FINITE CYCLICITY OF QUADRATIC SLOW-FAST DARBOUX SYSTEMS WITH A TWO-SADDLE LOOP

MARCEL BOBIEŃSKI AND LUBOMIR GAVRILOV

Abstract. We prove that the cyclicity of a quadratic slow-fast integrable system of Darboux type with a double heteroclinic loop, is finite and uniformly bounded.

1. Introduction

Let $\mathcal{F}_\varepsilon$, be an analytic family of analytic real plane foliations (or vector fields), depending on a small parameter $\varepsilon$, and having for all $\varepsilon > 0$ a bounded period annulus (a nest of periodic orbits) $\Pi_\varepsilon$. Consider a further multi parameter analytic deformation $\mathcal{F}_{\varepsilon,\delta}$ of $\mathcal{F}_\varepsilon$, and denote by $Z(\varepsilon) = \text{Cycl}(\mathcal{F}_{\varepsilon,\delta}, \bar{\Pi}_\varepsilon)$ the maximal number of limit cycles which bifurcate from the closure $\bar{\Pi}_\varepsilon$ for sufficiently small $\|\delta\|$. The number $Z(\varepsilon)$ is therefore the cyclicity of the closed period annulus $\bar{\Pi}_\varepsilon$ with respect to the deformed foliation $\mathcal{F}_{\varepsilon,\delta}$ [7]. The purpose of the paper is to prove the $\varepsilon$-uniform boundedness of $Z(\varepsilon)$ in the case, when $\mathcal{F}_\varepsilon$, $\varepsilon > 0$, is a Darboux integrable plane foliation, and $\mathcal{F}_0$ has a curve of singular points (slow manifold), as on fig. 1.1. The family $\mathcal{F}_\varepsilon$ will be referred to as a slow-fast integrable system of Darboux type.

Another motivation for our result comes from the program of proving uniform finiteness of the number of limit cycles of plane quadratic vector fields [3, 7] (the existential Hilbert’s 16th problem). Indeed, a
family of plane quadratic systems can be slow-fast, with a degenerate graphics as on fig. 1.1.

The main difficulty in the study of $F_{\varepsilon, \delta}$ is on the one hand the "turning point" of $F_0$, and on the other hand the double heteroclinic loop of $F_{\varepsilon}$ ($\varepsilon > 0$). The zeros of pseudo-Abelian integrals appearing in the usual Poincaré-Melnikov method does not detect the so called alien limit cycles [4]. Nevertheless, in recent papers [1, 2], it has been shown that the number of zeros of pseudo Abelian integrals, which arise as a first approximation of the first return map of our slow-fast system, have the desired uniform finiteness property.

In this paper we replace the pseudo-Abelian integrals studied in [1, 2] by the true Dulac maps defining the limit cycles as fixed points. Our main result is the uniform finiteness of the number of limit cycles of the slow-fast Darboux system under consideration. The analytic deformations $F_{\varepsilon, \delta}$ which we consider are arbitrary and can depend on an arbitrary given number of parameters.

Our method makes a strong use of the properties of the foliation $F_{\varepsilon, \delta}$ in a complex domain, where we apply a technique derived from the so called "Petrov trick", along the lines of [5, 6].

2. STATEMENT OF THE RESULT

Using the notations of [2], let $P_0 = P_0(x, y), P_1 = P_1(x, y)$ be real bivariate polynomials and consider the differential system

\[(2.1) \quad P'_1 = \varepsilon P_1, \quad P'_0 = -P_0, \quad \varepsilon \in \mathbb{R}^+\]

which induces the foliation

\[(2.2) \quad F_{\varepsilon} : \varepsilon P_1 dP_0 + P_0 dP_1 = 0.\]

It has a Darboux type first integral $H = P_0^2 P_1$, and for $\varepsilon = 0$ a curve of singular points $\{(x, y) : P_0(x, y) = 0\}$. For $\varepsilon$ close to zero the curve is the slow manifold of the slow-fast system (2.1). In the present paper we are interested in the simplest possible slow-fast Darboux system, shown on fig[1.1]. More precisely, assume that

\[P_0 = y - x^2, \quad P_1 = 1 - y.\]

Consider further the following perturbed slow-fast Darboux integrable foliation

\[(2.3) \quad F_{\varepsilon, \delta} : \varepsilon P_1 dP_0 + P_0 dP_1 + P dx + Q dy = 0.\]

where $P = P(x, y, \delta), \quad Q = Q(x, y, \delta)$ are real polynomials in $x, y$ depending analytically on $\delta \in (\mathbb{R}^N, 0)$, and such that

\[P(x, y, 0) = Q(x, y, 0) \equiv 0.\]

For every fixed sufficiently small $\varepsilon > 0$, denote by $Z(\varepsilon)$ the maximal number of limit cycles of $F_{\varepsilon, \delta}$, which bifurcate from the compact region $\Pi$ bound by the curves $\{P_0 = 0\}, \{P_1 = 0\}$, for sufficiently small $\|\delta\|$. 

The number $Z(\varepsilon)$ is therefore the cyclicity of the closed period annulus $\bar{\Pi}$ of $F_{\varepsilon,0}$ under the deformation $F_{\varepsilon,\delta}$. The main result of the paper is the following theorem

**Theorem 2.1.** The cyclicity $Z(\varepsilon)$ is finite and uniformly bounded in $\varepsilon > 0$.

**Remark 2.2.** Authors expect that the theorem 2.1 remains true in the more general case, under the following assumptions on the unperturbed Darboux integrable foliation 2.2. Assume that the real curves $\{P_0 = 0\}$, $\{P_1 = 0\}$ are smooth, intersect transversally and bound a compact region $\Pi$ in which the foliation (2.2) has no singular points. Assume also that $\{P_0 = 0\}$ is transverse to the foliation $dP_1 = 0$ except at one point of simple tangency. Additional tools are needed to investigate the isoclines structures in this more general case.

3. **The integrable foliation $F_{\varepsilon,0}$ where $\varepsilon > 0$**

In this section we assume that $\delta = 0$ and $\varepsilon > 0$ is a sufficiently small fixed real parameter. The foliation $F_{\varepsilon,0}$ has a Darboux type first integral $H = (y - x^2)^{\varepsilon}(1 - y)$ and its phase portrait is shown on fig. 3.1. It has a real nest of cycles bounded by the parabola $y - x^2 = 0$ and a line $y - 1 = 0$. The center point is located at

$$p_c = (0, y_c), \ y_c = \frac{\varepsilon}{\varepsilon + 1}.$$
Figure 3.2. Every complex leaf of the foliation $\mathcal{F}_{\varepsilon,0}$, which is not the line $\{y = 1\}$, is a double covering of the Riemann surface $C_{\varepsilon}$.

3.1. The complex leaves of $\mathcal{F}_{\varepsilon,0}$. The leaves of dimension one of $\mathcal{F}_{\varepsilon,0}$ are connected open Riemann surfaces, on which the function

$$(y - x^2)^{\varepsilon}(1 - y)$$

is constant. Every leaf intersects the cross-section $\{x = 0\}$ at at least one point $(0, y_0)$. The Riemann surface $C_{\varepsilon}$ of the multi-valued analytic function

$$(3.1) \quad f(y) := y(1 - y)^{1/\varepsilon}, \ y \neq 1$$

is conformally equivalent either to $\mathbb{C} (\varepsilon \notin \mathbb{Q})$, or to $\mathbb{C}^* (\varepsilon \in \mathbb{Q})$ and on the leaf through $(0, y_0)$ holds

$$(3.2) \quad x^2 = (f(y) - f(y_0))(1 - y)^{-1/\varepsilon}.$$ 

This implies the following

**Proposition 3.1.** Every complex leaf of the integrable foliation $\mathcal{F}_{\varepsilon,0}$ which does not contain the center point $(0, y_c)$, or is not the line $\{y = 1\}$, is a double covering of $C_{\varepsilon}$ given by the formula (3.2). The $y$-coordinates of the ramification points of the covering are the solution of the equation $f(y) = f(y_0)$. The leaf of $\mathcal{F}_{\varepsilon,0}$ through the center point $p_c = (0, y_c)$ has a single singular point at $p_c$ which is a normal crossing,
and otherwise is a double covering of \( C_\varepsilon \) given by the formula (3.2), with ramification points \((0,y)\) satisfying \( f(y) = f(y_\varepsilon) \).

### 3.2. Analytic continuation of the Dulac maps

Consider the cross-section \( \{ x = 0 \} \) parameterized by \( y \), as well the two Dulac maps \( D_1 = D_2 \) shown on figure 3.1. One obvious extension of \( D_1,2 \) is

\[
D_{1,2} : (0,1) \to (0,1)
\]

which is a real-analytic involution

\[
D_{1,2}(\bar{y}) = \overline{D_{1,2}(y)}, \quad D_{1,2}^2 = id, \quad D_{1,2}(y_\varepsilon) = y_\varepsilon.
\]

For the needs of the present paper we do not need a global description of the Dulac maps, but only an appropriate domain of analiticity in which we shall apply later the argument principle. This domain is described as follows.

Let \( C_{\pm \pi}, C_{\pm 0} \) be the real curves in the complex \( y \)-plane, defined in polar coordinates as follows

\[
C_{+0} = \{ \rho = \frac{\sin(\varepsilon \varphi)}{\sin(\varphi + \varepsilon \varphi)} : \pi \frac{1+\varepsilon}{1+\varepsilon} < \varphi < \pi \frac{1+\varepsilon}{1+\varepsilon} \}
\]

\[
C_{-0} = \{ \rho = \frac{\sin(\varepsilon \varphi)}{\sin(\varphi + \varepsilon \varphi)} : 0 < \varphi < \pi \frac{1+\varepsilon}{1+\varepsilon} \}
\]

\[
C_{-\pi} = \{ \rho = \frac{\sin(\varepsilon \varphi + \varepsilon \pi)}{\sin(\varphi + \varepsilon \varphi + \varepsilon \pi)} : 0 < \varphi < \pi(1-\varepsilon) \frac{1+\varepsilon}{1+\varepsilon} \}
\]

\[
C_{\pi} = \{ \rho = \frac{\sin(\varepsilon \varphi - \varepsilon \pi)}{\sin(\varphi + \varepsilon \varphi - \varepsilon \pi)} : -\pi(1-\varepsilon) \frac{1+\varepsilon}{1+\varepsilon} < \varphi < 0 \}.
\]

Let \( D_1 \) be the open complex domain delimited by the curves \( C_{\pm 0} \) and \( C_{\pm \pi} \), and \( D_0 \) be the complex domain delimited by the curves \( C_{\pm 0} \) and the segment \((-\infty,0)\), see fig. 3.3.

**Proposition 3.2.** The real Dulac maps (3.4) allow an extension to bi-holomorphic maps

\[
D_{1,2} : D_0 \cup D_1 \cup C_{+0} \cup C_{-0} \cup \{ y_\varepsilon \} \to D_0 \cup D_1 \cup C_{+0} \cup C_{-0} \cup \{ y_\varepsilon \}
\]

where

\[
D_{1,2}(D_1) = D_0, \quad D_{1,2}(C_{+0}) = C_{-0}, D_{1,2}(y_\varepsilon) = y_\varepsilon.
\]

They can be further analytically continued to a suitable open neighborhood of \( C_{\pi}, C_{-\pi} \) and

\[
D_{1,2}(C_{-\pi}) = D_{1,2}(C_{\pi}) = (-\infty,0).
\]

The limit of \( D_{1,2} \) at \( y = 1 \) exists and \( D_{1,2}(1) = 0 \).
Proof. The function $H(0, y) = y^\varepsilon (1 - y)$ allows an analytic continuation in $\mathbb{C} \setminus (-\infty, 0]$. The real Dulac map satisfies $H(0, y) = H(0, D_{1,2}(y))$ so does its complex extension, when it exists. Consider the isoclines

$$C_\theta = \{y \in D_0 \cup D_1 : \arg(y^\varepsilon (1 - y)) = \varepsilon \theta\}$$

or equivalently

$$C_\theta = \{y : \varepsilon \arg(y) + \arg(1 - y) = \varepsilon \theta\}.$$ 

which implies in polar coordinates

$$C_\theta = \{(\rho, \phi) : \rho = \sin(\varepsilon \varphi - \theta) \sin(\varphi + \varepsilon \varphi - \theta)\}.$$ 

Thus $D_1 \cup D_0$ is an union of the isoclines $C_\theta$

$$D_1 \cup D_0 = \{C_\theta : -\pi < \theta < \pi\}$$

where each $C_\theta$ has exactly two connected components, contained respectively in $D_1$ or $D_0$, see fig. 3.3. For $\theta = 0$ the Dulac map exchanges the segments $(0, y_c)$ and $(y_c, 1)$, by continuity the same holds true for all $\theta$. Moreover, by a local analysis of the Dulac map near 1 we have

$$\lim_{y \to 1} D_{1,2}(y) = 0$$
and the identity \(|y^f(1-y)| = |D_{1,2}(y)^f(1-D_{1,2}(y))|\) implies that when \(|y|\) tends to infinity along \(C_\theta\), then so does \(D_{1,2}(y)\). As \(D_{1,2}'(y) \neq 0\) then \(D_{1,2}\) is a bi-analytic map between connected components of \(C_\theta\), a bijection between \(D_1\) and \(D_0\), and finally a bi-holomorphic map between \(D_1\) and \(D_0\). The claims about the behavior of \(D_{1,2}\) along the border of the domains \(D_1, D_0\) are straightforward. □

Consider the continuous family of orbits \(\{\gamma_{y}^{1,2}\}_{y}, y \in (\varepsilon/(1+\varepsilon), 1)\) defining \(D_{1,2}(y)\) on fig. 3.1 respectively. Each \(\gamma_{y}^{1,2}\) is a real path starting at \(y\) and terminating at \(D_{1,2}(y)\).

**Definition 3.3.** We shall say that the analytic functions \(D_{1,2}\) defined in the domain \(D_1\) allow a geometric realization, provided that the continuous families of real paths \(\{\gamma_{y}^{1,2}\}_{y}, y \in (y_c, 1)\) allow an extension to continuous families of paths \(\{\gamma_{y}\}_{y \in D_1}\), contained in the complex leaves of the foliation \(F_{\varepsilon,0}\), such that each \(\gamma_{y}^{1,2}\) starts at \(y\) and terminates at \(D_{1,2}(y)\).

Of course, the families of paths \(\{\gamma_{y}\}_{y}\) are defined up to a homotopy. We note that although \(D_1 = D_2\) they allow non-equivalent geometric realizations.

**Proposition 3.4.** The real Dulac maps \(D_{1,2}\) allow a geometric realization in the domain \(D_1\).

**Proof.** Consider the projection

\[
\pi : \mathbb{C} \to \mathbb{C} : (x, y) \mapsto y
\]

and let \(\Gamma_y\) be the connected component of a leaf of the foliation \(F_{\varepsilon,0}\) through the point \((0, y)\) which is contained in the pre-image under \(\pi\) of the domain \(\overline{D}_0 \cup D_1\). If \(c = y^f(1-y)\), then along \(\Gamma_y\) holds

\[
(3.5) \quad x^2 = y + c(1-y)^{-\frac{1}{2}}
\]

which shows that \(\pi : \Gamma_y \to \overline{D}_0 \cup D_1\) is a double covering, ramified at the points \(y\) and \(D_1(y) = D_2(y)\). The leaf \(\Gamma_y\) is therefore a smooth open Riemann surface, except \(\Gamma_{y_c}\) which has a normal crossing at \((0, y_c)\), where \(y_c = D_1(y_c) = D_2(y_c)\).

Let \(\sigma_y\) be a path in \(\overline{D}_0 \cup D_1\) connecting \(y\) to \(D_1(y) = D_2(y)\). Such a path can be lifted in \(\Gamma_y\) under \(\pi\) in two different ways. To every continuous family of paths \(\sigma_y\) correspond therefore two continuous families of lifts in \(\gamma_{y} \subset \Gamma_y\). We apply now these considerations to the continuous family of segments \(\sigma_y = [y, D_1,2(y)] \subset \mathbb{R}, y \in (y_c, 1)\). The lift of \(\sigma_y\) will be the real orbit \(\gamma_{y}^{1}\) or \(\gamma_{y}^{2}\) defining \(D_1(y)\) or \(D_2(y)\), and shown on fig. 3.1.

To complete the proof we

- extend first the segments \(\sigma_y = [y, D(y)] \subset \mathbb{R}\) to a continuous family of paths \(\sigma_y \subset D_1 \cup D_0, y \in D_1\) which connect \(y \in D_1\) to
$\mathcal{D}(y) \in \mathbb{D}_0$, as it is illustrated on fig. 3.4.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.4.png}
\caption{The continuous family of paths $\{\sigma_y\}_y$}
\end{figure}

- Second, to each path $\sigma_y$, $y \in \mathbb{D}_1$, we associate its lift $\gamma_y$ with respect to $\pi$ with initial point $(0, y)$. For $y \in (y_c, 1)$ the lift $\gamma_y$ was already defined, by continuity it will be defined without ambiguity for all $y \in \mathbb{D}_1$. For $y \in (y_c, 1)$ the end point of $\gamma_y$ is $(0, \mathcal{D}(y))$. As the end point of $\gamma_y$ depends analytically on $y$, then it is $(0, \mathcal{D}(y))$ for all $y \in \mathbb{D}_1$.

\[ \square \]

Remark 3.5. The above considerations show that the Riemann surface $\Gamma_y$ is a topological cylinder, which is a double covering of the topological disc $\overline{\mathbb{D}}_0 \cup \mathbb{D}_1$, ramified over $y$ and $\mathcal{D}(y)$. The closed loop $\gamma_y^1 \circ (\gamma_y^2)^{-1}$ is the generator of the fundamental group of the cylinder.

Remark 3.6. It can be shown, by making use of Proposition 3.1 that $\mathcal{D}_{1,2}$ allows also an analytic extension to $\mathbb{C} \setminus \{1\}$ with countably many algebraic singularities. More precisely, if $y_0 \neq 1$ is a singular point, then $y_0$ belongs to the leaf through the center point $p_c$. The curve on the $y$-plane containing the possible singular points is defined therefore by the equation

\[ \{ y : |f(y)| = |f(y_c)| \}. \]

This curve is easily analyzed and it is shown on fig. 3.5.
3.3. **Geometric variation.** We describe a geometric construction of the variation. It will be used later, to control intersection points of curves \( \{ \text{Im} D_1 = 0 \} \) and \( \{ \text{Im} D_2 = 0 \} \).

In the following proposition the dot denotes product in the path grupoid, i.e. \( \gamma^1 \cdot \gamma^2 \) denotes path consisting of \( \gamma^1 \) followed by \( \gamma^2 \), where the end point of \( \gamma_1 \) coinicide with the starting point of \( \gamma_2 \). The analog meaning for the inverse.

**Proposition 3.7.** Let \( \gamma_1^{1,2} \) be a geometric realization of Dulac maps \( D_{1,2} \) and let \( y_\pm \in C_{\pm\pi} \) and \( y_- = \overline{y_+} \). The path \( \gamma_8 = (\gamma_{y_+}^1 \cdot (\gamma_{y_-}^2)^{-1}) \cdot (\gamma_{y_+}^1 \cdot (\gamma_{y_-}^2)^{-1}) \) is a closed loop that projects on the y-plane to the loop shown on figure 3.6. It can be homotopically deformed to a loop located in finite (\( \epsilon \)-independent) distance from the slow parabola \( y - x^2 = 0 \).
At the limit $\varepsilon \to 0$ it is the figure eight loop in the leaf of fast foliation – see figure 3.7.

Proof. By definition of the curves $C_{\pm\pi}$, $D_{1,2}(y_{\pm}) \in \mathbb{R}_-$; due to the conjugation relation $y_- = \overline{y_+}$ and the Schwartz reflection principle, the images of $y_{\pm}$ coincide. Thus, the path $(\gamma_1^+: (\gamma_2^-)^{-1})$ joins $Y_+$ with $y_-$ and passes through the third ramification point $D_{1,2}(y_{\pm})$ of the double covering 3.5. It can be deformed homotopically to the path avoiding the ramification point $D_{1,2}(y_{\pm})$ from the left.

The next path is a complex conjugacy of the first one. It is the same composition of $\gamma$ paths with indices $(1, 2)$ exchanged and the direction reversed. The composition is as shown on the figure 3.6 after homotopical deformation that avoiding the ramification points $y_{\pm}$. □

4. BLOWING UP THE TURNING POINT

![Figure 4.1. The real level sets of $e^{-y}(y - x^2)$](image)

When $\varepsilon$ tends to zero, the center $p_c$ of the integrable foliation tends to the contact point $p_0 = (0, 0)$ of the slow manifold $\{y = x^2\}$ with the leaves $y = \text{const}$. The point $p_0$ is therefore the turning point of our slow-fast Darboux foliation $F_{\varepsilon,0}$, and the study of the Dulac map near $p_0$ when $\varepsilon$ tends to zero will be studied, as explained in detail in [2, section 3], by a weighted blow up in the $(x, y, \varepsilon)$-space. Namely, the
rescaling
\[(4.1)\]
\[x \to \sqrt{\varepsilon} x, \ y \to \varepsilon y, \ \varepsilon \to \varepsilon\]
sends the center point \(y_c\) to \(1/(1 + \varepsilon)\), leaves the parabola \(y = x^2\) invariant and transforms the first integral \([(1 - y)(y - x^2)^{1/\varepsilon}]\) of the foliation \(\mathcal{F}_{\varepsilon,0}\) to the form
\[\varepsilon(1 - \varepsilon y)^{1/2}(y - x^2) = \varepsilon e^{-y+O(\varepsilon)}(y - x^2) = \varepsilon e^{-y}(y - x^2) + O(\varepsilon^2).\]

The blown up foliation has therefore, in every compact neighborhood of the origin, an analytic first integral, uniformly in \(x, y, O(\varepsilon)\)-close to \(e^{-y}(y - x^2)\)
see fig. 4.1. Finally, the curves \(C_{\pm \pi}\) are transformed to curves on a finite distance from the origin \(y = 0\), see fig. 4.2.

Figure 4.2. The cross-section \(\{x = 0\}\) in a neighborhood of the turning point after the weighted blow up of the \((x, y, \varepsilon)\)-space

5. THE PERTURBED FOLIATION \(\mathcal{F}_{\varepsilon, \delta}\) WHERE \(\varepsilon > 0\) AND \(\|\delta\|\) IS MUCH SMALLER THAN \(\varepsilon\)

In the preceding section we considered two Dulac maps \(\mathcal{D}_1\) and \(\mathcal{D}_2\) which coincide as analytic functions, but were defined geometrically in two different ways. We have respectively two non-equivalent geometric realizations \(\{\gamma_y^{1, 2}\}_{y \in \mathcal{D}_1}\) of the same analytic function \(\mathcal{D} = \mathcal{D}_1 = \mathcal{D}_2\).

In this section we suppose that \(\varepsilon > 0\) is fixed, \(\delta \in \mathbb{R}^N\), and \(\|\delta\|\) is sufficiently small with respect to \(\varepsilon\). Let \(p_c = (x_c, y_c)\) be the singular point (a focus) of \(\mathcal{F}_{\varepsilon, \delta}\), close to the center of \(\mathcal{F}_{\varepsilon, 0}\). We consider the cross-section \(\{x = x_c\}\) and as before define geometric realizations of the corresponding Dulac maps \(\mathcal{D}_1\) and \(\mathcal{D}_2\). Indeed, when \(\delta \neq 0\), the continuous family of paths \(\{\gamma_y^{1, 2}\}_{y \in \mathcal{D}_1}\), contained in the leaves of the foliation \(\mathcal{F}_{\varepsilon, 0}\) persist under a small perturbation, at least when the
initial point \((0, y)\) belongs to a relatively compact domain \(K \cup D_1\), and the leaves of the foliation \(F_{\varepsilon, 0}\) are transverse to the line \(\{x = 0\}\) at \((0, y)\), as well at \((0, D(y))\). We obtain in this way two continuous families of paths in the leaves of the perturbed foliation \(F_{\varepsilon, \delta}\), which begin at a point \((0, y)\) and terminate at a point \((0, D_1(y))\) or \((0, D_2(y))\) respectively.

The above holds true also in a neighborhood of the center point \(p_c\) (which becomes a focus after the perturbation), but also in a neighborhood of the singular point \((0, 1) \in \{x = 0\}\). Indeed, after the perturbation a saddle point persists, and the Dulac map near such a point is defined in every fixed sector, centered at the singular point. The image of the real analytic curves \(C_{\pm \pi}^1\) under the Dulac map is the negative real semi-axes, see Proposition 3.2. Therefore these curve can be also defined by the condition that \(\{y : \text{Im}(D_1(y)) = 0, \text{Im}(D_2(y)) = 0\}\). We denote for a further use these two curves by \(C_{\pm \pi}^{1,\pm \pi} = C_{\pm \pi}^{1,\pm \pi}(\varepsilon, \delta)\) and \(C_{\pm \pi}^{2,\pm \pi} = C_{\pm \pi}^{2,\pm \pi}(\varepsilon, \delta)\) respectively (when there is no ambiguity, the dependence on \(\varepsilon\) and \(\delta\) is omitted).

As shown in [5, 6], the curves \(C_{\pm \pi}^{1,\pm \pi}\) are analytic, including at the singular points of the Dulac maps, corresponding to the saddles \(s_1\) and \(s_2\). Consider the closed complex domain \(D\), which is bounded by \(C_{\pm \pi}^{1,\pm \pi}\), and the line \(\Re y > y_c\), as on fig 5.2.

**Proposition 5.1.** The number of the limit cycles of \(F_{\varepsilon, \delta}\), bifurcating from the closed period annulus of \(F_{\varepsilon, 0}\), is bounded by the variation of the argument of the analytic function \(D_1 - D_2\) divided by \(2\pi\), along the border \(\partial D\) of the domain shown on fig. 5.2.

**Proof.** The Dulac maps have analytic continuations in \(D\), as their geometric realizations persist under sufficiently small perturbation. The limit cycles are in a one-to-one correspondence with the fixed points \(y, D_1(y) = D_2(y)\) of the first return map. Therefore it is enough to bound the zeros of \(D_1 - D_2\) by making use of the argument principle, as explained in [6 section 2.1]. ☐
Corollary 5.2. For every fixed $\varepsilon > 0$ and for every sufficiently small $\delta$, the Dulac maps $D_1, D_2$ are real analytic functions in a suitable neighborhood of $D \setminus \{s_i\}$ where $s_i \sim 1$ are the singular points of the Dulac maps.

Our intension is to apply the argument principle to the analytic function $D_1 - D_2$ in $D$, in order to bound its complex zeros (equivalently, fixed points of the return map, or complex limit cycles). For this we note that although the Dulac map is singular at $s_i$, it is continuous at these points. We shall bound uniformly one one side, the variation of the argument of $D_1 - D_2$ along the segment $\{\Re y > y_c\} \cap D$, and on the other hand the number of the zeros of the imaginary part of $D_1 - D_2$ along $\{C_{\pm 1}^\pm \cup C_{\pm 2}^\pm \} \cap D$. By definition of the curves $C_{\pm 1}^\pm, C_{\pm 2}^\pm$, these zeros are just the intersection points $C_{\pm 1}^+ \cap C_{\pm 2}^+ + C_{\pm 1}^- \cap C_{\pm 2}^-$, counted with multiplicity. The curves $C_{\pm 1}^\pm, C_{\pm 2}^\pm$ depend, however, on $\varepsilon, \delta$, and their behavior when the parameters $\varepsilon, \delta$ tend to zero is crucial. When $\delta = 0, \varepsilon > 0$ the curves are explicit and tend to the real axes as $\varepsilon \to 0$.

6. Proof of Theorem 2.1

Suppose first that $\varepsilon > 0$ belongs to a sufficiently small but fixed neighborhood of the origin. Consider the segment corresponding to the part of $\partial D$, contained in the line $\{\Re y = y_c\}$. It follows from Proposition 3.3 that along this segment the functions $D_1, D_2$ have a geometric
realization and hence are analytic, both in \( y \) and \( \varepsilon, \delta \), provided that \( \varepsilon \neq 0 \) and \( ||\delta|| \) is much smaller than \( \varepsilon \). To prove the analyticity in a neighborhood of \( \varepsilon = 0 \) we consider the rescaling (4.1) and the domain \( D_1 \) shown on fig. 4.2. It can be shown along the same lines as in Proposition 3.4 that the Dulac maps have a geometric realization in \( D_1 \) again, and hence it is also analytic in \( \sqrt{\varepsilon}, \delta \), in a neighborhood of \( \varepsilon = 0, \delta = 0 \). This implies the analyticity for \( \varepsilon, \delta \) close to zero (after the rescaling). Therefore the variation of the argument along the compact segment, corresponding to the part of \( \partial D \), contained in the line \( \{ \Re y = y_c \} \), is uniformly bounded.

Along the remaining part of \( \partial D \) the displacement map \( D_1 - D_2 \) is analytic, except at the singular points \( s_i \). For this purpose we bound the increase of the argument of \( D_1 - D_2 \) by the number of the zeros of its imaginary part (the so called "Petrov trick") along \( \partial D \). Clearly, these zeros are exactly the intersection points of the curves \{Im \( D_1 = 0 \)\}, \{Im \( D_2 = 0 \)\}, that is to say \( C_{1-\pi} \cap C_{2-\pi} \) and \( C_{1+\pi} \cap C_{2+\pi} \). The intersection points have a transparent geometric meaning: they correspond to complex limit cycles intersecting the cross-section, or fixed points of the holonomy map \( \mathcal{H}ol \) along the "figure height loop" which we recall now (see [6, 5, 2]). If the holonomy map \( \mathcal{H}ol \) were analytic with respect to the parameters too, this would imply an uniform bound for the number of fixed points of \( Z(\varepsilon) \), when \( \varepsilon > 0 \) belongs to a sufficiently small neighborhood of the origin.

Indeed, the one-dimensional leaves of the foliation \( \mathcal{F}_{0,0} \) are the punctured discs \( \{(x, y) : y = c\} \setminus \{ (\pm \sqrt{c}, c) \} \), where \( c \neq 0 \). The geometric realization of \( \mathcal{H}ol = \text{id} \) is then explained in proposition 3.7 and shown on figures 3.6, 3.7 from which the analyticity follows, except along the
leaf \( \{ y = 0 \} \) through the turning point \((0,0)\). In a neighborhood of the turning point we use the rescaling (4.1) and describe the geometric realization of \( \text{Hol} \) as follows. Let \( y \in C_{-\pi} \) on fig. 6.1 and note that \( \bar{y} \in C_{+\pi} \). Consider the paths in the leaves of the foliation \( \mathcal{F}_{0,0} \)

\[ \gamma_y^1, (\gamma_y^1)^{-1}, \gamma_y^2, (\gamma_y^2)^{-1}. \]

connecting the points \( y, D_1(y), \bar{y}, D_2(y), \bar{y} \). These paths can be composed and the resulting closed path defines the holonomy map \( \text{Hol} \) which is then obviously analytic in \( \varepsilon, \delta \). The projection of these four paths on the cross-section, in the case of a fixed point of \( \text{Hol} \) are shown on fig. 6.1.

In a completely similar way we may prove the uniform boundedness of \( Z(\varepsilon) \) for \( \varepsilon > 0 \) in a neighborhood of \( \varepsilon = \infty \). Suffice it to exchange the roles of \( P_0 \) and \( P_1 \), and consider the foliation with a first integral \( P_0 P_1^{1/\varepsilon} \), where \( 1/\varepsilon > 0 \) belongs to a sufficiently small neighborhood of the origin.

Finally, when \( \varepsilon \) belongs to a compact subset of \((0, \infty)\), the uniform boundedness of \( Z(\varepsilon) \) follows from [6]. The Theorem is proved.

**REFERENCES**

[1] Marcin Bobieński, Pavao Mardešić, and Dmitry Novikov. Pseudo-Abelian integrals: unfolding generic exponential. *J. Differential Equations*, 247(12):3357–3376, 2009.

[2] Marcin Bobieński, Pavao Mardešić, and Dmitry Novikov. Pseudo-Abelian integrals on slow-fast Darboux systems. *Ann. Inst. Fourier (Grenoble)*, 63(2):417–430, 2013.

[3] F. Dumortier, R. Roussarie, and C. Rousseau. Hilbert’s 16th problem for quadratic vector fields. *J. Differential Equations*, 110(1):86–133, 1994.

[4] Freddy Dumortier and Robert Roussarie. Abelian integrals and limit cycles. *J. Differential Equations*, 227(1):116–165, 2006.

[5] Lubomir Gavrilov. On the number of limit cycles which appear by perturbation of Hamiltonian two-saddle cycles of planar vector fields. *Bull. Braz. Math. Soc. (N.S.)*, 42(1):1–23, 2011.

[6] Lubomir Gavrilov. On the number of limit cycles which appear by perturbation of two-saddle cycles of planar vector fields. *Funktsional. Anal. i Prilozhen.*, 47(3), 2013.

[7] Robert Roussarie. *Bifurcation of planar vector fields and Hilbert’s sixteenth problem*, volume 164 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1998.

**INSTITUTE OF MATHEMATICS, WARSAW UNIVERSITY, UL. BANACHA 2, 02-097 WARSAW, POLAND**  
E-mail address: mbobi@mimuw.edu.pl  

**INSTITUT DE MATHÉMATIQUES DE TOULOUSE, UMR 5219, UNIVERSITÉ DE TOULOUSE, CNRS, UPS IMT, F-31062 TOULOUSE CEDEX 9, FRANCE**