VERMA MODULES OVER RESTRICTED QUANTUM $\mathfrak{sl}_3$ AT A FOURTH ROOT OF UNITY

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Abstract. For a semisimple Lie algebra $\mathfrak{g}$ of rank $n$, let $\mathcal{U}_\zeta(\mathfrak{g})$ be the restricted quantum group of $\mathfrak{g}$ at a primitive fourth root of unity. This quantum group admits a natural Borel-induced representation $V(\mathbf{t})$, with $\mathbf{t} \in (\mathbb{C}^\times)^n$ determined by a character on the Cartan subalgebra. Ohtsuki showed that for $\mathfrak{g} = \mathfrak{sl}_2$, the braid group representation determined by tensor powers of $V(\mathbf{t})$ is the exterior algebra of the Burau representation. In this paper, we begin work on the $\mathfrak{g} = \mathfrak{sl}_3$ case. This includes a generalization of the decomposition for $V(\mathbf{t}) \otimes V(\mathbf{s})$ used in Ohtsuki’s work, which we expect to hold for any $\mathfrak{g}$. We also define a stratification of $(\mathbb{C}^\times)^4$ whose points $(\mathbf{t}, \mathbf{s})$ in the lower strata are associated to representations $V(\mathbf{t}) \otimes V(\mathbf{s})$ which do not have a homogeneous cyclic generator. Moreover, we characterize exactly when the isomorphism $V(\mathbf{t}) \otimes V(\mathbf{s}) \cong V(\lambda \mathbf{t}) \otimes V(\lambda^{-1} \mathbf{s})$ holds.

1. Introduction

Ohtsuki describes the exterior algebra of the Burau representation as Turaev-type $R$-matrix actions on a family of representations dependent on a complex parameter $t \in \mathbb{C}^\times$, see [Oht02]. The universal $R$-matrix for this action arises from a version of quantum $\mathfrak{sl}_2$ at a fourth root of unity $\zeta$ called the unrolled restricted quantum group $\mathcal{U}_H^{\zeta}(\mathfrak{sl}_2)$. J. Murakami was the first to describe the (unrolled) restricted quantum group and showed its relation to the Alexander polynomial using the representations $V(t)$ [Mur92, Mur93]. These representations $V(t)$ are finite dimensional Verma modules of highest weight $t$. In other words, they are induced by characters on the Borel subalgebra.

In this paper, we begin an investigation of higher rank restricted quantum groups $\mathcal{U}_\zeta(\mathfrak{g})$. An important aspect in understanding the structure of their associated braid group representations is the tensor product decomposition for generalizations of the modules $V(t)$. For example, this decomposition allows us to determine spectral properties of the braid operators and therefore skein relations. We hope that they will assist in developing a geometric interpretation of these representations in analogy with Burau’s construction, and in finding relations to classical invariants such as Reidemeister torsion and the Alexander polynomial.

We start with basic definitions and properties of $\mathcal{U}_\zeta(\mathfrak{g})$ and its induced modules for a semisimple Lie algebra of rank $n$. However, the goal of this paper is to provide a thorough investigation of the $\mathfrak{sl}_3$ case and properties of its tensor products, and set foundations for higher rank.

We recall, in Section 2, the construction of quantum groups at roots of unity from Lusztig’s divided powers algebra. In contrast to the small quantum group $u_q(\mathfrak{g})$ described in [Lus90], the restricted quantum group $\mathcal{U}_\zeta(\mathfrak{g})$ considered here is infinite dimensional. For $q = \zeta$, a primitive fourth root of unity, we give a generators and relations description and a PBW
basis of $\mathcal{U}_\zeta(g)$ for each Lie type. In the following, $\Phi^+$ denotes the nonzero positive roots of $\mathcal{U}_\zeta(g)$, and $\Lambda^+ \subseteq \Phi^+$ the subset of nonzero positive simple roots.

**Proposition 1.1.** The restricted quantum group $\mathcal{U}_\zeta(g)$ is the $\mathbb{Q}_q$-algebra generated by $E_{\alpha_i}, F_{\alpha_i}$, for $\alpha_i \in \Lambda^+$, and $K_j$ for $1 \leq j \leq n$ with relations:

\[
K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad [E_{\alpha_i}, F_{\alpha_j}] = \delta_{ij} [K_i]_{d_i}, \quad (1)
\]
\[
K_i E_{\alpha_j} = q^{d_i \langle \alpha_j, \alpha_i \rangle} E_{\alpha_j} K_i, \quad K_i F_{\alpha_j} = q^{-d_i \langle \alpha_j, \alpha_i \rangle} F_{\alpha_j} K_i, \quad (2)
\]
\[
E_{\alpha_i} E_{\alpha_j} = E_{\alpha_j} E_{\alpha_i} \quad \text{for } |i - j| > 1, \quad (3)
\]
\[
F_{\alpha_i} F_{\alpha_j} = F_{\alpha_j} F_{\alpha_i} \quad \text{for } |i - j| > 1, \quad (4)
\]
\[
E_{\alpha_i}^2 = F_{\alpha_i}^2 = 0 \quad \text{for } \alpha_i \in \Phi^+. \quad (5)
\]

**Corollary 1.2.** The restricted quantum group $\mathcal{U}_\zeta(g)$ has a PBW basis

\[
\{ E^{\psi} F^{\psi'} K^k : \psi, \psi' \in \{0, 1\}^{\Phi^+} \text{ and } k \in \mathbb{Z}^n \},
\]

where $E^{\psi} = \prod_{\alpha \in \Phi^+} E^{\psi}_{\alpha}$, $F^{\psi} = \prod_{\alpha \in \Phi^+} F^{\psi}_{\alpha}$, $K^k = \prod_{i=1}^n K_i^{k_i}$, and products are ordered with respect to $<_{br}$.

We then consider the group of characters $\mathcal{P}$ on the Cartan torus of the restricted quantum group. A character $\overline{t}$ is determined by the images $t_i \in \mathbb{C}^\times$ of Cartan generators $K_i$ for $1 \leq i \leq n$. Thus, $\overline{t}$ can be identified with a tuple $(t_1, \ldots, t_n) \in (\mathbb{C}^\times)^n$. Multiplication in $\mathcal{P}$ is given by $\overline{t} \overline{s} = (t_1 s_1, \ldots, t_n s_n)$ with identity $\overline{1} = (1, \ldots, 1)$. The character $\overline{t}$ extends to a character $\gamma_{\overline{t}}$ on the Borel subalgebra $B$ by taking the value zero off the Cartan torus.

Let $V_{\overline{t}} = \langle v_0 \rangle$ be the 1-dimensional left $B$-module determined by $\gamma_{\overline{t}}$, i.e. for $b \in B$, $bv_0 = \gamma_{\overline{t}}(b)v_0$. We then define the representation $V(\overline{t})$ to be the induced module

\[
V(\overline{t}) = \text{Ind}_B^\mathcal{U}_\zeta(g)(V_{\overline{t}}) = \mathcal{U}_q(g) \otimes_B V_{\overline{t}}. \quad (7)
\]

If $g$ is type $ADEG$ and $q = \zeta$ is a fourth root of unity, then the quantum group representations $V(\overline{t})$ each have dimension $2^{|\Phi^+|}$, with $|\Phi^+|$ equal to the number of positive roots.

For the remainder of the introduction, assume $g = sl_3$ unless noted otherwise. Consider the subsets of $\mathcal{P}$:

\[
\mathcal{X}_1 = \{ \overline{t} \in \mathcal{P} : t_1^2 = 1 \}, \quad \mathcal{X}_2 = \{ \overline{t} \in \mathcal{P} : t_2^2 = 1 \} \quad (8)
\]
\[
\mathcal{C} = \mathcal{X}_1 \cap \mathcal{X}_2 \quad \mathcal{H} = \{ \overline{t} \in \mathcal{P} : (t_1 t_2)^2 = -1 \}. \quad (9)
\]

Reducibility of $V(\overline{t})$ is determined by whether $\overline{t}$ belongs to the algebraic set

\[
\mathcal{R} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{H}. \quad (10)
\]

**Lemma 1.3.** The representations $V(\overline{t})$ are indecomposable and non-isomorphic for each $\overline{t} \in \mathcal{P}$. Further, $V(\overline{t})$ is irreducible if and only if $\overline{t} \in \mathcal{P} \setminus \mathcal{R}$.

Let $\Sigma$ denote the weights of $V(\overline{1})$ with multiplicity, so that $[\sigma \overline{t} : \sigma \in \Sigma]$ lists the weights of $V(\overline{t})$ with multiplicity. A pair of characters $(\overline{t}, \overline{s}) \in \mathcal{P}^2$ is called non-degenerate if $V(\sigma \overline{t} \overline{s})$ is irreducible for each $\sigma \in \Sigma$. Our first main result is a decomposition rule for $V(\overline{t}) \otimes V(\overline{s})$, given that the pair $(\overline{t}, \overline{s})$ is non-degenerate.
Theorem 1.4 (Tensor Product Decomposition). Let \((\mathbf{t}, \mathbf{s})\) be a non-degenerate pair. The tensor product \(V(\mathbf{t}) \otimes V(\mathbf{s})\) decomposes as a direct sum of irreducibles according to the formula
\[
V(\mathbf{t}) \otimes V(\mathbf{s}) \cong \bigoplus_{\sigma \in \Sigma} V(\sigma \mathbf{t} \mathbf{s}).
\]   
(11)

Note that Theorem 1.4 generalizes the formula given in [Oht02] for the \(sl_2\) case, having taken \(\Sigma = [1, -1]\). We also have that non-degenerate tensor product representations only depend on the product of \(\mathbf{t}\) and \(\mathbf{s}\), thus motivating the following definition. We call an isomorphism
\[
V(\mathbf{t}) \otimes V(\mathbf{s}) \cong V(\lambda \mathbf{t}) \otimes V(\lambda^{-1} \mathbf{s}),
\]   
(12)

for some \(\lambda \in \mathcal{P}\), a transfer.

Here, we give a description of all transfers on tensor product representations \(V(\mathbf{t}) \otimes V(\mathbf{s})\) of \(U(\mathfrak{sl}_3)\). Our approach is to first find representations generated by a single weight vector under the action of non-Cartan elements, we call such a representation \(\text{homogeneous cyclic}\). If \(V(\mathbf{t}) \otimes V(\mathbf{s})\) is homogeneous cyclic then it is characterized by the weight \(-\mathbf{t} \mathbf{s}\) of its generator. The values of \((\mathbf{t}, \mathbf{s})\) for which cyclicity fails determine the \(\text{acyclicity locus} \ \mathcal{A}\).

Theorem 1.5 (Homogeneous Cyclic Tensor Product Representations). The \(\text{acyclicity locus} \ \mathcal{A} \subseteq \mathcal{P}^2\) is given by
\[
\mathcal{X}^2_1 \cup \mathcal{X}^2_2 \cup \mathcal{H}^2 \cup (\mathcal{H} \times \mathcal{C}) \cup (\mathcal{C} \times \mathcal{H}).
\]   
(13)

Let
\[
\hat{\Delta}(\mathcal{H}) = \{ (\mathbf{t}, \mathbf{s}) \in \mathcal{H}^2 : (t_1s_1)^2 = 1 \} = \{ (\mathbf{t}, \mathbf{s}) \in \mathcal{P}^2 : (t_1s_1)^2 = 1, (t_1t_2)^2 = (s_1s_2)^2 = -1 \}.
\]   
(14)

Let \(\mathcal{Y}_i = \mathcal{X}_i \cap \mathcal{H}\). Then \(\mathcal{P}^2\) is stratified according to the filtration \(\mathcal{P}_0^2 \subset \mathcal{P}_1^2 = \mathcal{A} \subset \mathcal{P}_2^2 = \mathcal{P}^2\), with
\[
\mathcal{P}_0^2 = \mathcal{C}^2 \cup \mathcal{Y}_1^2 \cup \mathcal{Y}_2^2 \cup \hat{\Delta}(\mathcal{H}) \cup ((\mathcal{Y}_1 \cup \mathcal{Y}_2) \times \mathcal{C}) \cup (\mathcal{C} \times (\mathcal{Y}_1 \cup \mathcal{Y}_2))
\]   
(16)

given by the union of pairwise intersections of distinct algebraic sets together with \(\hat{\Delta}(\mathcal{H})\).

We illustrate the inclusion \(\mathcal{P}_0^2 \subseteq \mathcal{P}_1^2\) in Figure 1 below.

![Figure 1](image-url)

**Figure 1.** The inclusion of \(\mathcal{P}_0^2\) in \(\mathcal{P}_1^2\).

We may also define an action of \(\mathcal{P}\) on \(\mathcal{P}^2\) as follows. Let \(\lambda \in \mathcal{P}\) and \((\mathbf{t}, \mathbf{s}) \in \mathcal{P}^2\), then
\[
\lambda \cdot (\mathbf{t}, \mathbf{s}) = (\lambda \mathbf{t}, \lambda^{-1} \mathbf{s}).
\]   
(17)

The swapping of coordinates is replicated by the action of \(\lambda = t^{-1}s\). As \(\mathcal{P}_0^2\) and \(\mathcal{P}_1^2\) are preserved under the exchange of \(\mathbf{t}\) and \(\mathbf{s}\), the stratification respects the equivalence determined by a braiding. We group the defining subsets of \(\mathcal{P}_0^2\) and \(\mathcal{P}_1^2\) so that they are preserved by
swaps, and we refer to the resulting subsets as symmetrized. Under this grouping, we have the following theorem.

**Theorem 1.6** (Transfer Principle). Suppose \((t, s)\) belongs to a symmetrized subset in the \(n\)-stratum. Then

\[
V(t) \otimes V(s) \cong V(\lambda t) \otimes V(\lambda^{-1}s)
\]

if and only if \(\lambda \cdot (t, s)\) belongs to the same symmetrized subset in the \(n\)-stratum.

Relations between quantum invariants and the Alexander polynomial are studied in [KS91, KP17, Sar15, Vir07]. These papers consider invariants from quantum supergroups and \(\overline{U}^H_\zeta(\mathfrak{sl}_2)\). In [BCGP16], the authors construct a family of TQFTs based on the non-semisimple category of unrolled restricted quantum \(\mathfrak{sl}_2\) representations. At a fourth root of unity, their invariant recovers the Reidemeister torsion. These results make use of previous work [CGP17], which studies the representation theoretic properties of unrolled restricted quantum \(\mathfrak{sl}_2\). However, little is known about representations of higher rank unrolled restricted quantum groups and their invariants.

### 1.1. Future Work

As described above, we define the representations \(V(t)\) for any restricted quantum group. What is the structure of these representations? We conjecture that Theorem 1.4 holds for any restricted quantum group at a fourth root of unity.

**Conjecture 1.7.** Let \(\Sigma\) denote the weights of the representation \(V(\mathbf{1})\) with multiplicities of \(\overline{U}^H_\zeta(\mathfrak{g})\) for any semisimple Lie algebra \(\mathfrak{g}\). Then for any tuple \((t, s)\) such that \(V(\sigma ts)\) is irreducible for each \(\sigma \in \Sigma\),

\[
V(t) \otimes V(s) \cong \bigoplus_{\sigma \in \Sigma} V(\sigma ts).
\]

The arguments given here are expected to extend to higher rank, but we would also like to characterize reducibility and non-degeneracy more precisely. Moreover, characterizing transfers in higher rank will require developing new techniques.

At other roots of unity, the orders of the \(E_i\) and \(F_i\) are different and the Serre relations consist of more than far commutativity. We are interested in understanding the representation theory of these algebras and if similar results hold.

For semisimple \(\mathfrak{g}\), we expect that there exist indecomposable tensor product representations of \(\overline{U}^H_\zeta(\mathfrak{g})\) which are reducible. In rank 1, we prove in Corollary D.8 that such representations are given by \(V(t) \otimes V(t^{-1})\) for generic \(t\). In this work, we have shown that the representation \(V(t) \otimes V(t^{-1})\) of \(\overline{U}^H_\zeta(\mathfrak{sl}_2)\) is homogeneous cyclic and the hypotheses of the decomposition theorem are not met.

**Question 1.8.** Let \(t\) be such that \(V(t) \otimes V(t^{-1})\) is a homogeneous cyclic representation of \(\overline{U}^H_\zeta(\mathfrak{g})\). Is it true that \(V(t) \otimes V(t^{-1})\) is reducible and indecomposable?

Recall that the Alexander polynomial is obtained by coloring knots by representations \(V(t)\) of the unrolled restricted quantum group \(\overline{U}^H_\zeta(\mathfrak{sl}_2)\). We will study the knot invariant obtained from quantum \(\mathfrak{sl}_3\) representations \(V(\mathbf{t})\) in a later work. We also consider the generalization of other \(\overline{U}^H_\zeta(\mathfrak{sl}_2)\) quantum invariants to higher rank.
1.2. **Structure of Paper.** In Section 2 we recall the quantum group $U_q(g)$ according to Lusztig [Lus90] and define the restricted quantum group $\overline{U}_q(g)$. We define the induced representations $V(\mathfrak{t})$ for any root of unity before assuming $q = \zeta$ and $g = \mathfrak{sl}_3$ for the remainder of the paper.

In Section 3 we show that the representations $V(\mathfrak{t})$ are indecomposable for all choices of $\mathfrak{t}$, but are reducible in some cases. The results on irreducibility are then applied in Section 4 to find a direct sum decomposition for sufficiently generic tensor product representations.

Sections 5, 6, and 7 are concerned with finding homogeneous cyclic representations, and transfer isomorphisms which are not implied by Theorem 1.4. Section 5 sets up the language and the method used for finding cyclic representations, we also characterize cyclicity in the generic case. Sections 6 and 7 each study cyclicity for some non-generic choice of characters.

Gathering the conclusions of these sections yields the cyclicity theorem and transfer principle for representations $V(\mathfrak{t}) \otimes V(\mathfrak{s})$, stated in Section 8.

General computations which are referenced throughout the paper are compiled in Appendix A.

In Appendix B we give an informal discussion of the unrolled quantum group $U^H_q(g)$.

The latter two sections of the Appendix include information on induced representations used in proving Theorem 1.4, and a discussion of the $\mathfrak{sl}_2$ theory.

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2. **Restricted Quantum Groups and the Representations $V(\mathfrak{t})$**

In this section, we recall the definition of the quantum group for a semisimple Lie algebra $g$. Following Lusztig [Lus90], we obtain a restricted quantum group by setting the deformation parameter $q$ to a root of unity in the context of the divided powers algebra. We call the restricted quantum group at a primitive fourth root of unity $\overline{U}_\zeta(\mathfrak{g})$. We give a generators and relations description, and a PBW basis of $\overline{U}_\zeta(\mathfrak{g})$. We then define the representations $V(\mathfrak{t})$ as being induced by $\overline{U}_\zeta(\mathfrak{g})$ with respect to a character on the Borel subalgebra. Specializing to $g = \mathfrak{sl}_3$, $V(\mathfrak{t})$ is an 8-dimensional vector space and we characterize the $\overline{U}_\zeta(\mathfrak{sl}_3)$ action on it.

Let $q$ be a formal parameter and let $g$ be Lie algebra with $n \times n$ Cartan matrix $(A_{ij})$ symmetrized by the vector $(d_i)$ with entries in \{1, 2, 3\}. Let $\Phi^+$ be the space of positive root vectors, and $\Delta^+$ the positive simple roots of $g$. We define the quantum group $U_q(g)$ following [Lus90] and refer the reader there for additional details. We set

$$[N]_d! = \prod_{j=1}^{N} \frac{q^d_j - q^{-d_j}}{q^d_j - q^{-d_j}}, \quad \left[ M + N \right]_d = \frac{[M + N]_d!}{[M]_d! [N]_d!},$$

(20)
and
\[ [x]_d = \frac{x - x^{-1}}{q^d - q^{-d}}, \] (21)

omitting subscripts when \( d = 1 \).

**Definition 2.1.** Let \( U_q(\mathfrak{g}) \) be the algebra over \( \mathbb{Q}(q) \) generated by \( E_i, F_i \), and \( K_i^{\pm} \) for \( 1 \leq i \leq n \) subject to the relations:

\[
K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad [E_i, F_j] = \delta_{ij} [K_i]_{d_i}, \quad (22)
\]
\[
K_iE_j = q^{d_i A_{ij}} E_j K_i, \quad K_iF_j = q^{-d_i A_{ij}} F_j K_i, \quad (23)
\]
\[
\sum_{r+s=1-A_{ij}} (-1)^s [1 - A_{ij}]_{d_i} E_i^r E_j E_i^s = 0, \quad \text{for } i \neq j, \quad (24)
\]
\[
\sum_{r+s=1-A_{ij}} (-1)^s [1 - A_{ij}]_{d_i} F_i^r F_j F_i^s = 0, \quad \text{for } i \neq j. \quad (25)
\]

Equations (24) and (25) are called the *quantum Serre relations*.

**Definition 2.2.** Let \( U_q^{\text{div}}(\mathfrak{g}) \) be the subalgebra of \( U_q(\mathfrak{g}) \) generated by

\[
E_i^{(N)} = \frac{E_i^N}{[N]_{d_i}!}, \quad F_i^{(N)} = \frac{F_i^N}{[N]_{d_i}!}, \quad K_i^{\pm},
\]

over \( \mathbb{Z}[q, q^{-1}] \) for \( N \geq 0 \). We call \( U_q^{\text{div}}(\mathfrak{g}) \) the *divided-powers algebra*.

The Hopf algebra structure on \( U_q(\mathfrak{g}) \) is defined by the maps below for \( 1 \leq i \leq n \), and extends to the entire algebra via their (anti-)homomorphism properties:

\[
\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad S(E_i) = -E_i K_i^{-1}, \quad \epsilon(E_i) = 0 \quad (26)
\]
\[
\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i, \quad S(F_i) = -K_i F_i, \quad \epsilon(F_i) = 0 \quad (27)
\]
\[
\Delta(K_i) = K_i \otimes K_i, \quad S(K_i) = K_i^{-1}, \quad \epsilon(K_i) = 1. \quad (28)
\]

According to [Lus90], powers of \( K_i \) and the collection of maps \( \psi : \Phi^+ \to \mathbb{Z}_{\geq 0} \), together with the braid group action on the quantum group defined therein determine a PBW basis of \( U_q(\mathfrak{g}) \). Let \( \mathbb{Q}_l \) be the quotient of \( \mathbb{Q}[q, q^{-1}] \) by the ideal generated by the \( l \)-th cyclotomic polynomial. Let \( l_i \) be the order of \( q^{2d_i} \) in \( \mathbb{Q}_l \). For each \( \alpha \in \Phi^+ \) set \( l_\alpha = l_i \) if \( \alpha \) is in the Weyl orbit of \( \alpha_i \in \Delta^+ \). The divided powers elements \( E_\alpha^{(N)} \) and \( F_\alpha^{(N)} \) for \( \alpha \in \Phi^+ \) and \( 0 \leq N \leq l_\alpha - 1 \), together with \( K_i^{\pm} \) for \( 1 \leq i \leq n \) generate a \( \mathbb{Q}_l \)-subalgebra \( U_q^{\text{div}}(\mathfrak{g}) \subseteq U_q^{\text{div}}(\mathfrak{g}) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Q}_l \). As elements of \( U_q^{\text{div}}(\mathfrak{g}) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Q}_l \),

\[
E_\alpha^{l_\alpha} = [l_\alpha]_{d_\alpha}! E_\alpha^{(l_\alpha)} = 0 \quad \text{and} \quad F_\alpha^{l_\alpha} = [l_\alpha]_{d_\alpha}! F_\alpha^{(l_\alpha)} = 0. \quad (29)
\]

**Definition 2.3.** The *restricted quantum group* \( \overline{U}_q(\mathfrak{g}) \) is the \( \mathbb{Q}_l \)-algebra generated by \( E_i, F_i \), and \( K_i^{\pm} \) inside \( U_q^{\text{div}}(\mathfrak{g}) \). The Hopf algebra structure on \( \overline{U}_q(\mathfrak{g}) \) is inherited from the one carried by \( U_q(\mathfrak{g}) \).

**Remark 2.4.** In contrast to the small quantum group \( u_q(\mathfrak{g}) \) described in [Lus90], the restricted quantum group \( \overline{U}_q(\mathfrak{g}) \) considered here is infinite dimensional.
Lemma 2.5. Let $k \in \mathbb{Z}^n$ and $K^k = \prod_{i=1}^{n} K_i^{k_i}$ be a basis vector of the Cartan torus. Then $K^k$ is central if and only if
\[(d_iA_{ij})k \in (l\mathbb{Z})^n.\]

Proof. Recall that $(d_iA_{ij})$ is the symmetrized Cartan matrix of $g$. Since
\[K^kE_j = q^{\sum_l(i_dA_{ij})}E_jK^k = q^{(d_iA_{ij})k_j}E_jK^k,\]
the result follows. \qed

In this paper, we study the representations of the restricted quantum group where $q = \zeta$ is a primitive fourth root of unity. We then denote this quantum group by $\overline{U}_\zeta(g)$. Discussion for the unrolled restricted quantum group can be found in Appendix B.

Recall that the symmetrizing vector $d \in \{1, 2, 3\}^n$ in each type is given by:
\[
A_n, D_n, E_{6,7,8} : \quad d = [1 \ldots 1] \quad (30)
\]
\[
B_n : \quad d = [2 \ldots 2 1] \quad (31)
\]
\[
C_n : \quad d = [1 \ldots 1 2] \quad (32)
\]
\[
F_4 : \quad d = [2 2 1 1] \quad (33)
\]
\[
G_2 : \quad d = [1 3] \quad (34)
\]
and that $l_i$ is the order of $q^{2d_i}$ in $\mathbb{Q}_l$. At a fourth root of unity, we have $l = 4$. If $d_i = 2$, then $l_i = 1$ and implies $E_\alpha = 0$. Otherwise, $l_i = 2$ and
\[
E_\alpha^2 = 0 \quad \text{and} \quad F_\alpha^2 = 0. \quad (35)
\]
Moreover, the Serre relations found in equations (24) and (25) reduce to “far commutativity,” as in equations (3) and (4) below.

Definition 2.6. A root vector for which $d_i = 2$ is called negligible. The collection of negligible positive roots is denoted by $\Phi_0^+ \subset \Phi^+$. We set $\Delta_0^+ = \Delta^+ \cap \Phi_0^+$, $\Phi^+ = \Phi^+ \setminus \Phi_0^+$, and $\overline{\Delta}^+ = \Delta^+ \setminus \Delta_0^+$. In addition, $\Phi^+$ is equipped with some ordering $<_{br}$ according the braid group action mentioned above.

We refer the reader to [Lus90] for more details on $<_{br}$. In the $\mathfrak{sl}_3$ case, we have the ordering
\[
\alpha_1 <_{br} \alpha_3 <_{br} \alpha_2. \quad (36)
\]

Proposition 1.1. The restricted quantum group $\overline{U}_\zeta(g)$ is the $\mathbb{Q}_4$-algebra generated by $E_\alpha$, $F_\alpha$, for $\alpha \in \overline{\Delta}^+$, and $K_j$ for $1 \leq j \leq n$ with relations:
\[
K_jK_i^{-1} = K_i^{-1}K_j = 1, \quad [E_\alpha, F_\alpha] = \delta_{ij} [K_i]_{d_i}, \quad (1)
\]
\[
K_jE_\alpha = q^{d_iA_{ij}}E_\alpha K_j, \quad K_jF_\alpha = q^{-d_iA_{ij}}F_\alpha K_j, \quad (2)
\]
\[
E_\alpha E_\beta = E_\beta E_\alpha \quad \text{for } |i - j| > 1, \quad (3)
\]
\[
F_\alpha F_\beta = F_\beta F_\alpha \quad \text{for } |i - j| > 1, \quad (4)
\]
\[
E_\alpha^2 = F_\alpha^2 = 0 \quad \text{for } \alpha \in \overline{\Phi^+}. \quad (5)
\]

The Hopf algebra structure on $\overline{U}_\zeta(g)$ is inherited from $U_q(g)$, and described in equations (26)-(28). We now state a modification of Theorem 8.3 from [Lus90].
Corollary 1.2. The restricted quantum group $\overline{U}_\zeta(g)$ has a PBW basis

$$\{E^\psi F^{\psi'} K^k : \psi, \psi' \in \{0, 1\}^{\Phi^+} \text{ and } k \in \mathbb{Z}^n\},$$

where $E^\psi = \prod_{\alpha \in \Phi^+} E_{\alpha}^{\psi_{\alpha}}, F^\psi = \prod_{\alpha \in \Phi^+} F_{\alpha}^{\psi_{\alpha}}, K^k = \prod_{i=1}^n K_i^{k_i}$, and products are ordered with respect to $<_{br}$.

Remark 2.7. Recall the Cartan matrices for the Lie algebras of types $A_2, B_2, C_3,$ and $F_4$.

$$A_2 : \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad B_2 : \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} \quad C_3 : \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix} \quad F_4 : \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

(37)

(38)

It can be shown that the following isomorphisms hold:

$$\overline{U}_\zeta(b_2) \cong \overline{U}_\zeta(sl_2)[K_2^+]/(K_2 E + EK_2, K_2 F + FK_2)$$

$$\overline{U}_\zeta(f_4) \cong \overline{U}_\zeta(c_3)[K_3^+] \cong \overline{U}_\zeta(sl_3)[K_3^+] \cong \overline{U}_\zeta(sl_3)[K_3^+]/(K_3 E_2 + E_2 K_3, K_3 F_2 + F_2 K_3).$$

(39)

(40)

Consider the subalgebras of $\overline{U}_\zeta(g)$:

$$U^0 = \langle K_i^+ : 1 \leq i \leq n \rangle, \quad U^+ = \langle E_{\alpha} : \alpha \in \Phi^+ \rangle, \quad \text{and} \quad U^- = \langle F_{\alpha} : \alpha \in \Phi^+ \rangle.$$  

(41)

We denote the Borel subalgebra by $B = U^+ \otimes U^0$. It follows that $\overline{U}_\zeta(g) \cong U^- \otimes B$.

We now define the representation $V(\mathfrak{t})$ as a Verma module over $U_q(g)$ at a primitive $l$-th root of unity. Note that the group of characters $P$ on $U^0$ is isomorphic to $(\mathbb{C}^\times)^n$. Each character $\mathfrak{t} = (t_1, \ldots, t_n)$ is determined by the images $t_i$ of $K_i$ in $\mathbb{C}^\times$. The character $\mathfrak{t}$ extends to a character $\gamma(\mathfrak{t}) : B \to \mathbb{C}$ by

$$\gamma(\mathfrak{t})(K_i) = t_i, \quad \gamma(\mathfrak{t})(E_i) = 0.$$ 

(42)

Definition 2.8. Let $\gamma(\mathfrak{t}) : B \to \mathbb{C}$ be a character as in (42). Let $V_\mathfrak{t} = \langle v_0 \rangle$ be the 1-dimensional left $B$-module determined by $\gamma(\mathfrak{t})$, i.e. for $b \in B$, $bv_0 = \gamma(\mathfrak{t})(b)v_0$. We define the representation $V(\mathfrak{t})$ to be the induced module

$$V(\mathfrak{t}) = \text{Ind}^B_B U_q(g)(V_\mathfrak{t}) = \overline{U}_q(g) \otimes_B V_\mathfrak{t}.$$ 

(43)

See Appendix C for more details on induced modules.

Remark 2.9. Note that $V(\mathfrak{t})$ and $U^-$ are isomorphic as vector spaces. When $q = \zeta$ a primitive fourth root of unity, $V(\mathfrak{t})$ has dimension $2^{\left| \Phi^+ \right|}$ with basis determined by Corollary 1.2. In types $ADEG$, $\overline{\Phi^+} = \Phi^+$ and so $V(\mathfrak{t})$ has dimension $2^{\left| \Phi^+ \right|}$.

In addition to assuming $q = \zeta$, we focus on the case $g = sl_3$. We use $\overline{U}$ to denote the restricted quantum group $\overline{U}_\zeta(sl_3)$. The simple roots of $\overline{U}$ are $\alpha_1$ and $\alpha_2$, there is a single non-simple root $\alpha_3$, and $d_i = 1$ for $1 \leq i \leq n$. We define $E_3$ and $F_3$ using the braid group action described in [Lus90] as

$$E_3 = -(E_1 E_2 + \zeta E_2 E_1) \quad \text{and} \quad F_3 = -(F_2 F_1 - \zeta F_1 F_2).$$ 

(44)
Observe that $E_3^2 = 0$ implies $E_1 E_2 E_1 E_2 = E_2 E_1 E_2 E_1$; similarly for the $F$ terms.

For each $\psi = (\psi_1, \psi_2, \psi_3) \in \{0, 1\}^3$, we define

$$F^\psi = F^{\psi_1, \psi_2, \psi_3} = F_{1}^{\psi_1} F_{3}^{\psi_3} F_{2}^{\psi_2}. \quad (45)$$

By Corollary 1.2, we have a basis

$$B = \{1, F_1, F_2, F_1 F_2, F_3, F_1 F_3, F_2 F_3, F_1 F_2 F_3\} \quad (46)$$

of $U^-$, which is equipped with a lexicographical ordering $\prec$. We say that $\psi \prec \psi'$ if for some index $j$, $\psi(j) < \psi'(j)$ and for each $i > j$, $\psi(i) = \psi'(i)$. Moreover, $\prec$ is a total ordering on $\{0, 1\}^3$. Denote by $1 \in \{0, 1\}^3$ the covector which evaluates to one on each $\alpha \in \Phi^+$. The space $\{0, 1\}^3$ is presented in Figure 2 below.

![Figure 2](image_url)

**Figure 2.** The space $\{0, 1\}^3$, which determines a basis for $U^-$. It is visualized as a cube.

Let $\pi$ be the simple root components of a root $\alpha \in \Phi^+$. In particular, $\pi_3 = \pi_1 + \pi_2$. This map induces

$$P : \{0, 1\}^3 \to \mathbb{Z}^{\Delta^+}$$

$$(\psi_1, \psi_2, \psi_3) \mapsto (\psi_1 + \psi_2 + \psi_3)$$

which can be seen as a projection of the cube in Figure 2 into the plane and the deletion of segments associated to adding (001). We use $\mathcal{B}$ to determine a basis of $V(\mathfrak{t})$ by tensoring $v_0$, and introduce the notation $v_\psi = F^\psi v_0$ so that $v_{(000)} = v_0$. In addition, we use $v_h = v_0$ and $v_l = v_1$ to denote “highest” and “lowest” weight vectors. All vector expressions in $V(\mathfrak{t})$ will be expressed using the basis $\mathcal{B}$.

Let $\alpha \in \Phi^+$ and define $\psi^\alpha \in \{0, 1\}^3$ so that $\psi^\alpha(\alpha) = 1$ and is zero otherwise.

**Lemma 2.10.** For each $\alpha \in \Phi^+$ and $\psi \in \{0, 1\}^3$ such that $\psi(\alpha) = 0$, there exist coefficients $a_\psi \in \mathbb{Q}$ such that

$$F^\alpha F^\psi = \sum_{P(\psi') = P(\psi^\alpha) + P(\psi)} a_\psi F^\psi' \neq 0. \quad (49)$$
For each $\alpha \in \Delta^+$ and $\psi \in \{0, 1\}^{\Phi^+}$, there exist coefficients $b_\psi, c_\psi \in \mathbb{Q}_4$ such that
\[ [E_\alpha, F^\psi] = \sum_{P(\psi') = -P(\psi) + P(\psi)} b_{\psi'} F^{\psi'} [K_i] + c_{\psi'} F^{\psi'} [\zeta K_i]. \] (50)

The lemma follows from direct computation, see [Lus90] for the precise relations in (49).

**Remark 2.11.** The actions of $K_1$ and $K_2$ break $V(t)$ into weight spaces. Recall that multiplication in $\mathcal{P}$ is entrywise. To each $\psi \in \{0, 1\}^{\Phi^+}$ we associate $\sigma_\psi \in \mathcal{P}$ so that the equality $K_i v_\psi = \sigma_\psi t(K_i) v_\psi$ holds. More precisely,
\[ \sigma_\psi t = \left( \zeta^{2\psi(1)+\psi(2)+3\psi(3)} t_1, \zeta^{\psi(1)+2\psi(2)+3\psi(3)} t_2 \right). \] (51)

Thus, the $V(t)$ weight spaces are labeled by $\{\sigma_\psi t : \psi \in \{0, 1\}^{\Phi^+}\}$.

**Definition 2.12.** Let $\Sigma$ be the weights of $V(1) = V(1,1)$ with multiplicity, i.e. $\Sigma \subseteq \mathcal{P}$ is the list of characters
\[ [(1, 1), (-1, \zeta), (\zeta, -1), (-\zeta, -\zeta), (-\zeta, -\zeta), (\zeta, 1), (1, \zeta), (-1, -1)]. \] (52)

Without explicit reference to $\psi$, we may also write
\[ K_i v_\sigma = \sigma t(K_i) v_\sigma \quad \text{and} \quad F^\sigma v_0 = v_\sigma \] (53)
for $\sigma \in \Sigma$. Using the weight space data above, we describe the remaining actions of the induced $\mathcal{U}$-module. Each $F^\psi$ acts on the basis of $V(t)$ as it would on $\mathcal{B}$ by left multiplication. In particular, the action of $F_1$ and $F_2$ is independent of $t$. Recall the definition of $[x]$ from (21). The actions of $E_1$ and $E_2$ are defined to be zero on $v_0$, but the commutation relations $[E_i, F_j] = \delta_{ij} [K_i]$ determine a non-trivial action on the other basis vectors. These non-zero actions are given explicitly in Table 1. Note that the representation $V(t)$ has coefficients in $\mathbb{Q}_4[t_1^\pm, t_2^\pm]$.

We describe the representation in terms of basis vectors and maps between them by presenting the action of this module on weight spaces, as seen in Figure 4. Each solid vertex indicates a one dimensional weight space of $V(t)$, and the “dotted” vertex indicates the two dimensional weight space spanned by $F^{(110)} v_h$ and $F^{(001)} v_h$. An edge is drawn between vertices if the action of either $E_1$ or $E_2$ is nonzero between the associated weight
spaces. We do not assign edges to matrix elements of \( F_1 \) and \( F_2 \), since they are independent of \( t \). However, for non-generic choices of the parameter \( t \), edges are deleted from the graph because matrix elements of \( E_1 \) and \( E_2 \) vanish. We orient the graph so that \( F_1 \) acts downward left and \( F_2 \) acts downward right at each vertex. Each \( E_i \) acts in the opposite direction of the corresponding \( F_i \).

| Table 1. Nonzero actions of \( E_1 \) and \( E_2 \) in \( V(t) \) expressed using the basis determined by \( \mathcal{B} \). |
|---------------------------------|
| \( E_1 F^{(100)} v_h = |t_1| F^{(000)} v_h \) |
| \( E_1 F^{(110)} v_h = |t_1| F^{(010)} v_h \) |
| \( E_1 F^{(001)} v_h = \zeta t_1 F^{(010)} v_h \) |
| \( E_1 F^{(101)} v_h = \zeta t_1 F^{(110)} v_h - |t_1| F^{(001)} v_h \) |
| \( E_1 F^{(111)} v_h = |t_1| F^{(111)} v_h \) |
| \( E_2 F^{(010)} v_h = |t_2| F^{(000)} v_h \) |
| \( E_2 F^{(110)} v_h = |t_2| F^{(100)} v_h \) |
| \( E_2 F^{(001)} v_h = t_2^{-1} F^{(100)} v_h \) |
| \( E_2 F^{(011)} v_h = t_2^{-1} F^{(110)} v_h + |t_2| F^{(001)} v_h \) |
| \( E_2 F^{(111)} v_h = |t_2| F^{(101)} v_h \) |

**Figure 4.** The action of \( \mathcal{U} \) on the weight spaces of \( V(t) \).

### 3. Properties of the Representations \( V(t) \)

Note that when either \( t_1 \) or \( t_2 \) is a fourth root of unity, terms may vanish from the expressions describing the action of \( E_1 \) and \( E_2 \) on \( V(t) \), seen in Table 1. These vanishings are related to the reducibility of the representation. In Proposition 3.8, we show that \( V(t) \) is reducible if and only if \( t \) belongs to the algebraic set \( \mathcal{R} \). We also prove that every representation \( V(t) \) is indecomposable and each \( t \in \mathcal{P} \) determines \( V(t) \) up to isomorphism.

**Lemma 3.1.** For any nonzero vector \( u \in U^- \), there exists \( \psi' \in \{0, 1\}^{\Phi^+} \) such that \( F^{\psi'} u \) is a nonzero multiple of \( F^1 \). Similarly for \( U^+ \).

**Proof.** Fix \( u \in U^- \). We express \( u \) using the PBW basis,

\[
u = \sum_{\psi \in \{0, 1\}^{\Phi^+}} u_{\psi} F^{\psi}.
\]
Let $\psi$ be the least $\psi \in \{0, 1\}^{\Phi^+}$ with respect to $<$ such that $u_\psi$ is nonzero. Let $\overline{\psi} = 1 - \psi$. It follows from Lemma 2.10 that $F\overline{\psi}F^\psi \in \langle F^1 \rangle$ is nonzero. For $\psi' > \psi$, either $P(\overline{\psi}) + P(\psi')$ has a component with value greater than two, or both $\overline{\psi}(\alpha_3)$ and $\psi'(\alpha_3)$ are nonzero. Therefore, $F\overline{\psi}F^\psi' = 0$ for $\psi' > \psi$. Since $\psi$ is associated with the smallest nonzero component of $u$, we have

$$F\overline{\psi}u = F\overline{\psi}\left(\sum_{\psi \geq \psi} u_\psi F^\psi\right) = u_{\overline{\psi}} F\overline{\psi}F^\psi \in \langle F^{\langle 111 \rangle} \rangle$$

is nonzero. □

**Proposition 3.2.** For all $\mathfrak{t} \in \mathcal{P}$, $V(\mathfrak{t})$ is an indecomposable representation.

**Proof.** Suppose that $V(\mathfrak{t})$ admits a direct sum decomposition $W_1 \oplus W_2$ with respect to the $U$ action. Fix non-zero vectors $w_1 \in W_1$ and $w_2 \in W_2$. By Lemma 3.1, there exists $\overline{\psi} \in \{0, 1\}^{\Phi^+}$ so that $F\overline{\psi}w_1$ is a nonzero multiple of $v_1$. Thus, $v_1 \in W_1$. The same argument applies to $w_2$, and so $v_1 \in W_2$. Hence, $\langle v_1 \rangle$ is a subspace in $W_1 \cap W_2$, which contradicts the existence of a direct sum decomposition. Thus, $V(\mathfrak{t})$ is indecomposable. □

**Remark 3.3.** For every pair of distinct characters $\mathfrak{t}, \mathfrak{s} \in \mathcal{P}$, $V(\mathfrak{t}) \not\cong V(\mathfrak{s})$. This is clear since the highest weight determines the representation. Thus, the representations $V(\mathfrak{t})$ form an infinite family of representation classes.

**Definition 3.4.** The irreducibility vector of a representation $V(\mathfrak{t})$ is the vector

$$\Omega = E_1^1 v_1 = E_1^1 F_1 v_h.$$

We now work towards characterizing irreducibility by showing that $V(\mathfrak{t})$ is reducible if and only if $\Omega$ vanishes.

**Remark 3.5.** If $\Omega$ is non-zero, then it is a highest weight vector.

**Proposition 3.6.** We have the equality

$$\Omega = -\zeta \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} v_h.$$

**Proof.** We compute $\Omega$ using equations (112) and (117):

$$E_1 E_3 E_2 F_1 F_3 F_2 v_h = E_3 E_2 [K_1] F_3 F_2 v_h = -\zeta \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} v_h. \boxdot$$

We see that $\Omega$ vanishes on the following subsets of $\mathcal{P}$:

$$\mathcal{X}_1 = \{ \mathfrak{t} \in \mathcal{P} : t_1^2 = 1 \}, \quad \mathcal{X}_2 = \{ \mathfrak{t} \in \mathcal{P} : t_2^2 = 1 \}, \quad \mathcal{H} = \{ \mathfrak{t} \in \mathcal{P} : (t_1 t_2)^2 = -1 \}.$$ (54)

Let $\mathcal{R}$ be the union $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{H}$.

**Lemma 3.7.** The collection of vectors

$$\mathcal{E} = \{ E^\psi v_1 : \psi \in \Phi^+ \}$$

forms a basis for $V(\mathfrak{t})$ if and only if $\Omega \neq 0$.

**Proof.** The last vector in $\mathcal{E}$, with respect to $<$, is $\Omega$. Thus, $\Omega = 0$ implies $\mathcal{E}$ is not a basis.
Suppose now that $\mathcal{E}$ does not form a basis. Then there exists a linear dependence
\[ \sum_{\psi \geq \underline{\psi}} c_{\psi} E^\psi v_l = 0 \]
for some constants $c_{\psi} \in \mathbb{Q}[t_1^+, t_2^+, \zeta]$, with $c_{\underline{\psi}}$ nonzero. By Lemma 3.1, there exists $\overline{\psi} \in \{0, 1\}^{\Phi^+}$ and $c \neq 0$ such that $c E^{\overline{\psi}} E^\psi = E^1$ and $c E^{\overline{\psi}} E^\psi = 0$ for each $\psi' > \underline{\psi}$. Then
\[ \Omega = c E^{\overline{\psi}} E^\psi v_l = \frac{c E^{\overline{\psi}}}{c_{\underline{\psi}}} \left( \sum_{\psi \geq \underline{\psi}} c_{\psi} E^\psi v_l \right) = 0. \]
Thus, proving the claim.

**Proposition 3.8.** The following are equivalent:
- $V(t)$ is irreducible
- $\Omega \neq 0$
- $t \not\in \mathbb{R}$.

**Proof.** Irreducibility holds if and only if any nonzero $v \in V(t)$ is a cyclic vector for the module. That is to say, the action of $U$ on $v$ generates $V(t)$. Fix any $v \neq 0$. By Lemma 3.1, we may assume $v = v_l$. Raising this lowest weight vector $v_l$ by each $E^\psi$, we obtain the vectors of $\mathcal{E}$, which we claim to be a basis of $V(t)$. Equivalently, by Lemma 3.7, we check that $\Omega$ is nonzero. From Proposition 3.6, $\Omega = 0$ exactly when $t \in \mathbb{R}$. It follows that $U$ acting on $v$ generates $V(t)$ if and only if $t \not\in \mathbb{R}$. Since this holds for every non-zero $v \in V(t)$, we have proven the claim.

**Remark 3.9.** By Lemma 2.5, the central Cartan elements of $U$ are generated by $K_1^1$ and $K_2^1$. Let $C$ be the category of finite dimensional representations on which each $K_i$ acts diagonally. Let $a \in \mathcal{P}$, and let $C_a \subseteq C$ be the subcategory on which $K_i^1 = a(K_i)1$. Then
\[ C = \bigoplus_{a \in \mathcal{P}} C_a. \] (55)

Since $K_1$ and $K_2$ are group-like, we have
\[ C_a \otimes C_b \subseteq C_{ab} \] (56)
for every $a, b \in \mathcal{P}$. The category $C_a$ contains the representations \{ $V(t) : t^4 = a$ \}. Therefore, each $C_a$ with $a_1 = 1$, $a_2 = 1$, or $a_1 a_2 = 1$ is non-semisimple. For each $a \in \mathcal{P}$,
\[ C_1 \otimes C_a \subseteq C_a, \] (57)
and so each $C_a$ is non-semisimple.

**Remark 3.10.** Depending on how irreducibility fails, different arrows vanish from Figure 4. The basic cases can be seen in Figure 5. In this figure, the segments which are not present in (a) indicate that the action of $E_1$ is zero on the weight spaces of $F_1 v_h$ and $F^4 v_h$. Similarly in (b). In (c) we assume $t \in \mathcal{H}$. In this case, the image of $E_1$ and $E_2$ coincide in the subspace spanned by $F^{(110)} v_h$ and $F^{(001)} v_h$, thus disconnecting the graph.

By considering the minimal subrepresentation containing $v_l$, we see that each set $\mathcal{X}_1, \mathcal{X}_2$ and $\mathcal{H}$ corresponds to a distinct family of 4-dimensional irreducible subrepresentations: left zig-zag, right zig-zag, and diamond. If $t \in \mathcal{P}$ belongs to two such sets, then these
correspond to a family of 1-dimensional subrepresentations, or one of two 3-dimensional subrepresentations. This correspondence is given by associating a highest weight vector to the varieties containing \( \mathbf{t} \), as shown in Figure 6. A highest weight in the subspace \( \langle F^{(110)} v_h, F^{(001)} v_h \rangle \) is associated to the algebraic sets \( \mathcal{H} \), \( \mathcal{X}_1 \cap \mathcal{H} \), and \( \mathcal{X}_2 \cap \mathcal{H} \). The dashed lines partition the varieties by dimension of their associated irreducible representation: 1, 3, 4, or 8.

**Figure 6.** The correspondence between subsets of \( \mathcal{P} \) and the highest weight vector generated by \( F^1 v_h \) in \( V(\mathbf{t}) \).

4. **Semisimple Tensor Product Representations**

In this section, we introduce the notion of non-degeneracy to characterize the decomposability of representations \( V(\mathbf{t}) \otimes V(\mathbf{s}) \). Given a tensor product representation, we may express it as a direct sum provided that the representations \( V(\sigma \mathbf{t} \mathbf{s}) \) are irreducible for each choice of \( \sigma \in \Sigma \). Recall that \( \Sigma \) consists of the weights of \( V(1) \), as given in Definition 2.12. The goal of this section is to prove Theorem 1.4, which we recall following the definition of non-degeneracy.
Definition 4.1. A pair \((t, s) \in P^2\) is called non-degenerate if \(\sigma t s\) is irreducible for all \(\sigma \in \Sigma\). We call \(V(t) \otimes V(s)\) a non-degenerate representation if \((t, s)\) non-degenerate.

Theorem 1.4 (Tensor Product Decomposition). Let \((t, s)\) be a non-degenerate pair. The tensor product \(V(t) \otimes V(s)\) decomposes as a direct sum of irreducibles according to the formula

\[
V(t) \otimes V(s) \cong \bigoplus_{\sigma \in \Sigma} V(\sigma t s).
\]

Our proof of the theorem relies on finding highest weight vectors in the tensor product and looking at their orbits under the \(\bar{U}\) action. Each of these cyclic subspaces is a copy of an induced representation and is identified by the weight of its generating vector. These vectors can be easily described in \(\text{Ind}_{\bar{B}}^{\bar{U}} \left( V_{\bar{L}} \otimes \text{Ind}_{\bar{B}}^{\bar{U}} (V_{\bar{s}}) \right)\), which by Proposition C.4 is isomorphic to \(V(t) \otimes V(s)\). The desired result follows from this intermediate isomorphism. Denote \(V(t) \hat{\otimes} V(s) = \text{Ind}_{\bar{B}}^{\bar{U}} \left( V_{\bar{L}} \otimes \text{Ind}_{\bar{B}}^{\bar{U}} (V_{\bar{s}}) \right)\).

Recall that \(\gamma_{\bar{t}}\) is the character which determines the action of \(B\) on \(V_{\bar{t}}\). For \(a \in \bar{U}\), let \(\Delta(a) = a' \otimes a''\) be the coproduct of \(a\) with the implicit summation notation. By definition,

\[
V(t) \hat{\otimes} V(s) = \bar{U} \otimes_{B} (V_{\bar{L}} \otimes (\bar{U} \otimes_{B} V_{\bar{s}})) \cong (\bar{U} \otimes (V_{\bar{L}} \otimes (\bar{U} \otimes V_{\bar{s})))) / Q \cong (\bar{U} \otimes \bar{U}) / Q'
\]

with

\[
Q = \langle a_1 b_1 \otimes (v_h \otimes (a_2 b_2 \otimes v_h)) - a_1 \otimes (b'_1. v_h \otimes (b''_1 a_2 \otimes b_2 . v_h)) : a_i \in \bar{U}, b_i \in B \rangle
\]

\[
\cong \langle a_1 (b_1 v_h \otimes a_2 b_2 v_h) - \gamma_{\bar{t}}(b'_1)\gamma_{\bar{s}}(b_2) a_1 v_h \otimes b''_1 a_2 v_h : a_i \in \bar{U}, b_i \in B \rangle
\]

\[
= Q'
\]

and the above isomorphisms suppress tensoring of 1-dimensional vector spaces \(V_{\bar{t}}\) and \(V_{\bar{s}}\). We include \(v_h\) in the notation for vectors in \(V(t) \hat{\otimes} V(s)\) to avoid confusion with the algebra \(\bar{U} \otimes \bar{U}\) i.e. a vector \(v = a_1 \otimes (v_h \otimes (a_2 \otimes v_h)) \in V(t) \hat{\otimes} V(s)\) will be denoted by \(a_1 (v_h \hat{\otimes} a_2 v_h)\) under the identification in (58). The action of \(\bar{U}\) is by left multiplication on the first tensor factor, which may then be simplified. An example of the action is provided below.

Example 4.2. The action of \(F_1 E_1\) on \(v_h \hat{\otimes} v_{(110)}\) is given as follows:

\[
F_1 E_1 (v_h \hat{\otimes} v_{(110)}) = F_1 (\gamma_{\bar{t}}(E_1) v_h \hat{\otimes} K_1 v_{(110)} + v_h \hat{\otimes} E_1 v_{(110)})
\]

\[
= v_{(100)} \hat{\otimes} [\zeta s_1] v_{(010)}.
\]

We fix a basis

\[
\mathcal{B}' = \{ F^\psi v_0 \hat{\otimes} F^{\psi'} v_0 : \psi, \psi' \in \{0, 1\}^{\Phi^+} \} = \{ v_\psi \hat{\otimes} v_{\psi'} : \psi, \psi' \in \{0, 1\}^{\Phi^+} \}
\]

doing \(V(t) \hat{\otimes} V(s)\) by the taking the basis \(\mathcal{B}\), from (46), in each tensor factor. We extend the notion of Lemma 2.10 to \(V(t) \hat{\otimes} V(s)\).

Lemma 4.3. For each \(\alpha \in \Phi^+\) and \(\psi_1, \psi_2 \in \{0, 1\}^{\Phi^+}\) such that \(\psi_1(\alpha)\) is nonzero, there exist coefficients \(a_\psi \in \mathbb{Q}_4\) such that

\[
F_\alpha F^{\psi_1} v_h \hat{\otimes} F^{\psi_2} v_h = \sum_{P(\psi') = P(\psi^\alpha) + P(\psi_1)} a_{\psi'} F^{\psi'} v_h \hat{\otimes} F^{\psi_2} v_h \neq 0.
\]
For each $\alpha \in \Delta^+$ and $\psi \in \{0, 1\}^{\Phi^+}$, there exist coefficients $b_\psi, c_\psi \in \mathbb{Q}_4$ such that

$$E_\alpha F_\psi v_h \otimes F_\psi^2 v_h = \sum_{P(\psi') = -P(\psi^\alpha) + P(\psi_2)} c_\psi F_\psi v_h \otimes F_\psi^2 v_h + \sum_{P(\psi') = -P(\psi^\alpha) + P(\psi_1)} c_\psi F_\psi v_h \otimes F_\psi^2 v_h.$$  \hspace{1cm} (66)

Equation (66) follows from the intermediate step

$$E_\alpha F_\psi v_h \otimes F_\psi^2 v_h = F_\psi E_\alpha (v_h \otimes F_\psi^2) + \sum_{P(\psi') = -P(\psi^\alpha) + P(\psi_1)} c_\psi F_\psi v_h \otimes F_\psi^2 v_h.$$  \hspace{1cm} (67)

Lemma 4.4. Let $(t, s) \in \mathcal{P}$ be a non-degenerate pair and $\sigma \in \Sigma$. Then the subspace

$$V_\sigma = \langle F^\psi \Omega \otimes v_\sigma : \psi \in \{0, 1\}^{\Phi^+} \rangle \subseteq V(t) \otimes V(s)$$  \hspace{1cm} (68)

and $V(\sigma ts)$ are isomorphic as $U$-modules.

Proof. Suppose $(\sigma_1, \sigma_2) = \sigma \in \Sigma$. Recall $v_\sigma$ as in (53), and $\Omega \otimes v_\sigma = E^1 (v_1 \otimes F_\sigma v_h)$. By Proposition 3.6, the $v_h \otimes F_\sigma^2 v_h$ component of $\Omega \otimes F_\sigma^2 v_h$, as expressed in the basis $\mathcal{B}'$, is

$$-\zeta [\sigma_1 t_1 s_1] [\sigma_2 t_2 s_2] [\zeta \sigma_1 \sigma_2 t_1 t_2 s_1 s_2] v_h \otimes F_\sigma^2 v_h$$

and all other nonzero components of $\Omega \otimes F_\sigma^2 v_h$ have some non-trivial $F_\psi$ in the first tensor factor. Having assumed non-degeneracy, $\Omega \otimes F_\sigma^2 v_h$ is non-zero. Hence, $\Omega \otimes F_\sigma^2 v_h$ is a highest weight vector of weight $\sigma ts$ and $V_\sigma$ is an irreducible subrepresentation of $V(t) \otimes V(s)$. Thus, by the irreducibility of $V(\sigma ts)$, the map which sends $v_h \in V(\sigma ts)$ to $\Omega \otimes F_\sigma^2 v_h \in V_\sigma$ determines an isomorphism $V(\sigma ts) \cong V_\sigma$. \hfill \Box

Lemma 4.5. Let $(t, s)$ be a non-degenerate pair. Then $V(t) \otimes V(s)$ is isomorphic to the direct sum $\bigoplus_{\sigma \in \Sigma} V_\sigma$.

Proof. Observe that the non-degeneracy assumption on $(t, s)$ implies the irreducibility of each $V_\sigma$. Every $V_\sigma = V_\sigma^\psi$ includes into $V(t) \otimes V(s)$ as the subspace generated by $U^-$ acting on the highest weight vector $\Omega \otimes F_\sigma^\psi v_h$. Since each $\Omega \otimes F_\sigma^\psi v_h$ is distinct and each $V_\sigma^\psi$ is irreducible, $V_\sigma^\psi \cap V_\sigma^\psi' = \langle 0 \rangle$ for $\psi \neq \psi'$. Hence, $\bigoplus_{\sigma \in \Sigma} V_\sigma$ injects into $V(t) \otimes V(s)$. By dimensionality, this injection is a surjection and, therefore, an isomorphism. \hfill \Box

Let $\Theta$ denote the isomorphism $\bigoplus_{\sigma \in \Sigma} V_\sigma \cong V(t) \otimes V(s)$ described in Lemma 4.5. Using the aforementioned lemmas we prove the first main theorem.

Proof of Theorem 1.4. We construct an intertwiner $\Gamma$ in the following diagram when $(t, s)$ is a non-degenerate tuple.

$$\begin{array}{ccc}
V(t) \otimes V(s) & \xrightarrow{\Psi} & V(t) \otimes V(s) \\
\downarrow \Theta & & \downarrow \Theta \\
\bigoplus_{\sigma \in \Sigma} V(\sigma ts) & \xrightarrow{\Gamma} & \bigoplus_{\sigma \in \Sigma} V(\sigma ts)
\end{array}$$

We see that $\Gamma = \Theta \circ \Psi$ is given by a composition of isomorphisms. The above lemmas establish that $\Theta$ is an isomorphism for non-degenerate tuples. Moreover, $\Psi$ is an
isomorphism by Proposition C.4, which is independent of \( t \) and \( s \). This proves the theorem.

**Remark 4.6.** We see that a tensor product of indecomposable, but reducible, representations may decompose into a direct sum of irreducibles. For example, \( V(t_1, 1) \otimes V(1, s_2) \) is a non-degenerate tensor product representation for generic \( t_1 \) and \( s_2 \).

Observe that the isomorphism class of a non-degenerate tensor product depends only on the product \( ts \). We define an action of \( \mathbb{P} \) on \( \mathbb{P}^2 \) which preserves the product \( ts \) as follows. Let \( \lambda, t, s \in \mathbb{P} \) and set

\[
\lambda \cdot (t, s) = (\lambda t, \lambda^{-1} s).
\]

**Corollary 4.7.** Let \( (t, s) \) be a non-degenerate tuple. For any \( \lambda \in \mathbb{P} \) such that \( \lambda \cdot (t, s) \) is also non-degenerate, then

\[
V(t) \otimes V(s) \cong V(\lambda t) \otimes V(\lambda^{-1} s).
\]

**5. Cyclicity in the Generic Case**

We begin this section by defining a homogeneous cyclic representation. One goal for the remainder of this paper is to describe which representations \( V(t_1) \otimes V(s_2) \) are homogeneous cyclic and extend Corollary 4.7 to those representations. The second goal is to give a complete description of when equation (70) holds. In this section, we establish the methods used to find homogeneous cyclic vectors. By noting how a representation fails to be cyclically generated, we sort tensor product representations into families on which equation (70) holds for some \( \lambda \in \mathbb{P} \).

**Definition 5.1.** A homogeneous cyclic vector for a \( U \)-module \( M \) is a weight vector \( \tilde{v} \in M \) such that

\[
M = \langle E^\psi F^{\psi'} \tilde{v} : \psi, \psi' \in \{0, 1\}^{\Phi^+} \rangle.
\]

We say that \( M \) is generated by \( \tilde{v} \) and call \( M \) a homogeneous cyclic representation.

Let \( \tilde{w} \) be the generator of a 1-dimensional \( U^0 \)-module such that \( K_i \tilde{w} = \tau(K_i) \tilde{w} \) for some \( \tau = (\tau_1, \tau_2) \in \mathbb{P} \). Define \( W_\tau \) to be the induced representation \( U \otimes_{U^0} \tilde{w} \). We suppress the \( U^0 \) subscript on the tensor product when it is clear that we are referring to vectors in \( W_\tau \).

**Remark 5.2.** We find that \( W_\tau \) is 64-dimensional as a vector space with PBW basis

\[
\{ F^\psi E^{\psi'} \otimes \tilde{w} : \psi, \psi' \in \{0, 1\}^{\Phi^+} \} = (1 \otimes \tilde{w}, F^{(100)} \otimes \tilde{w}, ..., F^1 E^1 \otimes \tilde{w}).
\]

Hence, \( W_\tau \) is a homogeneous cyclic representation with generator \( 1 \otimes \tilde{w} \).

**Lemma 5.3.** Let \( M \) be a homogeneous cyclic representation with generator \( \tilde{v} \in M \) for which

\[
\{ E^\psi F^{\psi'} \tilde{v} : \psi, \psi' \in \{0, 1\}^{\Phi^+} \}
\]

is a basis, i.e. \( M \) has dimension 64. Then \( M \) is isomorphic to \( W_\tau \) for some \( \tau \in \mathbb{P} \).

**Proof.** Let \( \tau \in \mathbb{P} \) such that \( K_i \tilde{v} = \tau(K_i) \tilde{v} \). The map which sends \( 1 \otimes \tilde{w} \) to \( \tilde{v} \) determines an isomorphism between \( W_\tau \) and \( M \).

\( \square \)
Corollary 5.4. If $V(t) \otimes V(s)$ and $V(w) \otimes V(z)$ are both homogeneous cyclic representations, then they are isomorphic if and only if $ts = wz$.

We introduce the notation $wt(\mathbf{A})$ to denote the $\mathbf{A}$ weight space of $V(t) \hat{\otimes} V(s)$. Recall the basis $B'$ of $V(t) \hat{\otimes} V(s)$ as in (64), which will be used throughout the remainder of the paper.

Lemma 5.5. A vector $F^{\psi_1}v_h \hat{\otimes} F^{\psi_2}v_h \in V(t) \hat{\otimes} V(s)$ belongs to $wt(\mathbf{A})$ if and only if

$$wt(\mathbf{A}) = \langle F^{\psi_1}v_h \hat{\otimes} F^{\psi_2}v_h \in V(t) \hat{\otimes} V(s) : P(\psi) + P(\psi') = P(\psi_1) + P(\psi_2) \rangle.$$  

(73)

Proof. Since $K_1$ and $K_2$ are group-like, the weight only depends on $P(\psi_1) + P(\psi_2)$ and can be determined from (51). \qed

Lemma 5.6. A homogeneous cyclic vector $\tilde{v}$ for $V(t) \hat{\otimes} V(s)$, if one exists, must belong to the $-ts$ weight space and have a nonzero $v_h \hat{\otimes} v_l$ component.

Proof. Suppose $V(t) \hat{\otimes} V(s)$ admits a homogeneous cyclic vector $\tilde{v} \in wt(\mathbf{A})$ for some $\mathbf{A} \in \mathcal{P}$. Since $E^1 \tilde{v} \neq 0$, there is a nonzero component $F^{\psi_1}v_h \hat{\otimes} F^{\psi_2}v_h$ of $\tilde{v}$ such that

$$E^1(F^{\psi_1}v_h \hat{\otimes} F^{\psi_2}v_h) \neq 0.$$  

In particular, by Lemma 4.3,

$$P(\psi_1) + P(\psi_2) \geq (22).$$

On the other hand, $F^2 \tilde{v} \neq 0$ and so $\tilde{v}$ has a nonzero component $v_h \hat{\otimes} F^{\psi_3}v_h$. By Lemma 5.5,

$$P(\psi_3) = P(\psi_1) + P(\psi_2) \geq (22).$$

Thus, $\psi_3 = 1$, $\lambda = -ts$, and $v_h \hat{\otimes} v_l$ is a nonzero component of $\tilde{v}$. \qed

Let $\pi$ denote the projection in $V(t) \hat{\otimes} V(s)$ to the subspace $\langle v_h \hat{\otimes} F^{\psi}v_h : \psi \in \{0,1\}^\Phi^+ \rangle$ and $\pi_\psi$ the projection to $\langle v_h \hat{\otimes} F^{\psi}v_h \rangle$, both taken with respect to the basis $B'$. Let $d_\psi$ denote the scalar part of the projection $\pi_\psi$. Then, for every $v \in V(t) \hat{\otimes} V(s)$,

$$\pi_\psi(v) = d_\psi(v)v_h \hat{\otimes} F^{\psi}v_h.$$  

(74)

Definition 5.7. Let $\psi \in \{0,1\}^\Phi^+$. A vector $v \in wt(-ts)$ is effective at level $\psi$ if there exists $x \in U$ such that $\pi_\psi(xv) \neq 0$, otherwise $v$ is not effective at level $\psi$.

Informally, a $-ts$ weight vector is effective at level $\psi$ if any vector in its image under $U^+$ has a nonzero $v_h \hat{\otimes} F^{\psi}v_h$ component. A vector which is effective for each $\psi$ is a cyclic vector.

Recall the lexicographical ordering on $\{0,1\}^\Phi^+$ given after (46). The following lemma is a consequence of Lemma 4.3.

Lemma 5.8. Let $\psi_1, \psi_2, \psi_3 \in \{0,1\}^\Phi^+$ such that $\psi_2 < \psi_3$. Then $F^{\psi_1}v_h \hat{\otimes} F^{\psi_2}v_h$ is not effective at level $\psi_3$.

Proof. We consider the actions of $U^+$ and $U^-$ on $F^{\psi_1}v_h \hat{\otimes} F^{\psi_2}v_h$. Since $U^+$ does not belong to the Borel subalgebra, the actions of $F_1$ and $F_2$ are only on the first tensor factor. For some coefficients $a_\psi \in \mathbb{Q}_4$, we have

$$F_i(F^{\psi_1}v_h \hat{\otimes} F^{\psi_2}v_h) = \sum_{P(\psi) = P(\psi_1) + P(\psi_2)} a_\psi F^{\psi}v_h \hat{\otimes} F^{\psi_2}v_h.$$
The following are equivalent in Proposition 5.9.

On the other hand, $E_1$ and $E_2$ act according to Lemma 2.10. Thus, there exist coefficients $b_\psi, c_\psi, \in \mathbb{Q}_4[t_1^\pm, t_2^\pm, s_1^\pm, s_2^\pm]$ so that

$$E_i.(F^{\psi_1} v_h \otimes F^{\psi_2} v_h) = \sum_{P(\psi) = P(\psi') + P(\psi'')} b_\psi F^{\psi_1} v_h \otimes F^{\psi_2} v_h + \sum_{P(\psi) = -P(\psi') + P(\psi'')} c_\psi F^{\psi} v_h \otimes F^{\psi_2} v_h.$$  

In either case, each nonzero component of the resulting expression is a vector $F^{\psi} v_h \otimes F^{\psi'} v_h$ with $\psi' \leq \psi_2 < \psi_3$. Thus, $F^{\psi_3} v_h$ cannot occur in the second tensor factor from the action of $U^+$ or $U^-$, and so $F^{\psi_3} v_h \otimes F^{\psi_2} v_h$ is not effective at level $\psi_3$.

The natural guess for a cyclic generator $\tilde{v} \in V(\hat{t}) \otimes V(\hat{s})$ is $v_h \otimes v_l$; however, it may not be the case that it generates the entire module. Indeed, multiplication by elements $F^{\psi} E^{\psi'}$ on $v_h \otimes v_l$ is given by

$$F^{\psi} E^{\psi'}(v_h \otimes v_l) = F^{\psi}(v_h \otimes E^{\psi'} v_l) = (F^{\psi} v_h) \otimes (E^{\psi'} v_l).$$  

(75)

An instance of this can be seen in Example 4.2 and the expression vanishes if $s_1 = \zeta$.

Figure 7 shows the subspace of $V(\hat{t}) \otimes V(\hat{s})$ generated by $v_h \otimes v_l$ under the action of $U^+$, assuming that it is a cyclic vector. To distinguish diagrams for $V(\hat{t}) \otimes V(\hat{s})$ from those of $V(\hat{t})$, each vertex is labeled with a $\otimes$, and the multiplicity two weight space is labeled by $\otimes$. As before, each edge corresponds to a nonzero matrix element of either $E_1$ or $E_2$.

However, in later diagrams, these may depend on the choice of generator $\tilde{v} \in wt(-t\mathbf{s})$. We will assume $\tilde{v}$ is chosen maximally, in the sense that all possible nonzero matrix elements are present in the diagram. Since the action of $F_1$ and $F_2$ is independent of the choice of parameters, we do not include edges corresponding to their action.

**Proposition 5.9.** The following are equivalent in $V(\hat{t}) \otimes V(\hat{s})$:

- $v_h \otimes v_l$ is a homogeneous cyclic vector
- $v_h \otimes v_l$ is effective at level (000)
- $s \notin \mathcal{R}$.

**Proof.** The first two statements are seen to be equivalent by considering

$$\{v_h \otimes v_l, v_h \otimes E^{(100)} v_l, v_h \otimes E^{(010)} v_l, v_h \otimes E^{(110)} v_l, v_h \otimes E^{(001)} v_l, v_h \otimes E^{(101)} v_l, v_h \otimes E^{(011)} v_l, v_h \otimes E^{1} v_l\}.$$
As in Lemma 3.7, this is a linearly independent set if and only if
\[ E^1(v_h \hat{\otimes} v_l) = v_h \hat{\otimes} \Omega \neq 0. \]
This is equivalent to \( v_h \hat{\otimes} v_l \) being a cyclic generator. The latter equivalence follows from Proposition 3.6, which shows that
\[ E^1(v_h \hat{\otimes} v_l) = v_h \hat{\otimes} \Omega = -\zeta [s_1] [s_2] [\zeta s_1 s_2] v_h \otimes v_h. \]
\( \square \)

We outline an informal algorithm which finds a homogeneous cyclic \( \tilde{v} \) vector for \( V(t) \hat{\otimes} V(s) \), if one exists, or tells one does not exist. This algorithm is a guide for the computations in Sections 6 and 7.

1. Suppose \( \tilde{v} = v_h \hat{\otimes} v_l \).
2. If \( \tilde{v} \) is a generator stop, otherwise find the greatest \( \psi \in \{0, 1\}^* \) such that \( \tilde{v} \) is not effective at level \( \psi \).
3. If there exists \( v \in wt(-ts) \) such that \( \tilde{v} + v \) is effective at levels \( \psi \) through 1, then replace \( \tilde{v} \) with \( \tilde{v} + v \) and return to (2). Otherwise, \( \tilde{v} \) cannot be made into a generator and the representation is not homogeneous cyclic, stop.

Proposition 5.9 tells us to proceed to step (3) of the algorithm if \( s \) belongs to \( X_1, X_2 \) or \( \mathcal{H} \). Each case corresponds to different levels for which \( v_h \hat{\otimes} v_l \) is not effective. Suppose a representation \( V(t) \hat{\otimes} V(s) \) is not cyclic and the algorithm produces a vector \( \tilde{v} \) which is not effective at level \( \psi \). The diagram we obtain as a result is similar to the one in Figure 7, but with some edges, corresponding to the zero actions of \( E_1 \) and \( E_2 \), deleted. In contrast to Figure 5, a disconnected graph implies a direct sum decomposition of the representation. Moreover, \( v_h \otimes F^e \otimes v_h \) belongs to the head of \( V(t) \hat{\otimes} V(s) \). The head of \( V(t) \hat{\otimes} V(s) \) together with the product \( ts \) is enough to determine \( V(t) \hat{\otimes} V(s) \) up to isomorphism. We will be able to determine which algebraic sets contain \( (t, s) \) from the diagrams we construct in the following sections, and therefore isomorphism classes of representations \( V(t) \hat{\otimes} V(s) \).

6. Cyclicity for \( s \in X_1 \cup X_2 \)

The cases with \( s \in X_1 \cup X_2 \) are easier to manage than those with \( s \in \mathcal{H} \), and so they are treated first. By the symmetry of the computations in this section, we only show the cases when \( s \in X_1 \) and when \( s \in C \). The conclusion of this section is that if \( (t, s) \) belongs to any of \( X_1^2, X_2^2 \), or \( R \times C \) then \( V(t) \hat{\otimes} V(s) \) is not cyclic. Throughout this section we assume \( s_1^2 = 1 \) unless stated otherwise.

**Lemma 6.1.** For \( \tilde{v} \) to be effective at level (011), the \( F_1 v_h \hat{\otimes} F_3 F_2 v_h \) component of \( \tilde{v} \) must be nonzero. Moreover,
\[ d_{(011)}(E_{11}.(F_1 v_h \hat{\otimes} F_3 F_2 v_h)) = s_1 [t_1]. \] (76)

**Proof.** We have already shown that \( v_h \hat{\otimes} v_l \) is not a homogeneous cyclic vector. More precisely,
\[ E^1(v_h \hat{\otimes} v_l) = v_h \hat{\otimes} E^1 v_l = -[s_1] v_h \hat{\otimes} F_3 F_2 v_h = 0, \]
having referred to Table 1 and as \( |1| = [-1] = 0 \). We wish to find a vector \( \tilde{v} \) such that \( E^1 \tilde{v} \) is effective at level (011). It follows from Lemma 5.8 that \( \tilde{v} \) must have a nonzero \( F_1 v_h \hat{\otimes} F_3 F_2 v_h \) component. We apply \( E_1 \) to it, and by equation (102), \( E_1 \) commutes with \( F_3 F_2 \),
\[ E_1.(F_1 v_h \hat{\otimes} F_3 F_2 v_h) = [t_1 s_1] v_h \hat{\otimes} F_3 F_2 v_h = s_1 [t_1] v_h \hat{\otimes} F_3 F_2 v_h. \]
\( \square \)
Corollary 6.2. The pair \((t, s)\) is not effective at level \((011)\) if and only if it belongs to \(X_1^2\), and is not effective at level \((101)\) if and only if it belongs to \(X_2^2\).

Let \(\rho_B\) and \(\rho_{U_0}\) be the projections in \(U\) to \(B\) and \(U^0\) in the PBW basis, respectively.

Lemma 6.3. Suppose \(s_2\) is not a fourth root of unity. Then

\[
\begin{align*}
   d_{(000)}(E_1 E_3 E_2 (v_h \hat{\otimes} v_l + F_1 v_h \hat{\otimes} F_3 F_2 v_h)) &= \zeta |s_2| |\zeta s_2| |t_1|, \\
   \text{and} &
\end{align*}
\]

(77)

and \(v_h \hat{\otimes} v_l + F_1 v_h \hat{\otimes} F_3 F_2 v_h\) is an effective vector for level \((000)\) if and only if \(t \notin X_1\).

Proof. By equation (117),

\[
\rho_{U_0} E_3 E_2 F_3 F_2 = -\zeta |K_2| |\zeta K_1 K_2|
\]

and by equation (112),

\[
E_1 E_3 E_2 F_1 v_h \hat{\otimes} F_3 F_2 v_h = E_3 E_2 [K_1] v_h \hat{\otimes} F_3 F_2 v_h = -\zeta |s_2| |\zeta s_2| |t_1| v_h \otimes v_h.
\]

Indeed, when \(s_2\) is not a fourth root of unity, effectiveness of \(F_1 v_h \hat{\otimes} F_3 F_2 v_h\) at level \((000)\) is equivalent to \(t_1 \notin \{\pm 1\}\). \( \Box \)

Similarly, if \(s_1\) is not a fourth root of unity and \(s \in X_2\), then \(v_h \hat{\otimes} v_l + F_2 v_h \hat{\otimes} F_1 F_3 v_h\) is a homogeneous cyclic vector if and only if \(t \notin X_2\). Next, we suppose that both \(s_1\) and \(s_2\) are fourth roots of unity which square to 1. The case when \(s_1\) and \(s_2\) are fourth roots of unity and exactly one has square \(-1\) is considered in the next section.

Lemma 6.4. Let \(s \in X_1 \cap X_2\). The representation \(V(t) \hat{\otimes} V(s)\) is cyclic if and only if \(t \notin R\).

Proof. By the above,

\[
v_h \hat{\otimes} v_l + F_1 v_h \hat{\otimes} F_3 F_2 v_h + F_2 v_h \hat{\otimes} F_1 F_3 v_h
\]

is not effective at level \((000)\) under the assumption \(s \in C\). Based on the previous computation, effectiveness only needs to be shown at level \((000)\) and only by appending \(v_l \hat{\otimes} v_h\) may we obtain a cyclic vector. By Proposition 3.6,

\[
E^1(v_l \hat{\otimes} v_h) = \Omega \hat{\otimes} v_h = -\zeta |t_1| |t_2| |\zeta t_1 t_2| v_l \hat{\otimes} v_h.
\]

To summarize the results of this section, we have the following corollary.

Corollary 6.5. If \((t, s)\) belongs to any of \(X_1^2\), \(X_2^2\), or \(R \times (X_1 \cap X_2)\) then \(V(t) \hat{\otimes} V(s)\) is not cyclic.

Generically, each of these cases can be illustrated by omissions from the representation graph of Figure 7, they can be seen in Figure 8.

7. Cyclicity for \(s \in H\)

In addition to the cases considered in Section 6, \(v_h \hat{\otimes} v_l\) is not a homogeneous cyclic vector when \((s_1 s_2)^2 = -1\). This section requires more work than the last because the computations involve the multiplicity two weight space occurring at levels \((110)\) and \((001)\). Throughout this section, we assume that \(s \in H\) unless specified otherwise.

Lemma 7.1. The vector subspace generated by \(E_1 E_2 v_h \hat{\otimes} v_l\) and \(E_2 E_1 v_h \hat{\otimes} v_l\) has dimension 1.
Proof. The proof is a computation of the vectors $E_1 E_2 v_l$ and $E_2 E_1 v_l$ into simplified terms which involve only the $F^\psi$, and showing they are multiples of each other. Using Table 1, we express these vectors in the basis $\langle F^{(110)} v_h, F^{(001)} v_h \rangle$:

$$E_1 E_2 v_l = \lfloor s_2 \rfloor E_1 F^{(101)} v_h = \lfloor s_2 \rfloor (\zeta s_1 F^{(110)} v_h - \lfloor \zeta s_1 \rfloor F^{(001)} v_h) = \lfloor s_2 \rfloor \begin{bmatrix} \zeta s_1 \\ - \lfloor \zeta s_1 \rfloor \end{bmatrix}$$

and

$$E_2 E_1 v_l = \lfloor s_1 \rfloor E_2 F^{(011)} v_h = \lfloor s_1 \rfloor (s_2^{-1} F^{(110)} v_h + \lfloor s_2 \rfloor F^{(001)} v_h) = \lfloor s_1 \rfloor \begin{bmatrix} s_2^{-1} \\ \lfloor s_2 \rfloor \end{bmatrix}.$$

The linear dependence is exhibited by computing the determinant of the matrix of coefficients, ignoring scale factors:

$$\begin{vmatrix} \zeta s_1 & s_2^{-1} \\ - \lfloor \zeta s_1 \rfloor & \lfloor s_2 \rfloor \end{vmatrix} = [\zeta s_1 s_2] = 0. \quad \Box$$

Corollary 7.2. The vectors $E^{(101)} v_h \overset{\wedge}{\otimes} v_l$ and $E^{(011)} v_h \overset{\wedge}{\otimes} v_l$ are zero when $s \in \mathcal{H}$. Hence, $v_h \overset{\wedge}{\otimes} v_l$ is not effective at the levels (010) and (100).

As in Lemmas 6.3 and 6.4, we determine which $\underline{-ts}$ weight vectors must be added to $v_h \overset{\wedge}{\otimes} v_l$ to yield a cyclic vector $\tilde{v}$ and state when no such vectors exist. We first produce a vector which is effective at either level (110) or (001), and does not belong to the $\langle E^{(110)} v_h \overset{\wedge}{\otimes} v_l \rangle$ subspace. Recall that $\pi$ denotes the projection to the subspace $\langle v_h \overset{\wedge}{\otimes} F^\psi v_h : \psi \in \{0, 1\}^{\Phi^+} \rangle$ with respect to the basis $\mathcal{B}^\prime$.

Definition 7.3. A vector $v \in wt(\underline{-ts})$ is said to have the spanning property if

$$\text{span}\{\pi E_1 E_2 v, \pi E_3 v\} = \text{span}\{v_h \overset{\wedge}{\otimes} F_1 F_2 v_h, v_h \overset{\wedge}{\otimes} F_3 v_h\}. \quad (78)$$

We consider four vectors:

$$F_1 F_2 v_h \overset{\wedge}{\otimes} F_1 F_2 v_h, \quad F_1 F_2 v_h \overset{\wedge}{\otimes} F_3 v_h, \quad F_3 v_h \overset{\wedge}{\otimes} F_1 F_2 v_h, \quad \text{and} \quad F_3 v_h \overset{\wedge}{\otimes} F_3 v_h. \quad (79)$$
whose linear combinations are candidates for producing a vector with the spanning property. We denote the span of these vectors by Λ.

Let \( \Delta(\mathcal{H}) \) be the subset of \( \mathcal{H}^2 \) given by

\[
\{(t, s) \in \mathcal{H}^2 : (t_1 s_1)^2 = -1\} \equiv \{(t, s) \in \mathcal{P}^2 : (t_1 s_1)^2 = (t_2 s_2)^2 = (t_1 t_2)^2 = -1\}.
\]

(80)

Note that \( \Delta(\mathcal{H}) \) is preserved by the action of \( \mathcal{P} \) on \( \mathcal{P}^2 \) given in (69).

**Lemma 7.4.** There exists a vector \( v \in \Lambda \) effective at either level (110) or (001) if and only if \( (t, s) \notin \Delta(\mathcal{H}) \).

**Proof.** We show that effectiveness of all vectors \( v \in \Lambda \) fails if and only if \( t_1 s_1 \in \{\pm i\} \) and \( t_2 s_2 \in \{\pm i\} \). We determine when the basis vectors of \( \Lambda \) are all simultaneously ineffective at levels (110) and (001). We focus on computing the relevant components of these vectors when acted on by \( E_1 E_2 \) and \( E_3 \). Among the eight vectors to compute, we begin by computing \( E_1 E_2 F_1 F_2, E_1 E_2 F_3, E_3 F_1 F_2, E_3 F_3 \) and project them via \( \rho_{v_0} \). We refer to equations (103), (104), (107), and (99) to obtain:

\[
\rho_{v_0} E_1 E_2 F_1 F_2 = [K_1] [K_2], \quad \rho_{v_0} E_1 E_2 F_3 = -[K_1] K_2^{-1},
\]

\[
\rho_{v_0} E_3 F_1 F_2 = -\zeta K_1^{-1} [K_2], \quad \rho_{v_0} E_3 F_3 = [K_1 K_2].
\]

It follows that for \( \psi \in \{(110), (001)\} \), we have:

\[
d_\psi(E_1 E_2 (F_1 F_2 v_h \otimes F^\psi v_h)) = [\zeta t_1 s_1] [\zeta t_2 s_2], \quad d_\psi(E_1 E_2 (F_3 v_h \otimes F^\psi v_h)) = \zeta [\zeta t_1 s_1] (t_2 s_2)^{-1},
\]

\[
d_\psi(E_3 (F_1 F_2 v_h \otimes F^\psi v_h)) = (t_1 s_1)^{-1} [\zeta t_2 s_2], \quad d_\psi(E_3 (F_3 v_h \otimes F^\psi v_h)) = - [t_1 s_1 t_2 s_2].
\]

The vanishing of \( [\zeta t_1 s_1] \) and \( [\zeta t_2 s_2] \) implies all the above expressions vanish. Thus, all of the above vectors vanish exactly when \( (t_1 s_1)^2 = -1 \) and \( (t_2 s_2)^2 = -1 \). In which case, all vectors in \( \Lambda \) are ineffective for levels (110) and (001). Since \( s \in \mathcal{H} \), we have proven the claim. \( \square \)

**Lemma 7.5.** There exists \( v \in \Lambda \) with the spanning property if and only if

\[
(t_1 s_1)^2 \neq -1, \quad (t_2 s_2)^2 \neq -1, \quad \text{and} \quad (t_1 t_2)^2 \neq 1.
\]

(81)

**Proof.** Let \( v \in \Lambda \) be the linear combination with coefficients in \( \mathbb{Q}[t_1^\pm, t_2^\pm, s_1^\pm, s_2^\pm] \),

\[
c_1 F_1 F_2 v_h \otimes F_2 v_h + c_2 F_1 F_2 v_h \otimes F_3 v_h + c_3 F_3 v_h \otimes F_1 F_2 v_h + c_4 F_3 v_h \otimes F_3 v_h.
\]

Let \( v_{12} = \pi E_1 E_2 v \) and \( v_3 = \pi E_3 v \). The \( v_h \otimes F_1 F_2 v_h \) component of \( v_{12} \) only comes from \( c_1 F_1 F_2 v_h \otimes F_1 F_2 v_h \) and \( c_2 F_3 v_h \otimes F_1 F_2 v_h \), while its \( v_h \otimes F_3 v_h \) component only from \( c_2 F_1 F_2 v_h \otimes F_3 v_h \) and \( c_4 F_3 v_h \otimes F_3 v_h \). Similarly for \( v_3 \). Moreover, each of these components have already been computed in the proof of Lemma 7.4. In our current notation, we write \( v_{12} \) and \( v_3 \) as vectors in the basis \( \langle v_h \otimes F_1 F_2 v_h, v_h \otimes F_3 v_h \rangle \):

\[
v_{12} = [\zeta t_1 s_1] \left[ c_1 [\zeta t_2 s_2] + c_3 (t_2 s_2)^{-1} \right], \quad v_3 = [c_1 (t_1 s_1)^{-1} [\zeta t_2 s_2] - c_3 [t_1 s_1 t_2 s_2] - c_2 (t_1 s_1)^{-1} [t_2 s_2] - c_4 [t_1 s_1 t_2 s_2]].
\]

We compute the determinant of the matrix whose columns are the vectors \( v_{12} \) and \( v_3 \):

\[
\begin{vmatrix} v_{12} & v_3 \\ v_{12} + t_1 s_1 v_{12} & v_3 + t_1 s_1 t_2 s_2 \end{vmatrix} = -\zeta [\zeta t_1 s_1] [\zeta t_1 s_1 t_2 s_2] [\zeta t_2 s_2] (c_1 c_4 - c_2 c_3). \quad \square
\]
The previous lemma has shown the spanning property independently of \(v_h \otimes_v t \) being present. By adding other \(-ts\) weight vectors to the vectors of \(\Lambda\) just considered, the spanning property may hold more generally. We proceed by assuming at least one of the conditions in (81) is not met.

**Lemma 7.6.** Suppose \((t_1s_1)^2 = -1\). There exists \(v \in wt(-ts)\) with the spanning property if and only if

\[
(t, s) \notin \mathcal{A}(H) \quad \text{and either} \quad t \notin X_2 \quad \text{or} \quad s \notin (X_2 \cap H).
\]

**Proof.** By Lemma 7.4, we require \((t_2s_2)^2 \neq -1\) in order for a vector from the subspace \(\Lambda\) to contribute to the spanning set. Let \(v_{12}\) and \(v_3\) be as in the proof of Lemma 7.5. Under the present assumptions, \(v_{12}\) is zero and under a relabeling

\[
v_3 = \begin{bmatrix} c_1(t_1s_1)^{-1} \zeta t_2 s_2 - c_3 t_1s_1 t_2 s_2 \\ c_2(t_1s_1)^{-1} \zeta t_2 s_2 - c_4 t_1s_1 t_2 s_2 \end{bmatrix} = \begin{bmatrix} c_1 \zeta t_2 s_2 - c_3 t_1s_1 t_2 s_2 \\ c_2 \zeta t_2 s_2 - c_4 t_1s_1 t_2 s_2 \end{bmatrix}
\]

in the basis \(\langle v_h \hat{\otimes} F_1 F_2 v_h, v_h \hat{\otimes} F_3 v_h \rangle\).

Together with \(v_3\), only the vectors \(E_i E_2 \cdot (v_h \hat{\otimes} v_1)\) and \(\pi E_1 E_2 \cdot (F^{(010)} v_h \hat{\otimes} F^{(101)} v_h)\) may contribute to the spanning set, as \(\pi E_1 E_2 \cdot (F^{(100)} v_h \hat{\otimes} F^{(011)} v_h) = 0\). More explicitly, by equations (108) and (100), those vectors are

\[
E_1 E_2 \cdot (v_h \hat{\otimes} v_1) = v_h \hat{\otimes} (E_1 E_2 F_1 F_2 v_h | s_2) = [s_2] \begin{bmatrix} \zeta s_1 \\ -[\zeta s_1] \end{bmatrix}
\]

and

\[
\pi E_1 E_2 \cdot (F_2 v_h \hat{\otimes} F_1 F_3 v_h) = \pi E_1 \cdot [K_2] \cdot (v_h \hat{\otimes} F_1 F_3 v_h) = [t_2 s_2] \begin{bmatrix} \zeta s_1 \\ -[\zeta s_1] \end{bmatrix}.
\]

Both of these vectors are zero when \(s_2 = 1\) and \(t_2 = 1\), in which case \(\text{span}\{\pi E_1 E_2 v, \pi E_3 v\}\) is at most 1-dimensional.

**Lemma 7.7.** Suppose \((t_2s_2)^2 = -1\). There exists \(v \in wt(-ts)\) with the spanning property if and only if \((t, s) \notin \mathcal{A}(H)\).

**Proof.** By Lemma 7.4, we require \((t, s) \notin \mathcal{A}(H)\). However, for any such \((t, s)\) it can be shown that \(E_3 \cdot (v_h \hat{\otimes} v_1)\) is non-zero and forms a spanning set with \(v_{12}\).

**Lemma 7.8.** Suppose \((t_1t_2)^2 = 1\). There exists \(v \in wt(-ts)\) with the spanning property if and only if \((t, s) \notin (X_1 \cap X_2) \times (X_2 \cap H)\).

**Proof.** Let \(v_{12}\) and \(v_3\) be as above so that

\[
v_{12} = [\zeta t_1 s_1] \begin{bmatrix} c_1 \zeta t_2 s_2 + c_3 (t_2 s_2)^{-1} \\ c_2 \zeta t_2 s_2 + c_4 (t_2 s_2)^{-1} \end{bmatrix} \quad \text{and} \quad v_3 = \begin{bmatrix} c_1(t_1s_1)^{-1} \zeta t_2 s_2 - c_3 t_1 t_2 s_1 s_2 \\ c_2(t_1s_1)^{-1} \zeta t_2 s_2 - c_4 t_1 t_2 s_1 s_2 \end{bmatrix}
\]

in the basis \(\langle v_h \hat{\otimes} F_1 F_2 v_h, v_h \hat{\otimes} F_3 v_h \rangle\). Note that \(v_{12}\) and \(v_3\) are linearly dependent, by Lemma 7.5. However, if \((t_1s_1)^2 \neq -1\), then for any vector \(w\) belonging to \(\text{span}\{v_h \hat{\otimes} F_1 F_2 v_h, v_h \hat{\otimes} F_3 v_h\}\), there are choices of \(c_i\) so that \(v_{12}\) equals to \(w\) and similarly for \(v_3\). In particular, \((t_1s_1)^2 \neq -1\) implies \(v_{12}\) and \(E_3 \cdot (v_h \hat{\otimes} v_1)\) form a spanning set shown in the above proof. If \((t_1s_1)^2 = -1\), then \((t_2s_2)^2 = 1\). According to Lemma 7.6, we then require \(s_2 \neq 1\) in order to form a spanning set between \(v_3\) and \(E_1 E_2 \cdot (v_h \hat{\otimes} v_1)\). Hence, there is a spanning set if and only if \((t_1s_1)^2 \neq -1\) or \(s_2 \neq 1\). 

\(\square\)
Corollary 7.9. If \((t, s)\) belongs to \(\Delta(\mathcal{H})\) or \(\mathcal{X}_2 \times (\mathcal{X}_2 \cap \mathcal{H})\) then \((t, s)\) is not homogeneous cyclic.

Assuming that we have appended a vector \(\tilde{v}\) which generates the subspace \(\langle v_h \hat{\otimes} F_1 F_2 v_h, v_h \hat{\otimes} F_3 v_h \rangle\), it remains to show that \(\tilde{v}\) recovers the entire module. As noted in Corollary 7.2,

\[ E^{(101)}(v_h \hat{\otimes} v_t) = E^{(011)}(v_h \hat{\otimes} v_t) = 0. \]

As such we may neglect the \(v_h \hat{\otimes} v_t\) component of the cyclic vector at this point. In fact, the vectors \(F_1 v_h \hat{\otimes} F_3 F_2 v_h\) and \(F_2 v_h \hat{\otimes} F_1 F_3 v_h\) do not contribute to effectiveness beyond levels \((110)\) and \((001)\). We move our attention to effectiveness at levels \((010)\) and \((100)\). We first determine whether \(F_1 F_3 v_h \hat{\otimes} F_2 v_h\) and \(F_3 F_2 v_h \hat{\otimes} F_1 v_h\) are effective before considering vectors in \(\Lambda\).

Lemma 7.10. The vector \(F_1 F_3 v_h \hat{\otimes} F_2 v_h\) is effective at level \((010)\) if and only if

\[ (t_1 s_1)^2 \neq -1 \quad \text{and} \quad (t_1 t_2)^2 \neq -1, \quad (84) \]

and \(F_3 F_2 v_h \hat{\otimes} F_1 v_h\) is effective at level \((100)\) if and only if

\[ (t_2 s_2)^2 \neq -1 \quad \text{and} \quad (t_1 t_2)^2 \neq -1. \quad (85) \]

Proof. First, we find \(\rho_{v_0} E_1 E_3 F_1 F_3\) given by equations \((111)\), \((99)\), and \((104)\),

\[ \rho_{v_0} E_1 E_3 F_1 F_3 = \rho_{v_0} (E_3 [\zeta K_1] + E_1 E_2 K_1^{-1} F_3) = -\zeta [K_1] [\zeta K_1 K_2]. \]

Therefore, \(d_{(010)}(E_1 E_3 (F_1 F_3 v_h \hat{\otimes} F_2 v_h)) = -\zeta [\zeta t_1 s_1] [t_1 s_1 t_2 s_2]\). Hence, \(F_1 F_3 v_h \hat{\otimes} F_2 v_h\) is effective if and only if \((t_1 s_1)^2 \neq -1\) and \((t_1 t_2)^2 \neq -1\). A similar computation shows

\[ d_{(100)}(E_3 E_2 (F_3 F_2 v_h \hat{\otimes} F_1 v_h)) = -\zeta [\zeta t_2 s_2] [t_1 s_1 t_2 s_2]. \]

\[ \square \]

Lemma 7.11. There does not exist a vector effective for levels \((100)\) and \((010)\) if and only if \((t, s) \in \mathcal{H}^2\).
Corollary 7.12. The representation ineffective for levels (100) and (010). This proves the claim. □

Lemma 7.13. We have the following equalities:

\[
\begin{align*}
d_{(000)}(E^1: (F_1 F_2 v_h \widehat{\otimes} F_1 F_2)) &= -[t_1 s_1] [s_2] [\zeta t_2 s_2] & \quad & \quad & \quad & \text{(86)} \\
&= d_{(000)}(E^1: (F_1 F_2 v_h \widehat{\otimes} F_2)) = s_1 [t_1 s_1] [t_2] & \quad & \quad & \quad & \text{(87)} \\
&= d_{(000)}(E^1: (F_3 v_h \widehat{\otimes} F_1 F_2)) = -s_1 s_2 [s_2] ([s_1] [\zeta t_2] - [t_1 s_1] t_2^{-1}) & \quad & \quad & \quad & \text{(88)} \\
&= d_{(000)}(E^1: (F_3 F_1 v_h \widehat{\otimes} F_3)) = s_1 [\zeta t_2] [s_1] - \zeta [s_1 s_2] [t_1 s_1] t_2^{-1} s_2^{-1} & \quad & \quad & \quad & \text{(89)} \\
&= d_{(000)}(E^1: (F_3 F_1 v_h \widehat{\otimes} F_2)) = s_1 s_2 [s_2] [t_1 t_2] [t_1 s_1] & \quad & \quad & \quad & \text{(90)} \\
&= d_{(000)}(E^1: (F_3 F_2 v_h \widehat{\otimes} F_1)) = \zeta s_1 s_2 [s_1] [t_1 t_2] [t_2 s_2]. & \quad & \quad & \quad & \text{(91)}
\end{align*}
\]

Figure 10. Representation graph of \( V(t) \widehat{\otimes} V(s) \) generated by \( U^+ \) acting on \( \tilde{v} \in wt(\xi) \) when \((t, s)\), is assumed to be generic, and belongs to the indicated subset of \( P^2 \).

Proof. After Lemma 7.10, it remains to compute the actions of \( E_1 E_3 \) and \( E_4 E_2 \) on \( F^\psi v_h \widehat{\otimes} F^{\psi'} v_h \) for \( \psi, \psi' \in \{(110), (001)\} \). We compute by equations (113) and (114):

\[
\rho_B E_1 E_3 F_2 = -\zeta E_1 [K_1 K_2], \quad \rho_B E_1 E_3 F_3 = E_1 [K_1 K_2].
\]

By equations (109) and (110),

\[
\rho_B E_3 E_2 F_1 F_2 = 0 \quad \rho_B E_3 E_2 F_3 = \zeta E_2 [K_1 K_2].
\]

Observe that for each \( \psi \in \{(110), (001)\} \),

\[
[K_1 K_2] (v_h \widehat{\otimes} F^{\psi'} v_h) = -[t_1 s_1 t_2 s_2] v_h \widehat{\otimes} F^{\psi'} v_h.
\]

and each action is zero exactly when \( (t_1 t_2)^2 = -1 \), in which case there is no effective vector in \( \Lambda \). If \( \pi \in \mathcal{H} \), by Lemma 7.10, the vectors \( F_1 F_3 \widehat{\otimes} F_2 v_h \) and \( F_3 F_2 v_h \widehat{\otimes} F_1 v_h \) are also ineffective for levels \((100)\) and \((010)\). This proves the claim. □

Corollary 7.12. The representation \( V(t) \widehat{\otimes} V(s) \) is not homogeneous cyclic due to ineffectiveness at levels \((100)\) and \((010)\) if and only if \((t, s) \in \mathcal{H}^2 \).

Lastly, we investigate the \( v_h \widehat{\otimes} v_h \) level by considering the action of \( E^1 \).
Proof. We compute by equations (115), (116), (112), and (110),
\[
\rho_B E_1 E_3 E_2 F_1 F_2 = -\zeta E_1 E_2 [\zeta K_1] K_2 + E_3 [\zeta K_1] K_2
\]
\[
\rho_B E_1 E_3 E_2 F_3 = \zeta E_1 E_2 [K_1 K_2] - E_3 [\zeta K_1] K_2^{-1}
\]
\[
\rho_B E_1 E_3 E_2 F_1 F_3 = -\zeta E_2 [K_1 K_2] [\zeta K_1]
\]
\[
\rho_B E_1 E_3 E_2 F_3 F_2 = E_1 [K_1 K_2] [\zeta K_2].
\]
We include the first computation here, the others are similar:
\[
d_{(000)}(E^1.(F_1 F_2 v_h \hat{\otimes} F_1 F_2 v_h)) = ([s_1] [s_2]) [t_1 s_1] (t_2 s_2) - \zeta(s_1^{-1} [s_2]) [t_1 s_1] [-\zeta t_2 s_2]
\]
\[= -\zeta s_1 s_2 [t_1 s_1] [s_2] [t_2]. \]

**Corollary 7.14.** The representation \( V(\mathfrak{t}) \hat{\otimes} V(\mathfrak{g}) \) is not effective for level \((000)\) if and only if \((\mathfrak{t}, \mathfrak{g})\) belongs to any of \(X_1 \times (X_1 \cap \mathcal{H}), \hat{\Delta}(\mathcal{H}), X_2 \times (X_2 \cap \mathcal{H}), \) or \((X_1 \cap X_2) \times \mathcal{H}\).

**Proof.** Recall the underlying assumption that \((s_1 s_2)^2 = -1\). Proposition 3.6 implies that \(v_l \hat{\otimes} v_h\) is not effective for level \((000)\) if and only if
\[
(t_1 s_1)^2 = 1, \quad (t_2 s_2)^2 = 1, \quad \text{or} \quad (t_1 t_2)^2 = 1.
\]
We assume at least one such equality holds, otherwise \(v_l \hat{\otimes} v_h\) can be taken as a non-zero component of \(\hat{v}\) to produce a vector effective at level \((000)\). The vectors considered in Lemma 7.13 may be used as a non-zero component of \(\hat{v}\). We determine when these vectors are all ineffective.

Suppose only \((t_1 s_1)^2 = 1\) then \(F_3 v_h \hat{\otimes} F_1 F_2 v_h, F_3 v_h \hat{\otimes} F_3 v_h,\) and \(F_3 F_2 v_h \hat{\otimes} F_1 v_h\) are effective at level \((000)\). Observe that
\[
d_{(000)}(E^1.(F_3 v_h \hat{\otimes} F_3 v_h)) = 0
\]
only if \(s_1^2 = 1\) or \((t_2 s_2)^2 = 1\). We assume \(s_1^2 = 1\), which implies \(t_2^2 = 1\) and \(s_2 = -1\), and all vectors vanish. Hence, each \((\mathfrak{t}, \mathfrak{g}) \in X_1 \times (X_1 \cap \mathcal{H})\) is not effective at level \((000)\).

If \((t_1 s_1)^2 = 1\) and \((t_2 s_2)^2 = 1\), then \((t_1 t_2)^2 = -1\) and all vectors are zero. Thus, each \((\mathfrak{t}, \mathfrak{g}) \in \hat{\Delta}(\mathcal{H})\) is not effective at level \((000)\).

If we allow only \((t_2 s_2)^2 = 1\), then \(d_{(000)}(E^1.(F_1 F_2 v_h \hat{\otimes} F_3 v_h))\) vanishes only if \(t_2^2 = 1\). Thus, \(s_2 = 1\) and \(s_1^2 = -1\). At this stage, all vectors vanish. Hence, a pair \((\mathfrak{t}, \mathfrak{g}) \in X_2 \times (X_2 \cap \mathcal{H})\) is not effective.

So far we have not considered the \(t_1 t_2\). Thus, we assume only \((t_1 t_2)^2 = 1\). Again, \(F_1 F_2 v_h \hat{\otimes} F_3 v_h\) vanishes only if \(t_2^2 = 1\). Assuming this, then \(t_1^2 = 1\) and all vectors vanish. Therefore, if \((\mathfrak{t}, \mathfrak{g}) \in (X_1 \cap X_2) \times \mathcal{H}\). then \(V(\mathfrak{t}) \hat{\otimes} V(\mathfrak{g})\) does not have a vector effective at level \((000)\). This proves the claim.

8. The Cyclicity Theorem and Transfer Principle

We have considered each of the cases identified in Proposition 5.9. Gathering the results of Corollaries 6.2, 6.5, 7.9, 7.12, and 7.14 we may concisely characterize the existence of a homogeneous cyclic vector and the transfer principle.
Definition 8.1. The acyclicity locus $\mathcal{A}$ is defined to the subset of $P^2$ for which $V(t) \otimes V(s)$ is not homogeneous cyclic.

Let $\mathcal{C} = X_1 \cap X_2$ and $\mathcal{Y}_i = X_i \cap H$ for $i = 1, 2$.

Theorem 1.5 (Homogeneous Cyclic Tensor Product Representations). The acyclicity locus $\mathcal{A} \subseteq P^2$ is given by

$$X_1^2 \cup X_2^2 \cup H^2 \cup (H \times C) \cup (C \times H).$$

Note that $\mathcal{A}$ can be partitioned according to the intersections of its defining varieties, with the exception of $\hat{\Delta}(H)$. Recall that Figure 1 gives an illustration of these inclusions. We define a stratification of $\mathcal{A}$ according to these inclusions.

Definition 8.2. The cyclicity stratification of $P^2$ is defined by the filtration

$$P^2_0 \subset P^2_1 = A \subset P^2_2 = P^2,$$

with

$$P^2_0 = C^2 \cup Y_1^2 \cup Y_2^2 \cup \hat{\Delta}(H) \cup ((Y_1 \cup Y_2) \times C) \cup (C \times (Y_1 \cup Y_2))$$

Remark 8.3. The maximal irreducible subspace generated by some $\tilde{v} \in wt(-ts)$ for $(t, s) \in H^2 \cup (H \times C) \cup (C \times H)$ has two highest weight vectors as seen in Figures 8b, 9b, and 11c.

Remark 8.4. Non-degenerate implies homogeneous cyclic, with a homogeneous cyclic vector given by summing appropriate $-ts$ weight vectors from each direct summand.

Definition 8.5. A transfer is an isomorphism of representations determined by the action of $\lambda$ on $(t, s)$,

$$V(t) \otimes V(s) \cong V(\lambda t) \otimes V(\lambda^{-1}s).$$

A transfer is called trivial if $\lambda \in C$. If $\lambda = t^{-1}s$ acts on $(t, s)$ then the transfer is called a swap.
We group the defining subsets of \( \mathcal{P}^2_0 \) and \( \mathcal{P}^2_1 \) so that they are preserved by swaps, and we refer to the resulting subsets as symmetrized. That is to say, we identify
\[
(\mathcal{Y}_1 \times \mathcal{C}) \cup (\mathcal{C} \times \mathcal{Y}_1) \quad \text{and} \quad (\mathcal{Y}_2 \times \mathcal{C}) \cup (\mathcal{C} \times \mathcal{Y}_2)
\]
as two, rather than four, algebraic sets in order to be preserved under swaps.

**Corollary 8.6.** If \((\mathbf{t}, \mathbf{s}) \in \mathcal{P}^2 \setminus \mathcal{A}\) and its image under \(\lambda\) also belongs to \(\mathcal{P}^2 \setminus \mathcal{A}\), then \(\lambda\) determines a transfer.

**Theorem 1.6** (Transfer Principle). Suppose \((\mathbf{t}, \mathbf{s})\) belongs to a symmetrized subset in the \(n\)-stratum. Then
\[
V(\mathbf{t}) \otimes V(\mathbf{s}) \cong V(\lambda\mathbf{t}) \otimes V(\lambda^{-1}\mathbf{s})
\]
if and only if \(\lambda.(\mathbf{t}, \mathbf{s})\) belongs to the same symmetrized subset in the \(n\)-stratum.

**Proof.** The \(n = 2\) case implies \(V(\mathbf{t}) \otimes V(\mathbf{s})\) is a homogeneous cyclic representation. This case is a restatement of Corollaries 5.4 and 8.6.

Suppose \(n = 1\). Figures 8c, 8a, 8b, 11c, and 10a show that \(\mathcal{X}_1, \mathcal{X}_2, \mathcal{H}, \) and \((\mathcal{H} \times \mathcal{C}) \cup (\mathcal{C} \times \mathcal{H})\) determine non-isomorphic representations. These representations are generated by two or three vectors whose weights are determined by \(-\mathbf{ts}\), the weight of \(\hat{v}\). Since \(-\mathbf{ts}\) and the representation diagram are invariant under some \(\lambda \in \mathcal{P}\), such a \(\lambda\) determines a transfer. Note that the representation diagrams for \(\mathcal{A}(\mathcal{H})\) are different from those of \(\mathcal{H}\), see Figure 9b, but \(\mathcal{A}(\mathcal{H}) \subseteq \mathcal{H}\) is preserved by \(\lambda\).

In the \(n = 0\) case, we only need to consider trivial transfers and swaps on \(\mathcal{P}^2_0\). However, the same argument applies. \(\square\)

**Appendix A. Commutation Relations**

In this section we gather the general computations used throughout the paper.

\[
\begin{align*}
[E_1, F_3] &= \zeta F_2 K_1 & (97) \\
[E_2, F_3] &= -F_1 K_2^{-1} & (98) \\
[E_3, F_3] &= [K_1 K_2] & (99) \\
[E_1, F_1 F_3] &= \zeta F_1 F_2 K_1 - F_3 [\zeta K_1] & (100) \\
[E_2, F_1 F_3] &= 0 & (101) \\
[E_1, F_3 F_2] &= 0 & (102) \\
[E_1 E_2, F_1 F_3] &= [K_1] [K_2] + F_2 E_2 [K_1] + F_1 E_1 [K_2] & (103) \\
[E_1 E_2, F_3] &= F_2 E_2 K_1 - (F_1 E_1 + [K_1]) K_2^{-1} & (104) \\
[E_3, F_1] &= E_2 K_1^{-1} & (105) \\
[E_3, F_2] &= \zeta E_1 K_2 & (106) \\
[E_3, F_1 F_2] &= \zeta F_1 E_1 K_2 - \zeta([K_2] + F_2 E_2) K_1^{-1} & (107) \\
[E_1 E_2, F_1 F_3] &= F_1 F_2 E_2 K_1 - F_3 E_2 [K_1] & (108) \\
[E_3 E_2, F_1 F_2] &= -\zeta F_1 E_1 E_2 K_2 + F_1 E_3 [K_2] & (109) \\
E_3 E_2 F_3 &= E_3(F_3 E_2 - F_1 K_2^{-1}) & (110)
\end{align*}
\]
\[ E_3 E_2 F_3 = (F_3 E_3 E_2 + E_2 [\zeta K_1 K_2] - F_1 E_3 K^{-1}_2 - E_2 K^{-1}_1 K^{-1}_2) F_2 \]

\[ = F_3 E_3 (F_2 E_2 + [K_2]) + E_2 F_2 [K_1 K_2] + F_1 E_3 F_2 K^{-1}_2 - \zeta E_2 F_2 K^{-1}_1 K^{-1}_2 \]

\[ = F_3 F_2 E_3 E_2 + \zeta F_3 E_1 E_2 K_2 + F_3 E_3 [K_2] + F_2 E_2 [K_1 K_2] + [K_2] [K_1 K_2] \]

\[ + F_1 F_2 E_3 K^{-1}_2 + \zeta F_1 E_1 - \zeta F_2 E_2 - \zeta [K_2] K^{-1}_1 K^{-1}_2 \]

**Appendix B. Unrolled Restricted Quantum Group and R-Matrix**

In this section, we informally discuss the unrolled quantum group, which is obtained from the restricted quantum group by adjoining \( H_i \) for \( 1 \leq i \leq n \). In the unrolled case, upon passing to power series in \( H \), the category of representations is braided. We recall the description of the braiding in terms of the \( R \)-matrix given in [CP95]. We also comment on the relation between the representations of the restricted and unrolled restricted quantum groups.

**Definition B.1.** Let \( U^H_q(g) \) denote the unrolled quantum group, given by \( U_q(g)[H_1, \ldots, H_n] \) with relations:

\[ H_i K_j^\pm = K_j^\pm H_i, \quad [H_i, E_j] = A_{ij} E_j, \quad [H_i, F_j] = -A_{ij} F_j \]

in addition to the relations of Definition 2.1.

The elements \( H_i \) are primitive:

\[ \Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \quad S(H_i) = -H_i \quad \epsilon(H_i) = 0. \]

It can be shown that this Hopf algebra structure for \( H_i \) is compatible with \( U_q(g) \subseteq U^H_q(g) \), which is motivated by regarding \( K_i \) as \( q^{H_i} \). The construction of the unrolled restricted quantum group follows from the earlier methods applied to \( U^H_q(g) \). Equivalently, we append the \( H_i \) to the restricted quantum group.

**Definition B.2.** Let \( \overline{U}^H_q(g) \) denote the unrolled restricted quantum group, given by \( \overline{U}_q(g)[H_1, \ldots, H_n] \) modulo the relations in (118).
The unrolled quantum group has a Hopf algebra structure determined on generators by equations (26), (27), (28), and (119).

At an $l$-th root of unity, the representation $V(\mathfrak{t})$ extends from the restricted quantum group to the unrolled restricted quantum group. Fix a character $\mathfrak{t}$ and an $l$-th root of unity $q = e^{2\pi i m/l}$. Choose $\alpha_i$ such that $q^{\alpha_i} = e^{2\pi i \alpha_i/l} = t_i$. Since $t_i \neq 0$ there are infinitely many choices of $\alpha_i$. We use the notation $V_\alpha$, as in [CGP17], to denote representations of $\overline{U}_q^H(\mathfrak{sl}_2)$ for which $H$ acts by $\alpha$, and restrict to $V(t)$ on $\overline{U}_q^H(\mathfrak{sl}_2)$. More generally, we have $V_\alpha$ extending $V(\mathfrak{t})$. Moreover, there is a restriction map

$$\mathcal{F} : V_\alpha \to V(q_\alpha)$$

given by forgetting the action of $H_i$ for $1 \leq i \leq n$, and $q_\alpha = (q^{\alpha_1}, \ldots, q^{\alpha_n})$ denotes entrywise exponentiation.

We consider the representations with $H_i v_0 = \alpha_i v_0$. Once given an action of $H_i$, we can define the $R$-matrix action on tensor product representations. It can be shown that the $R$-matrix can be normalized to depend only on $\zeta^\alpha$ and not $\alpha$ itself. The formula for the universal $R$-matrix is given explicitly in [CP95, Theorem 8.3.9] for simple $\mathfrak{g}$, and is truncated under the assumptions $q = \zeta$. For weight representations $V$ and $W$, we define the map $R : V \otimes W \to V \otimes W$ by

$$R = \zeta^{\sum \alpha_{ij}(A^{-1})_{ij}H_i \otimes H_j} \prod_{\alpha \in \Phi^+} (1 \otimes 1 + (\zeta - \zeta^{-1})E_\alpha \otimes F_\alpha),$$

with the action of $\zeta^{\sum \alpha_{ij}(A^{-1})_{ij}H_i \otimes H_j}$ on weight vectors $v$ and $w$ given by

$$\zeta^{\sum \alpha_{ij}(A^{-1})_{ij}H_i \otimes H_j}(v \otimes w) = \zeta^{\sum \alpha_{ij}(A^{-1})_{ij}E_\mu \otimes F_\nu}(v \otimes w),$$

with $H_i \otimes H_j(v \otimes w) = \mu_{ij} E_\mu \otimes F_\nu$, and the product over $\Phi^+$ follows the ordering $<_{br}$. Note that the expression $\zeta^{\sum \alpha_{ij}(A^{-1})_{ij}H_i \otimes H_j}$ does not belong to $\overline{U}_\zeta^H(\mathfrak{g}) \otimes \overline{U}_\zeta^H(\mathfrak{g})$, as it is defined in terms of power series in $H_i$. The category of weight representations of $\overline{U}_\zeta^H(\mathfrak{g})$ has a braiding given by $P \circ R$, with $P$ the map which swaps tensor factors.

**Appendix C. The Ind Functor**

In this appendix we define a general induced module. This construction is used to study tensor products of $V(\mathfrak{t})$ for $U$ and sets the foundation for proving Theorem 1.4.

**Definition C.1.** Let $A$ be an algebra and $B \subseteq A$ a subalgebra. Define $\text{Ind}^A_B : B\text{-mod} \to A\text{-mod}$ the induction functor on $B$-modules by

$$M \mapsto \text{Ind}^A_B(M) := A \otimes_B M = A \otimes M/(ab \otimes m - a \otimes b.m : a \in A, b \in B, m \in M).$$

Then $\text{Ind}^A_B(M)$ is indeed an $A$-module, with action given by multiplication in the left tensor factor. On $B$-equivariant maps, $\text{Ind}^A_B$ produces an $A$-equivariant map:

$$f \in \text{Hom}_B(M, N) \mapsto \text{Ind}^A_B(f) := id_A \otimes f \in \text{Hom}(\text{Ind}^A_B(M), \text{Ind}^A_B(N)).$$
The $A$-equivariance of $\text{Ind}^A_B(f)$ is straightforward to verify.

Consider the $B$-modules $M$ and $N$, with $B$ a subbialgebra of a bialgebra $A$. There are two types of induced representations on the tensor product of $M$ and $N$, namely

$$\text{Ind}^A_B(M) \otimes \text{Ind}^A_B(N) = (A \otimes_B M) \otimes (A \otimes_B N) \quad (125)$$

and

$$\text{Ind}^A_B(M \otimes \text{Ind}^A_B(N)) = A \otimes_B (M \otimes (A \otimes_B N)) \quad (126)$$

Since $\text{Ind}^A_B(M) \otimes \text{Ind}^A_B(N)$ is a tensor product of $A$-modules, $A$ acts via the coproduct action. Whereas $A$ acts by left multiplication on $\text{Ind}^A_B(M \otimes \text{Ind}^A_B(N))$, only elements of $B$ pass to $M \otimes \text{Ind}^A_B(N)$ which then utilize the coproduct.

**Lemma C.2.** Let $A$ be a Hopf algebra and $M$ an $A$-module. Define

$$\varphi : A \otimes M \to A \otimes M \quad (127)$$

$$a \otimes m \mapsto a' \otimes S(a'')m,$$

under the implied summation convention. Then the map $\varphi$ is an isomorphism with inverse $\varphi^{-1}(a \otimes m) = a' \otimes a''m$.

**Remark C.3.** Note that $\varphi$ satisfies the following commutative diagram.

$$\begin{array}{ccc}
A \otimes M & \xrightarrow{\varphi} & A \otimes M \\
L_{\Delta(x)} \downarrow & & \downarrow L_x \otimes \text{id} \\
A \otimes M & \xrightarrow{\varphi} & A \otimes M
\end{array}$$

Here $L_{\Delta(x)}$ denotes left multiplication of $x' \otimes x''$ on the tensor product, and $L_x$ is left multiplication by $x$.

**Proposition C.4.** Let $A$ be a Hopf algebra, $B \subseteq A$ a subalgebra, and $M$ a $B$-module. Define

$$\Psi : \text{Ind}^A_B(M) \otimes \text{Ind}^A_B(N) \to \text{Ind}^A_B(M \otimes \text{Ind}^A_B(N)) \quad (128)$$

$$[a_1 \otimes_B m] \otimes [a_2 \otimes_B n] \mapsto [a'_1 \otimes_B (m \otimes [S(a'')a_2 \otimes_B n])],$$

under the implied summation convention. Then $\Psi$ defines a natural isomorphism of $A$-modules with inverse

$$\Psi^{-1} ([a_1 \otimes_B (m \otimes [a_2 \otimes_B n])]) = [a'_1 \otimes_B m] \otimes [a''a_2 \otimes_B n]. \quad (129)$$

**Proof.** It is left to the reader to check that $\Psi$ and $\Psi^{-1}$ are indeed inverses. Observe that

$$\text{Ind}^A_B(M) \otimes \text{Ind}^A_B(N) \cong (A \otimes M) \otimes (A \otimes N)/R_1$$

with

$$R_1 = \langle (a_1b_1 \otimes m) \otimes (a_2b_2 \otimes n) - (a_1 \otimes b_1.m) \otimes (a_2 \otimes b_2.n) : a_1, a_2 \in A, b_1, b_2 \in B, m \in M, n \in N \rangle$$

and

$$\text{Ind}^A_B(M \otimes \text{Ind}^A_B(N)) \cong A \otimes (M \otimes (A \otimes N))/R_2$$
with
\[ R_2 = (a_1b_1 \otimes (m \otimes (a_2b_2 \otimes n)) - a_1 \otimes (b_1' \cdot m \otimes (b_1'' a_2 \otimes b_2.n)) : a_1, a_2 \in A, b_1, b_2 \in B, m \in M, n \in N). \]

We prove well definedness by showing that \( \Psi \) and \( \Psi^{-1} \) sends two representatives of the same class to the same class:

\[
\Psi([a_1b_1 \otimes_B m] \otimes [a_2b_2 \otimes_B n] - [a_1 \otimes_B b_1 \cdot m] \otimes [a_2 \otimes_B b_2.n])
\]
\[
= [(a_1b_1)' \otimes_B (m \otimes [S((a_1b_1)'')a_2b_2 \otimes_B n]) - [a_1' \otimes_B (b_1'm \otimes [S(a_1'')a_2 \otimes_B b_2.n])]
\]
\[
= [a_1'b_1 \otimes_B (m \otimes [(S(b_1'')S(a_1'')a_2 \otimes_B n)]) - [a_1' \otimes_B (b_1'm \otimes [b_1'(S(b_1')S(a_1'') a_2 \otimes_B b_2.n)])
\]
\[
= 0
\]

and

\[
\Psi^{-1}([a_1b_1 \otimes_B (m \otimes [a_2b_2 \otimes_B n])] - [a_1 \otimes_B (b_1'm \otimes [b_1'' a_2 \otimes_B b_2.n])])
\]
\[
= [(a_1b_1)' \otimes_B m] \otimes [(a_1b_1)'a_2b_2 \otimes_B n] - [a_1' \otimes_B b_1'm] \otimes [(a_1b_1)'a_2 \otimes_B b_2.n]
\]
\[
= 0.
\]

It now remains to show commutativity of the following diagram for any choice of \( B \)-equivariant maps \( f : M \to M' \) and \( g : N \to N' \):

\[
\begin{array}{ccc}
\text{Ind}^A_B(M) \otimes \text{Ind}^A_B(N) & \overset{\Psi}{\longrightarrow} & \text{Ind}^A_B(M \otimes \text{Ind}^A_B(N)) \\
\text{Ind}^A_B(f \otimes \text{Ind}^A_B(g)) & \downarrow & \text{Ind}^A_B(f \otimes \text{Ind}^A_B(g)) \\
\text{Ind}^A_B(M') \otimes \text{Ind}^A_B(N') & \overset{\Psi}{\longrightarrow} & \text{Ind}^A_B(M' \otimes \text{Ind}^A_B(N'))
\end{array}
\]

We compute

\[
\text{Ind}^A_B(f \otimes \text{Ind}^A_B(g)) \circ \Psi((a_1 \otimes m) \otimes (a_2 \otimes n)) = a_1' \otimes (f(m) \otimes S(a_1'')a_2 \otimes g(n)),
\]

which agrees with

\[
\Psi \circ (\text{Ind}^A_B(f) \otimes \text{Ind}^A_B(g))((a_1 \otimes m) \otimes (a_2 \otimes n)) = \Psi((a_1 \otimes f(m)) \otimes (a_2 \otimes g(n))).
\]

Hence, proving the proposition. \( \square \)

**Appendix D. Results for \( \overline{U}_\zeta(\mathfrak{sl}_2) \)**

Here, we prove results for irreducibility and cyclicity of the representations \( V(t) \) and \( V(t)^{\otimes 2} \) of \( \overline{U}_\zeta(\mathfrak{sl}_2) \). A summary of these properties are given at the end of this section. We will use the language from Sections 2 and 5. In this section, let \( \mathcal{P} \) be the group of characters on \( U^0 \), which is isomorphic to \( \mathbb{C}^\times \).

**Definition D.1.** The 2-dimensional representation \( V(t) \) is defined on generators by:

\[
E = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix}
\]

expressed in the standard basis \( (v_0, v_1) = (v_0, Fv_0) \).

**Proposition D.2.** The representation \( V(t) \) is irreducible if and only if \( t^2 \neq 1 \).

**Proof.** We compute \( \Omega = EFv_0 = [t] v_0 \). This vector vanishes when \( t^2 = 1 \). \( \square \)
**Theorem D.3** ([Oht02]). The tensor product of irreducible representations decomposes as a direct sum of irreducible representations according to
\[ V(t) \otimes V(s) \cong V(ts) \oplus V(-ts). \]  
(131)
A basis of \( V(t) \otimes V(s) \) which determines a basis of \( V(ts) \oplus V(-ts) \) is
\[ (v_0 \otimes v_0, \Delta(F)v_0 \otimes v_0, \Delta(E)v_1 \otimes v_1, v_1 \otimes v_1). \]
(132)
This basis can be found in Appendix A.3 of [Oht02]. Let
\[ \mathcal{I} = \{(t, s) \in \mathcal{P}^2 : (ts)^2 = 1\} \quad \text{and} \quad \mathcal{C} = \{t \in \mathcal{P} : t^2 = 1\}. \]
(133)
The direct sum decomposition holds for some representations which are not necessarily irreducible. The following is a refinement of Theorem D.3, as \( t \) and \( s \) are not assumed to be generic.

**Proposition D.4.** The tensor decomposition in (131) holds if and only if
\[ (t, s) \in (\mathcal{P}^2 \setminus \mathcal{I}) \cup \mathcal{C}^2. \]
(134)

**Proof.** Since
\[ \Delta(F)v_0 \otimes v_0 = Fv_0 \otimes v_0 + K^{-1}v_0 \otimes Fv_0 = v_1 \otimes v_0 + t^{-1}v_0 \otimes v_1 \]
and
\[ \Delta(E)v_1 \otimes v_1 = Ev_1 \otimes Kv_1 + v_1 \otimes Ev_1 = -[t]sv_0 \otimes v_1 + [s]v_1 \otimes v_0, \]
the vectors in (132) do not form a basis either when \((t, s) \in \mathcal{C}\) so that \( \Delta(E)v_1 \otimes v_1 \)
vanishes, or when \((t, s) \in \mathcal{I}\) so that the vectors \( \Delta(E)v_1 \otimes v_1 \) and \( \Delta(F)v_0 \otimes v_0 \) are linearly dependent. We consider each case separately.

If \( \Delta(E)v_1 \otimes v_1 = 0 \), then any combination of \( v_0 \otimes v_1 \) and \( v_1 \otimes v_0 \) which is linearly independent from \( \Delta(F)v_0 \otimes v_0 \) can be used in place of \( \Delta(E)v_1 \otimes v_1 \) in (132). Thus, proving the isomorphism in (131) for \((t, s) \in \mathcal{C}\).

In the latter case, let \((t, s) \in \mathcal{I} \setminus \mathcal{C}\). Assume \( V(t) \otimes V(s) \cong W_1 \oplus W_2 \), and \( W_1 \) contains \( \Delta(F)v_0 \otimes v_0 \neq 0 \). Since \( \Delta(E)v_1 \otimes v_1 \neq 0 \) and is proportional to \( \Delta(F)v_0 \otimes v_0 \), both \( v_0 \otimes v_0 \)
and \( v_1 \otimes v_1 \) belong to \( W_1 \). Thus \( W_1 \) is at least 3-dimensional. Consider any vector \( v \) in the 
\(-ts\) weight space. Then \( v \) can be expressed as a linear combination of \( \Delta(F)v_0 \otimes v_0 \) and \( v_1 \otimes v_0 \), and
\[ \Delta(F)v \in \langle \Delta(F)v_1 \otimes v_0 \rangle = \langle v_1 \otimes v_1 \rangle \subseteq W_1. \]
Thus, \( v \in W_1 \); and in particular, \( v_1 \otimes v_0 \) belongs to \( W_1 \). Therefore, \( V(t) \otimes V(s) \cong W_1 \) is indecomposable for \((t, s) \in \mathcal{I} \setminus \mathcal{C}\).

Let \( V(t) \otimes V(s) \) denote the induced representation \( \text{Ind}^\mathcal{T}_c^{(sl_2)}_B \left( V_t \otimes \text{Ind}^\mathcal{T}_c^{(sl_2)}_B (V_s) \right) \). Using the methods of Section 5, we determine the existence of a cyclic vector for \( V(t) \otimes V(s) \). In this case \( v_h \bar{\otimes} v_l = v_0 \bar{\otimes} v_1 \).

**Proposition D.5.** The following are equivalent in \( V(t) \otimes V(s) \):

- \( v_h \bar{\otimes} v_l \) is a homogeneous cyclic vector
- \( s \notin \mathcal{C} \).

**Proof.** It is enough to compute \( Ev_h \bar{\otimes} v_l = v_h \bar{\otimes} Ev_l = [s] v_h \bar{\otimes} v_h. \)
Proposition D.6. Suppose \( s \in \mathbb{C} \). There exists a homogeneous cyclic vector for \( V(t) \hat{\otimes} V(s) \) if and only if \( t \notin \mathbb{C} \).

Proof. A generating vector must have weight \(-ts\) and we have already proven that \( v_h \otimes v_l \) is not sufficient for cyclicity under the present assumptions. Thus, we compute

\[
Ev_l \hat{\otimes} v_h = [ts] v_h \hat{\otimes} v_h = s [t] v_h \hat{\otimes} v_h.
\]

This shows \( v_l \hat{\otimes} v_h + v_h \hat{\otimes} v_l \) is a generating vector for \( V(t) \hat{\otimes} V(s) \) if and only if \( t \notin \mathbb{C} \). \( \square \)

Corollary D.7. The acyclicity locus is \( \mathcal{A} = \mathbb{C}^2 \).

Corollary D.8. The representations \( V(t) \otimes V(t^{-1}) \) for \( t \notin \mathbb{C} \) are homogeneous cyclic, reducible, and indecomposable.

We plot the acyclicity locus, denoted by circles, and the curves \( ts = 1 \) and \( ts = -1 \) in Figure 12 below. The other curves drawn denote single isomorphism classes of representations. It is enough to plot only the first and second quadrants by considering sign transfers. Each curve \( ts = c \) for \( c^2 \neq 1 \) corresponds to an isomorphism class of \( V(t) \hat{\otimes} V(s) \). If \( c^2 = 1 \), then each curve \( ts = c \) determines an isomorphism class of \( V(t) \hat{\otimes} V(s) \) for \( s^2 \neq 1 \). Each point along the curve \( s = |t| \) determines a unique decomposable tensor product representation. On the other hand, each point on the curve \( s = c \) determines a unique homogeneous cyclic representation whenever \( c^2 \neq 1 \).

![Figure 12. Isomorphism classes of representations \( V(t) \otimes V(s) \).](image)

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