Perturbative and Non-Perturbative Results for $N = 2$ Heterotic Strings

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In this paper we summarize our recent work about perturbative and non-perturbative effects in four-dimensional heterotic strings with $N = 2$ space-time supersymmetry.

1. Introduction

During the last year some major progress has been obtained in the understanding of the non-perturbative dynamics of $N = 2$ Yang-Mills theories as well as $N = 2$ superstrings in four dimensions. In this paper we report on some work concerning the computation of perturbative couplings for $N = 2$ heterotic string theories \[2\] as well as concerning the non-perturbative monodromies \[2\]. We will show that in the rigid limit the perturbative and also the non-perturbative monodromies lead to the monodromies discussed by Seiberg and Witten in \[3\]. Related perturbative results were independently obtained in \[4\]. Our non-perturbative monodromies should be compared \[6\] with the results \[7\] obtained from the string-string duality between the heterotic and type II strings with $N = 2$ space-time supersymmetry.

2. Classical results

In this section we collect some results about $N = 2$ heterotic strings and the related classical prepotential; we will in particular work out the relation between the enhanced gauge symmetries, the duality symmetries and Weyl transformations. We will consider four-dimensional heterotic vacua which are based on compactifications of six-dimensional vacua on a two-torus $T_2$. The moduli of $T_2$ are commonly denoted by $T$ and $U$ where $U$ describes the deformations of the complex structure, $U = (\sqrt{G} - iG_{12})/G_{11}$ ($G_{ij}$ is the metric of $T_2$), while $T$ parametrizes the deformations of the area and of the antisymmetric tensor, $T = 2(\sqrt{G} + iB)$. (Possibly other existing vector fields will not play any role in our discussion.) The scalar fields $T$ and $U$ are the spin-zero components of two $U(1)$ $N = 2$ vector supermultiplets. All physical properties of the two-torus compactifications are invariant under the group $SO(2,2,\mathbb{Z})$ of discrete target space duality transformations. It contains the $T \leftrightarrow U$ exchange, with group element denoted by $\sigma$ and the $PSL(2,\mathbb{Z})_T \times PSL(2,\mathbb{Z})_U$ dualities, which act on $T$ and $U$ as

\begin{align}
(T, U) \rightarrow \left(\frac{aT - ib}{icT + d}, \frac{aU - ib'}{icU + d'}\right),
\end{align}

The classical monodromy group, which is a true symmetry of the classical effective Lagrangian, is generated by the elements $\sigma$, $g_1$, $g_2$: $T \rightarrow 1/T$ and $g_2$: $T \rightarrow 1/(T - i)$. The transformation $t$: $T \rightarrow T + i$, which is of infinite order, corresponds to $t = g_2^{-1}$. Whereas $PSL(2,\mathbb{Z})_T$ is generated by $g_1$ and $g_2$, the corresponding elements in $PSL(2,\mathbb{Z})_U$ are obtained by conjugation with $\sigma$, i.e. $g_1' = \sigma^{-1} g_1 \sigma$.

The $N = 2$ heterotic string vacua contain two further $U(1)$ vector fields, namely the graviphoton field, which has no physical scalar partner, and the dilaton-axion field, denoted by $S$. Thus the full Abelian gauge symmetry we consider is given by $U(1)^2_L \times U(1)^2_R$. At special lines in the $(T, U)$ moduli space, additional vector fields become massless and the $U(1)^2_L$ becomes enlarged to a non-Abelian gauge symmetry. Specifically,

\begin{itemize}
\item \textsuperscript{*}Talk presented by D. Lüst
\end{itemize}
there are four inequivalent lines in the moduli space where two charged gauge bosons become massless: $U = T, \ U = 1/T, \ U = T - i, \ U = T + i$. The quantum numbers of the states that become massless can be easily read of from the holomorphic mass formula \[ M = m_2 - \text{int} U + \text{int} T - n_2 T U, \] (2)

where $n_i, m_i$ are the winding and momentum quantum numbers associated with the $i$-th direction of the target space $T_2$. At each of the four critical lines the $U(1)_L^2$ is extended to $SU(2)_L \times U(1)_L$. Moreover, these lines intersect one another in two inequivalent critical points (for a detailed discussion see \[ \Box \]). At $(T, U) = (1, 1)$ the first two lines intersect. The four extra massless states extend the gauge group to $SU(2)_L \times SU(2)_R$. $SU(2)_R$ are the Weyl reflections of the enhanced \[ SU(2) \] groups \[ SU(2) \] are the Weyl reflections of $SU(2) \times SU(2)$. At $(T, U) = (1, 1), (\rho, \bar{\rho})$ respectively, have the following action \[ \delta T = \rho e^{i \pi / 6} \] \[ \delta U = \bar{\rho} \]

The Weyl groups of the enhanced gauge groups $SU(2)^2$ and $SU(3)$, realized at $(T, U) = (1, 1), (\rho, \bar{\rho})$ respectively, have the following action on $T$ and $U$ \[ \Box \]

| Weyl Reflections | $T \rightarrow T'$ | $U \rightarrow U'$ |
|------------------|------------------|------------------|
| $w_1$            | $T \rightarrow U$ | $U \rightarrow T$ |
| $w_2$            | $T \rightarrow \frac{1}{T}$ | $U \rightarrow \frac{1}{U}$ |
| $w_1'$           | $T \rightarrow \frac{1}{T}$ | $U \rightarrow \frac{1}{T}$ |
| $w_2'$           | $T \rightarrow U + i$ | $U \rightarrow T - i$ |
| $w_0'$           | $T \rightarrow \frac{U}{T}$ | $U \rightarrow \frac{T}{U}$ |

$w_1, w_2$ are the Weyl reflections of $SU(2)_{(1)} \times SU(2)_{(2)}$, whereas $w_1'$ and $w_2'$ are the fundamental Weyl reflections of the enhanced $SU(3)$. For later reference we have also listed the $SU(3)$ Weyl reflection $w_0' = w_2^{-1} w_1 w_2'$ at the hyperplane perpendicular to the highest root of $SU(3)$. Note that $w_2 = w_1'$. All these Weyl transformations are target space modular transformations and therefore elements of the monodromy group. All Weyl reflections can be expressed in terms of the generators $g_1, g_2, \sigma$ and, moreover, all Weyl reflections are conjugated to the mirror symmetry $\sigma$ by some group element \[ \Box \]

\[ w_1 = \sigma, \quad w_2 = g_1 \sigma g_1 = g_1^{-1} \sigma g_1 \] \[ w_2' = \sigma \sigma^{-1} = (g_1^{-1} g_2)^{-1} \sigma (g_1^{-1} g_2), \]

\[ w_0' = w_1^{-1} w_1' w_1'. \] \[ (5) \]

As already mentioned the four critical lines are fixed under the corresponding Weyl transformation. Thus it immediately follows that the numbers of additional massless states agrees with the order of the fixed point transformation at the critical line, points respectively \[ \Box \].

Let us now express the moduli fields $T$ and $U$ in terms of the field theory Higgs fields whose non-vanishing vacuum expectation values spontaneously break the enlarged gauge symmetries $SU(2)^2, SU(3)$ down to $U(1)^2$. First, the Higgs field of $SU(2)_{(1)}$ is given by $a_1 \propto (T - U)$. Taking the rigid field theory limit $\kappa = \frac{8 \pi}{M_{\text{Planck}}^2} \rightarrow 0$ we will expand $T = T_0 + \kappa \delta T, \ U = T_0 + \kappa \delta U$. Then, at the linearized level, the $SU(2)_{(1)}$ Higgs field is given as $a_1 \propto (\delta T - \delta U)$. Analogously, for the enhanced $SU(2)_{(2)}$ the Higgs field is $a_2 \propto (T - 1/U)$. Again, we expand as $T = T_0 + \kappa \delta T$, $U = \frac{T}{T_0}(1 + \delta U)$ which leads to $a_2 \propto \delta T + \delta U$. Finally, for the enhanced $SU(3)$ we obtain as Higgs fields $a_1 \propto \delta T + \delta U, \ a_2 \propto \rho^2 \delta T + \rho^{-2} \delta U$, where we have expanded as $T = \rho + \delta T, \ U = \rho^{-1} + \delta U$ (see section \[ \Box \] for details).

The classical vector couplings are determined by the holomorphic prepotential which is a homogeneous function of degree two of the fields $X^I \ (I = 1, \ldots, 3)$. It is given by \[ \Box \]

\[ F = i \frac{X^1 X^2 X^3}{X^0} = -STU, \] (6)

where the physical vector fields are defined as $S = \frac{X^1}{X^0}, \ T = -\frac{X^2}{X^0}, \ U = -\frac{X^3}{X^0}$ and the graviphoton corresponds to $X^0$. As explained in \[ \Box \], the period vector $(X^I, i F_I)$ $(F_I = \frac{\delta F}{\delta X^I})$, that follows from the prepotential \[ \Box \], does not lead to classical gauge couplings which all become small in the limit of large $S$. Specifically, the gauge
couplings which involve the $U(1)_S$ gauge group are constant or even grow in the string weak coupling limit $S \to \infty$. In order to choose a ‘physical’ period vector one has to replace $F^S_{\mu
u}$ by its dual which is weakly coupled in the large $S$ limit. This is achieved by the following symplectic transformation $(X^I, iF_I) \to (P^I, iQ_I)$ where $P^I = iF_I$, $Q_1 = iX^1$ and $P^i = X^i$, $Q_i = F_i$ for $i = 0, 2, 3$. In this new basis the classical period vector takes the form

$$\Omega^T = (1, TU, iT, iU, iSTU, iS, -SU, -ST),$$

(7)

where $X^0 = 1$. One sees that in this new basis all electric vector fields $P^I$ depend only on $T$ and $U$, whereas the magnetic fields $Q_I$ are all proportional to $S$.

The basis $\Omega$ is also well adapted to discuss the action of the target space duality transformations and, as particular elements of the target space duality group, of the four inequivalent Weyl reflections given in (3). These transformations are given by symplectic $Sp(8, \mathbb{Z})$ transformations, which act on the period vector $\Omega$ as

$$\left(\begin{array}{c} P^I \\ iQ_I \end{array}\right) \to \Gamma \left(\begin{array}{c} P^I \\ iQ_I \end{array}\right) = \left(\begin{array}{cc} U & Z \\ W & V \end{array}\right) \left(\begin{array}{c} P^I \\ iQ_I \end{array}\right),$$

(8)

where the $4 \times 4$ sub-matrices $U, V, W, Z$ have to satisfy the symplectic constraints $U^T V - W^T Z = V^T U - Z^T W = 1, U^T W = W^T U, Z^T V = V^T Z$. Invariance of the lagrangian implies that $W = Z = 0$, $VU^T = 1$. In case that $Z = 0$, $W \neq 0$ and hence $VU^T = 1$ the action is invariant up to shifts in the $\theta$-angles; this is just the situation one encounters at the one-loop level. The non-vanishing matrix $W$ corresponds to non-trivial one-loop monodromy due to logarithmic singularities in the prepotential. (This will be the subject of section 3.) Finally, if $Z \neq 0$ then the electric fields transform into magnetic fields; these transformations are the non-perturbative monodromies due to logarithmic singularities induced by monopoles, dyons or other non-perturbative excitations (see section 4).

The classical action is completely invariant under the target space duality transformations. Thus the classical monodromies have $W, Z = 0$. The matrices $U$ (and hence $V = U^T, -1 = U^* \text{ are given by}$

$$U_\sigma = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right),$$

$$U_{g_1} = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right),$$

$$U_{g_2} = \left(\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right).$$

(9)

At the classical level the $S$-field is invariant under these transformations. The corresponding symplectic matrices for the four inequivalent Weyl reflections then immediately follow from the previous equations (4) and (5).

3. Perturbative results

Let us first review the main results about the one-loop perturbative holomorphic prepotential which were derived in [6,7]. Using simple power counting arguments it is clear that the one-loop prepotential must be independent of the dilaton field $S$. The same kind of arguments actually imply that there are no higher loop corrections to the prepotential in perturbation theory. Thus the perturbative, i.e. one loop prepotential takes the form

$$F = F^{(\text{Tree})}(X) + F^{(1\text{-loop})}(X)$$

$$= i \frac{X^1 X^2 X^3}{X^0} + (X^0)^2 f(T, U)$$

$$= -STU + f(T, U).$$

(10)

Since the target space duality transformations are known to be a symmetry in each order of perturbation theory, the tree level plus one-loop effective action must be invariant under these transformations, where however one has to allow for
discrete shifts in the various $\theta$ angles due to monodromies around semi-classical singularities in the moduli space where massive string modes become massless. Instead of the classical transformation rules, in the quantum theory, $(P^I, iQ_I)$ transform according to

$$P^I \rightarrow U^I_J P^J, \quad iQ_I \rightarrow V_I^J iQ_J + W_IJ P^J$$

where

$$V = (U^T)^{-1}, \quad W = V \Lambda, \quad \Lambda = \Lambda^T \quad (12)$$

and $U$ belongs to $SO(2,2,\mathbb{Z})$. Classically, $\Lambda = 0$, but in the quantum theory, $\Lambda$ is a real symmetric matrix, which should be integer valued in some basis.

Besides the target space duality symmetries, the effective action is also invariant, up to discrete shifts in the $\theta$-angles, under discrete shifts in the $S$-field, $D$: $S \rightarrow S - i$. Thus the full perturbative monodromies contain the following $Sp(8,\mathbb{Z})$ transformation:

$$V_S = U_S = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad W_S = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad Z_S = 0 \quad (13)$$

Invariance of the one-loop action up to discrete $\theta$-shifts then implies that

$$F^{(1-loop)}(X) \rightarrow F^{(1-loop)}(X) - \frac{i}{2} \Lambda_{IJ} P^I P^J \quad (14)$$

This reads in special coordinates like

$$f(T,U) \rightarrow (i\epsilon T + d)^{-2} \left( f(T,U) + \Psi(T,U) \right) \quad (15)$$

for an arbitrary $PSL(2,\mathbb{Z})_T$ transformation. $\Psi(T,U)$ is a quadratic polynomial in $T$ and $U$.

As explained in [8], the dilaton is not any longer invariant under the target space duality transformations at the one-loop level. Indeed, the relations [11] imply

$$S \rightarrow S + \frac{V_I^J (F^{(1-loop)}_J - i\Lambda_{JK} P^K)}{U^I J} \quad (16)$$

Near the singular lines the one-loop prepotential exhibits logarithmic singularities and is therefore not a singlevalued function when transporting the moduli fields around the singular lines. For example around the singular $SU(2)_{(1)}$ line $T = U \neq 1, \rho$ the function $f$ must have the following form

$$f(T,U) = \frac{1}{\pi} (T - U)^2 \log(T - U) + \Delta(T,U) \quad (17)$$

where $\Delta(T,U)$ is finite and singlevalued at $T = U \neq 1, \rho$. At the remaining three critical lines $f(T,U)$ takes an analogous form. Moreover at the intersection points the residue of the singularity must change in agreement with the number of states which become massless at these critical points. (These residues are of course just given by the $N = 2$ pure Yang-Mills $\beta$-functions for $SU(2)$, $SU(2)^2$ and $SU(3)$ (there are no massless additional flavors at the points of enhanced symmetries).) Specifically at the point $(T,U) = (1,1)$ the prepotential takes the form

$$f(T,U = 1) = \frac{1}{\pi} (T - 1) \log(T - 1)^2 + \Delta'(T) \quad (18)$$

and around $(T,U) = (\rho, \bar{\rho})$

$$f(T,U = \rho) = \frac{1}{\pi} (T - \rho) \log(T - \rho)^3 + \Delta''(T) \quad (19)$$

where $\Delta'(T)$, $\Delta''(T)$ are finite at $T = 1$, $T = \rho$ respectively. $f(T,U)$ is not a true modular form, but has non-trivial monodromy properties; remarkably, a closed, analytic expression for $f(T,U)$ in terms of the trilogarithm function was recently derived in [4]. However the third derivative transforms nicely under target space duality transformations, and using the informations about the order of poles and zeroes one can uniquely determine \( F_T^3 \) and \( F_U^3 \) .

$$
\partial_T^3 f(T,U) \propto \frac{+1}{2\pi} \frac{E_4(iT) E_4(iU) E_6(iU) \eta^{-24}(iU)}{j(iT) - j(iU)}
$$

$$
\partial_U^3 f(T,U) \propto \frac{-1}{2\pi} \frac{E_4(iT) E_6(iU) \eta^{-24}(iT) E_4(iU)}{j(iT) - j(iU)}
$$

This result has recently proved to be important to support the hypotheses that the quantum vector moduli space of the $N = 2$ heterotic string is given by the tree level vector moduli space of a dual type II, $N = 2$ string, compactified on a suitably chosen Calabi-Yau space. In addition to eq. (20) one can also deduce that

$$\partial_T \partial_U f = -\frac{2}{\pi} \log(jiT - jiU) + h(T, U), \quad (21)$$

where $h(T, U)$ is finite in the entire modular region. $\partial_T \partial_U f$ provides the gauge threshold corrections for the $U(1)^2$ gauge groups. It has precisely the right property that the coefficient of the logarithmic singularity is proportional to the number of generically massive states that become massless. The function $h(T, U)$ enters the definition of a dilaton field which is invariant under target space duality transformations. $S^{\text{inv}} = S - \frac{1}{2}h(T, U)$.

3.1. Perturbative $SU(2)_{(1)}$ monodromies

Let us now consider the element $\sigma$ which corresponds to the Weyl reflection in the first enhanced $SU(2)_{(1)}$. Under the mirror transformation $\sigma, T \leftrightarrow U$, $T - U \rightarrow e^{-i\pi}(T - U)$, and the $P$ transform classically and perturbatively as

$$P^0 \rightarrow P^0, \quad P^1 \rightarrow P^1, \quad P^2 \rightarrow P^3, \quad P^3 \rightarrow P^2 \quad (22)$$

The one-loop correction $f(T, U)$ transforms as

$$f(T, U) \rightarrow f(U, T) = f(T, U) - i(T - U)^2$$

$$f_T(U, T) = f_U(U, T) + 2i(T - U),$$

$$f_{\bar{U}}(U, U) = f_{\bar{T}}(U, U) - 2i(T - U) \quad (23)$$

The $f$ function must then have the following form for $T \rightarrow U$

$$f(T, U) = \frac{1}{\pi} (T - U)^2 \log(T - U) + \Delta(T, U) \quad (24)$$

with derivatives

$$f_T(T, U) = \frac{2}{\pi} (T - U) \log(T - U)$$

$$+ \frac{1}{\pi} (T - U) + \Delta_T$$

$$f_U(U, U) = -\frac{2}{\pi} (T - U) \log(T - U) - \frac{1}{\pi} (T - U) + \Delta_U \quad (25)$$

$\Delta(T, U)$ has the property that it is finite as $T \rightarrow U \neq 1, \rho$ and that, under mirror symmetry $T \leftrightarrow U, \Delta_T \leftrightarrow \Delta_U$. The 1-loop corrected $Q_2$ and $Q_3$ are thus given by

$$Q_2 = iSU - \frac{2i}{\pi} (T - U) \log(T - U)$$

$$- \frac{i}{\pi} (T - U) - i\Delta_T$$

$$Q_3 = iST + \frac{2i}{\pi} (T - U) \log(T - U)$$

$$+ \frac{i}{\pi} (T - U) - i\Delta_U \quad (26)$$

It follows from (21) that, under mirror symmetry $T \leftrightarrow U$, the dilaton $S$ transforms as

$$S \rightarrow S + i \quad (27)$$

Then, it follows that perturbatively

$$\begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} \rightarrow \begin{pmatrix} Q_3 \\ Q_2 \end{pmatrix} + \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} T \\ U \end{pmatrix} \quad (28)$$

Thus, the section $\Omega$ transforms perturbatively as $\Omega \rightarrow \Gamma_\infty^w \Omega$, where

$$\Gamma_\infty^w = \begin{pmatrix} U & 0 \\ UA & U \end{pmatrix}, \quad U = \begin{pmatrix} I & 0 \\ 0 & \eta \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} \eta & 0 \\ 0 & C \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

(29)

3.2. Truncation to the rigid case of Seiberg/Witten

In order to truncate the monodromies of local $N = 2$ supergravity to the case of rigid $N = 2$ super Yang-Mills one has to take the limit $M_{pl} \rightarrow \infty$.

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\textsuperscript{3} Note that one can always add polynomials of quadratic order in the moduli to a given $f(T, U)$ [3]. This results in the conjugation of the monodromy matrices. Hence, all the monodromy matrices given in the following are unique up to conjugation.
in an appropriate way. Specifically, one has to freeze, i.e. get rid off the dilaton and graviphoton degrees of freedom. This amounts to truncate the $8 \times 8$, $Sp(8)$ monodromy matrices (in case of two Higgs fields) of local supergravity to $4 \times 4$, $Sp(4)$ monodromy matrices of rigid supersymmetry. These rigid $Sp(4)$ matrices act within the four-dimensional subspace spanned by the $(P^2, P^3, iQ_2, iQ_3)$ which is related to the two Higgs fields $a_1$, $a_2$ and their duals $a_{D1}$, $a_{D2}$. However this truncation procedure requires some care since in the string case the dilaton as well as the graviphoton are in general not invariant under the Weyl transformations. It follows that the truncated $Sp(4)$ matrices are not simply given by the $4 \times 4$ submatrices of the local monodromies as we will show in the following.

In order to truncate the perturbative $SU(2)_{\{1\}}$ monodromy $\Gamma_{\infty}^{w_1}$ to the rigid one of Seiberg/Witten, we will expand

$$T = T_0 + \kappa \delta T, \quad U = T_0 + \kappa \delta U$$

Here we have expanded the moduli fields $T$ and $U$ around the same vev $T_0 \neq 1, \rho$. Both $\delta T$ and $\delta U$ denote fluctuating fields of mass dimension one. We will also freeze in the dilaton field to a large vev, that is we will set $S = \langle S \rangle \rightarrow \infty$. Then, the $Q_2$ and $Q_3$ given in (29) can be expanded as

$$Q_2 = i\langle S \rangle T_0 + \kappa \tilde{Q}_2 \quad , \quad Q_3 = i\langle S \rangle T_0 + \kappa \tilde{Q}_3$$

$$\tilde{Q}_2 = i\langle S \rangle \delta U - \frac{2i}{\pi}(\delta T - \delta U) \log \kappa^2(\delta T - \delta U)$$

$$- \frac{i}{\pi}(\delta T - \delta U) - i\Delta_T(\delta T, \delta U)$$

$$\tilde{Q}_3 = i\langle S \rangle \delta U + \frac{2i}{\pi}(\delta T - \delta U) \log \kappa^2(\delta T - \delta U)$$

$$+ \frac{i}{\pi}(\delta T - \delta U) - i\Delta_U(\delta T, \delta U)$$

Next, one has to specify how mirror symmetry is to act on the vev’s $T_0$ and $\langle S \rangle$ as well as on $\delta T$ and $\delta U$. We will take that under mirror symmetry $T_0 \rightarrow T_0$, $\delta T \leftrightarrow \delta U$, $\langle S \rangle \rightarrow \langle S \rangle$ (32) Note that we have taken $\langle S \rangle$ to be invariant under mirror symmetry. This is an important difference to (27). Using (32) and that $\delta T - \delta U \rightarrow e^{-i\pi} (\delta T - \delta U)$, it follows that the truncated quantities $\tilde{Q}_2$ and $\tilde{Q}_3$ transform as follows under mirror symmetry

$$\begin{pmatrix} \tilde{Q}_2 \\ \tilde{Q}_3 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} \delta T \\ \delta U \end{pmatrix}$$

Defining a truncated section $\tilde{\Omega} = (\tilde{P}^2, \tilde{P}^3, i\tilde{Q}_2, i\tilde{Q}_3) = (i\delta T, i\delta U, i\tilde{Q}_2, i\tilde{Q}_3)$, it follows that $\tilde{\Omega}$ transforms as $\tilde{\Omega} \rightarrow \tilde{\Gamma}_\infty^{w_1} \tilde{\Omega}$ under mirror symmetry (22) where

$$\tilde{\Gamma}_\infty^{w_1} = \begin{pmatrix} \tilde{U} & 0 \\ \tilde{\Lambda} \tilde{U} & \tilde{\Lambda} \end{pmatrix}, \quad \tilde{U} = \eta,$$

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\Lambda} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

Note that, because of the invariance of $\langle S \rangle$ under mirror symmetry, $\tilde{\Lambda} \neq -C$, contrary to what one would have gotten by performing a naive truncation of (24) consisting in keeping only rows and columns associated with $(P^2, P^3, iQ_2, iQ_3)$.

Finally, in order to compare the truncated $SU(2)$ monodromy (23) with the perturbative $SU(2)$ monodromy of Seiberg/Witten, one has to perform a change of basis from moduli fields to Higgs fields, as follows

$$\begin{pmatrix} a \\ a_D \end{pmatrix} = M \tilde{\Omega}, \quad M = \begin{pmatrix} m \\ m^* \end{pmatrix},$$

$$m = \frac{\gamma}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

where $\gamma$ denotes a constant to be fixed below. Then, the perturbative $SU(2)$ monodromy in the Higgs basis is given by

$$\tilde{\Gamma}_\infty^{Higgs} = M \Gamma_{\infty}^{w_1} M^{-1}$$

$$= \begin{pmatrix} m \tilde{U} m^{-1} & 0 \\ m^* \tilde{U} \tilde{\Lambda} m^{-1} & m^* \tilde{U} m^T \end{pmatrix}$$

(36)
which is computed to be

\[
\hat{\Gamma}_{\text{Higgs},w_1} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
\frac{4}{7} & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]  

(37)

Note that (37) indeed correctly shows that, under the Weyl reflection in the first SU(2), the second SU(2) is left untouched. The fact that reproduces this behaviour can be easily traced back to the fact that we have assumed that remains invariant under the mirror transformation \(\delta T \leftrightarrow \delta U\). Finally, comparing with the perturbative SU(2) monodromy of Seiberg/Witten [2] yields that \(\gamma^2 = 2\), whereas comparison with the perturbative SU(2) monodromy of Klemm et al [15,16] gives that \(\gamma^2 = 1\).

The discussion about the other Weyl twist is completely analogous to \(w_1\) and can be found in [2].

4. Non-perturbative monodromies

In order to obtain some information about non-perturbative monodromies in \(N = 2\) heterotic string compactifications, we will follow Seiberg/Witten’s strategy in the rigid case [3] and try to decompose the perturbative monodromy matrices \(\Gamma_\infty\) into \(\Gamma_\infty = \Gamma_M \Gamma_D\) with \(\Gamma_M\) (\(\Gamma_D\)) possessing monopole like (dyonic) fixed points. Thus each semi-classical singular line will split into two non-perturbative singular lines where magnetic monopoles or dyons respectively become massless. In doing so we will work in the limit of large dilaton field \(S\) assuming that in this limit the non-perturbative dynamics is dominated by the Yang-Mills gauge forces. Nevertheless, the monodromy matrices we will obtain are not an approximation in any sense, since the monodromy matrices are of course field independent. They are just part of the full quantum monodromy of the four-dimensional heterotic string.

Let us now precisely list the assumptions we will impose when performing the split of any of the semiclassical monodromies into the non-perturbative ones:

1. \(\Gamma_\infty\) must be decomposed into precisely two factors.

   \[
   \Gamma_\infty = \Gamma_M \Gamma_D
   \]  

(38)

2. \(\Gamma_M\) and therefore \(\Gamma_D\) must be symplectic.

3. \(\Gamma_M\) must have a monopole like fixed point. For the case of \(w_1\), for instance, it must be of the form

   \[
   (N,-M) = (0,0,N^2,-N^2,0,0,0,0)
   \]

(39)

4. \(\Gamma_D\) must have a dyonic fixed point. For the case of \(w_1\), for instance, it must be of the form

   \[
   (N,-M) = (0,0,N^2,-N^2,0,0,-M_2,M_2)
   \]

(40)

where \(N^2\) and \(M_2\) are proportional.

5. \(\Gamma_M\) and \(\Gamma_D\) should be conjugated, that is, they must be related by a change of basis, as it is the case in the rigid theory.

6. The limit of large \(S\) should be respected. This means that \(S\) should only transform into a function of \(T\) and \(U\) (for at least one of the four SU(2) lines, as will be discussed in the following).

In the following we will show that under these assumptions the splitting can be performed in a consistent way. We will discuss the non-perturbative monodromies for the SU(2) \((1)\) case in big detail. Unlike the rigid case, however, where the decomposition of the perturbative monodromy into a monopole like monodromy and a dyonic monodromy is unique (up to conjugation), it will turn out that there are several distinct decompositions, depending on four (discrete) parameters. Only a subset of these distinct decompositions should be, however, the physically correct one. One way of deciding which one is the physically correct one is to demand that, when truncating this decomposition to the rigid case, one recovers the rigid non-perturbative monodromies of Seiberg/Witten. This, however, requires one to have a reasonable prescription of taking the flat limit, and one such prescription was given in section (3.3).
4.1. Non-perturbative monodromies for \( SU(2)_{(1)} \)

The non-perturbative part \( f^{NP} \) of the prepotential will depend on the S-field. We will make the following ansatz for the prepotential

\[
F = i \frac{X^1 X^2 X^3}{X^0} + (X^0)^2 \left( f(T, U) + f^{NP}(S, T, U) \right)
\]  

(41)

In order to find a decomposition of \( \Gamma^\omega M \), \( \Gamma^\omega D = \Gamma^\omega M \Gamma^\omega D \), we will now make the following ansatz: \( \Gamma^\omega M \) has a peculiar block structure in that the indices \( j = 0, 1 \) of the section \( (P_j, iQ_j) \) are never mixed with the indices \( j = 2, 3 \). We will assume that \( \Gamma^\omega M \) and \( \Gamma^\omega D \) also have this structure. This implies that the problem can be reduced to two problems for \( 4 \times 4 \) matrices. Furthermore, we will take \( \Gamma^\omega M \) to be the identity matrix on its diagonal. The existence of a basis where the non-perturbative monodromies have this special form will be a posteriori justified by the fact that it leads to a consistent truncation to the rigid case.

Taking into account all these assumptions yields \( \Gamma^\omega M \) the following 8×8 non-perturbative monodromy matrices that depend on four parameters \( x, y, v, p \) and consistently describe the splitting of the \( T = U \) line

\[
\Gamma^\omega M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -2/3 & 2/3 \\
0 & 0 & 0 & 1 & 0 & 2/3 & -2/3 \\
x & y & 0 & 0 & 1 & 0 & 0 & 0 \\
y & v & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & p & p & 0 & 0 & 1 & 0 \\
0 & 0 & p & p & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]  

(42)

\( \Gamma^\omega M \) immediately follows from eq. (43).

The associated fixed points have the form

\[
(N, -M) = (0, 0, N^2, -N^2, 0, 0, 0, 0)
\]  

for the monopole and

\[
(N, -M) = \left( 0, 0, N^2, -N^2, 0, 0, \frac{3}{2} N^2, -\frac{3}{2} N^2 \right)
\]  

(44)

for the dyon.

4.2. Truncating the \( SU(2)_{(1)} \) monopole monodromy to the rigid case

The monopole monodromy matrix for the first \( SU(2) \), given in equation (42), depends on 4 undetermined parameters, namely \( x, v, y \) and \( p \neq 0 \). Note that demanding the monopole monodromy matrix to be conjugated to the dyonic monodromy matrix led to the requirement \( p \neq 0 \). On the other hand, it follows from (42) that

\[
S \rightarrow S - i (y + v(TU - f_S^{NP}))
\]  

(45)

Consider now the 4×4 monopole subblock which acts on \( (P^2, P^3, iQ_2, iQ_3) \)

\[
\Gamma_M^{Higgs,w} = M \Gamma_M^{w} M^{-1}
\]  

(46)

\[
\Gamma_M^{Higgs,w} = \begin{pmatrix}
1 & 0 & 2\alpha & -2\alpha \\
0 & 1 & 2\alpha & -2\alpha \\
p & p & 1 & 0 \\
p & p & 0 & 1 \\
\end{pmatrix}
\]  

\( \alpha = \frac{1}{2}, p \neq 0 \). Rotating it into the Higgs basis gives that

\[
\begin{pmatrix}
1 & 0 & -4\alpha^2 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 2\gamma & 0 & 1 \\
\end{pmatrix}
\]  

(47)

where \( \alpha = \frac{1}{3}, p \neq 0 \) and \( M \) is given in equation (43). In the rigid case, on the other hand, one expects to find for the rigid monopole monodromy
9

matrix in the Higgs basis that

\[
\tilde{\Gamma}^{Higgs,w}_{M} = \begin{bmatrix}
1 & 0 & -4\tilde{\alpha}\gamma^2 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 2\tilde{\alpha}\gamma & 0 & 1
\end{bmatrix}
\] (48)

(\tilde{\alpha} = \frac{1}{4}, \tilde{\rho} = 0.) The first and third lines of (48) are, for \(\tilde{\alpha} = \frac{1}{4}\), nothing but the monodromy matrix for one \(SU(2)\) monopole \((\gamma^2 = 2\) in the conventions of Seiberg/Witten \([3]\), and \(\gamma^2 = 1\) in the conventions of Klemm et al \([13]\)).

Thus, truncating the monopole monodromy matrix (42) to the rigid case appears to produce jumps in the parameters \(p \rightarrow \tilde{p} = 0\) and \(\alpha \rightarrow \tilde{\alpha}\) as given above. In \([2]\) we presented a field theoretical explanation for the jumps occurring in the parameters \(p\) and \(\alpha\) when taking the rigid limit. This explanation also determines, as a bonus, the values of the parameters \(v, y\) and \(p': y = \frac{8}{3}, p = \frac{4}{3}\), \(v = 0\). Moreover, one can show that, in order to decouple the four \(U(1)'s\) at the non-perturbative level, one has to have \(x = v\) and consequently \(x = 0\). Note that \(v = 0\) ensures that \(S \rightarrow S - iy\) under the \(SU(2)_{(1)}\) monopole monodromy.

5. Conclusions

In this paper we have first discussed the properties of the perturbative prepotential of \(N = 2\) heterotic strings. The derivatives of the holomorphic prepotential determine the low energy effective Lagrangian of the \(N = 2\) heterotic strings, such as the effective gauge couplings or the Kähler potential. At the one-loop level the effective couplings are related to automorphic functions in a very interesting way. In addition we have shown that the semiclassical monodromies associated with lines of enhanced gauge symmetries can be consistently split into pairs of non-perturbative lines of massless monopoles and dyons. Furthermore, all monodromies obtained in the string context allow for a consistent truncation to the rigid monodromies of \([3],[13],[4]\). Recently the non-perturbative monodromies for the models with the two fields \(S\) and \(T\) and their rigid limits were also determined by using the string-string duality between heterotic strings and four-dimensional type II strings, compactified on a suitably chosen Calabi-Yau space. The perturbative monodromy (in this case \(T \rightarrow \frac{1}{r}\)) and its decomposition into non-perturbative monopole and dyon monodromies, as computed from the type II Calabi-Yau side, agree with our perturbative and non-perturbative monodromies after introducing a compensating shift for the dilaton, \(S \rightarrow S - i\), which is generated by \(\Gamma_S\) in (13). We will come back to these issues in more detail in a future paper [6].

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