POWER SERIES RINGS AND PROJECTIVITY

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Abstract. We show that a formal power series ring $A[[X]]$ over a noetherian ring $A$ is not a projective module unless $A$ is artinian. However, if $(A, m)$ is any local ring, then $A[[X]]$ behaves like a projective module in the sense that $\text{Ext}^p_A(A[[X]], M) = 0$ for all $m$-adically complete $A$-modules. The latter result is shown more generally for any flat $A$-module $B$ instead of $A[[X]]$. We apply the results to the (analytic) Hochschild cohomology over complete noetherian rings.

1. Introduction

By a classical result of Baer [1], the ring $\mathbb{Z}[[X]]$ of formal power series is not a projective $\mathbb{Z}$-module. In this paper we show that more generally for any noetherian ring $A$ of dimension at least 1 the ring $A[[X]]$ is not a projective $A$-module, see Theorem 2.1. On the other hand we will deduce in Theorem 2.3 that any flat module $B$ over a local ring $(A, m)$ behaves like a projective module with respect to $m$-adically complete modules $M$ in the sense that $\text{Ext}^p_A(B, M) = 0$ for all $p \geq 1$.

Our motivation for these results comes from Hochschild cohomology. If $A \to B$ is a flat morphism of (commutative) rings then the usual bar resolution $\mathcal{B}_*$ provides a flat resolution of $B$ as a module over the enveloping algebra $B^{(e)} := B \otimes_A B$, and by definition Hochschild cohomology $\text{HH}^p(B/A, M)$ is the cohomology of the complex $\text{Hom}_{B^{(e)}}(\mathcal{B}_*, M)$. If $B$ is projective as an $A$-module then $\mathcal{B}_*$ is even a complex of projective $B^{(e)}$-modules and so

$$\text{HH}^p(B/A, M) \cong \text{Ext}^p_{B^{(e)}}(B, M).$$

In Proposition 3.1 we apply Theorem 2.3 to show that these isomorphisms still prevail, as long as $B$ is flat over $A$ and $M$ is a complete module over $B^{(e)}$.

Next assume that $A \to B$ is a flat morphism of complete local rings with isomorphic residue fields. Using complete tensor products one can introduce the completed Bar-resolution $\hat{B}_*$ to define an analytic version of Hochschild cohomology $\hat{\text{HH}}^p(B/A, M)$ as the cohomology of the complex $\text{Hom}_{\hat{B}^{(e)}}(\hat{B}_*, M)$. As an application of the preceding results we will show in Section 3 that in analogy with the isomorphisms above there are as well isomorphisms

$$\hat{\text{HH}}^p(B/A, M) \cong \text{Ext}^p_{\hat{B}^{(e)}}(B, M)$$

for complete $\hat{B}^{(e)}$-modules $M$.

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2. Complete modules and projectivity

By a result of Baer [Ba] the ring $\mathbb{Z}[[X]]$ is not a projective $\mathbb{Z}$-module. Using a variant of his idea of proof we can show the following result.

**Theorem 2.1.** If $A$ is a commutative noetherian ring of dimension $\geq 1$, then $A[[X]]$ is not a projective $A$-module.

**Proof.** As $A$ is noetherian by assumption, every ideal $a$ of $A$ is finitely generated and so $A[[X]]/aA[[X]] \cong (A/a)[[X]]$. Moreover, if $A[[X]]$ is projective as $A$-module so is $A[[X]]/aA[[X]]$ as a module over $A/a$. Replacing $A$ by $A/p$ for a minimal prime $p$ of $A$ we may, and will, henceforth assume that $A$ is a domain.

Now fix a maximal ideal $m$ of $A$ and consider the subset $Z \subseteq A[[X]]$ of all power series $F = \sum a_i X^i$ with $\lim_i a_i = 0$ in the $m$-adic topology. Clearly, $Z$ is an $A$-submodule of $A[[X]]$ that contains the polynomial ring $A[X] \subseteq A[[X]]$. We first show that

$$Z = A[X] + mZ.$$  

Indeed, if $F = \sum a_i X^i$ is in $Z$ then there is a sequence of numbers $n_i$ with $a_i \in m^{n_i}$ and $\lim n_i = \infty$. We may assume that $n_i \geq 1$ for $i \geq k$. Thus, if $f_1, \ldots, f_n$ is a system of generators of $m$ as an $A$-module, then for $i \geq k$ we can write $a_i = \sum_{j=1}^n a_{ij} f_j$ with $a_{ij} \in m^{n_i-1}$, whence

$$F = \sum_{i=0}^{k-1} a_i X^i + \sum_{j=1}^n F_j f_j,$$

where $F_j := \sum_{i \geq k} a_{ij} X^i$ is again in $Z$.

Now assume $A[[X]]$ were a projective $A$-module. It is then in particular contained in a free $A$-module $A^{(I)}$. Restricting the inclusion to $Z$ will then yield

$$Z \hookrightarrow A^{(I)}.$$  

As $A[X]$ is free on a countable basis, the image of $A[X]$ is contained in some direct summand $A^{(J)}$ of $A^{(I)}$, where $J \subseteq I$ is a countable subset. With $M := Z/A[X]$, there is hence an induced $A$-linear map

$$\varphi : M \to A^{(I \setminus J)}.$$  

By $(\ast)$ above, $mM = M$, whereas $\bigcap_{k \geq 0} m^k A^{(I \setminus J)} = 0$, whence $\varphi$ is necessarily the zero map and so already $Z \subseteq A^{(J)}$.

On the other hand, if $t \in m$ is a nonzero element, then the $A$-linear map

$$A[[X]] \to Z \quad \text{with} \quad F = \sum a_i X^i \mapsto \sum a_i t^i X^i$$

is injective, whence $A[[X]]$ can as well be realized as a submodule of $A^{(J)}$. We now show that this is impossible.

Indeed, if $A$ is uncountable, then the formula for Vandermonde’s determinant yields that the power series

$$F_a := \sum_{i=0}^{\infty} a^i X^i, \quad a \in A \setminus \{0\},$$

form $A$-linearly independent elements of $A[[X]]$. Hence, with $K$ the field of fractions of $A$, the vector space $A[[X]] \otimes_A K$ has uncountable dimension over $K$ contradicting the fact that it can be embedded into the countably generated vector space $A^{(J)} \otimes_A$.
\(K \cong K^{(J)}\). Finally, if \(A\) is countable then so is \(A^{(J)}\), while \(A[[X]]\) is uncountable using Cantor’s argument. Hence we obtain the desired contradiction concluding the proof that \(A[[X]]\) cannot be projective. \(\square\)

Remarks 2.2. 1. A minor variant of the argument shows as well that for an analytic algebra \(A\) over a valued field, the convergent power series ring \(A\{X\}\) is neither projective as an \(A\)-module. For \(A = \mathbb{C}\{t\}\), this was already observed by Wolffhardt \[Wol\ Satz 9\].

2. As pointed out by Avramov, if \(R\) is a non-local domain and if \(\hat{R}\) is its completion with respect to some maximal ideal \(m \subseteq R\), then \(\hat{R}\) is as well not projective as an \(R\)-module. In fact, if \(a \in R \setminus m\) is not a unit then \(R/\text{aR} \neq 0\) whereas \(\hat{R} \otimes_R (R/\text{aR}) \cong \hat{R}/\text{a}\hat{R} = 0\). Hence \(\hat{R}\) is not faithful as an \(R\)-module and so cannot be projective.

This shows in particular directly that \(A[[X]]\), or \(A\{X\}\) in the analytic case, is never a projective \(R = A[X]\)-module.

As was shown in \[Da\], the group \(\text{Ext}_A^1(\mathbb{Z}[[X]], T)\) vanishes whenever \(T\) is a finitely generated torsion \(\mathbb{Z}\)-module or, even more generally, \(T\) is of bounded torsion. We will supplement this observation by the following fact.

**Theorem 2.3.** Let \(A\) be a ring and \(m \subseteq A\) a maximal ideal. If \(B\) is a flat \(A\)-module then \(\text{Ext}_A^p(B, M) = 0\) for all \(p \geq 1\) and each \(m\)-adically complete \(A\)-module \(M\).

We first recall the following well-known result; see \[AC II.3.Cor.2 to Prop.5\]; and give its simple proof.

**Lemma 2.4.** Every flat module is free over a local ring \((A, m)\) whose maximal ideal is nilpotent, that is, \(m^n = 0\) for some \(n \geq 1\).

**Proof.** The lemma is certainly true if \(m = 0\), as then \(A = A/m\) is a field. In the general case, consider a flat \(A\)-module \(B\) and elements \(F_i \in B, i \in I\), whose residue classes form a basis of \(B/mB\) as a vector space over \(A/m\). Let us show that these elements then form a basis of \(B\) as an \(A\)-module. If \(B’\) is the submodule generated by these elements then \(B/B’ = m \cdot (B/B’)\) and so

\[B/B’ = m \cdot B/B’ = m^2 \cdot B/B’ = \cdots = m^n \cdot B/B’ = 0.\]

Thus, the elements \(F_i\) generate \(B\), and we now show that they are also linearly independent. For \(B\) is flat, and, if \(K = \ker(A^{(I)} \to B)\) denotes the kernel of the map defined by the \(F_i\), tensoring the exact sequence

\[0 \to K \to A^{(I)} \to B \to 0\]

with \(A/m\) over \(A\) results in the exact sequence

\[0 \to K/mK \to (A/m)^{(I)} \xrightarrow{\cong} B/mB \to 0.\]

This shows first \(K/mK = 0\) and then the same argument as before yields \(K = 0\). Thus \(B \cong A^{(I)}\) is free over \(A\), as claimed. \(\square\)

**Lemma 2.5.** Let \(A\) be a ring and \(m \subseteq A\) a maximal ideal. If \(B\) is a flat \(A\)-module, then \(\text{Ext}_A^p(B, M) = 0\) for all \(p \geq 1\) and every \(A\)-module \(M\) with \(m^n M = 0\) for some \(n \geq 0\).
Proof. As $B$ is flat over $A$, we have

\[(*) \quad \text{Ext}^p_A(B, M) \cong \text{Ext}^p_{A/m^n}(B/m^nB, M) \quad \forall p \geq 0.\]

Recall the simple argument: if $F_\bullet$ is a projective resolution of $B$ as an $A$-module then by flatness of $B$ the complex $F_\bullet/m^nF_\bullet$ is a projective resolution of $B/m^nB$ as an $A/m^n$-module. Hence $\text{Ext}^p_{A/m^n}(B/m^nB, M)$ can be computed from the complex

\[
\text{Hom}_{A/m^n}(F_\bullet/m^nF_\bullet, M) \cong \text{Hom}_A(F_\bullet, M).
\]

As the complex on the right has cohomology $\text{Ext}^p_A(B, M)$, the claim $(*)$ follows. Flatness of $B$ over $A$ implies that $B/m^nB$ is flat as an $A/m^n$-module, whence the lemma is a consequence of $(*)$ and Lemma 2.4.

Finally, recall the following simple fact.

Lemma 2.6. Let $A$ be a ring and \{\(f_n : H_{n+1} \to H_n\)\}_{n \geq 0} an inverse system of $A$-modules with surjective transition maps $f_n$. Setting $H := \prod_{n \geq 0} H_n$, the map $f : H \to H$ with $(h_n)_n \mapsto (h_n - f_n(h_{n+1}))_n$ is surjective and has kernel $\lim H_n$. \hfill \Box

Now we turn to the

Proof of Theorem 2.3. For $M_n := M/m^{n+1}M$, Lemmata 2.4 and 2.5 give that

\[
\text{Ext}^p_A(B, M_n) = 0 \quad \text{for all} \quad p \geq 1.
\]

Taking the direct product $H := \prod_{n \geq 0} M_n$ this implies

\[
\text{Ext}^p_A(B, H) = 0 \quad \text{for all} \quad p \geq 1,
\]

since the formation of Ext is compatible with direct products in the second component. As $M$ is complete as an $A$-module there is an exact sequence

\[
0 \to M \to H = \prod_{n \geq 0} M_n \xrightarrow{f} H = \prod_{n \geq 0} M_n \to 0,
\]

see Lemma 2.6. Applying $\text{Hom}_A(B, -)$ gives the long exact Ext-sequence

\[0 \to \text{Hom}_A(B, M) \to \text{Hom}(B, H) \xrightarrow{f^*} \text{Hom}_A(B, H) \to \text{Ext}^1_A(B, M) \to \cdots
\]

As $\text{Ext}^p_A(B, H) = 0$ for $p \geq 1$ it follows that $\text{Ext}^p_A(B, M) = 0$ for $p \geq 2$. It remains to show that $\text{Ext}^1_A(B, M) = 0$ or, equivalently, that the map

\[
f^* : \text{Hom}_A(B, H) \cong \prod_{n \geq 0} \text{Hom}_A(B, M_n) \to \text{Hom}_A(B, H) \cong \prod_{n \geq 0} \text{Hom}_A(B, M_n)
\]

is surjective. However, this is immediate from Lemma 2.6 as the transition maps

\[
\text{Hom}_A(B, M_{n+1}) \to \text{Hom}_A(B, M_n)
\]

are all surjective by Lemma 2.6.

Corollary 2.7. If $A$ is a complete local noetherian ring and $B$ is a flat $A$-module then $\text{Ext}^p_A(B, M) = 0$ for all $p \geq 1$ and all finite $A$-modules $M$. \hfill \Box

Remarks 2.8. 1. The corollary itself is not new, indeed Jensen established in [Je2, Thm.8.1]: If $A$ is commutative noetherian then it is a product of a finite number of complete local rings if and only if $\text{Ext}^i_A(B, M) = 0$, for $i \geq 1$, whenever $B$ is flat and $M$ is finite over $A$.\hfill \Box
2. In a similar vein, Frankild establishes in [Fr, Cor. 3.7]: For a local noetherian ring \((A, \mathfrak{m})\), an \(A\)-module \(B\) of finite projective dimension and an \(A\)-module \(M\) that is \(\mathfrak{m}\)-adically complete, \(\operatorname{Ext}^i_A(B, M) = 0\) for \(i > \operatorname{depth} A - \operatorname{depth} B\).

As any flat \(A\)-module is of finite projective dimension; see [Je1, p. 164]; with its depth either equal to the depth of the ring or infinite, this result specializes to Corollary 2.7.

3. For non-complete modules \(M\), the problem as to whether \(\operatorname{Ext}^p_A(B, M)\) vanishes for \(p \geq 0\) is much more intricate. For instance, the famous Whitehead problem asked whether the vanishing of \(\operatorname{Ext}^1_Z(B, Z)\) implies that the module \(B\) is free. As was shown by Shelah, this depends on the model of set theory used, see [Sh].

4. We do not know whether a result similar to Corollary 2.7 also holds in the analytic category. More precisely, let \(A \to A'\) be a homomorphism of analytic algebras over a valued field and \(B\) a finite \(A'\)-module that is flat over \(A\). It is natural to ask whether then \(\operatorname{Ext}^p_A(B, M) = 0\) for all finite \(A\)-modules \(M\) and all \(p \geq 1\).

### 3. Hochschild cohomology of complete algebras

Let us first recall the definition of Hochschild cohomology of a morphism of commutative rings \(A \to B\). Denoting \(B^{\otimes n}\) the usual \(n\)-fold tensor product over \(A\), the bar resolution \(\mathcal{B}_\bullet : \cdots \to B^{\otimes n} \to \cdots \to B^{\otimes 2} \to B \to 0\) provides a resolution of \(B\) as a \(B^{(e)}\): \(= B^{\otimes 2}\)-module, see [CE] or [Lo]. Note that this resolution is flat, respectively projective over \(B^{(e)}\) provided \(B\) has the same property over \(A\).

For any \(B^{(e)}\)-module \(M\) the modules \(\operatorname{HH}^p(B/A, M) = H^p(\mathcal{B}_\bullet \otimes_{B^{(e)}} M)\) and \(\operatorname{HHP}(B/A, M) = H^p(\operatorname{Hom}_{B^{(e)}}(\mathcal{B}_\bullet, M))\) are called, respectively, the Hochschild homology and cohomology of \(B/A\) with values in \(M\).

If \(\mathcal{P}_\bullet \to B \to 0\) is a projective resolution of \(B\) over \(B^{(e)}\), then there exists a comparison map \(\mathcal{P}_\bullet \to \mathcal{B}_\bullet\) over the identity of \(B\) that is a homomorphism of complexes and unique up to homotopy. Accordingly, for any \(B^{(e)}\)-module \(M\) there are homomorphisms \(\operatorname{HH}^p(B/A, M) \to \operatorname{Ext}^p_{B^{(e)}}(B, M)\), that are functorial in \(M\). These maps are isomorphisms as soon as \(B\) is a projective \(A\)-module, but an application of Theorem 2.3 shows that these maps are also isomorphisms in the following wider context.

**Proposition 3.1.** If \(A \to B\) is a flat homomorphism of commutative rings and \(M\) a \(B^{(e)}\)-module that is \(\mathfrak{m}\)-adically complete for some maximal ideal \(\mathfrak{m} \subseteq B^e\), then the natural maps \(\operatorname{HH}^p(B/A, M) \to \operatorname{Ext}^p_{B^{(e)}}(B, M)\) are isomorphisms for each \(p \geq 0\).

**Proof.** Consider the pair of spectral sequences with the same limit

\[
\begin{align*}
\' E_1^{pq} &= \operatorname{Ext}^q_{B^{(e)}}(\mathcal{B}_p, M) \Rightarrow \mathbb{E}^{p+q} \\
\" E_2^{pq} &= \operatorname{Ext}^p_{B^{(e)}}(\mathcal{H}^{q}(\mathcal{B}_\bullet), M) \Rightarrow \mathbb{E}^{p+q}.
\end{align*}
\]
As \( \mathcal{E}_\bullet \) is a resolution of \( B \), the second of these spectral sequences degenerates and identifies the limit as \( \mathbb{E}^n \cong \text{Ext}^n_{B^{\langle e \rangle}}(B, M) \). Concerning the first spectral sequence, each term \( \mathcal{E}_p \) is a flat \( B^{\langle e \rangle} \)-module, and so Theorem 2.3 shows that \( \mathcal{E}_1^{pq} = 0 \) for \( q \neq 0 \) and any \( p \). Hence the first spectral sequence degenerates as well, identifying the limit as \( \mathbb{E}^n \cong \text{HH}^n(B/A, M) \). \( \square \)

For the rest of this section, let us assume that \( A \to B \) is a homomorphism of complete local rings such that the extension of residue fields \( A/\mathfrak{m}_A \to B/\mathfrak{m}_B \) is an isomorphism. In the usual construction of Hochschild (co-)homology, as sketched above, one can then replace the ordinary tensor product by the complete one, \( B^{\otimes n} \), that is the completion of \( B^{\otimes n} \) with respect to the maximal ideal \( \ker(B^{\otimes n} \to (B/\mathfrak{m}_B)^{\otimes n} \cong B/\mathfrak{m}_B) \). The result is the analytic Bar-resolution \( \hat{\mathcal{B}}_\bullet \) that gives rise to analytic Hochschild (co-)homology

\[
\text{HH}_p(B/A, M) = H_p(\hat{\mathcal{B}}_\bullet \otimes_{\hat{B}^{\langle e \rangle}} M)
\]

and

\[
\text{HH}^p(B/A, M) = H^p(\text{Hom}_{\hat{B}^{\langle e \rangle}}(\hat{\mathcal{B}}_\bullet, M)).
\]

There is obviously a canonical morphism \( \mathcal{E}_\bullet \to \hat{\mathcal{B}}_\bullet \), whence there are induced maps

\[
\alpha_p : \text{HH}_p(B/A, M) \to \text{HH}_p(B/A, M)
\]

and

\[
\beta^p : \text{HH}^p(B/A, M) \to \text{HH}^p(B/A, M).
\]

As an application of the results of section 2 we obtain the following proposition that was actually the motivation for this paper.

**Proposition 3.2.** If \( B \) is a flat \( A \)-algebra and \( M \) is a complete \( \hat{B}^{\langle e \rangle} \)-module, then there is a natural isomorphism

\[
\text{HH}^p(B/A, M) \cong \text{Ext}^p_{\hat{B}^{\langle e \rangle}}(B, M)
\]

for each \( p \geq 0 \).

**Proof:** The analytic Bar-resolution \( \hat{\mathcal{B}}_\bullet \) provides a flat resolution of \( B \) as a \( \hat{B}^{\langle e \rangle} \)-module. Using again Theorem 2.3 and the same spectral sequence argument as above shows that the cohomology groups of the complex

\[
\text{Hom}_{\hat{B}^{\langle e \rangle}}(\hat{\mathcal{B}}_\bullet, M)
\]

can be identified with the \( \text{Ext}^p_{\hat{B}^{\langle e \rangle}}(B, M) \) as claimed. \( \square \)

**Remarks 3.3.**

1. As is well known, the map \( \alpha_p \) is not an isomorphism, in general. For instance, \( \text{HH}_1(B/A, B) \cong D_A(B) = \Omega^1_{B/A} \) is the universal module of differentials for \( A \to B \) that is not finitely generated, in general. On the other hand, \( \text{HH}_1(B/A, B) \cong D_A(B) = \Omega^1_{B/A} \) is the universally finite module of differentials, which is always a finite module (see, e.g., [SS] or [EGA IV]).

2. We do not know whether the maps \( \beta^p \) are isomorphisms when \( A \to B \) is flat and \( M \) is complete. For this it would be necessary to know whether \( \mathcal{E}_\bullet \otimes_{B^{\langle e \rangle}} \hat{B}^{\langle e \rangle} \) is a resolution of \( B \). However, \( B^{\langle e \rangle} \) is not noetherian, in general, whence it is not clear whether \( B^{\langle e \rangle} \to \hat{B}^{\langle e \rangle} \) is always a flat map.

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