The Distribution Function of the Longest Head Run

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Abstract

In this paper, the open problem of finding a closed analytical expression for the
distribution function of the length of the longest pure head run in coin tosses of
a possibly biased coin is solved by studying the closely related Markov chain of
current head runs.

1 Introduction

The question of longest head runs in coin tosses was posed for the first time by the
Hungarian researcher T.Varga in connection with a teaching experiment that he con-
ducted with secondary school children as Csörgo and Révész explain in [1]. Inspired
by this problem, in 1975 Erdős and Révész [2] published a result that gave a descrip-
tion of the almost sure asymptotic behaviour of the length of longest head runs in coin
tosses. Since then, this topic has widely been studied by many other authors. In 1990
e.g. Schilling [4] found a recursive formula to compute the distribution function of the
length of the longest head run. However, finding an explicit closed analytical expression
for the distribution function of the longest head run is still an open question.

In what follows a brief overview about how the paper is organized will be given. In
Section 2 the notion of coin tosses and of current and longest head runs are formalized,
moreover, notational conventions are fixed that will be used throughout the whole paper.
In Section 3 the main result will be presented, in particular, an explicit closed analytical
expression for the distribution function of the longest head run is presented. The proof
of the main result will be postponed since auxiliary results have to be provided first.
These auxiliary results stem from studying the closely related Markov chain of current
head runs which will be covered by Section 4. The proof of the main result will be given
in Section 5. Finally, in Section 6 we consider some examples and compare the results
with Schillings recursive formula for $F_n(3)$, $n = 0, 1, \ldots, 8$.

2 Preliminaries

Throughout the whole paper the following notions and notational conventions will be
adopted without stating and restating them again.

Coin tosses. Let $\{X_n\}_{n \geq 1}$ be an infinite sequence of independent and identically
Bernoulli distributed random variables with $\mathbb{P}(X_n = 1) = p \in (0, 1)$ for all integers
$n \geq 1$. We think of a sequence of tosses of a possibly biased coin. The event $\{X_n = 1\}$
can be thought of tossing head at the $n$-th coin toss and accordingly, $p$ is the probability

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of tossing head and $q = 1 - p$ the probability of tossing tail at the $n$-th coin toss.

**Current Head Run.** Let $H_n$ denote the length of the current (pure) head run at the $n$-th (i.e. at the current) coin toss. In plain words, $H_n$ is the maximal number of consecutive head tosses counting from the current toss backwards. To include the case $n = 0$ when no coin has been tossed yet, we set $H_0 := 0$. For any integer $n \geq 1$, we set $H_n := 0$ if $X_n = 0$, otherwise, if $X_n = 1$, we set $H_n = 1 + M_n$ where $M_n$ is the largest integer $M$, $0 \leq M \leq n - 1$, such $X_n = X_{n-1} = \cdots = X_{n-M} = 1$. The sequence $\{H_n\}_{n \geq 0}$ of the length of the current head run is a (homogeneous discrete) Markov chain with transition probabilities as displayed in Figure 1 of Section 4.

**Longest Head Run.** Furthermore, let $L_n$ be the length of the longest (pure) head run among the first $n$ coin tosses, i.e. define $L_n := \max\{H_j | 0 \leq j \leq n\}$ for any integer $n \geq 0$.

**Integer and the Non-Integer Part of a Number.** For any number $x$, let $\lfloor x \rfloor$ denote the greatest integer $m$ satisfying $m \leq x < m + 1$ and let $\langle x \rangle := x - \lfloor x \rfloor$ denote the uniquely determined non-integer part of $x$ satisfying $0 \leq \langle x \rangle < 1$.

**Specific Sequence of Functions.** Last, for a given $0 < p < 1$, we define a sequence of real-valued functions $\tilde{F}_n : \mathbb{R} \to \mathbb{R}$ with integer running index $n$ by

$$x \mapsto \tilde{F}_n(x) := 1 - p^x (1 + (n - x)q),$$

where $q := 1 - p$.

**Hungarian convention for distribution functions.** We follow the convention in the Hungarini literature and define for any random variable $Z$ the distribution function of $Z$ as $F(x) := P(Z < x)$ instead of the more common convention $P(Z \leq x)$.

Since the ‘success’ probability $0 < p < 1$ is arbitrary but fixed once and for all, dependences of functions, constants etc. on the parameter $p$ will consistently be omitted throughout the whole paper to avoid overloading formulae with indices and to enhance readability.

### 3 Main Result

**Theorem 3.1 (Longest Head Run Distribution Function).** For any integer $n \geq 0$, let $F_n(k) = P(L_n < k)$ be the distribution function of the longest head run’s length $L_n$. Then, we have

$$F_n(k) = \begin{cases} 
q^n & \text{for } k = 1 \\
\tilde{F}_n(k) & \text{for } 2 \leq k \leq n \text{ and } k > n - \frac{1 - p^k}{aq^x} \\
1 & \text{for } k > n.
\end{cases}$$

Furthermore, the function $\tilde{F}_n : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable and its restriction to half-open interval $(-\infty, n + 1]$ is strictly increasing. It satisfies $\tilde{F}_n(x) < 1$ for all $x \leq n + 1$ and $\tilde{F}_n(x) > 0$ for all $x > n - \frac{1 - p^x}{aq^x}$. 
4 On the Current Head Run

Before proving the main theorem we need to collect some facts about the current pure head run. To this end, consider the transition graph of the Markov chain \( \{ H_n \}_{n \geq 0} \) in Figure 1 below. As the length of the current head run \( H_0 \), when no coin has been tossed yet, must be zero, we start at state 0, i.e. \( H_0 = 0 \). If the current head run has length \( k \) at the \( n \)-th toss, i.e. \( H_n = k \), with the next toss \( n + 1 \) the current head run can reach length \( k + 1 \) with probability \( p \), however, if the next toss is a tail, the length of the current head run is zero (i.e. falling back in state 0) which can happen with probability \( q = 1 - p \). Writing \( p_{ij} = P(H_n = j | H_{n-1} = i) \) as usual, obviously the transition probabilities \( p_{ij} \) for the homogenous Markov chain \( \{ H_n \}_{n \geq 0} \) depicted in Figure 1 for all integers \( i, j \geq 0 \), are given as follows:

\[
(4.1) \quad p_{ij} = \begin{cases} 
  p & \text{if } j = i + 1 \\
  q & \text{if } j = 0 \\
  0 & \text{else}.
\end{cases}
\]

The initial distribution \( \{ \pi_k(0) \}_{k \geq 0} \) is given by \( \pi_0(0) = 1 \) and \( \pi_k(0) = 0 \) for \( k \geq 1 \). One easily verifies that this Markov chain is irreducible, aperiodic and positive recurrent and thus, there is a unique limiting distribution which coincides with the invariant distribution (e.g. see [3]). The irreducibility and aperiodicity are obvious. As the Markov chain is irreducible, for confirming positive recurrence, it suffices to check that an invariant distribution \( \{ \pi_k \}_{k \geq 0} \) exists (e.g. see [3]). One easily verifies that the distribution given by \( \pi_k = qp^k \) for \( k \geq 0 \) is invariant, i.e. it satisfies \( \pi_k = p \pi_{k-1} \) for \( k \geq 1 \) and \( \pi_0 = q \sum_{k \geq 0} \pi_k \).

Theorem 4.1 (Current Head Run Distribution). The Markov chain \( \{ H_n \}_{n \geq 0} \) with transition probabilities given by (4.1) is irreducible, aperiodic and positive recurrent. Let \( \{ \pi_k(n) \}_{k \geq 0} \) be the state distribution at the \( n \)-th toss where \( n \) is a non-negative integer and \( \{ \pi_k \}_{k \geq 0} \) be its limiting distribution. Then the probability that the current head run
has length $k$, where $n$ represents the current toss, is given for all integers $k \geq 0$ by

$$
\pi_k(n) = \mathbb{P}(H_n = k) = \begin{cases} 
qp^k & \text{for } k < n \\
 p^n & \text{for } k = n \\
0 & \text{for } k > n.
\end{cases}
$$

Furthermore, the unique limiting distribution is given by $\pi_k = qp^k$ for all $k \geq 0$.

**Proof.** Straightforward routine computation using the transition probabilities and the Chapman-Kolmogorov equation. \( \square \)

**Remark 4.1.** In particular, for $n > k$ the probability $\pi_k(n)$ that the current head run has length $k$ at the $n$-th toss has already reached its stationary limiting value $\pi_k = qp^k$.

**Theorem 4.2** (Current Head Run Distribution Function). Let $G_n(k) = \mathbb{P}(H_n \leq k)$ be the distribution function of $H_n$. Then for all integers $n \geq 0$ and for all integers $k > 0$ the following holds true

$$
G_n(k-1) = \mathbb{P}(H_n < k) = \begin{cases} 
1 - p^k & \text{for } k \leq n \\
1 & \text{for } k > n.
\end{cases}
$$

**Proof.** The case $k > n$ is obvious as the length of the current head run cannot exceed the current number of tosses. The case $k \leq n$ follows immediately by applying Theorem 4.1 to $\mathbb{P}(H_n < k) = 1 - \mathbb{P}(H_n \geq k) = 1 - \sum_{j=0}^{n-k} \mathbb{P}(H_n = k + j)$ and a straightforward computation using the identity for geometric series $\sum_{k=0}^{n-1} a^k = \frac{1-a^n}{1-a}$, $a \neq 1$. \( \square \)

**Theorem 4.3** (Current Head Run Conditional Probabilities). The conditional probabilities $\varrho_{k,l} := \mathbb{P}(H_{k+1} < k \mid H_{k+l-1} < k, H_{k+l-2} < k, \ldots, H_k < k)$ are given by

$$
\varrho_{k,l} = 1 - \frac{qp^k}{1 - p^k - (l-1)qp^k}
$$

for all positive integers $k, l > 0$ with $1 \leq l \leq n - k$ and $l < \frac{1-p^k}{qp^k}$.

**Proof.** Let $k, l$ be integers $\geq 1$.

Case $l = 1$: consider the tree diagram in Figure 2 and apply Theorems 4.1 and 4.2.

![Probability tree diagram for $\varrho_{k,1} = \mathbb{P}(H_{k+1} < k \mid H_k < k)$](image)

Figure 2: Probability tree diagram for $\varrho_{k,1} = \mathbb{P}(H_{k+1} < k \mid H_k < k)$. 
Resolving $1 - p^k = (1 - p^k)q_{k,1} + p^kq$ for $q_{k,1} = \mathbb{P}(H_{k+1} < k \mid H_k < k)$ yields

$$q_{k,1} = 1 - \frac{qp^k}{1 - p^k} = 1 - \frac{p^k - qp^k}{1 - p^k}$$

as desired.

Case $l = 2$: consider the tree diagram in Figure 3 and apply Theorems 4.1 and 4.2 again and the $q_{k,1}$ from case $l = 1$.

![Figure 3: Probability tree diagram for $q_{k,2} = \mathbb{P}(H_{k+2} < k \mid H_{k+1} < k, H_k < k)$.](image)

Resolving

$$1 - p^k = (1 - p^k) \left(1 - \frac{qp^k}{1 - p^k}\right) q_{k,2} + (1 - p^k) \frac{q^2p^k}{1 - p^k} + qp^k + qp^{k+1}$$

for $q_{k,2}$ yields

$$q_{k,2} = 1 - \frac{p^k - 2qp^k}{1 - p^k - qp^k} = 1 - \frac{qp^k}{1 - p^k - qp^k}$$

as desired. Proceeding in that fashion, one easily verifies that using the previous computed $q_{k,1}, q_{k,2}, \ldots, q_{k,l-1}$ and applying Theorems 4.1 and 4.2 one obtains the subsequent remaining conditional probabilities $q_{k,l}$.

### 5 Proof of the Main Result

Now, we are ready for the **Proof of Theorem 3.1**.

**Proof.** Let $n \geq 0$ be an integer. The statements $F_n(1) = q^n$ and $F_n(k) = 1$ for $k > n$ are obvious.
Next, we show $F_n(k) = \tilde{F}_n(k)$ for all integers $k, n$ with $2 \leq k \leq n$ and $k > n - \frac{1-p^k}{qp^k}$. From the definition of $L_n = \max\{H_j \mid 0 \leq j \leq n\}$ it is clear that for any integer $2 \leq k \leq n$ we have

$$\{L_n < k\} = \bigcap_{j=0}^{n}\{H_j < k\}$$

(5.1)

In plain words, (5.1) says that the length of the longest head run $L_n$ among the first $n$ tosses is less than $k$ if and only if the length of the current head run $H_j$ at any toss $j \leq n$ has never reached $k$.

Applying Theorem 4.3 to (5.1) and using $P(H_n < k) = 1$ for $n < k$ and the general fact that $P(A \mid B) = P(A)$ for any event $B$ with $P(B) = 1$ yields for any integer $k$ with $2 \leq k \leq n$ and $k > n - \frac{1-p^k}{qp^k}$

$$P(L_n < k) = \prod_{l=1}^{n-k} \theta_{k,l} = (1-p^k) \cdot \left(1 - \frac{qp^k}{1-p^k}\right) \cdots \left(1 - \frac{qp^k}{1-p^k-(n-k-1)qp^k}\right)$$

$$= 1 - p^k - (n-k)qp^k = 1 - p^k(1 + (n-k)q) = \tilde{F}_n(k)$$

as desired.

The statements about the functions $\tilde{F}_n(k)$ remains to be shown. Obviously, the functions $\tilde{F}_n(x)$ are infinitely differentiable for any $x \in \mathbb{R}$. Let $\ln(x)$ denote the natural logarithm for $x > 0$. Then, for the first derivative of $\tilde{F}_n(x)$, we have

$$\tilde{F}_n'(x) = \ln\left(\frac{1}{p}\right) p^x \left(1 + (n-x)q\right) + qp^x > \ln\left(\frac{1}{p}\right) p^{x+1} > 0$$

for any $x \leq n + 1$. This shows that $\tilde{F}_n$ is strictly increasing on $(-\infty, n + 1]$. From $\tilde{F}_n(n+1) = 1 - p^{n+2} < 1$ and $\tilde{F}_n(x) > 0$ for $x \leq n + 1$ is follows that $\tilde{F}_n(x) < 1$ for all $x \leq n + 1$. One easily verifies that $\tilde{F}_n(x) > 0$ if and only if $x > n - \frac{1-p^k}{qp^k}$.

### 6 Examples

Now, we consider two examples to gain some trust into this formula. For $p = 1/2$, the condition $k > n - \frac{1-p^k}{qp^k}$ reduces to

(6.1) $k > n + 2 - 2^{k+1}$.

**Example 6.1.** Let us look at the example of $n = 4$ tosses. For $k \geq 2$, we have $k \geq 2 > 6 - 2^{2+1} = -2$. Thus, the condition is meet for all $k \geq 2$ and the formula in Theorem 3.1 can be applied.

$$F_4(k) = P(L_4 < k) = \begin{cases} 
\frac{1}{2^k} & \text{for } k = 1 \\
1 - \frac{1}{2^k} \left(1 + \frac{4-k}{2}\right) & \text{for } 2 \leq k \leq 4 \\
1 & \text{for } k > 4.
\end{cases}$$

In the following table the distribution function for $n = 4$ fair coin tosses are computed.
On the Current and the Longest Head Run

| length $k$ | 1  | 2  | 3  | 4  | 5  |
|------------|----|----|----|----|----|
| $\mathbb{P}(L_4 < k)$ | $\frac{1}{16}$ | $\frac{1}{2}$ | $\frac{13}{16}$ | $\frac{15}{16}$ | 1 |

Table 1: Distribution function for $n = 4$ fair coin tosses.

Example 6.2. In his paper [4], Schilling presents a table with the number of sequences $A_n(3) = 2^n \cdot \mathbb{P}(L_n < 4)$ of length $n$ in which the longest head run for tosses of a fair coin does not exceed 3, the table looks like this:

| $n$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----|----|----|----|----|----|----|----|----|----|
| $A_n(3)$ | 1  | 2  | 4  | 8  | 15 | 29 | 56 | 108 | 208 |

Table 2: Number of sequences $A_n(3)$ in which the longest head run does not exceed length 3. Table taken from Schilling [4], values computed via his recursive formula.

Let us compare this with our explicit formula. We check condition (6.1) for $k = 4$:

$$n < k - 2 + 2^{k+1} = 2 + 2^5 = 34.$$  

This means in theory, we could compute $F_n(4) = \mathbb{P}(L_n < 4)$ up to $n \leq 33$. We compute $F_n(4)$ for $n = 0, 1, \ldots, 8$ matching the number of tosses of Schilling’s table. By virtue of Theorem 3.1 we have

$$F_n(4) = \mathbb{P}(L_n < 4) = \begin{cases} 
1 & \text{for } n < 4 \\
1 - \frac{1}{2^{n}}(1 + \frac{n-4}{2}) & \text{for } n \geq 4 
\end{cases}$$

| $n$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----|----|----|----|----|----|----|----|----|----|
| $F_n(4)$ | 1  | 1  | 1  | 1  | $\frac{15}{27}$ | $\frac{29}{27}$ | $\frac{7}{27}$ | $\frac{27}{27}$ | $\frac{13}{27}$ |

Table 3: Probabilities $F_n(4) = \mathbb{P}(L_n \leq 3)$ computed via Theorem 3.1

One easily verifies that multiplying the probabilities $F_n(4)$ with the factor $2^n$ yields the values $A_n(3)$ of Schilling’s table.
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