Riemannian Manifolds with Diagonal Metric. The Lamé and Bourlet Systems

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Abstract

We discuss a Lie algebraic and differential geometry construction of solutions to some multidimensional nonlinear integrable systems describing diagonal metrics on Riemannian manifolds, in particular those of zero and constant curvature. Here some special solutions to the Lamé and Bourlet type equations, determining by \( n \) arbitrary functions of one variable are obtained in an explicit form. For the case when the sum of the diagonal elements of the metric is a constant, these solutions are expressed as a product of the Jacobi elliptic functions and are determined by \( 2n \) arbitrary constants.

1 Introduction

The classical differential geometry serves as an injector of many equations integrable in this or that sense. Among them, the Lamé and Bourlet equations play especially remarkable role. These equations arise, in particular, in the following way.

Let \((U; z^1, \ldots, z^n)\) be a chart on a Riemannian manifold \((M, g)\), such that the metric tensor \( g \) has on \( U \) the form

\[
g|_U = \sum_{i=1}^{n} \beta_i^2 \, dz^i \otimes dz^i. \tag{1.1}
\]
In such a situation the metric tensor \( g \) is said to be *diagonal* with respect to the coordinates \( z^i \). The functions \( \beta_i \) are called the *Lamé coefficients*. Introduce an orthonormal basis in the space of 1-forms on \( U \) defining

\[
\theta^i = \beta_i dz^i.
\]

Here and in what follows we use for the indices referring to the orthonormal basis the same letters as for ones referring to the coordinate basis, but supply them with a hat. Note that for an orthonormal basis there is no actual distinction between lower and upper indices. In terms of the forms \( \theta^i \) the metric tensor is written as

\[
g|_U = \sum_{i=1}^n \theta^i \otimes \theta^i.
\]

Let us find the curvature two-forms \( \Omega^i_{\hat{j}\hat{k}} \) of the Levi-Civita connection corresponding to the metrics given by (1.1). The simplest way here is to use the second Cartan structural equation

\[
\Omega^i_{\hat{j}\hat{k}} = d\omega^i_{\hat{j}\hat{k}} + \sum_{k=1}^n \omega^i_{\hat{k}\hat{j}} \wedge \omega^k_{\hat{j}}.
\]

(1.2)

where \( \omega^i_{\hat{j}} \) are connection 1-forms related to the connection coefficients \( R^i_{\hat{j}\hat{k}} \) by the equality

\[
\omega^i_{\hat{j}} = \sum_{k=1}^n R^i_{\hat{j}\hat{k}} \theta^k.
\]

For the connection coefficients in the case of an orthonormal basis one has the expression

\[
R^i_{\hat{j}\hat{k}} = \frac{1}{2}(e^k_{\hat{i}\hat{j}} + e^j_{\hat{i}\hat{k}} - e^i_{\hat{j}\hat{k}}),
\]

where the functions \( e^j_{\hat{i}\hat{k}} \) are determined by the relation

\[
d\theta^i = -\frac{1}{2} \sum_{i,j=1}^n c^j_{\hat{i}\hat{k}} \theta^i \wedge \theta^k.
\]

Defining the so called *rotation coefficients*

\[
\gamma_{ij} = \frac{1}{\beta_i} \partial_i \beta_j, \quad i \neq j, \quad (1.3)
\]
we find
\[d\theta^i = \frac{1}{\beta_i} \sum_{j=1}^{n} \gamma_{ji} \theta^j \wedge \theta^i.\]

From this relation it follows that
\[c^i_{jk} = \frac{1}{\beta_i} (\gamma_{ki} \delta_{ij} - \gamma_{ij} \delta_{ik}),\]
therefore, in the case under consideration one has
\[\Gamma^i_{jk} = \frac{1}{\beta_k} (\gamma_{jk} \delta_{ki} - \gamma_{ik} \delta_{kj}),\]
and we come to the following expression for the connection forms
\[\omega^i_{j} = \gamma_{ji} dz^i - \gamma_{ij} dz^j.\]

Substituting this expression in (1.2), we obtain
\[\Omega^i_{j} = - \sum_{k \neq i,j} (\partial_k \gamma_{ji} - \gamma_{ki} \gamma_{jk}) dz^i \wedge dz^k - \sum_{k \neq i,j} (\partial_k \gamma_{ij} - \gamma_{ik} \gamma_{kj}) dz^k \wedge dz^j - (\partial_i \gamma_{ij} + \partial_j \gamma_{ji} + \sum_{k \neq i,j} \gamma_{ki} \gamma_{kj}) dz^i \wedge dz^j. \quad (1.4)\]

Using (1.4) we conclude that the Riemannian submanifold \((U, g|_U)\) of the Riemannian manifold \((M, g)\) is flat if and only if the rotation coefficients \(\gamma_{ij}\) satisfy the following system of partial differential equations
\[\partial_k \gamma_{ij} = \gamma_{ik} \gamma_{kj}, \quad i \neq j \neq k, \quad (1.5)\]
\[\partial_i \gamma_{ij} + \partial_j \gamma_{ji} + \sum_{k \neq i,j} \gamma_{ki} \gamma_{kj} = 0, \quad i \neq j, \quad (1.6)\]
where the notation \(i \neq j \neq k\) means that \(i, j, k\) are distinct. From the other hand, let \((U; z^1, \ldots, z^n)\) be a chart on the manifold \(M\), and we have a solution \(\gamma_{ij}\) of equations (1.3), (1.4). Let us rewrite (1.3) in the form
\[\partial_i \beta_j = \gamma_{ij} \beta_i,\]
and consider these equalities as equations for the functions \(\beta_i\). It is easy to show that due to (1.3), the integrability conditions for these equations are
satisfied, and we can find the functions $\beta_i$ which play the role of the Lamé coefficients having $\gamma_{ij}$ as the corresponding rotation coefficients. If we define a metric tensor on $U$ by

$$g = \sum_{i=1}^{n} \beta_i^2 \, dz^i \otimes dz^i,$$

then submanifold $U$ becomes a flat Riemannian manifold whose metric is diagonal with respect to the coordinates $z^i$.

Equations (1.5), (1.6) are called the Lamé equations. With the so called Egoroff property, $\gamma_{ij} = \gamma_{ji}$, equations (1.6) are equivalent to the following ones:

$$\left( \sum_{k=1}^{n} \frac{\partial}{\partial k} \right) \gamma_{ij} = 0, \quad i \neq j. \quad (1.8)$$

The corresponding solutions are represented in the form $\beta_i^2 = \partial_i F$ where $F$ is some function of the coordinates $z^i$. The system consisting of equations (1.5) and (1.8) is called sometimes the Darboux-Egoroff equations.

If $(U, g|_U)$ is a Riemannian manifold of constant curvature with the sectional curvature $k$, then we have \[\Omega^i_j = k \theta^i \wedge \theta^j = k \beta_i \beta_j \, dz^i \wedge dz^j.\]

Therefore, taking into account (1.4), one sees that the Riemannian submanifold $(U, g|_U)$ is of constant curvature with the sectional curvature $k$ if and only if the rotation coefficients satisfy the equations

$$\partial_k \gamma_{ij} = \gamma_{ik} \gamma_{kj}, \quad i \neq j \neq k; \quad (1.9)$$

$$\partial_i \gamma_{ij} + \partial_j \gamma_{ji} + \sum_{k \neq i, j} \gamma_{ki} \gamma_{kj} + k \beta_i \beta_j = 0, \quad i \neq j. \quad (1.10)$$

We call equations (1.9), (1.10) and (1.3) the Bourlet type equations. The Bourlet equations in the precise sense correspond to the case with $k = 1$ and $\sum_{i=1}^{n} \beta_i^2 = 1$, see, for example, [2, 3].

Sometimes it is suitable to rewrite at least a part of equations (1.9), (1.10) in a ‘Laplacian’ type form. Impose the condition

$$k \sum_{i=1}^{n} \beta_i^2 = c, \quad (1.11)$$
where \( c \) is a constant. It is convenient to allow the functions \( \beta_i \), and hence the functions \( \gamma_{ij} \), to take complex values. Therefore, we will assume that \( c \) is an arbitrary complex number. One can easily get convinced by a direct check with account of (1.3) and (1.11) that there takes place the relation

\[
\partial_i \beta_i = - \sum_{j \neq i} \gamma_{ij} \beta_j.
\]

Now, using the same calculations as those in [4, 5], and introducing, as there, the operators

\[
\Delta(i) = \sum_{j \neq i} \partial_j^2 - \partial_i^2,
\]

we obtain from equations (1.10)

\[
\Delta(i) \beta_i = \sum_{j \neq i} \beta_i[(\beta_i^{-1} \partial_i \beta_j)^2 - (\beta_j^{-1} \partial_j \beta_i)^2] - 2 \sum_{j \neq k \neq i} \beta_j \beta_k^{-2} (\partial_k \beta_i)(\partial_k \beta_j) + \beta_i(k \beta_i^2 - c).
\]

For \( n = 2 \) equations (1.9) and (1.3) are absent; equation (1.10) is reduced to the Liouville and sine-Gordon equations for \( \beta_1^2 + \beta_2^2 \) equals 0 and 1, respectively; while (1.6) is the wave equation. This is why for higher dimensions the Bourlet type equations with a nonzero constant \( c \) in (1.11), with \( c = 0 \), and the Lamé equations are called sometimes multidimensional generalisations of the sine-Gordon, Liouville, and wave equations, respectively, see, for example, [4, 5, 6, 7].

In the beginning of eighties an interest to the Lamé and Bourlet type equations was revived. In particular, it was shown that system (1.9), (1.10), (1.11) with \( k = 1 \) and \( c = 1 \) provides the necessary and sufficient conditions for the construction of a local immersion of the Lobachevsky space \( L_n \) into \( \mathbb{R}^{2n-1} \) [4], see also [3]. Further, as it follows from the results of works [8, 4], the problems of description of \( n \)-orthogonal curvilinear coordinate systems and of the classification of integrable Hamiltonian systems of hydrodynamic type [10] are almost equivalent. Note that system (1.9) is a natural generalisation [11] of the three wave system which is a relevant object in nonlinear optics. The Lamé equations also arise very naturally in the context of the Cecotti-Vafa equations describing topological-antitopological fusion, see [12].
and references therein, and in those of the multidimensional generalisations of the Toda type systems [13].

Probably the most interesting modern area where the Lamé equations with the corresponding initial conditions appear to be quite relevant, is related to the theory of Frobenius manifolds in the spirit of B. A. Dubrovin [12, 14], N. Hitchin [13] and Yu. I. Manin [16]. In particular, in the last very remarkable paper semisimple Frobenius manifolds are related to solutions of the Schlesinger equations, constrained by some special initial conditions. In our notations it corresponds to solutions of the Lamé system satisfying the Egoroff property, condition (1.11) with \( c = 0 \), which together automatically provide the validity of (1.3), since here \( (\sum \partial_k) \beta_i = 0 \); and also a rather restrictive requirement which is equivalent to \( (\sum z^k \partial_k) \beta_i \sim \beta_i \).

Finally note that the classification and description of diagonal metrics seems to be relevant for some modern problems of supergravity theories, including their elementary and solitonic supersymmetric \( p \)-brane solutions, see, for example, [17] and references therein.

The very fact of integrability of the equations in question has been established for quite a long time ago; the general solution is defined by \( n(n - 1)/2 \) functions of two variables for the Lamé system, and by \( n(n - 1) \) functions of one variable and \( n \) constants for the Bourlet system. However, an explicit form of the solutions for higher dimensions remained unknown.

In the present work we obtain in an explicit and rather simple form some special class of the solutions to the Lamé equations and to the Bourlet type equations with and without condition (1.11). If one does not impose condition (1.11), then our solutions are determined by \( n \) arbitrary functions of one variable, while with condition (1.11) the obtained solutions of the Bourlet equations are expressed as products of Jacobi elliptic functions and are determined by \( 2n \) arbitrary constants. The derivation of the solutions to both of these systems is given by using two different methods. One is based on the geometrical interpretation of the corresponding equations. Another approach uses a zero curvature (Lax type) representation of the Lamé and Bourlet type equations.

The Lax type representations of the Lamé and Bourlet type equations, different from ours, were considered in [11] and [10], respectively; see also [13]. In particular, the author of [11], using a multidimensional generalisation of the Zakharov-Shabat dressing method [19], succeeded to obtain some explicit solutions of the Lamé equations parametrised by \( n \) functions of one
variable, which are in a sense complementary to those presented below. In the beginning of November 1996, we were informed by V. E. Zakharov that he extended the results of [11] to the Bourlet system.

2 Bourlet type equations

We begin with the description of the zero curvature representation of the Bourlet type equations following [5]. Consider the case \( k > 0 \). Here the zero curvature representation is based on the Lie group \( O(n+1, \mathbb{R}) \). For the case \( k < 0 \) we should use the Lie group \( O(n, 1) \). Actually, we can complexify the Bourlet equations allowing the functions \( \beta_i \) and \( \gamma_{ij} \) to take complex values. In such a case we should use for the construction of the zero curvature representation the complex Lie group \( O(n+1, \mathbb{C}) \) and here positive and negative \( k \) can be considered simultaneously. It is clear that without any loose of generality we can take \( k = 1 \).

Let \( M_{ab} \) be the elements of the Lie algebra \( \mathfrak{o}(n+1, \mathbb{R}) \) of the Lie group \( O(n+1, \mathbb{R}) \) defined as

\[
(M_{ab})_{cd} = \delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad},
\]

The commutation relations for these elements have the standard form

\[
[M_{ab}, M_{cd}] = \delta_{ad}M_{bc} + \delta_{bc}M_{ad} - \delta_{ac}M_{bd} - \delta_{bd}M_{ac},
\]

and any element \( X \) of \( \mathfrak{g} \) can be represented as

\[
X = \sum_{a,b=1}^{n+1} x_{ab}M_{ab}.
\]

Such a representation is unique if we suppose that \( x_{ab} = -x_{ba} \).

In what follows we assume that the indices \( a, b, c, \ldots \) run from 1 to \( n+1 \), while the indices \( i, j, k, \ldots \) run from 1 to \( n \). Let \( (U; z^1, \ldots, z^n) \) be a chart on some smooth manifold \( M \). Consider the connection \( \omega = \sum_{i=1}^{n} \omega_i dz^i \) on the trivial principal fibre bundle \( U \times O(n+1, \mathbb{R}) \) with the components given by

\[
\omega_i = \sum_{k=1}^{n} \gamma_{ki}M_{ik} + \beta_i M_{i,n+1}.
\]
One can get convinced that the Bourlet type equations (1.3), (1.9) and (1.10) are equivalent to the zero curvature condition for the connection $\omega$, which, in terms of the connection components, has the form

$$\partial_i \omega_j - \partial_j \omega_i + [\omega_i, \omega_j] = 0.$$  \hspace{1cm} (2.2)

Identify the Lie group $O(n, \mathbb{R})$ with the Lie subgroup of $O(n+1, \mathbb{R})$ formed by the matrices $A \in O(n + 1, \mathbb{R})$, such that

$A_{i,n+1} = 0, \quad A_{n+1,j} = 0, \quad A_{n+1,n+1} = 1.$

Similarly, identify the Lie algebra $\mathfrak{o}(n, \mathbb{R})$ with the corresponding subalgebra of $\mathfrak{o}(n + 1, \mathbb{R})$.

Let the connection $\omega$ with the components of form (2.1) satisfies the zero curvature condition (2.2). Suppose that $U$ is simply connected, then there exists a mapping $\varphi$ from $U$ to $O(n + 1, \mathbb{R})$, such that

$$\omega_i = \varphi^{-1} \partial_i \varphi.$$

Parametrise $\varphi$ in the following way

$$\varphi = \xi \chi,$$  \hspace{1cm} (2.3)

where $\chi$ is a mapping from $U$ to $O(n, \mathbb{R})$ and the mapping $\xi$ has the form

$$\xi = e^{\psi_1 M_{12}} e^{\psi_2 M_{23}} \ldots e^{\psi_{n-1} M_{n-1,n}} e^{\psi_{n} M_{n,n+1}}.$$  \hspace{1cm} (2.4)

Here $\psi_i$ are some functions on $U$ having the meaning of the generalised Euler angles [20]. For the connection components $\omega_i$ one obtains the expression

$$\omega_i = \chi^{-1} (\xi^{-1} \partial_i \xi) \chi + \chi^{-1} \partial_i \chi.$$

Relation (2.4) gives

$$\xi^{-1} \partial_i \xi = \sum_{j=1}^{n-1} \partial_i \psi_j \sum_{k=j+1}^{n} \mu_{jk}(\psi) M_{jk} + \sum_{j=1}^{n} \partial_i \psi_j \nu_j(\psi) M_{j,n+1},$$

where

$$\mu_{j-1,j}(\psi) = \cos \psi_j, \quad 1 < j \leq n,$$  \hspace{1cm} (2.5)

$$\mu_{jk}(\psi) = \left( \prod_{l=j+1}^{k-1} \sin \psi_l \right) \cos \psi_k, \quad 1 < j + 1 < k \leq n,$$  \hspace{1cm} (2.6)

$$\nu_j(\psi) = \prod_{l=j+1}^{n} \sin \psi_l, \quad 1 \leq j < n, \quad \nu_n(\psi) = 1.$$  \hspace{1cm} (2.7)
Now, using the evident equalities
\[
\chi^{-1} \partial_i \chi = \frac{1}{2} \sum_{j,k,l=1}^{n} \chi_{lj} \partial_i \chi_{lk} M_{jk}, \quad \chi^{-1} M_{i,n+1} \chi = \sum_{j=1}^{n} \chi_{ij} M_{j,n+1}, \tag{2.8}
\]
one comes to the expressions
\[
\omega_i = \frac{1}{2} \sum_{j,k,l=1}^{n} \chi_{lj} \partial_i \chi_{lk} M_{jk}
+ \sum_{j,k=1}^{n} \sum_{l=1}^{n-1} \sum_{m=l+1}^{n} \mu_{lm}(\psi) \chi_{lj} \chi_{mk} M_{jk} + \sum_{j,l=1}^{n} \partial_i \psi_l \nu_l(\psi) \chi_{lj} M_{j,n+1}. \tag{2.9}
\]
Comparing (2.9) and (2.1), we have, in particular,
\[
\sum_{l=1}^{n} \partial_i \psi_l \nu_l(\psi) \chi_{lj} = \beta_i \delta_{ij}. \tag{2.10}
\]
Note that the geometrical meaning of the functions $\beta_i$ do not allow them to take zero value. Therefore, from (2.10) it follows that for any point $p \in U$ we have
\[
\det(\partial_i \psi_j(p)) \neq 0, \quad \nu_i(\psi(p)) \neq 0. \tag{2.11}
\]
Since the matrix $(\chi_{ij})$ is orthogonal, one easily obtains
\[
\chi_{ij} = \frac{1}{\beta_j} \partial_j \psi_i \nu_i(\psi),
\]
and, using again the orthogonality of $(\chi_{ij})$, one sees that
\[
\beta_i^2 = \sum_{l=1}^{n} \left( \partial_i \psi_l \nu_l(\psi) \right)^2. \tag{2.12}
\]
Therefore, we have
\[
\chi_{ij} = \frac{\partial_j \psi_i \nu_i(\psi)}{\sqrt{\sum_{l=1}^{n} \left( \partial_j \psi_l \nu_l(\psi) \right)^2}}. \tag{2.13}
\]
Thus, the matrix \((\chi_{ij})\), and hence the mapping \(\chi\), is completely determined by the functions \(\psi_i\), and its orthogonality is equivalently realised by the relation

\[
\sum_{l=1}^{n} \partial_i \psi_l \nu_l^2(\psi) \partial_j \psi_l = 0, \quad i \neq j. \tag{2.14}
\]

Suppose now that a set of functions \(\psi_i\) satisfies relations (2.11) and (2.14). Consider the mapping \(\varphi\) defined by (2.3) with the mapping \(\xi\) having form (2.4) and the mapping \(\chi\) defined by (2.13). Show that the mapping \(\varphi\) generates the connection with the components of form (2.1). First of all, with \(\beta_i\) of form (2.12) we can get convinced that in the case under consideration relation (2.10) is valid. Taking into account (2.14), one can write the relation

\[
\sum_{l=1}^{n} \partial_j \psi_l \nu_l^2(\psi) \partial_k \psi_l = \beta_j^2 \delta_{jk},
\]

whose differentiation with respect to \(z^i\) gives

\[
\sum_{l=1}^{n} \partial_j \psi_l \nu_l^2(\psi) \partial_k \partial_k \psi_l = - \sum_{l=1}^{n} \partial_l \partial_j \psi_l \nu_l^2(\psi) \partial_k \psi_l - 2 \sum_{l=1}^{n} \partial_j \psi_l \nu_l(\psi) \partial_k \nu_l(\psi) \partial_k \psi_l + 2 \beta_j \partial_l \beta_j \delta_{jk}.
\]

Since the left hand side of this equality is symmetric with respect to the transposition of the indices \(i\) and \(k\), its right hand side must also be symmetric with respect to this transposition, and, therefore, we have

\[
\sum_{l=1}^{n} \partial_j \psi_l \nu_l^2(\psi) \partial_k \partial_k \psi_l = - \sum_{l=1}^{n} \partial_l \partial_j \psi_l \nu_l^2(\psi) \partial_k \psi_l - 2 \sum_{l=1}^{n} \partial_j \psi_l \nu_l(\psi) \partial_k \nu_l(\psi) \partial_k \psi_l + 2 \beta_j \partial_l \beta_j \delta_{ij}.
\]

Using this equality, it is not difficult to show that

\[
\sum_{l=1}^{n} \chi_{ij} \partial_i \chi_{lk} = \gamma_{kj} \delta_{ij} - \gamma_{jk} \delta_{ik}
\]

\[
- \frac{1}{\beta_j \beta_k} \sum_{l=1}^{n} [\partial_j \psi_l \nu_l(\psi) \partial_k \nu_l(\psi) \partial_i \psi_l - \partial_k \psi_l \nu_l(\psi) \partial_j \nu_l(\psi) \partial_i \psi_l], \tag{2.15}
\]
where the functions $\gamma_{ij}$ are defined by (1.3).

Using the concrete form of the functions $\mu_{ij}(\psi)$ and $\nu_i(\psi)$, we can get convinced in the validity of the equalities

$$\frac{\partial \nu_j(\psi)}{\partial \psi_i} = 0, \quad 1 \leq i \leq j, \quad \mu_{ij}(\psi) = \frac{1}{\nu_j(\psi)} \frac{\partial \nu_i(\psi)}{\partial \psi_j},$$

which allow to show that

$$\sum_{l=1}^{n-1} \partial_i \psi_l \sum_{m=l+1}^{n} \mu_{lm}(\psi) \chi_{lj} \chi_{mk} = \frac{1}{\beta_j \beta_k} \sum_{l=1}^{n} \partial_j \psi_l \nu_i(\psi) \partial_k \nu_l(\psi) \partial_l \psi_i. \quad (2.16)$$

Substituting (2.15), (2.16) and (2.10) into (2.9), we come to expres sion (2.1). Thus, any set of functions $\psi_i$ satisfying (2.11) and (2.14) allows to construct a connection of form (2.1) satisfying the zero curvatu re condition (2.2) which is equivalent to the Bourlet type equations. Therefore, the general solution to the Bourlet type equations is described by (2.12) where the functions $\psi_i$ satisfy (2.11) and (2.14). In the simplest case we can satisfy (2.14) assuming that

$$\partial_i \psi_j = 0, \quad i \neq j; \quad (2.17)$$

in other words, each function $\psi_i$ depends on the corresponding coordinate $z^i$ only. In this case we obtain the following expressions for the functions $\beta_i$:

$$\beta_i = \partial_i \psi_i \prod_{j=i+1}^{n} \sin \psi_j, \quad 1 \leq i < n, \quad \beta_n = \partial_n \psi_n. \quad (2.18)$$

The corresponding expressions for the functions $\gamma_{ij}$ can be easily found and we do not give here their explicit form. Note here that since $\gamma_{ij} = 0$ for $i < j$, our solutions do not satisfy the Egoroff property.

There is a transparent geometrical interpretation of the results obtained above. Recall that solutions of the Bourlet type equations are associated with diagonal metrics in Riemannian spaces of constant curvature. Namely, let $(M, g)$ be a Riemannian manifold of constant curvature with the sectional curvature $k$, and $(U; z^1, \ldots, z^n)$ be such a chart on $M$ that the metric $g$ has on $U$ form (1.1). Then the Lamé and the corresponding rotation coefficients satisfy the Bourlet type equations. From the other hand, let $(U; z^1, \ldots, z^n)$ be a chart on a manifold $M$, and we have a solution of the Bourlet type
equations. Supply the open submanifold $U$ with metric (1.7); then $(U, g)$ becomes a Riemannian manifold of constant curvature with the sectional curvature $k$.

The simplest example of a manifold of constant curvature is an $n$-dimensional sphere in $\mathbb{R}^{n+1}$ with the metric induced by the standard metric on $\mathbb{R}^{n+1}$. Here if the radius of the sphere is $R$, then the sectional curvature is $1/R^2$. Let us show that the corresponding metric is diagonal with respect to the spherical coordinates.

Begin with the consideration of the standard metric in $\mathbb{R}^n$. Denoting the standard coordinates on $\mathbb{R}^n$ by $x_i$ and the spherical coordinates by $r$ and $\theta_1, \ldots, \theta_{n-1}$, one has

$$x^1 = r \prod_{k=1}^{n-1} \sin \vartheta_k, \quad x^i = r \cos \vartheta_{i-1} \prod_{k=1}^{n-1} \sin \vartheta_k, \quad 1 < i < n, \quad x^n = r \cos \vartheta_{n-1}.$$  

The standard metric on $\mathbb{R}^n$ has the form

$$g_{(n)} = \sum_{i=1}^n dx_i \otimes dx_i.$$  

Denote the functions describing the dependence of the coordinates $x^i$ on $r$ and $\theta_1, \ldots, \theta_{n-1}$, by $f_i(r, \theta)$. For any $n$ one has

$$\sum_{i=1}^n f_i^{(n)} = r^2. \quad (2.19)$$

Taking external derivative of this equality, we obtain

$$\sum_{i=1}^n f_i^{(n)} d f_i^{(n)} = r dr. \quad (2.20)$$

It is easy to get convinced that

$$f_i^{(n)} = f_i^{(n-1)} \sin \theta_{n-1}, \quad 1 \leq i < n, \quad f_n^{(n)} = r \cos \theta_{n-1}.$$

These relations imply

$$d f_i^{(n)} = \sin \theta_{n-1} d f_i^{(n-1)} + f_i^{(n-1)} \cos \theta_{n-1} d \theta_{n-1}, \quad 1 \leq i < n,$$

$$d f_n^{(n)} = \cos \theta_{n-1} dr - r \sin \theta_{n-1} d \theta_{n-1}.$$
Substituting these equalities into the relation
\[
g^{(n)} = \sum_{l=1}^{n} d f_l \otimes d f_l
\]
and using (2.19), (2.20) we obtain
\[
g^{(n)} = \left( g^{(n-1)} - d r \otimes d r \right) \sin^2 \theta_{n-1} + r^2 d \theta_{n-1} \otimes d \theta_{n-1} + d r \otimes d r.
\]
This equality gives
\[
g^{(n)} = r^2 \left[ \sum_{l=1}^{n-2} \left( \prod_{m=l+1}^{n-1} \sin^2 \theta_m \right) d \theta_l \otimes d \theta_l + d \theta_{n-1} \otimes d \theta_{n-1} \right] + d r \otimes d r. \tag{2.21}
\]
Consider now the unit \( n \)-dimensional sphere \( S^n \) in \( \mathbb{R}^{n+1} \). Denote the spherical coordinates in \( S^n \) by \( z_1, \ldots, z_n \). As it follows from (2.21), the explicit expression for the metric on \( S^n \) in terms of the spherical coordinates has the form
\[
g = \sum_{l=1}^{n-1} \left( \prod_{m=l+1}^{n} \sin^2 z_m \right) d z_l \otimes d z_l + d z_n \otimes d z_n.
\]
So we have a diagonal metric. Note that it can be written in the form
\[
g = \sum_{l=1}^{n} \nu_l^2(z) d z_l \otimes d z_l, \tag{2.22}
\]
where the functions \( \nu_l \) are given by (2.7). Let \( \psi \) be a diffeomorphism from \( S^n \) to \( S^n \). It is clear that \((S^n, \psi^*g)\) is also a Riemannian manifold of constant curvature with the sectional curvature equal to 1. Denoting \( \psi^* z^i = \psi_i \), one gets
\[
\psi^* g = \sum_{j,k,l=1}^{n} \partial_j \psi_l \nu_l^2(\psi) \partial_k \psi_l d z^j \otimes d z^k.
\]
Therefore, the metric \( \psi^* g \) is diagonal with respect to the coordinates \( z^i \) if and only if the functions \( \psi_i \) satisfy relations (2.14). In particular, if the
functions $\psi_i$ satisfy relations (2.17) we obtain the diagonal metrics with the Lamé coefficients given by (2.18).

In general, starting from some fixed diagonal metric in the space of constant curvature with the unit sectional curvature, one gets the family of explicit solutions to the Bourlet type equations parametrised by a set of $n$ functions each depending only on one variable. In terms of equations (1.3), (1.9) and (1.10) themselves, we formulate this observation as follows. Let the functions $\beta_i$, $\gamma_{ij}$ satisfy the Bourlet type equations; then for any set of functions $\psi_i$, such that

$$\partial_i \psi_j = 0, \quad i \neq j,$$

the functions

$$\beta_i'(z) = \beta_i(\psi(z)) \partial_i \psi_i(z), \quad \gamma_{ij}'(z) = \gamma_{ij}(\psi(z)) \partial_j \psi_j(z'),$$

(2.23)

where $\psi(z)$ stands for the set $\psi_1(z), \ldots \psi_n(z)$, also satisfy the Bourlet type equations.

Note that our considerations can be easily generalised to the case of complex metrics. In this case the zero curvature representation of the Bourlet type equations should be based on the Lie group $O(n + 1, \mathbb{C})$.

Return to the consideration of solutions (2.18) to the Bourlet type equations. If one imposes condition (1.11) where $c$ is an arbitrary zero or nonzero constant, then the arbitrary functions $\psi_i(z)$ satisfy the equation

$$\sum_{l=1}^{n} \left( \prod_{m=l+1}^{n} \sin^2 \psi_m \right) (\partial_l \psi_l)^2 = c,$$

thereof for some constants $c_i$, $i = 0, \ldots, n$, such that $c_0 = 0$ and $c_n = c$, one gets

$$\partial_i \psi_i = \sqrt{c_i - c_{i-1} \sin^2 \psi_i}.$$  

(2.24)

Hence, solution (2.18) takes the form

$$\beta_i = \sqrt{c_i - c_{i-1} \sin^2 \psi_i} \prod_{j=i+1}^{n} \sin \psi_j,$$

(2.25)

where the functions $\psi_i$ are determined by the ordinary differential equations (2.24). Suppose that all constants $c_i$, $i = 1, \ldots, n$, are different from zero.
With appropriate conditions on the constants $c_i$, in accordance with (2.24) one has

$$z_i + d^i = \int_0^{\psi_i} \frac{d\psi_i}{\sqrt{c_i - c_{i-1} \sin^2 \psi_i}},$$

where $d^i$ are arbitrary constants. Therefore,

$$\sqrt{c_i(z^i + d^i)} = F\left(\psi_i, \sqrt{\frac{c_{i-1}}{c_i}}\right),$$

where $F(\phi, k)$ is the elliptic integral of the first kind,

$$F(\phi, k) = \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}.$$

Thus, using Jacobi elliptic functions, we can write

$$\sin \psi_i(z^i) = \text{sn}\left(\sqrt{c_i(z^i + d^i), \sqrt{\frac{c_{i-1}}{c_i}}}\right),$$

$$\cos \psi_i(z^i) = \text{cn}\left(\sqrt{c_i(z^i + d^i), \sqrt{\frac{c_{i-1}}{c_i}}}\right).$$

Now, with the evident relation

$$\partial_i \psi_i(z^i) = \frac{\partial_i \sin \psi_i(z^i)}{\cos \psi_i(z^i)} = \sqrt{c_i} \text{dn}\left(\sqrt{c_i(z^i + d^i), \sqrt{\frac{c_{i-1}}{c_i}}}\right),$$

we write our solution as the product of elliptic functions,

$$\beta_i(z) = \sqrt{c_i} \text{dn}\left(\sqrt{c_i(z^i + d^i), \sqrt{\frac{c_{i-1}}{c_i}}} \prod_{j=i+1}^n \text{sn}\left(\sqrt{c_j(z^j + d^j), \sqrt{\frac{c_{j-1}}{c_j}}}\right)\right).$$

(2.26)

The case when some of the constants $c_i$ are equal to zero can be analysed in a similar way. Note that, taking into account the relations

$$\text{sn}(u, 1) = \tanh u, \quad \text{dn}(u, 1) = \frac{1}{\cosh u}, \quad \text{sn}(u, 0) = \sin u, \quad \text{dn}(u, 0) = 1,$$

with an appropriate choice of the constants $c_i$, we can reduce some of the elliptic functions entering the obtained solution to the trigonometric or hyperbolic ones.
It is clear from the solution in form (2.25) or (2.26), that it does not depend on the variable $z^1$ at all, since among the functions $\beta_i$, only $\beta_1$ depends on $\psi_1$ and only as $\partial_1 \psi_1$, while $\psi_1 = c_1 z^1 + d^1$.

In the simplest case $n = 2$ and $c_2 = 1$ with the parametrisation $\beta_1 = \cos(u/2)$, $\beta_2 = \sin(u/2)$, system (1.3), (1.9) and (1.10) is reduced to the sine-Gordon equation

$$\partial^2_1 u - \partial^2_2 u + \sin u = 0,$$

and one gets the evident solution $\sin(u/2) = \text{dn}(z^2 + d^2, \sqrt{c_1})$.

### 3 Lamé equations

The zero curvature representation of the Lamé equations is based on the Lie group $G$ of rigid motions of the affine space $\mathbb{R}^n$. This Lie group is isomorphic to the semidirect product of the Lie groups $O(n, \mathbb{R})$ and $\mathbb{R}^n$, where the linear space $\mathbb{R}^n$ is considered as a Lie group with respect to the addition operation. The standard basis of the Lie algebra $\mathfrak{g}$ of the Lie group $G$ consists of the elements $M_{ij}$ and $P_i$ which satisfy the commutation relations

\[
[M_{ij}, M_{kl}] = \delta_{il} M_{jk} + \delta_{jk} M_{il} - \delta_{ik} M_{jl} - \delta_{jl} M_{ik},
\]

\[
[M_{ij}, P_k] = \delta_{jk} P_i - \delta_{ik} P_j, 
\]

\[
[P_i, P_j] = 0.
\]

Let $(U; z^1, \ldots, z^n)$ be a chart on the manifold $M$. Consider the connection $\omega = \sum_{i=1}^n \omega_i dz^i$ on the trivial principal fibre bundle $U \times G$ with the components given by

$$\omega_i = \sum_{k=1}^n \gamma_{ki} M_{ik} + \beta_i P_i.$$  \hspace{1cm} (3.1)

It can be easily verified that equations (1.3)–(1.6) are equivalent to the zero curvature condition for the connection $\omega$. It is well known that the Lie algebra $\mathfrak{g}$ can be obtained from the Lie algebra $\mathfrak{o}(n+1, \mathbb{R})$ by an appropriate Inönü-Wigner contraction. Unfortunately, this fact does not give us a direct procedure for obtaining solutions of the Lamé equations from solutions of the Bourlet type equations. Therefore, we will consider the procedure for obtaining solutions of the Lamé equations independently.

Let the connection $\omega$ with the components of form (3.1) satisfies the zero curvature condition. Restricting to the case of simply connected $U$, write for
the connection components $\omega_i$ the representation

$$\omega_i = \varphi^{-1} \partial_i \varphi,$$

where $\varphi$ is some mapping from $U$ to $G$. Parametrise $\varphi$ in the following way:

$$\varphi = \xi \chi$$

where $\chi$ is a mapping from $U$ to $O(n, \mathbb{R})$, and the mapping $\xi$ has the form

$$\xi = e^{\psi_1 P_1} e^{\psi_2 P_2} \ldots e^{\psi_{n-1} P_{n-1}} e^{\psi_n P_n}.$$  \hspace{1cm} (3.3)

For the connection components $\omega_i$ one obtains the expression

$$\omega_i = \frac{1}{2} \sum_{j,k,l=1}^{n} \chi_{lj} \partial_i \chi_{lk} M_{jk} + \sum_{j,l=1}^{n} \partial_i \psi_l \chi_{lj} P_j.$$  \hspace{1cm} (3.4)

From the comparison of (3.4) and (3.1) we see that

$$\chi_{ij} = \frac{\partial_j \psi_i}{\sqrt{\sum_{l=1}^{n} (\partial_j \psi_l)^2}},$$  \hspace{1cm} (3.5)

and the functions $\psi_i$ satisfy the relation

$$\sum_{l=1}^{n} \partial_i \psi_l \partial_j \psi_l = 0, \quad i \neq j.$$  \hspace{1cm} (3.6)

The functions $\beta_i$ are connected with the functions $\psi_i$ by the formula

$$\beta_i^2 = \sum_{l=1}^{n} (\partial_l \psi_l)^2,$$  \hspace{1cm} (3.7)

and from the geometrical meaning of $\beta_i$ it follows that

$$\det(\partial_l \psi_j(a)) \neq 0.$$  \hspace{1cm} (3.8)

Suppose now that a set of functions $\psi_i$ satisfies relations (3.6) and (3.8). Consider the mapping $\varphi$ defined by (3.2) with the mapping $\xi$ having form (3.3) and the mapping $\chi$ defined by (3.5). It can be shown that the mapping $\varphi$ generates the connection with the components of form (3.1). Here the
functions $\beta_i$ are defined from (3.7), and the functions $\gamma_{ij}$ are given by (1.3). Thus, any set of functions $\psi_i$ satisfying (3.8) and (3.6) allows to construct a connection of form (3.1) satisfying the zero curvature condition which is equivalent to the Lamé equations, and in such a way we obtain the general solution.

Assuming that the functions $\psi_i$ satisfy (2.17), we have

$$
\beta_i = \partial_i \psi_i.
$$

(3.9)

It is clear that in this case $\gamma_{ij} = 0$. So one ends up with a trivial solution of the Lamé equations. To get nontrivial solutions one should consider different parametrisations of the mapping $\varphi$. For example, let us represent the mapping $\varphi$ in form (3.2) where the mapping $\chi$ again takes values in $O(n, \mathbb{R})$, while the mapping $\xi$ has the form

$$
\xi = e^{\psi_1 M_{12}} e^{\psi_2 M_{23}} \cdots e^{\psi_{n-1} M_{n-1,n}} e^{\psi_n P_n}.
$$

With such a parametrisation of $\xi$, one gets

$$
\omega_i = \frac{1}{2} \sum_{j,k,l=1}^n \chi_{lj} \partial_i \chi_{lk} M_{jk}
$$

$$
+ \sum_{j,k,l=1}^n \sum_{m=l+1}^{n-1} \sum_{j,l=1}^n \mu_{lm}(\psi) \chi_{lj} \chi_{mk} M_{jk}
$$

$$
+ \sum_{j,l=1}^n \partial_i \psi_l \nu_l(\psi) \chi_{lj} P_j,
$$

where

$$
\mu_{j-1,j}(\psi) = \cos \psi_j, \quad 1 < j < n; \quad \mu_{n-1,n}(\psi) = 1; \quad (3.10)
$$

$$
\mu_{jk}(\psi) = \left( \prod_{l=j+1}^{k-1} \sin \psi_l \right) \cos \psi_k, \quad 1 < j + 1 < k < n; \quad (3.11)
$$

$$
\mu_{jn}(\psi) = \prod_{l=j+1}^{n-1} \sin \psi_l, \quad 1 < j + 1 < n; \quad (3.12)
$$

$$
\nu_j(\psi) = \left( \prod_{k=j+1}^{n-1} \sin \psi_k \right) \psi_n, \quad 1 \leq j < n - 1; \quad (3.13)
$$

$$
\nu_{n-1}(\psi) = \psi_n; \quad \nu_n(\psi) = 1. \quad (3.14)
$$
Using these relations we come to the following description of the general solution to the Lamé equations. Let functions \( \psi_i \) satisfy the relations

\[
\sum_{l=1}^{n} \partial_i \nu_l^2(\psi) \partial_j \psi_l = 0, \quad i \neq j,
\]

and for any point \( p \in U \) one has

\[
\det(\partial_i \psi_j(p)) \neq 0, \quad \nu_i(\psi(p)) \neq 0.
\]

Then the functions \( \beta_i \) determined from the equality

\[
\beta_i^2 = \sum_{l=1}^{n} (\partial_i \nu_l(\psi))^2,
\]

and the corresponding functions \( \gamma_{ij} \) defined by (1.3) give the general solution of the Lamé equations. If the functions \( \psi_i \) satisfy (2.17), we get the following expressions for the functions \( \beta_i \)

\[
\beta_i = \partial_i \psi_i \left( \prod_{j=i+1}^{n-1} \sin \psi_j \right) \psi_n, \quad 1 \leq i < n - 1, \quad (3.15)
\]

\[
\beta_{n-1} = \partial_{n-1} \psi_{n-1} \psi_n, \quad \beta_n = \partial_n \psi_n. \quad (3.16)
\]

The geometrical interpretation of the obtained solutions to the Lamé equations is similar to one given in the previous section. Recall that solutions of the Lamé equations are associated with flat diagonal metrics in flat Riemannian spaces. The simplest case here is the standard metric in \( \mathbb{R}^n \),

\[
g = \sum_{l=1}^{n} dz^l \otimes dz^l.
\]

Applying a diffeomorphism \( \psi \) one gets the metric

\[
\psi^* g = \sum_{j,k,l=1}^{n} \partial_j \psi_l \partial_k \psi_l dz^j \otimes dz^k,
\]

which is diagonal if and only if the functions \( \psi_i \) satisfy (3.6). Here the functions \( \psi_i \) which obey (2.17) give the Lamé coefficients described by (3.3).
A more nontrivial example is provided by the metric arising after the transition to the spherical coordinates in $\mathbb{R}^n$. Introducing the notations $z^i = \vartheta_i$, $i = 1, \ldots, n-1$, and $z^n = r$, we rewrite (2.21) as

$$g = (z^n)^2 \left[ \sum_{l=1}^{n-2} \left( \prod_{m=l+1}^{n-1} \sin^2 z^m \right) dz^l \otimes dz^l + d z^{n-1} \otimes d z^{n-1} \right] + dz^n \otimes dz^n.$$

Therefore, the metric $g$ has form (2.22) with the functions $\nu_i$ given by (3.13), (3.14). A diffeomorphism $\psi$ with the functions $\psi_i = \psi^* z^i$ satisfying (2.17) gives the metric with the Lamé coefficients (3.15), (3.16).

In conclusion note that relations (2.23) describe the symmetry transformations not only of the Bourlet type equations, but also of the Lamé equations, and actually the existence of such transformations allows us to construct the solutions of the equations under consideration parametrised by $n$ arbitrary functions each depending on one variable.

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**References**

[1] Kobayashi, S., Nomizu K.: Foundations of differential geometry, Vol. 1. New York: Interscience, 1963.

[2] Bianchi, L.: Lezioni di geometria differenziale, Vol. 2, Part 2. Bologna: Zanichelli 1924; Sisteme tripli ortogonali, Opere, Vol. 3. Roma: Cremonese 1955.
[3] Darboux, G.: Leçons sur les systèmes orthogonaux et les coordonnées curvilignes. Paris: Gauthier-Villars 1910; Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal, Vols. 1–4. Paris: Gauthier-Villars 1887–1896.

[4] Aminov, Yu. A.: Isometric immersions of domains of \( n \)-dimensional Lobachevsky space in \((2n - 1)\)-dimensional euclidean space. Math. USSR Sbornik 39(3), 359–386 (1981).

[5] Saveliev, M. V.: Multidimensional nonlinear systems. Theor. Math. Phys. 69, 1234–1240 (1986); Multidimensional nonlinear dynamical systems, in M. A. Markov, V. I. Man’ko, V. V. Dodonov (eds.), “Group Theoretical Methods in Physics” Vol. 1, 113–127. Utrecht: VNU Science Press 1986.

[6] Tenenblat, K., Terng, C.-L.: Bäcklund theorem for \( n \)-dimensional submanifolds of \( R^{2n-1} \). Annals Math. 111, 477–490 (1980); Terng, C.-L.: A higher dimensional generalization of the Sine-Gordon equation and its soliton theory. Annals Math. 111, 491–510 (1980).

[7] Ablowitz, M. J., Beals, R., Tenenblat, K.: On the solution of the generalized sine-Gordon equations. Stud. Appl. Math. 74, 177–203 (1986).

[8] Tsarev, S. P.: The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method. Math. USSR Izvestija 37(2), 397-419 (1991).

[9] Tsarev, S. P.: Classical differential geometry and integrability of systems of hydrodynamic type. Proc. NATO Advanced Research Workshop on Applications of Analytical and Geometric Methods to Nonlinear Differential Equations (Exeter, England, 14–19 July 1992). [hep-th/9303092](http://arxiv.org/abs/hep-th/9303092).

[10] Dubrovin, B. A., Novikov, S. P.: Hamiltonian formalism of one-dimensional systems of hydrodynamic type, and the Bogolyubov-Whitham averaging method. Sov. Math. Doklady 27(3), 665–669 (1983).

[11] Zakharov, V. E.: Description of the \( n \)-orthogonal curvilinear coordinate systems and Hamiltonian integrable systems of hydrodynamic type. Part I. Integration of the Lamé equations. Submitted to Duke Math. Journal.

[12] Dubrovin, B.: Geometry and integrability of topological-antitopological fusion. Commun. Math. Phys. 152, 539–564 (1993).
[13] Razumov, A. V., Saveliev, M. V.: Multidimensional Toda type systems. Preprint IHEP 96–68, Protvino. [hep-th/9609031]

[14] Dubrovin, B.: Geometry of 2D topological field theories. In: Integrable Systems and Quantum Groups. Francaviglia, M., Greco S. (eds.) Lecture Notes in Mathematics vol. 1620. Berlin: Springer 1996, p. 120.

[15] Hitchin, N.: Frobenius manifolds (Notes by D. Calderbank). Preprint, 1996.

[16] Manin, Yu. I.: Frobenius manifolds, quantum cohomology, and moduli spaces (Chapters I, II, III). Preprint Max-Planck-Institut für Mathematik, Bonn, 1996.

[17] Lü, H., Pope, C. N., Sezgin, E., Stelle, K. S.: Stainless super p-branes. Nucl. Phys. B456, 669–698 (1995).

[18] Dubrovin, B. A.: Differential geometry of strongly integrable systems of hydrodynamic type. Funct. Anal. and its Appl. 24(4), 280–285 (1990).

[19] Zakharov, V. E., Shabat, A. B.: A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. Funct. Anal. and its Appl. 8(3), 226–235 (1974); Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II. Funct. Anal. and its Appl. 13(3), 166–174 (1979).

[20] Vilenkin, N. Ya.: Fonctions spéciales et théorie de la représentation des groupes, Paris: Dunod 1969.