EFFICIENT QUEUE-BASED CSMA WITH COLLISIONS

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Recently there has been considerable interest in the design of efficient carrier sense multiple access (CSMA) protocol for wireless network starting works by [10] [11] [8] [12] [7]. The basic assumption underlying these results is availability of perfect carrier sense information. This allows for design of continuous time algorithm under which collisions are avoided.

The primary purpose of this note is to show how these results can be extended in the case when carrier sense information may not be perfect, or equivalently delayed. Specifically, an adaptation of algorithm in [11] [12] is presented here for time slotted setup with carrier sense information available only at the end of the time slot. To establish its throughput optimality, in addition to method developed in [11] [12], understanding properties of stationary distribution of a certain non-reversible Markov chain as well as bound on its mixing time is essential. This note presents these key results.

A longer version of this note will provide detailed account of how this gets incorporated with methods of [11] [12] to provide positive recurrence of underlying network Markov process. In addition, these results will help design optimal rate control in conjunction with CSMA in presence of collision building upon method of [7].

1. Setup. We consider a single-hop wireless network of n queues. Queues receive work as per exogeneous arrivals and work leaves the system upon receiving service. Time is slotted and indexed by $\tau \in \{0, 1, \ldots\}$. Arrival process is assumed to be discrete time and brings unit sized packets. Let $Q_i(\tau) \in \mathbb{N}$ be number of packets waiting at the $i$th queue in the beginning of time slot $\tau$. Let $A_i(\tau)$ be the total number of packets arrived to queue $i$ till the end of time slot $\tau$. For convenience, we shall assume that in a given time slot, arrivals happen at the end of the time slot. Also assume $A_i(\cdot)$ is a Bernoulli i.i.d. process with rate $\lambda_i$, i.e. $\lambda_i = \Pr(A_i(\tau) - A_i(\tau - 1) = 1)

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and $A_i(\tau) - A_i(\tau - 1) \in \{0, 1\}$ for all $i, \tau \geq 1$. Let $Q(\tau) = [Q_i(\tau)]_{1 \leq i \leq n}$ and initially $\tau = 0$, $Q(0) = 0$

The work from queues is served at the unit rate, but subject to interference constraints. Specifically, let $G = (V, E)$ denote the interference graph between the $n$ queues, represented by vertices $V = \{1, \ldots, n\}$ and edges $E$: an $(i, j) \in E$ implies that queues $i$ and $j$ can not transmit simultaneously since their transmission interfere with each other. Formally, let $\sigma_i(\tau) \in \{0, 1\}$ denotes whether the queue $i$ is transmitting at time $\tau$, i.e. work in queue $i$ is being served at unit rate at time $\tau$ and $\sigma(\tau) = [\sigma_i(\tau)]$. Then, it must be that for $\tau \in \mathbb{N}$,

$$\sigma(\tau) \in \mathcal{I}(G) \triangleq \{ \rho = [\rho_i] \in \{0, 1\}^n : \rho_i + \rho_j \leq 1 \text{ for all } (i, j) \in E \}.$$ 

We shall assume that if a non-empty queue $i$ is served in time slot $\tau$, i.e. $Q_i(\tau) \geq 1$ and $\sigma_i(\tau) = 1$ then a packet departs from it near the end of the time slot $\tau$, but before arrival happens. In summary, queueing dynamics: for any $\tau \geq 0$ and $1 \leq i \leq n$,

$$Q_i(\tau + 1) = Q_i(\tau) - \sigma_i(\tau) I_{Q_i(\tau) > 0} + A_i(\tau).$$

1.1. Scheduling constraints. The scheduling algorithm decides the schedule $\sigma(\tau) \in \mathcal{I}(G)$ in the beginning of each time slot, possibly using $Q(\tau)$ and past history. This decision is made in a distributed manner by nodes. Specifically, in the beginning of each time slot, each node makes a decision to transmit or not. At the end of the time slot, node knows the following:

$\circ$ if it attempted to transmit, whether its attempt was successful;

$\circ$ if it did not attempt to transmit, whether any of its neighbor attempting to transmit was successful.

In summary, each node has delayed carrier sense information that is available at the end of the time slot.

1.2. Capacity region. From the perspective of network performance, we would like the scheduling algorithm to be such that the queues in network remain as small as possible for the largest possible range of arrival rate vectors. To formalize this notion of performance, we define the capacity region. Let $\Lambda$ be the capacity region defined as

$$\Lambda = \text{Conv}(\mathcal{I}(G))$$

$$\{ y \in \mathbb{R}_+^n : y \leq \sum_{\sigma \in \mathcal{I}(G)} \alpha_\sigma \sigma, \text{ with } \alpha_\sigma \geq 0, \text{ and } \sum_{\sigma \in \mathcal{I}(G)} \alpha_\sigma \leq 1 \}.$$ 

$^1$Bold letters are reserved for vectors; $0$, $1$ represent vectors of all $0$s & all $1$s respectively.
Definition 1 (throughput optimal) A scheduling algorithm is called throughput optimal, or stable, or providing 100% throughput, if for any \( \lambda \in \Lambda^o \) the (appropriately defined) underlying network Markov process is positive (Harris) recurrent.

2. Our algorithm. We present a randomized algorithm that is direct adaptation of the algorithm in [11, 12] for the discrete time setting.

In the beginning of each time slot, say \( \tau \), each node (or queue) does the following. With probability \( 1/2 \), independent of everything else, it does nothing. Otherwise, it executes the following:

1. If \( \sigma_i(\tau - 1) = 1 \), that is its transmission at time \( \tau - 1 \) was successful, then it decides to transmit with probability \( 1 - \frac{1}{W_i(\tau)} \).
2. If at time \( \tau - 1 \), any of its neighbor’s transmission was successful, then does not attempt to transmit with probability 1.
3. Otherwise, it attempts transmission with probability 1.

Few remarks about the algorithm. In case 1, we choose

\[
W_i(\tau) = \exp \left( \max \left\{ f(Q_i(\tau)), \sqrt{f(Q_{\max}(\tau))} \right\} \right),
\]

where \( f : \mathbb{R}_+ \rightarrow [0, \infty) \) is a strictly increasing function with \( f(0) = 0 \), \( \lim_{x \to \infty} f(x) = \infty \) and satisfies the property

\[
\lim_{x \to \infty} \exp(f(x)) \cdot f' \left( f^{-1}(\delta f(x)) \right) = 0, \quad \text{for any } \delta \in (0, 1).
\]

For example, any strictly increasing function with \( f(0) = 0 \) and \( f(x) = o(\log x) \) will have this property, e.g. \( f(x) = \sqrt{\log(x + 1)} \), \( \log \log(x + e) \), etc.

In above \( Q_{\max}(\cdot) = \max_i Q_i(\cdot) \), that is the maximum of all queue sizes. Of course, knowing this instantly is not possible. However, knowledge of \( Q_{\max}(\cdot) = O(1) \) suffices and a simple scheme to achieve this is presented in [11]. Of course, authors strongly believe that explicit information exchange for knowing such an estimate is needed.

Finally, it is assumed that if a node tries to attempt as part of the above algorithm, then it must send some data irrespective of the value of \( Q_i(t) \).

3. Properties of algorithm. To establish throughput optimality of the algorithm described above building upon method of [11, 12] will require us to understand property of the stationary distribution of a certain Markov chain of the space of independent sets \( \mathcal{I}(G) \) as well as its mixing time. We
study these two properties here. Relation of this Markov chain to algorithm of Section 2 is explained.

As mentioned earlier, a longer version of this note will provide detailed proof of throughput optimality using these properties.

3.1. A Markov chain & its mixing time. Consider a graph $G = (V, E)$ of $n = |V|$ nodes with node weights $W = [W_i] \in \mathbb{R}_{\geq 1}^n$ where $\mathbb{R}_{\geq 1} = \{x \in \mathbb{R} : x \geq 1\}$. We consider Markov chain on the space of independent sets of $G$, $\mathcal{I}(G)$ based on $W$ with certain qualitative properties. In what follows we define what are feasible transitions as part of the chain and provide properties of the corresponding transition probabilities. This may not lead to an exact definition of the Markov chain, i.e. a class of Markov chains can satisfy these properties. However, as we shall show that all Markov chains with these properties have desired properties in terms of stationary distribution and their mixing times.

Now we describe what sorts of transitions are allowed and properties of the corresponding transition probabilities. Suppose the Markov chain is currently in the state $\sigma \in \mathcal{I}(G)$. With abuse of notation, let $\sigma$ denote the subset of $V$ that $\{i \in V : \sigma_i = 1\}$. Then, under the Markov chain of interest, transition from $\sigma$ to $\sigma'$ is allowed if and only if $\sigma' = \sigma \cup S_2 \setminus S_1$ where $S_1 \subset \sigma$ and $S_2 \subset V$ such that $\sigma \cup S_2 \in \mathcal{I}(G)$. The probability of this transition, say $P_{\sigma\sigma'}$ is such that

$$P_{\sigma\sigma'} \propto \left( \prod_{i \in S_1} \frac{1}{W_i} \right) p(S_2),$$

where $2^{-n} \leq p(S_2) \leq 1$. Let $P = [P_{\sigma\sigma'}] \in [0, 1]^{|\mathcal{I}(G) \times \mathcal{I}(G)|}$ denote the transition probability matrix.

Under this Markov chain, there is strictly positive probability to reach empty set, $0$, from any other state $\sigma \in \mathcal{I}(G)$ and vice versa; empty set has a self loop. Therefore, the Markov chain is irreducible, aperiodic. It is finite state and hence it has unique stationary distribution, say $\pi$. We claim the following two properties of the Markov chain $P$: first is about $\pi$ and the second is about its mixing time.

**Lemma 1** For any $W \in \mathbb{R}_{\geq 1}^n$,

$$\mathbb{E}_\pi \left[ \sum_i \sigma_i \log W_i \right] \geq \left( \max_{\rho \in \mathcal{I}(G)} \sum_i \rho_i \log W_i \right) - O(n^2).$$

**Proof.** To start with, it is clear that the stationary distribution $\pi$ of the Markov chain $P$ has $\mathcal{I}(G)$ as its support. That is, $\pi = [\pi_\sigma]_{\sigma \in \mathcal{I}(G)}$ with
\( \pi_\sigma > 0 \) for all \( \sigma \in \mathcal{I}(G) \). Therefore, we can write
\[
\pi_\sigma \propto \exp \left( U(\sigma) \right),
\]
for some \( U : \mathcal{I}(G) \to \mathbb{R}_+ \) where \( \mathbb{R}_+ = \{ x \in \mathbb{R} : x \geq 0 \} \). We will show that for all \( \sigma \in \mathcal{I}(G) \)
\[
\left| U(\sigma) - \sum_i \sigma_i \log W_i \right| = O\left( n2^n \right).
\]
Assuming (4), we shall conclude the result of Lemma 1. For this, we wish to utilize the following proposition that is a direct adaptation of the known results in literature (cf. [5] or see [12]).

**Proposition 2** Let \( T : \Omega \to \mathbb{R} \) and let \( \mathcal{M}(\Omega) \) be space of all distributions on \( \Omega \). Define \( F : \mathcal{M}(\Omega) \to \mathbb{R} \) as
\[
F(\mu) = E_{\mu}(T(x)) + H_{ER}(\mu),
\]
where \( H_{ER}(\mu) \) is the standard discrete entropy of \( \mu \). Then, \( F \) is uniquely maximized by the distribution \( \nu \), where
\[
\nu_x \propto \exp \left( T(x) \right), \quad \text{for any} \quad x \in \Omega.
\]
Further, with respect to \( \nu \), we have
\[
E_{\nu}[T(x)] \geq \left[ \max_{x \in \mathcal{X}} T(x) \right] - \log |\Omega|.
\]
Now by applying Proposition 2 with \( \nu \) replaced by \( \pi \), \( \Omega \) replaced \( \mathcal{I}(G) \) and \( T \) replaced by \( F \), we have that
\[
E_{\pi} \left[ F(\sigma) \right] \geq \left[ \max_{\rho \in \mathcal{I}(G)} F(\sigma) \right] - \log |\mathcal{I}(G)|
\]
\[
\geq \left[ \max_{\rho \in \mathcal{I}(G)} F(\sigma) \right] - n,
\]
since \( |\mathcal{I}(G)| \leq 2^n \). Using (4) and (5), it follows that
\[
E_{\pi} \left[ \sum_i \sigma_i \log W_i \right] \geq \left[ \max_{\rho \in \mathcal{I}(G)} \sum_i \rho_i \log W_i \right] - O\left( n2^n \right).
\]
To complete the proof of Lemma 1 we shall establish the remaining claim (4). To this end, consider a different Markov chain on \( \mathcal{I}(G) \) with transition...
probability matrix \( Q = [Q_{\sigma\sigma'}] \) such that \( Q_{\sigma\sigma'} > 0 \) if and only if \( P_{\sigma\sigma'} > 0 \). Now if \( P_{\sigma\sigma'} > 0 \), then it must be that there are \( S_1 \subset \sigma \), \( S_2 \subset V \) so that \( \sigma \cup S_2 \in \mathcal{I}(G) \) and in this case, we define \( Q_{\sigma\sigma'} \) as

\[
Q_{\sigma\sigma'} \propto \frac{1}{2^n} \prod_{i \in S_1} \frac{1}{W_i}.
\]

Thus, we have that for all \( \sigma, \sigma' \in \mathcal{I}(G) \) with \( P_{\sigma\sigma'}, Q_{\sigma\sigma'} > 0 \),

\[
2^{-n} \leq \frac{P_{\sigma\sigma'}}{Q_{\sigma\sigma'}} \leq 1.
\]

It can be checked that \( Q \), like \( P \), is irreducible and aperiodic Markov chain on \( \mathcal{I}(G) \). Let \( \hat{\pi} \) be the unique stationary of \( Q \) on \( \mathcal{I}(G) \). We claim that

\[
\hat{\pi}_{\sigma} \propto \prod_{i : \sigma_i = 1} W_i = \exp \left( \sum_{i} \sigma_i \log W_i \right).
\]

To establish this, note that if transition from \( \sigma \) to \( \sigma' \) is feasible under \( Q \) (equivalently under \( P \)) then so is from \( \sigma' \) to \( \sigma \). Specifically, let \( \sigma = S_0 \cup S_1 \) and \( \sigma' = S_0 \cup S_2 \), where \( S_0, S_1, S_2 \) are disjoint sets and \( S_0 \cup S_1 \cup S_2 \) is an independent set of \( G \). Then,

\[
\hat{\pi}_\sigma Q_{\sigma\sigma'} = \left( \prod_{i : \sigma_i = 1} W_i \right) \times \left( \prod_{k \in S_1} \frac{1}{W_k} \right) \times 2^{-n}
\]

\[
= \left( \prod_{i \in S_0 \cup S_1} W_i \right) \times \left( \prod_{k \in S_1} \frac{1}{W_k} \right)
\]

\[
= \left( \prod_{i \in S_1} W_i \right) \times 2^{-n}
\]

\[
= \left( \prod_{i \in S_0 \cup S_2} W_i \right) \times \left( \prod_{k \in S_2} \frac{1}{W_k} \right) \times 2^{-n}
\]

\[
= \hat{\pi}_{\sigma'} Q_{\sigma' \sigma}.
\]

The (10) establishes that \( Q \) is reversible and satisfies detailed balance equation with \( \hat{\pi} \) as its stationary distribution. This establishes (7).

Given (7), to establish (4) as desired, it is sufficient to show that for any \( \sigma \in \mathcal{I}(G) \),

\[
2^{-n^2} \leq \frac{\pi_{\sigma}}{\hat{\pi}_{\sigma}} \leq 2^{n^2}.
\]
To establish this, we shall use the characterization of stationary distributions for any irreducible, aperiodic finite state Markov chain given through what is known as the ‘Markov chain tree theorem’ (cf. see [1]). To this end, define a directed graph \( \mathcal{G} = (I(G), \mathcal{E}) \) with \( I(G) \) as vertices and directed edge \( (\sigma, \sigma') \in \mathcal{E} \) if and only if \( P_{\sigma, \sigma'} > 0 \) (equivalently \( Q_{\sigma, \sigma'} > 0 \)). Let \( \mathcal{T}_\sigma \) be the space of all directed spanning trees of \( \mathcal{G} \) rooted at \( \sigma \in I(G) \). Define weight of a tree \( T \in \mathcal{T}_\sigma \) with respect to transition matrix \( P \), denoted as \( w(T, P) \), as

\[
w(T, P) = \prod_{(\rho, \rho') \in T} P_{\rho, \rho'}.
\]

Similarly, define weight of \( T \in \mathcal{T}_\sigma \) with respect to \( Q \), denoted as \( w(T, Q) \), as

\[
w(T, Q) = \prod_{(\rho, \rho') \in T} Q_{\rho, \rho'}.
\]

Then, the Markov Tree Theorem states that for any \( \sigma \in I(G) \),

\[
\pi_\sigma \propto \sum_{T \in \mathcal{T}_\sigma} w(T, P).
\]

(12)

And, similarly for \( \sigma \in I(G) \),

\[
\hat{\pi}_\sigma \propto \sum_{T \in \mathcal{T}_\sigma} w(T, Q).
\]

(13)

Since the number of edges in each spanning tree is no more than \( |I(G)| \leq 2^n \), by (8), (12) and (13), it follows that for all \( \sigma \in I(G) \)

\[
2^{-n^2} \leq \frac{\pi_\sigma}{\hat{\pi}_\sigma} \leq 2^n 2^{2n}.
\]

(14)

This completes the proof of [11] and subsequently that of Lemma [11] .

□

Now we will obtain a mixing rate (or time) of the non-reversible Markov chain \( P \). To this end, we present a bound of the matrix norm of \( P^* \) since it crucially determines the mixing rate of \( P \) (cf. [9]). Here, \( P^* \) is the adjoint matrix of \( P \) and the matrix norm \( \|P^*\| \) is defined as

\[
\|P^*\| = \sup_{v \in \mathcal{E}, ||v|| = 0} \frac{\|P^*v\|_2}{||v||_2},
\]

where \( ||u||_2 = \sqrt{\sum_{\sigma \in I(G)} \pi_\sigma (u_\sigma)^2} \) for any \( u \in \mathbb{R}^{I(G)} \).
Lemma 3 Given $P$ described above, let $P^*$ be its adjoint. Then,

$$\|P^*\| \leq 1 - \frac{1}{2^{4n(2n+2)+2}(W_{\text{max}})^{4n}}. \tag{15}$$

Proof. We shall use Cheeger’s inequality to bound spectral gap for reversible Markov chain defined by $PP^*$ and then use it to bound $\|P^*\|$ using standard result (cf. [9]).

To that end, let $\lambda_2$ and $\lambda_{|I(G)|}$ be the second-largest and smallest eigenvalues of $PP^*$, respectively. It is known [9] that

$$\|P^*\| = \sqrt{\max \{ |\lambda_2|, |\lambda_{|I(G)|}| \}}
= \max \left\{ \sqrt{1 - (1 - |\lambda_2|)}, \sqrt{1 - (1 - |\lambda_{|I(G)|}|)} \right\}
\leq \max \left\{ 1 - \frac{1 - |\lambda_2|}{2}, 1 - \frac{1 - |\lambda_{|I(G)|}|}{2} \right\}. \tag{16}$$

First observe that $PP^* \geq \frac{1}{2n}I$ (component-wise) since $P, P^* \geq \frac{1}{2}I$. From this, it is easy to check that

$$\lambda_{|I(G)|} \geq 2 \times \frac{1}{2n} - 1. \tag{17}$$

Therefore, it suffices to obtain the bound of $\lambda_2$ for the desired bound of $\|P^*\|$ in Lemma 3. In general, in the absence of such bound one can use ‘lazy’ version of the Markov chain, i.e. add self loop to all states with probability $1/2$, to make all eigenvalues non-negative and hence need to bound $\lambda_2$ only.

Next, we will use the Cheeger’s inequality [2, 4, 6, 3, 13], it is well known that

$$\lambda_2 \leq 1 - \frac{\Phi^2}{2}.$$  

Here, $\Phi$ is the conductance of $R := PP^*$, defined as

$$\Phi = \min_{S \subset I(G)} \frac{Q(S, S^c)}{\min\{\pi(S), \pi(S^c)\}},$$

where $S^c = I(G) \setminus S$, $Q(S, S^c) = \sum_{\sigma \in S, \sigma' \in S^c} \pi(\sigma)R(\sigma, \sigma')$. Now we will consider the following naive bounds for $\pi$ and $R$ to derive the desired bound

$^2PP^*$ is reversible, hence all eigenvalues are real and in the interval $[-1, 1]$. 
of $\Phi$ and $\lambda_2$.

$$\min_{\sigma \in \mathcal{I}(G)} \pi_{\sigma} \geq \frac{1}{2^n 2^n} \pi_{\sigma} \geq \frac{1}{2^n 2^n} \times \frac{1}{2^n(W_{\text{max}})^n} = \frac{1}{2^n(2^n+1)(W_{\text{max}})^n};$$

(18)

where (a) and (b) follows from (14) and (9), respectively. In addition,

$$\min_{R(\sigma, \sigma') \neq 0} R(\sigma, \sigma') \geq \min_{P(\sigma, \sigma') \neq 0} P(\sigma, \sigma') \times \min_{P(\sigma, \sigma') \neq 0} P^*(\sigma, \sigma') \geq \min_{P(\sigma, \sigma') \neq 0} P(\sigma, \sigma') \times \left( \min_{\sigma \in \mathcal{I}(G)} \pi_{\sigma} \times \min_{P(\sigma, \sigma') \neq 0} P(\sigma, \sigma') \right) \geq \frac{1}{2^n} \times \left( \frac{1}{2^n(2^n+1)(W_{\text{max}})^n} \times \frac{1}{2^n} \right) = \frac{1}{2^n(2^n+3)(W_{\text{max}})^n};$$

(19)

where (a) and (b) follows from (14) and (9), respectively. In addition,

$$\Phi \geq \min_{S \subseteq \mathcal{I}(G)} Q(S, S^c) \geq \min_{R(\sigma, \sigma') \neq 0} \pi_{\sigma} R(\sigma, \sigma') \geq \min_{\sigma \in \mathcal{I}(G)} \pi_{\sigma} \times \min_{R(\sigma, \sigma') \neq 0} R(\sigma, \sigma') \geq \frac{1}{2^n(2^n+2)(W_{\text{max}})^n};$$

(20)

Therefore, from the Cheeger’s inequality and (20), (21)

$$1 - \lambda_2 \geq \Phi^2 / 2 \geq \frac{1}{2^{4n(2^n+2)+1}(W_{\text{max}})^4n}.$$

The desired bound of $\|P^*\|$ follows from (16), (17), (21) and the property $W_{\text{max}} \geq 1$. This completes the proof of Lemma 3. □

3.2. Relation to Algorithm. Here is a quick explanation of why Markov chain $P$ described in Section 3.1 arises naturally as part of the algorithm described in Section 2. To that end, the weight vector $W = W(\tau)$ is time
varying and function of $Q(\tau)$ as per \cite{2}. And, transition of the set of successfully transmitting nodes $\sigma(\tau - 1)$ at time slot $\tau - 1$ to the set of successfully transmitting nodes $\sigma(\tau)$ at time $\tau$ is as per the transition matrix $P = P(\tau)$, where $P = P(\tau)$ has properties described above with weight $W = W(\tau)$.

To see this, consider the $\sigma(\tau - 1)$. Then, a subset $S_1 \subset \sigma(\tau - 1)$ can decide to stop transmitting at time $\tau$ and these decisions are taken with probability proportional to $\prod_{i \in S_1} \frac{1}{W_i(\tau)}$. Clearly, no nodes in neighborhood of $\sigma(\tau - 1)$ will attempt transmission as per the algorithm. Therefore, new nodes attempting transmission must be such that they are not neighbors of any of the nodes in $\sigma(\tau - 1)$. For any subset $S_2$ such that $\sigma(\tau - 1) \cup S_2 \in \mathcal{I}(G)$, it is possible to have $\sigma(\tau)$ include $S_2$. This is because, nodes in $S_2$ attempt transmission and all of their neighbors (which by definition are not part of $\sigma(\tau - 1)$) do not attempt transmission – this happens with probability proportional to $2^{-|S_2| - |\Gamma(S_2)|}$, where $\Gamma(S_2)$ are neighbors of $S_2$. Indeed, for $\sigma(\tau)$ to transit exactly to $\sigma(\tau - 1) \cup S_2 \setminus S_1$, the overall probability can be argued in a similar manner to be proportional to the following:

$$\left( \prod_{i \in S_1} \frac{1}{W_i(\tau)} \right) \times p'(S_2),$$

$2^{-n} \leq p'(S_2) \leq 1$. This completes the explanation of relation between the algorithm and the Markov chain described in Section 3.1.

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