Counting non isomorphic maximal independent sets in the cycle graph

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Maximal independent sets of $C_n$

The non-isomorphic MISs of $C_n$

Some properties of $\text{Aut}(C_n)$

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Summary
The $n$-cycle graph

Définition

- Let $G = (V, E)$ be a simple undirected graph, with vertex set $V$ and edge set $E$.
- The $n$-cycle $C_n$ is the graph consisting of a cycle with $n$ vertices.
- The set of vertices of $C_n$ are labelled either clockwise or counterclockwise. The index set of the vertices is assumed to be the cyclic group $\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}$ with addition modulo $n$. 
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Maximal independent sets

• An independent set of $G$ is a subset $X$ of $V$ such that no two vertices in $X$ are adjacent.

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Counting the MIS of $C_n$

- It was shown by Füredi (1987) that the total number of MIS’s of $C_n$ is given by the $n$th term $p(n)$ of the Perrin sequence (Sloane’s A001608),
- $p(n)$ is defined recursively as $p(1) = 0$, $p(2) = 2$, $p(3) = 3$, and
  \[ p(n) = p(n - 2) + p(n - 3), \quad (n \geq 4). \]
  Hence the 12-cycle admits $p(12) = 29$ different MISs.
- As a corollary we note that it is possible to generate the MISs of $C_n$ in a univocal way from both the MISs of $C_{n-3}$ and the MISs of $C_{n-2}$.  

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The automorphisms group of the $n$-cycle

- An isomorphism from a graph $G$ to itself is called an automorphism of $G$. The set of automorphisms of a digraph $G$ is a group called the automorphism group of $G$ and denoted $\text{Aut}(G)$.

- It is well known that the group $\text{Aut}(C_n)$ is isomorphic to the dihedral group of order $2n$,

$$D_{2n} = \{1, \sigma, \ldots, \sigma^{n-1}, \tau, \sigma \tau, \ldots, \sigma^{n-1} \tau\},$$

where $\sigma$ is the rotation and $\tau$ is the reflection with the properties $\sigma^n = 1$, $\tau^2 = 1$, and $\tau \sigma = \sigma^{-1} \tau$.

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Non-isomorphic MISs of the $n$-cycle

- The action of the automorphism group $\text{Aut}(C_n)$ of $C_n$ on the family of MIS’s of $C_n$ gives rise to a partition of this family into orbits, each containing isomorphic (i.e., identical up to a rotation and a reflection) MIS’s.

- For instance, it is easy to verify that the 29 MISs of $C_{12}$ can be grouped into four orbits: an orbit with two isomorphic MIS’s of size 6, an orbit with three isomorphic MIS’s of size 4 and two orbits with each one 12 isomorphic MIS’s of size 5.
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The four non isomorphic MISs of the 12–cycle
Some properties of $\text{Aut}(C_n)$

• $X_n \subseteq 2^{V_n}$, the set of MISs of $C_n$, is a family of subsets of $V_n$ with the property that if $X \in X_n$ then $g(X) \in X_n$ for all $g \in D_{2n}$.

• For any $X \in X_n$, the orbit and the stabilizer of $X$ under the action of $D_{2n}$ are respectively defined as

\[
\text{Orb}(X) = \{g(X) : g \in D_{2n}\},
\]

\[
\text{Stab}(X) = \{g \in D_{2n} : g(X) = X\}.
\]

• $\text{Stab}(X)$ is either a cyclic or a dihedral subgroup of $D_{2n}$ and by the orbit-stabilizer theorem, we have

\[
|\text{Orb}(X)| \times |\text{Stab}(X)| = |D_{2n}| = 2n,
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which implies that both $|\text{Orb}(X)|$ and $|\text{Stab}(X)|$ divide $2n$. 

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MIS orbit decomposition

• Let \( \mathcal{O}_n = \mathcal{X}_n / D_{2n} \) denote the set of orbits of \( \mathcal{X}_n \) under the action of \( D_{2n} \). \( \mathcal{X}, \mathcal{X}' \in \mathcal{X}_n \) are isomorphic, and we write \( \mathcal{X} \sim \mathcal{X}' \), if \( \text{Orb}(\mathcal{X}) = \text{Orb}(\mathcal{X}') \).

• For any divisor \( d \geq 1 \) of \( 2n \) (we write \( d|2n \)), denote by \( \mathcal{O}_d^n \) the set of orbits of \( \mathcal{X}_n \) of cardinality \( \frac{2n}{d} \). \( \mathcal{O}_d^n \) is also the set of orbits whose elements have a stabilizer of cardinality \( d \).

• Thus:

\[
\begin{align*}
\text{orb}(n) &= |\mathcal{O}_n|, \\
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- Thus:

$$\text{orb}(n) = |O_n|,$$
$$\text{orb}_d(n) = |O^d_n| \quad (d|2n),$$
MIS orbit decomposition (continue)

- The previous formulation leads to

\[
\text{orb}(n) = \sum_{d \mid 2n} \text{orb}_d(n).
\]

- Since the orbits partition the set \(X_n\) we immediately have

\[
|X_n| = \sum_{d \mid 2n} \frac{2^n}{d} \text{orb}_d(n).
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Proposition

For any \(d \mid 2n\), we have

\[
\text{orb}_d(n) = \begin{cases} 
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| n | MIS orb(n) | Size of the stabilizer |
|---|------------|------------------------|
| 1 | 1          | 1                      |
| 2 | 2          | 1                      |
| 3 | 3          | 1                      |
| 4 | 2          | 1                      |
| 5 | 5          | 1                      |
| 6 | 5          | 1                      |
| 7 | 7          | 1                      |
| 8 | 10         | 2                      |
| 9 | 12         | 2                      |
| 10 | 17       | 3                      |
| 11 | 22        | 2                      |
| 12 | 29        | 4                      |
| 13 | 39        | 3                      |
| 14 | 51        | 5                      |
| 15 | 68        | 5                      |
| 16 | 90        | 7                      |
| 17 | 119       | 6                      |
| 18 | 158       | 10                     |
| 19 | 209       | 9                      |
| 20 | 277       | 14                     |
| 21 | 357       | 14                     |
| 22 | 486       | 20                     |
| 23 | 644       | 20                     |
| 24 | 853       | 30                     |
| 25 | 1130      | 31                     |
| 26 | 1497      | 44                     |
| 27 | 1993      | 46                     |
| 28 | 2527      | 67                     |
| 29 | 3480      | 74                     |
| 30 | 4510      | 84                     |
| 31 | 6107      | 117                    |
| 32 | 8090      | 161                    |
| 33 | 10717     | 186                    |
| 34 | 14197     | 224                    |
| 35 | 16807     | 262                    |
| 36 | 24914     | 407                    |
| 37 | 33004     | 489                    |
| 38 | 43721     | 654                    |
| 39 | 57918     | 801                    |
| 40 | 75725     | 1064                   |
| 41 | 101639    | 1315                   |
| 42 | 134643    | 1742                   |
| 43 | 178364    | 2174                   |
| 44 | 236262    | 2667                   |
| 45 | 313007    | 3513                   |
| 46 | 414646    | 4747                   |
Irregular MIS of $C_{15}$
A very useful result

The Padovan sequence \((q(n))_{n \in \mathbb{N}}\) (shifted Sloane’s A000931), is defined as \(q(1) = 0, q(2) = 1, q(3) = 1\), and

\[
q(n) = q(n - 2) + q(n - 3) \quad (n \geq 4),
\]

Proposition

- \(\text{orb}(n)\) is the number of cyclic compositions of \(n\) in which each term is either 2 or 3, where a clockwise writing is not distinguished from its counterclockwise counterpart.
- \(q(n)\) is the number of (linear) compositions of \(n\) in which each term is either 2 or 3.
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Let \((r(n))_{n \in \mathbb{N}}\) be the sequence defined by

\[
r(n) = \begin{cases} 
q(k), & \text{if } n = 2k - 1, \\
q(k + 2), & \text{if } n = 2k.
\end{cases}
\]

**Proposition**

- \(r(n)\) is the number of cyclic and palindromic compositions of \(n\) in which each term is either 2 or 3.
- \(r(n)\) is also the number of orbits of \(X_n\) whose elements have a stabilizer not included in \(\langle a \rangle\).
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Some useful definitions

- The *Dirichlet convolution product* of two sequences \((f(n))_{n \in \mathbb{N}}\) and \((g(n))_{n \in \mathbb{N}}\) is the sequence \(((f \ast g)(n))_{n \in \mathbb{N}}\) defined as

\[
(f \ast g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).
\]

- Consider the following integer sequences:

\[
\begin{align*}
\mu(n) & = A008683(n) \quad (\text{Möbius function}), \\
A113788(n) & = \frac{1}{n} (p \ast \mu)(n), \\
1(n) & = 1, \\
e_k(n) & = \begin{cases} 1, & \text{if } n = k, \\
0, & \text{else,} \\
\end{cases} \quad (k \in \mathbb{N}).
\end{align*}
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\mu(n) &= A008683(n) \quad \text{(Möbius function)}, \\
A113788(n) &= \frac{1}{n} (p \ast \mu)(n), \\
1(n) &= 1, \\
e_k(n) &= \begin{cases} 1, & \text{if } n = k, \\ 0, & \text{else,} \end{cases} \quad (k \in \mathbb{N}).
\end{align*}
\]
Our main result

Theorem
There holds

\[
\begin{align*}
\text{orb} &= r + \text{orb}_1 \ast \mathbf{1}, \\
2\text{orb}_1 &= A113788 - r \ast \mu, \\
\text{orb}_2 &= r \ast \mu + \text{orb}_1 \ast e_2.
\end{align*}
\]
Summary

$r(n)$ is the number of non-isomorphic MIS’s of $C_n$ having at least one symmetry axis, where two MIS’s are isomorphic if they are identical up to a rotation and a reflection. It is also the number of cyclic and palindromic compositions of $n$ in which each term is either 2 or 3.

$$r(n) = \begin{cases} q(k), & \text{if } n = 2k - 1, \\ q(k + 2), & \text{if } n = 2k. \end{cases}$$
For any $d|2n$, $orb_d(n)$ gives the number of non-isomorphic MIS’s of $C_n$ having $\frac{2n}{d}$ isomorphic representatives. This sequence can always be expressed from one of the sequences $orb_1(n)$ and $orb_2(n)$.

\[
orb_1(n) = \frac{1}{2} (A113788(n) - (r \ast \mu)(n))
\]

\[
orb_2(n) = (r \ast \mu)(n) + (orb_1 \ast e_2)(n).
\]
Summary

orb(n) gives the number of non-isomorphic MIS's of $C_n$. It is also the number of cyclic compositions of $n$ in which each term is either 2 or 3, where a clockwise writing is not distinguished from its counterclockwise counterpart.

$$\text{orb}(n) = r(n) + (\text{orb}_1 \ast 1)(n) = \frac{1}{2} \left( r(n) + \frac{1}{n} (p \ast \phi)(n) \right).$$

where $\phi = A000010$ is the Euler totient function.
(orb₁ * 1)(n) is the number of non-isomorphic MIS’s of Cₙ having no symmetry axis. It is also the number of cyclic and non-palindromic compositions of n in which each term is either 2 or 3, where a clockwise writing is not distinguished from its counterclockwise counterpart.

(orb₁ * 1)(n) = orb(n) − r(n).
| $n$ | $p(n)$ | $q(n)$ | $r(n)$ | $\text{orb}(n)$ | $\text{orb}_1(n)$ | $\text{orb}_2(n)$ | $\text{orb}^\sigma(n)$ | $\text{orb}_1^\sigma(n)$ |
|-----|--------|--------|--------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1   | 0      | 0      | 0      | 0               | 0               | 0               | 0               | 0               |
| 2   | 2      | 1      | 1      | 1               | 0               | 1               | 1               | 1               |
| 3   | 3      | 1      | 1      | 1               | 0               | 1               | 1               | 1               |
| 4   | 2      | 1      | 1      | 1               | 0               | 0               | 1               | 0               |
| 5   | 3      | 2      | 1      | 1               | 0               | 1               | 1               | 1               |
| 6   | 3      | 2      | 2      | 2               | 0               | 0               | 2               | 0               |
| 7   | 7      | 3      | 1      | 1               | 0               | 1               | 1               | 1               |
| 8   | 10     | 4      | 2      | 2               | 0               | 1               | 2               | 1               |
| 9   | 12     | 5      | 2      | 2               | 0               | 1               | 2               | 1               |
| 10  | 17     | 7      | 3      | 3               | 0               | 1               | 3               | 1               |
| 11  | 22     | 9      | 2      | 2               | 0               | 2               | 2               | 2               |
| 12  | 29     | 12     | 4      | 4               | 0               | 2               | 4               | 2               |
| 13  | 39     | 16     | 3      | 3               | 0               | 3               | 3               | 3               |
| 14  | 51     | 21     | 5      | 5               | 0               | 3               | 5               | 3               |
| 15  | 68     | 28     | 4      | 5               | 1               | 2               | 6               | 4               |
| 16  | 90     | 37     | 7      | 7               | 0               | 5               | 7               | 5               |
| 17  | 119    | 49     | 5      | 6               | 1               | 5               | 7               | 7               |
| 18  | 158    | 65     | 9      | 10              | 1               | 6               | 11              | 8               |
| 19  | 209    | 86     | 7      | 9               | 2               | 7               | 11              | 11              |
| 20  | 277    | 114    | 12     | 14              | 2               | 9               | 16              | 13              |

**Table:** First 20 values of the main sequences
| $n$ | $p(n)$ | $q(n)$ | $r(n)$ | orb$(n)$ | orb$_1(n)$ | orb$_2(n)$ | orb$^\sigma(n)$ | orb$_1^\sigma(n)$ |
|-----|--------|--------|--------|----------|-----------|-----------|----------------|----------------|
| 21  | 367    | 151    | 9      | 14       | 5         | 7         | 19             | 17             |
| 22  | 486    | 200    | 16     | 20       | 4         | 13        | 24             | 21             |
| 23  | 644    | 265    | 12     | 20       | 8         | 12        | 28             | 28             |
| 24  | 853    | 351    | 21     | 30       | 9         | 16        | 39             | 34             |
| 25  | 1130   | 465    | 16     | 31       | 15        | 15        | 46             | 45             |
| 26  | 1497   | 616    | 28     | 44       | 16        | 24        | 60             | 56             |
| 27  | 1983   | 816    | 21     | 48       | 27        | 19        | 75             | 73             |
| 28  | 2627   | 1081   | 37     | 67       | 30        | 32        | 97             | 92             |
| 29  | 3480   | 1432   | 28     | 74       | 46        | 28        | 120            | 120            |
| 30  | 4610   | 1897   | 49     | 104      | 54        | 44        | 159            | 151            |
| 31  | 6107   | 2513   | 37     | 117      | 80        | 37        | 197            | 197            |
| 32  | 8090   | 3329   | 65     | 161      | 96        | 58        | 257            | 250            |
| 33  | 10717  | 4410   | 49     | 188      | 139       | 46        | 327            | 324            |
| 34  | 14197  | 5842   | 86     | 254      | 167       | 81        | 422            | 414            |
| 35  | 18807  | 7739   | 65     | 302      | 237       | 63        | 539            | 537            |
| 36  | 24914  | 10252  | 114    | 407      | 292       | 104       | 700            | 687            |
| 37  | 33004  | 13581  | 86     | 489      | 403       | 86        | 892            | 892            |
| 38  | 43721  | 17991  | 151    | 654      | 501       | 145       | 1157           | 1145           |
| 39  | 57918  | 23833  | 114    | 801      | 687       | 110       | 1488           | 1484           |
| 40  | 76725  | 31572  | 200    | 1064     | 862       | 189       | 1928           | 1911           |

**Tab.:** 21 - 40 values of the main sequences