COUNTING USING HALL ALGEBRAS II. EXTENSIONS FROM QUIVERS

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ABSTRACT. We count the \( F_q \)-rational points of GIT quotients of quiver representations with relations. We focus on two types of algebras – one is one-point extended from a quiver \( Q \), and the other is the Dynkin \( A_2 \) tensored with \( Q \). For both, we obtain explicit formulas. We study when they are polynomial-count. We follow the similar line as in the first paper but algebraic manipulations in Hall algebra will be replaced by corresponding geometric constructions.

INTRODUCTION

We continue our development on algorithms to count the points of various representation varieties of a quiver with relations. In [5], we applied several counting characters to the Harder-Narasimhan identity (2.1) in the Hall algebra of a quiver and obtained several interesting formulas. All characters that we considered are originated from Reineke’s counting character \( \int \) from the Hall algebra to certain quantum power series ring. Unfortunately \( \int \) fails to be an algebra morphism for non-hereditary algebras, though Harder-Narasimhan identity exists quite generally. However, applying the same map \( \int \) to the HN-identity can still generate effective counting formulas. We will follow the similar line as the first paper. The only change is that we replace algebraic manipulations in the Hall algebras by corresponding geometric constructions.

We first state the main results of this notes. Let \( k \) to be the finite field \( F_q \) with \( q \) elements and \( A \) be any basic algebra presented by \( A = kQ/I \). Fix a slope function \( \mu \), and we denote by \( \text{Rep}_\mu^\alpha(A) \) the variety of \( \alpha \)-dimensional \( \mu \)-semistable representations of \( A \), and by \( \text{Mod}_\mu^\alpha(A) \) its GIT quotient.

**Lemma 0.1.** \( |\text{Rep}_\mu^\alpha(A)| = \sum (-1)^{s-1} |\text{Frep}_{\alpha_1+\cdots+\alpha_s}(A)| \), where the sum runs over all decomposition \( \alpha_1 + \cdots + \alpha_s = \alpha \) of \( \alpha \) into non-zero dimension vectors such that \( \mu(\sum_{i=1}^s \alpha_i) < \mu(\alpha) \) for \( k < s \).

We will define the key varieties \( \text{Frep}_{\alpha_1+\cdots+\alpha_s}(A) \) in Section 1. In particular, if all Frep varieties can be effectively counted, then so are \( \text{Rep}_\mu^\alpha(A) \). The map \( \int \) have so-called \( \Delta \) and \( S \) analogs. They are defined in [10] as \( \int_\Delta \) and \( \int_S \). Here, \( \Delta \) and \( S \) are related to the comultiplication and the antipode in the Hall algebra. In our geometric setting, Lemma 0.1 and Frep varieties have \( \Delta \) and \( S \) analogs as well.

Recall that a variety \( X \) is called polynomial-count (or has a counting polynomial) if there exists a (necessarily unique) polynomial \( P_X = \sum a_i t_i \in \mathbb{C}[t] \) such that...
for every finite extension $\mathbb{F}_{q^r}/\mathbb{F}_q$, we have $|X(\mathbb{F}_{q^r})| = P_X(q^r)$. We are especially interested in when all Frep varieties are polynomial-count. If this is the case, it is clear that each $\text{Mod}^\mu_n(A)$ is polynomial-count when it is a geometric quotient.

In this notes, we will mainly focus on a class of algebras called one-point extensions from a quiver. Let $Q$ be any finite quiver and $E$ a representation of $Q$. The one-point extension of $Q$ by $E$ is the triangular algebra $kQ[E] := \langle \overset{E}{kQ} \rangle$. We also interested in the tensor product algebra $kA_n(Q) := kA_n \otimes kQ$, where $A_n$ is the Dynkin quiver of type $A_n$.

**Lemma 0.2.** For $A = kQ[E]$ or $kA_n(Q)$, we have explicit counting formulas for Frep varieties and their $\Delta$ and $S$ analogs.

**Theorem 0.3.**

1. For $A = kQ[E]$, all GIT quotients $\text{Mod}^\mu_n(A)$ can be explicitly counted in terms of quiver Grassmannians of $E$. If $E$ is add-poly-count, then all $\text{Mod}^\mu_n(A)$ are polynomial-count.
2. For $A = kA_n(Q)$, all $\text{Mod}^\mu_n(A)$ have counting polynomials, which can be explicitly computed.
3. If $E$ is add-poly-count, $\text{Mod}^\mu_n(kA_2 \otimes kQ[E])$ is polynomial-count for certain choice of $\alpha$ and $\mu$.

This notes are organized as follows. In Section 1 we provide necessary background on the representation theory of quivers with relations and points counting. In Section 2 we introduce the Frep variety and the notion of F-poly-count. After recalling the Harder-Narasimhan identity in the Hall algebra, we conclude our key lemma (Lemma 2.5). In Section 3 we first review the trivial extension of algebras in general, then specialize to the case of one-point extensions from a quiver. We describe the relations of these algebras from the projective presentation of $E$. In Section 4 we show in Lemma 4.1 that their Frep varieties can be counted in terms of the usual representation varieties. In Section 5 we show in Lemma 5.1 that these usual representation varieties can be counted in terms of the Grassmannians of $\pi E$. Motivated by this, we introduce add-poly-count property for a representation. We conclude by our fist main result – Theorem 5.5 (Theorem 0.3(1)). Many examples will follow in Section 7. In Section 6 we apply our algorithm to count homological strata on the geometric quotients. The method is outlined in Theorem 6.4. In Section 8 we work with the algebra $kA_2(Q)$. Our second main result – Theorem 8.4 (Theorem 0.3(2)) gives an analogous counting formula, which is independent of Grassmannians of representations. In Section 9 we consider the $\Delta$-analog of counting. We introduce the $\Delta$-analog of the Frep varieties. Lemma 9.1 is the $\Delta$-analog of Lemma 4.1. We conclude by our third main results – Theorem 9.2 (Theorem 0.3(3)). Finally in Section 10 we consider the $S$-analog of counting. Our final main results is Theorem 10.2 which removes the assumption of being a geometric quotient in our previous results.

Most of our constructions can be easily generalized to the motivic setting. Since the main application of this theory will be in the quantum algebra, we will not pursue that generality. The geometry of these moduli spaces will be studied in another series of notes [7].
1. Preliminary

1.1. Quivers with Relations. Let $Q$ be a finite quiver with the set of vertices $Q_0$ and the set of arrows $Q_1$. If $a \in Q_1$ is an arrow, then $ta$ and $ha$ denote its tail and its head respectively. Fix a dimension vector $\alpha$, the space of all $\alpha$-dimensional representations over a field $k$ is

$$\text{Rep}_\alpha(Q) := \bigoplus_{a \in Q_1} \text{Hom}(k^{\alpha(ta)}, k^{\alpha(ha)}).$$

The group $G = \text{GL}_\alpha := \prod_{v \in Q_0} \text{GL}_{\alpha(v)}$ acts on $\text{Rep}_\alpha(Q)$ by the natural base change. Two representations $M, N \in \text{Rep}_\alpha(Q)$ are isomorphic if they lie in the same $\text{GL}_\alpha$-orbit.

Let $kQ$ be the path algebra of $Q$ over $k$, then $M \in \text{Rep}_\alpha(Q)$ is naturally a (right) $kQ$-module. Fix a set $R$ of homogeneous elements in $kQ$ with respect to the bigrading: $kQ = \bigoplus_{v,w \in Q_0} vkQw$. Here, we abuse the vertex $v$ for the trivial path $e_v$. If $M(r) = 0$ for all $r \in R$, then we say $M$ is a representation of $Q$ with relations $R$. The path algebra of $Q$ with relations $R$ is the quotient algebra $A := kQ/\langle R \rangle$. A representation of $Q$ with relations $R$ naturally becomes an $A$-module.

The assignment $M \mapsto M(r)$ defines a polynomial map $ev(r) : \text{Rep}_\alpha(Q) \to \text{Hom}(k^{\alpha(tr)}, k^{\alpha(hr)})$, which is represented by an $\alpha(hr) \times \alpha(tr)$ matrix with entries in $k[\text{Rep}_\alpha(Q)]$. Let $\bar{R} \subseteq k[\text{Rep}_\alpha(Q)]$ be the ideal generated by the entries of all $ev(r)$ for which $r \in R$. The representation space $\text{Rep}_\alpha(A)$ is the scheme $\text{Spec}(k[\text{Rep}_\alpha(Q)]/\bar{R})$. As a variety, $\text{Rep}_\alpha(A)$ consists of all $\alpha$-dimensional representations of $A$.

1.2. Stability and GIT Quotients. A weight $\sigma$ is an integral linear functional on $\mathbb{Z}^{Q_0}$. A slope function $\mu$ is a quotient of two weights $\sigma/\theta$ with $\theta(\alpha) > 0$ for any dimension vector $\alpha$.

**Definition 1.1.** A representation $M$ is called $\mu$-semi-stable (resp. $\mu$-stable) if $\mu(\overline{L}) \leq \mu(\overline{M})$ (resp. $\mu(\overline{L}) < \mu(\overline{M})$) for every non-trivial subrepresentation $L \subset M$.

We denote by $\text{Rep}^\mu_\alpha(A)$ the variety of $\alpha$-dimensional $\mu$-semistable representations of $A$. By the standard GIT construction [9], there is a categorical quotient $q : \text{Rep}^\mu_\alpha(A) \to \text{Mod}^{\mu}_{\text{st}}(A)$ and its restriction to the stable representations $\text{Rep}^{\mu}_{\text{st}}_\alpha(A)$ is a geometric quotient.

A slope function $\mu$ is called coprime to $\alpha$ if $\mu(\gamma) \neq \mu(\alpha)$ for any $\gamma < \alpha$. So if $\mu$ is coprime to $\alpha$, then there is no strictly semistable (semistable but not stable) representation of dimension $\alpha$. In this case, $\text{Mod}^\mu_\alpha(A)$ must be a geometric quotient.

Note that the semi-stable objects with a fixed slope $\mu_0$ form an exact subcategory $\text{mod}_{\mu_0}(A)$. For any dimension vector $\alpha$, we can always modify $\mu$ to get a new slope function $\mu_\alpha$ such that $\mu_\alpha(\alpha) = 0$ and $\text{Rep}^{\mu_\alpha}_\alpha(A) = \text{Rep}^\mu_\alpha(A)$. If $\mu(\alpha) = \frac{\sigma(\alpha)}{\theta(\alpha)}$, then we can take $\mu_\alpha = \frac{\sigma}{\theta(\alpha)}$, where $\sigma_\alpha = \theta(\alpha)\sigma - \sigma(\alpha)\theta$.

**Lemma 1.2.** [3 Proposition 3.3]

(1) Harder-Narasimhan filtration: Every representation $M$ has a unique filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_{m-1} \subset M_m = M$ such that $N_i = M_i/M_{i+1}$ is $\mu$-semi-stable and $\mu(N_i) > \mu(N_{i+1})$. 


When the flag is only 2-step, we may use the usual Grassmannian notation. For combinations of isomorphism classes\(^\text{denote}\) a limit\(^\text{M}_{i}\) of \(X\) of Lemma 1.3.\(^\text{Lemma 1.4.}\)

\[
\prod_{\alpha} \text{flags of subspaces of dimension } k
\]

For any decomposition of dimension vector \(\lambda\), there is an action \(\lambda: X^{k} \times X \rightarrow X\) such that for every \(x \in X\) the limit \(\lim_{t \rightarrow 0} \lambda(t, x)\) exists. Assume in addition that \(X^{k}\) is projective, then \(Z\) is l-pure.

**Lemma 1.3.** \(^2\) Proposition A.2] Assume that \(X\) is smooth quasi-projective and that there is an action \(\lambda: k^{r} \times X \rightarrow X\) such that for every \(x \in X\) the limit \(\lim_{t \rightarrow 0} \lambda(t, x)\) exists. Assume in addition that \(X^{k}\) is projective, then \(Z\) is l-pure.

**Lemma 1.4.** \(^1\) Proposition 6.1] If \(X\) is counted by a rational function \(P_{X}\), then \(P_{X}\) must lie in \(\mathbb{Z}[t]\). Its specialization at \(q = 1\) gives the l-adic Euler characteristic of \(X\).

**Definition 1.5.** The Poincaré polynomial \(P(X, q) \in \mathbb{Z}[q^{1/2}]\) of \(X\) is

\[
P(X, q) = \sum_{i \geq 0} (-1)^{i} \dim \text{H}_{c}^{i}(X, \mathbb{Q}_{l}) q^{i/2}.
\]

**Lemma 1.6.** \(^2\) Lemma A.1] Assume that \(X\) is l-pure and polynomial-count. Then \(P_{X}(q) = P(X, q)\). In particular, \(P_{X}(t) \in \mathbb{N}[t]\).

2. HN-Filtration Identity

Let \(A\) be any basic algebra presented by \(A = kQ/I\) for \(k\) the finite field \(\mathbb{F}_{q}\). For any decomposition of dimension vector \(\alpha = \sum_{i=1}^{s} \alpha_{i}\), we define \(\text{Fl}_{\alpha} := \prod_{\nu \in Q_{0}} \text{Fl}_{\nu}^{\alpha_{i}(v) - \alpha_{i}(v)}\), where \(\text{Fl}_{\nu}^{\alpha_{i}(v) - \alpha_{i}(v)}\) is the usual flag variety parameterizing flags of subspaces of dimension \(\alpha_{i}(v) < \alpha_{i+1}(v) < \cdots < \alpha_{1}(v) + \cdots + \alpha_{s-1}(v)\) in \(k^{\alpha(v)}\). To simplify the notation, we denote \(\alpha_{i} := \sum_{j=1}^{i} \alpha_{j}\).

**Definition 2.1.** We define the \(\text{Frep}\) variety:

\[
\text{Frep}_{\alpha} := \{ (M, L_{1}, \ldots, L_{s-1}) \in \text{Rep}_{\alpha}(A) \times \text{Fl}_{\alpha} | L_{1} \supset \cdots \supset L_{s} = M \}.
\]

Let \(r: \text{Frep}_{\alpha} \rightarrow \text{Rep}_{\alpha}(A)\) be the projection, the \(\text{flag variety}\ \text{Fl}_{\alpha}(M)\) of \(M\) is the fibre \(r^{-1}(M)\), and its subvarieties \(\text{Fl}_{N_{1}, \ldots, N_{s}}(M)\) is

\[
\{(L_{1}, \ldots, L_{s-1}) \in \text{Fl}_{\alpha} | L_{i}/L_{i-1} \cong N_{i} \}.
\]

When the flag is only 2-step, we may use the usual Grassmanian notation. For example, \(\text{Gr}_{1}(\alpha) := \text{Fl}_{\beta \gamma}\) and \(\text{Gr}_{1}(M) := \text{Fl}_{\beta \gamma}(M)\), where \(\beta = \alpha - \gamma\).

For any three \(A\)-modules \(U, V\) and \(W\) with dimension vector \(\beta, \gamma\) and \(\alpha = \beta + \gamma\), the Hall number \(F_{U}^{V}\) is by definition \(|\text{Fl}_{U}(W)|\). For any module \(M\), we denote \(a_{M} := |\text{Aut}_{Q}(M)|. \) Let \(H(A)\) be the space of all formal (infinite) linear combinations of isomorphism classes \([M]\) in \(\text{A-mod}\).
Lemma 2.2. [14] Proposition 1.1] The completed Hall algebra $H(A)$ is the associative algebra with multiplication

$$[U][V] := \sum_{[W]} F^W_{UV}[W],$$

and unit $\eta : \mathbb{C} \mapsto \mathbb{C}[0]$.

We fix a slope function $\mu$. For a dimension vector $\alpha$, let $\chi_\alpha = \sum_{M \in \operatorname{mod}^\mu(A)} [M]$ and $\chi^\mu_\alpha = \sum_{M \in \operatorname{mod}^\mu(A)} [M]$. Our convention is that they contain zero representation $[0]$. We consider a simple counting map $\int : H(A) \to \mathbb{Q}(q)$ defined by $[M] \mapsto a_M^{-1}$. Since $\frac{1}{a_M} = |\mathcal{O}_M| / |GL_\alpha|$, we have that $\int \chi_\alpha = \frac{|\operatorname{Rep}_\alpha(A)|}{|GL_\alpha|}$. We denote the function $\frac{|\operatorname{Rep}_\alpha(A)|}{|GL_\alpha|}$ by $r_\alpha(A, q)$. In general, this function may not be rational in $q$.

The existence of the Harder-Narasimhan filtration yields the following identity in the Hall algebra $H(A)$.

Lemma 2.3. [11] Proposition 4.8]

$$\chi_\alpha = \sum \chi^\mu_{\alpha_1} \cdots \chi^\mu_{\alpha_s},$$

where the sum running over all decomposition $\alpha_1 + \cdots + \alpha_s = \alpha$ of $\alpha$ into non-zero dimension vectors such that $\mu(\alpha_1) < \cdots < \mu(\alpha_s)$. In particular, solving recursively for $\chi^\mu_\alpha$, we get

$$\chi^\mu_\alpha = \sum (-1)^{s-1} \chi_{\alpha_1} \cdots \chi_{\alpha_s},$$

where the sum runs over all decomposition $\alpha_1 + \cdots + \alpha_s = \alpha$ of $\alpha$ into non-zero dimension vectors such that $\mu(\sum_{i=1}^k \alpha_i) < \mu(\alpha)$ for $k < s$.

The key observation is that

$$r_{\alpha_1 \cdots \alpha_s}(A) := \int \chi_{\alpha_1} \cdots \chi_{\alpha_s} = \frac{|\operatorname{Frep}_{\alpha_1 \cdots \alpha_s}(A)|}{|GL_\alpha|}.$$ So the problem boils down to counting these Frep varieties. In this paper, we will only focus on a class of algebras, for which these varieties can be effectively counted.

Definition 2.4. We say an algebra $A$ is polynomial-count if each $\operatorname{Rep}_\alpha(A)$ is polynomial-count. It is called $F$-polynomial-count if each $\operatorname{Frep}_{\alpha_1 \cdots \alpha_s}(A)$ is polynomial-count.

We do not know a single example where $A$ is polynomial-count but not $F$-polynomial-count. We conjecture that if each $\operatorname{Frep}_{\alpha_1, \cdots, \alpha_s}(A)$ is polynomial-count, then $A$ is $F$-polynomial-count.

Lemma 2.5.

$$|\operatorname{Rep}_\alpha^\mu(A)| = \sum (-1)^{s-1} |\operatorname{Frep}_{\alpha_1 \cdots \alpha_s}(A)|.$$

In particular, if $A$ is $F$-polynomial-count, then each $\operatorname{Mod}_\alpha^\mu(A)$ is polynomial-count when it is a geometric quotient.

Conjecture 2.6. The assumption of being a geometric quotient in the lemma can be dropped.
3. Trivial Extensions

Given two finite-dimensional $k$-algebras $A, B$ and a $A$-$B$-bimodule $E$, we get the trivial extension algebra $B[E] = \left( \begin{array}{cc} B & 0 \\ E & A \end{array} \right)$. In the meanwhile, we can form the category $\text{Rep}(AEB)$ of representations of the bimodule $AEB$ as follows: The objects are triples $(M_A, M_B, \varphi) \in \text{mod} A \times \text{mod} B \times \text{Hom}_B(M_A \otimes_A E, M_B)$. A morphism from $(M_A, M_B, \varphi)$ to $(N_A, N_B, \psi)$ is a pair $(f_A, f_B)$ making the following diagram commute:

\[
\begin{array}{ccc}
M \otimes_A E & \varphi & M_B \\
\downarrow f_A & & \downarrow f_B \\
N \otimes_A E & \psi & N_B
\end{array}
\]

**Lemma 3.1.** [A.2.7] The two categories $\text{Rep}(B[E])$ and $\text{Rep}(AEB)$ are equivalent.

**Proof.** The equivalence is given by $F(M) = (M_A, M_B, \varphi)$, where $M_A = M e_A, M_B = M e_B$ and $\varphi(m \otimes f) = m \left( \begin{array}{c} 0 \\ e \end{array} \right) e_B$. \(\square\)

In particular, if $M \in \text{Rep}(B[E])$ corresponds to $(M_A, M_B, \varphi)$, then the dimension vector of $M$ is given by $(M_A, M_B)$. One particular case we interested in is when $E$ is simply a right $B$-module. Think $E$ as a $k$-$B$-bimodule, the triangular algebra $(B \ 0 \\
E \ 1)$ is called (trivial) one-point extension of $B$ by $E$. There is an obvious dual notion of one-point coextension $B^\circ[E] := \left( \begin{array}{cc} k & 0 \\ E & 1 \end{array} \right)$.

**Lemma 3.2.**

- $\text{Rep}_{(n, \alpha)}(B[E])$ is the subvariety of $\text{Rep}_\alpha(B) \times \text{Hom}(nE, k^\alpha)$ defined as
  $\{(M, f) \in \text{Rep}_\alpha(B) \times \text{Hom}(nE, k^\alpha) \mid f \in \text{Hom}_B(nE, M)\}$.

- $\text{Rep}_{(\alpha, n)}(B^\circ[E])$ is the subvariety of $\text{Rep}_\alpha(B) \times \text{Hom}(k^\alpha, nE)$ defined as
  $\{(M, f) \in \text{Rep}_\alpha(B) \times \text{Hom}(k^\alpha, nE) \mid f \in \text{Hom}_B(M, nE)\}$.

Suppose that $E \in \text{Rep}(Q)$ is presented by $0 \rightarrow P_1 \xrightarrow{D} P_0 \rightarrow E \rightarrow 0$ with $P_1 = \bigoplus_i b_i P_v$ and $P_0 = \bigoplus_i b_i^0 P_v$, where $P_v$ is the indecomposable projective representation corresponding to the vertex $v$. Then the algebra $A = kQ[E]$ can be presented by a new quiver $Q(E)$, which is obtained from $Q$ by adjoining a new vertex “$-$” and for each $P_v$ in $P_0$ a new arrow from “$-$” to the vertex $v$. The relations are clearly given by the matrix $D$. In reality, the presentation is always chosen to be minimal. By abuse of notation, we also use $Q[E]$ to denote the new quiver $Q(E)$ with those new relations. The one-point coextension $kQ^\circ[E]$ can be similarly described using injective presentation of $E$. By convention, the newly adjoined vertex is denoted by “$+$”.

It is clear that $E$ is the first syzygy of $S_-$, so $0 \rightarrow P_1 \xrightarrow{D} P_0 \rightarrow P_-. E \rightarrow S_-$.

Moreover, a simple representation of $Q[E]$ is either $S_-$ or a simple representation of $Q$. So we conclude that $kQ[E]$ has global dimension 2. It is easy to compute the matrices $\mathcal{E}_A := (\text{Ext}_A^1(S_u, S_v))$. Let $\mathcal{E}_Q := (\text{Ext}_Q^1(S_u, S_v))$, then $\mathcal{E}_A^0 = \left( \begin{array}{cc} 1 & 0 \\ 0 & \mathcal{E}_Q^0 \end{array} \right), \mathcal{E}_A^1 = \left( \begin{array}{cc} 0 & \mathcal{E}_Q^1 \\ \mathcal{E}_Q^2 & 0 \end{array} \right), \mathcal{E}_A^2 = \left( \begin{array}{cc} 0 & \mathcal{E}_Q^0 \\ \mathcal{E}_Q^1 & 0 \end{array} \right)$, so the Euler matrix $\mathcal{E}_A$ of $A$ is $\left( \begin{array}{cc} 1 & \delta \\ \delta^T & 0 \end{array} \right)$, where $\delta = b^1 - b^0$ and $\mathcal{E}_Q$ is the Euler matrix of $Q$. Throughout this notes, $\langle -, - \rangle$ is the multiplicative Euler form of the quiver $Q$, that is, $\langle \alpha, \beta \rangle = q^{\alpha \mathcal{E}_Q \beta^T}$. Similarly, we define $\langle \alpha, \beta \rangle_i = q^{\alpha_i \mathcal{E}_Q \beta_i^T}$ for $i = 0, 1$. 

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4. Counting \( \text{Frep of } Q[E] \)

To simplify our notation, we always use letter with tilde to indicate that \( \tilde{M} \in \text{Rep}(Q[E]) \) can be represented by \( (M, f_M) \), where \( M \in \text{Rep}(Q) \) and \( f_M : nE \to M \). A dimension vector with tilde, say \( \tilde{\beta} \), consists of two components \((\beta_-, \beta_+)\), or \((\beta, \beta^-)\) for coextension.

**Lemma 4.1.** \( p : \text{Frep}_{\tilde{\beta}, \tilde{\gamma}}(Q[E]) \to \text{Fl}_{\tilde{\beta}, \tilde{\gamma}} \) is a fibre bundle with fibre

\[
\text{Rep}(\alpha - \gamma)(Q[E]) \times \text{Rep}_{\tilde{\beta}}(Q[E]) \times \prod_{a \in Q_1} \text{Hom}(k^{\beta(ta)}, k^{\gamma(ha)})
\]

So

\[
r_{\tilde{\beta}, \tilde{\gamma}}(Q[E]) := \frac{|\text{Frep}_{\tilde{\beta}, \tilde{\gamma}}(Q[E])|}{|\text{GL}_{\tilde{\alpha}}|} = \langle \beta, \gamma \rangle^{-1} \frac{1}{|\text{GL}_{\beta^-}|} |\text{GL}_{\beta^-}| r_{\tilde{\beta}}(Q[E]) r_{(\alpha - \gamma)}(Q[E])
\]

**Proof.** We will sketch the fibre bundle construction by a picture. After fixing an elements in \( \text{Fl}_{\tilde{\beta}, \tilde{\gamma}} \), we need to fill in the missing part for a \( \tilde{\alpha} \)-dimensional representation of \( Q[E] \). The missing part consists of a \( \tilde{\gamma} \)-dimensional representation \( S \), a \( \tilde{\beta} \)-dimensional representation \( T \), and a bunch of linear maps from \( T(ta) \) to \( S(ha) \), as indicated below.

\[
\text{We can stuff the space in the order below independently. The linear maps from } T_- \text{ together with all representations } S \text{ can be identified with } \text{Rep}_{(\alpha - \gamma)}(Q[E]); \text{ all representations } T \text{ can be identified with } \text{Rep}_{\tilde{\beta}}(Q[E]), \text{ and the rest of the linear maps are } \prod_{a \in Q_1} \text{Hom}(k^{\beta(ta)}, k^{\gamma(ha)}).
\]

For the last formula, we only need to notice that

\[
|\text{Fl}_{\tilde{\beta}, \tilde{\gamma}}| = \frac{|\text{Frep}_{\tilde{\beta}, \tilde{\gamma}}(Q[E])|}{|\text{GL}_{\tilde{\alpha}}|} = \frac{\langle \beta, \gamma \rangle^{-1} |\text{GL}_{\beta^-}|}{|\text{GL}_{\beta^-}|} |(\beta, \gamma)|_0.
\]

\( \square \)

The above 2-step case can be recursively generalized to the \( n \)-step case. We only state the analog for the last formula.

\[
(4.1) \quad r_{\tilde{\alpha_1} \cdots \tilde{\alpha_s}}(Q[E]) = \prod_{i=1}^{s} \left[ \frac{\alpha_i^-}{\alpha_i^-} \right] |\text{GL}_{\tilde{\alpha_i}}| \left| r_{(\alpha_1, \ldots, \alpha_i)}(Q[E]) \right|.
\]

We also state this formula for the dual case.

\[
r_{\tilde{\alpha_s} \cdots \tilde{\alpha_1}}(Q^*[E]) = \prod_{i=2}^{s} \left[ \frac{\alpha_i^+}{\alpha_i^+} \right] |\text{GL}_{\tilde{\alpha_i}}| \left| r_{(\alpha_i, \ldots, \alpha_s)}(Q^*[E]) \right|.
\]

Now the problem boils down to count those affine representation varieties \( \text{Rep}_{\alpha}(Q[E]) \).
5. Counting Affine

For any dimension vector $\beta$, we denote by $\text{Gr}^\beta(E)$ the variety parameterizing all $\beta$-dimensional quotient representations of $E$, and define

$$\text{Hom}_Q(E, \alpha) = \{(M, \phi, E_1, M_1) \in \text{Rep}_\alpha(Q) \times \text{Hom}(E, k^\alpha) \times \text{Gr}^\beta(E) \times \text{Gr}_\beta(\alpha) \mid \phi \in \text{Hom}_Q(E, M), E/\text{Ker} \phi = E_1, \text{Im} \phi = M_1\}.$$

Lemma 5.1. $p : \text{Hom}_Q(E, \alpha) \rightarrow \text{Gr}^\beta(E) \times \text{Gr}_\beta(\alpha)$ is a fibre bundle with fibre

$$\text{GL}_\beta \times \text{Rep}_{\alpha-\beta}(Q) \times \bigoplus_{a \in Q_i} (\text{Hom}(k^{(\alpha-\beta)(\alpha)}, k^{\beta(ha)})).$$

So

$$r_{(\alpha, \alpha)}(Q[E]) := \sum_{a = \gamma + \beta} \frac{[\text{Gr}^\beta(nE)]}{\langle \gamma, \beta \rangle} |\text{GL}_n| r_{\gamma}(Q).$$

Dually, we have a formula for the one-point coextension:

$$r_{(\alpha, \alpha)}(Q^* [E]) = \sum_{a = \gamma + \beta} \frac{[\text{Gr}^\beta(nE)]}{\langle \gamma, \beta \rangle} |\text{GL}_n| r_{\beta}(Q).$$

Proof: The fibre bundle construction is not hard to verify. Since we only need the last formula, we give a Hall algebra proof for that. We denote by $\text{Mon}(M, N)$ and $\text{Epi}_Q(M, N)$ the set of all monomorphisms and epimorphisms from $M$ to $N$ respectively. Fix a representation $W$, then the following identities clearly holds in the Hall algebra

$$\left( \sum_{[U]} [U] \right) \left( \sum_{[V]} \text{Epi}_Q(M, V)[[V]] \right) = \sum_{[W]} |\text{Hom}_Q(M, W)|[[W]].$$

Let $\mathcal{P}_Q$ be the completed quantum polynomial algebra $Q(q)[x]$, where the multiplication rule is $x^\alpha x^\beta = \langle \alpha, \beta \rangle^{-1} x^{\alpha+\beta}$. Then the map $\int : H(Q) \rightarrow \mathcal{P}_Q$ sending $[M] \rightarrow a_M^{-1} x^M$ is an algebra morphism [11]. Here, we use the slanted $\int$ to distinguish the one with target $Q(q)$. Apply $\int$ to both sides, we get

$$\int \sum_{[U]} [U] \int \sum_{[V]} \text{Epi}_Q(M, V)[[V]] = \sum_{[W]} |\text{Hom}_Q(M, W)|[[W]]$$

$$\Leftrightarrow \sum_{\gamma} r_{\gamma}(Q) x^\gamma \sum_{[V]} a_V^{-1} |\text{Epi}_Q(M, V)[x^V] = \sum_{[W]} a_W^{-1} |\text{Hom}_Q(M, W)[x^W]|$$

$$\Leftrightarrow \sum_{\gamma} r_{\gamma}(Q) x^\gamma \sum_{\beta} \text{Gr}^\beta(M) x^\beta = \sum_{[W]} \frac{|\mathcal{O}_W|}{|\text{GL}_\alpha|} |\text{Hom}_Q(M, W)[x^\alpha]| (\alpha = \overline{W})$$

$$\Leftrightarrow \sum_{\beta+\gamma = \alpha} \langle \gamma, \beta \rangle^{-1} r_{\gamma}(Q) \text{Gr}^\beta(M) = \sum_{W \in \text{Rep}_\alpha(Q)} \frac{|\mathcal{O}_W|}{|\text{GL}_\alpha|} |\text{Hom}_Q(M, W)|.$$
Remark 5.2. Let $R(Q)$ be the generating functions $R(Q) := \sum r_\alpha(Q)$, and

$$F^\bullet(E) := \sum \text{Gr}^\beta(E)x^\beta, \quad \text{and} \quad F_\bullet(E) := \sum \text{Gr}_\gamma(E)x^\gamma.$$ 

If we set

$$F^\infty(E) = \sum_{n=1}^{\infty} \frac{F^\bullet(nE)x^n}{|\text{GL}_n|} \quad \text{and} \quad F^\infty(E) = \sum_{n=1}^{\infty} \frac{F_\bullet(nE)x^n}{|\text{GL}_n|},$$

then Lemma 5.1 can be rewritten as

$$R(Q[E]) = R(Q)F^\infty(E), \quad \text{and} \quad R(Q^\circ[E]) = F^\infty(E)R(Q).$$

Definition 5.3. A representation $E \in \text{Rep}(Q)$ is called polynomial-count, if all its Grassmannians $\text{Gr}_\gamma(E)$ are polynomial-count. It is called add-polynomial-count, if each $nE$ is polynomial-count.

Corollary 5.4. $nE$ is polynomial-count if and only if $\text{Rep}_{(n, \alpha)}(Q[E])$ is polynomial-count for any $\alpha$.

If $E$ is add-polynomial-count, then $kQ[E]$ is F-polynomial-count by (4.1). It follows from Lemma 2.5, 4.1, and 5.1 that

Theorem 5.5. $\text{Rep}_\alpha(Q[E])$ can be explicitly counted in terms of $\text{Gr}_\gamma(nE)$’s. In particular, if $E$ is add-polynomial-count, then each $\text{Mod}_\alpha^n(Q[E])$ is polynomial-count when it is a geometric quotient.

We will see in the last section that the assumption of being a geometric quotient can be dropped. Polynomial-count representations of quivers will be studied in detail in [8]. This class includes rigid representations because of [5, Corollary 6.4], but there are many more.

Question 5.6. Is there a representation, which is polynomial-count but not add-polynomial-count?

6. APPLICATION TO HOMOLOGICAL STRATIFICATION

Definition 6.1. For any representation $E$, the $E$-homological stratification of $\text{Rep}_\alpha(Q)$ is the decomposition of $\text{Rep}_\alpha(Q)$ into (finite) disjoint union of locally closed subvarieties $\text{Rep}_\alpha(Q; E, h)$, where

$$\text{Rep}_\alpha(Q; E, h) = \{ M \in \text{Rep}_\alpha(Q) \mid \text{hom}_Q(E, M) = h \}.$$ 

By Lemma 3.2 we know that for $n \geq 0$,

$$|\text{Rep}_{(n, \alpha)}(Q[E])| = \sum_h |\text{Rep}_\alpha(Q; E, h)|q^{nh}.$$ 

The coefficient matrix of above linear system is a non-degenerated Vandermonde-type matrix, so we can solve all $\text{Rep}_\alpha(Q; E, h)$. In particular, $\text{Rep}_\alpha(Q; E, h)$ is polynomial-count if and only if $E$ is add-polynomial-count. We will see that the above is still true for

$$\text{Rep}_\alpha^\circ(Q; E, h) = \{ M \in \text{Rep}_\alpha^\circ(Q) \mid \text{hom}_Q(E, M) = h \}.$$
Definition 6.2. Let $\mu = \frac{\sigma}{\theta}$ be any slope function for $Q$. The (negative) extension $\mu_-$ to $Q[E]$ with respect to $E$ is defined by $\frac{\sigma}{\theta - \epsilon}$, where $\theta_-(n, \alpha) = n + \theta(\alpha)$ and $\sigma_-(n, \alpha) = \epsilon_n + \sigma_n(\alpha)$ for some sufficiently small positive $\epsilon \in \mathbb{Q}$. Similarly, we define the (positive) extension of $\mu_+$ to $Q^\delta[E]$ with respect to $\alpha$ as $\frac{\sigma_+}{\theta + \epsilon}$, where $\theta_+(\alpha, n) = \theta(\alpha) + n$, and $\sigma_+(\alpha, n) = \sigma_n(\alpha) + \epsilon n$.

The following lemma was proved in [4] Theorem 5.2 for $E$ projective, but the argument goes through for any $E$.

Lemma 6.3. We have the following identity in $\mathcal{P}_Q(\alpha)$:

\[
\left( \sum_{\beta} r_{\mu}(Q) x^\beta \right) \left( \sum_{\gamma} r_{\mu_-}(Q[E]) x^\gamma \right) = \sum_{\alpha} \left( \sum_{M \in \text{Rep}_Q(\alpha)} |\mathcal{O}_M||\text{Hom}_Q(nE, M)| \right) x^\alpha.
\]

Theorem 6.4. $|\text{Rep}_Q^\mu(Q, E, h)|$ can be explicitly computed from $\text{Gr}_\gamma(E)$. When $E$ is add-polynomial-count and $\text{Mod}_Q^\mu(Q)$ is a geometric quotient, each homological strata on $\text{Mod}_Q^\mu(Q)$ is polynomial-count.

Proof. According to Theorem [5, 6], all $r_{\mu_-}^\mu(n, \alpha)$’s can be computed from $\text{Gr}_\gamma(E)$, and so does the right hand side of (6.1). Notice that

\[
\sum_{M \in \text{Rep}_Q^\mu(\alpha)} |\mathcal{O}_M||\text{Hom}_Q(nE, M)| = \sum_{h} |\text{Rep}_Q^\mu(Q, E, h)| q^{\alpha h}.
\]

We can invert the same Vandermonde-type matrix as before to solve $|\text{Rep}_Q^\mu(Q, E, h)|$.

7. 3-VERTEX EXAMPLES

Consider the $n$-arrow Kronecker quiver $K_n$ and its extension by an $(m, d)$-dimensional representation $E$. Then we can view the algebra $kK_n[E]$ as an algebra coextended from $K_m$ by a $(d, n)$-dimensional representation $E^\circ$.

It follows from Remark 5.2 that

Proposition 7.1. $F^\infty(E)$ and $F^\infty(E^\circ)$ are related by

$R(K_n)F^\infty(E) = R(K_n|E]) = F^\infty(E^\circ)R(K_m)$.

In particular, if $E$ is add-polynomial-count, then so is $E^\circ$.

Let $A := kK_n^\infty[E]$ be the algebra coextended from $K_m$ by a representation $E$ of dimension $d$. For any dimension vector $\alpha = (\alpha_1, \alpha_2)$ of $K_m$, there is a unique choice of weight $\sigma$ up to scalar such that $\sigma(\alpha) = 0$. For the rest of this section, we always take $\mu = \frac{\sigma}{\theta}$ for different $\alpha$’s.

The first two isomorphisms below can be easily established by Lemma 3.2

Proposition 7.2. Mod$^\mu_{(\gamma_1, 1)}(A) \cong \text{Gr}_\gamma(E)$ and Mod$^\mu_{(\gamma_1, 1), 1}(A) \cong \text{Gr}_\gamma(1, 1)(E)$.

Assume that $E$ is not too special so that $\text{Gr}_{(n, 1)}(E)$ is empty.

\[
|\text{Mod}^\mu_{(1, 2), 1}(A)| = |\text{Gr}_{(1, 2)}(E) + ([m - 1] - [\epsilon_2 - 1])| \text{Gr}_{(1, 1)}(E)|,
\]

\[
|\text{Mod}^\mu_{(2, 2), 1}(A)| = |\text{Gr}_{(2, 2)}(E) + ([2m - 1] - [\epsilon_2 - 1])| \text{Gr}_{(2, 1)}(E)|,
\]

and so forth.

where $[n]$ is the quantum number.
Example 7.3. Consider the quiver

\[
\begin{array}{ccc}
1 & \overset{a}{\rightarrow} & 2 \\
\downarrow & & \downarrow \overset{b}{\rightarrow} 3
\end{array}
\]

with relation \(ab = 0\). The corresponding algebra \(A\) is one-point-extended from the Dynkin quiver \(A_2\) by the simple \(S_2\). So \(|\text{Rep}_{(n,\alpha)}(A)|\) can be computed by Lemma 7.5. Note that \(\text{Gr}^\delta(nS_1)\) is just the usual Grassmannian variety \(\text{Gr}^\beta(n)\). It follows that the quiver

\[
\begin{array}{ccc}
1 & \overset{a,c}{\rightarrow} & 2 \\
\downarrow & & \downarrow \overset{b,d}{\rightarrow} 3
\end{array}
\]

with relations \(ab = 0, cd = 0\) is polynomial-count. This algebra is extended from the Kronecker quiver \(K_2\) by a decomposable non-rigid representation of dimension \((2,2)\).

Example 7.4. Fix \(n \in \mathbb{N}\), we consider the quiver \(A_2(2, n)\)

\[
\begin{array}{ccc}
1 & \overset{a,b}{\rightarrow} & 2 \\
\downarrow & \overset{x_1 \ldots x_n}{\rightarrow} & 3
\end{array}
\]

with relation \(AX = 0\), where \(A = (a,b)\) and \(X = (x_1 \ x_2 \ldots x_{n-1})\). It is extended form \(K_n\) by \(E_n\) presented by

\[
0 \to (n-1)P_3 \overset{X^T}{\rightarrow} 2P_2 \to E_n \to 0.
\]

It is also coextended from \(K_2\) by the exceptional \(E_n^c\) presented by

\[
0 \to E_n^c \to nI_2 \overset{B^T}{\rightarrow} (n-1)I_1 \to 0,
\]

where \(B = \begin{pmatrix} a & b & 0 & 0 & \cdots & 0 \\ 0 & a & b & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & a & b \end{pmatrix}\). Although \(E_n\) is not rigid, it follows from Proposition 7.4 that it is add-polynomial-count. \(|\text{Gr}_\gamma(E_n^c)|\) can be recursively computed by the cluster theory and Lemma 7.5. A closed formula was given in [15] Theorem 4.3.

\[
|\text{Gr}_\gamma(E_n^c)| = \begin{cases} 1 & \gamma = (0,0), (n+1,n) \\ \left[ \frac{n-\gamma}{\gamma_1-1} \right] \left[ \frac{n+1}{\gamma_2+1} \right] & \text{otherwise}, \end{cases}
\]

where \(\left[ \frac{n}{k} \right]\) is the quantum binomial coefficient.

Now we recall [5] Proposition 2.8. We also draw some easy consequences, which are useful for counting the Grassmanian of representations.

Lemma 7.5. Assume that \(\dim U = \alpha_1\) and \(\dim V = \alpha_2\).

\[
\sum_{\gamma_1 + \gamma_2 = \gamma} \langle \gamma_1, \alpha_2 - \gamma_2 \rangle |\text{Gr}_{\gamma_1}(U)||\text{Gr}_{\gamma_2}(V)| = \sum_{[W]} \frac{|\text{Ext}_Q(U,V)_W|}{|\text{Ext}_Q(U,V)|} |\text{Gr}_\gamma(W)|.
\]

Now suppose that \(\text{Ext}_Q(U,V) = 0\). Then

\[
F_*(U \oplus V) = \sum_{\gamma_1, \gamma_2} \langle \gamma_1, \alpha_2 - \gamma_2 \rangle \text{Gr}_{\gamma_1}(U) \text{Gr}_{\gamma_2}(V) x^{\gamma_1 + \gamma_2}.
\]

Hence, if both \(U\) and \(V\) are (add)-polynomial-count, then so is \(U \oplus V\). Moreover, if \(\text{Ext}_Q(V,U) = k^e\) and \(W\) is the only non-trivial middle term of the extensions, then

\[
(q^e - 1)F_*(W) = q^e \sum_{\gamma_1, \gamma_2} \langle \gamma_2, \alpha_1 - \gamma_1 \rangle \text{Gr}_{\gamma_2}(V) \text{Gr}_{\gamma_1}(U) x^{\gamma_1 + \gamma_2} - F_*(U \oplus V).
\]
Example 7.6. We add one arrow to Example 7.4

\[
\begin{array}{c}
\begin{tikzcd}
 a,b & 1 \\
 & x \rightarrow c \\
2 & 3
\end{tikzcd}
\end{array}
\]

Then it is extended from \(K_3\) by \(E_n \oplus P_3\), or coextended from \(K_2\) by \(E_n^0 \oplus I_1\). Let \(A_n = kK_3^2[E_n^0 \oplus I_1]\). Since \(\text{Ext}_{K_3}(E_n^0, I_1) = 0\), we can use Lemma 7.5 or compute directly \(r_n(A_n)\). So we are able to find all \(|\text{Mod}^\mu_{(1,1,1)}(A_n)|\). For example,

\[
\begin{align*}
|\text{Mod}^\mu_{(1,1,1)}(A_n)| &= q^2 + 2q + 1, \\
|\text{Mod}^\mu_{(1,1,1)}(A_n)| &= [n] + [3] - 1, \\
|\text{Mod}^\mu_{(2,2,1)}(A_n)| &= q^4 + 2q^3 + 4q^2 + 2q + 1.
\end{align*}
\]

However, all \(\text{Mod}^\mu_{(1,1,1)}(A_n)\) are different, they are Hirzebruch surfaces \(F_n\) [7].

Example 7.7. Consider quiver

\[
\begin{array}{c}
\begin{tikzcd}
1 & 2 & 3
\end{tikzcd}
\end{array}
\]

with relation \(xa + yb + zc = 0\). It is coextended from \(K_3\) by a rigid module presented by \(0 \rightarrow E \rightarrow 3I_2 \xrightarrow{(a,b,c)} I_1 \rightarrow 0\). Similar calculation as before gives

\[
\begin{align*}
|\text{Mod}^\mu_{(1,1,1)}(A)| &= [2][3], \\
|\text{Mod}^\mu_{(2,2,1)}(A)| &= |\text{Mod}^\mu_{(1,1,2)}(A)| = [3], \\
|\text{Mod}^\mu_{(1,2,1)}(A)| &= [3][5], \\
|\text{Mod}^\mu_{(2,2,1)}(A)| &= |\text{Mod}^\mu_{(1,2,2)}(A)| = |\text{Mod}^\mu_{(2,2,1)}(A)| = 3(1, 1, 3, 3, 3, 1, 1).
\end{align*}
\]

The first one is [7] a divisor \(D\) on \(\mathbb{P}^2 \times \mathbb{P}^2\) of bidegree \((1, 1)\), or equivalently the complete flag variety \(F_3\) of \(k^3\).

Now consider the deformation \(E' \oplus I_2\) of \(E\), where \(0 \rightarrow E' \rightarrow 2I_2 \xrightarrow{(a,b)} I_1 \rightarrow 0\). Since \(\text{Ext}_{\mathbb{Q}}(I_2, E') = k\) with \(E\) the only non-trivial middle term, we can compute \(F_*(E')\) using Lemma 7.5.

\[
F_*(E') = 1 + [2]x^{(1, 0)} + [2]x^{(1, 1)} + [2]x^{(2, 1)} + x^{(0, 2)} + [5]x^{(1, 2)} + \left[\frac{5}{2}\right]x^{(2, 2)} + \cdots.
\]

\[
\begin{align*}
|\text{Mod}^\mu_{(1,1,1)}(A)| &= (1, 3, 2, 1), \\
|\text{Mod}^\mu_{(2,1,1)}(A)| &= |\text{Mod}^\mu_{(1,1,2)}(A)| = (2, 2, 1), \\
|\text{Mod}^\mu_{(1,2,1)}(A)| &= [3][5], \\
|\text{Mod}^\mu_{(2,2,1)}(A)| &= |\text{Mod}^\mu_{(1,2,2)}(A)| = [3][5](1, 0, 1), \\
|\text{Mod}^\mu_{(1,2,2)}(A)| &= |\text{Mod}^\mu_{(2,2,1)}(A)| = 3(1, 1, 4, 4, 3, 1, 1).
\end{align*}
\]

Note that the first one is irreducible and singular by Lemma 1.6.

Example 7.8. Consider quiver

\[
\begin{array}{c}
\begin{tikzcd}
1 & 2 & 3
\end{tikzcd}
\end{array}
\]
with relation $AX = 0$, where $A = \begin{pmatrix} c & 0 & -\pi \\ -b & a & 0 \\ 0 & 0 & -c & b \end{pmatrix}$. It is coextended from $K_3$ by $E$ presented by the following base diagram. The black dots are a basis in $E_1$; while the white dots are a basis in $E_2$. The letter on an arrow means the identity map on the arrow of the same letter.

It is known [7] that for a general representation $E_g$ of dimension $(6, 3)$, $\text{Gr}(1, 1)(E_g)$ is an elliptic curve. So $E_g$ is not polynomial-count. However, for this special $E$, $\text{Gr}(1, 1)(E)$ is three $\mathbb{P}_1$’s intersecting at a point. With a little effort one can show that $E$ is actually polynomial-count.

8. The Universal Case: $A_2(Q)$

Let us consider a category, which is universal in the sense that it contains all one-point extensions of $Q$ as its full subcategories. It is clearly the module category of $kA_2(Q) := kQ \otimes kA_2$, where $A_2$ is the Dynkin quiver $1 \rightarrow 2$. The quiver of $kA_2(Q)$ is composed of two copies of $Q$ corresponding to two idempotents of $kA_2$, and morphism arrows connecting the same vertices in two different copies. The relations are obviously the commuting diagram relations. By abuse of notation, we use $A_2(Q)$ to denote such a quiver with relations. So the dimension vector of $A_2(Q)$ is composed of two dimension vectors of $Q$, say $(\alpha, \beta)$. By convention, $\alpha$ correspond to the quiver sending morphism arrows.

Let $V$ be an $\alpha$-dimensional $k$-vector space. We denote by $\text{In}_{c_d \cap e_e}(\alpha)$ the incidence variety

$$\{(C, D, E) \in \text{Gr}_c(V) \times \text{Fl}_{e-e}(V) \mid \dim(C \cap D) = c_d, \dim(C \cap E) = e_e\},$$

and by $\text{Gr}_{b}(\alpha)$ the incidence variety

$$\{(B, E) \in \text{Gr}_b(V) \times \text{Gr}_e(V) \mid V/B = B, V/E = E, \dim(B \cap E) = c - d + e \}.$$
Lemma 8.1. $p: \text{Frep}_{(\beta_u, \beta_d), (\gamma_u, \gamma_d)}(A_2(Q)) \to \text{Fl}_{(\beta_u, \beta_d), (\gamma_u, \gamma_d)}$ is a fibre bundle with fibre

\[
(8.1) \bigcup_{b,c,d,e,c,d,e} \text{In}^{c\to c}_{c\to c} (\gamma_d) \times \text{Gr}^{b\to c}_{e\to d} (\beta_u) \times \text{Gr}^{c\to c}_{e\to d} (\gamma_u) \times \text{Gr}^{b\to c}_{e\to d} (\beta_d) \times \text{GL}_b \times \text{GL}_c \times \text{GL}_d
\]

\[
(8.2) \times \prod_{a \in Q_1} \text{Hom}(k_{c_d(ta)}, k_{c_d(ha)}) \times \text{Hom}(k_{c_d-c_d(ta)}, k_{c_d(ha)}) \times \text{Hom}(k_{c-d_c(ta)}, k_{c(ha)})
\]

\[
(8.3) \times \text{Hom}(k_{c_d(ta)}, k_{c_d(ha)}) \times \text{Hom}(k_{c_d-c_d(ta)}, k_{c_d(ha)}) \times \text{Hom}(k_{c-d_c(ta)}, k_{c(ha)})
\]

\[
(8.4) \times \text{Hom}(k_{b(ta)}, k_{b+d-e}(ta)) \times \text{Hom}(k_{b(ta)}, k_{b-d}(ha)) \times \text{Hom}(k_{\beta_u(ta)}, k_{\beta_u-c}(ha)) \times \text{Hom}(k_{\beta_d(ta)}, k_{\gamma_u-c}(ha)) \times \text{Hom}(k_{\beta_d(ta)}, k_{\gamma_d-c}(ha))
\]

So $r_{(\beta_u, \beta_d), (\gamma_u, \gamma_d)}(A_2(Q)) := \frac{|\text{Frep}_{(\beta_u, \beta_d), (\gamma_u, \gamma_d)}(A_2(Q))|}{|\text{Fl}_{(\alpha_u, \alpha_d)}|}$ is equal to

\[
\sum_{b,c,d,e,c,d,e} t_{b,c,d,e,c,d,e} \cdot r_{(\beta_u, \beta_d), (\gamma_u, \gamma_d)}(A_2(Q)) \cdot r_{(\beta_u, \beta_d), (\gamma_u, \gamma_d)}(A_2(Q))^{-1} \prod_{a \in Q_1} \text{Hom}(k_{c_d(ta)}, k_{c_d(ha)}) \times \text{Hom}(k_{c_d-c_d(ta)}, k_{c_d(ha)}) \times \text{Hom}(k_{c-d_c(ta)}, k_{c(ha)})
\]

where $t_{b,c,d,e,c,d,e} = \frac{\text{Hom}(k_{b(ta)}, k_{b+d-e}(ta)) \times \text{Hom}(k_{b(ta)}, k_{b-d}(ha)) \times \text{Hom}(k_{\beta_u(ta)}, k_{\beta_u-c}(ha)) \times \text{Hom}(k_{\beta_d(ta)}, k_{\gamma_u-c}(ha)) \times \text{Hom}(k_{\beta_d(ta)}, k_{\gamma_d-c}(ha))}{(\beta_u, \beta_d, \gamma_u, \gamma_d)}(A_2(Q))^{-1}$.

Proof. We sketch the fibre bundle construction by a picture. After fixing an element in $\text{Fl}_{(\beta_u, \beta_d), (\gamma_u, \gamma_d)}$, we need to fill in the missing part for a $(\alpha_u, \alpha_d)$-dimensional representation of $A_2(Q)$. Similar to Lemma 8.1, the missing part consist of a $(\gamma_u, \gamma_d)$-dimensional representation $S$, a $(\beta_u, \beta_d)$-dimensional representation $T$, and a bunch of linear maps from $T(ta)$ to $S(ha)$, as indicated below.

The first step is to choose a configuration of image spaces of the vertical and diagonal morphism arrows. Let $b, c$ be the rank vector of the morphism arrows of $T$ and $S$ respectively, and $e$ be the rank vector of the diagonal morphism arrows. Let $d$ be the rank vector of the diagonal morphism arrows restricted on the kernel of the morphism arrows of $T$. The second step is to stuff in the following order – the lower part of $S$, the upper part of $T$, the rest part of $S$ and $T$, and other diagonal arrows. They corresponds to $(8.2, 8.3, 8.4, 8.5, 8.6)$ respectively. We leave the details to the readers.
To compute \( r_{(\beta_\alpha, \beta_\gamma)}(A_2(Q)) \), we only need to count the incidence varieties. We use the transitive action of \( \text{GL}_n \) and count the stabilizers. The following formulas are immediate.

\[
\frac{|F_{(\beta_u, \beta_d, \gamma_u, \gamma_d)}|}{|\text{GL}_{(\alpha_u, \alpha_d)}|} = \frac{|\beta_u, \gamma_u \rangle_{0}^{-1} \langle \beta_d, \gamma_d \rangle_{0}^{-1}}{|\text{GL}_{\gamma_u} \| \text{GL}_{\beta_u} \| \text{GL}_{\gamma_d} \| \text{GL}_{\beta_d}|} \quad \text{and} \quad \frac{|\text{In}_{e_d, e \beta_e}^{c_d, c_e}(\gamma_d)|}{|\text{GL}_{\gamma_d}|} = \frac{(c - c_e, c_e - c_d)_{0} \langle d - c_d, c_d \rangle_{0} \langle e - d - c_e + c_d, d + c_e - c_d \rangle_{0} \langle \gamma_d - c - e + c_e, c + e - c_e \rangle_{0}^{-1}}{(c - c_d, c_d)_{0} \langle GL \_d \| \text{GL}_{\epsilon - c} \| \text{GL}_{\rho - d - c} \| \text{GL}_{\gamma_d - c - e + c_e}|}
\]

\[
\frac{|\text{Gr}_{e_d}^{b} (\beta_u)|}{|\text{GL}_{\beta_d}|} = \frac{|\langle e - d, b + d - c \rangle_{0}^{-1} \langle b + d, \beta_u - b - d \rangle_{0}^{-1}}{|\text{GL}_{\beta_u} \| \text{GL}_{b} \| \text{GL}_{\rho - d}|}
\]

\[
\frac{|\text{Gr}_{\epsilon}^{\gamma_u} (\gamma_u) \| \text{Gr}_{b} (\beta_d)|}{|\text{GL}_{\gamma_u} \| \text{GL}_{\beta_d}|} = \frac{|\langle c, \gamma_u - c \rangle_{0}^{-1} \langle \beta_d - b, b \rangle_{0}^{-1}}{|\text{GL}_{\epsilon} \| \text{GL}_{\epsilon - c} \| \text{GL}_{b} \| \text{GL}_{\beta_d - b}|}
\]

Put the fibre bundle structure and these equations together, and we obtain what we desire.

This result can be generalized to the s-step Frep varieties. So we conclude that the algebra \( A_2(Q) \) is \( F \)-polynomial-count. For the 1-step case, it suffices to set \( \beta_u = \beta_d = 0 \).

Corollary 8.2.

\[
r_{(\alpha, \beta)}(A_2(Q)) = \sum_{\beta_1 \beta_2} \langle \beta_1, \beta_2 \rangle_{-1} \langle \beta_2, \alpha - \beta_2 \rangle_{-1} r_{\beta_1}(Q) r_{\beta_2}(Q) r_{\alpha - \beta_2}(Q).
\]

This formula has a dual version:

\[
r_{(\alpha, \beta)}(A_2(Q)) = \sum_{\alpha_1 \alpha_2} \langle \alpha_1, \alpha_2 \rangle_{-1} \langle \beta - \alpha_1, \alpha_1 \rangle_{-1} r_{\alpha_1}(Q) r_{\beta - \alpha_1}(Q) r_{\alpha_1}(Q).
\]

Remark 8.3. Alternatively, this corollary can be proved by a Hall algebra method similar to Lemma 5.1. Consider the following identity in the algebra \( \mathit{H}(Q) \otimes \mathit{H}(Q) \).

\[
\left( [0] \otimes \sum_{[U]} [U]\right) \left( \sum_{[M], [V]} |\text{Epi}_Q(M, V)| |M| \otimes |V| \right) = \sum_{[M], [W]} |\text{Hom}_Q(M, W)| |M| \otimes |W|.
\]

Applying the character \( \int \otimes \int \) to the both sides, we see the result immediately.

It follows from Lemma 2.5 and 8.1 that

Theorem 8.4. If \( \text{Mod}_{\alpha}^{\mu}(A_2(Q)) \) is a geometric quotient, then it has a counting polynomial, which can be explicitly computed.

We will see in the last section that the assumption of being a geometric quotient is unnecessary. This result is known [5] Theorem 4.3 for some special choices of \( \alpha \) and \( \mu \).

Example 8.5. Consider the 3-arrow Kronecker quiver \( K_3 \) with dimension vectors \( \alpha = (3, 4) \) and \( \gamma = (1, 3) \). Let \( M \) be a general representation of dimension \( \alpha \), then \( M \) has no subrepresentation of dimension \( (1, 2) \). So the projection \( \text{Gr}_\gamma(M) \rightarrow \text{Gr}_1(M_1) \cong \mathbb{P}^2 \) is an isomorphism. We can use the algorithm in [5] Corollary 4.4] to find that

\[
|\text{Mod}_{\alpha}^{\mu}(K_3)| = (1, 0, 1)^2(1, 1, 1, 3, 5, 3, 1, 1, 1),
\]

\[
|\text{Mod}_{(\gamma, \alpha)}^{\mu}(A_2(K_3))| = [3][2]^2(1, 4, 2, 8, 5, 8, 2, 4, 1),
\]
where $\hat{\mu} = \frac{\hat{\sigma}}{\sigma}$ is the slope function constructed in [5, Section 1]. Recall that $\hat{\sigma} = \epsilon(\gamma_1 + \gamma_2)$ for some sufficiently small $\epsilon$. Now we change $\hat{\sigma}$ to $\hat{\sigma} = \epsilon\gamma_1$, then

$$|\text{Mod}^\mu_{(\gamma_1,\alpha)}(A_2(K_3))| = |\mathbb{P}^\mu| \cdot |\text{Mod}^\mu_\sigma(K_3)|.$$

There is no difficulty to generalize the above results to $A_n(Q)$. In fact, we will do it more generally in [6].

Conjecture 8.6. If $E$ is $\text{F}$-polynomial-counting, then $kQ[E] \otimes kA_2$ is $\text{F}$-polynomial counting.

9. $\Delta$-ANALOG

Let us come back to general $A = kQ/I$. Consider the map $\int_{\Delta(\gamma)}[W] = \frac{|\text{Gr}_\gamma(W)|}{a_W}$ as in [5, Section 2]. If we apply this map to $\chi^\mu_\alpha$, we get

$$\int_{\Delta(\gamma)} \chi^\mu_\alpha = \sum_{W \in \text{mod}_\mu^\mu(A)} a_W^{-1} |\text{Gr}_\gamma(W)|.$$

We known from [5, Lemma 1.2] that when $\text{Mod}^\mu_\sigma(A)$ is a geometric quotient, this number is equal to $(q - 1)^{-1} |\text{Mod}^\mu_{(\gamma,\alpha)}(A \otimes kA_2)|$ for some slope $\hat{\mu}$. To compute $\int_{\Delta(\gamma)} \chi^\mu_\alpha$, we apply $\int_{\Delta(\gamma)}$ to (2.1) as before. We define

$\text{Frep}_{\alpha_1,\cdots,\alpha_s}(A) = \{(M, L_1, \ldots, L_{s-1}, S) \in \text{Frep}_{\alpha_1}(A) \times \text{Gr}_\gamma(A) \mid S \subset M\}$,

then

$$\int_{\Delta(\gamma)} \chi_{\alpha_1 \cdots \alpha_s} = |\text{Frep}_{\alpha_1,\cdots,\alpha_s}(A)|/|\text{GL}_\alpha|.$$

Let

$\text{Fl}^{\gamma_{1},\gamma_{2}}_{\alpha_1} = \{(M, L_1, \ldots, L_{s-1}, S) \in \text{Fl}_{\alpha_1}(M) \times \text{Gr}_\gamma(M) \mid \dim \pi_i(S \cap L_i) = \gamma_i\}$,

where $\pi_i: L_i \to L_i/L_{i-1}$ is the projection. Then $\text{Frep}_{\alpha_1,\cdots,\alpha_s}(A)$ is stratified by the locally closed subvarieties

$\text{Frep}_{\alpha_1,\cdots,\alpha_s,\gamma_{1},\gamma_{2}}(A) := \text{Frep}_{\alpha_1,\cdots,\alpha_s}(A) \cap \text{Fl}^{\gamma_{1},\gamma_{2}}_{\alpha_1}.$

When $s = 2$, for any decompositions $\alpha = \beta + \gamma$ and $a = b + c$, $\text{Fl}_{\beta,\gamma}^{b,c}$ is the same as the incidence variety

$$\text{Gr}_{\beta,\gamma}^{a}(\alpha) = \{(U, V) \in \text{Gr}_a(M) \times \text{Gr}_\gamma(M) \mid \dim(U \cap V) = c\}.$$

The proof of the following lemma is similar to that of Lemma 5.1 and 8.1 so we leave it for the readers.

Lemma 9.1. $p: \text{Frep}_{\beta,\gamma}^{b,c}(Q^o[\epsilon]) \to \text{Gr}_{\hat{\alpha},\gamma}^{\hat{\beta},\gamma}(\hat{\alpha})$ is a fibre bundle with fibre

$$\begin{align*}
\text{Rep}_{\beta-b,a_+}(Q^o[\epsilon]) \times \text{Rep}_{b,a_+}(Q^o[\epsilon]) \times \text{Rep}_{\gamma-c,\gamma_+}(Q^o[\epsilon]) & \times \prod_{a \in Q_1} \text{Hom}(k^{(\beta-b)(a)}(\gamma), \gamma_+) \times \text{Hom}(k^{(b)(a)}(\gamma), \gamma) \times \text{Hom}(k^{(\gamma-c)(a)}(\gamma), \gamma).
\end{align*}$$

So

$$r^{\gamma}_{\beta,\gamma}(Q^o[\epsilon]) = \sum_{b+c=a} t_{(\beta,\gamma,b,c)} \cdot r_{(\beta-b,a)(\gamma)} \cdot r_{(b,a)(\gamma)} \cdot r_{\gamma} \cdot r_{\gamma}^{\gamma},$$

where $t_{(\beta,\gamma,b,c)} = \frac{\alpha_+}{\gamma_+} \cdot \frac{\gamma_+}{(\beta-b)_{+}} \cdot \frac{\gamma_+}{(b,c)(\gamma-c)}_{+} |\text{GL}_{\gamma+b}^{a_+} \mid |\text{GL}_{\alpha_+} |$, and $r^{\gamma} = r_{\gamma}(Q^o[\epsilon])$. 
Readers can easily write out the formula for the dual case \(Q[E]\). This lemma can be recursively generalized to the \(s\)-step case: \(p : \text{Frep}_{\gamma_s, \ldots, \gamma_1}(A) \to \text{Fl}_{\gamma_s, \ldots, \gamma_1}\).

**Theorem 9.2.** If \(E\) is add-polynomial-count and \(\text{Mod}^{\mu}(Q[E])\) is a geometric quotient, then \(\sum_{M \in \text{Mod}^{\mu}(Q[E])} |\text{Gr}_{\gamma}(M)|\) is polynomial-count for any \(\gamma\).

As in [5, Section], we can also consider the \(t\)-step analog of \(\Delta(r)\):

\[
\int_{\Delta^t(\gamma_t, \ldots, \gamma_1)} [W] = aW^{-1}|\text{Fl}_{\gamma_t, \ldots, \gamma_1}(W)|.
\]

Everything can be generalized to this case without any essential difficulty.

### 10. S-analog

Finally we consider the map \(\mathcal{F}\) from the Hall algebra \(H(A)\) to the formal power series algebra \(\mathbb{Q}(q)[[x]]\) in \(|Q_0|\) variables as in [5, Section 8]:

\[
\mathcal{F}[W] = aW^{-1}\sum_{i=0} (-1)^{i+1} F_i(W)x^\alpha,
\]

where \(F_i(W)\) is the number of \(i\)-step filtrations of \(W\). We recall from [5, Lemma 8.3] that the number \(\sum_{i=0} (-1)^{i+1} F_i(W)\) has a neat formula in terms of the multiplicities of simple summands of \(W\).

Fix a slope function \(\mu\) and a slope \(\mu_0 \in \mathbb{Q}\). Let \(\text{mod}_{\mu_0}(A)\) be the abelian subcategory of all semistable representations with slope \(\mu_0\), and \(\chi_{\mu_0} = \sum_{M \in \text{mod}_{\mu_0}(A)} [M]\).

Let us denote by \(a_{\alpha}\) the number of \(\alpha\)-dimensional absolutely stable representations and \(m_{\alpha} = |\text{Mod}^{\mu}(A)|\).

**Definition 10.1.** The absolute (resp. relative) Poincaré series of \(\text{Rep}(A)\) at \(\mu_0\) is \(A_{\mu_0}(A) = \sum_{\mu(\alpha) = \mu_0} a_{\alpha}(q)x^\alpha\) (resp. \(M_{\mu_0}(A) = \sum_{\mu(\alpha) = \mu_0} m_{\alpha}(q)x^\alpha\)). Our convention is that the relative ones have constant term 1, but 0 for the absolute ones.

It was proved in [10, Theorem 4.1] that

\[
\mathcal{F}[\chi_{\mu_0}] = \text{Exp} \left( \frac{A_{\mu_0}(Q)}{1 - q} \right),
\]

where \(\text{Exp}\) is the plethystic exponential in the \(\lambda\)-ring \(\mathbb{Q}(q)[[x]]\) [10, Section 2]. Moreover, it is known [13, Theorem 8.3] that

\[
M_{\mu_0}(Q) = \text{Exp} \left( A_{\mu_0}(Q) \right).
\]

Actually, for both the argument would work for any algebra not necessarily hereditary.

To compute \(\mathcal{F}[\chi_{\mu_0}]\), we apply \(\mathcal{F}\) to each individual \(\chi_{\alpha}\) for \(\mu(\alpha) = \mu_0\) using (2.1).

### Theorem 10.2.

The assumption of being a geometric quotient in Theorem 5.5 and 8.4 can be dropped.
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