HOMOLOGY GROUP AUTOMORPHISMS OF RIEMANN SURFACES

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Abstract. If \( \Gamma \) is a finitely generated Fuchsian group such that its derived subgroup \( \Gamma' \) is co-compact and torsion free, then \( S = \mathbb{H}^2/\Gamma \) is a closed Riemann surface of genus \( g \geq 2 \) admitting the abelian group \( A = \Gamma/\Gamma' \) as a group of conformal automorphisms. We say that \( A \) is a homology group of \( S \). A natural question is if \( S \) admits unique homology groups or not, in other words, is there are different Fuchsian groups \( \Gamma_1 \) and \( \Gamma_2 \) with \( \Gamma_1' = \Gamma_2' \)? It is known that if \( \Gamma_1 \) and \( \Gamma_2 \) are both of the same signature \( (0; k_1, \ldots, k_n) \), for some \( k \geq 2 \), then the equality \( \Gamma_1' = \Gamma_2' \) ensures that \( \Gamma_1 = \Gamma_2 \). Generalizing this, we observe that if \( \Gamma_j \) has signature \( (0; k_j, \ldots, k_j) \) and \( \Gamma_j' = \Gamma_j' \), then \( \Gamma_1 = \Gamma_2 \). We also provide examples of surfaces \( S \) with different homology groups. A description of the normalizer in \( \text{Aut}(S) \) of each homology group \( A \) is also obtained.

1. Introduction

Let \( S \) be a closed Riemann surface of genus \( g \geq 2 \) and let \( \text{Aut}(S) \) be its group of conformal automorphisms. In 1890, Schwarz \([7]\) proved that \( \text{Aut}(S) \) is finite and later, in 1893, Hurwitz \([4]\) obtained the upper bound \( |\text{Aut}(S)| \leq 84(\gamma - 1) \). Since then, the study of groups of conformal automorphisms of closed Riemann surfaces has been of interest in the community of Riemann surfaces and related areas. In 1987, Nakayama \([6]\) proved that if \( A < \text{Aut}(S) \) is an abelian group, then \( |A| \leq 4(g + 1) \) (if \( A \) is a cyclic group, then \( |A| \leq 4g + 2 \) \([9]\)).

An abelian group \( A < \text{Aut}(S) \) is called a homology group of \( S \) if there is not a closed Riemann surface \( R \) of genus \( \gamma > g \) admitting an abelian group \( B \) of conformal automorphisms such that \( R/B \) and \( S/A \) are isomorphic as Riemann orbifolds. In this case, \( S \) is a homology Riemann surface and \( (S, A) \) is a homology Riemann pair.

In terms of Fuchsian groups, the above can be described as follows \((2)\). Let \( \mathbb{H}^2 \) be the hyperbolic plane, let \( \Gamma < \text{PSL}_2(\mathbb{R}) = \text{Aut}(\mathbb{H}^2) \) be a Fuchsian group such that \( S/A = \mathbb{H}^2/\Gamma \) (as Riemann orbifolds) and let \( \Gamma' \) be it derived subgroup. Then \( A \) is a homology group of \( S \) if and only if: (i) \( \Gamma' \) is torsion free, (ii) \( S = \mathbb{H}^2/\Gamma' \) and (iii) \( A = \Gamma/\Gamma' \).

Note that, as the index of \( \Gamma' \) in \( \Gamma \) is finite (since \( A \) is finite), conditions (ii) and (iii) necessarily assert that \( S/A \) has genus zero, so it has signature of the form \((0; k_1, \ldots, k_{n+1})\), for some \( n \geq 2 \) and \( k_j \geq 2 \). Condition (i) is equivalent to satisfies Maclachlan’s condition \([5]\)

\[ \text{lcm}(k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_{n+1}) = \text{lcm}(k_1, \ldots, k_{n+1}), \quad \forall j = 1, \ldots, n + 1, \]

where \( \text{lcm} \) denotes the “least common multiple”.

Both, (i) a description of those hyperelliptic homology Riemann surfaces and (ii) an algebraic representation of those homology Riemann pairs \((S, A)\) where \( S/A \) has a triangular signature, were given in \([2]\). In the same paper it was noticed that a homology group cannot be isomorphic to \( \mathbb{Z}_p \), for \( p \) prime, nor to \( \mathbb{Z}_2^2 \). But it can be isomorphic to a cyclic group of...
order different from a prime. For instance, if \( \Gamma = \langle x_1, x_2, x_3 : x_1^2 = x_2^3 = x_3^{10} = x_1x_2x_3 = 1 \rangle \), then \( A = \Gamma/\Gamma' \cong \mathbb{Z}_{10} \) is a homology group for the homology Riemann surface \( S = \mathbb{H}^2/\Gamma' \).

Let \( (S = \mathbb{H}^2/\Gamma', A = \Gamma/\Gamma') \) be a homology Riemann pair such that all the cone points of \( S/A \) have the same order \( k \geq 2 \), i.e., \( \Gamma \) has signature \( (0; k, n+1, k) \) for some \( k, n \geq 2 \), then \( A \cong \mathbb{Z}_n^k \). In this particular situation, we say that \( A \) is a \emph{generalized Fermat group} of type \((k, n)\) and that \( S \) is a \emph{generalized Fermat curve} of type \((k, n)\). By the Riemann-Hurwitz formula, the genus of \( S \) is \( g = 1 + k^{n-1}((n-1)(k-1)-2)/2 \), so (i) \((k-1)(n-1) > 2\) and (ii) \( n \) is uniquely determined by \( g \) and \( k \). An algebraic curve description for \( S \) was obtained in [1] (see Section 2.2).

In [3] it was proved that a homology Riemann surface \( S \) admits at most one generalized Fermat group of a fixed type \((k, n)\), equivalently, if \( \Gamma_1 \) and \( \Gamma_2 \) are both Fuchsian groups with the same signature \((0; k, n+1, k)\) such that \( \Gamma_1' = \Gamma_2' \), then \( \Gamma_1 = \Gamma_2 \).

This does not rule out the possibility for \( S \) to have generalized Fermat groups of different types. We start by observing that this is not the situation.

**Theorem 1.** A closed Riemann surface admits at most one generalized Fermat group.

In terms of Fuchsian groups, the above uniqueness result is equivalent to the following commutator rigidity property.

**Corollary 1.** For \( j = 1, 2 \), let \( \Gamma_j \subset \text{Aut}(\mathbb{H}^2) \) be a co-compact Fuchsian group with signature \((0; k, n+1, k_j)\), where \( k_j, n_j \geq 2 \) and \((n_j-1)(k_j-1) > 2\). If \( \Gamma_1' = \Gamma_2' \), then \( \Gamma_1 = \Gamma_2 \).

By Theorem 1, a homology Riemann surface admits at most one generalized Fermat group. One may wonder if this uniqueness property holds for general homology groups. As a consequence of the results in [8], the generic homology closed Riemann surface admits only one homology group. In Section 4, we show explicit examples to see that uniqueness of homology groups is not always true (and, moreover, they might be either normal or non-normal subgroups). In Example 2, the surface has genus two and it has two different conjugated homology groups isomorphic to \( \mathbb{Z}_6 \) (in particular, these homology groups are not normal subgroups). In Example 3, the surface is hyperelliptic of genus \( g \geq 2 \) even, and it admits two non-isomorphic homology groups, one isomorphic to \( \mathbb{Z}_6 \) and the other isomorphic to \( \mathbb{Z}_6 \times \mathbb{Z}_2 \), both of them being normal subgroups.

As noted from Example 2, a homology group \( A \) of \( S \) might not be a normal subgroup of \( \text{Aut}(S) \). We proceed to provide a description of the normalizers \( N_A \) of \( A \) in \( \text{Aut}(S) \). First, we need some definitions. Let \( \mu_{S,A} \) be the least common multiple of the branch orders of the conical points of \( S/A \). For each \( p \in S \), with a non-trivial \( A \)-stabilizer \( A_p \), set \( n_p := [\mu_{S,A}]/[A_p] \). Let \( \text{Aut}(S^{orb,A}) \) be the Riemann orbifold whose underlying Riemann surface is \( S \) and its cone points are those points \( p \in S \) with non-trivial \( A_p \) and \( n_p \geq 2 \) (which is the corresponding cone order). Let \( \text{Aut}(S^{orb,A}) \) be the group of conformal automorphisms of \( S \) keeping invariant the above cone points together their orders. One may see that \( N_A \leq \text{Aut}(S^{orb,A}) \) and, if all cone points of \( S/A \) have the same order, then \( S = S^{orb,A} \).

In general, for an abelian group (not necessarily a homology group) \( A < \text{Aut}(S) \), it might happen that \( N_A \neq \text{Aut}(S^{orb,A}) \). For instance, if we consider Klein’s surface of genus three, defined by \( S := \{ [x : y : z] \in \mathbb{P}^2 : y^7 = xz^4(x-z)^3 \} \), then \( \text{Aut}(S) \cong \text{PSL}_2(7) \) (of order 168, the maximum possible). If \( A = \{ [x : y : z] \mapsto [x : e^{2\pi i/7}y : z] \} \cong \mathbb{Z}_7 \), then \( S/A \) has signature \((0; 7, 7, 7)\). If \( F_7 := \{ [x : y : z] \in \mathbb{P}^2 : x^7 + y^7 + z^7 = 0 \} \) (the classical Fermat curve of degree \( 7 \), which has genus 15) and \( B = \{ [x : y : z] \mapsto [e^{2\pi i/7}x : y : z], [x : y : z] \mapsto [x : e^{2\pi i/7}y : z] \} \cong \mathbb{Z}_7^2 \), then \( S/A \) is isomorphic as orbifold to \( F_7/B \). So \( A \) is not a homology group of \( S \). In this case, \( S^{orb,A} = S \), so \( \text{Aut}(S^{orb,A}) = \text{Aut}(S) \cong \text{PSL}_2(7) \). As \( A \) is not a
normal subgroup, \( N_A \neq \text{Aut}(S^{\text{orb}, A}) \). In the next, we observe that this is not the case for \( A \) a homology group.

**Theorem 2.** If \( A \) is a homology group of the closed Riemann surface \( S \), then \( N_A = \text{Aut}(S^{\text{orb}, A}) \).

## 2. Preliminaries and known facts

### 2.1. Riemann orbifolds

A Riemann orbifold \( O \) is provided by a Riemann surface \( S \), called its underlying Riemann surface structure, together a discrete collection of points, say \( p_1, p_2, \ldots \in S \), called its cone points, where each of these cone points \( p_j \) has associated an integer \( k_j \geq 2 \), called its cone order. If \( S \) is a closed Riemann surface of genus \( g \) (we also say that the orbifold has genus \( g \)), then the number of its cone points is finite, say \( p_1, \ldots, p_n \in S \) and, in this case, the tuple \( (g; k_1, \ldots, k_n) \) is called the signature of \( O \).

A conformal homeomorphism between two Riemann orbifolds is a conformal homeomorphism between the corresponding Riemann surfaces sending cone points bijectively to cone points and preserving the cone orders. If both orbifolds are the same, then we talk of a conformal automorphism of \( O \) and we denote by \( \text{Aut}(O) \) its group of conformal automorphisms.

If \( O \) is a Riemann orbifold and \( H < \text{Aut}(O) \) acts discontinuously (in general it will be finite), then the quotient \( O/H \) is again a Riemann orbifold. Let us denote by \( \pi : O \to O/H \) the canonical quotient map. Let \( p \in O \) and \( H(p) \) be its \( H \)-stabilizer, say of order \( m \geq 1 \). If \( p \) is not a cone point of \( O \) and \( m \geq 2 \), then \( \pi(p) \) is a cone point of \( O/H \) of order \( m \). If \( p \) is a cone point of order \( n \geq 2 \), then \( \pi(p) \) is a cone point of \( O/H \) of order \( mn \).

### 2.2. Generalized Fermat curves

Let \( k \geq 2 \) and \( n \geq 2 \) be such that \((n-1)(k-1) > 2\). Let \( S \) be a generalized Fermat curve of type \((k,n)\) and let \( A \equiv \mathbb{Z}_k^n \) be a generalized Fermat group of type \((k,n)\) of \( S \). We may identify the quotient orbifold \( S/A \) with the Riemann sphere \( \mathbb{C} \) and its cone points being \( \infty, 0, 1, \ldots, \lambda_{n-2} \). Let \( \pi_{S,A} : S \to \mathbb{C} \) be a Galois branched covering induced by the action of \( A \). Below we summarize some of the previous results on these objects.

**Theorem 3** ([1, 3]). With the above notations, the following hold.

(a) \( A \) is the unique generalized Fermat group of type \((k,n)\) of \( S \), in particular, \( A \) is a normal subgroup of \( \text{Aut}(S) \).

(b) An algebraic model for \( S \) is the following non-singular projective algebraic curve (a fiber product of \((n-1)\) classical Fermat curves of degree \( k \))

\[
C_S : \left\{ \begin{array}{l}
x_1^k + x_2^k + x_3^k = 0 \\
\lambda_1 x_1^k + x_2^k + x_4^k = 0 \\
\vdots \\
\lambda_{n-2} x_1^k + x_2^k + x_{n+1}^k = 0
\end{array} \right\} \subseteq \mathbb{P}^n.
\]

(c) In this algebraic model, (i) \( A = \langle a_1, \ldots, a_n \rangle < \text{Aut}(C_S) \), where \( a_j \) is multiplication of the \( j \)-th coordinate by a primitive \( k \)-root of \( 1 \), and (ii) the Galois branched covering map \( \pi_{S,A} \), in this algebraic model, is given by

\[
\pi_{C_S} : C_S \to \mathbb{C} : [x_1 : \cdots : x_{n+1}] \mapsto \left( \frac{x_1}{x_1} \right)^k.
\]

(d) If \( a_{n+1} = (a_1 \cdot \cdots \cdot a_n)^{-1} \), then (i) every element of \( A \) acting with fixed points is a power of some \( a_j, j = 1, \ldots, n+1 \), and every fixed point of a non-trivial power of \( a_j \) is also a fixed point of \( a_j \).
As a consequence of the above result, and using the fact that $S$ is uniformized by the derived subgroup of the uniformizing Fuchsian group of the orbifold $S/A$, there is a short exact sequence $1 \to A \to \text{Aut}(S) \xrightarrow{\rho} \text{Aut}(S/A) \to 1$, where, under our identification, Aut$(S/A)$ is the subgroup of Möbius transformations keeping invariant the collection

$$\{p_1 = \infty, p_2 = 0, p_3 = 1, p_4 = \lambda_1, \ldots, p_{n+1} = \lambda_{n+2}\}.$$  

In the above, the surjective homomorphism $\rho$ is defined by: $\pi_{S/A} \circ a = \rho(a) \circ \pi_{S/A}$, for every $a \in A$. If $T \in \text{Aut}(S/A)$, then it defines a permutation $\sigma_T \in \Im_{n+1}$ of these points. If $a_T \in A$ is such that $\rho(a_T) = T$, then the conjugation action of $a_T$ on the collection $\{\alpha_1, \ldots, \alpha_{n+1}\}$ is again $\sigma_T$ [1].

**Remark 1.** As a generalized Fermat pair $(S, A)$ corresponds to the derived subgroup (which is a characteristic subgroup) of a Fuchsian group $\Gamma$, such that $S/A = \Im^2/\Gamma$, the following lifting property holds. Let $(S_1, A_1)$ and $(S_2, A_2)$ be two generalized Fermat pairs, both of the same type $(k, n)$. Let $\pi_{S_j, A_j}: S_j \to S_j/A_j$, $j = 1, 2$, be a regular (branched) covering with deck group $A_j$. Then, for any biholomorphism (of orbifolds) $\psi : S_1/A_1 \to S_2/A_2$ there is a biholomorphism $\eta : S_1 \to S_2$ such that $\pi_{S_j, A_j} \circ \eta = \psi \circ \pi_{S_1, A_1}$.

### 3. Proof of Theorems 1 and 2

#### 3.1. Proof of Theorem 1

Let $S$ be a homology closed Riemann surface admitting generalized Fermat groups $A \cong \mathbb{Z}^n_q$ and $B \cong \mathbb{Z}^n_p$, where $k, l \geq 2$. Let us recall that $n \geq 2$ (respectively, $m \geq 2$) is uniquely determined by the genus of $S$ and $k$ (respectively, $l$). As there is only one generalized Fermat group of a fixed type, if $k = l$, then $A = B$. So, let us assume that $k \neq l$.

As $A$ is a normal subgroup, the group $B$ induces an abelian group $\tilde{B}$ of conformal automorphisms of the Riemann orbifold $O_A = S/A$ of signature $(0; k, \ldots, k)$ (which can be identified with the Riemann sphere). So $\tilde{B}$ is either isomorphic to a cyclic group $\mathbb{Z}_q$, $q \geq 2$, or to the Klein group $\mathbb{Z}^2_2$. (i) If $\tilde{B} \cong \mathbb{Z}_q$, then $O_A/B$ has signature of the form $(0; k, \ldots, k, q, \hat{q}, \ldots, qk)$, where $\alpha \geq 1$, $\beta \in \{0, 1, 2\}$ and $n + 1 = \alpha q + \beta$. (ii) If $\tilde{B} \cong \mathbb{Z}^2_2$, then $O_A/B$ has signature of the form $(0; k, \ldots, k, 2k, \hat{p}, \ldots, 2\hat{p}, \hat{p} \delta)$, where $\alpha \geq 0$, $\beta_1, \beta_2 \in \{0, 1, 2\}$, $\beta_1 + \beta_2 = 3$ and $n + 1 = 4\alpha + 2\beta_1$.

Similarly, the group $A$ induces an abelian group $\tilde{A}$ of conformal automorphisms of the Riemann orbifold $O_B = S/B$ of signature $(0; l, \ldots, l)$. So $\tilde{A}$ is either isomorphic to a cyclic group $\mathbb{Z}_p$, $p \geq 2$, or to the Klein group $\mathbb{Z}^2_2$. (i) If $\tilde{A} \cong \mathbb{Z}_p$, then $O_B/A$ has signature of the form $(0; l, \hat{p}, \ldots, l, pl, \hat{p} \ldots, pl)$, where $\alpha \geq 1$, $\beta \in \{0, 1, 2\}$ and $m + 1 = \alpha p + \beta$. (ii) If $\tilde{A} \cong \mathbb{Z}^2_2$, then $O_B/A$ has signature of the form $(0; l, \ldots, l, 2l, \hat{p}, \ldots, 2\hat{p}, \hat{p} \delta)$, where $\alpha \geq 0$, $\beta_1, \beta_2 \in \{0, 1, 2\}$, $\beta_1 + \beta_2 = 3$ and $m + 1 = 4\alpha + 2\beta_1$.

As $O_A = O_B = S/(A, B)$, the two orbifolds must have the same cone points and respective cone orders. We proceed to check this in each of the possible cases.

1. If $\tilde{A} \cong \mathbb{Z}_p$ and $\tilde{B} \cong \mathbb{Z}_q$, then (as $k \neq l$) we must have $k = pl$ and $qk = l$, from which $pq = 1$, a contradiction.

2. If $\tilde{A} \cong \mathbb{Z}_p$ and $\tilde{B} \cong \mathbb{Z}^2_2$, then (as $k \neq l$) we must have that $k = pl$, $\alpha = \hat{p}$, and either:
   
   (a) $\beta_2 = 0$, $2k = l$, $\beta_1 = \hat{\alpha}$, $n + 1 = 4\alpha + 2\beta_1$ and $m + 1 = \hat{\alpha} p + \hat{\beta}$.
   
   (b) $\beta_1 = 0$, $l = 2$, $\hat{\alpha} = \beta_2$, $n + 1 = 4\alpha$ and $m + 1 = \hat{\alpha} p + \hat{\beta}$.

In case (a), as $k = pl$ and $2k = l$, we must have $2p = 1$, a contradiction. In case (b), $k = 2p$, $l = 2$, $n + 1 = 4\alpha$ and $m + 1 = 3p + \alpha$, where $\alpha \in [1, 2]$. The genus $g$ of $S$ has the
form  
\[ g = 1 + k^{n-1}(n-1)(k-1) - 2)/2, \quad g = 1 + 2^{m-1}(m-3)/2. \]

If \( a = 1 \), then it follows that \( 3 	imes 2^{3p} = 32p^2 \), which is not possible for \( p \geq 2 \). If \( a = 2 \), then \( 2^{3p} = 256p^2 \), which is neither possible.

(3) If \( \tilde{A} \cong \mathbb{Z}_2^2 \cong \tilde{B} \), then (as \( k \neq l \)) we have the following possibilities:
(a) \( k = 2l \), and \( 2k = l \), which is a contradiction as \( l \geq 2 \).
(b) \( k = 2l, 2k = 2l \), a contradiction as \( k \geq 2 \).
(c) \( k = 2, 2k = l, 2 = l \), a contradiction, as \( l \geq 2 \).
(d) \( k = 2, 2k = 2l, 2 = l \), from which \( k = l \), a contradiction.

3.2. Proof of Theorem 2. Let \( S \) be a closed Riemann surface of genus \( g \geq 2 \) and let \( A < \text{Aut}(S) \) be a homology group of \( S \). Then the homology orbifold \( O_{S,A} = S/A \) has signature \((0;k_1,...,k_{n+1})\), where we may assume \( 2 \leq k_1 \leq k_2 \leq \cdots \leq k_{n+1} \) (satisfying Maclachlan’s condition (1)).

Without loss of generality, we may assume that the cone points of \( O_{S,A} \) are given by \( \infty \) (of order \( k_1 \)), \( 0 \) (of order \( k_2 \)), \( 1 \) (of order \( k_3 \)), \( \lambda_1 \) (of order \( k_4,...,k_{n-2} \)), and \( \lambda_n \) (of order \( k_{n+1} \)). Let \( \mu = \text{lcm}(k_1,...,k_{n+1}) \) and let \( P_A : S \to \tilde{S} \) be a regular branched covering, with deck group \( A \), whose branch values are the above cone points.

Let us consider the homology orbifold \( O_{S,A}^{\mu} \) of signature \((0;\mu^{n+1},\mu)\) where the cone points are the same as for \( O_{S,A} \) (we have only changed the order of them). As previously observed, the homology cover of \( O_{S,A}^{\mu} \) is represented by the algebraic curve

\[
C_{S,A} = \left\{ \begin{array}{rl}
x_1^n + x_2^n + x_3^n &= 0 \\
\lambda_1 x_1^n + x_2^n + x_3^n &= 0 \\
\vdots & \vdots \\
\lambda_{n-2} x_1^n + x_2^n + x_{n+1}^n &= 0 
\end{array} \right\} \subseteq \mathbb{P}^n,
\]

and the corresponding homology group \( H_A \cong \mathbb{Z}_2^n \) (that is, \( C_{S,A}/H = O_{S,A}^{\mu} \)) is generated by the transformations \( a_j \) (multiplication of the \( j \)-coordinate by \( w_j = e^{2\pi i/j} \)). Let

\[
\pi_A : C_{S,A} \to \tilde{S} : [x_1 : \cdots : x_{n+1}] \mapsto -(x_2/x_1)^\mu,
\]

which is a regular branched covering, with deck group \( H_A \) and whose branch values are \( \infty, 0, 1, \lambda_1, \ldots, \lambda_{n-2} \).

Let \( K_A \triangleleft H_A \) be the subgroup generated by the elements \( a_1^{k_1}, a_2^{k_2}, \ldots, a_n^{k_n} \) and \((a_1 \cdots a_n)^{k_{n+1}}\). The orbifold \( C_{S,A}/K_A \) has a Riemann surface structure \( S^* \) admitting the Abelian group \( H_A/K_A \) as group of conformal automorphisms. By the construction \( H_A/K_A \) is isomorphic to \( A \) and \( S^* \) is a homology cover of \( O_{S,A} \). So, we may assume \( S = S^* \) and \( A = H_A/K_A \).

Let \( Q_A : C_{S,A} \to S \) be a regular branched covering, with deck group \( K_A \), such that \( \pi_A = P_A \circ Q_A \). In this case, \( S^{orb,A} = C_{S,A}/K_A \). Note that the subgroup \( K_A \) is uniquely determined by the branch values of the cone points of the orbifold \( O_{S,A} \).

Let \( \Gamma \) be a Fuchsian group such that \( \mathbb{H}^2/\Gamma = O_{S,A}^\mu \), so \( C_{S,A} = \mathbb{H}^2/\Gamma^* \). By the uniqueness of \( K_A \), there is a unique subgroup \( K \) of \( \Gamma \), containing \( \Gamma^* \) such that \( \mathbb{H}^2/K = C_{S,A}/K_A \). We observe that \( \Gamma^* \) is the smallest normal subgroup \( U \) of \( K \) such that \( K/U \) is isomorphic to \( K_A \). Let \( \phi \in \text{Aut}(C_{S,A}/K_A) = \text{Aut}(S^{orb,A}) \) and let \( \eta \in \text{Aut}(\mathbb{H}^2) \) be a lifting of \( \phi \) (so it normalizes \( K \)). By the uniqueness of \( \Gamma^* \) in \( K \), \( \eta \) keeps invariant it, so it descends to an automorphisms \( \psi \) of \( C_{S,A} \). By the uniqueness of the generalized Fermat group \( H_A = \Gamma^*/\Gamma^* [3] \), it is also invariant under conjugation by \( \psi \). As \( \psi \) is a lifting of \( \phi \), \( K_A \) is also invariant under \( \psi \). It follows that \( \phi \) normalizes \( A \).
4. Examples

Example 1. Let $S$ be the genus three hyperelliptic Riemann surface defined by the hyperelliptic curve $w^2 = u^8 - 1$. This surface admits the following automorphisms:

\[
\alpha(u, w) = \left(\frac{\sqrt{7} u}{u}, w\right), \quad \beta(u, w) = (-u, -w), \quad \gamma(u, w) = \left(\frac{1}{u}, iw/u^4\right),
\]

of respective orders 8, 2 and 4. If $A = \langle \alpha, \beta \rangle \cong \mathbb{Z}_8 \times \mathbb{Z}_2$, then $S/A$ has signature $(0; 2, 2, 8)$. It can be checked that $A$ is a homology group of $S$. In this case, the points in $S$ projecting to those of order 2 in $S/A$ are the eight Weierstrass points and these are the cone points of the orbifold $S^{\text{orb}_A}$, each of them with cone order 4, so $\text{Aut}(S^{\text{orb}_A}) = \text{Aut}(S)$. Theorem 2 asserts that $A$ is a normal subgroup of $\text{Aut}(S)$.

Example 2. Let $S$ be the genus two Riemann surface defined by the hyperelliptic curve $y^2 = x(x^4 - 1)$. This surface admits the order six automorphism

\[
\eta(x, y) = \left(i(1 + x)/(1 - x), 2(1 - iy)/(x - 1)^3\right)
\]

and $\eta^\prime(x, y) = (x, -y)$ is the hyperelliptic involution. If $A = \langle \eta \rangle \cong \mathbb{Z}_6$, then one may see that the quotient orbifold $S/A$ has signature $(0; 2, 2, 3, 3)$ and $A$ is a homology group of $S$. On $S$ we also have the order four automorphism $\rho(x, y) = (-x, iy)$. Then $B = \rho A \rho^{-1} \cong \mathbb{Z}_6$ is also a homology group of $S$. As

\[
\rho \circ \eta \circ \rho^{-1}(x, y) = \left(-i(1 - x)/(1 + x), -2(1 - iy)/(1 + x)^3\right) \notin A,
\]

we see that $A \neq B$. In particular, $S$ has two different homology groups, both isomorphic to $\mathbb{Z}_6$, and $N_A \neq \text{Aut}(S)$.

Example 3. Let $g \geq 2$ be an even integer and let $S$ be the genus $g$ Riemann surface defined by the hyperelliptic curve $y^2 = x^{2g+2} - 1$. This Riemann surface admits the order $2g + 2$ automorphism $\alpha(x, y) = (e^{2\pi i/(g+1)}x, y)$ and $\tau(x, y) = (x, -y)$ is its hyperelliptic involution. If $A = \langle \alpha, \tau \rangle \cong \mathbb{Z}_{2g+2} \times \mathbb{Z}_2$, then one may see that the quotient orbifold $S/A$ has signature $(0; 2, 2, g + 2, 2g + 2)$ and $A$ is a homology group of $S$. Similarly, if $B = \langle \alpha^2, \tau \rangle \cong \mathbb{Z}_{2g+2}$, then $S/B$ has signature $(0; 2, 2, 2g + 1, 2g + 1)$ and $B$ is again a homology group of $S$. In particular, $S$ has two different homology groups, one isomorphic to $\mathbb{Z}_{2g+2}$ and the other to $\mathbb{Z}_{2g+2} \times \mathbb{Z}_2$. Note that, by Theorem 2, $N_A = \text{Aut}(S)$ as the orbifold points of $S^{\text{orb}_A}$ are exactly the Weierstrass points, each one with cone order $g + 1$; in particular, $A$ is a normal subgroup. Similarly, it can be seen that $B$ is also a normal subgroup.

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