Sensitive Random Variables are Dense in Every $L^p(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mathbb{P})$

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Abstract

We show that, for every $1 \leq p < +\infty$ and for every Borel probability measure $\mathbb{P}$ over $\mathbb{R}$, every element of $L^p(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mathbb{P})$ is the $L^p$-limit of some sequence of random variables in $L^p(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mathbb{P})$ that are Lebesgue-almost everywhere differentiable with derivatives having norm greater than any pre-specified real number at every point of differentiability. In general, this result provides, in some direction, a finer description of an $L^p$-approximation for $L^p$ functions on $\mathbb{R}$.

Keywords: sensitive Borel random variables on the real line; differentiation; $L^p$-denseness; polygonal approximation; statistics

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1 Introduction

Regarding an $L^p$-approximation for $L^p$ functions with $1 \leq p < +\infty$, it is well-known, apart from the classical result of the $L^p$-denseness of simple measurable functions defined on an arbitrary measure space, that compactly supported continuous functions on a measure space whose ambient space is locally compact Hausdorff are $L^p$-dense.

Given any probability measure $\mathbb{P}$ defined on the Borel sigma-algebra $\mathcal{B}_\mathbb{R}$ of $\mathbb{R}$, we are interested in giving a more “controlled” $L^p$-approximation for every $1 \leq p < +\infty$, in terms of the local steepness of the graphs of the approximating random variables. For our purposes, a Borel random variable on $(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mathbb{P})$ is called $M$-sensitive if and only if the random variable is differentiable almost everywhere modulo Lebesgue measure and has the property that the normed derivative is $> M$ at every point of differentiability. Here $M$ is a given real number $\geq 0$. We stress that for a random variable on $\mathbb{R}$ to be $M$-sensitive is a function-theoretic property of the random variable itself; this conceptual clarification would be advisable as the conventional terminology in probability theory would lead the reader to instinctively take our $M$-sensitiveness as a distributional property of a random variable in analogy with “absolutely continuous random variables”. Now what we should like to prove is the possibility of $L^p$-approximating any given element of $L^p(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mathbb{P})$ by $M$-sensitive random variables.

We have communicated our intention mostly in a mathematically pure language, which is for an a priori concern of clarity. But our finding also admits a natural, application-oriented interpretation. As

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Using pure language can also be justified; probability theory is embedded in mathematics once we acknowledge that probability is viewed as a measure, and statistics, on the other hand, is embedded in mathematics once we interpret samples in terms of random variables. In this regard, studying properties of the Lebesgue spaces of random variables, being a fundamental workplace in probability theory (and statistics), advances the understanding of both fields.
a statistic is in the broad sense a Borel function from \( \mathbb{R}^n \) to \( \mathbb{R} \), our result says that, if \( 1 \leq p < +\infty \), then every statistic, being continuous (in the function-theoretic sense), of a single sample point drawn from a Borel distribution over \( \mathbb{R} \), such as a Gaussian one, can be \( L^p \)-approximated by statistics with finite \( p \)-th moment that are “as zigzagged as desired”.

2 Proof

We clarify, before the proof, some everyday terms present in the introductory discussion that might not always admit a uniform usage in the related literature. By a Borel probability measure over \( \mathbb{R} \) is meant a probability measure defined on \( \mathcal{B}_\mathbb{R} \); by a Borel random variable on \( \mathbb{R} \), with \( \mathbb{R} \) considered as a Borel probability space, we mean an \( \mathbb{R} \)-valued, Borel-measurable function defined on \( \mathbb{R} \). The set of all reals \( \geq 0 \) will be denoted by \( \mathbb{R}_+ \).

The reader is invited to mind that our definition of the concept of \( M \)-sensitiveness presumes Borel measurability; we should like to prove

**Theorem.** If \( 1 \leq p < +\infty \), and if \( \mathbb{P} \) is a Borel probability measure over \( \mathbb{R} \), then the \( M \)-sensitive random variables are \( L^p \)-dense in \( L^p(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mathbb{P}) \) for every \( M \in \mathbb{R}_+ \).

**Proof.** For simplicity, we abbreviate \( L^p(\mathbb{R}, \mathcal{B}_\mathbb{R}, \mathbb{P}) \) as \( L^p(\mathbb{P}) \) in our proof.

We begin by claiming that random variables of the form \( \sum_{j=1}^n x_j \mathbf{1}_{V_j} \), where \( n \in \mathbb{N} \) with \( x_1, \ldots, x_n \in \mathbb{R} \) and \( V_1, \ldots, V_n \subseteq \mathbb{R} \) being disjoint open intervals, are \( L^p \)-dense in \( L^p(\mathbb{P}) \). Indeed, since the simple Borel random variables are \( L^p \)-dense in \( L^p(\mathbb{P}) \), by Minkowski inequality it suffices to prove that the random variables of the desired form are \( L^p \)-dense in the simple Borel random variables. In turn, it suffices to show that for every \( B \in \mathcal{B}_\mathbb{R} \) and every \( \varepsilon > 0 \) there are some disjoint open intervals \( V_1, \ldots, V_N \subseteq \mathbb{R} \) such that \( |\mathbf{1}_{\bigcup_{j=1}^N V_j} - \mathbf{1}_B|_{L^p} < \varepsilon \); here \( | \cdot |_{L^p} \) denotes the in-context \( L^p \)-norm. Every Borel probability measure over a metric space is outer regular (e.g. Theorem 1.1, Billingsley [1]); so, given any \( \varepsilon > 0 \) and any \( B \in \mathcal{B}_\mathbb{R} \), there is some open (with respect to the usual topology of \( \mathbb{R} \), certainly) subset \( G \) of \( \mathbb{R} \) such that \( G \supset B \) and \( \mathbb{P}(G \setminus B) < (\varepsilon/2)^p \). Since \( G \) is also a countable union of disjoint open intervals \( V_1, V_2, \ldots \) of \( \mathbb{R} \), and since \( \mathbb{P} \) is a finite measure, there is some \( N \in \mathbb{N} \) such that \( \mathbb{P}(\bigcup_{j \geq N+1} V_j) < (\varepsilon/2)^p \). But then Minkowski inequality and the disjointness of these \( V_j \) together imply

\[
|\mathbf{1}_{\bigcup_{j=1}^N V_j} - \mathbf{1}_B|_{L^p} = |\mathbf{1}_{\bigcup_{j=1}^N V_j} - \mathbf{1}_{\bigcup_{j \geq N+1} V_j} - \mathbf{1}_B|_{L^p} \\
\leq |\mathbf{1}_{\bigcup_{j=1}^N V_j \setminus B}|_{L^p} + |\mathbf{1}_{\cup_{j \geq N+1} V_j \setminus B}|_{L^p} \\
< \varepsilon,
\]

and the claim follows.

Let \( X \in L^p(\mathbb{P}) \). If \( \varepsilon > 0 \), choose some random variable \( \phi_0 \) that is a linear combination of finitely many indicators of disjoint open intervals of \( \mathbb{R} \) such that \( |\phi_0 - X|_{L^p} < \varepsilon/2 \). Replace \( \phi_0 \) with its extension to \( \mathbb{R} \), if necessary, by assigning 0 to the complement of the finite union of disjoint open intervals corresponding to \( \phi_0 \); then i) the (resulting) random variable \( \phi_0 \) is Borel, ii) the \( L^p \)-distance between \( \phi_0 \) and \( X \) remains the same, iii) by construction, the set of all the points at which the random variable \( \phi_0 \) is not differentiable is finite; in particular, the random variable \( \phi_0 \) is differentiable almost everywhere modulo Lebesgue measure with \( D\phi_0 = 0 \) at every point of differentiability.

Let \( M \in \mathbb{R}_+ \). Let \( b := \lfloor 2\varepsilon^{-1}(M+1) \rfloor \), i.e. the least integer no less than the real number \( 2\varepsilon^{-1}(M+1) \). Define a function \( \phi_1 : \mathbb{R} \to \mathbb{R} \) by assigning 0 to each \( j/b \) with even \( j \in \mathbb{Z} \), assigning 1 to each \( j/b \) with
odd $j \in \mathbb{Z}$, and taking the continuous linear interpolations between all the points $j/b$ so that $|D\varphi_1| = b$ on $\mathbb{R}$ at every point of differentiability and $\varphi_1$ is continuous. Then $\varphi_1$ is Borel; moreover, since $|\varphi_1| \leq 1$ on $\mathbb{R}$, we have $\varphi_1 \in L^p(\mathbb{P})$. We would like to remark also that the set of all the points at which $\varphi_1$ is not differentiable is countable, and each element of the set is isolated (with respect to the standard topology of $\mathbb{R}$).

If $Y := \varphi_0 + 2^{-1} \varepsilon \varphi_1$, then $Y \in L^p(\mathbb{P})$ and

$$|Y - X|_{L^p} \leq |\varphi_0 - X|_{L^p} + 2^{-1} \varepsilon |\varphi_1|_{L^p} < \varepsilon.$$  

Further, the set of all the points at which $Y$ is not differentiable is by construction countable with each element being isolated. In particular, the random variable $Y$ is Lebesgue-almost everywhere differentiable, and we have

$$|DY| = |D\varphi_0 + 2^{-1} \varepsilon D\varphi_1|$$
$$= 2^{-1} \varepsilon |D\varphi_1|$$
$$= 2^{-1} \varepsilon b$$
$$\geq M + 1$$
$$> M$$

at every point where $Y$ is differentiable. Since $Y$ is then $M$-sensitive, the proof is complete. □

We draw some posterior remarks herewith:

Remarks.

- Our construction would be “amicable” in the sense that it does not depend on particularly deep results in analysis; and it is inspired by the general proof idea of the marvelous result that somewhere differentiable functions are meager in any given classical Wiener space (e.g. Theorem 12.10, Krantz [2]).

- In our setting, replacing the $L^p$-metric with the uniform metric is not necessarily possible as the involved functions are not necessarily even bounded.

- Our proof applies to every Borel finite measure over $\mathbb{R}$; the proof of Theorem 1.1. in Billingsley [1] is applicable also to Borel finite measures, and the value of corresponding $L^p$-norm of $\varphi_1$ is immaterial.

- Figuratively, the graph of a typical example of $Y$ constructed in our proof would be a “polygon” with “Lebesgue-few missing parts”. □

References

[1] Billingsley, P. (1999). Convergence of Probability Measures, second edition. John Wiley.

[2] Krantz, S.G. (1991). Real Analysis and Foundations, first edition. CRC Press.