Surprising phenomena in a rich new class of inflationary models

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Abstract. We report on a new class of fast-roll inflationary models. In a huge part of its parameter space, inflationary perturbations exhibit quite unusual phenomena such as scalar and tensor modes freezing out at widely different times, as well as scalar modes reentering the horizon during inflation.

One specific point in parameter space is characterized by extraordinary behavior of the scalar perturbations. Freeze-out of scalar perturbations as well as particle production at horizon crossing are absent. Also the behavior of the perturbations around this quasi-de Sitter background is dual to a quantum field theory in flat space-time. Finally, the form of the primordial power spectrum is determined by the interaction between different modes of scalar perturbations.

Keywords: inflation, cosmological perturbation theory

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1 Introduction

Inflation has become the standard paradigm for the description of the early universe, solving the monopole, horizon and flatness problems as well as providing for a mechanism to seed structure in the universe. Various explicit models for this period of exponential expansion, most relying on the dynamics of real scalar “inflaton” fields, mostly agree with current observations of perturbations of an isotropic background, such as cosmic microwave background (CMB) fluctuations and large scale structure (LSS) (although see [1] and references therein). When analyzing these models, we mostly rely on the assumption of “slow-roll” [2–4] — the slow evolution of the inflaton field — although fast-roll models have also been considered [5]. Slow roll is an approximation made to solve the perturbation equations, but not an absolute requirement for an extended period of accelerated expansion. Even though the scalar spectral index has arguably been measured by the WMAP satellite [6] to be smaller than unity, \( n_s \approx 0.96 \), this constrains the slow-roll parameters

\[
\epsilon \equiv -\frac{\partial_t H}{H^2}, \quad \eta \equiv 2 \frac{\partial_\phi^2 H}{H}, \quad \zeta^2 = \frac{1}{2} \frac{\partial_\phi H \partial_\phi^3 H}{H^2}, \quad (1.1)
\]

all in reduced Planck units \( M_p^2 = \frac{1}{8\pi G} = 1 \), only if they are small. This was exactly the assumption made when solving the perturbation equations, be it using Hankel functions [2], the \( \Delta N \) formalism [3, 7], or the WKB approximation [4, 8]. If the slow-roll parameters are large, then it is impossible to make any general statement about their values from the measurement of \( n_s \).

While it is true that \( 0 < \epsilon < 1 \) for successful inflation, higher order slow-roll parameters may be large, contrary to intuition. Besides exploring unknown corners of inflationary model
space, there are many additional motivations to study models with large $\eta$. For example, string theory motivated models of inflation are typically characterized by large values of $\eta$ [9]. Also, inflationary non-Gaussianities are suppressed by powers of the slow-roll parameters [10], so that one might expect large non-Gaussianities in fast-roll inflationary models.

As the problem of analyzing the behavior of fast-roll inflationary models is technically very challenging, so far those models received only limited attention. Essentially, there are two ways to study this largely unexplored realm. One of them is finding exact solutions to a coupled system of non-linear differential equations. Another approach is to solve these equations numerically. In our paper, we adopt both approaches, providing an exact solution for a subset of initial conditions as well as numerically charting the phase space of a new class of inflationary models with large acceleration, i.e. large $\eta$. This allows us to study the behavior of both background and perturbations in new and interesting regimes, where we find various unexpected and even downright bizarre phenomena.

While our model can be made to agree with observations for a small range of parameters, it possesses several most unusual features for parameter ranges that are not compatible with observations. Some of these features might seem counter-intuitive. For example, according to general lore, perturbation modes that exit the Hubble horizon, freeze out and remain classical until the present epoch. We find that is not true for a particular model of this family. The modes remain quantum mechanical at all times and wavelengths, even long after they cross the Hubble scale. Also, it is generally believed that the interactions among the different modes of scalar perturbations are suppressed by powers of the slow roll parameters, so that only insignificant amounts of non-Gaussianity are created during single-field inflation. Instead we show that it is the interaction between different modes that defines the very form of the scalar power spectrum. Other unusual features include the fact that tensor and scalar modes can freeze out at different times and even unfreeze during inflation. Finally, it turns out that for a specific member of this family, the theory of scalar perturbations is described by a quantum field theory in flat space time.

Our paper is organized as follows. In section 2 we present an explicit one-parameter family of fast-roll models whose background equation is exactly solvable for certain initial conditions. In section 3 we discuss the spectrum of scalar and tensor perturbations, which can be reliably computed despite the fact that the slow-roll parameter $\eta$ may be larger than unity. In section 4 we present an in-depth analysis of the perturbations for a special member of the family of models, where the scalar perturbations do not follow the usual behavior of oscillating, being stretched to the horizon, and freeze-out, but instead obey an exactly massless harmonic oscillator type equation. In section 5 we conclude and offer some ideas for future work on this and similar classes of models.

2 A most curious model

In this section we discuss in detail the shape of the potential, solve the background dynamics and analyze the phase space for our class of models.

2.1 Shape of potential

Consider a single real scalar field $\phi$ in a 4d Friedman-Robertson-Walker universe

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2,$$  

(2.1)
Figure 1. (a) $V(\phi)$ for $p = 0.04$. Inflation takes place close to the top of the potential, rolling down from $\phi \approx 0$ towards larger values of $|\phi|$. The potential is very shallow and turns negative only for $\phi > \sqrt{\frac{24}{p}} \approx 122$, i.e. about 30 e-folds after the end of inflation. For this model, the power spectrum of scalar perturbations has a spectral index that is compatible with observations, $n_s = 1 - p = 0.96$.

(b) $V(\phi)$ for $p = -2$. For negative $p$, the potential goes to $-\infty$ for $\phi \to \infty$. Inflation takes place rolling from small values of $|\phi|$ towards $\phi = 0$ and ends through tunneling. The scalar power spectrum possess some interesting features, see text.

where the scale factor $a(t)$ is related to the Hubble parameter $H(t) \equiv \frac{\dot{a}}{a}$. In units $M_p^2 \equiv \frac{1}{8\pi G} = 1$, the field’s potential is given by

$$ V(\phi) = H_0^2 e^{-\frac{p}{8} \phi^2} \left(3 - \frac{p^2}{8} \phi^2 \right), \quad (2.2) $$

where $H_0$ is some energy scale and $p$ is a real parameter. For positive $p$, the minimum of the potential is located at $\phi_{1,2} = \pm 2\sqrt{\frac{6}{p}},$ with $V(\phi_{1,2}) = -H_0^2 \frac{p}{2} e^{-\left(\frac{6}{p} + 1\right)}$ and its maximum is located at $\phi_3 = 0$, with $V(\phi_3) = 3H_0^2$. For negative $p$, the minima turn into maxima and the maximum turns into a local minimum. (See figure 1.) Note that for $p \leq -6$, the potential possess just a maximum and no minima, mimicking the behavior of hill-top models [11]. The potential is negative for field values $|\phi| > \sqrt{\frac{24}{|p|}}$ independent of the sign of $p$. For positive $p$, we use the form of the potential

$$ V = H^2(3 - \epsilon), \quad (2.3) $$

which is exactly equivalent to (2.7), to see that the potential turns negative when $\epsilon = 3$, i.e. when $\phi = \phi_- \equiv \sqrt{\frac{24}{p}}$. This occurs $\Delta N = \frac{1}{p} \ln 3$ e-foldings after the end of inflation. Thus, if the potential (2.7) holds for $\phi > \phi_-$, then the universe would subsequently rapidly contract without entering an AdS phase [12].

If on the other hand $p < 0$, inflation happens close to the local minimum near the origin. The field $\phi$ slowly rolls down the hill towards the minimum of its potential, so that inflation never ends in the classical approximation. The model $p = -2$ was first discussed by [13, 14] who reach somewhat different conclusions.

If quantum effects are taken into account, this conclusion changes. For $p < 0 \neq -2$, Hawking-Moss tunneling\(^{1}\) [15] through the potential barrier will terminate the inflationary phase in any given Hubble patch. Still, the average length of the inflationary stage will be

\(^{1}\)For $p = -2$, the Hawking-Moss rate is undefined since stochastic eternal inflation is absent, see section 4.
enormously large — the inverse Hawking-Moss tunneling rate (per Hubble volume $H_0^{-3}$) is exponentially suppressed

$$t_{\text{inf}} \sim H_0^{-1} \exp \left( \frac{\kappa M_p^2}{H_0^2} \right),$$

where $\kappa$ is a numerical factor of $O(1)$ and $H_0 \ll M_p$.

Another issue is that the true minima of the potential are located at $\phi \to \pm \infty$, $V = -\infty$. Because it is well-known [12] that the universe collapses rapidly for negative potentials, we need to avoid tunneling away from the local minimum to negative values of the potential. Thus we should define the potential piece-wise as e.g.

$$V = \begin{cases} H_0^2 e^{-\frac{4}{p} \phi^2} \left( 3 - \frac{p^2}{8} \phi^2 \right), & |\phi| < \phi_* \\ H_0^2 e^{-\frac{4}{p} \phi_*^2} \left( 3 - \frac{p^2}{8} \phi_*^2 \right), & |\phi| > \phi_* \end{cases},$$

where $\phi_*$ is any arbitrary position between the zero crossings

$$|\phi_*| < \frac{\sqrt{24}}{|p|}.$$  

This enables us to safely avoid tunneling to negative potential energies. However we point out that even though such tunneling events are rare, at late times the universe would be not necessarily be sufficiently flat. Curing this would require a second phase of inflation.

The universe would be automatically homogeneous if we introduced a coupling of $\phi$ to a second field $\chi$ which is massive during inflation and only becomes dynamically important after inflation, i.e. once $\phi$ settled near its minimum $\phi = 0$, similar to waterfall models.

### 2.2 Classical background evolution

The equations of motion for $\phi$ and the metric are of the usual form

$$\ddot{\phi} + 3H \dot{\phi} + \partial_\phi V = 0,$$

$$H^2 = \frac{1}{3} \left( \frac{1}{2} \dot{\phi}^2 + V \right), \quad \dot{H} = -\frac{1}{2} \dot{\phi}^2,$$

where $\dot{\phi} \equiv \partial_t \phi$. After switching the independent variable from $t$ to the number of e-folds $N$ ($dN = H dt$, with $N = 0$ at the beginning of inflation, $N > 0$ at later times – opposite to the standard definition of $N$), we find

$$\partial_N^2 \phi + (3 - \epsilon) \partial_N \phi + \frac{\partial_\phi V}{H^2} = 0,$$

$$H^2 \left( 3 - \frac{1}{2} \left( \partial_N \phi \right)^2 \right) = V, \quad \partial_N \ln H = -\frac{1}{2} \left( \partial_N \phi \right)^2.$$

If we want the universe to undergo a period of inflation/accelerated expansion, then we must have

$$0 < \ddot{a} = a \left( \dot{H} + H^2 \right) = a H^2 (1 - \epsilon).$$

Inflation thus happens whenever $0 < \epsilon < 1$. 

\[ \text{(2.9)} \]
Of course, it is quite impossible to solve this set of differential equations for general initial conditions. One exact solution for the subset of initial conditions (see the appendix for details)

\[
\begin{align*}
\phi(t=0) &= \frac{2\sqrt{2}c_0}{p}, \\
\dot{\phi}(t=0) &= H_0\sqrt{2c_0}e^{-\frac{t_0}{p}}, \\
H(t=0) &= H_0e^{-\frac{t_0}{p}}, 
\end{align*}
\] (2.10)

where we keep $a_0$ explicit for use in the next section, has the form

\[
a = a_0e^N, H = H_0e^{-\frac{t_0}{p}e^{pN}}, \epsilon = \epsilon_0e^{pN}, \phi = \frac{2\sqrt{2c_0}}{p}e^{\frac{c_0N}{2}},
\] (2.11)

Thus this solution has

\[
H(\phi) = H_0e^{-p\phi^2/8} \quad \text{and} \quad \epsilon = \frac{p^2}{8}\phi^2,
\] (2.12)

with initial values for $\phi, \dot{\phi}$, and $H$ fixed by the choice of $\epsilon_0$ for a model with given $p$. Note that the parameter $\eta$ is given by

\[
\eta = -\frac{p}{2} + \epsilon_0e^{pN} = -\frac{p}{2} + \epsilon,
\] (2.13)

that is, $|\eta| \approx \frac{p}{2}$ during most of the inflationary period which in principle can be arbitrarily large.

There are 2 qualitatively different regimes in this family of models, $p$ positive and $p$ negative.

For $p > 0$, $\epsilon$ grows with time, and inflation ends when $\epsilon = 1$. The field starts near the global maximum $\phi = 0$, rolls away and accelerates until inflation ends at $\phi_{\text{end}} = \frac{2\sqrt{2}}{p}$. Defining $N_f$ to be the end of inflation fixes $\epsilon_0 \equiv e^{-pN_f}$. If we arbitrarily fix $N = 0$ to be the onset of inflation, then $N_f$ is also the number of inflationary e-folds.

The phase space, shown in figure 2 for $p = 0.04$, has a single unstable fixed point at the origin.\footnote{This value of $p \approx 0.04$ leads to a power spectrum of scalar perturbations compatible with observations, see section 3.1.} The inflationary attractor is the slow roll attractor (solid red line) and seems identical to the background solution (2.11) (dashed black line). There are no fixed point attractors shown as the universe will collapse as soon as the potential turns negative for $|\phi| > \sqrt{2p}$ (not shown).

If $p < 0$, then $\epsilon$ decreases with time. As we assume that inflation starts at $N = 0$, we take $c_0 = 1$ for negative $p$. The time evolution according to (2.11) does not lead to an end of inflation at any particular final number of e-folds $N_f$. The field will reach the local minimum only in the infinite future. The field may start rolling from $|\phi| > |\phi_{\text{crit}}| = \frac{2\sqrt{2}}{p}$, but only for values $|\phi| < |\phi_{\text{crit}}|$ will inflation actually begin. Note that we can make $\eta$ large by making $p$ large and negative and still have an infinite number of e-folds of inflation. However, for $p < -6$, the background solution (2.11) is unstable. It corresponds to $\phi$ starting its evolution with just enough momentum to roll up the hill and come to a rest on the top.
Figure 2. (a) Phase portrait for $p = 0.04$. The black dashed line is the trajectory of the solution (2.11). (c) Zoom of panel (a) with the hypothetical slow roll attractor in solid red. In this case, slow-roll is an attractor, owing to the fact that $\eta = -0.02$ is small.

Figure 3. (a) Phase portrait for $p = -2$. The black dashed line is the trajectory of the solution (2.11). (c) Zoom of panel (a) with the hypothetical slow roll attractor in solid red. Obviously, slow-roll is not an attractor in this model, in agreement with the fact that $\eta = 1$ is large.

The phase portrait, shown in figure 3 for $p = -2$, has a single fixed point attractor at the origin, corresponding to the local minimum of the potential. Unlike chaotic $m^2\phi^2$ inflation, there are no circular trajectories around the attractive fix-point. In other words, the field does not oscillate around the local minimum. The local maxima of the potential at $\phi = \pm 2$ manifest themselves as unstable points in the phase space diagram. For $|\phi| > 2$, the trajectories run to regions where the potential is negative, leading to the collapse discussed by [12] (not shown in the figure). At late times, the solution (2.11) obviously approaches the attractor solution. Notice that the latter (dashed black line) differs from the slow-roll attractor (solid red line)

\[
\dot{\phi}_{\text{SR}} = -\frac{\partial V}{3H^2} \approx \frac{H_0}{12}p(6 + p)\phi = -\frac{2}{3}H_0\phi, \\
\dot{\phi}_{\text{exact}} = \frac{p}{2}H_0\phi e^{-\frac{p}{8}\phi^2} \approx \frac{p}{2}H_0\phi = -H_0\phi,
\]

demonstrating again that we are not dealing with a model of slow-roll inflation, see figure 3(b). Thus, it is justified to use (2.11) as an asymptotic solution when dealing with the late-time evolution and focus on its properties.
3 Linear perturbations

In this section we describe the behavior of linear scalar and tensor perturbations around the attractor solution found in the previous section and compute their power spectra for generic values of $p$. As it turns out, scalar and tensors freeze out at widely different times for large values of $|p|$.

3.1 Scalar power spectrum

Using the formalism of [2], it is straightforward to examine the scalar perturbations of the metric

$$u''_k(\tau) + \left( k^2 - \frac{z''}{z} \right) u_k(\tau) = 0, \quad (3.1)$$

where

$$u_k \equiv -\frac{a\dot{\phi}}{H} \left( \Psi - \frac{H}{\dot{\phi}} \delta \phi \right). \quad (3.2)$$

Here, $\delta \phi$ is the perturbation of the inflaton field $\phi$, $\Psi$ is the scalar perturbation of the spatial part of the metric, $\tau$ is conformal time, $u'_k \equiv \partial_\tau u_k = a\partial_t u_k$ and $z \equiv \frac{a\dot{\phi}}{H} = a\sqrt{2\epsilon}$ for any single-field inflation model. We assume that we are at late times and thus evolve along the attractor solution, coinciding with the exact solution (2.11), to obtain

$$z = \sqrt{2\epsilon_0} a_0 e^{(\frac{p}{2}+1)N}$$

and

$$\frac{z''}{z} = a^2 H^2 \left( \frac{p}{2} + 1 \right) \left( \frac{p}{2} + 2 - \epsilon \right), \quad (3.3)$$

leading to the equations of motion for scalar perturbations $u_k$

$$u''_k + \left[ k^2 - a^2 H^2 \left( \frac{p}{2} + 1 \right) \left( \frac{p}{2} + 2 - \epsilon \right) \right] u_k = 0. \quad (3.4)$$

Before we attempt to solve this equations, we note that $\frac{z''}{z}$ possesses a local maximum before the end of inflation, implying that some modes might reenter the horizon during inflation$^3$ (see figure 4) albeit on scales much smaller than observable in the CMB today. For $p < -2 - \sqrt{3}$ and $p > -2 + \sqrt{3}$, their wave numbers are given by values close to $k_{\ast} = \frac{z''}{z} (N_{\ast})$, with

$$N_{\ast} = \frac{1}{p} \ln \left( \frac{3 + p \pm \sqrt{p^2 + 4p + 1}}{2\epsilon_0} \right), \quad (3.5)$$

determined by the condition that $\frac{\partial}{\partial N} \frac{z''}{z} \bigg|_{N=N_{\ast}} = 0$. Similar effects were observed by [16, 17].

Let us return to solving the mode equation (3.4) where $\epsilon \ll 1$ around the time that the relevant scales for the CMB leave the horizon ($N = 0 \ldots 10$ e-folds, i.e. $50 \ldots 60$ e-folds before the end of inflation). Because $\epsilon \ll 1$, $H$ is approximately constant, $H \simeq H_0$. In this

$^3$If $k^2 > \frac{z''}{z}$, then the modes behave like free oscillators, and if $k^2 < \frac{z''}{z}$, the modes will freeze out.
Figure 4. (a) Comoving horizon \((\zeta'/\zeta)^{-1}\) for the scalar perturbations as a function of the number of e-folds \(N\), see (3.3) for the exact definition. Notice that there is a maximum comoving wavenumber \(k_*\) that will freeze out, depending on the value of \(p\), and that some modes will leave the horizon and come back into the horizon during inflation, i.e. while \(\epsilon < 1\). \(p = 1\) (dashed green), \(p = 1.5\) (dotted blue), \(p = 2\) (dot-dashed purple), \(p = -1\) (solid black). (b) Comparison of the comoving horizon for the scalar \((\zeta'/\zeta)^{-1}\) (dashed green) and tensor \((\alpha'/\alpha)^{-1}\) (dot-dashed blue) perturbations. For \(p = 2\), tensors freeze out about 1 e-fold after the scalars. Note that inflation starts at \(N = 0\) and ends (for \(p > 0\)) at \(N = 60\).

approximation, the solutions are given by (using the Bunch-Davis vacuum \(u_k = \frac{1}{\sqrt{2k}}e^{-ik\tau}\) as initial conditions)

\[
u_k(\tau) = e^{i((\frac{p}{4}+\frac{1}{2})\pi \frac{\sqrt{\pi}}{2}\sqrt{-\tau}H^{(1)}_{3+\epsilon}(\tau k),
\]

which are approximated at late times \((\tau \to 0)\), i.e. after freeze-out, as

\[
u_k(\tau) = -i2^{\frac{p+3}{2}}\sqrt{-\tau}^{-\frac{p+3}{2}}\pi \Gamma \left(\frac{p+3}{2}\right) e^{i((\frac{p}{4}+\frac{1}{2})\pi (-k\tau)^{-\frac{p+3}{2}}}.\]

Thus we can compute the power spectrum of scalar perturbations (we insert explicit factors of \(M_p\))

\[
\mathcal{P}_s = \frac{2^{p-1}}{\pi^3} \left[ \Gamma \left(\frac{p+3}{2}\right) \right]^2 \frac{H_0^2 a_0^2}{M_p^2} \epsilon_0 \left( \frac{k}{H_0}\right)^{-p} \]

\[= \frac{2^{p-1} 3.81 \times 10^{66p}}{\epsilon_0 \pi^3} \left[ \Gamma \left(\frac{p+3}{2}\right) \right]^2 \left( \frac{H_0}{M_p}\right)^{p+2} \left( \frac{k/a_0}{\text{Mpc}^{-1}}\right)^{-p},\]

where \(\epsilon_0 = e^{-p\Delta N}\) for a duration of inflation of \(\Delta N\) efolds, and \(a_0\) is the scale factor at the onset of inflation. The scalar spectral index \(n_s - 1 = -p\) is consistent\(^4\) with measurements [6] by the WMAP satellite 0.93 < \(n_s < 0.99\) (95%CL) for 0.01 < \(p < 0.07\), leading to values of \(H_0 \approx 10^{-4} \ldots 10^{-5} M_p\) for a scalar amplitude of \(A_s = 2.44 \times 10^{-9}\). As we show below, the ratio of amplitude of the tensor spectrum and the scalar spectrum can be tuned to be compatible with observations if inflation lasts sufficiently long.

\(^4\)Also, after inflation ends in this model, the universe will stay in a expanding phase for another \(\Delta N = 16 \ldots 110\) efolds.
3.2 Tensor power spectrum and freeze out

The equations of motion for the tensor perturbations $v_k$ are given exactly by

$$v_k'' + [k^2 - a^2 H^2 (2 - \epsilon)] v_k = 0,$$

where $v_k \equiv a \psi_{k,\lambda}$, with the tensor part of the metric perturbation $h_{ij} = \int \frac{d^3 k}{(2\pi)^3} \sum \lambda \psi_{k,\lambda}(\tau) e_{ij}(k, \lambda) e^{ik\cdot x}$ and transverse polarization tensor $e_{ij}$. They can be solved in an analogous manner to the scalar perturbations, again assuming $\epsilon$ is tiny at the times of interest, to obtain an exactly scale free spectrum

$$P_T = \frac{H_0^2}{4\pi^2 k^0},$$

and tensor-scalar ratio $r \equiv \frac{P_T}{P_S}$

$$r = \frac{\epsilon_0}{3.81 \times 10^{56}} \left[ \Gamma \left( \frac{p+3}{2} \right) \right]^2 2^{p+1} \left( \frac{H_0}{M_p} \right)^{-p} \left( \frac{k/a_0}{\text{Mpc}^{-1}} \right)^p,$$

which is $r < 0.4$ for inflation lasting 120 efolds and $p = 0.01$ and $r < 10^{-4}$ for inflation lasting 60 efolds and $p = 0.07$. The former is marginally excluded by observations, whereas the latter is well compatible with observations.

From the equations of motion, it is immediately clear that tensors freeze out at a different time than scalars (see figure 4). Due to the difference between $z''$ and $a''$, tensor and scalar modes of the same comoving wavelength cross their respective horizons at different times. Conversely, at each point in time, scalar and tensor modes of wavenumber difference $\Delta k^2$ cross their horizons, with

$$\sqrt{\Delta k^2} = \sqrt{\frac{z''}{z} - \frac{a''}{a}} = aH \sqrt{\frac{p}{2} \left( \frac{p}{2} + 3 - \epsilon \right)}.$$  

Increasing $|p|$ leads to larger and larger differences in the freeze-out time.\(^5\)

4 Special case without freeze-out

There are several models with specific values of $p$ that deserve special attention as they show unexpected behavior of the perturbation modes. First of all, when $-2 > p > -4$, the sign of $\frac{z''}{z}$ for the background solution/attractor flips. Thus the scalar perturbations obtain a positive mass squared and consequently do not freeze-out.

Secondly, when $p = -2$, then $z = \sqrt{2 \epsilon a}$ is identically constant at late times (corresponding to small $\phi$) as we see from the numerical analysis of the system for different initial conditions, see figure 5. As inflation lasts an exponentially large number of efolds (owing to the exponentially suppressed rate of tunneling out of the false minimum at $\phi = 0$), we can use this observation to study the generation of fluctuations assuming $z = \text{constant}$ along all phase space trajectories. Thus, the equation of motion for $u_k$ simplifies to

$$u_k'' + k^2 u_k = 0,$$

\(^5\)Note that large $p$ are inconsistent with observations.
indicating that a) particle creation by the gravitational field is absent and b) modes never freeze out, not even when the physical wavelength of a given mode exceeds the Hubble scale.\footnote{Note that for the case $p = -2$ the asymptotic form (3.7) does not hold.}

We stress that by freeze-out we refer to the regime of the perturbation equations (3.1) where the $k^2$ term becomes sub-dominant compared to effective negative mass-square term. Although there is only a finite amount of conformal time after a given mode leaves the horizon, this does not make the mode classical even at $\tau - > 0$ in our case, but only slows down the rate of oscillations in physical time.

The absence of a quasi-classical regime implies that there is no stochastic eternal inflation, as the modes $u_k$ never become classical. This is easy to understand\footnote{We thank Latham Boyle for pointing this out.} when looking at the action [10] for the curvature perturbations $\zeta$

$$S = \int d^4x \sqrt{-g} \epsilon^\mu_\nu \partial_\mu \zeta \partial_\nu \zeta.$$  \hfill (4.2)

For $p = -2$, the scale-factor dependence of $\sqrt{-g}, \epsilon$ and $g^\mu_\nu$ precisely cancel, making $\zeta$ “live” in Minkowski space.

In “ordinary” inflationary models, the resulting path integral would become classical as the action is growing exponentially due to the scale-factor dependence. In this particular model, the action will not grow like a power of $a$, thus the path integral

$$Z = \exp \left( \frac{iS}{\hbar} \right)$$  \hfill (4.3)

will remain dominated by quantum effects.

Let us first understand why $z$ approaches constant value at late times. As explained above, for almost arbitrary initial conditions $\phi(t = 0) = \phi_0$, $\dot{\phi}(t = 0) = \dot{\phi}_0$ the system reaches the attractor trajectory in finite time. For $p = -2$ and at small $\phi < M_P$ the attractor trajectory is given by the expression

$$\dot{\phi} = -H_0 \phi$$  \hfill (4.4)
Solving the background FRW equations for the trajectory (4.4), one can easily check that
\[ a\phi = \text{constant}, \tag{4.5} \]
in this regime. In other words, \( \phi \sim a^{-1} \) at late times: although \( \phi \) becomes exponentially small, classically it takes an infinite amount of time for the system to reach the false minimum \( \phi = 0 \).

Also note that (4.5) can be derived without using the asymptotic expression for the attractor (4.4). Indeed, the equation of motion for the variable \( U = a\phi \) is given by
\[ U'' - \frac{a''}{a} U + a^4 \frac{\partial V(U/a)}{\partial U} = 0, \tag{4.6} \]
where \( ' \) denotes derivative w.r.t. conformal time \( \tau \). Since inflation lasts infinitely long, one can use the expansion in powers of \( 1/a \) in order to find its asymptotic solution at late times.

To leading order in \( 1/a \), equation (4.6) acquires the form
\[ U'' = 0. \tag{4.7} \]
Note that this equation is independent of \( a \) and, as we will see below, the equation for the Fourier modes \( u_k \) also has non-trivial \( a \)-independent form in the limit \( 1/a \to \infty \). This is what makes our model special. Indeed, for a generic inflationary model one would have
\[ u_k'' + k^2 u_k + \left( m^2 a^2 - \frac{a''}{a} \right) u_k = 0, \tag{4.8} \]
which can be approximated to
\[ u_k'' + k^2 u_k + (m^2 - 2H^2) a^2 u_k = 0, \tag{4.9} \]
to leading order in \( 1/a \). There, the \( 1/a \) expansion would only illustrate the familiar fact that modes with effective mass \( m^2 > H^2 \) get amplified by inflation, while modes with \( m^2 < H^2 \) get exponentially damped. In our model with \( p = -2 \), the effective mass term proportional to \( a^2 \) cancels exactly.

Let us now discuss the dynamics of fluctuations around the background \( U = \text{constant} \). Since \( \ddot{z} z'' = 0 \) in our case, the leading effect influencing the dynamics of the modes \( u_k \) is the interaction between them.

To estimate the leading (third order in small perturbations) effect of this interaction, it is necessary to work in the constant curvature gauge fixed to second order in small perturbations [10]
\[ a\phi = U + u(\tau, x), u = a\delta \phi(\tau, x), h_{ij} = a^2 \tilde{h}_{ij}, \tag{4.10} \]
\[ \det \tilde{h} = 1, \tilde{h}_{ij} = \delta_{ij} + \tilde{\gamma}_{ij} + \frac{1}{2} \tilde{\gamma}_{il} \tilde{\gamma}_{lj} + \ldots, \tag{4.11} \]
\[ \tilde{\gamma}_{iii} = 0, \partial_i \tilde{\gamma}_{ij} = 0. \tag{4.12} \]

Keeping only the leading order in \( 1/a \), one finds that the scalar part of the ADM action for fluctuations has the form
\[ S = \int_{-1/H}^0 d\tau \int d^3 x \left( \frac{1}{2} (u')^2 - \frac{1}{2} (\partial_x u)^2 - W(u) \right), \tag{4.13} \]
where
\[ W(u) = h_u u^3, \quad h_u = \frac{H_0^2}{4M^2_P} U, \] (4.14)
in the third order w.r.t. small fluctuations. Note that the action is independent of the scale factor \( a \) in the limit \( a \to \infty \), e.g. \( |\tau| \ll H^{-1} \). The interaction terms do not get red-shifted, neither do the occupation numbers for the modes \( u_k \) nor the effective temperature of the perturbations \( T \). In other words, since \( U \) is expected to be constant at late times, we see that \( \text{in the limit } 1/a \to 0, \text{ the physics of fluctuations on around this quasi-de Sitter space is described by the quantum field theory (4.13) living in flat spacetime.} \)

Then, the question of initial conditions for the model becomes very important for the model (4.13). While it is natural to assume \( U = \text{constant} \) as initial condition for the background (trajectories corresponding to other choice of initial conditions approach the attractor \( U = \text{constant} \) in finite time), the initial conditions for the perturbation modes \( u_k \) can be chosen arbitrarily. Choosing a multi particle initial state such that
\[ n_k = \left( \frac{1}{2k} \langle |u_k'|^2 + k^2 u_k^2 \rangle \right) - \frac{1}{2} > 1, \] (4.15)
and \( \langle |u_k|^2 \rangle = \frac{n_k}{2\omega_k} \) is large for certain intervals of \( k \), one can expect that the distribution \( n_k \) will change with time, gradually approaching a thermal distribution
\[ n_k \sim \exp \left( -\frac{k^2}{T_{\text{therm}}} \right). \] (4.16)
The final value of the thermalization temperature \( T_{\text{therm}} \) can be easily estimated: since the equations of motion following from (4.13) have an integral of motion and preserve the total “energy” of the system
\[ E = \int d^3x \left( \frac{1}{2}(u')^2 + \frac{1}{2}(\nabla u)^2 + W(u) \right) + \frac{1}{2}(U')^2, \] (4.17)
the thermalization temperature is determined by the amount of “energy” (4.17) in the system at \( \tau \approx H^{-1} \)
\[ T_{\text{therm}} \sim E^{1/4}, \] (4.18)
where \( V \sim H_0^{-3} \) is the 3-volume of the initial Hubble patch. Note that the energy (4.17) is unbounded from below, if only second and third order terms w.r.t. small fluctuations are kept in the action (4.13). This issue is resolved by taking 4th order terms into account. No higher order terms survive in the \( 1/a \to 0 \) approximation.

The question of thermalization time scale is actually non-trivial. For example, if we choose an “initial” state at \( \tau \approx H^{-1} \) such that \( n_k \) is peaked at some \( k = k_i \) and zero at other \( k \), a careful analysis [18] shows that complete thermalization takes at least
\[ \tau_{\text{therm}} \sim \left( \frac{T_{\text{therm}}}{k_i} \right)^{2/5}. \] (4.19)
The physical reason for the long thermalization time scale is the following. The evolution of \( n_k \) is described by a system of coupled kinetic equations. Typically, when initial occupation numbers \( n_k \) are large, the general solution this system gets quickly attracted to (and spends
a long time in) a self-similar regime of weak wave turbulence [18], characterized by a power law spectrum

\[ n_k \sim k^{-3/2} \]  

(4.20)

and by a scaling behavior of occupation numbers with time.

Note on the other hand that the spectrum of perturbations has the Rayleigh-Jeans form [19]

\[ n_k \sim \frac{T}{\omega_k} \sim k^{-1} \]  

(4.21)

in the infrared part, if ultimate thermalization is achieved.

The spectra (4.20) and (4.21) are trivially related to the power spectrum of primordial perturbations:

\[ k^3 p_k = k^3 \left| \frac{u_k}{z} \right|^2 = \frac{k^3 n_k}{z^2 \omega_k}. \]  

(4.22)

However, the expression above holds interest for us if and only if inflation comes to an end — only in this case eq. (4.22) is related to CMB anisotropies that we observe in the sky.

As discussed in section 2.1, inflation can only end due to tunneling from near the false minimum at \( \phi = 0 \) to the true minima at \( \phi \rightarrow \pm \infty \). The duration of inflation can be estimated as

\[ \tau_{\text{inf}} \sim \frac{1}{H_0 a_0}. \]  

(4.23)

The effective action for the fluctuations (4.13) is only valid during a finite amount\(^8\) of conformal time \( -H^{-1} < \tau < 0 \). If the thermalization time scale is longer than this period, the perturbation spectrum will be given by (4.20). The crucial point is that — whether there is enough conformal time for the system to reach full thermalization or not — the modes keep interacting even after they leave the horizon, making the power spectrum change with time even for superhorizon modes.

Let us also briefly discuss the issue of backreaction of fluctuations \( u_k \) on the background \( U \). The presence of the third order term in the ADM action (4.13) leads to an instability through growth of small fluctuations \( u(\tau, x) \). Since the trilinear vertex is of the form

\[ W_3 \sim \frac{H_0^2}{M_P^2} U u^3, \]  

(4.24)

growth of \( u \) leads to decrease of the background field \( U \). Ultimately, \( u \) stops growing either when \( U \) reaches 0, so that the trilinear vertex (4.24) effectively disappears, or when the quartic term \( W_4 \sim \frac{H_0^2}{M_P^2} u^4 \) starts to dominate over the trilinear term in the effective potential \( W(u) \). While it would take an infinite amount of time for the classical, unperturbed system to reach the minimum of the potential, the background field will vanish in finite time if perturbations are taken into account.

Finally, we would like to remark that the ADM action (4.13) above contains only terms corresponding to interaction between scalar modes. Strictly speaking, interaction between tensor and scalar modes should also be taken into account, since it does not vanish in the limit \( a \rightarrow \infty \): namely, the term corresponding to decay of one scalar into two tensors

\[ W_{3,T} \sim H_0^2 U uu^2 \]  

(4.25)

\(^8\)We thank the referee for reminding us of this fact.
is of the same order of magnitude as $W_3$ in the limit $a \to \infty$. This fact has several important consequences. First of all, since tensors do freeze out even for $p = -2$, super-horizon tensor modes generate an effective term in the ADM effective action which is linear in $u$ and $U$, leading to a faster decay of the background field $U$. On the other hand, scalar fluctuations $u$ also effectively decay into tensor modes with $k > aH$. We leave the study of this issue for future work.

5 Conclusion and outlook

In this paper we discuss the behavior of the family of inflationary models

$$V = H_0^2 M^2 e^{-\frac{p}{4} \frac{\phi^2}{M^2}} \left( 3 - \frac{p^2}{8} \frac{\phi^2}{M^2} \right),$$

with arbitrary $p$ and $H_0 < M_{\text{pl}}$.

Exploring the phase space of this family of models for large $|p|$ reveals the existence of a non-slow-roll attractor which at late times can be approximated by

$$\dot{\phi} = \frac{p}{2} H_0 \phi.$$ 

For small values of $|p|$ this attractor coincides with the slow-roll attractor, see equation (2.14). For all values of $|p|$, this family of models is observationally ruled out. However, they turn out to exhibit many fascinating features, some quite counterintuitive. The reason why the standard lore based on the slow roll approximation fails in this case is that the value of $p$ essentially determines the value of the slow roll parameter $\eta = 2 \frac{\partial^2 \phi}{2 H}$, so that larger values of $|p|$ correspond to larger values of $\eta$, see equation (2.13).

Table 1 show the physical properties of our family of models for different values of the parameter $p$. The most interesting physics takes place in the model with $p = -2$ (see the figure 2b). First of all, to leading order in a $1/a$ expansion, the scalar perturbations in this model turn out to be described by a quantum field theory in flat spacetime (4.13). The linear scalar perturbations are free massless harmonic oscillators and thus never freeze out. Gravitational particle production, stochastic kicks uplifting the background value of the inflaton field and a regime of stochastic eternal inflation are completely absent.

If the initial state for the scalar fluctuations $u_k = a \delta \phi_k$ has a non-zero occupation number for a range of momenta, one finds that the corresponding occupation numbers $n_k$ typically approach a spectrum of form (4.20) due to interactions between the modes $u_k$, with a spectral index $n_s = \frac{3}{2}$. Interestingly, the effects of the interaction between the different modes $u_k$ do not get redshifted away.

As a curious aside, we notice that the shape of the potential (2.2) is strongly reminiscent of the SUGRA form

$$V_{\text{SUGRA}} = e^K \left( R^{ij} D_i W D_j \bar{W} - 3 |W|^2 \right),$$

where the Kähler potential $K$ is a real function of the complex fields $\phi^i, \bar{\phi}^i$, the superpotential $W$ is a holomorphic function of the complex fields $\phi^i$, the covariant derivative

---

9 In fact, models with $-4 < p < -2$ are also characterized by the absence of freeze out. Here, the scalar perturbations are harmonic oscillators with time-varying but strictly positive masses. We leave a thorough examination of this issue to future work.
\[
|\eta| > 1
\]

\[
\text{Table 1. Behaviour of our model } V = H_0^2 e^{-\frac{q}{4} \phi^2} (3 - \frac{q}{8} \phi^2) \text{ for different values of } p. 
\]

\[
D_i W = \partial_{\phi^i} W + \partial_{\phi^j} KW \text{ and } K^{ij} = (\partial_{\phi^i} \phi^j) K^{-1}. \text{ Assuming the Kähler potential } K = -\frac{q}{4} \phi^2 \text{ and the superpotential } W = H_0 = \text{constant, the potential can be computed to}
\]

\[
V_{\text{SUGRA}} = -H_0^2 e^{-\frac{q}{4} \phi^2} \left(3 + \frac{q}{4} |\phi|^2\right).
\]

For \( q = -2 \), this agrees with (5.1) for \( p = -2 \) up to an overall sign

\[
V^{q=-2}_{\text{SUGRA}} = -V^{p=-2},
\]

possibly indicating that this model may be embedded into string theory.

Finally, we remark that it might be possible to have both modes without freeze-out and a scalar power spectrum with \( n_s \approx 0.96 \) for a wider class of models, where the attractor solution has \( z = \text{constant} \) instead of \( \zeta' = \text{constant} \). In this case, it is immediately clear that the equations of motion for the perturbations \( u_k \) again simplify to those of a free harmonic oscillator.
What is not clear is whether in these models inflation actually takes place: the inflationary condition from \((2.9)\) corresponds to a highly non-trivial differential equation for the Hubble parameter \(H\). We leave a thorough examination of this class of models for future work.

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A ADM action for scalar fluctuations in the case \(p = -2\)

In this appendix we explicitly derive the 3rd order ADM action \((4.13)\) for the fluctuations \(u = a\delta\phi\) in the case \(p = -2\) (for the sake of convenience, we will denote the fluctuations \(\delta\phi\) here by \(\varphi\)).

Calculated in the constant curvature gauge \((4.12)\), the 3rd order ADM action has the form [10]

\[
S_3 = \int d\tau d^3x a^4 \left( -\frac{\dot{\varphi}}{H} \varphi^2 - a^{-2} \frac{\dot{\varphi}}{4H} \varphi (\partial\varphi)^2 - \dot{\varphi} \partial_i \chi \partial_i \varphi + \frac{3}{8H} \varphi^3 - \frac{\dot{\varphi}}{16H^3} \varphi^3 - \frac{\dot{V}''}{4H} \varphi^3 - \frac{V'''}{6} \varphi^3 + \frac{\dot{\varphi}^3}{4H^2} \varphi^2 \dot{\varphi} + \frac{\dot{\varphi}}{4H} (\varphi \partial_i \partial_j \chi \partial_i \partial_j \chi + \varphi \partial^2 \chi \partial^2 \chi) \right) \quad (A.1)
\]

where \(\chi\) satisfies the equation

\[
\partial^2 \chi = \frac{\dot{\varphi}^2}{2H^2} \frac{d}{dt} \left( -\frac{H}{\dot{\varphi}} \varphi \right). \quad (A.2)
\]

To estimate how different terms in \((A.1)\) behave at \(a \rightarrow \infty\), we note that in this regime all trajectories in the phase space approach the non-slow roll attractor \(\dot{\varphi} = -H_0 \varphi\). By definition, the variable \(u\)

\[
\varphi = \frac{u}{a} \sim \frac{1}{a}, \quad (A.3)
\]

since \(u\) remains finite on the attractor. Therefore,

\[
\delta\phi = \frac{u'}{a^2} - \frac{Hu}{a} \approx -\frac{Hu}{a} \quad (A.4)
\]

to leading order in \(1/a\) (here prime denotes the usual derivative with respect to conformal time \(\tau\)). Also note that in the vicinity of the attractor \(\dot{\varphi} = -H_0 \varphi\) the scalar \(\chi\) evolves according to

\[
\partial^2 \chi = \pm \frac{U u'}{2a^3}, \quad (A.5)
\]

so that \(\chi\) has at most the order \(1/a^3\) as \(a \rightarrow \infty\).

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Analyzing the 3rd order ADM action (A.1) term by term, we find that\(^{10}\)
\[-a^4 \frac{\dot{\phi} \varphi^2}{H} \sim O(1),\]  
\[-a^4 a^{-2} \frac{\dot{\phi} \varphi (\partial \varphi)^2}{H} \sim O(a^{-2}),\]  
\[-a^4 \phi \partial_i \chi \partial_i \varphi \sim O(a^{-1}),\]  
\[-a^4 \frac{3 \dot{\phi}^3}{8H} \varphi^3 \sim O(a^{-2}),\]  
\[-a^4 \frac{\dot{\phi}^5}{16H^3} \varphi^3 \sim O(a^{-4}),\]  
\[-a^4 \frac{\dot{\phi} V''}{4H} \varphi^3 \sim O(1),\]  
\[-a^4 \frac{V''}{6} \varphi^3 \sim O(1),\]  
\[-a^4 \frac{\dot{\phi}^3}{4H^2} \varphi^2 \dot{\varphi} \sim O(a^{-2}),\]  
\[-a^4 \frac{\dot{\phi}^2}{4H^2} \varphi^2 \partial^2 \chi \sim O(a^{-3})\]  
and finally
\[-a^4 \frac{\dot{\phi}}{4H} (-\varphi \partial_i \partial_j \chi \partial_i \partial_j \chi + \varphi \partial_i \chi \partial^2 \chi) \sim O(a^{-3}).\]  

As we find, only three terms (A.6), (A.11) and (A.12) survive out of 10 terms present in (A.1) when \(a \to \infty\). Moreover, one can explicitly show that the terms (A.11) and (A.12) cancel each other, so that the term ((A.6)) is the only one that contributes to the effective potential \(W(u)\) in (4.13).

### B Exact background solution and perturbations for a subset of initial conditions

By definition, the “slow-roll” parameter \(\epsilon\) is given by
\[
\epsilon = -\frac{d \ln H}{d N},
\]
where \(N\) is the number of e-folds from the end of inflation, \(dN = H dt\), where \(t\) is physical time.

Useful formula are
\[
a = e^N, \quad (B.2)
\]
\[
\frac{\partial \phi}{\partial N} = \sqrt{2\epsilon}, \quad (B.3)
\]
\[
V = H^2 (3 - \epsilon), \quad (B.4)
\]
\[
\eta = -\frac{1}{2\epsilon} \frac{\partial \epsilon}{\partial N} + \epsilon, \quad (B.5)
\]

\(^{10}\)For briefness, we only focus on the case \(\phi < 0, \dot{\phi} > 0\), generalization to the case \(\phi > 0, \dot{\phi} < 0\) is straightforward.
which are exact equations with no assumptions about slow-roll. Let us examine the one-parametric family of models \( \epsilon = \epsilon_0 e^{pN} \).

### B.1 Background

It is straightforward to integrate the above equations to obtain

\[
H = H_0 e^{-\frac{\epsilon_0 e^{pN}}{p}}, \tag{B.6}
\]
\[
\phi = \frac{2\sqrt{2\epsilon_0}}{p} e^{\frac{pN}{2}}, \tag{B.7}
\]
\[
\Rightarrow H = H_0 e^{\frac{p}{2}\phi^2}, \tag{B.8}
\]
\[
\epsilon = \frac{p^2}{8} \phi^2, \tag{B.9}
\]
\[
\eta = -\frac{p}{2} + \epsilon_0 e^{pN}, \tag{B.10}
\]
\[
V(\phi) = H_0^2 e^{\frac{p}{2}\phi^2} \left( 3 - \frac{p^2}{8} \phi^2 \right), \tag{B.11}
\]

where \( H_0 \) is a free integration constant. \(|\eta|\) is arbitrarily large for large \(|p|\), yet inflation proceeds as \( 0 < \epsilon = \epsilon_0 e^{p|N|} < 1 \). However, the potential has extrema at \( \frac{\partial V}{\partial \phi} = 0 \)

\[
\phi_1 = 0, \quad V(\phi_1) = 3H_0^2, \tag{B.12}
\]
\[
\phi_{2/3} = \pm \frac{2\sqrt{6} - p}{p}, \quad V(\phi_{2/3}) = H_0^2 \frac{p^6}{2} e^{\frac{6}{p} - 1}. \tag{B.13}
\]

### B.2 Perturbations

The perturbation equations for scalars/tensors are given by [2]

\[
u''_k + \left( k^2 - \frac{z''_{S,T}}{z_{S,T}} \right) u_k = 0, \tag{B.14}
\]

where \( t \) denotes derivative with respect to conformal time \( dt = \frac{dt}{a} \), and \( z_S = a\sqrt{2}\epsilon \) for scalar and \( z_T = a \) for tensor perturbations, where \( a \) is the scale factor of the FRW metric. Converting derivatives w.r.t. to \( \tau \) to derivatives w.r.t. \( N \), we use

\[
\frac{\partial}{\partial \tau} = a \frac{\partial}{\partial N} = a H \frac{\partial}{\partial N}, \tag{B.15}
\]
\[
\frac{\partial^2}{\partial \tau^2} = a^2 H^2 \left( \frac{\partial^2}{\partial N^2} + (1 - \epsilon) \frac{\partial}{\partial N} \right), \tag{B.16}
\]

to rewrite \( \frac{z''_{S,T}}{z_{S,T}} \)

\[
\frac{a''}{a} = \frac{1}{a} \left( a^2 H^2 \partial_N^2 a + a^2 H^2 (1 - \epsilon) \partial_N a \right), \tag{B.17}
\]
\[
\frac{a''}{a} = a^2 H^2 (2 - \epsilon), \tag{B.18}
\]
\[
z = \sqrt{2} \alpha = a_0 \sqrt{2} \epsilon e^{-\left( \frac{p}{2} + 1 \right) N}, \tag{B.19}
\]
\[
z'' = a^2 H^2 \left( \frac{p}{2} + 1 \right) \left( \frac{p}{2} + 2 - \epsilon \right), \tag{B.20}
\]
and to obtain for the tensor modes $v_k$ and scalar modes $u_k$

$$u_k'' + \left[ k^2 - a^2 H^2 \left( \frac{p}{2} + 1 \right) \left( \frac{p}{2} + 2 - \epsilon \right) \right] u_k = 0, \quad \text{(B.21)}$$

$$v_k'' + \left( k^2 - a^2 H^2 (2 - \epsilon) \right) v_k = 0. \quad \text{(B.22)}$$

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