Variational Approach to Dynamics of Quantum Fields

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October 30, 2001

1 Introduction

In the summer of 1995, Dominique Vautherin came to Kyoto and stayed three months at the Yukawa Institute of Kyoto University where he delivered a series of very informal, pedagogical lectures on the application of variational methods to quantum field theories. This initiated our long lasting enjoyable collaboration on the subject and Yasuhiko Tsue joined the force later. Throughout this collaboration Vautherin was always enthusiastic to explore new physics problems and always came up with new innovative ideas, much of which I suspect have origins in his expertise in nuclear many-body theory. We are very sorry that this fruitful collaboration with Vautherin came to end so soon by his untimely death and that we are no longer able to be inspired by his deep physics insights, charmed by his elegance, and most of all cheered up by his very warm presence. I like to dedicate these lectures to the memory of Dominique Vautherin from whom I learned most of the material presented below.

First, I like to say a few words about physics motivations of our works. In recent years, several authors have constructed non-trivial time-dependent solutions of classical field equations of effective meson fields. Such solutions are relevant in considering the fate of defects which might be produced in dynamical order-disorder phase transitions in ultrarelativistic nucleus-nucleus collisions or in the evolution of very early universe. We like to study the effect of quantum fluctuations which has been ignored in these classical analyses. We may consider the classical solution as a collective mode of underlying microscopic degrees of freedom. It is well-known in nuclear many-body

*Lectures delivered at the NATO Advanced Study Institute on QCD Perspectives on Hot and Dense Matter, Cargèse, Aug. 6 - 18, 2001.
theory that the internal microscopic motions of nucleons are influenced by the motion of the nuclear mean field such as rotation or vibration. One may expect similar effects would arise in the field theory.

Our method is based on the Schrödinger wave functional representation of the quantum field theories in which the time-evolution of the wave function is explicitly considered. Another advantage of using the Schrödinger picture is that one can introduce the variational method to construct approximate but non-perturbative solutions of the problem as in similar problems in quantum mechanics. The extension of the variational principle for time-dependent wave function has also been developed. The method can also be extended to deal with statistical ensembles described by Gaussian form of density matrices.

2 A Simple Exercise in Quantum Mechanics

We start with a very elementary example in quantum mechanics which all students learn in an introductory course of quantum mechanics: the harmonic oscillator problem. Although our interests lies in the time-dependent solution of quantum field theories, it would be instructive to first study the time-dependent solution of this simple exactly soluble problem in order to gain some physical insights into our more difficult problems since quantum field theories are nothing but an ensemble of infinite number of coupled harmonic oscillators.

Let us consider a simple one-dimensional harmonic oscillator whose Hamiltonian is given by

\[ H_0 = \frac{1}{2} \left( p^2 + \omega^2 x^2 \right), \tag{1} \]

where we set the mass of the particle \( m = 1 \) for simplicity. We look for solutions of the time-dependent Schrödinger equation:

\[ i \frac{\partial}{\partial t} \Psi(t) = H \Psi(t) \]

with the quantization condition: \([x, p] = i\).

There are two ways to solve this problem. The easier one is the algebraic method in which one introduces "creation" and "annihilation" operators:

\[ a^\dagger = \frac{1}{\sqrt{2\omega}} (-ip + \omega x), \quad a = \frac{1}{\sqrt{2\omega}} (ip + \omega x) \]

which satisfy a usual commutation relation: \([a, a^\dagger] = 1\) and diagonalize the Hamiltonian: \( H = \omega \left( a^\dagger a + \frac{1}{2} \right) \). The ground state of the Hamiltonian is given by the condition: \( a|0\rangle = 0 \). This method is usually transcribed to the quantization of fields and one obtains particle creation and annihilation operators as basic building blocks in describing physical processes. In this approach, one usually does not refer to the wave function of the system explicitly but instead concentrates on amplitudes of particular process
for given initial and final states specified by the particle number and other quantum numbers.

Alternatively, one can solve the problem by the analytic method in which one expresses the Schrödinger equation in coordinate representation with the differential operator \( p = -i\partial/\partial x \) and solve the resultant second order differential equation. The eigenfunction of this differential equation are Hermite’s polynomials with the Gaussian ground state wave function:

\[
\Psi_0(x) = \langle x|0 \rangle = \left(\frac{\omega}{\pi}\right)^{1/4} e^{-\frac{1}{2} \omega x^2}
\]

It is straightforward to write down the ground state wave functional of the quantized free field \( \varphi(x) \). For one component scalar field theory, it becomes just a product of the Gaussian wave functions of normal modes each specified by the momentum with the oscillator frequency \( \omega_k = \sqrt{m^2 + k^2} \):

\[
\Psi_0[\varphi(x)] = \mathcal{N} \exp \left[ -\frac{1}{2} \sum_k \omega_k \varphi_k^* \varphi_k \right],
\]

with a proper normalization condition.

**Time-dependent variational wave function:**

Now let us consider time-dependent solutions of the harmonic oscillator. We first modify the ground state wave function by adding extra complex phase factor \( e^{i(p_0 - i\omega x_0)x} \). One then obtains the Gaussian wave function with its the center shifted

\[
\Psi(x, t; x_0, p_0) = \exp \left[ -\frac{1}{2} \omega (x - x_0(t))^2 + ip_0(t)x \right]
\]

Here \( p_0(t), x_0(t) \) are time-dependent parameters which are to be determined by imposing that the above function is a solution of the time-dependent Schrödinger equation. Here we use the variational method to derive the equations of motion of \( p_0(t), x_0(t) \).

The Schrödinger equation can be obtained by imposing a stationary condition \( \delta S = 0 \) for the action:

\[
S[\Psi] = \int dt \langle \Psi(t)|H - i\frac{\partial}{\partial t}|\Psi(t)\rangle
\]

with respect to variation of the wave function \( \langle \Psi(t)\rangle \). For the variational wave function in the form of \( \Psi \), the integrand can be computed easily with \( \langle \Psi|p^2|\Psi\rangle = p_0^2 \), \( \langle \Psi|x^2|\Psi\rangle = x_0^2 \), \( \langle \Psi|i\frac{\partial}{\partial t}|\Psi\rangle = -\dot{p}_0 x_0 \):

\[
\langle \Psi|H - i\frac{\partial}{\partial t}|\Psi\rangle = p_0^2 + \omega^2 x_0^2 + \dot{p}_0 x_0.
\]

Taking stationary conditions of the action with respect to two time-dependent parameters, \( \delta S/\delta p_0 = \delta S/\delta x_0 = 0 \), one finds

\[
\dot{p}_0 = -\omega^2 x_0, \quad \dot{x}_0 = p_0
\]
which are just the classical equation of motion of the harmonic oscillator. The center-shifted Gaussian wave function may be obtained, beside the phase factor $e^{-ip_0 x}$, by operating a translation operator on the ground state:

$$|0; x_0, p_0⟩ \sim \exp[-i x_0 p] |0⟩ = \exp \left[ -i \sqrt{\omega} x_0 (a^\dagger - a) \right] |0⟩.$$  

It thus describes a phase-coherent mixture of infinite number of excited states $|n⟩$. The corresponding states in quantum field theory are called coherent states. One thus sees that classical field equations arises if one describes the wave functional of the quantized fields in terms of coherent states.

Keeping Gaussian form of variational wave function one can go one step further by introducing additional phase factor quadratic in $x$. The wave function now takes the form of

$$\Psi_{sq}(x, t) = \left( \frac{\mu}{\pi} \right)^{1/4} \exp \left[ -\frac{1}{2} (\mu + i\sigma)(x - x_0)^2 - ip_0 x \right]$$  

where we have introduced two new real parameters, $\mu, \sigma$, characterizing the width of the wave function, which we shall consider as time-dependent variational parameters. The time-dependent parameter $\mu(t)$ is related to the quantum fluctuation of the position of the particle around its mean $x_0 = ⟨x⟩$ by $⟨(x - x_0)^2⟩ = 1/μ(t)$. The another parameter $\sigma$ is related to the rate of change of $\mu$ as we shall see below. This modification of the wave function thus describes the breathing motion (squeezing and stretching) of the wave function centered at $x_0(t)$. In the quantum field theory, these generalized coherent states are called squeezed states.

With the new variational wave function the action integrand becomes

$$\langle \Psi(t)|H - i \frac{\partial}{\partial t}|\Psi(t)⟩_{sq} = \frac{1}{2} \left( \dot{p}_0^2 + \omega^2 x_0^2 \right) + \frac{1}{4} \left( \mu + \sigma^2 + \frac{\omega^2}{\mu} \right) + \dot{p}_0 x_0 - \frac{\dot{\sigma}}{4\mu}$$  

Taking the stationary conditions with respect to the variations of $\mu(t), \sigma(t)$, in addition to classical parameters, $x_0(t), p_0(t)$, one finds

$$\dot{\mu} = 2\sigma \mu, \quad \dot{\sigma} = \sigma^2 + \omega^2 - \mu^2;$$  

while the equations of motion of $x_0(t), p_0(t)$ are unchanged. We see that the imaginary part $\sigma$ of the Gaussian width parameter plays a role similar to the velocity of the motion of the center of the Gaussian $x_0$. For the harmonic oscillator Hamiltonian, coherent states and squeezed states are exact solutions of the Schrödinger equations. The classical motion and the quantum fluctuation decouple in this exactly soluble problem. This is not the case, however, when the potential is not quadratic in $x$.

**Anharmonicity:**

To illustrate the effect of non-harmonic part of the potential, we add a term quartic in $x$ in our Hamiltonian:

$$H_I = \frac{\lambda}{4!} x^4$$
In the scalar field theory, this corresponds to adding a $\varphi^4$ self-interaction term. This anharmonic term generates new terms in the integrand of the action:

$$\langle \Psi(t)|H_I|\Psi(t)\rangle_{sq.} = \frac{\lambda}{4!} \left( \frac{3}{4\mu^2} + 3\frac{x_0^2}{\mu} + x_0^4 \right)$$

which cause coupling between the classical motion of the mean of particle position and the quantum fluctuation around it:

$$\dot{x}_0 = p_0, \quad \dot{p}_0 = -\omega^2 x_0 - \frac{\lambda}{6} x_0^3 - \frac{\lambda}{4\mu} x_0,$$

$$\dot{\mu} = 2\sigma \mu, \quad \dot{\sigma} = \sigma^2 + \omega^2 - \mu^2 + \frac{\lambda}{4\mu} + \frac{\lambda}{2} x_0^2$$

Let us see how the time independent solution (the ground state) is modified by the anharmonic term. The conditions $\dot{x}_0 = \dot{\mu}_0 = 0$ demand that $p_0 = \sigma = 0$, while the remaining two conditions $\dot{p}_0 = \dot{\sigma}_0 = 0$ determine the values of $x_0$ and $\mu$. There are two types of solutions depending on the sign of $\omega^2$. When $\omega^2 > 0$ we have a ”normal” solution centered at the origin $x_0 = 0$ but the width is modified slightly as determined by:

$$\mu^2 = \omega^2 + \frac{\lambda}{4\mu}$$

This equation may be called the gap equation for the reason to be discussed in the next section. If $\omega^2 < 0$, the potential has double minima at $x_{\text{min.}} = \pm \sqrt{6/\lambda \omega}$ and the $x_0 = 0$ point becomes local maximum of the potential. In this case, there are two ”symmetry breaking” solutions centered at the two solutions of

$$\omega^2 + \frac{\lambda}{4\mu} + \frac{\lambda}{6} x_0^2 = 0$$

where the value of the width parameter $\mu$ is determined self-consistently with the modified gap equation:

$$\mu^2 = -2\omega^2 - \frac{\lambda}{2\mu}.$$

We note that the position of the center of our Gaussian variational wave function shift slightly toward the origin from the position of the minimum of the potential.

### 3 Scalar Field Theory: $\varphi^4$ model

Having been warmed up by a much simpler problem in quantum mechanics, it is now our task to transcribe the result to a problem in quantum field theories. We consider first a prototype scalar field theory with $\varphi^4$ self-interaction.

The Hamiltonian density of the theory is given by

$$H(x) = \frac{1}{2} \pi^2(x) + \frac{1}{2} (\nabla \varphi(x))^2 + \frac{m_0^2}{2} \varphi^2(x) + \frac{\lambda}{24} \varphi^4(x),$$

(8)
Table 1: Correspondence between quantum mechanics and scalar field theory

| quantum mechanics | scalar field theory |
|-------------------|---------------------|
| $x$               | $\varphi(x)$        |
| $\omega^2$       | $m_0^2 - \Delta$ (or $m_0^2 + k^2$) |
| $\frac{1}{4}x^4$ | $\frac{\lambda}{4}\varphi^4(x)$ |
| $x_0(t), p_0(t)$  | $\bar{\varphi}(x, t), \bar{\pi}(x, t)$ |
| $\mu(t), \sigma(t)$ | $\frac{1}{2}G^{-1}(x, y, t), 2\Sigma(x, y, t)$ |
| $\mu \xi$        | $G^{-1/2} \xi G^{-1/2}$ |

where $\pi(x)$ is an operator conjugate to the field $\varphi(x)$ and is expressed by a functional derivative $-i\delta/\delta \varphi(x)$. \(^1\)

We write a Gaussian time-dependent variational wave functional for this Hamiltonian formally as

$$\Psi[\varphi(x)] = \mathcal{N} \exp \left( i\langle \bar{\pi} | \varphi - \bar{\varphi} \rangle - \langle \varphi - \bar{\varphi} | \frac{1}{4G} + i\Sigma | \varphi - \bar{\varphi} \rangle \right),$$

(9)

where $G$, $\Sigma$, $\bar{\varphi}$, $\bar{\pi}$ define respectively the real and imaginary part of the kernel of the Gaussian width and its average position and momentum. We have used the short hand notation $\langle \bar{\pi} | \varphi \rangle = \int \bar{\pi}(x, t)\varphi(x)dx$. Although it looks a little horrible, it is just a straightforward generalization of (3). The correspondences between the previous quantum mechanical example and the present case are summarized in the Table 1.

The equations of motion are found to be

$$\dot{\bar{\varphi}} = -\bar{\pi},$$

$$\dot{\bar{\pi}} = \left( -\Delta + m_0^2 + \frac{1}{6}\bar{\varphi}^2(x) + \frac{\lambda}{2}G(x, x) \right) \bar{\varphi};$$

$$\dot{G} = 2(G\Sigma + \Sigma G),$$

$$\dot{\Sigma} = \frac{1}{8}G^{-2} - 2\Sigma^2 - \frac{1}{2} \left( -\Delta + m_0^2 + \frac{1}{2}\bar{\varphi}^2 + \frac{1}{2}G(x, x) \right).$$

(10)

where it is understood that $\bar{\varphi}$ and $\bar{\pi}$ denote vectors with components $\bar{\varphi}(x)$ and $\bar{\pi}(x)$, $G$ and $\Sigma$ denote matrices with matrix elements $G(x, y)$ and $\Sigma(x, y)$, and the matrix product $G\Sigma$ is given by $\int d\mathbf{z}G(x, \mathbf{z})\Sigma(\mathbf{z}, y)$.

For the vacuum, we should have time-independent solution so that $\bar{\pi} = \Sigma = 0$, $\bar{\varphi}(x) = \varphi_0$, and

$$G(x, y, t) = G_0(x - y) = \int d\mathbf{k} \frac{e^{i(x - y) \cdot \mathbf{k}}}{2\sqrt{\mu^2 + k^2}}.$$  

(11)

\(^1\)A care must be taken to give precise mathematical meanings for these expressions[3], but we do not go into such problems here.
where the effective mass $\mu$ is determined self-consistently by the non-linear integral equation

$$\mu^2 = m_0^2 + \frac{\lambda}{2} G_0(0) + \frac{\lambda}{2} \varphi_0^2. \quad (12)$$

which is usually called the gap equation because it determines the mass gap self-consistently. In the symmetric phase the expectation value of the field $\varphi_0$ vanishes while in the symmetry broken phase it must be such that

$$m_0^2 + \frac{\lambda}{2} G_0(0) + \frac{\lambda}{6} \varphi_0^2 = 0. \quad (13)$$

This last equation implies that $\mu^2 = \lambda \varphi_0^2 / 3$. The equations of motion (14) determine the time-evolution of the variational wave functional (9).

We comment on two well-known problems which are absent in quantum mechanics but are characteristic in quantum field theory: divergences and covariance. In the above expression the momentum integral in $G_0(0)$ is quadratically divergent. As well-known, this divergence originates from couplings of infinite degrees of freedom (all momentum modes) in the interaction of local fields. In the renormalizable field theories, these divergences (or cut-off dependence) may be absorbed into an appropriate redefinition of finite number of physical parameters, such as the mass and the coupling constant. This renormalization procedure works at least order by order in the power series expansion in terms of the coupling constant. In our non-perturbative calculation scheme with the variational method, we may absorb the cut-off dependence into the mass and the coupling constant.[10, 11] We encounter, however, well-known triviality problem: renormalized coupling constant $\lambda_R$ becomes zero as one sends the cut-off $\Lambda$ to infinity keeping the original coupling $\lambda$ positive for the stability of the ground state. Despite this well-known pathology, the model still makes sense as an effective theory with the finite cut-off.

The another problem is the lack of manifest covariance in our formulation: the time coordinate $t$ have been treated differently from spatial coordinates throughout calculations. This is an old problem which was originally solved in QED by Tomonaga, Schwinger, Feynman and Dyson who developed the covariant perturbation theory based on the interaction representation. Vautherin invented a new ingenious trick to derive manifest covariant form of the equations of motion which I shall now describe. [12, 14]

Mean field equations in the Hartree-Bogoliubov form:

A key ingredient of his method is the reduced density matrix defined by

$$\mathcal{M}(x,y;t) = \begin{pmatrix} i\langle \hat{\phi}(x)\hat{\pi}(y) \rangle - 1/2 & \langle \hat{\phi}(x)\hat{\phi}(y) \rangle \\ \langle \hat{\pi}(x)\hat{\pi}(y) \rangle & -i\langle \hat{\pi}(x)\hat{\phi}(y) \rangle - 1/2 \end{pmatrix}, \quad (14)$$

where $\hat{\phi} = \varphi - \bar{\varphi}$, $\hat{\pi} = \pi - \bar{\pi}$, and expectation values are calculated with the Gaussian functional $\Psi(t)$ and is given by

$$\mathcal{M} = \begin{pmatrix} -2iG\Sigma & G \\ \frac{1}{4}G^{-1} + 4\Sigma G\Sigma & 2i\Sigma G \end{pmatrix}. \quad (15)$$
Using (10), one can show that the equation of motion of the reduced density matrix can be cast into the Liouville-von Neumann form
\[
i\dot{\mathcal{M}} = [\mathcal{H}, \mathcal{M}],
\]
where the generalized Hamiltonian $\mathcal{H}$ is given by
\[
\mathcal{H} = \begin{pmatrix} 0 & 1 \\ \Gamma & 0 \end{pmatrix}.
\]

with
\[
\Gamma = -\Delta + m_0^2 + \frac{\lambda}{2} \bar{\rho}^2 + \frac{\lambda}{2} G(x, x).
\]

This form of equations is known in many-body theory as the time-dependent Hartree-Bogoliubov equations.

The reduced density matrix satisfies $\mathcal{M}^2 = \frac{1}{4} I$ so that it has two eigenvalues of $\pm 1/2$. We write the n-th eigenvector of $\mathcal{M}$ with eigenvalue $1/2$ as $(u_n(x), v_n(x))$:
\[
\mathcal{M} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u_n \\ v_n \end{pmatrix},
\]
then one can show that vectors $(u_n^*(x), -v_n^*(x))$ give eigenvectors for eigenvalue $-1/2$. The $u$ and $v$ components of eigenvectors are called mode functions. With these eigenvectors the reduced density matrix has a spectral decomposition as
\[
\mathcal{M} = \frac{1}{2} \sum_{n>0} \left[ \begin{pmatrix} u_n^* \\ v_n^* \end{pmatrix} (v_n^*, u_n) + \begin{pmatrix} u_n^* \\ -v_n^* \end{pmatrix} (-v_n, u_n) \right].
\]

The Liouville-von Neumann equation (16) can be rewritten in terms of the mode functions as
\[
i\partial_t \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Gamma & 0 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix}.
\]
Eliminating $v_n$ and inserting (18) we obtain a modified Klein-Gordon type equation for the mode functions
\[
\left( \Box + m_0^2 + \frac{\lambda}{2} \bar{\rho}^2 + \frac{\lambda}{2} G(x, x) \right) u_n = 0,
\]
where the spectral representation (20) implies
\[
G(x, x) = \langle x|G(t)|x \rangle = \frac{1}{2} \sum_n |u_n(x, t)|^2.
\]

To write the equation of motion fully covariant way, we introduce a Feynman propagator in terms of the mode functions
\[
\langle x|S|y \rangle = \theta(x_0 - y_0) \sum_{n>0} u_n^*(x, x_0) u_n(y, y_0)
\] 
\[
+ \theta(y_0 - x_0) \sum_{n<0} u_n^*(x, x_0) u_n(y, y_0).
\]

so that
\[ \langle x|G(x_0)|x \rangle = \langle x|S|x \rangle \quad (24) \]

Then we finally arrive at a fully covariant, self-consistent equations of motion for the mode functions\[^{12, 14}\] which reproduces the same equations obtained earlier by the functional integral method\[^{15}\].

This way of writing the equations of motion also paves a way to generalize the calculation at finite temperatures.

### 4 Statistical Ensembles

Foregoing discussions are limited to evolution of a single coherent (Gaussian) state. In realistic physical situations, we are more interested in the evolution of the statistical ensemble which is described by the density matrix:

\[ \hat{\rho}(t) = \sum_n |\Psi_n(t)\rangle p_n(t)\langle \Psi_n(t)| \quad (25) \]

where \( p_n(t) \) is a probability distribution specifying the ensemble hence it should satisfy \( \sum_n p_n(t) = 1 \). The expectation value of an observable \( O \) is given by \( \langle O \rangle = \text{Tr} \hat{\rho} O \) and the statistical entropy \( S \) is given by

\[ S = -\text{Tr} \hat{\rho} \ln \hat{\rho} = -\sum_n p_n \ln p_n \quad (26) \]

Hence if \( p_n \) is time-independent then the entropy is conserved. In equilibrium at temperature \( T = 1/\beta \), the density matrix is given by the canonical ensemble: \( \rho^{eq}_n(t) = e^{-\beta E_n}/Z \) where \( Z = \sum_n e^{-\beta E_n} \) which maximizes the entropy \( S(t) \) under the condition of fixed expectation value of the energy \( E = \langle H \rangle \). The density matrix with time-independent \( p_n \) obeys the Liouville-von Neumann equation: \( i\partial_t \hat{\rho} = [H, \hat{\rho}] \).

The variational method has been extended for the time-dependent density matrix by Eboli, Jackiw and Pi.\[^{8}\] Without going into detail, we illustrate the essence of their method in terms of simple quantum mechanical example with harmonic oscillator Hamiltonian\[^{1}\]. We first introduce the coordinate representation of the density matrix by

\[ \rho(x, y; t) = \langle x|\hat{\rho}(t)|y \rangle = \sum_n |\Psi_n(x, t)\rangle p_n(t)\langle \Psi_n^*(y, t)| \quad (27) \]

and observe that with single Gaussian variational wave function the density matrix is just a product of two Gaussian: \( \rho(x, y; t) \sim e^{-\omega(x^2+y^2)/2} \). One can show that in the other extreme limit of thermal equilibrium, the density matrix \( \rho(x, y; t) \) can also be expressed by a mixed Gaussian form. This is so because the equilibrium density matrix \( \hat{\rho}_{eq}(\beta) = e^{-\beta H} \) obeys the imaginary time Schrödinger equation (the Bloch equation): \( -\partial_{\tau} \hat{\rho}_{eq}(\tau) = H \hat{\rho}_{eq} \). Using \( \hat{\rho}_{eq}(x, y) = \langle x|\hat{U}(-i\tau)|y \rangle \) and well-known path-integral expression of the matrix elements of the unitary evolution operator \( \hat{U}(t) = e^{-itH} \), one
finds
\[ \rho_{eq.}(x, y) = \left( \frac{\omega}{2\pi \sinh \omega \beta} \right)^{1/2} \exp \left[ -\frac{\omega}{2 \sinh \omega \beta} \left\{ (x^2 + y^2) \cosh \omega \beta - 2xy \right\} \right] \] (28)
which is again Gaussian with an extra term containing a product of two coordinates \( xy \).

For more general mixed states we introduce a generalized Gaussian density matrix
\[ \rho(x, y; t) = N \exp \left[ ip_0(\hat{x} - \hat{y}) - \frac{\mu}{2} (\hat{x}^2 + \hat{y}^2 - 2\xi \hat{x} \hat{y}) + i \frac{\sigma}{2} (\hat{x}^2 - \hat{y}^2) \right] \] (29)
where we have used the time-dependent shifted coordinates: \( \hat{x} = x - x_0 \) and \( \hat{y} = y - x_0 \).

Equations of motion of the time-dependent parameters of the generalized Gaussian density matrix (29) can be derived from the Liouville equation of the density matrix and we obtain a set of equation similar to (30) with a small modification:
\[ \dot{\mu} = 2 \sigma \mu, \quad \dot{\sigma} = \sigma^2 + \omega^2 - (1 - \xi^2) \mu^2. \] (30)

The parameter \( \xi \) is called the mixing parameter which measures the degree of mixture of different pure states in the ensemble; it remains constant for an adiabatic evolution of the system. In equilibrium, \( p_0 = x_0 = \sigma = 0 \) and other two parameters are given by the specific functions of temperature:
\[ \mu_{eq.} = \omega \coth \omega \beta, \quad \xi_{eq.} = \cosh^{-1} \omega \beta \] (31)
as indicated by the formula (28). Extension of the Gaussian density matrix in quantum mechanics to that in quantum field theories is straightforward as indicated in the last row of the Table 1.

The reduced density matrices we have introduced in the previous section for a pure Gaussian state can be extended for a mixed state immediately by the replacements:
\[ \langle \hat{\phi}(x) \hat{\pi}(y) \rangle = \text{Tr}(\hat{\rho}(t) \hat{\phi}(x) \hat{\pi}(y)), \langle \hat{\phi}(x) \hat{\phi}(y) \rangle = \text{Tr}(\hat{\rho}(t) \hat{\phi}(x) \hat{\phi}(y)), \text{etc.} \] (32)
and one can derive equations of motion of the reduced density matrix similar to the pure state case. In the case of equilibrium distribution, this amounts to introduce a factor containing the occupation number in the sum over the mode functions.

5 Rotating chiral condensate in the \( O(N) \) sigma model

We briefly mention about an application of the above method to the \( O(N) \) sigma model which is composed of \( N \)-components coupled scalar fields \( \varphi_n \) with continuous \( O(N) \) symmetry. We expect that this global symmetry of the model is broken spontaneously at low temperatures, characterized by non vanishing expectation value of one component of fields, say \( \varphi_0 = \text{Tr} \hat{\rho} \varphi \neq 0 \), and the system exhibits an order-disorder phase
transition to a state with \( \bar{\varphi}_0 = 0 \). This is what is expected in QCD where the chiral symmetry is broken in vacuum and is expected to be restored at finite temperature. This chiral phase transition has been studied by an effective theory with pion and sigma fields with \( O(4) \) global symmetry.

We have applied our method to describe a special kind of time-dependent condensate which rotates in a subspace of the internal symmetry space:

\[
\begin{pmatrix}
\bar{\varphi}_1(x) \\
\bar{\varphi}_2(x)
\end{pmatrix} = \begin{pmatrix}
\cos(q \cdot x) & \sin(q \cdot x) \\
-\sin(q \cdot x) & \cos(q \cdot x)
\end{pmatrix} \begin{pmatrix}
\varphi_0 \\
0
\end{pmatrix} = \exp[i(q \cdot x)\tau_2] \begin{pmatrix}
\varphi_0 \\
0
\end{pmatrix}
\]

(33)

where \( \tau_2 = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix} \) and \( q_\mu = (\omega, q) \) is a four vector specifying the direction of rotation: for a spatially uniform rotation in time it is a pure time-like vector; while a static condensate with oscillation in space, it is a pure space-like vector. The matrix \( U(x) = \exp[i(q \cdot x)\tau_2] \) can be considered as a ”gauge transformation” to the local rest frame of the rotating condensate. In the rotating frame, the four derivative which appear in the equation of motion of the mode functions is ”gauge transformed” to \( U\partial_\mu U^\dagger = \partial_\mu - iq_\mu \tau_2 \) and the effect of the rotation is seen in this frame as an appearance of the apparent ”centrifugal force”. Indeed, our result of phase diagram for such dynamical condensates shows that the amplitude of the chiral condensate with uniform time-like rotation increases due to the centrifugal force. This effect has been observed in the classical solutions of Anselm and Ryskin [3]; our quantum generalization of their solution shows that this effect is amplified by the coupling of quantum fluctuations to rotations, which the static condensate with spatial oscillations are more suppressed by the quantum fluctuations.[13] A phase diagram of rotating condensate was obtained in [14] and the damping of the rotating condensate due to the symmetry breaking perturbation was computed by the method of the response function in [16].

### 6 Outlook

Vautherin started to work on the variational approach to quantum field theories many years ago with Arthur Kerman. They developed many important ideas in their unpublished works and tried to solve QCD non-perturbatively with their method with a hope to gain new insights in the quark confinement problem.[10] The Gaussian Ansatz for the variational wave functional however has difficulty of breaking the local gauge invariance and the projection to color singlet state destroys a nice feature of the Gaussian wave functional.[17] Vautherin continued to work on the problem with his students and brought a new insight into the problem again introducing a technique developed in nuclear many-body theory in his last paper.[18] His efforts in this direction may be carried over to study the dynamical evolution of the quark-gluon plasma in ultra-relativistic nucleus-nucleus collisions. Vautherin was also interested in the recent experimental breakthrough of creating weakly interacting Bose-Einstein condensates in well-controlled laboratory environments. Our method can be also applied to such
problem to investigate the effect of quantum fluctuations which are usually ignored in theoretical descriptions. [19]

I am much indebted to Yasuhiko Tsue as well as to Dominique Vautherin for our works quoted above. I thank them for sharing the joy of physics in our collaboration.

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