A damage mechanics based method for fatigue life prediction of the metal graded materials

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Abstract. Based on the continuum damage mechanics theory, the fatigue life prediction for TC4-TC11 graded material was conducted. At first, the damage evolution equation was derived, then the method to calibrate material parameters for TC4-TC11 graded material was proposed, and all the material parameters were obtained. A beam model with TC4-TC11 graded material was established by using the stratified method and finite element method. Finally, the fatigue life of TC4-TC11 graded beam was predicted.

Keywords: damage mechanics, fatigue life prediction, metal graded material, finite element

1. Introduction
In the year of 1984, Japan's MasyuhiNINO, ToshioHIRA and other materials scientists proposed the concept of "functionally graded materials" (FGM) [1]. Scientists from China and many other countries have also started to study the preparation technology, material properties and practical value of FGM. FGM has become a new star in the material world, which is with a strong development trend. The metal graded material is a kind of FGM, which can meet the requirement that different location has different mechanical property. So the metal graded material has a wide application in various fields, especially in the aerospace field [3-8]. However, because of its special inhomogeneity, the method to predict the life of uniform materials is not adapted to this new material. Therefore, a new method to predict the fatigue life of metal graded material is presented, which is based on the theory of continuum damage mechanics [9-11]. Firstly, according to the principle of thermodynamics, the damage evolution equation of metal graded materials is derived. Secondly, the damage evolution parameters for TC4-TC11 metal graded materials are calibrated according to the experimental data. Finally, a model of beam is established and the stress analysis and life prediction are carried out. The simulation results show that this method is feasible.

2. Theoretical model
In this paper, the TC4-TC11 metal graded materials are studied. The corresponding damage evolution equation is derived, based on the continuum damage mechanics theory. For isotropic materials, the damage variable D is introduced to describe the deterioration of materials, which can be given as the reduction of stiffness [12-13].
where $E$ is Young’s Modulus of material and $E_D$ is the effective Young’s Modulus with damage. As $E_D$ ranges from $E$ to 0, $D$ varies between 0 and 1.

Using Kronecker delta

$$
\{x\} = \begin{cases} 
1 & x > 0 \\
0 & x \leq 0
\end{cases}
$$

the damage evolution equation is expressed by [14]

$$
\frac{dD}{dN} = \alpha \left( \frac{Y_{max}^2 - Y_{th}^2}{(1-D)^m} \right)
$$

where $\alpha$, $m$, $Y_{th}$ are parameters related to the properties of the material and $Y$ is strain energy release rate. $Y_{max}$ is the maximum value and $Y_{th}$ is the threshold value, which means the damage occurs when $Y_{max} > Y_{th}$. According to thermodynamics of damaged materials, $Y$ is defined as

$$
Y = -\rho \frac{\partial \Psi}{\partial D}
$$

where $\rho$ is density and $\Psi$ is Helmholtz free energy function. For the case of isotropic damage, Equation (4) can be derived when stress is constant.

$$
Y = \frac{1}{2} C_{ijkl} e_i^e e_j^e = \frac{W^E}{1-D}
$$

where $W^E$ is strain energy density.

In the condition of uniaxial loading,

$$
W^E = \frac{1}{2} \sigma \epsilon^e, \quad \sigma = E(1-D)\epsilon^e
$$

then,

$$
Y = \frac{\sigma^2}{2E(1-D)^2}
$$

In the case of multiaxial loading,

$$
W^E = \frac{1}{2} \sigma_{ij} e_i^e e_j^e, \quad \sigma_{ij} = C_{ijkl} e_i^e e_j^e (1-D)
$$

then the strain energy release rate is

$$
Y = \frac{W^E}{1-D} = \frac{(\sigma_{ij}R_i)^{1/2}}{2E(1-D)^{1/2}}
$$

where

$$
R_i = \frac{2}{3} (1+v) + 3(1-2v) \left( \frac{\sigma_{ij}}{\sigma_m} \right)^2
$$

In order to obtain the uniform equation, the concept of damage equivalent stress is introduced. The equivalent stress can be denoted as:

$$
\sigma_e = \sigma_m (R_\sigma)^{1/2} = \left[ \frac{3}{2} s_{ij} s_{ij} \right]^{1/2} (R_\sigma)^{1/2}
$$

So the uniform equation of the strain energy release rate can be written as:

$$
Y = \frac{1}{2} \left( \frac{1}{1-D} \right)^{1/2} \frac{1}{E} \sigma_e^2
$$

From Equation (11), $Y_{max}$ can be expressed by the following formula:

$$
Y_{max} = \frac{1}{2} \left( \frac{1}{1-D} \right)^{1/2} \frac{1}{E} \sigma_{e,max}^2
$$
Here the relationship between the threshold value $Y_{th}$ and the D is assumed to satisfy Equation (11). Then,

$$Y_{th} = \frac{1}{2} \left( \frac{1}{1-D} \right)^{\frac{1}{2}} \frac{1}{E} \sigma_{e,th}^2 \tag{13}$$

where $\sigma_{e,th}$ is the threshold value of the equivalent stress.

Substituting Equation (12) and Equation (13) into Equation (2), the damage evolution equation can be obtained as:

$$\frac{dD}{dN} = a \left( \frac{1}{2E} \right)^{\frac{2}{m}} \frac{\left( \sigma_{e,max} - \sigma_{e,th} \right)^n}{(1-D)^{2m}} \tag{14}$$

Then Equation (14) could be rewritten as:

$$\frac{dD}{dN} = \beta \frac{\left( \sigma_{e,max} - \sigma_{e,th} \right)^n}{(1-D)^{2m}} \tag{15}$$

where

$$\beta = a \left( \frac{1}{2E} \right)^{\frac{2}{m}} \tag{16}$$

Separating the Equation (15), Equation (17) and Equation (18) can be obtained as follows

$$\int_{0}^{N_f} \left( \sigma_{e,max} - \sigma_{e,th} \right)^n dN = \left( \sigma_{e,max} - \sigma_{e,th} \right)^n N_f \tag{17}$$

$$\frac{1}{\beta} \left( \frac{1}{D_0} \right)^{2m} dD = \frac{1}{\beta(2m+1)} \left( 1-D_0 \right)^{2m+1} \tag{18}$$

where $D_0$ is the initial damage.

Equation (17) is equal to Equation (18) with the same value $C_1$. It is clear that different fatigue curves can be obtained when $D_0$ has different values. By choosing $D_0 = D_{0,m}$, the mean fatigue S-N curve can be obtained as:

$$\left( \sigma_{e,max} - \sigma_{e,th} \right)^n N_{f,m} = C_{1,m} \tag{19}$$

where $N_{f,m}$ is the mean fatigue life.

$$C_{1,m} = \frac{1}{\beta(2m+1)} \left( 1-D_{0,m} \right)^{2m+1} \tag{20}$$

Finally, the fatigue failure criterion of metal graded materials under load spectrum is studied. Here the simplified load spectrum (as shown in figure 1) is taken as the research object. The simplified load spectrum is composed of several load blocks with the constant amplitude, which have different maximum stress and corresponding cycles.

![Figure 1](image-url)
Processing Equation (15), it can be obtained as follows:

\[
\int_0^{N_f} \left( \sigma_{e,max} - \sigma_{e,th} \right)^m dN = \frac{1}{\beta} \int_0^{D_h} (1-D)^{2m} dD
\]  

(21)

For the load spectrum, it can be calculated as:

\[
\int_0^{N_f} \left( \sigma_{e,max} - \sigma_{e,th} \right)^m dN = \sum_{k=1}^k \left( \sigma_{e,max}^{(k)} - \sigma_{e,th} \right)^m N_k
\]

(22)

where,

\[N_f = \sum_{k=1}^k N_k\]  

(23)

And,

\[
\frac{1}{\beta} \int_0^{D_h} (1-D)^{2m} dD = \frac{1}{\beta(2m+1)}(1-D_h)^{2m+1} = C
\]

(24)

The following fatigue failure criterion can be derived by combining Equation (22) and Equation (24):

\[
\sum_{k=1}^k \left( \sigma_{e,max}^{(k)} - \sigma_{e,th} \right)^m N_k = C
\]

(25)

For each load block, the corresponding fatigue life can be calculated as:

\[
N_{f,k} = \frac{C}{\left( \sigma_{e,max}^{(k)} - \sigma_{e,th} \right)^m}
\]

(26)

where \(N_{f,k}\) is the corresponding fatigue life of metal graded material under the stress \(\sigma_{e,max}^{(k)}\).

In summary, the following form of fatigue failure criteria can be obtained [9,12]:

\[
\sum_{k=1}^k \frac{N_k}{N_{f,k}} = 1
\]

(27)

3. Parameter calibration for TC4-TC11 medal graded material

3.1. Calibration method

According to the characteristics of metal graded materials, it can be divided into two parts: the substrate region and the transition region. In this paper, it is assumed that the substrate zone and transition zone have the same form of damage evolution equation, but the specific parameters of the different regions are different. For parameter calibration in the substrate region, the conventional damage evolution model for homogeneous material is used. Then for parameter calibration in the transition region, the interpolation method is used for the rest part, after the calibration of the part with specific proportion has been done. In this paper, the parameters of the damage evolution equation have been identified according to the fatigue test results. Taking TC4-TC11 medal graded material as an example, the basic steps are as follows:

1. Fitting the S-N curve of two substrate regions materials and fitting the specific region which is made up of two base materials each 50%.
2. Deriving the damage evolution equation after determined the force situation as a uniaxial, biaxial or multiaxial force.
3. Parts of the parameters can be calibrate by comparing the S-N curve equation and the damage evolution equation and the rest material parameters can be obtained by a method of selecting the farthest point.

3.2. Results of calibration
Firstly, the S-N curve is fitted by the experimental data. The results of three kinds of materials are
shown in figure 2:

![S-N curves of fatigue test for standard parts of three kinds of materials](image)

The fitting equations are:

- TC4: \( S = 396.28 + 58173.465 \times N^{-0.56} \)
- TC11: \( S = 636.31 + 30053.545 \times N^{-0.64} \)
- TC4+TC11: \( S = 501.106 + 43958.758 \times N^{-0.57} \)

This paper takes the simplest situation of \( K_T = 1 \) as an example.

According to Equation (17) and Equation (18), the stress-life relationship of standard specimens under \( K_T = 1 \) can be written as follows:

\[
N_f = \frac{1}{\beta_k (2m_k + 1)} \left(1 - D_{0,k}\right)^{2m_k+1}
\]

where \( \sigma_{k,th}, m_k, \beta_k, D_{0,k} \) are material parameters.

For simplifying Equation (28), another parameter \( C_1 \) is defined as:

\[
C_1 = \frac{1}{\beta_k (2m_k + 1)} \left(1 - D_{0,k}\right)^{2m_k+1}
\]

Then the Equation (28) can be written by:

\[
N_f = \frac{C_1}{\left(\sigma_{k,\text{max}} - \sigma_{k,\text{th}}\right)^{m_k}}
\]

Logarithm of the upper equation:

\[
\log N_f = \log C_1 - m_k \log (\sigma_{k,\text{max}} - \sigma_{k,\text{th}})
\]

The equation form of the S-N curve is:

\[
\log N = C - m \log (\sigma_{\text{max}} - \sigma_0)
\]

Comparing the Equation (30) with Equation (31), the following equations can be obtained:

\[
\log C_1 = C \\
m_k = m \\
\sigma_{k,\text{th}} = \sigma_0
\]

Then two material parameters of the damage evolution equation \( \sigma_{k,\text{th}} \) and \( m_k \) can be identified.

Meanwhile, an equation about \( \beta_k \) and \( D_{0,k} \) can be obtained.

In order to identify the value of \( \beta_k \) and \( D_{0,k} \), the point which has the longest fatigue life has been
picked out, assuming the initial damage \( D_{0,k} = 0 \).

Then,

Figure 2. S-N curves of fatigue test for standard parts of three kinds of materials (\( K_T = 1, R = -1 \)).
\[
(\sigma_{\text{max}} - \sigma_{\text{th}})^n N_f = \frac{1}{\beta_k (2m_k +1)}
\]  

(33)

\(\beta_k\) can be obtained. Substituting \(\beta_k\) into Equation (29), \(D_{0,k}\) can be obtained.

At this point, all the material parameters of the damage evolution equation can be obtained.

The material parameters of other regions in transition zone can be obtained by interpolation with the identified data.

The results of the calibrated parameters in transition zone are shown in table 1.

| Percentage of TC4 | \(\sigma_{\text{th}}\)  | \(m_k\)  | \(D_{0,k}\) | \(\beta_k\) |
|-------------------|-----------------|-------|-----------|-------|
| 100               | 396.2765        | 1.7857 | 0.0163    | 0.6228|
| 90                | 414.8123        | 1.7923 | 0.0466    | 0.6056|
| 80                | 434.5631        | 1.7924 | 0.0739    | 0.5993|
| 70                | 455.5289        | 1.7862 | 0.0983    | 0.6039|
| 60                | 477.7099        | 1.7735 | 0.1196    | 0.6193|
| 50                | 501.1059        | 1.7544 | 0.1380    | 0.6457|
| 40                | 525.7171        | 1.7289 | 0.1535    | 0.6829|
| 30                | 551.5432        | 1.6969 | 0.1659    | 0.7310|
| 20                | 578.5845        | 1.6585 | 0.1754    | 0.7901|
| 10                | 606.8409        | 1.6137 | 0.1819    | 0.8600|
| 0                 | 636.3123        | 1.5625 | 0.1854    | 0.9408|

### 4. Fatigue life prediction of TC4-TC11 metal graded material

#### 4.1. Damage mechanics-finite element method

When predicting the fatigue life of components by using damage mechanics method [15-17], the stress field and damage field must be calculated. The process of calculation is summarized as follows:

1. Calculating the increment of the damage \(\Delta D_i\) for element after \(\Delta N\) cycles

\[
\Delta D_i = \alpha_i \left( \frac{Y_{\text{max}}^i - Y_{\text{th}}^i}{(1-D_{i-1})^n} \right) \Delta N
\]  

(34)

2. Updating the damage and element stiffness as shown in Equations (35) and (36).

\[
D_i = D_{i-1} + \Delta D_i
\]  

(35)

\[
E = E_0 \times (1 - D_i)
\]  

(36)

3. Repeating Steps1-2 until the damage extent at any element reaches 1 and the fatigue life is obtained.

#### 4.2. Stratified method and finite element analysis

##### 4.2.1. Stratified method

The properties of the metal graded materials vary uniformly in the transition region, thus in order to obtain a more precise stress field, the stratified method is used in the finite element analysis. The stratified method is referred to that the model is divided into many layers and assuming that the same layer has the same parameters, but the parameters are changed in different layers which abide by some certain rules. According to this assumption, the traditional finite element method is applied to the stress analysis in each layer. Based on the finite element analysis, it can be known that the more
numbers of layers, the higher the accuracy. However, at the same time, too many layers will result in a waste of computation time. So the proper number of layers is important. And in order to ensure continuity and alleviate the mutation between different layers, the material properties are assumed to change by a certain function.

4.2.2. Finite element analysis
A cantilever beam is designed with a size of 100mm*20mm*6mm, which is under the applied force of 2650N in the free end, as shown in figure 3.

From the Equation (28), it can be found that the material fatigue life is not only related to the stress, but also to the Young's modulus E. If E is small, the increment of each cycle is larger. Thus, the crack initiates may not occur at the position of maximum stress.

Based on the above consideration, a design that the upper and lower ends of the beam adopting TC11 with a higher Young's modulus and the middle part adopting TC4 with a lower Young's modulus is put forward as shown in figure 4.

To calculate the stress distribution, the stratified method is used [18-21]. The transition region is divided into ten layers with different layers having different material properties. And the assumption that the properties vary following the power exponential function is taken as the calculated results show it has a higher accuracy than following a linear function.

Solid185 element is chosen to carry out the mesh generation and stress analysis, and the calculated results are shown in figure 5 to figure 7.

As a contrast, the finite element model of TC4 homogeneous beam is established. The structure size and loading condition are the same as that of the graded beam. The results are shown in figure 8.
4.3. Life prediction results

The life prediction for homogeneous beam and graded beam is conducted, respectively. The results show that the maximum stress of homogeneous beam appears at the top and bottom, whose value is $\sigma = 640.667 \text{MPa}$. And the predicted life is $N_f = 28800$

The results of the maximum stress and fatigue life of graded beam are shown in table 2. The table only lists the results of the lower part of the beam considering its symmetry.

**Table 2.** Maximum stress and fatigue life of each regions of graded beam.

| Region                        | The lowest region | The bottom of the lower transition region | The upper of the lower transition region | The bottom of the middle layer region |
|-------------------------------|-------------------|------------------------------------------|----------------------------------------|-------------------------------------|
| Threshold value (MPa)         | 636.3             | 609.5                                    | 413.1                                  | 396.3                               |
| Maximum stress (MPa)          | 643.9             | 376.8                                    | 145.6                                  | 131.4                               |
| Fatigue life                  | 668000            | $N_f > 10^7$ (because the maximum is under the threshold value) |                                        |                                     |

From table 2, the fatigue life of the graded beam is $N_f = 668000$.

Comparing the calculated results of the two beams, it shows that although the maximum stress of metal graded beam is larger than homogeneous beam, the fatigue life of metal graded beam is longer than the homogeneous one. The calculated results are consistent with the practical situation, which indicates that the method is feasible.

5. Conclusions

Based on the continuum damage mechanics theory, the fatigue life prediction of metal graded materials is carried out, and the main conclusions are summarized as follows:

According to the principle of thermodynamics, the damage evolution equation for the fatigue life prediction of metal graded materials is derived.

The calibration method of the material parameters in the damage evolution equation is proposed. Taking the TC4-TC11 metal graded material as an example, the specific process of material parameter calibration is given.

The fatigue life prediction of TC4-TC11 graded beam is carried out by using the damage mechanics-finite element method and the results show that the method is feasible.

In this paper, the influence of environment, temperature, etc. hasn’t been considered when calculating the stress and damage. Besides, in the practical application, most of the metal graded materials are in the complex environment. Therefore, in the following research, analysis considering
the environmental temperature and mechanical environment synthetically will be conducted, which can make the results more close to the actual situation.

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