Symmetry in noncommutative quantum mechanics

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Abstract

We reconsider the generalization of standard quantum mechanics in which the position operators do not commute. We argue that the standard formalism found in the literature leads to theories that do not share the symmetries present in the corresponding commutative system. We propose a general prescription to specify a Hamiltonian in the noncommutative theory that preserves the existing symmetries. We show that it is always possible to choose this Hamiltonian in such a way that the energy spectrum of the standard and non-commuting theories are identical, so that experimental differences between the predictions of both theories are to be found only at the level of the detailed structure of the energy eigenstates.

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PACS: 02.40.Gh, 03.65.Ca, .
Keywords: Noncommutative quantum mechanics, gauge symmetry, Landau problem.

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1 Introduction

A lot of attention has been given recently to the formulation and possible experimental consequences of extensions of the standard formalism of quantum mechanics to allow for non-commuting position operators \[4–14\]. Much of this work has been inspired by ideas coming from the realms of string theory \[1\] and quantum field theory (for a review, see \[2\]). The standard way of constructing a noncommutative field theory is to replace the usual product of fields (appearing in the action in the functional integral) by the so called Moyal or star product, defined by

\[
(\phi_1 \star \phi_2)(x) = \exp \left( i\theta_{\mu\nu} \partial_x^\mu \partial_y^\nu \right) \phi_1(x) \phi_2(y) |_{x=y},
\]

where \(\theta_{\mu\nu}\) is an antisymmetric constant matrix. As \(\theta_{0i} \neq 0\) leads to a non-unitary theory \[3\], only the elements \(\theta_{ij}\), \(i, j = 1, 2, 3\), are allowed to be non-vanishing. This leads to the Moyal commutation relations between the spatial coordinate fields

\[
x_i \star x_j - x_j \star x_i = i\theta_{ij}.
\]

In order not to spoil the isotropy of space it is mandatory to choose \(\theta_{ij}\) proportional to the constant antisymmetric matrix \(\epsilon\), \(\theta_{ij} = \theta\epsilon_{ij}\), where \(\theta\), the noncommutativity parameter, is a constant with dimension of \((\text{length})^2\) and

\[
\epsilon = \begin{pmatrix}
0 & -1 & 1 \\
1 & 0 & -1 \\
-1 & 1 & 0
\end{pmatrix}.
\]

Inspired by this formalism, many authors have considered an extension of non-relativistic quantum mechanics, usually referred to in the literature as noncommutative quantum mechanics (NCQM). This extended theory is formulated in the same terms as the standard theory (SQM), that is, in terms of the same dynamical variables represented by operators in a Hilbert space and a state vector that evolves according to the Schrödinger equation,

\[
i\hbar \frac{d}{dt} |\psi\rangle = H_{nc} |\psi\rangle
\]

where \(H_{nc}\) is the Hamiltonian for a given system in the noncommutative theory. The crucial difference with the standard theory is that in the extended theory the operators representing the position of a particle, \(X_i\), are no longer assumed to commute among themselves, but instead the following non-canonical commutation relations are postulated:

\[
[X_i, X_j] = i\theta_{ij}, \quad [X_i, P_j] = i\hbar \delta_{ij}, \quad [P_i, P_j] = 0.
\]

with \(\theta_{ij} = \theta\epsilon_{ij}\) as before. To completely specify a particular NCQM system it is necessary to define the Hamiltonian \(H_{nc}\), which from here on we shall
simply denote by $H_\theta \equiv H_{nc}$. This Hamiltonian $H_\theta$ is to be chosen such that it reduces to the Hamiltonian $H$ for the standard theory in the limit $\theta \to 0$. Two approaches are found in the literature: (a) simply take $H_\theta = H$, so that the only difference between SQM and NCQM is the presence of a nonzero $\theta$ in the commutator of the position operators [4, 9, 11, 14]; or (b) naively derive the Hamiltonian from the Moyal analog of the standard Schrödinger wave equation, namely

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = H(p = \hbar \frac{\partial}{\partial x}) \ast \psi(x, t) \equiv H_\theta \psi(x, t), \quad (6)$$

where $H(p, x)$ is the same Hamiltonian as in the standard theory, so that the $\theta$-dependence enters now solely through the star product in the equation above [7, 8, 10]. In [10] it has been shown that, for a Hamiltonian of the type

$$H(p, x) = \frac{p^2}{2m} + V(x), \quad (7)$$

describing a non-relativistic particle moving in an external potential, the modified Hamiltonian $H_\theta$ can be simply obtained by a shift in the argument of the potential, namely,

$$H_\theta = \frac{p^2}{2m} + V(x_i - \frac{1}{2\hbar} \theta_{ij} p_j). \quad (8)$$

where $x, p$ are now canonical variables.

As we shall show in the next section, these two approaches actually lead to the same physical theories. However, these theories have to be taken with suspicion. In fact, claims have been presented in the recent literature [5] that a rigorous derivation of noncommutative quantum mechanics from noncommutative field theory does not lead to the simple Moyal Schrödinger equation (6). For instance, it is found in [5] that, at tree level in NCQED, there are no noncommutative corrections to the hydrogen atom spectrum, contradicting therefore the main conclusion of reference [14].

In this paper we will argue that the theories obtained by the approach just described are flawed in a more fundamental way: in general, they do not share the symmetries possessed by their commutative counterpart. We shall show nevertheless that there is natural way of making the transition to the noncommutative quantum theory, without spoiling the symmetries (such as rotational or gauge invariance) of the particular system being considered. Although the prescription to obtain the form of the deformed Hamiltonian $H_\theta$ corresponding to a given $H$ does not uniquely define $H_\theta$, it can be considered minimal in a sense discussed later on. We will see that this prescription will lead to dramatically different consequences than the standard approach, the most conspicuous one being that the energy spectrum of the standard and non-commuting theories are identical, so that experimental differences between both theories are to
be found only at the level of the detailed structure of the energy eigenstates, as manifested for instance through transition rates or expectation values involving the physical position operator.

2 Rotational symmetry in NCQM

In reference [14] it has been shown that the phase-space dynamical variables $X_i, P_j$ can be expressed linearly in terms of canonically commuting variables $x_i, p_j$ as

$$X_i = x_i - \frac{\theta}{2\hbar} \varepsilon_{ij} p_j, \quad P_i = p_i,$$

with the inverse transformation given by

$$x_i = X_i + \frac{\theta}{2\hbar} \varepsilon_{ij} P_j, \quad p_i = P_i.$$

We would like to emphasize that throughout this paper the capitalized symbols $X_i, P_j$ will always denote the physical position and momentum operators, both in SQM ($\theta = 0$) and NCQM ($\theta \neq 0$). On the other hand, the lowercase symbols $x_i, p_j$ will denote the canonically commuting auxiliary variables defined by (10). The momentum operators $p_i$ and $P_i$ can be interchanged, but $x_i$ coincides with the physical position operator $X_i$ only when $\theta = 0$.

From (9) it is clear that approaches (a) and (b), described in the introduction, lead to identical theories, since

$$\frac{P^2}{2m} + V(x_i - \frac{1}{2\hbar} \theta_{ij} p_j) = \frac{P^2}{2m} + V(X_i).$$

Let us now consider rotational invariance. Due to the non-commuting nature of the position operators $X_i$, the canonical operators $L_i = \varepsilon_{ijk} X_j P_k$, $i = 1, 2, 3$, do no longer satisfy the angular momentum algebra, $[L_i, L_j] = i \varepsilon_{ijk} L_k$. However, since the operators $x_i, p_j$ are canonically conjugated, it is immediate that the operators

$$J_i = \varepsilon_{ijk} x_j p_k = L_i + \frac{\theta}{2\hbar} (P_i P - P^2),$$

where $P \equiv P_1 + P_2 + P_3$ and $P^2 \equiv P_1^2 + P_2^2 + P_3^2$, do and therefore form a representation of the algebra of the rotation group.

Therefore, the NCQM theory defined by a Hamiltonian $H_{\theta}$ will describe a rotationally invariant system only if each of the generators $J_i$ commutes with $H_{\theta}$. This is clearly not the case if $H_{\theta}$ is of the type (11), even for a central potential.
\(V(R^2)\), with \(R^2 \equiv X_i X_i\), since

\[
R^2 = \left( x_i - \frac{\theta}{2\hbar} \varepsilon_{ij} p_j \right) \left( x_i - \frac{\theta}{2\hbar} \varepsilon_{ik} p_k \right)
= r^2 + \frac{\theta}{2\hbar} (J_1 + J_2 + J_3) + \frac{\theta^2}{2\hbar^2} \left( p_1^2 p_2^2 - p_1 p_2 - p_2 p_3 - p_3 p_1 \right),\]

(13)

and \([R^2, J_i] \neq 0\) for \(\theta \neq 0\).

Now, in terms of the auxiliary canonical variables \(x_i, p_j\), the construction of rotationally invariant operators proceeds as usual. For instance, the Hamiltonian

\[
H_\theta = \frac{p^2}{2m} + V(x^2)
\]

(14)

clearly commutes with each \(J_i\) and reduces to

\[
H = \frac{p^2}{2m} + V(X^2)
\]

(15)

in the limit \(\theta \to 0\). We see that the spectra of both Hamiltonians are identical, since these are completely determined by the commutation relations between the position and momentum variables. Does this mean that both theories, on the one hand the standard theory based on the Hamiltonian (15), and on the other the deformed theory based on the non-canonical commutation relations (5) and the Hamiltonian (14), are unitarily equivalent? It does not, since the coordinate transformation

\[
S^{-1} X_i S = x_i, \quad S^{-1} P_i S = p_i
\]

(16)

is not unitary, as it does not preserve the commutators (5) which evaluate to c-numbers.

Clearly the Hamiltonian (14) is not the most general rotationally invariant operator reducing to the standard SQM Hamiltonian (13) in the limit \(\theta \to 0\). But it can certainly considered to be “minimal” in its class.

As a particular example we consider the NCQM of the isotropic three-dimensional harmonic oscillator, whose dynamics we postulate to be governed by the rotationally invariant Hamiltonian

\[
H_\theta = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 r^2.
\]

(17)

The well-known spectrum of \(H_\theta\),

\[
E = \hbar \omega (n_1 + n_2 + n_3 + 3/2),
\]

(18)

has no dependence on \(\theta\) and is identical to the commutative case. We note that many authors [4, 9, 11, 12], have obtained \(\theta\)-dependent spectra for this system;
this is not contradictory with our result: it merely reflects the fact that these authors assume $H_\theta(X,P) = H_0(X,P)$, where $H_0$ is exactly the same as the Hamiltonian defining the commutative system. This choice, however, does not lead to a rotationally invariant theory, as we have pointed out.

$\theta$-dependent differences between both theories will arise when we consider, for instance, the expectation value of $R^2$, which measures the spatial width of the quantum state. With $R^2$ given by (13) and using, for the standard harmonic oscillator,

$$\langle r^2 \rangle = \frac{E}{m\omega^2}, \quad \langle p^2 \rangle = mE,$$

(19)

together with $\langle L_i \rangle = \langle p_i \rangle = 0$ and $\langle p_ip_j \rangle = 0$ for $i \neq j$, we find for the expectation value of $R^2$ in a energy eigenstate of energy $E$

$$\langle R^2 \rangle = \frac{E}{m\omega^2} \left( 1 + \frac{m^2 \omega^2 \theta^2}{2\hbar^2} \right).$$

(20)

This result can also be written as

$$\langle R^2 \rangle = \langle R^2 \rangle|_{\theta=0} \left[ 1 + \frac{1}{2} \left( \theta/l_\omega^2 \right)^2 \right],$$

(21)

where $l_\omega = \sqrt{\hbar/m\omega}$ is the oscillator length of the standard harmonic oscillator.

### 3 The Landau electron in SQM

In this section we review the standard quantum mechanical treatment of a non-relativistic electron moving in the background of a uniform external magnetic field, with special emphasis on the symmetries of the system. In SQM, such a system is described by the Hamiltonian

$$H = \frac{1}{2m} \left( \mathbf{P} - e\mathbf{A}(\mathbf{R}) \right)^2,$$

(22)

where the physical variables $\mathbf{R}, \mathbf{P}$ are canonically commuting (i.e. $\theta = 0$) and $\mathbf{A}(\mathbf{r})$ is some vector potential describing the uniform magnetic field, which we take to point in the $z$-direction, and we have for simplicity omitted Pauli’s spin term, since it is unchanged in the transition to the noncommutative theory.

Perhaps the most fundamental property of the Hamiltonian (22) is that the resulting theory is gauge invariant, that is, the physical consequences of the theory are unchanged by a gauge transformation of the vector potential,

$$\mathbf{A}(\mathbf{r}) \rightarrow \mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla \chi(\mathbf{r}).$$

(23)
(Here and throughout this section lowercase position variables such as \( \mathbf{r}, x, y, \) etc., will denote classical c-number variables and not operators.) This invariance can be explicitly represented in terms of a unitary operator
\[
T_\chi = \exp \left[ \frac{i}{\hbar} \chi (\mathbf{R}) \right],
\]
under which the fundamental dynamical variables of the theory transform as
\[
T_\chi^\dagger \mathbf{R} T_\chi = \mathbf{R},
\]
\[
T_\chi^\dagger \mathbf{P} T_\chi = \mathbf{P} + e \nabla \chi (\mathbf{R}),
\]
so that
\[
T_\chi^\dagger H [\mathbf{A}] T_\chi = H [\mathbf{A}'].
\]
Since the operator \( T_\chi \) is unitary, the transformation property (27) implies that the energy spectrum of the theory is gauge invariant.

Gauge invariance also plays a crucial role in establishing that the theory described by the Landau Hamiltonian (22) has the space symmetries of the background magnetic field, that is, translational symmetry and rotational symmetry around the magnetic field direction. Let us consider first translational symmetry. This symmetry is apparently broken once we make a particular gauge choice, for instance, the symmetric gauge,
\[
\mathbf{A} (\mathbf{r}) = -\frac{1}{2} \mathbf{r} \times \mathbf{B} = \frac{B}{2} (-y, x, 0),
\]
or Landau gauge,
\[
\mathbf{A} (\mathbf{r}) = B (-y, 0, 0).
\]
Clearly, the Hamiltonian (22) no longer commutes with all three generators of infinitesimal translations, \( P_x, P_y, P_z \). In fact,
\[
e^{-i \mathbf{P} \cdot \mathbf{a} / \hbar} H [\mathbf{A}(\mathbf{r})] e^{i \mathbf{P} \cdot \mathbf{a} / \hbar} = H [\mathbf{A}(\mathbf{r} + \mathbf{a})].
\]
However, in view of the gauge invariance (27), it is not actually necessary that the Hamiltonian \( H \) be unchanged under the translation \( \mathbf{r} \rightarrow \mathbf{r} + \mathbf{a} \); it is enough that the potential \( \mathbf{A}(\mathbf{r} + \mathbf{a}) \) be a gauge transform of the original potential \( \mathbf{A}(\mathbf{r}) \). But this is just the case if \( \mathbf{B} = \nabla \times \mathbf{A} \) is a uniform magnetic field, as can be seen by performing a Taylor expansion around \( \mathbf{r} \),
\[
A_i (\mathbf{r} + \mathbf{a}) = A_i (\mathbf{r}) + \frac{\partial A_i}{\partial x_j} (\mathbf{r}) a_j + \frac{1}{2} \frac{\partial^2 A_i}{\partial x_k \partial x_j} (\mathbf{r}) a_k a_j + \ldots,
\]
and using
\[
\frac{\partial A_i}{\partial x_j} = \frac{\partial A_j}{\partial x_i} - \epsilon_{ijk} B_k,
\]
so that
\[ \frac{\partial^{n+1} A_i}{\partial x_{k_0} \cdots \partial x_{k_n} \partial x_j} = \frac{\partial}{\partial x_i} \frac{\partial^n A_j}{\partial x_{k_0} \cdots \partial x_{k_n}} , \]
we indeed find that
\[ A(r + a) = A(r) + \nabla \chi_a(r), \]
where
\[ \chi_a(r) = a \cdot r \times B + a \cdot A + \frac{1}{2!} a_k \frac{\partial (a \cdot A)}{\partial x_k} + \frac{1}{3!} a_k a_l \frac{\partial (a \cdot A)}{\partial x_k \partial x_l} + \cdots . \]

An alternative way of looking at the same problem is to actually enforce the invariance of the Hamiltonian, but under a set of generalized translation generators, given by
\[ P_A = P + \epsilon (A(R) + R \times B) , \]
which also represent the algebra of the Euclidean translation group,
\[ [P_{A_i}, P_{A_j}] = 0 . \]

We note that under the gauge transformation (25) the generalized translation generators transform covariantly, that is,
\[ T_\chi P_A T_\chi = P_{A'} , \]
where \( A' = A + \nabla \chi. \)

The latter point of view can be also adopted in considering rotational invariance. We define a set of generalized rotation generators,
\[ L_A = L + \epsilon (A(R) + \frac{1}{2} R \times B) \]
which transform covariantly under gauge transformations and satisfy the canonical commutation relations
\[ [L_{A_i}, L_{A_j}] = i \hbar \varepsilon_{ijk} L_{A_k} , \]
provided the magnetic field \( B \) is uniform. Then the Hamiltonian (24) describing the motion of an electron in a uniform magnetic field \( B = B\hat{z} \) commutes with \( L_{A_3} , \)
\[ [H, L_{A_3}] = 0 . \]

This can be explicitly demonstrated, for instance, in the symmetric gauge (23), where \( L_A \) reduces to the canonical \( L = R \times P \), and \( L_z \) indeed commutes with the Hamiltonian, since in this gauge one has
\[ H = \frac{1}{2m} \left[ P^2 + \left( \frac{eB}{2} \right)^2 R^2 - eBL_z \right] . \]
4 The Landau electron in NCQM

All the symmetries described above for the motion of an electron in the background of a uniform magnetic field will be preserved in the noncommutative theory if we choose the Hamiltonian of the same form as in the standard theory, but with the physical variables \( R \) and \( P \) replaced by the auxiliary canonical variables \( r \) and \( p \):

\[
H_\theta = \frac{1}{2m} (p - eA(r))^2
\]

(43)

\[
= \frac{1}{2m} \left( P - e \left( R + \frac{\theta}{2\hbar} \tilde{P} \right) \right)^2 ,
\]

(44)

where we have used the customary notation \( \tilde{P}_i \equiv \varepsilon_{ij} P_j \). The spectrum of \( H_\theta \) is exactly the same as that of the original Hamiltonian (22) defining the standard theory, that is, the well known Landau level spectrum

\[
E(p_z, n) = \frac{p_z^2}{2m} + \hbar \omega_B (n + \frac{1}{2}) ,
\]

(45)

where \( \omega_B = eB/m \) is the electron cyclotron frequency. However, as in the case of the harmonic oscillator studied in section 2, the precise shape of the stationary quantum states is different in both theories, a fact that is exemplified, for instance, by computing the radius of the orbit corresponding to a given Landau level. To show this, we will use the formalism developed in [15] to compute the expectation value of

\[
\rho_x = x - \xi_x = \frac{1}{2} \sqrt{\frac{\ell_B^2}{\hbar}} p_y - \frac{\ell_B^2}{\hbar} \left( 1 - \frac{\theta}{2\ell_B^2} \right) p_y
\]

(47)

\[
\rho_y = y - \xi_y = \frac{1}{2} \sqrt{\frac{\ell_B^2}{\hbar}} (1 - \frac{\theta}{2\ell_B^2}) p_x
\]

(48)

where we have set \( p_z = 0 \) for simplicity. As shown in [15], the canonical operators \( x, y, p_x, p_y \) can be written in terms of two pairs of independent harmonic oscillator operators, \((a, a^\dagger)\) and \((b, b^\dagger)\), so that \( \rho_x, \rho_y \) take the form

\[
\rho_x = \frac{\ell_B}{\sqrt{2}} \left( 1 - \frac{\theta}{4\ell_B^2} \right) i(a - a^\dagger) + \frac{\theta}{4\sqrt{2}\ell_B} (b + b^\dagger)
\]

(49)

\[
\rho_y = \frac{\ell_B}{\sqrt{2}} \left( 1 - \frac{\theta}{4\ell_B^2} \right) (a + a^\dagger) + \frac{\theta}{4\sqrt{2}\ell_B} i(b - b^\dagger)
\]

(50)
and hence
\[ \rho_x^2 + \rho_y^2 = \ell_B^2 \left( 1 - \frac{\theta}{4\ell_B^2} \right)^2 (1 + 2a^\dagger a) + \frac{\theta^2}{16\ell_B^2} (1 + 2b^\dagger b) \]
\[ + \frac{\theta}{2} \left( 1 - \frac{\theta}{4\ell_B^2} \right) i(ab - a^\dagger b^\dagger). \quad (51) \]

In terms of the operators \( a, a^\dagger, b, b^\dagger \) the Hamiltonian and the angular momentum operator \( l_z \) read
\[ H = \hbar \omega_B (a^\dagger a + \frac{1}{2}) \quad \text{and} \quad l_z = \hbar (b^\dagger b - a^\dagger a), \quad (52) \]
so that in a state with definite numbers of \( a \)- and \( b \)-quanta, \( n \) and \( n' \), respectively, such that the energy is \( E = \hbar \omega_B (n + 1/2) \) and the angular momentum is \( l_z = \hbar l \), we have
\[ \langle \rho_x^2 + \rho_y^2 \rangle_{n,l} = \ell_B^2 (2n + 1) \left[ 1 - \frac{\theta}{2\ell_B^2} + \frac{\theta^2}{8\ell_B^2} \left( 1 + \frac{l}{2n + 1} \right) \right] \quad (53) \]

5 Gauge invariance in NCQM

The arguments of section (3) can be repeated line by line to show that the Hamiltonian \( H_\theta (43) \) enjoys the property
\[ T^\dagger_\theta \chi H_\theta [A] T_{\theta \chi} = H_\theta [A'], \quad (54) \]
where \( A' = A + \nabla \chi \) as before, but now
\[ T_{\theta \chi} = \exp \left[ \frac{e}{\hbar} \chi (r) \right] = \exp \left[ i \frac{e}{\hbar} \chi \left( R + \frac{\theta}{2\hbar} \hat{P} \right) \right]. \quad (55) \]

In terms of the physical variables \( R \) and \( P \), the corresponding symmetry operation can be written as
\[ T^\dagger_{\theta \chi} X_i T_{\theta \chi} = X_i - \frac{e}{2\hbar} \varepsilon_{ij} \partial_j \chi \left( R + \frac{\theta}{2\hbar} \hat{P} \right) \quad (56) \]
\[ T^\dagger_{\theta \chi} P_i T_{\theta \chi} = P_i + e \partial_i \chi \left( R + \frac{\theta}{2\hbar} \hat{P} \right). \quad (57) \]

It is clear that this transformation reduces to the standard one (22) in the limit \( \theta \to 0 \). We note in passing that it is impossible to maintain the standard transformation (23) in the noncommutative theory, since the canonical commutator \( [X_i, P_j] = i\hbar \delta_{ij} \) would not be preserved:
\[ i\hbar \delta_{ij} = T^\dagger [X_i, P_j] T = [X_i, P_j + e \partial_j \chi (R)] = i\hbar \delta_{ij} + e[X_i, \partial_j \chi (R)]. \quad (58) \]
where the last term is non-vanishing in NCQM.

Acknowledgments

This work was supported by CONICYT (Chile) under Grants Fondecyt PLC-8000017 and 1000710, and by a Cátedra Presidencial. P.G. would like to thank I. Schmidt for his support.

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