Exact evaluations and reciprocity theorems for finite trigonometric sums

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Abstract

We evaluate in closed form several classes of finite trigonometric sums. Two general methods are used. The first method uses contour integration and extends a previous method used by two of the authors. In the second, we work in two cyclotomic fields to evaluate new sums involving roots of unity, which lead to the evaluations of several sums involving trigonometric functions. Reciprocity theorems for certain trigonometric sums are also established.

Keywords: Finite trigonometric sums, Characters, Reciprocity theorems, Gauss sums

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1 Introduction

Sums involving the exponential function or trigonometric functions have been used extensively in a variety of settings within mathematics. In particular, they appear in many problems in number theory. In most instances, the relevant sums cannot be evaluated in closed form, and in such cases one is content to provide either upper or lower bounds for these sums. Here, we focus on certain classes of trigonometric sums with the aim of obtaining exact evaluations.
Another important goal of the paper is to provide reciprocity theorems for certain trigonometric sums. Besides their intrinsic elegance, these reciprocity theorems can also be interpreted as a decisive step toward the exact evaluation of the corresponding sums. This is similar to how the law of quadratic reciprocity can be applied repeatedly in order to compute the Legendre symbol in any given concrete case.

We continue the studies of trigonometric sum evaluations that were made in earlier papers \[2,5,6\] by two of the present authors and their collaborators. Since then, several other authors, for example, da Fonseca et al. \[15,16\], Wang and Zheng \[30\], and He \[22\], have made excellent contributions to the study and evaluations of trigonometric sums.

Many reciprocity theorems are found in the literature. In particular, one of the reciprocity theorems involving cotangents is equivalent to the reciprocity theorem for Dedekind sums \[24,25\]. In addition to papers cited above, see also papers of Chu \[9\], Fukuhara \[17\], and Shibukawa \[29\], wherein very general reciprocity theorems for trigonometric sums, in particular, for cotangent sums, are proved. Later, we provide more details.

Many new reciprocity theorems are proved in the present paper. For example, Theorems 3.2 and 3.3 provide two new reciprocity theorems for sums involving sines and cotangents. Moreover, our methods yield reciprocity theorems involving \[2^n\] sums, where \( n \geq 1 \). See Theorem 3.4 for a reciprocity theorem involving 4 trigonometric sums.

Several of our evaluations of trigonometric sums involve Dirichlet characters. For example, in Theorems 4.1 and 4.2, we evaluate certain analogues of Gauss sums.

The summands for most evaluations in the literature involve a ‘small’ number of trigonometric functions. Another feature of our paper is the evaluation of certain trigonometric sums with an arbitrary number of distinct trigonometric functions. Theorem 5.1 provides an example.

One of our primary goals is to evaluate sums, in which the trigonometric functions have two linearly independent periods, that is periods, say \( p \) and \( q \), where \( (p, q) = 1 \) and \( p, q > 1 \). Theorem 6.1 is our most general theorem on evaluating sums of trigonometric functions with two linearly independent periods. Corollary 10.2 (which is not a corollary of Theorem 6.1) provides a particularly elegant example of an evaluation of a sum having trigonometric functions with two linearly independent periods. To the best of our knowledge, there are no previous explicit evaluations of trigonometric sums with two distinct periods in the literature. However, McIntosh \[23, p. 202\] expressed one such sum in terms of generalized Dedekind sums. Although his representation is not an ‘evaluation’ in the sense that we use the term in this paper, his result was of importance in motivating the research in our paper.

Beginning in Sect. 7, we introduce a new method for evaluating trigonometric sums. The method features logarithmic differentiation, partial fractions, and the evaluations of several necessary sums involving roots of unity. Although our primary motivation was to use these identities to evaluate trigonometric sums, particularly, involving sines and cosines, we think that the theorems involving roots of unity are interesting in themselves and potentially useful in other investigations. Besides being useful, several evaluations and identities are elegant in themselves. In particular, Lemmas 9.4, 9.6, 11.3, and 11.4 provide beautiful reciprocity theorems for certain sums of roots of unity.

Our approach rests upon four logarithmic differentiations. It can be extended with the use of \( n \) successive logarithmic differentiations, for any natural number \( n \). However,
the calculations become increasingly laborious, and a computer algebra system, such as Mathematica, would be advisable.

2 Evaluations of two trigonometric sums

Theorem 2.1 Let $k$ denote an odd positive integer, and let $a$ denote a positive integer. Then

$$
\sum_{0 < n < k/2} \frac{\cos^2(\pi n/k)}{\cos^{2a+2}(2\pi n/k)} = -\frac{1}{2} - \pi i \left( R_{k/4} + R_{3k/4} \right),
$$

where $R_{k/4}$ and $R_{3k/4}$, respectively, denote the residues of the function

$$
F(z) := \frac{\cos(\pi z/k)}{\cos^{2a+2}(2\pi z/k)(e^{2\pi iz} - 1)}
$$

at the poles $k/4$ and $3k/4$.

Proof Let $C_N$ denote an indented, positively oriented rectangle with horizontal sides passing through $\pm iN$, $N > 0$, and vertical sides passing through 0 and $k$. The left vertical side is indented by a semi-circle $C_0$ of radius $\epsilon$, $0 < \epsilon < 1$, in the left half-plane. The right vertical line is indented at $z = k$ by the semi-circle $C_{0+k}$. Let $I(C_N)$ denote the interior of $C_N$. Let $R_\alpha$ denote the residue of a pole of $F(z)$ at $z = \alpha$ on $I(C_N)$. We apply the residue theorem to $F(z)$, integrated over $C_N$. On $I(C_N)$, there are simple poles at $z = 0, 1, 2, \ldots, k-1$ and poles of order $2a+2$ at $z = k/4, 3k/4$. We easily see that

$$
R_n = \frac{1}{2\pi i} \frac{\cos^2(\pi n/k)}{\cos^{2a+2}(2\pi n/k)}, \quad n = 0, 1, 2, \ldots, k-1.
$$

(2.2)

Also observe that

$$
R_n = R_{k-n}, \quad 0 < n < k/2.
$$

(2.3)

Hence, by the residue theorem, (2.2), and (2.3),

$$
\int_{C_N} F(z) dz = 1 + 2 \sum_{0 < n < k/2} \frac{\cos^2(\pi n/k)}{\cos^{2a+2}(2\pi n/k)} + 2\pi i \left( R_{k/4} + R_{3k/4} \right).
$$

(2.4)

We now directly calculate $\int_{C_N} F(z) dz$. Observe that $F(z) = F(z+k)$. Hence, the integrals over the vertical sides of $C_N$ cancel. Let $C_{NT}$ and $C_{NB}$, respectively, denote those portions of $C_N$ over the top and bottom sides. It is easy to see that the integrals over $C_{NT}$ and $C_{NB}$ tend to 0 as $N \to \infty$. Hence,

$$
\lim_{N \to \infty} \int_{C_N} F(z) dz = 0.
$$

Hence, by (2.4),

$$
1 + 2 \sum_{0 < n < k/2} \frac{\cos^2(\pi n/k)}{\cos^{2a+2}(2\pi n/k)} + 2\pi i \left( R_{k/4} + R_{3k/4} \right) = 0,
$$

(2.5)

which is easily seen to be equivalent to (2.1).
Corollary 2.2  For any positive odd integer $k$,

$$
\sum_{0 < n < k/2} \frac{\cos^2(\pi n/k)}{\cos^4(2\pi n/k)} = \begin{cases} 
\frac{1}{2} + \frac{k}{24}(3 + 4k + 3k^2 + 2k^3), & \text{if } k \equiv 1 \pmod{4}, \\
-\frac{1}{2} + \frac{k}{24}(-3 + 4k - 3k^2 + 2k^3), & \text{if } k \equiv 3 \pmod{4}. 
\end{cases}
$$

(2.6)

Proof  Let $a = 1$ in Theorem 2.1. To calculate $R_{k/4}, R_{3k/4}$ we use Mathematica by expanding

$$
F(z) := \frac{\cos^2(\pi z/k)}{\cos^4(2\pi z/k)(e^{2\pi i z} - 1)}
$$

into power series about $k/4$ and $3k/4$, respectively, and recording the associated residues. First,

$$
R_{k/4} = \begin{cases} 
\frac{ki}{48\pi} ((3 - 3i) + 4k + 3k^2 + 2k^3), & \text{if } k \equiv 1 \pmod{4}, \\
\frac{ki}{48\pi} ((-3 - 3i) + 4k - 3k^2 + 2k^3), & \text{if } k \equiv 3 \pmod{4}. 
\end{cases}
$$

(2.7)

Secondly,

$$
R_{3k/4} = \begin{cases} 
\frac{ki}{48\pi} ((3 + 3i) + 4k + 3k^2 + 2k^3), & \text{if } k \equiv 1 \pmod{4}, \\
\frac{ki}{48\pi} ((-3 + 3i) + 4k - 3k^2 + 2k^3), & \text{if } k \equiv 3 \pmod{4}. 
\end{cases}
$$

(2.8)

It follows from (2.7) and (2.8) that

$$
2\pi i (R_{k/4} + R_{3k/4}) = \begin{cases} 
-\frac{k}{12} (3 + 4k + 3k^2 + 2k^3), & \text{if } k \equiv 1 \pmod{4}, \\
-\frac{k}{12} (-3 + 4k - 3k^2 + 2k^3), & \text{if } k \equiv 3 \pmod{4}. 
\end{cases}
$$

(2.9)

Using (2.1) and (2.9), we find that

$$
\sum_{0 < n < k/2} \frac{\cos^2(\pi n/k)}{\cos^4(2\pi n/k)} = \begin{cases} 
\frac{1}{2} + \frac{k}{24}(3 + 4k + 3k^2 + 2k^3), & \text{if } k \equiv 1 \pmod{4}, \\
\frac{1}{2} + \frac{k}{24}(-3 + 4k - 3k^2 + 2k^3), & \text{if } k \equiv 3 \pmod{4}, 
\end{cases}
$$

which completes the proof of Corollary 2.2. \hfill \Box

Example 2.3  Let $k = 5$ in Corollary 2.2. An easy calculation shows that

$$
\left( \frac{\cos \frac{\pi}{5}}{2\cos^2 \frac{2\pi}{5}} \right)^2 + \left( \frac{\cos \frac{2\pi}{5}}{2\cos^2 \frac{2\pi}{5}} \right)^2 = 18.
$$
Example 2.4 Let $k = 7$ in Corollary 2.2. Then, after a moderate computation, we find that
\[
\left( \frac{\cos \frac{\pi}{7}}{2 \cos^2 \frac{2\pi}{7}} \right)^2 + \left( \frac{\cos \frac{2\pi}{7}}{2 \cos^2 \frac{3\pi}{7}} \right)^2 + \left( \frac{\cos \frac{3\pi}{7}}{2 \cos^2 \frac{\pi}{7}} \right)^2 = 41.
\]
This past example was also established by Örs Rebák [27] in the course of solving a longstanding open problem in Ramanujan’s lost notebook [26].

Theorem 2.5 Let $k$ denote an odd positive integer, and let $a$ denote a positive integer. Then,
\[
\sum_{0 < n < k/2} \frac{\sin^2 a(\pi n/k)}{\sin^2 a + 2(2\pi n/k)} = -\pi\left( R_0 + R_{k/2} \right), \tag{2.10}
\]
where $R_0$ and $R_{k/2}$, respectively, denote the residues of the function
\[
G(z) := \frac{\sin^2 a(\pi z/k)}{\sin^2 a + 2(2\pi z/k)(e^{2\pi iz} - 1)} \tag{2.11}
\]
at the poles $0$ and $k/2$.

Proof As in the previous proof, let $C_N$ denote the indented, positively oriented rectangle with horizontal sides passing through $\pm iN$, $N > 0$, and vertical sides passing through $0$ and $k$. Let $I(C_N)$ denote the interior of $C_N$. Let $R_\alpha$ denote the residue of a pole of $G(z)$ at $z = \alpha$ on $I(C_N)$. We apply the residue theorem to $G(z)$, integrated over $C_N$. On $I(C_N)$, there are simple poles at $z = 1, 2, \ldots, k - 1$, a pole of order $3$ at $z = 0$, and a pole of order $2a + 2$ at $z = k/2$. We easily see that
\[
R_\alpha = \frac{1}{2\pi i} \frac{\sin^2 a(\pi n/k)}{\sin^2 a + 2(2\pi n/k)}, \quad n = 1, 2, \ldots, k - 1. \tag{2.12}
\]
Also observe that
\[
R_n = R_{k-n}, \quad 0 < n < k/2. \tag{2.13}
\]
Hence, by the residue theorem, (2.12), and (2.13),
\[
\int_{C_N} G(z)dz = 2 \sum_{0 < n < k/2} \frac{\sin^2 a(\pi n/k)}{\sin^2 a + 2(2\pi n/k)} + 2\pi i \left( R_0 + R_{k/2} \right). \tag{2.14}
\]
We now directly calculate $\int_{C_N} F(z)dz$. The details are analogous to those in the proof of Theorem 2.1. Thus,
\[
\lim_{N \to \infty} \int_{C_N} G(z)dz = 0. \tag{2.15}
\]
Hence, by (2.14) and (2.15),
\[
2 \sum_{0 < n < k/2} \frac{\sin^2 a(\pi n/k)}{\sin^2 a + 2(2\pi n/k)} + 2\pi i \left( R_0 + R_{k/2} \right) = 0,
\]
which is easily seen to be equivalent to (2.10).
By setting $a = 1$ in Theorem 2.5, we obtain the following corollary.

**Corollary 2.6** If $k$ is an odd positive integer, then

$$
\sum_{1 \leq n < k/2} \frac{\sin^2(\pi n/k)}{\sin^2(2\pi n/k)} = \frac{k^4 + 6k^2 - 7}{96}. 
$$

(2.16)

**Example 2.7** Let $k = 7$ in Corollary 2.6. Then, by an elementary calculation,

$$
\sum_{1 \leq n \leq 3} \frac{\sin^2(\pi n/7)}{\sin^2(2\pi n/7)} = 28. 
$$

(2.17)

The identity (2.17) was first established by Berndt and Zhang [7, p. 233] “by a direct laborious computation.”

Replace $k$ by $2k + 1$ in (2.16). Thus,

$$
\sum_{n=1}^{k} \frac{\sin^2(\pi n/(2k + 1))}{\sin^2(2\pi n/(2k + 1))} = \frac{(2k + 1)^4 + 6(2k + 1)^2 - 7}{96} = \frac{16k^4 + 32k^3 + 48k^2 + 32k}{96} = \frac{k(k^3 + 2k^2 + 3k + 2)}{6} = \frac{k(k + 1)(k^2 + k + 2)}{6}. 
$$

(2.18)

Recall that the $k$th triangular number $T(k)$ is defined by

$$
T(k) := \frac{k(k + 1)}{2}. 
$$

Now,

$$
T(T(k)) = \frac{k(k + 1)}{4} \left( \frac{k(k + 1)}{2} + 1 \right) = \frac{k(k + 1)(k^2 + k + 2)}{8}. 
$$

(2.19)

Comparing (2.18) with (2.19), we conclude that

$$
\sum_{n=1}^{k} \frac{\sin^2(\pi n/(2k + 1))}{\sin^2(2\pi n/(2k + 1))} = \frac{4}{3} T(T(k)).
$$

(2.20)

The beautiful identity (2.20) was established by Örs Rebák [28]. Is (2.20) an example of a yet identified class of trigonometric sums that can be represented in terms of triangular numbers?

The more general, related sum

$$
\sum_{j=1}^{k-1} \frac{\sin^2(\pi aj/k)}{\sin^2(\pi aj/k)},
$$

where $n$, $k$, and $a$ are positive integers with $a < k$, has been studied and evaluated by other authors. See [5, p. 379] for one of these proofs and for further references.
3 Reciprocity theorems

We establish some apparently new reciprocity theorems for cotangent sums. We indicate how to prove reciprocity theorems for $2n$ sums with $2n$ distinct periods, although we work out the details for only $n = 1, 2$.

The following theorem was given by McIntosh [23, Theorem 3, p. 199]. We provide another proof, because it leads to a generalization mentioned after the proof.

**Theorem 3.1** Let $p$ and $q$ be distinct, positive, odd integers, each $\geq 3$. Then,

$$
\sum_{n=1}^{p-1} \cot(\pi n/p) \cot(\pi np/q) \overline{\sin^2(\pi n/q)} + q \sum_{n=1}^{p-1} \cot(\pi n/p) \cot(\pi nq/p) \overline{\sin^2(\pi n/p)} = \frac{p^4 + q^4 - 5p^2q^2 + 3}{45}.
$$

**Proof** Let

$$f(z) := \frac{\cot(\pi z/q) \cot(\pi pz/q)}{(e^{2\pi iz}) - 1) \sin^2(\pi z/q)} \tag{3.1}\text{.}$$

Integrate $f(z)$ over a rectangle $C_N$, with horizontal sides $z = x \pm i\epsilon, 0 \leq x \leq q$, and with vertical sides $z = 0 + iy, q + iy, -N \leq y \leq N$. The contour contains semi-circular indentations of radius $\epsilon$, $0 < \epsilon < 1$, at 0 and $q$, with the former lying in the left-half plane, and the latter lying in the half-plane $x \leq q$. On the interior of $C_N$, $f(z)$ has simple poles at $z = n, 1 \leq n \leq q - 1$; simple poles at $z = nq/p, 1 \leq n \leq p - 1$; and a pole of order 5 at the origin. Let $R_\alpha$ denote the residue of a pole $\alpha$ of $f(z)$. By the residue theorem,

$$
\int_{C_N} f(z) dz = 2\pi i \sum_{n=1}^{q-1} R_n + 2\pi i \sum_{n=1}^{p-1} R_{nq/p} + 2\pi i R_0 = \sum_{n=1}^{q-1} \cot(\pi n/q) \cot(\pi np/q) \overline{\sin^2(\pi n/q)} + 2q \sum_{n=1}^{p-1} \frac{\cot(\pi n/p)}{(e^{2\pi iz}) - 1) \sin^2(\pi n/p)} + 2\pi i R_0. \tag{3.2}\text{.}
$$

Now,

$$
\frac{\cot(\pi n/p)}{(e^{2\pi iz}) - 1) \sin^2(\pi n/p)} = \frac{\cot(\pi n/p)}{\sin(\pi nq/p) \sin^2(\pi n/p)}
\frac{e^{\pi iz} nq/p)}{(e^{2\pi iz}) - 1) \sin^2(\pi n/p)} = \frac{nq/p)}{(e^{2\pi iz}) - 1) \sin^2(\pi n/p)}
\sin(\pi nq/p) \sin^2(\pi n/p)} = \frac{i}{2} \frac{\cos(\pi nq/p) \cot(\pi n/p)}{\sin(\pi nq/p) \sin^2(\pi n/p)} - \frac{1}{2} \frac{\sin(\pi nq/p) \cot(\pi n/p)}{\sin(\pi nq/p) \sin^2(\pi n/p)} \tag{3.3}\text{.}
$$

When the far right side of (3.3) is substituted into (3.2), the contribution of the latter quotient in (3.3) will be equal to 0, since the terms with index $n$ and $p - n', 0 < n < p/2$, have opposite signs and so cancel. Hence, we find that (3.2) can be written in the form

$$
\int_{C_N} f(z) dz = \sum_{n=1}^{q-1} \cot(\pi n/q) \cot(\pi np/q) \overline{\sin^2(\pi n/q)} + q \sum_{n=1}^{p-1} \frac{\cot(\pi n/p) \cot(\pi nq/p)}{\sin^2(\pi n/p)} + 2\pi i R_0. \tag{3.4}\text{.}
$$

Expanding $f(z)$ from (3.1) in a Laurent series about $z = 0$ with the aid of *Mathematica*, we find that

$$
2\pi i R_0 = -\frac{p^4 + q^4 - 5p^2q^2 + 3}{45p}. \tag{3.5}\text{.}
$$
Putting (3.5) in (3.4), we deduce that

\[
\int_{CN} f(z)\,dz = \sum_{n=1}^{q-1} \frac{\cot(\pi n/q) \cot(\pi np/q)}{\sin^2(\pi n/q)} + \frac{q}{p} \sum_{n=1}^{p-1} \frac{\cot(\pi n/p) \cot(\pi nq/p)}{\sin^2(\pi n/p)} - \frac{p^4 + q^4 - 5p^2q^2 + 3}{45p}.
\] (3.6)

By a now familiar argument, a direct calculation gives

\[
\lim_{N \to \infty} \int_{CN} f(z)\,dz = 0.
\] (3.7)

Combining (3.6) and (3.7) and then multiplying both sides by \(p\), we conclude that

\[
\frac{q}{p} \sum_{n=1}^{p-1} \frac{\cot(\pi n/p) \cot(\pi nq/p)}{\sin^2(\pi n/p)} = \frac{p^4 + q^4 - 5p^2q^2 + 3}{45},
\]

which completes the proof of Theorem 3.1. \(\blacksquare\)

Generalizing our argument, we can derive a reciprocity theorem for

\[
\sum_{n=1}^{p-1} \frac{\cot(\pi n/p) \cot(\pi nq/p)}{\sin^{2m}(\pi n/p)}
\]

for any positive integer \(m\). The next two results give these reciprocity theorems for \(m = 2, 3\), respectively.

**Theorem 3.2** Let \(p\) and \(q\) be distinct, positive, odd integers, each \(\geq 3\). Then,

\[
P \sum_{n=1}^{q-1} \frac{\cot(\pi n/q) \cot(\pi np/q)}{\sin^4(\pi n/q)} + \frac{q}{p} \sum_{n=1}^{p-1} \frac{\cot(\pi n/p) \cot(\pi nq/p)}{\sin^4(\pi n/p)} = \frac{2p^6 + 2q^6 - 7p^4q^2 - 7p^2q^4 + 3p^4 + 7q^4 - 35p^2q^2 + 31}{945}.
\]

**Theorem 3.3** Let \(p\) and \(q\) be distinct, positive, odd integers, each \(\geq 3\). Then,

\[
P \sum_{n=1}^{q-1} \frac{\cot(\pi n/q) \cot(\pi np/q)}{\sin^6(\pi n/q)} + \frac{q}{p} \sum_{n=1}^{p-1} \frac{\cot(\pi n/p) \cot(\pi nq/p)}{\sin^6(\pi n/p)} = \frac{3p^8 + 3q^8 + 20p^6 + 20q^6 + 56p^4 + 56q^4 - 280p^2q^2 - 10p^6q^2 - 10p^2q^6 - 7p^4q^4 + 289}{14175}.
\]
**Theorem 3.4** (Four-sum reciprocity relation) Let \( p, q, r, \) and \( s \) be distinct, odd positive integers, relatively prime in pairs. Then,

\[
\frac{1}{q} \sum_{n=1}^{q-1} \cot(\pi n/q) \cot(\pi pn/q) \cot(\pi rn/q) \cot(\pi sn/q) \sin^2(\pi n/q)
\]

\[
+ \frac{1}{p} \sum_{n=1}^{p-1} \cot(\pi n/p) \cot(\pi pn/p) \cot(\pi rn/p) \cot(\pi sn/p) \sin^2(\pi n/p)
\]

\[
+ \frac{1}{r} \sum_{n=1}^{r-1} \cot(\pi n/r) \cot(\pi pn/r) \cot(\pi qn/r) \cot(\pi sn/r) \sin^2(\pi n/r)
\]

\[
+ \frac{1}{s} \sum_{n=1}^{s-1} \cot(\pi n/s) \cot(\pi pn/s) \cot(\pi rn/s) \cot(\pi qn/s) \sin^2(\pi n/s)
\]

\[
= \frac{1}{1890pqr} \left\{ 35(p^2 q^2 r^2 + q^2 r^2 s^2 + q^2 r^2 s^2 + p^2 r^2 s^2) 
\right.
\]

\[
- 7(p^4 q^2 + p^4 r^2 + q^4 p^2 + r^4 p^2 + s^4 p^2 + q^4 s^2 + q^4 r^2 + r^4 q^2 + s^4 q^2 + s^4 r^2)
\]

\[
+ r^4 s^2 + q^4 s^2) + 2(p^6 + q^6 + r^6 + s^6) - 21(p^2 + q^2 + r^2 + s^2) + 20 \right\}. \tag{3.8}
\]

**Proof** Let

\[
f(z) := \frac{\cot(\pi z/q) \cot(\pi pz/q) \cot(\pi rz/q) \cot(\pi sz/q)}{\sin^2(\pi z/q)(e^{2\pi iz} - 1)}, \quad z = x + iy. \tag{3.9}
\]

Let \( C_N \) denote a positively oriented rectangle with vertical sides passing through \( x = 0 \) and \( x = q \), horizontal sides \( 0 \leq x \leq q, y = N, \) and \( 0 \leq x \leq q, y = -N. \) Furthermore, the left vertical line is indented by a semicircle of radius \( \epsilon < 1 \) around \( z = 0 \) and lying in the left half-plane, and lastly the right vertical line has an indentation of radius \( \epsilon < 1 \) around \( z = q \) and lying in the half-plane \( x \leq q. \) On the interior of \( C_N, f(z) \) has a pole of order 7 at the origin, and simple poles at \( z = n, 1 \leq n \leq q - 1; z = qn/p, 1 \leq n \leq p - 1; z = qn/r, 1 \leq n \leq r - 1; \) and \( z = qn/s, 1 \leq n \leq s - 1. \)

We apply the residue theorem to \( f(z) \) on \( C_N. \) Let \( R_a \) denote the residue of \( f(z) \) at a pole \( a. \) First,

\[
R_a = \frac{\cot(\pi n/q) \cot(\pi pn/q) \cot(\pi rn/q) \cot(\pi sn/q)}{2\pi i \sin^2(\pi n/q)}, \quad 1 \leq n \leq q - 1.
\]

Thus, from the residue theorem, we obtain the contribution

\[
\frac{q-1}{\sum_{n=1}^{q-1} \cot(\pi n/q) \cot(\pi pn/q) \cot(\pi rn/q) \cot(\pi sn/q)} \sin^2(\pi n/q). \tag{3.10}
\]
Secondly, for $1 \leq n \leq p - 1$,

$$R_{qn/p} = \frac{\cot(\pi n/p) \cot(\pi pn/p) \cot(\pi sn/p)}{(\pi p/q)(e^{i\pi qn/p} - 1) \sin^2(\pi n/p)}$$

$$= \frac{q \cot(\pi n/p) \cot(\pi pn/p) \cot(\pi sn/p) (\cos(\pi qn/p) - i \sin(\pi qn/p))}{2\pi ip \sin(\pi qn/p) \sin^2(\pi n/p)}$$

$$= \frac{q \cot(\pi n/p) \cot(\pi pn/p) \cot(\pi sn/p) \cot(\pi qn/p)}{2\pi p \sin^2(\pi n/p)}. \quad (3.11)$$

Hence, from the residue theorem, the contributions from (3.11) total

$$\frac{q}{p} \sum_{n=1}^{p-1} \left\{ \frac{\cot(\pi n/p) \cot(\pi pn/p) \cot(\pi sn/p) \cot(\pi qn/p)}{\sin^2(\pi n/p)} - i \frac{\cot(\pi n/p) \cot(\pi pn/p) \cot(\pi sn/p)}{\sin^2(\pi n/p)} \right\}$$

$$= \frac{q}{p} \sum_{n=1}^{p-1} \frac{\cot(\pi n/p) \cot(\pi pn/p) \cot(\pi sn/p) \cot(\pi qn/p)}{\sin^2(\pi n/p)}, \quad (3.12)$$

because in the latter sum on the left-hand side of (3.12), the terms with index $n$ and $p - n$, $1 \leq n \leq (p - 1)/2$, cancel.

By analogous calculations, the contributions of the residue theorem from the poles $z = qn/r$, $1 \leq n \leq r - 1$, and $z = qn/s$, $1 \leq n \leq s - 1$, are, respectively,

$$\frac{q}{r} \sum_{n=1}^{r-1} \cot(\pi n/r) \cot(\pi pn/r) \cot(\pi qn/r) \cot(\pi sn/r) \frac{1}{\sin^2(\pi n/r)}, \quad (3.13)$$

and

$$\frac{q}{s} \sum_{n=1}^{s-1} \cot(\pi n/s) \cot(\pi pn/s) \cot(\pi qn/s) \cot(\pi sn/s) \frac{1}{\sin^2(\pi n/s)}. \quad (3.14)$$

Lastly, harkening back to (3.9) and expanding $f(z)$ in a Laurent series about $z = 0$ with the help of Mathematica, we find that

$$2\pi i R_0 = -\frac{1}{1890 p r s} \left\{ 35(p^2 q^2 r^2 + p^2 q^2 s^2 + q^2 r^2 s^2 + p^2 r^2 s^2) 
- 7(p^4 q^2 + p^4 r^2 + q^4 p^2 + r^4 p^2 + s^4 p^2 + q^4 s^2 + q^4 r^2 + r^4 q^2 + s^4 q^2 + s^4 r^2
+ r^4 s^2 + p^4 s^2) + 2(p^6 + q^6 + r^6 + s^6) - 21(p^2 + q^2 + r^2 + s^2) + 20 \right\}. \quad (3.15)$$
In summary, bringing together (3.10), (3.12), (3.13), and (3.14), and dividing both sides by \( q \), we deduce that

\[
\frac{1}{q} \int_{C_N} f(z) \, dz = \frac{1}{q} \sum_{n=1}^{q-1} \frac{\cot(\pi n/q) \ cot(\pi pn/q) \ cot(\pi rn/q) \ cot(\pi sn/q)}{\sin^2(\pi n/q)} \\
+ \frac{1}{p} \sum_{n=1}^{p-1} \frac{\cot(\pi n/p) \ cot(\pi qn/p) \ cot(\pi rn/p) \ cot(\pi sn/p)}{\sin^2(\pi n/p)} \\
+ \frac{1}{r} \sum_{n=1}^{r-1} \frac{\cot(\pi n/r) \ cot(\pi pn/r) \ cot(\pi qn/r) \ cot(\pi sn/r)}{\sin^2(\pi n/r)} \\
+ \frac{1}{s} \sum_{n=1}^{s-1} \frac{\cot(\pi n/s) \ cot(\pi pn/s) \ cot(\pi qn/s)}{\sin^2(\pi n/s)} = -\frac{1}{1890pqrs} \left\{ 35(p^2q^2r^2 + p^2q^2s^2 + q^2r^2s^2 + p^2r^2s^2) \\
- 7(p^4q^2 + p^4r^2 + q^4p^2 + q^4r^2 + s^4p^2 + s^4r^2 + q^4p^2 + q^4r^2 + s^4q^2 \\
+ s^4r^2 + r^4s^2 + p^4s^2) \\
+ 2(p^6 + s^6 + r^6 + q^6) - 21(p^2 + r^2 + s^2 + q^2) + 20 \right\}.
\] (3.16)

By a familiar argument, we can easily show that

\[
\lim_{N \to \infty} \int_{C_N} f(z) \, dz = 0.
\] (3.17)

If we now combine (3.16) and (3.17), we readily deduce (3.8) to complete the proof of Theorem 3.4.

Let \( p_1, p_2, \ldots, p_{2m} \) denote \( 2m \) positive odd integers, relatively prime in pairs. By an extension of the argument that we employed to prove Theorem 3.4, we can prove a \( 2m \)-term reciprocity theorem for the \( 2m \) sums

\[
\frac{1}{p_r} \sum_{n=1}^{p_r-1} \frac{\cot(\pi n/p_r) \ cot(\pi np_1/p_r) \cdots \cot(\pi np_{r-1}/p_r) \ cot(\pi np_{r+1}/p_r) \cdots \cot(\pi np_{2m}/p_r)}{\sin^2(\pi n/p_r)},
\]

where \( 1 \leq r \leq 2m \).

It is clear that we can also obtain a \( 2m \)-term reciprocity theorem if we replace \( \sin^2(\pi n/p_r) \) by \( \sin^2(\pi n/p_r) \), \( 1 \leq r \leq 2m \), for any positive integer \( N \).

### 4 Two theorems on analogues of Gauss sums

Let \( \chi \) denote a Dirichlet character modulo \( k \). Define the Gauss sum,

\[
G(z, \chi) := \sum_{n=1}^{k} \chi(n)e^{2\pi iz/k}, \quad z \in \mathbb{C}.
\] (4.1)
Trivially, from (4.1),
\[
G(z, \chi) = \begin{cases} 
\sum_{n=1}^{k} \chi(n) \cos(2\pi nz/k), & \text{if } \chi \text{ is even,} \\
i \sum_{n=1}^{k} \chi(n) \sin(2\pi nz/k), & \text{if } \chi \text{ is odd.} 
\end{cases}
\] (4.2)

For each integer \( n \) and \( \chi \) non-principal, we have the factorization theorem [4, p. 9]
\[
G(n, \chi) = \chi(n)G(1, \chi) =: \chi(n)G(\chi).
\] (4.3)

Recall also that if \( \chi \) is a non-principal character modulo \( k \) [4, p. 10],
\[
G(\chi) = \begin{cases} 
\sqrt{k}, & \text{if } \chi \text{ is even,} \\
i\sqrt{k}, & \text{if } \chi \text{ is odd.} 
\end{cases}
\] (4.4)

Set
\[
M(k, \chi) := \sum_{j=1}^{k} j\chi(j).
\] (4.5)

We denote the class number of the imaginary quadratic field \( \mathbb{Q}(\sqrt{-k}) \) by \( h(-k) \). For \( k \geq 7 \) and odd, the classical formula for the class number is given by [1, p. 299]
\[
h(-k) = -\frac{1}{k}M(k, \chi).
\] (4.6)

In this section, very roughly, we prove two general theorems in which for odd \( \chi \), sine is replaced by a quotient of sines, and which for even \( \chi \), cosine is replaced by a quotient of cosines in (4.2).

**Theorem 4.1** Let \( \chi \) denote an even, non-principal character modulo \( p \). Let \( a > 1 \) denote an odd positive integer. Assume that \( a - 3 < 2p \). Then,
\[
S(a, p, \chi) := \sum_{0 < n < p/2} \chi(n) \frac{\sin(a\pi n/p)}{\sin(\pi n/p)} = C_0 p,
\] (4.7)

where \( C_0 \) is the constant defined by
\[
C_0 := \frac{1}{G(\chi)} \sum_{m>0,n>1,r\geq 0} \frac{\chi(n)}{\mu^{2m+2n+2ar=a-1}} \sum_{r=0}^{p-1} \beta_r \mu^{-2ar} \sum_{m=0}^{\infty} \mu^{-2m}, |\mu| < 1,
\] (4.8)

where \( \beta_0 = 1 \) and \( \beta_1 = -1 \).
Proof. Let
\[ f(z) := \frac{G(z, \chi)}{G(\chi)} \frac{\sin(a\pi z/p)}{(e^{2\pi iz} - 1) \sin(\pi z/p)}. \] (4.9)

Integrate over the positively oriented indented rectangle \( C_N \) with vertical sides passing through the origin and \( p \), and horizontal sides \( z = x \pm iN, 0 \leq x \leq p \). The indented rectangle \( C_N \) contains a semi-circular indentation of radius \( \epsilon, 0 < \epsilon < 1 \), about 0 in the left half-plane. The rectangle also possesses an analogous indentation about \( p \), lying in the half-plane to the left of the line \( z = p + iy, -N \leq y \leq N \). Since \( G(0, \chi) = 0 \), the function \( f \) has a removable singularity at the origin. Thus, \( f(z) \) is analytic on \( C_N \) and its interior except for simple poles at \( z = 1, 2, \ldots, p-1 \). Their residues \( R_n \) are easily seen to be
\[ R_n = \frac{G(n, \chi)}{G(\chi)} \frac{\sin(a\pi n/p)}{2\pi i \sin(\pi n/p)}, \quad 1 \leq n \leq p - 1. \]

Therefore, by the residue theorem and (4.3),
\[ \int_{C_N} f(z) \, dz = \sum_{n=1}^{p-1} \chi(n) \frac{\sin(a\pi n/p)}{\sin(\pi n/p)}. \] (4.10)

Since \( \chi \) is even and \( a \) is odd,
\[ \chi(p-n) \frac{\sin(a\pi (p-n)/p)}{\sin(\pi (p-n)/p)} = \chi(n) \frac{\sin(a\pi n/p)}{\sin(\pi n/p)}, \quad 0 < n < p/2. \]

Hence, we can rewrite (4.10) in the form
\[ \int_{C_N} f(z) \, dz = 2 \sum_{0 < n < p/2} \chi(n) \frac{\sin(a\pi n/p)}{\sin(\pi n/p)} = 2S(a, p, \chi), \] (4.11)

by (4.7).

We next directly integrate \( \int_{C_N} f(z) \, dz \). Using the definition (4.9), we easily see that \( f(z) = f(z + p) \), and so the integrals over the vertical sides of \( C_N \) cancel. Let \( C_{NT} \) and \( C_{NB} \) denote the upper and lower sides of \( C_N \), respectively. For \( z = x + iN \), set
\[ \mu := e^{-\pi ix/p} e^{\pi N/p}. \]

We examine \( f(z) \) on each portion of \( C_{NT} \). First,
\[ \frac{1}{e^{2\pi i(x+iN)} - 1} = \frac{1}{\mu^{-2p} - 1} = -1 + O(\mu^{-2p}), \quad N \to \infty. \] (4.12)

Second, we readily see that
\[ G(x + iN, \chi) = \sum_{n=1}^{p-1} \frac{\chi(n)}{\mu^{2n}}. \] (4.13)

Third, an elementary calculation gives
\[ \sin(\pi a(x + iN)/p) = -\frac{1}{2i} \mu^a(1 - \mu^{-2a}). \] (4.14)
Fourth,
\[
\frac{1}{\sin(\pi (x + iN)/p)} = - \frac{2i}{\mu(1 - \mu^{-2})} = - \frac{2i}{\mu} \sum_{m=0}^{\infty} \mu^{-2m}. \tag{4.15}
\]

Gathering (4.12)–(4.15) together, we find that
\[
f(x + iN) = \frac{1}{G(\chi)} \sum_{n=1}^{p-1} \frac{\chi(n)}{\mu^{2n}} \mu^a (1 - \mu^{-2a}) \mu^{-1} \sum_{m=0}^{\infty} \mu^{-2m} (-1 + O(\mu^{-2p}))
\]
\[
= \frac{1}{G(\chi)} \sum_{n=1}^{p-1} \frac{\chi(n)}{\mu^{2n}} \mu^a - 1 \sum_{r=0}^{1} \beta_r \mu^{-2ar} \sum_{m=0}^{\infty} \mu^{-2m} (-1 + O(\mu^{-2p})) \tag{4.16}
\]
\[
= b_d e^{\pi Nd/p} + b_{d-1} e^{\pi N(d-1)/p} + \ldots + b_0 + b_{-1} e^{-N\pi/p} + \ldots, \tag{4.17}
\]
where
\[
\beta_0 = 1, \quad \beta_1 = -1, \tag{4.18}
\]
and where \(d = a - 3\).

Now, let \(z = x - iN, \ 0 \leq x \leq p\). Set
\[
v := e^{\pi is/p} e^{\pi N/p}.
\]

We examine each of the terms comprising \(f(x - iN)\). First,
\[
\frac{1}{e^{2\pi i(x - iN)} - 1} = \frac{1}{v^{2p} - 1} = v^{-2p} + O(\mu^{-4p}), \quad N \to \infty. \tag{4.19}
\]

Second, replacing \(n\) by \(p - n\) below, we find that
\[
G(x - iN, \chi) = \sum_{n=1}^{p-1} \chi(n)v^{2n} = \sum_{n=1}^{p-1} \chi(p - n)v^{2(p - n)} = v^{2p} \sum_{n=1}^{p-1} \frac{\chi(n)}{v^{2n}}, \tag{4.20}
\]

since \(\chi\) is even. Third, by employing the same kind of calculations that produced (4.14) and (4.15), we find that
\[
\sin(\pi a(x - iN)/p) = \frac{v^{a}}{2i}(1 - v^{-2a}) \tag{4.21}
\]
and
\[
\frac{1}{\sin(\pi (x - iN)/p)} = \frac{2i}{v} \sum_{m=0}^{\infty} v^{-2m}. \tag{4.22}
\]

Bringing (4.19)–(4.22) together, we arrive at
\[
f(x + iN) = \frac{1}{G(\chi)} v^{2p} \sum_{n=1}^{p-1} \frac{\chi(n)}{v^{2n}} v^{a}(1 - v^{-2a}) v^{-1} \sum_{m=0}^{\infty} v^{-2m} v^{-2p} (1 + O(v^{-2p}))
\]
\[
= \frac{1}{G(\chi)} v^{2p} \sum_{n=1}^{p-1} \frac{\chi(n)}{v^{2n}} v^{a - 1} \sum_{r=0}^{1} \beta_r v^{-2ar} \sum_{m=0}^{\infty} v^{-2m} (1 + O(v^{-2p})) \tag{4.23}
\]
\[
= c_d e^{\pi Nd/p} + c_{d-1} e^{\pi N(d-1)/p} + \ldots + c_0 + c_{-1} e^{-N\pi/p} + \ldots, \tag{4.24}
\]
where $\beta_r, r = 0, 1$, is defined in (4.18), and, as before, $d = a - 3$.

Recall that $C_{NT}$ and $C_{NB}$ denote those portions of $C_N$ in which $z = x + iN$ and $z = x - iN$, $0 \leq x \leq N$, respectively. By combining like powers of $e^{\pi N/p}$, and beginning with $b_d + c_d$, we successively conclude that

$$
\lim_{N \to \infty} \left( \int_{C_{NT}} b_j dx + \int_{C_{NB}} c_j dx \right) = 0, \quad d \geq j \geq 1.
$$

Trivially, as $N \to \infty$,

$$
\lim_{N \to \infty} \left( \int_{C_{NT}} b_j dx + \int_{C_{NB}} c_j dx \right) = 0, \quad j \leq -1.
$$

Hence, in conclusion, there remains only one integral to investigate, namely,

$$
\lim_{N \to \infty} \left( \int_{C_{NT}} b_0 dz + \int_{C_{NB}} c_0 dz \right) = \int_0^p b_0 dx + \int_0^p c_0 dx = -pb_0 + pc_0 = -2pb_0,
$$

(4.25)

since, from (4.16) and (4.23), $b_0 = -c_0$.

Finally, from (4.11) and (4.25),

$$
S(a, p, \chi) = -pb_0,
$$

(4.26)

where $b_0$ is the constant term in (4.16). Note that the constant term arises only from terms when

$$
a - 1 - 2m - 2n - 2ar = 0.
$$

(4.27)

Examining (4.12), (4.17), (4.19), and (4.24), we see that we are ignoring secondary terms arising from the Laurent expansions of $1/(e^{2\pi i z} - 1)$ on $C_{NT}$ and $C_{NB}$. Thus, from (4.27), we need to make the assumption that $a - 3 < 2p$. From (4.16) and (4.8), we see that $C_0 = -b_0$, and so using (4.26), we complete the proof of (4.7).

As indicated above, the hypothesis $a - 3 < 2p$ can be relaxed if further terms are employed from the Laurent series expansions (4.12) and (4.19).

In a similar manner, we can prove the following theorem. Since the details of the proof are like those of Theorem 4.1, we will forego many of them.

**Theorem 4.2** Let $\chi$ denote an even, non-principal character modulo $p$, where $p$ is odd. Let $a > 1$ denote an odd positive integer. Assume that $a - 3 < 2p$. Then,

$$
S(a, p, \chi) := \sum_{0 < n < p/2} \chi(n) \frac{\cos(a\pi n/p)}{\cos(\pi n/p)} = C_0 p,
$$

(4.28)

where $C_0$ is the constant defined by

$$
C_0 := \frac{1}{G(\chi)} \sum_{m \geq 0, n \geq 1, r \geq 0} \frac{p-1}{2m+2n+2ar=a-1} \sum_{r=0}^{1} \sum_{|\mu| < 1} \mu^{-2ar} (-1)^m \mu^{-2m} \sum_{n=0}^{\infty} \beta_r \mu^{-2ar} \mu^{-2m}.
$$

(4.29)

where $\beta_0 = \beta_1 = 1$. 


Proof Let

\[ f(z) := \frac{G(z, \chi)(\cos(a\pi z/p))}{G(\chi)(e^{2\pi i z} - 1) \cos(\pi z/p)}. \]

Integrate over the same positively oriented indented rectangle \( C_N \) as in the proof of Theorem 4.2. We see that \( f(z) \) is analytic on \( C_N \) and its interior except for simple poles at \( z = 1, 2, \ldots, p - 1 \). The residue \( R_n, 1 \leq n \leq p - 1 \), is easily seen to be

\[ R_n = \frac{G(n, \chi)(\cos(a\pi n/p))}{G(\chi)(2\pi i \cos(\pi n/p))}, \quad 1 \leq n \leq p - 1. \]

Note that because \( a \) is odd, \( f(z) \) does not have a pole at \( z = \frac{1}{2}p \). Therefore, by the residue theorem and (4.3),

\[ \int_{C_N} f(z)\,dz = \sum_{n=1}^{p-1} \chi(n) \frac{\cos(a\pi n/p)}{\cos(\pi n/p)}. \quad (4.30) \]

Since \( \chi \) is even, \( R_n = R_{p-n}, 0 < n < p/2 \). Hence, we can rewrite (4.30) in the form

\[ \int_{C_N} f(z)\,dz = 2 \sum_{0 < n < p/2} \chi(n) \frac{\cos(a\pi n/p)}{\cos(\pi n/p)} = 2S(a, p, \chi), \]

by (4.28).

We next directly calculate \( \int_{C_N} f(z)\,dz \). By periodicity, the integrals of \( f(z) \) over the vertical sides of \( C_N \) cancel. Let \( C_{NT} \) and \( C_{NB} \) denote the upper and lower sides of \( C_N \), respectively. Set

\[ \mu := e^{-\pi ix/p} e^{\pi N/p}. \]

We examine each portion of \( f(z) \) on \( C_{NT} \), where we put \( z = x + iN \). First,

\[ \frac{1}{e^{2\pi i(x+iN)} - 1} = \frac{1}{\mu^{-2p} - 1} = -1 + O(\mu^{-2p}), \quad N \to \infty. \quad (4.31) \]

Second, we readily see that

\[ G(x + iN, \chi) = \sum_{n=1}^{p-1} \frac{\chi(n)}{\mu^{2n}}. \quad (4.32) \]

Third, an elementary calculation gives

\[ \cos(\pi a(x + iN)/p) = \frac{1}{2} \mu^a(1 + \mu^{-2a}). \quad (4.33) \]

Fourth,

\[ \frac{1}{\cos(\pi(x + iN)/p)} = \frac{2}{\mu(1 + \mu^{-2})} = \frac{2}{\mu} \sum_{m=0}^{\infty} (-1)^m \mu^{-2m}. \quad (4.34) \]
Using (4.31)–(4.34), we find that

\[ f(x + iN) = \frac{1}{G(\chi)} \sum_{n=1}^{p-1} \frac{\chi(n)}{\mu^{2n}} \mu^a(1 + \mu^{-2a})\mu^{-1} \sum_{m=0}^{\infty} (-1)^m \mu^{-2m} (1 + O(\mu^{-2p})) \]

\[ = \frac{1}{G(\chi)} \sum_{n=1}^{p-1} \frac{\chi(n)}{\mu^{2n}} \mu^{a-1} \sum_{r=0}^{1} \frac{1}{\mu^{-2ar}} \sum_{m=0}^{\infty} (-1)^m \mu^{-2m} (1 + O(\mu^{-2p})) \]

\[ = b_d e^{\pi N d/p} + b_{d-1} e^{\pi N (d-1)/p} + \cdots + b_0 + b_{-1} e^{-N\pi/p} + \cdots, \quad (4.35) \]

where

\[ \beta_0 = \beta_1 = 1 \]

and where \( d = a - 3 \).

Arguing as we did in the previous proof, we find that

\[ f(x - iN) = \frac{1}{G(\chi)} \sum_{n=1}^{p-1} \frac{\chi(n)}{\mu^{2n}} \mu^a(1 + \mu^{-2a})\mu^{-1} \sum_{m=0}^{\infty} (-1)^m \mu^{-2m} (1 + O(\mu^{-2p})) \]

\[ = c_d e^{\pi N d/p} + c_{d-1} e^{\pi N (d-1)/p} + \cdots + c_0 + c_{-1} e^{-N\pi/p} + \cdots. \quad (4.36) \]

With the use of (4.35) and (4.36), the remainder of the proof follows along the same lines as the proof for Theorem 4.1.

In [2, p. 354, Theorem 5.1], the following analogue of Theorem 4.1 is proved. Suppose that \( a, b, \) and \( k \) are positive integers with \( b \) odd and greater than 1. Then,

\[ \sum_{0 < n < k/2} \frac{\sin^2(2\pi bn/k)}{\sin^2(2\pi n/k)} = -\frac{1}{2} b^a + k \sum_{m,n,r \geq 0} (-1)^n \binom{a}{m} \binom{a-1+m}{m}, \]

where in the case \( r = 0 \), the terms are to be multiplied by \( \frac{1}{2} \).

**Example 4.3** Let \( a = 3 \). There is one term satisfying the conditions \( m \geq 0, n \geq 1, r \geq 0, 2 = 2m + 2n + 2r \), namely, when \((m, n, r) = (0, 1, 0)\). Thus, using (4.8) and (4.29), respectively, and also (4.4), we conclude that

\[ \sum_{0 < n < p/2} \frac{\chi(n) \sin(3\pi n/p)}{\sin(\pi n/p)} = \sqrt{p} \]

and

\[ \sum_{0 < n < p/2} \frac{\chi(n) \cos(3\pi n/p)}{\cos(\pi n/p)} = \sqrt{p}. \]

**Example 4.4** Let \( a = 5 \). There are three terms \((m, n, r)\) satisfying the conditions \( 4 = 2m + 2n + 2r \), namely, for \((m, n, r) = (0, 2, 0), (1, 1, 0), (0, 1, 1)\). Thus,

\[ \sum_{0 < n < p/2} \frac{\chi(n) \sin(5\pi n/p)}{\sin(\pi n/p)} = \left[ \chi(2) + 1 - 1 \right] \sqrt{p} \]

\[ = \begin{cases} \sqrt{p}, & \text{if } p \equiv 1, 7 \pmod{8}, \\ -\sqrt{p}, & \text{if } p \equiv 3, 5 \pmod{8}. \end{cases} \]
and
\[ \sum_{0 < n < p/2} \chi(n) \frac{\cos(5\pi n/p)}{\cos(\pi n/p)} = \left\{ \begin{array}{l} \chi(2) - 1 + 1 \sqrt{p} \quad \text{if } p \equiv 1, 7 \pmod{8}, \\ -\sqrt{p} \quad \text{if } p \equiv 3, 5 \pmod{8}. \end{array} \right. \]

5 Sums with multiple sines

Theorem 5.1 Let \( \chi \) denote an even, non-principal character modulo the odd prime \( p \). Let \( a_1, a_2, \ldots, a_k \) denote \( k \) odd positive integers, with each \( > 1 \). Assume that \( a_1 + a_2 + \cdots + a_k - k < 2p \). Then
\[ S(a_1, a_2, \ldots, a_k, p, \chi) := \sum_{0 < n < p/2} \chi(n) \frac{\sin(a_1 \pi n/p) \sin(a_2 \pi n/p) \cdots \sin(a_k \pi n/p)}{\sin^k(\pi n/p)} = C_0p, \quad (5.1) \]
where \( C_0 \) is the constant defined by
\[ C_0 := \frac{1}{G(\chi)} \sum_{m \geq 0, n \geq 1, 0 < r \leq a_1 + a_2 + \cdots + a_k} \frac{\sum_{n=1}^{p-1} \chi(n) \mu^{2n}}{\mu^{2n}} \times \sum_{r=0}^{a_1 + a_2 + \cdots + a_k} \beta_r \mu^{-2r} \mu^{a_1 + a_2 + \cdots + a_k - k} \sum_{m=0}^{\infty} \left( \frac{k + m - 1}{k - 1} \right) \mu^{-2m}, \quad |\mu| < 1. \quad (5.2) \]

Proof We provide a sketch of the proof, since the details are similar to those in the proof of Theorem 4.1 in which \( k = 1 \). Let
\[ f(z) := \frac{G(z, \chi) \sin(a_1 \pi z/p) \sin(a_2 \pi z/p) \cdots \sin(a_k \pi z/p)}{G(\chi) (e^{2\pi i z} - 1) \sin^k(\pi z/p)}. \]

Since \( f(z) \) is analytic at the origin, its only poles on the interior of \( C_N \) are at \( z = n, 1 \leq n \leq p - 1 \). The residue of \( f \) at \( z = n, 1 \leq n \leq p - 1 \), is given by
\[ R_n = \frac{G(n, \chi) \sin(a_1 \pi n/p) \sin(a_2 \pi n/p) \cdots \sin(a_k \pi n/p)}{2\pi i \sin^k(\pi n/p)}. \]

Thus, by the residue theorem and (4.3),
\[ \int_{C_N} f(z)dz = \sum_{n=1}^{p-1} \chi(n) \frac{\sin(a_1 \pi n/p) \sin(a_2 \pi n/p) \cdots \sin(a_k \pi n/p)}{\sin^k(\pi n/p)}. \quad (5.3) \]

Since \( \chi \) is even and \( a_j \) is odd, \( 1 \leq j \leq k \),
\[ \chi(p - n) \frac{\sin(a_1 \pi (p - n)/p) \sin(a_2 \pi (p - n)/p) \cdots \sin(a_k \pi (p - n)/p)}{\sin(\pi (p - n)/p)} = \chi(n) \frac{\sin(a_1 \pi n/p) \sin(a_2 \pi n/p) \cdots \sin(a_k \pi n/p)}{\sin^k(\pi n/p)}, \quad 0 < n < p/2. \]
Hence, we can rewrite (5.3) in the form

\[
\int_{C_N} f(z) dz = 2 \sum_{0 < n < p/2} \chi(n) \frac{\sin(a_1 \pi n/p) \sin(a_2 \pi n/p) \cdots \sin(a_k \pi n/p)}{\sin^4(\pi n/p)}
\]

\[
= 2S(a_1, a_2, \ldots, a_k, p, \chi),
\]

(5.4)

by (5.1).

We now integrate \( f(z) \) over \( C_N \). First, consider \( C_{NT} \). The procedure is the same as that for Theorem 4.1. Let

\[
\mu := e^{-\pi ix/p} e^{\pi N/p}.
\]

We examine each portion of \( f(z) \) on \( C_{NT} \), where we put \( z = x + iN \). First,

\[
\frac{1}{e^{2\pi i (x+iN)} - 1} = \frac{1}{\mu^{-2p} - 1} = -1 + O(\mu^{-2p}), \quad N \to \infty.
\]

(5.5)

Also,

\[
G(x + iN, \chi) = \sum_{n=1}^{p-1} \frac{\chi(n)}{\mu^{2n}}.
\]

(5.6)

As in (4.14), we have

\[
\sin(\pi a_j (x + iN)/p) = -\frac{1}{2i} \mu^{a_j} (1 - \mu^{-2a_j}), \quad 1 \leq j \leq k.
\]

(5.7)

From (4.15),

\[
\frac{1}{\sin(\pi (x + iN)/p)} = -\frac{2i}{\mu(1 - \mu^{-2})}.
\]

(5.8)

Lastly, by the binomial theorem,

\[
(1 - \mu^{-2})^{-k} = \sum_{m=0}^\infty \binom{k + m - 1}{k - 1} \mu^{-2m}.
\]

(5.9)

Employing (5.5)–(5.9), we conclude that

\[
f(x + iN) = \frac{1}{G(\chi)} \sum_{n=1}^{p-1} \frac{\chi(n)}{\mu^{2n}} \prod_{j=1}^{k} \mu^{a_j} (1 - \mu^{-2a_j}) \mu^{-k} \sum_{m=0}^\infty \binom{k + m - 1}{k - 1} \mu^{-2m} (-1 + O(\mu^{-2p})).
\]

(5.10)

By the same procedure that we used in obtaining (4.23), we can derive a formula for \( f(x - iN) \) analogous to that in (5.10). Proceeding as before, we obtain the evaluation

\[
\lim_{N \to \infty} \int_{C_N} f(z) dz = 2pC_0,
\]

(5.11)

where the value of \( C_0 \) is given (5.2). Combining (5.4) with (5.11), we conclude our proof of Theorem 5.1. \( \Box \)
At the conclusion of the proof of Theorem 4.1, we remarked that the hypothesis $a - 3 < 2p$ could be relaxed. Note that in the proof of Theorem 5.1 to ensure that the error term does not affect the constant term, we need to make the assumption that $a_1 + a_2 + \cdots + a_k - k - 2 < 2p$. Thus, by refining the error term in (5.10), the condition $a_1 + a_2 + \cdots + a_k - k - 2 < 2p$ can be relaxed.

Theorem 5.1 is a generalization of [2, p. 346, Cor. 4.5]. Namely, if $k$ is an odd positive integer and $\chi$ is an even character modulo $k$, then

$$\sum_{0 < n < k/2} \chi(n) \frac{\sin(3\pi n/k) \sin(5\pi n/k) \sin(7\pi n/k)}{\sin^3(\pi n/k)} = \sqrt{k} \sum_{m=0}^{6} \chi(m) \left\{ \binom{8-m}{2} - \binom{5-m}{2} - \binom{3-m}{2} \right\}.$$  

6 A sum with characters and trigonometric functions with two distinct periods

**Theorem 6.1** Let $p$ and $q$ be relatively prime, positive integers. Let $\chi_1$ and $\chi_2$ denote non-principal characters modulo $p$ and $q$, respectively. Suppose that $a_1$ and $a_2$ are odd positive integers, and that $b_1$ and $b_2$ denote even positive integers. Assume that

$$q(a_1b_1 - a_1 - 3) + p(a_2b_2 - a_2 - 3) < 2pq. \quad (6.1)$$

Define

$$S(a_1, a_2, b_1, b_2, p, q, \chi_1, \chi_2) := \sum_{0 < n \leq pq-1 \atop n \neq kp, kq} \chi_1(n) \chi_2(n) \frac{\sin^{a_1}(b_1 \pi n/p) \sin^{a_2}(b_2 \pi n/q)}{\sin^{a_1+1}(\pi n/p) \sin^{a_2+1}(\pi n/q)}.$$  

Then,

$$S(a_1, a_2, b_1, b_2, p, q, \chi_1, \chi_2) = pq(-C_0 + C'_0) + 4 \frac{M(p, \chi_1)}{G(\chi_1)} \frac{M(q, \chi_2)}{G(\chi_2)} b_1^{a_1} b_2^{a_2} - 2i b_1^{a_1} M(p, \chi_1) G(\chi_1) \chi_2(p) \sum_{k=1}^{q-1} \chi_2(k) \frac{\sin^{a_2}(b_2 \pi kp/q)}{\sin^{a_2+1}(\pi kp/q)} - 2i b_2^{a_2} M(q, \chi_2) G(\chi_2) \chi_1(q) \sum_{k=1}^{p-1} \chi_1(k) \frac{\sin^{a_1}(b_1 \pi kp/p)}{\sin^{a_1+1}(\pi kp/p)}, \quad (6.2)$$

where $M(k, \chi)$ is defined in (4.5), and where $C_0$ and $C'_0$ are defined by (6.29) and (6.30), respectively.

If $\chi_1$ and $\chi_2$ are odd, then the two sums on the right-hand side of (6.2) have been evaluated by two of the present authors in [6]. It was assumed in [6] that $\chi_1$ and $\chi_2$ are odd, so that the sums could be evaluated in terms of class numbers (4.6). However, the evaluations can be expressed in terms of the sums (4.5), and so the restriction that the characters be odd is unnecessary.

**Proof** Define

$$F(z) := \frac{G(z, \chi_1) G(z, \chi_2) \sin^{a_1}(b_1 \pi z/p) \sin^{a_2}(b_2 \pi z/q)}{G(\chi_1) G(\chi_2) \sin^{a_1+1}(\pi z/p) \sin^{a_2+1}(\pi z/q) e^{2\pi iz} - 1}. \quad (6.3)$$
Let $C_N, N > 0$, denote the indented rectangle with horizontal sides passing through $\pm iN$ and vertical sides passing through 0 and $pq$, with semicircular indentations around $z = 0$ of radius $\epsilon < 1$ in the left half-plane and around $z = pq$ of radius $\epsilon < 1$ lying immediately to the left of the line $x = pq$. On the interior of $C_N$, $F(z)$ has a simple pole at $z = 0$. At $n = 1, 2, \ldots, pq - 1$, $F(z)$ also has poles. If $n \neq kp, kq$, for any integer $k$, then the poles are simple, since $(p, q) = 1$. Suppose that $n = kp, 1 \leq k \leq q - 1$. Then, $G(n, \chi_1) = 0$. Hence, $kp$ is a simple pole. Similarly, if $n = kq, 1 \leq k \leq p - 1$, then $kq$ is also a simple pole.

Let $R_n$ denote the residue of $F(z)$ at a pole $z = \alpha$. First,

$$R_0 = \lim_{z \to 0} \frac{G(z, \chi_1)}{G(\chi_1)(e^{2\pi iz} - 1)} \frac{G(z, \chi_2)}{G(\chi_2) \sin(\pi z/p) \sin(\pi z/q)} \quad \text{for any integer } n.$$  

Next, by (4.3),

$$R_n = \frac{\chi_1(n) \chi_2(n) \sin a_1(b_1 \pi n/p) \sin a_2(b_2 \pi n/q)}{2\pi i} \frac{\sin a_1+1(\pi n/p) \sin a_2+1(\pi n/q)}{(\pi/q)G(\chi_1)} \left(1 - e^{2\pi iz}\right),$$  

Third, let $n = kp, 1 \leq k \leq q - 1$. Then,

$$R_{kp} = \lim_{z \to kp} \frac{G(z, \chi_1)}{G(\chi_1) \sin(\pi z/p)} \frac{G(z, \chi_2)}{G(\chi_2) \sin(\pi z/q) \sin(\pi z/p)} \frac{z - kp}{e^{2\pi iz} - 1}$$  

Now,

$$\lim_{z \to kp} \frac{G(z, \chi_1)}{G(\chi_1) \sin(\pi z/p)} = \lim_{z \to kp} \frac{\chi_1(j) \chi_2(j) e^{2\pi iz/p} e^{2\pi iz/q}}{G(\chi_1) \sin(\pi z/p) \cos(\pi z/q)},$$  

by (4.5). Next, by (4.3),

$$\lim_{z \to kp} \frac{G(z, \chi_2)}{G(\chi_2) \sin(\pi z/q) \sin(\pi z/p)} = \chi_2(p) \chi_2(k) \frac{\sin(\pi kp/q) \sin(\pi k/p)}{\sin(\pi z/p) \sin(\pi z/q)}.$$  

Next,

$$\lim_{z \to kp} \frac{\sin a_1(b_1 \pi z/p)}{\sin a_1(\pi z/p)} = (-1)^k b_1 a_1,$$  

since $b_1$ is even. Lastly,

$$\lim_{z \to kp} \frac{z - kp}{e^{2\pi iz} - 1} = \frac{1}{2\pi i}.$$  

Gathering (6.7)–(6.10) and putting them in (6.6), we conclude that

$$R_{kp} = \frac{b_1 a_1 M(p, \chi_1)}{\pi G(\chi_1)} \chi_2(p) \chi_2(k) \frac{\sin a_2(b_2 \pi kp/q) \sin(\pi k/p)}{\sin a_2(\pi kp/q) \sin(\pi z/q)}.$$
By analogy,

\[ R_{kq} = \frac{b_2^a M(q, \chi_2)}{\pi G(\chi_2)} \chi_1(q) \chi_1(k) \frac{\sin^{a_1}(b_1 \pi kq / p)}{\sin^{a_1+1}(\pi kq / p)}, \quad 1 \leq k \leq p - 1. \]  

(6.12)

Finally, using (6.4), (6.5), (6.11), and (6.12), and the residue theorem, we conclude that

\[
\int_{C_N} F(z)dz = -4 \frac{M(p, \chi_1) M(q, \chi_2)}{G(\chi_1) G(\chi_2)} b_1^{a_1} b_2^{a_2} \\
+ \sum_{0 < n < pq - 1} \chi_1(n) \chi_2(n) \frac{\sin^{a_1}(b_1 \pi n / p)}{\sin^{a_1+1}(\pi n / p)} \frac{\sin^{a_2}(b_2 \pi n / q)}{\sin^{a_2+1}(\pi n / q)} \\
+ 2i b_1^{a_1} M(p, \chi_1) \chi_2(p) \sum_{k=1}^{q-1} \chi_2(k) \frac{\sin^{a_2}(b_2 \pi kp / q)}{\sin^{a_2+1}(\pi kp / q)} \\
+ 2i b_2^{a_2} M(q, \chi_2) \chi_1(q) \sum_{k=1}^{p-1} \chi_1(k) \frac{\sin^{a_1}(b_1 \pi kq / p)}{\sin^{a_1+1}(\pi kq / p)} \\
= I(a_1, a_2, b_1, b_2, p, q, \chi_1, \chi_2). 
\]  

(6.13)

We now calculate the integral on the left-hand side of (6.13) directly. First observe from the definition (6.3) that because \(a_1\) and \(a_2\) are odd and \(b_1\) and \(b_2\) are even, the integrals over the two vertical sides of \(C_N\) cancel.

We examine the integral over the top side, \(C_{NT}\), of \(C_N\). Let \(z = x + iN, 0 \leq x \leq pq\), and set

\[ \mu = e^{\pi i x / pq} e^{\pi N / pq}. \]

We express the functions comprising \(F(z)\) in terms of \(\mu\). First,

\[
G(z, \chi_1) = \sum_{n=0}^{p-1} \chi_1(n) \mu^{-2qn} \quad \text{and} \quad G(z, \chi_2) = \sum_{n=0}^{q-1} \chi_2(n) \mu^{-2qn}. \]  

(6.14)

Second, since \(a_1\) is odd,

\[
\sin^{a_1}(b_1 \pi z / p) = \left( \frac{\mu^{-b_1 q} - \mu^{b_1 q}}{2i} \right)^{a_1} = -\frac{1}{(2i)^{a_1}} \mu^{a_1 b_1 q} \sum_{m=0}^{a_1} (-1)^m \binom{a_1}{m} \mu^{-2b_1 q m}. 
\]  

(6.15)

Analogously,

\[
\sin^{a_2}(b_2 \pi z / q) = \left( \frac{\mu^{-b_2 p} - \mu^{b_2 p}}{2i} \right)^{a_2} = -\frac{1}{(2i)^{a_2}} \mu^{a_2 b_2 p} \sum_{m=0}^{a_2} (-1)^m \binom{a_2}{m} \mu^{-2b_2 p m}. 
\]  

(6.16)
Thus, as $a_1$ is odd,

$$
\sin^{-a_1-1}(\pi z/p) = \left(\frac{\mu^{-q} - \mu^q}{2i}\right)^{-a_1-1}
$$

$$
= (2i)^{a_1+1} \mu^{-(a_1+1)q} (1 - \mu^{-2q})^{-a_1-1}
$$

$$
= (2i)^{a_1+1} \mu^{-(a_1+1)q} \sum_{r=0}^{\infty} (-1)^r \binom{-(a_1+1)}{r} \mu^{-2qr}
$$

$$
= (2i)^{a_1+1} \mu^{-(a_1+1)q} \sum_{r=0}^{\infty} \left( (a_1 + 1) - 1 + r \right) \mu^{-2qr}, \quad (6.17)
$$

and, analogously,

$$
\sin^{-a_2-1}(\pi z/q) = \left(\frac{\mu^{-p} - \mu^p}{2i}\right)^{-a_2-1}
$$

$$
= (2i)^{a_2+1} \mu^{-(a_2+1)p} \sum_{r=0}^{\infty} \left( (a_2 + 1) - 1 + r \right) \mu^{-2pr}, \quad (6.18)
$$

Fourth,

$$
\frac{1}{e^{2\pi i z} - 1} = -1 + O(\mu^{-2pq}), \quad (6.19)
$$

as $\mu \to \infty$.

Using (6.14)–(6.19), we deduce that

$$
F(z) = F(x + iN) = \left\{ \begin{array}{l}
2i G(x_1) \mu^{q(a_1b_1-a_1-1)} \sum_{n_1=1}^{p-1} \frac{\chi_1(n_1)}{\mu^{2p_1n_1}} \sum_{m_1=0}^{a_1} \frac{m_1}{(a_1)} \mu^{-2b_1m_1} \\
\times \sum_{r_1=0}^{\infty} \left( \frac{a_1 + r_1}{r_1} \right) \mu^{-2qr_1} \\
\times \frac{2i}{G(x_2)} \mu^{p(a_2b_2-a_2-1)} \sum_{n_2=1}^{q-1} \frac{\chi_2(n_2)}{\mu^{2p_2n_2}} \sum_{m_2=0}^{a_2} \frac{m_2}{(a_2)} \mu^{-2b_2m_2} \\
\times \sum_{r_2=0}^{\infty} \left( \frac{a_2 + r_2}{r_2} \right) \mu^{-2pr_2} \\
\end{array} \right\} (-1 + O(\mu^{-2pq})), \quad (6.20)
$$

as $\mu$ tends to infinity. For brevity, we write

$$
F(z) = F(x + iN) = A_D(x)e^{D\pi N/(pq)} + A_{D-1}(x)e^{(D-1)\pi N/(pq)} + \cdots + A_0(x) + O(e^{-\pi N/(pq)}),
$$

where $A_D(x), A_{D-1}(x), \ldots, A_1(x), A_0(x)$ are functions of $x$, and

$$
D = q(a_1b_1-a_1-1) - 2p + p(a_2b_2-a_2-1) - 2q. \quad (6.21)
$$

Thus,

$$
\int_{C_{NT}} F(z)dz = B_D e^{D\pi N/(pq)} + B_{D-1} e^{(D-1)\pi N/(pq)} + \cdots + B_0 + O(e^{-\pi N/(pq)}), \quad (6.22)
$$
where $B_D, B_{D-1}, \ldots, B_1, B_0$ are constants.

The calculation of the integral over $C_{NB}$, the lower edge of $C_N$, follows along the same lines as above. Set $z = x - iN$ and define

$$\mu = e^{\pi x/(pq)} e^{\pi N/(pq)}.$$ 

Note that (6.19) needs to be replaced by

$$\frac{1}{e^{2\pi x} - 1} = \mu^{-2pq} + O(\mu^{-4pq}), \quad |\mu| \to \infty.$$ 

Then,

$$F(z) = F(x - iN) = \left\{ \frac{2i}{G(x_1)} \mu^{q(a_1b_1 - a_1 - 1)} \sum_{n_1=1}^{p-1} \frac{X_1(n_1)}{\mu^{-2qn_1}} \sum_{m_1=0}^{a_1} (-1)^{m_1} \left( \frac{a_1}{m_1} \right) \mu^{-2bn_1} 
\times \sum_{r_1=0}^{\infty} \left( \frac{a_1 + r_1}{r_1} \right) \mu^{-2qr_1} \right\}$$

$$\times \left\{ \frac{2i}{G(x_2)} \mu^{q(a_2b_2 - a_2 - 1)} \sum_{n_2=1}^{q-1} \frac{X_2(n_2)}{\mu^{-2qn_2}} \sum_{m_2=0}^{a_2} (-1)^{m_2} \left( \frac{a_2}{m_2} \right) \mu^{-2pb_2m_2} 
\times \sum_{r_2=0}^{\infty} \left( \frac{a_2 + r_2}{r_2} \right) \mu^{-2qr_2} \right\} \mu^{-2pq} + O(\mu^{-4pq}),$$

(6.23)

as $\mu$ or $N$ tends to infinity. Again, for brevity, write

$$F(z) = F(x - iN) = A'_H(x)e^{H\pi N/(pq)} + A'_{H-1}(x)e^{(H-1)\pi N/(pq)} + \cdots + A'_0(x) + O(e^{-\pi N/(pq)}),$$

where $A'_H(x), A'_{H-1}(x), \ldots, A'_0(x)$ are functions of $x$, and

$$H = q(a_1b_1 - a_1 - 1) + 2q(p - 1) + p(a_2b_2 - a_2 - 1) + 2p(q - 1) - 2pq$$

$$= q(a_1b_1 - a_1 - 1) - 2p + p(a_2b_2 - a_2 - 1) - 2q + 2pq.$$ 

(6.24)

From (6.21) and (6.24), we observe that $H = D + 2pq$. Thus,

$$\int_{C_{NB}} F(z)dz = B'_H e^{H\pi N/(pq)} + B'_{H-1} e^{(H-1)\pi N/(pq)} + \cdots + B'_0 + O(e^{-\pi N/(pq)}),$$

(6.25)

where $B'_H, B'_{H-1}, \ldots, B'_0$ are constants. Define $B_H = B_{H-1} = \cdots = B_{D+1} = 0$.

In summary, by (6.22) and (6.25), we have shown that

$$\int_{C_N} F(z)dz = (B_H + B'_H)e^{H\pi N/(pq)} + (B_{H-1} + B'_{H-1})e^{(H-1)\pi N/(pq)}$$

$$+ \cdots + (B_0 + B'_0) + O(e^{-\pi N/(pq)}),$$

(6.26)

as $N \to \infty$. Combining (6.26) and (6.13) and recalling the definition (6.13), we deduce that

$$I(a_1, a_2, b_1, b_2, p, q, \chi_1, \chi_2)$$

$$= (B_H + B'_H)e^{H\pi N/(pq)} + (B_{H-1} + B'_{H-1})e^{(H-1)\pi N/(pq)} + \cdots + (B_0 + B'_0) + O(e^{-\pi N/(pq)}),$$

(6.27)
Hence, from (6.27) and (6.28),
\[ B_H + B'_H = B_{H-1} + B'_{H-1} = \cdots = B_1 + B'_1 = 0. \] (6.28)

In summary,
\[ I(a_1, a_2, b_1, b_2, p, q, \chi_1, \chi_2) = B_0 + B'_0. \]

Returning to (6.20) and (6.22), we observe that \( B_0 = -pqC_0 \), where
\[
C_0 = \frac{4}{G(\chi_1)G(\chi_2)} \sum_{n_1,m_1,r_1,n_2,m_2,r_2 \geq 0} \mu(q(a_1 b_1 - a_1 - 1) - 2q_{n_1} - 2b_1 m_1 - 2q_{r_1} + p(a_2 b_2 - a_2 - 1) - 2p_{n_2} - 2b_2 m_2 - 2p_{r_2} = 0)
\times \sum_{r_1=0}^{\infty} \left( \frac{a_1 + r_1}{r_1} \right) \mu^{-2q_{r_1}}
\times \left\{ \mu^p(a_2 b_2 - a_2 - 1) \sum_{n_2=1}^{q-1} \frac{\chi_2(n_2)}{\mu^{2p_{n_2}}} \sum_{m_2=0}^{a_2} (-1)^{m_2} \frac{a_2}{m_2} \mu^{-2p_{b_2} m_2} \right\}
\times \sum_{r_2=0}^{\infty} \left( \frac{a_2 + r_2}{r_2} \right) \mu^{-2p_{r_2}} \right]. \] (6.29)

Returning to (6.23) and (6.25), we furthermore observe that \( B'_0 = pqC'_0 \), where
\[
C'_0 = -\frac{4}{G(\chi_1)G(\chi_2)} \sum_{n_1,m_1,r_1,n_2,m_2,r_2 \geq 0} \mu(q(a_1 b_1 - a_1 - 1) + 2q_{n_1} - 2b_1 m_1 - 2q_{r_1} + p(a_2 b_2 - a_2 - 1) + 2p_{n_2} - 2b_2 m_2 - 2p_{r_2} - 2pq = 0)
\times \sum_{r_1=0}^{\infty} \left( \frac{a_1 + r_1}{r_1} \right) \mu^{-2q_{r_1}}
\times \left\{ \mu^p(a_2 b_2 - a_2 - 1) \sum_{n_2=1}^{q-1} \frac{\chi_2(n_2)}{\mu^{2p_{n_2}}} \sum_{m_2=0}^{a_2} (-1)^{m_2} \frac{a_2}{m_2} \mu^{-2p_{b_2} m_2} \right\}
\times \sum_{r_2=0}^{\infty} \left( \frac{a_2 + r_2}{r_2} \right) \mu^{-2p_{r_2}} \right]. \] (6.30)

In summary,
\[ I(a_1, a_2, b_1, b_2, p, q, \chi_1, \chi_2) = pq(-C_0 + C'_0), \] (6.31)

where \( C_0 \) and \( C'_0 \) are given by (6.29) and (6.30), respectively.
Lastly, in our calculations of both $C_0$ and $C'_0$, we ignored the big-$O$ terms arising from expanding
\[
\frac{1}{e^{2\pi iz} - 1}
\] (6.32)
in powers of $\mu$. In each case, to do this, we require that
\[
q(a_1b_1 - a_1 - 3) + p(a_2b_2 - a_2 - 3) < 2pq. 
\] (6.33)

If we combine (6.31) with (6.13), we arrive at (6.2).

In our proof, we used only the first term when expanding (6.32) in a geometric series. If we employed additional terms, this would enable us to weaken the hypothesis (6.33), (6.1) and prove a stronger theorem.

One could also prove a theorem involving a sum of products of $n$ sines. The ‘evaluation’ would be in terms of sums with the quotients of $1, 2, \ldots, n - 1$ sines in the summands. One of the technical difficulties in the proof would be the calculation of the residue at $z = 0$. Also, there would be poles at integers with different numbers of prime factors in them, corresponding to sums with different numbers of sine products in their summands.

7 Preliminary lemmas

In the remainder of this paper, we offer another approach for evaluating and finding reciprocity theorems for trigonometric sums. Previous authors have employed partial fractions to evaluate trigonometric sums. However, our systematic approach through sums involving roots of unity is different and leads to new results. Although some of our trigonometric evaluations and reciprocity theorems in the following sections can be established by either of the two principal methods in this paper, there are others, such as Theorems 10.1 and 12.1, for which the approach by contour integration might not be feasible, or at least not as facile.

**Lemma 7.1** Let $k$ be an odd positive integer and $z_n = e^{2\pi in/k}$. If $k \equiv 1 \pmod{4}$, then
\[
\sum_{n=1}^{k} \frac{1}{z_n \pm i} = \pm \frac{k}{i \pm 1},
\]
\[
\sum_{n=1}^{k} \frac{1}{(z_n \pm i)^2} = \frac{k(k - 1)i}{i \pm 1} \pm \frac{k^2i}{2},
\]
\[
\sum_{n=1}^{k} \frac{1}{(z_n \pm i)^3} = \frac{\mp k(k - 1)(k - 2) - k^3i}{2(i \pm 1)} \pm \frac{3k^2(k - 1)}{4},
\]
\[
\sum_{n=1}^{k} \frac{1}{(z_n \pm i)^4} = \frac{-k(k - 1)(k - 2)(k - 3)i}{6(i \pm 1)} \pm \frac{k^3(k - 1)}{i \pm 1} \pm \frac{k^2(k - 1)(7k - 11)i}{12} - \frac{k^4}{4}.
\]
If $k \equiv 3 \pmod{4}$, then we have
\[
\sum_{n=1}^{k} \frac{1}{z_n \pm i} = \pm \frac{k}{i \mp 1},
\]
\[
\sum_{n=1}^{k} \frac{1}{(z_n \pm i)^2} = \frac{k(k - 1)i}{i \pm 1} \pm \frac{k^2i}{2},
\]
Proof\ Let\ \(k\)\ be\ a\ positive\ integer\ and\ \(z_n = e^{2\pi i n/k}\),\ we\ obtain\ the\ identities,\n
\[
\sum_{n=1}^{k} \frac{1}{z_n + 1} = \frac{k(1 - 2)k^3 - 3k^2(k - 1)}{2i + 1} - \frac{3k^2(k - 1)}{4},
\]
\[
\sum_{n=1}^{k} \frac{1}{(z_n + 1)^2} = -k(1 - 2)k^3i + \frac{k^3(k - 1)}{i + 1} + \frac{k^2(k - 1)(7k - 11)i}{12} - \frac{k^4}{4}.
\]

**Lemma 7.2** Let \(k\) be an odd positive integer and \(z_n = e^{2\pi i n/k}\). Then,
\[
\sum_{n=1}^{k-1} \frac{1}{z_n + 1} = \frac{k - 1}{2}, \quad \sum_{n=1}^{k-1} \frac{1}{(z_n + 1)^2} = -\frac{(k - 1)^2}{4}, \quad \sum_{n=1}^{k-1} \frac{1}{(z_n + 1)^3} = \frac{(k - 1)(3k - 1)}{8}, \quad \sum_{n=1}^{k-1} \frac{1}{(z_n + 1)^4} = \frac{(k - 1)(k^3 + k^2 - 21k + 3)}{48}.
\]

**Proof** Setting \(x = -1\) in the identities in (7.1), we can derive
\[
\sum_{n=1}^{k-1} \frac{1}{z_n + 1} = \frac{k - 1}{2}, \quad \sum_{n=1}^{k-1} \frac{1}{(z_n + 1)^2} = -\frac{(k - 1)^2}{4}, \quad \sum_{n=1}^{k-1} \frac{1}{(z_n + 1)^3} = \frac{(k - 1)(3k - 1)}{8}, \quad \sum_{n=1}^{k-1} \frac{1}{(z_n + 1)^4} = \frac{(k - 1)(k^3 + k^2 - 21k + 3)}{48}.
\]

This completes our proof.\n
**Lemma 7.3** Let \(k\) be a positive integer and \(z_n = e^{2\pi i n/k}\). Then,
\[
\sum_{n=1}^{k-1} \frac{1}{z_n - 1} = -\frac{k - 1}{2}, \quad \sum_{n=1}^{k-1} \frac{1}{(z_n - 1)^2} = -\frac{(k - 1)(k - 5)}{12},
\]

Proof Setting \(x = i\) and \(x = -i\) in these identities, we derive the identities of Lemma 7.1.\n
This completes our proof.
\[
\sum_{n=1}^{k-1} \frac{1}{(zn - 1)^3} = \frac{(k - 1)(k - 3)}{8}, \quad \sum_{n=1}^{k-1} \frac{1}{(zn - 1)^4} = \frac{(k - 1)(k^3 + k^2 - 109k + 251)}{720}.
\]

**Proof** Defining
\[
g(x) := \frac{x^k - 1}{x - 1} = (x - z_1) \cdots (x - z_{k-1}) = \sum_{j=0}^{k-1} x^j,
\]
we find that
\[
g(1) = k, \quad g'(1) = \frac{k(k - 1)}{2}, \quad g''(1) = \frac{k(k - 1)(k - 2)}{3},
\]
\[
g'''(1) = \frac{k(k - 1)(k - 2)(k - 3)}{4}, \quad g^{(4)}(1) = \frac{k(k - 1)(k - 2)(k - 3)(k - 4)}{5}.
\]
Therefore, we see that
\[
\sum_{n=1}^{k-1} \frac{1}{zn - 1} = \left[ \frac{g'''}{g} \right]_{x=1} = \frac{k - 1}{2},
\]
\[
\sum_{n=1}^{k-1} \frac{1}{(zn - 1)^2} = \left[ \left( \frac{g'}{g} \right)' \right]_{x=1} = \left[ \frac{(g')^2 - gg''}{g^2} \right]_{x=1} = \frac{(k - 1)(k - 5)}{12},
\]
\[
\sum_{n=1}^{k-1} \frac{1}{(zn - 1)^3} = \frac{1}{2} \left[ \left( \frac{g'}{g} \right)'' \right]_{x=1} = \left[ \frac{g''' - 2g'g''}{g^3} \right]_{x=1} = \frac{(k - 1)(k - 3)}{8},
\]
\[
\sum_{n=1}^{k-1} \frac{1}{(zn - 1)^4} = \frac{1}{6} \left[ \left( \frac{g'}{g} \right)''' \right]_{x=1} = \left[ \frac{g'''' - 3g'g''' + (g')^3}{6g^4} \right]_{x=1} = \frac{(k - 1)(k - 3)(k - 4)}{720}.
\]

This completes the proof. \(\square\)

### 8 Explicit evaluations of several trigonometric sums

We now demonstrate that the lemmas in the previous two sections can be used to derive several explicit evaluations of trigonometric sums. We first provide a second proof of Corollary 2.2.

**Proof** We let
\[
S = \sum_{n=1}^{k} \frac{\cos^2(\pi n/k)}{\cos^4(2\pi n/k)}, \quad z_n = e^{2\pi i n/k}, \quad \text{and} \quad R(x) = 4 \frac{x^3(x + 1)^2}{(x^2 + 1)^4}.
\]

Note that
\[
z_n + \frac{1}{z_n} = 2 \cos(2\pi n/k) = 4 \cos^2(\pi n/k) - 2.
\]

Thus,
\[
R(z_n) = 4 \frac{z_n + 2 + 1/z_n}{(z_n + 1/z_n)^4} = \frac{\cos^2(\pi n/k)}{\cos^4(2\pi n/k)}, \quad \text{and} \quad S = \sum_{n=1}^{k} R(z_n).
\]
We can check that $R(x)$ has the following partial fraction decomposition:

\[
R(x) = \frac{i}{4(x+i)} - \frac{1 - 3i}{4(x+i)^2} + \frac{1 + 2i}{2(x+i)^3} + \frac{1}{2(x+i)^4} - \frac{i}{4(x-i)} - \frac{1 + 3i}{4(x-i)^2} + \frac{1 - 2i}{2(x-i)^3} + \frac{1}{2(x-i)^4}.
\]

Therefore, we have

\[
S = \sum_{n=1}^{k} R(z_n) = \frac{i}{4} \sum_{n=1}^{k} \left( \frac{1}{z_n + i} - \frac{1}{z_n - i} \right) - \frac{1}{4} \sum_{n=1}^{k} \left( \frac{1}{(z_n + i)^2} + \frac{1}{(z_n - i)^2} \right) + \frac{3i}{4} \sum_{n=1}^{k} \left( \frac{1}{(z_n + i)^3} - \frac{1}{(z_n - i)^3} \right) + \frac{1}{2} \sum_{n=1}^{k} \left( \frac{1}{(z_n + i)^4} + \frac{1}{(z_n - i)^4} \right).
\]

By Lemma 7.1, for $k \equiv 1 \pmod{4}$,

\[
S = \frac{ik}{4} \left( \frac{1}{i+1} + \frac{1}{i-1} \right) - \frac{k(k-1)}{4} \left( \frac{1}{i+1} + \frac{1}{i-1} \right) - \frac{3k(k-1)}{4} \left( \frac{1}{i+1} - \frac{1}{i-1} \right) + \frac{3k^2}{4} \left( \frac{1}{i+1} + \frac{1}{i-1} \right) - \frac{k(k-1)(k-2)}{4} \left( \frac{1}{i+1} - \frac{1}{i-1} \right) - \frac{k^3}{4} \left( \frac{1}{i+1} + \frac{1}{i-1} \right) + \frac{k^3(k-1)}{2} \left( \frac{1}{i+1} - \frac{1}{i-1} \right) - \frac{k(k-1)(k-2)(k-3)i}{12} \left( \frac{1}{i+1} + \frac{1}{i-1} \right) + \frac{k^3(k-1)}{2} \left( \frac{1}{i+1} - \frac{1}{i-1} \right) - \frac{k^4}{4}.
\]

\[
= \frac{k}{4} + \frac{k^2}{3} + \frac{k^3}{4} + \frac{k^4}{6}.
\]

For $k \equiv 3 \pmod{4}$, the argument is similar, and so we forego it.

This completes our proof. \qed

**Corollary 8.1** If $k$ is an odd positive integer, then

\[
\sum_{n=1}^{k} \cos^4\left(\frac{2\pi n}{k}\right) = \sum_{n=1}^{k} \cos^4\left(\frac{n\pi}{k}\right) = \frac{k^2(k^2+2)}{3}.
\]

**Proof** Since $k$ is odd, we can easily check that

\[
\sum_{n=1}^{k} \cos^4\left(\frac{2\pi n}{k}\right) = \sum_{n=1}^{k} \cos^4\left(\frac{n\pi}{k}\right).
\]
Similarly, we let
\[ z_n = e^{2\pi i n/k} \quad \text{and} \quad R(x) = \frac{16x^2}{(x + 1)^4}. \]

Then, by (8.1) and Lemma 7.2, we deduce that
\[
\begin{align*}
\sum_{n=1}^{k} \frac{1}{\cos^4(\pi n/k)} &= \sum_{n=1}^{k} R(z_n) = 16 \sum_{n=1}^{k} \left\{ \frac{1}{(z_n + 1)^2} - \frac{2}{(z_n + 1)^3} + \frac{1}{(z_n + 1)^4} \right\} \\
&= 16 \left\{ \frac{(k - 1)^2}{4} + \frac{(k - 1)(3k - 1)}{4} + \frac{(k - 1)(k^2 + k^2 - 21k + 3)}{48} + \frac{1}{16} \right\} \\
&= \frac{k^2(k^2 + 2)}{3}.
\end{align*}
\]

\[ \square \]

**Corollary 8.2** If \( k \) is an odd positive integer, then
\[
\sum_{n=1}^{k} \frac{\sin^2(\pi n/k)}{\cos^4(2\pi n/k)} = \begin{cases} 
\frac{k^4}{6} - \frac{k^3}{4} + \frac{k^2}{3} - \frac{k}{4}, & \text{if } k \equiv 1 \pmod{4}, \\
\frac{k^4}{6} + \frac{k^3}{4} + \frac{k^2}{3} + \frac{k}{4}, & \text{if } k \equiv 3 \pmod{4}.
\end{cases}
\]

**Proof** This result is an immediate consequence of Corollaries 2.2 and 8.1. \[ \square \]

We provide a brief sketch of another proof of Corollary 2.6. Set
\[
S = \sum_{n=1}^{k-1} \frac{\sin^2(\pi n/k)}{\sin^4(2\pi n/k)} , \quad z_n = e^{2\pi i n/k}, \quad \text{and} \quad R(x) = -\frac{4x^3}{(x - 1)^2(x + 1)^3}.
\]

Observe that
\[
\cos^2(\pi n/k) = \frac{(z_n + 1)^2}{4z_n}, \quad \sin^2(\pi n/k) = 1 - \cos^2(\pi n/k) = -\frac{(z_n - 1)^2}{4z_n}.
\]

This implies that
\[
\frac{\sin^2(\pi n/k)}{\sin^4(2\pi n/k)} = -\frac{4z_n^3}{(z_n - 1)^2(z_n + 1)^4} = R(z_n).
\]

Thus, using partial fractions, we find that
\[
S = \sum_{n=1}^{k-1} R(z_n) = \sum_{n=1}^{k-1} \left\{ \frac{1}{4(z_n + 1)} + \frac{3}{4(z_n + 1)^2} - \frac{2}{(z_n + 1)^3} + \frac{1}{(z_n + 1)^4} \\
- \frac{1}{4(z_n - 1)} - \frac{1}{4(z_n - 1)^2} \right\}.
\]

Employing Lemmas 7.2 and 7.3 in (8.3), we arrive at
\[
S = \frac{k - 1}{4} - \frac{3(k - 1)^2}{16} + \frac{(k - 1)(3k - 1)}{4} + \frac{(k - 1)(k^3 + k^2 - 21k + 3)}{48} \\
+ \frac{(k - 1)(k - 5)}{48} = \frac{k^4}{48} + \frac{k^2}{8} - \frac{7}{48}.
\]

This completes our second proof.
Corollary 8.3 Let \( k \) be a positive integer. Then,

\[
\sum_{n=1}^{k-1} \frac{1}{\sin^4(\pi n/k)} = \frac{1}{45}(k^4 + 10k^2 - 11).
\]

Proof We define

\[ z_n = e^{2\pi in/k} \quad \text{and} \quad R(x) = \frac{16x^2}{(x-1)^4}. \]

Note that

\[ R(x) = 16 \left\{ \frac{1}{(x-1)^2} + \frac{2}{(x-1)^3} + \frac{1}{(x-1)^4} \right\}, \]

so that by Lemma 7.3 and (8.2),

\[
\sum_{n=1}^{k-1} \frac{1}{\sin^4(\pi n/k)} = \sum_{n=1}^{k-1} R(z_n) = 16 \sum_{n=1}^{k-1} \left\{ \frac{1}{(z_n-1)^2} + \frac{2}{(z_n-1)^3} + \frac{1}{(z_n-1)^4} \right\}
\]

\[
= -\frac{4(k-1)(k-5)}{3} + 4(k-1)(k-3)
\]

\[
+ \frac{1}{45}(k-1)(k^3 + k^2 - 109k + 251)
\]

\[
= \frac{1}{45}(k^4 + 10k^2 - 11).
\]

\( \square \)

Corollary 8.4 If \( k \) is an odd positive integer, then

\[
\sum_{n=1}^{k-1} \frac{1}{\sin^4(2\pi n/k)} = \frac{1}{45}(k^4 + 10k^2 - 11).
\]

Proof Corollary 8.4 follows from Corollary 8.3.

\( \square \)

From Corollaries 2.6 and 8.4, we can easily derive the following corollary.

Corollary 8.5 If \( k \) is an odd positive integer, then

\[
\sum_{n=1}^{k-1} \frac{\cos^2(\pi n/k)}{\sin^4(2\pi n/k)} = \frac{1}{720}(k^4 + 70k^2 - 71).
\]

Corollary 8.6 If \( k \) is a positive integer, then

\[
\sum_{n=1}^{k-1} \frac{1}{\sin^2(\pi n/k)} = \frac{k^2 - 1}{3}.
\]
Proof If we define
\[ z_n = e^{2\pi i n/k} \quad \text{and} \quad R(x) = -\frac{4x}{(x - 1)^2}, \]
then by (8.2),
\[ \sum_{n=1}^{k-1} \frac{1}{\sin^2(\pi n/k)} = \sum_{n=1}^{k-1} R(z_n). \]

Upon expanding \( R(z_n) \) into partial fractions and using Lemma 7.3, we may easily finish the proof. \( \square \)

See [5, p.365] for another proof of Corollary 8.6.

**Corollary 8.7** For each positive integer \( k \),
\[ S_3(1,k) := -\frac{3}{8k^3} \sum_{n=1}^{k-1} \cot^2(\pi n/k) \csc^2(\pi n/k) = -\frac{(k^2 - 1)(k^2 - 4)}{120k^3}. \]

**Proof** From Corollaries 8.6 and 8.3, it follows that
\[
S_3(1,k) = -\frac{3}{8k^3} \sum_{n=1}^{k-1} \frac{\cos^2(\pi n/k)}{\sin^2(\pi n/k)} = -\frac{3}{8k^3} \sum_{n=1}^{k-1} \left\{ \frac{1}{\sin^3(\pi n/k)} - \frac{1}{\sin^2(\pi n/k)} \right\} = \frac{1}{8k^3} \left\{ \frac{1}{15} (k^4 + 10k^2 - 11) - (k^2 - 1) \right\} = -\frac{(k^2 - 1)(k^2 - 4)}{120k^3}. \]
\( \square \)

An evaluation related to that in Corollary 8.7 can be found in [5, p.367, Corollary 2.7].

**Corollary 8.8** For each odd positive integer \( k \),
\[ S_3(2,k) := -\frac{3}{8k^3} \sum_{n=1}^{k-1} \cot(2\pi n/k) \cot(\pi n/k) \csc^2(\pi n/k) = -\frac{(k^2 - 1)(k^2 - 19)}{240k^3}. \]

**Proof** Analogously, by Corollaries 8.6 and 8.3, we have
\[
S_3(2,k) = -\frac{3}{8k^3} \sum_{n=1}^{k-1} \frac{\cos(2\pi n/k) \cos(\pi n/k)}{\sin(2\pi n/k) \sin^3(\pi n/k)} = -\frac{3}{8k^3} \sum_{n=1}^{k-1} \frac{\cos(2\pi n/k)}{2 \sin^3(\pi n/k)} \\
= -\frac{3}{8k^3} \sum_{n=1}^{k-1} \left\{ \frac{1}{2 \sin^4(\pi n/k)} - \frac{1}{\sin^2(\pi n/k)} \right\} = \frac{1}{8k^3} \left\{ \frac{1}{30} (k^4 + 10k^2 - 11) - (k^2 - 1) \right\} = -\frac{(k^2 - 1)(k^2 - 19)}{240k^3}. \]
\( \square \)
In fact, $S_3(1, k)$ and $S_3(2, k)$ are generalized Dedekind sums, which we now define. Let $B_3(x)$ denote the third Bernoulli polynomial. Then, the generalized Dedekind sum $S_3(h, k)$ is defined by [23, p. 198]

$$S_3(h, k) = \sum_{n=1}^{k-1} \frac{n}{k} B_3 \left( \frac{hn}{k} - \left\lfloor \frac{hn}{k} \right\rfloor \right)$$

$$= -\frac{3}{8k^3} \sum_{n=1}^{k-1} \cot(h\pi n/k) \cot(\pi n/k) \csc^2(\pi n/k),$$

(8.4)

where $[x]$ is the greatest integer $\leq x$. We have therefore used McIntosh’s formulation involving generalized Dedekind sums [23, p. 199, Equation (6)] in our expressions of Corollaries 8.7 and 8.8.

**Theorem 8.9** If $k$ is an odd positive integer, then

$$\sum_{n=1}^{k} \frac{1}{\cos^2(\pi n/k)} = k^2.$$  

**Proof** Define

$$z_n := e^{2\pi in/k} \quad \text{and} \quad R(x) := \frac{4x}{(x+1)^2}.$$  

If we employ Lemma 7.2 to $R(z_n)$, we readily complete the proof \(\square\)

The problem of finding a general method for evaluating

$$\sum_{n=1}^{k-1} \frac{1}{\cos^{2m}(\pi n/k)}, \quad m \geq 1,$$

(8.5)

was apparently first raised by Gardner [18] and Fisher [14]. For each integer $m \geq 1$, He [22] established a generating function for (8.5), providing their values in terms of polynomials in $k$. Moreover, Y. He also showed that

$$\sum_{n=1}^{k-1} \frac{1}{\sin^{2m}(\pi n/k)}$$

(8.6)

can be evaluated in terms of a certain sequence of polynomials in the variable $k$. Earlier, Chu and Marini [11] established a generating function for the general sum (8.5). See also further papers by Chu [8], [10] and Hassan [21].

da Fonseca et al. [15], [16] developed methods for determining values of (8.5) and (8.6). These authors study further trigonometric sums, and their paper also provides an excellent survey of a broad expanse of related literature.

Via generating functions, Cvijović [12] also showed how to evaluate

$$\sum_{j=1}^{k} \csc^{2n} \left( \frac{(2j-1)\pi}{4k} \right) = \sum_{j=1}^{k} \sec^{2n} \left( \frac{(2j-1)\pi}{4k} \right), \quad n \geq 1.$$  

See a paper by Wang and Zheng [30] for evaluations of several further classes of trigonometric sums.
Corollary 8.10 For each odd positive integer \( k \),
\[
\sum_{n=1}^{k} \tan^2(\pi n/k) \sec^2(\pi n/k) = \frac{k^2(k^2 - 1)}{3}.
\]

Proof Using Corollaries 8.9 and 8.1, we derive
\[
\sum_{n=1}^{k} \tan^2(\pi n/k) \sec^2(\pi n/k) = \sum_{n=1}^{k} \frac{\sin^2(\pi n/k) \cos^4(\pi n/k)}{\cos^6(\pi n/k)}
= \frac{k^2(k^2 + 2)}{3} - k^2 = \frac{k^2(k^2 - 1)}{3}.
\]

Corollary 8.11 For each odd positive integer \( k \),
\[
\sum_{n=1}^{k} \cos(2\pi n/k) \sec^4(\pi n/k) = -\frac{k^2(k^2 - 4)}{3}.
\]

Proof From Corollaries 8.9 and 8.1, it follows that
\[
\sum_{n=1}^{k} \cos(2\pi n/k) \sec^4(\pi n/k) = \sum_{n=1}^{k} \frac{2}{\cos^2(\pi n/k)} - \frac{1}{\cos^4(\pi n/k)}
= 2k^2 - \frac{k^2(k^2 + 2)}{3} = -\frac{k^2(k^2 - 4)}{3}.
\]

9 Further lemmas on sums of rational functions of roots of unity
Applying Lemma 7.3 and partial fraction decompositions, we can readily derive the following identities.

Lemma 9.1 If \( k \) is a positive integer and \( z_n = e^{2\pi in/k} \), then
\[
\sum_{n=1}^{k-1} \frac{z_n}{z_n - 1} = \frac{k - 1}{2},
\]
\[
\sum_{n=1}^{k-1} \frac{z_n^2}{(z_n - 1)^2} = \frac{(k - 1)(k - 5)}{12},
\]
\[
\sum_{n=1}^{k-1} \frac{z_n^2}{(z_n - 1)^3} = \frac{(k - 1)(k + 1)}{24},
\]
\[
\sum_{n=1}^{k-1} \frac{z_n}{(z_n - 1)^4} = \frac{(k - 1)(k + 1)(k^2 - 19)}{720},
\]
\[
\sum_{n=1}^{k-1} \frac{z_n^3}{(z_n - 1)^4} = \frac{(k - 1)(k + 1)(k^2 - 19)}{720},
\]
\[
\sum_{n=1}^{k-1} \frac{z_n^4}{(z_n - 1)^4} = \frac{(k - 1)(k^3 + k^2 - 109k + 251)}{720}.
\]
Lemma 9.2 \textit{Let} \( p \) \textit{and} \( q \) \textit{be positive integers with} \( p, q \geq 2 \) \textit{and} \((p, q) = 1\). \textit{If} \( \omega_j = e^{2\pi ij/p} \) \textit{and} \( \xi_j = e^{2\pi ij/q} \), \textit{then}

\[
\frac{x(x^p + 1)(x^q + 1)}{(x - 1)^2(x^p - 1)(x^q - 1)} = \frac{x^2 + q^2 + 1}{3pq(x - 1)^2} + \frac{p^2 + q^2 + 13}{3pq(x - 1)^3} + \frac{8}{pq(x - 1)^4} + \frac{4}{pq(x - 1)^5} + \frac{2}{p} \sum_{j=1}^{p-1} \frac{a_j^2(\omega_j^q + 1)}{(\omega_j - 1)^2(\omega_j^q - 1)(x - \omega_j)} + \frac{2}{q} \sum_{j=1}^{q-1} \frac{\xi_j^2(\xi_j^p + 1)}{(\xi_j - 1)^2(\xi_j^p - 1)(x - \xi_j)}.
\]

\textbf{Proof} \textit{We first note that by taking logarithmic derivatives, we can easily derive}

\[
\frac{1}{x^p - 1} = \frac{1}{p} \sum_{j=1}^{p} \frac{\omega_j}{x - \omega_j} \quad \text{and} \quad \frac{1}{x^q - 1} = \frac{1}{q} \sum_{j=1}^{q} \frac{\xi_j}{x - \xi_j}.
\]

\textit{So, we have}

\[
\frac{x(x^p + 1)(x^q + 1)}{(x - 1)^2(x^p - 1)(x^q - 1)} = \frac{x}{(x - 1)^2} \left(1 + \frac{2}{p} \sum_{j=1}^{p} \frac{\omega_j}{x - \omega_j}\right) \left(1 + \frac{2}{q} \sum_{j=1}^{q} \frac{\xi_j}{x - \xi_j}\right)
\]

\[
= \frac{x}{(x - 1)^2} \left(1 + \frac{2}{p} \sum_{j=1}^{p} \frac{\omega_j}{x - \omega_j}\right) + \frac{2}{p} \sum_{j=1}^{p} \frac{\omega_j x}{(x - 1)^2(x - \omega_j)} + \frac{2}{q} \sum_{j=1}^{q-1} \frac{\xi_j x}{(x - 1)^2(x - \xi_j)} + \frac{4}{pq} \sum_{j=1}^{p-1} \frac{\omega_j x}{(x - 1)^3(x - \omega_j)} + \frac{4}{pq} \sum_{j=1}^{q-1} \frac{\xi_j x}{(x - 1)^3(x - \xi_j)} + \frac{2}{pq} \sum_{j=1}^{p-1} \frac{\omega_j \xi_j x}{(x - 1)^2(x - \omega_j)(x - \xi_j)}.
\]

\textit{We divide the far right side of (9.2) into six parts,} \( A_1, A_2, A_3, A_4, A_5, A_6 \), \textit{and examine each separately.}

\textit{First, we consider the following partial fraction decompositions:}

\[
\frac{x}{(x - 1)^2} = \frac{1}{x - 1} + \frac{1}{(x - 1)^2},
\]

\[
\frac{x}{x(x - a)} = \frac{a}{(a - 1)^2(x - 1)} - \frac{1}{(a - 1)(x - 1)^2} + \frac{a}{(a - 1)^2(x - a)},
\]

\[
\frac{x}{(x - 1)^3(x - a)} = -\frac{1}{(a - 1)^2(x - 1)^3} + \frac{a}{(a - 1)^2(x - a)},
\]

\[
\frac{x}{(x - 1)^2(x - a)(x - b)} = -\frac{ab - 1}{(a - 1)^2(b - 1)^2(x - 1)} + \frac{1}{(a - 1)(b - 1)(x - 1)^2} + \frac{a}{(a - 1)^2(a - b)(x - a)} + \frac{b}{(b - 1)^2(b - a)(x - b)}.
\]
By (9.3),

\[ A_1 := \frac{x}{(x-1)^2} + \frac{2x}{p(x-1)^3} + \frac{2x}{q(x-1)^3} + \frac{4x}{pq(x-1)^4} \]
\[ = \frac{1}{x-1} + \left(1 + \frac{2}{p} + \frac{2}{q}\right) \frac{1}{(x-1)^2} + \left(\frac{2}{p} + \frac{2}{q}\right) \frac{1}{(x-1)^3} + \frac{1}{pq(x-1)^4}. \]  

(9.7)

Next, using (9.4), we have

\[ A_2 := \frac{2}{p} \sum_{j=1}^{p-1} \frac{\omega_j x}{(x-1)^2(x-\omega_j)} \]
\[ = \frac{2}{p} \sum_{j=1}^{p-1} \left\{ -\frac{\omega_j^2}{(\omega_j-1)^2(x-1)} - \frac{\omega_j}{(\omega_j-1)(x-1)^2} + \frac{\omega_j^2}{(\omega_j-1)^2(x-\omega_j)} \right\}, \]

(9.8)

and

\[ A_3 := \frac{2}{q} \sum_{j=1}^{q-1} \frac{\xi_j x}{(x-1)^2(x-\xi_j)} \]
\[ = \frac{2}{q} \sum_{j=1}^{q-1} \left\{ -\frac{\xi_j^2}{(\xi_j-1)^2(x-1)} - \frac{\xi_j}{(\xi_j-1)(x-1)^2} + \frac{\xi_j^2}{(\xi_j-1)^2(x-\xi_j)} \right\}. \]

(9.9)

By (9.5) and (9.6), we successively obtain

\[ A_4 := \frac{4}{pq} \sum_{j=1}^{p-1} \frac{\omega_j x}{(x-1)^3(x-\omega_j)} \]
\[ = \frac{4}{pq} \sum_{j=1}^{p-1} \left\{ -\frac{\omega_j^2}{(\omega_j-1)^3(x-1)} - \frac{\omega_j^2}{(\omega_j-1)^2(x-1)^2} - \frac{\omega_j}{(\omega_j-1)(x-1)^3} + \frac{\omega_j^2}{(\omega_j-1)^2(x-\omega_j)} \right\}, \]

(9.10)

\[ A_5 := \frac{4}{pq} \sum_{j=1}^{q-1} \frac{\xi_j x}{(x-1)^3(x-\xi_j)} \]
\[ = \frac{4}{pq} \sum_{j=1}^{q-1} \left\{ -\frac{\xi_j^2}{(\xi_j-1)^3(x-1)} - \frac{\xi_j^2}{(\xi_j-1)^2(x-1)^2} - \frac{\xi_j}{(\xi_j-1)(x-1)^3} + \frac{\xi_j^2}{(\xi_j-1)^2(x-\xi_j)} \right\}, \]

(9.11)

and

\[ A_6 := \frac{4}{pq} \sum_{j=1}^{p-1} \sum_{l=1}^{q-1} \frac{\omega_l \xi_j x}{(x-1)^2(x-\omega_l)(x-\xi_j)} = \frac{4}{pq} \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} \left\{ -\frac{\omega_i \xi_j}{(\omega_i-1)(\xi_j-1)(x-1)^2} + \frac{\omega_i^2 \xi_j}{(\omega_i-1)^2(\xi_j-1)(x-\omega_i)} + \frac{\omega_i \xi_j^2}{(\xi_j-1)^2(\xi_j-\omega_i)(x-\xi_j)} \right\}. \]

(9.12)
We now substitute (9.7)–(9.12) into (9.2). Then, we calculate the partial fraction decompositions of \( \frac{1}{(x - 1)^n} \), \( n = 1, 2, 3, 4 \), and \( \frac{1}{x - \omega_j}, \frac{1}{x - \xi_j} \), all of which now appear in the new representation of (9.2).

First, by Lemma 9.1, the coefficient of \( \frac{1}{x - 1} \) is

\[
1 - \frac{2}{p} \sum_{j=1}^{p-1} \frac{\omega_j^2}{(\omega_j - 1)^2} - \frac{2}{q} \sum_{j=1}^{q-1} \frac{\xi_j^2}{(\xi_j - 1)^2} - \frac{4}{pq} \sum_{j=1}^{p-1} \frac{\omega_j^2}{(\omega_j - 1)^3} - \frac{4}{pq} \sum_{j=1}^{q-1} \frac{\xi_j^2}{(\xi_j - 1)^3} + \frac{4}{pq} \left\{ \sum_{i=1}^{p-1} \frac{\omega_i}{(\omega_i - 1)^2} \sum_{j=1}^{q-1} \frac{\xi_j^2}{(\xi_j - 1)^2} - \sum_{i=1}^{p-1} \frac{\omega_i}{(\omega_i - 1)^3} \sum_{j=1}^{q-1} \frac{\xi_j}{(\xi_j - 1)^2} \right\} = \frac{p^2 + q^2 + 1}{3pq}.
\]

(9.13)

Similarly, we see that the coefficient of \( \frac{1}{(x - 1)^2} \) is

\[
1 + \frac{2}{p} + \frac{2}{q} - \frac{2}{p} \sum_{j=1}^{p-1} \frac{\omega_j}{\omega_j - 1} - \frac{2}{q} \sum_{j=1}^{q-1} \frac{\xi_j}{\xi_j - 1} - \frac{4}{pq} \sum_{j=1}^{p-1} \frac{\omega_j^2}{(\omega_j - 1)^2} - \frac{4}{pq} \sum_{j=1}^{q-1} \frac{\xi_j^2}{(\xi_j - 1)^2} + \frac{4}{pq} \sum_{i=1}^{p-1} \frac{\omega_i}{\omega_i - 1} \sum_{j=1}^{q-1} \frac{\xi_j}{\xi_j - 1} = \frac{p^2 + q^2 + 13}{3pq},
\]

(9.14)

and the coefficient of \( \frac{1}{(x - 1)^3} \) is

\[
\frac{2}{p} + \frac{2}{q} + \frac{4}{pq} - \frac{4}{pq} \sum_{j=1}^{p-1} \frac{\omega_j}{\omega_j - 1} - \frac{4}{pq} \sum_{j=1}^{q-1} \frac{\xi_j}{\xi_j - 1} = \frac{8}{pq}.
\]

(9.15)

It is easy to see that the coefficient of \( \frac{1}{(x - 1)^4} \) is

\[
\frac{4}{pq}.
\]

(9.16)

Next, for each \( 1 \leq j \leq p - 1 \), the coefficient of \( \frac{1}{x - \omega_j} \) is

\[
\frac{2\omega_j^2}{p(\omega_j - 1)^2} + \frac{4\omega_j^2}{pq(\omega_j - 1)^3} + \frac{4\omega_j^2}{pq(\omega_j - 1)^2} \sum_{i=1}^{q-1} \frac{\xi_i}{\omega_j - \xi_i}.
\]

\[
= \frac{2\omega_j^2}{p(\omega_j - 1)^2} \left\{ 1 + \frac{2}{q(\omega_j - 1)} + \frac{2\omega_j}{q} \sum_{i=1}^{q-1} \frac{1}{\omega_j - \xi_i} - \frac{2(q - 1)}{q} \right\}.
\]

From the identity

\[
\sum_{j=1}^{k-1} \frac{1}{x - z_j} = kx^{k-1} \frac{1}{x^k - 1} - \frac{1}{x - 1},
\]

(9.17)
it follows that the coefficient of \( \frac{1}{x - \omega_j} \) is equal to

\[
\frac{2\omega_j^2}{p(\omega_j - 1)^2} \left\{ 1 + \frac{2}{q(\omega_j - 1)} + \frac{2\omega_j}{q} \left( \frac{q\omega_j^{q-1}}{\omega_j^q - 1} - \frac{1}{\omega_j - 1} \right) - \frac{2(q - 1)}{q} \right\} = \frac{2\omega_j^2(\omega_j^q + 1)}{p(\omega_j - 1)^2(\omega_j^q - 1)}.
\]

(9.18)

Analogously, for each \( 1 \leq j \leq q - 1 \), the coefficient of \( \frac{1}{x - \xi_j} \) is

\[
\frac{2\xi_j^2(\xi_j^p + 1)}{q(\xi_j - 1)^2(\xi_j^p - 1)}.
\]

(9.19)

Putting (9.13)–(9.19) into (9.2), we complete the proof of Lemma 9.2.

\[ \square \]

**Lemma 9.3** Let \( p \) and \( q \) be positive integers with \( q \geq 2 \) and \((p, q) = 1\). If \( \xi_j = e^{2\pi ij/q} \), then

\[
\sum_{j=1}^{q-1} \frac{\xi_j}{(\xi_j - 1)^2(\xi_j^p - 1)} = \frac{q^2 - 1}{24}.
\]

**Proof** We first note that by symmetry

\[
\sum_{n=1}^{q-1} \cot \left( \frac{\pi np}{q} \right) \csc^2 \left( \frac{\pi n}{q} \right) = 0.
\]

On the other hand,

\[
\sum_{n=1}^{q-1} \cot \left( \frac{\pi np}{q} \right) \csc^2 \left( \frac{\pi n}{q} \right) = -4i \sum_{n=1}^{q-1} \frac{\xi_n^p + 1}{\xi_n^p - 1} \frac{\xi_n}{(\xi_n - 1)^2}
\]

\[
= -4i \sum_{n=1}^{q-1} \left( 1 + \frac{2}{\xi_n^p - 1} \right) \frac{\xi_n}{(\xi_n - 1)^2}.
\]

Therefore, by Lemma 9.1, we arrive at

\[
\sum_{n=1}^{q-1} \frac{\xi_n}{(\xi_n - 1)^2(\xi_n^p - 1)} = -\frac{1}{2} \sum_{n=1}^{q-1} \frac{\xi_n}{(\xi_n - 1)^2} = \frac{q^2 - 1}{24}.
\]

\[ \square \]

**Lemma 9.4** Let \( p \) and \( q \) be positive integers with \( p, q \geq 2 \) and \((p, q) = 1\). If \( \omega_j = e^{2\pi ij/p} \) and \( \xi_j = e^{2\pi ij/q} \), then

\[
p \sum_{j=1}^{q-1} \frac{\xi_j}{(\xi_j - 1)^3(\xi_j^p - 1)} + q \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j - 1)^3(\omega_j^q - 1)}
\]

\[
= \frac{1}{720} (p^4 + q^4 - 5p^2q^2 - 15p^2q - 15pq^2 + 15p + 15q + 3).
\]
Proof. We first observe that by (9.1), (9.5), and Lemma 9.1,

\[
\frac{qx}{(x-1)^3(x^q-1)} = \frac{x}{(x-1)^3} + \frac{x}{(x-1)^3} \sum_{i=1}^{q-1} \frac{1}{x - \xi_i}
\]

\[
= \frac{1}{(x-1)^3} + \frac{1}{(x-1)^4} - \sum_{i=1}^{q-1} \frac{\xi_i^2}{(\xi_i-1)^3(x-1)} - \sum_{i=1}^{q-1} \frac{\xi_i^2}{(\xi_i-1)^2(x-1)^2}
\]

\[
- \sum_{i=1}^{q-1} \frac{\xi_i}{(\xi_i-1)^3} + \sum_{i=1}^{q-1} \frac{\xi_i^2}{(\xi_i-1)^3(x-\xi_i)}
\]

\[
= \frac{(q-1)(q+1)}{24} \frac{1}{x-1} + \frac{(q-1)(q-5)}{12} \frac{1}{(x-1)^2} - \frac{q-3}{2} \frac{1}{(x-1)^3}
\]

Thus, it follows that

\[
q \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j-1)^3(\omega_j^q-1)} = \frac{(q-1)(q+1)}{24} \sum_{j=1}^{p-1} \frac{1}{\omega_j-1} + \frac{(q-1)(q-5)}{12} \sum_{j=1}^{p-1} \frac{1}{(\omega_j-1)^2}
\]

\[
- \frac{q-3}{2} \sum_{j=1}^{p-1} \frac{1}{(\omega_j-1)^3} + \sum_{j=1}^{p-1} \frac{1}{(\omega_j-1)^4}
\]

\[
+ \sum_{j=1}^{p-1} \sum_{i=1}^{q-1} \frac{\xi_i^2}{(\xi_i-1)^3(\omega_j-\xi_i)}. \tag{9.20}
\]

Using (9.17), we deduce that

\[
\sum_{j=1}^{p-1} \sum_{i=1}^{q-1} \frac{\xi_i^2}{(\xi_i-1)^3(\omega_j-\xi_i)} = \sum_{i=1}^{q-1} \frac{\xi_i^2}{(\xi_i-1)^3} \left\{ \frac{1}{\xi_i-1} - \frac{p}{\xi_i} - \frac{p}{\xi_i(\xi_i^p-1)} \right\}
\]

\[
= \sum_{i=1}^{q-1} \frac{\xi_i^2}{(\xi_i-1)^4} - p \sum_{i=1}^{q-1} \frac{\xi_i}{(\xi_i-1)^3} - p \sum_{i=1}^{q-1} \frac{\xi_i}{(\xi_i-1)^3(\xi_i^p-1)}. \tag{9.21}
\]

Hence, by Lemmas 7.3 and 9.1, (9.20), and (9.21),

\[
q \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j-1)^3(\omega_j^q-1)} = \frac{1}{720}(p^4 + q^4 - 5p^2q^2 - 15p^2q - 15pq^2 + 15p + 15q + 3)
\]

\[
- p \sum_{i=1}^{q-1} \frac{\xi_i}{(\xi_i-1)^3(\xi_i^p-1)}
\]

which completes the proof. □

We mention that Gessel [19] proved a reciprocity theorem involving roots of unity that is in the spirit of Lemma 9.4. Let \(m, n,\) and \(r\) denote positive integers such that \((m, n) = 1\)
and \(0 \leq r < m + n\). Then,

\[
\frac{1}{m} \sum_{\lambda^{m-1} \neq \lambda} \frac{\lambda^{r+1}}{(\lambda^n - 1)(\lambda - 1)} + \frac{1}{n} \sum_{\lambda^{n-1} \neq \lambda} \frac{\lambda^{r+1}}{(\lambda^m - 1)(\lambda - 1)} = -\frac{1}{12} \left( \frac{m}{n} + \frac{n}{m} + \frac{1}{mn} \right) + \frac{1}{4} \left( \frac{1}{m} + \frac{1}{n} - 1 \right) + \frac{r}{2} \left( \frac{1}{m} + \frac{1}{n} - \frac{1}{mn} \right) - \frac{r^2}{2mn}.
\]

(9.22)

A generalization of (9.22) was established by Beck et al. [3, Theorem 8].

**Lemma 9.5** Let \(k > 1\) be an integer and \(z_j = e^{2\pi ij/k}\). Then,

\[
\frac{x}{(x-1)^2(x^n-1)^2} = -\frac{(k-1)(k^2-4k+1)}{12k^2(x-1)} + \frac{(k-1)(5k-13)}{12k^2(x-1)^2} - \frac{k-2}{k^2(x-1)^3}
\]

\[
+ \frac{1}{k^2(x-1)^4} - \frac{1}{k^2} \sum_{j=1}^{k-1} \left\{ \frac{kz_j^2}{(z_j-1)^2} + \frac{2z_j^2}{(z_j-1)^3} \right\} \frac{1}{x-z_j}
\]

\[
+ \frac{1}{k^2} \sum_{j=1}^{k-1} \frac{z_j^3}{(z_j-1)^2(x-z_j)^2}.
\]

(9.23)

**Proof** We first observe that by (9.1),

\[
\frac{k^2x}{(x-1)^2(x^n-1)^2} = \frac{x}{(x-1)} \left( \frac{1}{x-1} + \sum_{j=1}^{k-1} \frac{z_j}{x-z_j} \right)^2
\]

\[
= \frac{x}{(x-1)^3} + \frac{2x}{(x-1)^3} \sum_{j=1}^{k-1} \frac{z_j}{x-z_j} + \frac{x}{(x-1)^2} \left\{ \sum_{j=1}^{k-1} \frac{z_j^2}{(x-z_j)^2} + \sum_{i,j=1}^{k-1} \frac{z_iz_j}{(x-z_i)(x-z_j)} \right\}.
\]

(9.24)

Applying (9.5), (9.6), and the following partial fraction decomposition

\[
\frac{x}{(x-1)^2(x-a)^2} = \frac{a+1}{(a-1)^3(x-1)} - \frac{a+1}{(a-1)^3(x-a)} + \frac{1}{(a-1)^2(x-1)^2} + \frac{a}{(a-1)^2(x-a)^2},
\]

we deduce that

\[
\frac{k^2x}{(x-1)^2(x^n-1)^2} = \frac{1}{(x-1)^3} + \frac{1}{(x-1)^4}
\]

\[
-2 \sum_{j=1}^{k-1} \left\{ \frac{z_j^2}{(z_j-1)^3(x-1)^2} + \frac{z_j^2}{(z_j-1)^2(x-1)^2} + \frac{z_j}{(z_j-1)^3(x-1)} - \frac{z_j^2}{(z_j-1)^3(x-z_j)} \right\}
\]

\[
+ \sum_{j=1}^{k-1} \left\{ \frac{z_j^2(z_j+1)}{(z_j-1)^3(x-1)^2} - \frac{z_j^2(z_j+1)}{(z_j-1)^2(x-j)^2} + \frac{z_j^2}{(z_j-1)^3(x-z_j)^2} + \frac{z_j^3}{(z_j-1)^2(x-z_j)^2} \right\}
\]

\[
+ \sum_{i,j=1}^{k-1} \frac{z_iz_j}{(z_i-1)^3(z_j-1)^2(x-1)^2} + \frac{z_iz_j}{(z_i-1)^2(z_j-1)^3(x-1)^2} + \frac{z_iz_j}{(z_i-1)^2(z_j-1)(x-1)^2} \frac{2z_iz_j^2}{(z_j-1)^2(x-z_j)^2}
\]

(9.24)
We now calculate the coefficients of \( \frac{1}{(x-1)^n} \), \( 1 \leq n \leq 4 \), say, \( E_1, E_2, E_3, E_4 \), respectively, and also those of \( \frac{1}{(x-z_j)^n} \), \( n = 1, 2 \), say, \( E_5, E_6 \), respectively, in (9.24).

First, we find that the coefficient of \( \frac{1}{x-1} \) is

\[
-2 \sum_{j=1}^{k-1} \frac{z_j^2}{(z_j - 1)^3} + \sum_{j=1}^{k-1} \frac{z_j^3 + z_j^2}{(z_j - 1)^3} + \sum_{i,j=1 \atop i \neq j}^{k-1} \frac{z_i z_j^2 - z_i z_j}{(z_i - 1)^2(z_j - 1)^2}
\]

\[
= \sum_{j=1}^{k-1} \frac{z_j^3}{(z_j - 1)^3} - \sum_{j=1}^{k-1} \frac{z_j^2}{(z_j - 1)^3} + \left( \sum_{j=1}^{k-1} \frac{z_j^2}{(z_j - 1)^2} \right)^2 - \sum_{j=1}^{k-1} \frac{z_j^4}{(z_j - 1)^4}
\]

\[
- \left( \sum_{j=1}^{k-1} \frac{z_j}{(z_j - 1)^2} \right)^2 + \sum_{j=1}^{k-1} \frac{z_j^2}{(z_j - 1)^4},
\]

which reduces by Lemma 9.1 to

\[
E_1 = -\frac{(k-1)(k^2 - 4k + 1)}{12}.
\] (9.25)

Similarly, by Lemma 9.1 the coefficient of \( \frac{1}{(x-1)^2} \) is

\[
E_2 = -2 \sum_{j=1}^{k-1} \frac{z_j^2}{(z_j - 1)^2} + \sum_{j=1}^{k-1} \frac{z_j^3}{(z_j - 1)^2} + \sum_{i,j=1 \atop i \neq j}^{k-1} \frac{z_i z_j}{(z_i - 1)(z_j - 1)}
\]

\[
= -2 \sum_{j=1}^{k-1} \frac{z_j^2}{(z_j - 1)^2} + \left( \sum_{j=1}^{k-1} \frac{z_j}{(z_j - 1)} \right)^2 = \frac{(k-1)(5k - 13)}{12}.
\] (9.26)

Also, the coefficient of \( \frac{1}{(x-1)^3} \) is

\[
E_3 = 1 - 2 \sum_{j=1}^{k-1} \frac{z_j}{(z_j - 1)} = 2 - k,
\] (9.27)

and the coefficient of \( \frac{1}{(x-1)^4} \) is

\[
E_4 = 1.
\] (9.28)
Next, for each \(1 \leq j \leq k-1\), the coefficient of \(\frac{1}{x-z_j}\) is

\[
E_5 = \frac{2x_j}{(z_j-1)^3} - \frac{z_j^3 + z_j^2}{(z_j-1)^2} + \frac{2x_j}{(z_j-1)^2} \sum_{i \neq j}^{k-1} \frac{z_i}{(z_j - z_i)}
\]

\[
= -\frac{z_j^2}{(z_j-1)^2} + \frac{2x_j}{(z_j-1)^2} \sum_{i \neq j}^{k-1} \left( \frac{z_j}{(z_j - z_i)} - 1 \right)
\]

\[
= -\frac{(2k-3)z_j^2}{(z_j-1)^2} + \frac{2z_j^3}{(z_j-1)^2} \left\{ \sum_{i \neq j}^{k-1} \left( \frac{1}{z_j - z_i} - \frac{1}{z_j - 1} \right) \right\}
\]

\[
= -\frac{(2k-3)z_j^2}{(z_j-1)^2} + \frac{2z_j^3}{(z_j-1)^2} \sum_{i \neq j}^{k-1} \frac{1}{1 - z_i}
\]

\[
= -\frac{kz_j^2}{(z_j-1)^2} - \frac{2x_j^2}{(z_j-1)^3}
\]

(9.29)

where we applied Lemma 7.3.

Lastly, for each \(1 \leq j \leq k-1\), we can easily see that the coefficient of \(\frac{1}{(x-z_j)^2}\) is

\[
E_6 = \frac{z_j^3}{(z_j-1)^2}
\]

(9.30)

Putting (9.25)–(9.30) in (9.24), we finish the proof of Lemma 9.5. \qed

Lemma 9.6 Let \(p\) and \(q\) be positive integers with \(p, q \geq 2\) and \((p, q) = 1\). If \(\omega_j = e^{2\pi ij/p}\) and \(\xi_j = e^{2\pi ij/q}\), then

\[
p^2 \sum_{j=1}^{q-1} \frac{\xi_j}{(\xi_j - 1)^2(\xi_j^p - 1)^2} - q^2 \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j - 1)^2(\omega_j^q - 1)^2}
\]

\[
= -\frac{1}{720}(3p^4 - q^4 - 25p^2 + 25q^2 - 5p^2q^2 - 30p^2q + 30q + 3)
\]

\[
+ 2q \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j - 1)^3(\omega_j^q - 1)}.
\]

(9.31)

We remark that using the symmetry of the left-hand side of the equation (9.31), we can easily derive Lemma 9.4 as a corollary of Lemma 9.6.
Proof We note that applying Lemma 9.5 with $x = \xi_j$, $k = p$, and $z_j = \omega_j$ and summing over $1 \leq j \leq q - 1$ yields

$$
p^2 \sum_{j=1}^{q-1} \frac{\xi_j}{(\xi_j - 1)^2(\xi_j^p - 1)^2} = -\frac{(p - 1)(p^2 - 4p + 1)}{12} \sum_{j=1}^{q-1} \frac{1}{\xi_j - 1} + \frac{(p - 1)(5p - 13)}{12} \sum_{j=1}^{q-1} \frac{1}{(\xi_j - 1)^2} - \frac{(p - 2)}{12} \sum_{j=1}^{q-1} \frac{1}{(\xi_j - 1)^3} + \sum_{j=1}^{q-1} \frac{1}{(\xi_j - 1)^4}
- \sum_{i=1}^{p-1} \left\{ \frac{p\omega_i^2}{(\omega_i - 1)^2} + \frac{2\omega_i^2}{(\omega_i - 1)^3} \right\} \sum_{j=1}^{q-1} \frac{1}{(\xi_j - \omega_i)} + \sum_{i=1}^{p-1} \frac{\omega_i^3}{(\omega_i - 1)^2} \sum_{j=1}^{q-1} \frac{1}{(\xi_j - \omega_i)^2}. \tag{9.32}
$$

By Lemma 7.3, the sum of the first four summations in (9.32) can be written as

$$
\frac{1}{720} (q^4 + 30p^3q - 30p^3 - 25p^2q^2 + 25p^2 + 5q^2 - 30pq + 30p - 6). \tag{9.33}
$$

Next, we consider that by (9.17) and Lemmas 9.1 and 9.3,

$$
\sum_{i=1}^{p-1} \left\{ \frac{p\omega_i^2}{(\omega_i - 1)^2} + \frac{2\omega_i^2}{(\omega_i - 1)^3} \right\} \sum_{j=1}^{q-1} \frac{1}{(\xi_j - \omega_i)}
= \sum_{i=1}^{p-1} \left\{ \frac{p\omega_i^2}{(\omega_i - 1)^2} + \frac{2\omega_i^2}{(\omega_i - 1)^3} - \frac{p\omega_i}{(\omega_i - 1)^2} - \frac{2q\omega_i}{(\omega_i - 1)^3} \right\}
= \frac{1}{360} (p + 1)(p - 1)(p^2 + 15pq - 15p - 30q + 11)
- 2q \sum_{i=1}^{p-1} \frac{\omega_i}{(\omega_i - 1)^3(\omega_i^q - 1)}. \tag{9.34}
$$

Lastly, taking the derivative of both sides of the identity in (9.17), we see that

$$
\sum_{j=1}^{k-1} \frac{1}{(x - z_j)^2} = -\frac{1}{(x - 1)^2} - \frac{k(k - 1)x^{k-2}}{x^k - 1} + \frac{k^2 x^{2k-2}}{(x^k - 1)^2}. \tag{9.35}
$$
From the identities and Lemmas 9.1 and 9.3, we have

\[
\sum_{j=1}^{p-1} \frac{\omega_j^3}{(\omega_j - 1)^2} \sum_{j=1}^{q-1} \frac{1}{(\xi_j - \omega_j)^2} = - \sum_{i=1}^{p-1} \frac{\omega_i^3}{(\omega_i - 1)^2} \left\{ \frac{1}{(\omega_i - 1)^2} + \frac{q(q - 1)\omega_i^{q-2}}{\omega_i^q - 1} - \frac{q^2 \omega_i^{2q-2}}{(\omega_i^q - 1)^2} \right\} \\
\sum_{i=1}^{p-1} \frac{\omega_i^3}{(\omega_i - 1)^2} \frac{q(q - 1)}{\omega_i^{q-2}} \left[ \frac{2\omega_i}{(\omega_i - 1)^2} + \frac{\omega_i}{(\omega_i - 1)^2(\omega_i^q - 1)} \right] + q^2 \sum_{i=1}^{p-1} \frac{\omega_i}{(\omega_i - 1)^2(\omega_i^q - 1)^2} = - \frac{1}{720}(p^2 - 1)(p^2 - 30q^2 + 30q - 19) + \sum_{i=1}^{p-1} \frac{q^2 \omega_i}{(\omega_i - 1)^2(\omega_i^q - 1)^2}. \tag{9.36}
\]

From (9.32), (9.33), (9.34), and (9.36), the lemma follows. □

10 The determination of a trigonometric sum with two independent periods

Theorem 10.1 Let \( p \) and \( q \) be positive integers with \( p, q \geq 2 \) and \( (p, q) = 1 \), and let \( \omega_j = e^{2\pi i j/p} \). Then,

\[
\sum_{n=1}^{pq-1} \frac{\cot \left( \frac{\pi n}{p} \right) \cot \left( \frac{\pi n}{q} \right)}{\sin^2 \left( \frac{\pi n}{pq} \right)} = \frac{1}{45p}(p^4 q^3 - 5p^4 q + 4q^3 - 5p^2 q^3 + 55p^2 q + 30p^2 - 50q - 30) + \frac{32}{p} \sum_{j=1}^{pq-1} \frac{\omega_j}{(\omega_j - 1)^2(\omega_j^q - 1)} + \frac{32q}{p} \sum_{j=1}^{pq-1} \frac{\omega_j}{(\omega_j - 1)^2(\omega_j^q - 1)^2}.
\]

Proof We define

\[ z_n = e^{2\pi in/pq}, \quad f(x) = \frac{(x - 1)(x^{pq} - 1)}{(x^p - 1)(x^q - 1)} = \prod_{n=1}^{pq-1} (x - z_n), \]

and

\[ R(x) = \frac{4x(x^p + 1)(x^q + 1)}{(x - 1)^2(x^p - 1)(x^q - 1)}. \]

From the identities

\[
\sin^2 \left( \frac{\pi n}{pq} \right) = - \frac{(z_n - 1)^2}{4z_n}, \quad \cot \left( \frac{\pi n}{p} \right) = \frac{z_n^p + 1}{z_n^p - 1}, \quad \cot \left( \frac{\pi n}{q} \right) = \frac{z_n^q + 1}{z_n^q - 1}. \tag{10.1}
\]
it follows that

$$\sum_{n=1}^{pq-1} \frac{\cot \left( \frac{\pi H}{p} \right) \cot \left( \frac{\pi H}{q} \right)}{\sin^2 \left( \frac{\pi H}{pq} \right)} = \sum_{n=1}^{pq-1} R(z_n).$$

(10.2)

By Lemma 9.2,

$$\sum_{n=1}^{pq-1} R(z_n) = 4 \sum_{n=1}^{pq-1} \left\{ \frac{p^2 + q^2 + 1}{3pq(z_n - 1)} + \frac{p^2 + q^2 + 13}{3pq(z_n - 1)^2} + \frac{8}{pq(z_n - 1)^3} + \frac{4}{pq(z_n - 1)^4} \right\}$$

$$+ \sum_{j=1}^{p-1} \frac{2\omega_j^2 (\omega_j^q + 1)}{p(\omega_j - 1)^2(\omega_j^q - 1)(z_n - \omega_j)} + \sum_{j=1}^{q-1} \frac{2\xi_j^q (\xi_j^p + 1)}{q(\xi_j - 1)^2(\xi_j^p - 1)(z_n - \xi_j)}.$$

(10.3)

where \( \xi_j = z_j^p = e^{2\pi ij/q} \).

Now, we examine

$$\sum_{n=1}^{pq-1} \frac{1}{(x - z_n)^4} = \frac{pqx^{pq-1}}{x^{pq} - 1} + \frac{1}{x - 1} - \sum_{a=pq} ax^{a-1}.$$
\[
\sum_{n=1}^{pq-1} \frac{1}{z_n - 1} = -\frac{(p - 1)(q - 1)}{2},
\]

\[
\sum_{n=1}^{pq-1} \frac{1}{(z_n - 1)^2} = -\frac{(p^2 - 1)(q^2 - 1)}{12} + \frac{(p - 1)(q - 1)}{2},
\]

\[
\sum_{n=1}^{pq-1} \frac{1}{(z_n - 1)^3} = \frac{1}{8} (pq + p + q - 3)(pq - p - q + 1),
\]

\[
\sum_{n=1}^{pq-1} \frac{1}{(z_n - 1)^4} = \frac{1}{720} (p^4 q^4 - 110p^2 q^2 - p^4 + 110p^2 - q^4 + 110q^2 - 109)
\]

\[+ \frac{1}{2} (p - 1)(q - 1).\] \hspace{1cm} (10.4)

Next, note that

\[
f' \left( \frac{f}{x} \right) = p q x^{pq-1} + \frac{1}{x - 1} - \frac{px^{p-1}}{x^p - 1} - \frac{qx^{q-1}}{x^q - 1}
\]

\[= p (q - 1)x^p - x^{p(q-1) - \cdots - x^2p - x^p}
\]

\[= \frac{1}{x - 1} - \frac{q x^{q-1}}{x^q - 1}
\]

Therefore, it follows that

\[
\sum_{n=1}^{pq-1} \frac{1}{z_n - \omega_j} = -\left[ f' \right]_{\omega_j} = -\frac{p(q - 1)}{2\omega_j} - \frac{1}{\omega_j - 1} + \frac{q \omega_j^{q-1}}{\omega_j - 1}. \] \hspace{1cm} (10.5)

Analogously, we have

\[
\sum_{n=1}^{pq-1} \frac{1}{z_n - \xi_j} = -\left[ f' \right]_{\xi_j} = -\frac{q(p - 1)}{2\xi_j} - \frac{1}{\xi_j - 1} + \frac{p \xi_j^{p-1}}{\xi_j - 1}. \] \hspace{1cm} (10.6)

Now, by (10.4), we obtain

\[
\sum_{n=1}^{pq-1} \left\{ \frac{p^2 + q^2 + 1}{3pq(z_n - 1)} + \frac{p^2 + q^2 + 13}{3pq(z_n - 1)^2} + \frac{8}{pq(z_n - 1)^3} + \frac{4}{pq(z_n - 1)^4} \right\}
\]

\[= \frac{1}{180pq} (p^4 q^4 - 5p^4 q^2 - 5p^2 q^4 + 4p^4 + 4q^4 + 15p^2 q^2 - 10q^2 - 10p^2 + 6). \] \hspace{1cm} (10.7)
Also, with the use of (10.5) and Lemmas 9.1 and 9.3, we derive

\[
\sum_{n=1}^{pq-1} \sum_{j=1}^{p-1} \frac{2\omega_j^2 (\omega_j^q + 1)}{p(\omega_j - 1)^2 (\omega_j^q - 1) (\varepsilon_n - \omega_j)} \\
= \sum_{j=1}^{p-1} \left\{ \frac{2\omega_j^2}{p(\omega_j - 1)^2} + \frac{4\omega_j^2}{p(\omega_j - 1)^2 (\omega_j^q - 1)} \right\} \left\{ -\frac{p(q - 1)}{2\omega_j} - \frac{1}{\omega_j - 1} + \frac{q\omega_j^{q-1}}{\omega_j^q - 1} \right\} \\
= -(q - 1) \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j - 1)^2} - 2 \sum_{j=1}^{p-1} \frac{\omega_j^2}{(\omega_j - 1)^3} + 2q \sum_{j=1}^{p-1} \frac{\omega_j^{q+1}}{\omega_j - 1)^2 (\omega_j^q - 1)} \\
- 2(q - 1) \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j - 1)^2 (\omega_j^q - 1)} - 4 \sum_{j=1}^{p-1} \frac{\omega_j^2}{(\omega_j - 1)^3 (\omega_j^q - 1)} + 4q \sum_{j=1}^{p-1} \frac{\omega_j^{q+1}}{(\omega_j - 1)^2 (\omega_j^q - 1)^2} \\
= \frac{1}{12p} (p^2 - 1)(q - 1) - \frac{4}{p} \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j - 1)^3 (\omega_j^q - 1)} + \frac{4q}{p} \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j - 1)^2 (\omega_j^q - 1)^2}.
\]

(10.8)

Analogously, by (10.6) and Lemma 9.1, we find that

\[
\sum_{n=1}^{pq-1} \sum_{j=1}^{q-1} \frac{2\xi_j^2 (\xi_j^p + 1)}{q(\xi_j - 1)^2 (\xi_j^p - 1) (\varepsilon_n - \xi_j)} \\
= \frac{1}{12q} (q^2 - 1)(p - 1) - \frac{4}{q} \sum_{j=1}^{q-1} \frac{\xi_j}{(\xi_j - 1)^3 (\xi_j^p - 1)} + \frac{4q}{q} \sum_{j=1}^{q-1} \frac{\xi_j^2}{(\xi_j - 1)^2 (\xi_j^p - 1)^2} \\
= \frac{1}{180pq} (-4p^4 + 10p^2 + 25p^2 q^2 + 45p^2 q - 25q^2 - 45q - 6) \\
+ \frac{12}{p} \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j - 1)^3 (\omega_j^q - 1)} + \frac{4q}{p} \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j - 1)^2 (\omega_j^q - 1)^2}.
\]

(10.9)

where we employed Lemmas 9.4 and 9.6. Referring to (10.2) and (10.3), and adding the identities in (10.7), (10.8) and (10.9), we complete our proof. \qed
McIntosh [23, p. 202], and Greaves et al. [20, p. 65, Equation (29)] established a representation for

\[ \sum_{n=1}^{pq-1} \frac{\cot \left( \frac{\pi n}{p} \right) \cot \left( \frac{\pi n}{q} \right)}{\sin^2 \left( \frac{\pi n}{pq} \right)} \]  

in terms of certain generalized Dedekind sums \( S_3(p, q) \) and \( S_3(q, p) \), defined in (8.4).

**Corollary 10.2** Let \( p \) and \( q \) be positive integers such that \( p, q \geq 2 \) and \( q \equiv \pm 1 \pmod{p} \). Then,

\[ \sum_{n=1}^{pq-1} \frac{\cot \left( \frac{\pi n}{p} \right) \cot \left( \frac{\pi n}{q} \right)}{\sin^2 \left( \frac{\pi n}{pq} \right)} = \begin{cases} \frac{1}{45p} (p^2 - 1)(p^2 - 4)(q - 1)^2(q + 2), & \text{if } q \equiv 1 \pmod{p}, \\ \frac{1}{45p} (p^2 - 1)(p^2 - 4)(q + 1)^2(q - 2), & \text{if } q \equiv -1 \pmod{p}. \end{cases} \]

**Proof** Suppose that \( q \equiv 1 \pmod{p} \). Then, Theorem 10.1 and Lemma 9.1 imply

\[ \sum_{n=1}^{pq-1} \frac{\cot \left( \frac{\pi n}{p} \right) \cot \left( \frac{\pi n}{q} \right)}{\sin^2 \left( \frac{\pi n}{pq} \right)} = \frac{1}{45p} (p^4q^3 - 5p^4q + 4q^3 - 5p^2q^3 + 55p^2q + 30p^2 - 50q - 30) + \frac{32}{p} \sum_{j=1}^{p-1} \omega_j^{q^2} \sum_{j=1}^{p-1} \omega_j^{q^2} + \frac{32q}{p} \sum_{j=1}^{p-1} \omega_j^{q^2} \sum_{j=1}^{p-1} \omega_j^{q^2} \]

\[ = \frac{1}{45p} (p^4q^3 - 5p^4q - 5p^2q^3 - 5p^2q + 4q^3 + 10q) + \frac{32(q + 1)}{p} \cdot (p - 1)(p + 1)(p^2 - 19) + \frac{720}{p} \]

\[ = \frac{1}{45p} (p^2 - 1)(p^2 - 4)(q - 1)^2(q + 2). \]

The proof for \( q \equiv -1 \pmod{p} \) follows along similar lines. \( \square \)

### 11 Further lemmas on sums involving roots of unity

From Lemma 7.2, we can easily derive the following lemma.

**Lemma 11.1** Let \( k \) be an odd positive integer and \( z_n = e^{2\pi in/k} \). Then, we have

\[ \sum_{n=1}^{k-1} z_n = \frac{k - 1}{2}, \quad \sum_{n=1}^{k-1} (z_n + 1)^2 = \frac{k^2 - 1}{4}, \]

\[ \sum_{n=1}^{k-1} \frac{z_n^2}{(z_n + 1)^2} = \frac{(k - 1)^2}{4}, \quad \sum_{n=1}^{k-1} \frac{z_n}{(z_n + 1)^3} = \frac{k^2 - 1}{8}, \]
\[
\begin{align*}
\sum_{n=1}^{k-1} \frac{x^2_n}{(x_n+1)^3} &= \frac{k^2-1}{8}, \\
\sum_{n=1}^{k-1} \frac{x_n}{(x_n+1)^4} &= \frac{(k^2-1)(k^2-3)}{48}, \\
\sum_{n=1}^{k-1} \frac{x^3_n}{(x_n+1)^4} &= \frac{(k^2-1)(k^2-3)}{48}.
\end{align*}
\]

**Lemma 11.2** Let \( p \) and \( q \) be positive integers with \( p, q \geq 2 \) and \( p, q \neq 1 \). If \( \omega_j = e^{2\pi ij/p} \) and \( \xi_j = e^{2\pi i q/j} \), then

\[
\frac{x(x^p+1)(x^q+1)}{(x+1)^2(x^p-1)(x^q-1)} = \frac{1}{pq(x-1)} + \frac{1}{pq(x-1)^2} + \frac{2}{p} \sum_{j=1}^{p-1} \frac{\omega_j}{x - \omega_j} + \frac{2}{q} \sum_{j=1}^{q-1} \frac{\xi_j}{x - \xi_j}.
\]

**Proof** We first use (9.1) to obtain

\[
\frac{x(x^p+1)(x^q+1)}{(x+1)^2(x^p-1)(x^q-1)} = \frac{x}{(x+1)^2} \left( 1 + \frac{2}{x^p-1} \right) \left( 1 + \frac{2}{x^q-1} \right)
\]

\[
= \frac{x}{(x+1)^2} \left( 1 + \frac{2}{p} \sum_{j=1}^{p-1} \frac{\omega_j}{x - \omega_j} \right) \left( 1 + \frac{2}{q} \sum_{j=1}^{q-1} \frac{\xi_j}{x - \xi_j} \right)
\]

\[
= \frac{x}{(x+1)^2} + \frac{2x}{p(x-1)} + \frac{2x}{q(x+1)^2(x-1)} + \frac{2}{pq} \sum_{j=1}^{p-1} \frac{\omega_j x}{x - \omega_j} + \frac{2}{pq} \sum_{j=1}^{q-1} \frac{\xi_j x}{x - \xi_j}
\]

\[
+ \frac{4}{pq} \sum_{j=1}^{p-1} \frac{\omega_j x}{(x+1)^2(x-\omega_j)} + \frac{4}{pq} \sum_{j=1}^{q-1} \frac{\xi_j x}{(x+1)^2(x-\xi_j)}.
\]

(11.1)

Now, we consider the following partial fraction decompositions:

\[
\frac{x}{(x+1)^2} = \frac{1}{x+1} - \frac{1}{(x+1)^2},
\]

\[
\frac{x}{(x+1)^2(x-a)} = -\frac{a}{(a+1)^2(x+1)} + \frac{1}{(a+1)(x+1)^2} - \frac{a}{(a+1)(x+a)^2} + \frac{a}{(a+1)(b-a)(x-a)} + \frac{b}{(a+1)(b-a)(x-b)}.
\]

(11.5)
We divide the right-hand side of (11.1) into six parts, \( B_1, \ldots, B_6 \), and find the partial
decomposition of each. By (11.2), (11.3), and (11.4),

\[
B_1 := \frac{x}{(x+1)^2} + \frac{2x}{p(x+1)^2(x-1)} + \frac{2x}{q(x+1)^2(x-1)} + \frac{4x}{pq(x+1)^2(x-1)^2} \\
= \frac{1}{x+1} - \frac{1}{(x+1)^2} + \left( \frac{2}{p} + \frac{2}{q} \right) \left\{ -\frac{1}{4(x+1)} + \frac{1}{2(x+1)^2} + \frac{1}{4(x-1)} \right\} \\
+ \frac{1}{pq} \left( \frac{1}{(x-1)^2} - \frac{1}{(x+1)^2} \right) \\
= \frac{(p+q)}{2pq} \frac{1}{(x-1)} + \frac{1}{pq(x-1)^2} \left( \frac{2pq - p - q}{(x+1)} \right) \\
- \frac{(p-1)(q-1)}{pq} \frac{1}{(x+1)^2}. \quad (11.6)
\]

From (11.3), it follows that

\[
B_2 := \frac{2}{p} \sum_{j=1}^{p-1} \frac{\omega_j x}{(x+1)^2(x - \omega_j)} \\
= \frac{2}{p} \sum_{j=1}^{p-1} \left\{ \frac{-\omega_j^2}{(\omega_j+1)^2(x+1)} + \frac{\omega_j}{(\omega_j+1)(x+1)^2} + \frac{\omega_j^2}{(\omega_j+1)^2(x-\omega_j)} \right\}, \quad (11.7)
\]

and

\[
B_3 := \frac{2}{q} \sum_{j=1}^{q-1} \frac{\xi_j x}{(x+1)^2(x - \xi_j)} \\
= \frac{2}{q} \sum_{j=1}^{q-1} \left\{ \frac{-\xi_j^2}{(\xi_j+1)^2(x+1)} + \frac{\xi_j}{(\xi_j+1)(x+1)^2} + \frac{\xi_j^2}{(\xi_j+1)^2(x-\xi_j)} \right\}. \quad (11.8)
\]

Similarly, using (11.5), we obtain

\[
B_4 := \frac{4}{pq} \sum_{j=1}^{p-1} \frac{\omega_j x}{(x+1)^2(x-1)(x - \omega_j)} \\
= \frac{4}{pq} \sum_{j=1}^{p-1} \left\{ \frac{\omega_j(\omega_j - 1)}{4(\omega_j+1)^2(x+1)} - \frac{\omega_j}{2(\omega_j+1)(x+1)^2} \right\} \\
+ \frac{\omega_j}{4(1 - \omega_j)(x-1)} + \frac{\omega_j^2}{(\omega_j+1)^2(\omega_j - 1)(x-\omega_j)}, \quad (11.9)
\]

\[
B_5 := \frac{4}{pq} \sum_{j=1}^{p-1} \frac{\xi_j x}{(x+1)^2(x-1)(x - \xi_j)} \\
= \frac{4}{pq} \sum_{j=1}^{p-1} \left\{ \frac{\xi_j(\xi_j - 1)}{4(\xi_j+1)^2(x+1)} - \frac{\xi_j}{2(\xi_j+1)(x+1)^2} \right\} \\
+ \frac{\xi_j}{4(1 - \xi_j)(x-1)} + \frac{\xi_j^2}{(\xi_j+1)^2(\xi_j - 1)(x-\xi_j)}, \quad (11.10)
\]
Next, we substitute (11.6)–(11.11) into (11.1) and calculate the coefficient of each term. First, by Lemma 9.1, we see that the coefficient of \( \frac{1}{x - 1} \) is

\[
\frac{(p + q)}{2pq} + \frac{1}{pq} \sum_{j=1}^{p-1} \frac{\omega_j}{(1 - \omega_j)} + \frac{1}{pq} \sum_{j=1}^{q-1} \frac{\xi_j}{(1 - \xi_j)} = \frac{1}{pq}
\]

and the coefficient of \( \frac{1}{(x - 1)^2} \) is \( \frac{1}{pq} \). Also, the coefficient of \( \frac{1}{x + 1} \) is equal to

\[
\frac{(2pq - p - q)}{2pq} + \frac{(p - 1)^2}{2p} + \frac{(q - 1)^2}{2q} - \frac{1}{4pq} \left((p - 1)^2 + (p^2 - 1)\right) - \frac{1}{4pq} (q - 1)^2 + \frac{1}{4pq} (p - 1)^2(q - 1)^2 (p - 1)(q^2 - 1) = 0,
\]

where we used Lemma 11.1. Analogously, the coefficient of \( \frac{1}{(x + 1)^2} \) is

\[
- \frac{(p - 1)(q - 1)}{pq} + \frac{2}{p} \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j + 1)} + \frac{2}{q} \sum_{j=1}^{q-1} \frac{\xi_j}{(\xi_j + 1)} - \frac{2}{pq} \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j + 1)}
\]

\[
- \frac{2}{pq} \sum_{j=1}^{q-1} \frac{\xi_j}{(\xi_j + 1)} - \frac{4}{pq} \sum_{i=1}^{p-1} \frac{\omega_i}{(\omega_i + 1)} \sum_{j=1}^{q-1} \frac{\xi_j}{(\xi_j + 1)} = 0.
\]

Lastly, using the same argument in (9.18), we find that the coefficient of \( \frac{1}{x - \omega_j}, 1 \leq j \leq p - 1 \), is

\[
\frac{2\omega_j^2}{p(\omega_j + 1)^2} \left\{1 + \frac{2}{q} \sum_{i=1}^{\omega_j - 1} \frac{\xi_j}{(\omega_j - \xi_i)}\right\} = \frac{2\omega_j^2(\omega_j^q + 1)}{p(\omega_j + 1)^2(\omega_j^q - 1)},
\]

and the coefficient of \( \frac{1}{x - \xi_j}, 1 \leq j \leq p - 1 \), is

\[
\frac{2\xi_j^2(\xi_j^p + 1)}{q(\xi_j + 1)^2(\xi_j^p - 1)}.
\]
Putting all these together in (11.1), we finish our proof. \hfill \Box

**Lemma 11.3** Let $p$ and $q$ be relatively prime positive integers such that $q \geq 3$ and $q$ is odd. If $\xi_j = e^{2\pi ij/q}$, then

$$
\sum_{j=1}^{q-1} \frac{\xi_j}{(\xi_j + 1)^2(\xi_j^p - 1)} = -\frac{q^2 - 1}{8}.
$$

**Proof** We observe that by symmetry

$$
\sum_{n=1}^{q-1} \cot \left( \frac{\pi np}{q} \right) \sec^2 \left( \frac{\pi n}{q} \right) = 0.
$$

Also,

$$
\sum_{n=1}^{q-1} \cot \left( \frac{\pi np}{q} \right) \sec^2 \left( \frac{\pi n}{q} \right) = 4i \sum_{n=1}^{q-1} \left( 1 + \frac{2}{\xi_n^p - 1} \right) \frac{\xi_n}{(\xi_n + 1)^2}.
$$

Hence, by Lemma 11.1, we have

$$
\sum_{n=1}^{q-1} \frac{\xi_n}{(\xi_n + 1)^2(\xi_n^p - 1)} = -\frac{1}{2} \sum_{n=1}^{q-1} \frac{\xi_n}{(\xi_n + 1)^2} = -\frac{q^2 - 1}{8}.
$$

\hfill \Box

**Lemma 11.4** Let $p$ and $q$ be relatively prime positive integers such that $p, q \geq 2$. If $\omega_j = e^{2\pi ij/p}$ and $\xi_j = e^{2\pi ij/q}$, then

$$
\sum_{j=1}^{p-1} \frac{1}{(\omega_j - 1)(\omega_j^p - 1)} + \sum_{j=1}^{q-1} \frac{1}{(\xi_j - 1)(\xi_j^p - 1)} = -\frac{1}{12} (p^2 + q^2 - 9pq + 3p + 3q + 1).
$$

**Proof** From (9.1), (9.17), and Lemmas 7.3 and 9.1, we can derive

$$
\begin{align*}
\sum_{j=1}^{p-1} \frac{1}{(\omega_j - 1)(\omega_j^p - 1)} &= \sum_{j=1}^{p-1} \left\{ \frac{1}{(\omega_j - 1)^2} + \sum_{i=1}^{q-1} \frac{\xi_i}{(\omega_j - 1)(\omega_j - \xi_i)} \right\} \\
&= -\frac{(p-1)(p-5)}{12} + \sum_{i=1}^{q-1} \frac{\xi_i}{(1-\xi_i)} \sum_{j=1}^{p-1} \left\{ \frac{1}{\omega_j - 1} - \frac{1}{\omega_j - \xi_i} \right\} \\
&= -\frac{(p-1)(p-5)}{12} + \frac{(q-1)(p-1)}{2} - \sum_{i=1}^{q-1} \frac{\xi_i}{(1-\xi_i)} \left\{ \frac{1}{\xi_i - 1} - \frac{p}{\xi_i - \xi_i(\xi_i^p - 1)} \right\} \\
&= -\frac{1}{12} (p^2 + q^2 - 9pq + 3p + 3q + 1) - p \sum_{j=1}^{q-1} \frac{1}{(\xi_j - 1)(\xi_j^p - 1)}.
\end{align*}
$$

This completes our proof. \hfill \Box
Lemma 11.5  Let $p$ and $q$ be relatively prime positive integers such that $p, q \geq 2$. If $\omega_n = e^{2\pi in/p}$ and $\xi_n = e^{2\pi in/q}$, then

\[
q \sum_{n=1}^{p-1} \frac{1}{(\omega_n + 1)(\omega_n^q - 1)} + p \sum_{n=1}^{q-1} \frac{1}{(\xi_n + 1)(\xi_n^p - 1)} = -\frac{1}{4}(3pq - p - q - 1).
\]

Proof  Using (9.1) and Lemmas 7.2 and 7.3, we have

\[
q \sum_{n=1}^{p-1} \frac{1}{(\omega_n + 1)(\omega_n^q - 1)} = \sum_{j=1}^{p-1} \frac{1}{\omega_n - 1} + \sum_{j=1}^{q-1} \frac{1}{\xi_n + 1} + \sum_{j=1}^{q-1} \frac{1}{\xi_n^p - 1}.
\]

Now, by (9.17),

\[
q \sum_{j=1}^{q-1} \frac{\xi_j}{\xi_j + 1} + \sum_{n=1}^{p-1} \frac{1}{\omega_n - \xi_j} = \sum_{j=1}^{p-1} \frac{1}{\xi_j + 1} \left\{ \frac{1}{\xi_j - 1} - \frac{p}{\xi_j (\xi_j^p - 1)} \right\}
\]

\[
= \frac{1}{2} \sum_{j=1}^{q-1} \left( \frac{1}{\xi_j + 1} + \frac{1}{\xi_j - 1} \right) - \sum_{j=1}^{q-1} \frac{1}{\xi_j + 1} - \sum_{j=1}^{q-1} \frac{1}{\xi_j^p - 1}.
\]

Lemma 11.6  Let $p$ and $q$ be relatively prime odd positive integers such that $p, q \geq 3$. If $\omega_n = e^{2\pi in/p}$ and $\xi_n = e^{2\pi in/q}$, then

\[
q \sum_{n=1}^{p-1} \frac{1}{(\omega_n + 1)^2(\omega_n^q - 1)} + p \sum_{n=1}^{q-1} \frac{1}{(\xi_n + 1)^2(\xi_n^p - 1)} = \frac{1}{8}(p^2q + pq^2 + p + q - 6pq + 2).
\]

Proof  Applying Lemmas 11.3 and 11.5, we obtain
\[
\begin{aligned}
p^{-1} & \sum_{n=1}^{q} \frac{1}{(\omega_n + 1)^2(\omega_n^q - 1)} + p^{-1} \sum_{n=1}^{q} \frac{1}{(\xi_n + 1)^2(\xi_n^p - 1)} \\
&= q^{-1} \sum_{n=1}^{q} \left\{ \frac{1}{(\omega_n + 1)(\omega_n^q - 1)} - \frac{\omega_j}{(\omega_n + 1)^2(\omega_n^q - 1)} \right\} \\
&+ p^{-1} \sum_{n=1}^{q} \left\{ \frac{1}{(\xi_n + 1)(\xi_n^p - 1)} - \frac{\xi_j}{(\xi_n + 1)^2(\xi_n^p - 1)} \right\} \\
&= -\frac{1}{4}(3pq - p - q - 1) + \frac{q}{8}(p^2 - 1) + \frac{p}{8}(q^2 - 1) \\
&= \frac{1}{8}(p^2 q + pq^2 + p + q - 6pq + 2),
\end{aligned}
\]

which completes the proof. \(\square\)

**Lemma 11.7** Let \(k > 1\) be an integer and \(z_n = e^{2\pi i n/k}\). Then,

\[
\begin{aligned}
\frac{x}{(x+1)^2(x^k - 1)^2} &= -\frac{k - 1}{4k^2(x - 1)} + \frac{1}{4k^2(x - 1)^2} - \frac{(k - 1)}{4(x + 1)^2} - \frac{1}{4(x + 1)^2} \\
&- \frac{1}{k^2} \sum_{j=1}^{k-1} \left\{ \frac{zk_j^2}{(z_j - 1)^2} - \frac{2z_j}{z_j + 1} \right\} \frac{1}{(x - z_n)} + \frac{1}{k^2} \sum_{n=1}^{k-1} z_n^2 \frac{1}{(z_n + 1)^2(x - z_n)^2}.
\end{aligned}
\]

**Proof** From (9.23), we can deduce that

\[
\begin{aligned}
k^2x \\
(x + 1)^2(x^k - 1)^2 &= \frac{x}{(x+1)^2(x^k - 1)^2} + \frac{2x}{(x+1)^2(x - 1)^2} \sum_{j=1}^{k-1} \frac{z_j}{x - z_j} \\
&+ \frac{x}{(x + 1)^2} \left\{ \sum_{j=1}^{k-1} \frac{z_j^2}{(x - z_j)^2} + \sum_{i,j=1}^{k-1} \frac{z_i z_j}{(x - z_i)(x - z_j)} \right\}.
\end{aligned}
\]

Employing (11.4), (11.5) and the following partial fraction decomposition

\[
\begin{aligned}
x \\
(x+1)^2(x-a)^2 &= \frac{a - 1}{(a + 1)^3(x + 1)} - \frac{1}{(a + 1)^3(x + 1)^2} - \frac{a - 1}{(a + 1)^3(x - a)} + \frac{a}{(a + 1)^2(x - a)^2},
\end{aligned}
\]

we derive

\[
\begin{aligned}
k^2x \\
(x + 1)^2(x^k - 1)^2 &= \frac{1}{4(x - 1)^2} - \frac{1}{4(x + 1)^2} \\
&+ \sum_{j=1}^{k-1} \left\{ \frac{z_j(z_j - 1)}{2(z_j + 1)^2(x + 1)} - \frac{z_j}{(z_j + 1)(x + 1)^2} + \frac{z_j}{2(1 - z_j)(x - 1)} \\
&+ \frac{2z_j^2}{(z_j + 1)^2(z_j - 1)(x - z_j)} \right\}.
\end{aligned}
\]
\[ + \sum_{j=1}^{k-1} \left\{ \frac{z_j^2(z_j - 1)}{(z_j + 1)^3(x + 1)} - \frac{z_j^2}{(z_j + 1)^2(x + 1)^2} - \frac{z_j^2(z_j - 1)}{(z_j + 1)^3(x - z_j)} \right\} \\
+ \sum_{j=1}^{k-1} \frac{z_j^3}{(z_j + 1)^2(x - z_j)^2} \right\} + \sum_{i,j=1}^{k-1} \left\{ \frac{z_i z_j (z_i z_j - 1)}{(z_i + 1)^2(z_j + 1)^2(x + 1)} - \frac{z_i z_j}{(z_i + 1)(z_j + 1)(x + 1)^2} \right\} \\
+ \sum_{i,j=1}^{k-1} \frac{2z_i^2 z_j}{(z_i + 1)^2(z_j - z_j)(x - z_i)} \right\}, \quad (11.14) \]

Now, we write the right-hand side of (11.14) as

\[ \frac{H_1}{x - 1} + \frac{H_2}{(x - 1)^2} + \frac{H_3}{x + 1} + \frac{H_4}{(x + 1)^2} + \sum_{n=1}^{k-1} \frac{H_{5,n}}{x - z_n} + \sum_{n=1}^{k-1} \frac{H_{6,n}}{(x - z_n)^2} \quad (11.15) \]

and calculate each \( H_i \) with \( 1 \leq i \leq 4 \) and \( H_{j,n} \) with \( j = 5, 6 \) and \( 1 \leq n \leq k - 1 \).

We can easily see that by Lemma 9.1,

\[ H_1 = \sum_{j=1}^{k-1} \frac{z_j}{2(1 - z_j)} = -\frac{k - 1}{4}, \quad \text{and} \quad H_2 = \frac{1}{4}. \]

Next, using Lemma 11.1, we have

\[ H_3 = \sum_{j=1}^{k-1} \left\{ \frac{z_j(z_j - 1)}{2(z_j + 1)^2} + \frac{z_j^2(z_j - 1)}{(z_j + 1)^3} \right\} + \sum_{i,j=1}^{k-1} \left\{ \frac{z_i z_j (z_i z_j - 1)}{(z_i + 1)^2(z_j + 1)^2} \right\} + \sum_{i,j=1}^{k-1} \frac{z_i z_j}{(z_i + 1)(z_j + 1)(x + 1)^2} \]

\[ = \sum_{j=1}^{k-1} \left\{ \frac{z_j^2 - z_j}{2(z_j + 1)^2} + \frac{z_j^3 - z_j^2}{(z_j + 1)^3} \right\} + \left( \sum_{j=1}^{k-1} \frac{z_j^2}{(z_j + 1)^2} \right)^2 - \sum_{j=1}^{k-1} \frac{z_j^4}{(z_j + 1)^4} \]

\[ - \left( \sum_{j=1}^{k-1} \frac{z_j}{(z_j + 1)^2} \right)^2 + \sum_{j=1}^{k-1} \frac{z_j^2}{(z_j + 1)^4} \]

\[ = -\frac{k^2(k - 1)}{4}, \]

and

\[ H_4 = -\frac{1}{4} - \sum_{j=1}^{k-1} \frac{z_j}{z_j + 1} - \sum_{j=1}^{k-1} \frac{z_j^2}{(z_j + 1)^2} - \sum_{i,j=1}^{k-1} \frac{z_i z_j}{(z_i + 1)(z_j + 1)} \]

\[ = -\frac{1}{4} - \sum_{j=1}^{k-1} \frac{z_j}{z_j + 1} - \left( \sum_{j=1}^{k-1} \frac{z_j}{z_j + 1} \right)^2 = -\frac{k^2}{4}. \]
Now, for each \(1 \leq n \leq k - 1\),
\[
H_{5,n} = \frac{2z_n^2}{(z_n + 1)^2(z_n - 1)} - \frac{z_n^2(z_n - 1)}{(z_n + 1)^3} + \sum_{j=1}^{k-1} \frac{2z_n^2z_j}{(z_n + 1)^2(z_n - z_j)}
\]
\[
= \frac{2z_n^2}{(z_n + 1)^2(z_n - 1)} - \frac{z_n^2 - z_n^2}{(z_n + 1)^3} + \frac{2z_n^2}{(z_n + 1)^2} \sum_{j=1}^{k-1} \frac{z_j}{z_n - z_j}. \tag{11.16}
\]

Observe that
\[
\sum_{j=1}^{k-1} \frac{z_j}{z_n - z_j} = \sum_{j=1}^{k-1} \left( \frac{z_n}{z_n - z_j} - 1 \right) = \sum_{j=1}^{k-1} \frac{z_n}{z_n - z_j} - \frac{z_n}{z_n - 1} - (k - 2)
\]
\[
= \sum_{s=1}^{k-1} \frac{1}{1 - z_s} - \frac{z_n}{z_n - 1} - (k - 2) = -\frac{(k - 3)}{2} - \frac{z_n}{z_n - 1}, \tag{11.17}
\]
where we used Lemma 7.3. Putting (11.17) into (11.16) yields
\[
H_{5,n} = \frac{2z_n^2}{(z_n + 1)^2(z_n - 1)} - \frac{z_n^2 - z_n^2}{(z_n + 1)^3} - \frac{(k - 3)z_n^2}{(z_n + 1)^2} - \frac{2z_n^3}{(z_n + 1)^2(z_n - 1)}
\]
\[
= -\frac{(k - 1)z_n^2}{(z_n + 1)^2} - \frac{z_n^3 - z_n^3}{(z_n + 1)^3} = -\frac{kz_n^2}{(z_n + 1)^2} + \frac{2z_n^2}{(z_n + 1)^3}.
\]

Clearly, we see that for each \(1 \leq n \leq k - 1\),
\[
H_{6,n} = \frac{z_n^3}{(z_n + 1)^2}.
\]

Putting all these together into (11.15), we complete our proof. \(\square\)

**Lemma 11.8** Let \(p\) and \(q\) be relatively prime odd positive integers such that \(p, q \geq 3\), and let \(\omega_n = e^{2\pi i n/p}\) and \(\xi_n = e^{2\pi i n/q}\). Then,
\[
q^2 \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j + 1)^2(\omega_j^q - 1)^2} - p^2 \sum_{j=1}^{q-1} \frac{\xi_j}{(\xi_j + 1)^2(\xi_j^p - 1)^2}
\]
\[
= 1 - \frac{1}{48} (3p^2q^2 + 6p^2q + 5p^2 - 5q^2 - 6q - 3) + 2q \sum_{n=1}^{p-1} \frac{\omega_n}{(\omega_n + 1)^3(\omega_n^q - 1)}. \tag{11.18}
\]

**Proof** We first apply Lemma 11.7 to obtain
\[
p^2 \sum_{j=1}^{q-1} \frac{\xi_j}{(\xi_j + 1)^2(\xi_j^p - 1)^2} = -\frac{p-1}{4} \sum_{j=1}^{q-1} \frac{1}{(\xi_j - 1)} + \frac{1}{4} \sum_{j=1}^{q-1} \frac{1}{(\xi_j - 1)^2}
\]
\[
- \frac{p^2(p-1)}{4} \sum_{j=1}^{q-1} \frac{1}{(\xi_j + 1)} - \frac{p^2}{4} \sum_{j=1}^{q-1} \frac{1}{(\xi_j + 1)^2} - \sum_{n=1}^{p-1} \frac{p\omega_n^2}{(\omega_n + 1)^2}
\]
\[
- \frac{2\omega_n^2}{(\omega_n + 1)^3} \sum_{j=1}^{q-1} \frac{1}{(\xi_j - \omega_n)} + \sum_{n=1}^{p-1} \omega_n^3 \sum_{j=1}^{q-1} \frac{1}{(\xi_j - \omega_n)^2}. \tag{11.18}
\]
Using Lemmas 7.2 and 7.3, we can calculate the sum of the first four summations of (11.18) as follows:

\[
\frac{(p - 1)(q - 1)}{8} - \frac{(q - 1)(q - 5)}{48} - \frac{p^2(p - 1)(q - 1)}{8} + \frac{p^2(q - 1)^2}{16} = -\frac{1}{48}(q - 1)(6p^2 - 3p^2q - 3p^2 - 6p + q + 1).
\] (11.19)

Next, by (9.17) and Lemmas 7.2, 7.3, 11.1 and 11.3,

\[
\sum_{n=1}^{p-1} \left\{ \frac{p\omega_n^2}{(\omega_n + 1)^2} - \frac{2\omega_n^2}{(\omega_n + 1)^3} \right\} \sum_{j=1}^{q-1} \frac{1}{(\xi_j - \omega_n)}
\]

\[
= \sum_{n=1}^{p-1} \left\{ \frac{p\omega_n^2}{(\omega_n + 1)^2} - \frac{2\omega_n^2}{(\omega_n + 1)^3} \right\} \left\{ \frac{1}{\omega_n - 1} - \frac{q}{\omega_n} - \frac{q}{\omega_n(\omega_n^q - 1)} \right\}
\]

\[
= \frac{1}{4}(p - 1)(-p^2q + p^2 + pq + p + 2q) + 2q \sum_{n=1}^{p-1} \frac{\omega_n}{(\omega_n + 1)^3(\omega_n^q - 1)}.
\] (11.20)

Similarly, we use (9.35) to obtain

\[
\sum_{n=1}^{p-1} \frac{\omega_n^3}{(\omega_n + 1)^2} \sum_{j=1}^{q-1} \frac{1}{(\xi_j - \omega_n)^2}
\]

\[
= -\sum_{n=1}^{p-1} \frac{\omega_n^3}{(\omega_n + 1)^2(\omega_n - 1)^2} - q(q - 1) \sum_{n=1}^{p-1} \frac{\omega_n}{(\omega_n + 1)^2(\omega_n^q - 1)}
\]

\[
+ q^2 \sum_{n=1}^{p-1} \frac{\omega_n}{(\omega_n + 1)^2(\omega_n^q - 1)^2}
\]

\[
= -\sum_{n=1}^{p-1} \left\{ \frac{1}{2(\omega_n + 1)} - \frac{1}{4(\omega_n + 1)^2} + \frac{1}{2(\omega_n - 1)} + \frac{1}{4(\omega_n - 1)^2} \right\}
\]

\[
- q(q - 1) \sum_{n=1}^{p-1} \left\{ \frac{\omega_n}{(\omega_n + 1)^2} + \frac{\omega_n}{(\omega_n + 1)^2(\omega_n^q - 1)} \right\}
\]

\[
+ q^2 \sum_{n=1}^{p-1} \left\{ \frac{\omega_n}{(\omega_n + 1)^2} + \frac{2\omega_n}{(\omega_n + 1)^2(\omega_n^q - 1)} + \frac{\omega_n}{(\omega_n + 1)^2(\omega_n^q - 1)^2} \right\}
\]

\[
= -\frac{1}{24}(p^2 - 1)(3q^2 - 3q + 1) + q^2 \sum_{n=1}^{p-1} \frac{\omega_n}{(\omega_n + 1)^2(\omega_n^q - 1)^2}.
\] (11.21)

where we applied Lemmas 7.2, 7.3, 11.1 and 11.3.
Putting (11.19)–(11.21) into (11.18), we arrive at

\[
p^2 \sum_{j=1}^{q-1} \frac{\xi_j}{(\xi_j + 1)^2(\xi_j^p - 1)^2} = -\frac{1}{48}(q-1)(6p^3 - 3p^2q - 3p^2 - 6p + q + 1)
\]

\[
- \frac{1}{8}(p-1)(-p^2q + p^2 + pq + p + 2q) - 2q \sum_{n=1}^{p-1} \frac{\omega_n}{(\omega_n + 1)^3(\omega_n^q - 1)}
\]

\[
- \frac{1}{24}(p^2 - 1)(3q^2 - 3q + 1) + q^2 \sum_{n=1}^{p-1} \frac{\omega_n}{(\omega_n + 1)^2(\omega_n^q - 1)^2}
\]

\[
= -\frac{1}{48}(3p^2q^2 + 6p^3q + 5p^2 - 5q^2 - 6q - 3) - 2q \sum_{n=1}^{p-1} \frac{\omega_n}{(\omega_n + 1)^3(\omega_n^q - 1)}
\]

\[
+ q^2 \sum_{n=1}^{p-1} (\omega_n + 1)^2(\omega_n^q - 1)^2.
\]

This completes the proof of the lemma. \(\square\)

13 A second trigonometric sum with two independent periods

**Theorem 12.1** Let \(p\) and \(q\) be odd positive integers with \(p, q \geq 3\) and \((p, q) = 1\), and let \(\omega_j = e^{2\pi ij/p}\). Then,

\[
\sum_{\substack{n=1 \atop p \nmid n, q \nmid n}}^{pq-1} \frac{\cot \left( \frac{\pi n}{p} \right) \cot \left( \frac{\pi n}{q} \right)}{\cos^2 \left( \frac{\pi n}{pq} \right)} = \frac{2}{3p}(p^2 - 1)(5q + 3)
\]

\[
+ 32 \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j + 1)^3(\omega_j^q - 1)} - 32q \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j + 1)^2(\omega_j^q - 1)^2}.
\]

**Proof** We set

\[
z_n = e^{2\pi in/(pq)}, \quad f(x) = \frac{(x - 1)(x^{pq} - 1)}{(x^p - 1)(x^q - 1)} = \prod_{n=1 \atop p \nmid n, q \nmid n}^{pq-1} (x - z_n),
\]

and

\[
R(x) = \frac{x(x^p + 1)(x^q + 1)}{(x + 1)^2(x^{pq} - 1)(x^{q} - 1)}.
\]

Then, by the identities in (10.1) we see that

\[
\sum_{\substack{n=1 \atop p \nmid n, q \nmid n}}^{pq-1} \frac{\cot \left( \frac{\pi n}{p} \right) \cot \left( \frac{\pi n}{q} \right)}{\cos^2 \left( \frac{\pi n}{pq} \right)} = -4 \sum_{\substack{n=1 \atop p \nmid n, q \nmid n}}^{pq-1} R(z_n).
\] (12.1)
Now, by Lemma 11.2,

$$
\sum_{n=1}^{pq-1} R(z_n) = \frac{1}{pq} \sum_{n=1}^{pq-1} \left\{ \frac{1}{(z_n - 1)} + \frac{1}{(z_n - 1)^2} \right\} 
\quad + \frac{2}{p} \sum_{n=1}^{pq-1} \sum_{j=1}^{p-1} \frac{\omega_j^2(\omega_j^q + 1)}{(\omega_j + 1)^2(\omega_j^q - 1)(z_n - \omega_j)} 
\quad + \frac{2}{q} \sum_{n=1}^{pq-1} \sum_{j=1}^{q-1} \frac{\xi_j^2(\xi_j^p + 1)}{(\xi_j + 1)^2(\xi_j^p - 1)(z_n - \xi_j)}.
$$

(12.2)

where $\xi_j = e^{2\pi ij/q}$. Applying the first two identities in (10.4) yields

$$
\frac{1}{pq} \sum_{n=1}^{pq-1} \left\{ \frac{1}{(z_n - 1)} + \frac{1}{(z_n - 1)^2} \right\} = -\frac{(p^2 - 1)(q^2 - 1)}{12pq}.
$$

(12.3)

Also, by (10.5) we obtain

$$
\frac{2}{p} \sum_{n=1}^{pq-1} \sum_{j=1}^{p-1} \frac{\omega_j^2(\omega_j^q + 1)}{(\omega_j + 1)^2(\omega_j^q - 1)(z_n - \omega_j)} 
\quad = \frac{2}{p} \sum_{j=1}^{p-1} \left\{ \frac{\omega_j^2}{(\omega_j + 1)^2} + \frac{2\omega_j^2}{(\omega_j + 1)^2(\omega_j^q - 1)} \right\} \left\{ -p(q - 1) - \frac{1}{\omega_j - 1} + \frac{q\omega_j^{q-1}}{\omega_j - 1} \right\} 
\quad = -(q - 1) \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j + 1)^2} - \frac{2}{p} \sum_{j=1}^{p-1} \frac{\omega_j^2}{(\omega_j + 1)^2(\omega_j^q - 1)} + \frac{2q}{p} \sum_{j=1}^{p-1} \frac{\omega_j^{q+1}}{(\omega_j + 1)^2(\omega_j^q - 1)} 
\quad - 2(q - 1) \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j + 1)^2(\omega_j^q - 1)} - \frac{4}{p} \sum_{j=1}^{p-1} \frac{\omega_j^2}{(\omega_j + 1)^2(\omega_j^q - 1)} 
\quad + \frac{4q}{p} \sum_{j=1}^{p-1} \frac{\omega_j^{q+1}}{(\omega_j + 1)^2(\omega_j^q - 1)^2} 
\quad = -(q - 1) \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j + 1)^2} - \frac{2}{p} \sum_{j=1}^{p-1} \left\{ \frac{3\omega_j + 1}{4(\omega_j + 1)^2} + \frac{1}{4(\omega_j - 1)} \right\} 
\quad + \frac{2q}{p} \sum_{j=1}^{p-1} \left\{ \frac{\omega_j}{(\omega_j + 1)^2} + \frac{\omega_j}{(\omega_j + 1)^2(\omega_j^q - 1)} \right\} - 2(q - 1) \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j + 1)^2(\omega_j^q - 1)} 
\quad - \frac{4}{p} \sum_{j=1}^{p-1} \left\{ \frac{3\omega_j}{4(\omega_j + 1)^2(\omega_j^q - 1)} + \frac{1}{4(\omega_j + 1)^2(\omega_j^q - 1)} + \frac{1}{4(\omega_j - 1)(\omega_j^q - 1)} \right\} 
\quad + \frac{4q}{p} \sum_{j=1}^{p-1} \left\{ \frac{\omega_j}{(\omega_j + 1)^2(\omega_j^q - 1)} + \frac{\omega_j}{(\omega_j + 1)^2(\omega_j^q - 1)^2} \right\}.
$$
\[-\frac{1}{2p} (2pq - 2p - 4q + 3) \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j + 1)^2} - \frac{1}{2p} \sum_{j=1}^{p-1} \frac{1}{(\omega_j + 1)^2} - \frac{1}{2p} \sum_{j=1}^{p-1} \frac{1}{(\omega_j - 1)^2} \]

\[-\frac{1}{p} (2pq - 2p - 6q + 3) \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j + 1)^2(\omega_j^p - 1)} - \frac{1}{p} \sum_{j=1}^{p-1} \frac{1}{(\omega_j + 1)^2(\omega_j^p - 1)} \]

\[-\frac{1}{p} \sum_{j=1}^{p-1} \frac{1}{(\omega_j - 1)(\omega_j^p - 1)} + \frac{4q}{p} \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j + 1)^2(\omega_j^p - 1)^2} \]

\[-\frac{1}{8p} (2pq - 2p - 4q + 3)(p^2 - 1) + \frac{1}{8p} (p - 1)^2 + \frac{1}{4p} (p - 1) \]

\[-\frac{1}{8p} (2pq - 2p - 6q + 3)(p^2 - 1) - \frac{1}{p} \sum_{j=1}^{p-1} \frac{1}{(\omega_j + 1)^2(\omega_j^p - 1)} \]

\[-\frac{1}{p} \sum_{j=1}^{p-1} \frac{1}{(\omega_j - 1)(\omega_j^p - 1)^2} + \frac{4q}{p} \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j + 1)^2(\omega_j^p - 1)^2} \]

\[= -\frac{1}{8p} (p^2 - 1)(2q - 1) - \frac{1}{p} \sum_{j=1}^{p-1} \frac{1}{(\omega_j + 1)^2(\omega_j^p - 1)} - \frac{1}{p} \sum_{j=1}^{p-1} \frac{1}{(\omega_j - 1)(\omega_j^p - 1)} \]

\[+ \frac{4q}{p} \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j + 1)^2(\omega_j^p - 1)^2} \]

(12.4)

where we employed Lemmas 7.2, 7.3, 11.1 and 11.3.

Analogously, we can deduce that

\[2 \sum_{n=1}^{p^{q-1}} \sum_{p^{1/n}q^n} \frac{\xi_j^2(\xi_j^p - 1)}{(\xi_j + 1)^2(\xi_j^p - 1)(\xi_n - \xi_j)} = -\frac{1}{8q} (q^2 - 1)(2p - 1) \]

\[-\frac{1}{q} \sum_{j=1}^{q-1} \xi_j - \frac{1}{q} \sum_{j=1}^{q-1} \frac{\xi_j}{(\xi_j + 1)^2(\xi_j^p - 1)} + \frac{4p}{q} \sum_{j=1}^{q-1} \frac{\xi_j}{(\xi_j + 1)^2(\xi_j^p - 1)^2} \]

(12.5)

From (12.2)–(12.5), with the aid of Lemmas 11.4, 11.6, and 11.8, it follows that

\[\sum_{n=1}^{p^{q-1}} R(\xi_n) = \frac{1}{24pq} (8p^2 + 8q^2 - 14p^2q^2 + 3p^2q + 3pq^2 - 3p - 3q - 2) \]

\[-\frac{1}{pq} \left\{ p \sum_{j=1}^{p-1} \frac{1}{(\omega_j - 1)(\omega_j^p - 1)} + p \sum_{j=1}^{q-1} \frac{1}{(\xi_j - 1)(\xi_j^p - 1)} \right\} \]

\[-\frac{1}{pq} \left\{ q \sum_{j=1}^{p-1} \frac{1}{(\omega_j + 1)^2(\omega_j^p - 1)} + q \sum_{j=1}^{q-1} \frac{1}{(\xi_j + 1)^2(\xi_j^p - 1)} \right\} \]
\[ B. C. \text{ Berndt et al. Res Math Sci (2023) 10:40} \]

By (12.1), we complete our proof.  

\[ \sum_{\substack{n=1 \\ p \nmid n, q \nmid n}}^{pq-1} \frac{\cot \left( \frac{\pi n}{p} \right) \cot \left( \frac{\pi n}{q} \right)}{\cos^2 \left( \frac{\pi n}{pq} \right)} = \begin{cases} \frac{2}{p} (p^2 - 1)(q - 1), & \text{if } q \equiv 1 \pmod{p}, \\ \frac{2}{p} (p^2 - 1)(q + 1), & \text{if } q \equiv -1 \pmod{p}. \end{cases} \]

**Proof** Suppose that \( q \equiv 1 \pmod{p} \). Then, Theorem 12.1 implies that

\[ \sum_{\substack{n=1 \\ p \nmid n, q \nmid n}}^{pq-1} \frac{\cot \left( \frac{\pi n}{p} \right) \cot \left( \frac{\pi n}{q} \right)}{\cos^2 \left( \frac{\pi n}{pq} \right)} = \frac{2}{3p} (p^2 - 1)(5q + 3) \]

\[ + \frac{32}{p} \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j + 1)^3(\omega_j - 1)} - \frac{32q}{p} \sum_{j=1}^{p-1} \frac{\omega_j}{(\omega_j + 1)^2(\omega_j - 1)} \]

\[ = \frac{2}{3p} (p^2 - 1)(5q + 3) + \frac{4}{p} \sum_{j=1}^{p-1} \left\{ - \frac{1}{\omega_j + 1} - \frac{2}{(\omega_j + 1)^2} + \frac{4}{(\omega_j + 1)^3} + \frac{1}{\omega_j - 1} \right\} \]

\[ - \frac{8q}{p} \sum_{j=1}^{p-1} \left\{ \frac{1}{(\omega_j - 1)^2} - \frac{1}{(\omega_j + 1)^2} \right\} \]

\[ = \frac{2}{3p} (p^2 - 1)(5q + 3) - \frac{4(p^2 - 1)}{p} - \frac{4q(p^2 - 1)}{3p} = \frac{2}{p} (p^2 - 1)(q - 1), \]

where we applied Lemmas 7.2 and 7.3.

For \( q \equiv -1 \pmod{p} \), the proof is similar. \( \square \)
14 Reciprocity theorems

Theorem 13.1 Let $p$ and $q$ be relatively prime positive integers with $p, q \geq 2$, and let $\xi_j = e^{2\pi i j/q}$. Then,

$$
\sum_{n=1}^{q-1} \cot \left( \frac{\pi n p}{q} \right) \cot \left( \frac{\pi n}{q} \right) = q - 1 - 4 \sum_{n=1}^{q-1} \frac{1}{(\xi_n - 1)(\xi_n^p - 1)}
$$

Proof Employing the identities

$$
\cot \left( \frac{\pi n}{q} \right) = i \frac{\xi_n + 1}{\xi_n - 1}, \quad \cot \left( \frac{\pi n p}{q} \right) = i \frac{\xi_n^p + 1}{\xi_n^p - 1}
$$

and Lemma 7.3, we deduce that

$$
\sum_{n=1}^{q-1} \cot \left( \frac{\pi n p}{q} \right) \cot \left( \frac{\pi n}{q} \right) = -\sum_{n=1}^{q-1} \frac{\xi_n^p + 1}{\xi_n^p - 1} \frac{\xi_n + 1}{\xi_n - 1}
$$

$$
= -\sum_{n=1}^{q-1} \left[ 1 + \frac{2}{\xi_n^p - 1} \right] \left[ 1 + \frac{2}{\xi_n - 1} \right]
$$

$$
= -\sum_{n=1}^{q-1} \left[ 1 + \frac{2}{\xi_n - 1} + \frac{2}{\xi_n^p - 1} + \frac{4}{(\xi_n - 1)(\xi_n^p - 1)} \right]
$$

$$
= q - 1 - 4 \sum_{n=1}^{q-1} \frac{1}{(\xi_n - 1)(\xi_n^p - 1)}
$$

where we used the assumption $(p, q) = 1$ for the penultimate equality.

\[\square\]

Corollary 13.2 Let $p$ and $q$ be relatively prime positive integers with $p, q \geq 2$. Then,

$$
p \sum_{n=1}^{q-1} \cot \left( \frac{\pi n p}{q} \right) \cot \left( \frac{\pi n}{q} \right) + q \sum_{n=1}^{p-1} \cot \left( \frac{\pi n q}{q} \right) \cot \left( \frac{\pi n}{p} \right) = \frac{1}{3} (p^2 + q^2 - 3pq + 1).
$$

(13.2)

Proof By Theorem 13.1 and Lemma 11.4, we have

$$
p \sum_{n=1}^{q-1} \cot \left( \frac{\pi n p}{q} \right) \cot \left( \frac{\pi n}{q} \right) + q \sum_{n=1}^{p-1} \cot \left( \frac{\pi n q}{q} \right) \cot \left( \frac{\pi n}{p} \right)
$$

$$
= p(q - 1) - 4p \sum_{n=1}^{q-1} \frac{1}{(\xi_n - 1)(\xi_n^p - 1)} + q(p - 1) - 4q \sum_{j=1}^{p-1} \frac{1}{(\omega_j - 1)(\omega_j^q - 1)}
$$

$$
= 2pq - p - q + \frac{1}{3} (p^2 + q^2 - 9pq + 3p + 3q + 1)
$$

$$
= \frac{1}{3} (p^2 + q^2 - 3pq + 1).
$$

\[\square\]
The reciprocity relation (13.2) is equivalent to the reciprocity theorem for Dedekind sums. Let

\[(x) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & \text{if } x \text{ is not an integer}, \\ 0, & \text{if } x \text{ is an integer}. \end{cases} \]

Then, for integers \(p, q\) with \((p, q) = 1\), the Dedekind sum \(s(p, q)\) is defined by

\[s(p, q) \equiv \sum_{n=1}^{q} \left( \left( \frac{pn}{q} \right) \left( \frac{n}{q} \right) \right).\]

The Dedekind sum \(s(p, q)\) satisfies a famous reciprocity theorem [25, p. 4]

\[s(p, q) + s(q, p) = -\frac{1}{4} + \frac{1}{12} \left( \frac{p}{q} + \frac{1}{pq} + \frac{q}{p} \right). \tag{13.3}\]

Rademacher [24], [25, p. 18] proved that

\[s(p, q) = \frac{1}{4q} \sum_{n=1}^{q-1} \cot \left( \frac{\pi pn}{q} \right) \cot \left( \frac{\pi n}{q} \right), \tag{13.4}\]

which he used to give a proof of (13.3). Thus, from (13.4), we see that the reciprocity theorem (13.2) is equivalent to the reciprocity theorem for Dedekind sums (13.3). The beautiful book [25] by Rademacher and Grosswald gives an accessible account of (13.3), (13.4), and, more generally, of Dedekind sums. Other proofs of (13.2) have been devised by Berndt and Yeap [5, pp. 370, 371] and Fukuhara [17]. Further references can be found in both aforementioned papers.

**Theorem 13.3** Let \(p\) and \(q\) be relatively prime positive integers with \(p, q \geq 2\), and let \(\xi_j = e^{2\pi ij/q}\). Then,

\[
\sum_{n=1}^{q-1} \cot \left( \frac{\pi np}{q} \right) \cot^3 \left( \frac{\pi n}{q} \right) = \frac{(q - 1)^2}{2} + \sum_{n=1}^{q-1} \frac{12}{(\xi_n - 1)^2(\xi_n^p - 1)} + \sum_{n=1}^{q-1} \frac{16}{(\xi_n - 1)^3(\xi_n^p - 1)}. 
\]

**Proof** Applying (13.1) and Lemmas 7.3 and 9.3, we derive

\[
\sum_{n=1}^{q-1} \cot \left( \frac{\pi np}{q} \right) \cot^3 \left( \frac{\pi n}{q} \right) = \sum_{n=1}^{q-1} \frac{\xi_n^p + 1}{\xi_n - 1} \left( \frac{\xi_n + 1}{\xi_n - 1} \right)^3 
\]

\[
= \sum_{n=1}^{q-1} \left( 1 + \frac{2}{\xi_n^p - 1} \right) \left( 1 + \frac{2}{\xi_n - 1} \right)^3 
\]

\[
= \sum_{n=1}^{q-1} \left( 1 + \frac{6}{\xi_n - 1} + \frac{12}{(\xi_n - 1)^2} + \frac{8}{(\xi_n - 1)^3} + \frac{2}{\xi_n^p - 1} \right. 
\]

\[
+ \frac{12}{(\xi_n - 1)(\xi_n^p - 1)} + \frac{24}{(\xi_n - 1)^2(\xi_n^p - 1)} + \frac{16}{(\xi_n - 1)^3(\xi_n^p - 1)} \bigg) 
\]

\[
= \frac{(q - 1)^2}{2} + \sum_{n=1}^{q-1} \frac{12}{(\xi_n - 1)^2(\xi_n^p - 1)} + \sum_{n=1}^{q-1} \frac{16}{(\xi_n - 1)^3(\xi_n^p - 1)}. 
\]

This completes our proof. \(\square\)
Lemma 13.4  Let $p$ and $q$ be relatively prime positive integers with $p, q \geq 2$. Then,

$$
p \sum_{n=1}^{q-1} \frac{1}{(\xi_n - 1)^2(\xi_n^p - 1)} + q \sum_{n=1}^{p-1} \frac{1}{(\omega_n - 1)^2(\omega_n^q - 1)} = \frac{1}{24} (p^2q + pq^2 + 2p^2 + 2q^2 - 18pq + 5p + 5q + 2).
$$

Proof  From Lemmas 9.3 and 11.4, we can easily obtain

$$
p \sum_{n=1}^{q-1} \frac{1}{(\xi_n - 1)^2(\xi_n^p - 1)} + q \sum_{n=1}^{p-1} \frac{1}{(\omega_n - 1)^2(\omega_n^q - 1)}
$$

$$
= p \sum_{n=1}^{q-1} \left( \frac{\xi_n}{(\xi_n - 1)^2(\xi_n^p - 1)} - \frac{1}{(\xi_n - 1)(\xi_n^p - 1)} \right)
+ q \sum_{n=1}^{p-1} \left( \frac{\omega_n}{(\omega_n - 1)^2(\omega_n^q - 1)} - \frac{1}{(\omega_n - 1)(\omega_n^q - 1)} \right)
$$

$$
= \frac{1}{24} (p^2q + pq^2 + 2p^2 + 2q^2 - 18pq + 5p + 5q + 2).
$$

Lemma 13.5  If $p$ and $q$ are relatively prime positive integers with $p, q \geq 2$, then

$$
p \sum_{n=1}^{q-1} \frac{1}{(\xi_n - 1)^3(\xi_n^p - 1)} + q \sum_{n=1}^{p-1} \frac{1}{(\omega_n - 1)^3(\omega_n^q - 1)}
$$

$$
= \frac{1}{720} (p^4 + q^4 - 5p^2q^2 - 45p^2q - 45pq^2 - 60p^2 - 60q^2 + 540pq - 135p - 135q - 57).
$$

Proof  Similarly, from Lemmas 9.4 and 13.4, it follows that

$$
p \sum_{n=1}^{q-1} \frac{1}{(\xi_n - 1)^3(\xi_n^p - 1)} + q \sum_{n=1}^{p-1} \frac{1}{(\omega_n - 1)^3(\omega_n^q - 1)}
$$

$$
= p \sum_{n=1}^{q-1} \left( \frac{\xi_n}{(\xi_n - 1)^3(\xi_n^p - 1)} - \frac{1}{(\xi_n - 1)^2(\xi_n^p - 1)} \right)
+ q \sum_{n=1}^{p-1} \left( \frac{\omega_n}{(\omega_n - 1)^3(\omega_n^q - 1)} - \frac{1}{(\omega_n - 1)^2(\omega_n^q - 1)} \right)
$$

$$
= \frac{1}{720} (p^4 + q^4 - 5p^2q^2 - 15p^2q - 15pq^2 + 15p + 15q + 3)
- \frac{1}{24} (p^2q + pq^2 + 2p^2 + 2q^2 - 18pq + 5p + 5q + 2)
$$

$$
= \frac{1}{720} (p^4 + q^4 - 5p^2q^2 - 45p^2q - 45pq^2 - 60p^2 - 60q^2 + 540pq - 135p - 135q - 57).
$$

\square
Corollary 13.6 If \( p \) and \( q \) are relatively prime positive integers with \( p, q \geq 2 \), then

\[
P q^{-1} \sum_{n=1}^{q-1} \cot \left( \frac{\pi np}{q} \right) \cot^3 \left( \frac{\pi n}{q} \right) + q \sum_{n=1}^{p-1} \cot \left( \frac{\pi nq}{p} \right) \cot^3 \left( \frac{\pi n}{p} \right)
= \frac{1}{45} \left( p^4 + q^4 - 5p^2q^2 - 15p^2 - 15q^2 + 45pq - 12 \right).
\]

Proof By Theorem 13.3, we have

\[
P q^{-1} \sum_{n=1}^{q-1} \cot \left( \frac{\pi np}{q} \right) \cot^3 \left( \frac{\pi n}{q} \right) + q \sum_{n=1}^{p-1} \cot \left( \frac{\pi nq}{p} \right) \cot^3 \left( \frac{\pi n}{p} \right)
= \frac{p(q - 1)^2}{2} + 12p q^{-1} \sum_{n=1}^{q-1} \frac{1}{(\xi_n - 1)^2(\xi_n^p - 1)} + 16p \sum_{n=1}^{q-1} \frac{1}{(\xi_n - 1)^3(\xi_n^p - 1)}
\]

Employing Lemmas 13.4 and 13.5 in (13.5) yields

\[
P q^{-1} \sum_{n=1}^{q-1} \cot \left( \frac{\pi np}{q} \right) \cot^3 \left( \frac{\pi n}{q} \right) + q \sum_{n=1}^{p-1} \cot \left( \frac{\pi nq}{p} \right) \cot^3 \left( \frac{\pi n}{p} \right)
= \frac{p(q - 1)^2}{2} + \frac{q(p - 1)^2}{2} + \frac{1}{2} \left( p^2q + pq^2 + 2p^2 + 2q^2 - 18pq + 5p + 5q + 2 \right)
+ \frac{1}{45} \left( p^4 + q^4 - 5p^2q^2 - 45p^2q - 45pq^2 - 60p^2 - 60q^2 + 540pq - 135p - 135q - 57 \right)
= \frac{1}{45} \left( p^4 + q^4 - 5p^2q^2 - 15p^2 - 15q^2 + 45pq - 12 \right).
\]

\[\Box\]

Theorem 13.7 Let \( p \) and \( q \) be positive integers with \( q \geq 2 \) and \( (p, q) = 1 \), and let \( \xi_n = e^{2\pi in/q} \). Then,

\[
T(p, q) := q^{-1} \sum_{n=1}^{q-1} \cot \left( \frac{\pi np}{q} \right) \cot \left( \frac{\pi n}{q} \right) \sin^2 \left( \frac{\pi n}{q} \right) = \frac{(q^2 - 1)}{3} + 16 \sum_{n=1}^{q-1} \frac{\xi_n}{(\xi_n - 1)^3(\xi_n^p - 1)}.
\]

Proof We apply the identities (13.1) and

\[
\sin^2 \left( \frac{\pi n}{q} \right) = -(\xi_n - 1)^2 \frac{1}{4\xi_n}
\]

to obtain

\[
T(p, q) = 4 q^{-1} \sum_{n=1}^{q-1} \frac{\xi_n(\xi_n + 1)(\xi_n^p + 1)}{(\xi_n - 1)^3(\xi_n^p - 1)} = 4 q^{-1} \sum_{n=1}^{q-1} \left\{ \frac{\xi_n(\xi_n + 1)}{(\xi_n - 1)^3} + \frac{2\xi_n(\xi_n + 1)}{(\xi_n - 1)^3(\xi_n^p - 1)} \right\}
\]

\[
= 4 q^{-1} \sum_{n=1}^{q-1} \left\{ \frac{\xi_n}{(\xi_n - 1)^2} + \frac{2\xi_n}{(\xi_n - 1)^3} + \frac{2\xi_n}{(\xi_n - 1)^2(\xi_n^p - 1)} + \frac{4\xi_n}{(\xi_n - 1)^3(\xi_n^p - 1)} \right\}.
\]
Invoking Lemmas 9.1 and 9.3, we have

\[ T(p, q) = 4 \left\{ \frac{(q^2 - 1)}{12} - \frac{2(q^2 - 1)}{24} + \frac{(q^2 - 1)}{12} \right\} + 16 \sum_{n=1}^{q-1} \frac{\xi_n}{(\xi_n - 1)^3(\xi_n^p - 1)} \]

\[ = \frac{(q^2 - 1)}{3} + 16 \sum_{n=1}^{q-1} \frac{\xi_n}{(\xi_n - 1)^3(\xi_n^p - 1)}. \]

Hence, we complete the proof. \(\Box\)

Observe from Corollary 8.7 and (8.4) that

\[ T(p, q) = -\frac{8}{3} k^3 S_3(1, k). \]

A completely different proof of Corollary 13.8 was given by McIntosh [23, Theorem 3, p. 199].

**Corollary 13.8** Let \( p \) and \( q \) be positive integers with \( p, q \geq 2 \) and \( (p, q) = 1 \). Then,

\[ 45pT(p, q) + 45qT(q, p) = p^4 + q^4 - 5p^2q^2 + 3. \]

**Proof** Let \( \omega_n = e^{2\pi in/p} \) and \( \xi_n = e^{2\pi in/q} \). By Theorem 13.7, we have

\[ 45pT(p, q) + 45qT(q, p) = p^4 + q^4 - 5p^2q^2 + 3 \]

\[ = 15p(q^2 - 1) + 720p \sum_{n=1}^{q-1} \frac{\xi_n}{(\xi_n - 1)^3(\xi_n^p - 1)} + 15q(p^2 - 1) \]

\[ + 720q \sum_{n=1}^{p-1} \frac{\omega_n}{(\omega_n - 1)^3(\omega_n^q - 1)} \]

\[ = 15(p^2q + pq^2 - p - q) + (p^4 + q^4 - 5p^2q^2 - 15p^2q - 15pq^2 + 15p + 15q + 3) \]

\[ = p^4 + q^4 - 5p^2q^2 + 3, \]

where we employed Lemma 9.4 to the sums in the second equality above. \(\Box\)

**Theorem 13.9** Let \( p \) and \( q \) be relatively prime positive integers such that \( q \geq 3 \) is odd, and let \( \xi_j = e^{2\pi ij/q} \). Then,

\[ \sum_{n=1}^{q-1} \frac{\cos^2 \left( \frac{\pi n\xi_j}{q} \right)}{\cos^2 \left( \frac{\pi n}{q} \right)} = \frac{q-1}{8} \sum_{n=1}^{q-1} \frac{\xi_n}{(\xi_n^2 - 1)(\xi_n^p - 1)}. \]

**Proof** From (13.1) and the identity

\[ \cos^2 \left( \frac{\pi n}{q} \right) = \frac{(\xi_n + 1)^2}{4\xi_n}, \quad (13.6) \]
it follows that

\[
\sum_{n=1}^{q} \frac{q \cot \left( \frac{\pi np}{q} \right) \cot \left( \frac{\pi n}{q} \right)}{\cos^2 \left( \frac{\pi n}{q} \right)} = -4 \sum_{n=1}^{q} \frac{\xi_n (\xi_n^p + 1)}{(\xi_n^2 - 1)(\xi_n^2 + 1)(\xi_n^p - 1)}
\]

\[
= -4 \sum_{n=1}^{q} \frac{\xi_n}{(\xi_n^2 - 1)} \left( 1 + \frac{2}{\xi_n^p - 1} \right)
\]

\[
= -2 \sum_{n=1}^{q} \left( \frac{1}{\xi_n - 1} + \frac{1}{\xi_n + 1} \right) - 8 \sum_{n=1}^{q} \frac{\xi_n}{(\xi_n^2 - 1)(\xi_n^p - 1)}.
\]

By Lemmas 7.2 and 7.3, we see that the first summation in the final equality is equal to 0. This completes the proof. □

**Corollary 13.10** Let \( p \) and \( q \) be odd positive integers with \( p, q \geq 3 \) and \((p, q) = 1\). Then,

\[
\frac{q - 1}{p} \sum_{n=1}^{p} \frac{\frac{\xi_n - 1}{\xi_n}}{\xi_n - 1} + q \sum_{n=1}^{q} \frac{\frac{\xi_n - 1}{\xi_n}}{\xi_n - 1} = -\frac{1}{3} (p^2 + q^2 - 2).
\]

**Proof** Applying Theorem 13.9, and Lemmas 11.4 and 11.5, we obtain

\[
\frac{q - 1}{p} \sum_{n=1}^{p} \frac{\frac{\xi_n - 1}{\xi_n}}{\xi_n - 1} + q \sum_{n=1}^{q} \frac{\frac{\xi_n - 1}{\xi_n}}{\xi_n - 1} = -4p \sum_{n=1}^{p} \frac{\xi_n}{(\xi_n^2 - 1)(\xi_n^p - 1)} - 8q \sum_{n=1}^{q} \frac{\omega_n}{(\omega_n^2 - 1)(\omega_n^p - 1)}
\]

\[
= -4p \sum_{n=1}^{p} \left( \frac{1}{\xi_n - 1} + \frac{1}{\xi_n + 1} \right) \frac{1}{(\xi_n^p - 1)} - 4q \sum_{n=1}^{p} \left( \frac{1}{\omega_n - 1} + \frac{1}{\omega_n + 1} \right) \frac{1}{(\omega_n^p - 1)}
\]

\[
= \frac{1}{3} (p^2 + q^2 - 9pq + 3p + 3q + 1) + 3pq - p - q - 1
\]

\[
= \frac{1}{3} (p^2 + q^2 - 2).
\]

□

**Theorem 13.11** Let \( p \) and \( q \) be positive integers with \( p, q \geq 2 \) and \((p, q) = 1\). Then,

\[
\sum_{n=1}^{q} \frac{\frac{\xi_n - 1}{\xi_n}}{\xi_n - 1} = \frac{q^2 - 1}{3} + 16 \sum_{n=1}^{q} \frac{\xi_n}{(\xi_n^2 - 1)^3(\xi_n^p - 1)}.
\]
Similarly, by (13.1) and (13.6), we see that
\[ q^{-1} \sum_{n=1}^{q-1} \frac{\cot \left( \frac{\pi n p}{q} \right) \cot^3 \left( \frac{\pi n}{q} \right)}{\cos^2 \left( \frac{\pi n}{q} \right)} = 4 \sum_{n=1}^{q-1} \frac{\xi_n (\xi_n + 1)(\xi_n^p + 1)}{(\xi_n - 1)^3(\xi_n^p - 1)} \]
\[ = 4 \sum_{n=1}^{q-1} \left( 1 + \frac{2}{\xi_n^p - 1} \right) \left( 1 + \frac{2}{\xi_n - 1} \right) \frac{\xi_n}{(\xi_n - 1)^2} \]
\[ = 4 \sum_{n=1}^{q-1} \left\{ \frac{\xi_n}{(\xi_n - 1)^2} + \frac{2\xi_n}{(\xi_n - 1)^3} + \frac{2\xi_n}{(\xi_n - 1)^2(\xi_n^p - 1)} + \frac{4\xi_n}{(\xi_n - 1)^3(\xi_n^p - 1)} \right\} \]
\[ = \frac{q^2 - 1}{3} + 16 \sum_{n=1}^{q-1} \frac{\xi_n}{(\xi_n - 1)^3(\xi_n^p - 1)}, \]
where we applied Lemma 9.1 to the first two sums on the right-hand side of the third equality above, and Lemma 9.3 to the third sum on the right-hand side of the same equality.

\[ 15 \text{ Final remarks} \]
There exist further reciprocity theorems for cotangent sums, some of which are related to reciprocity theorems for analogues or generalizations of Dedekind sums. For example, see papers of Dieter [13] and Fukuhara [17]. These papers also contain several references to further work. Dedekind sums appear in the modular transformation formula for the Dedekind eta-function. Are any of the trigonometric sums appearing in our new reciprocity theorems related to modular transformation formulas for other functions?

In this paper, we have concentrated on the evaluations of sums of sines or cotangents. The literature contains many elegant evaluations of other kinds of trigonometric sums, for example, sums of cosines or cosecants. Perhaps the methods of the present paper have applications to these sums as well.
Evaluations of certain analogues of Gauss sums are given in Sect. 4. In this paper and in [6], further evaluations of trigonometric sums involving Dirichlet characters are derived. We feel that our techniques involving characters can be employed to evaluate several further trigonometric sums with characters.

We developed a theory of finite sums involving roots of unity in which powers up to 4 appear in the summands. One might extend our techniques to handle sums of roots of unity involving powers greater than 4.

At the end of Sect. 2, we remarked that the evaluation of a certain sum of sines has an arithmetic interpretation. It would be interesting to investigate the possibility of further arithmetic interpretations of trigonometric sum evaluations.

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Data Availability
Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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