Quantum Mechanics in Pseudotime

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Abstract

Based on some results on reparametrisation of time in Hamiltonian path integral formalism, a pseudo time formulation of operator formalism of quantum mechanics is presented. Relation of reparametrisation of time in quantum with super symmetric quantum mechanics is established. We show how some important concepts such as shape invariance and tools like isospectral deformation appear in pseudo time quantum mechanics.
1 Introduction

The classical trajectory of a point particle can be given in terms a parameter, other than time, to be called pseudotime in this paper. A Lagrangian formulation, called homogeneous formalism [1] is available and leads to a singular Lagrangian. A Hamiltonian formulation can be written down using Dirac’s treatment of such systems. Quantisation of such a system can then proceed by setting the fundamental commutator brackets equal to $i\hbar$ times the Dirac brackets. A relativistic particle, string theory, general relativity constitute important examples which are manifestly invariant under a reparametrisation of time[2]. Many investigations of reparametrisation in quantum mechanics start with the homogeneous formalism of classical mechanics and use Dirac quantisation.

Non relativistic point particle is not manifestly invariant under a reparametrisation of time. However it found important applications in the path integral formulation of quantum mechanics[3]. Duru and Kleinert [4] formulated reparameterisation of time within the Hamiltonian path integral approach and used it as a tool, along with K-S transformation[5], to obtain a path integral solution for H atom problem. Following this idea several authors used Lagrangian as well as Hamiltonian path integral to write path integral solution for several exactly solvable potential models.

In [6] a scheme of setting up Hamiltonian path integral (HPI) was proposed. Later quantisation scheme in arbitrary coordinates using reparametrised Hamiltonian path integral (RHPI) and it was found that correct quantisation could be carried out without a need to add $O(\hbar^2)$ terms [7]. Further results on RHPI and applications have been given in [8].

In this paper we wish to formulate reparametrisation of time in the operator
formalism of quantum mechanics closely following the ideas of reparametrisation in the path integral approach. The central results, that we need from earlier works, are Duru Kleinert formula and operator ordering. These are summarised in Sec. 3.

Path integral representation derives its importance from the fact that it can be used as scheme of quantisation too. It is this fact that will be of crucial importance to us in this paper.

In this article we are concerned with reparametrisation in operator formalism of nonrelativistic quantum mechanics. We will connect work on Hamiltonian path integral with some well known results in quantum mechanics of one dimensional systems. It should then be possible to extend those results to higher dimensional systems.

The aim of this paper is to propose a pseudo-time formalism of quantum mechanics and to link available results of Hamiltonian path integrals to corresponding results in the operator formalism and is organised as follows. The homogeneous formulation of classical mechanics of a point particle is briefly described in the next section. Section 3 summarises results about using reparametrisation of time in the Hamiltonian path integral approach. In next section several known results is established with pseudotime quantum mechanics. The reparameterisation of time appears to be intimately related to supersymmetric quantum mechanics (SUSYQM) [9]. We to establish a connection of shape invariance, isospectral deformation [10] with pseudo-time formalism. The quantum Hamilton Jacobi equation (QHJ) [11, 12] is related to pseudo time quantum mechanics in a simple and straight forward manner. A connection with exceptional polynomials [13] is briefly mentioned. In Sec. 5 the mathematical structure of quantum theory in pseudo-time is presented. Several possible different interpretations of the mathematical formulation of pseudo-time
quantum mechanics are outlined in Sec.6. Directions for possible further study and
concluding remarks are given the last section of this paper.

2 Reparametrisation in classical mechanics

In this section we briefly describe the homogeneous formalism\[1\]. In classical me-
chanics the trajectory of a point particle is found by solving the Hamilton’s equations
for generalised coordinates and momenta as functions of time.

\[ q = q(t), \quad p = p(t), \quad t_1 \leq t \leq t_2. \] (1)

These are parametrised curves in phase space. The trajectories can also be specified
by choosing another parameter \(\sigma\), to be called pseudo-time. In this description the
coordinates and momenta are to be solved as functions of \(\sigma\) and the answer is to be
written as

\[ q = q(\sigma), \quad p = p(\sigma), \quad \sigma_1 \leq \sigma \leq \sigma_2. \] (2)

To connect with (1), we require another equation of the form

\[ t = t(\sigma), \quad t(\sigma_1) = t_1, t(\sigma_2) = t_2. \] (3)

treating \(t\) like a coordinate. For this purpose, one adds an equation for evolution of
\(t\) in pseudotime

\[ \frac{dt}{d\sigma} = f(q(t)) \] (4)

where the function \(f\) is a positive function of coordinates \(q\) and will be called local
time scaling function (LTSF). A Hamiltonian formalism can be set up by using the
pseudo Hamiltonian

\[ \mathcal{H}_E = f(q)(H - E) \] (5)
The pseudo energy is a constant of motion for dynamics in pseudo time $\sigma$. The Hamilton’s equations in pseudo time, using pseudo Hamiltonian, give the dynamics correctly when the pseudo energy is set equal to zero. It may be noted that setting pseudo energy equal to zero means using the energy conservation equation $H(q,p) - E = 0$. Thus classical equations of motion can be set up equally well in pseudotime without changing physical content.

3 Reparametrisation in path integral formalism

In 1979 Duru and Kleinert used K-S transformation and Hamiltonian path integral representation in pseudo time to arrive at an exact path integral solution for H-atom problem [4]. Crucial to their derivation was a formula which we call Duru Kleinert formula, see (6). This formula related the path integral in time $t$ to Hamiltonian path integral in pseudo time, and was derived by Duru and Kleinert by means of formal manipulations. It was therefore a fortunate circumstance that it worked for H atom.

Later the same approach was employed and relation between path integrals in time $t$ and pseudo time $\sigma$ was carefully derived using the accepted rules for path integrals within the time slicing approach[14]. In general, terms of order $\bar{\hbar}^2$ were required to be added to the potential when writing setting up Duru-Kleinert formula. A number of authors used Hamiltonian, as well as Lagrangian, path integral formulations to obtain exact solution for the propagator for many problems [15]. It may be recalled that previous to the work by Duru and Kleinert, only a small class of quantum mechanical problems could be solved using the path integral representation [16]. Use of reparametrisation opened the way to obtain path integral solution for large class of problems known to be exactly solvable.
It may be mentioned that reparametrisation, articles on the path integral approach used a few other techniques, notably addition of new degrees of freedom, for arriving at the solutions of potential problems in quantum mechanics.

It has been known that a careful point transformation, and also use of Hamiltonian path integral to quantise a system in non-Cartesian coordinates require addition of $O(\hbar^2)$ terms to the classical Hamiltonian. It is to be noted that the same problem also appears when we attempt to use canonical quantisation directly in the non-Cartesian coordinates [6]. In a series of papers on Hamiltonian path integral quantisation, it was shown that $O(\hbar^2)$ terms were not required if one suitably combined reparametrisation with the Hamiltonian path integral. Thus reparametrisation of time seems to be intimately tied to point transformations in Hamiltonian path integrals.

We recall a few central points and quote a few results from a previous study of Hamiltonian path integral quantisation within time slicing approach.

**Hamiltonian path integral**

Given a classical Hamiltonian $H(q, p)$, a scheme of constructing a particular path integral has been suggested and investigated in [6]. Using a particular form, $(q_t \parallel q_{0t_0})$, for short time propagator, Hamiltonian path integral (HPI) $K_H(q_t, q_{0t_0})$ was defined as summation over all paths from $q_0$ at time $t_0$ to $q$ at time $t$. However this did not lead to correct Schrödinger equation in non-Cartesian coordinates.

In [7] another path integral representation was introduced by means for Duru-Kleinert formula

$$\mathcal{H}(q_t, q_{00}) \overset{\text{def}}{=} \int_{-\infty}^{+\infty} \left( \frac{dE}{2\pi \hbar} \right) \exp(-iEt/\hbar) \int_0^{\infty} [\sqrt{f(q)}f(q_0)] \times K_{\mathcal{H}}(q_\sigma, q_{00}) d\sigma. \quad (6)$$

Here the expression $K_{\mathcal{H}}(q_\sigma, q_{00})$, appearing in the right hand side is the HPI con-
constructed using the pseudo Hamiltonian $\mathcal{H}_E = f(q)(H - E)$ corresponding to the LTSF function $f(q)$ and will be called reparametrised Hamiltonian path integral (RHPI). It may be noted that RHPI reduces to HPI for the special case of LTSF $f(q) = 1$.

The $\mathcal{K}(q_t, q_0)$ defined by (6) will be called DK propagator and, to simplify the notation, its dependence on Hamiltonian $H(q, p)$ and LTSF $f(q)$ will not be shown explicitly.

We will now give a brief summary of main results obtained earlier.

**Operator ordering**

Here we list the operator ordering implicit in the above discussion for the special case of potential problems in $n$-dimension. Corresponding statements for more general case can be written down but will not be required here. The Hamiltonian function $H(x, p)$ will be assumed to be of the form.

$$H(x, p) = \frac{p^2}{2m} + V(x)$$  \hspace{1cm} (7)

As already remarked the HPI obeys the correct Schrödinger equation and the corresponding operator is

$$\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 + V(x).$$  \hspace{1cm} (8)

The RHPI $K_{\mathcal{H}_E}$, satisfies the Schrödinger equation with pseduo Hamiltonian operator $\hat{\mathcal{H}}_E$ given by

$$\hat{\mathcal{H}}_E = \frac{1}{2m} \nabla f(x) \nabla + f(x) \left( V(x) - E \right).$$  \hspace{1cm} (9)

\footnote{Here $x$ denotes Cartesian coordinates}
Normalisation of RHPI $K_{\mathcal{H}_E}$: In [8] it was shown that the path integral RHPI $K_{\mathcal{H}_E}$ appearing in the right hand side of (6) satisfies the Schrödinger equation

$$i\hbar \frac{d}{dt} K_{\mathcal{H}_E}(xt, x_0 t_0) = \hat{H}_E K_{\mathcal{H}_E}(xt, x_0 t_0)$$

(10)

and has the normalisation

$$\lim_{t \to t_0} K_{\mathcal{H}_E}(xt, x_0 t_0) = (f(x))^{-1} \delta^{(n)}(x - x_0).$$

(11)

Thus it is seen that out that the path integral RHPI constructed with pseudo Hamiltonian is has a normalisation different from what is required. This explains appearance factor $\sqrt{f(x)f(x_0)}$ in the right hand side of Duru-Kleinert formula.

DK Propagator:

$$\tilde{H}_E = \frac{1}{\sqrt{f}} \left( \frac{1}{2m} \nabla f(x) \nabla + f(x) \left( V(x) - E \right) \right) \frac{1}{\sqrt{f}}$$

(12)

Writing this last expression as $\tilde{H}_E \equiv \tilde{H} - E$, the expression for $\tilde{H} - E$ can be rearranged in alternate forms

$$\tilde{H} = \frac{1}{f} \frac{1}{2m} (\nabla - w) f(x) (\nabla - w) + V(x)$$

(13)

$$= \frac{1}{2m} (\nabla + w)(\nabla - w) + V(x)$$

(14)

$$= \frac{1}{2m} \nabla^2 + V(x) + \Delta V$$

(15)

where $w_k = \partial_k \Omega(x)$, $\Omega(x) = \frac{1}{2} \ln f(x)$ and

$$\Delta V = \frac{\hbar^2}{2m} \left\{ (\nabla \Omega(x))^2 - \nabla^2 \Omega(x) \right\}$$

(16)

The DK propagator $\mathcal{K}(xt, x_0 t_0)$ of (6) obeys the Schrödinger equation with $\tilde{H}$ as the Hamiltonian

$$i\hbar \frac{d}{dt} \mathcal{K}(xt, x_0 t_0) = \tilde{H} \mathcal{K}(xt, x_0 t_0)$$

(17)

and is normalised as

$$\mathcal{K}(xt, x_0 t_0)|_{t=t_0} = \delta(x - x_0).$$

(18)
Qunatisation in arbitrary coordinates:

Identifying HPI $K_H$ as propagator for the corresponding quantum problem leads to correct quantisation scheme in Cartesian coordinates. However the same scheme, when used for quantisation in non-Cartesian coordinates, did not give correct quantisation scheme. It was became necessary to add $O(h^2)$ terms to the Hamiltonian[6].

This problem of appearance of $O(h^2)$ terms is not specific to the scheme that was used in reference [6]. A need for these extra $O(h^2)$ terms has been well known in the Hamiltonian path integral literature. In fact the problem reappears even in canonical quanatisation scheme. In a general case this difficulty of canonical quanatisation is masked by ordering problems and can be seen most clearly in polar coordinates in two dimensions, a model not having any ordering problem for the Hamiltonian,[6][3].

It was demonstrated in reference [7], that the DK-propagator with the choice, $f(q) = \rho(q)$ as LTSF function, leads to correct quantisation in arbitrary coordinates without need to add any $O(h^2)$ terms. This scheme worked with the classical Hamiltonian $H(q,p)$ directly in arbitrary coordinates; setting up a Hamiltonian path integral in Cartesian coordinates and changing variables was not required.

4 Connection of RHPI with operator formalism

In this section we establish relationship of some of the important concepts in operator formalism and DK propagator.

Supersymmetric quantum mechanics

Supersymmetric quantum mechanics has been an active area of research for several decades and the concept of shape invariance continues to attract a great deal of attention. It is easy to see the place that supersymmetric partners and shape invar-
ance have in the pseudotime path integral framework. Starting from a free particle Hamiltonian $V(x) = 0$ to set up a DK propagator, scaling functions $f(x)$ and $1/f(x)$ lead to quantum system whose time evolution is governed by Hamiltonian $H_+$ and $H_-$, respectively which are given by

$$H_{\pm} = \frac{p^2}{2m} + \frac{\hbar^2}{2m} (w^2 \mp w'),$$

(19)

where $w = \frac{d\Omega}{dx}$ and $\Omega = \ln f$.

$$H(q, p) \xrightarrow{f} \text{DK Propagator for } H_+$$

(20)

$$H(q, p) \xrightarrow{1/f} \text{DK Propagator for } H_-$$

(21)

In the terminology of SUSYQM, the above two Hamiltonians will be recognised as supersymmetric partners with $w(x)$ playing the role of superpotential.

**Darboux transformation** The Darboux transformation [17] gives a relation between the eigenfunctions of SUSY partner Hamiltonians $H_{\pm}$. The propagators for $H_{\pm}$ are both related to free particle HPI in pseudo time with scaling functions $f(x)$ and $1/f(x)$ respectively, see (19)-(20).

The Darboux transformation and its generalisations by Crum and by Krien are powerful results that have found large number of applications to several areas including exactly integrable models. Darboux’s result is implicitly contained in Duru Kleinert formula. It is of interest to establish at a direct and explicit correspondence between Darboux transformation and DK formula.

**Isospectral deformation of a potential**

The isospectral deformation of a potential $V(x)$ generates a new potential having exactly same spectrum as the original potential $V(x)$. This process makes use of results from SUSYQM.
Consider a model with potential $V(x)$ corresponding to superpotential $w(x)$. We now consider a two RHPI with $H(q,p)$ as free particle with certain scaling functions $f(x)$ and $1/f(x)$. This will result in DK-propagator for potentials $V_{\pm}(x)$. Demanding that $V_{+}(x)$ coincide with $V(x)$, a solution for $f(x)$ will lead to $V_{-}(x)$ which will be the isospectral deformation of the original potential $V(x)$. The steps for arriving at the required solution for $f(x)$ will closely follow the steps known in the literature for the isospectral deformation and no further explanation is required.

**Exceptional Polynomials** In an an earlier paper [18], a systematic procedure for deformation of radial oscillator potential was given in the framework of QHJ was presented. It was found that demanding shape invariance be preserved under the deformation, led to the isospectral shift of the radial oscillator potential. All these steps can in principle be translated and followed in the pseudotime formalism as presented here. Starting directly form the differential equation for classical orthogonal polynomials and using using pseudo time framework, to will be interesting to find a direct route to the exceptional polynomials.

**Time dependent supersymmetry** Time dependent supersymmetry and time dependent Darboux transformation have been studied[19]. These studies will be connected with reparametrisation of time with LTSF which is a function of time, $\frac{dt}{d\sigma} = f(q,t)$. The basic equations in the Hamiltonian path integral formalism will then have correspondence with equations of [19].

**Quantum Hamilton Jacobi equation**

Quantum Hamilton Jacobi formalism provides a scheme of computing energy eigenvalues without solving for wave functions [11] and has been studied extensively [12] If
we substitute $\psi(x) = \exp(iS(x)/\hbar)$ the Schrödinger equation for a potential problem

$$V(x)$$

$$\frac{-\hbar^2}{2m}\nabla^2\psi(x) - V(x)\psi(x) = E\psi(x). \quad (22)$$

gets transformed into an equation for $S(x)$

$$\frac{1}{2m}(\nabla S(x))^2 + \frac{i\hbar}{2m}\nabla^2 S(x) + V(x) - E = 0. \quad (23)$$

This equation is known as the quantum Hamilton Jacobi (QHJ) equation for potential $V(x)$. Consider the Schrödinger equation

$$\frac{-\hbar^2}{2m}\nabla^2\psi(x) + V(x)\psi(x) = E\psi(x). \quad (24)$$

If we set up DK propagator and demand that LTSF $f(x)$ be such that $\psi(x) = \text{constant}$ is a solution of Schrodinger equation for $\tilde{H}$, we get the following equation for the scaling function $f(x)$

$$\frac{\hbar^2}{2m}[\left((\nabla \Omega(x))^2 - \nabla^2 \Omega(x)\right] + V(x) - E = 0. \quad (25)$$

Identifying $\Omega(x)$ with $iS(x)/\hbar$, the above equation becomes identical with the QHJ equation (23).

5 General structure of quantum mechanics in pseudo-time

In this section we will work with non-Cartesian coordinates coordinates and canonical momenta $q, p$ in place of Cartesian coordinates. The symbol $\rho(q)$ will denote the volume element defined by $dx = \rho(q)dq$. The states of system will be elements of Hilbert space $\mathcal{H}$ of all square integrable functions $f(q)$:

$$\int |\langle q|\psi\rangle|\rho(q) dq < \infty$$

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Given a system with classical dynamics governed by Hamiltonian function \( H(q, p) \) and a scaling function \( f(q) \), introduce pseudotime \( \sigma \) and pseudo Hamiltonian \( \mathcal{H}_E \)

\[
\frac{dt}{d\sigma} = f(q),
\]
\[
\mathcal{H}_E = f(q)(H(q, p) - E) = f(q)(\mathcal{H} - E),
\]

where \( \mathcal{H} = f(q)H(q, p) \).

We will work in the Heisenberg picture. Therefore the propagator will be given by \( \langle qt|q_0 t_0 \rangle \), where \( |qt\rangle \) is eigenvector of the position operator \( \hat{q}(t) \) at time \( t \):

\[
\hat{q}_k(t)|qt\rangle = q_k|qt\rangle, k = 1, 2, \ldots
\]

A path integral representation for propagator is constructed out of short time propagator \( (qt|q_0) \) in the usual fashion by using time slicing approach and summing over all paths.

We recall \( (6) \) and the comment on normalisation of RHPI. It has been noted that the HPI in real time \( t \) and RHPI in pseudo time \( \sigma \) are normalised differently, and that one needs to multiply by factor \( \sqrt{f(x)f(x_0)} \) at the end. We introduces a second set of position eigenvectors \( \mathcal{B} = \{ |q\sigma \rangle \} \) defined by

\[
|q\sigma \rangle = (f(q))^{-1/2}|q\rangle
\]

which are normalised differently:

\[
\langle q|q_0 \rangle = (f(q))^{-1}\rho^{-1}(q)\delta^{(n)}(q - q_0).
\]

The completeness relations now take the form

\[
\int \rho(q)|q\rangle\langle q|dq = \hat{1}, \quad \int \rho(q)f(q)|q\rangle\langle q| = \hat{\rho}.
\]
The new eigenvectors of position $|q\rangle$ span a different Hilbert space $\mathcal{H}_f$ with changed scalar product $\langle \psi|\phi \rangle$ defined as:

$$\int \psi(q)\phi(q)\rho(q)\,dq \rightarrow \int \psi(q)\phi(q)\rho(q)f(q)\,dq.$$  

(32)

Thus we interpret the change to basis $|q\rangle$ as a switch to different Hilbert space of functions square integrable with a new measure $\rho(q)f(q)$.

The process of setting up the RHPI with LTSF $f(q)$ can now be regarded as consisting of following steps.

Noting that $\langle q|\psi \rangle \in \mathcal{H}$ and $\langle q|\psi \rangle \in \mathcal{H}_f$ representing a state vector $|\psi \rangle$ in the two Hilbert spaces are related by

$$\langle q|\psi \rangle = (f(q))^{-1/2}\langle q|\psi \rangle,$$  

(33)

we define a mapping of the operators in the two Hilbert spaces by

$$\mathcal{X}_f = (f(q))^{-1/2} \tilde{X} (f(q))^{1/2}.$$  

(34)

The vector space equations will then be preserved when a transition is made from the original Hilbert space to another one labelled by $f(q)$.

If $\tilde{X}$ is written as an ordered expression $\tilde{X}(q, \frac{\partial}{\partial q})$, the above equation translates into

$$\mathcal{X}_f = X(q, D)$$  

(35)

where

$$D_k = \partial_k + \omega_k \quad \text{and} \quad \omega_k = \frac{1}{2} \frac{\partial (\ln f)}{\partial q_k}.$$  

(36)

Setting up of the the propagator as RHPI and using DK propagator can now be interpreted as a sequence of the following steps:
1. Change from original Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{H}_f$ labelled by the LTSF $f(q)$, and defined by a different square integrability requirement

$$\int |\psi(q)|^2 \rho(q) dq < \infty \rightarrow \int |\psi(q)|^2 \rho(q) f(q) dq < \infty.$$  

(37)

2. Set up the propagator in the new Hilbert space $\mathcal{H}_f$ as a path integral RHPI in pseudotime $\sigma$.

3. Revert back to original Hilbert $\mathcal{H}$ space by using DK propagator for time evolution in time $t$.

6 Interpretations of pseudotime quantum mechanics

So far we have looked at the mathematical structure of quantum theory as suggested by Hamiltonian path integral in pseudo time. Now we discuss a few different possible interpretations of our equations in a manner which incorporates reparametrisation of time in quantum theory.

The classical formulations in different pseudo times are all equivalent in the sense that they give rise to the same trajectory. In quantum theory they solutions of dynamical equations look different for different choices of pseudo times because of appearance of $O(\hbar^2)$ terms appearing in the potential. For example a free particle in real time $t$ will, in pseudo time, appear as a particle moving in a non constant potential $\Delta V$.

Simplest interpretation would of our equations will be that, for a particular choice of Hamiltonian $H(q, p)$, only DK propagator is to be regarded as having physical significance. The RHPI is introduced solely for technical purposes for setting up the propagator and that it appears in intermediate steps of quantisation and evaluation of the propagator using path integrals. A possibility along these lines is that for each
choice of coordinates $q$, there is one (or more) preferred choice(s) of LTSF $f(q)$ determined by the requirement $\Delta V = 0$, as is indicated by quantisation in arbitrary coordinates.

Another way to interpret the equations and $\Delta V$ term is that the quantum mechanical equations are not invariant under the reparametrisation of time. The DK propagator for different LTSF function $f(q)$ will give the same results if we subtract a $\Delta V$ term from the potential while writing the pseudo Hamiltonian. This would not be a desirable feature for theories which have reparametrisation invariance built in the classical theory and the symmetry needs to be preserved in the quantum theory too.

In a third approach to interpretation, one can view the quantum theory as fundamental and interactions being linked in some way to choice of physical time, or to a particular frame of reference. In this case one would still need to discover a manifestly covariant form of equations in quantum theory.[21]

**A gauge covariant formalism for pseudo time dynamics:** There is yet another approach to setting up equations in pseudo time. The states and operators representing dynamical variables transform according to the rule in (33) and (34). In coordinate representation this is like a nonunitary, but invertible, gauge transformation

$$\psi(x) \rightarrow \psi_f(x) = \exp(\Omega(x))\psi(x),$$

where $\Omega$ is a real function of $x$.

Recall that we are free to choose the overall normalisation of the wave function in any fashion we like, $\psi(x)$ and $N\psi(x)$ give rise to same physics. The transformation (38) amounts to gauging this freedom in choice of normalisation constant. This, in
fact, is close to the original suggestion of Hermann Weyl and was used by him in a different context and with a different motivation.\[22\]

Thus one can think of introducing a gauge field required to maintain the new gauge invariance. Such a 'potential field' can always be introduced as a mathematical construction to do book keeping and the potential having different values in different gauges (pseudo times). Whether this 'field' can be assigned dynamical properties can only decided by further investigation and confirmation of its existence by experiments.

7 Concluding remarks

It is obvious that SUSYQM has intimate connection with reparametrisation of time in quantum mechanics. The tower of SUSY partners correspond to choices LTSF $f^n(x)$ for different values of $n$. It has been shown in this paper that this connection becomes transparent when one uses the Hamiltonian path integral formalism of quantum mechanics. Since the path integral approach is not restricted to one dimension, it offers a possibility of formulating SUSYQM in higher dimensions. In \[20\] the concept of shape invariance has been extended to quantum mechanics in arbitrary dimensions. However, a generalisation of a few other results such as intertwining of partner potentials, is needed in order to carry forward work done in SUSYQM one dimension. Darboux transformation is powerful tool and has a wide ranging applications in different areas of mathematical physics and exactly solvable models. It will be useful to further study connection of Darboux transformation with RHPI.

An area where results of our work on reparamterisation will be useful is to the systems where the action has singularities \[21\] and standard path integral needs to be regularised. Working in pseudotime, with a suitable choice of LTSF, can be used
to regularize singularities. Use of path integral to quantise and shifting to operator formalism should make it technically easier as compared to using the path integral formalism alone.

One of the important approaches to reparametrisation is using the homogeneous formalism and applying Dirac quantisation. The Dirac canonical quantisation fixes commutators, leaving questions related to ordering open. The Hamiltonian path integral quantisation approach followed here is a powerful framework and goes beyond canonical quantisation. It determines an operator ordering too. Besides fixing operator ordering, the formalism proposed here gives a way of relating results of pseudotime quantum mechanics with those in time $t$; an explicit formula, DK formula, has been written down to relate the time evolutions in time $t$ and pseudotime. Though the ordering scheme depends on the way Hamiltonian path integral is set up, the scheme of [7, 8] appears to be useful for establishing connections with SUSYQM and other areas in the literature. Further investigation of RHPI and its correspondence with operator formalism appears to be promising for a study of several areas of exactly solvable systems and systems with reparametrisation invariance.

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