TOWARDS COMMUTATOR THEORY FOR RELATIONS

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Abstract. In a general algebraic setting, we state some properties of commutators of reflexive admissible relations.

After commutator theory in Universal Algebra has been discovered about thirty years ago [12], many important results and applications have been found. An introduction to commutator theory for congruence modular varieties can be found in [2] and [3]. Shortly after, results valid for larger classes of varieties have been obtained in [4] and [5]. More recent results, as well as further references, can be found, among others, in [1, 6, 10, 11].

Present-day theory deals with commutators of congruences. However, the possibility of a commutator theory for compatible reflexive relations has been voiced already in [5, p. 186]. Indeed, as noticed in [9] (in part independently in [5]), some notions from classical commutator theory can be extended to relations.

If $A$ is any algebra, and $R, S$ are compatible and reflexive relations, define $M(R, S)$ to be the set of all matrices of the form

$$
\begin{pmatrix}
  t(\overline{a}, \overline{b}) & t(\overline{a}', \overline{b}') \\
  t(\overline{a}', \overline{b}) & t(\overline{a}', \overline{b}')
\end{pmatrix}
$$

where $\overline{a}, \overline{a}' \in A^n$, $\overline{b}, \overline{b}' \in A^m$, for some $m, n \geq 0$, $t$ is an $m + n$-ary term operation of $A$, and $\overline{a}Ra', \overline{b}Sb'$.

We define $[R, S]$ to be the smallest congruence that centralizes $R$ modulo $S$, that is, the smallest congruence $\delta$ such that $z\delta w$ whenever $x\delta y$ and $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in M(R, S)$.

In our present setting, another commutator operation is more useful (cf. also [7]). Let $[R, S|1]$ be the transitive closure of the set

$$\left\{ (z, w) \left| \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in M(R, S) \right. \right\}$$

Notice that $[R, S]$ is, in general, much larger than $[R, S|1]$. $[R, S|1]$ is reflexive and compatible. If $S$ is a tolerance, then $[R, S|1]$ is a congruence.

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Clearly, \([R, S|1]\) is monotone in both arguments. Moreover, \([R, S|1] \subseteq S^*\), and \([R, S|1] \subseteq Cg(R);\) actually, \([R, S|1] \subseteq (S \cap (R^- \circ R))^*\).

For a relation \(R\) on some algebra, let \(R^\circ\) denote the smallest tolerance containing \(R\), and let \(R^-\) denote the converse of \(R\). \(R^*\) is the transitive closure of \(R\), and \(Cg(R)\) is the smallest congruence containing \(R\).

Note the following easy but useful properties of \([R, S|1]\).

**Lemma 1.** For \(R, R_1, R_2, S, T, U\) reflexive and admissible relations on some algebra, the following hold:

(i) \([R_1 \circ R_2, S|1] \subseteq (S \cap (R_2^- \circ (S \cap (R_1^- \circ R_1)) \circ R_2))^*\).

(ii) \([R, S \circ T|1] \subseteq ((R^- \circ R) \cap ((S \cap (R^- \circ (S \cap T^-) \circ R)) \circ (T \cap (R^- \circ (S^- \cap T) \circ R)))^*\).

(iii) \([R, S\circ T\circ U|1] \subseteq ((R^- \circ R) \cap (S \circ (T \cap (R^- \circ (T \cap (S^- \circ U^-)) \circ R)) \circ U))^*\).

**Proof.** Just draw a diagram. \(\square\)

Notice that it is possible to get a common generalization of (i) and (ii), as well as of (i) and (iii). However, we get rather long formulae.

**Theorem 2.** For every algebra \(A\), each of the following conditions imply all conditions below it:

(i) \(R \subseteq [R, R^\circ|1]\) for every reflexive compatible relation \(R\).

(ia) \(R^* \subseteq [R, R^\circ|1]\) for every reflexive compatible relation \(R\).

(ib) \(R^- \subseteq [R, R^\circ|1]\) for every reflexive compatible relation \(R\).

(ic) \(R^\circ \subseteq [R, R^\circ|1]\) for every reflexive compatible relation \(R\).

(id) \(Cg(R) = [R, R^\circ|1]\) for every reflexive compatible relation \(R\).

(ii) \(R \cap T \subseteq [R, T|1]\) for every tolerance \(T\) and every reflexive compatible relation \(R\).

(iii) \((R_1 \circ R_2) \cap T \subseteq (T \cap (R_2^- \circ (T \cap (R_1^- \circ R_1)) \circ R_2))^*\) for every tolerance \(T\) and all reflexive compatible relations \(R_1, R_2\).

(iv) \(R_1 \cap (T \circ R_2) \subseteq (T \cap (R_2^- \circ (T \cap (R_1^- \circ R_1)) \circ R_2))^* \circ R_2\) for every tolerance \(T\) and all reflexive compatible relations \(R_1, R_2\).

(v) \(\beta \cap (T \circ S) \subseteq (T \cap (S \circ (T \cap \beta) \circ S))^* \circ S\) for every congruence \(\beta\) and tolerances \(T, S\).

(vi) \(\beta \cap (T \circ \gamma) \subseteq \gamma \cup (T \cap \beta)^*\)
for every congruences \( \beta, \gamma \) and tolerance \( T \).

Conditions (i), (ia), (ib), (ic), (id) and (ii) above are equivalent for every algebra.

Proof. (i) \( \Rightarrow \) (ia) \( R \subseteq [R, R^\circ |1] \) implies \( R^* \subseteq [R, R^\circ |1]^* = [R, R^\circ |1] \).

(ia) \( \Rightarrow \) (i) is trivial, since \( R \subseteq R^* \).

The proof that (i) and (ib)-(id) are equivalent is similar.

(i) \( \Rightarrow \) (ii) Apply (i) with \( R \cap T \) in place of \( R \), thus getting

\[
R \cap T \subseteq [R \cap T, (R \cap T)^\circ |1] \subseteq [R, T^\circ |1] = [R, T |1]
\]

since \([, |1]\) is monotone.

(ii) \( \Rightarrow \) (i) By taking \( T = R^\circ \) in (ii) we obtain (i).

(iii) \( \Rightarrow \) (iv) First notice that \( R_1 \cap (T \circ R_2) \subseteq ((R_1 \circ R_2^-) \cap T) \circ R_2 \), since

if \((a, c) \in R_1 \cap (T \circ R_2)\) then \(a R_1 c\) and there is \(b\) such that \(a T b R_2 c\), thus

\(a R_1 c R_2^- b\) and \((a, b) \in T \cap (R_1 \circ R_2^-)\).

The conclusion follows by applying (iii) to \((R_1 \circ R_2^-) \cap T\).

(iv) \( \Rightarrow \) (v) is trivial.

(v) \( \Rightarrow \) (vi) is trivial: take \( S = \gamma \) and notice that \((T \cap (S \circ (T \cap \beta) \circ S))^* \subseteq (S \circ (T \cap \beta) \circ S)^* = \gamma \lor (T \cap \beta)^* \).

By a similar argument, strengthening condition (i) in Theorem 2, we get:

Theorem 3. For every algebra \( A \), each of the following conditions imply all conditions below it:

(i) \( R \subseteq [R, R |1] \) for every reflexive compatible relation \( R \).

(ia) \( R^* \subseteq [R, R |1] \) for every reflexive compatible relation \( R \).

(ii) \( R \cap T \subseteq [R, T |1] \) for all reflexive compatible relations \( T, R \).

(iii) \( (R_1 \circ R_2) \cap T \subseteq (T \cap (R_2^- \circ (T \cap (R_1^- \circ R_1)) \circ R_2))^* \) for all reflexive compatible relations \( R_1, R_2, T \).

(iv) \( R_1 \cap (T \circ R_2) \subseteq (T \cap (R_2 \circ (T \cap (R_1 \circ R_1)) \circ R_2^-))^* \circ R_2 \) for all reflexive compatible relations \( R_1, R_2, T \).

(v) \( \beta \cap (T \circ S) \subseteq (T \cap (S \circ (T \cap \beta) \circ S))^* \circ S \) for every congruence \( \beta \), tolerance \( S \) and reflexive compatible relation \( T \).

(vi) \( \beta \cap (T \circ \gamma) \subseteq (\gamma \circ (T \cap \beta))^* \) for every congruences \( \beta, \gamma \) and reflexive compatible relation \( T \).

Conditions (i), (ia) and (ii) above are equivalent for every algebra.
Proposition 4. For every algebra \( A \), the following conditions are equivalent:

(i) \( R \subseteq [R, R] \) and \( R^\gamma \subseteq [R, R] \)

for every reflexive compatible relation \( R \).

(ii) \( Cg(R) = [R, R] \)

for every reflexive compatible relation \( R \).

There are other interesting consequences of condition (i) in Theorem 2, respectively. For example, we can apply conditions (ii) and (iii) in Lemma 3. As an example, we show:

Theorem 5. (i) If an algebra \( A \) satisfies \( R \subseteq [R, R^\gamma] \) for every reflexive compatible relation \( R \), then \( A \) satisfies

\[
R \cap (S \circ T) \subseteq (S \cap (R \circ (S \cap T) \circ R)) \circ (T \cap (R \circ (S \cap T) \circ R))^* 
\]

for all reflexive compatible relations \( R, S, T \). In particular, if \( \gamma \) is a congruence and \( \gamma \supseteq S \cap T^\gamma \) then \( \gamma \cap (S \circ T) \cap (T \circ S^\gamma) \subseteq ((\gamma \cap S) \circ (\gamma \cap T))^* \).

(ii) If an algebra \( A \) satisfies \( R \subseteq [R, R] \) for every reflexive compatible relation \( R \), then \( A \) satisfies

\[
R \cap (S \circ T) \subseteq ((S \cap (R \circ (S \cap T) \circ R)) \circ (T \cap (R \circ (S \cap T) \circ R))^* 
\]

for all reflexive compatible relations \( R, S, T \). In particular, if \( \gamma \) is a congruence and \( \gamma \supseteq S \cap T^\gamma \) then \( \gamma \cap (S \circ T) \subseteq ((\gamma \cap S) \circ (\gamma \cap T))^* \).

Proof. (ii) By the assumption with \( U = R \cap (S \circ T) \) in place of \( R \), we have \( U \subseteq [U, U^\gamma] \subseteq [R, S \circ T^\gamma] \). Now, apply Lemma 4(ii).

The proof of (i) is similar. Take \( U = R \cap (S \circ T) \cap (T \circ S^\gamma) \) in place of \( R \), thus getting \( U \subseteq [U, U^\gamma] \subseteq [R, S \circ T^\gamma] \), since \( U^\circ \subseteq ((S \circ T) \cap (T \circ S^\gamma))^\circ = (S \circ T) \cap (T \circ S^\gamma) \subseteq S \circ T \). Again, apply Lemma 4(ii).

By using a more refined notation (already introduced in \( \mathbf{9} \)), we can improve Lemma 4.

For \( R, S, T \) compatible and reflexive relations, let

\[
K(R, S; T) = \left\{ (z, w) \mid M(R, S), xTy \right\}
\]

Thus, \( [R, S] \) is the transitive closure of \( K(R, S; 0) \), and \( [R, S] \) is the smallest congruence \( \gamma \) such that \( K(R, S; \gamma) \leq \gamma \). Hence, the importance of the operator \( K \) stems from the fact that any two elements congruent modulo \( [R, S] \) can be obtained by a finite number of applications of \( K(R, S; -) \) and of transitive closure and converse.

Lemma 6. For \( R, R_1, R_2, S, T, U, V \) reflexive and admissible relations on some algebra, the following hold:

(i) \( K(R_1 \circ R_2, S; V) \subseteq K(R_2, S; K(R_1, S; V)) \).
In particular, we shall deal with the following properties

\[(ii) K(R, S \circ T; V) \subseteq K(R, S; S \cap (V \circ T^-)) \circ K(R, T; T \cap (S^- \circ V)).\]

\[(iii) K(R, S \circ T \circ U; V) \subseteq (R^- \circ V \circ R) \cap (S \circ K(R, T; T \cap (S^- \circ V \circ U^-)) \circ U).\]

Since, trivially, \(K(R, S; V) \subseteq S \cap (R^- \circ (S \cap V) \circ R)\), Lemma 8 can be obtained as an immediate consequence of Lemma 6, taking \(V = 0\).

**Problem 7.** Which of the following conditions are equivalent in a variety?

(i) \(R \subseteq [R, R^c]\) for every reflexive compatible relation \(R\);

(ii) \(R \subseteq [R, R^c]\) for every reflexive compatible relation \(R\);

(iii) \(R \subseteq [R, R]\) for every reflexive compatible relation \(R\);

(iv) \(R \subseteq [R, R]\) for every tolerance \(R\);

(v) \(R \subseteq [R, R]\) for every tolerance \(R\).

Notice that, for every algebra, \(R \subseteq [R, R]\) is equivalent to \(R \cap T \subseteq [R, T]\). Announced results by K. Kearnes and E. Kiss suggest the possibility that (i) and (v) above are not equivalent. Thus, probably, commutator theory for relations has stronger consequences than commutator theory for tolerances, if we define the commutator to be \([R, S]\) rather than \([R, S]\).

In a sequel to this paper we shall derive consequences from the existence of a difference term and of a weak difference term for \([R, R^c]\) and for \([R, R]\). In particular, we shall deal with the following properties

(a) \(R \subseteq [R, R^c] \circ R^-\),

(b) \(R \subseteq [R, R^c] \circ R^- \circ [R, R^c]\),

(c) \(R \circ R \subseteq [R, R^c] \circ R\),

(d) \(R \circ R \subseteq [R, R^c] \circ R \circ [R, R^c]\),

(e) \(Cg(R) = [R, R^c] \circ R\),

(f) \(Cg(R) = [R, R^c] \circ R \circ [R, R^c]\),

(g) \(R \subseteq [R, R^c] \circ R^-\),

(h) \(R \subseteq [R, R^c] \circ R^- \circ [R, R]\),

(i) \(R \circ R \subseteq [R, R^c] \circ R\),

(j) \(R \circ R \subseteq [R, R^c] \circ R \circ [R, R]\),

(k) \(Cg(R) = [R, R^c] \circ R\),

(l) \(Cg(R) = [R, R^c] \circ R \circ [R, R]\).

We shall also deal with a weaker commutator

\([R, S]_w = \left\{ (x, w) \mid \begin{vmatrix} x & x \\ x & w \end{vmatrix} \in M(R, S) \right\}\)

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