Improved algorithms for Correlation Clustering with local objectives

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Abstract

Correlation Clustering is a powerful graph partitioning model that aims to cluster items based on the notion of similarity between items. An instance of the Correlation Clustering problem consists of a graph \( G \) (not necessarily complete) whose edges are labeled by a binary classifier as "similar" and "dissimilar". Classically, we are tasked with producing a clustering that minimizes the number of disagreements: an edge is in disagreement if it is a "similar" edge and is present across clusters or if it is a "dissimilar" edge and is present within a cluster. Define the disagreements vector to be an \( n \) dimensional vector indexed by the vertices, where the \( v \)-th index is the number of disagreements at vertex \( v \). Recently, Puleo and Milenkovic (ICML '16) initiated the study of the Correlation Clustering framework in which the objectives were more general functions of the disagreements vector. In this paper, we study algorithms for minimizing \( \ell_q \) norms \((q \geq 1)\) of the disagreements vector for both arbitrary and complete graphs. We present the first known algorithm for minimizing the \( \ell_q \) norm of the disagreements vector on arbitrary graphs and also provide an improved algorithm for minimizing the \( \ell_q \) norm \((q \geq 1)\) of the disagreements vector on complete graphs. Finally, we compliment these two algorithmic results with some hardness results.

1 Introduction

A basic task in machine learning is that of clustering items based on similarity between them. This task can be elegantly captured by Correlation Clustering, a clustering framework first introduced by Bansal, Blum, and Chawla (2004). In this model, we are given access to items and the similarity/dissimilarity between them in the form of a graph \( G \) on \( n \) vertices. The edges of \( G \) represent whether the items are similar or dissimilar and are labelled as "+" and "−" respectively. The goal is to produce a clustering that agrees with the labeling of the edges as much as possible, i.e., to group positive edges in the same cluster and place negative edges across different clusters (a positive edge that is present across clusters or a negative edge that is present within the same cluster is said to be in disagreement). The Correlation Clustering problem can be viewed as an agnostic learning problem, where we are given noisy examples and the task is to fit a hypothesis as best as possible to these examples. Co-reference resolution (see e.g., Cohen and Richman (2001, 2002)), spam detection (see e.g., Ramachandran et al. (2007); Bonchi et al. (2014)) and image segmentation (see e.g., Wirth (2017)) are some of the application to which Correlation Clustering has been applied to in practice.

This task is made trivial if the labeling given is consistent (transitive): if \((u, v)\) and \((v, w)\) are similar, then \((u, w)\) is similar for all vertices \(u, v, w\) in \( G \) (the connected components on similar edges would give an optimal clustering). Instead, it is assumed that the given labeling is inconsistent, i.e., it is possible that \((u, w)\) are dissimilar even though \((u, v)\) and \((v, w)\) are similar. For such a triple \(u, v, w\), every possible clustering incurs a disagreement on at least one edge and thus, no perfect clustering exists. The optimal clustering is the one which minimizes the disagreements. Moreover, as the number of clusters is not predefined, the optimal clustering can use anywhere from 1 to \( n \) clusters.

The classical metric that has been used in Correlation Clustering is that of minimizing the total number of edges in disagreement. Let the disagreements vector be an \( n \) dimensional vector indexed by the vertices where the \( v \)-th coordinate equals the number of disagreements at vertex \( v \). Thus, minimizing the total number of disagreements is equivalent to minimizing the \( \ell_1 \) norm of the disagreements vector. Puleo and Milenkovic (2016) initiated the study of more local objectives in the Correlation Clustering framework. They focus on complete graphs and study the minimization of \( \ell_q \) norms \((q \geq 1)\) of the disagreements vector. Charikar, Gupta, and Schwartz (2017) studied the problem of minimizing the \( \ell_\infty \) norm of the disagreements vector (also known as Min Max Correlation Clustering) for general graphs. For higher values of \( q \) (particularly \( q = \infty \)), a clustering optimized for minimizing the \( \ell_q \) norm prioritizes reducing the disagreements at vertices that are worst off. Thus, such metrics are very unforgiving in most cases as it is possible that in the optimal clustering there is only one vertex with high disagree-
ments while every other vertex has low disagreements. Thus, one is forced to infer the most pessimistic picture about the overall clustering. The $\ell_2$ norm is a solution to this tension between the $\ell_1$ and $\ell_\infty$ objectives. The $\ell_2$ norm of the disagreements vector takes into account the disagreements at each vertex while also penalizing the vertices with high disagreements more heavily. Thus, a clustering optimized for the minimum $\ell_2$ norm gives a more balanced clustering as it takes into consideration both the global and local picture.

Our contributions. In this paper, we provide positive and negative results for Correlation Clustering with the $\ell_q$ objective. We first study the problem of minimizing disagreements on arbitrary graphs. We present the first approximation algorithm minimizing any $\ell_q$ norm ($q \geq 1$) of the disagreements vector.

Theorem 1.1. There exists a polynomial time $O(n^{\frac{1}{2} - \frac{1}{q}} \cdot \log^{\frac{1}{2} + \frac{1}{q}} n)$ approximation algorithm for the minimum $\ell_q$ disagreements problem on general weighted graphs.

For the $\ell_2$ objective, the above algorithm leads to an approximation ratio of $O(n^{\frac{1}{2} / q})$, thus providing the first known approximation ratio for optimizing the clustering for this version of the objective. Note that the above algorithm matches the best approximation guarantee of $O(\log n)$ for the classical objective of minimizing the $\ell_1$ norm of the disagreements vector. For the $\ell_\infty$ norm, our algorithm matches the guarantee of the algorithm by Charikar, Gupta, and Schwartz (2017) up to log factors. Fundamental combinatorial optimization problems like Multicut, Multiway Cut and s-t Cut can be framed as special cases of Correlation Clustering. Thus, Theorem 1.1 leads to the first known algorithms for Multicut, Multiway Cut and s-t Cut with the $\ell_q$ objective when $q \neq 1$ and $q \neq \infty$. We can also use the algorithm from Theorem 1.1 to obtain $O(n^{\frac{1}{2} - \frac{1}{q}} \cdot \log^{\frac{1}{2} + \frac{1}{q}} n)$ bi-criteria approximation for Min $\alpha$-Balanced Partitioning with the $\ell_q$ objective (we omit details here).

Next, we study the case of complete graphs. For this case, we present an improved 5 approximation algorithm for minimizing any $\ell_q$ norm ($q \geq 1$) of the disagreements vector.

Theorem 1.2. There exists a polynomial time 5 approximation algorithm for the minimum $\ell_q$ disagreements problem on complete graphs.

We also study the case of complete bipartite graphs where disagreements need to be bounded for only one side of the bipartition, and not the whole vertex set. We give an improved 5 approximation algorithm for minimizing any $\ell_q$ norm ($q \geq 1$) of the disagreements vector.

Theorem 1.3. There exists a polynomial time 5 approximation algorithm for the minimum $\ell_q$ disagreements problem on complete bipartite graphs where disagreements are measured for only one side of the bipartition.

Our algorithm for the minimum $\ell_q$ disagreements problem is based on rounding the natural convex programming relaxation for this problem. We show that our result is best possible according to this relaxation by providing an almost matching integrality gap. The integrality gap example we provide is for the minimum $\ell_q$ $s-t$ cut problem (a special case of correlation clustering) and show the following result.

Theorem 1.4. The natural convex programming relaxation for the minimum $\ell_q$ disagreements problem has an integrality gap of $\Omega(n^{\frac{1}{2} - \frac{1}{q}})$ on arbitrary graphs.

Finally, we present a hardness of approximation result for minimum $\ell_\infty$ $s-t$ cut.

Theorem 1.5. There is no $\alpha$-approximation algorithm for the minimum $\ell_\infty$ $s-t$ cut problem for $\alpha < 2$ unless $P = NP$.

Previous work. Bansal, Blum, and Chawla (2004) showed that it is NP-hard to find a clustering that minimizes the total disagreements, even on complete graphs. They give a constant-factor approximation algorithm to minimize disagreements and a PTAS to maximize agreements on complete graphs. For complete graphs, Ailon, Charikar, and Newman (2008) presented an elegant randomized algorithm with an approximation guarantee of 3 to minimize total disagreements. They also gave a 2.5 approximation algorithm based on LP rounding. This factor was improved to slightly less than 2.06 by Chawla, Makarychev, Schramm, and Yaroslavtsev (2015). Since, the natural LP is known to have an integrality gap of 2, the problem of optimizing the classical objective is almost settled with respect to the natural LP. For arbitrary graphs, the best known approximation ratio is $O(\log n)$ (see Charikar, Guruswami, and Wirth 2003; Demaine, Emmanuel, Fiat, and Immorlica 2006). Assuming the Unique Games Conjecture, there is no constant-factor approximation algorithm for Correlation Clustering is possible (see Chawla et al. 2006). Puleo and Milenkovic (2016) first studied Correlation Clustering with more local objectives. For minimizing $\ell_q$ ($q \geq 1$) norms of the disagreements vector on complete graphs, their algorithm achieves an approximation guarantee of 48. This was improved to 7 by Charikar, Gupta, and Schwartz (2017). Charikar et al. (2017) also studied the problem of minimizing the $\ell_\infty$ norm of the disagreements vector on general graphs. They showed that the natural LP/SDP has an integrality gap of $n^{1/2}$ for this problem and provided a $O(\sqrt{n})$ approximation algorithm for minimum $\ell_\infty$ disagreements. Puleo and Milenkovic
2 Preliminaries

We now formally define the Correlation Clustering with \( \ell_q \) objective problem. We will need the following definition. Consider a set of points \( V \) and two disjoint sets of edges on \( V \): positive edges \( E^+ \) and negative edges \( E^- \). We assume that every edge has a weight \( w_{uv} \). For every partition \( P \) of \( V \), we say that a positive edge is in disagreement with \( P \) if the endpoints \( u \) and \( v \) belongs to different parts of \( P \); and a negative edge is in disagreement with \( P \) if the endpoints \( u \) and \( v \) belongs to the same part of \( P \). The vector of disagreements, denoted by \( \text{disagree}(P, E^+, E^-) \), is a \( |V| \) dimensional vector indexed by elements of \( V \). Its coordinate \( u \) equals

\[
\text{disagree}_u(P, E^+, E^-) = \sum_{v: (u,v) \in E^+ \cup E^-} w_{uv} \mathbf{1}\left( (u, v) \text{ is in disagreement with } P \right).
\]

That is, \( \text{disagree}_u(P, E^+, E^-) \) is the weight of disagreeing edges incident to \( u \). We similarly define a cut vector for a set of edges \( E \):

\[
\text{cut}_u(P, E) = \sum_{v: (u,v) \in E} w_{uv} \mathbf{1}(u \text{ and } v \text{ are separated by } P).
\]

We use the standard definition for the \( \ell_q \) norm of a vector \( x \): \( \|x\|_q = \left( \sum_x x^q \right)^{1/q} \) and \( \|x\|_\infty = \max_x x_u \). For a partition \( P \), we denote by \( P(u) \) the piece that contains vertex \( u \).

**Definition 1.** In the Correlation Clustering problem with \( \ell_q \) objective, we are given a graph \( G \) on a set \( V \) with two disjoint sets of edges \( E^+ \) and \( E^- \) and a set of weights \( w_{uv} \). The goal is find a partition \( P \) that minimizes the \( \ell_q \) norm of the disagreements vector, \( \| \text{disagree}(P, E^+, E^-) \|_q \).

In our algorithm for Correlation Clustering on arbitrary graphs, we will use a powerful technique of padded metric space decompositions (see e.g., Bartal (1996), Rao (1999); Fakcharoenphol and Talwar (2003); Gupta, Krauthgamer, and Lee (2003)).

**Definition 2 (Padded Decomposition).** Let \((X,d)\) be a metric space on \( n \) points, and let \( \Delta > 0 \). A probabilistic distribution of partitions \( P \) of \( X \) is called a padded decomposition if it satisfies the following properties:

- Each cluster \( C \in P \) has diameter at most \( \Delta \).
- For every \( u \in X \) and \( \varepsilon > 0 \),

\[
\Pr(\text{Ball}(u, \varepsilon) \not\subset P(u)) \leq D \cdot \frac{\delta}{\Delta}
\]

where Ball \((u, \delta) = \{v \in X : d(u, v) \leq \delta\}\).

**Theorem 2.1** (Fakcharoenphol, Rao, and Talwar (2004)). Every metric space \((X,d)\) on \( n \) points admits a \( D = O(\log n) \) separating padded decomposition. Moreover, there is a polynomial-time algorithm that samples a partition from this distribution.

3 Convex Relaxation

In both algorithms for arbitrary and complete graphs, we use a convex relaxation given in Figure 3.1. Our convex relaxation for Correlation Clustering is fairly standard. It is similar to relaxations used in the papers by Garg, Vazirani, and Yannakakis (1996); Demaine, Emanuel, Fiat, and Immorlica (2006); Charikar, Guruswami, and Wirth (2003). For every pair of vertices \( u \) and \( v \), we have a variable \( x_{uv} \) that is equal to the distance between \( u \) and \( v \) in the “multicut metric”. Variables \( x_{uv} \) satisfy the triangle inequality constraints (P3). They are also symmetric (P4) and \( x_{uv} \in [0, 1] \) (P5). Thus, the set of vertices \( V \) equipped with the distance function \( d(u, v) = x_{uv} \) is a metric space.

Additionally, for every vertex \( u \in V \), we have variables \( y_u \) and \( z_u \) (see constraints (P1) and (P2)) that lower bound the number of disagreeing edges incident to \( u \). The objective of our convex program is to minimize \( \max_{y,z} \left( \sum_y y_u \right) \). Note that all constraints in the program (P) are linear; however, the objective function of (P) is not convex as is. So in order to find the optimal solution, we raise the objective function to the power of \( q \) and find feasible \( x,y,z \) that minimizes the objective \( \max_{y,z} \left( \sum_y y_u \right)^q \).

Let us verify that program (P) is a relaxation for Correlation Clustering. Consider an arbitrary partitioning \( P \) of \( V \). In the integral solution corresponding to \( P \), we set \( x_{uv} = 0 \) if \( u \) and \( v \) are in the same cluster in \( P \); and \( x_{uv} = 1 \) if \( u \) and \( v \) are in different clusters in \( P \). In this solution, \( x_{uv} \) satisfy triangle inequality constraints (P3) and \( x_{uv} = x_{vu} \) (P4). Observe that a positive edge \((u,v) \in E^+\) is in disagreement with \( P \) if \( x_{uv} = 1 \); and a negative edge \((u,v) \in E^-\) is in disagreement if \( x_{uv} = 0 \). Thus, in this integral solution, \( y_u = \text{disagree}_u(P, E^+, E^-) \) and moreover, \( z_u \leq y_u^q \). Therefore, in the integral solution corresponding to \( P \), the objective function of (P) equals \( \| \text{disagree}_u(P, E^+, E^-) \|_q \). Of course, the cost of the optimal fractional solution to the problem may be less than...
minimize $\max \left( \|y\|_q, \left( \sum_{u \in V} z_u \right)^{\frac{1}{q}} \right)$

subject to $y_u = \sum_{v:(u,v) \in E^+} w_{uv} x_{uv} + \sum_{v:(u,v) \in E^-} w_{uv} (1 - x_{uv})$ for all $u \in V$ \hspace{1cm} (P1)

$z_u = \sum_{v:(u,v) \in E^+} w_{uv}^q x_{uv} + \sum_{v:(u,v) \in E^-} w_{uv}^q (1 - x_{uv})$ for all $u \in V$ \hspace{1cm} (P2)

$x_{v_1v_2} + x_{v_2v_3} + x_{v_1v_3} \geq x_{v_1v_3}$ for all $v_1, v_2, v_3 \in V$ \hspace{1cm} (P3)

$x_{uv} = x_{vu}$ for all $u, v \in V$ \hspace{1cm} (P4)

$x_{uv} \in [0, 1]$ for all $u, v \in V$ \hspace{1cm} (P5)

Figure 3.1: Convex relaxation for Correlation Clustering with min $\ell_q$ objective for $q < \infty$.

We remark that our algorithm also works for $q = \infty$. Indeed, the behaviour of the algorithm does not depend on $q$ (in fact, $q$ is not even a part of the algorithm’s input). Hence, inequality (1) holds for any $q < \infty$. In the limit as $q$ tends to infinity, we get the following result.

Corollary 4.2. The following inequality holds for a random partition $P$ from Theorem 4.1:

$$E \left[ \|\text{cut}(P, E)\|_\infty \right] \leq C n^{\frac{1}{q}} \log^{\frac{q}{2}} n \cdot \left[ \max_{(u,v) \in E} w \cdot \mathbf{1}(d(u,v) \neq 0) + \max_{w \in V} \left( \sum_{v:(u,v) \in E} w_{uv} \frac{d(u,v)}{\Delta} \right) \right].$$ (2)

We will need the following definition.

Definition 3. Let $(X, d)$ be a metric space. The $\varepsilon$-neighborhood of a set $S \subset X$ is the set of points at distance at most $\varepsilon$ from $S$:

$$N_\varepsilon(S) = \{ u \in X : \exists v \in S \text{ such that } d(u,v) \leq \varepsilon \}.$$

The $\varepsilon$-neighborhood of the boundary of a partition $P$ is the set of points

$$N_\varepsilon(\partial P) = \bigcup_{P \in \mathcal{P}} (N_\varepsilon(P) \setminus P) = \{ u \in X : \exists v \in X \text{ s.t. } d(u,v) \leq \varepsilon \text{ and } P(u) \neq P(v) \}.$$

We first describe an algorithm which succeeds with probability at least $1/2$ and fails with probability at most $1/2$. If the algorithm succeeds it outputs a random partition $P$ of $X$ such that the diameter of every set $P$ in $P$ is at most $\Delta$ and for every $q$ and every weighted...
Algorithm 4.1 succeeds when ∆ = 1. Thus, for the sake of analysis, we assume that ∆ = 1. Denote this event by E. We present a pseudo-code for our algorithm in Section 2). First, the algorithm picks a random padded graph using a simple deterministic procedure which we describe in the end of this section.

Our algorithm is based on the procedure for generating bounded padded stochastic decompositions (see Section 2). First, the algorithm picks a random padded decomposition P of the metric space X. Then, it finds the ε-neighborhood Nε(∂P) of the boundary of P. Finally, it outputs P if |Nε(∂P)| ≤ 2Dε/∆ and fails otherwise. We present a pseudo-code for our algorithm in Figure 4.1.

4.2 Analysis

Our algorithm is scale invariant i.e., its output does not change if we multiply all distances in the metric space (X, d) and the parameter ∆ by some positive number λ. Thus, for the sake of analysis, we assume that ∆ = 1. Algorithm 4.1 succeeds when Nε(P) has size at most M. Denote this event by E. We first show that Pr(E) ≥ 1/2.

Figure 4.1: Metric decomposition algorithm.

graph G = (X, E, w), we have

$$\begin{align*}
\mathbb{E}\left[\|\text{cut}(P, E)\|_q \mid \text{algorithm succeeds}\right] & \leq C'n^{\frac{q-1}{q}} \log^{\frac{q-1}{q}} n \cdot \left(\sum_{u \in X} \sum_{v:(u,v) \in E} w_{uv}^q \frac{d(u,v)}{\Delta}\right)^{1/q} +\\ & + \left(\sum_{u \in X} \left(\sum_{v:(u,v) \in E} w_{uv}^q \frac{d(u,v)}{\Delta}\right)^q\right)^{1/q}.
\end{align*}$$

To obtain a valid partition with probability 1, we repeat our algorithm for at most \( \lceil \log_2 n \rceil \) iterations till it succeeds and output the obtained solution. If the algorithm does not succeed after \( \lceil \log_2 n \rceil \) iterations (which happens with probability at most \( 1/n \)), we partition the graph using a simple deterministic procedure which we describe in the end of this section.

Lemma 4.3. Algorithm 4.1 succeeds with probability at least 1/2.

Proof. Let \( \bar{E} \) be the complement of the event \( E \). We need to show that \( \Pr(\bar{E}) \leq 1/2 \). To this end, we bound the expected size of the set \( N_{\varepsilon}(P) \) using the second property of padded decompositions:

$$\begin{align*}
\mathbb{E}[|N_{\varepsilon}(\partial P)|] & = \sum_{u \in X} \Pr(u \in N_{\varepsilon}(\partial P)) \\
& = \sum_{u \in X} \Pr(Ball(u, \varepsilon) \not\subset P(u)) \\
& \leq \sum_{u \in X} D_{\varepsilon} = D_{\varepsilon}n.
\end{align*}$$

Here, we used that \( u \in N_{\varepsilon}(\partial P) \) if and only if \( Ball(u, \varepsilon) \not\subset P(u) \). Now, by Markov's inequality,

$$\Pr(\bar{E}) = \Pr(|N_{\varepsilon}(\partial P)| > 2D_{\varepsilon}n) \leq \frac{D_{\varepsilon}n}{2D_{\varepsilon}n} = \frac{1}{2}.$$  

Let \( X_{uv} \) be the indicator of the event \{ \( P(u) \neq P(v) \) \} i.e., the event that points \( u \) and \( v \) are separated by the partition \( P \). By the second property of padded stochastic decompositions, we have \( \mathbb{E}(X_{uv}) = \Pr(P(u) \neq P(v)) \) \leq D \cdot d(u, v) \). Since \( \Pr(\bar{E}) \geq 1/2 \), for each \( (u, v) \in E \), we have

$$\mathbb{E}[X_{uv} \mid E] \leq \frac{\mathbb{E}[X_{uv}]}{\mathbb{E}(P)} \leq 2\mathbb{E}[X_{uv}] \leq 2D \cdot d(u, v).$$

Consequently,

$$\begin{align*}
\mathbb{E}[w_{uv} X_{uv}^q \mid E] & \leq 2D \cdot w_{uv} d(u, v) \quad \text{and} \quad (4) \\
\mathbb{E}[w_{uv}^q X_{uv}^q \mid E] & \leq 2D \cdot w_{uv}^q d(u, v). \quad (5)
\end{align*}$$

We split all edges \( E \) into two groups: short edges, which we denote by \( E_{\leq \varepsilon} \), and long edges, which we denote by \( E_{> \varepsilon} \). Short edges are edges of length at most \( \varepsilon \); long edges are edges of length greater than \( \varepsilon \). Note that \( \text{cut}(P, E) = \text{cut}(P, E_{\leq \varepsilon}) + \text{cut}(P, E_{> \varepsilon}) \).

For every subset \( E' \subset E \) (in particular, for \( E' = E_{\leq \varepsilon} \) and \( E' = E_{> \varepsilon} \)), we have

$$\begin{align*}
\mathbb{E}\left[\|\text{cut}(P, E')\|_q^q \mid E\right] & = \sum_{u \in X} \mathbb{E}\left[(\sum_{v:(u,v) \in E'} w_{uv} X_{uv})^q \mid E\right].
\end{align*}$$

We separately upper bound \( \mathbb{E}[\|\text{cut}(P, E_{\leq \varepsilon})\|_q^q \mid E] \) and \( \mathbb{E}[\|\text{cut}(P, E_{> \varepsilon})\|_q^q \mid E] \) using the formula above and inequalities (4), (5) and then use the triangle inequality for \( \ell_q \) norms to bound \( \mathbb{E}[\|\text{cut}(P, E)\|_q^q \mid E] \).

**Long edges.** Fix a vertex \( u \) and consider long edges incident to \( u \). Their total weight is upper bounded by

$$\sum_{v:(u,v) \in E_{> \varepsilon}} w_{uv} \leq \sum_{v:(u,v) \in E_{> \varepsilon}} w_{uv} \frac{d(u,v)}{\varepsilon} \geq 1.$$
Thus,
\[
\left( \sum_{v:(u,v) \in E_{>\varepsilon}} w_{uv} X_{uv} \right)^q \leq \left( \sum_{v:(u,v) \in E_{>\varepsilon}} w_{uv} \right)^{q-1} \left( \sum_{v:(u,v) \in E_{>\varepsilon}} w_{uv} X_{uv} \right) \leq \left( \sum_{v:(u,v) \in E_{>\varepsilon}} w_{uv} d(u,v) \right)^{q-1} \left( \sum_{v:(u,v) \in E_{>\varepsilon}} w_{uv} X_{uv} \right).
\]
Plugging this expression into formula (6) with $E = E_{\leq \varepsilon}$ and using inequality (4), we get the following upper bound on $\mathbb{E}[\| \text{cut}(\mathcal{P}, E_{>\varepsilon}) \|_q | \mathcal{E}]$:
\[
\mathbb{E}[\left( \sum_{v:(u,v) \in E_{>\varepsilon}} w_{uv} X_{uv} \right)^q | \mathcal{E}] \leq M^{q-1} \sum_{v:(u,v) \in E_{>\varepsilon}} \mathbb{E}[w_{uv}^q X_{uv}^q | \mathcal{E}] \leq M^{q-1} \sum_{v:(u,v) \in E_{>\varepsilon}} w_{uv}^q d(u,v).
\]
Finally, by Jensen’s inequality, we have
\[
\mathbb{E}[\| \text{cut}(\mathcal{P}, E_{>\varepsilon}) \|_q | \mathcal{E}] \leq \left( \mathbb{E}[\| \text{cut}(\mathcal{P}, E_{>\varepsilon}) \|_q | \mathcal{E}] \right)^{\frac{q}{q-1}} \mathbb{E}[\| \text{cut}(\mathcal{P}, E_{>\varepsilon}) \|_q | \mathcal{E}] \leq 2D M^{q-1} \sum_{u \in X} \left( \sum_{v:(u,v) \in E_{>\varepsilon}} w_{uv} d(u,v) \right)^{q/2}. \tag{7}
\]

**Short edges.** To bound $\| \text{cut}(\mathcal{P}, E_{\text{short}}) \|_q$, we will make use of the following lemma.

**Lemma 4.4.** Consider non-negative (dependent) random variables $X_1, \ldots, X_n$. Suppose that at most $M$ of them are non-zero with probability 1. Then, for every $q \geq 1$, the following bound holds:
\[
\mathbb{E}\left[ X_1^q + \cdots + X_n^q \right] \leq M^{q-1} \sum_{i=1}^n \mathbb{E}[X_i^q].
\]

**Proof.** Let $x_1, \ldots, x_m$ be the non-zero random variables in a certain sampling of $X_1, \ldots, X_n$ for some $m \leq M$. Suppose that $m \neq 0$. Using Jensen’s inequality, we have
\[
\left( \frac{x_1 + \cdots + x_m}{m} \right)^q \leq \frac{1}{m} \sum_{j=1}^m x_j^q,
\]
and, therefore,
\[
\left( x_1 + \cdots + x_m \right)^q \leq m^{q-1} \sum_{j=1}^m x_j^q \leq M^{q-1} \sum_{j=1}^m x_j^q.
\]
The inequality above also holds when $m = 0$. Thus, the expectation of the left-hand side is upper bounded by the expectation of the right-hand side. This concludes the proof. \[\square\]

Fix a vertex $u$. Observe that if $(u,v)$ is a short edge which is cut by $\mathcal{P}$ then $v$ must belong to $N_\varepsilon(\partial \mathcal{P})$. Thus, the number of non-zero random variables $X_{uv}$ for a given $u$ and $(u,v) \in E_{\leq \varepsilon}$ is upper bounded by $|N_\varepsilon(\partial \mathcal{P})|$. If the algorithm succeeds, then $|N_\varepsilon(\partial \mathcal{P})| \leq M$. Thus, by Lemma 4.4,
\[
\mathbb{E}[\left( \sum_{v:(u,v) \in E_{\leq \varepsilon}} w_{uv} X_{uv} \right)^q | \mathcal{E}] \leq M^{q-1} \sum_{v:(u,v) \in E_{\leq \varepsilon}} \mathbb{E}[w_{uv}^q X_{uv}^q | \mathcal{E}] \leq M^{q-1} \sum_{v:(u,v) \in E_{\leq \varepsilon}} w_{uv}^q d(u,v).
\]
Finally, by Jensen’s inequality, we have
\[
\mathbb{E}[\| \text{cut}(\mathcal{P}, E_{\leq \varepsilon}) \|_q | \mathcal{E}] \leq \left( \mathbb{E}[\| \text{cut}(\mathcal{P}, E_{\leq \varepsilon}) \|_q | \mathcal{E}] \right)^{\frac{q}{q-1}} \mathbb{E}[\| \text{cut}(\mathcal{P}, E_{\leq \varepsilon}) \|_q | \mathcal{E}] \leq 2D M^{q-1} \sum_{u \in X} \left( \sum_{v:(u,v) \in E_{\leq \varepsilon}} w_{uv} d(u,v) \right)^{q/2}. \tag{8}
\]

To obtain the desired bound (3), we substitute $D = O(\log n)$, $\varepsilon = 1/\sqrt{2Dn}$, and $M = 2D\varepsilon n/\Delta$ in bounds (7) and (8) and then apply the triangle inequality for the $\ell_q$ norm.

To finish the proof of Theorem 1.1, we describe what we do in the unlikely event that Algorithm 4.1 fails $\lceil \log_2 n \rceil$ times.

**Lemma 4.5.** There exists a polynomial-time deterministic algorithm that given a metric space $(X,d)$ on $n$ points and parameter $\Delta$ returns a partition $\mathcal{P}$ of $X$ such that the diameter of every set $P$ in $\mathcal{P}$ is at most $\Delta$ and for every $q$ and every weighted graph $G = (X,E,w)$, we have
\[
\| \text{cut}(\mathcal{P}, E) \|_q \leq n \left( \sum_{u \in X} \left( \sum_{v:(u,v) \in E} \frac{w_{uv} d(u,v)}{\Delta} \right)^q \right)^{1/q}.
\]

**Proof.** Consider a graph $\tilde{G} = (X, \tilde{E})$ on $X$ with edges $\tilde{E} = \{(u,v) \in X \times X : d(u,v) \leq \Delta/n\}$. The algorithm partitions $\tilde{G}$ into connected components and outputs the result. Note that the diameter of each connected component $P$ is less than $\Delta$, since the length of every edge in $\tilde{G}$ is less than $\Delta/n$. Let $E_{\text{cut}}$ be the set of cut edges in graph $G$. If two vertices $(u,v)$ are separated by $\mathcal{P}$, then $d(u,v) \geq \Delta/n$. Hence, for every cut edge
\((u, v) \in E_{\text{cut}}\), we have \(nd(u, v)/\Delta \geq 1\). Thus,

\[
\|\text{cut}(P, E)\|_q = \left( \sum_{u \in X} \left( \sum_{v: (u,v) \in E} w_{uv} \right)^q \right)^{1/q}
\]

\[
\leq \left( \sum_{u \in X} \left( \sum_{v: (u,v) \in E} w_{uv} \frac{n \cdot d(u, v)}{\Delta} \right)^q \right)^{1/q}
\]

\[
= n \left( \sum_{u \in X} \left( \sum_{v: (u,v) \in E} \frac{d(u, v)}{\Delta} \right)^q \right)^{1/q}.
\]

\(\square\)

5 Correlation Clustering on Complete Graphs

In this section, we present our algorithm for Correlation Clustering on complete graphs and its analysis. Our algorithm achieves an approximation ratio of 5 and is an improvement over the approximation ratio of 7 by Charikar, Gupta, and Schwartz (2017).

5.1 The Algorithm

Our algorithm is based on rounding an optimal solution to the convex relaxation \((P)\). We start with considering the entire vertex set of unclustered vertices. At each step \(t\) of the algorithm, we select a subset of vertices as a cluster \(C_t\) and remove it from unclustered vertices. Thus, each vertex is assigned to a cluster exactly once and is never removed from a cluster once it is assigned.

For each vertex \(w \in V_t\), let \(\text{Ball}(w, \rho) = \{ u \in V : x_{uw} \leq \rho \}\) be the set of vertices within a distance of \(\rho\) from \(w\). For \(r = 1/5\) the quantity \(r - x_{uw}\) where \(u \in \text{Ball}(w, r)\) represents the distance from \(u\) to the boundary of the ball of radius 1/5 around \(w\). Let \(V_t \subseteq V\) be the set of unclustered vertices at step \(t\), and define

\[
L_t(w) = \sum_{u \in \text{Ball}(w, r) \cap V_t} r - x_{uw}.
\]

At each step \(t\), we select the vertex \(w_t\) that maximizes the quantity \(L_t(w)\) over all unclustered vertices \(w \in V_t\) and select the set \(\text{Ball}(w_t, 2r)\) as a cluster. We repeat this step until all the nodes have been clustered. A pseudo-code for our algorithm is given in Figure 6.1

5.2 Analysis

In this section, we present an analysis of our algorithm.

Theorem 5.1. Algorithm 2 gives a 5-approximation for Correlation Clustering on complete graphs.

For an edge \((u, v) \in E\), let \(LP(u, v)\) be the LP cost of the edge \((u, v)\): \(LP(u, v) = x_{uv}\) if \((u, v) \in \mathcal{E}^+\) and \(LP(u, v) = 1 - x_{uv}\) if \((u, v) \in \mathcal{E}^-\). Let \(\text{ALG}(u, v) = 1((u, v)\) in disagreement \).

Define

\[
\text{profit}(u) = \sum_{(u, v) \in E} LP(u, v) - r \sum_{(u, v) \in E} \text{ALG}(u, v),
\]

negative edges \((u, v) \in \mathcal{E}^-\), we have \(x_{uv} \leq 1/2\) and \(1 - x_{uv} \geq 1/2\). Thus, disagreee\(_t\)(\(P, \emptyset, \mathcal{E}^-\)) \(\leq 2y_u\) for every \(u\), and \(E\|\text{disagree}(P, \emptyset, \mathcal{E}^-)\|_q \leq 2\|y\|_q \leq 2LP\). This completes the proof.

Theorem 4.6. There exists a randomized polynomial-time \(O(n^{\frac{3}{2}} \log \frac{n}{\Delta})\) approximation algorithm for Correlation Clustering with the \(\ell_q\) objective (\(q \geq 1\)).

We remark that the same algorithm gives \(O(\sqrt{n \log n})\) approximation for the \(\ell_{\infty}\) norm. We omit the details in the conference version of the paper.

Proof. Our algorithm first finds the optimal solution \(x, y, z\) to the convex relaxation \((P)\) presented in Section 4.3. Then, it defines a metric \(d(u, v) = x_{uv}\) on the vertices of the graph. Finally, it runs the metric space partitioning algorithm with \(\Delta = 1/2\) from the previous section (see Theorem 4.1) and output the obtained partitioning \(P\).

Let us analyze the performance of this algorithm. Denote the cost of the optimal solution \(x, y, z\) by \(LP\). We know that the cost of the optimal solution \(OPT\) is lower bounded by \(LP\) (see Section 4.3 for details). By Theorem 4.1 applied to the graph \(G = (V, \mathcal{E}^+)\) (note: we ignore negative edges for now),

\[
E\left[\|\text{cut}(P, \mathcal{E}^+)\|_q\right] \leq \frac{C}{\Delta} n^{\frac{q-1}{2q}} \log \frac{n+1}{2r} n \cdot \left( \left( \sum_{u \in V} y_u^q \right)^{\frac{1}{q}} + \left( \sum_{u \in V} z_u^q \right)^{\frac{1}{q}} \right) \leq 4Cn^{\frac{q-1}{2q}} \log \frac{n+1}{2r} n \cdot LP. \quad (9)
\]

Recall that a positive edge is not in agreement if and only if it is cut. Hence, disagreee\(_t\)(\(P, \mathcal{E}^+, \emptyset\)) = cut\(_t\)(\(P, \mathcal{E}^+\)), and the bound above holds for \(E\|\text{disagree}(P, \mathcal{E}^+, \emptyset)\|_q\). By the triangle inequality, \(E\|\text{disagree}(P, \mathcal{E}^+, \mathcal{E}^-)\|_q \leq E\|\text{disagree}(P, \mathcal{E}^+, \emptyset)\|_q + E\|\text{disagree}(P, \emptyset, \mathcal{E}^-)\|_q\). Hence, to finish the proof, it remains to upper bound \(E\|\text{disagree}(P, \emptyset, \mathcal{E}^-)\|_q\).

Observe that the diameter of every cluster returned by the algorithm is at most \(\Delta = 1/2\). For all disagreeing
where \( r = 1/5 \). We show that for each vertex \( u \in V \), we have \( \text{profit}(u) \geq 0 \) (see Lemma 5.2 below) and, therefore, the number of disagreeing edges incident to \( u \) is upper bounded by \( 5y(u) \):

\[
\text{ALG}(u) = \sum_{v \in E} \text{ALG}(u, v) \leq \frac{1}{r} \sum_{v \in E} \text{LP}(u, v) = 5y(u).
\]

Thus, \( \|\text{ALG}\|_q \leq 5\|y\|_q \) for any \( q \geq 1 \). Consequently, the approximation ratio of the algorithm is at most 5 for any norm \( \ell_q \).

**Lemma 5.2.** For every \( u \in V \), we have \( \text{profit}(u) \geq 0 \).

At each step \( t \) of the algorithm, we create a new cluster \( C_t \) and remove it from the graph. We also remove all edges with at least one endpoint in \( C_t \). Denote this set of edges by

\[
\Delta E_t = \{(u, v) : u \in C_t \text{ or } v \in C_t \}.
\]

Now let

\[
\text{profit}_t(u, v) = \begin{cases} 
\text{LP}(u, v) - r\text{ALG}(u, v), & \text{if } (u, v) \in \Delta E_t \\
0, & \text{otherwise}
\end{cases}
\]

\[
\text{profit}_t(u) = \sum_{v \in V_t} \text{profit}_t(u, v)
\]

(10)

As all sets \( \Delta E_t \) are disjoint, \( \text{profit}(u) = \sum_t \text{profit}_t(u) \).

Thus, to prove Lemma 5.2 it is sufficient to show that \( \text{profit}_t(u) \geq 0 \) for all \( t \). Note that we only need to consider \( u \in V_t \) as \( \text{profit}_t(u) = 0 \) for \( u \notin V_t \).

Consider a step \( t \) of the algorithm and vertex \( u \in V_t \). Let \( w = w_t \) be the center of the cluster chosen at this step. First, we show that since the diameter of the cluster \( C_t \) is \( 4r \), for all negative edges \( (u, v) \in E^- \) with \( u, v \in C_t \), we can charge the cost of disagreement to the edge itself, that is, \( \text{profit}_t(u, v) \) is nonnegative for \( (u, v) \in E^- \) (see Lemma 5.3). We then consider two cases: \( x_{uw} \in [0, r] \cup [3r, 1] \) and \( x_{uw} \in (r, 3r] \).

The former case is fairly simple since disagreeing positive edges \( (u, v) \in E^+ \) (with \( x_{uw} \in [0, r] \cup [3r, 1] \)) have a “large” LP cost. In Lemma 5.4 and Lemma 5.5 we prove that the cost of disagreement can be charged to the edge itself and hence \( \text{profit}_t(u) \geq 0 \).

We then consider the latter case. For vertices \( u \) with \( x_{uw} \in (r, 3r] \), \( \text{profit}_t(u, v) \) for some disagreeing positive edges \( (u, v) \) might be negative. Thus, we split the profit at step \( t \) for such vertices \( u \) into the profit they get from edges \( (u, v) \) with \( v \in \text{Ball}(w, r) \cap V_t \) and from edges with \( v \in V_t \setminus \text{Ball}(w, r) \). That is,

\[
\text{profit}_t(u) = \left( \sum_{v \in \text{Ball}(w, r)} \text{profit}_t(u, v) \right) + \left( \sum_{v \in V_t \setminus \text{Ball}(w, r)} \text{profit}_t(u, v) \right).
\]

Denote the first term by \( P_{\text{high}}(u) \) and the second term by \( P_{\text{low}}(u) \).

We show that \( P_{\text{low}}(u) \geq -L_t(u) \) (see Lemma 5.9) and \( P_{\text{high}} \geq L_t(w) \) (see Lemma 5.7) and conclude that \( \text{profit}_t(u) = P_{\text{high}}(u) + P_{\text{low}}(u) \geq 0 \) since \( L_t(w) = \max_{w \in V_t} L_t(w') \geq L_t(u) \).

In the following claim, we show that we can charge the cost of disagreement of a negative edge to the edge itself.

**Claim 5.3.** For a negative edge \( (u, v) \in E^- \), \( \text{profit}_t(u, v) \) is always nonnegative.

**Proof.** The only case when \( (u, v) \) is in disagreement is when both \( u \) and \( v \) belong to the new cluster. In this case, they lie in the ball of radius \( 2r \) around \( w \) (and thus \( x_{uw}, x_{uw} \leq 2r \)). Thus the distance \( x_{uw} \) between them is at most \( 4r \) (because \( x_{uw} \leq x_{uw} + x_{uw} \leq 4r \)).

The LP cost of the edge \( (u, v) \) is at least \( L_p(u, v) = 1 - x_{uw} \geq 1 - 4r = r \). Thus, \( \text{profit}_t(u, v) = L_p(u, v) - r\text{ALG}(u, v) = L_p(u, v) - r \geq 0 \).  

In Lemma 5.4 and Lemma 5.5 we consider the case when \( x_{uw} \in [0, r] \cup (3r, 1] \).

**Lemma 5.4.** If \( x_{uw} \leq r \), then \( \text{profit}_t(u, v) \geq 0 \) for all \( v \in V_t \).

**Proof.** If \( x_{uw} \in E^- \), then \( \text{profit}_t(u, v) \geq 0 \) by Claim 5.3. Assume that \( x_{uw} \in E^+ \). Since \( x_{uw} \leq r \), \( u \) belongs to
the cluster $C_t$. Thus, $(u, v)$ disagrees only if $v$ does not belong to that cluster. In this case, $x_{uv} \geq 2r$ and by
the triangle inequality $x_{uv} \geq x_{vw} - x_{uw} \geq r$. Therefore, $\text{profit}_t(u, v) = x_{u,v} - r \geq 0$.

Lemma 5.5. If $x_{uw} \geq 3r$, then $\text{profit}_t(u, v) \geq 0$ for all $v \in V_t$.

Proof. As in the previous lemma, we can assume that $x_{uw} \in E^+$. If $x_{uw} \geq 3r$, then $u$ does not belong to
the new cluster $C_t$. Thus, $(u, v)$ disagrees only if $v$ belongs to $C_t$. In this case, $x_{uw} \leq 2r$ and by
the triangle inequality $x_{uw} \geq x_{uw} - x_{vw} \geq r$. Therefore, $\text{profit}_t(u, v) = x_{u,v} - r \geq 0$.

We next consider $u$ such that $x_{uw} \in (r, 3r]$. First, we
show that the profit we obtain from every edge $(u, v)$
with $v \in \text{Ball}(w, r)$ is at least $r - x_{uw}$, regardless of
whether the edge is positive or negative.

Claim 5.6. If $x_{uw} \in (r, 3r]$ and $v \in \text{Ball}(w, r) \cap V_t$, then $\text{profit}_t(u, v) \geq r - x_{uw}$.

Proof. First consider $u$ such that $x_{uw} \in (r, 2r]$. Note
that $x_{uw} \geq x_{uw} - x_{vw} \geq r - x_{uw}$. Moreover, $x_{uw} \leq x_{uw} + x_{vw} \leq 2r + x_{vw}$. Thus, if $(u, v) \in E^+$, then $\text{profit}_t(u, v) \geq r - x_{uw}$. Otherwise, $\text{profit}_t(u, v) \geq (1 - 2r - x_{vw}) - r \geq 2r - x_{uw}$.

For $u \in (2r, 3r]$, note that $x_{uw} \geq x_{uw} - x_{uw} \geq 2r - x_{uw}$. Moreover, $x_{uw} \leq x_{uw} + x_{vw} \leq 3r + x_{vw}$. Thus, if $(u, v) \in E^+$, then $\text{profit}_t(u, v) \geq (2r - x_{uw}) - r \geq x_{uw} - r$. Otherwise, $\text{profit}_t(u, v) \geq (3 - 3r - x_{vw}) \geq 2r - x_{uw}$.

Using the above claim, we can sum up the profits from all
vertices $v$ in $\text{Ball}(w, r)$ and lower bound $P_{\text{high}}(u)$ as
follows.

Lemma 5.7. If $x_{uw} \in (r, 3r]$, then $P_{\text{high}}(u) \geq L_t(w)$.

Proof. By Claims 5.5, 5.6 we have $\text{profit}_t(u, v) \geq r - x_{uw}$ for
all $v \in V_t$. Thus,

$$
P_{\text{high}}(u) = \sum_{v \in \text{Ball}(w, r) \cap V_t} \text{profit}_t(u, v) \geq \sum_{v \in \text{Ball}(w, r) \cap V_t} (r - x_{uv}) = L_t(w).
$$

We now lower bound $P_{\text{low}}(u)$. To this end, we estimate
each term $\text{profit}_t(u, v)$ in the definition of $P_{\text{low}}$.

Claim 5.8. If $x_{uw} \in (r, 3r]$ and $v \in V_t \setminus \text{Ball}(w, r)$, then $\text{profit}_t(u, v) \geq \min(x_{uw} - r, 0)$.

Proof. By Claim 5.5 if $(u, v)$ is a negative edge, then $\text{profit}_t(u, v) \geq 0$. The profit is 0 if $x_{uv} \notin \Delta E_t$ (i.e.,
neither $u$ nor $v$ belong to the new cluster). So let us
assume that $(u, v)$ is a positive edge in $\Delta E_t$. Then, the
profit obtained from $(u, v)$ is $x_{uv}$ if $(u, v)$ is in agreement
and $x_{uw} - r$ if $(u, v)$ is in disagreement. In any case,
$\text{profit}_t(u, v) \geq x_{uw} - r \geq \min(x_{uw} - r, 0)$.

Lemma 5.9 is an immediate corollary of Claim 5.8.

Lemma 5.9. If $x_{uw} \in (r, 3r]$, then $P_{\text{low}}(u) \geq L_t(u)$.

Proof. By Claim 5.8 we have $\text{profit}_t(u, v) \geq \min(x_{uw} - r, 0)$ for all $v \in V_t$. Thus,

$$
P_{\text{low}}(u) = \sum_{v \in V_t \setminus \text{Ball}(w, r)} \text{profit}_t(u, v) \geq \sum_{v \in V_t \setminus \text{Ball}(w, r)} \min(x_{uw} - r, 0) \geq \sum_{v \in V_t \setminus \text{Ball}(w, r)} x_{uw} - r = -L_t(u).
$$

Here we used that (a) all terms $\min(x_{uw} - r, 0)$ are non-
positive, and (b) $\min(x_{uw} - r, 0) = 0$ if $v \notin \text{Ball}(u, r)$.

6 Integrality gap

In this section, we present an integrality gap example
for the convex program (P). We describe an instance
of the convex cut problem on $\Theta(n^\frac{1}{2} - \frac{1}{2})$. In our integrality gap
example, we describe a layered graph with $\Theta(n^\frac{1}{2})$
layers, with each layer consisting of a complete bipartite
graph on $\Theta(n^\frac{1}{2})$. Between each layer $i$ and $i + 1$,
there is a terminal $s_i$ which connects these two layers.
Finally, the terminals $s$ and $t$ are located at opposite
ends of this layered graph. We will observe that for any
integral cut separating $s$ and $t$, there will be at least one
vertex such that a large fraction of the edges incident to
it are cut. We will show that there is a corresponding
fractional solution that is cheaper compared to any integral
cut as the fractional solution can “spread” the cut
equally across the layers, thus not penalizing any
individual layer too harshly. In doing so, we will prove
the following theorem.

Theorem 6.1. The integrality gap for the convex relaxation
(P) is $\Omega(n^\frac{1}{2} - \frac{1}{2})$.

Proof. We now give a more formal description of the
layered graph discussed above. The construction has
two parameters $a$ and $b$, so we will call such a graph $G_{a,b}$.
The graph consists of $b$ layers with each layer consisting
of the complete bipartite graph $K_{a,a}$. We refer to layer $i$
of the graph as $G^i_{a,b}$ and refer to the left and right hand of the bipartition as $L(G^i_{a,b})$ and $R(G^i_{a,b})$ respectively. In addition to these layers, the graph consists of $b + 1$ terminals $\{s, t, s_1, \ldots, s_{b-1}\}$ (we will refer to $s$ as $s_0$ and $t$ as $s_b$ interchangeably). For each $i \in \{1, \ldots, b - 1\}$, the vertex $s_i$ is connected to all the vertices in $R(G^i_{a,b})$ and $L(G^i_{a,b})$. Finally, $s$ is connected to all the vertices in $L(G^1_{a,b})$ and $t$ is connected to all the vertices in $R(G^b_{a,b})$.

Consider any integral cut separating $s$ and $t$ in the graph $G_{a,b}$. Any such cut must disconnect at least one pair of consecutive terminals (if all pairs of consecutive terminals are connected, then $s$ is still connected to $t$). Thus let $j \in \{0, 1, \ldots, b\}$ be such that $s_{j-1}$ is disconnected from $s_j$ and consider the subgraph induced on $\{s_{j-1} \cup s_j \cup G^j_{a,b}\}$. We will show that this induced subgraph contains a vertex such that $\Omega(a^{j+2})$ of its incident edges are cut. Intuitively, since $s_{j-1}$ are not cut, then $s_j$ are cut. If $s_j$ cut $s_{j-1}$, then $s_j$ must have many neighbors in $L(G^j_{a,b})$ and $R(G^j_{a,b})$ respectively. As $G^j_{a,b}$ is highly connected, in order for $s_{j-1}$ to be separated from $s_j$, there must be a vertex in $G^j_{a,b}$ with many incident edges which are cut. If cut$(s_{j-1})$ or cut$(s_j)$ is at least $a/2$, then we are done. Otherwise, $s_j$ is connected to at least $a/2$ vertices in $R(G^j_{a,b})$, so every $u$ adjacent to $s_{j-1}$ must have at least $a/2$ incident edges which are cut. Therefore, $OPT^q \geq \Omega(a^q)$.

We now present a fractional cut separating $s$ and $t$. If an edge $e$ connects $s_i$ to a vertex in $R(G^j_{a,b})$ for some $i \in \{1, \ldots, b\}$, set the length of the edge to be $1/b$; otherwise set the edge length to be 0. We let $x_{uv}$ be the shortest path metric in this graph. It is easy to see that such a solution is feasible. We now analyze the quality of this solution. For each $i \in \{1, \ldots, b\}$, we have $y_{s_i} = a/b$ and for each $u \in R(G^i_{a,b})$, we have $y_u = 1/b$. Thus

$$LP^q = ab \left(\frac{1}{b}\right)^q + b \left(\frac{a}{b}\right)^q.$$ 

If $b > a$, then

$$LP^q \leq ab \left(\frac{1}{b}\right)^q + b \left(\frac{a}{b}\right)^q = 2a$$

and if $b > a$, then

$$LP^q \leq ab \left(\frac{1}{b}\right)^q + b \left(\frac{a}{b}\right)^q \leq a^q \left(a^{-(q-1)} + b^{-(q-1)}\right).$$

Setting $a = b = \Omega(n^{1/2})$ gives

$$\frac{OPT^q}{LP^q} = \Omega(n^{1/2 – 1/2q}),$$

so the integrality gap is $\frac{OPT}{LP} = \Omega(n^{1/2 – 1/2q})$.

### 7 Hardness of approximation

In this section, we prove the following hardness result.

**Theorem 7.1.** It is NP-hard to approximate the min $\ell_\infty$ s-t cut problem within a factor of $2 - \varepsilon$ for every positive $\varepsilon$.

**Proof.** The proof follows a reduction from 3SAT. We will describe a procedure that reduces every instance of a 3CNF formula $\phi$ to a graph $G_\phi$ such that the minimum $\ell_\infty$ s-t cut for $G_\phi$ has a certain value if and only if the formula $\phi$ is satisfiable.

**Reduction from 3SAT:** Given a 3CNF instance $\phi$ with $n$ literals and $m$ clauses, we describe a graph $G_\phi$ with $(2 + 4n + 5m)$ vertices and $(6n + 8m)$ edges. We refer to the vertex and edge set of $G_\phi$ as $V(G_\phi)$ and $E(G_\phi)$. For every literal $x_i$, $i \in \{1, \ldots, n\}$, we have four nodes, $x_i^T, x_i^F, x_i^1$, and $x_i^2$. Additionally, we have a “False” and a “True” node. For every $i \in \{1, \ldots, n\}$, we connect “False” with $x_i^T$ and “True” with $x_i^F$ using an infinite weight edge. Both $x_i^T$ and $x_i^F$ are connected to $x_i^1$ and $x_i^2$ using edges of weight 1.

For every clause $C$ in $\phi$, we will create a gadget in $G_\phi$ consisting of five nodes. We will refer to the subgraph induced by these nodes as $G_\phi[C]$. Let the clause $C = (y_1 \lor y_2 \lor y_3)$. We have a node in the gadget for each $y_i$, $i \in \{1, 2, 3\}$, and two additional nodes $C_a$ and $C_b$. We connect $y_2$ and $y_3$ to $C_b$, and $y_1$ and $C_b$ to $C_a$, all using unit weight edges.

We connect the gadget $G_\phi[C]$ for clause $C = (y_1 \lor y_2 \lor y_3)$ to the main graph as follows. For each $i \in \{1, 2, 3\}$, connect the vertex for the literal $y_i$ to the vertex $y_i^2$ with a unit weight edge. Finally connect the node $C_a$ to the “True” vertex using an infinite weight edge. An example of a 3CNF formula $\phi$ and the corresponding $G_\phi$ is given in Figure 7.

**Fact 1.** Consider the gadget $G_\phi[C]$ for the clause $C = (y_1 \lor y_2 \lor y_3)$. If all three nodes $y_1, y_2,$ and $y_3$ need to be disconnected from $C_a$, then either $\{cut_{C_a} = 2$ or $cut_{C_b} = 2$. If at most two of the three nodes $y_1, y_2,$ and $y_3$ need to be disconnected from $C_a$, then there is a cut that separates those nodes from $C_a$ such that both $cut_{C_a}$ and $cut_{C_b}$ are at most 1.

**Lemma 7.2.** Given a 3CNF formula $\phi$, consider the graph $G_\phi$ constructed according to the reduction described above. The formula $\phi$ is satisfiable if and only if the minimum $\ell_\infty$ True-False cut $P$ for the graph $G_\phi$ has value 1, that is, $\|cut_P\|_\infty = 1$.

**Proof.** Let $0 \Rightarrow \min \ell_\infty$ True-False cut: If the 3CNF formula $\phi$ is satisfiable, then the graph $G_\phi$ has a minimum $\ell_\infty$ s-t cut of value exactly 1. This can be seen as follows. Given a satisfying assignment $x^*$, we
will construct a cut $E_P$ (and corresponding partition $P$) such that for every vertex $u \in V(G_\phi)$, $\text{cut}_P(u) \leq 1$. For every $i \in \{1, \ldots, n\}$, if $x_i^* \in P$, then include $(x_i^T, x_i^F)$ and $(\bar{x}_i^T, x_i^F)$ as part of the cut $E_P$, else include $(x_i^T, x_i^F)$ and $(\bar{x}_i^T, x_i^F)$ as part of the cut $E_P$. Note that this cuts exactly one edge incident to each vertex $x_i^T, x_i^F, \bar{x}_i^T$ and $x_i^T$ for $i \in \{1, \ldots, n\}$. Since $\phi$ has a satisfiable assignment, each clause $C$ has at least one literal which is True, and hence the node corresponding to this literal is not connected to the vertex False in $G_\phi - E_P$. Thus, each clause $C$ has at most two literals that are False, and thus there are at most two False-True paths that go through this gadget. From Fact 1, we can know that we can include edges from $E(G_\phi[C])$ in $E_P$ such that both cut$_P(C_a)$ and cut$_P(C_b)$ are at most 1 and the False-True paths through this gadget are disconnected. Thus, $E_P$ disconnects True from False such that $||\text{cut}_P(G_\phi)||_\infty = 1$.

**minimum $\ell_\infty$ True-False cut $\Rightarrow$ 3SAT**: Let $G_\phi$ be the graph constructed for the 3CNF formula $\phi$ such that there is a cut $E_P \subseteq E(G_\phi)$ (and corresponding partition $P$) such that $P$ separates True from False and $||\text{cut}_P(G_\phi)||_\infty = 1$. We will construct a satisfying assignment $x^*$ from the formula $\phi$. Since $\text{cut}_P(u) \leq 1$ for every $u \in V(G_\phi)$, none of the $(\text{True}, x_i^T)$, $(x_i^F, \text{False})$ edges are part of the cut $P$ for $i \in \{1, \ldots, n\}$. In order for True to be separated from False, either the edges $(x_i^T, x_i^F)$ and $(\bar{x}_i^T, x_i^F)$ are part of the cut $E_P$, or the edges $(x_i^T, x_i^F)$ and $(x_i^T, \bar{x}_i^F)$ are part of the cut $E_P$. This gives us our assignment: for each $i \in \{1, \ldots, n\}$, if $(x_i^T, x_i^F) \in E \setminus E_P$, then assign $x_i^*$ as True and $\bar{x}_i^*$ as False. Otherwise $(x_i^T, x_i^F) \in E \setminus E_P$, so assign $x_i^*$ as False and $\bar{x}_i^*$ as True. Now, we show that $x^*$ is a satisfiable assignment for $\phi$. To see this, note that for each clause $C$, there exists at least one literal $y_i$ such that the node corresponding to $y_i$ is still connected to $C_a$. As the cut $E_P$ separates True and False, $(y_i^T, y_i^F) \in E \setminus E(G_\phi)$ and hence $y_i^* = \text{True}$. Thus, the assignment $x^*$ is satisfiable for $\phi$.

Thus, we can conclude Theorem 5.1 from the reduction procedure provided and Lemma 5.2.

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Figure 7.1: $G_\phi$ for the 3CNF formula $\phi = (x_1 \lor \bar{x}_2 \lor x_3) \land (x_2 \lor \bar{x}_4 \lor \bar{x}_5) \land (\bar{x}_1 \lor x_4 \lor x_5)$. 
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**A Correlation Clustering on Complete Bipartite Graphs**

Let \( (V = L \cup R, E) \) be a complete bipartite graph with \( L \) and \( R \) being the bipartition of the vertex set. In this section, we provide and analyze an algorithm for correlation clustering on complete graphs with an approximation guarantee of 5 for minimizing the mistakes on one side of the bipartition (which without loss of generality will be \( L \)). The algorithm and analysis for complete graphs is very similar to the algorithm and analysis for complete graphs. At each step \( t \) of our algorithm, we select a cluster center \( w_t \in L \) and a cluster \( C_t \subseteq (L \cup R) \) and remove it from the graph. This clustering step is repeated until all vertices in \( L \) are part of some cluster. If there are any remaining vertices in \( R \) which are unclustered, we put them in a single cluster.

Similar to the definition of \( \text{Ball}(w, \rho) \) in Section 5, we define \( \text{Ball}_S(w, \rho) = \{ u \in S : x_{uw} \leq \rho \} \). We select the cluster centers \( w_t \) in step \( t \) as follows. Let \( V_t \subseteq V \) be the set of unclustered vertices at the start of step \( t \). We redefine \( L_t^S(w) = \sum_{u \in \text{Ball}_{V_t \cap S}(w, \rho)} r - x_{uw} \). We select \( w_t \) to maximize \( L_t(w) \). We then select \( \text{Ball}_{L \cup R}(w, 2r) \) as our cluster and repeat. A pseudocode for the above algorithm is provided in Figure A.1.

**A.1 Analysis**

In this section, we present an analysis of our algorithm.

**Theorem A.1.** Algorithm 3 gives a 5-approximation for Correlation Clustering on complete bipartite graphs.
Input: Optimal solution $x$ to the linear program (P).
Output: Clustering $C$.

1. Let $V_0 = V$, $r = 1/5$, $t = 0$.
2. while $(V_t \cap L \neq \emptyset)$
   - Find $w_t = \arg \max_{w \in L} \frac{L_t^R(w)}{\bar{r}}$.
   - Create a cluster $C_t = Ball_{L \cup R}(w_t, 2r) \cap V_t$.
   - Set $V_{t+1} = V_t \setminus C_t$ and $t = t + 1$.
3. Let $C_L = (C_0, \ldots, C_{t-1})$.
4. if $(r \cap V_t \neq \emptyset)$
   - Let $C_R = R \cap V_t$.
5. Return $C = C_L \cup \{C_R\}$.

Figure A.1: Algorithm for Correlation Clustering on complete bipartite graphs.

where disagreements are measured on only one side of the bipartition.

The proof of this theorem is almost identical to the proof of Theorem 5.1. We define $LP(u, v)$, $ALG(u, v)$, $\text{profit}_t(u, v)$ for every edge $(u, v)$ and $\text{profit}(u)$, $\text{profit}_t(u)$ for every vertex $u$ as in Section 5. We then show that for each vertex $u \in L$, we have $\text{profit}(u) \geq 0$ and, therefore, the number of disagreeing edges incident to $u$ is upper bounded by $5y(u)$:

$$ALG(u) = \sum_{v(u,v) \in E} ALG(u, v) \leq \frac{1}{r} \sum_{v(u,v) \in E} LP(u, v) = 5y(u).$$

Thus, $\|ALG\|_q \leq 5\|y\|_q$ for any $q \geq 1$. Consequently, the approximation ratio of the algorithm is at most 5 for any norm $\ell_q$.

**Lemma A.2.** For every $u \in L$, we have $\text{profit}(u) \geq 0$.

As in Lemma 5.2, we need to show that $\text{profit}_t(u) \geq 0$ for all $t$. Note that we only need to consider $u \in V_t \cap L$ as $\text{profit}_t(u) = 0$ for $u \notin V_t$.

Consider a step $t$ of the algorithm and vertex $u \in V_t \cap L$. Let $w = w_t$ be the center of the cluster chosen at this step. First, we show that since the diameter of the cluster $C_t$ is $4r$, for all negative edges $(u, v) \in E^-$ with $u, v \in C_t$, we can charge the cost of disagreement to the edge itself, that is, $\text{profit}_t(u, v)$ is nonnegative for $(u, v) \in E^-$ (see Lemma 5.3). We then consider two cases: $x_{uw} \in [0, r] \cup [3r, 1]$ and $x_{uw} \in (r, 3r]$.

The former case is fairly simple since disagreeing positive edges $(u, v) \in E^+$ with $x_{uw} \in [0, r] \cup [3r, 1]$ have a “large” LP cost. In Lemma 5.4 and Lemma 5.5, we prove that the cost of disagreement can be charged to the edge itself and hence $\text{profit}_t(u) \geq 0$.

We then consider the latter case. Similarly to Lemma 5.2, we split the profit at step $t$ for vertices $u$ with $x_{uw} \in (r, 3r]$ into the profit they get from edges $(u, v)$ with $v$ in $\text{Ball}_R(w, r) \cap V_t$ and from edges with $v$ in $V_t \setminus \text{Ball}_R(w, r)$. That is,

$$\text{profit}_t(u) = \sum_{v \in \text{Ball}_R(w, r) \cap V_t} \text{profit}_t(u, v) + \sum_{v \in V_t \setminus \text{Ball}_R(w, r)} \text{profit}_t(u, v).$$

Denote the first term by $P_{\text{high}}(u)$ and the second term by $P_{\text{low}}(u)$. We show that $P_{\text{low}}(u) \geq -L_t^R(u)$ (see Lemma A.6) and $P_{\text{high}} \geq L_t^R(u) = \sum_{v \in \text{Ball}_R(w, r) \cap V_t} r - x_{vw}$ (see Lemma A.5) and conclude that $\text{profit}_t(u) = P_{\text{high}}(u) + P_{\text{low}}(u) \geq L_t^R(u) - L_t^R(u) \geq 0$ since $L_t^R = \max_{u' \in V_t} L_t^R(u') \geq L_t^R(u)$.

Consider $u$ such that $x_{uw} \in (r, 3r]$. First, we show that we obtain from every edge $(u, v)$ with $v \in \text{Ball}_R(w, r)$ is at least $r - x_{uw}$, regardless of whether the edge is positive or negative.

**Claim A.3.** If $x_{uw} \in (r, 3r]$ and $v \in \text{Ball}_R(w, r) \cap V_t$, then $\text{profit}_t(u, v) \geq r - x_{uw}$.

**Proof.** First consider $u$ such that $x_{uw} \in (r, 2r]$. Note that $x_{uw} \geq x_{uw} - x_{uw} \geq r - x_{uw}$. Moreover, $x_{uw} \leq x_{uw} + x_{uv} \leq 2r + x_{uv}$. Thus, if $(u, v) \in E^+$, then $\text{profit}_t(u, v) \geq r - x_{uw}$. Otherwise, $\text{profit}_t(u, v) \geq 1 - 2r - x_{uv}$. For $u \in (2r, 3r]$, note that $x_{uw} \geq x_{uw} - x_{uw} \geq 2r - x_{uv}$. Moreover, $x_{uw} \leq x_{uw} + x_{uv} \leq 3r + x_{uv}$. Thus, if $(u, v) \in E^+$, then $\text{profit}_t(u, v) \geq (2r - x_{uw}) - r \geq r - x_{uw}$. Otherwise, $\text{profit}_t(u, v) \geq (1-3r-x_{uv}) \geq 2r - x_{uv}$. Using the above claim, we can sum up the profits from all vertices $v$ in $\text{Ball}_R(w, r)$ and lower bound $P_{\text{high}}(u)$ as follows.

**Lemma A.4.** If $x_{uw} \in (r, 3r]$, then $P_{\text{high}}(u) \geq L_t^R(u)$.

**Proof.** By Claim , we have $\text{profit}_t(u, v) \geq r - x_{uw}$ for all $v \in R \cap V_t$. Thus,

$$P_{\text{high}}(u) = \sum_{v \in \text{Ball}_R(w, r) \cap V_t} \text{profit}_t(u, v) \geq \sum_{v \in \text{Ball}_R(w, r) \cap V_t} r - x_{uw} = L_t^R(u).$$
We now lower bound \( P_{\text{low}}(u) \). To this end, we estimate each term \( \text{profit}_t(u, v) \) in the definition of \( P_{\text{low}} \).

**Claim A.5.** If \( x_{uv} \in (r, 3r] \) and \( v \in V_t \setminus \text{Ball}_R(w, r) \), then \( \text{profit}_t(u, v) \geq \min(x_{uv} - r, 0) \).

**Proof.** By Claim 5.3 if \((u, v)\) is a negative edge, then \( \text{profit}_t(u, v) \geq 0 \). The profit is 0 if \( x_{uv} \notin \Delta E_t \) (i.e., neither \( u \) nor \( v \) belong to the new cluster). So let us assume that \((u, v)\) is a positive edge in \( \Delta E_t \). Then, the profit obtained from \((u, v)\) is \( x_{uv} \) if \((u, v)\) is in agreement and \( x_{uv} - r \) if \((u, v)\) is in disagreement. In any case, \( \text{profit}_t(u, v) \geq x_{uv} - r \geq \min(x_{uv} - r, 0) \). \( \square \)

Lemma A.6 is an immediate corollary of Claim A.5

**Lemma A.6.** If \( x_{uw} \in (r, 3r] \), then \( P_{\text{low}}(u) \geq -L^R_t(u) \).

**Proof.** By Claim 5.8 we have \( \text{profit}_t(u, v) \geq \min(x_{uv} - r, 0) \) for all \( v \in V_t \). Thus,

\[
P_{\text{low}}(u) = \sum_{v \in V_t \setminus \text{Ball}_R(w, r)} \text{profit}_t(u, v)
\geq \sum_{v \in V_t \setminus \text{Ball}_R(w, r)} \min(x_{uv} - r, 0)
\geq a \sum_{v \in V_t} \min(x_{uv} - r, 0)
=b \sum_{v \in \text{Ball}_R(u, r) \cap V_t} x_{uv} - r
=-L^R_t(u).
\]

Here we used that (a) all terms \( \min(x_{uv} - r, 0) \) are non-positive, and (b) \( \min(x_{uv} - r, 0) = 0 \) if \( v \notin \text{Ball}(u, r) \). \( \square \)