These lecture notes review current progress on the class of conformal theories which may be studied by quantizing the conformal Toda dynamics. After summarizing recent developments of the quantum Liouville theory, one recalls how two-dimensional black holes come out from the non-abelian Toda systems, and reviews the geometrical interpretation of the $A_n$-Toda theories, just put forward, that relate W geometries with the external geometry of particular (W) surfaces.

1 Introduction

The connection between conformally invariant field theories (CIFT) and integrable models has proven to be more and more useful in recent years. A particular role is played in this connection by the conformal Toda theories that generate, upon quantization, a whole class of the most important CIFT’s. Two-dimensional gravity in the conformal gauge is notoriously equivalent to the Liouville theory which is the conformal Toda theory associated with the Lie algebra $A_1$. A step to generalize this situation was made in [3] where it was shown that the Toda theory associated with any given simple Lie algebra gives two Noether realizations of the corresponding W-algebra. Thus, if the above $A_1$-Liouville scheme is repeated for the other Lie algebras, there should exist generalizations of two-dimensional gravity (called W-gravities) which are invariant by generalized diffeomorphisms, and coincide with the conformal Toda theories when a particular local coordinate frame is used.
These lecture notes cover recent developments.

In section 2, Liouville theory is discussed using the operator approach [3, 12, 14, 15, 17], where the quantum-group structure plays a fundamental role. After recalling some background material, one reviews recent results:

1) The complete derivation [20] of the holomorphic operator-algebra (braided category) where the role of the quantum group structure is fully established. The main new result here is that the fusion and braiding are not completely given by the quantum-group 3-j or 6-j symbols, as is usually assumed: There are coupling constants, which are not trigonometrical functions, and are not determined solely by the quantum-group symmetry.

2) The derivation [17] of the gravity-matter coupling in the weak-coupling regime, where the present method is found to agree with matrix-model calculations. One crucial point here is to show that the continuation in the number of screening operators which is made in the Coulomb gas picture is equivalent to the symmetry between quantum-group spin $J$ and $-J - 1$ put forward in my previous study [13, 14] of the strong-coupling regime.

The remaining sections cover recent progress about the more general Toda theories, considered at the purely classical level.

In section 3, it is first recalled how Toda theories are associated with any given embedding of $\mathfrak{sl}(2)$ into a given simple algebra. This defines a gradation. If the subgroup of gradation zero is non-abelian, the Toda theory is called non-abelian. The point of section 3 is to review the recent discovery [24] of the black-hole background metric of these non-abelian Toda theories.

The basic point of section 4 is the recent proposal [1, 2] that one can regard the W-geometries as the extrinsic geometry of particular two dimensional surfaces (W-surface) embedded into target spaces that are higher dimensional Kähler manifolds (We will restrict ourselves to the simplest particular situation, i.e. our target space is $\mathbb{C}P^n$ which corresponds to the $A_n$-type W-geometry). Instead of introducing higher-spin gauge generators, our approach makes use of the extrinsic curvatures of the embedded surface at its regular points, and relates it with the Toda dynamics mentioned above. The main virtue of our approach is that it is very simple to begin with. A W surface is characterized by the specific chiral structure of its embedding which we call chiral for short.

Section 5 deals with the geometry of the Toda hierarchy. The dynamical variables of this hierarchy are shown to give particular coordinates of the higher dimensional Kähler manifolds. They are dealt with by means of the free-fermion formalism, which is shown to be deeply connected with the concept of analytic continuation and with its W-generalizations. An important point here is to show that this allows us to extend the W transformations as reparametrizations of the target space, so that they become linear.

Finally, in section 6, we first reformulate our approach in terms of the intrinsic geometry of the family of associated surfaces in the Grassmannians $G_{n+1,k+1}$, $k =
1, \cdots, n. This is needed to study singular points and global aspects of W-surfaces following the general scheme of refs. [1, 2], where these surfaces were shown to be instanton solutions of the associated non-linear $\sigma$ models. The aim is to establish the generalization of the Gauss-Bonnet theorem to the W-surfaces discussed above. The instanton-number associated with each mapping are the global invariant of W-geometry. They are connected with the singularity indices of the W-surface.

2 Two dimensional gravity

2.1 The classical structure

Let us first recall some feature of the classical Liouville dynamics. In the conformal gauge, it is governed by the action:

$$S = \frac{1}{4\pi} \int d^2 x \sqrt{\hat{g}} \left\{ \frac{1}{2} \hat{g}^{ab} \partial_a \Phi \partial_b \Phi + e^{2\sqrt{\gamma} \Phi} + \frac{1}{2\sqrt{\gamma}} R_0 \Phi \right\}$$  \hspace{1cm} (2.1)

$\hat{g}_{ab}$ is the fixed background metric. We work for fixed genus, and do not integrate over the moduli. As is well known, one can choose a local coordinate system such that $\hat{g}_{ab} = \delta_{ab}$. Thus we are reduced to the action

$$S = \frac{1}{4\pi} \int d\sigma d\tau \left( \frac{1}{2} \left( \frac{\partial \Phi}{\partial \sigma} \right)^2 + \frac{1}{2} \left( \frac{\partial \Phi}{\partial \tau} \right)^2 + e^{2\sqrt{\gamma} \Phi} \right)$$ \hspace{1cm} (2.2)

where $\sigma$ and $\tau$ are the local coordinates. The complex structure is assumed to be such that the curves with constant $\sigma$ and $\tau$ are everywhere tangent to the local imaginary and real axis respectively. In a typical situation, one may work on the cylinder $0 \leq \sigma \leq 2\pi$, $-\infty \leq \tau \leq \infty$ obtained by an appropriate mapping from one of the handles of a general Riemann surface, and we shall do so in the present article. The action $2.2$ corresponds to a conformal theory such that $\exp(2\sqrt{\gamma} \Phi) d\sigma d\tau$ is invariant. The classical equivalent of the chiral vertex operators may be obtained very simply[4, 5, 15] by using the fact that the field $\Phi(\sigma, \tau)$ satisfies the equation

$$\frac{\partial^2 \Phi}{\partial \sigma^2} + \frac{\partial^2 \Phi}{\partial \tau^2} = 2\sqrt{\gamma} e^{2\sqrt{\gamma} \Phi}$$ \hspace{1cm} (2.3)

if and only if

$$e^{-\sqrt{\gamma} \Phi} = i \sqrt{\gamma} \sum_{j=1,2} f_j(x_+) g_j(x_-); \quad x_{\pm} = \sigma \mp i\tau$$ \hspace{1cm} (2.4)

where $f_j$ (resp. $g_j$), which are functions of a single variable, are solutions of the same Schrödinger equation

$$- f_j'' + T(x_+) f_j = 0, \quad \text{resp.} \quad - g_j'' + T(x_-) g_j.$$ \hspace{1cm} (2.5)
The solutions are normalized so that their Wronskians $f_1 f_2 - f_1 f_2'$ and $g_1 g_2 - g_1 g_2'$ are equal to one. The proof of this basic fact is straightforward [4, 5, 15]. The potentials $T(x_+)$ and $\overline{T}(x_-)$ are the two components of the stress-energy tensor, and, after quantization, Eqs. (2.3) become the Virasoro Ward-identities associated with the vanishing of the singular vector at the second level. As a result the Liouville theory also describes minimal models provided the coupling constant $\gamma$ is taken to be negative. This is how, we shall treat the matter fields. For the dynamics associated with the action Eq. (2.2), $\tau$ is the time variable, and the canonical Poisson brackets are

$$\{\Phi(\sigma_1, \tau), \frac{\partial}{\partial \tau} \Phi(\sigma_2, \tau)\}_{P.B.} = 4 \pi \delta(\sigma_1 - \sigma_2), \quad \{\Phi(\sigma_1, \tau), \Phi(\sigma_2, \tau)\}_{P.B.} = 0 \quad (2.6)$$

The cylinder $0 \leq \sigma \leq 2\pi, -\infty \leq \tau \leq \infty$ may be mapped on the complex plane of $z = e^{\tau + i\sigma}$, and the above Poisson brackets lead to the usual radial quantization.

A priori, any two pairs $f_j$ and $g_j$ of linearly independent solutions of Eq. 2.3 are suitable. In this connection, it is convenient to rename the functions $g_j$ by letting $\overline{f}_1 = -g_2$, $\overline{f}_2 = g_1$. Then one easily sees that Eq. 2.4 is left unchanged if $f_j$ and $\overline{f}_j$ are replaced by $\sum_k M_{jk} f_k$ and $\sum_k M_{kj} \overline{f}_k$, respectively, where $M_{jk}$ is an arbitrary constant matrix with determinant equal to one. Eq. 2.4 is $sl(2, C)$-invariant with $f_j$ transforming as a representation of spin $1/2$. At the quantum level, the $f_j$’s and $\overline{f}_j$’s become operators that do not commute, and the group $sl(2)$ is deformed to become the quantum group $U_q(sl(2))$. This structure plays a crucial role at the quantum level, and we now elaborate upon the classical $sl(2)$ structure where the calculations are simple.

At the classical level, it is trivial to take Eq. 2.4 to any power. For positive integer powers $2J$, one gets (letting $\beta = i \sqrt{2}$)

$$e^{-2J \sqrt{\gamma}} \Phi = \sum_{M=-J}^{J} \frac{\beta^{2J} (-1)^{J-M} (2J)!}{(J+M)! (J-M)!} \left( f_1(x_+) \overline{f}_2(x_-) \right)^{J-M} \left( f_2(x_+) \overline{f}_1(x_-) \right)^{J+M}. \quad (2.7)$$

It is convenient to put the result under the form

$$e^{-2J \sqrt{\gamma}} \Phi = \beta^{2J} \sum_{M=-J}^{J} (-1)^{J-M} f_M^{(J)} (x_+) \overline{f}_M^{(J)}(x_-). \quad (2.8)$$

where $J \pm M$ run over integer. The $sl(2)$-structure has been made transparent by letting

$$f_M^{(J)} \equiv \left( \frac{2J}{J+M} \right)^{J-M} (f_1)^{J-M} (f_2)^{J+M}, \quad \overline{f}_M^{(J)} \equiv \left( \frac{2J}{J+M} \right)^{J-M} (\overline{f}_1)^{J+M} (\overline{f}_2)^{J-M}. \quad (2.9)$$

The notation anticipates that $f_M^{(J)}$ and $\overline{f}_M^{(J)}$ form representations of spin $J$. This is indeed true since $f_1, f_2$ and $\overline{f}_1, \overline{f}_2$ span 1/2 representations, by construction. Explicitly one finds

$$I_\pm f_M^{(J)} = \sqrt{(J \pm M)(J \pm M + 1)} f_M^{(J)}, \quad I_0 f_M^{(J)} = M f_M^{(J)}.$$
\[ T_{\pm}^{(j)} M = \sqrt{(J \pm M)(J \pm M + 1)} T_{M \pm 1}^{(j)}, \quad T_{3}^{(j)} M = M T_{M}^{(j)}, \]  
(2.10)

where \( T_{\ell} \) and \( \overline{T}_{\ell} \) are the infinitesimal generators of the \( x_+ \) and \( x_- \) components respectively. Moreover, one sees that

\[ (I_{\ell} + \overline{T}_{\ell}) e^{-2J\sqrt{\gamma}\Phi} = 0 \]  
(2.11)

so that the exponential of the Liouville field are group invariants.

### 2.2 The basic chiral operator-algebras

Denote by \( C \) the central charge of gravity. The standard screening charges \( -\alpha_{\pm} \) of the Liouville theory[6, 18] are such that

\[ \alpha_{\pm} = \frac{1}{2} \left( \sqrt{\frac{C - 1}{3}} \pm \sqrt{\frac{C - 25}{3}} \right), \]
\[ \alpha_{\pm} = \frac{Q}{2} \pm \alpha_0, \quad Q = \sqrt{\frac{C - 1}{3}}, \quad \alpha_0 = \frac{1}{2} \sqrt{\frac{C - 25}{3}} \]  
(2.12)

\( Q, \) and \( \alpha_0 \) are introduced so that they agree with the standard notations. Kac’s formula may be written as

\[ \Delta_{Kac}(J, \hat{J}; C) = -\frac{1}{2} \beta(J, \hat{J}; C) \left( \beta(J, \hat{J}; C) + Q \right), \quad \beta(J, \hat{J}; C) = J\alpha_- + \hat{J}\alpha_+, \]  
(2.13)

where \( 2J \) and \( 2\hat{J} \) are positive integers. We shall deal with generic values of \( C \) in order to avoid the complications of quantum group representations. As a result, we have to deal with \( h \) and \( \hat{h} \) on the same basis. This is in fact crucial for dealing with the strong-coupling regime[18, 13, 14]. There are two quantum-group parameters \( h = \pi(\alpha_-)^2/2, \) with \( q = e^{ih}, \) and \( \hat{h} = \pi(\alpha_+)^2/2, \) with \( \hat{q} = e^{i\hat{h}}. \) Thus the number of fields doubles with respect with the classical case just recalled. The holomorphic operator will have a hat, or not, if they are related with \( \hat{h}, \) or \( h, \) respectively; and the antiholomorphic operators are distinguished by an additional bar. According to Eq.2.13, the most general Liouville field is to be written as \( \exp(-(J\alpha_- + \hat{J}\alpha_+)\Phi). \) These fields have decompositions onto holomorphic and antiholomorphic operators to which we shall come below.

First we concentrate on holomorphic fields which are only functions of \( x = \sigma - i\tau. \) Then the whole structure may be described on the unit circle \( \tau = 0. \) The holomorphic operators associated with \( h \) form a subfamily which we next describe following refs.[12, 14, 20]. There are two useful basis of operators. First, the vertex operators \( V_m^{(j)} \), with \( -J \leq m \leq J, \) and \( 2J \) a positive integer. They diagonalize the monodromy matrix

\[ V_m^{(j)}(\sigma + 2\pi) = e^{2ihm\varphi} e^{2ihm^2} V_m^{(j)}(\sigma), \]  
(2.14)
where $\varpi$ is the Liouville quasi-momentum (zero mode) around the cylinder. The meaning of the index $m$ is to specify the shift of $\varpi$:

$$V_m^{(J)}(\sigma) \varpi = (\varpi + 2m) V_m^{(J)}(\sigma). \quad (2.15)$$

The Liouville momentum has been normalized so that this shift is integer. $\varpi$ characterizes the Verma module. We shall work on the sphere, where the spectrum of $\varpi$ is of the form $\varpi_J = \varpi_0 + 2J$. The momentum $\varpi_0 = 1 + 2\pi/h$ is the one of the $sl(2,C)$ invariant state. There is a Verma modules $\mathcal{H}_J$ for each $\varpi_J$. The Moore Seiberg (MS) chiral vertex-operators connect three specified Verma modules and are thus of the form $\phi^{J_3}_{J_1,J_2}$. The operators $V_m^{(J)}$, on the contrary, act in the direct sum $\mathcal{H} = \oplus_J \mathcal{H}_J$. It is quite obvious, according to Eq.(2.15), that the two are related by the projection operator $\mathcal{P}_J$:

$$\mathcal{P}_J \mathcal{H} = \mathcal{H}_J, \quad \mathcal{P}_J V_m^{(J_2)} \equiv \phi^{J_2}_{J_3,J_3+m}(2.16)$$

The $V$ fields are such that $< \varpi_2|V_m^{(J)}|\varpi_1>$ is equal to one of $\varpi_1 = \varpi_3 + 2m$, and is equal to zero otherwise. This normalization is required by the symmetry between three legs (sphere with three punctures). The complete fusion and braiding algebras take the form:

$$\mathcal{P}_K V_{m_1}^{(J_1)} V_{m_2}^{(J_2)} = \sum_{J=|J-J_2|}^{J_1+J_2} F_{K+m_1,J}^{[J_1 \ K \ K+m_1+J_2]} \sum_{\nu} \mathcal{P}_K V_{m_1+m_2}^{(J_2)} < \varpi_J,\nu| V_{m_2}^{(J_1)}|\varpi_{J_2}> \quad (2.17)$$

where $\mathcal{P}_K$ is the projector onto $\mathcal{H}_K$, and,

$$\mathcal{P}_K V_{m_1}^{(J_1)} V_{m_2}^{(J_2)} = \sum_{n_2} \sum_{J_2=|J_2-J_1|} B_{K+m_1,J_2}^{[J_1 \ K \ K+m_1+J_2]} \mathcal{P}_K V_{n_2}^{(J_2)} V_{m_1+m_2-n_2}^{(J_1)} \quad (2.18)$$

For simplicity of notation we omitted the dependence upon the world-sheet variables, which is standard. Using Eq.(2.10) one may verify that these expressions have the general MS form. In ref.[20], it was shown that

$$F_{J_23,J_123}^{[J_1 \ J_2 \ J_3]} = \frac{g_{J_1J_2}^{J_3} g_{J_2J_3}^{J_1} g_{J_3J_1}^{J_2}}{g_{J_1J_2}^{J_3} g_{J_2J_3}^{J_1} g_{J_3J_1}^{J_2}} \{J_1, J_2, J_3; J_123\} \quad (2.19)$$

where $\{J_1, J_2, J_3; J_123\}$ denote the 6-j symbols of $U_q(sl(2))$, with $q = e^{ih}$. This term was of course expected, in view of the quantum-group structure previously exhibited, in particular, in refs.[12, 13]. However, there appear, in addition, coupling constants $g_{J_1J_2}^{J}$, which are not trigonometrical functions of $h$. Their general expression is

$$g_{J_1J_2}^{J} = \prod_{n=1}^{J_1+J_2-J} \frac{G(J_1-n/2)G(J_2-n/2)G(J+n/2)}{G((n-1)/2)} \quad (2.20)$$
where
\[ G(J) \equiv \frac{\Gamma \left( 1 + (1 + 2J)h/\pi \right)}{\Gamma \left( -1 + (1 + 2J)h/\pi \right)}. \] (2.21)

\( \Gamma(x) \) is the usual (not \( q \)-deformed) \( \Gamma \)-function. The relation between braiding and fusion matrices is standard[19], and one may verify that the appearance of the coupling constant does not spoil the consistency conditions (pentagonal, and so on) satisfied by the 6-j symbols. In terms of the \( V \) fields the operator-algebra is entirely expressed in terms of quantum group invariant. Thus, it is clearly symmetric, but one does not see how the quantum group acts. This last feature is exhibited[11] by going to another basis of operators denoted \( \xi^{(j)}_M \) (the quantum group basis) where \( M \) will really be the “third” component of the quantum-group spin. The change of basis is of the form[11, 12]
\[ \mathcal{P}_K \xi^{(j)}_M(\sigma) := \sum_m C_{KK+2m}^{J_1J_2M',M}(J,M) \mathcal{P}_K V^{(j)}_m(\sigma). \] (2.22)

where the coefficient \( C_{KK+2m}^{J_1J_2M',M}(J,M) \) is proportional to a \( q \)-deformed hypergeometric function. The fusion properties become[20]
\[ \xi^{(J_1)}_M \xi^{(J_2)}_N = \sum_{J,J_1,J_2} \delta_{J_1,J_2} \langle J_1, M_1; J_2, M_2 | J \rangle \sum_{\nu} \xi^{(J_1J_2)}_{M_1+M_2} \langle \nu | V^{(J_1)}_{J_1,J_2} | \nu \rangle, \] \[ \langle \nu | V^{(J_1)}_{J_1,J_2} | \nu \rangle = \sum_{J,J_1,J_2} \delta_{J_1,J_2} \langle J_1, M_1; J_2, M_2 | J \rangle \sum_{\nu} \xi^{(J_1J_2)}_{M_1+M_2} \langle \nu | V^{(J_1)}_{J_1,J_2} | \nu \rangle, \] \[ \xi^{(J_1)}_M \xi^{(J_2)}_N = \sum_{N_1N_2} (J_1, J_2)^{N_2N_1}_{M_1M_2} \xi^{(J_2)}_{M_2} \xi^{(J_1)}_{M_1}. \] (2.24)

The braiding relations are[12]
\[ \xi^{(J_1)}_M \xi^{(J_2)}_N = \sum_{N_1N_2} (J_1, J_2)^{N_2N_1}_{M_1M_2} \xi^{(J_2)}_{M_2} \xi^{(J_1)}_{M_1}. \] (2.24)

The braiding matrix \( (J_1, J_2)^{N_2N_1}_{M_1M_2} \) is given by the matrix element of the universal \( R \)-matrix of \( U_q(sl(2)) \):
\[ (J_1, J_2)^{N_2N_1}_{M_1M_2} = \left( (\langle J_1, M_1 | \otimes \langle J_2, M_2 |) \mathcal{R} (|J_1, N_1 >> \otimes |J_2, N_2 >>) \right) . \] (2.25)

\( |J, M >> \) are group theoretic states which span the representation of spin \( J \) of \( U_q(sl(2)) \), and \( \mathcal{R} \) is the universal \( R \)-matrix:
\[ \mathcal{R} = e^{-2ihJ_3 \otimes J_3} \sum_{n=0}^{\infty} \frac{(1 - e^{2ih})^n}{[n]} e^{i\hbar n/2} e^{i\hbar J_3(J_+)^n} e^{i\hbar J_3(J_-)^n}, \] (2.26)

\( J_\pm, J_3 \) are the quantum-group generators, which satisfy
\[ [J_+, J_] = [2J_3], \quad [J_3, J_\pm] = \pm J_\pm. \] (2.27)

\(^3\)We only consider one of the two possible orderings on the unit circle
We define $\lfloor r \rfloor \equiv \sin(hr)/\sinh$ in general. The fusion and braiding of the $\xi$ fields are covariant under the quantum group action

$$J_3 \xi_M^{(J)} = M \xi_M^{(J)}, \quad J_\pm \xi_M^{(J)} = \sqrt{[J \mp M][J \pm M + 1]} \frac{\xi_M^{(J)}}{\xi_{M\pm 1}}. \quad (2.28)$$

Indeed, first in the the operator-product Eq.2.23, the term $\langle \varpi_J, \{ \nu \} | V_{J_2}^{(J_1)} | \varpi_{J_2} \rangle >$ is a quantum-group invariant since it does not involves the indices $M_1$, or $M_2$. Thus, by the basic property of the Clebsch-Gordan coefficients, it follows that $\xi_M^{(J_1)} \xi_M^{(J_2)}$ also gives a representation of the quantum group algebra \[2.27\] with the co-product generators

$$\Lambda_\pm = J_\pm \otimes e^{ihJ_3} + e^{-ihJ_3} \otimes J_\pm, \quad \Lambda_3 = J_3 \otimes 1 + 1 \otimes J_3. \quad (2.29)$$

The tensor product is defined so that

$$(A \otimes B) \left( \xi_M^{(J_1)} (\sigma) \xi_M^{(J_2)} (\sigma') \right) := (A \xi_M^{(J_1)} (\sigma)) (B \xi_M^{(J_2)} (\sigma')), \quad (2.30)$$

Roughly speaking the operator-product correspond to “adding” the quantum group spins. Since the result of the fusion should not depend upon the ordering, the non-commutativity of the $\xi$ fields as quantum operators should precisely cancel the lack of symmetry of the co-product, and Clebsch-Gordan coefficients. This is why the braiding relations are given by the universal R-matrix. Thus the mathematical deformation of $sl(2)$, is precisely governed by the truly quantum mechanical effects of the theory. Second, since the universal $R$-matrix which gives the braiding commutes with the co-product, the braiding relation Eq.2.24 is also covariant. This quantum-group symmetry, does not prevent invariant coupling constants $g_{J_1 J_2}$ to appear\[4\]. The value of these coupling constants makes the difference between Liouville theory and the $SU(2)$ WZNW model at this level.

An other important feature is that the fusion and braiding matrices do not depend upon the Verma module to which the fusion and braiding relations are applied. This is in contrast with the usual MS expressions. In the latter case, this means that the fusion and braiding matrices are not c-numbers since they depend upon the eigenvalue of the Liouville momentum. The corresponding solution of the Yang-Baxter equation thus takes a form that differs from the solutions obtained from the R-matrix of a quantum group. It is only when (and if) this dependence upon the Verma module may be removed that a direct connection with quantum-group representations is established. Note that, for $A_n$-W-algebras ($n \neq 2$), the same procedure works but gives\[14\] a deformation of $sl(n)$ that differs from the standard one.

2.3 The Liouville field, and matter-gravity couplings

We consider closed surfaces following ref.\[17\]. The above $\xi$ fields together with their antiholomorphic counterparts $\bar{\xi}$ allow us to reconstruct the fields $\exp -J\alpha_\Phi(\sigma, \tau)$.

\[4\] we simply see here an application of the well-known Wigner-Eckart theorem, which is so much used, for instance, in atomic physics
Imposing locality, and conservation of the winding number determines this operator completely. One finds
\[ e^{-J\alpha_-\Phi(\sigma,\tau)} = \frac{1}{\sqrt{\omega}} \sum_{M=-J}^{J} (-1)^{M-J} e^{ih(J-M)} \xi^{(J)}_M(x_+) \xi^{(J)}_{-M}(x_-) \sqrt{\omega} \]  
(2.31)

The quantum-group action on the $\bar{\xi}$ field is given by:
\[ \mathcal{J}_3 \bar{\xi}_M^{(J)} = M \bar{\xi}_M^{(J)}, \quad \mathcal{J}_\pm \bar{\xi}_M^{(J)} = \sqrt{[J \mp M] [J \pm M + 1]} \bar{\xi}_M^{(J)}. \]  
(2.32)

Thus if we define
\[ \mathcal{J}_\pm = J_\pm e^{-ih\mathcal{J}_3} + e^{ih\mathcal{J}_3} J_\pm, \quad \mathcal{J}_3 = J_3 + \mathcal{J}_3, \]  
(2.33)
we obtain a representation of the quantum-group algebra Eq.2.28. Then one easily checks that
\[ \mathcal{J}_\pm \exp(-J\alpha_-\Phi) = \mathcal{J}_3 \exp(-J\alpha_-\Phi) = 0, \]  
(2.34)
so that the quantized Liouville field is a quantum-group invariant. This is the quantized version of Eq.2.11. Thus the $sl(2)$ symmetry recalled in section 2 has been deformed by the quantization in the mathematically standard way. The most general Liouville field $\exp(-(J\alpha_- + J\alpha_+)\Phi)$, is given by a similar expression in terms of the fields $\xi, \tilde{\xi},$ and $\bar{\xi}$ (see ref[17] for details).

Next we consider the dressing of minimal models with central charge $D$ by Liouville with central charge $C$. The balance of central charge requires
\[ C + D = 26. \]  
(2.35)
We shall be concerned with the case $D < 1$, where the Liouville theory is in its weakly coupled regime $C > 25$. The quantum structure of the Liouville theory just recalled is basically a consequence of operator differential equations which are equivalent to the Virasoro Ward-identities that describe the decoupling of null vectors. Thus the same operator algebra, with appropriate quantum deformation parameters also describes the matter with $D < 1$. We will thus have another copy of the quantum-group structure recalled above. It will be distinguished by primes. Thus we let
\[ D = 1 + 6\left(\frac{h'}{\pi} + \frac{\pi}{h'} + 2\right) = 1 + 6\left(\frac{\hat{h}'}{\pi} + \frac{\pi}{\hat{h}'} + 2\right), \quad \text{with} \quad h'\hat{h}' = \pi^2, \]  
(2.36)

The appropriate dressing by gravity is such that one is concerned with matrix elements of operators of the type
\[ \mathcal{V}_{J', \hat{J}}(\sigma, \tau) \equiv e^{((J' + 1)\alpha_- - J'\alpha_+)\Phi(\sigma, \tau)} e^{-(J'\alpha_+ + J'\alpha'_+)X(\sigma, \tau)}, \]  
(2.37)
where $X$ is the matter field that is similar to the Liouville field. In ref.\[17\], the present approach was used to determine the three-point gravity-matter couplings:

$$\langle \prod_{\ell=1}^{3} \psi_{J_{\ell}}, \bar{\psi}_{J_{\ell}}(z_{\ell}, \bar{z}_{\ell}) \rangle = C_{1,2,3}/\left( \prod_{k,l} |z_k - z_l|^2 \right).$$  (2.38)

For positive $J'$, Eq. 2.37 involves negative quantum-group spins. This difficulty is equivalent with the need for negative number of dressing operator that is encountered in the Coulomb gas approach. Remarkably, this problem is precisely solved by symmetry $J \rightarrow -J - 1$, which was found in refs.\[13, 14\]. Finally the result is very simple

$$C_{1,2,3} = \prod_l \frac{B_{J_l, \bar{J}_l}}{\Gamma \left( 1 + 2\bar{J}_l + (1 + 2J_l)h/\pi \right)^2}.$$  (2.39)

It factorises, and thus gives back results of matrix models for ratios of correlators that do not depend upon the normalizations. The outcome of ref.\[17\] is that the symmetry between quantum-group spins $J$ and $-J - 1$ is the explanation of the continuation in the number of screening operators discovered by Goulian and Li. Moreover, and contrary to the previous discussions of this problem, the present approach clearly separates the emission operators for each leg. This clarifies the structure of the dressing by gravity. It is shown, in particular that the end points are not treated on the same footing as the mid point. Since the outcome is completely symmetric this suggests the possibility of a picture-changing mechanism.

Before leaving this topic we quote an older result\[21\] concerning open surfaces. The equivalent of Eq. 2.31 is

$$e^{-\alpha - J\Phi(\sigma)} = \left( \frac{h}{16\pi^3} \right)^J \sum_{M,N} <J, M| A|J, N > \xi^{(J)}_M(\sigma) \xi^{(J)}_N(2\pi - \sigma)$$  (2.40)

where

$$A = e^{-ihJ_{\bar{J}}^2} \sum_{r,s=0}^{\infty} e^{ih(r+s)J_{\bar{J}}^2} e^{ishr/2} \frac{[r]! [s]!}{[r+s]!}.$$  (2.41)

The matrix $A$ is the universal reflection matrix associated with $U_q(sl(2))$, a concept that is gradually recognized as very important for integrable systems with boundaries.

### 3 BLACK HOLES FROM TODA THEORIES

In the present section, as well as in the coming ones, we restrict ourselves to classical Toda field theories, that describe the classical limits of conformal theories. Although this is a much simpler problem than the quantum one, it already teaches us a lot, since the appropriate operator quantization\[3, 21\] does preserve the key features of
the classical Toda theories. The basic point of these theories is that their dynamical
equations, no matter how complicated, are derivable from flatness conditions

\[ [\partial_+ - A_+, \partial_- - A_-] \equiv \partial_- A_+ - \partial_+ A_- + [A_+, A_-] = 0, \quad (3.1) \]

which allow to obtain the general solutions in closed form\[^{22}\]\[^{23}\]. We use \( \partial_+ \) and \( \partial_- \) as short hands for \( \partial/\partial x_+ \) and \( \partial/\partial x_- \), respectively. The Lax pair \( A_\pm \) is systematically constructed once the Lie algebra, the gradation, and the grading spectrum of \( A_\pm \) are chosen\[^{22}\]\[^{23}\]. Consider a finite-dimensional Lie group \( G \), with Lie algebra \( G \), and a gradation \( G = \oplus_{m \in \mathbb{Z}} G_m \), with \( [G_m, G_n] \in G_{m+n} \). There is a grading operator \( H \), such that \( [H, G_m] = 2m G_m \). The basic idea is to consider the flatness condition Eq.(3.1) taking \( A_\pm \in G_0 \oplus G_{\pm 1} \). The (abelian) \( A_N \)-Toda theories considered in the coming sections are associated with the principal grading of \( A_N \) where \( G_0 \) is the Cartan subalgebra, and where \( G_{\pm 1} \) is generated by the simple (positive, resp. negative) roots. It is derivable from the Lagrangian

\[ -\mathcal{L} = \frac{1}{2} \sum_{a,b=1}^{N} K_{ab}^{(A_N)} \partial_+ \phi^a \partial_- \phi^b + \sum_{a=1}^{N} \prod_{b=1}^{N} \exp \left( K_{ab}^{(A_N)} \phi^b \right), \quad (3.2) \]

The matrix \( K_{ab}^{(A_N)} \) is the Cartan matrix of \( A_N \). The fields \( \phi^a(x_+, x_-) \) are the bosonic Toda fields. We shall come back to this model in the next section. At the present time let us simply note that the fields \( \phi^a \) may be regarded as components of a string-position in a \( N \)-dimensional target space with a metric \( G_{ab} = K_{ab}^{(A_N)} \). It is somewhat trivial from this view point since this metric is constant. This feature is no more true for non-abelian Toda theories, namely those for which the subalgebra \( G_i \) is non-abelian. This was shown in ref\[^{24}\] in the examples\[^{7}\] of the Lie algebras \( B_{N-1} \). Associated with a maximal subalgebra \( D_{N-1} \) (of maximal rank) in the Lie algebra \( B_{N-1} \), we have the Lagrangian

\[ -\mathcal{L} = \frac{1}{2} \left[ \sum_{j,k=1}^{N-1} K_{jk}^{(D_{N-1})} \partial_+ \phi^j \partial_- \phi^k - \tanh^2 \left( \frac{\phi^{N-1} - \phi^{N-2}}{2} \right) \frac{\partial \phi^N}{\partial z_+} \frac{\partial \phi^N}{\partial z_-} \right] \]

\[ + \sum_{i=1}^{N-1} \prod_{j=1}^{N-1} \exp \left( K_{ij}^{(D_{N-1})} \phi^j \right). \quad (3.3) \]

In this case, the zero-grading part is given by \( G_0 = gl(1) \oplus \cdots \oplus gl(1) \oplus A_1 \), where the one-dimensional linear algebra \( gl(1) \) appears \( N - 1 \) times. The \( \phi \)-dependent part of the metric involves the hyperbolic-tangent-square function which is familiar in the 2D black-hole game. In particular if we choose \( N = 3 \), we find

\[ -\mathcal{L} = \frac{1}{2} \left[ 2 \sum_{i=1}^{2} \partial_+ \phi^i \partial_- \phi^i - \tanh^2 \left( \frac{\phi^1 - \phi^2}{2} \right) \partial_+ \phi^3 \partial_- \phi^3 \right] + \sum_{i=1}^{2} \exp(2\phi^i) \quad (3.4) \]

\[^{5}\text{Note that there is no useful non-abelian version of the Toda lattice for the series } A_N \]
In order to clarify the black-hole aspect, it is convenient to change field variables. Let
\[ \Phi = (\phi^1 + \phi^2)/2, \quad r = (\phi^1 - \phi^2)/2, \quad \theta = \phi^3/2. \] (3.5)

One gets
\[ -\mathcal{L} = \partial_+ \Phi \partial_- \Phi + \partial_+ r \partial_- r - \tanh^2(r) \partial_+ \theta \partial_- \theta + \cosh(2r)e^{2\theta}. \] (3.6)

The target-space metric is
\[ G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\tanh^2 r \end{pmatrix}. \] (3.7)

One sees that the first component, which corresponds to the \( \phi \) variable, defines a subspace that decouples from the rest from the viewpoint of Riemannian geometry. In the \((r, \theta)\) space we have \( G = \begin{pmatrix} 1 & 0 \\ 0 & -\tanh^2 r \end{pmatrix} \), which coincides with Witten’s black-hole metric\([25]\) exactly.

The outcome of ref.\([24]\) is that non-abelian Toda theories provide exactly solvable conformal systems in the presence of a black hole. They correspond to gauged WZNW models where the gauge group is nilpotent, and are thus basically different from the ones currently considered, following Witten. The non-abelian Toda potential gives a cosmological term which may be integrated exactly at the classical level.

4 GEOMETRY OF CHIRAL SURFACES

4.1 The background

The present discussion is concerned with the generalization of the situation described in section 2 for 2D gravity. On the one hand, the above Liouville dynamics is a particular case of the Toda dynamics\([4]\) which, as just recalled, exists for each simple Lie algebra. The Liouville case is associated with \( A_1 \). It was shown in ref.\([3]\) that general Toda systems are related, through Noether theorem to the non-linear extensions of the Virasoro algebra called W-algebras. Thus going from Liouville to general Toda should correspond, in general coordinates to going from Einstein gravity to W gravity. As a first step towards formulating these fascinating theories, we next unravel the geometrical meaning of Toda systems. This is, at least a way to see the geometrical meaning of W algebras. We will show that it corresponds to the extrinsic geometry of embeddings of special \((W)\) surfaces, a viewpoint which is natural since in general, we expect that conformal systems and their W generalizations are to be connected with string theories. The material covered until the end is a summary of refs.\([1, 2]\).

\*\*\*from now on we only consider the principal grading
4.2 Chiral embeddings

The basic objects are two-dimensional surfaces embedded in Kähler manifolds. We shall only consider $C^n$ here explicitly. The case of $C P^n$ is treated by using homogeneous coordinates in $C^{n+1}$. We call $z$ and $\bar{z}$ the surface parameters. One may think of $z$ as an ordinary complex number, in which case the parametrization is Euclidean, and $\bar{z}$ is the complex conjugate of $z$; or take $z$, and $\bar{z}$ to be real, in which case $x_0 \equiv (z + \bar{z})/2,$ and $x_1 \equiv (z - \bar{z})/2$ are coordinates of the Minkowsky type. The adjective chiral means function of a single variable $z$ or $\bar{z}$ (if $z$ is a complex variable this is equivalent to analytic or anti-analytic). We parametrize $C^n$ by coordinates $X^A$, $\bar{X}^A$, $1 \leq A, \bar{A} \leq n$. A chiral embedding is defined by equations of the form

$$X^A = f^A(z), \quad A = 1, \ldots, n, \quad \bar{X}^A = \bar{f}^A(\bar{z}), \quad \bar{A} = 1, \ldots, n.$$  \hspace{1cm} (4.1)

where $f$ and $\bar{f}$ are arbitrary functions. We call a $W$-surface the corresponding manifold $\Sigma$. We shall see that its extrinsic geometry is directly related to $W$ transformations. It is convenient to introduce the matrix of inner products:

$$g_{ij} \equiv \sum_A f^{(i)A}(z) \bar{f}^{(j)A}(\bar{z}), \quad 1 \leq i, j \leq n.$$  \hspace{1cm} (4.2)

We use $\partial$ and $\bar{\partial}$ as short hands for $\partial/\partial z$ and $\partial/\partial \bar{z}$ respectively. $f^{(i)A}$ stands for $(\partial)^i f^A$, and $\bar{f}^{(j)A}$ stands for $(\bar{\partial})^j \bar{f}^A$. Later on we shall exhibit a particular parametrization of $C^n$, called $W$-parametrization, where the derivatives with $i$ or $j$ larger than one will be replaced by first order derivatives in other variables, so that the covariance properties of the present discussion will become more transparent. This section is concerned with generic regular points of $\Sigma$ where the Taylor expansions of $f^A$ and $\bar{f}^A$ give linearly independent vectors. Then $f^{(a)}$, and $\bar{f}^{(a)}$, $a = 1, \ldots, n$, span the following

**Definition 1 Moving frame in $C^n$.** Consider the vectors $e_a$, and $\bar{e}_a$, $a = 1, \ldots, n$, with components

$$e_a^A = \frac{1}{\sqrt{\Delta_a \Delta_{a-1}}} \begin{vmatrix} g_{11} & \cdots & g_{a1} \\ \vdots & \ddots & \vdots \\ g_{1a} & \cdots & g_{aa} \end{vmatrix}, \quad e^A_a = 0,$$  \hspace{1cm} (4.3)

$$\bar{e}_a^\bar{A} = 0, \quad \bar{e}_a^\bar{A} = \frac{1}{\sqrt{\Delta_{a} \Delta_{a-1}}} \begin{vmatrix} g_{11} & \cdots & g_{a1} \\ \vdots & \ddots & \vdots \\ g_{1a} & \cdots & g_{aa} \end{vmatrix}, \quad \bar{e}^\bar{A}_a = 0.$$  \hspace{1cm} (4.4)

$\Delta_a$ is the determinant

$$\Delta_a \equiv \begin{vmatrix} g_{11} & \cdots & g_{a1} \\ \vdots & \ddots & \vdots \\ g_{1a} & \cdots & g_{aa} \end{vmatrix}.$$  \hspace{1cm} (4.5)
One may verify that the moving frame defined above is orthonormal, that is,

\[(\textbf{e}_a, \textbf{e}_b) = (\bar{\textbf{e}}_a, \bar{\textbf{e}}_b) = 0, \quad (\textbf{e}_a, \bar{\textbf{e}}_b) = \delta_{a,b}.\]  

(4.6)

The vectors \(\textbf{e}_1\) and \(\bar{\textbf{e}}_1\) are tangents to the surface, while the other vectors are clearly normals. Thus the Gauss-Codazzi equations may be derived by studying their derivatives along the W-surface \(\Sigma\). One derives equations of the form

\[
\partial \textbf{e}_a = \sum_b \omega^b_{za} \textbf{e}_b, \quad \bar{\partial} \textbf{e}_a = \sum_b \bar{\omega}^b_{za} \bar{\textbf{e}}_b, \quad \partial \bar{\textbf{e}}_a = \sum_b \bar{\omega}^b_{za} \textbf{e}_b, \quad \bar{\partial} \bar{\textbf{e}}_a = \sum_b \omega^b_{za} \bar{\textbf{e}}_b. 
\]

(4.7)

which may be regarded as generalized Frenet-Serret formulae. Next we recall the

**Definition 2** \(\mathbb{C}P^n\) **target space.** The complex projective space \(\mathbb{C}^n\) is defined to be the quotient of the space \(\mathbb{C}^{n+1}\) by the equivalence relation

\[X \sim Y, \quad \text{if} \quad X^A = Y^A \rho(Y), \quad \text{and} \quad \bar{X}^\bar{A} = \bar{Y}^\bar{A} \bar{\rho}(\bar{Y}).\]  

(4.8)

where \(\rho\) and \(\bar{\rho}\) are arbitrary functions of \(n+1\) variables.

Thus, the modification to go from \(\mathbb{C}^n\) to \(\mathbb{C}P^n\) is to use \(n+1\) coordinates, so that now \(A\) and \(\bar{A}\) run from 0 to \(n\), and to write formulae that are covariant under rescaling. This is achieved by letting the indices of the matrix \(g_{ij}\) run from 0 to \(n\) in the \(\mathbb{C}P^n\) formulae for the moving frame. for this one includes derivatives of order 0. Our basic result is the following:

**Theorem 1** Gauss-Codazzi equations.

Define Toda fields by

\[\phi_a = -\ln(\tau_a), a = 1, \ldots, n; \tau_a \equiv \begin{vmatrix} g_{00} & \cdots & g_{a0} \\ \vdots & \ddots & \vdots \\ g_{0a} & \cdots & g_{aa} \end{vmatrix}.\]  

(4.9)

The integrability conditions of the Frenet-Serret equations for the embedding in \(\mathbb{C}P^n\) (analogous to Eqs.2.7) coincide with the Toda equations associated with \(A_n\):

\[\partial \bar{\partial} \phi_i + \exp \left( \sum_{j=1}^{n} K_{ij} \phi_j \right) = 0.\]  

(4.10)

\(K\) is the Cartan matrix associated with \(A_n\). The functions \(\tau_a\) relevant for \(\mathbb{C}P^n\) are similar to the \(\Delta_a\) of Eq.2.3 except that they include the derivatives of zeroth order.

In conclusion: The \(A_n\) Toda dynamics describes the extrinsic geometry of W surfaces.
4.3 Some basic facts about $A_n$ Toda theories

Its general solution is of the form

$$e^{-\phi_k} = \sum_{i_1<\ldots<i_k} \left| \begin{array}{ccc} \chi^{i_1} & \ldots & \chi^{i_k} \\ \chi^{(1)}_1 & \ldots & \chi^{(1)}_k \\ \vdots & \ldots & \vdots \\
\chi^{(k-1)}_1 & \ldots & \chi^{(k-1)}_k \end{array} \right| \left| \begin{array}{ccc} \bar{\chi}^{i_1} & \ldots & \bar{\chi}^{i_k} \\ \bar{\chi}^{(1)}_1 & \ldots & \bar{\chi}^{(1)}_k \\ \vdots & \ldots & \vdots \\
\bar{\chi}^{(k-1)}_1 & \ldots & \bar{\chi}^{(k-1)}_k \end{array} \right| . \quad (4.11)$$

where $k$ runs from 1 to $n$. It is expressed in terms of $n$ functions of $z$ $\chi^1, \ldots, \chi^n$, and $n$ functions of $\bar{z}$ $\bar{\chi}^1, \ldots, \bar{\chi}^n$. Upper indices in between parenthesis denotes derivatives. These $n$ functions $\chi^k$ (resp. $\bar{\chi}^k$) are restricted to be solution of the same differential equation $\chi^{(n+1)k} - \sum_{\ell=0}^{n-1} U_{n+1-\ell} \chi^{(\ell)k} = 0$ (resp. $\bar{\chi}^{(n+1)k} - \sum_{\ell=0}^{n-1} \bar{U}_{n+1-\ell} \bar{\chi}^{(\ell)k} = 0$). The set of potentials $\{U_\ell, \ell = 2, \ldots, n+1\}$ and $\{\bar{U}_\ell, \ell = 2, \ldots, n+1\}$, each generate a realization of the $A_n$ W algebra by Poisson brackets. These are non-linear generalizations of the Virasoro algebra (conformal transformations). The Toda dynamics is non-chiral, and this is why the W algebra appears twice (for the holomorphic and anti-holomorphic components).

It follows that from the above geometrical derivation of the Toda equations, we may discuss the geometrical meaning of the W transformations.

4.4 Connection with the WZNW models

It is known, in general, that there is one W algebra associated with each simple Lie algebra $\mathcal{G}$. This appears in several ways. First, as we have already seen, there is a Toda theory, and, hence, two PB realizations of the associated W-algebra for any given $\mathcal{G}$. On the other hand, consider the affine Lie algebra $\tilde{\mathcal{G}}$ associated with $\mathcal{G}$. The associated non-chiral theory is the WZNW model whose quantum solutions are given by representations of $\tilde{\mathcal{G}}$. It is possible to derive the Toda theory from the WZNW model by conformal reduction. Here we have the

**Theorem 2 Gauss decomposition from moving frame.** The moving-frame equations may be written as

$$e_a = \sum_{b \leq a} C_{ab}(z, \bar{z}) \sqrt{\frac{\tau_a}{\tau_{a+1}}} f^{(b)}(z), \quad \text{with } C_{aa} = 1, \quad (4.12)$$

$$e_a = \sum_{b \leq a} A_{ba}(z, \bar{z}) \sqrt{\frac{\tau_a}{\tau_{a+1}}} f^{(b)}(\bar{z}), \quad \text{with } A_{aa} = 1. \quad (4.13)$$

The matrix $\theta = g^{-1}$ $s$ such that

---

The non-simple Lie algebras are trivially reduced to the simple case by separating the invariant subalgebras.
1) It has the Gauss decomposition
\[ \theta = ABC, \]
where \( A \) and \( C \), which appear in eqs. 2.11 and 2.12 are triangular with diagonal elements equal to one, and
\[ B = \exp \left( \sum_i h_i \phi_{i+1} \right). \]
\( h_i \) are the Cartan generators.

2) It is a solution of the conformally reduced WZNW model associated with \( A_n \).

5 Geometry of Toda hierarchy

The Toda equations are a subsystem of the Toda hierarchy\[7\]. (This is the non-chiral version of the fact that the Virasoro algebra is identical with the second Poisson bracket of KdV, and that W algebras are obtained from KP hierarchies and Gelfand Dicki brackets). Introduce the additional variables as coordinates in our geometrical embedding problem. This is best done using the free-fermion formalism. Let

\[
\begin{align*}
[\psi_n, \psi_m]_+ &= \left[ \psi_n^+, \psi_m^+ \right]_+ = 0, \\
\psi_n^+ \langle \emptyset \rangle &= 0, & \langle \emptyset | \psi_n^+ &= 0 \quad \forall n.
\end{align*}
\]

We use the semi-infinite indices \( n = 0, 1, 2, \cdots, \infty \) for the fermion-operators. The vacuum states \( |\emptyset\rangle \) and \( \langle \emptyset| \) correspond to the no-particle states. The \( n \)-particle ground state is created from them in the standard way:
\[ |n\rangle = \psi_n^{+\dagger} \cdots \psi_0^{+\dagger} |\emptyset\rangle, \quad \langle n| = \langle \emptyset| \psi_0 \cdots \psi_{n-1}. \]

The current operators,
\[
J_n = \sum_{s=0}^\infty \psi_{n+s}^+ \psi_s, \quad \bar{J}_n = \sum_{s=0}^\infty \psi_s^+ \psi_{n+s},
\]
will be taken as Hamiltonians as one does for the KP hierarchy. The rôle of these fermions may be understood as follows. Take the case where \( z \) is a complex variable. Then the embedding functions \( f^A \) are analytic, and each of them is entirely determined by its Taylor expansion around a single point of its analyticity domain. Its behaviour at any other point of its Riemann surface is fixed by analytic continuation. The following free-fermion formalism realizes this continuation automatically. Consider the Taylor expansions at the point \( z \):
\[ f^A(z + x) = \sum_{s=0}^\infty f^{(s)}(z) \frac{x^s}{s!}, \quad \bar{f}^A(z + \bar{x}) = \sum_{s=0}^\infty \bar{f}^{(s)}(z) \frac{\bar{x}^s}{s!}. \]
To these developments, we associate the free-fermion operators,

\[ \psi f^A(z) = \sum_{s=0}^{\infty} f^{(s)A}(z) \psi_s, \quad \psi^+ f^A(\zbar) = \sum_{s=0}^{\infty} \zbar f^{(s)A}(\zbar) \psi^+_s. \]  

(5.6)

The basic property of these operators are

**Proposition 1 Fermionic representation of chiral functions.**

1) Any change of the Taylor-expansion point \( z, \zbar \) can be absorbed by the action of the Hamiltonians \( J_1, J_1^{-1} \). In particular, one has

\[ \psi f^A(z) = e^{-J_1 z} \psi f^A(0) e^{J_1 z}, \quad \psi^+ f^A(\zbar) = e^{\zbar J_1^{-1} z} \psi^+ f^A(0) e^{-\zbar J_1^{-1} z}. \]  

(5.7)

2) The embedding functions are represented by the fermion expectation-values

\[ f^A(z) = \langle 0 | \psi f^A(z_0) e^{J_1(z-z_0)} | 1 \rangle, \quad f^A(\zbar) = \langle 1 | e^{J_1(z-z_0)} \psi^+ f^A(\zbar_0) | 0 \rangle. \]  

(5.8)

**Definition 3 KP-parametrization of \( CP^n \).**

Given a chiral surface embedded into \( CP^n \), the associated KP-parameters of the target space are \( n+1 \) variables \( z^{(0)}, z^{(1)} = z, z^{(2)}, \ldots, z^{(n)} \), noted \([z]\), and \( n+1 \) variables \( \zbar^{(0)}, \zbar^{(1)} = \zbar, \zbar^{(2)}, \ldots, \zbar^{(n)} \), noted \([\zbar]\). The change of coordinates from \( X^A \), \( X^\zbar \) to \([z]\), \([\zbar]\) is defined by

\[ X^A = f^A([z]), \quad X^\zbar = f^\zbar([\zbar]). \]  

(5.9)

where \( f^A([z]) \), and \( f^\zbar([\zbar]) \), are the solutions of the equations

\[ \frac{\partial}{\partial z^{(i)}} f^A([z]) = \frac{\partial f^A([z])}{\partial z^{(i)}}, \quad \frac{\bar{\partial}}{\bar{\partial} \zbar^{(i)}} f^\zbar([\zbar]) = \frac{\partial f^\zbar([\zbar])}{\partial \zbar^{(i)}} \]  

(5.10)

with the initial conditions \( f^A([z]) = f^A(z) \) for \( z^{(0)}, z^{(2)}, \ldots, z^{(n)} = 0 \), and \( f^\zbar([\zbar]) = f^\zbar(\zbar) \) for \( \zbar^{(0)}, \zbar^{(2)}, \ldots, \zbar^{(n)} = 0 \).

These coordinates coincide with the higher variables of the KP hierarchy. Indeed, their definition is most natural in the free-fermion language, where it is easy to see that

\[ f^A([z]) = \langle 0 | \psi f^A e^{-\sum_i J_i z^{(i)}} | 1 \rangle, \quad f^\zbar([\zbar]) = \langle 1 | e^{\sum_i J_i \zbar^{(i)}} \psi^+ f^\zbar | 0 \rangle. \]  

(5.11)

The dependence in \([z]\) and \([\zbar]\) is dictated by the action of the higher currents \( J, \zbar \), defined by Eq.5.4, that is, \( J_1 z \rightarrow \sum_{i=0}^{n} J_i z^{(i)}, \zbar J_\zbar \zbar \rightarrow \sum_{i=0}^{n} J_i \zbar^{(i)} \) in Eq.5.8. The basic virtue of the KP coordinates is that they allow us to extend the previous discussion away from the W surface (they parametrize at least a neighborhood of it) in such a way that is becomes covariant. In particular, the metric \( g \) has the following extension

\[ g_{ij}([z],[\zbar]) = \sum_A \partial f^A([z]) \bar{\partial} f^A([\zbar]), \quad \partial_i \equiv \frac{\partial}{\partial z^{(i)}}, \quad \bar{\partial}_i \equiv \frac{\bar{\partial}}{\bar{\partial} \zbar^{(i)}}. \]  

(5.12)

Now, only first-order derivatives appear. This expression coincides with the true Riemannian metric with respect to the KP coordinates.
5.1 W transformations

A general infinitesimal W-transformation is a change of embedding functions which takes the form

$$\delta W f^A(z) = \sum_{j=0}^{n} w^j(z) \partial^{(j)} f^A(z), \quad \delta W \bar{f}^A(\bar{z}) = \sum_{j=0}^{n} \bar{w}^j(\bar{z}) \partial^{(j)} \bar{f}^A(\bar{z}),$$

(5.13)

It is not difficult to show there exists a unique extension such that the differential equation 5.10 is left invariant. It is of the form

$$\delta_W f^A([z]) = \sum_r W^r([z]) \partial_r f^A([z]),$$

$$\delta_W \bar{f}^A([\bar{z}]) = \sum_r \bar{W}^r([\bar{z}]) \partial_r \bar{f}^A([\bar{z}])$$

(5.14)

where $W^r$ and $\bar{W}^r$ are functionals of $w^j$ and $\bar{w}^j$ respectively. Only first order derivative appear. Thus the W transformation become extended as reparametrizations

$$\delta_W z^{(r)} = W^r([z]), \quad \delta_W \bar{z}^{(r)} = \bar{W}^r([\bar{z}]).$$

(5.15)

They become particular diffeomorphisms of $CP^n$. Thus they are extended as linear transformations.

5.2 Dynamical equations

The followings topics are discussed in ref.[2]

The above functions $\tau_a$ when extended become tau-functions in the sense of Miwa-Jimbo-Sato[7].

The KP coordinates are related with a generalized moving frame, whose integrability condition is equivalent to the the well-known Zakharov-Shabat of the $A_n$ Toda hierarchy.

The extension of the associated WZNW model gives solutions of a 2n dimensional generalization of the WZNW equations where the currents are replaced by the Christoffel symbols of the KP coordinates.

6 SINGULAR POINTS, GLOBAL STRUCTURE

At this point, it is useful to change the viewpoint, and make use of Grassmannians. This part is draws a lot of inspiration from ref.[10]

6.1 Associated mappings

Definition 4 Associated mappings.
Consider the family of osculating hyperplanes with contact of order \( k \) denoted \( \mathcal{O}_k \) \((k = 1, \cdots, n)\) to the original W-surface. With \( CP^n \) as the target space, this family defines an embedding into the Grassmannian \( G_{n+1,k+1} \), which we call the \( k \)th associated mappings to the original W-surface.

This formulation looks different, but is equivalent to the construction of the moving frame and only uses the intrinsic geometries of the induced metrics for \( k = 1, \cdots, n \). In practice, what this means is that, instead of forming moving-frame vectors \( e_k \) out of \( f, \cdots, f^{(k)} \) \((k = 1, \cdots, n)\), we consider the nested osculating planes \( \mathcal{O}_1 \subset \mathcal{O}_2 \subset \cdots \subset \mathcal{O}_n \). It is obvious that those two have the same information. It is well-known that the Grassmannians are Kähler manifolds. The induced metric on the \( k \)th associated surface in \( G_{n+1,k+1} \) is simply,

\[
 g^{(k)}_{z\bar{z}} = \partial \bar{\partial} \ln \tau_{k+1}(z, \bar{z}), \quad g^{(k)}_{zz} = g^{(k)}_{\bar{z}\bar{z}} = 0, \tag{6.1}
\]

so that the Toda field \( \phi_{k+1} \equiv -\ln(\tau_{k+1}) \) appears naturally. Thus \( -\phi_{k+1} \) is equal to the Kähler potential of the \( k \)th associated surface. At this point, it is very clear that by considering the associated surfaces, we can restrict ourselves to intrinsic geometries.

In the discussion of section 4, the Toda equation came out from the Gauss-Codazzi equation. Here, it is equivalent to the local Plücker formula

\[
 R^{(k)}_{z\bar{z}} \sqrt{g^{(k)}_{z\bar{z}}} = -g^{(k+1)}_{z\bar{z}} + 2g^{(k)}_{z\bar{z}} - g^{(k-1)}_{z\bar{z}}. \tag{6.2}
\]

where \( R^{(k)}_{z\bar{z}} \) is the only non-vanishing component of the intrinsic Riemann tensor on the \( k \)th surface.

### 6.2 The instanton-numbers of a W-surface

A key point in the coming discussion is to use topological quantities that are instanton-numbers. W-surfaces are instantons of the associated non-linear \( \sigma \)-model. The general situation is as follows. W-surfaces are characterized by their chiral parametrizations which thus satisfy the Cauchy-Riemann relations. These are self-duality equations so that the coordinates of a W-surface define fields that are solutions of the associated non-linear \( \sigma \) model, with an action equal to the topological instanton-number. For a general Kähler manifold \( M \) with coordinates \( \xi^\mu \) and \( \xi^{\bar{\mu}} \), and metric \( h_{\mu\bar{\nu}} \), the action associated with 2D manifolds of \( M \) with equations \( \xi^\mu = \varphi^\mu(z, \bar{z}) \), and \( \xi^{\bar{\mu}} = \bar{\varphi}^{\bar{\mu}}(z, \bar{z}) \) is

\[
 S = \frac{1}{2} \int d^2 x \ h_{\mu\bar{\nu}} \partial_j \varphi^\mu \partial_j \bar{\varphi}^{\bar{\nu}}. \tag{6.3}
\]

In this short digression we let \( z = x_1 + ix_2 \), and \( \partial_j = \partial/\partial x_j \). The instanton-number is defined by

\[
 Q = \frac{i}{2\pi} \int d^2 x \ \epsilon_{jk} \ h_{\mu\bar{\nu}} \partial_j \varphi^\mu \partial_k \bar{\varphi}^{\bar{\nu}}. \tag{6.4}
\]
For W-surfaces and their associated surfaces, $\bar{\partial}\phi^\mu = \partial\bar{\phi}^\bar{\mu} = 0$, and one has $S = \pi Q$. $Q$ is proportional to the integral of the determinant of the induced metric, that is $Q = \frac{i}{2\pi} \int d^2 x \partial\bar\partial \ln \tau_1$.

Moreover we may also apply formula Eq. 6.4 to the $k$th associated surface. This gives

**Definition 5 Higher instanton-numbers of the W-surface.**

The $k$th instanton number of the W-surface $Q_{k+1}$ is defined by,

$$Q_{k+1} \equiv \frac{i}{2\pi} \int_{\Sigma} dzd\bar{z} g^{(k)}_{z\bar{z}}$$

Its topological nature is obvious from Eq. 6.1 which shows that the integrand is indeed a total derivative. The collection of the ($k$-th) instanton-numbers together with the original one $Q \equiv Q_1$ gives a set of topological quantities which characterize the global properties of the original W-surface.

### 6.3 Global classification of W surfaces

In section 4, we have constructed the moving frames at the point where the tau-functions are regular. When those functions become irregular, we meet an obstruction to derive the moving frames. In the WZNW language, this signals that there appears a global obstruction to the Gauss decomposition. Toda equations should be modified at these points. In the following, we study the structure of such singularities.

### 6.4 Gauss-Bonnet Theorem for W-surfaces

For isolated singularities\(^8\), the obstruction to constructing the moving frame may be reduced to the vanishing of certain terms in the Taylor expansion. The latter is characterized by the ramification indices $\beta_k$ which are integers. Apply the Gauss-Bonnet theorem for each of the $k$th associated surfaces by computing $\int_{\Sigma} R^{(k)}_{zz} \sqrt{g^{(k)}_{zz}}$. The integral is first computed over $\Sigma$, where small neighborhoods of singularities are removed. The ramification indices at singularity was defined so that at a singular point the induced metric of the $k$th associated surface behaves as

$$g^{(k)}_{zz} \sim (z - z_0)^{\beta_k(z_0)}(\bar{z} - \bar{z}_0)^{\bar{\beta}_k(\bar{z}_0)} g^{(k)}_{z\bar{z}},$$

where $g^{(k)}_{z\bar{z}}$ is regular at $z_0, \bar{z}_0$. Since we do not assume that $f(z) = \bar{f}(\bar{z})$, $\beta_k(z_0)$ and $\bar{\beta}_k(\bar{z}_0)$ may be different. By letting $\epsilon \to 0$, one sees, that the contribution of the singularities to the Gauss-Bonnet formula is proportional to the $k$-th ramification index

$$\beta_k \equiv \frac{1}{2} \sum_{(z_0, \bar{z}_0) \in \Sigma} \left( \beta_k(z_0) + \bar{\beta}_k(\bar{z}_0) \right).$$

\(^8\)if there is a cut with a finite number of sheets, one takes the appropriate covering
The contribution of the regular part does not depend upon $k$, since changing $k$ is equivalent to using a different complex structure, while the result is equal to the Euler characteristic that does not depend upon it. The Gauss-Bonnet theorem for the $k$th associated surface finally gives

$$
\frac{i}{2\pi} \int_{\Sigma} dzd\bar{z}R^{(k)}_{z\bar{z}}\sqrt{g_{z\bar{z}}} = 2 - 2g + \beta_k.
$$

Combining these last relations with Eqs.\ref{eq:6.2}, one arrives at the

**Theorem 3 Global Plücker formulae** The genus $g$ of a $W$-surface is related to its instanton-numbers and ramification-indices by the relations

$$2 - 2g + \beta_k = 2Q_k - Q_{k+1} - Q_{k-1}, \quad \begin{cases} \kappa_n = 1, \cdots, n \\ Q_{n+1} \equiv 0, Q_0 \equiv 0 \end{cases} \tag{6.9}$$

### 6.5 Simple example

Consider the case of Liouville theory, for which one has $n = 1$. The simplest chiral surface corresponds to

$$f^0 = 1, \quad f^1 = z; \quad \bar{f}^0 = 1, \quad \bar{f}^1 = \bar{z}; \quad \tau_1 = 1 + z\bar{z}, \quad \tau_2 = 1.$$

The instanton number is thus $Q = (i/2\pi) \int dzd\bar{z}/(1 + z\bar{z})^2 = 1$. On the other hand, one has

$$R_{z\bar{z}}\sqrt{g_{z\bar{z}}} = -\partial\bar{\partial}\ln(g_{z\bar{z}}) = \frac{2}{(1 + z\bar{z})^2},$$

so that

$$\frac{i}{2\pi} \int dzd\bar{z}R_{z\bar{z}}\sqrt{g_{z\bar{z}}} = 2Q = 2,$$

and one verifies that the above formulae indeed hold with vanishing genus and ramification index. In this example, the Liouville solution coincides with the metric of the Lobachevki half-plane. Upon quantization, it corresponds to the $SL(2,\mathbb{C})$ invariant vacuum of the Liouville theory.

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