Global-in-time stability of 2D MHD boundary layer in the Prandtl-Hartmann regime

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Abstract: In this paper, we prove global existence of solutions with analytic regularity to the 2D MHD boundary layer equations in the mixed Prandtl and Hartmann regime derived by formal multi-scale expansion in [10]. The analysis shows that the combined effect of the magnetic diffusivity and transversal magnetic field on the boundary leads to a linear damping on the tangential velocity field near the boundary. And this damping effect yields the global in time analytic norm estimate in the tangential space variable on the perturbation of the classical steady Hartmann profile.

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1 Introduction

The following mixed Prandtl and Hartmann boundary layer equations from the classical incompressible MHD system were derived in [10] for flat boundary in two space dimensions (2D) when the physical parameters such as Reynolds number, magnetic Reynolds number and the Hartmann number satisfy some constraints in the high Reynolds numbers limit:

\[
\begin{align*}
\partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 &= \partial_y b_1 + \partial_y^2 u_1, \\
\partial_y u_1 + \partial_y^2 b_1 &= 0, \\
\partial_x u_1 + \partial_y u_2 &= 0, \\
& \quad x \in \mathbb{R}, \; y \in \mathbb{R}^+.
\end{align*}
\]

Here, \((u_1, u_2)\) denotes the velocity field of the boundary layer and \(b_1\) is the corresponding tangential magnetic component.

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For the classic Hartmann boundary layer system, there is a family of steady solutions called Hartmann layer. It turns out that Hartmann layer is also a solution to the above system. In this paper, we will study the global-in-time stability of the Hartmann layer in the analytic function space. For this, consider the system (1.1) with initial data
\[ u_1(t = 0, x, y) = u_{10}(x, y), \] (1.2)
and the no-slip boundary conditions
\[ u_1|_{y=0} = 0, \quad u_2|_{y=0} = 0. \] (1.3)
And the far field is taken as a uniform constant state. Consequently, the pressure term vanishes in equation of (1.1). Denote
\[ \lim_{y \to +\infty} u_1 = \bar{u}, \quad \lim_{y \to +\infty} b_1 = \bar{b}. \] (1.4)
Integrating the equation of (1.1) over \( y \) yields
\[ -u_1(t, x, y) + \bar{u} = \partial_y b_1. \] (1.5)
Thus, the equations (1.1) can be written as
\[
\begin{aligned}
\partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 &= -u_1(t, x, y) + \bar{u} + \partial_y^2 u_1, \\
\partial_x u_1 + \partial_y u_2 &= 0.
\end{aligned}
\] (1.6)
Recall that the classical Hartmann boundary layer is given by
\[ u_1 = (1 - e^{-y})\bar{u}, \quad u_2 = 0, \] (1.7)
which is a steady solution to (1.6). Without loss of generality, set \( \bar{u} = 1 \) and denote the perturbation by \((u, v)\):
\[ u_1 = (1 - e^{-y}) + u, \quad u_2 = v. \] (1.8)
Obviously, \((u, v)\) satisfies
\[
\begin{aligned}
\partial_t u + (1 - e^{-y} + u)\partial_x u + v\partial_y (1 - e^{-y} + u) &= -u + \partial_y^2 u, \\
\partial_x u + \partial_y v &= 0,
\end{aligned}
\] (1.9)
with initial data
\[ u_0(x, y) = u_{10}(x, y) - (1 - e^{-y}), \] (1.10)
and boundary condition
\[ u|_{y=0} = 0, \quad v|_{y=0} = 0. \] (1.11)
Before we state the main result, let us first introduce the following weighted analytic regularity function spaces. For some \( r > 1 \), denote an analytic weight \( M_m \) by
\[
M_m = \frac{(m+1)^r}{m!}.
\]
With a parameter \( \tau > 0 \), set
\[
X_m = \| e^{\alpha y} \partial_y^m g \|_{L^2(\mathbb{R}^2_+)} \tau^m M_m, \quad Z_m = \| e^{\alpha y} \partial_y^m g \|_{L^2(\mathbb{R}^2_+)} \tau^m M_m, \\
Y_m = \| e^{\alpha y} \partial_y^m g \|_{L^2(\mathbb{R}^2_+)} \tau^{-1/2} m^{1/2} M_m, \quad D_m = \| e^{\alpha y} \partial_x^m g \|_{L^\infty_y L^2_x} \tau^m M_m,
\]
and define
\[
\| g \|_{X_0,\alpha}^2 = \sum_{m \geq 0} X_m^2, \quad \| g \|_{X_{\tau,\alpha}}^2 = \sum_{m \geq 0} Z_m^2, \\
\| g \|_{Y_0,\alpha}^2 = \sum_{m \geq 0} Y_m^2, \quad \| g \|_{D_0,\alpha}^2 = \sum_{m \geq 0} D_m^2.
\]
Here, \( \tau \) denotes the analytic radius.

**Theorem 1.1** Let the initial data \( u_{10}(x,y) \) be a small perturbation of the Hartmann profile \((\bar{u}(1 - e^{-y}), 0)\) satisfying the compatibility conditions and
\[
\| \partial_y u_{10} + u_{10} - \bar{u} \|_{X_{\tau_0,\alpha}} \leq \delta_0,
\]
with \( r > 1 \), \( 0 < \alpha < \sqrt{2}/2 \) for some small constant \( \delta_0 > 0 \). Then there exists a unique global-in-time solution \( (u_1, u_2) \) to the problem (1.1)-(1.4) satisfying
\[
\| g \|_{X_{\tau(t),\alpha}} \leq e^{-2(1-2\alpha^2)t} \delta_0, \quad \text{with} \quad \tau(t) > \tau_0/2,
\]
for all time \( t \geq 0 \), where \( g = \partial_y u_1 + u_1 - \bar{u} \).

**Remark 1.1** The initial analytic radius \( \tau_0 \) and the size of initial perturbation \( \delta_0 \) should satisfy the constraint of (3.19).

Now let us briefly review the background and some related works. First of all, the Prandtl equations derived by Prandtl [27] in 1904 describe the fluid phenomena near a boundary with no-slip boundary condition through the high Reynolds number limit of the incompressible Navier-Stokes equations. For this system, so far the mathematical theories are basically limited to two space dimensions except in the framework of analytic or Gevrey function spaces or under some structure constraints. And the justification of the Prandtl ansatz has been extensively investigated with only a few results, cf. [14, 22, 28, 29] and references therein. In fact, under the monotonicity condition on the tangential velocity component in the normal direction, Oleinik firstly obtained the local existence of classical solutions in the two space dimensions by using the Crocco transformation, cf. [25]. This result together with some other works in this direction are presented in Oleinik-Samokhin’s classical book [26]. Recently, this well-posedness result
was re-proved by using energy method in the framework of weighted Sobolev spaces in [11] and [23] independently, by observing the cancellation mechanism in the system. By imposing an additional favorable pressure condition, a global in time weak solution was obtained by Xin and Zhang in [30].

When this monotonicity condition is violated, separation of the boundary layer is expected. For this, E-Engquist constructed a finite time blowup solution to the Prandtl equations in [5]. And this kind of blowup result is extended to the van Dommelen-Shen type singularity in [16] when the outer Euler flow is spatially periodic. In addition, some interesting ill-posedness (or instability) phenomena of solutions to both the linear and nonlinear Prandtl equations around a shear flow have been studied, cf. [7, 8, 12, 13] and the survey paper [9].

In the framework of analytic functions, Sammartino and Caflisch [28, 29, 3] proved the local well-posedness result of the Prandtl system and justified the Prandtl ansatz. The analyticity requirement in the normal variable $y$ was later removed by Lombardo, Cannone and Sammartino in [21]. The main argument used in [21, 28] is to apply the abstract Cauchy-Kowalewskaya theorem. For recent development of mathematical theories in Gevrey function space to Prandtl equations, one can refer to [9, 17].

A natural question then is whether global existence of smooth (or strong) solution can be achieved for the Prandtl equations in either analytic or Gevrey regularity function spaces. However, the answer to this is still not known. In this direction, a lower bound of the life-span for small analytic solution to the Prandtl equations with small perturbation analytic initial data was given by Zhang and Zhang in [31]. Precisely, when the outflow velocity is of the order of $\varepsilon^{5/3}$, and the initial perturbation is of the order of $\varepsilon$, then the Prandtl system admits a unique analytic solution with life-span greater than $\varepsilon^{-4/3}$. On the other hand, when the initial data is a small perturbation of a Guassian error function, almost global existence for the Prandtl equations is obtained by Ignatova and Vicol in [15], where the cancellation observed in [23] and the monotonicity of background solution are essentially used to have a linear time decay damping effect.

Back to the MHD system, it is believed that suitable magnetic field can stabilize the boundary layer in some physical regime [24, 2, 4, 11]. One can also refer to some very recent results in [10, 18, 19, 20] for the derivation of MHD boundary layer equations, stability analysis of magnetic field on the boundary layer from the mathematical point of view.

The purpose of this paper is to prove a global-in-time existence of solution to a mixed Prandtl and Hartmann MHD boundary layer equations derived in [10]. The key observation is that the combined effect of the magnetic diffusivity and transversal magnetic field on the boundary leads to a linear damping on the tangential velocity field near the boundary. This damping effect yields a time exponential decay in analytic regularity norm of the solution when we consider perturbation near the classical Hartmann profile.

Finally, the rest of the paper is organized as follows. We will reformulate the problem by using the cancellation mechanism observed in [11, 23] for the study of Prandtl equations in Sobolev space in Section 2. The uniform estimates on the solution in analytic norm will be given in Section 3. Based on the uniform estimates, the global existence and uniqueness of solution to (1.1) will be proved in the last section. Throughout the paper, $C$, $\bar{C}$, $C_0$ and $C_1$ are used to denote some generic constants.
2 Preliminaries

As for the classical Prandtl equation, one needs to use the cancellation mechanism in the system to overcome the loss of derivative in order to close the a priori estimate. For this, note that the vorticity in the boundary layer \( \omega = \partial_y u \) satisfies
\[
\partial_t \omega + (1 - e^{-y} + u)\partial_x \omega - ve^{-y} + v\partial_y \omega = -\omega + \partial^2_y \omega. \tag{2.1}
\]
As in [1, 23], one can use the vorticity to cancel some term with essential difficulty in the equation for \( u \). Precisely, by noticing the Hartmann layer \((u_1, 0)\) has the property \( u_1yy = -1 \), set
\[
g = \omega + u, \tag{2.2}
\]
then the new unknown function \( g \) satisfies
\[
\partial_t g + (1 - e^{-y} + u)\partial_x g + v\partial_y g = -g + \partial^2_y g. \tag{2.3}
\]
The relation between the new unknown function \( g \) and \( u \) is
\[
u = e^{-y} \int_0^y e^z g(t, x, z)dz, \tag{2.4}
\]
and the initial data of \( g \) is
\[
g(0, x, y) = \partial_y u_{10}(x, y) + u_{10}(x, y) - 1. \tag{2.5}
\]
As for the boundary condition on \( g \), note that from (1.9) and (1.11), we have
\[
(\partial_y \omega - u)|_{y=0} = 0,
\]
which implies
\[
(\partial_y g - g)|_{y=0} = 0. \tag{2.6}
\]
In the rest of the paper, we consider the reformulated problem (2.3)-(2.6).

Remark 2.1 If \( \|g\|_{X_{r, \alpha}} < \infty \), then it follows from (2.4) that \( \|u\|_{X_{r, \alpha'}} < \infty \) and \( \|u\|_{Z_{r, \alpha'}} < \infty \) with \( 0 \leq \alpha' < \alpha \). In addition, \( \|u\|_{D_{r, \alpha}} < \infty \).

3 Uniform Estimates

In this section, we will derive the uniform estimates on the solution to (2.3)-(2.6) in analytic regularity norm through energy method.

Applying the operator \( \partial_x^m \) on (2.3), multiplying the resulting equation by \( e^{2\alpha y} \partial_x^m g \) and integrating it over \( \mathbb{R}_+^2 \) yield
\[
\int_{\mathbb{R}_+^2} \partial_x^m (\partial_t g + (1 - e^{-y} + u)\partial_x g + v\partial_y g + g - \partial^2_y g) e^{2\alpha y} \partial_x^m g dx dy = 0. \tag{3.1}
\]
We estimate the above equation term by term. Firstly,
\[ \int_{\mathbb{R}^2} \partial_x^m \partial_t e^{2\alpha y} \partial_x^m g dx dy = \frac{1}{2} \frac{d}{dt} \| e^{\alpha y} \partial_x^m g \|_{L^2}^2, \]  
(3.2)
and
\[ \int_{\mathbb{R}^2} \partial_x^m e^{2\alpha y} \partial_x^m g dx dy = \| e^{\alpha y} \partial_x^m g \|_{L^2}^2, \]  
(3.3)

and
\[ -\int_{\mathbb{R}^2} \partial_x^2 \partial_y \partial_x e^{2\alpha y} \partial_x^m g dx dy \]
\[ = \int_{\mathbb{R}} \partial_y \partial_x^m g(t, x, 0) \partial_x^m g(t, x, 0) dx + \| e^{\alpha y} \partial_y \partial_x^m g \|_{L^2}^2 + 2\alpha \int_{\mathbb{R}^2} \partial_y \partial_x^m e^{2\alpha y} \partial_x^m g dx dy \]
\[ = \| \partial_x^m g(t, x, 0) \|_{L^2}^2 + \| e^{\alpha y} \partial_y \partial_x^m g \|_{L^2}^2 - \alpha \| \partial_x^m g(t, x, 0) \|_{L^2}^2 - 2\alpha \| e^{\alpha y} \partial_x^m g \|_{L^2}^2 \]
\[ = (1 - \alpha) \| \partial_x^m g(t, x, 0) \|_{L^2}^2 + \| e^{\alpha y} \partial_y \partial_x^m g \|_{L^2}^2 - 2\alpha \| e^{\alpha y} \partial_x^m g \|_{L^2}^2, \]  
(3.4)
where in the second equality, we have used the boundary condition (2.6). For the two mixed nonlinear terms in (3.1), we firstly have
\[ \int_{\mathbb{R}^2} \partial_x^m ((1 - e^{-y} + u) \partial_x^m g) e^{2\alpha y} \partial_x^m g dx dy \]
\[ = \sum_{j=0}^{[m/2]} \left( \begin{array}{c} m \\ j \end{array} \right) \int_{\mathbb{R}^2} \partial_x^{m-j} u \partial_x^{j+1} e^{2\alpha y} \partial_x^m g dx dy \triangleq R_1, \]
and
\[ |R_1| \leq \sum_{j=0}^{[m/2]} \left( \begin{array}{c} m \\ j \end{array} \right) \| \partial_x^{m-j} u \|_{L^2 L_y^\infty} \| e^{\alpha y} \partial_x^{j+1} g \|_{L^\infty L_y^2} \| e^{\alpha y} \partial_x^m g \|_{L^2} \]
\[ + \sum_{j=[m/2]+1}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) \| \partial_x^{m-j} u \|_{L^\infty L_y^\infty} \| e^{\alpha y} \partial_x^{j+1} g \|_{L^2} \| e^{\alpha y} \partial_x^m g \|_{L^2}. \]

For \( 0 \leq j \leq [m/2] \), by (2.4), one has
\[ \| \partial_x^{m-j} u \|_{L^2 L_y^\infty} = \| \partial_x^{m-j} \int_0^t e^{-(y-z)} g(t, x, z) dz \|_{L^2 L_y^\infty} \]
\[ \leq \| e^{\alpha y} \partial_x^{m-j} \int_0^y e^{-(y-z)} g(t, x, z) e^{\alpha z} e^{-\alpha z} dz \|_{L^2 L_y^\infty} \]
\[ = \| \int_0^y e^{-(1-\alpha)(y-z)} \partial_x^{m-j} g(t, x, z) e^{\alpha z} dz \|_{L^2 L_y^\infty} \leq C \| e^{\alpha y} \partial_x^{m-j} g \|_{L^2}, \]
provided that \( \alpha < 1 \). Using the Agmon inequality gives that
\[ \| e^{\alpha y} \partial_x^{j+1} g \|_{L^\infty L_y^2} \leq C \| e^{\alpha y} \partial_x^{j+1} g \|_{L^2}^{1/2} \| e^{\alpha y} \partial_x^{j+2} g \|_{L^2}^{1/2}. \]
For \([m/2] + 1 \leq j \leq m\),

\[
\|\partial_x^{m-j} u\|_{L^\infty_y L_2^x} = \|\partial_x^{m-j} \int_0^y e^{-(y-z)} g(t, x, z) dz\|_{L^\infty_y L_2^x} \\
\leq \|e^{\alpha_y} \partial_x^{m-j} \int_0^y e^{-(y-z)} g(t, x, z) e^{\alpha_z} e^{-\alpha z} dz\|_{L^\infty_y L_2^x} \\
= \|\int_0^y e^{-(1-\alpha)(y-z)} \partial_x^{m-j} g(t, x, z) e^{\alpha_z} e^{-\alpha z} dz\|_{L^\infty_y L_2^x} \\
\leq C \|e^{\alpha_y} \partial_x^{m-j} g\|_{L^2_y} \|e^{\alpha_y} \partial_x^{m-j+1} g\|_{L^2_y}.
\]

Consequently,

\[
\sum_{m \geq 0} |R_1| \tau^{2m} M_m^2 \\
\leq \frac{C}{(\tau(t))^{1/2}} \left\{ \sum_{m \geq 0} \sum_{j=0}^{[m/2]} X_{m-j} Y_{1/2}^{1/2} Y_{1/2}^2 M_j \frac{M_j M_j^{1/2}}{M_{m-j} M_{j+1}^{1/2}} (m_j) M_m \right\}^{1/2} (m_j) M_m \frac{M_j M_j^{1/2}}{M_{m-j} M_{j+1}^{1/2}} (m_j) M_m
\]

\[
+ \sum_{m \geq 0} \sum_{j=[m/2]+1}^m X_{m-j}^{1/2} Y_{m-j+1} Y_{1/2} M_j \frac{M_j M_j^{1/2}}{M_{m-j} M_{j+1}^{1/2}} (m_j) M_m \frac{M_j M_j^{1/2}}{M_{m-j} M_{j+1}^{1/2}} (m_j) M_m
\]

(3.5)

The second nonlinear term can be estimated as follows. Note that

\[
\int_{\mathbb{R}^2_+} \partial_x^m (v \partial_y g) e^{2\alpha y} \partial_x^m g \, dxdy \\
= \sum_{j=0}^m (m_j) \int_{\mathbb{R}^2_+} \partial_x^{m-j} v \partial_x^j \partial_y g e^{2\alpha y} \partial_x^m g \, dxdy \triangleq R_2,
\]

and

\[
|R_2| \leq \sum_{j=0}^{[m/2]} (m_j) \|\partial_x^{m-j} v\|_{L^2_y L^\infty_x} \|e^{\alpha_y} \partial_x^j \partial_y g\|_{L^\infty_y L^2_x} \|e^{\alpha_y} \partial_x^m g\|_{L^2_x} \\
+ \sum_{j=[m/2]+1}^m (m_j) \|\partial_x^{m-j} v\|_{L^\infty_y L^2_x} \|e^{\alpha_y} \partial_x^j \partial_y g\|_{L^2_x} \|e^{\alpha_y} \partial_x^m g\|_{L^2_x}.
\]

For \(0 \leq j \leq [m/2]\),

\[
\|\partial_x^{m-j} v\|_{L^2_y L^\infty_x} = \|\int_0^y (e^{-z} \int_0^z e^{s} \partial_x^{m-j+1} g(t, x, s) ds) \, dz\|_{L^2_y L^\infty_x} \\
= \|\int_0^y e^{-\alpha z} (\int_0^z e^{(1-\alpha)(s-z)} \partial_x^{m-j+1} g(t, x, s) e^{\alpha s} ds) \, dz\|_{L^2_y L^\infty_x}
\]

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\[ \leq C \| e^{\alpha y} \partial_x^{m-j+1} g \|_{L^2}, \]

where we have used the fact that \( 0 < \alpha < 1 \). Note that
\[ \| e^{\alpha y} \partial_x^{j} \partial_y g \|_{L^\infty L^2} \leq C \| e^{\alpha y} \partial_x^{j+1} \partial_y g \|^{1/2}_{L^2} \| e^{\alpha y} \partial_x^{j+1} \partial_y g \|^{1/2}_{L^2}. \]

For \( [m/2] + 1 \leq j \leq m, \)
\[ \| \partial_x^{m-j} v \|_{L^\infty} = \| \int_0^y \left( e^{-z} \int_0^z e^s \partial_x^{m-j+1} g(t, x, s) ds \right) dz \|_{L^\infty} \]
\[ = \| \int_0^y e^{-\alpha z} \left( \int_0^z e^{(1-\alpha)(s-z)} \partial_x^{m-j+1} g(t, x, s) e^{\alpha s} ds \right) dz \|_{L^\infty} \]
\[ \leq C \| e^{\alpha y} \partial_x^{m-j+1} g \|^{1/2}_{L^2} \| e^{\alpha y} \partial_x^{m-j+2} g \|^{1/2}_{L^2}. \]

Consequently,
\[ \sum_{m \geq 0} |R_2| \tau^{2m} M_m^2 \]
\[ \leq \frac{C}{(\tau(t))^{1/2}} \left\{ \sum_{m \geq 0} \sum_{j=0}^{[m/2]} \frac{m^{[m/2]} Y_{m-j+1} Z_{j}^{1/2} Z_{j+1}^{1/2} Y_{m}}{M_{m-j+1} M_{j+1}^{1/2} M_{j+1}^{1/2} (m-j+1)^{1/2} m^{1/2}} \right\} (3.6) \]
\[ + \sum_{m \geq 0} \sum_{j=[m/2]+1}^{m} \frac{m^{[m/2]} Y_{m-j+1}^{1/2} Y_{m-j+2}^{1/2} Z_{j} Y_{m}}{M_{m-j+1} M_{m-j+2}^{1/2} M_{j}^{1/2} (m-j+1)^{1/4} (m-j+2)^{1/4} m^{1/2}} \frac{m^{[m/2]} M_m}{(m-j+1)^{1/4} (m-j+2)^{1/4} m^{1/2}} \right\} \]

Combining the above estimates and using the definitions of \( X_{r, \alpha}^r, Z_{r, \alpha}^r \) and \( Y_{r, \alpha}^r \) give
\[ \frac{1}{2} \frac{d}{dt} \| g \|^2_{X_{r, \alpha}} - \tau \| g \|^2_{Y_{r, \alpha}} + \| g \|^2_{Y_{r, \alpha}} + (1 - 2\alpha^2) \| g \|^2_{X_{r, \alpha}} + (1 - \alpha) \| g(t, x, 0) \|^2_{X_{r, \alpha}} \]
\[ \leq \sum_{m \geq 0} |R_1| \tau^{2m} M_m + \sum_{m \geq 0} |R_2| \tau^{2m} M_m^2. \]

Since
\[ \frac{m^{[m/2]} M_m}{M_{m-j+1} M_{j+1}^{1/2} M_{j+1}^{1/2} (j+1)^{1/4} (j+2)^{1/4} m^{1/2}} \]
\[ = \frac{(m+1)^{r}(j+1)(j+2)^{1/2}}{(m-j+1)^{r}(j+2)^{r/2}(j+3)^{r/2}(j+1)^{1/4} (j+2)^{1/4} m^{1/2}} \]
\[ \leq C(1+j)^{1/2-r} \]
\[ (3.8) \]

for all \( 0 \leq j \leq [m/2] \), and
\[ \frac{m^{[m/2]} M_m}{M_{m-j+1} M_{m-j+1}^{1/2} M_{j+1}^{1/2} (j+1)^{1/4} m^{1/2}} \]

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for all \([m/2] + 1 \leq j \leq m\). Consequently,

\[
\sum_{m \geq 0} |R_1| \tau^{2m} M_m \leq \frac{C}{(\tau(t))^{1/2}} \left\{ \sum_{m \geq 0} \sum_{j=0}^{[m/2]} X_{m-j} Y_{j+1}^{1/2} Y_{j+1} \eta_{m+j+1} (1+j)^{1/2-r} \right. \\
+ \sum_{m \geq 0} \sum_{j=[m/2]+1}^{m} X_{m-j}^{1/2} X_{m-j+1} Y_j Y_m (m-j+1)^{1/2-r} \right\} \\
\leq \frac{C}{(\tau(t))^{1/2}} \|g\|_{X_{r,\alpha}} \|g\|_{Y_{r,\alpha}}^{2}, \tag{3.10}
\]

Note that in the above second inequality, we have used the following discrete Young’s inequality

\[
\|\zeta \cdot (\eta * \xi)\|_{l^1} \leq C\|\zeta\|_{l^2} \|\eta\|_{l^1} \|\xi\|_{l^2}
\]

with \(\zeta_k = Y_k, \eta_k = Y_{k+1}^{1/2} Y_{k+1}^{1/2-r}, \xi_k = X_k\) for the first term on the right hand side in (3.10), and then Hölder inequality for \(\|\eta\|_{l^1} \leq C\|g\|_{Y_{r,\alpha}}\), provided \(r > 1\). And for the second term on the right hand side of (3.10), we can choose of \(\zeta_k = Y_k, \eta_k = X_k^{1/2} X_{k+1}^{1/2}(k+1)^{1/2-r}, \xi_k = Y_{k+1}\) with \(\|\eta\|_{l^1} \leq C\|g\|_{X_{r,\alpha}}\), provided \(r > 1\). In conclusion, (3.10) holds if \(r > 1\).

Similarly,

\[
\sum_{m \geq 0} |R_2| \tau^{2m} M_m \leq \frac{C}{(\tau(t))^{1/2}} \left\{ \sum_{m \geq 0} \sum_{j=0}^{[m/2]} Y_{m-j+1} Z_{j+1}^{1/2} Z_j Y_m (1+j)^{1/2-r} \right. \\
+ \sum_{m \geq 0} \sum_{j=[m/2]+1}^{m} Y_{m-j+1}^{1/2} Y_{m-j+2} Z_j Y_m (m-j+1)^{1/2-r} \right\} \\
\leq \frac{C}{(\tau(t))^{1/2}} \|g\|_{Z_{r,\alpha}} \|g\|_{Y_{r,\alpha}}^{2}, \tag{3.11}
\]

provided that \(r > 1\).

It follows, from (3.7), (3.10) and (3.11), that

\[
\frac{1}{2} \frac{d}{dt} \|g\|_{X_{r,\alpha}}^{2} + \|g\|_{Z_{r,\alpha}}^{2} + (1 - 2\alpha^2) \|g\|_{X_{r,\alpha}}^{2} + (1 - \alpha) \|g(\cdot, 0)\|_{X_{r,\alpha}}^{2} \\
\leq (\hat{\tau} + \frac{C}{(\tau(t))^{1/2}} (\|g\|_{X_{r,\alpha}} + \|g\|_{Z_{r,\alpha}})) \|g\|_{Y_{r,\alpha}}^{2}. \tag{3.12}
\]

Set

\[
\hat{\tau} + \frac{C}{(\tau(t))^{1/2}} (\|g\|_{X_{r,\alpha}} + \|g\|_{Z_{r,\alpha}}) = 0. \tag{3.13}
\]
Then we have
\[ \frac{1}{2} \frac{d}{dt} \|g\|^2_{X^{r,\alpha}} + \|g\|_{Z^{r,\alpha}}^2 + (1 - 2\alpha^2) \|g\|_{X^{r,\alpha}}^2 + (1 - \alpha) \|g(\cdot, 0)\|^2_{X^{r,\alpha}} \leq 0. \]  
(3.14)

When \(0 < \alpha < \sqrt{2}/2\), we have
\[ \frac{1}{2} \frac{d}{dt} \|g\|^2_{X^{r,\alpha}} + \|g\|_{Z^{r,\alpha}}^2 + (1 - 2\alpha^2) \|g\|_{X^{r,\alpha}}^2 \leq 0. \]  
(3.15)

It follows that
\[ e^{2(1-2\alpha^2)t} \frac{d}{dt} \|g\|^2_{X^{r,\alpha}} + 2e^{2(1-2\alpha^2)t} \|g\|_{Z^{r,\alpha}}^2 + 2(1-2\alpha^2)e^{2(1-2\alpha^2)t} \|g\|_{X^{r,\alpha}}^2 \leq 0, \]  
(3.16)

that implies
\[ e^{2(1-2\alpha^2)t} \|g\|^2_{X^{r,\alpha}} + \int_0^t 2e^{2(1-2\alpha^2)s} \|g(s)\|_{Z^{r,\alpha}}^2 ds \leq \|g(0)\|^2_{X^{r,\alpha}}. \]  
(3.17)

From (3.13), one has
\[ \tau(t)^{3/2} - \tau_0^{3/2} = -C \int_0^t (\|g(s)\|_{X^{r,\alpha}} + \|g(s)\|_{Z^{r,\alpha}}) ds \]
\[ \geq -C \int_0^t \|g(0)\|_{X^{r,\alpha}} e^{-(1-2\alpha^2)s} ds - C \int_0^t \|g(s)\|_{Z^{r,\alpha}} e^{(1-2\alpha^2)s} e^{-(1-2\alpha^2)s} ds \]
\[ \geq -C_1 \|g(0)\|_{X^{r,\alpha}}. \]  
(3.18)

Hence, if the initial perturbation data is suitably small such that
\[ \frac{\tau_0}{K} > C_1^{2/3} \|g(0)\|_{X^{r,\alpha}}^{2/3}, \]  
(3.19)

with \(K = (2\sqrt{2})^{2/3}/(2\sqrt{2} - 1)^{2/3}\). Then (3.18) implies that \(\tau(t) > \tau_0/2\) for all \(t \geq 0\). Consequently, we have

**Proposition 3.1** Under the same conditions of Theorem 1.1, suppose that \(g\) is a solution to (2.3)-(2.6) with analytic regularity in the norm \(X^{r,\alpha}\), then
\[ e^{2(1-2\alpha^2)t} \|g\|^2_{X^{r,\alpha}} + \int_0^t 2e^{2(1-2\alpha^2)s} \|g(s)\|^2_{Z^{r,\alpha}} ds \leq \|g(0)\|^2_{X^{r,\alpha}}. \]  
(3.20)

with \(\tau(t) > \tau_0/2\) for all \(t \geq 0\).

**4 The Proof of Theorem 1.1**

By the uniform estimates on the solution to (2.3)-(2.6) given in Proposition 3.1 and the local existence of solutions in analytic function space that can be obtained by using the argument used in [15], the global existence of solution to (2.3)-(2.6) follows. Then by the relation (2.4), the
global existence of $u$ to the initial-boundary value problem (1.9)-(1.11) is proved. In addition, according to Remark 2.1 and Proposition 3.1, it follows that $\|u\|_{X_{r_0}^{\alpha}} + \|\partial_y u\|_{X_{r_0}^{\alpha'}} < \infty$ with $0 \leq \alpha < \alpha'$ and $\tau > \tau_0/2$. As consequence, the proof of global existence part in Theorem 1.1 is completed.

We now turn to prove the uniqueness. For this, it suffices to show the uniqueness of solution to (2.3)-(2.6). Assume that there are two solutions $g_1$ and $g_2$ to (2.3)-(2.6) with the same initial data $g_0$ satisfying $\|g_0\|_{X_{r_0}^{\alpha}} \leq \delta_0$. Denote the radii of analytic regularity for $g_1$ and $g_2$ by $\tau_1(t)$ and $\tau_2(t)$ respectively. Define $\tau(t)$ by

$$\dot{\tau} + \frac{C}{(\tau(t))^{1/2}}(\|g_1\|_{X_{r_1}^{\alpha}} + \|g_1\|_{Z_{r_1}^{\alpha}}) = 0$$

with initial data $\tau(0) = \frac{\tau_0}{4}$. By the estimates given in Section 3, the analyticity radius $\tau(t)$ satisfies

$$\frac{\tau_0}{8} \leq \tau(t) \leq \frac{\tau_0}{4} \leq \min\{\tau_1(t),\tau_2(t)\}, \quad \text{for} \ t \geq 0. \quad (4.2)$$

Note that $\bar{g} = g_1 - g_2$ satisfies

$$\partial_t \bar{g} + (1 - e^{-y} + u_1) \partial_x \bar{g} + (v_1 - v_2)\partial_y g_1 = -\bar{g} + \partial_y^2 \bar{g} + R,$$

with

$$R = -(u_1 - u_2) \partial_x g_2 - v_2 \partial_y g_1. \quad (4.3)$$

The initial data and the boundary condition for $\bar{g}$ are

$$\bar{g}(t = 0, x, y) = 0, \quad (\partial_y \bar{g} - \bar{g})|_{y=0} = 0. \quad (4.5)$$

Following the arguments used in Section 3, we have

$$\frac{1}{2} \frac{d}{dt} \|\bar{g}\|^2_{X_{r_1}^{\alpha}} + \|\bar{g}\|^2_{Z_{r_1}^{\alpha}} + (1 - 2\alpha^2)\|\bar{g}\|^2_{X_{r_1}^{\alpha}} + (1 - \alpha)\|\bar{g}\|_{X_{r_1}^{\alpha}} \leq (\dot{\tau} + \frac{C}{(\tau(t))^{1/2}}(\|g_1\|_{X_{r_1}^{\alpha}} + \|g_1\|_{Z_{r_1}^{\alpha}}))\|\bar{g}\|^2_{Y_{r_1}^{\alpha}} + \frac{C_0}{(\tau(t))^{1/2}}\|g_2\|_{Y_{r_1}^{\alpha}}\|\bar{g}\|^2_{X_{r_1}^{\alpha}} + \|\bar{g}\|^2_{Z_{r_1}^{\alpha}}. \quad (4.6)$$

From (4.1), we have

$$\dot{\tau} + \frac{C}{(\tau(t))^{1/2}}(\|g_1\|_{X_{r_1}^{\alpha}} + \|g_1\|_{Z_{r_1}^{\alpha}}) \leq 0, \quad (4.7)$$

where we have used the facts that $\tau(t) \leq \tau_1(t)$ and the norms $X_{r_1}^{\alpha}$ and $Z_{r_1}^{\alpha}$ are increasing with respect to $\tau$. Moreover,

$$\frac{C_0}{(\tau(t))^{1/2}}\|g_2\|_{Y_{r_1}^{\alpha}} \leq \frac{C_0}{\tau}\|g_2\|_{Z_{r_1}^{\alpha}} \leq \frac{C_0}{\tau}\|g_2\|_{X_{r_2}^{\alpha}} \leq \frac{C_1}{\tau_0}\|g(0)\|_{X_{r_10}^{\alpha}} e^{-2(1-2\alpha^2)t} \leq \tilde{C}_0^{1/3}, \quad (4.8)$$

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where in the second inequality $2\tau \leq \tau_2$ is used, and in the third inequality (3.19) and (3.20) are used. By choosing $\delta_0$ suitably small, it follows, from (4.6)-(4.8), that
\[
\frac{d}{dt} \|\bar{g}\|_{X_{r,\alpha}}^2 + \eta \|\bar{g}\|_{X_{r,\alpha}}^2 \leq 0,
\]
with $\eta$ being a small positive constant. This implies uniqueness of solution to (2.3) for all $t > 0$. Then, the proof of uniqueness part in Theorem 1.1 is completed.

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