Linear Symmetries of the Unsquared Measurement Variety

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Abstract

We introduce a new family of algebraic varieties, \( L_{d,n} \), which we call the unsquared measurement varieties. This family is parameterized by a number of points \( n \) and a dimension \( d \). These varieties arise naturally from problems in rigidity theory and distance geometry. In those applications, it can be useful to understand the group of linear automorphisms of \( L_{d,n} \). Notably, a result of Regge implies that \( L_{2,4} \) has an unexpected linear automorphism. In this paper, we give a complete characterization of the linear automorphisms of \( L_{d,n} \) for all \( n \) and \( d \). We show, that apart from \( L_{2,4} \), the unsquared measurement varieties have no unexpected automorphisms. Moreover, for \( L_{2,4} \) we characterize the full automorphism group.

1 Introduction

Many questions in graph rigidity and distance geometry can be answered by studying an object called the squared measurement variety. Given a configuration \( p \) of \( n \geq d + 2 \) ordered points in \( \mathbb{R}^d \), we can measure the \( N := \binom{n}{2} \) ordered squared Euclidean distances between each pair of the points and consider this as a single point in \( \mathbb{R}^N \). (We associate each point in \( p \) with a vertex of the complete graph, \( K_n \), and each of the \( N \) vertex pairs with an edge of \( K_n \). Under this association, we refer to each of the \( N \) measurements as the squared distance of an edge.) If we take the union of the measurement points over all possible configurations of \( n \) points in \( \mathbb{R}^d \), we obtain a subset of \( \mathbb{R}^N \) which we call the Euclidean squared measurement set (of the complete graph \( K_n \)) denoted as \( M_{d,n}^E \). Under complexification and the Zariski-closure of this measurement set, we obtain a variety which we call the squared measurement variety (of the complete graph) denoted as \( M_{d,n} \subset \mathbb{C}^N \). (Formal definitions are below.) This has also been called the Cayley-Menger variety. This variety is linearly isomorphic to \( S_{d}^{n-1} \), the variety of complex, symmetric \((n-1) \times (n-1)\) matrices of rank \( d \) or less, which is a well understood variety.

In [5], Boutin and Kemper showed that one can uniquely reconstruct (up to congruence) a generic configuration \( p \) in \( \mathbb{R}^d \) given its \( N \) unlabeled pairwise squared distances. By unlabeled, we mean that we are not told which distance measurement corresponds to which pair of points. This central result has many applications in rigidity and distance geometry [7, 19]. The key to their result is showing that there are no permutations of the coordinate axes of \( \mathbb{C}^N \) (called edge permutations) that map \( M_{d,n} \) to itself, except for the permutations that are consistent with a permutation of the indices of the \( n \) points. Such permutations are said to be induced by a vertex relabeling.

One can, of course, expand the question and ask about all the non-singular linear maps on \( \mathbb{C}^N \) that map \( M_{d,n} \) to itself. We call these linear automorphisms of \( M_{d,n} \). Due to the linear relationship between \( M_{d,n} \) and \( S_{d}^{n-1} \), classifying the linear automorphisms of \( M_{d,n} \) boils down to looking at the linear automorphisms of \( S_{d}^{n-1} \). This is a classical question, and it is well known that the linear automorphisms of \( S_{d}^{n-1} \) are all linear maps with a “factored” form \( B^T G B \), where \( G \in S_{d}^{n-1} \) and \( B \) is any \((n-1) \times (n-1)\) non-singular matrix (see, e.g., [3]).
An even more general and daunting distance geometry problem is to reconstruct an \( n \) point configuration in \( \mathbb{R}^d \) given an unlabeled set of Euclidean lengths of \( N \) paths through the configuration \([10, 31]\). By path, we mean an ordered sequence of vertices, and we define its length to be the sum of the Euclidean lengths of each edge along the path. Importantly, a path-length is defined as a sum of Euclidean edge lengths, not a sum of squared Euclidean edge lengths. As such, the relevant algebraic variety to study should represent lengths instead of squared lengths. Also, in the unlabeled setting, we are not even given the information as to which combinatorial paths were measured in the first place. Thus we are not just concerned with coordinate permutations of \( \mathbb{C}^n \) but with more general linear maps acting on this variety (such as those arising from sums of lengths).

To this end, we define the squaring map \( s(\cdot) \) be the map from \( \mathbb{C}^N \) onto \( \mathbb{C}^N \) that acts by squaring each of the \( N \) coordinates of a point. We then define the unsquared measurement variety of \( n \) points in \( d \) dimensions, \( L_{d,n} \), as the preimage of \( M_{d,n} \) under the squaring map. (Each point in \( M_{d,n} \) has \( 2^N \) preimages in \( L_{d,n} \), arising through coordinate negations). We prove below (Theorem 2.11) that for \( d \geq 2 \), the variety \( L_{d,n} \) is irreducible (for \( d = 1 \) it is actually a reducible arrangement of linear subspaces).

In this paper, we wish to understand the set of linear automorphisms of \( L_{d,n} \), ie. the non-singular linear maps on \( \mathbb{C}^N \) that map \( L_{d,n} \) to itself. Any coordinate permutation that is induced by a vertex relabeling must be an automorphism. Also due to the squaring construction, any coordinate negation will be an automorphism. We call the group of automorphisms generated by vertex relabelings and coordinate the signed vertex relabelings.

By homogeneity, any uniform scale on \( \mathbb{C}^N \) will also be an automorphism. Let us call the group of automorphisms generated by signed vertex relabelings and uniforms scalings the expected automorphisms of \( L_{d,n} \).

We then ask: are there any “unexpected” linear automorphisms of \( L_{d,n} \)? Recall that \( M_{d,n} \) has many linear automorphisms that are not permutations of any type. By analogy, there is no a priori restriction on what the linear automorphisms of \( L_{d,n} \) can be.

In, fact, \( L_{2,4} \) does have an unexpected linear automorphism. Regge [27] (see also, Roberts [28]) showed that the following linear map always takes the Euclidean lengths \( l \) of the edges of any 4-point configuration in \( \mathbb{R}^2 \) to those, \( l' \), of some different 4-point configuration in \( \mathbb{R}^2 \).

\[
\begin{align*}
l'_{13} &= l_{13} \\
l'_{24} &= l_{24} \\
l'_{12} &= (-l_{12} + l_{23} + l_{34} + l_{14})/2 \\
l'_{23} &= (l_{12} - l_{23} + l_{34} + l_{14})/2 \\
l'_{34} &= (l_{12} + l_{23} - l_{34} + l_{14})/2 \\
l'_{14} &= (l_{12} + l_{23} + l_{34} - l_{14})/2
\end{align*}
\]

(*)

This “Regge symmetry” gives rise to an unexpected linear automorphism of \( L_{2,4} \). So the plot has thickened.

The first main result of this paper is that \( L_{2,4} \) is the only unsquared measurement variety with an unexpected linear automorphism.

**Theorem 1.1.** Let \( d \geq 1 \) and let \( n \geq d + 2 \). Assume that \( \{d, n\} \neq \{2, 4\} \). Then any linear automorphism \( A \) of \( L_{d,n} \) is a scalar multiple of a signed vertex relabeling.

This theorem is proven by combining the three cases proven below in Theorems 5.2, 5.7 and 5.27.
The second main result of this paper is to fully characterize the group of linear automorphisms of $L_{2,4}$. The details for this statement require a few definitions.

**Definition 1.2.** Define $\text{Aut}(L_{2,4})$ to be the linear automorphisms of $L_{2,4}$. Let the group $\mathbb{P} \text{Aut}(L_{2,4})$ be induced on the equivalence classes of $A \in \text{Aut}(L_{2,4})$ under the relation “$A'$ is a complex scale of $A$”.

We also consider the real subgroup $\text{Aut}_\mathbb{R}(L_{2,4})$. This has a counterpart $\mathbb{P} \text{Aut}_\mathbb{R}(L_{2,4})$ of equivalence classes up to real scale, and $\mathbb{P}_+ \text{Aut}_\mathbb{R}(L_{2,4})$, on equivalence classes defined up to positive scale. It is well-defined to refer to an element of $\mathbb{P}_+ \text{Aut}_\mathbb{R}(L_{2,4})$ as being non-negative, since any equivalence class containing a non-negative $A$ consists entirely of non-negative matrices.

**Theorem 1.3.** The group $\mathbb{P} \text{Aut}(L_{2,4})$ is of order $11520 = 768 \cdot 15$. It is generated by linear automorphisms that are represented by matrices with rational elements.

The group $\mathbb{P}_+ \text{Aut}_\mathbb{R}(L_{2,4})$ is of order 23040 and is isomorphic to the Weyl group $D_6$. The subset of non-negative elements of $\mathbb{P}_+ \text{Aut}_\mathbb{R}(L_{2,4})$ is a subgroup of order 24 and acts by relabeling the vertices of $K_4$.

(This is proven as Theorem 5.15 below.)

We will also see that The group $\mathbb{P}_+ \text{Aut}_\mathbb{R}(L_{2,4})$ is in fact generated by the edge permutations induced by vertex relabeling, sign flip matrices, and the one Regge symmetry of $(\ast)$.

Our proof of this second theorem is computer aided.

**Remark 1.4.** That $\mathbb{P}_+ \text{Aut}_\mathbb{R}(L_{2,4})$ contains a subgroup isomorphic to $D_6$ is based on conversations with Dylan Thruston (see [30]) and has antecedents in [8]. See [1, 32] for other geometric connections.

The central step for the proof of Theorem 1.1 is understanding which linear projection maps acting on $L_{d,n}$ can have deficient dimensions. This is done in Theorem 4.2 below. That result can also be of independent interest in unlabeled rigidity problems [10]. Additionally, in Appendix B, we study the large linear subspaces contained in $L_{2,4}$, which can also be of independent use in unlabeled rigidity [10].

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## 2 Measurement Varieties

We start by establishing our basic terminology. We relegate our needed definitions and theorems from algebraic geometry to Appendix A.

**Definition 2.1.** Fix positive integers $d$ and $n$. Throughout the paper, we will set $N := \binom{n}{2}$, $C := \binom{d+1}{2}$, and $D := \binom{d+2}{2}$.

These constants appear often because they are, respectively, the number of pairwise distances between $n$ points, the dimension of the group of congruences in $\mathbb{R}^d$, and the number of edges in a complete $K_{d+2}$ graph.
Definition 2.2. A configuration, \( \mathbf{p} = (p_1, \ldots, p_n) \) is a sequence of \( n \) points in \( \mathbb{R}^d \). (If we want to talk about points in \( \mathbb{C}^d \), we will explicitly call this a complex configuration.) The affine span of a configuration need not be all of \( \mathbb{R}^d \).

We think of the integers in \([1, \ldots, n]\) as the vertices of an abstract complete graph \( K_n \). An edge, \( \{i,j\} \), is an unordered distinct pair of vertices. The complete edge set of \( K_n \) has cardinality \( N \).

Fixing a configuration \( \mathbf{p} \) in \( \mathbb{R}^d \), we define the length of an edge \( \{i,j\} \) to be the Euclidean distance between the points \( p_i \) and \( p_j \), a real number.

Next we will study the basic properties of two related families of varieties, the squared and unsquared measurement varieties.

The squared variety is very well studied in the literature, but the unsquared variety is much less so. Since we are interested in integer sums of unsquared edge lengths, we wish to understand the structure of this unsquared variety.

Definition 2.3. Let us index the coordinates of \( \mathbb{C}^N \) as \( ij \), with \( i < j \) and both between \( 1 \) and \( n \). We also fix an ordering on the \( ij \) pairs to index the coordinates of \( \mathbb{C}^N \) as \( i \) with \( i \) between \( 1 \) and \( N \).\(^1\)

Let us begin with a complex configuration \( \mathbf{p} \) of \( n \) points in \( \mathbb{C}^d \) with \( d \geq 1 \). We will always assume \( n \geq d + 2 \). There are \( N \) vertex pairs (edges), along which we can measure the complex squared length as

\[
m_{ij}(\mathbf{p}) := \sum_{k=1}^{d} (p_{ki}^k - p_{kj}^k)^2
\]

where \( k \) indexes over the \( d \) dimension-coordinates. Here, we measure complex squared length using the complex square operation with no conjugation. We consider the vector \([m_{ij}(\mathbf{p})]\) over all of the vertex pairs, with \( i < j \), as a single point in \( \mathbb{C}^N \), which we denote as \( m(\mathbf{p}) \).

Definition 2.4. Let \( M_{d,n} \subset \mathbb{C}^N \) be the the image of \( m(\cdot) \) over all \( n \)-point complex configurations in \( \mathbb{C}^d \). We call this the squared measurement variety of \( n \) points in \( d \) dimensions.

When \( n \leq (d + 1) \), then \( M_{d,n} = \mathbb{C}^N \).

Definition 2.5. If we restrict the domain to be real configurations, then we call the image under \( m(\cdot) \) the Euclidean squared measurement set denoted as \( M^E_{d,n} \subset \mathbb{R}^N \). This set has real dimension \( dn - C \).

The following theorem reviews some basic facts. Most of the ideas are discussed in [2], but we include a detailed proof here for completeness and ease of reference.

Theorem 2.6. Let \( n \geq d + 2 \). The set \( M_{d,n} \) is linearly isomorphic to \( S^{n-1} \), the variety of complex, symmetric \( (n-1) \times (n-1) \) matrices of rank \( d \) or less. Thus, \( M_{d,n} \) is a variety. It is irreducible. Its dimension is \( dn - C \). Its singular set \( \text{Sing}(M_{d,n}) \) consists of squared measurements of configurations with affine spans of dimension strictly less than \( d \).

\(^1\)This ordering choice does not matter as long as we are consistent. It is there to let us switch between coordinates indexed by edges of \( K_n \) and indexed using flat vector notation. For \( n = 4, N = 6 \) we will use the order: \( 12, 13, 23, 14, 24, 34 \).
Proof. Such an isomorphism is developed in [33] and further, for example, in [13], see also [11, Section 7]. The basic idea is as follows. We can, w.l.o.g., translate the entire complex configuration \( p \) in \( \mathbb{C}^d \) such that the last point \( p_n \) is at the origin. We can then think of this as a configuration of \( n - 1 \) vectors in \( \mathbb{C}^d \). Any such complex configuration gives rise to a symmetric \((n - 1) \times (n - 1)\) complex Gram matrix (where no conjugation is used), \( G(p) \), of rank at most \( d \). Conversely, any symmetric complex matrix \( G \) of rank \( d \) or less can be (Tagaki) factorized, giving rise to a complex configuration of \( n - 1 \) vectors in \( \mathbb{C}^d \), which, along with the origin, gives us an \( n \)-point complex configuration \( p \) so that \( G = G(p) \).

With this in place, let \( \varphi \) be the invertible linear map from the space of \((n - 1) \times (n - 1)\) symmetric complex matrices \( G \), to \( \mathbb{C}^N \) (indexed by vertex pairs \( ij \), with \( i < j \)) defined as \( \varphi(G)_{ij} := G_{ii} + G_{jj} - 2G_{ij} \) (where \( G_{in} \) and \( G_{nj} \) is interpreted as 0). (For invertibility see [11, Lemma 7].)

When \( G = G(p) \) is the gram matrix of a complex configuration \( p \) in \( \mathbb{C}^d \), then \( \varphi(G) \) computes the squared edge lengths of \( p \). Since any symmetric matrix of rank at most \( d \) arises as the Gram matrix, \( G(p) \) from some complex configuration \( p \) in \( \mathbb{C}^d \), we see that the image of \( \varphi \) acting on \( S^{n-1}_d \), is contained in \( M_{d,n} \). Conversely, since every point in \( M_{d,n} \) arises from a complex configuration \( p \), and \( p \) gives rise to a Gram matrix \( G(p) \), we see that the image of \( \varphi \) acting on rank constrained matrices is onto \( M_{d,n} \). This gives us our isomorphism of varieties (Lemma A.4.)

Irreducibility of \( M_{d,n} \) follows from the fact that it is the image of an affine space (complex configuration space) under a polynomial (the squared-length map). The dimension follows from the dimension of \( S^{n-1}_d \) which is \( d(n - 1) - \binom{d}{2} \) (this is consistent with a degree of freedom count; see e.g., [17] for details).

For the description of the singular set of determinantal varieties of rank-constrained matrices, see for example [16, Page 184] (which can also be applied to the symmetric case). Meanwhile, we know that \( G = G(p) \) has rank \( < d \) iff \( p \) has a deficient affine span in \( \mathbb{C}^d \) (see for example [11, Lemma 26]). For an explicit statement about the singular set of \( M_{d,n} \), see [2, Proposition 4.5]. □

Remark 2.7. We note, but will not need, the following: For \( d \geq 1 \), the smallest complex variety containing \( M_{d,n}^\mathbb{R} \) is \( M_{d,n} \).

We note the following minimal instances where \( n = d + 2 \). In these cases, the variety has codimension 1.

The variety \( M_{1,3} \subset \mathbb{C}^3 \) is defined by the vanishing of the simplicial volume determinant, that is, the determinant of the following matrix

\[
\begin{pmatrix}
2m_{13} & (m_{13} + m_{23} - m_{12}) & \\
(m_{13} + m_{23} - m_{12}) & 2m_{23} & \\
&&
\end{pmatrix}
\]

where we use \((m_{12}, m_{13}, m_{23})\) to represent the coordinates of \( \mathbb{C}^3 \). This is the Gram matrix, \( \varphi^{-1}(m(p)) \), described in the proof of Theorem 2.6.

The variety \( M_{2,4} \subset \mathbb{C}^6 \) is defined by the vanishing of the determinant of the matrix

\[
\begin{pmatrix}
2m_{14} & (m_{14} + m_{24} - m_{12}) & (m_{14} + m_{34} - m_{13}) & \\
(m_{14} + m_{24} - m_{12}) & 2m_{24} & (m_{24} + m_{34} - m_{23}) & \\
& (m_{24} + m_{34} - m_{23}) & 2m_{34} & \\
&&&
\end{pmatrix}
\]

The variety \( M_{3,5} \subset \mathbb{C}^{10} \) is defined by the vanishing of the determinant of the matrix

\[
\begin{pmatrix}
2m_{15} & (m_{15} + m_{25} - m_{12}) & (m_{15} + m_{35} - m_{13}) & (m_{15} + m_{45} - m_{14}) & \\
(m_{15} + m_{25} - m_{12}) & 2m_{25} & (m_{25} + m_{35} - m_{23}) & (m_{25} + m_{45} - m_{24}) & \\
(m_{15} + m_{35} - m_{13}) & (m_{25} + m_{35} - m_{23}) & 2m_{35} & (m_{35} + m_{45} - m_{34}) & \\
(m_{15} + m_{45} - m_{14}) & (m_{25} + m_{45} - m_{24}) & (m_{35} + m_{45} - m_{34}) & 2m_{45} & \\
& & & &
\end{pmatrix}
\]
These same polynomial calculations can be done by constructing the Cayley-Menger determinants.

When \( n > d + 2 \), then \( M_{d,n} \) has higher codimension, and requires the simultaneous vanishing of more than one minor, characterizing the rank \( d \).

Next we move on to unsquared lengths.

**Definition 2.8.** Let the squaring map \( s(\cdot) \) be the map from \( \mathbb{C}^N \) onto \( \mathbb{C}^N \) that acts by squaring each of the \( N \) coordinates of a point. Let \( L_{d,n} \) be the preimage of \( M_{d,n} \) under the squaring map. (Each point in \( M_{d,n} \) has \( 2^N \) preimages in \( L_{d,n} \), arising through coordinate negations). We call this the unsquared measurement variety of \( n \) points in \( d \) dimensions.

**Definition 2.9.** We can define the Euclidean length map of a real configuration \( p \) as

\[
 l_{ij}(p) := \sqrt{\sum_{k=1}^{d} (p_i^k - p_j^k)^2 }
\]

where we use the positive square root. We call the image of \( p \) under \( l \) the Euclidean unsquared measurement set denoted as \( L_{d,n}^E \subset \mathbb{R}^N \). Under the squaring map, we get \( M_{d,n}^E \). We denote by \( l(p) \), the vector \( [l_{ij}(p)] \) over all vertex pairs. We may consider \( l(p) \) either as a point in the real valued \( L_{d,n}^E \) or as a point in the complex variety \( L_{d,n} \).

Indeed, \( L_{d,n}^E \) is the set we are often interested in for applications, but it will be easier to work with the whole variety \( L_{d,n} \).

**Remark 2.10.** The locus of \( L_{2,4} \) where the edge lengths of a triangle, \((l_{12}, l_{13}, l_{23})\), are held fixed is studied in beautiful detail in [6], where this is shown to be a Kummer surface.

The following theorem is the main result of this section.

**Theorem 2.11.** Let \( n \geq d + 2 \). \( L_{d,n} \) is a variety. It has pure dimension \( dn - C \). Assuming that \( d \geq 2 \), we also have the following: \( L_{d,n} \) is irreducible.

The proof is in the next subsection. The non-trivial part will be showing irreducibility, which we will do in Proposition 2.24 below. Indeed, in one dimension, the variety \( L_{1,3} \) is reducible.

**Remark 2.12.** We note, but will not need the following: For \( d \geq 2 \), the smallest complex variety containing \( L_{d,n}^E \) is \( L_{d,n} \).

Returning to our minimal examples: The variety \( L_{1,3} \subset \mathbb{C}^3 \) is defined by the vanishing of the determinant of the following matrix

\[
 \begin{vmatrix}
 2l_{13}^2 & (l_{13}^2 + l_{23}^2 - l_{12}^2) \\
 (l_{13}^2 + l_{23}^2 - l_{12}^2) & 2l_{23}^2 \\
 \end{vmatrix}
\]

where we use \((l_{12}, l_{13}, l_{23})\) to represent the coordinates of \( \mathbb{C}^3 \).

The variety \( L_{2,4} \subset \mathbb{C}^6 \) is defined by the vanishing of the determinant of the matrix

\[
 \begin{vmatrix}
 2l_{14}^2 & (l_{14}^2 + l_{24}^2 - l_{12}^2) & (l_{14}^2 + l_{24}^2 - l_{13}^2) \\
 (l_{14}^2 + l_{24}^2 - l_{12}^2) & 2l_{24}^2 & (l_{24}^2 + l_{34}^2 - l_{23}^2) \\
 (l_{14}^2 + l_{24}^2 - l_{13}^2) & (l_{24}^2 + l_{34}^2 - l_{23}^2) & 2l_{34}^2 \\
 \end{vmatrix}
\]

The variety \( L_{3,5} \subset \mathbb{C}^{10} \) is defined by the vanishing of the determinant of the matrix

\[
 \begin{vmatrix}
 2l_{15}^2 & (l_{15}^2 + l_{25}^2 - l_{12}^2) & (l_{15}^2 + l_{25}^2 - l_{13}^2) & (l_{15}^2 + l_{35}^2 - l_{14}^2) \\
 (l_{15}^2 + l_{25}^2 - l_{12}^2) & 2l_{25}^2 & (l_{25}^2 + l_{35}^2 - l_{23}^2) & (l_{25}^2 + l_{35}^2 - l_{24}^2) \\
 (l_{15}^2 + l_{25}^2 - l_{13}^2) & (l_{25}^2 + l_{35}^2 - l_{23}^2) & 2l_{35}^2 & (l_{35}^2 + l_{45}^2 - l_{34}^2) \\
 (l_{15}^2 + l_{35}^2 - l_{14}^2) & (l_{25}^2 + l_{35}^2 - l_{24}^2) & (l_{35}^2 + l_{45}^2 - l_{34}^2) & 2l_{45}^2 \\
 \end{vmatrix}
\]
Remark 2.13. It turns out that $L_{1,3}$ is reducible and consists of the four hyperspaces defined, respectively, by the vanishing of one of the following equations:

\[
\begin{align*}
&l_{12} + l_{23} - l_{13} \\
&l_{12} - l_{23} + l_{13} \\
&-l_{12} + l_{23} + l_{13} \\
&l_{12} + l_{23} + l_{13}
\end{align*}
\]

This reducibility can make the one-dimensional case quite different from dimensions 2 and 3.

Notice that the first octant of the real locus of 3 of these hyperspaces arises as the Euclidean lengths of a triangle in $\mathbb{R}^1$ (that is, these make up $L^R_{1,3}$). The specific hyperplane is determined by the order of the 3 points on the line.

2.1 Proof

We will now develop the proof of Theorem 2.11. The main issue will be proving the irreducibility of $L_{d,n}$. The special case of $n = d + 2$ follows from [9], but we are interested in the general case, $n \geq d + 2$. The basic idea we will use is that a variety whose smooth locus is connected must be irreducible. More specifically, our strategy is to define a “good” locus of points in $L_{d,n}$, and show that this locus is connected, made up of smooth points, and with its Zariski closure equal to $L_{d,n}$. This, along with Theorem A.9, will prove irreducibility. Note that when the word “Zariski” is not attached to a topological term, you can interpret the term in the standard topology.

We will show connectivity using a specific path construction. This will rely centrally on the complex setting that we have placed ourselves in. Showing (algebraic) smoothness will mostly be a technical matter.

Definition 2.14. Let the zero locus $Z$ of $\mathbb{C}^N$ be the points where at least one coordinate vanishes. Let the bad locus $\text{Bad}(M_{d,n})$ of $M_{d,n}$ be the union of its singular locus $\text{Sing}(M_{d,n})$ together with the points in $M_{d,n}$ that are in $Z$. We will call the remaining locus $\text{Good}(M_{d,n})$ good.

Let the bad locus $\text{Bad}(L_{d,n})$ of $L_{d,n}$ be the preimage of the bad locus of $M_{d,n}$ under the squaring map $s$. We will call the remaining locus $\text{Good}(L_{d,n})$ good.

We refer to points on the good locus as good points, and analogously for bad points.

Lemma 2.15. $\text{Good}(M_{d,n})$ is path-connected.
Proof. Let \( m_1 \) and \( m_2 \) be any two good points in \( M_{d,n} \). These correspond to two configurations \( p \) and \( q \). A path in configuration space, connecting \( p \) to \( q \), will remain, under \( m(\cdot) \), on \( \text{Good}(M_{d,n}) \) when the affine span of the configuration does not drop in dimension, and no edge between any two points has zero squared length. This can always be done, as we have \( n \geq d + 2 \) points. (This is even true for one-dimensional configurations, in the complex setting, as a zero squared length is a condition that has complex-codimension of at least 1, and thus the bad locus is non-separating.) \( \square \)

We next record a lemma that follows from basic results of covering space theory. See [26, Sections 53, 54] for more details.

Definition 2.16. A path \( \tau \) on a space \( X \) is a continuous map from the unit interval to \( X \). A loop is a path with \( \tau(0) = \tau(1) \). Let \( p \) be a map from a space \( \tilde{X} \) to \( X \). A lift \( \tilde{\tau} \) of \( \tau \) (under \( p \)) is a map such that \( p(\tilde{\tau}) = \tau \). It is a path on \( \tilde{X} \).

Intuitively, a lift is just tracing out the path \( \tau \) in the preimage through \( p \). In what follows, \( \mathbb{C}^\times \) is the punctured complex plane.

Lemma 2.17. Let \( p \) be the map \( \mathbb{C}^\times \rightarrow \mathbb{C}^\times \) given by \( z \mapsto z^2 \). Let \( x := p(z) \). A loop \( \tau \) starting at \( x \) uniquely lifts to a loop \( \tilde{\tau} \) starting at \( z \) if \( \tau \) winds around the origin an even number of times, and otherwise it lifts to a path that ends at \( -z \).

Proof Sketch. See [26, Chapters 53, 54] for definitions. The map \( \mathbb{C}^\times \rightarrow \mathbb{C}^\times \) given by \( z \mapsto z^2 \) is a covering map. Call the base \( B \) and the cover \( F \) and the covering map \( p \). Each loop \( \tau \) in \( B \), starting at \( x \), lifts uniquely to a path \( \tilde{\tau} \) in \( F \), starting at \( z \). The path \( \tilde{\tau} \) ends at a uniquely defined point \( z' \in p^{-1}(x) \) under the lifting correspondence. In our case the fiber is \( \{z, -z\} \). Moreover every \( z' \) in the fiber can be reached under the lifting of some loop \( \tau \) (see [26, Theorem 54.4]).

The fundamental group of the base is \( \pi_1(B) = \pi_1(\mathbb{C}^\times) \cong \mathbb{Z} \). The covering map determines an induced map \( p_\ast : \pi_1(F) \rightarrow \pi_1(B) \). The image of the induced map consists of loops that wind around the origin an even number of times in \( F \) so it is isomorphic to \( 2\mathbb{Z} \). The lifting correspondence induces a bijective map from the group \( \pi_1(B)/p_\ast(\pi_1(F)) \cong \mathbb{Z}_2 \) to the fiber above \( x \), and (only) loops in \( p_\ast(\pi_1(F)) \) lift to loops in \( F \). (see [26, Theorem 54.6]).

Thus, this lift, starting from \( z \), is a path from \( z \) to \( -z \) if and only if \( \tau \) winds around the origin an odd number of times. \( \square \)

Looking at the product space \( (\mathbb{C}^\times)^N \), we can also view the squaring map \( s \) as a covering map mapping this product space to itself, and we can apply Lemma 2.17 coordinate-wise.
Lemma 2.18. Assume $d \geq 2$. Suppose $l$ and $l'$ are two points in $L_{d,n}$ that differ only by a negation along one coordinate. Then, there is a path that connects $l$ to $l'$ and stays in $\text{Good}(L_{d,n})$.

Proof. W.l.o.g., we will negate the coordinate corresponding to the edge lengths between vertices 1 and 2. But first, we need to develop a little gadget.

Let $q$ be a special configuration with the following properties: $q_1$ is at the origin, $q_2$ is placed one unit along the first axis of $\mathbb{C}^d$; and the remaining points are arranged so that they all lie within $\epsilon$ of the second axis in $\mathbb{C}^d$, but such that they are greater than one unit apart along the second axis from each other and also from $q_1$. (Note that this step requires that $d \geq 2$.) Moreover we choose the remaining points so that $q$ has a full $d$-dimensional affine span. This configuration has the following property: the squared distances of all of the edges are dominated by the contribution from the second coordinate, except for the squared distance along the edge ${1, 2}$, which is dominated by the contribution from its first coordinate. See Figure 2.

Let $a(t)$ be the path in configuration space, parameterized by $t \in [0, \pi]$ where, for each $i$, we multiply the first coordinate of $q_i$ by $e^{-t\sqrt{-1}}$. This path ends at $a(\pi)$, a configuration which is a reflection of $q$. Under $m$, this gives us a loop $\tau := m(a)$ in $M_{d,n}$ that starts and ends at the point $y := m(q)$. By construction, the loop $\tau$ avoids any singularities or vanishing coordinates. Fixing one point $z$ in $s^{-1}(y)$, the loop $\tau$ lifts to a path $\tilde{\tau}$ in $L_{d,n}$ that ends at some point $z'$ in the fiber $s^{-1}(y)$. Moreover, this path remains in $\text{Good}(L_{d,n})$.

If we project $\tau$ onto the coordinate of $\mathbb{C}^N$ corresponding to the edge ${1, 2}$, we see that the image maps to a loop that winds around the origin of $\mathbb{C}$ exactly once. If we project this loop onto any of the other coordinates, we obtain a loop that cannot wind about the origin of $\mathbb{C}$ at all. See Figure 3. By Lemma 2.17, the lifted loop $\tilde{\tau}$ in $L_{d,n}$ must end at the point $z'$ that arises from $z$ by negating the first coordinate.

Going now back to our problem, let $p$ be any configuration such that $m(p) = s(l)$. Let $w$ be a configuration path from $p$ to our special $q$. Let $\omega := m(w)$. From Lemma 2.15 this path can be chosen to avoid any singular points or points where a coordinate vanishes. Let the concatenated path $\sigma$ be $\omega^{-1} o \tau o \omega$. This is a loop in $M_{d,n}$ that starts and ends at $m(p)$. The projection of $\sigma$ onto the coordinate of $\mathbb{C}^N$ corresponding to the edge ${1, 2}$, defined by forgetting all other coordinates, winds around the origin exactly once (any loops due to $\omega$ cancel out), while the other coordinate

\[ \int d\xi^2 \] complex squared distance on edge $\{1, 2\}$

\[ \int dy^2 \] complex squared distance on other edge

Figure 3: Since the squared length along edge $\{1, 2\}$ arises from its $x$ component, our path along this edge measurement winds once about the origin in $\mathbb{C}$. For any other edge, the $x$ component of the squared distance is dominated by the other coordinates and the resulting path stays far from the origin in $\mathbb{C}$.
projections are simply connected in $\mathbb{C}^\times$ (any loops due to $\omega$ cancel out). Thus, fixing the point $l$ in $L_{d,n}$, from Lemma 2.17, $\sigma$ must lift to a path $\tilde{\sigma}$ that ends at $\gamma$. Moreover, this path stays in the good locus.

**Lemma 2.19.** For $d \geq 2$, $\text{Good}(L_{d,n})$ is path-connected.

**Proof.** Let $l_1$ and $l_2$ be two good points in $\text{Good}(L_{d,n})$. Define $m_i := s(l_i)$. Let $\tau$ be a path in $M_{d,n}$ from $m_1$ to $m_2$ that avoids the singular set of $M_{d,n}$, and such that no coordinate ever vanishes (as guaranteed by 2.15). Fixing $l_1$, the path $\tau$ lifts to a path $\tilde{\tau}$ in $L_{d,n}$ that remains in the good locus and that connects $l_1$ to some point $l'_2$ in the fiber $s^{-1}(s(l_2))$. The only remaining issue is that $l'_2$ may have some of its coordinates negated from our desired target point $l_2$. This can be solved by repeatedly applying the good negating paths guaranteed by Lemma 2.18. 

We now move on to the technical matters of smoothness.

**Lemma 2.20.** Every point $l \in \text{Good}(L_{d,n})$ is smooth and with $\text{Dim}_l(L_{d,n}) = dn - C$. Every point in $\text{Bad}(L_{d,n}) - Z$ is singular.

**Proof.** Every good point in $M_{d,n}$ is (algebraically) smooth, and thus, from Theorem A.11, is analytically smooth of dimension $dn - C$. Also, from Theorem A.11, every singular point in $M_{d,n}$ is not analytically smooth.

The differential $ds$ of the squaring map $s$ on $\mathbb{C}^N$ is represented by an $N \times N$ Jacobian matrix $J$ at each point in $\mathbb{C}^N$. At points in $\mathbb{C}^N$ where none of the coordinates vanish, $J$ is invertible. Thus, from the inverse function theorem, every good point in $L_{d,n}$ is analytically smooth of dimension $dn - C$. Also every bad point in $L_{d,n} - Z$ is not analytically smooth.

Again using Theorem A.11, we have each good point (algebraically) smooth and with $\text{Dim}_l(L_{d,n}) = dn - C$. Similarly, we also have that every bad point in $L_{d,n} - Z$ is singular.

Note that there may be some bad points of $L_{d,n}$ in $Z$ that are still smooth.

**Remark 2.21.** The above lemma can be proven directly using more machinery from algebraic geometry. In particular, away from $Z$, the squaring map from $\mathbb{C}^N$ to itself is an “étale morphism” [24, page 18]. This property transfers to the map $s(\cdot)$ acting on $L_{d,n} - Z$, as this property transfers under a “base change”. The results then follow immediately.

**Lemma 2.22.** The Zariski closure of $\text{Good}(L_{d,n})$ is $L_{d,n}$.

**Proof.** Recall the following principle: Given any point $z$ in $\mathbb{C}^\times$, we can always find a neighborhood $B$ of $z^2$, so that there is a well defined, single valued, continuous square root function from $B$ to $\mathbb{C}$, with $\sqrt{z^2} = z$.

Returning to our setting, let $l$ be any point in $L_{d,n}$, and let $m := s(l)$ be its image in $M_{d,n}$ under the coordinate squaring map. The good points of $M_{d,n}$ are dense in $M_{d,n}$. (Letting $m = m(p)$ for some $p$, there is always a nearby configuration $p'$ with a full span and no edge with vanishing squared length. Moreover, the map $m(\cdot)$ is continuous.) Thus we can always find an arbitrarily close point $m'$ that is in $\text{Good}(M_{d,n})$.

Next we argue that we can find a point $l'$ such that $s(l') = m'$ (putting it in $\text{Good}(L_{d,n})$) with $l'$ is arbitrarily close to $l$. Given $m'$, in order to determine $l'$ we need to select a “sign” for the square-root on each coordinate $ij$. When $l_{ij} \neq 0$ then using the above principle, we can pick a sign so that $l'_{ij}$ is near to $l_{ij}$. When $l_{ij} = 0$ then we can use any sign to obtain an $l'_{ij}$ that is sufficiently close to 0.
Since this can be done for each \( l \), then \( L_{d,n} \) is in the standard-topology closure of \( \text{Good}(L_{d,n}) \). Thus, from Theorem A.3, \( L_{d,n} \) is in the Zariski closure of \( \text{Good}(L_{d,n}) \). Since \( L_{d,n} \) itself is closed and contains \( \text{Good}(L_{d,n}) \), we are done.

\[
\square
\]

**Lemma 2.23.** Every component of \( L_{d,n} \) is of dimension equal to \( dn - C \).

**Proof.** From Lemma 2.20 each good point has a local dimension of \( dn - C \). Thus, the good locus is covered by a set of components of \( L_{d,n} \), all of dimension \( dn - C \). The Zariski closure of \( \text{Good}(L_{d,n}) \) is \( L_{d,n} \) (Lemma 2.22). Thus, no new components need to be added during the Zariski closure. \( \square \)

We can now prove irreducibility.

**Proposition 2.24.** For \( d \geq 2 \), \( L_{d,n} \) is irreducible.

**Proof.** From Lemma 2.20, all of the points in \( \text{Good}(L_{d,n}) \) are smooth. From Lemma 2.19, \( \text{Good}(L_{d,n}) \) is path-connected, and thus connected as a subspace of \( \mathbb{C}^n \).

Now we show that all of \( \text{Good}(L_{d,n}) \) lies in a single irreducible component \( V \) of \( L_{d,n} \). Fix an irreducible component \( V \), such that \( G = \text{Good}(L_{d,n}) \cap V \) is non-empty. Notice that \( G \) is a closed subspace of \( \text{Good}(L_{d,n}) \) (Theorem A.3). Now let \( W \) be the union of all the remaining irreducible components of \( L_{d,n} \). By similar reasoning \( H = W \cap \text{Good}(L_{d,n}) \) is closed in \( \text{Good}(L_{d,n}) \).

From Theorem A.9, \( G \) and \( H \) are disjoint. On the other hand, \( V \cup W = L_{d,n} \), so \( G \cup H = \text{Good}(L_{d,n}) \). Because \( \text{Good}(L_{d,n}) \) is connected and \( G \) is closed and not empty, its complement \( H \) must be empty to be closed. Hence, \( G = \text{Good}(L_{d,n}) \).

To finish the proof, recall that Lemma 2.22 says that the Zariski closure of \( \text{Good}(L_{d,n}) \) is \( L_{d,n} \). This closure must be contained in any variety, such as \( V \), that contains \( \text{Good}(L_{d,n}) \). Since we also have \( V \subseteq L_{d,n} \), equality holds and we get irreducibility. \( \square \)

And now we can complete the proof of our theorem:

**Proof of Theorem 2.11.** \( L_{d,n} \) can be seen to be a variety by pulling back the defining equations of the variety \( M_{d,n} \) through \( s \). Dimension is Lemma 2.23. Irreducibility is Proposition 2.24. \( \square \)

### 3 Automorphisms of \( M_{d,n} \)

**Definition 3.1.** A linear automorphism of a variety \( V \) in \( \mathbb{C}^N \) is a non-singular linear transform on \( \mathbb{C}^N \) (that is, a non-singular \( N \times N \) complex matrix \( A \)) that bijectively maps \( V \) to itself.\(^2\)

**Definition 3.2.** An \( N \times N \) matrix \( P \) is a permutation if each row and column has a single non-zero entry, and this entry is 1. A matrix \( P' = DP \), where \( D \) is diagonal and invertible, is a generalized permutation. Each row and column has exactly one non-zero entry. A generalized permutation has uniform scale if it is a scalar multiple of a permutation matrix.

**Definition 3.3.** A generalized permutation acting on an edge set is induced by a vertex relabeling when it has the same non-zero pattern as an edge permutation that arises from a vertex relabeling.

We now present the following slight generalization of [5, Lemma 2.4]. Here we deal with generalized permutations instead of permutations, but the same proof applies.

\(^2\)In our setting, \( V \) will always be a cone, so linear isomorphisms (as opposed to affine ones) are natural.
**Theorem 3.4** ([5, Lemma 2.4]). Suppose that \( A \) is a generalized permutation that is a linear automorphism of \( M_{d,n} \). Then \( A \) is induced by a vertex relabeling.

The following material will help us slightly strengthen Theorem 3.4, and will also be used later in Section 4.

First we define the combinatorial notion of infinitesimally dependent and independent sets of edges in \( d \) dimensions.

**Definition 3.5.** Let \( d \) be some fixed dimension and \( n \) a number of vertices. Let \( E := \{E_1, \ldots, E_k\} \) be an ordered subset of the \( N \) edges. The ordering on the edges of \( E \) fixes an association between each edge in \( E \) and a coordinate axis of \( \mathbb{C}^k \). Let \( m_E(p) \) be the map from \( d \)-dimensional configuration space to \( \mathbb{C}^k \) measuring the squared lengths of the edges of \( E \).

We denote by \( \pi_E \) the linear map from \( \mathbb{C}^N \) to \( \mathbb{C}^k \) that forgets the edges not in \( E \), and is consistent with the ordering of \( E \). Specifically, we have an association between each edge of \( K_n \) and an index in \( \{1, \ldots, N\} \), and thus we can think of each \( E_i \) as simply its index in \( \{1, \ldots, N\} \). Then, \( \pi_E \) is defined by the conditions: \( \pi_E(e_j) = 0 \) when \( j \in E \) and \( \pi_E(e_j) = e'_j \) when \( E_i = j \), where \( \{e_1, \ldots, e_N\} \) denotes the coordinate basis for \( \mathbb{C}^N \) and \( \{e'_1, \ldots, e'_k\} \) denotes the coordinate basis for \( \mathbb{C}^k \). We call \( \pi_E \) an edge forgetting map.

With this notation, the map \( m_E(\cdot) \) is simply the composition of the complex measurement map \( m(\cdot) \) and \( \pi_E \).

**Definition 3.6.** We say the an edge set \( E \) is infinitesimally independent in \( d \) dimensions if there exists a complex configuration \( p \) in \( \mathbb{C}^d \), where we can differentially vary each of the \( |E| \) squared lengths independently, by appropriately differentially varying our configuration \( p \).

Formally, this means that the image of the differential of \( m_E(\cdot) \) at \( p \) is \( |E| \)-dimensional. This exactly coincides with the notion of infinitesimal independence from graph rigidity theory [20].

We call such a configuration \( p \), \( E \)-regular. Every configuration in some appropriate neighborhood of an \( E \)-regular point is also \( E \)-regular (by semi-continuity). This neighborhood must include configurations with full affine spans and no coincident points.

For any configuration \( p \) with full affine span, \( m(p) \) is smooth (Theorem 2.6). Thus for any \( E \)-regular configuration \( p \), with full affine span, using the chain rule, the differential image of \( \pi_E \) at the point \( m(p) \) is \( |E| \)-dimensional. We call such a point of \( M_{d,n}, E \)-regular. Such points must exist when \( E \) is infinitesimally independent.

For any smooth point \( x \) of \( M_{d,n} \) with no zero coordinates, all of its preimages under the squaring map, \( s(\cdot) \), are smooth in \( L_{d,n} \) (Lemma 2.20). Thus for any preimage \( l \) of an \( E \)-regular point \( x \), with no zero coordinates, the differential image of \( \pi_E \) at the point \( l \) is \( |E| \)-dimensional. (As the Jacobian of \( s(\cdot) \) at \( l \) is diagonal and bijective.) We call such a point of \( L_{d,n}, E \)-regular. Such points must exist when \( E \) is infinitesimally independent.

An edge set that is not infinitesimally independent in \( d \) dimensions is called infinitesimally dependent in \( d \) dimensions.

The following is a standard result from rigidity theory (see, e.g., [14, Corollary 2.6.2]).

**Proposition 3.7.** Let \( E \) be an edge set (with all its edges distinct). Suppose \( |E| \leq \binom{d+2}{2} \) and \( E \) is infinitesimally dependent in \( d \) dimensions. Then \( |E| = \binom{d+2}{2} \) and \( E \) consists of the edges of a \( K_{d+2} \) subgraph (in some order).

**Proof Sketch.** Assume, w.l.o.g., that \( E \) is infinitesimally dependent and inclusion-wise minimal with this property. If \( E \) does not consist of the edges of a \( K_{d+2} \) subgraph, then it has a vertex \( v \) of degree at most \( d \). Let \( p \) be in general affine position. This means, in particular, that \( p_v \) is not in the affine
span of its neighbors. Hence, the \( \leq d \) squared lengths of each edge in edge set \( E' \) incident on \( v \) can be differentially varied independently (by exercising the \( d \) degrees of freedom in \( p_v \)). Thus the edges of \( E' \) can be removed from \( E \) leaving the remainder, \( E \setminus E' \), still infinitesimally dependent. This contradicts the assumed minimality of \( E \). \( \square \)

**Lemma 3.8.** Any linear automorphism \( A \) of \( M_{d,n} \) is a linear automorphism of \( M_{1,n} \).

**Proof.** The singular set of \( M_{d,n} \) is \( M_{d,n-1} \) by Theorem 2.6. Thus, from Theorem A.8, \( A \) must be a linear automorphism of \( M_{d-1,n} \). We then see, by induction, that \( A \) is also a linear automorphism of \( M_{1,n} \). \( \square \)

In fact, this kind of induction has been recently used to greatly strengthen Boutin and Kemper’s unique reconstructability result [5] to apply to a much larger class of graphs than just the complete graphs [12].

**Lemma 3.9.** Let \( m_{12}, m_{13} \) and \( m_{23} \) be the squared edge lengths of a 1-dimensional triangle, and suppose that \( s_{12}, s_{13} \) and \( s_{23} \) are scalars such that the simplicial volume determinant

\[
\det \begin{pmatrix}
2m_{13} & (m_{13} + m_{23} - m_{12}) \\
(m_{13} + m_{23} - m_{12}) & 2m_{23}
\end{pmatrix} = 2(m_{12}m_{13} + m_{12}m_{23} + m_{13}m_{23}) - (m_{12}^2 + m_{13}^2 + m_{23}^2)
\]

(see Section 2) is mapped to a multiple of itself under the scaling \( m_{ij} \mapsto s_{ij}m_{ij} \). Then the \( s_{ij} \) are all equal.

**Proof.** The hypothesis means that the desired statement holds for any specialization of the \( m_{ij} \). Consider the case where \( m_{23} = 0 \). The presence of the monomials \( m_{12}^2 \) and \( m_{12}m_{13} \) then imply that \( s_{12}^2 = s_{12}s_{13} \), that is, \( s_{12} = s_{13} \). Continuing the same way, we see that \( s_{12} = s_{23} \). \( \square \)

Now we can state the following slight strengthening of Theorem 3.4.

**Theorem 3.10.** Suppose that \( A \) is a generalized permutation that is a linear automorphism of \( M_{d,n} \). Then \( A \) is induced by a vertex relabeling and has uniform scale.

**Proof.** Theorem 3.4 tells us that \( A \) is induced by a vertex relabeling. Next we need to prove uniform scale. From Lemma 3.8, we can look at \( A \) as an automorphism of \( M_{1,n} \).

Let \( \pi_K \) be an edge forgetting map that ignores all of the edges in the complement of an edge set \( K \), consisting of the edges of a fixed triangle. Under any ordering of the edges of \( K \), we have \( \pi_K(M_{1,n}) = M_{1,3} \). (which is cut out from \( \mathbb{C}^3 \) by the simplicial volume determinant as in Lemma 3.9).

We know that we can factor \( A \) into \( DP \), where \( D \) is diagonal and \( P \) is a permutation induced by a vertex relabeling. Since a vertex relabeling is a linear automorphism of \( M_{1,n} \), then so too is \( D \).

Since \( D \) is diagonal, and \( \pi_K \) is an edge forgetting map, then \( \pi_K D \) is \( D' \pi_K \) for an appropriate \( 3 \times 3 \) diagonal scaling matrix \( D' \), making \( D' \) an automorphism of \( M_{1,3} \). So it has to send the simplicial volume determinant to a multiple of itself. This is the situation of Lemma 3.9, and we conclude that the scaling on each triangle is uniform.

That \( A \) has a uniform scale then follows from applying the above argument repeatedly to overlapping triangles until we have determined the scale on every edge. \( \square \)
4 Linear maps from $L_{d,n}$ to $\mathbb{C}^D$

In this section, which forms the technical heart of this paper, we will study how linear projections act on $L_{d,n}$.

Let $d \geq 1$. Recall that $D := \binom{d+2}{2}$. In this section, $E$ will be a $D \times N$ matrix representing a rank $r$ linear map from $L_{d,n}$ to $\mathbb{C}^D$, where $r$ is some number $\leq D$. Our goal is to study linear maps where the dimension of the image is strictly less than $r$. In particular this will occur when $E(L_{d,n}) = L_{d,d+2}$.

**Definition 4.1.** We say that $E$ has $K_{d+2}$ support if it depends only on measurements supported over the $D$ edges corresponding to a $K_{d+2}$ subgraph of $K_n$. Specifically, all the columns of the matrix $E$ are zero, except for at most $D$ of them, and these non-zero columns index edges contained within a single $K_{d+2}$.

The main result of this section is:

**Theorem 4.2.** Let $E$ be a $D \times N$ matrix with rank $r$. Suppose that the image $E(L_{d,n})$, a constructible set, is not of dimension $r$. Then $r = D$ and $E$ has $K_{d+2}$ support.

**Remark 4.3.** Theorem 4.2 does not hold when $L_{d,n}$ is replaced by $M_{d,n}$. As described in the introduction, the linear automorphism group of $S_{n-1}^d$ is quite large, and thus provides automorphisms $A$ of $M_{d,n}$ that have dense support. Thus, even if some $E$ has $K_{d+2}$ support the composite map $EA$ would not, and it could still have a low-dimensional image.

The proof relies (crucially) on the more technical, linear-algebraic Proposition 4.4, proved below. The idea leading to it is as follows.

If a point $l$ is smooth in $L_{d,n}$ then so is any $l'$ obtained by negating various coordinates of $l$. Thus, the collection of complex analytic tangent spaces to $L_{d,n}$, $T_lL_{d,n}$, at $l$ and its orbit under coordinate negations gives us an arrangement $\mathcal{T}$ of $2^N$ linear spaces (related through coordinate negation). Any $E$ meeting the hypothesis of Theorem 4.2 necessarily drops rank on every subspace in $\mathcal{T}$. This would not be possible if $E$ or the collection of tangent spaces $T_lL_{d,n}$ were sufficiently general. On the other hand, we know that the geometry of our situation is special enough that when $E$ has rank $D$ and $K_{d+2}$ support, then $E$ does drop rank on each of the $T_lL_{d,n}$. Proposition 4.4 asserts that this is the only possibility. This proof relies on the negation-based symmetry of $L_{d,n}$ and on the fact that $K_{d+2}$ is the only graph on $D$ or fewer edges that is infinitesimally dependent (Proposition 3.7).

First we present the proof of Theorem 4.2, which effectively reduces our problem to the linear situation covered in Proposition 4.4.

**Proof of Theorem 4.2.** Clearly, the image of the map must be contained in an $r$-dimensional linear space spanned by the columns of $E$. Suppose that either $r < D$, or $E$ does not have $K_{d+2}$ support. Then, from Proposition 4.4 below, there must be a smooth point $l'$ such that $\dim(E(T_{l'}L_{d,n})) = r$. Then, from the Local Submersion Theorem for smooth maps [15, page 20], the map must be locally surjective onto the $r$-dimensional linear space. Thus the image (a constructible set) cannot have smaller dimension. \(\square\)

We are now ready to state the key technical result in this section.

**Proposition 4.4.** Let $E$ be a $D \times N$ matrix with rank $r$. Suppose that either $r < D$ or $E$ does not have $K_{d+2}$ support. Then there is a smooth point $l' \in L_{d,n}$ with the property that $\dim(E(T_{l'}L_{d,n})) = r$. 

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4.1 Proof of Proposition 4.4

The rest of the section is occupied with the proof, which we break down into steps. We use a technical lemma about coordinate negation and determinants that is relegated to its own Section 4.2.

Definition 4.5. A sign flip matrix $S$ is a diagonal matrix with $\pm 1$ on the diagonal. A coordinate flip of a point or subspace is its image under a sign flip matrix.

Definition 4.6. Let $m$ be a smooth point in $M_{d,n}$, and $T_m M_{d,n}$ be its complex analytic tangent. We can describe $T_m M_{d,n}$ by a $(dn-C) \times N$ complex matrix $T_m$. (The row ordering is not relevant).

Referring back to Definition 3.6, if $E$ is an infinitesimally independent edge set, then the columns of $T_m$ corresponding to $E$ at an $E$-regular point of $M_{d,n}$, are linearly independent. The same is true of the matrix $T_1$ that expresses the tangent space $T_1 L_{d,n}$ at an $E$-regular point of $L_{d,n}$. Such points must exist when $E$ is infinitesimally independent.

The first step is to restrict to an interesting range of $n$.

Lemma 4.7. Proposition 4.4 holds when $n < d + 2$.

Proof. When $n \leq d + 1$, $T_1 L_{d,n}$ is equal to the full embedding space, and thus $\dim(E(T_1 L_{d,n})) = r$. Proposition 4.4 is then trivial in this case. □

Thus, from now on, we may assume that $n \geq d + 2$.

Let $T$ be a $(dn-C) \times N$ matrix with rows spanning the tangent space $T_1 L_{d,n}$ at some smooth point $l$. The complex analytic tangent space at a smooth point of a variety with pure dimension has the same dimension as the variety, which explains the shape of $T$.

Block form and column basis Each column of $E$ and $T$ corresponds to an edge in $K_n$. We are going to make use of edge-permuted versions of these matrices that have particular block structures. To this end, we are now going to look at the columns of $E$ and determine which subsets can form a basis, $E_2$, of a linear space of dimension $r$. So we permute and then partition the columns of $E$ into a block form

$$(E_1 \quad E_2),$$

where $E_1$ is $D \times (N-r)$ and $E_2$ is $D \times r$. We define a column basis, $E_2$ of $E$, to be good when $r = D$ and the columns of $E_2$ correspond to the edges of a $K_{d+2}$. Any other column basis $E_2$ will be called bad. We denote by $E_2$ the edge set corresponding to the columns of $E_2$.

Suppose that $E$ has $K_{d+2}$ support and $r = D$, then the $r$ columns of $E$ corresponding to the edges of this $K_{d+2}$ must form the only column basis of $E$. Moreover, it is good.

Lemma 4.8. If $E$ does not have $K_{d+2}$ support or $r < D$, then there is a bad column basis for $E$.

Proof. If $r < D$, then by definition, no column basis can be good. From now on, then, assume that $r = D$.

If $E$ is supported on only $D$ columns, there is a unique column basis $E_2$. Thus in this case, non-$K_{d+2}$ support for $E$ will imply that the unique column basis is bad.

Suppose instead there are more than $D$ non-zero columns of $E$. Thus, starting from, say, a good basis $E_2$, we can exchange a non-zero column of $E_1$ with an appropriate one from $E_2$ to obtain another basis which is bad: removing an edge from a $K_{d+2}$ and replacing it with any other edge results in a graph that cannot be a $K_{d+2}$ (it has more vertices). □
Remark 4.9. In light of the paragraph preceding this lemma, Lemma 4.8 can be made into an “if and only if” statement.

Going back to $T$ and applying the same column used obtain $(E_1 \ E_2)$, we get a block form

$$(T_1 \ T_2)$$

where $T_1$ is $(dn - C) \times (N - r)$ and $T_2$ is $(dn - C) \times r$.

**Lemma 4.10.** Assuming that $E_2$ is a bad basis of $E$ and $l$ is $E_2$-regular, the matrix $T_2$ has rank $r$ (and in particular has linearly independent columns)

**Proof.** Since $(E_1, E_2)$ arises from a bad basis, and we have only applied column permutations, the columns of $T_2$ corresponds to a subgraph $G$ of $K_n$ with at most $D$ edges which is not $K_{d+2}$. Proposition 3.7 tells us that the edges of $G$ are infinitesimally independent. So, by $E_2$-regularity of $l$, these columns of $T$ are linearly independent (Definition 4.6). □

**Row rank**

**Lemma 4.11.** Assuming that $E_2$ is a bad basis of $E$ and $l$ is $E_2$-regular. Then the block matrix $(T_1 \ T_2)$ contains $r$ rows, $(T'_1 \ T'_2)$, such that $T'_2$ forms a non-singular matrix.

**Proof.** Since we have a bad basis, from Lemma 4.10, $T_2$ has $r$ linearly independent columns and thus $r$ linearly independent rows. We can select any set of rows corresponding to a row basis of $T_2$. □

Similarly, we have

**Lemma 4.12.** Let $E_2$ be a column basis for $E$. Then the block matrix $(E_1 \ E_2)$ contains $r$ rows, $(E'_1 \ E'_2)$, such that $E'_2$ forms a non-singular matrix.

Next, we derive an implication of $E$ dropping rank on the tangent space.

**Lemma 4.13.** Suppose there is a smooth point $l \in L_{d,n}$ such that $l$ and all of its coordinate flips $l'$ have the property that $\text{Dim}(E(T_l L_{d,n})) < r$. Let $E_2$ be a bad basis for $E$. Let $S_1$ be any $(N - r) \times (N - r)$ sign flip matrix, and $S_2$, any $r \times r$ sign flip matrix. Then the $r \times r$ matrix $Z := E'_1 S_1 T'_1^\top + E'_2 S_2 T'_2^\top$ is singular.

**Proof.** Let $S$ be the $N \times N$ be the sign flip matrix with the $S_i$ as its diagonals. Let $l'$ be the point obtained from $l$ under the sign flips of $S$. Because $L_{d,n}$ is symmetric under coordinate negations, then $T_l L_{d,n}$ is spanned by the columns of $ST^\top$. Then we have $\text{Dim}(E(T_l L_{d,n})) = \text{rank}(E ST^\top) = \text{rank}(E'_1 S_1 T'_1^\top + E'_2 S_2 T'_2^\top) \geq \text{rank}(E'_1 S_1 T'_1^\top + E'_2 S_2 T'_2^\top)$.

If for some $S$, the matrix $Z$ were non-singular, then we would have a certificate that $E$ does not drop rank on that coordinate flip of the tangent space, in contradiction to the hypothesis on $\text{Dim}(E(T_l L_{d,n})))$. □

**Remark 4.14.** The rank of $Z$ may change as the $S_i$ do, but it cannot rise to $r$. 

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Conclusion of the proof Assume that $E$ does not have $K_{d+2}$ support or $r < D$. From Lemma 4.8, there is a bad column basis $E_2$ for $E$. From Lemma 4.11, for an $E_2$-regular $l$, $T_2'$ is a non-singular matrix.

Suppose that at this $l$, we had for all of its coordinate flips $l'$, the property that $\dim(E(T_L l_{d,n})) < r$. Then from Lemma 4.13, for any choice of $S_2$, the matrix $Z$ would be singular. Since $E_2$ is a basis, $E_2'$ is non-singular matrix (Lemma 4.12), thus $Z' := S_2 E_2^{-1} = S_2 (E_2' - 1 E_1 S_1 T_1^T) + T_2' \top$ would be singular for any choice of $S_2$. Thus, Lemma 4.15 on determinants and sign flips (below) would apply to $Z'$, and we would conclude that $T_2'$ is singular.

From the resulting contradiction, we can deduce that for an $E_2$-regular point $l$, one of its coordinate flips $l'$ must have $\dim(E(T_L l_{d,n})) = r$. By regularity, $l$ is smooth, and so too is any coordinate flip such as $l'$. $\square$

4.2 Determinants and flips

In this section, we will establish a technical lemma about determinants and sign flips.

Lemma 4.15. Suppose that $Z = SX + Y$ is an $r \times r$ matrix and $\det(Z) = 0$ for all choices of sign flips, $S$. Then $\det(Y) = 0$.

Proof. Multilinearity of the determinant allows us to express $\det(Z)$ as $\det(Z^S) + \det(Z^{S'})$, where $Z^S$ is the matrix $Z$ with its first row replaced by the first row of $SX$, and where $Z^{S'}$ is the matrix $Z$ with its first row replaced by the first row of $Y$. We can likewise expand out each of $\det(Z^S)$ and $\det(Z^{S'})$ by splitting their second rows. Applying this decomposition recursively we ultimately get:

$$\det(SX + Y) = \sum_{I \subseteq [r]} \det(Z_I^S)$$

where $[r] = \{1, 2, \ldots, r\}$, and $Z_I^S$ is the matrix that has the rows indexed by $I$ from $SX$ and the rest from $Y$.

Now sum the above over the $2^r$ choices of $S$ and rearrange

$$\sum_S \det(SX + Y) = \sum_S \sum_{I \subseteq [r]} \det(Z_I^S) = \sum_{I \subseteq [r]} \sum_S \det(Z_I^S)$$

For fixed $I$, each $\det(Z_I^S) = (-1)^{\sigma(S,I)} \det(Z_I^1)$, where $\sigma(S, I)$ is the number of rows corresponding to $I$ where $S$ has a diagonal entry of $-1$. Thus, for each $I$, $(\ast)$ is

$$2^{r-|I|} \cdot \left( \sum_{k=0}^{|I|} \frac{|I|}{k} (-1)^k \right) \cdot \det(Z_I^1)$$

(The power of two factor accounts for all of the sign choices in $S$ over the complement of $I$.) The coefficient of $\det(Z_I^1)$ equals $2^r$ when $I$ is empty. Otherwise it is zero since the inner term is simply the binomial expansion of $(1 - 1)^{|I|}$. Thus,

$$\sum_S \det(SX + Y) = 2^r \det(Y)$$

Since this sum vanishes by hypothesis, we get $\det(Y) = 0$. $\square$
5 Automorphisms of $L_{d,n}$

In this section we will characterize the linear automorphisms of $L_{d,n}$ for all $d$ and $n$. One key feature will be that we are no longer restricted to the case of edge permutations.

We will need to consider a few distinct cases for $d$ and $n$.

**Definition 5.1.** Set $N := \binom{n}{2}$ and identify the rows and columns of an $N \times N$ matrix with the edges of $K_n$.

A signed permutation is an $N \times N$ matrix $P'$ that is the product $SP$ of a sign flip matrix $S$ and a permutation matrix $P$.

A signed permutation $P' := SP$ is induced by a vertex relabeling if $P$ is induced by a vertex relabeling of $K_n$.

### 5.1 Automorphisms of $L_{d,n}$, $n \geq d + 3$

Let $d \geq 1$. This section will be concerned with $L_{d,n}$ where $n$ is larger than the minimal value, $d + 2$.

**Theorem 5.2.** Let $n \geq d + 3$. Then any linear automorphism $A$ of $L_{d,n}$ of is a scalar multiple of a signed permutation that is induced by a vertex relabeling.

The plan is to use machinery from Section 4 to show that the automorphism must be in the form of a generalized edge permutation. We will then be able to switch over to the $M_{d,n}$ setting, where we can apply Theorem 3.10.

**Definition 5.3.** Let $A$ be an $N \times N$ matrix. We identify the rows and columns of $A$ with the edges of $K_n$. This induces a map $\tau_A$ from subgraphs of $K_n$ to subgraphs of $K_n$ by mapping the subgraph associated with a collection of rows to the column support of this sub-matrix.

**Lemma 5.4.** Let $n \geq d + 2$ and suppose that $A$ is a linear automorphism of $L_{d,n}$. Then the associated combinatorial map $\tau_A$ induces a permutation on $K_{d+2}$ subgraphs of $K_n$.

**Proof.** If $E$ is any $D \times N$ matrix of rank $D$, with $E(L_{d,n}) \subset L_{d,d+2}$, then the map $EA$ also has these properties. Thus, by Theorem 4.2 both $E$ and $EA$ have $K_{d+2}$ support. There is such an $E$ for each $K_{d+2}$ subgraph: simply take the matrix of the edge forgetting map $\pi_K$, where $K$ is an edge set comprising the edges of this $K_{d+2}$. This situation is only possible if $\tau_A(T)$ maps each $K_{d+2}$ subgraph $T$ to another $K_{d+2}$ subgraph.

If the map on $K_{d+2}$ subgraphs induced by $\tau_A$ is not injective, then the matrix $A$ would have more than $D$ rows supported by only $D$ columns, and thus $A$ would be singular. Since $A$ is a linear automorphism of $L_{d,n}$ it has to be invertible, and the resulting contradiction completes the proof.

This lets us prove the following.

**Lemma 5.5.** Let $n \geq d + 3$ and let $A$ be a linear automorphism of $L_{d,n}$. Then $A$ is a generalized permutation.

**Proof.** Suppose, w.l.o.g., that the row corresponding to the edge $e := \{1, 2\}$ has two non-zero entries corresponding to edges $\{i, j\}$ and $\{k, \ell\}$. By Lemma 5.4, any $K_{d+2}$ subgraph $T$ containing the edge $e$ must be mapped by $\tau_A$ to a $K_{d+2}$ subgraph $T'$ that contains the vertex set $X := \{i, j\} \cup \{k, \ell\}$.

Since $|X| \geq 3$ there are at most $\binom{n-3}{d-1}$ choices for $T'$. Meanwhile, there are $\binom{n-2}{d}$ choices for $T$. Since $n \geq d + 3$, we have $\binom{n-2}{d} > \binom{n-3}{d-1}$, contradicting the permutation of $K_{d+2}$ subgraphs guaranteed by Lemma 5.4.
Thus each row of $A$ can have at most one non-zero entry. As a non-singular matrix, this makes $A$ a generalized permutation.

At this point, we want to move back to the setting of $M_{d,n}$, which we do with this next result.

**Lemma 5.6.** Let $A := DP$ be a generalized permutation, where $D$ is an invertible diagonal matrix and $P$ is a permutation matrix. If $A$ is a linear automorphism of $L_{d,n}$ then $D^2P$ is a linear automorphism of $M_{d,n}$.

**Proof.** Let $l^2$ denote the vector of coordinate-wise square of a vector $l \in \mathbb{C}^N$; in this proof squares of vectors are coordinate-wise. Now we check that

\[
\begin{align*}
 l^2 &\in M_{d,n} \implies l \in L_{d,n} \implies DP l \in L_{d,n} \implies (A \text{ is an automorphism}) \\
 (DP l)^2 &\in M_{d,n} \implies D^2 (P l)^2 \in M_{d,n} \implies (D \text{ is diagonal}) \\
 (D^2P) l^2 &\in M_{d,n} \quad \text{(P is a permutation)}
\end{align*}
\]

\[\square\]

**Proof of Theorem 5.2.** From Lemma 5.5, any linear automorphism $A$ of $L_{d,n}$ with $n \geq d + 3$ is a generalized permutation $A = DP$. Lemma 5.6 implies that $A$ gives rise to a generalized edge permutation $D^2P$ that is a linear automorphism of $M_{d,n}$. Theorem 3.10 then tells us that $D^2P = s^2P$ has uniform scale and also is induced by a vertex relabeling. Finally $A$ is then a scalar multiple of a signed permutation (Lemma 5.6 “forgets” the signs) as required.

\[\square\]

### 5.2 Automorphisms of $L_{d,d+2}$, with $d \geq 3$

Our next case is when $n$ is minimal, but we will only deal with the case of $d \geq 3$.

**Theorem 5.7.** Let $d \geq 3$. Then any linear automorphism $A$ of $L_{d,d+2}$ is a scalar multiple of a signed permutation that is induced by a vertex relabeling.

The plan is to use some of the structure of the singular locus of $L_{d,d+2}$ to reduce our problem to that of $L_{d-1,d+2}$. Then we can directly apply Theorem 5.2.

**Lemma 5.8.** Let $d \geq 3$. $L_{d-1,d+2}$ is an irreducible subvariety of $\text{Sing}(L_{d,d+2})$.

**Proof.** Looking first at the squared measurement variety, from Theorem 2.6, we know that $\text{Sing}(M_{d,d+2}) = M_{d-1,d+2}$.

Let $Z$ be the locus of $\mathbb{C}^N$ where at least one coordinate vanishes, and let $S := L_{d-1,d+2} - Z$. Thus from Lemma 2.20, the points in $S$, are (algebraically) singular in $L_{d,d+2}$. So $S$ is contained in $\text{Sing}(L_{d,d+2})$.

From Theorem 2.11, when $d \geq 3$, we have $L_{d-1,d+2}$ is irreducible. The set $S$ is obtained from $L_{d-1,d+2}$ by removing a strict subvariety, which must be of lower dimension due to irreducibility. Thus $S$ is a full-dimensional constructible subset of the irreducible $L_{d-1,d+2}$. Thus the Zariski closure of $S$ is $L_{d-1,d+2}$.

Since $\text{Sing}(L_{d,d+2})$ is an algebraic variety, it must contain the Zariski closure of $S$ which is $L_{d-1,d+2}$.

\[\square\]
Lemma 5.9. \(L_{d-1,d+2}\) has a full-dimensional affine span.

Proof. Since \(L_{d-1,d+2}\) contains \(L_{1,d+2}\), we just need to show that this smaller variety has a full-dimensional affine span.

For a fixed \(i\), let us look at configuration \(p\) of \(d + 2\) points with \(p_i\) placed at 1 and the rest of the points placed at the origin. Then \(l := l(p)\) has all zero coordinates except for the \(d + 1\) edges connecting \(p_i\) to the other points. Under the symmetry of \(L_{1,d+2}\) under sign negation, we can find points in \(L_{1,d+2}\) with the signs of the \(l\) flipped at will. Thus using affine combinations of these flipped points we can produce a point on the \(l_{ij}\) axis, for any \(j\). Iterating over the \(i\) gives us our result. \(\square\)

Now we wish to explore the decomposition of \(\text{Sing}(L_{d,d+2})\) into its irreducible components.

For each \(ij\), Let \(Z_{ij}\) be the subvariety of \(\text{Sing}(L_{d,d+2})\) with a zero-valued \(ij\)th coordinate. As discussed above in Lemma 2.20 any singular point that is not contained in \(L_{d-1,d+2}\) must have at least one zero coordinate (in order to be in the “bad locus” described there). Thus we can write \(\text{Sing}(L_{d,d+2})\) as the union of \(L_{d-1,d+2}\) and the \(Z_{ij}\).

For \(d \geq 3\), \(L_{d-1,d+2}\) is irreducible, and thus from Lemma A.6 (applied to the union of components of \(\text{Sing}(L_{d,d+2})\)) it must be fully contained in at least one component \(C\) of \(\text{Sing}(L_{d,d+2})\). And, again from from Lemma A.6 (applied to the union of \(L_{d-1,d+2}\) and the \(Z_{ij}\)), \(C\) must be fully contained in either \(L_{d-1,d+2}\) or one of the \(Z_{ij}\). Meanwhile, \(L_{d-1,d+2}\) it is not contained in any \(Z_{ij}\). Thus we can conclude that:

Lemma 5.10. Let \(d \geq 3\). \(L_{d-1,d+2}\) is a component of \(\text{Sing}(L_{d,d+2})\).

From Lemma A.6 (applied to the union of \(L_{d-1,d+2}\) and the \(Z_{ij}\)), any other component of \(\text{Sing}(L_{d,d+2})\) must be contained in one of the \(Z_{ij}\). Thus, we can also conclude:

Lemma 5.11. Let \(d \geq 3\). Any component of \(\text{Sing}(L_{d,d+2})\) that is not \(L_{d-1,d+2}\) cannot have a full-dimensional affine span.

Now with this understanding of \(\text{Sing}(L_{d,d+2})\) established we can move on to the automorphisms.

Lemma 5.12. Let \(d \geq 3\). Any linear automorphism \(A\) of \(L_{d,d+2}\) must be a linear automorphism of \(L_{d-1,d+2}\).

Proof. From Theorem A.8, \(A\) must be a linear automorphism of \(\text{Sing}(L_{d,d+2})\). And from Theorem A.5 must map components of \(\text{Sing}(L_{d,d+2})\) to components of \(\text{Sing}(L_{d,d+2})\).

From Lemma 5.10, \(L_{d-1,d+2}\) is a component of this singular set and from Lemma 5.9 it has a full-dimensional affine span. Meanwhile, from Lemma 5.11, no other component can have a full-dimensional affine span. Thus, as a bijective linear map, \(A\) must map \(L_{d-1,d+2}\) to itself. \(\square\)

And we can finish the proof.

Proof of Theorem 5.7. The theorem now follows by combining Lemma 5.12 together with Theorem 5.2. \(\square\)
5.3 Automorphisms of $L_{2,4}$

The method of the previous section fails for $L_{2,4}$ as $L_{1,4}$ is reducible. In fact, the theorem itself fails in this case. The group of linear automorphisms is, in fact, larger than expected.

In particular, Regge [27] (see also, Roberts [28]) gave a linear map that always takes the Euclidean lengths of the edges of a tetrahedral configuration in $\mathbb{R}^2$ to those of a different tetrahedral configuration in $\mathbb{R}^2$. See Equation $(\star)$ in the introduction.

Below we will fully characterize the automorphism group of $L_{2,4}$. When we restrict our automorphisms to have only non-negative entries, only the expected symmetries will remain.

**Definition 5.13.** A linear automorphism $A$ of $L_{2,4}$ is **real** if its matrix has only real entries, **rational** if its matrix has only rational entries, and **non-negative** if its matrix contains only real and non-negative entries.

Clearly there are 24 linear automorphism that arise by simply permuting the 4 vertices. There are also the 32 linear automorphisms that arise from optionally negating up to 5 of the coordinate axes in $\mathbb{C}^6$. Combining these gives us a discrete group of 768 linear automorphisms.

Because any global scale will be an automorphism, the group of linear automorphisms of $L_{2,4}$ is not a discrete group. We now define several groups that will play a role in our analysis.

**Definition 5.14.** Define $\text{Aut}(L_{2,4})$ to be the linear automorphisms of $L_{2,4}$. Let the group $\mathbb{P} \text{Aut}(L_{2,4})$ be induced on the equivalence classes of $A \in \text{Aut}(L_{2,4})$ under the relation “$A'$ is a complex scale of $A$”. We define $\mathbb{P} \text{Aut(Sing}(L_{2,4}))$ via a similar construction. Importantly, we will see below that $\mathbb{P} \text{Aut(Sing}(L_{2,4}))$ is the automorphism group of a projective subspace arrangement and thus is a discrete group. Also, we have $\mathbb{P} \text{Aut}(L_{2,4}) < \mathbb{P} \text{Aut(Sing}(L_{2,4}))$. Thus, all the “projectivized” groups we define are discrete.

We also consider the real subgroup $\text{Aut}_\mathbb{R}(L_{2,4})$. This has a counterpart $\mathbb{P} \text{Aut}_\mathbb{R}(L_{2,4})$ of equivalence classes up to real scale, and $\mathbb{P}^+ \text{Aut}_\mathbb{R}(L_{2,4})$, on equivalence classes defined up to positive scale. It is well-defined to refer to an element of $\mathbb{P}^+ \text{Aut}_\mathbb{R}(L_{2,4})$ as being non-negative, since any equivalence class containing a non-negative $A$ consists entirely of non-negative matrices.

The main theorem of this section characterizes the linear automorphisms of $L_{2,4}$ as follows. The proof is in the next subsections.

**Theorem 5.15.** The group $\mathbb{P} \text{Aut}(L_{2,4})$ is of order $11520 = 768 \cdot 15$ and is generated by linear automorphisms of $L_{2,4}$ that are rational.

The group $\mathbb{P}^+ \text{Aut}_\mathbb{R}(L_{2,4})$ is of order $23040$ and is isomorphic to the Weyl group $D_6$. The subset of non-negative elements of $\mathbb{P}^+ \text{Aut}_\mathbb{R}(L_{2,4})$ is a subgroup of order 24 and acts by relabeling the vertices of $K_4$.

**Remark 5.16.** The group $\mathbb{P}^+ \text{Aut}_\mathbb{R}(L_{2,4})$ is in fact generated by the edge permutations induced by vertex relabeling, sign flip matrices, and the one Regge symmetry of $(\star)$ from the introduction (see supplemental script).

The rest of this section develops the proof of Theorem 5.15.

**The Singular Locus of $L_{2,4}$** In this section, we will study the singular locus of $L_{2,4}$. This will be used for the proof of Theorem 5.15, which characterizes the linear automorphisms of $L_{2,4}$. In particular, a linear automorphism of a variety must also be a linear automorphism of its singular locus.
**Theorem 5.17.** The singular locus \( \text{Sing}(L_{2,4}) \) consists of the union of 60 3-dimensional linear subspaces. These subspaces can be partitioned into three types, which we call I, II and III.

**Type I:** There are 32 subspaces of this type. They arise from configurations of 4 collinear points, and together make up \( L_{1,4} \). They are each defined by (the vanishing of) three equations of the following form:

\[
\begin{align*}
    l_{12} - s_{13}l_{13} + s_{23}l_{23} \\
    l_{12} - s_{14}l_{14} + s_{24}l_{24} \\
    s_{13}l_{13} - s_{14}l_{14} + s_{34}l_{34}
\end{align*}
\]

where each \( s_{ij} \) takes on the values \( \{-1, 1\} \).

**Type II:** There are 24 subspaces of this type. They arise when one pair of vertices is collapsed to a single point. For example, if we collapse \( p_1 \) with \( p_2 \), we get the equations:

\[
\begin{align*}
    l_{12} \\
    l_{13} - s_{23}l_{23} \\
    l_{14} - s_{24}l_{24}
\end{align*}
\]

This gives us 4 subspaces, and we obtain this case by collapsing any of the 6 edges.

**Type III:** There are 4 subspaces of this type. They arise by setting the three edges lengths of one triangle to zero. For example:

\[
\begin{align*}
    l_{12} \\
    l_{13} \\
    l_{23}
\end{align*}
\]

**Proof.** The singular locus of a variety \( V \) is defined by adding to the ideal \( I(V) \), the equations that express a rank-drop in the Jacobian matrix of a set of equations generating \( I(V) \).

We first verify in the Magma CAS that the ideal defined by our single simplicial volume determinant equation is radical.\(^3\) This also follows from [9].

In Magma, we calculate the Jacobian of this equation to express the singular locus. Magma is then able to factor this algebraic set into components (that are irreducible over \( \mathbb{Q} \)), and in this case outputs the above decomposition. (See supplemental script.) \( \square \)

**Flats and intersection graph**  Theorem A.8 tells us that any linear automorphism of \( L_{2,4} \) must be a linear automorphism of its singular set, and so must map each of its singular three-dimensional subspaces to some three-dimensional singular subspace. As a linear automorphism, it must also preserve the intersection lattice of the three-dimensional singular subspace arrangement. Therefore, by finding the set of linear automorphisms that preserve the intersection lattice of these subspaces, we can constrain our search for automorphisms of \( L_{2,4} \) to just that set. Combinatorial descriptions of an intersection lattice of a subspace arrangement can be constructed in many ways. Here, it suffices to consider a partial description that comprises the three-dimensional singular subspaces and their one-dimensional intersections.

**Definition 5.18.** We denote by \( V_3 \) the set of singular three-dimensional subspaces of \( L_{2,4} \). We denote by \( V_1 \) the set of one-dimensional subspaces created as the intersections of all pairs and triples of spaces in \( V_3 \).

\(^3\)Magma does this check over the field \( \mathbb{Q} \), but since \( \mathbb{Q} \) is a perfect field, this implies that the ideal is also radical under any field extension [23, Page 169].
Lemma 5.19. The set of one-dimensional subspaces $V_1$ consists of 46 elements. These come in 3 classes:

Type I: There are 6 one-dimensional subspaces of this type. They are generated by vectors of the form

$$e_i$$

where $e_i$ is one of the coordinate axes of $\mathbb{C}^6$.

Type II: There are 24 one-dimensional subspaces of this type. They are generated by vectors of the form

$$e_i \pm e_j \pm e_k \pm e_l$$

where $i, j, k, l$ correspond to the four edges of a 4-cycle. These measurements correspond to collapsing two sets of two vertices that are connected by four edges.

Type III: There are 16 one-dimensional subspaces of this type. They are generated by vectors of the form

$$e_i \pm e_j \pm e_k$$

where $i, j, k$ correspond to three edges incident to one vertex. These measurements correspond to collapsing one triangle.

Proof. This follows directly from calculating the intersections of all pairs and triples of the 60 singular subspaces of $L_{2,4}$. This has been done in the Magma CAS. (See supplemental script.) \qed

Definition 5.20. We define $\Delta$ as the bipartite graph that has one set of vertices corresponding to the three-dimensional singular subspaces of $L_{2,4}$ (one vertex for each three-dimensional subspace), the other set of vertices corresponding to the one-dimensional intersection subspaces $V_1$ (one vertex for each one-dimensional subspace), and an edge between vertex $i$ of the first set and vertex $j$ of the second set whenever the $i$th three-dimensional subspace includes the $j$th one-dimensional subspace.

Definition 5.21. A graph automorphism of a bipartite (two-colored) graph is a permutation $\rho$ of the vertex set such that the color of vertex $i$ is the same as the color of $\rho(i)$, and vertices $(i, j)$ form an edge if and only if $(\rho(i), \rho(j))$ also form an edge.

By finding the automorphisms of the graph $\Delta$ we can constrain our search for automorphisms of $\{V_3, V_1\}$, and thus of $L_{2,4}$.

Lemma 5.22. The bipartite graph $\Delta$ has 11520 automorphisms. Under this automorphism group, the graph has three orbits. One orbit corresponds to the set of 60 three-dimensional singular subspaces. Another orbit corresponds to the subset of 30 one-dimensional subspaces in $V_1$ of type I and II. A third orbit corresponds to the subset of 16 one-dimensional subspaces of type III.

Proof. We have computed this using Nauty [22] within Magma. (See supplemental script.) \qed

Graph automorphisms to arrangement automorphisms

A priori, it might be the case that some of these graph automorphisms do not arise from a linear transform of $\mathbb{C}^6$ acting as an automorphism on the subspace arrangement $\{V_3, V_1\} \subset \mathbb{C}^6$. We rule this out.
Lemma 5.23. Each of the graph automorphisms of $\Delta$ gives rise to a unique linear automorphism of the arrangement $\{\mathcal{V}_3, \mathcal{V}_1\}$ on $L_{2,4}$, up to a global scale. Each equivalence class of such linear maps contains a rational-valued matrix.

Proof. Each graph automorphism $\rho$ gives rise to a permutation of the spaces in $\mathcal{V}_3$. A $6 \times 6$ matrix $A$ describing a linear transform that maps the three-dimensional subspaces in the same manner must satisfy $540 = 60 \cdot 9$ linear homogeneous constraints, nine for each pair $(i, \rho(i)), i \in \mathcal{V}_3$.

Magma gives us a generating set of size 6 for the group of graph automorphisms. For each of the 6 generators of the graph automorphism group, we write out the system of linear constraints. When doing so, we discover that this system always has a solution that is unique, up to a global scale. The $540 \times 36$ constraint matrix can always be written as a rational-valued matrix, since the subspace arrangement $\{\mathcal{V}_3, \mathcal{V}_1\}$ can be defined using rational-valued coefficients. (See supplemental script.)

Arrangement automorphisms are $L_{2,4}$ automorphisms

It might also be possible that there are linear transforms which preserve the subspace arrangement $\{\mathcal{V}_3, \mathcal{V}_1\}$, but do not preserve the entire $L_{2,4}$ variety. We rule this out as well.

Lemma 5.24. Each of the graph automorphisms of $\Delta$ gives rise to a unique linear automorphism on $L_{2,4}$, up to a global scale. Each equivalence class of such linear maps contains a rational-valued matrix.

Proof. From Lemma 5.23, each of the graph automorphisms gives rise to a, unique up to scale, rational-valued linear automorphism of our arrangement. When we pull back the single defining equation of $L_{2,4}$ through each such invertible linear map, we verify that we recover said equation. Thus this map is a linear automorphism of $L_{2,4}$.

Reflection group

Next, we make a definition that will be helpful in establishing the connection between $\mathbb{P}_+ \mathrm{Aut}_\mathbb{R}(L_{2,4})$ and the Weyl group $D_6$. For definitions, see [18].

Definition 5.25. We define the reflection group $W$ as the real matrix group generated by the set of reflections in $\mathbb{R}^6$ across the 30 hyperplanes that are orthogonal to the 30 one-dimensional real intersection subspaces of type I and II.

The following lemma was based on conversions with Dylan Thurston.

Lemma 5.26. The reflection group $W$ is of order 23040, and is isomorphic to the Weyl group $D_6$. The reflection group leaves the variety $L_{2,4}$ invariant.

Proof. From the 30 vectors that generate $W$, we generate a larger set of 60 vectors $\phi$ that has the same reflection group as follows: For each vector $f$ in the original 30-set, we create two vectors $\pm 2f/\|f\|$ in the 60-set. Next, we verify that the set $\phi$ is a (reduced, crystallographic) root system by: i) applying each generator of the group $W$ to the set $\phi$ and verifying that it leaves the set invariant; and ii) verifying that the set satisfies the integrality condition $\forall f, g \in \phi, 2(f \cdot g)/\|f\| \in \mathbb{Z}$.

A reflection group of a root system is a Weyl group. To prove the first part of the lemma, we need only classify the root system (and thus the Weyl group) according to the finite catalog of rank 6 possibilities. We use the procedure described in [18, page 48], which we summarize here.
We begin by choosing any vector $h \in \mathbb{Q}^6$ that is not proportional or perpendicular to a vector in $\phi$, and then we identify the subset of positive roots $\phi^+ := \{ f : f \in \phi, (h \cdot f) > 0 \}$. Since $\phi$ is a root system, it will be the case that $|\phi^+| = |\phi|/2 = 30$. Among the positive roots, we identify the subset of simple roots as the vectors $f \in \phi^+$ that cannot be decomposed as $g_1 + g_2$ for some $g_i \in \phi^+$. By construction, simple roots form a basis for the embedding vector space, so in the present case there will be 6 of them. Finally, we can classify the group by examining the pattern of pairwise angles between simple roots.

Applying this calculation to our root system, we find that the pairwise angles between the simple roots are $0$ or $2\pi/3$. We draw a Dynkin diagram that has one vertex for each simple root and an edge $(i,j)$ whenever the angle between roots $i$ and $j$ is $2\pi/3$. Doing so, we find that this diagram is of type $D_6$. This means that the reflection group is isomorphic to the Weyl group $D_6$, which is of order 23040. This proves the first part of the lemma.

To prove the second part of the lemma, we use the fact that the reflection group $W$ is generated by the 6 reflections from the simple roots. We pull back the single defining equation of $L_{2,4}$ through each of these 6 linear maps, and we verify that we recover said equation.

Note that the group could also be identified from its computed order. (See supplemental script.)

□

Proof

The proof of our theorem is now nearly complete.

Proof of Theorem 5.15. From Theorem A.8, a linear automorphism of $L_{2,4}$ must be a linear automorphism of its singular set $V_3$, and thus must preserve the incidence structure of $\{V_3, V_1\}$. Any linear automorphism of this incidence structure must give rise to a graph automorphism of $\Delta$. By Lemma 5.22, there are 11520 graph automorphisms of $\Delta$, and from Lemma 5.24, each gives rise to a rational valued linear automorphism of $L_{2,4}$, unique up to scale. Summarizing, we have shown that $\mathbb{P}\text{Aut}(L_{2,4}) = \mathbb{P}\text{Aut}(\text{Sing}(L_{2,4}))$, and that both of these groups are isomorphic to the automorphism group of the graph $\Delta$. Lemma 5.24 also implies that each equivalence class in $\mathbb{P}\text{Aut}(L_{2,4})$ contains a rational representative, so this group can be generated by rational matrices.

Because of the rational generators mentioned above, the group $\mathbb{P}\text{Aut}_\mathbb{R}(L_{2,4})$ is isomorphic to the others. It then follows that the order of $\mathbb{P}+\text{Aut}_\mathbb{R}(L_{2,4})$ is 23040 = $2 \cdot 11520$.

Next, we deal with the classification of $\mathbb{P}+\text{Aut}_\mathbb{R}(L_{2,4})$. By Lemma 5.26 (specifically the second statement), the elements of $W$ generate some subgroup $G$ of $\mathbb{P}+\text{Aut}_\mathbb{R}(L_{2,4})$. In fact, no two elements of $W$ are related by a positive scale, so $W$ is isomorphic to this $G$. The first part of Lemma 5.26 says that $W$ has the same order as $\mathbb{P}+\text{Aut}_\mathbb{R}(L_{2,4})$, so $W$ and $\mathbb{P}+\text{Aut}_\mathbb{R}(L_{2,4})$ are isomorphic.

For the third part of the theorem, we need only test 23040 matrices and retain those that have only non-negative entries. This has been done in the Magma CAS, and indeed, it yields only the 24 edge permutations induced by vertex relabeling. (See supplemental script.) This is, in particular, a subgroup of $\mathbb{P}+\text{Aut}_\mathbb{R}(L_{2,4})$.

□

5.4 Automorphisms of $L_{1,3}$

Theorem 5.27. Any linear automorphism $A$ of $L_{1,3}$ is a scalar multiple of a signed permutation that is induced by a vertex relabeling.
Proof. $L_{1,3}$ comprises 4 hyperplanes. Each permutation on these 4 planes gives us at most a single linear automorphism of $L_{1,3}$ up to scale. Thus $\mathbb{P} \text{Aut}(L_{1,3})$ is isomorphic to a subgroup of $S_4$ and, in particular, has order at most 24.

Meanwhile $\mathbb{P} \text{Aut}(L_{1,3})$ contains a subgroup of order 24 generated by vertex relabeling and sign flips. By the above, this must be the whole group. \qed

Remark 5.28. If we want to see $S_4$ acting by sign flips and coordinate permutations, we can observe that these maps are symmetries of the cube that permute the opposite corner diagonals.

A Algebraic Geometry Preliminaries

We summarize the needed definitions and facts about complex algebraic varieties. For more see [16].

In this section $N$ and $D$ will represent arbitrary numbers.

Definition A.1. A (complex embedded affine) variety (or algebraic set), $V$, is a (not necessarily strict) subset of $\mathbb{C}^N$, for some $N$, that is defined by the simultaneous vanishing of a finite set of polynomial equations with coefficients in $\mathbb{C}$ in the variables $x_1, x_2, \ldots, x_N$ which are associated with the coordinate axes of $\mathbb{C}^N$.

A variety can be stratified as a union of a finite number of complex analytic submanifolds of $\mathbb{C}^N$.

A finite union of varieties is a variety. An arbitrary intersection of varieties is a variety.

The set of polynomials that vanish on $V$ form a radical ideal $I(V)$, which is generated by a finite set of polynomials.

A variety is reducible if it is the proper union of two varieties $V_1$ and $V_2$. (Proper means that $V_1$ is not contained in $V_2$ and vice versa.) Otherwise it is called irreducible. A variety has a unique decomposition as a finite proper union of its maximal irreducible subvarieties called components. (Maximal means that a component cannot be contained in a larger irreducible subvariety of $V$.)

A variety $V$ has a well defined (maximal) dimension $\text{Dim}(V)$, which will agree with the largest $D$ for which there is an open subset of $V$, in the standard topology, that is a $D$-dimensional complex submanifold of $\mathbb{C}^N$.

The local dimension $\text{Dim}_l(V)$ at a point $l$ is the dimension of the highest-dimensional irreducible component of $V$ that contains $l$. If all components of $V$ have the same dimension, we say it has pure dimension.

Any strict subvariety $W$ of an irreducible variety $V$ must be of strictly lower dimension.

Definition A.2. A constructible set $S$ is a set that can be defined using a finite number of varieties and a finite number of Boolean set operations.

The Zariski closure of $S$ is the smallest variety $V$ containing it. The set $S$ has the same dimension as its Zariski closure $V$.

The image of a variety $V$ of dimension $D$ under a polynomial map is a constructible set $S$ of dimension at most $D$. If $V$ is irreducible, then so too is the Zariski closure of $S$. (We say that $S$ is irreducible.)

Theorem A.3. Any variety $V$ is a closed subset of $\mathbb{C}^N$ in the standard topology. If a subset $S$ of $\mathbb{C}^N$ is standard-topology dense in a variety $V$, then $V$ is the Zariski closure of $S$.

We will need the following easy lemmas.

Lemma A.4. Let $A$ be a bijective linear map on $\mathbb{C}^N$. The image under $A$ of a variety $V$ is a variety of the same dimension. If $V$ is irreducible, then so too is this image.
Proof. The image $S := A(V)$ must be a constructible set.

Since $A$ is bijective, then there is also map $A^{-1}$ acting on $\mathbb{C}^N$, and $S$ must be the inverse image of $V$ under this map. Thus, by pulling back the defining equations of $V$ through $A^{-1}$, we see that $S$ must also be a variety.

The dimension follows from the fact that maps cannot raise dimension, and our map is invertible. □

Theorem A.5. If $A$ is a bijective linear map on $\mathbb{C}^N$ that acts as bijection between two reducible varieties $V$ and $W$, then it must bijectively map components of $V$ to components of $W$.

Proof. From Lemma A.4, $A$ must map irreducible varieties to irreducible varieties. As a bijection, it also must preserve subset relations (which define maximality). □

Lemma A.6. Let $V = V_1 \cup V_2$ be a union of varieties. Then any irreducible subvariety $W$ of $V$ must be fully contained in at least one of the $V_i$.

Proof. If $W$ was not fully contained in either $V_i$, then it could be written as the proper union of varieties $W = \bigcup_i(W \cap V_i)$ contradicting its irreducibility. □

There are two approaches for defining smooth and singular points. One comes from our algebraic setting, while the other comes from the more general setting of complex analytic varieties (which we will explicitly refer to as “analytic”). It will turn out that (algebraic) smoothness implies analytic smoothness, and that analytic smoothness implies (algebraic) smoothness.

Definition A.7. The Zariski tangent space at a point $l$ of a variety $V$ is the kernel of the Jacobian matrix of a set of generating polynomials for $I(V)$ evaluated at $l$.

A point $l$ is called (algebraically) smooth in $V$ if the dimension of the Zariski tangent space equals the local dimension $\dim_l(V)$. Otherwise $l$ is called (algebraically) singular in $V$. The locus of singular points of $V$ is denoted $\text{Sing}(V)$. The singular locus is itself a strict subvariety of $V$.

Theorem A.8. If $A$ is a bijective linear map on $\mathbb{C}^N$ that acts as a bijection between two irreducible varieties $V$ and $W$, then it must map singular points to singular points.

This is a special case of the more general setting of “regular maps” and “isomorphisms of varieties” [16, Page 175].

Theorem A.9. If a point $l$ is contained in two distinct components of $V$, then $l$ cannot be a smooth point in $V$.

See [29, II. 2. Theorem 6].

Definition A.10. If a point $l$ in a variety $V$ has a neighborhood in $V$ that is a complex submanifold of $\mathbb{C}^N$ with some dimension $D$, then we call the point analytically smooth of dimension $D$ in $V$, or just analytically smooth in $V$. Otherwise we call the point analytically singular in $V$.

The following theorem tells us that there is no difference between these two notions of smoothness.

Theorem A.11. An (algebraically) smooth point $l$ in a variety $V$ must be an analytically smooth point of dimension $\dim_l(V)$ in $V$.

A point $l$ that is analytically smooth of dimension $D$ in $V$ must be an (algebraically) smooth point $l$ in $V$ with $\dim_l(V) = D$.

For discussions on this theorem see [16, Exercise 14.1], [25, Page 13]. See [21, Page 14] for the setting where one does not assume irreducibility, or even pure dimension.

Note that the second direction does not have a corresponding statement in the setting of real algebraic varieties.
B Fano Varieties of $L_{2,4}$

This section contains a bonus result about the linear subsets in $L_{2,4}$. Though it is not needed for the rest of the paper, it can be of use for unlabeled rigidity problems [10].

**Definition B.1.** Given an affine algebraic cone $V \subset \mathbb{C}^N$ (an affine variety defined by a homogeneous ideal), its Fano-$k$ variety $\text{Fano}_k(V)$ is the subset of the Grassmanian $\text{Gr}(k+1,N)$ corresponding to $k+1$-dimensional linear subspaces that are contained in $V$.

**Theorem B.2.** The only 3-dimensional linear subspaces that are contained in $L_{2,4}$ are the 60 3-dimensional linear spaces comprising its singular locus. Moreover, there are no linear subspaces of dimension $\geq 4$ contained in $L_{2,4}$.

**Proof.** This proposition is proven by calculating the Fano-2 variety of $L_{2,4}$ in the Magma CAS [4], and comparing it to the the Fano-2 variety of the singular locus of $L_{2,4}$.

We use the approach described in [16, Page 70] to compute the Fano-2 variety of $L_{2,4}$. We summarize this approach here. We shall order the coordinates of $\mathbb{C}^6$ in the order $(l_1, l_2, l_3, l_4, l_5, l_6)$. Let us specify a point in $\mathbb{C}^6$ as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \\ \lambda_7 & \lambda_8 & \lambda_9 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

where the $\lambda_i$ are variables that specify a three-dimensional linear subspace of $\mathbb{C}^6$, and the $t_j$ are variables that specify a point on that subspace. Note that this can only represent an affine open subset of the Grassmanian; it cannot represent three-dimensional linear subspaces that are parallel to the first three coordinate axes.

We can compute the polynomial in $[\lambda_i, t_j]$ vanishing when the associated points in $\mathbb{C}^6$ are also in $L_{2,4}$. We can then look at all of the coefficients (polynomials in $\lambda_i$) of the monomials in $t_j$. These coefficient polynomials vanish identically iff the linear subspace specified by the $\lambda_i$ is in $L_{2,4}$. Thus these coefficients generate an affine open subset of $\text{Fano}_2(L_{2,4})$.

To study the whole Fano variety, we must also look at the other affine subsets of the Grassmanian. Due to the vertex symmetry of $L_{2,4}$, we only need to consider the additional two matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \\ \lambda_7 & \lambda_8 & \lambda_9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & 1 & 0 \\ \lambda_4 & \lambda_5 & \lambda_6 \\ \lambda_7 & \lambda_8 & \lambda_9 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These three matrices represent the triplet of coordinate axes corresponding to, respectively, a triangle, a chicken-foot, and a simple open path. Thus, these 3 open subsets of $\text{Fano}_2(L_{2,4})$, together with vertex relabeling, cover the full Fano variety.

We compute these 3 open subsets of $\text{Fano}_2(L_{2,4})$ in Magma, and verify that, in each of these open subsets, $\text{Fano}_2(L_{2,4})$ is 0-dimensional and $|\text{Fano}_2(L_{2,4})| = |\text{Fano}_2(\text{Sing}(L_{2,4}))|$. As $\text{Fano}_2(L_{2,4}) \supset \text{Fano}_2(\text{Sing}(L_{2,4}))$, we can conclude that $\text{Fano}_2(L_{2,4}) = \text{Fano}_2(\text{Sing}(L_{2,4}))$ (see supplemental script).

As Fano-2 variety is discrete, the higher Fano varieties of $L_{2,4}$ must also be empty. □
Remark B.3. We have been unable to fully compute any of the Fano varieties of $L_{3,5}$ in any computer algebra system, but partial results do not look promising. We have been able to verify that $\text{Fano}_6(L_{3,5})$ is not empty (see supplemental script). This together with our (partial) understanding of $\text{Sing}(L_{3,5})$ suggests that $L_{3,5}$ indeed contains 6-dimensional linear spaces that are not contained in its singular locus.

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