Framework for quantum modeling of fiber-optical networks: Part I

(Rev. 0.2.5: suggestions and corrections welcome)

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Abstract

We formulate quantum optics to include frequency dependence in the modeling of optical networks. Entangled light pulses available for quantum cryptography are entangled not only in polarization but also, whether one wants it or not, in frequency. We model effects of the frequency spectrum of faint polarization-entangled light pulses on detection statistics. For instance, we show how polarization entanglement combines with frequency entanglement in the variation of detection statistics with pulse energy.

Attention is paid not only to single-photon light states but also to multi-photon states. These are needed (1) to analyze the dependence of statistics on energy and (2) to help in calibrating fiber couplers, lasers and other devices, even when their desired use is for the generation of single-photon light.

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## Contents

### PART I

1. Introduction .......................................................... 1
   A. Quantum modeling ................................................. 2
   B. Aims in developing a framework ............................... 3
   C. Approach .......................................................... 4

2. Modes, commutation rules, and light states .................... 5
   A. Single-photon state spread over multiple modes .......... 9
   B. Single-mode, multi-photon states .............................. 9
   C. Broad-band coherent states ...................................... 10
   D. General state ..................................................... 11
   E. Density matrices and traces ..................................... 12
   F. Partial traces of light states ................................... 13
   G. Bi-photons: excitation in each of two orthogonal modes .. 15

3. Projections .......................................................... 16
   A. Action of single-mode projections on multi-mode states .. 17
   B. Multi-mode n-photon projector .................................. 17
   C. Number operator .................................................. 18

4. Loss and frequency dispersion .................................. 19
   A. Loss cannot evade “no cloning” ............................... 19

5. Local quantum fields ............................................... 20
   A. Temporally local hermitian fields .............................. 21
   B. Time, space, and dispersion ..................................... 22
   C. Projections in terms of local operators ....................... 23

6. Scattering matrix .................................................... 23
   A. Network without frequency mixing .............................. 24

7. Polarized and entangled light states ............................. 25
   A. Fiber splice (without extraneous modes) ..................... 26
   B. Coupler ........................................................... 27
   C. Entangled states .................................................. 27
| Section                                                                 | Page |
|------------------------------------------------------------------------|------|
| D. Polarization-entangled states                                       | 28   |
| 8. Detection                                                           | 29   |
|   A. Simple examples                                                   | 30   |
|   B. Model of APD detector for quantum cryptography                    | 31   |
|   C. Detection probabilities                                           | 33   |
|   D. Effect of time bounds on detection                                | 36   |
|   E. Detection, energy, and photon subspaces                           | 36   |
|   F. Preceding the APD detector by a beam-splitter                     | 37   |
| 9. Polarization-entangled light for QKD                                | 39   |
|   A. Bi-photon light states                                            | 39   |
|   B. Effect of a beam splitter                                         | 41   |
|   C. Effect of polarization rotation                                   | 41   |
| PART II                                                                |      |
| 10. Modeling polarization-entangled QKD                                | 43   |
|   A. Outcomes and probabilities                                        | 45   |
|   B. Light state                                                       | 48   |
|   C. Energy profile                                                    | 50   |
|   D. Calculation of probabilities                                      | 51   |
|   E. Case I: No frequency entanglement                                 | 56   |
|   F. Case II: Limit of extreme frequency entanglement as $\zeta \to \pm \infty$ | 58   |
|   G. Example numbers                                                   | 59   |
| Appendix A. Background                                                 | 59   |
| Appendix B. Operator Lemmas                                            | 60   |
| Appendix C. Algebra of frequency-entangled operators                   | 64   |
| Appendix D. Fourier transforms in space and time                       | 76   |
| Appendix E. Expansion of light states in tensor products of broad-band | 77   |
|   coherent states                                                      |      |
| Appendix F. MATLAB programs for Section 10                            | 78   |
| References                                                             | 97   |
PART I

1. INTRODUCTION

The complexities of quantum optics, with its multiple integrals over frequency and wave vectors, tempt one to simplify, and indeed the groundbreaking equations that launched quantum key distribution (QKD) were simplified rather drastically, often leaving out altogether the frequency spectrum of the light involved. While on one hand the QKD equations have involved simplifications, on the other hand they invoke concepts of quantum decision theory, little used in quantum optics, such as trace distances between density operators as a measure of their distinguishability. The motivation for putting the complications of frequency spectra back into the equations by which we model the faint light used in QKD comes from recognizing that both in implementing QKD systems and in designing eavesdropping attacks against them, frequency spectra play a crucial role.

This report adapts quantum optics to deal as directly as possible with pulses of weak light propagating through optical fibers. The equations introduced here to model faint light give expression to frequency spectra, including frequency-entanglement; they also define and show examples of relevant partial traces of density operators for entangled, frequency-dependent light, needed to make use of an exceedingly useful relation between entangled-state QKD and QKD implemented without entanglement.

The report grew from notes on techniques, some borrowed, others developed from scratch, needed to model a version of BB84 that uses polarization-entangled light. Polarization-entangled light from available sources is also frequency-entangled, and the driving question was how this frequency entanglement modulates the dependence of polarization-entangled QKD detection probabilities on mean photon number.

Some subsequent papers dealing with frequency effects in polarization-entangled QKD, such as Ref. [1], use the techniques and results of this report, and, in particular use certain convolution integrals that are described in Sec. 10 and investigated in detail in Appendix C, with accompanying MATLAB programs given in Appendix F.
A. Quantum modeling

By definition, quantum modeling invokes equations constrained in form to those of quantum mechanics, expressing a joint probability distribution of (theoretical) outcomes in terms of an initial density operator $\rho$ at time $t = 0$, a hamiltonian evolution operator $H$, and a resolution of the identity consisting of a set of non-negative operators $M_j$ satisfying $\sum_j M_j \leq 1$. These engender a probability of a (theoretical) outcome $j$ [2]:

$$\Pr(j) = \text{Tr}[e^{-iHt/\hbar} \rho e^{iHt/\hbar} M_j].$$  \hspace{1cm} (1.1)

Whether or not one makes it explicit, at $\rho$, $H$, and $M_j$ are functions of parameters that one views as under experimental control; all that a model can say is said in terms of how the probabilities depend, via $\rho$, $H$, and $M_j$, on these parameters. A system of equations for modeling particular devices includes equations that specify properties of the $H$, $\rho$, and $M_j$, thus specializing the probability distribution $\Pr(j)$. The outcome $j$ can be a list of components, e.g. one component for each of several detectors, in which case $\Pr(j)$ is viewed as a joint probability for the components of the outcome.

In choosing quantum equations to model experiments with light, one expresses light by one or another density operator $\rho$, and one expresses detecting devices by operators $M_j$, possibly augmented by probe particles, as discussed in [2]. As is well known, the boundary between preparation of light and its detection is chosen by the modeler, and can be pushed around [3]. Different choices of $\rho$, $H$, or $M_j$ set up different quantum models. Implicitly or explicitly, $\rho$ is a function of variables that express the setting of various knobs on the laser and/or other devices that generate the light, and $M_j$ is a function of variables that express knob settings on the detectors, such as those that control polarizing filters. To claim that a set of equations of quantum mechanics describes an arrangement of devices is to claim that the probabilities calculated from the equations more or less fit relative frequencies of experimental outcomes obtained, for some range of knob settings, from the arrangement of devices.

As discussed in [2], choosing equations to model an experiment takes guesswork, indeed, two layers of guesswork. From experiments with devices and a first layer of guesswork [4, 5], one abstracts experimental relative frequencies that can be compared with modeled probabil-
ities. Demanding an approximate fit to experimental relative frequencies puts a constraint on probabilities of outcomes as functions of knob settings, and hence establishes a property of states and operators that can be judged as fitting an experimental situation. Still, diverse systems of equations can always be found to agree with any given set of experimental relative frequencies and yet disagree among themselves in probabilities that they generate for arrangements of devices not yet explored. For this reason, arriving at equations of quantum mechanics by which to describe the measured behavior of QKD devices requires reaching beyond logic to make what may as well be called a guess. For this reason, the sensible use of equations in modeling QKD is hardly to ‘prove security’ but instead to help in achieving transmission of keys in the face of practical obstacles and to design eavesdropping attacks.

By recognizing an irreducible freedom in choosing systems of equations to describe an arrangement of devices, we can clarify the mathematics of modeling, unencumbered by physical interpretations that are always subject to choice. Besides helping with QKD, the techniques of modeling presented here can also serve other uses of faint light that generate interesting joint detection statistics expressible in quantum mechanics but unknown to classical physics, correlations that allow the invention of new kinds of “cameras” with which to see and respond to the physical world.

B. Aims in developing a framework

Picture an experimental network involving light sources and detectors linked by fibers as shown in Fig. 1, where the blob in the middle can include phase shifters, fiber couplers, and attenuators, as well as conversions from fiber to free space and back again to fiber. Mainly I discuss so-called single-mode fibers; (most of these actually support two polarizations). The various fibers of a network need not be alike; for instance they can vary in their propagation constants and in their attenuation. Although most of the discussion is in terms of fiber, free-space links can also be included. The mathematical framework offered allows modeling the variety of responses encountered experimentally in light detection [2]; in particular, the response to single-photon states need not be binary.
The aim in developing this framework for the analysis of optical networks is this:

1. Provide equations to express violations of Bell inequalities.

2. Provide for modeling networks assembled from smaller pieces, like tinker toys, by splicing fibers of one to fibers of another.

3. Provide for convenient expression of pieces of networks that are free of frequency conversion but that have diverse fibers with diverse propagation constants at any single frequency. I.e provide mathematics convenient for expressing single-frequency modes for networks (or parts of networks) containing fibers that differ in their propagation constants.

C. Approach

We split up the task of modeling a quantum network into the following modules:

1. Develop mathematics for the quantum mechanics of a set of uncoupled, lossless transmission lines, each line expressed by a set of modes, where each mode supports a range of frequencies propagating in two directions, denoted “+” and “−”.

   (a) Corresponding to each mode, introduce a creation operator, and define single-mode quantum states as the vacuum state acted on by superpositions of these operators over some frequency band.
FIG. 2: Dissipation modeled by coupling of desired modes to loss modes.

(b) Extend to many modes.

(c) Define projection operators, each of which corresponds to a subspace of $n$ photon states on a Hilbert space that is a tensor product of some number of modes.

   i. Use lattice of projections to define subspaces.

(d) Define detection operators on such states.

(e) Perform calculations by use of commutation relations.

2. Use scattering theory to approximate interacting transmission lines by operators that convert an in-state on non-interacting lines to an out-state on non-interacting lines. (See Fig. 1.)

3. Model loss and dispersion by introducing coupling of desired modes to extraneous modes, as illustrated in Fig. 2.

4. Model frequency preserving interactions among transmission lines by unitary transformations acting on creation operators for all desired and extraneous modes.

5. Sketch two applications to a network for quantum key distribution, in which these concepts and techniques work together.

2. Modes, Commutation Rules, and Light States

In analogy with classical electromagnetics, we will model a path, such as an optical fiber, as a system of modes, with each mode supporting a continuous range of frequencies. We assume
a vacuum state $|0\rangle$, normalized so that

$$\langle 0 | 0 \rangle = 1. \quad (2.1)$$

Other states are defined by various creation operators acting on the vacuum state. These in turn are defined in terms of more singular single-frequency creation operators, as follows. For modes $a, b, \ldots$, one could introduce annihilation operators $\hat{a}, \hat{b}, \ldots$; however we avoid the clutter of the “hats,” so that, for example, $b$ expresses a mode name in some contexts but in other contexts expresses an annihilation operator for that mode. Each mode is bi-directional; we call one direction “+” and the other “−”. Let $a_+^\dagger(\omega)$ be the creation operator for excitation at an angular frequency $\omega$ in the “+” direction, and let $a_-^\dagger(\omega)$ be the creation operator for propagation in the “−” direction. The operators $a_+$ and $a_-$ are defined for non-negative $\omega$. It is often convenient to combine the “+” and “−” operators into a single operator. Since these are both defined for $\omega \geq 0$, we can define $a(\omega)$ for $-\infty < \omega < \infty$ by

$$a(\omega) = \begin{cases} a_+^\dagger(\omega) & \text{for } \omega > 0, \\ a_-^\dagger(-\omega) & \text{for } \omega < 0. \end{cases} \quad (2.2)$$

For any given mode, a single-photon state is any state of the form

$$a_f^\dagger |0\rangle, \quad (2.3)$$

where the creation operator $a_f^\dagger$ is defined by

$$a_f^\dagger \equiv \int_{-\infty}^{\infty} d\omega \ f(\omega) a_+^\dagger(\omega), \quad (2.4)$$

and $f$ is any square-integrable complex-valued function of $\omega$, normalized so that

$$\int d\omega |f(\omega)|^2 = 1, \quad (2.5)$$

where in this and the following integrals, the integration limits are $-\infty$ and $\infty$ unless otherwise specified; the role of negative frequencies will be discussed shortly. For the given mode, $a_+^\dagger(\omega)$ is the creation operator for excitation at an angular frequency $\omega$. It is the adjoint of an annihilation operator $a(\omega)$, and the commutation relation between the two is

$$[a(\omega), a_+^\dagger(\omega')] = \delta(\omega - \omega'), \quad (2.6)$$

$$[a(\omega), a(\omega')] = 0. \quad (2.7)$$
This relation is a simplification appropriate to fiber modes of the commutation relation in [6]. The annihilation operator is required to satisfy the rule

$$\langle \forall \omega \rangle \ a(\omega)|0\rangle = 0,$$

which from the adjoint of Eq. (2.4) implies that

$$a_f|0\rangle = 0. \quad (2.9)$$

From Eq. (2.6) follows the commutation rule for any functions $f$ and $g$:

$$[a_g, a_f^\dagger] = \int d\omega d\omega' g^*(\omega)f(\omega')\delta(\omega - \omega').$$

$$= \int d\omega' g^*(\omega)f(\omega') \delta(\omega - \omega')$$

$$= \int d\omega g^*(\omega)f(\omega), \quad (2.10)$$

where the asterisk denotes the complex conjugate. This can be written more compactly as

$$[a_g, a_f^\dagger] = (g, f), \quad (2.11)$$

where we define the inner product of functions

$$(g, f) = \int d\omega g^*(\omega)f(\omega). \quad (2.12)$$

From this with normalized functions $g = f$, it follows that $(f, f) = 1$ by Eq. (2.5), and we see that $a_f^\dagger$ and $a_f$ are boson creation and annihilation operators satisfying $[a_f, a_f^\dagger] = 1$. As a result, we can calculate the norm of a single-photon state $a_f^\dagger|0\rangle$ to be

$$\|a_f^\dagger|0\rangle\| = (\langle 0|a_f^\dagger a_f|0\rangle)^{1/2}$$

$$= [(\langle 0|(a_f^\dagger a_f + 1)|0\rangle)]^{1/2} = \langle 0|0\rangle^{1/2} = 1,$$

where the next-to-last equality follows from Eq. (2.9). For probabilities to make sense, we must require a finite inner product on quantum states, which requires normalizable states. This makes some bandwidth necessary: there can be no normalizable states at a single frequency. Because $a_f^\dagger(\omega)|0\rangle$ has no norm, we call $a_f^\dagger(\omega)$ an improper operator.
The Hamiltonian operator for the mode (without zero-point energy) is

$$H = \hbar \int d\omega |\omega| a^\dagger(\omega)a(\omega). \quad (2.14)$$

Example 1: The expectation energy of a single-photon state $a_f^\dagger|0\rangle$ defined by the energy operator of Eq. (2.14) is:

$$\langle 0|a_f H a_f^\dagger|0\rangle = \hbar \langle 0|\int \int \int d\omega_1 d\omega_2 d\omega_3 f^*(\omega_1)a(\omega_1)|\omega_2|a^\dagger(\omega_2)a(\omega_2)f(\omega_3)a^\dagger(\omega_3)|0\rangle$$

$$= \hbar \int d\omega |\omega| |f(\omega)|^2, \quad (2.15)$$

where the second equality is obtained using Eq. (2.6) and integrating out the $\delta$-functions. Thus if $f(\omega)$ is concentrated around some $\omega_0$, one finds by this prescription a photon energy of about $\hbar|\omega_0|$.

Commutation rules enable the calculation of probabilities of the form of Eq. (1.1). For instance, for a pure state, the right-hand side of Eq. (1.1) takes the form

$$\langle 0|\int d\omega_1 \cdots d\omega_n g(\omega_1, \ldots, \omega_n) Pol(a_i(\omega_j), a_i^\dagger(\omega_j), b_\ell(\omega_k), b_\ell^\dagger(\omega_k), \ldots)|0\rangle, \quad (2.16)$$

where $Pol$ is a polynomial in creation and annihilation operators subject to $\delta$-function commutation relations, and every annihilation operator acting from the left on the vacuum state $|0\rangle$ gives 0, as does every creation operator acting on the right of $\langle 0|$. The standard method of evaluating such a probability is to use the commutation relations to put $Pol$ in normal order. In the cases of interest, every power of every creation operator is paired with the same power of the corresponding annihilation operator, so that the only terms that contribute after the commutations have the form

$$\langle 0|\int d\omega_1 \cdots d\omega_n g(\omega_1, \ldots, \omega_n)Q(\omega_1, \ldots, \omega_n)|0\rangle$$

$$= \int d\omega_1 \cdots d\omega_n g(\omega_1, \ldots, \omega_n)Q(\omega_1, \ldots, \omega_n), \quad (2.17)$$

where $Q$ is a sum of products of $\delta$-functions, and the equality follows from Eq. (2.1).
A. Single-photon state spread over multiple modes

Two modes $a$ and $b$ are called orthogonal if and only if $[a, b^\dagger] = [a, b] = 0$. Superpositions of single-photon states across orthogonal modes $a$ and $b$ have the form

$$ (c_1 a_f + c_2 b_g)^\dagger |0\rangle, \quad (2.18) $$

where $\sum_j |c_j|^2 = 1$.

B. Single-mode, multi-photon states

To construct the most general single-mode $n$-photon states, we introduce notation for multi-photon operators in which there are as many frequency variables as there are operator factors:

$$ h: a^n \stackrel{\text{def}}{=} \int \cdots \int d\omega_1 \cdots d\omega_n h(\omega_1, \ldots, \omega_n) a^\dagger(\omega_1) \cdots a^\dagger(\omega_n). \quad (2.19) $$

Because the creation operators commute with one another, all that matters about $h$ is the part of it symmetric under interchange of arguments, denoted

$$ S(\omega_1, \ldots, \omega_n) h(\omega_1, \ldots, \omega_n) \stackrel{\text{def}}{=} \frac{1}{n!} \sum_{\pi \in S_n} h(\omega_{\pi 1}, \ldots, \omega_{\pi n}), \quad (2.20) $$

where $S_n$ denotes the permutation group of order $n$. Thus we have

$$ h: a^n = S(\omega_1, \ldots, \omega_n) h: a^n. \quad (2.21) $$

The normalization condition

$$ \int \cdots \int d\omega_1 \cdots d\omega_n |S(\omega_1, \ldots, \omega_n) h(\omega_1, \ldots, \omega_n)|^2 = 1 \quad (2.22) $$

assures unit norm for the single-mode $n$-photon state

$$ \frac{1}{\sqrt{n!}} (h: a^n) |0\rangle. \quad (2.23) $$

The adjoint works according to

$$ [(h: a^n) |0\rangle]^\dagger = (0) (h^\ast: a^n). \quad (2.24) $$

9
1. **Inner product of two single-mode, multi-photon states**

Consider two modes that need not be orthogonal, such as two linearly polarized modes \(a\) and \(b\) with an angle \(\theta\) between them, so the commutation relation between them is \([b(\omega), a^\dagger(\omega')] = \cos \theta \delta(\omega - \omega')\). For symmetric functions \(h\) of \(n\) arguments and \(h'\) of \(m\) arguments, the usual manipulations show that

\[
\frac{1}{\sqrt{n!m!}} \langle 0| (h'^*: b^m)(h : a^{\dagger n})|0\rangle = \langle \mathcal{S}(\omega_1, \ldots, \omega_m) h', \mathcal{S}(\omega_1, \ldots, \omega_n) h \rangle \cos^n \theta, \tag{2.25}
\]

where we define the inner product of the multi-variable functions \(h\) and \(h'\) as

\[
(h', h) = \begin{cases} 
0, & \text{if } h \text{ and } h' \text{ are unlike in number of arguments}, \\
\int \cdots \int d\omega_1 \cdots d\omega_n h'^* n \omega_n (\omega_1, \ldots, \omega_n) h_n(\omega_1, \ldots, \omega_n), & \text{otherwise}.
\end{cases} \tag{2.26}
\]

Thus for \(h = h'\) and \(h\) normalized per Eq. (2.22), the inner product of the two \(n\)-photon states is just \(\cos^n \theta\).

2. **Energy of single-mode, \(n\)-photon state**

For \(h\) normalized per Eq. (2.22) and the energy operator \(H\) of Eq. (2.14), the expectation energy for the state \(n!^{-1/2} (h : a^{\dagger n})|0\rangle\) is

\[
\frac{1}{n!} \langle 0| (h^*: a^n) H (h : a^{\dagger n})|0\rangle = n \hbar \omega_h, \tag{2.27}
\]

where we define an average angular frequency

\[
\omega_h \overset{\text{def}}{=} \int \cdots \int d\omega_1 \cdots d\omega_n |\omega_1| |\mathcal{S}(\omega_1, \ldots, \omega_n) h(\omega_1, \ldots, \omega_n)|^2. \tag{2.28}
\]

C. **Broad-band coherent states**

As a special case of a two-photon state, one can choose any normalized function \(f\) and define \(h(\omega_1, \omega_2) = 2^{-1/2} f(\omega_1) f(\omega_2)\) to produce a two-photon state \(2^{-1/2} (a_{f}^\dagger)^2|0\rangle\). This immediately generalizes to higher powers of \(a_{f}^\dagger\), leading to a ‘broad-band’ coherent state. From Eq. (2.10) we see that \(a_{f}^\dagger\) and \(a_{f}\) are boson creation and annihilation operators satisfying
$[a_f, a_f^\dagger] = 1$, so that Louisell’s discussion of coherent states [7, Sec. 3.2] applies to them. Hence we can define $n$-photon $f$-states by powers of $a_f^\dagger$ acting on the vacuum state $n$ times, producing the normalized state $(n!)^{-1/2}(a_f^\dagger)^n|0\rangle$. Coherent $f$-states can be defined as superpositions of these states, just as in Louisell [7, p. 104]:

$$|\alpha, a_f\rangle \overset{\text{def}}{=} \exp\left(-\frac{1}{2} |\alpha|^2\right) \sum_{n=0}^{\infty} (n!)^{-1/2} (\alpha a_f^\dagger)^n |0\rangle$$

This is an example of a calculation that proceeds exactly as if frequency dependence were collapsed, so that $a_f$ works like a simple oscillator annihilation operator. In contrast, frequency dependence matters in the commutation rule

$$[a_g, a_f^\dagger] = \int d\omega g^* (\omega) f(\omega),$$

which follows from Eqs. (2.6) and (2.7).

To calculate the energy of the coherent state, note first that the commutation rules Eqs. (2.6) and (2.7) imply for the coherent state defined by Eq. (2.29)

$$a(\omega)|\alpha, a_f\rangle = \alpha f(\omega)|\alpha, a_f\rangle.$$  

From this and Eqs. (2.14) and (2.5) follows the expectation energy

$$\langle \alpha, a_f | H | \alpha, a_f \rangle = \hbar \omega_f |\alpha|^2,$$

where we have defined a mean absolute angular frequency

$$\omega_f \overset{\text{def}}{=} \int d\omega |\omega||f(\omega)|^2.$$  

D. General state

The general state is a sum of terms, not necessarily normalized, each of the form

$$|\psi\rangle = f: \prod_{j=1}^{J} a_{j m_j}^\dagger |0\rangle.$$  

11
Symmetries under interchange of variables are best expressed in a notation intermediate between that of writing out the integrals and the compact “colon” notation. We expand the shorthand \( f: a^{tn} \) to \( f(\omega): a^{tn}(\omega) \), understanding that with an exponent \( n \) involved, \( (\omega) \) is short for a list of \( n \) frequency variables and \( a^{tn}(\omega) \) is short for \( a^{\dagger}(\omega_1) \cdots a^{\dagger}(\omega_n) \). Written in this notation, Eq. (2.34) becomes

\[
|\psi\rangle = f(\omega_1, \ldots , \omega_J): \prod_{j=1}^{J} a^{\dagger n_j}(\omega_j)|0\rangle; \tag{2.35}
\]

where \( (\omega_j) \) is a list of \( n_j \) frequency variables. Symmetry under interchange of creation operators implies, as in Eq. (2.21),

\[
f(\omega): a^{\dagger n}(\omega) = [S(\omega)f(\omega)]: a^{\dagger n}(\omega), \tag{2.36}
\]

where the symmetry operator \( S \) is defined in Eq. (2.20); this generalizes to

\[
f(\omega_1, \ldots , \omega_J): \prod_{j=1}^{J} a^{\dagger n_j}(\omega_j)|0\rangle = [S(\omega_1) \cdots S(\omega_J)f(\omega_1, \ldots , \omega_J)]: \prod_{j=1}^{J} a^{\dagger n_j}(\omega_j)|0\rangle. \tag{2.37}
\]

Sometimes we write such expressions a little more compactly, using the convention that \( \omega_n = (\omega_1, \ldots , \omega_J) \) as

\[
f(\omega_n): \prod_{j=1}^{J} a^{\dagger n_j}(\omega_j)|0\rangle = \left[ \prod_{j=1}^{J} S(\omega_j) \right] f(\omega_n) : \prod_{j=1}^{J} a^{\dagger n_j}(\omega_j)|0\rangle. \tag{2.38}
\]

We can distinguish one set of modes from another by replacing some of the \( a_j \) by other labels, such as \( b_j \). For example, \( a \)-modes can refer to Alice while \( b \)-modes refer to Bob. This leads to a general term in a state expansion of the form

\[
|\psi\rangle = f(\omega_n, \bar{\omega}_n): \left( \prod_{j=1}^{J} a^{\dagger n_j}(\omega_j) \right) \left( \prod_{k=1}^{K} b^{\dagger m_k}(\bar{\omega}_k) \right)|0\rangle. \tag{2.39}
\]

E. Density matrices and traces

A density matrix is a sum

\[
\rho = \sum_n w_n|\Psi(n)\rangle \langle \Psi(n)|, \tag{2.40}
\]
where the $|\Psi(n)\rangle$ are unit vectors, $w_n \geq 0$, and $\sum_n w_n = 1$; or an integral

$$\rho = \int w(u) \, du \, |\Psi(u)\rangle\langle\Psi(u)|,$$

with the $|\Psi(u)\rangle$ unit vectors, $w(u) \geq 0$ and $\int du \, w(u) = 1$. More generally, a density matrix can be any convex sum of these discrete and continuous types.

We need the trace of a density operator multiplied by a bounded operator $M$. Although in the context of infinite-dimensional spaces the trace is sometimes defined only for positive operators [8, 9], we want to apply a trace to terms that occur when a pure-state density operator is expanded (i.e., $|\Psi\rangle\langle\Psi| = \sum_{m,n} c_m c_n^* |\Psi_m\rangle\langle\Psi_n|$. For this we define

$$\text{Tr}[(|\psi\rangle\langle\phi|) M] = \langle\phi|M|\psi\rangle,$$

and define more general traces by linearity. We then find

$$\text{Tr}(\rho M) \overset{\text{def}}{=} \sum_n w_n \langle \Psi(n) | M | \Psi(n) \rangle \quad \text{(discrete case)},$$

$$\text{Tr}(\rho M) \overset{\text{def}}{=} \int \, du \, w(u) \langle \Psi(u) | M | \Psi(u) \rangle \quad \text{(continuous case)}.$$

**F. Partial traces of light states**

Probabilities for detection of light states often involve partial traces, defined for an operator on a tensor-product space $\mathcal{H}_A \otimes \mathcal{H}_B$ by linearity from the following special case. For $|a\rangle, |a'\rangle \in \mathcal{H}_A$ and $|b\rangle, |b'\rangle \in \mathcal{H}_B$,

$$\text{Tr}_B(|a\rangle|b\rangle \langle b'|\langle a'|) = (\langle b'|b\rangle)|a\rangle\langle a'|,$$

$$\text{Tr}_A(|a\rangle|b\rangle \langle b'|\langle a'|) = (\langle a'|a\rangle)|b\rangle\langle b'|.$$ (2.44)

From this it follows that for any bounded operator $M_A$ that acts only on $\mathcal{H}_A$, we have

$$\text{Tr}_A[(|a\rangle|b\rangle \langle b'|\langle a'|) M_A] = (\langle a'|M_A|a\rangle)|b\rangle\langle b'|.$$ (2.45)

Here is an example. If a detection operator is of the form $M_{A,j} \otimes 1_B$, then

$$\text{Tr}_{AB}[\rho_{AB}(M_{A,j} \otimes 1_B)] = \text{Tr}_A(\rho M_{A,j}).$$ (2.46)
\[ \rho' = \text{Tr}_B(\rho_{AB}). \] (2.47)

The partial trace over \( b \)-modes of any light state \( \rho \) is calculated from the commutation relations applied to the inner product of \( b \)-mode factors in the usual way, supported by the notation developed in the preceding subsection. For example, one can deal with the case of a single \( a \)-mode and a single \( b \)-mode as follows. Suppose we have

\[ |\psi_{mn}\rangle = \int d\omega \ d\bar{\omega} \ g(\omega, \bar{\omega}) a^\dagger_m(\omega) b^\dagger_n(\bar{\omega}) |0\rangle, \] (2.48)

where we abbreviate \( \omega = \omega_1, \ldots, \omega_m, \ \bar{\omega} = \bar{\omega}_1, \ldots, \bar{\omega}_n, \ a^\dagger_m(\omega) = a^\dagger(\omega_1) \cdots a^\dagger(\omega_m), \) etc.

Similarly, suppose

\[ |\phi_{m'n'}\rangle = \int d\omega' \ d\bar{\omega}' h(\omega', \bar{\omega}') a^\dagger_{m'}(\omega') b^\dagger_{n'}(\bar{\omega}') |0\rangle. \] (2.49)

Then, for an operator \( M_j(a) \) that commutes with \( b(\bar{\omega}) \) and \( b^\dagger(\bar{\omega}) \),

\[ \text{Tr}_{ab}[|\psi_{mn}\rangle \langle \phi_{m'n'}| M_j(a)] = \text{Tr}_a[\rho' M_j(a)], \] (2.50)

where

\[ \rho' = \text{Tr}_b(|\psi_{mn}\rangle \langle \phi_{m'n'}|) \]
\[ = \int d\omega d\omega' \left[ \int d\bar{\omega} d\bar{\omega}' g(\omega, \bar{\omega}) h^*(\omega', \bar{\omega}) \langle 0_b | b^\dagger(\bar{\omega}') b^\dagger(\bar{\omega}) |0_b \rangle \right] a^\dagger_m(\omega) |0_a\rangle \langle 0_a | a^\dagger_m(\omega') \]
\[ = n! \delta_{m'n'} \int d\omega d\omega' \left[ \int d\bar{\omega} g(\omega, \bar{\omega}) S(\bar{\omega}) h^* \langle 0_a | a^\dagger_{m'}(\omega') a^\dagger_m(\omega) |0_a \rangle \right] b^\dagger_{n'}(\bar{\omega}') |0\rangle \langle 0 | b^\dagger_{n'}(\bar{\omega}') \] (2.51)

where the second equality follows from Lemma (B16) of Appendix B applied to the \( b \)-modes.

Similarly, one calculates for the trace over the \( a \)-modes

\[ \text{Tr}_a(|\psi_{mn}\rangle \langle \phi_{m'n'}|) \]
\[ = \int d\bar{\omega} d\bar{\omega}' \left[ \int d\omega d\omega' g(\omega, \bar{\omega}) h^*(\omega', \bar{\omega}) \langle 0_a | a^\dagger_m(\omega') a^\dagger_{m'}(\omega) |0_a \rangle \right] b^\dagger_{n}(\bar{\omega}) |0_b\rangle \langle 0_b | b^\dagger_{n}(\bar{\omega}) \]
\[ = m! \delta_{mm'} \int d\omega d\omega' \left[ \int d\omega' g(\omega, \omega') S(\omega) h^* \langle 0_a | a^\dagger_{n'}(\omega') a^\dagger_n(\omega) |0_a \rangle \right] b^\dagger_{n}(\bar{\omega}) |0\rangle \langle 0 | b^\dagger_{n}(\bar{\omega}). \] (2.52)

The more general case involves complications, but here it is. Let \( n = \sum_{j=1}^{J} n_j \) and \( m = \sum_{k=1}^{K} m_k \) and similarly for primed quantities; let

\[ |\psi\rangle = h(\omega_n, \bar{\omega}_m): \left( \prod_{j=1}^{J} a^\dagger_{n_j}(\omega_j) \right) \left( \prod_{k=1}^{K} b^\dagger_{m_k}(\bar{\omega}_k) \right) |0\rangle, \]
\[ |\psi'\rangle = h'(\omega'_n, \bar{\omega}'_m): \left( \prod_{j=1}^{J'} a^\dagger_{n'_j}(\omega'_j) \right) \left( \prod_{k=1}^{K'} b^\dagger_{m'_k}(\bar{\omega}'_k) \right) |0\rangle. \] (2.53)
On expanding the ‘colon’ notation and abbreviating using $d\omega_n$ for $d\omega_1 d\omega_2 \cdots d\omega_J$ and $d\tilde{\omega}_m$ for $d\tilde{\omega}_1 d\tilde{\omega}_2 \cdots d\tilde{\omega}_K$, one finds

$$\text{Tr}_a (|\psi\rangle\langle\psi'|)$$

$$= \int d\tilde{\omega}_m d\tilde{\omega}'_{m'} \left\{ \int d\omega_n d\omega'_{n'} h(\omega_n, \tilde{\omega}_m) h^*(\omega'_{n'}, \tilde{\omega}'_{m'}) \right\}$$

$$\langle 0_a | \left( \prod_{j=1}^{J'} a_j^{n_j'}(\omega_j') \right) \left( \prod_{j=1}^{J} a_j^{n_j}(\omega_j) \right) | 0_a \rangle \left( \prod_{k=1}^{K'} b_k^{m_k'}(\tilde{\omega}_k') \right)$$

$$\delta_{J,J'} \left( \prod_{j=1}^{J} \delta_{n_j,n_j'} \right) \int d\omega_n d\omega'_{n'} \left\{ \int d\omega_n h(\omega_n, \tilde{\omega}_m) \left( \prod_{j=1}^{J} S(\omega_j) \right) h^*(\omega_n, \tilde{\omega}'_{m'}) \right\}$$

$$\times \left( \prod_{k=1}^{K} b_k^{m_k}(\tilde{\omega}_k) \right) | 0_b \rangle \langle 0_b | \left( \prod_{k=1}^{K'} b_k^{m_k'}(\tilde{\omega}_k') \right). \quad (2.54)$$

The last equation follows from Lemma (B15) of Appendix B.] Because the $b$-creation operators commute among themselves, as do the $b$-annihilation operators, there is one more symmetry:

$$\text{Tr}_a (|\psi\rangle\langle\psi'|)$$

$$= \delta_{J,J'} \left( \prod_{j=1}^{J} \delta_{n_j,n_j'} \right) \int d\omega_n d\omega'_{n'} \left\{ \int d\omega_n \left[ \left( \prod_{k=1}^{K} S(\tilde{\omega}_k) \right) \left( \prod_{j=1}^{J} S(\omega_j) \right) h(\omega_n, \tilde{\omega}_m) \right] \right\}$$

$$\times \left( \prod_{k=1}^{K} S(\tilde{\omega}_k') \right) \left( \prod_{j=1}^{J} S(\omega_j) \right) h^*(\omega_n, \tilde{\omega}'_{m'}) \left( \prod_{k=1}^{K'} b_k^{m_k}(\tilde{\omega}_k) \right)$$

$$\prod_{k=1}^{K} b_k^{m_k'}(\tilde{\omega}_k'). \quad (2.55)$$

G. Bi-photons: excitation in each of two orthogonal modes

For a state that exhibits a single photon in each of two orthogonal modes, $a$ and $b$, whether in a single fiber or in different fibers, the general form is

$$\langle h:a^\dagger b^\dagger |0\rangle, \quad (2.56)$$

where we extend our notation by defining

$$\langle h:a^\dagger b^\dagger \rangle \overset{\text{def}}{=} \int d\omega d\tilde{\omega} h(\omega, \tilde{\omega}) a^\dagger(\omega) b^\dagger(\tilde{\omega}), \quad (2.57)$$
with the normalization requirement that
\[ 1 = \int \int d\omega \, d\tilde{\omega} \, |h(\omega, \tilde{\omega})|^2. \] (2.58)

Note: the normalization requirement rules out \( h(\omega, \tilde{\omega}) \) of the form \( f(\omega)\delta(\omega - \tilde{\omega}) \).

Other states, including multi-mode, multi-photon states will be introduced after we have the machinery of projections.

3. PROJECTIONS

To deal efficiently with polarization-entangled states, we need to characterize subspaces of states by the projections that leave them invariant. These are provided here. It is instructive to compare and contrast the projections for the function space of states used here with the case of a single oscillator with its basis states \( |n\rangle, n = 0, 1, \ldots \) [7]. With our range of frequencies there are countless 1-photon states (and countless \( n \)-photon states), as described in Sec. 2 B, in contrast to the single \( n \)-photon state \( |n\rangle \) for given \( n \) of an oscillator; however, there is a one-to-one correspondence between a set of projection operators for photon number and the projections \( |n\rangle\langle n| \) for the oscillator.

For both the single oscillator and the states that are characterized by functions of frequency, one writes \( |0\rangle\langle 0| \) for the vacuum projector. To the oscillator projection \( |1\rangle\langle 1| \) corresponds the projection that leaves invariant single-photon states while killing all states of more or fewer photons:
\[ P_1(a) = \int d\omega \, a^\dagger(\omega)|0\rangle\langle 0|a(\omega). \] (3.1)

It is easy to check that for any \( f \) one gets \( P_1(a)(a^\dagger_f|0\rangle) = a^\dagger_f|0\rangle \), and that for any of the \( n \)-photon states \( |\psi_n\rangle \) with \( n \neq 1 \), one finds \( P_1(a)|\psi_n\rangle = 0 \). Similarly the projection for two-photon states is
\[ P_2(a) = \frac{1}{2} \int \int d\omega_1 \, d\omega_2 \, a^\dagger(\omega_1)a^\dagger(\omega_2)|0\rangle\langle 0|a(\omega_1)a(\omega_2). \] (3.2)

The general case is
\[ P_n(a) = \frac{1}{n!} \int \cdots \int d\omega_1 \cdots d\omega_n \, a^\dagger(\omega_1) \cdots a^\dagger(\omega_n)|0\rangle\langle 0|a(\omega_1) \cdots a(\omega_n). \] (3.3)
It follows that any state in the space of superpositions over all $n$ is unchanged by the operator obtained by summing over all $P_n(a)$, and hence this sum is the unit operator

$$
\sum_{n=0}^{\infty} P_n(a) = 1.
$$

Equation (3.3) implies a relation among the projections for adjacent values of $n$

$$
P_{n+1}(a) = \frac{1}{n+1} \int d\omega a^\dagger(\omega) P_n(a) a(\omega).
$$

Although this mirrors the situation for a single oscillator, the projections $P_n(a)$ cannot be expressed as the outer product of a vector with its adjoint; indeed the $n$-photon subspaces are infinite dimensional.

From the commutation Eqs. (2.6) and (2.7) it follows that

$$
P_n(a) \prod_{j=1}^{m} a^\dagger(\omega_j) |0\rangle = \delta_{n,m} \prod_{j=1}^{m} a^\dagger(\omega_j) |0\rangle;
$$

furthermore, this holds if one inserts into both sides of the product the same polynomial in operators for modes orthogonal to $a$. Also from Eq. (3.6) follows the relation

$$
P_n(a)(h:a^tm)|0\rangle = \delta_{n,m}(h:a^tm)|0\rangle.
$$

A. Action of single-mode projections on multi-mode states

In the context of two orthogonal modes $a$ and $b$, the symbol $P_n(a)$ is re-used as shorthand for $P_n(a) \otimes 1_b$, where $1_b$ is the unit operator on the $b$-factor of a tensor product. Two projections for differing values of $n$ for a given mode are mutually orthogonal, but not when one projection is for one mode and the other projection for another; e.g. $P_1(a_1)P_3(a_1) = 0$ but $P_1(a_1)P_3(a_2) \neq 0$. Similar remarks apply to the case of more than two mutually orthogonal modes.

B. Multi-mode $n$-photon projector

To study polarization we will need the idea of an $n$-photon state that can be distributed over two modes. We denote the projector for $n$ photons distributed arbitrarily among modes $a_1$ and
\( a_2 \) by

\[
P_n(a_1, a_2) = \sum_{k=0}^{n} P_k(a_1)P_{n-k}(a_2).
\]  

(3.8)

**Note:** All these projectors can be subdivided into “+” and “−” parts, e.g.

\[
P_n(a) = P_{n,+}(a) + P_{n,-}(a).
\]  

(3.9)

### C. Number operator

Similarly to the single oscillator case, one constructs an operator that has as its expectation value for a given state the mean photon number for that state. This is the number operator for the mode under discussion:

\[
\hat{N}(a) \overset{\text{def}}{=} \sum_{n=0}^{\infty} nP_n(a).
\]  

(3.10)

By use of Eqs. (3.5) and (3.4), this is transformed into the more convenient form:

\[
\hat{N}(a) = \sum_{n=1}^{\infty} nP_n(a) = \sum_{n=1}^{\infty} \frac{1}{n} \int d\omega a_\dagger(\omega) P_{n-1}(a) a(\omega)
\]

\[
= \int d\omega a_\dagger(\omega) \left( \sum_{n=1}^{\infty} P_{n-1}(a) \right) a(\omega) = \int d\omega a_\dagger(\omega)a(\omega).
\]  

(3.11)

It is also useful to express the number operator for the “+” and “−” directions:

\[
\hat{N}_\pm(a) = \int_{0}^{\infty} d\omega a_\pm(\omega)a_\pm(\omega).
\]  

(3.12)

For a space (or subspace) spanned by two modes \( a_1 \) and \( a_2 \), define

\[
\hat{N}(a_1, a_2) = \sum_{n=0}^{\infty} nP_n(a_1, a_2).
\]  

(3.13)

As sketched in Lemma (B21) of Appendix B, a calculation similar to that for one mode shows

\[
\hat{N}(a_1, a_2) = \int d\omega \{ a_1(\omega)a_1(\omega) + a_2(\omega)a_2(\omega) \}.
\]  

(3.14)

Here again, we can pick out directions, as in Eq. (3.12). There is no difficulty in extending to more than two modes.
4. LOSS AND FREQUENCY DISPERSION

We partition the modes to be analyzed into “desired modes” and “extraneous modes,” as illustrated in Fig. 2. By considering coupling of desired modes to extraneous loss modes, one can model a wide variety of loss mechanisms. The choice of model for loss is tied to the choice of model of detection.

To model loss of a mode to be detected by a binary detector, neglecting memory effects in the detector, for now we succumb to the charm of simplicity in the approach of Mandel [10, p. 640]. In this approach a mode $b'$ prior to loss is related to a mode $b$ after loss by the following equation for the respective annihilation operators:

$$b' (\omega) = \eta_{\text{loss}} (\omega) b (\omega) + (1 - |\eta_{\text{loss}} (\omega)|^2)^{1/2} c (\omega),$$

(4.1)

where $c$ expresses an undetected mode into which all the $b'$-energy spills except for a fraction. In the case where $\eta_{\text{loss}}$ is independent of frequency, this un-lost fraction is just $|\eta_{\text{loss}}|^2$.

An alternative is to consider the coupling of desired modes to a heat bath. This leads to so-called master equations. The simplest form that expresses the essential features makes the Markhoff approximation [7, p. 347, Eq. (6.2.61)]. With further simplifications, including that of a zero-temperature heat bath, we obtain the time behavior of a density operator that at time $t_0$ expresses the state $a_f^\dagger |0\rangle$ to be, in the Schrödinger picture (SP), a density operator of the form

$$\rho_f = \int \int d\omega d\omega' \exp \{-[\gamma (\omega) + \gamma (\omega')|t} f (\omega) f^* (\omega') e^{i(\omega - \omega')t} a^\dagger (\omega)|0\rangle \langle 0| a (\omega')$$

$$+ \left(1 - \int d\omega \exp \{-[\gamma (\omega) + \gamma (\omega')|t} |f (\omega)|^2 \right) |0\rangle \langle 0|.$$  

(4.2)

[Need to work this out for multi-photon states.]

A. Loss cannot evade “no cloning”

I believe (and need to check) that coupling of desired modes to thermal modes cannot make two states more distinguishable. More formally, suppose the desired modes are $A$ and the extraneous modes are $B$, and the in-state is expressed by either some tensor product $\rho_{A,\text{in},1} \otimes$
\(\rho_{B,\text{in}}\) or by \(\rho_{A,\text{in},2} \otimes \rho_{B,\text{in}}\). After a unitary evolution \(U\) acting on the tensor-product space of \(A\) and \(B\), the out-state is \(U\rho_{A,\text{in},j} \otimes \rho_{B}U^{\dagger}\), with \(j = 1\) or \(2\). The reduced density matrix for \(A\) is obtained by the partial trace over \(B\):

\[
\rho_{A,\text{out},j} = \text{Tr}_{B}(U\rho_{A,\text{in},j} \otimes \rho_{B}U^{\dagger}).
\] (4.3)

(Note that, unlike traces, partial traces of a product depend on the order of the factors.) My guess is that in all cases

\[
\text{Tr}(\rho_{A,\text{out},1}^{1/2}\rho_{A,\text{out},2}^{1/2}) \geq \text{Tr}(\rho_{A,\text{in},1}^{1/2}\rho_{A,\text{in},2}^{1/2}).
\] (4.4)

For this it is necessary and sufficient to prove for all density operators \(\rho_1\) and \(\rho_2\) acting on the tensor-product space of \(A\) and \(B\) that

\[
\text{Tr}_{A}[(\text{Tr}_{B} \rho_1)^{1/2}(\text{Tr}_{B} \rho_2)^{1/2}] \geq \text{Tr}(\rho_1^{1/2}\rho_2^{1/2}).
\] (4.5)

5. LOCAL QUANTUM FIELDS

Improper operators introduced so far have been integrated over frequency to produce proper operators. We want also to construct proper operators by integrating over space and/or time, which leads us to take Fourier transforms, as outlined in Appendix D. In analogy with quantum electrodynamics for propagation in vacuum, we introduce (improper) local annihilation field operators in the Heisenberg picture:

\[
a_{+}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} d\omega a_{+}(\omega)e^{-i[\omega t-k(\omega)x]},
\]

\[
a_{-}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} d\omega a_{-}(\omega)e^{-i[\omega t+k(\omega)x]},
\] (5.1)

where \(k(\omega)\) is an experimentally determined propagation factor, defined so that \(\text{sgn} k(\omega) = \text{sgn} \omega\). We assume

1. \(k(-\omega) = -k(\omega)\). (5.2)

2. \(\lim_{\omega \to \infty} \frac{k(\omega)}{\omega} > 0\). (5.3)
\[ \frac{dk(\omega)}{d\omega} > 0. \] (5.4)

4. Kramers-Kronig relations connect this \( k(\omega) \) to the loss coefficient \( \gamma(\omega) \) of Eq. (4.2) [11].

The operator \( a^\dagger_\pm(x, t) \) is ‘improper’ in that it takes a normalized state to an unnormalizable state; a proper operator can be defined by averaging \( a^\dagger_\pm(x, t) \) over a spacetime region. (This averaging takes place automatically in time-dependent perturbation theory [7, p. 257].) Drawing on Eq. (2.2), we can define

\[
a(x, t) \overset{\text{def}}{=} a_+(x, t) + a_-(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \, a(\omega) e^{-i|\omega|t - k(\omega)x}, \] (5.5)

and from Eqs. (5.1) and (2.6) follows the commutation relation

\[
[a(x, t), a^\dagger(\omega)] = \frac{1}{\sqrt{2\pi}} \exp\{-i|\omega|t - k(\omega)x\}. \] (5.6)

A. Temporally local hermitian fields

From the non-hermitian operator \( a(x, t) \) can be constructed the two non-commuting hermitian quadrature operators

\[
q(x, t) \overset{\text{def}}{=} a^\dagger(x, t) + a(x, t)
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \, \{ a^\dagger(\omega) e^{i|\omega|t - k(\omega)x} + a(\omega) e^{-i|\omega|t - k(\omega)x} \}, \] (5.7)

\[
p(x, t) \overset{\text{def}}{=} i[a^\dagger(x, t) - a(x, t)]
= \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \, \{ a^\dagger(\omega) e^{i|\omega|t - k(\omega)x} - a(\omega) e^{-i|\omega|t - k(\omega)x} \}. \] (5.8)

Note \( p(x, t) \) is like Louisell’s voltage operator. Fourier-transforms show

\[
\int_{-\infty}^{\infty} dt \, a^\dagger_\pm(x, t) a_\pm(x, t) = \int_{0}^{\infty} d\omega \, a^\dagger_\pm(\omega) a_\pm(\omega). \] (5.9)

This will be used in constructing operators to model detection.
B. Time, space, and dispersion

Because the variation of $k(\omega)$ with $\omega$ is non-linear, the operator that is convenient for space localization must differ from that which is convenient for time localization. Consider operators such as the Hamiltonian and the number operator (shortly to be introduced) of the form $\hat{O} = \int d\omega f(\omega) a^\dagger(\omega) a(\omega)$. In the case of the number operator defined as $\hat{N}_\pm(a) = \int_0^\infty d\omega a^\dagger_\pm(\omega) a_\pm(\omega) = \int dt a^\dagger_\pm(x,t) a_\pm(x,t)$, the local operator $a^\dagger_\pm(x,t) a_\pm(x,t)$ acts as a kind of density in time for $\hat{N}_\pm(a)$; however, the space integral of $a^\dagger_\pm(x,t) a_\pm(x,t)$ is something else:

$$\int_{-\infty}^\infty dx a^\dagger_\pm(x,t) a_\pm(x,t) = \int_0^\infty d\omega_1 \int_0^\infty d\omega_2 a^\dagger_\pm(\omega_1) a_\pm(\omega_2) e^{i(\omega_1 - \omega_2)t} \delta[k(\omega_1) - k(\omega_2)], \quad (5.10)$$

which, with the relation

$$\delta[k(\omega) - k(\omega')] = \left|\frac{dk(\omega)}{d\omega}\right|^{-1} \delta(\omega - \omega'), \quad (5.11)$$

becomes

$$\int_{-\infty}^\infty dx a^\dagger_\pm(x,t) a_\pm(x,t) = \int_0^\infty d\omega a^\dagger_\pm(\omega) a_\pm(\omega) \left|\frac{dk(\omega)}{d\omega}\right|^{-1}. \quad (5.12)$$

Note that the “extra factor” $1/|dk/d\omega|$ expresses a group velocity [12].

With this in mind, we construct an energy density operator $a'_\pm(x,t)$ by

$$a'_\pm(x,t) \overset{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \left(\omega \frac{dk(\omega)}{d\omega}\right)^{1/2} a_\pm(\omega) e^{-i|\omega|t - k(\omega)x}. \quad (5.13)$$

One can check that this has the property

$$\int_{-\infty}^\infty dx a'^\dagger_\pm(x,t) a'_\pm(x,t) = \int_0^\infty d\omega |\omega| a'^\dagger_\pm(\omega) a_\pm(\omega) = H/\hbar. \quad (5.14)$$

There are lots of other possibilities. For instance, define a field

$$\tilde{a}(x,t) \overset{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty d\omega \frac{dk(\omega)}{d\omega} a(\omega) e^{-i|\omega|t - k(\omega)x}; \quad (5.15)$$

then

$$H/\hbar = i \int_{-\infty}^\infty dx \tilde{a}^\dagger(x,t) \frac{\partial a(x,t)}{\partial t}, \quad (5.16)$$

so that the two distinct fields $a$ and $\tilde{a}$ both enter.
In order to deal with finite fibers, we explore the operator obtained by making the limits of integration over $x$ finite; however, the resulting integral is no longer independent of $t$. Further, while the integral obtained from putting finite $x$-limits in (5.14) is at least hermitian, even this modest property fails for the integral obtained by putting finite $x$-limits in (5.16). Nonetheless, in some cases such a truncated operator can be a useful approximation to the hamiltonian.

### C. Projections in terms of local operators

It will be interesting to examine approximations to projections in terms of operators that are, so to speak, confined in time and space. For reference, here we express some projections in terms of local operators. From Eq. (3.1) and the inverse Fourier transform in time of Eq. (5.1), one computes

$$P_1(a) = \int_{-\infty}^{\infty} dt \ a^\dagger(x,t)|0\rangle \langle 0|a(x,t)$$

(independent of $x$). Similarly, from Eq. (3.2) one computes

$$P_2(a) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \ a^\dagger(x,t)a^\dagger(x',t')|0\rangle \langle 0|a(x,t)a(x',t'),$$

and one can keep on going. This procedure also works to express the number operator of Eq. (3.11) in terms of local operators:

$$\hat{N}(a) = \int d\omega \ a^\dagger(\omega)a(\omega) = \int_{-\infty}^{\infty} dt \ a^\dagger(x,t)a(x,t).$$

### 6. SCATTERING MATRIX

Couplers and other networks can be analyzed in terms of a scattering matrix. For this, one supposes that there is time $t_1$ before which the network acts as a set of uncoupled fibers and perhaps a quantum memory uncoupled to these fibers; one supposes a later time $t_2$ after which the interaction is over, so that again one has a set of uncoupled fibers and a quantum memory in isolation. Prior to $t_1$, the light is modeled by an in-state as an integral over frequency of a polynomial in in-mode creation operators acting on the vacuum. For time $t > t_2$ the light
is modeled by an out-state, again as an integral over frequency of a polynomial in out-mode creation operators acting on the vacuum.

For example, to analyze a coupler, we typically express state preparation in terms of in-state creation operators while we express detection in terms of out-state creation operators. Hence calculating the probability of an outcome calls for expressing in-state operators in terms of out-state operators, or vice-versa, according to whichever is more convenient.

The use of scattering theory in this context is to relate the creation operators for in-state modes to creation operators for out-state modes, and this relation is constrained to be a unitary transformation. Because of unitarity, commutation relations are preserved and so are inner products. A complication is that unitarity holds only for a transform over all the modes involved, “extraneous” as well as “desired.”

In general the creation operator for an out-mode at one frequency depends on creation operators for in-modes at all frequencies; i.e., the operator mixes frequencies, as in parametric down conversion. For linear networks we get the major simplification that out-mode operators for a given frequency $\omega$ are unitary transforms of in-mode operators for the same frequency.

After a scattering transformation is obtained, either by guessing or by calculating it from some more detailed model, one can check whether the in-states and out-states are consistent with particular values of $t_1$ and $t_2$. (For a network with strong internal reflections, the interval $t_2 - t_1$ must be large enough to allow reverberations to die out.)

A. Network without frequency mixing

For the special case of a network that is linear in that it mixes no frequencies, if the extraneous modes are accounted for, then the out-state annihilation operators $a_{j,\text{out}}(\omega), j = 1, 2, \ldots$, are some unitary transform of the in-state annihilation operators. That is, let

$$\vec{a}_{\text{out}} = \begin{bmatrix} a_{1,\text{out}} \\ a_{2,\text{out}} \\ \vdots \end{bmatrix}.$$  \hspace{1cm} (6.1)
and similarly define $\vec{a}_{\text{in}}$. Then there is some frequency-dependent, unitary matrix $U(\omega)$, having dimension equal to the number of out-modes (by assumption equal to the number of in-modes), such that

$$\vec{a}_{\text{out}}(\omega) = U(\omega)\vec{a}_{\text{in}}(\omega). \quad (6.2)$$

Taking hermitian conjugates and multiplying the result by $U(\omega)$ on the right gives a relation between the row vectors $\vec{a}_{\text{out}}^\dagger$ and $\vec{a}_{\text{in}}^\dagger$:

$$\vec{a}_{\text{out}}^\dagger = \vec{a}_{\text{in}}^\dagger U^\dagger(\omega). \quad (6.3)$$

To understand the application of scattering formalism to modeling fiber interactions illustrated in Fig. 1, first model light pulses as quantum states expressed by creation operators acting on vacuum. We think of an interaction that, to some approximation, starts at $t_1$ or later and is complete by $t_2$ or earlier. Prior to $t_1$, the incoming light is modeled as involving only in-state creation operators. After $t_2$, the light is modeled as a different state, involving only out-state creation operators, related to the in-state creation operators by an equation of the form (6.3).

### 7. POLARIZED AND ENTANGLED LIGHT STATES

It is possible to make a fiber, for instance with an elliptical cross section, that propagates only a single mode; however, the fibers used in the Quantum Network all carry two polarization modes. We follow convention by misnaming fibers, so that when we say a ‘single-mode’ fiber we mean a fiber that propagates not a single spatial mode $a$ but one that propagates two polarized modes $a_1$ and $a_2$, mutually orthogonal in that the $[a_1, a_2^\dagger] = 0$, with propagation constants $k_{a1}(\omega)$ and $k_{a2}(\omega)$, respectively. This means that for a network (as in Fig. 1) with $n$ ‘single-mode’ fibers there are $2n$ in-modes and $2n$ out-modes, so that, neglecting extraneous modes, the single-frequency scattering matrix $U(\omega)$ has dimension $2n$.

(For special materials, one may need to separate out the two directions to allow $k_{aj^+} \neq k_{aj^-}$.) Note that a linear superposition of the two modes with different propagation constants has no definable propagation constant.

In describing modeling approaches for fibers that support two polarizations we make (and
try to state) various simplifying assumptions; throughout we assume frequency conservation. Taking polarization into account doubles the number of modes relative to the number of so-
called single-mode fibers, for the simple reason that these are misnamed.

For a polarized fiber having orthogonal modes $a_1$ and $a_2$, the form of a single-photon state is [see Eq. (2.18)]:

$$|1\text{-photon}\rangle = (c_1 a_{1,f} + c_2 a_{2,g})^\dagger |0\rangle,$$

(7.1)

for any normalized $f$ and $g$, assuming numerical constants $|c_1|^2 + |c_2|^2 = 1$. The same operator generates the coherent state of the form of Eq. (2.29):

$$|\alpha, (c_1 a_{1,f} + c_2 a_{2,g})\rangle \overset{\text{def}}{=} e^{-|\alpha|^2/2} e^{\alpha (c_1 a_{1,f} + c_2 a_{2,g})^\dagger} |0\rangle = |c_1 \alpha, a_{1,f}\rangle \otimes |c_2 \alpha, a_{2,g}\rangle,$$

(7.2)

where the second equation follows from $\exp[(c_1 a_{1,f} + c_2 a_{2,g})^\dagger] = \exp(c_1 a_{1,f}^\dagger) \exp(c_2 a_{2,g})^\dagger$, which is a consequence of the commutativity $[a_1^\dagger, a_2^\dagger] = 0$.

[**The whole business seems to assume a reference position $x = 0$; other values of $x$

A. Fiber splice (without extraneous modes)

Consider the situation shown in Fig. 3, where a fiber $a$ on the left is spliced to a fiber $b$ on the right. We suppose the splice is centered at $x = 0$. For such a heterogeneous spliced fiber, the local field operator $a(x,t)$ cannot satisfy a field equation that exhibits translational symmetry; instead, the field equation stems from a hamiltonian that has a change at the splice; the situation is reminiscent of a potential problem in quantum mechanics.
Without trying to model the details, and neglecting coupling to extraneous modes, the effect of the splice is to convert in-mode operators to out-mode operators, as expressed by a unitary matrix $U(\omega)$. Recognizing polarizations, we have modes $a_1$ and $a_2$ in the $a$-fiber and modes $b_1$ and $b_2$ in the $b$-fiber, with propagation constants $k_{a_1}$, $k_{a_2}$, $k_{b_1}$, and $k_{b_2}$, respectively. This makes the four in-modes $a_{j+}$ and $b_{j-}$, $j = 1, 2$, along with the four out-modes $a_{j-}$ and $b_{j+}$; hence if we neglect loss, the scattering matrix $U(\omega)$ is 4-by-4:

$$\begin{bmatrix}
a^\dagger_{1-}(\omega), & b^\dagger_{1+}(\omega), & a^\dagger_{2-}(\omega), & b^\dagger_{2+}(\omega)
\end{bmatrix} = \begin{bmatrix}
b^\dagger_{1-}(\omega), & a^\dagger_{1+}(\omega), & b^\dagger_{2-}(\omega), & a^\dagger_{2+}(\omega)
\end{bmatrix} U^\dagger(\omega). \quad (7.3)$$

Under the simplifying assumption that the splice has no coupling between polarizations, the equation factors into two 2-by-2 pieces, one piece for each polarization. With $U(\omega)$ defined this way, the perfectly homogeneous situation in which $k_{a_j}(\omega) = k_{b_j}(\omega)$, thus making the splice invisible, corresponds to $U(\omega) = 1$.

(To account for loss, we can add extraneous dimensions and then trace them out, thereby getting a matrix that is not unitary, so that the out-power can be less than the in-power.) [*For single-frequency matrices, we use power rather than energy, because energy is definable only for non-zero bandwidth.]

**B. Coupler**

Fiber couplers, analogous to beam splitters, have four fibers, $a_j, b_j, c_j, d_j$, with $j = 1, 2$ (see Fig. 4). Each of the eight modes comes in two directions, “+” and “−”. Without loss, $U(\omega) \in SU(8)$.

**C. Entangled states**

Whether entangled in frequency or in polarization or in both, entangled states have to do with tensor products of vector spaces. Let a vector space $V$ be a tensor product of vector spaces $V_\alpha$, where $\alpha$ ranges over some index set. Relative to this factorization, a generic vector in $v \in V$ is a sum of tensor products of vectors $v_\alpha \in V_\alpha$; unless it can be written as a single product, not a sum of products, $v$ is called entangled (relative to the factorization).
We stress relative to a factorization because vector spaces for light states can be factored in more than one way, and to speak sensibly of ‘entangled states’ one must know which factorization is meant. The vector space for a single mode involves tensor products over subspaces for frequency bands; this factorization is relevant to frequency-entangled states. Within any frequency band, there is an infinite tensor product over single frequencies, which will not be used here in speaking of ‘entanglement’. Polarization-entanglement involves a vector space for several modes, factored by mode.

Frequency-preserving, unitary transformations of the single-frequency creation operators carry an in-state that is a tensor product of broad-band coherent states to an out-state that is also a tensor product of broad-band coherent states. This is a special property; indeed the only case I know in which it works is that of coherent states. [* Work out proof] (In generic cases, a tensor product of single-photon in-states is carried to an entangled, multi-photon state.) This makes broad-band coherent states of special interest as an (overcomplete) basis for studying light in the context of frequency-preserving splices and couplers. See Appendix E.

D. Polarization-entangled states

A polarization-entangled state involves four modes; we think of two orthogonal modes \(a_1\) and \(a_2\) in a left fiber to Alice and two orthogonal modes \(b_1\) and \(b_2\) in a right fiber to Bob. (One can think of 1 as horizontal and 2 as vertical.) Then there are four creation operators \(a_1^\dagger(\omega)\), \(a_2^\dagger(\omega)\), \(b_1^\dagger(\omega)\), and \(b_2^\dagger(\omega)\); everything commutes except that \([a_i(\omega), a_j^\dagger(\omega')] = \delta_{i,j} \delta(\omega - \omega')\), where \(\delta_{i,j} = 1\) if \(j = i\) and otherwise is 0.
The defining property of a state \(|\phi\rangle\) characterized by a single \(a\)-photon combined with a single \(b\)-photon is

\[
P_1(a_1, a_2)P_1(b_1, b_2)|\phi\rangle = |\phi\rangle. \tag{7.4}
\]

A necessary and sufficient condition for this is that \(|\phi\rangle\) have the form

\[
|\phi\rangle = \sum_{i,j=1}^{2} C_{i,j} (h: a_i^\dagger b_j^\dagger)|0\rangle, \tag{7.5}
\]

where

\[
\sum_{i,j=1}^{2} |C_{i,j}|^2 = 1, \tag{7.6}
\]

where we use the definition stated in Eq. (2.57), and the four otherwise arbitrary functions \(h_{i,j}\) satisfy the normalization condition (in which no symmetry is assumed):

\[
\int \int d\omega d\tilde{\omega} |h_{i,j}(\omega, \tilde{\omega})|^2 = 1. \tag{7.7}
\]

To think about quantum key distribution, we will need to consider states of both fewer and more photons. A state with \(m\) photons in \(a\)-modes, \(j\) of which are in mode \(a_1\), and with \(n\) photons in \(b\)-modes, \(k\) of which are in mode \(b_1\), has the form

\[
\frac{1}{\sqrt{j!(m-j)! k!(n-k)!}} (a_1^j a_2^{m-j} b_1^k b_2^{n-k})_h |0\rangle
\]

\[
= \frac{1}{\sqrt{j!(m-j)! k!(n-k)!}} \int \cdots \int d\omega_1 d\tilde{\omega}_1 \cdots d\omega_n d\tilde{\omega}_n h(\omega_1, \tilde{\omega}_1, \ldots, \omega_n, \tilde{\omega}_n)
\]

\[
\times a_1^j(\omega_1) \cdots a_1^j(\omega_j) a_2^k(\omega_{j+1}) \cdots a_2^k(\omega_m) b_1^k(\tilde{\omega}_1) \cdots b_1^k(\tilde{\omega}_k) b_2^k(\tilde{\omega}_{k+1}) \cdots b_2^k(\tilde{\omega}_n)|0\rangle. \tag{7.8}
\]

There is no loss of generality in requiring \(h\) to be symmetric under each interchange \(\omega_j \leftrightarrow \omega_k\) for which \(\omega_j\) and \(\omega_k\) pertain to the same mode. When \(h\) with this symmetry is normalized as in Eq. (2.22), the state defined by Eq. (7.8) has unit norm. The most general state is a weighted sum of such states.

8. DETECTION

A light detector as used in an experimental setup will be modeled by a positive operator-valued measure (POVM), \(\{M_j\}\), so the probability of outcome \(j\) is \(\text{Tr}(M_j \rho)\), where \(\rho\) is a
density operator, and $\sum_j M_j = 1$ with $M_j \geq 0$. In the simplest models to be discussed here, the density operator is for the light; in more complex models discussed elsewhere [2], the density operator can be for a pure or mixed state of not only light to be detected but also probe particles. One could also include the generation of backward-propagating light, but here we do not get into this level of complication.

A. Simple examples

Example 2: As a first theoretical example, a detector as narrow-band as possible that will register outcome 1 with certainty, given the one-photon state $a_g^\dagger |0\rangle$, has $M_1 = a_g^\dagger |0\rangle \langle 0|a_g$. This ‘filtered’ detector discriminates well against any other one-photon state $a_f^\dagger |0\rangle$ if $f$ is unlike $g$; i.e., the probability for outcome 1 for a state $a_f^\dagger |0\rangle$ is $\text{Tr}(a_g^\dagger |0\rangle \langle 0|a_g a_f^\dagger |0\rangle \langle 0|a_f) = |\langle 0|a_g a_f^\dagger |0\rangle|^2 = |\int d\omega g^*(\omega)f(\omega)|^2$.

Given a set of mutually orthogonal functions $g_j$, $j = 1, \ldots, N$, there can be a detector with $N$ possible outcomes, with $M_j = a_g^\dagger |0\rangle \langle 0|a_g$. Thus a single-photon detector is certainly not restricted to giving a yes-no outcome, and discrimination among different single-photon states is possible. Herein lies a caution to designers who hope that distinct lasers generate distinct states that differ in polarization.

Example 3: An example of a model of detectors preceded by filters used for joint detection of a two-mode state is $M_1 = a_f^\dagger b_g^\dagger |0\rangle \langle 0|b_g a_f$. Then outcome 1 would be described as ‘finding the state $a_f^\dagger |0\rangle$ in measuring mode $a$ jointly with finding the state $b_g^\dagger |0\rangle$ in measuring mode $b$.’ For the two-mode state (2.56), the probability of this outcome is readily calculated to be $|\int \int d\omega d\omega' f^*(\omega)g^*(\omega')h(\omega,\omega')|^2$.

Example 4: An interesting and perhaps novel application of probe particles involves detection at two coordinated locations using probe particles that have previously become entangled. This can produce an example that swaps the detection operator $M_1$ and the state of Example 2, leaving the probability invariant. I.e., the same probabilities and outcomes arise from measuring an entangled state with an unentangled detector as from measuring an unentangled state with an entangled detector [2].
B. Model of APD detector for quantum cryptography

For multi-photon states, a simple approach is to model a detector as a POVM for a state involving only the light to be detected. This assumes that memory effects in the detector, such as those often attributed to trapped carriers in photo-diodes, are insignificant (for instance because the network design enforces enough hardware dead time); it also assumes teetering in the detector plays no significant role [2]. For simplicity, here we add the assumption that the detector and its pulse-shaping circuitry choose an outcome of ‘yes’ or ‘no’ without any additional detail. With these assumptions, all one has left is dark-count and efficiency; one obtains a class of detector models that respond to $n$ photon states according to

$$\Pr(\text{Detect}|n\text{-photon state}) = d_n,$$

with $0 \leq d_n \leq 1$. With the additional assumption that the detector is flat in its frequency response over the bandwidth of the light pulses to be detected, the detection operator that generates these probabilities is

$$M_1(a) = \sum_{n=0}^{\infty} d_n P_{n,+}(a),$$

where $P_n(a)$ is defined in Eq. (3.3) and we use $a_+(\omega)$ in place of $a(\omega)$, so the integrals are over positive frequencies only. [For a detector of both polarization modes $a_1$ and $a_2$ of a fiber, one has instead,

$$M_1(a_1, a_2) = \sum_{n=0}^{\infty} d_n P_{n,+}(a_1, a_2),$$

with $P_{n,+}(a_1, a_2)$ defined by Eq. (3.8) with $a_+(\omega)$ in place of $a(\omega)$.
]

To model the use of avalanche photo-diode (APD) detectors in the DARPA Quantum Network, we explore in detail a specialization of this type of model. We will use this model in Sec. 9 to study the variation in detection statistics as the energy of transmitted pulses is raised above that of a single-photon state.

An APD detector for an optical fiber responds to both polarization modes of the fiber; however, by placing a polarizing beam splitter before the detector, one can effectively eliminate the light from one mode. We assume this case, so that only one polarization mode is relevant.
Then we suppose that the probability of the detector responding to any single-photon state is \(\eta_{\text{det}}\), assumed independent of frequencies over the range of frequencies relevant to the light state. We need to model the probability of detecting multi-photon states. By the assumption already made, the probability of failing to detect any single-photon state is \(1 - \eta_{\text{det}}\). Assume that the probability of failing to respond to an \(n\)-photon state is just the single-photon failure probability raised to the \(n\)-th power: \((1 - \eta_{\text{det}})^n\). Then the probability assigned by this model to registering a detection, given an \(n\)-photon state and temporarily ignoring dark counts, is

\[
\Pr(\text{Detect}|n\text{-photon state}) = 1 - (1 - \eta_{\text{det}})^n. \tag{8.4}
\]

To allow for dark counts we replace this by

\[
\Pr(\text{Detect}|n\text{-photon state}) = 1 - (1 - p_{\text{dark}})(1 - \eta_{\text{det}})^n. \tag{8.5}
\]

To deal with states that are superpositions over varying numbers of photons, we recall that the operator for an ideal detector that responds always to any \(n\)-photon state, with no probability of detection for a state that has no \(n\)-photon component, is the projection \(P_{n,\pm}\) defined in Sec. 5. The following detection operator \(M_1(a)\) invokes \(P_{n,\pm}\) to provide the desired probabilities for a detector of a single \(a\)-mode [13]:

\[
M_1(a) = \sum_{n=0}^{\infty} [1 - (1 - p_{\text{dark}}(a))(1 - \eta_{\text{det}})^n] P_{n,+}(a), \tag{8.6}
\]

where \(P_{n}(a)\) is defined in Eq. (3.3) and we use \(a_{\pm}(\omega)\) in place of \(a(\omega)\), so the integrals are over positive frequencies only. [For a detector of both polarization modes \(a_1\) and \(a_2\) of a fiber, one has instead,

\[
M_1(a_1, a_2) = \sum_{n=0}^{\infty} [1 - (1 - p_{\text{dark}}(a_1, a_2))(1 - \eta_{\text{det}})^n] P_{n,+}(a_1, a_2), \tag{8.7}
\]

with \(P_{n,+}(a_1, a_2)\) defined by Eq. (3.8) with \(a_{\pm}(\omega)\) in place of \(a(\omega)\).]

As applied to states with photon-number components negligible except when \(\eta_{\text{det}} n \ll 1\), we notice

\[
1 - (1 - p_{\text{dark}})(1 - \eta_{\text{det}})^n \approx p_{\text{dark}} + (1 - p_{\text{dark}})\eta_{\text{det}} n, \tag{8.8}
\]
from which we obtain for this case an approximation that simplifies Eq. (8.6)

\[ M_1(a) \approx \sum_{n=0}^{\infty} [p_{\text{dark}}(a) + (1 - p_{\text{dark}}(a))\eta_{\text{det}}] P_{n,+}(a) \]

\[ = p_{\text{dark}}(a) + (1 - p_{\text{dark}}(a))\eta_{\text{det}} \int_0^{\infty} d\omega a^\dagger(\omega)a(\omega). \] (8.9)

C. Detection probabilities

The most general state involving \( k \) modes can be written as

\[ |\psi\rangle = \sum_{n=0}^{\infty} h_n : \text{Pol}_n(a_1^\dagger, \ldots, a_k^\dagger)|0\rangle, \] (8.10)

where \( h_n \) is a function of \( n \) frequency variables and \( \text{Pol}_n(a_1^\dagger, \ldots, a_k^\dagger) \) is a homogeneous polynomial of degree \( n \) in the \( k \) creation operators without any annihilation operators.

Let \( x \) stand for any of the modes \( a_j \). The operator for ‘detect’ for mode \( x \) is \( M_1(x) \) and the operator for ‘no-detect’ is obtained from Eqs. (8.6) and (3.4) as

\[ M_0(x) \overset{\text{def}}{=} 1 - M_1(x) = (1 - p_{\text{dark}}(x)) \sum_{n=0}^{\infty} (1 - \eta_{\text{det}})^n P_{n,+}(x). \] (8.11)

Any theoretical outcome produced by APD detectors as modeled here is specified by two lists of modes, a list \( J_0 \) for which APD detectors register ‘no-detect,’ and a list \( J_1 \) for which APD detectors register ‘detect.’ The operator for ‘detect’ for modes in the list \( J_1 \) and ‘no-detect’ for modes in the list \( J_0 \) is \( M_1(J_1)M_0(J_0) \), where we define

\[ M_0(J_0) = \prod_{x \in J_0} M_0(x), \]
\[ M_1(J_1) = \prod_{x \in J_1} M_1(x). \] (8.12)

The corresponding probability is then

\[ \Pr(J_0, J_1) = \langle \psi | M_1(J_1)M_0(J_0)|\psi\rangle. \] (8.13)

For \( |\psi\rangle \) expressed in the form of Eq. (8.10) and the APD model that invokes \( M_1 \) as defined in Eq. (8.6), calculating this probability is surprisingly simple. From Eqs. (8.11) and (3.6)
we obtain the effect of $M_0(a_j)$ on a state vector defined by an integral over frequencies of a monomial in creation operators

$$M_0(a) \prod_{j=1}^{n} a^\dagger(\omega_j) |0\rangle = (1 - p_{d\text{ark}}(a)) (1 - \eta_{d\text{et}})^n \prod_{j=1}^{n} a^\dagger(\omega_j) |0\rangle$$

$$= (1 - p_{d\text{ark}}(a)) \prod_{j=1}^{n} [(1 - \eta_{d\text{et}}) a^\dagger(\omega_j)] |0\rangle. \quad (8.14)$$

From this we arrive at the following important

**Proposition:** The effect of an operator $M_0(a)$ for ‘no-detection’ on a general state $|\psi\rangle$ is to multiply the state by $(1 - p_{d\text{ark}}(a))$ and to replace every instance of a creation operator $a^\dagger(\omega)$ by $(1 - \eta_{d\text{et}}) a^\dagger(\omega_j)$. Further this holds if creation operators for modes orthogonal to $a$ enter the polynomial $\text{Pol}_n(a^\dagger_1, \ldots, a^\dagger_k)$.

This generalizes to ‘no-detection’ of more modes.

**Proposition:** The effect of a product of creation operators $M_0(a_1) M_0(a_2) \cdots M_0(a_\ell)$ on a general state $|\psi\rangle$ is to multiply the state by

$$\prod_{j=1}^{\ell} (1 - p_{d\text{ark}}(a_j)), \quad (8.15)$$

and to replace every instance of a creation operator $a^\dagger_j(\omega)$, for $j = 1, \ldots, \ell$, by $(1 - \eta_{d\text{et}}(a_j)) a^\dagger_j(\omega_j)$.

For purposes of calculating probability terms $\langle \psi | M_0(L) | \psi \rangle$, where $L$ is an arbitrary list of mode names (as in Sec. 9), we can put this in a symmetric form. Because only like powers of creation and annihilation operators appear in terms that contribute to probabilities, we can distribute the factor $(1 - \eta_{d\text{et}}(a_j))$ evenly between the creation and the annihilation operators, as follows.

**Proposition:** For any list $L$ of mode names, one has

$$\langle \psi | M_0(L) | \psi \rangle = \prod_{x \in L} (1 - p_{d\text{ark}}(x)) \langle \psi' | \psi' \rangle, \quad (8.16)$$

where $|\psi'\rangle$ is the expression obtained from $|\psi\rangle$ defined in Eq. (8.10) by replacing every instance of a creation operator $a^\dagger_j(\omega)$, for $j = 1, \ldots, \ell$, by $[1 - \eta_{d\text{et}}(a_j)]^{1/2} a^\dagger_j(\omega_j)$, and $\langle \psi' |$ is obtained from $\langle \psi |$ by the corresponding replacement of $a(\omega)$ by $[1 - \eta_{d\text{et}}(a_j)]^{1/2} a_j(\omega_j)$. 

34
Caution: This and the preceding two propositions require that the polynomial in Eq. (8.10) contain no annihilation operators.

For evaluating \( M_1(J_1) \), the story is more complicated. What makes Proposition (8.16) work is that the substitution of \([1 - \eta_{\text{det}}(a_j)]^{1/2}a_j^\dagger(\omega_j)\) for \(a_j^\dagger(\omega_j)\) commutes with products, and this does not hold for the analogous rule for \(M_1\). From Eqs. (8.14) and (8.11) we find
\[
M_1(a) \prod_{j=1}^{n} a_j^\dagger(\omega_j)|0\rangle = \left[1 - (1 - p_{\text{dark}})(1 - \eta_{\text{det}})^n\right]\prod_{j=1}^{n} a_j^\dagger(\omega_j)|0\rangle. \tag{8.17}
\]
This proves useful in Sec. 9; however its use is constrained because it cannot be interchanged with the taking of products of operators. For this reason we benefit from the following method of evaluating \( M_1(L) \) in terms of terms of the form \( M_0(L_j) \). Equations (8.11) and (8.12) imply, for any set \(L\) of mode names,
\[
M_1(L) = \prod_{x \in L} [1 - M_0(x)]. \tag{8.18}
\]
With this, we express the product of \(M_1\) factors in Eq. (8.12) by
\[
M_1(L) = \sum_{X \subseteq L} (-1)^{\#(X)} M_0(X), \tag{8.19}
\]
where for any set \(S\), \(\#(S)\) denotes the number of elements in \(S\), the sum is over all subsets of \(L\), including both \(L\) itself and the empty set \(\phi\), and we adopt the convention that
\[
M_0(\phi) = 1. \tag{8.20}
\]
Example:
\[
M_1(a_1, b_2) = 1 - M_0(a_1) - M_0(b_2) + M_0(a_1, b_2). \tag{8.21}
\]
Thus we arrive at an equation for the detection operators in Eq. (8.13):
\[
M_0(J_0)M_1(J_1) = \sum_{X \subseteq J_1} (-1)^{\#(X)} M_0(J_0\|X), \tag{8.22}
\]
where \(J_0\|X\) denotes the concatenation of the two lists of mode names. This implies
\[
\langle \psi | M_0(J_0)M_1(J_1) | \psi \rangle = \sum_{X \subseteq J_1} (-1)^{\#(X)} \langle \psi | M_0(J_0\|X) | \psi \rangle
\]
\[
= (-1)^{\#(J_0)} \sum_{X \subseteq J_1} (-1)^{\#(J_0\|X)} \langle \psi | M_0(J_0\|X) | \psi \rangle. \tag{8.23}
\]

35
D. Effect of time bounds on detection

In many applications a detector is gated on only briefly. Approximating the turn-on and
turn-off as perfectly abrupt, we model the effect of gating the detector on for a duration $T$
centered at a time $t_g$ by use of Eq. (5.9) with the infinite limits replaced by finite times $t_g - T/2$
and $t_g + T/2$. For instance in Eq. (8.9) we replace

$$
\int_0^{\infty} d\omega a_j^\dagger(\omega)a_j(\omega)
$$

by

$$
\int_{t_g-T/2}^{t_g+T/2} dt a_{j\pm}(x,t)a_{j\pm}(x,t)
$$

for whichever sense of the $\pm$ sign corresponds to propagation toward the detector. Substitution
from Eq. (5.1) and carrying out the time integration yield

$$
\int_{t_g-T/2}^{t_g+T/2} dt a_{j\pm}(x,t)a_{j\pm}(x,t) = \int_0^{\infty} d\omega \int_0^{\infty} d\omega' a_{j\pm}(\omega)a_{j\pm}(\omega')e^{i[(\omega-\omega')t_g+(k(\omega)-k(\omega'))x]} \frac{1}{\pi} \frac{\sin[(\omega-\omega')T/2]}{\omega-\omega'}.
$$

[* Look at Mandel [10], first at pp. 573ff; then pp. 691ff. Check in Mandel, p. 696 (14.2–
17), for arguments that dark counts are in principle unavoidable, due to vacuum fluctuations.]

E. Detection, energy, and photon subspaces

Let $\mathcal{H}$ be the vector space of light states generated by integrals over monomials in creation
operators acting on the vacuum state $|0\rangle$. Then for a mode $a$, $P_n(a)\mathcal{H}$ is the subspace of states
generated by a weighted integral over $n$ factors, $\prod_{k=0}^{n} a_j^\dagger(\omega_k)$, along with any number of factors
of creation operators for modes orthogonal to $a$, all acting on the vacuum state $|0\rangle$. Any vector
$|v\rangle$ in $\mathcal{H}$ can be written as a sum over terms in mutually orthogonal subspaces:

$$
|v\rangle = \sum_{n=0}^{\infty} C_n |v_{n,a}\rangle,
$$

with $|v_{n,a}\rangle \in P_n(a)\mathcal{H}$ and of unit norm. Then we have $\langle v_{n,a}|v_{m,a}\rangle = \delta_{m,n}$. Further, we have
$a(\omega)|v_{0,a}\rangle = 0$ and, for $n > 0$, $a(\omega)|v_{n,a}\rangle \in P_{n-1}(a)\mathcal{H}$. It follows that

$$
(\forall \omega, \omega') \quad \langle v|a_j^\dagger(\omega)a_j(\omega')|v\rangle = \sum_{n=0}^{\infty} \langle v_{n,a}|a_j^\dagger(\omega)a_j(\omega')|v_{n,a}\rangle.
$$
That is, there are no cross terms. The probability of detection of a mode \( a \), as well as the energy in the mode, are often expressed by integrals over products \( a^\dagger(\omega)a(\omega) \) or, per Eqs. (8.24) and (8.25), \( a^\dagger(\omega)a(\omega') \), with the result that both the energy for the mode \( a \) and the probability of detection can be expressed as sums with no cross terms between terms with differing numbers of \( a \) photons. That is, the expectation energy for energy in the \( a \)-mode is

\[
\hbar \sum_{n=0}^\infty n |C_n|^2 \omega_n,
\]

where \( \omega_n \) is an averaged frequency that is straightforward to work out. Similarly, using any of the detection models discussed above, the probability of detecting the state \(|v\rangle\) by use of a detector involving only the mode \( a \) is

\[
\Pr(\text{Detection of } |v\rangle) = \sum_{n=0}^\infty |C_n|^2 \Pr(\text{Detection of } |v_{n,a}\rangle).
\]

F. Preceding the APD detector by a beam-splitter

The simplest model of beam splitting (which neglects frequency dependence, reflection, and mixing of polarizations) expresses an in-mode \( a \) in terms of mutually orthogonal out-modes \( b \) and \( c \) by an SU(2) transformation:

\[
a^\dagger(\omega) = (\eta_{\text{trans}})^{1/2}b^\dagger(\omega) + (1 - \eta_{\text{trans}})^{1/2}c^\dagger(\omega).
\]  

(8.28)

Consider an \( n \)-photon \( a \)-mode in-state defined in Eq. (2.19),

\[
|\psi_n\rangle = (n!)^{-1/2}h_n:a^\dagger|0\rangle,
\]  

(8.29)

where, without loss of generality, \( h_n \) is symmetric under permutations of its \( n \) arguments, normalized as in Eq. (2.22). The out-state from the SU(2) transformation expresses the state downstream of a beam-splitter as

\[
|\psi_n\rangle = (n!)^{-1/2} \sum_{k=0}^n \binom{n}{k} \eta_{\text{trans}}^{k/2} (1 - \eta_{\text{trans}})^{(n-k)/2} |\psi_{nk}\rangle,
\]  

(8.30)

where we define the unnormalized state vector

\[
|\psi_{nk}\rangle := (h_n:b^k c^{(n-k)}|0\rangle).
\]  

(8.31)

Now calculate the probability of ‘no-detect’ for this state by a detector of the \( b \) mode, described by the operator \( M_0(b) \) per the preceding APD model:

\[
\Pr(\text{no detect } b) = \langle \psi_n | M_0(b) | \psi \rangle
\]
\begin{align*}
\frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k}^2 \eta_{\text{trans}}^k (1 - \eta_{\text{trans}})^{(n-k)} & \times \langle 0 | (h_n^* : b^k c^{n-k}) M_0(b) (h_n : b^{k*} c^i (n-k)) | 0 \rangle. \\
\text{(8.32)}
\end{align*}

By Proposition (8.16), this becomes
\begin{align*}
\langle \psi_n | M_0(b) | \psi \rangle &= (1 - p_{\text{dark}}) \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k}^2 \eta_{\text{trans}}^k (1 - \eta_{\text{trans}})^{(n-k)} (1 - \eta_{\text{det}})^k \\
& \times \langle 0 | (h_n^* : b^k c^{n-k}) (h_n : b^{k*} c^i (n-k)) | 0 \rangle. \\
\text{(8.33)}
\end{align*}

From Lemmas (B18), (B17) of Appendix B, along with the normalization of \( h_n \), we have
\begin{align*}
\langle 0 | (h_n^* : b^k c^{n-k}) (h_n : b^{k*} c^i (n-k)) | 0 \rangle &= k!(n - k)!, \\
\text{(8.34)}
\end{align*}
whence we obtain
\begin{align*}
\langle \psi_n | M_0(b) | \psi \rangle &= (1 - p_{\text{dark}}) \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k}^2 \eta_{\text{trans}}^k (1 - \eta_{\text{trans}})^{(n-k)} (1 - \eta_{\text{det}})^k k!(n - k)! \\
&= (1 - p_{\text{dark}}) \sum_{k=0}^{n} \binom{n}{k} [\eta_{\text{trans}} (1 - \eta_{\text{det}})]^k (1 - \eta_{\text{trans}})^{(n-k)} \\
&= (1 - p_{\text{dark}}) [\eta_{\text{trans}} (1 - \eta_{\text{det}}) + 1 - \eta_{\text{trans}}]^n \\
&= (1 - p_{\text{dark}}) (1 - \eta_{\text{trans}} \eta_{\text{det}})^n. \\
\text{(8.35)}
\end{align*}

This is just the probability of ‘no-detect’ for the in-state by an \( \alpha \)-mode detector, modified by replacing \( \eta_{\text{det}} \) by the product \( \eta_{\text{trans}} \eta_{\text{det}} \). Thus we have arrived at the

**Proposition:** For the APD model described above, we consider a beam splitter for which the free input is a vacuum state, and which passes a frequency-independent fraction of energy \( \eta_{\text{trans}} \); then the probability ‘no-detect’ downstream of the splitter is obtained from the expression for the probability upstream by replacing the detector efficiency \( \eta_{\text{det}} \) by a reduced efficiency \( \eta_{\text{trans}} \eta_{\text{det}} \).

**Caution:** This proposition holds specifically for the APD model; there are certainly other models, such as photon counting, to which it does not apply.
9. POLARIZATION-ENTANGLED LIGHT FOR QKD

Some early models of quantum key distribution assumed bi-photon states [14]. A year ago John Schlafer asked how the energy of the light pulse—its multi-photon content—affects the statistics of detection. As preparation for Sec. 10, where we start to answer this question, here we show some of the possible multi-photon states that have to be considered.

A. Bi-photon light states

The general bi-photon light state is given by Eq. (7.5). Here we confine ourselves to a special case in which detection statistics are invariant whenever the same polarization transform is performed on both the $a$ and $b$ fibers leaving the source. (It will be interesting to check experimentally for this invariance.) The polarization transformations are the group SU(2), and we refer to bi-SU(2) invariance as invariance of detection probabilities under the following frequency-independent transformation, for arbitrary complex $u, v$ such that $|u|^2 + |v|^2 = 1$,

\[
\begin{bmatrix}
a_1(\omega)
a_2(\omega)
\end{bmatrix} \rightarrow \begin{bmatrix} ua_1(\omega) + va_2(\omega) \\
-v^*a_1(\omega) + u^*a_2(\omega)
\end{bmatrix}, \quad
\begin{bmatrix}
b_1(\tilde{\omega})
b_2(\tilde{\omega})
\end{bmatrix} \rightarrow \begin{bmatrix} ub_1(\tilde{\omega}) + vb_2(\tilde{\omega}) \\
-v^*b_1(\tilde{\omega}) + u^*b_2(\tilde{\omega})
\end{bmatrix}.
\]  

(9.1)

By analyzing the case $u = 0$, $v = 1$, one sees that bi-SU(2) invariance requires that $h$ in Eq. (7.5) satisfy

\[
\begin{bmatrix}
h_{11}(\omega, \tilde{\omega}) & h_{12}(\omega, \tilde{\omega}) \\
h_{21}(\omega, \tilde{\omega}) & h_{22}(\omega, \tilde{\omega})
\end{bmatrix} = g(\omega, \tilde{\omega}) \begin{bmatrix} 0 & 1 \\
-1 & 0
\end{bmatrix},
\]

(9.2)

where $(g, g) = 1$ but $g$ is an otherwise arbitrary function. Any such state is a single $a$-photon together with a single $b$-photon. Sufficiency of this condition to assure bi-photon invariance is shown by the substitution defined in Eq. (9.1).

To explore the dependence of detection probabilities on energy, we need to analyze multi-photon states. We confine ourselves to an opening step in this direction by choosing multi-photon states that have bi-SU(2) invariance of the form

\[
|\psi\rangle = \sum_{n=0} C_n |\psi_n\rangle,
\]

(9.3)
\[ |\psi_n\rangle = f : (a_1^\dagger b_2^\dagger - a_2^\dagger b_1^\dagger)^n |0\rangle \]
\[ = \int d\omega_1 d\bar{\omega}_1 \cdots d\omega_n d\bar{\omega}_n \ f(\omega_1, \bar{\omega}_1, \ldots, \omega_n, \bar{\omega}_n) \prod_{j=1}^{n} \left( a_1^\dagger (\omega_j) b_2^\dagger (\bar{\omega}_j) - a_2^\dagger (\omega_j) b_1^\dagger (\bar{\omega}_j) \right) |0\rangle. \]  
(9.4)

Because the factors under the product all commute with one another, what matters about \( f \) is the part that is invariant under permutations that swap \((\omega_j, \bar{\omega}_j)\) with \((\omega_k, \bar{\omega}_k)\). We think of a vector \( \vec{\omega}_j = (\omega_k, \bar{\omega}_k) \) and denote the average of \( f \) over permutations of these vectors by
\[ S(\vec{\omega}_1, \ldots, \vec{\omega}_n) f(\vec{\omega}_1, \ldots, \vec{\omega}_n) \overset{\text{def}}{=} \frac{1}{n!} \sum_{\pi \in S_n} f(\omega_{\pi 1}, \bar{\omega}_{\pi 1}, \ldots, \omega_{\pi n}, \bar{\omega}_{\pi n}). \] \hspace{1cm} (9.5)

This is the first of several symmetries that will be seen to couple polarization entanglement to frequency entanglement in a way that affects how detection statistics depend on light energy.

The terms \( |\psi_n\rangle \) have the further decomposition
\[ |\psi_n\rangle = \sum_{m=0}^{n} |\psi_{nm}\rangle, \] \hspace{1cm} (9.6)

where the unnormalized states \( |\psi_{nm}\rangle \) are defined by
\[ |\psi_{nm}\rangle = (-1)^{n-m} \binom{n}{m} \int d\omega_1 d\bar{\omega}_1 \cdots d\omega_m d\bar{\omega}_m \]
\[ \times [S(\omega_1, \ldots, \omega_m) S(\omega_{m+1}, \ldots, \omega_n) S(\bar{\omega}_1, \ldots, \bar{\omega}_n) f(\omega_1, \bar{\omega}_1, \ldots, \omega_n, \bar{\omega}_n)] \]
\[ \times \left( \prod_{j=1}^{m} a_1^\dagger (\omega_j) b_2^\dagger (\bar{\omega}_j) \right) \left( \prod_{j=m+1}^{n} a_2^\dagger (\omega_j) b_1^\dagger (\bar{\omega}_j) \right) |0\rangle. \] 
(9.7)

with the convention on products defined in Eq. (B13), and with the additional symmetry operations that act on \( \omega_j \) without acting on \( \bar{\omega}_j \), so that \( S(\omega_1, \ldots, \omega_m) \) is defined for \( m < n \) by
\[ S(\omega_1, \ldots, \omega_m) f(\omega_1, \bar{\omega}_1, \ldots, \omega_n, \bar{\omega}_n) \]
\[ = \frac{1}{m!} \sum_{\pi \in S_{n-m}} f(\omega_{\pi 1}, \bar{\omega}_{\pi 1}, \ldots, \omega_{\pi m}, \bar{\omega}_{\pi m}, \omega_{m+1}, \bar{\omega}_{m+1}, \ldots, \omega_n, \bar{\omega}_n). \] \hspace{1cm} (9.8)

Similarly, \( S(\omega_{m+1}, \ldots, \omega_n) \) operates on the arguments \( \omega_j \) for \( j = m + 1, \ldots, n \).
Remark: Although $S$ as defined in Eq. (2.20) is not a quantum operator, it is a linear operator on a function space, indeed a projection, so that, as an operator, $S^2 = S$. If $h$ is a function with arguments operated on by $S$ then $Sh$ is invariant under the group of permutations over which $S$ averages. In particular, $Sh$ is invariant under the swapping of any two of the arguments listed in $S$. That is the first point. The second point is that a transposition of $\tilde{\omega}_j$ and $\tilde{\omega}_k$ followed by a transposition of $\omega_j$ and $\omega_k$ is a transposition of $\tilde{\omega}_j$ and $\tilde{\omega}_k$.

B. Effect of a beam splitter

Generalizing on Eq. (8.28), if we neglect loss, reflections, and unwanted polarization couplings, we model the effect of a non-polarizing beam splitter (or fiber coupler) that splits say mode $a_j$ into modes $a_{j1}$ and $a_{j2}$ by

$$a_j^\dagger = ua_{j1}^\dagger + va_{j2}^\dagger$$  \hspace{1cm} (9.9)

for some complex $u$, $v$ such that $|u|^2 + |v|^2 = 1$. To see the effect of this splitting of $a_1$ on the state $|\psi_{nm}\rangle$, one makes the substitution for $a_1$ in Eq. (9.7) defined by Eq. (9.9). The important observation is that no further permutation symmetries enter, so the effect is only to replace

$$\prod_{j=1}^{m} a_1^\dagger(\omega_j)$$  \hspace{1cm} (9.10)

by

$$\sum_{k=0}^{m} \binom{m}{k} u^k v^{m-k} \left( \prod_{j=1}^{k} a_{11}^\dagger(\omega_j) \right) \left( \prod_{j=k+1}^{m} a_{12}^\dagger(\omega_j) \right).$$  \hspace{1cm} (9.11)

C. Effect of polarization rotation

Unlike beam splitting, the effect of a polarization rotation involves an added permutation symmetry, and accounting for this symmetry is essential in calculating the inner products needed to arrive at probabilities of detection. This additional symmetry occurs because polarization rotation mixes, for example, an $a_1$-mode with an $a_2$-mode.
[Part II of this paper contains Section 10, Appendices A through F, and the References, as outlined in the Table of Contents.]