TWO SIMPLE CRITERION TO OBTAIN EXACT CONTROLLABILITY AND STABILIZATION OF A LINEAR FAMILY OF DISPERSIVE PDE’S ON A PERIODIC DOMAIN

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Abstract. In this work, we use the classical moment method to find a practical and simple criterion to determine if a family of linearized Dispersive equations on a periodic domain is exactly controllable and exponentially stabilizable with any given decay rate in \( H^s(T) \) with \( s \in \mathbb{R} \). We apply these results to prove that the linearized Smith equation, the linearized dispersion-generalized Benjamin-Ono equation, the linearized fourth-order Schrödinger equation, and the Higher-order Schrödinger equations are exactly controllable and exponentially stabilizable with any given decay rate in \( H^s(T) \) with \( s \in \mathbb{R} \).

1. Introduction

In this work, we consider a family of linear one-dimensional dispersive equations on the periodic domain \( T := \mathbb{R}/(2\pi\mathbb{Z}) \), and investigate its control properties from the point of view of distributed control. Specifically, we consider the family of equations

\[
\partial_t u - \partial_x A u = f(x,t), \quad x \in T, \ t \in \mathbb{R},
\]

where \( u = u(x,t) \) denotes a real or complex-valued function of two real variables \( x \) and \( t \), the forcing term \( f = f(x,t) \) is added to the equation as a control input supported in a given open set \( \omega \subset T \), and \( A \) denotes a linear Fourier multiplier operator. We assume that the multiplier \( A \) is of order \( r - 1 \), for some \( r \in \mathbb{R} \), with \( r \geq 1 \), that is, the symbol \( a : \mathbb{Z} \to \mathbb{R} \) is given by

\[
\hat{A}u(k) := a(k)\hat{u}, \ k \in \mathbb{Z},
\]

where \( \hat{u} \) stands for the Fourier transform of \( u \) (see (2.2)), and

\[
|a(k)| \leq C|k|^{r-1}, \quad |k| \geq k_0,
\]

for some \( k_0 \geq 0 \) and some positive constant \( C \).

Equation (1.1) encompass a wide class of linear dispersive equations. For instance, the well-known linearized Korteweg-de Vries equation \( (A = -\partial_x^2) \), the Schrödinger equation \( (A = i\partial_x) \), the Benjamin-Ono equation \( (A = \mathcal{H}\partial_x, \text{where } \mathcal{H} \text{ stands for the Hilbert transform}) \), and the Benjamin equation \( (A = \partial_x^2 + \alpha\mathcal{H}\partial_x, \text{where } \alpha \text{ is a positive constant}) \). In the literature there is a wide range of references studying controllability and stabilization properties of linear and nonlinear dispersive equations. Specifically, for the Korteweg-de Vries equation (KdV) equation, the results regarding controllability and stabilization can be found in

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For the Schrödinger equation we refer the reader to [21, 22, 9, 33, 34]. Also, the study on the controllability and stabilization for the Benjamin and Benjamin-Ono (BO) equations have received attention in the last decade, see [30, 31] and [25, 24, 26], respectively. So, our main goal in this paper is to study all these equations in a unified way.

Under the above conditions, the linear operator $A$ commutes with derivatives and may be seen as a self-adjoint operator on $L^2_p(T)$ (see Section 2 for notations). Note also that solutions of the homogeneous equation (1.1) ($f = 0$) with initial data $u(0) = u_0$ conserve the “mass” in the sense that

$$2\pi \hat{u}(0, t) = 2\pi \hat{u}_0(0),$$

for all $t \in \mathbb{R}$, where $\hat{u}$ stands for the Fourier transform of $u$ in the space variable (see (2.2)).

Before proceeding let us make clear the problems we are interested in.

**Exact controllability problem:** Let $s \in \mathbb{R}$ and $T > 0$ be given. Let $u_0$ and $u_1$ in $H^s_p(T)$ be given with $\hat{u}_0(0) = \hat{u}_1(0)$. Can one find a control input $f$ such that the unique solution $u$ of the initial-value problem (IVP)

$$\begin{cases}
\partial_t u - \partial_x Au = f(x, t), & x \in T, \ t \in \mathbb{R},

u(x, 0) = u_0(x)
\end{cases}$$

(1.4)

is defined until time $T$ and satisfies $u(x, T) = u_1(x)$ for all $x \in T$?

**Asymptotic stabilizability problem:** Let $s \in \mathbb{R}$ and $u_0 \in H^s_p(T)$ be given. Can one define a feedback control law $f = Ku$, for some linear operator $K$, such that the resulting closed-loop system

$$\begin{cases}
\partial_t u - \partial_x Au = Ku, & x \in T, \ t \in \mathbb{R}^+,

u(x, 0) = u_0(x),
\end{cases}$$

(1.5)

is globally well-defined and asymptotically stable to an equilibrium point as $t \to +\infty$?

In the present manuscript we use the classical Moment method (see [38]) and a generalization of Ingham’s inequality see (see [20, Theorem 4.6] and [15]), to find a practical criterion regarding the eigenvalues associated with the operator $\partial_x A$ to determine if equation (1.1) is exactly controllable and exponentially stabilizable. Therefore, we were able to extend the techniques used by the authors in [25, 30, 33] to a wide class of linearized dispersive equations on a periodic domain.

**Remark 1.1.** Generalizing these techniques to linear systems of two or more equations require additional efforts because the mixed dispersive terms present in the equations generally induce a modification of the orthogonal basis we are considering on $L^2_p(T) \times L^2_p(T)$ (see for instance [5, Proposition. 2.2]). This usually implies a loss of regularity of the considered controls (see also [29, Theorem 2.23]).

As usual in control theory for dispersive models (see [25, 40, 23, 30]), in order to keep the mass of (1.4) conserved, we define a bounded linear operator $G : H^s_p(T) \to H^s_p(T)$ in the following way: let $g$ be a real non-negative function in $C^\infty_p(T)$ such that

$$2\pi \tilde{g}(0) = \int_0^{2\pi} g(x) \, dx = 1,$$

(1.6)
and assume \( \text{supp } g = \omega \subset \mathbb{T} \), where \( \omega = \{ x \in \mathbb{T} : g(x) > 0 \} \) is an open interval. The operator \( G \) is then defined as
\[
G(\phi) := g\phi - g \langle \phi, g \rangle, \quad \phi \in H^s_p(\mathbb{T}),
\]
where the first product must be understood in the periodic distributional sense and \( \langle \cdot, \cdot \rangle \) denotes the pairing between \( \mathcal{S}' \) and \( \mathcal{S} \) (see notations below).

The control input \( f \) is then chosen to be of the form \( f(\cdot, t) = G(h(\cdot, t)), t \in [0, T] \). As a consequence, the function \( h \in L^2([0, T]; H^s_p(\mathbb{T})) \) is now viewed as the new control function.

**Remark 1.2.** Some remarks concerning the operator \( G \) are in order.

(1) It is not difficult to see that \( G \) is self-adjoint in \( L^2_p(\mathbb{T}) \) (see [30, Proposition 3.2]). In addition, the authors in [25, Remark 2.1] and [29, Lemma 2.20] showed that for any \( s \in \mathbb{R} \) the operator \( G \) acting from \( L^2([0, T]; H^s_p(\mathbb{T})) \) into \( L^2([0, T]; H^s_p(\mathbb{T})) \) is linear and bounded.

(2) When \( s \geq 0 \) we may write
\[
G(\phi)(x) = g(x) \left[ \phi(x) - \int_0^{2\pi} \phi(y)g(y) \, dy \right],
\]
which is exactly the operator defined, for instance, in [25, 40, 23, 30].

(3) By recalling that for any \( \phi \in \mathcal{S}' \) and \( g \in \mathcal{S} \) (see [17, Corollary 3.167])
\[
\langle \phi, g \rangle = 2\pi \sum_{k \in \mathbb{Z}} \hat{\phi}(k)\hat{g}(-k),
\]
in view of (1.6), we obtain
\[
\hat{G}(\phi)(0) = \hat{\phi} \ast \hat{g}(0) - \hat{g}(0) \langle \phi, g \rangle = \sum_{j \in \mathbb{Z}} \hat{\phi}(j)\hat{g}(-j) - \hat{g}(0) \langle \phi, g \rangle = 0,
\]
where the convolution of two sequences of complex numbers \((\alpha_k)_{k \in \mathbb{Z}} \) and \((\beta_k)_{k \in \mathbb{Z}} \) is the sequence \((\alpha * \beta)_k \) defined by
\[
(\alpha * \beta)_k = \sum_{j \in \mathbb{Z}} \alpha_j \beta_{k-j}.
\]
This implies that any solution \( u \) of (1.4) (with \( f(x, t) = G(h(x, t)) \)) conserves the quantity \( 2\pi \hat{u}(0, t) \). In particular, if \( \hat{u}_0(0) = 0 \) then \( 2\pi \hat{u}(0, t) = 0 \), for any \( t \in [0, T] \).

Next, we turn attention to our criteria to obtain the controllability and stabilization of equation (1.1). As we will see, they directly link these problems with some specific properties of the eigenvalues and eigenfunctions associated to the operator \( \partial_x \mathcal{A} \). To derive our first criterion regarding exact controllability, we assume that \( \partial_x \mathcal{A} \) has a countable number of eigenvalues that are all simple, except by a finite number that have finite multiplicity. Specifically, we will assume that the following hypotheses hold:

\( (H1) \) \( \partial_x \mathcal{A} \psi_k = i\lambda_k \psi_k \), where \( \psi_k \) is defined in (2.1) and \( \lambda_k = ka(k), \) for all \( k \in \mathbb{Z} \).

Note we are counting multiplicities, implying that the eigenvalues in the sequence \( \{i\lambda_k\}_{k \in \mathbb{Z}} \) are not necessarily distinct. For each \( k_1 \in \mathbb{Z} \), we set \( I(k_1) := \{ k \in \mathbb{Z} : \lambda_k = \lambda_{k_1} \} \) and \( m(k_1) := |I(k_1)| \), where \( |I(k_1)| \) denotes the number of elements in \( I(k_1) \). Concerning the quantity \( m(k_1) \), we assume the following:
(H2) \( m(k_1) \leq n_0 \), for some \( n_0 \in \mathbb{N} \) and for all \( k_1 \in \mathbb{Z} \),

and

(H3) there exists \( k_1^* \in \mathbb{N} \) such that \( m(k_1) = 1 \), for all \( k_1 \in \mathbb{Z} \) with \( |k_1| \geq k_1^* \).

Assumptions (H2) and (H3) together say that all eigenvalues \( i\lambda_k \) have finite multiplicity. In addition, they are simple eigenvalues for sufficiently large indices.

If we count only distinct eigenvalues, we may obtain a sequence \( \{\lambda_k\}_{k \in \mathbb{I}} \), \( \mathbb{I} \subseteq \mathbb{Z} \), with the property that \( \lambda_k \neq \lambda_{k_2} \), for any \( k_1, k_2 \in \mathbb{I} \) with \( k_1 \neq k_2 \). Our main result at this point reads as follows.

**Theorem 1.3** (Criterion I). Let \( s \in \mathbb{R} \) and assume (H1), (H2), and (H3). Suppose that

\[
\gamma := \inf_{k, n \in \mathbb{Z}} |\lambda_k - \lambda_n| > 0 \tag{1.8}
\]

and

\[
\gamma' := \sup_{S \subseteq \mathbb{I}, n \in \mathbb{Z}} \inf_{k \in S \setminus \lambda_k} |\lambda_k - \lambda_n| > 0, \tag{1.9}
\]

where \( S \) runs over all finite subsets of \( \mathbb{I} \). Then for any \( T > \frac{2\pi}{\gamma} \) and for each \( u_0, u_1 \in H^s_p(\mathbb{T}) \) with \( \tilde{u}_0(0) = \tilde{u}_1(0) \), there exists a function \( h \in L^2([0, T]; H^s_p(\mathbb{T})) \) such that the unique solution \( u \) of the non-homogeneous system

\[
\begin{align*}
\partial_t u(t) &= \partial_x Au(t) + G(h)(t) \in H^{s-r}_p(\mathbb{T}), \quad t \in (0, T), \\
u(0) &= u_0 \in H^s_p(\mathbb{T}),
\end{align*}
\tag{1.10}
\]

satisfies \( u(T) = u_1 \). Furthermore,

\[
\|h\|_{L^2([0, T]; H^s_p(\mathbb{T}))} \leq \nu (\|u_0\|_{H^s_p(\mathbb{T})} + \|u_1\|_{H^s_p(\mathbb{T})}),
\tag{1.11}
\]

for some positive constant \( \nu \equiv \nu(s, g, T) \).

**Remark 1.4.** Note that if \( \gamma' \) defined in (1.9) is infinite (\( \gamma' = +\infty \)), then (1.10) is exactly controllable for any positive time \( T \). In particular, if (H1) holds and

(i) \( a(-k) = a(k) \), for all \( k \in \mathbb{I} \);

(ii) \( \lim_{|k| \to +\infty} |(k + 1)a(k + 1) - ka(k)| = +\infty \), where \( k \) runs over \( \mathbb{I} \),

then system (1.10) is exactly controllable for any \( T > 0 \). In fact, from (i) we infer that \( \lambda_k = -\lambda_{-k} \) for all \( k \in \mathbb{I} \). On the other hand, property (ii) yields that \( \gamma' = +\infty \) and the real sequence \( \{\lambda_k\}_{k \in \mathbb{I}} \) is strictly increasing/decreasing for \( k \in \mathbb{I} \) with \( |k| > k_1^* \) for some \( k_1^* \), implying that (H2)-(H3) hold. Also, since terms of the sequence \( \{\lambda_k\}_{k \in \mathbb{I}} \) are distinct, it is clear that (1.8) holds.

Property (ii) in Remark 1.4 implies the so called “asymptotic gap condition” for the eigenvalues \( \{i\lambda_k\}_{k \in \mathbb{I}} \) associated with the operator \( \partial_x A \). This property is crucial to obtain the exact controllability for any \( T > 0 \). It appears that many dispersive models hold the properties (i) and (ii). For instance, the linearized KdV equation \([40, 23]\), the linearized Benjamin-Ono equation \([25]\), and the linearized Benjamin equation \([30]\). See Figure 1 for an illustrative figure.

Next we shall prove that even when we have an infinity quantity of repeated eigenvalues associated with \( \partial_x A \) in a particular form, we can still obtain an exact
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Figure 1. Dispersion of $\lambda_k$'s for KdV, Benjamin-Ono and Benjamin equations

controllability result. This will provide our second criterion. For this, we will assume (H1) and

(H4) there are $n_0, k_1^* \in \mathbb{N}$ such that $m(k_1) \leq n_0$, for all $k_1 \in \mathbb{Z}$ with $|k_1| < k_1^*$. In addition, $m(k_1) = 2$ for all $|k_1| \geq k_1^*$.

and

(H5) $a(-k) = -a(k)$, for all $k \in \mathbb{Z}$ with $|k| \geq k_1^*$.

Assumption (H4) says that, except near the origin, all eigenvalues are double. Moreover, in view of (H5), $\lambda_k = \lambda_{-k}$, for all $|k| \geq k_1^*$. This implies that $I(k_1) = \{-k_1, k_1\}$ for $|k_1| \geq k_1^*$.

As before, if we are interested in counting only the distinct eigenvalues we can obtain a set

$$J \subset \{-k_1^* + 1, -k_1^* + 2, \ldots\}$$

such that the sequence $\{\lambda_k\}_{k \in J}$ has the property that $\lambda_{k_1} \neq \lambda_{k_2}$, for any $k_1, k_2 \in J$, with $k_1 \neq k_2$.

Our second result regarding controllability reads as follows.

**Theorem 1.5 (Criterion II).** Let $s \in \mathbb{R}$ and assume (H1), (H4), and (H5). Suppose

$$\tilde{\gamma} := \inf_{k, n \in J \setminus k \neq n} |\lambda_k - \lambda_n| > 0$$

and

$$\tilde{\gamma}' := \sup_{S \subseteq \mathbb{J}} \inf_{k, n \in \mathbb{J} \setminus S \setminus k \neq n} |\lambda_k - \lambda_n| > 0,$$

where $S$ runs over the finite subsets of $\mathbb{J}$. Then for any $T > \frac{2\pi}{\tilde{\gamma}}$ and for each $u_0, u_1 \in H^s_p(\mathbb{T})$ with $\hat{u}_0(0) = \hat{u}_1(0)$, there exists a function $h \in L^2([0, T]; H^s_p(\mathbb{T}))$ such that
the unique solution $u$ of the non-homogeneous system (1.10) satisfies $u(T) = u_1$. Moreover, there exists a positive constant $\nu \equiv \nu(s, g, T)$ such that (1.11) holds.

Remark 1.6. If hypotheses $(H1)$, $(H4)$ and $(H5)$ hold with
$$\lim_{k \to +\infty} |(k + 1)a(k + 1) - ka(k)| = +\infty,$$
where $k$ take values in $\mathcal{J}$, then the system (1.10) is exactly controllable for any $T > 0$.

It is not difficult to see that the linear Schrödinger equation holds the assumptions in Theorem 1.5 and Remark 1.6. See Figure 2 for an illustration of the eigenvalues. Actually, the exact controllability and exponential stabilization for the linear (and nonlinear cubic) Schrödinger equations were proved in [33], where the authors used $f(x, t) = Gh(x, t) := g(x)h(x, t)$ as a control input. Here we show that the control input as described in (1.7) also serves to prove the exact controllability. The advantage of using this control input is that it allow us to get a controllability and stabilization result for the linear Schrödinger equation in the Sobolev space $H^s_p(\mathbb{T})$ for any $s \in \mathbb{R}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{disp_lambda.png}
\caption{Dispersion of $\lambda_k$’s for Schrödinger equation}
\end{figure}

Attention is now turned to our stabilization results. In what follows, $G^*$ denotes the adjoint operator of $G$. We will prove that if one chooses the feedback law $Ku = -GG^*u$ then the closed-loop system (1.5) is exponentially stable. More precisely, we have the following.

**Theorem 1.7.** Let $g$ be as in (1.6) and let $s \in \mathbb{R}$ be given. Under the assumptions of Theorem 1.3 or Theorem 1.5, there exist positive constants $M = M(g, s)$ and $\alpha = \alpha(g)$ such that for any $u_0 \in H^s_p(\mathbb{T})$ the unique solution $u$ of the closed-loop system
$$\begin{aligned}
\begin{cases}
u(0) = u_0 \in H^s_p(\mathbb{T}), \\
&\|u(\cdot, t) - \hat{u}_0(0)\|_{H^s_p(\mathbb{T})} \leq Me^{-\alpha t}\|u_0 - \hat{u}_0(0)\|_{H^s_p(\mathbb{T})}, \text{ for all } t \geq 0.
\end{cases}
\end{aligned}$$

satisfies
The feedback law $Ku = -GG^*u$ in Theorem 1.7 is the simplest one providing the exponential decay with a fixed exponential rate. However, by changing the feedback law one is able to show that the resulting closed-loop system actually has an arbitrary exponential decay rate. More precisely,

**Theorem 1.8.** Let $s \in \mathbb{R}$, $\lambda > 0$, and $u_0 \in H^s_p(\mathbb{T})$ be given. Under the assumptions of Theorem 1.3 or Theorem 1.5, there exists a bounded linear operator $K_\lambda$ from $H^s_p(\mathbb{T})$ to $H^s_p(\mathbb{T})$ such that the unique solution $u$ of the closed-loop system

$$
\begin{aligned}
&u(t) = \partial_t u(t) + K_\lambda u(t), \\
&t > 0,
\end{aligned}
$$

(1.14)

satisfies

$$
\|u(\cdot, t) - \hat{u}_0(0)\|_{H^s_p(\mathbb{T})} \leq M e^{-\lambda t}\|u_0 - \hat{u}_0(0)\|_{H^s_p(\mathbb{T})},
$$

for all $t \geq 0$, and some positive constant $M = M(g, \lambda, s)$.

The paper is organized as follows: In section 2 a series of preliminary results that will be used throughout this work are recalled. In Section 3 we prove well-posedness results. The main results regarding controllability and stabilization are proved in Sections 4 and 5, respectively. In Section 6, we apply our general criteria to establish the corresponding results regarding exact controllability and exponential stabilization for the linearized Smith equation, the linearized dispersion-generalized Benjamin-Ono equation, the fourth-order Schrödinger and a higher-order Schrödinger equation. Finally, in Section 7 some concluding remarks and future works are presented.

## 2. Preliminaries

In this section we introduce some basic notations and recall the main tools to obtain our results. We denote by $\mathcal{P}$ the space $C^\infty_p(\mathbb{T})$ of all $C^\infty$ functions that are $2\pi$-periodic. By $\mathcal{P}'$ (the dual of $\mathcal{P}$) we denote the space of all periodic distributions. By $L^2_p(\mathbb{T})$ we denote the standard space of the square integrable $2\pi$-periodic functions. It is well-known that the sequence $\{\psi_k\}_{k \in \mathbb{Z}}$ given by

$$
\psi_k(x) := \frac{e^{ikx}}{\sqrt{2\pi}}, \quad k \in \mathbb{Z}, \quad x \in \mathbb{T}
$$

(2.1)

is an orthonormal basis for $L^2_p(\mathbb{T})$. The Fourier transform of $v \in \mathcal{P}'$ is defined as

$$
\hat{v}(k) = \frac{1}{2\pi} \langle v, e^{-ikx} \rangle, \quad k \in \mathbb{Z}.
$$

(2.2)

Next we introduce the periodic Sobolev spaces. For a more detailed description and properties of these spaces, we refer the reader to [17]. Given $s \in \mathbb{R}$, the (periodic) Sobolev space of order $s$ is defined as

$$
H^s_p(\mathbb{T}) = \left\{ v \in \mathcal{P}' \mid \| v \|_{H^s_p(\mathbb{T})} := 2\pi \sum_{k = -\infty}^{\infty} (1 + |k|)^{2s} |\hat{v}(k)|^2 < \infty \right\}.
$$

We consider the space $H^s_p(\mathbb{T})$ as a Hilbert space endowed with the inner product

$$
(h, v)_{H^s_p(\mathbb{T})} = 2\pi \sum_{k \in \mathbb{Z}} (1 + |k|)^{2s} \hat{h}(k) \overline{\hat{v}(k)},
$$

(2.3)
For any $s \in \mathbb{R}$, $(H^s_p(T))'$, the topological dual of $H^s_p(T)$, is isometrically isomorphic to $H^{-s}_p(T)$, where the duality is implemented by the pairing

$$\langle h, v \rangle_{H^{-s}_p(T) \times H^s_p(T)} = 2\pi \sum_{k \in \mathbb{Z}} \hat{h}(k) \hat{v}(k),$$

for all $v \in H^s_p(T)$, $h \in H^{-s}_p(T)$.

**Remark 2.1.** It is well-known that any distribution $v \in \mathcal{D}'$ may be written as (see, for instance, [17, page 188])

$$v = \sqrt{2\pi} \sum_{k \in \mathbb{Z}} \hat{v}(k) \psi_k,$$

(2.4)

where the series converges in the sense of $\mathcal{D}'$. In particular, any $v \in H^s_p(T)$, $s \in \mathbb{R}$, can be written in the form (2.4).

We also consider the closed subspace

$$H^s_0(T) := \{ v \in H^s_p(T) | \hat{v}(0) = 0 \}.$$

It can be seen that if $s_1, s_2 \in \mathbb{R}$ with $s_1 \geq s_2$, then $H^{s_1}_0(T) \hookrightarrow H^{s_2}_0(T)$, where the embedding is dense. We denote $H^s_0(T)$ by $L^2_0(T)$. In particular, $L^2_0(T)$ is a closed subspace of $L^2_p(T)$.

We continue with some characterization of Riesz basis in Hilbert spaces (see [13] for more details). In what follows, $J$ represents a countable set of indices which could be finite or infinite.

**Theorem 2.2.** Let $\{x_n\}_{n \in J}$ be a sequence in a Hilbert space $H$. Then the following statements are equivalent.

1. $\{x_n\}_{n \in J}$ is a Riesz basis for $H$ (see [13, Definition 7.9]).
2. $\{x_n\}_{n \in J}$ is complete in $H$ (see [13, Definition 1.25]) and there exist constants $A, B > 0$ such that

   $$\text{for all } c_1, ..., c_N \text{ scalars, } A \sum_{n=1}^{N} |c_n|^2 \leq \| \sum_{n=1}^{N} c_n x_n \|_H^2 \leq B \sum_{n=1}^{N} |c_n|^2.$$  

3. There is an equivalent inner product $(\cdot, \cdot)$ for $H$ such that $\{x_n\}_{n \in J}$ is an orthonormal basis for $H$ with respect to $(\cdot, \cdot)$.
4. $\{x_n\}_{n \in J}$ is a complete Bessel sequence (see [13, Definition 7.1]) and possesses a biorthogonal system $\{y_n\}_{n \in J}$ (see [13, Definition 4.10]) that is also a complete Bessel sequence.

**Proof.** See [13, Theorem 7.13].

Finally, we recall the generalized Ingham’s inequality.

**Theorem 2.3.** Let $\{\lambda_k\}_{k \in J}$ be a family of real numbers, satisfying the uniform gap condition

$$\gamma = \inf_{k, n \in J, k \neq n} |\lambda_k - \lambda_n| > 0.$$  

Set

$$\gamma' = \sup_{S \subset J} \inf_{k, n \in J \setminus S, k \neq n} |\lambda_k - \lambda_n| > 0,$$

where $S$ runs over all finite subsets of $J$. 
If $I$ is a bounded interval of length $|I| > \frac{2\pi}{\gamma}$, then there exist positive constants $A$ and $B$ such that
\[ A \sum_{k \in J} |c_k|^2 \leq \int_I |f(t)|^2 dt \leq B \sum_{k \in J} |c_k|^2, \]
for all functions of the form $f(t) = \sum_{k \in J} c_k e^{i\lambda_k t}$ with square-summable complex coefficients $c_k$.

Proof. See [20, page 67]. \[ \square \]

For further generalizations of the Ingham inequality (see [15]) we refer the reader to [3] and [20].

3. Well-posedness

In this section we establish a global well-posedness result for system (1.10). We start with some results concerning the homogeneous equation. This results are quite standard but for the sake of completeness we bring the main steps.

Proposition 3.1. Let $r$ be as in (1.3). For any $u_0 \in H^r_p(\mathbb{T})$, the homogeneous problem
\[
\begin{cases}
    u \in C(\mathbb{R}; H^r_p(\mathbb{T})) \cap C^1(\mathbb{R}, L^2_p(\mathbb{T})), \\
    \partial_t u = \partial_x A u \in L^2_p(\mathbb{T}), \quad t \in \mathbb{R}, \\
    u(0) = u_0,
\end{cases}
\]
has a unique solution.

Proof. First note that from Plancherel’s identity, for any $\varphi, \psi \in D(\partial_x A) = H^r_p(\mathbb{T})$, we have
\[
(\partial_x A \varphi, \psi)_{L^2_p(\mathbb{T})} = 2\pi \left( \sum_{k=-\infty}^{+\infty} \partial_x A \varphi(k) \overline{\psi(k)} \right)
= 2\pi \left( \sum_{k=-\infty}^{+\infty} ika(k) \hat{\varphi}(k) \overline{\hat{\psi}(k)} \right)
= -2\pi \left( \sum_{k=-\infty}^{+\infty} \hat{\varphi}(k)(-ik)a(k) \overline{\hat{\psi}(k)} \right)
= - (\varphi, \partial_x A \psi)_{L^2_p(\mathbb{T})},
\]
which implies that $\partial_x A$ is skew-adjoint. Hence, Stone’s theorem gives that $\partial_x A$ generates a strongly continuous unitary group $\{U(t)\}_{t \in \mathbb{R}}$ on $L^2_p(\mathbb{T})$. Therefore, Theorem 3.2.3 in [6] yields the desired result. \[ \square \]

Proposition 3.1 provides the well-posedness theory for (3.1) only for initial data in $H^r_p(\mathbb{T})$. However, we can still obtain the well-posedness for initial data in $H^s_p(\mathbb{T})$ for any $s \in \mathbb{R}$. To do so, one needs a more accurate description of the unitary group $\{U(t)\}_{t \in \mathbb{R}}$. At least in a formal level, by taking Fourier’s transform in the spatial variable, it is not difficult to see that the solution of (3.1) may be written as
\[
\hat{u}(t)(k) = e^{ika(k)t} \hat{u}_0(k), \quad k \in \mathbb{Z},
\]
or, by taking the inverse Fourier transform,
\[
u(t) = \left(e^{ika(k)t} \hat{u}_0(k)\right) \vee, \quad t \in \mathbb{R}. \]

This means that
\[ u(x, t) = \sum_{k \in \mathbb{Z}} e^{ik\alpha(t)} x \hat{u}_0(k) e^{ikx}, \quad t \in \mathbb{R}, \]  
must be the unique solution of (3.1).

The above calculation suggests that, in a rigorous way, we may define the family of linear operators \( U : \mathbb{R} \to \mathcal{L}(H^s_p(T)) \) by
\[ t \to U(t) \psi := e^{\partial_x At} \psi = (e^{ik\alpha(t)} x \hat{\psi}(k))^{v'}, \]  
in such a way that the solution of (3.1) now becomes \( u(t) = U(t)u_0, \quad t \in \mathbb{R}. \)

From the growth condition (1.3) and classical results on the semigroup theory (see for instance [6], [32] or [17] for additional details), we can show that the family of linear operators \( \{U(t)\}_{t \in \mathbb{R}} \) given by (3.5) indeed defines a strongly continuous one-parameter unitary group on \( H^s_p(T) \), for any \( s \in \mathbb{R} \). Additionally, if \( u(t) = U(t)u_0 \) with \( u_0 \in H^s_p(T) \), then
\[ \lim_{h \to 0} \left\| \frac{u(t + h) - u(t)}{h} - \partial_x Au \right\|_{H^{s-r}(T)} = 0, \]  
uniformly with respect \( t \in \mathbb{R} \). In particular, the following result holds.

**Theorem 3.2.** Let \( s \in \mathbb{R} \) and \( u_0 \in H^s_p(T) \) be given. Then the homogeneous problem
\[ \begin{cases} u \in C(\mathbb{R}; H^s_p(T)), \\ \partial_t u = \partial_x Au \in H^{s-r}(T), t \in \mathbb{R}, \\ u(0) = u_0, \end{cases} \]  
has a unique solution.

Next, we deal with the well-posedness of the non-homogenous linear problem (1.10).

**Lemma 3.3.** Let \( 0 < T < \infty, s \in \mathbb{R}, \) \( u_0 \in H^s_p(T) \), and \( h \in L^2([0, T]; H^s_p(T)) \). Then, there exists a unique mild solution \( u \in C([0, T], H^s_p(T)) \) for the IVP (1.10).

**Proof.** This is a consequence of Corollary 2.2 and Definition 2.3 in [32, page 106], and the fact that \( G(h) \in L^1([0, T]; H^s_p(T)) \). Furthermore, the unique (mild) solution of (1.10) is given by
\[ u(t) = U(t)u_0 + \int_0^t U(t - t')Gh(t') dt', \quad t \in [0, T]. \]  
This completes the proof of the lemma. \( \square \)

4. **Proof of the Control Results**

In this section we use the classical moment method (see [38]) to show the criteria I and II regarding exact controllability for (1.10). First of all, by replacing \( u_1 \) by \( u_1 - U(T)u_0 \) if necessary, we may assume without loss of generality that \( u_0 = 0 \) (see [30, page 10]), implying that \( \tilde{u}_1(0) = \tilde{u}_0(0) = 0 \). Consequently, if we write \( u_1(x) = \sum_{k \in \mathbb{Z}} c_k \psi_k(x) \) with \( \psi_k \) as in (2.1) then \( c_0 = 0 \).

Our first result is a characterization to get the exact controllability for (1.10). Its proof is similar to the proof of Lemma 4.1 in [30], passing to the frequency space when necessary; so we omit the details.
Lemma 4.1. Let $s \in \mathbb{R}$ and $T > 0$ be given. Assume $u_1 \in H^s_p(\mathbb{T})$ with $\widehat{u_1}(0) = 0$. Then, there exists $h \in L^2([0,T];H^s_p(\mathbb{T}))$ such that the solution of the IVP (1.10) with initial data $u_0 = 0$ satisfies $u(T) = u_1$ if and only if
\[
\int_0^T \langle Gh(\cdot,t), \varphi(\cdot,t) \rangle_{H^s_p \times (H^s_p)'} \, dt = \langle u_1, \varphi_0 \rangle_{H^s_p \times (H^s_p)'} ,
\] (4.1)
for any $\varphi_0 \in (H^s_p(\mathbb{T}))'$, and $\varphi$ is the solution of the adjoint system
\[
\left\{
\begin{array}{l}
\varphi \in C([0,T]; (H^s_p(\mathbb{T}))'), \\
\partial_t \varphi = \partial_x A \varphi \in H^{-s-r}_p(\mathbb{T}), \quad t > 0, \\
\varphi(T) = \varphi_0.
\end{array}
\right.
\] (4.2)

Next corollary is a consequence of Lemma 4.1. Having in mind its importance, we write the proof.

Corollary 4.2. Let $s \in \mathbb{R}$, $T > 0$, and $u_1 \in H^s_p(\mathbb{T})$ with $\widehat{u_1}(0) = 0$ be given. Then, there exists $h \in L^2([0,T];H^s_p(\mathbb{T}))$, such that the unique solution of the IVP (1.10) with initial data $u_0 = 0$ satisfies $u(T) = u_1$ if and only if there exists $\delta > 0$ such that
\[
\int_0^T \|G^* U(\tau)^* \phi^* \|^2_{(H^s_p(\mathbb{T}))'}(\tau) \, d\tau \geq \delta^2 \|\phi^*\|^2_{(H^s_p(\mathbb{T}))'},
\] (4.3)
for any $\phi^* \in (H^s_p(\mathbb{T}))'$.

Proof. (\Rightarrow) Let $T > 0$ and define the linear map $F_T : L^2([0,T]; H^s_p(\mathbb{T})) \to H^s_p(\mathbb{T})$ by $F_T(h) = u(T)$, where $u$ is the (mild) solution of (1.10) with $u(0) = 0$. From the hypothesis, the map $F_T$ is onto, and, given $u_1 \in H^s_p(\mathbb{T})$, we have
\[
F_T(h) = u_1 = \int_0^T U(T - s)(G(h))(s) \, ds ,
\] (4.4)
for some $h \in L^2([0,T];H^s_p(\mathbb{T}))$. Therefore,
\[
\|F_T(h)\|_{H^s_p(\mathbb{T})} \leq \int_0^T \|U(T - s)(G(h))(s)\|_{H^s_p(\mathbb{T})} \, ds \leq c \int_0^T \|h\|_{H^s_p(\mathbb{T})} \, ds \leq cT^{\frac{1}{2}} \|h\|_{L^2([0,T];H^s_p(\mathbb{T}))},
\]
for some constant $c$ depending on $s$. So, $F_T$ is a bounded linear operator. Thus, $F_T^{-1}$ exists, is a bounded linear operator, and it is one-to-one (see Rudin [37, Corollary b) page 99]). Also, from Theorem 4.13 in [37] (see also [7, page 35]), we have that there exists $\delta > 0$ such that
\[
\|F_T(\phi^*)\|_{(L^2([0,T];H^s_p(\mathbb{T})))'} \geq \delta \|\phi^*\|_{(H^s_p(\mathbb{T}))'}, \text{ for all } \phi^* \in (H^s_p(\mathbb{T}))'.
\] (4.5)

From Lemma 4.1, we have that the solution $u$ of (1.10) with $u_0 = 0$ satisfies
\[
\int_0^T \langle Gh(\cdot,t), \varphi(\cdot,t) \rangle_{H^s_p \times (H^s_p)'} \, dt - \langle u_1, \varphi_0 \rangle_{H^s_p \times (H^s_p)'} = 0 ,
\] (4.6)
for any $\varphi_0 \in (H^s_p(\mathbb{T}))'$, and $\varphi$ the solution of the adjoint system (4.2). By noting that $\varphi(\cdot, t) = U(T - t)^* \varphi_0$, it follows from (4.6) that
\[
\int_0^T \langle h(\cdot,t), G^* U(T - t)^* \varphi_0 \rangle_{H^s_p(\mathbb{T}) \times (H^s_p(\mathbb{T}))'} \, dt = \langle u(T), \varphi_0 \rangle_{H^s_p(\mathbb{T}) \times (H^s_p(\mathbb{T}))'}
\]
\[
= \langle F_T(h), \varphi_0 \rangle_{H^s_p(\mathbb{T}) \times (H^s_p(\mathbb{T}))'}
\]
\[
= \langle h, F_T^* \varphi_0 \rangle_{L^2([0,T];H^s_p(\mathbb{T})) \times (L^2([0,T];H^s_p(\mathbb{T}))').
\]
Identifying \( L^2([0,T]; H^p_\tau(T)) \) with its dual one infers \( F_T^* = G^*U(T-t)^* \), and using (4.5), we have
\[
\|G^*U(T-t)^*(\phi^*)\|_{L^2([0,T]; H^p_\tau(T))} \geq \delta \|\phi^*\|_{(H^p_\tau(T))'}, \quad \text{for all } \phi^* \in (H^p_\tau(T))',
\]
or, equivalently,
\[
\int_0^T \|G^*U(T-t)^*(\phi^*(x))\|_{(H^p_\tau(T))'}^2 \, dt \geq \delta^2 \|\phi^*\|^2_{(H^p_\tau(T))'}, \quad \text{for all } \phi^* \in (H^p_\tau(T))'.
\]
The change of variables \( \tau = T - t \) yields (4.3).

(\( \Rightarrow \)) If (4.3) holds, then \( F_T^* = G^*U(T-t)^* \) is onto. It is easy to prove that \( F_T^* \) is bounded from \((H^p_\tau(T))'\) into \( (L^2([0,T]; H^p_\tau(T)))' \). Therefore, \( F_T \) is onto. From computations similar to those above we obtain that (4.6) holds. Then Lemma 4.1 implies the result and we conclude the proof of the corollary.

The following characterization is fundamental to prove the existence of control for (1.10) with initial data \( u_0 = 0 \). It provides a method to find the control function \( h \) explicitly.

**Lemma 4.3 (Moment Equation).** *Let* \( s \in \mathbb{R} \) *and* \( T > 0 \) *be given. If*
\[
u_1(x) = \sum_{l \in \mathbb{Z}} c_l \psi_l(x) \in H^p_\tau(T),
\]
is a function such that \( \hat{\nu}_1(0) = 0 \), *then the solution* \( u \) *of* (1.10) *with initial data* \( u_0 = 0 \) *satisfies* \( u(T) = u_1 \) *if and only if there exists* \( h \in L^2([0,T]; H^p_\tau(T)) \) *and*
\[
\int_0^T (Gh(x,t), e^{-i\lambda_k(T-t)}\psi_k(x))_{L^2_\tau(T)} \, dt = c_k, \quad \forall \, k \in \mathbb{Z},
\]
*where* \( \lambda_k := ka(k) \).

**Proof.** (\( \Rightarrow \)) By taking \( \varphi_0 = \psi_k \in (H^p_\tau(T))' \) in (4.2), identity (3.3) implies that
\[
\varphi(x,t) = \left( e^{-i\lambda_k(T-t)}\hat{\psi}_k(l) \right)^{\vee} = \sum_{l \in \mathbb{Z}} e^{-i\lambda_k(T-t)}\hat{\psi}_k(l) e^{ilx} = e^{-i\lambda_k(T-t)}\psi_k(x),
\]
where in the last identity we used that \( \hat{\psi}_k(l) = \frac{1}{\sqrt{2\pi}} \delta_{kl} \), with \( \delta_{kl} \) being the Kronecker delta. Now, using (4.1) one gets
\[
\int_0^T (Gh(x,t), \varphi(x,t))_{L^2_\tau(T)} \, dt - \left( \sum_{l \in \mathbb{Z}} c_l \psi_l(x), \varphi_0(x) \right)_{L^2_\tau(T)} = 0.
\]
Therefore, for any \( k \in \mathbb{Z} \),
\[
\int_0^T (Gh(x,t), e^{-i\lambda_k(T-t)}\psi_k(x))_{L^2_\tau(T)} \, dt = 2\pi \sum_{j \in \mathbb{Z}} \left( \sum_{l \in \mathbb{Z}} c_l \psi_l(x) \right) \overline{\psi_k(j)} = 2\pi \sum_{l \in \mathbb{Z}} c_l \hat{\psi}_l(k) \frac{1}{\sqrt{2\pi}} = c_k,
\]
as required.
Now, suppose that there exists $h \in L^2([0, T]; H^s_p(\mathbb{T}))$ such that (4.7) holds. With similar calculations as above, we obtain
\[
\int_0^T \left( Gh(x, t), e^{-i\lambda_k(T-t)} \psi_k(x) \right)_{L^2_p(\mathbb{T})} dt - (u_1(x), \psi_k(x))_{L^2_p(\mathbb{T})} = 0, \quad k \in \mathbb{Z}. \tag{4.8}
\]
For any $\varphi_0 \in C^\infty_p(\mathbb{T})$ we may write
\[
\varphi_0(x) = \sum_{k \in \mathbb{Z}} \sqrt{2\pi} \hat{\varphi}_0(k) \psi_k(x),
\]
where the series converges uniformly. Thus, using the properties of the inner product and (4.8), we get
\[
\int_0^T (Gh(x, t), \varphi(x, t))_{L^2_p(\mathbb{T})} dt = (u_1(x), \varphi_0(x))_{L^2_p(\mathbb{T})}, \tag{4.9}
\]
where we used that the solution of (4.2) may be expressed as
\[
\varphi(x, t) = \sum_{k \in \mathbb{Z}} e^{-i\lambda_k(T-t)} \hat{\varphi}_0(k) e^{ikx}
\]
with the series converging uniformly. By density, (4.9) holds for any $\varphi_0 \in (H^s_p(\mathbb{T}))*'$. An application of Lemma 4.1 then gives the desired result. \qed

**Lemma 4.4.** For $\psi_k$ as in (2.1) and $G$ as in (1.7), define
\[
m_{j,k} := \hat{G}(ejx)(k) = \int_0^{2\pi} G(\psi_j(x)) \overline{\psi_k(x)} dx, \quad j, k \in \mathbb{Z}. \tag{4.10}
\]
Given any finite sequence of nonzero integers $k_j, j = 1, 2, 3, \ldots, n$, let $M_n$ be the $n \times n$ matrix,
\[
M_n := \begin{pmatrix}
  m_{k_1,k_1} & \cdots & m_{k_1,k_n} \\
  m_{k_2,k_1} & \cdots & m_{k_2,k_n} \\
  \vdots & \ddots & \vdots \\
  m_{k_n,k_1} & \cdots & m_{k_n,k_n}
\end{pmatrix}.
\]
Then
(i) there exists a constant $\beta > 0$, depending only on $g$, such that
\[
m_{k,k} \geq \beta, \quad \text{for any } k \in \mathbb{Z} - \{0\}.
\]
(ii) $m_{j,0} = 0$, $j \in \mathbb{Z}$.
(iii) $M_n$ is invertible and hermitian.
(iv) there exists $\delta > 0$, depending only on $g$, such that
\[
\delta_k = \|G(\psi_k)\|_{L^2_p(\mathbb{T})}^2 > \delta > 0, \quad \text{for all } k \in \mathbb{Z} - \{0\}. \tag{4.11}
\]
(v) $m_{-k,k} = m_{k,-k} = m_{k,k}$.

**Proof.** The proof of parts (i) and (iii) can be found in [29, page 296]. Part (iv) was proved in [40, page 3650] (see also [25, page 213]). Parts (ii) and (v) are direct consequences of the definition in (4.10). \qed

Now we give the proof of our first criterion regarding controllability of nonhomogenous linear system (1.10) stated in Theorem 1.3.
Proof of Theorem 1.3. As we already discussed, it suffices to assume \( u_0 = 0 \). Let us start by performing a suitable decomposition of \( Z \). Indeed, in view of \((H3)\) there are only finitely many integers in \( \mathbb{I} \), say, \( k_j, j = 1, 2, \ldots, n_0^* \), for some \( n_0^* \in \mathbb{N} \), such that one can find another integer \( k \neq k_j \) with \( \lambda_k = \lambda_{k_j} \). By setting

\[
\mathbb{I}_j := \{ k \in \mathbb{Z} : k \neq k_j, \lambda_k = \lambda_{k_j} \}, \quad j = 1, 2, \ldots, n_0^*,
\]

we then get the pairwise disjoint union,

\[
\mathbb{Z} = \mathbb{I} \cup \mathbb{I}_1 \cup \mathbb{I}_2 \cup \cdots \cup \mathbb{I}_{n_0^*}. \tag{4.12}
\]

We now prove the theorem in six steps.

**Step 1.** The family \( \{ e^{-i\lambda_k t} \}_{k \in \mathbb{I}} \), with \( \lambda_k = ka(k) \), is a Riesz basis for \( H := \text{span}\{ e^{-i\lambda_k t} : k \in \mathbb{I} \} \) in \( L^2([0, T]) \).

In fact, since \( L^2([0, T]) \) is a reflexive separable Hilbert space so is \( H \). In addition, by definition, it is clear that \( \{ e^{-i\lambda_k t} \}_{k \in \mathbb{I}} \) is complete in \( H \). On the other hand, from \((1.8)-(1.9)\) and Theorem 2.3, there exist positive constants \( A \) and \( B \) such that

\[
A \sum_{n \in \mathbb{I}} |b_n|^2 \leq \int_0^T |f(t)|^2 dt \leq B \sum_{n \in \mathbb{I}} |b_n|^2, \tag{4.13}
\]

for all functions of the form \( f(t) = \sum_{n \in \mathbb{I}} b_n e^{-i\lambda_n t}, t \in [0, T], \) with square-summable complex coefficients \( b_n \). In particular, if \( b_1, \ldots, b_N \) are \( N \) arbitrary constants we have

\[
A \sum_{n=1}^N |b_n|^2 \leq \left\| \sum_{n=1}^N b_n e^{-i\lambda_n t} \right\|_H^2 \leq B \sum_{n=1}^N |b_n|^2.
\]

Hence, an application of Theorem 2.2 gives the desired property.

**Step 2.** There exists a unique biorthogonal basis \( \{ q_j \}_{j \in \mathbb{I}} \subseteq H^* \) to \( \{ e^{-i\lambda_k t} \}_{k \in \mathbb{I}} \).

Indeed, Step 1 and Theorem 2.2 implies that \( \{ e^{-i\lambda_k t} \}_{k \in \mathbb{I}} \) is a complete Bessel sequence and possesses a biorthogonal system \( \{ q_j \}_{j \in \mathbb{I}} \) which is also a complete Bessel sequence. Moreover, Corollary 5.22 in [13, page 171] implies that \( \{ q_j \}_{j \in \mathbb{I}} \) is also a basis for \( H \) (after identifying \( H^* \) and \( H \)). So, from Lemma 5.4 [13, page 155], we get that \( \{ e^{-i\lambda_k t} \}_{k \in \mathbb{I}} \) is a minimal sequence in \( H \); and, hence, exact (see [13, Definition 5.3]). Finally, Lemma 5.4 in [13, page 155] gives that \( \{ q_j \}_{j \in \mathbb{I}} \) is the unique biorthogonal basis to \( \{ e^{-i\lambda_k t} \}_{k \in \mathbb{I}} \). Note that an immediate consequence is that

\[
(e^{-i\lambda_k t}, q_j)_H = \int_0^T e^{-i\lambda_k t} q_j(t) dt = \delta_{kj}, \quad k, j \in \mathbb{I}, \tag{4.14}
\]

where \( \delta_{kj} \) represents the Kronecker delta.

**Step 3.** Here we will define the appropriate control function \( h \).

In fact, let \( \{ q_j \}_{j \in \mathbb{I}} \) be the sequence obtained in Step 2. The next step is to extend the sequence \( q_j \) for \( j \) running on \( \mathbb{Z} \). In view of \((4.12)\) it remains to define this sequence for indices in \( \mathbb{I}_j, j = 1, \ldots, n_0^* \). Furthermore, \((H2)\) gives that \( \mathbb{I}_j \) contains at most \( n_0 - 1 \) elements. Without loss of generality, we may assume that all multiple eigenvalues have multiplicity \( n_0 \); otherwise we may repeat the procedure below according to the multiplicity of each eigenvalue. Thus we write

\[
\mathbb{I}_j = \{ k_j,1, k_j,2, k_j,3, \ldots, k_j,n_0-1 \}, \quad j = 1, 2, \ldots, n_0^* \tag{4.15}
\]

To simplify notation, here and in what follows we use \( k_{j,0} \) for \( k_j \). Given \( k_{j,l} \in \mathbb{I}_j \) we define \( q_{k_{j,l}} := q_{k_{j,l}} \). At this point recall that \( \lambda_{k_{j,l}} = \lambda_{k_j} \) for any \( j = 1, 2, \ldots, n_0^* \) and \( l = 0, 1, 2, \ldots, n_0 - 1 \).
Having defined $q_j$ for all $j \in \mathbb{Z}$, we now define the control function $h$ by

$$h(x,t) = \sum_{j \in \mathbb{Z}} h_j \overline{\psi}_j(t) \psi_j (x),$$  \hspace{1cm} (4.16)

for suitable coefficients $h_j$’s to be determined later. From the definition of $G$, we obtain

$$T \int_0^T \left( G(h)(x,t), e^{-i\lambda_k(T-t)} \psi_k(x) \right)_{L^2_p(T)} dt = T \int_0^T \left( \sum_{j \in \mathbb{Z}} h_j \overline{\psi}_j(t) G(\psi_j)(x,t), e^{-i\lambda_k(T-t)} \psi_k(x) \right)_{L^2_p(T)} dt$$

$$= \sum_{j \in \mathbb{Z}} h_j \int_0^T \overline{\psi}_j(t) e^{i\lambda_k(T-t)} dt \left( G(\psi_j)(x), \psi_k(x) \right)_{L^2_p(T)}$$

$$= \sum_{j \in \mathbb{Z}} h_j e^{i\lambda_k T} m_{j,k} \int_0^T \overline{\psi}_j(t) e^{-i\lambda_k t} dt,$$  \hspace{1cm} (4.17)

with $m_{j,k}$ defined in (4.10).

**Step 4.** Here we find $h_j$’s such that $h$ defined by (4.16) serves as the required control function.

First of all, note that in order to prove the first part of the theorem, identity (4.17) and Lemma 4.3 yield that it suffices to choose $h_j$’s satisfying

$$c_k = \sum_{j \in \mathbb{Z}} h_j e^{i\lambda_k T} m_{j,k} \int_0^T \overline{\psi}_j(t) e^{-i\lambda_k t} dt,$$  \hspace{1cm} (4.18)

where $u_1(x) = \sum_{n \in \mathbb{Z}} c_n \psi_n(x)$.

We will show now that we may indeed choose $h_j$’s satisfying (4.18). To see this, first observe that, since $c_0 = 0$, part (ii) in Lemma 4.4 implies that (4.18) holds for $k = 0$ independently of $h_j$’s. In particular, we may choose $h_0 = 0$. Next, from (4.14), if

$$k \in \tilde{I} := \mathbb{I} - \{k_1, \ldots, k_{n_0} \}$$

we see that (4.18) reduces to

$$c_k = h_k m_{k,k} e^{i\lambda_k T}.$$  \hspace{1cm} (4.19)

Hence, in view of part (iii) in Lemma 4.4, we have

$$h_k = \frac{c_k e^{-i\lambda_k T}}{m_{k,k}}, \quad k \in \tilde{I}.$$  \hspace{1cm} (4.19)

On the other hand, if $k \in \mathbb{Z} - \tilde{I}$ then $k = k_{j,l_0}$ for some $j \in \{1, \ldots, n_0^* \}$ and $l_0 \in \{0, 1, \ldots, n_0 - 1 \}$. Since $\lambda_k = \lambda_{k_{j,l_0}} = \lambda_{k_j}$, the integral in (4.18) is zero, except for those indices in $\mathbb{I} \cup \{k_j \}$. In particular, (4.18) reduces to

$$c_{k_{j,l_0}} = c_k = \sum_{l=0}^{n_0 - 1} h_{k_{j,l}} m_{k_{j,l},k_{j,l_0}} e^{i\lambda_{k_{j,l_0}} T}.$$  \hspace{1cm} (4.20)
When \( l_0 \) runs over the set \( \{0, 1, \ldots, n_0 - 1\} \), the equations in (4.20) may be seen as a linear system for \( h_{kj,l} \) (with \( j \) fixed) whose unique solution is
\[
\begin{pmatrix}
h_{kj,0} \\
h_{kj,1} \\
\vdots \\
h_{kj,n_0 - 1}
\end{pmatrix}^T = \begin{pmatrix}
c_{kj,0} e^{-i\lambda_{kj,0} T} \\
c_{kj,1} e^{-i\lambda_{kj,1} T} \\
\vdots \\
c_{kj,n_0 - 1} e^{-i\lambda_{kj,n_0 - 1} T}
\end{pmatrix}^T \begin{pmatrix}
M_j
\end{pmatrix}^{-1}, \quad \text{for } j = 1, 2, \ldots, n_0^*,
\]
where
\[
M_j = \begin{pmatrix}
m_{kj,0,0,0} & m_{kj,0,0,1} & \cdots & m_{kj,0,0,n_0 - 1} \\
m_{kj,1,0,0} & m_{kj,1,0,1} & \cdots & m_{kj,1,0,n_0 - 1} \\
\vdots & \vdots & \ddots & \vdots \\
m_{kj,n_0 - 1,0,0} & m_{kj,n_0 - 1,0,1} & \cdots & m_{kj,n_0 - 1,0,n_0 - 1}
\end{pmatrix}.
\]

Since from Lemma 4.4 the matrix \( M_j \) is invertible, equation (4.21) makes sense. Consequently, for any \( j \in \mathbb{Z} = \mathbb{I} \cup \mathbb{I}_1 \cup \mathbb{I}_2 \cup \cdots \cup \mathbb{I}_{n^*_0} \), we may choose \( h_j \)'s according to (4.19) and (4.21).

**Step 5.** The function \( h \) defined by (4.16) with \( h_0 = 0 \) and \( h_k, k \neq 0 \), given by (4.19) and (4.21) belongs to \( L^2([0, T]; H^s_p(\mathbb{T})) \).

Indeed, recall from Step 2 that \( \{q_j\}_{j \in \mathbb{I}} \) is a Riesz basis for \( H \). Thus, from Theorem 2.2 part (3), it follows that \( \{q_j\}_{j \in \mathbb{I}} \) is a bounded sequence in \( L^2([0, T]) \). Consequently, \( \{q_j\}_{j \in \mathbb{Z}} \) is also bounded in \( L^2([0, T]) \). Hence, by using the explicit representation in (4.16), we deduce
\[
\|h\|^2_{L^2([0, T]; H^s_p(\mathbb{T}))} = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} (1 + |k|)^{2s} |h_k|^2 \int_0^T |q_k(t)|^2 \, dt
\leq C \sum_{k \in \mathbb{Z}} (1 + |k|)^{2s} |h_k|^2,
\]
for some positive constant \( C \). Thus, from identity (4.19) and Lemma 4.4 part (ii) we obtain
\[
\|h\|^2_{L^2([0, T]; H^s_p(\mathbb{T}))} \leq C \sum_{k \in \mathbb{I}, k \neq 0} (1 + |k|)^{2s} \left| \frac{c_k e^{-i\lambda_k T}}{m_{k,k}} \right|^2 + C \sum_{k \in \mathbb{Z} - \mathbb{I}} (1 + |k|)^{2s} |h_k|^2
\leq C \beta^2 \sum_{k \in \mathbb{I}, k \neq 0} (1 + |k|)^{2s} |c_k|^2 + C \sum_{k \in \mathbb{Z} - \mathbb{I}} (1 + |k|)^{2s} |h_k|^2.
\]
(4.23)

Since \( u_1 \in H^s_p(\mathbb{T}) \) the above series converges. In addition, since the set \( \mathbb{Z} - \mathbb{I} \) is finite we conclude that the right-hand side of (4.23) is finite, implying that \( h \) belongs to \( L^2([0, T]; H^s_p(\mathbb{T})) \).

In order to complete the proof of the theorem it remains to establish (1.11).

**Step 6.** Estimate (1.11) holds.

From Step 5 we see that we need to estimate the second term on the right-hand side of (4.23). So, fix some nonzero \( k \in \mathbb{Z} - \mathbb{I} \). We may write \( k = kj, l \) for some \( l = 0, 1, 2, \ldots, n_0 - 1 \) and \( j = 1, 2, \ldots, n_0^* \). From (4.21) we infer
\[
|h_{kj,l}|^2 \leq \sum_{m=0}^{n_0 - 1} |h_{kj,m}|^2 \leq \left( \sum_{m=0}^{n_0 - 1} |c_{kj,m} e^{-i\lambda_{kj,m} T}|^2 \right) \|M_j^{-1}\|^2 \leq \|M_j^{-1}\|^2 \sum_{m=0}^{n_0 - 1} |c_{kj,m}|^2,
\]
where $\|M_j^{-1}\|$ is the Euclidean norm of the matrix $M_j^{-1}$. This implies that

\[
(1 + |k_{j,l}|)^{2s}|h_{j,l}|^2 \leq \sum_{m=0}^{n_0-1} \|M_j^{-1}\|^2 \frac{(1 + |k_{j,l}|)^{2s}}{(1 + |k_{j,m}|)^{2s}}(1 + |k_{j,m}|)^{2s}|c_{k_{j,m}}|^2
\]

\[
\leq C(s) \sum_{m=0}^{n_0-1} (1 + |k_{j,m}|)^{2s}|c_{k_{j,m}}|^2,
\]

with

\[
C(s) = \max_{j=1,2,\ldots,n_0} \left\{ \|M_j^{-1}\|^2 \frac{(1 + |k_{j,l}|)^{2s}}{(1 + |k_{j,m}|)^{2s}} \right\}.
\]

Therefore,

\[
\sum_{k \in \mathbb{Z}^3} (1 + |k|^2)|h_k|^2 = \sum_{j=1}^{n_0} \sum_{l=0}^{n_0-1} (1 + |k_{j,l}|)^{2s}|h_{k_{j,l}}|^2
\]

\[
\leq C(s)n_0 \sum_{m=1}^{n_0} \sum_{l=0}^{n_0-1} (1 + |k_{j,m}|)^{2s}|c_{k_{j,m}}|^2
\]

\[
= C(s)n_0 \sum_{k \in \mathbb{Z}^3} (1 + |k|^2)|c_k|^2.
\]  \hfill (4.24)

Gathering together (4.23) and (4.24), we obtain

\[
\|h\|_{L^2([0,T];H_p^m(T))}^2 \leq \frac{C}{\beta^2} \sum_{k \in \mathbb{Z}^3, k \neq 0} (1 + |k|^2)|c_k|^2 + CC(s)n_0 \sum_{k \in \mathbb{Z}^3} (1 + |k|^2)|c_k|^2
\]

\[
\leq \nu^2\|u_1\|_{H_p^m(T)}^2,
\]  \hfill (4.25)

where $\nu^2 = \max \left\{ \frac{C}{\beta^2}, n_0CC(s) \right\}$.

This completes the proof of the theorem. \qed

Now we proof our second criterion regarding controllability of non-homogenous linear system (1.10) stated in Theorem 1.5.

**Proof of Theorem 1.5.** The proof is similar to that of Theorem 1.3. So we bring only the necessary changes and estimates. As before, we assume $u_0 = 0$. In view of (H4), (H5) we may find finitely many integers in $\mathbb{J}$, say, $k_j$, $j = 1, 2, \ldots, n_0^*$, for some $n_0^* \in \mathbb{N}$, with $n_0^* \leq 2k_j^* - 1$, such that one can find another integer $k \neq k_j$ with $\lambda_k = \lambda_{k_j}$. Let

\[
\mathbb{J}_j := \{ k \in \mathbb{Z} : k \neq k_j, \lambda_k = \lambda_{k_j} \}, \quad j = 1, 2, \ldots, n_0^*,
\]

and

\[
\mathbb{J}^- := \{ k \in \mathbb{Z} : k \leq -k_j^* \}.
\]

Then we obtain the pairwise disjoint decomposition

\[
\mathbb{Z} = \mathbb{J}^- \cup \mathbb{J} \cup \mathbb{J}_1 \cup \mathbb{J}_2 \cup \ldots \cup \mathbb{J}_{n_0^*}.
\]  \hfill (4.26)

Again, we may prove the theorem into six steps.

**Step 1.** The family $\{e^{-i\lambda_k t}\}_{k \in \mathbb{J}}$, with $\lambda_k = k\alpha(k)$, is a Riesz basis for $H := \text{span}\{e^{-i\lambda_k t} : k \in \mathbb{J}\}$ in $L^2([0,T])$. 

In fact, this is a consequence of (1.12)-(1.13), Theorem 2.3, and Theorem 2.2.

**Step 2.** There exists a unique biorthogonal basis \( \{ q_j \}_{j \in J} \subseteq H^* \) to \( \{ e^{-i\lambda_k t} \}_{k \in J} \).

This is a consequence of Theorem 2.2, Corollary 5.22 in [13], and Lemma 5.4 [13]. Furthermore, we have that

\[
(e^{-i\lambda_k t}, q_j)_H = \int_0^T e^{-i\lambda_k t} q_j(t) \, dt = \delta_{k,j}, \quad k, j \in J.
\]

**Step 3.** Here we will define an adequate control function \( h \).

As in (4.16), for suitable coefficients \( h_j \) to be determined later we set

\[
h(x, t) = \sum_{j \in Z} h_j \varphi_j(t) \psi_j(x),
\]

where, according to the decomposition (4.26), the sequence \( \{ q_k \}_{k \in Z} \) is defined as follows: if \( k \in J \) then \( q_k \) is given in Step 2; if \( k \in J_j \) for some \( j \in \{1, \ldots, n_0\} \) then by writing (assuming that all multiple eigenvalues have multiplicity \( n_0 \))

\[
J_j = \{ k_{j,1}, k_{j,2}, k_{j,3}, \ldots, k_{j,n_0-1} \},
\]

and denoting \( k_j \) by \( k_{j,0} \) we set

\[
q_k = q_{k_{j,0}} := q_{k_{j,0}} = q_{k_j}.
\]

Finally, if \( k \in J^- \) then we set

\[
q_k = q_{-k}.
\]

With this choice of \( \{ q_k \}_{k \in Z} \), as in (4.17) we have

\[
\int_0^T \left( G(h)(x, t), e^{-i\lambda_k (T-t)} \psi_k(x) \right)_{L^2(\Omega)} dt = \sum_{j \in Z} h_j e^{i\lambda_k T} m_{j,k} \int_0^T \varphi_j(t) e^{-i\lambda_k t} dt.
\]

**Step 4.** In this step we find \( h_j \)'s such that \( h \) defined by (4.28) serves as the required control function.

By writing \( u_1(x) = \sum_n c_n \psi_n(x) \), it is enough to consider \( h_j \)'s satisfying

\[
c_k = \sum_{j \in Z} h_j e^{i\lambda_k T} m_{j,k} \int_0^T \varphi_j(t) e^{-i\lambda_k t} dt.
\]

From Lemma 4.4 part (ii) we may take \( h_0 = 0 \). To see that we can choose \( h_j \) such that (4.30) holds let us start by defining the following sets of indices

\[
J^+ := \{ k \in Z : k \geq k^*_1 \},
\]

\[
\bar{J} = \{ k \in Z : k = k_{j,l}; \ l = 0, 1, 2, \ldots, n_0 - 1, \ j = 1, 2, \ldots, n_0^* \},
\]

and

\[
\bar{I} = \{ k \in Z : k \notin J^+ \cup J^- \text{ and } k \notin \bar{J} \}.
\]

It is clear that \( Z = \bar{I} \cup \bar{J} \cup J^+ \cup J^- \). In addition, note that \( \bar{I} \) is nothing but the set of those indices for which the corresponding eigenvalue is simple. Without loss of generality we will assume that \( \bar{I} \) is nonempty; otherwise, this part has no contribution and these indices do not appear in (4.30).

The idea now is to obtain \( h_k \) according to \( k \in \bar{I}, k \notin \bar{J}, \) or \( k \in J^+ \cup J^- \). From (4.27) we see that (4.30) reduces to

\[
c_k = h_k m_{k,k} e^{i\lambda_k T}, \quad k \in \bar{I}.
\]
Indeed, from Lemma 4.4 part (v) we infer that
\[ h_k = \frac{c_k e^{-i\lambda k T}}{m_{k,k}}, \quad k \in \mathbb{I}. \quad (4.31) \]

Next, if \( k \in \mathbb{J}_- \), then \( k = k_j l_0 \) for some \( j \in \{1, \ldots, n_0^*\} \) and \( l_0 \in \{0, 1, \ldots, n_0 - 1\} \). Thus, as in Step 4 of Theorem 1.3, we see that
\[ ck_j l_0 = \sum_{l=0}^{n_0-1} h_{k_j, l} m_{k_j, j, k_j l_0} e^{i\lambda k_j l_0 T}, \]

By solving the above system for \( h_{k_j, l} \) (with \( j \) fixed and running \( l_0 \) over \( \{0, 1, \ldots, n_0 - 1\} \)) we find
\[ \begin{pmatrix} h_{k_j, 0} \\ h_{k_j, 1} \\ \vdots \\ h_{k_j, n_0 - 1} \end{pmatrix} = \begin{pmatrix} c_{k_j, 0} e^{-i\lambda k_j, 0 T} \\ c_{k_j, 1} e^{-i\lambda k_j, 1 T} \\ \vdots \\ c_{k_j, n_0 - 1} e^{-i\lambda k_j, n_0 - 1 T} \end{pmatrix} \tilde{M}_j^{-1}, \quad j = 1, 2, \cdots, n_0, \quad (4.32) \]

where
\[ \tilde{M}_j = \begin{pmatrix} m_{k_j, 0, k_j, 0} & m_{k_j, 0, k_j, 1} & \cdots & m_{k_j, 0, k_j, n_0 - 1} \\ m_{k_j, 1, k_j, 0} & m_{k_j, 1, k_j, 1} & \cdots & m_{k_j, 1, k_j, n_0 - 1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{k_j, n_0 - 1, k_j, 0} & m_{k_j, n_0 - 1, k_j, 1} & \cdots & m_{k_j, n_0 - 1, k_j, n_0 - 1} \end{pmatrix}. \]

Finally, if \( k \in \mathbb{J}^+ \) we have \(-k \in \mathbb{J}^-\) and \( I(k) = \{k, -k\} \). We deduce from (4.30) that
\[ \begin{cases} c_k = h_k e^{i\lambda k T} m_{k,k} + h_{-k} e^{i\lambda k T} m_{-k,k}, \\ c_{-k} = h_k e^{i\lambda -k T} m_{k,-k} + h_{-k} e^{i\lambda -k T} m_{-k,-k}. \end{cases} \]

Solving this system for \( h_k \) and \( h_{-k} \) we obtain
\[ \begin{pmatrix} h_k \\ h_{-k} \end{pmatrix} = \begin{pmatrix} c_k e^{-i\lambda k T} \\ c_{-k} e^{-i\lambda -k T} \end{pmatrix} \tilde{M}^{-1}, \quad k \in \mathbb{J}^+ \quad (4.33) \]

where
\[ \tilde{M}^{-1} = \frac{1}{d_k} \begin{pmatrix} m_{-k,-k} & -m_{k,-k} \\ -m_{-k,k} & m_{k,k} \end{pmatrix}, \quad d_k = m_{k,k} m_{-k,-k} - m_{k,-k} m_{-k,k}. \]

Summarizing the above construction, we see that, for any \( k \in \mathbb{Z} \), we may choose \( h_k \) according to (4.31), (4.32), and (4.33).

Next we observe that the matrix \( \tilde{M}^{-1} \) is bounded uniformly with respect to \( k \). Indeed, from Lemma 4.4 part (v) we infer that \( d_k = |m_{k,k}|^2 - |m_{k,-k}|^2 \). Now, from the definition of \( G \),
\[ m_{k,-k} = \frac{1}{2\pi} \left[ \int_0^{2\pi} g(x) e^{ikx} dx - \left( \int_0^{2\pi} g(x) e^{ikx} dx \right) \left( \int_0^{2\pi} g(y) e^{iky} dy \right) \right] = \tilde{g}(-k) - 2\pi [\tilde{g}(-k)]^2. \]

Since \( g \) is smooth, using the Riemann-Lebesgue lemma we obtain
\[ \lim_{k \to \pm \infty} |m_{k,-k}| = 0. \]
On the other hand, in view of (1.6), for any \( k \neq 0 \),
\[
m_{k,k} = \frac{1}{2\pi} \left( \int_0^{2\pi} g(x) dx - \left| \int_0^{2\pi} g(x)e^{ikx} dx \right|^2 \right) = \frac{1}{2\pi} - |\hat{g}(-k)|^2,
\]
and hence,
\[
\lim_{k \to +\infty} m_{k,k} = \frac{1}{2\pi}.
\]

Since
\[
\lim_{k \to +\infty} d_k = \frac{1}{4\pi^2},
\]
we may assume, without loss of generality, that \( d_k > \frac{1}{4\pi^2} \), for any \( k \geq k^* \). Therefore, there exists \( D > 0 \), independent of \( k \in \mathbb{J}^+ \), such that
\[
\|M^{-1}\| \leq D, \tag{4.34}
\]
where \( \|M^{-1}\| \) is the Euclidean norm of the matrix \( M^{-1} \).

**Step 5.** The control function \( h \) defined by (4.28) with \( h_0 = 0 \), and \( h_k, k \neq 0 \), given by (4.31), (4.32), and (4.33) belongs to \( L^2([0,T];H^s_p(\mathbb{T})) \).

Indeed, as in (4.22) we obtain
\[
\|h\|_{L^2([0,T];H^s_p(\mathbb{T}))} \leq C \sum_{k \in \mathbb{Z}} (1 + |k|)^{2s} |h_k|^2,
\]
for some positive constant \( C \). Next, in the series above we split the sum according to \( k \in \mathbb{I}, k \in \mathbb{J} \) or \( k \in \mathbb{J}^+ \cup \mathbb{J}^- \). Thus, we may write
\[
\|h\|_{L^2([0,T];H^s_p(\mathbb{T}))} \leq C \sum_{k \in \mathbb{I}} (1 + |k|)^{2s} |h_k|^2 + \sum_{k \in \mathbb{J}} (1 + |k|)^{2s} |h_k|^2 + C \sum_{k \in \mathbb{J}^+ \cup \mathbb{J}^-} (1 + |k|)^{2s} |h_k|^2. \tag{4.35}
\]

The first two terms on the right-hand side of (4.35) may be estimated as in Theorem 1.3 (see (4.24)). Thus,
\[
\|h\|_{L^2([0,T];H^s_p(\mathbb{T}))} \leq \frac{C}{\beta^2} \sum_{k \in \mathbb{I}} (1 + |k|)^{2s} |c_k|^2 + C C(s) \sum_{k \in \mathbb{J}} (1 + |k|)^{2s} |c_k|^2 + C \sum_{k \in \mathbb{J}^+ \cup \mathbb{J}^-} (1 + |k|)^{2s} |h_k|^2, \tag{4.36}
\]
where \( C(s) = \max_{j=1,2,\ldots,n^*_a} \left\{ \frac{\|M_j^{-1}\|^2}{(1 + |k_j,l|)^{2s}} \right\} \).

For the last term in (4.36), identity (4.33) and (4.34) imply that for any \( k \in \mathbb{J}^+ \),
\[
|h_k|^2 \leq \left( |c_k|^2 + |c_{-k}|^2 \right) \|M^{-1}\|^2 \leq \left( |c_k|^2 + |c_{-k}|^2 \right) D^2,
\]
and
\[
(1 + |k|)^{2s} |h_k|^2 \leq D^2 (1 + |k|)^{2s} |c_k|^2 + D^2 (1 + |k|)^{2s} |c_{-k}|^2. \tag{4.37}
\]
Since the right-hand side of (4.37) is symmetric with respect to \( k \), the same estimate holds for \( k \in \mathbb{J}^- \), from which we deduce that
\[
\sum_{k \in \mathbb{J}^+ \cup \mathbb{J}^-} (1 + |k|)^{2s} |h_k|^2 \leq 2D^2 \sum_{k \in \mathbb{J}^+ \cup \mathbb{J}^-} (1 + |k|)^{2s} |c_k|^2. \tag{4.38}
\]
Combining (4.36) with (4.37) we get that \( h \) belongs to \( L^2([0,T];H^s_p(T)) \).

**Step 5.** Estimate (1.11) holds.

In view of (4.36) and (4.38), we obtain

\[
\|h\|_{L^2([0,T];H^s_p(T))} \leq \frac{C}{\beta} \sum_{k \in \mathbb{N}} (1 + |k|) |c_k|^2 + CC(s)n_0 \sum_{k \in \mathbb{N}} (1 + |k|) |c_k|^2 + 2D^2C \sum_{k \in J^+ \cup J^-} (1 + |k|) |c_k|^2
\]

\[
\leq \nu^2 \|u_1\|^2_{H^s_p(T)},
\]

where \( \nu^2 = \max \left\{ \frac{C}{\beta}, n_0 CC(s), 2CD^2 \right\} \).

This completes the proof of the theorem.

**Remark 4.5.** The dependence of \( \nu \) with respect to \( T \) is implicit in the constant \( C \) which may depend on the time \( T \).

As an immediate consequence of Theorems 1.3 and 1.5 we get the following corollary.

**Corollary 4.6.** For \( s \in \mathbb{R} \) and \( T > \frac{2\pi}{\gamma'} \) given, there exists a unique bounded linear operator

\[
\Phi : H^s_p(T) \times H^s_p(T) \rightarrow L^2([0,T];H^s_p(T))
\]

such that

\[
u_1 = U(T)u_0 + \int_0^T U(T-s)(G(\Phi(u_0,u_1)))(\cdot,s) \, ds \] (4.39)

and

\[
\|\Phi(u_0,u_1)\|_{L^2([0,T];H^s_p(T))} \leq \nu (\|u_0\|_{H^s_p(T)} + \|u_1\|_{H^s_p(T)}), \] (4.40)

for some positive constant \( \nu \).

We end this section recalling Corollary 4.2 to obtain the observability inequality, which in turn plays a fundamental role to get the exponential asymptotic stabilization with arbitrary decay rate.

**Corollary 4.7.** Let \( s \in \mathbb{R} \) and \( T > \frac{2\pi}{\gamma'} \) be given. There exists \( \delta > 0 \) such that

\[
\int_0^T \|G^*U(\tau)^* \phi^*\|_{(H^s_p(T))'}^2 \, d\tau \geq \delta^2 \|\phi^*\|_{(H^s_p(T))'}^2,
\]

for any \( \phi^* \in (H^s_p(T))' \).

**Remark 4.8.** If \( \gamma' = +\infty \) or \( \tilde{\gamma}' = +\infty \) then Corollaries 4.6 and 4.7 are valid for any positive time \( T \).

**5. Proof of Theorems 1.7 and 1.8**

This section is devoted to prove the exponential stabilization results. Once we have the observability inequality in Corollary 4.7 it is well known that this implies the stabilization. So, we just give the main steps. Fist recall we are dealing with the equation

\[
\partial_t u = \partial_x A u + Gh. \] (5.1)

Since any solution of (5.1) preserves its mass, without loss of generality, one can assume that the initial data \( u_0 \) satisfies \( \tilde{u}_0(0) = 0 \) (otherwise, we perform the change
of variables $\hat{u} = u - \hat{u}(0))$. Thus, it is enough to study the stabilization problem in $H^s_0(\mathbb{T})$, $s \in \mathbb{R}$.

The idea to prove Theorems 1.7 and 1.8 is to show the existence of a bounded linear operator, say, $K_1$ on $H^s_0(\mathbb{T})$ such that
\[ h = K_1 u \]
serves as the feedback control law. So, we study the stabilization problem for the system
\[
\begin{aligned}
\partial_t u &= \partial_x Au + GK_1 u \in H^{s-r}_0(\mathbb{T}), \quad t > 0, \\
u(0) &= u_0 \in H^s_0(\mathbb{T}),
\end{aligned}
\]  

First, we prove that system (5.2) is globally well-posed in $H^s_0(\mathbb{T})$, $s \in \mathbb{R}$.

**Theorem 5.1.** Let $u_0 \in H^s_0(\mathbb{T})$, with $r$ as in (1.3). Then the IVP (5.2) has a unique solution
\[ u \in C([0, \infty); H^s_0(\mathbb{T})) \cap C^1([0, \infty); L^2_0(\mathbb{T})). \]
Moreover, if $u_0 \in H^s_0(\mathbb{T})$, then we have $u \in C([0, \infty); H^s_0(\mathbb{T}))$, for any $s \in \mathbb{R}$.

**Proof.** Since $\partial_x A$ is the infinitesimal generator of a $C_0$-semigroup $\{U(t)\}_{t \geq 0}$ in $H^s_0(\mathbb{T})$ and $GK_1$ is a bounded linear operator on $H^s_0(\mathbb{T})$, we have that $\partial_x A + GK_1$ is also an infinitesimal generator of a $C_0$-semigroup on $H^s_0(\mathbb{T})$ (see, for instance, [32, page 76]). Thus this a consequence of the semigroup theory. □

As we will see, Theorems 1.7 and 1.8 are consequences of the following result.

**Theorem 5.2.** Let $s \in \mathbb{R}$ be given and $g$ as in (1.6). For any given $\lambda > 0$, there exist a bounded linear operator $K_1$ on $H^s_0(\mathbb{T})$ such that the unique solution of the closed-loop system
\[
\begin{aligned}
\partial_t u &= \partial_x Au + GK_1 u, \\
u(0) &= u_0,
\end{aligned}
\]  

satisfies
\[ \|u(\cdot, t)\|_{H^s_0(\mathbb{T})} \leq Me^{-\lambda t}\|u_0\|_{H^s_0(\mathbb{T})}, \quad \text{for all } t \geq 0, \]  

where the positive constant $M$ depends on $s$ and $G$ but is independent of $u_0$.

**Proof.** This is a consequence of Corollary 4.7 and the classical principle that exact controllability implies exponential stabilizability for conservative control systems (see Theorem 2.3/Theorem 2.4 in [27] and Theorem 2.1 [42]). To be more precise, according to [42, 27], one can choose
\[ K_1 = -G^* L_{T,\lambda}^{-1}, \]
where, for some $T > \frac{2\pi}{\lambda}$,
\[ L_{T,\lambda} = \int_0^T e^{-2\lambda \tau} U(-\tau)GG^*U(-\tau)^* \phi d\tau, \quad \phi \in H^s_0(\mathbb{T}), \]
and $U(t)$ is the $C_0$-semigroup generated by $\partial_x A$ (see Lemma 2.4 in [23] for more details). In addition, if one simply chooses $K_1 = -G^*$ then there exists $\alpha > 0$ such that estimate (5.4) holds with $\lambda$ replaced by $\alpha$. □

Finally, observe that Theorem 1.7 and Theorem 1.8 are direct consequences of Theorem 5.2 just by taking $K_1 = -G^*$ and $K_1 = -G^* L_{T,\lambda}^{-1}$, respectively.
6. Applications

As an application of our results, we will establish the controllability and stabilization for some linearized dispersive equations of the form (1.1).

6.1. The linearized Smith equation. The nonlinear Smith equation posed on the entire real line reads as

$$\partial_t u - \partial_x A u + u \partial_x u = 0, \quad x \in \mathbb{R}, \ t \in \mathbb{R},$$

where $u = u(x,t)$ denotes a real-valued function and $A$ is the nonlocal operator defined by

$$\hat{A}u(\xi) := 2\pi \left( \sqrt{\xi^2 + 1} - 1 \right) \hat{u}(\xi).$$

Here the hat stands for the Fourier transform on the line. Equation (6.1) was derived by Smith in [43] and it governs certain types of continental-shelf waves. From the mathematical viewpoint, the well-posedness of the IVP associated to (6.1) in $H^s(\mathbb{R})$ has been studied for instance in [1], [16], and [17]. In [1, Theorems 7.1 and 7.7] the authors proved that (6.1) is globally well-posed in $H^s(\mathbb{R})$ for $s = 1$ and $s \geq 3/2$. In [16] the author established a global well-posedness result in the weighted Sobolev space $H^s(\mathbb{R}) \cap L^2((1 + |x|^2)^s \, dx)$ for $s > 3/2$.

The control equation associated with the linearized Smith equation on the periodic setting reads as

$$\partial_t u - \partial_x A u = Gh, \quad x \in \mathbb{T}, \ t \in \mathbb{R},$$

where $A$ is such that

$$\hat{A}u(k) := 2\pi \left( \sqrt{k^2 + 1} - 1 \right) \hat{u}(k), \quad k \in \mathbb{Z},$$

so that $a(k) = 2\pi \left( \sqrt{k^2 + 1} - 1 \right)$.

In what follows we will show that Criterion I can be applied to prove that (6.2) is exactly controllable in any positive time $T > 0$ and exponentially stabilizable with any given decay rate in the Sobolev space $H^s_p(\mathbb{T})$, with $s \in \mathbb{R}$. Indeed, first of all note that clearly,

$$|a(k)| \leq C|k|,$$

for some positive constant $C$ and any $k \in \mathbb{Z}$. In addition the quantity $2\pi \hat{u}(0,t)$ is invariant by the flow of (6.2).

Using the Fourier transform, it is easy to check that $(H1)$ holds with $\lambda_k = 2\pi (k\sqrt{k^2 + 1} - k)$. See Figure 3 for an illustrative picture. By noting that $y \mapsto 2\pi \left( y \sqrt{y^2 + 1} - y \right)$ is a strictly increasing function we then deduce that all eigenvalues $\lambda_k$ are simple, giving $(H2)$ and $(H3)$. Additionally, observe that $a(-k) = a(k)$ for any $k \in \mathbb{Z}$ and

$$\lim_{|k| \to +\infty} |(k+1)a(k+1) - ka(k)| = +\infty.$$
6.2. The fourth-order Schrödinger equation. Here we consider the control equation associated with the linear fourth-order Schrödinger equation
\begin{equation}
i\partial_t u + \partial_x^2 u + \mu \partial_x^4 u = 0,
\end{equation}
where $u$ is a complex-valued function and $\mu \neq 0$ is a real constant. Equation (6.4) is the linearized version, for instance, of the fourth-order cubic nonlinear equation
\begin{equation}
i\partial_t u + \partial_x^2 u + \mu \partial_x^4 u + |u|^2 u = 0,
\end{equation}
which was introduced in [18] and [19] to describe the propagation of intense laser beams in a bulk medium with Kerr nonlinearity when small fourth-order dispersion are taken into account. Several results concerning well-posedness for (6.5) may be found in [10] (see also subsequent references). Control and stabilization for (6.5) have already appeared in [4].

Equation (6.4) also serves as the linear version of the more general equation
\begin{equation}
i\partial_t u + \partial_x^2 u + \mu \partial_x^4 u + F = 0,
\end{equation}
with
\begin{equation*}
F = \frac{1}{2}|u|^2 u + \mu \left(\frac{3}{8}|u|^4 u + \frac{3}{2}(\partial_x u)^2 u | + |\partial_x u|^2 u + \frac{1}{2}u^2 \partial_x^2 u + 2 |u|^2 \partial_x^2 u\right),
\end{equation*}
which describes the 3-dimensional motion of an isolated vortex filament embedded in an inviscid incompressible fluid filling an infinite region. Sharp results concerning local well-posedness in Sobolev spaces were proved in [14].

In order to set (6.4) as in (1.1) we define $A = i(\partial_x + \mu \partial_x^3)$, so that $a(k) = -k + \mu k^3$. Thus we may consider the equation
\begin{equation}
\partial_t u - \partial_x A u = G h.
\end{equation}
We promptly see that the mass is also conserved by the flow of (6.7) and
\begin{equation*}
|a(k)| \leq C |k|^3
\end{equation*}
for some constant $C > 0$ and any $k \in \mathbb{Z}$. Also, we easily check that (H1) holds and $\lambda_k = -k^2 + \mu k^4$. See Figure 3. Note if $\mu < 0$ then the even polynomial $p(y) = -y^2 + \mu y^4$ has no nontrivial roots, implying that $\lambda_k, k \neq 0$ are double eigenvalues and (H4) holds with $n_0 = 1$ and $k^*_1 = 1$. On the other hand, if $\mu > 0$ then $p(y)$ has the nontrivial roots $\pm 1/\sqrt{\mu}$; hence, if $k^*_1$ is the less integer satisfying $1/\sqrt{\mu} \leq k^*_1$, we see that (H4) holds with $n_0 = 4$.

It is also clear that (H5) also holds. Even more, we may check that
\begin{equation*}
\lim_{k \to +\infty} |(k+1)a(k+1) - ka(k)| = +\infty.
\end{equation*}
As a consequence, we may now apply Remark 1.6 to conclude that Theorem 1.5 holds for any $T > 0$. Consequently, Theorems 1.7 and 1.8 also hold.

6.3. The linearized dispersion-generalized Benjamin-Ono equation. In this subsection we investigate the control and stabilization properties of the linearized dispersion-generalized Benjamin-Ono (LDGBO) equation, which contains fractional-order spatial derivatives on a periodic domain,
\begin{equation}
\partial_t u + \partial_x D^\alpha u = 0, \quad x \in \mathbb{T}, \quad t \in \mathbb{R},
\end{equation}
where $\alpha > 0$, $u$ is a real-valued function and the Fourier multiplier operator $D^\alpha$ is defined as
\begin{equation}
\hat{D}^\alpha u(k) = |k|^\alpha \hat{u}(k), \quad \text{for all } k \in \mathbb{Z}.
\end{equation}
When \( \alpha \in (1, 2) \), the dispersion generalized Benjamin-Ono (DGOB) equation

\[
\partial_t u + \partial_x D^\alpha u + u \partial_x u = 0, \quad x \in \mathbb{R}, \quad t > 0,
\]

(6.10)
defines a family of equations which models vorticity waves in coastal zones [41]. The end points \( \alpha = 1 \) and \( \alpha = 2 \) corresponds to the well-known Benjamin-Ono and KdV equations, respectively. In this sense (6.10) defines a continuum of equations of dispersive strength intermediate to two celebrated models. Regarding control and stabilization properties, the author in [11] proved that the LDGBO equation with \( \alpha \in (1, 2) \) is exactly controllable in \( H^s(T) \) with \( s \geq 0 \) and exponentially stabilizable in \( L^2(T) \). Here we extend these results to the (periodic) Sobolev space \( H^s_p(T) \) with \( s \in \mathbb{R} \), for any \( \alpha > 0 \).

In fact, we consider the operator \( \mathcal{A} \) in (1.1) defined by \( \mathcal{A} = -D^\alpha \). Therefore, \( a(k) = -|k|^\alpha \) and it is easy to verify that

\[
|a(k)| \leq |k|^\alpha,
\]

and

\[
a(k) = a(-k),
\]

for any \( k \in \mathbb{Z} \). Hence, (H1) holds with \( \lambda_k = -k|k|^\alpha \). Using the L’Hospital rule we can prove that

\[
\lim_{y \to +\infty} y^{\alpha+1} \left( \left( 1 + \frac{1}{y} \right)^{\alpha+1} - 1 \right) = +\infty, \quad \text{for any } \alpha > 0.
\]

From this, we conclude that

\[
\lim_{k \to +\infty} |(k + 1)a(k + 1) - ka(k)| = \lim_{k \to +\infty} ((k + 1)^{\alpha+1} - k^{\alpha+1}) = +\infty.
\]

Thus, we can apply Remark 1.4 to infer that Theorem 1.3 holds for any \( T > 0 \). Consequently, Theorems 1.7 and 1.8 also hold in this particular case.

**Figure 3.** Dispersion of \( \lambda_k \)'s for the Smith and fourth-order Schrödinger equations with \( \mu > 0 \).
Finally, we point out the authors in [12] developed a dissipation-normalized Bourgain-type space, which simultaneously gains smoothing properties from the dissipation and dispersion present in the equation, to show that the nonlinear DGBO equation on a periodic setting is well-posed and local exponentially stable in $L^2_p(T)$. Extending these results to the Sobolev space $H^s_p(T)$ with $s > 0$ is a challenging task. This is an open problem.

6.4. Higher-order Schrödinger equation. In this section we consider the following higher-order Schrödinger equations

$$i\partial_t u + \alpha_2 \partial_x^2 u + \alpha_4 \partial_x^4 u + \ldots + \alpha_{2m} \partial_x^{2m} u = 0$$  \hspace{1cm} (6.11)

and

$$i\partial_t u + \alpha_2 \partial_x^2 u - i\alpha_3 \partial_x^3 u + \alpha_4 \partial_x^4 u - i\alpha_5 \partial_x^5 u + \ldots - i\alpha_{2m+1} \partial_x^{2m+1} u = 0,$$ \hspace{1cm} (6.12)

where $m$ is a positive integer and $\alpha_j$ are real constants with $\alpha_2 \neq 0$, $\alpha_{2m} \neq 0$, and $\alpha_{2m+1} \neq 0$. Equations (6.11) and (6.12) are the linearized versions of an infinite hierarchy of nonlinear Schrödinger equations (see [2]). Thus, here we consider the control equation

$$\partial_t u - \partial_x A_{2m+j} u = Gh,$$  \hspace{1cm} (6.13)

where

$$A_{2m+j} = \begin{cases} i\alpha_2 \partial_x + i\alpha_4 \partial_x^3 + \ldots + i\alpha_{2m} \partial_x^{2m-1}, & \text{if } j = 0, \\ i\alpha_2 \partial_x + \alpha_3 \partial_x^2 + i\alpha_4 \partial_x^2 + \alpha_5 \partial_x^4 + \ldots + \alpha_{2m+1} \partial_x^{2m}, & \text{if } j = 1. \end{cases}$$

The symbol associated with $A_{2m+j}$ is

$$a_{2m+j}(k) = \begin{cases} -\alpha_2 k + \alpha_4 k^3 + \ldots + \alpha_{2m+1}(1)^m k^{2m-1}, & \text{if } j = 0, \\ -\alpha_2 k - \alpha_2 k^2 + \alpha_4 k^3 + \alpha_5 k^4 + \ldots + \alpha_{2m+1}(1)^m k^{2m}, & \text{if } j = 1. \end{cases}$$

It is clear that

$$|a_{2m+j}(k)| \leq C|k|^{2m-1+j},$$

for some $C > 0$ and $|k|$ large enough.

Let us show that in the cases $j = 0$ and $j = 1$ we can apply Theorems 1.5 and 1.3, respectively. Indeed, assume first $j = 0$. It is easy to see that (H1) holds where the eigenvalues $i\lambda_k$ are such that

$$\lambda_k = -\alpha_2 k^2 + \alpha_4 k^4 + \ldots + (-1)^m \alpha_{2m} k^{2m}.$$  

The polynomial $p_{2m}(y) = -\alpha_2 y^2 + \alpha_4 y^4 + \ldots + (-1)^m \alpha_{2m} y^{2m}$ is even and goes to either $+\infty$ or $-\infty$ as $|y| \to +\infty$ (according to $m$ and the sign of $\alpha_{2m}$). Thus (H4) and (H5) holds with $n_0 = 2m$ and $k_1^*$ sufficiently large.

Assume now $j = 1$. In this case we have

$$\lambda_k = -\alpha_2 k^2 - \alpha_2 k^3 + \alpha_4 k^4 + \alpha_5 k^5 + \ldots + \alpha_{2m+1}(1)^m k^{2m+1}.$$  

Note that the polynomial $p_{2m+1}(y) = -\alpha_2 y^2 - \alpha_2 y^3 + \alpha_4 y^4 + \alpha_5 y^5 + \ldots + \alpha_{2m+1}(1)^m y^{2m+1}$

has different limits ($+\infty$ or $-\infty$) as $y \to +\infty$ or $y \to -\infty$ (according to $m$ and the sign of $\alpha_{2m+1}$). Hence (H2) and (H3) holds with $n_0 = 2m + 1$ and $k_1^*$ sufficiently large.

Furthermore, either in the case $j = 0$ or $j = 1$, it can be showed that the eigenvalues $\{i\lambda_k\}$ satisfies the "asymptotic gap condition". Hence, we obtain the exact controllability for any $T > 0$ and Theorems 1.7–1.8 hold as well.
In this work, we have showed two different criteria to prove that a linearized family of dispersive equations on a periodic domain is exactly controllable and exponentially stabilizable with any given decay rate in the Sobolev space $H^s_p(\mathbb{T})$ with $s \in \mathbb{R}$. We have applied these results to prove exact controllability and exponential stabilization for the linearized Smith equation and Schrödinger-type equations on a periodic domain. In a forthcoming paper we plan to use these results to prove some fundamental properties like the propagation of compactness, the unique continuation property and the propagation of smoothness for the solutions of the nonlinear Smith equation in order to show that it is exactly controllable and exponentially stabilizable on a periodic domain. That is the adequate approach to prove exact controllability and exponential stabilization for nonlinear PDE’s of dispersive type (see [23, 22, 26, 24, 31]). However, the symbol of the linear part associated to the Smith equation creates extra difficulty to prove the unique continuation property on a periodic domain. This work is in progress.

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