I study a class of Wilsonian formulations of non-Abelian gauge theories in algebraic noncovariant gauges where the Wilsonian infrared cutoff \( \Lambda \) is inserted as a mass term for the propagating fields. In this way the Ward-Takahashi identities are preserved to all scales. Nevertheless the BRS-invariance is broken and the theory is gauge-dependent and unphysical at \( \Lambda \neq 0 \). Then I discuss the infrared limit \( \Lambda \to 0 \). I show that the singularities of the axial gauge choice are avoided in planar gauge and in light-cone gauge. Finally the rectangular Wilson loop of size \( 2L \times 2T \) is evaluated at lowest order in perturbation theory and a noncommutativity between the limits \( \Lambda \to 0 \) and \( T \to \infty \) is pointed out.

1 Introduction and conclusion

The aim of this talk is to present in a few words the results of Ref. 1, concerning the consistency of the Exact Renormalization Group Approach of Wilson in its application to gauge theories. In particular we refer here to its most recent formulation in which the Wilsonian cutoff \( \Lambda \) is interpreted as an infrared regulator and the evolution of the 1PI effective action \( \Gamma(\Phi, \Lambda) \) is studied from the ultraviolet scale \( \Lambda = \Lambda_0 \) down to the physical scale \( \Lambda = 0 \). The application of this framework to gauge theories is problematic since there is no way of inserting the infrared cutoff consistently with gauge-invariance (we mean with BRS-invariance): as a consequence the \( \Lambda \neq 0 \) theory is gauge-dependent and unphysical and the limit \( \Lambda \to 0 \) has to be taken strictly.

In particular one can ask three simple questions: i) is the \( \Lambda \to 0 \) limit of the 1PI Green functions regular? ii) is the \( \Lambda \to 0 \) limit of the would be physical quantities regular? iii) in the computation of the interquark potential from a Wilson loop of size \( 2L \times 2T \) with \( T \gg L \), is the limit \( T \to \infty \) commuting with the limit \( \Lambda \to 0 \)?

These questions can be fully answered in perturbation theory with the following results: i) the regularity of the \( \Lambda \to 0 \) limit strongly depends on the gauge choice; in particular in the axial gauge it is not possible to define the off-shell Green functions perturbatively, even at one-loop level and for an Abelian theory; ii) the \( \Lambda \to 0 \) limit is delicate even for would-be physical quantities like the Wilson loop: in particular the limit does not exist in the axial gauge at order \( O(g^4) \) in perturbation theory, as proved by A. Panza and R. Soldati (these proceedings and\(^4\)); iii) the definition of the interquark potential is a rather subtle question and in general the limits \( T \to \infty \) and \( \Lambda \to 0 \) do not commute. In the planar gauge case this is manifest even at \( O(g^2) \)
in perturbation theory.

In general gauges, including the usual covariant gauges, the noncommutativity of the limits is expected to hold at order $O(g^4)$ in perturbation theory, connected with a non-exponentiation of the Wilson loop at $\Lambda \neq 0$. We plan to elucidate this question in a future work. The rest of this talk is devoted to a discussion of the results i)-ii)-iii).

2 Massive linear gauges

As it is well known, in order to define a perturbative quantum field theory from a classical gauge-invariant field theory one is forced to break gauge invariance through the addition of a gauge-fixing term to the action. This procedure replaces the original local gauge symmetry with a new global graded symmetry, the celebrated BRS symmetry. The choice of the gauge-fixing term is at large extent arbitrary; however, in order to have independence of the gauge-invariant observables from the gauge choice, the gauge-fixing term, depending on gauge-fields, ghost fields and auxiliary fields, must be a BRS-cocycle. However, when an infrared regulator inconsistent with the BRS-symmetry is introduced, then the would be gauge-invariant observables become gauge-dependent. The natural expectation is that the gauge-dependent terms are vanishing as the infrared regulator is removed. However, this is a quite delicate point since, even if the recovering of the gauge symmetry at $\Lambda = 0$ has been proven for proper vertices at non-exceptional configurations of momenta\footnote{erg2: submitted to World Scientific on November 1, 2018}, there are no theorems guaranteeing the infrared safety of important physical quantities as the interquark potential, which is the first thing one would like to compute in the Wilson renormalization group approach\footnote{erg2: submitted to World Scientific on November 1, 2018}. This motivate our detailed study.

For concreteness here we restrict to linear gauge-fixings of kind

$$F(A) = L^\mu(p)A_\mu(p), \quad L_\mu(p) = an_\mu + bp_\mu ,$$

where $a$ and $b$ are numerical parameters and $n^\mu$ is a fixed four-vector in the Minkowsky space. With the notations of\footnote{erg2: submitted to World Scientific on November 1, 2018} the gauge-fixed action reads

$$S_F = \int_x -\frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} + \lambda \cdot F(A) - \bar{C} \cdot \frac{\partial F}{\partial A_\mu} D_\mu C + \frac{1}{2} \xi_2 \lambda \cdot \lambda ,$$

or equivalently, after elimination of the auxiliary fields $\lambda^a$,

$$S_{\text{lin.g.}}^{\text{red}} = \int_x -\frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} - \bar{C} \cdot L^\mu D_\mu C + \frac{1}{2} \xi_2 L^\mu A_\mu \cdot L^\nu A_\nu .$$

To this reduced action, we add an infrared cutoff as a mass-like term\footnote{erg2: submitted to World Scientific on November 1, 2018}

$$S_{\text{lin.g.}}^{\text{red}}(A, C, \bar{C}, A) = S_{\text{lin.g.}}^{\text{red}}(A, C, \bar{C}) + \int_x \frac{1}{2} \Lambda^2 A^2 .$$

Now we can define an invertible massive propagator for the gauge fields. The explicit form of the propagator in Euclidean space and in the $\xi_2 \to 0$ limit is

$$D_{\lambda,\mu\nu}(p_E) \overset{\xi_2 \to 0}{=} \frac{1}{p_E^2 + \Lambda^2} \delta_{\mu\nu} - \frac{p_E \cdot L (E_{\mu\nu} L + L_{\mu\nu})}{L^2 (p_E^2 + \Lambda^2 + L^2 \Lambda^2)} + \frac{L^2 L_{\mu\nu} E_{\mu\nu}}{(p_E^2 + \Lambda^2 + L^2 \Lambda^2)} - \frac{L^2 L_{\mu\nu} L_{\mu\nu}}{(p_E^2 + \Lambda^2 + L^2 \Lambda^2)} .$$

\footnote{erg2: submitted to World Scientific on November 1, 2018}
We notice that in limit $\xi_2 \to 0$ the propagator is both transverse
\[ L^\mu D_{\Lambda,\mu\nu}(p_E) \xrightarrow{\xi_2 \to 0} 0, \quad \forall \Lambda \] and invariant up to rescaling of the gauge-fixing $L(p_E) \to CL(p_E)$. In particular for $L_\mu \propto n_{E,\mu}$, with $n_{E,\mu}^2 > 0$, we find the massive axial gauge whereas for light-like vectors $L_\mu = n_{E,\mu}$ such as $n_{E}^2 = 0$ we find the massive light-cone gauge. The characteristic feature of these gauges is that the non-Abelian gauge symmetry can be expressed in terms of Abelian-like Ward-Takahashi identities, which hold for any $\Lambda$. This is the advantage with respect to the usual Wilsonian formulation in covariant gauges with a generic cutoff function; in this latter case there is a highly nontrivial fine-tuning problem to solve in order to implement correctly the (modified) Slavnov-Taylor identities.

In the axial gauge case for $n_{E}^2 = (0, 0, 1, 0)$ the propagator contains factors $1/(p_3^2 + \Lambda^2)$ which generate spurious divergences in the physical limit $\Lambda \to 0$; these divergences which will be discussed in the next section. In the light-cone case the gauge vector has the form $n_{E}^2 = (0, 0, 1, i)$ and there are spurious poles of kind $1/(p_E \cdot n_E)$ for any $\Lambda$. However, after Wick rotation in Minkowsky space $p_0 = -ip_4$, we see that these poles are regularized with the well known Mandelstam-Leibbrandt (ML) prescription which has very good analytical properties. Therefore the light-cone gauge perturbative expansion is expected to be safe to all orders. To be explicit, the Minkowskian propagator reads
\[ -D_{\Lambda,\mu\nu}(p) = \frac{1}{p^2 - \Lambda^2 + i\epsilon} \left\{ g_{\mu\nu} - \frac{n_\mu p_\nu + n_\nu p_\mu}{\|p \cdot n\|^2} + \frac{\Lambda^2 n_\mu n_\nu}{\|p \cdot n\|^2} \right\}, \] the spurious poles are regularized with the prescription
\[ \frac{1}{\|p \cdot n\|^2} = \frac{p \cdot n^*}{(p \cdot n)(p \cdot n^*) + i\epsilon}, \quad n = (1, 0, 0, 1), \quad n^* = (-1, 0, 0, 1) \] and the propagator reduces to the standard one at $\Lambda \to 0$ when the term proportional to $n_\mu n_\nu$ can be neglected. We point out that this is not always possible and that certainly the $n_\mu n_\nu$ term cannot be neglected in the computation of infrared singular quantities.

Finally, there is another algebraic noncovariant gauges which has been studied in the literature, the planar gauge. This gauge choice is not very convenient since the effective action does not satisfy simple Ward-Takahashi identities (the reason being the presence of derivatives in the gauge-fixing term). Nevertheless the computation of one-loop Feynman diagrams in planar gauge is simpler than in light-cone gauge and its study is useful for sake of comparison with the other gauge choices.

We found that the more convenient way to insert the infrared cutoff in order to have a simple propagator is to modify the massless BRS action as
\[ S_{\text{planar}}(\Lambda) = \int_x \left[ -\frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} - C \cdot n^{\mu} D_\mu C + \lambda \cdot n^{\mu} A_\mu + \frac{1}{2} \Lambda^2 A_\mu \cdot A^\mu + \frac{1}{2} \lambda \cdot \frac{n^2}{\partial^2 + \Lambda^2} \lambda \right]. \] This gives as reduced action
\[ S_{\text{planar}}^{\text{red}}(\Lambda) = \int_x \left[ -\frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} + \frac{1}{2} \Lambda^2 A_\mu \cdot A^\mu + \frac{\lambda}{2\pi^2} n^{\mu} A_\mu \cdot (\partial^2 + \Lambda^2) n_\mu A^\mu - C \cdot n^{\mu} D_\mu C \right]. \]
Notice that the “mass” $\Lambda^2$ multiplies the term
\[
\frac{1}{2} A^\mu \left( g_{\mu \nu} - \frac{n_\mu n_\nu}{n^2} \right) A^\nu = \frac{1}{2} A^i g_{ij} A^j, \quad i, j \in \{0, 1, 2\}
\]
and thus only transverse (with respect to $n_\mu$) degrees of freedom are screened. There are other possible ways of introducing the infrared cutoff in the planar gauge, but this is the more interesting one in the sense that one obtains a propagator which is identical to the propagator of the light-cone gauge, except for the fact that the gauge vector $n^\mu$ is space-like ($n^2 = -n_E^2 = -1$). As a consequence, there are spurious divergences at $p \cdot n = 0$ for any $\Lambda$. Therefore we need an explicit prescription to manage them. The simplest choice is the CPV prescription

\[
\frac{1}{|p \cdot n|} \equiv \lim_{\varepsilon \to 0} \frac{p \cdot n}{(p \cdot n)^2 + \varepsilon^2}.
\]

We remind that in higher order computations the CPV prescription could be problematic; however this issue is not relevant for the one-loop computations discussed here.

### 3 Problems of the pure axial gauge

The standard massless version of the pure axial gauge where the spurious divergences are managed with the CPV prescription is affected by many serious problems [13] (for a recent attempt of solving these problems in a nonstandard version of the axial gauge see for instance [11]); in particular it is impossible to define a renormalizable perturbation theory to all orders. The situation is similar in the Wilsonian formulation, at least at the perturbative level [4].

We found two major infrared problems: i) the Fourier transform of the propagator is divergent at $\Lambda \to 0$; ii) one loop Feynman diagrams which are infrared finite in covariant gauges becomes divergent in axial gauge for all configurations of momenta. Moreover, even for would be gauge-invariant quantities as finite size Wilson loops, the limit $\Lambda \to 0$ does not exist at order $O(g^4)$ in perturbation theory [4], therefore the axial gauge is definitely ruled out for perturbative applications. The origin of all problems comes from the part of the propagator proportional to $p_\mu p_\nu$,

\[
D^{pp}_{\Lambda, \mu \nu} = \frac{p_\mu p_\nu n^2}{(p^2 + \Lambda^2)((p \cdot n)^2 + n^2\Lambda^2)}.
\]

This quantity is a messy source of infrared divergencies due to the identity (in the sense of distributions)

\[
\frac{1}{p_3^2 + \Lambda^2} \xrightarrow{\Lambda \to 0} \frac{\pi}{\Lambda} \delta(p_3)
\]

*However, it should be noticed that the problems of the perturbative expansion of Green functions are absent in other kinds of expansions of the evolution equation, like for instance the expansion in covariant derivatives. As a matter of fact, the axial gauge Exact Renormalization Group equation is well defined before the expansion in powers of the gauge fields, as stressed by J. Pawlowski at this conference. Therefore the axial gauge formulation can be useful for nonperturbative applications.*
This means that for any regular function $f(p_3, \Lambda)$ such as $f(0,0) \neq 0$ we have
\[
\lim_{\Lambda \to 0} \int_{p_3} \frac{f(p_3, \Lambda)}{p_3^2 + \Lambda^2} = \lim_{\Lambda \to 0} \frac{f(0,0)}{2\Lambda} = \infty.
\tag{14}
\]
The problems relative to (13) are discussed at length in \[3\]. Here we simply notice that in planar and light-cone gauges the double pole is absent and therefore these problems are avoided. In particular it is possible to prove that the $x-$space propagator is regular in both gauges and that the one-loop Feynman diagrams are well defined and with a regular limit for off-shell configurations of momenta. At higher loops, the situation for the planar gauge is dubious, whereas the light-cone gauge is expected to be both infrared safe for non-exceptional configurations of momenta and renormalizable to all orders.

4 The Wilson loop test

The Wilson loop is the simplest physical quantity where the effects and the problems of the infrared regularization can be studied. In particular our scope here is to study the subtleties of the $\Lambda \to 0$ limit and to test how the essential property of the gauge-independence of the Wilson loop is recovered when the infrared cutoff is removed. In concrete in this section we compute the Wilson up to order $O(g^2)$ in perturbation theory. This is enough for elucidating various important features of axial, planar and light-cone gauges and it is a first step versus a more comprehensive computation at order $O(g^4)$ in perturbation theory.

For definiteness, we shall consider a rectangular Wilson loop $\Gamma_{LT}$ of size $2L \times 2T$, with $T \gg L$,
\[
W_{\Gamma_{LT}} = \frac{1}{N_c} < \text{Tr} \{ P \exp \left( ig \int_{\Gamma_{LT}} A_\mu dx^{\mu} \right) \} >
\tag{15}
\]
where $P$ denotes the path ordering on the loop $\Gamma_{LT}$, $\text{Tr}$ is the trace in the fundamental representation of $SU(N_c)$ and the average is evaluated via a perturbative expansion of the Euclidean functional integral. The Wilson loop can easily be computed at order $O(g^2)$ in perturbation theory and is given by the explicit expression
\[
W_{\Gamma_{LT}}^{(2)} = -g^2 C_F \int_{-1}^{1} ds_1 \int_{-1}^{1} ds_2 \frac{d\Gamma^{\mu}}{ds_1} \frac{d\Gamma^{\nu}}{ds_2} D_{\mu\nu}(\Gamma(s_1) - \Gamma(s_2)),
\tag{16}
\]
with $C_F = (N_c^2 - 1)/(2N_c)$. This formula can be further simplified in the $T \to \infty$ limit since various contributions are subleading. Moreover, due to the identity (which holds since $\Gamma_{LT}$ is a closed path)
\[
\int_{\Gamma_{LT}} dx^{\mu} dy^{\nu} D_{\mu\nu}(x - y) = \int_{\Gamma_{LT}} dx^{\mu} dy^{\nu} D_{T\mu\nu}^{T}(x - y),
\tag{17}
\]
actually only the transverse part of the propagator contributes. At zero mass $D_{T\mu\nu}^{T}(p)$ is the same in all gauges and this is the reason why the final result is gauge-independent; however at $\Lambda \neq 0$ there is a dependence on the gauge vector $n_\mu$ coming from the part of the propagator proportional to $\Lambda^2$. It is this part which is problematic, in the sense that there could be a problem of commutativity between
the limits $\Lambda \to 0$ and $T \to \infty$. This happens in the planar gauge case, where we have a gauge-dependent contribution to the interquark potential of the kind

$$
\frac{2g^2 N_c C_F}{3} \Lambda^2 T^2 \frac{\exp(-\Lambda \cdot 2L)}{4\pi \cdot 2L}.
$$

(18)

We see that the gauge-dependent contribution is quadratically vanishing in the limit $\Lambda \to 0$, at finite $T$, but quadratically divergent for $T \to \infty$ at finite $\Lambda$. The $T \to \infty$ limit cannot be taken before the $\Lambda \to 0$ limit. This is the crucial point of our analysis. We expect this feature become manifest for any gauge at higher orders in perturbation theory.

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