Vanishing viscosity limit for viscous magnetohydrodynamic equations with a slip boundary condition

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Abstract

We consider the evolutionary MHD systems, and study the the regularity and vanishing viscosity limit of the 3-D viscous system in a class of bounded domains with a slip boundary condition. We derive the convergence is in $H^{2k+1}$, for $k \geq 1$, if the initial date holds some sufficient conditions.

Key words: Magnetohydrodynamic system; slip boundary condition; vanishing viscosity limit.

1 Introduction and results

Let $\Omega$ be an open bounded domain in $R^3$. We consider the initial and boundary value problem for the system of viscous MHD equations

$$
\begin{align*}
\partial_t u - \nu \Delta u + (u \cdot \nabla) u - (H \cdot \nabla) H + \nabla p &= 0 \text{ in } \Omega, \\
\nabla \cdot u &= 0 \text{ in } \Omega, \\
\partial_t H - \mu \Delta H + (u \cdot \nabla) H - (H \cdot \nabla) u &= 0 \text{ in } \Omega, \\
\nabla \cdot H &= 0 \text{ in } \Omega, \\
\mathbf{u} &= \mathbf{u}_0, H = H_0, \text{ at } t = 0,
\end{align*}
$$

(1)

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with the following slip without friction boundary conditions

\[
\begin{align*}
\mathbf{u} \cdot \mathbf{n} &= 0, \nabla \times \mathbf{u} \cdot \mathbf{\tau} = 0, \mathbf{H} \cdot \mathbf{n} &= 0, \nabla \times \mathbf{H} \cdot \mathbf{\tau} = 0 \quad \text{on } \partial \Omega
\end{align*}
\]

(2)

where \(\nabla\cdot\) and \(\nabla\times\) denote the \(\text{div}\) and \(\text{curl}\) operators, \(\mathbf{n}\) the outward normal vector and \(\mathbf{\tau}\) any unit tangential vector of \(\partial \Omega\).

The corresponding ideal MHD system is usually equipped with the slip boundary condition, namely

\[
\begin{align*}
\partial_t \mathbf{u}^0 + (\mathbf{u}^0 \cdot \nabla)\mathbf{u}^0 - (\mathbf{H}^0 \cdot \nabla)\mathbf{H}^0 + \nabla p^0 &= 0 \quad \text{in } \Omega, \\
\nabla \cdot \mathbf{u}^0 &= 0 \quad \text{in } \Omega, \\
\partial_t \mathbf{H}^0 + (\mathbf{u}^0 \cdot \nabla)\mathbf{H}^0 - (\mathbf{H}^0 \cdot \nabla)\mathbf{u}^0 &= 0 \quad \text{in } \Omega, \\
\nabla \cdot \mathbf{H}^0 &= 0 \quad \text{in } \Omega, \\
\mathbf{u}^0 &= \mathbf{u}_0, \mathbf{H}^0 = \mathbf{H}_0, \quad \text{at } t = 0,
\end{align*}
\]

(3)

\[
\mathbf{u}^0 \cdot \mathbf{n} = 0, \mathbf{H}^0 \cdot \mathbf{n} = 0 \quad \text{on } \Omega.
\]

(4)

Our aim is to investigate strong convergence, up to the boundary, of the solution \((\mathbf{u}, \mathbf{H})\) of the MHD (1) to the solution \((\mathbf{u}^0, \mathbf{H}^0)\) of the ideal MHD system (3), as \((\nu, \mu) \to 0\).

The boundary conditions (2) are a special Navier-type slip boundary conditions, which allow the fluid to slip at a slip velocity proportional to the shear stress introduced by Navier [1]. This type of boundary conditions has been used in many fluid problems (see e.g. [2], [3], [4], [5]).

The viscous MHD system in the whole space or with non-slip boundary conditions has been studied extensively (see e.g. [6], [7], [8], [9], [10]). The solvability, regularity of the 3-D viscous MHD system with a slip boundary condition, we refer to [11].

The issue of vanishing viscosity limits of the Navier-Stokes equations is classical and fundamental importance in fluid dynamics and turbulence theory (see e.g. [12], [13], [14], [15], [16], [17], [18]).

In flat boundary case, the 3-D inviscid limit for solution \((\mathbf{u}, \mathbf{H})\) to the slip boundary problem (1) and (2) has been considered in [11]. In [11], they state the following result. Assume \(\nabla \cdot \mathbf{u}_0 = 0, \nabla \cdot \mathbf{H}_0 = 0\), and \((\mathbf{u}_0, \mathbf{H}_0) \in \mathbf{H}^3\) satisfy the boundary conditions (2). Then, as \((\nu, \mu) \to 0\),

\[
(\mathbf{u}, \mathbf{H}) \to (\mathbf{u}^0, \mathbf{H}^0) \quad \text{in } L^p(0, T; \mathbf{H}^3(\Omega)) \cap C([0, T]; \mathbf{H}^2(\Omega)),
\]

(5)

for some \(T > 0\) and any \(p \in [1, +\infty)\), where \((\mathbf{u}^0, \mathbf{H}^0)\) is the solution to the ideal MHD equations (3) and (4).
It should be noted that the approach encounters great difficulties for general domains as pointed out by [16]. Thus, following [16], we restrict the problem to a cubic domain $Q = [0, 1]^2_{\text{per}} \times (0, 1)$ with the boundary conditions on two opposite faces $z = 0$ and $z = 1$, and others be assumed periodic, which was called flat boundary case.

Our approach here is motivated by the idea introduced in [19] to study the same problems for the Navier-Stokes equations. We prove the following result.

**Theorem 1.1** Let the initial data $u_0 \in V^{2k-1} \cap H^{2k+1}, H_0 \in V^{2k-1} \cap H^{2k+1}$, $k \geq 1$. Then there exist strong solution of the MHD equation (1) and (2) in the "cubic domain" (flat boundary case) on some time interval $[0, T]$, s.t.

$$\|u\|_{L^\infty(0,T;H^{2k+1})} + \|H\|_{L^\infty(0,T;H^{2k+1})} \leq C,$$

$$\|\partial_t u\|_{L^2(0,T;H^{2k})} + \|\partial_t H\|_{L^2(0,T;H^{2k})} \leq C. \tag{6}$$

And

$$(u, H) \to (u^0, H^0) \text{ in } C([0, T]; H^{2k}), \text{ as } (\nu, \mu) \to 0, \tag{7}$$

where $(u^0, H^0)$ is the unique solution of the ideal MHD equations (3) and (4).

Further, denoting $\omega^0 = \nabla u^0$ and $\zeta^0 = \nabla H^0$, if $\|\partial_n^{2k} \omega^0\|_{L^\infty(0,T;C^2(\partial \Omega))} \leq C$, $\|\partial_n^{2k} \zeta^0\|_{L^\infty(0,T;C^2(\partial \Omega))} \leq C$, $\|\partial_t \partial_n^{2k} \omega^0\|_{L^2(0,T;C^1(\partial \Omega))} \leq C$, $\|\partial_t \partial_n^{2k} \zeta^0\|_{L^2(0,T;C^1(\partial \Omega))} \leq C$, then

$$(u, H) \to (u^0, H^0) \text{ in } C([0, T]; H^{2k+1}), \text{ as } (\nu, \mu) \to 0. \tag{8}$$

The paper is organized as follows. Some tools are drawn in section 2. A priori estimates to the MHD systems are given in section 3. The results of vanishing viscosity limit and the convergence rate are presented in section 4.

## 2 Notations and preliminaries

Throughout the rest of this paper, denote by $v_\tau = v \cdot \tau$ and $v_n = v \cdot n$ on the boundary $\partial \Omega$. For the flat boundary case, $v \cdot n = 0$ and $\nabla \times v = 0$ are equivalent to $v_n = 0$ and $\partial_n v_\tau = 0$ on $\partial Q$. And $\partial Q = \{(x, y, z); z = 0, z = 1\} \cap \overline{Q}$. For convenience, $\Omega$ and $Q$ may be omitted when we write these spaces without confusion.

We begin our analysis with a formula of integration by parts.
Let $\Omega$ be a regular open, bounded set in $\mathbb{R}^3$. Then, for sufficiently regular vector fields $v$,

$$
- \int_{\Omega} \Delta v \cdot v \, dx = \| \nabla v \|^2_{L^2} - \int_{\partial \Omega} \partial_n v \cdot v \, d\sigma. \tag{9}
$$

It is easily shown that if $v$ is, sufficiently regular, vector fields in a flat boundary domain then

$$
\partial_n v \cdot v = \partial_n v_\tau \cdot v_\tau + \partial_n v_n \cdot v_n. \tag{10}
$$

It follows that $\partial_n v \cdot v$ vanishes on the boundary if either of the following conditions is satisfied,

(a) $v \cdot n = 0, \nabla \times v \times n = 0$ on $\partial \Omega$,

(b) $v \times n = 0$ on $\partial \Omega, \nabla \cdot u = 0$ in $\Omega$.

To study functions with either of above boundary conditions, we introduce series of function sets.

Let

$$
H = \{ v \in H^1; \nabla \cdot v = 0 \text{ in } \Omega \},
$$

$$
V^{-1} = \{ v \in H; v_n = 0 \text{ on } \partial \Omega \},
$$

$$
V^0 = \{ v \in H; v_\tau = 0 \text{ on } \partial \Omega \},
$$

$$
V^{2k} = \{ v \in H^{2k+1}; \partial_n^{2j} v \in V^0, j = 0, 1, \ldots, k \},
$$

$$
V^{2k+1} = \{ v \in H^{2k+2}; v \in V^{-1}, \partial_n^{2j+1} v \in V^0, j = 0, 1, \ldots, k \}.
$$

Then, the following propositions are easily obtained

**Proposition 2.1** Let $k \geq 1$. Then $V^k \subseteq V^{k-2}$.

**Proposition 2.2** Let $k \geq -1$, and $v \in V^k$. Then $\partial_n^j v \in V^{k-j}, j = 0, 1, \ldots, k+1$.

Rewrite (11) with the new notations,

**Lemma 2.1** Let $k \geq 0$ and $v \in V^k$. Then $\partial_n v \cdot v = 0$ on $\partial \Omega$.

It should be considered that when $v$ is not in $V^0$ or $V^1$. For energy estimates, we construct a boundary layer to fill the gap.
Lemma 2.2 In the flat boundary case, assume \( \| h_r \|_{C^1(\partial \Omega)} \leq C \) for \( k \geq 1 \). Then, for any \( \epsilon << 1 \), there is a \( v^\epsilon \in V^{2k-1} \), \( \chi^\epsilon = \nabla \times v^\epsilon \), such that \( \nabla \times v^\epsilon \equiv 0 \) as \( \epsilon^2 \leq z \leq 1 - \epsilon^2 \), furthermore,

\[
\chi^\epsilon \in C^{2k+1}(\overline{\Omega}), \chi^\epsilon_n \in C^{2k}(\overline{\Omega}),
\]

\[
\partial_n^2 \chi^\epsilon = h_r, \partial_n^2 \chi^\epsilon_n = 0 \text{ on } \partial \Omega,
\]

\[
\| z^i (1 - z)^j \partial_n^{2k+1} \chi^\epsilon \|_{L^p} \leq C \epsilon^{\frac{1}{2p} + \frac{i+j}{2}}, \quad (12)
\]

\[
\| z^i (1 - z)^j \partial_n^{2k} \chi^\epsilon_n \|_{L^p} \leq C \epsilon^{\frac{1}{2p} + \frac{i+j}{2}},
\]

\[
\| \partial_n^{2k} \chi^\epsilon \|_{L^p} \leq C \epsilon^{\frac{1}{2p} + \frac{3}{2}},
\]

for \( i \in \mathbb{R}^+, 1 \leq p < +\infty \).

**Proof.** It’s trivial to find a function \( \varphi(z) \in C^1[0, \infty) \), s.t.

\[
\begin{cases}
\varphi(z) = 1 & \text{at } z = 0, \\
\varphi(z) = 0 & \text{at } z \geq 1, \\
\int_0^1 F^j(\varphi)(s)ds = 0, j = 0, 1, \cdots, 2k - 1,
\end{cases}
\]

where \( F \) is an integrate operator from \( C[0, \infty) \) to \( C^1[0, \infty) \), and \( F(f)(z) = \int_0^z f(s)ds \), \( F^0(f) = f \), \( F^j = F(F^{j-1}) \), \( j \geq 1 \).

Denote by \( \varphi^\epsilon(z) = \varphi\left(\frac{z}{\epsilon^2}\right) \). Then,

\[
\| z^i \partial^j \varphi^\epsilon \|_{L^p} \leq C \epsilon^{\frac{1}{2p} + \frac{i+j}{2}} \text{ for } i \in \mathbb{R}^+, j \leq 2, 1 \leq p \leq +\infty.
\]

Set \( \psi^\epsilon(z) = h_r(0)\varphi^\epsilon(z) + h_r(1)\varphi^\epsilon(1 - z) \), and \( \psi_3^\epsilon = -\int_0^z \nabla \cdot \psi(x, y, s)ds \). It follows that

\[
\nabla \cdot \psi^\epsilon = 0 \in \Omega, \psi_3^\epsilon = h_r \text{ on } \partial \Omega.
\]

Next, set

\[
\chi^\epsilon = F^{2k}(\psi^\epsilon).
\]

Since \( F^j(\varphi^\epsilon) = 0 \) on \( \partial \Omega \), for \( j = 1, 2, \cdots, 2k \), it follows that \( \partial^2 \chi^\epsilon = 0, \partial^{2k+1} \chi^\epsilon_n = 0 \) on \( \partial \Omega \), for \( j = 0, 1, \cdots, 2k - 1 \). Furthermore, \( \nabla \cdot \chi^\epsilon = 0 \in \Omega \). In other words, \( \chi^\epsilon \in V^{2k-2} \). Therefore, \( \int \chi^\epsilon = 0 \).

Finally, let \( \zeta^\epsilon \) satisfy the following equations

\[
\begin{cases}
- \Delta \zeta^\epsilon = \chi^\epsilon \text{ in } \Omega, \\
\zeta_\tau^\epsilon = 0, \partial_\tau \zeta_3^\epsilon = 0 \text{ on } \partial \Omega.
\end{cases}
\]

(14)
The necessary condition \( \int \chi^3 = 0 \) of existence holds by classical elliptic theories. Applying \( \text{div} \) to equation (14), together with \( \text{div}\xi^\varepsilon = 0 \) on \( \partial \Omega \), then \( \nabla \cdot \xi^\varepsilon = 0 \) in \( \Omega \).

Set \( \nu^\varepsilon = \nabla \times \xi^\varepsilon \) and notice that \( \nabla \times \nabla \xi^\varepsilon = -\Delta \chi^\varepsilon \), then the proof is completed after a simple calculation. ■

Now, we derive some results of nonlinearities.

**Lemma 2.3** \((\mathbf{u} \cdot \nabla)\mathbf{v}\) is normal to the boundary, if either of the following conditions holds

\[
(a) \quad \mathbf{u} \in \mathbf{V}^0, \mathbf{v} \in \mathbf{V}^1, \\
(b) \quad \mathbf{u} \in \mathbf{V}^{-1}, \mathbf{v} \in \mathbf{V}^0. 
\]

**Lemma 2.4** Let \( j \geq 0, \mathbf{u}, \mathbf{v} \in \mathbf{V}^j \). Then, \((\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{n} = 0 \) on \( \partial \Omega \).

The proof is left to the reader.

**Theorem 2.1** Let \( \mathbf{u}, \mathbf{v} \in \mathbf{V}^{2k+1}, k \geq 1 \). Then, for \( 0 \leq j \leq 2k+1 \),

\[
\begin{cases}
\partial_n^i(\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{n} = 0 & \text{on} \ \partial \Omega, \quad \text{if} \ j \ \text{is even}, \\
\partial_n^i(\mathbf{u} \cdot \nabla)\mathbf{v} \times \mathbf{n} = 0 & \text{on} \ \partial \Omega, \quad \text{if} \ j \ \text{is odd}.
\end{cases}
\]

**Proof.** It’s easily derived,

\[
\partial_n^i(\mathbf{u} \cdot \nabla)\mathbf{v} = \sum_{i=0}^{j} \partial_n^i(\mathbf{u} \cdot \nabla) \partial_n^{j-i} \mathbf{v}.
\]

If \( j \) is odd, \( i \) and \( j - i \), or \( j - i \) and \( i \), are odd and even, respectively. Recalling Proposition 2.1 \( \partial_n^i \mathbf{u} \in \mathbf{V}^1 \) and \( \partial_n^{j-i} \mathbf{v} \in \mathbf{V}^0 \), or \( \partial_n^i \mathbf{u} \in \mathbf{V}^0 \) and \( \partial_n^{j-i} \mathbf{v} \in \mathbf{V}^1 \). It follows Lemma 2.3 \( \partial_n^i(\mathbf{u} \cdot \nabla) \partial_n^{j-i} \mathbf{v} \times \mathbf{n} = 0 \) on \( \partial \Omega \), and the desired result is obtained. If \( j \) is even, \( i \) and \( j - i \) are all odd or even. Recalling proposition 2.1 and Lemma 2.4 \( \partial_n^i(\mathbf{u} \cdot \nabla) \partial_n^{j-i} \mathbf{v} \cdot \mathbf{n} = 0 \). And the desired result is obtained. ■

Finally, denote \( \nabla \times (\mathbf{u} \cdot \nabla)\mathbf{v} - (\mathbf{u} \cdot \nabla)(\nabla \times \mathbf{v}) \) by \( F(Du, Dv) \). By appealing to Theorem 2.1 the following results can be obtained.

**Corollary 2.1** Let \( \mathbf{u}, \mathbf{v} \in \mathbf{V}^{2k+1}, k \geq 1 \). Then, for \( 0 \leq j \leq k \),

\[
\partial_n^{2j}(\mathbf{u} \cdot \nabla)(\nabla \times \mathbf{v}) \times \mathbf{n} = 0, \partial_n^{2j}F(Du, Dv) \times \mathbf{n} = 0 \ \text{on} \ \partial \Omega.
\]
3 A priori estimates

Now, we derive formal energy estimates assuming that \( u_0, u, H_0, H \) are sufficiently regular. As pointed out in \([11]\) and \([16]\), the key in studying the vanishing viscosity limit is to control the vorticity created on the boundary.

Set

\[
\omega = \nabla \times u, \zeta = \nabla \times H.
\]

Recalling the boundary conditions \([2]\) together with the notations introduced in section \([2]\),

\[
u, \omega, \zeta, (u \cdot \nabla) \omega, (H \cdot \nabla) \zeta, (u \cdot \nabla)(H \cdot \nabla) \zeta, (\nabla \times u), (\nabla \times H), (\nabla D u), (\nabla D H), (\nabla D H), (\nabla D H), (\nabla D H), (\nabla D H)
\]

are all normal to boundary. Then, by equations (18) and (19), \( \partial_n^2 \omega \) and \( \partial^2_n \zeta \) are normal to \( \partial \Omega \). It follows that \( \omega \in V^2, \zeta \in V^2, u \in V^3 \) and \( H \in V^3 \).

Similarly, applying operator \( \partial^2_n \) to both sides of equations (18) and (19). Step by step, the following result is obtained.

**Lemma 3.1** Let \( u \) and \( H \) be sufficient regularity. Then, for \( k \in N \),

\[
u, \omega, \zeta, (u \cdot \nabla) \omega, (H \cdot \nabla) \zeta, (u \cdot \nabla)(H \cdot \nabla) \zeta, (\nabla \times u), (\nabla \times H), (\nabla D u), (\nabla D H), (\nabla D H), (\nabla D H), (\nabla D H)
\]

are all normal to boundary. Then, by equations (18) and (19), \( \partial^2_n \omega \) and \( \partial^2_n \zeta \) are normal to \( \partial \Omega \). It follows that \( \omega \in V^2, \zeta \in V^2, u \in V^3 \) and \( H \in V^3 \).

Applying the operator \( \partial^\alpha_{x,y,z} \) (\( \alpha \) is a multi-index and \( |\alpha| \leq 2k \)) to both sides of equations (18) and (19), one gets

\[
v, u, \omega, \zeta, (u \cdot \nabla) \omega, (H \cdot \nabla) \zeta, (u \cdot \nabla)(H \cdot \nabla) \zeta, (\nabla \times u), (\nabla \times H), (\nabla D u), (\nabla D H), (\nabla D H), (\nabla D H), (\nabla D H)
\]

are all normal to boundary. Then, by equations (18) and (19), \( \partial^2_n \omega \) and \( \partial^2_n \zeta \) are normal to \( \partial \Omega \). It follows that \( \omega \in V^2, \zeta \in V^2, u \in V^3 \) and \( H \in V^3 \).

Similarly, applying operator \( \partial^2_n \) to both sides of equations (18) and (19). Step by step, the following result is obtained.
\partial_t \partial_{x,y,z}^\alpha \zeta - \mu \Delta \partial_{x,y,z}^\alpha \zeta + (u \cdot \nabla) \partial_{x,y,z}^\alpha \omega - (H \cdot \nabla) \partial_{x,y,z}^\alpha \omega \\
+ \sum_{|\beta| = 1, \beta + \gamma = \alpha} (\partial_{x,y,z}^\beta u \cdot \nabla) \partial_{x,y,z}^\gamma \zeta - \sum_{|\beta| = 1, \beta + \gamma = \alpha} (\partial_{x,y,z}^\beta H \cdot \nabla) \partial_{x,y,z}^\gamma \omega \\
+ \sum_{|\beta| \geq 1, \beta + \gamma = \alpha} (\partial_{x,y,z}^\beta u \cdot \nabla) \partial_{x,y,z}^\gamma \zeta - \sum_{|\beta| \geq 1, \beta + \gamma = \alpha} (\partial_{x,y,z}^\beta H \cdot \nabla) \partial_{x,y,z}^\gamma \omega \\
+ F(D \partial_{x,y,z}^\alpha u, DH) + F(Du, D \partial_{x,y,z}^\alpha H) - F(D \partial_{x,y,z}^\alpha H, Du) - F(DH, D \partial_{x,y,z}^\alpha u) \\
+ \sum_{1 \leq |\beta| \leq |\alpha| - 1, \beta + \gamma = \alpha} F(D \partial_{x,y,z}^\beta u, D \partial_{x,y,z}^\beta H) - \sum_{1 \leq |\beta| \leq |\alpha| - 1, \beta + \gamma = \alpha} F(D \partial_{x,y,z}^\beta H, D \partial_{x,y,z}^\beta u) = 0.

Next, multiplying both sides of the above equations by \partial_{x,y,z}^\alpha \omega and \partial_{x,y,z}^\alpha \zeta, respectively, integrating in \Omega, and summing them up. Note that \partial_{x,y,z}^\alpha \omega \nu \in V^0 \text{ and } \partial_{x,y,z}^\alpha \zeta \nu \in V^1 \text{ and } \partial_{x,y,z}^\alpha \zeta \nu \in V^1 \text{ for } |\alpha| \leq 2k,

\int (H \cdot \nabla) \partial_{x,y,z}^\alpha \omega \cdot \partial_{x,y,z}^\alpha \zeta + \int (H \cdot \nabla) \partial_{x,y,z}^\alpha \zeta \cdot \partial_{x,y,z}^\alpha \omega = 0,

\|\nabla u\|_{L^\infty} + \|\nabla H\|_{L^\infty} \leq C\|u\|_{H^3} + C\|H\|_{H^3} \leq C\|\omega\|_{H^{2k}} + C\|\zeta\|_{H^{2k}},

and

\|u\|_{W^{1,4}} + \|H\|_{W^{1,4}} \leq C\|\omega\|_{W^{1,4}} + C\|\zeta\|_{W^{1,4}} \leq C\|\omega\|_{H^1} + C\|\zeta\|_{H^1}, 2 \leq j \leq 2k.

By Lemma 2.1 and summing up for all \|\alpha\| \leq 2k, one obtains

\frac{1}{2} \frac{d}{dt} (\|\omega\|_{H^{2k}}^2 + \|\zeta\|_{H^{2k}}^2) + \nu \|\nabla \omega\|_{H^{2k}}^2 + \nu \|\nabla \zeta\|_{H^{2k}}^2 \leq C\|\omega\|_{H^{2k}}^3 + C\|\zeta\|_{H^{2k}}^3.

Comparing with the ordinary differential equation

\begin{equation}
\begin{cases}
y'(t) = Cy^2, \\
y(0) = \|\omega_0\|_{H^{2k}}^2 + \|\zeta_0\|_{H^{2k}}^2,
\end{cases}
\end{equation}

where \omega_0 = \nabla \times u_0 \text{ and } \zeta_0 = \nabla \times H_0, \text{ then denoting by } T^* \text{ the blow up time, it follows that a priori estimates hold, for } T < T^*,

\|\omega\|_{L^\infty(0,T;H^{2k})} + \|\zeta\|_{L^\infty(0,T;H^{2k})} \leq C.

Thus, we have the following result.

**Theorem 3.1** Let \( u_0 \in \mathbf{V}^{2k-1} \cap \mathbf{H}^{2k+1} \text{ and } H_0 \in \mathbf{V}^{2k-1} \cap \mathbf{H}^{2k+1}, k \geq 1. \text{ Then there exist } T \text{ and } C(\|u_0\|_{H^{2k+1}}, T), \text{ s.t.}

\|u\|_{L^\infty(0,T;H^{2k+1})} + \|H\|_{L^\infty(0,T;H^{2k+1})} \leq C. \tag{23}
Taking the inner product \((18), \partial_t \omega \) and \((19), \partial_t \zeta \), one obtains that
\[
\| \partial_t \omega \|_{L^2(0,T;H^{2k-1})} + \| \partial_t \zeta \|_{L^2(0,T;H^{2k-1})} \leq C.
\] It follows that
\[
\| \partial_t u \|_{L^2(0,T;H^{2k})} + \| \partial_t H \|_{L^2(0,T;H^{2k})} \leq C.
\]

According to \(\| \omega_0 \|_{H^{2k}} \leq C\) and \(\| \zeta_0 \|_{H^{2k}} \leq C\), then by equations \((18)\) and \((19)\),
\[
\| \partial_t \omega \|_{t=0} \leq H^{2k-2} + \| \partial_t \zeta \|_{t=0} \leq H^{2k-2} \leq C.
\]

Similarly, applying operator \(\partial_t \partial^x_{y,z} \) to both sides of equations \((18)\) and \((19)\),
for \(|\alpha| \leq 2k - 2\), and multiplying \(\partial_t \partial^x_{y,z} \omega\) and \(\partial_t \partial^\alpha \), respectively, we have,
\[
\| \partial_t u \|_{L^\infty(0,T;H^{2k-1})} + \| \partial_t H \|_{L^\infty(0,T;H^{2k-1})} \leq C.
\] (24)

Thus, we can conclude

**Theorem 3.2** Let the conditions of Theorem 3.1 be satisfied, then for \(s \leq k\)
\[
\| \partial_t^s u \|_{L^\infty(0,T;H^{2k+1-2s})} + \| \partial_t^s H \|_{L^\infty(0,T;H^{2k+1-2s})} \leq C,
\] (25)
\[
\| \partial_t^{s+1} u \|_{L^2(0,T;H^{2k-2s})} + \| \partial_t^{s+1} H \|_{L^2(0,T;H^{2k-2s})} \leq C.
\]
where \(C = C(\|u_0\|_{H^{2k+1}}, T)\).

Then, the regularity of the solution of MHD equations \((1)\) and \((2)\) is investigated,

**Theorem 3.3** Let the conditions of Theorem 3.1 be satisfied. Then for \(s \leq k\),
there exist a time \(T\) depending on the initial date and unique classical solution of
MHD equations \((1)\) with boundary condition \((2)\). In addition,
\[
\| \partial_t^s u \|_{L^\infty(0,T;H^{2k+1-2s})} + \| \partial_t^s H \|_{L^\infty(0,T;H^{2k+1-2s})} \leq C,
\] (26)
\[
\| \partial_t^{s+1} u \|_{L^2(0,T;H^{2k-2s})} + \| \partial_t^{s+1} H \|_{L^2(0,T;H^{2k-2s})} \leq C,
\]
where \(C = C(\|u_0\|_{H^{2k+1}}, \|H_0\|_{H^{2k+1}}, T)\).

4 The vanishing viscosity limit

This section focuses on the vanishing viscosity limit of the MHD system for the
flat boundary case.

**Theorem 4.1** Let the conditions of Theorem 3.1 be satisfied for \(k \geq 1\). Then
as \((\nu, \mu) \to 0\), \((u, H)\) converge to the unique solution \((u^0, H^0)\) of the ideal MHD
system with the same initial date in the sense
\[
(u, H) \to (u^0, H^0) \text{ in } C(0, T; H^{2k}).
\] (27)
Proof. It follows from Theorem 3.3 that
\[ u(\nu, \mu), H(\nu, \mu) \] is uniformly bounded in \( L^\infty(0, T; H^{2k+1}) \),

and
\[ \partial_t u(\nu, \mu), \partial_t H(\nu, \mu) \] is uniformly bounded in \( L^2(0, T; H^{2k}) \),

for all \( \nu > 0 \) and \( \mu > 0 \). By the standard compactness result, there exist a subsequence \( \nu_k \) of \( \nu \), \( \mu_k \) of \( \mu \) and vector functions \( u^0 \) and \( H^0 \), such that
\[
(u(\nu_k, \mu_k), H(\nu_k, \mu_k)) \rightarrow (u^0, H^0) \text{ in } C(0, T; H^{2k}),
\]
as \( (\nu, \mu) \rightarrow 0 \). Passing to the limit, we can find \( (u^0, H^0) \) solves the ideal MHD equations \( (3) \) and \( (4) \). Together with the uniqueness of the strong solution of the ideal MHD systems, we then show the convergence of whole sequence. \( \blacksquare \)

Now, we present the convergence rate.

**Theorem 4.2** Let the conditions of Theorem 3.1 be satisfied for \( k \geq 1 \). Then,
\[
\|u - u^0\|_{L^\infty(0,T;H^{2k-1})} + \|H - H^0\|_{L^\infty(0,T;H^{2k-1})} \leq C\nu + C\mu.
\]

Proof. Set \( \omega^0 = \nabla \times u^0 \) and \( \zeta^0 = \nabla \times H^0 \). Recalling Lemma 3.1 and Theorem 4.1 one obtains
\[
\begin{align*}
 u^0(t) &\in X^{2k-1} \cap H^{2k+1}, H^0(t) \in X^{2k-1} \cap H^{2k+1}, \\
 \omega^0(t) &\in X^{2k-2} \cap H^{2k}, \zeta^0(t) \in X^{2k-2} \cap H^{2k}.
\end{align*}
\]

Set \( \tilde{u} = u - u^0 \), \( \tilde{\omega} = \nabla \times \tilde{u}, \tilde{H} = H - H^0 \), \( \tilde{\zeta} = \nabla \times \tilde{H} \). We can find \( \tilde{u}, \tilde{\omega} \) solve
\[
\begin{align*}
 \partial_t \tilde{\omega} - \nu \Delta \tilde{\omega} + (u \cdot \nabla)\tilde{\omega} + (\tilde{u} \cdot \nabla)\omega^0 - (H \cdot \nabla)\tilde{\zeta} - (\tilde{H} \cdot \nabla)\zeta^0 \\
 + F(Du, D\tilde{u}) + F(D\tilde{u}, Du^0) - F(DH, D\tilde{H}) - F(D\tilde{H}, DH^0) = \nu \Delta \omega^0, \\
 \partial_t \tilde{\zeta} - \mu \Delta \tilde{\zeta} + (u \cdot \nabla)\tilde{\zeta} + (\tilde{u} \cdot \nabla)\zeta^0 - (H \cdot \nabla)\tilde{\omega} - (\tilde{H} \cdot \nabla)\omega^0 \\
 + F(Du, D\tilde{H}) + F(D\tilde{u}, D\omega^0) - F(DH, D\tilde{u}) - F(D\tilde{H}, Du^0) = \mu \Delta \omega^0,
\end{align*}
\]
Noting that \( \tilde{\omega}^0 \in V^{2k-2}, \tilde{\zeta}^0 \in V^{2k-2}, \|\Delta \omega\|_{H^{2k-2}} \leq C \) and \( \|\Delta \zeta\|_{H^{2k-2}} \leq C \), the same argument in proof of Theorem 3.1 can be followed. Taking the inner products \( (33), (34) \) and \( (31) \), one obtains the desired result can be obtained. \( \blacksquare \)

There is a gap between \( (\partial^2_k \omega_n, \partial^2_k \zeta_n) \) and 0. In other words, \( \omega^0 \) and \( \zeta^0 \) are not in \( V^{2k} \). Assuming \( \|\partial^2_k \omega_n\|_{L^\infty(0,T;C^2(\partial\Omega))} \leq C, \|\partial^2_k \zeta_n\|_{L^\infty(0,T;C^2(\partial\Omega))} \leq C, \)
\[ \| \partial_t \partial_n^{2k} \omega_n \|_{L^2(0,T;C^2(\partial \Omega))} \leq C, \quad \| \partial_t \partial_n^{2k} \zeta_n \|_{L^2(0,T;C^2(\partial \Omega))} \leq C, \] by Lemma 2.2 there exist \( v^1 \in V^{2k-1}, \) \( \lambda^1 = \nabla \times v^1 \in V^{2k-2} \) \((i = 1, 2), \) s.t. \( \partial_n^{2k} \lambda_{\tau} = -\partial_n^{2k} \omega_n, \) \( \partial_n^{2k} \chi_{\tau} = -\partial_n^{2k} \zeta_n \) on \( \partial \Omega, \) \( \| z^i (1-z)^j \partial_n^{2k+1} \chi^i \|_{L^\infty(0,T;L^2)} \leq C \nu^{\frac{2j+1}{2}}, \) and further \( \| \partial_t \chi^i \|_{L^2(0,T;H^{2k})} \leq C \nu^{\frac{i}{2}}, \) \( \| \partial_t \lambda^i \|_{L^2(0,T;H^{2k})} \leq C \nu^{\frac{i}{2}}, \) \( i = 1, 2, j = 0, 1. \)

Set \( \hat{u} = u - u^0 - v^1, \) \( \hat{H} = H - H^0 - v^2 \) \( \hat{\omega} = \nabla \times \hat{u} = \omega - \omega^0 - \chi^1, \) \( \hat{\zeta} = \nabla \times \hat{H} = \zeta - \zeta^0 - \chi^2. \)

From equations (33) and (34), one obtains,
\[
\begin{align*}
\partial_t \hat{\omega} - \nu \triangle \hat{\omega} + (u \cdot \nabla) \hat{\omega} + (\hat{u} \cdot \nabla) \omega^0 - (H \cdot \nabla) \hat{\zeta} - (\hat{H} \cdot \nabla) \zeta^0 \\
+ F(Du, D\hat{u}) + F(D\hat{u}, Du^0) - F(DH, D\hat{H}) + F(D\hat{H}, DH^0) \\
= \nu \triangle \omega^0 - \partial_t \chi^1 + \nu \triangle \chi^0 - (u \cdot \nabla) \chi^1 - (v^1 \cdot \nabla) \omega^0 + (H \cdot \nabla) \chi^2 + (v^2 \cdot \nabla) \zeta^0 \\
- F(Du, Dv^1) - F(Dv^1, Du^0) + F(DH, Dv^2) + F(Dv^2, DH^0),
\end{align*}
\]
\[
\begin{align*}
\partial_t \hat{\zeta} - \nu \triangle \hat{\zeta} + (u \cdot \nabla) \hat{\zeta} + (\hat{u} \cdot \nabla) \zeta^0 - (H \cdot \nabla) \hat{\omega} - (\hat{H} \cdot \nabla) \omega^0 \\
+ F(Du, D\hat{H}) + F(D\hat{H}, DH^0) - F(D\hat{H}, Du^0) - F(D\hat{H}, Du^0) \\
= \nu \triangle \zeta^0 - \partial_t \chi^2 + \nu \triangle \chi^0 - (u \cdot \nabla) \chi^2 - (v^1 \cdot \nabla) \omega^0 + (H \cdot \nabla) \chi^1 + (v^2 \cdot \nabla) \zeta^0 \\
- F(Du, Dv^2) - F(Dv^2, DH^0) + F(Dv^2, Du^0),
\end{align*}
\]

Then, taking the inner products (35), \( \hat{\omega}, \hat{\zeta} \) \( H^{2k} \) \( (36), \hat{\zeta} \) \( H^{2k}, \hat{\zeta} \)

Note that
\[
\begin{align*}
| \nabla \hat{\omega} |_{L^2} & \leq \| (u \cdot \nabla) \partial_n^{2k} \chi^1 \|_{L^2} + \| u_3 \|_{L^\infty} \| z(1-z) \|_{L^\infty} \| z(1-z) \partial_n^{2k+1} \chi^1 \|_{L^2} \leq C \nu^{\frac{1}{4}} \\
\| \hat{\omega} \|_{L^\infty(0,T;H^{2k})} + \| \hat{\zeta} \|_{L^\infty(0,T;H^{2k})} & \leq C \nu^{\frac{1}{4}} + C \mu^{\frac{1}{4}}.
\end{align*}
\]

The following result is concluded,

**Theorem 4.3** Let the conditions of Theorem 3.1 be satisfied for \( k \geq 1. \) Assume the solution \((u^0, H^0)\) of the ideal MHD equations (3) and (4) satisfy \( \| \partial_n^{2k} \omega^0 \|_{L^\infty(0,T;C^2(\partial \Omega))} \leq C, \) \( \| \partial_n^{2k} \zeta^0 \|_{L^\infty(0,T;C^2(\partial \Omega))} \leq C, \) \( \| \partial_t \partial_n^{2k} \omega^0 \|_{L^2(0,T;C^2(\partial \Omega))} \leq C, \) \( \| \partial_t \partial_n^{2k} \zeta^0 \|_{L^2(0,T;C^2(\partial \Omega))} \leq C, \) then,
\[
\| u - u^0 \|_{L^\infty(0,T;H^{2k+1})} + \| H - H^0 \|_{L^\infty(0,T;H^{2k+1})} \leq C \nu^{\frac{1}{4}} + C \mu^{\frac{1}{4}}.
\]
Finally, we give two remarks.

**Remark 4.1** If the conditions of Theorem 4.2 are all satisfied, then for $s \leq k - 1$,\n\[
\| \partial_s^s u - \partial_s^s u_0 \|_{L^\infty(0,T;H^{2k-1-2s})} + \| \partial_s^s H - \partial_s^s H_0 \|_{L^\infty(0,T;H^{2k-1-2s})} \leq C \nu + C \mu. \tag{38}
\]

**Remark 4.2** If the conditions of Theorem 4.3 are all satisfied, then for $s \leq k$,\n\[
\| \partial_s^s u - \partial_s^s u_0 \|_{L^\infty(0,T;H^{2k+1-2s})} + \| \partial_s^s H - \partial_s^s H_0 \|_{L^\infty(0,T;H^{2k+1-2s})} \leq C \nu^\frac{1}{4} + C \mu^\frac{1}{4}. \tag{39}
\]

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