Asymptotically split extensions and $E$-theory

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Abstract

We show that the $E$-theory of Connes and Higson can be formulated in terms of $C^*$-extensions in a way quite similar to the way in which the $KK$-theory of Kasparov can. The essential difference is that the role played by split extensions should be taken by asymptotically split extensions. We call an extension of a $C^*$-algebra $A$ by a stable $C^*$-algebra $B$ asymptotically split if there exists an asymptotic homomorphism consisting of right inverses for the quotient map. An extension is called semi-invertible if it can be made asymptotically split by adding another extension to it. Our main result is that there exists a one-to-one correspondence between asymptotic homomorphisms from $SA$ to $B$ and homotopy classes of semi-invertible extensions of $S^2A$ by $B$.

1 Introduction

Connes and Higson introduced in [4] a construction which produces an asymptotic homomorphism out of an extension of $C^*$-algebras. The Connes-Higson construction is the backbone of $E$-theory and gives us a way to study $C^*$-extensions via asymptotic homomorphisms. Such a translation can be quite powerful within the territory of $KK$-theory, where the $C^*$-extensions are semi-split, i.e. admit a completely positive contraction as a right-inverse for the quotient map. It is namely known that the Connes–Higson construction sets up a bijection between homotopy classes of semi-split extensions and completely positive asymptotic homomorphisms. This bijection is particularly useful because completely positive asymptotic homomorphisms are easier to handle than general ones, and because the powerful homotopy invariance results of Kasparov, [8], allows one to translate homotopy information to more algebraic information about the $C^*$-extensions. This well-behaved correspondence between semi-split $C^*$-extensions and homotopy classes of completely positive asymptotic homomorphisms was used in [4] to obtain a better understanding of the short exact sequence of the UCT-theorem by identifying the kernel of the map from $KK(A, B) = \text{Ext}^{-1}(SA, B)$ to $KL(A, B)$ as the group arising from the weakly quasi-diagonal extensions of $SA$ by $B\otimes K$.

The present paper originated in the desire to extend the nice relation between $C^*$-extensions and asymptotic homomorphisms beyond the case of semi-split extensions. The key problem in this connection is (at least for the moment) to decide if the Connes–Higson construction

\textsuperscript{1}Partially supported by RFFI, grant No 99-01-01201
is injective in general. In other words, the problem is to decide if two $C^*$-extensions - with stable and maybe suspended ideals - which give rise to homotopic asymptotic homomorphisms must themselves be homotopic. From [7] we know that this is the case when both extensions are suspensions and the result of the present paper shows that it is also the case when both extensions are what we call semi-invertible and the quotient $C^*$-algebra is a double extension. But in general we still don’t know the answer. Nonetheless, we shall show here that there is a way to faithfully represent $E$-theory by use of $C^*$-extensions which does not require infinitely many suspensions as in [7] or longer decomposition series as in [5].

To describe this, let $A$ and $B$ be separable $C^*$-algebras and assume for simplicity that $B$ is stable. We call an extension of $A$ by $B$ asymptotically split when there is a family $(\pi_t)_{t \in [1, \infty)}$ of right-inverses for the quotient map such that $(\pi_t)_{t \in [1, \infty)}$ is an asymptotic homomorphism. An extension is then semi-invertible when it can be made asymptotically split by adding another extension to it. We prove that

1) Every asymptotic homomorphism $S^2A \to B$ is homotopic to the Connes–Higson construction of a semi-invertible extension of $SA$ by $B$.

2) Two semi-invertible extensions of $S^2A$ by $B$ are homotopic (as semi-invertible extensions) if and only if the Connes–Higson construction applied to them give homotopic asymptotic homomorphisms.

These results show that the $E$-theory of Connes and Higson can be formulated in terms of $C^*$-extensions in a way quite similar to the way in which the $KK$-theory of Kasparov can. The essential difference is that the role played by split extensions should be taken by asymptotically split extensions. It is our hope that this parallel between the way $KK$-theory and $E$-theory can be described in terms of $C^*$-extensions can be strengthened even further. In particular it would be nice if some of the suspensions occurring in 1) and 2) could be removed and if one could substitute homotopy with a more algebraic relation in the description of $E$-theory.

## 2 Asymptotically split extensions and $\text{Ext}^{-1/2}$

In the following $A$ and $B$ are separable $C^*$-algebras, $B$ stable, i.e. $B = B \otimes K$, where $K$ denotes the $C^*$-algebra of compact operators. As usual, we denote by $C_0(X)$ the $C^*$-algebra of continuous functions on $X$ vanishing at infinity, and $SA = C_0(0, 1) \otimes A$ denotes the suspension $C^*$-algebra over $A$. Let $M(B)$ denote the multiplier algebra of $B$, $Q(B) = M(B)/B$ the corresponding corona algebra and $q_B : M(B) \to Q(B)$ the quotient map. We shall identify the set of extensions of $A$ by $B$ with $\text{Hom}(A, Q(B))$ in the standard way, [3]. Two extensions $\varphi, \psi : A \to Q(B)$ are unitarily equivalent when there is a unitary $w \in M(B)$ such that $\text{Ad} q_B(w) \circ \varphi = \psi$. As is wellknown the set of unitary equivalence classes of extensions of $A$ by $B$ form a semi-group and we denote this semi-group by $\text{Ext}(A, B)$.

Recall that an asymptotic homomorphism from $A$ to $B$ is a family $\varphi = \{\varphi_t\}_{t \in [1, \infty)} : A \to B$ of maps such that $t \mapsto \varphi_t(a)$ is continuous for any $a \in A$ and the $\varphi_t$’s behave like a $*$-
homomorphism asymptotically as $t \to \infty$, \[4\]. Namely, for any $a, b \in A$, $\lambda \in \mathbb{C}$ one has
\[
\lim_{t \to \infty} \|\varphi_t(a^*) - \varphi_t(a)^*\| = 0,
\lim_{t \to \infty} \|\varphi_t(\lambda a + b) - \lambda \varphi_t(a) - \varphi_t(b)\| = 0,
\lim_{t \to \infty} \|\varphi_t(ab) - \varphi_t(a)\varphi_t(b)\| = 0.
\]
Two asymptotic homomorphisms $\varphi$ and $\psi$ are equivalent when $\lim_{t \to \infty} \|\varphi_t(a) - \psi_t(a)\| = 0$ for all $a \in A$ and are homotopic when there exists an asymptotic homomorphism $\phi = \{\phi_t\}_{t \in [1, \infty)} : A \to C[0, 1] \otimes B$ such that the compositions with the evaluation maps at 0 and 1 coincide with $\varphi$ and $\psi$, respectively. The semi-group of homotopy classes of asymptotic homomorphisms we denote by $[[A, B]]$.

An extension $\varphi : A \to Q(B)$ is called \textit{asymptotically split} when there is an asymptotic homomorphism $\pi = \{\pi_t\}_{t \in [1, \infty)} : A \to M(B)$ such that $q_B \circ \pi_t = \varphi$ for all $t$. An extension $\varphi : A \to Q(B)$ is called \textit{semi-invertible} when there is an extension $\psi$ such that $\varphi \oplus \psi : A \to Q(B)$ is asymptotically split. Two semi-invertible extensions are called \textit{stably equivalent} when they become unitarily equivalent after addition by asymptotically split extensions.

Stable equivalence is an equivalence relation on the subset of semi-invertible extensions in $\text{Hom}(A, Q(B))$ and the corresponding equivalence classes form an abelian group which we denote by $\text{Ext}^{-1/2}(A, B)$. $\text{Ext}^{-1/2}$ is a bifunctor which is contravariant in the first variable, $A$, and covariant with respect to quasi-unital $\ast$-homomorphisms in the second variable, $B$. It is easy to see that the Connes–Higson construction, \[4\], annihilates asymptotically split extensions and therefore gives rise to a group homomorphism
\[
CH : \text{Ext}^{-1/2}(A, B) \to [[SA, B]].
\]

Two semi-invertible extensions
\[
0 \to B \to E_1 \to A \to 0
\]
and
\[
0 \to B \to E_2 \to A \to 0
\]
are called \textit{homotopic} when there is a commuting diagram
\[
\begin{array}{cccccc}
0 & \to & B & \to & E_1 & \to & A & \to & 0 \\
\pi_0 \uparrow & & \uparrow & & \| & & \| & \\
0 & \to & C[0, 1] \otimes B & \to & E & \to & A & \to & 0 \\
\pi_1 \downarrow & & \downarrow & & \| & & \| & \\
0 & \to & B & \to & E_2 & \to & A & \to & 0
\end{array}
\]
of semi-invertible extensions. The $\ast$-homomorphisms $\pi_0, \pi_1 : C[0, 1] \otimes B \to B$ are here the surjections obtained from evaluation at the endpoints of $[0, 1]$.

The main tool in this paper is the map $E$ introduced in \[10\], cf. \[11\]. We recall the construction here. Given an asymptotic homomorphism $\varphi = \{\varphi_t\}_{t \in [1, \infty)} : A \to B$ we choose a sequence $1 \leq t_1 \leq t_2 \leq t_3 \leq \cdots$ such that
\[
\lim_{i \to \infty} t_i = \infty \quad \text{and} \quad \lim_{i \to \infty} \sup_{t \in [t_i, t_{i+1}]} \|\varphi_t(a) - \varphi_{t_i}(a)\| = 0
\]
for all $a \in A$. Let $e_{ij}, i, j \in \mathbb{Z}$ denote the standard matrix units, which act on the standard Hilbert $B$-module $l_2(\mathbb{Z}) \otimes B$ in the obvious way. Then

$$\Phi(a) = \sum_{i \geq 1} \varphi_{t_i}(a)e_{ii}$$

defines a map $\Phi : A \to \mathcal{L}_B(l_2(\mathbb{Z}) \otimes B)$, where $\mathcal{L}_B(l_2(\mathbb{Z}) \otimes B)$ is the $C^*$-algebra of bounded adjointable operators on the Hilbert $C^*$-module $l_2(\mathbb{Z}) \otimes B$. We identify $\mathcal{K} \otimes B$ with the ideal of $B$-compact operators in $\mathcal{L}_B(l_2(\mathbb{Z}) \otimes B)$ and observe that $\Phi$ is a $\ast$-homomorphism modulo $\mathcal{K} \otimes B$. Furthermore, $\Phi(a)$ commutes modulo $\mathcal{K} \otimes B$ with the two-sided shift $T = \sum_{j \in \mathbb{Z}} e_{j,j+1}$. So we get in this way a $\ast$-homomorphism

$$E(\varphi) : C(T) \otimes A \to Q(\mathcal{K} \otimes B) = \mathcal{L}_B(l_2(\mathbb{Z}) \otimes B) / \mathcal{K} \otimes B$$

such that

$$E(\varphi)(f \otimes a) = f(T)\Phi(a), \quad f \in C(T), \ a \in A.$$ 

Here and in the following we denote by $\underline{S}$ the image in $Q(\mathcal{K} \otimes B) = \mathcal{L}_B(l_2(\mathbb{Z}) \otimes B) / \mathcal{K} \otimes B$ of an element $S \in \mathcal{L}_B(l_2(\mathbb{Z}) \otimes B)$. It can be checked directly that the map $E$ is well-defined and does not depend on the choice of a discretization.

**Lemma 2.1** $E(\varphi) \in \text{Ext}^{-1/2}(C(T) \otimes A, \mathcal{K} \otimes B)$.

**Proof.** Let $-E(\varphi) : C(T) \otimes A \to Q(\mathcal{K} \otimes B)$ be the extension which results when we in the construction of $E(\varphi)$ use

$$\Psi(a) = \sum_{i \leq 0} \varphi_{t_{-i+1}}(a)e_{ii}$$

instead of $\Phi$. Then $-E(\varphi) \oplus E(\varphi)$ is unitary equivalent to an extension $\psi : C(T) \otimes A \to Q(\mathcal{K} \otimes B)$ such that $\psi(f \otimes a) = \pi_t(f \otimes a)$ for all $t \in [1, \infty)$, $f \in C(T), a \in A$, where

$$\pi = \{\pi_t\}_{t \in [1,\infty)} : C(T) \otimes A \to \mathcal{L}_B(l_2(\mathbb{Z}) \otimes B)$$

is an asymptotic homomorphism obtained by convex interpolation of maps $\pi_n, n \in \mathbb{N}$, with the property that

$$\pi_n(f \otimes a) - f(T)\left(\sum_{|i| \leq n} \varphi_{t_n}(a)e_{ii} + \sum_{i > n} \varphi_{t_i}(a)e_{ii} + \sum_{i < -n} \varphi_{t_{-i+1}}(a)e_{ii}\right) \in \mathcal{K} \otimes B$$

and

$$\lim_{n \to \infty} \pi_n(f \otimes a) - f(T)\left(\sum_{|i| \leq n} \varphi_{t_n}(a)e_{ii} + \sum_{i > n} \varphi_{t_i}(a)e_{ii} + \sum_{i < -n} \varphi_{t_{-i+1}}(a)e_{ii}\right) = 0,$$

$f \in C(T), a \in A$. But $\psi$ is obviously asymptotically split. \hfill $\square$

Let $\text{Ext}^{-1/2}(A, B)_h$ denote the abelian group of homotopy classes of semi-invertible extensions of $A$ by $B$. $\text{Ext}^{-1/2}(A, B)_h$ is then a quotient of $\text{Ext}^{-1/2}(A, B)$. By homotopy invariance of the Connes–Higson construction we get a map

$$CH : \text{Ext}^{-1/2}(A, B)_h \to [[SA, B]].$$
Thanks to Lemma 2.1 we get from the above construction a well-defined map
\[ E : [[A, B]] \to \text{Ext}^{-1/2}(C(T) \otimes A, B)_h, \]
cf. [11]. By pulling back along the canonical inclusion \( SA \subseteq C(T) \otimes A \) we can also consider \( E \) as a map
\[ E : [[A, B]] \to \text{Ext}^{-1/2}(SA, B)_h. \]

Our main result can now be formulated as follows.

**Theorem 2.2**

a) The map \( CH : \text{Ext}^{-1/2}(SA, B) \to [[S^2 A, B]] \) is surjective.

b) The map \( E : [[SA, B]] \to \text{Ext}^{-1/2}(S^2 A, B)_h \) is an isomorphism.

Let \( \chi : [[SA, B]] \to [[S^2 C(T) \otimes A, B]] \) be the map obtained by taking the exterior product product with the asymptotic homomorphism \( S^2 C(T) \to S \otimes K \) which is the suspension of the asymptotic homomorphism \( S^2 \to K \) obtained by applying the Connes–Higson construction to the Toeplitz extension. The composition of \( \chi \) with the obvious map \( [[S^2 C(T) \otimes A, B]] \to [[S^3 A, B]] \) will be denoted by \( \chi_0 \). To prove a) we use the following statement, cf. [11, 11].

**Lemma 2.3** The diagram
\[ \begin{array}{ccc}
\text{Ext}^{-1/2}(S^2 A, B)_h & \xleftarrow{E} & [[SA, B]] \\
\downarrow{\chi} & & \downarrow{\chi} \xrightarrow{CH} [[S^3 A, B]], \\
\end{array} \tag{2.1} \]
commutes.

**Proof.** We are going to prove commutativity of the diagram
\[ \begin{array}{ccc}
\text{Ext}^{-1/2}(C(T) \otimes A, B)_h & \xleftarrow{E} & [[A, B]] \\
\downarrow{\chi} & & \downarrow{\chi} \xrightarrow{CH} [[SC(T) \otimes A, B]], \\
\end{array} \]
which immediately implies commutativity of (2.1).

To describe \( \chi \) choose a sequence of continuous functions \( \kappa_n : [1, \infty) \to [0, 1] \), \( n \in \mathbb{N} \), such that
\[ \kappa_n(1) = 1, \quad \lim_{t \to \infty} \kappa_n(t) = 0, \quad n \in \mathbb{N}, \tag{2.2} \]
and
\[ \lim_{n \to \infty} \sup_{t \in [1, \infty)} \| \kappa_{n+1}(t) - \kappa_n(t) \| = 0. \tag{2.3} \]

One way of constructing such a sequence of functions is to set \( a_n = \sum_{i=1}^n \frac{1}{i} \) and let \( \kappa_n \) be the function
\[ \kappa_n(t) = \begin{cases} 
1, & t \in [1, a_n], \\
a_n + 1 - t, & t \in [a_n, a_n + 1], \\
0, & t \in [a_n + 1, \infty), 
\end{cases} \]
but the actual choice is not important as soon as (2.2) and (2.3) are satisfied. Put \( K(t) = \sum_{i\in \mathbb{N}} \kappa_i(t) \). Denote by \( P \) the projection \( \sum_{i\in \mathbb{N}} e_{ii} \) in \( l_2(\mathbb{Z}) \). Then \( PTP \) is a one-sided shift of index one. The asymptotic homomorphism \( \chi \) is then determined by the condition that

\[
\lim_{t \to \infty} \| \chi_t(f \otimes e^{2\pi i x}) - f(K(t))PTP \| = 0,
\]

where \( f \in C_0(0, 1) \) and \( e^{2\pi i x} \) is a generator for \( C(T) \).

Let \( \varphi = \{ \varphi_t \}_{t \in [1, \infty)} : A \to B \) be an asymptotic homomorphism. Then \( CH \circ E[\varphi] \) is equivalent to the asymptotic homomorphism \( \psi = \{ \psi_t \}_{t \in [1, \infty)} : SC(T) \otimes A \to B \) defined by

\[
\psi_t(f \otimes e^{2\pi i x} \otimes a) = T \sum_{i \in \mathbb{N}} f(\kappa_i(t))\varphi_i(a)e_{ii}.
\]

Define another asymptotic homomorphism \( \psi' = \{ \psi'_t \}_{t \in [1, \infty)} : SC(T) \otimes A \to B \) by

\[
\psi'_t(f \otimes e^{2\pi i x} \otimes a) = PTP \sum_{i \in \mathbb{N}} f(\kappa_i(t))\varphi_i(a)e_{ii}.
\]

Since

\[
\lim_{t \to \infty} \| \psi_t(f \otimes e^{2\pi i x} \otimes a) - \psi'_t(f \otimes e^{2\pi i x} \otimes a) \| = 0
\]

for any \( f \in C_0(0, 1) \), \( a \in A \), it follows that the asymptotic homomorphisms \( \psi \) and \( \psi' \) are equivalent. By using the freedom in the choice of the \( \kappa_i \)'s we can arrange that there is a sequence \( 0 < m_1 < m_2 < \ldots \) in \( \mathbb{N} \) such that

\[
\kappa_i(t) = 0 \quad \text{for all} \quad t \in [t_j, t_{j+1}], \quad i = 1, 2, \ldots, m_j
\]

and

\[
\kappa_i(t) = 1 \quad \text{for all} \quad t \in [t_j, t_{j+1}], \quad i \geq m_{j+1}.
\]

Define a new sequence \( s_1 \leq s_2 \leq s_3 \leq \ldots \) in \([1, \infty)\) such that

\[
\begin{align*}
  s_i &= 0, \quad 0 \leq i < m, \\
  s_{m_1} &= s_{m_1 + 1} = \ldots = s_{m_2 - 1} = t_1, \\
  s_{m_2} &= s_{m_2 + 1} = \ldots = s_{m_3 - 1} = t_2,
\end{align*}
\]

and so on. Then \( \{ \varphi_{s_n} \} \) is also a discretization for \( \varphi \), so \( \psi' \) is homotopic to \( \psi'' \), where

\[
\psi''_t(f \otimes e^{2\pi i x} \otimes a) = PTP \sum_{i \in \mathbb{N}} f(\kappa_i(t))\varphi_{s_i}(a)e_{ii}
\]

asymptotically as \( t \to \infty \). Since

\[
[\chi \otimes \varphi]_t(f \otimes e^{2\pi i x} \otimes a) = PTP \sum_{i \in \mathbb{N}} f(\kappa_i(t))\varphi_i(a)e_{ii},
\]

asymptotically as \( t \to \infty \), we find that

\[
\lim_{t \to \infty} \sup_{i} \| \psi''_t(f \otimes e^{2\pi i x} \otimes a) - [\chi \otimes \varphi]_t(f \otimes e^{2\pi i x} \otimes a) \|
\leq \lim_{t \to \infty} \| f(\kappa_i(t))\varphi_{s_i}(a) - f(\kappa_i(t))\varphi_i(a) \| = 0
\]

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for any \( f \in C_0(0, 1), a \in A \). Since elements of the form \( f \otimes e^{2\pi i z} \otimes a \) generate \( SC(T) \otimes A \) as a \( C^* \)-algebra, it follows that \( \lim_{t \to \infty} \| \psi_t''(z) - [\chi \otimes \varphi](z) \| = 0 \) for all \( z \in SC(T) \otimes A \). Consequently the asymptotic homomorphisms \( CH \circ E[\varphi] \) and \( \chi \otimes \varphi \) are homotopic. \( \Box \)

Since \( \chi \) is an isomorphism, it follows that \( CH : \text{Ext}^{-1/2}(S^2A, B) \to [[S^3A, B]] \) is surjective. But the inverse in \( E \)-theory of the asymptotic homomorphism defining \( \chi \) is a genuine \(*\)-homomorphism \( \mu : SA \to S^3A \otimes M_2 \) and the naturality of the Connes–Higson construction gives us a commuting diagram

\[
\begin{array}{ccc}
\text{Ext}^{-1/2}(SA, B) & \xrightarrow{\mu^*} & \text{Ext}^{-1/2}(S^3A \otimes M_2, B) \\
\downarrow CH & & \downarrow CH \\
[[S^2A, B]] & \xrightarrow{\mu^*} & [[S^3A \otimes M_2, B]]
\end{array}
\]

We see that this proves a) of Theorem 2.2.

To complete the proof Theorem 2.2 it now suffices to show that the \( CH \)-map of diagram (2.1) is injective. The rest of the paper is devoted to this.

3 Proof of b) of Theorem 2.2

Given two commuting unitaries \( S, T \) in a \( C^* \)-algebra, we define a projection \( P(S, T) \) in the \( 2 \times 2 \) matrices over the \( C^* \)-algebra generated by \( S \) and \( T \) in the following way. Let \( s, c_0, c_1 : [0, 1] \to \mathbb{R} \) be the functions

\[
c_0(t) = |\cos(\pi t)|1_{[0, \frac{1}{2}]}(t), \quad c_1(t) = |\cos(\pi t)|1_{(\frac{1}{2}, 1]}(t), \quad s(t) = \sin(\pi t),
\]

where \( 1_{[0, \frac{1}{2}]} \), \( 1_{(\frac{1}{2}, 1]} \) are the characteristic functions of the corresponding segments. Set \( \tilde{g} = sc_0 \), \( \tilde{h} = sc_1 \) and \( \tilde{f} = s^2 \). Since \( \tilde{f}, \tilde{g} \) and \( \tilde{h} \) are continuous and 1-periodic they give rise to continuous functions, \( f, g, h \), on \( \mathbb{T} \). Set

\[
P(S, T) = \begin{pmatrix} f(S) & g(S) + h(S)T \\ h(S)T^* + g(S) & 1 - f(S) \end{pmatrix},
\]

cf. [3]. In particular, this gives us a projection \( P \in C(\mathbb{T}^2) \otimes M_2 \) when we apply the recipe to the canonical generating unitaries of \( C(\mathbb{T}^2) \). Note that \( P \) is an element of \( M_2((SC(T))^+) \subseteq M_2(C(\mathbb{T}^2)) \). In general, \( P(S, T) \) is in the range of \( \text{id}_{M_2} \otimes \lambda \), where \( \lambda : (SC(T))^+ \to C^*(S, T) \) is the unital \(*\)-homomorphism with

\[
\lambda((1 - e^{2\pi i z}) \otimes 1) = 1 - S, \quad \lambda((1 - e^{2\pi i z}) \otimes e^{2\pi i y}) = T - ST.
\]

Consider also the projection

\[
P_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_2 \subseteq C(\mathbb{T}^2) \otimes M_2.
\]
We can then define a map
\[ \text{Bott}_A : \text{Ext}^{-1/2}(C(T^2) \otimes A, B)_h \to \text{Ext}^{-1/2}(A, B)_h \]
such that
\[ [\varphi] \mapsto [(\text{id}_{M_2} \otimes \varphi) \circ b_A] - [(\text{id}_{M_2} \otimes \varphi) \circ b_0], \]
where \( b_A, b_0 : A \to M_2(C(T^2)) \otimes A \) are the maps \( b_A(a) = P \otimes a \) and \( b_0(a) = P_0 \otimes a \), respectively. The main part of the proof will be to establish the following.

**Proposition 3.1** Let \( i : SA \to C(T) \otimes A \) be the canonical embedding, \( e : C(T) \otimes A \to A \) the map obtained from evaluation at \( 1 \in T \) and \( c : A \to C(T) \otimes A \) the \( * \)-homomorphism which identifies \( A \) with the constant \( A \)-valued functions over \( T \). Then
\[ -\text{Bott}_{SA} \circ E \circ \text{CH}([\psi] - e^* \circ c^*[\psi]) = i^* [\psi] \]
in \( \text{Ext}^{-1/2}(SA, B)_h \) for every semi-invertible extension \( \psi \in \text{Hom}(C(T) \otimes A, Q(B)) \).

To begin the proof of Proposition 3.1, observe that \( c^*([\psi] - e^* \circ c^*[\psi]) = 0 \) in \( \text{Ext}^{-1/2}(A, B) \). We can therefore add an asymptotically split extension \( \chi \) to \( c^*([\psi] - e^* \circ c^*([\psi])) \) such that the resulting extension is asymptotically split. It follows that
\[ \psi' = \psi - e^* \circ c^*([\psi]) + e^*(\chi) \]
is a semi-invertible extension of \( C(T) \otimes A \) by \( B \) such that \( i^*[\psi'] = i^*[\psi] \) and \( c^*(\psi') \) is an asymptotically split extension of \( A \) by \( B \). Since \( \text{CH}(e^*(\chi)) = (Se)^* \circ (\text{CH}(\chi)) = 0 \) because \( \chi \) is asymptotically split, it suffices (by using \( \psi' \) instead of \( \psi \)) to consider a semi-invertible extension \( \psi \in \text{Hom}(C(T) \otimes A, Q(B)) \) with the property that \( c^*(\psi) \) asymptotically splits, and show that \( \text{Bott}_{SA} \circ E \circ \text{CH}([\psi]) = i^*[\psi] \). So let \( \psi \) be such an extension and set \( \varphi = \psi \circ i \).

**Lemma 3.2** Let \( e^{2\pi ix} \) denote the identity function of the circle \( T \). There is a unitary \( U \in M(M_2(B)) \) such that
\[ \left( \begin{array}{cc} \psi(e^{2\pi ix} f \otimes a) & \psi(f \otimes a) \\ 0 & 0 \end{array} \right) = q_{M_2(B)}(U) \left( \begin{array}{cc} \psi(f \otimes a) & 0 \end{array} \right) \]
for all \( f \in C(T), a \in A \).

**Proof.** We use the well-known fact, [12], that a surjective \( * \)-homomorphism between separable \( C^* \)-algebras admits a surjective unital extension to a \( * \)-homomorphism between the multiplier algebras. The \( * \)-homomorphism \( \psi \) extends to a unital \( * \)-homomorphism
\[ \hat{\psi} : M(C(T) \otimes A) \to M(\psi(C(T) \otimes A)), \]
Then \( V = \hat{\psi}(e^{2\pi ix} \otimes 1_A) \) is a unitary in \( M(\psi(C(T) \otimes A)) \) (\( 1_A \) means here the unit in \( M(A) \) and hence \( e^{2\pi ix} \otimes 1_A \) is really just the identity function of \( T \) considered as a unitary multiplier of \( C(T) \otimes A \)). Set \( D = q_B^{-1}(\psi(C(T) \otimes A)) \subseteq M(B) \). Since \( q_B \) maps \( D \) onto \( \psi(C(T) \otimes A)) \)
it extends to a surjective unital $*$-homomorphism $\widehat{q}_B : M(D) \to M(\psi(C(T) \otimes A))$. Since 
\[
\begin{pmatrix}
V \\
V^* 
\end{pmatrix}
\]
is in the connected component of 1 in $M_2(M(\psi(C(T) \otimes A)))$, there is a unitary $U \in M_2(M(D))$ such that
\[
\text{id}_{M_2} \otimes \widehat{q}_B(U) = \begin{pmatrix}
V \\
V^* 
\end{pmatrix}.
\]
Note that $M(D) \subseteq M(B)$ since $B$ is an essential ideal in $D$. We can therefore regard $U$ as a unitary in $M_2(M(B))$. It is then clear that $U$ has the stated property. \hfill \Box

It follows from Lemma 3.3 that after adding 0 to $\psi$ and $\varphi$, we may assume that there is a unitary $w \in M(B)$ such that
\[
q_B(w)\psi(f \otimes a) = \psi(e^{2\pi i x} f \otimes a), \quad f \in C(T), \quad a \in A.
\] (3.4)

**Lemma 3.3** Let \( \{u_t\}_{t \in [1, \infty)} \) be a continuous approximate unit for $B$ such that \( \lim_{t \to \infty} u_t \pi_1(a) - \pi_1(a) = 0 \) for all $a \in A$. There is then an increasing continuous function $r : [1, \infty) \to [1, \infty)$ such that $r(t) \geq t$ for all $t \in [1, \infty)$ and $\lim_{t \to \infty} f(u_{r(t)})\pi_1(a) - f(u_{r(t)})\pi_1(a) = 0$ for all $a \in A$, $f \in C_0(0,1)$.

**Proof.** By the Bartle–Graves selection theorem \cite{2}, there is a continuous function $\chi : A \to M(B)$ such that $\chi(a) - \pi_1(a) \in B$ for all $A$. The same selection theorem also provides us with an equicontinuous asymptotic homomorphism $\pi' = (\pi'_t) : A \to M(B)$ such that
\[
\lim_{t \to \infty} \pi'_t(a) - \pi'_t(a) = 0 \quad \text{for all } a \in A.
\]
Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ be a sequence of finite subsets with dense union in $A$. By using that \( \{\pi_t(a) - \chi(a) : t \in [1, n], \quad a \in F_n\} \) is a compact subset of $B$ for all $n$, it is then straightforward to construct an $r$ with $r(t) \geq t$ such that $\lim_{t \to \infty} u_{r(t)}\pi_t(a) - u_{r(t)}\chi(a) = 0$ for all $a \in \bigcup_n F_n$. It follows that $\lim_{t \to \infty} u_{r(t)}\pi'_t(a) - u_{r(t)}\chi(a) = 0$ for all $a \in \bigcup_n F_n$, and by continuity of $\chi$ and equicontinuity of $\{\pi'_t\}$ it follows that this actually holds for all $a \in A$. But then $\lim_{t \to \infty} u_{r(t)}\pi_t(a) - u_{r(t)}\pi_1(a) = 0$ since $\chi(a) - \pi_1(a) \in B$ for all $a \in A$. The fact that $\lim_{t \to \infty} f(u_{r(t)})\pi_1(a) - f(u_{r(t)})\pi_1(a) = 0$ for all $a \in A$, $f \in C_0(0,1)$, then follows from Weierstrass’ theorem. \hfill \Box

It follows from Lemma 3.3 that $CH[\psi] \in [[SC(T) \otimes A, B]]$ is represented by an asymptotic homomorphism $CH(\psi)$ such that
\[
\lim_{t \to \infty} CH(\psi)_t(f \otimes g \otimes a) - f(u_{r(t)})g(w)\pi_t(a) = 0
\]
for all $a \in A$, $g \in C(T)$, $f \in C_0(0,1)$. By choosing the approximate unit $\{u_t\}$ in Lemma 3.3 appropriately \cite{1} we may assume that $\lim_{t \to \infty} u_{r(t)}\pi_t(a) - \pi_t(a)u_{r(t)} = 0$, $\lim_{t \to \infty} (1 - u_{r(t)})w\pi_t(a) - \pi_t(a)w = 0$ for all $a \in A$, and $\lim_{t \to \infty} u_{r(t)}w - wu_{r(t)} = 0$. We can therefore find a discretization $CH(\psi)_i, i \in N$, of $CH(\psi)$ such that

1) $\lim_{t \to \infty} \pi_t(a) - \pi_{t+1}(a) = 0$ for all $a \in A$,

2) $\lim_{t \to \infty} u_{r(t_i)} - u_{r(t_{i+1})} = 0$, 

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3) $\lim_{i \to \infty} wu_{r(t_i)} - u_{r(t_i)}w = 0$,
4) $\lim_{i \to \infty} (1 - u_{r(t_i)})(w\pi_t(a) - \pi_t(a)w) = 0$ for all $a \in A$.

To simplify notation, set $\pi_n = \pi_{t_n}$ and $u_n = u_{r(t_n)}$. Set $\pi_n = \pi_{-n}$ when $n < 0$ and $\pi_0 = \pi_1$.

We find that $E \circ CH[\psi] \in \operatorname{Ext}^{-1/2}(SC(T^2) \otimes A, B)_h$ is represented by a $*$-homomorphism $\Phi$, where $\Phi : SC(T^2) \otimes A \to \mathcal{L}_B(l^2(\mathbb{Z}) \otimes B)$ is a map such that

$$\Phi(f \otimes g \otimes h \otimes a) = \left( \sum_{n \geq 0} f(u_n)e_{nm} \right) \left( \sum_{n \in \mathbb{Z}} g(w)e_{mn} \right) h(T) \left( \sum_{n \in \mathbb{Z}} \pi_n(a)e_{nm} \right)$$

modulo $K \otimes B$ for all $f \in C_0(0, 1)$, $g, h \in C(T)$, $a \in A$.

Set

$$W = \sum_{n \in \mathbb{Z}} we_{nm}, \quad U = \sum_{n \geq 0} u_n.$$ 

Then $W, T$ and $U$ commute modulo $K \otimes B$. Define $\bar{\pi} : A \to Q(K \otimes B)$ such that $\bar{\pi}(a) = \sum_n \pi_n(a)e_{nm}$. Then $\bar{\pi}$ is a $*$-homomorphism which commutes with $U$ and $T$.

Let $Q \in M_2(Q(B))$ be the projection

$$Q = \begin{pmatrix} s^2(U) & sc_0(U) + sc_1(U)W \\ sc_1(U)W^* + sc_0(U) & (c_0 + c_1)^2(U) \end{pmatrix}.$$ 

**Lemma 3.4** - $\operatorname{Bott}_{SA} \circ E \circ CH[\psi]$ is represented in $\operatorname{Ext}^{-1/2}(SA, B)_h$ by an extension $\lambda : SA \to M_2(Q(B))$ such that

$$\lambda((1 - e^{2\pi ix}) \otimes a) = Q \begin{pmatrix} 1 - T & \bar{\pi}(a) \\ 1 - T & 1 \end{pmatrix},$$ 

$a \in A$.

**Proof.** To simplify notation, set

$$\tilde{U} = \begin{pmatrix} (1 - e^{2\pi ix})(U) & (1 - e^{2\pi ix})(U) \\ (1 - e^{2\pi ix})(U) & (1 - e^{2\pi ix})(U) \end{pmatrix}.$$ 

By definition $\operatorname{Bott}_{SA} \circ E \circ CH[\psi] = [\lambda_+] - [\lambda_-]$, where $\lambda_{\pm} : SA \to M_2(Q(B))$ are $*$-homomorphisms such that

$$\lambda_+((1 - e^{2\pi ix}) \otimes a) = P(T, W)\tilde{U} \begin{pmatrix} \bar{\pi}(a) \\ \bar{\pi}(a) \end{pmatrix}$$

and

$$\lambda_-((1 - e^{2\pi ix}) \otimes a) = P_0\tilde{U} \begin{pmatrix} \bar{\pi}(a) \\ \bar{\pi}(a) \end{pmatrix}.$$ 

Set

$$X = 1 - (1 - P(T, W)\tilde{U})(1 - P_0\tilde{U}^*).$$
Then \([\lambda_+] - [\lambda_-] = [\lambda']\), where \(\lambda' : SA \to M_2(Q(B))\) is given by

\[
\lambda'((1 - e^{2\pi i x}) \otimes a) = X \left( \begin{array}{c} \overline{\pi(a)} \\ \overline{\pi(a)} \end{array} \right).
\]

Note that \(X\) is an element in the \(2 \times 2\) matrices over the \(C^*\)-algebra generated by \(1 - T, W\) and \((1 - e^{2\pi i x})(U)\). In fact, if we define \(\Lambda : S \otimes C(T) \otimes S \to Q(B)\) such that

\[
\Lambda((1 - e^{2\pi i x}) \otimes e^{-2\pi iy} \otimes (1 - e^{2\pi i z})) = (1 - e^{2\pi i x})(U) W (1 - T),
\]

there is a quasi-unitary \(d \in M_2(S \otimes C(T) \otimes S)\) such that \(\text{id}_{M_2} \otimes \Lambda(d) = X\). Here \(S\) is shorthand for the \(C^*\)-algebra \(C_0(0, 1)\). Also we remind the reader that a quasi-unitary is an element \(d\) of a \(C^*\)-algebra \(D\) such that \(1 - d\) is unitary in \(D^+\). Alternatively, it is a normal element with spectrum in \(\{1 - z : z \in T\}\). Then

\[
\text{id}_{M_2} \otimes \Lambda \otimes \left( \begin{array}{c} \overline{\pi} \\ \overline{\pi} \end{array} \right) : S \otimes C(T) \otimes S \otimes A \to M_2(Q(B))
\]

is semi-invertible, with the inverse given by the \(*\)-homomorphism which results when one replaces \(U\) with \(\sum_{n < 0} u_{-n}\) in the definition of \(\Lambda\). Define

\[
\nu : SA \to M_2(S \otimes C(T) \otimes S \otimes A)
\]

such that \(\nu((1 - e^{2\pi i x}) \otimes a) = d \otimes a\) and note that \(\lambda' = (\text{id}_{M_2} \otimes \Lambda \otimes \left( \begin{array}{c} \overline{\pi} \\ \overline{\pi} \end{array} \right)) \circ \nu\). Let \(\alpha\) be the automorphism of \(M_2(S \otimes C(T) \otimes S \otimes A)\) which exchanges the two suspensions by a \(\pi/2\) rotation of \(R^2\). Then

\[-[\lambda] = \left[ \left( \text{id}_{M_2} \otimes \Lambda \otimes \left( \begin{array}{c} \overline{\pi} \\ \overline{\pi} \end{array} \right) \right) \circ \alpha \circ \nu \right]
\]

in \(\text{Ext}^{-1/2}(S \otimes C(T) \otimes S \otimes A, B)_h\). It follows that

\[-[\lambda] = \left[ \left( \text{id}_{M_2} \otimes \Lambda \otimes \left( \begin{array}{c} \overline{\pi} \\ \overline{\pi} \end{array} \right) \right) \circ \alpha \circ \nu \right]
\]

in \(\text{Ext}^{-1/2}(SA, B)_h\). Set

\[
Y = 1 - \left( 1 - Q \left( \begin{array}{cc} 1 - T & 1 - T \end{array} \right) \right) \left( 1 - P_0 \left( \begin{array}{cc} 1 - T^* & 1 - T^* \end{array} \right) \right),
\]

and note that \(\left( \text{id}_{M_2} \otimes \Lambda \otimes \left( \begin{array}{c} \overline{\pi} \\ \overline{\pi} \end{array} \right) \right) \circ \alpha \circ \nu = \mu\), where \(\mu : SA \to M_2(Q(B))\) is such that

\[
\mu((1 - e^{2\pi i x}) \otimes a) = Y \left( \begin{array}{c} \overline{\pi(a)} \\ \overline{\pi(a)} \end{array} \right).
\]

It follows that \([\mu] = [\lambda] - [\lambda']\), where \(\lambda'((1 - e^{2\pi i x}) \otimes a) = (1 - T) \overline{\pi(a)}\). It is easily seen that \(\mu'\) is asymptotically split. Therefore \([\mu] = [\lambda]\). 

\[
X = \left( \begin{array}{cc} \frac{s(U)}{c_0(U) + c_1(U)W^*} & -c_0(U) - c_1(U)W \frac{s(U)}{s(U)} \\ c_0(U) + c_1(U)W^* & \frac{s(U)}{c_0(U) + c_1(U)W^*} \end{array} \right).
\]
and
\[ Z = \begin{pmatrix} iW_+ & 0 \\ 0 & -iW_+ \end{pmatrix}, \]
where
\[ W_+ = \sum_{n \geq 0} w_{en} + \sum_{n < 0} e_{nn} \in \mathcal{L}_2(l_2(\mathbb{Z}) \otimes B). \]

Then \( Z \) and \( X \) are unitaries in \( M_2(Q(B)) \). Let \( T_0 : l_2(\mathbb{Z}) \otimes B \to l_2(\mathbb{Z}) \otimes B \) be the unitary
\[ T_0 = \sum_{n \in \mathbb{Z}\setminus\{-1\}} e_{n,n+1} + w_{-1,0}. \]

We can then define an extension \( \lambda_1 : SA \to Q(B) \) such that
\[ \lambda_1((1 - e^{2\pi i x}) \otimes a) = (1 - T_0) \pi(a). \]

**Lemma 3.5** Let \( \lambda : SA \to M_2(Q(B)) \) be the extension of Lemma 3.4. Then
\[ \text{Ad } X^* \circ \lambda = \text{Ad } Z \circ \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix}. \]

**Proof.** Note that \( \lambda \) and \( \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix} \) both extend to unital *-homomorphisms \( C(T) \otimes A \to M_2(Q(B)) \) defined such that they send \( 1 \otimes a \) to \( \begin{pmatrix} \pi(a) & 0 \\ 0 & \pi(a) \end{pmatrix} \), \( a \in A \). By considering these extensions we see that it suffices to show that
\[ X^* \begin{pmatrix} \pi(a) & 0 \\ 0 & \pi(a) \end{pmatrix} X = Z \begin{pmatrix} \pi(a) & 0 \\ 0 & \pi(a) \end{pmatrix} Z^*, \quad (3.5) \]
and
\[ V^* \left( Q \begin{pmatrix} T & T^* \\ T^* & T^t \end{pmatrix} + Q^\perp \right) \begin{pmatrix} \pi(a) \\ \pi(a) \end{pmatrix} V \]
\[ = Z \begin{pmatrix} T_0 & 1 \\ 1 & T_0 \end{pmatrix} \begin{pmatrix} \pi(a) \\ \pi(a) \end{pmatrix} Z^* \]
\[ = Z \begin{pmatrix} 1 & 0 \\ 0 & T_0 \end{pmatrix} \begin{pmatrix} T_0 & T_0^* \\ T_0^* & T_0^t \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \pi(a) \\ \pi(a) \end{pmatrix} Z^*. \quad (3.6) \]

To simplify the verification, observe that \( W_+ T_0 = TW_+ \) from which it follows that
\[ Z \begin{pmatrix} T_0 & T_0^* \\ T_0^* & T_0^t \end{pmatrix} Z^* = \begin{pmatrix} T & T^t \\ T^t & T^* \end{pmatrix}. \]
Since \( X \) clearly commutes with \( \begin{pmatrix} T & T^t \\ T^t & T^* \end{pmatrix} \) and \( Z \) with \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) we see that (3.6) will follow from (3.3) and
\[ X^* Q X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.7) \]
Thus we need only check (3.3) and (3.7), and we leave that to the reader. \( \square \)
All in all we now have that \(-\text{Bott}_{SA} \circ E \circ CH[\psi] = [\lambda_1] \) in \(\text{Ext}^{-1/2}(SA, B)_h\). Define
\[
\kappa : SA \to Q(B) \quad \text{by} \quad \kappa((1 - e^{2\pi ix}) \otimes a) = (1 - T) \overline{a}(a).
\]
The extension \(\kappa\) is asymptotically split and hence \([\lambda_1] = [\lambda_1] - [\kappa]\). Since \([\lambda_1] - [\kappa] = [\mu]\), where \(\mu : SA \to Q(B)\) is given by \(\mu((1 - e^{2\pi ix}) \otimes a) = (1 - T_0T^*)\overline{a}(a)\) and since \(T_0T^* = \sum_{n \neq -1} e_m + w_{e_{-1,-1}}\), we see that \([\lambda_1] = [\phi]\). This completes the proof of Proposition 3.1.

**Corollary 3.6** The map
\[
CH : \text{Ext}^{-1/2}(SA, B)_h \to [[S^2 A, B]]
\]
is injective on \(i^*(\text{Ext}^{-1/2}(C(T) \otimes A, B)_h)\).

**Proof.** Let \(\psi \in \text{Ext}^{-1/2}(C(T) \otimes A, B)\) and assume that \(CH(i^*[\psi]) = 0\). By the naturality of the Connes–Higson construction this implies that
\[
(Si)^*(CH[\psi] - (Se)^* \circ (Sc)^*(CH[\psi])) = CH(i^*(\psi)) = 0
\]
in \([[S^2 A, B]]\). But the split exactness of the functor \([[S-, B]]\), \([3]\), implies then that
\[
0 = CH[\psi] - (Se)^* \circ (Sc)^*(CH[\psi]) = CH([\psi] - e^* \circ e^*[\psi]).
\]
And then \(i^*[\psi] = 0\) by Proposition 3.1.

**Lemma 3.7** The map
\[
(Si)^* : \text{Ext}^{-1/2}(SC(T) \otimes A, B)_h \to \text{Ext}^{-1/2}(S^2 A, B)_h
\]
is surjective.

**Proof.** To prove this we shall identify \(S^2 = C_0(\mathbb{R}^2)\) with \(C_0(D)\), where \(D = \mathbb{R}^2 \setminus \{(0, y) \in \mathbb{R}^2 : y \geq 0\}\) and \(SC(T)\) with \(C_0(\mathbb{R}^2 \setminus \{0\})\). It is easy to see that there is a continuous map \(F : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}^2\) such that \(F(0, -)\) is a homeomorphism \(\mu = F(0, -) : \mathbb{R}^2 \to D\); \(F(1, z) = z, \ z \in \mathbb{R}^2\), and \(F^{-1}(K)\) is compact for every compact subset \(K\) of \(D\). It follows that \(f \mapsto f \circ \mu^{-1}\) is an endomorphism of \(C_0(D)\) which is homotopic to id\(_{C_0(D)}\). Hence if \(\varphi \in \text{Hom}(S^2 A, Q(B))\) is a semi-invertible extension, \([\varphi] = [\chi]\) in \(\text{Ext}^{-1/2}(S^2 A, B)\), where \(\chi(f) = \varphi(f \circ \mu^{-1})\). Define \(\psi : SC(T) \to Q(B)\) by \(\psi(g) = \varphi(g \circ \mu^{-1})\). Then \((Si)^*[\psi] = [\varphi]\).

Lemma 3.7 and Corollary 3.6 in combination prove that the \(CH\)-map of diagram (2.1) is injective. This completes the proof of b) of Theorem 2.2.
References

[1] Arveson W. Notes on extensions of $C^*$-algebras. Duke Math. J. 44 (1977), 329–355.

[2] Bartle R. G., Graves L. M. Mappings between function spaces. Trans. Amer. Math. Soc. 72 (1952), 400–413.

[3] Busby R. Double centralizers and extensions of $C^*$-algebras. Trans. Amer. Math. Soc. 132 (1968), 79–99.

[4] Connes A., Higson N. Déformations, morphismes asymptotiques et $K$-théorie bivariante. C. R. Acad. Sci. Paris Sér. I Math. 311 (1990), 101–106.

[5] Cuntz J. A general construction of bivariant $K$-theories on the category of $C^*$-algebras. Operator algebras and operator theory (Shanghai, 1997), 31-43, Contemp. Math., 228, Amer. Math. Soc., Providence, RI, 1998.

[6] Dădărlat M., Loring T. A. $K$-homology, asymptotic representations and unsuspended $E$-theory. J. Funct. Anal. 126 (1994), 367–383.

[7] Houghton-Larsen T., Thomsen K. Universal (co)homology theories. $K$-theory 16 (1999), 1–27.

[8] Kasparov G. G. The operator $K$-functor and extensions of $C^*$-algebras. Izv. Akad. Nauk SSSR ser. Mat. 44 (1980), 571–636.

[9] Loring T. A. $K$-theory and asymptotically commuting matrices. Canad. J. Math. 40 (1988), 197–216.

[10] Manuilov V. M., Mishchenko A. S. Asymptotic and Fredholm representations of discrete groups. Russian Acad. Sci. Sb. Math. 189 (1998), 1485–1504.

[11] Manuilov V. M., Thomsen K. Quasidiagonal extensions and sequentially trivial asymptotic homomorphisms. Preprint, 1998.

[12] Pedersen G. K. $C^*$-algebras and their automorphism groups. Academic Press, London – New York – San Francisco, 1979.

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