RATIONAL DECAY RATES FOR A PDE HEAT–STRUCTURE INTERACTION: A FREQUENCY DOMAIN APPROACH

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Abstract. In this paper, we consider a simplified version of a fluid–structure PDE model—in fact, a heat–structure interaction PDE-model. It is intended to be a first step toward a more realistic fluid–structure PDE model which has been of longstanding interest within the mathematical and biological sciences [33, p. 121], [17], [19]. This physically more sound and mathematically more challenging model will be treated in [13]. The simplified model replaces the linear dynamic Stokes equation with a linear n-dimensional heat equation (heat–structure interaction). The entire dynamics manifests both hyperbolic and parabolic features. Our main result is as follows: Given smooth initial data—i.e., data in the domain of the associated semigroup generator—the corresponding solutions decay at the rate \( o(t^{-\frac{1}{2}}) \) (see Theorem 1.3 below). The basis of our proof is the recently derived resolvent criterion in [15]. In order to apply it, however, suitable PDE-estimates need to be established for each component by also making critical use of the interface conditions. A companion paper [6] will sharpen Lemma 5.8 of the present work by use of a lengthy and technical microlocal argument as in [26, 29, 30, 31], to obtain the optimal value \( \alpha = 1 \); hence, the optimal decay rate \( o(t^{-1}) \). See Remarks 1.2,1.3.

1. Introduction and statement of main result. Introduction. We proceed to describe the canonical heat–structure PDE model of the present paper. This is the first step toward the more realistic fluid–structure PDE model which has the more challenging dynamic Stokes equation in place of the n-dimensional heat equation in (1.1a) below [33, p. 121], [19]. It will be treated in a subsequent publication [13]. The presence of the pressure is responsible for significant additional mathematical challenges already at the level of establishing semigroup well posedness [7], [10], [4]. For the present problem of rational decay, the present treatment of a simplified model offers a strategy, a template to be followed also in the original fluid–structure model. Due to the presence of the pressure, however, serious additional mathematical challenges need to be overcome which require a lengthy

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Such reference yields the same decay \( o(t^{-1/2}) \) as in the present paper. Throughout, \( \Omega_f \subseteq \mathbb{R}^n, n = 2 \) or 3, will denote the bounded domain on which the heat component of the coupled PDE system evolves. Its boundary will be denoted here as \( \partial \Omega_f = \Gamma_s \cup \Gamma_f, \Gamma_s \cap \Gamma_f = \emptyset \), with each boundary piece being sufficiently smooth. Moreover, the geometry \( \Omega_s \), immersed within \( \Omega_f \), will be the domain on which the structural component evolves with time. As configured then, the coupling between the two distinct fluid and elastic dynamics occurs across boundary interface \( \Gamma_s = \partial \Omega_s \); see Figure 1. In addition, the unit normal vector \( \nu(x) \) will be directed away from \( \Omega_f \), and so toward \( \Omega_s \). (This specification of the direction of \( \nu \) will influence the computations to be done below.)

![Fig. 1: The Fluid–Structure Interaction](image)

On this geometry in Figure 1, we thus consider the following heat–structure PDE model in solution variables \( u = [u_1(t,x), u_2(t,x), \ldots, u_n(t,x)] \) (the heat component here replacing the usual fluid velocity field), and \( w = [w_1(t,x), w_2(t,x), \ldots, w_n(t,x)] \) (the structural displacement field):

\[
\begin{align*}
\text{(PDE)} & \quad \begin{cases} 
& u_t - \Delta u = 0 \quad \text{in } (0,T) \times \Omega_f, \\
& w_{tt} - \Delta w + w = 0 \quad \text{in } (0,T) \times \Omega_s; \quad (1.1a) \\
\end{cases} \\
\text{(BC)} & \quad \begin{cases} 
& u|_{\Gamma_f} = 0 \quad \text{on } (0,T) \times \Gamma_f, \\
& u|_{\Gamma_s} = w_t \quad \text{on } (0,T) \times \Gamma_s, \\
& \frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} \quad \text{on } (0,T) \times \Gamma_s; \quad (1.1c) \\
\end{cases} \\
\text{(IC)} & \quad [w(0,\cdot), w_t(0,\cdot), u(0,\cdot)] = [w^*_0, w^*_1, u^*_0] \in \mathbf{H}, \quad (1.1f)
\end{align*}
\]

where space of well-posedness is taken to be the finite energy space

\[
\mathbf{H} \equiv H^1(\Omega_s) \times L^2(\Omega_s) \times L^2(\Omega_f), 
\]

for the variable \([w, w_t, u]\). (We are using the common notation \( \mathbf{H}^* \equiv [H^*]^n \). \( \mathbf{H} \) is a Hilbert space with the following norm inducing inner product, where \((f,g)_\Omega \equiv \)
\[ \int_{\Omega} f \tilde{y} d\Omega: \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}, \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{f} \end{bmatrix} \in H \]

\[ = (\nabla v_1, \nabla \tilde{v}_1)_{\Omega_s} + (v_1, \tilde{v}_1)_{\Omega_s} + (v_2, \tilde{v}_2)_{\Omega_s} + (f, \tilde{f})_{\Omega_f}. \]

**Semigroup well-posedness.** As was done in [5], [7]–[11], [4] for a fluid–structure system in which Stokes flow is used to describe the fluid component of the dynamics, one can provide a non-trivial semigroup formulation so as to describe the time-evolving PDE model (1.1a)–(1.1f). In fact (as a very special case of the above references, as no pressure term appears now), one can define a modeling generator \( A : H \to H \) as follows:

\[ A \equiv \begin{bmatrix} 0 & I & 0 \\ \Delta - I & 0 & 0 \\ 0 & 0 & \Delta \end{bmatrix}, \]

with domain \( D(A) \subset H \) being composed of all \([w_0, w_1, u_0] \in H\) which satisfy the following:

(A.1) \([w_0, w_1, u_0] \in H^1(\Omega_s) \times H^1(\Omega_s) \times H^1(\Omega_f)\).

(A.2) On the boundary portion \( \Gamma_f \), the fluid component \( u_0 |_{\Gamma_f} = 0 \).

(A.3) The structural component \( w_0 \) satisfies \( \Delta w_0 \in L^2(\Omega_s) \). In consequence, elliptic theory provides that \( \partial w_0 / \partial \nu \mid_{\Gamma_s} \) is well-defined as an element of \([H^{-\frac{1}{2}}(\Gamma_s)]^n\); see e.g., [20, p. 71, Thm. 3.8.1], [10, Lem. 3.1, p. 426].

(A.4) The fluid component \( u_0 \) satisfies \( \Delta u_0 \in L^2(\Omega_f) \). In consequence, elliptic theory provides that \( \partial u_0 / \partial \nu \mid_{\Gamma_s} \) is well-defined as an element of \([H^{-\frac{1}{2}}(\partial \Omega_f)]^n\); see e.g., [20, p. 71, Thm. 3.8.1], [10, Lem. 3.1, p. 426].

(A.5) The components obey the following relation on boundary interface \( \Gamma_s \):

\[ u_0 = w_1 \quad \text{on} \quad \Gamma_s, \quad \text{in} \quad H^{\frac{1}{2}}(\Gamma_s). \]

(A.6) The components \([u_0, w_0] \) obey the following relation on the boundary interface \( \Gamma_s \):

\[ \partial u_0 / \partial \nu = \partial w_0 / \partial \nu \quad \text{on} \quad \Gamma_s, \quad \text{in} \quad H^{-\frac{1}{2}}(\Gamma_s). \]

In regard to well-posedness, one can proceed as in [7]—or as in [4] and [9], these being inf–sup Babuska–Brezzi approaches to well-posedness—to establish the following:

**Theorem 1.1.** (i) The operator \( A : D(A) \subset H \) is dissipative

\[ \text{Re}(Ax, x)_H = - \int_{\Omega_f} |\nabla f|^2 d\Omega_f, \quad x = [v_1, v_2, f] \in D(A); \]

in fact, maximal dissipative, and thus it generates a contraction \( C_0 \)-semigroup \( \{e^{At}\}_{t \geq 0} \) on \( H \). Thus, given \([w_0, w_1, u_0] \in H\), the solution \([w, w_t, u] \in C([0, T]; H)\) of (1.1a–f) is given by

\[ [w(t), w_t(t), u(t)] = e^{At}[w_0, w_1, u_0]. \]

Moreover, the fluid component satisfies the additional regularity, \( u \in L^2(0, T; H^1(\Omega_f)) \).
(ii) The closed right half-plane $\mathbb{C}^+ = \{ \lambda \in \mathbb{C} : \text{Re} \; \lambda \geq 0 \}$ belongs to the resolvent set $\rho(A)$ of $A$: $\mathbb{C}^+ \subset \rho(A)$; in particular, $i \mathbb{R} \subset \rho(A)$.

(iii) The resolvent operator $R(\lambda, A)$, $\lambda \in \mathbb{C}^+$ is not compact on the state space $H$. More precisely, the component of $R(\lambda, A)$ on the space component $H^1(\Omega_s)$ for $w_0$ of $[w_0, w_1, u_0]$ is not compact.

(iv) The semigroup $e^{At}$ is strongly stable on $H$: $e^{At} x \to 0$ as $t \to +\infty$, $\forall x \in H$.

Remark 1.1. If one replaces the interface condition (1.1e) by the following dissipative condition: $\partial w/\partial \nu = \partial w/\partial t$ in $(0,T) \times \Gamma_S$, then the corresponding problem still generates a s.c. semigroup $e^{At}$ which, moreover, is now uniformly (exponentially) stable in $L^2$-domain by using suitable energy methods [8], [12], respectively, without assumed geometrical conditions on $\Omega_s$.

**Statement of main result.** Part (ii) of Theorem 1.1 allows for the inference of strong decay of the fluid–structure model (1.1a)–(1.1f), as asserted in Part (iv). This can be done by invoking [2], [35], or else [16], see [7] and also [10]; while the Nagy–Foias–Foguel approach [32] fails because of the lack of compactness inferred in part (iii). The main topic of the present paper deals rather with the more advanced notion of rational decay of the semigroup $e^{At}$ (which readily then implies its strong stability, as $\mathcal{D}(A)$ is dense in $H$). To achieve this result, one may seek to pursue either an analysis in the $t$-domain by using suitable energy methods [36], [40, Thm. 6.2, p. 694], or else an analysis in the $\lambda$-domain. In this paper we will follow the second approach. More precisely, our main result of rational decay for solutions of (1.1a–f) will ultimately invoke the following operator-theoretic (and sharp) recent result in [15].

**Theorem 1.2.** (See [15].) Let $\{T(t)\}_{t \geq 0}$ be a bounded $C_0$-semigroup on a Hilbert space $H$ with generator $A$ such that $i \mathbb{R} \subset \rho(A)$. Then for fixed $\alpha > 0$ the following are equivalent:

\begin{align}
(i) \quad & \|R(is; A)\| = \mathcal{O}(|s|^\alpha), \quad |s| \to \infty; \\
(ii) \quad & \|T(t)x\|_H = o(t^{-\frac{k}{2}})\|x\|_{\mathcal{D}(A)}.
\end{align}

In order to apply Theorem 1.2 to problem (1.1a–f), we shall need to establish suitable PDE-estimates for each component of the system, by also making critical use of the interface conditions (1.1d–f). In the present paper, we shall succeed to apply Theorem 1.2 with $\alpha = 2$ to our original semigroup $e^{At}$ of Theorem 1.1. [The exponent $\alpha = 2$ is not optimal, see Remark 1.3 below.] As a consequence, we will have the following result of rational decay. The main result of this paper is:

**Theorem 1.3.** For initial data $[w_0^*, w_1^*, u_0^*] \in \mathcal{D}(A)$, the corresponding solution $[w, w_1, u]$ of (1.1a)–(1.1f) obeys the following decay rate for large time $t > 0$:

\begin{align}
\left\| \begin{bmatrix} w(t) \\ w_1(t) \\ u(t) \end{bmatrix} \right\|_H = \left\| e^{At} \begin{bmatrix} w_0^* \\ w_1^* \\ u_0^* \end{bmatrix} \right\|_H \leq o\left(\frac{1}{t^2}\right) \left\| \begin{bmatrix} w_0^* \\ w_1^* \\ u_0^* \end{bmatrix} \right\|_{\mathcal{D}(A)} , \quad t > 0, \quad t \to \infty.
\end{align}
Remark 1.2. To our knowledge e.g., [37], Theorem 1.2 has so far been employed to obtain rational decay of solutions only in the case of some 1-dimensional, single (uncoupled) PDEs, using also a spectral analysis, in particular a Riesz-basis property available for such 1-d models, e.g., [1]. The present paper appears to be the first one where such $\lambda$-analysis ultimately based on Theorem 1.2 is successful in the case of complicated multi-dimensional systems of two strongly coupled (at the interface) PDEs of different type. Thus, the analysis provided here may serve as a template for other complicated multi-dimensional systems of coupled PDEs.

In contrast, the literature on polynomial/rational decay is mostly based on a $t$-domain analysis, by using energy methods. A precursor paper which we wish to recall is paper [36], reported also in [40, Thm. 6.2, p. 695]. More recent works include [38], [39], etc. However, a first relevant reference to our present paper in [45], where a $t$-analysis for the same heat–wave system (1.1a–f) produces the rational decay (1.9) with $\alpha = 6$, i.e., with rate $(t^{-\frac{1}{6}})$ [45, Eqn. (7.3), Thm. 11, p. 88]; that is, "$\frac{1}{3}$," worse than in the present Theorem 1.3. This result of [45] is improved to $\alpha = 1 - 2\epsilon, \epsilon > 0$, i.e. with rate $(t^{-(1-\epsilon)})$ in [20], again by using a time-domain approach.

Remark 1.3. As noted above the statement of Theorem 1.3, our result with $\alpha = 2$ is not optimal. The optimal parameter is, in fact, $\alpha = 1$, yielding therefore the optimal decay rate $(t^{-\frac{1}{2}})$ instead of $(t^{-\frac{1}{2}})$ in (1.10). The treatment of the present paper is optimal up until the analysis culminating with Lemma 5.7. This is noted in Remark 5.1. To obtain the optimal parameter one needs to revisit estimate (5.38) of Lemma 5.8 involving the "last term" of the analysis $\langle \partial w/\partial \nu, \partial f/\partial \nu \rangle_{\Gamma_s}$. In a subsequent paper [6]—which is built upon the present one up to Lemma 5.7—we shall provide the noted improvement to $\alpha = 1$. This, however, will require performing a lengthy technical microlocal analysis argument as in [26, 29, 30, 31]. Space restrictions do not permit it to include this argument here. We do not believe that it is possible to obtain $\alpha = 1$ unless microlocal estimates are used in estimating the term in Lemma 5.8.

2. The resolvent equation and orientation of the proof of Theorem 1.3.

The resolvent equation. Let $\beta \in \mathbb{R}$, so that $(i\beta - A)^{-1} \in \mathcal{L}(H)$ by Theorem 1.1(ii). Then, with $y^* = [v_1^*, v_2^*, f^*] \in H$, with pre-image $x = [v_1, v_2, f] \in D(A)$, we consider the resolvent equation $(i\beta I - A)x = y^*$, that is

\[
(i\beta I - A) \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ f^* \end{bmatrix}, \quad x = \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} = (i\beta I - A)^{-1} \begin{bmatrix} v_1^* \\ v_2^* \\ f^* \end{bmatrix} = (i\beta I - A)^{-1} y^*.
\]

From the definition of the domain for (1.4), and as noted for the Stokes–wave system of [7, p. 28], the relation (2.1) gives the explicit relations:

\[
i\beta v_1 - v_2 = v_1^* \in H^1(\Omega_s);
\]

\[
\Delta v_1 + (\beta^2 - 1)v_1 = -i\beta v_1^* - v_2^* \in L^2(\Omega_s),
\]

\[
\frac{\partial v_1}{\partial \nu} = \frac{\partial f}{\partial \nu} \in H^{-\frac{1}{2}}(\Gamma_s);
\]

(2.3a)
with argument $x$.

\textbf{Theorem 1.3 by applying Theorem 1.2.}

Our goal and guiding idea in proving Theorem 1.3 will be as follows: We seek the lowest positive number $\alpha$ and a positive number $\sigma$, with $0 \leq \sigma \leq 2\alpha$, such that

\[
X^2 \leq 2C_{\beta_0} \left\{ |\beta|^\alpha \|\nabla f\|_{L^2(\Omega)} + |\beta|^\sigma (Y^*)^2 \right\}, \quad \text{for all } |\beta| \text{ large, say } |\beta| \geq \beta_0 \geq 1.
\]

(2.6)

Indeed, once the estimate (2.6) is established in Proposition 6.2 below—by (6.5), with $\alpha = 2$ and $\sigma = \frac{1}{2}$—the proof of Theorem 1.3 will be completed via the following argument.

\textbf{Completion of the proof of Theorem 1.3 under the validity of (2.6).}

With $x \in D(A)$ and $y^* = [v_1^*, v_2^*, f^*] \in H$ as above in (2.1), we introduce the notation

\[
X = \|x\|_H = \|(v_1, v_2, f)\|_H; \quad X^2 = \|v_1\|_{H^1(\Omega)}^2 + \|v_2\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2;
\]

\[Y^* = \|y^*\|_H = \|(v_1^*, v_2^*, f^*)\|_H; \quad (Y^*)^2 = \|v_1^*\|_{H^1(\Omega)}^2 + \|v_2^*\|_{L^2(\Omega)}^2 + \|f^*\|_{L^2(\Omega)}^2.
\]

(2.5a)

(2.5b)

Completion of the proof of Theorem 1.3 under the validity of (2.6). With $x \in D(A)$ and $y^*$ as in (2.1), we have by invoking the dissipativity relation (1.7) and (2.1) and recalling (2.5a–b):

\[
\|\nabla f\|_{L^2(\Omega)}^2 = -\text{Re}(Ax, x)_H = \text{Re}((i\beta - A)x, x)_H = \text{Re}(y^*, x)_H \leq \|x\|_H \|y^*\|_H = XY^*.
\]

(2.7)

Using (2.7) in (2.6) then yields for all $|\beta| \geq \beta_0 \geq 1$, $\epsilon > 0$:

\[
X^2 \leq 2X(C_{\beta_0} |\beta|^\alpha Y^*) + 2C_{\beta_0} |\beta|^\sigma (Y^*)^2 \leq \epsilon X^2 + C_{\beta_0, \epsilon} |\beta|^{2\alpha} (Y^*)^2;
\]

after recalling $\sigma \leq 2\alpha$; or, with $\tilde{C}_{\beta_0, \epsilon} = C_{\beta_0, \epsilon}/(1 - \epsilon)$:

\[
X^2 \leq \tilde{C}_{\beta_0, \epsilon} |\beta|^{2\alpha} (Y^*)^2 \quad \text{or} \quad \left\| \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} \right\|_H \leq \sqrt{\tilde{C}_{\beta_0, \epsilon} |\beta|^{\alpha}} \left\| \begin{bmatrix} v_1^* \\ v_2^* \\ f^* \end{bmatrix} \right\|_H,
\]

(2.9)

for all $|\beta| \geq \beta_0 \geq 1$, recalling again (2.5a–b). The estimate (2.9) now gives the desired inequality via (2.1):

\[
\|\mathcal{R}(i\beta, A)\|_{L(H)} = O(|\beta|^{\alpha}), \quad \text{for all } |\beta| \geq \beta_0 \geq 1.
\]

(2.10)

Below we shall derive estimate (2.6) with $\alpha = 2$ and $\sigma = \frac{1}{2}$, thereby proving Theorem 1.3 by applying Theorem 1.2.
3. **An auxiliary system and corresponding static energy identity.** We start by introducing the “Dirichlet” map on $\Omega_s$ [27, p. 181], [34]:

\[
\mathcal{D}g = h \iff \{ \Delta h = 0 \text{ in } \Omega_s; \ h|_{\Gamma_s} = g \text{ on } \Gamma_s \}; \tag{3.1a}
\]

\[
\mathcal{D} \in \mathcal{L}(\mathcal{H}^r(\Gamma_s); \mathcal{H}^{r+\frac{1}{2}}(\Omega_s)), \text{ for all } r \in \mathbb{R}. \tag{3.1b}
\]

Following past strategies employed in the theory of boundary control—see e.g., [42], [25, 26, 27, 28]—we seek to homogenize the $v_1$-boundary value problem (2.3a–b). To this end, we introduce a new variable

\[
\omega = v_1 - \mathcal{D}(v_1|_{\Gamma_s}) \in \mathcal{H}^2(\Omega_s), \tag{3.2}
\]

so that it satisfies the following two homogeneous boundary value problems:

\[
\begin{align*}
\Delta \omega + (\beta^2 - 1) \omega &= \omega^* \text{ in } \Omega_s, & (3.3a) \\
\omega|_{\Gamma_s} &= 0 \text{ on } \Gamma_s; & (3.3b)
\end{align*}
\]

where

\[
\omega^* = -(\beta^2 - 1) \mathcal{D}(v_1|_{\Gamma_s}) - i\beta v_1^* - v_2^* \in \mathcal{L}^2(\Omega_s). \tag{3.4}
\]

A bound on $\omega^*$ will be given in (4.3) below. As such, we have by elliptic theory (see [34]) that

\[
\left\{ \omega, \frac{\partial \omega}{\partial \nu} \right\} \in \mathcal{H}^2(\Omega_s) \times \mathcal{H}^{1/2}(\Gamma_s),
\]

with continuous dependence on the data. (Note that, in contrast, the solution of (2.3a–b) satisfies only $v_1 \in \mathcal{H}^1(\Omega_s)$ and $\partial v_1/\partial \nu \in \mathcal{H}^{-1/2}(\Gamma_s)$.) Thus, via component-wise application of Proposition A.1 of the Appendix, the following energy identities are suitable for the $\omega$-problem (3.3a–b), rather than the $v_1$-problem (2.3a–b). To this end, we recall the following conventional notation [41].

**Convention.** $\nabla \omega = n \times n$ matrix whose $i^{th}$ column is $\nabla \omega_i$; for an $n$-vector $h$, then $h \cdot \nabla \omega = n \times n$ matrix whose $i^{th}$ column is $h \cdot \nabla \omega_i$. If $A = (a_{jk})_{j,k=1}^n$, $B = (b_{jk})_{j,k=1}^n$ are $n \times n$ matrices, then we set $A \cdot B = \sum_{j,k=1}^n a_{jk}b_{jk}$, while $AB$ is written for the usual matrix product.

**Proposition 1.** (a) Given a $[C^2(\Omega_s)]^n$-vector field $\mathbf{h}(x)$ with symmetric, positive semidefinite Jacobian matrix $\mathcal{H}(x)$, then the solution of the homogeneous problem on the LHS of (3.3a–b) satisfies the following identities:

\[
\begin{align*}
(a) \int_{\Omega_s} \mathcal{H}(x) \nabla \omega \cdot \nabla \omega \, d\Omega_s &= - \text{Re} \int_{\Gamma_s} \frac{\partial \omega}{\partial \nu} \cdot (\mathbf{h} \cdot \nabla \omega) \, d\Gamma_s + \frac{1}{2} \int_{\Gamma_s} |\nabla \omega|^2 \mathbf{h} \cdot \nu d\Gamma_s \\
&\quad + \frac{1}{2} \int_{\Omega_s} \left\{ |\nabla \omega|^2 + (1 - \beta^2) |\omega|^2 \right\} \text{div}(\mathbf{h}) \, d\Omega_s - \text{Re} \int_{\Omega_s} \omega^* \cdot (\mathbf{h} \cdot \nabla \omega) \, d\Omega_s. \tag{3.5}
\end{align*}
\]

\[
(b) \int_{\Omega_s} \left\{ |\nabla \omega|^2 + (1 - \beta^2) |\omega|^2 \right\} \text{div}(\mathbf{h}) \, d\Omega_s
\]

\[
= - \text{Re} \int_{\Omega_s} |\nabla \text{div}(\mathbf{h}) \cdot \nabla \omega| \, d\Omega_s - \text{Re} \int_{\Omega_s} \omega^* \cdot \nabla \text{div}(\mathbf{h}) \, d\Omega_s. \tag{3.6}
\]
\[(c)\] Combining (a) and (b), we obtain
\[
\int_{\Omega_s} H(x) \nabla \omega \cdot \nabla \omega d\Omega_s
\]
\[
= -\text{Re} \int_{\Gamma_s} \frac{\partial \omega}{\partial \nu} \cdot (h \cdot \nabla \omega) d\Gamma_s + \frac{1}{2} \int_{\Gamma_s} |\nabla \omega|^2 h \cdot \nu d\Gamma_s
\]
\[
- \frac{1}{2} \text{Re} \int_{\Omega_s} |\nabla \text{div}(h) \cdot \nabla \omega| \cdot \omega d\Omega_s
\]
\[
- \frac{1}{2} \text{Re} \int_{\Omega_s} \omega^* \cdot \omega \text{div}(h) d\Omega_s - \text{Re} \int_{\Omega_s} \omega^* \cdot (h \cdot \nabla \omega) d\Omega_s. \tag{3.7}
\]

**Proof.** The relations (a) and (b) follow immediately from the identities (A.1), (A.2), (A.3) of Proposition A.1 of the Appendix, which holds true with \(\omega_i = z, \eta = (1 - \beta^2), \omega_1^* = -F^c\), upon summing up the component relations relative to vector \(\omega\), and using the fact that \(\omega|_{\Gamma_s} = 0\). Relation (c) corresponds to identity (A.4).

We next specify vector field \(h(x) = (x - x_0)\), where \(x_0 \in \mathbb{R}^n\)—viz., \(h(x)\) is radial—so that the solution \(\omega\) of problem (3.3a–b) (LHS) satisfies
\[
h \cdot \nabla \omega = (h \cdot \nu) \frac{\partial \omega}{\partial \nu}; \quad |\frac{\partial \omega}{\partial \nu}| = |\nabla \omega| \quad \text{on } \Gamma_s; \quad H(x) = I_n; \quad \text{div}(h) = n; \quad \nabla(\text{div}(h)) = 0. \tag{3.8}
\]
Then, upon using these relations (3.8), the identity (3.7) becomes the following:

**Proposition 2.** (i) The solution of the homogeneous problem on the RHS of (3.3a–b) satisfies the relation,
\[
\int_{\Omega_s} |\nabla \omega|^2 d\Omega_s = -\frac{1}{2} \int_{\Gamma_s} \frac{\partial \omega}{\partial \nu} \cdot h \cdot \nu d\Gamma_s + \text{(IT)}, \tag{3.9}
\]
where the interior terms (IT) are given by
\[
\text{(IT)} = -\frac{n}{2} \text{Re} \int_{\Omega_s} \omega^* \cdot \omega d\Omega_s - \text{Re} \int_{\Omega_s} \omega^* \cdot (h \cdot \nabla \omega) d\Omega_s \tag{3.10a}
\]
\[
= \frac{n}{2} \text{Re} \int_{\Omega_s} \omega^* \cdot \omega d\Omega_s + \text{Re} \int_{\Omega_s} \omega \cdot (h \cdot \nabla \omega^*) d\Omega_s. \tag{3.10b}
\]

We note that all the terms on the RHS of (3.9) and (3.10a) are well defined as \(L^2\)-inner products, as \(\omega \in \mathbf{H}^2(\Omega_s)\), while the last integral term on the RHS of (3.10b) is to be interpreted as a duality pairing between \(\omega \in \mathbf{H}_0^1(\Omega_s)\) and \(h \cdot \nabla \omega^* \in \mathbf{H}^{-1}(\Omega_s)\), as \(\omega^* \in \mathbf{L}^2(\Omega_s)\) by (3.4). (ii) The interior terms (IT) in (3.10b) satisfy for \(|\beta| \geq 1\) the estimate
\[
|\text{(IT)}| \leq \epsilon \left( ||\beta \omega||^2_{\mathbf{L}^2(\Omega_s)} + ||\nabla \omega||^2_{\mathbf{L}^2(\Omega_s)} \right)
\]
\[
+ C_{h, \epsilon} \left( ||v_1^*||^2_{\mathbf{H}^1(\Omega_s)} + ||v_2^*||^2_{\mathbf{L}^2(\Omega_s)} + \|\beta + 1\|\mathbb{D}(v_1|_{\Gamma_s})\|^2_{\mathbf{H}^1(\Omega_s)} \right). \tag{3.11}
\]

**Proof.** (i) The relation (3.9) is immediate upon combining (3.7) and (3.8). The relation (3.10b) follows from (3.10a) via Green’s Theorem with \(\omega|_{\Gamma_s} = 0\) by (3.3b).
(ii) We have then from (3.10b), recalling $\omega^*$ in (3.4):

$$
|||\text{IT}||| = \left| \text{Re} \left( \mathbf{h} \cdot \nabla \omega^* + \frac{n}{2} \omega^*, \omega \right) \right|_{\Omega_s} 
\leq \left| \left( \mathbf{h} \cdot \nabla v_1^*, \beta \omega \right)_{\Omega_s} + \left( \mathbf{h} \cdot \nabla v_2^*, \omega \right)_{\Omega_s} + \left( \beta + 1 \right) \mathbf{h} \cdot \nabla \mathcal{D}(v_1|_{\Gamma_s}), \beta \omega \right)_{\Omega_s} 
+ \frac{n}{2} \left( iv_1^*, \beta \omega \right)_{\Omega_s} + \left( iv_2^*, \omega \right)_{\Omega_s} + \left( \beta + 1 \right) \mathcal{D}(v_1|_{\Gamma_s}), \beta \omega \right)_{\Omega_s} \right|
$$

(3.12)

$$
|||\text{IT}||| \leq \epsilon \left( \| \beta \omega \|^2_{L^2(\Omega_s)} + \| \nabla \omega \|^2_{L^2(\Omega_s)} \right) 
+ C_{h,\epsilon} \left( \| v_1^* \|^2_{H^1(\Omega_s)} + \| v_2^* \|^2_{L^2(\Omega_s)} + \| [\beta + 1] \nabla \mathcal{D}(v_1|_{\Gamma_s}) \|^2_{L^2(\Omega_s)} \right), \ |\beta| \geq 1,
$$

(3.13)

via duality pairing on $\left( \mathbf{h} \cdot \nabla v_2^*, \omega \right)_{\Omega_s}$ between $H^{-1}(\Omega_s)$ and $H^1_0(\Omega_s)$; or else by applying a Green Theorem. \hfill \Box

The estimate (3.13) for $|||\text{IT}|||$ is sufficient for the time being; it will be taken to completion in Section 4, Eqn. (4.8). The heart of the matter in this work is Section 5, which deals with an appropriate estimation of $\| \partial_\nu \omega \|_{L^2(\Gamma_s)}$.

4. A preliminary estimate. The goal of this section is to establish the following basic estimate:

**Proposition 3.** The solution variables $f$ and $\omega$ of (2.4a–b) and (3.3a–b), respectively, satisfy say for $|\beta| \geq 3$:

$$
\| f \|^2_{L^2(\Omega, I)} + \| \beta \omega \|^2_{L^2(\Omega, I)} + \| \nabla \omega \|^2_{L^2(\Omega, I)} 
\leq C_h \left( \left\| \frac{\partial \omega}{\partial \nu} \right\|^2_{L^2(\Gamma_s)} + \| \nabla f \|^2_{L^2(\Omega, I)} + \| v_1^* \|^2_{H^1(\Omega_s)} + \| v_2^* \|^2_{L^2(\Omega_s)} \right).
$$

(4.1)

**Proof of Proposition 4.1. Step 1.**

**Lemma 4.1.** (i) With reference to the structural variable $v_1$ in (2.3a–b), and the Dirichlet map in (3.1a), we have for $|\beta| \geq 1$:

$$
\| (\beta + 1) \mathcal{D}(v_1|_{\Gamma_s}) \|^2_{H^1(\Omega_s)} + \left\| \frac{\partial}{\partial \nu} \mathcal{D}(v_1|_{\Gamma_s}) \right\|^2_{H^{-\frac{1}{2}}(\Gamma_s)} 
\leq C \left( \| \nabla f \|^2_{L^2(\Omega_s)} + \| v_1^* \|^2_{H^1(\Omega_s)} \right),
$$

(4.2)

where positive constant $C$ is independent of $\beta$, for say, $|\beta| \geq 1$.

(ii) With reference to the forcing term $\omega^*$ in $L^2(\Omega_s)$ in (3.4), we have for $|\beta| \geq 1$:

$$
\| \omega^* \|^2_{L^2(\Omega_s)} \leq C \left\{ \| v_2^* \|^2_{L^2(\Omega_s)} + \| \nabla f \|^2_{L^2(\Omega_s)} + \| v_1^* \|^2_{H^1(\Omega_s)} \right\}.
$$

(4.3)

**Proof.** (i) Using critically the boundary interface condition in (2.4b) we have

$$
v_1|_{\Gamma_s} = \frac{f|_{\Gamma_s} + v_1^*|_{\Gamma_s}}{i\beta} \text{ on } \Gamma_s.
$$

(4.4)
Subsequently, we invoke the regularity of $D$ in (3.1b), elliptic theory, and (4.4) to obtain the estimate
\[
\|D(v_1|_{\Gamma_*})\|_{H^1(\Omega_\ast)} + \left\| \frac{\partial}{\partial v} D(v_1|_{\Gamma_*}) \right\|_{H^{-\frac{1}{2}}(\Gamma_*)} 
\leq C \left\| v_1|_{\Gamma_*} \right\|_{H^\frac{1}{2}(\Gamma_*)} = \frac{C}{|\beta|} \| f|_{\Gamma_*} + v_1^*|_{\Gamma_*} \|_{H^\frac{1}{2}(\Gamma_*)} 
\leq \frac{C}{|\beta|} \left[ \| \nabla f \|_{L^2(\Omega_f)} + \| v_1^* \|_{H^1(\Omega_\ast)} \right],
\]
which proves (4.2) for $|\beta| \geq 1$.

(ii) Applying the estimate (4.2) for $D(v_1|_{\Gamma_*})$ on the right-hand side of $\omega^*$ in (3.4), we have a fortiorti the estimate (4.3).

**Step 2.** Applying Lemma 4.1(i) to the right-hand side of the relation (3.13) for the interior term (IT), we obtain the following corollary.

**Corollary 1.** (i) The following estimate holds true for the interior term (IT) defined in (3.10):
\[
|\langle IT \rangle | \leq \epsilon \left( \| \beta \omega \|_{L^2(\Omega_\ast)}^2 + \| \nabla \omega \|_{L^2(\Omega_\ast)}^2 \right) 
+ C_{h,e} \left( \| \nabla f \|_{L^2(\Omega_f)}^2 + \| v_1^* \|_{H^1(\Omega_\ast)}^2 + \| v_2^* \|_{L^2(\Omega_\ast)}^2 \right), \quad |\beta| \geq 1. \tag{4.6}
\]

(ii) Our definitive estimate on the (IT)-term (3.10) is, say for $|\beta| \geq \sqrt{3}$:
\[
|\langle IT \rangle | \leq \epsilon \| \nabla \omega \|_{L^2(\Omega_\ast)}^2 + C_{h,e} \left( \| \nabla f \|_{L^2(\Omega_f)}^2 + \| v_1^* \|_{H^1(\Omega_\ast)}^2 + \| v_2^* \|_{L^2(\Omega_\ast)}^2 \right). \tag{4.7}
\]

(iii) The solution $\omega$ of the boundary value problems (3.3a–b) satisfies the estimate, say for $|\beta| \geq \sqrt{3}$:
\[
\| \nabla \omega \|_{L^2(\Omega_\ast)}^2 \leq C_h \left[ \left\| \frac{\partial \omega}{\partial v} \right\|_{L^2(\Gamma_\ast)}^2 + C_h \left( \| \nabla f \|_{L^2(\Omega_f)}^2 + \| v_1^* \|_{H^1(\Omega_\ast)}^2 + \| v_2^* \|_{L^2(\Omega_\ast)}^2 \right) \right]. \tag{4.9}
\]

**Proof.** (i) Multiplying both sides of the differential equation on the RHS of (3.3a) by $\overline{\omega}$ and subsequently integrating by parts via Green Theorem and the B.C. $\omega|_{\Gamma_*} = 0$ in (3.3b), gives
\[
(\beta^2 - 1) \| \omega \|_{L^2(\Omega_\ast)}^2 = \| \nabla \omega \|_{L^2(\Omega_\ast)}^2 + \left( \frac{1}{\beta^2} \omega^*, \beta \omega \right)_{\Omega_\ast}. \tag{4.10}
\]

Upon combining the estimate (4.3) for $\omega^*$ with the basic inequality $|ab| \leq a^2 + C_b b^2$ yields
\[
(\beta^2 - 1) \| \omega \|_{L^2(\Omega_\ast)}^2 \leq \| \nabla \omega \|_{L^2(\Omega_\ast)}^2 + \epsilon |\beta| \| \omega \|_{L^2(\Omega_\ast)}^2 
+ C \left( \| \nabla f \|_{L^2(\Omega_f)}^2 + \| v_1^* \|_{H^1(\Omega_\ast)}^2 + \| v_2^* \|_{L^2(\Omega_\ast)}^2 \right). \tag{4.11}
\]
Then (4.11) readily implies (4.7) for $\epsilon > 0$ small, and say, $\beta^2 \geq (\beta^2 - 1 - \epsilon)$, or say for $|\beta|^2 \geq 3$.

(ii) Using estimate (4.7) on the RHS of estimate (4.6) yields estimate (4.8).

(iii) Returning to identity (3.9) and invoking here estimate (4.8) with $\epsilon > 0$ small, readily yields estimate (4.9).

\textbf{Step 4.} To conclude the proof of Proposition 3, we add the term $\|f\|_{L^2(\Omega_f)}^2 + \|\nabla \omega\|_{L^2(\Omega_f)}^2$ to both sides of the inequality (4.7). On the RHS of the resulting inequality, we invoke estimate (4.9) for $\|\nabla \omega\|^2$, as well as the Poincaré Inequality so as to estimate $\|f\|_{L^2(\Omega_f)}^2 \leq C\|\nabla f\|_{L^2(\Omega_f)}^2$. In this way, upon taking $\epsilon > 0$ small enough, we obtain the sought-after inequality (4.1) of Proposition 4.1.

The next result will be invoked in Section 5 twice, in obtaining estimates (5.37) as well as (5.39).

\textbf{Lemma 4.3.} The solution $\omega$ of problems (3.3a–b) satisfies the following estimate a-fortiori:

$$\|\omega\|_{H^2(\Omega_f)} \leq C \beta^2 \left[ \|\beta v_1\|_{L^2(\Omega_s)}^2 + \|v_2^*, f^*\|_{H^2}^2 \right].$$

\textbf{Proof.} Invoking elliptic theory—see, e.g., [34]—to the boundary value problem on the RHS of (3.3a–b) immediately gives (4.12).

\textbf{5. An estimate for} $\|\frac{\partial \omega}{\partial n}\|_{L^2(\Gamma_f)}$. With reference to estimate (4.1), the objective of the present section is to derive the following estimate:

\textbf{Theorem 5.1.} (i) The normal derivative of the solution $\omega$ of the boundary value problem on the RHS of (3.3a–b) satisfies the estimate, say for $|\beta| \geq 1$:

$$\left\| \frac{\partial \omega}{\partial n} \right\|_{L^2(\Gamma_f)}^2 \leq \epsilon \|\beta v_1\|_{L^2(\Omega_s)}^2$$

$$+ C \epsilon \left[ \beta^2 \|\nabla f\|_{L^2(\Omega_f)}^2 + |\beta| \|f^*\|_{L^2(\Omega_f)}^2 + \|v_1^*, v_2^*, f^*\|_{H^2}^2 \right].$$

(ii) The solution variables $f$ and $\omega$ of (2.4) and (3.3), respectively, satisfy the estimate, say for $|\beta| \geq 3$:

$$\|f\|_{L^2(\Omega_f)}^2 + \|\beta \omega\|_{L^2(\Omega_s)}^2 + \|\nabla \omega\|_{L^2(\Omega_s)}^2$$

$$\leq \epsilon \|\beta v_1\|_{L^2(\Omega_s)}^2 + C \epsilon \left[ \beta^2 \|\nabla f\|_{L^2(\Omega_f)}^2 + |\beta| \|f^*\|_{L^2(\Omega_f)}^2 + \|v_1^*, v_2^*, f^*\|_{H^2}^2 \right].$$

\textbf{Proof of Theorem 5.1(i).} \textbf{Step 1.} It will be critical in our analysis to split the solution of the fluid system (2.4a–b) into three components; i.e.,

$$f = f_1 + f_2 + f_3 \text{ in } \Omega_f,$$

where the $f_i$ satisfy the respective problems (here again, $\partial \Omega_f = \Gamma_s \cup \Gamma_s$):

$$\begin{align*}
\Delta f_1 & = i\beta f \quad \text{ in } \Omega_f, \\
\Delta f_2 & = -f^* \quad \text{ in } \Omega_f, \\
f_1|_{\partial \Omega_f} & = 0 \quad \text{ on } \partial \Omega_f, \\
f_2|_{\partial \Omega_f} & = 0 \quad \text{ on } \partial \Omega_f;
\end{align*}$$

$$\begin{align*}
\Delta f_3 & = 0 \quad \text{ in } \Omega_f, \\
f_3|_{\Gamma_f} & = f|_{\Gamma_f} \quad \text{ on } \Gamma_f; \\
f_3|_{\Gamma_s} & = 0 \quad \text{ on } \Gamma_s.
\end{align*}$$
Step 2. The relevant information concerning the three problems in (5.4) are given next.

Lemma 5.2. (a) With reference to the $f_1$-problem in (5.4), we have

(i) \[
\|\nabla f_1\|_{L^2(\Omega_f)} \leq C \left( \|\nabla f\|_{L^2(\Omega_f)} + \|f^*\|_{L^2(\Omega_f)} \right); \tag{5.5}
\]

(ii) \[
\left\| \frac{\partial f_1}{\partial \nu} \right\|_{H^{\frac{1}{2}}(\partial \Omega_f)} \leq C \|f_1\|_{H^2(\Omega_f)} \leq C |\beta|^{\frac{1}{2}} \left[ \|\nabla f\|_{L^2(\Omega_f)} + \|f^*\|_{L^2(\Omega_f)} \right]; \tag{5.6}
\]

(iii) \[
\left\| \frac{\partial f_1}{\partial \nu} \right\|_{L^2(\partial \Omega_f)} \leq C |\beta|^{\frac{1}{2}} \left[ \|\nabla f\|_{L^2(\Omega_f)} + \|f^*\|_{L^2(\Omega_f)} \right]. \tag{5.7}
\]

(b) The following estimate holds (an improvement over a straightforward majorization obtained by Poincaré’s Inequality):

\[
\|f\|_{L^2(\Omega_f)} \leq \frac{C}{|\beta|^{\frac{1}{2}}} \left[ \|\nabla f\|_{L^2(\Omega_f)} + \|f^*\|_{L^2(\Omega_f)} \right]. \tag{5.8}
\]

Proof. (a)(i) Using the $f_1$-equation in (5.4) we have

\[
(\Delta f_1, f_1)_{\Omega_f} = (i\beta f, f_1)_{\Omega_f} = (\Delta f + f^*, f_1)_{\Omega_f},
\]

after invoking the fluid system (2.4). Applying Green’s First Theorem to both sides of the relation, and using $f_1|_{\partial \Omega_f} = 0$ will then yield via Poincaré Inequality,

\[
\|\nabla f_1\|_{L^2(\Omega_f)}^2 = (\nabla f, \nabla f_1)_{\Omega_f} - (f^*, f_1)_{\Omega_f} \leq \|\nabla f\|_{L^2(\Omega_f)} \|\nabla f_1\|_{L^2(\Omega_f)} + C \|f^*\|_{L^2(\Omega_f)} \|\nabla f_1\|_{L^2(\Omega_f)}, \tag{5.9}
\]

which establishes (5.5) by using $2ab \leq eb^2 + \frac{1}{b^2}$.

(a)(ii) The $f_1$-equation in (5.4a) furthermore yields $f = \frac{\Delta f_1}{\beta}$. Hence

\[
\|f\|_{H^{-1}(\Omega_f)} = \frac{1}{|\beta|} \|\Delta f_1\|_{H^{-1}(\Omega_f)}; \text{ i.e., } \|f\|_{H^{-1}(\Omega_f)} \leq C \frac{\|\nabla f_1\|_{L^2(\Omega_f)}}{|\beta|}; \tag{5.10}
\]

(again using Poincaré’s Inequality via (5.4b)). But $L^2(\Omega_f) = H^0(\Omega_f) = [H^1(\Omega_f), H^{-1}(\Omega_f)]$—see [34, Lemma 12.1, p. 73]—and this, along with the interpolation inequality of [34, Proposition 2.3, p. 19] will yield via (5.11) and Poincaré Inequality

\[
\|f\|_{L^2(\Omega_f)} \leq C \|f_1\|_{H^{-1}(\Omega_f)} \|f\|_{H^1(\Omega_f)} \leq \frac{C}{|\beta|^{\frac{1}{2}}} \|\nabla f_1\|_{H^1(\Omega_f)} \|\nabla f\|_{L^2(\Omega_f)}; \tag{5.12}
\]

In addition, a direct elliptic estimate for the $f_1$-problem in (5.4) is

\[
\left\| \frac{\partial f_1}{\partial \nu} \right\|_{L^2(\partial \Omega_f)} \leq C \|f_1\|_{H^1(\Omega_f)} \leq C |\beta| \|f\|_{L^2(\Omega_f)}. \tag{5.13}
\]
Substituting the estimate (5.12) into the right-hand side of (5.13), followed by the estimate (5.5), we then have for $|\beta| \geq 1$,

$$\left\| \frac{\partial f_1}{\partial \nu} \right\|_{H^1/2(\Gamma_s)} \leq C \| f_1 \|_{H^2(\Omega_f)}$$

(5.14)

$$\leq C |\beta|^{\frac{1}{2}} \|
abla f_1 \|_{L^2(\Omega_f)}^{\frac{1}{2}} \| \nabla f \|_{L^2(\Omega_f)}^{\frac{1}{2}}$$

(5.15)

(by (5.5))

$$\leq C |\beta|^{\frac{1}{2}} \left[ \| \nabla f \|_{L^2(\Omega_f)} + \| f^* \|_{L^2(\Omega_f)} \right].$$

(5.16)

This is the estimate (5.6).

(a)(iii) We use the trace moment inequality and subsequently invoke the estimate (5.6) for $\| f_1 \|_{H^2(\Omega_f)}$, and estimate (5.5) for $\| f_1 \|_{H^1(\Omega_f)}$. Thereby we obtain

$$\left\| \frac{\partial f_1}{\partial \nu} \right\|_{L^2(\partial \Omega_f)} \leq C \| f_1 \|_{H^2(\Omega_f)} \| f_1 \|_{H^1(\Omega_f)}$$

(5.17)

(by (5.6), (5.5))

$$\leq C \left[ |\beta|^{\frac{1}{2}} \| \nabla f \|_{L^2(\Omega_f)} + |\beta|^{\frac{1}{2}} \| f^* \|_{L^2(\Omega_f)} \right]$$

(5.18)

$$\leq C \left[ |\beta|^{\frac{1}{2}} \| \nabla f \|_{L^2(\Omega_f)} + |\beta|^{\frac{1}{2}} \| f^* \|_{L^2(\Omega_f)} \right]$$

(5.19)

This establishes (5.7).

(b) Substituting the estimate (5.5) into the right-hand side of (5.12) yields

$$\| f \|_{L^2(\Omega_f)} \leq \frac{C}{|\beta|^{\frac{1}{2}}} \left[ \| \nabla f \|_{L^2(\Omega_f)} + \| f^* \|_{L^2(\Omega_f)} \right] \| \nabla f \|_{L^2(\Omega_f)}^{\frac{1}{2}},$$

(5.21)

from which (5.8) readily follows.

By a immediate application of elliptic and Sobolev Trace Theory, we also have the following lemmas.

**Lemma 5.3.** With reference to the $f_2$-problem in (5.4) we have

$$\left\| \frac{\partial f_2}{\partial \nu} \right\|_{H^{\frac{1}{2}}(\partial \Omega_f)} \leq C \| f_2 \|_{H^2(\Omega_f)} \leq C \| f^* \|_{L^2(\Omega_f)}.$$

(5.22)
Lemma 5.4. With reference to the $f_3$-problem in (5.4) we have

$$\left\| \frac{\partial f_3}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\partial \Omega_f)} \leq C \| f_3 \|_{H^1(\Omega_f)} \leq C \| f_3 \|_{H^1(\Gamma_s)}$$

$$= C \| f \|_{H^1(\Gamma_s)} \leq C \| \nabla f \|_{L^2(\Omega_f)}.$$  \hspace{1cm} (5.23)

Step 3. From $\omega = v_1 - D(v_1|_{\Gamma_s}) \in H^2(\Omega_s)$ in (3.2), we obtain

$$\frac{\partial \omega}{\partial \nu}|_{\Gamma_s} = \frac{\partial v_1}{\partial \nu} |_{\Gamma_s} - \frac{\partial D(v_1|_{\Gamma_s})}{\partial \nu} |_{\Gamma_s}.$$  \hspace{1cm} (5.24)

where we note that $\frac{\partial v_1}{\partial \nu} |_{\Gamma_s}, \frac{\partial D(v_1|_{\Gamma_s})}{\partial \nu} |_{\Gamma_s} \in H^{-\frac{1}{2}}(\Gamma_s)$ only, by (2.3b) (in the definition of $D(A)$) and (3.1). We have then, by (5.24) in estimating the critical boundary term in (4.1),

$$\left\| \frac{\partial \omega}{\partial \nu} \right\|_{L^2(\Gamma_s)}^2 = \left( \frac{\partial \omega}{\partial \nu} |_{\Gamma_s} \frac{\partial \omega}{\partial \nu} |_{\Gamma_s} \right)_{\Gamma_s}$$

(by (5.24))

$$= \left\langle \frac{\partial \omega}{\partial \nu} |_{\Gamma_s}, \frac{\partial v_1}{\partial \nu} |_{\Gamma_s} \right\rangle - \left\langle \frac{\partial \omega}{\partial \nu} |_{\Gamma_s}, \frac{\partial D(v_1|_{\Gamma_s})}{\partial \nu} |_{\Gamma_s} \right\rangle_{\Gamma_s}$$  \hspace{1cm} (5.26)

(by (2.3b))

$$= \left\langle \frac{\partial \omega}{\partial \nu} |_{\Gamma_s}, \frac{\partial f_1}{\partial \nu} |_{\Gamma_s} \right\rangle - \left\langle \frac{\partial \omega}{\partial \nu} |_{\Gamma_s}, \frac{\partial D(v_1|_{\Gamma_s})}{\partial \nu} |_{\Gamma_s} \right\rangle_{\Gamma_s}.$$  \hspace{1cm} (5.27)

after using (2.3b). (Here, we have used the notation $\langle \cdot, \cdot \rangle_{\Gamma_s}$ to signify duality pairings between $\frac{\partial \omega}{\partial \nu} |_{\Gamma_s} \in H^2(\Gamma_s)$ and $\frac{\partial v_1}{\partial \nu} |_{\Gamma_s}, \frac{\partial D(v_1|_{\Gamma_s})}{\partial \nu} |_{\Gamma_s} \in H^{-\frac{1}{2}}(\Gamma_s)$.) Invoking the decomposition in (5.3) above, we then have from (5.27),

$$\left\| \frac{\partial \omega}{\partial \nu} |_{\Gamma_s} \right\|_{L^2(\Gamma_s)}^2 = \sum_{i=1}^{3} \left\langle \frac{\partial \omega}{\partial \nu} |_{\Gamma_s}, \frac{\partial f_i}{\partial \nu} |_{\Gamma_s} \right\rangle_{\Gamma_s} - \left\langle \frac{\partial \omega}{\partial \nu} |_{\Gamma_s}, \frac{\partial D(v_1|_{\Gamma_s})}{\partial \nu} |_{\Gamma_s} \right\rangle_{\Gamma_s}.$$  \hspace{1cm} (5.28)

Lemma 5.5. Regarding the first term on the right-hand side of (5.28), we have

$$\left\| \frac{\partial \omega}{\partial \nu} \right\|_{L^2(\Gamma_s)}^2 \leq \epsilon \left\| \frac{\partial \omega}{\partial \nu} \right\|_{L^2(\Gamma_s)}^2 + C \left[ |\beta|^{\frac{1}{2}} \| \nabla f \|_{L^2(\Omega_f)}^2 + |\beta|^{\frac{1}{2}} \| f^* \|_{L^2(\Omega_f)}^2 \right].$$  \hspace{1cm} (5.29)
Proof. We invoke the estimate (5.7) in majorizing as follows:

\[
\left\| \left( \frac{\partial \omega}{\partial \nu}_{\Gamma_s}^{\cdot}, \frac{\partial f_1}{\partial \nu}_{\Gamma_s}^{\cdot} \right) \right\|_{L^2(\Gamma_s)} \leq \left\| \frac{\partial \omega}{\partial \nu}_{\Gamma_s}^{\cdot} \right\|_{L^2(\Gamma_s)} \left\| \frac{\partial f_1}{\partial \nu}_{\Gamma_s}^{\cdot} \right\|_{L^2(\Gamma_s)}
\]

(5.30)

\[
\leq \frac{\epsilon}{2} \left\| \frac{\partial \omega}{\partial \nu}_{\Gamma_s}^{\cdot} \right\|_{L^2(\Gamma_s)}^2 + \frac{\epsilon}{2} \left\| \frac{\partial f_1}{\partial \nu}_{\Gamma_s}^{\cdot} \right\|_{L^2(\Gamma_s)}^2
\]

(5.31)

(by (5.7)) \leq \frac{\epsilon}{2} \left\| \frac{\partial \omega}{\partial \nu}_{\Gamma_s}^{\cdot} \right\|_{L^2(\Gamma_s)}^2 + C_\epsilon \left\| f^* \right\|_{L^2(\Omega_f)}^2.
\]

(5.32)

Lemma 5.6. Regarding the second term on the right-hand side of (5.28), we have

\[
\left\| \left( \frac{\partial \omega}{\partial \nu}_{\Gamma_s}^{\cdot}, \frac{\partial f_2}{\partial \nu}_{\Gamma_s}^{\cdot} \right) \right\|_{L^2(\Gamma_s)} \leq \epsilon \left\| \frac{\partial \omega}{\partial \nu}_{\Gamma_s}^{\cdot} \right\|_{L^2(\Gamma_s)}^2 + C_\epsilon \left\| f^* \right\|_{L^2(\Omega_f)}^2.
\]

(5.33)

Proof. The estimate (5.33) follows at once a-fortiori from (5.22).

Lemma 5.7. Regarding the fourth term on the right-hand side of (5.28), we have

\[
\left\| \left( \frac{\partial \omega}{\partial \nu}_{\Gamma_s}^{\cdot}, \frac{\partial \mathcal{D}(v_1)|_{\Gamma_s}}{\partial \nu} \right) \right\|_{L^2(\Gamma_s)} \leq \epsilon \|v_1\|_{L^2(\Omega_s)}^2 + C_\epsilon \left[ \|v_1^*, v_2^*, f^*\|_{H}^2 + \|\nabla f\|_{L^2(\Omega_f)}^2 \right] .
\]

(5.34)

Proof. We exploit the duality pairing \langle \cdot, \cdot \rangle_{\Gamma_s} and invoke estimate (4.2) for \| \frac{\partial \mathcal{D}(v_1)|_{\Gamma_s}}{\partial \nu} \|_{H^{-\frac{1}{2}}(\Gamma_s)} as well as Sobolev trace theory and estimate (4.12). By these means we then obtain for \| \beta \| \geq 1

\[
\left\| \left( \frac{\partial \omega}{\partial \nu}_{\Gamma_s}^{\cdot}, \frac{\partial \mathcal{D}(v_1)|_{\Gamma_s}}{\partial \nu} \right) \right\|_{L^2(\Gamma_s)} \leq \left\| \frac{\partial \omega}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\Gamma_s)} \left\| \frac{\partial \mathcal{D}(v_1)|_{\Gamma_s}}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\Gamma_s)}
\]

(5.35)

(by (4.2)) \leq \frac{C}{|\beta|} \|\omega\|_{H^2(\Omega_s)} \left[ \|\nabla f\|_{L^2(\Omega_f)} + \|v_1^*\|_{H^1(\Omega_s)} \right] \]

(5.36)

(by (4.12)) \leq C \left[ \|\beta v_1\|_{L^2(\Omega_s)} + \|v_1^*, v_2^*, f^*\|_{H} \right] \left[ \|\nabla f\|_{L^2(\Omega_f)} + \|v_1^*\|_{H^1(\Omega_s)} \right] .
\]

(5.37)

From this inequality (5.37), then (5.34) readily follows.

Remark 5.1. Up to this point, that is, neglecting for the moment the third term \langle \frac{\partial \omega}{\partial \nu}, \frac{\partial f_3}{\partial \nu} \rangle_{\Gamma_s} on the RHS of (5.28), then according to (2.6) in the orientation in Section 2, the analysis so far carried out would yield the value of the parameter in Theorem 1.2 to be \alpha = \frac{1}{2}, as dictated by (5.29); while (5.33) and (5.34) would yield “\alpha = 0.” This is seen by substituting estimates (5.29), (5.33), (5.34) on the RHS of identity (5.28) [with the above third term neglected] and using the resulting estimate for \| \frac{\partial \omega}{\partial \nu} \|_{L^2(\Gamma_s)} on the RHS of the basic estimate (4.1) of Proposition 4.1.
It is the estimate of the third so far ‘neglected’ term—the one in the subsequent Lemma 5.8—which is responsible for obtaining $\alpha = 2$ in the sought-after estimate (2.6), when substituting the estimate for the RHS of (5.28) in (4.1). Lemma 5.8 below will be sharpened in [6] by use of a lengthy and technical microlocal argument as in [26, 29, 30, 31] to ultimately obtain $\alpha = 1$, the optimal value.

Lemma 5.8. Regarding the third term on the right-hand side of (5.28), we have

$$\left\langle \frac{\partial \omega}{\partial \nu}, \frac{\partial f_3}{\partial \nu} \right\rangle_{\Gamma_s} \leq \epsilon \| \beta v_1 \|^2_{L^2(\Omega_s)} + C \epsilon \left[ \beta^2 \| \nabla f \|^2_{L^2(\Omega_f)} + \|[v^*_1, v^*_2, f^*]_H\|^2 \right].$$

(5.38)

Proof. By invoking the duality pairing $\langle \cdot, \cdot \rangle_{\Gamma_s}$, trace theory, and estimates (5.23) and (4.12), we obtain

$$\left\langle \frac{\partial \omega}{\partial \nu}, \frac{\partial f_3}{\partial \nu} \right\rangle_{\Gamma_s} \leq \epsilon \| \beta v_1 \|^2_{L^2(\Omega_s)} + C \epsilon \left[ \beta^2 \| \nabla f \|^2_{L^2(\Omega_f)} + \|[v^*_1, v^*_2, f^*]_H\|^2 \right].$$

(5.42)

This estimate (5.43) now yields (5.1), upon a rescaling of $\epsilon$, small enough.

(ii) In turn, applying the estimate (5.1) to the right-hand side of (4.1) yields (5.2).

This concludes the proof of Theorem 5.1.
6. Completion of the proof of Theorem 1.3: From \( \omega \) back to \( v_1 \).

**Step 1. Proposition 6.1.** With reference to the variables \( v_1 \) and \( f \) of problems (2.3), (2.4), we have for \( |\beta| \geq \sqrt{3} \):

\[
\|f\|^2_{L^2(\Omega_f)} + \|\beta v_1\|^2_{L^2(\Omega_v)} + \|\nabla v_1\|^2_{L^2(\Omega_v)} \\
\leq \epsilon \|\beta v_1\|^2_{L^2(\Omega_v)} + C_{\epsilon} \left[ \beta^2 \|\nabla f\|^2_{L^2(\Omega_f)} + |\beta| \frac{1}{2} \|f^*\|^2_{L^2(\Omega_f)} + \|v_1^*, v_2^*, f^*\|^2_{H} \right].
\]

(6.1)

**Proof.** Using the expression \( \omega = v_1 - \mathcal{D}(v_1|_{\Gamma_\omega}) \) in (3.2), we obtain *a fortiori* from estimate (4.2),

\[
\|\beta v_1\|^2_{L^2(\Omega_v)} \leq \|\beta \omega\|^2_{L^2(\Omega_v)} + |\beta| \|\mathcal{D}(v_1|_{\Gamma_\omega})\|^2_{L^2(\Omega_v)}
\]

(by (4.2)) \leq \|\beta \omega\|^2_{L^2(\Omega_v)} + C \left[ \|\nabla f\|^2_{L^2(\Omega_f)} + \|v_1^*\|^2_{H^1(\Omega_v)} \right].

(6.2)

In the same way, starting from (3.2):

\[
\|\nabla v_1\|^2_{L^2(\Omega_v)} \leq \|\nabla \omega\|^2_{L^2(\Omega_v)} + \|\nabla \mathcal{D}(v_1|_{\Gamma_\omega})\|^2_{L^2(\Omega_v)}
\]

(by (4.2)) \leq \|\nabla \omega\|^2_{L^2(\Omega_v)} + C \left[ \|\nabla f\|^2_{L^2(\Omega_f)} + \|v_1^*\|^2_{H^1(\Omega_v)} \right].

(6.3)

Next, square estimates (6.2) and (6.3) and add them up; moreover, add the term \( \|f\|^2_{L^2(\Omega_f)} \) to both sides of the resulting inequality. We then obtain for \( |\beta| \geq 1 \),

\[
\|f\|^2_{L^2(\Omega_f)} + \|\beta v_1\|^2_{L^2(\Omega_v)} + \|\nabla v_1\|^2_{L^2(\Omega_v)} \\
\leq \|f\|^2_{L^2(\Omega_f)} + 2 \|\beta \omega\|^2_{L^2(\Omega_v)} + 2 \|\nabla \omega\|^2_{L^2(\Omega_v)}
\]

\[
+ C \left[ \|\nabla f\|^2_{L^2(\Omega_f)} + \|v_1^*\|^2_{H^1(\Omega_v)} \right]
\]

(by (5.2)) \leq \epsilon \|\beta v_1\|^2_{L^2(\Omega_v)}

\[
+ C_{\epsilon} \left[ \beta^2 \|\nabla f\|^2_{L^2(\Omega_f)} + |\beta| \frac{1}{2} \|f^*\|^2_{L^2(\Omega_f)} + \|v_1^*, v_2^*, f^*\|^2_{H} \right].
\]

(6.4)

where in the last step we have invoked the estimate (5.2). This establishes the estimate (6.1).

**Step 2. Proposition 6.2.** Recalling the term \( \mathbf{X} \) of (2.5a), the following estimate holds true for all \( |\beta| \geq \sqrt{3} \):

\[
\mathbf{X}^2 = \|v_1\|^2_{H^1(\Omega_v)} + \|v_2\|^2_{L^2(\Omega_v)} + \|f\|^2_{L^2(\Omega_f)}
\]

\[
\leq C \left[ \beta^2 \|\nabla f\|^2_{L^2(\Omega_f)} + |\beta| \frac{1}{2} \|f^*\|^2_{L^2(\Omega_f)} + \|v_1^*, v_2^*, f^*\|^2_{H} \right].
\]

(6.5)
Proof. We return to the estimate (6.1) and add \( \frac{1}{4} \| v_2 \|_{L^2(\Omega_s)}^2 \) to both sides thereof, so as to have
\[
\| f \|_{L^2(\Omega_f)}^2 + \| \beta v_1 \|_{L^2(\Omega_s)}^2 + \| \nabla v_1 \|_{L^2(\Omega_s)}^2 + \frac{1}{4} \| v_2 \|_{L^2(\Omega_s)}^2 \\
\leq \epsilon \| \beta v_1 \|_{L^2(\Omega_s)}^2 + \epsilon \| \nabla v_1 \|_{L^2(\Omega_s)}^2 + \frac{1}{4} \| v_2 \|_{L^2(\Omega_s)}^2 \\
+ \frac{\| v_2 \|_{L^2(\Omega_s)}^2}{4},
\]
(6.6)

On the right-hand side, we recall from (2.2) that \( v_2 = i \beta v_1 - v_1^* \); and hence
\[
\frac{1}{4} \| v_2 \|_{L^2(\Omega_s)}^2 \leq \frac{1}{2} \left[ \| \beta v_1 \|_{L^2(\Omega_s)}^2 + \| v_1^* \|_{L^2(\Omega_s)}^2 \right].
\]
(6.7)

Applying (6.6) to (6.5) now yields
\[
\| f \|_{L^2(\Omega_f)}^2 + \left(1 - \frac{1}{2} - \epsilon \right) \| \beta v_1 \|_{L^2(\Omega_s)}^2 + \| \nabla v_1 \|_{L^2(\Omega_s)}^2 + \frac{\| v_2 \|_{L^2(\Omega_s)}^2}{4} \\
\leq \epsilon \beta^2 \| \nabla f \|_{L^2(\Omega_f)}^2 + \| \beta \|_{L^2(\Omega_f)}^2 \| f^* \|_{L^2(\Omega_f)}^2 + \| [v_1^*, v_2^*, f^*] \|_{H}^2,
\]
(6.8)

thereby leading to (6.5). \( \square \)

Recalling (2.5b), with \((Y)^2 = \| v_1^* \|_{H^1(\Omega_s)}^2 + \| v_2^* \|_{L^2(\Omega_s)}^2 + \| f^* \|_{L^2(\Omega_f)}^2\), we see that the estimate (6.5) gives us the requisite inequality (2.6), with therein \( \alpha = 2 \) and \( \sigma = \frac{1}{2} < 2 \alpha = 4 \). Thus, the argument immediately below (2.6) applies, and yields the desired estimate (2.10) with \( \alpha = 2 \).

This concludes the proof of Theorem 1.3.

Appendix A. The static wave (elliptic) identity.

Proposition 4. Let (scalar-valued) function \( z \) be a “smooth enough” solution to the following equation (with no boundary conditions imposed):
\[
\eta z - \Delta z = F \quad \text{in } \Omega_s,
\]
(A.1)

where real-valued forcing term \( F \in L^2(\Omega_s) \), and parameter \( \eta \in \mathbb{R} \). Let, moreover, \( h(x) \) be any \([C^2(\Omega_s)]^n\)-vector field. Then \( z \) obeys the following basic identities:

(i)
\[
\int_{\Omega_s} H(x) \nabla z \cdot \nabla z d\Omega_s \\
= - \int_{\Gamma_s} \frac{\partial z}{\partial n} (h \cdot \nabla z) d\Gamma_s + \frac{\eta}{2} \int_{\Gamma_s} z^2 h \cdot \nu d\Gamma_s + \frac{1}{2} \int_{\Gamma_s} |\nabla z|^2 h \cdot \nu d\Gamma_s \\
+ \frac{1}{2} \int_{\Omega_s} \left( |\nabla z|^2 + \eta z^2 \right) \text{div}(h) d\Omega_s + \int_{\Omega_s} F (h \cdot \nabla z) d\Omega_s,
\]
(A.2)

where \( H \) above is the Jacobian of vector field \( h \).
(ii) \[
\int_{\Omega_s} \left\{ |\nabla z|^2 + \eta z^2 \right\} \text{div}(\mathbf{h}) \, d\Omega_s \\
= - \int_{\Gamma_s} \frac{\partial z}{\partial \mathbf{n}} \text{div}(\mathbf{h}) \, d\Gamma_s \\
- \int_{\Omega_s} \nabla z \cdot [z \text{div}(\mathbf{h})] \, d\Omega_s + \int_{\Omega_s} Fz \text{div}(\mathbf{h}) \, d\Omega_s. \tag{A.3}
\]

(iii) Combining (A.2) and (A.3),
\[
\int_{\Omega_s} H(x) \nabla z \cdot \nabla z \, d\Omega_s \\
= - \int_{\Gamma_s} \frac{\partial z}{\partial \mathbf{n}} (\mathbf{h} \cdot \nabla z) \, d\Gamma_s + \frac{\eta}{2} \int_{\Gamma_s} z^2 \mathbf{h} \cdot \mathbf{n} \, d\Gamma_s + \frac{1}{2} \int_{\Gamma_s} |\nabla z|^2 \mathbf{h} \cdot \mathbf{n} \, d\Gamma_s \\
- \frac{1}{2} \int_{\Gamma_s} \frac{\partial z}{\partial \mathbf{n}} \text{div}(\mathbf{h}) \, d\Omega_s - \frac{1}{2} \int_{\Omega_s} \nabla z \cdot [z \text{div}(\mathbf{h})] \, d\Omega_s \\
+ \frac{1}{2} \int_{\Omega_s} Fz \text{div}(\mathbf{h}) \, d\Omega_s + \int_{\Omega_s} F(\mathbf{h} \cdot \nabla z) \, d\Omega_s. \tag{A.4}
\]

**Proof.** The above identities (A.2) and (A.3) for the solution \( z \) of Eqn. (A.1) are nothing but the static wave versions corresponding to the well-known dynamic identities [24, Eqn. (2.27), p. 157], [25, Eqn. (2.18), Eqn. (2.20), p. 255], [43, p. ], [28, p. 959]. They are proved the same way. Thus, identity (A.2) and (A.3) follow by multiplying Eqn. (A.1) by \( \mathbf{h} \cdot \nabla z \) and \( z \text{div}(\mathbf{h}) \), respectively, and integrating by parts. For details, we refer to the dynamic case of these references.

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