ON THE K-THEORY OF TRUNCATED POLYNOMIAL ALGEBRAS,
REVISITED

MARTIN SPEIRS

1. INTRODUCTION

The algebraic K-theory of truncated polynomial algebras over perfect fields of positive characteristic was first evaluated by Hesselholt and Madsen [6]. Their proof relied on a delicate analysis of the facet structure of regular cyclic polytopes. We present a new proof that only uses the homology of the cyclic bar construction together with Connes’ operator.

**Theorem 1.1.** [6, Theorem A] Let k be a perfect field of positive characteristic. Then there is an isomorphism

\[ K_{2r-1}(k[x]/(x^e), (x)) \cong W_r(k)/V_r \mathcal{W}_r(k) \]

and the groups in even degrees are zero.

We briefly summarize the method. Let k be a perfect field of characteristic \( p > 0 \) and let \( A = k[x]/(x^e) \) and \( I = (x) \) the ideal generated by the variable. The k-algebra \( A \) is the pointed monoid algebra for the pointed monoid \( \Pi_e = \{0, 1, x, \ldots, x^{e-1}\} \) determined by \( x^e = 0 \). There is a canonical equivalence of cyclotomic spectra

\[ \text{THH}(A) \simeq \text{THH}(k) \otimes \mathcal{B}^{\mathcal{Y}}(\Pi_e) \]

where the Frobenius morphism on the right is the tensor product of the usual Frobenius and the unstable Frobenius on the cyclic bar construction of \( \Pi_e \). Using the theory of cyclic sets one obtains a \( \mathbb{T} \)-equivariant splitting of the cyclic bar construction,

\[ \mathcal{B}^{\mathcal{Y}}(\Pi_e) \simeq \bigvee_{m \geq 0} B(m) \]

into simpler \( \mathbb{T} \)-spaces \( B(m) \). The singular homology and Connes’ operator of these \( \mathbb{T} \)-spaces is easily determined and reduces to computations
of the Hochschild homology of $A$ first carried out in [2] and [9]. The answer is simple enough that the Atiyah-Hirzebruch spectral sequence degenerates allowing us to directly determine the homotopy groups of $\text{THH}(k) \otimes B(m)$. From [11] the topological cyclic homology of $A$ is given by the equalizer

$$\text{TC}(A; p) \to \text{TC}^{-}(A) \xrightarrow{\text{can}} \text{TP}(A)$$

so using the above splitting this reduces to computing $(\text{THH}(k) \otimes B(m))^{hT}$ and $(\text{THH}(k) \otimes B(m))^{tT}$. We achieve this by an inductive procedure, making use of the highly co-connective Frobenius map

$$\varphi : (\text{THH}(k) \otimes B(m))^{hT} \to (\text{THH}(k) \otimes B(pm))^{tT}$$

and the periodicity of $(\text{THH}(k) \otimes B(m))^{tT}$. Assembling the answers for varying $m$ then yields the TC-calculation. Applying McCarthy’s theorem one obtains the result.

We note that the method used here has recently been applied by Hesselholt and Nikolaus [8] to compute the $K$-theory of cuspidal curves over $k$, thereby verifying the conjectural result from [4]. We consider this method a first step towards making topological cyclic homology as easy to compute as Connes’ cyclic homology $HC$.

1.1. Acknowledgements. I would like to thank Lars Hesselholt for his generous and valuable guidance while working on this project. I would also like to thank Ryo Horiuchi and Malte Leip for several useful conversations during the writing of this paper.

2. Witt vectors, big and small

The purpose of this short section is to show the following well-known splitting. Let $s = s(p, r, d)$ be the unique positive integer such that

$$p^{s-1}d \leq r < p^s d$$

if it exists, or else $s = 0$. Let $e = p^u e'$ with $(p, e') = 1$.

**Lemma 2.1.** Let $k$ be a perfect field of characteristic $p > 0$. There is an isomorphism

$$\mathbb{W}_{re}(k)/\mathbb{V}_{e}(k) \simeq \prod W_n(k)$$
where the product is indexed over $1 \leq m' \leq re$ with $(p, m') = 1$ and with $h = h(p, r, e, m')$ given by

$$h = \begin{cases} 
s & \text{if } e' \nmid m' \\ 
\min\{u, s\} & \text{if } e' | m' 
\end{cases}$$

where $s = s(p, re, m')$ is the function defined above.

**Proof.** We use the isomorphism

$$\mathcal{W}_r(k) \xrightarrow{\langle l_d \rangle} \prod \mathcal{W}_s(k)$$

(natural with respect to $\mathbb{Z}_{(p)}$-algebras) where the product runs over $d$ such that $(p, d) = 1$ and $1 \leq d \leq r$ and where $s = s(p, r, d)$, see for example [5, Prop. 1.10 and Example 1.11]. The $d'$th component of this map is the composite

$$l_d : \mathcal{W}_r(k) \xrightarrow{F_d} \mathcal{W}_{[r/d]}(k) \xrightarrow{pr_{e'}} \mathcal{W}_s(k)$$

where $F_d$ is the Frobenius map. If $m' = e'd$ with $d \leq r$ then one readily checks that $s(p, re, m') = s(p, r, d) + u$ and that the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{W}_r(k) & \xrightarrow{l_d} & \mathcal{W}_s(k) \\
\downarrow\psi & & \downarrow\psi' \\
\mathcal{W}_{re}(k) & \xrightarrow{l_{m'}} & \mathcal{W}_{s+u}(k)
\end{array}
$$

This corresponds to the case $u \leq s$. Since $(p, e') = 1$ we have

$$\mathcal{W}_{s+u}(k)/(e'\mathcal{V}_{p^u}\mathcal{W}_s(k)) \cong \mathcal{W}_u(k).$$

Thus, we get an isomorphism

$$\mathcal{W}_{re}(k)/\mathcal{V}_{e}\mathcal{W}_r(k) \xrightarrow{\sim} \prod \mathcal{W}_{u}(k) \times \prod \mathcal{W}_{s}(k) \times \prod \mathcal{W}_{h}(k)$$

where in the middle term, the first product is indexed over $1 \leq d \leq r$ with $(p, d) = 1$, the second product is indexed over $1 \leq m' \leq re$ with $e' \mid m'$ and with $u > s$, the third product is indexed over $1 \leq m' \leq re$ with $e' \nmid m'$ and with $(p, m') = 1$. In the last term, the product is indexed over $1 \leq m' \leq re$ with $(p, m') = 1$. \qed
3. Hochschild homology of truncated polynomial algebras

In this section we review the results of \cite{2} and \cite{9} on cyclic homology of algebras of the form \( A = k[x]/f(x) \). We work over a general commutative unital base ring \( k \). The Hochschild homology of \( A \) over \( k \) is the homology of the associated chain complex for the cyclic \( k \)-module

\[
B^c(A/k)[n] = A^\otimes n + 1
\]

where the tensor product is over \( k \). The cyclic structure maps are given as follows

\[
d_i(a_0 \otimes \cdots \otimes a_n) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & 0 \leq i < n \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & i = n \end{cases}
\]

\[
s_i(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n
\]

\[
t_n(a_0 \otimes \cdots \otimes a_n) = a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}.
\]

The Hochschild homology then is the homology \( \text{HH}_*(A/k) \) of the associated chain complex with differential given by the alternating sum of the face maps.

**Proposition 3.1.** Let \( A = k[x]/(x^e) \) where \( k \) is a commutative unital ring. There is an isomorphism

\[
\text{HH}_*(A/k) = \begin{cases} A & \text{if } * = 0 \\ k[1] \oplus k[x, \ldots, x^{e-1}] & \text{if } * > 0 \text{ even} \\ k[1, x, \ldots, x^{e-1}] \oplus k/ek[x^{e-1}] & \text{if } * > 0 \text{ odd} \end{cases}
\]

where \( k \) denotes the \( e \)-torsion elements of \( k \).

The proof uses a common technique for such rings, namely the construction of a small and computable complex. The task is then to show that this complex is quasi-isomorphic to the Hochschild complex. For a \( k \)-algebra \( A \) of the form \( A = k[x]/(f(x)) \), assuming it is flat as a \( k \)-module then the Hochschild homology may be calculated as \( \text{Tor}^A_*(A, A) \) where \( A^e = A \otimes A^{op} \). So it suffices to find a small \( A-A \)-bimodule resolution of \( A \). Given such a resolution \( R(A)_* \rightarrow A \) one now tensors over \( A^e \) with \( A \) to get a complex, \( \overline{R}(A)_* \) computing \( \text{HH}_*(A/k) \). For an appropriate choice of resolution the corresponding complex \( \overline{R}(A)_* \) has the following form

\[
0 \leftarrow A \leftarrow A^f(x) \leftarrow A \leftarrow A^f(x) \leftarrow A \leftarrow \cdots
\]
from which the result readily follows.

We now introduce a splitting of the Hochschild homology of the $k$-algebra $A = k[x]/(x^e)$. Equip $A$ with a “weight” grading by declaring $x^m$ have weight $m$. This induces a grading on the tensor powers of $A$ and we let

$$B^C(A/k;m)[n] \subseteq B^C(A/k)[n] = A^C_{n+1}$$

be the sub $k$-module of weight $m$. It is generated by those tensor monomials whose weight is equal to $m$. This forms a sub cyclic $k$-module of $B^C(A/k)[-]$ and so we obtain a splitting

$$B^C(A/k)[-] \simeq \bigoplus_{m \geq 0} B^C(A/k;m)[-]$$

of cyclic $k$-modules, and of the associated chain complexes. Taking homology then gives a splitting as well,

$$HH_*(A/k) \simeq \bigoplus_{m \geq 0} HH_*(A/k;m).$$

In the following lemma, let $d = d(e, m) = \left\lfloor \frac{m-1}{e} \right\rfloor$ be the largest integer less that $(m-1)/e$.

**Lemma 3.1.** Let $k$ and $A$ be as in Proposition 3.1. If $m$ is not a multiple of $e$ then $HH_*(A/k;m)$ is concentrated in degrees $2d$ and $2d + 1$ where it is free of rank 1 as a $k$-module. In this case Connes’ $B$-operator takes the generator in degree $2d$ to $m$ times the generator in degree $2d + 1$, up to a sign. If $m$ is a multiple of $e$ then $HH_*(A/k;m)$ is concentrated in degree $2d + 1$ and $2d + 2$. The group in degree $2d + 1$ is isomorphic to $k/ek$ while the group in degree $2d + 2$ is isomorphic to $ek$. In this case Connes’ operator acts trivially.

**Proof.** First we prove that the groups are as stated. We follow the proof given in [7] Section 7.3]. Consider the resolution of $A$ as an $A \otimes A$-module constructed by [2], denoted $R(A)_*$ having the form

$$\cdots \xrightarrow{\Delta} A \otimes A \xrightarrow{\delta} A \otimes A \xrightarrow{\Delta} A \otimes A \xrightarrow{\delta} A \otimes A \xrightarrow{\mu} A \rightarrow 0$$

where

$$\Delta = \frac{x^e \otimes 1 - 1 \otimes x^e}{x \otimes 1 - 1 \otimes x} \quad \text{and} \quad \delta = 1 \otimes x - x \otimes 1$$

In [2] a quasi-isomorphism $\psi : R(A/k)_* \longrightarrow B(A/k)_*$ with the bar-resolution is constructed. Since $\Delta$ increases the weight by $e - 1$ and $\delta$ by 1,
and since the differential $b'$ of the bar resolution preserves weight, we see (by induction on $j$) that $\psi_{2j}$ increases weight by $je$, whereas $\psi_{2j+1}$ increases weight by $je + 1$. Tensoring over $A^e$ with $A$ gives a quasi-isomorphism $\overline{\psi} : \overline{R}(A/k) \rightarrow B^{cy}(A/k)$, which has the same weight shift. The result now follows from Proposition 3.1.

For the statements about Connes’ operator, this follows by an explicit choice of a quasi-isomorphism $\psi$ (and its inverse). This is done in [2, Section 1] and in [2, Proposition 2.1.] the computation of Connes’ operator is given.

4. TOPOLOGICAL HOCHESSLHD HOMOLOGY AND THE CYCLIC BAR CONSTRUCTION

Let $\Pi_e = \{0, 1, x, \ldots, x^{e-1}\}$ be the pointed monoid determined by setting $x^e = 0$. Then the truncated polynomial algebra $A$ is the pointed monoid ring $k(\Pi_e) = k[\Pi_e]/k[0]$. The cyclic bar construction of $\Pi_e$ is the cyclic set $B^{cy}(\Pi_e)[-]$ with

$$B^{cy}(\Pi_e)[k] = \Pi_e^{\wedge(k+1)}$$

and with the usual Hochschild-type structure maps. We write $B^{cy}(\Pi_e)$ for the geometric realization of $B^{cy}(\Pi_e)[-]$. The space $B^{cy}(\Pi_e)$ admits a natural $T$-action where $T$ is the circle group, as does the geometric realization of any cyclic set. Furthermore it is an unstable cyclotomic space, i.e. there is a map

$$\psi_p : B^{cy}(\Pi) \rightarrow B^{cy}(\Pi)^{C_p}$$

which is equivariant when the domain is given the natural $T/C_p$-action. For a construction of this map see [1, Section 2] or, for a review in our setup, see [12, Section on cyclic bar construction].

To every non-zero $n$-simplex $\pi_0 \wedge \cdots \wedge \pi_n \in B^{cy}(\Pi_e)[n]$ we associate its weight as follows, each $\pi_i$ is equal to $x^{m_i}$ for some $0 \leq m_i \leq e - 1$. Let

$$w(\pi_0 \wedge \cdots \wedge \pi_n) = \sum_{i=0}^{n} m_i.$$  

The weight is preserved by the cyclic structure maps and so we obtain a splitting of pointed cyclic sets

$$B^{cy}(\Pi_e)[-] = \bigvee_{m \geq 0} B^{cy}(\Pi_e;m)[-].$$
where \( B^\Sigma(\Pi_e; m)[\cdot] \subseteq B^\Sigma(\Pi_e)[\cdot] \) consists of all simplicies with weight \( m \). Let \( B(m) \) denote the geometric realization of \( B^\Sigma(\Pi_e; m)[\cdot] \). So we have a splitting of pointed \( T \)-spaces

\[
B^\Sigma(\Pi_e) \simeq \bigvee_{m \geq 0} B(m).
\]

By [12, Splitting lemma] we have \( \text{THH}(k(\Pi_e)) \simeq \text{THH}(k) \otimes B^\Sigma(\Pi_e) \) as cyclotomic spectra. Here the Frobenius on the right hand side is the tensor product of the usual Frobenius on \( \text{THH}(k) \) (as constructed in [11, Section III.2]) and the Frobenius on \( \Sigma^\infty B^\Sigma(\Pi_e) \) arising from the unstable Frobenius (see [12, Section on cyclic bar construction]). The relative \( \text{THH} \) corresponds to simply cutting out the weight zero part, i.e. we have an equivalence of \( T \)-spectra

\[
\text{THH}(A, I) \simeq \bigoplus_{m \geq 1} \text{THH}(k) \otimes B(m)
\]

where \( I = (x) \) is the ideal generated by the variable.

Given any pointed monoid \( \Pi \) there is an isomorphism of cyclic \( k \)-modules

\[
w : k(B^\Sigma(\Pi)[\cdot]) \longrightarrow B^\Sigma(k(\Pi)/k)[\cdot]
\]

which map \( \pi_0 \wedge \cdots \wedge \pi_n \) to \( \pi_0 \otimes \cdots \otimes \pi_n \). Note that \( k(B^\Sigma(\Pi)[\cdot]) \) is the cellular complex for the space \( B^\Sigma(\Pi) \). In particular the associated homology \( H_*(k(B^\Sigma(\Pi)[\cdot])) \) computes the cellular homology of \( B^\Sigma(\Pi) \).

In the following lemma, we let \( d = d(e, m) = \left\lfloor \frac{m-1}{e} \right\rfloor \) for any \( m \geq 1 \).

**Lemma 4.1.** ([7, Lemma 7.3]) Let \( k \) and \( A \) be as in Proposition 3.1 and let \( B(m) \subseteq B^\Sigma(\Pi_e) \) be as described above.

1. If \( e \nmid m \) then \( \tilde{H}_*(B(m); Z) \) is free of rank 1 if \( *= 2d, 2d+1 \) and zero, otherwise. The Connes’ operator takes a generator in degree \( 2d \) to \( m \) times a generator in degree \( 2d + 1 \).
2. If \( e \mid m \) then \( \tilde{H}_*(B(m); Z) \) is isomorphic to \( k/ek \) if \( *= 2d + 1 \), to \( e^k \) if \( *= 2d + 2 \), and zero otherwise.

**Proof.** We use the isomorphism of cyclic \( k \)-modules

\[
w : k(B^\Sigma(\Pi_e)[\cdot]) \rightarrow B^\Sigma(A/k)[\cdot]
\]
This map preserves the weight decomposition, mapping \( k(B(m)[-]) \) isomorphically to \( B^\Sigma(A/k;m)[-] \). Furthermore the map commutes with the Connes operator, as shown in the proof of [13, Proposition 1.4.5]. Now by Lemma 3.1 we can read off what

\[ HH_*(A/k;m) = \tilde{H}_*(B(m);k) \]

is and how Connes’ operator acts. □

Note that in particular \( \tilde{H}_{2d+2}(B(m);k) \) is free of rank 1 over \( k \), when \( e \) is zero in \( k \). Thus there is room for a non-trivial Connes’ operator in this case. However, it follows again from Lemma 3.1 that it is trivial in this case.

**Lemma 4.2.** Let \( T \) be a bounded below \( C_p \)-spectrum and \( X \) a finite pointed \( C_p \)-CW-complex. Then the lax symmetric monoidal structure map

\[ T^{IC_p} \otimes (\Sigma^\infty X)^{IC_p} \longrightarrow (T \otimes \Sigma^\infty X)^{IC_p} \]

is an equivalence.

**Proof.** See [12, Lemma 3.5.1]. □

**Proposition 4.1.** Let \( A = k[x]/(x^e) \). There is a \( T \)-equivariant equivalence of spectra

\[ THH(A) \simeq \bigoplus_{m \geq 0} THH(k) \otimes B(m). \]

Under this equivalence the Frobenius morphism \( THH(A) \to THH(A)^{IC_p} \) restricts to the map

\[ THH(k) \otimes B(m) \longrightarrow THH(k)^{IC_p} \otimes B(pm)^{IC_p} \longrightarrow (THH(k) \otimes B(pm))^{IC_p} \]

where the second map is the lax symmetric monoidal structure on the Tate-\( C_p \)-construction. This second map is an equivalence, while the restricted Frobenius \( \tilde{\phi} : \Sigma^\infty B(m) \to (\Sigma B(pm))^{IC_p} \) is a \( p \)-adic equivalence.

**Proof.** The proof follows that of the similar statement in [12, Proposition 3.5.1]. Taking \( T = THH(k) \) and \( X = B(m) \) in Lemma 4.2 yields the claim about the lax symmetric monoidal structure map. To see that the restricted Frobenius is a \( p \)-adic equivalence, one factors it accordingly as follows.
S \otimes B(m) \xrightarrow{\phi} (S \otimes B(pm))^{tC_p}

\Delta_p \otimes \Delta_p

S^{tC_p} \otimes (sd_p B(pm))^{C_p} \xrightarrow{(1.)} (S \otimes sd_p B(pm))^{C_p} \xrightarrow{D_p} (S \otimes B(pm)^{C_p})^{tC_p}

Now the Segal conjecture says that $\Delta_p : S \to S^{tC_p}$ is a $p$-adic equivalence. The map labelled (1.) is an equivalence since $sd_p B(pm)^{C_p}$ is a finite $C_p$-CW-complex with trivial $C_p$-action, and since $(-)^{tC_p}$ is exact. The map $D_p$ is the equivalence from the $p$-subdivision of $B(pm)$ to $B(pm)$ itself. Finally, the map labelled (2.) is an equivalence since $(-)^{tC_p}$ is trivial on free $C_p$-CW-complexes, cf. [7, Lemma 9.1].

**Corollary 4.1.** The restricted Frobenius map

$\varphi(m) : \text{THH}(k) \otimes B(m) \to (\text{THH}(k) \otimes B(pm))^{tC_p}$

induces an isomorphism in degrees $\geq 2d + 1$ when $e \nmid m$, and induces an isomorphism in degrees $\geq 2d + 2$ when $e | m$.

**Proof.** This follows readily from Proposition 4.1 and Lemma 4.1 using the Atiyah-Hirzebruch spectral sequence. □

5. Negative and Periodic Topological Cyclic Homology

We compute TP and TC$^-$ using an inductive procedure based on the $p$-adic valuation of the integer $m$ in indexing the T-space $B(m)$. We choose generators for the homology of the spaces $B(m)$. If $e \nmid m$ let $y_m$ and $z_m$ be generators for the homology in degree $2d$ and $2d + 1$, respectively. If $e | m$ and $p | e$ then we let $z_m$ and $w_m$ be generators of the homology in degree $2d + 1$ and $2d + 2$, respectively.

**Lemma 5.1.** In the Tate spectral sequence for $\pi_*(\text{THH}(k) \otimes B(m))$ the class $z_m$ is an infinite cycle for all $m$.

**Proof.** Although the statement does not seem to require it, we must deal with the cases $e | m$ and $e \nmid m$ separately. In both cases we use the T-equivariant map $HZ_p \to \text{THH}(k)$. One way of getting such a map is by using the calculation $\tau_{\geq 0} \text{TC}(k) = HZ_p$ since then we have the T-equivariant
map

\[ HZ_p \simeq \tau_{\geq 0} TC(k) \to TC(k) \to TC^-(k) \to THH(k). \]

This map induces a map of Tate spectral sequences from \( \pi_*(HZ_p \otimes B(m)) \) to \( \pi_*(THH(k) \otimes B(m)) \).

Suppose first that \( e \nmid m \). Then from Lemma 4.1 we may compute the \( E^2 \)-page of the Tate spectral sequence for \( HZ_p \otimes B(m) \) to be

\[ E^2 = \mathbb{Z}_p[t^{\pm 1}][y_m, z_m] \Rightarrow \pi_*(HZ_p \otimes B(m))^\mathbb{T} \]

where \( |y_m| = (0, m-1) \) and \( |z_m| = (0, m) \). The differential structure is determined by \( d^2(y_m) = mtz_m \), and so \( E^3 = E^\infty = \mathbb{Z}_p/m\mathbb{Z}_p[t^{\pm 1}][z_m] \) so \( z_m \) is an infinite cycle. It follows that \( z_m \in k[t^{\pm 1}, x][y_m, z_m] \) (the \( E^2 \) page for the target spectral sequence) is an infinite cycle.

Now suppose \( e \mid m \). Then from Lemma 4.1 we may compute the \( E^2 \)-page of the Tate spectral sequence for \( HZ_p \otimes B(m) \) to be \( \mathbb{Z}_p[t^{\pm 1}][z_m] \) with \( |z_m| = (0, m) \) from which it follows immediately that \( z_m \) is an infinite cycle. \( \square \)

**Lemma 5.2.** Let \( X \) be a \( \mathbb{T} \)-spectrum such that the underlying spectrum is an \( HZ \)-module. The \( d^2 \) differential of the \( \mathbb{T} \)-Tate spectral sequence is given by \( d^2(\alpha) = td(\alpha) \) where \( d \) is Connes’ operator.

**Proof.** See [3, Lemma 1.4.2] \( \square \)

**Proposition 5.1.** If \( e \nmid m \) then

\[ \pi_{2r+1}(THH(k) \otimes B(p^vm'))^{\mathbb{T}} \simeq \mathbb{W}_v(k) \]

for all \( r \in \mathbb{Z} \), and

\[ \pi_{2r+1}(THH(k) \otimes B(p^vm'))^{\mathbb{T}} \simeq \begin{cases} \mathbb{W}_{v+1}(k) & \text{if } d \leq r \\ \mathbb{W}_v(k) & \text{if } r < d \end{cases} \]

The even homotopy groups are trivial.

**Proof.** We proceed by induction on \( v \geq 0 \). Suppose \( v = 0 \), so \( m = m' \), and consider the Tate spectral sequence

\[ E^2 = k[t^{\pm 1}, x][y_m', z_m'] \Rightarrow \pi_*(THH(k) \otimes B(m'))^{\mathbb{T}} \]
By Lemma 5.1 the only possible non-zero differentials are those beginning at \( y_m \). Furthermore
\[
d^2(y_m) = m'td(y_m) = m'tz_m
\]
by Lemma 5.2 and Lemma 4.1. Since \( m' \) is a unit in \( k \), \( d^2 \) is an isomorphism. In summary, the \( E^2 \)-page looks as follows (shifted up by 2 in the horizontal direction).

|   | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|---|---|
|   | \( x^3 y_m \) | \( x^2 z_m \) | \( x^2 y_m \) | \( xz_m \) | \( xy_m \) | \( z_m \) | \( t^2 y_m \) |
|   | \( t y_m \) | \( y_m \) | \( t^{-1} y_m \) | \( t^{-2} y_m \) |
|   | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |

Thus \( E^3 = E^\infty = 0 \) is trivial, as claimed. To determine the \( T \)-homotopy fixed points, we truncate the Tate spectral sequence, removing the first quadrant. The classes \( z_m, x^n \) are no longer hit by differentials and so \( E^3 = E^\infty = k[x][z_m] \) where \( z_m \) has degree \( 2d + 1 \). This proves the claim for \( v = 0 \).

Suppose the claim is known for all integers less than or equal to \( v \). By Proposition 4.1 the Frobenius
\[
\varphi(p^vm') : \pi_*(\Thh(k) \otimes B(p^vm'))^hT \to \pi_*(\Thh(k) \otimes B(p^{v+1}m'))^T
\]
is an isomorphism in high degrees. The induction hypothesis then implies that the domain is isomorphic to \( W_{v+1}(k) \) when \( * = 2r + 1 \geq 2d + 1 \). By periodicity we conclude that \( \pi_*(\Thh(k) \otimes B(p^{v+1}m'))^T \) is concentrated in odd degrees where,
\[
\pi_{2r+1}(\Thh(k) \otimes B(p^{v+1}m'))^T \simeq W_{v+1}(k)
\]
for any \( r \in \mathbb{Z} \). Considering again the Tate spectral sequence we see that we must have

\[
d^{2v+2}(y_{p^{v+1}m'}) = tz_{p^{v+1}m'}(xt)^v
\]

and so \( E^{2v+3} = E^\infty \). Truncating the spectral sequence to obtain the homotopy fixed-point spectral sequence, we now see that

\[
\pi_{2r+1}(\text{THH}(k) \otimes B(p^{v}m'))^kT \simeq \begin{cases} 
W_{u+2}(k) & \text{if } d \leq r \\
W_{v+1}(k) & \text{if } r < d 
\end{cases}
\]

This completes the proof. \( \square \)

To deal with the case where \( e \) does divide \( m \) we factor \( e = p^u e' \) where \((p, e') = 1\). Thus \( e | p^v m' \) if and only if \( v \geq u \) and \( e' | m' \).

**Proposition 5.2.** If \( e | m \) then

\[
\pi_{2r+1}(\text{THH}(k) \otimes B(p^{v}m'))^kT \simeq W_u(k)
\]

and

\[
\pi_{2r+1}(\text{THH}(k) \otimes B(p^{v}m'))^kT \simeq W_u(k)
\]

for all \( r \in \mathbb{Z} \).

**Proof.** We use induction on \( v \geq u \). Suppose \( v = u \). Then

\[
\pi_* (\text{THH}(k) \otimes B(p^{v-1}m'))^{kT} \xrightarrow{\text{periodicity}} \pi_* (\text{THH}(k) \otimes B(p^{v}m'))^{kT}
\]

is an isomorphism in high enough degrees. The domain was evaluated in Proposition 5.1; it is \( W_u(k) \) in odd degrees greater than \( 2d + 1 \). By periodicity we conclude the result of the codomain. Now suppose the result has been verified for all integers greater than \( u \) and strictly less than \( v \). Again using the Frobenius we conclude that

\[
\pi_{2r+1}(\text{THH}(k) \otimes B(p^{v}m'))^kT \simeq W_u(k)
\]

for all \( r \in \mathbb{Z} \).

Consider the Tate spectral sequence with \( E^2 \)-page \( k[t^{\pm 1}, x][z_m, w_m] \). Since \( z_m \) is an infinite cycle the only possible way that this sequence collapses to yield the correct result is if

\[
d^{2u}(w_m) = (tx)^u z_m
\]
Thus \( E^{2u+1} = E^{\infty} \). As before, by truncating the first quadrant, we get the spectral sequence for the homotopy \( T \)-fixed points whose \( E^{2u} \)-page clearly shows the result.

\[ \square \]

6. Topological cyclic homology

We now prove Theorem 1.1. By McCarthy’s result [10, Main Theorem] it suffices to prove the following.

**Theorem 6.1.** Let \( k \) be a perfect field of positive characteristic. Then there is an isomorphism

\[
TC_{2r-1}(k[x]/(x^e), (x)) \simeq W_{re}(k)/V_e W_r(k)
\]

and the groups in even degrees are zero.

**Proof.** In view of Lemma 2.1 it suffices to give an isomorphism

\[
TC_{2r-1}(k[x]/(x^e), (x)) \simeq \prod W_h(k)
\]

where the product is indexed over \( 1 \leq m' \leq re \) with \( (p, m') = 1 \) and with \( h = h(p, r, e, m') \) given by

\[
h = \begin{cases} 
s & \text{if } e' \nmid m' \\ 
\min\{u, s\} & \text{if } e' \mid m'
\end{cases}
\]

where \( s = s(p, re, m') \) is such that \( p^{s-1}m' \leq re < p^s m' \). Now \( TC(A, I) \) is given as the equalizer of \( \text{TP}(A, I) \) and \( \prod_{v \geq 0} TC_{p^v m'} \). This map splits as

\[
\prod_{m' \geq 1} \prod_{(p, m') = 1} \text{TC}^{-}(p^v m') \xrightarrow{q-can} \prod_{m' \geq 1} \prod_{(p, m') = 1} \text{TP}(p^v m')
\]

By Proposition 5.1 and Proposition 5.2 both \( \text{TC}^{-}(p^v m') \) and \( \text{TP}(p^v m') \) are concentrated in odd degrees, so the long exact sequence calculating \( TC \) splits into short exact sequences

\[
0 \to TC_*(m') \to \prod_{v \geq 0} \text{TC}^{-}(p^v m') \xrightarrow{q-can} \prod_{v \geq 0} \text{TP}_*(p^v m') \to 0
\]
Now if $e' \nmid m'$ then from Proposition 5.1 we have a map of short exact sequences

$$0 \longrightarrow \prod_{v \geq s} W_v(k) \longrightarrow \prod_{v \geq 0} \text{TC}_{2r+1}^{-}(p^v m') \longrightarrow \prod_{0 \leq v < s} W_{v+1}(k) \longrightarrow 0$$

$$0 \longrightarrow \prod_{v \geq s} W_v(k) \longrightarrow \prod_{v \geq 0} \text{TP}_{2r+1}(p^v m') \longrightarrow \prod_{0 \leq v < s} W_v(k) \longrightarrow 0$$

where $s = s(p,r,d(p^v m'))$. The left hand vertical map is an isomorphism (since in this range $\text{can}$ is an isomorphism and $\varphi$ is divisible by powers of $p$) and the right hand vertical map is an epimorphism with kernel $W_s(k)$. Thus $\text{TC}_{2r+1}(m') = W_s(k)$. Note that in this case $h = s$.

If $e' | m'$ then we must distinguish between two cases. First, if $s < u$ then again we get a map of short exact sequences

$$\prod_{s \leq v < u} W_v(k) \times \prod_{u \leq v} W_u(k) \longrightarrow \prod_{v \geq 0} \text{TC}_{2r+1}^{-}(p^v m') \longrightarrow \prod_{0 \leq v < s} W_{v+1}(k)$$

$$\prod_{s \leq v < u} W_v(k) \times \prod_{u \leq v} W_u(k) \longrightarrow \prod_{v \geq 0} \text{TP}_{2r+1}(p^v m') \longrightarrow \prod_{0 \leq v < s} W_v(k)$$

so in this case $\text{TC}_{2r+1}(m') = W_s(k)$. Since $u > s$ we have $h = s$ as claimed.

If instead, $u \leq s$ then the map of short exact sequences looks as follows

$$\prod_{v \geq s} W_v(k) \longrightarrow \prod_{v \geq 0} \text{TC}_{2r+1}^{-}(p^v m') \longrightarrow \prod_{0 \leq v < u} W_{v+1}(k) \times \prod_{u \leq v < s} W_u(k)$$

$$\prod_{v \geq s} W_v(k) \longrightarrow \prod_{v \geq 0} \text{TP}_{2r+1}(p^v m') \longrightarrow \prod_{0 \leq v < u} W_v(k) \times \prod_{u \leq v < s} W_u(k)$$

so in this case $\text{TC}_{2r+1}(m') = W_u(k)$. Since $u \leq s$ we see that $u = h$ in this case. This completes the proof. \qed

References

[1] M. Bökstedt, W. C. Hsiang, and I. Madsen, The cyclotomic trace and algebraic K-theory of spaces, Invent. Math., 111 (1993), pp. 465–539.

[2] J. A. Guccione, J. J. Guccione, M. J. Redondo, A. Solotar, and O. E. Villamayor, Cyclic homology of algebras with one generator, K-theory, 5 (1991), pp. 51–69.

[3] L. Hesselholt, On the p-typical curves in Quillen’s K-theory, Acta Math., 177 (1996), pp. 1–53.
[4] ——, On the K-theory of planar cuspal curves and a new family of polytopes, Algebraic Topology: Applications and New Directions, 620 (2014), p. 145.

[5] ——, The big de Rham–Witt complex, Acta Mathematica, 214 (2015), pp. 135–207.

[6] L. Hesselholt and I. Madsen, Cyclic polytopes and the K-theory of truncated polynomial algebras, Invent. Math., 130 (1997), pp. 73–97.

[7] ——, On the K-theory of finite algebras over Witt vectors of perfect fields, Topology, 36 (1997), pp. 29–101.

[8] L. Hesselholt and T. Nikolaus, Topological cyclic homology. in preparation.

[9] M. Larsen and A. Lindenstrauss, Cyclic homology of dedekind domains, K-theory, 6 (1992), pp. 301–334.

[10] R. McCarthy, Relative algebraic K-theory and topological cyclic homology, Acta Mathematica, 179 (1997), pp. 197–222.

[11] T. Nikolaus and P. Scholze, On topological cyclic homology, ArXiv: 1707.01799, (2017).

[12] M. Speirs, On the K-theory of coordinate axes in affine space, preprint, (2018).