Topological Kac-Moody Algebra and Wakimoto Representation

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Abstract

It is shown, using the Wakimoto representation, that the level zero SU(2) Kac-Moody conformal field theory is topological and can be obtained by twisting an $N=2$ superconformal theory. Expressions for the associated $N=2$ superconformal generators are written down and the Kac-Moody generators are shown to be BRST exact.

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Recently there has been a great deal of interest in topological field theories [1]-[7]. These theories are characterised by the existence of a BRST charge such that the energy momentum tensor of the theory is BRST exact [1]. Two dimensional version of the theories, e.g., the topological conformal field theories (TCFT’s) have also been studied extensively [3][4][6] because of their connection to two dimensional quantum gravity and matrix models [5]. Supersymmetric as well as higher genus generalization of TCFT’s have also been reported [7]-[11].

In an illuminating paper, the topological nature of a class of $c = 0$ conformal field theories (CFT’s) was shown by Eguchi and Yang [3]. This result was proved using the free boson, viz., Feigin-Fuchs realization of the CFT’s. Topological nature of $c = 0$ theories have also been shown for $W_N$ theories [12] as well as for a class of coset models [13].

In this note, we extend the results of [3] to level zero $SU(2)$ Kac-Moody Algebra. We show, using the Wakimoto realization of $SU(2)$ Kac-Moody and associated conformal algebra, that this theory is topological. We now start by reviewing the essential features of [3]. In Feigin-Fuchs construction, the expression for the energy momentum tensor $T(z)$ is

$$T(z) = -\frac{1}{2}(\partial\phi(z))^2 + i\alpha_0\partial^2\phi(z)$$

where $\phi(z)$ is a free boson with the two point function:

$$\langle \phi(z)\phi(\omega) \rangle = -\log(z - \omega).$$
The central charge of the theory, \( c = 1 - 12\alpha_0^2 \), vanishes for \( \alpha_0 = \frac{1}{2\sqrt{3}} \). The screening charge of the Feigin-Fuchs construction, \( Q = \oint G(z)dz = \oint e^{i\sqrt{3}\phi(z)}dz \), satisfies the nilpotency condition \( Q^2 = 0 \). Moreover, using the operator product expansions (OPE’s) it can be shown that for \( \alpha_0 = \frac{1}{2\sqrt{3}} \)

\[
\{Q, \bar{G}(z)\} = \mathcal{T}(z) \tag{3}
\]

where

\[
\bar{G}(z) = e^{-i\sqrt{3}\phi(z)}. \tag{4}
\]

Therefore if \( Q \) is identified as the BRST charge, the energy-momentum tensor is BRST exact, and the theory is topological. This topological field theory is in fact a twisted \( N = 2 \) superconformal theory. To see this, we write the energy-momentum tensor as

\[
\mathcal{T}(z) = \mathcal{T}_{N=2}(z) + \frac{1}{2}J(z) \tag{5}
\]

where

\[
\mathcal{T}_{N=2}(z) = -\frac{1}{2}(\partial\phi(z))^2 \tag{6}
\]

and

\[
J(z) = \frac{i}{\sqrt{3}} \partial\phi(z). \tag{7}
\]

It is now straightforward to show that the operators \( \mathcal{T}_{N=2}(z), J(z), G(z), \) and \( \bar{G}(z) \) satisfy the \( N = 2 \) superconformal algebra with central charge \( c = 1 \).
We now extend these results to the SU(2) Kac-Moody CFT using the
Wakimoto realization [14][15]. In this realization[15], the SU(2) genera-
tors are written as
\[ J^+(z) = \omega^+(z) \]
(8)
\[ J^0(z) = -i(\omega(z)\omega^+(z) + \frac{1}{2\alpha_0}\partial\phi(z)) \]
(9)
\[ J^- = \omega(z)\omega(z)\omega^+(z) + ik\partial\omega(z) + \frac{1}{\alpha_0}\partial\phi(z)\omega(z) \]
(10)
where \( \omega(z) \), \( \omega^+(z) \) and \( \phi(z) \) are free boson fields with the two point functions:
\[ \langle \omega(z_1)\omega^+(z_2) \rangle = -\langle \omega^+(z_1)\omega(z_2) \rangle = \frac{i}{(z_1 - z_2)} \]
(11)
\[ \langle \phi(z_1)\phi(z_2) \rangle = -\log(z_1 - z_2). \]
(12)

In this realization, the level \( k \) of the \( SU(2) \) Kac-Moody algebra [13] is
\[ k = -2 + \frac{1}{2\alpha_0^2}. \]
(13)

The Sugawara construction leads to the following expression for energy-
momentum:
\[ T(z) = -\frac{1}{2}(\partial\phi(z))^2 + i\alpha_0\partial^2\phi(z) + i\omega^+(z)\partial\omega(z). \]
(14)
The corresponding central charge is
\[ c = 3 - 12\alpha_0^2 = \frac{3k}{k+2}. \]
(15)
We now specialize to the case of level zero Kac-Moody algebras. In this case, using eqs. (13) and (15), we get $\alpha_0 = \frac{1}{2}$ and $c = 0$. Also, the eqs. (8-10) and (14) become

\[ J^+(z) = \omega^+(z) \]  \hspace{1cm} (16)
\[ J^0(z) = -i(\omega(z)\omega^+(z) + \partial\phi(z)) \]  \hspace{1cm} (17)
\[ J^-(z) = \omega(z)\omega(z)\omega^+(z) + 2\partial\phi(z)\omega(z) \]  \hspace{1cm} (18)

and

\[ T(z) = -\frac{1}{2}(\partial\phi(z))^2 + \frac{i}{2}\partial^2\phi(z) + i\omega^+(z)\partial\omega(z) \]  \hspace{1cm} (19)

respectively. Here we would like to remark that although the algebra satisfied by the generators (16)-(19) has level $k = 0$, but it is not a classical algebra. This is because OPE's of the generators involve multi-contractions.

Now we show that the Kac-Moody CFT represented by eqs. (16-19) is topological and is a twisted version of an $N = 2$ superconformal theory. To illustrate the $N = 2$ structure we rewrite

\[ T(z) = T_{N=2}(z) + \frac{1}{2}J^{U(1)}(z) \]  \hspace{1cm} (20)

where

\[ J^{U(1)}(z) = i\partial\phi(z) \]  \hspace{1cm} (21)

and

\[ T_{N=2}(z) = -\frac{1}{2}(\partial\phi(z))^2 + i\omega^+(z)\partial\omega(z). \]  \hspace{1cm} (22)

The supercharges of the $N = 2$ theory can also be obtained from the knowledge of the operator content of the level zero Kac-Moody algebra. The
screening operator of the SU(2) Kac-Moody algebra is given by \[ \Phi_+(z) = \omega^+(z)e^{i\phi(z)} \]. We identify it with one of the supercharges:

\[ G(z) = \omega^+(z)e^{i\phi(z)}. \]  

(23)

It can be shown that the operator

\[ \tilde{G}(z) = 2i\partial\omega(z)e^{-i\phi(z)} \]  

(24)

is the other supercharge. The operators (21)-(24) satisfy an N=2 superconformal algebra with central charge \( c = 3 \).

The topological nature of the original \( k = 0 \) Kac-Moody theory can now be shown in the usual manner \[ \mathbb{B} \mathbb{E} \] . The BRST charge, defined by

\[ Q = \oint G(z)dz \]  

(25)

is nilpotent. And using the operator algebra

\[ G(z_1)\tilde{G}(z_2) = \frac{2}{(z_1 - z_2)^3} + \frac{2J(z_2)}{(z_1 - z_2)^2} + \frac{2T(z_2) + \partial J(z_2)}{(z_1 - z_2)}. \]  

(26)

we have

\[ \{Q, \tilde{G}(z)\} = T(z). \]  

(27)

Therefore \( T(z) \) is BRST exact and the theory is topological.
Now, by defining the operators,

\[ j^\pm(z) = e^{-i\phi(z)} \]  \hspace{2cm} (28)
\[ j^0(z) = -i\omega(z)e^{-i\phi(z)} \]  \hspace{2cm} (29)
\[ j^-(z) = \omega^2(z)e^{-i\phi(z)} \]  \hspace{2cm} (30)

one obtains, using the OPE’s:

\[ \{ Q, j^{\pm,0}(z) \} = J^{\pm,0}(z). \]  \hspace{2cm} (31)

Therefore \( J^{\pm,0}(z) \) are BRST exact and \( j^{\pm,0}(z) \) are their BRST partners.

The primary fields of the \( SU(2) \) Kac-Moody CFT in the Wakimoto representation are given by \[ 15 \]

\[ \Phi^j_m(z) = (\omega(z))^{j-m}e^{-2i\alpha_0\phi(z)}, \quad m = -j, \ldots, j. \]  \hspace{2cm} (32)

The conformal weight of the primary fields \( \Phi^j_m(z) \) is given by \( \Delta^j_m = \frac{j(j+1)}{k+2} \).

It is known that due to unitarity considerations \[ 10 \], \( j \) is restricted to \( 0 \leq j \leq \frac{k}{2} \) and hence in the present case only \( j = 0 \) primary state survives and has weight zero. Moreover, due to BRST exactness of \( T(z) \) and \( J^{\pm,0}(z) \), Virasoro as well as Kac-Moody secondary states are absent from the BRST cohomology.

At this point we would like to remark that in the untwisted \( N=2 \) theory with \( c = 3 \), i.e., \( \alpha_0 = 0 \), the \( SU(2) \) currents in eqs.(8)-(10) are not well defined. This is unlike the case of superconformal extensions of the TCFT.
where the extra supersymmetry generator is well defined in the untwisted theory as well. This enables one to write the full algebra of the untwisted theory in these cases as $N > 2$ extended superconformal algebras [8][9].

Finally, we comment on the relation of our work with ref. [13]. In ref. [13] the topological nature of the cosets $\frac{G_0 \otimes G_k}{G_k}$ was shown which apparently seems to contain our case. However, one notices that the BRST charge in eqs.(5), (12), and (13) of ref. [13] is not well defined for $k=0$. But, as we have seen, Wakimoto realization overcomes this difficulty.

To conclude, we have proved that the level zero Kac-Moody theory is topological. One can possibly generalize our work, using Wakimoto kind of realization for other groups [17] as well as coset constructions. Generalization to super Kac-Moody algebras using the corresponding free field realization [18] should also be possible.

NOTE ADDED: After the completion of this work we received a preprint [19] where $N = 2$ supersymmetry of $(b, c)$ and $(\beta, \gamma)$ ghost system was used to construct a topological SL(2) Kac-Moody algebra. However, we believe that for generalization to other groups Wakimoto type of realization should be more useful.
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