Super-polynomial convergence and tractability of multivariate integration for infinitely times differentiable functions

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Abstract
We investigate multivariate integration for a space of infinitely times differentiable functions \( F_{s,u} := \{ f \in C^\infty[0,1]^s \mid \|f\|_{F_{s,u}} < \infty \} \), where \( \|f\|_{F_{s,u}} := \sup_{\alpha \in \mathbb{N}_0^s} \|f^{(\alpha)}\|_{L^1} / \prod_{j=1}^s u_j^{\alpha_j} \), \( f^{(\alpha)} := \partial^{\alpha_1} \cdots \partial^{\alpha_s} f \) and \( u = \{u_j\}_{j \geq 1} \) is a sequence of positive decreasing weights. Let \( e(n,s) \) be the minimal worst-case error of all algorithms that use \( n \) function values in the \( s \)-variate case. We prove that for any \( u \) and \( s \) considered \( e(n,s) \leq C(s) \exp(-c(s)(\log n)^2) \) holds for all \( n \), where \( C(s) \) and \( c(s) \) are constants which may depend on \( s \). Further we show that if the weights \( u \) decay sufficiently fast then there exist some \( 1 < p < 2 \) and absolute constants \( C \) and \( c \) such that \( e(n,s) \leq C \exp(-c(\log n)^p) \) holds for all \( s \) and \( n \). These bounds are attained by quasi-Monte Carlo integration using digital nets. These convergence and tractability results come from those for the Walsh space into which \( F_{s,u} \) is embedded.

Keywords: numerical integration, tractability, super-polynomial convergence, quasi-Monte Carlo, Walsh spaces, digital nets

1. Introduction
In this paper we approximate the integral on an \( s \)-dimensional unit cube

\[
\int_{[0,1]^s} f(\mathbf{x}) \, d\mathbf{x}
\]

by a quasi-Monte Carlo (QMC) algorithm which uses \( n \) function values of the form

\[
A_{n,s}(f) := \sum_{i=1}^n \frac{1}{n} f(t_i) \quad \text{for } t_i \in [0,1]^s.
\]
One classical issue is the optimal rate of convergence with respect to $n$. Another important issue is the dependence on the number of variables $s$, since $s$ can be hundreds or more in computational applications. The latter issue is related to the notion of tractability if we require no exponential dependence on $s$.

A large number of studies have been devoted to numerical integration on the unit cube for various function spaces. One typical case is that functions are only finitely many times differentiable, e.g., functions with bounded variation, periodic functions in the Korobov space and non-periodic functions in the Sobolev space, see [14, 18, 16, 5] and the references therein. For these cases, it is known that the rate of convergence is $O(n^{-\alpha})$ for some $\alpha > 0$ and thus we have polynomial convergence. Another interesting case is when the functions are smooth, i.e., infinitely times differentiable. Dick [2] gave reproducing kernel Hilbert spaces based on Taylor series for which higher order QMC rules achieve a convergence of $O(n^{-\alpha})$ with $\alpha > 0$ arbitrarily large. The spaces were later generalized in [23]. Further results were proved in [4, 10], where it is shown that exponential convergence holds for the Korobov space of periodic functions whose Fourier coefficients decay exponentially fast. Exponential convergence means that the integration error converges as $O(q^{n^p})$ for some $q \in (0, 1)$, $p > 0$. Note that exponential convergence was also shown for Hermite spaces on $\mathbb{R}^s$ with exponentially fast decaying Hermite coefficients [9].

In this paper we focus on a weighted normed space of non-periodic smooth functions

$$F_{s,u} := \left\{ f \in C^\infty[0, 1]^s \bigg| \|f\|_{F_{s,u}} := \sup_{\alpha=(\alpha_1, \ldots, \alpha_s) \in \mathbb{N}_0^s} \|f^{(\alpha)}\|_{L_1} \prod_{j=1}^s u_j^{\alpha_j} < \infty \right\}$$

with a sequence of positive weights $u = \{u_j\}_{j \geq 1}$, where $f^{(\alpha)} = \frac{\partial |\alpha|}{\partial x_1^{\alpha_1} \cdots \partial x_s^{\alpha_s}} f$. It is easy to check that all functions in $F_{s,u}$ are analytic from Taylor’s theorem. For $s = 1$, it is known that the integration error using Gaussian quadrature with $n$ points converges factorially, see, for instance, [1, (6.53)]. Our interest is the multivariate QMC rules using so-called digital nets. This is motivated by the results by Yoshiki [22] and is closely related to the notion of Walsh figure of merit (WAFOM) [12, 20, 22] first introduced by Matsumoto, Saito and Matoba. WAFOM is a criterion for numerical integration using digital nets and is computable in a reasonable time. Hence we can search for good digital nets with respect to WAFOM by computer, see [12, 7, 6] for numerical experiments. We observe that generalized WAFOM works well for the space $F_{s,u}$, see Remark 6.4.

The first purpose of this paper is to show that the integration error of QMC rules using digital nets for $F_{s,u}$ can achieve super-polynomial convergence as $O(\exp(-c(\log n)^2)) \asymp O(n^{-c \log n})$ for all $s$ and $u$ considered. Here the hidden constant and $c$ may depend on $s$. We remark that this convergence behavior was first observed in [13] as the decay of the lowest-WAFOM value and that the combination of [13] and [22] implies the convergence result for $F_{s,(1/2)^s,1}^2$. We also consider tractability for $F_{s,u}$. Let us briefly recall the notion of tractability (see [13, 16, 17] for more information). Let $n(\varepsilon, s)$ be the information
complexity, i.e., the minimal number \( n \) of function values which are required in order to approximate the \( s \)-variate integration within \( \varepsilon \). An integration problem is said to be tractable if \( n(\varepsilon, s) \) does not grow exponentially in \( \varepsilon \) nor \( s \). In particular, two notions of tractability have been mainly considered: polynomial tractability, i.e., \( n(\varepsilon, s) \leq C\varepsilon^{-\tau_1} s^{\tau_2} \), and strong polynomial tractability, i.e., \( n(\varepsilon, s) \leq C\varepsilon^{-\tau_1} \). A common way to obtain tractability is to consider weighted function spaces as introduced by Sloan and Woźniakowski [10]. Weighted spaces here mean that the dependence on the successive variables can be moderated by weights. Our weights \( u \) play the same role. For tractability results for spaces of smooth functions, see also [8].

The second purpose of this paper is to give a sufficient condition to achieve super-polynomial convergence with strong tractability. We show that if the weights \( u \) decay sufficiently fast then the integration error of QMC rules using digital nets for \( F_{s,u} \) can achieve dimension-independent super-polynomial convergence as \( O(\exp(-c(\log n)^p)) \), where the hidden constant and \( c \) are independent of \( s \) and \( n \), and \( 1 < p < 2 \) is determined from the decay of \( u \). This implies \( n(\varepsilon, s) \leq C\exp(c(\log \varepsilon^{-1})^{1/p}) \) for some \( C, c \geq 0 \).

These convergence and tractability results are also shown for the so-called Walsh space \( W_{s,a,b} \) into which \( F_{s,u} \) is embedded, which implies those for \( F_{s,u} \).

For \( W_{s,a,b} \), we show that the rate of convergence is of order \( \exp(\Theta((-\log n)^2)) \) for all \( a \) considered and the strong tractability result is equivalent to a typical decay of \( a \).

The rest of the paper is organized as follows. In Section 2 we give the necessary background including Walsh functions, the Dick weight, definitions of our function spaces and embeddings among them. In Section 3 we give precise definitions of the notions of convergence and tractability used in this paper. In Section 4 we present Theorems 1.1 and 1.2 which are the summary of all results in this paper. Necessary and sufficient conditions for convergence and tractability are given in Sections 5 and 6 respectively.

2. Preliminaries

Throughout this paper, we shall use the following notation. Let \( \mathbb{N} \) be the set of positive integers and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). For a positive integer \( b \geq 2 \), let \( \mathbb{Z}_b \) be a cyclic group with \( b \) elements, which we identify with the set \( \{0, 1, \ldots, b-1\} \), equipped with addition modulo \( b \). The operators \( \oplus \) and \( \ominus \) denote the digitwise addition and subtraction modulo \( b \), respectively. That is, for \( k, k' \in \mathbb{N}_0 \) whose \( b \)-adic expansions are \( k = \sum_{i=1}^{\infty} \kappa_i b^{i-1} \) and \( k' = \sum_{i=1}^{\infty} \kappa_i' b^{i-1} \) with \( \kappa_i, \kappa_i' \in \mathbb{Z}_b \) for all \( i \), \( \oplus \) and \( \ominus \) are defined as

\[
k \oplus k' = \sum_{i=1}^{\infty} \eta_i b^{i-1} \quad \text{and} \quad k \ominus k' = \sum_{i=1}^{\infty} \eta'_i b^{i-1},
\]

where \( \eta_i = \kappa_i + \kappa_i' \pmod{b} \) and \( \eta'_i = \kappa_i - \kappa_i' \pmod{b} \), respectively. In case of vectors in \( \mathbb{N}_0^n \), the operators \( \oplus \) and \( \ominus \) are applied componentwise.
2.1. Walsh functions

In this subsection, we introduce Walsh functions and Walsh coefficients, which are widely used in analyzing the integration error, see [5, Appendix A] for general information. We first give the definition of Walsh functions for the one-dimensional case and then generalize it to the higher-dimensional case.

**Definition 2.1.** Let \( b \geq 2 \) be a positive integer and let \( \omega_b = \exp(2\pi \sqrt{-1}/b) \). We denote the \( b \)-adic expansion of \( k \in \mathbb{N}_0 \) by \( k = \kappa_1 + \kappa_2 b + \cdots + \kappa_i b^{-i} \) with \( \kappa_1, \ldots, \kappa_i \in \mathbb{Z}_b \). Then the \( k \)-th \( b \)-adic Walsh function \( \text{wal}_k: [0, 1) \rightarrow \{1, \omega_b, \ldots, \omega_b^{b-1}\} \) is defined as

\[
\text{wal}_k(x) := \omega_b^{\kappa_1 \xi_1 + \cdots + \kappa_i \xi_i},
\]

for \( x \in [0, 1) \) whose \( b \)-adic expansion is given by \( x = \xi_1 b^{-1} + \xi_2 b^{-2} + \cdots \), which is unique in the sense that infinitely many of the \( \xi_i \) are different from \( b - 1 \).

**Definition 2.2.** Let \( b, s \in \mathbb{N} \) with \( b \geq 2 \). Let \( x = (x_1, \ldots, x_s) \in [0, 1)^s \) and \( k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s \). The \( k \)-th \( b \)-adic Walsh function \( \text{wal}_k: [0, 1)^s \rightarrow \{1, \omega_b, \ldots, \omega_b^{b-1}\} \) is defined as

\[
\text{wal}_k(x) := \prod_{j=1}^s \text{wal}_{k_j}(x_j).
\]

Since we shall always use Walsh functions in a fixed base \( b \), we omit the subscript and simply write \( \text{wal}_k \) or \( \text{wal}_k(x) \) in this paper. Some important properties of Walsh functions, used in this paper, are described below, see [5, Appendix A.2] for the proof.

**Proposition 2.3.** The following holds true:

1. For all \( k \in \mathbb{N}_0^s \), we have

\[
\int_0^1 \text{wal}_k(x) \, dx = \begin{cases} 
1 & \text{if } k = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

2. For all \( k, l \in \mathbb{N}_0^s \), we have

\[
\int_{[0,1)^s} \text{wal}_k(x) \overline{\text{wal}_l(x)} \, dx = \begin{cases} 
1 & \text{if } k = l, \\
0 & \text{otherwise}.
\end{cases}
\]

3. For all \( k, k' \in \mathbb{N}_0^s \) and \( x \in [0, 1)^s \), we have

\[
\text{wal}_{k \oplus k'}(x) = \text{wal}_k(x) \text{wal}_{k'}(x),
\]

\[
\text{wal}_{k \oplus k'}(x) = \text{wal}_{k}(x) \overline{\text{wal}_{k'}(x)}.
\]

4. The system \( \{\text{wal}_k \mid k \in \mathbb{N}_0^s\} \) is a complete orthonormal system in \( L^2[0, 1)^s \) for any positive integer \( s \).
We define the Walsh coefficients as follows.

**Definition 2.4.** Let \( k \in \mathbb{N}_0 \) and \( f : [0,1)^s \to \mathbb{C} \). The \( k \)-th Walsh coefficient of \( f \) is defined as

\[
\hat{f}(k) := \int_{[0,1)^s} f(x) \text{wal}_k(x) \, dx.
\]

The Walsh series of the function \( f \) is given by

\[
f(x) \sim \sum_{k \in \mathbb{N}_0} \hat{f}(k) \text{wal}_k(x)
\]

for any \( f \in L^2([0,1)^s) \). We note that all functions considered in this paper are equal to their Walsh series, see also the next subsection.

### 2.2. Function spaces and embeddings

In this subsection, we introduce the function spaces \( F_{s,u}, W_{s,a,b} \) and \( \tilde{W}_{s,a,b} \) considered in this paper and give embeddings from \( F_{s,u} \) to \( W_{s,a,b} \).

The space of smooth functions \( F_{s,u} \) is defined as in (1). Throughout the paper, we always assume that \( u_1 \geq u_2 \geq \cdots > 0 \). (2)

It is shown in [22] for \( b = 2 \) and [21] for the general case that Walsh coefficients of functions in \( F_{s,u} \) decay sufficiently fast, as given in Theorem 2.6 below.

To state the theorem, we define the Hamming weight \( v(k) \), the generalized Dick weight \( \mu(a;k) \) and the modified Dick weight \( \tilde{\mu}(a;k) \) for \( k \in \mathbb{N}_0, a \in \mathbb{R}^s \) and \( \kappa \in \mathbb{N}_0 \). Note that the Dick weight is originally defined for \( a = 0 \) in [12].

**Definition 2.5.** Let \( a \in \mathbb{R} \) and \( k \in \mathbb{N} \) with \( b \)-adic expansion \( k = \sum_{i=1}^{\infty} \kappa_i b^{i-1} \) with \( \kappa_i \in \mathbb{Z}_b \). Define the function \( h \) as \( h(0) = 0 \) and \( h(1) = 1 \) for \( k \neq 0 \). The Hamming weight \( v(k) \) is defined as the number of nonzero digits for the \( b \)-adic expansion of \( k \), i.e.,

\[
v(k) := \sum_{i=1}^{\infty} h(\kappa_i).
\]

We define the generalized Dick weight \( \mu(a;k) \) and the modified Dick weight \( \tilde{\mu}(a;k) \) for the 1-dimensional case as

\[
\mu(a;k) := \sum_{i=1}^{\infty} (i + a) h(\kappa_i) \quad \text{and} \quad \tilde{\mu}(a;k) := \sum_{i=1}^{\infty} \max(i + a, 1) b(\kappa_i).
\]

For the \( s \)-dimensional case, let \( a = (a_1, \ldots, a_s) \in \mathbb{R}^s \) and \( k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s \) and we define \( \mu(a;k) \) and \( \tilde{\mu}(a;k) \) as

\[
\mu(a;k) := \sum_{j=1}^{s} \mu(a_j;k_j) \quad \text{and} \quad \tilde{\mu}(a;k) := \sum_{j=1}^{s} \tilde{\mu}(a_j;k_j).
\]
We can now give the decay of Walsh coefficients which appears in [21, Corollary 3.10].

**Theorem 2.6.** Put \( m_b := 2 \sin(\pi/b) \) and \( M_b := 2 \sin(|b/2|\pi/b) \). Assume \( f \in \mathcal{F}_{s,u} \). Then it follows that

\[
|\hat{f}(k)| \leq \|f\|_{\mathcal{F}_{s,u}} b^{-\nu(0,k)} \prod_{j=1}^{s} (m_b^{-1}u_j)^{v(k_j)} D_b^{\min(1,v(k_j))},
\]

where \( D_b = 2 \) for \( b = 2 \) and \( D_b = M_b + bm_b/(b - M_b) \) otherwise.

This decay motivates us to define Walsh spaces \( W_{s,a,b} \) and \( \tilde{W}_{s,a,b} \) of Walsh series whose Walsh coefficients are controlled by the generalized (resp. modified) Dick weight. Let \( a = (a_j)_{j \geq 1} \) be a sequence of real-valued weights. Throughout the paper, we assume

\[
a_1 \leq a_2 \leq a_3 \leq \cdots,
\]

which corresponds to [22]. We first define \( W_{s,a,b} \) as

\[
W_{s,a,b} := \left\{ f: [0,1)^s \to \mathbb{R} \mid f(x) = \sum_{k \in \mathbb{N}_0^s} \hat{f}(k)\text{wal}_k(x) \text{ and } \|f\|_{W_{s,a,b}} < \infty \right\}
\]

equipped with the norm

\[
\|f\|_{W_{s,a,b}} := \sup_{k \in \mathbb{N}_0^s} |\hat{f}(k)| b^{\mu(a;k)}
\]

and \( \tilde{W}_{s,a,b} \) as

\[
\tilde{W}_{s,a,b} := \left\{ f \in W_{s,a,b} \mid \|f\|_{\tilde{W}_{s,a,b}} := \sup_{k \in \mathbb{N}_0^s} |\hat{f}(k)| b^{\mu(a;k)} < \infty \right\}.
\]

Note that all Walsh series in \( W_{s,a,b} \) and \( \tilde{W}_{s,a,b} \) converge. Indeed, for all \( X \in (-1,1) \) and a positive integer \( l \), we have

\[
\sum_{k \in \mathbb{N}_0^s, k_j < b^j \forall j} X^{\mu(a;k)} = \sum_{k \in \mathbb{N}_0^s} \prod_{j=1}^{s} \prod_{i=1}^{l} X^{i+a_j} h(k_j,i) = \prod_{j=1}^{s} \prod_{i=1}^{l} \sum_{k_j,i=0}^{b-1} X^{i+a_j} h(k_j,i) = \prod_{j=1}^{s} \prod_{i=1}^{l} (1 + (b-1)X^{i+a_j}),
\]

where we denote the \( b \)-adic expansion of \( k_j \) by \( k_j = \sum_{i=1}^{l} \kappa_{j,i} b^{i-1} \) with \( \kappa_{j,i} \in \mathbb{Z}_b \) in the first equality, and the right-most product converges for \( l \to \infty \) if \( |X| < 1 \).
This is also true for the modified Dick weight with $\mu(\alpha; \mathbf{k})$ and $i + a_j$ replaced by $\tilde{\mu}(\alpha; \mathbf{k})$ and $\max(i + a_j, 1)$. Hence we have

$$\sum_{\mathbf{k} \in \mathbb{N}^n_0} X^{\tilde{\mu}(\alpha; \mathbf{k})} = \prod_{j=1}^{s} \prod_{i=1}^{\infty} (1 + (b - 1)X^{i + a_j}) \quad \text{for all } |X| < 1, \quad (4)$$

$$\sum_{\mathbf{k} \in \mathbb{N}^n_0} X^{\mu(\alpha; \mathbf{k})} = \prod_{j=1}^{s} \prod_{i=1}^{\infty} (1 + (b - 1)X^{\max(i + a_j, 1)}) \quad \text{for all } |X| < 1. \quad (5)$$

Thus all functions in $\mathcal{W}_{s,a,b}$ and $\mathcal{W}'_{s,a,b}$ converge.

We now give embeddings from $\mathcal{F}_{s,u}$ to $\mathcal{W}_{s,a,b}$. From Theorem 2.6 we have

$$|\tilde{f}(\mathbf{k})| \leq \|f\|_{\mathcal{F}_{s,u}} \prod_{j=1}^{s} D_b^{\min(1, u(k_j))} b^{-\mu(\log_b(m_b^{-1}u_j); \mathbf{k})}$$

$$\leq \|f\|_{\mathcal{F}_{s,u}} \prod_{j=1}^{s} b^{-\mu(\log_b(D_b m_b^{-1}u_j); \mathbf{k})}$$

for $f \in \mathcal{F}_{s,u}$. Thus we obtain continuous embeddings

$$\mathcal{F}_{s,u} \subset \mathcal{W}_{s,u',b} \quad \text{with} \quad \|f\|_{\mathcal{W}_{s,u',b}} \leq \|f\|_{\mathcal{F}_{s,u}}, \quad (6)$$

$$\mathcal{F}_{s,u} \subset \mathcal{W}_{s,u'',b} \quad \text{with} \quad \|f\|_{\mathcal{W}_{s,u'',b}} \leq D_b^s \|f\|_{\mathcal{F}_{s,u}}, \quad (7)$$

where $u' = (- \log_b(D_b m_b^{-1}u_j))_{j \geq 1}$ and $u'' = (- \log_b(m_b^{-1}u_j))_{j \geq 1}$. Note that all functions in $\mathcal{F}_{s,u}$ are equal to their Walsh expansions, see [3, Section 3.3] or [5, Theorem A.20]. Embedding (6) implies that good algorithms for $\mathcal{W}_{s,u',b}$ are also good for $\mathcal{F}_{s,u}$. Thus we mainly consider $\mathcal{W}_{s,a,b}$ in the following sections.

The Walsh space $\mathcal{W}'_{s,a,b}$ is considered instead of $\mathcal{W}_{s,a,b}$ in Section 6, since the modified Dick weight does not take negative values and thus is easier to treat. Actually, $\mathcal{W}_{s,a,b}$ and $\mathcal{W}'_{s,a,b}$ are norm equivalent. Indeed, we have

$$\mu(\alpha; \mathbf{k}) \leq \tilde{\mu}(\alpha; \mathbf{k}) \leq \mu(\alpha; \mathbf{k}) + \sum_{j=1}^{s} \sum_{i \in \mathcal{N}_j} (1 - (i + a_j))$$

for all $\mathbf{k} \in \mathbb{N}^n_0$, where $\mathcal{N}_j$ is defined as $\mathcal{N}_j := \{i \in \mathbb{N} \mid i + a_j \leq 1\}$. Thus

$$\|f\|_{\mathcal{W}_{s,a,b}} \leq \|f\|_{\mathcal{W}'_{s,a,b}} \leq b^{\sum_{j=1}^{s} \sum_{i \in \mathcal{N}_j} (1 - (i + a_j))} \|f\|_{\mathcal{W}_{s,a,b}}, \quad (8)$$

where the empty sum equals 0, which implies the norm-equivalence of $\mathcal{W}_{s,a,b}$ and $\mathcal{W}'_{s,a,b}$. Furthermore, we can consider $\mathcal{W}_{s,a,b}$ instead of $\mathcal{W}_{s,a,b}$ for convergence results. In Section 6.3, we consider tractability results, we shall assume a condition on the weights which implies that the constant factor in (8) is bounded independently of $s$.  

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3. Integration

Let $\mathcal{H} = \mathcal{F}_{s,u}, \mathcal{W}_{s,a,b}$ or $\mathcal{W}_{s,a,b}$. We consider multivariate integration

$$I(f) = \int_{[0,1]^s} f(x) \, dx \quad \text{for all } f \in \mathcal{H}. $$

We approximate $I(f)$ by algorithms

$$A_{n,s}(f) = \phi_{n,s}(f(t_1), \ldots, f(t_n)),$$

where $\phi_{n,s} : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary function and $t_i \in [0,1)^s$ for $i = 1, \ldots, n$. The worst-case error of the algorithm $A_{n,s}$ in the space $\mathcal{H}$ is defined by

$$e_{\text{wor}}(A_{n,s}, \mathcal{H}) = \sup_{f \in \mathcal{H}, \|f\| \leq 1} |I(f) - A_{n,s}(f)|.$$

Let $e(n, s, \mathcal{H})$ be the $n$-th minimal worst-case error,

$$e(n, s) = e(n, s, \mathcal{H}) = \inf_{A_{n,s}} e_{\text{wor}}(A_{n,s}, \mathcal{H}),$$

where the infimum is extended over all algorithms using $n$ function values. For $n = 0$, we approximate $I(f)$ by a real number. Since $\mathcal{H}$ is symmetric, i.e., $f \in \mathcal{H}$ implies $-f \in \mathcal{H}$, the zero algorithm is the best for $n = 0$, and thus we have $e(0, s, \mathcal{H}) = 1$. Hence the integration problem is well normalized for all $s$.

For $\varepsilon \in (0,1)$, we define the information complexity of integration

$$n(\varepsilon, s) = n(\varepsilon, s, \mathcal{H}) = \min \{ n \in \mathbb{N} \mid e(n, s, \mathcal{H}) \leq \varepsilon \}$$

as the minimal number of function values needed to obtain an $\varepsilon$-approximation.

We are interested in the convergence of the minimal worst-case error of the form

$$e(n, s) \leq C(s) e^{-c(s)(\log n)^p} \quad \text{for all } s, n \in \mathbb{N},$$

where $C(s)$ and $c(s)$ are positive real numbers which may depend on $s$ and where $p > 1$. The condition $p > 1$ implies that this convergence is super-polynomial.

We note that if (9) holds, then for all $s \in \mathbb{N}$ and $\varepsilon \in (0,1)$ we have

$$n(\varepsilon, s) \leq \left\lceil \exp \left( \frac{\log C(s) + \log \varepsilon^{-1}}{c(s)} \right)^{1/p} \right\rceil, \quad (10)$$

where $(X)_+ := \max(X, 0)$ for $X \in \mathbb{R}$. Furthermore, (10) for $e = C(s) e^{-c(s)(\log n)^p}$ implies (9). This means that (9) is equivalent to (10), which shows that asymptotically $n(\varepsilon, s)$ increases with order $\exp(O((\log \varepsilon^{-1})^{1/p}))$ with respect to $\varepsilon$. However, how does $n(\varepsilon, s)$ depend on $s$? This, of course, depends on $C(s)$ and $c(s)$ and is the subject of tractability. Tractability means that we control the behavior of $C(s)$ and $c(s)$ and rule out the cases for which $n(\varepsilon, s)$ depends exponentially on $s$. In this paper we consider two convergence behaviors as

$$e(n, s) \leq C \exp(As) e^{-c(\log n)^p} \quad \text{for all } s, n \in \mathbb{N}, \quad (11)$$
and
\[ e(n, s) \leq C e^{-c(\log n)^p} \quad \text{for all } s, n \in \mathbb{N}, \] (12)
for some \( p > 1 \) and \( A, C, c > 0 \). Applying (10) to (11) and using the inequality
\[ (X + Y)^{1/p} \leq X^{1/p} + Y^{1/p} \quad \text{for } X, Y \geq 0, \] (11) implies that
\[ n(\varepsilon, s) \leq C' \exp(c'(\log \varepsilon^{-1})^{1/p}) \exp(A's^{1/p}) \quad \text{for all } s \in \mathbb{N}, \varepsilon \in (0, 1) \] (13)
for some \( A', C', c' \geq 0 \). Conversely, (13) implies (11) for some \( A, C, c \geq 0 \), which
follows from the inequality \( 2^{-1+1/p}(X^{1/p} + Y^{1/p}) \leq (X + Y)^{1/p} \) for \( X, Y \geq 0 \).
Similarly (12) is equivalent to the fact that
\[ n(\varepsilon, s) \leq C' \exp(c'(\log \varepsilon^{-1})^{1/p}) \quad \text{for all } s \in \mathbb{N}, \varepsilon \in (0, 1) \] (14)
for some \( C', c' \geq 0 \), which can be regarded as super-polynomial convergence with strong tractability. Although (13) shows that the information complexity may depend super-polynomially on \( s \), we will show that (11) is equivalent to (12) for the Walsh space.

4. Main results

In this section, we present the main results of this paper. The following theorems give the super-polynomial convergence and tractability results for \( W_{s, a, b} \) and \( F_{s, u} \).

**Theorem 4.1.** Consider integration defined over the Walsh space \( W_{s, a, b} \) with a weight sequence \( a \) satisfying (3). Then we have the following.

1. For fixed \( a \), there exist positive constants \( c_{i,s} (i = 1, 2, 3, 4) \) which may depend on \( s \) and \( a \) such that it holds that
\[ \exp \left( -\frac{(\log n)^2}{2s \log b} - c_{1,s} \log n - c_{2,s} \right) \leq e(n, s, W_{s, a, b}) \]
\[ \leq c_{3,s} \exp(-c_{4,s}(\log n)^2) \]
for all \( s \) and \( n \). In particular \( e(n, s, W_{s, a, b}) \) is of order \( \exp(\Theta(-\log n)^2)) \).
2. For any \( a \) considered, there do not exist \( A, C, c \geq 0 \) such that \( (11) \) holds for \( p = 2 \).
3. Let \( 1 < p < 2 \). Then the following are equivalent.
   (a) The sequence \( a \) satisfies \( \lim \inf_{j \to \infty} a_j/j^{(p-1)/(2-p)} > 0 \).
   (b) There exist constants \( A, C, c \geq 0 \) such that for all \( s, n \in \mathbb{N} \) we have
\[ e(n, s) \leq C \exp(As)e^{-c(\log n)^p}. \]
   (c) There exist constants \( C, c \geq 0 \) such that for all \( s, n \in \mathbb{N} \) we have
\[ e(n, s) \leq Ce^{-c(\log n)^p}. \]
Theorem 4.2. Consider integration defined over $F_{s,u}$ with a weight sequence $u$ satisfying (4). Then we have the following.

1. There exist positive constants $c_{0,s}$ and $c_{6,s}$ depending on $s$ and $u$ such that
   $$e(n, s, F_{s,u}) \leq c_{5,s}\exp(-c_{6,s}(\log n)^2) \quad \text{for all } s, n \in \mathbb{N}.$$

2. Let $1 < p < 2$ be a real number. If the weight sequence $u$ satisfies
   $$\liminf_{j \to \infty} \log(u_j^{-1})/j^{(p-1)/(2-p)} > 0,$$
   then there exist constants $C, c \geq 0$ such that
   $$e(n, s) \leq Ce^{-c(\log n)^p} \quad \text{for all } s, n \in \mathbb{N}.$$

These theorems follow from Theorems 5.3–5.5 and Corollaries 6.10 and 6.15.

5. Lower bounds

We prove the following lower bound on $e(n, s, W_{s,a,b})$ similarly to [4, Theorem 1], which treats the Korobov space.

Lemma 5.1. Let $A$ be a finite subset of $\mathbb{N}^*_0$. Then for all $n < |A|$ we have

$$e(n, s, W_{s,a,b}) \geq \left(\max_{k,k^* \in A} b_{\mu(\alpha; k \oplus k^*)}\right)^{-1}.$$

Proof. Take an arbitrary algorithm $A_{n,s}(f) = \phi_{n,s}(f(t_1), \ldots, f(t_n))$. Define

$$g_1(x) = \sum_{k \in A} c_k \text{wal}_k(x)$$

for $c_k \in \mathbb{C}$ such that $g_1(t_i) = 0$ for all $i = 1, 2, \ldots, n$. Since we have $n$ homogeneous linear equations and $|A| > n$ unknowns $c_k$, there exists a nonzero vector of such $c_k$’s, and we can normalize the $c_k$’s by assuming that

$$\max_{k \in A} |c_k| = c_{k^*} = 1 \quad \text{for some } k^* \in A.$$

Define the function

$$g_2(x) := C g_1(x) \text{wal}^*_k(x) = C \sum_{k \in A} c_k \text{wal}_{k \oplus k^*}(x),$$

where $C$ is defined as $C := (\max_{k,k^* \in A} b_{\mu(\alpha; k \oplus k^*)})^{-1}$. Then we have

$$\|g_2\|_{W_{s,a,b}} = C \max_{k \in A} |c_k b_{\mu(\alpha; k \oplus k^*)}|$$

$$\leq C \max_{k \in A} b_{\mu(\alpha; k \oplus k^*)} \leq C \max_{k,k^* \in A} b_{\mu(\alpha; k \oplus k^*)} = 1,$$

where $\|\cdot\|_{W_{s,a,b}}$ is naturally extended to complex-valued Walsh series.

We now define a real-valued function $f(x) := (g_2(x) + \overline{g_2(x)})/2$, where $\overline{g_2(x)} := \overline{g_2(x)}$. Note that $\|\overline{g_2}\|_{W_{s,a,b}} = \|g_2\|_{W_{s,a,b}}$ since $\mu(\alpha; k \oplus k^*) = \mu(\alpha; k^* \oplus k)$ for all $k$. The norm of $f$ is bounded by

$$\|f\|_{W_{s,a,b}} \leq (\|g_2\|_{W_{s,a,b}} + \|\overline{g_2}\|_{W_{s,a,b}})/2 = \|g_2\|_{W_{s,a,b}} \leq 1.$$

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We note that \( A_{n,s}(f) = \phi_{n,s}(0, \ldots, 0) \) and \( I(f) = Cc_k = C \). Hence,
\[
e(n, s, W_s, a, b) \geq |I(f) - A_{n,s}(f)| \geq |C - \phi_{n,s}(0, \ldots, 0)|.
\]
Further we consider the function \(-f\). We have \( \| -f \|_{W_s, a, b} \leq 1 \) and \( A_{n,s}(-f) = \phi_{n,s}(0, \ldots, 0) \). Hence,
\[
e(n, s, W_s, a, b) \geq |I(-f) - A_{n,s}(-f)| \geq |C + \phi_{n,s}(0, \ldots, 0)|.
\]
Combining these two inequalities, we have
\[
e(n, s, W_s, a, b) \geq \max(|C - \phi_{n,s}(0, \ldots, 0)|, |C + \phi_{n,s}(0, \ldots, 0)|) \geq C.
\]
Since this holds for arbitrary algorithm \( A_{n,s} \), we conclude that \( e(n, s) \geq C \), as claimed.

For a non-negative integer \( d \), we now define
\[
A_{s,d} = \{ k \in \mathbb{N}_0^s | k_j < b^d \text{ for all } j = 1, 2, \ldots, s \}.
\]
The cardinality of the set \( |A_{s,d}| \) is clearly \( b^{sd} \). If \( a_j \geq 0 \) holds for all \( j \), then
\[
\max_{k, k^* \in A_{s,d}} b^{\mu(a, k \land k^*)} = \left( \max_{k \in A_{s,d}} b^{\mu(a, k)} \right)^{-1} = b^{-\sum_{j=1}^{d} (i + a_j)} = b^{-\sum_{j=1}^{d} (d+1)/2+a_jd},
\]
where we use \( k \land k^* \in A_{s,d} \) for all \( k, k^* \in A_{s,d} \) for the first equality. This implies the following corollary.

**Corollary 5.2.** Let \( d \in \mathbb{N} \) and assume \( a_j \geq 0 \) for all \( j \). Then we have
\[
e(n, s, W_s, a, b) \geq b^{-\sum_{j=1}^{d} (d+1)/2+a_jd} \quad \text{for all } n < b^{sd}.
\]

We prove a lower bound for the worst-case error and a necessary condition to achieve (\text{11}) for \( W_s, a, b \) in the following three theorems with the following notation. For \( x \in \mathbb{R} \), we define \( \exp_x(x) := b^x \) and \( \max(a, x) := (\max(a, x))_{j=1}^s \).

**Theorem 5.3.** Let \( a = (a_j)_{j=1}^s \in \mathbb{R}^s \) and \( a' = (a'_j)_{j=1}^s := \max(a, 0) \). Then we have
\[
e(n, s, W_s, a, b) \geq \exp_b \left( -\frac{(\log n)^2}{2s(\log b)^2} - \left( \frac{3s}{2} + \sum_{j=1}^{s} a'_j \right) \frac{\log n}{s \log b} - \left( s + \sum_{j=1}^{s} a'_j \right) \right).
\]

**Proof.** Let \( n \in \mathbb{N} \). Put \( d = \lceil \log n/(s \log b) \rceil + 1 \), so that \( b^{(d-1)} \leq n < b^{sd} \). It follows from Corollary 5.2 and the embedding \( W_s, a, b \subset W_s, a', b \) that
\[
e(n, s, W_s, a, b) \geq e(n, s, W_s, a', b) \geq b^{-\sum_{j=1}^{d} (d+1)/2+a'_j} \]
\[
\geq \exp_b \left( -\frac{s}{2} \left( \frac{\log n}{s \log b} + 1 \right)^2 - \left( \frac{3s}{2} + \sum_{j=1}^{s} a'_j + 1/2 \right) \right).
\]
which proves the result.

**Theorem 5.4.** For any \( a \) considered, there do not exist \( A, C, c \geq 0 \) such that \( (11) \) holds for \( p = 2 \).

**Proof.** We will argue by contradiction. Let \( a' = \max(a, 0) \). Suppose that \( (11) \) holds for some \( A, C, c \geq 0 \). Then this assumption and Theorem 5.3 imply that

\[
\exp_b \left( -\frac{(\log n)^2}{2s(\log b)^2} - \left( \frac{3s}{2} + \sum_{j=1}^{s} a'_j \right) \frac{\log n}{s \log b} - \left( s + \sum_{j=1}^{s} a'_j \right) \right) \\
\leq e(n, s, W_{s,a',b}) \leq c(n, s, W_{s,a,b}) \leq C \exp(A s) e^{-c(\log n)^2}
\]

holds for all \( s \) and \( n \). Taking the limit as \( n \) goes to infinity, we have \( 1/(2s \log b) \geq c \) for all \( s \). This is a contradiction.

**Theorem 5.5.** Consider integration defined over \( W_{s,a,b} \) under \( (9) \). Let \( 1 < p < 2 \) and assume that \( (11) \) holds for this \( p \) and some \( A, C, c \geq 0 \). Put \( r := (p-1)/(2-p) \). Then we have

\[
\liminf_{j \to \infty} \frac{a'_j}{j^r} > 0.
\]

**Proof.** Let \( a' = \max(a, 1) \) and \( N = N(s) := b^{s\lfloor a'_s \rfloor} - 1 \). Corollary 5.2 implies

\[
e(N, s, W_{s,a',b}) \geq \exp_b \left( -\sum_{j=1}^{s} \left( \frac{|a'_j|^2}{2} + \left( a'_j + \frac{1}{2} \right) |a'_s| \right) \right) \geq \exp_b(-3s|a'_s|^2),
\]

where we use \( a'_j + 1/2 \leq a'_s + 1/2 \leq 5|a'_s|/2 \) in the second inequality. Combining this with the assumption that \( (11) \) holds for \( W_{s,a,b} \), for all \( s \) we have

\[
\exp_b(-3s|a'_s|^2) \leq C \exp(A s) e^{-c(\log N(s))^p}.
\]

By taking the logarithm and using \( b^{(s-1)|a'_s|} \leq N(s) \) and \( C \leq \exp(C s) \), we have

\[
-3s(\log b)|a'_s|^2 \leq (C + A)s - c((s-1)|a'_s| \log b)^p,
\]

and thus

\[
-\frac{(C + A)}{s^{p-1}|a'_s|^p} + \frac{(s-1)^p}{s^p} c(\log b)^p \leq 3 \log b \left( \frac{|a'_s|}{s^r} \right)^{2-p}.
\]

holds for all \( s \). Taking the limit inferior as \( s \) goes to infinity, we have

\[
\liminf_{j \to \infty} \frac{a'_j}{j^r} \geq \left( \frac{c(\log b)^{p-1}}{3} \right)^{1/(2-p)} > 0,
\]

which implies the desired result.
6. Upper bounds

In this section, motivated by [13] and its generalization [20], we prove the existence of good QMC algorithms which achieve super-polynomial convergence and tractability in Sections 6.2 and 6.3, respectively. Such QMC algorithms are given by digital nets. Digital nets are point sets which have the structure of a \( \mathbb{Z}_b \)-module introduced by Niederreiter, see for instance [14]. We introduce the notion of digital nets in the following subsection.

6.1. Digital nets

For a positive integer \( m \) and a non-negative integer \( k \) with its \( b \)-adic expansion \( k = \sum_{i=1}^{\infty} \kappa_i b^{i-1} \), we define the \( m \)-digit truncated vector \( \text{tr}_m(k) \in \mathbb{Z}^m_b \) as

\[
\text{tr}_m(k) = (\kappa_1, \kappa_2, \ldots, \kappa_m)^\top.
\]

**Definition 6.1.** Let \( G_1, \ldots, G_s \in \mathbb{Z}^{l \times d}_b \) be \( l \times d \) matrices over \( \mathbb{Z}_b \) with \( d \leq l \). Let \( 0 \leq k < b^d \). For \( 1 \leq j \leq s \) and \( 1 \leq i \leq l \), define \( y_{i,k,j} \in \mathbb{Z}_b \) as

\[
(y_{1,k,j}, \ldots, y_{l,k,j})^\top = G_j \text{tr}_d(k),
\]

where the matrix vector multiplication is over \( \mathbb{Z}_b \). Then we define

\[
x_{k,j} = \frac{y_{1,k,j}}{b} + \frac{y_{2,k,j}}{b^2} + \cdots + \frac{y_{l,k,j}}{b^l} \in [0, 1)
\]

for \( 1 \leq j \leq s \). In this way we obtain the \( k \)-th point \( x_k = (x_{k,1}, \ldots, x_{k,s}) \). We define \( P = P(G_1, \ldots, G_s) := \{x_0, \ldots, x_{b^d-1}\} \) (\( P \) is considered as a multiset) and call it a digital net over \( \mathbb{Z}_b \) with precision \( l \), or simply a digital net.

The dual net of a digital net, which is defined as follows, plays an important role in the subsequent analysis.

**Definition 6.2.** Let \( P = P(G_1, \ldots, G_s) \) be a digital net over \( \mathbb{Z}_b \). The dual net of \( P \), denoted by \( P^\perp = P^\perp(G_1, \ldots, G_s) \), is defined as

\[
P^\perp := \{k = (k_1, \ldots, k_s) \in \mathbb{N}^s_0 \mid G_1^\top \text{tr}_1(k_1) + \cdots + G_s^\top \text{tr}_1(k_s) = 0\}.
\]

The next lemma, which is a slight generalization of [5, Lemma 4.75] to our context, connects a digital net with Walsh functions.

**Lemma 6.3.** Let \( P \) be a digital net over \( \mathbb{Z}_b \) and \( P^\perp \) its dual net. Then we have

\[
|P|^{-1} \sum_{x \in P} \text{wal}_k(x) = \begin{cases} 1 & \text{if } k \in P^\perp, \\ 0 & \text{otherwise}. \end{cases}
\]

From now on, we consider integration defined over \( \widetilde{W}_{s,a,b} \). We use QMC algorithms over digital nets. That is, for a digital net \( P \), we use \( P(f) := \)
\[ |P|^{-1} \sum_{x \in P} f(x) - I(f) = |P|^{-1} \sum_{x \in P} \sum_{k \in \mathbb{N}_0^s} \hat{f}(k) \text{wal}_k(x) - I(f) \]
\[ = \sum_{k \in \mathbb{N}_0^s} \hat{f}(k) |P|^{-1} \sum_{x \in P} \text{wal}_k(x) - I(f) \]
\[ = \sum_{k \in P^\perp} \hat{f}(k) - \hat{f}(0) \]
\[ = \sum_{k \in P^\perp \setminus \{0\}} \hat{f}(k). \]

Hence we have
\[ |P|^{-1} \sum_{x \in P} f(x) - I(f) \leq \sum_{k \in P^\perp \setminus \{0\}} |\hat{f}(k)| \leq \|f\|_{\tilde{W}_{s,a,b}} \sum_{k \in P^\perp \setminus \{0\}} b^{-\tilde{\mu}(a;k)}. \quad (15) \]

**Remark 6.4.** WAFOM, a criterion for digital nets, is defined as a truncated version of the sum on the rightmost side of (15) for \( a = 0 \) in [12, 20] and for \( a = 1 \) in [22, 0]. Note that \( \tilde{\mu}(a;k) = \mu(a;k) \) in these cases. Thus the sum (and the sum with \( \tilde{\mu}(a;k) \) replaced by \( \mu(a;k) \)) can be regarded as an untruncated version of WAFOM generalized by weights \( a \), which has not been considered as far as the author knows. We can say that \( \mathcal{W}_{s,a} \), \( \mathcal{W}_{s,a,b} \) and \( \tilde{\mathcal{W}}_{s,a,b} \) are function spaces for which WAFOM is a suitable quality criterion.

We now define the minimal weight of \( P^\perp \) by
\[ \delta_{P^\perp} := \inf_{k \in P^\perp \setminus \{0\}} \tilde{\mu}(a;k). \]

Then the rightmost side of (15) is bounded by \( \|f\|_{\tilde{W}_{s,a,b}} \sum_{k \in \mathbb{N}_0^s} b^{-\tilde{\mu}(a;k)} \), where the sum is extended over all \( k \in \mathbb{N}_0^s \) with \( \tilde{\mu}(a;k) \geq \delta_{P^\perp} \). This argument implies the following lemma.

**Lemma 6.5.** Let \( P \) be a digital net. Then we have
\[ e^{wot}(P, \tilde{W}_{s,a,b}) \leq \sum_{k \in \mathbb{N}_0^s} b^{-\tilde{\mu}(a;k)}. \quad (16) \]

The right-hand side of (16) will be evaluated in the following sections.

We now prove a lemma which gives the existence of digital nets whose minimal weight is large, which generalizes [13, Proposition 2] and [20, Proposition 4].

First we define
\[ \text{vol}_{s,a}(M) := |\{k \in \mathbb{N}_0^s | \tilde{\mu}(a;k) \leq M\}|. \]
Lemma 6.6. Let $M$ be a real number and $\rho_b$ be the smallest prime factor of $b$. Let $d, l$ be positive integers with $l \geq M - a_1 - 1$. If $\text{vol}_{s,a}(M) \leq \rho_b^d b^L$ holds, then there exists a digital net $P$ over $\mathbb{Z}_b$ with precision $l$ satisfying $|P| = b^d$ and $\delta_{p,\ast} \geq M$.

Proof. Let $G_1, \ldots, G_s \in \mathbb{Z}_b^{l \times d}$ be matrices. Recall

$$k \in P^\perp(G_1, \ldots, G_s) \iff G_1 \text{tr}(k_1) + \cdots + G_s \text{tr}(k_s) = 0.$$ 

Thus, for a given $0 \neq k \in \mathbb{N}_0^d$ with $k_j < b'$ for all $j$, we have

$$|\{(G_1, \ldots, G_s) \in (\mathbb{Z}_b^{l \times d})^s \mid k \in P^\perp(G_1, \ldots, G_s)\}| \leq b^{sd} / \rho_b^d,$$

where $\rho_b$ is the smallest prime factor of $b$. Hence we have

$$|\{(G_1, \ldots, G_s) \in (\mathbb{Z}_b^{l \times d})^s \mid \min_{k \in P^\perp(G_1, \ldots, G_s) \setminus \{0\}} \hat{\mu}(\alpha; k) > M\}|$$

$$= b^{sd} - |\{(G_1, \ldots, G_s) \in (\mathbb{Z}_b^{l \times d})^s \mid \min_{k \in P^\perp(G_1, \ldots, G_s) \setminus \{0\}} \hat{\mu}(\alpha; k) \leq M\}|$$

$$\geq b^{sd} - \sum_{\substack{k \neq 0, k_j < b' \forall j \geq 0}} |\{(G_1, \ldots, G_s) \in (\mathbb{Z}_b^{l \times d})^s \mid k \in P^\perp(G_1, \ldots, G_s)\}|$$

$$> b^{sd} - \text{vol}_{s,a}(M)b^{sd} / \rho_b^d.$$

Thus, if $\text{vol}_{s,a}(M) \leq \rho_b^d b^L$ holds, there exists $(G_1, \ldots, G_s) \in (\mathbb{Z}_b^{l \times d})^s$ with

$$\inf_{k \in P^\perp(G_1, \ldots, G_s) \setminus \{0\}} \hat{\mu}(\alpha; k) \geq M. \quad (17)$$

Furthermore, from the assumption $l \geq M - a_1 - 1$ we have

$$\min\{\hat{\mu}(\alpha; k) \mid k \in \mathbb{N}_0^d, k_j \geq b' \exists j \geq 0\} \geq \max(1, a_1 + l + 1) \geq M. \quad (18)$$

Combining (17) and (18), we obtain the result. 

6.2. Super-polynomial convergence results

In this subsection, we prove super-polynomial convergence for $\mathcal{F}_{s,a}, W_{s,a,b}$ and $\tilde{W}_{s,a,b}$. Taking (6) and (5) into account, we have only to consider $\mathcal{W}_{s,a,b}$.

First we prove a bound on $\text{vol}_{s,a}(M)$ along [11, Exercise 3(b), p.332] and its modifications [13, 20], which treat the case of $a = 0$. Since $\text{vol}_{s,a}(M) \leq 1$ holds if $M < 1$, we assume that $M \geq 1$. We have

$$\text{vol}_{s,a}(M) = \sum_{k \in \mathbb{N}_0^d} 1 \leq \sum_{k \in \mathbb{N}_0^d} X^{\hat{\mu}(\alpha; k) - M} \leq \sum_{k \in \mathbb{N}_0^d} X^{\hat{\mu}(\alpha; k) - M}$$
for all $X \in (0, 1)$, and the right-most expression is equal to $\prod_{i=1}^{s} \prod_{j=1}^{\infty} (1 + (b - 1)X^{\max(i + a_j, 1)})/X^{M}$ from (4). By taking the logarithm on both sides and using the well-known inequality $\log(1 + X) \leq X$, for all $X \in (0, 1)$ we have

$$
\log \text{vol}_{s,a}(M) \leq \sum_{j=1}^{s} \sum_{i=1}^{\infty} (b - 1)X^{\max(i + a_j, 1)} + M \log X^{-1}.
$$

(19)

We proceed to bound $\sum_{i=1}^{\infty} X^{\max(i + a_j, 1)}$. If $a_j \geq 0$, it is equal to $X^{a_j+1}/(1 - X)$. Otherwise, we have

$$
\sum_{i=1}^{\infty} X^{\max(i + a_j, 1)} = \sum_{i: i + a_j \leq 1} X^{1} + \sum_{i: i + a_j > 1} X^{i + a_j}
\leq \sum_{i: i + a_j \leq 1} 1 + \sum_{i' = 1}^{\infty} X^{i'}
= n_j + X/(1 - X),
$$

where $n_j := |\mathcal{N}_j| = |\{i \in \mathbb{N} \mid i + a_j \leq 1\}|$. Thus, in both cases, we obtain

$$
\sum_{i=1}^{\infty} X^{\max(i + a_j, 1)} \leq n_j + \frac{X}{1 - \min(X^{a_j}, 1)}.
$$

Applying this inequality to (19), we have

$$
\log \text{vol}_{s,a}(M) \leq (b - 1) \sum_{j=1}^{s} \left( n_j + \frac{X}{1 - \min(X^{a_j}, 1)} \right) + M \log X^{-1}
\leq (b - 1) \sum_{j=1}^{s} \left( n_j + (\log X^{-1})^{-1} \min(X^{a_j}, 1) \right) + M \log X^{-1}.
$$

(20)

Putting $X = 1/\exp(\sqrt{(b - 1)s/M})$ and using $\min(X^{a_j}, 1) \leq 1$, we obtain

$$
\log \text{vol}_{s,a}(M) \leq N_s + 2\sqrt{(b - 1)sM},
$$

where we define $N_s := (b - 1) \sum_{j=1}^{s} n_j$. We have thus proved the following.

**Lemma 6.7.** For all $M \geq 0$ we have

$$
\text{vol}_{s,a}(M) \leq \exp \left( N_s + 2\sqrt{(b - 1)sM} \right).
$$

We note that Lemma 6.7 and the fact that $\text{vol}_{s,a}(M) \leq 1$ if $M < 1$ implies

$$
\text{vol}_{s,a}(M) \leq \exp \left( \left( N_s + 2\sqrt{(b - 1)s} \right) \sqrt{M} \right).
$$

(21)
Now we give a bound on the right-hand side of (16). From Lemma 6.7 we have

\[ \sum_{k \in \mathbb{N}_0} b^{-\tilde{\mu}(\alpha; k)} \geq M b - \sum_{i=0}^{\infty} \sum_{k \in \mathbb{N}_0, M+i \leq \tilde{\mu}(\alpha; k) < M+i+1} b^{-(M+i)} \leq \sum_{i=0}^{\infty} \text{vol}_{s,a}(M+i+1) b^{-(M+i)} \leq \sum_{i=0}^{\infty} \exp \left( N_s + 2\sqrt{(b-1)s(M+i+1)} \right) b^{-(M+i)} \]

for all \( M \geq 0 \). We can easily check \( \sqrt{x} \leq x/(2\sqrt{B}) + \sqrt{B}/2 \) for all \( x, B \geq 0 \).

Applying this inequality with \( x = M+i+1 \), the right-hand side of (22) is bounded by

\[ \sum_{i=0}^{\infty} \exp \left( N_s + 2\sqrt{(b-1)sB(M+i+1)} \right) b^{-(M+i)} \]

Taking \( B \) as \( \sqrt{(b-1)sB} = (\log b)/2 \), we obtain a bound on the right-hand side of (16) by

\[ \sum_{k \in \mathbb{N}_0, \tilde{\mu}(\alpha; k) \geq M} b^{-\tilde{\mu}(\alpha; k)} \leq C_s \exp(-M/2), \]

where the positive constant \( C_s \) is defined by

\[ C_s = \exp(N_s + (\log b)/2 + 2(b-1)s/\log b) (1 - \exp(-\log b/2))^{-1}. \]

Hence Lemma 6.5 implies the following lemma.

**Lemma 6.8.** Let \( P \) be a digital net. Then we have

\[ e_{\text{wor}}(P, \tilde{W}_{s,a,b}) \leq C_s \exp(-\delta_{P^*}(\log b)/2). \]

Put \( C_s' := N_s + 2\sqrt{(b-1)s} \). From (21), the condition of Lemma 6.6 is satisfied if \( \exp(C_s' \sqrt{M}) \leq \rho_b^d \), which is equivalent to \( M \leq (d \log \rho_b/C_s')^2 \). Therefore the following bound on the worst-case error follows from Lemmas 6.6 and 6.7.

**Theorem 6.9.** Let \( d \) be a positive integer. Then there exists a digital net \( P \) over \( \mathbb{Z}_b \) with precision \( l \) with \( |P| = b^d \) and \( l \geq (\log \rho_b/C_s')^2 d^2 - 1 - a_1 \) such that

\[ e_{\text{wor}}(P, \tilde{W}_{s,a,b}) \leq C_s \exp \left( -C_s'' d^2 \right), \]

where \( C_s'' := (\log \rho_b)^2/(2C_s^2) \).
In particular, \( e(b^d, s, \tilde{W}_{s,a,b}) \) is bounded by the right-hand side of (23). Therefore we have \( e(n, s, W_{s,a,b}) \leq C_s \exp \left(-C_s' \log n \right)^2 \). Thus embeddings (6) and (8) imply the following convergence result.

**Corollary 6.10.** There exist constants \( C_{i,s} \) and \( C_{i,s}' \) \((i = 1, 2, 3)\) which depend on \( s \) and the weights \( u \), or \( a \) such that for all \( n \) we have
\[
e(n, s, \tilde{W}_{s,a,b}) \leq C_{1,s} \exp \left(-C_{1,s}' \left( \log n \right)^2 \right),
\]
\[
e(n, s, W_{s,a,b}) \leq C_{2,s} \exp \left(-C_{2,s}' \left( \log n \right)^2 \right),
\]
\[
e(n, s, F_{s,u}) \leq C_{3,s} \exp \left(-C_{3,s}' \left( \log n \right)^2 \right).
\]

**6.3. Tractability results**

We have proved super-polynomial convergence for the function spaces, but this convergence depends heavily on \( s \). In this subsection, we prove a tractability result under the assumption of the sufficient condition from Theorem 4.1. That is, let \( r > 0 \) and assume that the sequence \( a \) satisfies \( \lim inf_{j \to \infty} a_j/j^r > 0 \). This implies that there exist a positive real number \( a \) and a non-negative integer \( A \) such that
\[
a_j \geq aj^r \quad \text{for all} \quad j > A.
\]
Hence hereafter we assume (24). Under this assumption, \( N_j = \{ i \in \mathbb{N} \mid i + a_j \leq 1 \} \) is empty for sufficiently large \( j \). Hence the constant factor in (8) is independent of \( s \) and thus the tractability result for \( W_{s,a,b} \) follows from that for \( \tilde{W}_{s,a,b} \). We also note that \((b - 1) \sum_{j=1}^{\infty} n_j\) is finite and we denote it by \( N \). We also show Item 2 of Theorem 4.2 using the embedding (6). The following arguments are parallel to those in Section 6.2.

First we prove a bound on \( \text{vol}_{s,a}(M) \) under the assumption (24). We need the following lemma to bound \( \sum_{j=1}^{s} X^{a_j} \).

**Lemma 6.11.** For all \( 0 < X < 1 \), we have
\[
\sum_{j=1}^{s} X^{a_j} \leq r^{-1} \Gamma(1/r)(a \log X^{-1})^{-1/r},
\]
where \( \Gamma(z) := \int_{0}^{\infty} t^{z-1} \exp(-t) \, dt \) is the Gamma function.

**Proof.** Since \( X^{ax^r} \) is a monotonically decreasing function of \( x \), we have
\[
\sum_{j=1}^{s} X^{a_j} \leq \int_{0}^{\infty} X^{ax^r} \, dx \leq \int_{0}^{\infty} \exp(-ax^r \log X^{-1}) \, dx.
\]
Now we consider the substitution \( ax^r \log X^{-1} = z \). Then we have \( dx = r^{-1} (a \log X^{-1})^{-1/r} z^{1-r/r} \, dz \), and thus
\[
\int_{0}^{\infty} \exp(-ax^r \log X^{-1}) \, dx = r^{-1} (a \log X^{-1})^{-1/r} \int_{0}^{\infty} z^{1-r/r} \exp(-z) \, dz
\]
\[
= r^{-1} \Gamma(1/r)(a \log X^{-1})^{-1/r},
\]
which proves the lemma.
Combining (20) and Lemma 6.11, for all $X \in (0, 1)$ we have

$$\log \text{vol}_{s,a}(M) \leq (b - 1) \left( \sum_{j=1}^{A} \left( n_j + \frac{1}{\log X^{-1}} \right) + \sum_{j=A+1}^{s} \frac{X^{ar_j}}{\log X^{-1}} \right) + M \log X^{-1}$$

$$\leq (b - 1) \left( \frac{A}{\log X^{-1}} + \frac{r^{-1} \Gamma(1/r)a^{-1/r}}{(\log X^{-1})^{1+1/r}} \right) + N + M \log X^{-1}.$$

Putting $X = 1/\exp(M^{-r/(2r+1)})$ and using $M \geq 1$, we obtain

$$\log \text{vol}_{s,a}(M) \leq c_1 M^{(r+1)/(2r+1)},$$

where $c_1 = (b - 1)(A + r^{-1} \Gamma(1/r)a^{-1/r}) + N + 1$. We have thus proved the following lemma.

**Lemma 6.12.** Assume (24). Then for all $M \geq 0$ we have

$$\text{vol}_{s,a}(M) \leq \exp(c_1 M^{(r+1)/(2r+1)}).$$

Note that the bound on $\text{vol}_{s,a}(M)$ from this lemma is weaker than Lemma 6.7 with respect to $M$ but independent of $s$ instead.

In the following, we bound the right-hand side of (16) along the lines of Section 6.2. For $M \geq 0$ we have

$$\sum_{k \in \mathbb{N}_0^s} b^{-\tilde{\mu}(a,k)} \leq \sum_{i=0}^{\infty} \exp(c_1 (M + i + 1)^{(r+1)/(2r+1)} b^{-(M+i)}). \quad (25)$$

We can easily check the inequality

$$x^{(r+1)/(2r+1)} \leq \frac{r+1}{2r+1} B^{r-x} + \frac{r}{2r+1} B^{r+1} \quad \text{for all } x, B \geq 0.$$

Applying this inequality with $x = M + i + 1$, the right-hand side of (25) is bounded by

$$\sum_{i=0}^{\infty} \exp \left( c_1 \frac{r+1}{2r+1} B^{-r}(M + i + 1) + c_1 \frac{r}{2r+1} B^{r+1} \right) b^{-(M+i)}$$

$$= b \exp \left( c_1 \frac{r}{2r+1} B^{r+1} \right) \sum_{i=0}^{\infty} \exp \left( \left( c_1 \frac{r+1}{2r+1} B^{r} - \log b \right) (M + i + 1) \right).$$

Now we choose $B$ such that

$$c_1 \frac{r+1}{2r+1} B^{-r} = \frac{\log b}{2}.$$

Thus we have a bound on the right-hand side of (16) as

$$\sum_{k \in \mathbb{N}_0^s, \tilde{\mu}(a,k) \geq M} b^{-\tilde{\mu}(a,k)} \leq c_2 \exp(-M(\log b)/2),$$

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where the positive constant $c_2$ is defined as

$$c_2 = \exp \left( \frac{\log b}{2} + \frac{c_1 r}{2r+1} \left( \frac{2c_1 (r+1)}{(2r+1) \log b} \right)^{(r+1)/r} \right) \frac{1}{1 - \exp\left(-\frac{(\log b)}{2}\right)}.$$

Hence Lemma 6.5 implies the following lemma.

**Lemma 6.13.** Assume (24). If $P$ is a digital net, we have

$$e_{wor}(P, \tilde{W}_{s,a,b}) \leq c_2 \exp\left(-\delta_{P^{\perp}} \frac{(\log b)}{2}\right).$$

Now we prove the existence of good digital nets. By Lemma 6.12, the condition of Lemma 6.6 is satisfied if $\exp\left(c_1 M \frac{(r+1)}{(2r+1)}\right) \leq \rho b$, which is equivalent to $M \leq (d \log \rho / c_1)^{(2r+1)/(r+1)}$. Therefore we have the following bound on the worst-case error independent of $s$.

**Theorem 6.14.** Let $d \in \mathbb{N}$ and put $c_3 = (\log \rho / c_1)^{(2r+1)/(r+1)}$. Assume (24). Then there exists a digital net $P$ over $\mathbb{Z}_b$ with precision $l$ with $|P| = b^d$ and $l \geq c_3 d^{(2r+1)/(r+1)} - 1 - a_1$ such that

$$e_{wor}(P, \tilde{W}_{s,a,b}) \leq c_2 \exp\left(-\frac{c_3 \log b}{2} \left( d^{(2r+1)/(r+1)} \right) \right). \quad (26)$$

In particular, $e(b^d, s)$ is bounded by the right-hand side of (26). Therefore we have $e(n, s, \tilde{W}_{s,a,b}) \leq c_2 \exp\left(-c_4 \log b \frac{d^{(2r+1)/(r+1)}}{(r+1)}\right)$, where $c_4 := \frac{c_3 \log b}{2}$. Thus embeddings (11) and (8) imply the following tractability result.

**Corollary 6.15.** Assume (24). Then there exist constants $C_i$ and $C_i'$ ($i = 1, 2$) which are independent of $s$ such that for all $s$ and $n$ we have

$$e(n, s, \tilde{W}_{s,a,b}) \leq C_1 \exp\left(-C_1' \frac{\log n}{(2r+1)/(r+1)}\right),$$

$$e(n, s, W_{s,a,b}) \leq C_2 \exp\left(-C_2' \frac{\log n}{(2r+1)/(r+1)}\right).$$

Assume that the weight sequence $u$ satisfies $\liminf_{j \to \infty} \log(u_{j-1}^{-1})/j^r > 0$. Then there exists constants $C_3, C_3'$ independent of $s$ such that for all $s$ and $n$ we have

$$e(n, s, F_{s,u}) \leq C_3 \exp\left(-C_3' \frac{\log n}{(2r+1)/(r+1)}\right).$$

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