Generating lightwave-photon-and-magnon entanglement with a mechanical oscillator as a “cold reservoir”

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We present a scheme to implement a steady lightwave-photon-and-magnon entanglement in a hybrid photon-magnon system by adiabatically eliminating the auxiliary microwave cavity and effectively laser cooling a delocalized Bogoliubov mode. The system consists of magnons, lightwave and microwave photons, and phonons. The magnons are embodied by a collective motion of a large number of spins in a macroscopic ferrimagnet. To achieve an entangling interaction between magnons and lightwave photons, we drive optical cavity and magnon at the red and blue sideband associated with the mechanical resonator. In particular, optimizing the relative ratio of effect couplings, rather than simply increasing their magnitudes, is essential for achieving strong entanglement. Unlike typical dissipative entanglement schemes, our results cannot be described by treating the effects of the entangling reservoir via a Lindblad master equation.

Introduction.—In quantum science, information processing [1–3] is inseparable from quantum states preparation and their evolution. They act as an important “quantum resource” and are undoubtedly the most important ones among all kinds of quantum states. With the rapid development of quantum information processing, quantum entanglement is widely used for quantum key distribution [4–6], quantum dense coding [7, 8], quantum teleportation [9, 10], quantum entangled code [11, 12], quantum computation [13, 14], quantum communication [15, 16] and so on. As is all known that entangled states of multiple sub-systems have many properties that two subsystems do not hold. This is not only related to the nature of quantum entanglement, but also helps people to understand the basic theory of quantum mechanics, and to develop many magical applications.

In recent years, the preparation of entangled states in ferromagnetic systems, especially with the help of yttrium iron garnet (YIG) spheres has attracted great interest. It has been found that the Kittel mode [17] of YIG sphere can be strongly coupled with microwave photons in a high-quality cavity, leading to the cavity polaritons [18–20] and the vacuum Rabi splitting. The phenomenon is then called magnon cavity quantum electrodynamics (QED) [21–25]. Since then, a number of interesting developments such as observation of bistability [26] and coupling of a single superconducting qubit to the Kittel mode [17, 27] have been made. Obviously, the magnon-photon system can provide a new platform for us to study the unique effect of strong coupling QED. This is very similar to the platforms provided by superconducting qubits [28], semiconductor qubits [29] and double quantum dots [30].

Besides, YIG has very low loss for various different information carriers, including magnon, acoustic phonon, and microwave photon. Moreover, as a dielectric, YIG is transparent for infrared light. All these properties would enable further integration of the magnonic systems with opto- or electromechanical elements, providing an excellent platform for quantum state transfer among different physical systems. The magnon excitons formed by the interaction between lightwave photons and magnons, as an important meta-excitation, have attracted great attention because they are expected to realize quantum storage and quantum sensor in the field of quantum information [31].

Here we present a model of the hybrid cavity-magnon system, and prove that it is possible to implement quantum entanglement between magnon and lightwave photon via reservoir engineering mechanism. Specifically, we prove that, the use of an auxiliary microwave cavity mode is beneficial to our system, not only for cooling the oscillator to the quantum ground state, but also for providing a base magnetic field for the magnon. The dissipative entanglement scheme that we describe is related to optomechanical cavity-cooling schemes [32, 33] which have been used successfully to cool mechanical resonators to ground states. Our results show that by optimizing the ratio of effect coupling strengths, this laser-cooling mechanism can be used to yield large amounts of time-independent entanglement, rather than simply increasing their magnitudes. The amount of entanglement is far greater than the maximum possible entanglement allowed by a coherent parametric interaction in previous studies.

The model.—The physical model in our studies is shown in Fig. 1, which consists of an optical cavity coupled with a microwave (MW) cavity A via a movable mirror and a YIG sphere confined in the MW cavity. The magnon (a quantized spin wave) is a collective excitations of a large number of spins inside the massive YIG sphere.
which is simultaneously in a uniform bias magnetic field and near the maximum magnetic field of the auxiliary WM cavity $A$ with a fixed mirror and a movable one. We consider the movable mirror is a quantum mechanical harmonic oscillator and the size of the YIG sphere is much smaller than the MW wavelengths. Hence we can neglect any other radiation pressure on the sphere induced by the WM cavity field. In addition, a small YIG sample embedded in a WM cavity, the magnon in the YIG sphere are seen as dipolar spin waves under magnetostatic approximation, where the magnetic dipolar interactions dominate both the electric and exchange interactions [18, 19, 34]. In this case the wave number $K$ of the magnon mode satisfies the relation $K_0 \ll K \ll \sqrt{1/\Lambda_{xx}}$ with $K_0 = \omega_0 / \mu_0\mu_s$ being the wave number of the microwave field propagating in the YIG material and the exchange constant $\Lambda_{xx} = 3 \times 10^{-16} \text{ m}^2$. Moreover, the microwave magnetic field $\mathbf{h}$ related to a magnon mode of the YIG sphere satisfies the magnetostatic equation [34]

$$\nabla \times \mathbf{h} = \frac{\partial \mathbf{D}}{\partial t} = \frac{K_0^2 K \mathbf{m}}{K_0^2 - K^2}, \quad (1)$$

where $\mathbf{D}$ is the microwave electric displacement vector and the magnetization $\mathbf{m}$ is excited by $\mathbf{h}$. The right side of Eq. (1) is approximately zero when $K \gg K_0$, leading to $\partial \mathbf{D}/\partial t \approx 0$, which implies that magnon modes are essentially static. Therefore, the whole system Hamiltonian can be written as ($h = 1$) [35]

$$H = \omega_a a^\dagger a + \omega_b b^\dagger b + \omega_{\text{cool}} A^\dagger A + g_{ab} a^\dagger a (b^\dagger + b) + g_{B} B_z S_z + e^{i\omega_1 t} g_{Am} (A^\dagger S_- + AS_+) + g_{Am} A^\dagger A (b^\dagger + b), \quad (2)$$

where $\omega_a$ ($\omega_b$, $\omega_{\text{cool}}$) is the resonance frequency of the cavity mode (mechanical mode, auxiliary cavity mode), $a$ ($b$, $A$) and $a^\dagger$ ($b^\dagger$, $A^\dagger$) are, respectively, the annihilation and creation operators of the cavity mode (mechanical mode, auxiliary cavity mode), and satisfying $[O, O^\dagger] = 1$ and $O = a, b, A$. $g_{ab} (g_{Am})$ denotes the coupling strength between the cavity (auxiliary cavity) mode and the movable mirror. $g_B$ denotes the electron $g$-factor, $\mu_0$ is Bohr magneton, $B_z$ is the effective magnetic fields, and $\tilde{g}_Am$ is the coupling strength of the cavity photon to a single spin in the magnon mode, and $\varphi$ is a relative phase. Magnon collective spin operators are given by $S = (S_x, S_y, S_z)$, with $S_\pm = S_x \pm iS_y$. It is noted that these collective spin operators are related to the magnon annihilation and creation operators $m$ and $m^\dagger$ via Holstein-Primakoff transformation [36]: $S_+ = m^\dagger (\sqrt{2N}m - m^\dagger m)$, $S_- = m (\sqrt{2N}m - m^\dagger m)$ with $N$ being the spin number of the corresponding collective spin operator and $s = \sqrt{2}$ being the spin number of the ground state $Fe^{3+}$ ion in YIG.

To achieve our goal, we use two laser beams to drive the optical cavity field and the magnon mode at the red and blue sideband respectively. The two frequencies of the laser satisfy [37]: $\omega_1 = \omega_i - \omega_d$ and $\omega_2 = \omega_i + \omega_d$ with $\omega_i = g_B\mu_B B_z$ being the resonance frequencies of the magnon mode, which can be adjusted in a large range by altering the external bias magnetic field $\tilde{H}$ via $\omega_i = \gamma_0 \tilde{H}$ with the magnetic field $\tilde{H} = i\omega_d - B_z (e^{i\omega_2 t} + e^{-i\omega_2 t})$ with amplitude $B_z$ and frequency $\omega_d$. For the low-lying excitations ($\langle m^\dagger m \rangle \ll 2N$), we have $S_+ \approx \sqrt{\Delta N} m^\dagger$ and $S_- \approx \sqrt{\Delta N} m$, the drive Hamiltonian leading to $H_{\text{dri}} \approx i\Omega_1 (a^\dagger e^{-i\omega_1 t} - a e^{i\omega_1 t}) + i\Omega_2 (m^\dagger e^{i\omega_2 t} - m e^{-i\omega_2 t})$.

The Hamiltonian in the rotating frame with respect to the two driving lasers $\omega_1 a^\dagger a + \omega_2 m^\dagger m + \omega_{\text{cool}} A^\dagger A$ is (set $\omega_{\text{cool}} = \omega_2$ and $\varphi = \frac{\pi}{2}$)

$$H = H_{\text{free}} + H_{\text{int}} + H_{\text{dri}},$$

$$H_{\text{free}} = \omega_a a^\dagger a - \omega_b b^\dagger b + \omega_{\text{cool}} A^\dagger A + g_{ab} a^\dagger a (b^\dagger + b) + g_{Am} A^\dagger A (b^\dagger + b) + g_{Am} (A^\dagger m + Am^\dagger) + g_{Am} A^\dagger A (b^\dagger + b), \quad (3)$$

where $\omega_a$ ($\omega_b$, $\omega_{\text{cool}}$) is the resonance frequency of the cavity mode (mechanical mode, auxiliary cavity mode), $a$ ($b$, $A$) and $a^\dagger$ ($b^\dagger$, $A^\dagger$) are, respectively, the annihilation and creation operators of the cavity mode (mechanical mode, auxiliary cavity mode), and satisfying $[O, O^\dagger] = 1$ and $O = a, b, A$. $g_{ab} (g_{Am})$ denotes the coupling strength between the cavity (auxiliary cavity) mode and the movable mirror. $g_B$ denotes the electron $g$-factor, $\mu_0$ is Bohr magneton, $B_z$ is the effective magnetic fields, and $\tilde{g}_Am$ is the coupling strength of the cavity photon to a single spin in the magnon mode, and $\varphi$ is a relative phase. Magnon collective spin operators are given by $S = (S_x, S_y, S_z)$, with $S_\pm = S_x \pm iS_y$. It is noted that these collective spin operators are related to the magnon annihilation and creation operators $m$ and $m^\dagger$ via Holstein-Primakoff transformation [36]: $S_+ = m^\dagger (\sqrt{2N}m - m^\dagger m)$, $S_- = m (\sqrt{2N}m - m^\dagger m)$ with $N$ being the spin number of the corresponding collective spin operator and $s = \sqrt{2}$ being the spin number of the ground state $Fe^{3+}$ ion in YIG.

The dissipative dynamics of the system can be described by a set of nonlinear quantum Langevin equations
\[ \dot{a} = -i(\omega_b + \frac{\kappa_a}{2})a - ig_{ab}a(b^{\dagger} + b) + \Omega_1 + \sqrt{\kappa_a}\delta a^{in}(t), \]
\[ \dot{m} = (i\omega_b - \frac{\kappa_m}{2})m + g_{Am}A + \Omega_2 + \sqrt{\kappa_m}\delta m^{in}(t), \]
\[ \dot{b} = -i(\omega_b + \frac{\gamma}{2})b - ig_{ab}a^{\dagger}a - ig_{Ab}A^{\dagger}A + \sqrt{\kappa_b}\delta b^{in}(t), \]
\[ \dot{A} = -\frac{\kappa_A}{2}A - ig_{Ab}A(b^{\dagger} + b) + g_{Am}m + \sqrt{\kappa_A}\delta A^{in}(t), \]
where \( L^{in}(t) = \{a, m, b, A\} \) is the vacuum input noise operator, with the only nonzero correlation function
\[ \langle L^{in}(t)L^{in\dagger}(t') \rangle = (n_L + 1)\delta(t - t'), \]
\[ \langle L^{in\dagger}(t)L^{in}(t') \rangle = n_LT(t - t'), \]
where \( n_L \) being equilibrium mean thermal occupancies of target mode.

Since the cavity mode and the magnon mode are driven by a strong WM source, we have \(|I| \gg 1\) \((I = a, m, A)\) at its steady state. In this case, we can linearize the Hamiltonian around the steady-state average values by writing cavity and magnon operator as linear sum of the classical mean \((O)\) and the quantum fluctuation \(\delta O_j\), i.e. \(O = \langle O \rangle + \delta O (O = a, m, b, A)\), and neglect small second-order fluctuation terms and the small terms that contain \(\langle b \rangle\). By substituting \(O = \langle O \rangle + \delta O\) into Eq. (4), we have the linearized QLEs for the quantum fluctuations
\[ \delta \dot{a} = -(i\omega_b + \frac{\kappa_a}{2})\delta a - ig_{ab}a(\delta b^{\dagger} + \delta b) + \sqrt{\kappa_a}\delta a^{in}(t), \]
\[ \delta \dot{m} = (i\omega_b - \frac{\kappa_m}{2})\delta m + g_{Am}A + \sqrt{\kappa_m}\delta m^{in}(t), \]
\[ \delta \dot{b} = -(i\omega_b + \frac{\gamma}{2})\delta b - ig_{ab}(a^{\dagger}\delta a + \delta a) + ig_{ab}a(\delta b^{\dagger} + \delta b) - ig_{Ab}(\delta A^{\dagger} + \delta A) + \sqrt{\kappa_b}\delta b^{in}(t), \]
\[ \delta \dot{A} = -\frac{\kappa_A}{2}\delta A + g_{Am}\delta m - ig_{Ab}(\delta b^{\dagger} + \delta b) + g_{Ab}A + \sqrt{\kappa_A}\delta A^{in}(t), \]
In addition, if \(\kappa_A \gg g_{Ab}\langle A\rangle\), \(g_{Am}\) in Eq. (6), then we have \(\delta A = 0\), leading to adiabatically eliminating the terms that contain \(\delta A\) and obtaining the direct coupling between phonon and magnon, whose corresponding linearized system Hamiltonian
\[ H_{lin} = \omega_b\delta a^{\dagger}\delta a - \omega_m\delta m^{\dagger}\delta m + \omega_b\delta b^{\dagger}\delta b + \langle G_m\delta m^{\dagger}\delta m + G_a\delta a^{\dagger}\delta a \rangle + H.c., \]
Here \(G_m = \frac{g_{Am}}{\kappa_A}\langle A\rangle = \frac{8g_{Am}G_a}{\kappa_A\kappa_m - 4G_m^2}\) \((G_A = g_{Ab}\langle A\rangle)\) and \(G_a = g_{ab}\langle a\rangle = \frac{g_{ab}\langle a\rangle}{\kappa_A}\) (assumed to be real in the following).
In the interaction picture of \(\omega_b\delta a^{\dagger}\delta a - \omega_m\delta m^{\dagger}\delta m + \omega_b\delta b^{\dagger}\delta b\), we can neglect the fast oscillating terms under the rotating-wave approximation to get an effective Hamiltonian
\[ H_{eff} = (G_a\cosh r - G_m\sinh r)\beta_a^{\dagger}\beta_a + H.c., \]
where \(G = \sqrt{G_m^2 - G_a^2}\) represents the effective coupling between the mode \(\delta b\) and the squeezed mode \(\beta_a\), which denotes the delocalized Bogoliubov mode operators and are defined as unitary transformations of the target modes with a two-mode squeezing operator, respectively.
\[ \beta_a = S(r)\delta a S(r)^\dagger = \delta a \cosh r + \delta m^{\dagger} \sinh r, \]
\[ \beta_m = S(r)\delta m S(r)^\dagger = \delta m \cosh r + \delta a^{\dagger} \sinh r, \]
with \(S(r) = \exp[r(\delta a \delta m - H.c.)], r = \tanh^{-1}(G_m/G_a)\).

The joint ground state of \(\beta_a\) is the two-mode squeezed vacuum state \([r] = S(r)[0, 0]\), where \([0, 0]\) is the vacuum of the target modes.

The mechanism.—Now let’s discuss the interaction mechanism of the scheme. The mode \(\beta_m\) in Eq. (8) is a mechanically dark mode \([38, 39]\), which completely decouples from the mechanical mode in the good-cavity limit and can be neglected. It implies that the squeezed mode \(\beta_a\) has a simple beam-splitter interaction with the mechanical mode \(\delta b\). In this case, \(H_{free} + H_{eff}\) can be trivially diagonalized, resulting in hybridized modes \(\beta_{\pm} = \beta_a \pm \delta b\) with energies \(\omega_{\pm} \pm G\). The existence of three distinct eigenmodes (two hybrid, one dark) can be useful to understand entanglement in the case where the mechanical mode is driven by excessive thermal noise \([40, 41]\). Besides, we can see that Eq. (8) is a standard cavity cooling process \([32, 33]\), so if we introduce another auxiliary cavity with cold reservoir and couple it to the mechanical oscillator, then the beam-splitter coupling \(H_{eff}\) can be used to cool \(\beta_a\) towards vacuum, resulting in a stationary entangled state. In the procedure, the auxiliary mode is used to laser cool thermal occupancy of the mechanical mode towards the ground state by providing a source of cold damping. Hence, the mechanical damping rate \(\gamma\) includes large contribution of the optical damping, i.e., an ideal oscillator should have high frequencies and low-Q.

We further use input-output theory to derive the Heisenberg equation of motion for describing the cooling potential of \(H_{eff}\) based on Eq. (8). It is
\[ \delta \dot{a} = -(i\omega_b + \frac{\kappa_a}{2})\delta a - iG_{a}\delta b - \sqrt{\kappa_a}\delta a^{in}, \]
\[ \delta \dot{m} = -(i\omega_b + \frac{\kappa_m}{2})\delta m + iG_{m}\delta b - \sqrt{\kappa_m}\delta m^{in}, \]
\[ \delta \dot{b} = -(i\omega_b + \frac{\gamma}{2})\delta b - i(G_a\delta a + G_m\delta m^{\dagger}) + \sqrt{\kappa_b}\delta b^{in}. \]

The equation can be used to find the correlation and steady-state occupancy of Bogoliubov modes. For the sake of simplicity, we set \(\kappa_a = \kappa_m = \kappa\) for the convenience of calculation in the following passage.

Imagine that at the beginning each mode in the system is isolated. For fixed \(r > 0\), even at zero temperature, the thermal occupancy of \(\beta_a\) and \(\beta_m\) shown in Eq. (9) is still non-zero. It implies that vacuum noise driving the cavity and magnon acts as effective thermal noise for \(\beta_a, \beta_m\). Therefore, when the system is in the initial state (i.e.,
Also shows that large values \( \delta_b \) can be achieved by simply cooling the Bogoliubov mode \( \beta \). For the equivalent three-mode Gaussian system, if 

\[
\langle \beta_a \beta_b \rangle = N_a = n_m \cosh^2 r + (n_a + 1) \sinh^2 r, \\
\langle \beta_m \beta_m \rangle = N_m = n_a \cosh^2 r + (n_m + 1) \sinh^2 r, \\
i.e., N_a = N_m = \sinh^2 r, (n_a = n_m = 0).
\]

With the increase of the squeezing parameter \( r \), the effective heating of the \( \beta_a \) modes becomes exponentially large, so the sub-system will not be entanglement. When the system in the long time limit, which only contains the role of \( H_{eff} \) (the dark-mode \( \beta_m \) is unaffected), the occupancy of \( \beta_a \) is modified as [32, 33]

\[
\langle \beta_a \beta_a \rangle = \frac{\kappa}{\Gamma_{opt} + \kappa} \frac{\gamma + \Gamma_{opt} + \kappa}{\gamma + \kappa} N_a + \frac{\Gamma_{opt} \gamma}{(\Gamma_{opt} + \kappa)(\gamma + \kappa)} n_b,
\]

where \( n_b \) represents the temperature of the mechanical bath (which includes the cooling auxiliary cavity). Therefore, we can see that both the thermal occupancy of the oscillator and the auxiliary cavity are contained in \( \beta_a \) mode. And the effective “cold damping rate” of \( \beta_a \) by the mechanics is defined as \( \Gamma_{opt} = 4G^2/\gamma \). This equation is very similar to the cavity cooling equation, where now the mechanical oscillator plays the role of a cold reservoir. Therefore, although the vacuum thermal noise will heat Bogoliubov mode to a higher effective temperature, the cold reservoir acting on the oscillator can be used to cool \( \beta_a \) under the interaction of Eq. (8). We can also use the Duan criterion [42] to prove that the magnon-and-lightwave-photon entanglement will exist if one cools \( \beta_a \) so that (cf. Appendix A)

\[
\langle \beta_a \beta_a \rangle \leq \sinh^2 r.
\]

It’s noted that \( N_a \) of the initial state of the system at zero temperature is exactly \( \sinh^2 r \), and so the state is not an entanglement state not up to standard of Duan criterion. However, any amount of cooling described in Eq. (12) will then lower the value of \( \sinh^2 r \), leading to an entanglement of the two target modes. So, although the Bogoliubov mode \( \beta_a \) is decoupled from mechanical mode \( \delta b \), that is, it cannot be cooled by the oscillator, the system cannot achieve an ideal two-mode squeezed vacuum state. Nevertheless, a strong steady-state entanglement between lightwave photons and magnons can also be achieved by simply cooling the Bogoliubov mode \( \beta_a \). For the equivalent three-mode Gaussian system, if

\[
G_a > \sqrt{G_m^2 - \frac{\kappa \gamma}{4}}, \text{i.e., } G_a \gtrsim G_m
\]

then our linear system is always stable (cf. Appendix B and Fig. 2).

**Numerical simulation and analysis.**—To rigorously quantify the magnon-and-lightwave-photon entanglement, we compute and discuss the log negativity \( E_N \) [43], which is a function of the covariance matrix (cf. Appendix C). In Fig. 3, we display the peak values of the magnon-and-lightwave-photon entanglement in the long-time limit as nonmonotonic functions of the entangling interaction \( G_m \) in most sets of parameters of \( G_a \) and take a maximum for a specific \( G_a \). One find that the maximum entanglement depend not only on the strength of the coupling, but also on their ratio (in fact, in the experiment, the coupling strength is generally less than 2 MHz). For a fixed \( G_a, G_m \), increases have two competing effects. On the one hand, it can increase the squeezing parameter \( r = \tanh^{-1}(G_m/G_a) \) and the delocalization of the Bogoliubov modes (enhancing entanglement). On the other hand, it can reduce the effective coupling \( G = \sqrt{G_a^2 - G_m^2} \) between the mode \( \beta_a \) and the “cold” mode \( \delta b \) (or increases the effective temperature of these modes), which is harmful for the cooling effect (reducing entanglement). As is well known, both the squeezing parameter and the cooling effect play an important role in the generation of quantum entanglement, but their influences are different. Fig. 3 also shows that large values of \( E_N \) are possible even when \( n_b \neq 0 \) (the solid black line shown in Fig. 3). The maximum entanglement is achieved by carefully balancing the opposing tendencies described above, which occurs at a non-zero dissipation strength, and at zero mechanical temperature is simply given by \( \gamma = 2G \). This value simply minimizes the occupancy of \( \beta_a \), and corresponds to a simple impedance matching condition (i.e., the rate with which the \( \beta_a \) mode and mechanics exchange energy matches the rate at which the mechanics and its bath exchange energy). More relevant to experiment is to consider \( \kappa \) and \( \gamma \) fixed, and optimize the entanglement over coupling strength. Focusing on the cooperativity \( G_a^2/\kappa \gamma \gg 1 \), we find that for fixed \( G_m \), the \( G_a \) optimal analytical solutions \( G_a^{opt} \) is
given by

\[ G_{a}^{\text{opt}} \approx G_{m} + \sqrt{\frac{\kappa^{2} + 4\kappa^{2} + 4\gamma^{3}}{8\kappa}}, \quad (15) \]

and the \( G_{a}^{\text{opt}} \) is depicted by the black dotted line in Fig. 3, which corresponds to the path of the maximum entanglement at any value of \( G_{m} \). Note that for large cooperativity, this optimal value can easily correspond to a strong interaction \( G > \kappa, \gamma \). Thus, in this optimal regime, the effects of our “engineered reservoir” (cold mechanical resonator) on the target modes cannot be described by a Markovian master equation; this is in stark contrast to standard dissipation-by-entanglement schemes. For the \( G_{a}^{\text{opt}} \) above, the entanglement takes simple forms in two relevant limits. If we hold \( \gamma \) fixed while taking the limit \( \gamma/\kappa \to \infty \), the \( E_{N} \) asymptotic analytical solutions \( E_{N}^{\text{asy}} \) is given by

\[ E_{N}^{\text{asy}} \approx 2r - 2\ln 2. \quad (16) \]

To verify the previous derivation of \( E_{N}^{\text{asy}} \), we drew Fig. 4. In Fig. 4, we can find that for a fixed \( G_{a}^{\text{opt}} \), \( E_{N} \) is a monotonous function of \( G_{m} \). For \( n_{b} = 0 \) and in the strong coupling limit, the cooling factor of \( \beta_{a} \) in Eq. (12) saturates to a value \( \kappa/(\gamma + \kappa) \). Therefore, the maximal cooling of \( \beta_{a} \) is consequently set by the ratio \( \gamma/\kappa \), i.e., increasing \( \gamma/\kappa \) increases the amount that one can cool the delocalized \( \beta_{a} \) mode, and hence enhances entanglement. For a fixed value, the larger the ratio \( \gamma/\kappa \), the closer the value of \( E_{N} \) is to the entanglement asymptotically analytical solution \( E_{N}^{\text{asy}} \) under the long-time limit. Therefore, under the good cavity limit, this more concise analytical solution is more favorable for our research and the error of the equivalent model in the context of weak coupling is very small (shown in the inset in Fig. 4). While the parameters needed for such \( E_{N} \) may be out of reach in current-generation experiments, they may be more feasible by implementing a superconducting circuit realization of our scheme [44, 45].

In fact, in this paper we have used parameters similar to those achieved in recent state-of-the-art experiments on microwave-circuit systems [46, 47]. We assume that a with \( \omega_{b}/2\pi = 10\text{MHz} \) and \( \gamma/2\pi = 0.8\text{MHz} \) mechanical resonator is first cavity cooled to near its ground state, which is predominantly due to the cold optical damping used for the cooling. By balancing the ratio of the effective coupling, we can get the maximum entanglement \( E_{N} \sim 1.7 \). This exceeds by an order-of-magnitude the intracavity entanglement obtained in previous studies of the same system [37], as well as the maximum of ln 2 possible with a coherent two-mode squeezing interaction. If this entanglement was used for a teleportation experiment, the maximum possible fidelity would be 0.72 [48, 49]; this reduces the error by a factor of 2.4 compared to what would be possible with \( E_{N} \sim \ln 2 \).

Note that the above results are valid only when the magnon excitation number \( \langle m^{1}m \rangle \ll 2Ns = 5N \). For a 250-\( \mu \text{m} \)-diameter YIG sphere, the number of spins \( N \simeq 3.5 \times 10^{16} \), and \( G_{m}/2\pi = 2\text{MHz} \) corresponds to \( \langle m \rangle \simeq 6.876 \times 10^{6} \), and \( \Omega_{2} \simeq 4.3475 \times 10^{14} \text{Hz} \), leading to \( \langle m^{1}m \rangle \simeq 7.5 \times 10^{14} \ll 5N = 1.8 \times 10^{17} \), which is well satisfied. The unwanted nonlinear effects due to the Kerr nonlinear term \( Km^{\text{mm}}m \) in the Hamiltonian [26, 50], where \( K \) is the Kerr coefficient, which is inversely proportional to the volume of the sphere. For a 1-\( \mu \text{m} \)-diameter YIG sphere used in Refs. [26, 50], \( K/2\pi \approx 10^{-10} \text{Hz} \), and thus for what we use a 250-\( \mu \text{m} \)-diameter sphere, \( K/2\pi \approx \).
In order to keep the Kerr effect negligible, $K |\langle m \rangle|^3 \ll \Omega$ must hold. For the parameters that we selected $K |\langle m \rangle|^3 \approx 3.56 \times 10^{13} Hz \ll \Omega \approx 7.1 \times 10^{14} Hz$, so implying that the nonlinear effects are negligible and the linearization treatment of the model is a good approximation.

Conclusions.—We have presented a general method for the dissipative generation of lightwave-photon and magnon entanglement in a hybrid cavity-magnon system. The entanglement generated here could be verified by measuring the covariance matrix of the two target modes using homodyne techniques [51]. Our hybrid power system shows four advantages: (i) A cold auxiliary cavity mode is introduced to change the oscillator into a "cold" mode, and then the oscillator is used as the "engineered reservoir" of the whole system to cool Bogoliubov mode $\beta_a$; (ii) We only need to cool one Bogoliubov mode $\beta_a$ to get the magnon-and-lightwave-photon entanglement, which is very close to the two-mode squeezing entanglement $E_N \sim 2r$; (iii) By fixing $\gamma/\kappa$ and balancing the two opposite effects, we can obtain an ideal curve $G_{\text{opt}}$ in Fig. 3. When $G_a = G_{\text{opt}}^a$, on which the maximum entanglement can be obtained regardless of any value of $G_m$; (iv) And finally, by fixing $G_{\text{opt}}^a$ we get an entangled function of $\gamma/\kappa$, and for a fixed $\gamma$, the larger the ratio $\gamma/\kappa$ the closer $E_N$ to $E_N^{\text{asy}}$, when $\gamma/\kappa \rightarrow 0$, the two curves coincide. In this limit, $E_N^{\text{asy}}$ can be approximated as an expression for entanglement.

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Appendix A: The Duan criterion of the Bogoliubov mode occupancy

According to Eq. (4), we can get

$$\langle \{\beta_a^+, \beta_a^-\} \rangle \equiv 2N_a + 1 = \langle \{\delta a^+ \delta a\} \rangle \cosh^2 r + \langle \{\delta m^+ \delta m\} \rangle \sinh^2 r + \frac{1}{2} \langle \{\delta a \delta m\} + \{\delta a^\dagger \delta m^\dagger\} \rangle \sinh 2r. \quad (A1)$$

The $\{\cdots\}$ denotes anti-commutator and according to generalized Duan inequality [42], any entangled state will necessarily satisfy

$$D \equiv \langle (a X_a + \frac{1}{a} X_m^2)^2 \rangle + \langle (Y_a - \frac{1}{a} Y_m)^2 \rangle \leq \sigma^2 + \frac{1}{\sigma^2}, \quad (A2)$$

where $X_j = (\delta j + \delta j^\dagger)/\sqrt{2}$, $Y_j = (\delta j - \delta j^\dagger)/i\sqrt{2}$ and $\sigma$ is an arbitrary nonzero real number. If we set $\sigma^2 = \cosh r$, we can easily verify that

$$2N_a + 1 = \frac{D}{2} \sinh 2r. \quad (A3)$$

For this choice of $\sigma$, using the Duan criterion on $D$ thus yields that for any entangled state, the occupancy of the $\beta_a$ mode must satisfy:

$$N_a \leq \sinh^2 r. \quad (A4)$$

Appendix B: System stability condition

The linearized Heisenberg-Langevin equations for our three-mode system are given in Eq. (10). If we drop the noise terms, these equations take the form $\dot{\vec{v}} = \mathbf{M} \cdot \vec{v}$ where $\vec{v} = (\delta a, \delta m, \delta b)$ and $\mathbf{M}$ is a $3 \times 3$ matrix. For the system to be stable, we require that the eigenvalues of $\mathbf{M}$ all have a negative real part. This requirement leads to the well-known Routh-Hurwitz stability conditions [52]. In the simple case $\kappa_a = \kappa_m = \kappa$, the stability conditions reduce to the following necessary and sufficient condition:

$$G_a > \sqrt{\frac{C^2}{m} - \frac{\kappa \gamma}{4}}. \quad (B1)$$

Thus, if $\kappa_a = \kappa_m = \kappa$ and $G_a \geq G_m$, the system is always stable (regardless of the magnitude of $\kappa$ and $\gamma$).

Appendix C: Definition of the logarithmic negativity

In this paper, unless specified, we use the logarithmic negativity $E_N$ to quantify the degree of entanglement; this quantity is a rigorous entanglement monotone, and is zero for separable states. For two-mode Gaussian states of sort realized by the lightwave photon mode $\delta a$ and the magnon mode $\delta m$ in our system, it can be calculated using the expression [43]

$$E_N \equiv \max[0, -\ln 2 \eta^-], \quad (C1)$$

with

$$\eta^- = \frac{1}{\sqrt{2}} \sqrt{\Sigma - \sqrt{\Sigma^2 - 4 \det V}} \quad (C2)$$

and

$$\Sigma = \det B + \det B^\dagger - 2 \det C. \quad (C3)$$

Here $V$ is the $4 \times 4$ CM of the two modes of interest, defined via $V_{ij} = \frac{i}{2}(\Delta \xi_j \Delta \xi_i + \Delta \xi_i \Delta \xi_j)$, with $\Delta \xi_j = \xi_j - \langle \xi_j \rangle$ and $\xi = \{X_a, Y_a, X_m, Y_m\}$. Here $X_i = (\delta i + \delta i^\dagger)/\sqrt{2}$ and $Y_i = (\delta i - \delta i^\dagger)/i\sqrt{2}$ and $[\delta i, \delta j^\dagger] = \delta_{ij}$. The matrix $B, B^\dagger$ and $C$ are $2 \times 2$ matrices related to the covariance matrix $V$ as

$$V = \begin{bmatrix} B & C \\ C^T & B^\dagger \end{bmatrix}. \quad (C4)$$

Therefore, a Gaussian state is entangled if and only if $\eta^- < \frac{1}{2}$, which is equivalent to Simons necessary and sufficient entanglement nonpositive partial transpose criterion for Gaussian states [53], which can be written as $4 \det V < \Sigma - \frac{1}{4}$.
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