A deletion-contraction long exact sequence for chromatic symmetric homology

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Abstract

In [4], the authors generalize Stanley’s chromatic symmetric function [7] to vertex-weighted graphs. In this paper we find a categorification of their new invariant extending the definition of chromatic symmetric homology to vertex-weighted graphs. We prove the existence of a deletion-contraction long exact sequence for chromatic symmetric homology which gives a useful computational tool and allow us to answer two questions left open in [2]. In particular, we prove that, for a graph $G$ with $n$ vertices, the maximal index with nonzero homology is not greater than $n - 1$. Moreover, we show that the homology is non-trivial for all the indices between the minimum and the maximum with this property.

Introduction

The chromatic symmetric function $X_G$ of a graph $G$, defined by Stanley in [7], is a remarkable combinatorial invariant which refines the chromatic polynomial. In [6], Sazdanovic and Yip categorified this invariant by defining a new homological theory, called the chromatic symmetric homology of $G$. This construction, inspired by Khovanov’s categorification of the Jones polynomial [1], is obtained by assigning a graded representation of the symmetric group to every subgraph of $G$, and a differential to every cover relation in the Boolean poset of subgraphs of $G$. The chromatic symmetric homology $H_{s,t}(G)$ is then defined as the homology of this chain complex; its bigraded Frobenius series $\text{Frob}_G(q,t)$, when evaluated at $q = t = 1$, reduces to Stanley’s chromatic symmetric function expressed in the Schur basis. This categorification has interesting properties which have been investigated in [2] and [3].

In [4], Logan Crew and Sophie Spirkl generalize Stanley’s chromatic symmetric function [7] to vertex-weighted graphs $(G, w)$ with the definition of the weighted chromatic symmetric function $X_{(G,w)}$. One of the primary motivations for extending the chromatic symmetric function to vertex-weighted graphs is the existence of a deletion-contraction relation in this setting, which, as known, holds for the chromatic polynomial, but doesn’t hold for the chromatic symmetric function, as observed by Stanley in [7].

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In this paper we generalize chromatic symmetric homology to vertex-weighted graphs. We obtain in this way a categorification of the weighted chromatic symmetric function that we call \textit{weighted chromatic symmetric homology} and we denote by $H_{*,*}(G, w)$. The weighted chromatic symmetric homology specializes to the chromatic symmetric homology if $w = 1$ is the function assigning weight 1 to each vertex, i.e. if $G$ is an unweighted graph.

Moreover, we prove the existence of a deletion-contraction long exact sequence for the weighted chromatic symmetric homology which lifts to homology the deletion-contraction relation that holds for the function defined by Crew and Spirkl.

In particular, we prove that

**Theorem.** Let $(G, w)$ be a vertex-weighted graph and let $e$ be an edge of $G$. For each $j \geq 0$, there is a long exact sequence in homology

$$
\rightarrow H_{i,j}(G \setminus e, w) \rightarrow H_{i,j}(G, w) \rightarrow H_{i-1,j}(G/e, w/e) \rightarrow H_{i-1,j}(G \setminus e, w) \rightarrow \ldots,
$$

where $G \setminus e$ denotes the graph $G$ with the edge $e$ removed, $G/e$ denotes the graph $G$ with the edge $e$ contracted to a point, and $w/e$ denotes the weight function on $G/e$ defined in Section 1.

The long exact sequence in homology gives a useful computational tool and allow us to answer two questions left open in [2].

Let $\text{span}_0(G)$ denote the homological span of the degree 0 chromatic symmetric homology of $G$. In [2], the authors formulate the following two conjectures.

**Conjecture (C.5).** Given any graph $G$, chromatic symmetric homology groups $H_{i,0}(G; C)$ are non-trivial for all $0 \leq i \leq \text{span}_0(G) - 1$.

**Conjecture (C.6).** Let $G$ be a graph with $n$ vertices and $m$ edges, and let $b$ denote the number of blocks of $G$. Then $n - b \leq \text{span}_0(G) \leq n - 1$.

Using the deletion-contraction long exact sequence for chromatic symmetric homology we show that Conjecture C.5 and a part of Conjecture C.6 are true, also for the case of vertex-weighted graphs.

In particular, denoting by $k_{\text{max}}^i(G, w)$ the largest index $k$ such that $H_{k,i}(G, w) \neq 0$ and by $k_{\text{min}}^i(G, w)$ the smallest one ($k_{\text{min}}^0(G, w)$ is always 0 in the case of simple graphs), we prove that

**Theorem.** Given any graph $(G, w)$, chromatic symmetric homology groups $H_{i,j}(G, w; C)$ are non-trivial for all $k_{\text{min}}^i(G, w) \leq i \leq k_{\text{max}}^i(G, w)$, $j \geq 0$. 

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Theorem. Let \((G, w)\) be a graph with \(n\) vertices and \(m\) edges. Then \(k^j_{\text{max}}(G, w) \leq n - 1\) for all \(j \geq 0\). Moreover, if \(m \geq 1\), \(k^0_{\text{max}}(G, w) \leq n - 2\), so span0(G) \(\leq n - 1\).

The paper is organized as follows. In Section 1 we recall the definition and some basic properties of the weighted chromatic symmetric function. In Section 2 we build our categorification and prove the existence of a long exact sequence in homology that lifts the deletion-contraction relation for the weighted chromatic symmetric function. Finally, in Section 3, we present some applications of the mentioned sequence and we prove the last two theorems above.

1 Weighted chromatic symmetric function

Let \(G\) be a graph. Then \(G \setminus e\) denotes the graph \(G\) with the edge \(e\) removed and \(G/\!\!/e\) denotes the graph \(G\) with the edge \(e\) contracted to a point.

**Definition 1.** Define a **vertex-weighted graph** \((G, w)\) to be a graph \(G = (V(G), E(G))\) together with a vertex-weight function \(w : V(G) \to \mathbb{N}\). The **weight** of a vertex \(v \in V(G)\) is \(w(v)\).

**Remark 2.** Let \(G\) be any graph. Then \(G\) can be viewed as the vertex-weighted graph \((G, 1)\), where 1 is the function assigning weight 1 to each vertex.

**Definition 3.** Given a vertex-weighted graph \((G, w)\), we say that \(F \subseteq V(G)\) is a **state** of \(G\), and we define the **total weight** \(w(F)\) of \(F\) to be \(\sum_{v \in F} w(v)\). Moreover, we define the total weight \(w(G)\) of \(G\) to be the total weight of \(V(G)\).

The set \(Q(G)\) of all the states of \(G\) has a structure of Boolean lattice, ordered by reverse inclusion. In the Hasse diagram of \(Q(G)\), we direct an edge \(\epsilon(F, F')\) from a subgraph \(F\) to a subgraph \(F'\) if and only if \(F'\) can be obtained by removing an edge from \(F\).

In [4], Logan Crew and Sophie Spirkl generalize Stanley’s chromatic symmetric function [7] to vertex-weighted graphs with the following definition:

**Definition 4.** Let \((G, w)\) be a vertex-weighted graph. Then the **weighted chromatic symmetric function** is

\[
X_{(G, w)}(x_1, x_2, \ldots) = \sum_{\kappa} \prod_{v \in V(G)} x^{w(v)}_{\kappa(v)},
\]

where the sum ranges over all proper colorings \(\kappa : V(G) \to \mathbb{N}\) of \(G\).
Remark 5. If $G$ has a loop, then $X_{(G,w)} = 0$ for every $w : V(G) \to \mathbb{N}$. Moreover, if $e_1, e_2$ are edges of $G$ with the same endpoints, then $X_{(G,w)} = X_{(G\setminus e_1,w)} = X_{(G\setminus e_2,w)}$ for every $w : V(G) \to \mathbb{N}$.

Remark 6. Note that $X_{(G,1)} = X_G$, where $X_G$ is the usual chromatic symmetric function.

Recall that, if $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of a positive integer $n$, i.e. a non-increasing sequence of positive integers whose sum is $n$, the power sum symmetric function $p_\lambda$ is defined as

$$p_\lambda(x_1, x_2, \ldots) = p_{\lambda_1}(x_1, x_2, \ldots) \cdots p_{\lambda_k}(x_1, x_2, \ldots),$$

where $p_r(x_1, x_2, \ldots) = x_1^r + x_2^r + \ldots$, for $r \in \mathbb{N}$.

Let $\Lambda_n$ be the $\mathbb{Z}$-module of the homogeneous symmetric functions of degree $n$. Then $\{p_\lambda \mid \lambda \text{ partition of } n\}$ is a basis for $\Lambda_n$. Another basis for $\Lambda_n$ is given by the Schur symmetric functions $\{s_\lambda \mid \lambda \text{ partition of } n\}$. Moreover, let $\Lambda^\mathbb{C} = \bigoplus_{n \geq 0} \Lambda_n$ denote the space of symmetric functions in the indeterminates $x_1, x_2, \ldots$.

Definition 7. Given a vertex-weighted graph $(G, w)$, and $F \subseteq E(G)$, we define $\lambda(G, w, F)$ to be the partition of $w(G)$ whose parts are the total weights of the connected components of $(G', w)$, where $G' = (V(G), F)$.

Lemma 8 ([4], Lemma 3). Let $(G, w)$ be a vertex-weighted graph. Then

$$X_{(G,w)} = \sum_{F \subseteq E(G)} (-1)^{|F|} p_{\lambda(G,w,F)}.$$

One of the primary motivations for extending the chromatic symmetric function to vertex-weighted graphs is the existence of a deletion-contraction relation in this setting.

Definition 9. Let $(G, w)$ be a vertex-weighted graph, and let $e = (v_1, v_2) \in E(G)$. We define $w/e : V(G/e) \to \mathbb{N}$ to be the modified weight function on $G/e$ such that $w/e = w$ if $e$ is a loop, and otherwise $(w/e)(v) = w(v)$ if $v \neq v_1, v_2$, and for the vertex $v^*$ of $G/e$ formed by the contraction, $(w/e)(v^*) = w(v_1) + w(v_2)$.

We have the following:

Theorem 10 ([4], Lemma 2). Let $(G, w)$ be a vertex-weighted graph, and let $e \in E(G)$ be any edge. Then

$$X_{(G,w)} = X_{(G\setminus e,w)} - X_{(G/e,w/e)}.$$

Note that the deletion-contraction relation of Theorem 10 does not give a similar relation for the ordinary chromatic symmetric function, since if we contract a non loop edge we do not get an ordinary chromatic symmetric function.
2 Weighted chromatic symmetric homology

Now we build a categorification of the invariant just introduced.

In this section we assume that the set of edges of $G$ is ordered.
Let $\mathfrak{S}_n$ denote the symmetric group on $n$ elements. The irreducible representations of $\mathfrak{S}_n$ over $\mathbb{C}$ are indexed by the partitions of $n$, and are called Specht modules. Let $S^\lambda$ denote the Specht module indexed by $\lambda$.

The Grothendieck group $R_n$ of representations of $\mathfrak{S}_n$ is the free abelian group on the isomorphism classes $[S^\lambda]$ of irreducible representations of $\mathfrak{S}_n$, modulo the subgroup generated by all $[V \oplus W] - [V] - [W]$. Let $R = \bigoplus_{n \geq 0} R_n$. If $[V] \in R_a$ and $[W] \in R_b$, define a multiplication in $R$ by

$$[V] \circ [W] = [\text{Ind}_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_{a+b}} V \otimes W].$$

Here the tensor product $V \otimes W$ is regarded as a representation of $\mathfrak{S}_n \times \mathfrak{S}_m$ in the obvious way: $(\sigma \times \tau) \cdot (v \otimes w) = \sigma \cdot v \otimes \tau \cdot w$; and $\mathfrak{S}_n \times \mathfrak{S}_m$ is regarded as a subgroup of $\mathfrak{S}_{n+m}$ with $\mathfrak{S}_n$ acting on the first $n$ integers and $\mathfrak{S}_m$ acting on the last $m$ integers. The induced representation can be defined quickly by the formula

$$\text{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}} = \mathbb{C}[\mathfrak{S}_{n+m}] \otimes_{\mathbb{C}[\mathfrak{S}_n \times \mathfrak{S}_m]} (V \otimes W).$$

It is straightforward to verify that this product is well defined and makes $R$ into a commutative, associative, graded ring with unit.

The morphism of graded rings given by sending the Specht modules to the Schur functions

$$ch : R \to \Lambda^\mathbb{C}, [S^\lambda] \to s_\lambda$$

is an isomorphism.
Moreover, for $n \in \mathbb{N}$, we have

$$ch^{-1}(p_n) = \sum_{i=0}^{n-1} (-1)^i [S^{(n-i,1)}]. \quad (1)$$

For the proofs of these two last facts see [5], Section 7.3.
With the notation of [6], we define:

**Definition 11.** Let $(G, w)$ be a vertex-weighted graph. Suppose $F \subseteq E(G)$ is a state with $r$ connected components of total weights $b_1^w, \ldots, b_r^w$ respectively. To $F$, we assign the graded $\mathfrak{S}_{w(G)}$-module

$$M_F^w = \text{Ind}_{\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_r}}^{\mathfrak{S}_{w(G)}} (L_{b_{i_1}^w} \otimes \cdots \otimes L_{b_{i_r}^w}), \quad (2)$$
where $L_a$ denotes the $q$-graded $S_a$-module
\[
L_a = \bigoplus_{j=0}^{a-1} S^{(a-j,1^j)},
\]
and $S^{(a-j,1^j)}$ is the Specht module related to the partition $(a-j,1^j)$ of the positive integer $a$. The grading is given by the index $j$.

**Definition 12.** For $i \geq 0$, the $i$-th weighted chain module for $(G,w)$ is
\[
C_i(G,w) = \bigoplus_{|F|=i} M^w_F.
\]
More precisely, since $M^w_F = \bigoplus_{j \geq 0} (M^w_F)_j$ is graded, then for $i,j \geq 0$, we define
\[
C_{i,j}(G,w) = \bigoplus_{|F|=i} (M^w_F)_j.
\]

**Remark 13.** Observe that $(M^w_F)_j = 0$ if $j \geq b^w_t$ for all $t = 1, \ldots, r$.

Since the differential defined in [6] depends only on the $b_i$'s, we can define a differential in the same way, replacing the $b_i$'s with the $b^w_i$'s.

Let $F$ be a state of $G$. Suppose $F' = F - e$ where $e \in E(G)$. We define the $S_w(G)$-modules morphism $d^{(G,w)}_e : M^w_F \rightarrow M^w_{F'}$, i.e. the per-edge maps, in the following way.

There are two cases to consider:

**Case 1** The edge $e$ is incident to vertices in the same connected component of $F'$.
Since $M^w_F$ and $M^w_{F'}$ are equal, we define $d_e : M^w_F \rightarrow M^w_{F'}$ to be the identity map.

**Case 2** The edge $e$ is incident to vertices in different connected components of $F'$.
First, consider the simplest case where $F$ consists of one connected component and $F'$ consists of two components $A$ and $B$. Suppose $w(A) = a$ and $w(B) = b$, so that $a + b = w(G)$. Since, by Frobenius Reciprocity, $\text{Hom}_{S_w(G)}(M^w_F, M^w_{F'}) \cong \text{Hom}_{\mathbb{S}_a \times \mathbb{S}_b}(\Lambda^*T \oplus (\Lambda^*T)[1], \Lambda^*T)$, where
\[
T = (S^{(a-1,1)} \otimes 1_{\mathbb{S}_b}) \oplus (1_{\mathbb{S}_a} \otimes S^{(b-1,1)})(\text{see [6], Lemma 2.6}),
\]
we choose the element $d_e \in \text{Hom}_{S_w(G)}(M^w_F, M^w_{F'})$ to be the map that corresponds to the $(\mathbb{S}_a \times \mathbb{S}_b)$-module map that is the identity on $\Lambda^*T$ and zero on $(\Lambda^*T)[1]$. 

In the general case when \( F \) has more than one connected component, the definition of the per-edge map is achieved by recursion on the two-component case.

Suppose \( F \) is a state with \( r \) connected components \( B_1, \ldots, B_r \) of total weights \( b_1^w, \ldots, b_r^w \). Further suppose that the removal of the edge \( e \in E(G) \) decomposes \( B_r \) into two components \( A \) and \( B \) of total weights \( a \) and \( b \) respectively \((a + b = b_r^w)\). Let \( d_\zeta : L_{b^w} \to \text{Ind}_{S_{b^w}} \circ \text{Ind}_{S_a \times S_b} (L_a \otimes L_b) \) be the per-edge map defined previously (note that \( M_{b^w} = L_{b^w} \), since \( B_r \) is connected), and let \( N = L_1^w \otimes \cdots \otimes L_{b_r^w} \). The map \( d_\epsilon : M_F \to M_F' \) is chosen to be

\[
d_\epsilon = \text{Ind}_{S_{b^w}} \circ \text{Ind}_{S_{a^w}} \circ \text{id}_N \otimes d_\zeta
\]

**Definition 14.** Let \( F \) and \( F' \) be states of \( G \). Assume that \( F' = F \setminus e, e \in E(F) \). The sign of \( \epsilon = \epsilon(F, F') \), \( \text{sgn}(\epsilon) \), is defined as \((-1)^k\), where \( k \) is the number of edges of \( F \) less than \( e \).

**Definition 15.** For \( i \geq 0 \), define \( d_i^{(G,w)} : C_i(G, w) \to C_{i-1}(G, w) \) letting

\[
d_i^{(G,w)} = \sum \text{sgn}(\epsilon) d_\epsilon^{(G,w)},
\]

where the sum is over all edges \( \epsilon \) in the Hasse diagram of \( Q(G) \) joining a state with \( i \) edges to a state with \( i - 1 \) edges. We also define \( d_{i,j}^{(G,w)} : C_{i,j}(G, w) \to C_{i-1,j}(G, w) \) to be the map \( d_i^{(G,w)} \) in the \( j \)-th grading.

**Proposition 16.** The maps \( d_i^{(G,w)} \) form a differential on the chain complex \( C_\ast(G, w) \).

**Proof.** The proof is completely analogous to that of Proposition 2.10 of [6] replacing the \( b_i \)'s with the \( b_i^w \)'s. \(\square\)

**Definition 17.** For \( i, j \geq 0 \), the \((i,j)\)-th weighted chromatic symmetric homology of \((G,w)\) is

\[
H_{i,j}(G, w) = \ker d_{i,j}^{(G,w)} / \text{im} d_{i+1,j}^{(G,w)}.
\]

Moreover, we define

\[
H_i(G, w) = \bigoplus_{j \geq 0} H_{i,j}(G, w).
\]
Remark 18. $H_{*,*}(G, 1) = H_{*,*}(G)$, where $H_{*,*}(G)$ is the usual chromatic symmetric homology.

Example 19. Let $(K_2, w)$ be the segment with a vertex $v_1$ of weight 1 and the other $v_2$ of weight 2. The labels of the vertices indicate their weights.

We have

- $C_{1,0}(K_2, w) = (M_F^w)_0 = S^{(3)}$;
- $C_{0,0}(K_2, w) = (M_F^w)_0 = Ind_{S_w(v_1) \times S_w(v_2)} S(2) \otimes S^{(1)} = S^{(3)} \oplus S^{(2,1)}$;
- $C_{1,1}(K_2, w) = (M_F^w)_1 = S^{(2,1)}$;
- $C_{0,1}(K_2, w) = (M_F^w)_1 = Ind_{S_w(v_1) \times S_w(v_2)} S(2,1) \otimes S^{(1)} = S^{(2,1)} \oplus S^{(3)}$;
- $C_{1,2}(K_2, w) = (M_F^w)_2 = S^{(1)}$;
- $C_{0,2}(K_2, w) = 0$.

Therefore, $H_{1,0}(K_2, w) = H_{1,1}(K_2, w) = H_{0,2}(K_2, w) = 0$, $H_{0,0}(K_2, w) = S^{(2,1)}$, $H_{0,1}(K_2, w) = H_{1,2}(K_2, w) = S^{(3)}$.

In general,

- $C_{1,0}(K_2, w) = (M_F^w)_0 = S^{w(v_1) + w(v_2)}$;
- $C_{0,0}(K_2, w) = (M_F^w)_0 = Ind_{S_{w(v_1)} \times S_{w(v_2)}} S^{w(v_1) + w(v_2)} \otimes S^{w(v_2)}$
  \[= S^{w(v_1) + w(v_2)} \oplus \bigoplus_{\lambda} (S^\lambda)^{m\lambda}.\]
We don’t give the details about the $S^λ$’s which appear in the last formula and their multiplicities. You can find an explanation of it in [5], Section 7.3. We say only that they are all different from $S^{(w(v_1)+w(v_2))}$. Therefore, we have $H_{1,0}(K_2, w) = 0$ and $H_{0,0}(K_2, w) \neq 0$. Moreover, $H_{i,0}(K_2, w) = 0$ for any $i \geq 2$, since $K_2$ does not have any states with more than one edge.

**Definition 20.** The bigraded Frobenius series of $H_{*,*}(G, w) = \bigoplus_{i,j \geq 0} H_{i,j}(G, w)$ is

$$Frob_{(G, w)}(q, t) = \sum_{i,j \geq 0} (-1)^{i+j}t^iq^jch(H_{i,j}(G, w)).$$

**Example 21.** Let’s consider the vertex-weighted graph of the previous example. We have

$$Frob_{(K_2, w)}(q, t) = -(q + t q^2)s_{(1^3)} + s_{(2,1)}.$$

**Lemma 22.** For any vertex-weighted graph $(G, w)$,

$$\sum_{i,j \geq 0} (-1)^{i+j}ch(H_{i,j}(G, w)) = \sum_{i,j \geq 0} (-1)^{i+j}ch(C_{i,j}(G, w)).$$

**Proof.** Let $n$ be any positive integer. Any short exact sequence of $\mathfrak{S}_n$-modules $0 \to A \to B \to C \to 0$ is split exact, so $B \cong A \oplus C$ and $ch(B) = ch(A) + ch(C)$. Let $Z_{i,j}(G, w) = \ker d_{i,j}^{(G, w)}$ and $B_{i,j}(G, w) = \text{im} d_{i+1,j}^{(G, w)}$. For $i,j \geq 0$, we have short exact sequence $0 \to Z_{i,j}(G, w) \to C_{i,j}(G, w) \to B_{i-1,j}(G, w) \to 0$ and $0 \to B_{i,j}(G, w) \to Z_{i,j}(G, w) \to H_{i,j}(G, w) \to 0$, where $B_{-1,j}(G, w)$ is understood to be zero. Thus

$$ch(C_{i,j}(G, w)) = ch(Z_{i,j}(G, w)) + ch(B_{i-1,j}(G, w))$$

$$= ch(H_{i,j}(G, w)) + ch(B_{i,j}(G, w)) + ch(B_{i-1,j}(G, w)).$$

If we multiply this by $(-1)^{i+j}$ and we sum over all $i,j \geq 0$, we get:

$$\sum_{i,j \geq 0} (-1)^{i+j}ch(C_{i,j}(G, w)) = \sum_{i,j \geq 0} (-1)^{i+j}ch(H_{i,j}(G, w)) + \sum_{i,j \geq 0} (-1)^{i+j}ch(B_{i,j}(G, w)) +$$

$$\sum_{i,j \geq 0} (-1)^{i+j}ch(B_{i-1,j}(G, w)) = \sum_{i,j \geq 0} (-1)^{i+j}ch(H_{i,j}(G, w)) + \sum_{i,j \geq 0} (-1)^{i+j}ch(B_{i,j}(G, w))$$

$$- \sum_{i,j \geq 0} (-1)^{i+j}ch(B_{i,j}(G, w)) = \sum_{i,j \geq 0} (-1)^{i+j}ch(H_{i,j}(G, w)).$$

\[\square\]
Theorem 23. Weighted chromatic symmetric homology categorifies the weighted chromatic symmetric function. That is, for any vertex-weighted graph \((G, w)\),

\[
\text{Frob}_{(G, w)}(1, 1) = X_{(G, w)}.
\]

Proof. Using Lemma 22 and Lemma 8, we have

\[
\text{Frob}_{(G, w)}(1, 1) = \sum_{i, j \geq 0} (-1)^i \text{j-ch}(H_{i, j}(G, w)) = \sum_{i \geq 0} \left( \sum_{j \geq 0} (-1)^i \text{j-ch}(C_{i, j}(G, w)) \right)
\]

\[
= \sum_{i \geq 0} \left( \sum_{F \subseteq E(G), |F| = i} p_{\lambda(G, w, F)} \right) = X_{(G, w)}.
\]

Now we want to lift to homology the result of Theorem 10.

Proposition 24. Let \((G, w)\) be a vertex-weighted graph and let \(e\) be an edge of \(G\). For each \(i, j \geq 0\), there is a short exact sequence of \(S_{w(G)}\)-modules

\[
0 \rightarrow C_{i, j}(G \setminus e, w) \rightarrow C_{i, j}(G, w) \rightarrow C_{i-1, j}(G/e, w/e) \rightarrow 0.
\]

Proof. By definition

\[
C_{i, j}(G \setminus e, w) = \bigoplus_{|F| = i, F \subseteq E(G \setminus e)} (M^w_F)_j
\]

\[
= \bigoplus_{|F| = i, F \subseteq E(G), e \notin F} (M^w_F)_j \subseteq \bigoplus_{|F| = i, F \subseteq E(G)} (M^w_F)_j = C_{i, j}(G, w).
\]

Therefore, there is a short exact sequence

\[
0 \rightarrow C_{i, j}(G \setminus e, w) \overset{\iota_j}{\rightarrow} C_{i, j}(G, w) \overset{\pi_j}{\rightarrow} C_{i, j}(G, w) \rightarrow 0,
\]

where \(\iota_j\) is the inclusion and \(\pi_j\) is the projection to the quotient.

We have that

\[
\frac{C_{i, j}(G, w)}{C_{i, j}(G \setminus e, w)} = \bigoplus_{|F| = i, F \subseteq E(G)} (M^w_F)_j \cong \bigoplus_{|F| = i, F \subseteq E(G), e \notin F} (M^w_F)_j.
\]

Since, if \(F\) is a state of \((G, w)\) with \(i\) edges such that \(e \in F\), then \(M^w_F = M^w_{F/e}\), because the contraction does not change the total weight of the connected components of \(F\), and \(F/e\) is a state of \((G/e, w/e)\) with \(i - 1\) edges, we have that
\[ \bigoplus_{|F|=i,F \subseteq E(G), \epsilon \in F} (M^w_F)_{ij} = C_{i-1,j}(G/e, w/e), \]

and the theorem follows. \( \square \)

**Remark 25.** If \( G \) is an unweighted graph, for each \( i, j \geq 0 \), we have the following short exact sequence of \( \mathcal{S}_{|V(G)|} \)-modules

\[ 0 \to C_{i,j}(G \setminus e) \to C_{i,j}(G) \to C_{i-1,j}(G/e, 1/e) \to 0. \]

**Proposition 26.** Let \((G, w)\) be a vertex-weighted graph and let \( e \) be an edge of \( G \). For each \( j \geq 0 \), there is a short exact sequence of chain complexes

\[ 0 \to C_{*,j}(G \setminus e, w) \to C_{*,j}(G, w) \to C_{*,j}(G/e, w/e) \to 0. \]

**Proof.** With the notation of the proof of Proposition \( \text{24} \), we have to show that, for each \( i \geq 0 \), \( d^{(G, w)}_i \circ t_i = t_{i-1} \circ d^{(G \setminus e, w)}_i \) and \( d^{(G/e, w/e)}_{i-1} \circ \pi_i = \pi_{i-1} \circ d^{(G, w)}_i \). It is clear that the first equality holds. Let’s look at the second.

If \( i = 0, 1 \), we have 0 on both sides. Consider \( i \geq 2 \). Since, if \( F \) is a state of \((G, w)\) with \( i \) edges such that \( e \in F \), then \( M^w_F = M^{w/e}_F \), \( \pi_i \) is the map such that

\[ \pi_i|_{M^w_F} = \begin{cases} id & \text{if } e \in F, \\ 0 & \text{if } e \notin F. \end{cases} \]

Therefore,

\[ \pi_{i-1} \circ d^{(G, w)}_i = \sum_{e} sgn(e) \pi_{i-1} \circ d^{(G, w)}_i = \sum_{e} sgn(e') d^{(G, w)}_{e'} \]

where the last sum is over all the \( e' \) in the Hasse diagram of \( Q(G, w) \) joining a state of \((G, w)\) with \( i \) edges that contains \( e \) to a state of \((G, w)\) with \( i - 1 \) edges that also contains \( e \).

On the other hand, \( d^{(G/e, w/e)}_{i-1} \circ \pi_i = \sum_{e''} sgn(e'') d^{(G/e, w/e)}_{e''} \), where the sum is over all the \( e'' \) in the Hasse diagram of \( Q(G/e, w/e) \) joining a state of \((G/e, w/e)\) with \( i - 1 \) edges to a state of \((G/e, w/e)\) with \( i - 2 \) edges.

We know that, if \( F \) is a state of \( G \) with \( i \) edges such that \( e \in F \), then \( M^w_F = M^{w/e}_F \) and \( F/e \) is a state of \((G/e, w/e)\) with \( i - 1 \) edges. Therefore, if \( e' \) is an edge in the Hasse diagram of \( Q(G, w) \) connecting a state \( F \) of \((G, w)\) with \( i \) edges that contains \( e \) with a state \( F' \) of \((G, w)\) with \( i - 1 \) edges that also contains \( e \),

\[ d^{(G, w)}_{e'} : M^w_F \to M^w_{F/e} \to M^w_{F/e} \]

coincides with \( d^{(G/e, w/e)}_{e''} \), where \( e'' \) is an edge in the Hasse diagram of \( Q(G/e, w/e) \) joining the state \( F/e \) of \((G/e, w/e)\) with \( i - 1 \) edges to the state \( F'/e \) of \((G/e, w/e)\) with \( i - 2 \) edges.
Since there is a bijection between the states of \( G \) with \( i \) edges that contains \( e \) and the states of \((G/e, w/e)\) with \( i - 1 \) edges, we have that the two sums coincide. Therefore,

\[
d_{i-1}^{(G/e, w/e)} \circ \pi_i = \pi_{i-1} \circ d_i^{(G, w)}.
\]

Therefore, we have:

**Theorem 27.** Let \((G, w)\) be a vertex-weighted graph and let \( e \) be an edge of \( G \). For each \( j \geq 0 \), there is a long exact sequence in homology

\[
\cdots \rightarrow H_{i,j}(G \setminus e, w) \rightarrow H_{i,j}(G, w) \rightarrow H_{i-1,j}(G / e, w / e) \xrightarrow{\gamma^*} H_{i-1,j}(G \setminus e, w) \rightarrow \cdots \quad (4)
\]

**Proof.** The short exact sequences of chain complexes in Proposition 26 induce for each \( j \geq 0 \) a long exact sequence in homology. \( \square \)

**Remark 28.** The specialization of the Frobenius series at \( q = t = 1 \) recovers the deletion-contraction relation of Theorem 10.

**Remark 29.** The description for \( \gamma^* \) follows from the standard diagram chasing argument in the zig-zag lemma and the result is as follows. It is the linear extension of the map that, given a state of \((G/e, w/e)\) with \( i - 1 \) edges, where \( e = (v_e, w_e) \) is an edge of \( G \) that has been contracted to a point, expands \( v_e = w_e \) by adding \( e \) with weight \( w(v_e) \) at the vertex \( v_e \) and \( w(w_e) \) at the vertex \( w_e \) and then deletes \( e \). In this way we get a state of \((G \setminus e, w)\) with \( i - 1 \) edges.

**Remark 30.** If \( G \) is an unweighted graph, for each \( j \geq 0 \), we have the following long exact sequence in homology

\[
\cdots \rightarrow H_{i,j}(G \setminus e) \rightarrow H_{i,j}(G) \rightarrow H_{i-1,j}(G / e, 1 / e) \xrightarrow{\gamma^*} H_{i-1,j}(G \setminus e) \rightarrow \cdots.
\]

### 2.1 Properties of \( H_{*,*}(G, w) \)

The deletion-contraction long exact sequence allows us to give a different and faster proof of the following two properties of chromatic symmetric homology, contained in [6], and to extend them to the case of vertex-weighted graphs.

**Proposition 31.** If \((G, w)\) contains a loop, then \( H_{*,*}(G, w) = 0 \).

**Proof.** Let \((G, w)\) be a graph with a loop \( l \). The exact sequence for \((G, w)\) with respect to \( l \) is
Using our description of the snake map $\gamma^*$ in Remark 29, we get that the map $H_{i,j}(G/l, w/l) \xrightarrow{\gamma^*} H_{i,j}(G\setminus l, w) \to H_{i,j}(G, w) \to H_{i-1,j}(G/l, w/l) \xrightarrow{\gamma^*} H_{i-1,j}(G\setminus l, w) \to \ldots$.

Therefore, from now on we assume that $G$ is simple, so without loops or multiple edges.

Given two vertex-weighted graphs $(A, w_A)$ and $(B, w_B)$, let $(A + B, w_{A+B})$ denote their disjoint union, where

$$w_{A+B}(v) = \begin{cases} w_A(v), & \text{if } v \in V(A), \\ w_B(v), & \text{if } v \in V(B). \end{cases}$$

**Proposition 33.** For $i,j \geq 0$,

$$H_{i,j}(A + B, w_{A+B}) = \bigoplus_{p+q=i, r+s=j} \text{Ind}_{\mathbb{S}^{w_A(A) + w_B(B)}}^{\mathbb{S}^{w_A(A) \times w_B(B)}} \left( H_{p,q}(A, w_A) \otimes H_{r,s}(B, w_B) \right).$$

**Proof.** The proof is completely analogous to the unweighted case. See [6], Proposition 3.3.

**Remark 34.** If $(G, w)$ is a graph with homology $H_{i,j}(G, w) = \bigoplus_{\lambda} (S^{\lambda})^{\oplus m_{\lambda}}$, then the homology of the disjoint union of $G$ with a single vertex with weight $w_\bullet$ is

$$H_{i,j}(G + \bullet) = \bigoplus_{\mu} (S^{\mu})^{\oplus m_{\mu}},$$

where the sum is over all partitions $\mu$ which can be obtained by adding $w_\bullet$ boxes to the partitions $\lambda$ indexing the irreducible factors of $H_{i,j}(G, w)$. 
3 Applications

The deletion-contraction long exact sequence in homology has proved to be a useful computational tool. Moreover, we can use it to compute weighted chromatic symmetric homology starting from unweighted chromatic symmetric homology.

Example 35. Let \((K_2, w)\) be the segment with a vertex of weight 1 and the other of weight 2. We can compute its homology using the deletion-contraction long exact sequence.

Let \(G = P_3\) be the graph made of two segments with a vertex in common, and let \(e \in E(G)\). We have that \((K_2, w) = G/e\) and \(G \setminus e\) is the disjoint union of \(K_2\) and an isolated vertex.

We have \(H_{0,0}(G \setminus e) = H_{1,1}(G \setminus e) = \mathbf{S}^{(2,1)} \oplus \mathbf{S}^{(1^3)}\) and \(H_{1,0}(G \setminus e) = 0\). Moreover, we have \(H_0(G) = H_{2,2}(G) = \mathbf{S}^{(1^3)}\), \(H_{1,1}(G) = \mathbf{S}^{(2,1)} \oplus \mathbf{S}^{(2,1)} \oplus \mathbf{S}^{(1^3)}\) and \(H_{0,1}(G) = H_{2,0}(G) = H_{2,1}(G) = 0\).

For \(j = 0\), we have the following long exact sequence in homology:

\[
0 \longrightarrow H_{1,0}(K_2, w) \longrightarrow 0 \longrightarrow 0 \longrightarrow H_{0,0}(K_2, w) \longrightarrow \mathbf{S}^{(2,1)} \oplus \mathbf{S}^{(1^3)} \longrightarrow H_{1,0}(K_2, w) \longrightarrow 0,
\]

from which we can conclude that \(H_{1,0}(K_2, w) = 0\) and \(H_{0,0}(K_2, w) = \mathbf{S}^{(2,1)}\).

For \(j = 1\), we have the following long exact sequence in homology:

\[
0 \longrightarrow H_{1,1}(K_2, w) \longrightarrow \mathbf{S}^{(2,1)} \oplus \mathbf{S}^{(1^3)} \longrightarrow \mathbf{S}^{(2,1)} \oplus \mathbf{S}^{(1^3)} \longrightarrow H_{0,1}(K_2, w) \longrightarrow 0,
\]

from which we can conclude that \(H_{1,1}(K_2, w) = 0\) and \(H_{0,1}(K_2, w) = \mathbf{S}^{(1^3)}\).

For \(j = 2\), we have the following long exact sequence in homology:

\[
0 \longrightarrow \mathbf{S}^{(1^3)} \longrightarrow H_{1,2}(K_2, w) \longrightarrow 0 \cdots \longrightarrow 0,
\]

from which we can conclude that \(H_{1,2}(K_2, w) = \mathbf{S}^{(1^3)}\) and \(H_{0,2}(K_2, w) = 0\).

Now, given a graph \((G, w)\), let \(\text{span}_0(G, w)\) denote the homological span of the degree 0 weighted chromatic symmetric homology of \((G, w)\), i.e. of \(H_{i,0}(G, w)\). We have \(\text{span}_0(G, w) = k + 1\) where \(k\) is maximal among indices such that \(H_{k,0}(G, w) \neq 0\), since we are assuming that \(G\) has no loops, so \(H_{0,0}(G, w)\) is always nonzero.

In [2], the authors left open the following
Conjecture (C.6). Let $G$ be a graph with $n$ vertices and $m$ edges, and let $b$ denote the number of blocks of $G$. Then $n - b \leq \text{span}_0(G) \leq n - 1$.

We denote by $k^i_{\text{max}}(G, w)$ the largest index $k$ such that $H_{k,i}(G,w) \neq 0$ and by $k^j_{\text{min}}(G, w)$ the smallest one. As observed earlier, $k^0_{\text{min}}(G, w)$ is always 0.

Using the deletion-contraction long exact sequence for weighted chromatic symmetric homology,[4] we can prove that

**Theorem 36.** Let $(G, w)$ be a graph with $n$ vertices and $m$ edges. Then $k^j_{\text{max}}(G, w) \leq n - 1$ for all $j \geq 0$. Moreover, if $m \geq 1$, $k^0_{\text{max}}(G, w) \leq n - 2$, so $\text{span}_0(G) \leq n - 1$.

**Proof.** We prove that, if $i \geq 0$ is an index such that $H_{i,j}(G,w) \neq 0$, then we have $i \leq n - 1$.

We proceed by induction on the number $m \geq 0$ of edges of $G$. If $m = 0$, we have that the homology $H_{i,j}(G,w)$ is trivial for all $i > 0$, since we don’t have any states with more than zero edges. Therefore, the first inequality holds.

Furthermore, if we require $m \geq 1$, at the base step we have to consider the case $m = 1$. It follow from Remark [34] that we can assume without loss of generality that $G$ is connected, so, if $m = 1$, then $G$ is a segment with two vertices and an edge between them. It follows from Example [19] that $k^0_{\text{max}}(G, w) = 0$, so the second part of the theorem holds.

We now assume the statement true for any graph with $m - 1$ edges. Let $\nu(G)$ denote the number of vertices of $G$ and $e(G)$ the number of edges of $G$. We have that $\nu(G \setminus e) = \nu(G)$ and $e(G \setminus e) = e(G) - 1 = m - 1$. Moreover, we have that $\nu(G/e) = \nu(G) - 1$ and $e(G/e) = e(G) - 1 = m - 1$.

Let $i > \nu(G) - 2$. Since $\nu(G \setminus e) = \nu(G)$, we have also that $i > \nu(G \setminus e) - 2$. By inductive hypothesis, we have $H_{i,j}(G \setminus e, w) = 0$. Moreover, since $i - 1 > \nu(G) - 3 = e(G/e) - 2$, by inductive hypothesis, we have $H_{i-1,j}(G/e, w/e) = 0$ and $H_{i,j}(G/e, w/e) = 0$.

From the deletion-contraction long exact sequence [4]

\[ \cdots \rightarrow H_{i,j}(G/e, w/e) \rightarrow H_{i,j}(G \setminus e, w) \rightarrow H_{i,j}(G, w) \rightarrow H_{i-1,j}(G/e, w/e) \rightarrow \cdots, \]

it follows that $H_{i,j}(G, w) = 0$. \qed

In [2], the authors left open also the following

**Conjecture (C.5).** Given any graph $G$, chromatic symmetric homology groups $H_{i,0}(G; \mathbb{C})$ are non-trivial for all $0 \leq i \leq \text{span}_0(G) - 1$, $j \geq 0$.

Using the deletion-contraction long exact sequence, we can prove the following
Theorem 37. Let \((G, w)\) be a graph. Then \(H_{i,j}(G, w; \mathbb{C})\) is non-trivial for all \(k_{\min}^j(G, w) \leq i \leq k_{\max}^j(G, w)\), \(j \geq 0\).

Since \(k_{\min}^0(G, w)\) is always 0, Theorem\textsuperscript{37} shows in particular that Conjecture C.5 is true.

Proof. We proceed by induction on the number \(m \geq 0\) of edges of \(G\). If \(m = 0\), we have that the homology \(H_{i,j}(G, w)\) is trivial for all \(i > 0\), since we don’t have any states with more than zero edges. Therefore, the result holds.

Now assume the statement true for any graph with \(m-1\) edges.

If \(k_{\max}^j(G \setminus e, w) \geq k_{\max}^j(G, w)\), since \(G \setminus e\) has \(m-1\) edges, by inductive hypothesis, we have that \(H_{k_{\max}^j(G, w), j}^j(G \setminus e, w) \neq 0\). If \(H_{k_{\max}^j(G, w), j}^j(G/e, w/e) = 0\), then by inductive hypothesis, it is also \(H_{k_{\max}^j(G, w), j}^j(G/e, w/e) = 0\). Therefore, by the deletion-contraction long exact sequence\textsuperscript{4}

\[
\ldots \rightarrow H_{k_{\max}^j(G, w), j}^j(G/e, w/e) \rightarrow H_{k_{\max}^j(G, w), j}^j(G \setminus e, w) \rightarrow H_{k_{\max}^j(G, w), j}^j(G, w) \rightarrow \ldots,
\]

we have \(H_{k_{\max}^j(G, w), j}^j(G \setminus e, w) \cong H_{k_{\max}^j(G, w), j}^j(G, w)\).

Otherwise, \(H_{k_{\max}^j(G, w), j}^j(G/e, w/e) \neq 0\), so \(k_{\max}^j(G/e, w/e) \geq k_{\max}^j(G, w) - 1\).

If instead \(k_{\max}^j(G \setminus e, w) < k_{\max}^j(G, w)\), we have \(H_{k_{\max}^j(G, w), j}^j(G \setminus e, w) = 0\) and \(H_{k_{\max}^j(G, w), j}^j(G, w) \neq 0\). Therefore, by the deletion-contraction long exact sequence\textsuperscript{4}

\[
\ldots \rightarrow H_{k_{\max}^j(G, w), j}^j(G \setminus e, w) \rightarrow H_{k_{\max}^j(G, w), j}^j(G, w) \rightarrow H_{k_{\max}^j(G, w), j}^j(G/e, w/e) \rightarrow \ldots,
\]

we have that the map from \(H_{k_{\max}^j(G, w), j}^j(G, w)\) to \(H_{k_{\max}^j(G, w), j}^j(G/e, w/e)\) is injective. Hence, \(H_{k_{\max}^j(G, w), j}^j(G, w)\) is isomorphic to the image of this map, which is a non-trivial submodule of \(H_{k_{\max}^j(G, w), j}^j(G/e, w/e)\). It follows that

\[
H_{k_{\max}^j(G, w), j}^j(G/e, w/e) \neq 0 \text{ and } k_{\max}^j(G/e, w/e) \geq k_{\max}^j(G, w) - 1.
\]

Now assume \(k_{\min}^j(G, w) \leq i \leq k_{\max}^j(G, w)\) and prove that \(H_{i,j}(G, w)\) is non-trivial. As observed above, we have three cases to consider:

(i) \(k_{\max}^j(G \setminus e, w) \geq k_{\max}^j(G, w)\) and \(H_{k_{\max}^j(G, w), j}^j(G \setminus e, w) \cong H_{k_{\max}^j(G, w), j}^j(G, w)\);

(ii) \(k_{\max}^j(G \setminus e, w) \geq k_{\max}^j(G, w)\) and \(k_{\max}^j(G/e, w/e) \geq k_{\max}^j(G, w) - 1\);
(iii) \( k^i_{\text{max}}(G \setminus e, w) < k^i_{\text{max}}(G, w) \) and \( k^i_{\text{max}}(G / e, w / e) \geq k^i_{\text{max}}(G, w) - 1 \).

In case (i), \( k^i_{\text{max}}(G \setminus e, w) \geq k^i_{\text{max}}(G, w) \) and \( H^j_{k^i_{\text{max}}(G, w)}(G \setminus e) \cong H^j_{k^i_{\text{max}}(G, w)}(G, w) \), so by inductive hypothesis we have that \( H_{i,j}(G \setminus e, w) \) is non-trivial. It follows from \( \ref{4} \) and for how the maps are defined, that also \( H_{i,j}(G, w) \) is non-trivial.

In case (ii), if \( k^i_{\text{max}}(G \setminus e, w) \geq k^i_{\text{max}}(G, w) \) and \( k^i_{\text{max}}(G / e, w / e) \geq k^i_{\text{max}}(G, w) - 1 \), then \( i - 1 \leq k^i_{\text{max}}(G, w) - 1 \leq k^i_{\text{max}}(G / e, w / e) \). Therefore, by induction, \( H_{i-1,j}(G / e) \) is non-trivial. Moreover, by induction, also \( H_{i,j}(G \setminus e, w) \) is non-trivial. It follows from \( \ref{4} \) and for how the maps are defined, that also \( H_{i,j}(G, w) \) is non-trivial.

Finally, we consider the case (iii) with \( k^i_{\text{max}}(G \setminus e, w) < k^i_{\text{max}}(G, w) \). We just have to see what happens if \( k^i_{\text{max}}(G \setminus e, w) < i \leq k^i_{\text{max}}(G, w) \), since, if \( i \leq k^i_{\text{max}}(G \setminus e, w) < k^i_{\text{max}}(G, w) \), as in the previous case, both \( H_{i-1,j}(G / e, w / e) \) and \( H_{i,j}(G \setminus e, w) \) are non-trivial, and so it is \( H_{i,j}(G, w) \neq 0 \). If \( k^i_{\text{max}}(G \setminus e, w) < i \leq k^i_{\text{max}}(G, w) \), we have that \( H_{i,j}(G \setminus e, w) = 0 \). From the deletion-contraction long exact sequence \( \ref{4} \),

\[
\cdots \rightarrow H_{i,j}(G \setminus e, w) \rightarrow H_{i,j}(G, w) \rightarrow H_{i-1,j}(G / e, w / e) \rightarrow \ldots,
\]

it follows that the map from \( H_{i,j}(G, w) \) to \( H_{i-1,j}(G / e, w) \) is injective. Moreover, since \( i - 1 \leq k^i_{\text{max}}(G, w) - 1 \leq k^i_{\text{max}}(G / e, w / e) \), as proved above, by induction, \( H_{i-1,j}(G / e, w / e) \) is non-trivial. Hence, for how the maps are defined, \( H_{i,j}(G, w) \) is non-trivial. \( \square \)

### 3.1 Future directions

Chandler, Sazdanovic, Stella and Yip in \[2\] investigated the properties of chromatic symmetric homology with integer coefficients. They conjectured that a graph \( G \) is non-planar if and only if its chromatic symmetric homology in bidegree \((1,0)\) contains \( \mathbb{Z}_2 \)-torsion. In \[3\], the authors showed that the chromatic symmetric homology of a finite non-planar graph contains \( \mathbb{Z}_2 \)-torsion in bidegree \((1,0)\). We hope that these new tools will help to understand if this conjecture is true also in the other direction.

Moreover, we think that the deletion-contraction long exact sequence could simplify the computation of the homology, even in the unweighted case, and allow to study it better.
Acknowledgments

I thank Salvatore Stella and Luca Moci for suggesting me to work on this topic and for many helpful conversations about it; without them this paper would not have been possible. I thank Alex Chandler for reading the article and for his valuable advise. Finally, I am grateful to the reviewers for their precise and useful comments.

References

[1] Dror Bar-Natan. “On Khovanov’s categorification of the Jones polynomial”. In: Algebr. Geom. Topol. 2 (2002), pp. 337–370. ISSN: 1472-2747. DOI: 10.2140/agt.2002.2.337 URL: https://doi.org/10.2140/agt.2002.2.337.

[2] Alex Chandler, Radmila Sazdanovic, Salvatore Stella, and Martha Yip. On the Strength of Chromatic Symmetric Homology for graphs. 2019. DOI: 10.48550/ARXIV.1911.13297 URL: https://arxiv.org/abs/1911.13297.

[3] Azzurra Ciliberti and Luca Moci. “On Chromatic Symmetric Homology and Planarity of Graphs”. In: Electron. J. Combin. 30.1 (2023), Paper No. 1.15. DOI: 10.37236/11397 URL: https://doi.org/10.37236/11397.

[4] Logan Crew and Sophie Spirkl. “A deletion-contraction relation for the chromatic symmetric function”. In: European J. Combin. 89 (2020), pp. 103143, 20. ISSN: 0195-6698. DOI: 10.1016/j.ejc.2020.103143 URL: https://doi.org/10.1016/j.ejc.2020.103143.

[5] William Fulton. Young tableaux. Vol. 35. London Mathematical Society Student Texts. With applications to representation theory and geometry. Cambridge University Press, Cambridge, 1997, pp. x+260. ISBN: 0-521-56144-2; 0-521-56724-6.

[6] Radmila Sazdanovic and Martha Yip. “A categorification of the chromatic symmetric function”. In: J. Combin. Theory Ser. A 154 (2018), pp. 218–246. ISSN: 0097-3165. DOI: 10.1016/j.jcta.2017.08.014 URL: https://doi.org/10.1016/j.jcta.2017.08.014.

[7] Richard P. Stanley. “A symmetric function generalization of the chromatic polynomial of a graph”. In: Adv. Math. 111.1 (1995), pp. 166–194. ISSN: 0001-8708. DOI: 10.1006/aima.1995.1020 URL: https://doi.org/10.1006/aima.1995.1020.