Inflation on a single brane – exact solutions

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Algorithms are developed for generating a class of exact braneworld cosmologies, where a self–interacting scalar field is confined to a positive–tension brane embedded in a bulk containing a negative cosmological constant. It is assumed that the five–dimensional Planck scale exceeds the brane tension but is smaller than the four–dimensional Planck mass. It is shown that the field equations can be expressed as a first–order system. A number of solutions to the equations of motion are found. The potential resulting in the perfect fluid model is identified.

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I. INTRODUCTION

The possibility that our observable universe may be viewed as a domain wall embedded in a higher–dimensional space has received considerable attention recently. Motivated by developments in superstring and M–theory, it is assumed in this scenario that the standard model interactions are confined to a (3 + 1)–dimensional hypersurface, but that gravity may propagate through the ‘bulk’ dimensions perpendicular to the brane. This change in viewpoint has important consequences for early universe cosmology and, in particular, for the inflationary paradigm.

Various cosmological aspects of branes embedded in five dimensions have been investigated. The effects of including a scalar field in the bulk have also been studied recently. In this paper, we consider a co–dimension 1 brane with positive tension, ρ, embedded in vacuum Einstein gravity with a negative cosmological constant. This corresponds to the scenario introduced by Randall and Sundrum where the extra dimension is infinite. We focus on the region of parameter space defined by $\lambda^{1/4} \ll m_5 \ll m_4$, where $m_{4,5}$ represent the four– and five–dimensional Planck scales, respectively.

One approach to determining the expansion of the brane is to project the five–dimensional metric onto the brane world–volume. In effect, this is equivalent to solving Einstein’s field equations, together with appropriate jump conditions, for a negative cosmological constant and an energy–momentum tensor restricted to the brane. The field equations admit a first integral that may be interpreted as a generalized Friedmann equation.

We emphasize, however, that the dynamics could differ in a compactified scenario, due to the effects of radion stabilization. If the cosmological constant and brane tension are related in an appropriate fashion, this equation takes the form:

$$H^2 = \frac{4\pi}{3\lambda m_4^2} \rho \left( \rho + 2\lambda \right) + \frac{\epsilon}{a^4},$$

where $\rho$ represents the energy density of matter on the brane, $\epsilon$ is a constant and we have assumed that the four–dimensional cosmological constant is zero. The last term on the right–hand side of Eq. (1.1) behaves as ‘dark radiation’ and arises due to the backreaction of the bulk gravitational degrees of freedom on the brane. At sufficiently low energies, $\rho \ll \lambda$, the standard cosmic behaviour is recovered and the primordial nucleosynthesis constraint is satisfied provided that $\lambda \geq (1\,\text{MeV})^4$.

Quantum gravitational effects become important if the energy density on the brane exceeds the five–dimensional Planck scale. In this case, the assumption that matter is confined to the brane may become unreliable. However, if $\rho \ll m_4^4$, there is a region of parameter space, corresponding to $\lambda \ll \rho \ll m_5^4$, where the classical solution is still valid, but where the quadratic correction becomes significant. In particular, this term can play an important role during inflation. Cosmological inflation has played a central role in studies of the very early universe. (For recent reviews, see, e.g., Refs. [4,8,10].) It is therefore important to study the inflationary dynamics of the braneworld scenario. In the standard inflationary cosmology, the universe is dominated by a scalar ‘infla–ton’ field self–interacting through a potential, $V(\phi)$, with an energy density $\rho = \dot{\phi}^2/2 + V$. Inflation proceeds if the potential energy of the field dominates its kinetic energy. Maartens et al. have recently considered the case where a

*In this paper we assume that the world–volume metric of the three–brane is the spatially flat, Friedmann–Robertson–Walker (FRW) line element with scale factor, $a(t)$, and Hubble parameter, $H \equiv \dot{a}/a$. A dot and prime denote differentiation with respect to cosmic time, $t$, and the scalar field, $\phi$, respectively.
single inflaton field is confined to the brane and have
derived the necessary criteria for successful inflation when
the slow–roll approximation is valid [8]. In general, the
quadratic term in the Friedmann equation (1.1) results in
an enhanced friction on the field, implying that inflation
is possible for a wider region of parameter space than in
the standard cosmology [6,8,10,11].

When the scalar field is confined to the brane, energy–
momentum conservation implies that its equation of mo-
tion has the standard form:

\[ \dot{\phi} + 3H \dot{\phi} + V'(\phi) = 0. \]  

(1.2)

Eq. (1.2) can be expressed in terms of the energy density
such that

\[ \dot{\rho} = -3H \rho \phi^2 \]  

(1.3)

and it follows that a given inflationary braneworld model
is specified by a solution to the Friedmann–scalar field
equations (1.1) and (1.3).

In general, however, it is very difficult to solve this
system of equations, even when the standard slow–roll
assumptions, \( \dot{\phi} \ll V \) and \( |\phi| \ll H |\dot{\phi}| \), are made. It
is therefore important to develop generating techniques
for finding exact solutions to Eqs. (1.1) and (1.2) and
this is the purpose of the present work. During inflation,
the dark radiation redshifts rapidly and soon becomes
dynamically negligible and we therefore consider models
where \( \epsilon = 0 \). On the other hand, we do not assume that
the slow–roll approximation is necessarily valid. Thus,
such a study has direct applications to the final stages
of inflation, where the kinetic energy of the inflaton field
in fact becomes significant. Depending on the form
of the potential, this may happen when the quadratic
approximation correction to the Friedmann equation (1.1)
is still impor-
tant. The search for exact scalar field braneworlds is also
important because it allows one to classify the possible
types of behaviour that arise in these universes and to
uncover the generic characteristics of such models.

II. FIRST–ORDER FIELD EQUATIONS

From a particle physics perspective, it is natural to
begin by specifying the functional form of the poten-
tial. However, even for simple choices, such as expo-
nential or power law potentials, analytical progress can
not be made. An alternative route, originally introduced
within the context of standard chaotic inflation [18], is
to specify the time–dependence of the scale factor, \( a(t) \).
In the braneworld scenario, given \( a(t) \), one may deduce
\( \rho(t) \) from Eq. (1.1) and hence \( \dot{\phi}(t) \) from Eq. (1.3).
Integrating yields \( \phi(t) \). The time–dependence of the poten-
tial follows immediately from the definition of the energy
density and the form of \( V(\phi) \) is deduced by inverting \( \dot{\phi}(t) \).
The drawback of this approach is that such an inversion
is not always possible and moreover a realistic potential
does not necessarily result.

In principle, these problems are partially avoided by
first specifying an invertible form for \( \phi(t) \) [19], although
we do not explore this possibility further here. Our
approach is to note that during inflation, the scalar field
rolls monotonically down its potential. Thus, Eq. (1.3)
may be expressed in the form

\[ \rho' = -3H \dot{\phi}. \]  

(2.1)

The formal limit, \( \lambda \to \infty \), corresponds to the standard in-
flationary scenario. In this case, Eqs. (1.1) and (2.1) can
be written in the particularly simple ‘Hamilton–Jacobi’
form [20,21]

\[ H' a' = -4\pi \frac{H a}{m^4} \]  

(2.2)

\[ H' = \frac{4\pi}{m^2} \dot{\phi} \]  

(2.3)

\[ (H')^2 - \frac{12\pi}{m^4} H^2 = \frac{32\pi^2}{m^4} V, \]  

(2.4)

where the Hubble parameter is viewed implicitly as a
function of the scalar field. Eq. (2.3) may be integrated
to yield the scale factor:

\[ a(\phi) = \exp \left[ -\frac{4\pi}{m^4} \int_0^\phi d\phi \frac{H}{H'} \right] \]  

(2.5)

and this implies that the cosmological dynamics is de-
termined, up to a single quadrature, once the functional
form of \( H(\phi) \) has been specified. It has been suggested
that \( H(\phi) \) should be viewed as the solution generating
function when analysing inflationary cosmologies [22].
The advantage of such an approach is that the form of the
potential is readily deduced from Eq. (2.4). The ques-
tion that naturally arises within the braneworld context,
therefore, is whether there exists an analogous generating
function when the quadratic correction in the Friedmann
equation (1.1) is relevant.

To proceed, we define a new function \( y(\phi) \):

\[ \rho = \frac{2\lambda y^2}{1 - y^2}, \]  

(2.6)

where the restriction \( y^2 < 1 \) must be imposed for the
weak energy condition to be satisfied. Substituting Eq.
(2.6) into the Friedmann equation (1.1) implies that the
Hubble parameter is given by

\[ H(\phi) = \frac{16\pi \lambda}{3m^4} \left( \frac{y}{1 - y^2} \right)^{1/2} \]  

(2.7)

The right–hand side of Eq. (2.7) can be expanded as a
geometric progression:

\[ H = \left( \frac{16\pi \lambda}{3m^4} \right)^{1/2} \left[ y + y^3 + y^5 + \ldots \right] \]  

(2.8)
implying that \( y \) is proportional to the Hubble parameter in the low–energy limit, \( y \rightarrow 0 \) \((\rho/\lambda \rightarrow 0)\).

Substitution of Eq. (2.6) into the scalar field equation (2.1) implies that

\[ \dot{\phi} = -\left(\frac{\lambda m_4^2}{3\pi}\right)^{1/2} \frac{y'}{1 - y'^2} \] (2.9)

and the potential is given in terms of \( y(\phi) \) by combining Eqs. (2.6) and (2.9):

\[ V(\phi) = \frac{2\lambda y^2}{1 - y'^2} - \frac{\lambda m_4^2}{6\pi} \left(\frac{y'}{1 - y'^2}\right)^2. \] (2.10)

It follows from Eqs. (2.7) and (2.9) that the scale factor satisfies

\[ y' a' = -\frac{4\pi}{m_4^2} y a. \] (2.11)

Eq. (2.11) is formally identical to Eq. (2.2) and may be integrated to yield the scale factor in terms of a single quadrature with respect to the scalar field:

\[ a(\phi) = \exp \left[ -\frac{4\pi}{m_4^2} \int_0^\phi d\phi\frac{y}{y'} \right]. \] (2.12)

Finally, the dependence of the scalar field on cosmic time is deduced by evaluating the integral

\[ t - t_0 = \left(\frac{3\pi}{\lambda m_4^2}\right)^{1/2} \int_\phi^{0} d\phi \frac{y'^2 - 1}{y'}. \] (2.13)

and inverting the result, where \( t_0 \) is an arbitrary integration constant.

Thus, we have reduced the second–order system of equations (2.1) and (2.2) to the non–linear, first–order system (2.11) and (2.13) by employing the scalar field as dynamical variable.

Further insight may be gained by defining a second new function \( b(\phi) \):

\[ y = \tanh b. \] (2.14)

It follows immediately from Eqs. (2.7) and (2.9) that the Hubble parameter is given by

\[ H = \left(\frac{4\pi \lambda}{3m_4^2}\right)^{1/2} \sinh 2b \] (2.15)

and that the scalar field varies as

\[ \dot{\phi} = -\left(\frac{\lambda m_4^2}{3\pi}\right)^{1/2} b'. \] (2.16)

Eq. (2.16) is formally equivalent to Eq. (2.3). Hence, the scale factor is given by

\[ a = \exp \left[ -\frac{2\pi}{m_4^2} \int_0^{\phi} d\phi \frac{\sinh 2b}{y'} \right]. \] (2.17)

and the potential takes the simple form

\[ V = 2\lambda \sinh^2 b - \frac{\lambda m_4^2}{6\pi} b'^2. \] (2.18)

The time–dependence of the scalar field is determined by the integral

\[ t - t_0 = -\left(\frac{3\pi}{\lambda m_4^2}\right)^{1/2} \int_{\phi_0}^{\phi} d\phi \frac{1}{y'}. \] (2.19)

The function \( b(\phi) \) plays the equivalent role to that of the Hubble parameter in the field equation (2.3). Indeed, \( b(\phi) \propto H(\phi) \) in the low–energy limit. It follows from Eq. (2.10) that \( b(t) \) is a monotonically decreasing function of cosmic time.

To summarize this Section, we have found that the braneworld field equations can be rewritten after suitable redefinitions in a way that directly extends the Hamilton–Jacobi form of the standard scalar field cosmology. However, the physical interpretation of the variables is different in the two cases. Nevertheless, this correspondence implies that similar techniques may be employed to find exact braneworld cosmologies. In particular, when the functional form of the parameter \( y(\phi) \) is known, the potential is determined in terms of this function and its first derivative. The function \( a(\phi) \) follows from Eq. (2.12) and \( \phi(t) \) follows by evaluating Eq. (2.13) and inverting the result. An alternative method for solving the field equations is to specify the form of \( b(\phi) \) and to then evaluate the two integrals in Eqs. (2.17) and (2.19). This is equivalent to determining the kinetic energy of the scalar field as a function of the field itself. A related technique was recently employed in the standard inflationary scenario \( b^3 \). The advantage of this approach is that the integrand in Eq. (2.17) can be expanded as a power series in \( b \).

### III. EXACT BRANEWORLDS

We now find exact braneworld models by employing the above techniques. Firstly, we consider the ansatz

\[ y = \text{sech}\left(\frac{\sqrt{2\pi C}}{m_4} \phi \right), \] (3.1)

where \( C \) is an arbitrary constant. Substitution of Eq. (2.1) into Eq. (2.10) implies that the potential is given by

\[ V = \frac{\lambda}{3}(6 - C^2) \coth^2 \left(\frac{\sqrt{2\pi C}}{m_4} \phi \right). \] (3.2)

Thus, we require \( C^2 < 6 \) for the potential to be positive–definite and we consider this region of parameter space in what follows. Evaluating Eq. (2.13) implies that
\[ t - t_0 = \left( \frac{3}{4 \pi \lambda} \right)^{1/2} \frac{m_4}{C^2} \cosh \left( \frac{\sqrt{2\pi C}}{m_4} \phi \right) \]  

(3.3)

and substituting Eqs. (3.1) and (3.3) into Eq. (2.12) implies that the scale factor is given by

\[ a(t) = \left[ \frac{4\pi \lambda C^4}{3m_4^2} (t - t_0)^{2 - 1} \right]^{1/C^2}. \]  

(3.4)

Without loss of generality, we may choose \( t_0 = -\left[3m_4^2/(4\pi \lambda C^4)\right]^{1/2} \) such that the origin of time corresponds to a vanishing scale factor.

It can be verified by direct substitution that the solution [3.3] is a scaling solution, in the sense that the kinetic and potential energies of the scalar field redshift at the same rate as the brane expands. The field behaves as a perfect fluid with an effective equation of state, \( p = \omega \rho \), where the barotropic index is given by

\[ \omega = \frac{C^2 - 3}{3}. \]  

(3.5)

Thus, we have found the scalar field model equivalent to the perfect fluid cosmology presented in Ref. [4]. The solution is interesting because it reduces to the power–law cosmology driven by an exponential potential in the low–energy limit. Inflation proceeds indefinitely into the future if \( C^2 < 2 \) and the expansion decelerates for \( C^2 > 2 \). However, at early times, the asymptotic behaviour is \( a \propto t^{1/C^2} \), and inflation proceeds for a finite time for \( C^2 < 1 \). In this limit, the potential is of the form \( V \propto \phi^{-2} \), where the constant of proportionality determines the power of the expansion.

At late times the function \( y \) given in Eq. (3.3) asymptotes to an exponential form. It is therefore of interest to consider a second ansatz

\[ y = \exp \left( -\sqrt{2\pi C} \phi / m_4 \right), \]  

(3.6)

that is valid for all time, where we assume implicitly that \( \phi > 0 \) and that \( C^2 < 6 \). Integrating Eqs. (2.9) and (2.12) implies that

\[ \phi = \frac{m_4}{\sqrt{2\pi C}} \ln \left( T + \sqrt{T^2 - 1} \right), \]  

(3.7)

\[ a = \left( T + \sqrt{T^2 - 1} \right)^{2/C^2}, \]  

(3.8)

where we have introduced a rescaled time variable

\[ T \equiv \left( \frac{\pi \lambda C^4}{3m_4^2} \right)^{1/2} (t - t_0) \]  

(3.9)

and the origin of time corresponds to \( T = 1 \). The potential of the scalar field is deduced by substituting Eq. (3.3) into Eq. (2.10):

\[ V = \frac{2\lambda y^2}{1 - y^2} \left[ 1 - \frac{C^2}{6} \frac{1}{1 - y^2} \right]. \]  

(3.10)

The potential is negative for \( y^2 > (6 - C^2)/6 \), has a single maximum located at \( y^2 = (6 - C^2)/(6 + C^2) \) and exponentially decays to zero from above as \( y^2 \to 0 \). Although the potential is negative to the right of the maximum, the solution exists for all \( \phi > 0 \) because the initial magnitude of the field’s kinetic energy is sufficiently large for it to move over the maximum and reach \( \phi \to +\infty \).

We now consider the formulation of the cosmological brane equations summarized in Eqs. (2.14)–(2.19). One ansatz that can be invoked is

\[ b = \left( \frac{3\pi A^2}{\lambda m_4^2} \right)^{1/2} \phi, \]  

(3.11)

where \( A \) is a constant. From Eq. (2.16), this represents a model where the kinetic energy of the scalar field is a constant for all time:

\[ \phi = \phi_0 - A (t - t_0). \]  

(3.12)

The scale factor and potential of the field are readily deduced from Eqs. (2.17) and (2.18), respectively:

\[ a = \exp \left[ -\frac{\lambda}{3A^2} \cosh \left( \sqrt{\frac{12\pi^2 A^2}{\lambda m_4^2}} \phi \right) \right], \]  

(3.13)

\[ V = 2\lambda \sinh^2 \left( \sqrt{\frac{3\pi A^2}{\lambda m_4^2}} \phi \right) - \frac{A^2}{2}. \]  

(3.14)

Early times correspond to the limit \( \phi \gg \lambda^{1/2}m_4/A \), where the potential has an asymptotically exponential form and inflation may proceed for a wide range of parameter space. Inflation ends within a finite time, however, and the field eventually falls into its minimum at \( V_{\text{min}} = -A^2/2 \). The expansion of the universe is then reversed and the subsequent collapse enables the field to move up the other side of its potential.

Another solvable model is defined by

\[ b \equiv p_2 \phi^2, \]  

(3.15)

where \( p_2 \) is an arbitrary constant. Substituting Eq. (3.15) into Eq. (2.18) implies that the potential is given by

\[ V = 2\lambda \sinh^2 \left( p_2 \phi^2 \right) - \frac{2\lambda p_2^2 m_4^2}{3\pi} \phi^2. \]  

(3.16)

This potential has a single maximum at \( V(\phi = 0) = 0 \) and two minima at

\[ \sinh(2p_2 \phi^2) = \frac{p_2 m_4^2}{3\pi}. \]  

(3.17)

Integration of Eq. (2.16) implies that the time–dependence of the scalar field is given by

\[ \phi = \phi_0 \exp \left[ -\left( \frac{4\lambda p_2^2 m_4^2}{3\pi} \right)^{1/2} (t - t_0) \right]. \]  

(3.18)
Finally, the growth in the scale factor can be expressed as a power series by substituting Eq. (3.15) into Eq. (2.17) and integrating:

\[ a = \exp \left[ -\frac{\pi}{2p_2 m_4^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+1)!} (2p_2 \phi^2)^{2n+1} \right]. \]

(3.19)

Before concluding, we develop in the next Section an algorithm that generates a new braneworld cosmology from a known solution such as those presented above.

**IV. A THIRD ALGORITHM**

Since \( b(t) \) is a monotonically decreasing function, Eq. (2.16) may be rewritten as

\[ \phi^2 = -\left( \frac{\lambda m_4^2}{3 \pi} \right)^{1/2} b. \]

(4.1)

Defining a new function

\[ c \equiv \exp \left[ -\int t \, dt' b(t') \right] \]

(4.2)

and a new time parameter

\[ \eta \equiv \int t \, dt' c(t') \]

(4.3)

implies that Eq. (4.1) may be then expressed in the form of a one–dimensional Helmholtz equation:

\[ \left[ \frac{d^2}{d\eta^2} - U(\eta) \right] c = 0, \]

(4.4)

where the effective potential is uniquely determined by the kinetic energy of the scalar field:

\[ U(\eta) \equiv \left( \frac{3 \pi}{\lambda m_4^2} \right)^{1/2} \left( \frac{d\phi}{d\eta} \right)^2. \]

(4.5)

The importance of Eq. (4.4) is that if a particular solution, \( c_1(\eta) \), is known, the general solution can be written in terms of a single quadrature with respect to this solution [24]:

\[ c = c_1 \left( \kappa + \int^{\eta} \frac{dz}{c_1^2(z)} \right), \]

(4.6)

where \( \kappa \) is an arbitrary constant. It is this feature that form the basis of the algorithm. Suppose a particular solution to the braneworld field equations has already been found, i.e., that \( \{b(t), \phi(t)\} \) are known. The function, \( c_1(\eta) \), can in principle be evaluated from Eqs. (1.2) and (1.3). Eq. (4.6) then yields the new solution for \( c(\eta) \) and, hence, \( b(\eta) \) from the definition (4.3). The form of \( \phi(\eta) \) is identical in both solutions, but the potential of the scalar field is different in the new solution. It is given by

\[ V[\eta(\phi)] = 2 \lambda \sin^2 \left[ b(\eta) \right] - \frac{1}{2} c^2(\eta) \left( \frac{d\phi}{d\eta} \right)^2, \]

(4.7)

or, equivalently, by

\[ V[\eta(\phi)] = 2 \lambda \left[ \sin \left( \frac{dc}{d\eta} \right) \right]^2 - \left( \frac{\lambda m_4^2}{12 \pi} \right)^{1/2} c \frac{d^2c}{d\eta^2}. \]

(4.8)

The form of \( V(\phi) \) follows from Eqs. (4.4) and (4.8) and the scale factor is deduced as before from Eq. (2.17).

In effect, each particular braneworld cosmology is twinned with a second solution. In practice, it may not always be possible to evaluate the integrals (4.2) and (1.3), but it would be interesting to explore models of this nature further. We also remark that in the low–energy limit, \( c \) varies as a power of the scale factor of the brane. Thus, the above discussion is also relevant to standard scalar field cosmology.

**V. CONCLUSIONS**

In conclusion, we have presented algorithms for solving the braneworld Friedmann equation for a single, self–interacting scalar field confined to a brane with positive tension3 embedded in five–dimensional, vacuum Einstein gravity. The formalism does not assume the slow–roll approximation and is valid during inflation when the field is monotonically rolling down its potential. It does not apply if the field is oscillating about a minimum. A number of new exact solutions were found, including the scalar field model that corresponds to a perfect fluid. The algorithms may be viewed as generalizations of the Hamilton–Jacobi formalism that has played a central role in analyses of the standard chaotic inflationary scenario [21]. In the latter case, the Hamilton–Jacobi formalism provides the necessary framework for establishing the precise correspondence between the potential of the scalar field and the scalar and tensor perturbation spectra that are generated during inflation [17]. It would be interesting to employ the techniques developed above to establish the equivalent correspondence in the braneworld scenario.

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