ON RELATIONS AMONG MULTIPLE ZETA VALUES OBTAINED IN KNOT THEORY

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Abstract. This paper focuses on linear and algebraic relations among multiple zeta values which were obtained in knot theory. It is shown that they can be derived from the associator relations, i.e. the pentagon equation and the shuffle relation.

0. Introduction

The multiple zeta value (MZV in short) is the real number defined by the following power series

\[
\zeta(k) := \sum_{0 < n_1 < \cdots < n_m} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}}
\]

for \( k = (k_1, \ldots, k_m) \) with \( m, k_1, \ldots, k_m \in \mathbb{N}(= \mathbb{Z}_{>0}) \) which is admissible, i.e. \( k_m > 1 \) (its convergent condition). The MZV's were studied allegedly first by Euler [E] for \( m = 1 \) and 2. They have recently undergone a huge revival of interest due to their appearance in various branches of mathematics and physics. In connection with motive theory ([An, Br, DG]), linear and algebraic relations among MZV's are particularly important.

As far as the author knows, there are 4 relations among MZV's which were obtained in knot theory:

(A) Le-Murakami relation (1995) in [LM1] Lemma 3.2.1 and Theorem 3.3.1:

For any integer \( N \geq 2 \) (actually \( N \) can be a variable),

\[
N \frac{\sinh h}{\sinh Nh} = 1 + \sum_{k: \text{admissible}} \frac{(-1)^{\text{dp}(k)}(1 - N^2)(2N)^{\text{wt}(k)}}{N^2\text{ht}(k)(2\pi\sqrt{-1})^{\text{wt}(k)}} \zeta(k) h^{\text{wt}(k)}
\]

holds in \( \mathbb{C}[[h]] \). Here \( \text{wt}(k) := k_1 + \cdots + k_m, \text{dp}(k) := m \) and \( \text{ht}(k) := \sharp\{i \mid k_i > 1\} \).

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(B) *Le-Murakami relation (1996)* in [LM2] Theorem 4.6: For any integer \( N \geq 3 \) (actually \( N \) can be a variable),

\[
\frac{N \sinh h}{\sinh(N-1)h + \sinh h} = 1 + \sum_{k: \text{admissible}} \frac{(-1)^{dp(k)}2^{\text{wt}(k)}g(k)}{(2\pi \sqrt{-1})^{\text{wt}(k)}} \zeta(k)h^{\text{wt}(k)}
\]

holds in \( \mathbb{C}[[h]] \) where

\[
g(k_1, \ldots, k_m) := (0, 1, 0) \cdot uv^{k_1-1}uv^{k_2-1} \cdots uv^{k_m-1} \cdot (1, 1, N)^t \in \mathbb{Z}
\]

with

\[
u := \begin{pmatrix} N-1 & 1 & -1 \\ 1 & N-1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad v := \begin{pmatrix} N-1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & N-1 \end{pmatrix}.
\]

(C) *Takamuki relation* in [T] Proposition 5, 6 and Theorem 3: For any integer \( N \geq 2 \) (actually \( N \) can be a variable),

\[
N^2 \frac{\sinh h}{\sinh Nh} \cdot \frac{\cosh Nh}{\cosh h} = N + (N^2 - 1)N.
\]

\[
\sum_{n=1}^{\infty} \sum_{k=1}^{n} \sum_{(p, q, r) \in I_2n, k} (-1)^{\text{wt}(r)}2^{n-k} A(p, q, r; N) \left( \frac{\zeta(\tau(p + r, q))}{(2\pi \sqrt{-1})^{2n}} \right)
\]

\[
\sum_{2 \leq l, m \leq 2n-2} \sum_{i=1}^{[l/2]} \sum_{j=1}^{[m/2]} \sum_{(p, q, r) \in I_{li}, (s, t, u) \in I_{mj}} (-1)^{\text{wt}(u) + \text{wt}(r) + m} 2^{n-i-j-1} B(p, q, r, s, t, u; N)
\]

\[
\cdot \left( \begin{pmatrix} p + r \\ s + u \end{pmatrix} \left( \begin{pmatrix} r \\ u \end{pmatrix} \right) \left( \begin{pmatrix} \zeta(\tau(p + r, q)) \zeta(\tau(s + t, u)) \end{pmatrix} \right) \right) h^{2n}
\]

holds in \( \mathbb{C}[[h]] \). For \( p = (p_1, \ldots, p_k), q = (q_1, \ldots, q_k) \in \mathbb{Z}^k_{>0} \) we put

\[
\begin{pmatrix} p + q \\ q \end{pmatrix} := \prod_{i=1}^{r} \begin{pmatrix} p_i + q_i \\ q_i \end{pmatrix} \quad \text{and} \quad \tau(p, q) := \begin{pmatrix} 1^{p_1-1}, q_1+1, \ldots, 1^{p_k-1}, q_k+1 \end{pmatrix}
\]

where \( 1^n \) means the index where 1 repeats \( n \)-times. We also put

\[
I_{n,k} := \{(p, q, r) \mid p, q, r \in \mathbb{Z}^k_{>0}, \text{wt}(p+q+r) = n, q_i \geq 1, p_i+q_i \geq 1, p_i \geq 1\},
\]

\[
A(p, q, r; x) := x^{\text{wt}(q)-k} \left\{(x-1)^{p_1} - (x+1)^{p_1}\right\} \left\{(x-1)^{\text{wt}(r)} - (x+1)^{\text{wt}(r)}\right\}
\]

\[
\cdot \prod_{a=2}^{i} \left\{ (x-1)^{p_i+1} + (x+1)^{p_i+1}\right\},
\]

\[
B(p, q, r, s, t, u; x) := \frac{A(p, q, r; x)A(s, t, u; x)\{(x-1)^{\text{wt}(r+u)} - (x+1)^{\text{wt}(r+u)}\}}{\{(x-1)^{\text{wt}(r)} - (x+1)^{\text{wt}(r)}\}} \left\{ (x-1)^{\text{wt}(u)} - (x+1)^{\text{wt}(u)}\right\},
\]
for \((p, q, r) \in I_{l,i}\) and \((s, t, u) \in I_{m,j}\).

(D) Ihara-Takamuki relation in [II] Theorem 2:

\[
\frac{7^2 \cdot [6]_q [4]_q}{[12]_q [7]_q [2]_q} = 7 + \sum_{p, q \text{ dp}(p) = \text{dp}(q)} \frac{(-1)^{\text{wt}(p)} w(p, q)}{(2\pi \sqrt{-1})^{\text{wt}(p)+\text{wt}(q)}} \zeta(\tau(p, q)) h^{\text{wt}(p)+\text{wt}(q)}
\]

holds in \(\mathbb{C}[[h]]\) where

\[
[n]_q := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}
\]

for an integer \(n \geq 1\) with \(q := e^h\) and

\[
w(p, q) := -(7, 7, 7, 7) \cdot x^{p_1} y^{q_1} \cdots x^{p_k} y^{q_k} \cdot (27, 7, 14, 0)^t \in \mathbb{Q}
\]

for \(p = (p_1, \ldots, p_k), q = (q_1, \ldots, q_k) \in \mathbb{Z}_{>0}^k\) with

\[
x := \begin{pmatrix} -14 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
y := \begin{pmatrix} 5/14 & -9/14 & -9/14 & 27/14 \\ -3/14 & 1/14 & 3/14 & 1/14 \\ 3/7 & 1/7 & 1/7 & 1/7 \end{pmatrix}.
\]

Note that (C) is an algebraic relation.

The purpose of this paper is to clarify that the above 4 relations can be derived from the associator relations, namely from the pentagon relation and the shuffle relation (see Definition 1.1 and Theorem 1.2).

**Theorem 0.1.** Let \((\mu, \varphi)\) be any associator (see Definition 1.1 below). Then the relations replacing \(\zeta(k_1, \ldots, k_m)\) with \(\zeta_{\varphi}(k_1, \ldots, k_m)\) and \(2\pi \sqrt{-1}\) with \(\mu\) in (A)-(D) hold.

Here for each series \(\varphi \in \mathbb{C}\langle\langle X_0, X_1 \rangle\rangle\) we denote the coefficient of \(X_0^{k_{m-1}} X_1 \cdots X_0^{k_{1-1}} X_1\) \((m, k_1, \ldots, k_m \in \mathbb{N} \text{ and } k_m > 1)\) in \(\varphi\) multiplied with \((-1)^m\) by

\[
\zeta_{\varphi}(k_1, \ldots, k_m) \in \mathbb{C}.
\]

The contents of this paper go as follows: [II] is a review of the formalism of associators and the Grothendieck-Teichmüller group. In [II] we show auxiliary lemmas to prove Theorem 0.1 in [III].

1. **ASSOCIATORS AND THE GROTHENDIECK-TEICHMÜLLER GROUP**

Denote by \(U\mathfrak{F}_2 = \mathbb{C}\langle\langle X_0, X_1 \rangle\rangle\) the non-commutative formal power series ring, which is regarded as the completion of the universal enveloping algebra of the free Lie algebra \(\mathfrak{F}_2\) with two variables \(X_0\) and \(X_1\). It is equipped with a structure of Hopf algebra whose coproduct \(\Delta : U\mathfrak{F}_2 \to U\mathfrak{F}_2 \hat{\otimes} U\mathfrak{F}_2\) is given by

\[
\Delta(X_0) = X_0 \otimes 1 + 1 \otimes X_0 \text{ and } \Delta(X_1) = X_1 \otimes 1 + 1 \otimes X_1,
\]
whose unit \( \epsilon : U\mathfrak{F}_2 \to U\hat{\mathfrak{F}}_2 \otimes U\hat{\mathfrak{F}}_2 \) is given by
\[
\epsilon(X_0) = 0 \text{ and } \epsilon(X_1) = 0,
\]
and whose antipode \( S : U\hat{\mathfrak{F}}_2 \to U\hat{\mathfrak{F}}_2 \) is the anti-automorphism such that
\[
S(X_0) = -X_0 \text{ and } S(X_1) = -X_1.
\]
For any algebra homomorphism \( \iota : U\hat{\mathfrak{F}}_2 \to S \), the image \( \iota(\varphi) \in S \) is denoted by \( \varphi(\iota(X_0), \iota(X_1)) \).

**Definition 1.1** ([Dr]). A pair \((\mu, \varphi)\) with a non-zero element \( \mu \) in \( \mathbb{C} \) and a series \( \varphi = \varphi(X_0, X_1) \in U\mathfrak{F}_2 \) is called an associator if it satisfies the associator relations (1.1)–(1.4), that is, the shuffle product \(^1\)
\[
(1.1) \quad \Delta(\varphi) = \varphi \otimes \varphi \text{ and } \varphi(0, 0) = 1,
\]
one pentagon equation
\[
(1.2) \quad \varphi(t_{12}, t_{23} + t_{24}) \varphi(t_{13} + t_{23}, t_{34}) = \varphi(t_{23}, t_{34}) \varphi(t_{12} + t_{13}, t_{24} + t_{34}) \varphi(t_{12}, t_{23})
\]
in \( U\mathfrak{a}_4 \) and two hexagon equations
\[
(1.3) \quad \exp\left\{ \frac{\mu(t_{13} + t_{23})}{2} \right\} = \varphi(t_{13}, t_{12}) \exp\left\{ \frac{\mu t_{13}}{2} \right\} \varphi(t_{13}, t_{23})^{-1} \exp\left\{ -\frac{\mu t_{23}}{2} \right\} \varphi(t_{12}, t_{23}),
\]
\[
(1.4) \quad \exp\left\{ \frac{\mu(t_{12} + t_{13})}{2} \right\} = \varphi(t_{23}, t_{13})^{-1} \exp\left\{ \frac{\mu t_{13}}{2} \right\} \varphi(t_{12}, t_{13}) \exp\left\{ \frac{\mu t_{12}}{2} \right\} \varphi(t_{12}, t_{23})^{-1}
\]
in \( U\mathfrak{a}_3 \) where we denote by \( U\mathfrak{a}_3 \) (resp. \( U\mathfrak{a}_4 \)) the completion of the universal enveloping algebra of the pure braid Lie algebra \( \mathfrak{a}_3 \) (resp. \( \mathfrak{a}_4 \)) with 3 (resp. 4) strings, which is generated by \( t_{ij} \) \((1 \leq i, j \leq 3 \text{ (resp. 4)})\) with defining relations
\[
t_{ii} = 0, \quad t_{ij} = t_{ji}, \quad [t_{ij}, t_{ik} + t_{jk}] = 0 \quad (i, j, k: \text{ all distinct})
\]
and \([t_{ij}, t_{kl}] = 0 \quad (i, j, k, l: \text{ all distinct}).\)

The equations (1.2)–(1.4) reflect the three axioms of braided monoidal categories [JS]. Associators are essential for construction of quasi-triangular quasi-Hopf quantized universal enveloping algebras (cf. [Dr]) and also for a reconstruction of universal Vassiliev knot invariant (the Kontsevich invariant [Ko, B1]) in a combinatorial way (cf. Le and Murakami [LM3], Bar-Natan [B2], Kassel and Turaev [KT]).

\(^1\)See its equivalent formulation ([21]). It is also known as the group-like condition because it is equivalent to saying \( \varphi \in \exp \hat{\mathfrak{F}}_2 \).
Drinfeld \cite{Dr} proved that such a pair \((\mu, \varphi)\) always exists for any given \(\mu\). Note that \(\left(1, \varphi\left(\frac{X_0}{\mu}, \frac{X_1}{\mu}\right)\right)\) is an associator if and only if \((\mu, \varphi)\) is so. Any associator satisfies the so-called 2-cycle relation (cf. \cite{Dr}).

\[(1.5) \quad \varphi(X_0, X_1)\varphi(X_1, X_0) = 1,\]

which is a consequence of (1.1) and (1.2) (cf. \cite{F3}). Actually the two hexagon equations are a consequence of the one pentagon equation:

**Theorem 1.2** (\cite{F3}). Let \(\varphi = \varphi(X_0, X_1)\) be an element of \(U\tilde{\mathfrak{S}}_2\) satisfying (1.1) and (1.2). Then there always exists \(\mu \in \mathbb{C}\) (unique up to signature) such that the pair \((\mu, \varphi)\) satisfies (1.3) and (1.4).

Several different proofs of the above theorem were obtained in \cite{AT, BD, W}. For various aspects of associators, consult \cite{F5}.

The Grothendieck-Teichmüller group was introduced by Drinfeld \cite{Dr} in his study of deformations of quasi-triangular quasi-Hopf quantized universal enveloping algebras. It was defined to be the set of degenerated associators. The construction of the group was also stimulated by the previous idea of Grothendieck, *un jeu de Teichmüller-Lego*, which was posed in his research proposal *Esquisse d’un programme* \cite{G}.

**Definition 1.3** (\cite{Dr}). A degenerated associator is a group-like series \(\varphi \in U\tilde{\mathfrak{S}}_2\) satisfying the defining equations (1.2), (1.4) of associators with \(\mu = 0\). The *Grothendieck-Teichmüller group* \(GRT_1(\mathbb{C})\) is defined to be the set of degenerated associators.

It indeed forms a group \cite{Dr} by the multiplication

\[(1.6) \quad \varphi_1 \circ \varphi_2 := \varphi_2 \cdot \varphi_1(X_0, \varphi_2^{-1}X_1\varphi_2).\]

It was also shown that the set of associators with each fixed \(\mu \in \mathbb{C}^\times\) forms a right \(GRT_1(\mathbb{C})\)-torsor by (1.6).

**Example 1.4** (\cite{Dr}). By using the KZ (Knizhnik-Zamolodchikov) equation, Drinfeld constructed a series \(\Phi_{KZ} = \Phi_{KZ}(X_0, X_1) \in U\tilde{\mathfrak{S}}_2\) called the *KZ associator* (a.k.a Drinfeld associator) and he showed that the pair \((2\pi \sqrt{-1}, \Phi_{KZ})\) satisfies the associator relations (1.1)–(1.4) in \cite{Dr}.

One of the most important properties of \(\Phi_{KZ}\) is that MZV’s appear as its coefficients:

\[(1.7) \quad \zeta_{\Phi_{KZ}}(k_1, \cdots, k_m) = \zeta(k_1, \cdots, k_m)\]

\(^2\) It is denoted by \(GRT_1\) because it is a *unipotent* part of the graded version of the Grothendieck-Teichmüller group.
for $k_m > 1$. Actually general coefficients of $\Phi_{KZ}$ are calculated to be
linear combinations of MZV’s by the formula (2.2) below.³

Since $\Phi_{KZ}$ satisfies the associator relations and MZV’s appear as its
coefficients, lots of algebraic relations among MZV’s are obtained.

2. Auxiliary lemmas

We denote $\shuffle : U\mathfrak{F}_2 \hat{\otimes} U\mathfrak{F}_2 \to U\mathfrak{F}_2$ to be a shuffle product of $U\mathfrak{F}_2$
which is an associative and commutative product recursively defined
by $W \shuffle 1 = 1 \shuffle W = W$ and

$$UW \shuffle VW' = U(W \shuffle VW') + V(UW \shuffle W') \text{ with } U, V \in \{A, B\}.$$ 

for any word (a monic monomial element in $U\mathfrak{F}_2$) $W$ and $W'$. It can
be identified with the dual of the coproduct $\Delta : U\mathfrak{F}_2 \to U\mathfrak{F}_2 \hat{\otimes} U\mathfrak{F}_2$ by
the identification $U\mathfrak{F}_2^* \simeq U\mathfrak{F}_2$. For each series $\varphi \in U\mathfrak{F}_2$, we denote
the linear map sending each word to its coefficient in $\varphi$ by $I_\varphi : U\mathfrak{F}_2 \to \mathbb{C}$, 
whence

$$I_\varphi(X_0^{k_m-1}X_1 \cdots X_0^{k_1-1}X_1) = (-1)^m \zeta_\varphi(k_1, \ldots, k_m)$$

for $k_m > 1$. We note that (1.1) for $\varphi$ is equivalent to

(2.1) \hspace{1cm} I_\varphi(W) \cdot I_\varphi(W') = I_\varphi(W \shuffle W') \text{ and } I_\varphi(1) = 1

for any $W$ and $W' \in U\mathfrak{F}_2$.

Lemma 2.1. Let $\varphi \in U\mathfrak{F}_2$ be group-like without linear terms i.e. a
series satisfying (1.1) and $I_\varphi(X_0) = I_\varphi(X_1) = 0$. Suppose that $W$
is written as $X_1^rVX_0^s$ ($r, s \geq 0$, $V \in X_0 \cdot U\mathfrak{F}_2 \cdot X_1$ or $V = 1$). Then

(2.2) \hspace{1cm} I_\varphi(W) = \sum_{0 \leq a \leq r \atop 0 \leq b \leq s} \begin{pmatrix} a+b \\ a \end{pmatrix} (-1)^a I_\varphi\left(\pi(X_1^a \shuffle X_1^{r-a}VX_0^{s-b} \shuffle X_0^b)\right).$

Here $\pi : U\mathfrak{F}_2 \to U\mathfrak{F}_2$ is the natural projection $U\mathfrak{F}_2 \to \mathbb{C} + X_0 \cdot U\mathfrak{F}_2 \cdot X_1$\nannihilating $X_1 \cdot U\mathfrak{F}_2$ and $U\mathfrak{F}_2 \cdot X_0$.

Proof. Let $x$ and $y$ be commutative variables and consider

$$\mathbb{C}\langle\langle X_0, X_1 \rangle\rangle[[x_0, x_1]] := \mathbb{C}\langle\langle X_0, X_1 \rangle\rangle \hat{\otimes} \mathbb{C}[[x_0, x_1]].$$

Let

$$g_1 : \mathbb{C}\langle\langle X_0, X_1 \rangle\rangle \to \mathbb{C}\langle\langle X_0, X_1 \rangle\rangle[[x_0, x_1]]$$

be the algebra homomorphism sending $X_0, X_1$ to

$$X_0 - x_0 := X_0 \otimes 1 - 1 \otimes x_0, \hspace{0.5cm} X_1 - x_1 := X_1 \otimes 1 - 1 \otimes x_1$$

³An essentially same formula appeared in [LM2] Theorem A.9 though it seems
to include an error on the signature which was corrected in [F1] Proposition 3.2.3.
respectively and let
\[ g_2 : \mathbb{C}\langle\langle X_0, X_1 \rangle\rangle[[x_0, x_1]] \rightarrow \mathbb{C}\langle\langle X_0, X_1 \rangle\rangle \]
be the well-defined \( \mathbb{C} \)-linear map sending \( W \otimes x_0^p x_1^q \) to \( X_1^q W X_0^p \) for each word \( W \) and \( p, q \geq 0 \). Then by definition
\[ g_2 \circ g_1 (V X_0) = g_2 \circ g_1 (X_1 V) = 0 \]
for any \( V \in \mathbb{C}\langle\langle X_0, X_1 \rangle\rangle \). So we have
\[ g_2 \circ g_1 \circ \pi = g_2 \circ g_1 \]
(cf. the arguments in [F2] Lemmas 3.32-3.38. and in [LM2] Appendix A.)

By (1.1) and our assumption, we have
\[ \varphi(X_0 - x_0, X_1 - x_1) = \varphi(X_0, X_1) \]
in \( \mathbb{C}\langle\langle X_0, X_1 \rangle\rangle[[x_0, x_1]] \). Namely
\[ \varphi = (g_2 \circ g_1)(\varphi). \]
By combining the above two equations, we get
\[ \varphi = (g_2 \circ g_1 \circ \pi)(\varphi). \]
By comparing its coefficient of \( W = X_1^q V X_0^p \), we obtain (2.2).

The following lemma will be used in the proof of our main theorem.

**Lemma 2.2.** Assume that \( \varphi \in U\hat{\mathfrak{g}}_2 \) satisfies (1.1) and (1.5). Then the duality relation
\[ (2.3) \quad \zeta_\varphi(\tau(p, q)) = \zeta_\varphi(\tau(q^*, p^*)) \]
holds for any \( p = (p_1, \ldots, p_k), \quad q = (q_1, \ldots, q_k) \in \mathbb{Z}_{>0}^k \) where \( r^* := (r_k, \ldots, r_1) \) for \( r = (r_1, \ldots, r_k) \).

**Proof.** By definitioin,
\[ \zeta_\varphi(\tau(p, q)) = (-1)^{w_t(p)} I_\varphi(X_0^{q_1} X_1^{p_1} \cdots X_0^{q_k} X_1^{p_k}). \]
By the fundamental property of universal enveloping algebra, \( S(x) = -x \) for \( x \in \hat{\mathfrak{g}}_2 \), hence
\[ S(\varphi) = \varphi^{-1}. \]
Then by combining it with (1.5), we obtain (2.3).
3. **Proof of Theorem 0.1**

Let $X$ be a 1-dimensional compact oriented real manifold with ordered connected components. Denote $\mathcal{A}(X)$ to be a $\mathbb{C}$-vector space of chord diagrams there. A finite number (which yields a degree in $\mathcal{A}(X)$) of unordered pairs of distinct interior points on $X$ regarded up to orientation and component preserving homeomorphisms subject to 4T-relation. We denote its completion with respect to the degree by the same symbol $\mathcal{A}(X)$. Put $\mathcal{A}_0(X)$ to be its further quotient by FI-relation. Here the 4T-relation stands for the 4 terms relation defined by

$$D_1 - D_2 + D_3 - D_4 = 0$$

where $D_j$ are chord diagrams with four chords identical outside a ball in which they differ as illustrated in Figure 3.1.

\[
\begin{align*}
D_1 & \quad - \quad D_2 \\
D_3 & \quad + \quad D_4
\end{align*}
\]

Figure 3.1. 4T-relation

The FI-relation stands for the frame independent relation where we put

$$D = 0$$

for any chord diagrams $D$ with an isolated chord as illustrated in Figure 3.2. In more detail for those notions, consult [B1, Ko].

\[
\begin{align*}
\begin{array}{c}
\quad \\
\quad \\
\end{array}
\end{align*}
\]

Figure 3.2. FI-relation

**Proof of Theorem 0.1** Let $\gamma_{KZ}$ be the element in $\mathcal{A}(S^1)$ depicted as in Figure 3.3 where $S^1$ means the oriented circle.
By (1.7), its image $\bar{\gamma}_{KZ}$ under the projection $\mathcal{A}(S^1) \to \mathcal{A}_0(S^1)$ is described below (cf. [LM1] Theorem 3.1.3):

$$\bar{\gamma}_{KZ} = \emptyset + \sum_{k: \text{admissible}} \frac{(-1)^{wt(k)+dp(k)}}{(2\pi\sqrt{-1})^{wt(k)}} \zeta(k)D_k.$$  

Here $D_{(k_1,\ldots,k_m)}$ is the chord diagram depicted in Figure 3.4.
(A) Le-Murakami relation (1995): They constructed a weight system (denoted $W_r$ in [LM1] §3.2), a degree preserving linear map $W_{95}^N : A_0(S^1) \to \mathbb{C}[[h]]$

by using the fundamental representation of the Lie algebra $\mathfrak{sl}(N)$ for each $N \geq 2$. In [LM1] Lemma 3.2.1, it was shown that

$W_{95}^N(D_k) = \frac{1 - N^2}{N^2 h(k) - 1} (-2Nh)^{wt(k)}.$

In [LM1] Theorem 2.3.1, they showed that

$N^{-2}W_{95}^N(\bar{\gamma}_{KZ})W_{95}^N(\hat{Z}(K)) = P_K(e^h - e^{-h}, e^{-Nh})$

for any link $K$, where $\hat{Z}(K) \in A_0(S^1)$ is the Kontsevich invariant ([Ko]) and $P_K(a, z)$ is the HOMFLY-PT polynomial, the Laurent polynomial in two variables $a$ and $z$ with integer coefficients uniquely defined by the skein relation

$z^{-1}P_{K_+}(a, z) - zP_{K_-}(a, z) = aP_{K_0}(a, z)$

and the initial condition $P_{\emptyset}(a, z) = 1$. Here $K_+, K_-, K_0$ are all identical except within a ball in which they differ as illustrated below

$\begin{array}{ccc}
\hat{\top} & \hat{\top} & \hat{\top} \\
K_+ & K_- & K_0
\end{array}$

and $\emptyset$ means the trivial knot. For the 2-components trivial link $K = \emptyset \emptyset$, we have

$P_{\emptyset \emptyset}(a, z) = \frac{1}{a} \left\{ \frac{1}{z} - z \right\}$

by the above skein relation. While the left hand side of (3.3) for $K = \emptyset \emptyset$ was calculated to be

$N^{-2}W_{95}^N(\bar{\gamma}_{KZ})W_{95}^N(\hat{Z}(\emptyset \emptyset)) = N^{-2}W_{95}^N(\bar{\gamma}_{KZ})W_{95}^N(\gamma_{KZ}^{-1})^2 = N^2W_{95}^N(\gamma_{KZ})^{-1}$

by $\hat{Z}(\emptyset) = \gamma_{KZ}^{-1}$ and [LM1] Proposition 2.1.3 (or it can be directly derived from [LM1] Theorem 2.1.4). Thus the following can be deduced from (3.3) (cf. [LM1] Theorem 3.3.1)

$W_{95}^N(\bar{\gamma}_{KZ}) = N^2 \sinh \frac{h}{N^2h}.$

This is how the relation was shown for $(2\sqrt{-1}, \Phi_{KZ})$.

The following is a key lemma which is a direct consequence of the result shown in [LM3].
Lemma 3.1. Let \((\mu, \varphi)\) be any associator. Put \(\gamma(\mu, \varphi)\) to be the element in \(\mathcal{A}(S^1)\) replacing \(\Phi_{KZ}(\frac{X_0}{2\pi \sqrt{-1}}, \frac{X_1}{2\pi \sqrt{-1}})\) with \(\varphi(\frac{X_0}{\mu}, \frac{X_1}{\mu})\) in Figure 3.3. Then
\[
\bar{\gamma}(\mu, \varphi) = \bar{\gamma}_{KZ}
\]
holds in \(\mathcal{A}_0(S^1)\).

Proof. We may assume \(\mu = 1\) because \(\gamma(\mu, \varphi) = \gamma(1, \varphi(X_0/\mu, X_1/\mu))\) by definition. As is explained in [LM3], our \(\gamma(1, \varphi)\) is nothing but the framed link pre-invariant \(Z_f^F(U)\) of the so-called U-knot \(U\) depicted in Figure 3.5. Here \(F\) is a ‘twistor’ from \(\Phi_{KZ}(\frac{X_0}{2\pi \sqrt{-1}}, \frac{X_1}{2\pi \sqrt{-1}})\) to \(\varphi\), that is, a certain element \(F\) in \(\mathcal{A}(\uparrow\uparrow)\) such that
\[
\varphi = F^{2, 3} \cdot F^{1, 23} \cdot \Phi_{KZ}(\frac{X_0}{2\pi \sqrt{-1}}, \frac{X_1}{2\pi \sqrt{-1}}) \cdot (F^{12, 3})^{-1} \cdot (F^{1, 2})^{-1}
\]
holds in \(\mathcal{A}(\uparrow\uparrow\uparrow)\) (for precise, see [LM3] §7). According to [LM3] Theorem 8 and its corollary, \(Z_f^F(K)\) is an independent of any choice of a twistor \(F\), namely, any choice of an associator \((1, \varphi)\). Thus
\[
\gamma(\mu, \varphi) = \gamma_{KZ}
\]
in \(\mathcal{A}(S^1)\), from which we obtain the claim. □

By Lemma 3.1 we may replace \(\bar{\gamma}_{KZ}\) with \(\bar{\gamma}(\mu, \varphi)\) in (3.4). Therefore Le-Murakami relation (1995) holds for any associator \((\mu, \varphi)\) because the equality (3.1) replacing \(\zeta(k_1, \ldots, k_m)\) with \(\zeta(\varphi(k_1, \ldots, k_m))\) and \(2\pi \sqrt{-1}\) with \(\mu\) holds. □

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4 So we have \(\gamma_{KZ} = \gamma(2\pi \sqrt{-1}, \Phi_{KZ})\).

5 It is not an invariant of framed links but it will be so after one multiply it with a suitable powers of \(\gamma(1, \varphi)\) (cf. [LM3] Theorem 2.1).

6 Hence hereafter we will denote it simply by \(Z_f(K)\).
(B) Le-Murakami relation (1996): They constructed a weight system (denoted $W$ in [LM2] §3.2), a linear map

$$W^\text{96}_N : \mathcal{A}(S^1) \to \mathbb{C}[[h]]$$

by using the fundamental representation of the Lie algebra $\mathfrak{so}(N)$ for each $N \geq 3$. In [LM2] Section 3, they introduced an invariant of unoriented framed links which is given by

$$\kappa(L) := \frac{N^{s_L} - 2 W^\text{96}_N(Z_f(L))}{W^\text{96}_N(\gamma_{\text{KZ}})^{s_L - 1}} \in \mathbb{C}[[h]]$$

for framed link diagrams $L$ with $s_L$ maximal points. They showed that it satisfies the followings:

$$\kappa(L_r) = e^{(N-1)h} \kappa(L), \quad \kappa(L_l) = e^{-(N-1)h} \kappa(L),$$

$$\kappa(L_+) - \kappa(L_-) = \{ \exp(h) - \exp(-h) \} \cdot \{ \kappa(L_0) - \kappa(L_\infty) \},$$

$$\kappa(\text{O}) = 1.$$

Here $L_r$ (resp. $L_l$) is the same framed link diagram to $L$ with a right-handed (resp. left-handed) curl added (using a type I Reidemeister move) and $L_+, L_-, L_0, L_\infty$ are all identical except within a ball in which they differ as illustrated below

![Diagram](image)

and $\text{O}$ means the trivial unoriented knot. For the 2-components trivial unoriented link $L = \text{O} \text{O}$, the above equations yield

$$\kappa(\text{O} \text{O}) = \frac{e^{(N-1)h} - e^{-(N-1)h}}{e^h - e^{-h}} + 1$$

While the left hand side of (3.5) for $L = \text{O} \text{O}$ was calculated to be

$$\kappa(\text{O} \text{O}) = \frac{W^\text{96}_N(\text{O} \text{O})}{W^\text{96}_N(\gamma_{\text{KZ}})} = \frac{N^2}{W^\text{96}_N(\gamma_{\text{KZ}})}$$

by [LM2] Section 3.2. Thus the following is obtained

$$W^\text{96}_N(\gamma_{\text{KZ}}) = \frac{N^2 \sinh h}{\sinh(N-1)h + \sinh h}.$$
and in [LMZ] Proposition 4.5 showed that

\begin{equation}
W_N^{96} \circ \psi(D_k) = (-2)^{\text{wt}(k)} N g(k) k^{\text{wt}(k)}.
\end{equation}

The above equations with (3.1) immediately yields the Le-Murakami relation (1996) for \((2\pi \sqrt{-1}, \Phi_{KZ})\).

Again by Lemma 3.1, we may replace \(\bar{\gamma}_{KZ}\) with \(\bar{\gamma}_{(\mu, \varphi)}\) in (3.6). Thus the validity of the relation for any associator \((\mu, \varphi)\) follows.

\((C)\) Takamuki relation: Let \(\varphi\) be any invertible series in \(\mathbb{C}(\langle X_0, X_1 \rangle)\). Consider the chord diagram \(\delta_{\varphi} \in \mathcal{A}(S^1)\) defined as in Figure 3.6

Assume that \(\varphi\) satisfies (1.1) and (1.5). Then by Lemma 2.1

\begin{equation}
\varphi = 1 + \sum_{k=1}^{\infty} \sum_{p,q,r,s \in \mathbb{Z}_{\geq 0}} (-1)^{\text{wt}(p)+\text{wt}(s)} \zeta_p(\tau(p + r, q + s)) \binom{p + r}{r} \binom{q + s}{s} \cdot X_1^{\text{wt}(r)} X_0^{q_k} X_1^{p_k} \cdots X_0^{q_1} X_1^{p_1} X_0^{\text{wt}(s)}.
\end{equation}

Figure 3.6. \(\delta_{\varphi}\)
By Lemma 2.2

\begin{equation}
\varphi = 1 + \sum_{k=1}^{\infty} \sum_{p,q,r,s \in \mathbb{Z}_{\geq 0}} (-1)^{wt(q)+wt(r)} \zeta_{\varphi}(\tau(p, q, r + s)) \left( \frac{p + r}{r} \right) \left( \frac{q + s}{s} \right) \cdot X_1^{wt(s)} X_0^{p_1} X_1^{q_1} \cdots X_0^{p_k} X_1^{q_k} X_0^{wt(r)}.
\end{equation}

By Lemma 2.2 and (1.5),

\begin{equation}
\varphi^{-1} = 1 + \sum_{k=1}^{\infty} \sum_{p,q,r,s \in \mathbb{Z}_{\geq 0}} (-1)^{wt(q)+wt(r)} \zeta_{\varphi}(\tau(p, q, r + s)) \left( \frac{p + r}{r} \right) \left( \frac{q + s}{s} \right) \cdot X_0^{wt(s)} X_1^{p_1} X_0^{q_1} \cdots X_1^{p_k} X_0^{q_k} X_1^{wt(r)}.
\end{equation}

They are equal to the formulae in [T] § 5 when \( \varphi(A, B) = \Phi_{KZ}(\frac{A}{2\pi\sqrt{-1}}, \frac{B}{2\pi\sqrt{-1}}) \).

By the equations (3.9) and (3.10), the following can be deduced (cf. [T] Proof of Proposition 5 and 6)

\begin{equation}
W_{N}^{95}(\bar{\delta}_{\varphi}) = e^{-Nh} \left\{ (e^h + e^{-h})R_{\varphi} + N^2(e^h - e^{-h}) \right\}
\end{equation}

where \( R_{\varphi} \) means the right hand side of Takamuki relation replacing \( \zeta(k) \) with \( \zeta_{\varphi}(k) \) and \( 2\pi\sqrt{-1} \) with 1.

**Lemma 3.2.** Let \( (\mu, \varphi) \) be any associator with \( \mu = 1 \). Then

\[ \bar{\delta}_{\varphi} = \bar{\gamma}(\mu, \varphi) \]

in \( A_0(S^1) \).

**Proof.** Again as is explained in [T], the quotient \( \delta_{\varphi} \in A_0(S^1) \) is nothing but Kontsevich's ([Ko]) knot (pre-)invariant \( Z^{F}(K_0) \) in [LM3] of the unknot \( K_0 \) represented in Figure 3.7 with a twistor \( F \) from \( \Phi_{KZ}(\frac{x_0}{2\pi\sqrt{-1}}, \frac{x_1}{2\pi\sqrt{-1}}) \) to \( \varphi \). Because \( K_0 \) has two maximums and the unknot diagram \( \circ \) has

\[ \text{Figure 3.7. } K_0 \]

a maximum with \( Z^{F}(\circ) = \bar{\gamma}(\mu, \varphi) \) and they give equivalent knots, we have \( \delta_{\varphi} \cdot (\bar{\gamma}(\mu, \varphi))^{-2} = (\bar{\gamma}(\mu, \varphi))^{-1} \).
Therefore by combining (3.11) with (3.4), we obtain Takamuki relation for any associator \((1, \varphi)\) by the above lemma. Hence the relation also holds for any associator \((\mu, \varphi)\) because \((1, \varphi(X_0/\mu, X_1/\mu))\) forms an associator. \(\Box\)

(D) Ihara-Takamuki relation: They constructed a normalized weight system (denoted \(\hat{W}_0\) in [IT] §3), a linear map

\[ W^0_{IT}: A_0(S^1) \to \mathbb{C}[[h]] \]

by using the 7-dimensional irreducible standard representation \(\Gamma_{1,0}\) of the exceptional simple Lie algebra of type \(G_2\). In [IT] Proposition 4, they showed

\[
W^0_{IT}(D\tau(p,q)) = w(p,q)h^{wt(p)+wt(q)}
\]

and

\[
w(p,q) = w(q^*, p^*).
\]

From [IT] Proposition 2, 3 and 5 where they related \(W^0_{IT}(\bar{\gamma}_{KZ})\) with Kuperberg’s computation [Ku] on the quantum dimension of the associated quantum representation, it follows

\[
W^0_{IT}(\bar{\gamma}_{KZ}) = \frac{7^2 \cdot [6]_q[4]_q}{[12]_q[7]_q[2]_q}.
\]

Then by combining the above three relation with the duality relation (cf. Lemma 2.2) among MZV’s

\[
\zeta(\tau(p,q)) = \zeta(\tau(q^*, p^*))
\]

we obtain Ihara-Takamuki relation for \((2\pi\sqrt{-1}, \Phi_{KZ})\) since (3.1) can be reformulated as

\[
\bar{\gamma}_{KZ} = \mathcal{O} + \sum_{p,q} \frac{(-1)^{wt(q)}}{(2\pi\sqrt{-1})^{wt(p)+wt(q)}} \zeta(\tau(p,q)) D\tau(p,q).
\]

Suppose that \((\mu, \varphi)\) be any associator. Again by Lemma 3.1, we may replace \(\bar{\gamma}_{KZ}\) with \(\bar{\gamma}_{(\mu, \varphi)}\) in (3.13). The validity of the duality relation for \((\mu, \varphi)\) is verified in Lemma 2.3, hence from which we obtain Ihara-Takamuki relation for \((\mu, \varphi)\). \(\Box\)

Remark 3.3. There is another proof of the derivation of Le-Murakami relation (A) from the associator relation: The relation (A) is generalized to Ohno-Zagier relation ([OZ]), but which was shown in [Li] to be a consequence of the regularized double shuffle relation. Whereas in [F4] it was shown that the last relation can be derived from the associator relation.
Though we do not have a precise definition of relations among MZV’s in knot theory, the author expects that all such relations should be derived from the associator relations i.e. the pentagon equation and the shuffle relation.

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