STRICHARTZ ESTIMATES FOR THE SCHRÖDINGER PROPAGATOR IN WIENER AMALGAM SPACES

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ABSTRACT. In this paper we study the Strichartz estimates for the Schrödinger propagator in the context of Wiener amalgam spaces which, unlike the Lebesgue spaces, control the local regularity of a function and its decay at infinity separately. This separability makes it possible to perform a finer analysis of the local and global behavior of the propagator. Our results improve some of the classical ones in the case of large time.

1. INTRODUCTION

Consider the following Cauchy problem for the Schrödinger equation

\[
\begin{cases}
  i\partial_t u + \Delta u = 0, \\
  u(x, 0) = f(x),
\end{cases}
\]

with \((x, t) \in \mathbb{R}^n \times \mathbb{R}, n \geq 1\). Applying the Fourier transform to (1.1), the solution \(u(x, t)\) is given by

\[
e^{it\Delta}f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi - t|\xi|^2)} \hat{f}(\xi) d\xi.
\]

(1.2)

Here the Fourier multiplier \(e^{it\Delta}\) is called the Schrödinger propagator.

The following space-time integrability of (1.2) in \(L^p\) spaces has been intensively studied in the last forty years:

\[
\|e^{it\Delta}f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2}
\]

(1.3)

for \((q, r)\) Schrödinger admissible, i.e., for

\[
q, r \geq 2, \quad \frac{2}{q} + \frac{n}{r} = \frac{n}{2}, \quad (q, r, n) \neq (2, \infty, 2).
\]

(1.4)

See [13, 7, 10, 9] and references therein.

In this paper we consider these space-time estimates, known as Strichartz estimates, in Wiener amalgam spaces which, unlike the \(L^p\) spaces, control the local regularity of a function and its decay at infinity separately. This separability makes it possible to perform a finer analysis of the local and global behavior of the solution.
These spaces were first introduced by Feichtinger [4] and have already appeared as a technical tool in the study of partial differential equations ([14]).

To begin with, let us recall the definition of Wiener amalgam spaces. Let \( \varphi \in C_0^\infty \) be a test function satisfying \( \| \varphi \|_{L^2} = 1 \). Let \( 1 \leq p, q \leq \infty \). Then the Wiener amalgam space \( W(L^p, L^q) \) is defined as the space of functions \( f \in L^p_{loc} \) equipped with the norm

\[
\| f \|_{W(L^p, L^q)} = \left\| \| \tau_x \varphi \|_{L^p} \right\|_{L^q},
\]

where \( \tau_x \varphi(\cdot) = \varphi(\cdot - x) \). Here different choices of \( \varphi \) generate the same space and yield equivalent norms. The Wiener amalgam space can be also seen as a natural extension of \( L^p \) space in view of \( W(L^p, L^p) = L^p \). More generally, the Wiener amalgam space \( W(A, B) \) for Banach spaces \( A \) and \( B \) is defined in the same way.

In [3], Cordero and Nicola established the following estimates for Schrödinger admissible \( (q, r) \):

\[
\| e^{i t \Delta} f \|_{W(L^\infty, L^q), W(L^2, L^r)} \lesssim \| f \|_{L^2}.
\]  
(1.5)

By complex interpolation (see (2.4)) between (1.5) and (1.3), they obtained further estimates

\[
\| e^{i t \Delta} f \|_{W(L^{\tilde{q}}, L^q), W(L^{\tilde{r}}, L^r)} \lesssim \| f \|_{L^2}
\]  
(1.6)

for \( (\tilde{q}, \tilde{r}) \) and \( (q, r) \) satisfying \( 1 \leq \tilde{q}, \tilde{r} \leq \infty \), \( 2 \leq q, r \leq \infty \),

\[
\tilde{r} \leq r, \quad \frac{2}{\tilde{q}} + \frac{n}{r} \leq \frac{n}{2} \leq \frac{2}{\tilde{q}} + \frac{n}{\tilde{r}}.
\]

\( \tilde{r}, r \) \( < \infty \) if \( n = 2 \), and if \( n \geq 3 \), \( \tilde{r} \leq 2n/(n - 2) \). As mentioned in [3], these estimates say that the analysis of the local regularity of the Schrödinger propagator is quite independent of its decay at infinity since there are no relations between the pairs \( (\tilde{q}, \tilde{r}) \) and \( (q, r) \) other than \( \tilde{r} \leq r \). (See also [2, 11] for related results.)

Our goal in this paper is to provide a picture of the Strichartz estimates in Wiener amalgam spaces for the Schrödinger propagator on initial data with regularity. We attempt to obtain

\[
\| e^{i t \Delta} f \|_{W(L^{\tilde{q}}, L^q), W(L^{\tilde{r}}, L^r)} \lesssim \| f \|_{\dot{H}^\sigma}
\]  
(1.7)

with the homogeneous Sobolev norm \( \| f \|_{\dot{H}^\sigma} = \| \nabla^\sigma f \|_{L^2} \), \( \sigma > 0 \). In the case of the Lebesgue space estimates,

\[
\| e^{i t \Delta} f \|_{L^q_t L^\sigma_x} \lesssim \| f \|_{\dot{H}^\sigma},
\]  
(1.8)

where \( 0 < \sigma < n/2 \), \( q \geq 2 \) and

\[
\frac{2}{q} + \frac{n}{r} = \frac{n}{2} - \sigma,
\]

(1.9)

one can proceed by first considering initial data which are frequency localized to annuli and then use Littlewood-Paley theory to obtain the desired estimates for general data. It seems difficult to proceed in this way in the case of the generalized estimates (1.7). Here we bypass Littlewood-Paley theory to obtain directly the estimates (1.7). The key ingredient in our approach is the availability of estimates for the integral kernel of the Fourier multiplier \( e^{i t \Delta |\nabla|^{-\sigma}} \). Our main result is the following theorem.
Theorem 1.1. Let \( n \geq 1 \). Let \( 2 \leq \tilde{q} < q < \infty \), \( 2 \leq \tilde{r}, r \leq \infty \) and \( \max\{0, (n-2)/4\} < \sigma < n/2 \). Assume that \((\tilde{q}, \tilde{r})\) and \((q, r)\) satisfy
\[
\frac{2}{\tilde{q}} + \frac{n-1}{\tilde{r}} > \frac{n}{2} - \sigma \tag{1.10}
\]
and
\[
\frac{2}{q} + \frac{n}{r} = \frac{n}{2} - \sigma - \frac{n-1}{\tilde{r}}. \tag{1.11}
\]
Then we have
\[
\|e^{it\Delta}f\|_{W(L^{\tilde{q}}, L^{q})_{x}} \lesssim \|f\|_{\dot{H}^{\sigma}}. \tag{1.12}
\]

Remark 1.2. In particular, when \( \tilde{r} = \infty \), the condition (1.11) becomes the condition (1.9) and this case is then comparable to the classical estimate (1.8). Roughly speaking, the estimate (1.12) in this case shows that the \( W(L^\infty, L^r)_{x} \)-norm of the solution has a \( L^{q_t} \)-decay at infinity. Hence, our result is better than the classical one for large time since the classical \( L^{r} \)-norm in (1.8) is rougher than the \( W(L^\infty, L^r)_{x} \)-norm, although locally the classical \( L^{q_t} \) regularity is replaced by \( L^{\tilde{q}_t} \) with \( \tilde{q} < q \).

Remark 1.3. From complex interpolation (see (2.4)) between bilinear form estimates given from (1.12) and (1.6), we can obtain further estimates. See Section 4 for details. In a different way, one can also easily obtain further estimates by the interpolation between (1.12) and (1.8) with the same \( \sigma \). We omit the details. Finally, we can trivially increase \( q, r \) and diminish \( \tilde{q}, \tilde{r} \) in (1.12) by using the inclusion relation (see (2.1)) of Wiener amalgam spaces.

The outline of this paper is as follows: In Section 2 we prove Theorem 1.1 assuming Proposition 2.2 which shows fixed-time estimates for the integral kernel of the Fourier multiplier \( e^{it\Delta} |\nabla|^{-\sigma} \). Proposition 2.2 is proved in Section 3. We consider Remark 1.3 in Section 4.

Throughout this paper, the letter \( C \) stands for a positive constant which may be different at each occurrence. We also denote \( A \lesssim B \) to mean \( A \leq CB \) with unspecified constants \( C > 0 \).

2. Proof of Theorem 1.1

In this section we prove Theorem 1.1. First we list some basic properties of Wiener amalgam spaces which will be frequently used in the sequel. We refer to [4, 5, 6, 8] for details:

Lemma 2.1. Let \( 1 \leq p_i, q_i \leq \infty \) for \( i = 0, 1, 2 \). Then the followings hold:
• Inclusion: if \( p_1 \geq p_2 \) and \( q_1 \leq q_2 \),
  \[
  W(L^{p_1}, L^{q_1}) \subset W(L^{p_2}, L^{q_2}). \tag{2.1}
  \]
• Convolution: if \( 1/p_2 + 1 = 1/p_0 + 1/p_1 \) and \( 1/q_2 + 1 = 1/q_0 + 1/q_1 \),
  \[
  W(L^{p_0}, L^{q_0}) \ast W(L^{p_1}, L^{q_1}) \subset W(L^{p_2}, L^{q_2}). \tag{2.2}
  \]
More generally, if $B_0 * B_1 \subset B_2$ and $C_0 * C_1 \subset C_2$,
\[ W(B_0, C_0) * W(B_1, C_1) \subset W(B_2, C_2) \quad (2.3) \]
for Banach spaces $B_i$ and $C_i$, $i = 0, 1, 2$.

- **Duality**: if $1 \leq p, q \leq \infty$,
\[ W(L^p, L^q)' = W(L^{p'}, L^{q'}) \]
Here, $p', q'$ are conjugate exponents.

- **Complex interpolation**: if $q_0 < \infty$ or $q_1 < \infty$,
\[ [W(L^{p_0}, L^{q_0}), W(L^{p_1}, L^{q_1})]_{\theta} = W(L^p, L^q) \quad (2.4) \]
whenever
\[ \frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}, \quad \frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}, \quad 0 < \theta < 1. \quad (2.5) \]

### 2.1. Proof of Theorem 1.1

Now we turn to the proof of Theorem 1.1. By duality, the desired estimate (1.12) is equivalent to
\[ \left\| \int_{\mathbb{R}} e^{-is\Delta} |\nabla|^{-\sigma} F(\cdot, s) ds \right\|_{L^2_\tau} \lesssim \| F \|_{W(L^{p'}, L^{r'})_{t} W(L^{r'}, L^{\infty})_{x}}. \quad (2.6) \]

By the standard TT* argument, (1.12) is now equivalent to
\[ \left\| \int_{\mathbb{R}} e^{it\Delta} |\nabla|^{-2\sigma} F(\cdot, s) ds \right\|_{W(L^{p'}, L^{r'})_{t} W(L^{r'}, L^{\infty})_{x}} \lesssim \| F \|_{W(L^{p'}, L^{r'})_{t} W(L^{r'}, L^{\infty})_{x}}. \quad (2.7) \]
To obtain (2.7), we first write the integral kernel $K_t(x)$ of the multiplier $e^{it\Delta} |\nabla|^{-2\sigma}$ as
\[ K_t(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi - t|\xi|^2} \frac{d\xi}{|\xi|^{2\sigma}}. \quad (2.8) \]
Then (2.7) is rephrased as follows:
\[ \left\| \int_{\mathbb{R}} (K_{t-s} \ast F(\cdot, s)) ds \right\|_{W(L^{p'}, L^{r'})_{t} W(L^{r'}, L^{\infty})_{x}} \lesssim \| F \|_{W(L^{p'}, L^{r'})_{t} W(L^{r'}, L^{\infty})_{x}}. \quad (2.9) \]

From now on, we will obtain (2.9). By Minkowski’s inequality and the convolution relation (2.2), it follows that
\[ \left\| \int_{\mathbb{R}} (K_{t-s} \ast F(\cdot, s)) ds \right\|_{W(L^{p'}, L^{r'})_{t} W(L^{r'}, L^{\infty})_{x}} \]
\[ \leq \left\| \int_{\mathbb{R}} \| K_{t-s} \ast F(\cdot, s) \|_{W(L^{p'}, L^{r'})_{t} W(L^{r'}, L^{\infty})_{x}} ds \right\|_{W(L^{p'}, L^{r'})}, \]
\[ \leq \left\| \int_{\mathbb{R}} \| K_{t-s} \|_{W(L^{\frac{n}{2}}, L^{\frac{n}{2}})} \| F(\cdot, s) \|_{W(L^{p'}, L^{r'})_{t} W(L^{r'}, L^{\infty})_{x}} ds \right\|_{W(L^{p'}, L^{r'})}. \quad (2.10) \]
Recall the Hardy-Littlewood-Sobolev fractional integration theorem (see e.g. [12], p. 119) in dimension 1:
\[ L^p(\mathbb{R}) \ast L^{1/\alpha, \infty}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R}) \quad (2.11) \]
for $0 < \alpha < 1$ and $1 \leq p < q < \infty$ with $\frac{1}{q} + 1 = \frac{1}{p} + \alpha$. Applying (2.11) with $p = q'$ and $\alpha = \frac{\sigma}{q}$, and Young’s inequality, the convolution relations (2.3) then give

$$W(L^{\frac{q}{2}}, L^{\frac{q}{\infty}}) \ast W(L^{q'}, L^{q'}) \subset W(L^{\tilde{q}}, L^{\tilde{q}})$$

for

$$2 \leq \tilde{q} \leq \infty \quad \text{and} \quad 2 < q < \infty.$$  \hfill (2.12)

Hence we get

$$\left\| \int_{\mathbb{R}} \| K_{t-s} \|_{W(L^{\frac{q}{2}}, L^{\frac{q}{\infty}})} \| F(\cdot, s) \|_{W(L^{q'}, L^{q'})} ds \right\|_{W(L^{\frac{q}{2}}, L^{\frac{q}{\infty}})} \lesssim \| K_{t} \|_{W(L^{\frac{q}{2}}, L^{\frac{q}{\infty}}), W(L^{q'}, L^{q'})} \| F \|_{W(L^{q'}, L^{q'}), W(L^{q'}, L^{q'})}.$$  \hfill (2.13)

Combining (2.10) and (2.13), we now obtain the desired estimate (2.9) if

$$\| K_{t} \|_{W(L^{\frac{q}{2}}, L^{\frac{q}{\infty}}), W(L^{q'}, L^{q'})} < \infty$$  \hfill (2.14)

for $(\tilde{q}, \tilde{r})$ and $(q, r)$ satisfying the same conditions as in Theorem 1.1. To show (2.14), we use the following fixed-time estimates for the integral kernel which will be proved in Section 3.

**Proposition 2.2.** Let $n \geq 1$. Let $2 \leq \tilde{r}, \tilde{r} \leq \infty$ and $0 < \sigma < n/2$. Assume that

$$\frac{n-1}{\tilde{r}} + \frac{n}{r} < \sigma \quad \text{if} \quad 0 < \sigma \leq n/4,$$

and

$$\frac{n-1}{\tilde{r}} + \frac{n}{r} < \frac{n}{2} - \sigma \quad \text{if} \quad n/4 \leq \sigma < n/2.$$  \hfill (2.15)

Then we have

$$\| K_{t} \|_{W(L^{\frac{q}{2}}, L^{\frac{q}{\infty}})} \lesssim \begin{cases} |t|^{-\frac{n}{2} + \sigma + \frac{n-1}{r}} & \text{if} \quad 0 < |t| \leq 1, \\ |t|^{-\frac{n}{2} + \sigma + \frac{n-1}{r} + \frac{\sigma}{q}} & \text{if} \quad |t| \geq 1. \end{cases}$$  \hfill (2.16)

To begin with, we set $h(t) = \| K_{t} \|_{W(L^{\frac{q}{2}}, L^{\frac{q}{\infty}})}$ and choose $\varphi(t) \in C_{0}^{\infty}(\mathbb{R})$ supported on $\{ t \in \mathbb{R} : |t| \leq 1 \}$. To calculate $\| h \|_{W(L^{\frac{q}{2}}, L^{\frac{q}{\infty}})}$ using (2.17), we divide $\| h \varphi \|_{L^{\frac{q}{2}}}$ into three cases, $|k| \leq 1, 1 \leq |k| \leq 2$ and $|k| \geq 2$.

First we consider the case $|k| \leq 1$. By using (2.17) and the support condition of $\varphi$,

$$\| h \varphi \|_{L^{\frac{q}{2}}}^{1/2} \lesssim \int_{0 \leq |t| \leq 1} |t|^{\frac{n}{2} - \sigma + \frac{n-1}{r}} dt + \int_{1 \leq |t| \leq |k| + 1} |t|^{\frac{n}{2} - \sigma + \frac{n-1}{r} + \frac{\sigma}{q}} dt.$$  \hfill (2.18)

Since $\frac{n}{2} - \sigma + \frac{n-1}{r} + \frac{\sigma}{q} > 0$ by the condition (1.10), the first integral in the right-hand side of (2.18) is trivially finite. The second inequality is bounded as follows:

$$\int_{1 \leq |t| \leq |k| + 1} |t|^{\frac{n}{2} - \sigma + \frac{n-1}{r} + \frac{\sigma}{q}} dt \lesssim \frac{(|k| + 1)^{\frac{n}{2} - \sigma + \frac{n-1}{r} + \frac{\sigma}{q} + 1}}{\frac{n}{2} - \sigma + \frac{n-1}{r} + \frac{\sigma}{q} + 1} \lesssim |k|.$$  \hfill (2.19)
Indeed, since $\frac{2}{q}(-\frac{r}{q} + \sigma + \frac{n-1}{r} + \frac{p}{r}) < 0$ by the condition (1.11), the second inequality in (2.19) follows easily from the mean value theorem. Hence we get

$$\|h_{\tau}v\|_{L^q_t} \lesssim 1$$

when $|k| \leq 1$. The other cases $1 \leq |k| \leq 2$ and $|k| \geq 2$ are handled in the same way:

$$\|h_{\tau}v\|_{L^q_t} \lesssim \int_{|k|-1 \leq |t| \leq |k|+1} \left| t \right| \frac{2}{q}(-\frac{r}{q} + \sigma + \frac{n-1}{r} + \frac{p}{r}) dt$$

$$\lesssim (|k|+1)\frac{2}{q}(-\frac{r}{q} + \sigma + \frac{n-1}{r} + \frac{p}{r})+1 - (|k|-1)\frac{2}{q}(-\frac{r}{q} + \sigma + \frac{n-1}{r} + \frac{p}{r})+1$$

$$\lesssim (|k|-1)\frac{2}{q}(-\frac{r}{q} + \sigma + \frac{n-1}{r} + \frac{p}{r}).$$

Consequently, we get

$$\|h_{\tau}v\|_{L^q_t} \lesssim \begin{cases} 1 & \text{if } |k| \leq 2, \\ ((|k|-1)\frac{2}{q}(-\frac{r}{q} + \sigma + \frac{n-1}{r} + \frac{p}{r})+1 & \text{if } |k| \geq 2. \end{cases}$$

By (2.20), $\|h_{\tau}v\|_{L^q_t}$ belongs to $L^q_t$ since we are assuming the condition (1.11) which is equivalent to $\frac{2}{q} = \frac{n}{r} - \sigma - \frac{n-1}{r} - \frac{p}{r}$. This implies

$$\|h\|_{W(L^q_t L^q \to L^q_t \to L^q_{t\to \infty})} < \infty$$

for $(\tilde{q}, \tilde{r})$ and $(q, r)$ satisfying the conditions (2.12), $2 \leq \tilde{r}, r \leq \infty$, (1.10), (1.11), (2.15), (2.16).

Combining (1.11) and (2.15), we see $2/q > n/2 - 2\sigma$. Since $2 < q < \infty$ by (2.12), this implies the restriction $\sigma > (n - 2)/4$. On the other hand, the conditions (1.11) and (2.16) imply $2/q > 0$. There is no restriction in this case. Finally, combining (1.10) and (1.11), we see $2/\tilde{q} > 2/q + n/r$. Hence $q > \tilde{q}$. Therefore, we get the desired estimate (2.14) for $(\tilde{q}, \tilde{r})$ and $(q, r)$ satisfying $2 \leq \tilde{q} < q < \infty$, $2 \leq \tilde{r}, r \leq \infty$, (1.10), (1.11) when $(n - 2)/4 < \sigma < n/2$. This completes the proof of Theorem 1.1.

3. Proof of Proposition 2.2

In this section, we prove Proposition 2.2 by making use of the following lemma. (As mentioned in [1], this lemma is seen to be sharp in the case $\gamma = n/2$.)
Lemma 3.1. ([1], Lemma 2.2) Let \( n \geq 1 \) and \( 0 < \gamma < n \). Then if \( t \neq 0 \)

\[
\left| \int_{\mathbb{R}^n} e^{i(x \cdot \xi - t |\xi|^2)} \frac{d\xi}{|\xi|^\gamma} \right| \lesssim \begin{cases} 
|t|^{-\left(\frac{n}{2} - \gamma\right)} & \text{if } |x| \leq \sqrt{t}, \\
\frac{|t|^{-\left(n/2 - \gamma\right)}}{(|x|^2 + |t|)^{\gamma/2}} & \text{if } |x| \geq \sqrt{t}.
\end{cases}
\] (3.1)

We prove (2.17) only for the case \( \tilde{r}, r < \infty \) because the other cases \( \tilde{r} = \infty \) or \( r = \infty \) follow clearly and more easily from the same argument. We divide cases into \( 0 < \sigma \leq n/4 \) and \( n/4 \leq \sigma < n/2 \).

3.1. The case \( 0 < \sigma \leq n/4 \). From (2.8) and (3.1) with \( \gamma = 2\sigma \), we see

\[
|K_t(x)| \lesssim \begin{cases} 
|t|^{-\left(\frac{n}{2} - 2\sigma\right)} |t|^{-\sigma} & \text{if } |x| \leq \sqrt{t}, \\
|t|^{-\left(\frac{n}{2} - 2\sigma\right)} |x|^{-2\sigma} & \text{if } |x| \geq \sqrt{t}.
\end{cases}
\] (3.2)

To calculate \( \|K_t\|_{W(L^2, L^\gamma)} \) using (3.2), we choose \( \varphi(x) \in C_0^\infty(\mathbb{R}^n) \) supported on \( \{x \in \mathbb{R}^n : |x| \leq 1\} \) and divide \( \|K_t\varphi\|_{L^{\gamma/2}} \) into two cases, \(|y| \leq 1 + \sqrt{t}\) and \(|y| \geq 1 + \sqrt{t}\).

First we consider the case \(|y| \leq 1 + \sqrt{t}\). By using (3.2) and the support condition of \( \varphi \),

\[
|t|^{\frac{n}{2} - 2\sigma}\|K_t\varphi\|_{L^{\gamma/2}} \lesssim \int_{|y| \leq 1} |t|^{-\frac{n}{2} - \sigma}\int_{|x| \leq \sqrt{t}} |t|^{-\frac{n}{2}} dx + \int_{\sqrt{t} \leq |x| \leq |y| + 1} |x|^{-\sigma} dx
\]

\[
= |t|^{-\frac{n}{2}} \int_{|y| - 1}^{\sqrt{t}} \rho^{n-1} d\rho + \int_{|y| + 1}^{\sqrt{t}} \rho^{-\sigma + n - 1} d\rho
\]

\[
= |t|^{-\frac{n}{2}} \sqrt{t}^n - (|y| - 1)^n + (|y| + 1)^{-\sigma + n - \sqrt{t} - \sigma + n - 1}.
\]

Since \( n - 1 \geq 0 \) and \(-\sigma + n - 1 < 0\) from (2.16), by applying the mean value theorem as before, we now see

\[
|t|^{\frac{n}{2} - 2\sigma}\|K_t\varphi\|_{L^{\gamma/2}} \lesssim |t|^{\frac{n}{2} - \left(\sigma + \frac{n-1}{2}\right)} \left(\sqrt{t} - |y| + 1\right) + |t|^{\frac{n}{2} - \left(\sigma + \frac{n-1}{2}\right)} \left(|y| + 1 - \sqrt{t}\right)
\]

\[
= 2|t|^{\frac{n}{2} - \left(\sigma + \frac{n-1}{2}\right)}
\] (3.3)

when \(|y| \leq 1 + \sqrt{t}\). The other case \(|y| \geq 1 + \sqrt{t}\) is handled in the same way:

\[
|t|^{\frac{n}{2} - 2\sigma}\|K_t\varphi\|_{L^{\gamma/2}} \lesssim \int_{|y| - 1}^{|y| + 1} |t|^{|y| - 1} d\rho
\]

\[
= \frac{(|y| + 1)^{-\sigma + n} - (|y| - 1)^{-\sigma + n}}{-\sigma + n}
\]

\[
\lesssim (|y| - 1)^{\sigma + \frac{n-1}{2}}.
\] (3.4)
By (3.3) and (3.6), it follows now that

$$
\|K_t\|^{r/2}_{W(L^2_x,L^2_y)} = \|K_t\|_{L^{r/2}_y}^{r/2} \\
\lesssim \int_{|y| \leq 1 + \sqrt{t}} |t|^{\frac{n-1}{2} - \frac{\sigma}{2} + \gamma} dy \\
+ \int_{|y| \geq 1 + \sqrt{t}} |(y| - 1)^{r(-\sigma + \frac{n-1}{r})}|t|^{-\frac{\sigma}{2}} |t|^{-\frac{n}{2} - \gamma} dy.
$$

(3.5)

The first integral in the right-hand side of (3.5) is bounded as

$$
\int_{|y| \leq 1 + \sqrt{t}} |t|^{\frac{n-1}{2} - \frac{\sigma}{2} + \gamma} dy = |t|^{\frac{n-1}{2} - \frac{\sigma}{2} + \gamma} \int_{0}^{1 + \sqrt{t}} \rho^{n-1} d\rho \\
\lesssim |t|^{\frac{n-1}{2} - \frac{\sigma}{2} + \gamma} (1 + \sqrt{t})^{n-1}.
$$

(3.6)

For the second inequality, we use the binomial theorem with the binomial coefficients $C_{n,k}$ to obtain

$$
\int_{|y| \geq 1 + \sqrt{t}} |(y| - 1)^{r(-\sigma + \frac{n-1}{r})}|t|^{-\frac{\sigma}{2}} |t|^{\frac{n}{2} - \gamma} dy = \int_{\sqrt{t}}^{\infty} \rho^{-\sigma + \frac{n-1}{r} + \gamma} \rho^{n-1} d\rho \\
= \sum_{k=0}^{n-1} C_{n,k} \int_{\sqrt{t}}^{\infty} \rho^{-\sigma + \frac{n-1}{r} + \gamma + k} d\rho \\
\lesssim \sqrt{t}^{-\sigma + \frac{n-1}{r} + 1} \sum_{k=0}^{n-1} C_{n,k} \sqrt{t}^{\frac{n}{2}} \\
= |t|^{-\frac{\sigma}{2} + \frac{n-1}{r} + \frac{1}{2} (1 + \sqrt{t})^{n-1}}.
$$

(3.7)

Here, for the third inequality, we used the fact that $r(-\sigma + \frac{n-1}{r}) + k + 1 < 0$ for all $0 \leq k \leq n - 1$. Indeed, this fact follows from the condition (2.15).

Combining (3.5), (3.6) and (3.7), we have

$$
\|K_t\|_{W(L^2_x,L^2_y)} \lesssim \int \left[ t^{-\frac{\sigma}{2} + \frac{n-1}{r} + \frac{1}{2}} (1 + \sqrt{t})^n + |t|^{-\frac{\sigma}{2}} (1 + \sqrt{t})^{n-1} \right] dy.
$$

Hence we get (2.17) as desired.

3.2. The case $n/4 \leq \sigma < n/2$. The estimate (2.17) in this case is proved in the same way as in the previous case. From (2.8) and (3.1) with $\gamma = 2\sigma$, we see

$$
|K_t(x)| \lesssim \begin{cases} 
|t|^{-\frac{\sigma}{2} + \frac{n}{2}} & \text{if } |x| \leq \sqrt{t}, \\
|t|^{-n+2\sigma} & \text{if } |x| \geq \sqrt{t}.
\end{cases}
$$

(3.8)

To calculate $\|K_t\|_{W(L^2_x,L^2_y)}$ using (3.8), we choose $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ supported on $\{x \in \mathbb{R}^n : |x| \leq 1\}$ and divide $\|K_t\|_{L^{r/2}_y}$ into two cases, $|y| \leq 1 + \sqrt{t}$ and $|y| \geq 1 + \sqrt{t}$, as before.

First we consider the case $|y| \leq 1 + \sqrt{t}$. By using (3.8) and the support condition of $\varphi$,
\[\|K_t \tau_0 \varphi\|_{L^r_t(L^s_x)}^{\frac{r}{2}} \leq \int_{|y| - 1 \leq x \leq \sqrt{t}} |t|^{-\frac{\sigma}{2} + \frac{a}{r}} dx + \int_{\sqrt{t} \leq |x| \leq |y| + 1} |x|^{-\frac{n+2\sigma}{2}} dx\]

\[= |t|^{-\frac{\sigma}{2} + \frac{a}{r}} \int_{|y| - 1}^{\sqrt{t}} \rho^{-n-1} d\rho + \int_{\sqrt{t}}^{|y| + 1} \rho^{-\frac{\sigma}{2} + \frac{a}{r}} d\rho\]

\[= |t|^{-\frac{\sigma}{2} + \frac{a}{r}} \frac{\sqrt{t}^n - (|y| - 1)^n}{n} + \frac{(|y| + 1)^{\frac{\sigma}{2} + \frac{a}{r} + \frac{n-1}{r}} - (|y| - 1)^{\frac{\sigma}{2} + \frac{a}{r} + \frac{n-1}{r}} + 1}{\frac{\sigma}{2} + \frac{a}{r} + \frac{n-1}{r}}.\]

Since \(n - 1 \geq 0\) and \(\frac{\sigma}{2} + \frac{a}{r} + \frac{n-1}{r} < 0\) from (3.16), by applying the mean value theorem as before, we now see

\[\|K_t \tau_0 \varphi\|_{L^r_t(L^s_x)}^{\frac{r}{2}} \leq |t|^{-\frac{\sigma}{2} + \frac{a}{r}} (\sqrt{t} - |y| + 1) + |t|^{-\frac{\sigma}{2} + \frac{a}{r}} (|y| + 1 - \sqrt{t})\]

(3.9)

when \(|y| \leq 1 + \sqrt{t}\). Similarly, for the other case \(|y| \geq 1 + \sqrt{t}\), we get

\[\|K_t \tau_0 \varphi\|_{L^r_t(L^s_x)}^{\frac{r}{2}} \leq \int_{|y| - 1 \leq x \leq |y| + 1} |x|^{-\frac{n+2\sigma}{2}} dx\]

\[= \int_{|y| - 1}^{1} \rho^{\frac{\sigma}{2} + \frac{a}{r} + \frac{n-1}{r}} d\rho\]

\[= \frac{(|y| + 1)^{\frac{\sigma}{2} + \frac{a}{r} + \frac{n-1}{r}} - (|y| - 1)^{\frac{\sigma}{2} + \frac{a}{r} + \frac{n-1}{r}} + 1}{\frac{\sigma}{2} + \frac{a}{r} + \frac{n-1}{r}}.\]

(3.10)

By (3.9) and (3.10), it follows now that

\[\|K_t\|_{W(L^s_x(L^r_t))}^{\frac{r}{2}} = \|K_t \tau_0 \varphi\|_{L^r_t(L^s_x)}^{\frac{r}{2}}\]

\[\leq \int_{|y| \leq 1 + \sqrt{t}} |t|^{-\frac{\sigma}{2} + \frac{a}{r} + \frac{n-1}{r}} dy + \int_{|y| \geq 1 + \sqrt{t}} (|y| - 1)^{\frac{\sigma}{2} + \frac{a}{r} + \frac{n-1}{r}} dy.\]

(3.11)

The first integral in the right-hand side of (3.11) is bounded as

\[\int_{|y| \leq 1 + \sqrt{t}} |t|^{-\frac{\sigma}{2} + \frac{a}{r} + \frac{n-1}{r}} dy = |t|^{-\frac{\sigma}{2} + \frac{a}{r} + \frac{n-1}{r}} \frac{1}{\sqrt{t}} \rho^{-n-1} d\rho\]

\[\leq |t|^{-\frac{\sigma}{2} + \frac{a}{r} + \frac{n-1}{r}} (1 + \sqrt{t})^n.\]

(3.12)
For the second inequality, we use the binomial theorem with the binomial coefficients $C_{n,k}$ to obtain

$$
\int_{|y| \geq 1 + \sqrt{t}} (|y| - 1)^{(-\frac{n}{2} + \frac{\sigma}{2} + \frac{n-1}{2})} dy = \int_{\sqrt{t}}^{\infty} \rho^{(-\frac{n}{2} + \frac{\sigma}{2} + \frac{n-1}{2})} (\rho + 1)^{n-1} d\rho
\leq \sum_{k=0}^{n-1} C_{n,k} \int_{\sqrt{t}}^{\infty} \rho^{(-\frac{n}{2} + \frac{\sigma}{2} + \frac{n-1}{2})+k} d\rho
\leq \sqrt{t}^{-\frac{n}{2} + \frac{\sigma}{2} + \frac{n-1}{2}+1} \sum_{k=0}^{n-1} C_{n,k} \sqrt{t}^k
= |t|^{\frac{\sigma}{2} + \frac{n-1}{2}+\frac{1}{2}} (1 + \sqrt{t})^{n-1}. \tag{3.13}
$$

Here, for the third inequality, we used the fact that $\rho r^{(-\frac{n}{2} + \frac{\sigma}{2} + \frac{n-1}{2})+k+1} < 0$ for all $0 \leq k \leq n - 1$. Indeed, this fact follows from the condition (2.16).

Combining (3.11), (3.12) and (3.13), we have

$$
\|K_t\|_{W(L^q, L^r)} \lesssim |t|^{-\sigma + \frac{n-1}{2}} ((1 + \sqrt{t})^n + |t|^{\frac{1}{2}} (1 + \sqrt{t})^{n-1})^\frac{1}{2}. \tag{4.1}
$$

Hence we get (2.17) as desired.

4. Concluding remarks

In this final section, we discuss Remark 1.3 in detail. As mentioned there, we can obtain further estimates by complex interpolation (see (2.4)) between bilinear form estimates given from (1.6) and (1.12). Here we explain this only for the particular case where we use (1.5) and (1.12) with $\bar{r} = \infty$ instead of (1.6) and (1.12), respectively. This is strictly intended to make the argument shorter, and one could adapt the same argument from this case to handle the other cases as well.

**Corollary 4.1.** Let $n \geq 1$ and $\max\{0, (n-2)/8\} < \sigma < n/4$. Assume that $(q, r)$ satisfy (1.9),

$$
\frac{2}{q} > \frac{n}{4} - \sigma, \quad 0 < \frac{1}{q} < \frac{1}{q} + \frac{1}{4} \leq \frac{1}{2} \quad \text{and} \quad 2 \leq r \leq \infty. \tag{4.1}
$$

Here, $r \neq \infty$ if $n = 2$. Then we have

$$
\|e^{it\Delta} f\|_{W(L^q, L^r)_{\sigma}} \lesssim \|f\|_{\dot{H}^\sigma}. \tag{4.2}
$$

**Remark 4.2.** When $\bar{r} = 4$, the possible range of $\bar{q}$ in Theorem 1.1 is $2 \leq \bar{q} < 8/(n - 4\sigma + 1)$. On the other hand, the possible range of $\bar{q}$ in the above corollary is $4 \leq \bar{q} < 8/(n - 4\sigma)$ if $\sigma > \max\{0, (n-2)/4\}$. Since $8/(n - 4\sigma + 1) < 8/(n - 4\sigma)$, it gives further estimates which do not follow from Theorem 1.1. For fixed $q$, the exponent $r$ given from (1.9) is smaller than $r$ given from (1.11) with $\bar{r} = 4$. From this observation and the inclusion relation (2.1), we also note that (1.2) is stronger than the estimate (1.12) with $\bar{r} = 4$ in Theorem 1.1. Similarly for fixed $r$. 
Proof of Corollary 4.1. Firstly, we recall from Subsection 2.1 that the standard TT\* argument gives that
\[ \|e^{it\Delta}f\|_{W(L^6, L^6), W(L^8, L^8)_*} \lesssim \|f\|_{H^s} \]
is equivalent to the estimate (2.7) which is in turn equivalent to the following bilinear form estimate
\[ |T(F, G)| \lesssim \|F\|_{W(L^6, L^6), W(L^8, L^8)_*} \|G\|_{W(L^6, L^6), W(L^8, L^8)_*}, \quad (4.3) \]
where
\[ T(F, G) := \int_{\mathbb{R}} \int_{\mathbb{R}} (e^{-i\theta \Delta} |\nabla|^{-\sigma} G(x, t), e^{-i\theta \Delta} |\nabla|^{-\sigma} G(x, t)) \, ds \, dt. \]
Next, using the Cauchy-Schwarz inequality,
\[ |T(F, G)| = \left| \int_{\mathbb{R}} \int_{\mathbb{R}} (e^{-i\theta \Delta} |\nabla|^{-\sigma} F(x, s), e^{-i\theta \Delta} |\nabla|^{-\sigma} G(x, t)) \, ds \, dt \right| \leq \left\| \int_{\mathbb{R}} e^{-i\theta \Delta} |\nabla|^{-\sigma} F(x, s) \, ds \right\|_{L^2} \left\| \int_{\mathbb{R}} e^{-i\theta \Delta} G(x, t) \, dt \right\|_{L^2}. \quad (4.4) \]
For the first \( L^2 \) norm in (4.3), we use the following dual estimate of (1.12) with \( \tilde{r} = \infty \)
and \( \sigma \) replaced by \( 2\sigma \),
\[ \left\| \int_{\mathbb{R}} e^{-i\theta \Delta} |\nabla|^{-\sigma} F(x, s) \, ds \right\|_{L^2} \lesssim \|F\|_{W(L^{\tilde{q}^*}, L^{\tilde{q}^*}), W(L^{r_1}, L^{r_1})_*}, \quad (4.5) \]
where \( 2 \leq \tilde{q}^* < q^* < \infty, 2 \leq r_1 \leq \infty, \max\{0, (n-2)/8\} < \sigma < n/4, \)
\[ \frac{2}{\tilde{q}^*} > \frac{n}{2} - 2\sigma \quad \text{and} \quad \frac{2}{q^*} + \frac{n}{r_1} = \frac{n}{2} - 2\sigma. \quad (4.6) \]
For the second \( L^2 \) norm in (4.4), we use the following dual estimate of (1.5),
\[ \left\| \int_{\mathbb{R}} e^{-it\Delta} G(x, t) \, dt \right\|_{L^2} \lesssim \|G\|_{W(L^{q_1}, L^{q_1}), W(L^{r_2}, L^{r_2})_*}, \quad (4.7) \]
where \( (q_2, r_2) \) is Schrödinger admissible (see (1.4)). Combining (4.5), (4.6) and (4.7),
we then have
\[ |T(F, G)| \lesssim \|F\|_{W(L^{q_1}, L^{q_1}), W(L^{r_2}, L^{r_2})_*} \|G\|_{W(L^{q_1}, L^{q_1}), W(L^{r_2}, L^{r_2})_*}, \quad (4.8) \]
and by symmetry
\[ |T(F, G)| \lesssim \|F\|_{W(L^{q_1}, L^{q_1}), W(L^{r_2}, L^{r_2})_*} \|G\|_{W(L^{q_1}, L^{q_1}), W(L^{r_2}, L^{r_2})_*}, \quad (4.9) \]
for \( \tilde{q}_1, (q_1, r_1) \) and \( (q_2, r_2) \) given as above. Finally, by applying the complex interpolation (2.3) with \( \theta = 1/2 \) between (4.8) and (4.9), we obtain (4.3) for
\[ \frac{1}{\tilde{q}} = \frac{1}{2\tilde{q}_1}, \quad \frac{1}{q} = \frac{1}{2} \left( \frac{1}{q_1} + \frac{1}{q_2} \right), \quad \frac{1}{r} = \frac{1}{4}, \quad \frac{1}{r_1} = \frac{1}{4} \left( 1 + \frac{1}{r_2} \right). \]
Combining the second condition in (4.6) and \( 2/q_2 + n/r_2 = n/2 \) implies the condition (1.9). From the first condition in (4.6), we see the first condition in (4.1). Since \( \tilde{q}_1 < q_1 < \infty \) and \( 2 \leq q_2 \leq \infty, \) \( 0 < 1/q < 1/4 + 1/(2\tilde{q}_1) = 1/4 + 1/\tilde{q}_1. \) Since \( \tilde{q}_1 \geq 2, \) it follows also that \( \tilde{q} \geq 4. \) Hence we see the second condition in (4.1). From the
conditions $2 \leq r_1 \leq \infty$ and $2 \leq r_2 \leq \infty$, we finally see $2 \leq r \leq \infty$. Here, $r \neq \infty$ if $n = 2$ since $r_2 \neq \infty$ if $n = 2$. This determines the last condition in (4.1). □

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