Modular architecture of isospin, color, and generation

David Ritz Finkelstein

Georgia Institute of Technology

(Dated: 24 December 2014)

Starting from the vacuum, iterated Grassmann-algebra formation consecutively introduces vector spaces and groups with the structure first of charge, then of isospin, then of color.

PACS numbers: 12.90.+b

\[ \text{Electronic mail: finkelstein@gatech.edu} \]
I. QUANTUM SET THEORY

The cellular model of the internal variables of the Standard Model constructed here began as a finite quantum relativistic version of the von Neumann cellular automaton[1,2]. The usual language for the description of finite systems is finite set theory. But set theory incorporates Boolean algebra, and quantum theory revises Boolean algebra at the quantum level, so that quantum theory must also revise set theory at that level. A quantum set theory is reviewed here and applied to the internal variables of the Standard Model.

Quantum sets turn out to be spins of associated orthogonal groups. Therefore this work is also in the line of earlier proposals of Feynman[3] and Penrose[4], composing the world out of spins. It adds the key element of modular architecture[5], from which a world of quantum bits or spins arises naturally. I recapitulate quantum set theory (Section [II]), and apply it to isospin and color (Section [III]), generation (Section [IV]), and orbital variables (Section [V]).

II. QUANTUM SET THEORY

A quantum set is a Fermi-Dirac (or odd) combination of identical quantum systems. More fully put:

Let \( V \) be the input vector space of a quantum system[6]. Let \( \iota \) be the unitizer:

\[
\iota v := \{v\} \quad \text{for all } v \in V,
\]

modulo the identification

\[
\iota (av + v') \equiv a\iota v + \iota v'
\]

that makes the mapping \( \iota : V \to \iota'V \) linear. Let \( \iota'V \) designate the vector space of unit sets of elements of \( V \). Let \( \bigwedge \iota'V \) be the Grassmann algebra over \( \iota'V \). Let \( S \) be the Grassmann algebra over its own unitization that is minimal in that regard:

\[
S = \bigwedge \iota' \cdot S = \bigcup_{r=1}^{\infty} S^r, \quad S^r := \left[ \bigwedge \iota' \right]^r \mathbb{C}.
\]

The classical correspondent of \( S \) is the set \( S_c \) of all sets of finite rank and empty foundation.

Then the quantum set of rank \( r \) is the hypothetical quantum system with input vector space \( S^r \subset S \). Ranks nest: \( S^0 \subset S^1 \subset S^2 \subset \ldots \).
Isospin, color, generation

Suppose that the history from input to output of the quantum system under study is a quantum set of some rank \( R \) to be determined. Provisionally, \( R \approx 6 \) suffices for quantum field theories (Section V).

This gives algebraic expression, for example, to statements that a fermion field includes \( (\supset) \) elementary fermions, and that an elementary fermion contains \( (\ni) \) a spin, an isospin, and a color.

The modules of this modular architecture are the unit quantum sets of rank \( \leq R \).

The input space of the generic quantum set is infinite-dimensional, but modular quantum theory currently stops at rank 6, whose dimensionality is only

\[
d = 2^{(264K)}.
\] (4)

The non-semisimple algebras and infinite spectra of present physics arise in quantum set theory only as singular limits.

Table I below samples a basis of quantum sets, the standard basis, and assigns serial numbers to its basic sets in the order of generation.

Beneath each set \( s_n \) is its serial number \( n \).

Table I can also serve as a table of monads (unit sets, first-grade Grassmann elements) if a bar is imagined over every entry. This is useful since the fundamental fermions and their modules are monads.

Elements of the standard basis of rank \( r \) that are not elements of rank \( r - 1 \) are said to have proper rank \( r \) and belong to \( S^{[r]} \). Unlike ranks, proper ranks are disjoint, except for the common element 0.

I cast Cartan’s construction of spinors into the present terms:

For any complex vector space \( \mathcal{V} \), the dual space dual \( \mathcal{V} := \widetilde{\mathcal{V}} \) is the space of linear mappings \( \mathcal{V} \rightarrow \mathbb{R} \). The duplex space is

\[
\text{dup} \mathcal{V} := \text{dual} \mathcal{V} \oplus \mathcal{V}.
\] (5)

Define the duplex form \( \|w\|_{\text{dup}} \) on \( \mathcal{W} = \text{dup} \mathcal{V} \) by the condition

\[
\forall v \in \mathcal{V}, u \in \widetilde{\mathcal{V}} : \quad \|u + v\|_{\text{dup}} := u \circ v,
\] (6)

the value of \( u \) on \( v \). The duplex form is neutral. It is invariant under a group \( \text{SO}(2n\mathbb{C}) \) of complex matrices, where \( n = \dim \mathcal{V} \), but this group is reduced by the condition (5).
Isospin, color, generation

TABLE I. Basic sets and fermion kinds. $r =$ proper rank, $s_n =$ n-th basic set vector.

Define the spinors of $\text{so}(2n)$ as the elements of $\bigwedge V$. They support the spin representation of $\text{so}(2n\mathbb{C})$.

The vectors of the Grassmann algebra $S^r$ are thus spinors of the real Lie algebra $\text{so}(\text{dup} S^{r-1})$. There is thus a tautologous connection between statistics and spin, but the spin is not restricted to the Lorentz group yet.

III. ISOSPIN AND COLOR

The 16 monads of rank 4 in Table II are clearly partitioned into four proper ranks 1, 2, 3, and 4, with respective multiplicities 1, 1, 2, and 12. The contents of these monads are all the entries of proper ranks 0, 1, 2, and 3 respectively.
Isospin, color, generation

The 16 fundamental left-handed fermions of the Standard Model have a similar pattern: the left-handed antineutrino is one isospin singlet, the left-handed positron is a second, the left-handed neutrino and the left-handed electron form an isospin doublet, and 12 left-handed quarks repeat this pattern in three colors, \( 12 = 3 \times (1 + 1 + 2) \).

This suggests a structural correspondence between fermion kinds and monads of Table I:

C1. The two lepton singlets correspond to the two monads of proper ranks 2 and 1.

C2. The lepton isospin doublet corresponds to the two monads of proper rank 3.

C3. The 12 quarks correspond to the 12 monads of proper rank 4.

These fermions are entered next to the contents of their monad in Table II, not the monads themselves. The 24-dimensional Standard Model internal group,

\[
G_{\text{SM}} := SU(3) \times SU(2) \times U(1) \tag{7}
\]

then acts on \( S^3 \) as follows.

Proper rank 2, being two-dimensional, defines an SU(2) group that blends the two basis vectors of proper rank 2; that is, maps each into a superposition of both. Since according to the assignment C1, proper rank 2 represents a lepton isospin doublet, this proper-rank-2 SU(2) must be identified with isospin SU(2).

If this assignment is correct, the 12 monadic dimensions of quark proper rank \( S^{[4]} \) should then resolve into 3 copies of lepton space \( C = 1 + 1 + 2 \), one for each color, coherently mixed by a color SU(3) group.

Proper rank \( S^{[4]} \) consists of polynomials in the unitizations of the vectors of rank 3.

Assume that \( \iota : S^{[3]} \to S^{[4]}, S^{[4]} \to S^{[5]} \) commutes with \( SU(2) : S^{[3]} \to S^{[3]}, S^{[4]} \to S^{[4]} \).

This fixes the action of SU(2) on \( S^{[4]} \).

Inspection of Table II shows that the \( 12 = 3 \times 4 \) monads of proper rank 4 are constructed by inserting any one of the three elements \( s_4, s_8, s_{12} \) of proper rank 3 into any one of the four lepton monads of rank 3. The three vectors

\[
c_1 := s_4, \quad c_2 := s_8, \quad c_3 = s_{12} \tag{8}
\]

are distinguished algebraically as the non-constant monomials in the unit sets of proper rank 3 only. The constant monomial is 1; its insertion into the lepton merely reproduces the lepton.
Isospin, color, generation

Therefore it is permissible to identify \( s_4, s_8, s_{12} \) with three independent input vectors for a *chromon*—“piece of color”, a quantum module that when inserted into a lepton module converts it to a quark module. It is not a particle, since it lacks orbital variables. It is two ranks below the particle rank in the hierarchy of rank. It has no operators but those of the color algebra \( \mathbb{C}(3) \).

Infer that a quark is an elementary fermion that contains a quark module, and that a quark module is a lepton module with a chromon inserted. A chromon is an element of a quark, not a part of it. A quark is still an elementary particle, with no parts.

Now it is meaningful to ask whether the strong interactions proceed by chromon exchange, and whether leptons are inert to strong interactions because they contain no chromons to exchange.

A gluon cannot be a unit quantum set, which has odd statistics. It can be a pair, or a quadruple, or . . . . If a gluon is a quark pair\(^7\), two gluons might bind by chromon exchange. The interaction between a gluon and a quark might likewise be mediated by chromon exchange. Such possibilities need further study.

The space \( S^3 \) indeed splits into three isomorphs of \( 3S^2 \). For these to be the three quark color spaces, they should be superposed by a color \( SU(3) \) group that commutes with the proposed isospin \( SU(2) \).

In fact the color Lie algebra \( su(3) \) is that generated by the eight antihermitian traceless sesquilinear combinations of chromon input operators \( c_i \)—creation operators—and their dual output operators \( \bar{c}_i \)—annihilation operators:

\[
c_1\bar{c}_1 - \bar{c}_1 c_1, \quad c_1\bar{c}_2 - \bar{c}_2 c_1, \quad \text{and} \quad i(c_1\bar{c}_2 + \bar{c}_2 c_1), \quad \ldots
\]

\[(9)\]

**IV. GENERATION**

The modular structure iterates. Just as the monads of proper rank 4 consist of \( 2^2 - 1 = 3 \) isomorphs of the \( 2^2 \) monads of proper rank 3 (the first-generation quark kinds), the monads of proper rank 5 consist of \( 2^4 - 1 = 15 \) isomorphs of the \( 2^4 = 16 \) monads of rank 4. They are generated by inserting, into each of the 16 monads of rank 4 in turn, each of the \( 2^4 - 1 = 15 \) non-constant monomials in the four monads of proper rank 4.

Models of generation being scarce, it is tempting to infer, at least provisionally and tentatively, that these 15 copies include the two observed higher generations of the first-
Isospin, color, generation

generation fermion kinds:

C4 The three generations of fundamental fermions correspond to three of the 16 isomorphs of the rank-4 monads in the rank-5 monads.

Since only three generations are found experimentally, one must then infer that the other 13 kinds are too massive to produce or too unstable to be detected. Perhaps the three monomials of rank 4 that contribute their genes to the three generations are the first three,

\[ s_{16} = \iota^41, \quad s_{32} = \iota(\iota^31 \wedge \iota1), \quad s_{48} = s_{32}s_{16} \]  

merely on the grounds that they are the simplest.

V. THE RANK OF THE FERMION

\( S^r \) has complex dimension designated by \( \text{hexp } r \):

\[
\dim S^r = 1 \ 2 \ 4 \ 16 \ 2^{16} \ 2^{(2^{16})} \ 2^{(2^{(2^{16})})} \ \ldots
\]
for \( r = 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ \ldots \) 

The corresponding table for the number of monads of rank \( r \) is

\[
\dim S^r_1 = 1 \ 2 \ 4 \ 16 \ 2^{16} \ 2^{(2^{16})} \ 2^{(2^{(2^{16})})} \ \ldots
\]
for \( r = 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ \ldots \) 

If \( L \) in time units is the Planck time \( T_P = 5 \times 10^{-44} \) s and \( T_U \) is the present age of the universe, \( T_U \approx 4 \times 10^{17} \) s, then the observed age in Planck units is \( N \sim 10^{60} \). The number of Planck cells in a history is roughly \( 10^{240} \).

Thus \( S^5 \) is much too small for field space; while \( S^6 \) is much larger than present physics needs for field space. Despite our abysmal ignorance of the microcosmos, it is reasonable to conclude that fermion field space has rank \( \gtrsim 6 \) and arbitrary grade. If rank 5 is used inefficiently for the generations, the particle rank may be raised to at least 7.

VI. ASSEMBLING CELLS

The quantification process defined in this section is used to assemble space-time from cells in the next section.
Isospin, color, generation

Every element of $S^{r+1}$ is an essentially unique Grassmann polynomial in the unitizations of the basis elements of $S^4$. Therefore every infinitesimal operator $S \in \text{op} S^r$ has a well-defined representation in $\text{op} S^{r+1}$, called its *cumulation* and written $\sum_{r}^{r+1} S$, defining a homomorphism $\sum_{r}^{r+1} : \text{op} S^r \to \text{op} S^{r+1}$.

Thus every operator in $\text{op} S^3$ has a cumulation in $\text{op} S^4$, and so forth to the particle rank $\text{op} S^R$, defining a sequence of Clifford homomorphisms

$$C^0 \xrightarrow{\Sigma^0} C^1 \xrightarrow{\Sigma^1} \cdots \xrightarrow{\Sigma^{R-1}} C^R \xrightarrow{\Sigma^R}$$ \hspace{1cm} (13)

Write their product from rank $r$ to $r'$ as

$$\sum_{r}^{r'} = \sum_{r-1}^{r'} \cdots \sum_{r+1}^{r+2} \sum_{r}^{r+1} : \text{op} S^r \to \text{op} S^{r'}.$$ \hspace{1cm} (14)

Write $\Sigma$ for the Lie algebra homomorphism induced by the Clifford homomorphism $\sum$. If $S$ is a spin component in $\text{op} S^r$, $\Sigma_{r'}^r S$ is the sum of corresponding components of all the spins of the cells making up the set of rank $r'$, multiplied as usual by unwritten identity operators.

**VII. ORBITAL MODULE**

Canonical quantization enormously reduced the degree of singularity of the algebra of observables of classical mechanics and field theory, as measured by the radical of the Lie algebra. Yang’s space-time quantization completely eliminated the remaining radical. In Yang’s quantized space-time $\mathcal{Y}$, which is actually a quantized phase space, the Poincaré group is reformed to $\text{spin}(5,1)$ or $\text{spin}(3,3)$. Other quantum commutation relations have been proposed for orbital variables, but Yang’s Lie algebra is the nearest simple one to the standard Lie algebra $a(x,p,L,i)$ of canonical relativistic quantum mechanics.

Yang represents this simple Lie algebra, however, by differential operators on a Hilbert space of functions on an underlying 6-dimensional continuum $\{\xi^a\}$. The Lie algebra of this Hilbert space is still singular.

To regularize the representation too, use spin operators instead of differential operators. The $\text{spin}(3,3)$ spinors have 8 components, 4 of left-handed chirality and 4 of right-handed. Identify them with the spinors in a suitable 8-dimensional subspace of the 16-spinor space $S^3$; and single out a 10-dimensional subalgebra $a_Y \subset a(S^3)$ of the 16-dimensional algebra $a(S^3) \cong \text{spin}(4,4)$ of $S^3 \sim 16\mathbb{C}$, leaving the meaning of the other six dimensions for later.
Isospin, color, generation

$S^4 \sim 16\mathbb{C}$ is the quantum space of the prototypical unit cell of the world cellular automaton. Write the quantum unit of time as $t_Q$. Dynamical considerations suggest that $t_Q \gg t_P$, the Planck time. Use $cht_Q$ units unless otherwise noted.

Then the spin angular momentum in $\text{op} S^3$ is

$$S_{n'n} = \frac{[\gamma_{n'}, \gamma_n]}{2}, \quad \gamma_n \in \text{op} S^3, \ n, n' = 1, \ldots, 8. \quad (15)$$

and obeys the $\text{so}(N_+, N_-)$ Lie-algebra relations

$$[S_{n''n''}, S_{n'n}] = -S_{n'n'g_{n'n'}} + S_{n'n'g_{n'n}} - S_{n'n}g_{n'n'} \quad (16)$$

Suppose that the fundamental fermion is a unit set of rank 6, as Section $\nabla$ indicates, founded on a cell of rank 3:

$$S_{n'n} := \sum_{6}^{3} \frac{[\gamma_{n'}, \gamma_n]}{2} \in \text{op} S^6. \quad (17)$$

Yang reformed the commutator Lie algebra of space, time, energy, momentum, angular momentum, boosts, and the complex numbers into one simple orbital algebra with symmetry group $\text{spin}(5, 1)$ or $\text{spin}(3, 3)$. Call the chosen symmetry group of these two the Yang group. For brevity take the Yang group to be $\text{spin}(3, 3)$ for the present. The peculiar three timelike dimensions reduce to one in the organization of the complex plane.

Following Feynman and Yang, quantize Yang orbital space further by taking its orbital variables $S_{n'n} \in \text{op} S^6$ to be components of the cumulative $\text{spin}(4, 4)$ angular momentum $S_{n'n}$ of the fundamental fermion, rather than differential operators.

Write $a \leftarrow \hat{a}$ and $\hat{a} \rightarrow a$ when the standard construct $a$ is a singular limit of the regular construct $\hat{a}$ of quantum set algebra. The limiting process in general including a self-organization that restricts quantum vectors to some sector of the quantum vector space.

Let the Clifford vector with components $\gamma^{n} \in \text{op} S^6$ obey the Clifford Clause

$$\{\gamma^{n'}, \gamma^{n}\} = 2g^{n'n}, \quad \text{where} \quad n, n' = 1, 2, \ldots, 6. \quad (18)$$

Set as usual

$$\gamma^{n'n} := \frac{1}{2}[\gamma^{n'}, \gamma^{n}] \in \mathcal{T}^3. \quad (19)$$

Then in an adapted spinor frame, the regularized Yang space $\hat{\mathcal{Y}}$ has the following basic orbital
Isospin, color, generation

operators on $S^6$, expressed as spin operators with $\hbar$, $c$, and the fundamental time $\tau$ as units:

\[
\begin{align*}
S_{n'n} & := \frac{1}{2} \sum_6 \gamma^{n'n}, \\
p_m & := \frac{1}{N} S_{6m}, \\
\hat{x}^m & := S_{6m}, \\
i & := \frac{2}{N} S^{65}, \\
\hat{L}_{m'm} & := S_{m'm};
\end{align*}
\] (20)

here $\sum$ is the iterated quantifier defined in (14); $S_{n'n}$ is the quantified angular momentum tensor of the representation of the cell algebra spin(4,4) supported by $S^6$; and $N/2$ is the maximum value of any component of $S_{n'n}$. Factors of $i$ have been absorbed into $\hat{x}^m$, $\hat{p}^m$, and $\hat{L}_{m'm}$ so that they are anti-Hermitian with respect to the spinor form $\beta$ of $S^6$.

This quantizes not only space-time $\{x^\mu\}$ but the orbital space $\{x^\mu, p_\mu, L_{\mu'\mu}, i\}$ as well. Phase space and orbital space are already quantum spaces in the standard quantum theory, in that momentum and position do not commute. The singular canonical commutation relations of Heisenberg among the operators $x, p, L, i$ of ordinary relativistic mechanics are all limits of regular angular momentum commutation relations of the usual form $[L, L] \sim L$ among components of the higher-dimensional Yang angular momentum $L_{y'y}$.

VIII. DISCUSSION

These assignments of physical meaning to some quantum sets are the opposite of experimental prediction. They do not extract experimental information from the theory but put it in, an indispensable first phase of theory construction.

Since set theory is a universal language, it is hardly surprising that its quantum version can be used to express parts of the Standard Model, but the expression turns out simpler than we had any right to expect, unless the theory has some correctness. If it extends at least approximately to the rest of the Standard Model, it removes some of the mystery of the Standard Model at the same time that it regularizes it. It answers Rabi’s question about the muon, “Who ordered this?” What ordered the muon is Nature’s modular architecture.

A finite quantum set theory of the orbital variables of the fermion is already known. The synthesis of the orbital and internal algebras is under study.
IX. ACKNOWLEDGMENT

I owe S. Alexander, G. D’Ariano, G. F. R. Ellis, S. R. Finkelstein, Tenzin Gyatso, H. Saller, and F. Tony Smith for discussions and information, and FQXi, the Templeton Foundation, and Dartmouth College for supporting some of the presentations of this work.

REFERENCES

1. D. Finkelstein. Space-time code. *Phys. Rev.* 184:1261-1271 (1969)
2. D. Finkelstein. Space-time structure in high energy interactions. In T. Gudehus, G. Kaiser, A. Perlmutter (ed.), *Fundamental Interactions at High Energy*. New York: Gordon & Breach, pp. 324-338 (1969)
3. R. P. Feynman. Personal communication ca. 1961. Feynman did this work ca. 1941.
4. R. Penrose. Angular momentum: an approach to combinatorial space-time. In T. Bastin (ed.), *Quantum Theory and Beyond*, 151–180. Cambridge 1971. Penrose kindly shared much of this seminal work with me ca 1960, long before publication.
5. H. Simon. The architecture of complexity. *Proc. Amer. Philosophical Soc.* 106:6 (1962).
6. V. A. Fock. *Foundations of Quantum Mechanics*. First edition. Moscow (1931). Second edition, Mir, Moscow (1978). In this historic first textbook on quantum theory, Fock replaced the single concept of present state employed in classical mechanics and field theory, by twin operational concepts of past input and future output operation. I do the same and call input and output vectors collectively *ports*. A port is a quantum vector as a mosquito is a malaria vector: it carries the quantum.
7. H. Saller. Gauge fields as bound states of subcanonical fermion fields. *Il Nuovo Cimento A* Series 11, 24, 391 (1974).
8. I. E. Segal. A class of operator algebras which are determined by groups. *Duke Mathematical Journal* 18:221–265 (1951). Especially §6A.
9. C. N. Yang. On quantized space-time. *Phys. Rev.* 72:874 (1947).