Hyperelliptic values of the Gamma function

Jan Lügering

Abstract
We show how to calculate particular values of the Gamma function for specific rational arguments in the interval (0,1) without using the Elliptic K-function. By introducing hyperelliptic generalisations of the trigonometric functions, we obtain a halving formula. We show that the constants used in this formula are transcendental in specific rational cases. We generalise Euler’s arcsine proof and state a useful formula by Königsberger. Finally, we apply the Gamma values to various equations like the area of superellipses.

Keywords: Gamma function, hyperelliptic integral, elliptic integral

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1 Introduction
The Gamma function can be defined for $\text{Re}(x) > 0$ as

$$\Gamma(x) := \int_0^\infty t^{x-1} \cdot e^{-t} \, dt$$

(1)

Recall the Legendre duplication formula for $x > 0$:

$$\Gamma(x) \cdot \Gamma(x + \frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2x-1}} \cdot \Gamma(2x)$$

(2)

and Gauss multiplication formula for $x \in \mathbb{C} \setminus \mathbb{N}_0$:

$$\forall n \in \mathbb{N} : \Gamma(x) \cdot \Gamma(x + \frac{1}{n}) \cdot \ldots \cdot \Gamma(x + \frac{n - 1}{n}) = \frac{(2\pi)^{n-1}}{n^{nx-\frac{1}{2}}} \cdot \Gamma(nx)$$

(3)
The Euler Beta function can be defined for $\text{Re}(w) > 0$ and $\text{Re}(z) > 0$ as:

$$B(w, z) := \int_0^1 t^{w-1} \cdot (1 - t)^{z-1} \, dt \quad (4)$$

The central identity in the theory of the Beta function is [5]:

$$\forall \text{Re}(w), \text{Re}(z) > 0 : B(w, z) = \frac{\Gamma(w) \cdot \Gamma(z)}{\Gamma(z + w)} \quad (5)$$

We also have the Euler relation:

$$\forall x \in \mathbb{C} \setminus \mathbb{Z} : \Gamma(x) \cdot \Gamma(1 - x) = \frac{\pi}{\sin(\pi x)} \quad (6)$$

The lemniscatic arcsine function is defined as [7]:

$$\text{arcsl}(x) := \int_0^x \frac{1}{\sqrt{1 - r^4}} \, dr \quad (7)$$

with the lemniscate constant

$$\varpi := 2 \cdot \int_0^1 \frac{1}{\sqrt{1 - r^4}} \, dr \quad (8)$$

in analogy to

$$\text{arcsin}(x) := \int_0^x \frac{1}{\sqrt{1 - r^2}} \, dr \quad (9)$$

and

$$\pi := 2 \cdot \int_0^1 \frac{1}{\sqrt{1 - r^2}} \, dr \quad (10)$$

We have the fascinating integral identity [2]:

$$A \cdot B := \left( \int_0^1 \frac{1}{\sqrt{1 - r^4}} \, dr \right) \cdot \left( \int_0^1 \frac{r^2}{\sqrt{1 - r^4}} \, dr \right) = \frac{\pi}{4} \quad (11)$$

Our goal is to generalize these functions and identities and apply this knowledge to calculate fractional values of the Gamma function.
2 Particular values of the Gamma function

Definition 1 We define for $s \geq 1$:
\[
\pi_s := 2 \cdot \int_0^1 \frac{1}{\sqrt{1 - x^s}} \, dx \quad (12)
\]

Definition 2 The generalised arcsine is defined for $n \in \mathbb{N}$ as:
\[
\text{arcs}_n(x) := \int_0^x \frac{1}{\sqrt{1 - y^n}} \, dy, \quad x \in V_n := \begin{cases}
[-1, 1] & \text{for } n \text{ even} \\
(-\infty, 1) & \text{for } n \text{ odd}
\end{cases} 
\quad (13)
\]

Proposition 3
\[
\forall |x| < 1 : \text{arcs}_n(x) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{x^{nk+1}}{4^k \cdot (nk + 1)}
\quad (14)
\]
and in particular we have
\[
\frac{\pi_n}{2} = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{4^k \cdot (nk + 1)}
\quad (15)
\]

Proof Let $|x| \leq r < 1$ be arbitrary.
\[
\text{arcs}_n(x) = \int_0^x \sum_{k=0}^{\infty} \binom{-1/2}{k} \cdot (-1)^k \cdot y^{nk} \, dy = \int_0^x \sum_{k=0}^{\infty} \binom{2k}{k} \cdot \frac{y^{nk}}{4^k} \, dy
\]
Since $\sum_{k=0}^{\infty} \binom{2k}{k} \cdot \frac{y^{nk}}{4^k}$ converges uniformly for $|y| \leq r$ we obtain:
\[
= \sum_{k=0}^{\infty} \binom{2k}{k} \frac{x^{nk+1}}{4^k \cdot (nk + 1)}, \forall |x| \leq r
\]
For every fixed $|x| < 1$ and in particular for $x = 1$, the series converges by comparison with the arcsine series. Abel’s limit theorem yields the result. □

Our next goal is to find a halving formula by proving an auxiliary formula.

Proposition 4
\[
\forall n \in \mathbb{N} : \frac{\Gamma\left(\frac{1}{n}\right)^2 \cdot 2^{\frac{2n}{n}-1}}{\Gamma\left(\frac{2}{n}\right) \cdot n} = \frac{\pi_n}{2}
\quad (16)
\]

Proof Let $m, n \in \mathbb{N}$.
On the one hand we have:
\[
B\left(\frac{1}{2}, \frac{m}{n}\right) = \frac{\sqrt{\pi} \cdot \Gamma\left(\frac{m}{n}\right)}{\Gamma\left(\frac{m}{n} + \frac{1}{2}\right)}
\]
On the other hand:

\[ B\left(\frac{1}{2}, \frac{m}{n}\right) = B\left(\frac{m}{n}, \frac{1}{2}\right) = \int_0^1 t^{\frac{m}{n} - 1} \cdot (1 - t)^{-\frac{1}{2}} \, dt \]

Substituting \( t := u^n \) yields:

\[ = n \cdot \int_0^1 \frac{u^{m-1}}{\sqrt{1-u^n}} \, du \]

Therefore

\[ \frac{\sqrt{\pi} \cdot \Gamma\left(\frac{m}{n}\right)}{\Gamma\left(\frac{m}{n} + \frac{1}{2}\right)} = n \cdot \int_0^1 \frac{u^{m-1}}{\sqrt{1-u^n}} \, du = \frac{n \cdot \pi_n}{2} \]

For \( m = 1 \) we obtain:

\[ \frac{\sqrt{\pi} \cdot \Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right) \cdot n} = \int_0^1 \frac{1}{\sqrt{1-u^n}} \, du = \frac{n \cdot \pi_n}{2} \]

\[ \implies \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)} = \frac{n \cdot \pi_n}{\Gamma\left(\frac{1}{n}\right) \cdot 2} \]

Rearranging the Legendre duplication formula for \( x > 0 \) yields:

\[ \frac{\sqrt{\pi}}{\Gamma\left(x + \frac{1}{2}\right)} = \frac{\Gamma(x) \cdot 2^{2x-1}}{\Gamma(2x)} \]

Setting \( x = \frac{1}{n} \):

\[ \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right)} = \frac{\Gamma\left(\frac{1}{n}\right) \cdot 2^{\frac{2}{n}-1}}{\Gamma\left(\frac{2}{n}\right)} \]

Comparing both equations implies:

\[ \frac{n \cdot \pi_n}{\Gamma\left(\frac{1}{n}\right) \cdot 2} = \frac{\Gamma\left(\frac{1}{n}\right) \cdot 2^{\frac{2}{n}-1}}{\Gamma\left(\frac{2}{n}\right)} \]

\[ \square \]

**Theorem 5** For \( s \geq 1 \) we have:

\[ \Gamma\left(\frac{1}{2s}\right) = \sqrt{\frac{\pi_{2s} \cdot \Gamma\left(\frac{1}{2}\right) \cdot 2^s}{2^{\frac{s}{2}}}} \]  \hspace{1cm} (17)

**Proof** Let \( l \in \mathbb{N} \).

In Proposition 4 set \( n = 2l \) and rearrange. Observe that its proof also holds for \( n \in \mathbb{R}_{\geq 1} \). \[ \square \]

**Corollary 6**

\[ \forall n \in \mathbb{N} : \Gamma\left(\frac{1}{2n}\right) = \prod_{k=1}^{n-1} (2^{n-1} \cdot \pi_{2k}) \cdot \frac{1}{2^{n-k+1}} \]  \hspace{1cm} (18)
Proof Let \( n \in \mathbb{N} \).

Initial case) For \( n = 1 \): \( \Gamma(\frac{1}{2}) = \sqrt{\pi} = \sqrt{\pi} \)

Induction hypothesis) Suppose that our claim holds for arbitrary, but once chosen, fixed \( n \in \mathbb{N} \).

Induction step) For \( n \mapsto n + 1 \):

\[
\Gamma(\frac{1}{2n+1}) = \sqrt{\pi_{2n+1} \cdot \Gamma(\frac{1}{2n}) \cdot 2^n \cdot 2^{2n-1}}
\]

\[
= \sqrt{\pi_{2n+1} \cdot 2^{n+1-1}} \cdot \sqrt{\prod_{k=1}^{n}(2^{n-1} \cdot \pi_{2k})^{\frac{1}{2(n-k+1)}}}
\]

The 2 at the left now transfers the powers \( \sum_{k=1}^{n} (\frac{1}{2})^k = 1 - \frac{1}{2^n} \) into the product.

\[
= \sqrt{\pi_{2n+1} \cdot 2^n} \cdot \sqrt{\prod_{k=1}^{n}(2^{n} \cdot \pi_{2k})^{\frac{1}{2(n-k+1)}}} = \prod_{k=1}^{n+1}(2^{n} \cdot \pi_{2k})^{\frac{1}{2(n-k+2)}}
\]

By the principle of induction, our claim holds for all \( n \in \mathbb{N} \). \( \square \)

Example 7

\[
\Gamma(\frac{1}{2}) = \sqrt{\pi} = \sqrt{\pi}
\]

(19)

\[
\Gamma(\frac{1}{4}) = \sqrt{2\pi} \sqrt{\frac{1}{2}} = \sqrt{2\pi} \sqrt{2\pi}
\]

(20)

The Euler relation yields:

\[
\Gamma(\frac{3}{4}) = \frac{\pi}{\sqrt{2\pi}}
\]

(21)

The pattern continues:

\[
\Gamma(\frac{1}{8}) = \sqrt{4\pi} \sqrt{4\pi} \sqrt{4\pi} \sqrt{4\pi}
\]

(22)

In general we obtain for the "quarter-length" of the "unit curve" \( q_n := \frac{\pi}{2^n} \):

\[
\Gamma(\frac{1}{2n}) = \sqrt{2^n \cdot q_{2^n} \cdot \sqrt{2^n \cdot q_{2^{(n-1)}} \cdot \sqrt{\ldots \sqrt{2^n \cdot q_2}}}}
\]

(23)

We will now try to get a more general division formula, again by proving an auxiliary formula. But first, we need to introduce auxiliary constants.

Definition 8 For \( l, n, k \in \mathbb{N} \) and \( l \geq 2 \) and \( k < l \) we define

\[
\pi_{l,n,k} := 2 \cdot \int_{0}^{1} \frac{1}{\sqrt{(1-t^n)^k}} dt
\]

(24)
Proposition 9

\[
\frac{l^{\frac{1}{n}} - \frac{1}{l} \cdot \Gamma\left(\frac{1}{n}\right)^2 \cdot \Gamma\left(\frac{k}{l}\right) \cdot \prod_{j=1,j\neq k}^{l-1} \Gamma\left(\frac{1}{n} + \frac{j}{l}\right)}{(2\pi)^{\frac{l-1}{2}} \cdot \Gamma\left(\frac{l}{n}\right) \cdot n} = \frac{\pi_{l,n,l-k}}{2}
\] (25)

For \(n, k, l \in \mathbb{N}\) and \(l \geq 2\), where \(1 \leq k \leq l-1\) fixed.

Proof Let \(n, k, l \in \mathbb{N}\), \(l \geq 2\), \(1 \leq k \leq l-1\) fixed.

On the one hand we have:

\[
B\left(\frac{1}{n}, \frac{k}{l}\right) = \frac{\Gamma\left(\frac{1}{n}\right) \cdot \Gamma\left(\frac{k}{l}\right)}{\Gamma\left(\frac{1}{n} + \frac{k}{l}\right)}
\]

On the other hand, substituting \(u^n := t\):

\[
B\left(\frac{1}{n}, \frac{k}{l}\right) = \int_0^1 t^{\left(\frac{1}{n} - 1\right)} \cdot (1 - t)^{\left(\frac{k}{l} - 1\right)} \, dt
\]

\[
= n \cdot \int_0^1 \frac{1}{\sqrt{(1 - u^n)^{l-k}}} \, du = \frac{\pi_{l,n,l-k}}{2}
\]

Therefore we have:

\[
\frac{2 \cdot \Gamma\left(\frac{1}{n}\right) \cdot \Gamma\left(\frac{k}{l}\right)}{n \cdot \pi_{l,n,l-k}} = \Gamma\left(\frac{1}{n} + \frac{k}{l}\right)
\]

Rearranging the Gaussian multiplication formula and setting \(x := \frac{1}{n}\) yields:

\[
\frac{(2\pi)^{\frac{l-1}{2}} \cdot l^{\frac{1}{n}} - \frac{1}{l} \cdot \Gamma\left(\frac{1}{n}\right)^2 \cdot \Gamma\left(\frac{k}{l}\right) \cdot \prod_{j=1,j\neq k}^{l-1} \Gamma\left(\frac{1}{n} + \frac{j}{l}\right)}{\Gamma\left(\frac{l}{n}\right) \cdot 2 \cdot \prod_{j=1,j\neq k}^{l-1} \Gamma\left(\frac{1}{n} + \frac{j}{l}\right)} = \Gamma\left(\frac{1}{n} + \frac{k}{l}\right)
\]

By comparing both equations we get the desired result. \(\square\)

Theorem 10

\[
\Gamma\left(\frac{1}{l \cdot n}\right) = \sqrt{\frac{\Gamma\left(\frac{1}{n}\right) \cdot l \cdot n \cdot (2\pi)^{\frac{l-1}{2}} \cdot l^{\frac{1}{n}} - \frac{1}{l} \cdot \pi_{l,n,l-k} \cdot \Gamma\left(\frac{k}{l}\right) \cdot \prod_{j=1,j\neq k}^{l-1} \Gamma\left(\frac{1}{n} + \frac{j}{l}\right)}{\Gamma\left(\frac{k}{l}\right) \cdot 2 \cdot \prod_{j=1,j\neq k}^{l-1} \Gamma\left(\frac{1}{n} + \frac{j}{l}\right)}}
\] (26)

For \(n, k, l \in \mathbb{N}\) and \(l \geq 2\), where \(1 \leq k \leq l-1\) is fixed.

Proof In Proposition 9 replace \(n\) by \(l \cdot n\) and rearrange. \(\square\)

A useful special case in the proof of Proposition 9 is the following...
Corollary 11
\[ \forall n \in \mathbb{N} : \frac{\Gamma\left(\frac{1}{n}\right)^2}{\Gamma\left(\frac{2}{n}\right) \cdot n} = \frac{\pi_{n,n,n-1}}{2} \] (27)

**Proof** In the proof of Proposition 9, we deduced
\[ \frac{2 \cdot \Gamma\left(\frac{1}{m}\right) \cdot \Gamma\left(\frac{k}{l}\right)}{n \cdot \pi_{l,n,l-k}} = \Gamma\left(\frac{1}{n} + \frac{k}{l}\right) \]
Now set \( l = n \) and \( k = 1 \). \( \square \)

This provides important information about dependencies between the auxiliary constants and the hyperelliptic constants. Therefore, the auxiliary constants should **not** be seen as a generalisation of the hyperelliptic constants.

Corollary 12
\[ \forall n \in \mathbb{N} : \pi_{2n} = \frac{\pi_{2n,2n,2n-1}}{2^{\left(1 - \frac{1}{n}\right)}} \] (28)

**Proof** Let \( n \in \mathbb{N} \).
Corollary 11 also yields a halving formula:
\[ \frac{\Gamma\left(\frac{1}{n}\right)^2}{\Gamma\left(\frac{2}{n}\right) \cdot n} = \pi_{2n,2n,2n-1} \]
whereas Proposition 4 yields the halving formula:
\[ \frac{\Gamma\left(\frac{1}{2n}\right)^2}{\Gamma\left(\frac{1}{n}\right) \cdot n} = \frac{\pi_{2n}}{2^{\left(\frac{1}{n} - 1\right)}} \]
By comparing both halving formulas, the result follows. \( \square \)

Remark 13
\[ \Gamma\left(\frac{1}{3}\right) = \sqrt[3]{\frac{\sqrt[3]{2}}{2}} \cdot \pi_3 \cdot \pi_2 \cdot \sqrt{2} \] (29)

**Proof** In the proof of Proposition 4 we deduced:
\[ \frac{\sqrt{\pi} \cdot \Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n} + \frac{1}{2}\right) \cdot n} = \int_0^1 \frac{1}{\sqrt{1 - u^n}} \, du \] (30)
Setting \( n = 3 \):
\[ \frac{\sqrt{\pi} \cdot \Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{6}\right) \cdot 3} = \int_0^1 \frac{1}{\sqrt{1 - u^3}} \, du \]
By Legendres duplication formula we have:
\[ \Gamma\left(\frac{5}{6}\right) = \Gamma\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{\sqrt{\pi} \cdot \Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{5}{3}\right)} \]
Applying the Euler relation, we get in total:

\[
\Gamma\left(\frac{1}{3}\right)^3 = \int_0^1 \frac{1}{\sqrt{1-u^3}} \, du = \frac{\pi_3}{2}
\]  

(31)

The integral identities 31 and 30 were remarked by Konrad Königsberger and can be found in his book [4].

In the book [3] by Freitag and Busam, we find the identity

\[
\Gamma\left(\frac{1}{6}\right) = 2^{-\frac{1}{2}} \cdot \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \cdot \Gamma\left(\frac{1}{3}\right)^2
\]  

(32)

Applying the halving formula to \(\frac{1}{3}\) we obtain

\[
\Gamma\left(\frac{1}{6}\right) = \sqrt[2]{\pi_6 \cdot 3 \cdot \sqrt{4 \cdot \sqrt{2} \cdot \sqrt{3} \cdot \pi_2 \cdot \pi_3}}
\]  

(33)

plugging in the values into the identity, we obtain

Remark 14

\[
\pi_6 = \sqrt{3} \cdot \frac{\pi_3}{2}
\]  

(34)

This is plausible, because by definition

Remark 15

\[
\lim_{n \to \infty} q_n = \lim_{n \to \infty} \frac{\pi_n}{2} = 1
\]  

(35)

3 Table of Values

Note that all the appearing constants \(\pi_n\) are transcendental and that the fractional constant \(\pi_{5/2}\) is also transcendental.

\[
\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi_2} = \sqrt{\pi}
\]  

(36)

\[
\Gamma\left(\frac{1}{3}\right) = \sqrt[3]{\sqrt{\frac{\pi_3}{2}} \cdot \pi_2 \cdot \sqrt{2}}
\]  

(37)

\[
\Gamma\left(\frac{1}{4}\right) = \sqrt{2\pi_4 \sqrt{2\pi_2}} = \sqrt{2\pi_4 \sqrt{2\pi}}
\]  

(38)

\[
\Gamma\left(\frac{1}{5}\right) = \sqrt[5]{\frac{5^3 \cdot \pi_5^2 \cdot \pi_{5/2} \cdot \pi_2}{2^{\frac{11}{8}} \cdot \sqrt{5-\sqrt{5}}}}
\]  

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\[
\Gamma\left(\frac{1}{6}\right) = \sqrt[3]{\frac{3}{2}} \cdot \pi_3 \cdot 3 \cdot \sqrt[3]{4 \cdot \sqrt{2} \cdot \sqrt[3]{3} \cdot \pi_2 \cdot \pi_3}
\]

(40)

\[
\Gamma\left(\frac{1}{8}\right) = \sqrt[4]{\frac{\pi_8}{4 \pi_4 \sqrt{4 \pi_2}}}
\]

(41)

4 Transcendence

Now what is the nature of the constants \(\pi_s\)? Are they irrational, or even transcendental? Thanks to a theorem by Theodor Schneider in [6], we immediately conclude the answer for the rational case and in particular for \(n \geq 2\).

**Schneider’s Theorem 16** For all \(a, b \in \mathbb{Q} \setminus \mathbb{Z}\) it holds, that \(B(a, b)\) is transcendental.

**Corollary 17** For all \(n \geq 2\) it holds, that \(\pi_n\) is transcendental.

**Proof** Let \(n \geq 2\) be arbitrary.

In the proof of proposition 4, we deduced:

\[
B\left(\frac{1}{2}, \frac{m}{n}\right) = n \cdot \frac{1}{\sqrt{1 - u^n}} du
\]

\[
\Rightarrow \frac{2}{n} \cdot B\left(\frac{1}{2}, \frac{1}{n}\right) = \pi_n
\]

Suppose, \(\pi_n\) would be algebraic.

\[
\Rightarrow \pi_n \cdot \frac{n}{2} = B\left(\frac{1}{2}, \frac{1}{n}\right)
\]

algebraic, which contradicts Schneider’s theorem. \(\Box\)

5 Generalising Euler’s Arcsine Proof

**Theorem 18** For \(n \in \mathbb{N}\) we have:

\[
\frac{\pi_n^2}{8 \cdot \int_0^1 \frac{t}{\sqrt{1 - t^n}} dt} = 1 + \sum_{l=0}^{\infty} \frac{(2l + 1)!!}{(2l + 2)!!} \cdot \frac{1}{n \cdot (l + 1) + 1} \cdot \prod_{k=0}^{l} \frac{2 \cdot (nk + 2)}{n + 2 \cdot (nk + 2)}
\]

(42)

**Proof** We have the Taylor series expansion

\[
\arcs_n(t) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{t^{nk+1}}{4^k \cdot (nk + 1)}
\]

(43)

In other words:
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\[ \text{arcs}_n(t) = t + \sum_{k=1}^{\infty} \frac{(2k-1) \ldots 5 \cdot 3 \cdot 1}{2k \ldots 4 \cdot 2} \cdot \left( \frac{t^{nk+1}}{(nk+1)} \right) \] (44)

Observing that:

\[ \frac{1}{2} \cdot (\text{arcs}_n(x))^2 = \int_0^x \frac{\text{arcs}_n(t)}{\sqrt{1-t^n}} \, dt \] (45)

We obtain:

\[ \frac{1}{2} \cdot (\text{arcs}_n(x))^2 = \int_0^x \frac{t}{\sqrt{1-t^n}} + \sum_{k=1}^{\infty} \frac{(2k-1) \ldots 5 \cdot 3 \cdot 1}{2k \ldots 4 \cdot 2} \cdot \left( \frac{1}{nk+1} \right) \cdot \int_0^x \frac{t^{nk+1}}{\sqrt{1-t^n}} \, dt \] (46)

Next, define

\[ I_l(x) = \int_0^x \frac{t^{l+n}}{\sqrt{1-t^n}} \, dt = \int_0^x \frac{t^{l+1}}{\sqrt{1-t^n}} \cdot \frac{t^{n-1}}{\sqrt{1-t^n}} \, dt \] (47)

Integrating by parts and multiplying the integrand with \( \sqrt{\frac{1-t^n}{1-t^l}} \):

\[ I_l(x) = -\frac{2}{n} \cdot x^{l+1} \cdot \sqrt{1-x^n} + \frac{2(l+1)}{n} \cdot \int_0^x \frac{t^l \cdot (1-t^n)}{\sqrt{1-t^n}} \, dt \] (48)

Setting \( x = 1 \) we obtain:

\[ \int_0^1 \frac{t^{l+n}}{\sqrt{1-t^n}} \, dt = \frac{2(l+1)}{n + 2(l+1)} \cdot \int_0^1 \frac{t^l}{\sqrt{1-t^n}} \, dt \] (49)

Now, mapping \( l \mapsto (nl + 1) \) the result follows.

This proof suggests that a number \( g \leq n \) exists, such that the system of integrals

\[ 0 \leq i \leq g : \int_0^1 \frac{x^i}{\sqrt{1-x^n}} \, dx \] (50)

is "complete" or "enough".

**Corollary 19** For \( n = 2 \) we conclude:

\[ \frac{\pi^2}{8} = \sum_{l=0}^{\infty} \frac{1}{(2l+1)^2} \] (51)

And therefore:

\[ \zeta(2) = \frac{\pi^2}{6} \] (52)
To compute the integral \( \int_0^1 \frac{t}{\sqrt{1-t^n}} \, dt \) for \( n = 3 \) the following relation by Leo Königsberger might be useful: [11]

**Theorem 20** For \( p \in \mathbb{N} \):

\[
\prod_{s=0}^{2p-1} (\zeta_{2p+1}^{s+1} - 1) \cdot \int_0^1 \frac{x^s}{\sqrt{x^{2p+1} - 1}} = \frac{(2\pi)^p \cdot \sqrt{2p+1}}{(2p+1)^p \cdot (2p-1)!!}
\]

(53)

**6 Further identities**

We now turn our attention to the mysterious identity involving the curva elastica. It seems, that the ratio of powers might determine which hyperelliptic constant appears.

**Proposition 21**

\[
\forall n \in \mathbb{N} : \ A_{2n} \cdot B_{\frac{n}{2n}} := (\int_0^1 \frac{1}{\sqrt{1-t^{2n}}} \, dt) \cdot (\int_0^1 \frac{t^n}{\sqrt{1-t^{2n}}} \, dt) = \frac{\pi}{2n} \quad (54)
\]

\[
\iff \forall n \in \mathbb{N} : \int_0^1 \frac{t^n}{\sqrt{1-t^{2n}}} \, dt = \frac{\pi^2}{n \cdot \pi_{2n}} \quad (55)
\]

**Proof** Using the taylor series of \( \text{arcs}_n \) in Proposition 3:

\[
(\int_0^1 \frac{1}{\sqrt{1-t^{2k}}} \, dt) \cdot (\int_0^1 \frac{t^k}{\sqrt{1-t^{2k}}} \, dt) = (\sum_{k=0}^{\infty} \binom{2k}{k} \cdot \frac{1}{4^k \cdot (2nk+1)}) \cdot (\sum_{k=0}^{\infty} \binom{2k}{k} \cdot \frac{1}{4^k \cdot (2nk+n+1)})
\]

Taking the Cauchy product, one verifies:

\[
= \frac{1}{n} \cdot \sum_{k=0}^{\infty} \binom{2k}{k} \cdot \frac{1}{4^k \cdot (2k+1)}
\]

\( \square \)

We may now conclude

**Theorem 22**

\[
\forall k \in \mathbb{N} : \frac{\Gamma(k+1)}{2k} = \frac{2k \cdot \Gamma(k+1)}{\Gamma(\frac{1}{2k})} = \frac{\sqrt{\pi} \cdot 2}{\pi_{2k}}
\]

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Proof On the one hand by the integral formula in Proposition 21:
\[
\int_0^1 \frac{t^k}{\sqrt{1-t^{2k}}} dt = \frac{\pi}{k \cdot \pi_{2k}}
\]

On the other hand by the identity in the proof of Proposition 4:
\[
\int_0^1 \frac{u^{m-1}}{\sqrt{1-u^n}} du = \frac{\sqrt{\pi} \cdot \Gamma\left(\frac{m}{n}\right)}{\Gamma\left(\frac{m}{n} + \frac{1}{2}\right) \cdot n}
\]

For \( m = k + 1, \ n = 2k:\)
\[
\frac{\Gamma\left(\frac{k+1}{2k}\right)}{\Gamma\left(\frac{2k+1}{2k}\right)} = \frac{\sqrt{\pi_2} \cdot 2}{\pi_{2k}}
\]

□

If we had more integral formulas, we would have more identities for values of the Gamma function by the same proof.

Corollary 23
\[
\forall n \geq 2 : \Gamma\left(\frac{2^{n-1} + 1}{2^n}\right) = \frac{1}{2^{n-1}} \cdot \frac{\sqrt{\pi_2}}{\pi_{2^n}} \cdot \Gamma\left(\frac{1}{2^{n}}\right) \tag{57}
\]

Proof In Theorem 22, set \( k = 2^{n-1} \) and use Corollary 6. □

For the sake of completion, by Corollary 6 and the Euler relation and knowing that
\[
\forall n \geq 2 : \sin\left(\frac{\pi}{2^n}\right) = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{\ldots}}} \over 2} \tag{58}
\]

with \((n - 1)\) nested square-roots, we have for \( n \geq 2:\)

Remark 24
\[
\Gamma\left(\frac{2^n - 1}{2^n}\right) = \frac{2 \cdot \pi_2}{\left(\sqrt{2^{n-1} \cdot \pi_{2^n}} \cdot \sqrt{2^{n-1} \cdot \pi_{2^{(n-1)}}} \cdot \ldots \cdot \sqrt{2^{n-1} \cdot \pi_2}\right) \cdot \left(\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{\ldots}}} \over 2}\right)} \tag{59}
\]

For computation, note that \( \Gamma\left(\frac{1}{2^n}\right) \) has \( n \) nested square-roots, whereas \( \sin\left(\frac{\pi}{2^n}\right) \) only has \((n - 1)\) nested square-roots.

Theorem 25
\[
\forall n \in \mathbb{N} : \Gamma\left(\frac{2^n + n}{2^n}\right) = \frac{\Gamma\left(\frac{1}{n}\right) \cdot 2 \cdot \sqrt{\pi}}{n \cdot \pi_n} \tag{60}
\]
Proof Let \( n \in \mathbb{N} \) be arbitrary.

By the Legendre duplication formula for \( x = \frac{1}{n} \) we have:

\[
\Gamma\left(\frac{2}{n}\right) = \frac{\Gamma\left(\frac{1}{n}\right) \cdot \Gamma\left(\frac{2+n}{2n}\right) \cdot 2^{(\frac{2}{n}-1)}}{\sqrt{\pi}}
\]

Comparison with the duplication formula in Proposition 4 yields:

\[
\frac{\Gamma\left(\frac{1}{n}\right)}{n \cdot \pi_n} = \frac{\Gamma\left(\frac{2+n}{2n}\right) \cdot 2^{-1}}{\sqrt{\pi}}
\]

\( \square \)

7 Miscellanea

It is well known, that the singular values \( K(\lambda^*(n)) \) of the complete elliptic integral, where \( \lambda^*(n) := \sqrt{\lambda(i \cdot \sqrt{n})} \) and \( \lambda \) the modular lambda function, are expressible via the Gamma function [8]. We obtain the following values:

\[
K(\lambda^*(1)) = K\left(\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}}{2}
\]

(61)

\[
K(\lambda^*(3)) = K\left(\frac{\sqrt{3} - \sqrt{2}}{4}\right) = \frac{\sqrt{3} \cdot \sqrt{5}}{4} \cdot \pi_3
\]

(62)

\[
K(\lambda^*(4)) = K\left((\sqrt{2} - 1)^2\right) = \frac{\sqrt{2}}{\sqrt{8}} + \frac{\sqrt{2}}{4}
\]

(63)

\[
K(\lambda^*(9)) = K\left(\frac{(\sqrt{3} - 1) \cdot (\sqrt{2} - \sqrt{3})}{2}\right) = \sqrt{\frac{\sqrt{3}}{9} + \frac{1}{6} \cdot \sqrt{2}}
\]

(64)

\[
K(\lambda^*(16)) = K\left((\sqrt{2} + 1)^2 \cdot (\sqrt{2} - 1)^4\right) = \frac{(\sqrt{2} + 1)^2 \cdot \sqrt{2}}{2^3}
\]

(65)

\[
K(\lambda^*(25)) = K\left(\frac{(\sqrt{10} - 2 \cdot \sqrt{2}) \cdot (3 - 2 \cdot \sqrt{5})}{2}\right) = \sqrt{\frac{2}{5} \cdot \sqrt{2}} + \frac{1}{\sqrt{10}} \cdot \sqrt{2}
\]

(66)

Certain limits of the arithmetic-geometric-mean are also expressible via the Gamma function [9]. We obtain the following values:

\[
AGM(1, \sqrt{2}) = \frac{\pi}{\sqrt{2}}
\]

(67)

\[
AGM\left(2, \sqrt{2} + \sqrt{3}\right) = \frac{4 \cdot \pi_2}{\sqrt{2^7} \cdot \pi_3}
\]

(68)

\[
AGM\left(1 + \sqrt{3}, \sqrt{8}\right) = \frac{12 \cdot \sqrt{2^7} \cdot \pi_2}{\sqrt{3 \cdot 4} \cdot \pi_6 \cdot \sqrt{3} \cdot \pi_3}
\]

(69)

Since

\[
K(k) = \frac{\pi}{2 \cdot AGM(1, \sqrt{1 - k^2})}
\]
Hyperelliptic values of the Gamma function

and

\[ \lambda^*(x)^2 + \lambda^*(\frac{1}{x})^2 = 1 \]

we have for \( n \in \mathbb{N} \)

\[ \text{AGM}(1, \lambda^*(\frac{1}{n})) = \frac{\pi}{2 \cdot K(\lambda^*(n))} \quad (70) \]

It is possible, that hyperelliptic constants \( \pi_n \) appear for certain values with sums of type:

\[ \sum_{k=0}^{\infty} \frac{1}{(nk + 1)^2}, \quad n \in \mathbb{N} \quad (71) \]

(On updating and researching for this paper, reading between the lines, I noticed that C.L. Siegel implicitly treated \( \zeta(3) \) as an "independent" constant that cannot be expressed by any known constants whatsoever and I think that it is more of a "logarithmic nature".)

By the functional equation

\[ \zeta(s) = 2^s \cdot \pi^{s-1} \cdot \sin(\frac{\pi s}{2}) \cdot \Gamma(1 - s) \cdot \zeta(1 - s) \]

we compute for \( s = \frac{3}{4} \)

\[ \frac{\zeta(\frac{3}{4})}{\zeta(\frac{1}{4})} = \sqrt{(2 + \sqrt{2}) \cdot 2\sqrt{2}} \quad (72) \]

Now setting \( s = \frac{2^n - 1}{2^n} \) we have

\[ \frac{\zeta(\frac{2^n - 1}{2^n})}{\zeta(\frac{1}{2^n})} = \sqrt[2^n]{2 + \sqrt[2^n]{2 + \sqrt[2^n]{\ldots}}} \cdot \sqrt[2^n]{2^{n-1} \cdot \pi_{2n} \cdot \sqrt[2^n]{\ldots} \cdot \sqrt[2^{n-2}]{2^{n-1} \cdot \pi_4 \cdot \sqrt[2^{n-2}]{\ldots}}} \]

therefore

\[ \forall \ n \in \mathbb{N} : \frac{\zeta(\frac{2^n - 1}{2^n})}{\zeta(\frac{1}{2^n})} = \sqrt{2 + \sqrt{2 + \sqrt{\ldots}}} \cdot \sqrt{2^{n-1} \cdot \pi_{2n} \cdot \sqrt{\ldots} \cdot \sqrt{2^{n-2} \cdot \pi_4 \cdot \sqrt{\ldots}}} \quad (73) \]

with \( n \) nested square-roots, respectively.

A superellipse is defined for \( n \in \mathbb{N} \) and \( a, b > 0 \) as

\[ C_n : \left| \frac{x}{a} \right|^n + \left| \frac{y}{b} \right|^n = 1 \quad (74) \]
It is well known, that the area is given by

\[ A_{C_n} = 4 \cdot ab \cdot \frac{\Gamma\left(\frac{n+1}{n}\right)^2}{\Gamma\left(\frac{n+2}{n}\right)} = 4 \cdot ab \cdot \frac{1}{2n} \cdot \frac{\Gamma\left(\frac{1}{n}\right)^2}{\Gamma\left(\frac{2}{n}\right)} \]  

(75)

For the case \( n = 2^\nu \) we calculate

\[ A_{C_{2^\nu}} = 4 \cdot ab \cdot \frac{1}{2^\nu+1} \cdot \frac{2^{\nu-1} \cdot \pi_{2^\nu} \cdot \sqrt{2^{\nu-1} \cdot \pi_{2^{\nu-1}} \cdot \sqrt{\ldots}}}{\sqrt{2^{\nu-2} \cdot \pi_{2^{\nu-1}} \cdot \sqrt{\ldots}}} \]

With \((\nu - 1)\) nested square-roots, we obtain:

\[ \forall \nu \in \mathbb{N} : A_{C_{2^\nu}} = \pi_{2^\nu} \cdot ab \cdot \sqrt{2 \sqrt{2 \sqrt{2 \sqrt{\ldots}}}} \]  

(76)

where 0 nested square-roots are to be understood as the factor 1.

Finally, I want to add that the constant \( \pi_3 \) has a particularly beautiful interpretation as the area of the ”butterfly curve” \( y^6 = x^2 - x^6 \), which is why I call it the ”butterfly constant”.

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