POSITIVE SOLUTIONS OF A FRACTIONAL BOUNDARY VALUE PROBLEM WITH A FRACTIONAL DERIVATIVE BOUNDARY CONDITION

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Abstract. In this paper, we apply Krasnosel’skii’s cone expansion and compression fixed point theorem to show the existence of at least one positive solution to the nonlinear fractional boundary value problem

\[ \frac{D^\alpha}{0^+} u + a(t)f(u) = 0, \quad 0 < t < 1, \quad 1 < \alpha \leq 2, \]

satisfying boundary conditions \( u(0) = D^\beta_{0^+} u(1) = 0, \quad 0 \leq \beta \leq 1. \)

1. Introduction. For \( \alpha \in (1, 2], \) we consider the fractional boundary value problem

\[ \frac{D^\alpha}{0^+} u + a(t)f(u) = 0, \quad 0 < t < 1, \tag{1} \]

satisfying the boundary conditions

\[ u(0) = D^\beta_{0^+} u(1) = 0, \tag{2} \]

where \( \beta \in [0, 1], \) \( D^\alpha_{0^+}, D^\beta_{0^+} \) are Riemann-Liouville derivatives of order \( \alpha \) and \( \beta \) respectively, \( f : [0, \infty) \to [0, \infty) \) is a continuous function, and \( a : [0, 1] \to [0, \infty) \) is an integrable function on \( [0, 1]. \) In this paper, we will put certain growth restrictions on \( f \) and then, by applying Krasnosel’skii’s cone expansion and compression fixed point theorem, show the existence of at least one positive solution to the boundary value problem (1), (2).

Recently, there has been much research done on the existence of positive solutions of fractional boundary value problems. For some examples, see \cite{1, 2, 3, 5, 7, 13, 14, 15} and the references therein. This work looks to generalize the work of Kaufmann and Mboumi \cite{8} by letting the condition on the solution at the right endpoint be \( D^\beta_{0^+} u(1) = 0, \) where \( \beta \in [0, 1]. \) If \( \beta = 0, \) these boundary conditions become the Dirichlet boundary conditions \( u(0) = u(1) = 0. \) When \( \beta = 1, \) these boundary conditions are the right focal boundary conditions \( u(0) = u'(1) = 0, \) which is what Kaufmann and Mboumi considered in \cite{8}. For the theory of fractional derivatives, we refer the reader to the books by Diethelm \cite{4}, Kilbas, Srivastava, and Trujillo \cite{9}, and Samko, Kilbas, and Marichev \cite{12}.

2. Preliminary Definitions and Theorems. We start with the definition of the Riemann-Liouville fractional integral and fractional derivative.

Definition 2.1. Let \( \nu > 0. \) The Riemann-Liouville fractional integral of a function \( u \) of order \( \nu, \) denoted \( I^\nu_{0^+} u, \) is defined as

\[ I^\nu_{0^+} u(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} u(s)ds, \]
provided the right-hand side exists. Moreover, let $n$ denote a positive integer and assume $n - 1 < \alpha \leq n$. The Riemann-Liouville fractional derivative of order $\alpha$ of the function $u : [0, 1] \to \mathbb{R}$, denoted $D^\alpha_0 u$, is defined as

$$D^\alpha_0 u(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t (t - s)^{n-\alpha-1} u(s)ds = D^n I^n_{0+} u(t),$$

provided the right-hand side exists.

In the analysis, we will make use of a cone in a Banach space.

**Definition 2.2.** Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathcal{K}$ of $\mathcal{B}$ is said to be a cone provided

(i) $\alpha u + \beta v \in \mathcal{K}$, for all $u, v \in \mathcal{K}$ and all $\alpha, \beta \geq 0$, and

(ii) $u \in \mathcal{K}$ and $-u \in \mathcal{K}$ implies $u = 0$.

The following is the well-known cone expansion and compression fixed point theorem.

**Theorem 2.3 (Krasnosel’skii [10]).** Let $\mathcal{B}$ be a Banach space and let $\mathcal{K} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that $\Omega_1, \Omega_2$ are open with $0 \in \Omega_1$, $\Omega_1 \subset \Omega_2$. Let $T : \mathcal{K} \cap (\Omega_2 \setminus \Omega_1) \to \mathcal{K}$ be a completely continuous operator such that either

(i) $\|Tu\| \leq \|u\|$, $u \in \mathcal{K} \cap \partial \Omega_1$, and $\|Tu\| \geq \|u\|$, $u \in \mathcal{K} \cap \partial \Omega_2$, or

(ii) $\|Tu\| \geq \|u\|$, $u \in \mathcal{K} \cap \partial \Omega_1$, and $\|Tu\| \leq \|u\|$, $u \in \mathcal{K} \cap \partial \Omega_2$.

Then $T$ has a fixed point in $\mathcal{K} \cap (\Omega_2 \setminus \Omega_1)$.

3. The Green’s Function. The Green’s function for $-D^\alpha_0 u = 0$ satisfying the boundary conditions (2) is given by (see [6])

$$G(t, s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s < t < 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}, & 0 \leq t \leq s < 1. \end{cases}$$

Therefore, $u$ is a solution of (1), (2) if and only if

$$u(t) = \int_0^1 G(t, s)a(s)f(u(s))ds, \quad 0 \leq t \leq 1.$$

The following Lemma gives properties of the Green’s function that are integral to our work. These properties can be utilized when applying many different fixed point theorems, such as the Leggett-Williams fixed point theorem [11], to (1),(2), making this result the major contribution of the paper.

**Lemma 3.1.** Let $\gamma \in (0, 1)$ be fixed. $G(t, s)$ satisfies the following properties:

$$G(t, s) \geq 0, \quad (t, s) \in [0, 1] \times [0, 1);$$

$$\max_{0 \leq t \leq 1} \int_0^1 G(t, s)ds = \frac{(\alpha-1)^{\alpha-1}}{\alpha(\alpha-\beta)\Gamma(\alpha)};$$

and

$$\min_{\gamma \leq s \leq 1} G(t, s) \geq [1 - (1 - \gamma)^{\beta}] \gamma^{\alpha-1} G(s, s) \text{ for all } \gamma \leq s < 1.$$

**Proof.** Notice that (4) holds trivially.

Next, we show (5) holds. Define $g_1(t, s) = t^{\alpha-1}(1-s)^{\alpha-1-\beta} - (t-s)^{\alpha-1}$, $0 \leq s \leq t$, and $g_2(t, s) = t^{\alpha-1}(1-s)^{\alpha-1-\beta}$, $t \leq s < 1$. Then

$$\int_0^t g_1(t, s)ds = \frac{\alpha t^{\alpha-1} - (\alpha-\beta)t^{\alpha}}{\alpha(\alpha-\beta)} - \frac{t^{\alpha-1}(1-t)^{\alpha-\beta}}{\alpha-\beta},$$

and

$$\int_0^t g_2(t, s)ds = \frac{t^{\alpha-1}(1-t)^{\alpha-\beta}}{\alpha-\beta} - \frac{\alpha t^{\alpha-1} - (\alpha-\beta)t^{\alpha}}{\alpha(\alpha-\beta)}.$$
and
\[ \int_t^1 g_2(t,s)ds = \frac{t^{\alpha-1}(1-t)^{\alpha-\beta}}{\alpha-\beta}. \]

So
\[ \int_0^1 G(t,s)ds = \frac{\alpha t^{\alpha-1} - (\alpha - \beta) t^{\alpha}}{\alpha(\alpha - \beta) \Gamma(\alpha)} \]

which has a maximum value at \( t = \frac{\alpha - 1}{\alpha - \beta} \). Therefore, (5) holds.

Last, we show (6) holds. First, note that since \( \alpha - 2 \leq 0 \),
\[ \left(1 - \frac{s}{t}\right)^{\alpha-2} \geq (1-s)^{\alpha-2} \geq (1-s)^{\alpha-1 - \beta}. \]

Therefore
\[ \frac{\partial g_1(t,s)}{\partial t} = (\alpha - 1) t^{\alpha-2} \left[(1-s)^{\alpha-1 - \beta} - \left(1 - \frac{s}{t}\right)^{\alpha-2}\right] \leq 0. \]

Hence \( g_1(t,s) \) is a decreasing function of \( t \). So for \( \gamma \leq s < 1 \),
\[ \frac{1}{\Gamma(\alpha)} g_1(t,s) \geq \frac{1}{\Gamma(\alpha)} g_1(1,s) \]
\[ = \frac{1}{\Gamma(\alpha)} [\gamma^{\alpha-1 - \beta} - (1-s)^{\alpha-1} - (1-s)^{\alpha-1}] \]
\[ = \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-1 - \beta} [1 - (1-s)^{\beta}] \]
\[ \geq \frac{1}{\Gamma(\alpha)} (1-s)^{\alpha-1 - \beta} [1 - (1-\gamma)^{\beta}] \]
\[ \geq [1 - (1-\gamma)^{\beta}] \gamma^{\alpha-1} \frac{1}{\Gamma(\alpha)} s^{\alpha-1} (1-s)^{\alpha-\beta-1} \]
\[ = [1 - (1-\gamma)^{\beta}] \gamma^{\alpha-1} G(s,s). \]

Next, notice that
\[ \frac{\partial g_2(t,s)}{\partial t} = (\alpha - 1) t^{\alpha-2} (1-s)^{\alpha-1 - \beta} \geq 0. \]

So \( g_2(t,s) \) is an increasing function of \( t \) and thus, for \( \gamma \leq t \leq s < 1 \),
\[ \frac{1}{\Gamma(\alpha)} g_2(t,s) \geq \frac{1}{\Gamma(\alpha)} g_2(\gamma,s) \]
\[ = \frac{1}{\Gamma(\alpha)} \gamma^{\alpha-1} (1-s)^{\alpha-1 - \beta} \]
\[ \geq [1 - (1-\gamma)^{\beta}] \gamma^{\alpha-1} \frac{1}{\Gamma(\alpha)} s^{\alpha-1} (1-s)^{\alpha-\beta-1} \]
\[ = [1 - (1-\gamma)^{\beta}] \gamma^{\alpha-1} G(s,s). \]

Therefore, (6) holds.

4. Positive Solutions. We define the Banach space \( \mathcal{B} = C[0,1] \) to be the space of all continuous functions on \([0,1]\) endowed with the supremum norm, \( \| u \| = \| u \|_0 = \max_{0 \leq t \leq 1} |u(t)| \).

Define the operator \( T : \mathcal{B} \to \mathcal{B} \) by
\[ Tu(t) = \int_0^1 G(t,s) a(s) f(u(s)) ds. \]

Therefore, if \( u \) is a fixed point of the operator \( T \), then \( u \) solves (1), (2).

We make three assumptions on the functions \( f \) and \( a \).

(H1) \( f : [0,\infty) \to [0,\infty) \) is continuous;
(H2) $a : [0, 1] \to [0, \infty)$ with $a \in L^\infty[0, 1]$; and
(H3) there exists a $\gamma \in (0, 1)$ and an $m > 0$ such that $a(t) \geq m$ a.e. on $[\gamma, 1]$.

The following result uses the compression result of Theorem 2.3 to give the existence of at least one positive solution of (1), (2).

**Theorem 4.1.** Suppose that (H1)-(H2) are satisfied and suppose $\gamma \in (0, 1)$ is such that (H3) is satisfied. Let $M = |a|_\infty$ and let $A, B \in \mathbb{R}$ with

$$0 < A \leq \frac{\alpha(\alpha - \beta)^{\alpha} \Gamma(\alpha)}{(\alpha - 1)^{\alpha - 1} M} \quad \text{and} \quad B \geq \left( [1 - (1 - \gamma)\beta]^{\alpha - 1} m \int_\gamma^1 G(s, s) ds \right)^{-1}.$$

If there exist positive constants $r$ and $R$, where $0 < r < R$ and $Br < AR$, such that $f$ satisfies

(i) $f(x) \leq AR$ for all $x \in [0, R]$, and
(ii) $f(x) \geq Br$ for all $x \in [0, r]$,

then (1), (2) has at least one positive solution $u$ with $r \leq \|u\| \leq R$.

**Proof.** Define the cone

$$\mathcal{K} = \{ u \in \mathcal{B} : u(t) \geq 0 \text{ for all } t \in [0, 1] \}.$$

By (4), $T : \mathcal{K} \to \mathcal{K}$. A standard application of the Arzelà-Ascoli Theorem gives that $T$ is completely continuous.

Define the open set $\Omega_2 = \{ u \in \mathcal{B} : \|u\| < R \}$. Let $u \in \mathcal{K} \cap \partial \Omega_2$. Then assumption (i) and (5) give

$$\|Tu\| = \max_{0 \leq t \leq 1} \int_0^1 G(t, s) a(s) f(u(s)) ds$$

$$\leq MAR \max_{0 \leq t \leq 1} \int_0^1 G(t, s) ds$$

$$\leq MAR \frac{(\alpha - 1)^{\alpha - 1}}{\alpha(\alpha - \beta)^{\alpha} \Gamma(\alpha)}$$

$$\leq R$$

$$= \|u\|.$$

So $\|Tu\| \leq \|u\|$ for all $u \in \partial \mathcal{K} \cap \Omega_2$.

Next, define the open set $\Omega_1 = \{ u \in \mathcal{B} : \|u\| < r \}$. Let $u \in \mathcal{K} \cap \partial \Omega_1$. Then, using (H1)-(H3), assumption (ii), and (6),

$$Tu(t) = \int_0^1 G(t, s) a(s) f(u(s)) ds$$

$$\geq \int_\gamma^1 G(t, s) a(s) f(u(s)) ds$$

$$\geq mBr \int_\gamma^1 G(t, s) ds$$

$$\geq mBr [1 - (1 - \gamma)\beta]^{\alpha - 1} \int_\gamma^1 G(s, s) ds$$

$$\geq r$$

$$= \|u\|.$$

Therefore, $\|Tu\| \geq \|u\|$ for all $u \in \mathcal{K} \cap \partial \Omega_1$. Therefore, since $0 \in \Omega_1 \subset \Omega_2$, part (ii) of Theorem 2.3 gives the existence of at least one fixed point of $T$ in $\mathcal{K} \cap (\overline{\Omega_2 \setminus \Omega_1})$. So there exists at least one solution $u$ of (1), (2) with $r \leq \|u\| \leq R$. \qed
The final result uses the expansive result of Theorem 2.3 to give at least one solution to (1), (2).

**Theorem 4.2.** Suppose that (H1)-(H2) are satisfied and suppose $\gamma \in (0,1)$ is such that (H3) is satisfied. Let $M = |a|\infty$ and let $A, B \in \mathbb{R}$ with

$$0 < A \leq \frac{\alpha(\alpha - \beta)\Gamma(\alpha)}{(\alpha - 1)\alpha - 1 M}$$

and

$$B \geq \left( [1 - (1 - \gamma) \beta] \gamma^\alpha - 1 \int_\gamma^1 G(s, s) \right) - 1.$$

If there exist positive constants $r$ and $R$, where $r < R$ and $Ar > BR$, such that $f$ satisfies

(i) $f(x) \geq BR$ for all $x \in [0,R]$, and

(ii) $f(x) \leq Ar$ for all $x \in [0,r]$,

then (1), (2) has at least one positive solution $u$ with $r < \|u\| \leq R$.

**Proof.** Again, we define the cone

$$\mathcal{K} = \{ u \in \mathcal{B} : u(t) \geq 0 \text{ for all } t \in [0,1] \}$$

and note that $T : \mathcal{K} \to \mathcal{K}$ and $T$ is completely continuous.

Define the open set $\Omega_1 = \{ u \in \mathcal{B} : \|u\| < r \}$. Let $u \in \mathcal{K} \cap \partial \Omega_1$. Then assumption (i) and (5) give

$$\|Tu\| = \max_{0 \leq t \leq 1} \int_0^t G(t, s) a(s) f(u(s)) ds$$

$$\leq MAr \max_{0 \leq t \leq 1} \int_0^1 G(t, s) ds$$

$$\leq MAr \frac{(\alpha - 1)\alpha - 1}{\alpha(\alpha - \beta)\Gamma(\alpha)}$$

$$\leq r$$

$$= \|u\|.$$

So $\|Tu\| \leq \|u\|$ for all $u \in \mathcal{K} \cap \partial \Omega_1$.

Next, define the open set $\Omega_2 = \{ u \in \mathcal{B} : \|u\| < R \}$. Let $u \in \mathcal{K} \cap \partial \Omega_2$. Then, by (H1)-(H3), assumption (ii), and (6),

$$Tu(t) = \int_0^1 G(t, s) a(s) f(u(s)) ds$$

$$\geq \int_\gamma^1 G(t, s) a(s) f(u(s)) ds$$

$$\geq mBR \int_\gamma^1 G(t, s) ds$$

$$\geq mBR[1 - (1 - \gamma) \beta] \gamma^\alpha - 1 \int_\gamma^1 G(s, s) ds$$

$$\geq R$$

$$= \|u\|.$$

Therefore, $\|Tu\| \geq \|u\|$ for all $u \in \mathcal{K} \cap \partial \Omega_2$. Therefore, since $0 \in \Omega_1 \subset \Omega_2$, part (i) of Theorem 2.3 gives the existence of at least one fixed point of $T$ in $\mathcal{K} \cap (\Omega_2 \setminus \Omega_1)$. So there exists at least one solution $u$ of (1), (2) with $r < \|u\| \leq R$. \qed
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