Explicit conversion from the Casimir force to Planck’s law of radiation

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Abstract

The Casimir force has its origin in finite modification of the infinite zero-point energy induced by a specific boundary condition for the spatial configuration. In terms of the imaginary-time formalism at finite temperature, the root of Planck’s law of radiation can be traced back to finite modification of the infinite vacuum energy induced by the periodic boundary condition in the temporal direction. We give the explicit conversion from the Casimir force to Planck’s law of radiation, which shows the apparent correspondence between the system bounded by parallel conducting plates and the thermodynamic system. The temperature inversion symmetry and the duality relation in the thermodynamics are also discussed. We conclude that the effective temperature characterized by the spatial extension should no longer be regarded as genuine temperature.

Key words: Casimir force; Planck’s law of radiation; temperature inversion symmetry; duality; thermodynamic functions

The history of physics has passed through a number of memorable breakthrough points. Unacceptable as they might have looked at a first glance, many apparently unusual concepts permeate among us today after tremendous efforts. The discovery of celebrated Planck’s law of radiation by Planck [1] in 1900 and the recognition of the Casimir force by Casimir [2] in 1948 should, of course, be enumerated as monumental achievements. In the language of the imaginary-time formalism of the finite-temperature field theory [3], Planck’s formula can be perceived as the energy associated with the assembly of an infinite number of free oscillators with the periodic boundary condition in the temporal (thermal) direction. The phrase “an infinite number of oscillators” would remind us of the Casimir force that arises from finite discrepancy of the

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zero-point energy in the presence of a constraining boundary condition. The precise measurement of the Casimir force demonstrated in 1997 [4] confirmed the Casimir effect experimentally, even though no attempt using parallel plates results in successful measurement so far yet.

These two phenomena are closely related to each other by the O(4) symmetry of the Euclidean space-time. Thus writing the thermodynamic functions at finite temperature $T$ and extension $l$ in terms of a dimensionless parameter $\xi = Tl$, we can readily observe the symmetry under the exchange $\xi \leftrightarrow 1/4\xi$, which was first noted by Brown and Maclay for the scaled free energy [5]. Santos and Tort have recently shown the extended symmetry in the system confined in a conducting rectangular cavity [6]. A concise overview on this topic is available from Ref. [7]. Many authors are still paying attention to this symmetric property, that is named temperature inversion symmetry. This property is a typical manifestation of the equivalence under the exchange between the role of $\beta = \hbar c/k_B T$ and that of $l$. The numerical coefficient is required to compensate for the difference of the boundary conditions: the temporal direction is periodic while the spatial one is fixed. This means that the spatial counterpart of the Matsubara frequency is half smaller. It is most often the case that authors confirmed the realization of the temperature inversion symmetry simply by looking over the resultant expressions with some cutoff scheme at finite temperature and extension. In this paper we will demonstrate the explicit conversion to the same form as Planck’s law of radiation beginning with the definition of the Casimir force, which is the simplest example of the temperature inversion symmetry. The significant points here are that we do not resort to the intuitionally obtuse imaginary-time formalism and also that our conversion does not necessitate any regularization scheme. We believe that our calculation would shed light upon the deeper insight towards the temperature inversion symmetry owing to the transparency of each procedure and absence of any cutoff added by hand.

Furthermore on occasions one might encounter controversies in understanding the results acquired at finite temperature and those at finite extension, or the results in the Euclidean space-time and those in the Minkowskian space-time. It is often the case that the topological object existing in the Euclidean world cannot be interpreted as a physical object in the Minkowskian world from the thermodynamic point of view (e.g. see [8]). Also some subtleties might come from the Wick rotation from one world to the other, especially when the theory contains fermionic fields. Thus it would be informative to see the concrete correspondence between Planck’s law and the Casimir force without resorting to the O(4) symmetry of the Euclidean space-time. Our conclusion will be that the construction of thermodynamics in terms of the effective temperature (spatial extension) cannot be achieved in spite of the O(4) symmetry.
Let us consider the following configuration: there are two square conducting metal plates with each side sized $L$, one of which is located at $z = 0$ in the $x$-$y$ plane and the other located at $z = l$ parallel to the $x$-$y$ plane, where $l \ll L$. In the present case where $L$ is regarded as quite large, we can treat the wave numbers along the $x$ and $y$ directions as continuous. As for the $z$ direction the fixed boundary condition imposed by the metals obliges the wave number to take the discrete values,

$$k_z = \frac{\pi n}{l}, \quad n = 0, 1, 2, \ldots \quad (1)$$

Then noting that the energy quanta of the zero-point oscillation is given by $\frac{1}{2} \hbar c k$ and that the number of the degrees of freedom corresponding to the polarization is two except for the zero mode, we can readily write down the zero-point energy as

$$U_0 = \hbar c L^2 \int \frac{dk_x \, dk_y}{(2\pi)^2} \left( \frac{1}{2} k_\perp + \sum_{n=1}^{\infty} k_n \right) \equiv \hbar c L^2 \int \frac{dk_x \, dk_y}{(2\pi)^2} I(k_x, k_y), \quad (2)$$

where

$$k_\perp = \sqrt{k_x^2 + k_y^2}, \quad k_n = \sqrt{k_\perp^2 + \left( \frac{\pi n}{l} \right)^2}. \quad (3)$$

Cauchy’s integral theorem enables us to rewrite $I(k_x, k_y)$ in the form of the contour integration in the complex plane,

$$I(k_x, k_y) = -\oint \frac{dk}{2\pi} \left( \frac{k^2}{k^2 + k_\perp^2} + \sum_{n=1}^{\infty} \frac{2k^2}{k^2 + k_n^2} \right), \quad (4)$$

where the integration contour is a semi-circle with infinitely large radius whose diameter is on the real axis. As is often performed, we make use of the formula,

$$\frac{\coth z}{z} = \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{2}{z^2 + \pi^2 n^2}, \quad (5)$$

to evaluate the summation over the Matsubara frequency, i.e.

$$I(k_x, k_y) = -\oint \frac{dk}{2\pi} \frac{l k^2}{\sqrt{k^2 + k_\perp^2}} \coth(l \sqrt{k^2 + k_\perp^2})$$

$$= -\oint \frac{dk}{2\pi} \frac{l k^2}{\sqrt{k^2 + k_\perp^2}} - \oint \frac{dk}{2\pi} \frac{2 l k^2}{\sqrt{k^2 + k_\perp^2} (e^{2l \sqrt{k^2 + k_\perp^2}} - 1)}. \quad (6)$$
Up to now nothing new appears in our procedure to evaluate the summation [9]. Here it will be worth noting again what is most important in our evaluation is that no term is dropped off in spite of the presence of apparently divergent (ill-defined) terms (of course, we can rigorously define them using some proper regularization only if we do not mind making the expressions a bit more jumbled). We comment upon that the expression (6) mathematically corresponds to the Abel-Plana formula,

$$\sum_{n=0}^{\infty} f(n) = \int_{0}^{\infty} dx f(x) + \frac{1}{2}f(0) + i \int_{0}^{\infty} dx \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1}. \quad (7)$$

Let us first consider about the second term since it is more easily simplified. $\sqrt{k^2 + k_\perp^2}$ may take twofold values in the complex plain. We must specify which one to take the square root before going on our discussion. Because the integrand of (6) is an even function with respect to $\sqrt{k^2 + k_\perp^2}$, we can choose it in such a way that the real-part of $\sqrt{k^2 + k_\perp^2}$ becomes positive. Then the contribution from the arched path located infinitely far away vanishes exponentially. What is left is only the contribution from the path on the real axis, that can be integrated by parts into (we write $k_z$ here instead of $k$ for later convenience)

$$-\int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \frac{2tk_z^2}{\sqrt{k_z^2 + k_\perp^2}(e^{2t\sqrt{k_z^2 + k_\perp^2}} - 1)} = \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \ln(1 - e^{-2t\sqrt{k_z^2 + k_\perp^2}}). \quad (8)$$

Then we will proceed towards the evaluation of the first term of (6). Since the singularities of the integrand lie on the imaginary axis, we can modify the path

![Fig. 1. The modification of the paths in the first and second quadrants.](image-url)
in the first quadrant into the line apart from the imaginary axis by $+\epsilon$ and the path in the second quadrant into the line apart from the imaginary axis by $-\epsilon$ (see Fig. 1). The important point is that the branch cut on the imaginary axis in the region $\text{Im} k > k_\perp$ corresponds to our way how to specify the sign of the square root. As a result, the integrations over the region $\text{Im} k < k_\perp$ cancel out each other and the remaining integrations result in

$$
\int_{i\infty-\epsilon}^{i\epsilon-\epsilon} \frac{dk}{2\pi} \frac{lk^2}{\sqrt{k^2 + k_\perp^2}} - \int_{i\epsilon+\epsilon}^{i\infty+\epsilon} \frac{dk}{2\pi} \frac{lk^2}{\sqrt{k^2 + k_\perp^2}} = l \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \sqrt{k_z^2 + k_\perp^2},
$$

where we changed the integration variable from $k$ to $k_z = -i\sqrt{k^2 + k_\perp^2}$.

Thus without any obscurity the expression (6) is transformed into a considerably concise form, that is

$$
I(k_x, k_y) = l \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} k + \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \ln(1 - e^{-2lk_z}).
$$

Then the zero-point energy is expressed as

$$
U_0 = \frac{\hbar c L^2}{l} \int \frac{d^3k}{(2\pi)^3} \left\{ k + \frac{1}{l} \ln(1 - e^{-2lk}) \right\}.
$$

The finite difference from the continuum counterpart given by

$$
\Delta U_0 = U_0 - U_0^{\text{cont}} = \frac{\hbar c L^2}{l} \int \frac{d^3k}{(2\pi)^3} \ln(1 - e^{-2lk})
$$

or the energy density

$$
u = \frac{\Delta U_0}{L^2l} = \frac{\hbar c}{l} \int \frac{d^3k}{(2\pi)^3} \ln(1 - e^{-2lk})
$$

generates the Casimir force, that is an attractive force acting on the plate per unit area,

$$
p = -\frac{\partial (lu)}{\partial l} = -2\hbar c \int \frac{d^3k}{(2\pi)^3} \frac{k}{e^{2lk} - 1}.
$$

On the other hand the thermodynamic functions at finite temperature $T$ are given by the followings:
\begin{align*}
p &= -f = -\frac{2\hbar c}{\beta} \int \frac{d^3k}{(2\pi)^3} \ln(1 - e^{-\beta k}), \\
u &= f + Ts = f - T \frac{\partial f}{\partial T} = -\frac{\partial (\beta p)}{\partial \beta} = 2\hbar c \int \frac{d^3k}{(2\pi)^3} \frac{k}{e^{\beta k} - 1},
\end{align*}

where \( p, f, u \) and \( s \) are the pressure, the Helmholtz free energy density, the internal energy density and the entropy density, respectively. \( \beta \) stands for the inverse temperature \( \beta = \frac{\hbar c}{k_B T} \). The expression of \( u \) is nothing but Planck’s law of radiation. Looking at the expressions (14) and (16) we can recognize the duality relations,

\[2l \iff \beta = \frac{\hbar c}{k_B T}, \quad p \iff -u. \quad (17)\]

The reason why the appearance of the additional coefficient in front of \( l \) is that we adopted the fixed boundary condition in the spatial direction as mentioned at the beginning of this paper. This duality relation is an embodiment of the symmetry under the exchange of the temporal axis and the spatial axis, which corresponds to the swap of the electric field (temporal component) and the magnetic field (spatial component) in the electrodynamics, that is, the electro-magnetic duality.

As long as concerned with the O(4) symmetry, one would regard \( l \) as the (inverse) effective temperature for the system. As is obvious in the calculation of the partition function in the functional integral method [10], each mathematical procedure is absolutely symmetric. Nevertheless physics is different, or, actually the latter relation in (17) prevents us from accepting \( l \) as the genuine temperature in a thermodynamic sense. For instance the entropy density in the thermodynamics can be written in terms of the pressure \( p \) and the internal energy density \( u \) as

\[s = \frac{p + u}{T}, \quad (18)\]

where the explicit expressions are derived from the equations (15) and (16) as \( p = \frac{\pi^2 \hbar c}{45\beta^4} \) and \( u = \frac{\pi^2 \hbar c}{15\beta^4} \). We can immediately confirm ourselves that the thermodynamic relation,

\[\frac{\partial s}{\partial u} = \frac{1}{T}, \quad (19)\]

is satisfied, which is the definition of the absolute temperature. Once we admit the duality relation (17), the dual entropy density at finite extension is given
by

\[ s = -\frac{2k_B l}{\hbar c}(u + p). \]  

(20)

Then the thermodynamic relation becomes

\[ \frac{\partial s}{\partial u} = \frac{6k_B l}{\hbar c} \neq \frac{2k_B l}{\hbar c}, \]  

(21)

which shows the inconsistency for the thermodynamic relations. In fact the canonical ensemble in the statistical mechanics is based upon the relation (19). The collapse of the relation (19) means that we cannot consider any thermodynamic system in terms of the effective temperature \( \hbar c/2k_B l \).

Thus we have seen the explicit conversion from the Casimir force to Planck’s law of radiation, as is evident in the duality relation (17). The prominent feature we would like to stress here is that we could establish the correspondence between the Casimir force and Planck’s law of radiation by resorting neither to the subtle O(4) symmetry nor to any artificial regularization. As far as we know, no one had ever expressed the Casimir force in the form plainly comparable with Planck’s law, like our goal (16). After the momentum integration for the Casimir force, we reach the well-known functional form of \( l^{-4} \), whose counterpart in the thermodynamics, of course, is the Stefan-Boltzmann law. What should be noted here is that the correspondence is between the pressure and the internal energy density, that have the same dimension. This is the reason why the system cannot be described in the language of thermodynamics by using the effective temperature, regardless of the almost trivial realization of the temperature inversion symmetry in the framework of the Euclidean functional method owing to the O(4) symmetry. We believe that our contribution presented here will provide an intuitive view in the forefront of physics.

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