Another Introduction to Geometric Algebra
with some Comments on Moore-Penrose Inverses

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Abstract. With his theory of extensions Hermann Grassmann gave algebra a substantial different shape: He indeed had invented a generalized version of Pauli and Dirac Algebra which can be applied not only to re-model our mathematical view on space and time, but also to re-model mathematics – and especially the relation between geometry and algebra – itself. Therefore this didactical version of generalized Pauli and Dirac Algebra is called Geometric Algebra.

In the following an introduction to Geometric Algebra is given and applications of Geometric Algebra in physics and mathematics are discussed. Special emphasis is given to the mathematics of pure and mixed sandwich products and to a different view on solving systems of linear equations. At the end matrix inverses of non-square matrices (e.g. Moore-Penrose generalized inverses) are discussed.

1. Time scales of mathematical inventions: The history of solving systems of linear equations
One of the earliest cultural techniques mankind had invented has been mathematics: “as far as our evidence goes, 'mathematics' precedes writing” [1]. And some of the earliest mathematical techniques mankind had invented have been strategies to solve systems of equations: systems of linear equations and mixed systems of linear and quadratic equations as well.

The main tool of solving systems of linear equations in these early times has been the algorithm we nowadays call “Gaussian elimination” [2, 3]. This indicates that time scales of mathematical inventions can sometimes be very or even very, very long: Several thousands of years ago mathematicians may already have been able to solve simple systems of equations, but next to nothing is known about this time as writing was invented only later.

Then at about two thousand years BC mathematicians of the Old Babylonian Period fixed the solution of systems of equations at cuneiform tablets for the first time. After that we again had to wait for another two thousand years before new progress was made: At about 0 BC Diophantus and his colleagues (not even the century is known in which Diophantus lived [4]) invented symbolic algebra and the use of variables.

And the next step of mathematical progress in solving systems of linear equations came again only nearly two thousand years later. After the invention of determinants by Seki (in Edo/Tokyo), who “was not only the discoverer but … had a much broader idea than that of his great German contemporary” [36] Leibniz (in Hanover), Cramer wrote down his rule in 1750 and opened the route to Grassmann and to “another annus mirabilis, 1844, the birthyear of the Ausdehnungslehre, one of the supreme landmarks in the history of the human mind” [5].
2. The theory of extensions: What has Grassmann said?

The way Grassmann solved systems of linear equations is of enormous conceptual importance. And he surely was right, telling his readers that “es werde ... die Algebra eine wesentlich veränderte Gestalt gewinnen” [6] – Algebra will gain a substantial different shape by using outer products.

Therefore it makes sense to start this introduction to Geometric Algebra by analyzing what Grassmann had to say and how he explained his solution strategy. Read the classics! If we only look on copies, which are carefully cleaned, polished, sometimes dramatically modified or even rebuild in a totally different way, we will only get a blurred and hazy look on the supreme landmarks in the history of the human mind. To really understand these landmarks, we should directly confront ourselves with them.

So let us start with a system of three linear equations with three unknown variables x, y, z

\[
\begin{align*}
    a_1 x + b_1 y + c_1 z &= r_1 \\
    a_2 x + b_2 y + c_2 z &= r_2 \\
    a_3 x + b_3 y + c_3 z &= r_3
\end{align*}
\]

(1)

(in contrast to Grassmann who immediately discussed the general case with an arbitrary number n of variables) and let us try to follow Grassmann’s explanation in German: “Hier können wir die Zahlenkoeffizienten, welche verschiedenen Gleichungen angehören, sofern wir diese Verschiedenheit an ihrem Begriff noch festhalten, als verschiedenartig ansehen, und zwar alle als an sich verschieden-artig, d. h. als unabhängig in dem Sinne unserer Wissenschaft, die einer und derselben Gleichung als unter sich in derselben Beziehung gleichartig” [6].

Even as a native German speaker it is extremely difficult to grasp the meaning of this one sentence as Grassmann’s train of thought not only seems to have been light-years away from standard mathematical reasoning of most contemporaries of his time, but also from standard mathematical reasoning we are used to today.

Perhaps a translation into English will help to clarify his ideas: “Here we can consider the coefficients which are part of different equations as different, if we still grasp this difference by its inner idea and indeed all of them as inherently different, this means, as independent in the sense of our science, (and the coefficients) of one and the same equation as of the same kind with respect to the same relation.”

A shorter version of these lines might reveal the core of the theory of extensions: Coefficients of the same equation (coefficients of a row) are of the same kind. And coefficients of different equations (coefficients of a column) are of different kind. But what is the inner meaning of this statement? What has Grassmann done by implementing this concept? The answer is quite simple: Grassmann anticipated geometrization. Grassmann, as a physicist, did what later Albert Einstein and Hermann Minkowski did to describe special relativity. Grassmann pre-invented the mathematics of relativity by attaching entities of different kind, which we nowadays call directions, to different equations.

Thus a shorter version of his explanation can be in modern words: Coefficients of a row point into identical directions, and coefficients of a column point into different directions.

Therefore the first equation of (1) is connected with a first direction, the second equation of (1) is connected with a second direction, and the third equation of (1) is connected with a third direction. Usually we describe these directional relations today by multiplying the equations by different base vectors.

\[
\begin{align*}
    (a_1 x + b_1 y + c_1 z) \text{ (base vector into x-direction)} &= r_1 \text{ (base vector into x-direction)} \\
    (a_2 x + b_2 y + c_2 z) \text{ (base vector into y-direction)} &= r_2 \text{ (base vector into y-direction)} \\
    (a_3 x + b_3 y + c_3 z) \text{ (base vector into z-direction)} &= r_3 \text{ (base vector into z-direction)}
\end{align*}
\]

(2)
But what are base vectors? How can we describe them mathematically? In the course of time Grassmann developed a clear and very helpful picture of base vectors: “Ich will den Verein dreier Strecken e₁, e₂, e₃, die zu einander senkrecht sind, und deren Länge und deren äusseres Produkt [e₁e₂e₃] gleich Eins sind, einen Normalverein nennen” [7]. “I would like to call the (set) of three line segments e₁, e₂, e₃ that are perpendicular to each other and whose lengths and exterior product [e₁e₂e₃] equal one a normal set” [8]. These are old words, and a “Verein” is a typical German construction, which can be translated as “union” or more mathematically as “set”. They form the basic building blocks of what we nowadays call a coordinate system with three axes which point into the directions of the line segments e₁, e₂, e₃. Thus eq. (2) can simply be written as:

\[
\begin{align*}
(a₁ x + b₁ y + c₁ z) e₁ &= r₁ e₁ \\
(a₂ x + b₂ y + c₂ z) e₂ &= r₂ e₂ \\
(a₃ x + b₃ y + c₃ z) e₃ &= r₃ e₃
\end{align*}
\]

(3)

From the perspective of physics, these eqs. (3) show an old-fashioned and really trumpery way of writing systems of linear equations embedded in a coordinate system which was trumped up by algebra fixed foss ignoring or even denying geometry. According to Hestenes (and many others) geometry links the algebra to the physical world, and “the power of GA” – Geometric Algebra based on Grassmann’s ideas – “derives from • the simplicity of the grammar,
• the geometric meaning of multiplication,
• the way geometry links the algebra to the physical world” [9].

The simplicity of the grammar has already been noticed and was emphasized by Grassmann who in later papers gave his “Bedingungsgleichungen” – “conditional equations” [10] describing base vectors of three-dimensional space

\[
e₁e₂ = -e₂e₁, \quad e₂e₃ = -e₃e₂, \quad e₃e₁ = -e₁e₃
\]

(4)

He then proposed to create “das mittlere Product, als aus dem äusseren und inneren zusammengesetzt” [10] – “the middle product, which is composed of outer and inner product”. This middle product is identical to the geometric product of Timerding [11] and Hestenes [9] who described the “canononical decomposition” of the product of two vectors \( a = a₁e₁ + a₂e₂ + a₃e₃ \) and \( b = b₁e₁ + b₂e₂ + b₃e₃ \) as

\[
ab = a \cdot b + a \wedge b
\]

(5)

The geometric meaning of multiplication is visualized in fig. 1. If two different vectors are multiplied, we will get an oriented parallelogram which consists of a symmetric scalar part or inner product

\[
a \cdot b = b \cdot a = \frac{1}{2} (ab + ba) = [a \mid b]
\]

(Grassmann’s style of writing on the right-hand side),

(6)

and an anti-symmetric bivector part or outer product

\[
a \wedge b = -b \wedge a = \frac{1}{2} (ab - ba) = [ab]
\]

(Grassmann’s style of writing on the right-hand side).

(7)

Thus the product of two different base vectors results in a base bivector, which represents an oriented unit square (or an oriented unit rectangle, e.g. \( \frac{1}{2}e₁, 2e₂ \)) with disappearing inner product \( e₁ \cdot e₂ = 0 \)

\[
e₁e₂ = e₁ \cdot e₂ + e₁ \wedge e₂ = e₁ \wedge e₂
\]

(8)

having right angles (fig. 1).

Of course the orientation then depends on the order of the multiplication. If we first make a step into the direction of the x-axis and then a step into the direction of the y-axis, we will get a positive or
counter-clockwise orientation. If we first make a step into the direction of the y-axis instead and then a step into the direction of the x-axis, we will get a negative or clockwise orientation. This change of orientation is codified in the negative signs of eqs. (4).

Again it is time for the bizarre announcement of Cambridge scientists Gull, Lasenby, and Doran: “We have now reached the point which is liable to cause the greatest intellectual shock” [12]. Grassmann’s equations (4) are identical to the algebra of Pauli matrices:

\[
\begin{align*}
\sigma_x \sigma_y &= - \sigma_y \sigma_x, \\
\sigma_y \sigma_z &= - \sigma_z \sigma_y, \\
\sigma_z \sigma_x &= - \sigma_x \sigma_z \\
\sigma_x^2 &= \sigma_y^2 = \sigma_z^2 = [\sigma_i \mid \sigma_j] = 1
\end{align*}
\] (9)

The historical conclusion is clear and conclusive: Grassmann had already invented Pauli Algebra (and Dirac Algebra by the way). Therefore eqs. (3) can be written in a more physics oriented, modern form as:

\[
\begin{align*}
(a_1 x + b_1 y + c_1 z) \sigma_x &= r_1 \sigma_x \\
(a_2 x + b_2 y + c_2 z) \sigma_y &= r_2 \sigma_y \\
(a_3 x + b_3 y + c_3 z) \sigma_z &= r_3 \sigma_z
\end{align*}
\] (10)

Grassmann’s next step to solve this system of linear equations has been, to add all equations and to rearrange the different terms.

\[
\begin{align*}
(a_1 x + b_1 y + c_1 z) \sigma_x + (a_2 x + b_2 y + c_2 z) \sigma_y + (a_3 x + b_3 y + c_3 z) \sigma_z \\
\end{align*}
\]

\[
\begin{align*}
(a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z) x + (b_1 \sigma_x + b_2 \sigma_y + b_3 \sigma_z) y + (c_1 \sigma_x + c_2 \sigma_y + c_3 \sigma_z) z = r_1 \sigma_x + r_2 \sigma_y + r_3 \sigma_z
\end{align*}
\] (11)

Now the three coefficient vectors

\[
\begin{align*}
a &= a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z \\
b &= b_1 \sigma_x + b_2 \sigma_y + b_3 \sigma_z \\
c &= c_1 \sigma_x + c_2 \sigma_y + c_3 \sigma_z
\end{align*}
\] (12)

and the resulting vector of constant terms

\[
r = r_1 \sigma_x + r_2 \sigma_y + r_3 \sigma_z
\] (13)

form the system of linear equations (11)

\[
a x + b y + c z = r
\] (14)

As the outer product of two identical or two parallel vectors disappears, a simple outer multiplication by two of the three coefficient vectors will yield an equation which possesses one unknown variable only. Or in Grassmann’s voluptuous German: “Aus dieser Gleichung, welche die Stelle jener Gleichungen vertritt, lässt sich nun auf der Stelle jede der Unbekannten (…) finden, wenn wir die beiden Seiten mit dem äusseren Produkte aus den Koefficienten der übrigen Unbekannten äusserlich multipliciren (…). Da nämlich, wenn man die Glieder der linken Seite einzeln multiplicirt, nach dem Begriff des äusseren Produktes (…) alle Produkte wegfallen, welche zwei gleiche Faktoren enthalten” [6].
Such a vaccine might have the following didactical core: an impression whose operands are created by Grassmann’s own view who once commented: „IOP Conf. Series: Journal of Physics: Conf. Series 1071 (2018) 012012 doi:10.1088/1742-6596/1071/1/012012

\[
(a \wedge b \wedge c) x = r \wedge b \wedge c \\
(a \wedge b \wedge c) y = a \wedge r \wedge c \\
(a \wedge b \wedge c) z = a \wedge b \wedge r
\]

The solution of the system of linear equations can then be found by dividing by the oriented volume of the mathematical object already identified by Grassmann (“begrifflich, dass [a b c] gleich dem Parallelepipedon (Spat) ist, in welchem 3 sich aneinander schließende Kanten gleich a, b und c sind” [10]) as a parallelepiped

\[
a \wedge b \wedge c = \det A \sigma_x \sigma_y \sigma_z
\]

which some mathematicians also call determinant [13]. If the vectors a, b, c are linearly independent and a, b, c, r are linearly dependent, pre-division and post-division will get identical scalar results:

\[
x = (a \wedge b \wedge c)^{-1} (r \wedge b \wedge c) \\
y = (a \wedge b \wedge c)^{-1} (a \wedge r \wedge c) \\
z = (a \wedge b \wedge c)^{-1} (a \wedge b \wedge r)
\]

These are Grassmann’s results of § 45 [6], or later reformulated on page 385 of [10]. And please kindly note that eq. (16) includes a definition for determinants of non-square matrices

\[
\det A = (a \wedge b \wedge c \wedge \ldots \wedge n) \sigma_1 \ldots \sigma_3 \sigma_5 \sigma_1
\]

which perfectly agrees with Arnold’s emotional call to define determinants geometrically as “the oriented volume of the parallelepiped whose edges are its columns … a secret … carefully hidden in purified algebraic education” [13].

To teach outer products (or in other words: determinants) is an important part of teaching Geometric Algebra. And to teach inner products is an important part of teaching Geometric Algebra. But of similar or of even higher importance might be to teach how to avoid outer and inner products at all as the following section suggests.

### 3. Intermezzo: Mathematical viruses

How can we deal with mathematical viruses? David Hestenes identified several severe viruses “which can infect the mind – the mind of anyone doing mathematics, from young student to professional mathematician. (…)” A mathematical virus (MV) is a preconception about the structure, function or method of mathematics which impairs one’s ability to do mathematics. Just as a CV is program which impairs the operating system of a computer, an MV is an idea which impairs the conceptualization of mathematics in the mind” [14].

One of these mathematical viruses Hestenes called “Grassmann Virus (MV/G), because it is a distortion of Grassmann’s own view” [14]. This virus has infected mathematicians who think that calculations using only outer products are more fundamental than calculations using the complete geometric product. Or in short: “Grassmann Algebra is more fundamental than Clifford Algebra,” which simply is wrong.

Thus an anti-MV/G vaccine is required. Such a vaccine might have the following didactical core: Let us replace outer products by geometric products. To prevent students from believing that calculations using only outer products are more fundamental than calculations using the complete geometric product, it is helpful to discuss the solution of systems of linear equations using complete geometric products.

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1 Physicists seem to be more immune to MV/G compared to mathematicians, as they surely admire Feynman who once commented: „So there is a connection, ultimately, between algebra and geometry. (…) the most remarkable formula in mathematics: \( e^{i\Theta} = \cos \Theta + i \sin \Theta \). This is our jewel” [15]. This jewel can be read as a simpler expression of the geometric product \( a \wedge b = a \cdot b + a \wedge b \) (5). And a jewel will not be split into pieces.
The solutions (17) of the system of linear equations (1) or (14) will then be presented as

\[ x = (c \, a \, b - b \, a \, c)^{-1} (c \, b \, r - b \, c \, r) \quad y = (a \, b \, c - c \, b \, a)^{-1} (a \, r \, c - c \, r \, a) \quad z = (b \, c \, a - a \, c \, b)^{-1} (b \, r \, a - a \, r \, b)^{-1} \]

Mixed sandwich products indeed play a special role in these equations. Therefore it is worth the trouble to take a good look at sandwich products.

4. Pure sandwich products – What is stronger: physics or mathematics?

If a mathematical object is pre- and post-multiplied by a second mathematical object, the result can be called a sandwich product. The first mathematical object is sandwiched by the second from left and right.

As an example we can sandwich the system of linear equations (14) by the geometric product of two coefficient vectors and its reversed geometric product to reach yet another path towards a solution

\[ b \, c \, r \, b \, c - c \, b \, r \, c \quad \Rightarrow \quad x = (b \, c \, a \, c \, b \, a \, c)^{-1} (b \, c \, a \, c \, b \, a \, c) \quad y = (c \, a \, c \, b \, a \, c)^{-1} (c \, a \, c \, b \, a \, c) \quad z = (a \, b \, a \, b \, c)^{-1} (a \, b \, a \, b \, c) \]

In Geometric Algebra pure sandwich products by k-vectors (by multivectors of constant grade k) can be used to model a very basic mathematical operation: reflections! I don’t know whether it is allowed to call the sandwich products of eqs. (20) reflections in a parallelogram. But it surely opens new conceptual and new didactical views on a subject, if these subjects are renamed in an unorthodox manner.

The idea that nature might follow reflection symmetry, is a basic physical concept. Therefore the mathematics of reflections is important, and I simply will repeat the Geometric Algebra equations describing reflections [16], [17] in the following.

| Reflection in a point (represented by scalar l): | Reflection in a line (represented by vector n): | Reflection in a plane (represented by bivector N): |
|---------------------------------------------|---------------------------------------------|---------------------------------------------|
| Scalars: \( k_{ref} = l \, k \, T^{-1} \) | Vectors: \( k_{ref} = n \, k \, n^{-1} \) | Bivectors: \( k_{ref} = N \, k \, N^{-1} \) |
| Vectors: \( r_{ref} = -l \, r \, T^{-1} \) | \( r_{ref} = n \, r \, n^{-1} \) | Vectors: \( r_{ref} = -N \, r \, N^{-1} \) |
| Bivectors: \( A_{ref} = l \, A \, T^{-1} \) | \( A_{ref} = n \, A \, n^{-1} \) | Bivectors: \( A_{ref} = N \, A \, N^{-1} \) |
| Trivectors: \( V_{ref} = -l \, V \, T^{-1} \) | \( V_{ref} = n \, V \, n^{-1} \) | Trivectors: \( V_{ref} = -N \, V \, N^{-1} \) |
| Quadvectors: \( Q_{ref} = l \, Q \, T^{-1} \) | \( Q_{ref} = n \, Q \, n^{-1} \) | Quadvectors: \( Q_{ref} = N \, Q \, N^{-1} \) |
| Pentavectors: \( P_{ref} = -l \, P \, T^{-1} \) | \( P_{ref} = n \, P \, n^{-1} \) | Pentavectors: \( P_{ref} = -N \, P \, N^{-1} \) |

Reflection in a 3d space or a reduced 3d spacetime (represented by trivector T):

| Reflection in a 4d hyperspace or spacetime (represented by quadvector Q): | Reflection in a 5d hyperspace or spacetime (represented by pentavector P): |
|---------------------------------------------|---------------------------------------------|
| Scalars: \( k_{ref} = T \, k \, T^{-1} \) | Vectors: \( k_{ref} = Q \, k \, Q^{-1} \) | Vectors: \( k_{ref} = P \, k \, P^{-1} \) |
| Vectors: \( r_{ref} = T \, r \, T^{-1} \) | \( r_{ref} = -Q \, r \, Q^{-1} \) | \( r_{ref} = -P \, r \, P^{-1} \) |
| Bivectors: \( A_{ref} = T \, A \, T^{-1} \) | \( A_{ref} = Q \, A \, Q^{-1} \) | \( A_{ref} = P \, A \, P^{-1} \) |
| Trivectors: \( V_{ref} = T \, V \, T^{-1} \) | \( V_{ref} = -Q \, V \, Q^{-1} \) | \( V_{ref} = P \, V \, P^{-1} \) |
| Quadvectors: \( Q_{ref} = T \, Q \, T^{-1} \) | \( Q_{ref} = Q \, Q \, Q^{-1} \) | \( Q_{ref} = P \, Q \, P^{-1} \) |
| Pentavectors: \( P_{ref} = T \, P \, T^{-1} \) | \( P_{ref} = -Q \, P \, Q^{-1} \) | \( P_{ref} = -P \, P \, P^{-1} \) |

(21)

(22)
Reflections in higher dimensional spaces or spacetimes (or spacetimevelocities\(^2\)) can be written in a similar way. Thus three quarters of all reflection formulae are positive sandwich products, and a quarter of all reflection formulae are negative sandwich products.

As an amateur philosopher I ask amateur questions: What is stronger: physics or mathematics? The idea, that nature might follow reflection symmetry, is a basic physical concept. The idea, that the left-hand side of an equation will remain identical to the right-hand side of this equation, if we multiply both sides of the equation by the same factors, is a basic mathematical concept. So nature sometimes has to make a decision if naïve physicists try to do experiments with reflections. Should nature follow the idea of reflection symmetry or should nature follow the idea of mathematical consistency and equivalence, if multivectors are transformed?

The most general multivector \( \mathbf{M} \) existing in three-dimensional space is the following linear combination:

\[
\mathbf{M} = k + \mathbf{r} + \mathbf{A} + \mathbf{V}
\]

(with \( k = k \cdot 1; \mathbf{r} = r_{11} \sigma_1 + r_{22} \sigma_2 + r_{33} \sigma_3 \)); \( \mathbf{A} = A_{12} \sigma_1 \sigma_2 + A_{23} \sigma_2 \sigma_3 + A_{31} \sigma_3 \sigma_1; \mathbf{V} = V_1 \sigma_1 \sigma_3 + V_2 \sigma_2 \sigma_1 \); \( V_1 = 0 \).

Surely the experimenting physicist will find it possible to get a sandwich product of multivector \( \mathbf{M} \) if she or he experiments with reflections in an axis as the ‘mathematical’ and the ‘physical’ outcome should be identical.

\[
\mathbf{M}_{\text{sand}} = n \mathbf{M} n^{-1} = n k n^{-1} + n \mathbf{r} n^{-1} + n \mathbf{A} n^{-1} + n \mathbf{V} n^{-1} = k_{\text{ref}} + \mathbf{r}_{\text{ref}} + \mathbf{A}_{\text{ref}} + \mathbf{V}_{\text{ref}} = \mathbf{M}_{\text{ref}}
\]

But what will happen, if an experimenting physicist experiments with reflections in a plane? Now nature has to make a decision as ‘mathematical’ and ‘physical’ outcome clearly are different.

\[
\mathbf{M}_{\text{sand}} = \mathbf{N} \mathbf{M} \mathbf{N}^{-1} = \mathbf{N} k \mathbf{N}^{-1} + \mathbf{N} \mathbf{r} \mathbf{N}^{-1} + \mathbf{N} \mathbf{A} \mathbf{N}^{-1} + \mathbf{N} \mathbf{V} \mathbf{N}^{-1} = k_{\text{ref}} - \mathbf{r}_{\text{ref}} + \mathbf{A}_{\text{ref}} - \mathbf{V}_{\text{ref}} \neq \mathbf{M}_{\text{ref}}
\]

Will nature decide for mathematical consistency and confuse physicists with additional minus signs by flipping over vectors and trivectors? Or will nature decide for unlimited and unrestricted perfect reflection symmetry at the experiments of physicists?

There are clear indications and an obituary postcard by Pauli (“Es ist uns eine bange Pflicht, bekannt zu geben, dass unsere langjährige, liebe Freundin PARITY am 19. Januar 1957 nach kurzem Leiden bei weiteren experimentellen Eingriffen sanft entschlafen ist.” [20]) which make it reasonable to suppose that nature has decided for mathematical consistency and against absolute reflection symmetry.

Unfortunately there is a parity confusion in the literature. Some authors explain parity as a reflection in a two-dimensional plane, e.g. Penrose: “This operation of spatial reflection (reflection in a mirror) is referred as P (which stands for parity)” [21]. Other authors explain parity as a reflection in a point, e.g. Kirch et al: “Die Paritätstransformation (P) entspricht einer Punktspiegelung am Ursprung des dreidimensionalen Raums” [22].

Of course all this is confusing. The reflection in a point is different from a reflection in an axis. And they are different from a reflection in a mirror. And all these reflections are of course different from a reflection in a three-dimensional space or an n-dimensional hyper-space. Therefore we should invent and use different names for these different reflections. If they are all called parity, then different parities of different types exist.

But good news of course is that nature also has decided to model Lorentz transformations by spacetime rotations and thus by two succeeding reflections. Therefore the minus signs will cancel in special relativistic experiments with Lorentz boosts:

\[
\mathbf{M}_{\text{double-sand}} = \mathbf{N}_2 \mathbf{N}_1 \mathbf{M} \mathbf{N}_1^{-1} \mathbf{N}_2^{-1} = k_{\text{rot}} + \mathbf{r}_{\text{rot}} + \mathbf{A}_{\text{rot}} + \mathbf{V}_{\text{rot}} = \mathbf{M}_{\text{rot}}
\]

\(^2\)Five-dimensional spacetimevelocity of Carmeli’s cosmological special relativity models our world by three spacelike dimensions \( x, y, z \) and two timelike dimensions \( v, c \) with Hubble time constant \( \tau = 1/H_0 \) [18, 19].
5. Mixed sandwich products

If a mathematical object is pre-multiplied by a second mathematical object and post-multiplied by a third mathematical object, this operation can be called a mixed sandwich product. The first mathematical object is sandwiched by two different mathematical objects from left and right.

While three quarters of all pure sandwich products can be interpreted as reflections and a quarter of all pure sandwich products as anti-reflections (a reflection followed by a change of signs), mixed sandwich products can be interpreted as sort of interchanges (German: Vertauschung) – or more precisely as exchanges (German: Austauschung) or replacements (German: Ersetzung), respectively. These are no perfect interchanges as the result depends on the order of pre- and post-multiplied factors.

As a simple example we first exchange or replace the x-axis by the y-axis by pre-multiplying a sandwiched vector \( \mathbf{r} \) by \( \sigma_x \) and post-multiplying it by \( \sigma_y \) (27). And we exchange or replace the y-axis by the x-axis by pre-multiplying a sandwiched vector \( \mathbf{r} \) by \( \sigma_y \) and post-multiplying it by \( \sigma_x \) (28).

\[
\begin{align*}
\sigma_x \mathbf{r} \mathbf{r}_y &= (\sigma_x \mathbf{r}_1 \mathbf{r}_2 \sigma_y + \mathbf{r}_2 \mathbf{r}_3 \sigma_x) \mathbf{r}_y = \mathbf{r}_2 \mathbf{r}_x + \mathbf{r}_1 \mathbf{r}_y - \mathbf{r}_2 \mathbf{r}_x \mathbf{r}_y \\
\sigma_y \mathbf{r} \mathbf{r}_x &= (\sigma_y \mathbf{r}_1 \mathbf{r}_2 \sigma_y + \mathbf{r}_2 \mathbf{r}_3 \sigma_y) \mathbf{r}_x = \mathbf{r}_2 \mathbf{r}_x + \mathbf{r}_1 \mathbf{r}_y + \mathbf{r}_2 \mathbf{r}_x \mathbf{r}_y 
\end{align*}
\]

(27) (28)

Both transformations interchange the x- and y-axes. But these operations have different effects on axes perpendicular to \( \sigma_x \) and \( \sigma_y \). Perpendicular vectorial axes become trivectorial axes with different orientations. The right-handed Euclidean coordinate system of the original vector \( \mathbf{r} \) with base units squaring to one (9) becomes a left-handed (27) or right-handed (28) hyperbolic Pseudo-Euclidean coordinate system with two real and one imaginary coordinate axes as the base units of (27) & (28) now square differently (29).

\[
\sigma_x^2 = \sigma_y^2 = 1 \quad (\sigma_x \sigma_y \sigma_z)^2 = -1
\]

(29)

This change of dimensionality can now be used to solve systems of linear equations. To solve a system of three linear equations, vector \( \mathbf{r} \) is now seen as a linear combination of coefficient vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \) (14). Subtracting mixed sandwich multiplications of \( \mathbf{r} \) with two of the three coefficient vectors (e.g. \( \mathbf{b} \) and \( \mathbf{c} \))

\[
\mathbf{c} \mathbf{r} \mathbf{b} = \mathbf{c} (\mathbf{a} + \mathbf{b} + \mathbf{c}) \mathbf{b} = \mathbf{c} \mathbf{a} \mathbf{b} + \mathbf{c} \mathbf{b}^2 + \mathbf{b} \mathbf{c}^2 \quad \mathbf{b} \mathbf{r} \mathbf{c} = \mathbf{b} \mathbf{a} \mathbf{c} + \mathbf{b} \mathbf{c}^2 + \mathbf{b} \mathbf{c}^2
\]

(30)

immediately results in the solution equations already given in (19).

But a more interesting conceptual feature of mixed sandwich products is important: If mixed sandwich products change the dimensionality, we should not only look on vectors, but on complete multivectors. Exchanging vectorial axes then is only one possible feature. It might also be possible to exchange or replace two objects which have different dimensions.

A one-dimensional axis might be exchanged or replaced by a two-dimensional plane (31). Or a three-dimensional volume might be exchanged or replaced by a zero-dimensional point (32). Multivector \( \mathbf{M} \) (23) will then be transformed into:

\[
\begin{align*}
\mathbf{M}_{\mathbf{n}} \rightarrow \mathbf{N} &= \mathbf{n} \mathbf{M} \mathbf{N} = \mathbf{n} (\mathbf{k} + \mathbf{r} + \mathbf{A} + \mathbf{V}) \mathbf{N} \quad \text{or} \quad \mathbf{M}_{\mathbf{N}} \rightarrow \mathbf{n} = \mathbf{N} \mathbf{M} \mathbf{n} = \mathbf{N} (\mathbf{k} + \mathbf{r} + \mathbf{A} + \mathbf{V}) \mathbf{n} \\
\mathbf{M}_{\mathbf{T}} \rightarrow \ell &= \mathbf{T} \mathbf{M} \ell = \mathbf{T} (\mathbf{k} + \mathbf{r} + \mathbf{A} + \mathbf{V}) \ell \quad \text{or} \quad \mathbf{M}_{\ell} \rightarrow \mathbf{T} = \ell \mathbf{M} \mathbf{T} = \ell (\mathbf{k} + \mathbf{r} + \mathbf{A} + \mathbf{V}) \mathbf{T}
\end{align*}
\]

(31) (32)

These mixed sandwich products are combined exchanges and dilations. If the magnitudes of pre- and post-multiplied factors equal one (\( |\mathbf{n}| = 1; |\mathbf{N}| = 1; |\mathbf{T}| = 1 \), the mixed sandwich products describe exchanges perfectly.

What will then be the structural interpretation of reflected multivector \( \mathbf{M}_{\text{ref}} = \mathbf{n} \mathbf{M} \mathbf{n}^{-1} \) of eq. (24)? This reflection can be split into two exchanges or replacements of unit vector \( \mathbf{n} = \mathbf{n}^{-1} \) (or \( |\mathbf{n}| = 1 \)) by unit scalar \( \ell = 1 \):

\[
\mathbf{M}_{\text{ref}} = \mathbf{n} \mathbf{M} \mathbf{n} = 1 (\mathbf{n} \mathbf{M} \mathbf{n}) = (\mathbf{M}_{\rightarrow 1})_{1} \rightarrow \mathbf{n}
\]

(33)
Thus the reflection in an axis pointing into the direction of \( \mathbf{n} \) can be modeled as an exchange or replacement of the vectorial axis pointing into the direction of \( \mathbf{n} \) by the scalar axis pointing into the dimensionless direction of \( \ell = 1 \), followed by a second exchange or replacement of the scalar axis pointing into the dimensionless direction of \( \ell = 1 \) by the vectorial axis pointing into the direction of \( \mathbf{n} \).

Or in general: The reflection in an arbitrary volume (or in a hyper-volume) can be modeled as an exchange of the scalar axis with the volume of reflection (or the hyper-volume of reflection), followed by an exchange of the volume of reflection (or the hyper-volume of reflection) with the scalar axis.

6. Rotating axes in special relativity: Lorentz transformations, as usual

While Pauli matrices (9) represent base vectors of three-dimensional Euclidean space, Dirac matrices \( \gamma_0, \gamma_1, \gamma_2, \gamma_3 \) with

\[
\gamma_i \gamma_j = - \delta_{ij}, \quad \gamma_0 \gamma_i = - \gamma_i \gamma_0, \quad \gamma_0 \gamma_0 = - \gamma_0 \gamma_0, \quad \gamma_i \gamma_i = - \gamma_i \gamma_i, \quad \gamma_i \gamma_j = - \gamma_j \gamma_i
\]

\[\gamma_i^2 = 1, \quad \gamma_0^2 = \gamma_0^2 = \gamma_0^2 = -1 \tag{34}\]

represent base vectors of four-dimensional Pseudo-Euclidean spacetime \([23, 24, 25, 26, 27]\). It is well known (see references \([23–27]\)), that a spacetime rotation of a coordinate system with axes pointing into the directions of the base vectors \( \gamma_0, \gamma_1, \gamma_2, \gamma_3 \) into a new coordinate system with axes pointing into the directions of \( \gamma_{0\text{new}}, \gamma_{1\text{new}}, \gamma_{2\text{new}}, \gamma_{3\text{new}} \) equals a Lorentz transformation. This Lorentz transformation can be modeled as a first spacetime reflection in an axis pointing into the direction of the bisector \( \mathbf{n} \) of the angle between old and new time axes

\[
\gamma_{t\text{old}} = \gamma_t, \quad \gamma_{t\text{new}} = \cosh \alpha \gamma_t + \sinh \alpha \gamma_s \quad (\gamma_t^2 = \gamma_{t\text{new}}^2 = \cosh^2 \alpha - \sinh^2 \alpha = 1) \tag{35}\]

\[
\mathbf{n} = \mathbf{n}^{-1} = \frac{\gamma_t + \gamma_{t\text{new}}}{\sqrt{(\gamma_t + \gamma_{t\text{new}})^2}} = \frac{1 + \cosh \alpha \gamma_t + \sinh \alpha \gamma_s}{\sqrt{2 + 2 \cosh \alpha}} = \cosh \frac{\alpha}{2} \gamma_t + \sinh \frac{\alpha}{2} \gamma_s \tag{36}\]

followed by a second spacetime reflection in the new time axis

\[
\mathbf{m}_1 = \mathbf{m}_1^{-1} = \gamma_{s\text{new}} = \cosh \alpha \gamma_t + \sinh \alpha \gamma_s \tag{37}\]

Thus the four base vectors pointing into the directions of the coordinate axes are Lorentz transformed as usual \([28]\) according to

\[
\gamma_{t\text{new}} = \mathbf{m}_1 \gamma_0 \mathbf{n}_1^{-1} \mathbf{m}_1^{-1} = \cosh \alpha \gamma_t + \sinh \alpha \gamma_s \tag{38}\]

\[
\gamma_{s\text{new}} = \mathbf{m}_1 \gamma_s \mathbf{n}_1^{-1} \mathbf{m}_1^{-1} = \sinh \alpha \gamma_t + \cosh \alpha \gamma_s \tag{39}\]

\[
\gamma_{s\text{new}} = \mathbf{m}_1 \gamma_s \mathbf{n}_1^{-1} \mathbf{m}_1^{-1} = \gamma_s \tag{40}\]

\[
\gamma_{z\text{new}} = \mathbf{m}_1 \gamma_z \mathbf{n}_1^{-1} \mathbf{m}_1^{-1} = \gamma_z \tag{41}\]

Vectors transform into vectors, and this is important: In contrast to that it will be shown later, that other transformations might transform vectors into linear combinations of vectors and geometric objects of other dimensions.

To avoid confusion it is helpful not to speak about active transformations (… a coordinate system remains unchanged while the event at position vector \( \mathbf{r} \) rotates actively…) and not to speak about passive transformations (… the coordinate system is rotated into a new one while the position vector \( \mathbf{r} \) passively remains at the same spot…). These bracketed statements are not very helpful as always everything is transformed: An old coordinate system is transformed into a new one while the old position vector is transformed into a new one (fig. 2).

Therefore it makes more sense to speak about forward and backward transformations instead. The transformation just described transforms the original coordinate system into a new one with a negative, anti-clockwise rotation and thus will be a backward transformation.
If the first spacetime reflection in an axis pointing into the direction of the bisector $n$ is now followed by a second reflection in the old time axis

$$m_2 = m_2^{-1} = \gamma_t$$

(42)

the new coordinate axes will be rotated back into the original ones, now describing the inverse of the previous backward transformations of eqs. (38) – (41), thus giving a forward transformation:

$$\gamma_t = m_2 \ n \ \gamma_{t\text{-new}} \ n^{-1} \ m_2^{-1} = \cosh \alpha \ \gamma_{t\text{-new}} - \sinh \alpha \ \gamma_{x\text{-new}}$$

(43)

$$\gamma_x = m_2 \ n \ \gamma_{x\text{-new}} \ n^{-1} \ m_2^{-1} = - \sinh \alpha \ \gamma_{t\text{-new}} + \cosh \alpha \ \gamma_{x\text{-new}}$$

(44)

$$\gamma_y = m_2 \ n \ \gamma_{y\text{-new}} \ n^{-1} \ m_2^{-1} = \gamma_{y\text{-new}}$$

(45)

$$\gamma_z = m_2 \ n \ \gamma_{z\text{-new}} \ n^{-1} \ m_2^{-1} = \gamma_{z\text{-new}}$$

(46)

Therefore the position vector $r$ of an event

$$r = c t \ \gamma_t + x \ \gamma_x + y \ \gamma_y + z \ \gamma_z$$

(47)

is either transformed backwards into

$$r_{\text{back}} = m_1 \ n \ r \ n^{-1} \ m_1^{-1}$$

$$= (c t \ \cosh \alpha + x \ \sinh \alpha) \ \gamma_t + (x \ \cosh \alpha + c t \ \sinh \alpha) \ \gamma_x + y \ \gamma_y + z \ \gamma_z$$

(48)

or forwards into

$$r_{\text{for}} = m_2 \ n \ r \ n^{-1} \ m_2^{-1}$$

$$= (c t \ \cosh \alpha - x \ \sinh \alpha) \ \gamma_t + (x \ \cosh \alpha - c t \ \sinh \alpha) \ \gamma_x + y \ \gamma_y + z \ \gamma_z$$

(49)

$$= c t_{\text{for}} \ \gamma_t + x_{\text{for}} \ \gamma_x + y \ \gamma_y + z \ \gamma_z$$

Figure 2. Sketch of forward Lorentz transformation with positive, anti-clockwise rotation of the time axis (top) and backward Lorentz transformation with negative, clockwise rotation (at the bottom).
resulting into the well-known coordinate changes
\[ c_{\text{new}} = c_{\text{old}} = c \cosh \alpha - x \sinh \alpha \quad x_{\text{new}} = x_{\text{old}} = x \cosh \alpha - c \sinh \alpha \quad y_{\text{new}} = y \quad z_{\text{new}} = z \] \hspace{1cm} (50)

As an observer of the old coordinate system will see a person, who is motionless in his own new coordinate system \((x = y = z = 0)\), moving with constant velocity
\[ \frac{v}{c} = \frac{\Delta x}{\Delta t} = \frac{\sinh \alpha}{\cosh \alpha} = \tanh \alpha \] \hspace{1cm} (51)

the Lorentz factor will be as usual
\[ \cosh \alpha = \frac{1}{\sqrt{1 - \tanh^2 \alpha}} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \sinh \alpha = \frac{\tanh \alpha}{\sqrt{1 - \tanh^2 \alpha}} = \frac{\frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}} \] \hspace{1cm} (52)

and the transformation formulas (50) will become
\[ c_{t_{\text{new}}} = \frac{ct - \frac{v}{c} x}{\sqrt{1 - \frac{v^2}{c^2}}} \quad x_{\text{new}} = \frac{x - \frac{v}{c} t}{\sqrt{1 - \frac{v^2}{c^2}}} \quad y_{\text{new}} = y \quad z_{\text{new}} = z \] \hspace{1cm} (53)

Of course these Lorentz transformations can now be interpreted as one rotation. Or they can be interpreted as two successive reflections. Or they can be seen as four exchanges of axes, e.g.
\[ r_{\text{new}} = m \ n \ r \ n^{-1} \ m^{-1} = m \ n \ r \ n \ m \]
\[ = m \ (n \ r \ 1) \ n \ m \]
\[ = 1 \ (m \ (1 \ n \ r \ 1) \ n) \ 1 \ m \] \hspace{1cm} (54)

At the beginning an axis pointing into the direction of unit vector \(n\) is exchanged or replaced by a scalar axis pointing into the non-existing direction of unit scalar 1. Then a scalar axis pointing into the non-existing direction of unit scalar 1 is exchanged or replaced by an axis pointing into the direction of \(n\). Then an axis pointing into the direction of unit vector \(m\) is exchanged or replaced by a scalar axis pointing into the non-existing direction of unit scalar 1. And finally a scalar axis pointing into the non-existing direction of unit scalar 1 is exchanged or replaced by an axis pointing into the direction of \(m\).

Are these intermediate steps only artificial mathematical subtleties? Are they mathematical inventions which do not represent real situations?

Or do these intermediate equations have physical significance? Do they describe situations, which might happen in nature? Are they real? Is it possible, that a second observer will look at nature from a coordinate system which is constructed by exchanging or replacing axes of the coordinate system of a first observer?

The following sections will perhaps show that the answer of the last questions might be: Yes, it is.

7. Creating something out of nothing
A coordinate system in which events can be measured at position \(r = ct\ y_1 + x\ y_2 + y\ y_3 + z\ y_4 \) (47) is considered as a coordinate system of empty space in special relativity, because squaring eq. (47)
\[ r^2 = (ct)^2 - x^2 - y^2 + z^2 \quad \text{or} \quad dr^2 = r^2 \ dt^2 - dx^2 - dy^2 + dz^2 \] \hspace{1cm} (55)

clearly shows that non-diagonal elements of the metric tensor disappear
\[ g_{\mu\nu} = 0 \text{ if } \mu \neq \nu \quad \text{and} \quad g_{00} = y_1^2 = c^2 \quad g_{11} = g_{22} = g_{33} = y_2^2 = y_3^2 = y_4^2 = -1 \] \hspace{1cm} (56)
and the diagonal elements show the hyperbolic structure of empty space.

In the following, axes will be exchanged or replaced and it will be seen, that this event, which took place in an empty space, will look like an event which takes place at a space which is filled with sort of a field. An empty space seen from a first observer will look like a space with a field seen from a second observer. By exchanging or replacing axes we might create (or destroy sometimes) fields.

As already said in the last section: Vectors transform into vectors, bivectors transform into bivectors, trivectors transform into trivectors, and scalars transform into scalars if spacetime rotations are modeled according to Lorentz transformations.

But now we have a different tool on hand: We are able now to exchange or replace axes. As a first try we will exchange or replace the old time axis which points into the direction of unit vector $\gamma_t$ by a new time axis, which points into the direction of unit vector $\gamma_{t\text{ - new}}$ (35), (38) given above. Position vector $\mathbf{r}$ will then be transformed into

$$\mathbf{r}_\text{ex} = \gamma_t \mathbf{r} \gamma_{t\text{-new}}$$

and the four base vectors pointing into the directions of the coordinate axes will be transformed into

$$\gamma_{t\text{-ex}} = \gamma_t \gamma_{t\text{-new}} = \cosh \alpha \gamma_t + \sinh \alpha \gamma_x \quad (58)$$

$$\gamma_{s\text{-ex}} = \gamma_s \gamma_{s\text{-new}} = -\cosh \alpha \gamma_s - \sinh \alpha \gamma_t \quad (59)$$

$$\gamma_{y\text{-ex}} = \gamma_y \gamma_{y\text{-new}} = -\cosh \alpha \gamma_y - \sinh \alpha \gamma_z \gamma_t \quad (60)$$

$$\gamma_{z\text{-ex}} = \gamma_z \gamma_{z\text{-new}} = -\cosh \alpha \gamma_z - \sinh \alpha \gamma_y \gamma_t \quad (61)$$

**Now vectors transform into vectors** (58), (59) and **into linear combinations of vectors and trivectors** (60), (61). This is an interesting feature. To compare this strange behavior with Lorentz transformations, it makes sense to transform the now left-handed coordinate system (58) – (61) into a right-handed coordinate system and to get rid of the minus signs of eqs. (59), (60), (61). Therefore the exchange or replacement (57) will be followed by a reflection at the three-dimensional new space which is represented by the oriented volume element

$$\gamma_{x\text{-new}} \gamma_{y\text{-new}} \gamma_{z\text{-new}} = -\cosh \alpha (\cosh^2 \alpha + 3 \sinh^2 \alpha) \gamma_x \gamma_y \gamma_z - \sinh \alpha (3 \cosh^2 \alpha + \sinh^2 \alpha) \gamma_y \gamma_z \gamma_t$$

$$= -\cosh \alpha (2 \cosh (2\alpha) - 1) \gamma_x \gamma_y \gamma_z - \sinh \alpha (\cosh (2\alpha) + 1) \gamma_y \gamma_z \gamma_t \quad (62)$$

which of course is a pure trivector. The now complete exchange transformation is

$$\mathbf{r}_{\text{ex}} = \gamma_{x\text{-new}} \gamma_{y\text{-new}} \gamma_{z\text{-new}} \mathbf{r}_\text{ex} \gamma_{x\text{-new}} \gamma_{y\text{-new}} \gamma_{z\text{-new}}$$

and the backward transformations of the four base vectors will result in the following base vectors of the right-handed final coordinate system:

$$\gamma_{t\text{-ex}} = \gamma_{x\text{-new}} \gamma_{y\text{-new}} \gamma_{z\text{-new}} \gamma_{t\text{-ex}} \gamma_{x\text{-new}} \gamma_{y\text{-new}} \gamma_{z\text{-new}} = \cosh \alpha \gamma_t + \sinh \alpha \gamma_x \quad (64)$$

$$\gamma_{s\text{-ex}} = \gamma_{x\text{-new}} \gamma_{y\text{-new}} \gamma_{z\text{-new}} \gamma_{s\text{-ex}} \gamma_{x\text{-new}} \gamma_{y\text{-new}} \gamma_{z\text{-new}} = \cosh \alpha \gamma_s + \sinh \alpha \gamma_t \quad (65)$$

$$\gamma_{y\text{-ex}} = \gamma_{x\text{-new}} \gamma_{y\text{-new}} \gamma_{z\text{-new}} \gamma_{y\text{-ex}} \gamma_{x\text{-new}} \gamma_{y\text{-new}} \gamma_{z\text{-new}} = \cosh \alpha \gamma_y + \sinh \alpha \gamma_z \gamma_t \quad (66)$$

$$\gamma_{z\text{-ex}} = \gamma_{x\text{-new}} \gamma_{y\text{-new}} \gamma_{z\text{-new}} \gamma_{z\text{-ex}} \gamma_{x\text{-new}} \gamma_{y\text{-new}} \gamma_{z\text{-new}} = \cosh \alpha \gamma_z + \sinh \alpha \gamma_y \gamma_t \quad (67)$$

What are coordinates? What are base vectors? Can these four entities be called base vectors? And can be coefficients of these four entities be called coordinates? The first two vectors (64) & (65) surely are base vectors as they square in time-like manner to plus one or in space-like manner to minus one.

$$\gamma_{t\text{-ex}}^2 = 1 \quad \gamma_{s\text{-ex}}^2 = -1 \quad (68)$$

The last two entities (66) & (67) are linear combinations of vectors and trivectors, quite similar to a complex number which has a scalar component squaring to a positive value and an imaginary com-
ponent squaring to a negative value. But now both components square to minus one (which is called pseudo-complex by some authors)

\[ \gamma_i^2 = \gamma_x^2 = (\gamma_x \gamma_y \gamma_t)^2 = (\gamma_x \gamma_y \gamma_t)^2 = -1 \]  

(69)

and the unit vectors itself square to a value which can be considered as pseudo-real

\[ \gamma_y^{\text{EX}} = -cosh^2 \alpha - sinh^2 \alpha \gamma_x \gamma_y \gamma_t + 2 \cosh \alpha \sinh \alpha \gamma_x \gamma_t = -cosh (2 \alpha) + sinh (2 \alpha) \gamma_x \gamma_t \]  

(70)

\[ \gamma_x^{\text{EX}} = -cosh^2 \alpha - sinh^2 \alpha \gamma_x \gamma_y \gamma_t + 2 \cosh \alpha \sinh \alpha \gamma_x \gamma_t = -cosh (2 \alpha) + sinh (2 \alpha) \gamma_x \gamma_t \]  

(71)

because the base units of all terms square to plus one:

\[ 1^2 = (\gamma_y \gamma_t)^2 = 1 \]  

(72)

Are these base entities acting as base vectors? Of course they do: Their inner products disappear.

\[ \gamma_x^{\text{EX}} \cdot \gamma_t^{\text{EX}} = \gamma_y^{\text{EX}} \cdot \gamma_t^{\text{EX}} = \gamma_x^{\text{EX}} \cdot \gamma_y^{\text{EX}} = \gamma_y^{\text{EX}} \cdot \gamma_x^{\text{EX}} = \gamma_y^{\text{EX}} \cdot \gamma_t^{\text{EX}} = \gamma_x^{\text{EX}} \cdot \gamma_t^{\text{EX}} = 0 \]  

(73)

Therefore they can be considered as orthogonal to each other. All geometric products of these base vectors are pure bivectors, only the last one \( \gamma_y^{\text{EX}} \gamma_x^{\text{EX}} \) is a linear combination of a bivector and a quadvector.

The inverse transformations from the exchanged coordinate system back to the original coordinate system then are:

\[ \gamma_t = \cosh \alpha \gamma_x^{\text{EX}} - \sinh \alpha \gamma_y^{\text{EX}} \]  

(74)

\[ \gamma_x = \cosh \alpha \gamma_x^{\text{EX}} - \sinh \alpha \gamma_y^{\text{EX}} \]  

(75)

\[ \gamma_y = \cosh \alpha \gamma_x^{\text{EX}} - \sinh \alpha \gamma_y^{\text{EX} \gamma_y^{\text{EX}} \gamma_t} \]  

(76)

\[ \gamma_z = \cosh \alpha \gamma_x^{\text{EX}} - \sinh \alpha \gamma_y^{\text{EX} \gamma_z^{\text{EX}} \gamma_t} \]  

(77)

Thus the complete forward transformations can be found again by a first transformation (57), followed now by a reflection at the old time axis pointing into the direction of base vector \( \gamma_t \)

\[ \gamma_{\text{for}} = \gamma_t \gamma_x^{\text{EX}} \gamma_t = \cosh \alpha \gamma_t - \sinh \alpha \gamma_x \]  

(78)

\[ \gamma_{x\text{for}} = \gamma_t \gamma_x^{\text{EX}} \gamma_t = \cosh \alpha \gamma_x - \sinh \alpha \gamma_t \]  

(79)

\[ \gamma_{y\text{for}} = \gamma_t \gamma_y^{\text{EX}} \gamma_t = \cosh \alpha \gamma_y - \sinh \alpha \gamma_y \gamma_t \]  

(80)

\[ \gamma_{z\text{for}} = \gamma_t \gamma_z^{\text{EX}} \gamma_t = \cosh \alpha \gamma_z - \sinh \alpha \gamma_y \gamma_t \]  

(81)

An event, which takes place at position vector \( \mathbf{r} \) will now be measured in the forwards or backwards exchanged coordinate systems at

\[ \mathbf{r}_{\text{back}} = \mathbf{r}_{\text{EX}} = \gamma_{x\text{new}} \gamma_{y\text{new}} \gamma_{z\text{new}} \gamma_t \mathbf{r} \gamma_{t\text{new}} \gamma_{y\text{new}} \gamma_{z\text{new}} \gamma_{x\text{new}} \gamma_{y\text{new}} \gamma_{z\text{new}} \gamma_t \mathbf{r} \gamma_{t\text{new}} \gamma_{y\text{new}} \gamma_{z\text{new}} \gamma_t \mathbf{r} 
\]

\[ = (ct \cosh \alpha + x \sinh \alpha) \gamma_t + (x \cosh \alpha + ct \sinh \alpha) \gamma_x \]

\[ + y \cosh \alpha \gamma_y + y \sinh \alpha \gamma_y \gamma_t + z \cosh \alpha \gamma_z + z \sinh \alpha \gamma_y \gamma_t \]  

(82)

or at

\[ \mathbf{r}_{\text{for}} = \gamma_t \mathbf{r} \gamma_{t\text{new}} \gamma_t = \mathbf{r}_{\text{t\text{new}} \gamma_t} \]

\[ = (ct \cosh \alpha - x \sinh \alpha) \gamma_t + (x \cosh \alpha - ct \sinh \alpha) \gamma_x \]

\[ + y \cosh \alpha \gamma_y - y \sinh \alpha \gamma_y \gamma_t + z \cosh \alpha \gamma_z - z \sinh \alpha \gamma_y \gamma_t \]  

(83)

or at
Figure 3. Sketch of forward exchange transformation with positive, anti-clockwise exchange of the time axis (top) and backward exchange transformation (at the bottom).

The components of the time directions and of the x-directions of exchange transformations of eqs. (82) & (83) are identical to the equivalent components of Lorentz transformations of eqs. (48) & (49). Therefore the sketches of the ct-x-plane of these transformations (y = z = 0), which are shown in figures 2 and 3, are completely identical.

But the components perpendicular to this plane are of course different, and a relevant question with respect to these components of the y- and z-directions should be discussed: What are coordinates?

The four coordinates which are connected with the usual base vectors \( \gamma_t, \gamma_x, \gamma_y, \gamma_z \) now change according to

\[
\begin{align*}
ct_{\text{for}} &= ct \cosh \alpha - x \sinh \alpha \\
x_{\text{for}} &= x \cosh \alpha - ct \sinh \alpha \\
y_{\text{for}} &= y \cosh \alpha \\
z_{\text{for}} &= z \cosh \alpha
\end{align*}
\]

(84)

But now two additional terms pop up! There are more “coordinates”:

\[
\begin{align*}
Y_{\text{for}} &= -y \sinh \alpha \\
Z_{\text{for}} &= -z \sinh \alpha
\end{align*}
\]

(85)

An observer will now measure one time-related and three spatial distance-related and two spacetime volume-related coordinates. She or he will measure two additional entities which should be considered as fields. They are part of a complete description of the event seen from the forwards transformed coordinate system and they have an effect on the physics of this event.

Again a motionless, stationary person with \( x = y = z = 0 \) will be seen by a second observer from the other coordinate system having a constant velocity. The hyperbolic transformation factors are then again identical to the Lorentz factors:

\[
\begin{align*}
\tanh \alpha &= \frac{v}{c} \\
\cosh \alpha &= \frac{1}{\sqrt{1 - \tanh^2 \alpha}} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \\
\sinh \alpha &= \frac{\tanh \alpha}{\sqrt{1 - \tanh^2 \alpha}} = \frac{\frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}
\end{align*}
\]

(86)

Therefore the transformations (84) & (85) will become
As a consequence of these transformations (87), time dilation, length contraction and additional fields will be observed. But this time no boost is the reason for these effects. They are caused by exchanging a small fraction of the x-direction into the time-direction.

8. Creating sort of an electric field (first try)
As discussed four years ago in Bregenz [27] 3d spatial vectors or 4d spacetime bivectors

\[ E_x = E_x \gamma_x \gamma_t \quad E_y = E_y \gamma_y \gamma_t \quad E_z = E_z \gamma_z \gamma_t \] (88)

can be used to describe an electrical field and 3d spatial bivectors or 4d spacelike bivectors

\[ B_x = B_x \gamma_x \gamma_z = -B_y \gamma_y \gamma_z \quad B_y = B_y \gamma_y \gamma_x = -B_z \gamma_z \gamma_y \quad B_z = B_z \gamma_z \gamma_x = -B_x \gamma_x \gamma_y \] (89)

can be used to describe a magnetic field.

To transform an empty spacetime (55), (56) into a spacetime filled with an electrical field, the old time axis represented by unit vector \( \gamma_t \) will be exchanged or replaced by a new axis which is a linear combination of a time-related vector and an electric bivector.

\[ \gamma_{\text{old}} = \gamma_t \quad \gamma_{\text{new}} = \cos \alpha \gamma_t + \sin \alpha \gamma_x \gamma_t \] (90)

As now both base units are time-like base units squaring to one

\[ \gamma_t^2 = (\gamma_x \gamma_t)^2 = 1 \quad \Rightarrow \quad \gamma_{\text{old}}^2 = \gamma_{\text{new}}^2 = 1 \] (91)

the linear combination should be composed of sine and cosine terms to get a new unit vector again squaring to one. Position vector \( \mathbf{r} \) will then be transformed analogously to eq. (57) into

\[ \mathbf{r}_{\text{el}} = \gamma_t \mathbf{r} \gamma_{\text{new}} \] (92)

and the four base vectors pointing into the directions of the coordinate axes will be transformed into

\[ \gamma_{x\text{-el}} = \gamma_t \gamma_x \gamma_{\text{new}} = \cos \alpha \gamma_t + \sin \alpha \gamma_x \gamma_t = -\gamma_x \gamma_t \gamma_{x\text{-el}} \] (93)

\[ \gamma_{x\text{-el}} = \gamma_t \gamma_x \gamma_{\text{new}} = -\cos \alpha \gamma_x - \sin \alpha \] (94)

\[ \gamma_{y\text{-el}} = \gamma_t \gamma_y \gamma_{\text{new}} = -\cos \alpha \gamma_y - \sin \alpha \gamma_x \gamma_y = \gamma_x \gamma_y \gamma_{y\text{-el}} \] (95)

\[ \gamma_{z\text{-el}} = \gamma_t \gamma_z \gamma_{\text{new}} = -\cos \alpha \gamma_z - \sin \alpha \gamma_y \gamma_z = \gamma_x \gamma_y \gamma_{z\text{-el}} \] (96)

Now vectors transform into linear combinations of vectors, bivectors, or scalars. But the minus signs of these eqs. (93) – (96) show, that a left-handed coordinate system has emerged. To transform it again into a right-handed, backwards transformed coordinate system, a reflection at the three-dimensional new space which is represented by the oriented volume element

\[ \gamma_{x\text{-el}} \gamma_{y\text{-el}} \gamma_{z\text{-el}} = -\cos \alpha \gamma_y \gamma_z \gamma_x - \sin \alpha \gamma_x \gamma_z = \gamma_y \gamma_z \gamma_{x\text{-el}} \] (97)

will cancel the negative signs and will result in the following base vectors of the right-handed final coordinate system:
\[ \gamma_{x-EL} = \cos \alpha \gamma_t + \sin \alpha \gamma_x = \gamma_x \gamma_t \gamma_{x-el} \]  
(98)

\[ \gamma_{x-EL} = \cos \alpha \gamma_x + \sin \alpha \]  
(99)

\[ \gamma_{y-EL} = \cos \alpha \gamma_y + \sin \alpha \gamma_y = \gamma_y \gamma_y \]  
(100)

\[ \gamma_{e-EL} = \cos \alpha \gamma_z + \sin \alpha \gamma_z = \gamma_z \gamma_z \gamma_{e-el} \]  
(101)

And the inverse, forward transformations will be constructed again by reflecting eq. (92) at the old time axis into the direction of base vector \( \gamma_t \).

\[ \gamma_{t-for} = \gamma_t \gamma_{t-el} \gamma_{t} = \cos \alpha \gamma_t - \sin \alpha \gamma_y \]  
(102)

\[ \gamma_{x-for} = \gamma_t \gamma_{x-el} \]  
(103)

\[ \gamma_{y-for} = \gamma_t \gamma_{y-el} \gamma_{t} = \cos \alpha \gamma_y - \sin \alpha \gamma_y \gamma_y \]  
(104)

\[ \gamma_{z-for} = \gamma_t \gamma_{z-el} \gamma_{t} = \cos \alpha \gamma_z - \sin \alpha \gamma_z \gamma_z \]  
(105)

An event, which takes place at position vector \( \mathbf{r} \) will now be measured in both coordinate systems at

\[ \mathbf{r}_{\text{back}} = \mathbf{r}_{\text{EL}} = \gamma_{x-el} y_{\text{EL}-y} y_{\text{EL}-z} \]  
(106)

or at

\[ \mathbf{r}_{\text{for}} = \gamma_t \gamma_{t-el} \gamma_{t} \]  
(107)

The four coordinates which are connected with the usual base vectors \( \gamma_t, \gamma_x, \gamma_y, \gamma_z \) now change according to

\[ \mathbf{r}_{\text{new}} = \mathbf{r}_{\text{new}} = \mathbf{r}_{\text{new}} = \gamma_t \gamma_{t-el} \gamma_{t} \]  
(108)

And again additional terms pop up. These three bivector and one scalar coordinates of eqs. (107) & (108) will be named courageously as:

\[ E_x = \pm ct \sin \alpha \]  
\[ X_{\text{EL}} = \pm x \sin \alpha \]  
\[ B_y = \pm y \sin \alpha \]  
\[ B_z = \pm z \sin \alpha \]  
(109)

These additional components cannot be seen and cannot be measured by a person in the forwards transformed coordinate system

\[ \mathbf{r}_{\text{for}} = \mathbf{r}_{\text{for}} = \gamma_t \gamma_{t-for} + y \gamma_{y-for} + z \gamma_{z-for} + 0 \]  
(110)

or by a person in the backwards transformed coordinate system

\[ \mathbf{r}_{\text{back}} = \mathbf{r}_{\text{back}} = \gamma_t \gamma_{t-back} + y \gamma_{y-back} + z \gamma_{z-back} + 0 \]  
(111)

measuring only the usual coordinates of empty spacetime. But they can be seen and they can be measured by an observer of the original coordinate system with axes pointing into the directions of the base vectors \( \gamma_t, \gamma_x, \gamma_y, \) and \( \gamma_z \) when this observer is looking at the new coordinate systems (106 & 107).

These additional coordinates might represent an electrical field bivector, an interesting scalar part and two magnetic field bivectors. And the dynamical behavior of these additional components is surprising: The electrical field bivector is connected with the time coordinate \( t \). An observer simply waiting at the origin at \( x = y = z = 0 \) will observe an electrical field which becomes stronger and stronger with increasing time \( t \). This is indeed an unexpected feature of transformations of coordinates.
9. Creating sort of an electric field (second try)

To transform an empty empty spacetime (55), (56) into an electrical field in an alternative way, the old x-axis represented by unit vector $\gamma_x$ will be exchanged or replaced by a new axis which is a linear combination of a space-related vector and an electric bivector.

$$\gamma_{x-old} = \gamma_x \quad \gamma_{x-new} = \cosh \alpha \gamma_x + \sinh \alpha \gamma_x \gamma_t$$  \hspace{1cm} (112)

As now one base unit is a space-like base unit squaring to minus one and the other base unit is a timelike base unit squaring to minus one

$$\gamma^2 = -1 \quad \text{and} \quad (\gamma_x \gamma_t)^2 = 1 \quad \Rightarrow \quad \gamma_{x-old}^2 = \gamma_{x-new}^2 = -1$$  \hspace{1cm} (113)

the linear combination should be composed as usual of hyperbolic sine and hyperbolic cosine terms to get a new unit vector again squaring to minus one. Position vector $\mathbf{r}$ will then be transformed into

$$\mathbf{r}_{EL} = \gamma_x \mathbf{r} \gamma_{x-new}$$  \hspace{1cm} (114)

and the four base vectors pointing into the directions of the coordinate axes will be transformed into

$$\gamma_{\mathbf{r}-el} = \gamma_x \gamma_t \gamma_{x-new} = \cosh \alpha \gamma_t - \sinh \alpha$$  \hspace{1cm} (115)

$$\gamma_{\mathbf{x}-el} = \gamma_x \gamma_y \gamma_{x-new} = - \cosh \alpha \gamma_x - \sinh \alpha \gamma_x \gamma_t = - \gamma_x \gamma_t \gamma_{x-el}$$  \hspace{1cm} (116)

$$\gamma_{\mathbf{y}-el} = \gamma_x \gamma_y \gamma_{x-new} = \cosh \alpha \gamma_y + \sinh \alpha \gamma_x \gamma_t = \gamma_x \gamma_{x-el}$$  \hspace{1cm} (117)

$$\gamma_{\mathbf{z}-el} = \gamma_x \gamma_z \gamma_{x-new} = \cosh \alpha \gamma_z + \sinh \alpha \gamma_x \gamma_t = \gamma_x \gamma_{x-el}$$  \hspace{1cm} (118)

Now vectors transform again into linear combinations of vectors, bivectors, or scalars. Reflecting again at the axis pointing into the old x-direction represented by $\gamma_x$ will reverse the sign first negative sign of eq. (116) giving:

$$\gamma_{x-EL} = \gamma_x \gamma_{\mathbf{r}-el} \gamma_x = \cosh \alpha \gamma_x - \sinh \alpha$$  \hspace{1cm} (119)

$$\gamma_{x-EL} = \gamma_x \gamma_{\mathbf{x}-el} \gamma_x = \cosh \alpha \gamma_x - \sinh \alpha \gamma_x \gamma_t$$  \hspace{1cm} (120)

$$\gamma_{x-EL} = \gamma_x \gamma_{\mathbf{y}-el} \gamma_x = \cosh \alpha \gamma_x - \sinh \alpha \gamma_x \gamma_t$$  \hspace{1cm} (121)

$$\gamma_{x-EL} = \gamma_x \gamma_{\mathbf{z}-el} \gamma_x = \cosh \alpha \gamma_x - \sinh \alpha \gamma_x \gamma_t$$  \hspace{1cm} (122)

An event, which takes place at position vector $\mathbf{r}$ will now be measured at

$$\mathbf{r}_{EL} = \gamma_x \gamma_y \gamma_z \gamma_{x-new} \gamma_x = - \mathbf{r} \gamma_{x-new} \gamma_x$$

$$= \cosh \alpha (ct \gamma_t + x \gamma_x + y \gamma_y + z \gamma_z) - \sinh \alpha (ct + x \gamma_x \gamma_t + y \gamma_y \gamma_t + z \gamma_z \gamma_t)$$  \hspace{1cm} (123)

$$= ct_{EL} \gamma_t + x_{EL} \gamma_x + y_{EL} \gamma_y + z_{EL} \gamma_z + cT + E_x \gamma_x \gamma_t + E_y \gamma_y \gamma_t + E_z \gamma_z \gamma_t$$

The four coordinates which are connected with the usual base vectors $\gamma_t, \gamma_x, \gamma_y, \gamma_z$ now change according to

$$ct_{EL} = ct \cosh \alpha \quad x_{EL} = x \cosh \alpha \quad y_{EL} = y \cosh \alpha \quad z_{EL} = z \cosh \alpha$$  \hspace{1cm} (124)

And again additional terms pop up. These three bivector and one scalar coordinates of eq. (123)

$$cT = -ct \sinh \alpha \quad E_x = -x \sinh \alpha \quad E_y = -y \sinh \alpha \quad E_z = -z \sinh \alpha$$  \hspace{1cm} (125)

are indeed electrical field components only and it is reasonable to conclude, that with this second try sort of an electric field has indeed been constructed now.

These additional components cannot be seen and cannot be measured by a person in a coordinate system with axes pointing into the directions of $\gamma_{x-EL}, \gamma_{x-EL}, \gamma_{x-EL}, \gamma_{x-EL}$. But they can be seen and they can be measured by an observer in the original coordinate system with time and space axes pointing into the directions of of $\gamma_x, \gamma_x, \gamma_x$ (123).
10. Generalized matrix inverses

As announced in the abstract, matrix inverses of non-square matrices (e.g. Moore-Penrose generalized inverses) will be discussed at the end, which now will be reached.

Let's go back to solution (17) of an exactly constrained system of three linear equations with three unknown variables (1). This solution on the left side of eq. (17) can be split into a matrix multiplication because the resulting vector \( \mathbf{r} \) is a linear combination of the three base vectors (13).

\[
x = (a \wedge b \wedge c)^{-1} (a \wedge c \wedge b) \mathbf{r}_1 + (a \wedge b \wedge c)^{-1} (a \wedge \sigma_x \wedge c) \mathbf{r}_2 + (a \wedge b \wedge c)^{-1} (a \wedge \sigma_y \wedge c) \mathbf{r}_3
\]

\[
y = (a \wedge b \wedge c)^{-1} (a \wedge \sigma_x \wedge c) \mathbf{r}_1 + (a \wedge b \wedge c)^{-1} (a \wedge \sigma_y \wedge c) \mathbf{r}_2 + (a \wedge b \wedge c)^{-1} (a \wedge \sigma_z \wedge c) \mathbf{r}_3
\]

\[
z = (a \wedge b \wedge c)^{-1} (a \wedge \sigma_x \wedge c) \mathbf{r}_1 + (a \wedge b \wedge c)^{-1} (a \wedge \sigma_y \wedge c) \mathbf{r}_2 + (a \wedge b \wedge c)^{-1} (a \wedge \sigma_z \wedge c) \mathbf{r}_3
\]

These solution equations can be expressed with the scheme of Falk (127), clearly identifying the lead matrix as the usual inverse \( \mathbf{A}^{-1} \) of the coefficient matrix \( \mathbf{A} \).

\[
\begin{align*}
(a \wedge b \wedge c)^{-1} (a \wedge c \wedge b) & \quad (a \wedge b \wedge c)^{-1} (a \wedge \sigma_x \wedge c) & \quad (a \wedge b \wedge c)^{-1} (a \wedge \sigma_y \wedge c) & \quad (a \wedge b \wedge c)^{-1} (a \wedge \sigma_z \wedge c) \\
(a \wedge b \wedge c)^{-1} (a \wedge \sigma_x \wedge c) & \quad (a \wedge b \wedge c)^{-1} (a \wedge \sigma_y \wedge c) & \quad (a \wedge b \wedge c)^{-1} (a \wedge \sigma_z \wedge c) & \quad (a \wedge b \wedge c)^{-1} (a \wedge \sigma_x \wedge c) \\
(a \wedge b \wedge c)^{-1} (a \wedge \sigma_y \wedge c) & \quad (a \wedge b \wedge c)^{-1} (a \wedge \sigma_z \wedge c) & \quad (a \wedge b \wedge c)^{-1} (a \wedge \sigma_x \wedge c) & \quad (a \wedge b \wedge c)^{-1} (a \wedge \sigma_y \wedge c)
\end{align*}
\]

(127)

All this happens in a three-dimensional space. What will now happen if the coefficient vectors \( a, b, c \) and the resulting vector \( \mathbf{r} \) which possess three components pointing into the three directions x, y, z are placed into a higher dimensional space and are rotated into an oblique dimensional position, now pointing into the four directions w, x, y, z?

Of course all relations between them remain unchanged if all vectors are rotated in the same way. Especially eq. (14) \( a \mathbf{x} + b \mathbf{y} + c \mathbf{z} = \mathbf{r} \) will not change. But the coefficient vectors and the resulting vector will now have four components, thus describing an overconstrained system of linear equations of three variables with four linear equations.

Therefore the solution equations (17) will remain unchanged, too. The only difference is now, that the decomposition of the solution into a matrix multiplication will have more components, and the leading matrix will have more columns than rows. This can be expressed again with the scheme of Falk (128).

\[
\begin{align*}
(a \wedge b \wedge c)^{-1} (a \wedge \sigma_x \wedge c) & \quad (a \wedge b \wedge c)^{-1} (a \wedge \sigma_y \wedge c) & \quad (a \wedge b \wedge c)^{-1} (a \wedge \sigma_z \wedge c) & \quad (a \wedge b \wedge c)^{-1} (a \wedge \sigma_w \wedge c) \\
(a \wedge b \wedge c)^{-1} (a \wedge \sigma_x \wedge c) & \quad (a \wedge b \wedge c)^{-1} (a \wedge \sigma_y \wedge c) & \quad (a \wedge b \wedge c)^{-1} (a \wedge \sigma_z \wedge c) & \quad (a \wedge b \wedge c)^{-1} (a \wedge \sigma_w \wedge c) \\
(a \wedge b \wedge c)^{-1} (a \wedge \sigma_y \wedge c) & \quad (a \wedge b \wedge c)^{-1} (a \wedge \sigma_z \wedge c) & \quad (a \wedge b \wedge c)^{-1} (a \wedge \sigma_x \wedge c) & \quad (a \wedge b \wedge c)^{-1} (a \wedge \sigma_y \wedge c)
\end{align*}
\]

(128)

The lead matrix can be again clearly identified as the inverse \( \mathbf{A}^{-1} \) of the now non-square coefficient matrix \( \mathbf{A} \). As this matrix is found by using Geometric Algebra or generalized Pauli algebra, it might be called the Pauli algebra generalized matrix inverse.

Summarizing and generalizing these results, it is possible to say: The matrix inverse of a square matrix with dimension \( n \times n \) can be found with the Geometric Algebra equivalent of Cramer’s rule by substituting coefficient vector \( a_i \) (\( a_1 = a; a_2 = b; a_3 = c; \) etc.) of eq. (18) by unit vectors \( \sigma_i \)
\[ a_1 \wedge a_2 \wedge a_3 \wedge \ldots \wedge a_{j-1} \wedge \sigma_i \wedge a_{j+1} \wedge \ldots \wedge a_n = \det A_{ij} \sigma_i \sigma_2 \sigma_3 \ldots \sigma_n \]  

(129)

The oriented volumes of these hyper-parallelepipeds (or paralleloptops) thus represent the determinants of the transposed submatrices \( A_{ij} \) which are already attached with the correct cofactor signs, usually called adjoint matrices. Therefore the elements of the matrix inverse can be written as quotient of eqs. (129) and (18), resulting in the following matrix inverse:

\[ (A^{-1})_{ij} = (a_1 \wedge a_2 \wedge a_3 \wedge \ldots \wedge a_n)^{-1} (a_1 \wedge a_2 \wedge a_3 \wedge \ldots \wedge a_{j-1} \wedge \sigma_i \wedge a_{j+1} \wedge \ldots \wedge a_n) \]  

(130)

Obviously inverses of underconstrained (more variables than equations, \( m < n \)) systems of simultaneous linear equations do not exist, as coefficient vectors \( a_i \) then are linearly dependent and therefore the outer product (18) has to be zero.

For this reason it makes sense to only discuss matrix inverses of overconstrained and consistent systems of simultaneous linear equations (\( m > n \), \( \det A \neq 0 \)).

The elements of the matrix inverse of a non-square matrix [29, 30, 31] with dimension \( m \times n \) can then be found again by an equation nearly identical to eq. (130)

\[ (A^{-1})_{ij} = (a_1 \wedge a_2 \wedge a_3 \wedge \ldots \wedge a_n)^{-1} (a_1 \wedge a_2 \wedge a_3 \wedge \ldots \wedge a_{j-1} \wedge \sigma_i \wedge a_{j+1} \wedge \ldots \wedge a_n) \]  

(131)

now resulting in a non-square matrix inverse \( A^{-1} \) with dimension \( n \times m \), because \( i \in \{1, 2, 3, \ldots, n\} \) and \( j \in \{1, 2, 3, \ldots, m\} \) have different values now.

Alternatively the right side of eq. (17) can be used to construct Pauli algebra generalized matrix inverses. A post-division by the oriented volumes of the coefficient vector parallelepipeds

\[
\begin{array}{cccc}
\sigma_x \wedge b \wedge c & a \wedge b \wedge c & (a \wedge b \wedge c)^{-1} & (a \wedge b \wedge c)^{-1} \\
\sigma_y \wedge b \wedge c & (a \wedge b \wedge c)^{-1} & (a \wedge b \wedge c)^{-1} & (a \wedge b \wedge c)^{-1} \\
\sigma_z \wedge b \wedge c & (a \wedge b \wedge c)^{-1} & (a \wedge b \wedge c)^{-1} & (a \wedge b \wedge c)^{-1} \\
(a \wedge b \wedge \sigma_x) (a \wedge b \wedge c)^{-1} & (a \wedge b \wedge \sigma_y) (a \wedge b \wedge c)^{-1} & (a \wedge b \wedge \sigma_z) (a \wedge b \wedge c)^{-1} & (a \wedge b \wedge c)^{-1} \\
\end{array}
\]

(132)

will result again in a square matrix inverse which is identical to the lead matrix of eq. (127).

But the overconstrained, consistent system of linear equations will then have a slightly different Pauli algebra generalized matrix inverse.

\[
\begin{array}{cccc}
\sigma_x \wedge b \wedge c & (a \wedge b \wedge c)^{-1} & (a \wedge b \wedge c)^{-1} & (a \wedge b \wedge c)^{-1} \\
\sigma_y \wedge b \wedge c & (a \wedge b \wedge c)^{-1} & (a \wedge b \wedge c)^{-1} & (a \wedge b \wedge c)^{-1} \\
\sigma_z \wedge b \wedge c & (a \wedge b \wedge c)^{-1} & (a \wedge b \wedge c)^{-1} & (a \wedge b \wedge c)^{-1} \\
(a \wedge b \wedge \sigma_x) (a \wedge b \wedge c)^{-1} & (a \wedge b \wedge \sigma_y) (a \wedge b \wedge c)^{-1} & (a \wedge b \wedge \sigma_z) (a \wedge b \wedge c)^{-1} & (a \wedge b \wedge c)^{-1} \\
\end{array}
\]

(133)

The lead matrix can be again clearly identified as an inverse \( \bar{A}^{-1} \) of the now non-square coefficient matrix \( A \). Of course the scalar parts of the pre-multiplied non-square matrix inverse \( A^{-1} \) and the post-multiplied non-square matrix inverse \( \bar{A}^{-1} \) are identical. But the bivector parts of the two non-square matrix inverses \( A^{-1} \) and \( \bar{A}^{-1} \) have reversed signs.

In a similar way the elements of post-multiplied Pauli algebra generalized matrix inverses \( (\bar{A}^{-1})_{ij} \) of higher-dimensional non-square coefficient matrices \( A \) can be constructed.

It can easily be shown that Pauli algebra generalized matrix inverses are left sided matrix inverses only:
A \ A^{-1} = I \quad \text{and} \quad \Delta \ A^{-1} = I \tag{134}
A \ A^{-1} \neq I \quad \text{and} \quad A \ A^{-1} \neq I \tag{135}

A post-multiplication of Pauli algebra generalized matrix inverses from the right (135) will not result in an identity matrix I. (And by the way: the identity matrices I of eqs. (134) & (135) have different dimensions, as A now is a non-square coefficient matrix.)

Sandwiching or transposing eqs. (134) will then give the following three different algebraic relations:

\[ A \ A^{-1} \ A = A \ I = A \quad \text{and} \quad A \ A^{-1} \ A = A \ I = A \quad \text{and} \quad A \ A^{-1} \ A = A \ I = A \quad \text{and} \quad A \ A^{-1} = I \ A^{-1} = A^{-1} \tag{136}\n
\[ A^{-1} \ A \ A^{-1} = I \ A^{-1} = A^{-1} \quad \text{and} \quad A^{-1} \ A \ A^{-1} = I \ A^{-1} = A^{-1} \quad \text{and} \quad A^{-1} \ A \ A^{-1} = I \ A^{-1} = A^{-1} \quad \text{and} \quad (A^{-1} \ A)^{T} = I^{T} = I = A^{-1} \ A \tag{137}\n
But unfortunately there is:

\[ (A \ A^{-1})^{T} \neq A \ A^{-1} \quad \text{and} \quad (A \ A^{-1})^{T} \neq A \ A^{-1} \tag{139}\]

Nevertheless, the last relation (139) can be transformed into an identity if only the identical scalar parts of A^{-1} or A^{-1} are taken into account (see section 11).

Multiplicativity does not hold either. If the product of two non-square matrices A and B

\[ A \ B = D \tag{140}\]

is given, the product of the Pauli algebra generalized matrix inverses B^{-1} and A^{-1} will not result in the Pauli algebra generalized matrix inverse D^{-1} of the resulting matrix D.

\[ B^{-1} \ A^{-1} \neq D^{-1} \tag{141}\]

A simple counter-example will be presented as attachment (see section 12).

11. Moore-Penrose generalized matrix inverses

More and more introductory textbooks of business mathematics and of mathematical economics discuss Moore-Penrose generalized matrix inverses A^{+} as elementary part of the foundations of matrix algebra [34, 35]. These Moore-Penrose generalized matrix inverses are defined and discussed in introductory books usually only with respect to the four Moore-Penrose conditions (142) to (145).

\[ A \ A^{+} \ A = A \tag{142}\n\[ A^{+} \ A \ A^{+} = A^{+} \tag{143}\n\[ (A \ A^{+})^{T} = A \ A^{+} \tag{144}\n\[ (A^{+} \ A)^{T} = A^{+} \ A \tag{145}\]

This starting point is didactically problematic: Students then develop a strong algebraic perspective of Moore-Penrose generalized matrix inverses. But they fail to develop a geometric perspective of generalized matrix inverses.

The geometric perspective is inherent to the Geometric Algebra interpretation of generalized Pauli and Dirac algebras. If this perspective is not mentioned and not discussed, a relevant part of mathematics education is missing. Mathematics should be considered as composed of geometry and algebra. If the geometric part of mathematics is neglected or even denied, the picture of mathematics, students will get, will be partly incomplete. This is a conceptual shortcoming and a severe didactical deficit.

Therefore another didactical path to Moore-Penrose generalized matrix inverses is proposed in the following: Moore-Penrose matrix inverses should not (or not only) be defined by the four Moore-
Penrose conditions. With respect to what Grassmann had achieved it is more helpful to first develop a complete Geometric Algebra picture and to discuss Pauli algebra generalized matrix inverses first. Then the definition of Moore-Penrose generalized matrix inverses will be straightforward and very simple: **Moore-Penrose generalized matrix inverses are the scalar part of Pauli algebra generalized matrix inverses.** The scalar parts of Pauli algebra generalized matrix inverses then automatically comply with the four Moore-Penrose conditions (142) to (145).

To get the scalar parts of $A^{-1}$ or $\Delta^{-1}$ it is possible to just throw away the bivector parts.

$$A^+ = \langle A^{-1} \rangle_0 = \langle \Delta^{-1} \rangle_0$$  \hspace{1cm} (146)

But it will be more elegant to define a positive and a negative superposition of pre- and post-multiplied Pauli algebra generalized matrix inverses $A^{-1}$ and $\Delta^{-1}$.

$$A^+ = \frac{1}{2} (A^{-1} + \Delta^{-1})$$  \hspace{1cm} (147)

$$A^- = \frac{1}{2} (A^{-1} - \Delta^{-1})$$  \hspace{1cm} (148)

$A^+$ will then be the usual Moore-Penrose generalized matrix inverse with scalar elements which solves an overconstrained, consistent system of linear equations

$$A \ q = r$$  \hspace{1cm} (149)

by a simple pre-multiplication from the left side

$$A^+ \ r = A^{-1} \ r = \Delta^{-1} \ r = q$$  \hspace{1cm} (150)

while $A^-$ will be a non-square matrix with bivector elements which is orthogonal to the resulting vector $r$ of constant terms.

$$A^- \ r = 0$$  \hspace{1cm} (151)

Thus from a geometric viewpoint Moore-Penrose generalized matrix inverses are not complete. Geometrically complete generalized matrix inverses are composed of the scalar and bivector parts of Pauli algebra generalized matrix inverses:

$$A^{-1} = A^+ + A^- \quad \text{and} \quad \Delta^{-1} = A^+ - A^-$$  \hspace{1cm} (152)

Even if these equations are not jewels (compared to the one of the first footnote), but more ordinary stones only, we should not split them. They tell us a lot about symmetry – and symmetry we will lose if we neglect the bivector (or in 3d quaternion) part $A^-$. 

**Appendix: Counter-example to multiplicativity**

According to the Babylonian way of teaching which Feynman described [32, 33], students even today may learn by discussing and analyzing examples.

Therefore a very simple counter-example to multiplicativity will be presented in the following. It will show that the Pauli algebra generalized matrix inverse $D^{-1}$ is not identical to the product of the Pauli algebra generalized matrix inverses $B^{-1}$ and $A^{-1}$ (141) even if $A \ B = D$ (140).

---

3 "… die alten Babylonier (besaßen) nicht die Fähigkeit, das in mathematischer Form auszudrücken. – Heute besitzen wir nicht die Fähigkeit, einem Studenten zu zeigen, wie er Physik *physikalisch* verstehen kann. (…) Mangels einer Ausdrucksmöglichkeit ist der einzige Weg, Physik physikalisch zu verstehen, auch heute noch der langweilige Babylonische Weg, viele Beispiele zu machen.” [33].
The non-square matrices \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \), \( B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \), and \( A B = D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \) are given.

The Geometric Algebra coefficient vectors of these matrices then are:

\[
\begin{align*}
a_1 &= \sigma_1 + \sigma_4 \\
a_2 &= \sigma_2 \\
a_3 &= \sigma_3
\end{align*}
\]

\( a_1 = \sigma_1 + \sigma_4 \quad a_2 = \sigma_2 \quad a_3 = \sigma_3 \) (153)

\[
\begin{align*}
b_1 &= \sigma_1 + \sigma_3 \\
b_2 &= \sigma_2
\end{align*}
\]

\( b_1 = \sigma_1 + \sigma_3 \quad b_2 = \sigma_2 \) (154)

\[
\begin{align*}
d_1 &= \sigma_1 + \sigma_3 + \sigma_4 \\
d_2 &= \sigma_2
\end{align*}
\]

\( d_1 = \sigma_1 + \sigma_3 + \sigma_4 \quad d_2 = \sigma_2 \) (155)

The oriented volumes and reciprocal oriented volumes are:

\[
\begin{align*}
a_1 \wedge a_2 \wedge a_3 &= \sigma_1\sigma_2\sigma_3 + \sigma_2\sigma_3\sigma_4 \\
(a_1 \wedge a_2 \wedge a_3)^{-1} &= -\frac{1}{2} (\sigma_1\sigma_2\sigma_3 + \sigma_2\sigma_3\sigma_4) \\
b_1 \wedge b_2 &= \sigma_1\sigma_2 - \sigma_2\sigma_3 \\
(b_1 \wedge b_2)^{-1} &= -\frac{1}{2} (\sigma_1\sigma_2 - \sigma_2\sigma_3) \\
d_1 \wedge d_2 &= \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_2\sigma_4 \\
(d_1 \wedge d_2)^{-1} &= -\frac{1}{3} (\sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_2\sigma_4)
\end{align*}
\]

The adjoint oriented volumes are:

\[
\begin{align*}
\sigma_1 \wedge a_2 \wedge a_3 &= \sigma_1\sigma_2\sigma_3 \\
\sigma_2 \wedge a_2 \wedge a_3 &= 0 \\
\sigma_3 \wedge a_2 \wedge a_3 &= 0 \\
\sigma_4 \wedge a_2 \wedge a_3 &= \sigma_2\sigma_3\sigma_4
\end{align*}
\]

\( \sigma_1 \wedge a_2 \wedge a_3 = \sigma_1\sigma_2\sigma_3 \) (159)

\[
\begin{align*}
a_1 \wedge a_1 &= \sigma_1\sigma_4 \sigma_1 \\
a_1 \wedge a_2 \wedge a_3 &= \sigma_1\sigma_2\sigma_3 + \sigma_2\sigma_3\sigma_4 \\
a_1 \wedge a_3 &= 0 \\
a_1 \wedge a_4 &= -\sigma_1\sigma_2\sigma_1
\end{align*}
\]

\( a_1 \wedge a_1 = \sigma_1\sigma_4 \sigma_1 \) (160)

\[
\begin{align*}
a_1 \wedge a_2 \wedge a_3 &= -\sigma_4\sigma_1\sigma_2 \\
a_1 \wedge a_3 &= 0 \\
a_1 \wedge a_4 &= -\sigma_1\sigma_2\sigma_1
\end{align*}
\]

\( a_1 \wedge a_2 \wedge a_3 = -\sigma_4\sigma_1\sigma_2 \) (161)

\[
\begin{align*}
\sigma_1 \wedge b_2 &= \sigma_1\sigma_2 \\
\sigma_2 \wedge b_2 &= 0 \\
\sigma_3 \wedge b_2 &= -\sigma_2\sigma_3
\end{align*}
\]

\( \sigma_1 \wedge b_2 = \sigma_1\sigma_2 \) (162)

\[
\begin{align*}
b_1 \wedge a_2 &= \sigma_1\sigma_2 - \sigma_2\sigma_3 \\
b_1 \wedge a_3 &= -\sigma_3\sigma_1
\end{align*}
\]

\( b_1 \wedge a_2 = \sigma_1\sigma_2 - \sigma_2\sigma_3 \) (163)

\[
\begin{align*}
\sigma_1 \wedge d_2 &= \sigma_1\sigma_2 \\
\sigma_2 \wedge d_2 &= 0 \\
\sigma_3 \wedge d_2 &= -\sigma_2\sigma_3 \\
\sigma_4 \wedge d_2 &= -\sigma_2\sigma_4
\end{align*}
\]

The Pauli algebra generalized matrix inverses of the lead matrix \( A \) then are:

\[
A^{-1} = \frac{1}{2} \begin{bmatrix}
1 - \sigma_1\sigma_4 & 0 & 0 & 1 + \sigma_1\sigma_4 \\
0 - \sigma_1\sigma_2 - \sigma_2\sigma_4 & 2 & 0 & 0 + \sigma_1\sigma_2 + \sigma_2\sigma_4 \\
0 - \sigma_1\sigma_3 - \sigma_3\sigma_4 & 0 & 2 & 0 + \sigma_1\sigma_3 + \sigma_3\sigma_4
\end{bmatrix}
\]

\( A^{-1} = \frac{1}{2} \begin{bmatrix}
1 + \sigma_1\sigma_4 & 0 & 0 & 1 - \sigma_1\sigma_4 \\
0 + \sigma_1\sigma_2 + \sigma_2\sigma_4 & 2 & 0 & 0 - \sigma_1\sigma_2 - \sigma_2\sigma_4 \\
0 + \sigma_1\sigma_3 + \sigma_3\sigma_4 & 0 & 2 & 0 - \sigma_1\sigma_3 - \sigma_3\sigma_4
\end{bmatrix}
\)
Check of the result:

\[
\begin{array}{c|ccc}
 & 1 & 0 & 0 \\
\hline
1 - \sigma_1\sigma_4 & 0 & 0 & 1 + \sigma_1\sigma_4 \\
0 - \sigma_1\sigma_2 - \sigma_2\sigma_4 & 2 & 0 & 0 + \sigma_1\sigma_2 + \sigma_2\sigma_4 \\
0 - \sigma_1\sigma_3 - \sigma_3\sigma_4 & 0 & 2 & 0 + \sigma_1\sigma_3 + \sigma_3\sigma_4 \\
\end{array}
\]

The Moore-Penrose generalized matrix inverse then is:

\[
A^+ = \frac{1}{2} \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0
\end{bmatrix}
\] (168)

The orthogonal bivector matrix to the Moore-Penrose generalized matrix inverse then is:

\[
A^- = \frac{1}{2} \begin{bmatrix}
-\sigma_1\sigma_4 & 0 & 0 & \sigma_1\sigma_4 \\
-\sigma_1\sigma_2 - \sigma_2\sigma_4 & 0 & 0 & \sigma_1\sigma_2 + \sigma_2\sigma_4 \\
-\sigma_1\sigma_3 - \sigma_3\sigma_4 & 0 & 0 & \sigma_1\sigma_3 + \sigma_3\sigma_4
\end{bmatrix}
\] (169)

The Pauli algebra generalized matrix inverses of the lag matrix \( B \) then are:

\[
B^+ = \frac{1}{2} \begin{bmatrix}
1 - \sigma_1\sigma_3 & 0 & 1 + \sigma_1\sigma_3 \\
0 - \sigma_1\sigma_2 - \sigma_2\sigma_3 & 2 & 0 + \sigma_1\sigma_2 + \sigma_2\sigma_3
\end{bmatrix}
\] (170)

\[
B^- = \frac{1}{2} \begin{bmatrix}
1 + \sigma_1\sigma_3 & 0 & 1 - \sigma_1\sigma_3 \\
0 + \sigma_1\sigma_2 + \sigma_2\sigma_3 & 2 & 0 - \sigma_1\sigma_2 - \sigma_2\sigma_3
\end{bmatrix}
\] (171)

Check of the result:

\[
\begin{array}{c|ccc}
 & 1 & 0 & 0 \\
\hline
1 - \sigma_1\sigma_3 & 0 & 1 + \sigma_1\sigma_3 & 2 \\
0 - \sigma_1\sigma_2 - \sigma_2\sigma_3 & 2 & 0 + \sigma_1\sigma_2 + \sigma_2\sigma_3 & 0
\end{array}
\]

The Moore-Penrose generalized matrix inverse then is:

\[
B^+ = \frac{1}{2} \begin{bmatrix}
1 & 0 & 1 \\
0 & 2 & 0
\end{bmatrix}
\] (172)
The orthogonal bivector matrix to the Moore-Penrose generalized matrix inverse then is:

\[
\mathbf{B}^* = \frac{1}{2} \begin{bmatrix}
-\sigma_1\sigma_3 & 0 & \sigma_1\sigma_3 \\
-\sigma_1\sigma_2 - \sigma_2\sigma_3 & 0 & \sigma_1\sigma_2 + \sigma_2\sigma_3
\end{bmatrix}
\]  

(173)

The product of both Pauli algebra generalized matrix inverses then is:

\[
\mathbf{B}^{-1}\mathbf{A}^{-1} = \frac{1}{2} \begin{bmatrix}
1 - \sigma_1\sigma_3 - \sigma_1\sigma_4 - \sigma_3\sigma_4 & 0 & 1 + \sigma_1\sigma_3 & \sigma_1\sigma_4 + \sigma_3\sigma_4 \\
0 - 2\sigma_1\sigma_2 - 2\sigma_2\sigma_4 & 2 & 0 + \sigma_1\sigma_2 + \sigma_2\sigma_3 & \sigma_1\sigma_2 - \sigma_2\sigma_3 + 2\sigma_2\sigma_4
\end{bmatrix}
\]  

(174)

Check of the result:

|     | 1 | 0 |
|-----|---|---|
| 0   | 1 | 1 |
| 1   | 0 | 0 |
| 0   | 0 | 2 |

The Pauli algebra generalized matrix inverses of the resulting matrix \(\mathbf{D}\) then are:

\[
\mathbf{D}^{-1} = \frac{1}{3} \begin{bmatrix}
1 - \sigma_1\sigma_3 - \sigma_1\sigma_4 & 0 & 1 + \sigma_1\sigma_3 & \sigma_1\sigma_4 + \sigma_3\sigma_4 \\
0 - 2\sigma_1\sigma_2 - 2\sigma_2\sigma_4 & 2 & 0 + \sigma_1\sigma_2 + \sigma_2\sigma_3 & \sigma_1\sigma_2 - \sigma_2\sigma_3 + 2\sigma_2\sigma_4
\end{bmatrix}
\]  

(175)

\[
\mathbf{D}^{-1} = \frac{1}{3} \begin{bmatrix}
1 + \sigma_1\sigma_3 + \sigma_1\sigma_4 & 0 & 1 - \sigma_1\sigma_3 + \sigma_3\sigma_4 & 1 - \sigma_1\sigma_4 - \sigma_3\sigma_4 \\
0 + 2\sigma_1\sigma_2 + 2\sigma_2\sigma_3 + 2\sigma_2\sigma_4 & 3 & 0 - \sigma_1\sigma_2 - 2\sigma_2\sigma_3 + \sigma_2\sigma_4 & 0 - \sigma_1\sigma_2 + \sigma_2\sigma_3 - 2\sigma_2\sigma_4
\end{bmatrix}
\]  

(176)

Check of the result:

|     | 1 | 0 |
|-----|---|---|
| 0   | 1 | 1 |
| 1   | 0 | 0 |
| 0   | 0 | 3 |

The Moore-Penrose generalized matrix inverse then is:

\[
\mathbf{D}^+ = \frac{1}{3} \begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 3 & 0 & 0
\end{bmatrix}
\]  

(177)

The orthogonal bivector matrix to the Moore-Penrose generalized matrix inverse then is:

\[
\mathbf{D}^* = \frac{1}{3} \begin{bmatrix}
-\sigma_1\sigma_3 - \sigma_1\sigma_4 & 0 & \sigma_1\sigma_3 - \sigma_3\sigma_4 & \sigma_1\sigma_4 + \sigma_3\sigma_4 \\
-2\sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_2\sigma_4 & 0 & \sigma_1\sigma_2 + 2\sigma_2\sigma_3 - \sigma_2\sigma_4 & \sigma_1\sigma_2 - \sigma_2\sigma_3 + 2\sigma_2\sigma_4
\end{bmatrix}
\]  

(178)
As $B^{-1}A^{-1}$ of eq. (174) is not identical to $D^{-1}$ of eq. (175) multiplicativity fails and the example shows that statement (141) $B^{-1}A^{-1} \neq D^{-1}$ is indeed correct.

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