NON-SINGULAR EXTENSIONS OF MORSE FUNCTIONS ON DISCONNECTED SURFACES

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Abstract. In this paper, we study non-singular extensions of Morse functions on closed orientable surfaces. By a non-singular extension of such a Morse function, we mean an extension to a function without critical points on some compact orientable 3-manifold having as boundary the given surface. In 1977, Curley characterized the existence of non-singular extensions of non-singular boundary germs in terms of combinatorics on associated labeled Reeb graphs. We apply Curley’s result to show that every Morse function on a closed orientable (possibly disconnected) surface has a non-singular extension to a 3-manifold that is connected.

1. Introduction

In the field of differential topology, major achievements like Smale’s proof of the h-cobordism theorem [7] show that Morse theory is evidently a powerful tool for investigating the topology of manifolds. Recall that a Morse function on a smooth manifold is a smooth function which has only non-degenerate critical points. A Morse function on a closed manifold can always be factored through its Reeb graph, which is a one-dimensional complex that is fundamental for constructing and studying Morse functions. In general, the Reeb graph of a function is the quotient of the source space by the equivalence relation that identifies points when they lie in the same connected component of a level set. For instance, it is known that two Morse functions on a closed orientable surface are $C^\infty$-equivalent if and only if their Reeb graphs are topologically equivalent [2–4].

We are concerned with non-singular extensions of Morse functions on a closed orientable surface $M$. By a non-singular extension of a Morse function $f : M \to \mathbb{R}$, we mean a function $F : N \to \mathbb{R}$ on a compact orientable 3-manifold $N$ with boundary $\partial N = M$ such that $F|\partial N = f$ and $F$ does not have critical points. Apart from Morse functions, it seems convenient to consider also the non-singular function $g : M \times \{0, 1\} \to \mathbb{R}$ which is restricted to Morse functions $M \times \{0\} = M \to \mathbb{R}$ on the boundary. By a non-singular extension of a non-singular function on $M$, we mean a non-singular extension $N \to \mathbb{R}$ of the boundary Morse function $M \times \{0\} = M \to \mathbb{R}$ which induces the given non-singular function for a suitable choice of a tubular neighborhood of $M$ in $N$.

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Non-singular extensions have been studied since the 1970s. For instance, Curley [1] gave a necessary and sufficient condition for the existence of non-singular extensions of a non-singular function germ on $M$. His condition is stated in terms of the combinatorics of the Reeb graph of the boundary Morse function, which is called a diagram in Curley’s terminology. More precisely, Curley introduced a labeling of the vertices of the diagram by signs to reflect the behavior of the non-singular function germ near the critical points of the boundary Morse function. Then, by Theorem 1 in [1], the given function germ on $M$ has a non-singular extension if and only if its labeled diagram has an admissible collapse, which is a certain map to some unlabeled diagram. It follows from Curley’s theorem that not every non-singular function germ admits a non-singular extension. Furthermore, as our Example 4.1 shows, even if non-singular extensions of a non-singular function germ exist, it might not be possible to achieve that the underlying 3-manifold is connected.

More recently, Laroche [5] considered the existence of non-singular extensions of non-singular germs on a compact non-orientable surface to a non-singular function on compact non-orientable 3-manifold; Yamamoto [12] gave a necessary and sufficient condition for non-singular extensions of non-singular germs $f : \partial V \times [0, 1) \to \mathbb{R}$ to an orientation-preserving immersion $F : V \to \mathbb{R}^2$ on a compact connected oriented surface; and Seigneur [10, 11] gave a necessary condition for extending non-singular function germs on the sphere $S^n$ to non-singular functions on the closed unit ball $D^{n+1}$.

In this paper, we develop Curley’s theorem by showing that any Morse function on an arbitrary (possibly disconnected) closed orientable surface has a non-singular extension whose underlying 3-manifold is connected (see Example 4.2 and Theorem 5.2). In order to be able to use Curley’s theorem, we prove Lemma 5.1, which deals with the extension of Morse functions to non-singular function germs.

The paper is organized as follows. Section 2 provides fundamental definitions, and introduces the notions of diagrams and labeled diagrams. In Section 3, we state Curley’s theorem in detail, and explain the necessary notion of collapses, which are certain maps from labeled diagrams to another diagram. In Section 4, we give some examples to motivate our main theorem. Finally, the purpose of Section 5 is to state and prove our main result, Theorem 5.2.

Throughout this paper, manifolds and maps between manifolds are smooth of class $C^\infty$ unless otherwise specified.

2. Preliminaries

Let $M$ be a closed surface and $f : M \to \mathbb{R}$ a function.

**Definition 2.1.** For a point $c \in M$, if the differential $df_c$ of $f$ at $c$ vanishes, then we call $c$ a critical point of $f$. If $df_c \neq 0$, then we call $c$ a regular point of $f$. If $c$ is a critical point of $f$, then we call $f(c)$ a critical value of $f$. For $y \in \mathbb{R}$, if $f^{-1}(y)$ is either empty or contains only regular points, then we call $y$ a regular value of $f$.

For a point $c \in M$, let $\varphi : U_c \to W_c$ be a chart around $c$, where $U_c \subset \mathbb{R}^2$ is an open neighborhood of the origin of $\mathbb{R}^2$, $W_c \subset M$ is an open neighborhood of the point $c$, and $\varphi(0) = c$. 
**Definition 2.2.** For a point \( c \in M \),
\[
\Phi_f(c) = \begin{pmatrix}
\frac{\partial^2 (f \circ \varphi)}{\partial x^2}(0) & \frac{\partial^2 (f \circ \varphi)}{\partial x \partial y}(0) \\
\frac{\partial^2 (f \circ \varphi)}{\partial y \partial x}(0) & \frac{\partial^2 (f \circ \varphi)}{\partial y^2}(0)
\end{pmatrix}
\]
is called the Hesse matrix of \( f \) at \( c \), where \((x, y)\) are coordinates of \( \mathbb{R}^2 \).

It is known that the rank of \( \Phi_f(c) \) does not depend on the choice of a local parameterization \( \varphi \) as above (for example, see [6]).

**Definition 2.3.** For a critical point \( c \) of \( f \), if the rank of the Hesse matrix \( \Phi_f(c) \) is equal to two, then we say that the critical point \( c \) is **non-degenerate**. Otherwise, we say that the critical point \( c \) is **degenerate**.

**Definition 2.4.** If all critical points of \( f \) are non-degenerate and the critical values are all distinct, then we call the function \( f \) a **Morse function**.

**Remark 2.5.** For a Morse function on a closed surface, the number of critical points is finite (for example, see [6, 8]).

**Definition 2.6.** Two functions \( f_1 : M_i \to \mathbb{R}, \ i = 0, 1 \), on not necessarily compact surfaces are called \( C^\infty \)-equivalent if there exist a diffeomorphism \( \Psi : M_0 \to M_1 \) and an orientation-preserving diffeomorphism \( \psi : \mathbb{R} \to \mathbb{R} \) such that the following diagram commutes.

\[
\begin{array}{ccc}
M_0 & \xrightarrow{f_0} & \mathbb{R} \\
\downarrow \psi & & \downarrow \psi \\
M_1 & \xrightarrow{f_1} & \mathbb{R}
\end{array}
\]

The following theorem is well known (for example, see [2] and [9, pp. 13–16]).

**Theorem 2.7.** Let \( f : M \to \mathbb{R} \) be a Morse function on a closed orientable surface. Then, for every critical point \( c \) of \( f \), there exists \( \varepsilon > 0 \) such that \( f \) restricted to \( f^{-1}((f(c) - \varepsilon, f(c) + \varepsilon)) \) is \( C^\infty \)-equivalent to one of the functions depicted in Figure 1, namely maximum, minimum and saddle point.

In the following, let \( M \) be a closed orientable surface and \( f : M \to \mathbb{R} \) a Morse function. Given such an \( f \), we define an equivalence relation on the points of \( M \) as follows. Two points \( a \) and \( b \) in \( M \) are said to be equivalent if \( f(a) = f(b) \), and \( a \) and \( b \) belong to the same connected component of \( f^{-1}(f(a)) \). In this case, we write \( a \sim b \).

**Definition 2.8.** The **diagram of \( f \)**, denoted by \( \text{dgm}(f) \), is the quotient space \( M/\sim \) equipped with the quotient topology.

We also define \( \text{dgm}(G) \) of a function \( G : N \to \mathbb{R} \) on a compact orientable manifold \( N \) in the same way.

Let \( q_f : M \to \text{dgm}(f) \) be the quotient mapping, and \( \bar{f} : \text{dgm}(f) \to \mathbb{R} \) be the unique continuous function such that \( \bar{f} \circ q_f = f \). Then, these maps fit into a commutative diagram.
FIGURE 1. On some neighborhood of each critical level set, the Morse function $f$ is $C^\infty$-equivalent to one of the height functions pictured above.

as follows.

$$
\begin{array}{c}
M \\
\downarrow q_f \\
dgm(f)
\end{array}
\xrightarrow{f}
\begin{array}{c}
\mathbb{R} \\
\uparrow \bar{f}
\end{array}
$$

An analogous commutative diagram exists for the diagram $dgm(G)$ of a function $G: N \to \mathbb{R}$ on a compact smooth manifold $N$.

A simple example is shown in Figure 2.

Remark 2.9. (1) It is known that $dgm(f)$ is always a one-dimensional complex whose vertices correspond bijectively via $q_f$ to the critical points of $f$. Furthermore, depending on the type of critical point of Figure 1, the behavior of $q_f$ and $\bar{f}$ near a critical point of $f$ is depicted in Figure 3.

(2) The restriction $\bar{f}$ to each edge of $dgm(f)$ is an embedding (compare Figure 4).
FIGURE 1. On some neighborhood of each critical level set, the Morse function $f$ is $C^\infty$-equivalent to one of the height functions pictured above.

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FIGURE 2. The diagram of a height function on the embedded $S^2$.

Local maximum point

Local minimum point

Saddle point A

Saddle point B

FIGURE 3. The forms of the vertices of $\text{dgm}(f)$ are dependent on the types of critical points of $f$.

FIGURE 4. An example of an impossible function on an edge of the diagram $\text{dgm}(f)$. 
FIGURE 5. Factorization of the Morse function $g' = g|_{M \times \{0\}}$ through a labeled diagram induced by the non-singular function $g : M \times [0, 1) \to \mathbb{R}$.

Now, let $g : M \times [0, 1) \to \mathbb{R}$ be a non-singular function such that $g|_{M \times \{0\}}$ is a Morse function (i.e. for a point $c_1 \in M \times [0, 1)$, the differential $dg_{c_1}$ of $g$ at $c_1$ does not vanish, but for a point $c_2 \in M \times \{0\}$, the differential $d(g|_{M \times \{0\}})_{c_2}$ of $g|_{M \times \{0\}}$ at $c_2$ may vanish). In this case, $\text{dgm}(g)$ denotes the diagram of the Morse function $g|_{M \times \{0\}}$ and we call it the diagram of $g$.

**Definition 2.10.** We label each vertex of $\text{dgm}(g)$ by either $+$ or $-$ as follows. For each critical point $c$ of $g|_{M \times [0]}$, let $v$ be the outward normal vector of $M \times [0, 1)$ at $c \in M = M \times \{0\}$. Since $g$ is non-singular, it follows that $dg_c \neq 0$, and we label the vertex corresponding to $c$ by

1. $+$ if $d g_c(v) > 0$,
2. $-$ if $d g_c(v) < 0$.

The diagram with labels at each vertex is called the labeled diagram of $g$ and is denoted by $\text{dgm}^+(g)$.

A simple example can be found in Figure 5.

### 3. Curley’s theorem

In this section, we introduce the so-called collapse map of non-singular extension, and Curley’s theorem by following Curley’s paper (for the details, see [1]).

For a closed orientable surface $M$, let $g : M \times [0, 1) \to \mathbb{R}$ be a non-singular function as defined in the introduction. That is, $g : M \times [0, 1) \to \mathbb{R}$ is a non-singular function such that $g|_{M \times [0]}$ is a Morse function. Although Curley introduces his theorem for a non-singular function germ $[g]$ of $g$ in his paper, we introduce his theorem without using the function germ $[g]$.

**Definition 3.1.** Suppose $N$ is a compact orientable three-dimensional manifold with $\partial N = M$. We say that a non-singular function $F : N \to \mathbb{R}$ is a non-singular extension of $g$ if there is a collaring embedding $i : M \times [0, 1) \to N$ with $i|_{M \times \{0\}}$ being the identity map of $M$ such that $F \circ i = g$. 
Let $F : N \to \mathbb{R}$ be a non-singular extension of $g : M \times [0, 1) \to \mathbb{R}$. That is, suppose that the diagram

$$
\begin{array}{ccc}
M \times [0, 1) & \xrightarrow{g} & \mathbb{R} \\
\downarrow^{i} & \searrow^{F} \\
N & \searrow \quad & \\
\end{array}
$$

is commutative. Let $\text{dgm}^+(g)$ be the labeled diagram of $g$, and $\text{dgm}(F)$ the diagram of $F$. Then, we have the commutative diagram

$$
\begin{array}{ccc}
\text{dgm}^+(g) & \xrightarrow{\tilde{g}} & \mathbb{R} \\
\downarrow^{C} & \searrow^{F} \\
\text{dgm}(F) & \searrow \quad & \\
\end{array}
$$

where $C : \text{dgm}^+(g) \to \text{dgm}(F)$ is induced by the collaring map $i$. The map $C : \text{dgm}^+(g) \to \text{dgm}(F)$ becomes a collapse, which is defined in Definition 3.3.

**Remark 3.2.** A collapse $C$ satisfies the following properties (for the details, see [1]).
(1) By $C$, the vertices of $\text{dgm}^+(g)$ are mapped to those of $\text{dgm}(F)$ bijectively.
(2) Around each vertex of $\text{dgm}^+(g)$, $C$ is depicted in Figure 6.

Let us consider the existence of a compact orientable three-dimensional manifold $N$ and a non-singular extension $F : N \to \mathbb{R}$ of the given non-singular function $g : M \times [0, 1) \to \mathbb{R}$. Let $h : \text{dgm}(h) \to \mathbb{R}$ be a function on a 1-complex $\text{dgm}(h)$ such that the restriction of $h$ on each edge is an embedding.
Definition 3.3. A map $C : \text{dgm}^+(g) \to \text{dgm}(h)$ is called a collapse if

(1) $C$ satisfies the properties in Remark 3.2 with $\text{dgm}(F)$ replaced by $\text{dgm}(h)$,

(2) the diagram

\[
\begin{array}{ccc}
\text{dgm}^+(g) & \xrightarrow{\bar{g}} & \mathbb{R} \\
\downarrow{C} & & \downarrow{h} \\
\text{dgm}(h) & & \\
\end{array}
\]

is commutative.

We define a partial order for the vertices of $\text{dgm}(h)$ as follows. For two vertices $x$ and $y$ of $\text{dgm}(h)$, we write $x < y$ if $h(x) < h(y)$ and there is a strictly monotonic path $s : [0, 1] \to \text{dgm}(h)$ such that $s(0) = x$ and $s(1) = y$ (i.e. if $t_1 < t_2$ for $t_1, t_2 \in [0, 1]$, $h \circ s(t_1) < h \circ s(t_2)$). Let $G^+$ (respectively $G^-$) be the set of vertices of type $G^+$ (respectively $G^-$) in $\text{dgm}^+(g)$ in the sense of Figure 6.

Definition 3.4. If there is a one-to-one correspondence $\gamma : G^+ \to G^-$ such that $C \circ \gamma(a) < C(a)$ for each $a \in G^+$, we say that the collapse $C$ is allowable.

The following characterization was proved by Curley [1].

Theorem 3.5. (Curley [1], modified) Let $M$ be a closed orientable (possibly disconnected) surface and $g : M \times [0, 1) \to \mathbb{R}$ be a non-singular function such that $g|_{M \times [0]}$ is a Morse function. Then, there exists a compact orientable three-dimensional manifold $N$ with $\partial N = M$ and a non-singular extension $F : N \to \mathbb{R}$ of $g$, if and only if there is an allowable collapse $C : \text{dgm}^+(g) \to \text{dgm}(h)$ and a continuous function $h : \text{dgm}(h) \to \mathbb{R}$ such that the restriction of $h$ on each edge is an embedding. Furthermore, if there exists an allowable collapse $C$ as above, we can construct a manifold $N$ and a non-singular extension $F$ as above in such a way that $\text{dgm}(F) = \text{dgm}(h)$ and $h = \bar{F}$.

In Curley’s theorem [1], Curley treats Theorem 3.5 to a non-singular function germ $[g]$ of $g : M \times [0, 1) \to \mathbb{R}$ at $M$, but, by choosing a representative element of germ $[g]$ appropriately, we can use his theorem to the non-singular function $g$.

4. Examples

In this section, we introduce some examples of existence of non-singular extension and non-existence of non-singular extension.

Example 4.1. Let us consider the non-singular function $g : (S^2 \sqcup S^2) \times [0, 1) \to \mathbb{R}$ as depicted in Figure 7. We can show that $g$ cannot be extended to a non-singular function on a compact orientable three-dimensional manifold that is connected, as follows.

The labeled diagram $\text{dgm}^+(g)$ of $g$ is as depicted in Figure 8.

Let $v_1, v_2, v_3$ and $v_4$ be the vertices of $\text{dgm}^+(g)$ as in Figure 8. By property (1) in Definition 3.3 of a collapse, around each vertex of $\text{dgm}^+(g)$, a collapse $C : \text{dgm}^+(g) \to \text{dgm}(h)$ must necessarily behave as depicted in Figure 9.

Let $w_1, w_2, w_3$ and $w_4$ be the vertices of $\text{dgm}(h)$ such that $C(v_i) = w_i$ and $d_i$ the edge in $\text{dgm}(h)$ incident to $w_i$ for $i = 1, 2, 3$ and 4. Let us consider which edge is connected
Let us consider the non-singular function Example 4.1.

In this section, we introduce some examples of existence of non-singular extension and non-allowably, we can use his theorem to the non-singular function $g$ as depicted in Figure 7. We can show that $\gamma$ of $h$ on each edge is an embedding. Furthermore, if there exists an allowable collapse $C$ as in $dgm$ of $h$. Let $g$ be a partial order for the vertices of $dgm$. In Curley’s theorem, Curley treats Theorem 3.5 to a non-singular function germ.

$\{v_1, v_2, v_3, v_4\}$ be the vertices of $dgm$. For two vertices $v$ and $w$ of $dgm$, we say that the collapse $C$ is called a $\overline{G}$ such that $v \rightarrow w$. Let $\gamma$ be a Morse function on the surface and $g$.

$\gamma$ such that $\gamma_1(v_1) < \gamma_2(v_2) < \gamma_3(v_3) < \gamma_4(v_4)$.

$\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4$.

$\gamma_1(v_1) < \gamma_2(v_2) < \gamma_3(v_3) < \gamma_4(v_4)$.

$\gamma_1(v_1) < \gamma_2(v_2) < \gamma_3(v_3) < \gamma_4(v_4)$.

$\gamma_1(v_1) < \gamma_2(v_2) < \gamma_3(v_3) < \gamma_4(v_4)$.
with the edge $d_i$ for each $i$. For $i = 2$, $d_2$ is an upward edge, and there is no edge other than $d_1$ that lies above $d_2$. Thus, the edge $d_2$ must be connected with the edge $d_1$. By a similar argument, we see that $d_3$ must be connected with the edge $d_4$. Therefore, we have a unique map $C : \text{dgm}^+(g) \to \text{dgm}(h)$ that satisfies the conditions for a collapse, as depicted in Figure 10.

Since $\text{dgm}(h)$ is disconnected, so is the three-dimensional manifold for a non-singular extension. Thus, $g$ cannot be extended to a non-singular function on a compact orientable three-dimensional manifold that is connected.

**Example 4.2.** Let us consider the Morse function $f : S^2 \sqcup S^2 \to \mathbb{R}$ as depicted in Figure 11. We can show that $f$ can be extended to a non-singular function on a compact orientable three-dimensional manifold that is connected, as follows.

The diagram $\text{dgm}(f)$ of $f$ is as depicted in Figure 12.

Let $v_1$, $v_2$, $v_3$ and $v_4$ be the vertices of $\text{dgm}(f)$ as depicted in Figure 12. We label the vertices $v_1$, $v_2$, $v_3$ and $v_4$ with $+$, $+$, $-$ and $-$, respectively, and denote it by $\text{dgm}^+(f)$. Then, we construct the one-dimensional complex $\text{dgm}(h)$ and the map $C : \text{dgm}^+(f) \to \text{dgm}(h)$ as depicted in Figure 13.
Then, for the vertices in $\text{dgm}^+(f)$,

1. $v_1$ is of type $M^+$,
2. $v_2$ is of type $N^+$,
3. $v_3$ is of type $M^-$, and
4. $v_4$ is of type $N^-$.

Therefore, the map $C : \text{dgm}^+(f) \to \text{dgm}(h)$ satisfies the conditions for a collapse as depicted in Figure 13. By Lemma 5.1 in Section 5 and Theorem 3.5, there is a non-singular extension $F : N \to \mathbb{R}$ on a compact orientable three-dimensional manifold $N$ with $\partial N = S^2 \sqcup S^2$ such that $F|_{S^2 \sqcup S^2} = \bar{f}$ and $\text{dgm}(F) = \text{dgm}(h)$. Since $\text{dgm}(h)$ is connected, so is the three-dimensional manifold $N$ for this non-singular extension $F$. Thus, $f$ can be extended to a non-singular function on a compact orientable three-dimensional manifold that is connected in this example.

5. Main theorem

In this section, we prove our main theorem (see Theorem 5.2). For this purpose, we need the following lemma.
LEMMA 5.1. For a closed orientable surface $M$, let $f : M \to \mathbb{R}$ be a Morse function and $\text{dgm}(f)$ the diagram of $f$. Let $\text{dgm}^+(f)$ be the result of labeling each vertex of diagram $\text{dgm}(f)$ by one of the signs $+$ and $\cdot$. Then, there is a non-singular function $g : M \times [0, 1) \to \mathbb{R}$ such that $g\mid_{M \times \{0\}} = f$ and $\text{dgm}^+(g) = \text{dgm}^+(f)$.

Proof. Let $v_1, \ldots, v_n$ be a list of the critical points of $f$ such that $f(v_1) < \cdots < f(v_n)$ (compare Definition 2.4). By composing with an automorphism of $\mathbb{R}$, we may assume without loss of generality that $f(v_k) = 2k$ for all $k = 1, \ldots, n$. For every $k = 1, \ldots, n$, we set $M_k = f^{-1}([2k-1, 2k+1))$ and $M' = \bigcup_{i=1}^n M_i$. In particular, note that $M' = M$.

Let us prove the lemma by descending induction on $k$. Namely, for $k = n, \ldots, 1$, we construct iteratively non-singular functions $g_k : M' \times [0, 1) \to \mathbb{R}$ such that $g_k\mid_{M' \times \{0\}} = g_{k+1}$ (when $k < n$), $g_k\mid_{M' \times \{0\}} = f\mid_{M_i}$ and $\text{dgm}^+(g_k) = \text{dgm}^+(f\mid_{M_i})$. Then, the function $g = g_1$ will have all the desired properties. For the construction, we use the models and the functions shown in Figure 14.

The construction of $g_n$ is as follows. For $v_n$, since $M$ is compact, $v_n$ is a local maximum critical point of $f$. By Theorem 2.7, for sufficiently small $0 < \epsilon < 1$, we may assume that $f\mid_{f^{-1}([2n-\epsilon, 2n+\epsilon])} : f^{-1}([2n-\epsilon, 2n+\epsilon]) \to \mathbb{R}$ is given by $f\mid_{f^{-1}([2n-\epsilon, 2n+\epsilon])} = \pi_{\max} + 2n$, where $\pi_{\max}$ denotes the function $\pi_{\max} : M^{\max} \to \mathbb{R}$ as depicted in Figure 14. Furthermore, by using Ehresmann’s fibration theorem (for the details, see [9]), $f^{-1}([2n-1, 2n-\epsilon]) \cong f^{-1}(2n-\epsilon) \times [2n-1, 2n-\epsilon]$, and $f\mid_{f^{-1}([2n-1, 2n-\epsilon])} : f^{-1}(2n-\epsilon) \times [2n-1, 2n-\epsilon] \to \mathbb{R}$ is given by $f(x, t) = t$ for $x \in f^{-1}(2n-\epsilon)$, and $t \in [2n-1, 2n-\epsilon]$. Thus, $M_n \cong M^{\max} \cup \phi f^{-1}(2n-\epsilon) \times [2n-1, 2n-\epsilon]$, where $\phi : \partial M^{\max} \to f^{-1}(2n-\epsilon)$ is a suitable diffeomorphism, and $f\mid_{M_n}$ can be given by the function in Figure 15. If $q_f(v_n)$ is labeled $+$ (respectively $\cdot$), let the non-singular function $g_n$ be the height function (a) (respectively (b)) in Figure 15.

Next, let us construct $g_k$ for $k < n$, where we assume that there exist the non-singular functions $g_{k+1}, g_{k+2}, \ldots, g_n$ and that $g_{k+1}(2k+1) = \bigcup_{j=1}^l S^1 j \times [0, 1)$ for $l \in \mathbb{N}$.

![Figure 14. Functions on the models $M^{\max}, M^{\min}, M^{SA}$ and $M^{SB}$.](image-url)
or \( g_k^{-1}(2k + 1) = \emptyset \). Depending on the type of \( v_k \), by using Ehreman’s fibration theorem in the same way as for \( v_n \), \( f|_{M_k} \) is given by one of the functions shown in Figure 16. (Although \( M_k \) is different from \( M_n \) exactly, we can use Ehreman’s fibration theorem for \( M_k \) similarly. In the case of \( M_k \), \( M_k \) may have some connected manifolds which do not have critical points. So \( f|_{M_k} \) is given the function from the suitable model and some cylinder to \( R \) according to the connected component.) Let \( N^\beta = M^\beta \sqcup (S^1 \times [2k − 1, 2k + 1]) \sqcup (S^1 \times [2k − 1, 2k + 1]) \sqcup \cdots \sqcup (S^1 \times [2k − 1, 2k + 1]) \) for \( \beta \in \{\max, \min, \SA, \SB\} \). Then, \( N^\beta \) is diffeomorphism \( M_k \). If \( q_f(v_k) \) is labeled + (respectively -), let the non-singular function \( h_k : N^\beta \times [0, 1) \rightarrow \mathbb{R} \) be the appropriate height function shown in Figure 17.

Then, \( M'_k \times [0, 1) \cong N^\beta \times [0, 1) \cup_{\phi_2} M'_{k+1} \times [0, 1) \), where

\[
\phi_2 : g_k^{-1}(2k + 1) \rightarrow h_k^{-1}(2k + 1)
\]

is a suitable diffeomorphism. Let \( g_k \) be

\[
g_k(x) = \begin{cases} 
  g_{k+1}(x) & \text{if } x \in M'_{k+1} \times [0, 1), \\
  h_k(x) & \text{if } x \in M_k \times [0, 1). 
\end{cases}
\]

There is some possibility of \( \text{dgm}^+(g_k) \neq \text{dgm}^+(f|_{M'_i}) \) (for example, if \( \phi_2 \) is not defined correctly, the vertex of \( v_k \) in \( \text{dgm}^+(f) \) does not correspond with the vertex in \( \text{dgm}^+(g_k) \) nicely). By modifying the diffeomorphism \( \phi_2 \), we can see that \( \text{dgm}^+(g_k) = \text{dgm}^+(f|_{M'_i}) \). Then, \( g_k \) satisfies the above conditions.

Let \( M_i \) be connected orientable closed surfaces and \( M \) be the disjoint union \( \bigsqcup_{i=0}^k M_i \). Let \( f : M \rightarrow \mathbb{R} \) be a Morse function on \( M \) and \( f_i : M_i \rightarrow \mathbb{R} \) Morse functions for \( 1 \leq i \leq k \) defined by \( f|_{M_i} = f_i \). Based on the above assumption, we have the following main theorem.

**Theorem 5.2.** For any Morse function \( f : M \rightarrow \mathbb{R} \), there is a compact connected orientable three-dimensional manifold \( N \) with boundary \( \partial N = M \) and a non-singular extension \( F : N \rightarrow \mathbb{R} \) such that \( F|_{\partial N} = f \).

**Proof.** Let \( \text{dgm}(f_i) \) be the diagram of \( f_i \), \( c^j \in M \) (\( 1 \leq j \leq l \)) the critical points of \( f \), and \( \text{dgm}(f) \) the diagram of \( f \). Assume that each closed surface is disjoint. Let \( I_i \) be a
monotonic path in \( \text{dgm}(f_i) \) from a maximum vertex \( m_i \) to a minimum vertex \( n_i \). Then, there is exactly one maximum vertex \( m_{k_1} \in \{m_1, m_2, \ldots, m_k\} \) such that the critical value \( \tilde{f}(m_{k_1}) \) is the maximum value in \( \{\tilde{f}(m_1), \tilde{f}(m_2), \ldots, \tilde{f}(m_k)\} \), and there is exactly one minimum vertex \( n_{k_2} \in \{n_1, n_2, \ldots, n_k\} \) such that the critical value \( \tilde{f}(n_{k_2}) \) is the minimum value in \( \{\tilde{f}(n_1), \tilde{f}(n_2), \ldots, \tilde{f}(n_k)\} \). We label each vertex in \( \text{dgm}(f) \) as follows (for example, see Figure 18 and Figure 19):

1. \( m_{k_1} \) is labeled by +,
2. \( n_{k_2} \) is labeled by −,
3. \( m_1, m_2, \ldots, m_{k_1-1}, m_{k_1+1}, \ldots, m_k \) are labeled by −,
4. \( n_1, n_2, \ldots, n_{k_2-1}, n_{k_2+1}, \ldots, n_k \) are labeled by +,
5. the other maximum vertices are labeled by +,
6. the other minimum vertices are labeled by −,
7. the saddle points A are labeled by −, and
8. the saddle points B are labeled by +.

We denote the resulting labeled diagram by \( \text{dgm}^*(f) \).
FIGURE 16. Models and their functions.

monotonic path in dgm $(f)$ from a maximum vertex $m_k \in \{m_1, m_2, \ldots, m_k\}$ such that the critical value $\bar{f}(m_k)$ is the maximum value in $\{\bar{f}(m_1), \bar{f}(m_2), \ldots, \bar{f}(m_k)\}$, and there is exactly one minimum vertex $n_k \in \{n_1, n_2, \ldots, n_k\}$ such that the critical value $\bar{f}(n_k)$ is the minimum value in $\{\bar{f}(n_1), \bar{f}(n_2), \ldots, \bar{f}(n_k)\}$. We label each vertex in dgm $(f)$ as follows (for example, see Figure 18 and Figure 19):

1. $m_k$ is labeled by $+$,
2. $n_k$ is labeled by $-$,
3. $m_1, m_2, \ldots, m_k - 1, m_k + 1, \ldots, m_k$ are labeled by $-$,
4. $n_1, n_2, \ldots, n_k - 1, n_k + 1, \ldots, n_k$ are labeled by $+$,
5. the other maximum vertices are labeled by $+$,
6. the other minimum vertices are labeled by $-$,
7. the saddle points A are labeled by $-$,
8. the saddle points B are labeled by $+$.

We denote the resulting labeled diagram by $dgm^*(f)$.

FIGURE 17. Extending the function of Figure 16 to non-singular function germs.
Let $I$ be a new edge which connects $m_{k_1}$ with $n_{k_2}$ and $\bar{g} : (\text{dgm}(f) \cup I) \to \mathbb{R}$ be the natural extension of the projection $\hat{f} : \text{dgm}(f) \to \mathbb{R}$, where $\bar{g}|_I$ is embedding (see Figure 19). We define an equivalence relation in $\text{dgm}(f) \cup I$ as follows: two points $x$ and $y \in \text{dgm}(f) \cup I$ are equivalent if $x = y$, or $\bar{g}(x) = \bar{g}(y)$ ($x, y \in \bigcup_{i=1}^{k} I_i \cup I$). The equivalence relation will be denoted by $x \sim y$. Let $\text{dgm}(h)$ be the quotient space $(\text{dgm}(f) \cup I)/\sim$. We define $C : \text{dgm}^s(f) \to \text{dgm}(h)$ as $C(x) = [x]$ (for example, see Figure 20).

Then, for each vertex in $\text{dgm}^s(f)$:

1. $m_{k_1}^+$ is of type $M^+$,
2. $n_{k_2}^-$ is of type $N^-$,
3. $m_{-1}, m_2^-, \ldots, m_{k_1-1}^-, m_{k_1+1}^-, \ldots, m_k^-$ are of type $M^-$,
Extensions of Morse functions on surfaces

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure20}
\caption{Example of allowable collapse.}
\end{figure}

\begin{enumerate}
\item[(4)] $n_1^+, n_2^+, \ldots, n_{k-1}^+, n_{k+1}^+, \ldots, n_k^+$ are of type $N^+$,
\item[(5)] the other maximum vertices are of type $M^+$,
\item[(6)] the other minimum vertices are of type $N^-$,
\item[(7)] the saddle points $A$ are of type $J^-$, and
\item[(8)] the saddle points $B$ are of type $J^+$.
\end{enumerate}

Since no vertex of type $G^+$ or $G^-$ appears, thus $C$ is an allowable collapse (see Definition 3.4). By Lemma 5.1, we can construct a non-singular function $g : M \times [0, 1) \to \mathbb{R}$ such that $g|_{M \times \{0\}} = f$ and $\text{dg}_m^+(g) = \text{dg}_m^+(f)$. Thus, by Theorem 3.5, there exists a compact orientable three-dimensional manifold $N$ with $\partial N = M$ and a non-singular extension $F : N \to \mathbb{R}$ of $g$. Since $\text{dg}_m(F) = \text{dg}_m(h)$ is connected, the constructed manifold $N$ is also connected. \hfill \Box

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