UPPER BOUNDS FOR THE ATTRACTOR DIMENSION OF DAMPED NAVIER-STOKES EQUATIONS IN $\mathbb{R}^2$

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Abstract. We consider finite energy solutions for the damped and driven two-dimensional Navier–Stokes equations in the plane and show that the corresponding dynamical system possesses a global attractor. We obtain upper bounds for its fractal dimension when the forcing term belongs to the whole scale of homogeneous Sobolev spaces from $-1$ to $1$.

1. Introduction

The theory of global attractors for the 2-D Navier–Stokes system

$$\begin{cases}
\partial_t u + (u, \nabla_x)u + \nabla_x p = \nu \Delta_x u + g, \\
u u \big|_{t=0} = u_0, \quad \text{div} u = 0.
\end{cases}$$

has been a starting point of the theory of infinite dimensional dissipative dynamical systems and remains in the focus of this theory, see [4, 10, 14, 19, 26, 37, 36] and the references therein.

In the case of a bounded domain $\Omega$ the corresponding dynamical system has a global attractor in the appropriate phase space. The attractor has finite fractal dimension, measured in terms of the dimensionless number $G$ (the Grashof number), $G = \|g\|\Omega / \nu^2$.

The best known estimate in the case of the Dirichlet boundary conditions $u|_{\partial \Omega} = 0$ is (see [37])

$$\dim_f A \leq c_D G,$$

while in the case of a periodic domain $x \in [0, 2\pi L]^2$ the estimate can be significantly improved (see [15]):

$$\dim_f A \leq c_{\text{per}} G^{2/3} (\ln(1 + G))^{1/3},$$

and, moreover, this estimate sharp up to a logarithmic correction as shown in [28], see also [17] for the alternative proof of the upper bound.

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Finding explicit majorants for the constants $c_D$ and $c_{\text{per}}$ amounts to finding sharp or explicit constants in certain Sobolev inequalities and spectral Lieb–Thirring inequalities. For example, $c_D \leq (4\pi^{3/4})^{-1}$ \cite{23}, and the majorant for $c_{\text{per}}$ can be easily written down using the recent result \cite{7} on the sharp constant in the logarithmic Brezis–Gallouet inequality (which is essential for the attractor dimension estimate in the periodic case).

In unbounded channel-like domains the Navier-Stokes system with Dirichlet boundary conditions is still dissipative in view of the Poincaré inequality. In particular, if the case of finite energy solutions is considered, the associated semigroup possesses a compact global attractor similarly to the case of bounded domains. Up to the moment, there are two alternative ways to establish this fact. The first one is based on the weighted energy estimates and careful analysis of the Leray projection in weighted Sobolev spaces, see \cite{2, 29} and references therein and the second one utilizes the so-called energy method (which will be also used in our paper) and the energy equality, see \cite{5, 34, 37}.

However, in contrast to the case of bounded domains, the solution semigroup is no longer compact, but is rather asymptotically compact and this fact strongly affects the existing upper bounds for the dimension of the attractor. Indeed, the best known estimate for the case of channel-like domains obtained in \cite{34} can be written as follows

$$\dim_f \mathcal{A} \leq \frac{c_{\text{LT}} \|g\|^2}{2 \lambda_1^2 \nu^4}.$$ 

Here $\lambda_1 > 0$ is the bottom of the spectrum of the Stokes operator in the channel, and $c_{\text{LT}}$ is the universal Lieb–Thirring constant, see \cite{3,33}. For example, in a straight channel of width $d$ we have $\lambda_1 \geq \pi^2/d^2$, so that in this case we obtain

$$\dim_f \mathcal{A} \leq \frac{1}{4\sqrt{3}\pi^4} \frac{d^4\|g\|^2}{\nu^4}.$$ 

We observe that these estimates are proportional to $\nu^{-4}$ (unlike \eqref{1.1} which is proportional to $\nu^{-2}$). On the other hand, it is worth mentioning that, to the best of our knowledge, no growing as $\nu \to 0$ lower bounds for the dimension of the attractor are known for the case of Dirichlet boundary conditions, regardless whether the underlying domain is bounded or unbounded. So, in contrast to the case of periodic boundary conditions, the behaviour of the attractor’s dimension as $\nu \to 0$ remains unclear for the case of Dirichlet boundary conditions even in the case of bounded domains.
We mention also that keeping in mind the Poiseulle flows and the mean flux integral, it seems more natural to consider the infinite energy solutions for the Navier-Stokes system in a pipe. In this case, the system remains dissipative and the existence of the so-called locally compact attractor can be established, see [142]. The dimension of this attractor may be infinite in general, but will be finite if the external forces $g(x)$ decay to zero as $|x| \to \infty$ (no matter how slow this decay is), see [29, 42] and the references therein.

In the whole $\mathbb{R}^2$ the Laplacian is not positive-definite and the Navier–Stokes system is not dissipative at least in a usual sense even in the case of zero external forces and finite energy solutions, see e.g. [35], see also [21, 40] and the references therein concerning the decay properties of various types of solutions for the Navier-Stokes problem with zero external forces. The presence of external forces makes the problem more complicated and usually only growing in time bounds for the solutions are available. We mention here only the recent results concerning the polynomial growth in time for the so-called uniformly local norms of infinite-energy solutions obtained in [20, 43], see also the references therein.

Let us now consider the damped and driven Navier–Stokes system

$$\begin{cases}
\partial_t u + (u, \nabla_x) u + \nabla_x p + \alpha u = \nu \Delta_x u + g, \\
u|_{t=0} = u_0, \quad \text{div} u = 0.
\end{cases} \tag{1.2}$$

with additional dissipative term $\alpha u$. The drag/friction term $\alpha u$, where $\alpha > 0$ is the Rayleigh or Ekman friction coefficient (or the Ekman pumping/dissipation constant), models the bottom friction in two-dimensional oceanic models and is the main energy sink in large scale atmospheric models [31].

The analytic properties of system (1.2) (such as existence of solutions, their uniqueness and regularity, etc.) remain very close to the analogous properties of the classical Navier-Stokes equations or Euler equations if the inviscid case $\nu = 0$ is considered. However, the friction term $\alpha u$ is very essential for the dynamics since it removes the energy which piles up at the large scales and from the mathematical point of view compensates the lack of the Poincare inequality. This makes the Navier–Stokes system and even the limit Euler system dissipative whatever the domain is and allows to study its global attractors in various phase spaces. For instance, the so-called weak global attractor for the inviscid case is constructed in [22] for the case of finite energy solutions; its compactness in the strong topology related with
the $H^1$-norm is verified in [11] and [12] for the cases of finite and infinite energy solutions respectively; the inviscid limit $\nu \to 0$ is studied in [16] including the absence of the so-called anomalous dissipation of enstrophy; the existence of a locally compact global attractor in the uniformly local phase spaces is established for the viscous case $\nu > 0$ in [43]; see also [6], [41] for the existence and uniqueness results for the stationary problem and the stability of stationary solutions for (1.2) with $\nu = 0$.

From the point of view of the attractors and their dimension the system (1.2) in the case of the periodic domain $x \in [0, 2\pi L]^2$ was studied in [24], where it was shown that the corresponding dynamical system possesses a global attractor $A$ (in $L^2$) whose fractal dimension is finite and satisfies the following estimate

$$\dim_f A \leq \min \left( \sqrt{6} \frac{\|\text{curl} g\| L}{\nu \alpha}, \frac{3}{8} \frac{\|\text{curl} g\|^2}{\nu \alpha^3} \right),$$

(1.3)

where the values of the constants are updated in accordance with [25]. The first estimate is enforced in the regime $\nu/L^2 \gg \alpha$, while the second estimate is enforced in the opposite regime $\nu/L^2 \ll \alpha$. We observe that both estimates are of the order $1/\nu$ as $\nu \to 0^+$ if all the remaining parameters are fixed. It was also shown in [24] that this rate of growth of the dimension is sharp, and the upper bounds were supplemented with a lower bound of the order $1/\nu$, based on the instability analysis of generalized Kolmogorov flows. The finite-dimensionality of the global attractor in the uniformly local phase spaces under the assumption that the external forces $g(x)$ decay to zero as $|x| \to \infty$ has been proved recently in [32], but no explicit upper bounds for the dimension was given there.

We would also like to point out that starting from the paper [27] the Lieb–Thirring inequalities are an essential analytical tool in the estimates of global Lyapunov exponents for the Navier–Stokes equations. This fully applies to our case.

We now observe that the first estimate in (1.3) blows up as the size of the periodic domain $L \to \infty$. On the other hand, the second estimate survives (the homogeneous $H^1$-norm is scale invariant in two dimensions). Therefore, one might expect that this estimate holds for $L = \infty$, that is, for $x \in \mathbb{R}^2$, and a motivation of the present work is to show that this is indeed the case.

In this paper we study the damped and driven Navier–Stokes system (1.2) in $\mathbb{R}^2$ in the class of finite energy solutions and our main aim is to obtain explicit upper bounds for the attractor’s dimension in terms of the parameters $\nu$ and $\alpha$ and various norms of the external forces $g$. 
In section 2 we recall for the reader convenience the proof of the well-posedness and derive the energy equality. Then using the energy equality method \cite{30, 34} we establish the asymptotic compactness of the solution semigroup and, hence, the existence of the global attractor \( \mathcal{A} \).

In section 3 we consider the case when the right-hand side \( g \) belongs to the scale of homogeneous Sobolev spaces \( \dot{H}^s \), \( s \in [-1, 1] \) and derive the following estimate(s) for the fractal dimension of the attractor \( A \):

\[
\dim_f A \leq \frac{1 - s^2}{64\sqrt{3}} \left( \frac{1 + |s|}{1 - |s|} \right)^{|s|} \frac{1}{\alpha^{2+s} \nu^{2-s}} \|g\|_{\dot{H}^s}^2, \quad s \in [-1, 1].
\]

In particular, for \( s = 1 \) we obtain

\[
\dim_f A \leq \frac{1}{16\sqrt{3}} \frac{\|\text{curl} \ g\|^2}{\nu \alpha^3},
\]

which up to a constant coincides with the second estimate in (1.3), proving thereby our expectation.

In this paper we use standard notation. The \( L^2 \)-norm and the corresponding scalar product are denoted by \( \| \cdot \| \) and \( (\cdot, \cdot) \).

2. A PRIORI ESTIMATES, WELL-POSEDNESS AND ASYMPTOTIC COMPACTNESS

We study the damped and driven Navier-Stokes system (1.2) in \( \mathbb{R}^2 \). Here \( u(t, x) = (u^1, u^2) \) is the unknown velocity vector field, \( p \) is the unknown pressure, \( g(x) = (g^1, g^2) \) is the given external force (and without loss of generality we can and shall assume that \( \text{div} \ g = 0 \)), the advection term is

\[
(u, \nabla_x) v = \sum_{i=1}^{2} u^i \partial_{x_i} v,
\]

and \( \alpha > 0, \nu > 0 \) are given parameters. We restrict ourselves to considering only finite energy solutions, so we assume that

\[
g \in [L^2(\mathbb{R}^2)]^2, \quad u_0 \in \mathcal{H} := \{ u_0 \in [L^2(\mathbb{R}^2)]^2, \text{ div } u_0 = 0 \},
\]

and by definition \( u = u(t, x) \) is a weak solution of (1.2) if

\[
u \in C(0, T; \mathcal{H}) \cap L^2(0, T; [H^1(\mathbb{R}^2)]^2), \quad T > 0,
\]

and (1.2) is satisfied in the sense of distributions. This means that for every divergence free test function \( \varphi(t, x) = (\varphi^1, \varphi^2) \), \( \text{div} \varphi = 0 \),
\( \varphi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^2) \), the following integral identity holds

\[
- \int_{\mathbb{R}} (u, \partial_t \varphi) \, dt + \int_{\mathbb{R}} ((u, \nabla_x)u, \varphi) \, dt + \alpha \int_{\mathbb{R}} (u, \varphi) \, dt + \nu \int_{\mathbb{R}} (\nabla_x u, \nabla_x \varphi) \, dt = \int_{\mathbb{R}} (g, \varphi) \, dt.
\]  

(2.2)

The following fact concerning the global well-posedness of the Navier-Stokes equations in 2D is well-known, see, for instance, [37, 38].

**Theorem 2.1.** For any \( u_0 \in \mathcal{H} \) there exists a unique solution \( u \) of problem (1.2) satisfying (2.1) and (2.2). Moreover, this solution satisfies the following energy equality for almost all \( t \geq 0 \):

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + \nu \|
abla_x u(t)\|^2 + \alpha \|u(t)\|^2 = (g, u(t)).
\]  

(2.3)

**Proof.** For the convenience of the reader, we remind below the key steps of the proof. It is strongly based on the so-called Ladyzhenskaya interpolation inequality

\[
\|u\|_{L^4(\mathbb{R}^2)} \leq C \|u\|_{\|\nabla_x u\|} \tag{2.4}
\]

which holds for any \( u \in H^1(\mathbb{R}^2) \). Indeed, this inequality together with (2.1) implies that any weak solution \( u \) satisfies

\[
u \leq L^4(0, T; L^4(\mathbb{R}^2))
\]

and, integrating by parts and using the Cauchy-Schwartz inequality, we see that

\[
|\langle u, \nabla_x u, \varphi \rangle| = \left| \sum_{i,j=1}^2 (u_i u_j, \partial_{x_j} \varphi^i) \right| \leq C \|u\|_{L^4(\mathbb{R}^2)}^2 \|\varphi\|_{H^1}.
\]

Therefore,

\[
\int_{\mathbb{R}} \langle (u, \nabla_x u), \varphi \rangle \, dt \leq C \|u\|_{L^4(0, T; L^4(\mathbb{R}^2))}^2 \|\varphi\|_{L^2(0, T; H^1(\mathbb{R}^2))}
\]

and

\[
(u, \nabla_x)u \in L^2(0, T; \mathcal{H}^{-1}) = [L^2(0, T; \mathcal{H}^1)]^*, \tag{2.5}
\]

where, as usual, \( \mathcal{H}^1 \) is a subspace of \( [H^1(\mathbb{R}^2)]^2 \) which consists of divergence free vector fields and \( \mathcal{H}^{-1} := [\mathcal{H}^1]^* \). Thus, from (2.2), we see that \( \partial_t u \in L^2(0, T; \mathcal{H}^{-1}) \) as well, and all terms in (2.2) make sense for the test function \( \varphi = u \in L^2(0, T; \mathcal{H}^1) \), so by approximation arguments (and the fact that the divergent free vector fields with compact support are dense in \( \mathcal{H} \)) we may take \( \varphi = u \) in (2.2). Then, using the
well-known orthogonality relation

\[ ((u, \nabla_x)v, v) = -\frac{1}{2}(\text{div } u, |v|^2) = 0, \tag{2.6} \]

we end up with the desired energy equality \[2.3\], see, for instance, \cite{37} for the details.

The existence of a weak solution can be obtained in a standard way based on the energy equality \[2.3\] either directly by the Faedo-Galerkin method or by approximation the problem \((1.2)\) in \(\mathbb{R}^2\) by the corresponding problems in bounded domains (see e.g., \cite{37,38} for the details), so we leave this proof to the reader and remind below the proof of the uniqueness. Let \(u_1\) and \(u_2\) be two weak solutions of problem \((1.2)\) and let \(v = u_1 - u_2\). Then \(v\) solves

\[ \partial_t v + (u_1, \nabla_x)v + (v, \nabla_x)u_2 + \alpha v + \nabla_x \bar{p} = \nu \Delta_x v, \quad \text{div } v = 0. \]

Multiplying this equation by \(v\) and integrating over \(x \in \mathbb{R}^2\), analogously to the energy equality, we have

\[ \frac{1}{2} \frac{d}{dt} \|v(t)\|^2 + \alpha \|v(t)\|^2 + \nu \|\nabla_x v(t)\|^2 = -((u_1, \nabla_x v), v) - ((v, \nabla_x)u_1, v). \tag{2.7} \]

The first term on the right-hand side of this equality vanishes in view of \(2.6\) and the second term can be estimated using the Ladyzhenskaya inequality as follows:

\[ \left| ((v, \nabla_x)u_2, v) \right| \leq \|v\|^2 \|\nabla_x u_2\| \leq C \|v\| \|\nabla_x v\| \|\nabla_x u_2\| \leq \nu \|\nabla_x v\|^2 + C^2 \nu^{-1} \|\nabla_x u_2\|^2 \|v\|^2. \]

Inserting this estimate into the right-hand side of \(2.7\) we obtain

\[ \frac{1}{2} \frac{d}{dt} \|v(t)\|^2 \leq C^2 \nu^{-1} \|\nabla_x u_2(t)\|^2 \|v(t)\|^2, \]

and the Gronwall inequality gives

\[ \|u_1(t) - u_2(t)\|^2 \leq e^{2C^2 \nu^{-1} \int_0^t \|\nabla_x u_2(s)\|^2 ds} \|u_1(0) - u_2(0)\|^2. \]

Thus, the uniqueness is proved and the theorem is also proved. \(\square\)

**Corollary 2.2.** The weak solution \(u(t)\) of problem \((1.2)\) satisfies the following dissipative estimate:

\[ \|u(t)\|^2 \leq \|u_0\|^2 e^{-\alpha t} + \alpha^{-2} \|g\|^2, \]

\[ 2\nu \int_t^{t+T} \|\nabla_x u(s)\|^2 ds \leq \alpha^{-1} T \|g\|^2 + \|u(t)\|^2. \tag{2.8} \]
Proof. Using the Cauchy–Schwartz inequality, the first estimate follows from (2.3) by the Gronwall inequality, and the second estimate is proved by integrating (2.3).

Thus, equation (1.2) defines a solution semigroup \( S(t) : \mathcal{H} \to \mathcal{H} \):

\[
S(t)u_0 := u(t), \quad u_0 \in \mathcal{H}
\]

where \( u(t) \) is a weak solution of equation (1.2) with the initial data \( u(0) = u_0 \). Moreover, this semigroup is dissipative according to estimate (2.8) and is Lipschitz continuous: for any \( u_1, u_2 \in \mathcal{H} \)

\[
\|S(t)u_1 - S(t)u_2\| \leq Ce^{Kt}\|u_1 - u_2\|,
\]

where the constants \( C \) and \( K \) depend only on \( \|u_0\| \). Our next aim is to verify the existence of a global attractor for this semigroup. For the reader convenience, we first remind its definition, see [4, 10, 14, 37] for more details.

**Definition 2.3.** Let \( S(t), t \geq 0, \) be a semigroup acting in a Banach space \( \mathcal{H} \). Then the set \( A \subset \mathcal{H} \) is a global attractor of the semigroup \( S(t) \) if

1) The set \( A \) is compact in \( \mathcal{H} \).
2) It is strictly invariant: \( S(t)A = A \).
3) It attracts the images of bounded sets in \( \mathcal{H} \) as \( t \to \infty \), i.e., for every bounded set \( B \subset \mathcal{H} \) and every neighborhood \( \mathcal{O}(A) \) of the set \( A \) in \( \mathcal{H} \) there exists \( T = T(B, \mathcal{O}) \) such that

\[
S(t)B \subset \mathcal{O}(A)
\]

for all \( t \geq T \).

To state the abstract theorem on the existence of a global attractor, we need more definitions.

**Definition 2.4.** Let \( S(t) \) be a semigroup in a Banach space \( \mathcal{H} \). Then, a set \( B \subset \mathcal{H} \) is an absorbing set of \( S(t) \) if, for every bounded sequence \( u_n \) in \( \mathcal{H} \) and for any sequence \( t_n \to \infty \), the sequence \( S(t_n)u_n \) is pre-compact in \( \mathcal{H} \).

A semigroup \( S(t) \) is asymptotically compact if for any bounded sequence \( u_n^0 \) in \( \mathcal{H} \) and for any sequence \( t_n \to \infty \), the sequence \( S(t_n)u_n^0 \) is pre-compact in \( \mathcal{H} \).

In order to verify the existence of a global attractor we will use the following criterion, see [4, 10, 26, 36, 37] for its proof.
Proposition 2.5. Let $S(t)$ be a semigroup in a Banach space $H$. Suppose that

1) $S(t)$ possesses a bounded closed absorbing set $B \subset H$;
2) $S(t)$ is asymptotically compact;
3) For every fixed $t \geq 0$ the map $S(t) : B \to H$ is continuous.

Then the semigroup $S(t)$ possesses a global attractor $A \subset B$. Moreover, the attractor $A$ has the following structure:

$$A = K\big|_{t=0},$$

where $K \subset L^\infty(\mathbb{R}, H)$ is the set of complete trajectories $u : \mathbb{R} \to H$ of semigroup $S(t)$ which are defined for all $t \in \mathbb{R}$ and bounded.

The next theorem which establishes the existence of a global attractor for the solution semigroup $S(t)$ associated with equation (1.2) is the main result of this section.

Theorem 2.6. The solution semigroup $S(t)$ of the damped Navier-Stokes problem (1.2) possesses a global attractor $A$ in $H$.

Proof. We will check the assumptions of Proposition 2.5. Indeed, the first assumption is satisfied due to the dissipative estimate (2.8) and the desired bounded and closed absorbing set can be taken as

$$B := \{ u_0 \in H : \|u_0\|_{L^2}^2 \leq 2\alpha^{-2}\|g\|_{L^2}^2 \}. \quad (2.10)$$

The third assumption is also satisfied due to estimate (2.9). Thus, we only need to check the asymptotic compactness. We will use the so-called energy method (see [3, 30, 34]) in order to do this.

Indeed, let $u^0_n \in H$ be a bounded sequence of the initial data. Then, due to estimate (2.3), we may assume without loss of generality that $u^0_n \in B$. Let $u_n(t), t \geq -t_n, t_n \to \infty$, be the sequence of solutions of

$$\partial_t u_n + (u_n, \nabla_x) u_n + \alpha u_n + \nabla_x p_n = \nu \Delta_x u_n + g, \quad \text{div} u_n = 0, \quad u_n\big|_{t=-t_n} = u^0_n.$$ 

Then, $u_n(0) = S(t_n) u^0_n$ and we only need to verify that the sequence $\{u_n(0)\}_{n=0}^\infty$ is precompact in $H$. In order to do so, we first verify that there exists a subsequence (which we also denote by $u_n$ for simplicity) such that

$$u_n(0) \to u(0) \quad (2.11)$$

converges weakly in $H$ to $u(0)$, where $u(t)$ is a complete bounded trajectory $u \in K$. We first note that due to the dissipative estimate (2.8),

$$\|u_n\|_{L^\infty(-T,T;H)} + \sup_{t \geq -T} \|u_n\|_{L^2(t,t+1;H^1)} \leq C, \quad (2.12)$$

where $K \subset L^\infty(\mathbb{R}, H)$ is the set of complete trajectories $u : \mathbb{R} \to H$ of semigroup $S(t)$ which are defined for all $t \in \mathbb{R}$ and bounded.
where $T \leq t_n$ and $C$ is independent of $n$ and $T$. Moreover, from the Ladyzhenskaya inequality \cite{2.4} we conclude also that $u_n$ is bounded in $L^4(-T, T; L^4)$. Thus, by the Banach-Alaoglu theorem, we may assume without loss of generality that

$$u^n \to u \text{ weak-star in } L^\infty(-T, T; \mathcal{H})$$

and weakly in $L^2(-T, T; H^1) \cap L^4(-T, T; L^4)$ for every $T \in \mathbb{N}$ to some function $u \in L^\infty(\mathbb{R}; \mathcal{H}) \cap L^2_{loc}(\mathbb{R}, H^1)$ which also satisfies estimate \cite{2.12}. Moreover, analogously to \cite{2.5}, we conclude that $\partial_t u_n$ is uniformly bounded in $L^2(t, t + 1; H^{-1})$. Thus, without loss of generality

$$\partial_t u^n \to \partial_t u$$

in the space $L^2(t, t + 1; H^{-1})$ for all $t \in \mathbb{R}$. Using the embedding

$$L^2(t, t + 1; H^{-1}) \cap L^2(t, t + 1; H^1) \subset C(t, t + 1; \mathcal{H}),$$

see \cite{37}, Lemma III.1.2, we see that $u_n(0) \to u(0)$. Thus, to verify \cite{2.11}, we only need to check that $u(t), t \in \mathbb{R}$ is a weak solution of \cite{1.2}. In other words, we need to pass to the limit as $n \to \infty$ in the analogue of \cite{2.2}:

$$-\int_{\mathbb{R}} (u_n, \partial_t \phi) dt + \int_{\mathbb{R}} ((u_n, \nabla_x) u_n, \varphi) dt + \alpha \int_{\mathbb{R}} (u_n, \varphi) dt +$$

$$+ \nu \int_{\mathbb{R}} (\nabla_x u_n, \nabla_x \varphi) dt = \int_{\mathbb{R}} (g, \varphi) dt,$$

(2.13)

where $\varphi(t, x) \in C_0^\infty(\mathbb{R}^3)^2$ is an arbitrary fixed divergence free function. Since passing to the limit in the linear terms is evident, we only need to pass to the limit $n \to \infty$ in the non-linear term. Integrating by parts and rewriting the nonlinear term in the form

$$((u_n(t), \nabla_x) u_n(t), \varphi(t)) = -\sum_{i,j=1}^2 (u_i^n(t) u_j^n(t) \partial_{x_i} \varphi^j(t))$$

and using the fact that the support of $\varphi$ is finite, we conclude that, for passing to the limit in the nonlinear term, it is enough to verify that, for every fixed $R, T > 0$,

$$u_n \to u \text{ strongly in } L^2(-T, T; L^2(B_0^R))),$$

(2.14)

where $B_0^R$ stands for the ball of radius $R$ in $\mathbb{R}^2$ centered at the origin. To verify the convergence \cite{2.14}, we recall that the sequence $u_n$ is bounded in the space $L^2(-T, T; H^1(B_0^R))$ due to the dissipative estimate \cite{2.8}. Moreover, analogously to \cite{2.5} but using the test functions
ϕ \in L^2(-T, T; \mathcal{H}_0^1(B_R^0))$, we see that $\partial_t u_n$ is bounded in the space $L^2(-T, T; \mathcal{H}^{-1}(B_R^0))$. Thus, since

$$\mathcal{H}^1(B_R^0) \subset \mathcal{H}(B_R^0) \subset \mathcal{H}^{-1}(B_R^0)$$

and the first embedding is compact, the compactness theorem (see, for instance [37, Theorem III.2.1]) implies that the embedding

$$H^1(-T, T; \mathcal{H}^{-1}(B_R^0)) \cap L^2(-T, T; \mathcal{H}^1(B_R^0)) \subset L^2(-T, T; \mathcal{H})$$

is compact. Therefore, $u_n$ is precompact in $L^2(-T, T; L^2(B_R^0))$ for every $R > 0$, $T > 0$ and passing to a subsequence if necessary, we conclude that the convergence \((2.14)\) indeed holds. Thus, passing to the limit in the nonlinear term of \((2.13)\) is verified and $u$ is a weak solution of \((1.2)\) which is defined for all $t \in \mathbb{R}$ and bounded, so $u \in \mathcal{K}$. This means that the convergence \((2.11)\) is verified.

We are now ready to verify that

$$u_n(0) \to u(0) \text{ strongly in } \mathcal{H}$$

(2.15)

and finish the proof of the theorem. We multiply the energy equality \((2.3)\) for the solutions $u_n$ by $e^{2\alpha t}$ and integrate from $-t_n$ to 0:

$$\|u_n(0)\|^2 = -2 \int_{-t_n}^0 e^{2\alpha s} \|\nabla_x u_n(s)\|^2 ds + \|u_n(-t_n)\|^2 e^{-2\alpha t_n} + 2 \int_{-t_n}^0 e^{2\alpha s} (g, u_n(s)) ds. \quad (2.16)$$

We want to pass to the limit $n \to \infty$ in this equality. Indeed, using the weak convergence $u_n \to u$ in $L^2_{\text{loc}}(\mathbb{R}, H^1)$ implying that

$$\limsup_{n \to \infty} -2 \int_{-t_n}^0 e^{2\alpha s} \|\nabla_x u_n(s)\|^2 ds = -2 \liminf_{n \to \infty} \int_{-t_n}^0 e^{2\alpha s} \|\nabla_x u_n(s)\|^2 ds \leq -2 \int_{-\infty}^0 e^{2\alpha s} \|\nabla_x u(s)\|^2 ds$$

and the uniform bounds \((2.12)\), we see from \((2.16)\) that

$$\limsup_{n \to \infty} \|u_n(0)\|^2 \leq -2 \int_{-\infty}^0 e^{2\alpha s} \|\nabla_x u(s)\|^2 ds + 2 \int_{-\infty}^0 e^{2\alpha s} (g, u(s)) ds.$$

On the other hand, thanks to the energy equality, for the whole-line $L^2$-bounded solution $u \in \mathcal{K}$ we have

$$\|u(0)\|^2 = -2 \int_{-\infty}^0 e^{2\alpha s} \|\nabla_x u(s)\|^2 ds + 2 \int_{-\infty}^0 e^{2\alpha s} (g, u(s)) ds$$
and, therefore, taking into the account the weak convergence (2.11), we finally arrive at
\[
\limsup_{n \to \infty} \|u_n(0)\|^2 \leq \|u(0)\|^2 \leq \liminf_{n \to \infty} \|u_n(0)\|^2.
\]
Thus, \(\lim_{n \to \infty} \|u_n(0)\| = \|u(0)\|\), and the strong convergence (2.15) is proved:
\[
\lim_{n \to \infty} \|u_n(0) - u(0)\|^2 = \lim_{n \to \infty} \left(\|u_n(0)\|^2 + \|u(0)\|^2 - 2(u_n(0), u(0))\right) = 0,
\]
and the theorem is also proved. \(\square\)

To conclude the section, we also discuss the extra regularity of the attractor \(A\). We say that \(u : \mathbb{R} \to H^1\) is a strong solution of the damped Navier-Stokes equations (1.2) if
\[
u \|\Delta x u(t)\|^2 + \alpha \|\nabla x u(t)\|^2 = (g, \Delta x u(t)). \quad (2.17)
\]
The proof of this theorem is analogous to Theorem 2.1 and even simpler since the solution \(u\) is a priori more regular now, see, for instance, [37] for more details. We only mention here that the identity (2.17) follows by multiplication of equation (1.2) by \(\Delta x u\), integration over \(x \in \mathbb{R}^2\) and using the well-known orthogonality relation \(((u, \nabla x)u, \Delta x u) = 0\) which holds for any \(u \in [H^2(\mathbb{R}^2)]^2\), \(\text{div} \ u = 0\). In fact, integrating by parts and setting \(\omega := \text{curl} \ u = \partial_{x_1} u^2 - \partial_{x_2} u^1\), the divergence theorem gives
\[
((u, \nabla x)u, \Delta x u) = \frac{1}{2} \int_{\mathbb{R}^2} \text{div}(u \omega^2) \, dx = 0
\]
for a smooth \(u \in C_0^\infty(\mathbb{R}^2)^2\), \(\text{div} \ u = 0\). The general case follows by a standard approximation procedure.

The next corollary gives the dissipative estimate for the solutions of (1.2) in \(H^1\), and its proof is similar to that of (2.8).
Corollary 2.8. The strong solution satisfies the following estimate:
\[
\| \nabla_x u(t) \|^2 \leq \| \nabla_x u_0 \|^2 e^{-\alpha t} + (2\alpha \nu)^{-1} \| g \|^2,
\]
\[
\nu \int_t^{t+T} \| \Delta_x u(s) \|^2 ds \leq \nu^{-1} T \| g \|^2 + \| \nabla_x u(t) \|^2.
\]

(2.18)

In the following corollary we establish the smoothing property for the weak solutions of equation (1.2).

Corollary 2.9. For \( t > 0 \), \( u(t) \in H^1 \) and the following estimate holds:
\[
\| u(t) \|^2_{H^1} \leq C t^{-1} (\| u(0) \|^2 + \| g \|^2), \quad 0 < t \leq 1,
\]
where \( C = C(\alpha, \nu) \). In particular, the attractor \( \mathcal{A} \) is bounded in \( H^1 \).

Proof. Since \( \| \nabla_x u(t) \|^2 \) is integrable, for every \( 0 < t \leq 1 \), there exists \( \tau \leq T \) such that \( u(\tau) \in H^1 \) and
\[
\| \nabla_x u(\tau) \|^2 \leq \frac{1}{T} \int_0^T \| \nabla_x u(s) \|^2 ds \leq \frac{1}{2\nu T} \left( \| u(0) \|^2 + \alpha^{-1} T \| g \|^2 \right),
\]
where the second inequality follows from (2.8). By (2.18) with the initial time \( t = \tau \), we have
\[
\| \nabla_x u(T) \|^2 \leq \| \nabla_x u(\tau) \|^2 + (2\alpha \nu)^{-1} \| g \|^2 \leq C T^{-1} \left( \| u(0) \|^2 + \| g \|^2 \right).
\]
This and (2.8) prove (2.19) and guarantee that the \( H^1 \)-ball
\[
B_1 := \{ u_0 \in H^1, \quad \| u_0 \|^2_{H^1} \leq R \}
\]
is a bounded absorbing set for the solution semigroup \( S(t) \) if \( R \) is large enough. Thus, \( \mathcal{A} \subset B_1 \) is bounded in \( H^1 \) and the proof is complete. \( \square \)

Remark 2.10. Arguing analogously, it is not difficult to show that the smoothness of the global attractor \( \mathcal{A} \) is restricted by the smoothness of the external forces \( g \). In particular, the attractor will be \( C^\infty \)-smooth (analytic) if the external forces are \( C^\infty \)-smooth (analytic).

3. Fractal dimension of the attractor: upper bounds

In this section, we prove that the global attractor constructed above has finite fractal (box-counting) dimension and give explicit upper bounds for this dimension in terms of the physical parameters \( \nu \) and \( \alpha \) and the norms of the right-hand side \( g \) in homogeneous Sobolev spaces. To this end, we first need to remind some definitions, see [37, 4] for more detailed exposition.

Definition 3.1. Let \( \mathcal{A} \subset \mathcal{H} \) be a compact set in a metric space \( \mathcal{H} \). Then, by the Hausdorff criterion, for every \( \varepsilon > 0 \) it can be covered by the finite number of \( \varepsilon \)-balls in \( \mathcal{H} \). Let \( N_\varepsilon(\mathcal{A}, \mathcal{H}) \) be the minimal number
of such balls. Then, the fractal (box-counting) dimension of $A$ in $H$ is defined via the following expression:

$$\dim_f(A, H) := \lim_{\varepsilon \to 0} \sup \frac{\ln N_\varepsilon(A, H)}{\ln \frac{1}{\varepsilon}}.$$ 

The fractal dimension coincides with the usual dimension if the set $A$ is regular enough (for instance, for the case where $A$ is a Lipschitz manifold in $H$), but may be non-integer for irregular sets (for instance, for the standard ternary Cantor set $K \subset [0, 1]$ this dimension is $\frac{\ln 2}{\ln 3}$), see, for instance, [33] for more details.

We now estimate the dimension of the global attractor $A$ by means of the so-called volume contraction method [13], [37]. The solution semigroup $S_t$ is uniformly quasi-differentiable on the attractor in the sense that there exists (for a fixed $t \geq 0$) a linear bounded operator $DS(t, u_0)$ such that

$$\|S(t)u_1 - S(t)u_0 - DS(t, u_0) \cdot (u_1 - u_0)\| \leq h(\|u_1 - u_0\|), \quad (3.1)$$

where $h(r)/r \to 0$ as $r \to 0$, and $u_0, u_1 \in A$.

The quasi-differential $DS(t, u_0)$ is the solution operator $\xi \to v(t)$ of the following equation of variations:

$$\partial_t v = L(t, u_0)v := -\Pi((v, \nabla_x)u(t)+(u(t), \nabla_x)v) - \alpha v + \nu \Delta x v, \quad v(0) = \xi, \quad (3.2)$$

where $\Pi : [L^2(\mathbb{R}^2)]^2 \to H$ is the Leray ortho-projection onto the divergence free vector fields, and $u(t) = S(t)u_0$ is the solution lying on the attractor and parameterized by $u_0 \in A$. For the proof see [3], where it is also shown that the solution semigroup is even differentiable for all $u_0 \in H$ and the differential $DS(t, u_0)$ depends continuously on the point $u_0$.

We define for $m = 1, 2 \ldots$ the numbers $q(m)$ (the sums of the first $m$ global Lyapunov exponents)

$$q(m) = \limsup_{t \to \infty} \sup_{u_0 \in A} \sup_{\{v_j\}_{j=1}^m \in H^1} \frac{1}{t} \int_0^t \sum_{j=1}^m (L(\tau, u_0)v_j, v_j) d\tau,$$

where the supremum closest to the integral is taken with respect to all $L^2$-orthonormal families $\{v_j\}_{j=1}^m \in H^1$.

To define $q(m)$ for all real $m \geq 1$ we just linearly interpolate between $q(m)$ and $q(m + 1)$ so that $q(m)$ is now a piece-wise linear continuous function of $m$.

The following theorem is the key technical tool for estimating the dimension of the attractor via the so-called volume contraction method.
Theorem 3.2. Let $S(t): \mathcal{H} \to \mathcal{H}$ be the solution semigroup associated with problem (1.2) in a Hilbert space $\mathcal{H}$ and let $\mathcal{A}$ be a compact invariant set of $S_t$ in $\mathcal{H}$: $S(T)\mathcal{A} = \mathcal{A}$.

Suppose that the semigroup $S(t)$ is uniformly quasi-differentiable for every fixed $t$ on $\mathcal{A}$ in the sense of (3.1).

Suppose further that the quasi-differential $D S(t, u_0)$ depends continuously on the initial point $u_0 \in \mathcal{A}$ as a map $D S(t, \cdot) : u_0 \to \mathcal{L}(\mathcal{H}, \mathcal{H})$.

Suppose that there exists number $m > 0$ such that $q(m) < 0$. Then $\dim f \mathcal{A} < m$.

For the proof of this theorem see [13], [37] in the case of Hausdorff dimension and [8] for the fractal dimension.

Remark 3.3. The condition on the continuity of the quasi-differentials with respect to the initial point is redundant in the case of the Hausdorff dimension. It is also redundant in the case of the fractal dimension if the graph of $q(m)$ lies below the straight line joining the points $(m - 1, q(m - 1))$ and $(m, q(m))$, where $q(m) < 0$ and $q(m - 1) \geq 0$, see [10] [9].

Also, in applications to infinite dimensional dissipative dynamical systems an upper bound for $q(m)$ is usually found in the form

$$q(m) \leq -c_1 m^\gamma + c_2, \quad \gamma \geq 1.$$  

For example, as we shall shortly see, $\gamma = 1$ in our case. In this case, also without the continuity condition, we have

$$\dim f \mathcal{A} \leq (c_2/c_1)^{1/\gamma}.$$  

To apply this theorem for obtaining the fractal dimension of the attractor $\mathcal{A}$ of the semigroup $S(t)$ generated by damped Navier-Stokes equation (1.2) we need to state the Lieb-Thirring inequality which plays a fundamental role in estimating the quantities $q(m)$.

Lemma 3.4 (Lieb–Thirring inequality). Let $\{v_j\}_{j=1}^m \in H^1(\mathbb{R}^2)^2$ be a family of orthonormal vector-functions and let $\text{div} v_j = 0$. Then the following inequality holds for $\rho(x) = \sum_{k=1}^m |v_k(x)|^2$:

$$\|\rho\|^2 = \int_{\mathbb{R}^2} \left( \sum_{j=1}^m |v_j(x)|^2 \right)^2 dx \leq c_{LT} \sum_{j=1}^m \|\nabla x v_j\|^2, \quad c_{LT} \leq \frac{1}{2\sqrt{3}}. \quad (3.3)$$

Proof. The proof (see [23]) is a reduction to the scalar case (which works in two dimensions) [8] and the use of the main result in [18].

We first set $\nu = 1$ and $\alpha = 1$ and begin with estimating the $m$-trace of the operator $L(t, u_0)$ in the equation of variations (3.2) for
this particular case. The general case will be reduced later to the case \( \nu = \alpha = 1 \) by the proper scaling.

**Proposition 3.5.** Let \( \nu = 1 \) and \( \alpha = 1 \). Then, the following estimate holds:

\[
\limsup_{T \to \infty} \sup_{u_0 \in A} \sup_{v_j} \frac{1}{T} \int_0^T \sum_{j=1}^m (L(t, u_0) v_j, v_j) \, dt \leq -m + \frac{1}{16 \sqrt{3}} \limsup_{T \to \infty} \sup_{u_0 \in A} \frac{1}{T} \int_0^T \| \nabla_x u(t) \|^2 \, dt, \tag{3.4}
\]

where \( u(t) = S(t) u_0 \).

**Proof.** Let \( v_1, \ldots, v_m \in H^1 \) be an orthonormal family in \( H \). Then, integrating by parts and using that the vector fields \( v_i \) are divergence free, we get

\[
\sum_{j=1}^m (L(u(t)) v_j, v_j) = \sum_{j=1}^m \| \nabla_x v_j \|^2 - \sum_{j=1}^m \| v_j \|^2 - \int_{\mathbb{R}^2} \sum_{j=1}^m \sum_{i,k=1}^2 v_j^k \partial_{x_k} u^i v_j^i \, dx = \]

\[
= -m - \sum_{i=1}^m \| \nabla_x v_i \|^2 - \int_{\mathbb{R}^2} \sum_{j=1}^m \sum_{i,k=1}^2 v_j^k \partial_{x_k} u^i v_j^i \, dx.
\]

To estimate the right-hand side above, we use the following point-wise inequality

\[
\left| \sum_{k,i=1}^2 v^k \partial_{x_k} u^i v^i \right| \leq 2^{-1/2} \| \nabla_x u \| |v|^2 \tag{3.5}
\]

which holds for any \( v = (v^1, v^2) \in \mathbb{R}^2 \) and any Jacobi matrix \( \nabla_x u = (\partial_{x_i} u^j)_{i,j=1}^2 \in \mathbb{R}^4 \) such that \( \partial_{x_1} u^1 + \partial_{x_2} u^2 = 0 \), see [8, Lemma 4.1]. Indeed, setting

\[
v = (\xi, \eta), \quad \nabla_x u = A := \begin{pmatrix} a & b \\ c & -a \end{pmatrix},
\]

we have by the Cauchy-Schwartz inequality

\[
(A v, v) = a(\xi^2 - \eta^2) + (b + c)\xi \eta = a(\xi^2 - \eta^2) + ((b + c)/2) 2\xi \eta \leq \sqrt{a^2 + (b + c)^2} \sqrt{\xi^2 - \eta^2}^2 + 4b^2 \eta^2 = \sqrt{a^2 + (b + c)^2} \cdot |v|^2 \leq \sqrt{a^2 + (b^2 + c^2)/2} \cdot |v|^2 = \frac{1}{\sqrt{2}} \sqrt{a^2 + b^2 + c^2 + a^2} \cdot |v|^2,
\]

\[
\leq \sqrt{a^2 + (b^2 + c^2)/2} \cdot |v|^2 = \frac{1}{\sqrt{2}} \sqrt{a^2 + b^2 + c^2 + a^2} \cdot |v|^2,
\]

\[
\leq \sqrt{a^2 + (b^2 + c^2)/2} \cdot |v|^2 = \frac{1}{\sqrt{2}} \sqrt{a^2 + b^2 + c^2 + a^2} \cdot |v|^2,
\]

\[
\leq \sqrt{a^2 + (b^2 + c^2)/2} \cdot |v|^2 = \frac{1}{\sqrt{2}} \sqrt{a^2 + b^2 + c^2 + a^2} \cdot |v|^2,
\]
and estimate (3.5) is proved. Using this pointwise estimate and the Lieb–Thirring inequality for divergence free vector fields (3.3), we finally have

\[
\sum_{j=1}^{m} (L(u(t))v_j, v_j) \leq -m - \sum_{i=1}^{m} \| \nabla_x v_i \|^2 + \frac{1}{\sqrt{2}} \int_{\mathbb{R}^2} \rho(x)|\nabla_x u(x)| \, dx \leq \\
\leq -m - \sum_{i=1}^{m} \| \nabla_x v_i \|^2 + \frac{1}{\sqrt{2}} \| \rho \| \| \nabla_x u(t) \| \leq \\
\leq -m - \sum_{i=1}^{m} \| \nabla_x v_i \|^2 + \frac{1}{\sqrt{2}} \left( c_{LT} \sum_{j=1}^{m} \| \nabla_x v_j \|^2 \right)^{1/2} \| \nabla_x u(t) \| \leq \\
\leq -m + \frac{c_{LT}}{8} \| \nabla_x u(t) \|^2 = -m + \frac{1}{16\sqrt{3}} \| \nabla_x u(t) \|^2.
\]

Integrating this inequality for \( t \in [0, T] \) and taking the supremum over all \( u_0 \in \mathcal{A} \), we obtain (3.4) and finish the proof of the proposition. \(\Box\)

**Theorem 3.6.** Let the assumptions of Theorem 2.1 hold and let \( \nu = \alpha = 1 \). Then the fractal dimension of the global attractor \( \mathcal{A} \) of problem (1.2) satisfies the following estimate:

\[
\dim_f \mathcal{A} \leq \frac{1}{16\sqrt{3}} \limsup_{T \to \infty} \frac{1}{T} \sup_{u_0 \in \mathcal{A}} \int_{0}^{T} \| \nabla_x u(t) \|^2 \, dt, \tag{3.6}
\]

where \( u(t) = S(t)u_0 \).

**Proof.** This estimate is a corollary of Theorem 3.2 and estimate (3.4). \(\Box\)

Thus, we only need to estimate the integral in the RHS of (3.6). We assume below that the right-hand side \( g \) belongs to the homogeneous Sobolev space \( \dot{H}^s(\mathbb{R}^2) = (-\Delta_x)^{-s/2}L^2(\mathbb{R}^2) \), \( s \in \mathbb{R} \) with norm:

\[
\| u \|_{\dot{H}^s}^2 := \int_{\mathbb{R}^2} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi, \tag{3.7}
\]

where

\[
\hat{u}(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}^2} u(x)e^{-i\xi x} \, dx
\]

is the Fourier transform of \( u \), see [39] for more details. Then, obviously,

\[
\| u \|_{\dot{H}^0} = \| u \|, \quad \| u \|_{\dot{H}^1} = \| \nabla_x u \|, \quad \| u \|_{\dot{H}^2} = \| \Delta_x u \|, \tag{3.8}
\]
and the following interpolation inequalities immediately follow from definition (3.7) and the Cauchy–Schwartz inequality:

\[
\|u\|_{H^s} \leq \|u\|^{1-s} \|\nabla_x u\|^s,
\]

\[
\|\Delta_x u\|_{H^{1-s}} = \|\nabla_x u\|_{H^{1-s}} \leq \|\nabla_x u\|^s \|\Delta_x u\|^{1-s}, \quad s \in [0, 1].
\] (3.9)

**Corollary 3.7.** Let the assumptions of Theorem 2.1 hold and let, in addition, \(g \in \dot{H}^{-s}\) for some \(s \in [0, 1]\) and \(\nu = \alpha = 1\). Then the fractal dimension of the attractor \(A\) of problem (1.2) satisfies the following estimate:

\[
\dim_f A \leq \frac{1 - s^2}{64\sqrt{3}} \left(\frac{1 + s}{1 - s}\right)^s \|g\|^2_{H^{-s}}.
\] (3.10)

**Proof.** From the energy equality (2.3) with \(\nu = \alpha = 1\), interpolation inequalities (3.9) and the Young inequality, we get

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2_{L^2} + \|u\|^2_{L^2} + \|\nabla_x u\|^2_{L^2} = (g, u) \leq \|g\|_{H^{-s}} \|u\|_{H^s} \leq \|g\|_{H^{-s}} \|\nabla_x u\|_{L^2} \|u\|^1_{L^2} \leq \frac{\varepsilon^2}{2} \|g\|^2_{H^{-s}} + \frac{\delta p}{p} \|\nabla_x u\|^2_{L^2} + \gamma^q \|u\|^2_{L^2},
\]

where \(p = \frac{2}{s}, q = \frac{2}{1-s}\) and positive numbers \(\varepsilon, \delta, \gamma\) are such that \(\varepsilon \delta \gamma = 1\). Fixing now the parameter \(\gamma\) in such way that

\[
\frac{\gamma^q}{q} = 1 \quad \Rightarrow \quad \gamma = \left(\frac{1 - s}{2}\right)^{-\frac{1-s}{s}},
\]

and integrating over \(t \in [0, T]\), we arrive at

\[
\frac{1}{2T} \left(\|u(t)\|^2 - \|u(0)\|^2\right) + \left(1 - \frac{\delta p}{p}\right) \frac{1}{T} \int_0^T \|\nabla_x u(t)\|^2 dt \leq \frac{\varepsilon^2}{2} \|g\|^2_{H^{-s}}.
\]

From the dissipative estimate (2.8) we conclude that

\[
\lim_{T \to \infty} \sup_{u_0 \in A} \frac{1}{2T} \left(\|u(t)\|^2 - \|u(0)\|^2\right) = 0
\]

and, therefore,

\[
\lim_{T \to \infty} \sup_{u_0 \in A} \frac{1}{T} \sup_{u_0} \int_0^T \|\nabla_x u(t)\|^2 dt \leq \frac{\varepsilon^2}{2} \left(1 - \frac{\delta p}{p}\right)^{-1} \|g\|^2_{H^{-s}}
\] (3.11)

and we only need to optimize the coefficient in the RHS with respect to \(\varepsilon\) and \(\delta\). Indeed, since \(\varepsilon \delta \gamma = 1\), we conclude that

\[
\varepsilon \delta = \left(\frac{1 - s}{2}\right)^{\frac{1-s}{s}} \quad \Rightarrow \quad \varepsilon = \delta^{-1} \left(\frac{1 - s}{2}\right)^{\frac{1-s}{s}}
\]
and
\[ \frac{\varepsilon^2}{2} \left( 1 - \frac{\delta^p}{p} \right)^{-1} = \frac{1}{2} \left( \frac{1 - s}{2} \right)^{1-s} \frac{1}{x \left( 1 - \frac{s}{2} x^2 \right)}, \]
where \( x := \delta^2 \). Thus, it only remains to maximize the function
\[ f(x) := x \left( 1 - \frac{s}{2} x^2 \right) \]
on the interval \( x \geq 0 \):
\[ f'(x) = 1 - \frac{s + 1}{2} x^s = 0 \Rightarrow x = \left( \frac{s + 1}{s + 1} \right)^s \Rightarrow f(x) = \frac{1}{s + 1} \left( \frac{2}{s + 1} \right)^s. \]
Inserting the obtained estimates into the right-hand side of (3.11), we finally get
\[ \limsup_{T \to \infty} \frac{1}{T} \sup_{u_0 \in \mathcal{A}} \int_0^T \| \nabla_x u(t) \|^2 dt \leq \frac{1 - s^2}{4} \left( \frac{1 + s}{1 - s} \right)^s \| g \|_{\dot{H}^s}^2. \]
This estimate together with (3.6) completes the proof of the corollary. \( \square \)

The next corollary gives the analogous result for \( g \in \dot{H}^s \) with \( s \geq 0 \).

**Corollary 3.8.** Let the assumptions of Theorem 2.1 hold and let, in addition, \( g \in \dot{H}^s \) for some \( s \in [0, 1] \) and \( \nu = \alpha = 1 \). Then the fractal dimension of the attractor \( \mathcal{A} \) of problem (1.2) satisfies the following estimate:
\[ \dim_f \mathcal{A} \leq \frac{1 - s^2}{64 \sqrt{3}} \left( \frac{1 + s}{1 - s} \right)^s \| g \|_{\dot{H}^s}^2. \]  

**Proof.** From the second energy equality (2.17) with \( \nu = \alpha = 1 \), interpolation inequalities (3.9) and the Young inequality, we get
\[ \frac{1}{2} \frac{d}{dt} \| \nabla_x u \|^2 + \| \nabla_x u \|^2 + \| \Delta_x u \|^2 = (g, \Delta_x u) \leq \| g \|_{\dot{H}^s} \| \Delta_x u \|_{\dot{H}^s} \leq \| g \|_{\dot{H}^s} \| \nabla_x u \|^s \| \Delta_x u \|^{1-s} \leq \frac{\varepsilon^2}{2} \| g \|_{\dot{H}^s}^2 + \frac{\delta^p}{p} \| \nabla_x u \|^2 + \frac{\gamma^q}{q} \| \Delta_x u \|^2, \]
where the exponents \( p, q \) and the constants \( \varepsilon, \delta, \gamma \) are exactly the same as in the proof of the previous corollary. Thus, using the strong dissipative estimate (2.18) and arguing exactly as in the proof of the previous corollary, we end up with
\[ \limsup_{T \to \infty} \frac{1}{T} \sup_{u_0 \in \mathcal{A}} \int_0^T \| \nabla_x u(t) \|^2 dt \leq \frac{1 - s^2}{4} \left( \frac{1 + s}{1 - s} \right)^s \| g \|_{\dot{H}^s}^2. \]
which together with (3.6) finishes the proof of the corollary. □

Thus, combining estimates (3.10) and (3.12), we end up with

\[ \dim f A \leq \frac{1 - s^2}{64\sqrt{3}} \left( \frac{1 + |s|}{1 - |s|} \right)^{|s|} \|g\|_{\dot{H}^s}^2 \]  

(3.13)

which holds for the case \( \nu = \alpha = 1 \) if \( g \in \dot{H}^{-1} \cap \dot{H}^1 \) and \( s \in [-1, 1] \).

Finally, we need the analogue of estimate (3.13) for general \( \nu, \alpha > 0 \).

We reduce this general case to the particular case \( \nu = \alpha = 1 \) by the proper scaling of \( t, x \) and \( u \). Indeed, let \( u = u(t, x) \) be a solution of (1.2) with arbitrary \( \nu, \alpha > 0 \). Then, taking 

\[ t' := \alpha t, \quad x' = \left( \frac{1}{\nu} \right)^{1/2} x, \quad \tilde{u} = \frac{1}{(\alpha \nu)^{1/2}} u \]

we see that the function \( \tilde{u}(t', x') := \frac{1}{(\alpha \nu)^{1/2}} u(t, x) \) solves equation (1.2) with \( \nu = \alpha = 1 \) and the external forces \( \tilde{g}(x') = \alpha^{-1}(\alpha \nu)^{-1/2} g(x) \).

Since the fractal dimension of the attractor does not change under this scaling, using the obvious scaling properties of the \( \dot{H}^s \)-norm:

\[ \|g(\gamma \cdot)\|_{\dot{H}^s}^2 = \gamma^{2(s-1)} \|g\|_{\dot{H}^s}^2, \]

we have proved the following result.

**Theorem 3.9.** Let the assumptions of Theorem 2.1 hold (now with arbitrary positive \( \nu \) and \( \alpha \)) and let, in addition \( g \in \dot{H}^s \) for some \( s \in [-1, 1] \). Then, the fractal dimension of the attractor \( A \) in \( \mathcal{H} \) satisfies the following estimate:

\[ \dim f A \leq \frac{1 - s^2}{64\sqrt{3}} \left( \frac{1 + |s|}{1 - |s|} \right)^{|s|} \frac{1}{\alpha^2 \nu^2} \left( \frac{\nu}{\alpha} \right)^s \|g\|_{\dot{H}^s}^2. \]  

(3.14)

**Proof.** Indeed, the above estimate follows from (3.13) and the identity

\[ \|\tilde{g}\|_{\dot{H}^s}^2 = \frac{1}{\alpha^2 \nu^2} \left( \frac{\nu}{\alpha} \right)^s \|g\|_{\dot{H}^s}^2. \]

Thus, combining estimates (3.10) and (3.12), we end up with

\[ \dim f A \leq \frac{1 - s^2}{64\sqrt{3}} \left( \frac{1 + |s|}{1 - |s|} \right)^{|s|} \frac{1}{\alpha^2 \nu^2} \left( \frac{\nu}{\alpha} \right)^s \|g\|_{\dot{H}^s}^2. \]  

(3.13)

which holds for the case \( \nu = \alpha = 1 \) if \( g \in \dot{H}^{-1} \cap \dot{H}^1 \) and \( s \in [-1, 1] \).

If \( g \in \dot{H}^{-1} \cap \dot{H}^1 \), then the rate of growth of the estimate (3.13) with respect to \( \nu \) as \( \nu \to 0 \) is the smallest when \( s = 1 \). In this case we have

**Corollary 3.10.** Suppose that \( g \in \dot{H}^1 \). Then the fractal dimension of the attractor \( A \) satisfies

\[ \dim f A \leq \frac{1}{16\sqrt{3}} \frac{\|\text{curl } g\|_{L^2}^2}{\alpha^3 \nu}. \]  

(3.15)
Proof. Since $\text{div} \, g = 0$, it follows that
\[ \|g\|_{H^1}^2 = \|\nabla_x g\|^2 = \|\text{div} \, g\|^2 + \|\text{curl} \, g\|^2 = \|\text{curl} \, g\|^2. \]

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