On parameterizations of Teichmüller spaces of surfaces with boundary

Ren Guo

Department of Mathematics, Rutgers University, Piscataway, NJ, 08854, USA
Email: renguo@math.rutgers.edu

Abstract

In [5], Luo introduced a $\psi_{\lambda}$ edge invariant which turns out to be a coordinate of the Teichmüller space of a surface with boundary. And he proved that for $\lambda \geq 0$, the image of the Teichmüller space under $\psi_{\lambda}$ edge invariant coordinate is an open cell. In this paper we verify his conjecture that for $\lambda < 0$, the image of the Teichmüller space is a bounded convex polytope.

AMS Classification 57M50; 30F45, 30F60

Keywords Teichmüller space, ideal triangulation, right-angled hexagon, $\psi_{\lambda}$ edge invariant.

1 Introduction

Suppose $S$ is a compact connected surface of non-empty boundary and has negative Euler characteristic. It is well known that there are hyperbolic metrics with totally geodesic boundary on the surface $S$. Two such hyperbolic metrics are called isotopic if there is an isometry isotopic to the identity between them. The space of all isotopy classes of hyperbolic metrics on $S$, denoted by $T(S)$, is called the Teichmüller space of the surface $S$.

There are several known parameterizations of the Teichmüller spaces. In particular, using the 3-holed decomposition of a surface, Fenchel-Nielsen introduced a coordinate for $T(S)$, for more detail see the book Imayoshi & Taniguchi [3]. Bonahon [1] produced a parametrization of the Teichmüller spaces using the sheared coordinate. Penner [7, 8] introduced the “lambda length” coordinate and simplicial coordinate of the decorated Teichmüller space. Recently Luo [4, 5] introduced a family of coordinates of $T(S)$. To be more precise, for each real number $\lambda$, he introduced a $\psi_{\lambda}$ edge invariant associated to a hyperbolic metric which turns out to be a coordinate of the Teichmüller space $T(S)$. When $\lambda \geq 0$, he proved that the image of the Teichmüller space under the coordinate is an open convex polytope independent of $\lambda$. Luo [6] conjectured that
for $\lambda < 0$, the image of the Teichmüller space under $\psi_\lambda$ edge invariant coordinate is a bounded convex polytope. The purpose of this paper is to verify this conjecture.

Let us begin by recalling the $\psi_\lambda$ edge invariant coordinate introduced by Luo [5]. The coordinate depends on a fixed ideal triangulation of $S$. Recall that a colored hexagon is a hexagon with three non-pairwise adjacent edges labelled by red and the opposite edges labelled by black. Take a finite disjoint union of colored hexagons and identify all red edges in pairs by homeomorphisms. The quotient is a compact surface with non-empty boundary together with an ideal triangulation. The 2-cells in the ideal triangulation are quotients of the hexagons. The quotients of red edges (respectively black edges) are called the edges (respectively $A$-arcs) of the ideal triangulation. It it well known that every compact surface $S$ of non-empty boundary and negative Euler characteristic admits an ideal triangulation.

In a hyperbolic metric, any hexagon in an ideal triangulation is isotopic (leaving the boundary of a surface fixed) to a hyperbolic right-angled hexagon. It is well known that a hyperbolic right-angled hexagon is determined up to isometry preserving coloring by the lengths of three red edges. Furthermore, for any $l_1, l_2, l_3 \in \mathbb{R}_{>0}$, there exists a unique colored hyperbolic right-angled hexagon whose three red edges have lengths $l_1, l_2, l_3$, for a proof see Buser [2].

Given an ideally triangulated surface $S$ with $E$ the set of all edges, each hyperbolic metric $d$ on $S$ has a length coordinate $l_d : E \to \mathbb{R}_{>0}$ which assigns each edge $e$ the length of the shortest geodesic arc homotopic to $e$ relative to the boundary of $S$. On the other hand, given a function $l : E \to \mathbb{R}_{>0}$, we can produce a hyperbolic metric with totally geodesic boundary on $S$. This metric is constructed by making each 2-cell with red edges $e_i, e_j, e_k$ a colored hyperbolic right-angled hexagon the lengths of whose red edges are $l(e_i), l(e_j), l(e_k)$. Thus, the Teichmüller space $T(S)$ can be identified with the space $\mathbb{R}^E_{>0}$ by length coordinates.

In [5], Luo introduced the $\psi_\lambda$ edge invariant of a hyperbolic metric as $\psi_\lambda : E \to \mathbb{R}$ defined by

$$\psi_\lambda(e) = \int_0^{a+b-c} \cosh^\lambda(t)dt + \int_0^{a'+b'-c'} \cosh^\lambda(t)dt$$

where $e$ is an edge of an ideal triangulation shared by two hyperbolic right-angled hexagons and $a, b, c, a', b', c'$ are lengths of the A-arcs labelled as in Figure[1]. Now consider the map $\Psi_\lambda : T(S) \to \mathbb{R}^E$ sending a hyperbolic metric $l$ to its $\psi_\lambda$ edge invariant.
Figure 1: The lengths of A-arcs in the definition of $\psi_\lambda$ edge invariant are labeled.

The following two theorems are proved in Luo [5]. The special case of $\lambda = 0$ was proved in Luo [4]. We use $(S,T)$ to denote a surface $S$ with an ideal triangulation $T$.

**Theorem 1.1** (Luo) Suppose $(S,T)$ is an ideally triangulated surface. For any $\lambda \in \mathbb{R}$, the map $\Psi_\lambda : T(S) \to \mathbb{R}^E$ is a smooth embedding. In particular, each hyperbolic metric with geodesic boundary on $(S,T)$ is determined up to triangulation preserving isometry by its $\psi_\lambda$ edge invariant.

An edge path $(H_0, e_1, H_1, ..., e_n, H_n)$ is a collection of hexagons and edges in an ideal triangulation so that two distinct hexagons $H_{i-1}$ and $H_i$ sharing the edge $e_i$ for $i = 1, ..., n$. An edge path $(H_0, e_1, H_1, ..., e_n, H_n)$ is an edge cycle if $H_0 = H_n$. For example see Figure 3. A fundamental edge path (or fundamental edge cycle) is an edge path (or edge cycle) so that each edge in the ideal triangulation appears at most twice in the path (or cycle).

**Theorem 1.2** (Luo) Let $\lambda \geq 0$. For an ideal triangulated surface $(S,T)$, $\Psi_\lambda(T(S)) = \{ z \in \mathbb{R}^E | \text{ for each fundamental edge cycle } (H_0, e_1, H_1, ..., e_n, H_n = H_0), \sum_{i=1}^{n} z(e_i) > 0 \}$. Thus $\Psi_\lambda(T(S))$ is an open convex polytope independent of the parameter $\lambda \geq 0$.

In this paper we generalize Theorem 1.2 to any real number $\lambda$. The main result is the following.
Theorem 1.3  For an ideal triangulated surface \((S,T)\), \(\Psi_\lambda(T(S))\) is the set of points \(z \in \mathbb{R}^E\) satisfying

1. \(z(e) < 2 \int_0^\infty \cosh^\lambda(t)dt\) for each edge \(e\);
2. \(\sum_{i=1}^n z(e_i) > -2 \int_0^\infty \cosh^\lambda(t)dt\) for each fundamental edge path \((H_0, e_1, H_1, ..., e_n, H_n)\);
3. \(\sum_{i=1}^n z(e_i) > 0\) for each fundamental edge cycle \((H_0, e_1, H_1, ..., e_n, H_n = H_0)\).

Thus \(\Psi_\lambda(T(S))\) is an open convex polytope. And \(\Psi_{\lambda_1}(T(S)) \subset \Psi_{\lambda_2}(T(S)) \subset \Psi_0(T(S)) = \Psi_{\lambda_3}(T(S))\) for \(\lambda_1 < \lambda_2 < 0 < \lambda_3\). The intersection \(\cap_{\lambda=0}^{-\infty} \Psi_\lambda(T(S))\) is empty.

It is easy to see when \(\lambda \geq 0\) the conditions in Theorem 1.3 are reduced to the third one which is exactly the condition in Theorem 1.2. The proof of Theorem 1.3 follows the same strategy used in Luo’s proof of Theorem 1.2 [5].

In section 2 we investigate degenerations of a hyperbolic right-angled hexagon. In section 3 we prove the main result Theorem 1.3.

2  Degenerations of a hyperbolic hexagon

In this section we always assume a hyperbolic right-angled hexagon has three non-pairwise adjacent edges of lengths \(l_1, l_2, l_3\) and opposite A-arcs of lengths \(\theta_1, \theta_2, \theta_3\) labelled in Figure 2. And recall that the r-coordinate is defined as \(r_i = \frac{\theta_j + \theta_k - \theta_i}{2}\).

![Figure 2: An hyperbolic right-angled hexagon with lengths of edges and A-arcs labeled.](image)

We improve a lemma proved in Luo [5].
Lemma 2.1 Consider $r_i$ as a function of $(l_1, l_2, l_3)$. We have $\lim_{l_i \to 0} r_i = \infty$ so that the convergence is uniform in $(l_1, l_2, l_3)$.

Proof By the cosine law of a hyperbolic right-angled hexagon, we see that for $i \neq j \neq k \neq i$,
\begin{align*}
cosh \theta_j &= \frac{\cosh l_j + \cosh l_i \cosh l_k}{\sinh l_i \sinh l_k} \\
&> \frac{\cosh l_i \cosh l_k}{\sinh l_i \sinh l_k} \\
&\geq \frac{\cosh l_i}{\sinh l_i}.
\end{align*}
Hence we have $\lim_{l_i \to 0} \theta_j = \infty$. Thus $\lim_{l_i \to 0} \frac{\cosh \theta_i}{\sinh \theta_j} = 1$. By symmetry we have $\lim_{l_i \to 0} \frac{\cosh \theta_k}{\sinh \theta_j} = 1$.

On the other hand, by cosine law we see that for $i \neq j \neq k \neq i$,
\begin{align*}
\cosh l_i - \frac{\cosh \theta_j \cosh \theta_k}{\sinh \theta_j \sinh \theta_k} &= \frac{\cosh \theta_i}{\sinh \theta_j \sinh \theta_k} > \frac{2e^{\theta_i}}{e^{\theta_j} + e^{\theta_k}} = \frac{2}{e^{2r_i}}.
\end{align*}
Since the left hand side converges to 0 as $l_i \to 0$, we have $\lim_{l_i \to 0} r_i = \infty$.

To show the convergence is uniform, we consider the following formula called tangent law derived in Luo [5]. For $i \neq j \neq k \neq i$,
\begin{align*}
\tanh^2 \frac{l_i}{2} &= \frac{\cosh r_j \cosh r_k}{\cosh r_i \cosh (r_i + r_j + r_k)}.
\end{align*}
By the formula,
\begin{align*}
\tanh^2 \frac{l_i}{2} &= \frac{1}{\cosh r_i} \cdot \frac{1}{(1 + \tanh r_j \tanh r_k) \cosh r_i + (\tanh r_j + \tanh r_k) \sinh r_i} \\
&\geq \frac{1}{\cosh r_i} \cdot \frac{1}{(1 + 1) \cosh r_i + (1 + 1) \sinh r_i} \\
&\geq \frac{1}{4 \cosh^2 r_i}.
\end{align*}
It follows that
\begin{align*}
\cosh^2 r_i &\geq \frac{1}{4 \tanh^2 \frac{l_i}{2}}.
\end{align*}
Thus $r_i$ converges to $\infty$ uniformly. }
Lemma 2.2 The following holds for some positive finite numbers $f_1, f_2, f_3, f_4, f_5$:

1. if $(l_1, l_2, l_3)$ converges to $(\infty, f_1, f_2)$, then $(\theta_1, \theta_2, \theta_3)$ converges to $(\infty, f_3, f_4)$;
2. if $(l_1, l_2, l_3)$ converges to $(\infty, \infty, f_5)$, then $\theta_3$ converges to 0;
3. if $(l_1, l_2, l_3)$ converges to $(\infty, \infty, \infty)$, then we can choose a subsequence of $(l_1, l_2, l_3)$ such that at least two of $\theta_1, \theta_2$ and $\theta_3$ converge to 0.

Proof (1) By the cosine law we have

$$\cosh \theta_1 = \frac{\cosh l_1 + \cosh l_2 \cosh l_3}{\sinh l_2 \sinh l_3},$$

if $\lim (l_1, l_2, l_3) = (\infty, f_1, f_2)$, we have $\lim \cosh \theta_1 = \infty$, or $\lim \theta_1 = \infty$. And since $\lim \frac{\cosh l_1}{\sinh l_1} = 1$,

$$\lim \cosh \theta_2 = \lim \frac{\cosh l_2 + \cosh l_1 \cosh l_3}{\sinh l_1 \sinh l_3} = \frac{\cosh f_2}{\sinh f_2} > 1.$$

Thus $\lim \theta_2$ is a positive finite number. By symmetry $\lim \theta_3$ is a positive finite number.

(2) If $\lim (l_1, l_2, l_3) = (\infty, \infty, f_5)$, we have

$$\lim \cosh \theta_3 = \lim \frac{\cosh l_3 + \cosh l_1 \cosh l_2}{\sinh l_1 \sinh l_2} = \lim \frac{\cosh l_3}{\sinh l_1 \sinh l_2} + 1 = 1.$$

Thus $\lim \theta_3 = 0$.

(3) If $\lim (l_1, l_2, l_3) = (\infty, \infty, \infty)$, we have

$$\lim \cosh \theta_i = \lim \frac{\cosh l_i + \cosh l_j \cosh l_k}{\sinh l_j \sinh l_k} = \lim \frac{\cosh l_i}{\sinh l_j \sinh l_k} + 1$$

$$= \lim \frac{2e^{l_i}}{e^{l_j + l_k}} + 1 = \lim \frac{2e^{l_i - l_j - l_k}}{e^{l_j - l_k} + 1}.$$ 

Since $\lim e^{l_i - l_j - l_k} e^{l_j - l_i - l_k} = \lim e^{-2l_k} = 0$, by taking subsequence of $(l_1, l_2, l_3)$, we may assume $\lim e^{l_i - l_j - l_k}$ and $\lim e^{l_j - l_i - l_k}$ exist. Then one of $\lim e^{l_i - l_j - l_k}$ and $\lim e^{l_j - l_i - l_k}$ is 0. Hence at least two of $\lim \theta_1, \lim \theta_2$ and $\lim \theta_3$ are 0. □

3 Proof of Theorem 1.3

Lemma 3.1 If $a > 0$, then for any real number $x$, we have

$$\int_0^{a+x} \cosh^\lambda(t)dt + \int_0^{a-x} \cosh^\lambda(t)dt > 0.$$
Proof Let \( f(a) \) be the function of the left hand side of the inequality. We see
\[ f'(a) = \cosh^2(a + x) + \cosh^2(a - x) > 0. \]
And \( f(0) = 0 \). Hence \( f(a) > 0 \) for \( a > 0 \).

Proof of Theorem 1.3 We denote the polytope defined by the inequalities in condition 1, 2, 3 by \( P_\lambda \). First we claim \( \Psi_\lambda(T(S)) \subset P_\lambda \). Indeed, fix a hyperbolic metric \( l \in T(S) \). For any edge \( e \), let \( r, r' \) be the \( r \)-coordinates of \( A \)-arcs facing \( e \), then
\[ \psi_\lambda(e) = \int_0^r \cosh^2(t)dt + \int_r^{r'} \cosh^2(t)dt < 2 \int_0^\infty \cosh^2(t)dt. \]
Thus the condition 1 holds.

Given an edge path \( (H_0, e_1, H_1, ..., e_n, H_n) \), for \( i = 1, ..., n - 1 \), let \( a_i \) be the length of the \( A \)-arc in \( H_i \) adjacent to \( e_i \) and \( e_{i+1} \). Denote the lengths of \( A \)-arcs in \( H_i \) facing \( e_i \) and \( e_{i+1} \) by \( b_i \) and \( c_i \) respectively as labelled in Figure 3 (a).

![Figure 3: (a) An example of an edge path with lengths of A-arcs labeled. (b) An example of an edge cycle with lengths of A-arcs labeled.](image)

Then by definition
\[ \psi_\lambda(e_1) = \int_0^r \cosh^2(t)dt + \int_0^{a_1 + c_1 - b_1} \cosh^2(t)dt, \]
where \( r \) is the \( r \)-coordinate of the \( A \)-arc in \( H_0 \) facing \( e_1 \). For \( i = 2, ..., n - 1 \),
\[ \psi_\lambda(e_i) = \int_0^{a_{i-1} + b_{i-1} - c_{i-1}} \cosh^2(t)dt + \int_0^{a_i + c_i - b_i} \cosh^2(t)dt. \]
And
\[ \psi_\lambda(e_n) = \int_0^{a_n+b_n-c_n-1} \frac{\cosh(t)}{2} dt + \int_0^{r'} \cosh(t) dt \]
where \( r' \) is the r-coordinate of the A-arc in \( H_n \) facing \( e_n \).

Hence by Lemma 3.1,
\[ \sum_{i=1}^n \psi_\lambda(e_i) = \sum_{i=1}^{n-1} \left( \int_0^{a_i+b_i-c_i} \frac{\cosh(t)}{2} dt + \int_0^{r'} \cosh(t) dt \right) \]
\[ + \int_0^r \cosh(t) dt + \int_0^{r'} \cosh(t) dt > \int_0^r \cosh(t) dt + \int_0^{r'} \cosh(t) dt > 2 \int_{-\infty}^{\infty} \cosh(t) dt = -2 \int_0^{\infty} \cosh(t) dt. \]

Thus the condition 2 holds.

Furthermore, if \((H_0, e_1, H_1, ..., e_n, H_n = H_0)\) is an edge cycle, \( H_0 \) contains both \( e_1 \) and \( e_n \). Let \( a_0 \) be the length of A-arc in \( H_0 \) adjacent to \( e_1 \) and \( e_n \), \( b_0, c_0 \) be the lengths of A-arcs facing \( e_n \) and \( e_0 \) respectively as labelled in Figure 3 (b). Thus the r-coordinates are \( r = \frac{a_0+b_0-c_0}{2} \) and \( r' = \frac{a_0+c_0-b_0}{2} \). Hence
\[ \sum_{i=1}^n \psi_\lambda(e_i) > \int_0^r \cosh(t) dt + \int_0^{r'} \cosh(t) dt > 0 \]
by Lemma 3.1 Thus the condition 3 holds.

Now by Theorem 1.1, \( \Psi_\lambda : T(S) \to P_\lambda \) is an embedding. Therefore \( \Psi_\lambda(T(S)) \) is open in \( P_\lambda \). We only need to show it is also closed in \( P_\lambda \). This will finish the proof since \( P_\lambda \) is connected.

Take a sequence \( l^{(m)} \in T(S) \) so that \( \lim_{m \to \infty} \Psi_\lambda(l^{(m)}) = z \in P_\lambda \). By taking subsequence, we may assume that \( \lim_{m \to \infty} l^{(m)} \in [0, \infty]^E \) exists and the length of each A-arc converges into \([0, \infty]\). We only need to show that \( \lim_{m \to \infty} l^{(m)} \in (0, \infty)^E = T(S) \). This will finish the proof since \( z = \Psi_\lambda(\lim_{m \to \infty} l^{(m)}) \).

Suppose otherwise that there is an edge \( e \in E \) so that \( \lim_{m \to \infty} l^{(m)}(e) \in \{0, \infty\} \). We will discuss two cases.

Case 1, \( \lim_{m \to \infty} l^{(m)}(e) = 0 \) for some \( e \in E \). Let \( H, H' \) be the hexagons sharing \( e \) and \( r^{(m)}, r'^{(m)} \) be the r-coordinates of the A-arcs in \( H, H' \) facing \( e \).
Then by Lemma 2.1, \( \lim_{m \to \infty} r^{(m)} \to \infty, \lim_{m \to \infty} r^{(m)}' \to \infty \). Then

\[
z(e) = \lim_{m \to \infty} \left( \int_0^{r^{(m)}} \cosh^\lambda(t)dt + \int_0^{r^{(m)}}' \cosh^\lambda(t)dt \right) = 2 \int_0^\infty \cosh^\lambda(t)dt.
\]

This is impossible since \( z \in P_\lambda \) must satisfy the condition 1.

Due to case 1, we can assume \( \lim_{m \to \infty} l^{(m)} \in (0, \infty] \).

Case 2, \( \lim_{m \to \infty} l^{(m)}(e) = \infty \) for some \( e \in E \). Define the subset \( E_\infty = \{ e \in E | \lim_{m \to \infty} l^{(m)}(e) = \infty \} \). We construct a graph \( G \) as follows. A vertex of \( G \) is a hexagon with at least one edge in \( E_\infty \). There is a dual-edge in \( G \) joining two vertexes if and only if the two hexagons corresponding to the vertexes share an edge in \( E_\infty \). The degree of a vertex of the graph \( G \) can only be 1, 2 or 3. Actually a vertex of degree 1, 2 or 3 is corresponding to the hexagon of type (1), (2) or (3) in Lemma 2.2 respectively.

We smooth the graph \( G \) at vertexes as follows. At a vertex of degree 1, we replace the small neighborhood of the vertex in \( G \) by a short smooth curve tangent to the unique dual-edge incident to the vertex as in Figure 4 (a). At a vertex \( v \) of degree of 2 or 3, every two dual-edges \( \tau_1, \tau_2 \) incident to \( v \) correspond to two edges \( e_1, e_2 \) in a hexagon. If the length of the A-arc adjacent to \( e_1, e_2 \) converges to 0, we replace the small neighborhood of the vertex \( v \) in \( G \) by a short smooth curve tangent to \( \tau_1, \tau_2 \). According to Lemma 2.2, every vertex of degree 2 can be smoothed as in Figure 4 (b) and there are two cases for a vertex of degree 3 according to the lengths of 2 or 3 A-arcs converge to 0 as in Figure 4 (c).

We denote by \( G' \) the graph smoothed at vertexes and the dual-edges of \( G' \) are the dual-edges of \( G \). We claim that there exists a smooth closed curve in \( G' \) such that every dual-edge repeats at most twice in the closed curve. In fact, we give every dual-edge of \( G' \) an arbitrary orientation. Pick up any smooth closed curve in \( G' \) which may contains infinite number of dual-edges. If there exists a dual-edge \( \tau \) repeats with the same orientation in the closed curve, there is another smooth closed curve starting and ending at \( \tau \). By this procedure we can reduce the number of dual-edges of a closed curve. At last we obtain a smooth closed curve in \( G' \) such that every dual-edge repeats at most twice.

This smooth closed curve in \( G' \) corresponds a fundamental edge path or fundamental edge cycle in the ideal triangulation. First assume it is a fundamental edge path \( (H_0, e_1, H_1, ..., e_n, H_n) \). Since the degree of the vertex corresponding to \( H_0 \) (or \( H_n \)) is 1, the lengths of other two edges other than \( e_1 \) (or \( e_n \)) converge to positive finite numbers in the sequence of metric \( l^{(m)} \). By Lemma 2.2(1) the r-coordinate of the A-arc in \( H_0 \) (or \( H_n \)) facing \( e_1 \) (or \( e_n \)) converges
Figure 4: Smooth graph $G$ at a vertex of degree 1 (a), degree 2 (b), degree 3 (c).
to $-\infty$. By the construction of the edge path, the length of A-arc adjacent to $e_i$ and $e_{i+1}$ converges to 0 for $i = 1, \ldots, n - 1$. And we denote $b_i, c_i$ the limit of lengths of A-arcs in $H_i$ facing $e_i, e_{i+1}$ respectively, see Figure 3(a).

Hence

$$z(e_1) = \int_{0}^{c_1/b_1} \cosh^\lambda(t) dt + \int_{0}^{c_1/b_1} \cosh^\lambda(t) dt.$$

For $i = 2, \ldots, n - 1,$

$$z(e_i) = \int_{0}^{b_i-1-c_i-1} \cosh^\lambda(t) dt + \int_{0}^{c_i/b_i} \cosh^\lambda(t) dt.$$

And

$$z(e_n) = \int_{0}^{b_n-1-c_n-1} \cosh^\lambda(t) dt + \int_{0}^{c_0/b_0} \cosh^\lambda(t) dt.$$

Hence

$$\sum_{i=1}^{n} z(e_i) = 2 \int_{0}^{-\infty} \cosh^\lambda(t) dt + \sum_{i=1}^{n-1} (\int_{0}^{c_i/b_i} \cosh^\lambda(t) dt + \int_{0}^{c_0/b_0} \cosh^\lambda(t) dt)$$

$$= -2 \int_{0}^{\infty} \cosh^\lambda(t) dt.$$

This is impossible since $z \in P_\lambda$ must satisfy the condition 2.

If the smooth closed curve in $G'$ corresponds to a fundamental edge cycle $(H_0, e_1, H_1, \ldots, e_n, H_n = H_0)$, the length of A-arc in $H_0$ adjacent to $e_1$ and $e_n$ is 0. Denote $b_0, c_0$ the lengths of A-arcs facing $e_n$ and $e_0$, see Figure 3(b). Thus

$$z(e_1) = \int_{0}^{b_0-c_0} \cosh^\lambda(t) dt + \int_{0}^{c_1/b_1} \cosh^\lambda(t) dt,$$

$$z(e_n) = \int_{0}^{b_n-1-c_n-1} \cosh^\lambda(t) dt + \int_{0}^{c_0/b_0} \cosh^\lambda(t) dt.$$

And as in the case of fundamental edge path, for $i = 2, \ldots, n - 1,$

$$z(e_i) = \int_{0}^{b_i-1-c_i-1} \cosh^\lambda(t) dt + \int_{0}^{c_i/b_i} \cosh^\lambda(t) dt.$$

Hence $\sum_{i=1}^{n} z(e_i) = 0$. This is impossible since $z \in P_\lambda$ must satisfy the condition 3.

We finish the proof of $\Psi_\lambda(T(S)) = P_\lambda$. Since there are only finite many fundamental edge paths or fundamental edge cycles in an ideal triangulation, $\Psi_\lambda(T(S))$
is defined by finite many inequalities in condition 1, 2, 3. Thus it is a open convex polytope.

The statement $\Psi_{\lambda_1}(T(S)) \subset \Psi_{\lambda_2}(T(S)) \subset \Psi_{\lambda_3}(T(S))$ for $\lambda_1 < \lambda_2 < 0 < \lambda_3$ is obvious since the function $\int_0^\infty \cosh^\lambda(t)dt$ is increasing in $\lambda$ and it is $\infty$ when $\lambda \geq 0$.

Since $0 < \cosh^\lambda(t) < \cosh^{-1}(t)$ for $\lambda < -1$ and $\int_0^\infty \cosh^{-1}(t)dt < \infty$, by Lebesgue’s dominated convergence theorem, we have

$$\lim_{\lambda \to -\infty} \int_0^\infty \cosh^\lambda(t)dt = \int_0^\infty \lim_{\lambda \to -\infty} \cosh^\lambda(t)dt = 0.$$ 

Thus the intersection $\bigcap_{\lambda=0}^\infty \Psi_{\lambda}(T(S))$ is the set of points $z \in \mathbb{R}^E$ satisfying $z(e) < 0$ for each edge $e$ and $\sum_{i=1}^n z(e_i) > 0$ for each fundamental edge path $(H_0, e_1, H_1, \ldots, e_n, H_n)$. It is an empty set. 

Acknowledgement

The author would like to thank his advisor, Feng Luo, for suggesting this problem and helpful discussions. This work is partially supported by NSF Grant #0604352.

References

[1] F. Bonahon, *Shearing hyperbolic surfaces, bending pleated surfaces and Thurston’s symplectic form*, Ann. Fac. Sci. Toulouse Math. (6) 5 (1996), no. 2, 233–297.

[2] Peter Buser, *Geometry and spectra of compact Riemann surfaces*, Progress in Mathematics, 106. Birkhäuser Boston, Inc., Boston, MA, 1992.

[3] Y. Imayoshi & M. Taniguchi, *An introduction to Teichmüller spaces*, Translated and revised from the Japanese by the authors, Springer-Verlag, Tokyo, 1992.

[4] Feng Luo, *On Teichmuller space of surfaces with boundary*, arXiv:math.GT/0601364, to appear in Duke Mathematical Journal.

[5] Feng Luo, *Rigidity of polyhedral surfaces*, preprint, 2006.

[6] Feng Luo, *private communication*.

[7] R. C. Penner, *The decorated Teichmüller space of punctured surfaces*, Comm. Math. Phys. 113 (1987), no. 2, 299–339.

[8] R. C. Penner, *Decorated Teichmüller theory of bordered surfaces*, Comm. Anal. Geom. 12 (2004), no. 4, 793–820.