The universal envelope of the topological closed string BRST-complex

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Abstract

We construct a universal envelope for any Poisson- and Gerstenhaber algebra. While the deformation theory of Poisson algebras seems to be partially trivial, results from string and M-theory suggest a rich deformation theory of Gerstenhaber algebras. We apply our construction in this case to well known questions on the topological closed string BRST-complex. Finally, we find a similar algebraic structure, as for the universal envelope, in the $SU(2)$-WZW model.

1 Introduction

The BRST-complex of the topological closed string is known to have the structure of a differential Gerstenhaber algebra (see [HM] and the literature cited therein). In the first part (section 2) of this paper, we show how to
construct a universal envelope for any Poisson and Gerstenhaber algebra. The result is a trialgebra in the sense of [CT] and a graded version thereof (which we will call an odd super-trialgebra for reasons to be discussed below), respectively. This immediately implies the existence of a universal envelope for the topological closed string BRST-complex. Moreover it follows that the deformation theory of the odd-super trialgebra corresponds to the third bicomplex (the Gerstenhaber complex) of [HM] if the coproduct of the odd super-trialgebra is kept fixed. Using recent results of [LMN], we derive an argument from this suggesting that the conjecture of [HM] on the physical interpretation of the Gerstenhaber complex as relating to (1, 1) little string theory on the NS5 brane in type IIB string theory should reduce in a dual description on the $N = 2$ supersymmetric sine-Gordon model coupled to topological gravity to a categorified version of the Kazhdan-Lusztig theorem on representation theory of affine Lie algebras.

In section 3, we discuss several examples of Poisson algebras. We show that in all these cases the deformation theory of the Poisson algebra reduces to that of the underlying ordinary associative algebra or is trivial. One is always forced to consider the bracket not as independent algebraic structure but as the first order deformation of the product. This is the usual interpretation of Poisson brackets in deformation theory, of course, but for the abstract notion of Poisson algebra this gives a severe restriction on the deformation theory. We conjecture that such a property should generally hold true for Poisson algebras for mathematics, as well, as physics motivated reasons. This implies that the deformation theory of trialgebras, arising as universal envelopes of Poisson algebras, is heavily restricted. In contrast, the deformation theory of Gerstenhaber algebras implies that odd super-trialgebras have a rich deformation theory, i.e. superextensions seem to be a decisive ingredient for trialgebraic structures. This suggests that four dimensional topological field theories might only lead to nontrivial invariants of 4-manifolds if they possess in this sense a kind of supersymmetry (We would like to stress that we only refer to these special graded algebraic structures, here. Obviously, it does not make sense to speculate of supersymmetry in the usual sense, referring e.g. to a Lagrangian formulation, in this context, since topological field theories do, in general, not even satisfy the spin-statistics theorem).

In section 4, we consider strings moving on the $S^3$ in the transversal geometry of a flat NS5 brane which are described by the $SU(2)$-WZW model. We find that also in the $SU(2)$-WZW model an odd super-trialgebra ap-
pears, defined by the bicovariant differential calculus on the $q$-deformed fuzzy sphere.

Section 5 contains some concluding remarks.

2 The universal envelope of Poisson- and Gerstenhaber algebras

Recall that a Poisson algebra is defined as a commutative, associative algebra $(A, \cdot)$ equipped with a Lie bracket $[,]$ such that for all $a, b, c \in A$

$$[a, b \cdot c] = [a, b] \cdot c + b \cdot [a, c]$$

Similarly, a Gerstenhaber algebra is defined as a $\mathbb{Z}_2$-graded associative algebra $(A, \cdot)$ with graded commutative even product $\cdot$ and an odd Lie bracket $[,]$ such that for all $a, b, c \in A$

$$[a, b \cdot c] = [a, b] \cdot c + (-1)^{(|a|-1)|b|} b \cdot [a, c]$$

Remark 1 Observe that in contrast to the case of super-Lie algebras or graded Poisson algebra, the Lie bracket of a Gerstenhaber algebra is odd with respect to the grading and, correspondingly, the derivation property is odd from the sign rule.

Definition 1 A trialgebra $(A, *, \Delta, \cdot)$ with $*$ and $\cdot$ associative products on a vector space $A$ and $\Delta$ a coassociative coproduct on $A$ is given if both $(A, *, \Delta)$ and $(A, \cdot, \Delta)$ are bialgebras and the following compatibility condition between the products is satisfied for arbitrary elements $a, b, c, d \in A$:

$$(a * b) \cdot (c * d) = (a \cdot c) * (b \cdot d)$$

Trialgebras were first suggested in [CF] as an algebraic means for the construction of four dimensional topological quantum field theories. It was observed there that the representation categories of trialgebras have the structure of so called Hopf algebra categories (see [CF]) and it was later shown
explicitly in [CKS] that from the data of a Hopf category one can, indeed, construct a four dimensional topological quantum field theory. The first explicit examples of trialgebras were constructed in [GS 2000a] and [GS 2000b] by applying deformation theory, once again, to the function algebra on the Manin plane and some of the classical examples of quantum algebras and function algebras on quantum groups. In [GS 2001] it was shown that one of the trialgebras constructed in this way appears as a symmetry of a two dimensional spin system. Besides this, the same trialgebra can also be found as a symmetry of a certain system of infinitely many coupled $q$-deformed harmonic oscillators.

With the notion of a trialgebra at hand, we can now formulate a concept of a universal envelope of a Poisson algebra:

Let $(A, \cdot, [\cdot, \cdot])$ be a Poisson algebra. Let $\hat{U}(A)$ be the universal envelope of the Lie algebra $(A, [\cdot, \cdot])$ completed with respect to the inclusion of formal power series. We can obviously extend the product $\cdot$ from $A$ to the tensor algebra over $A$ (and to a completion of the tensor algebra by formal power series) by

\[ (a \otimes b) \cdot (c \otimes d) = (a \cdot c) \otimes (b \cdot d) \]  \hspace{1cm} (1)

for all $a, b, c, d \in A$.

**Lemma 1** The product defined by (1) induces a product on $\hat{U}(A)$ and together with the commutative Hopf algebra structure on $\hat{U}(A)$ this product gives the structure of a trialgebra to $\hat{U}_G(A)$ where we define $\hat{U}_G(A)$ to be given as the subalgebra generated by the group-like elements in $\hat{U}(A)$.

**Proof.** Recall that a group-like element in $\hat{U}(A)$ is defined by the property

\[ \Delta (a) = a \otimes a \]

The relations of a trialgebra are checked by calculation. Essential is the fact that the quotient relations of $\hat{U}(A)$ do involve three elements of $A$ at once which means that they do not interfere with the compatibility relation (1) of a trialgebra.

Finally, one has to check that the derivation property of a Poisson algebra does not interfere with the structure of a trialgebra. For this, observe that upon identifying

\[ c \leftrightarrow c \cdot 1 + 1 \cdot c \]

\hspace{1cm} (2)
and
\[ c \otimes 1 \leftrightarrow 1 \otimes c \leftrightarrow c \]
for all \( c \in A \), the derivation property can be derived from the compatibility relations of the trialgebra. Namely,
\[
[a \cdot b, c] = [a \cdot b, c \cdot 1 + 1 \cdot c]
\]
\[
= (a \cdot b) \otimes (c \cdot 1 + 1 \cdot c) - (c \cdot 1 + 1 \cdot c) \otimes (a \cdot b)
\]
\[
= (a \otimes c) \cdot (b \otimes 1) - (c \otimes a) \cdot (1 \otimes b)
\]
\[
+ (a \otimes 1) \cdot (b \otimes c) - (1 \otimes a) \cdot (c \otimes b)
\]
\[
= [a, c] \cdot b + a \cdot [b, c]
\]
where \( \otimes \) denotes the product in \( \hat{U}(A) \). Observe that the above identification does not interfere with the fact that nontrivial trialgebras are always non-unital (see [GS 2000b]) since we have not used a normalization in (2).

This concludes the proof. \( \blacksquare \)

In a completely similar way, we get a super-trialgebra \( \hat{U}_G(A) \) from a graded Poisson algebra \((A, \cdot, [\cdot, \cdot])\) and an odd super-trialgebra from a Gerstenhaber algebra where super-trialgebras and odd super-trialgebras have the obvious gradings. By construction of the universal envelope, an odd super-trialgebra has one of the two products of odd type with respect to the grading and the compatibility relation of the products obeys an odd sign rule. In conclusion, we have the following result:

**Lemma 2** The universal envelope \( \hat{U}_G(A) \) of a Gerstenhaber algebra \((A, \cdot, [\cdot, \cdot])\) has the structure of an odd super-trialgebra.

Here, as we will do in the sequel, we have called \( \hat{U}_G(A) \) the universal envelope of the Gerstenhaber (or Poisson-) algebra.

The BRST-complex of the topological closed string has the structure of a differential Gerstenhaber algebra, i.e. a Gerstenhaber algebra with a differential \( d \) obeying
\[
d^2 = 0
\]
(strictly speaking, only the cohomology has the structure of a differential Gerstenhaber algebra while the complex is only a homotopy Gerstenhaber
algebra but in a first approach we will neglect BRST exact terms in this paper which means that we can treat the comlex as a differential Gerstenhaber algebra, too). Lemma 2 immediately implies that the universal envelope of the BRST-complex of the topological closed string has the structure of an odd super-trialgebra with differential.

In [HM] it was shown that the deformation theory of the BRST-complex of the topological closed string is described by three different bicomplexes: First, there is the complex of \( A_\infty \)-deformations (\( d \) and \( \cdot \) are deformed but \( [,] \) is kept fixed) which is generated by deforming the closed string correlation functions by closed string bulk operators. This is the complex described by the WDVV-equations. Next, one has the complex of \( L_\infty \)-deformations (\( d \) and \( [,] \) are deformed but \( \cdot \) is kept fixed) which in physics corresponds to deformations by open membrane operators (the closed string viewed as sitting on the boundary of the open membrane). Finally, there is the Gerstenhaber complex (\( d \) is fixed but both \( \cdot \) and \( [,] \) are deformed) which is conjectured to correspond to \((1,1)\) little string theory, i.e. strings attached to an NS5 brane background in type IIB superstring theory (see [HM] for the details).

Using the results of [HM], Lemma 2 immediately has the following corollary, then:

**Corollary 3** Let \( (A, \cdot, [,]) \) be the Gerstenhaber algebra of a topological closed string BRST-complex. The deformation theory described by the Gerstenhaber complex of [HM] is completely equivalent to the deformation theory of the odd super-trialgebra \( \hat{U}_G(A) \) with the coproduct \( \Delta \) of \( \hat{U}_G(A) \) kept fixed.

Observe that by construction one can - as in the case of Lie algebras - reconstruct the Gerstenhaber- or the Poisson algebra from the given universal envelope. Especially, we really get a 1-1 correspondence of the deformation theories in the above corollary.

**Remark 2** The corollary shows that one can generalize the deformation theory of the Gerstenhaber algebra to the full deformation theory of the odd super-trialgebra by including deformations of the coproduct \( \Delta \). Suppose the conjecture of [HM] on a correspondence of the Gerstenhaber complex to \((1,1)\) little string theory on an NS5 brane in type IIB string theory would hold true.
Then, the full deformation theory of the universal envelope \( \hat{U}_G(A) \) should correspond to a noncommutative deformation of \((1,1)\) little string theory since the odd Lie-structure of the BRST-complex is replaced by a quantum algebra in such a deformation. It remains a task for future work to check if this produces exactly the noncommutative deformations of little string theory which are known to appear in the presence of a stack of NS5-branes (see [GMSS] and [Har]).

**Remark 3** The above corollary means that the deformation theory of Gerstenhaber algebras, enlarged as just pointed out to the full deformation theory of \( \hat{U}_G(A) \), should have strong stability properties corresponding to ultrarigidity which holds for odd super-trialgebras completely analogous to the case of trialgebras considered in [Sch].

We conclude this section by pointing out a possible application of the universal envelope construction of a Gerstenhaber algebra: It suggests that one can approach the proof of the conjecture of [HM] on a relation of the Gerstenhaber complex to \((1,1)\) little string theory in a representation theoretic way, at least in a special case.

Consider the NS5-brane configuration in type IIB string theory as used in [LMN] where the \((1,1)\) little string theory is suggested to correspond by S-duality to instantons on the S-dual D5. It was shown in [LMN] that these can be related by another duality transformation to the A-model on \( \mathbb{CP}^1 \) with gravitational descendants taken into account. It was further shown in [EHY] that the A-model on \( \mathbb{CP}^1 \) can - including the case of coupling to topological gravity - be related by mirror symmetry to a B-model, i.e. we can add another segment to the chain of dualities starting from the \((1,1)\) little string theory on the NS5 in type IIB. Concretely, the B-model is given as a Landau-Ginzburg model with potential given by an \( N = 2 \) supersymmetric sine-Gordon model.

For simplicity, let us consider the usual (bosonic) sine-Gordon model and dispense of the coupling to topological gravity. It has been known for a long time that the fusion ring of the sine-Gordon model cannot - in contrast to the case of two dimensional conformal and three dimensional topological field theories - be generated by the representation theory of Hopf algebra deformation of a compact group (see [FK], [LR]). Indeed, it can be shown that the hidden symmetry of the sine-Gordon model is given by a quantum
deformation of the affine Lie algebra $\hat{sl}_2$ with the restricted sine-Gordon model being related to the quantum deformation of the Virasoro algebra of [FR] (see [FL]). For the family of restricted sine-Gordon models it was shown (see [FL]) that the conformal limit is given by the minimal unitary series and that the deformation from the minimal model to the corresponding restricted sine-Gordon model (leading to the deformation of the Virasoro algebra) is generated by the $\phi(1,3)$ field of the minimal model, i.e. it corresponds to the deformation

$$S \mapsto S + \frac{\lambda}{2\pi} \int d^2z \, \phi(1,3) (z, \bar{z})$$

of the action.

Now, by definition, a deformation as in (3) is of WDVV-type, i.e. if the conjecture of [HM] holds true it can not be traced back along the above chain of dualities to the same complex on the (1, 1) little string theory side (because, there, the Gerstenhaber complex is expected to appear). In contrast, the fact that the deformations arising from turning on the background fields in (1, 1) little string theory can definitely not be described as WDVV-type deformations shows that the three bicomplexes introduced in [HM] are not kept separate from each other under dualities. Trying to prove the conjecture of [HM] by applying the above chain of dualities leads, especially, to the following question, then: Can we find a nontrivial odd-supertrialgebra relating to the sine-Gordon model?

Remember that by Kazhdan-Lusztig theory the representation theory of an affine Lie-algebra corresponds to the deformation theory of a compact quantum group. One might suspect that the same holds true one level of deformations higher: The representation theory of a quantum affine algebra should correspond to the representation theory of a compact trialgebra (trialgebraic deformation of a compact quantum group). We therefore conjecture that it should be possible to prove the conjecture of [HM] by proving that the fusion structure of the sine-Gordon model corresponds to the representation theory of an odd-super trialgebra and that this result persists to hold true for the $N = 2$ sine-Gordon model coupled to topological gravity.

We expect the appearance of an odd super-trialgebra - instead of a usual bosonic trialgebra - even for the case of the simple (bosonic) sine-Gordon model because of the enlarged set of generators appearing for the $q$-deformed Virasoro algebra (see [FR]).

The following lemma provides support for our above conjecture:
Lemma 4  On the cohomology of any affine Hopf algebra there is up to homo-


topy an action of the Hopf algebra \( H_{GT} \) of \([Sch]\).

Proof.  The analogous action of the Grothendieck-Teichmüller group \( GT \) on Hochschild cohomology is proved in \([KS]\). Using this result, ignoring compatibility of the product and coproduct of a Hopf algebra, we would get an action of \( GT \) on the cohomology of the product and a coaction of the Hopf algebra dual of \( GT \) on the cohomology of the coproduct. Extending the action of \( GT \) by linearity to one of its group algebra, we arrive at an action of the Drinfeld double \( D(GT) \) of \( GT \) on the complete cohomology (involving product and coproduct). One checks by calculation that imposing the compatibility between product and coproduct still respects the defining equations of a sub-Hopf algebra. So, we arrive at a sub-Hopf algebra \( \mathcal{H} \) as acting on the cohomology of affine Hopf algebras. By direct calculation one checks also that

\[
\mathcal{H} \simeq \mathcal{H}_{GT}
\]

This concludes the proof. \( \blacksquare \)

Remark 4  As we see from the proof it holds in greater generality than for affine Hopf algebras, only. We have restricted to the affine setting because e.g. for the classical quantum groups the action of \( \mathcal{H}_{GT} \) trivializes in part because it is always possible up to isomorphism to assume that either only deformations of the product or only deformations of the coproduct occur (see e.g. \([CP]\)).

In section 4, we will give an additional argument in support of the conjecture of \([HM]\) which provides another application of our universal envelope construction for Gerstenhaber algebras.

3  Deformations of Poisson algebras

Obviously, one can introduce the three bicomplexes introduced in \([HM]\) for a differential Gerstenhaber algebra also for a differential Poisson algebra. Let us consider the case of a pure Poisson algebra, i.e. we have a trivial differential which is not deformed. Then the three complexes correspond to:
1. Deforming \( \cdot \) while \([,]\) is kept fixed.

2. Deforming \([,]\) while \(\cdot\) is kept fixed.

3. Deforming both \(\cdot\) and \([,]\).

We will consider the third possibility in a number of examples, now. We will discover that in all of these examples the third possibility is never realized and even more severe restrictions occur. In this section, we will sometimes omit to explicitly write the symbol \(\cdot\) (as usual for a product).

**Example 1** Consider the polynomial algebra over the Euclidean plane, i.e. the real associative unital algebra with generators \(x, y\) and relation

\[ xy = yx \]

We can introduce the usual Poisson bracket on this algebra, i.e. we have

\[ \{x, y\} = 1 \]

It is straightforward to check that we get a Poisson algebra in this way. Assume we have a deformation of the algebra with commutator

\[ [x, y] = xy - yx = \lambda \in \mathbb{R} \] (4)

and

\[ \{x, y\} = 1 + \mu xy, \ \mu \in \mathbb{R} \] (5)

Observe that since the monomials \(x^n y^m\), \(n, m \in \mathbb{N}\) give a basis of the algebra, any deformation of \(\{,\}\) has to satisfy

\[ \{x, y\} = \sum_{n,m} a_{nm} x^n y^m \]

By bilinearity of \(\{,\}\) - up to the constant non-deformed term - only (5) remains, then. The requirement that the deformation should constitute a Poisson algebra, again, leads to

\[ \{xy, y\} = x \{y, y\} + \{x, y\} y \]

\[ = \{x, y\} y \]
and

\[
\{xy, y\} = \{yx + \lambda, y\} = \{yx, y\} + \lambda \{1, y\} = \{yx, y\} = y \{x, y\}
\]
i.e.

\[
\{x, y\} y = y \{x, y\}
\]

Using (5), we get

\[
\mu xy^2 = \mu yxy = \mu (xy - \lambda) y
\]
i.e.

\[
\mu \lambda = 0
\]

So, either \(\lambda = 0\) or \(\mu = 0\) and possibility 3.) in the deformation theory of a Poisson algebra does not occur. In consequence, for a noncommutative space of type (4) (i.e. \(\lambda \neq 0\)) only the WDVV-like deformations (in the case of a Gerstenhaber algebra, they correspond to the WDVV-deformations) with fixed Poisson bracket are possible. Especially, this means that, up to multiplication by the constant \(\lambda\), the Poisson bracket equals the commutator of the noncommutative product (4). This agrees with the usual interpretation of the Poisson bracket in deformation theory where it gives the first order contribution for the deformation of the product. So, starting from the more general notion of a Poisson algebra, we are automatically forced to this interpretation of the Poisson bracket. We will see that this holds true in all the examples to follow.

**Example 2** Consider, again, the Poisson algebra on the Euclidean plane. We use (4) for the deformation of the bracket but instead of the Heisenberg type deformation (4) for the product, we use a Lie algebra type deformation (as one does in physics e.g. on the fuzzy sphere):

\[
[x, y] = \lambda_1 x + \lambda_2 y, \quad \lambda_1, \lambda_2 \in \mathbb{R}
\]  

(6)
Using the derivation property for

\[ \{xy, y\} \]

and relation (6), we now get

\[ \{x, y\} y = y \{x, y\} + \lambda_1 \{x, y\} \]

and using (5)

\[ y + \mu xy^2 = y + \mu yxy + \lambda_1 + \lambda_1 \mu xy \]

i.e.

\[ 0 = \mu (\lambda_1 x + \lambda_2 y) y + \lambda_1 + \lambda_1 \mu xy \]

Using the basis property of the monomials, we get

\[ \lambda_1 = 0 \]

and

\[ \mu \lambda_2 = 0 \]

A similar calculation with \( x, y \) interchanged leads to

\[ \lambda_2 = 0 \]

So, there is no Lie algebra type deformation of \( \cdot \) which is compatible with a continuous deformation of \( \{,\} \) or even with the undeformed bracket. Again, we would be forced to choose the Poisson bracket as the first order deformation of the deformed product to achieve compatibility and can not consider \( \{,\} \) as an independent algebraic structure.

**Example 3** Again, we take the Poisson algebra on the Euclidean plane and (5). For the deformation of the product we now take the \( q \)-deformation type

\[ xy = qyx, \quad q \in \mathbb{R} \]  \hspace{1cm} (7)

Applying the derivation property and (7) to

\[ \{xy, y\} \]

now leads to

\[ \{x, y\} y = qy \{x, y\} \]
Inserting (5), we get

\[ y + \mu xy^2 = qy + q\mu xy \]
\[ = qy + \mu xy^2 \]

i.e.

\[ q = 1 \]

So, we reach the same conclusion as in the second example.

One might ask if the impossibility to deform the full Poisson algebra structure and the fact that one is restricted to considering the Poisson bracket as the first order part of the deformation of the product is an artefact of two dimensions or might possibly occur only for algebras on finite dimensional manifolds since there the formality theorem of [Kon] applies. To show that the same effect occurs in the infinite dimensional case, we consider another example (which is prototypical in physics).

**Example 4** Consider the infinite dimensional algebra with generators \( a_n, \ n \in \mathbb{Z} \) and relations

\[ a_n a_m = a_m a_n \]  
(8)

with the bracket

\[ \{a_n, a_m\} = i n \delta_{n+m,0} \]  
(9)

This is the well known algebra of Fourier coefficients for the solutions of the wave equation with pointwise product [8] and the usual Poisson bracket [9]. Let us consider the following deformation of (8) and (9) to

\[ [a_n, a_m] = a_n a_m - a_m a_n = \Theta_{nm} \in \mathbb{C} \]  
(10)

and

\[ \{a_n, a_m\} = i n \delta_{n+m,0} + \sum_k \lambda_{nm}^k a_k \]  
(11)

with \( \lambda_{nm}^k \in \mathbb{C} \). Using (10) and the derivation property for

\[ \{a_l a_n, a_m\} \]
we get
\[ \sum_k \lambda_{nm}^k \Theta_{kl} + \sum_k \lambda_{lm}^k \Theta_{nk} = 0 \]

Choose \( n = m \). Then
\[ \sum_k \lambda_{ln}^k \Theta_{kn} = 0 \quad (12) \]

since the \( \lambda_{nm}^k \) are antisymmetric in the lower indices. Obviously, if (12) has a solution it can be chosen to be independent of \( l \). But then for all \( l, n, k \)
\[ \lambda_{ln}^k = \lambda_{nn}^k = 0 \]

and we arrive at the same conclusion as in the previous examples.

We conjecture that the phenomenon observed above should hold generally for Poisson algebras: Poisson brackets should only be interpretable as first order contributions to the deformation of the product and there should be no nontrivial deformation theory of Poisson algebras in the sense of the third complex above. Besides the above examples, we have the following two general motivating arguments in support of this conjecture:

- In the hierarchy of little disc operads (see [HM], [KS]) associative algebras appear as 1-algebras, Gerstenhaber algebras as \( 2n \)-algebras for \( n \geq 1 \), and Poisson algebras as \( (2n + 1) \)-algebras for \( n \geq 1 \). The deformation theory of associative algebras was shown to be described by a string theory ([CaFe], [Kon]) while the deformation theory of a Gerstenhaber algebra is described by \( M \)-theory and string theory with background fields turned on ([HM]). The expectation that \( M \)-theory is complete as a physical theory and needs no further deformation is fully in accordance with the conjecture that the deformation theory of Poisson algebras just reduces to the simpler deformation theory of associative algebras.

- One could embed the hierarchy of \( n \)-algebras into a categorical hierarchy of higher algebras (using a universal envelope construction of the type given above), passing from algebras to bialgebras, to trialgebras, to quadralgebras (with two associative products and two coassociative coproducts, all pairwise compatible), etc.. In this context, the triviality
of the deformation theory of Poisson algebras should be a consequence of ultrarigidity (there are no nontrivial deformations of trialgebras into quadraalgebras, see [Sch]).

If our above conjecture holds true, there is a decisive difference between the highly restricted deformation theory of Poisson algebras and the graded case with the rich deformation theory of Gerstenhaber algebras. Considering trialgebraic structures arising as universal envelopes of Poisson- and Gerstenhaber algebras, this means the following: For trialgebras arising as universal envelopes of Poisson algebras we should not expect a deformation theory which goes much beyond the theory of bialgebras and Hopf algebras. For odd super-trialgebras, on the other hand, we expect a rich deformation theory of deep relevance for physics (connected to string- and $M$-theory). In other words: The super-extension might be essential for trialgebras.

In [CF] trialgebras have been introduced with the aim to construct four dimensional topological invariants. Up to now, only very simple invariants have been constructed in this way. There might be a lesson to draw from our above conjecture for four dimensional topological quantum field theory: Maybe one has to pass to odd super-trialgebras to get nontrivial invariants. In this sense, supersymmetry might be a necessary ingredient for four dimensional topological quantum field theory.

4 A trialgebra in the WZW-model

The deformation theory and moduli of $(1, 1)$ little string theory is a hardly accessible topic. This is one reason why a direct approach to the conjecture of [HM] seems to be very difficult. On the other hand, the behaviour of strings in the transversal geometry of an NS5 brane has a much more accessible description in terms of a rational conformal field theory ([CHS], [Rev]). One can therefore try to find support for the conjecture of [HM] by trying to find the algebraic properties of the Gerstenhaber complex in string theory on the transversal geometry. Using our results on the universal envelope from section 2, we can more concretely phrase the following question: Can we find an odd super-trialgebra in the string theory on the transversal geometry? We will see in this section that the answer is in the affirmative.
For a flat NS5 brane, the background is completely determined by (see e.g. [BS]) vanishing R-R fields and

\[ ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + e^{-2\phi} \left( dr^2 + r^2 ds_3^2 \right) \]

\[ e^{-2\phi} = e^{-2\phi_0} \left( 1 + \frac{k}{r^2} \right) \]

\[ H = dB = -kd\Omega_3 \]

where \( \mu, \nu = 0, 1, \ldots, 5 \) are directions tangent to the NS5 brane. Here, \( ds_3 \) and \( d\Omega_3 \) denote the line element and volume form, respectively, on the \( S^3 \). So, in the transversal geometry of the flat NS5 there is always contained an \( S^3 \). Studying strings on the transversal geometry, we will restrict to strings moving only on this \( S^3 \) which are described by the \( SU(2) \)-WZW model. Open strings and \( D \)-branes in the \( SU(2) \)-WZW model have been studied in detail in [ARS] and the \( D \)-brane world volume geometry was found there to correspond to the \( q \)-deformed fuzzy sphere of [GMS].

Consider now the bicovariant differential calculus on the \( q \)-deformed fuzzy sphere (see [GMS]; [KlSch] for a general introduction to bicovariant differential calculi on quantum groups). We have the following lemma, then:

**Lemma 5** The bicovariant differential calculus on a quantum group defines the structure of an odd super-trialgebra.

**Proof.** Use the fact that the bicovariant differential calculus has the structure of a super Hopf algebra with the product given by the tensor product (see [KlSch]). In addition, the fact that it is a bimodule over the quantum group algebra (see [KlSch], again) gives an additional realization of the quantum group product. Using bicovariance one checks the compatibility relations of a trialgebra by direct calculation. The fact that it is an odd super-trialgebra derives from the super-Hopf algebra structure of the bicovariant differential calculus and the behaviour of the quantum group product with respect to the grading. 

**Remark 5** Since the structure a Gerstenhaber algebra is closely related to BRST-cohomology, this result suggests viewing the bicovariant differential calculus on the \( q \)-deformed fuzzy sphere as a way to deform the BRST-complex along with the deformation of the algebra of functions (from the smooth functions on the sphere to the algebra of functions on the \( q \)-deformed fuzzy sphere).
In conclusion, we find an odd super-trialgebra related to the string theory on the $S^3$ in the transversal geometry of the NS5 brane. This gives additional support to the conjecture of [HM].

5 Conclusion

We have given a universal envelope construction for any Poisson- and Gerstenhaber algebra. By studying several different examples we have motivated the conjecture that the deformation theory of Poisson algebras should basically be those of associative algebras with the Poisson bracket giving the first order deformations, i.e. there should be no nontrivial deformation theory of Poisson algebras as an abstract algebraic structure. This is in accordance with general properties of the universal envelope (ultrarigidity). In contrast, in the case of Gerstenhaber algebras one expects from results in string- and $M$-theory a rich deformation theory. We have applied our construction of the universal envelope in this case to motivate a conjecture on the possibility of a representation theoretic proof of a conjecture of [HM] on $(1,1)$ little string theory. Besides this, we have used the construction to get support for the conjecture of [HM] from considering the string theory on the $S^3$ in the transversal geometry of the NS5 brane which is described by the $SU(2)$-WZW model. These different instances where odd super-trialgebras - which give the algebraic structure of the universal envelope of a Gerstenhaber algebra - appear suggest that these might be algebraic structures which appear in a quite universal way in string- and $M$-theory.

Beyond this, Lemma 4 has the following implication: The finite deformation theory of an affine Hopf algebra is described by two Maurer-Cartan equations (for the deformation theory of the product and of the coproduct) and one constraint (resulting from the compatibility requirement for the deformations of the product and the coproduct). One could consider this system of equations as the equations of motion of a classical field theory, generalizing the approach of [BCOV] (where such an approach is followed for holomorphic deformations). As a consequence of Lemma 4, the action functional of this classical field theory would have to be invariant under the action of the Hopf algebra $H_{GT}$. We plan to investigate the properties of this field theory in more detail in future work.

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