ON THE RIEMANNIAN AND EINSTEIN-WEYL GEOMETRY IN THEORY OF THE SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

Some properties of the 4-dim Riemannian spaces with metrics

\[ ds^2 = 2(za_3 - ta_4)dx^2 + 4(za_2 - ta_3)dxdy + 2(za_1 - ta_2)dy^2 + 2dxdz + 2dydt \]

associated with the second order nonlinear differential equations

\[ y'' + a_1(x, y)y'^3 + 3a_2(x, y)y'^2 + 3a_3(x, y)y' + a_4(x, y) = 0 \]

with arbitrary coefficients \(a_i(x, y)\) and 3-dim Einstein-Weyl spaces connected with dual equations

\[ b'' = g(a, b, b') \]

where the function \(g(a, b, b')\) satisfied the partial differential equation

\[
\begin{align*}
g_{aacc} + 2cg_{abcc} + 2gg_{accc} + c^2 g_{bccc} + 2cg_{cacc} + g^2 g_{cccc} + (g_a + cg_b)g_{eccc} - 4g_{abcc} - 4cg_{bccc} - cg_{cc}g_{accc} - 3gg_{bccc} - gcg_{acc} + 4gcg_{bccc} - 3gg_{bccc} + 6gg_{bccc} &= 0
\end{align*}
\]

are considered.

1 Introduction

The second order ODE’s of the type

\[ y'' + a_1(x, y)y'^3 + 3a_2(x, y)y'^2 + 3a_3(x, y)y' + a_4(x, y) = 0 \] (1)

are connected with nonlinear dynamical systems in the form

\[
\begin{align*}
\frac{dx}{dt} &= P(x, y, z, \alpha_i), \\
\frac{dy}{dt} &= Q(x, y, z, \alpha_i), \\
\frac{dz}{dt} &= R(x, y, z, \alpha_i),
\end{align*}
\]

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where $\alpha_i$ are parameters.

For example the Lorenz system

$$
\dot{X} = \sigma(Y - X), \quad \dot{Y} = rX - Y - ZX, \quad \dot{Z} = XY - bZ
$$

having chaotic properties at some values of parameters is equivalent to the equation

$$
y'' - \frac{3}{y}y'^2 + (\alpha y - \frac{1}{x})y' + \varepsilon xy^4 - \beta x^3 y^4 - \beta x^2 y^3 - \gamma y^3 + \delta \frac{y'^2}{x} = 0, \tag{2}
$$

where

$$
\alpha = \frac{b + \sigma + 1}{\sigma}, \quad \beta = \frac{1}{\sigma^2}, \quad \gamma = \frac{b(\sigma + 1)}{\sigma^2}, \quad \delta = \frac{(\sigma + 1)}{\sigma}, \quad \varepsilon = \frac{b(r - 1)}{\sigma^2},
$$

and for investigation of its properties the theory of invariants was first used in [1–5].

According to this theory [6–10] all equations of type (1) can be divided in two different classes

I. $\nu_5 = 0$,

II. $\nu_5 \neq 0$.

Here the value $\nu_5$ is the expression of the form

$$
\nu_5 = L_2(L_1L_{2x} - L_2L_{1x}) + L_1(L_2L_{1y} - L_1L_{2y}) - a_1L_1^3 + 3a_2L_1^2L_2 - 3a_3L_1L_2^2 + a_4L_2^3
$$

then $L_1, L_2$ are defined by formulas

$$
L_1 = \frac{\partial}{\partial y}(a_{4y} + 3a_2a_4) - \frac{\partial}{\partial x}(2a_{3y} - a_{2x} + a_{1a_4}) - 3a_3(2a_{3y} - a_{2x}) - a_4a_{1x},
$$

$$
L_2 = \frac{\partial}{\partial x}(a_{1y} - 3a_1a_3) + \frac{\partial}{\partial x}(a_{3y} - 2a_{2x} + a_{1a_4}) - 3a_2(a_{3y} - 2a_{2x}) + a_1a_{4y}.
$$

For the equations with condition $\nu_5 = 0$ R. Liouville discovered the series of semi-invariants starting from:

$$
w_1 = \frac{1}{L_1^4} \left[ L_1^3(\alpha'L_1 - \alpha''L_2) + R_1(L_1^2) - L_1^2R_1 + L_1R_1(a_3L_1 - a_4L_2) \right],
$$

where

$$
R_1 = L_1L_{2x} - L_2L_{1x} + a_2L_1^2 - 2a_3L_1L_2 + a_4L_2^2
$$

or

$$
w_2 = \frac{1}{L_2^4} \left[ L_2^3(\alpha'L_2 - \alpha L_1) - R_2(L_2^2) + L_2^2R_2 - L_2R_2(a_1L_1 - a_2L_2) \right],
$$

where

$$
R_2 = L_1L_{2y} - L_2L_{1y} + a_1L_1^2 - 2a_2L_1L_2 + a_3L_2^2
$$

and

$$
\alpha = a_{2y} - a_{1x} + 2(a_1a_3 - a_2^2), \quad \alpha' = a_{3y} - a_{2x} + a_1a_4 - a_2a_3,
$$

$$
\alpha'' = a_{4y} - a_{3x} + 2(a_2a_4 - a_3^2).
$$

It has the form

$$
w_{m+2} = L_1 \frac{\partial w_m}{\partial y} - L_2 \frac{\partial w_m}{\partial x} + mw_m \left( \frac{\partial L_2}{\partial x} - \frac{\partial L_1}{\partial y} \right).
$$
In case \( w_1 = 0 \) there are another series of semi-invariants
\[
i_{2m+2} = L_1 \frac{\partial i_{2m}}{\partial y} - L_2 \frac{\partial i_{2m}}{\partial x} + 2mi_{2m}\left(\frac{\partial L_2}{\partial x} - \frac{\partial L_1}{\partial y}\right).
\]

where
\[
i_2 = \frac{3R_1}{L_1} + \frac{\partial L_2}{\partial x} - \frac{\partial L_1}{\partial y}.
\]

and corresponding sequence for absolute invariants
\[
j_{2m} = \frac{i_{2m}}{i_2}.
\]

In case \( \nu_5 \neq 0 \) the semi-invariants have the form
\[
\nu_{m+5} = L_1 \frac{\partial \nu_m}{\partial y} - L_2 \frac{\partial \nu_m}{\partial x} + m\nu_m\left(\frac{\partial L_2}{\partial x} - \frac{\partial L_1}{\partial y}\right).
\]

and corresponding serie of absolute invariants
\[
[5t_m - (m - 2)t_{m-2}]\nu_5^{2/5} = 5(L_1 \frac{\partial t_{m-2}}{\partial y} - L_2 \frac{\partial t_{m-2}}{\partial x})
\]

where
\[
t_m = \nu_m\nu_5^{-m/5}
\]

\section{Riemannian spaces in theory of ODE’s}

Here we present the construction of the Riemannian spaces connected with the equations (1).

We start from the equations of geodesical lines of two-dimensional space \( A_2 \) equipped with affine (or Riemannian) connection. They have the form
\[
\ddot{x} + \Gamma_1^{11}x^2 + 2\Gamma_1^{12}\dot{x}\dot{y} + \Gamma_1^{22}y^2 = 0,
\]
\[
\ddot{y} + \Gamma_1^{21}x^2 + 2\Gamma_1^{22}\dot{x}\dot{y} + \Gamma_2^{22}y^2 = 0.
\]

This system of equations is equivalent to one equation
\[
y'' - \Gamma_2^{22}y^3 + (\Gamma_2^{22} - 2\Gamma_1^{12})y'^2 + (2\Gamma_1^{12} - \Gamma_1^{11})y' + \Gamma_1^{11} = 0
\]

of type (1) but with special choice of coefficients \( a_i(x, y) \).

The equations (1) with arbitrary coefficients \( a_i(x, y) \) may be considered as equations of geodesics of 2-dimensional space \( A_2 \)
\[
\ddot{x} - a_3x^2 - 2a_2\dot{x}\dot{y} - a_1y^2 = 0,
\]
\[
\ddot{y} + a_4x^2 + 2a_3\dot{x}\dot{y} + a_2y^2 = 0
\]
equipped with the projective connection with components
\[
\Pi_1 = \begin{vmatrix} -a_3 & -a_2 \\ a_4 & a_3 \end{vmatrix}, \quad \Pi_2 = \begin{vmatrix} -a_2 & -a_1 \\ a_3 & a_2 \end{vmatrix}.
\]
The curvature tensor of this type of connection is
\[ R_{12} = \frac{\partial \Pi_2}{\partial x} - \frac{\partial \Pi_1}{\partial y} + [\Pi_1, \Pi_2] \]

and it has the components
\[
R_{112}^1 = a_{3y} - a_{2x} + a_1 a_4 - a_2 a_3 = \alpha', \quad R_{212}^1 = a_2 y - a_1 x + 2(a_1 a_3 - a_2^2) = \alpha, \\
R_{112}^2 = a_{3x} - a_{4y} + 2(a_3^2 - a_2 a_4) = -\alpha'', \quad R_{212}^2 = a_{2x} - a_{3y} + a_3 a_2 - a_1 a_4 = -\alpha'.
\]

For construction of the Riemannian space connected with the equation of type (1) we use the notice of Riemannian extension \( W \) of space \( A_2 \) with connection \( \Pi_{ij}^k \) [12]. The corresponding metric is
\[
d^2 s^2 = -2\Pi_{ij}^k \xi_k dx^i dx^j + 2d\xi_i dx^i
\]
and in our case it takes the following form \( (\xi_1 = z, \xi_2 = \tau) \)
\[
d^2 s^2 = 2(za_3 - \tau a_4)dx^2 + 4(za_2 - \tau a_3)dx dy + 2(za_1 - \tau a_2)dy^2 + 2dx dz + 2dy d\tau. \quad (3)
\]

So, it is possible to formulate the following statement

**Proposition 1** For a given equation of type (1) there is exists the Riemannian space with metric (3) having integral curves of such type of equation as part of its geodesics.

Really, the calculation of geodesics of the space \( W^4 \) with the metric (3) lead to the system of equations
\[
\frac{d^2 x}{ds^2} - a_3 \left( \frac{dx}{ds} \right)^2 - 2a_2 \frac{dx}{ds} \frac{dy}{ds} - a_1 \left( \frac{dy}{ds} \right)^2 = 0,
\]
\[
\frac{d^2 y}{ds^2} + a_4 \left( \frac{dx}{ds} \right)^2 + 2a_3 \frac{dx}{ds} \frac{dy}{ds} + a_2 \left( \frac{dy}{ds} \right)^2 = 0,
\]
\[
\frac{d^2 z}{ds^2} + [z(a_{4y} - \alpha'') - \tau a_{4x}] \left( \frac{dx}{ds} \right)^2 + 2[z a_{3y} - \tau (a_{3x} - \alpha'')] \frac{dx}{ds} \frac{dy}{ds} + +[z(a_{2y} + \alpha) - \tau (a_{2x} + 2\alpha')] \left( \frac{dy}{ds} \right)^2 + 2[a_{3y} \frac{dx}{ds} - a_{4x} \frac{dy}{ds} - 2a_{2y} \frac{dy}{ds}] + [a_{3y} \frac{dx}{ds} - a_{4x} \frac{dy}{ds} - 2a_{2y} \frac{dy}{ds}] = 0.
\]
\[
\frac{d^2 \tau}{ds^2} + [z(a_{3y} - 2\alpha') - \tau (a_{3x} - \alpha'')] \left( \frac{dx}{ds} \right)^2 + 2[z(a_{2y} - \alpha) - \tau a_{2x}] \frac{dx}{ds} \frac{dy}{ds} + +[z a_{1y} - \tau (a_{1x} + \alpha)] \left( \frac{dy}{ds} \right)^2 + 2[a_{2y} \frac{dx}{ds} - a_{3y} \frac{dx}{ds}] + 2[a_{1y} \frac{dy}{ds} - a_{2y} \frac{dy}{ds}] = 0.
\]

This system of equations has the integral
\[ 2(za_3 - \tau a_4)\dot{x}^2 + 4(za_2 - \tau a_3)\dot{x}\dot{y} + 2(za_1 - \tau a_2)\dot{y}^2 + 2\dot{z}^2 + 2\dot{y}\dot{\tau} = 1. \]

Remark that first two equations of this system are equivalent to the equation (1).
Thus, we construct the four-dimensional Riemannian space with metric (3) and connection

\[
\Gamma_1 = \begin{pmatrix}
-a_3 & -a_2 & 0 & 0 \\
-\alpha a_4 & -a_2 & 0 & 0 \\
\alpha z(\alpha a_4 - \alpha''a_3) & z(\alpha a_3) & \alpha a_3 & a_3 \\
\alpha z(\alpha a_3 - \alpha''a_2) & z(\alpha a_2) & \alpha a_2 & a_2 \\
\end{pmatrix},
\]

\[
\Gamma_2 = \begin{pmatrix}
-a_2 & -a_1 & 0 & 0 \\
-\alpha a_3 & -a_1 & 0 & 0 \\
\alpha z(\alpha a_3 - \alpha''a_2) & z(\alpha a_1) & \alpha a_1 & a_1 \\
\alpha z(\alpha a_1 - \alpha''a_2) & z(\alpha a_2) & \alpha a_2 & a_2 \\
\end{pmatrix},
\]

\[
\Gamma_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
a_3 & a_2 & 0 & 0 \\
a_2 & a_1 & 0 & 0 \\
\end{pmatrix}, \quad \Gamma_4 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
a_4 & -a_3 & 0 & 0 \\
a_3 & -a_2 & 0 & 0 \\
\end{pmatrix}.
\]

The curvature tensor of this metric has the form

\[
R^1_{112} = -R^3_{312} = -R^2_{212} = R^4_{412} = \alpha', \quad R^1_{212} = -R^4_{312} = \alpha, \quad R^2_{112} = -R^3_{412} = -\alpha'',
\]

\[
R^1_{312} = R^1_{412} = R^2_{312} = R^2_{412} = 0,
\]

\[
R^3_{312} = 2z(\alpha a_2 - \alpha a_3) + 2\alpha(\alpha a_4 - \alpha a_3),
\]

\[
R^4_{212} = 2z(\alpha a_3 - \alpha a_2) + 2\alpha(\alpha a_4 - \alpha a_2),
\]

\[
R^3_{212} = z(\alpha_x - \alpha_y + a_1 \alpha'' - a_3 \alpha) + \tau(\alpha_x'' - a_x + a_4 \alpha - a_3 \alpha'),
\]

\[
R^4_{112} = z(\alpha_y - \alpha_x + a_1 \alpha'' - a_3 \alpha) + \tau(\alpha_x'' - a_x + a_4 \alpha - a_3 \alpha'),
\]

Using the expressions for components of projective curvature of space $A_2$

\[
L_1 = \alpha_y'' - \alpha_x' + a_2 \alpha'' + a_4 \alpha - 2a_3 \alpha',
\]

\[
L_2 = \alpha_y' - \alpha_x + a_1 \alpha'' + a_3 \alpha - 2a_2 \alpha',
\]

they can be presented in form

\[
R^4_{112} = z(L_2 + 2a_2 \alpha' - 2a_3 \alpha) - \tau(L_1 + 2a_3 \alpha' - 2a_4 \alpha),
\]

\[
R^3_{212} = z(-L_2 + 2a_1 \alpha'' - 2a_2 \alpha') + \tau(L_1 + 2a_3 \alpha' - 2a_2 \alpha''),
\]

\[
R_{13} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -\alpha & 0 & 0 \\
\alpha' & 0 & 0 & 0 \\
\end{pmatrix}, \quad R_{14} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \alpha'' & 0 & 0 \\
-\alpha'' & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
R_{23} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -\alpha & 0 & 0 \\
\alpha & 0 & 0 & 0 \\
\end{pmatrix}, \quad R_{24} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \alpha' & 0 & 0 \\
-\alpha' & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
R^i_{j34} = 0.
\]
The Ricci tensor $R_{ik} = R^j_{iik}$ of the space $W^4$ has the components
\[ R_{11} = 2\alpha'', \quad R_{12} = 2\alpha', \quad R_{22} = 2\alpha, \]
and scalar curvature $R = g^{in}g^{km}R_{nm}$ of the space $W^4$ is $R = 0$. Now we introduce the tensor
\[ L_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} = R_{ij;k} - R_{ik;j}. \]
It has the following components
\[ L_{112} = -L_{121} = 2L_1, \quad L_{221} = L_{212} = -2L_2 \]
and with help of them the invariants of equations (1) may be constructed using the covariant derivations of the curvature tensor and the values $L_1, L_2$.

The Weyl tensor of the space $M^4$ is
\[ C_{ijk} = R_{ijk} + \frac{1}{2}(g_{jl}R_{ik} + g_{ik}R_{jl} - g_{jk}R_{il} - g_{il}R_{jk}) + \frac{R}{6}(g_{jk}g_{il} - g_{jl}g_{ik}). \]
It has only one component
\[ C_{1212} = tL_1 - zL_2. \]

Using the components of Riemannian tensor which are not zero
\[ R_{1412} = \alpha'', \quad R_{2412} = \alpha', \quad R_{2312} = -\alpha, \quad R_{3112} = \alpha', \quad R_{1212} = z(\alpha_x - \alpha_y' + a_1\alpha'' - 2a_2\alpha' + a_3\alpha) + t(\alpha'' - \alpha'_x - a_4\alpha + 2a_3\alpha' - a_3\alpha'') \]
the equation
\[ |R_{ab} - \lambda g_{ab}| = 0 \]
may be investigate.

With help of the Weyl tensor the properties of invariants of the space $M^4$ may be investigated. In particular, all invariants of the second order of the space $M^4$ are equal to zero.

**Remark 1** The spaces with metrics (3) are flat for the equations (1) with conditions
\[ \alpha = 0, \quad \alpha' = 0, \quad \alpha'' = 0, \]
on coefficients $a_i(x, y)$.

Such type of equations have the components of projective curvature
\[ L_1 = 0, \quad L_2 = 0 \]
and they are reduced to the form $y'' = 0$ with help of points transformations.

On the other hand there are examples of equations (1) with conditions $L_1 = 0$, $L_2 = 0$ but
\[ \alpha \neq 0, \quad \alpha' \neq 0, \quad \alpha'' \neq 0. \]

For such type of equations the curvature of corresponding Riemannian spaces is not equal to zero.

In fact, the equation
\[ y'' + 2e^{\varphi}y'^3 - \varphi_y y'^2 + \varphi_x y' - 2e^{\varphi} = 0 \]
where the function $\varphi(x, y)$ is solution of the Wilczynski-Tzitzeika nonlinear equation integrable by the Inverse Transform Method.
\[ \varphi_{xy} = 4e^{2\varphi} - e^{-\varphi}. \]
has conditions $L_1 = 0$, $L_2 = 0$ but
\[ \alpha \neq 0, \quad \alpha' \neq 0, \quad \alpha'' \neq 0. \]
Remark 2 The properties of the Riemann spaces with metrics (3) for the equations (2) with chaotical behaviour at the values of coefficients \((\sigma = 10, \ b = 8/3, \ r > 24)\) have a special interest. The Riemannian spaces of this type are characterized by the special conditions on the curvature tensor and its invariants.

The study of the geodesic deviation equation

\[
\frac{d^2\eta^i}{ds^2} + 2\Gamma^i_{jm} \frac{dx^m}{ds} \frac{d\eta^j}{ds} + \frac{\partial\Gamma^i_{kl}}{\partial x^j} \frac{dx^k}{ds} \frac{dx^l}{ds} \eta^j = 0
\]

also may be usefull for that.

Let us consider some applications of soliton theory to the study of the properties of equations of type (1).

They are based on the presentation of the metrics (3) in the form

\[
ds^2 = 2z(a_3dx^2 + 2a_2dxdy + a_1dy^2) - 2\tau(a_4dx^2 + 2a_3dxdy + a_2dy^2) + 2dxdz + 2dyd\tau,
\]

or

\[
ds^2 = 2zds_1^2 - 2\tau ds_2^2 + 2dxdz + 2dyd\tau.
\]

Let us consider some examples.

For the equation

\[
y'' + H^2(x, y)y'^3 + 3y' = 0
\]

we have the metrics

\[
ds^2 = 2z(dx^2 + H^2dy^2) - 4\tau dxdy + 2dxdz + 2dyd\tau.
\]

containing two-dimensional part

\[
ds_1^2 = dx^2 + H^2dy^2,
\]

which is connected with theory of the KdV equation

\[
K_y + KK_x + K_{xxx} = 0
\]

on the curvature \(K(x, y)\) of the metric (4).

For the equation

\[
y'' + y'^3 + 3\cos H(x, y)y'^2 + y' = 0
\]

corresponding metric is connected with integrable equation

\[
H_{xy} = \sin H.
\]

With the equations of type

\[
y'' + a_4(x, y) = 0
\]

a 4-dim Riemann space with metrics

\[
ds^2 = -2\tau a_4dx^2 + 2dxdz + 2dyd\tau
\]

and geodesics in form

\[
\ddot{x} = 0, \quad \ddot{y} + a_4(x, y)(\dot{x})^2 = 0, \quad \ddot{\tau} + a_{4y}(\dot{x})^2\tau = 0
\]
\[
\ddot{z} - \tau a_{4x}(\dot{x})^2 - 2\tau a_{4y}\dot{x}\dot{y} - 2a_4\dot{x}\ddot{\tau} = 0
\]
are connected.

For the equations

\[
y'' + 3a_3(x,y)y' + a_4(x,y) = 0
\]
corresponding Riemann space has the metric

\[
ds^2 = 2(za_3 - \tau a_4)dx^2 - 4\tau a_3 dxdy + 2dxdz + 2dyd\tau
\]
and geodesics

\[
\ddot{x} - a_3\dot{x}^2 = 0, \quad \ddot{y} + 2a_3\dot{x}\dot{y} + a_4\dot{x}^2 = 0,
\]

\[
\ddot{\tau} - 2a_3\ddot{x}\dot{\tau} - a_3\dot{y}^2 z + (a_{4y} - 2a_3^2 - 2a_{3x})\dot{x}^2 \tau = 0,
\]

\[
\ddot{z} + 2a_3\dot{x}\dot{z} - 2(a_3\dot{y} + a_4\dot{x})\dot{\tau} + [(a_{3x} + 2a_3^2)x^2 + 2a_{3y}\dot{x}]z - [a_{4x}\dot{x}^2 + 2(a_{4y} - 2a_3^2)\dot{x}\dot{y} + 2a_{3y}\dot{y}^2]\dot{\tau} = 0.
\]

Let us consider the possibility to investigate the properties of the equations (1) using the facts from theory of embedding of Riemann spaces into the flat spaces.

For the Riemann spaces of the class one (which can be embedded into the 5-dimensional Euclidean space) the following conditions are fulfilled

\[
R_{ijkl} = b_{ik}b_{jl} - b_{il}b_{jk}
\]
and

\[
b_{ij,k} = b_{ik,j} = 0
\]
where \(R_{ijkl}\) are the components of curvature tensor of the space with metrics \(ds^2 = g_{ij}dx^idx^j\).

Applications of these relations for the spaces with the metrics (2) lead to the conditions on the values \(a_i(x,y)\).

For the spaces of the class two (which admits the embedding into 6-dim Euclidean space with some signature) the conditions are more complicated. They are

\[
R_{abcd} = e_1(\omega_{ac}\omega_{bd} - \omega_{ad}\omega_{bc}) + e_2(\lambda_{ac}\lambda_{bd} - \lambda_{ad}\lambda_{bc}),
\]

\[
\omega_{abc} - \omega_{acb} = e_2(t_c\lambda_{ab} - t_b\lambda_{ac}),
\]

\[
\lambda_{abc} - \lambda_{acb} = -e_1(t_c\omega_{ab} - t_b\omega_{ac}),
\]

\[
t_{a;bc} - t_{a;cb} = \omega_{ac}\lambda_{b}^c - \lambda_{ac}\omega_{b}^c.
\]

and lead to the relations

\[
e^{abcd}e^{nmrs}e^{pqik}R_{abnm}R_{cdpq}R_{rsik} = 0
\]
and

\[
e^{cdmn}R_{abcd}R_{mn} = -8e_1e_2\epsilon^{cdmn}t_{c;d}t_{m;n}
\]
3 On relation with theory of the surfaces

The existence of metrics for the equations (1) may be used for construction of the surfaces.

One possibility is connected with two-dimensional surfaces embedded in a given 4-dimensional space and which are the generalization of the surfaces of translation. The equations for coordinates $Z^i(x, y)$ of such type of the surfaces are

$$\frac{\partial^2 Z^i}{\partial x \partial y} + \Gamma^i_{jk} \frac{\partial Z^j}{\partial x} \frac{\partial Z^k}{\partial y} = 0.$$

From the condition of compatibility of this system it is possible to get the coefficients $a_i(x, y)$ and correspondent second order ODE’s.

Another possibility for studying of two-dimensional surfaces in space with metrics (3) is connected with the choice of section

$$x = x, \quad y = y, \quad z = z(x, y), \quad \tau = \tau(x, y)$$

in space with metrics (3).

Using the expressions

$$dz = z_x dx + z_y dy, \quad d\tau = \tau_x dx + \tau_y dy$$

we get the metric

$$ds^2 = 2(z_x + za_3 - \tau a_4) dx^2 + 2(\tau_x + z_y + 2za_2 - 2\tau a_3) dxdy + 2(\tau_y + za_1 - \tau a_2) dy^2.$$

We can use this presentation for investigation of particular cases of equations (1).

1. The choice of the functions $z, \quad \tau$ in form

$$z_x + za_3 - \tau a_4 = 0,$$

$$\tau_x + z_y + 2za_2 - 2\tau a_3 = 0,$$

$$\tau_y + za_1 - \tau a_2 = 0$$

are connected with flat surfaces and are reduced at the substitution

$$z = \Phi_x, \quad \tau = \Phi_y$$

to the system

$$\Phi_{xx} = a_4 \Phi_y - a_3 \Phi_x,$$

$$\Phi_{xy} = a_3 \Phi_y - a_2 \Phi_x,$$

$$\Phi_{yy} = a_2 \Phi_y - a_1 \Phi_x,$$

compatible at the conditions

$$\alpha = 0, \quad \alpha' = 0, \quad \alpha'' = 0.$$

2. The choice of the functions $z = \Phi_x, \tau = \Phi_y$ satisfying to the system of equations

$$\Phi_{xx} = a_4 \Phi_y - a_3 \Phi_x,$$

$$\Phi_{yy} = a_2 \Phi_y - a_1 \Phi_x$$
with the coefficients \( a_i(x, y) \) in form

\[
a_4 = R_{xxx}, \quad a_3 = -R_{xxy}, \quad a_2 = R_{xyy}, \quad a_1 = R_{yyy}
\]

where the function \( R(x, y) \) is the solution of WDVV-equation

\[
R_{xxx}R_{yyy} - R_{xxy}R_{xyy} = 1
\]

are corresponded to the equations (1)

\[
y'' - R_{yyy}y'^3 + 3R_{xxy}y'^2 - 3R_{xxy}y' + R_{xxx} = 0.
\]

The following choice of the coefficients \( a_i \)

\[
a_4 = -2\omega, \quad a_1 = 2\omega, \quad a_3 = \frac{\omega_x}{\omega}, \quad a_2 = -\frac{\omega_y}{\omega}
\]

lead to the system

\[
\Phi_{xx} + \frac{\omega_x}{\omega} \Phi_x + 2\omega \Phi_y = 0,
\]

\[
\Phi_{yy} + 2\omega \Phi_x + \frac{\omega_y}{\omega} \Phi_y = 0
\]

with condition of compatibility

\[
\frac{\partial^2 \ln \omega}{\partial x \partial y} = 4\omega^2 + \frac{\kappa}{\omega}
\]

which is the Wilczynski-Tzitzeika-equation.

**Remark 3** The linear system of equations for the WDVV-equation some surfaces in 3-dim projective space is determined. In the canonical form it becomes [13]

\[
\Phi_{xx} - R_{xxx} \Phi_y + \left( \frac{R_{xxy}}{2} - \frac{R_{xyy}^2}{4} - \frac{R_{xxx}R_{xyy}}{2} \right) \Phi = 0,
\]

\[
\Phi_{yy} - R_{yyy} \Phi_x + \left( \frac{R_{yyy}}{2} - \frac{R_{xyy}^2}{4} - \frac{R_{yyy}R_{xyy}}{2} \right) \Phi = 0,
\]

The relations between invariants of Wilczynski for the linear system are correspondent to the various types of surfaces. Some of them with solutions of WDVV equation are connected.

**Remark 4** From the elementary point of view the surfaces connected with the system of equations like the Lorenz can be constructed on such a way. From the assumption

\[
z = z(x, y)
\]

we get

\[
\sigma(y - x)z_x + (rx - y - zx)z_y = xy - bz.
\]

The solutions of this equation give us the examples of the surfaces \( z = z(x, y) \).
Remark 5  Let us consider the system of equations

\[ \xi_{i,j} + \xi_{j,i} = 2\Gamma^k_{ij} \xi_k \]

for the Killing vectors of metrics (3). It has the form

\[ \begin{align*}
\xi_{1x} &= -a_3 \xi_1 + a_4 \xi_2 + (zA - ta_4x) \xi_3 + (zE + tF) \xi_4, \\
\xi_{2y} &= -a_1 \xi_1 + a_2 \xi_2 + (zC + tD) \xi_3 + (za_1y - tH) \xi_4, \\
\xi_{1y} + \xi_{2x} &= 2[-a_2 \xi_1 + a_3 \xi_2 + (za_3y - tB) \xi_3 + (zG - ta_2x) \xi_4], \\
\xi_{1z} + \xi_{3x} &= 2[a_3 \xi_3 + a_2 \xi_4], \\
\xi_{2z} + \xi_{3y} &= 2[a_2 \xi_3 + a_1 \xi_4], \\
\xi_3 &= 0, \\
\xi_4 &= 0.
\end{align*} \]

In particular case \( \xi_i(x, y) \)

\[ \begin{align*}
\xi_3 &= \xi_4 = 0, \\
\xi_i &= \xi_i(x, y)
\end{align*} \]

we get the system of equations

\[ \begin{align*}
\xi_{1x} &= -a_3 \xi_1 + a_4 \xi_2, \\
\xi_{2y} &= -a_1 \xi_1 + a_2 \xi_2, \\
\xi_{1y} + \xi_{2x} &= 2[-a_2 \xi_1 + a_3 \xi_2]
\end{align*} \]

equivalent to the system for the \( z=z(x, y) \) and \( \tau = \tau(x, y) \)

Remark 6  The Beltrami-Laplace operator

\[ \Delta = g^{ij}(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial}{\partial x^k}) \]

can be used for investigation of the properties of the metrics (3). For example the equation

\[ \Delta \Psi = 0 \]

has the form

\[ (ta_4 - za_3) \Psi_{zz} + 2(ta_3 - za_2) \Psi_{zt} + (ta_2 - za_1) \Psi_{tt} + \Psi_{xx} + \Psi_{yy} = 0. \]

Some solutions of this equation with geometry of the metrics (3) are connected.

Putting the expression

\[ \Psi = \exp[zA + tB] \]

into the equation

\[ \Delta \Psi = 0 \]

we get the conditions

\[ A = \Phi_y, \quad B = -\Phi_x, \]

and

\[ \begin{align*}
a_4 \Phi_y^2 - 2a_3 \Phi_x \Phi_y + a_2 \Phi_x^2 - \Phi_y \Phi_{xx} + \Phi_x \Phi_{xy} &= 0, \\
a_3 \Phi_y^2 - 2a_2 \Phi_x \Phi_y + a_1 \Phi_x^2 - \Phi_y \Phi_{xx} + \Phi_x \Phi_{xy} &= 0,
\end{align*} \]
Using the equation
\[ g^{ij} \frac{\partial F}{\partial x^i} \frac{\partial F}{\partial x^j} = 0 \]
or
\[ F_x F_z + F_y F_t - (t a_4 - z a_3) F_x F_z - 2 (t a_3 - z a_2) F_x F_t - (t a_2 - z a_1) F_t F_t = 0. \]
it is possible to investigate the properties of isotropical surfaces in space with metrics (3).

In particular case the solutions of eikonal equation in form
\[ F = A(x, y) z^2 + B(x, y) z t + C(x, y) t^2 + D(x, y) z + E(x, y) t \]
are existed if the following conditions are fulfilled
\[
\begin{align*}
2 A A_x + B A_y - a_1 B^2 - 4 a_2 AB - 4 a_3 A^2 &= 0, \\
2 A B_x + B A_x + 2 C A_y + B B_y - 4 a_1 B C - a_2 (B^2 + 8 AC) + 4 a_4 A^2 &= 0, \\
2 C B_y + B C_y + 2 A C_x + B B_x - 4 a_1 C^2 + a_3 (B^2 + 8 AC) + 4 a_4 A B &= 0, \\
2 C C_y + B C_x + 4 a_2 C^2 + 4 a_3 B C + a_4 B^2 &= 0, \\
2 A D_x + D A_x + E A_y + B D_y - 2 a_1 B E - 2 a_2 (B D + 2 A E) - 4 a_3 A D &= 0, \\
2 C D_y + (B D)_x + 2 A E_x + (B E)_y - 4 a_4 A C - 4 a_2 C D + 4 a_3 A E + 4 a_4 A D &= 0, \\
2 C E_y + C E_y + D C_x + B E_x - 4 a_2 C E + 2 a_3 (B E + 2 C D) + 2 a_4 B D &= 0, \\
D D_x + E D_y - a_1 E^2 - 2 a_2 D E - a_3 D^2 &= 0, \\
E E_y + D E_x + a_2 E^2 + 2 a_3 D E + a_4 D^2 &= 0.
\end{align*}
\]

**Remark 7** The metric (3) has a tetradic presentation
\[ g_{ij} = \omega_i^a \omega_j^b \eta_{ab} \]
where
\[
\eta_{ab} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}.
\]

For example we get
\[ ds^2 = 2 \omega_1 \omega_3 + 2 \omega_2 \omega_4 \]
where
\[
\omega_1 = dx + dy, \quad \omega_2 = dx + dy + \frac{1}{t(a_2 - a_4)} (dz - dt),
\]
\[
\omega_4 = -t(a_4 dx + a_2 dy), \quad \omega_3 = z(a_3 dx + a_1 dy) + \frac{1}{(a_2 - a_4)} (a_2 dz - a_4 dt).
\]
and
\[ a_1 + a_3 = 2 a_2, \quad a_2 + a_4 = 2 a_3. \]
Remark 8 Some of equations on curvature tensors in space $M^4$ are connected with ODE’s. For example, the equation

$$R_{ij,k} + R_{j;k:i} + R_{k;i,j} = 0$$

lead to the conditions on coefficients $a_i(x, y)$

$$\alpha''_x + 2a_3 \alpha'' - 2a_4 \alpha' = 0,$$
$$\alpha'_y + 2a_1 \alpha' - 2a_2 = 0,$$
$$\alpha''_y + 2\alpha'_y + 4a_2 \alpha'' - 2a_4 \alpha - 2a_3 \alpha' = 0,$$
$$\alpha_x + 2\alpha'_y - 4a_3 \alpha + 2a_2 \alpha' + 2a_1 \alpha'' = 0.$$

The solutions of this system give us the the second order equations connected with the space $M^4$ with condition (5) on the Ricci tensor. The symplest examples are

$$y'' - \frac{3}{y} y'^2 + y^3 = 0, \quad y'' - \frac{3}{y} y'^2 + y^4 = 0.$$

The conditions

$$R_{ij,k} - R_{j;k:i} = R^{n}_{ij;k:n}$$

also are interested. They are connected with theory of nonvacuum Einstein spaces.

Remark 9 The contruction of the Riemannian extension of two-dimensional spaces connected with ODE’s of type (1) can be generalized for three-dimensional spaces connected with the equations of the form

$$y'' + c_0 + c_1 x' + c_2 y' + c_3 x'^2 + c_4 x'y' + c_5 y'^2 + y' (b_0 + b_1 x' + b_2 y' + b_3 x'^2 + b_4 x'y' + b_5 y'^2) = 0,$$
$$x'' + a_0 + a_1 x' + a_2 y' + a_3 x'^2 + a_4 x'y' + a_5 y'^2 + x' (b_0 + b_1 x' + b_2 y' + b_3 x'^2 + b_4 x'y' + b_5 y'^2) = 0,$$

where $a_i, b_i, c_i$ are the functions of variables $x, y, z$.

4 On the Einstein-Weyl geometry

The relation between the equations in form (1) and

$$b'' = g(a, b, b')$$

with function $g(a, b, b')$ satisfying the partial differential equation

$$g_{aacc} + 2cg_{abc} + 2gg_{acc} + c^2 g_{bccc} + 2cg_{bc} +$$
$$+ g^2 g_{c} + (g_a + cg_b)g_{cc} - 4g_{abc} - 4cg_{bc} - cg_{bccc} -$$
$$- 3gg_{cc} - 4g_{aacc} + 4g_{b}g_{bc} - 3gg_{cc} + 6gb = 0$$

from geometrical point of view was studied by E.Cartan [14].

He showed that Einstein-Weyl 3-folds parameterize the families of curves of equation (5) which is dual to the equation (1).

Some examples of solutions of equation (5) were obtained first in [2].
Here we consider some examples of the Einstein-Weyl spaces.

Main facts of the theory of Einstein-Weyl spaces are the following [15].

A Weyl space is a smooth manifold equipped with a conformal metric \( g_{ij}(x) \), and a symmetric connection

\[
G^k_{ij} = \Gamma^k_{ij} - \frac{1}{2}(\omega^k_i j + \omega^k_j i - \omega^k_l g^{kl} g_{ij})
\]

(9)

with condition on covariant derivation

\[
D_i g_{kj} = \omega_i g_{kj}
\]

where \( \omega_i(x) \) are components of vector field.

The Weyl connection \( G^k_{ij} \) has a curvature tensor \( W_{ijkl} \) and the Ricci tensor \( W_{ij} \), is not symmetrical \( W_{ij} \neq W_{ji} \) in general case.

A Weyl space satisfying the Einstein condition

\[
\frac{1}{2}(W_{jl} + W_{lj}) = \lambda(x) g_{jl}(x),
\]

with some function \( \lambda(x) \), is called Einstein-Weyl space.

Let us consider some examples.

The components of Weyl connection of 3-dim space:

\[
ds^2 = dx^2 + dy^2 + dz^2
\]

are

\[
2G_1 = \begin{vmatrix}
-\omega_1 & -\omega_2 & -\omega_3 \\
-\omega_2 & -\omega_1 & 0 \\
0 & -\omega_1 & -\omega_1 \\
\end{vmatrix}
\]

\[
2G_2 = \begin{vmatrix}
-\omega_2 & \omega_1 & 0 \\
-\omega_1 & -\omega_2 & -\omega_3 \\
0 & \omega_3 & -\omega_2 \\
\end{vmatrix}
\]

\[
2G_3 = \begin{vmatrix}
-\omega_3 & 0 & \omega_1 \\
0 & -\omega_3 & \omega_2 \\
-\omega_1 & -\omega_2 & -\omega_3 \\
\end{vmatrix}
\]

From the equations of Einstein-Weyl spaces

\[
W_{[ij]} = \frac{W_{ij} + W_{ji}}{2} = \lambda g_{ij}
\]

we get the system of equations

\[
\omega_{3x} + \omega_{1z} + \omega_{1}\omega_{3} = 0, \quad \omega_{3y} + \omega_{2z} + \omega_{2}\omega_{3} = 0, \quad \omega_{2x} + \omega_{1y} + \omega_{1}\omega_{2} = 0,
\]

\[
2\omega_{1x} + \omega_{2y} + \omega_{3z} - \frac{\omega_{3}^2 + \omega_{2}^2}{2} = 2\lambda, \quad 2\omega_{2y} + \omega_{1x} + \omega_{3z} - \frac{\omega_{1}^2 + \omega_{3}^2}{2} = 2\lambda,
\]

\[
2\omega_{3z} + \omega_{2y} + \omega_{1x} - \frac{\omega_{1}^2 + \omega_{2}^2}{2} = 2\lambda.
\]

Remark that the first three equations lead to the Chazy equation [16]

\[
R''' + 2RR'' - 3R'^2 = 0
\]

for the function

\[
R = R(x + y + z) = \omega_1 + \omega_2 + \omega_3
\]

where \( \omega_i = \omega_i(x + y + z) \) and in general case they are generalization of classical Chazy equation.
Einstein-Weyl geometry of the metric $g_{ij} = \text{diag}(1, -e^U, -e^U)$ and vector $\omega_i = (2U_x, 0, 0)$ is determined by the solutions of equation [17]

$$U_{xx} + U_{yy} = (e^U)_{zz}.$$ 

This equation is equivalent to the equation

$$U_r = (e^{U/2})_z$$

(after substitution $U = U(x + y = \tau, z)$) having many-valued solutions.

The consideration of the E-W structure for the metrics

$$ds^2 = dy^2 - 4dx^2 - 4adt^2$$

lead to the dispersionless KP equation [18]

$$(U_t - UU_x)_x = U_{yy}.$$ 

2. An Einstein-Weyl geometry of the four-dimensional Minkovskii space

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2$$

The components of Weyl connection are

$$2G_1 = \begin{vmatrix} -\omega_1 & -\omega_2 & -\omega_3 & -\omega_4 \\ \omega_2 & -\omega_1 & 0 & 0 \\ \omega_3 & 0 & -\omega_1 & 0 \\ -\omega_4 & 0 & 0 & -\omega_1 \end{vmatrix}, \quad 2G_2 = \begin{vmatrix} -\omega_2 & \omega_1 & 0 & 0 \\ -\omega_1 & -\omega_2 & -\omega_3 & -\omega_4 \\ 0 & 0 & -\omega_2 & 0 \\ 0 & -\omega_4 & 0 & -\omega_2 \end{vmatrix},$$

$$2G_3 = \begin{vmatrix} -\omega_3 & 0 & \omega_1 & 0 \\ 0 & -\omega_3 & \omega_2 & 0 \\ -\omega_1 & -\omega_2 & -\omega_3 & -\omega_4 \\ 0 & 0 & -\omega_4 & -\omega_3 \end{vmatrix}, \quad 2G_4 = \begin{vmatrix} -\omega_4 & 0 & 0 & \omega_1 \\ 0 & -\omega_4 & 0 & -\omega_2 \\ 0 & 0 & -\omega_2 & -\omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & -\omega_4 \end{vmatrix}.$$ 

The Einstein-Weyl condition

$$W_{[ij]} = W_{ij} + W_{ji} = \lambda g_{ij}$$

where

$$W_{ij} = W_{ij}^l$$

and

$$W_{\lambda ij} = \frac{\partial G^l_{ij}}{\partial x^l} - 2G^l_{ij} + G^k_{im}G^n_{ij} - G^k_{jn}G^n_{il}$$

lead to the system of equations

$$\omega_{3x} + \omega_{1z} + \omega_1 \omega_3 = 0, \quad \omega_{3y} + \omega_{2z} + \omega_2 \omega_3 = 0,$$

$$\omega_{2x} + \omega_{1y} + \omega_1 \omega_2 = 0, \quad \omega_{4x} + \omega_{1t} + \omega_1 \omega_4 = 0,$$

$$\omega_{4y} + \omega_{2t} + \omega_2 \omega_4 = 0, \quad \omega_{4z} + \omega_{3t} + \omega_3 \omega_4 = 0,$$

$$3\omega_{1x} + \omega_{2y} + \omega_3 \omega_2 - \omega_{4t} + \omega_4^2 - \omega_1^2 - \omega_2^2 = 2\lambda,$$

$$3\omega_{2y} + \omega_{1x} + \omega_3 \omega_1 - \omega_{4t} + \omega_4^2 - \omega_1^2 - \omega_3^2 = 2\lambda,$$

$$3\omega_{3z} + \omega_{2y} + \omega_1 \omega_2 - \omega_{4t} + \omega_3^2 - \omega_1^2 - \omega_2^2 = 2\lambda,$$

$$3\omega_{4t} - \omega_{2y} - \omega_1 \omega_2 + \omega_3^2 + \omega_1^2 + \omega_2^2 = 2\lambda.$$
5 On solutions of dual equations

Equation (8) can be written in compact form

\[ \frac{d^2g_{cc}}{da^2} - g_e \frac{dg_{cc}}{da} - 4 \frac{dg_{bc}}{da} + 4g_e g_{bc} - 3g_b g_{cc} + 6g_{bb} = 0 \]  \hfill (10)

with help of the operator

\[ \frac{d}{da} = \frac{\partial}{\partial a} + c \frac{\partial}{\partial b} + g \frac{\partial}{\partial c}. \]

It has many types of reductions and the symplest of them are

\[ g = c^\alpha \omega [ac^{2\alpha-1}], \quad g = c^\alpha \omega [bc^{2\alpha-2}], \quad g = a^{-\alpha} \omega [ca^{2\alpha-1}], \]

\[ g = b^{1-2\alpha} \omega [cb^{2\alpha-1}], \quad g = a^{-2\alpha} \omega (c - b/a), \quad g = a^{-3\alpha} \omega [b/a, b - ac], \quad g = a^{3\alpha/2} \omega [b^2/a^2, c^\alpha/a^{\alpha-\alpha}]. \]

To integrate a corresponding equations let us consider some particular cases

1. \[ g = g(a, c) \]

   From the condition (10) we get

   \[ \frac{d^2g_{cc}}{da^2} - g_e \frac{dg_{cc}}{da} = 0 \]  \hfill (11)

   where

   \[ \frac{d}{da} = \frac{\partial}{\partial a} + g \frac{\partial}{\partial c}. \]

   Putting into (11) the relation

   \[ g_{ac} = -gg_{cc} + \chi(g_c) \]

   we get the equation for \( \chi(\xi), \quad \xi = g_c \)

   \[ \chi(\chi'' - 1) + (\chi' - \xi)^2 = 0. \]

   It has solutions

   \[ \chi = \frac{1}{2} \xi^2, \quad \chi = \frac{1}{3} \xi^2 \]

   So we get two reductions of the equation (10)

   \[ g_{ac} + gg_{cc} - \frac{g_e^2}{2} = 0 \]

   and

   \[ g_{ac} + gg_{cc} - \frac{2g_e^2}{3} = 0. \]

**Remark 10** The first reduction of equation (8) is followed from its presentation in form

\[ g_{ac} + gg_{cc} - \frac{1}{2} g_e^2 + cg_{bc} - 2g_b = h, \]

\[ h_{ac} + gh_{cc} - g_c h_c + ch_{bc} - 3h_b = 0 \]

and was considered before.
In particular case \( h = 0 \) we get one equation

\[ g_{ac} + gg_{cc} - \frac{1}{2}g_c^2 + cg_{bc} - 2g_b = 0 \]

which is the equation (10) for the function \( g = g(a, c) \). It can be integrate with help of Legendre transformation (see [3]).

The solutions of the equations of type

\[ u_{xy} = uu_{xx} + \epsilon u_x^2 \]

was constructed in [19]. In work of [20] was showed that they can be present in form

\[ u = B'(y) + \int [A(z) - \epsilon y]^{(1-\epsilon)/\epsilon} dz, \]

\[ x = -B(y) + \int [A(z) - \epsilon y]^{1/\epsilon} dz. \]

To integrate above equations we apply the parametric representation

\[ g = A(a) + U(a, \tau), \quad c = B(a) + V(a, \tau). \tag{11} \]

Using the formulas

\[ g_c = \frac{g_{\tau}}{c_{\tau}}, \quad g_a = g_a + g_{\tau} \tau_a \]

we get after the substitution in (10) the conditions

\[ A(a) = \frac{dB}{da} \]

and

\[ U_{a\tau} - \left( \frac{V_a U_{\tau}}{V_{\tau}} \right)_{\tau} + U \left( \frac{U_{\tau}}{V_{\tau}} \right)_{\tau} - \frac{1}{2} \frac{U_{\tau}^2}{V_{\tau}} = 0. \]

So we get one equation for two functions \( U(a, \tau) \) and \( V(a, \tau) \). Any solution of this equation are determined the solution of equation (10) in form (11).

Let us consider the examples.

\[ A = B = 0, \quad U = 2\tau - \frac{a\tau^2}{2}, \quad V = a\tau - 2 \ln(\tau) \]

Using the representation

\[ U = \tau \omega_{\tau} - \omega, \quad V = \omega_{\tau} \]

it is possible to obtain others solutions of this equation.
6 Third-order ODE’s and the Weyl-geometry

In the work of E.Cartan was studied the geometry of the equation

$$b''' = g(a, b, b', b'')$$

with General Integral in form

$$F(a, b, X, Y, Z) = 0.$$ 

It was showed that there are a lot types of geometrical structures connected with this type of equations.

More recently [21,22] the geometry of Third-order ODE’s was considered in context of the null-surface formalism and was discovered that in the case the function $g(a, b, b', b'')$ is satisfied the conditions:

$$\frac{d^2g_r}{da^2} - 2g_r \frac{dg_r}{da} - 3\frac{dg_c}{da} + \frac{4}{9}g_r^3 + 2g_c g_r + 6g_b = 0 \tag{12}$$

$$\frac{d^2g_{rr}}{da^2} - \frac{dg_{cr}}{da} + g_b = 0 \tag{13}$$

where

$$\frac{d}{da} = \frac{\partial}{\partial a} + \frac{c}{\partial b} + \frac{r}{\partial c} + \frac{g}{\partial r}.$$ 

the Einstein-Weyl geometry in space of initial dates has been realized.

We present here some solutions of the equations (12, 13) which is connected with theory of the second order ODE’s.

In the notations of E.Cartan we study the Third-order differential equations

$$y''' = F(x, y, y', y'')$$

where the function $F$ is satisfied to the system of conditions

$$\frac{d^2 F_2}{dx^2} - 2 F_2 \frac{dF_2}{dx} - 3 \frac{dF_1}{dx} + \frac{4}{9} F_2^3 + 2 F_1 F_2 + 6 F_0 = 0 \tag{14}$$

$$\frac{d^2 F_{22}}{dx^2} - \frac{dF_{12}}{dx} + F_{02} = 0$$

where

$$\frac{d}{dx} = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + F \frac{\partial}{\partial y''}.$$ 

We consider the case of equations

$$y''' = F(x, y', y'')$$

In this case $F_0 = 0$ and from the second equation we have

$$H_{x2} + y'' H_{12} + F H_{22} = 0$$

where

$$F_{x2} + F F_{22} - \frac{F_{22}^2}{2} + y'' F_{12} - 2 F_1 = H$$
With the help of this the first equation give us the condition

$$H_x + y''(H_1 - F_{11}) - FF_{12} - \frac{1}{18}F_2^3 - F_{x1} = 0$$

In the case

$$H = H(F_2), \quad \text{and} \quad F = F(x, y'')$$

we get the condition on the function $F$

$$F_{x2} + FF_{22} - \frac{F_2^2}{3} = 0$$

The corresponding third-order equation is

$$y''' = F(x, y'')$$

and it is connected with the second-order equation

$$z'' = g(x, z').$$

Another example is the solution of the system for the function $F = F(x, y', y'')$ obeying to the equation

$$F_{x2} + FF_{22} - \frac{F_2^2}{2} + y''F_{12} - 2F_1 = 0$$

In this case $H = 0$ and we get the system of equations

$$F_{x2} + FF_{22} - \frac{F_2^2}{2} + y''F_{12} - 2F_1 = 0$$

$$y''F_{11} + FF_{12} + \frac{1}{18}F_2^3 + F_{x1} = 0$$

with the condition of compatibility

$$\left(\frac{F_2^2}{6} - F_1\right)F_{22} + 2F_2F_{12} + 3F_{11} = 0$$

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