Irreducible Modules of Finite Dimensional Quantum Algebras of type A at Roots of Unity

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1 Introduction

The quantum group $U_q(g)$ associated with a simple Lie algebra $g$ is an associative algebra over the rational function field $\mathbb{C}(q)$ ($q$ is an indeterminate) and we can define its "integral" form over the Laurent polynomial ring $\mathbb{C}[q, q^{-1}]$, which enables us to specialize $q$ to any non-zero complex number $\varepsilon$. We are going to see two types of such integral forms and accordingly, we obtain two types of specializations, one is called the 'restricted specialization' denoted by $U^\text{res}_\varepsilon$, and the other is called the 'non-restricted specialization' denoted by $U_\varepsilon$. Both coincide if $\varepsilon$ is transcendental. But we are interested in the case that $\varepsilon$ is the $l$-th primitive root of unity, where $l$ is an odd integer greater than 1. In the case, they do not so. The former is initiated by Lusztig [4],[5] and the latter is introduced in [3] by DeConcini and Kac. Their representation theories are quite different: Irreducible $U^\text{res}_\varepsilon$-modules are highest weight modules in some sense and the classification of the irreducible modules is same as the one for simple Lie algebras or ordinary quantum algebras (see Theorem 3.5 below). Furthermore, irreducible modules possess the remarkable property "tensor product theorem" (see Theorem 3.6 below), which claims that arbitrary irreducible highest weight module $V(\lambda)$ with the highest weight $\lambda$ is divided into tensor product of two irreducible modules $V(\lambda^{(0)})$ and $V(\lambda^{(1)})$ where $\lambda^{(0)}$ and $\lambda^{(1)}$ are as in Theorem 3.6. Here the module $V(\lambda^{(0)})$ is identified with the irreducible $U^\text{fin}_\varepsilon$-module, while $U^\text{fin}_\varepsilon$ is some finite dimensional subalgebra of $U^\text{res}_\varepsilon$ (see 2.2) and the module $V(\lambda^{(1)})$ can be identified with the irreducible highest weight $U(g)$-module $V(\lambda^{(1)})$, whose structure is known very well. Thus, if the structure of $V(\lambda^{(0)})$ is clarified, we can analyze the detailed feature of $V(\lambda)$. Indeed, the character of $V(\lambda)$ is given by the famous Kazhdan-Lusztig formula. But structures as a module, e.g., explicit descriptions of basis vectors or actions of the generators on them, are not still clear.

On the other hand, irreducible $U_\varepsilon$-modules are not necessarily highest or lowest weight modules. They are characterized by many continuous parameters and if they are "generic", their dimensions are all same (see [3],[4]). But if we specialize the parameters properly, the modules become reducible. In [2], Date, Jimbo, Miki and Miwa constructed such $U_\varepsilon$-modules for $A_n$-type explicitly, which is called the 'maximal cyclic representations' that is realized in the vector space $V := (\mathbb{C}^n)^\oplus n(n+1)$. They contain the continuous parameters and it is shown that if those parameters are generic, they are irreducible. Here we
2 Algebras at roots of unity

In this section, we review the algebras treated in this article.

2.1 Restricted integral forms and specializations

Let $C(q)$ be the rational function field in an indeterminate $q$ and denote the ring $C[q, q^{-1}]$ by $A$. We use the notations:

$$[a]_q := \frac{q^a-q^{-a}}{q-q^{-1}}, \quad [a]_q! := [a]_q[a-1]_q \cdots [2]_q[1]_q,$$

$$[m]_q! := \frac{[m]_q!}{[k]_q!/\lbrack m-k\rbrack_q!}.$$

Let $I := \{1, 2, \cdots, n\}$ be the index set and $(a_{ij})_{i,j \in I}$ be the Cartan matrix of type $A$, i.e., $a_{ii} = 2$ ($1 \leq i \leq n$), $a_{i,i+1} = a_{i+1,i} = -1$ ($1 \leq i \leq n-1$), and $a_{ij} = 0$ otherwise. Let us denote the set of roots (resp. positive roots) by $\Delta$ (resp. $\Delta_+$). Let $\{h_i\}_{i \in I}$ be the set of simple coroots and $\{\alpha_i\}_{i \in I}$ the set of simple roots. Define the weight lattice $P := \{\lambda \mid \langle h_i, \lambda \rangle \in \mathbb{Z}\}$ (resp. the set of dominant integral weights $P_+ := \{\lambda \mid \langle h_i, \lambda \rangle \in \mathbb{Z}_{\geq 0}\}$). Let $\{\Lambda_i\}_{i \in I}$ be the fundamental weights which satisfy $\langle h_i, \Lambda_j \rangle = \delta_{ij}$ and then $P = \oplus_i \mathbb{Z}\Lambda_i$. Let $W$ be the Weyl group of type $A_n$, which is generated by the simple reflections $s_i$ ($i \in I$). The quantum algebra $U_q(\mathfrak{g})$ is the associative algebra generated by $e_i, f_i, t_i^\pm$ ($i \in I$) and the relations

$$t_i t_i^{-1} t_i = t_i, \quad t_i t_j = t_j t_i, \quad (i \neq j), \quad (2.1)$$

$$t_i e_j t_i^{-1} = q^{a_{ij}} e_j, \quad (i < j), \quad (2.2)$$

$$t_i f_j t_i^{-1} = q^{-a_{ij}} f_j, \quad (i < j), \quad (2.3)$$

$$e_i f_j - f_j e_i = \frac{t_i t_j t_i^{-1} - t_i^{-1} t_j^{-1}}{q-q^{-1}}, \quad (2.4)$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^{e_i(k)} e_i^{(1-a_{ij}-k)} = \sum_{k=0}^{1-a_{ij}} (-1)^{f_j(k)} f_j^{(1-a_{ij}-k)} = 0 \quad (i \neq j), \quad (2.5)$$

where $e_i^{(k)} := \frac{e_i^k}{[k]_q!}$ and $f_j^{(k)} := \frac{f_j^k}{[k]_q!}$. Here we set

$$\left[ \begin{array}{c} t_i^p q \nonumber \\
 r \nonumber \\
 q \nonumber \\
 \end{array} \right]_q := \prod_{s=1}^r t_i q^{p+1-s} - t_i^{-1} q^{s-p-1}$$

$$\frac{q^s - q^{-s}}{q^s - q^{-s}}.$$
The algebra $U_{\text{res}}$ is the $\mathcal{A}$-subalgebra of $U_q(g)$ generated by $e_{i}^{(k)}$, $f_{i}^{(k)}$, $t_{i}^{\pm}$ and $\left[t_{i}, p_{i}\right]_k$ ($i \in I$, $p, k \in \mathbb{Z}$ and $k \geq 0$), which is called the restricted integral form.

Here we can define the restricted specializations for any $\varepsilon \in \mathbb{C}^\times$:

$$U_{\varepsilon}^\text{res} := U_{\text{res}} \otimes_{\mathcal{A}} \mathbb{C}_{\varepsilon},$$

(2.6)

where $\mathcal{A}$ acts on $\mathbb{C}_{\varepsilon} := \mathbb{C}$ by $f(q)c := f(\varepsilon)c$ ($c \in \mathbb{C}$).

2.2 Finite dimensional quantum algebra

For $\varepsilon \in \mathbb{C}^\times$ we use the notation

$$[a] := \frac{q^a - q^{-a}}{q - q^{-1}}, \quad [a]! := [a][a-1] \cdots [2][1], \quad \begin{bmatrix} m \\ k \end{bmatrix}_q := \begin{bmatrix} m \\ k \end{bmatrix}_{q=\varepsilon}.$$

Since $\begin{bmatrix} m \\ k \end{bmatrix}_q \in \mathbb{C}[q, q^{-1}]$, the definition of $\begin{bmatrix} m \\ k \end{bmatrix}_q$ is valid.

As for the specializations of $q$, we shall be interested in the case that $\varepsilon$ is a root of unity. So in what follows, suppose that:

$l$ is the odd integer greater than 1 and $\varepsilon$ is the primitive $l$-th root of unity.

Under this setting, we can find an interesting finite dimensional subalgebra $U_{\varepsilon}^\text{fin}$ of $U_{\varepsilon}^\text{res}$. $U_{\varepsilon}^\text{fin}$ is defined as the subalgebra of $U_{\varepsilon}^\text{res}$ generated by $e_{i}$, $f_{i}$ and $t_{i}^{\pm}$ ($1 \leq i \leq n$). We know that this algebra is finite dimensional over $\mathbb{C}$ with the dimension $2^nn_l^2 + 2n$ (see Proposition 2.2 below).

This $U_{\varepsilon}^\text{fin}$ is also defined by "generators and relations" as follows;

**Proposition 2.1 ([1],[4])** The algebra $U_{\varepsilon}^\text{fin}$ is isomorphic to the associative $\mathbb{C}$-algebra with generators $e_{\alpha}$, $f_{\alpha}$ ($\alpha \in \Delta_+$) and $t_{i}^{\pm}$ ($1 \leq i \leq n$) satisfying the following relations;

$$t_{i}t_{i}^{-1} = t_{i}^{-1}t_{i} = 1, \quad t_{i}t_{j} = t_{j}t_{i}, \quad (2.7)$$

$$t_{i}e_{j}t_{i}^{-1} = \varepsilon^{q_{ij}}e_{j}, \quad (2.8)$$

$$t_{i}f_{j}t_{i}^{-1} = \varepsilon^{-q_{ij}}f_{j}, \quad (2.9)$$

$$e_{i}f_{j} - f_{j}e_{i} = \delta_{ij} \frac{t_{i} - t_{i}^{-1}}{\varepsilon - \varepsilon^{-1}}. \quad (2.10)$$

If $(\alpha, \alpha) = 0$ and $i < g(\alpha)$,

$$e_{i}e_{\alpha} = e_{\alpha}e_{i}, \quad (2.11)$$

$$f_{i}f_{\alpha} = f_{\alpha}f_{i}. \quad (2.12)$$

If $(\alpha, \alpha) = -1$ and $i < g(\alpha)$,

$$e_{\alpha + \alpha_{i}} = \varepsilon^{-1}e_{\alpha}e_{i} - e_{i}e_{\alpha}, \quad (2.13)$$

$$\varepsilon e_{i}e_{\alpha + \alpha_{i}} = e_{\alpha + \alpha_{i}}e_{i}, \quad (2.14)$$

$$\varepsilon e_{\alpha + \alpha_{i}}e_{\alpha} = e_{\alpha}e_{\alpha + \alpha_{i}}, \quad (2.15)$$

$$f_{\alpha + \alpha_{i}} = \varepsilon f_{\alpha}f_{i} - f_{i}f_{\alpha}. \quad (2.16)$$
where we define $g(\alpha)$ ($\alpha \in \Delta_+$) to be the largest index satisfying $c_i \neq 0$ if we write $\alpha = \sum_i c_i \alpha_i$ and set $e_i := e_{\alpha_i}$ and $f_i := f_{\alpha_i}$.

Define $(U^{\text{fin}}_{\epsilon})^+$ (resp. $(U^{\text{fin}}_{\epsilon})^-$, $(U^{\text{fin}}_{\epsilon})^0$) to be the subalgebra of $U^{\text{fin}}_{\epsilon}$ generated by $e_i$ (resp. $f_i, t_i^\pm$). Fix a reduced expression $w_0 = s_{i_1}s_{i_2}\cdots s_{i_n}$ of the longest element \( w \) of the Weyl group $W$ and set $\beta_k := s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k})$ for $k \in \{1, \cdots, N\}$ where $N := \frac{1}{2}n(n+1)$ is the number of positive roots. Here we have the following Poincaré-Birkhoff-Witt type theorem:

**Proposition 2.2 ([1],[3])**

(i) The algebra $(U^{\text{fin}}_{\epsilon})^+$ is a finite dimensional $C$-vector space with the basis
\[
\{e_{\beta_{N-1}^1} \cdots e_{\beta_1}^1\}_{0 \leq r_1, \cdots, r_N < l}.
\]

(ii) The algebra $(U^{\text{fin}}_{\epsilon})^-$ is a finite dimensional $C$-vector space with the basis
\[
\{f_{\beta_{N-1}^1} \cdots f_{\beta_1}^1\}_{0 \leq r_1, \cdots, r_N < l}.
\]

(iii) The algebra $(U^{\text{fin}}_{\epsilon})^0$ is a finite dimensional $C$-vector space with the basis
\[
\{t_{r_{n-1}}^1 \cdots t_{r_1}^1\}_{0 \leq r_1, \cdots, r_n < 2l}.
\]

(iv) Multiplication defines an isomorphism of $C$-vector space:
\[
(U^{\text{fin}}_{\epsilon})^- \otimes (U^{\text{fin}}_{\epsilon})^0 \otimes (U^{\text{fin}}_{\epsilon})^+ \cong U^{\text{fin}}_{\epsilon}.
\]

2.3 Non-restricted specializations

Here we see another type of specialization of $q$ to a root of unity.

Introduce the elements
\[
[t_i;m] := \frac{t_i q^m - t_i^{-1} q^{-m}}{q - q^{-1}} \in U_q(\mathfrak{g}).
\]

The algebra $U_{\mathcal{A}}$ is the $\mathcal{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by the elements $e_i$, $f_i$, $t_i^\pm$ and $[t_i;0]$ ($1 \leq i \leq n$).

**Remark.** The defining relations for $U_{\mathcal{A}}$ are as in 2.1, but replacing (2.4) by
\[
e_i f_j - f_j e_i = \delta_{ij}[t_i;0].
\]

and add the relation $(q - q^{-1})[t_i;0] = t_i - t_i^{-1}$.

Now for arbitrary $\epsilon \in \mathbb{C}^\times$ we define the $\mathbb{C}$-algebra
\[
U_{\epsilon} := U_{\mathcal{A}} \otimes_{\mathbb{C}} \mathbb{C}_{\epsilon},
\]

where $\mathcal{A}$ acts on $\mathbb{C}_{\epsilon} = \mathbb{C}$ by $f(q)c = f(\epsilon)c$ ($c \in \mathbb{C}$). This $U_{\epsilon}$ is called the non-restricted specialization.
3 Representations

3.1 Maximal cyclic representations of $U_\varepsilon$

The representation theory of $U_\varepsilon$ is discussed in [3] in which the maximal dimension of irreducible representations for $A_n$ type is given by $l \frac{1}{2} n(n+1)$ in the case $\varepsilon$ is the $l$-th root of unity and in [2], it is constructed explicitly and called the 'maximal cyclic representations'. Here we modify the presentations in [2] subtly in order to simplify the arguments in the section 4.

Let $H$ be the group generated by $\{x_{ij}, z_{ij}\}_{1 \leq i \leq j \leq n}$ and the center $\varepsilon$ with the relations $z_{ij}x_{ij} = \varepsilon x_{ij}z_{ij}$ and all others commute each other, and set $W := \mathbb{C}[H]$ the group ring of $H$. For $r := (r_1, \cdots, r_n)$, $s := (s_1, \cdots, s_n) \in (\mathbb{C}^\times)^n$, we define the map $\varphi_{r,s} : U_\varepsilon \to W$ by (see [2]):

\[
\varphi_{r,s}(e_i) := \sum_{k=1}^{n} x_{ik} x_{i(k+1)} \cdots x_{in} \{r_i z_{ik} z_{i(k+1)}^{-1} \varepsilon_{i(k+1)-1} \varepsilon_{i(k+1)+1}^{-1}\}, \tag{3.1}
\]

\[
\varphi_{r,s}(f_i) := \sum_{k=1}^{l} x_{i+1-k} x_{i+2-k} \cdots x_{in}^{-1} \times \{s_i z_{i+1-k} z_{i+1-n-k}^{-1} \varepsilon_{i+1-n-k}^{-1}\}, \tag{3.2}
\]

\[
\varphi_{r,s}(t_i) := \frac{r_i}{s_i} z_{i+1-n-k}^{-1} \varepsilon_{i+1-n-k}^{-1}, \tag{3.3}
\]

where we use the notation $\{z\} = (z - z^{-1})/(\varepsilon - \varepsilon^{-1})$.

Let $* : W \to W$ be the $\mathbb{C}$-linear involution defined by

$x_{jk}^* := x_{k+1-j}^{-1}, \ z_{jk}^* := z_{k+1-j}^{-1}$

and set

$A_{ik} := x_{ik} x_{i(k+1)} \cdots x_{in}, \ B_{ik} := z_{ik} z_{i(k+1)}^{-1} \varepsilon_{i(k+1)-1} \varepsilon_{i(k+1)+1}^{-1}$.

Then, (3.1) and (3.3) can be written in the following forms:

\[
\varphi_{r,s}(e_i) = \sum_{k=1}^{n} A_{ik} \{r_i B_{ik}\}, \quad \varphi_{r,s}(f_i) = \sum_{k=1}^{l} A_{n+1-i-n+1-k}^* \{s_i B_{n+1-i-n+1-k}^*\}. \tag{3.4}
\]

**Proposition 3.1** The map $\varphi_{r,s}$ defines a $\mathbb{C}$-linear algebra homomorphism from $U_\varepsilon$ to $W$.

**Lemma 3.2** The following commutation relations hold (see [3]/(2.5)).

\[
A_{ij} B_{ik} = \varepsilon^{-2} B_{ik} A_{ij} \quad \text{if} \quad j < k, \\
= \varepsilon^{-1} B_{ik} A_{ij} \quad \text{if} \quad j = k, \\
= B_{ik} A_{ij} \quad \text{if} \quad j > k.
\]
Proof of Proposition 3.3: We have $A_{ik}B_{ik} = \epsilon^{-1}B_{ik}A_{ik}$ and then

$$\varphi_{r,s}(e_i^m) = \sum_{k=1}^{n} (r_i \epsilon^{-1} B_{ik}) A_{ik}, \quad \varphi_{r,s}(f_i) = \sum_{k=1}^{i} (s_i \epsilon^{-1} B_{n+1-i+1-k}) A_{n+1-i+1-k}.$$ 

This implies $\varphi_{r,s} = \rho_{\epsilon^{-1}r, \epsilon^{-1}s}$ ($\rho_{r,s}$ is given in (3)). Thus, by Theorem 2.2 in [2], we obtained the desired result.

**Proposition 3.3** For any $m \in \mathbb{Z}_{>0}$, we have

$$\varphi_{r,s}(e_i^m) = [m]! \sum_{p=1}^{m} \prod_{1 \leq kp < \cdots < k_1 \leq n}^{p} \prod_{r=1}^{p} A_{r,k_r}^{\nu_r-\nu_{r+1}}, \quad (3.5)$$

$$\varphi_{r,s}(f_i^m) = [m]! \sum_{p=1}^{m} \prod_{1 \leq kp < \cdots < k_1 \leq n}^{p} \prod_{r=1}^{p} A_{n+1-i+1-k_r}^{\nu_r-\nu_{r+1}} \prod_{r=1}^{p} s_i B_{n+1-i+1-k_r}^{\nu_r-\nu_{r+1}}, \quad (3.6)$$

where $\nu_{r+1} = 0$ and we set

$$\{a; b; c\} := \frac{\{a b\} \{a b^{-1}\} \cdots \{a b^{-c+1}\}}{|c|!}.$$

**Remark.** The definition of $\{a; b; c\}$ is invalid for $\epsilon$ such that $[c]! = 0$. But in the right hand-side of (3.5) and (3.6) we see that the term

$$\prod_{r=1}^{p} [\nu_r - \nu_{r+1}] \quad (1 \leq \nu_p < \cdots < \nu_1 = m).$$

is valid since $[m]! / \prod_{r=1}^{p} [\nu_r - \nu_{r+1}] \in \mathbb{Z}[q, q^{-1}]$.

**Proof.** In [2], the following formula is given

$$\rho_{r,s}(e_i^m) = [m]! \sum_{p=1}^{m} \prod_{1 \leq kp < \cdots < k_1 \leq n}^{p} \prod_{r=1}^{p} \left\{ \frac{r_i B_{k_r}; -\nu_{r+1}}{\nu_r - \nu_{r+1}} \right\} \prod_{r=1}^{p} A_{k_r}^{\nu_r-\nu_{r+1}}, \quad (3.7)$$

where $\nu_{r+1} = 0$. Since $\varphi_{r,s} = \rho_{\epsilon^{-1}r, \epsilon^{-1}s}$, it follows from (3.7)

$$\varphi_{r,s}(e_i^m) = \rho_{\epsilon^{-1}r, \epsilon^{-1}s}(e_i^m)$$

$$= [m]! \sum_{p=1}^{m} \prod_{1 \leq kp < \cdots < k_1 \leq n}^{p} \prod_{r=1}^{p} \left\{ \frac{\epsilon^{-1}r_i B_{k_r}; -\nu_{r+1}}{\nu_r - \nu_{r+1}} \right\} \prod_{r=1}^{p} A_{k_r}^{\nu_r-\nu_{r+1}} \quad (3.8)$$

Here by Lemma 3.2, we obtain for $i \leq k_1 < \cdots < k_n$ and $1 \leq r \leq p$

$$\left\{ \frac{\epsilon^{-1}r_i B_{k_r}; -\nu_{r+1}}{\nu_r - \nu_{r+1}} \right\} \prod_{r=1}^{p} A_{k_r}^{\nu_r-\nu_{r+1}} = \prod_{r=1}^{p} A_{k_r}^{\nu_r-\nu_{r+1}} \left\{ \frac{\epsilon^{-1+r_i +\nu_r+1-2\nu_{r+1}+1} r_i B_{k_r}; -\nu_{r+1}}{\nu_r - \nu_{r+1}} \right\} = \prod_{r=1}^{p} A_{k_r}^{\nu_r-\nu_{r+1}} \left\{ \frac{r_i B_{k_r}; \nu_r - 1}{\nu_r - \nu_{r+1}} \right\},$$
where we use \( \nu_{r+1} = 0 \). Thus, we obtain (3.3). Similarly we also get (3.4).

Let \( V_{ij} \) \( (1 \leq i \leq j \leq n) \) be a copy of the vector space \( C^i \) and set \( \mathcal{V} := \bigotimes_{1 \leq i \leq j \leq n} V_{ij} \). Let \( u_0, \ldots, u_{n-1} \) be the standard basis of \( C^i \). Now we define the representation \( (\psi_{a,b}, \mathcal{V}) \) of \( \mathcal{W} \) as follows: Let \( Z_{jk}, X_{jk} \in \text{End}(\mathcal{V}) \) be the matrices defined as \( Z_{jk}u_i = u_{i+1} \) and \( X_{jk}u_i = \varepsilon^i u_i \) on the component \( V_{jk} \) and as the identity on the other component. For non-zero parameters \( a := (a_{ij})_{1 \leq i \leq j \leq n} \) and \( b := (b_{ij})_{1 \leq i \leq j \leq n} \in (C^\times)^{n(n+1)/2} \), define \( \psi_{a,b}(x_{ij}), \psi_{a,b}(z_{ij}) \in \text{End}(\mathcal{V}) \) to be

\[
\psi_{a,b}(x_{ij}) = a_{ij}X_{ij}, \quad \psi_{a,b}(z_{ij}) = b_{ij}Z_{ij}.
\]

We can easily check that these define the representation of \( \mathcal{W} \):

\[
\psi_{a,b} : \mathcal{W} \rightarrow \text{End}(\mathcal{V}).
\]

Composing \( \varphi_{r,s} \) and \( \psi_{a,b} \),

\[
\Phi_{r,s,a,b} := \psi_{a,b} \circ \varphi_{r,s} : U_\varepsilon \xrightarrow{\varphi_{r,s}} \mathcal{W} \xrightarrow{\psi_{a,b}} \text{End}(\mathcal{V}).
\]

we obtain the representation of \( U_\varepsilon \) denoted by \( (\Phi_{r,s,a,b}, \mathcal{V}) \). The representation introduced in [2] is just as \( (\Phi_{r,s,a,b}, \mathcal{V}) \) in our notation since we have \( \varphi_{r,s} = \rho_{r,s}^{-1} \varepsilon_{r,s}^{-1} \) in the proof of Proposition [3.3].

In [2], it is shown that the central elements of \( U_\varepsilon \) take values in an open set of \( C^{n(n+2)} \) and then by [3], it turns out to be generically irreducible for the parameters \( r, s, a, b \). We are interested in specializations of these parameters so that the representation \( (\Phi_{r,s,a,b}, \mathcal{V}) \) is not necessarily irreducible.

### 3.2 Representations of \( U_\varepsilon^{\text{res}} \) and \( U_\varepsilon^{\text{fin}} \)

We review the representation theory of \( U_\varepsilon^{\text{res}} \). The classification of the irreducible representations of \( U_\varepsilon^{\text{res}} \) is given by Lusztig [4]. Before seeing it, let us recall the notions of highest weight modules.

**Definition 3.4** Let \( V \) be a \( U_\varepsilon^{\text{res}} \)-module of type 1 (as for “type”, see [3],[4]).

(i) The weight spaces \( V_\lambda \) \( (\lambda = \sum_i m_i \Lambda_i \in P) \) of \( V \) are defined by

\[
V_\lambda := \left\{ v \in V | t_i v = \varepsilon^{m_i^{(0)}} v, \quad \left[ \begin{array}{c} t_i \ 0 \\ l \end{array} \right] v = \left[ \begin{array}{c} m_i^{(1)} \\ l \end{array} \right] v \right\},
\]

where \( m_i = m_i^{(0)} + lm_i^{(1)} \) and \( 0 \leq m_i^{(0)} < l \).

(ii) \( V \) is a highest weight module if \( V \) is generated by a primitive vector, i.e., a vector \( v \in V_\lambda \) for some \( \lambda \in P \), such that \( e_i v = \varepsilon_i^{(1)} v = 0 \) for any \( i \in I \). In the case, \( \lambda \) is called the highest weight and \( v \) is called the highest weight vector of \( V \).

Let \( V(\lambda) \) be the irreducible highest weight \( U_\varepsilon(\mathfrak{g}) \)-module given by \( V(\lambda) = U_\varepsilon(\mathfrak{g})/I \) where \( \lambda \in P_+ \) and, \( I \) is the left ideal generated by \( e_i, f_i^{1+} \) and \( t_i - q^{(h_i, \lambda)} (1 \leq i \leq n) \). Here denote the generator of \( V(\lambda) \) by \( v_\lambda \). Let \( V_\lambda^{\text{res}} \) be the \( U_\lambda^{\text{res}} \)-submodule of \( V(\lambda) \) generated by \( v_\lambda \). Set \( W_\lambda^{\text{res}}(\lambda) := V_\lambda^{\text{res}}(\lambda) \otimes_\mathcal{A} C_\varepsilon \),

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which is naturally $U^\text{res}_\varepsilon$-module. Note that $W^\text{res}_\varepsilon(\lambda)$ is not necessarily irreducible. So, let $Y$ be its maximal proper submodule and define $V^\text{res}_\varepsilon(\lambda) := W^\text{res}_\varepsilon(\lambda)/Y$ to be the irreducible quotient, which is type $1$ highest weight module with the highest weight $\lambda$.

**Theorem 3.5** ([B]) *Arbitrary finite-dimensional irreducible $U^\text{res}_\varepsilon$-module $V$ of type 1 is isomorphic to $V^\text{res}_\varepsilon(\lambda)$ for a unique $\lambda \in P_+$.*

Note that arbitrary finite-dimensional irreducible $U^\text{res}_\varepsilon$-module $V$ of type 1 is a direct sum of its weight spaces.

**Theorem 3.6** ([B]) *For $\lambda = \sum_i m_i\Lambda_i \in P_+$, define $\lambda^{(0)} := \sum_i m_i^{(0)}\lambda_i$ and $\lambda^{(1)} := \sum_i m_i^{(1)}\lambda_i$ where $m_i = m_i^{(0)} + lm_i^{(1)}$ with $0 \leq m_i^{(0)} < l$ (and then $\lambda = \lambda^{(0)} + l\lambda^{(1)}$). The $U^\text{res}_\varepsilon$-module $V^\text{res}_\varepsilon(\lambda)$ is isomorphic to $V(\lambda^{(0)}) \otimes V(l\lambda^{(1)})$."

Here we call a weight $\lambda \in P_+$ satisfying $\lambda = \lambda^{(0)}$ a $l$-restricted weight. As we have stated in the introduction, the module $V(\lambda^{(0)})$ is irreducible $U^\text{fin}_\varepsilon$-module and $V(l\lambda^{(1)})$ is identified with the irreducible highest weight $\mathfrak{sl}_n$-module $V(\lambda^{(1)})$. Since we know the structure of irreducible $\mathfrak{sl}_n$-module well, this theorem implies that the structure of the module $V^\text{res}_\varepsilon(\lambda)$ can be clarified if we shall make clear the one for $V(\lambda^{(0)})$.

### 4 Primitive vectors

Let $l$ and $\varepsilon$ be same as in the previous section.

#### 4.1 Specializations of parameters

Let $M := \{m = (m_{ijk})_{1 \leq i \leq k \leq n} | 0 \leq m_{ijk} \leq l - 1\}$ be the index set of the standard basis of $\mathcal{V}$. We can consider the additive structure on $M$ via the natural identification $M \cong (\mathbb{Z}/l\mathbb{Z})^{n(n+1)}$. For $m \in M$ we write $u_m := \otimes_{1 \leq i \leq k \leq n} u_{m_{ijk}}$ ($u_{m_{ijk}} \in V_{jk}$).

Here we consider the following specialization of parameters $r, s, a, b$:

\[ a_{ik}a_{i+1,k+1} \cdots a_{in} = 1, \]
\[ r_ib_{k-1}b_{i,k-1}^{-1}b_{i,k+1}^{-1} = 1, \]
\[ r_i^2 s_i b_{i-1,k}^{-1}b_{i+1,k}^{-1} = \varepsilon^{\lambda_i}, \]

where integers $\{\lambda_i\}_{1 \leq i \leq n}$ satisfy $0 \leq \lambda_i < l$.

**Remark.** Here note that the set of parameters satisfying (4.1)-(4.3) is never empty. Indeed, if we set $a_{jk} = b_{jk} = 1$ for any $(j,k)$ and $r_i = 1$ and $s_i = \varepsilon^{-\lambda_i}$ for any $i$, it is trivial to see that these satisfy (4.1)-(4.3). (By (4.3), we have $a_{jk} = 1$ for all $1 \leq j \leq k \leq n$).

**Lemma 4.1** *Under the specialization (4.2) and (4.3), we have*

\[ s_ib_{i+1-kn-1-k}^{-1}b_{i+1-kn+1-k}^{-1}b_{i-kn+1-k}^{-1}b_{i-kn-k}^{-1} = \varepsilon^{-\lambda_i}. \]
Proof. Using (4.2), we have $r_i b_{ik} b_{ik-1}^{-1} b_{i-1,k-1}^{-1} b_{i+1,k}^{-1} = 1 = r_i b_{ik-1} b_{ik-2} b_{i-1,k-2}^{-1} b_{i+1,k-1}^{-1}$ and then $b_{ik} b_{ik-1}^{-1} b_{i+1,k}^{-1} = b_{ik-2} b_{i-1,k-2}^{-1} b_{i+1,k-1}^{-1}$. Changing $i \to i - k$ and $k \to n - k$, we get

$$b_{i+1,k}^{-1} b_{i+1,k+n+1-k}^{-1} b_{i-k,n+1-k}^{-1} b_{i-k-1,n-k-1}^{-1} = b_{i-k,n-k-1}^{-1} b_{i-k-1,n-k-1}^{-1} b_{i-k-1,n-k-1}^{-1}.$$  

By (4.3) with $k = n$ and (4.3), we have $s_i b_{ik}^{-1} b_{i-1,n}^{-1} b_{i-1,n-1} = \varepsilon^{-1}$, which is (4.4) in the case $k = 1$. Suppose that (4.4) holds and substitute (4.5) into (4.4). Then we obtain

$$s_i b_{ik}^{-1} b_{i-k,n-k-1}^{-1} b_{i-k-1,n-k-1}^{-1} = \varepsilon^{-1}.$$  

Thus, the induction on $k$ proceeds and then we prove (4.4) for any $k \in \{1, 2, \ldots, i\}$.

By (4.1) and (4.3) and this lemma, we have

$$\Phi_{r,s,a,b}(e_i) := \sum_{k=i}^{n} X_i k X_{i,k+1} \cdots X_{i,n} \{Z_i k Z_{i,k-1} Z_{i-1,k-1} Z_{i+1,k}^{-1}\},$$  

$$\Phi_{r,s,a,b}(f_i) := \sum_{k=1}^{i} X_{i+1-k,n+1-k}^{-1} X_{i+2-k,n+2-k} \cdots X_{i,n}^{-1} \times \{\varepsilon^{-1} Z_{i+1-k,n-k} Z_{i+1-k,n+1-k}^{-1} Z_{i-k,n+1-k}^{-1} Z_{i-k,n-k}^{-1}\},$$  

$$\Phi_{r,s,a,b}(t_i) := \varepsilon^i Z_{i,n}^{-1} Z_{i-1,n}^{-1} Z_{i,n}^{-1}.$$  

4.2 Primitive vectors in $V$

Under the specialization in 4.1, we get the following:

**Proposition 4.2** Under the specialization (4.1) and (4.4), $v \in V$ satisfies the condition

$$e_i v = 0 \quad \text{for any } i = 1, \ldots, n,$$  

if and only if $v = cu_\vec{0}$ ($c \in \mathbb{C}$) where $\vec{0} = (0, 0, \ldots, 0) \in M$.

**Proof.** By (4.1) and (4.2), the action of $e_i$ on $u_m \in V$ ($m = (m_{gh}) \in M$) is given by

$$e_i u_m = \sum_{i \leq k \leq n} \{m_{ik} + m_{i,k-1} - m_{i-1,k-1} - m_{i+1,k}\} u_m + e_i u_m,$$  

where $\epsilon_{j,k} \in M$ satisfies that the $(j, k)$-entry is 1 and all others are 0. If $m = \vec{0}$, we have $m_{ik} + m_{i,k-1} - m_{i-1,k-1} - m_{i+1,k} = 0$ for all $i \leq k \leq n$, which implies that $e_i u_\vec{0} = 0$ for any $i$.

Conversely, assume that $v = \sum_{m \in M} c_m u_m$ ($c_m \in \mathbb{C}$) satisfies (4.9). First, we have

$$0 = c_n v = X_n n (Z_{n-1,n}^{-1} Z_{n,n}) v = \sum_{m \in M} c_m [m_{nn} - m_{n-1,n}] u_m + e_n u_m.$$  

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This implies that
\[ m_{n-1}n-1 \neq m_{n}n \implies c_m = 0, \quad (4.12) \]
and then we have 
\[ v = \sum_{m \in M, m_{n-1}n-1 = m_{n}n} c_m v_m. \]
Next, by \( e_{n-1}v = 0 \) we have
\[ 0 = e_{n-1}v = (X_{n-1}n-1X_{n-1}n \{ Z_{n-1}n-1Z_{n-2}n-2^{-1} \} + X_{n-1}n \{ Z_{n-1}n Z_{n-1}n-1Z_{n-2}n-1Z_{n-1}n^{-1} \} )v \]
\[ = \sum_{m \in M, m_{n-1}n-1 = m_{n}n} c_m [m_{n-1}n-1 - m_{n-2}n-2]u_{m+\epsilon_{n-1}n} \]
\[ + c_m [m_{n-1}n + m_{n-1}n-1 - m_{n-2}n-1 - m_{n}n]u_{m+\epsilon_{n-1}n}. \]
This implies that
\[ c_m [m_{n-1}n-1 - m_{n-2}n-2] = c_m [m_{n-1}n + m_{n-1}n-1 - m_{n-2}n-1 - m_{n}n] = 0 \quad (4.13) \]
for any \( m \in M \) satisfying \( m_{n-1}n-1 = m_{n}n \) since all vectors appear in the summation are linearly independent under the condition \( m_{n-1}n-1 = m_{n}n \), that is, the index \( m + \epsilon_{n-1}n-1 + \epsilon_{n-1}n \) and \( m' + \epsilon_{n-1}n \) never coincide for arbitrary \( m, m' \) under the condition \( m_{n-1}n-1 = m_{n}n \). Thus, by \((4.12)\) and \((4.13)\) we have unless
\[ m_{n-2}n-2 = m_{n-1}n-1 = m_{n}n, \]
\[ m_{n-2}n-1 = m_{n-1}n, \]
\[ c_m = 0. \]
Here we assume that \( c_m = 0 \) in \( v \) unless
\[ m_{i}i = m_{i+1}i+1 = \ldots = m_{n}n, \]
\[ m_{i+1} = m_{i+1}i+2 = \ldots = m_{n-1}n, \]
\[ \ldots \]
\[ m_{i}n-1 = m_{i+1}n. \]
By \( e_{i}v = 0 \) we get
\[ 0 = \sum_{k=i}^{n} X_{i+k}X_{i+k+1} \cdots X_{i+n} \{ Z_{i+k}Z_{i+k-1}Z_{i+k-1}Z_{i+k}^{-1} \}v \]
\[ = \sum_{m \in M, m \text{ satisfies } (4.14)} m_{i}k \sum_{k=i}^{n} c_m [m_{i}k + m_{i+k-1} - m_{i-1}k-1 - m_{i+1}k]u_{m+\epsilon_{i,k}+\ldots+\epsilon_{i,n}} \]
\[ = \sum_{m \in M, m \text{ satisfies } (4.14)} (c_m [m_{i}i - m_{i-1}i-1]u_{m+\epsilon_{i,i}+\ldots+\epsilon_{i,n}} \]
\[ + c_m [m_{i}i+1 + m_{i}i - m_{i-1}i-1 - m_{i+1}i+1]u_{m+\epsilon_{i,i+1}+\ldots+\epsilon_{i,n} \}
\[ + \ldots + \]
\[ + c_m [m_{i}n + m_{i}n-1 - m_{i-1}n-1 - m_{i+1}n]u_{m+\epsilon_{i,n}} \} \]
It follows from \((4.14)\) that all vectors appear in the summation \((4.15)\) are linearly independent. Therefore, we obtain that \( c_m = 0 \), unless
\[ m_{i}i - m_{i-1}i-1 = 0, \]
\[m_{i+1} + m_i - m_{i-1} - m_{i+1} = 0,\]
\[\ldots \ldots\]
\[m_n + m_{n-1} - m_{n-2} - m_{n+1} = 0.\]

Thus, from this and \((4.14)\) we get \(c_m = 0\) unless
\[m_{i-1} = m_{i} = m_{i+1} + 1 = \ldots \ldots = m_{n},\]
\[m_{i} = m_{i+1} = m_{i+2} + 1 = \ldots \ldots = m_{n-1},\]
\[\ldots \ldots\]
\[m_{i-1,n-2} = m_{i,n-1} = m_{i+1,n},\]
\[m_{i-1,n-1} = m_{i,n}.
\]

Thus, using \(e_2v = e_3v = \ldots = e_{n-1}v = e_nv = 0\) we have \(c_m = 0\) unless
\[m_{11} = m_{22} = m_{33} = \ldots \ldots = m_{nn},\]
\[m_{12} = m_{23} = m_{34} = \ldots = m_{n-1,n},\]
\[\ldots \ldots\]
\[m_{1,n-2} = m_{2,n-1} = m_{3,n},\]
\[m_{1,n-1} = m_{2,n}.
\]

Finally, using \(e_1v = 0\), we have
\[0 = \sum_{k=1}^{n} X_{1k}X_{1,k+1} \cdots X_{1n} \{Z_{1k}Z_{1,k+1}Z_{2k}^{-1}\}v\]
\[= \sum_{m \in M, m \text{ satisfies } (4.17)} c_m [m_{1k} + m_{1,k-1} - m_{2k}]u_{m+\epsilon_1 v} + \ldots + c_m [m_{1n} + m_{1,n-1} - m_{2n}]u_{m+\epsilon_1 n}.
\]

Under the condition of \((4.17)\), we get that \(c_m = 0\), unless
\[0 = m_{11} = m_{12} = \ldots = m_{1n}.\]
\[\text{(4.19)}\]

Therefore, it follows from \((4.17)\) and \((4.19)\) that \(c_m = 0\) unless
\[0 = m_{11} = m_{22} = m_{33} = \ldots \ldots = m_{nn},\]
\[0 = m_{12} = m_{23} = m_{34} = \ldots = m_{n-1,n},\]
\[\ldots \ldots\]
\[0 = m_{1,n-2} = m_{2,n-1} = m_{3,n},\]
\[0 = m_{1,n-1} = m_{2,n},\]
\[\text{which implies that } v = cu_0.\]
\[\square\]
Remark. In the proof of the proposition, we see easily that we do not need (4.3) essentially. It is required for simplification of the proof or the presentations. So it is possible to proceed the same argument for generic $a_{jk}$'s.

The primitive vector $u_\bar{g}$ possesses the following property:

**Proposition 4.3** Under the condition $(4.3), (4.2)$ and $(4.1)$, we have $f_i^{\lambda_1+1} u_\bar{g} = 0$.

**Proof.** We obtain the explicit form of $f_i^{\lambda_1+1}$ on $\mathcal{V}$ by $(3.6)$ in Proposition 3.3, taking $m = \lambda_i + 1$. By Lemma 4.1, under the specialization $(4.1)$, $(4.2)$ and $(4.3)$, we have $s b_{i+1-k n-k} b_{i+1-k n+1-k} b_{i-k n+1-k} b_{i-k n-k} = \varepsilon^{-\lambda_i}$. Thus, on $u_\bar{g}$ we have

$$f_i^{\lambda_1+1} u_\bar{g} = [\lambda_i + 1]! \sum_{p=1}^{\lambda_i+1} \prod_{r=1}^{\lambda_i+1-k} A_{n+1-i n+1-k} \prod_{r=1}^{p} \left\{ \varepsilon^{-\lambda_i}; \nu_r - 1 \right\} u_\bar{g}.$$  

For any $p \in \{1, 2, \ldots, \lambda_i + 1\}$ and any $\nu_p, \ldots, \nu_1$ satisfying $1 \leq \nu_p < \cdots < \nu_1 = \lambda_i + 1$, there exists some $r \in \{1, 2, \ldots, p\}$ such that $\nu_r + 1 \leq \lambda_i < \nu_r$. Therefore, for such $r$ we have

$$\left\{ \varepsilon^{-\lambda_i}; \nu_r - 1 \right\} = \frac{\varepsilon^{-\lambda_i+\nu_r-1} \cdots \varepsilon^{-\lambda_i+\nu_1-1}}{[\nu_r - \nu_{r+1}]!} = 0.$$

Thus, we obtain $f_i^{\lambda_1+1} u_\bar{g} = 0.$

Here for $\lambda := (\lambda_1, \ldots, \lambda_n)$ $(0 \leq \lambda_i < l)$ we define the $U_\varepsilon$-submodule $L(\lambda)$ of $\mathcal{V}$ by $L(\lambda) := U_\varepsilon u_\bar{g}$.

Let $V(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module as in 3.2. Let $V_\varepsilon(\lambda)$ be the $U_\varepsilon$-submodule of $V(\lambda)$ generated by $v_\lambda$. Set $V(\lambda) := V_\varepsilon(\lambda) \otimes_\mathbb{C} V(\lambda)$, which is naturally $U_\varepsilon$-module. Note that $V_\varepsilon(\lambda)$ is not necessarily irreducible. By Proposition 4.3, $f_i^{\lambda_1+1} u_\bar{g} = 0$ $(i \in I)$, thus we have the following surjective $U_\varepsilon$-linear map $\pi : V_\varepsilon(\lambda) \rightarrow L(\lambda)$ given by $\pi : v_\lambda \mapsto u_\bar{g}$. It seems that the module $L(\lambda)$ is in the similar stream of the theory of $U_\varepsilon(\mathfrak{g})$-modules. Here we expect that $L(\lambda)$ is an irreducible highest weight $U_\varepsilon$-module. Surprisingly, in the next section we obtain more interesting results that $L(\lambda)$ can be seen as an irreducible $U_\varepsilon(\mathfrak{g})$-module directly from $U_\varepsilon$-modules.

### 4.3 Shifts of parameters

Let $r^{(0)} = (r_j^{(0)}), s^{(0)} = (s_j^{(0)}) \in (\mathbb{C}^*)^n$ and $a^{(0)} = (a_{jk}^{(0)}), b^{(0)} = (b_{jk}^{(0)}) \in (\mathbb{C}^*)^N$ be the parameters satisfying $(4.3), (4.2)$.

Fix a basis vector $u_\xi \in \mathcal{V}$ $(\xi = (\xi_{jk}) \in M)$ arbitrarily and set $b(\xi) := (\varepsilon^{-\xi_{jk}} b_{jk}^{(0)}) \in (\mathbb{C}^*)^N$. In this setting we obtain the following:

**Proposition 4.4** For any $\mu = (\mu_{jk}) \in M$ and any $X \in U_\varepsilon$, set

$$\Phi_{r^{(0)}, s^{(0)}, a^{(0)}, b^{(0)}} (X) u_\mu = \sum_{m \in M} C_m u_m.$$  

\vspace{1cm}

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Then we have
\[ \Phi_{r(0),a(0),\alpha(0),\beta(\xi)}(X)u_{\mu+\xi} = \sum_{m \in M} C_m u_{m+\xi} \] (4.22)

Proof. It is shown easily from the formula
\[
\psi_{\alpha(0),\beta(0)}(x_{jk})u_{\mu,jk} = a_{jk}^{(0)} u_{\mu,jk+1}, \quad \psi_{\alpha(0),\beta(0)}(z_{jk})u_{\mu,jk} = b_{jk}^{(0)} \varepsilon^{\mu,jk} u_{\mu,jk},
\]
\[
\psi_{\alpha(0),\beta(0)}(x_{jk})u_{\mu,jk+\xi,jk} = a_{jk}^{(0)} u_{\mu,jk+\xi,jk+1},
\]
\[
\psi_{\alpha(0),\beta(0)}(z_{jk})u_{\mu,jk+\xi,jk} = b_{jk}^{(0)} \varepsilon^{\mu,jk+\xi,jk} u_{\mu,jk+\xi,jk} = b_{jk}^{(0)} \varepsilon^{\mu,jk} u_{\mu,jk+\xi,jk}.
\]

By Proposition 4.4, we have the following result.

**Proposition 4.5** We consider the representation \( (\Phi_{r(0),a(0),\alpha(0),\beta(\xi)}, \mathcal{V}) \). A vector \( v \in \mathcal{V} \) satisfies the condition
\[ e_i v = 0 \quad \text{for any } i = 1, \ldots, n, \] (4.23)
if and only if \( v = cu_\xi \) (\( c \in \mathbb{C} \)) where \( u_\xi \in \mathcal{V} \) is the fixed basis vector as above.

By this proposition, if we take the parameters properly, any basis vector \( u_\xi \) in \( \mathcal{V} \) can be a primitive vector.

## 5 Irreducible \( U^\text{fin}_\varepsilon \)-module

Suppose that parameters \( r, s, a, b \) satisfy the conditions (1.1)–(4.3). In this case \( u_\xi \) is the unique (up to constant) primitive vector in \( \mathcal{V} \). As we defined in the last section, set \( \mathcal{L}(\lambda) := U_\varepsilon u_\xi (\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n), 0 \leq \lambda_i \leq \ell - 1) \), which is a \( U_\varepsilon \)-submodule of \( \mathcal{V} \). In this section we shall see several properties of this module and it amounts to an irreducible \( U^\text{fin}_\varepsilon \)-module.

### 5.1 Root vectors

First, we see higher root vectors in \( U_\varepsilon \). There are several definitions for them. We shall introduce two of them here and discuss their relations.

The first one is defined by using (2.13) and (2.16). We also denote them by \( e_\alpha \) and \( f_\alpha \) (\( \alpha \in \Delta_+ \)).

**Lemma 5.1** The root vectors \( e_\alpha \) and \( f_\alpha \in U_\varepsilon \) (\( \alpha \in \Delta_+ \)) defined by (2.13) and (2.14) satisfy the relations (2.14), (2.14), (2.14), (2.14), (2.14) and (2.18) in \( U_\varepsilon \).

Proof. The proof for the \( f_\alpha \) case is similar to the \( e_\alpha \) case, thus we shall only see the \( e_\alpha \) case. We shall show (2.14), (2.14) and (2.18) simultaneously by the induction on the height of roots. Set \( \alpha = \alpha_j + \alpha_{j+1} + \cdots + \alpha_k \), \( \beta = \alpha - \alpha_j \) and \( \gamma = \beta - \alpha_{j+1} \in \Delta_+ \) (\( j < k \)). If the condition (\( \alpha, \alpha_i \)) = 0 and \( i < g(\alpha)(= k) \) hold, we know that \( i \in \{1, 2, \ldots, j-2, j+1, j+2, \ldots, k-1\} \). If \( i \in \{1, 2, \ldots, j-2, j+2, \ldots, k-1\} \), by the hypothesis of the induction we have \( e_i e_\beta = e_\beta e_i \) and then
\[ e_i e_\alpha = e_i (e^{-1} e_j e_\beta - e_\beta e_j) = (e^{-1} e_j e_\beta - e_\beta e_j) e_i = e_\alpha e_i. \]
If $i = j + 1$, by the hypothesis of the induction we have $\varepsilon e_{j+1}e_\beta = e_\beta e_{j+1}$, $e_je_\gamma = e_\gamma e_j$ and then
\[
e_{j+1}e_\alpha = e_{j+1}(\varepsilon^{-1}e_1e_\beta - e_\beta e_j) = e_{j+1}(\varepsilon^{-1}e_1(\varepsilon^{-1}e_{j+1}e_\gamma - e_\gamma e_{j+1}) - e_\beta e_j)
\]
\[
= e^{-2}e_{j+1}e_1e_{j+1}e_\gamma - \varepsilon^{-1}e_{j+1}e_je_{j+1}e_\gamma - e_{j+1}e_\beta e_j
\]
\[
= e^{-1}e_\beta e_{j+1}e_j + e_\gamma e_{j+1}e_\beta e_{j+1} + \varepsilon^{-1}e_\beta e_\beta e_{j+1} - e_\beta e_{j+1}e_{j+1} - e_\gamma e_{j+1}e_{j+1} - \varepsilon^{-1}e_\beta e_{j+1}e_j
\]
\[
= e^{-1}e_\beta e_{j+1}e_j - e_\beta e_{j+1} = e_\alpha e_{j+1}.
\]
Here we used the formula:
\[
e_{j+1}e_je_{j+1}e_\gamma = \varepsilon e_\beta e_{j+1}e_j + \varepsilon^2e_\gamma e_{j+1}e_\beta e_{j+1} + \varepsilon e_je_\beta e_{j+1},
\]
\[
e_{j+1}e_je_{j+1}e_\gamma = e_\beta e_{j+1}e_j + e_\gamma e_{j+1}e_\beta e_{j+1}.
\]

Next, if the condition $(\alpha, \alpha_j) = -1$ and $i < g(\alpha)(= k)$ hold, we know that $i = j - 1$. In this case, we have
\[
e_{j-1}e_\alpha e_{\alpha_j} = e_{j-1}(\varepsilon^{-1}e_{j-1}e_\alpha - e_\alpha e_{j-1})
\]
\[
= e_{j-1}(\varepsilon^{-1}e_j e_\beta - e_\beta e_j) - (\varepsilon^{-1}e_j e_\beta - e_\beta e_j)e_{j-1}
\]
\[
= e^{-2}((2)e_{j-1}e_je_{j-1} - e_j e_{j-1}^2 - e^{-1}e_\beta e_{j-1}e_j - e_{j-1}(\varepsilon^{-1}e_j e_\beta - e_\beta e_j)e_{j-1}
\]
\[
= e^{-1}(\varepsilon^{-1}e_j e_{j-1}e_\alpha - e_\alpha e_{j-1})e_{j-1} = e^{-1}e_{\alpha_j}e_{j-1}e_\alpha.
\]

Finally, under the same condition as above, using $e_j e_\alpha = \varepsilon^{-1}e_\alpha e_j$, $\varepsilon e_\beta e_\alpha = \varepsilon e_\alpha e_\beta$, $e_{j-1}e_\beta = e_\beta e_{j-1}$ and $e_j e_\beta = e_\beta e_j + e_\gamma e_\beta e_j = e_j e_\beta - (2)\beta e_\beta e_j + \varepsilon e_\beta e_j = 0$ we have
\[
e_\alpha e_{j-1} = (\varepsilon^{-1}e_\alpha e_\beta - e_\beta e_\beta)e_{j-1} = e_j e_\alpha e_{j-1} - e_\beta e_\beta e_{j-1}
\]
\[
= e^{-1}e_j e_\beta e_{j-1}e_\alpha e_{j-1} = e_j e_\alpha e_{j-1}e_\beta - e_\beta e_\beta e_{j-1}
\]
\[
= e^{-2}e_j e_{j-1}e_\beta - e^{-1}e_\beta e_{j-1}e_\beta + e_\gamma e_\beta e_{j-1}e_\beta
\]
\[
= e_{j-1}(\varepsilon^{-2}e_j e_\beta e_\beta e_\beta + e_\gamma e_\beta e_{j-1}e_\beta - \varepsilon^{-1}e_\beta e_{j-1}e_\beta e_\beta + e_\gamma e_\beta e_{j-1}e_\beta)
\]
\[
= e_{j-1} e_\alpha + (2)e_\alpha e_{j-1}e_\alpha.
\]

and then we have $e_\alpha e_{\alpha_j} = e_\alpha e_{\alpha_j} = e_\alpha e_{\alpha_j}$.

Here we introduce the alternative definition of root vectors $\overline{e}_\alpha$. For roots $\alpha = \alpha_1 + \alpha_{i+1} + \cdots \alpha_j$ and $\beta = \alpha_j + \alpha_{j+2} + \cdots + \alpha_k (i < j < k)$, we define
\[
\overline{e}_\alpha = \overline{e}_\alpha e_\beta - e_\beta \overline{e}_\alpha = \overline{e}_\alpha e_\beta - e_\beta \overline{e}_\alpha,
\]
\[
\overline{e}_\beta = \overline{e}_\beta e_\alpha - e_\alpha \overline{e}_\beta = \overline{e}_\beta e_\alpha - e_\alpha \overline{e}_\beta.
\]
where we set $\overline{e}_\alpha := e_i$ and $\overline{f}_i := f_i$. Note that this definitions are well-defined, that is, these do not depend on the choice of $j$.

We obtain the following simple relations between two types of the root vectors:
Lemma 5.2 For any $\alpha \in \Delta_+$, we have
\[
\tau_\alpha = \varepsilon^{-\text{height}(\alpha)-1}c_\alpha, \quad \tau'_\alpha = \varepsilon^{-\text{height}(\alpha)+1}f_\alpha, \tag{5.3}
\]
\[
\tau^l_\alpha = e^l_\alpha, \quad \tau^l_\alpha = f^l_\alpha. \tag{5.4}
\]

Proof. The proof of (5.3) is done by using induction on the height of roots and (5.4) is immediate consequence of (5.3) since $\varepsilon^l = 1$.

\[ \square \]

5.2 $U^\text{fin}_{\varepsilon}$-module structure on $\mathcal{V}$

For $\alpha \in \Delta_+$ we define the actions of $e_\alpha$ and $f_\alpha$ recursively by using the formula (2.13) and (2.16) as in the previous subsection.

Proposition 5.3 For any $\alpha \in \Delta_+$ and $i = 1, \ldots, n$, we have
\[
e^l_\alpha = f^l_\alpha = 0 \text{ and } t^l_i = 1 \text{ on } \mathcal{V}. \tag{5.5}
\]

Proof. Since $\Phi_{r,s,a,b}(t_i) = \varepsilon^{h_i} Z_{i-1}^{1} \in \mathcal{V}$ and $Z^1_{ij} = 1$, it is trivial that $t^l_i = 1$. To show the nilpotency of $e_\alpha$ and $f_\alpha$ we see the following result in [3].

Proposition 5.4 ([2] Proposition 3.4) For $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$, the actions of $\tau_\alpha$ and $\tau^l_\alpha$ on $\mathcal{V}$ are given by
\[
\tau_\alpha = \frac{1}{(\varepsilon - \varepsilon^{-1})^l} \left( \sum_{k_1 \geq \cdots \geq k_{j-i} \geq j} (-1)^p \theta(k_1 \geq \cdots \geq k_p < \cdots < k_{j-i+1}) C_{k_1 \cdots k_p-1}(C_{i+p-1}k_p - C_{i+p-1}^{-1}k_p)C_{i+p}k_{p+1} \cdots k_{j-i+1}D_{i+j-1}k_{j-i+1} \right) \cdot \text{id},
\]
\[
\tau^l_\alpha = \frac{1}{(\varepsilon - \varepsilon^{-1})^l} \left( \sum_{k_1 \geq n+1 \cdots \geq k_{j-i} \geq n+1-i} (-1)^p \theta(k_1 \geq \cdots \geq k_p < \cdots < k_{j-i+1}) \bar{C}_{n+1-j \cdots k_{p-1}}(\bar{C}_{n-j+p} - \bar{C}_{n-j+p}^{-1})\bar{C}_{n-j+p+1} \cdots k_{j-i+1} \bar{D}_{n+1 \cdots j-i+1} \right) \cdot \text{id},
\]

where $\theta(X) = 1$ if $X$ is true and $\theta(X) = 0$ otherwise, and we set
\[
C_{i+k} := (\varepsilon^{-1}r_{i+k}b_{i+k}^{-1}b_{i-k}^{-1})^l, \tag{5.6}
\]
\[
\overline{C}_{i+k} := (\varepsilon^{-1}s_{i+k}b_{i+k}^{-1}b_{i-k}^{-1})^l, \tag{5.7}
\]
\[
D_{i+k} := \prod_{p=k}^n (a_{p+1-i})^l, \overline{D}_{i+k} := \prod_{p=k}^n (a_{p+1-i}^{-1})^l, \tag{5.8}
\]

and $\phi_{i_1 \cdots k_p} := \phi_{i_1} \cdots \phi_{i_p}$ for $\phi = C, \overline{C}, D, \overline{D}$.

Applying the specializations of the parameters (1.1), (4.2) and (4.3) to $C_{i+k}$, $\overline{C}_{i+k}$, $D_{i+k}$ and $\overline{D}_{i+k}$, we have $C_{i+k} = \overline{C}_{i+k} = 1$, which implies that $\bar{C}_{n-j+p}k_p - \bar{C}_{n-j+p}^{-1}k_p = 0$ and then $\tau^l_\alpha = \tau^{l'}_\alpha = 0$. Since we have $e^l_\alpha = \tau^l_\alpha$ and $f^l_\alpha = \tau^{l'}_\alpha$ by Lemma 5.2, we obtain that $e^l_\alpha = f^l_\alpha = 0$ on $\mathcal{V}$. \[ \square \]
Theorem 5.5  
(i) If we define the actions of $e_\alpha$ and $f_\alpha$ ($\alpha \in \Delta_+$) by using (2.13) and (2.16), the vector space $V$ becomes $U_\varepsilon^{\text{fin}}$-module.

(ii) The subspace $L(\lambda)$ ($\lambda = (\lambda_1, \cdots, \lambda_n), \lambda_i \in \{0, 1, \cdots, l - 1\}$) is the irreducible $U_\varepsilon^{\text{fin}}$-submodule of $V$.

Proof. To show the former half of the theorem, it suffices to check the relations (2.7)–(2.20) in Proposition 2.1. The relations (2.7)–(2.10) are satisfied since $V$ is originally $U_\varepsilon$-module. The relations (2.11)–(2.18) are obtained from Lemma 5.2. We have the relations (2.19) and (2.20) from Proposition 5.3. Thus, we have the well-defined actions of $U_\varepsilon^{\text{fin}}$ on $V$.

5.3 Proof of irreducibility

In order to show the irreducibility of $L(\lambda)$, we need the following:

Proposition 5.6 Any finite dimensional $U_\varepsilon^{\text{fin}}$-module contains a primitive vector.

To show the proposition, we shall show the following lemma

Lemma 5.7 Let $L > 0$ be a sufficiently large integer. For any $i_1, i_2, \cdots, i_L \in I$ we have in $U_\varepsilon^{\text{fin}}$,

$$e_{i_L} \cdots e_{i_2} e_{i_1} = 0.$$  \hspace{1cm} (5.9)

Proof. We define a $\mathbb{Z}$-gradation on $(U_\varepsilon^{\text{fin}})^+$ by the way that we have seen in Proposition 2.3. $(U_\varepsilon^{\text{fin}})^+$ has the basis

$$\{ e_{\beta_1}^{r_1} e_{\beta_2}^{r_2} \cdots e_{\beta_N}^{r_N}, 0 \leq r_1, \cdots, r_N < l \}.$$  \hspace{1cm} (5.10)

Using this, we define

$$ (U_\varepsilon^{\text{fin}})^+_d := \bigoplus_{r_1 \text{ht}(\beta_1) + \cdots + r_N \text{ht}(\beta_N) = d} \mathbb{C} e_{\beta_1}^{r_1} e_{\beta_2}^{r_2} \cdots e_{\beta_N}^{r_N}.$$  \hspace{1cm} (5.10)

where $\text{ht}(\beta)$ is the height of a root $\beta \in \Delta_+$. We have

$$ (U_\varepsilon^{\text{fin}})^+ = \bigoplus_d (U_\varepsilon^{\text{fin}})^+_d.$$  \hspace{1cm} (5.10)

An element in $(U_\varepsilon^{\text{fin}})^+_d$ is called a homogeneous element of degree $d$. Since all the relations in $(U_\varepsilon^{\text{fin}})^+$, that is, (2.7)–(2.11), (2.13)–(2.14) and (2.19) are homogeneous, it is well-defined and then we obtain $(U_\varepsilon^{\text{fin}})^+_d (U_\varepsilon^{\text{fin}})^+_e \subset (U_\varepsilon^{\text{fin}})^+_{d+e}$ for $d, e \in \mathbb{Z}_{\geq 0}$. Hence, $e_{i_L} \cdots e_{i_2} e_{i_1}$ is a homogeneous element of degree $L$. It immediately follows from Proposition 2.3 that the maximum degree is $(l - 1) \sum_{\alpha=1}^N \text{ht}(\beta_\alpha) := J$, which implies that if $L > J$, $(U_\varepsilon^{\text{fin}})^+_L = 0$. Thus, if $L$ is sufficiently large, a homogeneous element $e_{i_L} \cdots e_{i_2} e_{i_1}$ must vanish.

Proof of Proposition 5.6 Suppose that a finite dimensional $U_\varepsilon^{\text{fin}}$-module $V$ does not have any primitive vector. So any non-zero $v \in V$ there exists an infinite sequence $i_1, i_2, \cdots, i_k, \cdots$ ($i_j \in I$) such that all vectors $v, e_{i_1} v,$
where $e_{i_2}e_{i_1}v, \ldots, e_{i_k} \cdots e_{i_2}e_{i_1}v, \ldots$ never vanish. But this contradicts to Lemma 5.7.

Therefore, $V$ contains a primitive vector.

Let $W$ be a non-zero submodule of $L(\lambda)$. By Proposition 5.4, $W$ contains a primitive vector. By the uniqueness of the primitive vector in $V$ (Proposition 4.3), $W$ has to contain $u_\theta$. Therefore, $W = L(\lambda)$ and then $L(\lambda)$ is irreducible.

Here we completed the proof of Theorem 5.5(ii).

By Proposition 7.1 in [5], for a 'l-restricted weight' (see 3.2) $\lambda \in P_+$ the $U^\text{res}_\varepsilon(\lambda)$ is identified with the irreducible $U^\text{fin}_\varepsilon$-module, which is isomorphic to $L(\lambda)$. Accordingly, by Theorem 5.5 we realize the irreducible highest weight $U^\text{res}_\varepsilon(\lambda)$ with the $l$-restricted highest weight $\lambda$ in the vector space $V$.

Our further problem is to write down a basis of $L(\lambda)$ explicitly. Any basis vector of the irreducible $U_q(\mathfrak{g})$-module $V(\lambda)$ is parametrized by "Young tableaux" of shape $\lambda$. So by the construction of $V^\text{res}_\varepsilon(\lambda)$ in 3.2, we deduce that a basis vector of $L(\lambda)$ would be parametrized by 'restricted'(in some sense) Young tableaux. We would also like to see the structure of the quotient module $V/L(\lambda)$ or the tensor product of $V \otimes V$.

In [6], for the $B_n$, $C_n$ and $D_n$-cases the analogous presentations of the maximal cyclic representations are given explicitly. Thus, we can apply the procedure adopted here to them and might hope to obtain the irreducible $U^\text{fin}_\varepsilon(B_n)$, $U^\text{fin}_\varepsilon(C_n)$, $U^\text{fin}_\varepsilon(D_n)$-modules, which will be discussed elsewhere.

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