PERIODIC POINTS ON CHARACTER VARIETIES

JUNHO PETER WHANG

Abstract. Given a surface of positive genus with finitely many punctures, we classify the periodic points for the mapping class group action on the moduli space of complex special linear rank two local systems. For genus at least two, the periodic points correspond to the finite local systems. For genus one, they correspond to the finite or special dihedral local systems.

1. Introduction

1.1. Let Σ be a surface of genus g with n punctures. Its mapping class group acts naturally on the character variety

\[ X = \text{Hom}(\pi_1 \Sigma, \text{SL}_2(\mathbb{C})) \sslash \text{SL}_2(\mathbb{C}) \]

parametrizing the isomorphism classes of semisimple representations \( \pi_1 \Sigma \to \text{SL}_2(\mathbb{C}) \) of the fundamental group, or semisimple \( \text{SL}_2(\mathbb{C}) \)-local systems on Σ. By definition, the periodic points on \( X \) are the points with finite mapping class group orbits. In this paper, we give a characterization of the periodic points on character varieties for surfaces of positive genus. We first state our main result for \( g \geq 2 \).

Theorem A. Let Σ be a surface of genus \( g \geq 2 \) with \( n \geq 0 \) punctures. The periodic points in \( X \) correspond precisely to the finite representations. In particular, this gives a dynamical characterization of the finite \( \text{SL}_2(\mathbb{C}) \)-local systems on surfaces of genus \( g \geq 2 \). Let us next define an irreducible representation \( \pi_1 \Sigma \to \text{SL}_2(\mathbb{C}) \), or the corresponding local system, to be special dihedral if its image lies in the infinite dihedral group (recalled in Section 2) and there is a nonseparating simple closed curve \( a \) in Σ such that the restriction of the representation to the complement \( \Sigma \setminus a \) is diagonal. We state our main result for \( g = 1 \).

Theorem B. Let Σ be a surface of genus 1 with \( n \geq 0 \) punctures. The periodic points in \( X \) correspond precisely to the finite or special dihedral representations.

Theorems A and B show in particular that, for surfaces of genus \( g > 0 \), the representations of \( \pi_1 \Sigma \) with Zariski dense image in \( \text{SL}_2(\mathbb{C}) \) have infinite mapping class group orbits in the character variety. This suggests an analogy with the result of Previte-Xia [10] on the topological density of orbits for \( \text{SU}(2) \)-character varieties of positive genus surfaces. In contrast, for \( (g, n) = (0, 4) \) there are periodic points on the character variety that correspond to representations with Zariski dense image in \( \text{SL}_2(\mathbb{C}) \). This increase in complexity of periodic points may be attributed to the inverse change in complexity of mapping class groups.

Date: July 4, 2017.

Key words and phrases. character variety, surfaces, mapping class group, finite orbits.
1.2. We discuss the literature and motivation behind our results. Character varieties and their mapping class group actions appear in several different contexts, and the periodic points often hold special interest.

For \((g, n) = (0, 4)\) and \((1, 1)\), the periodic points are intimately connected to the algebraic solutions of Painlevé VI equations. These are a family of nonlinear second-order ordinary differential equations on the complex plane, whose solutions have three essential singularities at 0, 1, and \(\infty\) and have no other singularities except movable poles. The nonlinear monodromy of the solutions is related to the mapping class group action on the character variety for \((g, n) = (0, 4)\). Using this connection, Lisovyy-Tyhkyy [9] completely classified the periodic points in this case as well as the algebraic solutions to Painlevé VI. This was preceded by the work of Dubrovin-Mazzocco [4], on a special subfamily of equations, which essentially settled the periodic points for \((g, n) = (1, 1)\). See also the work of Hitchin [8] on explicit constructions of algebraic Painlevé VI’s arising from finite dihedral representations. Following Boalch [2], the classification of algebraic Painlevé VI may be viewed as a nonlinear analogue of Schwarz’s list classifying the algebraic solutions to the hypergeometric equation of Riemann. The study of periodic points on character varieties for general \((g, n)\) is a natural extension of this program.

Another source of interest for the finite mapping class group orbits lies in the Diophantine geometry of the character variety \(X\), or rather its subvarieties \(X_k\) fixing the traces \(k = (k_1, \cdots, k_n) \in \mathbb{C}^n\) of monodromy around punctures. For \((g, n) = (1, 1)\), the subvarieties \(X_k\) are (up to the coefficient 3) the level sets of the Markoff cubic \(x_1^2 + x_2^2 + x_3^2 - 3x_1x_2x_3\). Bourgain-Gamburd-Sarnak [3] examined the question of strong approximation for these affine cubic surfaces, which asks for the surjectivity of the reduction mod \(q\) map

\[X_k(\mathbb{Z}) \to X_k(\mathbb{Z}/q\mathbb{Z})\]

away from an exceptional locus. Their approach is based on showing the transitivity of the mapping class group action on the nonexceptional locus of \(X_k(\mathbb{Z}/q\mathbb{Z})\). A finite mapping class group orbit over \(\mathbb{C}\) appears in \(X_k(\mathbb{Z}/q\mathbb{Z})\) for infinitely many \(q\) by reduction, and hence must be accounted for as part of the exceptional locus. Classification of the periodic points therefore arises naturally as part of our program to investigate strong approximation for general character varieties. Similarly, the classification of higher-dimensional mapping class group invariant subvarieties of character varieties is an interesting problem, which leave to future work.

1.3. We outline the proofs of our results and the contents of this paper. We begin by recording background on subgroups of \(\text{SL}_2(\mathbb{C})\), character varieties, and mapping class groups in Section 2. We also introduce the notion of loop configurations as a tool to keep track of subsurfaces of a surface. In Section 3, we deduce the case...
(g, n) = (1, 1) of Theorem B from the results of Dubrovin-Mazzocco \cite{4}. In Section 4, we show that Zariski dense representations, i.e. representations with Zariski dense image in $\text{SL}_2(\mathbb{C})$, are not periodic in the character variety for surfaces of positive genus. We prove this using the case $(g, n) = (1, 1)$ and the following lemma, which may be of independent interest.

**Main Lemma.** Let $\Sigma$ be a surface of positive genus $g > 0$ with $n \geq 0$ punctures. A representation $\pi_1 \Sigma \to \text{SL}_2(\mathbb{C})$ is irreducible, resp. Zariski dense, if and only if its restriction to some embedded surface of genus 1 with 1 boundary curve is irreducible, resp. Zariski dense.

Finally, in Section 5, we verify Theorems A and B on the locus of representations that do not have Zariski dense image in $\text{SL}_2(\mathbb{C})$, thus completing the proof.

1.4. **Acknowledgements.** I thank Peter Sarnak for guidance and suggesting the problem of understanding the periodic points on character varieties. I also thank Phillip Griffiths and Sophie Morel for guidance and helpful discussions. I thank William Goldman for informing me about the work of Biswas et al \cite{11}.

**Contents**

1. Introduction \hspace{2cm} 1
2. Background \hspace{2cm} 3
3. Case of one-holed torus \hspace{2cm} 9
4. Zariski dense representations \hspace{2cm} 12
5. Sparse representations \hspace{2cm} 16
6. References \hspace{2cm} 20

2. **Background**

2.1. **Subgroups of** $\text{SL}_2(\mathbb{C})$. Let $G$ be a proper algebraic subgroup of $\text{SL}_2(\mathbb{C})$. Up to conjugation, $G$ satisfies one of the following \cite{12} Theorem 4.29:

1. $G$ is a subgroup of the standard Borel group
   \[ B = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a \in \mathbb{C}^*, b \in \mathbb{C} \right\}. \]

2. $G$ is a subgroup of the infinite dihedral group
   \[ D_\infty = \left\{ \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix} : c \in \mathbb{C}^* \right\} \cup \left\{ \begin{bmatrix} 0 & c \\ -c^{-1} & 0 \end{bmatrix} : c \in \mathbb{C}^* \right\}. \]

3. $G$ is one of the finite groups $BA_4$, $BS_4$, and $BA_5$, which are the preimages in $\text{SL}_2(\mathbb{C})$ of the finite subgroups $A_4$ (tetrahedral group), $S_4$ (octahedral group), and $A_5$ (icosahedral group) of $\text{PGL}_2(\mathbb{C})$, respectively.

**Remark.** The finite subgroups $BA_4$, $BS_4$, $BA_5$ of $\text{SL}_2(\mathbb{C})$ can be explicitly described as follows. First, let us identify the group of unit quaternions

\[ \text{Sp}(1) = \{ z = (a, b, c, d) = a + bi + cj + dk \in \mathbb{H} : \|z\| = a^2 + b^2 + c^2 + d^2 = 1 \} \]
as a subgroup of \( SL_2(\mathbb{C}) \) by the map
\[
z = (a, b, c, d) \mapsto \begin{bmatrix} a + bi & c + di \\ -c + di & a - bi \end{bmatrix}.
\]
Under the identification, the binary tetrahedral group \( BA_4 \) is given by
\[
BA_4 = \{ \pm 1, \pm i, \pm j, \pm k, (\pm 1 \pm i \pm j \pm k)/2 \}
\]
with all sign combinations taken in the above. The binary octahedral group \( BS_4 \) is the union of \( BA_4 \) with all quaternions obtained from \((\pm 1, \pm 1, 0, 0)/\sqrt{2}\) by all permutations of coordinates and all sign combinations. The binary icosahedral group \( BA_5 \) is the union of \( BA_4 \) with all quaternions obtained from \((0, \pm 1, \pm \varphi^{-1}, \pm \varphi)/2\) by an even permutation of coordinates and all possible sign combinations, where \( \varphi = (1 + \sqrt{5})/2 \) is the golden ratio.

The above classification has the following consequence. Given a subset \( S \subset G(\mathbb{C}) \) of an algebraic group \( G \), let \( Zcl(S) \) denote the Zariski closure of \( S \). If \( \pi \leq G(\mathbb{C}) \) is a group, then \( Zcl(\pi) \) is an algebraic subgroup of \( G \). We say that \( \pi \) is Zariski dense in \( G \) if \( Zcl(\pi) = G \).

**Lemma 1.** A torsion-free subgroup of \( SL_2(\mathbb{C}) \) is Zariski dense in \( SL_2(\mathbb{C}) \) if and only if it is not conjugate to a subgroup of the standard Borel group \( B \).

**Proof.** The only if direction is clear, and we prove the converse. Let \( \pi \) be a torsion-free subgroup of \( SL_2(\mathbb{C}) \) not conjugate to a subgroup of \( B \). This implies that \( \pi \) is nontrivial, and if \( \pi \) is not Zariski dense in \( SL_2(\mathbb{C}) \) then up to conjugation it is a subgroup of \( D_\infty, BA_4, BS_4, \) or \( BA_5 \). The last three cases are immediately ruled out by the condition that \( \pi \) is torsion-free. If \( \pi \) is conjugate to a subgroup of \( D_\infty \) and is not conjugate to a subgroup of \( B \), then there exists an element of \( \pi \) which is conjugate to a matrix of the form
\[
\begin{bmatrix}
0 & c \\
-c^{-1} & 0
\end{bmatrix}
\]
for some \( c \in \mathbb{C}^* \). But this matrix has order 4, contradicting the hypothesis that \( \pi \) is torsion-free. Hence, \( \pi \) must be Zariski dense in \( SL_2(\mathbb{C}) \). \( \square \)

We record the following well-known observations.

**Lemma 2.** Given subgroups \( \pi' \leq \pi \) of an irreducible algebraic group \( G \) such that \( [\pi : \pi'] < \infty \), we have \( Zcl(\pi) = G \) if and only if \( Zcl(\pi') = G \).

**Proof.** The if direction is clear. For the converse, let \( g_1, \cdots, g_N \in \Gamma \) be a complete set of coset representatives for \( \pi/\pi' \). Note that \( Zcl(g_i\pi') = g_i Zcl(\pi') \) for each \( i = 1, \cdots, N \), and hence
\[
Zcl(\pi) = Zcl \left( \bigcup_{i=1}^{N} g_i\pi' \right) = \bigcup_{i=1}^{N} Zcl(g_i\pi') = \bigcup_{i=1}^{N} g_i Zcl(\pi').
\]
Suppose that \( Zcl(\pi) = G \). Since \( G \) is assumed to be irreducible, we see that \( G \leq g_i Zcl(\pi') \) for some \( g_i \), which implies that \( Zcl(\pi') = G \), as desired. \( \square \)

**Lemma 3.** A pair \((a, b)\) of elements in \( SL_2(\mathbb{C}) \) has a common eigenvector in \( \mathbb{C}^2 \), or in other words lies in \( B \) up to simultaneous conjugation, if and only if \( \text{tr}([a, b]) = 2 \), where \([a, b] = aba^{-1}b^{-1} \).
Proof. After conjugating $a$ and $b$ by an element of $\text{SL}_2(\mathbb{C})$, we reduce to one of the following three cases.

(1) We have $a = \pm 1$, where $1 = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$ is the identity matrix. In this case, $(a, b)$ obviously has a common eigenvector and $\text{tr}(a, b) = \text{tr}(1) = 2$.

(2) We have $a = s \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right]$ where $s \in \{\pm 1\}$. Writing $b = \left[ \begin{array}{cc} b_1 & b_2 \\ b_3 & b_4 \end{array} \right]$, we have

$$\text{tr}(aba^{-1}b^{-1}) = \text{tr}\left( \left[ \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right] \left[ \begin{array}{cc} b_1 & b_2 \\ b_3 & b_4 \end{array} \right] \left[ \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right] \left[ \begin{array}{cc} b_1 & b_2 \\ b_3 & b_4 \end{array} \right]^{-1} \right) = 2 + b_3^2,$$

using the fact that $b_1b_4 - b_2b_3 = 1$. Hence, we have $\text{tr}([a, b]) = 2$ if and only if $b_3 = 0$, which is equivalent to saying that $(a, b)$ has a common eigenvector.

(3) We have $a = \lambda \left[ \begin{array}{cc} 0 & 1 \\ 0 & -1 \end{array} \right]$ for some $\lambda \in \mathbb{C} \setminus \{\pm 1\}$. Writing $b = \left[ \begin{array}{cc} b_1 & b_2 \\ b_3 & b_4 \end{array} \right]$, we have

$$\text{tr}(aba^{-1}b^{-1}) = \text{tr}\left( \left[ \begin{array}{cc} \lambda & 0 \\ 0 & -\lambda \end{array} \right] \left[ \begin{array}{cc} b_1 & b_2 \\ b_3 & b_4 \end{array} \right] \left[ \begin{array}{cc} \lambda^{-1} & 0 \\ 0 & -\lambda \end{array} \right] \left[ \begin{array}{cc} b_1 & b_2 \\ b_3 & b_4 \end{array} \right]^{-1} \right) = 2b_1b_4 - (\lambda^2 + \lambda^{-2})b_2b_3.$$

The condition $\text{tr}([a, b]) = 2$ is then equivalent to $b_2b_3(\lambda^2 + \lambda^{-2} - 2) = 0$. Since $\lambda^2 + \lambda^{-2} - 2 = (\lambda - \lambda^{-1})^2 \neq 0$ by the hypothesis that $\lambda \neq \pm 1$, we conclude that this is equivalent to the statement that $b_2 = 0$ or $b_3 = 0$, or in other words, $(a, b)$ has a common eigenvector. Thus, $(a, b)$ has a common eigenvector if and only if $\text{tr}([a, b]) = 2$. \hfill \square

2.2. Character varieties. Let $\pi$ be a finitely generated group. The representation variety $\text{Hom}(\pi, \text{SL}_2(\mathbb{C}))$ is by definition the affine scheme representing the functor

$$S \mapsto \text{Hom}(\pi, \text{SL}_2(\Gamma(S, \mathcal{O}_S))))$$

for every affine $\mathbb{C}$-scheme $S$. Given a set of generators of $\pi$ with $m$ elements, we have an explicit presentation of $\text{Hom}(\pi, \text{SL}_2(\mathbb{C}))$ as a closed subscheme of $\text{SL}_2(\mathbb{C})^m$ defined by equations coming from relations among the generators. We define the $(\text{SL}_2(\mathbb{C})\text{-})$character of $\pi$ variety to be the affine invariant theoretic quotient

$$X(\pi) = \text{Hom}(\pi, \text{SL}_2(\mathbb{C})) \sslash \text{SL}_2(\mathbb{C})$$

of the representation variety $\text{Hom}(\pi, \text{SL}_2(\mathbb{C}))$ by the conjugation action of $\text{SL}_2(\mathbb{C})$. The complex points of $X(\pi)$ parametrize semisimple representations $\rho: \pi \to \text{SL}_2(\mathbb{C})$ up to $\text{SL}_2(\mathbb{C})$-conjugacy, where a representation is semisimple if it is semisimple as a representation into $\text{GL}_2(\mathbb{C})$. The coordinate ring $R(\pi)$ of $X(\pi)$ has presentation (see \cite{11} Theorem 7.1)

$$R(\pi) = \mathbb{C}[\{a \in \pi\}/(1 - 2, [a][b] - [ab] - [ab^{-1}])$$

where $1 \in \pi$ is the identity, and $[a] \in R(\pi)$ corresponds to the regular function on $X(\pi)$ given by $\rho \mapsto \text{tr}(\rho(a))$ for any representation $\rho: \pi \to \text{SL}_2(\mathbb{C})$. The relations follow from the observation that $\text{tr}(1) = 2$ and that, given any two $2 \times 2$ matrices $a$ and $b$, we have the relation $\text{tr}(a)\text{tr}(b) = \text{tr}(ab) + \text{tr}(ab^*)$ where $b^*$ is the adjugate of $b$. We recall the following facts \cite{11} Fact 2.6, Corollary 6.4:

**Fact 4.** Let $\pi$ be a finitely generated group, and let $R(\pi)$ be the coordinate ring of its character variety.

(1) We have $[a] = [a^{-1}]$ and $[ab] = [ba]$ for every $a, b \in \pi$.

(2) We have $[aba^{-1}b^{-1}] = [a]^2 + [b]^2 + [ab]^2 - [a][b][ab] - 2$. 



**PERIODIC POINTS ON CHARACTER VARIETIES 5**
(3) Let \( a_1, \cdots, a_m \) be a generating set for \( \pi \). The ring \( R(\pi) \) is finitely generated as a \( \mathbb{C} \)-algebra by the elements 
\[
[a_{i_1} \cdots a_{i_k}]
\]
for \( i_1 < \cdots < i_k \) integers in \( \{1, \cdots, m\} \) and \( k \geq 1 \).

**Example 5.** See [6] for details in the examples below. Let \( F_m \) denote the free group in \( m \geq 1 \) generators \( a_1, \cdots, a_m \).

1. We have an identification \( [a_1] : X(F_1) \simeq \mathbb{A}^1 \).
2. We have an identification \( (x, y, z) = ([a_1], [a_2], [a_1a_2]) : X(F_2) \simeq \mathbb{A}^3 \) by Fricke. From Lemma 3, it follows that the locus of reducible representations in \( X(F_2) \) is an affine cubic surface given by the equation
\[
x^2 + y^2 + z^2 - xyz = 2.
\]

The construction of the character variety \( X(\pi) \) is functorial in \( \pi \). Given a morphism \( f : \pi \rightarrow \pi' \) of finitely generated groups, the morphism \( f^* : X(\pi') \rightarrow X(\pi) \) is given by \( \rho \mapsto \text{semisimplification of } \rho \circ f_* \). In particular, \( \text{Aut}(\pi) \) naturally acts on \( X(\pi) \). The action is defined on the coordinate ring by \( \varphi \cdot [a] = [\varphi(a)] \) for \( \varphi \in \text{Aut}(\pi) \) and \( a \in \pi \). Since \( [bab^{-1}] = [a] \) for every \( a, b \in \pi \), this action factors through the outer automorphism group \( \text{Out}(\pi) \). Consider the morphism \( i : \mathbb{C}^* \rightarrow \text{SL}_2(\mathbb{C}) \) given by \( a \mapsto \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \) and inclusion \( j : D_{\infty} \rightarrow \text{SL}_2(\mathbb{C}) \). They induce \( \text{Out}(\pi) \)-equivariant maps
\[
i_* : \text{Hom}(\pi, \mathbb{C}^*) \rightarrow X(\mathbb{C}) \quad \text{and} \quad j_* : \text{Hom}(\pi, D_{\infty})/D_{\infty} \rightarrow X(\mathbb{C}) \).

**Lemma 6.** We have the following.

1. The map \( i_* : \text{Hom}(\pi, \mathbb{C}^*) \rightarrow X(\mathbb{C}) \) has finite fibers.
2. The map \( j_* : \text{Hom}(\pi, D_{\infty})/D_{\infty} \rightarrow X(\mathbb{C}) \) has finite fibers.

**Proof.**

1. Let \( \rho_1, \rho_2 : \pi \rightarrow \mathbb{C}^* \) be characters such that \( gi_* (\rho_1)g^{-1} = i_* (\rho_2) \) for some \( g \in \text{SL}_2(\mathbb{C}) \). Without loss of generality, we may assume given some \( a \in \pi \) such that \( \rho_1(a) \notin \{ \pm 1 \} \), since otherwise the image of \( \rho_1 \) is finite and we are done. Writing \( g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and \( \rho_1(a) = \lambda \) with \( \lambda \in \mathbb{C}^* \setminus \{ \pm 1 \} \), we have
\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} = \begin{bmatrix} \lambda ad - \lambda^{-1} bc & (\lambda^{-1} - \lambda) ab \\ (\lambda - \lambda^{-1}) cd & \lambda^{-1} ad - \lambda bc \end{bmatrix}.
\]
For the matrix on the right hand side to be diagonal, we must thus have \( a = d = 0 \) or \( b = c = 0 \) since \( \lambda \neq \pm 1 \). If \( a = d = 0 \), then \( \rho_2 = \rho_1^{-1} \). If \( b = c = 0 \), then \( \rho_1 = \rho_2 \).

This proves (1).

2. Let \( \rho_1, \rho_2 : \pi \rightarrow D_{\infty} \) be representations such that \( gj_* (\rho_1)g^{-1} = j_* (\rho_2) \) for some \( g \in \text{SL}_2(\mathbb{C}) \). Without loss of generality, we may assume given \( a \in \pi \) with \( \text{tr} \rho_1(a) \notin \{ 0, \pm 2 \} \), since otherwise the image of \( \rho_1 \) is finite and we are done. Note that \( \rho_1(a) \) must be diagonal. The equation \( g\rho_1(a)g^{-1} = \rho_2(a) \) shows that, by the same computation as above, we have \( g \in D_{\infty} \). This proves (2). \( \square \)

We make the following definition.

**Definition 7.** A representation \( \pi \rightarrow \text{SL}_2(\mathbb{C}) \) is said to be:

1. Zariski dense if its image is Zariski dense in \( \text{SL}_2(\mathbb{C}) \),
2. diagonal if it factors through \( i : \mathbb{C}^* \rightarrow \text{SL}_2(\mathbb{C}) \),
3. dihedral if it factors through \( j : D_{\infty} \rightarrow \text{SL}_2(\mathbb{C}) \), and
4. finite if its image is finite.
2.3. Mapping class groups. Throughout this paper, a surface is the complement of a finite set of interior points in a connected compact oriented topological manifold of dimension 2 with or without boundary. Let Σ be a surface. A simple closed curve \(a\) in \(\Sigma \setminus \partial \Sigma\) is essential if it does not bound a disk, is not isotopic to a boundary curve, and is not the boundary of an embedded punctured disk around a puncture of \(\Sigma\). An essential curve \(a\) in \(\Sigma\) is separating if the complement \(\Sigma \setminus a\) is not connected, and nonseparating otherwise.

Let \(\Sigma\) be a surface of genus \(g\) with \(n\) punctures or boundary curves. Let \(\text{PMod}(\Sigma)\) be the mapping class group of \(\Sigma\). By definition, it is the group of isotopy classes of orientation preserving homeomorphisms of \(\Sigma\) fixing the punctures or boundary points individually. Given a simple closed curve \(a \subset \Sigma \setminus \partial \Sigma\), let \(\tau_a \in \text{PMod}(\Sigma)\) be the associated Dehn twist, defined as follows. Let \(\tau\) be the homeomorphism from \(S^1 \times [0, 1]\) to itself given by \((\theta, t) \mapsto (\theta + 2\pi t, t)\), where we identify the circle \(S^1\) with \(\mathbb{R}/2\pi\mathbb{Z}\). Choose a closed tubular neighborhood \(N\) of \(a\) in \(\Sigma\), and an orientation preserving homeomorphism \(f : N \to S^1 \times [0, 1]\). The Dehn twist \(\tau_a\) is given by

\[
\tau_a(x) = \begin{cases} 
  f^{-1} \circ \tau \circ f(x) & \text{if } x \in N, \\
  x & \text{otherwise}.
\end{cases}
\]

The class of \(\tau_a\) in \(\text{PMod}(\Sigma)\) is independent of the choices involved above, and it also depends only on the isotopy class of \(a\). It is a standard fact that \(\text{PMod}(\Sigma)\) is generated by Dehn twists along simple closed curves in \(\Sigma\) (see [5, Chapter 4]).

After Section 2.2, we define the character variety of \(\Sigma\) to be

\[
X = X(\Sigma) = X(\pi_1 \Sigma)
\]

and set \(R(\Sigma) = R(\pi_1 \Sigma)\). The complex points of \(X(\Sigma)\) can be seen as parametrizing the isomorphism classes of semisimple \(SL_2(\mathbb{C})\)-local systems on \(\Sigma\). We recall that, for any \(a, b \in \pi_1 \Sigma\), we have \([a] = [bab^{-1}]\) and \([a] = [a^{-1}]\). Hence, each simple closed curve \(a \subset \Sigma\) unambiguously defines a class \([a] \in R(\Sigma)\). The natural morphism \(\text{PMod}(\Sigma) \to \text{Out}(\pi_1 \Sigma)\) induces an action of \(\text{PMod}(\Sigma)\) on \(X(\Sigma)\).

Given a morphism between two surfaces \(\Sigma' \to \Sigma\), we have an induced morphism of character varieties \(X(\Sigma) \to X(\Sigma')\). If \(\Sigma' \subset \Sigma\) is a subsurface, the induced morphism on character varieties is \(\text{PMod}(\Sigma')\)-equivariant for the induced morphism \(\text{PMod}(\Sigma') \to \text{PMod}(\Sigma)\) of mapping class groups. In particular, if a semisimple \(SL_2(\mathbb{C})\)-local system on \(\Sigma\) has a finite \(\text{PMod}(\Sigma)\)-orbit in \(X(\Sigma)\), then its restriction to \(\Sigma'\) has a finite \(\text{PMod}(\Sigma')\)-orbit in \(X(\Sigma')\) for every subsurface \(\Sigma' \subset \Sigma\).

2.4. Loop configurations. Let \(\Sigma\) be a surface of genus \(g\) with \(n\) punctures. We fix a base point in \(\Sigma\).

Definition 8. A sequence \(L = (\ell_1, \cdots, \ell_m)\) of based loops on \(\Sigma\) is clean if each loop is simple and the loops pairwise intersect only at the base point.

Example 9. Recall the standard presentation of the fundamental group

\[
\pi_1 \Sigma = \langle a_1, b'_1, \cdots, a_g, b'_g, c_1, \cdots, c_n | [a_1, b'_1] \cdots [a_g, b'_g]c_1 \cdots c_n \rangle.
\]

We can choose (the based loops representing) the generators so that the sequence of loops \((a_1, b'_1, \cdots, a_g, b'_g, c_1, \cdots, c_n)\) is clean. For \(i = 1, \cdots, g\), let \(b_i\) be the based simple loop parametrizing the curve underlying \(b'_i\) with the opposite orientation. Note that \((a_1, b_1, a_2, b_2, c_1, \cdots, c_n)\) is a clean sequence with the property that any product of distinct elements preserving the cyclic ordering on the sequence, such as \(a_1b_g\) or \(a_1a_2b_2b_g\) or \(b_gc_na_1\), can be represented by a simple loop in \(\Sigma\). We
shall refer to \((a_1, b_1, \cdots, a_g, b_g, c_1, \cdots, c_n)\) as the optimal generators of \(\pi_1\Sigma\). See Figure 1 for an illustration of the optimal generators for \((g, n) = (2, 1)\).

**Definition 10.** A loop configuration is a planar graph consisting of a single vertex \(v\) and a finite cyclically ordered sequence of directed rays, equipped with a bijection between the set of rays departing from \(v\) and the set of rays arriving at \(v\). We denote by \(\Gamma_{g,n}\) the loop configuration whose sequence of rays is of the form
\[
(a_1, b_1, \bar{a}_1, \bar{b}_1, \cdots, a_g, b_g, \bar{a}_g, \bar{b}_g, c_1, \bar{c}_1, \cdots, c_n, \bar{c}_n),
\]
where \(a_i, b_i, c_i\) are the rays directed away from \(v\), respectively corresponding to the rays \(\bar{a}_i, \bar{b}_i, \bar{c}_i\) directed towards \(v\). See Figure 2 for an illustration of \(\Gamma_{2,1}\).

Given a clean sequence \(L = (\ell_1, \cdots, \ell_m)\) of loops on \(\Sigma\), we have an associated loop configuration \(\Gamma(L)\), obtained by taking a sufficiently small open neighbourhood of the base point and setting the departing and arriving ends of the loops \(\ell_i\) to correspond to each other. For example, if \((a_1, b_1, \cdots, a_g, b_g, c_1, \cdots, c_n)\) are the optimal generators of \(\pi_1\Sigma\), then
\[
\Gamma(a_1, b_1, \cdots, a_g, b_g, c_1 \cdots, c_n) \simeq \Gamma_{g,n}.
\]

![Figure 1. Optimal generators for \((g, n) = (2, 1)\)](image1)

![Figure 2. Loop configuration \(\Gamma_{2,1}\)](image2)

**Definition 11.** Let \(h\) and \(m\) be nonnegative integers. A sequence of based loops \(L = (\ell_1, \cdots, \ell_{2h+m})\) on \(\Sigma\) is said to be in \((h, m + 1)\)-position if it is homotopic term-wise to a clean sequence \(L' = (\ell'_1, \cdots, \ell'_{2h+m})\) such that \(\Gamma(L') \simeq \Gamma_{h,m}\). We denote by \(\Sigma(L) \subset \Sigma\) the (isotopy class of a) subsurface of genus \(h\) with \(m + 1\) boundary curves obtained by taking a small closed tubular neighbourhood of the union of the simple curves underlying \(\ell'_1, \cdots, \ell'_{2h+m}\) in \(\Sigma\).
Lemma 12. Let \((a, b, c)\) be a sequence of loops on a surface in \((1, 2)\)-position.

1. The following pairs are in \((1, 1)\)-position:
   \((a, b), (a, bc), (ca, b), (ab, bc), (ac, bc), (ca, ab)\).

2. The triple \((c^{-1}b^{-1}a, b, c)\) is in \((1, 2)\)-position.

Proof. This is seen by drawing the corresponding loop configurations of homotopic clean sequences. See Figure 3 for the loop configurations occurring in part (1), and Figure 4 for the loop configuration of \((c^{-1}b^{-1}a, b, c)\). 

3. Case of one-holed torus

3.1. Let \(\Sigma\) be a surface of genus one with one boundary curve. Let \((a_1, b_1, c_1)\) be the optimal generators of \(\pi_1 \Sigma\), as defined in Example 9. Let \(X = X(\Sigma)\) be the character variety of \(\Sigma\). Note that \(\pi_1 \Sigma\) is freely generated by \(a_1\) and \(b_1\). Therefore, as mentioned in Example 5, we have an isomorphism

\[(x_1, x_2, x_3) = ([a_1], [b_1], [a_1b_1]) : X(\Sigma) \xrightarrow{\sim} \mathbb{A}^3.\]

For simplicity, we shall denote the isotopy classes of simple curves lying in the free homotopy classes of \(a_1, b_1, a_1b_1\) by the same letters.

Lemma 13. We have the following Dehn twist actions on \(X(\Sigma) = \mathbb{A}^3\). 

\[
\tau_{a_1^*} : (x_1, x_2, x_3) \mapsto (x_1, x_3, x_1x_3 - x_2), \\
\tau_{b_1^*} : (x_1, x_2, x_3) \mapsto (x_1x_2 - x_3, x_2, x_1), \\
\tau_{a_1b_1^*} : (x_1, x_2, x_3) \mapsto (x_2, x_2x_3 - x_1, x_3).
\]
Proof. We have in general that \( \tau_a(a) = a \) for any simple closed curve \( a \) in \( \Sigma \). This shows in particular that \( \tau_{a_1}(x_1) = \tau_{a_1}([a_1]) = [a_1] = x_1 \), and similarly for \( \tau_b(x_2) \) and \( \tau_{a_1b_1}(x_3) \). Next, note that \( \tau_{a_1}(b_1) = a_1b_1 \) and \( \tau_{a_1}(a_1b_1) = a_1a_1b_1 \), which shows

\[
\tau_{a_1}(x_2) = \tau_{a_1}([b_1]) = [a_1b_1] = x_3,
\]

\[
\tau_{a_1}(x_3) = \tau_{a_1b_1}([a_1b_1]) = [a_1a_1b_1] = [a_1][a_1b_1] - [b_1] = x_1x_3 - x_2.
\]

This gives us a description of the action of \( \tau^*_{a_1} \) on \( \mathbb{A}^3 \). The other transformations \( \tau^*_{b_1} \) and \( \tau^*_{a_1b_1} \) are dealt with similarly. \( \square \)

Let \( \Pi \) be the group of polynomial automorphisms of \( \mathbb{A}^3 \) generated by \( \tau^*_{a_1}, \tau^*_{b_1}, \) and \( \tau^*_{a_1b_1} \). It is precisely the image of the mapping class group \( \Gamma(\Sigma) \) in the group of polynomial automorphisms of \( X(\Sigma) = \mathbb{A}^3 \). Let \( \Pi' \) be the group generated by \( \Pi \) together with the following transformations:

\[
\sigma_{12} : (x_1, x_2, x_3) \mapsto (-x_1, -x_2, x_3),
\]

\[
\sigma_{23} : (x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3),
\]

\[
\sigma_{13} : (x_1, x_2, x_3) \mapsto (-x_1, x_2, -x_3).
\]

It is easy to see that \( |\Pi' : \Pi| < \infty \). Hence, a point in \( \mathbb{A}^3 \) has finite \( \Pi' \)-orbit if and only if it has finite \( \Pi \)-orbit. Now, the group \( \Pi' \) contains the transformations

\[
\beta_1 = \sigma_{12}\tau^*_{a_1b_1}(\tau^*_{b_1}\tau^*_{a_1})^{-1} : (x_1, x_2, x_3) \mapsto (-x_1, x_3 - x_1x_2, x_2),
\]

\[
\beta_2 = \sigma_{23}\tau^*_{a_1}(\tau^*_{b_1}\tau^*_{a_1})^{-1} : (x_1, x_2, x_3) \mapsto (x_3, -x_2, x_1 - x_2x_3).
\]

whose joint orbits in \( \mathbb{A}^3 \) were studied in Dubrovin-Mazzocco [4] Theorem 1.6, in connection with algebraic solutions of special Painlevé VI equations. They defined a triple \((x_1, x_2, x_3) \in \mathbb{A}^3(\mathbb{C})\) to be admissible if it has at most one coordinate zero and \( x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 - 2 \neq 0 \). The result of [4] we shall use is the following.

**Theorem 14** (Dubrovin-Mazzocco). The following is a complete set of representatives for the finite \((\beta_1, \beta_2)\)-orbits of admissible triples in \( \mathbb{A}^3 \):

\[
(0, -1, -1), (0, -1, -\sqrt{2}), (0, -1, -\varphi), (0, -1, -\varphi^{-1}), (0, -\varphi, -\varphi^{-1})
\]

where \( \varphi = (1 + \sqrt{5})/2 \) is the golden ratio.
Corollary 15. If \((x_1, x_2, x_3) \in \mathbb{A}^3(\mathbb{C})\) is an admissible triple with finite PMod(\(\Sigma\))-orbit, then it corresponds to a representation \(\rho : \pi_1 \Sigma \to \text{SL}_2(\mathbb{C})\) with finite image.

Proof of the corollary. Replacing \((x_1, x_2, x_3)\) by another triple within its PMod(\(\Sigma\))-orbit if necessary, we may assume that \((x_1, x_2, x_3)\) is one of the triples in Theorem 14 or its image under one of the transformations \(\sigma_{12}, \sigma_{23}, \text{or } \sigma_{13}\). We shall show that

\[
(x_1, x_2, x_3) = (\text{tr} a_1, \text{tr} a_2, \text{tr}(a_1 a_2))
\]

where \(a_1, a_2 \in \text{SL}_2(\mathbb{C})\) are elements that together lie in one of the finite subgroups \(BA_4, BS_4, \text{or } BA_5\) of \(\text{SL}_2(\mathbb{C})\). Since the matrix \(-1\) is contained in every one of these groups, it suffices to treat the case where \((x_1, x_2, x_3)\) is one of the triples in Theorem 14. By explicit computation, we find that the triples in Theorem 14 respectively correspond to traces of the triples of matrices

\[
\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right), \left(\begin{array}{cc}
-\frac{i}{2} & -\frac{i}{2} \\
\frac{i}{2} & \frac{i}{2}
\end{array}\right), \left(\begin{array}{cc}
-\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{array}\right), \left(\begin{array}{cc}
\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & -\frac{1}{2}
\end{array}\right), \left(\begin{array}{cc}
-\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2}
\end{array}\right), \left(\begin{array}{cc}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2}
\end{array}\right), \left(\begin{array}{cc}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2}
\end{array}\right)
\]

where \(\varphi = (1 + \sqrt{5})/2\) is the golden ratio. The matrices for the first triple all lie in the binary tetrahedral group \(BA_4\), the matrices for the second triple all lie in the binary octahedral group \(BS_4\), and the matrices for the remaining three triples all lie in the binary icosahedral group \(BA_5\). In each triple, the third matrix is the product of the first two. Thus, each of the triples in Theorem 14 correspond to characters of representations \(\pi_1 \Sigma \to \text{SL}_2(\mathbb{C})\) (\(\Sigma\) a one holed torus) with finite image, proving the corollary. \(\square\)

Corollary 16. Let \(\Sigma\) be a one holed torus. If \(\rho : \pi_1 \Sigma \to \text{SL}_2(\mathbb{C})\) is a Zariski dense representation, then its mapping class group orbit in \(X(\Sigma)\) is infinite.

Proof. Let \(x = (x_1, x_2, x_3) \in \mathbb{A}^3\) lie in a finite PMod(\(\Sigma\))-orbit. By Corollary 15 above, if \(x\) is admissible then it corresponds to a representation with finite image. If \(x\) is not admissible, then by definition we have one of the following:

1. at least two coordinates of \(x\) are zero, or
2. \(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3 - 2 = 0\).

In case (1), suppose that \(x = (x_1, 0, 0)\). Let \(\lambda \in \mathbb{C}^*\) be a complex number such that \(\lambda + \lambda^{-1} = x_1\). We have

\[
(x_1, 0, 0) = \left(\text{tr} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \text{tr} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \text{tr} \begin{bmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{bmatrix}\right)
\]

showing that \(x\) corresponds to the character of a representation with image in \(D_{\infty}\). The cases where \(x = (0, x_2, 0)\) or \((0, 0, x_3)\) are treated similarly. Lastly, if case (2)
occurs, then by Lemma 3 the point $x$ corresponds to the character of a reducible representation. This gives the desired result. \hfill \square

4. Zariski dense representations

4.1. Let $\Sigma$ be a surface of genus $g > 0$ and $n \geq 0$ punctures. Given a pair $(a, b)$ of based loops in (1,1)-position on $\Sigma$, there is an embedding $\Sigma(a, b) \subset \Sigma$ of a genus 1 surface with 1 boundary curve, i.e. a one holed torus. Up to isotopy, every embedding of a one holed torus is of the form $\Sigma(a, b)$ for some choice of $(a, b)$. For a representation $\rho : \pi_1 \Sigma \to \mathrm{SL}_2(\mathbb{C})$, let $\rho|\Sigma(a, b)$ denote the representation $\pi_1\Sigma(a, b) \to \mathrm{SL}_2(\mathbb{C})$ obtained by restriction. We shall now restate and prove the irreducibility part of the Main Lemma.

**Proposition 17.** Let $\Sigma$ be a surface of genus $g > 0$ and $n \geq 0$ punctures. A representation $\rho : \pi_1 \Sigma \to \mathrm{SL}_2(\mathbb{C})$ is irreducible if and only if there is an embedded one holed torus $\Sigma(a, b) \subset \Sigma$ such that $\rho|\Sigma(a, b)$ is irreducible.

**Proof.** The if direction is clear, and we now prove the converse. Let us fix a representation $\rho : \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ whose restriction to every embedded one holed torus is reducible. In this proof, given $a \in \pi_1 \Sigma$ we shall also denote by $a$ the matrix $\rho(a) \in \mathrm{SL}_2(\mathbb{C})$ for simplicity. The statement that $\rho|\Sigma(a, b)$ is reducible for an embedding $\Sigma(a, b) \subset \Sigma$ associated to a pair $(a, b)$ of loops in (1,1)-position is equivalent to saying that the pair $(\rho(a), \rho(b))$ of matrices in $\mathrm{SL}_2(\mathbb{C})$ has a common eigenvector in $\mathbb{C}^2$.

Throughout, we shall be using the following observation: if $a \in \mathrm{SL}_2(\mathbb{C}) \setminus \{ \pm 1 \}$, and if $x, y, z \in \mathbb{C}^2$ are eigenvectors of $a$, then at least two of them are proportional; in notation, $x \sim y$, $x \sim z$, or $y \sim z$. First, we prove the following claim.

**Claim.** Any triple $(a, b, c)$ of loops on $\Sigma$ in (1,2)-position has a common eigenvector under the representation $\rho$.

Let $(a, b, c)$ be in (1,2)-position. Each of the following pairs is in (1,1)-position by part (1) of Lemma 12 and has a common eigenvector by our hypothesis on $\rho$:

$$(a, b), (a, bc), (ca, b), (ab, bc), (ca, cb), (ac, bc), (ca, ab).$$

If $ab = \pm 1$, then since $(a, b)$ has a common eigenvector we find that $(a, b, c)$ has a common eigenvector, as desired. Henceforth, assume that $ab \neq \pm 1$. Now $(a, b, ab)$ has a common eigenvector, say $x$; $(ab, bc)$ has a common eigenvector, say $y$; and $(ca, ab)$ has a common eigenvector, say $z$. Since $ab \neq \pm 1$, we must have

$$x \sim y, \quad x \sim z, \quad \text{or} \quad y \sim z$$

where the relation $\sim$ indicates that the two vectors are proportional. If the first case occurs, then $(a, b, bc)$ has a common eigenvector, and hence $(a, b, c)$ does, as required. If the second case occurs, then $(a, b, ca)$ has a common eigenvector, and hence $(a, b, c)$ does, again as required. Henceforth, assume that the third case occurs, so that $(bc, ca)$ has a common eigenvector.

Now, suppose first that $[a] = \text{tr} \rho(a) = \pm 2$. Up to conjugation, we have the following possibilities.

(1) We have $a = \pm 1$. Since $(b, c)$ has a common eigenvector, this implies that $(a, b, c)$ has a common eigenvector, as desired.
(2) We have 

\[
 a = \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.
\]

Since \((a, bc)\) and \((a, b)\) each have a common eigenvector, \(bc\) and \(b\) must be upper triangular, and hence \(c\) is upper triangular. Thus \((a, b, c)\) has a common eigenvector.

It remains to treat the case \([a] \neq \pm 2\). As \((a, bc)\) has a common eigenvector, by Lemma 8 we have

\[
[a]^2 + [bc]^2 + [abc]^2 - [a][bc][abc] - 2 = 2
\]

and this implies that we cannot have \([bc] = [abc] = 0\). After conjugation of \(\rho\) by an element of \(\text{SL}_2(\mathbb{C})\), we have one of the following three cases.

1. We have \(bc = \pm 1\). This implies that \(b = \pm c^{-1}\). Since \((a, b)\) has a common eigenvector, we are done.

2. We have

\[
 bc = \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.
\]

Since \((a, bc)\) and \((bc, ca)\) each have a common eigenvector, \(a\) and \(ca\) are upper triangular, and hence \(c\) is upper triangular. This in turn implies that \(b\) is upper triangular, and we are done.

3. We have

\[
 bc = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \quad \lambda \in \mathbb{C}^* \setminus \{\pm 1\}.
\]

Since \((a, bc)\), \((bc, ca)\), and \((ab, bc)\) each have a common eigenvector, at least two of \(a, ab, ca\) must be simultaneously upper or lower triangular. Let us assume that \(a\) is upper triangular; the case where \(a\) is lower triangular is dealt with similarly. If \(ab\) is also upper triangular, this implies that \(b\) is also upper triangular, in turn \(c\) is upper triangular, and we are done. Hence, we may assume \(ab\) is lower triangular. Similarly, we may assume that \(ca\) is lower triangular. Since we have

\[
(ab)(ca) = a(bc)a
\]

with left hand side lower triangular and right hand side upper triangular, \(abca\) is diagonal. Writing

\[
 a = \begin{bmatrix} \mu & x \\ 0 & \mu^{-1} \end{bmatrix},
\]

we compute

\[
 a(bc)a = \begin{bmatrix} \mu^2 \lambda & x(\mu \lambda + \lambda^{-1} \mu^{-1}) \\ 0 & \mu^{-2} \lambda^{-1} \end{bmatrix}
\]

which implies that \(x = 0\) or \([abc] = \text{tr}(abc) = \mu \lambda + \lambda^{-1} \mu^{-1} = 0\). If \(x = 0\), that is, \(a\) is diagonal, then we find that \(b\) and \(c\) are lower triangular, and we are done. The case \([abc] = 0\) still remains; note the hypothesis that \([bc] \neq \pm 2\) implies that \([a] \neq 0\), and hence we have \([a], [bc] \neq 0\).

Hence, we have shown that if \((a, b, c)\) is in \((1, 2)\)-position then \((a, b, c)\) has a common eigenvector, except possibly if \([a], [bc] \neq 0\) and \([abc] = 0\). But in
this exceptional case, since \((c^{-1}b^{-1}a, b, c)\) is in \((1, 2)\)-position by part (2) of Lemma \([12]\) and since
\[
([c^{-1}b^{-1}a], [bc], [c^{-1}b^{-1}abc]) = ([a][bc] - [abc], [bc], [a]) = ([a][bc], [bc], [a])
\]
has none of the coordinates zero, running the same argument as above with \((a, b, c)\) replaced by \((c^{-1}b^{-1}a, b, c)\) we find that \((c^{-1}b^{-1}a, b, c)\) has a common eigenvector. But then \((a, b, c)\) also has a common eigenvector, as required.

Thus, we have proved our claim. To prove the proposition, we use the following inductive argument. Let \((a_1, a_2, \ldots, a_{2g-1}, a_{2g}, a_{2g+1}, \ldots, a_{2g+n})\) be the optimal generators of \(\pi_1\Sigma\) as defined in Example \([9]\) with the presentation
\[
\pi_1\Sigma = \langle a_1, \ldots, a_{2g+n} | [a_1, a_2^{-1}] \cdots [a_{2g-1}, a_{2g}^{-1}]a_{2g+1} \cdots a_{2g+n} \rangle.
\]
(We have changed our notation of the loops slightly to better accommodate the proof.) We show that \((a_1, \ldots, a_{2g+n})\) has a common eigenvector. For simplicity of arguments we may assume that at least one element in each pair
\[
(a_1, a_2), \ldots, (a_{2g-1}, a_{2g})
\]
is not equal to \(\pm 1\); for if some pair is of the form \((a, b)\) with \(a, b \in \{\pm 1\}\) we may simply skip over that pair in the considerations below.

If \((g, n) = (1, 0)\), then we are done since every representation \(\pi_1\Sigma \to \text{SL}_2(\mathbb{C})\) is abelian. We thus assume \((g, n) \neq (1, 0)\). By our claim, \((a_1, a_2, a_3)\) has a common eigenvector. Assume next that \(4 \leq k \leq 2g + n\) and \((a_1, \ldots, a_{k-1})\) has a common eigenvector \(x \in \mathbb{C}^2\). We show that \((a_1, \ldots, a_k)\) has a common eigenvector. We consider the following cases.

1. \(k = 5, 7, \ldots, 2g - 1\), or \(k = 2g + 1, 2g + 2, \ldots, 2g + n\). The triple \((a_1, a_2, a_k)\) is in \((1, 2)\)-position, and hence has a common eigenvector \(y\) by our claim.

   Assume now first that \(a_1 \neq \pm 1\). Given any \(3 \leq j < k\), note that \((a_1, a_2a_j, a_k)\) is in \((1, 2)\)-position and hence has a common eigenvector by our claim, say \(y_j\). Since \(a_1 \neq \pm 1\), we must have
   \[
   x \sim y, \quad x \sim y_j, \quad \text{or} \quad y_j \sim y.
   \]
   If one of the first two occurs, then we conclude that \((a_1, \ldots, a_k)\) has a common eigenvector \(x\), as required. Thus, we are left with the case \(y_j \sim y\) for every \(3 \leq j < k\). But this implies that \((a_1, a_2, a_2a_j, a_k)\) has a common eigenvector \(y\), which is thus shared also by \(a_j\). Thus, \(y\) is a common eigenvector of \((a_1, \ldots, a_k)\), as desired.

   Now, suppose that \(a_1 = \pm 1\), and hence \(a_2 \neq \pm 1\). Given any \(3 \leq j < k\), we observe that \((a_2, a_2^{-1}a_2a_j, a_k)\) is in \((1, 2)\)-position and hence has a common eigenvector, say \(y_j\). Since \(a_2 \neq \pm 1\), we must have
   \[
   x \sim y, \quad x \sim y_j, \quad \text{or} \quad y_j \sim y.
   \]
   If one of the first two occurs, then we conclude that \((a_1, \ldots, a_k)\) has a common eigenvector \(x\), as required. Thus, we are left with the case \(y_j \sim y\) for every \(3 \leq j < k\). But this implies that \((a_1, a_2, a_1^{-1}a_2a_j, a_k)\) has a common eigenvector \(y\), which is thus shared also by \(a_j\). Thus, \(y\) is a common eigenvector of \((a_1, \ldots, a_k)\), as desired.

2. \(k = 4, 6, \ldots, 2g\). First, consider the case where \(a_{k-1} = \pm 1\). It then suffices to show that the sequence \((a_1, \ldots, a_{k-2}, a_k)\) has a common eigenvector. This can be shown by repeating the argument the previous case (1) above.
Thus, we may assume that \( a_{k-1} \neq \pm 1 \). Note that, for each integer \( m \) with \( 2m < k \), by our claim we have:

- a common eigenvector \( w_m \) of \( (a_{k-1}, a_k, a_{2m}) \), and
- a common eigenvector \( z_m \) of \( (a_{k-1}, a_k, a_{2m-1}) \).

Since \( a_{k-1} \neq \pm 1 \), we must have
\[
x \sim w_m, \quad x \sim z_m, \quad \text{or} \quad w_m \sim z_m.
\]

If one of the first two occurs, then conclude that \( (a_1, \ldots, a_k) \) has a common eigenvector \( x \), and we are done. Thus, we are left with the case where \( w_m \sim z_m \) for every \( 2m < k \). Note in this case that \( w_m \) is a common eigenvector of \( (a_{2m-1}, a_{2m}, a_{k-1}, a_k) \). Comparing the vectors \( x, w_m, \) and \( w_m' \) for different \( m, m' \), we are in turn reduced to the case \( w_m \sim w_m' \) for all \( m, m' \), in which case \( (a_1, \ldots, a_k) \) has a common eigenvector \( w_1 \), as desired.

Thus, \( (a_1, \ldots, a_k) \) has a common eigenvector. This completes the induction, and shows that \( (a_1, \ldots, a_{2g+n}) \) has a common eigenvector, proving the proposition. \( \Box \)

**Remark.** A corollary of the above proof is that, fixing a optimal set of generators of \( \pi_1 \Sigma \), we can (in principle) give an explicit finite set of equations for the locus of reducible characters on \( X(\Sigma) \), in the case where the genus of \( \Sigma \) is at least 1.

### 4.2. We recall the following lemma of Selberg.

**Lemma (Selberg).** Let \( K \) be a field of characteristic zero. Given a finitely generated subgroup \( \pi \leq \text{GL}_n(K) \), there exists a torsion-free finite index subgroup \( \pi' \leq \pi \).

**Corollary 18.** Let \( \Sigma \) be a surface, and let \( \rho : \pi_1 \Sigma \to \text{SL}_2(\mathbb{C}) \) be Zariski dense representation. There exists a finite covering \( f : \Sigma' \to \Sigma \) such that the pullback
\[
\rho' = \rho \circ f_* : \pi_1 \Sigma' \to \text{SL}_2(\mathbb{C})
\]
has Zariski dense image which is moreover torsion-free.

**Proof.** Applying Selberg’s lemma, there exists a finite-index subgroup \( \pi' \) of \( \rho(\pi_1 \Sigma) \) which is torsion-free. Since \( \rho(\pi_1 \Sigma) \) is Zariski dense in \( \text{SL}_2(\mathbb{C}) \) by hypothesis, \( \pi' \) is also Zariski dense in \( \text{SL}_2(\mathbb{C}) \) by Lemma [2]. The subgroup \( \rho^{-1}(\pi') \leq \pi_1 \Sigma \) has finite index, and hence there is a corresponding finite covering \( f : \Sigma' \to \Sigma \) with \( \pi_1 \Sigma' \cong \rho^{-1}(\pi') \). The representation \( \rho' = \rho \circ f_* \) satisfies the desired properties. \( \Box \)

Let \( \Sigma \) be a surface of genus \( g > 0 \) and \( n \geq 0 \) punctures. We restate and prove the Zariski density part of the Main Lemma.

**Proposition 19.** Let \( \Sigma \) be a surface of positive genus \( g > 0 \) with \( n \geq 0 \) punctures. A representation \( \rho : \pi_1 \Sigma \to \text{SL}_2(\mathbb{C}) \) is Zariski dense if and only if there is an embedded one holed torus \( \tau \subset \Sigma \) such that the restriction \( \rho|\tau \) is Zariski dense.

**Proof.** The if direction is clear. Let \( \rho : \pi_1 \Sigma \to \text{SL}_2(\mathbb{C}) \) be a representation with Zariski dense image. By Corollary [18] there exists a finite covering \( f : \Sigma' \to \Sigma \) such that \( \rho'' = \rho \circ f_* : \pi_1 \Sigma'' \to \text{SL}_2(\mathbb{C}) \) has Zariski dense image which is moreover torsion-free. By Proposition [17] there is an embedding \( i : \Sigma' \to \Sigma'' \) of a one holed torus such that \( i^*(\rho'') \) is irreducible. Since \( \Im i^*(\rho'') \leq \Im \rho'' \) is torsion-free, we conclude by Corollary [1] that \( i^*(\rho'') \) has Zariski dense image.

The composition \( f \circ i : \Sigma' \to \Sigma \) presents \( \Sigma' \) as a finite covering of the image \( \tau = f \circ i(\Sigma') \).
The induced map on the quotient topological spaces $\Sigma'/\partial\Sigma' \to \tau/\partial\tau$ is a branched covering of closed surfaces possibly ramified only at one point (corresponding to the boundary component of $\Sigma'$). By equipping the quotients suitably with complex structures and using the Riemann-Hurwitz formula, we conclude that $\tau/\partial\tau$ has genus one, and hence $\tau$ is a one holed torus. Writing $j : \tau \to \Sigma$ for the immersion, we have $f^*j^*(\rho) = i^*f^*(\rho)$ which is Zariski dense, and hence we conclude that $j^*(\rho)$ is Zariski dense, as desired. \hfill $\square$

4.3. Combining Corollary [16] and the Main Lemma, we obtain the following. Let $\Sigma$ be a compact oriented surface of genus $g > 0$ with $n$ boundary curves.

**Proposition 20.** If a representation $\rho : \pi_1\Sigma \to \text{SL}_2(\mathbb{C})$ is Zariski dense, then its mapping class group orbit in $X(\Sigma)$ is infinite.

**Proof.** Let $\rho : \pi_1\Sigma \to \text{SL}_2(\mathbb{C})$ be Zariski dense. By the Main Lemma, there exists an embedded one-holed torus $\Sigma' \subset \Sigma$ such that the restriction $\rho|\Sigma'$ is Zariski dense. By Corollary [16], the $\text{PMod}(\Sigma')$-orbit of $\rho|\Sigma'$ in $X(\Sigma')$ must be infinite. A fortiori, the $\text{PMod}(\Sigma)$-orbit of $\rho$ in $X(\Sigma)$ is infinite. \hfill $\square$

We also record the following consequence of the above analysis.

**Corollary 21.** Let $\Sigma$ be a surface of positive genus with finitely many punctures. The locus of Zariski dense representations in $X(\Sigma)$ is a Zariski open set.

**Proof.** Let us write $X(\Sigma)_{\text{Zar}} \subset X(\Sigma)$ for the locus of Zariski dense representations. Proposition [19] shows that $X(\Sigma)_{\text{Zar}} = \bigcup_j (j^*)^{-1}(X(\Sigma')_{\text{Zar}})$ where $j : \Sigma' \to \Sigma$ runs over all embeddings of one holed tori into $\Sigma$. Thus, we are reduced to proving the result for a one holed torus.

Suppose that $\Sigma$ is a one holed torus. It suffices to show that, for $G = B, D_\infty, BA_4, BS_4,$ or $BA_5$, the locus of characters of representations $\rho$ with $\rho(\pi_1\Sigma) \subseteq G$ (up to conjugation) is Zariski closed in $X(\Sigma)$. This is clear if $G$ is one of the finite groups mentioned, as the locus consists of finite points.

We have $([a_1], [b_1], [a_1b_1]) : X(\Sigma) \simeq \mathbb{A}^3$. Under this identification, the locus in question corresponding to $G = B$ is the closed subset of $\mathbb{A}^3$ defined by the equation

$$[a_1]^2 + [b_1]^2 + [a_1b_1]^2 - [a_1][b_1][a_1b_1] - 2 = 0.$$ 

On the other hand, for $G = D_\infty$, the locus in question is the union of three lines $\{[a_1] = [b_1] = 0\}, \{[a_1] = [a_1b_1] = 0\}$, and $\{[a_1b_1] = [a_1] = 0\}$, and hence is closed. This proves the corollary. \hfill $\square$

5. **Sparse representations**

5.1. Let $\Sigma$ be a surface of positive genus $g$ with $n$ punctures. Let $X$ denote the character variety of $\Sigma$. In this subsection, we prove Theorems A and B on the locus of points on $X$ corresponding to representations with non-Zariski-dense image. On the locus of finite representations, the claim is clear. In view of Lemma [6] it only remains to understand the finite mapping class group orbits on $\text{Hom}(\pi_1\Sigma, \mathbb{C}^*)$ and $\text{Hom}(\pi_1\Sigma, D_\infty)/D_\infty$.

**Lemma 22.** Assume $\Sigma$ has genus $g > 0$ with $n \geq 0$ punctures. A representation $\rho : \pi_1\Sigma \to \mathbb{C}^*$ has finite mapping class group orbit in $\text{Hom}(\pi_1\Sigma, \mathbb{C}^*)$ if and only if it has finite image.
Proof. The if direction is clear, and we now prove the converse. Let \( \rho : \pi_1 \Sigma \to \mathbb{C}^* \) be a representation with finite mapping class group orbit in \( \text{Hom}(\pi_1 \Sigma, \mathbb{C}^*) \). Let \((a_1, b_1, \cdots, a_g, b_g, c_1, \cdots, c_n)\) be the optimal generators of \( \pi_1 \Sigma \). Assume first that \( n = 0 \) or \( 1 \). We have a short exact sequence 
\[
0 \to \text{Hom}(\pi_1 \Sigma, \mathbb{Z}) \to \text{Hom}(\pi_1 \Sigma, \mathbb{C}) \xrightarrow{\exp(2\pi i \cdot \cdot)} \text{Hom}(\pi_1 \Sigma, \mathbb{C}^*) \to 0
\]
which is \( \text{PMod}(\Sigma) \)-equivariant. The mapping class group action on \( \text{Hom}(\pi_1 \Sigma, \mathbb{Z}) \) and \( \text{Hom}(\pi_1 \Sigma, \mathbb{C}) \simeq \text{Hom}(\pi_1 \Sigma, \mathbb{Z}) \otimes \mathbb{C} \) is linear, factoring through the quotient \( \text{PMod}(\Sigma) \to \text{Sp}(2g, \mathbb{Z}) \) where \( \text{Sp}(2g, \mathbb{Z}) \) is the integral symplectic group. Let us choose a lift \( \tilde{\rho} \in \text{Hom}(\pi_1 \Sigma, \mathbb{C}) \) of \( \rho \). Let \( \gamma \in \text{Sp}(2g, \mathbb{Z}) \) be an element such that none of its eigenvalues are roots of unity. By hypothesis, there exists \( k \geq 1 \) with \( \gamma^k \cdot \rho = \rho \), or in other words
\[
\gamma^k \cdot \tilde{\rho} \equiv \tilde{\rho} \mod \text{Hom}(\pi_1 \Sigma, \mathbb{Z}).
\]
Setting \( \rho' = \gamma^k \cdot \tilde{\rho} - \tilde{\rho} \in \text{Hom}(\pi_1 \Sigma, \mathbb{Z}) \), we see that \( \tilde{\rho} = (\gamma^k - 1)^{-1} \rho' \in \text{Hom}(\pi_1 \Sigma, \mathbb{Q}) \) since the integral matrix \( \gamma^k - 1 \) has nonzero integral determinant and is therefore invertible over \( \mathbb{Q} \). In particular, \( \exp(2\pi i \rho') = \rho \) must be torsion, showing that the image of \( \rho \) is finite if \( n \leq 1 \), as desired. Thus, only the case \( n \geq 2 \) remains. Since \((a_1, b_1, \cdots, a_g, b_g)\) is in \((g, 1)\)-position, the above analysis shows that \( \rho(a_1) \) and \( \rho(b_i) \) are roots of unity for \( i = 1, \cdots, g \). Similarly, we see that the sequence
\[
L_i = (a_1, b_1, \cdots, a_{g-1}, b_{g-1}, a_g, b_g, c_i)
\]
is in \((g, 1)\)-position for every \( i = 1, \cdots, n \). Since \( \rho \) restricted to the surface \( \Sigma(L_i) \) must have finite \( \text{PMod}(\Sigma(L_i)) \)-orbit, \( \rho(c_i) \) is a root of unity for \( i = 1, \cdots, n \) as well. This shows that \( \rho \) has finite image, as desired. \( \square \)

Lemma 23. Assume \( \Sigma \) has genus \( g \geq 2 \) with \( n \geq 0 \) punctures. A representation \( \rho : \pi_1 \Sigma \to D_\infty \) has finite mapping class group orbit in \( \text{Hom}(\pi_1 \Sigma, D_\infty)/D_\infty \) if and only if it has finite image.

Proof. The if direction is clear, and we now prove the converse. Let \( \rho : \pi_1 \Sigma \to D_\infty \) be a representation with finite mapping class group orbit in \( \text{Hom}(\pi_1 \Sigma, D_\infty)/D_\infty \). We have a short exact sequence \( 0 \to \mathbb{C}^* \to D_\infty \to \mathbb{Z}/2\mathbb{Z} \to 0 \), where the morphism \( D_\infty \to \mathbb{Z}/2\mathbb{Z} \) is given by
\[
\begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix} \mapsto 0 \text{ and } \begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix} \mapsto 1 \text{ for all } c \in \mathbb{C}^*.
\]
This gives us a \( \text{PMod}(\Sigma) \)-equivariant map \( \text{Hom}(\pi_1 \Sigma, D_\infty)/D_\infty \to \text{Hom}(\pi_1 \Sigma, \mathbb{Z}/2\mathbb{Z}) \). The fibre of this map above the zero homomorphism consists of those points given by diagonal representations, to which Lemma 22 applies, noting that the map \( \text{Hom}(\pi_1 \Sigma, \mathbb{C}^*) \to \text{Hom}(\pi_1 \Sigma, D_\infty)/D_\infty \) has finite fibers. It suffices to consider the case where \( \rho : \pi_1 \Sigma \to D_\infty \) is not in the fiber over the zero homomorphism. Let \((a_1, b_1, \cdots, a_g, b_g, c_1, \cdots, c_n)\) be the optimal generators of \( \pi_1 \Sigma \). Note that \( L = (a_1, b_1, \cdots, a_g, b_g) \) is in \((g, 1)\)-position, and we have a \( \text{PMod}(\Sigma(L)) \)-equivariant morphism
\[
\text{Hom}(\pi_1 \Sigma, \mathbb{Z}/2\mathbb{Z}) \to \text{Hom}(\pi_1 \Sigma(L), \mathbb{Z}/2\mathbb{Z}).
\]
The action of \( \text{PMod}(\Sigma(L)) \) on \( \text{Hom}(\pi_1 \Sigma(L), \mathbb{Z}/2\mathbb{Z}) \) factors through the projection \( \text{PMod}(\Sigma(L)) \to \text{Sp}(2g, \mathbb{Z}/2\mathbb{Z}) \). From the transitivity of \( \text{Sp}(2g, \mathbb{Z}/2\mathbb{Z}) \) on \( (\mathbb{Z}/2\mathbb{Z})^{2g} \)
It suffices to show that the entries of the matrices $\rho$ are roots of unity. Let $\rho(a_i) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

It follows from Lemma 22 that $\rho(a_i)$, $\rho(a_2)$, $\rho(b_2)$, $\cdots$, $\rho(a_g)$, $\rho(b_g)$, $\rho(c_1)$, $\cdots$, $\rho(c_n)$ are roots of unity. Let $i_1 < \cdots < i_q$ be precisely the indices in $\{1, \cdots, n\}$ such that $\rho(c_i) = 1$. Since $L' = (a_2, b_2, \cdots, a_g, b_g, c_{i_1}, \cdots, c_{i_q}, b_1)$ is in $(g - 1, g + 2)$-position and the restriction of $\rho$ to $\Sigma(L')$ is diagonal, we see that $\rho(a_1)$, $\rho(a_2)$, $\rho(b_2)$, $\cdots$, $\rho(a_g)$, $\rho(b_g)$, $\rho(c_1)$, $\cdots$, $\rho(c_n)$ are roots of unity, and hence the image of $\rho$ is finite, as desired.

**Definition 24.** A representation $\rho : \pi_1 \Sigma \to D_\infty$ is special dihedral if:

1. it is irreducible, and
2. there is a nonseparating curve $a$ in $\Sigma$ such that $\rho(\Sigma \setminus a)$ is diagonal.

**Lemma 25.** Let $\Sigma$ be a surface of genus 1 with $n > 0$ punctures. Let $\rho : \pi_1 \Sigma \to D_\infty$ be special dihedral. Given a pair of loops $(a, b)$ in $(1, 1)$-position on $\Sigma$, at least one of $\rho(a)$ and $\rho(b)$ is not diagonal.

**Proof.** Let $\rho \in \pi_1 \Sigma \to D_\infty$ be special dihedral. Assume toward contradiction that $(a_1, b_1)$ is a pair of loops in $(1, 1)$-position on $\Sigma$ with both $\rho(a_1)$ and $\rho(b_1)$ diagonal. We can complete $(a_1, b_1)$ to a sequence of optimal generators $(a_1, b_1, c_1, \cdots, c_n)$ of $\pi_1 \Sigma$. Since $\rho$ is special dihedral, the matrices $\rho(c_1), \cdots, \rho(c_n)$ must be diagonal (noting that diagonality of a matrix is not changed under conjugation by an element in $D_\infty$). This implies that $\rho$ is in fact diagonal, contradicting the hypothesis that $\rho$ is irreducible.

**Lemma 26.** Let $\Sigma$ be a surface of genus 1 with $n > 0$ punctures. A representation $\rho : \pi_1 \Sigma \to D_\infty$ has finite mapping class group orbit in Hom($\pi_1 \Sigma, D_\infty$)/$D_\infty$ if and only if it is finite or special dihedral.

**Proof.** The same argument as in Lemma 23 shows that if $\rho : \pi_1 \Sigma \to D_\infty$ has finite PMod($\Sigma$)-orbit in Hom($\pi_1 \Sigma, D_\infty$)/$D_\infty$ then $\rho$ is finite or special dihedral. Let now $\rho : \pi_1 \Sigma \to D_\infty$ be a special dihedral representation. We shall show that $\rho$ has finite...
mapping class group orbit in Hom(π₁Σ, D∞)/D∞, or equivalently in the character variety X(Σ).

The coordinate ring R(Σ) of the character variety is finitely generated by the trace functions [a] for a finite collection of essential curves a in Σ and the loops around the punctures, in view of Fact 4.(3) and the property of optimal generators (see Example 9). Therefore, it suffices to show that the set

{\text{tr } \rho(a) : a \subset Σ \text{ essential curve} \subseteq \mathbb{C}}

is finite. Let (a₁, b₁, \cdots, aₙ, bₙ, c₁, \cdots, cₙ) be optimal generators of π₁Σ. Since ρ is special dihedral, it follows that ρ(c₁), \cdots, ρ(cₙ) are diagonal matrices. Suppose a is a separating simple closed curve in Σ underlying a loop in the free homotopy class of cᵢ₁ \cdots cᵢₖ for some integers i₁ < \cdots < iₖ in \{1, \cdots, n\}. Since ρ(cᵢ) are diagonal for i = 1, \cdots, n it follows that \text{tr } ρ(a) lies in a finite set that only depends on the traces \text{tr } ρ(c₁), \cdots, \text{tr } ρ(cₙ). But now, every separating simple closed curve in Σ is sent to one of the above form by some mapping class group element. Since the mapping class group action preserves the special dihedral representations, and since it fixes the traces \text{tr } ρ(c₁), \cdots, \text{tr } ρ(cₙ), we conclude that \{\text{tr } ρ(a) : a \subset Σ \text{ separating curve} \} is finite.

Figure 5. Building new loops out of old ones

It remains to show that \{\text{tr } ρ(a) : a \subset Σ \text{ nonseparating curve} \} is finite. Let a₀ be a nonseparating curve in Σ such that the restriction ρ|(Σ \setminus a₀) is diagonal; such a curve exists since ρ is special dihedral. Let b be a nonseparating curve in Σ. Up to isotopy, we may assume that b intersects a only finitely many times. If b does not intersect a₀, then b \subset Σ \setminus a₀ and hence \text{tr } ρ(b) can take only finitely many values as ρ|(Σ \setminus a₀) is diagonal.

If b intersects a exactly once, then we must have \text{tr } ρ(b) = 0 by Lemma 25. Let us now assume that b intersects a₀ more than once. Let us choose a parametrization of b. Since b is nonseparating, there must be two neighbouring points of intersection of a₀ and b where the two segments of b have the same orientation, as in Figure 5. The operations as in Figure 5 produce for us a new simple closed curve b' which is also nonseparating, as well as a pair (c, c') of simple loops in (1,1)-position on Σ. We have the trace relation

\text{tr } ρ(b) = \text{tr } ρ(c) \text{tr } ρ(c') - \text{tr } ρ(b').

By Lemma 25, we have \text{tr } ρ(c) \text{tr } ρ(c') = 0, and therefore \text{tr } ρ(b) = - \text{tr } ρ(b'). Note furthermore that b' intersects a₀ in a smaller number of points than b does. Applying induction on the number of intersection points, we thus conclude that

\{\text{tr } ρ(a) : a \subset Σ \text{ nonseparating curve} \}

is finite. This completes the proof that special dihedral representations have finite mapping class group orbits in X. □
The above lemmas combine together to give us the following result.

**Proposition 27.** Let $\Sigma$ be a surface of positive genus $g > 0$ with $n \geq 0$ punctures. Let $\rho : \pi_1 \Sigma \rightarrow \text{SL}_2(\mathbb{C})$ be a semisimple representation whose image is not Zariski dense in $\text{SL}_2(\mathbb{C})$. Then $\rho$ has a finite mapping class group orbit in $X(\Sigma)$ if and only if one of the following holds:

1. $\rho$ is a finite representation, or
2. $g = 1$ and $\rho$ is special dihedral up to conjugation.

5.2. **Proof of Theorems A and B.** Theorems A and B of this paper immediately follow by combining Propositions 20 and 27.

**References**

[1] Biswas, Indranil; Koberda, Thomas; Mj, Mahan; Santharoubane, Ramanujan. *Representations of surface groups with finite mapping class group orbits.* preprint. arXiv:1702.03622

[2] Boalch, Phillip. *Geometry of moduli spaces of meromorphic connections on curves, Stokes data, wild nonabelian Hodge theory, hyperkahler manifolds, isomonodromic deformations, Painlevé equations, and relations to Lie theory.* Habilitation memoir, Paris-Sud (Orsay) 12/12/12. arXiv:1305.6593

[3] Bourgain, Jean; Gamburd, Alexander; Sarnak, Peter. *Markoff triples and strong approximation.* (English, French summary) C. R. Math. Acad. Sci. Paris 354 (2016), no. 2, 131-135.

[4] Dubrovin, B.; Mazzocco, M. *Monodromy of certain Painlevé-VI transcendentals and reflection groups.* Invent. Math. 141 (2000), no. 1, 55147.

[5] Farb, Benson; Margalit, Dan. *A primer on mapping class groups.* Princeton Mathematical Series, 49. Princeton University Press, Princeton, NJ, 2012.

[6] Goldman, William M. *Trace coordinates on Fricke spaces of some simple hyperbolic surfaces.* Handbook of Teichmüller theory. Vol. II, 611684, IRMA Lect. Math. Theor. Phys., 13, Eur. Math. Soc., Zürich, 2009.

[7] Goldman, William M.; Xia, Eugene Z. *Ergodicity of mapping class group actions on SU(2)-character varieties.* Geometry, rigidity, and group actions, 591608, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 2011.

[8] Hitchin, N. J. *Poncelet polygons and the Painlevé equations.* Geometry and analysis (Bombay, 1992), 151-185, Tata Inst. Fund. Res., Bombay, 1995.

[9] Lisovyy, Oleg; Tykhyy, Yuriy. *Algebraic solutions of the sixth Painlevé equation.* J. Geom. Phys. 85 (2014), 124-163.

[10] Previte, Joseph P.; Xia, Eugene Z. *Topological dynamics on moduli spaces. II.* Trans. Amer. Math. Soc. 354 (2002), no. 6, 24752494.

[11] Przytycki, Józef H.; Sikora, Adam S. *On skein algebras and $\text{SL}_2(\mathbb{C})$-character varieties.* Topology 39 (2000), no. 1, 115-148.

[12] van der Put, Marius; Singer, Michael F. *Galois theory of linear differential equations.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 328. Springer-Verlag, Berlin, 2003. xviii+438 pp. ISBN: 3-540-44228-6

**Department of Mathematics, Princeton University, Princeton NJ**

**E-mail address:** jwhang@math.princeton.edu