THE C*-ALGEBRA OF SYMMETRIC WORDS IN TWO
UNIVERSAL UNITARIES

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Abstract. We compute the $K$-theory of the $C^*$-algebra of symmetric words in two universal unitaries. This algebra is the fixed point $C^*$-algebra for the order-two automorphism of the full $C^*$-algebra of the free group on two generators which switches the generators. Our calculations relate the $K$-theory of this $C^*$-algebra to the $K$-theory of the associated $C^*$-crossed-product by $\mathbb{Z}_2$.

1. Introduction

This paper investigates an example of a $C^*$-algebra of symmetric words in noncommutative variables. Our specific interest is in the $C^*$-algebra of symmetric words in the two universal unitaries generating the full $C^*$-algebra $C^*(F_2)$ of the free group on two generators. Our main result is the computation of the $K$-theory of this algebra.

The two canonical unitary generators of $C^*(F_2)$ are denoted by $U$ and $V$. The $C^*$-algebra of symmetric words in two universal unitaries $U$, $V$ is precisely defined as the fixed point $C^*$-algebra $C^*(F_2)_1$ for the order-2 automorphism $\sigma$ which maps $U$ to $V$ and $V$ to $U$. Our strategy to compute the $K$-theory of $C^*(F_2)_1$ relies upon the work of Rieffel [6, Proposition 3.4] about Morita equivalence between fixed point $C^*$-algebras and $C^*$-crossed-products.

The first part of this paper describes the two algebras of interest: the fixed point $C^*$-algebra $C^*(F_2)_1$ and the crossed-product $C^*(F_2) \rtimes_{\sigma} \mathbb{Z}_2$. We also exhibit an ideal $J$ in $C^*(F_2) \rtimes_{\sigma} \mathbb{Z}_2$ which is strongly Morita equivalent to $C^*(F_2)_1$ and can be easily described as the kernel of an very simple $*$-morphism. This allows us to reduce the problem to the calculation of the $K$-theory of $C^*(F_2) \rtimes_{\sigma} \mathbb{Z}_2$.

The second part of this paper starts by this calculation. We use a standard result from Cuntz [4] to calculate the $K$-theory of $C^*(F_2) \rtimes_{\sigma}$
proves that $A - a$ if Morita equivalent to $C$. A morphism of $algebras $U$ two unitaries $Z$ of $C$-algebras $U$.

Let $A$ be a unital C*-algebra and $\sigma$ an order-two automorphism of $A$. The fixed-point C*-algebra of $A$ for $\sigma$ is the set $A_1 = \{a + \sigma(a) : a \in A\}$. Set $A_{-1} = \{a - \sigma(a) : a \in A\}$. Then $A = A_1 + A_{-1}$ and $A_1 \cap A_{-1} = \{0\}$.

**Proof.** Let $\omega \in A$. Since $\sigma^2 = 1$ we have $\sigma(\omega + \sigma(\omega)) = \sigma(\omega) + \omega \in A_1$. On the other hand, if $a \in A$ then $a = \frac{1}{2}(a + \sigma(a)) + \frac{1}{2}(a - \sigma(a))$. Hence if $a \in A$ then $a - \sigma(a) = 0$ and $a \in \{\omega + \sigma(\omega) : \omega \in A\}$. Moreover this proves that $A = A_1 + A_{-1}$. Last, if $a \in A_1 \cap A_{-1}$ then $a = \sigma(a) = -\sigma(a) = 0$.

We can use Lemma 2.1 to obtain a more concrete description of the fixed-point C*-algebra $C^*(\mathbb{F}_2)_1$ of symmetric words in two unitaries.

**Theorem 2.2.** Let $\sigma$ be the automorphism of $C^*(\mathbb{F}_2) = C^*(U,V)$ defined by $\sigma(U) = V$ and $\sigma(V) = U$. Then the fixed point C*-algebra of $\sigma$ is $C^* (\mathbb{F}_2)_1 = C^* (U^n + V^n : n \in \mathbb{N})$. 

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We conclude the paper by looking a little closer to the obstruction to the existence of unitaries of nontrivial $K$-theory in $C^*(\mathbb{F}_2)_1$. This amounts to comparing the structure of the ideal $J$ and an ideal in $C^*(\mathbb{F}_2)_1$ related to the same representation as $J$.

We also should mention that, in principle, using the results in \cite{B}, one could derive information on the representation theory of $C^* (\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ from the representations of $C^* (\mathbb{F}_2)_1$.

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### 2. The Fixed Point C*-algebra

Let $C^* (\mathbb{F}_2) = C^*(U,V)$ be the universal C*-algebra generated by two unitaries $U$ and $V$. We consider the order-2 automorphism $\sigma$ of $C^* (\mathbb{F}_2)$ uniquely defined by $\sigma(U) = V$ and $\sigma(V) = U$. These relations indeed define an automorphism by universality of $C^* (\mathbb{F}_2)$. Our main object of interest is the fixed point C*-algebra $C^* (\mathbb{F}_2)_1 = \{a \in C^* (\mathbb{F}_2) : \sigma(a) = a\}$ which can be seen as the C*-algebra of symmetric words in two universal unitaries. The C*-algebra $C^* (\mathbb{F}_2)_1$ is related \cite{B} to the C*-crossed-product $C^* (\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$, which we will consider in our calculations. Our objective is to gain some understanding of the structure of the unitaries and projections in $C^* (\mathbb{F}_2)_1$.

The first step in our work is to describe concretely the two C*-algebras $C^* (\mathbb{F}_2)_1$ and $C^* (\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$. Using \cite{B} Proposition 3.4, we also exhibit an ideal in the crossed-product $C^* (\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ which is Morita equivalent to $C^* (\mathbb{F}_2)_1$.

The following easy lemma will be useful in our work:

**Lemma 2.1.** Let $A$ be a unital C*-algebra and $\sigma$ an order-two automorphism of $A$. The fixed-point C*-algebra of $A$ for $\sigma$ is the set $A_1 = \{a + \sigma(a) : a \in A\}$. Set $A_{-1} = \{a - \sigma(a) : a \in A\}$. Then $A = A_1 + A_{-1}$ and $A_1 \cap A_{-1} = \{0\}$.

**Proof.** Let $\omega \in A$. Since $\sigma^2 = 1$ we have $\sigma(\omega + \sigma(\omega)) = \sigma(\omega) + \omega \in A_1$. On the other hand, if $a \in A$ then $a = \frac{1}{2}(a + \sigma(a)) + \frac{1}{2}(a - \sigma(a))$. Hence if $a \in A$ then $a - \sigma(a) = 0$ and $a \in \{\omega + \sigma(\omega) : \omega \in A\}$. Moreover this proves that $A = A_1 + A_{-1}$. Last, if $a \in A_1 \cap A_{-1}$ then $a = \sigma(a) = -\sigma(a) = 0$.

We can use Lemma 2.1 to obtain a more concrete description of the fixed-point C*-algebra $C^* (\mathbb{F}_2)_1$ of symmetric words in two unitaries.
Proof. Obviously $\sigma(U^n + V^n) = U^n + V^n$ for all $n \in \mathbb{Z}$. Hence
$$C^* (\{U^n + V^n : n \in \mathbb{Z}\}) \subseteq C^* (\mathbb{F}_2).$$
Conversely, $C^* (\mathbb{F}_2) = \{\omega + \sigma(\omega) : \omega \in C^* (\mathbb{F}_2)\}$ by Lemma 2.1. So $C^* (\mathbb{F}_2)$ is generated by elements of the form $\omega + \sigma(\omega)$ where $\omega$ is a word in $C^* (\mathbb{F}_2)$, since $C^* (\mathbb{F}_2)$ is generated by words in $U$ and $V$, i.e. by monomial of the form $U^{a_0}V^{a_1} \ldots U^{a_n}$ with $n \in \mathbb{N}$, $a_0, \ldots, a_n \in \mathbb{Z}$. It is thus enough to show that for any word $\omega \in C^* (\mathbb{F}_2)$ we have $\omega + \sigma(\omega) \in S$ where $S = C^* (U^n + V^n : n \in \mathbb{Z})$. Since, if $\omega$ starts with a power of $V$ then $\sigma(\omega)$ starts with a power of $U$, we may as well assume that $\omega$ always starts with a power of $U$ by symmetry. Since the result is trivial for $\omega = 1$ we assume that $\omega$ starts with a nontrivial power of $U$. Such a word is of the form $\omega = U^{a_0}V^{a_1} \ldots U^{a_n}$ with $a_0, \ldots, a_{n-1} \in \mathbb{Z} \setminus \{0\}$ and $a_n \in \mathbb{Z}$.

We define the order of such a word $\omega$ as the integer $o(\omega) = \sum |a_n|$ if $a_n \neq 0$ and $o(\omega) = n - 1$ otherwise. In other words, $o(\omega)$ is the number of times we go from $U$ to $V$ or $V$ to $U$ in $\omega$. The proof of our result follows from the following induction on $o(\omega)$.

By definition, if $\omega$ is a word such that $o(\omega) = 0$ then $\omega + \sigma(\omega) \in S$. Let us now assume that for some $m \geq 1$ we have shown that for all words $\omega$ starting in $U$ such that $o(\omega) \leq m - 1$ we have $\omega + \sigma(\omega) \in S$. Let $\omega$ be a word of order $m$ and let us write $\omega = U^{a_0}\omega_1$ with $\omega_1$ a word starting in a power of $V$. By construction, $\omega_1$ is of order $m - 1$. Let $\omega_2 = V^{a_0}\omega_1$. By construction, $o(\omega_2) = m - 1$ or $m - 2$. Either way, by our induction hypothesis, we have $\omega_1 + \sigma(\omega_1) \in S$ and $\omega_2 + \sigma(\omega_2) \in S$. Now:

$$\omega + \sigma(\omega) = U^{a_0}\omega_1 + V^{a_0}\sigma(\omega_1) = (U^{a_0} + V^{a_0})(\omega_1 + \sigma(\omega_1)) - (\omega_2 + \sigma(\omega_2))$$

hence $\omega + \sigma(\omega) \in S$ and our induction is complete. Hence $C^* (\mathbb{F}_2)_1 = S$ as desired. 

We wish to understand more of the structure of the fixed point $C^*$-algebra $C^* (\mathbb{F}_2)$. Using Morita equivalence, we can derive its $K$-theory. According to [6, Proposition 3.4], $C^* (\mathbb{F}_2)$ is strongly Morita equivalent to the ideal $\mathcal{J}$ generated in the crossed-product $C^* (\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ by the spectral projection $p = \frac{1}{2} (1 + W)$ of the canonical unitary $W$ in $C^* (\mathbb{F}_2) \rtimes \mathbb{Z}_2$ such that $WUW = V$. We first provide a simple yet useful description of $C^* (\mathbb{F}_2) \rtimes \mathbb{Z}_2$ in term of unitary generators, before providing a description of the ideal $\mathcal{J}$ which will ease the calculation of its $K$-theory.
Lemma 2.3. Let $\sigma$ be the automorphism defined by $\sigma(U) = V$ and $\sigma(V) = U$ on the universal $C^*$-algebra $C^* (\mathbb{F}_2)$ generated by two universal unitaries $U$ and $V$. By definition, $C^* (\mathbb{F}_2) \rtimes_{\sigma} \mathbb{Z}_2$ is the universal $C^*$-algebra generated by three unitaries $U, V$ and $W$ subjects to the relations $W^2 = 1$ and $WUW^* = V$. The $C^*$-crossed-product $C^* (\mathbb{F}_2) \rtimes_{\sigma} \mathbb{Z}_2$ is equal to $C^* (U, W)$, or equivalently is the universal $C^*$-algebra generated by two unitaries $U$ and $W$ with the relation $W^2 = 1$, or equivalently $C^* (\mathbb{F}_2) \rtimes_{\sigma} \mathbb{Z}_2$ is $^*$-isomorphic to $C^* (\mathbb{Z} \rtimes \mathbb{Z}_2)$.

Proof. The $C^*$-subalgebra $C^* (U, W)$ of $C^* (\mathbb{F}_2) \rtimes_{\sigma} \mathbb{Z}_2$ contains $WUW = V$ and thus equals $C^* (\mathbb{F}_2) \rtimes_{\sigma} \mathbb{Z}_2$.

Let us now prove that the $C^*$-subalgebra $C^* (U, W)$ is universal for the given relations. Let $u, w$ be two arbitrary unitaries in some arbitrary $C^*$-algebra such that $w^2 = 1$. Let $v = uwv \in C^* (u, w)$. By universality of the crossed-product $C^* (\mathbb{F}_2) \rtimes_{\sigma} \mathbb{Z}_2$ there exists a unique $^*$-morphism $\varphi : C^* (\mathbb{F}_2) \rtimes_{\sigma} \mathbb{Z}_2 \to C^* (u, w)$ such that $\varphi(U) = u$, $\varphi(V) = v$ and $\varphi(W) = w$. Thus $C^* (U, W)$ is universal for the proposed relations. In particular, it is $^*$-isomorphic (by uniqueness of the universal $C^*$-algebra for the given relations) to $C^* (\mathbb{Z} \rtimes \mathbb{Z}_2)$.

Now, the ideal $\mathcal{J}$ can be described as the kernel of a particularly explicit $^*$-morphism of $C^* (\mathbb{F}_2) \rtimes_{\sigma} \mathbb{Z}_2$.

Proposition 2.4. Let $\sigma$ be the automorphism defined by $\sigma(U) = V$ and $\sigma(V) = U$ on the universal $C^*$-algebra $C^* (\mathbb{F}_2)$ generated by two universal unitaries $U$ and $V$. Let $W$ be the canonical unitary of the crossed-product $C^* (\mathbb{F}_2) \rtimes_{\sigma} \mathbb{Z}_2$ such that $WUW = V$. The fixed point $C^*$-algebra $C^* (\mathbb{F}_2, 1)$ is strongly Morita equivalent to the kernel $\mathcal{J}$ in $C^* (\mathbb{F}_2) \rtimes_{\sigma} \mathbb{Z}_2$ of the $^*$-morphism $\varphi : C^* (\mathbb{F}_2) \rtimes_{\sigma} \mathbb{Z}_2 \to C(\mathbb{T})$ defined by $\varphi(W) = -1$ and $\varphi(U)(z) = \varphi(V)(z) = z$ for all $z \in \mathbb{T}$.

Proof. Let $\varphi$ be the unique $^*$-morphism from $C^* (U, W)$ into $C(\mathbb{T})$ defined using the universal property of Lemma 2.3 by: $\varphi(W)(z) = -1$ and $\varphi(U)(z) = z$ for all $z \in \mathbb{T}$. Since $\varphi(p) = 0$ by construction, $\mathcal{J} \subseteq \ker \varphi$. We wish to show that $\ker \varphi \subseteq \mathcal{J}$ as well.

Let $a \in \ker \varphi$. Let $\pi$ be a representation of $C^* (\mathbb{F}_2) \rtimes_{\sigma} \mathbb{Z}_2$ which vanishes on $\mathcal{J}$. Then $\pi(p) = 0$ so $\pi(W) = -1$. Hence, for any $b \in C^* (\mathbb{F}_2, 1)$ (see Lemma 2.3) then $\pi(b) = \pi(W) \pi(b) \pi(W) = \pi(-b)$ so that $\pi(b) = 0$. In particular, $\pi(U - V) = 0$ and thus $\pi(U) = \pi(V)$.

Thus, up to unitary equivalence, $\pi = s \circ \varphi$ where $s : C(\mathbb{T}) \to C(S)$ with $S$ the spectrum of $\pi(U)$ and $s$ is the canonical surjection. In particular, $\pi(a) = s \circ \varphi(a) = 0$. Since $\pi$ is arbitrary, we conclude that the image of $a$ in $C^* (\mathbb{F}_2) \rtimes_{\sigma} \mathbb{Z}_2 / \mathcal{J}$ is null, and thus $a \in \mathcal{J}$ as required.

Our goal now is to compute the $K$-theory of the fixed point C*-algebra $C^*(\mathbb{F}_2)_1$. Since $C^*(\mathbb{F}_2)_1$ and the ideal $\mathcal{J}$ are Morita equivalent, they have the same $K$-theory. By Proposition (2.4), we have the short exact sequence $0 \to \mathcal{J} \to C^*(\mathbb{F}_2)_1 \xrightarrow{\phi} C(\mathbb{T}) \to 0$, and it seems quite reasonable to use the six-term exact sequence of $K$-theory to deduce the $K$-groups of $\mathcal{J}$ from the $K$-groups of $C^*(\mathbb{F}_2)_1$, as long as the later can be computed. The next section precisely follows this path, starting by computing the $K$-theory of the crossed-product $C^*(\mathbb{F}_2)_1 \rtimes_\sigma \mathbb{Z}_2$.

3. $K$-theory of the C*-crossed product and the Fixed Point C*-algebra

We use the homotopy-based result in [4] to compute the $K$-theory of the crossed-product $C^*(\mathbb{F}_2)_1 \rtimes_\sigma \mathbb{Z}_2$.

**Proposition 3.1.** Let $C^*(\mathbb{F}_2) = C^*(U, V)$ be the universal C*-algebra generated by two unitaries $U$ and $V$ and let $\sigma$ be the order-2 automorphism of $C^*(\mathbb{F}_2)$ defined by $\sigma(U) = V$ and $\sigma(V) = U$. Then $K_0(C^*(\mathbb{F}_2)_1 \rtimes_\sigma \mathbb{Z}_2) = \mathbb{Z}^2$ is generated by the spectral projections of $W$ and $K_1(C^*(\mathbb{F}_2)_1 \rtimes_\sigma \mathbb{Z}_2) = \mathbb{Z}$ is generated by $U$.

**Proof.** By Lemma (2.3), the C*-crossed-product $C^*(\mathbb{F}_2)_1 \rtimes_\sigma \mathbb{Z}_2$ is *-isomorphic to

$$C^*(\mathbb{Z} \ast \mathbb{Z}_2) = C^*(\mathbb{Z}) \ast_{\mathbb{C}} C^*(\mathbb{Z}^2) = C(\mathbb{T}) \ast_{\mathbb{C}} \mathbb{C}^2$$

where the free product is amalgated over the C*-algebra generated by the respective units in each C*-algebra. More precisely, we embed $\mathbb{C}$ via, respectively, $i_1 : \lambda \in \mathbb{C} \mapsto \lambda 1 \in C(\mathbb{T})$ and $i_2 : \lambda \in \mathbb{C} \mapsto (\lambda, \lambda) \in \mathbb{C}^2$.

There are natural *-morphisms from $C(\mathbb{T})$ and from $\mathbb{C}^2$ onto $\mathbb{C}$ defined respectively by $r_1 : f \in C(\mathbb{T}) \mapsto f(1)$ and $r_2 : \lambda \mapsto \lambda$. Now, $r_1 \circ i_1 = r_2 \circ i_2$ is the identity on $\mathbb{C}$. By [4], we conclude that the following sequences for $\varepsilon = 0, 1$ are exact:

$$0 \to \mathbb{C} \xrightarrow{j_\varepsilon} K_\varepsilon(C(\mathbb{T}) \oplus \mathbb{C}^2) \xrightarrow{k_\varepsilon} K_\varepsilon(C(\mathbb{T}) \ast_{\mathbb{C}} \mathbb{C}^2) \to 0$$

where $j_\varepsilon = K_\varepsilon(i_1) \oplus K_\varepsilon(-i_2)$ and $k_\varepsilon = K_\varepsilon(k_1 + k_2)$ where $k_1$ is the canonical embedding of $C(\mathbb{T})$ into $C(\mathbb{T}) \ast_{\mathbb{C}} \mathbb{C}^2$ and $k_2$ is the canonical embedding of $\mathbb{C}^2$ into $C(\mathbb{T}) \ast_{\mathbb{C}} \mathbb{C}^2$.

Now, $K_1(C(\mathbb{T}) \oplus \mathbb{C}^2) = \mathbb{Z}$ generated by the identity $z \in \mathbb{T} \mapsto z \in C(\mathbb{T})$. Since $K_1(C) = 0$ we conclude that $K_1(C(\mathbb{T}) \ast_{\mathbb{C}} \mathbb{C}^2) = \mathbb{Z}$ generated by the canonical unitary generator of $C(\mathbb{T})$. On the other hand, $K_0(C(\mathbb{T}) \oplus \mathbb{C}^2) = \mathbb{Z}^3$ (where the first copy of $\mathbb{Z}$ is generated by the unit $1_{C(\mathbb{T})}$ of $C(\mathbb{T})$ and the two other copies are generated by each of the projections $(1, 0)$ and $(0, 1)$ in $\mathbb{C}^2$). Now, ran $j_\varepsilon$ is the subgroup of $\mathbb{Z}^3$.
generated by the class of $1_{C(T)} \oplus -1_{C^2}$ where $1_{C^2}$ is the unit of $C^2$. This class is $(1,-1,-1)$, so we conclude easily that $K_0(C(T) \oplus C^2) = \mathbb{Z}^2$ is generated by the two projections in $C^2$ (whose classes are $(0,1,0)$ and $(0,0,1)$).

Using the *-isomorphism between $C(T) \oplus C^2$ and $C^* \langle F_2 \rangle \rtimes_{\sigma} \mathbb{Z}_2$ we conclude that $K_0(C^* \langle F_2 \rangle \rtimes_{\sigma} \mathbb{Z}_2) = \mathbb{Z}^2$ is generated by the spectral projections of $W$ while $K_1(C^* \langle F_2 \rangle \rtimes_{\sigma} \mathbb{Z}_2) = \mathbb{Z}$ is generated by the class of $U$ (or $V$ as these are equal by construction).

**Remark 3.2.** It is interesting to compare our results to the $K$-theory of the $C^*$-crossed-product by $\mathbb{Z}$ instead of $\mathbb{Z}_2$. One could proceed with the standard six-terms exact sequence \cite[Theorem 10.2.1]{11}, but it is even simpler to observe the following similar result to Lemma (2.3): the $C^*$-crossed-product $C^* \langle F_2 \rangle \rtimes_{\sigma} \mathbb{Z}_2$ is *-isomorphic to $C^* \langle F_2 \rangle$. If $C^* \langle F_2 \rangle$ is generated by the two universal unitaries $U, V$ and $W$ is the canonical unitary in $C^* \langle F_2 \rangle \rtimes_{\sigma} \mathbb{Z}$ such that $WUW = V$ then once again $C^* \langle U, V, W \rangle = C^* \langle U, W \rangle$ and, following a similar argument as for Lemma (2.3) we observe that $C^* \langle U, W \rangle$ is the $C^*$-algebra universal for two arbitrary unitaries.

Hence, $K_0(C^* \langle F_2 \rangle \rtimes_{\sigma} \mathbb{Z}) = \mathbb{Z}$ is generated by the identity while $K_1(C^* \langle F_2 \rangle \rtimes_{\sigma} \mathbb{Z}) = \mathbb{Z}^2$ is generated by $U$ and $W$.

We can now deduce the $K$-theory of $C^* \langle F_2 \rangle_1$ from Proposition (2.4) and Theorem (3.1). The natural way to do so is by using the six-terms exact sequence in $K$-theory applied to the short exact sequence $J \hookrightarrow C^* \langle F_2 \rangle \rtimes_{\sigma} \mathbb{Z}_2 \xrightarrow{\varphi} C(T)$.

**Theorem 3.3.** Let $C^* \langle F_2 \rangle = C^* \langle U, V \rangle$ be the universal $C^*$-algebra generated by two unitaries $U$ and $V$ and let $\sigma$ be the order-2 automorphism of $C^* \langle F_2 \rangle$ defined by $\sigma(U) = V$ and $\sigma(V) = U$. Let $C^* \langle F_2 \rangle_1 = \{a \in C^* \langle F_2 \rangle : \sigma(a) = a \}$ be the fixed point $C^*$-algebra for $\sigma$. Then $K_0(C^* \langle F_2 \rangle_1) = \mathbb{Z}$ is generated by the identity in $C^* \langle F_2 \rangle_1$ and $K_1(C^* \langle F_2 \rangle_1) = 0$.

**Proof.** By \cite[Proposition 3.4]{6}, $C^* \langle F_2 \rangle_1$ is strongly Morita equivalent to the ideal $J$ generated in $C^* \langle F_2 \rangle \rtimes_{\sigma} \mathbb{Z}_2$ by the projection $p = \frac{1}{2}(1 + W)$. Using Proposition (2.4), we can apply the six-terms exact sequence to the short exact sequence

$$0 \longrightarrow J \xrightarrow{i} C^* \langle F_2 \rangle \rtimes_{\sigma} \mathbb{Z}_2 \xrightarrow{\varphi} C(T) \longrightarrow 0$$

where $i$ is the canonical injection and $\varphi$ is the *-morphism of Proposition (2.4).

We denote by $\Phi$ the quotient isomorphism from $C^* \langle F_2 \rangle \rtimes_{\sigma} \mathbb{Z}_2 / J$ onto $C(T)$ induced by $\varphi$. Since we know the $K$-theory of $C^* \langle F_2 \rangle \rtimes_{\sigma} \mathbb{Z}_2$ by
Theorem \(\text{(3.1)}\), including a set of generators of the \(K\)-groups, and the \(K\)-theory of \(C(\mathbb{T})\), we can easily deduce the \(K\)-theory of \(J\). Indeed, we have the following exact sequence [11 9.3 p. 67]:

\[
\begin{align*}
& K_0(J) \xrightarrow{\delta} K_0(C^*(\mathbb{F}_2) \rtimes_{\sigma} \mathbb{Z}_2) = \mathbb{Z}^2 \xrightarrow{K_0(\phi)} K_0(C(\mathbb{T})) \\
& K_1(C(\mathbb{T})) \xleftarrow{K_1(\phi)} K_1(C^*(\mathbb{F}_2) \rtimes_{\sigma} \mathbb{Z}_2) = \mathbb{Z} \xrightarrow{K_1(i)} K_1(J)
\end{align*}
\]

Each statement in the following argument follows from the exactness of \(\text{(3.1)}\). Trivially, \(K_0(\phi)\) is a surjection, so \(\beta = 0\). Hence \(K_1(i)\) is injective. Yet, as \(\varphi(U) : z \in \mathbb{T} \mapsto z\), we conclude that \(K_1(\varphi)\) is an isomorphism (since it maps a generator to a generator), and thus \(K_1(i) = 0\). Now \(K_1(i) = 0\) and \(\beta = 0\) implies that \(K_1(J) = 0\). Since \(K_1(\varphi)\) is surjective, \(\delta = 0\) and thus \(K_0(i)\) is injective. Its image is thus isomorphic to \(K_0(J)\) and coincide with \(\ker K_0(\phi)\). Now, \(K_0(\phi)(p) = 0\) and \(K_0(\phi)(1 - p) = 1\) (by Theorem \(\text{(3.1)}\), \(p\) and \(1 - p\) generate \(K_0(C^*(\mathbb{F}_2) \rtimes_{\sigma} \mathbb{Z}_2)\)). Hence the image of \(K_0(i)\) is isomorphic to the copy of \(\mathbb{Z}\) generated by \(1 - p\) in \(K_0(C^*(\mathbb{F}_2) \rtimes_{\sigma} \mathbb{Z}_2)\). \[
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We can go a little deeper in the structure of the fixed point C*-algebra \(C^*(\mathbb{F}_2)_1\). Of interest is to compare the ideal \(J\) and its natural restrictions to \(C^*(\mathbb{F}_2)\) and \(C^*(\mathbb{F}_2)_1\). The motivation for this comparison is to understand the obstruction to the existence of any nontrivial unitary in \(C^*(\mathbb{F}_2)_1\) in the sense of \(K\)-theory.

**Theorem 3.4.** Let \(\theta\) be the *-epimorphism \(C^*(U, V) \rightarrow C(\mathbb{T})\) defined by \(\varphi(U) = \varphi(V) : z \in \mathbb{T} \mapsto z\). Let \(I = \ker \theta\). Then \(K_0(I) = 0\) and \(K_1(I) = \mathbb{Z}\) where the generating unitary in \(I^+ = I + 1\) of the \(K_1\) group is \(UV^*\).

Let \(I_1 = C^*(\mathbb{F}_2)_1 \cap I\). We then have \(C^*(\mathbb{F}_2)_1 / I_1 = C(\mathbb{T})\). Then \(K_1(I_1) = 0\) while \(K_0(I_1) = \mathbb{Z}\).

**Proof.** We first calculate the \(K\)-theory of \(I\). This can be achieved in at least two natural ways: by means of exact sequences (see Remark \(\text{(3.1)}\)) or directly, by the following simple argument. Let \(K_1(i) : K_1(I) \rightarrow K_1(C^*(\mathbb{F}_2))\) be the \(K_1\)-lift of the canonical inclusion \(i : I \rightarrow C^*(\mathbb{F}_2)\). We first will identify the range of \(K_1(i)\) and then show that \(K_1(i)\) is injective.

Let \(Z \in M_n(I) + 1_n\) be a unitary for some \(n \in \mathbb{N}\). In particular, \(Z\) is a unitary in \(M_n(C^*(\mathbb{F}_2))\). Let \((k, k') \in \mathbb{Z}^2 = K_1(C^*(\mathbb{F}_2))\) be the class \([Z]_{C^*(\mathbb{F}_2)}\) of \(Z\) in \(K_1(C^*(\mathbb{F}_2))\). Now, by definition of \(\theta\) and \(I\), we have \(K_1(\theta)([Z]_{C^*(\mathbb{F}_2)}) = K_1([1]_{C(\mathbb{T})}) = 0\). Yet, since \(\theta(U) = \theta(V) = z \in \mathbb{T} \mapsto z\), we conclude that \(K_1(\theta)(k, k') = k + k'\). Hence
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\[ Z \vert_{C^*(\mathbb{F}_2)} = (k, -k) \text{ for some } k \in \mathbb{Z}. \] Moreover, \( UV^* - 1 \in \mathcal{I} \) by construction, and \( [UV^*]_{C^*(\mathbb{F}_2)} = (1, -1) \), so the range of \( K_1(i) \) is the subgroup generated by \((1, -1) \) in \( \mathbb{Z}^2 = K_1(C^*(\mathbb{F}_2)) \).

On the other hand, assume now that \( [Z]_{C^*(\mathbb{F}_2)} = 0 \), i.e. \( Z \oplus 1_{m-n} \) is connected to \( 1 \) in \( M_m(C^*(\mathbb{F}_2)) \) for some \( m \geq n \). To ease notation, let \( 1_k \) be the identity in \( M_k \) \((C^*(\mathbb{F}_2))\), let \( Y_1 = Z \oplus 1_{m-n} \) and let \((Y_t)_{t \in [0,1]} \) be the homotopy of unitaries joining \( Y_1 \) to \( Y_0 = 1_m \) in \( M_m(C^*(\mathbb{F}_2)) \).

Let \( \Xi \) be the the unique \(*\)-endomorphism of \( C \) that \( M \Xi(k) \) \( = k \) \( \mathcal{B}(C) \) \( = \) \( \mathcal{B}(Y) \) \( = U \). For any \(*\)-endomorphism \( \eta \) of \( C^*(\mathbb{F}_2) \) we let \( M_\eta \) be the canonical \(*\)-endomorphism of \( M_m(C^*(\mathbb{F}_2)) \) induced from \( \eta \). Set \( Y_t' = Y_t(M_n(\Xi)(Y_t)) \) \( = M_\eta \) for all \( t \in [0,1] \). Then we check immediately that \( M_n(\eta)(Y_t) \) \( = 1_m \) so \( Y_t - 1_m \in M_m(\mathcal{I}) \) by definition. Hence, \((Y_t')_{t \in [0,1]} \) is now an homotopy in \( M_m(\mathcal{I}) \) between \( Y_t' \) and \( Y_0' \).

Using the six-terms exact sequence and Theorem \( \text{(3.3)} \), we can compute the \( K \)-theory of the ideal \( \mathcal{I}_1 \):

\[
\begin{align*}
K_0(\mathcal{I}_1) & \xrightarrow{\delta} K_0(C^*(\mathbb{F}_2)_{\overline{1}}) = \mathbb{Z} \\
\delta & \leftarrow K_1(C^*(\mathbb{F}_2)_{\overline{1}}) = \mathbb{Z} \\
K_1(C^*(\mathbb{F}_2)_{\overline{1}}) & \xrightarrow{\delta} K_0(C^*(\mathbb{F}_2)_{\overline{1}}) = \mathbb{Z} \quad \beta
\end{align*}
\]

where \( i \) and \( q \) are again the canonical injection and surjection. Each subsequent argument follows from the exactness of the six-terms sequence. Since \( K_0(C^*(\mathbb{F}_2)_{\overline{1}}) \) is generated by the class of the unit in \( C^*(\mathbb{F}_2)_{\overline{1}} \) and \( q(1) = 1 \) generated \( K_0(C^*(\mathbb{F}_2)_{\overline{1}}) \), we conclude that \( K_0(q) \) is the identity, so \( K_0(i) = 0 \) and \( \beta = 0 \). Hence \( K_0(\mathcal{I}_1) = \delta(\mathbb{Z}) \) and \( \beta \) is injective. Since \( K_1(C^*(\mathbb{F}_2)_{\overline{1}}) = 0 \) and \( \beta = 0 \) we conclude that \( K_1(\mathcal{I}_1) = 0 \). Therefore \( K_1(q) = 0 \), so \( \delta \) is injective and we get \( K_0(\mathcal{I}_1) = \mathbb{Z} \).
Remark 3.5. There is an alternative calculation of the $K$-theory of the ideal $\mathcal{I}$ using the simple six-term exact sequence:

$$
\begin{align*}
K_0(\mathcal{I}) &= 0 \xrightarrow{K_0(i)} K_0(C^*(\mathbb{F}_2)) = \mathbb{Z} \\
\delta &= 0 \uparrow \\
K_1(C(\mathbb{T})) &= \mathbb{Z} \xleftarrow{K_1(q)} K_1(C^*(\mathbb{F}_2)) = \mathbb{Z}^2 \\
\beta &= 0 \\
K_1(\mathcal{I}) &= \mathbb{Z}
\end{align*}
$$

corresponding to the defining exact sequence $\mathcal{I} \hookrightarrow C^*(\mathbb{F}_2) \xrightarrow{q} C(\mathbb{T})$ with $i$ the canonical injection and $q$ the canonical surjection. Now, $K_0(C^*(\mathbb{F}_2))$ is generated by 1, and as $q(1) = 1$ we see that $K_0(q)$ is the identity. Hence $\delta = 0$ and $K_0(i) = 0$ so $K_0(\mathcal{I}) = \delta(\mathbb{Z})$. On the other hand, $K_1(C^*(\mathbb{F}_2))$ is generated by $U$ and $V$, respectively identified with $(1,0)$ and $(0,1)$ in $\mathbb{Z}^2$. We have $q(U) = q(V) : z \mapsto z$ which is the generator of $K_1(C(\mathbb{T}))$, so $K_1(q)$ is surjective and thus $\delta = 0$. Hence $K_0(\mathcal{I}) = 0$. On the other hand, ker $K_1(q)$ is the group generated by $(1,-1)$, the class of $UV^*$. Thus $K_1(i)$, which is an injection, is in fact a bijection from $K_1(\mathcal{I})$ onto its range ker $K_1(q)$ and thus $K_1(\mathcal{I}) = \mathbb{Z}$ generated by $UV^*$, as indeed $q(UV^* - 1) = 0$ and thus $UV^* - 1 \in \mathcal{I}$.

It is not too surprising that $K_1(\mathcal{I}) = 0$ since $K_1(\mathcal{I}) = \mathbb{Z}$ is generated by $UV^* - 1$ which is an element in $C^*(\mathbb{F}_2)_{-1}$ of class $(1,-1)$ in $K_1(C^*(\mathbb{F}_2))$, so it is not connected to any unitary in $C^*(\mathbb{F}_2)_1$. Of course, this is not a direct proof of this fact, as homotopies in $M_n(\mathcal{I}_1) + I_n$ ($n \in \mathbb{N}$) is a more restrictive notion than in $M_n(C^*(\mathbb{F}_2)_1)$ ($n \in \mathbb{N}$). But more remarkable is the fact that $K_0(\mathcal{I}_1)$ contains some nontrivial element. Of course, $\mathcal{I}_1$ is projectionless since $C^*(\mathbb{F}_2)$ is by $\mathbb{Z}$, so the projection generating $K_0(\mathcal{I}_1)$ is at least (and in fact, exactly in) $M_2(\mathcal{I}_1) + I_2$. We now turn to an explicit description of the generator of $K_0(\mathcal{I}_1)$ and we investigate why this projection is trivial in both $K_0(C^*(\mathbb{F}_2)_1)$ and $K_0(\mathcal{I})$ but not in $K_0(\mathcal{I}_1)$. By exactness of the six-terms exact sequence, this projection is exactly the obstruction to the nontriviality of $K_1(C^*(\mathbb{F}_2)_1)$.

Theorem 3.6. Let $\theta : C^*(\mathbb{F}_2) \rightarrow C(\mathbb{T})$ be the $\ast$-homomorphism defined by $\theta(U) = \theta(V) : z \in \mathbb{T} \mapsto z$. Let $\mathcal{I}_1 = \{a \in C^*(\mathbb{F}_2)_1 : \theta(a) = 0\}$. Let $Z = \frac{1}{2}(U + V) \in C^*(\mathbb{F}_2)_1$. Let $\beta$ be the projection in $M_2(\mathcal{I}_1) + I_2$ defined by:

$$
\beta = \left[ \begin{array}{cc}
Z^*Z & Z^* \left( \sqrt{1 - ZZ^*} \right) \\
\left( \sqrt{1 - ZZ^*} \right)^* Z & 1 - ZZ^*
\end{array} \right].
$$

Then the generator of $K_0(\mathcal{I}_1)$ is $[\beta]_{\mathcal{I}_1} - [p_2]_{\mathcal{I}_1}$, where $p_2 = \left[ \begin{array}{cc} 1 & 0 \\
0 & 0 \end{array} \right]$ and, for any $C^*$-algebra $A$, we denote by $[\varphi]_A$ the $K_0$-class of any projection.
For any \( n \in \mathbb{N} \), the projection \( \beta \) is homotopic to \( p_2 \) in \( M_2(\mathcal{I}) + I_2 \) and in \( M_2(C^*(\mathbb{F}_2)_1) \). Thus \([\beta]_I = 0\) in \( K_0(\mathcal{I})\) and \([\beta]_{C^*(\mathbb{F}_2)_1} = [p_2]_{C^*(\mathbb{F}_2)_1} = [1]_{C^*(\mathbb{F}_2)_1}\) in \( K_0(C^*(\mathbb{F}_2)_1)\).

A simple calculation shows that \( \beta \) is a projection. We organize the proof of Theorem 3.6 in several lemmas. We start with the two quick observations that \( \beta \) is homotopic to \( p_2 \) in \( M_2(C^*(\mathbb{F}_2)_1) \) and in \( M_2(\mathcal{I}) + I_2 \), and then we prove that \([\beta]_I \neq [p_2]_I\) in \( K_0(\mathcal{I})\).

**Lemma 3.7.** The projection \( \beta \) and the projection \( 1 - p_2 \) are homotopic in \( M_2(C^*(\mathbb{F}_2)_1)\). Thus \([\beta]_{C^*(\mathbb{F}_2)_1} = [1]_{C^*(\mathbb{F}_2)_1}\).

**Proof.** For all \( t \in [0,1] \) we set:

\[
\beta_t = \begin{bmatrix}
  t^2Z^*Z & tZ^*(\sqrt{1-tZZ^*}) & tZ^*(\sqrt{1-tZZ^*}) \\
  (\sqrt{1-tZZ^*})Z & 1 & tZZ^*
\end{bmatrix}.
\]

Then \((\beta_t)_{t \in [0,1]}\) is by construction an homotopy in \( M_2(C^*(\mathbb{F}_2)_1)\) between \( \beta_1 = \beta \) and \( \beta_0 = 1 - p_2 \). Trivially \( 1 - p_2 \) and \( p_2 \) are homotopic, and \([p_2]_{C^*(\mathbb{F}_2)_1} = [1]_{C^*(\mathbb{F}_2)_1}\), hence our result.

The important observation in the proof of Lemma 3.7 is that although \( \beta_1 \in M_2(C^*(\mathbb{F}_2)_1) \) for all \( t \in [0,1] \), we have

\[
\theta(tZ\sqrt{1-tZZ^*})(z) = tz\sqrt{1-t} \neq 0
\]

for \( t \in (0,1) \) and \( z \in \mathbb{T} \), so \( \beta_t \) does not belong to the ideals \( \mathcal{I} \) and \( \mathcal{I}_1 \).

Since \( K_0(\mathcal{I}) = 0 \) it is trivial that the class of \( \beta \) in \( K_0(\mathcal{I}) \) is null, but the exact reason why it is so is interesting as a way to contrast with the calculations of the class of \( \beta \) in \( K_0(\mathcal{I}_1)\).

**Lemma 3.8.** In \( M_2(\mathcal{I}) + I_2 \) the projection \( \beta \) is homotopic to \( p_2 \). Hence in \( K_0(\mathcal{I}) \) we verify that we have indeed \([\beta]_I = 0\).

**Proof.** The unitary equivalence in Lemma 3.3 does not carry to the unitization of the ideal \( \mathcal{I} \), but we can check that \( \beta \) is homotopic to \( p_2 \) in \( M_2(\mathcal{I}) + I_2 \). Set \( Z_t = \frac{1}{2}(tU + (1-t)V) \) and set:

\[
\beta_t = \begin{bmatrix}
  Z_t^*Z_t & Z_t^*(\sqrt{1-Z_tZ_t^*}) & Z_t^*(\sqrt{1-Z_tZ_t^*}) \\
  (\sqrt{1-Z_tZ_t^*})Z_t & 1 & Z_tZ_t^*
\end{bmatrix}
\]

for all \( t \in [0,1] \). As before, \( \beta_t \) is a projection for all \( t \in [0,1] \) since \( \|Z_t\| = 1 \) for all \( t \in [0,1] \). Now, \( \beta_0 = \beta_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) while \( \beta_1 = \beta \). Of course, \( t \in [0, \frac{1}{2}] \mapsto \beta_t \) is continuous. Moreover:

\[
(tU + (1-t)V)(tU + (1-t)V)^* = 1 + (t-t^2)(UV^* + VU^*)
\]

so \( \theta(1 - Z_tZ_t^*) = 0 \). Hence, \( \beta_t \in M_2(\mathcal{I}) + I_2 \) for all \( t \in [0,1] \). Hence \( \beta \) is homotopic to \( p_2 \) in \( M_2(\mathcal{I}) + I_2 \).
Unlike in the case of Lemma (3.7), the homotopy used in the proof of Lemma (3.9) is in $M_2(\mathcal{I})$, but it is not in $M_2(C^*(\mathbb{F}_2)_1)$ and hence not in $M_2(\mathcal{I}_1)$.

The crux of this matter is that $\beta$ is the obstruction to the existence of a nontrivial element in $K_1(C^*(\mathbb{F}_2)_1)$. In view of Lemmas (3.7) and (3.8), we wish to see a concrete reason why $\beta$ cannot have the same class as $p_2$ in $K_0(\mathcal{I}_1)$. We start with a useful calculation: since $\beta$ and $p_2$ are homotopic in $C^*(\mathbb{F}_2)_1$, they are unitarily equivalent as well, and we now explicit a unitary implementing this equivalence:

**Lemma 3.9.** Let $Y = \left[ \begin{array}{cc} Z^* & \sqrt{1-Z^*Z} \\ \sqrt{1-Z^*Z} & -Z \end{array} \right]$. Then $Y$ is a unitary in $M_2(C^*(\mathbb{F}_2)_1)$ such that $Yp_2Y^* = \beta$.

**Proof.** Observe that $Z^*(1-ZZ^*) = Z^* - Z^*ZZ^* = (1 - Z^*Z)Z^*$. Thus, for any $n \in \mathbb{N}$ we get by a trivial induction that $Z^*(1-ZZ^*)^n = (1 - Z^*Z)^nZ^*$. Hence, for any polynomial $p$ by linearity, we have $Z^*(p(1-ZZ^*)) = (p(1-Z^*Z))Z^*$. By Stone-Weierstrass, we deduce that $Z^*f(1-ZZ^*) = f(1-Z^*Z)Z^*$ for any continuous function $f$ on the spectrum of $1-ZZ^*$ and $1-Z^*Z$ which is the compact $[0,1]$, and in particular for the square root. Therefore:

$$Z^*\sqrt{1-ZZ^*} = \left(\sqrt{1-Z^*Z}\right)Z^*. \tag{3.2}$$

Now, we have:

$$YY^* = \left[ \begin{array}{cc} Z^* & \sqrt{1-Z^*Z} \\ \sqrt{1-Z^*Z} & -Z \end{array} \right] \left[ \begin{array}{cc} Z & \sqrt{1-ZZ^*} \\ \sqrt{1-ZZ^*} & -Z^* \end{array} \right] = \left[ \begin{array}{cc} Z^*Z + 1 - Z^*Z & 0 \\ 0 & 1 \end{array} \right]$$

using (3.2) since $Z^*\sqrt{1-ZZ^*} - (\sqrt{1-Z^*Z})Z^* = 0$. Similarly, we get $Y^*Y = 1_2$.

Now, we compute $Yp_2Y^*$:

$$\left[ \begin{array}{cc} Z^* & \sqrt{1-Z^*Z} \\ \sqrt{1-ZZ^*} & -Z \end{array} \right] \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right] \left[ \begin{array}{cc} Z & \sqrt{1-ZZ^*} \\ \sqrt{1-Z^*Z} & -Z^* \end{array} \right] = \left[ \begin{array}{cc} Z^* & \sqrt{1-ZZ^*} \\ \sqrt{1-ZZ^*} & -Z \end{array} \right] \left[ \begin{array}{cc} Z & \sqrt{1-ZZ^*} \\ \sqrt{1-Z^*Z} & -Z^* \end{array} \right] = \left[ \begin{array}{cc} Z^*Z & Z^*\sqrt{1-ZZ^*} \\ (\sqrt{1-ZZ^*})Z & 1-ZZ^* \end{array} \right] = \beta.$$

Last, we observe that $\sigma(Z) = Z$ by construction and thus $\sigma(Y) = Y$ as well: in other words, $Y \in M_2(C^*(\mathbb{F}_2)_1)$ (and we recover that $\beta$ is unitarily equivalent in $M_2(C^*(\mathbb{F}_2)_1)$ to $p_2$). □
Note that \( Z - \lambda I \not\in I \) for all \( \lambda \in \mathbb{C} \) and so \( Y \) does not belong to \( M_2(I) + 1_2 \). Indeed, the following lemma shows that \( \beta \) and \( p_2 \) do not have the same \( K \)-class in \( I \), precisely because the conjunction of the conditions of symmetry and being in the kernel of \( \theta \) make it impossible to deform one into the other, even though each condition alone does not create any obstruction.

**Lemma 3.10.** We have \([\beta]_I - [p_2]_I \neq 0 \) in \( K_0(I) \).

**Proof.** To prove Theorem (3.6), it remains to show that \([\beta]_I - [p_2]_I \) is a generator for \( K_0(I) \). Let \( \delta : K_1(C(T)) \to K_0(I) \) be the exponential map in the six-term exact sequence in \( K \)-theory induced by the exact sequence \( 0 \to I \to C^*(\mathbb{F}_2)_1 \to C(T) \to 0 \). Let us denote by \( z \) the canonical unitary \( z : \omega \in T \mapsto \omega \) in \( C(T) \). Let us also denote by \( \theta_2 \) the map induced by \( \theta \) on \( M_2(C^*(\mathbb{F}_2)) \). By [4, Proposition 9.2.3], if \( u \) is any unitary in \( M_2(C^*(\mathbb{F}_2)_1) \) such that \( \theta_2(u) = \begin{bmatrix} z & 0 \\ 0 & z^* \end{bmatrix} \), then

\[
\delta([z]_{C(T)}) = [up_2u^*]_I - [p_2]_I .
\]

In particular, \( \theta_2(Y) = \begin{bmatrix} z & 0 \\ 0 & z^* \end{bmatrix} \) so

\[
\delta([z]_{C(T)}) = [Y_2Y^*]_I - [p_2]_I = [\beta]_I - [p_2]_I.
\]

On the other hand, by Theorem (3.1), \( \delta \) is an isomorphism of group. Since \([z]_{C(T)} \) is a generator of \( K_1(C(T)) \) we conclude that \([\beta]_I - [p_2]_I \) is a generator of \( K_0(I) \). \( \blacksquare \)

We thus have proven Theorem (3.6) by identifying \([\beta]_I - [p_2]_I \) as the generator of \( K_0(I) \) and verifying that without the conjoint conditions of symmetry via \( \sigma \) and \( \theta \), the difference of the classes of \( \beta \) and \( p_2 \) is null in both \( K_0(C^*(\mathbb{F}_2)_1) \) and in \( K_0(I) \).

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