LOW DISTORTION EMBEDDINGS INTO ASPLUND BANACH SPACES

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ABSTRACT. We give a simple example of a countable metric space $M$ that does not embed bi-Lipschitz with distortion strictly less than 2 into any Asplund space. Actually, if $M$ embeds with distortion strictly less than 2 to a Banach space $X$, then $X$ contains an isomorphic copy of $\ell_1$. We also show that the space $M$ does not embed with distortion strictly less than 2 into $\ell_1$ itself but it does embed isometrically into a space that is isomorphic to $\ell_1$.

1. Introduction

We say that a Banach space $X$ is $D$-bi-Lipschitz universal if every separable metric space embeds into $X$ with distortion at most $D$. The results of [6], resp. [1], show that $c_0$ is 2-bi-Lipschitz universal, resp. is not $D$-bi-Lipschitz universal for any $D < 2$. In the recent preprint [3], F. Baudier raised the following question: given a $C(K)$ Banach space $X$, what is the least constant $D$ such that $X$ is $D$-bi-Lipschitz universal?

The goal of the present article is to prove that Asplund $C(K)$ spaces and, more generally, all Asplund Banach spaces can be $D$-bi-Lipschitz universal only if $D \geq 2$.

We obtain our $C(K)$ result as a slight modification of Baudier’s elaboration on Aharoni’s original argument that $\ell_1$ does not embed into $c_0$ with distortion strictly less than 2. The general result is then obtained by a direct application of the deep “Zippin’s lemma” [11, Theorem 1.2]. The general result follows also from a stronger observation: if $D < 2$ and $X$ is $D$-bi-Lipschitz universal, then $\ell_1 \subset X$, which we prove in the Appendix. We also show in the Appendix that although $M$ does not embed into $\ell_1$ with distortion strictly less than 2, it embeds isometrically into a space that is isomorphic to $\ell_1$.

An immediate corollary of our $C(K)$ result is that the $C(K)$ spaces which are $D$-bi-Lipschitz universal with $D < 2$ are isometrically universal for the class of separable Banach spaces. One may ask whether this is true in general.

Problem 1. Assume that $X$ is $D$-bi-Lipschitz universal with $D < 2$. Does then every separable Banach space linearly embed into $X$? Does at least $c_0$ linearly embed into $X$?

Let us also mention that some refined results concerning low distortion embeddings between $C(K)$ spaces can be found in our paper [9].

The notation we use is standard. A mapping $f : M \to N$ between metric spaces $(M,d)$ and $(N,\rho)$ is bi-Lipschitz if there are constants $C_1, C_2 > 0$ such that $C_1 d(x,y) \leq \rho(f(x), f(y)) \leq C_2 d(x,y)$ for all $x, y \in M$. The distortion $\text{dist}(f)$ of $f$ is defined as $\inf C_2/C_1$ where the infimum

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is taken over all constants $C_1, C_2$ which satisfy the above inequality. We say that $M$ embeds bi-Lipschitz into $N$ with distortion $D$ if there exists such $f : M \to N$ with $\text{dist}(f) = D$. In this case, if the target space $N$ is a Banach space, we may always assume (by changing $f$) that $C_1 = 1$. For the following notions and results, see [4]. A Banach space $X$ is called Asplund if every closed separable subspace $Y \subset X$ has separable dual. A Hausdorff compact $K$ is called scattered if there exists an ordinal $\alpha$ such that the Cantor-Bendixson derivative $K^{(\alpha)}$ is empty. A countable Hausdorff compact is necessarily scattered. If $K$ is a Hausdorff compact then $C(K)$ is Asplund iff $K$ is scattered.

2. Results

Let $M = \{\emptyset\} \cup \mathbb{N} \cup F$ where $F = \{A \subset \mathbb{N} : 1 \leq |A| < \infty\}$ is the set of all finite nonempty subsets of $\mathbb{N}$. We put an edge between two points $a, b$ of $M$ if $a = \emptyset$ and $b \in \mathbb{N}$ or $a \in \mathbb{N}, b \in F$ and $a \in b$ thus introducing a graph structure on $M$. The shortest path metric $d$ on $M$ is then given for $n \neq m \in \mathbb{N} \subset M$ and $A \neq B \in F$ by

$$d(\emptyset, n) = 1, \quad d(n, m) = 2, \quad d(n, A) = 1 \text{ if } n \in A, \quad d(n, A) = 3 \text{ if } n \notin A,$$

$$d(\emptyset, A) = 2, \quad d(A, B) = 2 \text{ if } A \cap B \neq \emptyset, \quad d(A, B) = 4 \text{ if } A \cap B = \emptyset.$$ 

Thus $(M, d)$ is a countable (in particular separable) metric space.

**Lemma 2.** Let $X = C(K)$ for some compact space $K$ and assume that there exist $D \in [1, 2)$ and $f : M \to X$ such that

$$d(x, y) \leq \|f(x) - f(y)\| \leq Dd(x, y).$$

Then $K$ is not scattered.

**Proof.** We may assume, without loss of generality, that $f(\emptyset) = 0$. We will show for any ordinal $\alpha$ that $K^{(\alpha)} \neq \emptyset$. Let $\eta = 4 - 2D > 0$. For any $i, j \in \mathbb{N}$ let $X_{i, j} = \{x^* \in K : |\langle x^*, f(i) - f(j) \rangle| \geq \eta\}$. These are closed subsets of $K$. We will show that for any disjoint $A, B \in F$ and any ordinal $\alpha$ we have

$$\bigcap_{a \in A, b \in B} X_{a, b} \cap K^{(\alpha)} \neq \emptyset.$$

Let us start with $\alpha = 0$. Let $A, B \in F$ be disjoint. We take $x^* \in K$ such that $|\langle x^*, f(A) - f(B) \rangle| = \|f(A) - f(B)\| \geq 4$. Then for any $a \in A$ and any $b \in B$ we have

$$|\langle x^*, f(a) - f(b) \rangle| \geq |\langle x^*, f(A) - f(B) \rangle| - |\langle x^*, f(A) - f(a) \rangle| - |\langle x^*, f(B) - f(b) \rangle|$$

$$\geq 4 - 2D = \eta$$

thus $x^* \in \bigcap_{a \in A, b \in B} X_{a, b} \cap K$.

Let us assume that we have proved the claim for every $\beta < \alpha$. If $\alpha$ is a limit ordinal then the closedness of $X_{i, j}$ implies the claim. Let us assume that $\alpha = \beta + 1$. Let us fix two disjoint sets $A, B \in F$. Let $N = 1 + \max A \cup B$. By the inductive hypothesis we know that for $N \leq i < j$ there is $x^*_{i,j} \in K$ s.t.

$$x^*_{i,j} \in K^{(\beta)} \cap \bigcap_{a \in A \cup \{i\}, b \in B \cup \{j\}} X_{a, b}.$$
We put $\Gamma := \{x_{i,j}^r : N \leq i < j\} \subset K$. We define $\Phi : N \cap [N,\infty) \to \ell_1(\Gamma)$ by $\Phi(i) := \langle (\gamma, f(i)) \rangle_{\gamma \in \Gamma}$. Then the image of $\Phi$ is an $\eta$-separated countably infinite bounded set. Indeed $\|\Phi(i)\|_\infty \leq \|f(i)\| \leq \text{dist}(i, \emptyset) = D$. Let $N \leq i < j$ then $\|\Phi(i) - \Phi(j)\|_\infty \geq \|x_{i,j}^r, f(i) - f(j)\| \geq \eta$. Thus $\Gamma$ is infinite and therefore $K^{(\beta)} \cap \bigcap_{a \in A, b \in B} X_{a,b}$ is infinite, too. Now the closedness of $\bigcap_{a \in A, b \in B} X_{a,b}$ and compactness of $K^{(\beta)}$ imply our claim. 

\section*{Corollary 3.} If $X = C(K)$ for some compact space $K$ and $(M, d)$ embeds into $X$ with distortion strictly less than 2, then $X$ is isometrically universal for all separable spaces.

\begin{proof}
Since $C(K)$ is not Asplund, $K$ is not scattered. By a result of Pelczyński and Semadeni [8] there is a continuous surjection of $K$ onto $[0,1]$. Thus $C(K)$ contains isometrically $C([0,1])$ as a closed subspace. The proof is thus finished by the application of Banach-Mazur theorem [2].
\end{proof}

\section*{Theorem 4.} Let $X$ be an Asplund space and assume that $(M, d)$ embeds into $X$ with distortion $D$. Then $D \geq 2$. Consequently, no Asplund space is universal for embeddings of distortions strictly less than 2 for all separable metric spaces.

Since $M$ is countable, we may assume that $X$ is separable. The proof is then based on Lemma 2 and the following theorem of Zippin, see [11, Theorem 1.2] or [10, Lemma 5.11].

\section*{Theorem 5.} Let $X$ be a separable Asplund space. Let $\varepsilon > 0$. Then there exist a Banach space $Z$, a countable Hausdorff (in particular scattered) compact $S$, a subspace $Y$ of $Z$ isometric to $C(S)$ and a linear embedding $i : X \to Z$ with $\|i\| \|i^{-1}\| < 1 + \varepsilon$ such that for any $x \in X$ we have

$$\text{dist}_Z(i(x), Y) \leq \varepsilon \|i(x)\|_Z$$

\begin{proof}[Proof of Theorem 4] Let us assume that $D < 2$. Let $\varepsilon > 0$ be small enough so that $D' = D(1 + \varepsilon) < 2$ and also that for $\eta := \varepsilon 2D'$ we have $\frac{1 + 4\varepsilon}{1 - 2\eta} D' < 2$. Then $(M, d)$ embeds into $Z$ with distortion $D' < 2$ via some embedding $g$ such that $d(x, y) \leq \|g(x) - g(y)\| \leq D'd(x, y)$. We may assume, without loss of generality, $g(\emptyset) = 0$. Thus for every $x \in M$ we have $\|g(x)\| \leq 2D'$. We know that for each $x \in M$ there is $f(x) \in Y$ such that $\|g(x) - f(x)\| \leq \eta$. This implies that $\|g(x) - g(y)\| - 2\eta \leq \|f(x) - f(y)\| \leq \|g(x) - g(y)\| + 2\eta$. Now since $1 \leq d(x, y)$ we have

$$d(x, y)(1 - 2\eta) \leq \|f(x) - f(y)\| \leq d(x, y)D'(1 + 4\varepsilon).$$

This proves that $f$ is a bi-Lipschitz embedding of $M$ into $C(S)$ with distortion strictly less than 2 which is impossible according to Lemma 2.
\end{proof}

\section*{3. Appendix}

\section*{Theorem 6.} Let $X$ be a Banach space and suppose that there exist $D \in [1, 2)$ and $f : M \to X$ such that $d(x, y) \leq \|f(x) - f(y)\| \leq Dd(x, y)$. Then $X$ contains a copy of $\ell_1$.

\begin{proof}
We plan to use the Rosenthal theorem [2]. Thus, we have to find a sequence $(x_k) \subset X$ such that none of its subsequences is weakly Cauchy. We claim that if we put $x_k := f(k)$, then $(x_k)$ will have this property. Indeed, let $(f_n) \subset \mathbb{N}$ be given. We define $A_N = \{k_{2n} : n \leq N\}$ and
\[ B_N = \{ k_{2n-1} : n \leq N \} \] for every \( N \in \mathbb{N} \). Let us put \( \varepsilon := 4 - 2D > 0 \). Similarly as in the proof of Lemma 2 we will put

\[ X_{a,b} = \{ x^* \in B_{X^*} : \langle x^*, f(a) - f(b) \rangle \geq \varepsilon \}, \]

and we will show that for every \( n \in \mathbb{N} \)

\[ K_n := \bigcap_{a \in A_n, b \in B_n} X_{a,b} \neq \emptyset. \]

Now observe that \( (K_n) \) is a decreasing sequence of non-empty \( w^* \)-compacts. Thus there exists \( x^* \in \bigcap_{n=1}^{\infty} K_n \). It is clear that \( \langle x^*, x_{k2n} - x_{k2n+1} \rangle \geq \varepsilon \) for all \( n \in \mathbb{N} \). Thus \( (x_k)_{i=1}^{\infty} \) is not weakly Cauchy.

Weirdly, \( M \) does not embed well into \( \ell_1 \) either. We will see this using Enflo’s generalized roundness. Let us recall the definition as presented in [7].

**Definition 7.** A metric space \( (X,d) \) is said to have generalized roundness \( q \), written \( q \in gr(X,d) \), if for every \( n \geq 2 \) and all points \( a_1, \ldots, a_n, b_1, \ldots, b_n \in X \) we have

\[ \sum_{1 \leq i < j \leq n} ((d(a_i,a_j))^q + d(b_i,b_j))^q \leq \sum_{1 \leq i,j \leq n} d(a_i,b_j)^q. \]

**Proposition 8.** If \( f : (M,d) \overset{\omega}{\rightarrow} (X,\delta) \) and there is \( 0 < q \in gr(X,\delta) \), then \( D \geq 2 \).

**Proof.** Let \( a_1, \ldots, a_n, b_1, \ldots, b_n \in M \). We have

\[ \sum_{1 \leq i < j \leq n} ((\delta(f(a_i), f(a_j)))^q + \delta(f(b_i), f(b_j))^q \leq \sum_{1 \leq i,j \leq n} \delta(f(a_i), f(b_j))^q \]

and so

\[ \sum_{1 \leq i < j \leq n} ((d(a_i,a_j))^q + d(b_i,b_j))^q \leq D^q \sum_{1 \leq i,j \leq n} d(a_i,b_j)^q. \]

If \( a_1, \ldots, a_n \) are arbitrary in the 2nd floor and \( b_i = \{ a_1, \ldots, a_n \} \setminus \{ a_i \} \) (in the 3rd floor), the above inequality evaluates to

\[ n(n-1)2^q \leq D^q n ((n-1) + 3^q) , \]

i.e. \( (\frac{2}{D})^q \leq 1 + \frac{3^q}{n-1} \), which is possible for all \( n \) only if \( D \geq 2 \). \( \square \)

**Corollary 9.** The metric space \( M \) does not embed with distortion strictly less than 2 into any \( L_1(\mu) \).

**Proof.** This follows immediately from Proposition 8 and from the fact that \( 1 \in gr(L_1(\mu)) \) which is proved in [7, Corollary 2.6]. \( \square \)

On the other hand \( M \) lives isometrically in a space that is isomorphic to \( \ell_1 \).

**Proposition 10.** There is an equivalent norm \( \| \cdot \| \) on \( \ell_1 \) such that \( M \) embeds isometrically into \( (\ell_1, \| \cdot \|) \).
Proof. For the definition and basic properties of Lipschitz-free spaces see [5] and the references therein. The space $M$ embeds isometrically into $\mathcal{F}(M)$ where $\mathcal{F}(M)$ is the Lipschitz-free space over $M$. The result thus follows from the following fact.

Fact: If $(U, d)$ is a countable uniformly discrete bounded metric space then $\mathcal{F}(U)$ is isomorphic to $\ell_1$.

First, it is easy to observe that if we equip $N_0 = \{0\} \cup \mathbb{N}$ with the distance $\rho(0, x) = 1$ for every $n \in \mathbb{N}$, and $\rho(x, y) = 2$ for every $x \neq y \in \mathbb{N}$, then $\mathcal{F}(N_0)$ is isometrically isomorphic to $\ell_1$. Second, it is clear that $(U, d)$ is Lipschitz homeomorphic to $(N_0, \rho)$. The free spaces $\mathcal{F}(U)$ and $\mathcal{F}(N_0)$ are thus linearly isomorphic. This finishes the proof of the fact and of the proposition.

We do not know whether the Banach-Mazur distance between $\mathcal{F}(M)$ and $\ell_1$ is 2. The above proof only shows that it is at most 4 while Corollary 9 shows that it is at least 2.

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