Eight-dimensional non-geometric heterotic strings with gauge algebra $E_8 \times E_7$ were constructed by Malmendier and Morrison as heterotic duals of F-theory on K3 surfaces with $\Lambda^{1,1} \oplus E_8 \oplus E_7$ lattice polarization. Clingher, Malmendier and Shaska extended these constructions to eight-dimensional non-geometric heterotic strings with gauge algebra $E_7 \times E_7$ as heterotic duals of F-theory on $\Lambda^{1,1} \oplus E_7 \oplus E_7$ lattice polarized K3 surfaces.

In this study, we analyze the points in the moduli of non-geometric heterotic strings with gauge algebra $E_7 \times E_7$, at which the non-Abelian gauge groups on the F-theory side are maximally enhanced. The gauge groups on the heterotic side do not allow for the perturbative interpretation at these points. We show that these theories can be described as deformations of the stable degenerations, as a result of coincident 7-branes on the F-theory side. From the heterotic viewpoint, this effect corresponds to the insertion of 5-branes. These effects can be used to understand nonperturbative aspects of nongeometric heterotic strings.

Additionally, we build a family of elliptic Calabi–Yau 3-folds by fibering elliptic K3 surfaces, which belong to the F-theory side of the moduli of non-geometric heterotic strings with gauge algebra $E_7 \times E_7$, over $\mathbb{P}^1$. We find that highly enhanced gauge symmetries arise on F-theory on the built elliptic Calabi–Yau 3-folds.
1 Introduction

F-theory/heterotic duality [1, 2, 3, 4, 5] states that F-theory [1, 2, 3] compactification on an elliptic K3 fibered Calabi–Yau (n + 1)-fold describes a theory physically equivalent to heterotic compactification [7] on an elliptic Calabi–Yau n-fold. Non-perturbative aspects of heterotic theory can be studied by utilizing this duality. F-theory/heterotic duality is strictly formulated when the stable degeneration limit [5, 17] is taken on the F-theory side, in which K3 fibers split into pairs of half K3 surfaces.

Recently, eight-dimensional non-geometric heterotic strings with unbroken $e_8 e_7$ are constructed by Malmendier and Morrison [24] by utilizing the F-theory/heterotic duality. The Narain space [25]

\[ D_{2,18} / O(\Lambda^{2,18}) \]

(1)
gives the moduli space of eight-dimensional heterotic strings, and the double cover of this space,

\[ D_{2,18} / O^+(\Lambda^{2,18}) \]

(2)
is equivalent to the moduli space of F-theory on elliptic K3 surfaces with a section. This is the statement of the F-theory/heterotic duality. They considered F-theory compactifications on elliptic K3 surfaces with $H \oplus E_8 \oplus E_7$ lattice polarization, namely elliptically fibered K3 surfaces with a type $II^*$ fiber and a type $III^*$ fiber with a global section, and they constructed the moduli of heterotic strings with unbroken $e_8 e_7$ as the heterotic duals of them on the 2-torus. The moduli space of the non-geometric heterotic strings with unbroken $e_8 e_7$ constructed in [24] is given by

\[ D_{2,3} / O^+(L^{2,3}) \]

(3)
$L^{2,3}$ denotes the orthogonal complement of $H \oplus E_8 \oplus E_7$ inside the K3 lattice $\Lambda_{K3}$, and the non-geometric heterotic strings constructed in [24] possess $O^+(L^{2,3})$-symmetry. $O^+(L^{2,3})$ mixes the complex structure moduli, the Kähler moduli and the moduli of Wilson line values. Therefore, the resulting heterotic strings do not have a geometric interpretation [4] for this reason, the resulting heterotic strings are called non-geometric heterotic strings. A single Wilson line expectation value is nonzero for the non-geometric heterotic strings with unbroken $e_8 e_7$ gauge algebra as constructed in [24]. The mathematical results of Kumar [28] and Clingher and Doran [29, 30], which gave the Weierstrass equations of elliptic K3 surfaces with a global section with $E_8 E_7$ singularity, the coefficients of which are Siegel modular forms of even weight, were used in their construction.

Clingher, Malmendier and Shaska [31] extended the construction of non-geometric heterotic strings by Malmendier and Morrison to non-geometric heterotic strings with unbroken

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1. Recent progress of heterotic strings can be found, for example, in [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].
2. Stable degenerations in F-theory/heterotic duality are recently studied, for example, in [18, 19, 20, 21].
3. [22, 23] discussed connections of K3 surfaces with lattice polarizations, non-geometric heterotic strings, and $O^+(A_2^2)$-modular forms.
4. [27] discussed non-geometric type II theories.
$\mathfrak{e}_7\mathfrak{e}_7$ algebra. F-theory compactifications on elliptic K3 surfaces with $H \oplus E_7 \oplus E_7$ lattice polarization, namely K3 surfaces with a global section with two type $III^*$ fibers, were considered, and eight-dimensional non-geometric heterotic strings on $T^2$ were obtained as the heterotic duals in their construction. The moduli of the resulting heterotic strings is parametrized by the space

$$D_{2,4}/O^+(L^{2,4}).$$

$D_{2,4}$ is the symmetric space of $O(2,4)$, namely, $D_{2,4}$ is defined as $O(2) \times O(4) \setminus O(2,4)$. $D_{2,4}$ is also referred to as the bounded symmetric domain of type IV. $L^{2,4}$ denotes the orthogonal complement of $H \oplus E_7 \oplus E_7$ in the K3 lattice $\Lambda_{K3}$. The complex structure moduli, Kähler moduli and the moduli of Wilson line expectation values are mixed under the symmetry $O^+(L^{2,4})$, thus the heterotic strings constructed in [31] do not have a geometric interpretation as well. Two Wilson line expectation values are nontrivial in non-geometric heterotic strings with unbroken $\mathfrak{e}_7\mathfrak{e}_7$. See also, [32, 33, 34, 35, 36, 37, 38, 39, 40], for recent progress of non-geometric heterotic strings.

In this note, we analyze theories that correspond to the points in the moduli of eight-dimensional non-geometric heterotic strings on the 2-torus $T^2$ constructed in [31], at which the ranks of the non-Abelian gauge groups are enhanced to 18 on the F-theory side in the moduli. These are the maximal enhancements of the non-Abelian gauge groups on the F-theory side. We mainly consider $E_8 \times E_8$ heterotic strings, rather than $SO(32)$ heterotic strings. (However, we consider some applications to $SO(32)$ heterotic strings.) Because only up to $E_8 \times E_8 \times U(1)^4$ gauge group can arise in eight-dimensional $E_8 \times E_8$ heterotic strings compactified on the 2-torus [39], a consideration on the ranks of the non-Abelian gauge groups reveals that the gauge groups of the dual heterotic theories of these F-theory models do not allow for the perturbative interpretation. These heterotic strings include the nonperturbative effects of 5-branes. We find that these theories can be described as the deformations of the heterotic strings from the stable degeneration limit, in which the F-theory/heterotic duality strictly holds, and that these deformations result from the coincident 7-branes on the F-theory side. In the heterotic language, the effect of coincident 7-branes corresponds to the presence of 5-branes.

When the non-Abelian gauge groups on F-theory on an elliptic K3 surface are enhanced to rank 18, K3 surfaces become extremal K3 surfaces. A K3 surface is called attractive, when it has the Picard number $\rho = 20$, which is the highest value for a complex K3 surface. When an attractive K3 surface has an elliptic fibration with a section with the singularity type of rank 18, the elliptic fibration is referred to as extremal. Owing to the classification result in [41], the complex structures of extremal K3 surfaces on which non-Abelian gauge groups on F-theory compactifications are enhanced to rank 18 in the moduli can be determined, and this enables to deduce the Weierstrass equations of attractive K3 surfaces. By analyzing the deduced Weierstrass equations, we study F-theory compactifications and the non-geometric heterotic duals at these special points in the moduli.

We also discuss applications to $SO(32)$ heterotic strings in this study. We deduce the Weierstrass equations of elliptic K3 surfaces appearing as the compactification spaces of the F-theory duals of some $SO(32)$ heterotic strings, which are obtained as the transformations of
non-geometric heterotic strings. We also provide an example of $SO(32)$ heterotic string, which can be seen as a deformation of $SO(16) \times SO(16)$ heterotic string.

Additionally, we consider fibered elliptic K3 surfaces that belong to the F-theory side of the moduli of eight-dimensional non-geometric heterotic strings, over $\mathbb{P}^1$, to build elliptically fibered Calabi–Yau 3-folds with a global section. We study F-theory compactifications on the resulting elliptic Calabi–Yau 3-folds. We find that highly enhanced gauge groups arise in these compactifications. Local F-theory model buildings have been mainly discussed in recent studies [60, 61, 62, 63]. However, the global aspects of the geometry need to be considered to discuss the issues of gravity. We investigate F-theory on elliptically fibered Calabi–Yau 3-folds from the global perspective in this study.

A similar organization can be found in [39].

This note is structured as follows: in Section 2, we briefly review F-theory compactifications, and we also review attractive K3 surfaces and extremal K3 surfaces that are technically necessary to analyze special points in the moduli of eight-dimensional non-geometric heterotic strings and F-theory duals. We also review the construction of non-geometric heterotic strings with unbroken $e_7$ in [31].

In Section 3, we discuss the special points in the eight-dimensional non-geometric heterotic moduli with unbroken $e_7$ at which the ranks of the non-Abelian gauge symmetries on the F-theory side are enhanced to 18. The gauge groups in the heterotic strings which correspond to these points do not allow for the perturbative interpretations. We demonstrate that these theories can be seen as deformations of the stable degenerations as a consequence of the coincident 7-branes on the F-theory side. We also discuss applications to $SO(32)$ heterotic strings. We derive the Weierstrass equations of K3 elliptic fibrations as the compactification spaces of the F-theory duals of some $SO(32)$ heterotic strings. We determine the gauge groups that arise on F-theory compactifications, and the global structure of the gauge groups. We also discuss an example of $SO(32)$ heterotic string which can be seen as a deformation of $SO(16) \times SO(16)$ heterotic string.

We build elliptically fibered Calabi–Yau 3-folds in Section 4 by fibering examples of elliptic K3 surfaces, which belong to the F-theory side of the moduli of eight-dimensional unbroken $e_7$ non-geometric heterotic strings, over $\mathbb{P}^1$. We analyze F-theory compactifications on the resulting elliptic Calabi–Yau 3-folds. First, we consider the higher dimensional analogue of the construction of genus-one fibered K3 surfaces without a global section to build genus-one fibered Calabi–Yau 3-folds without a section. This construction ensures that the resulting 3-folds in fact satisfy the Calabi–Yau condition. Similar constructions of genus-one fibered Calabi–Yau 4-folds without a section using double covers can be found in [75]. Taking the Jacobian fibration of the resulting genus-one fibered Calabi–Yau 3-folds yields elliptically

Recent discussions of F-theory compactifications on elliptic Calabi-Yau 3-folds can be found, e.g., in [12, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59]. [57, 58, 59] discuss F-theory on Calabi-Yau 3-folds with terminal singularities.

Recent studies of F-theory compactifications on genus-one fibered spaces lacking a global section can be found in, for example, [64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84].

[85] discusses the Jacobians of elliptic curves.
fibered Calabi–Yau 3-folds with a global section. K3 fibers of these elliptic Calabi–Yau 3-folds belong to the F-theory side of the moduli of eight-dimensional non-geometric heterotic strings with unbroken $e_7 e_7$. Therefore, the obtained elliptic Calabi–Yau 3-folds can be seen as the fibering of such K3 surfaces over the base curve $\mathbb{P}^1$. We deduce the gauge groups on F-theory compactifications on the elliptic Calabi–Yau 3-folds, and we find that some specific models do not have a $U(1)$ gauge field. We determine the Mordell–Weil groups of some models, and we obtain the global structures of the gauge groups of these models. We state our concluding remarks in Section 5.

2 Review of non-geometric heterotic strings with unbroken $e_7 e_7$, F-theory and extremal K3s

2.1 Review of F-theory compactifications

We briefly review F-theory compactifications on elliptic K3 surfaces. A similar review can be found in [39]. F-theory is compactified on spaces that admit a genus-one fibration. The complex structure of the genus-one fiber is identified with the axio-dilaton in F-theory compactification. This formulation allows the axio-dilaton to have $SL(2, \mathbb{Z})$ monodromy. Genus-one fibrations do not necessarily admit a global section; there are situations in which they have a global section, and those in which they do not. F-theory compactifications on elliptic fibrations with a global section have been investigated in recent studies, for example, in [86, 87, 88, 89, 91, 92, 93, 94, 95, 96, 97, 98, 21, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111]. Although Calabi–Yau genus-one fibration lacking a global section cannot be expressed in the Weierstrass form, when the Jacobian fibration of it exists, the Jacobian fibration yields an elliptic fibration with a global section. Calabi–Yau genus-one fibration $Y$ and the Jacobian fibration $J(Y)$ have the identical types of the singular fibers, and they have the same discriminant loci.

Genus-one fibers degenerate over the codimension 1 locus in the base space, and this locus is referred to as the discriminant locus. Such fibers are called the singular fibers. When genus-one fiber degenerates, it becomes either $\mathbb{P}^1$ with a single singularity, or a sum of smooth $\mathbb{P}^1$’s meeting in specific ways. The types of the singular fibers of genus-one fibered surfaces were classified by Kodaira [112, 113]. Methods to determine the singular fibers of elliptic surfaces can be found in [114, 115].

In F-theory compactifications on genus-one fibrations, the non-Abelian gauge groups that form on the 7-branes correspond to the singular fibers of genus-one fibrations [3, 116]. The correspondences of the singular fibers and the singularity types of the compactification spaces are shown in Table 1 below. The corresponding monodromies and j-invariants of the singular fibers are also presented in the table.

The types of singular fibers of elliptic surfaces can be determined from the vanishing orders of the coefficients of the Weierstrass equations. The correspondences of the fiber types and the vanishing orders of the Weierstrass coefficients are shown in Table 2.
| Fiber type | J-invariant | Monodromy | Order of Monodromy | Singularity Type |
|------------|-------------|-----------|--------------------|------------------|
| $I_0$      | regular     | $-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | 2                 | $D_4$            |
| $I_m$      | $\infty$   | $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ | infinite         | $A_{m-1}$        |
| $I_m^*$    | $\infty$   | $-\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ | infinite         | $D_{m+4}$        |
| II         | 0           | $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ | 6                 | none.            |
| II*        | 0           | $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ | 6                 | $E_8$            |
| III        | 1728        | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ | 4                 | $A_1$            |
| III*       | 1728        | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | 4                 | $E_7$            |
| IV         | 0           | $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ | 3                 | $A_2$            |
| IV*        | 0           | $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ | 3                 | $E_6$            |

Table 1: Monodromies, j-invariants and the corresponding types of the singularities of singular fibers. “Regular” for j-invariant of $I_0^*$ fiber means that j-invariant can take any finite value in $\mathbb{C}$ for $I_0^*$ fiber.

When an elliptic fibration has a global section, the set of sections form a group, known as the Mordell–Weil group. The rank of the Mordell–Weil group gives the number of the $U(1)$ gauge fields in F-theory compactification on the elliptic fibration [3].

The second integral cohomology group $H^2(S, \mathbb{Z})$ of K3 surface $S$ includes the information of the geometry of the K3 surface. This group has the lattice structure, and it is called the K3 lattice, $\Lambda_{K3}$. The K3 lattice is unimodular, even lattice of signature $(3,19)$, and it is isometric to the direct sum of two $E_8$’s and three hyperbolic planes [117]

$$\Lambda_{K3} \cong E_8^2 \oplus H^3.$$ (5)

The group of divisors (modulo algebraic equivalence) constitutes a sublattice inside the K3 lattice, called the Néron-Severi lattice $NS(S)$. When a K3 surface has an elliptic fibration with a global section, an elliptic fiber and a global section generate the hyperbolic plane $H$ inside the Néron-Severi lattice. K3 surface $S$ admitting an elliptic fibration with a section is equivalent to the condition that the Néron-Severi lattice $NS(S)$ contains the hyperbolic plane $H$ [118]. When an elliptic K3 surface has the singular fibers, the Néron-Severi lattice $NS(S)$ contains the $ADE$ lattices that correspond to the types of the singular fibers. For example,
Table 2: List of the types of the singular fibers, and the corresponding vanishing orders of the coefficients, \( f, g \), of the Weierstrass equation \( y^2 = x^3 + f x + g \), and the orders of the discriminant, \( \Delta \).

| Fiber Type | \( \text{Ord}(f) \) | \( \text{Ord}(g) \) | \( \text{Ord}(\Delta) \) |
|------------|----------------------|----------------------|----------------------|
| \( I_0 \)  | \( \geq 0 \)         | \( \geq 0 \)         | 0                    |
| \( I_n \)  | 0                    | 0                    | \( n \)              |
| \( II \)   | \( \geq 1 \)         | 1                    | 2                    |
| \( III \)  | 1                    | \( \geq 2 \)         | 3                    |
| \( IV \)   | \( \geq 2 \)         | 2                    | 4                    |
| \( I_0^* \) | \( \geq 2 \)         | 3                    | 6                    |
|            | 2                    | \( \geq 3 \)         |                      |
| \( I_m^* \) | 2                    | 3                    | \( m + 6 \)          |
| \( IV^* \) | \( \geq 3 \)         | 4                    | 8                    |
| \( III^* \)| 3                    | \( \geq 5 \)         | 9                    |
| \( II^* \) | \( \geq 4 \)         | 5                    | 10                   |

that a K3 surface \( S \) is \( H \oplus E_7 \oplus E_7 \)-lattice polarized means that the Néron-Severi lattice \( NS(S) \) includes the lattice \( H \oplus E_7 \oplus E_7 \), and this is equivalent to the condition that the K3 surface \( S \) is elliptically fibered with a section, the singular fibers of which include two type \( III^* \) fibers (or worse). K3 surfaces with \( H \oplus E_7 \oplus E_7 \) lattice polarization are parametrized by the bounded symmetric domain of type \( IV \), \( D_2,4 \), modded out by the symmetry of the orthogonal complement of the lattice \( H \oplus E_7 \oplus E_7 \) inside the K3 lattice \( \Lambda_{K3} \):

\[
D_{2,4}/O^+(L^{2,4}).
\]

\( L^{2,4} \) denotes the orthogonal complement of \( H \oplus E_7 \oplus E_7 \) inside the K3 lattice \( \Lambda_{K3} \). Because the K3 lattice \( \Lambda_{K3} \) is isometric to \( E_8^2 \oplus H^3 \), \( L^{2,4} \) can also be defined as the orthogonal complement of \( E_7 \oplus E_7 \) in the lattice \( E_8^2 \oplus H^3 \).

By utilizing the F-theory/heterotic duality, eight-dimensional non-geometric heterotic strings with unbroken \( e_7 \cdot e_7 \), the moduli space of which is equivalent to (6) were constructed in [31]. In this note, we study the points in the moduli of such non-geometric heterotic strings at which the non-Abelian gauge symmetries are enhanced to rank 18 on the F-theory side.
2.2 Construction of \(e_7 e_7\) non-geometric heterotic strings by Clingher, Malmendier and Shaska

We briefly review the construction of eight-dimensional non-geometric heterotic strings with unbroken \(e_7 e_7\) by Clingher, Malmendier and Shaska [31].

As stated previously, the moduli of elliptic K3 surfaces with a global section with two \(E_7\) singularities, namely the K3 surfaces with \(H \oplus E_7 \oplus E_7\) lattice polarization, are parameterized by the following space:

\[
D_{2,4}/O^+(L^{2,4}).
\]  

The bounded symmetric domain of type \(IV\), \(D_{2,4}\), is known to be isomorphic to \(H_2\) [119]:

\[
H_2 \cong D_{2,4}.  \tag{8}
\]

\(H_2\) is defined as

\[
H_2 := \left\{ \left( \begin{array}{cc} z_1 & z_2 \\ z_3 & z_4 \end{array} \right) \in M_2(\mathbb{C}) \mid 4 \text{Im} z_1 \text{Im} z_4 > |z_2 - z_3|^2 \text{ and } \text{Im} z_4 > 0 \right\}.  \tag{9}
\]

As mentioned in [31], \(H_2\) is a generalization of the Siegel upper-half space \(H_2\) in the following sense:

\[
H_2 = \left\{ \omega \in H_2 \mid \omega^t = \omega \right\}.  \tag{10}
\]

The modular group \(\Gamma\) acting on \(H_2\) is defined as

\[
\Gamma = \left\{ G \in GL_4(\mathbb{Z}[i]) \mid G^\dagger \left( \begin{array}{cc} 0 & 1_2 \\ -1_2 & 0 \end{array} \right) G = \left( \begin{array}{cc} 0 & 1_2 \\ -1_2 & 0 \end{array} \right) \right\}.  \tag{11}
\]

\(1_2\) denotes the \(2 \times 2\) identity matrix. \(\left( \begin{array}{cc} A & B \\ C & D \end{array} \right)\) in the modular group \(\Gamma\) acts on \(\omega \in H_2\) as

\[
\left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \cdot \omega = (A \omega + B)(C \omega + D)^{-1}.  \tag{12}
\]

There is an involution, \(T\), that acts on \(H_2\) as

\[
T \cdot \omega = \omega^t.  \tag{13}
\]

The group \(\Gamma_T\) is defined to be the semi-direct product of the modular group \(\Gamma\) and \(< T >\):

\[
\Gamma_T := \Gamma \rtimes < T >.  \tag{14}
\]

There is an isomorphism \(\Gamma_T \cong O^+(L^{2,4})\), and this induces the isomorphism (8) [119].

Under the isomorphism \(\Gamma_T \cong O^+(L^{2,4})\), the ring of \(O^+(L^{2,4})\)-modular forms corresponds to the ring of \(\Gamma_T\)-modular forms of even characteristic [120], generated by the five modular forms \(J_k\) of weights \(2k\), \(k = 2, \cdots, 6\) [31]. See [31] for definitions of the modular forms \(J_k\).
In a special situation, the modular forms \( J_2, J_3, J_5, J_6 \) restrict to Igusa’s generators \([121], \psi_4, \psi_6, \chi_{10}, \chi_{12} \) (and \( J_4 \) vanishes in this situation) \([31]\).

The periods of \( H \oplus E_7 \oplus E_7 \) lattice polarized K3 surfaces \( S \) in \( H^2(S, \mathbb{Z}) \) determine points in \( \mathbb{H}_2 \). The Weierstrass coefficients of such elliptically fibered K3 surfaces were given in terms of \( \Gamma_T \)-modular forms of even weights \([31]\).

The Weierstrass equation of a K3 surface with \( H \oplus E_7 \oplus E_7 \) lattice polarization is given by \([31]\):

\[
y^2 = x^3 + (e t^4 + c t^3 + a t^2) x + t^5 + g t^6 + (d e + f) t^5 + c d t^4 + b t^3.
\]

(15)

Up to some scale factors, the coefficients are given in terms of the modular forms \( J_2, J_3, J_4, J_5, J_6 \) \([31]\):

\[
c = -J_5(\omega), \quad d = -\frac{1}{3} J_4(\omega), \quad e = -3J_2(\omega) \\
f = J_6(\omega), \quad g = -2J_3(\omega) \\
a = -3d^2 = -\frac{1}{3} J_4(\omega)^2, \quad b = -2d^3 = \frac{2}{27} J_4(\omega)^3.
\]

(16)

The elliptically fibered K3 surface determines a point in \( D_{2,4} \), and this also determines a point in \( \mathbb{H}_2 \) under the isomorphism \([8]\), which we denote by \( \omega \).

Now, consider a manifold \( M \) and a line bundle \( \Lambda \) on \( M \), and choose sections \( a, b, c, d, e, f, g \) of the line bundles \( \Lambda^{\otimes 16}, \Lambda^{\otimes 24}, \Lambda^{\otimes 10}, \Lambda^{\otimes 8}, \Lambda^{\otimes 6} \) and \( \Lambda^{\otimes 4} \), respectively. When the sections \( a, b, c, d, e, f, g \) are identified as \([16]\), because \( \Gamma_T \) is isomorphic to \( O^+ (L_{2,4}) \), the compactification on \( M \) (which is the 2-torus when we consider 8D heterotic strings) gives a heterotic string theory with \( O^+ (L_{2,4}) \)-symmetry. The moduli space of eight-dimensional heterotic strings on the 2-torus \( T^2 \) decomposes into the product of the complex structure moduli, the Wilson line expectation values and Kähler moduli, in a suitable limit \([122]\). The complex structure moduli, the Wilson line expectation values and Kähler moduli are mixed under the \( O^+ (L_{2,4}) \)-symmetry. This represents the construction of non-geometric heterotic strings with \( e_7 e_7 \) gauge algebra in \([31]\).

The locus in the moduli in which the singularity ranks of elliptic K3 surfaces are enhanced satisfies 5-brane solutions on the heterotic side. The generic 5-brane solutions of non-geometric heterotic strings with \( e_7 e_7 \) gauge algebra are discussed in \([31]\).

Elliptic K3 surfaces with the lattice polarization \( H \oplus E_7 \oplus E_7 \) were described in \([29]\) as the minimal resolution of the quartic hypersurfaces in \( \mathbb{P}^3 \) given by the following equations:

\[
Y^2ZW - 4X^3Z + 3\alpha XZW^2 + \beta ZW^3 + \gamma XZ^2W - \frac{1}{2}(\zeta W^4 + \delta Z^2W^2) + \varepsilon XW^3 = 0,
\]

(17)

where \([X : Y : Z : W]\) are homogeneous coordinates on \( \mathbb{P}^3 \). \( \alpha, \beta, \gamma, \delta, \varepsilon, \zeta \) are parameters, and \((\gamma, \delta) \neq (0, 0), \) and \((\varepsilon, \zeta) \neq (0, 0)\).
Making the following substitutions

\begin{align*}
X &= tx \\
Y &= y \\
W &= 4t^3 \\
Z &= 4t^4
\end{align*}

(18)
yields the Weierstrass equation with two type III* fibers as follows [31] :

\begin{equation}
y^2 = x^3 + 4t^3 (\gamma t^2 - 3\alpha t + \varepsilon) x - 8t^5 (\delta t^2 + 2\beta t + \zeta).
\end{equation}

(19)

Type III* fibers are at \( t = 0 \) and at \( t = \infty \).

K3 surface with the lattice polarization \( H \oplus E_7 \oplus E_7 \) given by (17) always admits another fibration with a type II* fiber and a type \( I_2^* \) fiber, as shown in [31], and the Weierstrass equation of this fibration is:

\begin{equation}
y^2 = x^3 - \frac{1}{3} \left[ 9\alpha t^4 + 3(\gamma \zeta + \delta \varepsilon) t^3 + (\gamma \varepsilon)^2 t^2 \right] x \\
+ \frac{1}{27} \left[ 27 t^7 - 54 \beta t^6 + 27(\alpha \gamma \varepsilon + \delta \zeta) t^5 + 9 \gamma \varepsilon (\delta \varepsilon + \gamma \zeta) t^4 + 2(\gamma \varepsilon)^3 t^3 \right].
\end{equation}

(20)

The Weierstrass equation (20) was used in [31] to construct eight-dimensional non-geometric heterotic strings with unbroken \( e_7 e_7 \). (Compare the equation (20) with the equation (15).) Although the presence of two \( E_7 \) singularities in the Weierstrass equation is explicit in the equation (19), as stated in [31], the Weierstrass equation (19) does not necessarily extend over the entire parameter space. For this reason, the Weierstrass equation (20) was instead used to construct non-geometric heterotic strings with unbroken \( e_7 e_7 \) in [31].

The K3 surface with the lattice polarization \( H \oplus E_7 \oplus E_7 \) (17) also admits another elliptic fibration further, the singular fibers of which include a type \( I_8^* \) fiber (or worse) [31]. This alternate fibration relates to \( SO(32) \) heterotic string. The Weierstrass equation of this fibration is obtained by making the following substitutions into the equation (17) [31]:

\begin{align*}
X &= tx^3 \\
Y &= \sqrt{2}x^2 y \\
W &= 2x^3 \\
Z &= 2x^2 (-\varepsilon t + \zeta).
\end{align*}

(21)
The Weierstrass equation is [31] :

\begin{equation}
y^2 = x^3 + Ax^2 + Bx,
\end{equation}

(22)

where

\begin{align*}
A &= t^3 - 3\alpha t - 2\beta \\
B &= (\gamma t - \delta)(\varepsilon t - \zeta).
\end{align*}

(23)
The discriminant is given by

\begin{equation}
\Delta = B^2 (A^2 - 4B).
\end{equation}

(24)
2.3 Extremal K3 surfaces

By the Shioda–Tate formula \([123, 124, 125]\), the following equality holds for an elliptic surface \(S\) with a global section:
\[
\text{rk} \text{ADE} + \text{rk} \text{MW} + 2 = \rho(S).
\] (25)

We have used \(\text{rk} \text{ADE}\) to denote the rank of the singularity type of an elliptic surface \(S\). The Picard number \(\rho(S)\) ranges from 2 to 20 for an elliptic K3 surface with a section. Thus, the rank of the singularity of an elliptic K3 surface \(S\) with a section is bounded by:
\[
\text{rk} \text{ADE} = \rho(S) - 2 - \text{rk} \text{MW} \leq 18 - \text{rk} \text{MW}.
\] (26)

Therefore, the rank of the singularity of an elliptic K3 surface \(S\) with a section can be 18 at the highest, and this value is achieved precisely when the Picard number attains the highest value 20, and the Mordell–Weil rank is 0. Physically, this means that the rank of the gauge group on F-theory compactification on an elliptic K3 surface is at most 18, and when the non-Abelian gauge group has the rank 18, it does not have a \(U(1)\) gauge field.

K3 surfaces with the Picard number 20 are called the attractive K3 surface\(^8\). Attractive K3 surfaces are known to be parametrized by three integers. The transcendental lattice \(T(S)\) of a K3 surface \(S\) is the orthogonal complement of the Néron-Severi lattice \(\text{NS} \text{(}S\text{)}\) inside the K3 lattice \(\Lambda_{K3}\), and the transcendental lattices \(T(S)\) of attractive K3 surfaces are positive-definite, even \(2 \times 2\) lattices. The complex structure of an attractive K3 surface is determined by the transcendental lattice \([127, 128]\). The intersection form of the transcendental lattice of an attractive K3 surface can be transformed into the following form under the \(\text{GL}_2(\mathbb{Z})\) action:
\[
\begin{pmatrix}
2a & b \\
b & 2c
\end{pmatrix}.
\] (27)

\(a, b, c\) are integers, \(a, b, c \in \mathbb{Z}\), and they satisfy the relations:
\[
a \geq c \geq b \geq 0.
\] (28)

Thus, the triplet of integers, \(a, b, c\), parameterizes the complex structure moduli of the attractive K3 surfaces. We denote an attractive K3 surface, whose transcendental lattice has the intersection form \([27]\) as \(S_{[2a \ b \ 2c]}\) in this study.

An elliptic attractive K3 surfaces with a section is said to be extremal when it has the Mordell–Weil rank 0. This condition is equivalent to that an elliptic K3 surface with a section has the singularity rank 18. Thus, the non-Abelian gauge group on F-theory compactification on an elliptic K3 surface has the rank 18 precisely when the K3 surface is extremal. In Section \([3]\) we study the points in the moduli of eight-dimensional non-geometric heterotic strings with unbroken \(e_7\) at which the non-Abelian gauge groups are enhanced to rank 18 on the F-theory side.

\(^8\)We refer to complex K3 surfaces with the highest Picard number 20 as attractive K3 surfaces, following the convention for the term used in \([126]\).
Elliptically fibered K3 surfaces generally admit distinct elliptic fibrations\(^9\), and distinct elliptic fibrations have different singularity types and different Mordell–Weil groups. Physically, this means that the gauge groups and \(U(1)\) gauge fields that arise in F-theory compactification on an elliptic K3 surface with the fixed complex structure vary, because there still remains a freedom to choose a fibration structure among the distinct choices of elliptic fibrations of that elliptic K3 surface\(^10\).

The attractive K3 surface whose transcendental lattice has the intersection form \[
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\] is particularly relevant to the contents of this study. The types of the elliptic fibrations of the attractive K3 surface \(S_{[2,0,2]}\) were classified in \[130\], and there are 13 types. We list these 13 types of elliptic fibration of the attractive K3 surface \(S_{[2,0,2]}\) in Appendix A.

3 Special points in the moduli of eight-dimensional non-geometric heterotic strings and F-theory duals with enhanced gauge groups

3.1 Summary

There are finitely many points in the moduli of eight-dimensional non-geometric heterotic strings with unbroken \(e_7\), at which the non-Abelian gauge groups on the F-theory side are enhanced to rank 18.

In Section 3.2, we show that the heterotic strings at these special points in the moduli can be described as deformations of the stable degenerations, as a result of the coincident 7-branes on the F-theory side. This effect can be seen as the insertion of 5-branes in the heterotic language. These effects can be used to study nonperturbative aspects of non-geometric heterotic strings. We also discuss applications to \(SO(32)\) heterotic strings.

As stated in Section 2.3, K3 surfaces become extremal on the F-theory side at these points in the moduli. The complex structures of the extremal K3 surfaces were classified in \[41\], and using this result, the complex structures of the extremal K3 surfaces at these points in the moduli can be determined. This enables to determine the Weierstrass equations of the extremal K3 surfaces which appear as compactification spaces on the F-theory side in the moduli. By using this approach, we study the physics of the theories at the enhanced special points in the moduli.

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\(^9\)Genus-one fibered K3 surfaces in general admit both genus-one fibrations without a section, as well as elliptic fibrations with a section. However, as shown in \[129\], the attractive K3 surface with the discriminant four, \(S_{[2,0,2]}\), only admits elliptic fibrations with a global section. \[104\] discussed F-theory compactification on the surface \(S_{[2,0,2]}\), in relation to the appearances of \(U(1)\) factor.

\(^{10}\)This point is discussed in \[19\].
In eight-dimensional $E_8 \times E_8$ heterotic strings on the 2-torus $T^2$, only the gauge groups up to $E_8 \times E_8 \times U(1)^4$ can arise in the perturbative description [39]. This implies that the heterotic dual of F-theory on an extremal elliptic K3 with the non-Abelian gauge group of rank 18 does not allow for the perturbative interpretation of the gauge group. This can reflect some non-perturbative aspects of the non-geometric heterotic strings.

### 3.2 F-theory on extremal K3 surfaces and nongeometric heterotic duals in the moduli

We discuss the points in the moduli of non-geometric heterotic strings with unbroken $e_7 e_7$, at which the non-Abelian gauge symmetries on the F-theory side are enhanced to rank 18. K3 surfaces as compactification spaces on the F-theory side become extremal at these points. There are finitely many such points in the moduli, and the complex structures and the singularity types of the extremal K3 surfaces that appear in the moduli can be determined from Table 2 in [41]. Among these, the ones the singularity types of which include $E_8 E_7$ are studied in [39]. We do not discuss these extremal K3 surfaces in this note. We discuss the extremal K3 surfaces that belong to the moduli, the singularity types of which include $E_8^2$.

The singularity types of the extremal K3 surfaces in the moduli of K3 surfaces with $H \oplus E_7 \oplus E_7$ lattice polarization, which do not include $E_8$, are as follows [41]:

$$E_7^2 A_3 A_1, \ E_7^2 D_4, \ E_7^2 A_4, \ E_7^2 A_2. \ (30)$$

We study F-theory on the extremal K3 surfaces possessing the first two singularity types in this note.

Because the perturbative eight-dimensional heterotic strings on $T^2$ can have up to $E_8 \times E_8 \times U(1)^4$, the heterotic duals of F-theory on these extremal K3 surfaces do not allow for the perturbative interpretation of these gauge groups [39]. As we demonstrate in Section 3.2.1 and Section 3.2.2, these theories can be seen as deformations of the stable degenerations as a result of the coincident 7-branes on the F-theory side. These theories satisfy multiple 5-brane solutions on the heterotic side.

#### 3.2.1 Extremal K3 with $E_7^2 A_3 A_1$ singularity

The complex structure of the extremal K3 surface with $E_7^2 A_3 A_1$ singularity is uniquely determined, and its transcendental lattice has the intersection form as follows [41]:

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}. \ (31)$$

Therefore, the attractive K3 surface $S_{[4 \ 0 \ 2]}$ [12] admits an extremal fibration with the singularity type $E_7^2 A_3 A_1$, and F-theory on this extremal fibration has non-geometric heterotic dual with

---

[11] The singularity types of extremal K3 surfaces can also be enhanced to $E_8 D_7$, as discussed in [31].

[12] The elliptic fibrations and the Weierstrass equations of the attractive K3 surface $S_{[4 \ 0 \ 2]}$ were obtained in [31].
unbroken $\varepsilon_7 \varepsilon_7$. The Weierstrass form of this extremal fibration can be found in [23] as:

$$y^2 = x^3 - \frac{9}{16} (t^2 + s^2 + \frac{10}{3} ts) t^3 s^3 x + \frac{9}{4} t^5 s^5 \left( \frac{1}{4} t^2 + \frac{1}{4} s^2 + \frac{7}{18} ts \right).$$  \hspace{1cm} (32)

The above Weierstrass equation was obtained in [23] as the quadratic base change of an extremal rational elliptic surface. Geometrically, the quadratic base change of a rational elliptic surface is to glue a pair of identical rational elliptic surfaces. Extremal rational elliptic surfaces are the rational elliptic surfaces with a global section, the singularity types of which have rank 8. The types of the singular fibers of the extremal rational elliptic surfaces were classified [132]. The fiber types of the extremal rational elliptic surfaces are listed in Appendix [3].

The complex structures of extremal rational elliptic surfaces are uniquely specified by the fiber types, except those with two fibers of type $I_0^*$. The complex structures of extremal rational elliptic surfaces with two fibers of type $I_0^*$ depend on the j-invariants of the fibers. The j-invariant $j$ of an extremal rational elliptic surface with two type $I_0^*$ fibers is constant over the base, and the fixed $j$ specifies the complex structure [132]. In this study, we denote, for example, the extremal rational elliptic surface with a type $III^*$ fiber and a type $III$ fiber as $X_{[III, III^*]}$. We simply use $n$ to denote a singular fiber of type $I_n$, and $m^*$ to represent a fiber of type $I_m^*$. The extremal rational elliptic surface with a type $III^*$ fiber, a type $I_2$ fiber and a type $I_1$ fiber is denoted as $X_{[III^*, 2, 1]}$. Because the complex structure of an extremal rational elliptic surface with two $I_0^*$ fibers depends on the j-invariant of the elliptic fibers, we use $X_{[0^*, 0^*]}(j)$ to denote this extremal rational elliptic surface.

As deduced in [23], the K3 extremal fibration [32] is obtained as the quadratic base change of the extremal rational elliptic surface $X_{[III^*. 2, 1]}$ in which two type $I_2$ fibers and two type $I_1$ fibers collide. While the quadratic base change of a rational elliptic surface generally yields an elliptic K3 surface, the singular fibers of which are twice the number of those of the original rational elliptic surface, at the special limits at which singular fibers collide, the singularity type of the resulting K3 surface is enhanced. As discussed in [23], two identical extremal rational elliptic surfaces $X_{[III^*. 2, 1]}$ are glued together to yield an elliptic K3 surface, which we denote $S_1$, the singular fibers of which consist of two type $III^*$ fibers, two $I_2$ fibers and two $I_1$ fibers. In the special limit at which 7-branes over which type $I_2$ fiber lies coincide with those over which type $I_2$ fiber lies, and 7-brane over which type $I_1$ fiber lies coincides with 7-brane over which type $I_1$ fiber lies, two type $I_2$ fibers are enhanced to type $I_4$ fiber, and two type $I_1$ fibers are enhanced to type $I_2$ fiber. In this limit, the K3 surface $S_1$ deforms to give the extremal K3 surface [32]. In short, F-theory on the extremal K3 surface [32] can be seen as deformation of the stable degeneration as the consequence of the coincident 7-branes.

Because the singularity rank of a rational elliptic surface is up to 8, the non-Abelian gauge group that arises on F-theory on a generic K3 surface obtained as the reverse of the stable degeneration has the rank up to 16. Here, by generic we mean a situation in which singular fibers of rational elliptic surfaces do not collide, when they are glued together to yield an elliptic K3 surface. When the large radius limit is taken, the heterotic dual of this compactification admits a geometric interpretation. In special situations in which singular
fibers collide, 7-branes become coincident and the singularity ranks of the resulting K3 surfaces enhance to become greater than 16. The gauge groups of the heterotic duals of F-theory compactifications on these K3 surfaces do not allow for the geometric interpretation.

The Mordell–Weil group of the K3 extremal fibration (32) is isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). Thus, the gauge group that arises in F-theory compactification on this extremal K3 surface is \( E_7 \times E_7 \times SU(4) \times SU(2) / \mathbb{Z}_2 \).

Comparing the Weierstrass equation (32) with the equation (19), we find that the following substitutions:

\[
\begin{align*}
\alpha &= \frac{5}{32} \\
\beta &= -\frac{7}{128} \\
\gamma = \varepsilon &= -\frac{9}{64} \\
\delta = \zeta &= -\frac{9}{128}
\end{align*}
\]

into the equation (19) yield the Weierstrass equation (32). Plugging the substitutions (34) into the equation of the alternate fibration (20), we obtain the following equation:

\[
y^2 = x^3 - \frac{1}{3} (10t^4 + 8t^3 + t^2) x + t^7 + \frac{56}{27} t^6 + \frac{26}{9} t^5 + \frac{8}{9} t^4 + \frac{2}{27} t^3,
\]

with the discriminant:

\[
\Delta \sim t^{11} (t+2)^2 (27t+4).
\]

From the equations (35) and (36), we find that the fibration (35) has a type \( II^* \) fiber at \( t = \infty \), a type \( I_5^* \) fiber at \( t = 0 \), a type \( I_2^* \) fiber at \( t = -2 \), and a type \( I_1^* \) fiber at \( t = -4/27 \). Thus, the fibration (35) has the singularity type \( E_8 D_9 A_1 \), and because the singularity type has rank 18, we deduce that this fibration is also extremal. Therefore, we find that the attractive K3 surface \( S_{[4 \ 0 \ 2]} \) admits an extremal fibration (35) with the singularity type \( E_8 D_9 A_1 \). This agrees with the results in [41, 131].

3.2.2 Extremal K3 with \( E_7^2 D_4 \) singularity

The complex structure of the extremal K3 surface with the singularity type \( E_7^2 D_4 \) is uniquely determined, and the intersection form of the transcendental lattice is [41]:

\[
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}.
\]

The Weierstrass equation of this extremal fibration of the attractive K3 surface \( S_{[2 \ 0 \ 2]} \) is given as follows:

\[
y^2 = x^3 + 4t^3 (t-s)^2 s^3 x,
\]

(38)
with the discriminant
\[ \Delta \sim t^9 s^9 (t - s)^6. \] (39)

\([t : s]\) in the equation (38) denotes the homogeneous coordinate of the base \(\mathbb{P}^1\). Two type III\(^*\) are at \([t : s] = [0 : 1]\) and \([1 : 0]\), and a type \(I_1^*\) fiber is at \([t : s] = [1 : 1]\).

As shown in [23], the extremal K3 fibration (38) can be seen as deformation of the stable degeneration. Gluing two identical extremal rational elliptic surfaces \(X_{[III, III^*]}\) yields an elliptic K3 surface, \(S_2\), the singular fibers of which have two type III\(^*\) fibers and two type III fibers. This is technically given by a generic quadratic base change of the extremal rational elliptic surface \(X_{[III, III^*]}\), and this is the reverse of the stable degeneration. A special limit, at which two type III fibers collide, the elliptic K3 surface \(S_2\) deforms to yield the extremal fibration (38) of the attractive K3 surface \(S_{[2\ 0\ 2]}\) [23]. 7-branes over which type III fiber lies coincide with those over which type III fiber lies in this limit. Therefore, F-theory on the extremal K3 surface (38) can be seen as deformation of the stable degeneration as the consequence of the coincident 7-branes.

The Mordell–Weil group of the extremal elliptic fibration (38) is isomorphic to \(\mathbb{Z}_2\) [130, 41]. Therefore, the gauge group on F-theory compactification on the extremal fibration (38) is \(E_7 \times E_7 \times SO(8)/\mathbb{Z}_2\). (40)

Comparing the equation (38) with the equation (19), we find that the following substitutions
\[ \begin{align*}
\alpha &= \frac{2}{3} \\
\beta &= \delta = \zeta = 0 \\
\gamma &= \varepsilon = 1
\end{align*} \] (41)
into (19) yield the Weierstrass equation (38).

By plugging the substitutions (41) into the equation (20), we obtain the following Weierstrass equation:
\[ y^2 = x^3 - \frac{1}{3} t^2 (6t^2 + 1)x + \frac{1}{27} t^3 (27t^4 + 18t^2 + 2), \] (42)
with the discriminant
\[ \Delta \sim t^{12}(27t^2 + 4). \] (43)

We can confirm from the equations (42) and (43) that this alternate fibration in fact has a type II\(^*\) fiber at \(t = \infty\), a type \(I_6^*\) fiber at \(t = 0\), and two type \(I_1\) fibers at the roots of \(27t^2 + 4 = 0\). Thus, the singularity type of the alternate fibration is \(E_8D_{10}\), and we find that this fibration is also extremal. This gives the fibration no. 2 in Table 4 of the attractive K3 surface \(S_{[2\ 0\ 2]}\) in Appendix A.

Double cover of \(\mathbb{P}^1 \times \mathbb{P}^1\) ramified along a bidegree \((4,4)\) curve, given by the following equation:
\[ \tau^2 = (t - \alpha_1)^3(t - \alpha_2) x^4 + (t - \alpha_3)^3(t - \alpha_2) \] (44)
yields a genus-one fibered K3 surface lacking a global section, but admitting a bisection, and this K3 surface was considered in \cite{74} in the context of F-theory compactifications on genus-one fibrations without a global section. $x$ denotes the inhomogeneous coordinate of the first $\mathbb{P}^1$, and $t$ denotes the inhomogeneous coordinate of the second $\mathbb{P}^1$, respectively. $\alpha_1, \alpha_2, \alpha_3$ are distinct points in $\mathbb{P}^1$. $\alpha$’s are superfluous parameters, and these can be mapped to:

$$\alpha_1 = 0, \quad \alpha_2 = 1, \quad \alpha_3 = \infty$$  \hspace{1cm} (45)

under some appropriate automorphism of the base $\mathbb{P}^1$. The K3 genus-one fibration (44) has two type $III^*$ fibers at $t = \alpha_1, \alpha_3$, and a type $I_0^*$ fiber at $t = \alpha_2$ \cite{74}.

The Jacobian fibration of the K3 genus-one fibration (44) gives the extremal K3 elliptic fibration (38), as demonstrated in \cite{74}. Utilizing this fact, in Section 4 we build an elliptically fibered Calabi–Yau 3-fold, by fibering the K3 genus-one fibration (44) over the base $\mathbb{P}^1$, and taking the Jacobian fibration of it. It turns out that the resulting Calabi–Yau 3-fold is a fibration of the extremal K3 surface (38) over the base $\mathbb{P}^1$. We also build a family of elliptic Calabi–Yau 3-folds which includes this Calabi–Yau 3-fold. These constructions will be discussed in detail in Section 4.

3.3 Applications to $SO(32)$ heterotic strings

As reviewed in Section 2.2, the K3 surface (17) with $H \oplus E_7 \oplus E_7$ lattice polarization admits another elliptic fibration, the singular fibers of which include a type $I_8^*$ fiber, given by the Weierstrass equation:

$$y^2 = x^3 + (t^3 - 3\alpha t - 2\beta) x^2 + (\gamma t - \delta)(\varepsilon t - \zeta) x \hspace{1cm} (46)$$

with the discriminant

$$\Delta \sim (\gamma t - \delta)^2(\varepsilon t - \zeta)^2[(t^3 - 3\alpha t - 2\beta)^2 - 4(\gamma t - \delta)(\varepsilon t - \zeta)]. \hspace{1cm} (47)$$

Therefore, there is a birational map which transforms the elliptic fibration with two $III^*$ fibers \cite{19} to an alternate fibration \cite{46} with a type $I_8^*$ fiber (or worse). Using this birational map, we send the extremal K3 elliptic fibrations with two $E_7$ singularities studied in Section 3.2 to another fibrations with a type $I_8^*$ fiber (or worse). This relates to $SO(32)$ heterotic strings.

As we saw previously in Section 3.2.2, the Weierstrass equation of the extremal fibration of the attractive K3 surface $S_{[2\ 0\ 2]}$ with the singularity type $E_7^2D_4$ is given by \cite{38}, with

\[
\begin{align*}
\alpha &= \frac{2}{3} \\
\beta &= \delta = \zeta &= 0 \\
\gamma &= \varepsilon &= 1.
\end{align*}
\hspace{1cm} (48)
\]
By plugging these values (48) into the alternate fibration (46), we obtain the Weierstrass equation as:

\[ y^2 = x^3 + t(t^2 - 2)x^2 + t^2x, \]  

(49) 

with the discriminant

\[ \Delta = t^4(t^2 - 2)^2 - 4t^2 \]  

(50) 

\[ = t^8(t^2 - 4). \] 

In the homogeneous form, the discriminant is

\[ \Delta \sim t^8s^4(t^2 - 4s^2). \]  

(51) 

From the equations (49) and (51), we deduce that the alternate fibration (49) has a type $I_8^*$ fiber at $[t : s] = [1 : 0]$, a type $I_2^*$ fiber at $[t : s] = [0 : 1]$, and two type $I_1$ fibers at $[t : s] = [2 : 1], [-2 : 1]$. Thus, the alternate fibration (49) has the singularity type $D_{12}D_6$, and we find that this fibration is also extremal. We conclude that the alternate fibration (49) yields the fibration no. 8 in Table 4 in Appendix A of the attractive K3 surface $S_{[202]}$.

The Mordell–Weil group of the fibration (49) is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$; therefore, the gauge group on F-theory compactification on the fibration (49) is

\[ SO(24) \times SO(12)/\mathbb{Z}_2. \]  

(52) 

The attractive K3 surface $S_{[202]}$ has another extremal fibration with the singularity type $D_8^2A_1^2$ [130]. (This is fibration no.9 in Table 4 in Appendix A.) As deduced in [23], the Weierstrass equation of this extremal fibration is given as follows:

\[ y^2 = x^3 - 3t^2s^2(t^2 + s^2 - t^2s^2)x + (t^2 + s^2)t^3s^3(2t^4 - 5t^2s^2 + 2s^4), \]  

(53) 

with the discriminant

\[ \Delta \sim t^{10}s^{10}(t - s)^2(t + s)^2. \]  

(54) 

Type $I_8^*$ fibers are at $[t : s] = [1 : 0], [0 : 1]$, and type $I_2^*$ fibers are at $[t : s] = [1 : 1], [1 : -1]$.

As shown in [23], extremal fibration (53) can be seen as deformation of the stable degeneration in which two extremal rational elliptic surfaces $X_{[4^*, 1, 1]}$ are glued together. Gluing of two extremal rational elliptic surfaces $X_{[4^*, 1, 1]}$ yields an elliptic K3 surface, which we denote by $S_3$, the singular fibers of which are two type $I_4^*$ fibers and four type $I_1$ fibers. In a limit at which two pairs of type $I_1$ fibers collide, type $I_1$ fibers collide and they are enhanced to a type $I_2$ fiber. In this limit, K3 surface $S_3$ deforms to yield the attractive K3 surface with the extremal fibration (53) [23]. Therefore, extremal fibration (53) can be seen as deformation of the stable degeneration, as a result of coincident 7-branes over which type $I_1$ fibers lie.

The Mordell–Weil group of the fibration (53) is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ [130]; therefore, the gauge group on F-theory compactification on the fibration (53) is

\[ SO(16) \times SO(16) \times SU(2)^2/\mathbb{Z}_2 \times \mathbb{Z}_2. \]  

(55)
We saw previously in Section 3.2.1 that the attractive K3 surface $S_{[402]}$ admits an extremal fibration with the singularity type $E_7^2A_3A_1$, and the Weierstrass equation of this fibration is given by (32), with

$$\begin{align*}
\alpha &= \frac{5}{32}, \\
\beta &= -\frac{7}{128}, \\
\gamma = \varepsilon &= -\frac{9}{64}, \\
\delta = \zeta &= -\frac{9}{128}.
\end{align*}$$

(56)

By plugging these into the equation (46), we obtain an alternate fibration given by:

$$y^2 = x^3 + (t^3 - \frac{15}{32} t + \frac{7}{64}) x^2 + \left(\frac{9}{64}\right)^2 (t - \frac{1}{2})^2 x,$$

(57)

with the discriminant

$$\Delta \sim (t - \frac{1}{2})^7 (4t + 1)^2 (t + 1).$$

(58)

From the equations (57) and (58), we find that the alternate fibration has a type $I_8^*$ fiber at $t = \infty$, a type $I_1^*$ fiber at $t = \frac{1}{2}$, a type $I_2$ fiber at $t = -\frac{1}{4}$, and a type $I_1$ fiber at $t = -1$. Thus, the alternate fibration (57) has the singularity type $D_{12}D_5A_1$, and this is also extremal. This result agrees with the elliptic fibrations with a section of the attractive K3 surface $S_{[402]}$ obtained in [131].

The Mordell–Weil group of the alternate fibration (57) is isomorphic to $\mathbb{Z}_2$ [41, 131]. Therefore, the gauge group on F-theory compactification on the fibration (57) is

$$SO(24) \times SO(10) \times SU(2)/\mathbb{Z}_2.$$  

(59)

4 Jacobian Calabi-Yau 3-folds and F-theory compactifications

In this section, we fiber elliptic K3 surfaces over $\mathbb{P}^1$ to yield elliptically fibered Calabi–Yau 3-folds [13] with a global section, and we study six-dimensional F-theory compactifications with $N = 1$ supersymmetry on the constructed Calabi–Yau 3-folds. K3 fibers in this construction include an elliptic K3 surface which belongs to F-theory side of the moduli of eight-dimensional non-geometric heterotic strings with unbroken $e_7e_7$, discussed in Section 3.2.2.

To be concrete, we first consider genus-one fibered K3 surfaces lacking a global section, built as double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified along a $(4,4)$ curve. The built genus-one fibered K3 surfaces do not have a global section, but they have a bisection [74]. We consider higher

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13 [13] [134] [135] discuss elliptic fibrations of 3-folds.
dimensional analogs of these K3 surfaces to yield genus-one fibered Calabi–Yau 3-folds, built as double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified along a multidegree $(4,4,4)$ surface. As we will show in Section 4.1, the constructed Calabi–Yau 3-folds are genus-one fibered, but they lack a global section. These Calabi–Yau 3-folds still have a bisection \[64\]. The pullback of $\mathcal{O}(1)$ class in $\mathbb{P}^1$ yields a bisection \[77\].

Taking the Jacobian fibrations of these Calabi–Yau genus-one fibrations yields elliptically fibered Calabi–Yau 3-folds with a global section. These elliptic Calabi–Yau 3-folds are also K3 fibered. We deduce the non-Abelian gauge groups on F-theory compactifications on the Jacobian Calabi–Yau 3-folds. We also perform consistency check of the obtained gauge groups, by considering the symmetry that the elliptic fibers possess. Highly enhanced gauge groups arise when we choose specific coefficients of the defining equations of the Jacobian Calabi–Yau 3-folds. We determine the Mordell–Weil groups of some specific Calabi–Yau 3-folds, and we deduce the global structures of the gauge groups for F-theory on these spaces. We obtain some models without a $U(1)$ gauge field.

4.1 Calabi-Yau 3-folds as double covers, Jacobian fibrations, and the discriminant locus

Double covers of the product of projective lines, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, ramified over a $(4,4,4)$ surface have the trivial canonical bundles, $K = 0$, therefore they describe Calabi–Yau 3-folds. Projection of $\mathbb{P}^1 \times \mathbb{P}^1$ is a double cover of $\mathbb{P}^1$ branched over 4 points, namely, it is an elliptic curve. Thus, projection onto $\mathbb{P}^1 \times \mathbb{P}^1$ gives a genus-one fibration. Fiber of projection onto $\mathbb{P}^1$ is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified along a $(4,4)$ curve, which yields a genus-one fibered K3 surface. These K3 surfaces do not have a section, but they have a bisection \[74]. Therefore, projection onto $\mathbb{P}^1$ yields a K3 fibration.

In this note, we particularly consider the double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by the following equations:

$$\tau^2 = f_1(t)g_1(u) x^4 + f_2(t)g_2(u),$$

where $x$ is the inhomogeneous coordinate on the first $\mathbb{P}^1$, $t, u$ are the inhomogeneous coordinates on the second and the third $\mathbb{P}^1$’s in the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. $f_1, f_2$ are polynomials in the variable $t$ of degree 4, and $g_1, g_2$ are polynomials of degree 4 in the variable $u$. By splitting the polynomials $f_1, f_2, g_1, g_2$ into linear factors, the equation (60) can be rewritten as follows:

$$\tau^2 = \prod_{i=1}^{4}(t - \alpha_i) \cdot \prod_{j=1}^{4}(u - \beta_j) \cdot x^4 + \prod_{k=5}^{8}(t - \alpha_k) \cdot \prod_{l=5}^{8}(u - \beta_l).$$

K3 fiber of this genus-one fibered Calabi–Yau 3-fold is given by:

$$\tau^2 = \prod_{i=1}^{4}(t - \alpha_i) \cdot x^4 + \prod_{k=5}^{8}(t - \alpha_k).$$
As shown in [74], K3 fiber (62) is genus-one fibered, but it does not have a global section. K3 fiber (62) has a bisection [74].

Using an argument similar to that in [75], we can show that Calabi–Yau 3-fold (61) indeed lacks a rational section. Suppose it admits a rational section. Then, it restricts to a K3 fiber, and this gives a global section to the K3 fiber, leading to a contradiction. By an argument similar to those in [74, 75, 77], the genus-one fibered Calabi-Yau 3-fold (61) has a bisection [14].

We can consider a special situation in which:

\[ \alpha_1 = \alpha_2 = \alpha_3, \quad \alpha_4 = \alpha_8, \quad \alpha_5 = \alpha_6 = \alpha_7. \] (63)

This yields a genus-one fibered Calabi–Yau 3-fold

\[ \tau^2 = (t - \alpha_1)^3(t - \alpha_4) \cdot \Pi_{j=1}^4(u - \beta_j) \cdot x^4 + (t - \alpha_5)^3(t - \alpha_4) \cdot \Pi_{l=5}^8(u - \beta_l), \] (64)

and K3 fiber is given by

\[ \tau^2 = (t - \alpha_1)^3(t - \alpha_4) x^4 + (t - \alpha_5)^3(t - \alpha_4). \] (65)

We find that this is the genus-one fibered K3 surface (44) lacking a section, which we discussed in Section 3.2.2 [15].

The Jacobian fibrations of the genus-one fibered Calabi–Yau 3-folds (61) yield elliptically fibered Calabi–Yau 3-folds with a section. The Jacobian fibrations are given by [136]:

\[ \tau^2 = \frac{1}{4} x^3 - \Pi_{i=1}^8(t - \alpha_i) \cdot \Pi_{j=1}^8(u - \beta_j) \cdot x. \] (66)

Projection onto \( \mathbb{P}^1 \times \mathbb{P}^1 \) gives an elliptic fibration, and projection onto \( \mathbb{P}^1 \) yields a K3 fibration. K3 fiber of the projection onto \( \mathbb{P}^1 \) is given by

\[ \tau^2 = \frac{1}{4} x^3 - \Pi_{i=1}^8(t - \alpha_i) \cdot x. \] (67)

In the situation (63), the Weierstrass equation of the Jacobian fibration is:

\[ \tau^2 = \frac{1}{4} x^3 - (t - \alpha_1)^3(t - \alpha_5)^3(t - \alpha_4)^2 \cdot \Pi_{j=1}^8(u - \beta_j) x, \] (68)

and K3 fiber (67) becomes:

\[ \tau^2 = \frac{1}{4} x^3 - (t - \alpha_1)^3(t - \alpha_5)^3(t - \alpha_4)^2 x. \] (69)

\[ \text{14 Thus, a discrete } \mathbb{Z}_2 \text{ symmetry arises in six-dimensional F-theory compactifications on the genus-one fibered Calabi-Yau 3-folds (61).} \]

\[ \text{15 As we stated previously in Section 3.2.2, } \alpha_1, \alpha_4, \alpha_5 \text{ in the equation (65) are superfluous parameters, and they can be mapped to 0, 1, } \infty \text{ under some automorphism of the base } \mathbb{P}^1. \]
This is the Jacobian fibration of the K3 fiber (65), and this is the extremal fibration (38) of the attractive K3 surface $S_{[2\,0\,2]}$ which belongs to F-theory side of the moduli of the non-geometric heterotic strings with unbroken $e_7$.

Obtained Jacobian Calabi–Yau 3-folds (66) yield fibrations of K3 surfaces (67) over $\mathbb{P}^1$, and this family includes fibrations of the extremal K3 surface $S_{[2\,0\,2]}$, which we discussed in Section 3.2.2, over $\mathbb{P}^1$.

The discriminant locus of the Jacobian Calabi–Yau 3-fold (66) is given by the following equation:

$$\Delta \sim \Pi_{i=1}^8 (t - \alpha_i)^3 \cdot \Pi_{j=1}^8 (u - \beta_j)^3.$$  \hskip 1cm (71)

Genus-one fibered Calabi–Yau 3-fold (61) and the Jacobian fibration (66) have the identical discriminant loci, and the identical configurations of the singular fibers.

From the equation (71), we find that the discriminant components of the Jacobian Calabi–Yau 3-fold (66) are given as follows:

$$A_i = \{ t = \alpha_i \} \quad (i = 1, \cdots, 8)$$  \hskip 1cm (72)

$$B_j = \{ u = \beta_j \} \quad (j = 1, \cdots, 8).$$

In F-theory on the Jacobian Calabi–Yau 3-fold (66), 7-branes are wrapped on these components. Components $A_i$ are isomorphic to $\{\text{pt}\} \times \mathbb{P}^1$, and components $B_j$ are isomorphic to $\mathbb{P}^1 \times \{\text{pt}\}$. Therefore, these are isomorphic to $\mathbb{P}^1$. The types of the singular fibers and the non-Abelian gauge groups on the 7-branes will be discussed in Section 4.2.

4.2 Non-Abelian gauge groups

We determine the non-Abelian gauge groups on F-theory on the Jacobian Calabi–Yau 3-folds constructed in Section 4.1. We also check the consistency of the obtained gauge groups.

4.2.1 Singular fibers of the Jacobian Calabi-Yau 3-folds, and non-Abelian gauge groups

From the Weierstrass equation (66) of the Jacobian Calabi–Yau 3-fold and the discriminant (71), we find that for a generic situation in which the coefficients $\alpha$’s and $\beta$’s are mutually distinct, the types of the singular fibers on the components $A_i, \ i = 1, \cdots, 8$, and $B_j, \ j = 1, \cdots, 8$, are $III$. In this case, the non-Abelian gauge group that arises on F-theory compactification on the Jacobian Calabi–Yau 3-fold (66) is

$$SU(2)^{16}.$$  \hskip 1cm (73)

\footnote{Using a coordinate transformation, (69) can be replaced with

$$\tau^2 = x^3 + 4(t - \alpha_1)^3(t - \alpha_4)^2(t - \alpha_5)^3 x.$$  \hskip 1cm (70)

As stated previously, we can send $\alpha_1, \alpha_4, \alpha_5$ to 0, 1, $\infty$, and this yields (38).}
When two of the coefficients, \( \alpha_i \) and \( \alpha_k \), become coincident, a pair of type III fibers on the components \( A_i \) and \( A_k \) collides, and it is enhanced to a type \( I_0^* \) fiber. Because the polynomial

\[
x^3 - \prod_{j=1}^{8}(u - \beta_j) \cdot x
\]

(74)
splits into the linear factor and the quadratic factor as:

\[
x \left( x^2 - \prod_{j=1}^{8}(u - \beta_j) \right)
\]

(75)
type \( I_0^* \) fiber is semi-split \[116\]. The non-Abelian gauge group that arises on the 7-branes wrapped on the component \( A_i \) is thus enhanced to \( SO(7) \) in this situation \[17\].

When three of the coefficients, \( \alpha_i \), \( \alpha_k \) and \( \alpha_l \), become coincident, type III fibers on the components \( A_i, A_k, A_l \) collide, and they are enhanced to a type \( III^* \) fiber. Further enhancement breaks the Calabi–Yau condition, as stated in \[74, 75\]. An argument similar to that stated previously equally applies to \( \beta \)'s and the components \( B_j \). The results are presented in Table 3 below.

| Component | Fiber type | non-Abel. Gauge Group |
|-----------|------------|-----------------------|
| \( A_{1,\ldots,8} \) | \( III \) | \( SU(2) \) |
| \( A_{1,\ldots,8} \) | \( I_0^* \) | \( SO(7) \) |
| \( A_{1,\ldots,8} \) | \( III^* \) | \( E_7 \) |
| \( B_{1,\ldots,8} \) | \( III \) | \( SU(2) \) |
| \( B_{1,\ldots,8} \) | \( I_0^* \) | \( SO(7) \) |
| \( B_{1,\ldots,8} \) | \( III^* \) | \( E_7 \) |

Table 3: Fiber types and the gauge groups on the discriminant components.

As discussed in Section 4.1, K3 fiber becomes most enhanced when:

\[
\alpha_1 = \alpha_2 = \alpha_3, \quad \alpha_4 = \alpha_8, \quad \alpha_5 = \alpha_6 = \alpha_7.
\]

(77)

In this case, the non-Abelian gauge group that arises on F-theory compactification on the Jacobian Calabi–Yau 3-fold \[68\] is

\[
E_7^2 \times SO(7) \times SU(2)^8.
\]

(78)

\[17\]In a special situation in which there are four pairs of identical \( \beta \)'s, e.g. \( \beta_1 = \beta_5, \beta_2 = \beta_6, \beta_3 = \beta_7, \beta_4 = \beta_8 \), the polynomial splits into three linear factors:

\[
x \left( x - \prod_{j=1}^{4}(u - \beta_j) \right) \left( x + \prod_{j=1}^{4}(u - \beta_j) \right).
\]

(76)

In this special situation, type \( I_0^* \) fiber over the component \( A_i \) becomes split, and the gauge group that forms on the 7-branes wrapped on the component \( A_i \) is enhanced to \( SO(8) \).
K3 fiber becomes the attractive K3 surface $S_{[2 \cdot 0 \cdot 2]}$ with the singularity type $E_7D_4$ in this situation, as discussed in Section 3.2.2. The singularity type of the Jacobian Calabi–Yau 3-fold (66) is most enhanced, when the following equalities hold among coefficients $\beta$’s further:

$$\beta_1 = \beta_2 = \beta_3, \quad \beta_4 = \beta_8, \quad \beta_5 = \beta_6 = \beta_7.$$  

(79)

In this case, the Weierstrass equation of the Jacobian Calabi–Yau 3-fold becomes

$$\tau^2 = \frac{1}{4}x^3 - (t - \alpha_1)^3(t - \alpha_5)^3(t - \alpha_4)^2 \cdot (u - \beta_1)^3(u - \beta_5)^3(u - \beta_4)^2 x.$$  

(80)

The types of the singular fibers over the components, $A_1, A_5, B_1, B_5$, are enhanced to type $III^*$, and the types of the singular fibers over the components $A_4$ and $B_4$ are enhanced to type $I_{0}^*$. In this situation, the non-Abelian gauge group on the F-theory compactification of the Jacobian Calabi–Yau 3-fold (80) is enhanced to:

$$E_7^4 \times SO(7)^2.$$  

(81)

4.2.2 Consistency check of the gauge groups

By an argument similar to those given in [74, 75], the defining equations of the Calabi–Yau double covers (60) are invariant under the following transformation:

$$x \rightarrow i \cdot x.$$  

(82)

We find from this that the genus-one fibers of the Calabi–Yau double covers (60) possess complex multiplications of order 4; therefore, smooth genus-one fibers of the Calabi–Yau double cover (60) have j-invariant 1728. This requires the singular fibers to have j-invariant 1728 [74, 75]. Because the singular fibers of Calabi–Yau genus-one fibration and its Jacobian fibration are identical, this means that the singular fibers of the Jacobian Calabi–Yau 3-fold (66) have j-invariant 1728 as well.

According to Table 1 in Section 2.1, the types of the singular fibers with j-invariant 1728 are: $III, III^*$, and $I_{0}^*$. (j-invariant of type $I_{0}^*$ fiber can take the value 1728.) Thus, the Jacobian Calabi–Yau 3-fold (66) can have the singular fibers, only of types $III, I_{0}^*$ and $III^*$. This agrees with the results obtained in Section 4.2.1. The monodromies of orders 2 and 4 characterize the non-Abelian gauge groups that form on F-theory compactifications of the Jacobian Calabi–Yau 3-folds (66).

4.3 Mordell-Weil groups of some models, and models without U(1) gauge field

We determine the Mordell–Weil groups of some F-theory models on the Jacobian Calabi–Yau 3-folds (68). We deduce that they do not have a $U(1)$ gauge field.
We saw previously that the Weierstrass equation of the Jacobian Calabi–Yau 3-fold becomes (68):

$$\tau^2 = \frac{1}{4} x^3 - (t - \alpha_1)^3(t - \alpha_5)(t - \alpha_4)^2 \cdot \prod_{j=1}^{8}(u - \beta_j) x,$$

(83)

when the K3 fibers are most enhanced, namely when K3 fibers become the attractive K3 surface $S_{[2 \, 0 \, 2]}$ with the singularity type $E_7^2D_4$. The Mordell–Weil group of this extremal K3 surface is known [130, 41], and it is isomorphic to $\mathbb{Z}_2$. (See fibration no.4 in Table 4 in Appendix A.) Using an argument similar to those given in [75, 77], we consider the specialization to the K3 fiber to deduce that the Mordell–Weil group of the Jacobian Calabi–Yau 3-fold (83) is isomorphic to that of the K3 fiber. Therefore, we find that the Mordell–Weil group of the Jacobian Calabi–Yau 3-fold (83) is isomorphic to $\mathbb{Z}_2$:

$$MW \cong \mathbb{Z}_2.$$ 

(84)

Thus, the global structure [137, 138, 139] of the gauge group forming on the 7-branes is given as follows:

$$E_7^2 \times SO(7) \times SU(2)^8/\mathbb{Z}_2.$$ 

(85)

The F-theory on the Jacobian Calabi–Yau 3-folds (83) does not have a $U(1)$ gauge field.

5 Conclusions

In this study, we investigated the points in the eight-dimensional moduli of non-geometric heterotic strings with unbroken $e_7 e_7$, at which the ranks of the non-Abelian gauge groups on the F-theory side are enhanced to 18. The gauge groups at these points do not allow for the perturbative interpretation on the heterotic side. We demonstrated in this study that these theories can be seen as deformations of the stable degenerations owing to an effect of coincident 7-branes. This effect corresponds to the insertion of 5-branes from the heterotic viewpoint. We also discussed application to $SO(32)$ heterotic strings.

K3 surfaces on the F-theory side of the moduli become extremal, when the non-Abelian gauge groups are enhanced to rank 18. We studied the Weierstrass equations of the extremal K3 surfaces that appear in the eight-dimensional moduli of non-geometric heterotic strings with unbroken $e_7 e_7$. The points in the moduli at which the ranks of the non-Abelian gauge groups are enhanced to 17 on the F-theory side also do not allow for the perturbative interpretations of the gauge groups on the heterotic side. It can be interesting to study these points in the moduli, and this can be a direction of future studies.

We also built elliptically fibered Calabi–Yau 3-folds, by fibering an elliptic K3 surface, which belongs to the F-theory side of the eight-dimensional moduli of non-geometric heterotic strings with unbroken $e_7 e_7$, over $\mathbb{P}^1$. We analyzed six-dimensional F-theory compactifications on the built elliptic Calabi–Yau 3-folds. When we choose specific sets of the parameters for the defining equations of these elliptic Calabi–Yau 3-folds, highly enhanced gauge groups form on the 7-branes. Eight-dimensional F-theory compactified on the extremal K3 fibers $S_{[2 \, 0 \, 2]}$ of these specific Calabi-Yau spaces has non-geometric heterotic duals. Determining whether
this duality extends to six-dimensional theories, namely whether F-theory compactifications on the total Calabi–Yau 3-folds have dual non-geometric heterotic strings, is a likely direction of future studies.

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A Elliptic fibrations of attractive K3 $S_{[2,0,2]}$

[130] classified the types of the elliptic fibrations of the attractive K3 surface with the discriminant four, $S_{[2,0,2]}$, and computed the Mordell–Weil groups of the fibrations. We present in Table 4 the types of the elliptic fibrations and the Mordell–Weil groups of the attractive K3 surface $S_{[2,0,2]}$ determined in [130].

| Elliptic fibrations of $S_{[2,0,2]}$ | type of singularity | MW group |
|-------------------------------------|---------------------|----------|
| No.1                                | $E_6^2A_1^2$        | 0        |
| No.2                                | $E_8D_{10}$         | 0        |
| No.3                                | $D_{16}A_1^2$       | $Z_2$    |
| No.4                                | $E_2^2D_4$          | $Z_2$    |
| No.5                                | $E_7D_{10}A_1$      | $Z_2$    |
| No.6                                | $A_{17}A_1$         | $Z_3$    |
| No.7                                | $D_{18}$            | 0        |
| No.8                                | $D_{12}D_6$         | $Z_2$    |
| No.9                                | $D_6^2A_1^2$        | $Z_2 \oplus Z_2$ |
| No.10                               | $A_{15}A_3$         | $Z_4$    |
| No.11                               | $E_6A_{11}$         | $Z \oplus Z_3$ |
| No.12                               | $D_6^3$             | $Z_2 \oplus Z_2$ |
| No.13                               | $A_9^2$             | $Z_5$    |

Table 4: List of the singularity types of the elliptic fibrations of K3 surface $S_{[2,0,2]}$, and the Mordell–Weil groups of the fibrations.

B Types of the singular fibers of extremal rational elliptic surfaces

The types of the singular fibers of the extremal rational elliptic surfaces [132] are presented in Table 5. The complex structures of the extremal rational elliptic surfaces are uniquely specified by the types of the singular fibers, except rational elliptic surfaces with two type $I_0^*$ fibers, $X_{[0^*,0^*]}(j)$ [132].
| Extremal rational elliptic surface | Type of singular fiber | Type of singularity |
|----------------------------------|-----------------------|---------------------|
| $X_{[II, II^*]}$                | $II, II^*$            | $E_8$              |
| $X_{[III, III^*]}$              | $III, III^*$          | $E_7A_1$           |
| $X_{[IV, IV^*]}$                | $IV, IV^*$            | $E_6A_2$           |
| $X_{[9^*, 0^*](j)}$             | $I_0^*, I_0^*$        | $D_4^2$            |
| $X_{[II^*, 1,1]}$               | $II^* I_1 I_1$        | $E_8$              |
| $X_{[III^*, 2,1]}$              | $III^* I_2 I_1$       | $E_7A_1$           |
| $X_{[IV^*, 3,1]}$               | $IV^* I_3 I_1$        | $E_6A_2$           |
| $X_{[4^*, 1.1]}$                | $I_4^* I_1 I_1$       | $D_8$              |
| $X_{[2^*, 2,2]}$                | $I_2^* I_2 I_2$       | $D_6A_1^2$         |
| $X_{[1^*, 4,1]}$                | $I_1^* I_4 I_1$       | $D_5A_3$           |
| $X_{[9,1,1,1]}$                 | $I_9 I_1 I_1 I_1$     | $A_8$              |
| $X_{[8,2,1,1]}$                 | $I_8 I_2 I_1 I_1$     | $A_7A_1$           |
| $X_{[6,3,2,1]}$                 | $I_6 I_3 I_2 I_1$     | $A_5A_2A_1$        |
| $X_{[5,5,1,1]}$                 | $I_5 I_5 I_1 I_1$     | $A_4^2$            |
| $X_{[4,4,2,2]}$                 | $I_4 I_4 I_2 I_2$     | $A_4^2A_1^2$       |
| $X_{[3,3,3,3]}$                 | $I_3 I_3 I_3 I_3$     | $A_2^4$            |

Table 5: List of the types of the singular fibers of extremal rational elliptic surfaces.

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