PROBABILITY MEASURES AND Milyutin Maps Between Metric Spaces

VESKO VALOV

Abstract. We prove that the functor \( \hat{P} \) of Radon probability measures transforms any open map between completely metrizable spaces into a soft map. This result is applied to establish some properties of Milyutin maps between completely metrizable spaces.

1. Introduction

In this paper we deal with metrizable spaces and continuous maps. By a (complete) space we mean a (completely) metrizable space, and by a measure a probability Radon measure. Recall that a measure \( \mu \) on \( X \) is said to be:

- **probability** if \( \mu(X) = 1 \);
- **Radon** if \( \mu(B) = \sup \{ \mu(K) : K \subset B \text{ and } K \text{ is compact} \} \) for any Borel set \( B \subset X \);

The support \( \text{supp} \mu \) of a measure \( \mu \) is the intersection of all closed subsets \( A \) of \( X \) with \( \mu(A) = \mu(X) \). It is well known that the support of any measure is non-empty and separable.

Everywhere below \( \hat{P}(X) \) stands for the space of all probability Radon measures on \( X \) equipped with the weak topology with respect to \( C^*(X) \). Here, \( C^*(X) \) is the space of bounded continuous functions on \( X \) with the uniform convergence topology. According to [2], \( \hat{P} \) is a functor in the category of metrizable spaces and continuous maps. In particular, for any map \( f : X \to Y \) there exists a map \( \hat{P}(f) : \hat{P}(X) \to \hat{P}(Y) \). A systematic study of the functor \( \hat{P} \) can be found in [2] and [3]. We also consider the subspace \( P_\beta(X) \subset \hat{P}(X) \) consisting of all measures \( \mu \) such that \( \text{supp} \mu \) is compact.

This paper is devoted to some properties of Milyutin maps between metrizable spaces. We say that \( f : X \to Y \) is a Milyutin map if there

2000 Mathematics Subject Classification. Primary 54C60; 60B05; Secondary 54C65, 54C10.

Key words and phrases. Probability measures, support of measures, Milyutin maps, metric spaces.

The author was partially supported by NSERC Grant 261914-08.

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exists a map \( g : Y \to \hat{P}(X) \) such that \( \text{supp} \ g(y) \subset f^{-1}(y) \) for every \( y \in Y \). Such \( g \) is called a choice map associated with \( f \). According to \cite{3} Theorem 3.15, for any metrizable \( X \) there exists a barycentric map \( b_{\hat{P}(X)} : \hat{P}(\hat{P}(X)) \to \hat{P}(X) \) such that \( b_{\hat{P}(X)}(\nu) = \nu \) for all \( \nu \in \hat{P}(X) \). Hence, if \( g \) is a choice map associated with \( f \), then the map \( b_{\hat{P}(X)} \circ \hat{P}(g) : \hat{P}(Y) \to \hat{P}(X) \) is a right inverse of \( \hat{P}(f) \). Consequently, \( f \) is a Milyutin map if and only if \( \hat{P}(f) \) admits a right inverse.

Our first principal result concerns the question when \( \hat{P}(f) \) is a soft map. Recall that a map \( f : X \to Y \) is soft if for any space \( Z \) and its closed subset \( A \) and any maps \( g : Z \to Y \), \( h : A \to X \) with \((f \circ h)|A = g \) there exists a map \( \bar{g} : Z \to X \) such that \( \bar{g} \) extends \( h \) and \( f \circ \bar{g} = g \). It is easily seen that every soft map is surjective and open.

**Theorem 1.1.** Let \( f : X \to Y \) be a surjective open map between complete spaces. Then \( \hat{P}(f) : \hat{P}(X) \to \hat{P}(Y) \) is a soft map.

The particular cases of Theorem 1.1 when both \( X \) and \( Y \) are either compact or separable were established in \cite{3} and \cite{4}, respectively.

Since any soft map admits a right inverse, a map \( f \) satisfying the hypotheses of Theorem 1.1 is a Milyutin map. We apply Theorem 1.1 to obtain some results about atomless and exact Milyutin maps introduced in \cite{14}. If \( f : X \to Y \) is a Milyutin map and there exists a choice map \( g \) such that \( \text{supp} \ g(y) = f^{-1}(y) \) (resp., \( g(y) \) is an atomless measure on \( f^{-1}(y) \)) for each \( y \in Y \), i.e. \( g(y)(\{x\}) = 0 \) for all \( x \in f^{-1}(y) \), then \( f \) is said to be an exact (resp., atomless) Milyutin map. It was established in \cite{14} that, in the realm of Polish spaces \( X \) and \( Y \), \( f \) is exact Milyutin if and only if it is open. The classes of atomless exact Milyutin maps and atomless Milyutin maps between Polish spaces were characterized in \cite{1} Theorem 1.6]. The first class consists of all open maps possessing perfect fibers (i.e., without isolated points) \cite{1} Theorem 1.6], and the second one of all maps \( f : X \to Y \) such that for some Polish space \( X_0 \subset X \) the restriction \( f_0 = f|X_0 : X_0 \to Y \) is an open surjection whose fibers are perfect \cite{1} Theorem 1.7].

Next theorem is a non-separable analogue of \cite{1} Theorem 1.7].

**Theorem 1.2.** A continuous surjection \( f : X \to Y \) of complete spaces is an atomless Milyutin map if and only if there exists a complete subspace \( X_0 \subset X \) such that \( f_0 = f|X_0 : X_0 \to Y \) is an open surjection and all fibers of \( f_0 \) are perfect sets. Moreover, for any such \( f \) there exists a map \( f^* : P_\beta(Y) \to \hat{P}(X) \) such that any \( f^*(\mu) \) is atomless and \( \hat{P}(f)(f^*(\mu)) = \mu, \mu \in P_\beta(Y) \).
We do not know whether under the hypotheses in Theorem 1.2 there exists a map $f^*: \hat{P}(Y) \to \hat{P}(X)$ such that each $f^*(\mu)$ is atomless and $\hat{P}(f)(f^*(\mu)) = \mu, \mu \in \hat{P}(Y)$. But if we are interested in atomless maps defined on $Y$, we have the following:

**Theorem 1.3.** Every open surjection $f: X \to Y$ with perfect fibers is a densely atomless Milyutin map provided $X$ and $Y$ are complete spaces.

Here, a Milyutin map $f: X \to Y$ is densely atomless if

$$\{g \in Ch_f(Y, X) : g(y) \text{ is atomless for all } y \in Y\}$$

is a dense $G_\delta$-set in the space $Ch_f(Y, X)$ of all choice maps associated with $f$ equipped with the source limitation topology. A few words about this topology. If $(X, d)$ is a bounded (complete) metric space, then there exists a (complete) metric $\hat{d}$ on $\hat{P}(X)$ generating its topology and extending $d$, see [3]. Then $Ch_f(Y, X)$ is a subspace of the function space $C(Y, \hat{P}(X))$ with the source limitation topology whose local base at a given $h \in C(Y, \hat{P}(X))$ consists of all sets

$$B_{\hat{d}}(h, \alpha) = \{g \in C(Y, \hat{P}(X)) : \hat{d}(g(y), h(y)) < \alpha(y) \text{ for all } y \in Y\},$$

where $\alpha$ is a continuous map from $Y$ into $(0, \infty)$. It is well known that this topology does not depend on the metric $\hat{d}$ and it has the Baire property in case $\hat{P}(X)$ is complete. Similarly, $f$ is said to be densely exact provided the set

$$\{g \in Ch_f(Y, X) : supp g(y) = f^{-1}(y) \text{ for every } y \in Y\}$$

is a dense and $G_\delta$-set in $Ch_f(Y, X)$. When $f$ is both densely atomless and densely exact, it is called densely exact atomless.

**Theorem 1.4.** Let $f: X \to Y$ be an open surjection of complete spaces and $\pi: X \to M$ a map into a separable space $M$. Then all choice maps $h \in Ch_f(Y, X)$ such that $\pi(supp h(y))$ is dense in $\pi(f^{-1}(y))$ for every $y \in Y$ form a dense $G_\delta$-set in $Ch_f(Y, X)$.

It is interesting whether in Theorem 1.4 one can substitute the phrase ”$\pi(supp h(y))$ is dense in $\pi(f^{-1}(y))$” by ”$\pi(supp h(y)) = \pi(f^{-1}(y))$”.

Next corollary is a parametrization of the Parthasarathy [12] result that perfect Polish spaces admit atomless measures. It also provides a partial answer of the question [1] whether any open surjection $f$ of complete spaces is an exact atomless Milyutin map provided all fibers of $f$ are perfect Polish spaces.
Corollary 1.5. Let \( f: X \to Y \) be an open and closed surjection of complete spaces such that all fibers of \( f \) are separable (and perfect). Then \( f \) is densely exact (atomless) Milyutin map.

Finally, we generalize [14, Corollary 1.4] and [1, Corollary 1.9] as follows (below a continuous set-valued map means a map which is both lower and upper semi-continuous):

Corollary 1.6. Let \( X \) and \( Y \) be complete spaces and \( \Phi: Y \to X \) a continuous set-valued map such that all values \( \Phi(y) \) are closed separable subsets of \( X \). Then there exists a map \( h: Y \to \hat{P}(X) \) such that \( \text{supp } h(y) = \Phi(y) \) for every \( y \in Y \). If, in addition, all \( \Phi(y) \) are perfect sets, the map \( h \) can be chosen so that every \( h(y) \) is atomless.

Acknowledgements: The author wishes to thank the referee for his/her valuable remarks and suggestions which significantly improved the paper.

2. Preliminaries

In this section we provide some preliminary results and establish the proof of Theorem 1.1.

Probability Radon measures on a complete space \( X \) can be described as positive linear functionals \( \mu \) on \( C^*(X) \) such that \( ||\mu|| = 1 \) and \( \lim \mu(h_\alpha) = 0 \) for any decreasing net \( \{h_\alpha\} \subset C^*(X) \) which pointwisely converges to 0, see [15]. Under this interpretation, \( \text{supp } \mu \) coincides with the set of all \( x \in X \) such that for every neighborhood \( U_x \) of \( x \) in \( X \) there exists \( \varphi \in C^*(X) \) such that \( \varphi(X\setminus U_x) = 0 \) and \( \mu(\varphi) \neq 0 \). This representation of \( \text{supp } \mu \) easily implies that the set-valued map \( \text{supp}: \hat{P}(X) \to X \) (assigning to each \( \mu \) its support) is lower semi-continuous, i.e., \( \{\mu \in \hat{P}(X): \text{supp } \mu \cap U \neq \emptyset\} \) is open in \( \hat{P}(X) \) for any open \( U \subset X \). For every closed \( F \subset X \), we have \( \mu(F) = \inf\{\mu(\varphi): \varphi \in C(F)\} \) (see for example [7] in case \( X \) is compact), where \( C(F) = \{\varphi \in C^*(X): 0 \leq \varphi \leq 1 \text{ and } \varphi(F) = 1\} \).

According to [4], any compatible (complete) metric \( \hat{d} \) on \( X \) generates a compatible (complete) metric \( \hat{d} \) on \( \hat{P}(X) \) such that

\[
\hat{d}(t\mu + (1-t)\mu', tv + (1-t)v') \leq t\hat{d}(\mu, \mu') + (1-t)\hat{d}(v, v')
\]

for all \( t \in [0,1] \) and \( \mu, \mu', v, v' \in \hat{P}(X) \). It is easily seen that every ball (open or closed) with respect to \( \hat{d} \) is convex.

Let \( A_\varepsilon(X) \) denote the set of all \( \mu \in \hat{P}(X) \) such that \( \mu(\{x\}) \geq \varepsilon \) for some \( x \in \text{supp } \mu \). For any closed \( K \subset X \) there exists a closed embedding \( i: \hat{P}(K) \to \hat{P}(X) \) defined by \( i(\nu)(h) = \nu(h|K) \) for all \( \nu \in \hat{P}(K) \).
Lemma 2.2. Let $X$ be a complete space, $K$ a perfect closed subset of $X$ and $G$ a convex open subset of $\hat{P}(K)$. Then for every $\varepsilon > 0$ we have:

1. $A_\varepsilon(X)$ is a closed subset of $\hat{P}(X)$;
2. $A_\varepsilon(X) \cap \bar{G}$ is a nowhere dense set in the closure $\bar{G}$.

Proof. (1) Since $\hat{P}(X)$ is metrizable, it suffices to check that $\mu_0 = \lim \mu_n \in A_\varepsilon(X)$ for every convergent sequence $\{\mu_n\}_{n \geq 1}$ in $\hat{P}(X)$ with $\{\mu_n\} \subset A_\varepsilon(X)$. To this end, let $H$ be the closure in $X$ of the set $\bigcup_{n \geq 0} \text{supp } \mu_n$. Because every $\mu \in \hat{P}(X)$ has a separable support, $H$ is a Polish subset of $X$. Considering all $\mu_n$, $n \geq 0$, as elements of $\hat{P}(H)$, we have that the sequence $\{\mu_n\}_{n \geq 1}$ is contained in $A_\varepsilon(H)$ and converges to $\mu_0$. Therefore, by [12, Theorem 8.1], $\mu_0 \in A_\varepsilon(H)$. Consequently, there exists $x_0 \in H$ with $\mu_0(\{x_0\}) \geq \varepsilon$. Therefore, $A_\varepsilon(X)$ is closed in $\hat{P}(X)$.

(2) Since $A_\varepsilon(K) = A_\varepsilon(X) \cap \hat{P}(K)$, it suffices to show that $A_\varepsilon(K)$ is nowhere dense in $\hat{P}(K)$. Suppose $A_\varepsilon(K)$ contains an open subset $W$ of $\hat{P}(K)$ and let $P_\omega(K)$ be the set of all $\mu \in \hat{P}(K)$ having a finite support. Since $P_\omega(K)$ is dense in $\hat{P}(K)$, there exists $\mu_0 = \sum_{i=1}^{i=k} \lambda_i \delta_{x_i} \in P_\omega(K) \cap W$. Here, $\delta_{x_i}$ denotes Dirac’s measures at $x_i$ and $\lambda_i = \mu_0(\{x_i\})$. Moreover, $\lambda_i \geq \varepsilon$ for at least one $i$. For each $i \leq k$ and $n \geq 1$ choose a neighborhood $V_i \subset K$ of $x_i$ and $n$ different points $x_{i(1)}, \ldots, x_{i(n)} \in V_i$ such that the family $\{V_i : 1 \leq i \leq k\}$ is disjoint and $\text{dist}(x_i, x_{i(j)}) \leq 1/n$ for all $1 \leq j \leq n$. This can be done because $K$ is perfect, so every neighborhood of $x_i$ contains infinitely many points. Consider now the measures $\mu_n = \sum_{i=1}^{i=k} \sum_{j=1}^{j=n} \frac{\lambda_i \delta_{x_{i(j)}}}{n}$. Since $\lim \mu_n = \mu_0$, there exists $n_0$ such that $\mu_n \in W$ for all $n \geq n_0$. Consequently, for every $n \geq n_0$ there exists $i \leq k$ with $\lambda_i/n \geq \varepsilon$, a contradiction. \qed

Lemma 2.2. Let $f : X \rightarrow Y$ be an open surjection between complete spaces such that $\dim Y = 0$. Then $\hat{P}(f) : \hat{P}(X) \rightarrow \hat{P}(Y)$ is a soft map.

Proof. According to Theorem 1.3 from [4], it suffices to show that $f$ is everywhere locally invertible. The last notion is defined as follows: for any space $Z$, a point $a \in Z$, a map $g : Z \rightarrow Y$ and an open set $U \subset X$ with $g(a) \in f(U)$ there exist a neighborhood $V$ of $a$ in $Z$ and a map $h : V \rightarrow U$ such that $f \circ h = g|V$. Obviously, $f$ is everywhere locally invertible provided it satisfies the following condition:
(\ast) For any open \( U \subset X \) and \( a \in f(U) \) there exists a map \( g: V \to U \) with \( V \) being a neighborhood of \( a \) in \( Y \) such that \( f(g(y)) = y \) for all \( y \in V \).

To show \( f \) satisfies \((\ast)\), fix an open set \( U \subset X \) and \( a \in f(U) \). Since \( f \) is open, the set \( V = f(U) \subset Y \) is also open and the set-valued map \( \Phi: V \to U, \Phi(y) = f^{-1}(y) \cap U \), is lower semi-continuous with closed values. Moreover, \( U \) admits a complete metric because \( X \) is complete. Then, by the 0-dimensional selection theorem of Michael \cite{11}, \( \Phi \) has a continuous selection \( g \). Obviously, \( g \) is as required. 

\begin{proof}[Proof of Theorem 1.1] First, let us show that \( \hat{f} = \hat{P}(f) \mid \hat{P}(f)^{-1}(Y) \) is everywhere locally invertible. It suffices to show that \( \hat{f} \) satisfies condition \((\ast)\) from Lemma 2.2. Suppose that \( U \subset \hat{P}(f)^{-1}(Y) \) is open and \( y_0 \in \hat{f}(U) \). We need to find a map \( \alpha: V \to U \), where \( V \) is a neighborhood of \( y_0 \) in \( Y \), such that \( \hat{f}(\alpha(y)) = y \) for every \( y \in V \). To this end, choose a 0-dimensional complete space \( Z \) and a perfect Milyutin map \( g: Z \to Y \), see \cite{9} (recall that a map is perfect if it is closed and has compact fibers). Next, consider the pull-back \( T = \{(z, x) \in Z \times X : g(z) = f(x)\} \) of \( Z \) and \( X \) with respect to the maps \( g \) and \( f \), and let \( p_f: T \to Z, p_g: T \to X \) be the corresponding projections. Since \( f \) is open, so is \( p_f \). For any \( y \in Y \) we have \( p_f^{-1}(g^{-1}(y)) = p_f^{-1}(f^{-1}(y)) = g^{-1}(y) \times f^{-1}(y) \). Since \( g \) is Milyutin, there exists a map \( g^*: Y \to \hat{P}(Z) \) such that \( \text{supp } g^*(y) \subset g^{-1}(y) \) for all \( y \in Y \). Let \( \hat{p}_f = \hat{P}(p_f): \hat{P}(T) \to \hat{P}(Z) \) and \( \hat{g} = \hat{P}(p_g): \hat{P}(T) \to \hat{P}(X) \). Take an open set \( G \subset \hat{P}(X) \) with \( G \cap \hat{P}(f)^{-1}(Y) = U \) and let \( W = \hat{g}^{-1}(G) \). Pick \( \mu^* \in G \cap \hat{P}(f^{-1}(y_0)) \) and let \( \nu_0 = \mu_0 \times \mu^* \) be the product measure, where \( \mu_0 = g^*(y_0) \). Obviously, \( \nu_0 \in \hat{P}(g^{-1}(y_0) \times f^{-1}(y_0)) \subset \hat{P}(T) \). Moreover, \( \hat{p}_f(\nu_0) = \mu_0 \) and \( \nu_0 \in W \) because \( \hat{p}_g(\nu_0) = \mu_\ast \in G \).

Now we can complete the proof that \( \hat{f} \) is everywhere locally invertible. Let \( g_0: \{y_0\} \to \hat{P}(T) \) be the constant map \( g_0(y_0) = \nu_0 \). Since \( \hat{p}_f(\nu_0) = g^*(y_0) \) and, by Lemma 2.2, the map \( \hat{p}_f \) is soft, there exists a map \( \theta: Y \to \hat{P}(T) \) extending \( g_0 \) such that \( \hat{p}_f \circ \theta = g^* \). Obviously, \( V = \theta^{-1}(W) \) is a neighborhood of \( y_0 \), and define \( \alpha = \hat{p}_g \circ \theta \). Since for any \( y \in V \) we have \( \hat{p}_f(\theta(y)) = g^*(y) \), \( p_f(\text{supp } \theta(y)) = \text{supp } g^*(y) \subset g^{-1}(y) \) and \( \text{supp } \theta(y) \subset g^{-1}(y) \times f^{-1}(y) \). So, \( \text{supp } \alpha(y) = p_g(\text{supp } \theta(y)) \subset f^{-1}(y) \). Consequently, \( \hat{f}(\alpha(y)) = y \). Moreover, \( \alpha(y) \in U \) for all \( y \in V \).

Since \( \hat{f} \) is everywhere locally invertible, by \cite{4,13} Theorem 1.3, the map \( \hat{P}(\hat{f}): \hat{P}(Y) \to \hat{P}(Y) \) is soft, where \( \hat{Y} = \hat{f}^{-1}(Y) \). Moreover, \( \hat{P}(X) \subset \hat{P}(\hat{Y}) \subset \hat{P}(\hat{P}(X)) \) because \( X \subset \hat{Y} \subset \hat{P}(X) \). Therefore the
following diagram

\[ \begin{array}{ccc}
\hat{P}(\hat{Y}) & \xrightarrow{b_\hat{P}} & \hat{P}(X) \\
\hat{P}(f) \downarrow & & \downarrow \hat{P}(f) \\
\hat{P}(Y) & \xrightarrow{i_{\hat{P}(Y)}} & \hat{P}(Y)
\end{array} \]

is commutative. Here, \( b_\hat{P} \) denotes the restriction \( b_{\hat{P}^*(X)}|\hat{P}(\hat{Y}) \) of the barycentric map \( b_{\hat{P}^*(X)}: \hat{P}(\hat{P}(X)) \to \hat{P}(X) \), see [3], and \( i_{\hat{P}(Y)} \) is the identity on \( \hat{P}(Y) \). Since \( b_\hat{P} \) retracts each \( \hat{P}(\hat{f})^{-1}(\mu) \) onto \( \hat{P}(f)^{-1}(\mu) \), \( \mu \in \hat{P}(Y) \), and \( \hat{P}(\hat{f}) \) is soft, we finally obtain that \( \hat{P}(f) \) is also soft. The proof is completed.

3. Atomless Milyutin maps

In this section we provide the proofs of Theorems 1.2 and 1.3.

**Proof of Theorem 1.2.** Suppose that \( f: X \to Y \) is a surjective atomless Milyutin map with \( X \) and \( Y \) complete spaces. Then there exists a choice map \( h: Y \to \hat{P}(X) \) associated with \( f \) such that \( h(y) \) is an atomless measure for all \( y \in Y \). Let \( X_0 = \bigcup \{ \text{supp } h(y) : y \in Y \} \) and \( f_0 = f|X_0 \). Since \( f_0^{-1} = \text{supp} \circ h \) is lower semi-continuous, \( f_0 \) is open. Hence, by [1, Theorem 3.6], \( X_0 \) is complete. Moreover, all \( f_0^{-1}(y) \) are perfect sets because \( h(y) \) are atomless measures.

For the other implication, assume that \( f: X \to Y \) is a surjection between complete spaces and there exists a complete subspace \( X_0 \subset X \) such that \( f_0 = f|X_0 \) is an open surjection possessing perfect fibers. Considering \( X_0 \) and \( f_0|X_0 \), we may suppose that \( f \) is open and all of its fibers \( f^{-1}(y) \), \( y \in Y \), are perfect sets. Then, by Theorem 1.1, \( f \) is Milyutin because \( \hat{P}(f) \) has a right inverse as a soft map. To show \( f \) is atomless, as in the proof of Theorem 1.1 take a 0-dimensional complete space \( Z \) and a perfect Milyutin map \( g: Z \to Y \). Since \( g \) is Milyutin, there exists a map \( g^*: \hat{P}(Y) \to \hat{P}(Z) \) such that \( \hat{P}(g)(g^*(\mu)) = \mu \) for all \( \mu \in \hat{P}(Y) \). By Theorem 1.1, \( \hat{P}(f) \) is open (as a soft map). Hence, \( \hat{f}: \hat{P}(f)^{-1}(Y) \to Y \) is also open (as a restriction of an open map onto a preimage-set). So, the set-valued map \( \Phi: Z \to \hat{P}(f)^{-1}(Y) \), \( \Phi(z) = \hat{f}^{-1}(g(z)) \), is lower semi-continuous. Actually, \( \Phi(z) = \hat{P}(f^{-1}(g(z))) \) for every \( z \in Z \). Let \( A_n, n \geq 1 \), be the set of all \( \mu \in \hat{P}(X) \) such that \( \mu(\{x\}) \geq 1/n \) for some point \( x \in \text{supp } \mu \). Since the fibers \( f^{-1}(y) \) are perfect sets, by Lemma 2.1, \( A_n \) are closed in \( \hat{P}(X) \) and all intersections \( A_n \cap \hat{P}(f^{-1}(y)) \) are nowhere dense in \( \hat{P}(f^{-1}(y)) \), \( y \in Y \). Then, by [0, Theorem 1.2], \( \Phi \) admits a selection \( \theta: Z \to \hat{P}(f)^{-1}(Y) \) such that
\[ \theta(z) \in \Phi(z) \setminus \bigcup_{n=1}^{\infty} A_n, \ z \in Z. \] This means that each measure \( \theta(z) \in \hat{P}(f^{-1}(g(z))) \) is atomless. The selection \( \theta \) generates a regular operator \( u: C^*(X) \to C^*(Z), u(\phi)(z) = \theta(z)(\phi) \) for all \( \phi \in C^*(X) \) and \( z \in Z. \) Finally, for every \( \mu \in P_\beta(Y) \) let \( f^*(\mu) \in \hat{P}(X) \) be the measure defined by \( f^*(\mu)(\phi) = g^*(\mu)(u(\phi)), \phi \in C^*(X). \) It is easily seen that this definition is correct (i.e., \( f^*(\mu) \in \hat{P}(X) \)) and \( f^*: P_\beta(Y) \to \hat{P}(X) \) is a continuous map.

Let us show that \( \hat{P}(f)(f^*(\mu)) = \mu \) for every \( \mu \in P_\beta(Y). \) It suffices to prove that \( f^*(\mu)(\alpha \circ f) = \mu(\alpha) \) for any \( \alpha \in C^*(Y). \) And this is really true because \( \phi = \alpha \circ f \) is the constant \( \alpha(y) \) on each set \( f^{-1}(y), y \in Y. \) So, \( u(\phi)(z) = \theta(z)(\phi) = \alpha(y) \) for any \( z \in g^{-1}(y). \) Thus, \( u(\phi) = \alpha \circ g \) and \( f^*(\mu)(\alpha \circ f) = g^*(\mu)(\alpha \circ g). \) Finally, since \( \hat{P}(g)(g^*(\mu)) = \mu, \) we have \( g^*(\mu)(\alpha \circ g) = \mu(\alpha). \)

So, it remains to prove only that every \( f^*(\mu), \mu \in P_\beta(Y), \) is an atomless measure. To this end, fix \( \mu_0 \in P_\beta(Y), x_0 \in \text{supp } f^*(\mu_0) \) and \( \eta > 0. \) It suffices to find a function \( \phi_0 \in C^*(X) \) with \( 0 \leq \phi_0 \leq 1 \) such that \( \phi_0(x_0) = 1 \) and \( f^*(\mu_0)(\phi_0) \leq \eta. \) Since \( \theta(z)(\{x_0\}) = 0, \) for every \( z \in Z \) there exists \( \phi_z \in C^*(X) \) and a neighborhood \( U_z \) of \( z \) in \( Z \) such that \( 0 \leq \phi_z \leq 1, \phi_z(x_0) = 1 \) and \( \theta(z)(\phi_z) < \eta \) whenever \( z' \in U_z. \) Using the compactness of \( g^{-1}(\text{supp } \mu_0) \) (recall that \( \mu_0 \) has a compact support and \( g \) is a perfect map), we find neighborhoods \( U_{z(i)}, i = 1, ..., k, \) covering \( g^{-1}(\text{supp } \mu_0), \) and let \( \phi_0 = \phi_{z(i)} \cdot \phi_{z(2)} \cdot \cdots \cdot \phi_{z(k)}. \) Then \( \phi_0 \) is as required. Indeed, since \( \hat{P}(g)(g^*(\mu_0)) = \mu_0, \) \( g^{-1}(\text{supp } \mu_0) \) contains the support of \( g^*(\mu_0). \) Consequently, \( g^*(\mu_0)(u(\phi_0)) \leq \max\{u(\phi_0)(z) : z \in g^{-1}(\text{supp } \mu_0)\}. \) So, there exists \( z_0 \in g^{-1}(\text{supp } \mu_0) \) such that \( g^*(\mu_0)(u(\phi_0)) \leq u(\phi_0)(z_0). \) Next, choose \( j \) with \( z_0 \in U_{z(j)} \) and observe that \( \phi_0 \leq \phi_j \) implies \( u(\phi_0)(z_0) \leq u(\phi_j)(z_0) = \theta(z_0)(\phi_j). \) Therefore, \( f^*(\mu_0)(\phi_0) \leq \theta(z_0)(\phi_j) < \eta \) because \( z_0 \in U_{z(j)}. \) The proof is completed.

**Proof of Theorem 1.3.** Take a 0-dimensional complete space \( Z, \) a perfect Milyutin map \( g: Z \to Y \) and a map \( g^*: \hat{P}(Y) \to \hat{P}(Z) \) which is a right inverse of \( \hat{P}(g). \) We equip \( \hat{P}(X) \) with a convex metric \( \hat{d}, \) and let \( A_n, n \geq 1, \) be the closed subsets of \( \hat{P}(X) \) considered in the proof of Theorem 1.2. We need to show that the set \( A \) of all atomless choice maps form a dense \( G_\delta \)-subset of \( Ch_f(Y, X). \) Since each \( A_n \) is closed in \( \hat{P}(X), \) it is easily seen that the sets

\[ U_n = \{ h \in Ch_f(Y, X) : h(y) \notin A_n \text{ for all } y \in Y \} \]

are open in \( Ch_f(Y, X) \) and \( A = \bigcap_{n \geq 1} U_n \). To prove that \( A \) is dense in \( Ch_f(Y, X), \) fix \( h \in Ch_f(Y, X) \) and a function \( \eta: Y \to (0, \infty). \) We
are going to find a map $h' \in A$ such that $\hat{d}(h(y), h'(y)) \leq \eta(y)$ for all $y \in Y$.

Denote by $B(h(g(z)), \eta(g(z)))$ the open ball in $\hat{P}(X)$ (with respect to $\hat{d}$) which is centered at $h(g(z))$ and has a radius $\eta(g(z))$. Define the set-valued map $\Phi: Z \to \hat{P}(X)$, $\Phi(z) = \hat{P}(\hat{P}(f^{-1}(g(z)))) \cap B(h(g(z)), \eta(g(z)))$. This is a convex and closed-valued map because any ball in $\hat{P}(X)$ with respect to $\hat{d}$ is convex. Since $\hat{f} = \hat{P}(f)(\hat{P}(f)^{-1}(Y))$ is open (as a soft map, see Theorem 1.1), the set-valued map $z \mapsto \hat{P}(f^{-1}(g(z)))$ is lower semi-continuous. Hence, by [10, Proposition 2.5], so is $\Phi$. Moreover, each $\Phi(z)$ is the closure of the convex open set $\hat{P}(f^{-1}(g(z))) \cap B(h(g(z)), \eta(g(z)))$ in $\hat{P}(f^{-1}(g(z)))$. Hence, according to Lemma 2.1, $A_n \cap \Phi(z), n \geq 1$, are nowhere dense sets in $\Phi(z)$ for every $z \in Z$.

Then, by [9, Theorem 1.2], $\Phi$ has a continuous selection $\theta: Z \to \hat{P}(X)$ avoiding the set $\bigcup_{i=1}^{\infty} A_n$, i.e., with $\theta(z) \in \Phi(z) \setminus \bigcup_{i=1}^{\infty} A_n$ for every $z \in Z$. Following the notations from the proof of Theorem 1.2, we extend $\theta$ to a map $\bar{\theta}: P_\rho(Z) \to \hat{P}(X)$ by $\bar{\theta}(\nu)(\phi) = \nu(u(\phi)), \phi \in C^*(X)$. Now let $h': Y \to \hat{P}(X)$ be the composition $\bar{\theta} \circ g^*$. It follows from the proof of Theorem 1.2 that $h'(y)$ is atomless and $h'(y) \in \hat{P}(f^{-1}(y))$ for all $y \in Y$. So, $h' \in A$.

It remains to show that $\hat{d}(h(y), h'(y)) \leq \eta(y), y \in Y$. To this end, we fix $y \in Y$ and take a sequence $\{\nu_n\} \subset P_\rho(g^{-1}(y))$ converging to $g^*(y)$ such that each $\nu_n$ has a finite support. It is easily seen that if $\nu = \sum_{i=1}^{k} t_i \delta_{z(i)} \in P_\rho(g^{-1}(y))$ is a measure with a finite support, then $\bar{\theta}(\nu) = \sum_{i=1}^{k} t_i \theta(z(i))$. Since $\hat{d}(\theta(z(i)), h(y)) \leq \eta(y)$ for all $i$ and the metric $\hat{d}$ is convex, we have $\hat{d}(\bar{\theta}(\nu), h(y)) \leq \eta(y)$. In particular, $\hat{d}(\theta(\nu_n), h(y)) \leq \eta(y)$ for every $n$. This implies that $\hat{d}(h'(y), h(y)) \leq \eta(y)$ because $h'(y)$ is the limit of the sequence $\{\bar{\theta}(\nu_n)\}$.

4. Exact Milyutin maps

In this section the proofs of Theorem 1.4 and Corollaries 1.5-1.6 are established.

Lemma 4.1. Let $U \subset X$ be a non-empty open set in a space $X$. Then the set $\hat{U} = \{\nu \in \hat{P}(X) : \text{supp } \nu \cap U \neq \emptyset\}$ is open convex and dense in $\hat{P}(X)$.

Proof. Since the support map $\nu \to \text{supp } \nu$ is a lower semi-continuous map, $\hat{U} \subset \hat{P}(X)$ is open. To show it is dense, suppose there exists an open set $W = \{\nu \in \hat{P}(X) : |\nu(\phi_i) - \nu_0(\phi_i)| < \varepsilon, 1 \leq i \leq k\}$ in $\hat{P}(X)$ with $W \subset \hat{P}(X) \setminus \hat{U}$, where $\phi_i \in C^*(X)$ and $\varepsilon > 0$. We can
suppose that $\nu_0$ has a finite support (recall that the measures with a finite support form a dense set in $\hat{P}(X)$). Let $\nu_0 = \sum_{j=1}^{j=m} \lambda_j \delta_{x(j)}$ such that $\lambda_1 > 0$ and $\sum_{j=1}^{j=m} \lambda_j = 1$. Then $\text{supp} \ \nu_0 = \{x(j) : 1 \leq j \leq m\} \subset X \setminus U$. Now, let $\nu' = \lambda_0 \delta_{x(0)} + (\lambda_1 - \lambda_0) \delta_{x(1)} + \sum_{j=2}^{j=m} \lambda_j \delta_{x(j)}$, where $x_0 \in U$ and $0 < \lambda_0 < \lambda_1$ such that $\lambda_0 |\phi_0(x_0) - \phi_0(x_1)| < \epsilon$ for every $i = 1, 2, \ldots, k$. The choice of $\lambda_0$ yields that $\nu' \in W$. Consequently, $\nu' \notin \hat{U}$ and $\text{supp} \ \nu' \subset X \setminus U$. This contradicts $x_0 \in U \cap \text{supp} \ \nu'$.

To show $\hat{U}$ is convex, it suffices to prove that $\text{supp} \ (t\nu_1 + (1-t)\nu_2) = \text{supp} \ \nu_1 \cup \text{supp} \ \nu_2$ for any $\nu_1, \nu_2 \in \hat{P}(X)$ and any $t \in (0, 1)$. Obviously, $\text{supp} \ \nu_1 \cup \text{supp} \ \nu_2 \supseteq \text{supp} \ (t\nu_1 + (1-t)\nu_2)$. Assume $x \in \text{supp} \ \nu_1$. Then for every neighborhood $V_x$ of $x$ there exists a function $\phi_x \in C^*(X)$ with $\phi_x(X \setminus V_x) = 0$ and $\nu_1(\phi_x) \neq 0$. Since $\nu_1(\phi_x) = \nu_1(\phi_x^+) - \nu_1(\phi_x^-)$, where $\phi_x^+$ and $\phi_x^-$ are the positive and negative parts of $\phi_x$, we can suppose $\phi_x$ is non-negative. Then, $\nu(\phi_x) \geq \nu_1(\phi_x) > 0$ with $\nu = \nu = t\nu_1 + (1-t)\nu_2$. Hence, $x \in \text{supp} \ \nu$ which completes the proof.  

\textbf{Proof of Theorem 1.4.} Choose a countable base $\{V_n : n \geq 1\}$ for the topology of $M$, and let $B_n = \{\nu \in \hat{P}(X) : \text{supp} \ \nu \cap \pi^{-1}(V_n) = \emptyset\}$. By Lemma 4.1, each $B_n$ is closed in $\hat{P}(X)$. Let $\mathcal{B}$ be the set of all maps $h \in \text{Ch}_f(Y, X)$ such that $\pi(\text{supp} \ h(y))$ is dense in $\pi(f^{-1}(y))$ for any $y \in Y$. Obviously, $\mathcal{B} = \bigcap_{n \geq 1} \mathcal{G}_n$, where $\mathcal{G}_n = \{h \in \text{Ch}_f(Y, X) : h(y) \notin B_n \text{ for all } y \in Y\}$. It suffices to show that each $\mathcal{G}_n$ is open and dense in $\text{Ch}_f(Y, X)$ with respect to the source limitation topology.

\textbf{Claim 1.} Each $\mathcal{G}_n$ is open in $\text{Ch}_f(Y, X)$.

We can suppose that each $V_n$ is of the form $V_n = g_n^{-1}(0, \infty)$ for some non-negative function $g_n \in C^*(M)$. Then $\nu \in B_n$ if and only if $\nu(g_n \circ \pi) = 0$, $n \geq 1$. Obviously the equality $D_n(\mu, \mu') = \hat{d}(\mu, \mu') + |\mu(g_n \circ \pi) - \mu'(g_n \circ \pi)|$, where $\mu, \mu' \in \hat{P}(X)$ and $\hat{d}$ is a compatible metric on $\hat{P}(X)$, defines a compatible metric on $\hat{P}(X)$ for every $n \geq 1$. Given $h \in \mathcal{G}_n$ we consider the continuous function $\alpha : Y \to (0, \infty)$, $\alpha(y) = h(y)(g_n \circ \pi)/2$. We have $B_{D_n}(h, \alpha) \subset \mathcal{G}_n$. Indeed, if $h' \in B_{D_n}(h, \alpha)$, then $|h'(y)(g_n \circ \pi) - h(y)(g_n \circ \pi)| \leq D_n(h(y), h'(y)) < \alpha(y)$ for all $y \in Y$. The last inequality implies $h'(y)(g_n \circ \pi) > \alpha(y) > 0$, $y \in Y$. Hence, $h'(y) \notin B_n$ for all $y \in Y$. So, $h' \in \mathcal{G}_n$ which completes the proof of Claim 1.

To show that any $\mathcal{G}_n$ is dense in $\text{Ch}_f(Y, X)$, we fix $m \geq 1$, $h \in \text{Ch}_f(Y, X)$ and a function $\eta : Y \to (0, \infty)$. We are going to find a map $h' \in \mathcal{G}_n$ with $\hat{d}(h'(y), h(y)) \leq \eta(y)$ for all $y \in Y$. To this end, following the proof of Theorems 1.2 and 1.3, take a complete 0-dimensional space $Z$ and a perfect Milutin map $g : Z \to Y$ with
a right inverse \( g^*: Y \to P_\beta(Z) \). We also consider the lower semi-continuous convex and closed-valued map \( \Phi: Z \to \hat{P}(X) \), \( \Phi(z) = \hat{P}(f^{-1}(g(z))) \cap B(h(g(z)), \eta(g(z))) \). According to Lemma 4.1, \( B_m \cap \hat{P}(f^{-1}(g(z))) \) is a closed nowhere dense subsets of \( \hat{P}(f^{-1}(g(z))) \) for every \( z \in Z \). Hence, all \( B_m \cap \Phi(z) \) are closed and nowhere dense in \( \Phi(z) \).

Then, by \cite[Theorem 1.2]{13}, \( \Phi \) has a continuous selection \( \theta: Z \to \hat{P}(X) \) such that \( \theta(z) \in \Phi(z) \) for every \( z \in Z \). As in the proof of Theorem 1.3, let \( h': Y \to \hat{P}(X) \) be the composition \( \theta \circ g^* \), where \( \theta: P_\beta(Z) \to \hat{P}(X) \) is an extension of \( \theta \) defined by \( \theta(\nu)(\phi) = \nu(u(\phi)) \), \( \phi \in C^*(X) \). Following the arguments from Theorem 1.3, we can show that \( \hat{d}(h'(y), h(y)) \leq \eta(y) \) for all \( y \in Y \). Next claim completes the proof of Theorem 1.4.

Claim 2. \( h'(y) \notin B_m \) for any \( y \in Y \).

The proof of this claim is reduced to find a function \( \phi_y \in C^*(X) \) such that \( \phi_y(X \setminus \pi^{-1}(V_m)) = 0 \) and \( h(y)(\phi_y) \neq 0 \). Indeed, in such a case \( supp \ h(y) \cap \pi^{-1}(V_m) \neq \emptyset \). Since \( \theta(z) \notin B_m \) for all \( z \in g^{-1}(y) \), \( supp \ \theta(z) \cap \pi^{-1}(V_m) \neq \emptyset \). Consequently, for any \( z \in g^{-1}(y) \) there exists a function \( \phi_z \in C^*(X) \) with \( \phi_z(X \setminus \pi^{-1}(V_m)) = 0 \) and \( \theta(z)(\phi_z) \neq 0 \).

Considering the positive or negative parts of \( \phi_z \), we may assume each \( \phi_z \geq 0 \). Next, use the continuity of \( \theta \) and the compactness of \( g^{-1}(y) \) to find finitely many points \( z(i) \in g^{-1}(y) \), \( i = 1, 2, \ldots, k \), and neighborhoods \( U_{z(i)} \) such that \( \theta(z)(\phi_{z(i)}) > 0 \) provided \( z \in U_{z(i)} \). Finally, let \( \phi_y = \sum_{i=1}^{k} \phi_{z(i)}(z) \). Then \( \phi_y(X \setminus \pi^{-1}(V_m)) = 0 \) and \( u(\phi_y)(z) = \theta(z)(\phi_y) > 0 \) for any \( z \in g^{-1}(y) \). So, \( h(y)(\phi_y) \geq \min\{u(\phi_y)(z) : z \in g^{-1}(y)\} > 0 \) because \( g^{-1}(y) \) is compact. This completes the proof of the claim.

**Proof of Corollary 1.5.** Since \( f \) is closed with separable fibers, there exists a map \( \pi: X \to Q \) such that all restrictions \( \pi|f^{-1}(y) \), \( y \in Y \), are embeddings, see \cite{13}. Here, \( Q \) is the Hilbert cube. Then, by Theorem 1.4 (with \( M \) replaced by \( Q \)), \( f \) is densely exact. If, in addition, the fibers of \( f \) are perfect, both Theorems 1.3 and 1.4 imply that \( f \) is densely exact atomless.

**Proof of Corollary 1.6.** Consider the graph \( G(\Phi) = \bigcup \{\{y\} \times \Phi(y) : y \in Y\} \subset Y \times X \) of \( \Phi \) and the projection \( f: G(\Phi) \to Y \). Since \( \Phi \) is continuous, \( G(\Phi) \) is closed in \( Y \times X \) and \( f \) is both open and closed. Then \( G(\Phi) \) is a complete space. Now, by Corollary 1.5, there exists a map \( h': Y \to \hat{P}(G(\Phi)) \) with each \( h'(y) \in \hat{P}(f^{-1}(y)) \) being exact measure. Therefore, \( supp \ h'(y) = f^{-1}(y) \). Let \( h = \hat{P}(\pi) \circ h' \), where \( \pi: G(\Phi) \to X \) is the projection into \( X \). Since \( \pi \) embeds each \( f^{-1}(y) \) onto \( \Phi(y) \), \( h \) is a map from \( Y \) into \( \hat{P}(X) \) such that \( supp \ h(y) = \Phi(y) \) for every \( y \in Y \). If \( \Phi(y) \) are perfect sets, so are the fibers \( f^{-1}(y) \), and
$h'$ can be chosen to be atomless and exact. In such a case $h$ is also atomless.

**Note added in proof.** Recently T. Banakh informed the author that V. Bogachev and A. Kolesnikov \[5\] proved the following result: The map $\hat{P}(f)$ from Theorem 1.1 is open. This, in combination with Michael’s convex-valued selection theorem \[10\], provides another proof of Theorem 1.1.

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Department of Computer Science and Mathematics, Nipissing University, 100 College Drive, P.O. Box 5002, North Bay, ON, P1B 8L7, Canada

E-mail address: veskov@nipissingu.ca