Modular Construction of Free Hyperplane Arrangements

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In this article, we study freeness of hyperplane arrangements. One of the most investigated arrangement is a graphic arrangement. Stanley proved that a graphic arrangement is free if and only if the graph is chordal and Dirac showed that a graph is chordal if and only if the graph is obtained by “gluing” complete graphs. We will generalize Dirac’s construction to simple matroids with modular joins introduced by Ziegler and show that every arrangement whose associated matroid is constructed in the manner mentioned above is divisionally free.

Keywords: hyperplane arrangement, free arrangement, matroid, modular join, chordality

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1 Introduction

A (central) hyperplane arrangement \(\mathcal{A}\) over a field \(\mathbb{K}\) is a finite collection of subspaces of codimension 1 in a finite dimensional vector space \(\mathbb{K}^\ell\). A standard reference for arrangements is \([8]\). Let \(S\) denote the polynomial algebra \(\mathbb{K}[x_1, \ldots, x_\ell]\), where \((x_1, \ldots, x_\ell)\)

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is a basis for the dual space \((\mathbb{K}^l)\)^*. Let \(\text{Der}(S)\) denote the the module of derivations of \(S\), that is,

\[
\text{Der}(S) := \{ \theta : S \to S \mid \theta \text{ is } S\text{-linear and } \theta(fg) = f\theta(g) + \theta(f)g \text{ for any } f, g \in S \}.
\]

The module of logarithmic derivations \(D(A)\) is defined by

\[
D(A) := \{ \theta \in \text{Der}(S) \mid \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in A \},
\]

where \(\alpha_H\) is a linear form such that \(\text{ker}(\alpha_H) = H\).

**Definition 1.1.** An arrangement \(A\) is called **free** if \(D(A)\) is a free \(S\)-module.

Although the definition of free arrangements is algebraic, Terao’s celebrated factorization theorem [13, Main Theorem] shows a solid relation between algebra, combinatorics, and topology of arrangements.

One of typical family of arrangements is graphic arrangements. Let \(\Gamma = ([\ell], E)\) denote a simple graph, where \([\ell] := \{1, \ldots, \ell\}\). Define a graphic arrangement \(A(\Gamma)\) by

\[
A(\Gamma) := \{ \{ x_i - x_j = 0 \} \mid \{ i, j \} \in E \}.
\]

A simple graph is **chordal** if every cycle of length at least 4 has a chord, which is an edge connecting nonconsecutive vertices of the cycle. Freeness of graphic arrangements is characterized in terms of graphs as follows.

**Theorem 1.2** ([5, Theorem 3.3]). A graphic arrangement \(A(\Gamma)\) is free if and only if \(\Gamma\) is chordal.

Let \(K_n\) denote the complete graph on \(n\) vertices. Note that \(K_0\) means the null graph. A chordal graph is constructed by gluing complete graphs as follows.

**Theorem 1.3** (Dirac [4, Theorem 1 and 2]). The class of chordal graphs coincides with the smallest class of graphs satisfying the following conditions.

1. For every nonnegative integer \(n\), the complete graph \(K_n\) belongs to the class.
2. Let \(\Gamma\) be a simple graph on \(V = V_1 \cup V_2\). If the induced subgraph \(\Gamma[V_1 \cap V_2]\) is complete (including the null graph) and both \(\Gamma[V_1]\) and \(\Gamma[V_2]\) belong to the class, then \(\Gamma\) belongs to the class.

The purpose of this paper is to generalize the class consisting of chordal graphs in terms of matroids with conditions described in Theorem 1.3 and associate it with freeness of hyperplane arrangements.

Let \(L\) be a geometric lattice. An element \(X\) is called **modular** if

\[
r(X) + r(Y) = r_L(X \wedge Y) + r(X \vee Y) \quad \text{for all } Y \in L,
\]

where \(r\) denotes the rank function of \(L\). A flat \(X\) of a simple matroid \(M\) is called **modular** if \(X\) is modular in \(L(M)\), the lattice of flats of \(M\).
Definition 1.4. A simple matroid $M$ on the ground set $E$ is said to be a modular join if there exist two proper modular flats $E_1$ and $E_2$ of $M$ such that $E = E_1 \cup E_2$. We also say that $M$ is the modular join over $X$, denoted $M = P_X(E_1, E_2) = P_X(M_1, M_2)$, where $M_i := M/E_i$ for $i = 1, 2$ and $X := E_1 \cap E_2$.

Remark 1.5. Ziegler [15] introduced a modular join, which is a special case of a generalized parallel connection or a strong join investigated by Brylawski [3] and Lindström [7]. Our definition of a modular join is different from Ziegler’s one. However, they are equivalent (See [15, Proposition 3.3] and [3, Proposition 5.10]). In addition, note that $X$ is modular in $M_1$, $M_2$, and $M$.

Definition 1.6. A matroid $M$ is called round (or nonsplit) if the ground set is not the union of two proper flats. A flat $X$ of $M$ is round if the restriction $M|X$ is round.

It is well known that a simple graphic matroid is round if and only if the associated graph is complete (See [2, Theorem 4.2] for example). Our generalization of chordal graphs is defined as follows.

Definition 1.7. Let $\mathcal{C}$ be the minimal class of simple matroids which satisfies the following conditions.

(i) The empty matroid is a member of $\mathcal{C}$.

(ii) If a simple matroid $M$ has a modular coatom $X$ and $M|X \in \mathcal{C}$, then $M \in \mathcal{C}$.

(iii) Let $M$ be a modular join of $M_1$ and $M_2$ over a round flat. If $M_1, M_2 \in \mathcal{C}$, then $M \in \mathcal{C}$.

One can prove that a simple graphic matroid belongs to $\mathcal{C}$ if and only if the associated graph is chordal. Recently Suyama, Torielli, and the author [12] treated a similar class for signed graphs.

The linear independence of an arrangement $\mathcal{A}$ determines a simple matroid $M(\mathcal{A})$ on itself. The main theorem of this paper is as follows (See Definition 2.4 for the definition of divisional freeness).

Theorem 1.8. If $M(\mathcal{A}) \in \mathcal{C}$, then $\mathcal{A}$ is divisionally free.

The organization of this paper is as follows. In [2] we recall basic properties about simple matroids and geometric lattices, including modularity. In addition, we introduce divisional atoms and study them for modular joins. In [3] we give a proof of Theorem 1.8.

2 Preliminaries

2.1 Simple matroids and geometric lattices

It is well known that there exists a bijection between simple matroids and geometric lattices (up to isomorphism).
Theorem 2.1 (See [14, p.54 Theorem 2] for example). The correspondence between a simple matroid and its lattice of flats is a bijection between the set of simple matroids and the set of geometric lattices.

Thus any properties about simple matroids are translated into properties of geometric lattices, and vice versa. For example, the contraction and the restriction of matroids are just intervals in the lattice of flats as follows.

Proposition 2.2 ([9, Proposition 3.3.2]). Let $X$ be a flat of a matroid $M$. Then

1. $L(M/X) \simeq [X, \hat{1}] = \{ F \in L(M) \mid X \leq F \}$.
2. $L(M|X) \simeq [\hat{0}, X] = \{ F \in L(M) \mid F \leq X \}$.

Note that the contraction of a simple matroid is not simple in general. However, we can associate the simple matroid $\text{si}(M)$ with a matroid $M$ and this operation does not affect the lattice of flats, that is, $L(M) \simeq L(\text{si}(M))$.

If $A$ be an arrangement and $M(A)$ the simple matroid on $A$, then for any hyperplane $H \in A$, we have $M(A^H) \simeq \text{si}(M(A)/H)$, where $A^H$ denotes the restriction defined by $A^H := \{ K \cap H \mid K \in A \setminus \{ H \} \}$.

2.2 Divisionality

Let $L$ be a geometric lattice. The characteristic polynomial $\chi(L, t) \in \mathbb{Z}[t]$ is defined by

$$
\chi(L, t) := \sum_{X \in L} \mu(X) t^{r(\hat{1}) - r(X)},
$$

where $r$ denotes the rank function of $L$ and $\mu: L \to \mathbb{Z}$ denotes the one-variable Möbius function of $L$ defined recursively by

$$
\mu(X) := \begin{cases} 
1 & \text{if } X = \hat{0}, \\
- \sum_{Y \prec X} \mu(Y) & \text{otherwise}.
\end{cases}
$$

The characteristic polynomial of $M$ is defined by $\chi(M, t) := \chi(L(M), t)$.

The intersection lattice $L(A)$ of a central arrangement $A$ is defined by

$$
L(A) := \left\{ \bigcap_{H \in B} H \mid B \subseteq A \right\}
$$

with a partial order by reverse inclusion. Note that $L(A)$ is a geometric lattice and naturally isomorphic to $L(M(A))$. The characteristic polynomial $\chi(A, t) \in \mathbb{Z}[t]$ is defined by

$$
\chi(A, t) := \sum_{X \in L(A)} \mu(X) t^{\dim X}.
$$
Since the rank function of $L(A)$ is given by the codimension, $\chi(A, t) = t^{\ell-r} \chi(M(A), t)$, where $\ell$ is the dimension of the ambient space and $r$ is the rank of $L(A)$. When $\ell = r$ we say that $A$ is essential. It is well known that there exists an essential arrangement $A_0$ for every arrangement $A$ such that $L(A) = L(A_0)$ and $A$ is free if and only if $A_0$ is free (See [8] for details).

**Theorem 2.3** (Abe [1, Theorem 1.1 (Division theorem)]). An arrangement $A$ is free if there exists $H \in A$ such that $A^H$ is free and $\chi(A^H, t)$ divides $\chi(A, t)$.

**Definition 2.4. Divisional freeness** is defined recursively by the following conditions.

(i) Every empty arrangement is divisionally free.

(ii) Suppose that $A^H$ is divisionally free and $\chi(A^H, t)$ divides $\chi(A, t)$. Then $A$ is divisionally free.

Note that, by Theorem 2.3, every divisionally free arrangement is free.

**Definition 2.5.** An atom $e$ of a simple matroid $M$ is called divisional if $\chi(M/e, t)$ divides $\chi(M, t)$.

### 2.3 Modularity

We excerpt some conditions equivalent to modularity from Brylawski [3].

**Proposition 2.6** (Brylawski [3, Theorem 3.3]). Let $X$ be an element of a geometric lattice $L$. Then the following conditions are equivalent.

1. $X$ is modular.
2. For $Y \leq Z$ in $L$, $Y \vee (X \wedge Z) = (Y \vee X) \wedge Z$.
3. For all $Y \in L$, $[X \wedge Y, X] \simeq [Y, X \vee Y]$.
4. For any atom $e \not\leq X$, $[\hat{0}, X] \simeq [e, X \vee e]$ and $X \vee e$ is modular in $[e, \hat{1}]$.

Brylawski also study properties of modularity. The following proposition is one of them.

**Proposition 2.7** (Brylawski [3, Proposition 3.6]). Let $X$ and $Y$ be modular flats of a simple matroid. Then $X \cap Y$ is a modular flat.

We give some properties of modularity required in this article.

**Proposition 2.8** (Probert [10, Corollary 4.2.8]). Every modular flat of a round matroid is round.

**Proposition 2.9** (Kung [6, Lemma 1.1]). Let $X$ be a coatom of a simple matroid $M$ on the ground set $E$. Then $X$ is modular if and only if for any $e \in E \setminus X$, the restriction $M|X$ isomorphic to $\text{si}(M/e)$. 

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Theorem 2.10 (Stanley [11, Theorem 2]). If $X$ is a modular element of a geometric lattice $L$, then $\chi([0, X], t)$ divides $\chi(L, t)$.

Lemma 2.11. Let $M$ be a simple matroid on the ground set $E$ and $X$ a modular coatom of $M$. Then every atom $e \in E \setminus X$ is divisional and $\text{si}(M/e) = M|X$.

Proof. By Propositioin 2.9 it follows that $\text{si}(M/e) = M|X$. Moreover, by Theorem 2.10, the characteristic polynomial $\chi(M/e, t) = \chi(\text{si}(M/e), t) = \chi(M|X, t)$ divides $\chi(M, t)$ and hence $e$ is divisional. \qed

2.4 Modular joins

We review some properties of modular joins and will show a relation between modular joins and divisional atoms.

Proposition 2.12 (see also Ziegler [15, Lemma 3.10]). Let $X$ be a minimal flat of a simple matroid $M$ such that $M$ is a modular join $M = P_X(M_1, M_2)$ over $X$. If $M_1$ has a modular coatom, then $M_1$ has a divisional atom not belonging to $X$.

Proof. Let $Z \subseteq E_1$ be a modular coatom of $M_1$, where $E_1$ denotes the ground set of $M_1$. By Lemma 2.11, every element in $E_1 \setminus Z$ is a divisional atom of $M_1$. Assume that $E_1 \setminus Z \subseteq X$. Then $Z \cap X \subseteq X$ and $M$ is a modular join $M = P_{Z \cap X}(M|Z, M_2)$ over $Z \cap X$, which is a contradiction to the minimality of $X$. Hence $E_1 \setminus Z \not\subseteq X$ and every element $E_1 \setminus (Z \cup X)$ is a desired atom. \qed

Theorem 2.13 (Brylawski [3, Theorem 7.8]). Let $M = P_X(M_1, M_2)$ be a modular join. Then

$$\chi(M, t) = \frac{\chi(M_1, t) \chi(M_2, t)}{\chi(M|X, t)}.$$ \hspace{1cm}

The following proposition is essentially due to Brylawski for generalized parallel connections. However, since we treat the special case of modular joins, we give a proof of the proposition below.

Proposition 2.14 (Brylawski [3, Theorem 5.11.4]). Let $M$ be a modular join $M = P_X(M_1, M_2)$ and $e$ an atom of $M_1$ not belonging to $X$. Then $\text{si}(M/e)$ is isomorphic to a modular join of $\text{si}(M_1/e)$ and $M_2$, and

$$\chi(\text{si}(M/e), t) = \frac{\chi(\text{si}(M_1/e), t) \chi(M_2, t)}{\chi(M|X, t)}.$$ \hspace{1cm}

Proof. Let $E_1$ and $E_2$ be the ground sets of $M_1$ and $M_2$, which are modular flats of $M$. Take an atom $e \in E_1 \setminus X$. The matroid $\text{si}(M/e)$ corresponds the interval $[e, \hat{1}]$ of $L(M)$ under the correspondence mentioned in Proposition 2.4. Note that $E_1$ is modular in $[e, \hat{1}]$ by Proposition 2.6(2) and $E_2 \lor e$ is modular in $[e, 1]$ by Proposition 2.6(4).

The atoms of $\text{si}(M/e)$ are identified with the atoms of the interval $[e, \hat{1}]$. These atoms are coincide with $\{ e \lor e' \mid e' \in E \setminus \{e\} \}$, where $E = E_1 \cup E_2$ denotes the ground set.
Thus si(M/e) is a modular join of matroids corresponding to [e, E_1] and [e, E_2 ∨ e]. The matroid corresponding to [e, E_1] is isomorphic to si(M_1/e). By Proposition 2.6(3), [e, E_2 ∨ e] ≃ [0, E_2]. Hence the matroid corresponding to [e, E_2 ∨ e] is isomorphic to M_2. Thus si(M/e) is isomorphic to a modular join of si(M_1/e) and M_2.

By Proposition 2.5(2), E_1 ∧ (E_2 ∨ e) = (E_1 ∧ E_2) ∨ e = X ∨ e. Using Theorem 2.13 and Proposition 2.6(3), we have

\[\chi(\text{si}(M/e), t) = \frac{\chi([e, E_1], t) \chi([e, E_2 ∨ e], t)}{\chi([e, X ∨ e], t)} = \frac{\chi([e, E_1], t) \chi([0, E_2], t)}{\chi([0, X], t)} = \frac{\chi(\text{si}(M_1/e), t) \chi(M_2, t)}{\chi(M[X], t)}.\]

Lemma 2.15. Let M be a modular join M = P_X(M_1, M_2). Every divisional atom of M_1 not belonging to X is a divisional atom of M.

Proof. Let e be a divisional atom of M_1 such that e ∉ X. Then there exists an integer a such that \(\chi(M_1, t) = (t - a)\chi(\text{si}(M_1/e), t)\). Using Theorem Proposition 2.13, we have

\[\chi(M, t) = \frac{\chi(M_1, t) \chi(M_2, t)}{\chi(M[X], t)} = \frac{(t - a)\chi(\text{si}(M_1/e), t) \chi(M_2, t)}{\chi(M[X], t)} = (t - a)\chi(\text{si}(M/e), t).\]

Thus e is a divisional atom of M.

3 Proof of Theorem 1.8

Lemma 3.1. The class \(\mathcal{C}\) is closed under taking restrictions to modular flats.

Proof. Let M ∈ \(\mathcal{C}\) and X a modular flat of M. We proceed by induction on the rank of M. The case \(r(M) = 0\) is trivial. Hence we suppose that \(r(M) \geq 1\).

First assume that M has a modular coatom Z such that M[Z] ∈ \(\mathcal{C}\). If X ⊆ Z, then X is a modular flat of M[Z]. By the induction hypothesis, M[X] = (M[Z])|X ∈ \(\mathcal{C}\). Assume X ∉ Z. Then X ∨ Z = E, the ground set of M. By Proposition 2.7, X ∩ Z is a modular flat of M and hence M[Z]. By the induction hypothesis, M[(X ∩ Z)] ∈ \(\mathcal{C}\). Moreover, by the modularity,

\[r(X) - r(X ∩ Z) = r(X ∨ Z) - r(Z) = r(E) - r(Z) = 1.\]

Therefore X ∩ Z is a modular coatom of M[X] and hence M[X] ∈ \(\mathcal{C}\).

Next we suppose that M is a modular join M = P_Y(E_1, E_2) over a round flat Y with M[E_i] ∈ \(\mathcal{C}\) for i = 1, 2. If X ⊆ E_i for some i, then M[X] = (M[E_i])|X ∈ \(\mathcal{C}\) by the induction hypothesis. Otherwise, X_i := X ∩ E_i ≠ ∅ is a proper subset of X and M[X_i] = (M[E_i])|X_i ∈ \(\mathcal{C}\) by the induction hypothesis for i = 1, 2. Since both X_1 and X_2 are modular by Proposition 2.7, \(X = X_1 ∪ X_2\), it follows that M[X] is a modular join of M[X_1] and M[X_2]. Moreover X_1 ∩ X_2 = X ∩ Y is round by Proposition 2.8. Therefore M[X] ∈ \(\mathcal{C}\).
**Theorem 3.2.** Every nonempty simple matroid $M \in \mathcal{C}$ has a divisional atom $e$ such that $si(M/e) \in \mathcal{C}$.

**Proof.** We will proof the following claims by induction on the rank of $M$.

(i) If $M$ has a modular coatom, then there exists a divisional atom $e$ such that $si(M/e) \in \mathcal{C}$.

(ii) If $X$ be a minimal round flat of $M$ such that $M$ is a modular join $M = P_X(E_1, E_2)$. Then, for each $i = 1, 2$, there exists a divisional atom $e_i \in E_i \setminus X$ such that $si(M/e_i) \in \mathcal{C}$.

First suppose that $r(M) = 1$, that is, the ground set of $M$ is a singleton. Then only the case (ii) occurs and the atom of $M$ satisfies the assertion.

Now suppose that $r(M) \geq 2$. If $M$ has a modular coatom $X$, then every atom $e \in E \setminus X$ is divisional and $si(M/e) = M/X \in \mathcal{C}$ by Lemma 2.11 and 3.1. Thus the assertion holds.

Next we suppose that $M$ is a modular join. We assume that $X$ is a minimal round flat of $M$ such that $M = P_X(M_1, M_2)$. Since every modular flat in $X$ is also round by Proposition 2.8, $X$ is a minimal flat such that $M$ is a modular join over $X$.

We will show that $M_1$ has a divisional atom $e_1$ not belonging to $X$ such that $si(M_1/e_1) \in \mathcal{C}$. Note that $M_1$ is a member of $\mathcal{C}$ by Lemma 3.1. Assume that $M_1$ has a modular coatom. Then, by Proposition 2.12, $M_1$ has a divisional atom not belonging to $X$ such that $si(M_1/e_1) \in \mathcal{C}$. Hence we may assume that $M_1$ has a minimal round flat $Y$ such that $M_1$ is a modular join $M_1 = P_Y(F, F')$. Since $X$ is round, we have $F \supseteq X$ or $F' \supseteq X$. Without loss of generality, we may assume that $F' \supseteq X$. By the induction hypothesis, $M_1$ has a divisional atom $e_1 \in F \setminus Y$ such that $si(M_1/e_1) \in \mathcal{C}$. Assume that $e_1 \in X$. Then $e_1 \in F'$ and hence $e_1 \in F \cap F' = Y$, which contradicts $e_1 \notin Y$. Thus $M_1$ has a divisional atom $e_1$ not belonging to $X$ such that $si(M_1/e_1) \in \mathcal{C}$.

By Lemma 2.15, $e_1$ is a divisional atom of $M$. Moreover, by Proposition 2.14, $si(M/e_1)$ is isomorphic to a modular join of $si(M_1/e_1)$ and $M_2$, and hence $si(M/e_1) \in \mathcal{C}$.

**Proof of Theorem 3.8.** Let $\mathcal{A}$ be an arrangement such that $M(\mathcal{A}) \in \mathcal{C}$. We proceed by induction on $|\mathcal{A}|$. By definition $A$ is divisionally free if $|\mathcal{A}| = 0$. Therefore we suppose that $|\mathcal{A}| \geq 1$. Without loss of generality we may assume that $\mathcal{A}$ is essential and then $\chi(\mathcal{A}, t) = \chi(M(\mathcal{A}), t)$. By Theorem 3.2 there exists a hyperplane $H \in \mathcal{A}$ such that $\chi(\mathcal{A}^H, t)$ divides $\chi(\mathcal{A}, t)$ and $M(\mathcal{A}^H) = si(M(\mathcal{A})/H) \in \mathcal{C}$. By the induction hypothesis, $\mathcal{A}^H$ is divisionally free and hence $\mathcal{A}$ is also divisionally free.

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