Global and blow-up radial solutions for quasilinear elliptic systems arising in the study of viscous, heat conducting fluids

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Abstract
We study positive radial solutions of quasilinear elliptic systems with a gradient term in the form
\[
\begin{aligned}
\Delta_p u &= v^m |\nabla u|^{\alpha} \quad \text{in } \Omega, \\
\Delta_p v &= v^q |\nabla u|^{\beta} \quad \text{in } \Omega,
\end{aligned}
\]
where \( \Omega \subset \mathbb{R}^N \) (\( N \geq 2 \)) is either a ball or the whole space, \( 1 < p < \infty \), \( m, q > 0 \), \( \alpha \geq 0 \), \( 0 \leq \beta \leq m \) and \( (p - 1 - \alpha)(p - 1 - \beta) - qm \neq 0 \). We first classify all the positive radial solutions in case \( \Omega \) is a ball, according to their behavior at the boundary. Then we obtain that the system has non-constant global solutions if and only if \( 0 \leq \alpha < p - 1 \) and \( mq < (p - 1 - \alpha)(p - 1 - \beta) \). Finally, we describe the precise behavior at infinity for such positive global radial solutions by using properties of three component cooperative and irreducible dynamical systems.

Keywords: radial symmetric solutions, p-Laplace operator, asymptotic behaviour, cooperative and irreducible dynamical systems
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(Some figures may appear in colour only in the online journal)
1. Introduction

In this paper we investigate positive radial solutions for quasilinear elliptic systems of the form

\[
\begin{align*}
\Delta_p u &= v^m |\nabla u|^{\alpha} \quad \text{in } \Omega, \\
\Delta_p v &= v^\beta |\nabla u|^{q} \quad \text{in } \Omega,
\end{align*}
\]

(1.1)

where \(1 < p < \infty\), \(\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)\) stands for the standard \(p\)-Laplace operator and \(\Omega \subset \mathbb{R}^N (N \geq 2)\) is either a ball \(B_R\) centered at the origin and having radius \(R > 0\), or the whole space. The exponents in (1.1) satisfy

\[
1 < p < \infty, \quad m, q > 0, \quad \alpha \geq 0, \quad 0 \leq \beta \leq m,
\]

and

\[
\delta := (p - 1 - \alpha)(p - 1 - \beta) - qm \neq 0.
\]

(1.2)

In order to motivate our study for the system (1.1), let us first consider the underlying equation

\[
\Delta_p u = v^m |\nabla u|^{q} \quad \text{in } \mathbb{R}^N, N \geq 2.
\]

(1.3)

Solutions to (1.3) appear as steady states in models that investigate the influence of nonlinear diffusion in Hamilton–Jacobi equation

\[
u_t - \Delta_p u + H(x, u, \nabla u) = 0 \quad \text{in } \mathbb{R}^N \times (0, T).
\]

(1.4)

In such a context, time dependent solutions for the equation

\[
u_t - \Delta_p u + |\nabla u|^q = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty)
\]

(1.5)

are investigated in [17, 18]. It is easy to see that steady states of (1.5) lead to our underlying equation (1.3) in the particular case \(m = 0\).

A more general model that accounts for nonlinear diffusion was proposed in Vázquez [22, section 3.2.1] is

\[
u_t = \text{div}(A(u, |\nabla u|)\nabla u) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty).
\]

(1.6)

In dimension \(N = 1\), the equations

\[
u_t = (\Phi(u_t))_x \quad \text{and} \quad \nu_{xx} = (\Phi(\nu))_{xx}
\]

that stem from (1.6), were proposed by Barenblatt and Vázquez [2] to analyze the process of contour enhancement in image processing, based on the evolution model of Sethian and Malladi. For \(A(u, |\nabla u|) = mu^{m-1}\), equation (1.6) leads to the porous medium equation arising in plasma physics. For \(A(u, |\nabla u|) = |\nabla u|^p\), (1.6) yields the \(p\)-Laplace evolution equation which appears in non-Newtonian fluids, turbulent flows in porous media, glaciology and other contexts. For \(A(u, |\nabla u|) = u^{-1}|\nabla u|^{p-2}\) we derive the equation

\[
u_t - \Delta_p u + u^{-1}|\nabla u|^p = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty),
\]

whose steady states correspond to (1.3) in the particular case \(m = -1\) and \(q = p\).

The elliptic system (1.1) which is generated by (1.3) was considered so far only in the semilinear case \(p = 2, \alpha = \beta = 0, m = 1, q = 2\) by Díaz [9 et al] as a prototype model in the study of dynamics of a viscous, heat-conducting fluid. Considering a unidirectional flow, independent of distance in the flow direction, the speed \(u\) and the temperature \(\theta\) satisfy the coupled equations
\[
\begin{align*}
\begin{cases}
u_t - \Delta u = \theta & \text{in } \Omega \times (0,T), \\
\theta_t - \Delta \theta = |\nabla u|^2 & \text{in } \Omega \times (0,T).
\end{cases}
\end{align*}
\] (1.7)

The source terms $\theta$ and $|\nabla u|^2$ represent the buoyancy force and viscous heating, respectively. With the change of variable $v = -\theta$, steady states of (1.7) satisfy
\[
\begin{align*}
\begin{cases}
\Delta u = v & \text{in } \Omega, \\
\Delta v = |\nabla u|^2 & \text{in } \Omega,
\end{cases}
\end{align*}
\] (1.8)

which is the semilinear version of (1.1) in the particular case $p = 2$, $\alpha = \beta = 0$, $m = 1$ and $q = 2$. In [9] was obtained that system (1.8) admits a positive solution which blows up at the boundary of a ball; such a solution is also unique for fixed data. Further, it was observed in [9] that in case of small dimensions $N \leq 9$ there also exists a boundary blow-up solution of (1.8) that changes sign. The study in [9] was then carried over to time dependent systems in [10, 11]. Recently Singh [21], Filippucci and Vinti [14] extended the study of positive radial solutions in [9] to more general class of nonlinearities.

Recent results [7, 12, 13] have discussed the existence and nonexistence of positive solutions for systems of inequalities of the above type in the frame of general quasilinear differential operators. Quasilinear elliptic systems without gradient terms have been extensively investigated in the last three decades; see, e.g. the results in [1, 3–6, 8].

In this paper we study non-constant positive radial solutions of (1.1), that is, solutions $(u, v)$ which fulfill:

- $u, v \in C^2(\Omega)$ are positive and radially symmetric;
- $u$ and $v$ are not constant in any neighbourhood of the origin;
- $u$ and $v$ satisfy (1.1).

If $\Omega = \mathbb{R}^N$, solutions of (1.1) will be called global solutions.

The presence of the gradient terms $|\nabla u|^\alpha$ and $|\nabla u|^\beta$ in the right-hand side entails a rich structure of the solution set of (1.1) which we aim to investigate in the following. Throughout this work, we identify radial solutions $(u, v)$ by their one variable representant, that is, $u(x) = u(r), v(x) = v(r), r = |x|$. In the following, for a function $f : (0, R) \rightarrow \mathbb{R}$ we denote $f(R^-) = \lim_{r \rightarrow R^-} f(r)$, provided such a limit exists.

2. Main results

In our first result below we classify all non-constant positive radial solutions in a ball $B_R$ according to their behavior at the boundary. We have:

**Theorem 2.1.** Assume $\Omega = B_R$, $1 < p < \infty$, $m, q > 0$, $0 \leq \alpha < p - 1$, $0 \leq \beta \leq m$ and $\delta \neq 0$. Then

(i) There are no positive radial solutions $(u, v)$ with $u(R^-) = \infty$ and $v(R^-) < \infty$.

(ii) All positive radial solutions of (1.1) are bounded if and only if

$$mq < (p - 1 - \alpha)(p - 1 - \beta).$$

(iii) There are positive radial solutions $(u, v)$ of (1.1) with $u(R^-) < \infty$ and $v(R^-) = \infty$

if and only if
mq > mp + (p - \alpha)(p - 1 - \beta).

(iv) There are positive radial solutions \((u, v)\) of (1.1) with \(u(R^-) = v(R^-) = \infty\) if and only if

\[(p - 1 - \alpha)(p - 1 - \beta) < mq \leq mp + (p - \alpha)(p - 1 - \beta).
\]

Our next result concerns the existence of non-constant global positive radial solutions of (1.1). We obtain the following optimal result:

**Theorem 2.2.** Assume \(\Omega = \mathbb{R}^N, p > 1, m, q > 0, \alpha \geq 0, 0 \leq \beta \leq m\) and \(\delta \neq 0\). Then, (1.1) admits non-constant global positive radial solutions if and only if

\[0 \leq \alpha < p - 1 \quad \text{and} \quad mq < (p - 1 - \alpha)(p - 1 - \beta).
\]

We next discuss the behavior at infinity of global positive radial solutions of (1.1). Using properties of three-component irreducible dynamical systems we are able to extend the result in [21, theorem 2.7] where extra conditions on exponents are required. For the sake of completeness, we have stated in appendix all the important results from the theory of cooperative and irreducible dynamical systems we used in the present work.

**Theorem 2.3.** Assume \(\Omega = \mathbb{R}^N, 0 \leq \alpha < p - 1\) and \(\delta > 0\). Then, any non-constant global positive radial solution \((u, v)\) of (1.1) satisfies

\[
\lim_{|x| \to \infty} \frac{u(x)}{|x|^{1 + \frac{\alpha(m+1)-(1+\beta)}{2}}} = A \quad \text{and} \quad \lim_{|x| \to \infty} \frac{v(x)}{|x|^{\frac{p-1-\alpha}{2}}} = B,
\]

where \(A = A(N, p, q, \alpha, \beta) > 0\) and \(B = B(N, p, q, m, \alpha, \beta) > 0\) have explicit expressions given by (5.21) and (5.22).

The quantities \(A|x|^{1 + \frac{\alpha(m+1)-(1+\beta)}{2}}\) and \(B|x|^{\frac{p-1-\alpha}{2}}\) that appear in theorem 2.3 may be regarded as stabilizing profiles for the steady states solutions in the time-depending system that corresponds to (1.1).

We point out that the requirement \(\delta > 0\) in (1.2) is a classical condition on superlinearity of the system as described in [4]. Also, the value of the limits \(A\) and \(B\) in (2.2) depend decreasingly on the space dimension \(N \geq 2\). One can see that from their expressions in (5.21) and (5.22).

Using MATLAB we have plotted the non-constant global positive solution \((u, v)\) to (1.1) (see figure 1 below) over the interval \([0, 500]\) for \(p = 10, \alpha = \beta = 1, m = 2, q = 4\) and for various space dimensions \(N = 3, 10, 30, 60\). The solutions was normalized at the origin by \(u(0) = \nu(0) = 1\).

In our next result we show that given any pair \((a, b)\) \(\in (0, \infty) \times (0, \infty)\), there exists a unique positive global radial solutions of (1.1) that emanates from \((a, b)\).

**Theorem 2.4.** Assume \(\Omega = \mathbb{R}^N, 1 < p < N, 0 \leq \alpha < p - 1\) and \(\delta > 0\). Then for any \(a > 0, b > 0\) there exists a unique non-constant global positive radial solution of (1.1) such that \(u(0) = a\) and \(v(0) = b\).

Finally, let us discuss the single equation (1.3) that underlays our system (1.1). The case \(q = 0\) was discussed in [19]. Here we are interested in the case \(m, q > 0\). From theorems 2.1–2.4 we find:
Corollary 2.5. Assume \( m, q > 0, p > 1 \) and \( m + q \neq p - 1 \). Then (1.3) has a non-constant positive radial solution if and only if
\[
0 < q < p - 1 \quad \text{and} \quad m < p - q - 1. \tag{2.3}
\]
If (2.3) holds, then any non-constant positive radial solution \( u \) of (1.3) satisfies
\[
\lim_{|x| \to \infty} \frac{u(x)}{|x|^{p-q-1}} = C(N, m, p, q) > 0.
\]
Further, if \( 1 < p < N \) then from any \( a > 0 \) there exists a unique non-constant positive radial solution \( u \) of (2.3) such that \( u(0) = a \).

The remaining of the paper contains the proofs of the above four theorems.

3. Proof of theorem 2.1

Let \((u, v)\) be a non-constant positive radial solution of (1.1) in a ball \( B_R \). Then \((u, v)\) satisfies
\[
\begin{align*}
\left[ r^{N-1} u' |u'|^{p-2} \right]' &= r^{N-1} v^m |u'|^\alpha \quad \text{for all } 0 < r < R, \\
\left[ r^{N-1} v' |v'|^{p-2} \right]' &= r^{N-1} v^q |u'|^\beta \quad \text{for all } 0 < r < R, \\
u'(0) &= 0, u(r) > 0, v(r) > 0 \quad \text{for all } 0 < r < R.
\end{align*}
\tag{3.1}
\]
Thus, \( r \mapsto r^{N-1} u' |u'|^{p-2} \) and \( r \mapsto r^{N-1} v' |v'|^{p-2} \) are nondecreasing and vanish at \( r = 0 \). Since \((u, v)\) is non-constant, it follows that \( u'(r) > 0 \) and \( v'(r) > 0 \) for all \( 0 < r < R \), so \( u \) and \( v \) are increasing. Thus, (3.1) reads
\[
\begin{align*}
\left[ (u')^{p-1} \right]' + \frac{N-1}{r} (u')^{p-1} &= v^m (u')^\alpha \quad \text{for all } 0 < r < R, \\
\left[ (v')^{p-1} \right]' + \frac{N-1}{r} (v')^{p-1} &= v^q (u')^\beta \quad \text{for all } 0 < r < R, \\
u'(0) &= 0, u(r) > 0, v(r) > 0 \quad \text{for all } 0 < r < R.
\end{align*}
\tag{3.2}
\]
which further implies
\[
\begin{align*}
&\left\{ \begin{array}{ll}
\left( u' \right)^{p-1-\alpha} + \frac{\gamma}{\nu} u'^{p-1-\alpha} = \frac{p-1-\alpha}{p-1} v^m & \text{for all } 0 < r < R, \\
\left( v' \right)^{p-1} + \frac{N-1}{r} v'^{p-1} = \nu^\beta (u')^q & \text{for all } 0 < r < R,
\end{array} \right.
\end{align*}
\]

where
\[
\gamma = \frac{(N-1)(p-1-\alpha)}{p-1} > 0.
\]

We can rearrange (3.3) in the form
\[
\begin{align*}
&\left\{ \begin{array}{ll}
\left( r^\gamma (u')^{p-1-\alpha} \right)' = \frac{p-1-\alpha}{p-1} r^\gamma v^m(r) & \text{for all } 0 < r < R, \\
\left( r^{N-1}(v')^{N-1} \right)' = r^{N-1} v^\beta (r)(u')^q & \text{for all } 0 < r < R,
\end{array} \right.
\end{align*}
\]

Lemma 3.1. Any non-constant positive radial solution \((u, v)\) of (1.1) in \(B_R\) satisfies
\[
\begin{align*}
&\left( N + \frac{\alpha}{p-1-\alpha} \right) (u'(r))^{p-1-\alpha} < r v^m(r) \text{ for all } 0 < r < R, \\
&N(v'(r))^{p-1} < r v^\beta (r)(u')^q \text{ for all } 0 < r < R, \\
&\frac{p-1-\alpha}{N(p-1-\alpha) + \alpha} v^m(r) < \left( (u')^{p-1-\alpha} \right)'(r) < \frac{p-1-\alpha}{p-1} v^m(r) \text{ for all } 0 < r < R,
\end{align*}
\]

and
\[
\frac{v^\beta (r)(u')^q(r)}{N} \leq \left( (v')^{p-1} \right)'(r) \leq v^\beta (r)(u')^q(r) \text{ for all } 0 < r < R.
\]

Proof. For simplicity, let us write \(w = u'\) so (3.3) and (3.5) read
\[
\begin{align*}
&\left\{ \begin{array}{ll}
\left( w^{p-1-\alpha} \right)' + \frac{\gamma}{\nu} w^{p-1-\alpha} = \frac{p-1-\alpha}{p-1} v^m & \text{for all } 0 < r < R, \\
\left( w' \right)^{p-1} + \frac{N-1}{r} w'^{p-1} = \nu^\beta w^q & \text{for all } 0 < r < R,
\end{array} \right.
\end{align*}
\]

and
\[
\begin{align*}
&\left\{ \begin{array}{ll}
\left( r^\gamma w^{p-1-\alpha} \right)' = \frac{p-1-\alpha}{p-1} r^\gamma v^m(r) > 0 & \text{for all } 0 < r < R, \\
\left( r^{N-1}(v')^{N-1} \right)' = r^{N-1} v^\beta (r)w^q(r), & \text{for all } 0 < r < R,
\end{array} \right.
\end{align*}
\]

Integrating the first equation of (3.11) and using the fact that \(v\) is strictly increasing on \((0, R)\) we deduce
\[
r^\gamma w^{p-1-\alpha}(r) = \frac{p-1-\alpha}{p-1} \int_0^r \gamma v^m(t) dt \\
< \frac{p-1-\alpha}{p-1} v^m(r) \int_0^r t \, dt \\
= \frac{p-1-\alpha}{(p-1)(\gamma+1)} r^{\gamma+1} v^m(r) \text{ for all } 0 < r < R.
\]
Hence,
\[ w^{p-1-\alpha}(r) < \frac{p - 1 - \alpha}{(p - 1)(\gamma + 1)} r^\alpha \] for all \( 0 < r < R \),

which proves (3.6). Using this estimate in the first equation of (3.10) it follows that
\[ \frac{p - 1 - \alpha}{p - 1} \nu^\alpha(r) = (w^{p-1-\alpha})'(r) + \frac{2}{r} w^{p-1-\alpha}(r) \]
\[ < (w^{p-1-\alpha})'(r) + \frac{\gamma(p - 1 - \alpha)}{(\gamma + 1)(p - 1)} \nu^\alpha(r) \] for all \( 0 < r < R \),

which implies
\[ (w^{p-1-\alpha})'(r) > \frac{p - 1 - \alpha}{(\gamma + 1)(p - 1)} \nu^\alpha(r) = \frac{p - 1 - \alpha}{N(p - 1 - \alpha) + \alpha} \nu^\alpha(r) \] for all \( 0 < r < R \). (3.13)

Also, from (3.11) and the positivity of \( w \) we deduce
\[ (w^{p-1-\alpha})'(r) < \frac{p - 1 - \alpha}{p - 1} \nu^\alpha(r) \] for all \( 0 < r < R \). (3.14)

Now, (3.8) follows from (3.13) and (3.14). At this point, let us note that from (3.8) we have that \( (u')^{p-1-\alpha} \) is positive and strictly increasing so \( w = u' \) is also positive and strictly increasing. Using this fact and the same approach as above we derive (3.7) and (3.9).

\[ \Box \]

**Proof of Theorem 2.1.** The existence of a non-constant positive solution to (1.1) in a small ball \( B_\rho \) follows from similar arguments to [14, proposition A1] (see also [7, proposition 9]). Specifically, we employ a fixed point argument for the mapping
\[ T : C^1[0, \rho] \times C^1[0, \rho] \to C^1[0, \rho] \times C^1[0, \rho], \]

given by
\[ T[u, v](r) = \left[ T_1[u, v](r), T_2[u, v](r) \right], \]

where
\[
\begin{align*}
T_1[u, v](r) &= a + \int_0^r \left( \frac{p - 1 - \alpha}{p - 1} I^{-\gamma} \int_0^s s^\alpha \nu^\alpha(s) ds \right)^{1/(p - 1 - \alpha)} ds, \\
T_2[u, v](r) &= b + \int_0^r \left( \frac{p - 1 - \alpha}{p - 1} I^{-N} \int_0^s s^{N-1} \nu^\alpha(s) |u'(s)|^q ds \right)^{1/(p - 1 - \alpha)} ds,
\end{align*}
\]

where \( a, b > 0 \). With a standard approach, there exists a small radius \( \rho > 0 \) such that \( T \) has a fixed point \((u, v)\) which is a non-constant positive radially symmetric solution of (3.1). Now, the pair \((u_\lambda, v_\lambda)\) defined as
\[ u_\lambda(x) = \lambda^{\frac{\alpha(p-1)}{2}} u \left( \frac{x}{\lambda} \right), \quad v_\lambda(x) = \lambda^{\frac{(p-1-\alpha)q}{2}} v \left( \frac{x}{\lambda} \right), \]

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provides a non-constant positive radially symmetric solution of (3.1) in the ball $B_{\lambda'\rho}$. This shows that in any ball of positive radius there are non-constant positive radially symmetric solution of (3.1).

Let us assume now that $(u,v)$ is a non-constant positive radially symmetric solution of (3.1) in $\Omega = B_R$, $R > 0$ and set $z = (u')^{p-1-\alpha}$. Then by (3.8) and (3.9) in lemma 3.1 we have

$$CV^m(r) \leq z'(r) \leq V^m(r) \quad \text{for all } 0 < r < R,$$

and

$$CV^\beta(r)z^{\frac{p-1-\alpha}{m}}(r) \leq [(v')^{p-1}](r) \leq v^\beta(r)z^{\frac{p-1-\alpha}{m}}(r) \quad \text{for all } 0 < r < R,$$

for some constant $C = C(N, p, \alpha) \in (0, 1)$.

(i) Assume that $u(R^-) = \infty$ and $v(R^-) < \infty$. Since, $u'$ is increasing (observe from (3.18) that $z$ is increasing, which implies $u'$ is also increasing) we deduce that $u'(R^-) = z(R^-) = \infty$. Also, from (3.18) we find

$$C_1 < z'(r)(r) \leq C_2 \quad \text{for all } r \in (0, R),$$

for some positive constants $C_1, C_2$. Integrating over $[0, R]$ we reach a contradiction.

(ii)–(iv) Let $(u, v)$ be a positive radial solution of (1.1) with $v(R^-) = \infty$. It follows from (3.18) that $z'(R^-) = \infty$. Also, $v'$ is increasing and $v(R^-) = \infty$ imply $v'(R^-) = \infty$. Using (3.18) and (3.19) we have

$$z'(r) \leq v^m(r) \quad \text{for all } 0 < r < R,$$

and

$$CV^\beta(r)z^{\frac{p-1-\alpha}{m}}(r) \leq [(v')^{p-1}](r) \quad \text{for all } 0 < r < R.$$  

Multiplying (3.20) and (3.21) we obtain

$$z^{\frac{p-1-\alpha}{m}}(r)z'(r) \leq CV^m(\beta)(r)[(v')^{p-1}](r) \quad \text{for all } 0 < r < R.$$

Integrating over $[0, R]$ in the above estimate we have

$$z^{\frac{p+\alpha-1}{p-1-\alpha}}(r) \leq C \int_0^r v^m(t)[(v')^{p-1}](t)dt \leq CV^m(\beta)(r) \int_0^r [(v')^{p-1}](t)dt.$$

So,

$$z^{\frac{p+\alpha-1}{p-1-\alpha}}(r) \leq CV^m(\beta)(r)v'(r) \quad \text{for all } 0 < r < R.$$  

Multiplying (3.22) by $z'(r)$ and using (3.20) we have

$$z^{\frac{p+\alpha-1}{p-1-\alpha}}(r)z'(r) \leq CV^m(\beta)(r)v'(r) \quad \text{for all } 0 < r < R,$$

that is

$$z^{\frac{p+\alpha-1}{p-1-\alpha}}(r)z'(r) \leq CV^m(\beta)(r)v'(r) \quad \text{for all } 0 < r < R.$$

A further integration over $[0, r], 0 < r < R$, yields
\[
\zeta^\frac{p+q(p-1-\alpha)}{p+q-p+1-\beta} (r) \leq C v^\frac{mp + p - 1 - \beta}{m} (r) \quad \text{for all } 0 < r < R.
\]

Hence, by the first estimate in (3.18) we find
\[
z^\frac{p+q(p-1-\alpha)}{p+q-p+1-\beta} (r) \leq C v^\frac{mp + p - 1 - \beta}{m} (r) \leq C (z'(r))^{\frac{m}{m+q+1-\beta}} \quad \text{for all } 0 < r < R,
\]

which yields
\[
z'(r)z^{-\sigma}(r) \geq C \quad \text{for all } 0 < r < R,
\]

where
\[
\sigma = \frac{m}{p - 1 - \alpha} \cdot \frac{q + p(p - 1 - \alpha)}{mp + p - 1 - \beta} > 0.
\]

Now, we return to (3.19) to get
\[
[(v')^{p-1}]'(r) \leq v^\beta(r)z^{\frac{q}{p-1-\beta}}(r) \quad \text{for all } 0 < r < R.
\]

Multiplying by \(v'(r)\) and integrating over \([0, r]\), we have
\[
\frac{(p-1)}{p} (v')^p(r) \leq \int_0^r v^\beta(t)v'(t)z^{\frac{q}{p-1-\beta}}(t)dt \leq z^{\frac{q}{p-1-\beta}}(r) \int_0^r v^\beta(t)v'(t)dt.
\]

Hence
\[
v'(r)v^{\frac{\sigma}{1-\sigma}}(r) \leq C z^{\frac{q}{p-1-\beta}}(r) \quad \text{for all } 0 < r < R.
\]

Multiplying (3.25) by \(z'(r)\) and using \(z'(r) \geq C v^\sigma(r)\) we find
\[
v'(r)v^{\frac{\sigma}{1-\sigma}}(r) \leq C z^{\frac{q}{p-1-\beta}}(r)z'(r) \quad \text{for all } 0 < r < R.
\]

Further integration over \([0, r]\) yields
\[
v^\frac{q+q(p-1-\alpha)}{p} (r) - v^\frac{q}{p} (0) \leq C z^{\frac{q+q(p-1-\alpha)}{p}} (r) \quad \text{for all } 0 < r < R.
\]

Since \(v(R^-) = \infty\), there exists \(\rho \in (0, R)\) such that
\[
v(r) \leq C z^{\frac{q+q(p-1-\alpha)}{p}} (r) \quad \text{for all } \rho \leq r < R.
\]

This yields \(z(R^-) = \infty\). Then, using (3.18) and (3.27) we obtain
\[
z'(r) \leq v^\sigma(r) \leq C z^\sigma (r) \quad \text{for all } \rho \leq r < R,
\]

where the exponent \(\sigma\) is defined in (3.24). It follows from (3.23) and (3.28) that
\[
C_1 \leq z'(r)z^{-\sigma}(r) \leq C_2 \quad \text{for all } \rho \leq r < R,
\]
for some constants \(C_2 > C_1 > 0\) depending only on parameters \(p, q, \alpha, \beta\) and dimension \(N\). Since \(z(R^-) = \infty\), we deduce from (3.29) that

\[
\sigma > 1
\]  

(3.30)

and

\[
C_1(R - r) \leq \frac{1^{1 - \sigma(r)}}{\sigma - 1} \leq C_2(R - r) \quad \text{for all } \rho \leq r < R.  
\]  

(3.31)

Using \(z = (u')^{p-1-\alpha}\), we have

\[
C_1(R - r) \frac{1}{(1 - (1 - \sigma) \alpha)} \leq u'(r) \leq C_2(R - r) \frac{1}{(1 - (1 - \sigma) \alpha)} \quad \text{for all } \rho \leq r < R.
\]

Since

\[
u(R^-) = u(\rho) + \int_\rho^R u'(t)dt,
\]

we deduce that

\[
u(R^-) < \infty \iff \int_\rho^R (R - t)^{\frac{1}{\sigma - 1}} dt  
\]

\[
\iff \int_0^1 s^{-\frac{1}{\sigma - 1}} (p - 1 - \alpha) ds < 0
\]

\[
\iff \sigma > \frac{p - \alpha}{p - 1 - \alpha},
\]

(3.32)

and similarly

\[
u(R^-) = \infty \iff \sigma \leq \frac{p - \alpha}{p - 1 - \alpha},
\]

(3.33)

From (3.31)–(3.33) we deduce that:

- There are solutions \(u(R^-) < \infty\) and \(v(R^-) = \infty \iff \sigma > \frac{p - \alpha}{p - 1 - \alpha}\).
- There are solutions \(u(R^-) = v(R^-) = \infty \iff 1 < \sigma \leq \frac{p - \alpha}{p - 1 - \alpha}\).
- All solutions are bounded \(\iff \sigma \leq 1\). Since \(\delta \neq 0\), we rule out the possibility \(\sigma = 1\).
- Hence, all positive radial solutions of (1.1) are bounded if and only if \(\sigma < 1\).

Hence, we conclude (ii)–(iv).

\[\square\]

4. Proof of theorem 2.2

Assume first that (2.1) holds. As argued in the beginning of the proof of theorem 2.1 we are able to construct a non-constant positive radial solution in a maximal ball. By construction, each component of such solution is increasing and by theorem 2.1(ii) the solution is bounded. Thus, the maximal domain of existence must be the whole space \(\mathbb{R}^N\). 

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Conversely, assume that (2.1) does not hold and there exists a non-constant global positive radial solution \((U, V)\) of (1.1). In order to reach a contradiction, we discuss separately the following three cases.

**Case 1.** \(0 \leq \alpha < p - 1\) and \(mq > (p - 1 - \alpha)(p - 1 - \beta)\).

From theorem 2.1 there exists a positive radial solution of (1.1) such that \(v(1^-) = \infty\). For any \(\lambda > 0\) set

\[ U_\lambda(x) = \lambda^{\frac{1}{p-1}} \int_0^{\infty} r^{-(\gamma - 1)\frac{p-1}{p}} U(r) \, dr, \quad V_\lambda(x) = \lambda^{\frac{\gamma}{p-1}} V(\lambda x). \]

Then, \((U_\lambda, V_\lambda)\) is a non-constant global positive radial global solution of (1.1). Replacing \((U, V)\) by \((U_\lambda, V_\lambda)\) for \(\lambda > 0\) small enough, we may assume that \(V(0) > v(0)\).

Let

\[ R := \sup \left\{ r \in (0,1) : V(t) > v(t) \quad \text{in} \quad (0,r) \right\}. \tag{4.1} \]

Clearly, since \(V(0) > v(0)\), we have \(0 < R \leq 1\). We claim that \(R = 1\). Assuming the contrary, from (3.11), for all \(0 < r < R\) we obtain

\[ [r^\gamma W^{p-1-\alpha}]' = \frac{p - 1 - \alpha}{p-1} r^{\gamma} V'(r) > \frac{p - 1 - \alpha}{p-1} r^{\gamma} V'(r) = [r^\gamma W^{p-1-\alpha}]', \]

where, as in the previous section we denote \(W = U'\) and \(w = u'\). An integration over \([0, r]\), \(0 < r \leq R\), yields \(W > v\) on \([0, R]\), which together with the second equation of (3.11) implies

\[ [r^{N-1}(W')^{p-1}]' = r^{N-1}W'(r)V'(r) > r^{N-1}w'(r)v'(r) = [r^{N-1}(v')^{p-1}]' \quad \text{for all} \quad 0 < r \leq R. \]

As before, this leads to \(V' > v'\) on \([0, R]\) and then \(V > v\) on \([0, R]\) which contradicts the maximality of \(R\) in (4.1). Hence \(R = 1\), so \(V > v\) on \((0,1)\). This yields \(V(1^-) = \infty\) which is a contradiction with the fact that \(V\) is defined on the whole positive semiline.

**Case 2.** \(\alpha > p - 1\).

By letting \(W = U'\) as in the proof of lemma 3.1, we have that \(r \mapsto r^{N-1}W^{p-1}\) is nondecreasing, so there exists

\[ L := \lim_{r \to \infty} r^{N-1}W^{p-1}(r) \in (0, \infty]. \]

As in the proof of theorem 2.1. We rewrite the first equation of (3.1) as

\[ [r^\gamma W^{p-1-\alpha}]' = \frac{\gamma}{N-1} r^{\gamma} V'(r) \quad \text{for all} \quad r > 0, \tag{4.2} \]

where

\[ \gamma = \frac{(N - 1)(p - 1 - \alpha)}{p - 1} < 0. \]

Integrating in (4.2) over \([r, \infty], \quad r > 0\), we find

\[ r^\gamma W^{p-1-\alpha}(r) = L \frac{\gamma}{N-1} + \frac{\gamma}{N-1} \int_r^{\infty} t^{\gamma} V'(t) \, dt \quad \text{for all} \quad r > 0. \]

Now, using the fact that \(v\) is increasing we have
\[ r^{\gamma} W^{p-1-\alpha}(r) \geq \frac{|\gamma|}{N-1} V^m(r) \int_r^\infty t^\gamma dt \quad \text{for all } r > 0. \]

In particular, the integral must be convergent, so \( \gamma < -1 \) and we deduce
\[ W^{p-1-\alpha}(r) \geq \frac{\gamma}{(N-1)(\gamma+1)} r V^m(r) \quad \text{for all } r > 0. \] (4.3)

We now use (4.3) into (4.2). Since \( \gamma < 0 \) we find
\[ \frac{\gamma}{N-1} V^m(r) = \left[ W^{p-1-\alpha} \right]'(r) + \frac{\gamma}{r} W^{p-1-\alpha}(r) \]
\[ \leq \left[ W^{p-1-\alpha} \right]'(r) + \frac{\gamma^2}{(N-1)(\gamma+1)} V^m(r) \quad \text{for all } r > 0. \]

Hence,
\[ \left[ W^{p-1-\alpha} \right]'(r) \geq \frac{\gamma}{(N-1)(\gamma+1)} V^m(r) > 0 \quad \text{for all } r > 0. \]

This shows that \( W^{p-1-\alpha} \) is increasing, so \( W \) must be decreasing. Since \( W \geq 0 \) and \( W(0) = 0 \), it follows that \( W \equiv 0 \), that is, \( U \equiv U(0) > 0 \) is constant, contradiction.

\textbf{Case 3.} \( \alpha = p-1 \).

As above, \( U' > 0 \) and \( V' > 0 \) and from the first equation of (3.1) we find
\[ (p-1) \frac{U''(r)}{U'(r)} + \frac{N-1}{r} = V^m(r) \quad \text{for all } r > 0. \]

Integrating over \( [0, 1] \) we deduce
\[ \ln \left[ \frac{r^{N-1}(U')^{p-1}(r)}{U'(r)} \right]|_0^1 = \int_0^1 V^m(t) dt, \]

which is a contradiction since \( U'(0) = 0 \) and the right-hand side of the above equality is finite.

\textbf{Remark.} The approach in Case 3 above shows in fact that if \( \alpha = p-1 \) then system (1.1) has no non-constant positive radial solutions in any ball.

\section{5. Proof of theorem 2.3}

Assume \((u, v)\) is a non-constant global positive radial solution of (1.1). Let \( t = \ln(r) \in \mathbb{R} \) and define the new functions \( X, Y, Z, W \) by
\[ X(t) = \frac{ru'(r)}{u(r)}, \quad Y(t) = \frac{rv'(r)}{v(r)}, \quad Z(t) = \frac{rv^m(r)}{\left(u'(r)\right)^{p-1-\alpha}}, \quad \text{and} \quad W(t) = \frac{rv^\beta(r)u'^{\alpha}(r)}{v'^{p-1}(r)}. \] (5.1)

A direct calculation shows that \((X, Y, Z, W)\) satisfies
\[
\begin{align*}
X_t &= X\left(\frac{p-N}{p-1} - X + \frac{1}{p-1}Z\right) \quad \text{for all } t \in \mathbb{R}, \\
Y_t &= Y\left(\frac{p-N}{p-1} - Y + \frac{1}{p-1}W\right) \quad \text{for all } t \in \mathbb{R}, \\
Z_t &= Z\left(\frac{(p-1)N-(N-1)\alpha}{p-1} - \frac{p-1-\alpha}{p-1}Z + mY\right) \quad \text{for all } t \in \mathbb{R}, \\
W_t &= W\left(\frac{(p-1)N-q(N-1)}{p-1} + \beta Y + \frac{q}{p-1}Z - W\right) \quad \text{for all } t \in \mathbb{R}.
\end{align*}
\]  

(5.2)

By L’Hopital’s rule we have

\[
\lim_{t \to \infty} X(t) = \lim_{r \to \infty} \frac{n'(r)}{u'(r)} = \lim_{r \to \infty} \left(1 + \frac{n''(r)}{u'(r)}\right) = \lim_{t \to \infty} \left(\frac{1}{p-1}Z(t) + \frac{p-N}{p-1}\right),
\]

(5.3)

provided \(\lim_{t \to \infty} Z(t)\) exists. In the following we shall study the system consisting of the last three equations of (5.2) which we write

\[
\zeta_t = g(\zeta) \quad \text{in } \mathbb{R},
\]

(5.4)

where

\[
\zeta(t) = \begin{bmatrix} Y(t) \\ Z(t) \\ W(t) \end{bmatrix} \quad \text{and} \quad g(\zeta) = \begin{bmatrix} Y\left(\frac{p-N}{p-1} - Y + \frac{1}{p-1}W\right) \\ Z\left(\frac{(p-1)N-(N-1)\alpha}{p-1} - \frac{p-1-\alpha}{p-1}Z + mY\right) \\ W\left(\frac{(p-1)N-q(N-1)}{p-1} + \beta Y + \frac{q}{p-1}Z - W\right) \end{bmatrix}.
\]

(5.5)

Note that the system (5.4) is cooperative and irreducible as described in the appendix. Also, the only equilibrium point of (5.4) and (5.5) with all components being strictly positive is

\[
P_\infty = \begin{bmatrix} Y_\infty \\ Z_\infty \\ W_\infty \end{bmatrix},
\]

(5.6)

where

\[
\begin{align*}
\frac{p-N}{p-1} - Y_\infty + \frac{1}{p-1}W_\infty &= 0, \\
\frac{(p-1)N-(N-1)\alpha}{p-1} - \frac{p-1-\alpha}{p-1}Z_\infty + mY_\infty &= 0, \\
\frac{(p-1)N-q(N-1)}{p-1} + \beta Y_\infty + \frac{q}{p-1}Z_\infty - W_\infty &= 0.
\end{align*}
\]

(5.7)

Solving (5.7) we find

\[
\begin{align*}
Y_\infty &= \frac{p(p-1-\alpha)q}{\alpha}, \\
Z_\infty &= \frac{m(p-1)Y_\infty}{p-1-\alpha} + N + \frac{\alpha}{p-1-\alpha}, \\
W_\infty &= (p-1)Y_\infty + N - p.
\end{align*}
\]

(5.8)

**Lemma 5.1.** The equilibrium point \(P_\infty\) is asymptotically stable.

**Proof.** Using (5.7) we compute the linearized matrix of (5.4) at \(P_\infty\) as

\[
M_\infty = \begin{bmatrix}
-Y_\infty & 0 & \frac{1}{p-1}Y_\infty \\
0 & mZ_\infty - \frac{p-1-\alpha}{p-1}Z_\infty & 0 \\
\beta W_\infty & \frac{q}{p-1}W_\infty & -W_\infty
\end{bmatrix}.
\]

(5.9)
The characteristic polynomial of $M_\infty$ is
\[ P(\lambda) = \det(\lambda I - M) = \lambda^3 + a\lambda^2 + b\lambda + c, \]
where
\[
\begin{align*}
a &= Y_\infty + \frac{p-1-\alpha}{p-1} Z_\infty + W_\infty, \\
b &= \frac{p-1-\alpha}{p-1} Y_\infty Z_\infty + \frac{p-1-\beta}{p-1} Y_\infty W_\infty + \frac{p-1-\alpha}{p-1} Z_\infty W_\infty, \\
c &= \frac{1}{(p-1)^2} Y_\infty Z_\infty W_\infty.
\end{align*}
\]
Since $Y_\infty$, $Z_\infty$, $W_\infty > 0$ and $p - 1 - \beta > 0$ (which follows easily from $\delta > 0$) we have
\[
a \geq \frac{p - 1 - \beta}{p - 1} Y_\infty + \frac{p - 1 - \alpha}{p - 1} Z_\infty + \frac{p - 1 - \beta}{p - 1} W_\infty.
\]
Thus, by AM-GM inequality we find
\[
a \geq \frac{3}{p - 1} \left( \frac{p - 1 - \alpha}{p - 1} \right)^{\frac{1}{3}} \left( \frac{p - 1 - \beta}{p - 1} \right)^{\frac{1}{3}} \left( Y_\infty Z_\infty W_\infty \right)^{\frac{1}{3}}.
\]
Similarly, by AM-GM we obtain
\[
b \geq \frac{3}{p - 1} \left( \frac{p - 1 - \alpha}{p - 1} \right)^{\frac{1}{3}} \left( \frac{p - 1 - \beta}{p - 1} \right)^{\frac{1}{3}} \left( Y_\infty Z_\infty W_\infty \right)^{\frac{1}{3}}.
\]
We now multiply the above estimates to deduce
\[
ab \geq \frac{9}{(p - 1)^2} \left( \frac{p - 1 - \alpha}{p - 1} \right) \left( \frac{p - 1 - \beta}{(p - 1)^2} \right) \left( Y_\infty Z_\infty W_\infty \right) > 9c.
\]
We claim that all three roots $\lambda_1$, $\lambda_2$ and $\lambda_3$ of the characteristic polynomial $P(\lambda)$ of $M_\infty$ have negative real part. Indeed, if $\lambda_i \in \mathbb{R}$, for all $i = 1, 2, 3$ then, since $P(\lambda) > 0$ for all $\lambda \geq 0$ it follows that $\lambda_i < 0$ for all $i = 1, 2, 3$. If $P$ has exactly one real root, say $\lambda_1 \in \mathbb{R}$, then $\Re(\lambda_2) = \Re(\lambda_3)$. Using $P(-a) = -ab + c < 0$, it follows that $\lambda_1 > -a$. Since $\lambda_1 + \lambda_2 + \lambda_3 = -a$ we easily deduce that $\Re(\lambda_2) = \Re(\lambda_3) < 0$. This proves that $P_\infty$ is asymptotically stable.

The following result is crucial in our analysis to establish the behavior of $\zeta(t)$ as $t \to \infty$.

**Lemma 5.2.** For all $t \in \mathbb{R}$ we have
\[
0 < Y(t) < Y_\infty, \quad N + \frac{\alpha}{p - 1 - \alpha} < Z(t) < Z_\infty, \quad N < W(t) < W_\infty.
\]

**Proof.** We divide our arguments into four steps.

**Step 1.** Preliminary facts: $Z(t) > N + \frac{\alpha}{p - 1 - \alpha}, W(t) > N$ for all $t \in \mathbb{R}$ and $\lim_{t \to -\infty} Y(t) = 0$.

The lower bounds for $Z$ and $W$ follow from (3.6) and (3.7) in lemma 3.1. Since $v'(0) = 0$ and $v(0) > 0$ we have $\lim_{t \to -\infty} Y(t) = \lim_{t \to 0} \frac{v(t)}{v'(0)} = 0$.
Step 2. There exists \( T \in \mathbb{R} \) such that \( Z(t) < Z_\infty \) for all \( t \in (-\infty, T] \).

The conclusion of this Step follows immediately once we prove that

\[
\lim_{t \to -\infty} Z(t) = N + \frac{\alpha}{p - 1 - \alpha}.
\]

(5.10)

Let \( t \in (-\infty, 0) \) and \( r = t' \in (0, 1) \). We use the generalized mean value theorem\(^5\) [20, theorem 5.9, page 107] over the interval \([0, r]\). Thus, there exists \( c \in (0, r) \) such that

\[
Z(t) = \frac{rv^m(r)}{u^{p-1-\alpha}(r)} = \frac{d}{dr} \left[ \frac{rv^m(r)}{u^{p-1-\alpha}(r)}(c) \right] = \frac{v^m(c) + cmv^m-1(c)v'(c)}{(p - 1 - \alpha)u^{p-1-\alpha}(c)u^m(c)} = \frac{p - 1}{p - 1 - \alpha} \frac{v^m(c)u^{\alpha}(c) + cmv^m-1(c)v'(c)u^{\alpha}(c)}{(u'p-1)^{\alpha}(c)}.
\]

Using the first equation in (3.2) we find

\[
Z(t) = \frac{p - 1}{p - 1 - \alpha} \frac{v^m(c)u^{\alpha}(c)}{v^m(c)(1 + m\frac{u'(c)}{\alpha(c)})}.
\]

and so,

\[
Z(t) = \frac{p - 1}{p - 1 - \alpha} \frac{Z(\text{Inc})}{Z(\text{Inc}) - (N - 1)} \left( 1 + mY(\text{Inc}) \right).
\]

(5.11)

Since

\[
Z(t) > N + \frac{\alpha}{p - 1 - \alpha} \quad \text{for all} \quad t \in \mathbb{R},
\]

(5.12)

we have

\[
\frac{Z(\text{Inc})}{Z(\text{Inc}) - (N - 1)} < \left( N + \frac{\alpha}{p - 1 - \alpha} \right) \frac{p - 1}{p - 1}.
\]

Thus from (5.11) we find

\[
\limsup_{t \to -\infty} Z(t) \leq \left( N + \frac{\alpha}{p - 1 - \alpha} \right) \left( 1 + \lim_{t \to -\infty} Y(t) \right) = N + \frac{\alpha}{p - 1 - \alpha}.
\]

(5.13)

Now, combining (5.12) and (5.13) we obtain that (5.10) holds. It follows that there exists \( T \in \mathbb{R} \) such that \( Z(t) < Z_\infty \) for all \( t \leq T \).

Step 3: There exists a sequence \( t_j \to -\infty \) such that

\[
Y(t_j) < Y_\infty, \quad Z(t_j) < Z_\infty \quad \text{and} \quad W(t_j) < W_\infty \quad \text{for all} \quad j \geq 1.
\]

(5.14)

Assume the above assertion is not true. In view of the previous steps and by taking \( T \in \mathbb{R} \) found at Step 2 small enough, we may assume

---

\(^5\) Generalized mean value theorem (or Cauchy’s theorem) states that if \( f, g : [a, b] \to \mathbb{R} \) are differentiable functions on \((a, b)\) and continuous on \([a, b]\), then there exists \( c \in (a, b) \) such that \( \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \).
\[ Y(t) < Y_\infty, \quad Z(t) < Z_\infty \quad \text{and} \quad W(t) \geq W_\infty \quad \text{for all } t \in (-\infty, T). \] (5.15)

Using this fact and last equation in (5.2) we deduce \( W_t < 0 \) on \((-\infty, T]\). Hence, \( W \) is decreasing in a neighbourhood of \(-\infty\) and there exists

\[ L := \lim_{t \to -\infty} W(t) = \lim_{r \to 0} \frac{rv^\beta(r)u^{\eta(r)}}{v^{p-1}(r)}. \]

Let \( t \in (-\infty, T] \) and \( r = e^t \). Applying the generalized mean value theorem as in the previous step and using the second equation of (3.2) we find \( c \in (0, r) \) such that

\[
W(t) = \frac{rv^\beta(r)u^{\eta(r)}}{v^{p-1}(r)} = \frac{d}{dr} \left[ rv^\beta(r)u^{\eta(r)}(r) \right] (c) \\
= \frac{d}{dr} \left[ v^{p-1}(r) \right] (c) \]

\[
c = c^\beta(v(c)u'(c)) + \beta cv^\beta(c)v' c + qcv^\beta(c)u^{\eta-1}(c)u''(c).
\]

Using the first equation of (3.2) we further compute

\[
W_t = \frac{W(\ln c)}{W(\ln c)} - (N - 1) \left[ 1 + \beta Y(\ln c) + \frac{q}{p-1}Z(\ln c) - \frac{q(N-1)}{p-1} \right].
\] (5.16)

Recall that by step 1, we have \( Z > N \) and \( W > N \) so that right hand side of (5.16) is positive. Passing to the limit with \( t \to -\infty \) (note that this implies \( c \to 0 \)) and using \( \lim_{t \to -\infty} Z(t) < Z_\infty \) and \( \lim_{t \to -\infty} Y(t) = 0 \) we find from (5.16) that

\[ L = \lim_{t \to -\infty} W(t) \leq \frac{L}{L - (N - 1)} \left[ 1 + \frac{q}{p-1}Z_\infty - \frac{q(N-1)}{p-1} \right]. \] (5.17)

Hence, by (5.17) and the last equation of (5.7) we find

\[ L \leq N + \frac{q}{p-1}Z_\infty - \frac{q(N-1)}{p-1} \]

\[ < \frac{N(p-1) - q(N-1)}{p-1} + \beta Y_\infty + \frac{q}{p-1}Z_\infty = W_\infty. \]

Thus \( L < W_\infty \) which, in light of the fact that \( W \) is decreasing on \((-\infty, T]\) implies \( W(t) < W_\infty \) for all \( t \in (-\infty, T] \), a contradiction with (5.15).

\textbf{Step 4.} \textit{Conclusion of the proof.}

Using the comparison result in theorem A.2 on each of the intervals \([t_j, \infty)\) we deduce

\[ Y(t) < Y_\infty, \quad Z(t) < Z_\infty \quad \text{and} \quad W(t) < W_\infty \quad \text{for all } t \geq t_j. \] (5.18)

Since \( t_j \to -\infty \) it follows that the estimates in (5.18) hold for all \( t \in \mathbb{R} \) and this together with Step 1 proves (5.9).

\textbf{Proof of theorem 2.3 completed.} Let \((u, v)\) be a non-constant global positive radial solution of (1.1). Denote by \((X, Y, Z, W)\) the solution of (5.2) corresponding to \(u\) and \(v\) as described in (5.1). Then \( \zeta(t) = \begin{bmatrix} Y(t) \\ Z(t) \\ W(t) \end{bmatrix} \) is a solution of (5.4) and (5.5). Thus, by lemma 5.2 we have
\[ P_* := \begin{bmatrix} 0 \\ N + \frac{\alpha}{p - 1 - \alpha} \end{bmatrix} < \zeta(0). \]

By theorem A.8 there exists a set \( \Sigma \subset \mathbb{R}^3 \) of Lebesgue measure zero such that

\[ \omega(\tilde{P}) \subseteq E \quad \text{for all} \quad \tilde{P} \in [P_*, P_\infty] \setminus \Sigma, \quad (5.19) \]

where \( E \) is the set of equilibrium points associated with (5.4) and (5.5). For \( \tilde{P} \in [P_*, P_\infty] \setminus \Sigma \) denote by

\[ \Phi(t, \tilde{P}) = \begin{bmatrix} \tilde{Y}(t) \\ \tilde{Z}(t) \\ \tilde{W}(t) \end{bmatrix} \]

the flow of (5.4) associated with the initial data \( \tilde{P} \). Since \( \tilde{P} \geq P_* \), by the comparison result in theorem A.2 it follows that

\[ \Phi(t, \tilde{P}) \geq \begin{bmatrix} 0 \\ N + \frac{\alpha}{p - 1 - \alpha} \\ 0 \end{bmatrix} \quad \text{for all} \quad t \geq 0. \]

Therefore, the only equilibrium points that \( \omega(\tilde{P}) \) may approach must be non-negative and have the second component greater than or equal to \( N + \frac{\alpha}{p - 1 - \alpha} \). It follows that

\[ \omega(\tilde{P}) \subseteq \{P_1, P_2, P_3, P_\infty\}, \]

where

\[ P_1 = \begin{bmatrix} N + \frac{\alpha}{p - 1 - \alpha} \\ 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} N + \frac{\alpha}{p - 1 - \alpha} \\ \frac{q}{p - 1 - \alpha} \end{bmatrix}, \quad P_3 = \begin{bmatrix} \frac{p - N}{p - 1 - \alpha} \\ \frac{\alpha + m(p - N)}{p - 1 - \alpha} \end{bmatrix}, \quad P_\infty \]

and \( P_\infty \) is given by (5.6). Note, that \( P_3 \) has all components non-negative if and only of \( p \geq N \).

We claim that

\[ \omega(\tilde{P}) = \{P_\infty\} \quad \text{for all} \quad \tilde{P} \in [P_*, P_\infty] \setminus \Sigma. \quad (5.20) \]

First we note that if \( P_\infty \in \omega(\tilde{P}) \) then, since \( P_\infty \) is asymptotically stable, it follows that \( \omega(\tilde{P}) = \{P_\infty\} \). Assume in the following that \( P_\infty \notin \omega(\tilde{P}) \) so \( \omega(\tilde{P}) \subseteq \{P_1, P_2, P_3\} \).

If \( \{P_1, P_2\} \subset \omega(\tilde{P}) \) or \( \{P_2, P_3\} \subset \omega(\tilde{P}) \) then \( \tilde{W} \) converges along a subsequence to 0 and to \( N + \frac{q}{p - 1 - \alpha} \). By the intermediate value theorem we deduce that for all \( 0 < \tau < N + \frac{q}{p - 1 - \alpha} \), there exists a sequence \( t_j \to \infty \) such that \( \tilde{W}(t_j) = \tau \) which contradicts the fact that \( \omega(\tilde{P}) \) is finite. Similarly, if \( \{P_1, P_3\} \subset \omega(\tilde{P}) \) we deduce that \( p > N \) and for all \( 0 < \gamma < \frac{p - N}{p - 1} \) there exists a sequence \( t_j \to \infty \) such that \( \tilde{Y}(t_j) = \gamma \) which is again a contradiction.
It follows that $\omega(\tilde{P})$ is a singleton. Let us show that in this situation we again raise a contradiction. Indeed, if for instance $\omega(\tilde{P}) = \{P_2\}$ then, as $t \to \infty$, we have
\[
\tilde{Y}(t) \to 0, \quad \tilde{Z}(t) \to N + \frac{\alpha}{p - 1 - \alpha} \quad \text{and} \quad \tilde{W}(t) \to N + \frac{q}{p - 1 - \alpha}.
\]
Then, for large $t > 0$ one has
\[
\tilde{Y}_t = \tilde{Y} \left( \frac{p - N}{p - 1} - \frac{1}{p - 1} \tilde{W} \right) > 0
\]
so $\tilde{Y}$ is increasing in a neighbourhood of infinity. It follows that for large $t > 0$ we have $\tilde{Y}(t) \leq \lim_{s \to \infty} \tilde{Y}(s) = 0$, contradiction. Similarly, if $\omega(\tilde{P}) = \{P_1\}$ or if $\omega(\tilde{P}) = \{P_3\}$ we reach a contradiction. Hence, the claim (5.20) holds.

Take now $P \in [P_*, P_\infty] \cap \Sigma$ and let $\tilde{P} \in [P_*, P_\infty] \setminus \Sigma$ be such that $\tilde{P} < P$. By (5.20) and the dichotomy theorem A.5 we have:

- either $\{P_\infty\} = \omega(\tilde{P}) < \omega(P)$;
- or $\omega(P) = \omega(\tilde{P}) = \{P_\infty\}$.

The first alternative cannot hold since by the comparison result in theorem A.2 we have $\omega(P) \leq P_\infty$. It follows that $\omega(P) = \{P_\infty\}$ so,
\[
\omega(P) = \{P_\infty\} \quad \text{for all} \quad P \in [P_*, P_\infty].
\]
In particular, for $P = \zeta(0)$ we find
\[
\omega(\zeta(0)) = \{P_\infty\}.
\]
Thus, as $t \to \infty$ we have
\[
Y(t) \to Y_\infty, \quad Z(t) \to Z_\infty, \quad W(t) \to W_\infty.
\]

By (5.3) there exists
\[
X_\infty := \lim_{t \to \infty} X(t) = \frac{1}{p - 1} Z_\infty + \frac{p - N}{p - 1}.
\]
Observe that
\[
\frac{u(r)}{r^{p(m+1)-(1+\beta)+\delta}} = \frac{1}{Y^{m(p-1)}(t)Z^{p-1-\beta}(t)W^m(t)}X^{\delta}(t)
\]
for all $r > 0$, so
\[
\lim_{r \to \infty} \frac{u(r)}{r^{1 + \frac{2\delta(1+\beta)}{4}}} = A,
\]
where

\[
A = \frac{1}{Y^{m(p-1)}(t)Z^{p-1-\beta}(t)W^m(t)}X^{\delta}(t).
\]
\[ A = \frac{1}{Y_\infty^{\frac{m(p-1)}{p-1-\sigma}} Z_\infty^{\frac{q}{p-1-\sigma}}} \in (0, \infty). \]

Similarly, we find
\[ \lim_{r \to \infty} \frac{v(r)}{r^{(p-1-\alpha)+\alpha}} = B, \]

where
\[ B = \frac{1}{Y_\infty^{\frac{m(p-1)}{p-1-\sigma}} Z_\infty^{\frac{q}{p-1-\sigma}}} \in (0, \infty). \]

### 6. Proof of theorem 2.4

The existence of a non-constant global positive radial solution \((u, v)\) of (1.1) with \(u(0) = a > 0\) and \(v(0) = b > 0\) follows from theorem 2.1. First, there exists a non-constant local positive radial solution \((u, v)\) as above as a fixed point of the mapping given by (3.15)–(3.17). In light of theorem 2.1 (ii), such a solution must be global. We focus in the following on the uniqueness part.

For any non-constant positive global solution \((u, v)\) of system (1.1) we denote
\[ u(r) = U(t), \quad v(r) = V(t) \quad \text{where } r = t^\theta, \quad \theta = -\frac{p-1}{N-p} < 0. \] (6.1)

Then \((U, V)\) satisfies (throughout this section \(t\) denotes the derivative with respect to \(t\) variable)
\[
\begin{cases}
|U'(t)|^{p-\alpha-2}U''(t) &= \frac{p-1-\alpha}{p-1} \theta |\theta|^{p-1} |U^{\beta}(t)|^{\beta} U'(t) \\
|V'(t)|^{p-2}V''(t) &= |\theta|^{p-1} |V^{\beta}(t)|^{\beta} V'(t) \\
U'(t) &< 0, \quad V'(t) < 0, \quad U(t) > 0, \quad V(t) > 0 \quad \text{for all } t > 0,
\end{cases}
\]

(6.2)

Letting \(W(t) = |U'(t)|^{p-\alpha-2}U''(t)\) we transform (6.2) into
\[
\begin{cases}
W'(t) &= \frac{p-1-\alpha}{p-1} \theta |\theta|^{p-1} |W^{\beta}(t)|^{\beta} \\
|V'(t)|^{p-2}V''(t) &= |\theta|^{p-1} |V^{\beta}(t)|^{\beta} |W(t)|^{\beta} \\
W(t) &< 0, \quad V'(t) < 0, \quad V(t) > 0 \quad \text{for all } t > 0,
\end{cases}
\]

(6.3)

Let now \(a, b > 0\) and \((u, v), (\tilde{u}, \tilde{v})\) be two pairs of non-constant global positive radial solutions of (1.1) with \(u(0) = \tilde{u}(0) = a\) and \(v(0) = \tilde{v}(0) = b\). We want to show that \(u \equiv \tilde{u}\) and \(v \equiv \tilde{v}\).

Let \(\epsilon > 0\) and set
\[ \tilde{u}(r) = (1 + \epsilon)u(r), \quad \tilde{v}(r) = (1 + \epsilon)\frac{\epsilon^{\frac{p-1-\alpha}{p-1}}}{r^{\frac{p-1-\alpha}{p-1}}} v(r). \]

It follows that (with \(r = t^\theta\) from (6.1))
\[ \dot{U}(t) = \ddot{u}(r), \quad \dot{V}(t) = \ddot{v}(r) \text{ and } \dot{W}(t) = |\dot{U}'(t)|^{p-\alpha-2} \dot{U}'(t) \] 

satisfy

\[
\begin{align*}
\dot{W}'(t) &= \frac{p-1}{p-\alpha-1} |\theta| \rho(t^{\alpha-1}(p-\alpha)) \ddot{W}(t) \quad \text{for all } t > 0, \\
\left[ |\dot{V}'(t)|^{p-2} \dot{V}'(t) \right]' &= (1 + \epsilon)^2 |\theta|^{p-\alpha q(t^{\alpha-1})(p-\alpha)} \ddot{W}(t) |\dot{W}(t)|^{p-\alpha-2} \\ 
\ddot{W}(t) &= 0, \quad \dot{V}(t) < 0, \quad \dot{V}'(t) > 0 \\ 
\ddot{W}(\infty) &= 0, \quad \dot{V}(\infty) = 0, \quad \dot{V}'(\infty) = (1 + \epsilon)^{2-\alpha-2} v(0).
\end{align*}
\]

Through the same change of variable \( r = t^\beta \) given by (6.1), the functions

\[ \dot{U}(t) = \ddot{u}(r), \quad \dot{V}(t) = \ddot{v}(r) \text{ and } \dot{W}(t) = |\dot{U}'(t)|^{p-\alpha-2} \dot{U}'(t), \]

satisfy

\[
\begin{align*}
\dot{W}'(t) &= \frac{p-1}{p-\alpha-1} |\theta| \rho(t^{\alpha-1}(p-\alpha)) \ddot{W}(t) \quad \text{for all } t > 0, \\
\left[ |\dot{V}'(t)|^{p-2} \dot{V}'(t) \right]' &= |\theta|^{p-\alpha q(t^{\alpha-1})(p-\alpha)} \ddot{W}(t) |\dot{W}(t)|^{p-\alpha-2} \\ 
\ddot{W}(t) &= 0, \quad \dot{V}(t) < 0, \quad \dot{V}'(t) > 0 \\ 
\ddot{W}(\infty) &= 0, \quad \dot{V}(\infty) = 0, \quad \dot{V}'(\infty) = \ddot{W}(0) = h.
\end{align*}
\]

Since, \( \dot{V}(\infty) > \ddot{V}(\infty) \) it follows from the first equation of (6.5) and (6.7) that the set

\[ A := \{ t > 0 : \quad \dot{W} > \ddot{W} \text{ on } (t, \infty) \} \]

is nonempty. We claim that \( A = (0, \infty) \). Assuming the contrary, one has

\[ t_0 = \inf A > 0 \]

together with

\[ \ddot{W}'(t) > \ddot{W}(t) \quad \text{for all } t \in (t_0, \infty) \quad \text{and} \quad \ddot{W}'(t_0) = \ddot{W}(t_0). \]

Using the first equation in (6.5) and (6.7) it follows that

\[ \dot{V}(t) > \ddot{V}(t) \quad \text{for all } t \in (t_0, \infty) \quad \text{and} \quad \dot{V}(t_0) = \ddot{V}(t_0). \]

Integrating (6.8) we find (since \( \ddot{W}(\infty) = \ddot{W}(\infty) = 0 \)) that

\[ |\ddot{W}(t)| = -\ddot{W}(t) > -\ddot{W}(t) = |\ddot{W}(t)| \quad \text{for all } t \in (t_0, \infty). \]

Hence, from second equation of (6.5), (6.7) and from (6.9), (6.10) we deduce

\[ \left[ |\dot{V}'(t)|^{p-2} \dot{V}'(t) \right]' > \left[ |\dot{V}'(t)|^{p-2} \dot{V}'(t) \right]' \quad \text{for all } t \in (t_0, \infty). \]

An integration over \([t, \infty]\) in the above inequality yields

\[ |\dot{V}'(t)|^{p-1} = -|\dot{V}'(t)|^{p-2} \dot{V}'(t) > -|\dot{V}'(t)|^{p-2} \dot{V}'(t) = |\dot{V}'(t)|^{p-1} \quad \text{for all } t \in (t_0, \infty). \]

This implies

\[ -\dot{V}(t) > -\dot{V}(t) \quad \text{for all } t \in (t_0, \infty). \]

Integrating now over \([t_0, \infty]\) and using \( \ddot{V}(\infty) > \ddot{V}(\infty) \) we obtain
\[ \tilde{V}(t_0) > \tilde{V}(t_0) + (\tilde{V}(\infty) - \tilde{V}(\infty)) > \tilde{V}(t_0), \]

which contradicts (6.9). Hence \( A = (0, \infty) \) which shows that \( \tilde{W}'(t) > \tilde{W}'(t) \) for all \( t \in (0, \infty) \).

Integrating over \([t, \infty] \) we have
\[
|\tilde{W}(t)| = -\tilde{W}(t) > -\tilde{W}(t) = |\tilde{W}(t)| \quad \text{for all } t \in (0, \infty).
\]

Using this estimate and the expression of \( \hat{W} \) and \( \hat{W} \) in (6.4) and (6.6) respectively we find
\[
-\hat{U}'(t) = |\hat{U}'(t)| > |\hat{U}'(t)| = -\hat{U}'(t) \quad \text{for all } t \in (0, \infty).
\]

A further integration over \([t, \infty] \) yields
\[
\hat{U}(t) > \hat{U}(t) + \hat{U}(\infty) > \hat{U}(t) \quad \text{for all } t \in (0, \infty).
\]

This implies
\[
\tilde{u}(r) = (1 + \epsilon) u(r) > \tilde{u}(r) \quad \text{for all } r > 0.
\]

Passing to the limit with \( \epsilon \to 0 \) we find \( u \geq \tilde{u} \) in \( (0, \infty) \). Also, \( \tilde{W}' > \tilde{W}' \) in \( (0, \infty) \) together with (6.5) and (6.7) yield \( V > \tilde{V} \) in \( (0, \infty) \). So,
\[
(1 + \epsilon) \frac{\epsilon - 1}{\alpha} v(r) > \tilde{v}(r) \quad \text{for all } r > 0.
\]

This also entails (by letting \( \epsilon \to 0 \)) that \( v \geq \tilde{v} \) in \( (0, \infty) \). Now, we can replace \( u \) by \( \tilde{u}, v \) by \( \tilde{v} \) to deduce
\[
\tilde{u} \geq u, \quad \tilde{v} \geq v \quad \text{in } (0, \infty).
\]

Thus, \( u \equiv \tilde{u} \) and \( v \equiv \tilde{v} \). This concludes the proof.

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**Appendix. Some results for cooperative dynamical systems**

We recall here some results on dynamical systems that we used in the current work.

For any vectors \( x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3 \) we let
\[
\begin{align*}
x \leq y & \quad \text{if } x_i \leq y_i, \quad i = 1, 2, 3, \\
x < y & \quad \text{if } x_i < y_i, \quad i = 1, 2, 3.
\end{align*}
\]

We also define the the closed interval \([x, y] = \{ u \in \mathbb{R}^3 : x \leq u \leq y \} \) and the open interval \([x, y) = \{ u \in \mathbb{R}^3 : x < u \leq y \} \) with endpoints at \( x \) and \( y \).

A set \( X \subset \mathbb{R}^3 \) is said to be \( p \)-convex if for any \( x, y \in X \), the segment line joining \( x \) and \( y \) is a subset of \( X \). Throughout this section \( X \) will be an open \( p \)-convex subset of \( \mathbb{R}^3 \).

Let \( g : X \to \mathbb{R}^3 \) be a \( C^1 \)-vector field. For any \( P \in \mathbb{R}^3 \) we denote by \( \Phi(t, P) \) the maximally defined solution of the differential equation
\[
\frac{d\zeta}{dt} = g(\zeta)
\]  
(A.1)
subject to the initial condition \( \zeta(0) = P \). The collection of maps \( \{ \Phi(t, \cdot) \} \) is called the flow of the differential equation (A.1).

**Definition A.1.** A \( C^1 \)-vector field \( g : X \to \mathbb{R}^3 \) is said to be cooperative if at any point \( P \in X \) we have

\[
\frac{\partial g_i}{\partial x_j}(P) \geq 0 \quad \text{for any } i, j = 1, 2, 3, \ i \neq j.
\]

Cooperative systems enjoy a comparison property of the flows as stated below.

**Theorem A.2 (See [15]).** Assume the \( C^1 \)-vector field \( g : X \to \mathbb{R}^3 \) is cooperative and let \( \zeta, \xi : [0, a] \to \mathbb{R}, \ a > 0, \) be two solutions of (A.1) such that

\[
\zeta(0) < \xi(0) \quad \text{(resp. } \zeta(0) \leq \xi(0))\,.
\]

Then

\[
\zeta(t) < \xi(t) \quad \text{(resp. } \zeta(t) \leq \xi(t)) \quad \text{for all } t \in [0, a].
\]

**Definition A.3.** The equilibrium set of (A.1) is the set \( E \) of points \( P \in X \) such that \( g(P) = 0 \). Any such element is called an equilibrium point of (A.1). Obviously, \( \Phi(t, P) = P \) for any equilibrium point \( P \).

**Definition A.4.** Let \( P \in X \). The \( \omega \)-limit set \( \omega(P) \) is defined as the set of all points \( Q \in \mathbb{R}^3 \) such that there exists \( \{ t_j \}, \ t_j \to \infty \) (as \( j \to \infty \)) such that \( \Phi(t_j, P) \to Q \) (as \( j \to \infty \)).

The following dichotomy result obtained in [15] essentially states that the omega limit sets preserve the partial order between the elements of \( X \) or approach the equilibrium set \( E \).

**Theorem A.5 (Limit set dichotomy, see [15, theorem 3.8] [16, theorem 1.16]).** Assume the \( C^1 \)-vector field \( g : X \to \mathbb{R}^3 \) is cooperative and let \( P, Q \in X, \ P < Q \). Then the following alternative holds:

(i) either \( \omega(P) < \omega(Q) \);

(ii) or \( \omega(P) = \omega(Q) \subset E \).

**Definition A.6.** A \( C^1 \)-vector field \( g : X \to \mathbb{R}^3 \) is said to be irreducible if at any point \( P \in X \) its gradient \( \nabla g(P) \) is an irreducible matrix.

**Remark A.7.** Recall that a general \( n \times n \) matrix \( M \) is irreducible if one of the following equivalent conditions holds:

(i) for any nontrivial partition \( I \cup J \) of the set \( \{1, 2, \ldots, n\} \) there exists \( i \in I, \ j \in J \) such that \( M_{ij} \neq 0 \);

(ii) the digraph associated with \( M \), that is, the oriented graph with vertices at \( 1, 2, \ldots, n \) which connects \( (i, j) \) if and only if \( M_{ij} \neq 0 \), is strongly connected.

The compact omega limit sets of cooperative and irreducible vector fields have a particular property in the sense that they approach the equilibrium set for almost all points in \( X \). This is formulated in the result below.
Theorem A.8 (See [15, theorem 4.1]). Assume the $C^1$-vector field $g : X \to \mathbb{R}^3$ is cooperative and irreducible and that for all $P \in X$ the $\omega$-limit set $\omega(P)$ is compact. Then, there exists $\Sigma \subset X$ with zero Lebesgue measure such that

$$\omega(P) \subset E \quad \text{for all} \quad P \in X \setminus \Sigma.$$ 

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