Generalized Duality in Curved String-Backgrounds

Amit Giveon* and Martin Roček†

School of Natural Sciences
Institute for Advanced Study
Princeton, NJ 08540

Abstract
The elements of $O(d, d, \mathbb{Z})$ are shown to be discrete symmetries of the space of curved string backgrounds that are independent of $d$ coordinates. The explicit action of the symmetries on the backgrounds is described. Particular attention is paid to the dilaton transformation. Such symmetries identify different cosmological solutions and other (possibly) singular backgrounds; for example, it is shown that a compact black string is dual to a charged black hole. The extension to the heterotic string is discussed.

*Email: giveon@iassns.bitnet
† Permanent address: ITP, SUNY at Stony Brook, Stony Brook NY 11794-3840.
Email: rocek@dirac.physics.sunysb.edu
1. Introduction

In string theory, conformal field theories (CFT’s) correspond to classical vacua (for a review, see [1]). In general, a CFT can be deformed by truly marginal operators. The space of couplings to these operators is a connected subspace of the space of string vacua. The space of all string vacua has many such components (which may approach the same boundaries). Many tractable cases involve CFT’s that can be described by a sigma-model action, which can be thought of as a string moving in a metric, antisymmetric tensor, and dilaton background. In these theories, some marginal operators generate deformations of the background.

Different string theory backgrounds may correspond to the same CFT. The simplest example of this phenomenon is the $R \to 1/R$ duality for a free boson compactified to a circle [2]. For $d$-dimensional toroidal backgrounds, this duality generalizes to a discrete symmetry group isomorphic to $O(d, d, \mathbb{Z})$ (or $O(d, d + 16, \mathbb{Z})$ for the heterotic string) [3, 4, 5].

In this paper, we discuss the discrete symmetry group acting on the space of curved $D$-dimensional backgrounds that are independent of $d$ coordinates ($d < D$). Such backgrounds include many explicitly known string vacua: black holes [6], $p$-branes [7], cosmological solutions [8, 9], etc.

The discrete symmetries identify vacua with geometries that in general are radically different. This characteristic stringy property is a consequence of the possibility that strings can wind around compact coordinates. The geometries related by the discrete symmetries differ in their identification of the (local) momentum modes and the (non-local) winding modes. Thus, to an observer composed of, e.g., momentum modes, geometries mathematically equivalent as CFT’s will appear physically distinct.

The discrete symmetries in general act on the dilaton; in the flat case, the constant dilaton is transformed to a new constant such that the string coupling remains invariant [10, 5]. In the curved case, the dilaton transforms analogously [11]. The dilaton transformation plays an important role in the understanding of the propagation of a string in the $D = 2$ black hole geometry [12], and in the understanding of duality invariant cosmological solutions [3].

The structure of the moduli space that the discrete symmetries act on is not known in general. In the flat ($D$-dimensional) case, the moduli space is locally
isomorphic to $O(D, D, \mathbb{R})/(O(D, \mathbb{R}))^2$ \cite{13}. Studies of low energy effective actions suggest that there is a moduli subspace of backgrounds independent of $d$ coordinates isomorphic to $O(d, d, \mathbb{R})/G$ \cite{14, 15, 16}, where $G$ is at least the diagonal subgroup $O(d, \mathbb{R})$ of $(O(d, \mathbb{R}))^2$ (the maximal compact subgroup of $O(d, d, \mathbb{R})$). This is consistent with results from string field theory \cite{17}. This moduli subspace is only correct when the $d$ coordinates have the topology of a torus. Though in general the local structure of the moduli space is unknown, we can still identify a discrete symmetry group that acts on it.

The main result of this paper is that there is a discrete symmetry group transforming curved $D$-dimensional backgrounds independent of $d$ coordinates ($d < D$), which is isomorphic to $O(d, d, \mathbb{Z})$. This group is naturally embedded in $O(D, D, \mathbb{Z})$, and acts on the $D$-dimensional (metric + antisymmetric tensor) background by fractional linear transformations, together with a dilaton transformation that preserves the string coupling. Though the explicit transformations are valid only to leading order in the inverse string tension $\alpha'$, we generalize the result of \cite{18} to argue that the symmetry survives to all orders.

The basic example of a nontrivial CFT that has two curved spacetime interpretations related by duality is the $D = 2$ black hole \cite{12}. More general discussions of such duality have appeared in \cite{11, 19}.

The paper is organized as follows: In section 2, we generalize the result of \cite{18} and construct general (conformal) curved $D$-dimensional backgrounds that are independent of $d$ coordinates as abelian quotients \cite{20} of CFT’s with $(D + d)$-dimensional backgrounds. In section 3, we use the construction of section 2 to find the discrete symmetries of the space of these $D$-dimensional backgrounds. In section 4, we explore the group structure of these symmetries, and find a group isomorphic to $O(d, d, \mathbb{Z})$, as well as a simple expression for its action on the backgrounds. In section 5, we focus on the dilaton and its transformations. In section 6, based on analogies with the flat case, we conjecture how these results extend to the heterotic string. In section 7, we study two examples: duality between compact black strings and charged black holes in the bosonic string, and duality between neutral and charged black holes in the heterotic string. In section 8, we close with a few comments and discuss related open problems.
2. Curved target spaces independent of $d$ coordinates as quotients

In this section we construct a general (conformal) curved background in $D$ dimensions that is independent of $d$ coordinates as an abelian quotient of a CFT with a $(D + d)$-dimensional background. Generalizing the result of [18], we start with a CFT with $d$ abelian left handed currents $J^i$ and right handed currents $\bar{J}^i$. The action is

$$S_{D+d} = S_1 + S_a + S[x],$$

$$S_1 = \frac{1}{2\pi} \int d^2z \left[ \partial \theta^i_1 \bar{\partial} \theta^i_1 + \partial \theta^i_2 \bar{\partial} \theta^i_2 + 2\Sigma_{ij}(x) \partial \theta^i_j \bar{\partial} \theta^i_j + \Gamma^1_{ai}(x) \partial x^a \bar{\partial} \theta^i_1 + \Gamma^2_{ia}(x) \partial \theta^i_2 \bar{\partial} x^a \right]$$

$$S_a = \frac{1}{2\pi} \int d^2z \left[ \partial \theta^i_1 \bar{\partial} \theta^i_2 - \partial \theta^i_2 \bar{\partial} \theta^i_1 \right]$$

$$S[x] = \frac{1}{2\pi} \int d^2z \left[ \Gamma_{ab}(x) \partial x^a \bar{\partial} x^b - \frac{1}{4} \Phi(x) R^{(2)} \right],$$

where $i, j = 1, \ldots, d$ and $a, b = d + 1, \ldots, D$, and $\Sigma_{ij}, \Gamma^1_{ai}, \Gamma^2_{ia}, \Gamma_{ab}$ are components of arbitrary $x$-dependent matrices, such that, together with the dilaton $\Phi$, the theory described by the action $S_{D+d}$ is conformal.

The antisymmetric term $S_a$ is (locally) a total derivative, and therefore may give only topological contributions, depending on the periodicity of the coordinates $\theta^i$.

To specify the periodicity, we define

$$\theta^i = \theta^i_2 - \theta^i_1, \quad \bar{\theta}^i = \theta^i_1 + \theta^i_2,$$

such that

$$\theta^i \equiv \theta^i + 2\pi, \quad \bar{\theta}^i \equiv \bar{\theta}^i + 2\pi.$$

In these coordinates, $S_a$ becomes

$$S_a = \frac{1}{2\pi} \int d^2z \frac{1}{2} \left[ \partial \bar{\theta}^i \bar{\partial} \theta^i - \partial \theta^i \bar{\partial} \bar{\theta}^i \right],$$

which takes half-integer values, and therefore contributes to the path-integral.

The action $S_{D+d}$ (2.1) is invariant under the $U(1)^d_L \times U(1)^d_R$ affine symmetry *This term was omitted in [18]; it is needed for gauge invariance of the gauged action, see below.
generated by currents

\[ J^i = \partial \theta_i^1 + \Sigma_{ji} \partial \theta_j^2 + \frac{1}{2} \Gamma_{ai}^1 \partial x^a \]
\[ = \frac{1}{2} \left[ -(I - \Sigma)_{ji} \partial \theta^j + (I + \Sigma)_{ji} \partial \bar{\theta}^j + \Gamma_{ai}^1 \partial x^a \right], \]
\[ J^i = \bar{\partial} \theta_2^i + \Sigma_{ij} \bar{\partial} \theta_1^j + \frac{1}{2} \Gamma_{ia}^2 \bar{\partial} x^a \]
\[ = \frac{1}{2} \left[ (I - \Sigma)_{ij} \bar{\partial} \theta^j + (I + \Sigma)_{ij} \bar{\partial} \bar{\theta}^j + \Gamma_{ia}^2 \bar{\partial} x^a \right]. \]

(2.5)

We choose to gauge the \( d \) anomaly-free axial combinations of the symmetries \(^{[23]}\); other options are generated by discrete symmetries discussed later. The gauged action is \(^{[20]}\)

\[ S_{gauged} = S_{D+d} + \frac{1}{2\pi} \int \! d^2 z \left[ A^i J^i + \bar{A}^i \bar{J}^i + \frac{1}{2} A^i \bar{A}^j (I + \Sigma)_{ij} \right]. \]

(2.6)

The antisymmetric term \( S_a \) \(^{[2.4]}\) is needed to insure gauge invariance under large gauge transformations.\(^\dagger\) Integrating out the gauge fields \( A^i, \bar{A}^i \) gives:

\[ S_D = \frac{1}{2\pi} \int \! d^2 z \left[ E_{IJ}(x) \partial X^I \bar{\partial} X^J - \frac{1}{4} \phi(x) R^{(2)} \right] \]
\[ = \frac{1}{2\pi} \int \! d^2 z \left[ E_{ij}(x) \partial \theta^i \bar{\partial} \bar{\theta}^j + F_{ia}^2(x) \partial \theta^i \bar{\partial} x^a + F_{ai}^1(x) \partial x^a \bar{\partial} \theta^i \right. \]
\[ \left. + F_{ab}(x) \partial x^a \bar{\partial} x^b - \frac{1}{4} \phi(x) R^{(2)} \right], \]

(2.7)

where

\[ \{ X^I \}_{I=1...D} = \{ \theta^i, x^a \}_{i=1...d, a=d+1...D} \]

and

\[ E_{IJ} = G_{IJ} + B_{IJ} = \begin{pmatrix} E_{ij} & F_{ib}^2 \\ F_{ai}^1 & F_{ab} \end{pmatrix}. \]

(2.8)

We have split \( \mathcal{E} \) into its symmetric and antisymmetric parts \( \mathcal{G} \) and \( \mathcal{B} \). The components are

\[ E_{ij} = (I - \Sigma)_{ik}(I + \Sigma)_{kj}^{-1}, \]

(2.9)

\(^1\)In \(^{[18]}\), the designation of axial vs. vector was interchanged.

\(^\dagger\)Indeed, with the choice of \( S_a \) above, minimal coupling \( \partial_\alpha \bar{\theta}^i \to \partial_\alpha \bar{\theta}^i + A^i_\alpha \) gives the correct gauged model.
and

\[ F_{ia}^2 = (I + \Sigma)_{ij} \Gamma_{ja}^1, \quad F_{ai}^1 = -\Gamma_{aj}^1 (I + \Sigma)_{ji}^{-1}, \quad F_{ab} = \Gamma_{ab} - \frac{1}{2} \Gamma_{ai}^1 (I + \Sigma)_{ij}^{-1} \Gamma_{jb}^2. \]  

(2.10)

This \( D \)-dimensional target space background is independent of the \( d \) coordinates \( \theta^i \). Since the relations (2.9, 2.10) can be inverted to solve for \( (\Sigma, \Gamma) \) in terms of \( (E, F) \), it is the most general such background. Following the reasoning of [18], if the original model \( S_{D+d} \) (2.1) is conformal, then \( S_D \) is conformal to one loop order with\(^5\)

\[ \phi = \Phi + \ln \det(I + \Sigma). \]  

(2.11)

This relation is also invertible, and thus a \( D \)-dimensional field theory with a background that is independent of \( d \) coordinates can be described as a quotient of a \((D + d)\)-dimensional field theory with \( d \) chiral currents. If the \( D \)-dimensional theory is conformally invariant, then the \((D + d)\)-dimensional theory is as well [18], and hence any CFT with a background that is independent of \( d \) coordinates can be described as a quotient of a CFT with a \((D + d)\)-dimensional background.

This construction will allow us to understand the discrete symmetries of the moduli space of string vacua in backgrounds that are independent of \( d \) coordinates.

3. Discrete Symmetries

Different string theory backgrounds may describe the same conformal field theory. Here we study discrete symmetries that relate different but equivalent backgrounds \((E(x), F(x), \phi(x)) \) (2.7). We first discuss transformations of \((E(x), F(x), \phi(x)) \) that follow from manifest symmetries of the action \( S_{D+d} \) (2.1). We then combine them with transformations that are manifest symmetries of \( S_D \) (2.7) to find a discrete symmetry group isomorphic to \( O(d, d; \mathbb{Z}) \).

The action \( S_{D+d} \) is invariant under the transformations

\[ \theta_1 \to O_1 \theta_1, \quad \theta_2 \to O_2 \theta_2, \quad O_{1,2} \in O(d, \mathbb{Z}), \]  

(3.1)

\(^5\)However, as noted in [18], higher order corrections to the background that give an exact CFT with a \( D \)-dimensional background exist; these corrections come from the integration measure.
together with
\[ \Sigma \rightarrow O_2 \Sigma O_1^t, \quad \Gamma^1 \rightarrow \Gamma^1 O_1^t, \quad \Gamma^2 \rightarrow O_2 \Gamma^2, \quad \Gamma \rightarrow \Gamma \] (3.2)
such that
\[ \frac{1}{2}(O_1 \pm O_2)_{ij} \in \mathbf{Z}. \] (3.3)
Here \( O(d, \mathbf{Z}) \) is the group of matrices \( O \) with integer entries satisfying \( OO^t = I \).

These symmetries can be found as follows: The action \( S_{D+d} \) is invariant up to total derivatives under \( O(d, \mathbf{R}) \times O(d, \mathbf{R}) \) acting on \((\theta_1, \theta_2)\) as in (3.1), together with (3.2) for the backgrounds. The periodic coordinates \( \theta, \tilde{\theta} \) (2.2) transform as:
\[ \left( \begin{array}{c} \theta \\ \tilde{\theta} \end{array} \right) \rightarrow \frac{1}{2} \left( \begin{array}{cc} O_1 + O_2 & O_1 - O_2 \\ O_1 - O_2 & O_1 + O_2 \end{array} \right) \left( \begin{array}{c} \theta \\ \tilde{\theta} \end{array} \right) \] (3.4)
To preserve the periodicities of \( \theta, \tilde{\theta} \) (2.3), the condition (3.3) must be satisfied. In particular, this implies \( O_{1,2} \in O(d, \mathbf{Z}) \).

The total derivative comes from the transformation of \( S_a \) (2.4), and is
\[ S_a[O_1 \theta_1, O_2 \theta_2] - S_a[\theta_1, \theta_2] = \frac{1}{2\pi} \int d^2 z \left[ M_{ij}(\partial \theta^i \partial \tilde{\theta}^j - \partial \theta^i \partial \tilde{\theta}^j) + N_{ij}(\partial \theta^i \partial \tilde{\theta}^j - \partial \tilde{\theta}^i \partial \theta^j) \right], \] (3.5)
where
\[ M = \frac{1}{4}(O_1^t - O_2^t)(O_1 + O_2), \quad N = \frac{1}{4}(O_1^t - O_2^t)(O_1 - O_2). \] (3.6)
The condition (3.3) implies that \( M_{ij}, N_{ij} \in \mathbf{Z} \), and hence the total derivative is an integer, and does not contribute to the path integral. This concludes the proof that (3.1, 3.2) are symmetries of the action \( S_{D+d} \).

The transformations of the background (3.2) induce transformations of the background \((E(x), F(x), \phi(x))\) (2.9, 2.10, 2.11) in \( S_D \) (2.7):
\[ E \rightarrow E' = (I - O_2 \Sigma O_1^t)(I + O_2 \Sigma O_1^t)^{-1} \] (3.7)
\[ F^1 \rightarrow F^{1'} = -\Gamma^1 (O_1 + O_2 \Sigma)^{-1} \]
\[ F^2 \rightarrow F^{2'} = (O_2^t + \Sigma O_1^t)^{-1} \Gamma^2 \] (3.8)
\[ F \rightarrow F' = \Gamma - \frac{1}{2} \Gamma^1 (O_2^t O_1 + \Sigma)^{-1} \Gamma^2 \]
\[ \phi \rightarrow \phi' = \Phi + \ln \det(I + O_2 \Sigma O_1^t). \quad (3.9) \]

These transformations can be rewritten as

\[ E' = \left[ (O_1 + O_2)E + (O_1 - O_2) \right] \left[ (O_1 - O_2)E + (O_1 + O_2) \right]^{-1} \quad (3.10) \]

\[ F^{1t} = 2F^1 \left[ (O_1 - O_2)E + (O_1 + O_2) \right]^{-1} \]

\[ F^{2t} = \frac{1}{2} \left[ (O_1 + O_2) - E'(O_1 - O_2) \right] F^2 \quad (3.11) \]

\[ F' = F - F^1 \left[ (O_1 - O_2)E + (O_1 + O_2) \right]^{-1}(O_1 - O_2)F^2 \]

\[ \phi' = \phi + \frac{1}{2} \ln \left[ \frac{\det G}{\det G'} \right] = \phi + \frac{1}{2} \ln \left[ \frac{\det G}{\det G'} \right] \quad (3.12) \]

where \( G \) is the background metric as defined in (2.8) and \( G \) is the symmetric part of \( E \) (2.9). We discuss the dilaton transformation (3.12) in detail in section 5.

The transformations (3.1, 3.2) induce a non-trivial action on the currents (2.5), and hence \( S_D'(S_D \text{ with a transformed background } (E', F', \phi')) \) is derived from \( S_D \) (2.1) by a different quotient. For example, some symmetries simply change the sign of \( J_i \) without changing \( \bar{J}_i \); this corresponds to a vector gauging (as opposed to an axial gauging) of the \( i \)'th \( U(1) \).

We now consider the additional transformations that are manifest symmetries of the action \( S_D \) itself. The first are integer “\( \Theta \)”-parameters that shift \( E \):

\[ E_{ij} \rightarrow E_{ij} + \Theta_{ij}, \quad \Theta_{ij} = -\Theta_{ji} \in \mathbb{Z} \]

\[ F^1 \rightarrow F^1, \quad F^2 \rightarrow F^2, \quad F \rightarrow F. \quad (3.13) \]

We refer to the group generated by these transformations as \( \Theta(\mathbb{Z}) \); these are obviously symmetries, as they shift the action \( S_D \) (2.7) by an integer and thus do not contribute to the path integral.

The second type of transformations are given by homogeneous transformations of \( E, F^1, F^2 \) under \( A \in Gl(d, \mathbb{Z}) \):

\[ E \rightarrow A^tEA, \quad F^2 \rightarrow A^tF^2, \quad F^1 \rightarrow F^1A, \quad F \rightarrow F. \quad (3.14) \]
These are obviously symmetries of the theory described by $S_D$, as they generate a change of basis in the space of $\theta$’s that preserves their periodicities.

Neither the $\Theta(Z)$ nor the $Gl(d, Z)$ transformations affect the dilaton, since they do not change the integration measure of the path integral; consequently, they are symmetries of the exact CFT’s, and receive no higher order corrections.

The group generated by all the symmetries discussed is isomorphic to $O(d, d, Z)$. A natural embedding of $O(d, d, Z)$ in $O(D, D, Z)$ acts on the background $E$ by fractional linear transformations, as explained in the next section.

4. The action of $O(d, d, Z)$

We begin by establishing our notation following [14]. The group $O(d, d, \mathbb{R})$ can be represented as $2d \times 2d$-dimensional matrices $g$ preserving the bilinear form $J$:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad (4.1)$$

where $a, b, c, d, I$ are $d \times d$-dimensional matrices, and

$$g^t J g = J \quad \Rightarrow \quad a^t c + c^t a = 0, \quad b^t d + d^t b = 0, \quad a^t d + c^t b = I. \quad (4.2)$$

This has an obvious embedding in $O(D, D, \mathbb{R})$ as

$$\hat{g} = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix} \quad (4.3)$$

where $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ are $D \times D$-dimensional matrices of the form

$$\hat{a} = \begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix}, \quad \hat{b} = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{c} = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{d} = \begin{pmatrix} d & 0 \\ 0 & I \end{pmatrix} \quad (4.4)$$

(here $I$ is the $(D - d) \times (D - d)$-dimensional identity matrix).
We define the action of $\hat{g}$ on $E$ by fractional linear transformations:

$$
\hat{g}(E) = E' = (\hat{a}E + \hat{b})(\hat{c}E + \hat{d})^{-1}
$$

$$
= \begin{pmatrix}
E' \\
F^1(cE + d)^{-1}
\end{pmatrix}
= \begin{pmatrix}
(a - E'c)F^2 \\
F - F^1(cE + d)^{-1}cF^2
\end{pmatrix}
$$

(4.5)

where

$$
E' = (aE + b)(cE + d)^{-1}
$$

(4.6)

is a fractional linear transformation of $E$ under $O(d,d)$.

The group $O(d,d)$ is generated by $[14]$:

Gl($d$):

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = \begin{pmatrix}
A^t & 0 \\
0 & A^{-1}
\end{pmatrix}
\quad \text{s.t.} \quad A \in Gl(d).
$$

(4.7)

$\Theta$:

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = \begin{pmatrix}
I & \Theta \\
0 & I
\end{pmatrix}
\quad \text{s.t.} \quad \Theta = -\Theta^t.
$$

(4.8)

Factorized duality:

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = \begin{pmatrix}
I - e_1 & e_1 \\
e_1 & I - e_1
\end{pmatrix}
\quad \text{s.t.} \quad e_1 = diag(1,0,\ldots,0).
$$

(4.9)

The maximal compact subgroup of $O(d,d)$ is $O(d,d) \times O(d,d)$ embedded as

$$
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
o_1 + o_2 & o_1 - o_2 \\
o_1 - o_2 & o_1 + o_2
\end{pmatrix}
\quad \text{s.t.} \quad o_{1,2} \in O(d).
$$

(4.10)

This subgroup includes factorized duality (4.9).

We now turn from definitions to the actual symmetries of the CFT. These form an $O(d,d,\mathbb{Z})$ discrete subgroup of $O(d,d,\mathbb{R})$ that acts on the background as above. The elements of the subgroup $O(d,d,\mathbb{Z})$ are given by matrices $g$ of the form (4.1,4.2) with integer entries.

Just as in the continuous case, the discrete group is generated by $Gl(d,\mathbb{Z})$, $\Theta(\mathbb{Z})$, and factorized duality; these are given by (4.7,4.8,4.9) with integer entries. The subgroup $O(d,\mathbb{Z}) \times O(d,\mathbb{Z})$ is given by the matrices (4.10), again with integer entries.
Clearly, the $O(d, \mathbb{Z}) \times O(d, \mathbb{Z})$ symmetries (3.10, 3.11), the $\Theta(\mathbb{Z})$ symmetries (3.13), and the $Gl(d, \mathbb{Z})$ symmetries (3.14) that we found in the previous section act on the background by the $O(d, d, \mathbb{Z}) \subset O(D, D, \mathbb{Z})$ fractional linear transformations (4.5) with the matrices $a, b, c, d$ given by (4.10, with $o_{1,2} = O_{1,2}$), (4.8, with $\Theta \in \mathbb{Z}$), and (4.7, with $A \in Gl(d, \mathbb{Z})$), respectively.

These results are compatible with the known discrete symmetries of the space of flat $D$-dimensional toroidal backgrounds [3, 4, 5]. In that case, the $O(d, d, \mathbb{Z})$ symmetries described above are simply a subgroup of the full $O(D, D, \mathbb{Z})$ symmetry group acting as in (4.5), for any $\hat{g} \in O(D, D, \mathbb{Z})$. For curved $D$-dimensional backgrounds, we expect that some large symmetry group acts (analogous to $O(D, D, \mathbb{Z})$); here we have described the $O(d, d, \mathbb{Z})$ subgroup that is associated with a $d$-dimensional toroidal isometry of the background.

In the flat case, the fractional linear transformation is an exact map between equivalent backgrounds; in the curved case, in general one expects higher order corrections to the transformed background [11]. For $\Theta(\mathbb{Z})$ and $Gl(d, \mathbb{Z})$, the transformations are exact; however, the factorized duality receives corrections from the path-integral measure. Nevertheless, because the transformation is exact in the $(D + d)$-dimensional model, we know that non-perturbative correction must exist such that factorized duality is exact.

We close this section with a general remark. The group $O(d, d)$ has two disconnected components (of the generators given in (4.7, 4.8, 4.9), only factorized duality (4.9) has $\det = -1$, and hence is not connected to the identity). Therefore one expects that in general the submoduli space generated by $O(d, d)$ can be disconnected. In the flat case, it happens that the $O(d, d)$ acts on the moduli space by a double covering, and as a result, the moduli space is connected. However, in general one gets two disconnected components of backgrounds with different topologies: for example, in the $D = 2$, $d = 1$ curved case, the two-dimensional black-hole and its dual are in two disconnected components mapped into each other by $O(1, 1)$. The duality transformation identifies the two components as conformal field theories. This is a general feature: factorized duality (4.9) maps one component of the moduli space of backgrounds to the other.

*N = 4 supersymmetry can protect duality transformations, and there are examples where the one loop transformations are exact even in curved backgrounds.

†Factorized duality is similar to mirror symmetry [21], which also identifies two (possibly) disconnected components of moduli space corresponding to backgrounds with different topologies. For $N = 2$ toroidal (orbifold) backgrounds, mirror symmetry and factorized duality are identical [22].
5. The dilaton

To complete the previous discussion of the transformation of the background under the $O(d,d,\mathbb{Z})$ symmetries, we consider the transformation of the dilaton. This is summarized in eq. (3.12), which we derive by proving a theorem: The quantity

\[
\tilde{\phi} = \phi + \frac{1}{2} \ln \text{det} G
\]  

(5.1)

is invariant under $O(d,d,\mathbb{Z})$ transformations. This implies

\[
\phi' = \phi + \frac{1}{2} \ln \left( \frac{\text{det} G}{\text{det} G'} \right).
\]  

(5.2)

The second equality in (3.12) follows from the proof of (5.1) given below.

We begin with the identity

\[
\begin{pmatrix}
G_{ij} & G_{ib} \\
G_{aj} & G_{ab}
\end{pmatrix} = \begin{pmatrix}
G_{ik} & 0 \\
G_{ak} & I_{ac}
\end{pmatrix} \begin{pmatrix}
I_{kj} & (G^{-1})_{kl}G_{lb} \\
0 & G_{cb} - G_{ck}(G^{-1})_{kl}G_{lb}
\end{pmatrix}
\]  

(5.3)

which implies

\[
\text{det}(G) = \text{det}(G_{ij})\text{det}(G_{ab} - G_{ak}(G^{-1})_{kl}G_{lb}).
\]  

(5.4)

We next prove that the following two quantities are separately invariant under $O(d,d,\mathbb{Z})$ transformations:

the invariant fiber dilaton : \( \hat{\phi} = \phi + \frac{1}{2} \ln \text{det}(G_{ij}) \),

(5.5)

the quotient metric : \( G_{ab} - G_{ak}(G^{-1})_{kl}G_{lb} \).

(5.6)

Geometrically, a $D$-dimensional space whose metric is independent of $d$ coordinates $\theta^i$ can be thought of as a bundle $M$ with fiber coordinates $\theta$. The metric on the fiber is $G_{ij}$, and hence we refer to $\hat{\phi}$ as the “invariant fiber dilaton”. The induced metric on the quotient space $M/\{\theta^i\}$ is the quotient metric (5.6).

To prove the invariance of (5.3), we consider the action of the generators of $O(d,d,\mathbb{Z})$ separately. The $\text{Gl}(d,\mathbb{Z})$ and $\Theta(\mathbb{Z})$ transformations trivially leave $\phi$ and
\[ \hat{\phi} = \Phi + \ln \det(I + \Sigma) + \frac{1}{2} \ln \det \frac{1}{2} [(I - \Sigma)(I + \Sigma)^{-1} + (I + \Sigma')(I - \Sigma')] \]

(5.7)

To get the second equality, we need to split \( \ln \det(I + \Sigma) \) as \( \frac{1}{2} \ln \det(I + \Sigma) + \frac{1}{2} \ln \det(I + \Sigma') \). Under \( \Sigma \to O(2) \), \( (I - \Sigma)t \Sigma \to O(1)(I - \Sigma)t \Sigma \), and hence \( \hat{\phi} \) is invariant. This completes the proof of the invariance of the fiber dilaton \( \hat{\phi} \).

The proof of the invariance of (5.6) is entirely parallel; again, the \( \text{Gl}(d, \mathbb{Z}) \) and \( \Theta(\mathbb{Z}) \) transformations trivially leave (5.6) unchanged. To prove invariance under factorized duality, we express (5.6) explicitly in terms of \( \Sigma \), \( \Gamma_1 \), \( \Gamma_2 \), \( \Gamma \) (see 2.9, 2.10):

\[
G_{ab} - G_{ak}(G^{-1})_{kl}G_{lb} = \left[ \frac{1}{2}(\Gamma + \Gamma') - \frac{1}{4}(\Gamma^1(I + \Sigma)^{-1} \Gamma^2 + \Gamma^{2t}(I + \Sigma')^{-1} \Gamma^{1t}) \right. \\
+ \left. \frac{1}{2}(\Gamma^1(I + \Sigma)^{-1} + \Gamma^{2t}(I + \Sigma')^{-1}) \{ (I - \Sigma)(I + \Sigma)^{-1} + (I + \Sigma')(I - \Sigma') \}^{-1} \right. \\
\times \left. \{ (I + \Sigma)^{-1} \Gamma^2 + (I + \Sigma')^{-1} \Gamma^{1t} \} \right]_{ab} \\
= \left[ \frac{1}{2}(\Gamma + \Gamma') + \frac{1}{4}(\Gamma^1(I - \Sigma') \Sigma)^{-1}(\Gamma^{1t} + \Sigma' \Gamma^2) + \Gamma^{2t}(I - \Sigma' \Sigma)^{-1}(\Gamma^2 + \Sigma' \Gamma^{1t}) \right]_{ab}. \\
(5.8)
\]

The final expression is manifestly invariant under (3.2), which completes the proof of the invariance of the quotient metric (5.6).

The theorem (3.2) that \( \tilde{\phi} \) is invariant follows from the invariance of (5.5, 5.6) together with the identity (5.4). This is compatible with results in low-energy effective field theories [13, 16] and string field theory [17, 23], and physically implies that the string coupling constant \( g_{\text{string}}^{-1} = \langle e^{\tilde{\phi}} \rangle = \langle \sqrt{\det G} e^\phi \rangle \) is invariant under \( O(d, d, \mathbb{Z}) \). The invariance of (5.3, 5.6) actually prove that \( \sqrt{\det(G_{ij})} e^\phi \) and the quotient metric (5.3) are separately invariant; of course, this holds only for \( O(d, d, \mathbb{Z}) \), and not the full \( O(D, D, \mathbb{Z}) \) that it is embedded in.
In this section, we make a conjecture about the discrete symmetries of the heterotic string by requiring compatibility with the flat limit and with the bosonic case. We start with a curved heterotic background, which we assume is a consistent, conformally invariant, heterotic string theory, with an action:

$$S_{\text{het}} = \frac{1}{2\pi} \int d^2 z \left[ E_{IJ}(x) \partial X^I \bar{\partial} X^J + A_{IA}(x) \partial X^I \bar{\partial} Y^A + E_{AB} \partial Y^A \bar{\partial} Y^B - \frac{1}{4} \phi(x) R^{(2)} + \text{(fermionic terms)} \right], \quad (6.1)$$

and with the second-class constraints that the $Y^A$ are chiral bosons: $\partial Y^A = 0$. As before, $\{X^I\} = \{\theta^i, x^a\}$, with $i = 1 \ldots d$, and $a = d + 1 \ldots D$. In addition, we have $d_{\text{int}}$ internal chiral bosons $Y^A$: $A = 1 \ldots d_{\text{int}}$. In flat space, we have $d_{\text{int}} = 16$, but more generally, we may find other solutions \([24, 25]\). The spacetime background is given by $(E, \phi)$, as in the bosonic case \((2.8, 2.11)\), and, in addition, the gauge field $A$. We assume that this curved background is independent of the $d$ coordinates $\theta^i$.

The constant internal background is

$$E_{AB} = G_{AB} + B_{AB}, \quad (6.2)$$

where $G_{AB}$ is the metric on the internal lattice (one half the Cartan matrix of the internal symmetry group when the lattice is the root lattice of a group) and $B_{AB}$ is its antisymmetrization \([24]\), i.e., $E_{AB}$ is upper triangular. In the spacetime supersymmetric flat case, the symmetry group is $E_8 \times E_8$. In curved space, this group is in general different \([24, 25]\). Following \([3]\), the expected symmetry group is isomorphic to $O(d, d + d_{\text{int}}, \mathbb{Z}) \subset O(D, D + d_{\text{int}}, \mathbb{Z})$ as in section 4, acting by fractional linear transformations on the $(D + d_{\text{int}}) \times (D + d_{\text{int}})$ dimensional matrix

$$\Xi(x) = \begin{pmatrix} E_{IJ} & \frac{1}{3} A_{IA}(G^{-1})_{AB} A_{JB} & A_{IA} \\ 0 & E_{AB} \end{pmatrix} \quad (6.3)$$

where $G^{-1}$ is the inverse of $G_{AB}$ in \((6.2)\). The matrix $\Xi$ is the embedding of the heterotic background into a bosonic $(D + d_{\text{int}})$-dimensional background, and the Spin(32)/$\mathbb{Z}_2$ string can be described as the $E_8 \times E_8$ string with a particular gauge field background \([27]\).
group $O(D, D + d_{int})$ is the subgroup of $O(D + d_{int}, D + d_{int})$ that preserves the form (B.3). Note that the spacetime metric $G_{IJ}$ (the symmetric part of $E_{IJ}$ in (B.3)) is the quotient metric of the $(D + d_{int})$-dimensional space modulo $\{Y^A\}$ (here, $G_{AB}$ is the fiber metric). This leads to a simple expression for the transformation of the dilaton:

$$
\phi' = \phi + \frac{1}{2} \ln \left( \frac{\det G}{\det G'} \right);
$$

(6.4)

note that this is independent of the gauge fields.

7. Applications

In this section we explore a number of consequences of the discrete symmetries. We first discuss an exact $D = 3$ closed string background that is independent of $d = 2$ coordinates [28]. We then turn to $D = 2$ heterotic backgrounds [24, 25]. In both cases, we find that uncharged black compact objects (strings or holes) are equivalent to charged $D = 2$ black holes.

7.1. The closed string example

The simplest nontrivial example after the $D = 2$ black hole duality [12] is a compact black string given by attaching a circle to every point of the $D = 2$ black hole spacetime ($Sl(2, \mathbb{R})_k/U(1) \times U(1)$). To leading order, the action is

$$
S_{BlackString} = \frac{1}{2\pi} \int d^2z \left[ k(\partial x \bar{\partial} x + \tanh^2 x \partial \theta^1 \partial \bar{\theta}^1) + \alpha \partial \theta^2 \partial \bar{\theta}^2 - \frac{1}{4} \phi(x) R^{(2)} \right],
$$

(7.1)

where

$$
\phi(x) = \phi_0 + \ln(\cosh^2 x).
$$

(7.2)

The first term in $S_{BlackString}$ is the euclidean black hole metric [3], and the second term describes a circle of radius $\sqrt{\alpha}$ attached to each point. This $D = 3$ background is independent of $d = 2$ coordinates $\theta^i$, and is described by the matrix (2.8)

$$
E = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix},
$$

(7.3)

*While writing up our results, we found similar observations in [29].
where

\[ E = \begin{pmatrix} k \tanh^2 x & 0 \\ 0 & \alpha \end{pmatrix}, \quad F = k. \] \quad (7.4)

The group of generalized duality transforations \( O(2, 2, \mathbb{Z}) \) maps this background into other backgrounds that (in general) have different spacetime interpretations.

Since \( F_1 = F_2 = 0 \) in (7.3, cf. 2.8), \( O(2, 2) \) acts on the background by transforming only \( E \) and \( \phi \) as given in (4.6) and (5.2). A particularly interesting point on the trajectory of \( O(2, 2, \mathbb{Z}) \) is reached by acting with the element

\[ g = \begin{pmatrix} I & \Theta \\ 0 & I \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix}, \quad (7.5) \]

where

\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (7.6) \]

This transforms \( E \) and \( \phi \) to

\[
E' = g(E) = \frac{1}{1 + \alpha k \tanh^2 x} \begin{pmatrix} k \tanh^2 x & \alpha k \tanh^2 x \\ -\alpha k \tanh^2 x & \alpha \end{pmatrix} \\
= \frac{1}{\cosh^2 x - \lambda} \begin{pmatrix} k(1 - \lambda) \sinh^2 x & \lambda \sinh^2 x \\ -\lambda \sinh^2 x & \lambda \cosh^2 x / k \end{pmatrix} \]

\[
\phi'(x) = \phi_0 - \ln(1 - \lambda) + \ln(\cosh^2 x - \lambda), \]

where

\[ \lambda = \frac{k \alpha}{1 + k \alpha}. \quad (7.9) \]

This gives an action

\[
S_{\text{Charge}} = \frac{1}{2\pi} \int d^2 z \left[ k(\partial x \bar{\partial} x + \frac{(1 - \lambda) \sinh^2 x}{\cosh^2 x - \lambda} \partial \bar{\theta}^1 \bar{\partial} \theta^1) \\
+ \frac{\lambda \sinh^2 x}{\cosh^2 x - \lambda} (\partial \bar{\theta}^1 \partial \theta^2 - \partial \bar{\theta}^2 \partial \theta^1) + \frac{\lambda \cosh^2 x}{k(\cosh^2 x - \lambda)} \partial \theta^2 \bar{\partial} \theta^2 - \frac{1}{4} \phi'(x) R^{(2)} \right], \]

(7.10)
and, after wick rotating $\theta^1 \to it$, corresponds to a charged black hole of the type found in $[28]$ with mass $M$

$$M = (1 - \lambda) M_0 = \sqrt{\frac{2}{k}} e^{\phi_0}, \quad M_0 = \sqrt{\frac{2}{k}} e^{\phi'_0}, \quad (7.11)$$

and charge

$$Q = \sqrt{\lambda(1 - \lambda)} \frac{2M_0}{k} = \sqrt{\frac{\lambda}{1 - \lambda}} \frac{2M}{k}, \quad (7.12)$$

where $\phi'_0 = \phi_0 - \ln(1 - \lambda)$ is the constant part of the dual dilaton $(7.8)$. The action $(7.10)$ is related to the precise action for the coset $(Sl(2, R)_k \times U(1))/U(1))$ $[28]$ by rescaling $\theta^2 \to k\theta^2$.\[†\]

This shows that the charged black hole is equivalent to the compact black string as a conformal field theory. We now consider some particular limits of this solution.

The limits $\alpha \to 0, \infty$ in $(7.1)$ is the 2D black hole $\times$ a degenerate circle $[3]$, with the two limits related by $R \to 1/R$ duality. These limits correspond to the limits $\lambda \to 0, 1$ in $(7.9)$. In $S_{\text{Charge}} (7.10)$, $\lambda \to 0$ is precisely the 2D black hole $\times$ the same degenerate circle; however, the $\lambda \to 1$ limit gives the action (modulo an integer total derivative term)

$$S_{\lambda \to 1} = \frac{1}{2\pi} \int d^2z \left[ k \partial x \partial x + \frac{1}{k} \coth^2 x \partial \theta^2 \partial \theta^2 + (1 - \lambda) \partial \theta^1 \partial \theta^1 - \frac{1}{4} \phi'(x) R^{(2)} \right], \quad (7.13)$$

$$\phi'(x) = \phi_0 - \ln(1 - \lambda) + \ln(\sinh^2 x),$$

which corresponds to the dual 2D black hole $\times$ a degenerate circle $[12]$. In both cases, the degenerate limits are equivalent as CFT’s to a noncompact black string.

7.2. The heterotic string example

We focus on the example of $[25]$, which is a $D = 2$ heterotic string with internal degrees of freedom taking values in a standard 12-dimensional lattice (the vector weights of $SO(24))$. We find that a family of charged black holes (and naked

\[†\]Our $k$ matches $[1]$, which is $2k$ of $[28]$.\[‡\]More precisely, only spacetime bosons have internal quantum numbers in the vector representations of $SO(24)$; spacetime fermions have internal quantum numbers in the spinor representations of $SO(24)$.\[17\]
singularities) are dual to a neutral one, which is the exact CFT given by the heterotic $D = 2$ black hole \([25]\).

We start with a heterotic $D = 2$ action:

$$S_{\text{het}} = \frac{1}{2\pi} \int d^2z \left[ k(\partial x \bar{\partial} x + \tanh^2 x \partial \theta \bar{\partial} \theta) + \partial Y^A \bar{\partial} Y^A - \frac{1}{4} \phi(x) R^{(2)} + \text{(fermionic terms)} \right], \quad (7.14)$$

where $A = 1, \ldots, 12$, $k = 5/2$ (for criticality), and $\phi = \phi_0 + \ln(cosh^2 x)$. This action describes a neutral heterotic $D = 2$ black hole. The conformal field theory \((7.14)\) corresponds to a background \((6.3)\)

$$\Xi = \begin{pmatrix} k & 0 & 0 \\ 0 & k \tanh^2 x & 0 \\ 0 & 0 & I \end{pmatrix}, \quad (7.15)$$

where the internal background $I$ is the $12 \times 12$ identity matrix corresponding to the vector weights of $SO(24)$. Only a $2 \times 2$ block $E$ of the matrix $\Xi$ is affected by the discrete transformations we discuss in this example:

$$E = \begin{pmatrix} k \tanh^2 x & 0 \\ 0 & 1 \end{pmatrix}. \quad (7.16)$$

By transforming $E$ and $\phi$ with a group element $g_n \in O(1,2,\mathbb{Z}) \subset O(1,13,\mathbb{Z})$ (where $n$ is an arbitrary integer):

$$g_n = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & n \Theta \end{pmatrix} \begin{pmatrix} A_n^t & 0 \\ 0 & A_n^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & n \end{pmatrix} \begin{pmatrix} 1 & n \Theta \\ -n & 1 \Theta \end{pmatrix}, \quad (7.17)$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}, \quad (7.18)$$

one finds

$$g_n(E) = E_n' = \begin{pmatrix} (n^2 + k \tanh^2 x)^{-1} & -2n(n^2 + k \tanh^2 x)^{-1} \\ 0 & 1 \end{pmatrix}, \quad (7.19)$$

18
\[ \phi'(x) = \phi_0 + \ln(n^2 + k) + \ln(\cosh^2 x - \frac{k}{n^2 + k}). \quad (7.20) \]

After rescaling
\[ \theta \to \frac{k + n^2}{\sqrt{k}} t, \quad (7.21) \]
and defining \( r \) to be a linear function of the dilaton \( \phi' \) (7.20),
\[ Qr = \ln(\cosh^2 x - \frac{k}{n^2 + k}), \quad (7.22) \]

where \( Q \) is a constant determined below, the background (7.19) gives rise to an action
\[ S_{\text{Charge}} = \frac{1}{2\pi} \int d^2 z \left[ f(r) \partial t \partial t + f(r)^{-1} \partial r \partial r - A(r) \partial t \partial Y^1 + \partial Y^A \partial Y^A \right. \]
\[ \left. - \frac{1}{4} \phi'(r) R^{(2)} + \text{(fermionic terms)} \right], \quad (7.23) \]
with
\[ f(r) = 1 - 2m e^{-Qr} - q^2 e^{-2Qr}, \quad (7.24) \]
\[ A(r) = nQ + 2qe^{-Qr}, \quad \phi'(r) = Qr + \phi_0 + \ln(n^2 + k), \]

where \( Q = 2/\sqrt{k'} \) is determined by the normalization of \( G_{rr} \) in (7.23), and
\[ 2m = \frac{n^2 - k'}{n^2 + k'}, \quad q = \frac{n\sqrt{k'}}{n^2 + k'}. \quad (7.25) \]

Following [6] we have replaced \( k \) with \( k' = k - 2 = 1/2 \) in (7.25). Wick rotating \( t \to it \), along with \( q \to -iq \) (necessary to maintain hermiticity of the action), the theory (7.23 with \( |n| > 1 \)) describes a \( D = 2 \) charged black hole with mass and charge
\[ M = Q(n^2 - k')e^{\phi_0}, \quad Q = n\sqrt{8}e^{\phi_0}. \quad (7.26) \]

For \( n = -1, 0, 1 \), the theory (7.23) describes a naked singularity.

\[ ^{\text{b}} \text{Recall that } G_{tt} = \Xi_{tt} - \frac{1}{4} A^2 \text{ (see 5.3).} \]
We emphasize that these backgrounds, for all $n$, are different spacetime interpretations of the same CFT: the exact CFT given by the neutral heterotic $D = 2$ black hole.

8. Concluding Remarks and Open Problems

We have shown that $O(d, d, \mathbb{Z})$ acts on the space of backgrounds that are independent of $d$ coordinates. We expect that in general the full symmetry group acting on the space of $D$-dimensional curved backgrounds is larger. Some of these extra symmetry generators can be found by considering quotients of $(D + d)$-dimensional actions that are more general than (2.1); we hope to discuss this somewhere else. Ideally, one would like to find the complete symmetry group for the space of all curved backgrounds.

The $O(d, d, \mathbb{Z})$ subgroup already leads to interesting relations between different geometries. We have illustrated this with charged black hole examples; similar studies in the context of string cosmology may lead to surprising consequences.

Elements of $O(d, d, \mathbb{Z})$ with $\det = -1$ relate backgrounds with (possibly) different topologies. In the flat case, such transformations coincide \cite{22} with mirror symmetry for $N = 2$ superconformal backgrounds \cite{21}. It would be interesting to understand the relation between the two in the general case.

Another open problem is the issue of higher order corrections; this is a problem for the spacetime interpretation of quotients as well as for discrete symmetry transformations.

Acknowledgments: We would like to thank R. Dijkgraaf, C. Nappi, E. Verlinde, E. Witten, and B. Zwiebach for discussions. The work of AG is supported in part by DOE grant No. DE-FG02-90ER40542, and that of MR is supported by the John Simon Guggenheim Foundation.
References

[1] M.B. Green, J.H. Schwarz and E. Witten, “Superstring theory”, Cambridge Univ. Press, Cambridge, 1987.

[2] K. Kikkawa and M. Yamasaki, Phys. Lett. 149B (1984) 357.

[3] A. Giveon, E. Rabinovici and G. Veneziano, Nucl. Phys. B322 (1989) 167.

[4] A. Shapere and F. Wilczek, Nucl. Phys. B320 (1989) 669.

[5] A. Giveon, N. Malkin and E. Rabinovici, Phys. Lett. 220B (1989) 551.

[6] E. Witten, Phys. Rev. D44 (1991) 314.

[7] S.B. Giddings and A. Strominger, “Exact Black Fivebranes in Critical Superstring Theory”, preprint UCSB-TH-91-35, 1991; C.G. Callan, J.A. Harvey, and A. Strominger, “World Sheet Approach to Heterotic Instantons and Solitons”, preprint PUPT-1244, 1991; J.H. Horne and G.T. Horowitz, “Exact Black String Solutions in Three Dimensions”, preprint UCSBTH-91-39, 1991; G. Horowitz and A. Strominger, Nucl. Phys. B360 (1991) 197.

[8] R. Brandenberger and C. Vafa, Nucl. Phys. B316 (1989) 301; M. Mueller, Nucl. Phys. B337 (1990) 37; B. Greene, A. Shapere, C. Vafa and S.T. Yau, Nucl. Phys. B337 (1990) 1.

[9] A.A. Tseytlin and C. Vafa, “Elements of String Cosmology”, preprint HUTP-91/A049.

[10] P. Ginsparg and C. Vafa, Nucl. Phys. B289 (1987) 414.

[11] T. Buscher, Phys. Lett. 159B (1985) 127, Phys. Lett. 194B (1987) 59, Phys. Lett. 201B (1988) 466; E. Smith and J. Polchinski, Phys. Lett. 263B (1991) 59; A.A. Tseytlin, Mod. Phys. Lett. A6 (1991) 1721.

[12] A. Giveon, Mod. Phys. Lett. A6 (1991) 2843; R. Dijkgraaf, E. Verlinde, and H. Verlinde, “String Propagation in a Black Hole Geometry”, IAS preprint IASSNS-HEP-91/22.

[13] K.S. Narain, Phys. Lett. 169B (1986) 41.

[14] A. Giveon, N. Malkin and E. Rabinovici, Phys. Lett. 238B (1990) 57.
[15] A. Giveon and M. Porrati, Phys. Lett. B246 (1990) 54; A. Giveon and M. Porrati, Nucl. Phys. B355 (1991) 422.

[16] G. Veneziano, Phys. Lett. B265 (1991) 287; K.A. Meissner and G. Veneziano, Phys. Lett. B267B (1991) 33; M. Gasperini, J. Maharana and G. Veneziano, “From Trivial to Non-trivial Conformal String Background via $O(d,d)$ Transformations”, preprint CERN-TH-6214/91, 1991.

[17] A. Sen, “$O(d) \times O(d)$ Symmetry of the Space of Cosmological Solutions in String Theory, Scale Factor Duality, and Two Dimensional Black Holes”, preprint IC/91/195, TIFR/TH/91-35, 1991; A. Sen, “Twisted Black p-Brane Solutions in String Theory”, preprint TIFR/TH/91-37, 1991; S.F. Hassan and A. Sen, “Twisting Classical Solutions in Heterotic String Theory”, preprint TIFR/TH/91-40, 1991.

[18] M. Roček and E. Verlinde, “Duality, Quotients, and Currents”, preprint ITP-SB-91-53, IASSNS-HEP-91/68.

[19] E.B. Kiritsis, Mod. Phys. Lett. A6 (1991) 2871; I. Bars, “String Propagation on Black Holes”, USC-91HEP-B3, 1991; I. Bars and K. Sfetsos, “Generalized Duality and Singular Strings in Higher Dimensions”, preprint USC-91/HEP-B5, 1991.

[20] K. Bardakci, E. Rabinovici, and B. Säring, Nucl. Phys. B299 (1988) 151; K. Gawedski and A. Kupianen, Nucl. Phys. B320 (1989) 625.

[21] B.R. Greene and M.R. Plesser, Nucl. Phys. B338 (1990) 15.

[22] A. Giveon and D.-J. Smit, Nucl. Phys. B349 (1991) 168.

[23] T. Kugo and B. Zwiebach, “Target Space Duality as a String Field Symmetry”, preprint to appear.

[24] S. deAlwis, J. Polchinski, and R. Schimmrigk, Phys. Lett. B218B (1989) 449.

[25] M.D. McGuigan, C.R. Nappi and S.A. Yost, “Charged Black Holes in Two-Dimensional String Theory”, preprint IASSNS-HEP-91/57, 1991.

[26] S. Elitzur, E. Gross, E. Rabinovici, and N. Seiberg, Nucl. Phys. B283 (1987) 413.
[27] P. Ginsparg, Phys. Rev. D35 (1987) 648; K.S. Narain, M.H. Sarmadi, and E. Witten, Nucl. Phys. B279 (1987) 369.

[28] N. Ishibashi, M. Li and A.R. Steif, “Two Dimensional Charged Black Holes in String Theory”, preprint UCSBTH-91-28, 1991.

[29] J.H. Horne, G.T. Horowitz and A.R. Steif, “An Equivalence Between Momentum and Charge in String Theory”, preprint UCSBTH-91-53, 1991.