SKEW-SYMMETRIC VANISHING LATTICES
AND INTERSECTIONS OF SCHUBERT CELLS

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§1. Introduction and results

In the present paper we apply the theory of skew-symmetric vanishing lattices developed around 15 years ago by B. Wajnryb, S. Chmutov, and W. Janssen for the necessities of the singularity theory to the enumeration of connected components in the intersection of two open opposite Schubert cells in the space of complete real flags. Let us briefly recall the main topological problem considered in [SSV] and reduced there to a group-theoretical question solved below. Let $N^{n+1}$ be the group of real unipotent uppertriangular $(n+1) \times (n+1)$ matrices and $D_i$ be the determinant of the submatrix formed by the first $i$ rows and the last $i$ columns. Denote by $\Delta_i$ the divisor $\{D_i = 0\} \subset N^{n+1}$ and let $\Delta^{n+1}$ be the union $\bigcup_{i=1}^{n} \Delta_i$. Consider now the complement $U^{n+1} = N^{n+1} \setminus \Delta^{n+1}$. The space $U^{n+1}$ can be interpreted as the intersection of two open opposite Schubert cells in $SL_{n+1}(\mathbb{R})/B$.

In [SSV] we have studied the number of connected components in $U^{n+1}$. The main result of [SSV] can be stated as follows.

Consider the vector space $T^n(\mathbb{F}_2)$ of upper triangular matrices with $\mathbb{F}_2$-valued entries. We define the group $\mathfrak{G}_n$ as the subgroup of $GL(T^n(\mathbb{F}_2))$ generated by $\mathbb{F}_2$-linear transformations $g_{ij}$, $1 \leq i \leq j \leq n-1$. The generator $g_{ij}$ acts on a matrix $M \in T^n(\mathbb{F}_2)$ as follows. Let $M^{ij}$ denote the $2 \times 2$ submatrix of $M$ formed by rows $i$ and $i+1$ and columns $j$ and $j+1$ (or its upper triangle in case $i = j$). Then $g_{ij}$ applied to $M$ changes $M^{ij}$ by adding to each entry of $M^{ij}$ the $\mathbb{F}_2$-valued trace of $M^{ij}$, and does not change all the other entries of $M$. For example, if $i < j$, then $g_{ij}$ changes $M^{ij}$ as follows:

\[
\begin{pmatrix}
m_{ij} & m_{i,j+1} \\
m_{i+1,j} & m_{i+1,j+1}
\end{pmatrix}
\mapsto
\begin{pmatrix}
m_{i+1,j+1} & m_{i,j+1} + m_{ij} + m_{i+1,j+1} \\
m_{i+1,j} + m_{ij} + m_{i+1,j+1} & m_{ij}
\end{pmatrix}.
\]

All the other entries of $M$ are preserved. The above action on $T^n(\mathbb{F}_2)$ is called the first $\mathfrak{G}_n$-action.

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Proposition (Main Theorem of [SSV]). The number $\#_{n+1}$ of connected components in $U^{n+1}$ coincides with the number of orbits of the first $\mathfrak{S}_n$-action.

Here we calculate the number of orbits of the first $\mathfrak{S}_n$-action and prove the following result (conjectured in [SSV]).

Main Theorem. The number $\#_{n+1}$ of connected components in $U^{n+1}$ (or, equivalently, the number of orbits of the first $\mathfrak{S}_n$-action) equals $3 \times 2^n$ for all $n \geq 5$.

Cases $n+1 = 2, 3, 4$ or $5$ are exceptional and $\#_2 = 2$, $\#_3 = 6$, $\#_4 = 20$, $\#_5 = 52$.

The structure of the paper is as follows. In §2 we give a detailed description of the first $\mathfrak{S}_n$-action and its quotient by the subspace of invariants (called the second $\mathfrak{S}_n$-action). We formulate explicit conjectures about the types, number and cardinalities of orbits of both actions. In §3 we find linear invariants of these actions. In §4 we formulate a number of results about the monodromy group of a skew-symmetric vanishing lattice over $\mathbb{F}_2$. Using these results we count in §5 the number of orbits of the second $\mathfrak{S}_n$-action. Finally, in §6 we explain the relation of the original (first) $\mathfrak{S}_n$-action to the monodromy group of the corresponding vanishing lattice and prove the results stated in §2. The concluding §7 contains some final remarks and speculations about the origin and further applications of the above $\mathfrak{S}_n$-action in Schubert calculus.

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§2. Two $\mathfrak{S}_n$-actions on $\mathbb{F}_2$-valued uppertriangular matrices and their orbits. Main tables.

2.1. The first $\mathfrak{S}_n$-action and its invariants. Let $G$ be a group acting on a linear space $\mathcal{V}$ over $\mathbb{F}_2$. We say that $x \in \mathcal{V}$ is an invariant of the action if $g(x) = x$ for any $g \in G$, and that $f \in \mathcal{V}^*$ is a dual invariant if $(g(x), f) = (x, f)$ for any $x \in \mathcal{V}$, $g \in G$ (here $(\cdot, \cdot)$ is the standard coupling $\mathcal{V} \times \mathcal{V}^* \to \mathbb{F}_2$). Evidently, dual invariants are just the invariants of the conjugate action of $G$ on $\mathcal{V}^*$.

In what follows we identify $(T^n(\mathbb{F}_2))^*$ with the space of $\mathbb{F}_2$-valued uppertriangular matrices in such a way that for any pair $M \in T^n(\mathbb{F}_2)$, $M' \in (T^n(\mathbb{F}_2))^*$ one has $(M, M') = \sum_{1 \leq i \leq j \leq n} m_{ij} m'_{ij}$.

Let us define the matrices $R_i \in (T^n(\mathbb{F}_2))^*$, $1 \leq i \leq n$, as follows: all the entries of the rectangular submatrix of $R_i$ formed by the first $i$ rows and the last $n+1-i$ columns are ones, and all the other entries of $R_i$ are zeros. Thus, for any $M \in T^n(\mathbb{F}_2)$ the value $(M, R_i)$ is just the sum of the entries of $M \mod 2$ over the corresponding pattern $\rho_i$ (see Fig. 1).
Let $E_i \in T^n(\mathbb{F}_2)$, $1 \leq i \leq n$, be the matrix whose $i$th diagonal contains only ones, and all the other entries are zeros.

**Theorem 2.1.** (i) The subspace $I_n \subset T^n(\mathbb{F}_2)$ of invariants of the first $\mathfrak{G}_n$-action is an $n$-dimensional vector space. It has a basis consisting of the matrices $E_1, \ldots, E_n$.

(ii) The subspace $D_n \subset (T^n(\mathbb{F}_2))^*$ of dual invariants of the first $\mathfrak{G}_n$-action is an $n$-dimensional vector space. It has a basis consisting of the matrices $R_1, \ldots, R_n$.

Let $D_n^\perp \subset T^n(\mathbb{F}_2)$ be the subspace orthogonal to $D_n$ with respect to the standard coupling. The translation of $D_n^\perp$ by an arbitrary element $M \in T^n(\mathbb{F}_2)$ we call a stratum of the first $\mathfrak{G}_n$-action. By the definition, all the dual invariants are fixed at all the elements of a stratum. An $n$-dimensional vector $h^S = (h^S_1, \ldots, h^S_n)$ is said to be the height of a stratum $S$ (with respect to the basis $\{R_i\}$) if $(M, R_i) = h^S_i$ for any $M \in S$. A stratum is called symmetric if its height is symmetric with respect to its middle, that is, if $h^S_i = h^S_{n-i+1}$ for all $1 \leq i \leq n$.

Evidently, each stratum is a union of certain orbits of the first $\mathfrak{G}_n$-action. The structure of strata is described by Theorem 2.2 below. For the sake of simplicity, we omit the words “of the first $\mathfrak{G}_n$-action” in the formulation and write just “orbit” and “stratum”.

**Theorem 2.2.** (i) Let $n = 2k + 1 \geq 5$, then

- each of $2^{k+1}$ symmetric strata consists of one orbit of length $2^{2k^2+k-1} - \varepsilon_k 2^{k^2+k-1}$, one orbit of length $2^{2k^2+k-1} + \varepsilon_k 2^{k^2+k-1} - 2^k$, and $2^k$ orbits of length 1, where $\varepsilon_k = -1$ for $k = 4t + 1$ and $\varepsilon_k = 1$ otherwise;
- each of $2^n - 2^{k+1}$ nonsymmetric strata consists of two orbits of length $2^{2k^2+k-1}$.

(ii) Let $n = 2k \geq 6$, and let $\bar{h}$ denote the vector of length $n$ the first $k$ entries of which are equal to $(1, 0, 1, 0, \ldots)$ and the last $k$ entries vanish, then

- each of $2^k$ symmetric strata consists of two orbits of length $(2^{2k(k-1)} - 1)2^{k-1}$ and $2^k$ orbits of length 1;
- each of $2^k$ nonsymmetric strata $S$ such that $h^S - \bar{h}$ is symmetric with respect to its middle consists of one orbit of length $(2^{k(k-1)} - 1)2^{k^2-1}$ and one orbit of length $(2^{k(k-1)} + 1)2^{k^2-1}$.
each of $2^n - 2^{k+1}$ remaining nonsymmetric strata consists of two orbits of length $2^{2k^2-k-1}$.

The assertions of Theorem 2.2 can be summarized in the following table. Observe that the total number of orbits in both cases equals $3 \times 2^n$, and we thus get Main Theorem.

\[
\begin{array}{cccccc}
\text{type} & \text{cardinality} & \#_{\text{orb}} & \text{type} & \text{cardinality} & \#_{\text{orb}} \\
\hline
\text{trivial} & 1 & 2^{2k+1} & \text{trivial} & 1 & 2^{2k} \\
\text{standard} & 2^{2k^2+k-1} & 2^{2k+2} - 2^{k+2} & \text{standard} & 2^{2k^2-k-1} & 2^{2k+1} - 2^{k+2} \\
\text{type 1} & (2^{k^2} - \varepsilon_k)2^{k^2+k-1} & 2^{k+1} & \text{type 1} & (2^{k^2(k-1)} - 1)2^{k-1} & 2^{k+1} \\
\text{type 2} & (2^{k^2} - \varepsilon_k)(2^{k^2-1} + \varepsilon_k)2^{k} & 2^{k+1} & \text{type 2} & (2^{k(k-1)} - 1)2^{k^2-1} & 2^{k} \\
\text{type 3} & 0 & 0 & \text{type 3} & (2^{k-1} + 1)2^{k^2-1} & 2^{k} \\
\text{type 4} & 0 & 0 & \text{type 4} & 0 & 0 \\
\text{type 5} & 0 & 0 & \text{type 5} & 0 & 0 \\
\end{array}
\]

Table 1. The orbits of the first $\mathfrak{S}_n$-action

2.2. The second $\mathfrak{S}_n$-action and its invariants. Let us introduce a $\mathfrak{S}_n$-action on $T^{n-1}(\mathbb{F}_2)$ closely related to the first $\mathfrak{S}_n$-action on $T^n(\mathbb{F}_2)$, which we will call the *second $\mathfrak{S}_n$-action*. This action is induced by taking the quotient modulo the subspace $\mathcal{I}_n$ of the invariants of the first $\mathfrak{S}_n$-action. Recall that by Theorem 2.1(i) $\mathcal{I}_n$ is an $n$-dimensional subspace of all matrices having the same entry on each diagonal. One can suggest the following natural description of the second $\mathfrak{S}_n$-action.

Consider the linear map $\Psi_n : T^n(\mathbb{F}_2) \to T^{n-1}(\mathbb{F}_2)$ such that the $(i, j)$th entry in the image equals the sum of the $(i, j)$th and the $(i + 1, j + 1)$th entries in the inverse image (thus entry $(i, j)$ in the image can be considered as representing the submatrix $M^{ij}$ of the initial matrix). Evidently, $\ker \Psi_n = \mathcal{I}_n$, and we obtain the induced action of $\mathfrak{S}_n$ on $T^{n-1}(\mathbb{F}_2)$. For any $M \in T^{n-1}(\mathbb{F}_2)$ the generator $g_{ij}$ affects only the $3 \times 3$ submatrix of $M$ centered at $m_{ij}$ by adding $m_{ij}$ to all entries marked by asterisks in the shape below:

\[
\begin{pmatrix}
* & * & * \\
* & m_{ij} & * \\
* & * & *
\end{pmatrix}
\]

Here we use the convention that if the above $3 \times 3$-shape does not fit completely in the upper triangle, then we change only the entries that fit.

For any $i$, $1 \leq i \leq k = [n/2]$, we define a subset $\pi_i$ of the entries of the $(n - 1) \times (n - 1)$ triangular shape as follows. The subset $\pi_k$ coincides with the initial $(n - 1) \times (n - 1)$
triangular shape. To build $\pi_i$, $1 \leq i \leq k - 1$, cut off the three $i \times i$ triangular shapes placed in all the three corners of the initial shape and get a hexagonal shape $\chi^1_i$ with the sides of lengths $i + 1$ and $n - 2i - 1$. The shape $\chi^1_i$ is included into a nested family of $j_i = \min\{i + 1, n - 2i - 1\}$ hexagonal shapes $\chi^j_i$; the shape $\chi^1_i$ is obtained by peeling the external layer of $\chi^{j_i-1}_i$. The last shape $\chi^{j_i}_i$ degenerates to a triangle (upper if $j_i = i + 1$ and lower otherwise). The subset $\pi_i$ consists of the three $i \times i$ triangular shapes as above and $[j_i/2]$ layers of the form $\chi^{2j}_i \setminus \chi^{2j+1}_i$, $1 \leq j \leq [j_i/2]$; here we assume that $\chi^{j_i+1}_i = \emptyset$, and so for $j_i$ even the last layer consists of the whole triangle $\chi^{j_i}_i$.

We now define the matrices $P_i \in (T^{n-1}(F_2))^*$, $1 \leq i \leq k$, as follows: all the entries of $P_i$ that belong to $\pi_i$ are ones, and all the other entries of $P_i$ are zeros. Thus, for any $M \in T^{n-1}(F_2)$ the value $(M, P_i)$ is just the sum of the entries of $M$ mod 2 over the corresponding pattern $\pi_i$ (see Fig. 2).

**Fig. 2. Patterns of dual invariants for the second $\mathcal{G}_n$-action in case $n = 8$**

**Theorem 2.3.** (i) The subspace of invariants of the second $\mathcal{G}_n$-action is trivial.

(ii) The subspace $\mathfrak{D}_{n-1} \subset (T^{n-1}(F_2))^*$ of dual invariants of the second $\mathcal{G}_n$-action is a $k$-dimensional vector space, $k = [n/2]$. It has a basis consisting of the matrices $P_1, ..., P_k$.

It follows immediately from Theorem 2.3(ii) that $\tilde{P}_i = P_i + P_{i-1}$, where $P_0 = 0$, is also a basis of $\mathfrak{D}_{n-1}$. The strata of the second $\mathcal{G}_n$-action and their heights (with respect to the basis $\{\tilde{P}_1\}$), which we denote $\eta$, are defined in the same way as for the first $\mathcal{G}_n$-action. It is easy to see that $\Psi_n$ takes any stratum of the first $\mathcal{G}_n$-action to a stratum of the second $\mathcal{G}_n$-action. Denote by $\psi_n : F_2^n \to F_2^k$ the mapping that takes $h^S$ to $\eta^{\Psi_n(S)}$. The following statement gives an explicit description of $\psi_n$.

**Theorem 2.4.** The heights of the strata $S$ and $\Psi_n(S)$ with respect to the bases $\{R_i\}$ and $\{\tilde{P}_1\}$ satisfy the relations

$$\eta^{\Psi_n(S)}_i = h^S_i + h^S_{i+1} + h^S_{n-i} + h^S_{n-i+1} \quad \text{for} \quad 1 \leq i < k,$$

$$\eta^{\Psi_n(S)}_k = h^S_k + h^S_{n-k+1}.$$

The structure of the strata of the second $\mathcal{G}_n$-action is described in Theorem 2.5 below. For the sake of simplicity, we omit the words “of the second $\mathcal{G}_n$-action” in the formulation and write just “orbit” and “stratum”.

Theorem 2.5. (i) Let \( n = 2k + 1 \geq 5 \), then

the stratum at height \((0, \ldots, 0)\) consists of one orbit of length \(2^{2k^2-1} - \varepsilon_k 2^{k^2-1}\), one orbit of length \(2^{2k^2-1} + \varepsilon_k 2^{k^2-1} - 1\), and one orbit of length 1, where \(\varepsilon_k = -1\) for \(k = 4t+1\) and \(\varepsilon_k = 1\) otherwise;

each of the other \(2^k - 1\) strata is an orbit of length \(2^k\).

(ii) Let \( n = 2k \geq 6\), and let \(\bar{n}\) denote the vector of length \(k\) the first \(k - 1\) entries of which are equal to 1 and the last entry equals \(k \mod 2\), then

the stratum at height \((0, \ldots, 0)\) consists of one orbit of length \(2^{k(k-1)} - 1\) and one orbit of length 1;

the stratum at height \(\bar{n}\) consists of one orbit of length \((2^{k(k-1)} - 1)2^{k(k-1)-1}\) and one orbit of length \((2^k - 1)2^{k(k-1)-1}\);

each of the other \(2^k - 2\) strata is an orbit of length \(2^{k(k-1)}\).

The assertions of Theorem 2.5 can be summarized in the following table. Observe that the total number of orbits in both cases equals \(2^k + 2\).

| type   | cardinality | \(#_{\text{orb}}\) | type   | cardinality | \(#_{\text{orb}}\) |
|--------|-------------|------------------|--------|-------------|------------------|
| trivial | 1           | 1                | standard | \(2^{2k^2}\) | \(2^k - 1\)      | \(2^{k(k-1)}\) | \(2^k - 2\)     |
| type 1  | \((2^k - \varepsilon_k)2^{k^2-1}\) | 1                | type 2  | \((2^k - \varepsilon_k)(2^{k^2-1} + \varepsilon_k)\) | 1                |
| type 3  | 0           | 0                | type 4  | 0           | 0                | \((2^{k(k-1)} - 1)\) | 1                |
| type 5  | 0           | 0                | type 5  | 0           | 0                | \((2^{k(k-1)} + 1)2^{k(k-1)-1}\) | 1                |

Table 2. The orbits of the second \(\mathfrak{S}_n\)-action

Remark. The map \(\Psi_n\) sends the orbits of the first \(\mathfrak{S}_n\)-action to the orbits of the second \(\mathfrak{S}_n\)-action with the same name (i.e. trivial to trivial, standard to standard, etc).

§3. INVARIANTS AND DUAL INVARIANTS OF THE \(\mathfrak{S}_n\)-ACTIONS

In this section we prove Theorems 2.1, 2.3 and 2.4.

3.1. The first \(\mathfrak{S}_n\)-action. Recall that we have identified \((T^n(\mathbb{F}_2))^*\) with the space of \(\mathbb{F}_2\)-valued uppertriangular matrices in such a way that for any pair \(M \in T^n(\mathbb{F}_2), M' \in (T^n(\mathbb{F}_2))^*\) one has \((M, M') = \sum_{1 \leq i \leq j \leq n} m_{ij} m'_{ij}.\)
Lemma 3.1. The conjugate to the first $G_n$-action is given by

$$(g_{ij} M')_{kl} = \begin{cases} 
    m_{i+1,j} + m_{i,j+1} + m_{i+1,j+1} & \text{if } k = i, l = j, \\
    m_{i+1,j} + m_{i,j+1} + m'_{i,j} & \text{if } k = i + 1, l = j + 1, \\
    m_{k,l} & \text{otherwise.}
\end{cases}$$

Proof. Indeed, the action of $g_{ij}$ on $T^n(F_2)$ affects only $M^{ij}$, while the action of $g_{ij}$ on $(T^n(F_2))^*$ defined above affects only $(M')^{ij}$. Therefore, it suffices to consider the actions in the corresponding 4-dimensional spaces. The matrix of $g_{ij}$ in coordinates $m_{ij}, m_{i,j+1}, m_{i+1,j}, m_{i+1,j+1}$ is

$$\begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix},$$

hence its conjugate is exactly as asserted by the lemma. □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. (i) Follows immediately from the fact that $I_n$ is defined by the equations $m_{ij} + m_{i+1,j+1} = 0$, $1 \leq i \leq j \leq n - 1$.

(ii) By Lemma 3.1, $D_n$ is defined by the equations $m'_{ij} + m'_{i,j+1} + m'_{i+1,j} + m'_{i+1,j+1} = 0$, $1 \leq i \leq j \leq n - 1$ (with the same as above convention concerning the case when only a part of the submatrix fits into the uppertriangular shape). One can prove easily that these \((n-1)/2\) equations are linearly independent. Indeed, each of $m'_{1j}$, $1 < j < n$, enters exactly two equations, while $m'_{11}$ only one equation. It follows immediately that none of the equations involving $m'_{1j}$ may participate in a nontrivial linear combination. The rest of the equations correspond to the same situation in dimension $n - 1$, so their linear independence follows by induction. We thus get $\dim D_n = n$. It remains to show that $R_1, \ldots, R_n$ provide a basis for $D_n$. Evidently, all these matrices are linearly independent and satisfy the equations defining $D_n$. □

3.2. The second $G_n$-action. For any matrix entry $m_{ij}$ we define its neighbors as all the entries marked by asterisks in (1), or, more precisely, those of $m_{i-1,j-1}, m_{i-1,j}, m_{i,j-1}, m_{i+1,j}, m_{i+1,j+1}, m_{i+1,j+1}$ that fit in the uppertriangular shape. It is helpful to consider a graph $\mathcal{H}_{n-1}$ whose vertices are all the matrix entries and edges join each entry with its neighbors. It is easy to see that $\mathcal{H}_{n-1}$ is an equilateral triangle with the sides of length $n - 2$ on the triangular lattice (see Fig. 3). One can give the following description of the second $G_n$-action in terms of $\mathcal{H}_{n-1}$: $g_{ij}$ acts on the space of $F_2$-valued functions on $\mathcal{H}_{n-1}$ by adding the value at $m_{ij}$ to the values at all of its neighbors.
Lemma 3.2. The conjugate to the second $\mathfrak{G}_n$-action acts as follows: $g_{ij}$ adds to the value at $m_{ij}$ the sum of the values at all its neighbors.

Proof. Similarly to the proof of Lemma 3.1, it suffices to consider the action in the corresponding 7-dimensional subspaces. The details are straightforward. 

To prove Theorem 2.3, we need the following technical proposition. Consider the subspace $\mathfrak{D}^1_{n-1} \subset (T^{n-1}(\mathbb{F}_2))^*$ of all linear forms invariant under the subgroup of $\mathfrak{G}_n$ generated by $\{g_{ij}: i \geq 2\}$; evidently, $\mathfrak{D}_{n-1} \subseteq \mathfrak{D}^1_{n-1}$. Let $\omega: \mathfrak{D}^1_{n-1} \to \mathbb{F}_2^{n-1}$ denote the projection on the first row, and let $\text{Sym}^p \subset \mathbb{F}_2^p$ denote the space of all vectors symmetric with respect to their middle.

Lemma 3.3. (i) The $i$th row of an arbitrary matrix $M \in \mathfrak{D}^1_{n-1}$ belongs to $\text{Sym}^{n-i}$, $1 \leq i \leq n-1$.

(ii) $\dim \mathfrak{D}^1_{n-1} = n-1$.

(iii) The image of $\omega$ coincides with $\text{Sym}^{n-1}$.

Proof. For $n = 3$ an immediate check shows that $\mathfrak{D}^1_2$ consists of the following four matrices:

$$
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 \\
1 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix},
$$

and so all the assertions of the lemma hold true.

Consider now an arbitrary $M \in \mathfrak{D}^1_{n-1}$. Let $a = (a_1, \ldots, a_{n-1})$, $b = (b_1, \ldots, b_{n-2})$, $c = (c_1, \ldots, c_{n-3})$ be the first three rows of $M$. It follows easily from the definition of $\mathfrak{D}^1_{n-1}$ that $a, b, c$ satisfy equations $a_i = a_1 + b_1 + b_i + b_{i-1} + c_{i-1}$, $1 \leq i \leq n-1$, where $b_0 = c_0 = 0$. Evidently, the submatrix of $M$ obtained by deleting the first row belongs to $\mathfrak{D}^1_{n-2}$. By induction, we may assume that the rows of this matrix are symmetric, that is, $b_i = b_{n-1-i}$, $1 \leq i \leq n-2$, and $c_i = c_{n-2-i}$, $1 \leq i \leq n-3$. We thus have $a_{n-i} = a_1 + b_1 + b_{n-i} + b_{n-1-i} + c_{n-1-i} = a_i$, $1 \leq i \leq n-1$, and hence the first row of $M$ is symmetric as well. Moreover, it is clear that the solutions of the above equations form a 1-dimensional affine subspace, and hence $\dim \mathfrak{D}^1_{n-1} = \dim \mathfrak{D}^1_{n-2} + 1 = n-1$. To get the third statement it suffices to notice that the first two rows define $M$ uniquely, and that the dimension of the subspace spanned by these two rows is exactly $n-1$. 

Proof of Theorem 2.3. (i) Follows immediately from the definition.
(ii) Let us define the linear mapping \( \nu : \mathfrak{D}^1_{n-1} \to \mathbb{F}^{n-1}_2 \) by the following rule: \( \nu_i(M) \) is the sum of the neighbors of \( m_{1i} \). By definition, \( \mathfrak{D}_{n-1} = \ker \nu \), so in order to find \( \dim \mathfrak{D}_{n-1} \) it suffices to determine the image of \( \nu \). As in the proof of Lemma 3.3, denote by \( a \) and \( b \) the first two rows of \( M \). Clearly, \( \nu_i = a_{i+1} + a_i + b_i + b_{i-1} \), \( 1 \leq i \leq n-1 \), where \( a_0 = a_n = b_0 = 0 \). These equations define a linear mapping that takes the space \( \text{Sym}^{n-1} \) of the first rows to the image of \( \nu \). It is easy to see that the corank of this mapping is zero for \( n-1 = 2k \) and one for \( n-1 = 2k-1 \); in other words, it equals \([n/2] - [(n-1)/2] \). Since \( \dim \text{Sym}^{n-1} = [n/2] \), we get that the dimension of the image of \( \nu \) equals \([(n-1)/2] \). It follows immediately that \( \dim \mathfrak{D}_{n-1} = n-1 - [(n-1)/2] = [n/2] \).

The remaining assertion about the structure of dual invariants \( P_i \) is rather obvious. Its proof consists of the two simple checks: that \( P_i \)'s are linearly independent and that any matrix entry has an even number of nonzero neighbors (shown by shaded areas on Fig. 2). Both are straightforward. \( \square \)

Now we can prove Theorem 2.4.

**Proof of Theorem 2.4.** It follows easily from Theorem 2.1 that the space \( D^n(\mathbb{F}_2) \) of diagonal \( n \times n \) matrices is transversal to the strata of the first action; moreover, \( \Psi^n \) takes \( D^n(\mathbb{F}_2) \) to \( D^{n-1}(\mathbb{F}_2) \). Therefore it is enough to check the assertion of the theorem only for diagonal matrices. For \( M \in D^n(\mathbb{F}_2) \) and \( M' = \Psi_n(M) \) Theorems 2.1 and 2.3 give

\[
(M', \tilde{P}_i) = m'_{ii} + m'_{n-i,n-i} + m'_{n-i+1,n-i+1} =
\]

\[
(m_{ii} + m_{i+1,i+1}) + (m_{n-i,n-i} + m_{n-i+1,n-i+1}) =
\]

\[
(M, R_i) + (M, R_{i+1}) + (M, R_{n-i}) + (M, R_{n-i+1})
\]

for \( 1 \leq i < k \). For \( n = 2k+1 \) this equality holds also for \( i = k \), but now \( k+1 = n-k \), hence the two middle terms vanish and thus \( (M', \tilde{P}_k) = (M, R_k) + (M, R_{n-k+1}) \). Finally, for \( n = 2k \) we have \( (M', \tilde{P}_k) = m'_{kk} = (M, R_k) + (M, R_{k+1}) = (M, R_k) + (M, R_{n-k+1}) \). \( \square \)

### §4. Some results about skew-symmetric vanishing lattices

In this section we quote and prove a number of results concerning the natural action of a group generated by transvections preserving a given skew-symmetric bilinear form on a vector space over \( \mathbb{F}_2 \). Our main reference is [Ja]. The relation to our original problem is explained in details in the next section and is based on the fact that the conjugate to the second \( \mathfrak{S}_n \)-action is exactly of this kind. This allows us to describe completely its orbits, as well as the orbits of the second \( \mathfrak{S}_n \)-action itself.

#### 4.1. Vanishing lattices and their monodromy groups.

We assume that \( \mathcal{V} \) is a vector space over \( \mathbb{F}_2 \) equipped with a skew-symmetric bilinear form \( \langle \cdot, \cdot \rangle = \langle \cdot, L(\cdot) \rangle \), where \( L \) is a linear map \( L : \mathcal{V} \to \mathcal{V}^* \). A quadratic function \( q \) associated with \( \langle \cdot, \cdot \rangle \) is an arbitrary \( \mathbb{F}_2 \)-valued function on \( \mathcal{V} \) satisfying

\[
q(\lambda x + \mu y) = \lambda^2 q(x) + \mu^2 q(y) + \lambda \mu \langle x, y \rangle.
\]

It is clear that a quadratic function completely determines the corresponding bilinear form. With any basis \( B \) of \( \mathcal{V} \) we associate a unique quadratic function \( q_B \) by requiring it to take value 1 on all elements of \( B \).
Let $K = \ker L$ be the kernel of $\langle \cdot, \cdot \rangle$, $\kappa = \dim K$, and $(e_1, f_1, \ldots, e_m, f_m, g_1, \ldots, g_n)$ be a symplectic basis for $\langle \cdot, \cdot \rangle$, that is, $\langle x, y \rangle = \sum (x_i y'_i + y_i x'_i)$, where $x = \sum x_i e_i + \sum x'_i f_i + \sum x'_i g_i$ and $y = \sum y_i e_i + \sum y'_i f_i + \sum y'_i g_i$. If the restriction $q|_K$ vanishes, then one can define the Arf invariant of $q$ by \( \text{Arf}(q) = \sum_{i=1}^m q(e_i)q(f_i) \), see e.g. [Pf]. For fixed dimensions of $V$ and $K$ there exist, up to isomorphisms, at most three possibilities: (i) $q(K) = 0$, $\text{Arf}(q) = 1$; (ii) $q(K) = 0$, $\text{Arf}(q) = 0$; (iii) $q(K) = \mathbb{F}_2$ (and so $\kappa \geq 1$).

In the first case one has $|q^{-1}(1)| = 2^{2m+\kappa-1} + 2^{m+\kappa-1}$, $|q^{-1}(0)| = 2^{2m+\kappa-1} - 2^{m+\kappa-1}$, in the second case one has $|q^{-1}(1)| = 2^{2m+\kappa-1} - 2^{m+\kappa-1}$, $|q^{-1}(0)| = 2^{2m+\kappa-1} + 2^{m+\kappa-1}$, and in the third case one has $|q^{-1}(1)| = |q^{-1}(0)| = 2^{2m+\kappa-1}$.

For any $\delta \in V$ we define the symplectic transvection $T_\delta : V \to V$ by $T_\delta(x) = x - \langle x, \delta \rangle \delta$. (Notice that $T_\delta$ is an element of the group $\text{Sp}^V$ of the automorphisms of $(V, \langle \cdot, \cdot \rangle)$). Given any subset $\Delta \subseteq V$, we let $\Gamma_\Delta \subseteq \text{Sp}^V$ denote the subgroup generated by the transvections $T_\delta$, $\delta \in \Delta$.

The main object of this section is a vanishing lattice, that is, a triple $(V, \langle \cdot, \cdot \rangle, \Delta)$ satisfying the following three conditions: (i) $\Delta$ is a $\Gamma_\Delta$-orbit; (ii) $\Delta$ generates $V$; (iii) if rank $V > 1$, then there exist $\delta_1, \delta_2 \in \Delta$ such that $\langle \delta_1, \delta_2 \rangle = 1$. The group $\Gamma_\Delta$ is called the monodromy group of the lattice.

We say that a basis $B$ of $V$ is weakly distinguished if $\Gamma_B = \Gamma_\Delta$. In this case $\Gamma_B$ respects $q_B$, so, in particular, $q_B(\delta) = 1$ for all $\delta \in \Delta$. Bases $B$ and $B'$ are called equivalent if $B' \subset \Gamma_B \cdot B$ and $B \subset \Gamma_{B'} \cdot B'$. One can easily see that if $B$ and $B'$ are equivalent, then $\Gamma_B = \Gamma_{B'}$, $q_B = q_{B'}$, and if $B$ is a weakly distinguished basis for $(V, \langle \cdot, \cdot \rangle, \Delta)$, then so is $B'$.

Let $B = (b_1, \ldots, b_d)$ be a basis in $V$, $d = \dim V$. We define the graph $\text{gr}(B)$ of $B$ as follows: $B$ is its vertex set, and $b_i$ is connected by an edge with $b_j$ if $\langle b_i, b_j \rangle = 1$. It is easy to see that $\text{gr}(B)$ is connected if $B$ is weakly distinguished.

A basis is called special if it is equivalent to a basis $B = (b_1, \ldots, b_d)$ such that for some $k$, $1 \leq k \leq d$, we have $\langle b_i, b_j \rangle = 0$ if $i = j$ or $i, j \geq k + 1$, and nonspecial otherwise. A vanishing lattice admitting a special (resp. nonspecial) weakly distinguished basis and its monodromy group are called special (resp. nonspecial).

Nonspecial monodromy groups have an especially simple characterization (we are primarily interested in this case since for $n \geq 5$ the group $\mathfrak{G}_n$ introduced in §1 can be interpreted as a nonspecial monodromy group, see §5 for details.)

**Theorem 4.1** ([Ja], Th. 3.8). Let $(V, \langle \cdot, \cdot \rangle, \Delta)$ be a vanishing lattice admitting a nonspecial weakly distinguished basis $B$. Then $\Gamma_B$ coincides with the subgroup $O^V_B(q_B)$ of $\text{Sp}^V$ consisting of all automorphisms that preserve $q_B$.

It follows that the classification of nonspecial vanishing lattices reduces to the classification of the corresponding quadratic functions. We thus see that for given $m$ and $\kappa$ such that $\dim V = 2m + \kappa$, $\dim K = \kappa$ there exist exactly three nonspecial vanishing lattices, depending on the values of $q_B(K)$ and $\text{Arf}(q_B)$. They are denoted by $O^V_B(2m, \kappa, \mathbb{F}_2)$, $O^V_B(2m, \kappa, \mathbb{F}_2)$, and $O^V(2m, \kappa, \mathbb{F}_2)$, and correspond to the cases $\text{Arf}(q_B) = 1$, $\text{Arf}(q_B) = 0$, and $q_B(K) = \mathbb{F}_2$, respectively (see [Ja, 4.2] for details).

The following statement, which can be extracted easily from [Ja, §4], provides a sufficient condition for a vanishing lattice to be nonspecial.
Lemma 4.2. A vanishing lattice is nonspecial if it admits a weakly distinguished basis \( B \) such that \( \text{gr}(B) \) contains the standard Dynkin diagram of the Coxeter group \( E_6 \) as an induced subgraph.

4.2. Orbits of nonspecial monodromy groups and of the conjugate actions. First of all, let us find the number of orbits of the monodromy group in the nonspecial case.

Lemma 4.3. Let \( (\mathcal{V}, \langle \cdot, \cdot \rangle, \Delta) \) be a vanishing lattice admitting a nonspecial weakly distinguished basis \( B \). Then the number of orbits of \( \Gamma_B \) equals \( 2^\kappa + 2 \), where \( \kappa = \dim \mathcal{K} \). These orbits are the \( 2^\kappa \) points of \( \mathcal{K} \) and the sets \( q_B^{-1}(0) \backslash \mathcal{K} \) and \( q_B^{-1}(1) \backslash \mathcal{K} \).

Proof. Evidently, any group generated by transvections acts trivially on \( \mathcal{K} \). Next, in the nonspecial case \( q_B^{-1}(1) \backslash \mathcal{K} \) is an orbit of \( \Gamma_B \) by Theorem 3.5 of [Ja]. To prove that \( q_B^{-1}(0) \backslash \mathcal{K} \) is an orbit as well, take an arbitrary pair \( u, v \notin \mathcal{K} \) such that \( q_B(u) = q_B(v) = 0 \). It is easy to see that there exist \( u', v' \notin \mathcal{K} \) such that \( q_B(u') = q_B(v') = 1 \) and \( \langle u, u' \rangle = \langle v, v' \rangle = 1 \). Define \( \mathcal{D}_u = \{ w \in \mathcal{V} : \langle w, u \rangle = \langle w, u' \rangle = 0 \} \), \( \mathcal{D}_v = \{ w \in \mathcal{V} : \langle w, v \rangle = \langle w, v' \rangle = 0 \} \). Evidently, \( \mathcal{K} \) is a subspace of both \( \mathcal{D}_u \) and \( \mathcal{D}_v \), and \( \dim \mathcal{D}_u = \dim \mathcal{D}_v = d - 2 \). Let \( q_u \) and \( q_v \) be the restrictions of \( q_B \) to \( \mathcal{D}_u \) and \( \mathcal{D}_v \), respectively. If \( q_B(\mathcal{K}) = \mathbb{F}_2 \), then the same is true for \( q_u \) and \( q_v \), and so the quadratic spaces \( (\mathcal{D}_u, q_u) \) and \( (\mathcal{D}_v, q_v) \) are isometric. Otherwise, if \( q_B(\mathcal{K}) = 0 \), the Arf invariants for the forms \( q_B, q_u, q_v \) are defined. Moreover, \( \text{Arf}(q_u) = \text{Arf}(q_v) = 1 + \text{Arf}(q_B) \), and hence \( (\mathcal{D}_u, q_u) \) and \( (\mathcal{D}_v, q_v) \) are again isometric by the Arf theorem (see [Pf]). In both cases the isometry \( \mathcal{D}_u \to \mathcal{D}_v \) can be extended to an isometry of the entire \( \mathcal{V} \) by letting \( u \mapsto v, u' \mapsto v' \). By Theorem 4.1 this isometry belongs to \( \Gamma_B \). \( \square \)

Let us consider now the conjugate to the action of the monodromy group of a vanishing lattice. We start from the following well-known result.

Lemma 4.4. Let \( G \) be a finite group acting on a finite-dimensional space \( \mathcal{V} \) over a finite field \( \mathfrak{k} \). Then the number of orbits of the action equals the number of orbits of the conjugate action.

Proof. The number of orbits of a finite group action on any finite set \( X \) can be calculated using the Frobenius formula (see e.g. [Ke]):

\[
\sharp = \frac{1}{|G|} \sum_{g \in G} |X^g|,
\]

where \( X^g \) is the set of all \( g \)-stable points. But the set \( \mathcal{V}^0 \) of the stable points for the linear operator \( E_g \) coincides with \( \ker(E_g - E) \), where \( E \) is the identity operator. Therefore, \( |\mathcal{V}^0| = |\mathfrak{k}|^{\dim \ker(E_g - E)} \). The equality \( \dim \ker(E_g - E) = \dim \ker(E_g - E)^* = \dim \ker(E_g^* - E) \) proves the statement. \( \square \)

Consider now the relation between the orbits of a nonspecial monodromy group \( \Gamma \) and the orbits of the conjugate action. Choose some basis \( \{b_1, \ldots, b_\kappa\} \) of \( \mathcal{K} \) and a \( \kappa \)-tuple \( \eta = \{\eta_1, \ldots, \eta_\kappa\} \in \mathbb{F}_2^\kappa \). Denote by \( \mathcal{A}^0 \) the affine subspace of \( \mathcal{V}^* \) of codimension \( \kappa \) that consists of all elements \( x \in \mathcal{V}^* \) such that \( (b_j, x) = \eta_j, 1 \leq j \leq \kappa \). Evidently, \( \mathcal{A}^0 \) is invariant.
under the conjugate action of $\Gamma$ for any $\eta \in \mathbb{F}_2^\kappa$. Moreover, since $\langle \cdot, \cdot \rangle$ is skew-symmetric, we see that the image $L(V)$ of the map $L: V \to V^*$ coincides with $A^0$.

Our strategy is as follows. To study the structure of the conjugate action we use an additional construction. We introduce two subspaces $V_1 \subset V$ and $V_2 \subset V$, which are transversal to $K$ (therefore $\dim V_1 = \dim V_2 = 2m$) and $V_1 + V_2 = V$. To each subspace $V_i$ corresponds a subgroup $\Gamma_i \subset \Gamma$ (generated by transvections w.r.t. the elements in $V_i$). One can easily see that the set $\Gamma_1 \cup \Gamma_2$ generates $\Gamma$. We first study the orbits of $\Gamma_i$ and of the corresponding conjugate action separately (which is fairly simple), and then describe their interaction. We start with the case of one subspace.

Consider a subspace $V_1 \subset V$ transversal to $K$. The restriction of the form $\langle \cdot, \cdot \rangle$ is nondegenerate on $V_1$. Let $\Gamma_1 \subset \Gamma$ be the subgroup generated by the transvections w.r.t. elements of $V_1$.

**Lemma 4.5.** Let $\Gamma$ be nonspecial. Then for any $\eta \in \mathbb{F}_2^\kappa$ the affine subspace $A^\eta$ consists of three orbits of the conjugate $\Gamma_1$-action.

**Proof.** Observe first that for any $\eta \in \mathbb{F}_2^\kappa$ the intersection of $A^\eta$ and $V_1^\perp \subset V^*$ contains exactly one element, which we denote by $A_1^\eta$. Indeed, if $\{v_1, \ldots, v_{2m}\}$ is an arbitrary basis of $V_1$, then $A_1^\eta$ is the unique solution of the equations $(bj, A_1^\eta) = \eta_j$, $1 \leq j \leq \kappa$, $(vj, A_1^\eta) = 0$, $1 \leq j \leq 2m$.

Next, let $L_1$ be the restriction of $L$ to $V_1$, and $L_1^\eta$ be the composition of $L_1$ with the translation by $A_1^\eta$. We say that $A_1^\eta$ is the *shift* corresponding to $A^\eta$; observe that $A_1^0 = 0$, and thus $L_1^0 = L_1$. Evidently, $L_1^\eta$ provides an affine isomorphism between $V_1$ and $A^\eta$. Recall that the conjugate action of $\Gamma$ (and hence, of $\Gamma_1$) preserves $A^\eta$. Moreover, since $\Gamma_1$ preserves $V_1$, the element $A_1^\eta$ is a fixed point of the conjugate to the $\Gamma_1$-action, as the unique annihilator of $V_1$ in $A^\eta$. Therefore, the diagram

$$
\begin{array}{ccc}
V_1 & \xrightarrow{L_1^\eta} & A^\eta \\
\downarrow{g_1} & & \uparrow{g_1^*} \\
V_1 & \xrightarrow{L_1^0} & A^\eta
\end{array}
$$

is commutative for any $g_1 \in \Gamma_1$. Since $\Gamma$ is nonspecial, the same is true for $\Gamma_1$ (by Lemma 4.2). It remains to apply Lemma 4.3 with $\kappa = 0$. □

Let us now choose one more subspace $V_2 \subset V$ transversal to $K$ such that $V_1 + V_2 = V$ and $\Gamma_1 \cup \Gamma_2$ generates $\Gamma$. To study the orbits of the conjugate $\Gamma$-action we have to find which orbits of the conjugate $\Gamma_1$-action are glued together by the conjugate $\Gamma_2$-action. Let us define an affine isomorphism $I^n: V_1 \to V_2$ by $I^n = (L_2^\eta)^{-1} \circ L_1^\eta$, where $L_2$ is the restriction of $L$ to $V_2$.

**Lemma 4.6.** Let $\Gamma$ be nonspecial.

(i) For $\eta = 0$ the following alternative holds: the linear space $A^0$ consists of two orbits of the conjugate $\Gamma$-action if there exist $u, v \in V_1$ such that $q_B(u) = 0$, $q_B(v) = q_B(I^0(u)) = q_B(I^0(v)) = 1$, and of three orbits of the conjugate $\Gamma$-action otherwise.

(ii) For any $\eta \neq 0$ the following alternative holds: the affine subspace $A^\eta$ consists of one orbit of the conjugate $\Gamma$-action if there exist $u, v \in V_1$ such that $q_B(u) = 0$, $q_B(v) = q_B(I^0(u)) = q_B(I^0(v)) = 1$, and of two orbits of the conjugate $\Gamma$-action otherwise.
Proof. The first part of this lemma follows easily from Lemma 4.3 with \( \kappa = 0 \) together with Lemma 4.5. To prove the second part we notice first that \( I^n(0) \neq 0 \), and then we are done by the same reasons. \( \square \)

§5. Orbits of the second \( \mathfrak{S}_n \)-action

In this section we prove Theorem 2.5 describing the orbits of the second \( \mathfrak{S}_n \)-action on the space of upper-triangular \( (n-1) \times (n-1) \) matrices over \( \mathbb{F}_2 \). Observe that the basic convention in this section is different from that of §4, namely, the conjugate to the second \( \mathfrak{S}_n \)-action is the action preserving a natural skew-symmetric form, and we study the second \( \mathfrak{S}_n \)-action itself as the conjugate to its conjugate using Lemmas 4.4-4.6.

5.1. The conjugate to the second \( \mathfrak{S}_n \)-action. In this section we assume that \( \mathcal{V} = (T^{n-1} (\mathbb{F}_2))^* \). Let us introduce a bilinear form \( \langle \cdot, \cdot \rangle \) on \( \mathcal{V} \) by \( \langle M, N \rangle = \sum_{(i_1,j_1),(i_2,j_2)} m_{i_1,j_1} n_{i_2,j_2} \), where \((i_1,j_1),(i_2,j_2)\) runs over all pairs of neighbors in \( \mathcal{S}_{n-1} \), see §3. Evidently, \( \langle \cdot, \cdot \rangle \) is skew-symmetric.

Let \( B_{n-1} \) be the standard basis of \( \mathcal{V} \) consisting of matrix entries. Elements of \( B_{n-1} \) correspond bijectively to the vertices of \( \mathcal{S}_{n-1} \). Denote by \( Q \) the quadratic function \( q_{B_{n-1}} \) associated with \( \langle \cdot, \cdot \rangle \). To find the value \( Q(M) \) for an arbitrary matrix \( M \in \mathcal{V} \) we represent \( M = \sum_{b_i \in B_{n-1}} \alpha_i b_i \) and define \( supp \) \( M = \{ b_i \in B_{n-1} : \alpha_i = 1 \} \). Let \( gr(M) \) denote the subgraph of \( \mathcal{S}_{n-1} \) induced by the vertices corresponding to \( supp \) \( M \). The following statement follows easily from the definitions.

Lemma 5.1. The value \( Q(M) \) equals \( \mod 2 \) the number of vertices of \( gr(M) \) plus the number of its edges.

As a corollary, we obtain the values of \( Q \) on the dual invariants \( \tilde{P}_1, \ldots, \tilde{P}_k \).

Corollary 5.2. (i) Let \( n = 2k + 1 \), then \( Q(\tilde{P}_i) = 0 \) for \( 1 \leq i \leq k \).

(ii) Let \( n = 2k \), then \( Q(\tilde{P}_i) = 1 \) for \( 1 \leq i \leq k-1 \), and \( Q(\tilde{P}_k) = k \mod 2 \).

Proof. (i) It follows from the description of the patterns \( \pi_i \) that \( gr(P_i) \), \( 1 \leq i < k \), consists of three copies of \( \mathcal{S}_1 \), several disjoint cycles, and one more copy of \( \mathcal{S}_{l_1} \) with \( l = |n-3i-2| + 1 \), provided \( j_i = \min\{i+1, n-2i-1\} \) is even. Therefore, by Lemma 5.1, we have \( Q(P_i) = 3i(2i-1) \mod 2 \) for \( j_i = i+1 \), \( i \) even, and \( Q(P_i) = 3i(2i-1) + l(2l-1) \mod 2 \) otherwise. Since \( n \) is odd, we have \( l = i \mod 2 \), and hence \( Q(P_i) = 0 \), \( 1 \leq i < k \). Finally, for \( P_k \) we have \( Q(P_k) = (n-1)(2n-3) = 2k(4k-1) \mod 2 = 0 \). It remains to notice that \( Q(\tilde{P}_i) = Q(P_i) + Q(P_{i-1}) \).

(ii) In the same way as in (i) we see that if \( 1 \leq i < k \), then \( Q(P_i) = 3i(2i-1) + l(2l-1) \mod 2 \) for \( j_i = i+1 \), \( i \) odd, and \( Q(P_i) = 3i(2i-1) \mod 2 \) otherwise. Since \( n \) is even, we have \( l = i + 1 \mod 2 \), and hence \( Q(P_i) = 0 \) for even \( i < k \) and \( Q(P_i) = 1 \) for odd \( i < k \). Finally, for \( P_k \) we have \( Q(P_k) = (n-1)(2n-3) = (2k-1)(4k-3) \mod 2 = 0 \). As in (i), the result follows from \( Q(\tilde{P}_i) = Q(P_i) + Q(P_{i-1}) \). \( \square \)

Let \( \mathcal{K} \) be the kernel of \( \langle \cdot, \cdot \rangle \). Evidently, \( \mathcal{K} \) is the space of dual invariants of the second \( \mathfrak{S}_n \)-action, hence Theorem 2.3 gives an explicit description of \( \mathcal{K} \). Applying Corollary 5.2 we get the following statement.
Corollary 5.3. Let \( n = 2k \), then \( Q(K) = \mathbb{F}_2 \) and \( |Q^{-1}(0)| = |Q^{-1}(1)| = 2^{2k^2-k-1} \), \( |Q^{-1}(0) \cap K| = |Q^{-1}(1) \cap K| = 2^{k-1} \).

Proof. It follows immediately from Corollary 5.2(ii) that \( Q(K) = \mathbb{F}_2 \). The statement concerning the sizes of \( |Q^{-1}(0)| \) and \( |Q^{-1}(1)| \) follows easily from the general description (see §4.1) with \( m = k(k-1) \) and \( \kappa = k \). The last statement follows from the fact that the normal form of \( Q|_K \) in case \( Q(K) = \mathbb{F}_2 \) is \( x_1^2 \) (see [Pf]). □

In case \( n = 2k+1 \) the situation is far more complicated.

Lemma 5.4. Let \( n = 2k+1 \), then \( Q(K) = 0 \). If \( k = 4t+1 \), then \( \text{Arf}(Q) = 1 \) and \( |Q^{-1}(0)| = 2^{2k^2+k-1} - 2^{k^2+k-1} \), \( |Q^{-1}(1)| = 2^{2k^2+k-1} + 2^{k^2+k-1} \). Otherwise \( \text{Arf}(Q) = 0 \) and \( |Q^{-1}(0)| = 2^{2k^2+k-1} + 2^{k^2+k-1} \), \( |Q^{-1}(1)| = 2^{2k^2+k-1} - 2^{k^2+k-1} \).

Proof. It follows immediately from Corollary 5.2(i) that \( Q(K) = 0 \), and hence the Arf invariant exists. Observe that by general theory of quadratic spaces (see §4.1) this means that \( |Q^{-1}(0)| \neq |Q^{-1}(1)| \). Moreover, \( |Q^{-1}(0)| > |Q^{-1}(1)| \) implies \( \text{Arf}(Q) = 0 \), while \( |Q^{-1}(0)| < |Q^{-1}(1)| \) implies \( \text{Arf}(Q) = 1 \). Therefore, to find \( \text{Arf}(Q) \) it suffices to count the number of elements in \( \mathcal{V} \) on which \( Q \) vanishes.

Let \( \omega : \mathcal{V} \rightarrow \mathbb{F}_2^{n-1} \) denote the projection on the first row (see Lemma 3.3), and let \( \mathcal{V}_a = \omega^{-1}(a), a \in \mathbb{F}_2^{n-1} \). We say that \( \mathcal{V}_a \) is inessential if \( |Q^{-1}(0) \cap \mathcal{V}_a| = |Q^{-1}(1) \cap \mathcal{V}_a| \), and essential otherwise.

Let us choose \( M_a \in \mathcal{V}_a \) such that \( Q(M_a) = 0 \); if \( \text{gr}(a) \) contains an even number of connected components, one can take \( \text{supp} M_a = \text{supp} a \), otherwise \( \text{supp} M_a \) contains one more vertex, which corresponds to the matrix entry \((n,n)\). We thus have \( M_a = M^0_a + M^1_a \), where \( \text{supp} M^0_a = \text{supp} a \) and \( |\text{supp} M^1_a| \leq 1 \).

Let us define a function \( Q_a \) on \( \mathcal{V}_0 \) by \( Q_a(M) = Q(M + M_a) \) for any \( M \in \mathcal{V}_0 \); observe that \( Q_a \) is just a shift of the restriction \( Q|_{\mathcal{V}_a} \). Evidently, \( Q_a(M) + Q(M_a) \). Therefore, \( Q_a(M + N) - Q_a(M) - Q_a(N) = \langle M, N \rangle \), which means that \( Q_a \) and \( Q \) define the same bilinear form on \( \mathcal{V}_0 \) (observe that \( \mathcal{V}_0 \) is identified naturally with \( (T^{n-2}(\mathbb{F}_2))' \)). Let us evaluate \( Q_a \) on the kernel \( K_0 \); \( Q_a(K_0) = \mathbb{F}_2 \) would mean that \( Q \) vanishes exactly on a half of the elements of \( \mathcal{V}_a \), in other words, that \( \mathcal{V}_a \) is inessential, while \( Q_a(K_0) = 0 \) would mean that \( \mathcal{V}_a \) is essential.

Since both \( Q \) and \( Q_a \) define the same bilinear form on \( \mathcal{V}_0 \), we see that \( K_0 \) is the space of dual invariants of the second \( \mathfrak{S}_{n-1} \)-action; therefore, by Theorem 2.3, it has a basis \( \{ \tilde{P}_1, \ldots, \tilde{P}_k \} \). By Corollary 5.2(ii), \( Q_a(\tilde{P}_i) = 1 + \langle \tilde{P}_i, M_a \rangle = 1 + \langle \tilde{P}_i, M^0_a \rangle \) for \( 1 \leq i \leq k-1 \) and \( Q_a(\tilde{P}_k) = k + \langle \tilde{P}_k, M^1_a \rangle \). Therefore, \( \mathcal{V}_a \) is essential if and only if the entries of \( a \) satisfy equations \( a_{k-i} + a_{k+i+1} = k+i \mod 2 \) for \( 0 \leq i \leq k-1 \). It is easy to see that any solution of the above equations is represented as \( a = h + s \), where \( h \) is as defined in Theorem 2.2(ii) and \( s \in \text{Sym}^{n-1} \). It follows from Theorem 2.3 that for any \( s \in \text{Sym}^{n-1} \) there exists \( S \in K \) such that \( \omega(S) = s \). Observe that in this case \( M \mapsto M + S \) takes \( \mathcal{V}_a \) to \( \mathcal{V}_{a+s} \). Moreover, \( Q(M + M_a + S) = Q(M + M_a) \), since \( \langle M + M_a, S \rangle = 0 \) follows from \( S \in K \) and \( Q(S) = 0 \) by Corollary 5.2(i). Therefore, all the essential subspaces \( \mathcal{V}_a \) influence \( \text{Arf}(Q) \) in the same way, hence, \( \text{Arf}(Q; \mathcal{V}) = \text{Arf}(Q_h; \mathcal{V}_0) \).

To study \( \text{Arf}(Q_h) \) on \( \mathcal{V}_0 \) we reiterate the same process once more, that is, we decompose \( \mathcal{V}_0 \) into affine subspaces \( \mathcal{V}_{0b} \), where \( b \) is defined by the projection \( \omega_0 \) of \( \mathcal{V}_0 \) on the first row.

We choose \( M_{0b} \in \mathcal{V}_{0b} \) similarly to \( M_a \) and define a function \( Q_{0b} \) on \( \mathcal{V}_{00} \) by
$Q_k(M + M_{0b})$. As before, $\mathcal{V}_{00}$ is identified naturally with $(T^{n−3}(\mathbb{F}_2))^*$, and $Q_{0b}$ and $Q$ define the same bilinear form on $\mathcal{V}_{00}$. The corresponding kernel $\mathcal{K}_{00}$ is the space of dual invariants of the second $\mathfrak{S}_{n−2}$-action; its basis is $\{\tilde{P}_1, \ldots, \tilde{P}_{k−1}\}$. We thus get that $\mathcal{V}_{ob}$ is essential if and only if the entries of $b$ satisfy equations $b_{k−i} + b_{k+i} = 0$ for $1 \leq i \leq k−1$. Any solution of these equations belongs to $\text{Sym}^{n−2}$. As before, $M \mapsto M + S$ with $S \in \mathcal{K}_0$ and $\omega_0(S) = s$ takes $\mathcal{V}_{ob}$ to $\mathcal{V}_{0,b+s}$ and $Q_h(M + M_{0b} + S) = Q_h(M + M_{0b})$; identity $Q_h(S) = 0$ follows readily from $Q(\tilde{P}_i) = (\tilde{P}_i, M_{0h}^0)$, $1 \leq i \leq k−1$. We thus get $\text{Arf}(Q_h; \mathcal{V}_0) = \text{Arf}(Q_h; \mathcal{V}_{00})$.

Recall that on $\mathcal{V}_{00}$ one has $Q_h(M) = Q(M) + \langle M, M^1_h \rangle$. It is easy to see that for $k = 4t$ and $k = 4t + 3$ one has $M^1_h = 0$, and hence $Q_h \equiv Q$ on $\mathcal{V}_{00}$. This means that the Arf invariant is constant on each triple of the form $k = 4t + 2, 4t + 3, 4t + 4$.

Let now $k = 4t + 1$ or $k = 4t + 2$, and hence $M^1_h \neq 0$. We decompose $\mathcal{V}_{00}$ into four affine subspaces $\mathcal{V}^{00}_{00}, \mathcal{V}^{01}_{00}, \mathcal{V}^{10}_{00}$, and $\mathcal{V}^{11}_{00}$. The subspace $\mathcal{V}^{ij}_{00}$ consists of the matrices having $i$ at position $(n − 1, n − 1)$ and $j$ at position $(n − 1, n)$. It is easy to see that $Q_h \equiv Q$ on $\mathcal{V}^{ii}_{00}$; moreover, the involution $M \mapsto M + M^1_h$ reverses the value of $Q$ (and thus of $Q_h$) on these subspaces. Therefore, the subspaces $\mathcal{V}^{ii}_{00}$ are inessential. On the other hand, $Q_h \equiv Q + 1$ on $\mathcal{V}^{ij}_{00}$, $i \neq j$, and the involution $M \mapsto M + M^1_h$ in this case preserves the value of $Q$ (and thus of $Q_h$). Therefore, $\text{Arf}(Q_h; \mathcal{V}_{00}) = 1 + \text{Arf}(Q; \mathcal{V}_{00})$. This means that the Arf invariant reverses twice on each triple of the form $k = 4t, 4t + 1, 4t + 2$.

To complete the proof it is enough to check the value of the Arf invariant for $k = 1$.

The exact values for the lengths of orbits follow easily from the general description (see §4.1) with $m = k^2$ and $k = k$. □

Let us now find the relation between the conjugate to the second $\mathfrak{S}_n$-action and the theory of skew-symmetric vanishing lattices explained in §4.

**Lemma 5.5.** (i) The triple $(\mathcal{V}, \langle \cdot, \cdot \rangle, B_{n−1})$ is a vanishing lattice.

(ii) The conjugate to the second $\mathfrak{S}_n$-action coincides with the action of $\Gamma B_{n−1}$.

(iii) The basis $B_{n−1}$ is weakly distinguished and its graph $\text{gr}(B_{n−1})$ coincides with $\mathfrak{S}_{n−1}$.

**Proof.** (i) We have to check that $B_{n−1}$ satisfies conditions (i)–(iii) of the definition of the vanishing lattices. Condition (ii) is evident. To check it suffices to take for $\delta_1$ and $\delta_2$ any pair of adjacent vertices of $\mathfrak{S}_{n−1}$. Finally, to check (i) we take an arbitrary pair $b, b'$ of adjacent vertices of $\mathfrak{S}_{n−1}$ and find that $T_b T_{b'}(b) = b'$.

(ii) Follows immediately from Lemma 3.2.

(iii) Obvious. □

**Lemma 5.6.** The conjugate to the second $\mathfrak{S}_n$-action is the action of a nonspecial monodromy group for $n \geq 5$.

**Proof.** By Lemmas 4.2 and 5.5 we have to check that the Dynkin diagram of $E_6$ is an induced subgraph of $\mathfrak{S}_m$ for $m \geq 4$. Since $\mathfrak{S}_m$ is an induced subgraph of $\mathfrak{S}_l$ for $m < l$, it suffices to find an induced subgraph corresponding to $E_6$ in $\mathfrak{S}_4$, see Fig. 4. □
5.2. The orbits of the second $G_n$-action. First of all, let us find the number of orbits of the second $G_n$-action.

Lemma 5.7. The number of orbits of the second $G_n$-action equals $2^k + 2$ for $n \geq 5$.

Proof. Indeed, by Lemma 4.4 the number of orbits of the second $G_n$-action equals that of its conjugate. By Lemma 5.6 the conjugate to the second $G_n$-action is the action of a nonspecial monodromy group; hence, by Lemma 4.3 the number of its orbits equals $2^\kappa + 2$, where $\kappa$ is the dimension of $K$. Since $K$ is just the space of dual invariants of the second $G_n$-action, Theorem 2.3(ii) implies $\kappa = k$. □

Let us now define two subsets of the vertex set of $H_{n-1}$ as follows: $\Sigma_1$ consists of the vertices corresponding to the matrix entries $(i, n-1)$, $1 \leq i \leq k$, and $\Sigma_2$ of the vertices corresponding to $(n-i, n-i)$, $1 \leq i \leq k$, see Fig. 5.

Following the strategy described in §4, we choose the following two subspaces in $V = (T^{n-1}(\mathbb{F}_2))^*$:

$$\mathcal{V}_1 = \{ M \in \mathcal{V} : \text{supp} M \cap \Sigma_1 = \emptyset \},$$
$$\mathcal{V}_2 = \{ M \in \mathcal{V} : \text{supp} M \cap \Sigma_2 = \emptyset \}.$$

Let $\mathcal{K}$ denote the kernel of $\langle \cdot, \cdot \rangle$. It follows easily from the explicit description of $\mathcal{K}$ (see Theorem 2.3(ii)) that $\text{codim} \mathcal{V}_1 = \text{codim} \mathcal{V}_2 = \dim \mathcal{K}$ and $\mathcal{V}_1 \cap \mathcal{K} = \mathcal{V}_2 \cap \mathcal{K} = 0$, hence $\mathcal{V}_1$ and $\mathcal{V}_2$ as above satisfy the assumptions of §4.
Fix the basis $\bar{P}_1, \ldots, \bar{P}_k$ of $\mathcal{K}$. It is easy to see that the stratum of the second $\mathfrak{g}_n$-action at height $\eta$ is just the affine subspace $\mathcal{A}^\eta$ as defined in §4.2. Therefore, the statements of Theorem 2.5(i) concerning the number and the lengths of orbits are yielded by the following proposition.

**Lemma 5.8.** Let $n = 2k + 1 \geq 5$.

(i) The linear subspace $\mathcal{A}^0 \subset \mathcal{V}^*$ consists of three orbits of the second $\mathfrak{g}_n$-action. The lengths of the orbits are $2^{2k^2-1} - \varepsilon_k 2^{k^2-1}$, $2^{2k^2-1} + \varepsilon_k 2^{k^2-1} - 1$, and 1, where $\varepsilon_k = -1$ for $k = 4t + 1$ and $\varepsilon_k = 1$ otherwise.

(ii) For any nonzero $\eta \in \mathbb{F}_2^k$ the affine subspace $\mathcal{A}^\eta \subset \mathcal{V}^*$ is an orbit of the second $\mathfrak{g}_n$-action.

**Proof.** (i) Let us first find the number of orbits contained in $\mathcal{A}^0$. By Lemma 4.6(i) it suffices to prove that $Q(M) = Q(I^n(M))$ for any $M \in \mathcal{V}_1$. Define $M^K = M + I^n(M)$, and let $L: \mathcal{V} \to \mathcal{V}^*$ be the linear mapping associated with $\langle \cdot, \cdot \rangle$. Since by definition $L(M) = L(I^n(M))$ for any $M \in \mathcal{V}_1$, we have $L(M^K) = 0$, and thus $M^K \in \mathcal{K}$. Therefore $Q(I^n(M)) = Q(M) + Q(M^K)$, and since by Lemma 5.4 $Q(\mathcal{K}) = 0$, we are done.

It follows now from Lemmas 4.5 and 4.3 that the three orbits in question are isomorphic to $\{0\}$, $(Q^{-1}(0) \cap \mathcal{V}_1) \setminus \{0\}$ and $Q^{-1}(1) \cap \mathcal{V}_1$. Evidently, $\text{Arf}(Q) = \text{Arf}(Q|_{\mathcal{V}_1})$ and $(\mathcal{V}_1, Q|_{\mathcal{V}_1})$ is a nondegenerate quadratic space of dimension $2k^2$. Therefore the statement concerning the lengths of the orbits follows from Lemma 5.4 and the general description of quadratic spaces (see §4.1) with $m = k^2$ and $\kappa = 0$.

(ii) By Lemma 5.7 the total number of orbits is $2^k + 2$, and by part (i) of the present Lemma exactly three orbits are contained in $\mathcal{A}^0$. Therefore, each of the $2^k - 1$ remaining orbits coincides with the corresponding $\mathcal{A}^\eta$. □

The corresponding statements of Theorem 2.5(ii) are yielded by the following proposition.

**Lemma 5.9.** Let $n = 2k \geq 6$ and $\bar{\eta} = (Q(\bar{P}_1), \ldots, Q(\bar{P}_k))$.

(i) The affine subspace $\mathcal{A}^{\bar{\eta}} \subset \mathcal{V}^*$ consists of two orbits of the second $\mathfrak{g}_n$-action. The lengths of the orbits are $2^{2k(k-1)-1} - 2^{k(k-1)-1}$ and $2^{2k(k-1)-1} + 2^{k(k-1)-1}$.

(ii) The linear subspace $\mathcal{A}^0 \subset \mathcal{V}^*$ consists of two orbits of the second $\mathfrak{g}_n$-action. The lengths of the orbits are $2^{2k(k-1)-1} - 1$ and 1.

(iii) For any $\eta \in \mathbb{F}_2^k$, $\eta \neq 0$, $\bar{\eta}$, the affine subspace $\mathcal{A}^\eta \subset \mathcal{V}^*$ is an orbit of the second $\mathfrak{g}_n$-action.

**Proof.** (i) Let $A^0_1 \in \mathcal{V}^*$ be the shift corresponding to $\mathcal{V}_1$, as defined in the proof of Lemma 4.5, and $A^0_2$ be the similar shift corresponding to $\mathcal{V}_2$. Observe that $A^0_1 + A^0_2 \in \mathcal{A}^0$, and hence there exists $H^0_2 \in \mathcal{V}_2$ such that $L(H^0_2) = A^0_1 + A^0_2$.

For any $M \in \mathcal{V}_1$ put $M^K = M + I^n(M) + H^0_2$. It follows from the definition of $I^n$ that $L(I^n(M)) = L(M) + A^0_1 + A^0_2$, hence $L(M^K) = 0$, that is, $M^K \in \mathcal{K}$.

Let us now find $Q(I^n(M))$. Since $I^n(M) = M + H^0_2 + M^K$ and $(M, H^0_2) = (M, A^0_1 + A^0_2)$, one has $Q(I^n(M)) = Q(M) + Q(H^0_2) + Q(M^K) + (M, A^0_1 + A^0_2)$, where $(\cdot, \cdot)$ is the standard coupling between $\mathcal{V}$ and $\mathcal{V}^*$ defined in §2.

Since $M \in \mathcal{V}_1$, we get $(M, A^0_1 + A^0_2) = (M, A^0_2) = |\text{supp} M \cap \text{supp} A^0_1|$. It follows easily from Theorem 2.3(ii) that $\text{supp} A^0_2 = \{(n - i, n - i) \in \Sigma_2 : Q(\bar{P}_1) = 1\}$. We thus have $(M, A^0_1 + A^0_2) = |\{(n - i, n - i) \in \Sigma_2 \cap \text{supp} M : Q(\bar{P}_1) = 1\}|$. 


On the other hand, \( M^K = \sum \{ \tilde{P}_i : (n - i, n - i) \in \Sigma_2 \cap \text{supp} M^K \} \), and hence \( Q(M^K) = |\{(n - i, n - i) \in \Sigma_2 \cap \text{supp} M^K : Q(\tilde{P}_i) = 1\}| \). Finally, \( M + M^K = \bar{I}^0(M) + H_2^2 \in \mathcal{V}_2 \), and hence \( \Sigma_2 \cap \text{supp} M = \Sigma_2 \cap \text{supp} M^K \).

Thus, \( Q(\bar{I}^0(M)) - Q(M) = Q(H_2^2) \) does not depend on \( M \), and hence \( u, v \) as in Lemma 4.6(ii) do not exist. Therefore, \( \mathcal{A}^0 \) consists of the two orbits of the second \( \mathfrak{G}_n \)-action. For the same reason the trivial one-element orbit obtained from \( \{0\} \) is glued to the image of \( Q^{-1}(0) \setminus \{0\} \), and hence the lengths of the orbits are the sizes of \( Q^{-1}(0) \) and \( Q^{-1}(1) \) for the nondegenerate case. Thus \( \kappa = 0 \) and \( m = k(k - 1) \), which yield the required result.

(ii), (iii) Follows immediately from Lemma 5.7 and part (i) in the same way as in Lemma 5.8(ii). To find the lengths of the orbits in \( \mathcal{A}^0 \) observe that \( L_1(0) = L_2(0) = 0 \in \mathcal{A}^0 \), hence the two parts that are glued together are \( Q^{-1}(0) \setminus \{0\} \) and \( Q^{-1}(1) \), and the result follows. \( \square \)

To obtain Theorem 2.5(ii) it suffices to notice that by Corollary 5.2(ii) \( \bar{\eta} \) as defined in Lemma 5.9 coincides with \( \bar{\eta} \) as defined in Theorem 2.5(ii).

§6. Orbits of the First \( \mathfrak{G}_n \)-Action

In this section we prove Theorem 2.2 concerning the structure of orbits of the first \( G_n \)-action.

6.1. The structure of the strata of the first \( \mathfrak{G}_n \)-action. Let us introduce some notation. We write \( X \sim \bar{X} \) if the matrices \( X \) and \( \bar{X} \) belong to the same orbit. We do not specify in this notation which \( \mathfrak{G}_n \)-action is meant; this should be always clear from the context, since each space under consideration carries exactly one \( \mathfrak{G}_n \)-action.

Let \( \Omega = \omega_1\omega_2\ldots\omega_t \) denote an arbitrary word in alphabet \( \{g_{ij}\} \); we write \( \bar{\Omega} \) for the word \( \omega_t \ldots \omega_2\omega_1 \). For any subset \( \Sigma \) of the alphabet we define \( [\Omega : \Sigma] \) as the number of occurrences of \( g_\sigma \), \( \sigma \in \Sigma \), in \( \Omega \). We write \( \Omega X \) to denote the matrix obtained from \( X \) by applying to it the elements \( \omega_1, \omega_2, \ldots, \omega_t \) in this order. We say that \( \Omega \) is \( X \)-nonredundant if \( \{\omega_1\ldots\omega_s\}X \neq \{\omega_1\ldots\omega_{s-1}\}X \) for \( s = 1, \ldots, t \) (here \( \omega_0X = X \)). Evidently, \( X \)-nonreducandy is preserved under isomorphisms and under \( \Psi_n \).

It is easy to see that each stratum \( S^h \) of the first \( \mathfrak{G}_n \)-action is a covering of degree \( 2^k \) over the stratum \( \Psi_n(s^h) \) of the second \( \mathfrak{G}_n \)-action. We are going to find a family of linear functionals \( C^h : T^n(\mathbb{F}_2) \to \mathbb{F}_2 \) with the following property: let \( M, \bar{M} \in S^h \) be an arbitrary pair of matrices such that \( \Psi_n(M) = \Psi_n(\bar{M}) \), then \( M \sim \bar{M} \) if and only if \( C^h(M) = C^h(\bar{M}) \). The existence of such a family would imply immediately that each nontrivial orbit of the first \( \mathfrak{G}_n \)-action is a covering over the corresponding orbit of the second \( \mathfrak{G}_n \)-action of degree \( 2^{k-1} \), except for the cases when \( C^h \) is trivial; in the latter case the degree equals \( 2^k \).

For any \( M \in T^n(\mathbb{F}_2) \) we denote by \( S(M) \) the set of matrices \( \bar{M} \) such that \( \Psi_n(M) = \Psi_n(\bar{M}) \) and \( M, \bar{M} \) belong to the same stratum of the first \( \mathfrak{G}_n \)-action. Further, let \( \mathcal{W} \) denote the \( k \)-dimensional subspace of \( T^n(\mathbb{F}_2) \) generated by the entries \( (1, i) \), \( 1 \leq i \leq k \), and \( \tau : T^n(\mathbb{F}_2) \to \mathcal{W} \) denote the natural projection.

Lemma 6.1. The projection \( \tau \) provides an affine isomorphism between \( S(M) \) and \( \mathcal{W} \) for any \( M \in T^n(\mathbb{F}_2) \).
Proof. Indeed, $S(M) = M + (\ker \Psi_n \cap D_n^\perp) = M + (\mathcal{I}_n \cap D_n^\perp)$. It follows from the explicit description of $D_n$ (see Theorem 2.1(ii)) that $\dim(\mathcal{I}_n \cap D_n^\perp) = k$ and $\ker \tau|_{\mathcal{I}_n \cap D_n^\perp} = 0$. \Box

Let us introduce, in addition to $\Sigma_1$ and $\Sigma_2$, one more subset of the vertex set of $\mathcal{F}_{n-1}$: $\Sigma_3$, consisting of vertices corresponding to matrix entries $(1,i)$, $1 \leq i \leq k$, see Fig. 5. As in §5.2, we define the subspace $\mathcal{V}_3 \subset \mathcal{V} = (T^{n-1}(\mathbb{F}_2))^*$,

$$\mathcal{V}_3 = \{ M \in \mathcal{V} : \text{supp} M \cap \Sigma_3 = \emptyset \}.$$ 

Recall that $L$ provides an isomorphism between $\mathcal{V}_3$ and $\mathcal{A}^0 = D_{n-1}^\perp$, and $A_1^\eta + A_2^\eta \in \mathcal{A}^0$; therefore, there exists $H_3^\eta \in \mathcal{V}_3$ such that $L(H_3^\eta) = A_1^\eta + A_2^\eta$.

**Lemma 6.2.** $Q(H_3^\eta) + (A_2^\eta, H_3^\eta) = 0$ for any $\eta \in \mathbb{F}_2^k$.

**Proof.** Let us give an explicit description of $H_3^\eta$. We define the matrices $H_i \in \mathcal{V}$, $1 \leq i \leq k$, in the following way. Let $1 \leq i < k$, then $\text{supp} H_i$ is obtained from the initial shape by deleting the upper-left justified $(n-i-1) \times (n-i-1)$ triangle and upper-right justified $(i-1) \times (i-1)$ triangle. For $n = 2k+1$ we define $H_k$ in the same way, and for $n = 2k$ $\text{supp} H_k$ is the lower-right justified $(k-1) \times (k-1)$ triangle, see Fig. 6.

![Supp H1, H2, H3, H4](image)

**Fig.6. Supports of the matrices $H_i$ in case $n = 8$**

Evidently, $H_i \in \mathcal{V}_3$, $1 \leq i \leq k$. Besides, $Q(H_i) = 1$ for $1 \leq i < k$ and $Q(H_k) = n$ mod 2. Finally, $(H_i, H_j) = 1$ for $i \neq j$.

Observe now that $H_3^\eta = \sum \{H_i : (n-i, n-i) \in \text{supp} A_2^\eta\}$. Therefore, for $n = 2k+1$ we have $Q(H_3^\eta) = c + c(c-1)/2 = c(c+1)/2 \mod 2$, where $c = |\text{supp} A_2^\eta|$. On the other hand, $(A_2^\eta, H_3^\eta) = 1 + 2 + \cdots + c = c(c+1)/2 \mod 2$. Hence $Q(H_3^\eta) + (A_2^\eta, H_3^\eta) = c(c+1) \mod 2 = 0$.

Let now $n = 2k$. If $(n-k, n-k) \notin \text{supp} A_2^\eta$, then the above proof applies. Otherwise $Q(H_3^\eta) = c-1 + c(c-1)/2 = c-1(c+2)/2 \mod 2$ and $(A_2^\eta, H_3^\eta) = 1 + 2 + \cdots + c-1 + c-1 = (c-1)(c+2)/2 \mod 2$, and hence $Q(H_3^\eta) + (A_2^\eta, H_3^\eta) = (c-1)(c+2) \mod 2 = 0$. \Box

Let us define a family of functionals $f^n : \mathcal{K} \to \mathbb{F}_2$, $\eta \in \mathbb{F}_2^k$, as follows: $f^n(X) = Q(X) + (A_2^\eta, X)$. It is easy to see that $f^n$ is a linear functional for any $\eta$, since $Q$ is linear on $\mathcal{K}$. The following statement stems immediately from Corollary 5.2.

**Lemma 6.3.** (i) Let $n = 2k+1$, then $f^n$ is nontrivial if and only if $\eta \neq 0$. 
Lemma 5.9, for any \( V \) the pieces of the trajectory that lie entirely in \( W \) with the help of \( \theta \) by defining \( F^n(x) = f^n(\theta(x)) \) for any \( x \in V \).

It is easy to check that the standard coupling \((\cdot, \cdot)\) restricted to \( \Sigma_3 \) identifies \( K^* \) with \( V_1^* \subset V^* \). Let \( f^n = (f^n_1, \ldots, f^n_k) \in V_3^k \); we define a linear functional \( c^h = (c^h_1, \ldots, c^h_k) \in W^* \) as follows: \( c^h = \Lambda f^{\psi_n(h)} \), where \( \Lambda \) is the matrix in \( GL_k(F_2) \) such that \( \lambda_{ij} = 1 \) iff \( i \leq j \), and lift it to the whole \( T^n(F_2) \) with the help of \( \tau \) by defining \( C^h(M) = c^h(\tau(M)) \) for any \( M \in T^n(F_2) \).

Lemma 6.4. Let \( M \in T^n(F_2) \) be an arbitrary matrix in \( S^h, h \in F_2^n \). If \( \tilde{M} \in S(M) \) and \( \tilde{M} \sim M \), then \( C^h(\tilde{M}) = C^h(M) \).

Proof. Indeed, let \( h = \psi_n(h) \) and let \( M' = g_{ij}M, M' \neq M \). Then \( C^h(M') = C^h(M) \) if \( (i, j) \notin \Sigma_3 \) and \( C^h(M') - C^h(M) = c^h_j + c^h_{j+1} = f^n_j \) otherwise. Hence the total variation of \( C^h(M) \) along the trajectory defined by an arbitrary \( M \)-nonredundant word \( \Omega \) equals \( \text{var}_\Omega C^h = [\Omega : \text{supp } F^n] \mod 2 \). Therefore, to prove the Lemma it suffices to show that for any \( M \)-nonredundant word \( \Omega \) such that \( \tilde{M} = \Omega M \) the number \( [\Omega : \text{supp } F^n] \) is even.

Evidently, there exists a decomposition \( \Omega = \Omega_1\Omega_2 \ldots \Omega_r \), with \( \Omega_i \neq \emptyset \) for \( i \geq 2 \) such that all the elements of \( \Omega_{2i+1} \) belong to the subgroup \( \mathfrak{S}_n^1 \subset \mathfrak{S}_n \) generated by the elements of \( V_1 \), and all the elements of \( \Omega_{2i} \) belong to the subgroup \( \mathfrak{S}_n^2 \subset \mathfrak{S}_n \) generated by the elements of \( V_2 \). By Lemma 4.5, instead of looking at the trajectory of \( M_{2i} = \{\Omega_1 \ldots \Omega_{2i}\}\Psi_n(M) \) under \( \Omega_{2i+1} \) one can look at the trajectory of \( (L^*_1)^{-1}(M_{2i}) \) in \( V_1 \), and instead of the trajectory of \( M_{2i+1} = \Omega_{2i+1}M_{2i} \) under \( \Omega_{2i+2} \), at the trajectory of \( (L^*_2)^{-1}(M_{2i+1}) \) in \( V_2 \). These pieces are glued together with the help of the isomorphism \( I^n \), see Fig. 7. Let us find the total variation \( \text{var}_\Omega F^n \) of the functional \( F^n \) along the obtained trajectory.

![Fig.7. Trajectory defined by Ω in the charts V₁ and V₂](image)

It follows immediately from Lemma 3.2 that for any \( x \in V \) and \( x' = g_{ij}x \neq x \) one has \( F^n(x') - F^n(x) = 1 \) iff \( (i, j) \in \text{supp } F^n \). Therefore, the total variation of \( F^n \) along the pieces of the trajectory that lie entirely in \( V_1 \) or \( V_2 \) equals \( [\Omega : \text{supp } F^n] \mod 2 \).

Let us find the variation of \( F^n \) under the isomorphism \( I^n \). Similarly to the proof of Lemma 5.9, for any \( x \in V_1 \) one has \( I^n(x) = x + H^n_3 + X^K \) with \( X^K \in K \). Thus, \( F^n(I^n(x)) = F^n(H^n_3) + F^n(X^K) - F^n(x) \). Thus, \( F^n(X^K) = Q(X^K) + (A_2^1, X^K) \), since \( \theta(H^n_3) = 0 \).
On the other hand, $Q(I^n(X)) - Q(X) = Q(X^\kappa) + Q(H^Z_0) + (A^n_{ij} + A^n_{ij}, X)$. Besides, $(A^n_0, X) = 0$ since $X \in \mathcal{V}_1$, $(A^n_0, X + H^Z_0 + X^\kappa) = 0$ since $I^n(X) = X + H^Z_0 + X^\kappa \in \mathcal{V}_2$, and $Q(H^Z_0) + (A^n_0, H^Z_0) = 0$ by Lemma 6.2. Adding the last four equalities we get $Q(F^n(X)) - Q(X) = Q(X^\kappa) + (A^n_0, X^\kappa)$. Therefore, $F^n(I^n(X)) - F^n(X) = Q(I^n(X)) - Q(X)$.

Recall that by Lemma 4.3 $Q$ is constant on each orbit in $\mathcal{V}_1$ and $\mathcal{V}_2$. Therefore, $\text{var}_\Omega F^n = \text{var}_\Omega Q + [\Omega : \text{supp } F^n] \mod 2$. Since $\Psi_n(M) = [\Psi_n, \tilde{M}]$, the trajectory defined by $\tilde{\Omega}$ is a loop, and hence $\text{var}_\Omega F^n = \text{var}_\Omega Q = 0$, thus $\text{var}_\Omega C^h = [\Omega : \text{supp } F^n] \mod 2 = 0$.

Let us now prove the converse statement.

**Lemma 6.5.** Let $M \in T^n(\mathbb{F}_2) \setminus \mathcal{I}_n$ be an arbitrary matrix in $S^h$, $h \in \mathbb{F}_2^n$. If $\tilde{M} \in S(M)$ and $C^h(\tilde{M}) = C^h(M)$, then $\tilde{M} \sim M$.

**Proof.** In what follows we assume that $h$ is fixed and $\eta = \psi_n(h)$. First of all, for any $Z \in \mathcal{K}$ we choose two matrices $M^Z_0, M^Z_1 \in \mathcal{V}_1$ such that

$$\theta(I^n(M^Z_i)) - \theta(M^Z_i) = Z, \quad Q(M^Z_i) = i, \quad i = 0, 1.$$  

Such a pair exists for any $Z \in \mathcal{K}$, since the first condition defines a translation of $\mathcal{V}_1 \cap \mathcal{V}_2$, and $Q$ is nonlinear on $\mathcal{V}_1 \cap \mathcal{V}_2$, and hence nonconstant on any translation of $\mathcal{V}_1 \cap \mathcal{V}_2$. Moreover, we can further assume that $M^Z_i \neq 0$ and $I^n(M^Z_i) \neq 0$ for $i = 0, 1$.

Let us fix an arbitrary nontrivial matrix $X \in \mathcal{V}_1$. By Lemma 4.3, $X, X' = M^Z_{Q(X)}$, and $X'' = M^Z_{Q(X)}$ belong to the same $\Sigma^n_1$-orbit. Let $\Omega^Z_2$ be an $X$-nonredundant word in $\{g_{ij}\}, (i, j) \notin \Sigma_1$, such that $X' = \Omega^Z_2 X$. We denote $Y' = I^n(X'), Y'' = I^n(X'')$. By the proof of Lemma 6.4, the variation of $Q$ under $I^n$ equals the variation of $F^n$. Therefore, if $f^n(Z) = 0$, then $Q(Y') = Q(X') = Q(X) = Q(X'') = Q(Y'')$, and hence $Y', Y'' \in \mathcal{V}_2$ belong to the same $\Sigma^n_2$-orbit. Let $\Omega^Z_2$ denote a $Y'$-nonredundant word in $\{g_{ij}\}, (i, j) \notin \Sigma_2$, such that $Y'' = \Omega^Z_2 Y'$. Evidently, $\Omega_Z = \Omega^Z_2 \Omega^Z_2 \Omega^Z_0$ is $X$-nonredundant and $\Omega_Z X = X$.

By Lemma 6.1, it suffices to prove that the value of any linear functional other than $c^h$ can be changed along an appropriate $\Omega_Z$ with $f^n(Z) = 0$. Take any $W \in \mathcal{K}$, and let $f^W \in \mathcal{V}_Z$ be the conjugate to $W$ with respect to $(\cdot, \cdot)$ restricted to $\Sigma_3$; put $F^W = f^W \circ \theta$. Then, similarly to the proof of Lemma 6.4, $0 = \text{var}_{\Omega_Z} F^W = [\Omega_Z : \text{supp } F^W] + f^W(Z) \mod 2$. Therefore, for $C^W = A f^W$ and $C^W = c^W \circ \tau$ one has $\text{var}_{\Omega_Z} C^W = [\Omega_Z : \text{supp } F^W] \mod 2 = f^W(Z)$.

Let now $W = \sum_{i=1}^k w_i \tilde{P}_i$. Evidently, it is enough to consider the following two types of $W$: (i) $w_i = 0$ for $i \neq j$, $w_j = 1$, where $\eta_j = 0$, and (ii) $w_i = 0$ for $i \neq j_1, j_2$, $w_{j_1} = w_{j_2} = 1$, where $\eta_{j_1} = \eta_{j_2} = 1$. In order to get $\text{var}_{\Omega_Z} C^W = 1$ for $n = 2k + 1$ one takes $Z = \tilde{P}_j$ in the first case and $Z = \tilde{P}_{j_1} + \tilde{P}_{j_2}$ with $\eta_{j_3} = 1$ in the second case. For $n = 2k$ one takes either $Z = \tilde{P}_j + \tilde{P}_{j_3}$ with $\eta_{j_3} = 0$ or $Z = \tilde{P}_{j_3}$ with $\eta_{j_3} = 1$ in the first case and $Z = \tilde{P}_{j_1} + \tilde{P}_{j_3}$ with $\eta_{j_3} = 1$ in the second case.

**Proof of Theorem 2.2.** Let $n = 2k + 1$ and $\psi_n(h) \neq 0$. Then by Theorem 2.5 and Lemmas 6.3, 6.4, 6.5 the stratum $S^h$ consists of two orbits distinguished by $C^h$. Their lengths are equal since $C^h$ is linear and nontrivial. For $\psi_n(h) = 0$ Lemmas 6.3 and 6.5 imply that nontrivial orbits in $S^h$ are just coverings of degree $2k$ of nontrivial orbits in $\Psi_n(S^h) = \mathcal{D}_{n-1}$ described in Theorem 2.5. Besides, each matrix in $\mathcal{I}_n \cap S^h$ forms itself a trivial orbit. The case $n = 2k$ is treated in the same way.
6.2. The structure of the symmetric strata of the first \( S_n \)-action: an alternative approach. In §2.2 we have introduced the second \( S_n \)-action as the one induced by the first \( S_n \)-action on the quotient of \( T^n(\mathbb{F}_2) \) modulo the subspace \( I_n \) of the invariants of the first \( S_n \)-action. Now we apply the same construction in the conjugate space. That is, we consider the action induced by the conjugate to the first \( S_n \)-action on the quotient of \( (T^n(\mathbb{F}_2))^* \) modulo the subspace \( D_n \) of the dual invariants of the first \( S_n \)-action. One can suggest the following natural description of this induced action.

Consider the linear map \( \Phi_n: (T^n(\mathbb{F}_2))^* \to T^{n-1}(\mathbb{F}_2) \) such that the \((i,j)\)th entry in the image equals the sum of the entries \((i,j), (i,j+1), (i+1,j), \) and \((i+1,j+1)\) in the inverse image (as before, entry \((i,j)\) in the image can be considered as representing the submatrix \( M_{ij} \) of the initial matrix). Lemma 3.1 implies immediately that \( \ker \Phi_n = D_n \), and we thus obtain the induced action of \( S_n \) on \( T^{n-1}(\mathbb{F}_2) \).

The following observation gives a clue to the structure of the symmetric strata of the first \( S_n \)-action.

**Lemma 6.6.** The \( S_n \)-action on \( T^{n-1}(\mathbb{F}_2) \) induced by \( \Phi_n \) coincides with the second \( S_n \)-action.

**Proof.** Simple exercise in linear algebra. \( \square \)

Therefore, the above construction provides an alternative description of the second \( S_n \)-action. As an immediate corollary of this description we obtain the structure of the stratum \( D_n^\perp \) at height \((0, \ldots, 0)\).

**Lemma 6.7.** The orbits of the first \( S_n \)-action in the stratum \( D_n^\perp \) are isomorphic to the orbits of the conjugate to the second \( S_n \)-action.

**Proof.** Indeed, consider the conjugate mapping \( \Phi_n^*: (T^{n-1}(\mathbb{F}_2))^* \to T^n(\mathbb{F}_2) \). It is easy to check that the image of \( \Phi_n^* \) coincides with \( D_n^\perp \). Moreover, \( \ker \Phi_n^* = 0 \), and hence \( \Phi_n^* \) provides an isomorphism between \( (T^{n-1}(\mathbb{F}_2))^* \) and \( D_n^\perp \). Now Lemma 6.6 implies that the first \( S_n \)-action on \( D_n^\perp \) is isomorphic to the conjugate to the second \( S_n \)-action. \( \square \)

We thus get the part of Theorem 2.2 concerning the structure of symmetric strata.

**Lemma 6.8.** (i) Let \( n = 2k + 1 \geq 5 \), then each of \( 2^{k+1} \) symmetric strata consists of one orbit of length \( 2^{2k^2+k-1} - \varepsilon_k2^{k^2+k-1} \), one orbit of length \( 2^{2k^2+k-1} + \varepsilon_k2^{k^2+k-1} - 2^k \), and \( 2^k \) orbits of length 1, where \( \varepsilon_k = -1 \) for \( k = 4t + 1 \) and \( \varepsilon_k = 1 \) otherwise;

(ii) Let \( n = 2k \geq 6 \), then each of \( 2^k \) symmetric strata consists of two orbits of length \( (2^{2k(k-1)} - 1)2^{k-1} \) and \( 2^k \) orbits of length 1.

**Proof.** The additive group \( I_n \) acts on \( T^n(\mathbb{F}_2) \) by translations. Evidently, any such translation takes \( D_n^\perp \) to a symmetric stratum and any symmetric stratum is the translation of \( D_n^\perp \) by a vector in \( I_n \). Besides, the action of \( I_n \) commutes with the first \( S_n \)-action, and hence takes orbits to orbits. Therefore, the orbit structure of any symmetric stratum coincides with that of the stratum \( D_n^\perp \).

To find the lengths of the orbits we use Lemmas 6.7, 5.6, and 4.3. Since \( \kappa = k \), we immediately get \( 2^k \) orbits of length 1. The lengths of the other two orbits are obtained readily from Corollary 5.3 and Lemma 5.4. \( \square \)
§7. Final Remarks

7.1. The technique of the previous paper [SSV] is substantially generalized in a forthcoming paper joint with A. Zelevinsky to a wide class of intersections of pairs of Schubert cells and double Schubert cells. This generalization is based on the chamber ansatz developed for the flag varieties and semisimple groups in [BFZ, BZ, FZ]. For each reduced decomposition of an element $u$ in a classical Coxeter group of type A, D, E we define a group which acts by symplectic transvections on the $\mathbb{Z}$-module generated by all chamber sets associated with the chosen decomposition. We prove that the $\mathbb{F}_2$-reduction of the above action counts the number of connected components in in the intersection $B \cap B_u$ of two real open Schubert cells in $G/B$ taken in the split form. Analogous results are obtained for the intersections $G^{u,v} = BuB \cap B_-vB_-$ in a semisimple simplylaced group $G$.

These results lead to the following general setup. Given a connected undirected graph $\Gamma$, consider the $\mathbb{F}_2$-vector space $V_\Gamma$ generated by its vertices and define an $\mathbb{F}_2$-valued bilinear form $\langle p, q \rangle = \sum \{p_iq_j : i \text{ adjacent to } j\}$. Every vertex $\delta \in \Gamma$ determines the symplectic transvection $T_\delta : V_\Gamma \to V_\Gamma$ sending $p$ to $p - \langle p, \delta \rangle \delta$. Let us also choose some subset $B$ of vertices of $\Gamma$ and define the group $G_B$ generated by all $T_\delta$, $\delta \in B$.

**Problem.** Find the number of orbits of the $G_B$-action on $V_\Gamma$.

A substantial part of the results of the present paper are valid in this more general setup as well. The authors hope to solve the above Problem at least for the graphs (and their groups) arising from intersections of Schubert cells. Our preliminary considerations, supported by numerical evidence, suggest the following surprising conjecture.

**Conjecture.** The number of connected components in the intersection of two open Schubert cells in $SL_{n+1}(\mathbb{R})/B$ in relative position $w$ equals $3 \cdot 2^{n-1}$ for any generic $w$ and $n \geq 5$.

Here genericity means that the graph of bounded chambers introduced implicitly in [SSV] and studied in detail in the forthcoming paper with A. Zelevinsky contains an induced subgraph isomorphic to the Dynkin diagram of $E_6$.

7.2. Another very intriguing fact is that the group $G_n$ studied in the present paper is the $\mathbb{F}_2$-reduction of the monodromy group of the Verlinde algebra $su(3)_n$ introduced in [Zu] and studied recently in [GZV]. (The authors are obliged to S. Chmutov and S. M. Gusein-Zade for this observation.) Notice that the monodromy group of any $su(m)_n$ is interpreted in [GZV] as the monodromy group of the isolated singularity of a certain Newton polynomial expanded in elementary symmetric functions. This gives us a hope to both find a natural representation of Verlinde algebra $su(3)_n$ in the sections of some bundle over the intersection of open opposite Schubert cells and, more generally, to find a relation between the topology of intersections of Schubert cells and singularity theory of symmetric polynomials.

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