Relations between asymptotic and Fredholm representations

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Abstract

We prove that for matrix algebras $M_n$ there exists a monomorphism

$$(\prod_n M_n/ \oplus_n M_n) \otimes C(S^1) \to \mathcal{Q}$$

into the Calkin algebra which induces an isomorphism of the $K_1$-groups. As a consequence we show that every vector bundle over a classifying space $B\pi$ which can be obtained from an asymptotic representation of a discrete group $\pi$ can be obtained also from a representation of the group $\pi \times \mathbb{Z}$ into the Calkin algebra. We give also a generalization of the notion of Fredholm representation and show that asymptotic representations can be viewed as asymptotic Fredholm representations.

1 Asymptotic representations as representations into the Calkin algebra

Let $\pi$ be a discrete finitely presented group, and let $F \subset \pi$ be a finite subset. Denote by $U(n)$ the unitary group of dimension $n$ and fix a number $\varepsilon > 0$.

Definition 1.1 A map $\sigma : \pi \to U(n)$ is called an $\varepsilon$-almost representation with respect to $F$ if $\sigma(g^{-1}) = \sigma(g)^{-1}$ holds for all $g \in \pi$ and if

$$\|\sigma\|_F = \sup\{\|\sigma(gh) - \sigma(g)\sigma(h)\| : g, h, gh \in F\} \leq \varepsilon.$$

Let $\{n_k\}$ be a strictly increasing sequence of positive integers and let $\sigma = \{\sigma_k : \pi \to U(n_k)\}$ be a sequence of $\varepsilon_k$-almost representations. We assume that the groups $U(n_k)$ are embedded into the groups $U(n_{k+1})$ in the standard way, so it makes possible to compare almost representations for different $k$. Then we can consider the maps $\sigma_k \oplus 1 : \pi \to U(n_k) \oplus U(n_{k+1} - n_k) \to U(n_{k+1})$, which we also denote by $\sigma_k$.

Definition 1.2 A sequence of $\varepsilon_k$-almost representations is called an asymptotic representation of the group $\pi$ (with respect to the finite subset $F$ and a sequence $\{n_k\}$) if the sequences $\varepsilon_k$ and $\|\sigma_k(g) - \sigma_{k+1}(g)\| : g \in F \subset \pi$ tend to zero.

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It was shown in [7] that this definition is equivalent to the definition given in [3, 4]. It does not depend on the choice of the finite subset $F$ when $F$ is big enough and sets of generators and relations of the group $\pi$ are finite.

Two asymptotic representations $\sigma_0$ and $\sigma_1$ are called homotopic if there exists a family of asymptotic representations $\sigma_t = \{\sigma_{t,k}\}$ such that the functions $\sigma_{t,k}(g)$ are continuous for all $g \in \pi$ and $\lim_{k \to \infty} \max_t \|\sigma_{t,k}\|_F = 0$. It can be easily seen that every asymptotic representation is homotopic to some asymptotic representation corresponding to a given sequence $\{n_k\}$.

It is well known that the asymptotic representations are exact representations in some more compound $C^*$-algebras [4, 5]. Remember this construction. Let $M_n$ be the $n \times n$ matrix algebra. Consider the $C^*$-algebra $B = \prod_{k=1}^{\infty} M_{n_k}$ of norm-bounded sequences of matrices. We suppose that the sequence $n_k$ is strictly increasing. Denote by $B^+$ the $C^*$-algebra $B$ with adjoined unit. Both algebras $B$ and $B^+$ contain a $C^*$-ideal $I = \oplus_{k=1}^{\infty} M_{n_k}$ of sequences of matrices with norms tending to zero.

Denote the corresponding quotient algebras by $Q = B/I$ and $Q^+ = B^+/I$. Let $\bar{\alpha} : B^+ \to B^+$ be the right shift, $\bar{\alpha}(m_1, m_2, \ldots) = (0, m_1, m_2, \ldots)$, $(m_i) \in B^+$. As $\bar{\alpha}(I) \subset I$, so the homomorphism $\bar{\alpha}$ induces the homomorphism $\alpha : Q^+ \to Q^+$. Let

$$Q^+_\alpha = \{q \in Q^+ : \alpha(q) = q\} \subset Q^+$$

be the $\alpha$-invariant $C^*$-subalgebra. The adjoined unit gives the splittable short exact sequence

$$Q_\alpha \to Q^+_\alpha \to C.$$  \hspace{1cm} (1.1)

Let $e = (e_k) \in B$ be the sequence of diagonal matrices having unities at the first place and zeroes at the other places, $e_k \in M_{n_k}$.

**Lemma 1.3** The group $K_0(Q^+_\alpha)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ with generators $[e]$ and $[1]$; $K_0(Q_\alpha) \cong \mathbb{Z}$ with generator $[e]$; $K_1(Q_\alpha) = K_1(Q^+_\alpha) = 0$.

**Proof.** Let $p \in M_r(Q_\alpha)$ be a projection and let $p' \in M_r(B)$ be a projection which is the lift of $p$. Then $p' = (p_k)$ is a sequence of projections. This sequence is $\alpha$-invariant, hence the rank of $p_k$ is constant beginning from some $k$. Therefore the projection $p$ is a multiple of the projection $e$ for big enough $k$. The case of the group $K_1(Q_\alpha)$ can be considered by the similar way. \hspace{1cm} ●

Denote by $Q$ the Calkin algebra.

**Theorem 1.4** There exists a monomorphism

$$\psi : Q^+_\alpha \otimes C(S^1) \to Q$$  \hspace{1cm} (1.2)

such that the induced homomorphism

$$\psi_* : K_1(Q_\alpha \otimes C(S^1)) \to K_1(Q)$$  \hspace{1cm} (1.3)

is an isomorphism.
Proof. We start with defining the homomorphism (1.2). Let $V_k$ be a $n_k$-dimensional Hilbert space where the algebra $M_{n_k}$ acts, with a fixed basis $\{e^{(i)}_k\}$, $i \leq n_k$. Fix the embeddings

$$j_k : V_{k-1} \rightarrow V_k,$$

mapping the vectors $e^{(i)}_{k-1}$ into $e^{(i)}_k$. Denote by $W_{k-1}$ the orthogonal complements,

$$V_k = j_k(V_{k-1}) \oplus W_{k-1}.$$

For $k = 1$ let $W_0 = V_1$. Put

$$H_k = V_k \oplus (\oplus_m W_{k,m}); \quad W_{k,m} \cong W_k, \quad m \in \mathbb{N}; \quad H = \oplus H_k$$

and define a homomorphism $\psi$ of the $C^*$-algebra $Q^+_\alpha$ into the Calkin algebra $Q$ of the Hilbert space $H$. Let $q \in Q^+_\alpha$ and let $q' = (q_k) \in B^+$ be a lift of $q$. Denote by $(q_k)_{ij}$ the matrix elements of the matrix $q_k$. By definition the limits $\lim_{i \to \infty} (q_k)_{ii}$ exist and are equal to each other for all $k$. Denote the value of these limits by $\lambda(q)$. Define $\psi(q)$ as an operator on $H$ which acts on the spaces $V_k$ by multiplication by the matrices $q_k$, and on the spaces $W_{k,m}$ by multiplication by $\lambda(q)$. Such operator is defined up to compact operators, hence it gives an element $\psi(q) \in Q$. Consider now the $C^*$-algebra $Q^+_\alpha \otimes C(S^1)$ of continuous $Q^+_\alpha$-valued functions over a circle. The construction described above allows us to define a homomorphism $Q^+_\alpha \otimes C(S^1) \rightarrow Q$. Put $\psi(q \otimes 1_{C(S^1)}) = \psi(q)$. It remains now to define the image of the function $u = e^{2\pi it}$ from $C(S^1)$. Define a Fredholm operator $F$ (with zero index) on the Hilbert space $H$ by the following conditions:

i) $F$ maps every subspace $V_k$ (except $V_1$) onto the subspace $V_{k-1} \oplus W_{k-1,1}$ so that $F(e^{(i)}_k) = e^{(i)}_{k-1}$ for $i < n_{k-1}$;

ii) $F$ isomorphically maps the subspace $V_1$ onto $W_{0,1}$;

iii) $F$ isomorphically maps every subspace $W_{k,m}$ onto the subspace $W_{k,m+1}$.

Thus defined operator $F$ commutes with the image $\psi(Q^+_\alpha)$ modulo compacts, hence we get the needed homomorphism $\psi$ if we put $\psi(1_{Q^+_\alpha} \otimes e^{2\pi it}) = F$.

Denote the basis vectors of the subspaces $W_{k,m}$ by $w_{k,m}^{(j)}$ with $n_k - 1 + 1 \leq j \leq n_k$. Then we can represent the action of the operator $F$ by the diagram

$$\begin{array}{cccccccccccc}
... & \leftarrow & w_{1,2}^{(1)} & \leftarrow & w_{1,1}^{(1)} & \leftarrow & e_1^{(1)} & \leftarrow & e_2^{(1)} & \leftarrow & e_3^{(1)} & \leftarrow & ... \\
... & \leftarrow & w_{1,2}^{(n_1)} & \leftarrow & w_{1,1}^{(n_1)} & \leftarrow & e_1^{(n_1)} & \leftarrow & e_2^{(n_1)} & \leftarrow & e_3^{(n_1)} & \leftarrow & ...
\end{array}$$
It remains to show that the homomorphism (1.3) is an isomorphism. As $K_1(Q_\alpha^+) = 0$, so by the Künneth formula we have

$$K_1(Q_\alpha^+ \otimes C(S^1)) \cong K_0(Q_\alpha^+) \otimes K^1(S^1).$$

Let $[u] \in K^1(S^1)$ be a generator. Then the group $K_1(Q_\alpha^+ \otimes C(S^1))$ is generated by two elements: $[e] \otimes [u]$ and $[1] \otimes [u]$. As

$$\psi(1 \otimes u) = F \quad \text{and} \quad \text{ind } F = 0,$$

so we obtain $\psi_*([1] \otimes [u]) = 0$. Compute $\psi_*([e] \otimes [u])$. By [3] one has

$$[e] \otimes [u] = [(1 - e) \otimes 1 + e \otimes u],$$

hence

$$\psi_*([e] \otimes [u]) = \psi_*([e] \otimes [u]) = \psi_*([(1 - e) \otimes 1 + e \otimes u]) = [1 - \psi(e \otimes 1) + \psi(e \otimes u)] = [1 - \psi(e \otimes 1) + \psi(e \otimes 1)F].$$

The action of the operator

$$F' = 1 - \psi(e \otimes 1) + \psi(e \otimes 1)F$$

is given by the formula

$$F'(e_k^{(j)}) = \begin{cases} e_k^{(j)} & \text{for } j > 1, \\ e_{k-1}^{(j)} & \text{for } j = 1, k > 1, \\ 0 & \text{for } j = k = 1; \end{cases} \quad F'(w_{k,m}^{(j)}) = w_{k,m}^{(j)},$$

so the image of $F'$ coincides with the whole space $H$, and the kernel of $F'$ is one-dimensional and is generated by the vector $e_1^{(1)}$. Therefore $\text{ind } F' = 1$, hence the homomorphism $\psi_*$ maps the generator of the group $K_1(Q_\alpha \otimes C(S^1)) \cong \mathbb{Z}$ into the generator of the group $K_1(Q)$. 

Denote by $\mathcal{R}_a(\pi)$ the Grothendieck group of virtual asymptotic representations of the group $\pi$. Let $\mathcal{R}_a(\pi)$ denote the kernel of the map

$$\mathcal{R}_a(\pi) \rightarrow \mathcal{R}_a(e) \cong \mathbb{Z},$$

defined by the trivial representation (here $e$ is the trivial group). Let $B\pi$ be the classifying space for the group $\pi$. Remember that a construction was defined in [4], which allows to obtain a vector bundle over $B\pi$ starting from an asymptotic representation. This construction gives a homomorphism

$$\phi : \mathcal{R}_a(\pi) \rightarrow K^0(B\pi)$$

which can be described as follows. Let $C^*[\pi]$ be the group $C^*$-algebra of the group $\pi$ and let $\xi \in K^0_{C^*[\pi]}(B\pi)$ be the universal bundle. An asymptotic representation $\sigma$ defines a homomorphism

$$\overline{\sigma} : C^*[\pi] \rightarrow Q_\alpha^+,$$
which maps the universal bundle $\xi$ into some element $\sigma(\xi) \in K^0_{\mathbb{Q}^*}(B\pi)$. This homomorphism defines the homomorphism

$$\phi' : \mathcal{R}_a(\pi) \longrightarrow K^0_{\mathbb{Q}^*}(B\pi).$$

It follows from the lemma 1.3, from the Künneth formula and from (1.1) that the lower line of the diagram

$$\begin{array}{cccc}
\widetilde{\mathcal{R}}_a(\pi) & \longrightarrow & \mathcal{R}_a(\pi) & \longrightarrow \\
\downarrow \phi & \downarrow \phi' & \downarrow \phi' & \\
K^0_{\mathbb{Q}^*}(B\pi) & \longrightarrow & K^0_{\mathbb{Q}^*}(B\pi) & \longrightarrow \\
\end{array}$$

is exact, therefore the left vertical arrow is well-defined. As we have

$$K^0_{\mathbb{Q}^*}(B\pi) = K^0(B\pi) \otimes K_0(Q) \cong K^0(B\pi),$$

so we can define $\phi$ to be this left vertical arrow after identifying $K^0_{\mathbb{Q}^*}(B\pi)$ with $K^0(B\pi)$. Notice that the image of the homomorphism $\phi'_e$ coincides with the subgroup in $K^0(B\pi)$ generated by the trivial representations.

As there exists a natural isomorphism

$$j : K^0_*(B\pi \times S^1 \times S^1) \rightarrow K^0_{\mathbb{Q}^*}(B\pi \times S^1),$$

for any $C^*$-algebra $A$, so multiplication by the Bott generator $\beta \in K^0_*(S^1 \times S^1)$ defines an inclusion

$$\overline{\beta} : K^0_*(B\pi) \overset{\otimes \beta}{\longrightarrow} K^0_*(B\pi \times S^1 \times S^1) \overset{j}{\longrightarrow} K^0_{\mathbb{Q}^*}(B\pi \times S^1) \cong K^0_{\mathbb{Q}^*}(B(\pi \times \mathbb{Z})).$$

In the case $A = Q_\pi$ we will write $\overline{\beta}$ instead of $\overline{\beta}_{Q_\pi}$.

Now denote by $\mathcal{R}_Q(\pi)$ the group of (virtual) representations of the group $\pi$ into the Calkin algebra. It is easily seen that the homomorphism (1.4) allows us to define a homomorphism

$$\overline{\psi} : \mathcal{R}_a(\pi) \longrightarrow \mathcal{R}_Q(\pi \times \mathbb{Z})$$

given by the formula $\overline{\psi}(\overline{\sigma}) = \psi(\overline{\sigma} \otimes id)$ for $\overline{\sigma} \in \mathcal{R}_a(\pi)$ and

$$\overline{\sigma} \otimes id : C^*[\pi \times \mathbb{Z}] \cong C^*[\pi] \otimes C(S^1) \longrightarrow Q^*_\alpha \otimes C(S^1).$$

Let $\eta \in K^0_{C^*[\pi \times \mathbb{Z}]}(B(\pi \times \mathbb{Z}))$ be the universal bundle over $B(\pi \times \mathbb{Z})$. Then there exists also a homomorphism

$$f : \mathcal{R}_Q(\pi \times \mathbb{Z}) \longrightarrow K^0_*(B(\pi \times \mathbb{Z}))$$

defined as the image of $\eta$ over $B(\pi \times \mathbb{Z})$ under the representations into the Calkin algebra.

All these homomorphisms (1.3) – (1.6) can be represented by the diagram

$$\begin{array}{cccc}
\widetilde{\mathcal{R}}_a(\pi) & \phi & \longrightarrow & K^0_{\mathbb{Q}^*}(B\pi) \\
\downarrow & & \downarrow & \psi _* \\
\mathcal{R}_a(\pi) & \overline{\psi} & \longrightarrow & \mathcal{R}_Q(\pi \times \mathbb{Z}) \\
\end{array}$$

\begin{array}{cccc}
& f & \longrightarrow & K^0_*(B(\pi \times \mathbb{Z})) \\
\end{array}$$

where the left vertical arrow is the inclusion.
Theorem 1.5 The diagram \((1.8)\) is commutative.

Proof. Direct calculation. 

So now we have the homomorphism
\[
\bar{\varphi} : \widetilde{R}_a(\pi) \longrightarrow K^0_Q(B\pi \times S^1) \cong K_1(B\pi \times S^1).
\] (1.9)

Theorem 1.5 shows that every element of the group \(K^0(B\pi)\) which can be obtained by an asymptotic representation of the fundamental group \(\pi\) can be obtained also by a representation of the group \(\pi \times \mathbb{Z}\) into the Calkin algebra.

2 Asymptotic representations as Fredholm representations

Notice that the group \(K^0_Q(B\pi \times S^1)\) can be decomposed into direct sum:
\[
K^0_Q(B\pi \times S^1) = K^0_Q(B\pi) \oplus K^0_Q(B\pi \wedge S^1) \cong K_1(B\pi) \oplus K^0(B\pi)
\] (2.1)

induced by an inclusion map \(i : s_0 \rightarrow S^1\), where \(s_0 \in S^1\), and the image of the homomorphism \(\bar{\varphi} (1.9)\) lies only in the second summand of (2.1). Indeed, consider the composition of the map \(\varphi\) with the map \(i^* : K^0_Q(B\pi \times S^1) \longrightarrow K^0_Q(B\pi)\). But as the multiplication by the Bott generator is involved in the map \(\bar{\varphi}\), so its composition with \(i^*\) gives the zero map. Therefore the image of the group \(\widetilde{R}_a(\pi)\) lies in \(K^0(B\pi)\) and hence defines a (virtual) vector bundle over \(B\pi\).

On the other hand the image of the map \((1.7)\) need not be contained in the second summand of (2.1). It would be so if the representation of the group \(\pi \times \mathbb{Z}\) into the Calkin algebra would be a part of a Fredholm representation of the group \(\pi\). Using the notion of asymptotic representations we can now give a generalization of the Fredholm representations which would also ensure that the image of such representations would lie in the second summand of (2.1).

Let \(\rho : \pi \times \mathbb{Z} \longrightarrow \mathcal{Q}\) be a representation into the Calkin algebra and let \(F \subset \pi\) be a finite subset of \(\pi\) containing its generators. Denote by \(B(H)\) the algebra of bounded operators on a separable Hilbert space \(H\). Let \(q : B(H) \longrightarrow \mathcal{Q}\) be the canonical projection.

Definition 2.1 We call a map \(\tau : \pi \longrightarrow B(H)\) an \(\varepsilon\)-trivialization for \(\rho\) if
\[
i) \ ||\tau(gh) - \tau(g)\tau(h)|| \leq \varepsilon \text{ for any } g, h \in F \subset \pi,
\]
\[
ii) q(\tau(g)) = \rho(g; 0) \text{ for any } g \in \pi, \ (g; 0) \in \pi \times \mathbb{Z}.
\]

Definition 2.2 Suppose that for every \(\varepsilon > 0\) there exists an \(\varepsilon\)-trivialization \(\tau_\varepsilon\) for \(\rho\). Then the pair \((\tau_\varepsilon, \rho)\) is called an asymptotic Fredholm representation.
Let $u$ be a generator of the group $\mathbb{Z}$. Notice that the image of the group $\pi$ under $\varepsilon$-trivializations commutes with some Fredholm operator $F = \rho(0, u)$ modulo compacts. Denote the group of all asymptotic Fredholm representations by $\mathcal{R}_{aF}(\pi) \subset \mathcal{R}_Q(\pi \times \mathbb{Z})$.

**Proposition 2.3** The image of $\mathcal{R}_{aF}$ under the map $f$ [4] lies in the group $K^0(B\pi \wedge S^1)$.

**Proof.** It was described in the paper [7] how to construct a bundle over $B\pi$ with the fibers isomorphic to the Hilbert space $H$ and with the structural group $GL(H)$ starting from an almost representation $\tau_\varepsilon$ for small enough $\varepsilon$. To do so one should construct transition functions acting on the fibers $\xi_x$,

$$T_\varepsilon(x) : \xi_x \rightarrow \xi_{gx}$$

for $g \in \pi$, $x \in E\pi$. One should chose representatives $\{a_\alpha\}$ in each orbit of the set of vertices of $E\pi$ and define $T_\varepsilon(a_\alpha) = \tau_\varepsilon(g)$. Take now an arbitrary vertex $b \in E\pi$. Then there exists such $h \in \pi$ that $b = h(a_\alpha)$ and we should put $T_\varepsilon(b) = \tau_\varepsilon(gh)\tau_\varepsilon^{-1}(h)$. Further these transition functions should be extended by linearity to all simplices of $E\pi$. But obviously $q(T_\varepsilon(b)) = q(\tau_\varepsilon(gh)\tau_\varepsilon^{-1}(h)) = q(\tau_\varepsilon(g))$, hence after we pass to quotients, the transition functions $q(T_\varepsilon(x))$ would become constant and the bundle with the structural group being the invertibles of the Calkin algebra. But as any bundle with fibers $H$ is trivial, so the quotient bundle with fibers isomorphic to the Calkin algebra is trivial too and the projection of $\varphi(\mathcal{R}_{aF})$ onto the first summand of $K^0_\mathbb{Q}(B\pi)$ is equal to zero. 

**Remark.** In the section [4] we have seen that an asymptotic representation $\sigma = (\sigma_\alpha)$ defines a homomorphism $\rho : \pi \times \mathbb{Z} \rightarrow GL(\mathbb{Q})$ into the group of invertibles of the Calkin algebra. But the same asymptotic representation gives $\varepsilon$-trivializations of $\rho$ for any $\varepsilon > 0$. Indeed we can put

$$\tau_{\varepsilon n}(g) = (1, 1, \ldots, 1, \sigma_n(g), \sigma_{n+1}(g), \ldots).$$

So asymptotic representations define asymptotic Fredholm representations and we have an inclusion $\mathcal{R}_a(\pi) \subset \mathcal{R}_{aF}(\pi)$.

### 3 Asymptotic representations and extensions

It was shown in [4] that using a quasi central approximate unity one can construct an asymptotic representation out of an extension of $C^*$-algebras. We study how this construction is related to asymptotic representations of the initial $C^*$-algebra.

Let $A$ be a $C^*$-algebra such that there exists a homomorphism $q_A : A \rightarrow \mathbb{C}$ of $A$ into the complex numbers and let $F \subset A$ be a finite set of generators for $A$. We can repeat our definitions of asymptotic representations from the section [4] in the case of $C^*$-algebras instead of discrete groups. For simplicity sake we assume that dimension of an almost representation $\sigma_n$ equals to $n$.

We consider a sequence of maps $c_n : A \rightarrow M_n$ where $M_n$ acts on a finite-dimensional Hilbert space $V_n$. Take an infinite-dimensional Hilbert space $H_n \supset V_n$ and define a map $\sigma_n : A \rightarrow B(H)$ by $\sigma_n(a) = c_n(a) \oplus q_A(a)$ where $a \in A$ and $q_A(a)$ is a scalar on $V_n$. The
map $\sigma = \oplus_n \sigma_n$ gives an asymptotic representation of the $C^*$-algebra $A$ if (after identifying all $H_n$) the norms $\|\sigma_{n+1}(a) - \sigma_n(a)\|$ tend to zero for $a \in F$.

Consider the $C^*$-algebra $\mathcal{E}$ in the Hilbert space $\oplus_n H_n$ generated by $\sigma(a), a \in A$ and by the translation operator $F$ defined in the proof of the theorem [1.4]. Then one has a short exact sequence

$$\oplus_n M_n \rightarrow \mathcal{E} \rightarrow A \otimes C(S^1). \tag{3.1}$$

We consider a discrete version of the construction of [4]. If $e_n \in \oplus_n M_n$ is a quasi central approximate unity [1] then the exact sequence (3.1) defines an asymptotic representation given by the formula

$$\rho_n : A \otimes C(S^1) \otimes C_0(S^1) \xrightarrow{\text{as}} \oplus_n M_n \tag{3.2}$$

given by the formula

$$\rho_n(a \otimes g \otimes f) = (a \otimes g)' \cdot f(e_n),$$

where $(a \otimes g)' \in \mathcal{E}$ is a lift for $a \otimes g$. Denote by $K_n \subset \oplus_n H_n$ the subspace on which one has $e_n \neq 0, e_n \neq 1$. If $K_n$ is finite-dimensional, $k(n) = \dim K_n$, then we can get a sequence of finite-dimensional almost representations

$$\overline{\rho}_n : A \otimes C(S^1) \otimes C_0(S^1) \xrightarrow{\text{as}} \oplus_n M_{k(n)}$$

by the formula

$$\overline{\rho}_n(a \otimes g \otimes f) = P_n(a \otimes g)'P_n \cdot f(e_n),$$

where $P_n$ is the projection onto $K_n$, $g \in C(S^1), f \in C_0(S^1), f(0) = 0$.

There exists also a well-known asymptotic representation $\beta = (\beta_m)$ of the commutative $C^*$-algebra $C(S^1 \times S^1)$ into the matrix algebra $M_n$ given by

$$\beta_m(e^{2\pi i x}e^{2\pi i y}) = T_m U_m^t, \quad \beta_m : C(S^1) \otimes C(S^1) \xrightarrow{\text{as}} M_m,$$

where $T_m$ and $U_m$ are the $m$-dimensional Voiculescu matrices, $T_m$ is a translation and $U_m$ is a diagonal matrix [8]. This asymptotic representation realizes the Bott isomorphism.

The tensor product of $\overline{\sigma}$ by $\beta$ gives an asymptotic representation

$$\overline{\sigma} \otimes \beta_n : A \otimes C(S^1) \otimes C_0(S^1) \xrightarrow{\text{as}} M_{n \times m}.$$

Put $\hat{\sigma} = P_n \sigma_n P_n$.

**Theorem 3.1** There exists a quasi central approximate unity $e_n$ for the exact sequence (3.1) and a sequence of numbers $m(n)$ such that the asymptotic representations $\rho_n$ (3.2) and $\overline{\sigma}_n \otimes \beta_{m(n)}$ into the matrix algebra $M_{(n+m(n)) \times m(n)}$ are equivalent. Moreover for $a \in F \subset A$, $e^{2\pi i x} \in C(S^1)$ and $f \in C_0(S^1)$ the norms

$$\|\rho_n(a \otimes e^{2\pi i x} \otimes f) - \overline{\sigma}_n(a) \otimes \beta_{m(n)}(e^{2\pi i x} \otimes f)\|$$

tend to zero.

**Proof.** Define a sequence of integers $m(n)$ so that the following conditions would be satisfied:
i) $\max_{n \leq k \leq n+m(n)} \|\sigma_n(a) - \sigma_{n+k}(a)\|$ tend to zero for any $a \in F \subset A$,

ii) $m(n)$ tends to infinity.

As we now assume that dim $V_n = n$, so dim $W_{k,l} = 1$. We denote the basis vectors of $V_n$ by $v_n^{(j)}$, $j = 1, \ldots, n$ and the vectors of the subspaces $W_{k,l}$ by $v_{k,l}^{(j)}$. In these denotations the shift operator $F$ constructed in the section [I] acts by the simple formula $Fv_n^{(j)} = v_{n-1}^{(j)}$. Put

$$a_{n,i}^{(j)} = \begin{cases} 1, & \text{for } -n \leq i \leq n, \\ \frac{m(n)-i+n}{m(n)}, & \text{for } n < i \leq n + m(n) \text{ and } j \leq n + m(n), \\ 0, & \text{otherwise} \end{cases}$$

and define the diagonal quasi central approximate unity $e_n$ by the formula

$$e_n v_i^{(j)} = a_{n,i}^{(j)} v_i^{(j)}.$$ 

Notice that $e_n$ exactly commutes with $\sigma_n(a)$ (which are diagonal too) and for any $f \in C_0(S^1)$, $f(0) = 0$, $f(e_n)$ almost commutes with $F$.

Obviously the norms

$$\|\rho_n(a \otimes e^{2\pi i x} \otimes f) - \mathcal{P}_n(a \otimes e^{2\pi i x} \otimes f)\|$$

tend to zero, so we may deal with finite-dimensional asymptotic representations $\mathcal{P}_n$ instead of $\rho_n$. Direct calculations show that

$$\mathcal{P}_n(a \otimes e^{2\pi i x} \otimes f) = P_n a' F \mathcal{P}_n f(e_n) = P_n (\oplus_{i=n+1}^{m(n)} \mathcal{P}_n \sigma_i(a)) P_n F \mathcal{P}_n f(e_n) = \oplus_{i=n+1}^{m(n)} \tilde{\sigma}_i(a) F \mathcal{P}_n f(e_n),$$

where $a' = \oplus_{i=1}^{\infty} \mathcal{P}_i(a) \in \mathcal{E}$ (resp. $F$) is a lift for $a$ (resp. for $e^{2\pi i x}$), and $f(e_n)$ is the diagonal matrix with elements $\frac{m(n)-i+n}{m(n)}$ on the diagonal. It is easily seen that the norms

$$\|P_n F \mathcal{P}_n f(e_n) - T_{m(n)} f(e_n)\|$$

tend to zero and that

$$T_{m(n)} f(e_n) = \beta_{m(n)}(e^{2\pi i x} \otimes f).$$

So it remains to notice that by our choice of the sequence $m(n)$ the norms

$$\|((\oplus_{i=n+1}^{m(n)} \tilde{\mathcal{P}}_i(a) - (\oplus_{i=n+1}^{m(n)} \tilde{\mathcal{P}}_n(a)\|$$

tend to zero too.

**Remark.** Unfortunately in the case of discrete group $C^*$-algebras we do not know if it is possible to construct a natural map from the group $\mathcal{R}_a(C^*(\pi) \otimes C(S^1) \otimes C_0(S^1))$ into $K^0(B\pi)$ which would extend the map (1.4) and close the diagram

$$\begin{array}{ccc}
\tilde{\mathcal{R}}_a(\pi) & \xrightarrow{\delta} & \mathcal{R}_a(C^*[\pi] \otimes C(S^1) \otimes C_0(S^1)) \\
\lor & & \lor \\
& K^0(B\pi). & \end{array}$$
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