The Equivariant Chow Ring of SO(4)
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0. Introduction

Let $G$ be a reductive algebraic group. The algebraic analogue of $EG$ is attained by approximation. Let $V$ be a $\mathbb{C}$-vector space. Let $G \times V \to V$ be an algebraic representation of $G$. Let $W \subset V$ be a $G$-invariant open set satisfying:

(i) The complement of $W$ in $V$ is of codimension greater than $q$.
(ii) $G$ acts freely on $W$.
(iii) There exists a geometric quotient $W \to W/G$.

$W$ is an approximation of $EG$ up to codimension $q$. Let $e = \dim(W/G)$. The equivariant Chow groups of $G$ (acting on a point) are defined by:

$$A^G_{-j}(\text{point}) = A^e_{-j}(W/G)$$

for $0 \leq j \leq q$. An argument is required to check the Chow groups are well-defined (see [EG1]). The basic properties of equivariant Chow groups are established in [EG1]. In particular, there is a natural intersection ring structure on $A^G_i(\text{point})$. For notational convenience, a superscript will denote the Chow group codimension:

$$A^G_{-j}(\text{point}) = A^j_G(\text{point}).$$

Equation (1) becomes:

$$\forall \ 0 \leq j \leq q, \ A^j_G(\text{point}) = A^j(W/G).$$

$W/G$ is an approximation of $BG$. $A^*_G(\text{point})$ is called the (equivariant) Chow ring of $G$. $A^*_G(\text{point})$ is naturally isomorphic to the ring of algebraic characteristic classes of (étale locally trivial) principal $G$-bundles. The equivariant Chow ring of $G$ was first defined by B. Totaro in [T].

Consider now the orthogonal and special orthogonal algebraic groups (over $\mathbb{C}$). The Chow ring of $O(n)$ is generated by the Chern classes of the standard representation. The odd classes are 2-torsion:

$$A^*_O(n)(\text{point}) = \mathbb{Z}[c_1, \ldots , c_n]/(2c_1, 2c_3, 2c_5, \ldots ).$$

The Chow ring of $SO(n = 2k + 1)$ is also generated by the Chern classes of the standard representation. The odd classes are 2-torsion and $c_1 = 0$:

$$A^*_SO(n=2k+1)(\text{point}) = \mathbb{Z}[c_1, \ldots , c_n]/(c_1, 2c_3, 2c_5, \ldots ).$$

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$A^*_O(n)$ (point) was first computed by B. Totaro. Algebraic computations of $A^*_O(n)$ (point) and $A^*_{SO(2k+1)}$ (point) can be found in [P2]. The Chow ring of $SO(n)$ has been computed with $\mathbb{Q}$-coefficients in [EG2]. The integral Chow ring of $SO(n = 2k)$ is not known in general.

Since $BO(n)$ is approximated by the set of non-degenerate quadratic forms in $\text{Sym}^2 S^*$ (where $S \to G(n, \infty)$ is the tautological sub-bundle), the Chow ring can be analyzed by degeneracy calculations ([P2]). Algebraic $BSO(n)$ double covers $BO(n)$. If $n = 2k + 1$, there is a product decomposition $O(n) \cong \mathbb{Z}/2\mathbb{Z} \times SO(n)$. As a result, the double cover geometry for $n = 2k + 1$ is tractable and a computation of $A^*_{SO(2k+1)}$ (point) can be made. In case $n = 2k$, the double cover geometry is more complicated.

Since $SO(2) \cong \mathbb{C}^*$, the first non-trivial even case is $SO(4)$. In this paper, the ring $A^*_{SO(4)}$ (point) is determined. Let $c_1, c_2, c_3, c_4$ be Chern classes of the standard representation of $SO(4)$. Let $F$ be one of the two distinct irreducible 3-dimensional representations of $SO(4)$, and let $f_2$ be the second Chern class $F$.

**Theorem 1.** $A^*_{SO(4)}$ (point) is generated by the Chern classes $c_1, c_2, c_3, c_4,$ and $f_2$. Define $x \in A^2_{SO(4)}$ by $x = c_2 - f_2$.

$$A^*_{SO(4)}(\text{point}) = \mathbb{Z}[c_1, c_2, c_3, c_4, x]/(c_1, 2c_3, xc_3, x^2 - 4c_4)$$

Let $\tilde{F}$ be the other irreducible 3-dimensional representation of $SO(4)$. Let $\tilde{f}_2$ be the second Chern class of $\tilde{F}$. Since (see [FH])

$$F \oplus \tilde{F} \cong \wedge^2 V,$$

the relation $\tilde{f}_2 = 2c_2 - f_2$ is obtained. Hence, $c_2 - \tilde{f}_2 = -x$. The presentation in Theorem 1 does not depend upon the choice of 3-dimensional representation.

Thanks are due to D. Edidin, W. Fulton, W. Graham, and B. Totaro for conversations about $BSO(n)$. The $SO(4)$ calculation presented here is similar in spirit to the $SO(2k + 1)$ calculations of [P2]. In [P1] and [P2], equivariant Chow rings are used to compute ordinary Chow rings of certain moduli spaces of maps and Hilbert schemes of rational curves.

1. Ruled Quadric Surfaces

Let $V \cong \mathbb{C}^4$ be equipped with a non-degenerate quadratic form $Q$. A ruled quadric surface in $\mathbb{P}(V)$ is a pair $(X, r)$ where $X \subset \mathbb{P}(V)$ is a nonsingular quadric surface and $r$ is a choice of ruling. Let $\mathcal{X} \subset$
\( P(\text{Sym}^2 V^*) \) be the parameter space of nonsingular quadrics. The parameter space of ruled quadrics, \( \mathcal{X}_{\text{ruled}} \), is an étale double cover of \( \mathcal{X} \) via the natural map: \( \mathcal{X}_{\text{ruled}} \rightarrow \mathcal{X} \).

There are natural maps \( \text{SO}(V) \rightarrow \text{PSO}(V) \subset \text{PGL}(V) \) and \( \text{O}(V) \rightarrow \text{PO}(V) \subset \text{PGL}(V) \). Let \( \text{SO}(V) \) and \( \text{O}(V) \) act on \( \text{PGL}(V) \) on the right via these maps. There exist geometric quotients (see [P2]):

\[
\text{PGL}(V)/\text{SO}(V) \rightarrow \text{PGL}(V)/\text{O}(V).
\]

Consider the quadric surface \( (Q) \subset \text{P}(V) \) obtained from the quadratic form. The standard left action \( \text{PGL}(V) \times \text{P}(V) \rightarrow \text{P}(V) \) yields a transitive \( \text{PGL}(V) \)-action on the space of nonsingular quadric surfaces. The stabilizer of \( (Q) \) for this action is exactly \( \text{PO}(V) \subset \text{PGL}(V) \).

Hence, there is a canonical isomorphism

\[
\text{PGL}(V)/\text{O}(V) \cong \mathcal{X}.
\]

For the entire paper, fix a ruling \( r \) of \( (Q) \). Since \( \text{PGL}(V) \) acts transitively on the space of ruled quadrics and the stabilizer of \( ((Q), r) \) is exactly \( \text{PSO}(V) \subset \text{PGL}(V) \), there is a canonical isomorphism determined by \( ((Q), r) \):

\[
\text{PGL}(V)/\text{SO}(V) \cong \mathcal{X}_{\text{ruled}}.
\]

There is a canonical Plücker embedding \( G(2, V) \hookrightarrow \text{P}(\wedge^2 V) \). Let \( Z \subset G(3, \wedge^2 V) \) be the open locus of 2-planes in \( \text{P}(\wedge^2 V) \) which intersect \( G(2, V) \) transversely in a nonsingular conic curve.

**Lemma 1.** There is a canonical isomorphism \( Z \cong \mathcal{X}_{\text{ruled}} \).

**Proof.** The family of lines determined by a nonsingular plane conic \( C \subset G(2, V) \subset \text{P}(\wedge^2 V) \) sweeps out an irreducible degree 2 surface in \( \text{P}(V) \). There are three possibilities for this degree 2 surface: a double plane, a quadric cone, or a nonsingular quadric surface. If the conic \( C \) sweeps out a a double plane \( H \subset \text{P}(V) \), then \( C \subset P \subset G(2, V) \) where \( P \) is the plane of all lines contained in \( H \). If \( C \) sweeps out a quadric cone, then \( C \subset P \subset G(2, V) \) where \( P \) is the plane of all lines passing through the vertex of the cone. Hence, if \( C \) is the transverse intersection \( P \cap G(2, V) \) of a plane, then \( C \) must correspond to a ruling of a unique nonsingular quadric surface. Conversely, a ruling of a nonsingular quadric surface yields a conic curve in \( G(2, V) \) which is the transverse intersection of a unique 2-plane in \( \text{P}(V) \). These maps are easily seen to be algebraic. \qed

By Lemma 1, the ruled quadric \( ((Q), r) \) corresponds to a 3-dimensional subspace \( F \subset \wedge^2 V \). \( F \) is \( \text{SO}(V) \)-invariant since \( ((Q), r) \) is stabilized by \( \text{SO}(V) \). \( F \) is therefore a 3-dimensional representation of \( \text{SO}(V) \). Let
s be the other ruling of \((Q)\). \(((Q), s)\) similarly corresponds to an invariant 3-dimensional subspace \(\tilde{F} \subset \wedge^2 V\). The \(SO(V)\) representation \(\wedge^2 V\) decomposes as \(\wedge^2 V \cong F \oplus \tilde{F}\).

2. \(BSO(V)\)

Let \(V \cong \mathbb{C}^4\) be equipped with a non-degenerate quadratic form as before. Approximations to \(E SO(V)\) and \(BSO(V)\) are obtained via direct sums of the representation \(V^*\). Let \(m > > 0\) and let

\[
W_m \subset \bigoplus_1^m V^*
\]
denote the spanning locus. \(W_m\) is the locus of \(m\)-tuples of vectors of \(V^*\) which span \(V^*\). The natural action of \(SO(V)\) on \(W_m\) is free and has a geometric quotient (see \([P2]\)). The codimension of the complement of \(W_m\) in \(\bigoplus_1^m V^*\) is \(m - 3\). \(W_m\) is an approximation of \(E SO(V)\) up to codimension \(m - 4\). Therefore

\[
BSO(V) = \lim_{m \to \infty} W_m/\text{SO}(V),
\]

\[
A^*_\text{SO}(V)(\text{point}) = \lim_{m \to \infty} A^*(W_m/\text{SO}(V)).
\]

There is a scalar \(\mathbb{C}^*\)-action on \(W_m\). Let \(P(W_m) = W_m/\mathbb{C}^*\). Since this \(\mathbb{C}^*\)-action commutes with the \(SO(V)\)-action, there is diagram of quotients:

\[
\begin{array}{ccc}
W_m & \xrightarrow{\tau_1} & W_m/\text{SO}(V) \\
i_1 \downarrow & & \downarrow i_2 \\
P(W_m) = W_m/\mathbb{C}^* & \xrightarrow{\tau_2} & P(W_m)/\text{SO}(V)
\end{array}
\]

All the maps in (2) are quotient maps:

(i) \(i_1\) is a free \(\mathbb{C}^*\)-quotient.
(ii) \(i_2\) is a free \(\mathbb{C}^*/(\pm)\)-quotient.
(iii) \(\tau_1\) is a free \(SO(V)\)-quotient.
(iv) \(\tau_2\) is a free \(PSO(V)\)-quotient.

The existence of these quotients is easily deduced (see \([P2]\)).

First consider the space \(P(W_m)/\text{SO}(V)\). Let \(Q\) be the quadratic form on \(V\) and let \(r\) be the ruling of the quadric surface \((Q) \subset P(V)\) fixed in section \([P]\). An element \(f \in P(W_m)\) yields a canonical embedding

\[\mu_f : P(V) \hookrightarrow P(\mathbb{C}^m).\]

The image under \(\mu_f\) of \(((Q), r)\) is a ruled quadric surface in \(P(\mathbb{C}^m)\) associated canonically to \(f \in P(W_m)\). Since \(PSO(V) \subset PGL(V)\) is exactly the stabilizer of the ruled quadric \(((Q), r)\), it follows that \(P(W_m)/\text{SO}(V)\) is isomorphic to the parameter space of ruled quadric
surfaces in \( \mathbb{P}(\mathbb{C}^m) \). Since a ruled quadric surface in \( \mathbb{P}(\mathbb{C}^m) \) spans a unique 3-plane in \( \mathbb{P}(\mathbb{C}^m) \), the parameter space is fibered over \( G(4, m) \). By Lemma 3, the parameter space of ruled quadric surface in \( \mathbb{P}(\mathbb{C}^m) \) is canonically isomorphic to an open set
\[
Z \subset G(3, \wedge^2 S)
\]
where \( S \to G(4, m) \) is the tautological sub-bundle.

The Chow computations in section 2 will require two results about line bundles. We have seen \( \mathbb{P}(W_m)/SO(V) \) is canonically fibered over \( G(4, m) \). Let \( c_1 \) be the first Chern class of the tautological bundle \( S \) on \( G(4, m) \). Let \( c_1 \) also denote the pull-back of this class to \( \mathbb{P}(W_m)/SO(V) \). For \( m > 4 \), \( A^1(\mathbb{P}(W_m)) \cong \mathbb{Z} \) with generator \( c_1(\mathcal{O}_P(-1)) \) (which is the Chern class of the line bundle associated to the \( \mathbb{C}^* \)-bundle \( i_1 \).

**Lemma 2.** \( \tau_2^*(c_1) = c_1(\mathcal{O}_{\mathbb{P}}(-4)) \).

**Proof.** Elements of \( \mathbb{P}(W_m) \) correspond to embeddings of \( \mathbb{P}(V) \) in \( \mathbb{P}(\mathbb{C}^m) \). The class \( \tau_2^*(-c_1) \) is the divisor class of embeddings that meet a fixed \((m - 5)\)-plane in \( \mathbb{P}(\mathbb{C}^m) = \mathbb{P}^{m-1} \). This divisor class is determined by a \( 4 \times 4 \) determinant. Hence, \( \tau_2^*(-c_1) = c_1(\mathcal{O}_{\mathbb{P}}(4)) \). \( \square \)

The map \( i_2 : W_m/SO(V) \to \mathbb{P}(W_m)/SO(V) \) is a \( \mathbb{C}^*/(\pm) \)-bundle. Since there is an abstract isomorphism \( \mathbb{C}^*/(\pm) \cong \mathbb{C}^* \), \( i_2 \) is also a \( \mathbb{C}^* \)-bundle. Let \( N \) be the line bundle on \( \mathbb{P}(W_m)/SO(V) \) canonically associated to \( i_2 \).

**Lemma 3.** \( \tau_2^*(N) \cong \mathcal{O}_{\mathbb{P}}(-2) \).

**Proof.** Let \( i_1/(\pm) : W_m/(\pm) \to \mathbb{P}(W_m) \). The map \( i_1/(\pm) \) is a free \( \mathbb{C}^*/(\pm) \)-quotient. The line bundle associated to the \( \mathbb{C}^*/(\pm) \)-bundle \( i_1/(\pm) \) is \( \mathcal{O}_{\mathbb{P}}(-2) \). The map \( \tau_1/(\pm) : W_m/(\pm) \to W_m/SO(V) \) is \( \mathbb{C}^*/(\pm) \)-equivariant. Hence, \( \tau_2^*(N) \cong \mathcal{O}_{\mathbb{P}}(-2) \). \( \square \)

### 3. Chow Calculations

In this section, the Chow ring of \( W_m/SO(V) \) is determined (up to codimension \( m - 4 \)). Consider the parameter space of ruled quadrics in \( \mathbb{P}(\mathbb{C}^m) \):
\[
Z \subset G(3, \wedge^2 S) \to G(4, m).
\]
Let \( D \) be the complement of \( Z \) in \( G(3, \wedge^2 S) \). Following the notation of section 2, \( W_m/SO(V) \) is the \( \mathbb{C}^* \)-bundle associated to a line bundle \( N \to Z \). Therefore,
\[
A^*(W_m/SO(V)) \cong A^*(Z)/(c_1(N)) \cong A^*(G(3, \wedge^2 S))/(I_D, c_1(N))
\]
where $I_D \subset A^*(\mathbf{G}(3, \wedge^2 S))$ is the ideal generated by cycles supported on $D$ and $\mathbf{N}$ is any extension of $N$ to $\mathbf{G}(3, \wedge^2 S)$. The ideal $I_D$ is determined by constructing a well-behaved variety which surjects onto $D$.

Let $\mathbf{G}(2, S) \hookrightarrow \mathbf{P}(\wedge^2 S)$ be the canonical relative Plücker embedding. $D$ is exactly the locus of 2-planes in the fibers of $\mathbf{P}(\wedge^2 S)$ which do not meet $\mathbf{G}(2, S)$ transversely in a nonsingular conic curve. Equivalently, $D$ is the locus of 2-planes $P$ in the fibers of $\mathbf{P}(\wedge^2 S)$ which satisfy one of the following conditions:

(i) $P \cap \mathbf{G}(2, S)$ is a pair of distinct lines in $P$.
(ii) $P \cap \mathbf{G}(2, S)$ is a double line in $P$.
(iii) $P \cap \mathbf{G}(2, S) = P$.

$D$ is dominated by a canonical Grassmannian bundle over $\mathbf{G}(2, S)$. Let $B \to \mathbf{G}(2, S)$ be the tautological sub-bundle. By wedging, there is canonical surjective bundle map on $\mathbf{G}(2, S)$:

$$\wedge^2 S \otimes \wedge^2 B \to \wedge^4 S$$

which induces a canonical sequence on $\mathbf{G}(2, S)$:

(3) \[ 0 \to K \to \wedge^2 S \to \wedge^4 S \otimes (\wedge^2 B)^* \to 0. \]

There is a canonical inclusion $\wedge^2 B \subset K$ and a quotient sequence

(4) \[ 0 \to \wedge^2 B \to K \to E \to 0 \]

on $\mathbf{G}(2, S)$. The geometric interpretation of these sequences is as follows. Let $\xi \in \mathbf{G}(2, S)$. $\mathbf{P}(K_\xi) \subset \mathbf{P}(\wedge^2 S_\xi)$ is the projective tangent space to $\mathbf{G}(2, S_\xi)$ at $\xi$. $\mathbf{P}(\wedge^2 B_\xi)$ in $\mathbf{P}(\wedge^2 S_\xi)$ is the Plücker image of the point $\xi$. The fiber of the Grassmannian bundle $\mathbf{G}(2, E) \to \mathbf{G}(2, S)$ over $\xi$ corresponds to the 2-planes $P$ of $\mathbf{P}(\wedge^2 S_\xi)$ that are tangent to $\mathbf{G}(2, S)$ at $\xi$. There is a canonical map

$$\rho : \mathbf{G}(2, E) \to D$$

which is a surjection of algebraic varieties. Let $[P] \in D$. The fiber of $\rho$ over $[P]$ is simply the set of points of $P \cap \mathbf{G}(2, S)$ where $P$ is tangent to $\mathbf{G}(2, S)$. In case (i) above, the fiber is a point. In case (ii), the fiber is a straight line in $\mathbf{P}(\wedge^2 S)$. In case (iii), the fiber is 2-plane in $\mathbf{P}(\wedge^2 S)$. Hence, there is stratification of $D$ by intersection type (i-iii) where $\rho$ is a projective bundle over each stratum. The Chow groups of $\mathbf{G}(2, E)$ therefore surject upon the Chow groups of $D$. 
The ideal \( I_D \) is determined by calculating the push-forwards of the Chow classes of \( \mathbb{G}(2, E) \) to \( \mathbb{G}(3, \wedge^2 S) \). Consider the projection
\[
\pi : \mathcal{G} = \mathbb{G}(3, \wedge^2 S) \times_{\mathbb{G}(4, m)} \mathbb{G}(2, S) \to \mathbb{G}(3, \wedge^2 S). 
\]
The sequences (3) and (4) pull-back to \( \mathcal{G} \). Let \( F \to \mathbb{G}(3, \wedge^2 S) \) denote the tautological sub-bundle (and also let \( F \) denote the pull-back to \( \mathcal{G} \) of this bundle). There is a canonical inclusion
\[
\iota : \mathbb{G}(2, E) \hookrightarrow \mathcal{G}
\]
determined by the sequences (3) and (4). \( \mathbb{G}(2, E) \subset \mathcal{G} \) is the closed subvariety of points \( g \in \mathbb{G} \) where
\[
\wedge^2 B_g \subset F_g \subset K_g.
\]
The class \([\mathbb{G}(2, E)]\) in Chow ring of \( A^* (\mathcal{G}) \) is easily found by degeneracy calculations. Let \( c_1, c_2, c_3, c_4 \) be the Chern classes of \( S \to \mathbb{G}(4, m) \). Let \( b_1, b_2 \) be the Chern classes of \( B \to \mathbb{G}(2, S) \). Let \( f_1, f_2, f_3 \) be the Chern classes of \( F \to \mathbb{G}(3, \wedge^2 S) \). Since \( \mathcal{G} \) is a tower of Grassmanian bundles, these Chern classes \( c_i, b_j, f_k \) generate \( A^* (\mathcal{G}) \). Let \( Y \) be the locus of points \( g \in \mathcal{G} \) such that \( F_g \subset K_g \). \( Y \) is the nonsingular degeneracy locus of the canonical bundle map on \( \mathcal{G} \),
\[
F \to \wedge^4 S \otimes (\wedge^2 B)^*,
\]
obtained from the inclusion \( F \subset \wedge^2 S \) and sequence (3). By the Thom-Porteous formula on \( \mathcal{G} \) (see [F]),
\[
A^* (\mathcal{G}) \ni [Y] = c_3 (F^* \otimes \wedge^4 S \otimes (\wedge^2 B)^*) 
\]
\[= -f_3 + (c_1 - b_1) f_2 - (c_1 - b_1)^2 f_1 + (c_1 - b_1)^3.\]
\( Y \) is canonically isomorphic to the Grassmannian bundle \( \mathbb{G}(3, K) \to \mathbb{G}(2, S) \). There is natural bundle quotient sequence on \( Y \):
\[
0 \to F \to K \to K/F \to 0. 
\]
The locus \( \mathbb{G}(2, E) \subset Y \) is the set of points \( y \in Y \) such that \( \wedge^2 B_y \subset F_y \). \( \mathbb{G}(2, E) \subset Y \) is the nonsingular degeneracy locus of the canonical bundle map on \( Y \),
\[
\wedge^2 B \to K/F, 
\]
obtained from the sequences (4) and (5). By the Thom-Porteous formula on \( Y \),
\[
A^*(Y) \ni [\mathbb{G}(2, E)] = c_2 ((K/F) \otimes (\wedge^2 B)^*)
\]
\[= b_1^2 - c_1 b_1 + c_2^2 - 2c_1 f_1 + f_1^2 - f_2 + 2c_2.\]
The class \([G(2, E)] \in A^*(G)\) is there expressed by

\[
A^*(G) \ni [G(2, E)] = (-f_3 + (c_1 - b_1)f_2 - (c_1 - b_1)^2 f_1 + (c_1 - b_1)^3)
\]

\[
\cdot (b_1^2 - c_1 b_1 + c_1^2 - 2c_1 f_1 + f_1^2 - f_2 + 2c_2).
\]

Since \(G(2, E)\) is a Grassmannian bundle over \(G(2, S)\), the Chow ring of \(G(2, E)\) is generated over \(A^*(G(2, S))\) by the Chern classes \(h_1, h_2\) of the tautological sub-bundle \(H \to G(2, E)\). Via the embedding \(\iota : G(2, E) \hookrightarrow G\), \(H\) is isomorphic to \(\iota^*(F)/\iota^*(\Lambda^2 B)\). The Chern classes \(h_1\) and \(h_2\) can be expressed via \(\iota\) in terms of the classes \(b_j\) and \(f_k\). Therefore, the classes

\[
G(2, E) \cap M(c_1, c_2, c_3, c_4, b_1, b_2, f_1, f_2, f_3)
\]

(where \(M\) is monomial in the Chern classes) span the Chow ring of \(G(2, E)\). The ideal \(I_D \subset A^*(G(3, \Lambda^2 S))\) is generated by the \(\pi\) push-forwards of the classes \(f\):

\[
(6) \quad [G(2, E)] \cdot M(c_1, c_2, c_3, c_4, b_1, b_2, f_1, f_2, f_3) \in A^*(G)
\]

Since the classes \(c_i, f_k\) in \(A^*(G)\) are pull-backs from \(G(3, \Lambda^2 S)\), \(I_D\) is generated by the elements

\[
\pi_*([G(2, E)] \cdot M(b_1, b_2)).
\]

By the standard relations satisfied by the classes \(b_1\) and \(b_2\) over \(A^*(G(4, m))\), it follows that \(I_D\) is generated by:

\[
\begin{align*}
\pi_*([G(2, E)]) & \\
\pi_*([G(2, E)] \cdot b_1) & \\
\pi_*([G(2, E)] \cdot b_1^2) & \quad \pi_*([G(2, E)] \cdot b_2) \\
\pi_*([G(2, E)] \cdot b_1 b_2) & \\
\pi_*([G(2, E)] \cdot b_1^2 b_2) & \\
\pi_*([G(2, E)]) = 13c_1 - 2f_1.
\end{align*}
\]

\textbf{Lemma 4.} \textit{The pair} \((13c_1 - 2f_1, c_1(N))\) \textit{generates} \(A^1(G(3, \Lambda^2 S))\).

\textit{Proof.} Recall the notation of diagram \(2\):

\[\tau_2 : P(W_m) \to Z \subset G(3, \Lambda^2 S).\]

Let \(L = c_1(O_P(-1))\) be a generator of \(A^1(P(W_m))\). By Lemma 2, \(\tau_2^*(c_1) \cong 4L\). Since \(\tau_2^*[D] = \tau_2^*(13c_1 - 2f_1) = 0\), \(\tau_2^*(f_1) = 26L\). Therefore the image of \(\tau_2^*\) is the subgroup \(\mathbb{Z}(2L)\).
Since \([D] = 13c_1 - 2f_1\) is not divisible in \(A^1(G(3, \wedge^2 S))\), \(A^1(Z) \cong \mathbb{Z}\) and \(\tau_2^*\) is an isomorphism:

\[
\tau_2^* : A^1(Z) \xrightarrow{\sim} \mathbb{Z}(2L).
\]

By Lemma 3, \(\tau_2^*(c_1(N)) = 2L\). Therefore, \(c_1(N)\) generates \(A^1(Z)\). It now follows that the pair \((13c_1 - 2f_1, c_1(N))\) generates the group \(A^1(G(3, \wedge^2 S))\).

Therefore, \((I_D, c_1(N)) = (I_D, c_1, f_1)\).

By Lemma 4 it suffices to compute the five remaining push-forwards modulo the ideal \(J = (c_1, f_1)\). The results are (modulo \(J\)):

\[
\begin{align*}
\pi_*(\mathbb{G}(2, E)) : b_1 &= 0. \\
\pi_*(\mathbb{G}(2, E)) : b_2^2 &= -2f_3. \\
\pi_*(\mathbb{G}(2, E)) : b_2 &= c_3 - f_3. \\
\pi_*(\mathbb{G}(2, E)) : b_1 b_2 &= (c_2 - f_2)^2 - 4c_4. \\
\pi_*(\mathbb{G}(2, E)) : b_1^2 b_2 &= c_2 f_3 + f_2 c_3.
\end{align*}
\]

Hence \((I_D, c_1(N)) = (c_1, f_1, 2c_3, c_3 - f_3, (c_2 - f_2)^2 - 4c_4, (c_2 - f_2)c_3)\).

The ring \(A^*(G(4, m))\) is freely generated (up to codimension \(m - 4\)) by \(c_1, c_2, c_3, c_4\). The ring \(A^*(G(3, \wedge^2 S))\) has the following presentation (up to codimension \(m - 4\)):

\[
A^*(G(3, \wedge^2 S)) \cong \mathbb{Z}[c_1, c_2, c_3, c_4, f_1, f_2, f_3] / (t_4, t_5, t_6)
\]

where the \(t_4, t_5, t_6\) are the Chern classes of the tautological quotient bundle \(T : 0 \rightarrow F \rightarrow \wedge^2 S \rightarrow T \rightarrow 0\). We find (modulo \(J\)):

\[
\begin{align*}
t_4 &= (c_2 - f_2)^2 - 4c_4. \\
t_5 &= 2f_2 f_3 - 2c_2 f_3. \\
t_6 &= f_2 (-(c_2 - f_2)^2 + 4c_4) + f_3^2 - c_3^2.
\end{align*}
\]

There is a presentation:

\[
A^*(G(3, \wedge^2 S)) / (I_D, c_1(N)) \cong \mathbb{Z}[c_1, c_2, c_3, c_4, f_1, f_2, f_3] / (I_D, c_1, f_1, t_4, t_5, t_6)
\]

(7) (up to codimension \(m - 4\)). Surprisingly, the relations \(t_4, t_5, t_6\) are contained in the ideal \((I_D, c_1, f_1)\). By the limit procedure, \(A_{SO(4)}^*(\text{point}) \cong \mathbb{Z}[c_1, c_2, c_3, f_2] / (c_1, 2c_3, (c_2 - f_2)c_3, (c_2 - f_2)^2 - 4c_4)\).

The vector bundles \(S, F \subset \wedge^2 S\) on the approximation \(W_m/\text{SO}(V)\) are easily seen to be obtained from the principal \(\text{SO}(V)\)-bundle

\[
W_m \rightarrow W_m / \text{SO}(V)
\]

and the representations \(V, F \subset \wedge^2 V\) defined in section [4]. Define

\[
x = c_2 - f_2.
\]

Theorem 1 is proved.
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