Hadron mass scaling near the s-wave threshold

Tetsuo Hyodo
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan
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We study the influence of a two-hadron threshold when the hadron mass scales with respect to some QCD parameters. The general behavior of the energy of the bound and resonance state near the two-body threshold for a local potential is derived from the expansion of the Jost function around the threshold. We show that the same scaling holds for the non-local potential induced by the coupling to a bare state. In p- or higher partial waves, the scaling law of the stable bound state continues across the threshold describing the real part of the resonance energy. In contrast, the leading contribution of the scaling is forbidden by the nonperturbative dynamics near the s-wave threshold. As a consequence, the bound state energy is not continuously connected to the real part of the resonance energy. This universal behavior originates in the vanishing of the field renormalization constant of the zero-energy resonance in s wave. A proof is given for the vanishing of the field renormalization constant, together with the detailed discussion.

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I. INTRODUCTION

The properties of hadrons are determined by highly nonperturbative dynamics of QCD. In some cases, however, the mass of hadrons can be expressed by a systematic expansion of certain QCD parameters. The mass of hadrons with light quarks is expanded by the light quark mass \( m_q \) in chiral perturbation theory [1]. When the hadron contains one heavy quark, its mass can be given in powers of the inverse of the heavy quark mass \( 1/m_Q \), with the leading contribution of \( O(m_Q) \) [2]. It is also possible to express the mass of hadrons in the \( 1/N_c \) expansion [3]. These expansions dictate the scaling laws of the hadron mass as functions of the dimensionless parameter \( x = m_q/\Lambda, \Lambda/m_Q \), and \( 1/N_c \) with \( \Lambda \) being the nonperturbative energy scale of QCD. The leading contribution of the expansions is determined by the construction of hadrons. For instance, the mass of the Nambu-Goldstone bosons scales with \( (m_q/\Lambda)^{3/2} \), while the expansion of other light hadrons contains a constant term as the leading contribution. In the heavy sector, the mass of the singly heavy hadrons primarily scales as \( (\Lambda/m_Q)^{-1} \) for the heavy quark mass and \( (m_q/\Lambda)^0 \) for the light quark mass. The ordinary mesons and ordinary baryons behave as \( (1/N_c)^0 \) and \( (1/N_c)^{-1} \), respectively. The higher order corrections can also be calculated systematically. The mass scaling is useful, for instance, to extrapolate the results of the lattice QCD simulation to the physical quark mass region.

Because each hadron has its own scaling law, one may encounter the situation where a hadron mass goes across a two-hadron threshold with the same quantum numbers. For instance, the signal of the \( \Lambda \Lambda \) dibaryon is found as \( (1/\Lambda)^2 \) slightly above the \( \Lambda \Lambda \) threshold at the physical point [6].

In fact, many hadron resonances are obtained as stable bound states in the heavy \( m_q \) simulation, because the lowest two-hadron threshold usually contains the pion whose mass grows as \( \sqrt{m_q} \).

When the hadron mass moves across the threshold, one may naively expect that the same scaling law with the bound state energy describes the real part of the resonance energy above the threshold. It should be however noted that the threshold dynamics at the hadronic level is also highly nonperturbative [7]. For instance, the two-body scattering length in the s-wave channel diverges when the binding energy is sent to zero. Thus, we have to carefully examine the coupling effect to the two-hadron channel in the threshold energy region.

II. FORMULATION

In the following, we consider the behavior of the mass scaling when a hadron (hereafter called the bare state) approaches a two-body threshold from the lower energy side. We focus on the lowest energy two-body threshold for the same quantum numbers with the bare state. This two-body channel will be referred to as the scattering channel. We consider the near-threshold kinematics where the nonrelativistic treatment is applicable. We assume that there is no long range interaction.

The effect of the scattering channel is described by the coupled-channel Hamiltonian [8]

\[
\begin{pmatrix}
\hat{H}_0 & \hat{V} \\
\hat{V} & \hat{H}_{sc}
\end{pmatrix}
\begin{pmatrix}
\Psi
\end{pmatrix}
= E
\begin{pmatrix}
\Psi
\end{pmatrix},
\begin{pmatrix}
\Psi
\end{pmatrix}
= \begin{pmatrix}
c(E)|\psi_0\rangle \\
c(E)|\chi_{E}(p)|\langle p|
\end{pmatrix},
\]

where \( \hat{H}_0 (\hat{H}_{sc}) \) is the Hamiltonian for the bare state (scattering) channel, \( \hat{V} \) is the transition potential, and \( c(E)|\chi_{E}(p)| \) is the wavefunction for the bare state (scattering) channel component \( |\psi_0\rangle (|p\rangle) \). In the scattering channel, the eigenvalue is \( \hat{H}_{sc}|p\rangle = p^2/(2\mu)|p\rangle \) with

* hyodo@yukawa.kyoto-u.ac.jp
the reduced mass $\mu$ for the bare state channel, we have $H_0|\psi_0\rangle = M_0|\psi_0\rangle$ where $M_0$ is the energy of the bare state measured from the threshold in the absence of the scattering channel. We consider that the scaling of $M_0(x)$ is known with respect to the QCD parameter $x$. Our aim is to determine the scaling of the eigenenergy of the coupled-channel Hamiltonian $\hat{E}_h(x)$. This enables us to relate the eigenenergy $E_h$ and the bare state energy $M_0$.

We first consider the eigenenergy of the system (1) for a fixed $x$. To this end, we eliminate the bound state channel by the Feshbach method \[11, 12\]. The effective potential which acts on the scattering channel is given by

$$\hat{V}_{\text{eff}}(E) = \frac{\langle \psi_0 | \hat{V} | \psi_0 \rangle}{E - M_0 - \Sigma(E)}.$$  \hspace{1cm} (2)

Solving the Lippmann-Schwinger equation, we obtain the two-body scattering amplitude as

$$f(p, p', E) = -\frac{4\pi^2 \hat{V} \langle \psi_0 | \hat{V} | \psi_0 \rangle}{E - M_0 - \Sigma(E)},$$  \hspace{1cm} (3)

where the self-energy is defined as

$$\Sigma(E) = \int \frac{\langle \psi_0 | \hat{V} | q \rangle \langle q | \hat{V} | \psi_0 \rangle}{E - q^2/(2\mu) + i\epsilon} q^2 q.$$  \hspace{1cm} (4)

The eigenenergy $E_h$ of the Hamiltonian is identified from the pole of the amplitude \[9\], namely,

$$E_h - M_0 = \Sigma(E_h).$$  \hspace{1cm} (5)

For a sufficiently large $|E_h|$, the self-energy behaves as $\Sigma(E_h) \sim 1/E_h$ where the scattering state contribution is suppressed and we obtain $E_h \sim M_0$. This means that the effect of the scattering channel is negligible in the energy region far away from the threshold, and the scaling of the eigenenergy $E_h(x)$ can be well described by the scaling of the bare mass $M_0(x)$, as naively expected. Nontrivial behavior emerges near the threshold.

III. THRESHOLD BEHAVIOR FROM THE JOST FUNCTION

To focus on the near-threshold phenomena, we first adjust the QCD parameter $x$ such that the eigenenergy appears exactly on top of the threshold ($E_h = 0$). This corresponds to setting the bare mass as $M_0 = -\Sigma(0) > 0$.\[13\]

In this case, the scattering amplitude has a pole at zero energy. The pole of the amplitude is equivalent to the zero of the Jost function $\gamma_l(p)$ (Fredholm determinant) for the $l$-th partial wave with the eigenmomentum $p = \sqrt{2\mu E_h}$. The properties of the Jost function are summarized in Appendix A (see also Ref. [13]). From the expansion of the Jost function around $p = 0$ in Eq. (A6), when $\gamma_l(p) = 0$ at $p = 0$, it can be expanded as

$$\gamma_l(p) = \begin{cases} \alpha_0 p + \mathcal{O}(p^2) & l = 0, \\ \beta_l p^2 + \mathcal{O}(p^3) & l \neq 0. \end{cases}$$  \hspace{1cm} (6)

The real expansion coefficients $\alpha_0$ and $\beta_l$ are determined by the potential and the wavefunction. It is shown for a general local potential that $\gamma_l(p)$ goes to zero exactly as $p (p^2)$ for $l = 0 (l \neq 0)$ \[14\] so that $\alpha_0$ and $\beta_l$ are guaranteed to be nonzero. This means that the zero of the Jost function at the threshold is simple for $l = 0$, while it is double for $l \neq 0$. For the non-local potential [2], we cannot directly apply the result of Ref. [14]. We will nevertheless demonstrate in Secs. [15] and [16] that the same scaling law is derived for the potential [2], as long as the pole exists at the threshold.

We then shift the bare mass as $M_0 \rightarrow M_0 + \delta M$ by changing the QCD parameter $x$ and examine the modification of the eigenenergy. For a given $M_0$, it is always possible to consider a sufficiently small shift $\delta M \ll M_0$. The effective potential at zero energy is then modified by

$$\hat{V}_{\text{eff}} \rightarrow \left(1 + \frac{\delta M}{-M_0}\right)\hat{V}_{\text{eff}} \equiv (1 + \delta \lambda)\hat{V}_{\text{eff}},$$  \hspace{1cm} (7)

where $\delta \lambda = -\delta M/M_0$. Thus, the small shift of the bare mass results in the multiplicative modification of the strength of the effective potential. When the bare mass $M_0$ is decreased (increased), the strength of the potential is enhanced by $\delta \lambda > 0$ (reduced by $\delta \lambda < 0$) and we expect to have a bound (resonance) state. For a positive $\delta \lambda$, the eigenmomentum in the leading order of $\delta \lambda$ is given by (see Appendix A)

$$p = i(\alpha_0/\gamma_0)\delta \lambda \quad l = 0, \quad \beta_l p^2 = -(\alpha_l'/\beta_l)\delta \lambda \quad l \neq 0,$$  \hspace{1cm} (8)

with $\alpha_l' = d\alpha_l/d(\delta \lambda)|_{\delta \lambda = 0}$. Thus, the energy of the bound state is

$$E_h = \begin{cases} -F_0 \delta \lambda^2 = -\tilde{F}_0 \delta M^2 & l = 0, \\ -F_l \delta \lambda = \tilde{F}_l \delta M & l \neq 0. \end{cases}$$  \hspace{1cm} (9)

with the positive coefficients $F_0 = (\alpha_0^2)/(2\mu \gamma_0^2)$, $F_l = (\alpha_l^2)/(2\mu \gamma_l^2)$ ($l \neq 0$), $\tilde{F}_0 = F_0/M_0^2$, and $\tilde{F}_l = F_l/M_0$ ($l \neq 0$). We find that, with a small increase of the potential strength by the factor $1 + \delta \lambda$ (small decrease of the bare mass $\delta M$), the binding energy grows linearly in $\delta \lambda$ ($\delta M$) for $l \neq 0$ and quadratically for $l = 0$.

This result can be analytically continued to the negative $\delta \lambda$ region. For $l = 0$, the eigenenergy is negative. This solution corresponds to the virtual state, because the eigenmomentum has the opposite sign from the

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1. Possible residual two-body potential $\hat{V}_{\text{eff}}$ can be treated perturbatively by a proper field redefinition \[6, 10\].

2. For a finite coupling of the scattering state and the bare state, $\Sigma(0)$ must be nonzero. The case $E_h = M_0 = 0$, which corresponds to the vanishing of the coupling, will be separately discussed in Sec. [16].
bound state. On the other hand, for \( l \neq 0 \), the eigenenergy becomes complex, so the pole represents the resonance solution. The real part is determined by the same energy becomes complex, so the pole represents the resonance energy \((l \neq 0)\) and the energy of the virtual state \((l = 0)\), and the dashed line represents the imaginary part of the resonance energy \((l \neq 0)\).

FIG. 1. (Color online) Schematic illustration of the near-threshold eigenenergy \( E_h \) for \( l = 0 \) (a) and \( l \neq 0 \) (b) as a function of the variation of the potential \( \delta \lambda \) or the variation of the bare mass \( \delta M \). The solid lines represent the bound state energy, the dotted lines stand for the real part of the resonance energy \((l \neq 0)\) and the energy of the virtual state \((l = 0)\), and the dashed line represents the imaginary part of the resonance energy \((l \neq 0)\).

For positive (negative) \( 1/a_0 = 0 \) is simple, in accordance with the Jost function analysis. The eigenmomentum is found to be

\[
p = i/a_0. \tag{13}
\]

For positive (negative) \( 1/a_0 \), the eigenmomentum is positive (negative) pure imaginary, which corresponds to the bound (virtual) state solution. In order to obtain the resonance solution above the threshold, we need the contribution from the negative effective range \([15]\). Even in this case, the low energy behavior \( p \ll \sqrt{2/|a_0 r_0|} \) is governed by Eq. \((13)\).

For \( l \geq 1 \) (in three dimension), the effective range parameter cannot be neglected in the low energy because of the causality bound \([16]\). This is intuitively understood by the dominance of the \( p^2 \) term in comparison with the \( ip^{2l+1} \) term. The low-energy amplitude behaves as \( f_l(p) \propto (-1/a_1 + r_l p^2/2)^{-1} \) so the pole at \( p = 0 \) is double. This allows the direct transition from the bound state to the resonance for \( l \neq 0 \).

### IV. THRESHOLD BEHAVIOR AND FIELD RENORMALIZATION CONSTANT

We now show that the threshold formula \([10]\) is derived from the non-local potential \([2]\) by the expansion of Eq. \((5)\). Near the threshold, Eq. \((5)\) is given by

\[
E_h - M_0 - \delta M = \Sigma(E_h). \tag{14}
\]

We have found that \( E_h \) is of the order of \( \delta M \), so we may regard \( E_h \) is sufficiently small. Expanding \( \Sigma(E_h) \) around \( E_h = 0 \), we obtain

\[
E_h = \frac{1}{1 - \Sigma^\prime(0)} \delta M, \quad \Sigma^\prime(E) \equiv \frac{d\Sigma(E)}{dE}, \tag{15}
\]

in the leading order of \( E_h \). The derivative of the self-energy is related to the field renormalization constant \( Z \) which expresses the elementariness of the bound state \([15,21]\). To calculate \( Z \), we use the relation between the channel coefficients

\[
\chi_{E_h}(q) \left( E_h - \frac{q^2}{2\mu} \right) = c(E_h) \langle \psi_0 | \hat{V} | q \rangle, \tag{16}
\]

which is obtained from Eq. \((14)\). Because the wavefunction of the bound state is normalized, we also have a relation

\[
|c(E_h)|^2 + \int |\chi_{E_h}(q)|^2 d^3q = 1. \tag{17}
\]

Using these relations, we obtain the field renormalization constant \( Z(E_h) \) as the overlap of the bound state wave function with the purely bare state \( \psi_0 \) as

\[
Z(E_h) = \left| \langle \psi_0 | 0 \rangle \right|^2 = |c(E_h)|^2 = \frac{1}{1 - \Sigma^\prime(E_h)}. \tag{18}
\]

It is shown that \( Z \) takes the value \( 0 \leq Z \leq 1 \) \([21]\). Because of the normalization \((17)\), \( 1 - Z = \int |\chi_{E_h}(q)|^2 d^3q \) corresponds to the compositeness which expresses the probability of finding the scattering (two-body molecule)

\[3\] Here we assume that both \( a_0 \) and \( r_0 \) are finite. The case with infinitely large effective range will be discussed in Sec. \([7,9]\).
component in the bound state. Thus, in Eq. (15),
the leading contribution to \( E_h \) from the shift of the bare mass \( \delta M \) is given by the field renormalization constant at zero binding energy

\[
E_h = Z(0) \delta M.
\]  

(19)

As we will show in Sec. V A to have a pole at threshold for \( l = 0 \), \( Z(0) \) must vanish. Because this is a subtle problem, we present a detailed discussion for \( Z(0) = 0 \) in Sec. V [3]. In the present context, the vanishing of the field renormalization constant \( Z(0) \) forbids the contribution proportional to \( \delta M \). This ensures the s-wave scaling \( E_h \propto \delta M^2 \) in Eq. (19).

For \( l \neq 0 \), \( Z(0) \) expresses the elementariness of the zero energy bound state. When \( Z(0) = 1 \), the bound state is regarded as a purely elementary state which is decoupled from the scattering channel. This is natural because the eigenvalue is given by \( E_h = \delta M \) so that the scaling law of the bare mass is not modified by the threshold effect, as a consequence of the decoupling from the scattering channel. Comparing Eq. (15) with the expansion of the Jost function, we obtain

\[
\frac{\Sigma(0)}{1 - \Sigma'(0)} = - \frac{\alpha'_l}{2 \mu \beta_l} \quad \text{for } l \neq 0,
\]  

(20)

which relates the self-energy and the expansion coefficients of the Jost function.

We remark that the field-renormalization constant is a model-dependent quantity. At first glance, however, one may think that \( Z(0) \) for nonzero \( l \) can be extracted from the hadron mass scaling near the threshold using Eq. (19). This is unfortunately not the case, because the relation between the QCD parameter \( x \) and the bare mass \( \delta M \) inevitably specifies the basis to measure \( Z(0) \). In other words, the definition of the bare hadron mass \( \delta M \) in QCD is model dependent.

V. COMPOSITENESS THEOREM

We have shown in Sec. IV that vanishing of the field renormalization constant is essential for the mass scaling in s wave. Here we prove this “compositeness theorem”. The statement is as follows.

*If the s-wave scattering amplitude has a pole exactly at the threshold with a finite range interaction, then the field renormalization constant vanishes.*

It is important to recall the different nature of the pole at the threshold for \( l = 0 \) and for \( l \neq 0 \). The pole at the threshold is a ordinary bound state in \( l \neq 0 \) case, while the s wave pole represents the special state called zero energy resonance [12]. It follows from the Schrödinger equation that the wavefunction at zero energy behaves as \( 1/r^l \) at large \( r \). The wavefunction is therefore normalizable for \( l \neq 0 \), while with \( l = 0 \) the wavefunction is not square integrable and does not represent a bound state. In this case, even with the finite range interaction, the wavefunction spreads to infinity. This is caused by the divergence of the scattering length, which is essential for the low energy universality in few-body systems [2].

A naive interpretation of the theorem \( Z(0) = 0 \) would be that the zero energy resonance is a purely composite state. However, a finite elementary component \( |c(E_h)|^2 \) is not necessarily excluded from the wavefunction. In the \( B \to 0 \) limit, the wavefunction of the scattering state spreads to infinity. In this case, because of the normalization (17), the fraction of the finite elementary component is zero, in comparison with the infinitely large scattering component [4]. Thus, we obtain \( Z(0) = 0 \) even with any finite admixture of the elementary component, because of the property of the scattering state.

In the following, we first give a proof of the theorem for the non-local potential (2) in Sec. V A. In Secs. V B and V C, we show that the theorem is valid for a general local potential, using the effective range expansion and the pole counting argument, respectively.

We emphasize that the \( B \to 0 \) limit is qualitatively different from the finite \( B \) case. For instance, \( Z(B) = 0 \) with a finite \( B \) implies the complete exclusion of the elementary component, because the scattering component is also finite. We discuss the structure of the bound state for finite \( B \) in Sec. V D. We show that for finite \( B \), the value of \( Z(B) \) is in principle arbitrary. The connection of the finite \( B \) and \( B \to 0 \) limit becomes clear by considering the decoupling limit in Sec. V E.

A. Proof

We consider the field renormalization constant \( Z \) with the potential (2). As shown in Eq. (19), \( Z \) for the bound state with the binding energy \( B = -E_h > 0 \) is related to the derivative of the self-energy as

\[
Z(B) = \frac{1}{1 - \Sigma'(-B)}, \quad \Sigma'(-B) = - \frac{d\Sigma(-B)}{dB}.
\]  

(21)

The s-wave self-energy (3) is given by

\[
\Sigma(-B) = -4\pi \sqrt{2\mu} \int_0^\infty dE' \sqrt{E'} |F(E')|^2 \frac{E'}{E' + B},
\]  

(22)

where we define the spherical s-wave form factor of the bare state \( F(E') = \langle \psi_0 | \hat{V} | q \rangle \) with \( E' = |q|^2/(2\mu) \). In order to reproduce the low energy limit of the scattering amplitude \( f_0(p) \to (\text{const}) \) with Eq. (21), the factor

\[4\text{ The usual normalization } \langle \Psi | \Psi \rangle = 1 \text{ is not applicable to the state vector with an infinite norm, such as the zero-energy resonance. The normalization of resonances is nevertheless ensured by the use of the Gamow vectors in the rigged Hilbert space [22, 23].}\]
Eq. (22) is sufficiently suppressed. With these conditions, \( B \) with a constant at small \( E \) energy at the threshold can also be shown by the spectral weight of the self-energy.

We note that the divergence of the derivative of the self-energy is calculated as

\[
\Sigma'(B) \approx -4\pi \sqrt{\mu_B} \sum_{\text{max}} \int_0^{\text{max}} \frac{dE' \sqrt{E'} g_0^2 [1 + \mathcal{O}(E')]}{E' + B} \cdot \sqrt{E_{\text{max}} - \sqrt{B} \arctan \left( \frac{\sqrt{E_{\text{max}}}}{B} \right)} + \ldots
\]

\[
= g_0^2 \left[ (\text{const.}) + \mathcal{O}(B^{1/2}) \right]
\]

\[
\left. \frac{\Sigma'(B)}{B \rightarrow 0} \right|_{\text{(finite)}}.
\]

The derivative of the self-energy is calculated as

\[
\Sigma'(B) \propto g_0^2 \left[ \frac{1}{\sqrt{B}} \arctan \left( \frac{\sqrt{E_{\text{max}}}}{B} \right) + \ldots \right]
\]

\[
= g_0^2 \left[ \frac{\pi}{2\sqrt{B}} + \mathcal{O}(B^0) \right]
\]

\[
\left. \frac{\Sigma'(B)}{B \rightarrow 0} \right|_{\text{(finite)}}.
\]

Thus, we find that the field renormalization constant vanishes in the \( B \rightarrow 0 \) limit:

\[
Z(B) = \frac{1}{1 - \Sigma'(B)} \left. \frac{B \rightarrow 0}{\text{finite}} \right|_{\text{(finite)}}.
\]

We note that the divergence of the derivative of the self-energy at the threshold can also be shown by the spectral representation [24]. The essential point is that the term \( \sqrt{B} \arctan(1/\sqrt{B}) \) in Eq. (24) below the threshold is a consequence of the analytic continuation of the imaginary part of the self-energy above the threshold. Because the imaginary part of the self-energy is constrained by the dispersion relation, Eq. (25) always holds.

The only exception to the above argument is the case with \( g_0^2 \rightarrow 0 \) in the \( B \rightarrow 0 \) limit where \( \Sigma'(0) \) and \( Z(0) \) can be finite. However, the absence of the coupling to the scattering state also indicates that the bare state cannot affect the scattering amplitude. Namely, we obtain the scattering amplitude for non-interacting particles which does not have a pole at the threshold. This contradicts the assumption of having a pole at \( p = 0 \).

Thus, we are left with the case for nonzero \( g_0^2 \) and the theorem is proved by Eq. (20). We will come back to the \( g_0 \rightarrow 0 \) limit in Sec. V E to discuss the structure of the bound state.

\[\text{---}\]

5 The non-analytic term is in the denominator of the amplitude comes from the self-energy.

B. Effective range expansion and composite theorem

Next we discuss the bound state from a general local potential, for which the result of Ref. [14] is applicable. We use the expression of the scattering length and the effective range in the weak binding limit [15]

\[
a_0 = \frac{2(1 - Z) R}{2 - Z}, \quad r_0 = \frac{-Z}{1 - Z} R, \quad R = \frac{1}{\sqrt{2\mu_B}}.
\]

where the correction terms of the order of the typical length scale of the interaction are neglected. As shown in Ref. [21], this formula provides the criteria to judge the structure of near-threshold bound state:

\[
\begin{cases}
a_0 < -r_0 & Z \sim 1, \text{(elementary dominance)} \\
a_0 \sim R \gg r_0 & Z \sim 0, \text{(composite dominance)}
\end{cases}
\]

In the limit \( B \rightarrow 0 \), we have \( R \rightarrow \infty \). If there is no constraint on the value of \( Z(0) \), there are three possibilities:

\[
\begin{cases}
a_0 = \infty, r_0 = (\text{finite}) & Z(0) = 0 \\
a_0 = \infty, r_0 = -\infty & 0 < Z(0) < 1 \\
a_0 = (\text{finite}), r_0 = -\infty & Z(0) = 1
\end{cases}
\]

For \( 0 < Z(0) \leq 1 \), the effective range should diverge. Intuitively, it is unlikely that the finite range interaction provides the infinitely large effective range. More rigorously speaking, \( r_0 = -\infty \) modifies the linear dependence of the eigenmomentum into quadratic in \( p \). This contradicts the fact that the pole at the \( p = 0 \) is simple [14]. Thus, we are left with the case with \( Z(0) = 0 \). In this case, the composite dominance in Eq. (28) is always guaranteed by \( a_0 = \infty \) and finite \( r_0 \). We emphasize again that this is only the dominance of the composite component, not the complete exclusion of the elementary component.

C. Pole counting and composite theorem

The pole counting argument is also useful to understand the meaning of the theorem. Here we also deal with the local potential. In Refs. [25, 26], the structure of the bound state is related to the pole positions in different Riemann sheets of the complex energy plane. For a given bound state pole, if there is a nearby pole in the different Riemann sheet (the shadow pole [27]), then the bound state is dominated by the elementary component. This method is later related to the field renormalization constant [8, 13]. The denominator of the effective range amplitude is a quadratic function of the eigenmomentum \( p \). The pole positions can be analytically calculated as functions of \( a_0 \) and \( r_0 \). Using the relations [27], they can be expressed by the binding energy and \( Z \) as [8].

\[
p_1 = \sqrt{2\mu_B}, \quad p_2 = -i\sqrt{2\mu_B} - \frac{Z}{\sqrt{2\mu_B}}.
\]
The pole \( p_1 \) (\( p_2 \)) is in the first (second) Riemann sheet in the energy plane and corresponds to the bound state (shadow) pole. For \( Z \sim 1 \) (elementary dominance), two poles have a similar energy \( p_1^2/2\mu \sim p_2^2/2\mu \). For \( Z \sim 0 \) (composite dominance), the shadow pole \( p_2 \) goes away from \( p_1 \) and the bound state is essentially described by the pole \( p_1 \).

Now we consider the \( B \to 0 \) limit. If there is no constraint on the value of \( Z(0) \), there are two possibilities:

\[
\begin{align*}
\begin{cases}
p_1 &= 0, p_2 = -i(\text{finite}) \quad : Z(0) = 0 \\
p_1 &= p_2 = 0 \quad : 0 < Z(0) \leq 1 .
\end{cases}
\tag{31}
\end{align*}
\]

In the \( 0 < Z(0) \leq 1 \) case, the pole at the threshold is double. This contradicts the simple pole at the \( p = 0 \) \cite{14}, and we are left with the case with \( Z(0) = 0 \).

### D. Finite binding case

The above discussion is valid for the pole exactly at the threshold. This is an idealization of the physical hadronic states which have a finite binding energy \( B \neq 0 \). Here we consider the bound state with a small but finite binding energy.

For a given \( B \neq 0 \), it is always possible to tune the form factor \( \langle \nu_0 | \mathcal{V} | q \rangle \) and the bare mass \( M_0 \) such that the self-energy \( \Sigma(-B) \) and its derivative \( \Sigma'(-B) \) take arbitrary values. In other words, the value of \( Z(B) \) for \( B \neq 0 \) is in principle arbitrary. In the effective range expansion, for a finite scattering length, it is in principle possible to generate the effective range such that \( a_0 \approx -r_0 \) which leads to the elementary dominance of the bound state. It is only in the \( B \to 0 \) limit where the scattering length diverges and the nonzero \( Z \) is forbidden.

It is instructive to compare the bound state case and resonance case. The arbitrariness of \( Z \) for the bound state stems from the fact that the binding energy \( B \) does not determine both \( a_0 \) and \( r_0 \). In contrast, because the pole position of a near-threshold resonance contains two independent quantities (real and imaginary parts), \( a_0 \) and \( r_0 \) are uniquely determined only by the pole position \cite{15}. What is missing in the bound state case is the position of the shadow pole in the second Riemann sheet. If the position of the shadow pole is given in addition to \( B \), the field renormalization constant is uniquely determined for the bound state.

The weak binding formula \cite{27} relates the field renormalization constant to the observables \((a_0, r_0, \text{and } B)\). Because the observables do not depend on the specific model, it is sometimes mentioned that the structure of the weakly bound state is model-independently determined. Strictly speaking, to derive the weak binding formula \cite{27} one implicitly assumes the absence of the singularity of the inverse amplitude [called Castillejo-Dalitz-Dyson (CDD) pole \cite{28}] between the threshold and the bound state pole \cite{15}. Let us write the position of the closest CDD pole as \( E = -C \) \cite{7}. The effective range expansion breaks down at the singularity of the inverse amplitude closest to the threshold. Thus, if \( -B < -C < 0 \), then the bound state pole locating outside of the valid region of the effective range expansion. In this case, the formula \cite{27} is not applicable and the field renormalization constant cannot be related to the observables. On the other hand, when the effective range expansion is valid at the energy of the bound state pole \(( -C < -B < 0 )\), the field renormalization constant \( Z \) can be related to the observables. Naively, having the CDD pole in the region \( -B < E < 0 \) for a small \( B \) requires a fine tuning, although there is no general principle to exclude this possibility.

### E. Decoupling limit

The bound state pole disappears from the scattering amplitude in the \( g_0 \to 0 \) limit, so this case is not relevant to the study of the mass scaling. Nevertheless, a detailed analysis of this decoupling limit provides an insight on the structure of the bound state. In Sec. V A, the expression of \( Z(B) \) for a small \( B \) is found to be

\[
Z(B) \approx \frac{1}{1 - c B} , \tag{32}
\]

where \( c \) is a nonzero constant determined by kinematics. Taking the \( g_0 \to 0 \) limit with a fixed \( B > 0 \), we have

\[
Z(B) \xrightarrow{g_0 \to 0} 1 \quad \text{for } B > 0 . \tag{33}
\]

This indicates that the bound state in this limit is a purely elementary state. Intuitively, the composite component disappears because of the absence of the coupling to the scattering state. If we decrease \( g_0 \) with a fixed \( B > 0 \) with the potential \cite{25}, we will find that the bare mass \( M_0 \) approaches the bound state pole position. In the \( g_0 \to 0 \) limit, the bare pole locates exactly at \( E = -B \), without the admixture of the scattering state. This is illustrated in Fig. 2 (dotted line).

Although the scattering amplitude does not have the bound state pole, the bare state exists in the decoupled sector and is interpreted as an elementary particle. In other words, the purely elementary state with \( Z = 1 \)

\footnote{After the submission of this paper, Ref. \cite{28} appears on the web, which discusses the near-threshold scaling and its relation to the structure of the bound state. Ref. \cite{28} shows that the elementary dominance is realized by a “significant fine tuning”, and it is natural to expect that the composite (molecular) state appears for small \( B \).}

\footnote{Thus \( E = -B \) is the closest pole and \( E = -C \) is the closest zero of the amplitude.}
cannot appear in the scattering amplitude by definition, because such state does not have the scattering state component. Thus, \( Z = 1 \) state is not realized only in the decoupled sector.

Next we consider the \( g_0 \to 0 \) limit with \( B = 0 \). As shown in Sec. V A, \( Z(0) \) is always zero for finite \( g_0 \). Thus, taking the decoupling limit with keeping \( B = 0 \), we obtain

\[
Z(0) \to 0 \quad \text{for} \quad B = 0. \tag{34}
\]

This is also illustrated in Fig. 2 (solid line). If we compare this result with the \( B \to 0 \) limit of Eq. (38), we find that the two limits \( B \to 0 \) and \( g_0 \to 0 \) do not commute with each other. Namely,

\[
\lim_{B \to 0} \lim_{g_0 \to 0} Z(B) = 1, \tag{35}
\]

while

\[
\lim_{g_0 \to 0} \lim_{B \to 0} Z(B) = 0. \tag{36}
\]

Thus, the value of \( Z(B) \) at \( B = g_0 = 0 \) is indefinite. In fact, if we take the two limits simultaneously, the value of \( Z \) depends on how \( g_0^2 \) approaches zero:

\[
\lim_{g_0,B \to 0} Z(B) = \begin{cases} 
0 & \text{for } g_0^2 \sim B^{1/2-\epsilon} \\
\frac{1}{1-cD} & \text{for } g_0^2 \sim DB^{1/2} \\
1 & \text{for } g_0^2 \sim B^{1/2+\epsilon}
\end{cases} \tag{37}
\]

with a positive \( \epsilon \).

The ambiguity of the limit value of \( Z \) reflects the arbitrariness of \( Z \) with finite \( B \). As discussed in Sec. V D for \( B > 0 \), the bound state with arbitrary \( Z \) can be generated by tuning the model parameters such as \( g_0 \). During the \( B \to 0 \) process, we can continuously tune the parameters such that the value of \( Z \) remains the same. This eventually leads to \( g_0 \to 0 \) in the \( B \to 0 \) limit, otherwise we should have \( Z = 0 \). Thus, if we try to take the \( B \to 0 \) limit with keeping a finite \( Z \), the bound state pole must disappear from the amplitude at the end. In this way, the state with a finite \( Z \) can only be realized in the decoupled sector. In order to maintain the pole in the \( B \to 0 \) limit, \( g_0 \) must be kept finite and the field renormalization constant vanishes at the end.

VI. MODEL CALCULATION

It is illustrative to solve the eigenvalue equation by introducing a specific model for the interaction potential in the \( l \)-th partial wave as

\[
\langle q | \hat{V} | \psi_0 \rangle = \langle \psi_0 | \hat{V} | q \rangle = g_0 |q|^l \Theta(\Lambda - |q|), \tag{38}
\]

with the real coupling constant \( g_0 \) and the cutoff parameter \( \Lambda \). The \( |q|^l \) dependence is chosen to reproduce the low energy behavior of the amplitude \( f_1(p) \sim p^{2l} \). The step function is introduced to tame the ultraviolet divergence. The self-energies for \( l = 0 \) and \( l = 1 \) channels are

\[
\Sigma_0(E) = -8\pi\mu g_0^2 \left[ \Lambda - \sqrt{-2\mu E^+} \arctan \left( \frac{\Lambda}{\sqrt{-2\mu E^+}} \right) \right], \tag{39}
\]

\[
\Sigma_1(E) = -8\pi\mu g_1^2 \frac{\Lambda^3}{3} + 2\mu E \frac{g_1^2}{g_0^2} \Sigma_0(E), \tag{40}
\]

where \( E^+ = E + i0^+ \). We numerically solve the eigenvalue equation \( 14 \) for these self-energies. For \( \delta M < 0 \) (\( \delta M > 0 \)), we choose the first (second) Riemann sheet of the complex energy plane to obtain the bound state (virtual and resonance state) solution. In this setup, the cutoff \( \Lambda \) determines the scale of the system. We choose the coupling constants as \( g_0^2 = \Lambda/(100\mu^2) \) and \( g_1^2 = 1/(40\mu^2\Lambda) \). This leads to \( M_0 \approx 0.25\Lambda^2/\mu \) for \( l = 0 \) and \( M_0 \approx 0.21\Lambda^2/\mu \) for \( l = 1 \).

The near-threshold eigenenergies are shown in Fig. 3 (a) and (b). We find that the near-threshold behavior follows the general scaling in Eqs. 11 and 12: quadratic dependence on \( \delta M \) in s wave and linear dependence in p wave. As shown in Eq. 11, the slope of the binding energy in the p-wave case is determined by the field renormalization constant at zero energy \( Z(0) = [1 - \Sigma_1(0)]^{-1} \approx 0.44 \).

These behaviors are realized only near the threshold. If we increase \( \delta M \) further, the virtual state in s wave acquires a finite width\(^8\) and eventually goes above the threshold to become the resonance 13. This is demonstrated in Fig. 4(a). It is also shown in Fig. 4(b) that the

\[8\text{At the point where the imaginary part starts, the real part exhibits a cusp behavior. This non-analytic cusp structure is essentially the same with what is discussed in Ref. 31.}\]
FIG. 3. (Color online) Near-threshold eigenenergies as functions of $\delta M$ for $l = 0$ (a) and for $l = 1$ (b). Solid, dotted, and dashed lines represent the energy in the first Riemann sheet, the real part of the energy in the second Riemann sheet, and the imaginary part of the energy in the second Riemann sheet, respectively.

FIG. 4. (Color online) Eigenenergies for $l = 0$ as functions of $\delta M$ in the region $|\delta M| \leq 0.2\Lambda^2/\mu$ (a) and in $|\delta M| \leq 2\Lambda^2/\mu$ (b). Solid, dotted, and dashed lines represent the energy in the first Riemann sheet, the real part of the energy in the second Riemann sheet, and the imaginary part of the energy in the second Riemann sheet, respectively.

real part of the energy asymptotically approaches the linear scaling $E_h \sim \delta M$, and the imaginary part vanishes. It is clear from Fig. 4 (a) that the scaling of the bound state energy is not continuously connected to the real part of the resonance energy near the $s$-wave threshold, because of the existence of the virtual state. This discontinuity is unavoidable, because it originates in the universal near-threshold scaling [10] and [11]. The analysis with the effective range expansion shows that the energy region where the virtual state appears is determined essentially by the effective range parameter $r_0$. For instance, the deepest energy of the virtual state is $E_h = -1/(2\mu r_0^2)$, and the width of the virtual state when it turns into the resonance is given by $\text{Im} E_h = -1/(\mu r_0^2)$ [15]. This suggests that the size of the scaling violating region is determined by the inverse of the effective range parameter.

VII. DISCUSSION

A. Chiral extrapolation

Let us now consider the implication of the present result to the chiral extrapolation for the lattice QCD [10]. In a naive application of chiral perturbation theory, two-body

\[^9\] In the present model with the sharp cutoff [38], the imaginary part suddenly vanishes at $\text{Re } E_h \approx 1.0\Lambda^2/\mu$. Accordingly, the real part exhibits a cusp behavior. For a smooth regularization, the imaginary part asymptotically decreases and the cusp of the real part does not appear.

\[^{10}\] The present argument is based on the analyticity of the S-matrix which is not guaranteed in a finite volume where actual simulation is performed. Here we consider the results in the infinite volume limit.
loop effect is incorporated by perturbative calculations. This corresponds to approximate Eq. (5) as
\[ E_h = M_0 + \Sigma(M_0) + \cdots . \] (41)
In this case, the scaling near the s-wave threshold becomes \( E_h \propto \delta M \) and the universal result cannot be reproduced. We emphasize that the nonperturbative effect [self-consistent treatment in Eq. (5)] is essential for the universal behavior around the s-wave threshold. Indeed, inclusion of the nonperturbative dynamics through the dispersion relations [31] shows the \( m_q \) dependence consistent with the universal scaling. It is worth mentioning that the importance of the re-summation in chiral perturbation theory is known for the \( NN \) scattering [32] and the \( \bar{K}N \) scattering [33]. A common feature for these sectors is the existence of the near-threshold s-wave (quasi) bound state, deuteron in the \( NN \) scattering and \( \Lambda(1405) \) in the \( \bar{K}N \) scattering. We encounter the same situation during the mass scaling across the threshold, when the bound state pole approaches the s-wave threshold. Thus, the re-summation should be properly performed for the chiral extrapolation near an s-wave threshold.

In p- or higher partial waves, on the other hand, perturbative calculation [11] provides an estimate of the field renormalization constant \( Z(0) = [1 - \Sigma'(0)]^{-1} \approx 1 + \Sigma'(0) \), when the coupling of the bare state and the scattering state is small. The mass scaling for \( l \neq 0 \) can therefore be estimated by the usual perturbative calculation.

Our analysis shows that the mass of hadrons scales discontinuously near the s-wave threshold. This raises a caution on the use of the perturbative extrapolation formula when the physical state is expected to appear near the threshold. This problem may be avoided if one extrapolate the potential, which is continuous in \( \delta M \), instead of the eigenenergy.

B. Feshbach resonance of cold atoms

The near-threshold behavior in the bound region is also studied for the Feshbach resonance in cold atom physics [34]. The energy of a shallow two-body bound state is proportional to the inverse scattering length squared \( E_2 \propto a_0^{-2} \), and the scattering length near a Feshbach resonance is given by \( a_0(B^{em}) \propto [1 - \Delta B^{em}/(B^{em} - B_0^{em})] \) with the external magnetic field \( B^{em} \), its critical strength \( B_0^{em} \), and the width parameter \( \Delta B^{em} \) [34]. The leading contribution to the binding energy is
\[ E_2 \propto (B^{em} - B_0^{em})^2 + \cdots . \] (42)
This shows the quadratic dependence of the binding energy on the strength of the magnetic field. Because the mass difference of the different spin states \( \Delta M \) is proportional to \( B^{em} - B_0^{em} \), the leading contribution to the binding energy is
\[ E_2 \propto (\Delta M)^2 + \cdots . \] (43)
This is nothing but the scaling in Eq. (10). The field renormalization constant \( Z \) at small binding energy is also calculated as [34] [36]
\[ Z \propto \frac{1}{a_0} \propto \sqrt{|E_2|} \] (44)
which is fully consistent with the compositeness theorem in Sec. V.

C. Three-body bound state

We finally note that the threshold scaling is universal for the two-body bound state. It has been found that the s-wave three-body bound state directly turns into a resonance across the three-body break up threshold when the Efimov effect occurs [37, 38]. Three-body break up process is beyond the applicability of the present framework. To analyze such behavior, we need to establish the low energy expansion of the three-body amplitude. The study of the scaling and compositeness of three-body bound states deserves an interesting future work.

VIII. SUMMARY

We have discussed the near-threshold behavior of the hadron mass scaling. Using the expansion of the Jost function, we derive the general scaling law of the pole of the scattering amplitude for a local potential. By utilizing the property of the field renormalization constant \( Z \) in the zero binding limit, the same scaling is obtained for the non-local potential of Eq. (2). It is shown for the s wave that the scaling of the binding energy does not continuously connected to the real part of the resonance energy.

We present a detailed discussion on the field renormalization constant of the zero energy resonance in s wave. It is shown that, if there is a pole exactly at the threshold, the field renormalization constant should vanish. The vanishing of the field renormalization constant at zero energy guarantees the quadratic scaling of the binding energy in the s wave. This result is interpreted as a consequence of the infinitely large two-body scattering component in the zero binding limit, which overwhelms any finite admixture of the elementary component. If we take the zero binding limit with keeping finite \( Z \), then the bound state pole decouples from the amplitude.

The near-threshold scaling found here gives caution to the chiral extrapolation of the hadron mass across the s-wave threshold, because naive perturbative calculation does not reproduce the general scaling law. As in the case of the \( NN \) and \( \bar{K}N \) scattering in chiral perturbation theory, the nonperturbative re-summation is necessary to reproduce the correct threshold behavior.
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Appendix A: Jost function

Here we summarize the basic properties of the Jost function \[1\]. In the following, we consider the two-body scattering by the spherical local potential \(V(r)\) in the absence of the long range force (such as the Coulomb interaction) so that the standard scattering theory can be formulated.

We first consider the regular solution of the Schrödinger equation \(\phi_{l,p}(r)\) with the angular momentum \(l\) and momentum \(p\). This is the radial wavefunction

\[
\phi_{l,p}(r) = j_l(pr) + \int_0^r dr' g_{l,p}(r,r') U(r') \phi_{l,p}(r'),
\]

where \(U(r) = 2\mu V(r)\) and the free Green’s function is given by \(g_{l,p}(r,r') = [j_l(pr)\hat{n}_l(pr') - \hat{n}_l(pr)j_l(pr')]/p\).

The Jost function \(J_l(p)\) is defined by the asymptotic behavior at \(r \to \infty\) of the regular solution \(\phi_{l,p}(r)\) as

\[
\phi_{l,p}(r) \to \frac{i}{2} [J_l(p)\hat{h}_l^-(pr) - J_l(-p)\hat{h}_l^+(pr)],
\]

where \(\hat{h}_l^\pm(z) = \hat{n}_l(z) \pm i j_l(z)\) is the Riccati-Hankel function. Recalling the definition of the \(s\)-matrix \(s_l(p)\) and the partial wave scattering amplitude \(f_l(p)\), we can express these quantities by the Jost function as

\[
s_l(p) = \frac{J_l(p)}{J_l(-p)} , \quad f_l(p) = \frac{J_l(-p) - J_l(p)}{2ipJ_l(p)} \tag{A3}
\]

Because the Jost function appears in the denominator, the zero of the Jost function is equivalent to the pole of the scattering amplitude.

From the comparison of the asymptotic form of the integral equation \((A1)\) with Eq. \((A2)\), we obtain the expression for the Jost function

\[
J_l(p) = 1 + \frac{1}{p} \int_0^\infty dr \hat{h}_l^+(pr) U(r) \phi_{l,p}(r). \tag{A4}
\]

This is useful to expand the Jost function at small \(p\). For \(p \to 0\), the Riccati functions and the regular solution behave as

\[
\hat{j}_l \sim \phi_l \sim p^{l+1}, \quad \hat{n}_l \sim p^{-l}. \tag{A5}
\]

Thus, the expansion of the Jost function at small \(p\) is given by

\[
J_l(p) = 1 + \alpha_l + \beta_l p^2 + O(p^4) + i[\gamma_l p^{2l+1} + O(p^{2l+3})]. \tag{A6}
\]

The real expansion coefficients \(\alpha_l, \beta_l, \gamma_l, \ldots\) depend on the potential \(U\).

Let us now tune the potential \(U\) such that the bound state appears exactly at the threshold. The condition to have a zero at \(p = 0\) is

\[
1 + \alpha_l = 0. \tag{A7}
\]

In this case, the expansion leads to

\[
J_l(p) = \beta_l p^2 + O(p^4) + i[\gamma_l p^{2l+1} + O(p^{2l+3})]. \tag{A8}
\]

which indicates Eq. \((8)\). In fact, the scaling \((8)\) is shown on the general ground for a local potential \[1,2\], so that the leading coefficients \(\gamma_0\) and \(\beta_l\) \((l \neq 0)\) cannot vanish. Next we introduce a small parameter \(\delta \lambda\) to modify the potential as

\[
U \to (1 + \delta \lambda)U. \tag{A9}
\]

In this case, the expansion of the Jost function is given by

\[
J_l(p; \delta \lambda) = 1 + \alpha_l(\delta \lambda) + \beta_l(\delta \lambda)p^2 + O(p^4) + i[\gamma_l(\delta \lambda)p^{2l+1} + O(p^{2l+3})]. \tag{A10}
\]

with a condition \(\alpha_0(0) = -1\). Expanding the coefficients for small \(\delta \lambda\), we obtain

\[
J_l(p; \delta \lambda) = \begin{cases} 
\alpha'_0 \delta \lambda + i \gamma_0 p + O(p^2, \delta \lambda p, \delta \lambda^2) & l = 0 \\
\alpha'_l \delta \lambda + \beta_l p^2 + O(p^3, \delta \lambda p^2, \delta \lambda^2) & l \neq 0 
\end{cases} \tag{A11}
\]

\[
\alpha'_l = \left. \frac{d \alpha_l}{d(\delta \lambda)} \right|_{\delta \lambda = 0}, \quad \beta_l = \beta_l(0), \quad \gamma_0 = \gamma_0(0), \tag{A12}
\]

which leads to the eigenmomenta in Eqs. \((8)\) and \((9)\).

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