The Singular Locus of an Almost Distance Function ∗†

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Abstract

The aim of this article is to generalize the notion of the cut locus and to get the structure theorem for it. For this purpose, we first introduce a class of 1-Lipschitz functions on a Finsler manifold, each member of which is called an almost distance function. Typical examples of an almost distance function are the distance function from a point and the Busemann functions.

The generalized notion of the cut locus in this paper is called the singular locus of an almost distance function. The singular locus consists of the upper singular locus and the lower singular locus. The upper singular locus coincides with the cut locus of a point $p$ for the distance function from the point $p$, and the lower singular locus coincides with the set of all copoints of a ray $\gamma$ when the almost distance function is the Busemann function of the ray $\gamma$. Therefore, it is possible to treat the cut locus of a closed subset and the set of copoints of a ray in a unified way by introducing the singular locus for the almost distance function.

In this article, some theorems on the distance function from a closed set and the Busemann function are generalized by making use of the almost distance function.

1 Introduction

Since Poincaré [2] introduced the notion of the cut locus of a point in a compact convex surface in 1905, this notion has been generalized for a submanifold of an (arbitrary dimensional) Riemannian manifold, and a closed subset of a Riemannian manifold or a Finsler manifold.

The cut locus of a point of a Riemannian manifold is the most fundamental case when we consider the cut locus. The cut locus of a point $p$ in a complete Riemannian manifold $M$ is very closely related to the distance function $d_p$ from the point $p$. It is well known that the cut locus equals the closure of the set of all non-differentiable points of the function $d_p$ on $M \setminus \{p\}$.

This function $d_p$ has two important properties: One is 1-Lipschitz, i.e., for any points $x, y$ in $M$, $d_p(x) - d_p(y) \leq d(y, x)$. Here $d(\cdot, \cdot)$ denotes the Riemannian distance on $M$. The other one is that for any unit speed minimal geodesic segment $\gamma : [0, a] \to M$ emanating from $p$, $d_p(\gamma(t)) = t$ holds on $[0, a]$.

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If the geodesic segment $\gamma$ has no extensions as a minimal geodesic, then the end point $\gamma(a)$ is called a cut point of $p$ along $\gamma$, and the set of the cut points along all minimal geodesic segments issuing from $p$ is called the cut locus of $p$.

In this paper we generalize the notion of the cut locus. For this purpose, we first introduce an almost distance function on a Finsler manifold. We need the notion of $f$-geodesics for introducing the almost distance function $f$.

Let $f$ be a 1-Lipschitz function on a connected Finsler manifold $M$. A unit speed geodesic segment $\alpha : I \to M$ is called an $f$-geodesic if $f(\alpha(t)) - f(\alpha(s)) = t - s$ holds for any $s, t \in I$, where $I$ denotes an interval.

A 1-Lipschitz function $f$ on a connected Finsler manifold $M$ is called an almost distance function if for each point $p \in f^{-1}(\inf f, \sup f)$, there exist a neighborhood $U_p$ of $p$ and a positive constant $\delta(p)$ such that for each point $q \in U_p$, there exists an $f$-geodesic $\gamma_q : [0, \delta(p)] \to M$ with $q = \gamma_q(0)$ or $q = \gamma_q(\delta(p))$.

The distance function $d_p$ is an almost distance function on a connected Finsler manifold. Another typical example is the Busemann function $B_p$ of a ray $\gamma$ on a forward complete connected Finsler manifold. It is known that a ray $\sigma$ is a coray of a ray $\gamma$ if and only if the ray $\sigma$ is a $B_{\gamma}$-geodesic and that for any ray $\gamma$, there exists a coray of $\gamma$ emanating from each point of the manifold (see for example [Ni, Oh, Sa]).

The class of almost distance functions is very rich as the last example in Section 7 shows that the singular locus of an almost distance function can have a very complicated structure, which never happens for the distance function from a point in a Riemannian manifold. The singular locus $\mathcal{C}(f)$ of an almost distance function $f$ is defined as the set of the end points of all maximal $f$-geodesics. For example, the singular locus of the distance function $d_p$ from a point $p$ on a bi-complete Finsler manifold equals the union of the cut locus of $p$ and $\{p\}$.

In this article, we will prove three main theorems, Theorems A, B, and C which have been already proved in the case where the almost distance function $f$ is the distance function from a closed subset or a Busemann function.

The following theorem generalizes [Ko, T1] Theorem 2.5, [Sa] Theorem 1.2 and [ST] Theorem 2.3.

**Theorem A** Let $f$ be an almost distance function on a connected Finsler manifold $M$ and $p \in f^{-1}(\inf f, \sup f)$. Then $f$ is differentiable at $p$ if and only if $p$ admits a unique $f$-geodesic. Moreover, if $f$ is differentiable at a point $p$ in $f^{-1}(\inf f, \sup f)$, its differential $df_p$ at $p$ is given by

$$df_p(v) = g_{\gamma(f(p))}(\dot{\gamma}(f(p)), v)$$  \hspace{1cm} (1.1)

for any $v \in T_p M$, where $\gamma$ denotes the unique maximal $f$-geodesic through $p = \gamma(f(p))$ with canonical parameter, and $\dot{\gamma}(f(p))$ denotes the velocity vector of $\gamma$ at $p$. Here, the canonical parameter of the $f$-geodesic $\alpha$ means that the parameter $t$ satisfies $f \circ \alpha(t) = t$.

The following theorem generalizes Theorem B in [ST].

**Theorem B** Let $f$ be an almost distance function on a bi-complete 2-dimensional Finsler manifold $M$. Then, the singular locus $\mathcal{C}(f) \cap f^{-1}(\inf f, \sup f)$ in $f^{-1}(\inf f, \sup f)$ of $f$ satisfies the following properties.
1. The set $C(f) \cap f^{-1}(\inf f, \sup f)$ is a local tree and any two points in the same connected component can be joined by a rectifiable curve in $C(f) \cap f^{-1}(\inf f, \sup f)$.

2. The topology of $C(f) \cap f^{-1}(\inf f, \sup f)$ induced from the intrinsic metric $\delta$ coincides with the induced topology of $C(f) \cap f^{-1}(\inf f, \sup f)$ from $M$.

3. The set $C(f) \cap f^{-1}(\inf f, \sup f)$ is a union of countably many rectifiable Jordan arcs except for the end points of $C(f) \cap f^{-1}(\inf f, \sup f)$.

Remark 1.1 Nasu [N2] investigates the topological structure of the singular locus of a Busemann function on a finitely connected 2-dimensional Finsler manifold (more generally a Busemann G-space) of nonpositive curvature. K. Shiohama and the present author [ShT] proved Theorem B for the distance function from a compact subset of an Alexandrov surface and for the Busemann function on a noncompact Alexandrov surface.

The following theorem is called Sard theorem for an almost distance function. Here a point is called a critical point (in the sense of Clarke) of an almost distance function $f$ if the generalized differential of $f$ at $p$ contains the zero 1-form. The detailed definition is given in Section 6.

For the distance function $d_N$ from a closed smooth submanifold $N$ of an $n$-dimensional complete Riemannian manifold, it is proved that the set of all critical values of $d_N$ is of measure zero, i.e., J. Itoh and the present author [IT] proved it when $n < 5$ and L. Rifford [R] generalized it for any $n$.

**Theorem C** Let $f$ be an almost distance function on a bi-complete 2-dimensional Finsler manifold $M$. Then, the set $C_V(f)$ of critical values of $f$ is of measure zero.

**Remark 1.2** Theorem C is still true for an arbitrary dimensional Riemannian manifold if we add some assumptions to the function $f$ (see Theorem 6.11).

As a corollary to Theorem C, we get

**Corollary to Theorem C** For each $t \in (\inf f, \sup f) \setminus C_V(f)$, the level set $f^{-1}(t)$ consists of locally finitely many mutually disjoint curves which are locally bi-Lipschitz homeomorphic to an interval.

**Remark 1.3** S. Ferry [Fe] proved that the critical values of the distance function $d_N$ is of measure zero for a closed subset $N$ of $\mathbb{R}^n$, $n \leq 3$, and he constructed a closed subset $N$ of $\mathbb{R}^4$ such that the level set $d_N^{-1}(t)$ is not a submanifold for all $t \in (0, 1)$. Fu [Fu] improved his result.

Let us recall that a Finsler manifold $(M, F)$ is an $n$-dimensional differential manifold $M$ endowed with a norm $F : TM \to [0, \infty)$ such that

1. $F$ is positive and differentiable on $\tilde{TM} := TM \setminus \{0\}$;
2. $F$ is 1-positive homogeneous, i.e., $F(x, \lambda y) = \lambda F(x, y)$, $\lambda > 0$, $(x, y) \in TM$;
3. the Hessian matrix \( g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} \) is positive definite on \( \tilde{T}M \).

Here \( TM \) denotes the tangent bundle over the manifold \( M \). The Finsler structure is called absolute homogeneous if \( F(x, -y) = F(x, y) \) because this leads to the homogeneity condition \( F(x, \lambda y) = |\lambda| F(x, y) \), for any \( \lambda \in \mathbb{R} \).

By means of the Finsler fundamental function \( F \) one defines the indicatrix bundle (or the Finslerian unit sphere bundle) by \( SM := \bigcup_{x \in M} S_x M \), where \( S_x M := \{ y \in M \mid F(x, y) = 1 \} \).

On a Finsler manifold \( (M, F) \) one can define the integral length of curves as follows. If \( \gamma : [a, b] \to M \) is a \( C^1 \)-curve in \( M \), then the integral length of \( \gamma \) is given by

\[
L_\gamma := \int_a^b F(\gamma(t), \dot{\gamma}(t))dt, \tag{1.2}
\]

where \( \dot{\gamma} = \frac{d\gamma}{dt} \) denotes the tangent vector along the curve \( \gamma \).

For a \( C^1 \)-variation

\[
\bar{\gamma} : (-\varepsilon, \varepsilon) \times [a, b] \to M, \quad (u, t) \mapsto \bar{\gamma}(u, t) \tag{1.3}
\]

of the base curve \( \gamma(t) \), with the variational vector field \( U(t) := \frac{\partial \bar{\gamma}}{\partial u}(0, t) \), by a straightforward computation one obtains

\[
(L_{\dot{\gamma}})'(0) = g_{\dot{\gamma}(b)}(\dot{\gamma}(b), U(b))^b_a - \int_a^b g_{\dot{\gamma}}(D_{\dot{\gamma}} \dot{\gamma}, U)dt, \tag{1.4}
\]

where \( D_{\dot{\gamma}} \) is the covariant derivative along \( \gamma \) with respect to the Chern connection and \( \gamma \) is arc length parametrized (see [BCS], p. 123, or [S], p. 77 for details of this computation as well as for some basics on Finslerian connections).

A regular \( C^\infty \)-curve \( \gamma \) on a Finsler manifold is called a geodesic if \( (L_{\dot{\gamma}})'(0) = 0 \) for all \( C^1 \)-variations of \( \gamma \) that keep its ends fixed. In terms of Chern connection a constant speed geodesic is characterized by the condition \( D_{\dot{\gamma}} \dot{\gamma} = 0 \).

If the base curve \( \gamma \) is a geodesic for the variation \( \bar{\gamma} \) above, one obtains, by (1.4), the following first variation formula:

\[
(L_{\dot{\gamma}})'(0) = g_{\dot{\gamma}(b)}(\dot{\gamma}(b), U(b)) - g_{\dot{\gamma}(a)}(\dot{\gamma}(a), U(a)) \tag{1.5}
\]

which is fundamental for our present study.

Using the integral length of a curve, one can define the Finslerian distance between two points on \( M \). For any two points \( p, q \) on \( M \), let us denote by \( \Omega_{p,q} \) the set of all \( C^1 \)-curves \( \gamma : [a, b] \to M \) such that \( \gamma(a) = p \) and \( \gamma(b) = q \). The map

\[
d : M \times M \to [0, \infty), \quad d(p, q) := \inf_{\gamma \in \Omega_{p,q}} L_\gamma \tag{1.6}
\]

gives the Finslerian distance on \( M \). It can be easily seen that \( d \) is in general a quasi-distance, i.e., it has the properties
1. \( d(p, q) \geq 0 \), with equality if and only if \( p = q \);

2. \( d(p, q) \leq d(p, r) + d(r, q) \), with equality if and only if \( r \) lies on a minimal geodesic segment joining \( p \) to \( q \) (triangle inequality).

In the case where \((M, F)\) is absolutely homogeneous, the symmetry condition \( d(p, q) = d(q, p) \) holds and therefore \((M, d)\) is a metric space. We do not assume this symmetry condition in the present paper.

We recall ([BCS], p. 151) that a sequence of points \( \{x_i\} \subset M \), on a Finsler manifold \((M, F)\), is called a forward Cauchy sequence if for any \( \varepsilon > 0 \), there exists \( N = N(\varepsilon) > 0 \) such that for all \( N \leq i < j \) we have \( d(x_i, x_j) < \varepsilon \). Likely, a sequence of points \( \{x_i\} \subset M \) is called a backward Cauchy sequence if for any \( \varepsilon > 0 \), there exists \( N = N(\varepsilon) > 0 \) such that for all \( N \leq i < j \) we have \( d(x_j, x_i) < \varepsilon \). The Finsler space \((M, F)\) is called forward (backward) complete with respect to the Finsler distance \( d \) if and only if every forward (backward) Cauchy sequence converges, respectively. Moreover, a Finsler manifold \((M, F)\) is called forward (respectively backward) geodesically complete if and only if any geodesic \( \gamma: (a, b) \to M \) can be forwardly (respectively backwardly) extended to a geodesic \( \gamma: (a, \infty) \to M \) (respectively \( \gamma: (-\infty, b) \to M \)). The equivalence between forward completeness and geodesically completeness is given by the Finslerian version of Hopf-Rinow Theorem (see for eg. [BCS], p. 168).

Let us point out that in the Finsler case, unlike the Riemannian counterpart, forward completeness is not equivalent to backward one, except the case when \( M \) is compact. A Finsler metric that is forward and backward complete is called bi-complete.

Let us also recall that for a forward complete Finsler space \((M, F)\), the exponential map \( \exp_p : T_p M \to M \) at an arbitrary point \( p \in M \) is a surjective map (see [BCS], p. 152 for details).

A unit speed geodesic on \( M \) with initial conditions \( \gamma(0) = p \in M \) and \( \dot{\gamma}(0) = T \in S_p M \) can be written as \( \gamma(t) = \exp_p(tT) \). Even though the exponential map is quite similar with the correspondent notion in Riemannian geometry, we point out two distinguished properties (see [BCS], p. 127 for details):

1. \( \exp_x \) is only \( C^1 \) at the zero section of \( TM \), i.e. for each fixed \( x \), the map \( \exp_x y \) is \( C^1 \) with respect to \( y \in T_x M \), and \( C^\infty \) away from it. Its derivative at the zero section is the identity map ([W]);

2. \( \exp_x \) is \( C^2 \) at the zero section of \( TM \) if and only if the Finsler structure is of Berwald type. In this case \( \exp \) is actually \( C^\infty \) on entire \( TM \) ([AZ]).

2 1-Lipschitz functions and \( f \)-geodesics

In this section we will give an equivalent condition for a function on a connected Finsler manifold to be 1-Lipschitz and some important examples of a 1-Lipschitz function on a manifold.

From now on, \((M, F)\) denotes a connected Finsler manifold with Finslerian distance function \( d \).
Definition 2.1 A function $f$ on the manifold $M$ is said to be 1-Lipschitz if
\[ f(y) - f(x) \leq d(x, y) \] (2.1)
holds for any $x, y \in M$.

Let $f$ be a 1-Lipschitz function on the manifold $M$. By exchanging $x$ and $y$ in the equation (2.1), we get that
\[ -d(y, x) \leq f(y) - f(x) \] (2.2)
holds for all $x, y$ of $M$. From (2.1) and (2.2) it follows that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is locally Lipschitz for any local chart $(U, \varphi)$ of $M$.

From Rademacher’s theorem (for example, see [M]), $f \circ \varphi^{-1}$ is differentiable almost everywhere on $\varphi(U)$, and hence the Finslerian gradient $\nabla f$ of $f$ exists almost everywhere on $M$. See [S] for the definition and basic properties of the Finslerian gradient.

Choose any differentiable point $p$ of $f$ and any unit tangent vector $v$ at $p$. Let $\gamma : [0, a] \to M$ denote the minimal geodesic segment defined by $\gamma(t) := \exp_p(tv)$. From (2.1), we have $f(\gamma(t)) - f(\gamma(0)) \leq d(\gamma(0), \gamma(t)) = t$ for $t \in (0, a)$. Thus,
\[ df_p(v) = \lim_{t \searrow 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t} \leq 1. \] (2.3)
If $df_p \neq 0$, then we obtain, from (2.3),
\[ g_{\nabla f_p}(\nabla f_p, v) \leq 1 \] (2.4)
for all unit tangent vectors $v$ at $p$. By substituting $v = \nabla f_p/F(\nabla f_p)$ in the equation (2.4) we get $F(\nabla f_p) \leq 1$. If $df_p = 0$, then it is trivial that $F(\nabla f_p) = 0 \leq 1$. Therefore, $\nabla f$ exists almost everywhere and $F(\nabla f_p) \leq 1$ if $\nabla f_p$ exists.

We may prove the converse by imitating the proof of [KT1] Lemma 3.13, and hence we get

Proposition 2.2 Let $f$ be a function defined on a connected Finsler manifold $M$. Then, the Finslerian gradient $\nabla f$ of the function $f$ exists almost everywhere and $F(\nabla f) \leq 1$ almost everywhere if and only if $f$ is 1-Lipschitz.

Example 2.3 Let $N$ be a closed subset of a connected Finsler manifold $M$. We define a function $d_N$ on $M$ by $d_N(x) := \inf\{d(p, x) | p \in N\}$. The function $d_N$ is a 1-Lipschitz function on $M$. Indeed, choose any points $x, y$ of $M$ and any positive number $\epsilon$. There exists a point $p_\epsilon \in N$ such that
\[ d(p_\epsilon, y) - \epsilon < d_N(y). \] (2.5)
Moreover, it is trivial that
\[ d_N(x) \leq d(p_\epsilon, x). \] (2.6)
Hence, by (2.5), (2.6) and the triangle inequality, we get,
\[ d_N(x) - d_N(y) < d(p_\epsilon, x) - d(p_\epsilon, y) + \epsilon \leq d(y, x) + \epsilon. \]
Since $\epsilon$ is arbitrary, we have $d_N(x) - d_N(y) \leq d(y, x)$. In particular, if the subset $N$ is a single point $p$, then the distance function $d_p$ from the point $p$ is 1-Lipschitz.
Example 2.4 We can introduce the function $d^N$ similar to $d_N$ defined by $d^N(x) := \inf\{d(x,p)\mid p \in N\}$. This function is not always a 1-Lipschitz function on $M$, but $(-1)d^N$ is 1-Lipschitz.

Example 2.5 A unit speed geodesic $\gamma : [0, \infty) \to M$ in a forward complete Finsler manifold $M$ is called a ray if $d(\gamma(0), \gamma(t)) = t$ holds on $[0, \infty)$. Then, the function $B_\gamma$ on $M$ is defined by $B_\gamma(x) := \lim_{t \to \infty}\{t - d(x, \gamma(t))\}$ is 1-Lipschitz. This function is called the Busemann function of the ray $\gamma$.

Example 2.6 Let $f_1$ and $f_2$ be 1-Lipschitz functions on a connected Finsler manifold $M$. Then $f := \max(f_1, f_2)$ is 1-Lipschitz. Indeed, choose any points $x, y \in M$. Without loss of generality, we may assume that $f(x) = f_1(x)$. Since $f(y) \geq f_1(y)$, we obtain, $f(x) - f(y) \leq f_1(x) - f_1(y) \leq d(y, x)$. It is also easy to prove that $\min(f_1, f_2)$ is a 1-Lipschitz function.

Example 2.7 The Busemann function can be generalized as follows. Let us consider a divergent sequence of points $\{x_n\}$ on the forward complete noncompact Finsler manifold $(M, F)$, that is $\lim_{n \to \infty} d(x_1, x_n) = \infty$. Then the function $f : M \to \mathbb{R}$ given by

$$f(x) := \lim_{n \to \infty} \sup\{d(x_1, x_n) - d(x, x_n)\}$$

is an 1-Lipschitz function. Notice that the function $[d(x_1, x_n) - d(x, x_n)]$ is 1-Lipschitz for each $n$, since $(-1)d^\mathbb{R}$ is 1-Lipschitz. This function is called a horofunction in [Cu] for Riemannian case.

Example 2.8 The following function gives another interesting example of a 1-Lipschitz function on a Riemannian manifold, which was introduced in the paper [Wu]. Let $M$ be a complete noncompact Riemannian manifold and let us define

$$\eta : M \to \mathbb{R}, \quad \eta(x) := \lim_{t \to \infty} \sup\{t - d(x, S_p(t))\}, \tag{2.7}$$

where $S_p(t) := \{x \in M\mid d(p, x) = t\}$, (see [Wu] for a more general setting). The triangle inequality implies that $\eta(x) := t - d(x, S_p(t)) \leq d(p, x)$ for all $t > 0$, hence $\eta$ is well-defined. For any fixed $t > 0$ the function $\eta_t(x)$ is 1-Lipschitz, hence $\eta$ is also 1-Lipschitz.

From now on, $f$ always denote a 1-Lipschitz function on the connected Finsler manifold $M$.

Definition 2.9 A unit speed nonconstant geodesic $\gamma : I \to M$ called an $f$-geodesic if

$$f(\gamma(t)) - f(\gamma(s)) = t - s \tag{2.8}$$

for any $s, t \in I$, where $I$ denotes an interval.

Such an $f$-geodesic is called maximal if there are no $f$-geodesics containing $\gamma(I)$ as a proper subarc. Notice that any $f$-geodesic is minimal. This property is proved in Lemma 2.10. Moreover, a sufficient condition for a unit speed Lipschitz curve to be an $f$-geodesic is given in this lemma.
Lemma 2.10 If a unit speed Lipschitz curve \( \gamma : [a, b] \rightarrow M \), (i.e. \( |\dot{\gamma}(t)| = 1 \) a.e.) satisfies

\[
b - a = f(\gamma(b)) - f(\gamma(a))
\]

then \( \gamma \) is a minimal geodesic segment and an \( f \)-geodesic. In particular, any \( f \)-geodesic is minimal.

Proof. The length \( L(\gamma) \) of the curve \( \gamma \) is given by \( L(\gamma) = \int_a^b F(\dot{\gamma}(t))dt \). Since \( \gamma \) is unit speed, we get \( L(\gamma) = b - a \). Since \( f \) is 1-Lipschitz, we get

\[
L(\gamma) = b - a = f(\gamma(b)) - f(\gamma(a)) \leq d(\gamma(a), \gamma(b)).
\]

Hence the curve \( \gamma \) is a minimal geodesic segment.

In order to prove that \( \gamma \) is an \( f \)-geodesic, we will next prove that the function \( \varphi(t) := t - f(\gamma(t)) \) is an increasing function on \([a, b]\). Choose any \( a \leq s < t \leq b \). Since \( f \) is 1-Lipschitz and the geodesic \( \gamma \) is a unit speed minimal geodesic, we get \( \varphi(s) - \varphi(t) \leq s - t + d(\gamma(s), \gamma(t)) = 0 \). Thus, the function \( \varphi \) is increasing on \([a, b]\). By combining our assumption \( \varphi(a) = \varphi(b) \), we may conclude that \( \varphi \) is constant and the geodesic \( \gamma \) satisfies (2.8) for any \( s, t \in [a, b] \). \( \square \)

Example 2.11 Let \( \gamma : [0, a] \rightarrow M \) be a unit speed minimal geodesic segment emanating from a point \( p = \gamma(0) \) on a connected Finsler manifold \( M \). Then, it is clear that \( d_p(\gamma(t)) = t \) holds on \([0, a]\), where \( d_p(x) := d(p, x) \) for each \( x \in M \). Hence \( \gamma \) is a \( d_p \)-geodesic. There does not always exist a minimal geodesic joining the point \( p \) to any point of \( M \), since \( M \) is not assumed to be complete. The following lemma guarantees the local existence of a \( d_p \)-geodesic without assumption of the completeness of a Finsler manifold.

Lemma 2.12 Let \( N \) be a closed subset of a connected Finsler manifold \( M \). Then for each point \( q \notin N \), there exists a \( d_N \)-geodesic to \( q \). In particular, for each point \( q \), there exists a \( d_p \)-geodesic to \( p \), if the point \( p \) is distinct from \( q \).

Proof. Choose any point \( q \notin N \) and a small positive number \( \delta \), so that \( S := \{x \in M | d(x, q) = \delta\} \) is compact, and each point of \( S \) is connected to the point \( q \) by the unique minimal geodesic segment. Since \( S \) is compact and the function \( d_N \) is continuous, there exist a point \( x_0 \in S \) with \( d_N(x_0) = \min\{d_N(x) | x \in S\} \). Then it is easy to check that

\[
d_N(q) \geq d_N(x_0) + d(x_0, q). \quad (2.9)
\]

If \( \alpha : [0, \delta] \rightarrow M \) denotes the unit speed minimal geodesic segment emanating from \( x_0 \) to \( q \), the equation (2.9) implies that

\[
d_N(\alpha(\delta)) \geq d_N(\alpha(0)) + \delta,
\]

and hence

\[
d_N(\alpha(\delta)) = d_N(\alpha(0)) + \delta,
\]

since \( d_N \) is 1-Lipschitz. From Lemma 2.10 it follows that \( \alpha : [0, \delta] \rightarrow M \) is a \( d_N \)-geodesic to the point \( q \). \( \square \)
3 First variation formulas for almost distance functions

We will give the proof of Theorem A and some interesting examples of an almost distance function in this section. Let $f$ be a 1-Lipschitz function on a connected Finsler manifold $(M, F)$.

It is sometimes convenient to make use of the following parametrization of $f$-geodesics.

**Definition 3.1** The parameter of an $f$-geodesic $\alpha : [a, b] \to M$ is called *canonical* if $f(\alpha(t)) = t$ holds on $[a, b]$.

By definition, it is clear that an $f$-geodesic is preserved by any parallel translation of the parameter of the $f$-geodesic. Hence, we can introduce the canonical parameter for any $f$-geodesic.

**Lemma 3.2** Let $\{\gamma_i\}_i$ be a sequence of $f$-geodesics defined on a common interval $[0, a]$. If the following three limits

\[ p := \lim_{i \to \infty} \gamma_i(0) \quad (3.1) \]
\[ w_\infty := \lim_{i \to \infty} \dot{\gamma}_i(0) \quad (3.2) \]
\[ v^f := \lim_{i \to \infty} \frac{1}{F(\exp_p^{-1}(\gamma_i(0)))} \exp_p^{-1}(\gamma_i(0)) \quad (3.3) \]

exist, then

\[ g_{w_\infty}(w_\infty, v^f) \geq g_{\dot{\gamma}(f(p))}(\dot{\gamma}(f(p)), v^f) \quad (3.4) \]

holds for any $f$-geodesic $\gamma$ emanating from $p$ with canonical parameter.

Moreover, we have

\[ \lim_{i \to \infty} \frac{f(\gamma_i(0)) - f(p)}{d(p, \gamma_i(0))} = g_{w_\infty}(w_\infty, v^f). \quad (3.5) \]

Here $\exp_p^{-1}$ denotes the local inverse map of the exponential map $\exp_p$ around the zero vector.

**Proof.** We may assume that each $\gamma_i(0)$ is a point in a convex ball centered at $p$. For each $\gamma_i(0)$, let $\sigma_i : [0, d(p, \gamma_i(0))] \to M$ denote the unit speed minimal geodesic segment joining $p$ to $\gamma_i(0)$, and hence

\[ \dot{\sigma}_i(0) = \frac{1}{F(\exp_p^{-1}(\gamma_i(0)))} \exp_p^{-1}(\gamma_i(0)). \quad (3.6) \]

Let us choose a constant $\delta \in (0, a)$ in such a way that $\gamma_\infty(\delta)$ is a point of a strongly convex ball around $p$. Here $\gamma_\infty(t) := \exp(tw_\infty)$ denotes the limit geodesic of the sequence
\{\gamma_i\}. Since \(\gamma_i\) is an \(f\)-geodesic and \(f\) is 1-Lipschitz, we obtain, 
\[ f(\gamma_i(0)) - f(\gamma_i(\delta)) = -d(\gamma_i(0), \gamma_i(\delta)), \]
and
\[ f(\gamma_i(\delta)) - f(p) \leq d(p, \gamma_i(\delta)), \]
respectively. Hence,
\[ f(\gamma_i(0)) - f(p) \leq d(p, \gamma_i(\delta)) - d(\gamma_i(0), \gamma_i(\delta)). \] (3.7)
It follows from the first variation formula (1.5) that
\[ d(p, \gamma_i(\delta)) - d(\gamma_i(0), \gamma_i(\delta)) = -\int_0^{d_i} h'(t) \, dt = \int_0^{d_i} g_{w_i(t)}(w_i(t), \dot{\sigma}_i(t)) \, dt \] (3.8)
holds, where \(d_i := d(p, \gamma_i(0))\), \(h(t) := d(\sigma_i(t), \gamma_i(\delta))\), and \(w_i(t)\) denotes the unit velocity vector at \(\sigma_i(t)\) of the minimal geodesic segment joining \(\sigma_i(t)\) to \(\gamma_i(\delta)\). It is clear that for each small interval \([0, c]\), the two sequences \(\{w_i(t)\}\) and \(\{\dot{\sigma}_i(t)\}\) of vector valued functions uniformly converge to \(w_\infty(t)\) and \(\dot{\sigma}_\infty(t)\) on \([0, c]\) respectively, where \(\sigma_\infty(t) := \exp(tv^f)\), and \(w_\infty(t)\) denotes the unit velocity vector at \(\sigma_\infty(t)\) of the minimal geodesic segment joining \(\sigma_\infty(t)\) to \(\gamma_\infty(\delta)\). Therefore, by (3.8), we obtain,
\[ \lim_{i \to \infty} \frac{d(p, \gamma_i(\delta)) - d(\gamma_i(0), \gamma_i(\delta))}{d_i} = g_{w_\infty}(w_\infty, v^f). \] (3.9)
Combining (3.7) and (3.9), we get
\[ \limsup_{i \to \infty} \frac{f(\gamma_i(0)) - f(p)}{d_i} \leq g_{w_\infty}(w_\infty, v^f). \] (3.10)
Let \(\beta\) denote any \(f\)-geodesic emanating from \(p\) with canonical parameter. Since \(f\) is 1-Lipschitz, and \(\beta\) is an \(f\)-geodesic, we have \(f(\gamma(0)) - f(\beta(f(p) + \delta)) \geq -d(\gamma(0), \beta(f(p) + \delta))\), and \(f(\beta(f(p) + \delta)) - f(p) = d(p, \beta(f(p) + \delta))\), respectively, where \(\delta > 0\) is a sufficiently small positive number, so that \(\beta(f(p) + \delta)\) is a point in a strongly convex ball centered at \(p\). Hence, we obtain,
\[ f(\gamma_i(0)) - f(p) \geq d(p, \beta(f(p) + \delta)) - d(\gamma_i(0), \beta(f(p) + \delta)). \]
By making use of the same technique as above, we get
\[ \liminf_{i \to \infty} \frac{f(\gamma_i(0)) - f(p)}{d_i} \geq g_{\beta(f(p))}(\dot{\beta}(f((p)), v^f) \] (3.11)
for any \(f\)-geodesic \(\beta\) emanating from \(p\). In particular, by choosing the limit \(f\)-geodesic \(\beta = \gamma_\infty\), we get (3.5), and (3.4) follows from (3.10) and (3.11). \(\Box\)

**Lemma 3.3** Let \(\{\gamma_i\}_i\) be a sequence of \(f\)-geodesics defined on a common interval \([b, 0]\).
If the following three limits
\[ p := \lim_{i \to \infty} \gamma_i(0) \]
(3.12)
\[ w_\infty := \lim_{i \to \infty} \dot{\gamma}_i(0) \]
(3.13)
\[ v^f := \lim_{i \to \infty} \frac{1}{F(\exp_p^{-1}(\gamma_i(0))} \exp_p^{-1}(\gamma_i(0)) \]
(3.14)
exist, then
\begin{equation}
    g_{w_{\infty}}(w_{\infty}, v^f) \leq g_{\hat{\gamma}(f(p))}(\hat{\gamma}(f(p)), v^f)
\end{equation}
holds for any \textit{f-geodesic} \( \gamma \) to \( p \) with canonical parameter. Moreover, we have
\begin{equation}
    \lim_{i \to \infty} \frac{f(\gamma_i(0)) - f(p)}{d(p, \gamma_i(0))} = g_{w_{\infty}}(w_{\infty}, v^f).
\end{equation}

\textbf{Proof.} We may assume that each \( \gamma_i(0) \) is a point in a convex ball centered at \( p \). For each \( \gamma_i(0) \), let \( \sigma_i : [0, d(p, \gamma_i(0))] \to M \) denote the unit speed minimal geodesic segment joining \( p \) to \( \gamma_i(0) \), and hence
\begin{equation}
    \dot{\sigma}_i(0) = \frac{1}{F(\exp_p^{-1}(\gamma_i(0)))} \exp_p^{-1}(\gamma_i(0)).
\end{equation}
Let us choose a constant \( \delta \in (0, [b]) \) in such a way that \( \gamma_{\infty}(t) := \exp(t w_{\infty}) \) denotes the limit geodesic of the sequence \( \{\gamma_i\} \). Since \( \gamma_i \) is an \textit{f-geodesic} and \( f \) is 1-Lipschitz, we obtain, \( f(\gamma_i(0)) - f(\gamma_i(\delta)) = d(\gamma_i(\delta), \gamma_i(0)), \) and \( f(\gamma_i(\delta)) - f(p) \geq -d(\gamma_i(\delta), p) \). Hence,
\begin{equation}
    f(\gamma_i(0)) - f(p) \geq d(\gamma_i(\delta), \gamma_i(0)) - d(\gamma_i(\delta), p).
\end{equation}
It follows from the first variation formula (1.5) that
\begin{equation}
    d(\gamma_i(\delta), \gamma_i(0)) - d(\gamma_i(\delta), p) = \int_0^{d_i} h'(t)dt = \int_0^{d_i} g_{w_i(t)}(w_i(t), \dot{\sigma}_i(t))dt
\end{equation}
holds, where \( d_i := d(p, \gamma_i(0)), h(t) := d(\gamma_i(\delta), \sigma_i(t)) \), and \( w_i(t) \) denotes the unit velocity vector at \( \sigma_i(t) \) of the minimal geodesic segment joining \( \gamma_i(\delta) \) to \( \sigma_i(t) \). It is clear that for each small interval \([0, c]\), the two sequences \( \{w_i(t)\} \) and \( \{\dot{\sigma}_i(t)\} \) of vector valued functions uniformly converge to \( w_{\infty}(t) \) and \( \dot{\sigma}_{\infty}(t) \) on \([0, c]\) respectively, where \( \sigma_{\infty}(t) := \exp(t w^f) \) and \( w_{\infty}(t) \) denotes the unit velocity vector at \( \sigma_{\infty}(t) \) of the minimal geodesic segment joining \( \sigma_{\infty}(t) \) to \( \gamma_{\infty}(\delta) \). Therefore, by (3.19), we obtain,
\begin{equation}
    \lim_{i \to \infty} \frac{d(\gamma_i(\delta), \gamma_i(0)) - d(\gamma_i(\delta), p)}{d_i} = g_{w_{\infty}}(w_{\infty}, v^f).
\end{equation}
Combining (3.18) and (3.20), we get
\begin{equation}
    \liminf_{i \to \infty} \frac{f(\gamma_i(0)) - f(p)}{d_i} \geq g_{w_{\infty}}(w_{\infty}, v^f).
\end{equation}

Let \( \beta \) denote any \textit{f-geodesic} to \( p \) with canonical parameter. Since \( f \) is 1-Lipschitz, and \( \beta \) is an \textit{f-geodesic}, we have \( f(\gamma_i(0)) - f(\beta(f(p) - \delta)) \leq d(\beta(f(p) - \delta), \gamma_i(0)) \), and \( f(\beta(f(p) - \delta)) - f(p) = -d(\beta(f(p) - \delta), p) \), where \( \delta > 0 \) is a sufficiently small number, so that \( \beta(f(p) - \delta) \) is a point in a strongly convex ball at \( p \). Hence, we obtain, \( f(\gamma_i(0)) - f(p) \leq d(\beta(f(p) - \delta), \gamma_i(0)) - d(\beta(f(p) - \delta), \gamma_i(0)) \). By making use of the same technique as above, we get
\begin{equation}
    \limsup_{i \to \infty} \frac{f(\gamma_i(0)) - f(p)}{d_i} \leq g_{\hat{\beta}(f(p))}(\hat{\beta}(f((p))), v^f)
\end{equation}
for any \textit{f-geodesic} \( \beta \) to \( p \). In particular, by choosing the limit \textit{f-geodesic} \( \beta = \gamma_{\infty} \), we get (3.19), and (3.15) follows from (3.21) and (3.22). \( \square \)
Definition 3.4 A 1-Lipschitz function \( f \) on a connected Finsler manifold \( M \) called an \textit{almost distance function} if for each point \( p \in f^{-1}(\inf f, \sup f) \), there exists a neighborhood \( U_p \) of \( p \) and a positive constant \( \delta(p) \) such that for each point \( q \in U_p \), there exists an \( f \)-geodesic \( \gamma_q : [0, \delta(p)] \to M \) with \( q = \gamma_q(0) \) or \( q = \gamma_q(\delta(p)) \).

Remark 3.5 Observe that this definition guarantees that maximal \( f \)-geodesics do not shrink to a point. It is possible to define the almost distance function on a metric space such as an Alexandrov space or a Busemann E-space. Note that a Finsler manifold is a Busemann E-space (see [Bu1]). For example, Theorem B in the introduction would be true for an almost distance function on an Alexandrov surface. We, however, restrict ourselves to the almost distance function on a differentiable manifold in this paper.

Obviously, the distance function from a closed subset \( N \) and the Busemann function on a Finsler manifold are typical examples of almost distance functions (see Examples 3.6, 3.7 and 3.8 below).

Example 3.6 Let \( N \) denote a closed subset of a connected Finsler manifold \( M \). From the proof of Lemma 2.12 it is easy to check that for each \( p \notin d_{\infty}(\inf d_N) = d_N^{-1}(0) \), any point \( q \in B_{r(p)}(p) \) admits a \( d_N \)-geodesic \( \gamma_q : [0, r(p)] \to M \) with \( \gamma_q(r(p)) = q \). Here \( r(p) := d_N(p)/3 \), and \( B_{r(p)}(p) \) denotes the open ball centered at \( p \) with radius \( r(p) \). Thus, \( d_N \) is an almost distance function on \( M \).

Example 3.7 Let \( \gamma : [0, \infty) \to M \) denote a ray on a forward complete Finsler manifold \( M \). Then, a ray \( \sigma : [0, \infty) \to M \) on \( M \) is called a \textit{coray} of (or \textit{asymptotic ray to}) the ray \( \gamma \) if \( \sigma \) has a sequence \( \{ q_i \} \) of points of \( M \) and a sequence \( \{ t_i \} \) of positive numbers with \( \lim_{i \to \infty} t_i = \infty \) such that \( \sigma(0) = \lim_{i \to \infty} q_i \) and \( \dot{\sigma}(0) = \lim_{i \to \infty} \dot{\sigma}(t_i) \), where \( \dot{\sigma}(t) \) denotes a minimal geodesic segment joining \( q_i = \sigma_i(0) \) to \( \gamma(t_i) \). Note that Busemann [Bu2] introduced the notion of asymptotic rays on a Busemann E-space, and he proved in [Bu1, Theorem 11.2] that a ray \( \sigma : [0, \infty) \to M \) is a coray of a ray \( \gamma \) if and only if the ray \( \sigma \) is a \( B_\gamma \)-geodesic on a Busemann E-space \( M \). Note that a Finsler manifold is a Busemann E-space. It is checked again in [Sa] that each coray is an \( B_\gamma \)-geodesic, where \( B_\gamma \) denotes the Busemann function of \( \gamma \) defined in Example 2.5. Remark that this property is also checked in [Oh] under a little stronger definition of the coray on a Finsler manifold. Therefore, any Busemann function on a forward complete Finsler manifold is an almost distance function.

Example 3.8 Let us check that \( \eta \) is an almost distance function, where \( \eta \) is the function introduced in Example 2.8. It is proved in [Wu, Lemma 6] that for each point \( x \in M \) there exists a ray \( \gamma \) emanating from \( x \) satisfying \( \eta(\gamma(t)) = t + \eta(x) \). This implies that each point admits an \( \eta \)-geodesic which is a ray, hence \( \eta \) is an almost distance function. The horofunction \( f \) introduced in Example 2.7 is also an almost distance function. In fact, for each point \( p \) on a complete Riemannian manifold, there exists a ray emanating from \( p \) which is an \( f \)-geodesic. Remark that the \( f \)-geodesic is a limit of the sequence of the minimal geodesic segments from \( p \) to the point \( x_n \).

The definition of an almost distance function would lead to the following question:

Under what conditions an almost distance function becomes the distance function from a
closed set?

In Proposition 3.3, we will see a sufficient condition for an almost distance function to be the distance function from a closed subset.

Example 3.9  An almost distance function on a forward complete and connected Finsler manifold $M$ is called a generalized Busemann function if each point of $M$ admits an $f$-geodesic $\gamma : [0, \infty) \to M$ emanating from $p$. Remark that $\gamma$ is a ray by Lemma 2.10. Choose any two generalized Busemann functions $f_1$ and $f_2$ on a forward complete and connected Finsler manifold $M$, and put $f := \max(f_1, f_2)$. From Example 2.6, the function $f$ is 1-Lipschitz. We will prove that $f$ is a generalized Busemann function. Let $p$ be any point of $M$. We may assume that $f_1(p) \geq f_2(p)$. Let $\gamma_1 : [0, \infty) \to M$ denote the maximal $f_1$-geodesic issuing from $p$. Hence $f_1(\gamma_1(t)) = t + f_1(p)$ holds on $[0, \infty)$. Since $f_2$ is 1-Lipschitz, $f_2(\gamma_1(t)) - f_2(p) \leq t$ on $[0, \infty)$. Therefore, $f_2(\gamma_1(t)) \leq t + f_1(p) = f_1(\gamma_1(t))$ on $[0, \infty)$. This implies that $f(\gamma_1(t)) = f_1(\gamma_1(t)) = t + f(p)$ on $[0, \infty)$, and $\gamma_1$ is an $f$-geodesic issuing from $p$. In particular, the function $f$ is an almost distance function on $M$.

Proposition 3.10  Let $f$ be an almost distance function on a connected Finsler manifold $(M, F)$. Let $\alpha$ be a unit speed geodesic emanating from a point $p = \alpha(0)$ in $f^{-1}(\inf f, \sup f)$. If the point $p$ admits a unique maximal $f$-geodesic $\gamma$ with canonical parameter, then

$$\lim_{t \searrow 0} \frac{f \circ \alpha(t) - f \circ \alpha(0)}{t} = g_{\gamma(f(p))} (\dot{\gamma}(f(p)), \dot{\alpha}(0))$$

(3.23)

holds.

Proof.  Since $f$ is an almost distance function, there exist a neighborhood $U_p$ of $p$ and a constant $\delta(p) > 0$ such that for each point $q \in U_p$, there exists an $f$-geodesic $\gamma_q : [0, \delta(p)] \to M$ with $q = \gamma_q(0)$ or $\gamma_q(\delta(p))$. Hence, for each sufficiently small $t > 0$, there exists an $f$-geodesic $\gamma_t : [0, \delta(p)] \to M$ with $\alpha(t) = \gamma_t(0)$ or $\gamma_t(\delta(p))$. Since the geodesic segment $\gamma$ is a unique $f$-geodesic through $p$, both tangent vectors $w_{\infty}$ defined in Lemmas 3.2 and 3.3 equal $\dot{\gamma}(f(p))$. Therefore, we get (3.23).

Remark 3.11  Innami [In1] gets a similar result to Proposition 3.10 for Busemann functions on a Busemann G-space.

Proof of Theorem A  Suppose that the point $p$ admits a unique maximal $f$-geodesic. From Proposition 3.10, it follows that

$$\lim_{s \searrow 0} \frac{f(\alpha(s)) - f(\alpha(0))}{s} = g_{\gamma(f(p))} (\dot{\gamma}(f(p)), \dot{\alpha}(0))$$

(3.24)

holds for any unit speed geodesic $\alpha(s)$ emanating from $p = \alpha(0)$. Then, it follows from Lemma 2.4 in [ST1] that $f$ is differentiable at $p$, and it is trivial from (3.24) that (1.1) holds.

Suppose next that $f$ is differentiable at $p$. Let $\alpha$ be any $f$-geodesic through $p$. Since $f(\alpha(t)) = t$ holds on $[f(p) - \delta(p), f(p)]$ or $[f(p), f(p) + \delta(p)]$, we get

$$g_{\nabla f_p}(\nabla f_p, \dot{\alpha}(f(p))) = df_p(\dot{\alpha}(f(p))) = 1.$$

(3.25)
On the other hand, the Cauchy-Schwartz inequality for Finsler norms (see for instance [S, Lemma 1.2.3]) says that
\[ g_Y(Y, V) \leq F(Y)F(V), \]
holds for any nonzero vectors \( Y, V \) and equality holds if and only if \( Y = \lambda V \) for some \( \lambda > 0 \). By (3.25), equality holds in our case. Hence, \( \alpha(f(p)) = \nabla f_p \), that is, the point \( p \) admits a unique \( f \)-geodesic with the velocity vector \( \alpha(f(p)) = \nabla f_p \).

As corollaries to Theorem A, we get that

(i) a given point \( p \) of a connected and forward complete Finsler manifold \( M \) is a differentiable point of the Busemann function of a ray \( \gamma \) if and only if \( p \) admits a unique coray of \( \gamma \), and that

(ii) a point \( q \) of a connected and backward complete Finsler manifold \( M \) is differentiable for the squared distance function \( d_N^2 \) from a closed subset \( N \) of \( M \) if and only if \( q \) admits a unique \( d_N \)-geodesic from \( N \) or \( q \in N \).

Indeed, the first claim was proved for the Riemannian case in [KT1], and in [Sa] for the Finslerian case. Likewise, the second claim was proved for a closed subset of a complete Riemannian manifold in [MM], by Mantegazza and Mennucci, and in [ST] for the Finslerian case.

4 Characterizations of \( f \)-geodesics

In this section, some characterizations of \( f \)-geodesics are given. It was proved by Sabau [Sa] that a ray \( \sigma : [0, \infty) \to M \) is a \( B_\gamma \)-geodesic, i.e. a coray of \( \gamma \), if and only if the subarc of \( \sigma \) in \( B_\gamma^{-1}(-\infty, a] \) is a reverse \( B_\gamma^{-1}[a, \infty) \)-segment for any \( a > B_\gamma(\sigma(0)) \). Here \( B_\gamma \) denotes the Busemann function of \( \gamma \), and the definition of the reverse \( B_\gamma^{-1}[a, \infty) \)-segment is given for general closed subsets \( N \) in the following.

**Definition 4.1** A unit speed geodesic \( \gamma : [a, b] \to M \) is called a reverse \( N \)-segment for a closed subset \( N \) of a connected Finsler manifold \( M \), if \( \gamma \) is a \((-1)d_N \)-geodesic and \( \gamma(b) \in N \), i.e., \( d_N(\gamma(b - t)) = t \) holds on \([a, b]\), where \( d_N \) is the function defined in Example 2.4.

Remark that a \( d_N \)-geodesic \( \gamma : [a, b] \to M \) is called an \( N \)-segment in [ST], if \( \gamma(a) \in N \), i.e., \( d_N(\gamma(t)) = t - a \) on \([a, b]\), where the function \( d_N \) is defined in Example 2.3 and that our reverse \( N \)-segments are called forward \( N \)-segments in [Sa]. Notice also that if we reverse the parameter of a reverse \( N \)-segment and the Finsler structure \( F \), then we get an \( N \)-segment with respect to the reversed Finsler structure. Here the reversed Finsler structure \( \overline{F} \) of \( F \) means \( \overline{F}(v) := F(-v) \).

Let \( f : M \to \mathbb{R} \) be a 1-Lipschitz function on a connected Finsler manifold \((M, F)\).

**Lemma 4.2** Let \( \alpha : [a, t_0] \to M \) and \( \beta : [t_0, b] \to M \) denote two \( f \)-geodesics satisfying \( \alpha(t_0) = \beta(t_0) \). Then the broken geodesic \( \gamma : [a, b] \to M \) obtained from \( \alpha \) and \( \beta \) is an \( f \)-geodesic, i.e. \( \gamma \) is smooth at \( \gamma(t_0) = \alpha(t_0) = \beta(t_0) \). Hence, any \( f \)-geodesic emanating from an interior point of a maximal \( f \)-geodesic \( \gamma \) is a subarc of \( \gamma \).
Proof. Since \( \alpha \) and \( \beta \) are \( f \)-geodesics, we have
\[
f(\alpha(t_0)) - f(\alpha(a)) = f(\beta(t_0)) - f(\alpha(a)) = t_0 - a, \quad f(\beta(b)) - f(\beta(t_0)) = b - t_0,
\]
and by summing up these two relations we obtain
\[
f(\gamma(b)) - f(\gamma(a)) = f(\beta(b)) - f(\beta(t_0)) = b - a \tag{4.1}
\]
and the conclusion follows from Lemma 2.10.

Combining Theorem A and Lemma 4.2, we get that any \( f \)-geodesic \( \gamma \) is the unique solution of the differential equation \( \nabla f_{\gamma(t)} = \dot{\gamma}(t) \) on the interior points of \( \gamma \).

**Lemma 4.3** Let \( \alpha : [a, b] \to M \) be an \( f \)-geodesic with canonical parameter on the Finsler manifold \((M, F)\). For each \( t_0 \in (a, b) \) the following statements are true

1. \( \alpha|_{[a, t_0]} \) is an \( M_f^a \)-segment to \( \alpha(t_0) \), where \( M_f^a := f^{-1}(-\infty, a] \);
2. \( \alpha|_{[t_0, b]} \) is a reverse \( bM_f \)-segment emanating from \( \alpha(t_0) \), where \( bM_f := f^{-1}[b, \infty) \).

**Proof.**
1. Let us choose arbitrary \( t_0 \in (a, b) \). Let \( x \) be any point of the set \( M_f^a \), that is, \( x \in M \) satisfies \( f(x) \leq f(\alpha(a)) = a \). Taking into account that the function \( f \) is 1-Lipschitz and \( \alpha \) is an \( f \)-geodesic, it follows
\[
d(x, \alpha(t_0)) \geq d(\alpha(t_0)) - f(x) \geq f(\alpha(t)) - f(\alpha(a)) = d(\alpha(a), \alpha(t_0)). \tag{4.2}
\]
Thus, we have proved that \( d(x, \alpha(t_0)) \geq d(\alpha(a), \alpha(t_0)) \) for any \( x \in M_f^a \), and \( d_{M_f}(\alpha(t_0)) = d(\alpha(a), \alpha(t_0)) = t_0 - a \). Hence \( \alpha|_{[a, t_0]} \) is an \( M_f^a \)-segment by Lemma 2.10.

The proof of 2 is completely similar.

**Lemma 4.4** Let \( \alpha : [a, b] \to M \) be an \( f \)-geodesic with canonical parameter on the Finsler manifold \((M, F)\). Then \( \alpha \) is an \( M_f^a \)-segment ending at \( \alpha(b) \) and a reverse \( bM_f \)-segment emanating from \( \alpha(a) \). In particular, for any \( t_0 \in (a, b) \), \( \alpha|_{[a, t_0]} \) is a unique \( M_f^a \)-segment to \( \alpha(t_0) \), and \( \alpha|_{[t_0, b]} \) is a unique reverse \( bM_f \)-segment emanating from \( \alpha(t_0) \).

**Proof.** Let \( \{\varepsilon_i\}_i \) be a sequence of positive numbers convergent to zero. By Lemma 4.3, for each \( i \), \( \alpha|_{[a, b-\varepsilon_i]} \) is an \( M_f^a \)-segment ending at \( \alpha(b - \varepsilon_i) \), and \( \alpha|_{[a+\varepsilon_i, b]} \) is a reverse \( bM_f \)-segment emanating from \( \alpha(a + \varepsilon_i) \), respectively. By taking the limit of the sequence \( \{\varepsilon_i\}_i \), we can conclude that \( \alpha \) is an \( M_f^a \)-segment ending at \( \alpha(b) \), and a reverse \( bM_f \)-segment emanating from \( \alpha(a) \). Let us choose an arbitrary \( t_0 \in (a, b) \). Since \( \alpha(t_0) \) is an interior point of \( \alpha \) it follows that \( \alpha|_{[a, t_0]} \) is the unique \( M_f^a \)-segment and \( \alpha|_{[t_0, b]} \) is the unique reverse \( bM_f \)-segment.

**Lemma 4.5** Let \( \alpha : [a_1, b_1] \to M \) be an \( M_f^{a_1} \)-segment to \( \alpha(b_1) \). Suppose that for each \( t \in (a_1, b_1) \) which is sufficiently close to \( b_1 \), there exists an \( f \)-geodesic \( \gamma_t : [c, d] \to M \) to \( \alpha(t) = \gamma_t(d) \) which intersects \( M_f^{a_1} \). Then \( \alpha \) is an \( f \)-geodesic.
Proof. Choose any \( t \in (a_1, b_1) \) sufficiently close to \( b_1 \) so that there exists an \( f \)-geodesic \( \gamma_t \) to \( \alpha(t) \) which intersects \( M^{a_1}_f \). From Lemma 4.3, it follows that the subarc of \( \gamma_t \) lying in \( a_1 M_f = f^{-1}[a_1, \infty) \) is an \( M^{a_1}_f \)-segment to \( \alpha(t) \). Since \( \alpha \) is also an \( M^{a_1}_f \)-segment and \( \alpha(t) \) is an interior point of the \( M^{a_1}_f \)-segment \( \alpha \), \( \alpha|_{[a_1, b]} \) coincides with the subarc of \( \gamma_t \), and in particular \( \alpha|_{[a_1, t]} \) is an \( f \)-geodesic. Since the parameter value \( t \) can be chosen arbitrarily close to \( b_1 \), \( \alpha \) is an \( f \)-geodesic.

\[ \square \]

The following lemma is dual to Lemma 4.5. The proof is similar.

**Lemma 4.6** Let \( \alpha : [a_1, b_1] \to M \) be a reverse \( b_1 M_f \)-segment emanating from \( \alpha(a_1) \). Suppose that for each \( t \in (a_1, b_1) \) which is sufficiently close to \( a_1 \), there exists an \( f \)-geodesic \( \gamma_t : [c, d] \to M \) emanating from \( \alpha(t) = \gamma_t(c) \) which intersects \( b_1 M_f \). Then \( \alpha \) is an \( f \)-geodesic.

Combining Lemmas 4.4 and 4.5 we obtain,

**Proposition 4.7** Let \( \alpha : [a_1, b_1] \to M \) be a unit speed geodesic with \( f(\alpha(a_1)) = a_1 \). Then, \( \alpha \) is an \( f \)-geodesic if and only if \( \alpha \) is an \( M^{a_1}_f \)-segment which admits a small positive \( \delta_1 \) such that for each point \( t \in (b_1 - \delta_1, b_1) \subset (a_1, b_1) \), there exists an \( f \)-geodesic \( \gamma_t \) to \( \alpha(t) \) which intersects \( M^{a_1}_f \).

Combining Lemmas 4.4 and 4.6 we obtain,

**Proposition 4.8** Let \( \alpha : [a_1, b_1] \to M \) be a unit speed geodesic with \( f(\alpha(b_1)) = b_1 \). Then, \( \alpha \) is an \( f \)-geodesic if and only if \( \alpha \) is a reverse \( b_1 M_f \)-segment which admits a small positive \( \delta_1 \) such that for each point \( t \in (a_1, a_1 + \delta_1) \subset (a_1, b_1) \), there exists an \( f \)-geodesic \( \gamma_t \) to \( \alpha(t) \) which intersects \( b_1 M_f \).

5 The singular locus of almost distance functions

Before defining the *singular locus* of an almost distance function, let us review the definitions of a cut point and the cut locus of a point.

Let \( p \) be a point of a Finsler manifold \( M \). A point \( q \) is called a cut point of \( p \) along a minimal geodesic segment \( \gamma \) joining the point \( p \) to \( q \), if the geodesic \( \gamma \) has no extension beyond \( q \) as a minimal geodesic segment. The cut locus of the point \( p \) is, by definition, the set of cut points along all minimal geodesic segments emanating from \( p \). Notice that the cut locus is similarly defined for a closed subset \( N \) of a Finsler manifold by making use of an \( N \)-segment (see [ST]). In other words, a cut point of the point \( p \) is an end point of a maximal \( d_p \)-geodesic and the cut locus of \( p \) is the set of cut points with respect to all maximal \( d_p \)-geodesics. Thus, it is natural to define the singular locus \( \mathcal{C}(f) \) of an almost distance function \( f \) on a Finsler manifold \( M \) as follows.

**Definition 5.1** The set \( \mathcal{C}(f) := \{ q \in M \mid q \text{ is an end point of a maximal } f \text{-geodesic} \} \) is called the *singular locus* of the almost distance function \( f \).
For example, \( C(d_p) = C_p \cup \{p\} \), where \( C_p \) denotes the cut locus of \( p \), when \( d_p \) is the distance function from a point \( p \) on a forward complete Finsler manifold. For each cut point \( q \) of \( p \), there is no \( d_p \)-geodesic emanating from \( q \), and there is no \( d_p \)-geodesic ending at \( p \). From this observation, we notice that each point of the singular locus is divided into two types:

**Definition 5.2** The subset \( C_+(f) := \{ p \in C(f) \mid \text{there is no } f \text{-geodesic emanating from } p \} \) is called the upper singular locus of the almost distance function \( f \), and the subset \( C_-(f) := \{ p \in C(f) \mid \text{there is no } f \text{-geodesic ending at } p \} \) is called the lower singular locus of the almost distance function \( f \).

**Remark 5.3**

1. For the distance function \( d_p \) from a point \( p \in M \), \( C_+(d_p) \) equals the cut locus of \( p \), and \( C_-(d_p) = \{ p \} \).

2. It is easy to see that \( C_+(f) \cap C_-(f) = \emptyset \) by Lemma 4.2.

3. For the Busemann function \( B_\gamma \) on a bi-complete Finsler manifold, there is no \( B_\gamma \)-geodesic ending at each point of the singular locus \( C(B_\gamma) \) of \( B_\gamma \) which is called a copoint of the ray \( \gamma \). Hence, \( C_-(B_\gamma) = C(B_\gamma) \). Nasu [N1] introduced first the notion of copoint for a Busemann E-space. The copoint is called an asymptotic conjugate point in [N1] [N2].

One natural question to ask is when an almost distance function becomes a distance one. The answer to this question is given in the following proposition.

**Proposition 5.4** Let \( f \) be an almost distance function on a backward complete and connected Finsler manifold \( M \) such that \( C_-(f) \subset N := f^{-1}(c) \neq \emptyset \), where \( c := \inf f \). If \( f^{-1}[c, \sup f) \) is dense in \( M \), then

\[
 f = d_N + c, \tag{5.1}
\]

holds on \( M \), where \( d_N(x) := d(N, x) \).

**Proof.** Choose any \( x \in f^{-1}(c, \sup f) \). Since \( f \) is an almost distance function, there exists a maximal \( f \)-geodesic \( \gamma : I \to M \) through \( x \in \gamma(I) \). Let the parameter of \( \gamma \) be canonical, so that \( f \circ \gamma(t) = t \geq c \) holds on the interval \( I \). Since \( f \circ \gamma(t) = t \geq c \) for any \( t \in I \), \( \inf I \geq c \). Choose any \( s, t \in I \) with \( s < t \). By Lemma 2.10 we obtain \( d(\gamma(s), \gamma(t)) = t - s \). This implies that the sequence \( \{ \gamma(t_i) \} \) is a backward Cauchy sequence for each decreasing sequence \( \{ t_i \} \) of elements of \( I \) convergent to \( a := \inf I \). Therefore, \( \lim_{i \to a} \gamma(t) = \gamma(a) \in C_-(f) \) exists, since \( M \) is backward complete. Since \( N \supset C_-(f) \), we have \( \gamma(a) \in N \), and \( a = f(\gamma(a)) = c \). Since \( \gamma|_{[c, \sup f)} \) is an \( N \)-segment by Lemma 4.4, \( d_N(x) = L(\gamma|_{[c, \sup f)}) = f(x) - c \). Hence \( f = d_N + c \) holds on \( f^{-1}[c, \sup f) \). Since the subset \( f^{-1}[c, \sup f) \) is dense in \( M \), and since both functions \( d_N \) and \( f \) are continuous on \( M \), \( f = d_N + c \) holds on \( M \).

We specialize our discussion in order to obtain characterizations of the singular locus \( C(f) \) in terms of sublevel or superlevel sets of an almost distance function \( f \).
Lemma 5.5 Let $f$ be an almost distance function on a forward complete and connected Finsler manifold $M$. If $p$ is a point in $C_+(f) \cap f^{-1}(\inf f, \sup f)$, then no sequences of points of $C_+(f)$ converge to $p$, and hence there exist a neighborhood $W_p(\subset U_p)$ of $p$ and a number $\delta(p) > 0$ such that for each $q \in W_p$, there exists an $f$-geodesic $\gamma : [0, \delta(p)] \to M$ with $\gamma(\delta(p)) = q$.

Proof. Since $f$ is an almost distance function, the point $p$ admits a neighborhood $U_p$ and a number $\delta(p) > 0$ such that for each $q \in U_p$, there exists an $f$-geodesic $\gamma : [0, \delta(p)] \to M$ with $q = \gamma(0)$ or $q = \gamma(\delta(p))$. Since $p \in C_+(f)$, there exists an $f$-geodesic $\alpha : [0, \delta(p)] \to M$ with $\alpha(\delta(p)) = p$, and there does not exist any $f$-geodesic emanating from $p$.

Supposing that there exists a sequence $\{q_i\}$ of points in $C_-(f)$ converging to $p$, we will get a contradiction. Without loss of generality, we may assume that $q_i \in U_p$ for each $i$. Since $q_i \in C_-(f) \cap U_p$ for each $i$, there exists an $f$-geodesic $\gamma_i : [0, \delta(p)] \to M$ with $\gamma_i(0) = q_i$. Thus, the sequence $\{\gamma_i\}$ has a limit $f$-geodesic $\gamma_\infty : [0, \delta(p)] \to M$ emanating from $\gamma_\infty(0) = p$. This is a contradiction. Hence, there exists a neighborhood $W_p(\subset U_p)$ of $p$ such that for each $q \in W_p$, there exists an $f$-geodesic $\gamma : [0, \delta(p)] \to M$ with $q = \gamma(\delta(p))$.

$\Box$

The following lemma is dual to Lemma 5.5.

Lemma 5.6 Let $f$ be an almost distance function on a backward complete and connected Finsler Manifold $M$. If $p$ is a point in $C_-(f) \cap f^{-1}(\inf f, \sup f)$, then no sequences of points of $C_-(f)$ converge to $p$ and hence there exist a neighborhood $W_p(\subset U_p)$ of $p$ and a number $\delta(p) > 0$ such that for each $q \in W_p$, there exists an $f$-geodesic $\gamma : [0, \delta(p)] \to M$ with $\gamma(0) = q$.

Theorem 5.7 Let $f$ be an almost distance function on a forward complete and connected Finsler manifold $M$. If $p$ is a point in $C_+(f) \cap f^{-1}(\inf f, \sup f)$, then there exist a neighborhood $V_p(\subset W_p)$ of $p$ and a positive number $\delta(p)$ such that for any unit speed geodesic segment $\alpha : [a_0, b_0] \to M$ with $\alpha(b_0) \in V_p$ and with $\alpha(a_0) \in f^{-1}(a_0)$, $\alpha$ is an $f$-geodesic if and only if $\alpha$ is an $M_f^{a_0}$-segment. Here $a_0 := -1/2 \cdot \delta(p) + f(p)$. In particular

$$C(M_f^{a_0}) \cap V_p = C_+(f) \cap V_p.$$  (5.2)

Here $C(M_f^{a_0})$ denotes the cut locus of the closed subset $M_f^{a_0}$ of $M$.

Proof. By Lemma 5.5, there exist a neighborhood $W_p$ of $p$ and a constant $\delta(p) > 0$ such that for each $q \in W_p$, there exists an $f$-geodesic $\gamma_q : [0, \delta(p)] \to M$ to $q = \gamma_q(\delta(p))$. Thus the geodesic $\gamma_q$ intersects $M_f^{a_0}$, if $f(q) < a_0 + \delta(p)$.

Let us put

$$V_p := W_p \cap f^{-1}(a_0, a_0 + \delta(p)).$$

Let $\alpha : [a_0, b_0] \to M$ denote a unit speed geodesic segment with $\alpha(b_0) \in V_p$ and with $\alpha(a_0) \in f^{-1}(a_0)$. Since $V_p$ is open, it is clear that $\alpha(t) \in V_p$ if $t \in (a_0, b_0)$ is sufficiently close to $b_0$. Therefore, from Lemma 5.5 it follows that for each $t \in (a_0, b_0)$ sufficiently close to $b_0$, there exists an $f$-geodesic $\gamma_t : [0, \delta(p)] \to M$ with $\alpha(t) = \gamma_t(\delta(p))$, which intersects $M_f^{a_0}$, and it follows from Proposition 4.7 that $\alpha$ is an $M_f^{a_0}$-segment if and only if $\alpha$ is an $f$-geodesic. Hence, the equation (5.2) is clear.

$\Box$
Since the Finsler distance function \( d(\cdot, \cdot) \) is not always symmetric, we need the notions of the reverse cut points and the reverse cut locus of a closed subset \( N \). For a closed subset \( N \) of a connected Finsler manifold \( M \), a reverse cut point of \( N \) is, by definition, the end point of a maximal reverse \( N \)-segment emanating from this point, and the reverse cut locus of \( N \) is the set of all reverse cut points of \( N \) along all reverse \( N \)-segments. Thus, the reverse cut locus of \( N \) equals the lower singular locus of the almost distance function \((-1)d^N\).

**Theorem 5.8** Let \( f \) be an almost distance function on a backward complete and connected Finsler manifold \( M \). If \( p \) is a point in \( \mathcal{C}_+(f) \cap f^{-1}(\inf f, \sup f) \), then there exist a neighborhood \( V_p(\subset W_p) \) of \( p \) and a positive number \( \delta(p) \) such that for any unit speed geodesic segment \( \alpha : [a_0, b_0] \to M \) with \( \alpha(a_0) \in V_p \) and with \( \alpha(b_0) \in f^{-1}(b_0) \), \( \alpha \) is an \( f \)-geodesic if and only if \( \alpha \) is a reverse \( b_0 \)\( M_f \)-segment. Here \( b_0 := 1/2 \cdot \delta(p) + f(p) \). In particular,

\[
\mathcal{C}_{rev}(b_0 M_f) \cap V_p \subseteq \mathcal{C}_-(f) \cap V_p.
\]

(5.3)

Here \( \mathcal{C}_{rev}(b_0 M_f) \) denotes the reverse cut locus of the closed subset \( b_0 M_f \) of \( M \).

**Lemma 5.9** If \( p \in f^{-1}(\inf f, \sup f) \) is a point of a bi-complete and connected Finsler manifold \( M \), then there exists a neighborhood \( V_p \) of \( p \) such that for any point \( q \in V_p \), there exists an \( f \)-geodesic \( \gamma : [0, \delta(p)] \to M \) with \( q = \gamma(0) \) or \( \gamma(\delta(p)) \), which intersects \( b_0 \)\( M_f \) or \( M_f^{a_0} \) respectively. In particular, for each point \( q \in V_p \),

\[
d(M_f^{a_0}, q) = f(q) - a_0
\]

(5.4)

or

\[
d(q, b_0 \ M_f) = b_0 - f(q).
\]

(5.5)

Here \( a_0 := -1/2 \cdot \delta(p) + f(p) \), \( b_0 := 1/2 \cdot \delta(p) + f(p) \).

**Proof.** Since \( f \) is an almost distance function, there exist a neighborhood \( U_p \) of \( p \) and a constant \( \delta(p) > 0 \) such that for each \( q \in U_p \), there exists an \( f \)-geodesic \( \gamma_q : [0, \delta(p)] \to M \) with \( q = \gamma_q(0) \) or \( \gamma_q(\delta(p)) \). Hence if \( q \in V_p := U_p \cap f^{-1}(a_0, b_0) \), then the geodesic \( \gamma_q \) intersects \( M_f^{a_0} \) or \( b_0 \ M_f \).

Choose any point \( q \in V_p \) and fix it. Hence there exists an \( f \)-geodesic \( \gamma : [0, \delta(p)] \to M \) with \( q = \gamma(0) \) or \( \gamma(\delta(p)) \). If the geodesic segment \( \gamma \) satisfies \( \gamma(\delta(p)) = q \) (respectively \( \gamma(0) = q \)), then by Lemma 4.4 the subarc of \( \gamma \) lying \( f^{-1}[a_0, b_0] \) is an \( M_f^{a_0} \)-segment (respectively a reverse \( b_0 \)\( M_f \)-segment). Hence, the equation (5.4) or (5.5) holds for each \( q \in V_p \).

\[\square\]

**Remark 5.10** For some \( p \in f^{-1}(\inf f, \sup f) \setminus \mathcal{C}(f) \) the neighborhood \( V_p \) guaranteed in Lemma 5.9 can admit both points of the singular sets \( \mathcal{C}_+(f) \) and \( \mathcal{C}_-(f) \), even if we choose a smaller one. In Section 7, we will construct an almost distance function \( f \) on Euclidean plane which admits a point in \( \overline{\mathcal{C}_+(f) \cap \mathcal{C}_-(f)} \setminus \mathcal{C}(f) \). Here \( \overline{A} \) denotes the closure of the set \( A \).
Proposition 5.11 Let \( f \) be an almost distance function on a forward complete and connected Finsler manifold \( M \), then, the set

\[
C^{(2)}_+(f) := \{ p \in f^{-1}(\inf f, \sup f) \mid \text{there exist at least two } f\text{-geodesics to } p \}
\]

is a subset of \( C_+(f) \). Moreover, for each point \( p \in C_+(f) \cap f^{-1}(\inf f, \sup f) \) admitting a unique maximal \( f\text{-geodesic} \), there exists a sequence of points in \( C^{(2)}_+(f) \) converging to \( p \). The subset \( C^{(2)}_-(f) \) of \( C_-(f) \) corresponding to \( C^{(2)}_+(f) \) also has the same properties for an almost distance function \( f \) on a backward complete and connected Finsler manifold.

Proof. Choose any \( p \in C^{(2)}_+(f) \). Since \( p \in C^{(2)}_+(f) \), there exist at least two \( f\)-geodesics to \( p \). Choose one of them, say \( \alpha : [f(p) - \delta_1, f(p)] \to M \). Suppose that \( p \notin C_+(f) \). Thus, there exists an \( f\)-geodesic \( \gamma : [f(p), f(p) + \delta_2] \to M \) emanating from \( p = \gamma(f(p)) \). By Lemma 4.2 we get \( \dot{\alpha}(f(p)) = \dot{\gamma}(f(p)) \). This implies that the point \( p \) admits a unique maximal \( f\)-geodesic to \( p \), a contradiction. Hence \( p \in C_+(f) \), and \( C^{(2)}_+(f) \subseteq C_+(f) \).

Choose any \( p \in C_+(f) \cap f^{-1}(\inf f, \sup f) \) admitting a unique maximal \( f\)-geodesic. Suppose that there exists an open ball \( B_\delta(p) \subseteq f^{-1}(\inf f, \sup f) \) of radius \( \delta \) centered at \( p \) such that \( B_\delta(p) \cap C^{(2)}_+(f) = \emptyset \). From Lemma 5.5 \( p \) has a neighborhood whose each point admits a unique maximal \( f\)-geodesic. Thus, by Theorem A, \( f \) is \( C^2 \) around \( p \). By making use of the same argument in the proof of Proposition 2.5 in [ST], we get a contradiction.

Remark 5.12 Bishop [Bh] proved Proposition 5.11 for the distance function from a point on a complete connected Riemannian manifold. Moreover, it was proved by Sabau [Sa] for Busemann functions on a forward complete connected Finsler manifold.

6 The singular locus of an almost distance function on a 2-dimensional Finsler manifold

Throughout this section, \( M \) always denotes a connected 2-dimensional Finsler manifold, unless otherwise stated. We recall that a homeomorphism from the closed interval \([0, 1]\) into \( M \) is called a Jordan arc. A topological space \( T \) is called a local tree if for any point \( x \) in \( T \) and any neighborhood \( U \) of \( x \), there exists a neighborhood \( V \subset U \) of \( x \) such that any distinct two points in \( V \) can be joined by a Jordan arc in \( V \) which is unique in \( V \).

A continuous curve \( c : [a, b] \to M \) is called rectifiable if its length

\[
l(c) := \sup \left\{ \sum_{i=1}^{k} d(c(t_{i-1}), c(t_i)) \mid a = t_0 < t_1 < \cdots < t_{k-1} < t_k := b \right\}.
\]

is finite. Remark that it is known that \( l(c) \) equals \( \int_a^b F(\dot{c}(t))dt \) for a Lipschitz curve \( c : [a, b] \to M \) (for example, see [ST] Theorem 7.5).

The intrinsic metric \( \delta \) on \( C(f) \cap f^{-1}(\inf f, \sup f) \) is defined as:

\[
\delta(q_1, q_2) := \begin{cases} 
\inf\{l(c) \mid c \text{ is a rectifiable arc in } C(f) \cap f^{-1}(\inf f, \sup f) \text{ joining } q_1 \text{ and } q_2\}, \\
+\infty, 
\end{cases}
\]

if \( q_1, q_2 \in C(f) \) are in the same connected component,

otherwise.
**Proof of Theorem B**

From Theorem 5.7 (respectively Theorem 5.8) it follows that for each point \( p \in C_+(f) \cap f^{-1}(\inf f, \sup f) \), (respectively \( p \in C_-(f) \cap f^{-1}(\inf f, \sup f) \)) there exist a neighborhood \( V_p \subset f^{-1}(\inf f, \sup f) \) of \( p \) and a sublevel set \( M_f^{a_0} \) (respectively a superlevel set \( b_0M_f \) ) of \( f \) such that \( C_+(f) \cap V_p = C(M_f^{a_0}) \cap V_p \) (respectively \( C_-(f) \cap V_p = C_{\text{rev}}(b_0M_f) \cap V_p \)). Since \( M \) is separable, \( C(f) \cap f^{-1}(\inf f, \sup f) \) is covered by the union of countably many open sets \( V_{p_i}, p_i \in C(f) \cap f^{-1}(\inf f, \sup f) \). By applying Theorem B in [ST] to the sublevel sets \( M_f^{a_0} \) and the superlevel sets \( b_0M_f \) determined from each point \( p_i \) above, we obtain Theorem B. Note that from Theorem 6.4 in [ST] it follows that every Jordan arc in \( C(f) \cap f^{-1}(\inf f, \sup f) \) is rectifiable.

We need the inverse function theorem for a Lipschitz map which was proved by F.C. Clarke [CF1] to prove Theorem C and its corollary. The Lipschitz version of the inverse function theorem is a tool of nonsmooth analysis developed by him [CF1, CLSW].

Let \( \phi : U \to \mathbb{R}^n \) denote a locally Lipschitz map from an open subset \( U \) of \( \mathbb{R}^n \) into \( \mathbb{R}^n \). The *generalized differential* \( \partial \phi_p \) at a point \( p \in U \) is defined by

\[
\partial \phi_p := \text{co}\{ \lim_{i \to \infty} d\phi_{p_i} \} \quad \text{where} \quad \text{co}(A) \quad \text{denotes the convex hull of the set} \ A \quad \text{and} \quad d\phi_{p_i} \quad \text{exists for each} \ p_i,
\]

where \( \text{co}(A) \) denotes the convex hull of the set \( A \), when \( A \) is a subset of a linear space. Note that from Rademacher’s theorem the differential \( d\phi \) of the local Lipschitz map \( \phi \) exists almost everywhere.

**Definition 6.1** A point \( p \in U \) is called nonsingular if each element of \( \partial \phi_p \) is of maximal rank, otherwise, it is called singular.

The following theorem was proved by F. H. Clarke [CF1].

**Theorem 6.2** Let \( \phi : U \to \mathbb{R}^n \) be a Lipschitz map from an open subset \( U \) of \( \mathbb{R}^n \) into \( \mathbb{R}^n \). If a point \( p \in U \) is nonsingular for \( \phi \), then there exist neighborhoods \( U_{p}, V_{\phi(p)} \) of \( p \) and \( \phi(p) \) respectively such that \( \phi|_{U_p} \) is a bi-Lipschitz homeomorphism from \( U_p \) onto \( V_{\phi(p)} \). As a corollary to this theorem, we get the implicit function theorem for a Lipschitz function.

**Theorem 6.3** Let \( f \) be a Lipschitz function defined on a open subset \( U \) of \( \mathbb{R}^n \). If a point \( p \in U \) is nonsingular for the function \( f \), then there exists an open neighborhood \( U_p \subset U \) of \( p \) such that \( U_p \cap f^{-1}(f(p)) \) is a topological hypersurface which is bi-Lipschitz homeomorphic to an open subset of \( \mathbb{R}^{n-1} \).

The generalized differential is naturally defined for a locally Lipschitz map between smooth manifolds (see [KT2]). Some tools of nonsmooth analysis are introduced in differential geometry and used in the proof of differentiable sphere theorems (see [KT2]).

Now, let us return to our situation. Let \( f \) denote an almost distance function on a connected Finsler manifold \( M \). By definition, a point \( p \in M \) is singular for \( f \) if and only if \( \partial f_p \) has zero. A singular point of the almost distance function \( f \) is called critical in the sense of Clarke.
Definition 6.4 Let \( C_r(f) \) denote the set of all critical points of \( f \) in the sense of Clarke and \( C_V(f) := f(C_r(f)) \), each element of which is called a critical value of \( f \).

Let us determine explicitly the generalized differential \( \partial f_p \) at a point \( p \). From Theorem A, it follows that \( df_q(\cdot) = g_{\gamma(f(q))}(\gamma(f(q)), \cdot) \) for a differentiable point \( q \), where \( \gamma \) denotes the unique \( f \)-geodesic through \( q \) with canonical parameter. Thus, we get

\[
\partial f_p = \text{co}\{\omega_p(\gamma) | \gamma \text{ is an } f \text{-geodesic through } p\},
\]

where \( \omega_p(\gamma) := g_{\gamma(f(p))}(\gamma(f(p)), \cdot) \).

A linear combination \( \sum_{i=1}^k \lambda_i \omega_p(\gamma_i) \) of \( \omega_p(\gamma_i), 1 \leq i \leq k \), where each \( \gamma_i \) denotes an \( f \)-geodesic through \( p \), is an element of \( \partial f_p \), if \( \sum_{i=1}^k \lambda_i = 1 \) and \( \lambda_i \geq 0 \). Moreover, the set of all such linear combinations of \( \omega_p(\gamma) \), where \( \gamma \) denotes an \( f \)-geodesic, is convex. Therefore, we obtain,

Lemma 6.5 The generalized differential \( \partial f_p \) of \( f \) at \( p \) is given by

\[
\partial f_p = \left\{ \sum_{i=1}^k \lambda_i \omega_p(\gamma_i) \mid \sum_{i=1}^k \lambda_i = 1, \lambda_i \geq 0, \text{each } \gamma_i \text{ is an } f \text{-geodesic through } p \right\}.
\]

Lemma 6.6 Let \( c : (a, b) \to M \) be a continuous curve on a connected and bi-complete Finsler manifold \( M \), which is differentiable at some \( t_0 \in (a, b) \), and \( f \) an almost distance function on the manifold \( M \). If \( f \circ c \) is also differentiable at \( t_0 \), and if \( p := c(t_0) \in C(f) \cap f^{-1}(\inf f, \sup f) \), then \( (f \circ c)'(t_0) = \omega_p(\gamma)(c(t_0)) \) holds for any \( f \)-geodesic \( \gamma \) through \( p \). In particular, \( (f \circ c)'(t_0) = 0 \) if \( p \) is a critical point of \( f \) in the sense of Clarke.

Proof. Without loss of generality, we may assume that \( t_0 = 0 \). Since \( p \in C(f) \), it is clear that \( p \in C_+(f) \) or \( p \in C_-(f) \). We will prove our lemma by assuming \( p \in C_+(f) \). The other case can be similarly proved.

Choose any \( f \)-geodesic \( \gamma \) through \( p \) with canonical parameter. By Lemma 5.5 \( \gamma \) is defined on \([f(p) - \delta(p), f(p)]\) and \( \gamma(f(p)) = p \) holds. Since \( f \) is 1-Lipschitz and \( \gamma \) is an \( f \)-geodesic,

\[
f(c(s)) - f(p) \leq d(\gamma(f(p) - \delta(p)), c(s)) - d(\gamma(f(p) - \delta(p)), p)
\]

holds for any \( s \in (a, b) \). Hence for any \( s < t < 0 \) in \((a, b)\) the equations

\[
\frac{f(c(s)) - f(p)}{s} \geq \frac{d(\gamma(f(p) - \delta(p)), c(s)) - d(\gamma(f(p) - \delta(p)), p)}{s}
\]

and

\[
\frac{f(c(t)) - f(p)}{t} \leq \frac{d(\gamma(f(p) - \delta(p)), c(t)) - d(\gamma(f(p) - \delta(p)), p)}{t}
\]

hold. By taking the limits of the above equations with respect to \( s, t \) respectively and by [1.5], we get

\[
\omega_p(\gamma)(\dot{c}(0)) \geq (f \circ c)'(0) \geq \omega_p(\gamma)(\dot{c}(0)).
\]

Therefore, \((f \circ c)'(0) = \omega_p(\gamma)(\dot{c}(0))\) holds for any \( f \)-geodesic \( \gamma \) through \( p \).
Suppose that the point $p$ is a critical point of $f$. It follows from Lemma 6.7 that for any $\omega_p \in \partial f_p$, we have $(f \circ c)'(t_0) = \omega_p(\dot{c}(t_0))$. From our assumption, $\partial f_p$ contains the zero 1-form. Thus $(f \circ c)'(t_0) = 0$. 

We need the following lemma to prove Lemma 6.8, which is the Sard Theorem for a continuous function. The proof is given in [ShT, Lemma 3.2].

**Lemma 6.7** Let $h : (a, b) \to R$ be a continuous function. Then the set $h(D_0(h))$ is of (Lebesgue) measure zero, where $D_0(h) := \{ t \in (a, b) \mid h'(t) \text{ exists and equals } 0 \}$. 

**Lemma 6.8** Let $c : (a, b) \to C(f) \cap f^{-1}(\inf f, \sup f)$ be a unit speed Lipschitz curve. Then the set $(f \circ c)(C_r(c))$ is of measure zero, where 

$$C_r(c) := \{ t \in (a, b) \mid c(t) \in C_r(f) \}.$$ 

**Proof.** Let $ND(c)$ denote the set of all $t \in (a, b)$ at which $c$ is not differentiable and $ND(f \circ c)$ the set of all $t \in (a, b)$ at which $f \circ c$ is not differentiable.

By Rademacher’s theorem, it follows that both sets $ND(c)$ and $ND(f \circ c)$ are of measure zero. Choose any $t \in (a, b) \setminus (ND(c) \cup ND(f \circ c))$. Hence $c(t)$ and $(f \circ c)(t)$ exist. If $c(t)$ is a critical point of $f$ in the sense of Clarke, then by Lemma 6.6 $(f \circ c)'(t) = 0$. Thus, the set $C_r(c)$ is a subset of $ND(c) \cup ND(f \circ c) \cup D_0(f \circ c)$, where $D_0(f \circ c) = \{ t \in (a, b) \mid (f \circ c)'(t) \text{ exists and equals } 0 \}$. Since $f \circ c$ is a Lipschitz function and $ND(c) \cup ND(f \circ c)$ is of measure zero, its image by $f \circ c$ is also of measure zero. Therefore, by Lemma 6.7 $(f \circ c)(C_r(c))$ is of measure zero. 

**Proof of Theorem C**

From Theorem B, it follows that there exist a countably many unit speed Lipschitz curves $m_i : [a_i, b_i] \to C(f) \cap f^{-1}(\inf f, \sup f)$ such that $C(f) \cap f^{-1}(\inf f, \sup f) \setminus E = \bigcup_{i=1}^{\infty} m_i[a_i, b_i]$, where $E$ denotes the set of all end points of $C(f)$. Let $E^{(2)} \subset E$ denote the set of end points admitting more than one $f$-geodesic.

It follows from the proof of [ShT] Theorem A(3), Theorems 5.7 and 5.8 that the set $E^{(2)}$ is a countable set. It is clear that $C_V(f)$ is a subset of the union of $\{ \inf f, \sup f \}$, 

$$\bigcup_{i=1}^{\infty} \left( (f \circ m_i)(\tilde{C}_r(m_i)) \right) \quad \text{and} \quad f(E^{(2)}),$$

where $\tilde{C}_r(m_i) := C_r(m_i) \cup \{ a_i, b_i \}$. Hence, by Lemma 6.8 the set $C_V(f)$ is of measure zero. 

Now, **Corollary to Theorem C** is clear from Theorem 6.3.

**Remark 6.9** By Rademacher’s theorem it was proved that $ND(m_i)$ and $ND(f \circ m_i)$ are of measure zero. Furthermore one can conclude that for each $m_i : [a_i, b_i] \to C(f) \cap f^{-1}(\inf f, \sup f)$, $ND(m_i)$ and $ND(f \circ m_i)$ are countable. Indeed, it follows from the proofs of [ShT] Theorem A(3) and [ST] Lemma 9.1 that there exist at most countably many points on the curve $m_i$ admitting more than two $f$-geodesics, and from [ST] Propositions 2.1 and 2.2 it follows that $(f \circ m_i)'(t)$ and $\dot{m}_i(t)$ exist if $m_i(t)$ admits exactly two $f$-geodesics. This property is very close to [T] Corollary 10, which says that the distance
function to the cut locus of a closed submanifold $N$ of a complete 2-dimensional Riemannian manifold is differentiable except for a countably many points in the unit normal bundle of $N$.

**Remark 6.10** Theorem C is still true for an arbitrary dimensional Riemannian manifold, if the function $f$ is smooth ($C^\infty$) on $f^{-1}((\inf f, \sup f) \setminus C(f)$, and $C(f)$ is closed. In fact, by combining Theorems 5.7, 5.8 and [R, Theorem 1], we get:

**Theorem 6.11** Let $f$ be an almost distance function on a complete arbitrary dimensional Riemannian manifold $M$. If the function $f$ is smooth on $f^{-1}((\inf f, \sup f) \setminus C(f)$ and $C(f)$ is closed, then the set $C(f)$ of the critical values of $f$ is of measure zero.

## 7 Examples of almost distance functions

In this section, we construct almost distance functions $f$ on Euclidean plane which admits a point in $(f^{-1}(\inf f, \sup f) \setminus C(f)) \cap C_+(f) \setminus C_-(f)$. Here $\overline{A}$ denotes the closure of the set $A$. Let $\mathbb{R}^2$ denote Euclidean plane with canonical coordinates $(x, y)$ with the origin $o = (0, 0)$. Let $D_1$ denote the unit closed ball centered at the origin, so that $D_1 = \{(x, y) | x^2 + y^2 \leq 1\}$. Choose any strictly decreasing sequence $\{\theta_i\}_{i=1}^\infty$ with $\theta_1 < \pi$ convergent to zero. For each $i \geq 1$, put $p_i := (2 \cos \omega_i, 2 \sin \omega_i)$, where $\omega_i := (\theta_i + \theta_{i+1})/2$. It is clear that for each $i \geq 1$, both points $(\cos \theta_i, \sin \theta_i)$, and $(\cos \theta_{i+1}, \sin \theta_{i+1})$ lie on the common circle centered at $p_i$ with radius $r_i := d(p_i, (\cos \theta_i, \sin \theta_i))$. We define a closed subset $N$ of $\mathbb{R}^2$ by

$$N := D_1 \setminus \bigcup_{i=1}^\infty B_{r_i}(p_i),$$

where $B_{r_i}(p_i)$ denotes the open ball centered at $p_i$ with radius $r_i$. The function $d_N$ is an almost distance function and $p_i \in C_+(d_N)$ for each $i$. Thus, the point $(2,0)$ is in the closure of $C(d_N)$, but it is an interior point of the maximal $d_N$-geodesic, ${\{(t,0) | t \geq 1\}}$.

It is clear to see that the rays $R_\theta(o) := \{(r \cos \theta, r \sin \theta) | r \geq 1\}$, $\theta \in (\theta_1, 2\pi) \cup \{\theta_i | i \geq 1\}$ emanating from $N$ are maximal $N$-segments, and hence maximal $d_N$-segments. Note that $d_N(t,0) = |t| - 1$ for any $t$ with $|t| \geq 1$.

Next we will construct the function $\eta$ defined by a sequence of closed subsets $\{C_n\}_{n=3}$ (see Example 5.8 and [Wu]). For each $\theta_i$, and $n \geq 3$, put $q_i^{(n)} := (n \cos \theta_i, -n \sin \theta_i)$. Recall that $\{\theta_i\}^n$ denote the strictly decreasing sequence convergent to zero. Hence, for each $i$ and $n \geq 3$, the point $u_i := (2 \cos \omega_i, -2 \sin \omega_i)$, where $\omega_i = (\theta_i + \theta_{i+1})/2$ is equidistant from $q_i^{(n)}$ and $q_{i+1}^{(n)}$. For each $n \geq 3$, we define a closed subset $C_n$ by

$$C_n := \mathbb{R}^2 \setminus \bigcup_{i=1}^\infty \left( B_{r_i^{(n)}}(u_i) \cup B_n(o) \right),$$

where $r_i^{(n)} := d(u_i, q_i^{(n)})$. We define an almost distance function $\eta$ on $\mathbb{R}^2$ by

$$\eta(p) := \lim_{n \to \infty} (n - d(p, C_n)).$$
It is clear to see that the rays $R_\theta(o) := \{(r \cos \theta, r \sin \theta) | r \geq 0\}, \theta \in [0, 2\pi - \theta_1] \cup \{2\pi - \theta_i \mid i \geq 2\}$ emanating from $o$ are $\eta$-geodesics. For each $u_i$, the rays $R_\theta(u_i) := \{(r \cos \theta, r \sin \theta) + u_i | r \geq 0\}, \theta \in [2\pi - \theta_1, 2\pi - \theta_{i+1}]$ emanating from $u_i$ are $\eta$-geodesics. For each point $p$ on the line segment $ou_i$ joining $o$ to $u_i$, the two rays $R_\theta(p) := \{(r \cos \theta, r \sin \theta) + p | r \geq 0\}, \theta = 2\pi - \theta_i, 2\pi - \theta_{i+1}$ are $\eta$-geodesics. Hence $u_i \in C_-(\eta)$ for each $u_i$ and each line segment $ou_i$ is a subset of $C_-(\eta)$. In particular, the point $(2, 0)$ is in the closure of $C_-(\eta)$, since $\lim_{u_i \to \infty} u_i = (2, 0)$.

Now we will construct another almost distance function $f^N_\eta$ by combining $\eta$ and $d_N$. Since $\eta(x, 0) = |x|$ for all $x \in \mathbb{R}$, $d_N(x, 0) = 0$ for all $x$ with $|x| \leq 1$, and $\eta(x, 0) = d_N(x, 0) + 1 = |x|$ for all $x$ with $|x| \geq 1$, $\eta_1(x, 0) = d_N(x, 0) + 1$ holds for all real number $x$, where $\eta_1$ denotes a 1-Lipschitz function on $\mathbb{R}^2$ defined by $\eta_1(x, y) := \max\{\eta(x, y), 1\}$. Thus we may define a 1-Lipschitz function $f^N_\eta$ on $\mathbb{R}^2$ by $f^N_\eta(x, y) = d_N(x, y) + 1$ for $y \geq 0$, and $f^N_\eta(x, y) = \eta_1(x, y)$ for $y \leq 0$. Note that $\inf f^N_\eta = 1$ and $\sup f^N_\eta = \infty$. It is easy to check that the function $f^N_\eta$ is an almost distance function on $\mathbb{R}^2$ and that the point $(2, 0)$ is in the closure of $C_-(f^N_\eta)$ and the closure of $C_+(f^N_\eta)$.

By imitating the way above, it is possible to construct an almost distance function $f$ which admits infinitely many points in $(\overline{C_+(f) \cap C_-(f)}) \setminus C(f)$. Indeed, for each pair of positive numbers $a < b < \pi/2$, we proved that there exists an almost distance function $f_{ab}$ on $\mathbb{R}^2$ such that $(2 \cos \omega_1, 2 \sin \omega_1)$ in $(\overline{C_+(f_{ab}) \cap C_-(f_{ab})}) \setminus C(f_{ab})$ and each ray $R_\theta(o) := \{(r \cos \theta, r \sin \theta) | r \geq 1\}$ emanating from the unit circle $x^2 + y^2 = 1$ is an $f_{ab}$-geodesic if $\theta \in [0, 2\pi] \setminus (a, b)$. Thus, if a strictly decreasing sequence $\{\epsilon_n\}_{n=1}^\infty$ with $\epsilon_1 < \pi/2$ convergent to 0 is given, we get a sequence of almost distance functions $f_n := f_{\epsilon_n, \epsilon_{n+1}}$ on $\mathbb{R}^2$ such that the point $(2 \cos \omega_n, 2 \sin \omega_n)$, where $\omega_n := (\epsilon_n + \epsilon_{n+1})/2$, is in $(\overline{C_+(f_n) \cap C_-(f_n)}) \setminus C(f_n)$ and each ray $R_\theta(o) := \{(r \cos \theta, r \sin \theta) | r \geq 1\}$ emanating from the unit circle $x^2 + y^2 = 1$ is an $f_n$-geodesic if $\theta \in [0, 2\pi] \setminus (\epsilon_{n+1}, \epsilon_n)$. Therefore, it is easy to construct an almost distance function $f$ on $\mathbb{R}^2$ such that for each $n$ the point $(2 \cos \omega_n, 2 \sin \omega_n)$ is an element of $(\overline{C_+(f) \cap C_-(f)}) \setminus C(f)$ and each ray $R_\theta(o) := \{(r \cos \theta, r \sin \theta) | r \geq 1\}$ emanating from the unit circle $x^2 + y^2 = 1$ is an $f$-geodesic if $\theta \in [0, 2\pi] \setminus (0, \epsilon_1)$.

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