Ward Identities for Cooper Pairs *

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Abstract

Ward identities for Cooper pairs are derived. These give consistent description of electronic current vertex and thermal current vertex.

Key Words: Ward identity, Cooper pair, current vertex, thermal current vertex

The quest for the correct expression of the thermal current vertex for Cooper pairs has a long history but the present status is still controversial.

A phenomenological expression on the basis of Ginzburg-Landau (GL) theory [1, 2] is naively expected to be reliable. The GL theory relates the electronic current vertex $\vec{J}^e$ and the thermal current vertex $\vec{J}^Q$ for Cooper pairs as

$$\vec{J}^Q = \frac{1}{2e} \left( i\omega_m + \frac{i\omega_\nu}{2} \right) \vec{J}^e,$$

(1)

in the limits of long wavelength and low frequency where incoming and outgoing Cooper pairs have bosonic thermal frequencies $i\omega_m$ and $i\omega_m + i\omega_\nu$ and $e$ is the charge of an electron ($e < 0$).

On the other hand, the most recent works [3, 4] in this field relate these two vertices as

$$\vec{J}^Q = \frac{1}{e} \left( i\omega_m + \frac{i\omega_\nu}{2} \right) \vec{J}^e.$$

(2)

However the origin of the factor 2 has not been explained convincingly.

In this Short Note we try to obtain the correct relation between $\vec{J}^e$ and $\vec{J}^Q$ on the basis of Ward identities.

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First we review the derivation [5] of the Ward identity for electronic current vertex. The vertex function is defined as

$$\Lambda_{\mu}^{\nu}(x, y, z) = \langle T_r \{ j_{\mu}^{\nu}(z) \psi_\uparrow(x) \psi_\downarrow(y) \} \rangle, \quad (3)$$

and under the charge-current conservation ($\sum_{\mu=0}^{3} \frac{\partial}{\partial \tau_{\mu}} j_{\mu}^{\nu}(z) = 0$) its divergence is transformed into

$$-i \sum_{\mu=0}^{3} \frac{\partial}{\partial \tau_{\mu}} \Lambda_{\mu}^{\nu}(x, y, z) = \langle T_r \{ [j_0^{\uparrow}(z), \psi_\tau(x)] \psi_\uparrow(y) \} \rangle \delta(\tau_z - \tau_x)$$

$$+ \langle T_r \{ \psi_\tau(x) [j_0^{\downarrow}(z), \psi_\downarrow(y)] \} \rangle \delta(\tau_z - \tau_y), \quad (4)$$

where $\langle \cdots \rangle$ represents the thermal average, $T_r$ is the time-ordering operator with respect to the imaginary time $\tau_z = iz_0$ and $z = (\vec{z}, z_0)$ with the coordinate vector $\vec{z} = (z_1, z_2, z_3)$ and the real time $z_0$. Here $\psi_\tau(x)$ and $\psi_\tau(y)$ are annihilation and creation operators of $\uparrow$-spin electron. The zeroth component of the electron current $j_0^{\uparrow}(z)$ is given by $j_0^{\uparrow}(z) = e \psi_\uparrow(z) \psi_\uparrow(z) + e \psi_\downarrow(z) \psi_\downarrow(z)$. Using the commutation relation $[j_0^{\uparrow}(z), \psi_\uparrow(y)] = e \psi_\uparrow(z) \psi_\uparrow(y)$ and $[j_0^{\downarrow}(z), \psi_\downarrow(y)] = -e \psi_\downarrow(z) \psi_\downarrow(y)$ and introducing the Fourier transform, we obtain the Ward identity

$$\sum_{\mu=0}^{3} \frac{k_{\mu} \Gamma^{\mu}_{\nu}(p, k)}{\mu} = eG^{-1}(p) - eG^{-1}(p + k), \quad (5)$$

for the electronic current vertex where $\Lambda_{\mu}^{\nu}(p, k) = G(p) \Gamma^{\mu}_{\nu}(p, k)G(p + k)$ and $G(p)$ is the electron propagator with four-momentum $p = (\vec{p}, p_0)$. The zeroth components $p_0$ and $k_0$ are fermionic ($p_0 = -ie_\nu$) and bosonic ($k_0 = -i\omega_\nu$) frequencies.

The extension of this Ward identity to the case of Cooper pairs is straightforward. In the following we consider the Cooper pair of s-wave pairing in the case of local attractive interaction. Since we discuss the normal metallic phase ($T > T_c$), the propagator of Cooper pairs is a fluctuation propagator. Replacing $\psi_\tau(x)$ by $\psi_\uparrow(x)\psi_\uparrow(x)$ and $\psi_\downarrow(y)$ by $\psi_\uparrow(y)\psi_\downarrow(y)$ and using the commutation relation $[j_0^{\uparrow}(z), \psi_\uparrow(z)\psi_\uparrow(y)] = 2e \psi_\uparrow(z)\psi_\uparrow(y)$ and $[j_0^{\downarrow}(z), \psi_\downarrow(z)\psi_\downarrow(y)] = -2e \psi_\downarrow(z)\psi_\downarrow(y)$, we obtain the Ward identity

$$\sum_{\mu=0}^{3} \frac{k_{\mu} \Delta_{\mu}^{\nu}(q, k)}{\mu} = 2eD^{-1}(q) - 2eD^{-1}(q + k), \quad (6)$$

for Cooper pairs where $\Delta_{\mu}^{\nu}$ is the counterpart of $\Gamma^{\mu}_{\nu}$ and $D(q)$ is the Cooper-pair fluctuation propagator with four-momentum $q = (\vec{q}, q_0)$ whose zeroth component $q_0$ is a bosonic frequency ($q_0 = -i\omega_m$). It should be noted that the factor $2e$ represents the charge carried by a Cooper pair and is automatically taken into account by the commutation relations. This point was missed in an early guess [6] of the Ward identity for Cooper pairs.
Second we review the derivation [7] of the Ward identity for thermal current vertex. Since the essence of the derivation is the same as the electronic current vertex, this review is short. In the limit of vanishing external momentum, \( \vec{k} \rightarrow 0 \), the Ward identity is obtained as

\[
\sum_{\mu=0}^{3} k_{\mu} \Gamma^Q_{\mu}(p + k, p) = p_0 G^{-1}(p + k) - (p_0 + k_0) G^{-1}(p),
\]

(7)

for electrons. Here \( p_0 \) and \( p_0 + k_0 \) result from the Fourier transform of the time-derivative of annihilation and creation operators of \( \uparrow \)-spin electron. The time-derivative results from the commutation relation between the zeroth component of the thermal current and annihilation or creation operator, since the zeroth component for \( \vec{k} \rightarrow 0 \) is the Hamiltonian of the system.

The extension of this Ward identity to the case of Cooper pairs is also straightforward. The Ward identity for thermal current vertex for Cooper pairs is

\[
\sum_{\mu=0}^{3} k_{\mu} \Delta^Q_{\mu}(q + k, q) = q_0 D^{-1}(q + k) - (q_0 + k_0) D^{-1}(q).
\]

(8)

Finally our Ward identities eqs. (6) and (8) are consistent with the GL result eq. (1).

This work arose from discussions with Kazumasa Miyake, Yukinobu Fujimoto and Shinji Watanabe at Osaka University.

**APPENDIX**

In this APPENDIX the derivation of the Ward identity for Cooper pairs is explained in detail.

1 **Introduction**

The Ward identity for electric current vertex is explained in Schrieffer’s textbook [5].

The Ward identity for heat current vertex is derived in Ono’s paper [7].

If we understand these discussions well, we can reach the Ward identity for Cooper pairs with little effort. Thus I review these works first and cast the results into those for Cooper pairs.
2 Ward Identity for Electric Current Vertex

First we review the derivation of the Ward identity for electric current vertex. We mainly discuss the case of zero temperature and cast the zero-temperature result into that of finite temperature.

We consider the three-point function $\Lambda_\mu^e (\mu = 1, 2, 3, 0)$ defined as

$$\Lambda_\mu^e (x, y, z) = \langle T \{ j_\mu^e (z) \psi_\uparrow (x) \psi_\uparrow^\dagger (y) \} \rangle,$$

where $\langle \cdots \rangle$ represents the expectation value in the ground state, $T$ is the time-ordering operator and $z = (z', z_0)$ with coordinate vector $z' = (z_1, z_2, z_3)$ and time $z_0$. Here $\psi_\uparrow (x)$ and $\psi_\uparrow^\dagger (y)$ are annihilation and creation operators of $\uparrow$-spin electron. The electric current $j_\mu^e$ obeys the charge-conservation law

$$\sum_{\mu=0}^3 \frac{\partial}{\partial z_\mu} j_\mu^e (z) = 0.$$  \hspace{1cm} (10)

Especially the electric charge $j_0^e$ is given by

$$j_0^e (z) = e \psi_\uparrow^\dagger (z) \psi_\uparrow (z) + e \psi_\uparrow^\dagger (z) \psi_\downarrow (z),$$  \hspace{1cm} (11)

where $e$ is the charge of an electron ($e < 0$). The time ordering of three operators results in the summation of 3! terms as

$$\Lambda_\mu^e (x, y, z) = \langle j_\mu^e (z) \psi_\uparrow (x) \psi_\uparrow^\dagger (y) \rangle \theta (z_0 - x_0) \theta (x_0 - y_0)$$

$$- \langle j_\mu^e (z) \psi_\uparrow^\dagger (y) \psi_\uparrow (x) \rangle \theta (z_0 - y_0) \theta (y_0 - x_0)$$

$$+ \langle \psi_\uparrow (x) j_\mu^e (z) \psi_\uparrow^\dagger (y) \rangle \theta (x_0 - z_0) \theta (z_0 - y_0)$$

$$- \langle \psi_\uparrow^\dagger (y) j_\mu^e (z) \psi_\uparrow (x) \rangle \theta (y_0 - z_0) \theta (z_0 - x_0)$$

$$+ \langle \psi_\uparrow (x) \psi_\uparrow^\dagger (y) j_\mu^e (z) \rangle \theta (x_0 - y_0) \theta (y_0 - z_0)$$

$$- \langle \psi_\uparrow^\dagger (y) \psi_\uparrow (x) j_\mu^e (z) \rangle \theta (y_0 - x_0) \theta (x_0 - z_0),$$  \hspace{1cm} (12)

where $\theta (x)$ is the unit step function. Thus the time-derivative of $\Lambda_\mu^e$ results in

$$\frac{\partial}{\partial z_0} \Lambda_\mu^e (x, y, z) = \delta (z_0 - x_0) \left( \theta (x_0 - y_0) \langle j_0^e (z), \psi_\uparrow (x) \psi_\uparrow^\dagger (y) \rangle \right.$$  

$$- \theta (y_0 - x_0) \langle \psi_\uparrow^\dagger (y) j_0^e (z), \psi_\uparrow (x) \rangle \right)$$

$$+ \delta (z_0 - y_0) \left( \theta (x_0 - y_0) \langle j_0^e (z), \psi_\uparrow (x) \psi_\uparrow^\dagger (y) \rangle \right.$$  

$$- \theta (y_0 - x_0) \langle \psi_\uparrow^\dagger (y) j_0^e (z), \psi_\uparrow (x) \rangle \right)$$

$$+ \langle T \left\{ \frac{\partial j_0^e (z)}{\partial z_0} \psi_\uparrow (x) \psi_\uparrow^\dagger (y) \right\} \rangle.$$

\hspace{1cm} (13)
Using again the time ordering the divergence of $\Lambda^e_\mu$ is expressed as

$$
\sum_{\mu=0}^{3} \frac{\partial}{\partial z_\mu} \Lambda^e_\mu(x, y, z) = \langle T \{ [j^e_\mu(z), \psi^\dagger(x)] \psi^\dagger(y) \} \rangle \delta(z_0 - x)
+ \langle T \{ \psi^\dagger(x) [j^e_\mu(z), \psi^\dagger(y)] \} \rangle \delta(z_0 - y)
+ \langle T \left\{ \sum_{\mu=0}^{3} \frac{\partial j^e_\mu}{\partial z_\mu}(z) \psi^\dagger(x) \psi^\dagger(y) \right\} \rangle.
$$

(14)

The last term on the right-hand side vanishes due to the charge-conservation law, eq.(10). Only equal space-time commutation relations are non-vanishing,

$$
[j^e_0(z), \psi^\dagger(z)] = e \psi^\dagger(z),
$$

(15)

and

$$
[j^e_0(z), \psi(z)] = -e \psi(z),
$$

(16)

so that the non-vanishing contribution becomes

$$
\sum_{\mu=0}^{3} \frac{\partial}{\partial z_\mu} \Lambda^e_\mu(x, y, z) = -e \langle T \{ \psi^\dagger(x) \psi^\dagger(y) \} \rangle \delta^4(z - x)
+ e \langle T \{ \psi^\dagger(x) \psi^\dagger(y) \} \rangle \delta^4(z - y).
$$

(17)

Introducing the electron propagator $G(x, y)$ as

$$
G(x, y) = -i \langle T \{ \psi^\dagger(x) \psi^\dagger(y) \} \rangle,
$$

(18)

this relation is written into

$$
\sum_{\mu=0}^{3} \frac{\partial}{\partial z_\mu} \Lambda^e_\mu(x, y, z) = -ie G(x, y) \delta^4(z - x) + ie G(x, y) \delta^4(z - y),
$$

(19)

Assuming the translational invariance we set $y = 0$ and introduce the Fourier transform as

$$
\Lambda^e_\mu(p, k) = \int d^4xe^{-ipx} \int d^4ze^{-ikz} \langle T \{ j^e_\mu(z) \psi^\dagger(x) \psi^\dagger(0) \} \rangle,
$$

(20)

where the four-momentum is defined as $p = (\vec{p}, p_0)$ and $k = (\vec{k}, k_0)$. The left-hand side of eq. (19) is evaluated as

$$
\sum_{\mu=0}^{3} \frac{\partial}{\partial z_\mu} \Lambda^e_\mu(x, 0, z) = \int \frac{d^4p}{(2\pi)^4} e^{ipx} \int \frac{d^4k}{(2\pi)^4} e^{ikz} \sum_{\mu=0}^{3} ik_\mu \Lambda^e_\mu(p, k),
$$

(21)

and the right-hand side is transformed as

$$
\int d^4xe^{-ipx} \int d^4ze^{-ikz} \left( -G(x, 0) \delta^4(z - x) + G(x, 0) \delta^4(z) \right)
= -G(p + k) + G(p),
$$

(22)
where
\[ G(p) = \int d^4xe^{-ipx}G(x,0). \] (23)

Therefore we obtain
\[ \sum_{\mu=0}^{3} k_{\mu} \Lambda_{\mu}^{\nu}(p,k) = eG(p) - eG(p+k). \] (24)

The vertex function \( \Gamma_{\mu}^{\nu} \) is introduced as
\[ \Lambda_{\mu}^{\nu}(p,k) = iG(p) \cdot \Gamma_{\mu}^{\nu}(p,k) \cdot iG(p+k), \] (25)
in accordance with the definition of the Green function, eq. (18). Then the Ward identity for the electric current vertex is given by
\[ \sum_{\mu=0}^{3} k_{\mu} \Gamma_{\mu}^{\nu}(p,k) = eG^{-1}(p) - eG^{-1}(p+k). \] (26)

Since the Fourier transform is introduced as
\[ px = p_{1}x_{1} + p_{2}x_{2} + p_{3}x_{3} - \epsilon t, \] (27)
where \( \epsilon \) is the energy and \( t \) is the time, \( x_{0} = t \) and \( p_{0} = -\epsilon \) (and in the same manner \( k_{0} = -\omega \) with \( \omega \) being the energy of the external field) in the zero-temperature formalism.

Here we check the limiting case of eq. (26). If we replace the full Green function \( G(p) \) by the free Green function \( G_{0}(p) \) and set \( k_{0} = 0 \), we obtain
\[ \sum_{\mu=0}^{3} k_{\mu} \Gamma_{\mu}^{\nu}(p,k) = \frac{e}{m} \vec{k} \cdot (\vec{p} + \vec{k}) / 2, \] (28)
where the free dispersion \( \epsilon_{\vec{p}} = p^{2}/2m \) is employed with \( m \) being the mass of electron. This relation means
\[ \Gamma_{\mu}^{\nu}(p,0) = ev_{\mu}, \] (29)
where the right-hand side is the proper electric current vertex with the electron velocity \( \vec{v} = \vec{p}/m \).

In the finite-temperature formalism we employ the time-ordering operator \( T_{\tau} \) and consider the three-point function
\[ \Lambda_{\mu}^{\nu}(x,y,z) = \langle T_{\tau}\{j_{\mu}^{\nu}(z)\psi_{\uparrow}(x)\psi_{\uparrow}^{\dagger}(y)\}\rangle, \] (30)
where the real time \( z_{0} \) and the imaginary time \( \tau_{z} \) is related by \( \tau_{z} = iz_{0} \) and \( \langle \cdots \rangle \) represents the thermal average. Taking the charge-conservation law, eq. (10), into account we obtain
\[ -i \sum_{\mu=0}^{3} \frac{\partial}{\partial z_{\mu}} \Lambda_{\mu}^{\nu}(x,y,z) = \langle T_{\tau}\{[j_{\mu}^{\nu}(z),\psi_{\uparrow}(x)]\psi_{\uparrow}^{\dagger}(y)\}\rangle \delta(\tau_{z} - \tau_{x}) \]
\[ + \langle T_{\tau}\{\psi_{\uparrow}(x)[j_{\mu}^{\nu}(z),\psi_{\uparrow}^{\dagger}(y)]\}\rangle \delta(\tau_{z} - \tau_{y}), \] (31)
instead of eq. (14). Here we have used the relation
\[
\frac{\partial}{\partial \tau z} = -i \frac{\partial}{\partial z_0}
\] (32)
The finite-temperature vertex function \( \Gamma^e_\mu \) is introduced as
\[
\Lambda^e_\mu(p, k) = [-G(p)] \cdot \Gamma^e_\mu(p, k) \cdot [-G(p + k)],
\] (33)
in accordance with the definition of the thermal Green function
\[
G(x, y) = -\langle T_\tau \{ \psi_\uparrow(x) \psi_\uparrow^\dagger(y) \} \rangle.
\] (34)
Then the resulting Ward identity is the same form as eq. (26) in the case of zero temperature. For the finite-temperature Ward identity the zeroth component of the four-momentum is \( p_0 = -i \varepsilon_n \) with fermionic thermal frequency \( \varepsilon_n \) and \( k_0 = -i \omega_n \) with bosonic thermal frequency \( \omega_n \).

The extension of this Ward identity to the case of Cooper pairs is straightforward. In the following we consider the Cooper pair of s-wave pairing in the case of local attractive interaction. Replacing \( \psi_\uparrow(x) \) by \( \Psi(x) = \psi_\downarrow(x) \psi_\uparrow(x) \) and \( \psi_\uparrow^\dagger(y) \) by \( \Psi^\dagger(y) = \psi_\uparrow^\dagger(y) \psi_\downarrow^\dagger(y) \) we consider the three-point function \( M^e_\mu \) as
\[
M^e_\mu(x, y, z) = \langle T_\tau \{ j^e_\mu(z) \Psi(x) \Psi^\dagger(y) \} \rangle,
\] (35)
where \( \Psi(x) \) and \( \Psi^\dagger(y) \) are annihilation and creation operators of a Cooper pair which has a bosonic character. Using the commutation relation
\[
[j^e_\mu(z), \Psi^\dagger(z)] = 2e \Psi^\dagger(z),
\] (36)
and
\[
[j^e_\mu(z), \Psi(z)] = -2e \Psi(z),
\] (37)
the divergence of \( M^e_\mu \) is expressed as
\[
3 \sum_{\mu=0}^3 \frac{\partial}{\partial \tau z_\mu} M^e_\mu(x, y, z) = -2e \langle T_\tau \{ \Psi(x) \Psi^\dagger(y) \} \rangle \delta^4(z - x)
+ 2e \langle T_\tau \{ \Psi(x) \Psi^\dagger(y) \} \rangle \delta^4(z - y),
\] (38)
by repeating the same calculations as eqs. (12), (13) and (14). Here the difference between fermion and boson is handled solely by the time-ordering operator \( T \) so that the expression of the divergence is common to fermion and boson. Introducing the Cooper-pair propagator \( D(x, y) \) as
\[
D(x, y) = -i \langle T_\tau \{ \Psi(x) \Psi^\dagger(y) \} \rangle,
\] (39)
we obtain the Ward identity
\[
3 \sum_{\mu=0}^3 k_\mu \Delta^e_\mu(q, k) = 2e D^{-1}(q) - 2e D^{-1}(q + k),
\] (40)
for Cooper pairs where $\Delta^e_\mu$ is the counterpart of $\Gamma^e_\mu$ and $D(q)$ is the Fourier transform of $D(x, 0)$ with four-momentum $q = (\vec{q}, q_0)$.

Although the above derivation for Cooper pairs is formulated at zero temperature, eq. (40) also holds at finite temperature with $q_0$ being a bosonic thermal frequency ($q_0 = -i\omega_n$). We are mainly interested in the normal metallic phase ($T > T_c$), the Cooper-pair propagator is a fluctuation propagator in this case.

It should be noted that the factor $2e$ represents the charge carried by a Cooper pair and is automatically taken into account by the commutation relation.

3 Ward Identity for Heat Current Vertex

First we review the derivation [7] of the Ward identity for heat current vertex. We consider the three-point function $\Lambda^Q_\mu(x, y, z)$ defined as

$$\Lambda^Q_\mu(x, y, z) = \langle T\{j^Q_\mu(z)\psi_\uparrow(x)\psi_\uparrow\dagger(y)\}\rangle,$$

(41)

where $j^Q_\mu$ is the heat current. The heat current $j^Q_\mu$ obeys the energy-conservation law

$$\sum_{\mu=0}^3 \frac{\partial}{\partial z_\mu} j^Q_\mu(z) = 0.$$

(42)

Assuming the translational invariance we set

$$\Lambda^Q_\mu(x, y, z) = \Lambda^Q_\mu(x - y, z - x),$$

(43)

so that the Fourier transform becomes

$$\int d^4ze^{-ikz} \int d^4xe^{-ip'x} \int d^4ye^{ipy}\Lambda^Q_\mu(x, y, z) = \Lambda^Q_\mu(p, p-k)(2\pi)^4\delta((-k-p'+p)),$$

(44)

where

$$\Lambda^Q_\mu(p, p-k) = \int d^4(x-y)e^{-ip(x-y)} \int d^4(z-x)e^{-ik(z-x)}\Lambda^Q_\mu(x - y, z - x),$$

(45)

and $\delta(-k-p'+p)$ represents the conservation of four-momentum. The divergence of $\Lambda^Q_\mu$ becomes

$$\sum_{\mu=0}^3 \frac{\partial}{\partial z_\mu} \Lambda^Q_\mu(x, y, z) = \langle T\{j^Q_\mu(z)\psi_\uparrow(x)\psi_\uparrow\dagger(y)\}\rangle \delta(z_0 - x_0)$$

$$+ \langle T\{\psi_\uparrow(x)j^Q_\mu(z)\psi_\uparrow\dagger(y)\}\rangle \delta(z_0 - y_0),$$

(46)

under the energy-conservation law, eq. (42). The left-hand side is evaluated as

$$\sum_{\mu=0}^3 \frac{\partial}{\partial z_\mu} \Lambda^Q_\mu(x - y, z - x) = \int \frac{d^4p}{(2\pi)^4}e^{ip(x-y)} \int \frac{d^4k}{(2\pi)^4}e^{ik(z-x)} \sum_{\mu=0}^3 ik_\mu \Lambda^Q_\mu(p, p-k),$$

(47)
so that the Fourier transform satisfies
\[
\sum_{\mu=0}^{3} i k_{\mu} A^{Q}_{\mu}(p, p - k) = \int d(x_0 - y_0) e^{-i p_0 (x_0 - y_0)} \int d(z_0 - x_0) e^{-i k_0 (z_0 - x_0)}
\]
\[
\times \left( \langle T\{j^{Q}_{k}(x_0), a_{p-k}(x_0)\}a^{\dagger}_{p}(y_0)\rangle \delta(z_0 - x_0)
\right.
\]
\[
+ \langle T\{a_{p-k}(x_0)j^{Q}_{k}(y_0), a^{\dagger}_{p}(y_0)\} \rangle \delta(z_0 - y_0) \right),
\]
(48)

where
\[
j^{Q}_{0}(z) = \sum_{k} e^{i \vec{k} \cdot \vec{z}} j^{Q}_{k}(z_0),
\]
\[
\psi_{\uparrow}(x) = \sum_{\vec{p}} e^{i \vec{p} \cdot \vec{x}} a_{\vec{p}}(x_0),
\]
\[
\psi^{\dagger}_{\uparrow}(y) = \sum_{\vec{p}} e^{-i \vec{p} \cdot \vec{y}} a^{\dagger}_{\vec{p}}(y_0),
\]
with
\[
\sum_{\vec{k}} \equiv \int \frac{d\vec{k}}{(2\pi)^3}.
\]

Then we evaluate the commutation relations. For such a purpose we introduce the Hamiltonian density \( h(z) \) for an isotropic system as
\[
h(z) = h^{\text{kin}}(z) + h^{\text{int}}(z) \]
at time \( z_0 \) where
\[
h^{\text{kin}}(z) = \frac{1}{2m} \sum_{\sigma} \nabla \psi^{\dagger}_{\sigma}(z) \cdot \nabla \psi_{\sigma}(z),
\]
(51) and
\[
h^{\text{int}}(z) = \frac{1}{2} \sum_{\sigma} \sum_{\sigma'} \int d\vec{z}^{'} \psi^{\dagger}_{\sigma}(\vec{z}) \psi^{\dagger}_{\sigma'}(\vec{z}) V(\vec{z} - \vec{z}^{'}) \psi_{\sigma'}(\vec{z}^{'}) \psi_{\sigma}(\vec{z}),
\]
(52) with a general interaction strength \( V(\vec{z} - \vec{z}^{'}) \) of two-body interaction and spin \( \sigma, \sigma' = \uparrow, \downarrow \). Within this paragraph the time of all operators is \( z_0 \). This Hamiltonian density \( h(z) \) is the zeroth component of the heat current \( j^{Q}_{0}(z) \). The Fourier component
\[
j^{Q}_{k}(z) = j^{Q}_{0}(z_0) = \int d\vec{z} e^{-i \vec{k} \cdot \vec{z}} j^{Q}_{0}(z),
\]
(53) is decomposed as
\[
j^{Q}_{k} = j^{\text{kin}}_{k} + j^{\text{int}}_{k}
\]
and
\[
j^{\text{kin}}_{k} = \frac{1}{2m} \sum_{\vec{p}} (\vec{p} - \vec{k}) \cdot \vec{p} \left( a^{\dagger}_{\vec{p} - \vec{k}} a_{\vec{p}} + b^{\dagger}_{\vec{p} - \vec{k}} b_{\vec{p}} \right),
\]
(54)

where \( b_{\vec{p}} \) and \( b^{\dagger}_{\vec{p} - \vec{k}} \) are the Fourier components of annihilation and creation operators of \( \downarrow \)-spin electron. Comparing
\[
[j^{\text{kin}}_{k}, a_{\vec{p} - \vec{k}}] = \frac{(\vec{p} - \vec{k}) \cdot \vec{p}}{2m} a_{\vec{p}} \], \quad [j^{\text{kin}}_{k}, a^{\dagger}_{\vec{p} - \vec{k}}] = \frac{(\vec{p} - \vec{k}) \cdot \vec{p}}{2m} a^{\dagger}_{\vec{p} - \vec{k}}.
\]
(55)
with

\[ [j^{\text{kin}}_k, a_{\vec{p}}] = -\frac{\vec{p} \cdot \vec{p}}{2m} a_{\vec{p}}, \quad [j^{\text{kin}}_k, a^\dagger_{\vec{p}-\vec{k}}] = \frac{\left(\vec{p} - \vec{k}\right) \cdot \left(\vec{p} - \vec{k}\right)}{2m} a^\dagger_{\vec{p}-\vec{k}}, \]  

(56)

we can use the replacement

\[ [j^{\text{kin}}_k, a_{\vec{p}-\vec{k}}] \Rightarrow [j^{\text{kin}}_{\vec{k}=0}, a_{\vec{p}}], \quad [j^{\text{kin}}_k, a^\dagger_{\vec{p}}] \Rightarrow [j^{\text{kin}}_{\vec{k}=0}, a^\dagger_{\vec{p}}], \]  

(57)

in the limit of vanishing external momentum, \( \vec{k} \to 0 \). The similar argument holds for \( [j^{\text{int}}_\mu, a_{\vec{p}-\vec{k}}] \) and \( [j^{\text{int}}_\mu, a^\dagger_{\vec{p}}] \) so that we obtain

\begin{align*}
[j^{Q}_k, a_{\vec{p}-\vec{k}}] &\Rightarrow [H, a_{\vec{p}}], \quad [j^{Q}_k, a^\dagger_{\vec{p}}] \Rightarrow [H, a^\dagger_{\vec{p}-\vec{k}}],
\end{align*}

(58)

since \( j^{Q}_{\vec{k}=0} = H \) where

\[ H = \int h(\vec{z}) d\vec{z}, \]

(59)

is the Hamiltonian.

Thus the equation of motion

\[ [H, a_{\vec{p}}(x_0)] = -i \frac{\partial}{\partial x_0} a_{\vec{p}}(x_0), \quad [H, a^\dagger_{\vec{p}-\vec{k}}(y_0)] = -i \frac{\partial}{\partial y_0} a^\dagger_{\vec{p}-\vec{k}}(y_0), \]

(60)

can be applied to eq. (48) and the result is

\begin{align*}
\sum_{\mu=0}^{3} i k_\mu A^Q_\mu(p, p-k) &= \int d(x_0 - y_0) e^{-ip_0(x_0-y_0)} \int d(z_0 - x_0) e^{-ik_0(z_0-x_0)} \\
&\quad \times \left( \frac{\partial}{\partial x_0} G_{\vec{p}}(x_0 - y_0) \delta(z_0 - x_0) + \frac{\partial}{\partial y_0} G_{\vec{p}-\vec{k}}(x_0 - y_0) \delta(z_0 - y_0) \right),
\end{align*}

(61)

where

\[ G_{\vec{p}}(x_0 - y_0) = -i \langle T\{a_{\vec{p}}(x_0)a^\dagger_{\vec{p}}(y_0)\} \rangle. \]

(62)

Employing

\[ \frac{\partial}{\partial x_0} G_{\vec{p}}(x_0 - y_0) = \int \frac{dp'}{2\pi} e^{ip'(x_0-y_0)} i p_0' G_{\vec{p}}(p'), \]

(63)

the first term on the right-hand side of eq. (61) becomes \( ip_0 G(p) \) where \( G(p) = G_{\vec{p}}(p_0) \). The second term is integrated in the same manner and becomes \(-i(p_0 - k_0)G(p - k)\). Then shifting the four-momentum we obtain

\[ \sum_{\mu=0}^{3} k_\mu A^Q_\mu(p + k, p) = (p_0 + k_0)G(p + k) - p_0 G(p). \]

(64)

This relation is converted into the Ward identity

\[ \sum_{\mu=0}^{3} k_\mu \Gamma^Q_\mu(p + k, p) = p_0 G^{-1}(p + k) - (p_0 + k_0)G^{-1}(p), \]

(65)
for heat current vertex. Here we have used the same relation \( \Lambda^Q_\mu(p + k, p) = iG(p + k) \cdot \Gamma^Q_\mu(p + k, p) \cdot iG(p) \) as eq. (25).

In the case of finite temperature the same relation as eq. (65) holds.

If we only consider the local interaction, the derivation becomes very simple as follows. Since we are mainly interested in BCS-type local interaction, such a derivation is sufficient for our purpose. Taking into account the fact, 

\[
\mathcal{J}^Q_\mu(z) = h(z), \text{ eq. (46) is equivalent to}
\]

\[
\sum_{\mu=0}^3 \frac{\partial}{\partial z_\mu} \Lambda^Q_\mu(x, y, z) = \{T\{h(x), \psi_\uparrow(x)\}\psi_\uparrow(y)\}\delta^4(z - x) + \{T\{\psi_\uparrow(x) [h(y), \psi_\uparrow(y)]\}\}\delta^4(z - y). 
\]  

(66)

If the interaction among electrons is local, then \([h(x), \psi_\uparrow(x)] = [H, \psi_\uparrow(x)] \) and \([h(y), \psi_\uparrow(y)] = [H, \psi_\uparrow(y)]\). Thus using

\[
[h(x), \psi_\uparrow(x)] = -i \frac{\partial}{\partial x_0} \psi_\uparrow(x), \quad [h(y), \psi_\uparrow(y)] = -i \frac{\partial}{\partial y_0} \psi_\uparrow(y),
\]

(67)
eq (66) becomes

\[
\sum_{\mu=0}^3 \frac{\partial}{\partial z_\mu} \Lambda^Q_\mu(x, y, z) = \frac{\partial}{\partial x_0} G(x, y) \delta^4(z - x) + \frac{\partial}{\partial y_0} G(x, y) \delta^4(z - y),
\]

(68)

which has the similar structure as eq. (19). Assuming the translational invariance we set \(G(x, y) = G(x - y)\) and introduce the Fourier transform

\[
\frac{\partial}{\partial x_0} G(x - y) = \int \frac{d^4 p'}{(2\pi)^4} e^{i p'(x - y)} p'_0 G(p'),
\]

(69)
as eq. (63). Performing the Fourier transform in eq. (45) we obtain

\[
\sum_{\mu=0}^3 k_\mu \Lambda^Q_\mu(p, p - k) = p_0 G(p) - (p_0 - k_0) G(p - k),
\]

(70)

which becomes eq. (64) by shifting the four-momentum.

Here we check the limiting case of eq. (65). If we replace the full Green function \(G(p)\) by the free Green function \(G_0(p)\) and set \(k_0 = 0\), we obtain

\[
\sum_{\mu=1}^3 k_\mu \Gamma^Q_\mu(p + k, p) = -\frac{p_0}{m} \vec{k} \cdot (\vec{p} + \vec{k}),
\]

(71)

and via \(p_0 = -\epsilon\) this relation means

\[
\Gamma^Q_\mu(p, p) = \epsilon \nu_\mu.
\]

(72)

where the right-hand side is the proper energy current vertex.
The extension of this Ward identity to the case of Cooper pairs is also straightforward. The Ward identity for heat current vertex for Cooper pairs is
\[
\sum_{\mu=0}^{3} k_\mu \Delta_\mu^Q (q + k, q) = q_0 D^{-1}(q + k) - (q_0 + k_0) D^{-1}(q),
\]
where \( \Delta_\mu^Q \) is the counterpart of \( \Gamma_\mu^Q \).

It should be noted that the factor \( q_0 \) represents the energy carried by a Cooper pair and is automatically taken into account by the commutation relation.

4 Conclusion

I have explained the derivation of the Ward identity in detail. I hope that you can save time in understanding it by this APPENDIX.

Our main results of Ward identities for Cooper pairs are eqs. (40) and (73) whose counterparts for electrons are eqs. (26) and (65). The charge and energy carried by Cooper pairs are properly taken into account by the commutation relations.

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