ASYMPTOTICS FOR THE CONCENTRATED FIELD BETWEEN CLOSELY LOCATED HARD INCLUSIONS IN ALL DIMENSIONS

Zhiwen Zhao\textsuperscript{a,b} and Xia Hao\textsuperscript{a,*}

\textsuperscript{a}School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China
\textsuperscript{b}Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence
University of Groningen, PO Box 407, 9700 AK Groningen, The Netherlands

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Abstract. When hard inclusions are frequently spaced very closely, the electric field, which is the gradient of the solution to the perfect conductivity equation, may be arbitrarily large as the distance between two inclusions goes to zero. In this paper, our objectives are two-fold: first, we extend the asymptotic expansions of [26] to the higher dimensions greater than three by capturing the blow-up factors in all dimensions, which consist of some certain integrals of the solutions to the case when two inclusions are touching; second, our results answer the optimality of the blow-up rate for any $m, n \geq 2$, where $m$ and $n$ are the parameters of convexity and dimension, respectively, which is only partially solved in [29].

1. Background and main results.

1.1. Background. It is well known that field concentrations appear widely in nature and industrial applications. In heterogeneous media which consists of the fibers with extreme conductivities and a finite-conductivity matrix, there may appear high concentration of the electric field as the distance $\varepsilon$ between inclusions tends to zero. Much effort has been devoted to quantitative understanding of this high concentration since Babuška et al’s famous work [5], where the Lamé system is used and the authors computationally analyzed the damage and fracture in fiber composite materials and observed numerically that the size of the strain tensor remains bounded as the distance $\varepsilon$ between inclusions tends to zero. The subsequent work [31] completed by Li and Nirenberg demonstrated this observation for general divergence form elliptic systems including the Lamé system with piecewise H"older continuous coefficients in all dimensions. The corresponding results for scalar elliptic equations can refer to [13, 32]. Note that the estimates in [31, 32] show inexplicit dependence on the elliptic coefficients and the distance between fibers. By making use of Green's function method, Dong and Li [17] recently showed their clear dependence in the upper and lower estimates of the gradient of a solution to the conductivity equation in two dimensions. However, it is still unsolved for more general elliptic equations and systems in all dimensions. More details about these open problems can be

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* Corresponding author.
seen in page 894 of [31]. Additionally, Calo, Efendiev and Galvis [14] presented an asymptotic expansion of a solution to elliptic equations in terms of the contrast $k$ as $k$ is sufficiently small or large.

When the contrast $k$ degenerates to $\infty$, we consider the perfect conductivity problem as follows:

$$
\begin{align*}
\Delta u &= 0, & \text{in } \Omega, \\
uu &= C_i, & \text{on } \partial D_i, \ i = 1, 2, \\
n\nuu &\bigg|_+ = 0, & \ i = 1, 2, \\
uu &= \varphi, & \text{on } \partial D,
\end{align*}
$$

(1.1)

where the free constants $C_1$ and $C_2$ are determined by the third line of (1.1) and

$$
n\nuu \bigg|_+: = \lim_{\tau \to 0} \frac{u(x + \nu \tau) - u(x)}{\tau}.
$$

Here and below $\nu$ denotes the unit outer normal to the domains and the subscript $\pm$ represents the limit from outside and inside the domain, respectively. The existence, uniqueness and regularity of weak solutions to (1.1) have been established in [10]. By contrast with the case when the contrast $k$ is finite and bounded below from zero, the gradient of a solution to problem (1.1) may appear blow-up. It has been proved that the blow-up rates of the gradient are $\varepsilon^{-1/2}$ in two dimensions [4, 6, 10, 3, 6, 3, 34, 35, 20], $\ln \varepsilon$ in two dimensions [10, 33, 11, 24], and $\varepsilon^{-1}$ in higher dimensions [10], respectively. Similar results were extended to the Lamé system with partially infinite coefficients, see [8, 9, 30]. While these works are related to the gradient estimates, there is another direction of research to give a precise characterization in terms of the singular behavior of the gradient. Kang, Lim and Yun [21] used an explicit singular function to completely describe the singularities of the electric field for two disks in two dimensions. Ammari et al. [2] extended the asymptotic results in [21] to the case when inclusions are strictly convex simply connected domains in two dimensions by utilizing the method of disks osculating to convex domains. In three dimensions, Kang et al. [22] obtained an asymptotic formula of $\nabla u$ for two spherical perfect conductors with the same radii. Recently, Li, Li and Yang [26] established a precise calculation of the energy to derive an asymptotic formula in dimensions two and three for two arbitrarily 2-convex inclusions. Li [29] then studied the $m$-convex inclusions with zero-curvature and captured a blow-up factor different from that in [26]. For nonlinear $p$-Laplace equation, Gorb and Novikov [18] captured the stress concentration factor. Ciraolo and Sciammetta [15, 16] further extended to the Finsler $p$-Laplacian. For more related work, we refer to papers [1, 7, 23, 28, 12, 19] and the reference therein.

In this paper, we first decompose the gradient into two explicit parts: one of them is the regular part and decays exponentially fast in the shortest segment between two inclusions; the other is the singular part and appears blow-up. In order to clarify the singular part, by following an idea in [26] for the calculation of energy, we capture the blow-up factors in all dimensions consisting of some certain integrals of the solutions to the case when two inclusions are touching. Then using the decomposition, we present an asymptotic formula of the concentrated field which characterizes the singularities of such a high concentration in the presence of the generalized $m$-convex inclusions. Our asymptotic results in Theorem 1.1 below answer all the optimality of the blow-up rate for any $m, n \geq 2$, which is only solved in [29] for the case of $m \geq 2(n - 1)$ if $n \geq 2$ and the case of $n - 1 \leq m < 2(n - 1)$.
if $n \geq 3$. Finally, for the purpose of meeting the needs of industrial applications, we also give a precise characterization in terms of the singular behavior of the concentrated field for two close-to-touching disks and spheres in dimensions two and three.

1.2. Main results. To formulate our problem and state the main results in a precise manner, we first describe our domain and introduce some notations. Let $D \subseteq \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with $C^{2,\alpha}$ ($0 < \alpha < 1$) boundary. Assume that there is a pair of $C^{2,\alpha}$-subdomains $D_1^*$ and $D_2$ inside $D$, such that these two subdomains touch only at one point and they are far away from the external boundary $\partial D$. That is, by a translation and rotation of the coordinates, if necessary, $\partial D_1^* \cap \partial D_2 = \{0'\} \subset \mathbb{R}^{n-1}$, and

$$D_1^* \subset \{(x', x_n) \in \mathbb{R}^n | x_n > 0\}, \quad D_2 \subset \{(x', x_n) \in \mathbb{R}^n | x_n < 0\}.$$ Throughout the paper, we use superscript prime to denote $(n - 1)$-dimensional domains and variables. Via a translation, we set

$$D_1^0 := D_1^* + (0', \varepsilon),$$

where $\varepsilon > 0$ is a sufficiently small constant. For the sake of simplicity, we drop superscripts and denote

$$D_1 := D_1^0, \quad \Omega := D \setminus \overline{D_1} \cup \overline{D_2}.$$ We further assume that there exists a small constant $R > 0$ independent of $\varepsilon$, such that the portions of $\partial D_1$ and $\partial D_2$ near the origin are, respectively, the graphs of two $C^{2,\alpha}$ functions $\varepsilon + h_1$ and $h_2$, and $h_i$, $i = 1, 2$ satisfy that for $m \geq 2$ and $\beta > 0$,

$$(H1) \ h_1(x') - h_2(x') = \lambda |x'|^m + O(|x'|^{m+\beta}), \text{ if } x' \in B'_2R,$$

$$(H2) \ |
abla_j h_i(x')| \leq \kappa_1 |x'|^{m-j}, \text{ if } x' \in B'_2R, \ i, j = 1, 2,$$

$$(H3) \ |h_1|_{C^{2,\alpha}(B'_2R)} + |h_2|_{C^{2,\alpha}(B'_2R)} \leq \kappa_2,$$

where $\lambda$ and $\kappa_i$, $i = 1, 2$, are three positive constants independent of $\varepsilon$. We would like to explain that assumption condition (H1) implies that for $x' \in B'_2R$,

$$|(h_1 - h_2)(x') - \lambda |x'|^m| \leq C |x'|^{m+\beta}, \text{ for some } \varepsilon\text{-independent constant } C.$$ This means that these two inclusions may possess different convexity, such as $h_1 = |x'|^m$ and $h_2 = -|x'|^{m+\beta}$.

For $z' \in B'_R$, $0 < t \leq 2R$, denote

$$\Omega_t(z') := \{x \in \mathbb{R}^n \mid h(x') < x_n < \varepsilon + h_1(x'), \ |x' - z'| < t\}.$$ We will use the abbreviated notation $\Omega_t$ for the domain $\Omega_t(0')$. Before stating our main results, we first introduce a scalar auxiliary function $\tilde{u}_1 \in C^2(\mathbb{R}^n)$ such that $\tilde{u}_1 = 1$ on $\partial D_1$, $\tilde{u}_1 = 0$ on $\partial D_2 \cup \partial D$ and

$$\tilde{u}_1(x) = \frac{x_n - h_2(x')}{\varepsilon + h_1(x') - h_2(x')}, \text{ in } \Omega_{2R}. \quad |\tilde{u}_1|_{C^2(\Omega \setminus \Omega_R)} \leq C. \quad (1.2)$$ To simplify notations used in the following, we denote

$$\rho_{\alpha,m}(\varepsilon) = \begin{cases} \varepsilon^{1-\frac{n-1}{m}}, & m > n - 1, \\ |\ln \varepsilon|^{-1}, & m = n - 1, \end{cases} \quad (1.3)$$
and 
\[ \Gamma\left[\frac{n-1}{m}\right] = \begin{cases} \Gamma\left(1 - \frac{n-1}{m}\right) & m > n - 1, \\ 1 & m = n - 1, \end{cases} \]

where \( \Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t}dt, s > 0 \) is the Gamma function. Denote by \( \omega_{n-1} \) the area of the surface of unit sphere in \((n-1)\)-dimension. For \((x', x_n) \in \Omega_{2R} \), denote
\[ \delta(x') := \varepsilon + h_1(x') - h_2(x'). \]  
(1.4)

Let \( \Omega^* := D \setminus (D_1^* \cup D_2) \). We define
\[ b_1^* = \int_{\partial D_1^*} \frac{\partial v_0^*}{\partial \nu}, \quad b_2^* = \int_{\partial D_2} \frac{\partial v_0^*}{\partial \nu}, \quad Q_j^* = \int_{\partial D} \frac{\partial v_j^*}{\partial \nu}, \quad j = 1, 2, \]
(1.5)

where \( v_i^*, i = 0, 1, 2, \) verify
\[ \begin{cases} \Delta v_0^* = 0, & \text{in } \Omega^*, \\ v_0^* = 0, & \text{on } \partial D_1^* \cup \partial D_2, \\ v_0^* = \varphi(x), & \text{on } \partial D, \end{cases} \]
(1.6)

and
\[ \begin{cases} \Delta v_1^* = 0, & \text{in } \Omega^*, \\ v_1^* = 1, & \text{on } \partial D_1^* \setminus \{0\}, \\ v_1^* = 0, & \text{on } (\partial D_2 \setminus \{0\}) \cup \partial D, \end{cases} \]
\[ \begin{cases} \Delta v_2^* = 0, & \text{in } \Omega^*, \\ v_2^* = 1, & \text{on } \partial D_2 \setminus \{0\}, \\ v_2^* = 0, & \text{on } (\partial D_1^* \setminus \{0\}) \cup \partial D, \end{cases} \]  
(1.7)

respectively. For \( m < n - 1 \), define
\[ a_{ij}^* := \int_{\partial D_i^*} \frac{\partial v_i^*}{\partial \nu}, \]
(1.8)

where \( v_i^*, i = 1, 2 \) are defined by (1.7). Denote
\[ Q^*[\varphi] = b_1^* Q_2^* - b_2^* Q_1^*, \quad \Theta^* = -\mathcal{M}(Q_1^* + Q_2^*), \quad \mathcal{S}^* = -a_{11}^* Q_2^* + a_{12}^* Q_1^*, \]
(1.9)

where
\[ \mathcal{M} = \frac{(n-1)\omega_{n-1} \Gamma[s-1]}{m \lambda^{n-1}}. \]
(1.10)

Note that the definition of \( \mathcal{S}^* \) is only valid for \( m < n - 1 \). For the order of the rest term, we define
\[ r_\varepsilon = \begin{cases} \varepsilon^{\min\left\{\frac{m}{n}, \frac{1}{2}\right\}}, & m > n - 1 + \beta, \\ \varepsilon^{\min\left\{\frac{m}{n}, \frac{1}{2}\right\}} \ln \varepsilon, & m = n - 1 + \beta, \\ \varepsilon^{\min\left\{1 - \frac{n-1}{m}, \frac{1}{2}\right\}}, & n - 1 < m < n - 1 + \beta, \\ \ln \varepsilon^{-1}, & m = n - 1, \\ \varepsilon^{\min\left\{\frac{m}{n} - \frac{1}{m}, \frac{1}{2}\right\}}, & m < n - 1. \end{cases} \]
(1.11)

Unless otherwise stated, in what following \( C \) represents a constant, whose values may vary from line to line, depending only on \( \lambda, \kappa_1, \kappa_2, R \) and an upper bound of the \( C^{2,\alpha} \) norms of \( \partial D_1 \) and \( \partial D_2 \), but not on \( \varepsilon \). \( O(1) \) denotes some quantity satisfying \(|O(1)| \leq C \) for some \( \varepsilon \)-independent constant \( C \).

**Theorem 1.1.** Assume that \( D_1, D_2 \subset D \subseteq \mathbb{R}^n \) \((n \geq 2)\) are defined as above, conditions (H1)-(H3) hold. Let \( u \in H^1(D) \cap C^1(\overline{\Omega}) \) be the solution of (1.1). Then for a sufficiently small \( \varepsilon > 0 \) and \( x \in \Omega_R \), if \( Q^*[\varphi] \neq 0 \),
(i) for $m \geq n - 1$,
\[
\nabla u = \frac{Q^*[\varphi]}{\Theta^*}(1 + O(r_\varepsilon))\rho_{n,m}(\varepsilon)\nabla \bar{u}_1 + O(1)\delta^{\frac{m-n}{m-n-2}}\|\varphi\|_{C^0(\partial D)};
\]
(ii) for $m < n - 1$,
\[
\nabla u = \frac{Q^*[\varphi]}{\Theta^*}(1 + O(r_\varepsilon))\nabla \bar{u}_1 + O(1)\delta^{\frac{m-n}{m-n-2}}\|\varphi\|_{C^0(\partial D)},
\]
where $\bar{u}_1$ and $\rho_{n,m}(\varepsilon)$ are defined by (1.2) and (1.3), respectively, $\delta$ is defined by (1.4), $Q^*[\varphi]$, $\Theta^*$ and $\Theta^*$ are defined in (1.9), and $r_\varepsilon$ is defined in (1.11).

**Remark 1.2.** To begin with, by contrast with [26, 29], we capture all the blow-up factors, especially the factor $\Theta^*$ under case $m < n - 1$. Moreover, our results here are more general than that of [26, 29] owing to the higher order rest term $O(|x'|^{m+\beta})$ considered in condition (H1). Secondly, our asymptotic results are pointwise expressions in the narrow region $\Omega_R$ with its length parameter $R$ independent of $\varepsilon$. This is different from the results in [22, 27, 19, 18], where the length of the thin gap considered in [22, 27] is less than $C|\ln \varepsilon|^{-2}$ and the uniform norm $\|\nabla u\|_{L^\infty(\Omega_\varepsilon)}$ is studied in [19, 18].

**Remark 1.3.** It is worth emphasizing that the blow-up factor $Q^*[\varphi]$ can determine the blow-up will occur or not. For example, take $m = n = 2$ and let $Q^*[\varphi] = 0$, then we see from Lemma 3.1 below that $Q[\varphi] = O(\varepsilon^{0.7})$. This, together with decomposition (2.8), (2.19), Theorems 2.1 and 2.2, Lemma 3.3 below, yields that
\[
|\nabla u| = \left|\frac{Q[\varphi]}{\Theta^*}\rho_{2,2}(\varepsilon)\nabla v_1 + C_2\nabla (v_1 + v_2) + \nabla v_0\right| \leq C_\varepsilon\delta^{-1} + C\delta^{-1}e^{-\frac{1}{2\sqrt{\varepsilon^2}} \leq C},
\]
which means that there appears no blow-up. In addition, we would like to point out that some special examples in terms of the domains and the boundary data, which let $Q^*[\varphi] \neq 0$, were given in [10].

**Remark 1.4.** Recalling the definition of $\bar{u}_1$ in (1.2), we know that the major singularity of $|\nabla \bar{u}_1|$ is determined by $\partial_{x_1} \bar{u}_1 = \frac{1}{\varepsilon + \lambda_1|x'|^{m+O(|x'|^{m+\beta})}}$ with its largest blow-up rate $\varepsilon^{-1}$ attaining at the $(n-1)$-dimensional sphere $\{|x'| \leq \varepsilon^{1/m}\} \cap \Omega$. Then from the asymptotics of $\nabla u$ in Theorem 1.1, we see that the maximum of $|\nabla u|$ arrives at the origin with its blow-up rate being $\varepsilon^{-1}\rho_{n,m}(\varepsilon)$ in the case of $m \geq n - 1$ and $\varepsilon^{-1}$ in the case of $m < n - 1$. Additionally, when $m < n - 1$, the asymptotic expansion (1.13) also holds for the generalized $C^{2,\alpha}$-inclusions satisfying the following assumption:
\[
\lambda_1|x'|^m \leq h_1(x') - h_2(x') \leq \lambda_2|x'|^m, \quad \text{in } B_{2R}',
\]
where $\lambda_1$ and $\lambda_2$ are two positive constants independent of $\varepsilon$.

We now consider a special example of two neighbouring disks (see Figure 1 below) and spheres in dimensions two and three. To be specific, there exist two positive constants $r_1$ and $r_2$, independent of $\varepsilon$, such that
\[
D_1 := B_{r_1}(0', \varepsilon + r_1), \quad D_2 := B_{r_2}(0', -r_2).
\]
Note that in this example the constant $\mathcal{M}$ defined in the blow-up factor $\Theta^*$ of (1.10) becomes
\[
\mathcal{M} = \pi \lambda_0^{1-\frac{n}{2}}, \quad \text{with } \lambda_0 = \frac{1}{2} \left( \frac{1}{r_1} + \frac{1}{r_2} \right).
\]
By the same argument as in Theorem 1.1, we obtain
Corollary 1.5. Assume as above, condition (1.14) holds and $Q^* \phi \neq 0$. Let $u \in H^1(D) \cap C^1(\Omega)$ be the solution of (1.1). Then for a sufficiently small $\varepsilon > 0$ and $x \in \Omega_{r_0}$, $0 < r_0 < \min\{r_1, r_2\}$ is a small constant independent of $\varepsilon$,

(i) for $n = 2$,

$$\nabla u = \frac{Q^* \phi}{\Theta^*} \frac{\sqrt{\varepsilon}}{1 - M^*_1 \sqrt{\varepsilon}} \nabla \bar{u}_1 + O(1) \|\phi\|_{C^0(\partial D)}; \quad (1.16)$$

(ii) for $n = 3$,

$$\nabla u = \frac{Q^* \phi}{\Theta^*} \left( \frac{1}{|\ln \varepsilon| - M^*_3} + O(\varepsilon^{1/4} |\ln \varepsilon|^{-2}) \right) \nabla \bar{u}_1 + O(1) \|\phi\|_{C^0(\partial D)}, \quad (1.17)$$

where $\bar{u}_1$ is defined by (1.2), $\delta$ is defined by (1.4), $Q^* \phi$ and $\Theta^*$ are defined in (1.9), $M^*_n$, $n = 2, 3$ are defined by (4.8) below.

Remark 1.6. Although the asymptotic results in [26, 29] actually covers disks and spheres in dimensions two and three, we also give its explicit asymptotic expressions for the convenience of future application. Moreover, we see from (1.15) and the constant $M^*_n$ defined in (4.8) below that the dependence on radius is exhibited in the form of the sum of curvature. This dependence is also captured in [21, 22, 27].

![Figure 1. Two close-to-touching disks](image)

We now present an overview of the rest of this paper. In Section 2, we split the solution $u$ for problem (1.1) into $v_i$, $i = 0, 1, 2$, defined by (2.2) and (2.3) below, and we prove that $\nabla \bar{u}_1$ is the main term of $\nabla v_1$, $\nabla v_0$ and $\nabla (v_1 + v_2)$ exhibit exponential decay with respect to the distance between two inclusions. In Section 3, we obtain the asymptotics of $Q[\phi]$ and $\Theta$, which together with the results obtained in Section 2 verify that Theorem 1.1 holds. The proof Corollary 1.5 is given in Section 4.

2. Preliminary.

2.1. Solution split. As in [10, 26], the solution $u$ of (1.1) can be split as follows

$$u(x) = C_1 v_1(x) + C_2 v_2 + v_0(x), \quad \text{in } D \setminus D_1 \cup D_2, \quad (2.1)$$
where $v_i, i = 0, 1, 2$, verify

\[
\begin{cases}
\Delta v_0 = 0, & \text{in } \Omega, \\
v_0 = 0, & \text{on } \partial D_1 \cup \partial D_2, \\
v_0 = \varphi(x), & \text{on } \partial D,
\end{cases}
\]

and

\[
\begin{cases}
\Delta v_i = 0, & \text{in } \Omega, \\
v_i = \delta_{ij}, & \text{on } \partial D_j, \ i, j = 1, 2, \\
v_i = 0, & \text{on } \partial D,
\end{cases}
\]

respectively. Similarly as in [10], we denote

\[
a_{ij} = \int_{\partial D_i} \frac{\partial v_j}{\partial \nu}, \quad b_i = \int_{\partial D_i} \frac{\partial v_0}{\partial \nu}, \quad Q_j = \int_{\partial D} \frac{\partial v_j}{\partial \nu} \quad \text{i, j = 1, 2}.
\]

Utilizing integration by parts, we obtain

\[
a_{12} = a_{21}, \quad a_{11} + a_{21} = -Q_1, \quad a_{22} + a_{12} = -Q_2.
\]

Due to the third line of (1.1), we have

\[
\begin{cases}
a_{11}C_1 + a_{12}C_2 + b_1 = 0, \\
a_{21}C_1 + a_{22}C_2 + b_2 = 0.
\end{cases}
\]

Then it follows from Cramer’s rule that

\[
C_1 - C_2 = \frac{b_2(a_{11} + a_{12}) - b_1(a_{21} + a_{22})}{a_{11}a_{22} - a_{12}a_{21}}.
\]

Observe that

\[
a_{11}a_{22} - a_{12}a_{21} = a_{11}(a_{21} + a_{22}) - a_{21}(a_{11} + a_{12}).
\]

Then substituting (2.4) and (2.6) into (2.5), we have

\[
C_1 - C_2 = \rho_{n,m}(\varepsilon) \frac{b_1Q_2 - b_2Q_1}{\rho_{n,m}(\varepsilon)(a_{22}Q_2 + a_{12}Q_1)} := \rho_{n,m}(\varepsilon) \frac{Q[\varphi]}{\Theta}.
\]

Combining decomposition (2.1) and (2.7), we have

\[
\nabla u = \frac{Q[\varphi]}{\Theta} \rho_{n,m}(\varepsilon) \nabla v_1 + C_2 \nabla (v_1 + v_2) + \nabla v_0.
\]

In fact, the decomposition (2.8) splits $\nabla u$ into two parts in an explicit form: the first part is $\frac{Q[\varphi]}{\Theta} \rho_{n,m}(\varepsilon) \nabla v_1$ and exhibits the singularities, while the second part $C_2 \nabla (v_1 + v_2) + \nabla v_0$ is bounded and possesses the exponentially decaying property. See the following sections for precise statements of these results.

2.2. Main terms. We can prove that the leading term of $\nabla v_1$ is $\nabla \bar{u}_1$ in the following.

**Theorem 2.1.** Assume as above. Let $v_1 \in H^1(\Omega)$ be the weak solution of (2.3). Then, for a sufficiently small $\varepsilon > 0$,

\[
\nabla v_1 = \nabla \bar{u}_1 + O(1)\delta \frac{m^2}{\varepsilon}, \quad \text{in } \Omega_R,
\]

and

\[
\|\nabla v_1\|_{L^\infty(\Omega \setminus \Omega_R)} \leq C.
\]
Proof. Recalling the definition of $\tilde{u}_1$ and utilizing $(H1)$–$(H2)$, it follows from a straightforward computation that for $i = 1, ..., n - 1$, $x \in \Omega_{2R}$,
\[
|\partial_i \tilde{u}_1| \leq C\delta^{-1/m}, \quad \partial_n \tilde{u}_1 = \delta^{-1}, \quad |\Delta \tilde{u}_1| \leq C\delta^{-2/m}.
\] (2.10)
We now divide into three parts to prove Theorem 2.1.

Part 1. Let $v_1 \in H^1(\Omega)$ be a weak solution of (2.3). Then
\[
\int_{\Omega} |\nabla w_1|^2 \, dx \leq C.
\] (2.11)
For $x \in \Omega_{2R}$, we denote
\[
w_1 := v_1 - \tilde{u}_1,
\]
where $\tilde{u}_1$ is defined by (1.2). Then $w_1$ solves
\[
\begin{cases}
\Delta w_1 = -\Delta \tilde{u}_1, & \text{in } \Omega, \\
w_1 = 0, & \text{on } \partial \Omega.
\end{cases}
\] (2.12)
Taking the test function $w_1$ in (2.12) and using the Poincaré inequality and Sobolev trace embedding theorem, it follows from integration by parts, (1.2) and (2.10) that
\[
\int_{\Omega} |\nabla w_1|^2 = \int_{\Omega_R} w_1 \Delta \tilde{u}_1 + \int_{\Omega \setminus \Omega_R} w_1 \Delta \tilde{u}_1 - \sum_{i=1}^{n-1} \int_{\Omega_R} w_1 \partial_i \tilde{u}_1 + C \int_{\Omega \setminus \Omega_R} |w_1| \\
\leq C \|\nabla w_1\|_{L^2(\Omega_R)} \|\nabla \tilde{u}_1\|_{L^2(\Omega_R)} + C \|\nabla w_1\|_{L^2(\Omega \setminus \Omega_R)} \leq C \|\nabla w_1\|_{L^2(\Omega)},
\]
where in the second line we use the fact that $\partial_n \tilde{u}_1 = 0$ in $\Omega_R$. Consequently, (2.11) is established.

Part 2. Proof of
\[
\int_{\Omega(z')} |\nabla w_1|^2 \, dx \leq C\delta^{n+2-\frac{m}{2}},
\] (2.13)
where $\delta$ is defined by (1.4). For $0 < t < s < R$, let $\eta$ be a smooth cutoff function such that $\eta(x') = 1$ if $|x' - z'| < t$, $\eta(x') = 0$ if $|x' - z'| > s$, $0 \leq \eta(x') \leq 1$ if $t \leq |x' - z'| \leq s$, and $|\nabla \eta(x')| \leq \frac{2}{\delta s}$. Then multiplying the equation in (2.12) by $w_1 \eta^2$ and utilizing integration by parts, we obtain the iteration formula as follows:
\[
\int_{\Omega_{t}(z')} |\nabla w_1|^2 \, dx \leq \frac{C}{(s-t)^2} \int_{\Omega_{t}(z')} |w_1|^2 \, dx + C(s-t)^2 \int_{\Omega_{t}(z')} \Delta \tilde{u}_1 \, dx.
\] (2.14)
For $|z'| \leq R$, $\delta < s \leq \frac{2}{\delta} \min\{\frac{1}{s}, |z'|\}$, we deduce that $\frac{\delta(z')}{C} \leq \delta(z') \leq C\delta(z')$ in $\Omega_{t}(z')$. Then using (2.10), we obtain
\[
\int_{\Omega_{t}(z')} |\Delta \tilde{u}_1|^2 \leq Cs^{n-1} \delta^{-\frac{m-4}{2}}, \quad \int_{\Omega_{t}(z')} |w_1|^2 \leq C\delta^2 \int_{\Omega_{t}(z')} |\nabla w_1|^2,
\] (2.15)
where in the second inequality we use the fact that $w_1 = 0$ on $\Gamma^*_R$.

Write
\[
F(t) := \int_{\Omega_{t}(z')} |\nabla w_1|^2.
\]
From (2.14) and (2.15), we see that
\[
F(t) \leq \left( \frac{c\delta}{s-t} \right)^2 F(s) + C(s-t)^2 s^{n-1} \delta^{-\frac{m-4}{2}},
\] (2.16)
where $c$ and $C$ are some constants independent of $\varepsilon$. 

Choose \( k = \lfloor \frac{1}{4} c_2 \delta^{-1/m} \rfloor + 1 \) and \( t_i = \delta + 2ci\delta, \ i = 0, 1, 2, ..., k \). Then, (2.16), together with \( s = t_{i+1} \) and \( t = t_i \), yields that
\[
F(t_i) \leq \frac{1}{4} F(t_{i+1}) + C(i + 1)^{n-1} \delta^{n+2/\delta}.
\]
By \( k \) iterations and utilizing (2.13), we deduce that for a sufficiently small \( \varepsilon > 0 \),
\[
F(t_0) \leq C\delta^{n+2/\delta}.
\]

**Part 3.** Proof of
\[
|\nabla w_1(x)| \leq C\delta^{1-\frac{2}{\delta}}, \quad \text{in } \Omega_R. \tag{2.17}
\]

By using a change of variables in \( \Omega_{\delta}(z') \) as follows:
\[
\begin{align*}
x’ - z’ &= \delta y’, \\
x_n &= \delta y_n,
\end{align*}
\]
we rescale \( \Omega_{\delta}(z’) \) into \( Q_1 \), where, for \( 0 < r \leq 1 \),
\[
Q_r = \left\{ y \in \mathbb{R}^n \left| \frac{1}{\delta} h_2(\delta y' + z') < y_n < \frac{\varepsilon}{\delta} + \frac{1}{\delta} h_1(\delta y' + z'), |y'| < r \right. \right\}. \tag{2.18}
\]

We denote the top and bottom boundaries of \( Q_r \) by
\[
\hat{\Gamma}_r^+ = \left\{ y \in \mathbb{R}^n \left| y_n = h_1(y') := \frac{\varepsilon}{\delta} + \frac{1}{\delta} h_1(\delta y' + z'), |y'| < r \right. \right\},
\]
and
\[
\hat{\Gamma}_r^- = \left\{ y \in \mathbb{R}^n \left| y_n = h_2(y') := \frac{1}{\delta} h_2(\delta y' + z'), |y'| < r \right. \right\},
\]
respectively. Then
\[
\hat{h}_1(0') - \hat{h}_2(0') = 1,
\]
and in view of condition (H2), we have
\[
|\nabla_{z'}^{i-j} \hat{h}_1(0')| \leq C|z'|^{m-j}, \quad i, j = 1, 2.
\]

Since \( R \) is a small positive constant, \( ||\hat{h}_i||_{C^{1,1}((-1,1)^{n-1})} \), \( i = 1, 2 \), are small and thus \( Q_1 \) is of nearly unit size as far as applications of Sobolev embedding theorems and classical \( L^p \) estimates for elliptic equations are concerned.

Let
\[
W_1(y', y_n) = w_1(\delta y' + z', \delta y_n), \quad \bar{U}_1(y', y_n) = u_1(\delta y' + z', \delta y_n).
\]
Thus, \( W_1(y) \) satisfies
\[
\begin{align*}
\Delta W_1 &= -\Delta \bar{U}_1, \quad \text{in } Q_1, \\
W_1 &= 0, \quad \text{on } \hat{\Gamma}_1^+.
\end{align*}
\]

Due to the fact that \( W_1 = 0 \) on \( \hat{\Gamma}_1^+ \), it follows from the Poincaré inequality that
\[
||W_1||_{H^1(Q_1)} \leq C||\nabla W_1||_{L^2(Q_1)}.
\]

Combining the Sobolev embedding theorem and classical \( W^{2,p} \) estimates for elliptic systems, we obtain that for some \( p > n \),
\[
||\nabla W_1||_{L^\infty(Q_{1/2})} \leq C||W_1||_{W^{2,p}(Q_{1/2})} \leq C \left( ||\nabla W_1||_{L^2(Q_1)} + ||\Delta \bar{U}_1||_{L^\infty(Q_1)} \right).
\]

Then rescaling back to \( w_1 \) and \( \bar{u}_1 \), we have
\[
||\nabla w_1||_{L^\infty(\Omega_{1/2}(z'))} \leq \frac{C}{\delta} \left( \delta^{1-\frac{2}{\delta}} ||\nabla w_1||_{L^2(\Omega_{1}(z'))} + \delta^2 ||\Delta \bar{u}_1||_{L^\infty(\Omega_{1}(z'))} \right).
\]
Making use of (2.10) and (2.13), we deduce that for \(|z'| \leq R\),
\[
\delta^{-\frac{n}{2}} \|\nabla w_1\|_{L^2(\Omega_\delta(z'))} \leq C \delta^{1 - \frac{n}{4}}, \quad \delta \|\Delta \tilde{u}_1\|_{L^\infty(\Omega_\delta(z'))} \leq C \delta^{1 - \frac{n}{4}}.
\]
Therefore, for \(h(z') < z_n < \varepsilon + h_1(z')\),
\[
|\nabla w_1(z', z_n)| \leq C \delta^{1 - \frac{n}{2}}.
\]
That is, (2.17) is proved. On the other hand, by utilizing the standard interior and boundary estimates for the Laplace equation, we obtain
\[
\|\nabla v_1\|_{L^\infty(\Omega \setminus \Omega_R)} \leq C.
\]
Then, we complete the proof of Theorem 2.1.

Similarly as in [25], it turns out that \(\nabla v_0\) and \(\nabla (v_1 + v_2)\) possess exponentially decaying property as follows.

**Theorem 2.2.** Assume as above. Let \(v_i \in H^1(\Omega), i = 0, 1, 2\), be the weak solutions of (2.2) and (2.3), respectively. Then, for a sufficiently small \(\varepsilon > 0\),
\[
|\nabla v_0| + |\nabla (v_1 + v_2)| \leq C\delta^{-\frac{n}{2}} e^{-\frac{1}{2}C\delta^{1 - \frac{1}{m}}} \quad \text{in } \Omega_R.
\]

For readers’ convenience, the detailed proof of Theorem 2.2 is left in the Appendix below. Note that \(u = C_i\) on \(\partial D_i\) and \(\|u\|_{H^1(\Omega)} \leq C\) (independent of \(\varepsilon\)), we see from the trace embedding theorem that
\[
|C_1| + |C_2| \leq C. \quad (2.19)
\]

In light of Theorem 2.2 and (2.19), decomposition (2.8) becomes
\[
\nabla u = \frac{Q[\varphi]}{\Theta} \rho_{n,m}(\varepsilon) \nabla v_1 + O(1)\delta^{-\frac{n}{2}} e^{-\frac{1}{2}C\delta^{1 - \frac{1}{m}}}.
\]

Thus, to prove Theorem 1.1, it suffices to establish the following two aspects of asymptotic expansions due to decomposition (2.20) and Theorem 2.1:

(i) Asymptotic of \(Q[\varphi]\);

(ii) Asymptotic of \(\Theta\).

3. **Proof of Theorem 1.1.**

3.1. **Asymptotic of \(Q[\varphi]\).** To obtain the asymptotic of \(Q[\varphi]\), we need to expand \(b_i\) and \(Q_i, i = 1, 2\), respectively.

**Lemma 3.1.** Assume as in Theorem 1.1. Then, for a sufficiently small \(\varepsilon > 0\),
\[
b_i = b_i^* + O(1)\|\varphi\|_{C^0(\partial D)}\varepsilon^{\frac{1}{2}}, \quad Q_i = Q_i^* + O(1)\varepsilon^{\frac{1}{2}}, \quad i = 1, 2. \quad (3.1)
\]

Consequently,
\[
Q[\varphi] = Q^*[\varphi] + O(1)\|\varphi\|_{C^0(\partial D)}\varepsilon^{\frac{1}{2}},
\]
where \(b_i^*, Q_i^*\) and \(Q^*[\varphi]\) are defined by (1.5) and (1.9), respectively.

**Proof.** Take \(b_1\) for example. Other cases are the same. In view of the definitions of \(b_1\) and \(b_1^*\) and making use of integration by parts, we get
\[
b_1 - b_1^* = \int_{\partial D} \frac{\partial(v_1 - v_1^*)}{\partial\nu} \cdot \varphi(x).
\]
Introduce an auxiliary function $\tilde{u}_1^* \in C^2(\mathbb{R}^n)$ such that $\tilde{u}_1^* = 1$ on $\partial D_1^*$, $\tilde{u}_1^* = 0$ on $\partial D_2 \cup \partial D$ and

$$\tilde{u}_1^*(x) = \frac{x_n - h_2(x')}{h_1(x') - h_2(x')}, \quad \text{in } \Omega^*_R, \quad \|\tilde{u}_1^*\|_{C^2(\Omega^* \setminus \Omega^*_R)} \leq C, \quad (3.2)$$

where $\Omega^*_R := \Omega^* \cap \{|x'| < r\}, \ 0 < r \leq 2R$. Then applying Theorem 2.1 to (1.7), we obtain that for $x \in \Omega^*_R$,

$$|\nabla (v_1^* - \tilde{u}_1^*)| \leq C|x'|^{m-2}, \quad |\nabla x'v_1^*| \leq \frac{C}{|x'|}, \quad |\partial_{x_n} v_1^*| \leq \frac{C\varepsilon}{|x'|^m}. \quad (3.3)$$

It follows from ($H2$) that for $x \in \Omega^*_R$,

$$|\nabla x'(\tilde{u}_1 - \tilde{u}_1^*)| \leq \frac{C}{|x'|}, \quad |\partial_{x_n}(\tilde{u}_1 - \tilde{u}_1^*)| \leq \frac{C\varepsilon}{|x'|^m(\varepsilon + |x'|^m)}. \quad (3.4)$$

For $0 < r < R$, define

$$C_r := \left\{ x \in \mathbb{R}^n \mid |x'| < r, \ 2 \min_{|x'| \leq r} h_2(x') \leq x_n \leq \varepsilon + 2 \max_{|x'| \leq r} h_1(x') \right\} .$$

Note that $v_1 - v_1^*$ solves

$$\begin{align*}
\Delta(v_1 - v_1^*) &= 0, \quad \text{in } D \setminus (D_1 \cup D_2), \\
v_1 - v_1^* &= 1 - v_1^*, \quad \text{on } \partial D_1 \setminus D_1^*, \\
v_1 - v_1^* &= v_1 - 1, \quad \text{on } D_1^* \setminus (D_1 \cup \{0\}), \\
v_1 - v_1^* &= 0, \quad \text{on } \partial D_2 \cup \partial D.
\end{align*}$$

Recalling the definition of $v_1^*$, we have

$$|\partial_{x_n} v_1^*| \leq C, \quad \text{in } \Omega^* \setminus \Omega^*_R.$$ 

Then,

$$|v_1 - v_1^*| \leq C\varepsilon, \quad \text{for } x \in \partial D_1 \setminus D_1^*. \quad (3.5)$$

Due to (2.9), we deduce that for $x \in \partial D_1^* \setminus (D_1 \cup C_{\varepsilon^{\frac{1}{mn}}})$,

$$|v_1 - v_1^*| \leq C\varepsilon^{\frac{1}{2}}. \quad (3.6)$$

It follows from Theorem 2.1 and (3.3)–(3.4) that for $x \in \Omega^*_R \cap \{|x'| = \varepsilon^{\frac{1}{nm}}\}$,

$$|\partial_{x_n}(v_1 - v_1^*)| \leq |\partial_{x_n}(v_1 - \tilde{u}_1)| + |\partial_{x_n}(\tilde{u}_1 - \tilde{u}_1^*)| + |\partial_{x_n}(v_1^* - \tilde{u}_1^*)| \leq C,$$

which, together with $v_1 - v_1^* = 0$ on $\partial D_2$, leads to that

$$|(v_1 - v_1^*)(x', x_n)| = |(v_1 - v_1^*)(x', x_n) - (v_1 - v_1^*)(x', h_2(x'))| \leq C\varepsilon^{\frac{1}{2}}. \quad (3.7)$$

In view of $v_1 - v_1^* = 0$ on $\partial D_2 \cup \partial D$ and (3.5)–(3.7), we obtain

$$|v_1 - v_1^*| \leq C\varepsilon^{\frac{1}{2}}, \quad \text{on } \partial(D \setminus (D_1 \cup D_1^* \cup D_2 \cup C_{\varepsilon^{\frac{1}{mn}}})).$$

By utilizing the maximum principle, we derive

$$|v_1 - v_1^*| \leq C\varepsilon^{\frac{1}{2}}, \quad \text{in } D \setminus (D_1 \cup D_1^* \cup D_2 \cup C_{\varepsilon^{\frac{1}{mn}}}), \quad (3.8)$$

which, in combination with the standard interior and boundary estimates, yields that

$$|\nabla (v_1 - v_1^*)| \leq C\varepsilon^{\frac{1}{2}}, \quad \text{on } \partial D.$$
Proof. where

\[ a_{3.2}. \]

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which verifies that (3.1) holds. \( \square \)

3.2. Asymptotic of \( \Theta \). For later use we introduce a notation as follows:

\[
\bar{\varepsilon} = \begin{cases} 
\varepsilon^{\frac{2}{m}}, & m > n - 1 + \beta, \\
\varepsilon^{\frac{m}{n}} \ln \varepsilon, & m = n - 1 + \beta, \\
\varepsilon^{1 - \frac{1}{m}}, & n - 1 < m < n - 1 + \beta, \\
\ln \varepsilon^{-1}, & m = n - 1.
\end{cases}
\]  

(3.9)

Lemma 3.2. Assume as in Theorem 1.1. Then, for a sufficiently small \( \varepsilon > 0 \),

(i) for \( m \geq n - 1 \),

\[ a_{11} = \frac{\mathcal{M}(1 + O(\bar{\varepsilon}))}{\rho_{m,n}(\varepsilon)}; \]  

(ii) for \( m < n - 1 \),

\[ a_{11} = a_{11}^* + O(1) \varepsilon^{\min\left(\frac{1}{m}, \frac{1}{m-1}\right)}; \]  

where \( a_{11}^* \) is defined by (1.8), \( \mathcal{M} \) is defined by (1.10), and \( \bar{\varepsilon} \) is defined by (3.9).

Proof. Fix \( \gamma = \frac{1}{12m} \). To begin with, \( a_{11} \) can be split into three parts as follows.

\[ a_{11} = \int_{\Omega_{\gamma}} |\nabla v_1|^2 + \int_{\Omega_{\gamma}} |\nabla v_1|^2 + \int_{\Omega_{\gamma}} |\nabla v_1|^2 =: I + II + III. \]

Step 1. With regard to \( I \), it follows from the definition of \( \bar{u} \) and Theorem 2.1 that

\[ I = \int_{\Omega_{\gamma}} |\partial_{x_\gamma} \bar{u}_1|^2 + \int_{\Omega_{\gamma}} |\partial_{x_\gamma} \bar{u}_1|^2 + 2 \int_{\Omega_{\gamma}} \nabla \bar{u}_1 \cdot \nabla (v_1 - \bar{u}_1) + \int_{\Omega_{\gamma}} |\nabla (v_1 - \bar{u}_1)|^2 \]

\[ = \int_{|x'| < 2\bar{\varepsilon} + h_1(x') - h_2(x')} \frac{dx'}{\varepsilon + h_1(x') - h_2(x')} + O(1) \varepsilon^{(n+m-3)\gamma}. \]  

As for \( II \), we further split it as follows.

\[ \Pi_1 = \int_{(\Omega_R\setminus\Omega_{\gamma})\setminus(\Omega_R^*\setminus\Omega_{\gamma}^*)} |\nabla v_1|^2 + \int_{(\Omega_R\setminus\Omega_{\gamma})\setminus(\Omega_R^*\setminus\Omega_{\gamma}^*)} |\nabla (v_1 - v_1^*)|^2 + 2 \int_{(\Omega_R\setminus\Omega_{\gamma})\setminus(\Omega_R^*\setminus\Omega_{\gamma}^*)} \nabla v_1^* \cdot \nabla (v_1 - v_1^*), \]

\[ \Pi_2 = \int_{\Omega_R^*\setminus\Omega_{\gamma}^*} |\nabla v_1|^2. \]

For \( \varepsilon^{\frac{1}{12m}} \leq |z'| \leq R \), by utilizing a change of variable as follows:

\[ \begin{cases} 
x' = x', \\
x_n = |z'|^m y_n,
\end{cases} \]

we rescale \( \Omega_{|z'|+|z'|^m} \setminus \Omega_{|z'|} \) and \( \Omega_{|z'|+|z'|^m} \setminus \Omega_{|z'|}^* \) into two nearly unit-size squares (or cylinders) \( Q_1 \) and \( Q_1^* \), respectively. Let

\[ V_1(y) = v_1(z' + |z'|^m y', |z'|^m y_n), \text{ in } Q_1, \]

and

\[ V_1^*(y) = v_1^*(z' + |z'|^m y', |z'|^m y_n), \text{ in } Q_1^*. \]
Since $0 < V_1, V_1^* < 1$, from the standard elliptic estimate, we obtain
\[
|\nabla^2 V_1| \leq C, \quad \text{in } Q_1, \quad \text{and } |\nabla^2 V_1^*| \leq C, \quad \text{in } Q_1^*.
\]
Applying an interpolation with (3.8), we deduce
\[
|\nabla (V_1 - V_1^*)| \leq C\varepsilon^{\frac{1}{4}(1 - \frac{1}{2})} \leq C\varepsilon^{\frac{1}{4}}.
\]
Then back to $v_1 - v_1^*$ and in light of $\varepsilon^{\frac{1}{4}} |z'| \leq R$, we obtain
\[
|\nabla (v_1 - v_1^*)(x)| \leq C\varepsilon^{\frac{1}{4}} |z'|^{-m} \leq C\varepsilon^{\frac{1}{4}}, \quad x \in \Omega_1^* \cup |z'|^m \setminus \Omega_1^*.
\]
Therefore,
\[
|\nabla (v_1 - v_1^*)| \leq C\varepsilon^{\frac{1}{4}}, \quad \text{in } D \setminus (D_1 \cup D_1^* \cup D_2 \cup C \varepsilon^{\frac{1}{4}}).
\] (3.13)
In view of the fact that the thickness of $(\Omega_R \setminus \Omega_{\varepsilon^{\gamma}}) \setminus (\Omega_R^* \setminus \Omega_{\varepsilon^{\gamma}})$ is $\varepsilon$ and making use of (3.13), we derive
\[
|\Pi_1| \leq C\varepsilon^{\frac{1}{4}}. \quad (3.14)
\]
For $\Pi_2$, it follows from (3.3) that
\[
\Pi_2 = \int_{\Omega_R \setminus \Omega_{\varepsilon^{\gamma}}} |\nabla u_1^*|^2 + 2 \int_{\Omega_R \setminus \Omega_{\varepsilon^{\gamma}}} \nabla u_1^* \cdot \nabla (v_1^* - \bar{u}_1^*) + \int_{\Omega_R \setminus \Omega_{\varepsilon^{\gamma}}} |\nabla (v_1^* - \bar{u}_1^*)|^2
\]
\[
= \int_{\varepsilon^{\gamma} < |x'| < R} \frac{dx'}{h_1(x') - h_2(x')} + A_R + \int_{\Omega_R \setminus \Omega_{R}^*} |\nabla v_1^*|^2 + O(1)\varepsilon^{\min\left(\frac{1}{4}, \frac{a + m - 3}{4m}\right)},
\]
where
\[
A_R := \int_{\Omega_R \setminus \Omega_R^*} |\nabla v_1^*|^2 + 2 \int_{\Omega_R} \nabla u_1^* \cdot \nabla (v_1^* - \bar{u}_1^*)
\]
\[
+ \int_{\Omega_R \setminus \Omega_R^*} (|\nabla (v_1^* - \bar{u}_1^*)|^2 + |\partial_x' u_1^*|^2).
\] (3.15)
Hence, this, together with (3.14), reads that
\[
\Pi_2 = \int_{\varepsilon^{\gamma} < |x'| < R} \frac{dx'}{h_1(x') - h_2(x')} + A_R + \int_{\Omega_R \setminus \Omega_R^*} |\nabla v_1^*|^2 + O(1)\varepsilon^{\min\left(\frac{1}{4}, \frac{a + m - 3}{4m}\right)}. \quad (3.16)
\]
As for the last term $\Pi_3$, since $|\nabla v_1|$ is bounded in $D_1^* \setminus (D_1 \cup \Omega_R)$ and $D_1 \setminus D_1^*$ and the volume of $D_1^* \setminus (D_1 \cup \Omega_R)$ and $D_1 \setminus D_1^*$ is of order $O(\varepsilon)$, it follows from (3.8) again that
\[
\Pi_3 = \int_{\Omega \setminus (D_1 \cup D_1^* \cup D_2 \cup \Omega_R)} |\nabla v_1|^2 + O(1)\varepsilon
\]
\[
= \int_{\Omega \setminus (D_1 \cup D_1^* \cup D_2 \cup \Omega_R)} |\nabla v_1|^2 + 2 \int_{\Omega \setminus (D_1 \cup D_1^* \cup D_2 \cup \Omega_R)} \nabla v_1^* \cdot \nabla (v_1 - v_1^*)
\]
\[
+ \int_{\Omega \setminus (D_1 \cup D_1^* \cup D_2 \cup \Omega_R)} |\nabla (v_1 - v_1^*)|^2 + O(1)\varepsilon
\]
\[
= \int_{\Omega_R \setminus \Omega_R^*} |\nabla v_1^*|^2 + O(1)\varepsilon^{\frac{1}{4}},
\]
which, together with (3.14) and (3.16), leads to that
\[
\alpha_{11} = \int_{\varepsilon^{\gamma} < |x'| < R} \frac{dx'}{h_1(x') - h_2(x')} + \int_{|x'| < \varepsilon^{\gamma}} \frac{dx'}{\varepsilon + h_1(x') - h_2(x')}
\]
\[
+ A_R + O(1)\varepsilon^{\min\left(\frac{1}{4}, \frac{a + m - 3}{4m}\right)}, \quad (3.17)
\]
Step 2. We now calculate
\[ \int_{\varepsilon^2 < |x'|^2 < R} \frac{dx'}{h_1(x') - h_2(x')} + \int_{|x'| < \varepsilon} \frac{dx'}{\varepsilon + h_1(x') - h_2(x')} \]
(i) For \( m \geq n - 1 \),
\[ \int_{|x'| < R} \frac{1}{\varepsilon + h_1 - h_2} + \int_{|x'| < \varepsilon} \frac{1}{\varepsilon + (h_1 - h_2)(\varepsilon + h_1 - h_2)} \]
\[ = \int_{|x'| < R} \frac{1}{\varepsilon + |x'|^m} + \int_{|x'| < \varepsilon} \frac{1}{\varepsilon + h_1 - h_2 - \frac{1}{\varepsilon + |x'|^m} + O(1) \varepsilon^{\frac{m+n-1}{2(m+n)}}} \]
\[ = (n-1)\omega_{n-1} \int_0^R \frac{s^{n-2}}{\varepsilon + \lambda s^m} + O(1) \int_0^R \frac{s^{n-2+\beta}}{\varepsilon + \lambda s^m} \]
\[ = \mathcal{M}(1 + O(\tilde{r}_\varepsilon)) \]
where \( \mathcal{M} \) is defined by (1.10) and \( \tilde{r}_\varepsilon \) is defined by (3.9). That is, (3.10) holds.
(ii) For \( m < n - 1 \),
\[ \int_{|x'| < R} \frac{dx'}{h_1 - h_2} - \int_{|x'| < \varepsilon} (h_1 - h_2)(\varepsilon + h_1 - h_2) \]
\[ = \int_{|x'| < R} |\partial_x \tilde{u}_1|^2 + O(1) \varepsilon^{\frac{n+1-m}{2(m+n)}} \]
which, in combination with (3.17), yields that (3.11) holds.

A direct application of Lemma 3.1 and Lemma 3.2 yields the following expansion.

Lemma 3.3. Assume as in Theorem 1.1. Then, for a sufficiently small \( \varepsilon > 0 \),
(i) for \( m \geq n - 1 \),
\[ \Theta = \Theta^* + O(r_\varepsilon); \]
(ii) for \( m < n - 1 \),
\[ \Theta = \mathcal{S}^* + O(r_\varepsilon), \]
where \( \Theta^* \) and \( \mathcal{S}^* \) are defined in (1.9), and \( r_\varepsilon \) is defined by (1.11).

Proof. To begin with, we consider the case of \( m \geq n - 1 \). Recalling the definition of \( \Theta \) defined in (2.7), we see from (2.4), (3.1) and (3.10) that
\[ \Theta = \rho_{n,m}(\varepsilon)(-a_{11}Q_2 + a_{12}Q_1) = -\rho_{n,m}(\varepsilon)(a_{11}(Q_1 + Q_2) + (Q_1)^2) \]
\[ = -\mathcal{M}(1 + O(\tilde{r}_\varepsilon))(Q_1 + Q_2) - \rho_{n,m}(\varepsilon)(Q_1)^2 \]
\[ = -\mathcal{M}(1 + O(\tilde{r}_\varepsilon))(Q_1 + Q_2)(1 + O(\varepsilon^{\frac{1}{2}})) - \rho_{n,m}(\varepsilon)(Q_1)^2 + O(\varepsilon^{\frac{3}{2}})) \]
\[ = \Theta^*(1 + O(r_\varepsilon)), \]
where the constant \( \mathcal{M} \) is defined by (1.10), the blow-up factor \( \Theta^* \) is defined in (1.9), the rest terms \( r_\varepsilon \) and \( \tilde{r}_\varepsilon \) are defined by (1.11) and (3.9), respectively. That is, (3.18) holds. Similarly, for \( m < n - 1 \), it follows from (3.1) and (3.11) that (3.19) holds.

We now claim that \( \Theta^* \neq 0 \) and \( \mathcal{S}^* \neq 0 \). In fact, we see from the Hopf Lemma that
\[ \left| \frac{\partial v_1^*}{\partial \nu} \right|_{D_1 \setminus \{0\}} > 0, \quad \left| \frac{\partial v_2^*}{\partial \nu} \right|_{D_1 \setminus \{0\}} < 0, \quad \left| \frac{\partial v_2^*}{\partial \nu} \right|_{\partial D_1 \setminus \{0\}} < 0, \quad \left| \frac{\partial v_2^*}{\partial \nu} \right|_{\partial D} < 0. \]
Then, we have
\[ \Theta^* = -\mathcal{M} \left( \int_{\partial D} \frac{\partial v_1^*}{\partial \nu} + \int_{\partial D} \frac{\partial v_2^*}{\partial \nu} \right) > 0, \]
and
\[ \mathcal{G}^* = -\int_{\partial D^*_1} \frac{\partial v_1^*}{\partial v} \int_{\partial D^*_1} \frac{\partial v_2^*}{\partial v} + \int_{\partial D^*_1} \frac{\partial v_2^*}{\partial v} \int_{\partial D^*_1} \frac{\partial v_1^*}{\partial v} > 0, \]
where the constant \( \mathcal{M} \) is defined by (1.10).

**Proof of Theorem 1.1.** In light of decomposition (2.20), we see from Theorem 2.1, Lemmas 3.1 and 3.3 that for \( m \geq n - 1 \),
\[ \frac{Q[\varphi]}{\Theta} = \frac{Q^*[\varphi]}{\Theta^*} \left( 1 + \frac{1}{\Theta^*} \right) + \frac{Q[\varphi] - Q^*[\varphi]}{\Theta} = \frac{Q^*[\varphi]}{\Theta^*} (1 + O(r_\varepsilon)), \]
which yields that
\[ \nabla u = \frac{Q[\varphi]}{\Theta} \rho_{n,m}(\varepsilon) \nabla v_1 + O(1) \delta^{-\frac{2}{n}}e^{-\frac{1}{2c_1\varepsilon}} \]
\[ \frac{Q^*[\varphi]}{\Theta^*}(1 + O(\varepsilon^2))\rho_{n,m}(\varepsilon)(\nabla \bar{u}_1 + O(\delta^{m-2})) + O(1) \delta^{-\frac{2}{n}}e^{-\frac{1}{2c_1\varepsilon}} \]
\[ \frac{Q^*[\varphi]}{\Theta^*}(1 + O(\varepsilon^2))\rho_{n,m}(\varepsilon)\nabla \bar{u}_1 + O(1) \delta^{-\frac{2}{n}}\|\varphi\|_{C^0(\partial D)}; \]
for \( m < n - 1 \), we have
\[ \frac{Q[\varphi]}{\Theta} = \frac{Q^*[\varphi]}{\Theta^*} \left( 1 + \frac{1}{\Theta^*} \right) + \frac{Q[\varphi] - Q^*[\varphi]}{\Theta} = \frac{Q^*[\varphi]}{\Theta^*} (1 + O(\varepsilon^2)), \]
and thus,
\[ \nabla u = \frac{Q[\varphi]}{\Theta} \nabla v_1 + C_2 \nabla (v_1 + v_2) + \nabla v_0 \]
\[ \frac{Q^*[\varphi]}{\Theta^*}(1 + O(\varepsilon^2))\nabla \bar{u}_1 + O(1) \delta^{-\frac{2}{n}}e^{-\frac{1}{2c_1\varepsilon}} \]
\[ \frac{Q^*[\varphi]}{\Theta^*}(1 + O(\varepsilon^2))\nabla \bar{u}_1 + O(1) \delta^{-\frac{2}{n}}\|\varphi\|_{C^0(\partial D)}, \]
where \( \Theta^* \) and \( \Theta^* \) are defined in (1.9). Consequently, we complete the proof of (3.18)–(3.19).

### 4. Proof of Corollary 1.5.
We first note that by utilizing Taylor expansion, we have
\[ h_1(x') - h_2(x') = \lambda_0 |x'|^2 + O(|x'|^4), \quad \text{in } \Omega_{r_0}, \quad (4.1) \]
where \( \lambda_0 = \frac{1}{2} \left( \frac{1}{r_1^2} + \frac{1}{r_2^2} \right) \) and \( 0 < r_0 < \min\{r_1, r_2\}, \) \( r_0 \) is a small constant independent of \( \varepsilon. \)

**Lemma 4.1.** Assume as in Corollary 1.5. Then, for a sufficiently small \( \varepsilon > 0, \)
\[ a_{11} = \frac{\mathcal{M}}{\rho_{n,2}(\varepsilon)} + \mathcal{K}_n + O(1)\varepsilon^{\frac{n-1}{m}}, \]
where \( \mathcal{M} \) is defined by (1.10), \( \mathcal{K}_n \) is defined by (4.7) below.

**Proof.** Similar to (3.15), \( \mathcal{K}_n \) is defined by (4.7) below.
where
\[ A_{\rho_0}^* := \int_{\Omega \setminus \Omega_{\rho_0}} |\nabla v_1|^2 + 2 \int_{\Omega_{\rho_0}} \nabla u_1 \cdot \nabla (v_1^* - u_1^*) + \int_{\Omega_{\rho_0}} (|\nabla (v_1^* - u_1^*)|^2 + |\nabla x \cdot u_1^*|^2). \]

First, it follows from (4.1) that
\[ \int_{\xi < |x'| < \rho_0} \left( \frac{1}{h_1 - h_2} - \frac{1}{\lambda_0|x'|^2} \right) dx' = \int_{\xi < |x'| < \rho_0} O(1)dx' = C^* + O(1)\varepsilon^{\frac{n}{24}}, \]
where \( C^* \) depends on \( \lambda_0, n, \rho_0 \), but not on \( \varepsilon \). Then
\[ \int_{\xi < |x'| < \rho_0} \frac{dx'}{\varepsilon + h_1 - h_2} = \int_{\xi < |x'| < \rho_0} \frac{dx'}{\varepsilon + \lambda_0|x'|^2} + C^* + O(1)\varepsilon^{\frac{n}{24}}. \]

Similarly, we derive
\[ \int_{|x'| < \varepsilon} \frac{dx'}{\varepsilon + h_1 - h_2} = \int_{|x'| < \varepsilon} \frac{dx'}{\varepsilon + \lambda_0|x'|^2} + O(1)\varepsilon^{\frac{n}{24}}. \]

Then the energy \( a_{11} \) becomes
\[ a_{11} = \int_{\xi < |x'| < \rho_0} \frac{dx'}{\lambda_0|x'|^2} + \int_{|x'| < \varepsilon} \frac{dx'}{\lambda_0|x'|^2} + M_{\rho_0}^* + O(1)\varepsilon^{\frac{n}{24}}, \quad (4.3) \]
where \( M_{\rho_0}^* = A_{\rho_0}^* + C^* \). For the purpose of expanding the energy \( a_{11} \), it suffices to calculate the major singular part as follows:
\[ \int_{\xi < |x'| < \rho_0} \frac{dx'}{\lambda_0|x'|^2} + \int_{|x'| < \varepsilon} \frac{dx'}{\lambda_0|x'|^2}. \]

(i) if \( n = 2 \), then
\[ \int_{\xi < |x'| < \rho_0} \frac{dx_1}{\lambda_0|x_1|^2} + \int_{|x'| < \varepsilon} \frac{dx_1}{\varepsilon + \lambda_0|x_1|^2} = \int_{-\infty}^{\infty} \frac{1}{\varepsilon + \lambda_0|x_1|^2} - \int_{|x_1| > \rho_0} \frac{dx_1}{\lambda_0|x_1|^2} + \int_{|x_1| > \varepsilon} \frac{\varepsilon}{\varepsilon + \lambda_0|x_1|^2} = \frac{\pi}{\sqrt{\lambda_0 \sqrt{\varepsilon}}} - \frac{2}{\lambda_0 \rho_0} + O(1)\varepsilon^{\frac{n}{24}}; \quad (4.4) \]

(ii) if \( n = 3 \), then
\[ \int_{\xi < |x'| < \rho_0} \frac{dx'}{\lambda_0|x'|^2} + \int_{|x'| < \varepsilon} \frac{dx'}{\varepsilon + \lambda_0|x'|^2} = \int_{|x'| < \rho_0} \frac{\rho_0^{n-2}}{\varepsilon + \lambda_0|x'|^2} + \int_{|x'| < \varepsilon} \frac{\rho_0^{n-2}(\varepsilon + \lambda_0|x'|^2)}{\varepsilon + \lambda_0|x'|^2} = \pi \int_0^{\rho_0} \frac{s^{n-2}}{\varepsilon + \lambda_0 s^2} + O(1)\varepsilon^{\frac{n}{24}} = \frac{\pi}{\lambda_0} \ln \varepsilon + \frac{\pi (\ln \lambda_0 + 2 \ln \rho_0)}{\lambda_0} + O(1)\varepsilon^{\frac{n}{24}}; \quad (4.5) \]

Then combining (4.2)–(4.5), we obtain
\[ a_{11} = \frac{M}{\rho_{n,2}^*} + K_n + O(1)\varepsilon^{\frac{n}{24}}; \quad (4.6) \]
where \( M \) is defined by (1.15) and
\[ K_n = \begin{cases} M_{\rho_0}^* - \frac{2}{\lambda_0 \rho_0}, & n = 2, \\ M_{\rho_0}^* + \frac{2}{\pi (\ln \lambda_0 + 2 \ln \rho_0)}, & n = 3. \end{cases} \quad (4.7) \]
We now claim that the constant $K_n$ is independent of $r_0$. If not, suppose that there exist $K_n(r_1^*)$ and $K_n(r_2^*)$, $r_i^* > 0$, $i = 1, 2$, $r_1^* \neq r_2^*$, both independent of $\varepsilon$, such that (4.6) holds. Then, we derive

$$K_n(r_1^*) - K_n(r_2^*) = O(\varepsilon^{\frac{n-1}{n+1}},$$

which reads that $K_n(r_1^*) = K_n(r_2^*)$.

The proof of Corollary 1.5. It follows from (2.4), Lemmas 3.1 and 4.1 that

$$\Theta - \Theta^* = -\rho_{n,2}(\varepsilon)(a_{11}(Q_1 + Q_2) + (Q_1)^2) + M(Q_1^* + Q_2^*)$$

$$= -\rho_{n,2}(\varepsilon)(\kappa_1 + \rho_{n,2}(\varepsilon)) (Q_1^* + Q_2^*) - \rho_{n,2}(\varepsilon)\varepsilon \rho_{n,2}(\varepsilon))$$

$$= -\rho_{n,2}(\varepsilon) (\kappa_n + O(\varepsilon^{\frac{n-1}{n+1}})) (Q_1^* + Q_2^*) - \rho_{n,2}(\varepsilon)\varepsilon \rho_{n,2}(\varepsilon))$$

$$= -\rho_{n,2}(\varepsilon) (\kappa_n(Q_1^* + Q_2^*) + (Q_1^*)^2) + O(\varepsilon^{\frac{n-1}{n+1}} \rho_{n,2}(\varepsilon)).$$

Denote

$$\tilde{M}_n^* = -\frac{K_n}{M} + (Q_1^*)^2.$$ 

Thus,

$$\frac{Q[\varphi]}{\Theta} - \frac{Q^*[\varphi]}{\Theta^*} = \frac{Q^*[\varphi]}{\Theta^*} \frac{\Theta^* - \Theta}{\Theta^* - \Theta} + \frac{Q[\varphi] - Q^*[\varphi]}{\Theta}$$

$$= \frac{Q^*[\varphi]}{\Theta^*} \frac{\tilde{M}_n^* \rho_{n,2}(\varepsilon) + O(\varepsilon^{\frac{n-1}{n+1}} \rho_{n,2}(\varepsilon))}{1 - \tilde{M}_n^* \rho_{n,2}(\varepsilon) + O(\varepsilon^{\frac{n-1}{n+1}} \rho_{n,2}(\varepsilon))} + O(\varepsilon^{\frac{1}{n+1}})$$

$$= \frac{Q^*[\varphi]}{\Theta^*} \frac{\tilde{M}_n^* \rho_{n,2}(\varepsilon) + O(\varepsilon^{\frac{n-1}{n+1}} \rho_{n,2}(\varepsilon))}{1 - \tilde{M}_n^* \rho_{n,2}(\varepsilon) + O(\varepsilon^{\frac{n-1}{n+1}} \rho_{n,2}(\varepsilon))},$$

which indicates that

$$\frac{Q[\varphi]}{\Theta} = \frac{Q^*[\varphi]}{\Theta^*} \frac{1}{1 - \tilde{M}_n^* \rho_{n,2}(\varepsilon)} + O(\varepsilon^{\frac{n-1}{n+1}} \rho_{n,2}(\varepsilon)).$$

This, together with decomposition (2.20) and Theorem 2.1, yields that (1.16)–(1.17) holds. \qed

5. Appendix: The proof of Theorem 2.2. Take $|\nabla v_0|$ for instance. The proof of the decaying estimate of $|\nabla (v_1 + v_2)|$ is the same and thus omitted. For the sake of convenience, we set $\|\varphi\|_{C^0(\partial D)} = 1$.

\textbf{Step 1.} Let $v_0 \in H^1(\Omega)$ be a weak solution of (2.2). Then

$$\|\nabla v_0\|_{L^2(\Omega)} \leq C.$$ 

(5.1)

Multiplying the equation in (2.2) by $v_0$ and utilizing integration by parts, it follows from the standard elliptic theory that

$$\int_{\Omega} |\nabla v_0|^2 = \int_{\partial D} \frac{\partial v_0}{\partial \nu} \varphi \leq C.$$ 

\textbf{Step 2.} Proof of

$$\|\nabla v_0\|_{L^2(\Omega(\varepsilon^z))} \leq C_{\varepsilon} \frac{1}{2\varepsilon^z - 1 - m},$$ 

(5.2)
where $\delta$ is defined by (1.4). Consider the following boundary value problem:
\[
\begin{align*}
\Delta v_0 &= 0, \quad \text{in } \Omega_{2R}, \\
v_0 &= 0, \quad \text{on } \Gamma_{2R}^+ \cup \Gamma_{2R}^-,
\end{align*}
\]

where
\[
\Gamma_{2R}^+ = \{ x \in \mathbb{R}^n | x_n = \varepsilon + h_1(x'), |x'| < 2R \},
\]
\[
\Gamma_{2R}^- = \{ x \in \mathbb{R}^n | x_n = h_2(x'), |x'| < 2R \}.
\]

For $0 < t < s < R$, we introduce a smooth cutoff function $\eta$ such that $\eta = 1$ in $\Omega_t(z')$, $0 \leq \eta \leq 1$ in $\Omega_s(z') \setminus \Omega_t(z')$, $\eta = 0$ in $\Omega_{2R} \setminus \Omega_s(z')$ and $|\nabla \eta| \leq \frac{2}{s-t}$.

Multiplying $\eta^2 v_0$ on both sides of the equation in (5.3) and making use of integration by parts, we obtain
\[
\int_{\Omega_t(z')} |\nabla v_0|^2 dx \leq \frac{C}{(s-t)^2} \int_{\Omega_s(z')} |v_0|^2 dx.
\]

Similarly as before, for $|z'| \leq R$, $\delta < s \leq \frac{2}{3} \min\{\varepsilon, |z'|\}$, in view of the fact that $v_0 = 0$ on $\Gamma_{2R}^+$, we deduce that
\[
\int_{\Omega_s(z')} |v_0|^2 \leq C\delta^2 \int_{\Omega_s(z')} |\nabla v_0|^2.
\]

Then (5.4) becomes
\[
\int_{\Omega_t(z')} |\nabla v_0|^2 dx \leq \left( \frac{C\delta}{s-t} \right)^2 \int_{\Omega_s(z')} |\nabla v_0|^2 dx.
\]

Similarly as before, we denote
\[
F(t) := \int_{\Omega_t(z')} |\nabla v_0|^2.
\]

Thus, we have
\[
F(t) \leq \frac{c\delta}{s-t} \leq F(s).
\]

where $c$ is a positive constant independent of $\varepsilon$.

Let $k = \left[ \frac{1}{4s-t} \right] + 1$ and $t_i = \delta + 2ic\delta$, $i = 0, 1, 2, ..., k$. Then, (5.5), in combination with $s = t_i+1$ and $t = t_i$, reads that
\[
F(t_i) \leq \frac{1}{4} F(t_{i+1}).
\]

After $k$ iterations, we see from (5.1) that
\[
F(t_0) \leq \frac{1}{4k} G(t_k) \leq Ce^{-\frac{1}{c\varepsilon^{-1}}r/m}.
\]

**Step 3.** Proof of
\[
|\nabla v_0| \leq C\delta^{-\frac{n}{2}} e^{-\frac{1}{c\varepsilon^{-1}}r/m}, \quad \text{in } \Omega_R.
\]

Similarly as above, we let
\[
V_0(y', y_n) = v_0(\delta y' + z', \delta y_n), \quad y \in Q_1,
\]

where $Q_1$ is defined in (2.18). Then $V_0(y)$ satisfies
\[
\begin{align*}
\Delta V_0 &= 0, \quad \text{in } Q_1, \\
V_0 &= 0, \quad \text{on } \Gamma_{1}^+.
\end{align*}
\]
Analogously, it follows from the Poincaré inequality, the Sobolev embedding theorem and classical $W^{2,p}$ estimates for elliptic systems again that for some $p > n$,

$$\|\nabla V_0\|_{L^\infty(Q_{1/2})} \leq C \|V_0\|_{W^{2,p}(Q_{1/2})} \leq C\|\nabla V_0\|_{L^2(Q_1)}.$$

Rescaling back to $v_0$ and using (5.2), we obtain

$$\|\nabla v_0\|_{L^\infty(\Omega_{\delta/2}(z'))} \leq C \delta^{-\frac{n}{2}} \|\nabla v_0\|_{L^2(\Omega_{\delta}(z'))} \leq C \delta^{-\frac{n}{2}} e^{-\frac{1}{2C\delta^{1/1+m}}}.$$

This implies that for $z \in \Omega_R$,

$$|\nabla v_0(z)| \leq C \delta^{-\frac{n}{2}} e^{-\frac{1}{2C\delta^{1/1+m}}}.$$

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E-mail address: zwzhao@mail.bnu.edu.cn
E-mail address: xiahao@mail.bnu.edu.cn