Nonlinear dynamics of rectangular nano-shells

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Abstract. Mathematical model of non-linear vibrations of shallow, elastic, isotropic nano-shells with rectangular base subjected to transverse sign-variable load are constructed. Based on Kirchhoff-Love thin shell theory with von Kármán nonlinear strains and the modified couple stress theory (MCST), size-dependent governing equations and corresponding boundary conditions are established through Hamilton’s principle. The governing PDEs are reduced to ODEs by the second-order Finite Difference Method (FDM). The obtained system of equations is solved by the Runge–Kutta methods of second order accuracy. The Cauchy problem is solved by the Runge–Kutta fourth-order method. We analyzed the convergence of these solutions depending on the step of integration over time and spatial coordinate. It was revealed, that taking into account nano-effects increases area of harmonic vibrations and leads to the appearance of as chaotic and hyperchaotic vibrations. The carried out numerical experiment shows, that the transition of vibrations from harmonic to chaotic follows to Feigenbaum’s scenario. In particular, to analyze the character type of vibration computation of largest Lyapunov exponents are employed. We found that hyperchaotic vibrations are characterized by two positive Lyapunov exponents and chaotic vibration by one positive Lyapunov exponent. For nano-shells, this phenomenon was discovered for the first time. Lyapunov exponents spectra estimated by different algorithms, including Wolf’s, Rosenstein’s, Kantz’s, and Sawada. Numerical examples of the theoretical investigations are given.

1. Introduction
In recent years, one of the most intensively and dynamically developing areas of industry is the microsystem technology, which includes micro- and nano-electromechanical systems (MEMS and NEMS). Microelectromechanical systems (MEMS) are successfully used in accelerometers, digital micro mirrors, pressure sensors, micro pumps. Such devices are used in the oil and gas industry, mechanical engineering, instrument making, robotics, mechatronics and medicine, for example, atomic force microscopes [1-3], micro drives [4], sensors [5], solid state gyroscopes, etc. Flexible rectangular shallow shells are of interest for use as MEMS/NEMS resonators. The composite devices including MEMS/NEMS are typically exposed to the environment and operating in adverse conditions. Requires the use of new MEMS / NEMS and their structural elements to maintain the integrity of the sensors and ensure operation in harsh environments, including chaotic vibrations. One of the first studies in the field of nano-mechanical effects belongs to Wang Y.C. et al. [6]. He presented a theoretical analysis and experimental results on the dynamic behavior of a bistable resonator in a micro-electromechanical system and demonstrated the existence of a strange attractor and chaos. In the work of Reni [7] was noted that often the error of the solution is taken as chaos. This is possible when the approximate solution is obtained by one method. In [8], a methodology for determining the truth of chaotic vibrations in the study of spherical axisymmetric shells with round base was proposed in response. It is known that when the thickness of the shells about a micron and submicron mechanical
behavior and deformation characteristics depend on the size [9, 10]. Therefore, the influence of size should be taken into account when analyzing the mechanical behavior of nanostructures. Articles [11, 12] devoted to the study of mechanical nanostructures. Most of the works devoted to the study of chaotic vibrations, both in macro and in micro-dimensional structures, was considered only the phenomenon of chaotic vibrations. In the number of studies have revealed that in addition to chaotic vibrations, there are also hyper-chaotic [13] vibrations, as well as other types of vibrations. This work is aimed at studying the chaotic dynamics of nano-shells and determining the "true" vibrations. As well as the study of new phenomena, such as hyperchaos.

2. Problem statement
Let us consider a shallow rectangular nano-shell in plan (the flatness is formulated according to the Vlasov theory [14]) with dimensions a, b, h along axes x, y, z respectively. The origin of the coordinate system is located in the up-per left corner of the shell, on its middle surface. The axes x, y are parallel to the shell sides and the axis z is directed towards the shell curvature (Fig. 1). In the given coordinate system, the shell is treated as a 3D region Ω defined by

$$\Omega = \{ x, y, z | (x, y, z) \in [0, a] \times [0, b] \times [-h/2, h/2] \}.$$  

The shell middle surface \(z = 0\) is defined as \(\Gamma = \{ x, y | (x, y) \in [0, a] \times [0, b] \}\). The following hypotheses hold:

1. The shell material is isotropic and elastic;
2. Geometric nonlinearity is taken in Kármán’s form;
3. The Kirchhoff-Love kinematic shell model is used.

The mathematical model is yielded by the Hamilton’s variational principle, taking into account the modified couple stress theory and presented in the following form:

$$\left( \frac{Eh^3(1-\nu)}{12(1+\nu)(1-2\nu)} + \frac{l^2 Eh}{2(1+\nu)} \right) \nabla^4 w + \nabla^2 F - hL(w, F) + \rho h \frac{\partial^2 w}{\partial t^2} + 2\rho \frac{\partial w}{\partial t} - q(x, y, t) = 0, \quad (1)$$

$$\frac{1}{Eh} \nabla^4 F + \nabla^2 w + \frac{1}{2} L(w, w) = 0;$$

where \(L(w, w); L(w, F)\) are the nonlinear operators:

$$L(w, w) = \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2; L(w, F) = \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 F}{\partial x \partial y} - 2 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 F}{\partial x \partial y};$$

$$\nabla^2 w = k_x \frac{\partial^2 (w)}{\partial y^2} + k_y \frac{\partial^2 (w)}{\partial x^2}; \quad k_x = \frac{1}{R_x}; \quad k_y = \frac{1}{R_y}.$$  

The following boundary condition is supplemented to Eq. (1):

Simple support on flexible non-stretched ribs:

$$w = 0; w^* = 0; F = 0 \quad \text{if} \quad x = 0; a$$
$$w = 0; w^* = 0; F = 0 \quad \text{if} \quad x = 0; b$$
As the initial conditions, we take the distribution of the deflections, deflections velocities at the initial time instant \( t = 0 \):

Initial conditions:

\[
\begin{align*}
\left. w(x, y) \right|_{t=0} &= 0, \quad \left. \frac{\partial w}{\partial t} \right|_{t=0} = 0; \\
\end{align*}
\]

(3)

Using the theory of similarity and dimension, the system of differential Eqs. (1 – 3), can be recast to its counterpart in dimension-less form by means of the following relations:

- \( \lambda = \frac{a}{b} \) - geometric parameter;
- \( \gamma = \frac{l}{h} \) - size-dependent parameter;
- \( x = a\xi, y = b\eta; w = h\tilde{w} \) - deflection;
- \( F = Eh^3\tilde{F} \) - function of effort;
- \( t = \frac{ab}{h} \sqrt{\frac{g\gamma}{E}} \) - time;
- \( q = \frac{Fh^4}{a^4b^2\gamma} \) - external pressure;
- \( \varepsilon = \frac{ab}{h} \sqrt{\frac{gE}{\gamma}} \) - area damping coefficient,
- \( h \) - shell thickness,
- \( a, b \) - shell dimensions in plan,
- \( \nu \) - Poisson’s ratio,
- \( E \) - modulus of elasticity of the material,
- \( g \) - acceleration of gravity,
- \( l \) - characteristic shell size,
- \( \gamma_1 \) - specific weight of the material,
- \( \kappa_x, R_x, R_y \) - the radius of the shells,
- \( k_x = \frac{a^2}{hR_x}; k_y = \frac{b^2}{hR_y} \) - dimensionless curvature parameters.

**Table 1.** Investigation of the convergence of finite difference methods depending from the parameters \( k_x, k_y; n \); and parameter \( \gamma \)

| \( \gamma \) | \( k_x = 0; k_y = 12; n = 8 \times 8 \) | \( k_x = 0; k_y = 24; n = 8 \times 8 \) | \( k_x = 12; k_y = 12; n = 8 \times 8 \) |
|---|---|---|---|
| \( \gamma = 0 \) | ![Image](image1.png) | ![Image](image2.png) | ![Image](image3.png) |
| \( \gamma = 0.1 \) | ![Image](image4.png) | ![Image](image5.png) | ![Image](image6.png) |
| \( \gamma = 0.3 \) | ![Image](image7.png) | ![Image](image8.png) | ![Image](image9.png) |
| \( \gamma = 0.5 \) | ![Image](image10.png) | ![Image](image11.png) | ![Image](image12.png) |
### Table 2. Convergence of solutions

| $\gamma = 0$ | $k_x = 0; k_y = 12$ |
|-------------|---------------------|
| $n = 8 \times 8$ | $n = 16 \times 16$ |

| $k_x = 0; k_y = 24$ |
| $n = 8 \times 8$ | $n = 16 \times 16$ |

| $k_x = 12; k_y = 12$ |
| $n = 8 \times 8$ | $n = 16 \times 16$ |

### Table 3. Legend

| Color | Description          |
|-------|----------------------|
| Blue  | Harmonic vibrations  |
| Cyan  | Vibrations at 2 independent frequencies |
| White | Chaotic state        |
| Black | Damped vibrations    |
| Yellow| Bifurcation states   |
| Purple| Undefined state      |

### 3. Results and discussions

Equation (1-3) is reduced to the Cauchy problem by the finite-difference method of the second order accuracy. The Cauchy problem is solved via methods of the Runge-Kutta and Newmark type. The convergence of these methods by depending from the number of partition points by spatial coordinates and time, as well as from by depending the size-dependent parameter $(\gamma = 0; 0.1; 0.3; 0.5)$ for values $k_x = k_y = 12; k_x = 0, k_y = 12; k_x = 0, k_y = 24$ was studied and analyzed. The charts of the character...
vibrations by depending from the frequency of forced vibrations $\omega_p$ and the amplitude of the forced load $q_0 (q = q_0 \times \sin \omega_p \times t)$ were built.

![Graph showing the dependence of the maximum deflection in the center of the shell on the intensity of the transverse alternating load for the size-dependent parameter $\gamma = 0.1$.](image1)

**Figure 2.** Dependence of the maximum deflection in the center of the shell on the intensity of the transverse alternating load for the size-dependent parameter $\gamma = 0.1$.

![Graph showing the dependence of the maximum deflection in the center of the shell on the intensity of the transverse alternating load for the size-dependent parameter $\gamma = 0.3$.](image2)

**Figure 3.** Dependence of the maximum deflection in the center of the shell on the intensity of the transverse alternating load for the size-dependent parameter $\gamma = 0.3$.

Since each chart gives a graphic representation of vibration types, an important characteristic is the number of points in the chart (versions of computation) — the resolution of the chart. To determine the optimal resolution we obtained and analyzed charts with the following resolutions $(q_0 \times \omega_p)$: 100 $\times$ 100, 200 $\times$ 200, 300 $\times$ 300, and 400 $\times$ 400. From an analysis of the Fourier power spectra, Morlet wavelet spectra, Poincare section, autocorrelation functions the dependence of quality of data representation on the program running time, a resolution of 300 $\times$ 300 was chosen. To obtain one chart with this resolution, it was necessary to calculate and analyze $9 \times 10^4$ datasets. The result of a study of
complex nonlinear dynamic problems depends on the method used to reduce the dimension of the problem. These charts are obtained by partitioning area of integration by the method of finite differences of the second order of accuracy on \( n = 8 \times 8; n = 16 \times 16 \) parts. An analysis of these charts shows that the size-dependent parameter \( \gamma \) and partitioning the area of integration by the finite difference method has a significant effect on the convergence of the solution. It should also be noted that the size-dependent parameter \( \gamma \) significantly changes the natural frequency of the vibrations (tab. 1-3; Fig. 2-4).

To obtain reliable results, the spectrum of Lyapunov exponents is determined by several methods of Kantz [15], Wolf [16], Rosenstein [17] and Sawada [18]. In fig. (2-4) the dependences of the maximum deflection \( w_{\text{max}} \) in the center of the shell \( a = b, \lambda = 1, k_x = k_y = 12 \), on the intensity of the transverse alternating load \( q_0 \) and the scale character of the vibrations for the size-dependent parameter \( \gamma = 0.1; 0.3; 0.5 \). Frequency \( \omega_p \) corresponds to the natural frequency of linear vibrations.

![Figure 4](image)

**Figure 4.** Dependence of the maximum deflection in the center of the shell on the intensity of the transverse alternating load for the size-dependent parameter \( \gamma = 0.5 \).

4. **Conclusions**

1. A mathematical model of non-linear vibrations of rectangular in plan shallow nano-shells under the action of a transverse alternating load was developed. The mathematical model was constructed taking into account Kirchhoff-Love hypothesis, geometric nonlinearity von Kármán and modified couple stress theory. The required equations are obtained from the Hamilton’s variational principle.

2. Also, an algorithm and a software for numerical studies of the convergence of the method of finite differences of the second order of accuracy depending on the number of points of the division of the nano-shell according to \( O_x \) and \( O_y \) have been developed and considered. A few solutions of particular problems have been obtained by using alternative approaches (methods of the Runge-Kutta type from 4th to 6th order and the Newmark method). The results of the solutions have been illustrated and discussed.

3. It was revealed that with an increase in the size-dependent parameter \( \gamma \) the vibrations have a smaller amplitude, the transition to chaotic vibrations occurs at high loads than at \( \gamma = 0 \). Areas of control parameters \( \{q_0; \omega_p\} \) with increasing \( \gamma \) subarea of harmonic vibrations increases.
4. The sign of spectrum of Lyapunov exponents has been also determined by different algorithms (Kantz, Wolf, Rosenstein and Sawada). This allows us to conclude that our results are reliable and describe the behavior of the system of an infinite number of degrees of freedom.

5. The analysis of the scenarios of transition from periodic to chaotic vibrations has been carried out. For the first time discovered presence of chaos and hyperchaos regions for nonlinear vibrations of nano-shells.

5. Acknowledgments

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6. References

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