Relativistic Time of Arrival

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Abstract

We propose a covariant algorithm for relativistic ideal measurements and for relativistic continuous measurements, its non-relativistic limit results the algorithm of the Event-Enhanced Quantum Theory. Therefore an additional intrinsic parameter, the proper time, is used. As an application we compute the time of arrival of a particle at a detector and find good agreement between the expected values of the time of arrival for weak detectors and the results of the relativistic point-mechanic over a wide range. For very high momentums there is a small probability for a negative time of arrival, so the expected times are a bit smaller than the results of the relativistic mechanics.

1 Introduction

One can use the PDP-algorithm of the Event-Enhanced Quantum Theory (EEQT) as described by Blanchard and Jadczyk [1, 2, 3, 4] to simulate detections of a non-relativistic electron (for example [5, 6, 7]). In this paper we are interested in detections of a relativistic electron in an external electromagnetic field.

Trying to define states and a reduction postulate in a relativistic theory can imply a lot of paradoxes and difficulties (for example see Y. Aharonov and D.Z. Albert [8, 9, 10]).

One possibility to avoid (some) difficulties is to consider the wave function for relativistic particle not as a function on the space-time continuum but as a function on the set of flat, space-like hypersurfaces in Minkowski space (for example see the papers by Breuer and Petruccione [11, 12]).

Another possibility is the introduction of a supplementary, intrinsic time, the proper time (for example see the paper by Horwitz and Piron [13] or the review paper by Fanchi [14]).

Blanchard and Jadczyk [15] formulated a relativistic algorithm for events by using this proper time.

In the following we also use the proper time to propose a relativistic extension of the EEQT, but another definition for the state of the system and its dynamics.

The total system consists of a classical and a quantum part. At a given proper time $\tau$ the state of the total system is a pair $(\omega_\tau, \Psi_\tau)$, $\omega_\tau$ is the state of the classical and $\Psi_\tau$ is the state of the quantum part.

We assume, that the classical part has only a finite number $N_C$ of possible pure states, therefore a state $\omega_\tau$ of the classical part is a number $\omega_\tau \in \{0..N_C-1\}$.

A state $\Psi_\tau$ of the quantum part must have the following properties:

(i) $\Psi_\tau$ is continuously differentiable and a solution of the Dirac equation with an external electromagnetic field

$$\left(\gamma^\mu \partial_\mu - \frac{e}{\hbar c} \gamma^\mu A_\mu - \frac{mc}{\hbar}\right) \Psi_\tau = 0$$

(1)

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(ii) For all \( y = (y^0, \vec{y}) \in \mathbb{R}^4, \vec{a} \in \mathbb{R}^3 \) with \( |\vec{a}|^2 < 1 \) and \( \varphi \in \mathbb{R}^3 \) with \( |\varphi| < 2\pi \) it follows:

\[
\int_{\mathbb{R}^3} d\vec{x} \left| \Psi_{\tau}(y^0 + \vec{a} \cdot (R(\varphi)\vec{x}), \vec{y} + R(\varphi)\vec{x}) \right|^2 < \infty \tag{2}
\]

\[
\lim_{|\vec{x}| \to \infty} |\vec{x}| \left| \Psi_{\tau}(y^0 + \vec{a} \cdot (R(\varphi)\vec{x}), \vec{y} + R(\varphi)\vec{x}) \right|^2 = 0 \tag{3}
\]

In this paper we use the Dirac representation of the \( \gamma \)-matrices. Moreover \( R(\varphi) \in SO(3) \) should be the rotation of \( |\varphi| \) around the vector \( \vec{\varphi}/|\vec{\varphi}| \).

A quantum state is uniquely given by its values on a plane

\[
P_{(y^0, \vec{y}), \vec{a}, \varphi} = \{(y^0 + \vec{a} \cdot (R(\varphi)\vec{x}), \vec{y} + R(\varphi)\vec{x}) | \vec{x} \in \mathbb{R}^4\} \tag{4}
\]

with \( y = (y^0, \vec{y}) \in \mathbb{R}^4, \vec{a} \in \mathbb{R}^3, |\vec{a}|^2 < 1 \) and \( \varphi \in \mathbb{R}^3 \) with \( |\varphi| < 2\pi \).

We introduce the operator \( U^{-1}((y^0, \vec{y}), \vec{a}, \varphi) \), which reduces the quantum state to a "plane state":

\[
(U^{-1}((y^0, \vec{y}), \vec{a}, \varphi)\Psi)(\vec{x}) := \Psi(y^0 + \vec{a} \cdot (R(\varphi)\vec{x}), \vec{y} + R(\varphi)\vec{x}) \tag{5}
\]

The operator \( U^{-1}((y^0, \vec{y}), \vec{a}, \varphi) \) is invertible, we call the inverse operator \( U((y^0, \vec{y}), \vec{a}, \varphi) \). Both operators are unitary.

In the following, we examine how the state of the total system changes, if we change the reference frame \( K \to K' \) with \( \vec{x} = \Lambda \vec{x} + \vec{a} \). We look only at Poincaré-transformations \( (\Lambda, \vec{a}) \) which do not mirror the space or invert the direction of time.

The classical state and the proper time should be invariant. The quantum state changes in the following way

\[
\Psi \longrightarrow \tilde{\Psi}(\vec{x}) = S(\Lambda)\Psi(\Lambda^{-1}(\vec{x} - \vec{a})) \tag{6}
\]

\( S \) being a non-singular \( 4 \times 4 \) matrix with \( S(\Lambda)^\gamma \nu S^{-1}(\Lambda) = (\Lambda^{-1})^\mu \nu \gamma' \).

Now we introduce a scalar product between two quantum states by

\[
<\Psi_A|\Psi_B> = \int_{\mathbb{R}^3} d\vec{x} \Psi_A^\dagger(y^0 + \vec{a} \cdot (R(\varphi)\vec{x}), \vec{y} + R(\varphi)\vec{x}) \left[ 1 - \gamma^0 \gamma^\nu \gamma^\nu \right] \Psi_B(y^0 + \vec{a} \cdot (R(\varphi)\vec{x}), \vec{y} + R(\varphi)\vec{x}) \tag{7}
\]

with \( y = (y^0, \vec{y}) \in \mathbb{R}^4, \vec{a} \in \mathbb{R}^3 \) with \( |\vec{a}|^2 < 1 \) and \( \varphi \in \mathbb{R}^3 \) with \( |\varphi| < 2\pi \) arbitrary.

The scalar product is positive definite and well defined, it is independent of the choice of \( y^0, \vec{y}, \vec{a}, \varphi \). Note that the number of free parameters \( (10) \) equals the number of parameters of a Poincaré-transformation. Moreover it is covariant, its value is equal in all reference frames \( <\Psi_A|\Psi_B>_K = <\tilde{\Psi}_A|\tilde{\Psi}_B>K \).

Events like the preparation or the detection of an electron happen at a proper time \( \tau_i \) at a space-time point \( x_i \).

To preserve a kind of order, we assume the following: taking two events happen at \( \tau_1 \) and \( \tau_2 \) with \( \tau_1 < \tau_2 \), there must be a reference frame, in which the time of the first event \( x_1^0 \) is earlier than the time of the second event \( x_2^0 \) (we only allow Poincaré-transformations, which do not mirror the space or invert the direction of time).

Therefore no "later" event can take place in the backward light-cone of a previous event:

\[
\tau_1 < \tau_2 \Rightarrow (\|x_2 - x_1\|^2 \geq 0 \text{ and } x_1^0 < x_2^0) \text{ or } (\|x_2 - x_1\|^2 < 0) \tag{8}
\]

\( \|x\|^2 = \|(x^0, \vec{x})\|^2 = (x^0)^2 - |\vec{x}|^2 \) being the Minkowski-distance.

This condition is invariant under the allowable Poincaré-transformations.

First we want to formulate an algorithm to describe ideal, infinitesimal short measurements playing the role of the reduction-postulate in the non-relativistic quantum mechanics.
There should be $n$ measurements, which happen at the proper times $\tau_i$ at the space-time points $z_i$, $i = 1..n$. The $i$th measurement is represented by an observable

$$A_i = \sum_j \lambda_{i,j} \phi_{i,j}$$

(9)

$\Phi_{i,j}$ being eigenvectors of $A_i$ with $1 = \sum_j \langle \phi_{i,j} | \phi_{i,j} \rangle$ and $\langle \phi_{i,j} | \phi_{i,k} \rangle = \delta_{j,k}$.

We assume, that $\tau_i < \tau_j$ for $i < j$ and no "later" measurement take place in the backward light-cone of a previous measurement (see above eq. (8)).

We can now formulate a relativistic reduction-postulate for ideal measurements:

(i) The particle is prepared at a proper time $\tau_0$ with $\tau_0 < \tau_1$ at a space-time point $x_0$, the state of the quantum part should be $\Psi_{\tau_0}$ with $\langle \Psi_{\tau_0} | \Psi_{\tau_0} \rangle = 1$ and the classical state is $\omega_{\tau_0} = 0$. Set $i = 1$.

(ii) The quantum and classical state change only in case of measurement, they have no $\tau$-development if there is no measurement:

$$\Psi_\tau = \Psi_{\tau_{i-1}}$$

(10)

$$\omega_\tau = \omega_{\tau_{i-1}}$$

(11)

for $\tau_{i-1} \leq \tau \leq \tau_i$.

(iii) The $i$th measurement takes place at proper time $\tau_i$ at a space-time point $z_i$, we get the measurement result $\lambda_{i,j}$ with probability

$$p(\lambda_{i,j}) = |\langle \Phi_{i,j} | \Psi_{\tau_{i-1}} \rangle|^2$$

(12)

If $\lambda_{i,j}$ is the measurement result, the state of system after the measurement is

$$\Psi_{\tau_i} \longrightarrow \Phi_{i,j}$$

(13)

$$\omega_{\tau_i} \longrightarrow \langle j$$

(14)

(iv) We set $i \rightarrow i + 1$ and go to step (ii).

The probabilities generated by this algorithm are the same we get if we use the standard-non-covariant reduction-postulate with the Dirac-equation and assume, that space-like separated observable commute.

Now we formulate an algorithm for continuous relativistic measurements, indeed we will propose in the following an algorithm to describe detections of an electron.

The electron should be prepared in a point $x_0 = (x_0^0, \vec{x}_0)$. We consider $N$ detectors with trajectories $z_j(\tau)$, $j = 1..N$. The trajectories start at proper time $\tau = 0$ from the backward light cone of the ‘preparing event’ $x_0$, $\|x_0 - z_j(0)\|^2 = 0$, $z_j^0(0) \leq x_0^0$.

We allow detections, which happen in the past of the preparation time, but we do not allow detections, if the detection space-time point lies in the backward light-cone of the preparation event.

The coupling between the quantum and the classical system is given by operators $G_j(\tau)$. We set $\Lambda(\tau) = \sum_j G_j(\tau)^+ G_j(\tau)$, $G_j(\tau)^+$ being the adjoint operator.

An operator $G_j(\tau)$ is uniquely given by its operation on a projection of the quantum state on a plane $G_j(\tau) = U((y_0^0, \vec{y}), \vec{\alpha}, \vec{\phi}) g_j(\tau) U^{-1}((y_0^0, \vec{y}), \vec{\alpha}, \vec{\phi})$ with $(y_0^0, \vec{y}), \vec{\alpha}, \vec{\phi}$ arbitrary.

We define now the following algorithm:

(i) The particle is prepared in a point $x_0$, the state of the quantum part should be $\Psi_0$ with $\langle \Psi_0 | \Psi_0 \rangle = 1$ and the classical state is $\omega = 0$, $\tau = 0$.

(ii) Choose uniformly a random number $r \in [0, 1]$.
(iii) Propagate the quantum state forward in proper time by solving
\[
\frac{\partial}{\partial \tau} \Psi_\tau = -\frac{1}{2} \Lambda(\tau) \Psi_\tau
\]  \hspace{1cm} \text{(15)}

until \( \tau = \tau_1 \), where \( \tau_1 \) is defined by
\[
1 - < \Psi_{\tau_1} | \Psi_{\tau_1} > = \int_0^{\tau_1} d\tau < \Psi_\tau | \Lambda \Psi_\tau >= r
\]  \hspace{1cm} \text{(16)}

A detection happens at proper time \( \tau = \tau_1 \).

(iv) We choose the detector \( k \), which detects the particle with probability
\[
p_k = \frac{1}{N} < G_k(\tau_1) \Psi_{\tau_1} | G_k(\tau_1) \Psi_{\tau_1} >
\]  \hspace{1cm} \text{(17)}

with \( N = \sum_j < G_j(\tau_1) \Psi_{\tau_1} | G_j(\tau_1) \Psi_{\tau_1} > \).

(v) Let \( l \) be the detector, which detects the particle. The detection happens at the point \( z_l(\tau_1) \). The detection induces then the following change of the states:
\[
\Psi_{\tau_1} \rightarrow \frac{G_l(\tau_1) \Psi_{\tau_1}}{\sqrt{< G_l(\tau_1) \Psi_{\tau_1} | G_l(\tau_1) \Psi_{\tau_1} >}}
\]  \hspace{1cm} \text{(18)}
\[
\omega \rightarrow l
\]  \hspace{1cm} \text{(19)}

The algorithm starts again perhaps with other detectors at position (ii).

The non-relativistic limit of this algorithm is the PDP-algorithm of the EEQT. To prove this, we define
\[
\Omega(\tau, \vec{x}) := \left( U^{-1}((ct + x_0^0, \vec{x}_0^0), \vec{0}, \vec{0}) \Psi_\tau \right)(\vec{x}) = \Psi_\tau (ct + x_0^0, \vec{x} + \vec{x}_0^0)
\]  \hspace{1cm} \text{(20)}

We get
\[
\hbar \frac{\partial}{\partial \tau} \Omega(\tau, \vec{x}) = \left( \frac{\partial}{\partial x_0^0} \Psi_\tau \right) (ct + x_0^0, \vec{x} + \vec{x}_0^0) + \hbar \frac{\partial \Psi_\tau}{\partial \tau} (ct + x_0^0, \vec{x} + \vec{x}_0^0)
\]
\[
= \hbar c \left( -\gamma_0^k \gamma^k \frac{\partial \Psi_\tau}{\partial x^k} (ct + x_0^0, \vec{x} + \vec{x}_0^0) - \frac{e}{c \hbar} \gamma_0^0 \gamma^\mu A_\mu - \frac{mc^2}{\hbar} \gamma_0^0 \Psi_\tau (ct + x_0^0, \vec{x} + \vec{x}_0^0) \right)
\]
\[
- \frac{i \hbar}{2} \left( \sum_j \left( U^{-1} G_j^\dagger(\tau) U \right) \left( U^{-1} G_j(\tau) U \right) \right) \Omega
\]  \hspace{1cm} \text{(21)}

with
\[
U^{-1} = U^{-1}((ct + x_0^0, \vec{x}_0^0), \vec{0}, \vec{0}) \text{ and } U = U((ct + x_0^0, \vec{x}_0^0), \vec{0}, \vec{0}).
\]

We examine the non-relativistic limit of eq. (21) doing the assumption (compare to calculations of the limit using the Dirac-equation in textbooks)
\[
\Omega(\tau, \vec{x}) = \exp \left( -\frac{mc^2}{\hbar} \gamma_0^0 \right) \left( \begin{array}{c} \phi \\ \chi \end{array} \right)
\]  \hspace{1cm} \text{(22)}

Moreover we assume, that the detectors detect only particles
\[
g_j(\tau) = \left( \begin{array}{cc} \tilde{g}_j(\tau) & 0 \\ 0 & 0 \end{array} \right)
\]  \hspace{1cm} \text{(23)}
In the non-relativistic limit we get the modified equation of the EEQT

\[
\frac{i\hbar}{\partial \tau} \phi = \left[ \frac{1}{2m} \sum_l \left( \frac{\hbar}{i} \frac{\partial}{\partial x^l} - \frac{e}{c} A^l \right)^2 - \frac{e\hbar}{2mc} \vec{\sigma} \cdot \vec{B} + eA^0 - \frac{\hbar}{2} \sum_j \vec{g}_j^+(\tau)\vec{g}_j(\tau) \right] \phi
\]  

(24)

Noting, that \( < \Psi_\tau | \Psi_\tau > = \int d\vec{x} \Omega(\tau, \vec{x})\Omega^{\dagger}(\tau, \vec{x}) \) \( \xrightarrow{c \to \infty} \) \( \int d\vec{x} \Phi(\tau, \vec{x})\Phi^{\dagger}(\tau, \vec{x}) \) and set \( t := \tau \) we get the PDP-algorithm of the EEQT as the non-relativistic limit of the above relativistic algorithm.

2 Application: Time of Arrival

One application of the above algorithm is the detection of a particle by one detector, which is at rest. We look at the problem in 1 + 1 dimension.

We introduce the operator

\[
(U^{-1}(y^0, y^1)) \Psi)(x) := \Psi(y^0, y^1 + x)
\]  

(25)

and the inverse operator \( U(y^0, y^1) \). The operators are unitary.

The particle is prepared at \( (ct_0 \equiv 0, x_0 \equiv -1\AA) \) and the initial state of the particle should be \( \Psi_0 = U(ct_0, x_0)\psi_0 \) with

\[
\psi_0(x) = \frac{1}{(2\pi)^{1/4} \sqrt{\eta}} \cdot \exp \left( -\frac{x^2}{4\eta^2} + \frac{i p_0}{\hbar} x \right) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]  

(26)

and \( \eta = 0.1\AA \). We get

\[
\Psi_0(ct, x) =
\]

\[
\sqrt{2\eta} \int dp \frac{E + mc^2}{2E} \exp \left( -\eta^2 \left( p - p_0 \right)^2 \frac{1}{h^2} - \frac{i p_0}{\hbar} x_0 \right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \exp \left( \frac{p}{\hbar} x - \frac{E}{\hbar} t \right)
\]

\[
+ \sqrt{2\eta} \int dp \frac{pc}{2E} \exp \left( -\eta^2 \left( p + p_0 \right)^2 \frac{1}{h^2} + \frac{i p_0}{\hbar} x_0 \right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \exp \left( -\frac{p}{\hbar} x + \frac{E}{\hbar} t \right)
\]  

(27)

For illustration, Fig. 1 (a) shows the square of the first component (particle) and Fig. 1 (b) the square of the forth component (anti-particle) of the initial state \( \Psi_0(ct, x) \) for the momentum \( p_0 = 0.75mc \).

The detector is put at \( x_D = 0\AA \), its trajectory is \( z(\tau) = (c\tau + x_0, x_D) \).

The coupling operator of the detector is given by

\[
G(\tau) = U(z(\tau))g(x)U^{-1}(z(\tau))
\]  

(28)

with \( g(x) \) a function characterizing the sensitivity of the detector.

The detector should detect only particles and not anti-particles.

Using our algorithm, the total probability, that the detector detects the electron is given by

\[
P_\infty = \int_0^\infty d\tau < \Psi_\tau | \Lambda \Psi_\tau >
\]  

(29)

5
The probability density for a "proper time of arrival" at the detector is given by

\[ P(\tau) = \frac{1}{P_\infty} \langle \Psi_\tau | \Lambda | \Psi_\tau \rangle \]  

(30)

With this probability density we can calculate the probability density for the time of arrival and the expected value in each system, for example in a system in which the detector is at rest:

\[ p(t) = P(t - \frac{x_0}{c}) \]  

(31)

\[ T = \int dt \cdot t \cdot p(t) = \int d\tau \left( \tau + \frac{x_0}{c} \right) P(\tau) = \int d\tau \tau P(\tau) + \frac{x_0}{c} \]  

(32)

Now we want to examine how the same situation looks like in a reference frame \( \tilde{K} \), which moves with velocity \( v \) with respect to the system \( K \) in which the detector is at rest. The transformation has the following form:

\[ \tilde{x} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left( \frac{1}{c} \frac{v}{c} \right) x \]  

(33)

The normalized probability density for the time of arrival is given by

\[ \tilde{p}(\tilde{t}) = \sqrt{1 - \frac{v^2}{c^2}} p \left( \frac{1 - \frac{v^2}{c^2}}{\tilde{t}} \right) = \sqrt{1 - \frac{v^2}{c^2}} P \left( \frac{1 - \frac{v^2}{c^2}}{\tilde{t}} \tau \right) - \frac{x_0}{c} \]  

(34)

The expected value is given by

\[ \tilde{T} = \int d\tilde{t} \tilde{p}(\tilde{t}) = \sqrt{1 - \frac{v^2}{c^2}} \int d\tilde{t} \tilde{t} p \left( \frac{1 - \frac{v^2}{c^2}}{\tilde{t}} \right) \]  

\[ = \sqrt{1 - \frac{v^2}{c^2}} \int dt \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \sqrt{1 - \frac{v^2}{c^2}} P(t) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} T \]  

(35)

For computation of \( P(\tau) \) we define

\[ \Omega(\tau, x) := (U^{-1}(z(\tau))\Psi_\tau)(x) = \Psi_\tau(c\tau + x_0, x + x_D) \]  

(36)

and we note, that \( \langle \Psi_\tau | \Psi_\tau \rangle := \int dx |\Omega(\tau, x)|^2 \) or \( \langle \Psi_\tau | \Lambda | \Psi_\tau \rangle := \int dx \Omega^+(\tau, x)g^+(x)g(x)\Omega(\tau, x) \).

We must now solve

\[ i\hbar \frac{\partial}{\partial \tau} \Omega(\tau, x) = -i\hbar c\gamma^0 \gamma^1 \frac{\partial \Omega}{\partial x} + mc^2 \gamma^0 \Omega - \frac{\hbar}{2} g^+(x)g(x)\Omega \]  

(37)

with the initial condition \( \Omega(0, x) = \Psi_0(x_0, x + x_D) = \Psi_0(x_0, x) \).

The probability density for the "proper time of arrival" is now

\[ P(\tau) = \int_0^\infty d\tau \int dx \Omega^+(\tau, x)g^+(x)g(x)\Omega(\tau, x) \]  

\[ \cdot \int dx \Omega^+(\tau, x)g^+(x)g(x)\Omega(\tau, x) \]  

(38)

### 2.1 Time of arrival with wide detectors

The detector should be characterized by the sensitivity function

\[ g(x) = \sqrt{\frac{2W}{\hbar}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{cases} 0 & : x < -\frac{\Delta x_D}{2} \\ \exp \left( -\frac{(x + x_D - \epsilon) - x}{\epsilon^2 - (x - \frac{\Delta x_D}{2} + \epsilon)^2} \right) & : -\frac{\Delta x_D}{2} \leq x < -\frac{\Delta x_D}{2} + \epsilon \\ 1 & : -\frac{\Delta x_D}{2} + \epsilon \leq x < \frac{\Delta x_D}{2} - \epsilon \\ \exp \left( -\frac{(x + \frac{\Delta x_D}{2} - \epsilon) - x}{\epsilon^2 - (x + \frac{\Delta x_D}{2} + \epsilon)^2} \right) & : \frac{\Delta x_D}{2} - \epsilon \leq x < \frac{\Delta x_D}{2} \\ 0 & : \frac{\Delta x_D}{2} \leq x \end{cases} \]  

(39)
with $\epsilon = 0.002\text{Å}$. The eq. 37 with the initial condition $\Omega(0,x) = \Psi_0(x_0,x)$ is solved numerically. The time development of $\Omega$ is approximated by

$$
\Omega(\tau + \Delta \tau) = \exp \left( -\frac{\Delta \tau}{\hbar} \left( \frac{\hbar c}{\gamma_1} \frac{\partial}{\partial x^1} + mc^2 \gamma_0 \right) - \Delta \tau g^+(x)g(x) \right) \Omega(\tau)
$$

$$
\approx \exp \left( -\frac{\Delta \tau}{2} g^+(x)g(x) \right) \exp \left( -\frac{\Delta \tau}{\hbar} \left( \frac{\hbar c}{\gamma_1} \frac{\partial}{\partial x^1} + mc^2 \gamma_0 \right) \right)
$$

We now discretize the proper time and the space. The first and the last operator can then be computed directly, the second operator is discretized by using the method of Wessels, Caspers and Wiegel [16].

The boundary conditions are walls at $x = -6\text{Å}$ and at $x = 4\text{Å}$, the time and space steps depend on the particle momentum: $p_0 = 0.25 - 0.75 (\Delta \tau = \Delta x = 0.001), p_0 = 1.0 (\Delta \tau = \Delta x = 0.00075), p_0 = 1.25 (\Delta \tau = \Delta x = 0.00075), p_0 = 1.5 (\Delta \tau = \Delta x = 0.0005), p_0 = 1.75 (\Delta \tau = \Delta x = 0.00043), p_0 = 2.0 (\Delta \tau = \Delta x = 0.000375)$. Fig. 2 shows the resulting expected values of the time of arrival for different momentums $p_0$ in the system in which the detector is at rest. Moreover the times calculated by the relativistic mechanics of point-particle are shown.

The error is approximated by

$$
error(T) = \frac{1}{\lambda - 1} \left| T(\lambda \cdot \Delta \tau) - T(\Delta \tau) \right|
$$

with $\lambda = 1.5$.

We find a good coincidence between the simulated results and those of the mechanics of point-particles. Only for very high momentums the expected times of the simulations are a bit smaller than those of the point-mechanics, because there is a probability for negative times of arrival in the simulations (see below).

The expected values in different reference frames are connected by eq. 38. The time of arrival $\tilde{t}_{RM}$ of the relativistic mechanic is connected to the result for a detector at rest $t_{RM}$ of the mechanic by the same factor:

$$
\tilde{t}_{RM} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} t_{RM}
$$

So we also have a good coincidence between the simulated results and those deduced from the relativistic mechanics in all reference frames.

In Fig. 3 probability densities in the system in which the detector is at rest are shown. The probability for negative times of arrival is zero for small momentums, but for momentums with are greater than one, we find a small probability for negative times.

Fig. 4 show the probability density for different system velocities $v$ with fixed particle energy $p_0 = 2.0mc$. Another question is, how the expected times depend on parameters of the detector. The detector width should be fixed at $\Delta x_D = 0.01\text{Å}$ and the particle momentum is $p_0 = 0.75mc$. We find, that the expected values for this parameters are nearly independent of the detector height $W$ over a wide range, only the total detection probability depends on the detector height $W$.

The normalized probability densities are also independent of the detector height. Moreover we examine the dependence, if the width of the detector is changed with fixed detector height $W = 1 \times 10^{-5} mc^2$ and particle momentum $p_0 = 0.75mc$. The expected values also show nearly dependence, the total detection probability shows a dependence on the detector width. The form of the probability densities do not change, they only becomes wider.
2.2 Time of arrival with point-like detectors

In this section, we will examine the limit of a point-like detector. The detector sensitivity should be characterized by

\[ g^+(x)g(x) = \kappa \cdot \begin{pmatrix} \delta(x) & 0 & 0 \\ 0 & \delta(x) & 0 \\ 0 & 0 & 0 \end{pmatrix} \]  

(42)

A non-relativistic particle-detector modeled by a \( \delta \)-function can be found in [5].

By integration of eq. (37) with the detector function eq. (42): \( \int_{-\epsilon}^{\epsilon} dx \) and do \( \epsilon \to 0 \), we get the boundary conditions at \( x = 0 \):

\[ \Omega_1(\tau, 0^+) - \Omega_1(\tau, 0^-) = 0 \]  

(43)

\[ \Omega_2(\tau, 0^+) - \Omega_2(\tau, 0^-) = 0 \]  

(44)

\[ \Omega_3(\tau, 0^+) - \Omega_3(\tau, 0^-) = -\frac{\kappa}{2c} \Omega_2(\tau, 0) \]  

(45)

\[ \Omega_4(\tau, 0^+) - \Omega_4(\tau, 0^-) = -\frac{\kappa}{2c} \Omega_1(\tau, 0) \]  

(46)

A solution of eq. (37) with the detector function eq. (42) with these boundary conditions is now

\[ \Omega(\tau, x) = \begin{cases} \Omega_{IN}(\tau, x) + \Omega_{REF}(\tau, x) & : x < 0 \\ \Omega_{TRA}(\tau, x) & : x > 0 \end{cases} \]  

(47)

with

\[ \Omega_{IN}(\tau, x) = \int dp \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + A_-(p) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \exp \left( \frac{p}{\hbar}x - \frac{E}{\hbar} \right) + \int dp \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \exp \left( -\frac{p}{\hbar}x + \frac{E}{\hbar} \right) \]

\[ \Omega_{REF}(\tau, x) = \int dp \frac{(\kappa/2c)(E + mc^2)}{2pc - (\kappa/2c)(E + mc^2)} \begin{pmatrix} A_+(p) \\ A_+(p) \end{pmatrix} \cdot \exp \left( \frac{p}{\hbar}x - \frac{E}{\hbar} \right) + \int dp \frac{pc(\kappa/2c)}{2pc - (\kappa/2c)(E + mc^2)} \begin{pmatrix} B_+(p) \\ B_+(p) \end{pmatrix} \cdot \exp \left( -\frac{p}{\hbar}x + \frac{E}{\hbar} \right) \]

\[ \Omega_{TRA}(\tau, x) = \int dp \frac{2pc}{2pc + (\kappa/2c)(E + mc^2)} \begin{pmatrix} A_+(p) \\ A_+(p) \end{pmatrix} \cdot \exp \left( \frac{p}{\hbar}x - \frac{E}{\hbar} \right) + \int dp \frac{2(E + mc^2)}{pc(\kappa/2c) + 2(E + mc^2)} \begin{pmatrix} B_+(p) \\ B_+(p) \end{pmatrix} \cdot \exp \left( -\frac{p}{\hbar}x + \frac{E}{\hbar} \right) \]
At $\tau = 0$ the reflection part is approximated to be small, so the initial value is $\Omega_{IN}(0,x) = N \cdot \Psi_0(x_0,x)$ with $N$ a normalization factor. So we get

$$A_+(p) = N \cdot \frac{2\eta}{(2\pi)^{3/4}} \frac{E + mc^2}{2E} \exp \left(-\frac{\eta^2(p - p_0)^2}{h^2} - i\frac{p}{\hbar} x_0 - i\frac{Ex_0}{\hbar c}\right)$$  \hspace{1cm} (48)

$$A_-(p) = 0$$  \hspace{1cm} (49)

$$B_+(p) = N \cdot \frac{2\eta}{(2\pi)^{3/4}} \frac{pe}{2E} \exp \left(-\frac{\eta^2(p + p_0)^2}{h^2} + i\frac{p}{\hbar} x_0 + i\frac{Ex_0}{\hbar c}\right)$$  \hspace{1cm} (50)

$$B_-(p) = 0$$  \hspace{1cm} (51)

The probability density of the "proper time of arrival" is now easy calculated:

$$P(\tau) = \frac{1}{\int_0^\infty d\tau |\Omega_1(\tau,0)|^2} |\Omega_1(\tau,0)|^2 = \frac{1}{\int_0^\infty d\tau |\Omega_{TRA,1}(\tau,0)|^2} |\Omega_{TRA,1}(\tau,0)|^2$$  \hspace{1cm} (52)

Fig. 2 shows also the expected values for the parameter $\kappa = 1.0c/1\text{Å}$ and the limit $\kappa \to 0$. The probability density $P(\tau)$ remains finite in the limit $\kappa \to 0$ and the wave function becomes the undisturbed Gaussian function. The results of $\kappa \to 0$ equals nearly those with wide detectors.

The expected times are nearly independent of $\kappa$, only for high momentums the time of arrival decreases with increasing $\kappa$. The reason is an increasing of the probability of negative times of arrival with increasing $\kappa$ (see Fig. 3).

3 Conclusion

Summarizing we introduce a relativistic algorithm to describe ideal, infinitesimal short measurements playing the role of the reduction-postulate in the non-relativistic quantum mechanics.

Assuming that space-like separated observables commute, the probabilities generated by this algorithm are the same we get if we use the standard-non-covariant reduction-postulate with the Dirac-equation.

Moreover we introduce a relativistic algorithm for continuous measurements, its non-relativistic limit is the algorithm of the Event-Enhanced Quantum Theory.

We discuss an application of it, the time of arrival of an electron at a detector.

First we examine numerically the case, if the detector is characterized by a wide, finite high sensitivity function. In doing so we find a good coincidence between the expected values of the time of arrival which results by the algorithm and the results of the relativistic point-mechanics. For very high momentum we find a small probability for negative time of arrival, so the expected values of the algorithm in these cases are a bit smaller than the results expected by the relativistic point-mechanics.

The results do not depend sensitively on the detector parameters over a wide range.

Second we examine the limit, if the detector sensitivity is characterized by a point-like, infinitesimal high function $\sim k\delta(x)$. For weak detectors ($\kappa \to 0$) the same results as in the case of a wide detector are found. For high momentums and $\kappa > 0$, the expected time decreases with increasing detector "height" $\kappa$.

Acknowledgments

I would like to thank Ph. Blanchard for many helpful discussions.
References

[1] Ph. Blanchard and A. Jadczyk. Event-enhanced formalism of quantum theory or Columbus solution to the quantum measurement problem. In V.P. Belavkin et al., editor, Quantum Communication and Measurement. Plenum Press, New York, 1995.

[2] Ph. Blanchard and A. Jadczyk. Quantum mechanics with event dynamics. Reports on Math. Phys. 36 (1995) 235.

[3] Ph. Blanchard and A. Jadczyk. Events and piecewise deterministic dynamics in event-enhanced quantum theory. Phys. Lett. A 263 (1995) 260.

[4] Ph. Blanchard and A. Jadczyk. Event-enhanced quantum theory and piecewise deterministic dynamics. Ann. Phys. 4 (1995) 583.

[5] Ph. Blanchard and A. Jadczyk. Time of events in quantum theory. Helv Phys Acta 69 (1996) 613.

[6] Ph. Blanchard and A. Jadczyk. Time and events. Int. J. Theor. Phys. 37 (1998) 227-233.

[7] A. Ruschhaupt. Simulations of barrier traversal and reflection times based on event enhanced quantum theory. Phys. Lett. A 250 (1998) 249.

[8] Y. Aharonov and D.Z. Albert. States and observables in relativistic quantum field theories. Phys. Rev. D 21 (1980) 3316.

[9] Y. Aharonov and D.Z. Albert. Can we make sense out of the measurement process in relativistic quantum mechanics? Phys. Rev. D 24 (1981) 359.

[10] Y. Aharonov and D.Z. Albert. Is the usual notion of time evolution adequate for quantum-mechanical systems? II. relativistic considerations. Phys. Rev. D 29 (1984) 228.

[11] H.-P. Breuer and F. Petruccione. Relativistic formulation of quantum-state diffusion. J. Phys. A 31 (1998) 33-52.

[12] H.-P. Breuer and F. Petruccione. A Lorentz covariant stochastic wave function dynamics for open systems. Phys. Lett. A 242 (1998) 205-210.

[13] L.P. Horwitz and C. Piron. Relativistic dynamics. Helv. Phys. Acta 46 (1973) 316.

[14] J.R. Fanchi. Review of invariant time formulations of relativistic quantum theories. Found. Phys. 23 (1993) 487.

[15] Ph. Blanchard and A. Jadczyk. Relativistic quantum events. Found. Phys. 26 (1996) 1669.

[16] P.P.F. Wessels, W.J. Caspers, and F.W. Wiegel. Discretizing the one-dimensional Dirac equation. Europhys.Lett. 46 (1999) 123-126.
Figure 1:
Initial state for particle momentum $p_0 = 0.75$

Square of the first component of the wave function (particle part)

Square of the forth component of the wave function (anti-particle part)
Figure 2:
Mean time of arrival versus particle momentum $p_0$, relativistic simulation with wide detector (circles with errorbars): detector height $W = 1 \times 10^{-5} mc^2$, detector width $\Delta x_D = 0.01\text{Å}$, other parameters see text, point-like detector: $\kappa \to 0$ (big dotted line), $\kappa = 1.0c/1\text{Å}$ (dashed line), relativistic mechanics $1\text{Å} \cdot \sqrt{1 + \frac{1}{p_0}}$ (small dotted line); the figure inside is a zoom of the right lower area of the figure outside.

Figure 3:
Probability density for the time of arrival for different particle momentum $p_0$, detector height $W = 1 \times 10^{-5} mc^2$, detector width $\Delta x_D = 0.01\text{Å}$, $p_0 = 0.75mc$ (solid line), $p_0 = 1.5mc$ (dotted line), $p_0 = 2.0mc$ (dashed-dotted line).
Figure 4:
Probability density for the time of arrival in different reference frames, particle momentum $p_0 = 2.0 \, mc$, detector height $W = 1 \times 10^{-5} \, mc^2$, detector width $\Delta x_D = 0.01\AA$, $v = 0.0c$ (solid line), $v = 0.5c$ (dotted line), $v = 0.9c$ (dashed-dotted line).

Figure 5:
Probability density for the time of arrival for different particle momentum $p_0$ with point-like detector, $p0 = 0.75mc$, $\kappa \to 0$ (small solid line), $p0 = 0.75mc$, $\kappa = 1.0c/1\AA$ (dotted line), $p0 = 2.0mc$, $\kappa \to 0$ (big solid line), $p0 = 2.0mc$, $\kappa = 1.0c/1\AA$ (dashed line).