Combinatorial Applications of the Subspace Theorem

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1 Introduction

The Subspace Theorem is a powerful tool in number theory. It has appeared in various forms and been adapted and improved over time. It’s applications include diophantine approximation, results about integral points on algebraic curves and the construction of transcendental numbers. But its usefulness extends beyond the realms of number theory. Other applications of the Subspace Theorem include linear recurrence sequences and finite automata. In fact, these structures are closely related to each other and the construction of transcendental numbers.

The Subspace Theorem also has a number of remarkable combinatorial applications. The purpose of this paper is to give a survey of some of these applications including sum-product estimates and bounds on unit distances. The presentation will be from the point of view of a discrete mathematician. We will state a number of variants of the Subspace Theorem below but we will not prove any of them as the proofs are beyond the scope of this work. However we will give a proof of a simplified special case of the Subspace Theorem which is still very useful for many problems in discrete mathematics.

A number of surveys have been given of the Subspace Theorem highlighting its multitude of applications. Notable surveys include those of Bilu [4], Evertse and Schlickewei [16] and Corvaja and Zannier [8]. These give many proofs of results from number theory and algebraic geometry using the Subspace Theorem including those mentioned above.

Wolfgang M. Schmidt was the first to state and prove a variant of the Subspace Theorem in 1972 [23]. His theorem was extended and played a very important role in modern number theory. Before we state the Subspace Theorem we need some definitions. A linear form is an expression of the form $L(x) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ where $a_1, \ldots, a_n$ are constants and $x = (x_1, \ldots, x_n)$.

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A collection of linear forms is *linearly independent* if none of them can be expressed as a linear combination of the others. Given \( x = (x_1, \ldots, x_n) \) we define the maximum norm

\[
\|x\| = \max(|x_1|, \ldots, |x_n|).
\]

**Theorem 1** (Subspace Theorem I). Suppose we have \( n \) linearly independent linear forms \( L_1, L_2, \ldots, L_n \) in \( n \) variables with algebraic coefficients. Given \( \varepsilon > 0 \), the non-zero integer points \( x = (x_1, x_2, \ldots, x_n) \) satisfying

\[
|L_1(x)L_2(x)\ldots L_n(x)| < \|x\|^{-\varepsilon}
\]

lie in finitely many proper linear subspaces of \( \mathbb{Q}^n \).

It generalised the Thue-Siegel-Roth Theorem on the approximation of algebraic numbers [22] to higher dimensions.

Theorem 1 has been extended in various directions by many authors including Schmidt himself, Schlickewei, Evertse, Amoroso and Viada. Analogues have been proved using \( p \)-adic norms and over arbitrary number fields and bounds on the number of subspaces required have been found. These bounds depend on the degree of the number field and the dimension. For some of these results and more information see [16], [17] and [2].

Now we give a \( p \)-adic version of the Subspace Theorem that we will use in the next section. Given a prime \( p \), the \( p \)-adic absolute value is denoted \( |x|_p \) and satisfies \( |p|_p = 1/p \). \( |x|_\infty \) denotes the usual absolute value so \( |x|_\infty = |x| \). We may refer to \( \infty \) as the *infinite prime*. We define the height of a rational vector \( x = (x_1, \ldots, x_n) \) by

\[
H(x) = \prod_p \|x\|_p = \prod_p \max\{1, |x_1|_p, \ldots, |x_n|_p\}.
\]

Here the product extends over all primes including the infinite prime. Note that for any \( x \) only finitely many terms in the product are not 1.

**Theorem 2** (Subspace Theorem II). Suppose \( S = \{\infty, p_1, \ldots, p_t\} \) is a finite set of primes, including the infinite prime. For every \( p \in S \) let \( L_{i,p} \) be linearly independent linear forms in \( n \) variables with algebraic coefficients. Then for any \( \varepsilon > 0 \) the solutions \( x \in \mathbb{Z}^n \) of

\[
\prod_{p \in S} \prod_{i=1}^n |L_{i,p}(x)|_v \leq H(x)^{-\varepsilon}
\]

are contained in finitely many proper linear subspaces of \( \mathbb{Q}^n \).

The power and utility of the Subspace Theorem is already evident in the above forms but there is a corollary, often itself called the Subspace Theorem, which makes even more applications possible. This corollary was originally given by Evertse, Schlickewei and Schmidt [17]. We present the version with the best known bound due to Amoroso and Viada [2].
Theorem 3 (Subspace Theorem III). Given an algebraically closed field $K$ and a subgroup $\Gamma$ of $K$ of finite rank $r$, suppose $a_1, a_2, \ldots, a_n \in K^*$. Then the number of solutions of the equation
\[ a_1z_1 + a_2z_2 + \cdots + a_nz_n = 1 \] (1)
with $z_i \in \Gamma$ and no subsum on the left hand side vanishing is at most
\[ A(n, r) \leq (8n)^{4n^4(n+nr+1)}. \]

The Erdős unit distance problem is an important problem in combinatorial geometry. It asks for the maximum possible number of unit distances between $n$ points in the plane. This problem is still open but recently Frank de Zeeuw and the authors have made progress towards this problem when the distances considered come from certain groups.

The structure of this paper will be as follows. In the next section we give a number of well-known applications of the Subspace Theorem. In Section 3 we give combinatorial applications. In particular, Section 3.1 contains the special case of the Subspace Theorem via Mann’s Theorem, Section 3.2 gives unit distance bounds and Section 3.3 gives sum-product estimates.

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2 Number theoretic applications

2.1 Transcendental numbers

Adamczewski and Bugeaud showed that all irrational automatic numbers are transcendental using the Subspace Theorem. An automatic number is a number for which there exists an integer $b > 0$ such that when the number is written in $b$-ary form it is the output of a finite automaton with input the natural numbers written from right to left. For more details see [1] or [4].

Here we will use a method similar to the proof of Theorem 3.3 in [4] to show:

Theorem 4. The number $\alpha$ given by the infinite sum
\[ \alpha = \sum_{n \geq 1} \frac{1}{2^{2^n}} \]
is transcendental.

Kempner showed in the early twentieth century that a large class of numbers defined similarly to $\alpha$ are transcendental [19]. The Subspace Theorem provides a tidy proof of this fact.
Proof of Theorem 4 Consider the binary expansion:

\[ \alpha = \frac{1}{4} + \frac{1}{16} + \frac{1}{256} + \frac{1}{65536} + \cdots = 0.0101000100000001\ldots_2. \]

So the binary expansion of \( \alpha \) consists of sections of zeros of increasing length separating solitary ones. Thus the expansion is not periodic and hence \( \alpha \) is not rational. We let \( b_n \) be the string given by the first \( n \) digits of this expansion. One can check that each \( b_n \) has two disjoint substrings of zeros of length \( n/8 \).

Assume \( \alpha \) is not transcendental. Then it is algebraic. Now each \( b_n \) starts with a string \( AOBO \), where \( O \) is a string of zeroes, the length of \( O \) is at least \( n/8 \) and \( A \) and \( B \) might have length zero. We will use the rational number represented in base 2 by \( AOBO\ldots \) to approximate \( \alpha \). Call this number \( \pi \). Then

\[ \pi = \frac{M}{2^n(2^b - 1)} \]

where \( M \in \mathbb{Z} \) and \( a \) and \( b \) are the lengths of the strings \( A \) and \( OB \) respectively. Clearly \( b \geq n/8 \) and \( a + b \leq n \) since \( AOBO \) is a substring of \( b_n \). Since \( \alpha \) starts with \( b_n \) we have

\[ |\alpha - \pi| \leq \frac{1}{2^{a+b+n/8}} \implies |2^{a+b}\alpha - 2^a\alpha - M| \leq \frac{1}{2^{n/8}}. \]

Now we apply the Subspace Theorem. We let \( S = \{2, \infty\} \) and

\[ L_{1,\infty}(x) = x_1, \quad L_{2,\infty}(x) = x_2, \quad L_{3,\infty}(x) = \alpha x_1 - \alpha x_2 - x_3, \]

\[ L_{1,2}(x) = x_1, \quad L_{2,2}(x) = x_2, \quad L_{3,2}(x) = x_3. \]

Note that by our assumption that \( \alpha \) is not transcendental the linear form \( L_{3,\infty} \) has algebraic coefficients. Let \( x = (2^{a+b}, 2^b, M) \). Now \( |M| \leq 2^{a+b} \) since \( 0 < \pi < 1 \). So \(|x| \leq 2^{a+b} \leq 2^n \). Multiplying the absolute values of the linear forms together we get

\[ \prod_{p \in S} \prod_{i=1}^3 |L_{i,p}(x)| = |2^a|_2|2^a|_\infty|2^{a+b}|_2|2^{a+b}|_\infty|M|_2|2^{a+b}\alpha - 2^a\alpha - M|_\infty \]

\[ \leq \frac{1}{2^{n/8}} \]

\[ \leq \frac{1}{\|x\|^{1/8}} \leq H(x)^{-1/8}. \]

The first two inequalities hold because \(|\alpha - \pi| \leq 2^{-a-b-n/8} \) and \(|2^a|_2|2^b|_\infty = 1 \).

We can do this for each \( n \) and \( b = \hat{b}(n) \) increases as \( n \) increases since \( b \geq n/8 \). Thus infinitely many of the vectors \( x = x(n) \) are distinct. By Theorem 2 these vectors are contained in finitely many subspaces of \( \mathbb{Q}^3 \). Thus one of these subspaces contains infinitely many of them. That is, there exist \( c, d, e \in \mathbb{Q} \) such that

\[ c2^{a(n)} + d2^{a(n)+b(n)} + eM(n) = 0 \]

for infinitely many \( n \). \( e \) cannot be zero since \( \hat{b}(n) \to \infty \) as \( n \to \infty \). Dividing by \( 2^{a(n)}(2^{b(n)} - 1) \) and taking limits we get \( \alpha = -d/e \) so \( \alpha \) is rational. This is a contradiction. Thus \( \alpha \) must be transcendental. \( \square \)
2.2 Linear recurrence sequences

A linear recurrence sequence is a sequence of numbers where the first few terms are given and the higher order terms are given by a recurrence relation. A famous example is the Fibonacci sequence \( \{F_n\} \) where \( F_1 = F_2 = 1 \) and \( F_n = F_{n-1} + F_{n-2} \) for \( n > 2 \). More formally, a linear recurrence sequence consists of constants \( a_1, \ldots, a_k \) in a field \( K \) for some \( k > 0 \) along with a sequence \( \{R_n\}_{n=1}^{\infty} \) with \( R_i \in K \) for \( 1 \leq i \leq k \) and

\[
R_n = a_1R_{n-1} + a_2R_{n-2} + \cdots + a_kR_{n-k}, \quad \text{for } n > k.
\]

If \( \{R_n\} \) is not expressible by any shorter recurrence relation then it is said to have order \( k \). In this case each \( a_i \neq 0 \).

We are interested in the structure of the zero set of a linear recurrence sequence. This is the set

\[
S(\{R_n\}) = \{ i \in \mathbb{N} : R_i = 0 \}.
\]

The Skolem-Mahler-Lech Theorem states that this set consists of the union of finitely many points and arithmetic progressions. Schmidt has given a quantitative bound for this theorem using various tools including the Subspace Theorem.

We will show a special case of this theorem using Theorem 3. We will restrict our attention to simple nondegenerate linear recurrence sequences. To define such sequences we need to define the companion polynomial of the recurrence sequence. If \( \{R_n\} \) is given as above then the companion polynomial of \( \{R_n\} \) is

\[
C(x) = x^k - a_1x^{k-1} - \cdots - a_k = x - a_k.
\]

Suppose the roots of this polynomial are \( \alpha_1, \ldots, \alpha_k \) with multiplicity \( b_1, \ldots, b_k \) respectively. Clearly, each \( \alpha_i \) is nonzero. If the companion polynomial has \( k \) distinct roots it is called simple. If \( \alpha_i/\alpha_j \) is not a root of unity for any \( i \neq j \) then the sequence is called nondegenerate.

**Theorem 5.** Suppose \( \{R_n\} \) is a simple nondegenerate linear recurrence sequence of order \( k \). Then

\[
|S(\{R_n\})| \leq (8k)^{8k^6}.
\]

**Proof.** We can express the recurrence relation using a matrix equation:

\[
\begin{pmatrix}
    a_1 & a_2 & \ldots & a_{k-1} & a_k \\
    1 & 0 & \ldots & 0 & 0 \\
    0 & 1 & \ldots & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \ldots & 1 & 0
\end{pmatrix}^n
\begin{pmatrix}
    R_k \\
    R_{k-1} \\
    \vdots \\
    R_1
\end{pmatrix}
= \begin{pmatrix}
    R_{k+n} \\
    R_{k-1+n} \\
    \vdots \\
    R_{1+n}
\end{pmatrix}.
\]

We call the matrix above \( A \). The characteristic polynomial of \( A \) is given by

\[
\chi(\lambda) = \lambda^k - a_1\lambda^{k-1} - \cdots - a_k.
\]
This is the same as the companion polynomial of \( \{R_n\} \). Thus \( A \) has distinct nonzero eigenvalues and so can be diagonalized. Thus multiplying out the left hand side we can solve for \( R_n \) to get
\[
R_n = c_1 \alpha_1^n + c_2 \alpha_2^n + \cdots + c_k \alpha_k^n \quad \text{for every } n > k.
\]
Then applying Theorem 3 to the equation \( c_1 x_1 + c_2 x_2 + \cdots + c_k x_k = 0 \) with solutions from the group of rank at most \( k \) generated by \( \{\alpha_1, \ldots, \alpha_k\} \) we get that the number of solutions is at most
\[
A(k, k) = (8k)^{4k(k+k^2+1)} \leq (8k)^{8k^6}.
\]
Since the sequence is nondegenerate we cannot have two values \( n, n' \) giving the same value for \( \alpha_i^n \) and \( \alpha_i^{n'} \) for each \( i \), hence each solution corresponds to a unique value from \( S(\{R_m\}) \).

3 Combinatorial applications

3.1 A proof of a very special case of the Subspace Theorem

Theorem 3 gives a bound on the number of nondegenerate solutions of a linear equation from a multiplicative group with rank not too large. What happens if the group in question has rank zero. This corresponds to solutions that are roots of unity. The Subspace Theorem can then be seen as a generalisation of the following results which follows from a theorem of H.B. Mann from 1965.

**Theorem 6.** Given \( (a_1, \ldots, a_k) \in \mathbb{Q}^k \), consider the equation
\[
a_1 x_1 + a_2 x_2 + \cdots + a_k x_k = 0.
\]
The number of solutions \( (\omega_1, \ldots, \omega_k) \) of this equation with the \( \omega_i \)'s roots of unity and no vanishing subsum is at most \( (k : \Theta(2k))^k \) where
\[
\Theta(k) = \prod_{\substack{p \leq k \text{ prime}}} p.
\]
Note that the logarithm of the function \( \Theta \) above is an important function in number theory called the first Chebyshev function.

Theorem 6 along with Lemma 7 below were proved by Frank de Zeeuw and the authors in [25]. This theorem provides another starting point for the development of the Subspace Theorem. The roots of unity give a relatively simple example of an infinite multiplicative group. We will give the proof of Theorem 6 below. First we prove Lemma 7 which was Mann’s result mentioned above [21].

**Lemma 7 (Mann).** Suppose we have
\[
a_1 \omega_1 + a_2 \omega_2 + \cdots + a_k \omega_k = 0,
\]
with \( a_i \in \mathbb{Q} \), the \( \omega_i \) roots of unity, and no proper nontrivial subsum vanishing. Then for every \( i, j \), \( (\omega_i/\omega_j)^{\Theta(k)} = 1 \).
Proof. Dividing by an appropriate factor we may assume that our equation is of the form $1 + a_2\omega_2 + \cdots + a_k\omega_k = 0$. Then we just need to show that each $\omega_i^{\Theta(k)} = 1$. We let $m$ be the smallest value such that $\omega_i^{m} = 1$ for $1 \leq i \leq k$. The proof proceeds by showing that $m$ is squarefree and any prime that divides $m$ cannot be larger than $k$. This means $m \leq \Theta(k)$.

Suppose $m = p^j m'$ with $p$ prime and $(p, m') = 1$. Now we have $\omega_i = \rho^{\sigma_i} \cdot \omega_i^*$, with $\rho$ a primitive $p$-th root of unity. So rewriting the sum grouping powers of $\rho$ we get

$$0 = 1 + (a_2\omega_2 + \cdots + a_k\omega_k) = 1 + (\alpha_0 + \alpha_1\rho + \cdots + \alpha_{p-1}\rho^{p-1}),$$

where, for each $i, \alpha_i \in K := \mathbb{Q}(\omega_2^*, \ldots, \omega_k^*)$ satisfies

$$\alpha_i = \sum_{i \in I_i} a_i\omega_i^*, \quad \text{with} \quad I_i = \{i : \sigma_i = \ell\}.$$

Let $f(x) = \alpha_{p-1}x^{p-1} + \cdots + \alpha_1x + (1 + \alpha_0)$. Then $f$ is a polynomial of degree at most $p - 1$ over the field $K$ and $f(\rho) = 0$. If $f$ were identically zero then, by the minimality of $m$, we would have a vanishing subsum.

The degree of $\rho$ over $K$ gives us that $p$ divides $m$ only once. Specifically since $[K(\rho) : \mathbb{Q}] = [K(\rho) : K][K : \mathbb{Q}]$ we have

$$\deg_K(\rho) = [K(\rho) : K] = \frac{[K(\rho) : \mathbb{Q}]}{[K : \mathbb{Q}]} = \frac{\phi(m)}{\phi(m/p)}.$$ 

This is $p$ if $j > 1$ and $p - 1$ if $j = 1$. But the degree of $f$ is at most $p - 1$ so we must have $j = 1$ since $\rho$ is a root of $f$.

Now $f$ must be a multiple of the irreducible polynomial $m$ of $\rho$ over $K$. But $m(x) = x^{p-1} + x^{p-2} + \cdots + 1$ so $f(x) = cm(x)$ where $c$ is a nonzero constant. Thus $f$ has $p$ nonzero coefficients and thus so does the original sum giving $p \leq k$.

Proof of Theorem 7. We first show that if we are given $a \in \mathbb{C}^*$ and two sums $a_1\omega_1 + \cdots + a_k\omega_k = a$ and $a_1'\omega_1' + \cdots + a_k'\omega_k' = a$ with rational coefficients and no vanishing subsums then for any $\omega_j$, there is an $\omega_i$ such that $(\omega_j/\omega_i)^{\Theta(2k)} = 1$.

Since $a_1\omega_1 + \cdots + a_k\omega_k = a = a_1'\omega_1' + \cdots + a_k'\omega_k'$, we get $a_1\omega_1 + \cdots + a_k\omega_k - a_1'\omega_1' - \cdots - a_k'\omega_k' = 0$.

This sum may have vanishing subsums so we consider minimal vanishing subsums of the form

$$\sum_{i \in I_i} a_i\omega_i - \sum_{j \in I_j'} a_j'\omega_j' = 0.$$

Each $\omega_j'$ is contained in such a minimal subsum of length at most $2k$. This subsum also contains some $\omega_i$ otherwise the original sum would have a vanishing subsum. Now the previous lemma gives that $(\omega_j'/\omega_i)^{\Theta(2k)} = 1$.

Note that above we require $a \in \mathbb{C}^*$. If $a = 0$ then the original sums will count as vanishing subsums when we consider the combined equation so Lemma 7 does not apply.
Now we can prove the theorem. For $a \in \mathbb{C}^*$ and $k$ a positive integer define $S(a, k)$ as the set of $k$-tuples $(\omega_1, \ldots, \omega_k)$, where each $\omega_i$ is a root of unity, such that there are $a_i \in \mathbb{Q}$ satisfying $a_1 \omega_1 + \cdots + a_k \omega_k = a$ with no vanishing subsums.

We fix a $k$-tuple $(\omega_1, \ldots, \omega_k) \in S(a, k)$. Given an element of $S(a, k)$, for each $\omega'_j$ (the $j$-th coordinate of that element) there is an $i$ such that $\omega_i - \Theta(2k)(\omega'_j)\Theta(2k) = 1$. So $\omega'_j$ is a root of the polynomial $\omega_i - \Theta(2k)x\Theta(2k) = 1$. This polynomial has $\Theta(2k)$ roots. We have $k$ choices for $j$ so at most $k\Theta(2k)$ choices for each $\omega'_j$. This gives the required bound.

This theorem can be used to prove Theorem 8 from the next section. We will show how using the Subspace Theorem instead allows the proof of the stronger Theorem 9.

3.2 Unit distances

The unit distance problem was first posed by Erdős in 1946 [14]. It asks for the maximum number, $u(n)$, of pairs of points with the same distance in a collection of $n$ points in the plane. By scaling the point set one may assume that the most popular distance is one, hence the name of the problem. The problem seeks asymptotic bounds. Erdős gave a construction using a $\sqrt{n} \times \sqrt{n}$ portion of a square lattice giving

$$u(n) \geq n^{1+c/\log \log n}.$$  

Number theoretic bounds for the number of integer solutions of the equation $x^2 + y^2 = a$ give the above inequality. Erdős conjectured that the magnitude of $u(n)$ is close to this lower bound. The best known upper bound is $u(n) \leq cn^{4/3}$. A number of proofs have been given showing $u(n) \leq cn^{4/3}$ using tools such as cuttings, edge crossings in graphs and the Szemerédi-Trotter Theorem. The first proof was due to Spencer, Szemerédi and Trotter [28]. For more details of the problem see [5]. We will look at a special case of this problem when the distances considered come from a multiplicative group with rank not too large. This does not seem to be a huge limitation as the unit distances from the lower bound construction above come from such a group as will be explained below.

Using Theorem 8 Frank de Zeeuw and the authors were able to show the following theorem [25]. Two points in the plane are said to have rational angle if the angle that the line between these two points makes with the $x$-axis is a rational multiple of $\pi$.

**Theorem 8.** Let $\varepsilon > 0$. Given $n$ points in the plane, the number of unit distances with rational angle between pairs of points is less than $n^{1+\varepsilon}$.

These unit distances correspond to roots of unity. The proof proceeds by counting certain paths in the unit distance graph and using Mann’s Theorem to bound the number of edges.

Using the Subspace Theorem in place of Mann’s Theorem one can instead consider unit distances from a multiplicative group with rank not too large with
respect to the number of points \textsuperscript{24}. Note that a unit distance in the plane can (and will) be considered as a complex number of unit length. So all unit distances can be considered as coming from a subgroup of $\mathbb{C}^*$.

**Theorem 9.** Let $\varepsilon > 0$. There exist a positive integer $n_0$ and a constant $c > 0$ such that given $n > n_0$ points in the plane, the number of unit distances coming from a subgroup $\Gamma \subset \mathbb{C}^*$ with rank $r < c \log n$ is less than $n^{1+\varepsilon}$.

This is our first combinatorial application of the Subspace Theorem. The proof is given below.

Suppose $G = G(V, E)$ is a graph on $v(G) = n$ vertices and $e(G) = cn^{1+\alpha}$ edges. We denote the minimum degree in $G$ by $\delta(G)$.

Note that by removing vertices with degree less than $(c/2)n^\alpha$ we have a subgraph $H$ with at least $e(H) \geq (c/2)n^{1+\alpha}$ edges and $\delta(H) \geq (c/2)n^\alpha$. The number of vertices in $H$ is at least $v(H) = \sqrt{cn^{1+\alpha}/2}$. We will consider such a well behaved subgraph instead of the original graph.

**Proof of Theorem 9.** Let $G$ be the unit distance graph on $n$ points with unit distances coming from $\Gamma$ as edges. We show that there are less than $n^{1+\varepsilon}$ such distances, i.e. edges, for any $\varepsilon > 0$. We can assume that $e(G) \geq (1/2)n^{1+\varepsilon}$, $v(G) \geq n^{1/2+\varepsilon/2}$ and $\delta(G) \geq (1/2)n^\varepsilon$.

Consider a path in $G$ on $k$ edges $P_k = p_0p_1 \ldots p_k$. We denote by $u_i(P_k)$ the unit vector between $p_i$ and $p_{i+1}$. The path is nondegenerate if $\sum_{i \in I} u_i(P_k) = 0$ has no solutions where $I$ is a nonempty subset of $\{0, 1, \ldots, k-1\}$. Note that such a sum is a sum of roots of unity with no vanishing subsums. We will denote by $\mathcal{P}_k(v, w)$ the set of nondegenerate paths of length $k$ between vertices $v$ and $w$.

The number of nondegenerate paths of length $k$ from any vertex is at least

$$\prod_{\ell=0}^{k-1} (\delta(G) - 2^\ell + 1) \geq \frac{n^{k\varepsilon}}{2^{2k}}.$$

The first expression is true since if we consider a path $P_\ell$ on $\ell < k$ edges then all but $2^\ell - 1$ possible continuations give a path $P_{\ell+1}$ with no vanishing subsums. The inequality is true if we have assume $2^k \leq (1/2)n^\varepsilon$, which is true if $k < \varepsilon \log n / \log 2 - 1$, a fact we will confirm at the end of the proof. From this we get that the number of nondegenerate paths $P_k$ in the graph is at least $n^{1/2+(k+1)/2+\varepsilon}/2^{2k+1}$. So there exist vertices $v, w$ in $G$ with

$$|\mathcal{P}_k(v, w)| \geq \frac{n^{(k+1)/2+\varepsilon}/2^{3/2}}{4^k}.$$

Consider a path $P_k \in \mathcal{P}_k(v, w)$, $P_k = p_0p_1 \ldots p_k$. Let $a$ be the complex number giving the vector between $p_0$ and $p_k$. Since $P_k$ is nondegenerate we get a solution of $(1/a)x_1 + (1/a)x_2 + \cdots + (1/a)x_k = 1$ with no vanishing subsums. Thus Theorem \textsuperscript{3} gives

$$|\mathcal{P}_k(v, w)| \leq (8k)^{4k^4(k+kr+1)}.$$

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This with the lower bound give

\[(k + 1/2)\varepsilon - 3/2) \log n \leq k \log 4 + 4k^4(k + kr + 1) \log(8k) \leq 5r k^3 \log k,\]

where the last inequality holds for \(k\) large enough. So

\[\varepsilon \leq \frac{5r k^4 \log k}{\log n} + \frac{3}{2k}. \tag{2}\]

This inequality holds for \(k \geq \exp((1/5)W(5c_2 \log n/r))\) where \(W\) is the positive real-valued function that solves \(x = W(x) e^{W(x)}\). Note that the function \(W\) satisfies \((1/2) \log x \leq W(x) \leq \log x\) for \(x \geq e\).

Since \(r + 1 \leq c \log n\) we can choose

\[c' \left( \frac{\log n}{r} \right)^{1/5} \leq k \leq c'' \left( \frac{\log n}{r} \right)^{1/5}.\]

Then for each \(\varepsilon > 0\) there is a constant \(c > 0\) such that (2) above holds for \(n\) large enough. Earlier we assumed that \(k \leq \varepsilon \log n / \log 2 - 1\). This holds for the value of \(k\) given above for \(n\) large enough. So the number of unit distances from \(\Gamma\) is less than \(cn^{1+\varepsilon}\) for each \(\varepsilon > 0\).

Performing a careful analysis of Erdős’ lower bound construction one can show that all unit distances come from a group with rank at most \(c \log n / \log \log n\) for some \(c > 0\). This group is generated by considering solutions of the equation \(x^2 + y^2 = p\) where \(p\) is prime. Using the prime number theorem for arithmetic progressions we get the bound on such solutions and thus on the rank. For all the details see [24]. So Erdős’ construction satisfies the conditions of Theorem 9. A similar approach could be used for other types of lattices. So all the best known lower bounds for the unit distance problem have unit distances coming from a well structured group. It would be interesting to see if any configuration of points with maximum unit distances has such a structure.

### 3.3 Sum-product estimates

The theory of sum sets and product sets plays an important part in combinatorics and additive number theory. The goal of the field is to show that for any finite subset of the complex numbers either the sum set or the product set is large.

Formally, given a set \(A \subset \mathbb{C}\), the sum set, denoted by \(A + A\), and product set, denoted by \(AA\), are

\[A + A := \{a + b : a, b \in A\}, \quad AA := \{ab : a, b \in A\}.\]

The following long standing conjecture of Erdős and Szemerédi [15] has led to much work in the field.
Conjecture 10. Let $\varepsilon > 0$ and $A \subset \mathbb{Z}$ with $|A| = n$. Then

$$|A + A| + |AA| \geq Cn^{2-\varepsilon}.$$ 

This conjecture is still out of reach. The best known bound, which holds for real numbers and not just integers, is $Cn^{4/3 - o(1)}$ due to Solymosi [27]. A similar bound was proved recently by Konyagin and Rudnev in [20]. Chang showed that when the product set is small the Subspace Theorem can be used to show that the sum set is large [6]. The following reformulation of Chang’s observation is due to Andrew Granville.

Theorem 11. Let $A \subset \mathbb{C}$ with $|A| = n$. Suppose $|AA| \leq Cn$. Then

$$|A + A| \geq \frac{n^2}{2} + O_C(n).$$  

We will present the proof of Theorem 11 below. To use the Subspace Theorem we need a multiplicative subgroup with finite rank to work with. The following lemma of Freiman provides this [18].

Lemma 12 (Freiman). Let $A \subset \mathbb{C}$. If $|AA| \leq C|A|$ then $A$ is a subset of a multiplicative subgroup of $\mathbb{C}^*$ of rank at most $r(C)$.

Proof of Theorem 11. We consider solutions of $x_1 + x_2 = x_3 + x_4$ with $x_i \in A$. A solution of this equation corresponds to two pairs of elements from $A$ that give the same element in $A + A$. Let us suppose that $x_1 + x_2 \neq 0$ (there are at most $|A| = n$ solutions of the equation $x_1 + x_2 = 0$ with $x_1, x_2 \in A$.)

First we consider the solutions with $x_4 = 0$. Then by rearranging we get

$$\frac{x_1}{x_3} + \frac{x_2}{x_3} = 1. \quad (3)$$

By Lemma 12 and Theorem 3 there are at most $s_1(C)$ solutions of $y_1 + y_2 = 1$ with no subsum vanishing. Each of these gives at most $n$ solutions of (3) since there are $n$ choices for $x_3$. There are only two solutions of $y_1 + y_2 = 1$ with a vanishing subsum, namely $y_1 = 0$ or $y_2 = 0$, and each of these gives $n$ solutions of (3). So we have a total of $(s_1(C) + 2)n$ solutions of (3).

For $x_4 \neq 0$ we get

$$\frac{x_1}{x_4} + \frac{x_2}{x_4} - \frac{x_3}{x_4} = 1. \quad (4)$$

Again by Freiman’s Lemma and the Subspace Theorem, the number of solutions of this with no vanishing subsum is at most $s_2(C)n$. If we have a vanishing subsum then $x_1 = -x_2$ which is a case we excluded earlier or $x_1 = x_3$ and then $x_2 = x_4$, or $x_2 = x_3$ and then $x_1 = x_4$. So we get at most $2n^2$ solutions of (4) with a vanishing subsum (these are the $x_1 + x_2 = x_2 + x_1$ identities.)

So, in total, we have at most $2n^2 + s(C)n$ solutions of $x_1 + x_2 = x_3 + x_4$ with $x_i \in A$. Suppose $|A + A| = k$ and $A + A = \{\alpha_1, \ldots, \alpha_k\}$. We may assume that $\alpha_1 = 0$. Recall that we ignore sums $a_1 + a_2 = 0$. Let

$$P_i = \{(a, b) \in A \times A : a + b = \alpha_i\}.$$
Then
\[ \sum_{i=2}^{k} |P_i| \geq n^2 - n = n(n-1). \]

Also, a solution of \( x_1 + x_2 = x_3 + x_4 \) corresponds to picking two values from \( P_i \) where \( x_1 + x_2 = \alpha_i \). Thus
\[ 2n^2 + s(C)n \geq \sum_{i=2}^{k} |P_i|^2 \geq \frac{1}{k-1} \left( \sum_{i=2}^{k} |P_i| \right)^2 \geq \frac{n^2(n-1)^2}{k-1} \]
by the Cauchy-Schwarz inequality. The bound for \( k \) follows.

A number of other combinatorial results follow from the Subspace Theorem. We give one more of these, from combinatorial geometry. This is similar to a result due to Chang and Solymosi \[7\]. Given two lines \( L \) and \( M \) we denote their point of intersection by \( L \cap M \).

**Theorem 13.** Let \( C > 0 \). Then there exists \( c > 0 \) such that for any \( n+3 \) lines \( L_1, L_2, L_3, M_1, \ldots, M_n \) in \( \mathbb{C}^2 \), with \( L_1 \cap L_2, L_1 \cap L_3 \) and \( L_2 \cap L_3 \) distinct, if the number of distinct intersection points \( L_i \cap M_j, 1 \leq i \leq 3, 1 \leq j \leq n \), is at most \( C \sqrt{n} \) then any line \( L \notin \{L_1, L_2, L_3\} \) has at least \( cn \) distinct intersection points \( L \cap M_j, 1 \leq j \leq n \).

There are many structure results similar to Theorem 13 in discrete geometry. These include Beck’s Theorem \[3\], a structure theorem for lines containing many points of a cartesian product by Elekes \[11\] and generalisations of this line theorem to surfaces by Elekes and Rónyai \[12\], Elekes and Szabó \[13\] and Frank de Zeeuw and the authors \[26\]. The proofs of these results used the Szemerédi-Trotter Theorem and techniques from commutative algebra and algebraic geometry. These theorems have been used to prove various results including a conjecture of Purdy about the number of distinct distances between two sets of collinear points in the plane. For more details see \[10\], \[9\] and \[26\].

We do not prove Theorem 13 completely but only give a sketch of how it follows from the Subspace Theorem. We don’t try to find an efficient quantitative version here and we don’t explain the refereed theorems in detail. The techniques applied are standard methods in additive combinatorics. All the details can be found in the book of Tao and Vu, "Additive Combinatorics" \[29\].

Apply an affine transformation which moves \( L_1 \) to the \( x \)-axis, \( L_2 \) to the \( y \)-axis, and \( L_3 \) to the horizontal line \( y = 1 \). The three lines have distinct intersection points thus such a transformation exists. Let us denote the \( x \)-coordinates of \( L_1 \cap M_i \) and \( L_3 \cap M_j \) by \( x_i \) and \( y_j \) respectively. The two sets of \( x \)-coordinates are denoted by \( X \) and \( Y \). Define a bipartite graph with vertices given by the intersection points of lines \( M_i \) with \( L_1 \) and \( L_3 \) (with vertex sets \( X \) and \( Y \) without multiplicity.) Two points are connected by an edge in the graph if they are connected by a line \( M_j \). This is a bipartite graph on at most \( C \sqrt{n} \) vertices with \( n \) edges. Using Szemerédi’s Regularity Lemma one can find a
A regular (random-like) bipartite graph, $G$, with at least $c'n$ edges and vertex sets $V_1 \subset X$ and $V_2 \subset Y$. If $M_i \cap L_2$ is the point $(0, \alpha)$ then $x_i/y_i = \alpha/(1-\alpha)$, or equivalently $x_i = \alpha y_i/(1-\alpha)$. The Balog-Szemerédi Theorem and Freiman’s Lemma imply that there are large subsets $X' \subset V_1$ and $Y' \subset V_2$ so that $X'$ and $Y'$ are subsets of a multiplicative subgroup of $\mathbb{C}^*$ of rank at most $r(C)$. As $G$ is regular, the subgraph spanned by $X', Y'$ still has at least some $c''n$ edges. We show that the lines represented by these $c''n$ edges cannot have high multiplicity intersections outside of $L_1, L_2, L_3$. If $(a, b)$ is a point of $M_i$ connecting two points of $X'$ and $Y'$ then $(a-x_i)/(a-y_i) = b/(1-b)$, which gives the solution $(x_i, y_i)$ to the equation $cx + dy = 1$ if $a \neq 0, b \neq 0, 1$. Here $c, d$ depend on $a$ and $b$ only. As $x_i$ and $y_i$ are from a multiplicative group of bounded rank, we have a uniform bound, $B$, on the number of lines between $X'$ and $Y'$ which are incident to $(a, b)$. There are $c'n$ lines connecting at most $C\sqrt{n}$ points. No more than $C\sqrt{n}/2$ of them might be parallel to any given line. Any line intersects at least $c'n - C\sqrt{n}$ of them. Any intersection point outside of lines $L_1, L_2$, and $L_3$ is incident to at most $B$ lines, so there are at least $cn$ distinct intersection points $L \cap M_j, 1 \leq j \leq n$ with any other line.

We are unaware of any proof of this fact without the Subspace Theorem.

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