Dynamical History of the Uranian System

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Abstract

We numerically simulate the past tidal evolution of the five large moons of Uranus (Miranda, Ariel, Umbriel, Titania, and Oberon). We find that the most recent major mean–motion resonance (MMR) between any two moons, the Ariel–Umbriel 5:3 MMR, had a large effect on the whole system. Our results suggest that this resonance is responsible for the current 4°3 inclination of Miranda (instead of the previously proposed 3:1 Miranda–Umbriel MMR), and that all five moons had their inclinations excited during this resonance. Miranda experienced significant tidal heating during the Ariel–Umbriel 5:3 MMR, due to its eccentricity being excited by Ariel’s secular perturbations. This tidal heating draws energy from the shrinking of Miranda’s orbit, rather than Ariel’s outward evolution, and can generate heat flows in excess of 100 mW m^{-2}, sufficient to produce young coronae on Miranda. We find that this MMR was followed by a sequence of secular resonances, which reshuffled the moons’ eccentricities and inclinations. We also find that the precession of Oberon’s spin axis is close to a resonance with the precession of Umbriel’s orbital plane, and that this spin–orbit resonance was likely excited during the Ariel–Umbriel 5:3 MMR. After the exit from the MMR, subsequent Ariel–Umbriel secular resonance and Oberon–Umbriel spin–orbit resonance may be able to explain the current low inclinations of Ariel and Umbriel. The age of Miranda’s surface features tentatively suggests Uranian tidal Q ≈ 15,000–20,000, which can be further refined in future work.

Unified Astronomy Thesaurus concepts: Celestial mechanics (211); Orbital resonances (1181); Uranian satellites (1750)

1. Introduction

Uranus has five large satellites (Miranda, Ariel, Umbriel, Titania, and Oberon) as well as a number of smaller “ring moons” closer to the planet and irregular satellites at large distances. The orbital planes of the five major moons are relatively close to the equator of Uranus (Table 1), indicating that these moons originated in a disk of material around Uranus after the planet itself formed.

The largest departure from the circular and equatorial orbits among the five major moons is the four-degree inclination of Miranda, which implies strong dynamical perturbation of Miranda’s orbit at some point after its formation. Other indications of past orbital excitation include a dramatic pattern of surface features on Miranda (Smith et al. 1986), as well as indications of past resurfacing on Ariel (Plescia 1987), despite these moons’ present-day small orbital eccentricities and correspondingly low tidal heating rates. The most common explanation for past perturbations is that a pair of moons encountered mutual mean–motion resonances due to differential tidal evolution.

Currently, none of the five major moons are in any mean–motion resonance with each other, although Miranda’s mean motion is affected by its relative proximity to a three-body resonance with an argument \( \lambda_M - 3\lambda_A + 2\lambda_U \), where \( \lambda \) is mean longitude and subscripts refer to the moons’ initials (Greenberg 1975; Jacobson 2014). However, because Ariel’s outward tidal evolution is expected to be faster than that of Miranda (Equation (4.167) from Murray & Dermott (1999), assuming no frequency dependence of Uranus’s tidal parameters), the moons will only cross the exact three-body resonance in the future, so this resonance should have had little effect on the satellite orbits in the past.

The most recent major resonance that the moons of Uranus should have encountered in the course of their tidal evolution is the 5:3 mean motion resonance (MMR) between Ariel and Umbriel (Figure 1). This is a second-order resonance, so we would expect both eccentricities and inclinations of the two moons to be affected by resonance passage (Murray & Dermott 1999, Section 8.8.2). However, the only study of this resonance to date employed a semi-analytical approximation that assumed planar orbits, ignoring any effects on inclinations (Tittemore & Wisdom 1988). Three decades later, direct numerical integration is feasible and the Ariel–Umbriel 5:3 MMR passage clearly needs to be revisited.

Because the 5:3 Ariel–Umbriel MMR was not thought to have affected Miranda, the established hypothesis for the origin of Miranda’s inclination is the past 3:1 Miranda–Umbriel MMR, which should have predated the Ariel–Umbriel 5:3 MMR (Tittemore & Wisdom 1989; Malhotra & Dermott 1990). If Miranda was originally on a low-inclination orbit, capture into the pure inclination subresonance (i\(^2\) in the nomenclature of Murray & Dermott 1999) is very likely, and the inclination of Miranda would subsequently increase with time. It is thought that the Miranda–Umbriel 3:1 MMR was broken at Miranda’s inclination of 4°3 by a secondary resonance (Tittemore & Wisdom 1989; Malhotra & Dermott 1990; Verheylewegen et al. 2013).

An even earlier Ariel–Umbriel 2:1 MMR crossing probably never happened, as this resonance would be very difficult to break once established (Tittemore & Wisdom 1990). This enabled Tittemore & Wisdom (1990) to put the limit of

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Past mean-motion ratios between Miranda, Ariel, and Umbriel plotted against the semimajor axis of Ariel (as the moon with the fastest tidal evolution). Major resonances are labeled using open circles. Evolutions of these period ratios were calculated using Equation (4.167) from Murray & Dermott (1999), with satellite parameters from Table 1. We assumed frequency-independent tidal $Q$ for Uranus. Plot design inspired by Peale (1988).

**Figure 1.** Past mean-motion ratios between Miranda, Ariel, and Umbriel plotted against the semimajor axis of Ariel (as the moon with the fastest tidal evolution). Major resonances are labeled using open circles. Evolutions of these period ratios were calculated using Equation (4.167) from Murray & Dermott (1999), with satellite parameters from Table 1. We assumed frequency-independent tidal $Q$ for Uranus. Plot design inspired by Peale (1988).

**Table 1.** Sizes and Mean Orbital Parameters of Uranus’s Five Large Regular Satellites

| Moon   | Mass $(10^{20}$ kg) | Radius (km) | Semimajor Axis $(R_U)$ | Eccentricity  | Inclination $(^\circ)$ |
|--------|---------------------|-------------|------------------------|---------------|------------------------|
| Miranda| 0.66                | 236         | 5.080                  | 0.000 6       | 4.449                  |
| Ariel  | 13.53               | 579         | 7.469                  | 0.001 2       | 0.041                  |
| Umbriel| 11.72               | 585         | 10.407                 | 0.003 9       | 0.128                  |
| Titania| 35.27               | 789         | 17.070                 | 0.001 1       | 0.079                  |
| Oberon | 30.14               | 761         | 22.830                 | 0.001 4       | 0.068                  |

Note. Orbits of Ariel–Oberon (credited to Laskar & Jacobson 1987), as well as Miranda–Oberon masses (credited to Jacobson 2007) and radii (credited to Thomas 1988) were retrieved from the Jet Propulsion Laboratory’s Solar System Dynamics website ssd.jpl.nasa.gov 2016 December 10. The orbit of Miranda is taken from Showalter & Lissauer (2006). Semimajor axes are scaled using the reference value of Uranus’s equatorial radius, 25,559 km, and inclinations are measured relative to Uranus’s equator.

![Figure 2.](image) **Figure 2.** Evolution of semimajor axes of Ariel and Umbriel in a simulation of their 5:3 MMR crossing. In this simulation, we used $Q/k_2 = 4 \times 10^4$ for Uranus (i.e., evolution was accelerated about 10 times over our nominal case). Resonance is established about 10 Myr into the simulation and lasts for about 20 Myr, at which point it spontaneously breaks.

In the following sections, we describe the different kinds of resonant behavior that we find to have played a role in the evolution of the Uranian satellites. According to our narrative, substantial shaping of the system by the Ariel–Umbriel 5:3 MMR (Section 2) was followed by redistribution of eccentricity and inclination among the moons by secular resonances (Section 3), and inclinations may have been damped by a spin–orbit resonance between Oberon’s obliquity precession and a secular mode (Section 4). The modern eccentricity of Ariel may have arisen even later, due to a three-body resonance involving Ariel, Umbriel, and Titania (Section 5). General discussion is given in Section 6, and conclusions are presented in Section 7.

### 2. Ariel–Umbriel 5:3 MMR

In this work, we primarily use numerical integrator SIMPL (Čuk et al. 2016a), which uses a mixed-variable symplectic (Wisdom & Holman 1991) algorithm developed by Chambers et al. (2002). This program integrates full equations of motion, and models perturbations from the planet’s oblateness, other moons, the Sun, and the planets, and also approximates both planetary and satellite tides. For a full description of SIMPL, we refer the reader to Čuk et al. (2016a). In this paper, we assume everywhere that $Q/k_2$ for both Uranus and the satellites is not a function of frequency. In all of our integrations, we included the solar perturbations, but in this work, no planetary perturbations were included. For each integration, we will specify which $Q/k_2$ was used for Uranus; enhancing the tidal response of Uranus is the only method we use for accelerating our runs.

In most of our simulations, we obtain initial conditions by taking present-day satellite orbits, and modify them in two ways: we set the inclination of Miranda to be about 0°1, and we change Ariel’s semimajor axis so that it is immediately interior to the 5:3 MMR with Umbriel (in Figure 2, Ariel is initially...
only 0.04% interior to the exact commensurability). Different simulations of this resonant crossing featured in this paper are obtained by varying tidal parameters of Uranus and the moons, as well as the exact placement of Ariel interior to the resonance.

The tidal properties of Uranus can be constrained only once we have a unique solution for the satellites’ dynamical history, but it is clear from prior work by Tittemore & Wisdom (1990) that the tidal quality factor $Q$ of Uranus must be in the approximate $10^3 < Q < 10^5$ range. We also use tidal Love number $k_2 = 0.1$ for Uranus, which is a conveniently round quantity and very close to the results of Gavrilov & Zharkov (1977). We also assume that the tidal $Q$ of Uranus is not frequency-dependent. In practice, because the timing of the Ariel–Umbriel 5:3 MMR crossing we study here is determined by Ariel’s tidal evolution, by “tidal $Q$” we mean “tidal $Q$ at Ariel’s synodic frequency.” Whenever we discuss accelerated tidal evolution, the nominal “realistic” tidal dissipation within Uranus is taken to correspond to $Q = 4 \times 10^4$ (with $k_2 = 0.1$), which is roughly the smallest tidal $Q$ for which Miranda did not cross the 3:1 MMR with Umbriel according to Tittemore & Wisdom (1989). This “nominal” value of Uranus tidal $Q$ was selected at the very beginning of this project (when our preliminary runs indicated excitation of Miranda’s inclination in the Ariel–Umbriel 5:3 resonance), and has been used as a guide to what tidal evolution rates should be considered realistic. This nominal assumption should not be confused with the tidal $Q$ of Uranus derived from our results, which is presented and discussed in Section 6.

Tidal response of the satellites is even harder to estimate. Since our integrator uses only the ratio $Q/k_2$ to calculate tidal accelerations rather than treating $Q$ and $k_2$ separately, we will set the satellites’ $Q = 100$ for all simulations (a common choice for solid bodies; see Section 4.9 of Murray & Dermott 1999), and only vary their tidal Love numbers. According to Murray & Dermott (1999), when treated as solid icy bodies, Miranda and Ariel should have approximate tidal Love numbers of $k_{2,M} = 0.001$ and $k_{2,A} = 0.01$. More rigid response (possibly due to very low temperatures) could lead to a lower $k_2$, while a subsurface ocean could result in a very large Love number (Enceladus apparently has $Q/k_2 \approx 100$; see Lainey et al. 2012). Therefore, in different simulations we will use tidal Love number $k_2$ for Ariel ranging from $k_2 = 0.001$ (very cold interior) to $k_2 = 1$ (global ocean?).

Integrations of the five major Uranian moons using SIMPL on standard workstation processors require about a day of computation time for 2 Myr of simulation time. This makes simulations longer than 50–100 Myr rather cumbersome, as they require multiple months of processor time. Therefore, we ran a mixture of accelerated and realistic simulations, with the accelerated simulations enabling us to explore the phase space, while the few realistic simulations act as a check that the acceleration of tidal forces is not introducing major artifacts.

Figure 2 shows semimajor axes of Ariel and Umbriel in a simulation that has been accelerated about 10 times over our nominal tidal evolution (i.e., we used $Q/k_2 = 4 \times 10^4$ for Uranus). Here, the moons were assumed to be very rigid and nondissipative ($Q/k_2 = 10^5$ for Miranda, $Q/k_2 = 10^6$ for other moons). Figure 2 shows Ariel and Umbriel spending just under 20 Myr in their 5:3 resonance, before the resonant lock is broken. Figure 3 shows the eccentricities ($e$) and inclinations ($i$) of all five moons in the same simulation. Contrary to the usual expectation for a two-body MMR, the $e$ and $i$ of nonparticipating moons are also excited by the resonance. This is due to strong secular coupling between the satellites (Laskar & Jacobson 1987), which redistributes $e$ and $i$ between the moons’ orbits. The amount of orbital excitation is inversely proportional to the mass of the satellite: Miranda is most strongly affected, followed by Ariel and Umbriel, with Titania and Oberon having lowest eccentricities and inclinations.

In order to make sure that the dynamical picture we see is not an artifact of the enhanced tidal accelerations in our model, we also conducted a long-duration simulation of the Ariel–Umbriel 5:3 resonance crossing using more realistic tidal accelerations. Figure 4 shows an evolution that was accelerated approximately three times over nominal case ($Q/k_2 = 1.33 \times 10^5$ for Uranus). The satellites were still considered rigid, although somewhat more dissipative ($Q/k_2 = 10^7$ for Miranda, $Q/k_2 = 10^4$ for other moons). The $e$ and $i$ attained by all five moons in Figure 4 are comparable to those shown in Figure 3, and possibly even slightly higher. After running a number of simulations using a range of tidal parameters for the planet and the satellites, we conclude that the excitation of all five satellite orbits, followed by resonance breaking, is the guaranteed outcome when the tidal damping within the moons, especially Ariel, is low ($Q/k_2 \gtrsim 10^4$).

To illustrate the effect of strong tidal damping within Ariel, in Figure 5 we show a simulation in which we sharply reduced the $Q/k_2$ of Ariel after about 20 Myr in the resonance (i.e., 50 Myr into the this simulation). In this simulation, we used $Q/k_2 = 2 \times 10^4$ for Uranus (i.e., accelerated by about a factor of two). Up to 50 Myr into the simulation, the moons’ tidal parameters were $Q/k_2 = 10^5$ (Miranda) and $Q/k_2 = 10^4$ (other moons). At 50 Myr, we instantaneously switched tidal dissipation within Ariel to $Q/k_2 = 100$, with the intention of approximating large-scale melting. The eccentricity and inclination of Ariel and Umbriel drop instantaneously, and Ariel temporarily leaves the resonance. At 60–70 Myr, the resonance is re-established, but now with much less excited orbits for all five moons (Figures 5 and 6, middle and bottom panels). The exception is the inclination of Miranda, which keeps growing over time, while Miranda’s eccentricity appears to be slowly damped by its own satellite tides (Figure 6, top panels).

The possibility that the moons of Uranus once inhabited a damped resonant state is somewhat attractive, as it would explain the low inclinations of the four larger moons, while still allowing Miranda to attain a large inclination, as observed. However, in numerous simulations that we ran, we never saw any sign of Ariel and Umbriel being able to leave the resonance while at low eccentricities. In that respect, we are in agreement with Tittemore & Wisdom (1988), who find that eccentricities of about $e = 0.02$ are needed to exit the resonance. While it is impossible to prove the negative using numerical simulations, we have accumulated several Gyr of simulations of damped Ariel–Umbriel 5:3 resonance, using a variety of system parameters, and no resonance breaking was observed. While it is in principle possible that there is some aspect of the problem we have missed, we must conclude at this point that Ariel and Umbriel cannot leave their 5:3 MMR if their eccentricities are strongly damped.

We also considered the possibility that the resonance may have been broken by an impact. The most advantageous impact would be a head-on impact with Umbriel, which would move it to a lower orbit and out of the resonance. Alternatively, a
resonance-breaking impact with Ariel would have to be in the direction of orbital motion, reducing the relative velocity between the impactor and the moon (impacts that make the two moons’ orbits converge would lead them to soon re-enter the resonance). By varying Umbriel’s initial $a$ in many short simulations, we find that the relative change of Umbriel’s semimajor axis required to exit the resonance is $\Delta a/a = 2 \times 10^{-4}$. If we assume orbital velocity of 5 km/s for Umbriel and an impact velocity of 10 km/s (exactly opposite the orbital motion), we find that the impactor should be $0.5 \times 10^{-4}$ the mass of Umbriel, or about 25 times smaller in diameter. A 50 km impactor would make a 500+ km crater on Umbriel,\(^3\) which should be geologically young. While there are indications of a large basin on Umbriel (Thomas 1988), it appears to be geologically old, rather than recent. More problematic is the probability of such an impact: Zahnle et al. (2003) estimate that the rate of formation of 500 km craters on Umbriel is $2 \times 10^{-18} \text{ km}^{-2} \text{ yr}^{-1}$. Taking into account the surface area of Umbriel (about $4 \times 10^6$ km), we get a formation rate of

\(^3\) Using Keith Holsapple’s Web-based crater size calculator: http://keith.aa.washington.edu/craterdata/scaling/index.htm.
10^{-11} per year, which gives a few percent probability of this impact happening over the age of the solar system. Even more restrictive is the requirement that the impact had to happen within a few hundred Myr after the establishment of the resonance, or else the inclination of Miranda would grow well beyond the observed 4°.3. Even before considering the probability of the correct impact geometry, we conclude that the probability of an impact sufficient to break the resonance in the required timeframe is well below one percent. We will therefore concentrate on the dynamical means of breaking the resonance.

The dynamical excitation of all five orbits by the Ariel–Umbriel resonance is a direct consequence of the coupled dynamical nature of the Uranian satellite system. Due to the relative strength of satellite–satellite perturbations when compared to the perturbations from Uranus’s oblateness, mean–motion resonances between Uranian satellites are overwhelmingly chaotic (Dermott et al. 1988). Furthermore, the nonunique correspondence between satellites and secular modes (Laskar & Jacobson 1987; Malhotra et al. 1989) means that the eccentricities and inclinations of moons other than the resonant pair can be affected by the resonance. A good
illustration of the mixing of secular modes is the isolated resonance capture at the beginning of the resonant encounter, best seen in Figures 5 and 6. This subresonance is separated relatively well from the rest of the Ariel–Umbriel 5:3 multiplet, due to the fast precession rate of Miranda. For a brief period, only the inclinations of Ariel and Miranda grow, indicating a resonant argument $\omega - \omega_{\text{UA}}$ or more properly, $5\omega - 3\lambda_A - \Omega_A - \Omega_M$, where the numbers refer to secular modes. While the secular mode II is associated primarily with Miranda, dynamical coupling between Ariel and Miranda appears strong enough to make II a participant in a stably librating Ariel–Umbriel mean–motion resonant argument. The second and third secular modes (in both eccentricity and inclination) are generally associated with Ariel and Umbriel, respectively, but there is significant coupling between neighboring moons; fourth and fifth secular modes in $e$ and $i$ are shared approximately equally between Titania and Oberon, with no one-to-one moon–mode correspondence (Laskar & Jacobson 1987).

Figure 7 shows the amplitudes of each of the five secular modes of the Uranian satellites (in both eccentricity and inclination) at the end of six dissimilar numerical simulations of Ariel–Umbriel 5:3 MMR. These simulations all had similar initial orbital conditions but used different tidal parameters for the planet and the moons, with some runs also using slightly different masses for Ariel and Umbriel (see Table 2 for details of each simulation). We suspect that much of the variation in our results is purely stochastic and not due to different initial conditions, but many more (computationally expensive) integrations would be needed to confirm this. In our simulations, the eccentricity modes are generally overexcited compared to the present state, but since they can be subsequently damped by eccentricity tides, here we will concentrate on the inclination modes, which are more constant over time. We note that the inclination of Miranda (i.e., secular mode I1) is generally well-matched at the end of our simulations, obviating the need for the system to pass through the Miranda–Umbriel 3:1 MMR (Tittemore & Wisdom 1989; Malhotra & Dermott 1990). However, the amplitudes of the other inclination modes (mode I2 in particular) are usually well above the observed values, seemingly ruling out Ariel–Umbriel 5:3 MMR crossing from happening. Here, we face a conundrum: in order to obtain the large inclination of Miranda, some dynamical excitation event had to happen in the system’s history, and Ariel–Umbriel 5:3 MMR is by far most recent commensurability for any tidal model. However, inclinations of the moons other than Miranda that were generated in the Ariel–Umbriel 5:3 MMR passage appear inconsistent with the present system. We find that the resolution of this discrepancy lies in the dynamical evolution of the moons following the mean–motion resonance crossing, which is discussed in the following sections.

3. Secular Resonances

Uranian regular moons are similar in size to Saturn’s mid-sized icy moons, but their secular dynamics is different, as the
Uranians’ orbital precession is not determined only by perturbations from the planet’s oblateness, but also by significant moon–moon interactions (Laskar & Jacobson 1987; Dermott et al. 1988). Moon–moon perturbations are mostly secular, but also have a component that depends on the proximity to the mean-motion resonances (Malhotra et al. 1989). Near-resonant perturbations make the secular frequencies vary considerably near MMRs, raising a possibility of secular resonances very close to mean-motion ones. Previously, Ćuk et al. (2016a) found that Saturnian moons Tethys and Dione passed through a secular resonance immediately following the crossing of the Dione–Rhea 5:3 MMR. Unlike the secular modes of the solar system (Murray & Dermott 1999), fundamental precession frequencies of the Saturnian satellites are well-separated, even near mean-motion resonances. However, arguments of the type $\varpi + \Omega$ (i.e., the sum of the longitude of the pericenter and longitude of the ascending node) for each moon are changing very slowly, and can encounter resonances between each other near MMRs (Ćuk et al. 2016a). The eccentricity and inclination of a moon changing in unison is a clear signature of a secular resonance of this type, with the $e$ and $i$ of one moon involved in the resonance being anticorrelated with those of the other. In other words, in a secular resonance of this type, one moon’s orbit should get both more eccentric and inclined, while the other moon’s orbit should become less eccentric and inclined at the same time.

A clear example of a secular resonance is seen in Figure 3, where the $e$ and $i$ of Ariel and Umbriel show major changes.
Arguments are significant near-resonant perturbations on the end of the simulation.

\[ \Sigma_5 \times 10^3, 2 \times 10^3, 10^3, 10^4, 10^5, 10^5 \]

Table 2

Details of the Six Simulations Plotted in Figure 7, Including the Key to the Symbols Used to Plot the End States

| #  | Symbol | Duration | \((Q/k_s)_U\) | \((Q/k_s)_M\) | \((Q/k_s)_4\) | Masses | Secular Res. | Note |
|----|--------|----------|---------------|---------------|---------------|--------|-------------|------|
| 1  | Circle | 50 Myr   | \(4 \times 10^4\) | \(10^7\)       | \(10^5\)      | Table 1 | Modes 2-3   | Figures 2, 3 |
| 2  | Square | 200 Myr  | \(1.33 \times 10^5\) | \(10^5\)       | \(10^4\)      | Table 1 | Modes 3-4   | Figure 4 |
| 3  | Delta  | 50 Myr   | \(4 \times 10^4\) | \(10^7\)       | \(10^3\)      | Jacobson (2014) | None |
| 4  | Nabla  | 50 Myr   | \(4 \times 10^4\) | \(10^7\)       | \(10^4\)      | Jacobson (2014) | Modes 3-4? |
| 5  | Diamond| 50 Myr   | \(4 \times 10^4\) | \(10^6\)       | \(10^4\)      | Jacobson (2014) | None |
| 6  | Pentagon| 5 Myr + 86 Myr | \(4 \times 10^7, 2 \times 10^4, 10^3, 10^4, 10^5, 10^5\) | Table 1 | Several minor | Two parts |

Note. Apart from the tidal properties of Uranus, Miranda, and Ariel, this table also lists whether we used the Ariel and Umbriel masses from Table 1, or updated values from Jacobson (2014) that have Ariel less massive than Umbriel. We also list secular resonances that happened between the breaking of 5:3 MMR and the end of the simulations. Simulation #6 consisted of 5 Myr of highly accelerated evolution, followed by a longer simulation at a more realistic rate.

Figure 8. Difference between the precession rate of arguments \(\varpi_U + \Omega_U\) (where \(1 \leq N \leq 5\)) for each of the five Uranian system secular modes at the end of simulation #5 (see Table 2) and today. Precession rates for these arguments are significantly higher post-MMR in the case of modes 2 and 3, due to near-resonant perturbations on \(\varpi\). It is clear that the system must encounter secular resonances as it evolves from a near-resonant configuration to its present state.

Left: Post-MMR; Right: Now

The relevant resonant argument of \(\varpi_2 + \Omega_2 - \varpi_3 - \Omega_3\) (where the integer subscripts refer to secular modes, not the moons) can be deduced from D’Alembert’s rules (Murray & Dermott 1999); we confirm this by finding that its proxy \(\varpi_A + \Omega_A - \varpi_U - \Omega_U\) (where letter subscripts refer to satellites) is librating around 180° with a large amplitude between 37 and 41 Myr in the simulation. Similarly, Figure 4 features a secular resonance between modes 3 and 4 (i.e., with the argument \(\varpi_3 + \Omega_3 - \varpi_4 - \Omega_4\)) close to the end of the simulation. We see some kind of secular resonance of this type in most of our simulations that went beyond the Ariel–Umbriel 5:3 MMR breaking, and all moons except Miranda are affected by at least one of these secular resonances.

We also identify a simple \(\varpi_3 - \varpi_4\) (i.e., E3-E4 mode) secular resonance that can only occur before the Ariel–Umbriel 5:3 MMR (visible as a decrease in \(e_U\) around 5 Myr in Figure 3). This resonance makes the eccentricity of Umbriel drop while increasing the \(e\) of Titania and Oberon; it is caused by the retrograde adjustment to the precession of \(\varpi_3\) just before the MMR due to the Ariel–Umbriel near-resonant terms, making the precession of \(\varpi_3\) as slow as that of \(\varpi_4\). This resonance is unlikely to have been important in the real system, as it is likely that the moons’ eccentricities would have been largely damped at that time.

While untargeted secular resonances appear to be common in our simulations, it would be valuable to have a more systematic picture of secular resonance crossings. Simulations that happened to end almost immediately after the mean–motion resonance breaking are particularly useful cases for understanding shifts in secular frequencies. Using the state vectors at the end of simulation #5 in Table 2, we produced Figure 8, which compares the frequency relevant secular argument \(\varpi + \Omega\) for each secular mode at the end of the simulation to the present Uranian system according to Jacobson (2014). Figure 8 demonstrates that the precession of secular argument \(\varpi + \Omega\) for modes 2 and 3 is expected to slow down significantly as Ariel and Umbriel move away from their resonance, while modes 1, 4, and 5 are hardly affected. In the process, it is inevitable that modes 2 and 3 would each go through secular resonances with modes 4 and 5, and also with each other. The precession rate of the argument \(\varpi + \Omega\) for mode 1 (closely associated with Miranda) is negative and outside the range swept by other secular frequencies, so Miranda is unlikely to be affected by any secular resonances of this type.

Since the secular resonances discussed here involve commensurabilities between relatively slow-changing arguments, it is desirable to study them using nonaccelerated tidal evolution simulations. We find that full numerical simulations of secular resonances themselves are generally manageable, but that the tidal evolution between the resonances takes excessive time and arguably does not require full numerical integration. In Figure 8, it is obvious that the system is not very close to any secular resonances at the “post-MMR” stage. Therefore, we turn to another integration featuring the Ariel–Umbriel 5:3 MMR crossing, namely simulation #4 in Table 2. Figure 9...
shows the precession rates of argument $\varpi + \Omega$ for each secular mode at the end of simulation #4 and compares them to the present system.

Figure 9 suggests that modes 2 and 4 are close to a secular resonance. We therefore integrated the end state of simulation #4 for 50 Myr. We used $Q/k_2 = 4 \times 10^5$ for Uranus, $Q/k_2 = 10^5$ for Miranda, and $Q/k_2 = 10^5$ for the other four moons; these tidal parameters may underestimate satellite tides, but enable us to observe the secular resonance isolated from other dynamical effects. The results of this simulation are plotted in Figures 10 and 11.

As can be seen in Figure 10, secular resonance between modes 2 and 4 increases the $e$ and $i$ of Ariel while decreasing the inclinations of Titania and Oberon and shrinking the amplitude of the variation in eccentricity of those two moons. The dissimilar behavior of $e$ and $i$ of both Titania and Oberon in this simulation is a consequence of the two moons’ inclinations being dominated by secular mode I4, whereas their eccentricity is dominated by the secular mode E5; see Laskar & Jacobson (1987) for the normal mode component table. So while the amplitude of E4 decreases over time (Figure 11), the average eccentricities of Titania and Oberon do not change much, as they are determined by the amplitude of the mode E5. The nonangular orbital elements of Umbriel appear to be largely unaffected, as are those of Miranda (not plotted).

Both Figures 8 and 9 imply that secular modes 2 and 3 should also experience a mutual secular resonance at some point following the MMR. As secular modes 2 and 3 (both in eccentricity and inclination) are strongly correlated with the orbits of Ariel and Umbriel, respectively, it is easy to detect this secular resonance through its proxy $\varpi_A + \Omega_A - \varpi_U - \Omega_U$. We were therefore able to place Ariel and Umbriel just interior to this secular resonance and allow the moons to enter it. The results of this numerical experiment are shown in Figures 12 and 13.

Figure 12 shows the secular resonance between modes 2 and 3, in which the eccentricity and inclination of Ariel decrease while the $e$ and $i$ of Umbriel increase. At the end of the simulation, the inclination of Ariel is relatively low, below 0°3. Figure 13 plots the resonance in terms of secular modes, and shows that the largest contribution to Ariel’s final inclination comes from mode I3, as mode I2 then has an amplitude of only about 0°1. However, this is still about twice as large as the current amplitude of I2 found by Laskar & Jacobson (1987), and almost an order of magnitude larger than the value in Jacobson (2014), suggesting that more work is needed to establish whether secular resonance alone may not offer a full explanation for the present low amplitude of the I2 mode.

We note that, in order for secular resonance to evolve to very low amplitudes of I2, the amplitude of E2 must have been originally higher than that of I2 (in radians), or else the secular resonance would break early because Ariel’s eccentricity stopped being dominated by E2 (see Figure 13). Larger amplitude for E2 over I2 is a minority outcome in our simulations of Ariel–Umbriel 5:3 MMR, but so far we have too few simulations and too many unexplored aspects of the problem to be able to determine probability of this outcome.

The above examples demonstrate that secular resonances can redistribute eccentricity and inclination among the moons of Uranus in the aftermath of the Ariel–Umbriel 5:3 MMR. The sole exception is Miranda, which is not affected by any of these resonances. While we have a limited number of direct numerical integrations, simulations shown in Figures 10 and 12, combined with the timeline of secular resonances suggested by Figures 8 and 9, allow us to understand the post-MMR dynamical evolution of the Uranian system. It appears that secular resonances can heavily deplete secular modes 2 and 4, while exciting secular mode 3. We have simulated secular resonances of modes 2 and 3 with mode 5, and found they lead to small kicks in $e$ and $i$, rather than resonance capture, regardless of tidal evolution rates used. Figure 10 shows that the low inclinations of Titania and Oberon can be a result of secular resonance (such as between modes 2 and 4). Similarly, Figure 12 shows that the low inclination of Ariel may also largely be the result of a later secular resonance (such as between modes 2 and 3). Furthermore, it is likely that the eccentricities of Miranda and Ariel have been damped in the time since the Ariel–Umbriel 5:3 MMR (more in Section 6). The eccentricity of Umbriel is currently relatively high ($e_U = 0.004$), and is consistent with Umbriel gaining $e$ in a secular resonance with Ariel. The largest remaining discrepancy between our simulations of the Uranian moons’ dynamical history and their current orbits is that we predict the inclination mode I3 to be excited (to an order of a degree) while the present amplitude of I3 is only about 0°7. We consider potential solutions in the next section.

4. Spin–Orbit Resonances

In our solar system, there are multiple examples of secular spin–orbit resonances, in which the precession period of a body’s spin axis is commensurable with an orbital precession period in the system. Sometimes spin–orbit resonances lead to a chaotically varying obliquity for a planet, as is the case for Mars (Ward 1973; Touma & Wisdom 1993). At other times, the spin axis can get captured into a stable resonance with orbital precession. Some of the examples of spin–orbit resonance capture in the solar system include excitation of Saturn’s obliquity by the $f_8$ secular mode of the solar system (Hamilton & Ward 2004; Ward & Hamilton 2004), as well as...
the alignment of Koronis asteroid family spins due to a resonance with their nodal precession (Vokrouhlický et al. 2003). More recently, spin–orbit resonances have been proposed as an important feature of compact multiple exoplanet systems (Millholland & Laughlin 2019). In each case, a change in one of the precession frequencies is required to achieve capture into the resonance, which can be supplied by planetary migration (in the case of Saturn), YORP radiative spin-down torque (for asteroids), or tidal spin-down (for close-in exoplanets).

In case of planetary satellites, no spin–orbit resonances of this type are known to have been observed thus far. Most planetary satellites have tidally evolved spins, and currently rotate at the synchronous rate while their obliquities are purely forced and determined by the so-called Cassini states (Colombo 1966; Peale 1969). In Cassini states, as the moon’s orbital plane precesses (by definition) around the Laplace plane, the satellite has a forced obliquity that causes its spin axis to precess (relative to the Laplace plane) with the same period as the orbit. While the Cassini states have some ingredients of a spin–orbit resonance, they are usually not considered to be one. However, extensive past tidal evolution of the Moon placed it (billions of years ago) at a distance from Earth at which the spin and orbital precession frequencies were equal, leading to the switch between two Cassini states. The Moon must have had an extremely high forced inclination during and after this transition (Ward 1975; Ćuk et al. 2016b).

The Uranian system has some similarities with both other satellite and planetary systems. Like all close-in satellites, the five large Uranian moons are in synchronous rotation due to tides raised on them by the planet. Unlike the moons of Saturn, each Uranian moon’s orbital precession involves several secular modes with different frequencies (Laskar & Jacobson 1987), reminiscent of the planetary system (Murray & Dermott 1999), offering numerous candidate frequencies for spin–orbit resonances. The reason spin–orbit resonances are relevant for the system’s orbital history is that the obliquity tides can damp inclination, as long as the forced obliquity is sufficiently high (Chyba et al. 1989; Chen & Nimmo 2016). Therefore, spin–orbit resonances have the potential to reconcile the discrepancy between the excited inclinations suggested by our simulations and the observed low inclinations of the four largest Uranian moons.

Figure 10. Simulation of the secular resonance between modes 2 and 4, with the resonant frequency $\omega_2 + \Omega_2 - \omega_4 - \Omega_4 = 0$. This simulation continues Ariel–Umbriel 5:3 MMR simulation #4 from Table 2, and uses our nominal $Q/k = 4 \times 10^5$ for Uranus.
moments of inertia of a moon, and

\[ A, B, C \] are the smallest, intermediate, and largest

moments of inertia of a moon, and \( n \) its mean motion. For a

hydrostatic satellite, \( C - B = 0.25(C - A) \) (e.g., Garrick-

Bethell et al. 2006), therefore \( C - A = (8/5)J_2MR^2 \), where

\( M, R, \) and \( J_2 \) are the mass, radius, and the dimensionless

oblateness moment. For near-spherical uniform bodies

\[ C = (2/5)MR^2, \] giving us the final form of Equation (1).

Comparison with secular frequencies of the system (Laskar

& Jacobson 1987; Malhotra & Dermott 1990; Jacobson 2014)
suggests that Oberon’s axial precession period (125 yr) is close
to the secular mode I3 (\( \approx 130 \) yr). For the viability of resonant

capture, it is crucial that Oberon’s axial precession in the zero-

obliquity, zero-libration case is slightly faster than the

potentially resonant orbital perturbation, as obliquity forced

by the resonance can only slow down the axial precession.

In order to test the plausibility of a resonance between

Oberon’s axial precession and the secular mode I3, we use the

numerical integrator R-SISTEM that was developed by Ćuk et al.

(2016b) for the Earth–Moon system. This integrator has all the

capabilities of SIMPL, while also integrating the rotational

motion of a satellite using the Lie-Poisson algorithm of Touma

& Wisdom (1994). A more detailed description of R-SISTEM is

available in the Methods section of Ćuk et al. (2016b).

Figure 14 shows a numerical simulation of a spin–orbit

resonance between Oberon and mode I3 done using R-SISTEM.

We used the nominal orbits for the current Uranian system

(Table 1) with one exception: we increased the inclination

of Umbriel to 0°6 (to demonstrate that it can be damped). For the

rotation of Oberon, we assumed a synchronous rotation rate,

but we created an in-plane misalignment between the

orientation of the long axis of Oberon and the direction to the

planet of about 25°. We used \( Q/k_3 = 10^6 \) for Oberon. The

initial misalignment led to longitudinal librations that were

damping relatively fast (middle panel), until the spin–orbit

resonance was reached at about 10^5 yr, and the obliquity of

Oberon was excited (top panel). The resonant argument

\( \phi_O - \Omega_3 \) (where \( \phi_O \) is the longitude of the intersection of the

equatorial planes of Oberon and Uranus) is plotted in the

bottom panel, clearly demonstrating capture into resonance.

Oberon eventually settles into a stable resonant state with an

obliquity of about 13°.

The simulation shown in Figure 14 used Ariel and Umbriel

masses listed in Table 1. Since the spin–orbit resonance is

sensitively dependent on the frequency of the I3 mode, we also

performed a simulation with masses taken from Jacobson

(2014). The smaller mass of Ariel in Jacobson (2014) makes the

frequency of I3 slightly lower, requiring a higher forced

inclination for Oberon to maintain the resonance, and Figure 15

shows that this new inclination is about 19°. This higher forced

obliquity also leads to an even faster inclination damping for

secular mode I3 (Chyba et al. 1989). We conclude that the

spin–orbit resonance is likely to be robust against the

uncertainty in satellite masses.

We have also run longer spin–orbit resonance simulations to
demonstrate that obliquity tides in this spin–orbit resonance act
to damp the amplitude of the I3 mode. Due to the slower speed

of R-SISTEM relative to SIMPL, we had to sacrifice some of the

elements of the simulation for speed. We decided to remove

Miranda from the simulation, as it does not directly affect the

spin–orbit resonance of Oberon, and this modification enables

us to use a longer time step (as Miranda’s orbital period was the

shortest timescale in the problem). Also, we started the

simulation shown in Figure 16 with the highest tidal dissipation

Figure 11. Same simulation as shown in Figure 10, plotted in terms of secular

modes rather than real satellites. Top panel plots eccentricities. Middle panel

shows inclinations, with modes E2 and I2 plotted in black and modes E4 and I4

in gray. Bottom panel shows the resonant argument \( \varpi_3 + \Omega_2 - \varpi_4 - \Omega_4 \).

Normal mode conversion matrix was calculated from the system’s configuration

at 20 Myr into the simulation.

Unlike their rates of orbital precession, which are well-
determined, the axial precession rates of the Uranian moons
have not been directly observed. Theoretical predictions of
axial precession rates depend on the shapes and gravity fields
of the satellites. These are based solely on limb coordinates for
Miranda and Ariel, and are completely unconstrained for the
other moons (Thomas 1988). Chen et al. (2014) have
previously calculated the expected moments of inertia of
Uranian moons, under the assumption that they have uniform
density and are in long-term hydrostatic equilibrium.

Table 3 lists the rotational parameters of the Uranian
satellites, mostly based on data in Table 3 of Chen et al.
(2014). We added a column with the calculated axial
precession periods of the satellites, under the assumptions of
uniform density, synchronous rotation, hydrostatic equilibrium,
circular orbits, and negligible obliquity for all the satellites.

The precession rate was calculated using the expression
(Ward 1975):

\[ \alpha = \frac{3}{2n} \frac{(C - A)}{C} = 6nJ_2, \]  

(1)
within Oberon that still allowed spin–orbit resonance capture \((Q/k_2 = 3.3 \times 10^3)\), which was determined empirically.

Figure 16 shows the evolution of Umbriel’s inclination during a two-part four-moon simulation. In the first simulation described above (gray line in Figure 16), the inclination of Umbriel is declining monotonically, with an approximately constant \(i(d\ell/dt)\) (Chyba et al. 1989). The resonance breaks when the mode I3 (approximately equivalent to the mean inclination of Umbriel) reaches about 0°2. The resonance breaks because both the offset of the resonant argument’s libration center from 180° and the argument’s libration amplitude (Figure 16, bottom panel) are inversely proportional to the amplitude of the I3 mode (i.e., approximately Umbriel’s mean inclination). As the strength of the I3 mode is damped, librations of the argument \(\phi_0 - \Omega_3\) grow and shift away from 180°, until they exit the stable island and the resonance is broken. This end point of the resonance is not inevitable, but rather is dictated by our choice of Oberon’s tidal properties. The center of librations in the resonance is offset from \(\phi_0 - \Omega_3 = 180°\) because of the presence of dissipation, and the amount of offset is proportional to Oberon’s \(k_2/Q\) ratio. To explore how weaker tidal dissipation affects the resonance, we continued the simulation using \(Q/k_2 = 10^4\) for Oberon, and once again introduced a free rotational (longitudinal) libration of Oberon in order for resonance capture to occur. Oberon re-enters the spin–orbit resonance and the damping of Umbriel’s inclination continues (at a slower pace) until it is again broken when the amplitude of the I3 mode is \(i_3 = 0°07\) (black line), which is close to the current value.

How realistic is the capture into Oberon’s secular spin–orbit resonance in the real system? There are two plausible routes to resonance capture. One is collisional, in which an impact induces rotational librations of Oberon (within synchronous state) that are large enough to enable resonance capture during libration damping. Using the approach of Lissauer (1985), we calculate that an impactor large enough to induce 20° longitudinal librations of Oberon (once again assuming a hydrostatic shape; see Table 3) should create a \(D = 70\) km crater on Oberon. According to Zahnle et al. (2003), craters of this size are formed on Oberon by cometary impacts 1–2 times in a Gyr. Therefore, establishment of Oberon’s spin–orbit resonance after the Ariel–Umbriel 5:3 MMR (and associated secular resonances) excited the I3 mode through cometary impacts is plausible. Additionally, as Oberon is the outermost regular satellite of Uranus, it may experience collisions with the irregular satellites of Uranus (Gladman et al. 1998, 2000; Kavelaars et al. 2004), some of which may have long-term variations in eccentricity, due to solar perturbations (Carruba et al. 2002; Nesvorný et al. 2003). The rate of irregular satellite impacts on Oberon is currently unknown, but in principle this process could significantly increase the likelihood of secular spin–orbit resonance capture.

The other route to spin–orbit resonance capture is purely dynamical. The spin–orbit resonance shown in Figures 14–16 is regular and devoid of chaotic behavior. However, we know that all of the secular modes of the Uranian system experience chaotic behavior during the Ariel–Umbriel 5:3 MMR (Figures 3–4). In order to explore the effects of chaos on the spin of Oberon, we took a simulation of MMR crossing (very similar to the one shown in Figures 2–3) which still had Ariel and Umbriel in the resonance at 50 Myr, and continued it using R-SISTEM, with the spin dynamics of Umbriel taken into account. Oberon was initially given small obliquity and rotational libration amplitude. The results are plotted in Figure 17: Oberon’s inclination varies chaotically between about zero and 20° for the satellite masses used, the resonant obliquity outside the MMR is 13°, as in Figure 14. This is clearly a consequence of the mode I3 being affected by the MMR, which in turn couples the precession of Oberon to the Ariel–Umbriel resonance.

So far, we have not seen any exits from the Ariel–Umbriel 5:3 MMR in simulations that included Oberon’s rotation.

Figure 12. Simulation of the secular resonance between secular modes 2 and 3, with the resonant frequency \(2\Omega_2 + \Omega_3 - 2\Omega_3 - \Omega_3 = 0\). Initial conditions were chosen so that the system would go through the resonance immediately. We used our nominal \(Q/k_2 = 4 \times 10^3\) for Uranus.
Slower speed of integrations using R-SYSTEM and unpredictability of the resonance breaking makes it impractical to numerically simulate this process with our current resources.

We simply note that the obliquity of Umbriel is near the resonant value for a large part of the simulation shown in Figure 17. It is likely that, for some of these values of obliquity and resonant argument, Oberon will find itself in the stable spin–orbit resonance once the MMR is broken. Our simulations also indicate that, if Oberon’s obliquity is higher than resonant after MMR breaking, capture into spin–orbit resonance through obliquity damping appears to be certain (for realistic tidal properties of Oberon). Future work should be able to establish the probability of Oberon entering the stable spin–orbit resonance due to the breaking of Ariel–Umbriel 5:3 MMR.

If the spin–orbit resonance is established at the exit from the MMR, the secular mode I3 could be gradually damped, and this damping can be continued during the I2-I3 mode secular resonance. This parallel evolution within both the secular and spin–orbit resonance is plausible, as both have natural

| Table 3 | Rotation-related Parameters of the Major Moons of Uranus |
|----------|----------------------------------------------------------|
| Moon     | Radius (km) | Density (kg m\(^{-3}\)) | Rotation Period (d) | \(J_2\) | Precession Period (yr) |
| Miranda  | 235.8       | 1200                     | 1.412               | 6.10 \times 10^{-3} | 0.105 6 |
| Ariel    | 578.9       | 1665                     | 2.52                | 1.39 \times 10^{-3} | 0.827 |
| Umbriel  | 584.7       | 1399                     | 4.13                | 6.13 \times 10^{-4} | 3.07 |
| Titania  | 788.9       | 1714                     | 8.71                | 1.13 \times 10^{-4} | 35.2 |
| Oberon   | 761.4       | 1630                     | 13.47               | 4.93 \times 10^{-5} | 124.7 |

Note. Parameters are taken from Table 2 in Chen et al. (2014), except for the last column, which we calculated. The data on radii and orbits are from Thomas (1988) and Jacobson et al. (1992), respectively. The gravity moments \(J_2\) were calculated by Chen et al. (2014) (we corrected an obvious typo in \(J_2\) for Oberon) on the assumption of the hydrostatic equilibrium, and were used by us to calculate the moons’ axial precession periods (see Equation (1) and associated text).
timescales on the order of $10^{8}$ yr, but direct numerical integrations will be necessary to test this hypothesis.

5. Three-body Resonances

While we can be confident that Ariel–Umbriel 5:3 MMR was the last major mean–motion commensurability between Uranian moons, our work has uncovered a number of unexpected dynamical features, including secular and spin–orbit resonances, that we did not foresee at the start of the project. It is natural to ask whether there are any additional resonances (or other dynamical events) that happened between the Ariel–Umbriel 5:3 MMR and the present, which should be taken into account when comparing the present system to our simulations. We decided on a brute-force approach, dividing up the total orbital evolution of Ariel (measured by the Ariel–Umbriel period ratio) between the MMR and the present into ten sections, and covering each of them by a simulation with $Q/k_2 = 2 \times 10^4$ for Uranus. These runs were therefore accelerated about 20 times over our nominal $Q/k_2 = 4 \times 10^5$ case, and ten 10 Myr simulations approximate about 2 Gyr of the tidal evolution we would have in the nominal case.

In this sweep for more recent resonances, we found only one new dynamical feature of note. It happens when the Ariel–Umbriel 5:3 inequality has a period of about 86 days (the period of this inequality is very long just after the exit from 5:3 MMR, and about 62 days at the current epoch). At this location, we see temporary captures of Ariel, Umbriel, or both into an apparent eccentricity-type resonance, and we also see much smaller kicks to their inclinations. Figure 18 plots the eccentricity of Ariel in a somewhat slower ($Q/k_2 = 10^5$ for Uranus) simulation centered around the resonance. We find that this resonance is a combination between two two-body near-resonances: the Ariel–Umbriel 5:3 and Umbriel–Titania 2:1 inequalities. These two inequalities have opposite signs: while Ariel is past 5:3 MMR with Umbriel (such that their inequality is prograde), Umbriel is interior to the 2:1 MMR with Titania (which makes their near-resonant frequency retrograde). Therefore, this resonant argument’s mean–motion part is $4\lambda_u - 3\lambda_t + 2\lambda_f$, which make it a third-order three-body resonance (Nesvorný & Morbidelli 1998; Quillen 2011; Gallardo et al. 2016). For our purposes, this resonance is important because it is likely the most recent process to have increased the eccentricity of Ariel, and is therefore crucial for estimating Ariel’s eccentricity damping timescale (Section 6).

The detectability of the third-order three-body Ariel–Umbriel–Titania resonance postdating the Ariel–Umbriel 5:3 MMR raises the possibility of other three-body resonances being prominent in the past of the system. While we did not have the time or the resources for a thorough search for new resonances at earlier epochs, there is an obvious candidate based on the above results. Before the Ariel–Umbriel 5:3

Figure 15. Similar to the top panel in Figure 14, but now showing the Oberon–Umbriel secular spin–orbit resonance with satellite masses from Jacobson (2014), rather than those from Table 1. Lower mass of Ariel decreases the precession rate of mode $I_3$, necessitating a higher forced obliquity of Oberon for resonance capture.

Figure 16. Damping of Umbriel’s inclination (top panel, as a proxy for the amplitude of the secular mode $I_3$) through the secular spin–orbit resonance with Oberon’s axial precession. Bottom panel shows the (proxy) resonant argument $\phi_0 - \Omega_U$. Gray line plots a simulation that used $Q/k_2 = 3.3 \times 10^4$ for Oberon, which was the highest dissipation that allowed for resonance capture in our simulations. First simulation was terminated once the resonance was broken at about 14.5 Myr, with average final inclination $i_U = 0'2$. We then ran another simulation continuing the orbital evolution (black line), but with $Q/k_2 = 10^4$ for Oberon. Lower dissipation allowed the spin–orbit resonance to be maintained at lower inclinations of Umbriel, with the final $i_U \approx 0'07$, close to its present value.

Figure 17. Obliquity of Oberon (black line) during a simulation of Ariel–Umbriel 5:3 MMR. Gray line plots the resonant obliquity in the absence of MMR (see Figure 14). Mean–motion resonance clearly induces chaotic dynamics in the precession of secular mode $I_3$, which in turn appears to make the obliquity of Oberon chaotic through the Oberon–I3 secular spin–orbit resonance.
MMR, the associated near-resonant frequency was retrograde, just like the Umbriel–Titania 2:1 inequality. Therefore, the combination of these two near-resonances would have a mean-motion argument of $6\lambda_U - 3\lambda_A - 2\lambda_T$, and being of the first order, it could potentially be much stronger than its more recent cousin. Figure 19 shows an integration using Uranian $Q/k_3 = 4 \times 10^4$ through this resonance (assuming initially low-e and low-i orbits for all the moons). The first-order three-body resonance is clearly stronger, and the moons spend a much longer time in the resonance. However, since the resonance is also much wider at low eccentricity than its third-order relative, there is chaotic overlap between the three subresonances that are each related to the eccentricity of Ariel, Umbriel, or Titania. This produces chaotic variations in eccentricity for all three moons. The resonance breaks when the eccentricity of Umbriel reaches about $e_U = 0.008$, with the eccentricities of all five moons being of same order (Miranda appears to be affected by secular perturbations of Ariel, while Oberon shares Titania’s secular modes).

In terms of capture and escape, this resonance has similarities to the Ariel–Umbriel 5:3 MMR, as it is chaotic and can be broken when the moons’ eccentricities reach certain values. This implies that, if there is strong enough tidal dissipation within moons and eccentricity is kept in check, this resonance could in principle last a very long time. Unlike in the case of the second-order Ariel–Umbriel 5:3 MMR, this three-body resonance does not affect inclinations appreciably, so there is no limit from growing inclinations on how long the residence in resonance can last. Figure 19 shows that, while the three-body resonance is active, tidal expansion of Umbriel’s orbit is enhanced and is very close to that of Miranda. Therefore, time spent in this three-body resonance is almost completely “invisible” to calculations of the timing of the past Miranda–Umbriel 3:1 MMR (see Figure 1). This has direct implications for the constraints we may be able to place on the tidal properties of Uranus based on past MMR crossings.

6. Discussion

6.1. Tidal Heating of Miranda

Miranda has dramatic surface features that indicate extensive geologic activity, some of which is relatively recent (Plescia 1987). According to crater counts, some of the most prominent surface features are large coronae. Two coronae, Arden and Inverness, appear to be only about 1 Gyr old, while others are significantly older, according to heliocentric impactor flux modeling (Zahnle et al. 2003; see also Section 6.3). A recent study of the morphology of these features, coupled with modeling of Miranda’s interior (Beddingfield et al. 2015) has found that heat flows of 30–100 mW m$^{-2}$ are necessary to produce the Arden corona. Can the excitation of Miranda’s eccentricity during the Ariel–Umbriel 5:3 resonance provide sufficient tidal heating for the formation of the young coronae?

In our simulations, we mostly focused on the overall dynamics of the Ariel–Umbriel resonance, which has major effects on Miranda’s orbit but is only weakly affected by Miranda itself. Therefore, we did not explore the phase space of Miranda’s tidal proprieties in great detail. However, we did run...
simulations, at 50 Myr, we changed Miranda’s tidal response to $Q/k_2 = 10^3$, similar to that of Enceladus (Lainey et al. 2012). Miranda’s average eccentricity becomes somewhat lower ($e_M \approx 0.02$) but remains chaotically excited, while the semimajor axis of Miranda shrinks due to strong eccentricity tides. There is no appreciable effect on Ariel and other moons, nor on Miranda’s inclination, which all behave the same as in other Ariel–Umbriel 5:3 MMR simulations.

Figure 20. Two simulations of the Ariel–Umbriel 5:3 MMR during which we greatly increased the rate of tidal dissipation within Miranda, as a proxy for internal melting. At the start of the simulations, we used $Q/k_2 = 2 \times 10^5$ for Uranus and $Q/k_2 = 10^3$ (black line) and $Q/k_2 = 10^6$ (gray line) for Miranda. In both simulations, at 50 Myr, we changed Miranda’s tidal response to $Q/k_2 = 100$, similar to that of Enceladus (Lainey et al. 2012). Miranda’s average eccentricity becomes somewhat lower ($e_M \approx 0.02$) but remains chaotically excited, while the semimajor axis of Miranda shrinks due to strong eccentricity tides. There is no appreciable effect on Ariel and other moons, nor on Miranda’s inclination, which all behave the same as in other Ariel–Umbriel 5:3 MMR simulations.

some integrations in which the tidal response of Miranda was greatly enhanced (to simulate melting, similar to what was done in Figures 5 and 6 for Ariel), in order to see whether there are effects on the Ariel–Umbriel resonance. It turns out the resonance stays practically unaffected, but there are major consequences for Miranda.

Figure 20 shows the orbital elements of Miranda and Ariel in two simulations of Ariel–Umbriel 5:3 MMR in which the tidal dissipation of Miranda was abruptly increased to an Enceladus-like value. Chaotic interaction between Ariel and Miranda continues, even though the eccentricity of Miranda now does not rise above $e_M = 0.02$. There is no visible change in the behavior of Ariel’s orbit. However, Miranda’s semimajor axis now drops relatively rapidly, due to tidal dissipation within Miranda (see Kaula 1964; Efroimsky & Makarov 2014). This semimajor axis drop has no effect on the excitation of Miranda’s eccentricity and inclination by Ariel or other moons, which is driven by secular dynamics. This is a very important result, as it offers a mechanism of tidal heating that is not limited by the power supplied by the outward tidal evolution rate of the moons (Peterson et al. 2015).

Figure 20 suggests that the semimajor axis of Miranda can decrease by about $0.01 R_U$ (0.2%) over 3 Myr (about $10^{14}$ s). Because $(E/E) = -(a/a)$ and the orbital energy of Miranda is about $1.4 \times 10^{27}$ J, the power lost from the orbit through tidal dissipation is about $3 \times 10^{11}$ W, yielding an energy flux of $0.4$ W m$^{-2}$, which is well in excess of the $31$–$112$ mW m$^{-2}$ required by Beddingfield et al. (2015) for formation of Arden Corona. Therefore, even with $Q/k_2 = 10^3$, Miranda would produce enough heat to explain the formation of the youngest coronae (note that Miranda’s average eccentricity during Ariel–Umbriel MMR would likely be higher for larger $Q/k_2$, enhancing the tidal power available). The tidal heating available from Miranda’s orbital shrinking due to eccentricity tides, ultimately caused by chaotic secular perturbations from other moons, is two orders of magnitude higher than that available from Miranda’s own tidal orbital expansion, even if an optimistic $Q = 11,000$ is assumed for Uranus. Therefore, we conclude that the Ariel–Umbriel 5:3 MMR is almost certainly the cause of the most recent episode of tidal heating on Miranda. Given that the coronae on Miranda have a range of ages (Zahnle et al. 2003), it is likely that there were other dynamical events preceding the Ariel–Umbriel 5:3 MMR that have caused major tidal heating episodes for Miranda.

6.2. Tidal Heating of Ariel

In contrast to Miranda’s freedom to migrate inward due to tidal dissipation, Ariel is the main driver of the mean–motion resonance that is exciting the moons’ orbits, and any inward movement of Ariel would break the resonance (as seen in Figure 5). Therefore, power available for the tidal heating of Ariel is limited to that available from tidal expansion of Ariel’s orbit, which is on the order of $1$ mW m$^{-2}$ (Peterson et al. 2015). Therefore, the Ariel–Umbriel 5:3 MMR is almost certainly not the root cause of geological activity on Ariel, which requires $28$–$92$ mW m$^{-2}$ (Peterson et al. 2015). Lack of significant heating of Ariel is consistent with the dynamical requirement that Ariel’s tidal response must be weak in order to allow Ariel and Umbriel to eventually exit their 5:3 MMR (Section 2).

None of the terrains on Ariel appear to be as recent as the youngest features on Miranda (Zahnle et al. 2003), which further supports our results that the last episode of tidal heating was likely caused by Ariel–Umbriel 5:3 MMR, and that it had significant geophysical consequences on Miranda but not Ariel. At this point, it is not clear what past dynamical mechanisms could have resulted in enough tidal heating to affect Ariel’s
The next two columns list the moons tidal Love numbers \( k_2 \), and the last column shows the moons perturbations, rather than mean surface, but our results for Miranda offer us some clues. Contrary to common expectation, it appears that secular perturbations, rather than mean-motion ones, may hold more promise in generating sufficient tidal heating. This is because secular resonances are not restricted to the power generated by orbital expansion due to tides, as mean-motion resonances are. The complexities of orbit-orbit and spin-orbit secular resonances we found in Sections 3 and 4 hint that more interesting dynamical behavior may be discovered once an even more ancient past of the Uranian system is studied using direct numerical integration.

### 6.3. Tidal Response of Uranus

In this paper, we presented theoretical reconstruction of part of the dynamical history of the Uranian system. In particular, we focused on matching the current orbital inclinations of the moons, which are much harder to change without resonances—unlike eccentricities, which are routinely damped by satellite tides. In this section, we will address the absolute chronology of the resonances we studied and its implication for Uranus’s tidal response, as well as what current eccentricities of the moons can tell us about their tidal responses.

Early in this project, we settled on a nominal Uranian tidal dissipation rate \( Q_p = 40,000 \) because it was immediately clear that Miranda’s inclination could be generated in the Ariel–Umbriel 5:3 MMR, meaning that the Miranda–Umbriel 3:1 MMR may not have happened. Miranda would likely accumulate excessive inclination if the moons crossed both of these resonances, especially if the Miranda–Umbriel 3:1 MMR was not broken at moderate \( i_M \) by a secondary resonance (Tittermore & Wisdom 1989; Malhotra & Dermott 1990). Peale (1988) found that the Miranda–Umbriel 3:1 MMR is crossed if \( Q_p < 3,900 \). Thus, we took \( Q_p = 40,000 \), the smallest tidal \( Q \) of Uranus for which that resonance is not crossed, as a good reference value for tidal \( Q \) of Uranus.

Our results completely change the above argument about Uranian \( Q_p \) based on resonance crossing. First of all, the calculation in Peale (1988) that finds \( Q_p = 39,000 \) as critical for Miranda–Umbriel 3:1 MMR crossing was based on a now-outdated mass of Miranda. By the same method we used to generate Figure 1, we calculate that \( Q_p \approx 25,000 \) puts this resonance at about 4.5 Gyr ago. Second, we find that the time the system spends in resonances that enhance the tidal evolution of Umbriel (hundreds of Myr in the Ariel–Umbriel 5:3 MMR and an unknown amount of time in the preceding three-body resonance) further lowers the tidal \( Q_p \) for which the system can avoid the Miranda–Umbriel 3:1 MMR. A Uranian tidal lag angle \( \delta \approx Q_p^{-1} \) directly proportional to synodic frequency (as opposed to a constant one used here) would also push 3:1 Miranda–Umbriel MMR further to the past, relative to Ariel–Umbriel resonances. However, by far the most important change to our understanding of past resonance crossings comes from the shrinking of Miranda’s orbit during the Ariel–Umbriel 5:3 MMR. Because this MMR likely lasted hundreds of Myr, Miranda’s orbit should have contracted by several percent, with tens of percent not being impossible if the highest dissipation rates were present throughout the duration of the MMR. Therefore, before the Ariel–Umbriel 5:3 MMR, Miranda was likely orbiting much farther outside the location of the 3:1 resonance with Umbriel. Therefore, Miranda’s current near-commensurability with Umbriel does not imply that their resonance would have been crossed in the past during Miranda’s outward tidal migration.

A more straightforward way of estimating Uranus’s tidal \( Q \) is by connecting the cratering ages of the moons’ surfaces to the past dynamical events such as resonance crossings. We already proposed that the young coronae on Miranda, Arden, and Inverness probably formed during the Ariel–Umbriel resonance crossing. The crater-derived age of these features depends on what assumption is made about the size-frequency distribution of cometary impactors (Zahnle et al. 2003). Recent results from the New Horizons mission (Singer et al. 2019) clearly indicate that cometary impactors have a shallower size-frequency distribution (i.e., one with fewer small members) than most prior models predicted. Therefore, we will adopt Zahnle et al. (2003) “Case A” model (which assumes fewer impactors at small sizes) to estimate the age of the Uranian satellite’s surfaces. This yields an age of about 1 Gyr (or a little less) for the Arden and Inverness coronae (Zahnle et al. 2003).

Given the tidal evolution rates of Ariel and Umbriel as a function of Uranus’s tidal \( Q \) (Table 4) and the fact that currently we have \( n_{U} - 3n_{A} = \Delta n = 2\pi (62d)^{-1} \) (where \( n \) is mean motion), for a very recent resonance like Ariel–Umbriel 5:3 MMR involving only 1% change in Ariel’s \( a \), a linear estimate of age from current tidal evolution rates is accurate enough to be useful:

\[
T[\text{Gyr}] \approx \left( \frac{Q_p}{40,000} \right)^2 \frac{2\Delta n}{9n_{A}} \left( \frac{a_A - a_U}{a_U} \right)^{-1},
\]
where the $\dot{a}$ are calculated for $Q_p = 40,000$ (Table 4), and we assumed $n_U/n_A \approx 3/5$. Thus, we can say that the Ariel–Umbriel 5:3 resonance would have happened 2.2 Gyr ago for our nominal $Q_p = 40,000$, and the resonance breaking at 1 Gyr ago requires $15,000 < Q_p < 20,000$. Further work on the modeling of heliocentric impacts on Uranian satellites could help improve on this estimate, and better determination of the masses of Ariel and Umbriel should further reduce uncertainty. This value of tidal $Q$ of Uranus is consistent with the Ariel–Umbriel 2:1 MMR having never been crossed, but further work is necessary to establish this. One should keep in mind that the cratering age uncertainties are significant, and also there is no guarantee that the tidal properties of Uranus did not change over time.

**6.4. Tidal Response of the Uranian Moons**

Having an estimate of the absolute age of the Ariel–Umbriel 5:3 MMR crossing enables us to say more about the tidal properties of the moons. Miranda’s current eccentricity of $e_M = 0.0013$ (Jacobson 2014) may require very fast freezing of Miranda once the Ariel–Umbriel resonance is broken, as Miranda’s $Q/k_2 = 10^5$ required for sufficient tidal heating also gives an eccentricity damping timescale of a couple Myr (Table 4). Miranda would have to transition to a rigid-body-like $Q/k_2 = 10^5 – 10^6$ (Table 4; we assume $Q = 100$ for the moons, in absence of better information) in a few Myr in order to preserve some of the eccentricity from the resonance. Alternatively, Miranda may have obtained eccentricity more recently, through the same three-body resonance as Ariel (Figure 18); more thorough examination of this resonance is needed, especially with very low pre-resonance eccentricities expected from tidal damping.

Ariel, or more correctly, secular mode E2, usually has a low eccentricity ($e_2 = 0.001–0.002$) at the end of the secular resonance with Umbriel shown in Figure 12. As the mode’s current amplitude is $e_2 = 0.001$, survival of ancient eccentricity would require $Q/k_2 > 10^5$ for Ariel, which would make Ariel exceptionally nondissipative. Fortunately, the three-body resonance with Umbriel and Titania (Figure 18), which has an age that is only 30% of that of the 5:3 MMR with Umbriel, significantly relaxes this constraint. For Ariel, $10^4 < Q/k_2 < 10^5$ would result in a tidal damping timescale of a couple hundred Myr, with about one damping timescale since the three-body resonance, so tidal damping could reconcile the simulation in Figure 18 with Ariel’s current eccentricity. The same $10^4 < Q/k_2 < 10^5$ for Umbriel would result in $\tau_e > 1$ Gyr, which is in agreement with a substantial fraction of Umbriel’s (i.e., mode E3) eccentricity surviving from the MMR (and the subsequent secular resonance) with Ariel.

In Figure 16, we show that $Q/k_2 = 10^4$ for Oberon can reproduce the amplitude of the secular inclination mode I3. Caveats include possible subsequent action of the Ariel–Umbriel secular resonance, as well as the dependence of spin–orbit resonance breaking on the obliquity of Oberon forced by the resonance, which in turn is a function of the exact masses of the satellites. In any case, just as it is for Ariel and Umbriel, the rough estimate of Oberon’s tidal Love number $k_2$ (with the admittedly naive assumption of $Q = 100$) is firmly in between the values expected for ice and rock bodies that are both uniform and rigid, using the approach of Murray & Dermott (1999). Therefore, our results suggest that Ariel, Umbriel, and Oberon have been fully solid bodies during and after the Ariel–Umbriel 5:3 MMR, or in absolute terms, for longer than 2 Gyr. This agrees with the calculation of Peterson et al. (2015) that the 5:3 MMR with Umbriel could not have mobilized ice on Ariel, and that Umbriel and Oberon would have experienced even less heating. Miranda, in contrast, must have been at least partially molten during the Ariel–Umbriel 5:3 MMR passage, but is also likely to be fully solid by now.

Titania offers us fewer direct dynamical constraints, as its tidal dissipation is relatively slow but not necessarily negligible. Unlike the inner three moons, which have eccentricities dominated by a single secular mode, Titania and Oberon share secular modes E4 and E5, and any damping of those two modes is likely to be driven primarily by Titania’s satellite tides (Table 4). Figure 7 suggests that the amplitudes of models E4 and E5 are a few times higher at the end of Ariel–Umbriel 5:3 MMR than they are now, potentially indicating some degree of damping. Table 4 suggests that the eccentricity damping timescale for Titania can be $\tau_e < 1$ Gyr only for $Q/k_2 = 10^3$, which is significantly lower than what we estimated for Oberon on the basis of its spin–orbit resonance.

This may indicate that the response of Titania to Uranus’s tides is in excess of that of a rigid body, and that Titania may have an internal ocean. A future mission to the Uranian system may be the only way to conclusively answer this question.

**7. Conclusions and Future Work**

In this paper, we used direct numerical integration to study the past Ariel–Umbriel 5:3 mean–motion resonance, as well as subsequent secular, spin–orbit, and three-body resonances. We find that prior analytical work, such as Tittemore & Wisdom (1988), was unable to fully capture the richness of dynamics of Uranian moons, which only becomes evident in a full numerical simulation. The Ariel–Umbriel 5:3 resonance is strongly chaotic and eventually breaks when the eccentricities of the two moons reach $e = 0.02–0.03$ range.

Our most important finding is that Miranda was strongly affected by the Ariel–Umbriel resonance, despite not being a direct participant, due to the strongly coupled secular dynamics of the Uranian moons (Laskar & Jacobson 1987). Chaotic secular interactions with Ariel while it is in the MMR lead to excitation of both the eccentricity and inclination of Miranda. The excitation of Miranda’s eccentricity is the likely cause of the most recent geological activity on Miranda, including the formation of Arden and Inverness coronae (Zahnle et al. 2003; Beddingfield et al. 2015). The excitation of inclination typically leaves Miranda with an orbital tilt of several degrees, naturally explaining its current $4^\circ.3$ inclination.

After the Ariel–Umbriel 5:3 MMR is broken, the system goes through a sequence of secular resonances among the moons’ secular modes. These secular resonances transfer eccentricity and inclination from the orbits of Ariel, Titania, and Oberon to that of Umbriel. We also find that the expected equilibrium shape of Oberon places the rate of its axial precession very close to a spin–orbit resonance with the secular mode I3, associated with the orbital tilt of Umbriel. This secular spin–orbit resonance is excited during the Ariel–Umbriel 5:3 MMR, and may continue once the MMR is broken. We find that joint action of secular resonances and Oberon’s precessional spin–orbit resonance can in principle explain the pattern of orbital inclinations of the four largest moons of Uranus.
More recently, the system went through a three-body resonance involving Ariel, Umbriel, and Titania. We propose that this resonance may have generated the current small eccentricities of Ariel and possibly Miranda. We find that there were other three-body resonances in the system’s past, which can make it more difficult to reconstruct the system’s dynamical history.

During the tidal heating of Miranda in the Ariel–Umbriel 5:3 MMR, Miranda’s semimajor axis experiences significant contraction. As a consequence, Miranda and Umbriel are unlikely to have crossed their mutual 3:1 resonance in the past, and the Ariel–Umbriel 5:3 MMR may be the sole source of Miranda’s inclination. Based on the published $\sim 1$ Gyr cratering ages of the young coronae on Miranda, we adopt this age for the Ariel–Umbriel 5:3 MMR crossing and estimate the tidal $Q$ of Uranus to be about $15,000 < Q < 20,000$, assuming $k_2 = 0.1$ (Gavrilov & Zharkov 1977).

Based on their current eccentricities and likely past resonant dynamics, we find that Ariel, Umbriel, and Oberon behave as rigid bodies and are unlikely to have had internal oceans during the Ariel–Umbriel 5:3 MMR crossing. Our results allow, and may even require, Titania to have (or have had) a subsurface ocean. If it exists, this ocean would not be a product of the resonances we studied, but rather would need to have been present beforehand.

Older coronae on Miranda, as well as terrains on Ariel that are not heavily cratered, appear to predate the Ariel–Umbriel 5:3 MMR, indicating a separate, more ancient dynamical event. This is likely to be a secular resonance of some kind acting on Ariel, as mean–motion resonances among Uranian moons do not have the power to cause large-scale melting (see Peterson et al. 2015). Our results suggest that an excitation of Ariel’s orbit would likely be passed on to Miranda, removing the need for separate heating events for Miranda. We hope that this paper shows that the dynamics of the Uranian system is far more complex that we previously thought, and that more work is needed to fully understand the orbital and geological history of the moons of Uranus.

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References
Beddingfield, C. B., Burr, D. M., & Emery, J. P. 2015, Icar, 247, 35
Carruba, V., Burns, J. A., Nicholson, P. D., & Gladman, B. J. 2002, Icar, 158, 434
Chambers, J. E., Quintana, E. V., Duncan, M. J., & Lissauer, J. J. 2002, AJ, 123, 2884
Chen, E. M. A., & Nimmo, F. 2016, Icar, 275, 132
Chen, E. M. A., Nimmo, F., & Glatzmaier, G. A. 2014, Icar, 229, 11
Chyba, C. F., Jankowski, D. G., & Nicholson, P. D. 1989, A&A, 219, L23
Colombo, G. 1966, AJ, 71, 895
Čuk, M., Dones, L., & Nesvorný, D. 2016a, ApJ, 820, 97
Čuk, M., Dones, L., Nesvorný, D., & Walsh, K. J. 2018, MNRAS, 481, 5411
Čuk, M., Hamilton, D. P., Lock, S. J., & Stewart, S. T. 2016b, Natur, 539, 402
Dermott, S. F., Malhotra, R., & Murray, C. D. 1988, Icar, 76, 295
Efroimsky, M., & Makarov, V. V. 2014, ApJ, 795, 6
Gallardo, T., Costo, L., & Badano, L. 2016, Icar, 274, 83
Garrick-Bethell, I., Wisdom, J., & Zuber, M. T. 2006, Sci, 313, 652
Gavrilov, S. V., & Zharkov, V. N. 1977, Icar, 32, 443
Gladman, B., Kavelaars, J., Holman, M., et al. 2000, Icar, 149, 370
Gladman, B. J., Nicholson, P. D., Burns, J. A., et al. 1998, Natur, 392, 897
Greenberg, R. 1975, MNRAS, 173, 121
Hamilton, D. P., & Ward, W. R. 2004, AJ, 128, 2510
Jacobson, R. A. 2007, AAS Meeting Abstracts, 39, 23.06
Jacobson, R. A. 2014, AJ, 148, 76
Jacobson, R. A., Campbell, J. K., Taylor, A. H., & Synnott, S. P. 1992, AJ, 103, 2064
Kaula, W. M. 1964, RvGSP, 2, 661
Kavelaars, J. J., Holman, M. J., Grav, T., et al. 2004, Icar, 169, 474
Lainey, V., Karaetkin, O., Desmars, J., et al. 2012, ApJ, 752, 14
Laskar, J., & Jacobson, R. A. 1987, A&A, 188, 212
Lissauer, J. J. 1985, IGR, 90, 11,289
Malhotra, R., & Dermott, S. F. 1990, Icar, 85, 444
Malhotra, R., Fox, K., Murray, C. D., & Nicholson, P. D. 1989, A&A, 221, 348
Millholland, S., & Laughlin, G. 2019, NatAs, 3, 424
Murray, C. D., & Dermott, S. F. 1999, Solar System Dynamics (Cambridge: Cambridge Univ. Press)
Nesvorný, D., Alvarelos, J. L. A., Dones, L., & Levison, H. F. 2003, AJ, 126, 398
Nesvorný, D., & Morbidelli, A. 1998, CeMDA, 71, 243
Peale, S. J. 1969, AJ, 74, 483
Peale, S. J. 1988, Icar, 74, 153
Peterson, G., Nimmo, F., & Schenk, P. 2015, Icar, 250, 116
Plescia, J. B. 1987, Natur, 327, 201
Quillen, A. C. 2011, MNRAS, 418, 1043
Showalter, M. R., & Lissauer, J. J. 2006, Sci, 311, 973
Singer, K. N., McKinnon, W. B., Gladman, B., et al. 2019, Sci, 363, 955
Smith, B. A., Soderblom, L. A., Beebe, R., et al. 1986, Sci, 233, 43
Thomas, P. C. 1988, Icar, 73, 427
Tittemore, W. C., & Wisdom, J. 1988, Icar, 74, 172
Tittemore, W. C., & Wisdom, J. 1989, Icar, 78, 63
Tittemore, W. C., & Wisdom, J. 1990, Icar, 85, 394
Touma, J., & Wisdom, J. 1993, Sci, 259, 1294
Touma, J., & Wisdom, J. 1994, AJ, 107, 1189
Verheylewegen, E., Noyelles, B., & Lemaître, A. 2013, MNRAS, 435, 1776
Vokrouhlický, D., Nesvorný, D., & Bottke, W. F. 2003, Natur, 425, 147
Ward, W. R. 1973, Sci, 181, 260
Ward, W. R. 1975, Sci, 189, 377
Ward, W. R., & Hamilton, D. P. 2004, AJ, 128, 2501
Wisdom, J., & Holman, M. 1991, AJ, 102, 1528
Zahnle, K., Schenk, P., Levison, H., & Dones, L. 2003, Icar, 163, 263