Solvability of the heat equation with a nonlinear boundary condition

Kotaro Hisa and Kazuhiro Ishige *

Abstract

In this paper we obtain necessary conditions and sufficient conditions for the solvability of the problem

\[
\begin{align*}
\partial_t u &= \Delta u, & x \in \mathbb{R}_+^N, & t > 0, \\
\partial_\nu u &= u^p, & x \in \partial \mathbb{R}_+^N, & t > 0, \\
u(x, 0) &= \mu(x) \geq 0, & x \in D := \mathbb{R}_+^N,
\end{align*}
\]

where \( N \geq 1, p > 1 \) and \( \mu \) is a nonnegative measurable function in \( \mathbb{R}_+^N \) or a Radon measure in \( \mathbb{R}^N \) with \( \text{supp} \mu \subset D \). Our sufficient conditions and necessary conditions enable us to identify the strongest singularity of the initial data for the solvability for problem (P). Furthermore, as an application, we obtain optimal estimates of the life span of the minimal solution of (P) with \( \mu = \kappa \varphi \) as \( \kappa \to 0 \) or \( \kappa \to \infty \).

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1 Introduction

We are interested in finding necessary conditions and sufficient conditions on the initial data for the solvability of problem

\[
\begin{align*}
\partial_t u &= \Delta u, & x &\in \mathbb{R}_+^N, & t > 0, \\
\partial_v u &= u^p, & x &\in \partial \mathbb{R}_+^N, & t > 0,
\end{align*}
\]

with the initial condition

\[u(x, 0) = \mu(x) \geq 0, \quad x \in D := \overline{\mathbb{R}_+^N},\]

where \(\mu\) is a nonnegative measurable function in \(\mathbb{R}_+^N\) or a Radon measure in \(\mathbb{R}_+^N\) with \(\text{supp} \mu \subset D\). For the solvability of problem (1.1) with (1.2), sufficient conditions have been studied in many papers (see e.g., [1], [2], [4], [6], [7], [9], [11] and [12]). However little is known concerning necessary conditions and the strongest singularity of initial data for which problem (1.1) possesses a local-in-time nonnegative solution is still open as far as we know.

In 1985, Baras and Pierre [3] studied necessary conditions on the initial data for the existence of nonnegative solutions of

\[\partial_t u = \Delta u + u^q, \quad x \in \mathbb{R}_+^N, \quad t > 0,
\]

where \(N \geq 1\) and \(q > 1\). Recently, the authors of this paper [8] proved the existence and the uniqueness of the initial trace of a nonnegative solution of a fractional semilinear heat equation

\[
\partial_t u = -(-\Delta)^{\theta/2} u + u^q, \quad x \in \mathbb{R}_+^N, \quad t > 0,
\]

where \(0 < \theta \leq 2\) and \(q > 1\). Furthermore, they showed that, if problem (1.3) possesses a local-in-time nonnegative solution, then its initial trace \(\mu\) satisfies the following:

1. \[
\sup_{x \in \mathbb{R}_+^N} \mu(B(x, T^\theta)) \leq \gamma_1 T^{N - \frac{1}{q - 1}} \quad \text{if} \ 1 < q < q_\theta;
\]
2. \[
\sup_{x \in \mathbb{R}_+^N} \mu(B(x, \sigma)) \leq \gamma_1 \left[ \log \left( e + \frac{T^\theta}{\sigma} \right) \right]^{-\frac{N}{q}} \quad \text{for all} \ 0 < \sigma < T^\theta \quad \text{if} \ q = q_\theta;
\]
3. \[
\sup_{x \in \mathbb{R}_+^N} \mu(B(x, \sigma)) \leq \gamma_1 \sigma^{N - \frac{q}{q - 1}} \quad \text{for all} \ 0 < \sigma < T^\theta \quad \text{if} \ q > q_\theta.
\]

Here \(q_\theta := 1 + \theta/N\). In [8], developing the arguments in [10] and [14], they also obtained sufficient conditions on the initial data for the existence of the solution of (1.3) and identified the strongest singularity of the initial data for which the Cauchy problem to (1.3) possesses a local-in-time nonnegative solution.

In this paper, motivated by [8], we show the existence and the uniqueness of the initial trace of a nonnegative solution of (1.1) and obtain necessary conditions on the existence of nonnegative solutions of (1.1) and (1.2). We also obtain new sufficient conditions on the existence of nonnegative solutions of (1.1) and (1.2). Our necessary conditions and sufficient conditions enable us to identify the strongest singularity of initial data...
for which problem \((1.1)\) possesses a local-in-time nonnegative solution. Surprisingly, the strongest singularity depends on whether it exists on \(\partial \mathbb{R}^N_+\) or not (see Corollary 1.1 and Section 6). Furthermore, we study how the life span of the solution of problem \((1.1)\) with \((1.2)\) depends on the behavior of the initial data near the boundary and at the space infinity. See Section 6.

We introduce some notation and define a solution of \((1.1)\). Throughout this paper we often identify \(\mathbb{R}^{N-1}\) with \(\partial \mathbb{R}^N_+\). For any \(x \in \mathbb{R}^N\) and \(r > 0\), let

\[
B(x, r) := \left\{ y \in \mathbb{R}^N : |x - y| < r \right\}, \quad B_+(x, r) := \left\{ (y', y_N) \in B(x, r) : y_N \geq 0 \right\}.
\]

For any \(L \geq 0\), we set

\[
D_L := \{(x', x_N) : x' \in \mathbb{R}^{N-1}, x_N \geq L^{1/2}\},
\]

\[
D'_L := \{(x', x_N) : x' \in \mathbb{R}^{N-1}, 0 \leq x_N < L^{1/2}\}.
\]

We remark that \(D = D_0 = \mathbb{R}^N_+\). Let \(\Gamma_N = \Gamma_N(x, t)\) be the Gauss kernel on \(\mathbb{R}^N\), that is

\[
\Gamma_N(x, t) := (4\pi t)^{-N/2} \exp \left(-\frac{|x|^2}{4t}\right), \quad x \in \mathbb{R}^N, \ t > 0.
\] (4.4)

Let \(G = G(x, y, t)\) be the Green function for the heat equation on \(\mathbb{R}^N_+\) with the homogeneous Neumann boundary condition, that is

\[
G(x, y, t) := \Gamma_N(x - y, t) + \Gamma_N(x - y_s, t), \quad x, y \in D, \ t > 0,
\] (1.5)

where \(y_s = (y', -y_N)\) for \(y = (y', y_N) \in D\). For any Radon measure \(\mu\) in \(\mathbb{R}^N\) with \(\text{supp} \mu \subseteq D\), define

\[
[S(t)\mu](x) := \int_D G(x, y, t) \, d\mu(y), \quad x \in D, \ t > 0.
\]

For any locally integrable nonnegative function \(\phi\) on \(D\), we often identify \(\phi\) with the Radon measure \(\phi \, dx\). Then it follows that

\[
\lim_{t \to +0} \|S(t)\eta - \eta\|_{L^\infty(D)} = 0, \quad \eta \in C_0(D : [0, \infty)).
\] (1.6)

**Definition 1.1** Let \(u\) be a nonnegative and continuous function in \(D \times (0, T)\), where \(0 < T < \infty\).

(i) We say that \(u\) is a solution of \((1.1)\) in \((0, T)\) if \(u\) satisfies

\[
u(x, t) = \int_D G(x, y, \tau) u(y, t - \tau) \, dy + \int_\tau^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t - s) u(y', 0, s) \, dy' \, ds
\]

for \((x, t) \in D \times (\tau, T)\) and \(0 < \tau < T\).

(ii) Let \(\mu\) be a nonnegative measurable function in \(\mathbb{R}^N_+\) or a Radon measure in \(\mathbb{R}^N\) with \(\text{supp} \mu \subseteq D\). We say that \(u\) is a solution of \((1.1)\) and \((1.2)\) in \([0, T)\) if \(u\) satisfies

\[
u(x, t) = \int_D G(x, y, t) \, d\mu + \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t - s) u(y', 0, s) \, dy' \, ds
\]

(1.7)
for \((x,t) \in D \times (0,T)\). If \(u\) satisfies (1.8) with \(= \) replaced by \(\geq\), then \(u\) is said to be a supersolution of (1.1) and (1.2) in \([0,T)\).

(iii) Let \(u\) be a solution of (1.1) and (1.2) in \([0,T)\). We say that \(u\) is a minimal solution of (1.1) and (1.2) in \([0,T)\) if \(u(x,t) \leq v(x,t)\) in \(D \times (0,T)\) for any solution \(v\) of (1.1) and (1.2) in \([0,T)\).

Next we are ready to state our main results. In Theorem 1.1 we show the existence and the uniqueness of the initial trace of the solution of (1.1) and give necessary conditions on

\[ \text{Theorem 1.1} \]

Let \(u\) be a solution of (1.1) in \((0,T)\), where \(0 < T < \infty\). Then there exists a unique Radon measure \(\mu\) in \(\mathbb{R}^N\) with \(\text{supp} \mu \subset D\) such that

\[ \lim_{t \to +0} \int_D u(y,t)\phi(y) \, dy = \int_D \phi(y) \, d\mu(y), \quad \phi \in C_0(\mathbb{R}^N). \]  

Furthermore, for any \(\delta > 0\), there exists \(\gamma_1 > 0\) such that

1. \[ \sup_{x \in D} \exp \left( - (1 + \delta) \frac{x^2}{4T} \right) \mu(B(x,T^\frac{1}{2})) \leq \gamma_1 T^{\frac{N}{2} - \frac{1}{2(p-1)}} \quad \text{if } 1 < p < p_*; \]

2. \[ \sup_{x \in D} \exp \left( - (1 + \delta) \frac{x^2}{4\sigma^2} \right) \mu(B(x,\sigma)) \leq \gamma_1 \left[ \log \left( e + \frac{T^\frac{1}{2}}{\sigma} \right) \right]^{-N} \quad \text{for } 0 < \sigma < T^{\frac{1}{2}}; \]

3. \[ \sup_{x \in D} \exp \left( - (1 + \delta) \frac{x^2}{4\sigma^2} \right) \mu(B(x,\sigma)) \leq \gamma_1 \sigma^{-N - \frac{1}{p-1}} \quad \text{for } 0 < \sigma < T^{\frac{1}{2}} \quad \text{if } p > p_* .\]

In Theorem 1.2 we show that the initial trace of the solution of (1.1) and (1.2) coincides with its initial data.

\[ \text{Theorem 1.2} \]

Let \(\mu\) be a Radon measure in \(\mathbb{R}^N\) with \(\text{supp} \mu \subset D\).

(a) Let \(u\) be a solution of (1.1) and (1.2) in \([0,T)\) for some \(T > 0\). Then (1.9) holds.

(b) Let \(u\) be a solution of (1.1) in \((0,T)\) for some \(T > 0\). Assume (1.9). Then \(u\) is a solution of (1.1) and (1.2) in \([0,T)\).

Combining Theorem 1.1 with Theorem 1.2, we obtain necessary conditions on the initial data for the solvability of problem (1.1) with (1.2).

\[ \text{Remark 1.1} \]

(i) If \(1 < p \leq p_*\) and \(\mu \neq 0\) on \(D\), then problem (1.1) possesses no nonnegative global-in-time solutions. See [4] and [7].

(ii) Let \(u\) be a solution of (1.1) in \([0,\infty)\) and \(1 < p \leq p_*\). It follows from assertions (1) and (2) of Theorem 1.1 that the initial trace of \(u\) must be identically zero in \(D\). Then Theorem 1.2 leads the same conclusion as in Remark 1.1 (i).

Next we state our main results on sufficient conditions for the solvability of problem (1.1) with (1.2). In what follows, for any Radon measure in \(\mathbb{R}^N\) and any bounded Borel set \(E\), we denote by \(|E|\) the Lebesgue measure of \(E\) and set

\[ \int_E d\mu = \frac{1}{|E|} \int_E d\mu. \]
Theorem 1.3 Let $1 < p < p^*$, $T > 0$ and $\delta \in (0,1)$. Set $\lambda := (1 - \delta)/4T$. Then there exists $\gamma_2 = \gamma_2(N,p,\delta) > 0$ with the following property:

- If $\mu$ is a Radon measure in $\mathbb{R}^N$ with $\text{supp}\, \mu \subset D$ satisfying

$$
\sup_{x \in D} \int_{B(x,T^2)} e^{-\lambda y_N^2} d\mu(y) \leq \gamma_2 T^{-\frac{1}{2(p-1)}},
$$

(1.10)

then there exists a solution $u$ of (1.1) and (1.2) in $[0,T)$ such that

$$0 \leq u(x,t) \leq 2\left| S(t)\mu(x) \right|, \quad (x,t) \in D \times (0,T).
$$

Theorem 1.4 Let $1 < \alpha < p$, $T > 0$ and $\delta \in (0,1)$. Set $\lambda := (1 - \delta)/4T$. Then there exists $\gamma_3 = \gamma_3(N,p,\alpha,\delta) > 0$ with the following property:

- Let $\mu_1$ be a Radon measure in $\mathbb{R}^N$ such that $\text{supp}\, \mu_1 \subset D_T$ and

$$
\sup_{x \in D_T} \int_{B(x,T^2)} e^{-\lambda y_N^2} d\mu_1(y) \leq \gamma_3 T^{-\frac{1}{2(p-1)}}.
$$

(1.11)

Let $\mu_2$ be a nonnegative measurable function in $\mathbb{R}^N_+$ such that $\text{supp}\, \mu_2 \subset D'_T$ and

$$
\sup_{x \in D'_T} \left[ \int_{B(x,T^2)} \mu_2(y)^\alpha dy \right]^\frac{1}{\alpha} \leq \gamma_3 \sigma^{-\frac{1}{p-1}} \text{ for } 0 < \sigma < T^\frac{1}{2}.
$$

(1.12)

Then there exists a solution $u$ of (1.1) and (1.2) in $[0,T)$ with $\mu = \mu_1 + \mu_2$ such that

$$0 \leq u(x,t) \leq 2\left| S(t)\mu_1(x) + 2\left( \left| S(t)\mu_2(x) \right| \right) \right|^{\frac{1}{2}}, \quad (x,t) \in D \times (0,T).
$$

Theorem 1.5 Let $p = p^*$, $\beta > 0$, $T > 0$ and $\delta \in (0,1)$. Set $\lambda := (1 - \delta)/4T$ and

$$
\Phi_\beta(s) := s[\log(e + s)]^\beta, \quad \rho(s) := s^{-N}\left[ \log\left( e + \frac{1}{s} \right) \right]^{-N} \text{ for } s > 0.
$$

(1.13)

Then there exists $\gamma_4 = \gamma_4(N,\beta,\delta) > 0$ with the following property:

- Let $\mu_1$ be a Radon measure in $\mathbb{R}^N$ such that $\text{supp}\, \mu_1 \subset D_T$ and

$$
\sup_{x \in D_T} \int_{B(x,T^2)} e^{-\lambda y_N^2} d\mu_1(y) \leq \gamma_4 T^{-\frac{1}{2(p-1)}}.
$$

(1.14)

Let $\mu_2$ be a nonnegative measurable function in $\mathbb{R}^N_+$ such that $\text{supp}\, \mu_2 \subset D'_T$ and

$$
\sup_{x \in D'_T} \Phi_\beta^{-1} \left[ \int_{B(x,T^2)} \Phi_\beta(T^{-\frac{1}{2(p-1)}}\mu_2(y)) dy \right] \leq \gamma_4 \rho(T^{-\frac{1}{2}}) \text{ for } 0 < \sigma < T^\frac{1}{2}.
$$

(1.15)

Then there exists a solution of (1.1) and (1.2) in $[0,T)$ with $\mu = \mu_1 + \mu_2$ such that

$$0 \leq u(x,t) \leq 2\left| S(t)\mu_1(x) + d\Phi_\beta^{-1} \left( \left| S(t)\Phi_\beta(\mu_2) \right| \right) \right|, \quad (x,t) \in D \times (0,T),$$

where $d$ is a positive constant depending only on $p$ and $\beta$. 

5
As a corollary of our theorems, we have:

**Corollary 1.1** Let \( \delta \) be the Delta function in \( \mathbb{R}^N \) and \( x_0 \in D \). Let \( \mu(y) = \delta(y - x_0) \) in \( \mathbb{R}^N \). Then there exists a solution of (1.1) and (1.2) in \([0, T)\) for some \( T > 0 \) if and only if, either

(i) \( x_0 \in \partial \mathbb{R}^N_+ \) and \( 1 < p < p_* \)  
or  
(ii) \( x_0 \in \mathbb{R}^N_+ \) and \( p > 1 \).

See also Theorem 6.3.

We develop the arguments in [8] and prove our theorems. Let \( u \) be a solution of (1.1) in \((0, T)\) for some \( T > 0 \). By the same argument as in [8] we can prove the existence and the uniqueness of the initial trace of the solution \( u \). Furthermore, we study a lower estimate of the solution \( u \) near the boundary \( \partial D \) by the use of \( \|u(\tau)\|_{L^1(B_+(z, \rho))} \), where \( z \in D, \rho \in (0, T^{1/2}) \) and \( \tau \in (0, T) \). (See Lemma 3.1.) Combining this lower estimate with [4, Lemma 2.1.2], we complete the proof of Theorem 1.1 in the case \( p \neq p_* \). For the case \( p = p_* \), we obtain an integral inequality with respect to the quantity

\[ \int_{\partial D} \Gamma_{N-1}(y', t) u(y', 0, t) \, dy' \] 

(see [8, 15]). Then we apply a similar iteration argument as in [13, Section 2] to obtain \( \|u(\tau)\|_{L^1(B_+(z, \rho))} \), where \( z \in D, \rho \in (0, T^{1/2}) \) and \( \tau \in (0, T) \). This completes the proof of Theorem 1.1 in the case \( p = p_* \). Theorem 1.2 is proved by a similar argument as in the proof of [8, Theorem 1.2] with the aid of Theorem 1.1. Furthermore, we prove a lemma on an estimate of an integral related to the nonlinear boundary condition (see Lemma 5.1) and apply the arguments in [8, 10, 14] to prove Theorems 1.3–1.5.

The rest of this paper is organized as follows. In Section 2 we recall some properties of the kernel \( G = G(x, y, t) \) and prove some preliminary lemmas on the kernel \( G \). In Section 3 we study the existence and the uniqueness of the initial trace. Furthermore, we obtain necessary conditions for the solvability of the solutions of (1.1) and (1.2), and prove Theorem 1.1. In Section 4 we apply Theorem 1.1 to prove Theorem 1.2. In Section 5 we obtain sufficient conditions on the initial data for the solvability of the solution of (1.1) and (1.2), and prove Theorems 1.3–1.5. In Section 6, as an application of our theorems, we obtain some estimates of the life span of the solution of (1.1) and (1.2).

## 2 Preliminaries

In this section we recall some properties of the kernel \( G = G(x, y, t) \) and prove preliminary lemmas. By (1.3) we have

\[ \Gamma_N(x - y, t) \leq G(x, y, t) \leq 2\Gamma_N(x - y, t), \quad x, y \in D, \ t > 0. \]  

(2.1)

It follows from (1.4) and (1.5) that

\[ G(x', x_N, y', 0, t) = G(y', 0, x', x_N, t) = 2\Gamma_N(x' - y', x_N, t) \]

\[ = 2(4\pi t)^{-\frac{N}{2}} \exp \left( -\frac{x_N^2}{4t} \right) \Gamma_{N-1}(x' - y', t) \] 

(2.2)
for \( x \in D, y' \in \mathbb{R}^{N-1} \) and \( t > 0 \). By the semigroup property of \( S(t) \) we see that
\[
\int_D G(x, y, t)G(y, z, s) \, dy = G(x, z, t + s)
\] (2.3)
for \((x, t), (z, s) \in D \times (0, \infty)\). Furthermore, we have the following two lemmas. In what follows, by the letter \( C \) we denote generic positive constants and they may have different values also within the same line.

**Lemma 2.1** Let \( \mu \) be a Radon measure in \( \mathbb{R}^N \) with \( \text{supp} \mu \subset D \). If \( [S(T)\mu](x) < \infty \) for some \( x \in D \) and \( T > 0 \), then \( S(t)\mu \) is continuous in \( D \times (0, T) \).

**Proof.** Assume \( [S(T)\mu](x) < \infty \) for some \( x \in D \) and \( T > 0 \). Let \( 0 < T' < T \). It follows from (2.1) that
\[
\infty > [S(T)\mu](x) \geq \int_D \Gamma_N(x - y, T) \, d\mu(y)
\]
\[
= (4\pi T)^{-\frac{N}{2}} \int_D \exp\left(-\frac{|x - y|^2}{4T}\right) \, d\mu(y) \geq C \int_D \exp\left(-\frac{|y|^2}{4T'}\right) \, d\mu(y).
\]
Then, applying the Lebesgue dominated convergence theorem, by (2.1) we see that \( S(t)\mu \) is continuous in \( D \times (0, T') \). Since \( T' \) is arbitrary, the proof is complete. \( \square \)

**Lemma 2.2** Let \( T > 0 \) and \( \mu \) be a Radon measure in \( \mathbb{R}^N \). Let \( \lambda \geq 0 \) be such that \( 4\lambda T < 1 \). Assume that \( \text{supp} \mu \subset D_L \) for some \( L \geq 0 \). Then there exists \( \gamma > 0 \) such that
\[
[S(t)\mu](x) \leq \gamma t^{-\frac{N}{2}} \exp\left(-\frac{L}{\gamma t}\right) \sup_{z \in D_L} \int_{B(z, t^2)} e^{-\lambda y_N^2} \, d\mu(y)
\] (2.4)
for \( x \in \partial \mathbb{R}_+^N \) and \( 0 < t \leq T \).

**Proof.** Let \( x \in \partial \mathbb{R}_+^N \) and \( 0 < t \leq T \). Let \( \lambda \geq 0 \) be such that \( 4\lambda T < 1 \). By the Besicovitch covering lemma we can find an integer \( m \) depending only on \( N \) and a set \( \{x_{k,i}\}_{k=1,\ldots,m, i \in \mathbb{N}} \subset D_L \) such that
\[
B_{k,i} \cap B_{k,j} = \emptyset \quad \text{if} \quad i \neq j,
\]
\[
D_L \subset \bigcup_{k=1}^m \bigcup_{i=1}^\infty B_{k,i},
\] (5)
where \( B_{k,i} := \overline{B(x_{k,i}, t^2)} \). Then it follows from (2.1) that
\[
[S(t)\mu](x) \leq 2 \sum_{k=1}^m \sum_{i=1}^\infty \int_{D_L \cap B_{k,i}} \Gamma_N(x - y, t) \, d\mu(y)
\]
\[
\leq C t^{-\frac{N}{2}} \sup_{k=1,\ldots,m, i \in \mathbb{N}} \int_{B_{k,i}} e^{-\lambda y_N^2} \, d\mu(y) \sum_{k=1}^m \sum_{i=1}^\infty \sup_{y \in D_L \cap B_{k,i}} \exp\left(-\frac{|x - y|^2}{4t} + \lambda y_N^2\right).
\] (2.6)
On the other hand, for any \( y \in D_L \) and \( r > 0 \), there exists a set \( \{y_t\}_{t=1}^{m'} \subset D_L \) such that
\[
B(y, r) \cap D_L \subset \bigcup_{t=1}^{m'} B(y_t, r) \cap D_L,
\] (2.7)
where \( m' \) is an integer depending only on \( N \). This implies that
\[
\int_{B_{k,i}} e^{-\lambda y_N^2} d\mu(y) \leq m' \sup_{z \in D_L} \int_{B(z, t^2)} e^{-\lambda y_N^2} d\mu(y)
\] (2.8)
for any \( k \in \{1, \ldots, m\}, i \in \mathbb{N} \) and \( t \in (0, T] \). On the other hand, since \( x \in \partial \mathbb{R}^N_+ \) and
\[
|x' - y'|^2 \geq (|x' - z'| - |z' - y'|)^2 = |x' - z'|^2 - 2|x' - z'||z' - y'| + |z' - y'|^2
\]
\[
\geq \frac{1}{2}|x' - z'|^2 - |z' - y'|^2 \geq \frac{1}{2}|x' - z'|^2 - 4t,
\]
\[
y_N^2 \geq (z_N - |y_N - z_N|)^2 = z_N^2 - 2z_N|y_N - z_N| + |y_N - z_N|^2
\]
\[
\geq \frac{1}{2}z_N^2 - |y_N - z_N|^2 \geq \frac{1}{2}z_N^2 - 4t,
\]
for \( y, z \in B_{k,i} \), where \( \delta := 1 - 4\lambda T > 0 \). This together with (2.6) and (2.8) implies that
\[
[S(t)\mu](x) \leq Ct^{-N} \sup_{z \in D_L} \int_{B(z, t^2)} e^{-\lambda y_N^2} d\mu(y)
\]
\[
\times \sum_{k=1}^m \sum_{i=1}^\infty \int_{D_L \cap B_{k,i}} \exp\left(-\frac{|x' - z'|^2}{8t}\right) \exp\left(-\frac{z_N^2}{8t}\right) dz
\]
\[
\leq Ct^{-N} \sup_{z \in D_L} \int_{B(z, t^2)} e^{-\lambda y_N^2} d\mu(y) \int_{D_L} \exp\left(-\frac{|x' - z'|^2}{8t}\right) \exp\left(-\frac{z_N^2}{8t}\right) dz
\]
\[
\leq Ct^{-N} \exp\left(-\frac{L}{16t}\right) \sup_{z \in D_L} \int_{B(z, t^2)} e^{-\lambda y_N^2} d\mu(y).
\]

Therefore we obtain (2.9). Thus Lemma 2.2 follows. \( \square \)

**Lemma 2.3** Assume that there exists a supersolution \( v \) of (1.1) and (1.2) in \([0, T)\) for some \( T > 0 \). Then there exists a minimal solution of (1.1) and (1.2) in \([0, T)\).

**Proof.** Since \( v \) is a supersolution in \([0, T)\), we have \( \lim_{T'} \sup_{T < T'} v(0, T) \geq [S(T')\mu](0) \) for any \( T' \in (0, T) \). Then Lemma 2.1 implies that \( S(t)\mu \in C(D \times (0, T)) \).

Let \( n \in \{1, 2, \ldots\} \). Set \( u_{n,1}(x, t) := [S(t)\mu](x) \). Since \( S(t)\mu \in C(D \times (0, T)) \), we can define \( u_{n,2} \) by
\[
u_{n,2}(x, t) := [S(t)\mu](x) + \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t - s) \left( \min\{u_{n,1}(y', 0, s), n\} \right)^p dy's
\]
for \( (x, t) \in D \times (0, T) \). Then it follows that
\[
u_{n,2} \in C(D \times (0, T)), \quad \nu_{n,2}(x, t) \leq v(x, t) \quad \text{on} \quad D \times (0, T).
\]
By induction we define \( u_{n,k} \in C(D \times (0, T)) \) by

\[
u_{n,k}(x, t) := [S(t) \mu](x) + \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t - s) \left( \min\{u_{n,k-1}(y', 0, s), \gamma\} \right) dy'\]

for \((x, t) \in D \times (0, T)\), where \( k = 1, 2, \ldots \). Furthermore, we see that

\[
\begin{align*}
  u_{n,1}(x, t) &\leq u_{n,2}(x, t) \leq \cdots \leq u_{n,k}(x, t) \\
  u_{1,k}(x, t) &\leq u_{2,k}(x, t) \leq \cdots \leq u_{n,k}(x, t),
\end{align*}
\]

for \((x, t) \in D \times (0, T)\). Then we deduce that the sequence \( \{u_{n,k}\} \) is equibounded and equicontinuous with respect to \( k \) and \( n \) on any compact set \( K \subset D \times (0, T) \) (see e.g., [5, Section 6] and [9, Section 2]). By the Ascoli-Arzelà theorem and the diagonal argument we can find a function \( u \in C(D \times (0, T)) \) such that

\[
u(x, t) = [S(t) \mu](x) + \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t - s) u(y', 0, s) dy' \leq v(x, t)
\]

for \((x, t) \in D \times (0, T)\). This means that \( u \) is a solution of (1.1) and (1.2) in \([0, T)\). Furthermore, we easily see that \( u \) is a minimal solution of (1.1) and (1.2) in \([0, T)\). Thus Lemma 2.4 follows. \( \square \)

At the end of this section we state the following two lemmas on the initial trace of the solution of (1.1). These are proved by similar arguments as in the proofs of Lemmas 2.3 and 2.4 in [8], respectively, and we left the proofs to the reader.

**Lemma 2.4** Let \( u \) be a solution of (1.1) in \((0, T)\), where \( 0 < T < \infty \). Then

\[
\sup_{0 < t < T - \epsilon} \int_{B_r(0, R)} u(y, t) dy < \infty
\]

for \( R > 0 \) and \( 0 < \epsilon < T \). Furthermore, there exists a unique Radon measure \( \mu \in \mathbb{R}^N \) with \( \text{supp} \mu \subset D \) such that

\[
\lim_{t \to +0} \int_D \eta(y) u(y, t) dy = \int_D \eta(y) u(y) dy, \quad \eta \in C_c(\mathbb{R}^N).
\]

**Lemma 2.5** Let \( \mu \) be a Radon measure in \( \mathbb{R}^N \) with \( \text{supp} \mu \subset D \). Let \( u \) be a solution of (1.1) and (1.2) in \([0, T)\) for some \( 0 < T < \infty \). Then (2.10) holds for \( \eta \in C_c(\mathbb{R}^N) \).

### 3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. For this aim, we prepare the following lemma.

**Lemma 3.1** Let \( u \) be a solution of (1.1) in \((0, T)\), where \( 0 < T < \infty \). For any \( \epsilon \in (0, 1/2) \), there exists \( \gamma_\epsilon > 0 \) such that

\[
u(x + \zeta, (1 - \epsilon)T + \rho^2 + \tau) \geq \gamma_\epsilon \Gamma_N \left( x, \frac{T}{\gamma_\epsilon} \right) \exp \left( -\frac{1 + \epsilon z_N^2}{1 - \epsilon 4T} \right) \int_{B_r(\zeta, \rho)} u(y, \tau) dy
\]

for \( x, \zeta \in D \), \( \rho \in (0, (\epsilon T)^{1/2}) \) and \( \tau \in (0, (1 - \epsilon)T) \), where \( \zeta := (z', 0) \). Here the constant \( \gamma_\epsilon \) depends only on \( N \) and \( \epsilon > 0 \).
**Proof.** Let $z \in D$. We can assume, without loss of generality, that $z' = 0$ and $\overline{\tau} = 0$. Let $\epsilon \in (0, 1/2)$, $0 < \rho < (\epsilon T)^{1/2}$ and $\tau \in (0, (1-\epsilon)T)$. Since

$$
\min_{y \in B_+}\Gamma_N(x-y, (1-\epsilon)T + \rho^2) \geq (4\pi((1-\epsilon)T + \rho^2))^{-\frac{N}{2}} \exp\left(-\frac{(|x|+|z|+\rho)^2}{4((1-\epsilon)T + \rho^2)}\right)
$$

$$
\geq (4\pi((1-\epsilon)T + \rho^2))^{-\frac{N}{2}} \exp\left(-\frac{C|x|^2 + C\rho^2}{4((1-\epsilon)T + \rho^2)}\right) \exp\left(-\frac{(1+\epsilon)z_N^2}{4((1-\epsilon)T + \rho^2)}\right)
$$

$$
\geq C^{-1}\Gamma_N(x, CT) \exp\left(-\frac{1+\epsilon z_N^2}{1-\epsilon}\right),
$$

by (1.7) and (2.1) we obtain

$$
u(x, (1-\epsilon)T + \rho^2 + \tau) \geq \int_{B_+} \Gamma_N(x-y, (1-\epsilon)T + \rho^2)u(y, \tau) \, dy
$$

$$
\geq C^{-1}\Gamma_N(x, CT) \exp\left(-\frac{1+\epsilon z_N^2}{1-\epsilon}\right) \int_{B_+} u(y, \tau) \, dy
$$

for $x \in D$. This implies (3.1), and Lemma 3.1 follows. □

Next we recall the following lemma (see [2] Lemma 2.1.2).

**Lemma 3.2** Let $\mu \in C^1(D)$ be such that $\partial_x \mu \leq 0$ in $R_N^\mathbb{N}$. Assume that there exists a solution of (1.1) and (1.2) in $[0,T)$ for some $T > 0$. Then

$$
[S(t)\mu](x',0) \leq \gamma t^{-\frac{1}{2(1-p)}}
$$

holds for $x' \in R^{N-1}$ and $t \in (0,T)$, where $\gamma$ is a constant depending only on $N$ and $p$.

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1** By Lemma 2.4 we can find a unique Radon measure $\mu$ in $R^N$ with supp $\mu \subset D$ satisfying (1.9). So it suffices to prove assertions (1), (2) and (3).

Let $u$ be a solution of (1.1) in $(0,T)$ for some $T > 0$. Let $0 < \sigma < T^{1/2}$ and $0 < \epsilon < 1/2$.

Lemma 3.1 implies that

$$
u(x + \overline{\tau}, (1-\epsilon)\sigma^2 + \rho^2 + \tau) \geq \gamma_N \Gamma_N\left(x, \sigma^2\gamma_N^\overline{\tau} \exp\left(-\frac{1+\epsilon z_N^2}{1-\epsilon}\right) \int_{B_+} u(y, \tau) \, dy
$$

(3.2)

for $x, z \in D, \rho \in (0, \epsilon^{1/2}(\sigma^2))$ and $\tau \in (0, (1-\epsilon)\sigma^2)$, where $\gamma_N$ is as in Lemma 3.1.

**Proof of assertions (1) and (3).** Since $\tilde{u}(x,t) := u(x + \overline{\tau}, t + (1-\epsilon)\sigma^2 + \rho^2 + \tau)$ is a solution of (1.1) in $(0, \epsilon\sigma^2 - \rho^2 - \tau)$, by Lemma 3.3 and (3.2) we can find a minimal solution $w$ of (1.1) in $[0, \epsilon\sigma^2 - \rho^2 - \tau)$ with

$$
w(x,0) = \gamma_N \Gamma_N\left(x, \sigma^2\gamma_N^\overline{\tau} \exp\left(-\frac{1+\epsilon z_N^2}{1-\epsilon}\right) \int_{B_+} u(y, \tau) \, dy
$$

(3.3)

Then it follows from Lemma 3.2 that

$$
C t^{-\frac{1}{2(1-p)}} \geq [S(t)w(0)](0)
$$

$$
= \gamma_N\Gamma_N\left(0, t + \overline{\tau}, \sigma^2\gamma_N^\overline{\tau} \exp\left(-\frac{1+\epsilon z_N^2}{1-\epsilon}\right) \int_{B_+} u(y, \tau) \, dy
$$

(3.3)

$$
= \gamma(4\pi)^{-\frac{N}{2}} \left(t + \overline{\tau}, \sigma^2\gamma_N^\overline{\tau} \right) \exp\left(-\frac{1+\epsilon z_N^2}{1-\epsilon}\right) \int_{B_+} u(y, \tau) \, dy
$$

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for \(0 < t < \epsilon\sigma^2 - \rho^2 - \tau\).

Let \(0 < \rho' < \rho\). Let \(\zeta \in C_0(\mathbb{R}^N)\) be such that

\[
\zeta = 1 \quad \text{on} \quad B(z, \rho'), \quad 0 \leq \zeta \leq 1 \quad \text{in} \quad \mathbb{R}^N, \quad \zeta = 0 \quad \text{outside} \quad B(z, \rho).
\]

By Lemma 2.4 we have

\[
\limsup_{\tau \to +0} \int_{B_+(z, \rho)} u(y, \tau) \, dy \geq \limsup_{\tau \to +0} \int_{D} u(y, \tau) \zeta(y) \, dy = \int_{D} \zeta \, d\mu(y) \geq \int_{B_+(z, \rho')} \, d\mu(y). \quad (3.4)
\]

Since \(\rho'\) is arbitrary, by (3.3) and (3.4) we obtain

\[
\gamma_*(4\pi)^{-\frac{N}{2}} \left( t + \frac{\sigma^2}{\gamma_*} \right)^{-\frac{N}{2}} \exp \left( -\frac{1 + \epsilon \frac{z_N^2}{1 - \epsilon 4\sigma^2}}{1 - \epsilon 4\sigma^2} \right) \int_{B_+(z, \rho)} \, d\mu \leq Ct \frac{1}{\pi(p-1)} \quad (3.5)
\]

for \(z \in D, \rho \in (0, \epsilon^{1/2} \sigma)\) and \(0 < t < \epsilon\sigma^2 - \rho^2\). Setting \(\rho = (\epsilon/2)^{1/2} \sigma\) and \(t = \epsilon\sigma^2/4\), we obtain

\[
\exp \left( -\frac{1 + \epsilon \frac{z_N^2}{1 - \epsilon 4\sigma^2}}{1 - \epsilon 4\sigma^2} \right) \int_{B_+(z, (\epsilon/2)^{1/2} \sigma)} \, d\mu \leq C\sigma^N \quad (3.6)
\]

for \(z \in D\) and \(\sigma \in (0, T^{1/2})\).

On the other hand, for any \(z \in D\), we can find \(\{z_\ell\}_{\ell=1}^{m'} \subset D\) such that

\[
B_+(z, \sigma) \subset \bigcup_{\ell=1}^{m'} B_+(z_\ell, (\epsilon/2)^{1/2} \sigma). \quad (3.7)
\]

Here \(m'\) is independent of \(z\). We can assume, without loss of generality, that \(B_+(z, \sigma) \cap B_+(z_\ell, (\epsilon/2)^{1/2} \sigma) \neq \emptyset\). Then \((z_\ell)_N \leq z_N + 2\sigma\) and it follows that

\[
\exp \left( -\frac{(1 + \epsilon)^2 \frac{z_N^2}{1 - \epsilon 4\sigma^2} + 1 + \epsilon \frac{(z_\ell)_N^2}{1 - \epsilon 4\sigma^2}}{1 - \epsilon 4\sigma^2} \right) \leq \exp \left( C \frac{1 + \epsilon}{1 - \epsilon} \right) \leq C.
\]

This together with (3.6) and (3.7) implies that

\[
\exp \left( -\frac{(1 + \epsilon)^2 \frac{z_N^2}{1 - \epsilon 4\sigma^2}}{1 - \epsilon 4\sigma^2} \right) \int_{B_+(z, \sigma)} \, d\mu \leq \sum_{\ell=1}^{m'} \exp \left( -\frac{(1 + \epsilon)^2 \frac{(z_\ell)_N^2}{1 - \epsilon 4\sigma^2} + 1 + \epsilon \frac{(z_\ell)_N^2}{1 - \epsilon 4\sigma^2}}{1 - \epsilon 4\sigma^2} \right) \times \exp \left( -\frac{1 + \epsilon \frac{(z_\ell)_N^2}{1 - \epsilon \sigma^2}}{1 - \epsilon \sigma^2} \right) \int_{B_+(z_\ell, (\epsilon/2)^{1/2} \sigma)} \, d\mu \leq C\sigma^N \quad (3.8)
\]

for \(z \in D\) and \(0 < \sigma < T^{1/2}\).

Let \(\delta > 0\). Taking a sufficiently small \(\epsilon \in (0, 1/2)\) if necessary, we have \((1 + \epsilon)^2/(1 - \epsilon) \leq 1 + \delta\). Then (3.8) implies assertions (1) and (3).
Proof of assertion (2). Let \( p = p_* \). Set \( \rho \in (0, (\epsilon/2)^{1/2}\sigma) \) and \( \tau \in (0, (1 - \epsilon)\sigma^{1/2}) \). For any \( z = (z', z_N) \in D \), set
\[
v(x, t) := u(x + \overline{z}, t + (1 - \epsilon)\sigma^2 + \rho^2)
\]
for \( x \in D \) and \( t \in (0, T - (1 - \epsilon)\sigma^2 - \rho^2) \), where \( \overline{z} = (z', 0) \). Since \( v \) is a solution of (1.1) in \( (0, T - (1 - \epsilon)\sigma^2 - \rho^2) \), we have
\[
v(x, t) = \int_D G(x, y, t - \tau)v(y, \tau)\,dy + \int_\tau^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t - s)v(y', 0, s)^p\,dy'\,ds \tag{3.9}
\]
for \( x \in D \) and \( 0 < \tau < t < T - (1 - \epsilon)\sigma^2 - \rho^2 \). In particular, for any \( 0 < T' < (T - (1 - \epsilon)\sigma^2 - \rho^2)/2 \), by [2.2] we have
\[
\infty > v(0, 2T') \geq \int_\tau^{2T'} \int_{\mathbb{R}^{N-1}} G(0, y', 0, 2T' - s)v(y', 0, s)^p\,dy'\,ds \\
= 2 \int_\tau^{2T'} (4\pi(2T' - s))^{-\frac{N}{2}} \int_{\mathbb{R}^{N-1}} \Gamma_{N-1}(y', 2T' - s)v(y', 0, s)^p\,dy'\,ds \\
\geq 2 \int_\tau^{2T'} (4\pi(2T' - s))^{-\frac{N}{2}} (4\pi s)^{-\frac{N-1}{2}} \int_{\mathbb{R}^{N-1}} \Gamma_{N-1}(y', s)v(y', 0, s)^p\,dy'\,ds
\]
for \( 0 < \tau < T' \). This together with the Jensen inequality that
\[
\infty > 2 \int_\tau^{2T'} (4\pi(2T' - s))^{-\frac{N}{2}} (4\pi s)^{-\frac{N-1}{2}} \left( \int_{\mathbb{R}^{N-1}} \Gamma_{N-1}(y', s)v(y', 0, s)\,dy' \right)^p \,ds
\]
for \( 0 < \tau < T' \). Since \( T' \) is arbitrary, we see that
\[
V(t) := \int_{\mathbb{R}^{N-1}} \Gamma_{N-1}(y', t)v(y', 0, t)\,dy' < \infty \tag{3.10}
\]
for almost all \( t \in (0, (T - (1 - \epsilon)\sigma^2 - \rho^2)/2) \).
It follows from (3.2) that
\[
\int_D G(x, y, t - \tau)v(y, \tau)\,dy = \int_D G(x, y, t - \tau)u(y + \overline{z}, (1 - \epsilon)\sigma^2 + \rho^2 + \tau)\,dy \\
\geq \gamma_* \exp \left( \frac{1 + \epsilon z_N^2}{1 - \epsilon 4\sigma^2} \right) \int_{B_+(z, \rho)} u(y, \tau)\,dy \int_D G(x, y, t - \tau)\Gamma_N \left( y, \frac{\sigma^2}{\gamma_*} \right) \,dy \tag{3.11}
\]
for \( x \in D \) and \( 0 < \tau < t < T - (1 - \epsilon)\sigma^2 - \rho^2 \), where \( \gamma_* \) is as in Lemma 3.1. Setting
\[
M_\tau := \gamma_* \exp \left( \frac{1 + \epsilon z_N^2}{1 - \epsilon 4\sigma^2} \right) \int_{B_+(z, \rho)} u(y, \tau)\,dy,
\]
by (3.9), (3.10) and (3.11) we obtain
\[
\infty > V(t) \geq M_\tau \int_{\mathbb{R}^{N-1}} \Gamma_{N-1}(x', t)\Gamma_N \left( x', 0, t - \tau + \frac{\sigma^2}{\gamma_*} \right) \,dx' \\
+ \int_{\mathbb{R}^{N-1}} \int_\tau^t \int_{\mathbb{R}^{N-1}} G(x', 0, y', 0, t - s)\Gamma_{N-1}(x', t)v(y', 0, s)^p\,dy'\,ds \,dx'
\]
for almost all $0 < \tau < T - (1 - \epsilon)\sigma^2 - \rho^2)/2$. It follows from $0 < \rho^2 < \epsilon\sigma^2/2$ that

\[
\int_{\mathbb{R}^{N-1}} \Gamma_N(x', t) \Gamma_N \left( x', 0, t - \tau + \frac{\sigma^2}{\gamma_s} \right) dx' = \int_{\mathbb{R}^{N-1}} \Gamma_N(x', t) \left( 4\pi \left( t - \tau + \frac{\sigma^2}{\gamma_s} \right) \right)^{-\frac{1}{2}} \Gamma_N \left( x', t - \tau + \frac{\sigma^2}{\gamma_s} \right) dx' = \left( 4\pi \left( t - \tau + \frac{\sigma^2}{\gamma_s} \right) \right)^{-\frac{1}{2}} \Gamma_N \left( 0, 2t - \tau + \frac{\sigma^2}{\gamma_s} \right) \geq c t^{-\frac{N-1}{2}} \tag{3.13}
\]

for $0 < \tau < \epsilon\sigma^2/4 < t < (T - (1 - \epsilon)\sigma^2 - \rho^2)/2$, where $c$ is a positive constant depending only on $N$ and $\epsilon$. Furthermore, by (2.2) and the Jensen inequality we have

\[
\int_{\mathbb{R}^{N-1}} \int_{\tau}^{t} \int_{\mathbb{R}^{N-1}} G(x', 0, y', 0, t - s) \Gamma_N(x', t) v(y', 0, s)^p dy' ds dx' = \int_{\tau}^{t} \int_{\mathbb{R}^{N-1}} 2[4\pi(t - s)]^{-\frac{1}{2}} \times \left[ \int_{\mathbb{R}^{N-1}} \Gamma_N(x', t) \Gamma_N(x' - y', t - s) dx' \right] v(y', 0, s)^p dy' ds = \int_{\tau}^{t} \int_{\mathbb{R}^{N-1}} 2[4\pi(t - s)]^{-\frac{1}{2}} \Gamma_N(y', 2t - s) v(y', 0, s)^p dy' ds \geq \int_{\tau}^{t} \int_{\mathbb{R}^{N-1}} 2[4\pi(t - s)]^{-\frac{1}{2}} \left( \frac{s}{2t} \right)^{\frac{N-1}{2}} \Gamma_N(y', s) v(y', 0, s)^p dy' ds = 2^{-\frac{N-1}{2}} \int_{\tau}^{t} (t - s)^{-\frac{1}{2}} s^{\frac{N-1}{2}} V(s)^p ds.
\]

Therefore, by (3.12), (3.13) and (3.14) we obtain

\[
V(t) \geq c M_\epsilon t^{-\frac{N}{2}} + 2^{-\frac{N-1}{2}} \pi^{-\frac{1}{2}} t^{-\frac{N-1}{2}} \int_{\epsilon\sigma^2/4}^{t} (t - s)^{-\frac{1}{2}} s^{\frac{N-1}{2}} V(s)^p ds \tag{3.15}
\]

for $0 < \tau < \epsilon\sigma^2/4$ and almost all $t \in (\epsilon\sigma^2/4, (T - (1 - \epsilon)\sigma^2 - \rho^2)/2)$. Set $a_1 := c$ and $\omega_1(t) := a_1 M_\epsilon t^{-\frac{N}{2}}$. Define $\omega_{n+1}(t) := 2^{-\frac{N-1}{2}} \pi^{-\frac{1}{2}} t^{-\frac{N-1}{2}} \int_{\epsilon\sigma^2/4}^{t} (t - s)^{-\frac{1}{2}} s^{\frac{N-1}{2}} \omega_n(s)^p ds$, $n = 1, 2, \ldots$, for $t > \epsilon\sigma^2/4$. Then it follows that

\[
\infty > V(t) \geq \omega_{n+1}(t) \geq a_{n+1} M_\epsilon^n t^{-\frac{N}{2}} \left[ \log \left( \frac{4t}{\epsilon\sigma^2} \right) \right]^{\frac{p-1}{p-1}} \tag{3.16}
\]
for almost all \( t \in (\epsilon \sigma^2/4, (T - (1 - \epsilon)\sigma^2 - \rho^2)/2) \) and \( n = 0, 1, 2, \ldots \). Here \( \{a_n\} \) is a sequence defined by

\[
a_{n+1} := 2^{-\frac{N-1}{2}} \pi^{-\frac{1}{2}} a_n^p \frac{p-1}{p^p-1}, \quad n = 1, 2, \ldots
\]  

(3.17)

Indeed, (3.16) holds with \( n = 0 \). Furthermore, if (3.16) holds for some \( n \in \{0, 1, 2, \ldots \} \), then, by (3.15) we have

\[
\infty > V(t) \geq \omega_{n+2}(t) = 2^{-\frac{N-1}{2}} \pi^{-\frac{1}{2}} a_{n+1}^p \int_{\rho^2/4}^t (t-s)^{-\frac{N}{2}} \omega_{n+1}(s)^p \, ds
\]

\[
= 2^{-\frac{N-1}{2}} \pi^{-\frac{1}{2}} a_{n+1}^p M_{\tau}^{p^{n+1}} t^{-\frac{N}{2}} \int_{\rho^2/4}^t (t-s)^{-\frac{N}{2}} s^{-1} \left[ \log \left( \frac{4s}{\epsilon \sigma^2} \right) \right]^{\frac{n+1}{p-1}} ds
\]

\[
> 2^{-\frac{N-1}{2}} \pi^{-\frac{1}{2}} a_{n+1}^p M_{\tau}^{p^{n+1}} t^{-\frac{N}{2}} \left[ \log \left( \frac{4t}{\epsilon \sigma^2} \right) \right]^{\frac{n+1}{p-1}}
\]

\[
= a_{n+2} M_{\tau}^{p^{n+1}} t^{-\frac{N}{2}} \left[ \log \left( \frac{4t}{\epsilon \sigma^2} \right) \right]^{\frac{n+1}{p-1}}
\]

for almost all \( t \in (\epsilon \sigma^2/4, (T - (1 - \epsilon)\sigma^2 - \rho^2)/2) \). This means that (3.16) holds for \( n + 1 \). Thus (3.16) holds for all \( n \in \{0, 1, 2, \ldots \} \).

On the other hand, similarly to [13, Lemma 2.20 (i)] (see also (3.26) in [8]), we can find \( b > 0 \) such that

\[
a_n \geq b^n, \quad n = 1, 2, \ldots
\]

This together with (3.16) implies that

\[
\infty > V(t) \geq \omega_{n+1}(t) \geq b^{p^{n+1}} M_{\tau}^{p^n} t^{-\frac{N}{2}} \left[ \log \left( \frac{4t}{\epsilon \sigma^2} \right) \right]^{\frac{n+1}{p-1}}
\]

\[
= t^{-\frac{N}{2}} \left[ \log \left( \frac{4t}{\epsilon \sigma^2} \right) \right]^{-\frac{1}{p-1}} \left( b^p M_{\tau} \left[ \log \left( \frac{4t}{\epsilon \sigma^2} \right) \right]^{\frac{1}{p-1}} \right)^n
\]

for almost all \( t \in (\epsilon \sigma^2/4, (T - (1 - \epsilon)\sigma^2 - \rho^2)/2) \) and \( n = 1, 2, \ldots \). Then it follows that

\[
M_{\tau} \leq b^{-p} \left[ \log \left( \frac{4t}{\epsilon \sigma^2} \right) \right]^{-\frac{1}{p-1}} = b^{-p} \left[ \log \left( \frac{4t}{\epsilon \sigma^2} \right) \right]^{-N},
\]

which implies that

\[
\exp \left( -\frac{1 + \epsilon \gamma_z^2}{1 - \epsilon 4\sigma^2} \right) \int_{B_+(z, \rho)} u(y, \tau) \, dy \leq (b^p \gamma_z)^{-1} \left[ \log \left( \frac{4t}{\epsilon \sigma^2} \right) \right]^{-N}, \quad z \in D, \tau \in \left( 0, \frac{\epsilon \sigma^2}{4} \right),
\]

for \( t \in (\epsilon \sigma^2/4, (T - (1 - \epsilon)\sigma^2 - \rho^2)/2) \). Then, similarly to (3.4), we obtain

\[
\exp \left( -\frac{1 + \epsilon \gamma_z^2}{1 - \epsilon 4\sigma^2} \right) \int_{B_+(z, \rho)} d\mu(y) \leq (b^p \gamma_z)^{-1} \left[ \log \left( \frac{4t}{\epsilon \sigma^2} \right) \right]^{-N}
\]  

(3.18)
for \( z \in D \) and \( t \in (\epsilon \sigma^2/4, (T - (1 - \epsilon)\sigma^2 - \rho^2)/2) \).

Set \( \rho = (\epsilon/4)^{1/2} \sigma \). Consider the case where \( 0 < \sigma^2 \leq T/2 \). It follows that

\[
\frac{T - (1 - \epsilon)\sigma^2 - \rho^2}{2} > \frac{T - \sigma^2}{2} \geq \frac{T}{4}.
\]

Setting \( t = T/4 \), by (3.18) we have

\[
\exp \left( -\frac{1}{1 - \epsilon} \frac{z^2}{4\sigma^2} \right) \int_{B^+(z, (\epsilon/4)^{1/2} \sigma)} d\mu(y) \leq (b^\rho \gamma_*)^{-1} \left[ \log \left( \frac{T}{\epsilon \sigma^2} \right) \right]^{-N} \leq C \left[ \log \left( e + \frac{T^{1/2}}{\sigma} \right) \right]^{-N}, \quad z \in D.
\]

(3.19)

On the other hand, in the case where \( T/2 < \sigma^2 < T \), we have

\[
\frac{T - (1 - \epsilon)\sigma^2 - \rho^2}{2} \geq \frac{\epsilon \sigma^2 - \rho^2}{2} = \frac{3}{8} \epsilon \sigma^2, \quad 1 < \frac{T}{\sigma^2} < 2.
\]

Then, taking a sufficiently small \( \epsilon \in (0, 1/2) \) if necessary, we set

\[
t = \frac{5}{16} \epsilon \sigma^2 \in \left( \frac{\epsilon \sigma^2}{4}, \frac{(T - (1 - \epsilon)\sigma^2 - \rho^2)}{2} \right)
\]

and by (3.18) we obtain

\[
\exp \left( -\frac{1}{1 - \epsilon} \frac{z^2}{4\sigma^2} \right) \int_{B^+(z, (\epsilon/4)^{1/2} \sigma)} d\mu(y) \leq (b^\rho \gamma_*)^{-1} \left[ \log \left( \frac{5}{4} \right) \right]^{-N} \leq C \left[ \log \left( e + \frac{T^{1/2}}{\sigma} \right) \right]^{-N}, \quad z \in D.
\]

(3.20)

Combining (3.19) and (3.20) and applying the same argument as in (3.18), we obtain

\[
\exp \left( -\frac{(1 + \epsilon)^2}{1 - \epsilon} \frac{z^2}{4\sigma^2} \right) \mu(B(z, \sigma)) \leq C \left[ \log \left( e + \frac{T^{1/2}}{\sigma} \right) \right]^{-N}, \quad z \in D, \ \sigma \in (0, T^{1/2}).
\]

Finally, similarly to the proof of assertions (1) and (3), for any \( \delta > 0 \), we take a sufficiently small \( \epsilon \in (0, 1/2) \) to obtain assertion (2). Thus Theorem 1.1 follows. \( \Box \)

4 Proof of Theorem 1.2

We modify the proof of [8, Theorem 1.2] to prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 2.5 it suffices to prove Theorem 1.2 (b). Let \( u \) be a solution of (1.1) in \((0, T)\), where \( 0 < T < \infty \). By (3.3) there exists \( \gamma > 0 \) such that

\[
\exp \left( -4\gamma \frac{z^2}{T^2} \right) \int_{B^+(z, (T/4)^{1/2})} u(y, \tau) \, dy \leq CT^{N - \frac{1}{2(p-1)}} (p-1)
\]

(4.1)
for \( z \in D \) and \( \tau \in (0, T/8) \). Let \( t \in (0, T) \). For any \( n = 1, 2, \ldots, \) by the Besicovitch covering lemma we can find an integer \( m \) depending only on \( N \) and a set \( \{x_{k,i}\}_{k=1,\ldots,m, i \in \mathbb{N}} \subset D \setminus B(0, nt^{1/2}) \) such that

\[
B_{k,i} \cap B_{k,j} = \emptyset \quad \text{if} \quad i \neq j \quad \text{and} \quad D \setminus B(0, nt^{1/2}) \subset \bigcup_{k=1}^{m} \bigcup_{i=1}^{\infty} B_{k,i},
\]

(4.2)

where \( B_{k,i} := B_+(x_{k,i}, t^{1/2}) \). Since

\[
\sup_{y \in B_{k,i}} \exp \left( 4\gamma \left( \frac{(x_{k,i})^2}{N} \right) \right) G(y, t - \tau)
\]

\[
\leq 2(4\pi(t - \tau))^{\frac{N}{2}} \sup_{y \in B_{k,i}} \exp \left( 4\gamma \left( \frac{|z_N| + |x_{k,i}N - z_N|^2}{T} \right) \exp \left( - \frac{|y|^2}{4(t - \tau)} \right) \right)
\]

\[
\leq Ct^{\frac{N}{2}} \exp \left( 8\gamma \frac{z_N^2}{T} \right) \sup_{y \in B_{k,i}} \exp \left( - \frac{(|z| - |z - y|)^2}{4t} \right)
\]

\[
\leq Ct^{\frac{N}{2}} \exp \left( 8\gamma \frac{z_N^2}{T} \right) \exp \left( - \frac{|z|^2}{8t} \right)
\]

for \( z \in B_{k,i} \) and \( 0 < \tau < t/2 \), by (4.1) and (4.2) we obtain

\[
\sup_{0 < \tau < t/2} \int_{D \setminus B(0, nt^{1/2})} G(y, t - \tau)u(y, \tau) \: dy \leq \sum_{k=1}^{m} \sum_{i=1}^{\infty} \sup_{0 < \tau < t/2} \int_{B_{k,i}} G(y, t - \tau)u(y, \tau) \: dy
\]

\[
\leq C \sup_{0 < \tau < t/2} \sup_{z \in D} \exp \left( -4\gamma \frac{z_N^2}{T} \right) \int_{B_+(z, (T/4)^{1/2})} u(y, \tau) \: dy
\]

\[
\times \sum_{k=1}^{m} \sum_{i=1}^{\infty} \sup_{0 < \tau < t/2} \sup_{y \in B_{k,i}} \exp \left( 4\gamma \left( \frac{x_{k,i}^2}{N} \right) \right) G(y, t - \tau)
\]

\[
\leq CT^{\frac{N}{2}} \pi^{\frac{1}{2}} \tau^{\frac{N}{2}} \sum_{k=1}^{m} \sum_{i=1}^{\infty} \int_{B_{k,i}} \exp \left( 8\gamma \frac{z_N^2}{T} \right) \exp \left( - \frac{|z|^2}{8t} \right) \: dz
\]

for \( 0 < t < T/4 \) and \( 0 < \tau < t/2 \). Then, taking a sufficiently small \( t > 0 \), we see that

\[
\sup_{0 < \tau < t/2} \int_{D \setminus B(0, nt^{1/2})} G(y, t - \tau)u(y, \tau) \: dy \leq Ct^{-N} \sum_{k=1}^{m} \sum_{i=1}^{\infty} \int_{B_{k,i}} \exp \left( - \frac{|z|^2}{16t} \right) \: dz,
\]

which together with (4.2) implies that

\[
\sup_{0 < \tau < t/2} \int_{D \setminus B(0, nt^{1/2})} G(y, t - \tau)u(y, \tau) \: dy
\]

\[
\leq Ct^{-N} \int_{D \setminus B(0, (n-1)t^{1/2})} \exp \left( - \frac{|z|^2}{16t} \right) \: dz \to 0
\]

as \( n \to \infty \). Similarly, by using Theorem 1.1 instead of (4.3), we see that

\[
\lim_{n \to \infty} \sup_{0 < \tau < t/2} \int_{D \setminus B(0, nt^{1/2})} G(y, t - \tau) \: d\mu = 0
\]

(4.4)
for all sufficiently small $t > 0$.

Let $\eta_n \in C_0(\mathbb{R}^N)$ be such that

$$0 \leq \eta_n \leq 1 \quad \text{in} \quad \mathbb{R}^N, \quad \eta_n = 1 \quad \text{on} \quad B(0, nt^{1/2}), \quad \eta_n = 0 \quad \text{outside} \quad B(0, 2nt^{1/2}).$$

Then we have

$$\left| \int_D G(y, t - \tau)u(y, \tau) \, dy - \int_D G(y, t) \, d\mu(y) \right| \leq \left| \int_D G(y, t)u(y, \tau)\eta_n(y) \, dy - \int_D G(y, t)\eta_n(y) \, d\mu(y) \right|$$

$$+ \left| \int_D [G(y, t - \tau) - G(y, t)]u(y, \tau)\eta_n(y) \, dy \right|$$

$$+ \int_{D \setminus B(0, nt^{1/2})} G(y, t - \tau)u(y, \tau) \, dy + \int_{D \setminus B(0, nt^{1/2})} G(y, t) \, d\mu(y)$$

for $n = 1, 2, \ldots$ and $\tau \in (0, t/2)$. By Lemma 2.4 we see that

$$\lim_{\tau \to 0^+} \left[ \int_D G(y, t)u(y, \tau)\eta_n(y) \, dy - \int_D G(y, t)\eta_n(y) \, d\mu(y) \right] = 0. \quad (4.5)$$

Furthermore, by Lemma 2.4 we have

$$\lim_{\tau \to 0^+} \left| \int_D [G(y, t - \tau) - G(y, t)]u(y, \tau)\eta_n(y) \, dy \right|$$

$$\leq \sup_{y \in B(0, 2nt^{1/2}), s \in (t/2, t)} |\partial_t G(y, s)| \lim_{\tau \to 0^+} \left[ \int_{B(0, 2nt^{1/2})} u(y, \tau) \, dy \right] = 0. \quad (4.6)$$

By (4.5), (4.6) and (4.7) we see that

$$\lim_{\tau \to 0^+} \left[ \int_D G(y, t)u(y, \tau)\eta_n(y) \, dy - \int_D G(y, t)\eta_n(y) \, d\mu(y) \right] = 0.$$

This together with (4.3) and (4.4) implies that

$$\lim_{\tau \to 0^+} \left| \int_D G(y, t - \tau)u(y, \tau) \, dy - \int_D G(y, t) \, d\mu(y) \right| = 0.$$

This together with Definition 1.1 (i) implies that $u$ is a solution of (1.1) and (1.2) in $[0, T)$. Thus Theorem 1.2 (b) follows, and the proof is complete. □

5 Proof of Theorems 1.3, 1.4 and 1.5

We prove Theorems 1.3, 1.4 and 1.5 by modifying the arguments in [8, 10, 14]. Lemma 5.1 is a key lemma in our proofs.
Lemma 5.1 Let $\Psi$ be a nonnegative convex function on $[0, \infty)$. Let $\mu$ be a nonnegative measurable function in $D$ such that $[S(T)\Psi(\mu)](x_0) < \infty$ for some $x_0 \in D$ and $T > 0$. Define
\[
W(x, t) := \Psi^{-1}([S(t)\Psi(\mu)](x)), \quad w(x', t) := W(x', 0, t),
\]
\[
F(x, t) := \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t - s)w(y', s) dy' ds,
\]
for $x' \in \mathbb{R}^{N-1}, x \in D$ and $t \in (0, T)$. Then there exists $\gamma > 0$ such that
\[
F(x, t) \leq \gamma t^\frac{1}{2} \left\| \frac{\Psi(W(t))}{W(t)} \right\|_{L^\infty(D)} W(x, t)
\]
\[
\times \int_0^t (t - s)^{-\frac{3}{2}} s^{-\frac{1}{2}} \left\| \frac{w(s)^p}{\Psi(w(s))} \right\|_{L^\infty(\mathbb{R}^{N-1})} ds
\]
for $x \in D$ and $0 < t < T$.

Proof. Since $[S(T)\Psi(\mu)](x_0) < \infty$ for some $x_0 \in D$ and $T > 0$, by Lemma 2.1 we can define $w = w(x', t)$ and $F = F(x, t)$ for $x' \in \mathbb{R}^{N-1}, x \in D$ and $t \in (0, T)$. It follows that
\[
F(x, t) \leq \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t - s)[S(s)\Psi(\mu)](y', 0) dy' ds. \tag{5.2}
\]
On the other hand, by (5.1) we have
\[
\int_{\mathbb{R}^{N-1}} G(x, y', 0, t - s)[S(s)\Psi(\mu)](y', 0) dy'
\]
\[
= \int_{\mathbb{R}^{N-1}} G(x', x_N, y', 0, t - s) \int_D G(y', 0, z', z_N, s)\Psi(\mu(z)) dz dy'
\]
\[
= \int_D \int_{\mathbb{R}^{N-1}} 2(4\pi(t - s))^{-\frac{1}{2}} \Gamma_{N-1}(x' - y', t - s) \exp \left(-\frac{x_N^2}{4(t - s)}\right)
\]
\[
\times 2(4\pi s)^{-\frac{1}{2}} \exp \left(-\frac{x_N^2}{4s}\right) \Gamma_{N-1}(y' - z', s)\Psi(\mu(z', z_N)) dy' dz
\]
for $x \in \mathbb{R}^{N-1}$ and $0 < s < t < T$. Then we have
\[
\int_{\mathbb{R}^{N-1}} G(x, y', 0, t - s)[S(s)\Psi(\mu)](y', 0) dy'
\]
\[
\leq \exp \left(-\frac{x_N^2}{4t}\right) \int_D 2(4\pi(t - s))^{-\frac{1}{2}} \Gamma_{N-1}(x' - z', t)
\]
\[
\times 2(4\pi s)^{-\frac{1}{2}} \exp \left(-\frac{x_N^2}{4s}\right) \Psi(\mu(z', z_N)) dz
\]
\[
\leq \exp \left(-\frac{x_N^2}{4t}\right) \int_D 2(4\pi t)^{-\frac{1}{2}} \exp \left(-\frac{x_N^2}{4t}\right) \Gamma_{N-1}(x' - z', t)
\]
\[
\times (t - s)^{-\frac{3}{2}} 2(4\pi s)^{-\frac{1}{2}} \Psi(\mu(z', z_N)) dz
\]
\[
= \exp \left(-\frac{x_N^2}{4t}\right) (t - s)^{-\frac{3}{2}} 2(4\pi s)^{-\frac{1}{2}} \int_D G(x', 0, z', z_N, t)\Psi(\mu(z)) dz
\]
\[
= \exp \left(-\frac{x_N^2}{4t}\right) \pi^{-\frac{1}{2}} t^\frac{3}{2} (t - s)^{-\frac{1}{2}} s^{-\frac{1}{2}} [S(t)\Psi(\mu)](x', 0)
\]
for \( x \in \mathbb{R}^{N-1} \) and \( 0 < s < t < T \). This together with (5.2) implies that

\[
F(x, t) \leq Ct^\frac{1}{2} \exp \left( -\frac{x_N^2}{4t} \right) [S(t)\Psi(\mu)](x', 0) \\
\times \int_0^t \left\| w(s)^p \right\|_{L^\infty(\mathbb{R}^{N-1})} (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}} ds
\]

for \( x \in D \) and \( t \in (0, T) \). Furthermore, it follows from (2.1) and (2.2) that

\[
\exp \left( -\frac{x_N^2}{4t} \right) [S(t)\Psi(\mu)](x', 0) \\
= 2(4\pi t)^{-\frac{N}{2}} \int_D \exp \left( -\frac{|x' - y'|^2}{4t} - \frac{x_N^2}{4t} \right) \Psi(\mu) dy \\
\leq 2(4\pi t)^{-\frac{N}{2}} \int_D \exp \left( -\frac{|x' - y'|^2}{4t} - \frac{|x_N - y_N|^2}{4t} \right) \Psi(\mu) dy \\
= 2 \int_D \Gamma_N(x - y, t)\Psi(\mu) dy \leq 2 \int_D G(x, y, t)\Psi(\mu) dy \leq 2[S(t)\Psi(\mu)](x)
\]

for \( x = (x', x_N) \in D \) and \( t \in (0, T) \). This together with (5.3) implies that

\[
F(x, t) \leq Ct^\frac{1}{2} [S(t)\Psi(\mu)](x) \int_0^t \left\| w(s)^p \right\|_{L^\infty(\mathbb{R}^{N-1})} (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}} ds \\
\leq Ct^\frac{1}{2} \left\| \Psi(W(t)) \right\|_{L^\infty(D)} \int_0^t \left\| w(s)^p \right\|_{L^\infty(\mathbb{R}^{N-1})} (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}} ds
\]

for \( x \in D \) and \( t \in (0, T) \). Thus we obtain (5.1), and the proof is complete. \( \square \)

**Proof of Theorem 1.3.** It suffices to consider the case \( T = 1 \). Indeed, for any solution \( u \) of (1.1) in \( [0, t) \), where \( 0 < T < \infty \), \( T^{1/(p-1)}u(T^{1/2}, x, t) \) is a solution of (1.1) in \( [0, 1) \).

Let \( \delta \in (0, 1) \) and set \( \lambda := (1 - \delta)/4 \). Assume (1.10). Then [\( S(1)\mu](0) < \infty \) and Lemma 2.1 implies that \( S(t)\mu \in C(D \times (0, 1)) \). We define

\[
\overline{\mu}(x, t) := 2[S(t)\mu](x), \quad W(x, t) := [S(t)\mu](x), \quad w(x', t) := W(x', 0, t),
\]

for \( x \in D, x' \in \mathbb{R}^{N-1} \) and \( t \in (0, 1) \). It follows from Lemma 2.2 with \( L = 0 \) and (1.10) that

\[
\|w(t)\|_{L^\infty(\mathbb{R}^{N-1})} \leq Ct^{-\frac{N}{2}} \sup_{x \in D} \int_{B(x, t^{\delta/2})} e^{-\lambda y_N} d\mu(y) \leq C\gamma_2 t^{-\frac{N}{2}} \tag{5.4}
\]

for \( t \in (0, 1) \). Applying Lemma 5.1 with \( \Psi(\tau) = \tau \), by (5.4) we obtain

\[
\int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t-s)w(y', s)^p dy' \; ds \\
\leq Ct^{\frac{1}{2}}W(x, t) \int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}\|w(s)\|_{L^\infty(\mathbb{R}^{N-1})}^p ds \\
\leq C\gamma_2^p t^{\frac{1}{2}}W(x, t) \int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}s^{-\frac{N}{2}(p-1)} ds \leq C\gamma_2^p W(x, t)
\]

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for \( x = (x', x_N) \in D \) and \( t \in (0, 1) \). Taking a sufficiently small \( \gamma_2 > 0 \) if necessary, by \( (5.5) \) we obtain

\[
\int_D G(x, y, t) d\mu(y) + \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t-s)\overline{\mu}(y', 0, s)^p \, dy' \, ds
\]

\[
= [S(t)\mu_1](x) + 2^p \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t-s)w(y', s)^p \, dy' \, ds
\]

\[
\leq [S(t)\mu_1](x) + W(x, t) = 2[S(t)\mu_1](x) = \overline{\mu}(x, t)
\]

for \( (x, t) \in D \times (0, 1) \). This means that \( \overline{\mu} \) is a supersolution of \( (1.1) \) and \( (1.2) \) in \( [0, 1] \). Therefore, by Lemma 2.3 we can find a solution of \( (1.1) \) and \( (1.2) \) in \( [0, 1] \) such that

\[ 0 \leq u(x, t) \leq \overline{\mu}(x, t) = 2[S(t)\mu_1](x) \in D \times (0, 1) \] such that

Thus Theorem 1.3 follows. \( \square \)

**Proof of Theorem 1.4** Similarly to the proof of Theorem 1.3 it suffices to consider the case \( T = 1 \). Let \( \delta \in (0, 1) \) and set \( \lambda := (1 - \delta)/4 \). Assume \( (1.11) \) and \( (1.12) \). Then Lemma 2.1 implies that \( S(t)\mu_1, S(t)\mu_2^\alpha \in C(D \times (0, 1)) \) and we define

\[
\overline{\mu}(x, t) := 2[S(t)\mu_1](x) + 2([S(t)\mu_2^\alpha](x))^{1/\alpha},
\]

\[
W_1(x, t) := [S(t)\mu_1](x), \quad W_2(x, t) := ([S(t)\mu_2^\alpha](x))^{1/\alpha},
\]

\[
w_1(x', t) := W_1(x', 0, t), \quad w_2(x', t) := W_2(x', 0, t),
\]

for \( x \in D, x' \in \mathbb{R}^{N-1} \) and \( t \in (0, 1) \). Then it follows from the Jensen inequality that

\[
\int_D G(x, y, t) d\mu(y) + \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t-s)\overline{\mu}(y', 0, s)^p \, dy' \, ds
\]

\[
\leq [S(t)\mu_1](x) + 2^p \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t-s)w_1(y', s)^p \, dy' \, ds
\]

\[
+ ([S(t)\mu_2^\alpha](x))^{1/\alpha} + 2^p \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t-s)w_2(y', s)^p \, dy' \, ds
\]

for \( (x, t) \in D \times (0, 1) \).

On the other hand, by Lemma 2.2 and \( (1.11) \) we see that

\[
\|w_1(t)\|_{L^\infty(\mathbb{R}^{N-1})} \leq C\frac{t^{3/2}}{\gamma_2^3} \exp\left(-\frac{1}{Ct}\right) \sup_{z \in D_1} \int_{B(z, t^{1/2})} e^{-\lambda \gamma_2^3} \, d\mu_1(y)
\]

\[
\leq C\gamma_3 t^{3/2} \exp\left(-\frac{1}{Ct}\right) \leq C\gamma_3, \quad t \in (0, 1].
\]

Applying Lemma 5.1 with \( \Psi(x) = \tau \), by \( (5.7) \) we obtain

\[
\int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t-s)w_1(y', s)^p \, dy' \, ds
\]

\[
\leq Ct^2 W_1(x, t) \int_0^t (t-s)^{-3/2} s^{-1/2} \|w_1(s)\|_{L^\infty(\mathbb{R}^{N-1})}^{p-1} \, ds \leq C\gamma_3^{p-1} W_1(x, t)
\]

for \( (x, t) \in D \times (0, 1) \). Similarly, by Lemma 2.2 and \( (1.12) \) with \( \sigma = t^{1/2} \) we see that

\[
\|W_2(t)\|_{L^\infty(D)} \leq \left[ C \sup_{z \in D_1} \int_{B(z, t^{1/2})} e^{-\lambda \gamma_2^3} \, d\mu_2^\sigma(y) \right]^{1/\alpha} \leq C\gamma_3 t^{-3/2(p-1)}
\]

(5.9)
for $t \in (0, 1)$. Applying Lemma 5.1 with $\Psi(\tau) = \tau^\alpha$, by (5.9) we obtain

$$\int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t-s) w_2(y', s)^p \, dy' \, ds$$

$$\leq C t^{\frac{2}{3}} \|W_2(t)\|_{L^\infty(D)}^{\alpha-1} W_2(x, t) \int_0^t (t-s)^{-\frac{3}{2}} \|w_2(s)\|_{L^\infty(\mathbb{R}^{N-1})}^{p-\alpha} \, ds$$

$$\leq C \gamma_3 \gamma_3 \gamma_1 \frac{1}{t} \int_0^t (t-s)^{-\frac{3}{2}} \|w_2(s)\|_{L^\infty(\mathbb{R}^{N-1})}^{p-\alpha} \, ds \leq C \gamma_3 \gamma_1 W_2(x, t)$$

for $(x, t) \in D \times (0, 1)$. Taking a sufficiently small $\gamma_3 > 0$ if necessary, by (5.6), (5.8) and (5.10) we obtain

$$\int_D G(x, y, t) \, d\mu(y) + \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t-s) \overline{u}(y', 0, s)^p \, dy' \, ds$$

$$\leq \|S(t)\|_{L^1(D)} [x] + W_1(x, t) + ([S(t)\mu_2^2](x))^{\frac{1}{2}} + W_2(x, t) = \overline{u}(x, t)$$

for $(x, t) \in D \times (0, 1)$. This means that $\overline{u}$ is a supersolution of (1.1) and (1.2) in $[0, 1)$. Therefore, by Lemma 2.3 we can find a solution of (1.1) and (1.2) in $[0, 1)$ such that $0 \leq u(x, t) \leq \overline{u}(x, t)$ in $D \times (0, 1)$. Thus Theorem 1.5 follows. □

**Proof of Theorem 1.5.** It suffices to consider the case $T = 1$. Let $\Phi_\beta(s)$ and $\rho(s)$ be as in (1.14). Let $h \geq 0$ be such that

(a) $\Phi(\tau) := \tau [\log(h + \tau)]^\beta$ is positive and convex in $(0, \infty)$;

(b) $\rho^\alpha(\tau)$ and $\Phi(\tau)/\tau$ are monotone increasing in $(0, \infty)$.

Let $\delta \in (0, 1)$ and set $\lambda := (1 - \delta)/4$. Assume (1.14) and (1.15). Then Lemma 2.1 implies that $S(t)\mu_1, S(t)\Phi(\mu_2) \in C(D \times (0, 1))$. We define

$$\overline{u}(x, t) := 2\|S(t)\|_{L^1(D)} [x] + d \overline{u}(x, t) \|S(t)\Phi(\mu_2)](x)\),$$

$$W_1(x, t) := [S(t)\mu_1](x), \quad W_2(x, t) := \Phi^{-1}([S(t)\Phi(\mu_2)](x),$$

$$w_1(x', t) := W_1(x', 0, t), \quad w_2(x', t) := W_2(x', 0, t),$$

for $x \in D, x' \in \mathbb{R}^{N-1}$ and $t \in (0, 1)$. Here $d$ is a positive constant to be chosen later. Since

$$\overline{u}(x, t) \leq 2\|S(t)\|_{L^1(D)} [x] + c d W_2(x, t) \quad \text{in} \quad D \times (0, 1)$$

for some $c > 0$, similarly to (5.6), it follows from the Jensen inequality that

$$\int_{\mathbb{R}^{N}} G(x, y, t) \, d\mu(y) + \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t-s) \overline{u}(y', 0, s)^p \, dy' \, ds$$

$$\leq \|S(t)\|_{L^1(D)} [x] + 2^{p-1} \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t-s) w_1(y', s)^p \, dy' \, ds$$

$$+ W_2(x, t) + 2^{p-1} (cd)^p \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t-s) w_2(y', s)^p \, dy' \, ds$$

for $(x, t) \in D \times (0, 1)$. Furthermore, similarly to (5.8), we obtain

$$\int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t-s) w_1(y', s)^p \, dy' \, ds \leq C_\gamma \gamma_1 W_1(x, t)$$

(5.13)
for \((x, t) \in D \times (0, 1)\).

On the other hand, it follows from Lemma 2.2 that
\[
\|\Phi(W_2(t))\|_{L^\infty(D)} = \|S(t)\Phi(\mu_2)\|_{L^\infty(D)} \\
\leq C t^{-\frac{N}{2}} \sup_{z \in D} \int_{B(z,t^{\frac{4}{N}})} \Phi(\mu_2) \, dy \\
= C t^{-\frac{N}{2}} \sup_{z \in D} \int_{B(z,t^{\frac{4}{N}})} \Phi(\mu_2) \, dy \\
\leq C \Phi_{\beta} \left( \gamma_4 \rho(t^{\frac{4}{N}}) \right) \\
\leq C \Phi \left( \gamma_4 \rho(t^{\frac{4}{N}}) \right)
\]
for \(t \in (0, 1)\). This together with property (b) of \(\Phi\) implies that
\[
\|w_2(s)^p\|_{L^\infty(\mathbb{R}^{N-1})} \leq \|w_2(s)^p\|_{L^\infty(\mathbb{R}^{N-1})} \\
\leq \frac{[\Phi^{-1}(C \Phi(\gamma_4 \rho(s^{\frac{4}{N}})))]^p}{C \Phi(\gamma_4 \rho(s^{\frac{4}{N}}))}
\]
for \(s \in (0, 1)\). Furthermore, we have
\[
\Phi(\gamma_4 \rho(s^{\frac{4}{N}})) = \gamma_4 \rho(s^{\frac{4}{N}}) [\log(h + \gamma_4 \rho(s^{\frac{4}{N}}))]^\beta \\
\geq C^{-1} \gamma_4 s^{-\frac{N}{2}} \left[ \log \left( e + \frac{1}{s} \right) \right]^{-N+\beta},\]
for \(s > 0\). Since \(\Phi^{-1}(\tau) \leq C \tau [\log(e + \tau)]^{-\beta}\) for \(\tau > 0\), it follows that
\[
\Phi^{-1}(C \Phi(\gamma_4 \rho(s^{\frac{4}{N}}))) \leq C \gamma_4 s^{-\frac{N}{2}} \left[ \log \left( e + \frac{1}{s} \right) \right]^{-N},\]
for \(s \in (0, 1)\). By (5.15), (5.16) and (5.17) we obtain
\[
\|w_2(s)^p\|_{L^\infty(\mathbb{R}^{N})} \leq C \gamma_4 s^{-\frac{N}{2}} \left[ \log \left( e + \frac{1}{s} \right) \right]^{-1-\beta}
\]
for \(s \in (0, 1)\). Similarly, by (5.14) and property (b) we have
\[
\|\Phi(W_2(t))\|_{L^\infty(D)} \leq \frac{C \Phi(\gamma_4 \rho(t^{\frac{4}{N}}))}{\Phi^{-1}(C \Phi(\gamma_4 \rho(t^{\frac{4}{N}})))} \\
\leq C \left[ \log \left( e + \frac{1}{t} \right) \right]^{\beta}
\]
for \(t \in (0, 1)\). By (5.18) and (5.19) we apply Lemma 5.1 with \(\Psi(\tau) = \Phi(\tau)\) to obtain
\[
\int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t-s) w_2(y', s)^p \, dy' \, ds \\
\leq C \gamma_4 s^{-\frac{N}{2}} \left[ \log \left( e + \frac{1}{t} \right) \right]^{\beta} W_2(x, t) \int_0^t (t-s)^{-\frac{N}{2}} \left[ \log \left( e + \frac{1}{s} \right) \right]^{-1-\beta} \, ds
\]
for \((x, t) \in D \times (0, 1)\). Therefore, by (5.12), (5.13) and (5.20) we have
\[
\int_D G(x, y, t) \mu(y) \, dy + \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t-s) \pi(y', 0, s)^p \, dy' \, ds \\
\leq |S(t)\mu_1|(x) + C \gamma_4^{-1} W_1(x, t) + W_2(x, t) + C (cd)^p \gamma_4^{-1} W_2(x, t) \\
\leq [1 + C \gamma_4^{-1}] |S(t)\mu_1|(x) + c' [1 + C (cd)^p \gamma_4^{-1}] \Phi_{\beta}^{-1} ([S(t)\Phi(\mu_2)](x))
\]
in $D \times (0, 1)$ for some $\epsilon' > 0$. Setting $d = 2\epsilon'$ and taking a sufficiently small $\gamma_4 > 0$ if necessary, we obtain

$$
\int_{\mathbb{R}^N_+} G(x, y, t) \mu(y) \, dy + \int_0^t \int_{\mathbb{R}^{N-1}} G(x, y', 0, t - s) \Pi(y', 0, s)^p \, dy' \, ds \leq \Pi(x, t)
$$

in $D \times (0, 1)$. Therefore $\Pi$ is a supersolution of (1.1) and (1.2) in $[0, 1)$, and by Lemma 2.3 we can find a solution of (1.1) and (1.2) in $[0, 1)$ such that $0 \leq u(x, t) \leq \overline{u}(x, t)$ in $D \times (0, 1)$. Thus Theorem 1.5 follows.

\[\blacksquare\]

6 Life span of solutions

Since the minimal solution is unique, we can define the maximal existence time $T(\mu)$ of the minimal solution $u$ of (1.1) and (1.2). We call $T(\mu)$ the life span of the solution $u$.

6.1 Life span for large initial data

Let $\kappa > 0$ and $\varphi$ be a nonnegative measurable function in $D$. In this subsection we study the behavior of $T(\kappa\varphi)$ as $\kappa \to \infty$.

Firstly, by Theorems 1.1, 1.3 and 1.4 we easily obtain the following result (compare with [11, Theorem 5.1] and [12, Corollary 1.2]).

**Theorem 6.1** Let $p > 1$ and $\varphi$ be a nonnegative continuous function in $D$ such that

$$
0 < \|\varphi\|_{L^\infty(\mathbb{R}^{N-1} \times [0, \delta])} < \infty, \quad \int_D e^{-\Lambda y_2^2} \varphi(y) \, dy < \infty,
$$

for some $\delta > 0$ and $\Lambda > 0$. Then there exists $\gamma > 0$ such that

$$
\gamma^{-1} \kappa^{-2(p-1)} \leq T(\kappa\varphi) \leq \gamma \kappa^{-2(p-1)}
$$

for all sufficiently large $\kappa > 0$.

Next we consider the case of $\text{dist} (\text{supp } \varphi, \partial D) > 0$.

**Theorem 6.2** Let $p > 1$ and $\varphi$ be a nonnegative measurable function in $D$ such that

$$
\int_D e^{-\Lambda y_2^2} \varphi(y) \, dy < \infty \quad (6.1)
$$

for some $\Lambda \geq 0$. Assume that $L := \text{dist} (\text{supp } \varphi, \partial D) > 0$. Then

$$
\lim_{\kappa \to \infty} (\log \kappa) T(\kappa\varphi) = \frac{L^2}{4}. \quad (6.2)
$$

**Proof.** We write $T_\kappa = T(\kappa\varphi)$ for simplicity. For any $\epsilon > 0$, we can find $z = (z', z_N) \in D$ such that

$$
\text{dist} (z, \partial D) \leq L + \epsilon, \quad \Pi(z) := \lim_{r \to 0} \int_{B(z, r)} \varphi(y) \, dy > 0. \quad (6.3)
$$
Then, by Theorem 1.1, for any \( \delta_1 > 0 \), we can find \( \gamma_1 > 0 \) such that

\[
\exp \left( -\left(1 + \delta_1 \right) \frac{z^2}{4T_\kappa} \right) \int_{B_+ (z, T_\kappa^\frac{1}{2})} \kappa \varphi(y) \, dy \leq \gamma_1 T_\kappa^{-\frac{1}{2(p-1)}}. \tag{6.4}
\]

This implies that \( T_\kappa \to 0 \) as \( \kappa \to \infty \). Furthermore, by (6.3) and (6.4) we have

\[
\frac{\kappa}{2} \mathcal{Z}(z) \leq \gamma_1 T_\kappa^{-\frac{1}{2(p-1)}} \exp \left( \left(1 + \delta_1 \right) \frac{z^2}{4T_\kappa} \right) \leq \exp \left( \left(1 + 2\delta_1 \right) \frac{(L + \epsilon)^2}{4T_\kappa} \right)
\]

for all sufficiently large \( \kappa \). Then we obtain

\[
(1 - \delta_1) \log \kappa \leq \frac{1 + 2\delta_1}{1 - \delta_1} \frac{(L + \epsilon)^2}{4T_\kappa}
\]

for all sufficiently large \( \kappa \), which implies that

\[
\limsup_{\kappa \to \infty} (\log \kappa) T_\kappa \leq \frac{1 + 2\delta_1}{1 - \delta_1} \frac{(L + \epsilon)^2}{4T_\kappa}.
\]

Letting \( \delta_1 \to 0 \) and \( \epsilon \to 0 \), we deduce that

\[
\limsup_{\kappa \to \infty} (\log \kappa) T_\kappa \leq \frac{L^2}{4}. \tag{6.5}
\]

Let \( 0 < d < L \). Let \( \delta_2 \in (0, 1) \) be such that

\[
L > \frac{d}{(1 - 2\delta_2)^2}. \tag{6.6}
\]

Set \( \tilde{T}_\kappa := \frac{d^2}{4 \log \kappa} \) and \( \lambda := (1 - \delta_2)/4 \tilde{T}_\kappa \). Let \( x \in D \) be such that \( B(x, \tilde{T}_\kappa^{1/2}) \cap \text{supp} \varphi \neq \emptyset \). Then, by (6.1) we have

\[
\int_{B(x, \tilde{T}_\kappa^{1/2})} e^{-\lambda y_N^2} \varphi(y) \, dy = \int_{B(x, \tilde{T}_\kappa^{1/2})} e^{-\left(\lambda - \Lambda\right) y_N^2} e^{-\lambda y_N^2} \varphi(y) \, dy
\leq C \sup_{y \in B(x, \tilde{T}_\kappa^{1/2})} \exp \left( -\frac{1 - 2\delta_2}{4\tilde{T}_\kappa} y_N^2 \right) \leq C \exp \left( -\frac{(1 - 2\delta_2)^2L^2}{d^2} \log \kappa \right)
\leq C \exp \left( -\frac{(1 - 2\delta_2)^2L^2}{d^2} \log \kappa \right) \leq C \kappa^{-\frac{(1 - 2\delta_2)^2L^2}{d^2}}
\]

for all sufficiently large \( \kappa \). This together with (6.6) implies that

\[
\sup_{x \in D} \int_{B(x, T_\kappa^{1/2})} e^{-\lambda y_N^2} \kappa \varphi(y) \, dy \leq C \kappa^{-\frac{1}{1 - (2\delta_2)^2}} T_\kappa^{-\frac{1}{2(p-1)}} = o \left( T_\kappa^{-\frac{1}{p-1}} \right)
\]

as \( \kappa \to \infty \). Therefore, by Theorem 1.4 we see that \( T_\kappa \geq \tilde{T}_\kappa \) for sufficiently large \( \kappa \), and we obtain

\[
\liminf_{\kappa \to \infty} (\log \kappa) T_\kappa \geq \frac{d^2}{4}
\]

for all sufficiently large \( \kappa \). Then we obtain

\[
\liminf_{\kappa \to \infty} (\log \kappa) T_\kappa \geq \frac{d^2}{4}
\]

for all sufficiently large \( \kappa \). This together with (6.6) implies that

\[
\liminf_{\kappa \to \infty} (\log \kappa) T_\kappa \geq \frac{d^2}{4}
\]
Letting \( d \to L \), we obtain
\[
\liminf_{\kappa \to \infty} (\log \kappa) T_\kappa \geq \frac{L^2}{4}.
\]
This together with (6.5) implies (6.2). Thus Theorem (6.2) follows. □

Similarly, we have:

**Theorem 6.3** Let \( p > 1 \), \( z = (z', z_N) \in \mathbb{R}^N_+ \) and \( \delta_z(y) := \delta(y - z) \). Then
\[
\lim_{\kappa \to \infty} (\log \kappa) T(\kappa \delta_z) = \frac{z^2_N}{4}.
\]

In the following two theorems, we study the relationship between the behavior of the life span \( T(\kappa \varphi) \) for sufficiently large \( \kappa \) and the singularity of \( \varphi \) at \( 0 \in \partial D \). Compare with [11, Theorem 5.2].

**Theorem 6.4** Let \( \varphi \) be a nonnegative measurable function in \( D \) such that
\[
\varphi(y) \geq \gamma |y|^A \left( \log \left( e + \frac{1}{|y|} \right) \right)^{-B}, \quad y \in B_+(0, 1), \tag{6.8}
\]
for some \( \gamma > 0 \), where \( A > -N, B \in \mathbb{R} \) or \( A = -N, B > 1 \).

(i) Let \( 1 < p < p_* \). Then
\[
\limsup_{\kappa \to \infty} \frac{T(\kappa \varphi)}{[\kappa (\log \kappa)^{-B}]^{-2(p-1)/N(p-1)+1}} < \infty \quad \text{if} \quad A > -N, \ B \in \mathbb{R}, \tag{6.9}
\]
\[
\limsup_{\kappa \to \infty} \frac{T(\kappa \varphi)}{[\kappa (\log \kappa)^{-B+1}]^{-2(p-1)/N(p-1)+1}} < \infty \quad \text{if} \quad A = -N, B > 1. \tag{6.10}
\]

(ii) Let \( p > p_* \). If either \( A < -1/(p-1) \) and \( B \in \mathbb{R} \) or \( A = -1/(p-1) \) and \( B < 0 \), then \( T(\kappa \varphi) = 0 \) for all \( \kappa > 0 \). If \( A = -1/(p-1) \) and \( B = 0 \), then \( T(\kappa \varphi) = 0 \) for all sufficiently large \( \kappa > 0 \). Furthermore,
- if \( A > -1/(p-1) \), then (6.9) holds;
- if \( A = -1/(p-1) \) and \( B > 0 \), then \( \liminf_{\kappa \to \infty} \kappa^{-\frac{1}{N}} |\log T(\kappa \varphi)| > 0 \).

(iii) Let \( p = p_* \). If \( A = -N \) and \( B < N + 1 \), then \( T(\kappa \varphi) = 0 \) for all \( \kappa > 0 \). If \( A = -N \) and \( B = N + 1 \), then \( T(\kappa \varphi) = 0 \) for all sufficiently large \( \kappa > 0 \). Furthermore,
- if \( A > -N \), then (6.9) holds;
- if \( A = -N \) and \( B > N + 1 \), then \( \liminf_{\kappa \to \infty} \kappa^{-\frac{1}{B+N+1}} |\log T(\kappa \varphi)| > 0 \).

**Proof.** We write \( T_\kappa := T(\kappa \varphi) \) for simplicity. We prove assertion (i). Let \( 1 < p < p_* \), \( A > -N \) and \( B \in \mathbb{R} \). For any \( \epsilon \in (0, A+N) \), since
\[
r^{-\epsilon} \left[ \log \left( e + \frac{1}{r} \right) \right]^{-B} \quad \text{is monotone decreasing near} \ r = 0,
\]
we have
\[
\int_{B(0,\sigma)} |y|^A \left[ \log \left( e + \frac{1}{|y|} \right) \right]^{-B} dy \geq C \sigma^{-\epsilon} \left[ \log \left( e + \frac{1}{\sigma} \right) \right]^{-B} \int_0^\sigma r^{A+\epsilon+N-1} dr \\
\geq C \sigma^{A+N} \left[ \log \left( e + \frac{1}{\sigma} \right) \right]^{-B}
\]
for all sufficiently small \( \sigma > 0 \). Then it follows from Theorem 1.1 that \( T_\kappa \to 0 \) as \( \kappa \to \infty \) and
\[
C \gamma \kappa T_\kappa^{\frac{A+1}{2}} \left[ \log \left( e + T_\kappa^{\frac{1}{2}} \right) \right]^{-B} \leq \gamma T_\kappa^{\frac{A}{2(p-1)}}
\]
that is
\[
T_\kappa^{\frac{A}{2(p-1)}} \left[ \log \left( e + T_\kappa^{\frac{1}{2}} \right) \right]^{-B} \leq C \kappa^{-1} \quad (6.11)
\]
for all sufficiently large \( \kappa > 0 \). Let \( \tilde{T}_\kappa \) be such that
\[
T_\kappa^{\frac{A}{2(p-1)}} \left[ \log \left( e + \tilde{T}_\kappa^{-\frac{1}{2}} \right) \right]^{-B} = C \kappa^{-1}
\]
Since \( A > -N \) and \( 1 < p < p_* \),
\[
h(s) := s^{\frac{A(p-1)+1}{2(p-1)}} \left[ \log \left( e + s^{-\frac{1}{2}} \right) \right]^{-B}
\]
is monotone increasing for all sufficiently small \( s > 0 \). Then, by (6.11) we have
\[
T_\kappa \leq \tilde{T}_\kappa \leq C [\kappa^{-1} (\log \kappa)^B]^\frac{2(p-1)}{A(p-1)+1} \quad (6.12)
\]
for all sufficiently large \( \kappa > 0 \). This implies (6.9). In the case where \( A = -N \) and \( B > 1 \), then
\[
\int_{B(0,\sigma)} |y|^A \left[ \log \left( e + \frac{1}{|y|} \right) \right]^{-B} dy \geq C \int_0^\sigma r^{-1} \left[ \log \left( e + \frac{1}{r} \right) \right]^{-B+1} dr \\
\geq C \left[ \log \left( e + \frac{1}{\sigma} \right) \right]^{-B+1}
\]
for all sufficiently small \( \sigma > 0 \). Then, similarly to (6.11), we obtain
\[
T_\kappa^{\frac{A(p-1)+1}{2(p-1)}} \left[ \log \left( e + T_\kappa^{-\frac{1}{2}} \right) \right]^{-B+1} \leq C \kappa^{-1}
\]
for all sufficiently large \( \kappa > 0 \). Therefore, by a similar argument as in (6.12) we obtain (6.10). Thus assertion (i) follows.

By similar arguments as in assertion (i) we apply Theorem 1.1 to obtain assertions (ii) and (iii). (We leave the details of the proof to the reader.) Then Theorem 6.4 follows. \( \square \)
Theorem 6.5 Let \( \varphi \) be a nonnegative measurable function in \( D \) such that \( \text{supp} \, \varphi \subset B(0,1) \) and
\[
\varphi(y) \leq \gamma |y|^A \left[ \log \left( e + \frac{1}{|y|} \right) \right]^{-B}, \quad y \in B(0,1),
\]
for some \( \gamma > 0 \).
(i) Let \( 1 < p < p_* \). Then
\[
\liminf_{\kappa \to \infty} \frac{T(\kappa \varphi)}{[\kappa(\log \kappa)^{-B}]^{\frac{2(p-1)}{4(p-1)+1}}} > 0 \quad \text{if} \quad A > -N, \quad B \in \mathbb{R},
\]
\[
\liminf_{\kappa \to \infty} \frac{T(\kappa \varphi)}{[\kappa(\log \kappa)^{-B+1}]^{\frac{2(p-1)}{4(p-1)+1}}} > 0 \quad \text{if} \quad A = -N, \quad B > 1.
\]
(ii) Let \( p > p_* \).
\quad \bullet \quad \text{If} \ A > -1/(p-1), \text{then} \ (6.14) \ \text{holds:}
\quad \bullet \quad \text{If} \ A = -1/(p-1) \text{ and} \ B > 0, \text{then} \ \limsup_{\kappa \to \infty} \kappa^{-\frac{1}{p}} |\log T(\kappa \varphi)| < \infty.
(iii) Let \( p = p_* \).
\quad \bullet \quad \text{If} \ A > -N, \text{then} \ (6.14) \ \text{holds:}
\quad \bullet \quad \text{If} \ A = -N \text{ and} \ B > N + 1, \text{then} \ \limsup_{\kappa \to \infty} \kappa^{-\frac{1}{p}+1} |\log T(\kappa \varphi)| < \infty.

Proof. In the case where \( A > -N \) and \( B \in \mathbb{R} \), for any \( \epsilon \in (0, A + N) \),
\[
\sigma^\epsilon \left[ \log \left( e + \frac{1}{r} \right) \right]^{-B} \quad \text{is monotone increasing near} \quad r = 0
\]
and we have
\[
\int_{B(0,\sigma)} |y|^A \left[ \log \left( e + \frac{1}{|y|} \right) \right]^{-B} dy \leq C \sigma^\epsilon \int_0^\sigma \int_0^{sr^{-\epsilon}} r^{A-\epsilon+N-1} dr d\sigma
\]
\[
\leq C \sigma^{A+N} \int_0^\sigma \left[ \log \left( e + \frac{1}{\sigma} \right) \right]^{-B} dr
\]
for all sufficiently small \( \sigma > 0 \). In the case where \( A = -N \) and \( B > 1 \), we obtain
\[
\int_{B(0,\sigma)} |y|^A \left[ \log \left( e + \frac{1}{|y|} \right) \right]^{-B} dy \leq C \int_0^\sigma \int_0^{sr^{-1}} \left[ \log \left( e + \frac{1}{r} \right) \right]^{-B} dr d\sigma
\]
\[
\leq C \left[ \log \left( e + \frac{1}{\sigma} \right) \right]^{-B+1}
\]
for all sufficiently small \( \sigma > 0 \). Then Theorem 6.5 follows from Theorems 1.3, 1.4 and Theorem 1.5. We leave the details of the proof to the reader. (See also the proof of Theorem 6.4.) □
6.2 Life span for small initial data

Motivated by [13], we state two theorems on the behavior of \( T(\kappa \varphi) \) as \( \kappa \to 0 \). Theorems 6.6 and 6.7 follow from Theorem 1.1 and Theorems 1.3–1.5.

**Theorem 6.6** Let \( N \geq 1 \) and \( p > 1 \). Let \( A > 0 \) and \( \varphi \) be a nonnegative measurable function in \( D \) such that \( 0 \leq \varphi(x) \leq (1 + |x|)^{-A} \) for \( x \in D \).

(i) Let \( p = p_* \) and \( A \geq 1/(p - 1) = N \). Then there exists \( \gamma > 0 \) such that

\[
\log T(\kappa \varphi) \geq \begin{cases} 
\gamma \kappa^{-(p-1)} & \text{if } A > N, \\
\gamma \kappa^{-\frac{1}{p}} & \text{if } A = N,
\end{cases}
\]

for all sufficiently small \( \kappa > 0 \).

(ii) Let \( 1 < p < p_* \) or \( A < 1/(p - 1) \). Then there exists \( \gamma' > 0 \) such that

\[
T(\kappa \varphi) \geq \begin{cases} 
\gamma' \kappa^{-(p-1)} \left( \frac{1}{2(p-1)} \right)^{-\frac{1}{2} \min\{A,N\}} & \text{if } A \neq N, \\
\gamma' \left( \frac{1}{\log(\kappa^{-1})} \right)^{\left( \frac{1}{2(p-1)} \right)^{N}} & \text{if } A = N,
\end{cases}
\]

for all sufficiently small \( \kappa > 0 \).

**Theorem 6.7** Let \( N \geq 1 \) and \( p > 1 \). Let \( A > 0 \) and \( \varphi \) be a nonnegative \( L^\infty(D) \)-function such that \( \varphi(x) \geq (1 + |x|)^{-A} \) for \( x \in D \).

(i) Let \( p = p_* \) and \( A \geq 1/(p - 1) = N \). Then there exists \( \gamma > 0 \) such that

\[
\log T(\kappa \varphi) \leq \begin{cases} 
\gamma \kappa^{-(p-1)} & \text{if } A > N, \\
\gamma \kappa^{-\frac{1}{p}} & \text{if } A = N,
\end{cases}
\]

(6.15)

for all sufficiently small \( \kappa > 0 \).

(ii) Let \( 1 < p < p_* \) or \( A < 1/(p - 1) \). Then there exists \( \gamma' > 0 \) such that

\[
T(\kappa \varphi) \leq \begin{cases} 
\gamma' \kappa^{-(p-1)} \left( \frac{1}{2(p-1)} \right)^{-\frac{1}{2} \min\{A,N\}} & \text{if } A \neq N, \\
\gamma' \left( \frac{1}{\log(\kappa^{-1})} \right)^{\left( \frac{1}{2(p-1)} \right)^{N}} & \text{if } A = N,
\end{cases}
\]

(6.16)

for all sufficiently small \( \kappa > 0 \).

Theorems 6.6 and 6.7 are proved by similar arguments as in [8, Section 5], and we leave the details of the proofs to the reader.

Finally, we show that \( \lim_{\kappa \to 0} T(\kappa \varphi) = \infty \) does not necessarily hold for problem (1.1).

**Theorem 6.8** Let \( p > 1 \), \( \lambda > 0 \) and \( \varphi(x) := \exp(\lambda x_N^2) \). Then

\[
\lim_{\kappa \to +0} T(\kappa \varphi) = (4\lambda)^{-1}.
\]

(6.17)
Proof. Let $\kappa > 0$ and $\delta > 0$. Set $T_\kappa := T(\kappa \varphi)$. It follows from Theorem 1.1 that

$$\infty > \exp \left( -(1 + \delta) \frac{x^2}{4T_\kappa} \right) \int_{B(x,T_\kappa^2)} \kappa \varphi(y) \, dy$$

$$\geq C \exp \left( -(1 + \delta) \frac{x^2}{4T_\kappa} \right) \kappa T_\kappa^N \exp \left( \lambda(x_N - T_\kappa^\frac{1}{2})^2 \right)$$

$$\geq C \kappa T_\kappa^N \exp \left\{ \left( \lambda - \frac{1 + \delta}{4T_\kappa} \right) x^2_N \right\} \exp \left( -2\lambda T_\kappa^\frac{3}{2} x_N + \lambda T_\kappa \right)$$

for all $x \in D_{T_\kappa}$. This implies that $\lambda - (1 + \delta)/4T_\kappa \leq 0$. Since $\delta > 0$ is arbitrary, we obtain

$$\limsup_{\kappa \to +0} T_\kappa \leq (4\lambda)^{-1}. \quad (6.18)$$

On the other hand, it follows that

$$\int_{B(x,T_\delta^2)} \exp \left( -(1 - \delta) \frac{y^2}{4T_\delta} \right) \kappa \exp (\lambda y^2) \, dy = \kappa, \quad x \in D_{T_\delta},$$

where $T_\delta := (1 - \delta)/4\lambda$. Then we deduce from Theorem 1.4 that $T_\kappa \geq T_\delta$ for all sufficiently small $\kappa > 0$. Since $\delta > 0$ is arbitrary, we obtain $\liminf_{\kappa \to +0} T_\kappa \geq (4\lambda)^{-1}$. This together with (6.18) implies (6.17). Thus Theorem 6.8 follows. $\square$

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