THE SCATTERING PROBLEM IN RYCKMAN’S CLASS OF JACOBI MATRICES

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ABSTRACT. We give a complete solution of the scattering problem for Jacobi matrices from a class which was recently introduced by E. Ryckman. We characterize the scattering data for this class and illustrate the inverse scattering on some simple examples.

1. INTRODUCTION

In mid 70th Guseinov [8, 9] developed a scattering theory for infinite Jacobi matrices

\[ J = J(\{a_n\}, \{b_n\}) = \begin{bmatrix}
    b_1 & a_1 & 0 & \ldots \\
    a_1 & b_2 & a_2 & \ldots \\
    0 & a_2 & b_3 & \ldots \\
    \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \tag{1.1}
\]

\(a_n > 0, b_n = \bar{b}_n\), which can be viewed as a discrete version of the scattering theory for one-dimensional Schrödinger operator on the half-line by Marchenko–Faddeev. The basic assumption on \(J\) is the finiteness of the first moment

\[ \sum_{n=1}^{\infty} n(|a_n - 1| + |b_n|) < \infty. \tag{1.2} \]

We say that a Jacobi matrix \(J = J(\{a_n\}, \{b_n\})\) belongs to Guseinov’s class \(\mathcal{G}\) if its parameters satisfy (1.2). Later Geronimo [4, 5] (see also [6]) solved the spectral problem for Jacobi matrices in more general “weighted” Guseinov’s classes by using the inverse scattering technique. The main feature of his results is that the decay of the Jacobi parameters \(\{a_n - 1\}, \{b_n\}\) manifests itself in the decay of the Fourier coefficients of the absolutely continuous part (after suitable modifications) of the measure.

In the modern scattering theory of Jacobi operators (see, e.g., [3]) the various analogues of (1.2) (for much more complex backgrounds) appeared, and they seemed to be indispensable.

In 2007 Ryckman [14, 15, 16] came up with a new class of Jacobi matrices, for which he obtained a complete spectral description. To state his result we introduce some notations and definitions. Let us write

\[ \beta = \{\beta_n\} \in l^2_s, \quad s > 0 \quad \text{if} \quad ||\beta||^2 = \sum_n |\beta_n|^2 < \infty. \]

Date: February 14, 2010.

∗The work was supported by the University of Massachusetts Lowell Research and Scholarship Grant, project number: H50090000000010.

∗∗The work was supported by the Austrian Science Found FWF, project number: P22025–N18.
Definition 1.1. A Jacobi matrix $J = J(\{a_n\}, \{b_n\})$ belongs to Ryckman’s class $\mathcal{R}$, or its spectral measure $\sigma(J) \in \mathcal{R}$, if the series $\sum_n (a_n - 1)$ and $\sum_n b_n$ are conditionally summable, and

$$ \xi_n := - \sum_{k=n+1}^{\infty} b_k \in \ell_1^2, \quad \eta_n := - \sum_{k=n+1}^{\infty} (a_k - 1) \in \ell_1^2 $$

Definition 1.2. A function $g$ on the unit circle $\mathbb{T}$ is said to be in the Besov class $B_{2}^{1/2}$ if the sequence of its Fourier coefficients is in $\ell_1^2(\mathbb{Z})$

$$ g(t) = \sum_{n \in \mathbb{Z}} g_n t^n, \quad \sum_{n \in \mathbb{Z}} |n||g_n|^2 < \infty. \quad (1.3) $$

Also, a function $f$ on the interval $[-2, 2]$ is said to be in $B_{2}^{1/2}$ if $\hat{f}(t) := f(t + \frac{1}{2})$ is in $B_{2}^{1/2}$.

Note that $\hat{f}$ is a symmetric function, $\hat{f}(t) = \hat{f}(t)$, and conversely, each symmetric function has the form $\hat{f}$. If

$$ \hat{f}(t) = \sum_{n \in \mathbb{Z}} \hat{f}_n t^n, $$

then the symmetry of $\hat{f}$ implies $\hat{f}_{-n} = \hat{f}_n$. If $\hat{f}$ is in addition a real function, then $\hat{f}_{-n} = \overline{\hat{f}_n}$.

Theorem 1.3 (Ryckman). $J \in \mathcal{R}$ if and only if the spectral measure $\sigma(J)$ of $J$ has this structure:

- The absolutely continuous part is supported by $[-2, 2]$, and

$$ \sigma_{ac}(dx) = f(x, J)dx = \rho(x, J)\sqrt{4 - x^2} dx, \quad \rho(x, J) = \frac{\rho_0(x, J)}{(2 - x)^{\gamma_1}(2 + x)^{\gamma_2}} \quad (1.4) $$

with $\gamma_1, \gamma_2$ equal 0 or 1, and $\log \rho_0 \in B_{2}^{1/2}$.

- The singular part is

$$ \sigma_{s}(dx) = \sum_{k=1}^{N} \sigma_k(J)\delta(\lambda_k), \quad N = N(J) < \infty, \quad \sigma_k(J) > 0, \quad \lambda_k \in \mathbb{R}\setminus[-2, 2]. \quad (1.5) $$

Note that $\mathcal{G} \subset \mathcal{R}$, and the inclusion is proper. Indeed,

$$ \sum_{n=1}^{\infty} n|\xi_n|^2 = \sum_{n=1}^{\infty} n \left( \sum_{k=n+1}^{\infty} b_k \right)^2 = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \sum_{l=n}^{\infty} n|b_k||b_l| $$

$$ \leq \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \sum_{l=n}^{\infty} l|b_k||b_l| \leq \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} k|b_k||b_l| $$

$$ = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} k|b_k||b_l| = \sum_{k=1}^{\infty} k \sum_{l=1}^{\infty} |b_k||b_l| = \left( \sum_{k=1}^{\infty} k|b_k| \right)^2. $$

Similarly,

$$ \sum_{n=1}^{\infty} n|\eta_n|^2 \leq \left( \sum_{k=1}^{\infty} k|a_k - 1| \right)^2. $$
On the other hand,  \( J(\{a_n\}, \{b_n\}) \) with  
\[
a_n = 1 + (-1)^n n^{-1-\epsilon}, \quad b_n = (-1)^n n^{-1-\epsilon}, \quad 0 < \epsilon < 1
\]
belongs to \( \mathcal{R} \), but (1.2) is false.

Given \( \sigma(J) \in \mathcal{R} \), define two outer functions which are key ingredients in both spectral and scattering theory of the class \( \mathcal{R} \). First, we put
\[
D_0(z) = D_0(z, J) := \exp \left\{ \frac{1}{2} \int_\mathbb{T} \frac{t + z}{t - z} \log \hat{\rho}_0(t, J) m(dt) \right\}
\]
(1.6)
with
\[
u_0(z) = \int_\mathbb{T} P_z(t) \log \hat{\rho}_0(t, J) m(dt), \quad v_0(z) = \int_\mathbb{T} Q_z(t) \log \hat{\rho}_0(t, J) m(dt) = \tilde{v}_0(z)
\]
a harmonic conjugate to \( u_0 \),
\[
u_0(t) = \log \hat{\rho}_0(t, J) = \sum_{k \in \mathbb{Z}} r_k t^k; \quad v_0(t) = \frac{1}{i} \sum_{k \in \mathbb{Z}} (\text{sgn } k) r_k t^k;
\]
both \( u_0 \) and \( v_0 \) are real, \( u_0 \) is symmetric, and \( v_0 \) antisymmetric: \( v_0(t) = -v_0(t) \). So
\[
D_0(t) = \overline{D_0(t)}, \quad |D_0(t)|^2 = \hat{\rho}_0(t, J)
\]
(1.7)
auto everywhere on \( \mathbb{T} \). It is known (cf., e.g., [17, Proposition 6.1.5]) that \( \log \hat{\rho}_0 \in B_{2}^{1/2} \) implies \( (\hat{\rho}_0)^{\pm 1} \in L^p(\mathbb{T}) \) for \( p < \infty \), so \( D_{0}^{-1} \in H^p \) for such \( p \).

Secondly, we put
\[
D(z) = D(z, J) := \exp \left\{ \frac{1}{2} \int_\mathbb{T} \frac{t + z}{t - z} \log \hat{\rho}(t, J) m(dt) \right\} = \frac{D_0(z, J)}{(1 - z)^{1/2}(1 + z)^{1/2}}.
\]
(1.8)
Both \( D_0 \) and \( D \) are related to the absolutely continuous part of the spectral measure. The discrete part is completely determined by the set of eigenvalues \( \{\lambda_k\} \), or equivalently, by the set
\[
Z(J) := \left\{ z_k(J) : \lambda_k = z_k(J) + \frac{1}{z_k(J)}, \quad k = 1, 2, \ldots, N \right\},
\]
(1.9)
z_k(J) \in (-1, 1) \setminus \{0\}, and by the set of masses \( \{\sigma_k(J)\}_{k=1}^N \) in (1.5).

**Definition 1.4.** Given \( J \in \mathcal{R} \), under the scattering data for \( J \) we mean the following collection \( \{\gamma_1(J), \gamma_2(J); Z(J); \mu_1(J), \ldots, \mu_N(J); s(t, J)\} \)
(1) A pair \( (\gamma_1(J), \gamma_2(J)) \) from (1.4);
(2) The set \( Z(J) \) from (1.9), or equivalently, a finite Blaschke product
\[
B(z, J) = \prod_{k=1}^{N} \frac{z - z_k(J)}{|z_k(J)|} \frac{1 - z_k(J)z}{1 - z_k(J)}, \quad B(t, J) = \frac{1}{B(t, J)}, \quad t \in \mathbb{T};
\]
(1.10)
(3) \( N = N(J) \) positive numbers
\[
\mu_k(J) := \sigma_k(J) \left| \frac{B'(z_k(J))}{D(z_k(J))} \right|^2 |1 - z_k(J)|^{-2} > 0, \quad k = 1, 2, \ldots, N;
\]
(1.11)
The scattering function
\[ s(t, J) := \phi_0(t, J) \phi_0(\bar{t}, J), \]  

(1.12)

\( \phi_0 \) is the Jost function for \( J \) (see Section 2).

Compared to [4, 5] we move backward, from spectral to scattering. The goal of the present note is to obtain a complete characterization of the scattering data in Ryckman’s class, and so demonstrate that the scattering theory goes far beyond Guseinov’s class (1.2). We analyze the scattering data, prove the uniqueness theorem in Section 2, and solve the inverse scattering problem in Section 3. A few simple examples are given in Section 4.

2. Scattering data

The basic three-term recurrence relation for a Jacobi matrix \( J(\{a_n\}, \{b_n\}) \)
\[ a_{n-1}y_{n-1} + b_ny_n + a_ny_{n+1} = (z + z^{-1})y_n, \quad n = 1, 2, \ldots, \quad a_0 = 1 \]
has two “distinguished” solutions. The first one, known as the sine-type solution, is
\[ y_n = s_n(z) = p_{n-1} \left( z + \frac{1}{z} \right), \quad s_0 = 0, \quad s_1 = 1, \]
\( p_k \) are orthonormal polynomials with respect to the spectral measure \( \sigma(J) \). A fundamental result by Szegő concerns an asymptotic behavior of orthonormal polynomials with respect to “nice” measures with \( \text{supp}\sigma \subset [-2, 2] \). It was extended substantially in [7, 11] and [12], where a finite (respectively, infinite) number of mass points outside \([-2, 2] \) is allowed. In our notation the Szegő asymptotics states that for \( J \in \mathcal{R} \)
\[ Q(z) := \lim_{n \to \infty} z^n p_n \left( z + \frac{1}{z} \right) \frac{B(z, J)}{(1 - z^2)D(z, J)} \]  

(2.1)
uniformly on the compact subsets of the unit disk \( \mathbb{D} \).

The second solution is the Jost solution \( y_n = \phi_n \), defined by a specific asymptotic behavior at infinity
\[ \lim_{n \to \infty} z^{-n} \phi_n(z, J) = 1 \]
uniformly on the compact subsets of the unit disk \( \mathbb{D} \). The Jost solution exists under certain additional assumptions (cf. [18, formulae (13.9.2)–(13.9.4)]), which are met for \( J \in \mathcal{R} \). \( \phi_0 \) is called the Jost function. For an exhaustive treatment of the Szegő and the Jost asymptotics see [1], [18, Section 13.9].

The relation between the Szegő asymptotics and the Jost function is given by (see, e.g., [2], [18, Theorem 13.9.2])
\[ \phi_0(z) = (1 - z^2)Q(z) = \frac{B(z, J)}{D(z, J)}. \]  

(2.2)
Hence, for \( J \in \mathcal{R} \) \( \phi_0 \in H^p \) with \( p < \infty \), so the boundary values of \( \phi_0 \) exist a.e., and the scattering function \( s \) (1.12) is well defined. Moreover, by (1.8)
\[ s(t, J) = \frac{(1 - t)\gamma(J)(1 + t)\gamma(J)}{(1 - \bar{t})\gamma(J)(1 + \bar{t})\gamma(J)} \frac{D_0(\bar{t}, J)}{D_0(t, J)} B^2(t, J) \]
\[ = (-1)^{\gamma(J)} t^{\gamma(J) + \gamma(J)} \frac{D_0(\bar{t}, J)}{D_0(t, J)} B^2(t, J). \]  

(2.3)
Clearly, \( s(t) = s(t) = s^{-1}(t) \).

**Theorem 2.1.** The scattering function of a Jacobi matrix \( J \in \mathcal{R} \) belongs to \( B_{2}^{1/2} \) and admits representation

\[
s(t, J) = (-1)^{\gamma(t)} t^{M} e^{-i\nu(t)},
\]

(2.4)

where \( M = 2N + \gamma_1(J) + \gamma_2(J) \in \mathbb{Z}_+ \), \( \nu \) satisfies

\[
v(t) = \overline{v(t)} = -v(\overline{t}), \quad v \in B_{2}^{1/2}.
\]

(2.5)

**Proof.** We apply (2.3). By (1.6)

\[
\frac{D_0(t, J)}{D_0(t, J)} = \frac{|D_0(t, J)|^2}{|D_0(t, J)|^2} = e^{-i\nu_0(t)}.
\]

Since \( u_0 = \log \tilde{\rho}_0 \in B_{2}^{1/2} \), and the Hilbert transform is bounded (isometric) in \( B_{2}^{1/2} \), then \( v_0 \in B_{2}^{1/2} \) and antisymmetric. Next,

\[
B^2(t, J) = t^{2N} \left( \prod_{k=1}^{N} \frac{1 - z_k t}{1 - z_k \overline{t}} \right)^2 = t^{2N} e^{-i\nu_1(t)},
\]

and as \( z_k \in \mathbb{D} \) then \( v_1 \in B_{2}^{1/2} \) and antisymmetric. It remains to put \( v = v_0 + v_1 \). It is clear from (2.4) that \( s \in B_{2}^{1/2} \), as claimed. \( \square \)

**Remark 2.2.** Representation (2.4) for the scattering function of \( J \in \mathcal{R} \) is a direct consequence of (2.3) and \( \log \tilde{\rho}_0 \in B_{2}^{1/2} \). A slight refinement of Peller’s theorem [13, Corollary 7.8.2] states that an arbitrary function

\[
B^2(t, J) = t^{2N} \left( \prod_{k=1}^{N} \frac{1 - z_k t}{1 - z_k \overline{t}} \right)^2 = t^{2N} e^{-i\nu_1(t)},
\]

(2.6)

a.e. admits representation

\[
h(t) = (-1)^{\gamma} t^{j} e^{-i\nu(t)},
\]

where \( \gamma = 0 \) or 1, \( j \in \mathbb{Z} \) an integer number, \( w \) satisfies (2.5), and such representation is unique. A pair \((\gamma, j)\) can be viewed as an index of \( h \).

Let us turn to the numbers \( \mu_k(J) \) (1.11). In his version of the scattering theory for Jacobi matrices (1.2) Guseinov suggested the normalizing constants

\[
m_k(J) := \sum_{n=1}^{\infty} |\phi_n(z_k(J), J)|^2, \quad k = 1, 2, \ldots, N
\]

(2.6)

as a part of the scattering data. We show that these values agree.

**Proposition 2.3.** Let \( J \in \mathcal{R} \). Then \( \mu_k(J) = m_k(J) \), \( k = 1, 2, \ldots, N \).

**Proof.** It is known from the general theory of Jacobi matrices and orthogonal polynomials, that the vectors

\[
\Pi_k = \{ s_n(z_k(J)) \}_{n \geq 1} = \{ p_n(\lambda_k) \}_{n \geq 0} \in \ell^2,
\]

so \( \Pi_k \) are eigenvectors of \( J \) with the corresponding eigenvalues \( \lambda_k \). Furthermore,

\[
\frac{1}{\sigma_k(J)} = \sum_{n=1}^{\infty} |s_n(z_k(J))|^2 = \sum_{n=0}^{\infty} |p_n(\lambda_k)|^2.
\]

(2.7)
On the other hand, $\phi_0(z_k(J)) = 0$, and so $\Phi_k = \{\phi_n(z_k(J), J)\}_{n \geq 1}$ are also eigenvectors of $J$ for the same eigenvalues. Hence $\Phi_k = c_k \Pi_k$, and we find the constants $c_k$ from the initial data $s_1 = 1$, so that $c_k = \phi_1(z_k(J), J)$. By (2.6) and (2.7) $m_k(J) = |\phi_1(z_k(J), J)|^2 \sigma_k^{-1}(J)$.

It remains to express $\phi_1$ in terms of the spectral data. Once the Jost asymptotics exists for $J \in \mathcal{R}$, the Jost solution $\phi_n$ is proportional to the Weyl solution

$$w_n(z) := ((z + z^{-1} - J)^{-1}e_1, e_n), \quad n = 1, 2, \ldots, \quad w_0 = 1,$$

that is, $\phi_n = \phi_0 w_n$. In particular,

$$\phi_1(z, J) = \phi_0(z, J) w_1(z) = \phi_0(z, J) M(z, J),$$

where $M$ is the Weyl function for $J$

$$M(z, J) = (z + z^{-1} - J)^{-1}e_1, e_1 = \int_{\mathbb{R}} \frac{\sigma(d\lambda)}{z + z^{-1} - \lambda} = \frac{\sigma_k(J)}{z + z^{-1} - \lambda_k} + \tilde{M}(z),$$

$\tilde{M}$ is analytic at $z_k(J)$. So

$$\phi_1(z, J) = M(z, J) \frac{B(z, J)}{D(z, J)}.$$

Since $\lim_{z \to z_k}(z - z_k(J)) M(z, J) = \sigma_k(J)(1 - z_k^{-2}(J))^{-1}$, we finally have

$$\phi_1(z_k(J), J) = \frac{\sigma_k(J)}{1 - z_k^{-2}(J)} \frac{B'(z_k(J), J)}{D(z_k(J), J)},$$

as needed.

To complete the analysis of scattering data we prove the uniqueness theorem.

**Theorem 2.4.** Let $J_1 \in \mathcal{R}$, $l = 1, 2$, have the same scattering data. Then $J_1 = J_2$.

**Proof.** We want to make sure that $\sigma(J_1) = \sigma(J_2)$. It is clear from (2.3) that

$$\frac{D_0(t, J_1)}{D_0(t, J_1)} = \frac{D_0(t, J_2)}{D_0(t, J_2)}.$$

But in Ryckman’s class $D_0^{1,1} \in H^2$, so the latter means $D_0(J_2) = c D_0(J_1)$, $D(J_2) = c D(J_1)$ for some $c > 0$, and hence $\sigma_{ac}(J_2) = c^2 \sigma_{ac}(J_1)$. Next, $\mu_k(J_1) = \mu_k(J_2)$ implies by (1.11) $\sigma_k(J_2) = c^2 \sigma_k(J_1)$, and the normalizing condition

$$\int_{-2}^2 f(x, J_1) \, dx + \sum_{k=1}^N \sigma_k(J_1) = \int_{-2}^2 f(x, J_2) \, dx + \sum_{k=1}^N \sigma_k(J_2) = 1$$

gives $c = 1$, as needed. \qed

3. **Inverse Scattering**

Consider the following collection of data $\{\gamma_1, \gamma_2; Z; \mu_1, \ldots, \mu_N; s\}$:

1. a pair of numbers $\langle \gamma_1, \gamma_2 \rangle$ from $\{0, 1\} \times \{0, 1\}$;
2. an arbitrary set of $N$ distinct points $Z = \{z_k\}_{k=1}^N$ in $(-1, 1) \setminus \{0\}$;
3. an arbitrary set of $N$ positive numbers $\mu_k$;
4. a function $s \in B^{1/2}_2$, $|s| = 1$ a.e. on T, with the index $\langle \gamma_1, 2N + \gamma_1 + \gamma_2 \rangle$, i.e.,

$$s(t) = (-1)^{\gamma_1} t^{2N + \gamma_1 + \gamma_2} e^{-i\omega(t)},$$

where $\omega$ satisfies $\omega(t) = \frac{\omega(t)}{\omega(t)} = -\omega(t)$, $\omega \in B^{1/2}_2$. 

**Theorem 3.1.** There exists a unique Jacobi matrix $J \in \mathcal{R}$, for which the above collection is the scattering data.

**Proof.** As in the proof of Theorem 2.1 we can write
\[ s(t) = \frac{(1-t)\gamma_1 (1+t)\gamma_2}{(1-t)\gamma_1 (1+t)\gamma_2} B^2(t, Z)e^{-iv_0(t)}, \]
v_0 is subject to (2.5). The Fourier series for $v_0$ is
\[ v_0(t) = \sum_{n \in \mathbb{Z}} \hat{v}_0(n) t^n, \quad \hat{v}_0(-n) = \overline{\hat{v}_0(n)} = -\hat{v}_0(n), \]
so $\hat{v}_0(0) = 0$. Take $u_0$ such that $v_0$ is its harmonic conjugate. Then $u_0(t) = \overline{u_0(t)} = u_0(t)$ and
\[ u_0(t) = \sum_{n \in \mathbb{Z}} \hat{u}_0(n) t^n, \quad \hat{u}_0(-n) = \overline{\hat{u}_0(n)} = \hat{v}_0(n). \]
Note that $u_0$ is defined up to an additive real constant $\hat{u}_0(0)$, which will be chosen later on from the normalization condition.

Define a function $\rho_0$ on $[-2,2]$ by $\hat{\rho}_0 = e^{i\hat{u}_0}$, and put
\[ \rho(x) := \frac{\rho_0(x)}{(2-x)(2+x)^{\gamma_1(2+x)^{\gamma_2}}, \quad f(x) := \frac{1}{2\pi} \rho(x) \sqrt{4-x^2}, \]
both up to a factor $C = e^{i\hat{u}_0(0)}$. Next, write
\[ D_0(z) = \exp \left\{ \frac{1}{2} \int_0^z \frac{t+z}{t-z} u_0(t)m(dt) \right\} = \exp \left\{ \frac{u_0(z) + iv_0(z)}{2} \right\}, \]
\[ D(z) = \frac{D_0(z)}{(1-z)^\gamma_1 (1+z)^\gamma_2}, \]
and put
\[ \sigma_k := \mu_k \left| \frac{D(z_k)}{B^2(z_k)} \right|^2 \left| 1 - z_k^{-2} \right|^{-2} > 0, \quad k = 1, 2, \ldots, N, \]
the latter values are defined up to a factor $C$ above, which is now taken from
\[ \int_{-2}^{2} f(x)\,dx + \sum_{k=1}^{N} \sigma_k = 1. \]
Since $v_0 \in B_2^{1/2}$, then so is $u_0$, and by Ryckman’s theorem the measure $\sigma = \{f, \{\sigma_k\}\}$ is the spectral measure of some Jacobi matrix $J \in \mathcal{R}$. By construction, $\{\gamma_1, \gamma_2; Z; \mu_1, \ldots, \mu_N; s\}$ is the scattering data for $J$, and $J$ is unique by Theorem 2.4. The proof is complete. \qed

### 4. Examples

As we see, it is comparatively easy to restore the spectral measure from the scattering data. Assume first that $N = 0$, that is, $\text{supp } \sigma \subset [-2,2]$. To find the Jacobi parameters it seems reasonable now to carry the measure over from $[-2,2]$ to the unit circle (the inverse Szegő transform) $\sigma = S_2^{(o)}(\mu)$ \(^1\)

\[ \sigma(dx) = f(x)\,dx, \quad \mu(dt) = w(t)m(dt), \quad w(t) = \frac{\epsilon f(t)}{|1-t^2|}, \quad (4.1) \]

\(^1\)Due to the form of (1.4) it is convenient to use the modified Szegő transform
Example 4.2. For finally, by (4.2) µ and

\[ b_{n+1} = \alpha_{2n} (1 - \alpha_{2n+1}) - \alpha_{2n+2} (1 + \alpha_{2n+1}), \]
\[ a_{n+1}^2 = (1 - \alpha_{2n+3} (1 - \alpha_{2n+2}) (1 + \alpha_{2n+1}), \quad n = 0, 1, \ldots, \quad (4.2) \]

see [18, Section 13.2].

Example 4.1. Let \( \gamma_1 = \gamma_2 = 0, a \in [0, 1), \) and \( s(t) = (1 - at)(1 - at^{-1})^{-1}. \) Since \( B = 1 \) we have as in the proof of Theorem 3.1

\[ D(z) = \frac{c}{1 - az}, \quad \tilde{\rho}(t) = |D(t)|^2 = \frac{c}{|1 - at|^2}, \quad f(x) = \frac{c\sqrt{4 - x^2}}{1 - ax + a^2}, \]

so by (4.1)

\[ \mu(dt) = \frac{cm(dt)}{|1 - at|^2}. \]

Hence \( \mu \) is the Bernstein–Szegő measure, for which the Verblunsky coefficients are

\( \alpha_0 = a, \quad \alpha_1 = \alpha_2 = \ldots = 0. \)

Finally, by (4.2)

\( b_1 = a, \quad b_2 = b_3 = \ldots = 0, \quad a_1 = a_2 = \ldots = 1. \)

Example 4.2. For \( \gamma_1, \gamma_2, a \) as above put \( s(t) = (1 - at)(1 - at^{-1})^{-1}. \) We now have

\[ D(z) = c(1 - az), \quad \tilde{\rho}(t) = c|1 - at|^2, \quad f(x) = c(1 - ax + a^2)\sqrt{4 - x^2}, \]

and \( \mu(dt) = c|1 - at|^2 m(dt). \) The Verblunsky coefficients are (see [17, Example 1.6.4])

\[ \alpha_n = -\frac{a^{-1} - a}{a^{-n-2} - a^{n+2}}, \quad n = 0, 1, \ldots. \]

Finally,

\[ b_{n+1} = -a^{2n+1} \frac{(1 - a^2)^2}{(1 - a^{2n+2}) (1 - a^{2n+4})}, \quad a_{n+1}^2 = 1 - a^{2n+2} \frac{(1 - a^2)^2}{(1 - a^{2n+4})^2}. \]

It is worth pointing at the difference between the above examples. In the first one \( D^{-1} \) is a polynomial (of degree 1), so there are finitely many nonzero Verblunsky coefficients \( \alpha_n, \) and \( J \) is of finite support, i.e., \( a_n = 1 \) and \( b_n = 0 \) for all large enough \( n. \) In the second case \( D^{-1} \) is a rational function with the pole \( 1/a, \) so the Jacobi parameters tend to their limits exponentially fast.

Example 4.3. Let \( \gamma_1 = \gamma_2 = N = 0, a, b \in [0, 1). \) Put

\[ s(t) = \frac{(1 - at)(1 - bt)}{(1 - at)(1 - bt)}. \]

As above, the spectral measure is

\[ \sigma(dx) = \frac{c\sqrt{4 - x^2}}{(1 - ax + a^2)(1 - bx + b^2)} dx, \quad \mu(dt) = \frac{cm(dt)}{|1 - at|^2|1 - bt|^2}. \]

The latter is again the Bernstein–Szegő measure with the Verblunsky coefficients

\( \alpha_0 = \frac{a + b}{1 + ab}, \quad \alpha_1 = -ab, \quad \alpha_2 = \alpha_3 = \ldots = 0. \)
Finally,
\[ b_1 = a + b, \quad b_2 = b_3 = \ldots = 0; \quad a_1^2 = 1 - ab, \quad a_2 = a_3 = \ldots = 1. \]

**Remark 4.4.** For \( \gamma_1 = \gamma_2 = 0 \) the Szegő transform \( S_z^{(o)} \) is a right one in the sense of [14]. We can compute the same examples with \( \gamma_1, \gamma_2 = 0,1 \) by using all four Szegő transforms \( S_z^{(o)}, S_z^{(e)}, S_z^{(±)}, \) correspondingly.

To complete the examples section consider the case \( N = 1 \), that is, the spectral measure has a mass point outside \([-2, 2]\). Now the Szegő transforms do not work, so we proceed in two steps. First, we compute the Jacobi parameters \( \sigma \) with the values of normalizing constants \( \sigma \) and the corresponding orthonormal polynomials are known explicitly.

Example 4.3 (with \( a = b, a_1 = 0 \)).

Let \( \gamma_1 = \gamma_2 = 0, N = 1, z_1 \in (0, 1) \) and \( \mu_1 > 0 \) are given, and \( s(t) = t^2 \). We have

\[
B^2(t) = \left( \frac{t - z_1}{1 - z_1 t} \right)^2 = t^2 \left( \frac{1 - z_1 t}{1 - z_1 t} \right)^2,
\]

so

\[
s(t) = B^2(t) \frac{D(t)}{D(1)} \quad \text{and} \quad D(z) = \frac{c_0}{(1 - z_1 z)^2}.
\]

The spectral measure is of the form

\[
\sigma(dx) = \frac{c_0^2 \sqrt{4 - x^2}}{2\pi(1 - z_1 x + z_1^2)^2} dx + \sigma_1 \delta(1_1), \quad \lambda_1 = z_1 + \frac{1}{z_1}
\]

with

\[
\sigma_1 = \mu_1 \left| \frac{D(z_1)}{B(z_1)} \right|^2 \left| 1 - z_1^{-2} \right|^2 = c_0^2 \frac{\mu_1}{z_1^4}.
\]

The measure is determined from \( \sigma(R) = 1 \), or

\[
c_0^{-2} = \int_{-2}^{2} f_0(x) dx + \frac{\mu_1}{z_1^4}, \quad f_0(x) = \frac{\sqrt{4 - x^2}}{2\pi(1 - z_1 x + z_1^2)^2}.
\]

Take \( \sigma_0(dx) = c_0^2 f_0 dx \), the measure on \([-2, 2]\), \( \sigma_0(R) = 1 \). It is not hard to find the values of normalizing constants \( c_0, c_1 \):

\[
c_0^{-2} = \frac{1}{1 - z_1^2} + \frac{\mu_1}{z_1^4}, \quad c_1^2 = \int_{-2}^{2} \frac{\sqrt{4 - x^2}}{2\pi(1 - z_1 x + z_1^2)^2} dx = \frac{1}{\sqrt{1 - z_1^2}}.
\]

If \( \mu_1(dt) = w_1(t) m(dt) \) and \( \sigma_0 = S_z^{(o)}(\mu_1) \), then \( w_1 = c|1 - z_1 t|^{-4} \), and as in Example 4.3 (with \( a = b = z_1 \))

\[
b_1(\sigma_0) = 2z_1, \quad b_2(\sigma_0) = \ldots = 0; \quad a_1^2(\sigma_0) = 1 - z_1^2, \quad a_2(\sigma_0) = \ldots = 1. \quad (4.3)
\]

Note that \( \sigma_0 \) is the Bernstein–Szegő measure on \([-2, 2]\) (see \([19, \text{Section II.2.6}]\)), and the corresponding orthonormal polynomials are known explicitly

\[
p_n(\sigma_0, z + z^{-1}) = \frac{z^{n+1} (1 - z_1^2)^2 - z^{-n-1}(1 - z_1 z)^2}{\sqrt{1 - z_1^2 (z - z^{-1})}}, \quad n = 1, 2, \ldots, \quad p_0 = 1.
\]

In particular, \( p_k(\sigma_0, \lambda_1) = \sqrt{1 - z_1^2} z_1^{-k}, k = 1, 2, \ldots \). The Christoffel kernels are

\[
K_{n+1}(\sigma_0, \lambda_1) = \sum_{k=0}^{n} p_k^2(\sigma_0, \lambda_1) = z_1^{-2n}, \quad n = 0, 1, \ldots, \quad K_0 = 0.
\]
To apply Nevai’s formulae write the perturbed measure \( \sigma \) in a canonical form
\[
\sigma(dx) = \frac{\sigma_0(dx) + \varepsilon \delta(\lambda_1)}{1 + \varepsilon}, \quad \varepsilon = \frac{\epsilon^2}{\epsilon^2_0} - 1 = \frac{\mu_1}{\epsilon^1_1} (1 - \epsilon^1_1),
\]
so
\[
a_n^2(\sigma) = a_n^2(\sigma_0) \frac{(1 + \varepsilon K_{n-1}(\sigma_0, \lambda_1))(1 + \varepsilon K_{n+1}(\sigma_0, \lambda_1))}{(1 + \varepsilon K_n(\sigma_0, \lambda_1))^2},
\]
\[
b_n(\sigma) = b_n(\sigma_0) - a_{n-1}(\sigma_0) V_{n-1} + a_n(\sigma_0) V_n, \quad n = 1, 2, \ldots,
\]
where
\[
V_n = \frac{\varepsilon p_{n-1}(\sigma_0, \lambda_1) p_n(\sigma_0, \lambda_1)}{1 + \varepsilon K_n(\sigma_0, \lambda_1)} = \begin{cases}
\frac{\varepsilon (1 - \epsilon^1_1)}{\epsilon^1_1 (\epsilon^1_1 - 2 + \varepsilon)}, & n \geq 2; \\
\frac{\epsilon \sqrt{1 - \epsilon^1_1}}{\epsilon^1_1 (1 + \varepsilon)}, & n = 1.
\end{cases}, \quad V_0 = 0.
\]
Eventually, we have
\[
a_1^2(\sigma) = (1 - \epsilon^1_1)^2 \frac{1 + \mu_1 (1 - \epsilon^1_1)^{-2}}{(1 + \mu_1 \epsilon^1_1^{-4} (1 - \epsilon^1_1)^{-1})^2},
\]
\[
a_n^2(\sigma) = 1 + \frac{\varepsilon (1 - \epsilon^1_1)^2}{(\epsilon^2_1 + \varepsilon \epsilon^1_1)(1 + \varepsilon)} z_n^2, \quad n = 2, 3, \ldots,
\]
and
\[
b_n(\sigma) = \frac{\varepsilon (1 - \epsilon^1_1)^2}{(\varepsilon + \epsilon^2_1)(\varepsilon + \epsilon^2_1)} z_n^{2n-5}, \quad n \geq 3,
\]
\( \varepsilon \) from (4.4).

**Remark 4.6.** The Jost function is now
\[
\phi_0(z) = c(1 - \epsilon^1_1 z) \left(1 - \frac{z}{\epsilon^1_1}\right),
\]
but \( J \) is of infinite support. The Jacobi parameters tend to their limits exponentially fast, cf. [2, Remark 1.10].

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