\rho, h)(x)$. Thus we have (24). \hspace{1cm} $\square$

5.2 Optimality of the PIA

We present next a definition that will be useful for the next results.

Definition 5.4 For any $\Theta = (\mu, \mu_\phi) \in \mathcal{V}$, define

$$[\Theta]_t = (\mu, + t, \mu_\phi).$$ \hspace{1cm} (31)
Let us recall that the PDMP \(\{\widehat{X}(t)\}\) and its associated components: \(X(t), Z(t), N(t), \tau(t)\) have been introduced in section 2.1 (see in particular equation (1)). We need several auxiliary results (Propositions 5.5, 5.6 and Corollary 5.7) to show that the PIA actually provides an optimal solution for the average cost problem of the PDMP \(X(t)\).

**Proposition 5.5** For \(\hat{y} = (y, z, s, n) \in \widehat{E}\) and \(U = (u, u_0) \in \mathcal{M}(\mathbb{N} \times E \times \mathbb{R}_+; \mathcal{U}) \times \mathcal{M}(\mathbb{N} \times E; \mathcal{U})\), define \(\Gamma^U(n, z) = (u(n, z, .), u_0(n, z)) \in \mathcal{U}\). For \(\epsilon \in (0, c)\) introduce

\[
\widehat{w}^U(\hat{y}) = \epsilon L_{-\epsilon}f(y, \Gamma^U(n, z)) + H_{-\epsilon}\mathcal{T}(y, \Gamma^U(n, z)) \in \mathcal{U} \\
- b\mathcal{L}_{-\epsilon}(y, \Gamma^U(n, z)),
\]

where \(\mathcal{T} = c - \epsilon\). Then for all \(x \in E, U \in \mathcal{U}\), we have

\[
E^U_{(x,0)} \left[ \widehat{w}^U(\widehat{X}^U(t)) \right] \leq e^{-\epsilon t}g(x) + \frac{b}{\epsilon}[1 - e^{-\epsilon t}].
\]

**Proof:** For \(\hat{y} = (y, z, s, n) \in \widehat{E}\) and \(U = (u, u_0) \in \mathcal{M}(\mathbb{N} \times E \times \mathbb{R}_+; \mathcal{U}) \times \mathcal{M}(\mathbb{N} \times E; \mathcal{U})\), define \(\hat{f}^U(\hat{y}) = f(y, u(n, z, s)), \hat{\mathcal{T}}^U(\hat{y}) = \mathcal{T}(y, u_0(n, z)), \hat{g}(\hat{y}) = g(y)\), and for \(t \in \mathbb{R}_+\), \(\hat{\Lambda}^U(y, t) = \Lambda^U(x, n, t)\). It is easy to show that \(\widehat{w}^U \in \mathcal{M}(\widehat{E})\). Moreover, for \(\hat{y} = (y, z, s, n) \in \widehat{E}\) and \(U = (u, u_0) \in \mathcal{M}(\mathbb{N} \times E \times \mathbb{R}_+; \mathcal{U}) \times \mathcal{M}(\mathbb{N} \times E; \mathcal{U})\), satisfying \(\Gamma^U(n, z) \in \mathcal{V}(y)\) we have by using Corollary 3.11 in [5] with \(\alpha = -\epsilon\) that

\[
\mathcal{T}L_{-\epsilon}f(y, \Gamma^U(n, z)) + H_{-\epsilon}\mathcal{T}(y, \Gamma^U(n, z)) + G_{-\epsilon}g(y, \Gamma^U(n, z)) \leq g(y).
\]

Moreover, from Remark 3.8 ii),

\[
0 < \mathcal{L}_{-\epsilon}(y, \Gamma^U(n, z)) \leq \mathcal{L}_{-\epsilon}(y, \Gamma^U(n, z)) \leq K_\lambda.
\]

From now on, consider \(U = (u, u_0) \in \mathcal{U}\). Notice that for any \(\hat{x} = (x, x, 0, k) \in \widehat{E}\)

\[
\widehat{w}^U(\hat{x}) = \epsilon L_{-\epsilon}f(x, \Gamma^U(k, x)) + H_{-\epsilon}\mathcal{T}(y, \Gamma^U(k, x)) + G_{-\epsilon}g(x, \Gamma^U(k, x)) - b\mathcal{L}_{-\epsilon}(x, \Gamma^U(k, x))
\]

\[
= \int_0^{\tau^U(x)} e^{s-\Lambda^U(x,t)} \left[ -b + \mathcal{T}(\phi(x,s), \nu_k(s)) + \lambda(\phi(x,s), \nu_k(s))Qg(\phi(x,s), \nu_k(s)) \right] ds
\]

\[
+ e^{\epsilon t_*(x)-\Lambda^U(x,t_*(x))} \left[ Qg(\phi(x, t_*(x)), u_0(k, x)) + \mathcal{T}(\phi(x, t_*(x)), u_0(k, x)) \right],
\]

with \(\nu_k(.) = u(k, x, .)\). Since for all \(k \in \mathbb{N}, x \in E, \Gamma^U(k, x) \in \mathcal{V}(x)\), it follows from equation (34) that

\[
\widehat{w}^U(\hat{x}) \leq g(x).
\]

Moreover, since \(\Gamma^U(N(t), Z(t))\) \(\tau(t) \in \mathcal{V}(X(t))\), the inequality (35) implies that

\[
J^U_m(t, \hat{x}) := E^U_{(x,k)} \left[ \int_0^{t \wedge \tau_m} e^{s-\Lambda^U(\hat{X}(s)) - b} ds + \int_0^{t \wedge T_m} e^{s-\Lambda^U(\hat{X}(s-))} dp^*(s)
\]

\[
+ e^{(t \wedge \tau_m) - \Lambda^U(\hat{X}(t \wedge T_m))} \widehat{w}^U(\hat{X}(t \wedge T_m)) \right],
\]

14
is well defined for any \( \hat{x} = (x, x, 0, k) \in \hat{E} \).

Let us show by induction on \( m \in \mathbb{N} \) that \( J^U_m(t, \hat{x}) \leq g(x) \) for all \( t \in \mathbb{R}_+ \), \( \hat{x} = (x, x, 0, k) \in \hat{E} \). Clearly, we have that \( J^U_0(t, \hat{x}) = \hat{w}^U(\hat{x}) \). Consequently, from equation (37), we have that \( J^U_m(t, \hat{x}) \leq g(x) \) for all \( t \in \mathbb{R}_+ \), \( \hat{x} = (x, x, 0, k) \in \hat{E} \). Now assume that for \( m \in \mathbb{N} \) we have that \( J^U_m(t, \hat{x}) \leq g(x) \) for all \( t \in \mathbb{R}_+ \), \( \hat{x} = (x, x, 0, k) \in \hat{E} \). Following the same arguments as in the proof of Proposition 4.3 in [4], it is easy to show that for \( t \in \mathbb{R}_+ \),

\[
J^U_{m+1}(t, \hat{x}) \leq \int_0^{t \wedge t_*(x)} e^{s - \Lambda^k(x, s)} \left[ -b + \bar{e} f(\phi(x, s), \nu_k(s)) + \lambda(\phi(x, s), \nu_k(s)) Q g(\phi(x, s), \nu_k(s)) \right] ds \\
+ I_{\{t \leq t_*(x)\}} e^{t_*(x) - \Lambda^k(x, t_*(x))} \left[ Q g(\phi(x, t_*(x)), u_0(k, x)) + \bar{e} \phi(x, t_*(x)), u_0(k, x) \right] \\
+ I_{\{t < t_*(x)\}} e^{t - \Lambda^k(x, t)} \hat{w}^U(\hat{x}, t)).
\]  

(38)

Now if \( t < t_*(x) \), then by using the fact that \( \hat{\phi}(\hat{x}, t) = (\phi(x, t), x, t, k) \) we get that

\[
\hat{w}^U(\hat{x}, t) = \bar{e} L_\epsilon f(x, \Gamma^U(k, x)) + H_\epsilon \bar{e} \phi(x, \Gamma^U(k, x), t) + G_\epsilon g(x, \Gamma^U(k, x), t) \\
- b L_\epsilon f(x, \Gamma^U(k, x), t),
\]

and it follows, by applying Proposition 4.2 in [4], that

\[
\hat{w}^U(\hat{x}) = \int_0^t e^{s - \Lambda^k(x, s)} \left[ -b + \bar{e} f(\phi(x, s), \nu_k(s)) + \lambda(\phi(x, s), \nu_k(s)) Q g(\phi(x, s), \nu_k(s)) \right] ds \\
+ e^{t - \Lambda^k(x, t)} \hat{w}^U(\hat{x}, t)).
\]  

(39)

Therefore, combining equations (38) and (39) we get that \( J^U_{m+1}(t, \hat{x}) \leq \hat{w}^U(\hat{x}) \) and by using equation (37) we have that \( J^U_{m+1}(t, \hat{x}) \leq g(x) \).

If \( t \geq t_*(x) \), then equations (36) and (38) yields \( J^U_m(t, \hat{x}) \leq \hat{w}^U(\hat{x}) \). By using equations (37), we have \( J^U_m(t, \hat{x}) \leq g(x) \), showing the fact that for all \( m \in \mathbb{N} \), \( J^U_m(t, \hat{x}) \leq g(x) \) for all \( t \in \mathbb{R}_+ \), \( \hat{x} = (x, x, 0, k) \in \hat{E} \).

Consequently, this implies that \( -b E_{(x, 0)} \left[ \int_0^{t \wedge T_m} e^{s} ds \right] + E_{(x, 0)} \left[ \int_0^{t \wedge T_m} \hat{w}^U(\hat{X}^U(t \wedge T_m)) \right] \leq g(x) \).

Combining Fatou’s Lemma and Remark 2.4 we obtain that

\[
-\frac{b}{e} \left[ e^t - 1 \right] + e^t E_{(x, 0)} \left[ \hat{w}^U(\hat{X}^U(t)) \right] \leq g(x),
\]

(40)

showing the result. \( \square \)

**Proposition 5.6** For all \( x \in E, U \in \mathcal{U} \), we have that \( E_{(x, 0)} \left[ \hat{w}^U(\hat{X}^U(t \wedge T_m)) \right] \) exists in \( \mathbb{R}_+ \) for any \( (t, m) \in \mathbb{R}_+ \times \mathbb{N} \) and

\[
\lim_{t \to +\infty} \lim_{m \to +\infty} E_{(x, 0)} \left[ \hat{w}^U(\hat{X}^U(t \wedge T_m)) \right] = 0.
\]  

(41)

**Proof:** Clearly, we have

\[
E_{(x, 0)} \left[ \hat{w}^U(\hat{X}^U(t \wedge T_m)) \right] = E_{(x, 0)} \left[ I_{\{t < T_m\}} \hat{w}^U(\hat{X}^U(t)) \right] + E_{(x, 0)} \left[ I_{\{t \geq T_m\}} \hat{w}^U(\hat{X}^U(T_m)) \right],
\]

and thus by using Remark 3.8 ii),

\[
0 \leq E_{(x, 0)} \left[ \hat{w}^U(\hat{X}^U(t \wedge T_m)) \right] \leq E_{(x, 0)} \left[ \hat{w}^U(\hat{X}^U(t)) \right] + E_{(x, 0)} \left[ \hat{w}^U(\hat{X}^U(T_m)) \right] + b K_\lambda.
\]  

(42)
Iterating Assumption 3.6, we obtain that for all \( m \in \mathbb{N} \), \( E_{(x,0)}^{U} \left[ \hat{w}^{U} (\hat{X}^{U}(T_{m})) \right] \leq g(x) + \frac{K_{g}}{1 - k_{g}}. \)

Combining equations (43), (44) and the previous inequality, the result follows.

**Corollary 5.7** For all \( U \in \mathcal{U} \),

\[
\lim_{t \to +\infty} \frac{1}{t} \lim_{m \to -\infty} E_{(x,0)}^{U} \left[ h(X(t \wedge T_{m})) \right] \leq 0, \tag{43}
\]

and

\[
\lim_{t \to +\infty} \frac{1}{t} \lim_{m \to -\infty} E_{(x,0)}^{U_{\hat{U}_{\phi}}} \left[ h(X(t \wedge T_{m})) \right] = 0. \tag{44}
\]

**Proof:** From equation (25), it follows that for all \( x \in E, \Gamma \in \mathbb{V}(x), \)

\[-\rho \mathcal{L}(x, \hat{u}_{\phi}(x)) + Gh(x, \hat{u}_{\phi}(x)) \leq h(x) \leq Lf(x, \Gamma) + Hr(x, \Gamma) + Gh(x, \Gamma). \tag{45}\]

Consequently, by using Remark 3.8 ii), the definition of \( \hat{w} \) and Assumption 3.4 we obtain that there exist \( M_1 > 0 \) such that for any \( U \in \mathcal{U} \)

\[ h(X(t \wedge T_{m})) \leq M_1 \left[ \hat{w}^{U} (\hat{X}^{U}(t \wedge T_{m})) + bK_{\lambda} \right]. \]

Consequently, combining the previous equation and (41) we obtain equation (43).

Moreover, notice that \( \left[ F^{U_{\hat{U}_{\phi}}}(N(t), Z(t)) \right] \tau(t) = \hat{u}_{\phi}(X(t)) \) and so equation (45) implies

\[-\|h\|_{2} \left[ \hat{w}^{U_{\hat{U}_{\phi}}} (\hat{X}^{U_{\hat{U}_{\phi}}}(t \wedge T_{m})) + bK_{\lambda} \right] - \rho K_{\lambda} \leq h(X(t \wedge T_{m})). \]

By using equation (41), this yields that \( \lim_{t \to +\infty} \frac{1}{t} \lim_{m \to -\infty} E_{(x,0)}^{U_{\hat{U}_{\phi}}} \left[ h(X(t \wedge T_{m})) \right] \geq 0. \) Combining the previous inequality with (43), the result follows.

Finally, we can now present our second main result. It states that the measurable selector \( \hat{u}_{\phi} \) of the optimality equation (24) associated to \( (\rho, h) \) gives an optimal feedback control \( U_{\hat{U}_{\phi}} \) for the process \{X(t)\}.

**Theorem 5.8** The control \( U_{\hat{U}_{\phi}} \) is an optimal strategy for the long-run average control problem:

\[ \rho = \inf_{U \in \mathcal{U}} \mathcal{A}(U, x) = \mathcal{A}(U_{\hat{U}_{\phi}}, x), \]

for all \( x \in E. \)

**Proof:** From Proposition 5.6 we have that \( E_{(x,0)}^{U} \left[ h(X(t \wedge T_{m})) \right] = E_{(x,0)}^{U} \left[ h(X(t \wedge T_{m})) \right] \) is well defined. Therefore, following the same arguments as in Proposition 4.3 in [4] it can be shown that

\[
E_{(x,0)}^{U} \left[ \int_{0}^{t \wedge T_{m}} f(X(s), X(N(s), Z(s), \tau(s))) ds + \int_{0}^{t \wedge T_{m}} r(X(s), u_{\phi}(N(s), X(s))) dp^{*}(s) \right]
\]

\[+ E_{(x,0)}^{U} \left[ h(X(t \wedge T_{m})) \right] \geq E_{(x,0)}^{U} \left[ \rho [t \wedge T_{m}] + h(x) \right], \]

where \( U = (u, u_{\phi}) \in \mathcal{U}. \) From equation (43), it implies that

\[
\lim_{t \to +\infty} \frac{1}{t} E_{(x,0)}^{U} \left[ \int_{0}^{t} f(X(s), u(N(s), Z(s), \tau(s))) ds + \int_{0}^{t} r(X(s), u_{\phi}(N(s), X(s))) dp^{*}(s) \right] \geq \rho,
\]

16
showing that \( \inf_{U \in \mathcal{U}} A(U, x) \geq \rho \).

From equation (44), it can be shown by using the same arguments as in the proof of Proposition 4.4 in [4] that

\[
\lim_{t \to +\infty} \frac{1}{t} E_{(x,0)}^{U_{\lambda_0}} \left[ \int_0^t f(X(s), \widehat{u}(X(s))) \, ds + \int_0^t r(X(s-), \widehat{u}(X(s-))) \, d\mu^*(s) \right]
\leq \rho - \lim_{t \to +\infty} \frac{1}{t} \lim_{m \to \infty} E_{(x,0)}^{U_{\lambda_0}} \left[ h(X(t \wedge T_m)) \right] = \rho,
\]

implying that \( \inf_{U \in \mathcal{U}} A(U, x) \leq \rho \).

Therefore, it follows that \( \rho = \inf_{U \in \mathcal{U}} A(U, x) = A(U_{\lambda_0}, x) \) for all \( x \in E \). \( \square \)

References

[1] A. Arapostathis, V.S. Borkar, E. Fernández-Gaucherand, M.K. Ghosh, and S.I. Marcus. Discrete-time controlled Markov processes with average cost criterion: a survey. *SIAM J. Control Optim.*, 31(2):282–344, 1993.

[2] D.P. Bertsekas and S.E. Shreve. *Stochastic optimal control*, volume 139 of *Mathematics in Science and Engineering*. Academic Press Inc., New York, 1978. The discrete time case.

[3] V. S. Borkar. *Topics in controlled Markov chains*, volume 240 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1991.

[4] O.L.V. Costa and F. Dufour. Average control of piecewise deterministic Markov processes. *ArXiv*, 0809.0477v1, page 34, 2008. Available at http://arxiv.org/abs/0809.0477.

[5] O.L.V. Costa and F. Dufour. The vanishing approach for the average continuous control of piecewise deterministic Markov processes. *ArXiv*, 0812.0820v1, page 23, 2008. Available at http://arxiv.org/abs/0812.0820.

[6] O.L.V. Costa and F. Dufour. Relaxed long run average continuous control of piecewise deterministic Markov processes. In *Proceedings of the European Control Conference*, pages 5052–5059, Kos, Greece, July, 2007.

[7] M.H.A. Davis. *Markov Models and Optimization*. Chapman and Hall, London, 1993.

[8] E.B. Dynkin and A.A. Yushkevich. *Controlled Markov processes*, volume 235 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1979.

[9] X. Guo and U. Rieder. Average optimality for continuous-time Markov decision processes in polish spaces. *The Annals of Applied Probability*, 16:730–756, 2006.

[10] X. Guo and Q. Zhu. Average optimality for Markov decision processes in Borel spaces: A new condition and approach. *Journal of Applied Probability*, 43:318–334, 2006.

[11] O. Hernández-Lerma and J.B. Lasserre. *Discrete-time Markov control processes*, volume 30 of *Applications of Mathematics*. Springer-Verlag, New York, 1996. Basic optimality criteria.

[12] O. Hernández-Lerma and J.B. Lasserre. Policy iteration for average cost Markov control processes on borel spaces. *Acta Applicandae Mathematicae*, 47:125–154, 1997.
[13] O. Hernández-Lerma and J.B. Lasserre. *Further topics on discrete-time Markov control processes*, volume 42 of *Applications of Mathematics*. Springer-Verlag, New York, 1999.

[14] O. Hernández-Lerma, R. Montes-de-Oca, and R. Cavazos-Cadena. Recurrence conditions for Markov decision processes with Borel state space: a survey. *Ann. Oper. Res.*, 28(1-4):29–46, 1991.

[15] S. P. Meyn. The policy iteration algorithm for average reward Markov decision processes with general state space. *IEEE Trans. Automat. Control*, 42(12):1663–1680, 1997.

[16] M. L. Puterman. *Markov decision processes: discrete stochastic dynamic programming*. Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. John Wiley & Sons Inc., New York, 1994. A Wiley-Interscience Publication.

[17] M. Schäl. Conditions for optimality and for the limit of the $n$-stage optimal polices to be optimal. *Zeit. Wahrs. Verw. Geb.*, 32:179–96, 1975.