Inverse Boundary Value Problems for Systems of Partial Differential Equations.

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Abstract

We describe the main results and the ideas of the proofs in the papers [E] and [ER2] (see References). In addition, we simplify the construction of asymptotic solutions in [E], using the results of [ER2], and we simplify the proof of estimate (3.9) that was given in [ER2].

1 The Schrödinger Equation with an External Yang-Mills Potential.

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$. Consider the Dirichlet problem, $u = f$ on $\partial \Omega$, for a system of differential equations of the form

\begin{equation}
-\Delta u - 2i \sum_{k=1}^{n} A_k(x) \frac{\partial u}{\partial x_k} + Bu = 0, \; x \in \Omega,
\end{equation}

where $u = (u_1, \ldots, u_m)$, and $A_k, \; k = 1, \ldots, n$, and $B$ are smooth $m \times m$ matrix functions. We assume that $\Omega$ is such that the Dirichlet problem has a unique solution $u \in H_1(\Omega)$ for every $f \in H^{1/2}_2(\partial \Omega)$. Let $\Lambda$ be the Dirichlet-to-Neumann operator,

\[ \Lambda f = \frac{\partial u}{\partial \hat{n}} + i A \cdot \hat{n} u \text{ on } \partial \Omega, \]
where $\hat{n}$ is the exterior unit normal to $\partial \Omega$ and $u$ is the solution of (1.1) with $u = f$ on $\partial \Omega$. The inverse boundary value problem is to find the coefficients in (1.1) given $\Lambda$. When one rewrites (1.1) in the form

$$(-i \frac{\partial}{\partial x} + A(x))^2 u + V(x) u = 0$$

with $A = (A_1, \ldots, A_n)$ and $V = B - \sum_{k=1}^{n} A_k^2 + i \sum_{k=1}^{n} \frac{\partial A_k}{\partial x_k}$, it becomes the time-independent Schrödinger equation for a particle under the influence of the Yang-Mills potential $(A, V)$. We say that two Yang-Mills potentials $(A^{(1)}, V^{(1)})$ and $(A^{(2)}, V^{(2)})$ are gauge equivalent if there exists a smooth, invertible matrix function $g(x)$ on $\Omega$ such that

$$(1.3) \quad A^{(2)} = g^{-1} A^{(1)} g - ig^{-1} \frac{\partial g}{\partial x} \quad \text{and} \quad V^{(2)} = g^{-1} V^{(1)} g.$$ 

The following theorem was proven in [E].

**Theorem 1.1.** Let $L^{(j)} u = 0$, $j = 1, 2$, be two Schrödinger equations of the form (1.2) in $\Omega \subset \mathbb{R}^n$, $n \geq 3$, with Yang-Mills potentials $(A^{(j)}, V^{(j)})$, $j = 1, 2$, and let $\Lambda^{(j)}$, $j = 1, 2$, be their Dirichlet-to-Neumann operators. Assume that $\Omega$ is convex. Then $\Lambda^{(1)} = \Lambda^{(2)}$ if and only if $(A^{(1)}, V^{(1)})$ and $(A^{(2)}, V^{(2)})$ are gauge equivalent.

The proof of Theorem 1.1 is based on the method of complex exponential solutions with a large parameter that was introduced in [SU]. Let $\mu, \nu$ and $l$ be pairwise orthogonal vectors in $\mathbb{R}^n$ with $|\mu| = |\nu| = 1$. We set $\theta = \mu + i \nu$, and for $\tau \gg 0$ we set $\zeta = l/2 + (\tau^2 - |l|^2/4)^{1/2} \mu$ and $\delta = \zeta + i \tau \nu$. We look for solutions of (1.2) of the form $u = v \exp(ix \cdot \delta)$. Hence $v$ must satisfy

$$(1.4) \quad L_\delta v = \text{def.} \ (-i \frac{\partial}{\partial x} + \delta)^2 v - 2i A(x) \cdot (\frac{\partial}{\partial x} + i \delta) v + B(x) v = 0.$$ 

In order to solve systems of the form (1.4) or, more generally, inhomogeneous systems

$$(1.5) \quad L_\delta v = f \text{ in } \Omega_0,$$

where $\overline{\Omega} \subset \Omega_0$, we will need solutions of matrix equations of the form

$$(1.6) \quad i \theta \cdot \frac{\partial C(x, \theta)}{\partial x} = \theta \cdot A(x) C(x, \theta), \ x \in \Omega_0,$$
where $C$ is an invertible $m \times m$ matrix function. One can show that, when $A(x)$ is extended to a function of compact support in $\mathbb{R}^n$, there may be no solution of (1.6) which tends to the identity matrix as $|x| \to \infty$. However, there are always solutions which grow polynomially. The following lemma was proven in [E].

**Lemma 1.1.** There exists an invertible matrix function $C(x, \theta)$, solving (1.6) and depending smoothly on $(x, \theta)$ on the domain $\Omega_0 \times \{ \theta = \mu + iv : \mu \cdot v = 0, |\mu| = |v| = 1 \}$. This solution is not unique, but it can be chosen to satisfy $C(x, e^{i\omega \theta}) = C(x, \theta)$.

Using Lemma 1.1 we can prove (see [E]):

**Lemma 1.2.** For any $f \in L^2(\Omega_0)$ there is a $v(\tau) \in H^2(\Omega_0)$ such that for $\tau$ sufficiently large $v(\tau)$ solves (1.5) in $\Omega$, and

$$
\|v\|_{H^l(\Omega_0)} \leq \frac{C}{(1 + \tau)^{1-l}} \|f\|_{L^2(\Omega_0)}, \ l = 0, 1,
$$

where $C$ is independent of $\tau$ and $f$.

To prove Lemma 1.2 we proceed as follows. Let $C(x, \mu + iv)$ be as in Lemma 1.1 and let $c(x, D, \tau)$ be the pseudo-differential operator with symbol $C(x, \xi' + iv)\chi$, where $\xi' = \xi - (\xi \cdot v)v$ and $\chi$ is a suitable cutoff function in $(\zeta, \xi, \tau)$. We look for $v$ in the form

$$
(1.7) \quad v = c(x, D, \tau)E(\tau)g, \quad \text{where } E(\tau) = (-i \frac{\partial}{\partial x} + \delta)^{-2} \text{ and } g \in L^2(\Omega_0).
$$

Substituting (1.7) into (1.4) one gets $g + T(\tau)g = f$, where the norm of $T(\tau)$ goes to zero as $\tau \to \infty$. Therefore $I + T(\tau)$ is invertible for $\tau$ large. Note that the proof of Lemma 1.2 in [E] is a generalization of the method in [ER1] which treated the (scalar) case of electro-magnetic potentials. Lemma 1.2 is used in [E] to prove the following:

**Lemma 1.3.** For every vector of polynomials, $p(z)$, in the complex variable $z = \theta \cdot x$ there is a solution $v(\tau)$ of (1.4) satisfying

$$
(1.8) \quad \lim_{\tau \to \infty} v(\tau) = C(x, \theta)\Pi^+(C^{-1} p),
$$

where $C$ is the matrix function from Lemma 1.1 and $\Pi^+$ is a Toeplitz projection.
Using (1.7) to construct the leading order term in \(v(\tau)\) made both the proof of Lemma 1.3 and its applications somewhat complicated in [E]. In [ER2] we found a way to construct such solutions more simply with explicit higher order asymptotics as \(\tau \to \infty\), and this lead to simpler proofs. We will give this construction in §2.

Now we can complete the proof of Theorem 1.1. Following the strategy of [SU], we can use the assumption \(\Lambda^{(1)} = \Lambda^{(2)}\), Green’s formula and Lemma 1.3 to derive integral identities involving \(C(j)\) and \((A(j), V(j))\), \(j = 1, 2\). Then arguments involving \(\bar{\theta}\)-equations in the parameters in these identities (see §5 and §6 in [E]) lead to the proof of Theorem 1.1.

2 Construction of Complex Exponential Solutions.

To construct solutions of (1.1) of the form \(u = v \exp(ix \cdot \delta)\) we will proceed as follows. Substituting \(u = v \exp(ix \cdot \delta)\) into (1.1), one sees that \(v\) must satisfy

\[
L_\delta v = \text{def.} -\Delta v - 2i\delta \cdot \frac{\partial v}{\partial x} - 2iA \cdot (i\delta v + \frac{\partial v}{\partial x}) + Bv = 0.
\]

We will construct \(v\) in the form \(v = \sum_{k=0}^n v_k + \tilde{v}_n\), where \(v_k = O(\tau^{-k})\) and is explicit modulo solutions of (1.6), and \(\tilde{v}_n\) is \(O(\tau^{-n-1})\). We have

\[
\delta = \tau \theta + \frac{l}{2} + O(\tau^{-1}) = \text{def.} \tau \theta + \delta' \text{ and}
\]

\[
L_\delta = \text{def.} -2i\tau \theta \cdot \frac{\partial}{\partial x} + 2\tau \theta \cdot A + M_\delta.
\]

To solve (2.1) modulo terms of order \(\tau^{-n}\) we require

\[
2\tau (i\theta \cdot \partial_x - \theta \cdot A) v_k = M_\delta v_{k-1}
\]

for \(k = 0, \ldots, n\) with \(v_{-1} = 0\). We set \(v_0 = C_0(x, \theta)p(x \cdot \theta)\), where \(C_0\) is a solution of (1.6) with the properties described in Lemma 1.1 and \(p(z)\) is a vector of polynomials in the complex variable \(z\). Since we only require that \(v\) satisfy (2.1) on \(\Omega\), we can introduce a cut-off function \(\psi \in C_0^\infty(\Omega_0), \psi = 1\) on a neighborhood of \(\Omega\), and set

\[
v_k = -i(2\tau)^{-1} C_k(x, \theta)(\theta \cdot \partial_x)^{-1}(C_k^{-1}(\cdot, \theta) \psi M_\delta v_{k-1}), \ k = 1, \ldots, n.
\]
In (2.2) the operator \((\theta \cdot \partial_x)^{-1}\) multiplies the Fourier transform by \((i\theta \cdot \xi)^{-1}\), and \(C_k\) is again a solution of (1.6) as in Lemma 1.1 (in applications so far we have taken \(C_k = C_0\), but this is not necessary). Since \(M_\theta\) does not increase the order of a term in \(\tau\) and \(i(2\tau)^{-1}(\theta \cdot \partial_x)^{-1}\) adds a factor of \(\tau^{-1}\), we have \(v_k = O(\tau^{-k})\) and

\[
L\delta(v_0 + v_1 + \cdots + v_n) = M_\delta v_n = O(\tau^{-n})
\]
on the neighborhood of \(\Omega\) where \(\psi = 1\). Hence, taking \(\psi_0 \in C^\infty(\Omega_0)\) supported in the set where \(\psi = 1\) such that \(\psi_0 = 1\) on \(\Omega\), \(v = v_0 + \cdots + v_n + \tilde{v}_n\) will be a solution of (2.2) in \(\Omega\) if

(2.3) \[
L\delta \tilde{v}_n = -\psi_0 M_\delta v_n
\]
in \(\Omega_0\).

To solve (2.3) apply Lemma 1.2. Lemma 1.2 holds in a more general form (see [ER2]): if \(f \in H^k(\Omega_0), k \geq 0\), then \(v \in H^{k+2}(\Omega_0)\), and one has the estimate

\[
\|v\|_{H^{k+2}(\Omega_0)} \leq \frac{C}{(1 + \tau)^{1-l}}\|f\|_{H^k(\Omega_0)}, \quad l = 0, 1.
\]

Since \(v_n\) is bounded by \(C\tau^{-n}\), the solutions given by Lemma 1.2 will be bounded by \(C\tau^{-n-1}\) in \(H^k(\Omega_0)\) for \(k \geq 0\). Thus the \(v(\tau)\) that we have constructed is a solution of (2.4) whose asymptotics in \(H^k(\Omega)\) up to order \(\tau^{-n}\) are given by the asymptotics of \(v_0 + \cdots + v_n\). For the leading term in the asymptotics we have

(2.4) \[
v(\tau) = C_0(x, \theta)p(\theta \cdot x) + O(\tau^{-1}),
\]
where \(O(\tau^{-1})\) means bounded by \(C\tau^{-1}\) in \(H^k(\Omega), k \geq 0\). Since the limit of \(v(\tau)\) no longer involves a Toeplitz projection, one no longer needs one of the arguments (Lemma 5.1 in [E]).

3 The Equations of Isotropic Elasticity.

The construction presented in §2 can be used to construct solutions of the system of isotropic elasticity as well. Using subscripts for derivatives, this system is given by

(3.1) \[
\sum_{j=1}^{3}(\lambda w^{j}_{x_j})_{x_k} + \sum_{j=1}^{3}(\mu(w^{j}_{x_j} + w^{j}_{x_k}))_{x_j} = 0, \quad k = 1, 2, 3,
\]
where \( w = (w^1, w^2, w^3) \) is the deformation of an elastic body with “Lamé parameters” \( \lambda(x) \) and \( \mu(x) \). Let \( w \) be the solution of the Dirichlet problem for (3.1) with \( w = h \) on \( \partial \Omega \). Then the inverse boundary value problem for this system is to recover \( \lambda \) and \( \mu \) from the Dirichlet-to-Neumann map

\[ \Lambda(h^k) = \sum_{j=1}^{3}(\lambda w^j_x)n^k_j + \sum_{j=1}^{3}\mu(w^j_x + w^j_k)n^j_k, \quad k = 1, 2, 3. \]

There is as yet no proof that \( \Lambda \) determines \( \lambda \) and \( \mu \). Partial results are given in [NU2] and [ER2]. The system (3.1) is not in the form (1.1). However, Ang, Ikehata, Trong and Yamamoto in [AITY] show that \( w = \mu^{-1/2}u + \mu^{-1}\nabla f - f\nabla \mu^{-1} \) will satisfy (1.1) when the 4-vector \((u, f)\) satisfies the system

\[ \Delta \begin{pmatrix} u \\ f \end{pmatrix} + V_1(x) \begin{pmatrix} \nabla f \\ \nabla \cdot u \end{pmatrix} + V_0(x) \begin{pmatrix} u \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

Here

\[ V_1(x) = \begin{pmatrix} -2\mu^{1/2}\nabla^2 \mu^{-1} & -\mu^{-1}\nabla \mu \\ 0 & \frac{\lambda + \mu}{\lambda + 2\mu}\mu^{1/2} \end{pmatrix} \]

and \( \nabla^2 f \) denotes the Hessian matrix \( \partial^2 f/\partial x_j \partial x_k \). The matrix \( V_0 \) is a complicated expression in \( \lambda, \mu \) and their derivatives, but it vanishes when \( \mu \) is constant. Since (3.2) does have the form (1.1), the method above can be used to construct solutions \((u, f) = \exp(i\delta \cdot x)(r, s)\) of (3.1) with prescribed asymptotics as \( \tau \to \infty \).

Suppose that one has two elastic bodies occupying the region \( \Omega \) with Lamé parameters \((\lambda^{(j)}, \mu^{(j)})\), \( j = 1, 2 \). Let \( \Lambda^{(j)} \), \( j = 1, 2 \), be the corresponding Dirichlet-to-Neumann maps. Using Green’s formula, one can verify that \( \Lambda^{(1)} = \Lambda^{(2)} \) is equivalent to

\[ H(w^{(2)}, w^{(1)}) = \text{def.} \int_{\Omega} [(\lambda^{(2)} - \lambda^{(1)})(\nabla \cdot \overline{w^{(2)}})(\nabla \cdot w^{(1)}) + \frac{1}{2}(\mu^{(2)} - \mu^{(1)})\sum_{1 \leq j, k \leq 3}(w^{(2)}_k)^j w^{(2)}_j \cdot w^{(1)}_k w^{(1)}_j \cdot w^{(1)}_k dx = 0 \]

for all solutions to \( L^{(1)}w^{(1)} = 0, L^{(2)}w^{(2)} = 0 \), where now \( L^{(j)}w^{(j)} = 0 \) is the system (3.1) with \( \lambda = \lambda_j \) and \( \mu = \mu_j \). We can use (3.3) in the following way. The method of §2 yields solutions of (3.2) with prescribed asymptotics. For the problem corresponding to \( L^{(1)} \) we use these with \( \delta^{(1)} = \delta \) as defined earlier, but for the problem corresponding to \( L^{(2)} \) we set \( \delta^{(2)} = \delta - l \). Taking
\[ w^{(j)} = \mu_j^{-1/2} u^{(j)} + \mu_j^{-1} \nabla f^{(j)} - f^{(j)} \nabla \mu_j^{-1}, \]

we substitute \( w^{(1)} \) and \( w^{(2)} \) into (3.3). When we collect the terms of each order in \( \tau \), this gives

\[ 0 = H(w^{(2)}, w^{(1)}) = \tau^2 H_2 + \tau^1 H_1 + H_0 + \cdots + \tau^{-N} H_{-N} + O(\tau^{-N-1}), \]

where each \( H_j \) is independent of \( \tau \). How far one can continue this expansion depends on the choice of \( n \) in \( \S 2 \). Note that each \( H_j \) must vanish when \( \Lambda^{(1)} = \Lambda^{(2)} \). In particular, one has

\[ H_2 = \int \Omega e^{i\xi(\theta \cdot r_0^{(2)}, s_0^{(2)})} V(x, \theta)(\theta \cdot r_0^{(1)}, s_0^{(1)})^t dx = 0, \]

where

\[ V(x, \theta) = \begin{pmatrix} (\lambda_1 + \mu_1 - \lambda_2 - \mu_2)(\mu_1 \mu_2)^{1/2} \left( \frac{2(\mu_2^{-1} - \mu_1^{-1})}{2(\mu_2^{-1} - \mu_1^{-1}) \mu_1^{-1/2} \theta \cdot \partial x b_2} \right) & 2(\mu_2^{-1} - \mu_1^{-1}) \mu_2^{-1/2} \theta \cdot \partial x b_2 \\ 2(\mu_2^{-1} - \mu_1^{-1}) \mu_1^{-1/2} \theta \cdot \partial x b_1 & 2(\mu_2^{-1} - \mu_1^{-1})(b_1 a_1 + b_2 a_2) \end{pmatrix}. \]

The functions \( a_j \) and \( b_j \), \( j = 1, 2 \) are given by

\[ a_j = (\theta \cdot \partial_x)^2 \mu_j^{-1} \quad \text{and} \quad b_j = \frac{\mu_j}{2} \frac{\lambda_j}{\lambda_j + 2 \mu_j}. \]

The functions \( (r_0^{(j)}, s_0^{(j)}) \) are vector solutions of the following version of (1.6)

\[ -2\theta \cdot \partial_x \begin{pmatrix} r_0 \\ s_0 \end{pmatrix} = V_1 \begin{pmatrix} s_0 \theta^t \\ \theta \cdot r_0 \end{pmatrix}. \]

This is (1.6) with

\[ A = -\frac{i}{2} V_1 \begin{pmatrix} 0_{3 \times 3} & \theta^t \\ \theta & 0 \end{pmatrix}. \]

One can analyze \( H_2 \) by the techniques that were used for the Yang-Mills case in [E]. However, here that does not lead to the conclusion that \( \lambda_1 = \lambda_2 \) and \( \mu_1 = \mu_2 \). Instead one arrives at the identity (Theorem 1.3 of [ER2])

\[ b_1^{1/2}(\theta \cdot \partial_x)^2(b_1^{-1/2}) - b_1(\theta \cdot \partial_x)^2(\mu_1^{-1}) = b_2^{1/2}(\theta \cdot \partial_x)^2(b_2^{-1/2}) - b_2(\theta \cdot \partial_x)^2(\mu_2^{-1}). \]

Since this holds identically in \( \theta \), it is equivalent to a system of five partial differential equations satisfied by \( (\lambda_1, \mu_1, \lambda_2, \mu_2) \). However, even in the case that \( \lambda_1 \) and \( \mu_1 \) are constant, these equations have solutions with \( \lambda_2 \) and \( \mu_2 \) nonconstant.

A more readily useful result is the following (Theorem 2 of [ER2]):
Theorem 3.1. Let
\[ \|f\|_\alpha^2 = \int_\Omega |f|^2 e^{2\alpha|x|^2} \, dx. \]
Then, if \( \Lambda^{(1)} = \Lambda^{(2)} \), for \( \alpha > \alpha_0 \), one has
\[ \|\lambda_1 + \mu_1 - \lambda_2 - \mu_2\|_\alpha \leq \frac{C}{\alpha} \|\mu_1 - \mu_2\|_\alpha \]
with \( C \) independent of \( \alpha \).

This result comes from \( \overline{\partial} \)-equations that arise when one follows the method of §6 in [E]. Theorem 3.1 can be used to deduce uniqueness for constrained forms of the inverse problem. For instance, if one is given either that \( \lambda_1 = \lambda_2 \) or that \( \mu_1 = \mu_2 \), and that the Dirichlet-to-Neumann maps are equal, then both Lamé parameters must be equal.

The second way that one can use (3.4) is to choose \( \theta \) as a function of \( l/|l| \) keeping \( \theta(l/|l|) \cdot l = 0 \). This makes (3.4) (and each of the other equations in the sequence \( \{H_j = 0\}_{j=2}^{\infty} \)) equivalent to a pseudo-differential equation of the form
\[ P(x, D)(\lambda_2 - \lambda_1) + Q(x, D)(\mu_2 - \mu_1) = 0 \]
To prove uniqueness for the inverse boundary value problem we would like to use (3.6) to bound \( \lambda_2 - \lambda_1 \) in terms of \( \mu_2 - \mu_1 \) or to bound \( \mu_2 - \mu_1 \) in terms of \( \lambda_2 - \lambda_1 \). Boundary determination for this inverse problem (see [NU1]) implies that \( \lambda_2 - \lambda_1 \) and \( \mu_2 - \mu_1 \) vanish to infinite order on \( \partial \Omega \). Nonetheless, (3.6) does not imply any estimates of this kind without more information on the operators \( P \) and \( Q \). Note that the construction in §2 makes \( (r, s) \) a function of \( \theta \), so that it contributes to the symbols of \( P \) and \( Q \), and, since the solutions of (3.5) are not explicit in general, one usually does not know what \( P \) and \( Q \) are. However, (3.6) becomes useful when one assumes that \( \nabla \mu_j \) is small in \( C^\infty \). In that case all entries in \( V_1 \) except \( (V_1)_{44} \) become small and one can solve (3.5) explicitly modulo small terms. In fact (3.5) has a unique matrix solution tending to the identity as \( |x| \to \infty \) and given by
\[ C_{00}(x, \theta) = \begin{pmatrix} I_{3 \times 3} & 0 \\ \varphi \theta & 1 \end{pmatrix}, \]
modulo small terms, where \( \varphi = (\theta \cdot \partial_x)^{-1}(-\psi b^{-1}) \). Choosing
\[ p = (\text{Re}\{\theta(l/|l|)\}, 0) \]
in (2.4), we get

\[ (r^{(j)}, s^{(j)}) = (\text{Re}\{\theta(l/|l|)\}, 0) + (\tilde{r}_0^{(j)}, \tilde{s}_0^{(j)}) + O(\tau^{-1}), \]

where \( \tilde{r}_0^{(j)}, \tilde{s}_0^{(j)} \) are symbols of order zero and \( \tilde{r}_0^{(j)} \) is small. Since \( \theta \cdot \text{Re}\{\theta\} = 1 \), (3.8) makes \( P(x, D) \) in (3.6) simply multiplication by \( (\mu_1 \mu_2)^{1/2}(\lambda_1 + 2\mu_1)^{-1}(\lambda_2 + 2\mu_2)^{-1} \) plus an operator of order zero whose norm can be made arbitrarily small by taking \( \nabla \mu_j, j = 1, 2 \), sufficiently small in \( C^k(\Omega) \). Thus (3.6) implies

\[ \|\lambda_2 - \lambda_1\|_{H^k(\Omega)} \leq C_k \|\mu_2 - \mu_1\|_{H^k(\Omega)} \]

for all \( k \), when \( \nabla \mu_j, j = 1, 2 \), is sufficiently small.

Uniqueness for the inverse boundary value problem when \( \nabla \mu_j \) is small will follow from (3.9) if we can find another estimate bounding a norm of \( \mu_2 - \mu_1 \) by a small constant times a norm of \( \lambda_2 - \lambda_1 \). One can get this estimate by using \( H_0 = 0 \) in the following way. The contributions to \( H_0 \) come from the expansion of \( (r, s) \) in \( \tau \) up to order \( \tau^{-2} \), and are therefore quite complicated in general. Nonetheless, using (3.7) and taking \( p = (0, 0, 0, 1) \) in (2.4), we find that (3.5) has the solution \((\tilde{r}_0, \tilde{s}_0) = (0, 0, 0, 1) + (\tilde{r}_0, \tilde{s}_0)\), where \((\tilde{r}_0, \tilde{s}_0))\) and its derivatives can be made arbitrarily small by taking \( \nabla \mu_j, j = 1, 2 \), sufficiently small in \( C^k \)-norm. Moreover, checking further, one sees that the higher order terms in the expansion of \((r, s)\) that one constructs using (2.2) are also small under these hypotheses. When one uses this choice of \((r, s)\) in the construction of \( w^{(1)} \) and \( w^{(2)} \), one gets

\[ H_0 = \int_{\Omega} e^{i\frac{l}{|l|}x}[(\lambda_1 - \lambda_2)A(x, \theta(l/|l|), l) + (\mu_1 - \mu_2)[(2\mu_1 \mu_2)^{-1}|l|^4 + B(x, \theta(l/|l|), l)]]dx, \]

where \( A \) and \( B \) are small symbols in \( l \) of orders two and four respectively. Thus we have (3.6) with \( \|P(x, D)(\lambda_2 - \lambda_1)\|_{L^2(\Omega)} \) bounded by a small constant times \( \|\lambda_2 - \lambda_1\|_{H^2(\Omega)} \) and

\[ Q(x, D)(\mu_2 - \mu_1) = (2\mu_1 \mu_2)^{-1}(\Delta)^2(\mu_2 - \mu_1) + R(x, D)(\mu_2 - \mu_1), \]

where \( \|R(x, D)(\mu_2 - \mu_1)\|_{L^2(\Omega)} \) is bounded by a small constant times \( \|\mu_2 - \mu_1\|_{H^2(\Omega)} \). Thus we have

\[ \|\mu_2 - \mu_1\|_{H^4(\Omega)} \leq \varepsilon(\|\mu_2 - \mu_1\|_{H^4(\Omega)} + \|\lambda_1 - \lambda_2\|_{H^4(\Omega)}), \]

where \( \varepsilon \) can be taken arbitrarily small when \( \nabla \mu_j, j = 1, 2 \) is sufficiently small in \( C^k(\Omega) \). Combining (3.9) and (3.10), we have the following result.
Theorem 3.2. Given that \( \lambda_j, \mu_j \) and \( \mu_j^{-1}, j = 1, 2 \), belong to a bounded set, \( B \), in \( C^k(\Omega) \) for \( k \) sufficiently large, there is an \( \varepsilon(B) > 0 \) such that \( \| \nabla \mu_j \|_{C^{k-1}(\Omega)} < \varepsilon(B), j = 1, 2 \), implies \( (\lambda_1, \mu_1) = (\lambda_2, \mu_2) \), if \( \Lambda^{(1)} = \Lambda^{(2)} \).

This is the main result of [NU2] and it is also Theorem 1 of [ER2]. The first version of [ER2] (July 2001) contains Theorem 3.2 with additional hypothesis \( \| \nabla \lambda_j \|_{C^{k-1}(\Omega)} < \varepsilon(B), j = 1, 2 \). We did not notice that our proof did not use this hypothesis until we received a preprint of [NU2] (November 2001). We are grateful to G. Nakamura and G. Uhlmann for sending this to us.

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