THE UNIRATIONALITY OF THE MODULI SPACES OF CURVES OF GENUS \( \leq 14 \)

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0. Introduction In this paper we prove that the moduli space of complex curves of genus 14 is unirational. Our method applies more in general to the moduli space \( \mathcal{M}_g \) with \( g \leq 14 \), so we use it to give new proofs of the unirationality of \( \mathcal{M}_g \) for \( g = 11, 12, 13 \).

The proof relies on linkage of curves in the projective space and on Mukai’s description of canonical curves, of certain low genera, as linear sections of a homogeneous space, ([M]). From these results of Mukai we deduce the unirationality of the Hilbert schemes of non special, smooth, irreducible curves of degree \( d \) and genus \( g \leq 10 \) in \( \mathbb{P}^r \), (see section 1). Then we use this property, together with linkage, for proving our results.

To add some historical remarks we recall that the proof of the unirationality of \( \mathcal{M}_g \) goes back to Severi for \( g \leq 10 \), [Se]. The cases of genus 11, 12, 13 were first proved by Sernesi and by Chang and Ran, [S1] for \( g = 12 \) and [CR1] for \( g = 11, 13 \). Quite recently a proof which is in part computational was given by Tonoli and Schreyer for \( g = 11, 12, 13 \), [TS].

A conjecture of Harris and Morrison implies that \( \mathcal{M}_g \) has negative Kodaira dimension for \( g \leq 22 \). Our result implies such a property for \( \mathcal{M}_{14} \), this was known up to now for \( g \leq 13 \) and \( g = 15, 16 \), cfr. [HM2], [CR2], [FP]. Of course things are different in higher genus: due to the fundamental results of Eisenbud, Harris and Mumford, \( \mathcal{M}_g \) has non negative Kodaira dimension for \( g \geq 23 \) and it is of general type for \( g \geq 24 \). Recently Farkas showed that \( \mathcal{M}_{23} \) has Kodaira dimension \( \geq 1 \), [F].

Our starting point has been the following observation: fix a curve \( D \) of genus 14 with general moduli. On \( D \) there are finitely many line bundles \( L \) of degree 8 such that \( h^0(L) = 2 \). For each of them \( \omega_D(-L) \) is very ample and defines an embedding

\[ D \subset \mathbb{P}^6. \]

Now consider the vector space \( V \) of quadratic forms vanishing on \( D \): if \( D \) is projectively normal then \( V \) has dimension 5, hence

\[ D \subset Q_1 \cap \cdots \cap Q_5 \]

where \( Q_1 \ldots Q_5 \) are independent quadrics. If \( Q_1 \ldots Q_5 \) define a complete intersection, then

\[ Q_1 \cap \ldots Q_5 = C \cup D \]

where \( C \) is a curve of degree 14. If \( C \) is smooth and connected then its geometric genus is 8. In section 5 we show the existence of a complete intersection of 5 quadrics

\[ C_o \cup D_o \subset \mathbb{P}^6 \]

which satisfies all the previous assumptions. Using the general set up proved in sections 2, 3 and 4 we are also able to deduce that \( C_o \) is a non specially embedded, projectively normal curve and that the Petri map

\[ \mu : H^0(\omega_{D_o}(1)) \otimes H^0(\mathcal{O}_{D_o}(1)) \to H^0(\omega_{D_o}) \]

is injective. Then, in their corresponding Hilbert schemes, \( C_o \) and \( D_o \) admit irreducible open neighborhoods \( \mathcal{C} \) and \( \mathcal{D} \) parametrizing curves with the same properties. It follows from the results of section 1 that \( \mathcal{C} \) is unirational. On the other hand the injectivity of \( \mu \) implies that the natural map \( f : \mathcal{D} \to \mathcal{M}_{14} \) is dominant. On \( \mathcal{C} \) one can easily construct a Grassmann bundle

\[ G \]

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which is locally trivial in the Zariski topology and parametrizes pairs \((C, V)\) such that \(C \in \mathcal{C}\) and \(V \subset H^0(I_C(2))\) is a 5-dimensional subspace. Since \(\mathcal{C}\) is unirational the same holds for \(\mathcal{G}\). Finally, the existence of the above complete intersection \(C_o \cup D_o\) makes possible to define a rational map

\[ \phi : \mathcal{G} \to \mathcal{D} \]

sending the general pair \((C, V)\) to \(D\), where \(C \cup D\) is the scheme defined by \(V\) and \(D\) is a smooth, irreducible element of \(\mathcal{D}\). It turns out that \(\phi\) is birational, hence \(f : \phi : \mathcal{G} \to \mathcal{M}_{14}\) is dominant and \(\mathcal{M}_{14}\) is unirational. A more elaborated, from the technical point of view, version of this idea works for showing the unirationality of the universal 5-symmetric product over \(\mathcal{M}_{12}\), of the universal 6-symmetric product over \(\mathcal{M}_{11}\) and finally of \(\mathcal{M}_{13}\). So we are able to obtain in this way the unirationality of \(\mathcal{M}_g\), \(g = 11, 12, 13\).

To continue with the example of genus 14 we give some more details on the way we prove the unirationality of \(\mathcal{C}\). In the general case of the Hilbert scheme of non special curves of degree \(d\) and genus \(g \leq 10\) the proof of the unirationality is analogous, (see section 2). It follows from Mukai’s results that a general genus 8 canonical curve is a linear section of the Plücker embedding

\[ G \subset \mathbf{P}^{14} \]

of the Grassmannian of lines of \(\mathbf{P}^5\), cfr. [M]. Assume \(x = (x_1, \ldots, x_8) \in G^8\) is general and let \(P_x\) be the space spanned by \(x_1, \ldots, x_8\). Then

\[ C_x =: P_x \cap G \]

is such a general canonical curve, moreover \(C_x\) is endowed with the line bundle

\[ H_x =: \omega_{C_x}(x_1 + \cdots + x_4 - x_5 - \cdots - x_8) \in \mathrm{Pic}_{14}(C_x). \]

The pair \((C_x, H_x)\) defines a point in the universal Picard variety \(\mathrm{Pic}_{14,8}\) and a rational map

\[ G^8 \to \mathrm{Pic}_{14,8}. \]

It turns out that the latter is dominant, (see section 3), therefore \(\mathrm{Pic}_{14,8}\) is unirational. Then, with a little bit more of effort, the unirationality of \(\mathcal{C}\) also follows.

Some frequently used notations and conventions.

- \(\mathcal{H}_{d, g, r}\) denotes the restricted Hilbert scheme. This is the subscheme, in the Hilbert scheme of curves of degree \(d\) and genus \(g\) in \(\mathbf{P}^r\), parametrizing smoothable, connected, non degenerate curves.
- \(W^r_d(C)\) is the Brill-Noether locus of all line bundles \(H \in \mathrm{Pic}^d(C)\) such that \(h^0(H) \geq r + 1\). \(W^r_{d,g}\) is the universal Brill-Noether locus over \(\mathcal{M}_g\). \(\mathrm{Pic}_{d,g}\) is the universal Picard variety.
- To simplify notations \(X \cap Y\) will be the scheme theoretic intersection of the schemes \(X\) and \(Y\), unless differently stated.
- \(I_{X/Y}\) is the ideal sheaf of \(X\) in \(Y\). If \(V\) is a vector space of sections of a line bundle then \(|V|\) is the associated linear system. The dual of a vector bundle \(E\) is \(E^*\), \(\mathbf{P}(E) =: \mathrm{Proj} E^*\).
- A nodal curve is a curve having ordinary nodes as its only singularities.

(*) The present work has been supported by the European research program EAGER and by the Italian research program GVA (Geometria delle varietà algebriche)

1 Auxiliary unirationality results. In this section we show some results of unirationality for the Hilbert schemes \(\mathcal{H}_{d, g, r}\) in genus \(g \leq 10\). More precisely let

\[ \mathcal{H}^{ns}_{d, g, r} =: \{ C \in \mathcal{H}_{d, g, r} \mid C \text{ is smooth and } \mathcal{O}_C(1) \text{ is non special } \}, \]
where $3 \leq r \leq d - g$. As is well known the Zariski closure of $H_{d,g,r}^{ns}$ is the unique irreducible component of the Hilbert scheme which dominates $\mathcal{M}_g$. We will prove the following

(1.2) **THEOREM** $H_{d,g,r}^{ns}$ is unirational for $7 \leq g \leq 10$.

The extension of the theorem to genus $g \leq 6$ is easy: see 1.11. The theorem is an application of the following description, due to Mukai, of a general canonical curve of genus $g = 7, 8, 9, 10$ (cfr. [M]).

(1.3) **THEOREM** For $g = 7, 8, 9, 10$ there exists a rational homogeneous space $P_g \subset \mathbb{P}^{\dim P_g + g - 2}$ whose general curvilinear section is a general canonical curve.

Let $P_g$ be a homogeneous space as above, we consider the open subset

(1.4) $$U \subset P_g^g$$

of points $x = (x_1, \ldots, x_g)$ such that $x_i \neq x_j (i \neq j)$ and moreover:

(i) the linear span $\mathbb{P}_x = \langle x_1 \ldots x_g \rangle$ has dimension $g - 1$,

(ii) $C_x = \mathbb{P}_x \cap G$ is a smooth, irreducible canonical curve.

On $U$ we have a universal canonical curve

$$\pi : C \to U$$

with fibre $C_x$ over $x$. $C$ contains the divisors $D_i = s_i(U)$, where $s_i : U \to C$ is the section sending $x$ to $x_i$. For any $d \geq g + 3$ we fix non zero integers $n_1, \ldots, n_g$ such that

$$d = n_1 + \cdots + n_g + 2 - 2g.$$

Then we consider the sheaf

(1.5) $$\mathcal{H} = \omega_\pi (n_1 D_1 + \cdots + n_g D_g),$$

where $\omega_\pi$ denotes the relative cotangent sheaf of $\pi$. For any $x \in U$ we have

$$\mathcal{H} \otimes \mathcal{O}_{C_x} = \omega_{C_x} (n_1 x_1 + \cdots + n_g x_g).$$

Let $C$ be a curve of genus $g$, then the Abel map $a : C^g \to \text{Pic}^g(C)$ is surjective. As is well known this property generalizes as follows:

(1.6) **LEMMA** Let $n_1, \ldots, n_g$ be non zero integers, then the map

$$a_{n_1 \ldots n_g} : C^g \to \text{Pic}^d(C)$$

sending $(x_1, \ldots, x_g)$ to $\omega_C (n_1 x_1 + \cdots + n_g x_g)$ is surjective.

(1.7) **LEMMA** Let $x$ be general in $U$, then $\mathcal{H} \otimes \mathcal{O}_{C_x}$ is non special and very ample.
PROOF Let $C = C_x$, by Mukai’s result we can assume that $C$ has general moduli. By the previous lemma, each $L \in \text{Pic}^d(C)$ is isomorphic to $\omega_C(\Sigma_{i \in I} y_i)$, for some $y = (y_1, \ldots, y_g) \in C^g$. Therefore, keeping $C$ fixed and possibly replacing $x$ by $y$, we can assume that $\mathcal{H} \otimes \mathcal{O}_C$ is general in $\text{Pic}^d(C)$.

On a general curve of genus $g$ a general line bundle of degree $d \geq g + 3$ is very ample, as follows from [ACGH] Theorem 1.8 p.216. Moreover such a line bundle is also non special.

Finally, up to shrinking the open set $U$, we can assume that:

1. $\text{Proj}(\pi_* \mathcal{H}) = U \times \mathbf{P}^{d-g}$,
2. $\mathcal{C}$ is embedded in $U \times \mathbf{P}^{d-g}$ by the map associated to the tautological sheaf,
3. $\mathcal{H} \otimes \mathcal{O}_{C_x}$ is non special, $\forall x \in U$.

(1.8) LEMMA The natural morphism $r : U \to \text{Pic}_{d,g}$ is dominant.

PROOF Let $x \in U$, by definition $r(x)$ is the isomorphism class of $(C_x, \mathcal{H} \otimes \mathcal{O}_{C_x})$ in the universal Picard variety $\text{Pic}_{d,g}$. To prove that $r$ is dominant, it suffices to show that $r(U)$ intersects a general fibre of the forgetful map $f : \text{Pic}_{d,g} \to M_g$ along a dense subset. A general fibre of $f$ is $\text{Pic}^d(C)$, where $C$ is general of genus $g$. By Mukai’s theorem $C$ is birational to a section $C_o = \mathbf{P}_o \cap P_g$, for some point $o = (o_1, \ldots, o_g)$ in $C^g_3 \cap U$. Note that $r/C^g_3 \cap U : C^g_3 \to \text{Pic}^d(C)$ associates to $x = (x_1, \ldots, x_g)$ the line bundle $\mathcal{H} \otimes \mathcal{O}_{C_x} = \mathcal{O}_C(n_1x_1 + \cdots + n_gx_g)$. Then such a map extends to the surjection $a_{n_1 \ldots n_g}$ considered in lemma 2.6 and hence $r(C^g_3 \cap U)$ is dense.

Let

\[(1.9) \quad p : \mathcal{H}^{ns}_{d,g,d-g} \to \text{Pic}_{d,g}\]

be the morphism sending $C$ to the isomorphism class of the pair $(C, \mathcal{O}_C(1))$ and let

\[(1.10) \quad q : U \times \text{PGL}(d-g+1) \to \mathcal{H}^{ns}_{d,g,d-g}\]

be the morphism sending $(x, \alpha)$ to $\alpha(C_x)$, observe that

$$p \cdot q(U \times \{\text{id}\}) = r(U).$$

Therefore, by lemma 1.8, $p \cdot q : U \times \text{PGL}(d-g+1) \to \text{Pic}_{d,g}$ is dominant.

PROOF OF THEOREM 1.2 We first show the case $r = d - g$. We know from 1.8 and 1.10 that both $p(q(U \times \text{PGL}(r + 1)))$ and $p(\mathcal{H}^{ns}_{d,g,d-g})$ contain a non empty open set of $\text{Pic}_{d,g}$. Therefore

\[A =: p^{-1}(p((U \times \text{PGL}(d-g+1))) \cap p(\mathcal{H}^{ns}_{d,g,d-g}))\]

contains a non empty open set. Then, since $\mathcal{H}^{ns}_{d,g,r}$ is irreducible, it suffices to show that

\[A \subset q(U \times \text{PGL}(r + 1)).\]

Indeed this implies that $q$ is dominant and hence that $\mathcal{H}^{ns}_{d,g,r}$ is unirational. To prove the above inclusion we observe that: $C \in A \implies p(C) \in p(q(U \times \text{PGL}(d-g+1))) \implies C$ is smooth and $\mathcal{O}_C(1)$ is non special $\implies C$ is linearly normal $\implies p^{-1}(p(C)) = P_C$, where $P_C$ is the $\text{PGL}(d-g+1)$-orbit of $C$. But then there exists $\beta \in \text{PGL}(r + 1)$ such that $\beta(C) = q(x, \alpha)$, that is $C = q(x, \beta^{-1}\alpha)$. Let us complete the proof with the case $r < d - g$: consider the space $M_r$ of all linear projections.
\( \mathbb{P}^{d-g} \to \mathbb{P}^r \) and the map \( s : Y \times M_r \to \mathcal{H}_{d,g,r}^{ns} \), sending \((C, \alpha)\) to \( \alpha(C) \). It is easy to see that \( s \) is dominant, therefore \( \mathcal{H}_{d,g,r}^{ns} \) is unirational too.

(1.11) REMARK Let \( 4 \leq g \leq 6 \) and let \( P_g \subset \mathbb{P}^{g+1} \) be a fixed, general Fano threefold of index one and genus \( g \). A direct count of parameters shows that a general canonical curve of genus \( g \) is a

\[ \text{curvilinear section of } \mathcal{P}_g. \]

Using this property one can show, with exactly the same proof as above, that \( \mathcal{H}_{d,g,r}^{ns} \) is unirational if \( 4 \leq g \leq 6 \). We leave to the reader the extension of the result to the case \( g \leq 3 \).

2. General set up. In this section we build up a somehow general strategy to apply the previous unirationality results. The basic idea is to consider families of irreducible curves \( D \) of genus \( g' \) which are linked to a general \( C \in \mathcal{H}_{d,g,r}^{ns} \) by a complete intersection of fixed type \((f_1, \ldots, f_{r-1})\). These families are unirational. In some cases they dominate \( M_{g'} \). We start with the following

(2.1) DEFINITION Let \( 3 \leq r \leq d-g \) and let \( g \leq 10 \), then

\[ \mathcal{C}_{d,g,r}^{ns} =: \{ C \in \mathcal{H}_{d,g,r}^{ns} / \rho_f \text{ has maximal rank for each } f \} \]

where \( \rho_f : H^0(\mathcal{I}_C/\mathcal{P}_r(f)) \to H^0(\mathcal{O}_C(f)) \) is the restriction map.

By semicontinuity \( \mathcal{C}_{d,g,r}^{ns} \) is open, it is non empty because the maximal rank condition is generically satisfied in the cases we are considering. Due to the results of the previous section \( \mathcal{C}_{d,g,r}^{ns} \) is irreducible and unirational. We fix a sequence of integers

(2.2)

\[ \sigma = (f_1, \ldots, f_s, k_1, \ldots, k_s) \]

satisfying

\[ 1 \leq k_i \leq n_i \text{ and } k_1 + \cdots + k_s = r - 1 \]

where \( n_i \) is the constant value of \( h^0(\mathcal{I}_C/\mathcal{P}_r(f_i)) \) when \( C \) moves in \( \mathcal{C}_{d,g,r}^{ns} \). Then we consider the Ideal sheaf \( \mathcal{J} \) of the universal curve

\[ C \subset \mathcal{C}_{d,g,r}^{ns} \times \mathbb{P}^r \]

and the natural projections \( p_1 : C \to \mathcal{C}_{d,g,r}^{ns} \) and \( p_2 : C \to \mathbb{P}^r \). By Grauert theorem the sheaf

(2.3)

\[ \mathcal{F}_i =: p_1^*(\mathcal{J} \otimes p_2^*\mathcal{O}_{\mathbb{P}^r}(f_i)) \]

is a vector bundle on \( \mathcal{C}_{d,g,r}^{ns} \), the fibre at \( C \) is the space \( H^0(\mathcal{I}_C/\mathcal{P}_r(f_i)) \). For each \( k \geq 1 \) we can also consider the Grassmann bundle

(2.4)

\[ u_k : G(k, \mathcal{F}_i) \to \mathcal{C}_{d,g,r}^{ns} \]

defined by \( k \) and \( \mathcal{F}_i \).

(2.5) DEFINITION \( G_{d,g,r}^\sigma \) is the fibre product \( G(k_1, \mathcal{F}_1) \times u_{k_1} \cdots \times u_{k_s} G(k_s, \mathcal{F}_s) \) over \( \mathcal{C}_{d,g,r}^{ns} \). \( G_{d,g,r}^\sigma \) is birational to the product \( \mathcal{C}_{d,g,r}^{ns} \times G(k_1, n_1) \times \cdots \times G(k_s, n_s) \), therefore the next statement is immediate.

(2.6) PROPOSITION \( G_{d,g,r}^\sigma \) is irreducible and unirational.
A point of $\mathcal{G}_{d,g,r}^\sigma$ is a sequence $(C, V_1, \ldots, V_s)$, where $C \in C_{d,g,r}^{ns}$ and $V_i$ is a $k_i$-dimensional subspace of $H^0(I_C/\mathbb{P}^r(f_i))$. Let

$$B \subset \mathbb{P}^r$$

be the scheme defined by the set of homogeneous forms $V_1 \cup \ldots \cup V_s$. Since $\text{dim} \ V_1 + \cdots + \text{dim} \ V_s = r - 1$ it is possible that $B$ is a curve: in this case $B$ is a complete intersection.

(2.7) DEFINITION (1) A point $(C, V_1, \ldots, V_s) \in \mathcal{G}_{d,g,r}^\sigma$ is a key-point if

$$B = C \cup D$$

is a nodal curve and the component $D$ is smooth, irreducible, non degenerate.

(2) $\mathcal{G}_{d,g,r}^\sigma$ satisfies the key condition if the set of its key-points is non empty.

Clearly the set of the key-points of $\mathcal{G}_{d,g,r}^\sigma$ is open. Let $C \cup D$ be a nodal complete intersection as above, from now on we will keep the following notations:

(2.8)

$$d' = \deg(D), \ g' = p_a(D), \ n = \text{cardinality of Sing } B.$$

The numbers $d'$, $g'$ and $n$ can be readily computed from $(d, g, \sigma)$, we have:

$$d + d' = f_1^{k_1} \cdots f_s^{k_s},$$

$$(g - g') = \frac{1}{2}(k_1f_1 + \cdots + k_sf_s - r - 1)(d - d')$$

and

$$n = (k_1f_1 + \cdots + k_sf_s - r - 1)d + 2 - 2g,$$

see [Fu] p.159 example 9.1.12. Finally we fix the notations for some natural maps:

(2.9) DEFINITION Assume that $\mathcal{G}_{d,g,r}^\sigma$ satisfies the key condition, then

$$\gamma_{d,g,r} : \mathcal{G}_{d,g,r}^\sigma \to \mathcal{H}_{d',g',r}$$

is the map sending a point $(C, V_1, \ldots, V_s)$ as above to $D$.

The natural map from $\mathcal{H}_{d,g,r}$ to the universal Brill-Noether locus will be denoted as

(2.13)

$$\beta_{d,g,r} : \mathcal{H}_{d,g,r} \to \mathcal{W}_{d,g}^r$$

and the forgetful map from $\mathcal{W}_{d,g}^r$ to the moduli space will be denoted as

(2.14.)

$$\alpha_{d,g,r} : \mathcal{W}_{d,g}^r \to \mathcal{M}_g$$

The next statement only summarizes our program for showing the unirationality of some moduli spaces, the proof is immediate.

(2.10) PROPOSITION Assume $\mathcal{G}_{d,g,r}^\sigma$ satisfies the key condition and that the image of

$$\gamma_{d,g,r} : \mathcal{G}_{d,g,r}^\sigma \to \mathcal{H}_{d',g',r}$$
dominates $M_{g'}$, then $M_{g'}$ is unirational.

3. Some useful criteria I. In this section we prove sufficient conditions to ensure that the key condition holds for $G_{d,g,r}$. This criterion is an elementary version of more general known properties, we show it for completeness and some lack of reference. We recall that a subvariety

$$Y \subset P^r$$

is a scheme theoretic intersection of hypersurfaces of degree $f$ iff $I_{Y/P^r}(f)$ is globally generated. Throughout all the section we will assume that

(3.1) \[ Y =: Y^c \cup Z \]

where:
- $Y^c$ is an equidimensional variety of codimension $c$, which is locally complete intersection with at most finitely many singular points.
- $Z$ is disjoint from $Y^c$ and it is either smooth 0-dimensional or empty.

(3.2) PROPOSITION Assume that $I_{Y/P^r}(f)$ is globally generated and that $\dim Y \leq 3$. Then there exists a complete intersection of $c$ hypersurfaces $Q_1 \ldots Q_c$ of degree $f$ such that:
- either $Q_1 \cap \ldots \cap Q_c = Y$
- or $Q_1 \cap \ldots \cap Q_c = X \cup Y$ and moreover: (1) $X$ is smooth and contains $Z$, (2) $X \cap Y^c$ is smooth and equidimensional of codimension $c + 1$.

PROOF Let $I := H^0(I_{Y/P^r}(f))$, we denote by $G^c$ the Grassmannian of codimension $c$ subspaces of $I$ and by $G_c$ the Grassmannian of $c$ dimensional subspaces. We assume that $Y$ is not a complete intersection of $c$ hypersurfaces of degree $f$: otherwise there is nothing else to show. Let

$$\sigma : P \to P^r$$

be the blowing up of $Y^c$. Then the strict transform of $| I |$ by $\sigma$ is $| fH - E |$, where $E$ is the exceptional divisor of $\sigma$ and $H$ is the pull-back of a hyperplane. Since $I_{Y/P^r}(f)$ is globally generated and $Y = Y^c \cup Z$, the base locus of $| fH - E |$ is $\sigma^{-1}(Z)$. Let $V \in G_c$ be general and let $B_V$ be the base locus of the strict transform of $| V |$ on $P$. By Bertini theorem we can assume that $B_V$ is smooth: this follows because $\sigma^{-1}(Z)$ is smooth and finite. Moreover we can assume that $B_V$ intersects transversally the exceptional divisor $E$. Since $Y^c$ is locally complete intersection, $E$ is a projective bundle with fibre of dimension $c - 1$. Then, since $V$ is general of codimension $c$ and $\text{Sing } Y^c$ is finite, we can also assume that $B_V \cap \sigma^{-1}(\text{Sing } Y^c)$ is empty. We claim that

$$\sigma/B_V : B_V \to P^r$$

is an embedding: this is obvious on $B_V - (B_V \cap E)$. To complete the proof consider $p \in B_V \cap E$ and $F = \sigma^{-1}(\sigma(p))$. $B_V$ is the complete intersection of $c$ independent divisors $D_1, \ldots D_c$ of $| fH - E |$. Since $O_F(fH - E) \cong O_{P^{r-1}}(1)$ the intersection scheme $B_V \cap F$ is a linear space. This must be 0-dimensional because $B_V$ is transversal to $E$, hence $\sigma/B_V$ is an embedding at $p$. Let

$$X_V = \sigma(B_V)$$
then $X_V$ is smooth and moreover we have that

$$X_V \cup Y$$

is complete intersection of the $c$ hypersurfaces $\sigma(D_1), \ldots, \sigma(D_c) \in |\mathcal{I}_Y/\mathbb{P}^r(f)|$.

Finally we show that $X_V \cap Y^c$ is smooth. Let $y \in Y^c - \text{Sing } Y$ and let

$$I_y = \{ q \in I \mid q \in m_y^2 \}.$$ 

Since $Y$ is scheme theoretic intersection of hypersurfaces of degree $f$, the space $I_y$ has codimension $c$ in $I$. This defines a morphism

$$\phi : Y^c - \text{Sing } Y \to G^c$$

 sending $y$ to $I_y$. For any $V \in G_c$ we can consider the Schubert cycle

$$\sigma_V = \{ L \in G^c \mid \text{dim}(L \cap V) \geq 1 \}.$$ 

It is well known that the singular locus of $\sigma_V$ is

$$\text{Sing } \sigma_V = \{ L \in G^c \mid \text{dim}(L \cap V) \geq 2 \}$$

and moreover that $\text{Sing } \sigma_V$ has codimension 4 in $G_c$. Since $\text{dim } Y \leq 3$ we have

$$\text{Sing } \sigma_V \cap \phi(Y^c - \text{Sing } Y) = \emptyset$$

for a general $V$. Then, by the transversality of a general $\sigma_V$, we can assume that $\phi^{-1}(\sigma_V)$ is smooth of codimension $c + 1$. On the other hand it turns out that

$$\phi^{-1}(\sigma_V) = Y^c \cap X_V.$$ 

Indeed: $y \in Y^c \cap X_V \iff y \in Y^c - \text{Sing } Y$ and $B_V$ is singular at $y \iff y \in Y^c - \text{Sing } Y$ and $\text{dim } I_y \cap V = 1$.

In particular we will apply the lemma when $Y^c$ is a nodal curve. So we point out the following

(3.3) **PROPOSITION** Let $Y = C \cup Z$, where $C$ is a nodal curve and $Z$ is a smooth, 0-dimensional scheme disjoint from $C$. Assume that $\mathcal{I}_Y/\mathbb{P}^r(f)$ is globally generated, then:

(1) $Y$ lies in a smooth surface $S$ which is complete intersection of $r - 2$ hypersurfaces of degree $f$.

(2) $|fH - C|$ is base-point-free, so that a general $D \in |fH - C|$ is transversal to $C$. $D$ is connected if $D^2 > 0$. ($H$ = hyperplane section of $S$).

**PROOF** By proposition 3.2 there exists a nodal complete intersection of $r - 1$ hypersurfaces of degree $f$

$$C \cup D$$

such that $D$ is smooth and contains $Z$. Let $V = H^0(\mathcal{I}_{C \cup D}/\mathbb{P}^r(f))$ and let $x \in C \cup D$: if $x$ is smooth no $Q \in |V|$ is singular at $x$. If $x$ is singular then $x$ is a node and there exists exactly one $Q$ singular at $x$. Since $\text{Sing } C \cup D$ is finite, the base locus of a general hyperplane in $|V|$ is a smooth surface $S$ as required. This shows (1), (2) follows immediately.

(3.4) **PROPOSITION** Let $Y = C \cup Z$ be as in 3.3 and let $\sigma = (f, r - 1)$. Assume that:
4. Some useful criteria II. Now we want to give some sufficient conditions, on some of the unirational Grassmann bundles $G_{d,g,r}$, so that $G'_{d,g,r}$ dominates the moduli space $M_{d,g,r}$.

(4.1) **Lemma** Let $C \cup D \subset P^r$ be a nodal complete intersection of $r - 1$ hypersurfaces of degree $f = \frac{1 + g}{r - 2}$. Then:

1. $C$ is $f$-normal iff $D$ is linearly normal and $D$ is $f$-normal iff $C$ is linearly normal.
2. $O_C(1)$ is non special iff $h^0(\mathcal{I}_D/P^s(f)) = r - 1$.
3. $C$ is non degenerate iff $O_D(f)$ is non special and $D$ is non degenerate iff $O_C(f)$ is non special.

**Proof** As in 3.3 $C \cup D$ is contained in a smooth complete intersection $S$ of $r - 2$ hypersurfaces of degree $f$.

1. Let $H$ be a hyperplane section of $S$, the assumption $f = \frac{1 + g}{r - 2}$ simply means that $H$ is a canonical divisor. It follows from the standard exact sequence

$$0 \to O_S(H - D) \to O_S(H) \to O_D(H) \to 0$$

that $D$ is linearly normal iff $h^1(O_S(H - D)) = 0$. Since $H$ is canonical we have $h^1(O_S(H - D)) = h^1(O_S(D)) = h^1(O_S(fH - C))$. Hence $D$ is linearly normal iff $C$ is $f$-normal. Since $C + D \sim fH$, the second equivalence follows exchanging $D$ with $C$.

2. At first we remark that $h^0(\mathcal{I}_D/P^s(f)) = h^0(O_S(fH - D)) + r - 2 = h^0(O_S(C)) + r - 2$. Secondly the standard exact sequence

$$0 \to O_S(H - C) \to O_S(H) \to O_C(1) \to 0$$

yields $h^1(O_C(1)) = h^2(O_S(H - C)) - h^2(O_S(H)) = h^0(O_S(C)) - 1$. Hence: $h^0(\mathcal{I}_D/P^s(2)) = r - 1 \iff h^0(O_S(C)) = 1 \iff h^1(O_C(1)) = 0$.

3. Consider the standard exact sequence

$$0 \to O_S(C) \to O_S(fH) \to O_D(f) \to 0$$

and its associated long exact sequence. Since $f \geq 1$, we have $h^1(O_S(fH)) = h^2(O_S(fH)) = 0$ and hence $h^1(O_D(f)) = h^2(O_S(C)) = h^0(O_S(H - C))$. This implies the first equivalence. Again the second one follows with by exchanging $C$ and $D$.

(4.2) **Proposition** Let $\sigma = (f, r - 1)$, $f = \frac{1 + g}{r - 2}$ and $r = d - g$. If $G'_{d,g,r}$ satisfies the key assumption. Then the image of the map $\beta_{d',g',r} \cdot \gamma_{d,g,r} : G'_{d,g,r} \to W'_{d',g'}$ is open.

**Proof** Under the assumption a general $C \in \mathcal{C}_{d,g,r}$ is linked to a smooth, irreducible, non degenerate curve $D$ by a complete intersection of $r - 1$ hypersurfaces of degree $f$. We remark that $C$ is both $f$-normal and linearly normal. This follows because the restriction map

$$\rho_m : H^0(O_{P^r}(m)) \to H^0(O_C(m))$$
has maximal rank and \(h^0(\mathcal{I}_C/\mathbb{P}_r(f)) > 0\). Hence \(\rho(f)\) is surjective and \(C\) is \(f\)-normal. On the other hand \(r = d - g\) implies \(h^0(\mathcal{O}_C(1)) = r + 1\), because \(\mathcal{O}_C(1)\) is non special. Then \(\rho(1)\) is an isomorphism and \(C\) is linearly normal. \(D\) has the following properties:

(i) \(D\) is linearly normal, (ii) \(D\) is \(f\)-normal, (iii) \(h^0(\mathcal{I}_D/\mathbb{P}_r(f)) = r - 1\), (iv) \(\mathcal{O}_D(f)\) is non special.

This follows from the previous lemma 4.1. Using the same lemma it is easy to see that the set

\[
U' = \{ D' \in \mathcal{H}_{d',g',r} / D' \text{ is smooth, irreducible, non degenerate and satisfies (i), \ldots, (iv) } \}
\]

is open. Let \(D' \in U'\), assume that: (v) the scheme defined by \(V' =: H^0(\mathcal{I}_{D'}/\mathbb{P}_r(f))\) is a complete intersection \(C' \cup D'\), where \(C'\) is smooth, irreducible. Then, by lemma 4.1, \(C' \in \mathcal{C}_{d,g,r}\) and hence \(D' = \gamma_{d,g,r}(C', V')\). Conversely, if \(D' = \gamma_{d,g,r}(C', V')\), then \(D'\) satisfies (v). Thus condition (v) defines an open set \(U \subset U'\) and \(U\) is the image of \(\gamma_{d,g,r}\). \(U\) is invariant under the action of \(\text{PGL}(r+1)\), moreover each \(D \in U\) is linearly normal. This implies that \(\beta_{d',g',r}(U) = U/\text{PGL}(r+1)\) and that \(\beta_{d',g',r}(U)\) is open in \(\mathcal{W}_{d',g'}\).

However recall that \(\mathcal{W}_{d',g'}\) could be reducible, even if the Brill-Noether number \(\rho(d', g', r) \geq 0\). So it could happen that \(\gamma_{d,g,r} \cdot \beta_{d',g',r}\) is not dominant and that its image does not dominate \(\mathcal{M}_{g'}\).

\[\text{(4.3) DEFINITION} \quad \text{Let } x \in \mathcal{W}_{d',g'} \text{ be the moduli point of the pair } (D, L), L \in \text{Pic}^d(D). \text{ We will say that } x \text{ is Petri general if the Petri map } \mu : H^0(\omega_D(-L)) \otimes H^0(L) \rightarrow H^0(\omega_D) \text{ is injective.} \]

The main universal Brill-Noether locus is the open set

\[
\mathcal{U}_{d',g'} = \{ x \in \mathcal{W}_{d',g'} / x \text{ is Petri general } \}.
\]

Let \(\rho(d', g', r) \geq 0\), by the main theorems of the Brill-Noether theory \(\mathcal{U}_{d',g'}\) is irreducible and dominates \(\mathcal{M}_{g}\) via the natural map. This motivates the previous definition.

\[\text{(4.4) LEMMA} \quad \text{Let } C \cup D \subset \mathbb{P}^r \text{ be a nodal complete intersection of } r - 1 \text{ hypersurfaces of degree } f = \frac{d^2}{d - 2} \text{ and let } r = d - g. \text{ Assume that } C \in \mathcal{C}_{d,g,r} \text{ then the multiplication}
\]

\[
\mu : H^0(\omega_D(-1)) \otimes H^0(\mathcal{O}_D(1)) \rightarrow H^0(\omega_D).
\]

In particular \(\mu\) has maximal rank if the ideal of \(C\) is generated by forms of degree \(f\).

\[\text{PROOF} \quad \text{We already know that } C \cup D \subset S, \text{ where } S \text{ is a smooth canonical surface which is a complete intersection of } r - 2 \text{ hypersurfaces of degree } f. \text{ Then it holds } \omega_D(-1) \cong \mathcal{O}_D(D). \text{ Moreover the standard exact sequence}
\]

\[
0 \rightarrow \mathcal{O}_S \otimes H^0(\mathcal{O}_S(H)) \rightarrow \mathcal{O}_S(D) \otimes H^0(\mathcal{O}_S(H)) \rightarrow \mathcal{O}_D(D) \otimes H^0(\mathcal{O}_S(H)) \rightarrow 0
\]

induces the exact commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^0(\mathcal{O}_S(H)) & \longrightarrow & H^0(\mathcal{O}_S(D)) \otimes H^0(\mathcal{O}_S(H)) & \longrightarrow & H^0(\omega_D(-1)) \otimes H^0(\mathcal{O}_D(1)) & \longrightarrow & 0 \\
\downarrow_{\text{id}} & & \downarrow_{\mu} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(\mathcal{O}_S(H)) & \longrightarrow & H^0(\mathcal{O}_S(H + D)) & \longrightarrow & H^0(\omega_D) & \longrightarrow & 0
\end{array}
\]
Since the left vertical arrow is an isomorphism we have \( \text{rk } \mu_S = \text{rk } \mu + r + 1 \), hence \( \text{rk } \mu_S \) is maximal iff \( \text{rk } \mu \) is maximal. Now observe that \( I_{C/S}(f) \cong \mathcal{O}_S(D) \) and consider the exact sequence

\[
0 \rightarrow V \otimes I_{S/P^r}(f) \rightarrow V \otimes I_{C/P^r}(f) \rightarrow V \otimes \mathcal{O}_S(D) \rightarrow 0,
\]

where \( V = H^0(\mathcal{O}_{P^r}(1)) \). The sequence induces the exact diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & V \otimes H^0(I_{S/P^r}(f)) & \longrightarrow & V \otimes H^0(I_{C/P^r}(f)) & \longrightarrow & V \otimes H^0(\mathcal{O}_S(D)) & \longrightarrow & 0 \\
& & \downarrow \mu_C & & \downarrow \mu_S & & & & \\
0 & \longrightarrow & H^0(I_{S/P^r}(f + 1)) & \longrightarrow & H^0(I_{C/P^r}(f + 1)) & \longrightarrow & H^0(\mathcal{O}_S(H + D)) & \longrightarrow & 0
\end{array}
\]

Since \( S \) is a complete intersection of hypersurfaces of degree \( f \), the left vertical arrow is an isomorphism. Then \( \mu_S \) has maximal rank iff \( \mu_C \) has maximal rank and the first statement follows. The second statement is an obvious consequence: if the homogeneous ideal of \( C \) is generated by forms of degree \( f \) then \( \mu_C \) is surjective, hence \( \mu \) has maximal rank.

(4.5) THEOREM Let \( \sigma = (f, r - 1) \), \( f = \frac{d^2}{r - 2} \) and \( r = d - g \). Assume \( \rho(d', g', r) \geq 0 \) and that one of the following conditions holds:

1. The ideal of a general \( C \in \mathcal{C}_{d,g,r} \) is generated by forms of degree \( f \) and \( 2g' - 2 > d' > f^{r-2} \).

2. \( \mathcal{I}^r_{d,g,r} \) satisfies the key condition and \( \mathcal{W}^r_{d',g'} \) is irreducible.

Then both \( \mathcal{U}^r_{d',g',r} \) and \( \mathcal{M}_{g'} \) are unirational.

PROOF Assume condition (1) holds and consider a general \( (C, V) \in \mathcal{G}^r_{d,g,r} \). Then the scheme defined by \( V \) is a nodal complete intersection \( C \cup D \), where \( D \) is a smooth curve of arithmetic genus \( g' \) and degree \( d' \). It is easy to show that the assumption \( 2g' - 2 > d' > f^{r-2} \) implies that \( D \) is irreducible and non-degenerate. Hence \( \mathcal{G}^r_{d,f,r} \) satisfies the key condition. Notice also that

\[
\rho(d', g', r) \geq 0 \implies h^0(\mathcal{O}_D(1))h^0(\omega_D(-1)) \leq g',
\]

otherwise the Petri map \( \mu \) would never be injective. Condition (1) implies that the multiplication

\[
\mu_C : H^0(\mathcal{O}_{P^r}(1)) \otimes H^0(I_{C/P^r}(f)) \rightarrow H^0(I_{C/P^r}(f + 1))
\]

is surjective. Then, by lemma 4.4, \( \mu \) has maximal rank and hence it is injective. Therefore \( \beta_{d',g',r}(D) \) is a point of the main universal Brill-Noether locus \( \mathcal{U}^r_{d',g'} \) and the image of

\[
\beta_{d',g',r} \cdot \gamma_{d,g,r} : \mathcal{G}^r_{d,g,r} \rightarrow \mathcal{W}^r_{d',g'}
\]

is contained in \( \mathcal{U}^r_{d',g'} \). On the other hand it follows from proposition 4.2 that such a image is open. Since \( \mathcal{U}^r_{d',g'} \) is irreducible and dominates \( \mathcal{M}_{g'} \) the statement follows.

Finally assume that (2) holds. Then the image of the above map is open and also dense in \( \mathcal{W}^r_{d',g'} \). Moreover \( \mathcal{W}^r_{d',g'} \) dominates \( \mathcal{M}_{g'} \) because \( \rho(d', g', r) \geq 0 \). Hence the statement follows again.

5 Curves of degree 14 and genus 8 in \( P^6 \). Now we want to prove that the homogeneous ideal of a general curve

\[
C \in \mathcal{C}_{14,8,6}
\]

is generated by quadrics. We start with a smooth, non-degenerate Del Pezzo surface

(5.1) \( Y \subset P^6 \)
of degree 6. It is well known that $\mathcal{I}_{Y/P^6}(2)$ is globally generated. Then, by lemma 3.5, there exists a reducible, nodal complete intersection of 4 quadrics

\[(5.2) \quad X \cup Y\]

where $X$ is a smooth, irreducible surface of degree 10 and

\[(5.3) \quad F = X \cap Y\]

is a smooth curve. We have

\[
\mathcal{O}_{X \cup Y}(1) \cong \omega_{X \cup Y}
\]

for the dualizing sheaf of $X \cup Y$, moreover it holds

\[(5.4) \quad \omega_X(F) \cong \omega_{X \cup Y} \otimes \mathcal{O}_X, \quad \omega_Y(F) \cong \omega_{X \cup Y} \otimes \mathcal{O}_Y.
\]

Since $Y$ is a Del Pezzo, it follows that $F \in | \mathcal{O}_Y(2) \rangle$ is a quadratic section of $Y$. In particular $F$ is a smooth canonical curve of genus 7 in $\mathbb{P}^6$. We point out that $F$ is non trigonal. This follows because a trigonal canonical curve has infinitely many trisecant lines. Since $Y$ is intersection of quadrics, these lines would be contained in $Y$: a clear contradiction. Let $H_X \cup H_Y$ be a general hyperplane section of $X \cup Y$, with $H_X \in | \mathcal{O}_X(1) \rangle$ and $H_Y \in | \mathcal{O}_Y(1) \rangle$. Since $F$ has degree 12, it follows

\[17 = p_a(H_X \cup H_Y) = p_a(H_X) + p_a(H_Y) + 11\]

and hence $p_a(H_X) = 5$.

\[(5.5) \text{PROPOSITION} \quad X \text{ is rational and projectively normal. Moreover the homogeneous ideal of } X \text{ is generated by quadrics.}\]

**PROOF** To prove that a surface is projectively normal it suffices to show the same property for a smooth, hyperplane section. Now $H_X$ is a smooth, non degenerate curve of genus $p$ and degree $2p$ in $\mathbb{P}^p$, with $p = 5$. The projective normality of such a model of a genus $p$ curve is proved in [GL]. To prove that $X$ is rational observe that

\[K_X \sim H_X - F\]

by 5.4, hence $mH_X K_X = -2m$ and $P_m(X) = 0$ for $m \geq 1$. Moreover the exact sequence

\[0 \to \mathcal{O}_X \to \mathcal{O}_X(H_X) \to \mathcal{O}_{H_X}(H_X) \to 0\]

implies that $X$ is regular. Indeed the restriction map $H^0(\mathcal{O}_X(H_X)) \to H^0(\mathcal{O}_{H_X}(H_X))$ is surjective because $h^0(\mathcal{O}_{H_X}(1)) = 6$ and $X, H_X$ are not degenerate. Passing to the long exact sequence we get

\[0 \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X(H_X)) \to \ldots\]

On the other hand we compute $h^2(\mathcal{O}_X(H_X)) = h^0(\mathcal{O}_X(K_X - H_X)) = h^0(\mathcal{O}_X(-E)) = 0$. Then Riemann-Roch yields $h^1(\mathcal{O}_X(H_X)) = 0$. Hence $h^1(\mathcal{O}_X) = 0$ and $X$ is regular. To show that the homogeneous ideal of $X$ is generated by quadrics we consider the exact diagram

\[
\begin{array}{c}
0 \longrightarrow V \otimes H^0(\mathcal{I}_{X\cup Y}(2)) \longrightarrow V \otimes (H^0(\mathcal{I}_X(2) \oplus H^0(\mathcal{I}_Y(2)))) \longrightarrow V \otimes H^0(\mathcal{I}_F(2)) \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow H^0(\mathcal{I}_{X\cup Y}(3)) \longrightarrow H^0(\mathcal{I}_X(3)) \oplus H^0(\mathcal{I}_Y(2)) \longrightarrow H^0(\mathcal{I}_F(3)) \longrightarrow 0
\end{array}
\]
where the vertical arrows are the natural multiplication maps and $V = H^0(\mathcal{O}_{\mathbb{P}^6}(1))$. The construction of the diagram is easy: the starting point is diagram is tensoring by $V$ the Mayer-Vietoris sequence

$$0 \to \mathcal{I}_{X \cup Y}(2) \to \mathcal{I}_X(2) \oplus \mathcal{I}_Y(2) \to \mathcal{I}_F(2) \to 0.$$ 

Since $X \cup Y$ is a complete intersection, $h^1(\mathcal{I}_{X \cup Y}(m)) = 0$, $m \geq 1$, hence the diagram is exact. We already know from the proof of the previous proposition that $F$ is a not trigonal canonical curve. Hence the right vertical arrow is surjective. The same is true for the left arrow, because $X \cup Y$ is complete intersection. Thus the central arrow is surjective. But this is the direct sum of the multiplication maps $u_X(2)$ and $u_Y(2)$, therefore both $u_X(2)$ and $u_Y(2)$ are surjective. To complete the proof it suffices to show that the multiplication

$$u_X(m) : V \otimes H^0(\mathcal{I}_X(m)) \to H^0(\mathcal{I}_X(m+1))$$

is surjective for $m \geq 3$. By Castelnuovo-Mumford theorem $u_X(m)$ is surjective for $m \geq 3$ if $h^i(\mathcal{I}(3-i)) = 0$, $i > 0$. This follows from the standard exact sequence

$$0 \to \mathcal{I}_X(3-i) \to \mathcal{O}_{\mathbb{P}^6}(3-i) \to \mathcal{O}_X(3-i) \to 0,$$

using the projective normality of $X$ and the vanishing of $h^1(\mathcal{O}_X(2))$ and of $h^2(\mathcal{O}_X(1))$.

Let

(5.6) $f : X \to \mathbb{P}^4$

be the adjoint map defined by the linear system

$$| K_X + H_X | = | 2H_X - F |.$$

Since $X$ is projectively normal, the linear system defining $f$ is cut on $X$ by the quadrics containing $F$. We know from Noether’s theorem that the ideal of $F$ is generated by quadrics, since $F$ is not trigonal. Hence $| 2H_X - F |$ is base-point-free and $f$ is a morphism. Let

(5.7) $S = f(X),$

it is easy to compute $(K_X + H_X)^2 = 4$ and $p_a(H_X + K_X) = 1$. This implies that $f$ is birational onto $S$ and that $S$ is a quartic Del Pezzo surface. Applying to $f$ Reider’s theorem it follows that $f$ contracts exactly six lines to the distinct points

(5.8) $b_1 \ldots b_6 \in S.$

(5.9) **Lemma** One can assume that $b_1, \ldots, b_6$ are general points on $S$ and that $S$ is a general quartic Del Pezzo surface.

**Proof** In the Hilbert scheme of quartic Del Pezzo surfaces consider the open set $U$ parametrizing integral surfaces. Let $u : S \to U$ be the universal surface and let

$$u_6 : S^6 \to U$$
be the six times fibre product of $u$. $U$ is irreducible and the fibre of $u_6$ at $S$ is $S^6$, therefore $S^6$ is irreducible. Let $b = (b_1, \ldots, b_6)$, then $(b, S)$ is a point of $S^6$. Note that $(b, S)$ defines the 6-dimensional linear system 
\[ | \mathcal{I}_{Z_b/S}(2) | \]
where $Z_b =: \{ b_1 \ldots b_6 \}$. The associated map $f_b : S \rightarrow \mathbb{P}^6$ is just $f^{-1}$ and the image 
\[
(5.10) \quad X_b
\]
of $f_b$ is just $X$. For the point $(b, S)$ the surface $X_b$ is smooth, projectively normal and its homogeneous ideal is generated by quadrics. All these properties are preserved on an open neighborhood of $(b, S)$, therefore they are satisfied by $X_{b'}$ for a general $(b', S') \in S^6$. This implies the statement.

Next we recall that there exists a blowing down $\sigma : S \rightarrow \mathbb{P}^2$ of 5 disjoint lines $L_1, \ldots, L_5$ of $S$. By the lemma we can assume that $L_i \cap \{ b_1 \ldots b_6 \}$ is empty. Thus 
\[ \tau =: \sigma \cdot f \]
is the blowing up of 11 distinct points of $\mathbb{P}^2$, that is 
\[
(5.11) \quad l_i = \sigma(L_i), \ i = 1 \ldots 5 \text{ and } e_j = \tau(E_j), \ j = 1 \ldots 6,
\]
where $E_j$ is the exceptional line contracted by $f$ to $b_j$. Let $P \in | \sigma^* \mathcal{O}_{\mathbb{P}^2}(1) |$, note that 
\[
(5.12) \quad \text{Pic}X = \mathbb{Z}[P] \oplus \mathbb{Z}[L_1] \oplus \ldots \mathbb{Z}[L_5] \oplus \mathbb{Z}[E_1] \oplus \ldots \mathbb{Z}[E_6].
\]
It is easy to compute that 
\[
(5.13) \quad | H_X | = | 6P - 2(L_1 + \cdots + L_5) - (E_1 + \cdots + E_6) |.
\]

By the lemma we can assume that $l_1, \ldots, l_6, e_1, \ldots, e_6$ are sufficiently general, in particular that $l_1, l_2, e_1, e_2$ are the base points of an irreducible pencil of conics. The strict transform on $X$ of a conic of this pencil will be denoted by 
\[
(5.14) \quad R.
\]

It is clear that $| R |$ is irreducible and base-point-free, a general $R$ is a smooth rational curve in $\mathbb{P}^6$ of degree $6 = H_X R$. We assume from now on that $l_1 \ldots l_5 e_1 \ldots e_6$ are in general position in $\mathbb{P}^2$.

(5.15) **LEMMA** (1) $R$ is non degenerate.
(2) Let $R' \subset R$ be a proper irreducible component, $R'$ is linearly normal.

**PROOF** (1) Note that 
\[ H_X - R \sim 4P - 2(L_3 + L_4 + L_5) - (L_1 + L_2 + E_3 + E_4 + E_5 + E_6). \]
Therefore $| H_X - R |$ is non empty if and only if there exists a quartic curve $Q \subset \mathbb{P}^2$ passing through $l_1, \ldots, l_5, e_3, e_4, e_5, e_6$ and singular at $l_3, l_4, l_5$. This does not happen if these points are sufficiently general in $\mathbb{P}^2$.

(2) $R'$ is either the strict transform of a line joining two of the points $l_1, l_2, e_1, e_2$ or the strict transform of a smooth conic through $l_1, l_2, e_1, e_2$ and $o \in \{ l_3, l_4, l_5, e_3, e_4, e_5, e_6 \}$. Let $o = e_i, i = 3 \ldots 6$. Then $R'$ is a smooth rational quintic and it suffices to show that $\text{dim } | H_X - R + E_6 | = 0$. This is equivalent to say that there exists a unique plane quartic passing through $l_1, \ldots, l_5, e_3, e_4, e_5$ and singular at $l_3, l_4, l_5$: since $l_1, \ldots, l_5, e_3, e_4, e_5$ are general this is true. We omit further details.
Finally a curve

\[ C \in |2H_X - R| \]

has arithmetic genus 8 and degree 14. \( C \) is exactly the curve we are looking for:

(5.16) **Theorem** A smooth \( C \in |2H_X - R| \) belongs to \( C_{14,8,6} \), moreover its homogeneous ideal is generated by quadrics.

**Proof** A general \( R \) as above is a smooth, non degenerate rational sextic curve in \( \mathbb{P}^6 \). Hence the homogeneous ideal of \( R \) is generated by quadrics. From this and the projective normality of \( X \) it follows that

\[ |2H_X - R| \]

is base-point-free. Then, by Bertini’s theorem, a general \( C \in |2H - R| \) is smooth, and it is connected because \( C^2 > 0 \). Tensoring the standard exact sequence

\[
0 \to I_{X/\mathbb{P}^6}(2) \to I_{C/\mathbb{P}^6}(2) \to I_{C/X}(2) \to 0
\]

by \( V = H^0(\mathcal{O}_{\mathbb{P}^6}(1)) \) and passing to the long exact sequence we obtain

\[
0 \to V \otimes H^0(I_{X/\mathbb{P}^6}(2)) \to V \otimes H^0(I_{C/\mathbb{P}^6}(2)) \to V \otimes H^0(I_{C/X}(2)) \to 0.
\]

The multiplication \( l : V \otimes H^0(I_{X/\mathbb{P}^6}(2)) \to H^0(I_{X/\mathbb{P}^6}(3)) \) is surjective because the ideal of \( X \) is generated by quadrics. On the other hand we have \( I_{C/X}(2) \cong \mathcal{O}_X(R) \). Thus, if the multiplication

\[
r : V \otimes H^0(\mathcal{O}_X(R)) \to H^0(\mathcal{O}_X(H_X + R))
\]

has maximal rank, then the same is true for \( \mu_C : V \otimes H^0(I_{C/\mathbb{P}^6}(2)) \to H^0(I_{C/\mathbb{P}^6}(3)) \). From the exact sequence

\[
0 \to \mathcal{O}_X(H_X) \to \mathcal{O}_X(H_X + R) \to \mathcal{O}_R(H_X + R) \to 0
\]

we obtain \( h^0(\mathcal{O}_X(H_X + R)) = 14 \). Hence \( r \) has maximal rank iff \( r \) is an isomorphism. Now \( | R | \) is a base-point-free pencil, in particular it has no fixed divisor. Then, applying the base-point-free pencil trick as proved for curves in [ACGH] p.126, it follows that \( \text{Ker } r = H^0(\mathcal{O}_X(H_X - R)) \).

Since \( R \) is non degenerate \( \text{Ker } r = (0) \). Hence \( r \) is an isomorphism and \( \mu_C \) is surjective. Using this fact, and Castelnuovo-Mumford theorem as in 5.5, one deduces that the ideal of \( C \) is generated by quadrics if \( h^1(\mathcal{I}_{C/\mathbb{P}^6}(3 - i)) = 0 \) for \( i > 0 \). This follows from the long exact sequence of

\[
0 \to \mathcal{I}_{C/\mathbb{P}^6}(3 - i) \to \mathcal{O}_{\mathbb{P}^6}(3 - i) \to \mathcal{O}_C(3 - i) \to 0
\]

if \( h^1(\mathcal{I}_{C/\mathbb{P}^6}(2)) = h^1(\mathcal{O}_C(1)) = 0 \). Since \( h^1(\mathcal{O}_X(R)) = h^1(\mathcal{I}_{X/\mathbb{P}^6}(2)) = 0 \), we already have \( h^1(\mathcal{I}_{C/\mathbb{P}^6}(2)) = 0 \). To show that \( \mathcal{O}_C(1) \) is non special the long exact sequence of

\[
0 \to \mathcal{O}_X(R - H_X) \to \mathcal{O}_X(H_X) \to \mathcal{O}_C(1) \to 0.
\]

We have \( h^1(\mathcal{O}_X(H_X)) = 0 \) then it suffices to show \( h^2(\mathcal{O}_X(R - H_X)) = 0 \). By Serre duality this is \( h^0(\mathcal{O}_X(K_X + H_X - R)) \). In \( \text{Pic}(X) \) we have \( K_X + H_X - R \sim P - (L_3 + L_4 + L_5) + E_1 + E_2 \).

Since \( l_3, l_4, l_5 \) are not collinear points the latter divisor is not linearly equivalent to an effective one. Hence \( h^2(\mathcal{O}_X(R - H_X)) = 0 \). Finally \( C \) is projectively normal: indeed \( C \) is non degenerate because \( H_X(H_X - C) < 0 \), moreover the non speciality of \( \mathcal{O}_C(1) \) implies that \( C \) is linearly normal.
Since $C$ is also 2-normal the projective normality of $C$ follows, ([ACGH] p. 140 D-5). In particular we have also shown that $C$ is in $C_{14,8,6}$.

6 The unirationality of $\mathcal{M}_{14}$. In order to show the unirationality of $\mathcal{M}_{14}$ we consider our usual Grassmann bundle

$$u : \mathcal{G}^7_{14,8,6} \to C_{14,8,6}$$

where we put $\sigma = (2, 5)$. Then, from the formulae given in 2.10, we compute that

$$(d, g, r) = (14, 8, 6) \Leftrightarrow (d', g', r) = (18, 14, 6).$$

By theorem 5.16 the homogeneous ideal of a general $C \in C_{14,8,6}$ is generated by quadrics. Moreover the condition $2g' - 2 > d' > r - 2$ is satisfied and the Brill-Noether number $\rho(d', g', r)$ is 0. Then, applying theorem 4.5 to this case, it follows:

(6.1) **THEOREM** Both $\mathcal{M}_{14}$ and $U^6_{18,14}$ are unirational.

Note that, via Serre duality, the main Brill-Noether locus $U^6_{18,14}$ is biregular to $U^1_{8,14}$.

7 The unirationality of $\mathcal{M}_{12}$. We put again $\sigma = (2, 5)$ and apply a very similar argument.

(7.1) CLAIM $\mathcal{G}^7_{15,9,6}$ satisfies the key assumption.

We note that $(d, g, r) = (15, 9, 6) \Leftrightarrow (d', g', r) = (17, 12, 6)$. Under the claim the map

$$\beta_{17,12,6} \cdot \gamma_{15,9,6} : \mathcal{G}^7_{15,9,6} \to W_{17,12,6}$$

exists, by proposition 4.2 its image is open. Again the condition $2g' - 2 > d' > r - 2$ holds in this case and $\rho(d', g', r) = 5$ is positive. Via Serre duality $W^6_{17,12,6}$ is biregular to $W^0_{5,12}$. This is the universal 5-symmetric product hence it is irreducible. Then, applying theorem 4.5, it follows:

(7.2) **THEOREM** Both $W^0_{5,12}$ and $\mathcal{M}_{12}$ are unirational.

PROOF OF THE CLAIM Let

$$X \cup Y$$

be the reducible complete intersection of 4 quadrics considered in section 4. Keeping the assumptions and notations used there, we consider on $X$ the irreducible, base-point-free pencil of rational normal sextics $| R |$. This pencil contains the curve

$$(7.3) \quad D_1 + E_6 \in | R |,$$

where $D_1$ is the strict transform on $X$ of the irreducible conic passing through the points $l_1, l_2, e_1, e_2, e_6$. We know from lemma 5.15-(2) that $D_1$ is a smooth rational normal quintic spanning a hyperplane in $\mathbb{P}^6$. Now we consider the linear system

$$| 2H_X - D_1 |$$

of curves of genus 9 and degree 15. Since $D_1$ is a non degenerate rational normal quintic, the sheaf $\mathcal{I}_{D_1/\mathbb{P}^6}(2)$ is globally generated. Hence the image of the natural restriction map

$$\rho : | \mathcal{I}_{D_1/\mathbb{P}^6}(2) | \to | 2H_X - D_1 |$$

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is base-point-free. Since \((2H_X - D_1)^2 > 0\) it follows that a general \(C \in |2H_X - D_1|\) is a smooth, irreducible curve. Moreover we have:

\[(7.5) \text{THEOREM} \ C \text{ is projectively normal and } \mathcal{O}_C(1) \text{ is non special, so that } C \in \mathcal{C}_{17,12,6}.\]

PROOF Note that \(C \sim C' + E_6\), where \(C' \in |2H_X - R|\). We have already studied \(|2H_X - R|\): we know from theorem 5.16 and its proof that \(|2H_X - R|\) is base-point-free and that a general \(C' \in |2H_X - R|\) is a projectively normal element of \(\mathcal{C}_{14,8,6}\). In particular we can assume that such a general \(C'\) is transversal to \(E_6\). We also recall that \(E_6\) is a line and that \(Z =: E_6 \cap C'\) is supported on 2 points. The non speciality of \(\mathcal{O}_{C' \cup E_6}(1)\) follows from the long exact sequence of

\[
0 \to \mathcal{O}_{C' \cup E_6}(1) \to \mathcal{O}_{C'}(1) \oplus \mathcal{O}_{E_6}(1) \to \mathcal{O}_Z(1) \to 0.
\]

In a completely analogous way, the vanishing of \(h^1(\mathcal{I}_{C' \cup E_6}(m)), m \geq 1\), follows from the long exact sequence of

\[
0 \to \mathcal{I}_{C' \cup E_6}/\mathbb{P}^6(m) \to \mathcal{I}_{C'}(m) \oplus \mathcal{I}_{E_6}/\mathbb{P}^6(m) \to \mathcal{I}_Z/\mathbb{P}^6(m) \to 0.
\]

Then, by semicontinuity the same properties hold for a general \(C \in |C' + E_6|\).

Now we fix a general, smooth \(C \in |2H_X - D_1|\), then \(C\) is transversal to \(D_1\) and

\[
C \cup D_1
\]

is a nodal quadratic section of \(X\). \(X\) is a scheme theoretic intersection of quadrics, hence the same property holds for \(C \cup D_1\). Then, applying proposition 3.2, there exists a nodal complete intersection of 5 quadrics

\[(7.6) \quad C \cup D_1 \cup D_2 = Q_1 \cap \cdots \cap Q_5.
\]

From formulae 2.10 we compute that \(D_1 \cup D_2\) is a nodal curve of degree 17 and arithmetic genus 12. On the other hand the surface \(X\) is linked to a smooth sextic Del Pezzo \(Y\) by a complete intersection of 4 quadrics, so it is not restrictive to assume

\[
Q_1 \cap \ldots Q_4 = X \cup Y.
\]

But then \(D_2\) is a quadratic section of \(Y\), hence it is smoothable to a canonical curve of genus 7. Observe that \(\text{Sing } D_2 \cap (C \cup D_1)\) is empty because \(C \cup D_1 \cup D_2\) is nodal. Moreover \(B := D_1 \cap D_2\) is a set of six linearly independent points on the rational normal quintic \(D_1\). Then, keeping \(D_1\) fixed and moving \(D_2\) in \(|\mathcal{I}_{B/Y}(D_1)|\), we can smooth \(D_2\). This shows that there exists a flat family

\[
D_1 \cup D_{2,t}, \ t \in T,
\]

such that \(D_{2,t}\) is a smooth canonical curve for \(t \in T - o\) and \(D_{2,o} = D_2\). Up to shrinking \(T\) we can assume that \(h^0(\mathcal{I}_{D_1 \cup D_{2,t}}(2))\) is constantly equal to 5. This follows by semicontinuity from the next lemma.

\[(7.7) \text{LEMMA} \ h^1(\mathcal{I}_{D_1 \cup D_2}/\mathbb{P}^6(2)) = 0 \text{ and } h^0(\mathcal{I}_{D_1 \cup D_2}/\mathbb{P}^6(2)) = 5.
\]

PROOF \(D_2\) is a quadratic section of \(Y\), so we have the standard exact sequence of ideal sheaves

\[
0 \to \mathcal{I}_Y/\mathbb{P}^6(2) \to \mathcal{I}_{D_2}/\mathbb{P}^6(2) \to \mathcal{O}_Y(D_1) \to 0.
\]
The associated long exact sequence yields \( h^1(I_{D_2/P^6}(2)) = 0 \). Then the long exact sequence of

\[ 0 \to I_{D_1 \cup D_2/P^6}(2) \to I_{D_1/P^6}(2) \oplus I_{D_2/P^6}(2) \to I_{D_1 \cap D_2/P^6}(2) \to 0 \]

yields \( h^0(I_{D_1 \cup D_2,P^6}(2)) = 5 \).

Since \( h^0(I_{D_1 \cup D_2,P^6}(2)) = 5 \), the above complete intersection \( C \cup D_1 \cup D_2 \) deforms in a flat family of complete intersections of 5 quadrics:

\[ C_t \cup D_1 \cup D_2 = Q_{1,t} \cap \cdots \cap Q_{5,t}, \ t \in T. \]

Since \( C \in C_{15,9,6} \), a general \( C_t \) belongs to \( C_{15,9,6} \). Since a general \( D_t \) is smooth, we conclude that it is not restrictive to assume that \( D_2 \) is a smooth canonical curve of genus 7.

**8.9** **Lemma** \( h^1(T_{P^6} \otimes \mathcal{O}_{D_1 \cup D_2}) = 0 \) so that \( D_1 \cup D_2 \) is smoothable.

**Proof** Consider the Mayer-Vietoris sequence

\[ 0 \to T_{P^6} \otimes \mathcal{O}_{D_1 \cup D_2} \to T_{P^6} \otimes (\mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2}) \to T_{P^6} \otimes \mathcal{O}_{D_1 \cap D_2} \to 0. \]

As is well known \( h^1(T_{P^6} \otimes \mathcal{O}_{D_1}) = 0 \) for the curves we are considering. On the other hand \( D_1 \cap D_2 \) is a set of 6 linearly independent points and \( D_1 \) is a non degenerate rational normal curve. Hence the restriction map \( H^0(T_{P^6} \otimes \mathcal{O}_{D_1}) \to T_{P^6} \otimes \mathcal{O}_{D_1 \cap D_2} \) is surjective. These remarks, and the long exact sequence of 7.9, imply \( h^1(T_{P^6} \otimes \mathcal{O}_{D_1 \cup D_2}) = 0 \). Then \( D_1 \cup D_2 \) is smoothable, (cfr. [HH] 2.1)

We conclude the proof of our claim: let \( \{D_t, t \in T\} \) be a flat family of curves in \( P^6 \) such that \( D_t \) is smooth for \( t \in T - o \) and \( D_o = D_1 \cup D_2 \). Let \( V_t = H^0(I_{D_t,P^6}(2)) \). By lemma 7.8 we can assume \( dim \ V_t = 5 \) for each \( t \in T \). The scheme defined by \( V_o \) is \( C \cup D_1 \cup D_2 \). Hence \( V_t \) defines a complete intersection of quadrics \( C_t \cup D_t \), with \( C_t \in C_{15,9,6} \) and the claim follows.

**8 The Unirationality of \( M_{11} \).** In this case we shift to curves in \( P^4 \), but the arguments are the same. We definitely assume \( \sigma = (3,3) \) and consider the Grassmann bundle

\[ u : G_{13,9,4}^\sigma \to C_{13,9,4}. \]

For \( \sigma = (3,3) \) it turns out that

\[ (d, g, r) = (13, 9, 4) \iff (d', g', r) = (14, 11, 4). \]

**8.4** **Claim** \( G_{13,9,4}^\sigma \) satisfies the key assumption.

This will be shown in theorem 9.10 of the next section. Under the claim the map

\[ \gamma_{13,9,4} : G_{14,11,4}^\sigma \to W_{14,11}^d \]

exists. By proposition 4.4 the image of \( \beta_{14,11,4} : G_{13,9,4}^\sigma \to W_{14,11}^2 \) is open. Serre duality yields a biregular map between \( W_{14,11}^d \) and the universal 6-symmetric product \( W_{6,11}^0 \). Hence \( W_{14,11}^d \) irreducible. Applying theorem 4.5 it follows:

**8.5** **Theorem** Both \( W_{6,11}^0 \) and \( M_{11} \) are unirational.
9. Curves of degree 12 and genus 8 in \( P^4 \). In this section we prove our previous claim 8.4. Preliminarily we also show that a general curve of degree 12 and genus 8 in \( P^4 \) is a scheme theoretic intersection of cubics. This will be used in the next section.

Instead of our favourite rational surface of degree 10 in \( P^6 \), we use now a smooth septic \( X \subset P^4 \) having sectional genus 5 and birational to a K3 surface. This surface is very well known and its properties are described, (cfr. [DES]). We preliminarily recall some of them:

(9.1) \textbf{PROPOSITION} \( X \) is projectively normal and its ideal is generated by 3 cubic forms.

The geometric construction of \( X \) is also well known, (cfr. [Ba]):

(9.2) \textbf{PROPOSITION} Let \( X' \subset P^5 \) be any smooth complete intersection of 3 quadrics and let \( e \in X' \) be a point not on a line of \( X' \). Then the image of \( X' \) under the linear projection of center \( e \) is a smooth surface \( X \) as above.

Thus \( X \) is defined by the blowing up \( \sigma : X \to X' \) at \( e \) and \( K_X = \sigma^{-1}(e) \), we have

\[
\text{Pic } X = \sigma^* \text{Pic } X' \oplus \mathbb{Z}[K_X].
\]

To construct some curves of genus 8 and 9 we choose a suitable \( X' \) with Picard number two:

(9.3) \textbf{PROPOSITION} There exists a smooth complete intersection of 3 quadrics \( X' \subset P^5 \) such that

\[
\text{Pic } X' = \mathbb{Z}[L'] \oplus \mathbb{Z}[H'],
\]

where \( H' \) is a hyperplane section of \( X' \) and \( L' \) is a very ample curve of degree 10 and genus 3. Moreover any effective divisor on \( X' \) is very ample.

\textbf{PROOF} The existence of a K3 surface \( X' \) with Picard lattice as above is a standard consequence of the surjectivity of the periods map for K3 surfaces. Such a lattice does not contain non zero vectors \( v \) such that \( v^2 = 0, -2, 2 \). Indeed let \( v = xH' + yL' \), then \( v^2 = 4(2x^2 + y^2 + 5xy) \neq 2, -2, 0 \).

Let \( D \) be any effective divisor, then \( D^2 \geq 4 \) and \( \dim \mid D \mid \geq 3 \). Let \( F \) be a fixed irreducible component of \( \mid D \mid \), then \( \dim \mid F \mid = 0 \) and hence \( F^2 < 0 \): a contradiction. Since \( X' \) is a K3 surface, then \( \mid D \mid \) is base-point-free and irreducible, moreover \( \mid D \mid \) is very ample unless \( D^2 = 2 \) or there exists a curve \( F \) such that \( DF \leq 2 \) and \( F^2 \in \{0, -2\} \). Hence \( D \) is very ample. Up to changing their sign, we can assume that both the generators \( H' \) and \( L' \) of \( \text{Pic } X' \) are effective. In particular we can assume that \( X' \) is embedded in \( P^5 \) by \( H' \). Then either \( X' \) is a complete intersection of 3 quadrics or contains a pencil \( \mid F \mid \) of plane cubics. Since \( F^2 = 0 \) the latter case is excluded.

From now on we assume that \( X' \) is a K3 surface as in the previous statement. On \( X' \) we have the very ample linear system

\[
\mid 3H' - L' \mid.
\]

of curves of degree 14 and genus 9. Let \( e \in X' \) be a general point, due to the very ampleness of the linear systems we are considering we can assume that:
(1) There exists \( A'_e \in |3H' - L'| \) having an ordinary node at \( e \) and no other singular point.

(2) There exists \( B'_e \in |L'| \) having an ordinary node at \( e \) and no other singular point.

(3) The linear systems \(|L' - e|\) and \(|3H' - L' - e|\) have a unique, simple base point at \( e \).

Finally let

\[ \pi: X' \to X \subset P^4 \]

be the projection of center \( e \). \( \pi \) is the inverse of the blow up \( \sigma: X \to X' \) at \( e \). Let

\[ A \subset X \subset P^4 \]

be the strict transform of \( A'_e \) by \( \sigma \). Then \( A \) is a smooth, irreducible curve of genus 8 and degree 12. Let \( H := \sigma^*H', L := \sigma^*L' \) then

\[ A \in |3H - L - 2K_X| \, . \]

Unfortunately \( \mathcal{O}_A(1) \) is special: this happens to every curve in \( X \), since \( X \) is regular and \( h^1(\mathcal{O}_X(1)) = 1 \). Nevertheless we can use \( A \) to show the following

(9.7) **Theorem** A general \( C \in C_{12,4,8} \) is a scheme theoretic intersection of cubics.

**Proof** It is easy to see that the Hilbert scheme \( \mathcal{H}_{12,8,4} \) is irreducible, in particular \( C_{12,8,4} \) is dense in \( \mathcal{H}_{12,8,4} \). Hence there exists a flat family \( \{ C_t, t \in T \} \) such that \( C_t \in C_{12,8,4} \) if \( t \neq o \) and \( A = C_o \). Then, to prove the theorem, it suffices to show that: (1) \( h^0(\mathcal{I}_{C_t/P^4}(3)) \) is constant on \( T \), (2) \( A \) is scheme theoretic intersection of cubics. To show (1) it suffices to show that \( A \) is 3-normal. This follows from the long exact sequence of

\[ 0 \to \mathcal{I}_{X/P^4}(3) \to \mathcal{I}_{A/P^4}(3) \to \mathcal{O}_X(3H - A) \to 0 \]

observing that \( h^1(\mathcal{O}_X(3H - A)) = 0 \) and that \( X \) is 3-normal. To show (2) recall that the ideal of \( X \) is generated by cubics. Then, to prove that \( A \) is a scheme theoretic intersection of cubics, it suffices to show that \(|3H - A|\) is base-point-free. This follows because \(|3H - A|\) is the strict transform on \( X \) of \(|3H' - A' - e|\), whose unique base point is \( e \).

Now we turn to curves of degree 13 and genus 9: the linear system \(|3H - L - K_X|\) is just the strict transform by \( \sigma \) of \(|3H' - L' - e|\), hence a general

\[ C_o \in |3H - L - K_X| \]

is smooth, irreducible of degree 13 and genus 9. Let

\[ B \subset X \subset P^4 \]

be the strict transform of \( B'_e \), \( B \) is a smooth octic of genus 2. We can assume that \( C_o \cup B \) is nodal, notice also that \( C_o + B \) is a cubic section of \( X \). Since the ideal of \( X \) is generated by 3 cubics, the ideal of \( C_o \cup B \) is generated by 4. Then, by 3.2, there exists a nodal complete intersection

\[ F_1 \cap F_2 \cap F_3 = C_o \cup B \cup B_1 \]
where \( F_1, F_2, F_3 \) are cubics. By 3.3 we can also choose \( F_1, F_2 \) so that
\[
F_1 \cap F_2 = X \cup Y
\]
where \( Y \) is smooth. Then \( Y \) is a quadric and \( B_1 \) is a smooth curve of type \((3,3)\) on it. Using \( C_o \) and the previous remarks we can finally show that:

(9.10) **THEOREM** A general \( C \in C_{13,9,4} \) is linked to a smooth, irreducible curve by a complete intersection of 3 cubics. In particular \( G^\sigma_{13,9,4} \) satisfies the key condition.

**PROOF** It is easy to see that \( H_{13,9,4} \) is irreducible. Therefore there exists an irreducible flat family \( \{C_t, t \in T\} \) such that \( C_t \in C_{13,9,4} \) if \( t \neq o \) and \( C_t = C_o \) if \( t = o \). \( C_o \) is 3-normal: the proof is exactly the same used for the curve \( A \) in the proof of theorem 9.7. Let \( W_t = H^0(\mathcal{I}_{C_t/P^4}(3)) \), then \( W_t \) has constant dimension 4. Let \( V_o \subset W_o \) be the subspace defining the complete intersection \( C_o \cup B \cup B_1 \), then we can move \( V_o \) in a family \( \{V_t, t \in T\} \) of 3-dimensional subspaces \( V_t \subset W_t \). We can assume that \( V_t \) defines a nodal complete intersection of 3 cubics

\[
C_t \cup D_t
\]
and that \( D_t \) is nodal, non degenerate, of degree 14 and arithmetic genus 11. Let \( \mathcal{H} \) be the complete Hilbert scheme of \( D_t \), clearly there exists a rational map
\[
\gamma: G^\sigma_{13,9,4} \to \mathcal{H}
\]
sending a general \( (C,V) \in G^\sigma_{12,8,4} \) to \( D, C \cup D \) being the complete intersection defined by \( V \). However we only know that \( D_o = \gamma(C_o,V_o) \) is the nodal union of two smooth curves, so any \( D \) in the image of \( \gamma \) could be singular. To complete the proof we show that this does not happen:

We recall that, by 4.1, each \( D \) in the image of \( \gamma \) is linearly normal and satisfies \( h^0(\mathcal{I}_D(3)) = 3 \). Let \( D = \gamma(C,V) \), the latter property implies that \( V = H^0(\mathcal{I}_{D/P^4}(3)) \) and hence that \( \gamma \) is birational onto its image. So we can compare dimensions.

Let \( D \) be general in the image of \( \gamma \). \( D \) is a flat deformation of \( D_o \). \( D_o \) has 6 nodes, moreover \( D_o = B_o \cup B_1 \), where \( B_o, B_1 \) are smooth, irreducible and \( B_o \) is not degenerate. Assume that \( D \) is not smooth. Then, since \( D \) is general, we have the following cases: (1) \( D \) is the nodal union of two smooth, irreducible curves, one of them not degenerate. \( D \) has at most 6 nodes, (2) \( D \) is nodal, irreducible with at most 6 nodes. We discuss separately the two cases.

(1) Let \( f: \mathcal{D} \to T \) be a flat family such that \( \mathcal{D}_t \) is general in the image of \( \gamma \) and \( D_o = D_o \). We can assume that \( T \) is smooth, irreducible and that each \( \mathcal{D}_t \) satisfies the condition in (1). Let \( \mathcal{D} \) be irreducible, then the two irreducible components of a general \( \mathcal{D}_t \) must have the same degree and genus. This implies that the degree is 7 and the genus is 3. Let \( \mathcal{F} \subset \mathcal{H} \) be the family of all 6-nodal curves \( D_1 \cup D_2 \) such that \( D_i \) is a smooth septic of genus 3: we have \( \dim \mathcal{F} = 48 < \dim G^\sigma = 60 \). Hence the image of \( \gamma \) is not in \( \mathcal{F} \). Assume now that \( \mathcal{D} \) is reducible, it is easy to deduce that then a general \( \mathcal{D}_t \) is like \( D_o \) i.e. it has 6 nodes and it is the union of a smooth canonical curve of genus 3 and of a smooth octic of genus 3. Again this family of reducible curves has dimension \( < \dim G^\sigma_{13,9,4} \). (2) We can consider an analogous family \( f: \mathcal{D} \to T \). In this case \( \mathcal{D}_t \) is irreducible, nodal with \( \nu \leq 6 \) nodes if \( t \neq o \) and \( D_o = D_o \). Moreover \( \mathcal{D}_t \) is non degenerate and linearly normal. Let \( \mathcal{F}_\nu \) be the corresponding family of irreducible, nodal curves of genus \( 11 - \nu \) and degree 14. It suffices to compute that \( \dim \mathcal{F}_\nu < 60 \). This is a not difficult exercise.

**10. The unirationality of \( M_{13} \)** Let \( \sigma = (3,3) \), continuing in the same vein we first point out that \( G^\sigma_{12,14,4} \) satisfies the key condition as follows from theorem 9.7. Since
\[
(d, g, r) = (12, 8, 4) \iff (d', g', r') = (15, 14, 4)
\]

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a general \( C \in C_{12,8,4} \) is linked to a smooth, irreducible, non degenerate \( D \) of genus 14 by a complete intersection of three cubics:

\[
C \cup D = F_1 \cap F_2 \cap F_3.
\]

\( D \) has genus 14 and not 13, however we will turn very soon to curves \( D \) having exactly one node. We already know that

\[
(10.1)\quad C \cup D \subset S,
\]

where \( S \) is a smooth complete intersection of two cubics. Let \( H \) be a hyperplane section of \( S \): we know as well that, since \( C \) is a scheme theoretic intersection of cubics, then the linear system

\[
| 3H - C |
\]

is base-point-free. From \( h^0(I_C, \mathbb{P}^4(3)) = 6 \) it follows \( \dim | 3H - C | = 3 \). Notice also that \((3H - C)^2 = 11\). This implies that, for every \( D \in | 3H - C | \), the linear series \( | O_D(D) | \) is a complete \( g^2 \) with no base points. Since the degree is 11, the associated morphism

\[
f_D : D \to \mathbb{P}^2
\]

is birational onto its image. We will say that a curve is \emph{uninodal} if it is nodal with a unique node.

From now on we assume that \( D \in | 3H - C | \) is uninodal, moreover we will denote as

\[
(10.3)\quad D
\]

the family of all uninodal \( D \in \mathcal{H}_{15,14,4} \) such that there exists a complete intersection \( C \cup D \) as above. Let \( \nu : D' \to D \) be the normalization and let \( n = \nu^*\text{Sing } D \), the morphism \( f_D \cdot \nu : D' \in \mathbb{P}^2 \) is defined by the line bundle

\[
L_{D'} =: \nu^* O_D(D).
\]

Since \( | L_{D'} \) is base-point-free of degree 11, the curve \( f_D(D) \) is an element of the quasi-projective variety of all reduced, irreducible linearly normal plane curves of degree 11 and genus 13. Due to the fundamental results of the Brill-Noether theory, such a variety is irreducible and a non empty open set of it parametrizes nodal curves, (cfr. [HM1] p.40). Moreover its image

\[
(10.4)\quad U_{11,13}^2 \subset \mathcal{W}_{11,13}^2
\]

dominate the moduli space \( \mathcal{M}_{13} \): this follows because the Brill-Noether number \( \rho(11,13,2) \) is \( \geq 0 \).

The image of the element \( f_D(D) \) is just the moduli point of the pair \( (D', L_{D'}) \) in \( \mathcal{W}_{11,13}^2 \). Since

\[
L_{D'} = \omega_{D'}(n) \otimes \nu^* O_D(-1),
\]

we can define a morphism

\[
(10.5)\quad \phi : D \to U_{11,13}^2
\]

sending a uninodal \( D \) to the moduli point of \( (D', L_{D'}) \).

\[
(10.6)\text{ LEMMA } \phi \text{ is dominant.}
\]
PROOF We fix an irreducible flat family of pairs \( \{(D_t, L_t), t \in T\} \) such that: (1) \( D_t \) is a smooth, irreducible curve of genus 13 and \( L_t \in Pic^{11}(D) \) is globally generated with \( h^0(L_t) = 3 \), (2) \( T \) dominates \( \mathcal{U}_{11,13}^2 \) via the natural map, (3) for \( t = o \ (D_o, L_o) = (D', L_{D'}) \). Up to a finite base change we can also assume that: (4) on each \( D_t \) there exists a rationally determined effective divisor \( n_t \) which is contracted to a point by the morphism \( f_t : D_t \to \mathbb{P}^2 \) defined by \( L_t \), (5) \( n_o = n \).

Now we consider the other family of pairs

\[
(D_t, \omega_{D_t}(n_t) \otimes L_t^{-1})
\]

and the corresponding family of associated maps \( h_t : D_t \to \mathbb{P}^4 \). To show that \( \phi \) is dominant it suffices to show that \( h_t(D_t) \in \mathcal{D} \) for \( t \) general. Since \( D = h_o(D_o) \) is uninodal, the general \( h_t(D_t) \) is uninodal. Then it is immediate to compute that \( h^0(\mathcal{I}_{h_t(D_t)}/\mathbb{P}^4(3)) \geq 3 \). On the other hand we have \( h^0(\mathcal{I}_{D}/\mathbb{P}^4(3)) = 3 \), as follows applying to \( C \cup D \) proposition 4.1. Thus, by semicontinuity, the same property holds for a general \( D_t \). Finally let \( V_t = H^0(\mathcal{I}_{h_t(D_t)}/\mathbb{P}^4(3)) \): the scheme defined by \( V_o \) is the curve \( C \cup D \). Hence the scheme defined by a general \( V_t \) is a nodal curve \( C_t \cup D_t \), with \( C_t \in \mathcal{C}_{12,8,4} \). Then \( h_t(D_t) \in \mathcal{D} \) and \( \phi \) is dominant.

(10.7) \textbf{DEFINITION} A point \( (C, V) \in \mathcal{G}_{12,8,4}^7 \) is uninodal if \( D = \gamma_{12,8,4}(C, V) \) is uninodal. The family of all uninodal points \( (C, V) \) will be denoted by \( \mathcal{N} \).

(10.8) \textbf{LEMMA} \( \mathcal{N} \) and \( \mathcal{D} \) are unirational.

PROOF It is clear that \( \gamma_{12,8,4}(\mathcal{N}) = \mathcal{D} \). Hence it will be sufficient to show that \( \mathcal{N} \) is unirational. Fix a general pair \( (C, x) \in \mathcal{C}_{12,8,4} \times \mathbb{P}^4 \), then consider the family \( F(C, x) \) of all uninodal points \((C, V)\) such that \( x = \text{Sing } D \) and \( \gamma_{12,8,4}(C, V) = D \). It is easy to see that \( F(C, x) \) is birational to an open subset of the Grassmannian

\[
G(2, I_x/I_{2,x}),
\]

where \( I_x =: \{f \in I / f \in m_x\} \), \( I_{2,x} =: \{f \in I_x / f \in m_x^2\} \) and \( I =: H^0(\mathcal{I}_{C}/\mathbb{P}^4(3)) \). Moreover \( F(C, x) \) is the fibre at the point \( (C, x) \) of the morphism

\[
\pi : \mathcal{N} \to \mathcal{C}_{12,8,4} \times \mathbb{P}^4
\]

sending \((C, V)\) to \((C, x)\), with \( x = \text{Sing } D \) and \( D = \gamma_{12,8,4}(C, V) \). We prove that \( \pi(\mathcal{N}) \) is dense: let \( \pi(C, V) = (C, x) \), then there exists a nodal complete intersection of 3 cubics \( C \cup D \) such that \( \text{Sing } D = x \). This condition is open on \( (C, V) \) and on \( x \), hence it holds on open neighborhoods \( U_C \) of \( C \) and \( U_x \) of \( x \). Therefore \( U_C \times U_x \subset \pi(\mathcal{N}) \) and \( \pi(\mathcal{N}) \) is dense. On a non empty open set \( A \) of \( \pi(\mathcal{N}) \) the space \( I_x/I_{2,x} \) has constant dimension. It is standard to construct on \( A \) a vector bundle \( \mathcal{I} \) having fibre \( I_x/I_{2,x} \) at the point \((C, x)\). On the other hand the fibre \( F(C, x) \) of \( \pi \) is open in \( G(2, I_x/I_{2,x}) \). Hence \( \mathcal{N} \) is birational to an open set of the Grassmann bundle \( G(2, \mathcal{Q}) \). This is birational to \( \mathcal{C}_{12,8,4} \times \mathbb{P}^4 \times G(2, 4) \), therefore it is unirational.

As a straightforward consequence of the lemma we have:

(10.9) \textbf{THEOREM} \( \mathcal{M}_{13} \) is unirational as well as the Severi variety of nodal plane curves of degree 11 and genus 13.

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