On Minimal Critical Independent Sets of Almost Bipartite non-König-Egerváry Graphs

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Abstract

A set $S \subseteq V$ is independent in a graph $G = (V, E)$ if no two vertices from $S$ are adjacent. The independence number $\alpha(G)$ is the cardinality of a maximum independent set, while $\mu(G)$ is the size of a maximum matching in $G$. If $\alpha(G) + \mu(G)$ equals the order of $G$, then $G$ is called a König-Egerváry graph [6, 25]. The number $d(G) = \max\{|A| - |N(A)| : A \subseteq V\}$ is called the critical difference of $G$ [27] (where $N(A) = \{v : v \in V, N(v) \cap A \neq \emptyset\}$). It is known that $\alpha(G) - \mu(G) \leq d(G)$ holds for every graph [16, 23, 24].

A graph $G$ is (i) unicyclic if it has a unique cycle, (ii) almost bipartite if it has only one odd cycle.

Let $\ker(G) = \bigcap\{S : S$ is a critical independent set$\}$, core$(G)$ be the intersection of all maximum independent sets, and corona$(G)$ be the union of all maximum independent sets of $G$. It is known that $\ker(G) \subseteq \text{core}(G)$ is true for every graph [10], while the equality holds for bipartite graphs [19], and for unicyclic non-König-Egerváry graphs [20].

In this paper, we prove that if $G$ is an almost bipartite non-König-Egerváry graph, then $\ker(G) = \text{core}(G)$, corona$(G) \cup N(\text{core}(G)) = V(G)$, and $|\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G) + 1$.

Keywords: independent set, critical set, critical difference, almost bipartite graph, König-Egerváry graph.

1 Introduction

Throughout this paper $G = (V, E)$ is a finite, undirected, loopless graph without multiple edges, with vertex set $V = V(G)$ of cardinality $n(G)$, and edge set $E = E(G)$ of size $m(G)$. If $X \subseteq V$, then $G[X]$ is the subgraph of $G$ spanned by $X$. By $G - W$ we mean the subgraph $G[V - W]$, if $W \subseteq V(G)$. For $F \subseteq E(G)$, by $G - F$ we denote the subgraph of $G$ obtained by deleting the edges of $F$, and we use $G - e$, if $F = \{e\}$. If $A, B \subseteq V$ and $A \cap B = \emptyset$, then $(A, B)$ stands for the set $\{e = ab : a \in A, b \in B, e \in E\}$. The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, and $N(A) = \bigcup\{N(v) : v \in A\}$, $N[A] = A \cup N(A)$ for $A \subseteq V$. By $C_n, K_n$ we mean the
chordless cycle on \( n \geq 3 \) vertices, and respectively the complete graph on \( n \geq 1 \) vertices. In order to avoid ambiguity, we use \( N_G(v) \) instead of \( N(v) \), and \( N_G(A) \) instead of \( N(A) \).

A cycle is a trail, where the only repeated vertices are the first and last ones. The graph \( G \) is unicyclic if it has a unique cycle.

Let us define the trace of a family \( \mathcal{F} \) of sets on the set \( X \) as \( \mathcal{F}|_X = \{ F \cap X : F \in \mathcal{F} \} \).

A set \( S \) of vertices is independent if no two vertices from \( S \) are adjacent, and an independent set of maximum size will be referred to as a maximum independent set. The independence number of \( G \), denoted by \( \alpha(G) \), is the cardinality of a maximum independent set of \( G \). Let \( \Omega(G) = \{ S : S \text{ is a maximum independent set of } G \} \), \( \text{core}(G) = \bigcap \{ S : S \in \Omega(G) \} \) \cite{11}, and \( \text{corona}(G) = \bigcup \{ S : S \in \Omega(G) \} \) \cite{14}. Clearly, \( \alpha(G) \leq \alpha(G-e) \leq \alpha(G) + 1 \) holds for each edge \( e \). An edge \( e \in E(G) \) is \( \alpha \)-critical whenever \( \alpha(G-e) > \alpha(G) \).

The number \( d_G(X) = |X| - |N(X)| \) is the difference of the set \( X \subseteq V(G) \), and \( d(G) = \max \{ d_G(X) : X \subseteq V \} \) is called the critical difference of \( G \). A set \( U \subseteq V(G) \) is critical if
\[
d_G(U) = d(G) \quad \cite{27}.
\]
The number \( \text{id}(G) = \max \{ d_G(I) : I \in \text{Ind}(G) \} \) is called the critical independence difference of \( G \). If \( A \subseteq V(G) \) is independent and \( d_G(A) = \text{id}(G) \), then \( A \) is called critical independent \cite{27}. Clearly, \( d(G) \geq \text{id}(G) \) is true for every graph \( G \). It is known that the equality \( d(G) = \text{id}(G) \) holds for every graph \( G \) \cite{27}.

For a graph \( G \), let \( \ker(G) = \bigcap \{ S : S \text{ is a critical independent set} \} \).

**Theorem 1.1**

(i) \cite{16} \( \ker(G) \) is the unique minimal critical (independent) set of \( G \), and \( \ker(G) \subseteq \text{core}(G) \) is true for every graph.

(ii) \cite{17,27} If \( G \) is a bipartite graph, or a unicyclic non-König-Egerváry graph, then \( \ker(G) = \text{core}(G) \).

A matching (i.e., a set of non-incident edges of \( G \)) of maximum cardinality \( \mu(G) \) is a maximum matching of \( G \). It is well-known that
\[
\left\lfloor \frac{\mu(G)}{2} \right\rfloor + 1 \leq \alpha(G) + \mu(G) \leq n(G)
\]
holds for every graph \( G \). If \( \alpha(G) + \mu(G) = n(G) \), then \( G \) is called a König-Egerváry graph \cite{6,25}. Various properties of König-Egerváry graphs are presented in \cite{2,3,12,13,15}. It is known that every bipartite graph is a König-Egerváry graph \cite{8,9}. This class includes also non-bipartite graphs (see, for instance, the graph \( G \) in Figure \ref{fig:konig}).

![Figure 1: G is a König-Egerváry graph with core(G) = \{a, b, c\} and ker(G) = \{a, b\}.](image)

**Theorem 1.2**

If \( G \) is a König-Egerváry graph, then

(i) \cite{12} \( \text{corona}(G) \cup N(\text{core}(G)) = V(G) \);

(ii) \cite{21} \( |\text{core}(G)| + |\text{corona}(G)| = 2\alpha(G) \).

We call a graph \( G \) almost bipartite if it has a unique odd cycle, denoted \( C = (V(C), E(C)) \). Since \( C \) is unique, it is chordless, and there is no other cycle of \( G \).
sharing edges with $C$. For every $y \in V(C)$, let us define $D_y = (V_y, E_y)$ as the connected bipartite subgraph of $G - E(C)$ containing $y$, and
\[ N_1(C) = \{ v : v \in V(G) - V(C), N(v) \cap V(C) \neq \emptyset \}. \]

Clearly, every unicyclic graph with an odd cycle is almost bipartite.

**Proposition 1.3** If $G$ is almost bipartite with $C = (V(C), E(C))$ as its unique odd cycle, then $V(D_a) \cap V(D_b) = \emptyset$ for every two different vertices $a, b \in V(C)$.

**Proof.** Assume, to the contrary, that there exist $a, b \in V(C)$, such that $V(D_a) \cap V(D_b) \neq \emptyset$. Let $x \in V(D_a) \cap V(D_b)$. Thus, there exist some path containing $x$, and connecting $a$ and $b$. Let $P_1$ be a shortest one of this kind. On the other hand, there exist two paths, say $P_2$ and $P_3$, connecting $a$ and $b$, and containing only vertices belonging to $C$. Therefore, either $P_1$ and $P_2$, or $P_1$ and $P_3$, give birth to an odd cycle, different from $C$, and thus contradicting the fact that $C$ is the unique odd cycle of $G$. 

As a consequence of Proposition 1.3 we may infer that $\{V(D_y) : y \in V(C)\}$ is a partition of $V(G)$.

There exist König-Egerváry graphs $G$ with $\ker(G) \neq \core(G)$; for instance, the graph in Figure 1.

There are also almost bipartite König-Egerváry graph may have $\ker(G) \neq \core(G)$; e.g., the graphs in Figure 2 have $\core(G_1) = \{a\}$ and $\core(G_2) = \{u, v, w\}$.

![Figure 2: Almost bipartite König-Egerváry graphs with $\ker(G_1) = \emptyset$ and $\ker(G_2) = \{u, v\}$.](image)

If $H_j, j = 1, 2, ..., k$, are all the connected components of $G$, it is easy to see that
\[ \Omega(G) = \bigcup_{j=1}^{k} \Omega(H_j), \quad \core(G) = \bigcup_{j=1}^{k} \core(H_j), \]
\[ \corona(G) = \bigcup_{j=1}^{k} \corona(H_j) \quad \text{and} \quad \ker(G) = \bigcup_{j=1}^{k} \ker(H_j). \]

In this paper we show that for every almost bipartite graph $G$, the following hold:
(i) $\ker(G) = \core(G)$;
(ii) $\corona(G) \cup N(\core(G)) = V(G)$;
(iii) $|\corona(G)| + |\core(G)| = 2\alpha(G) + 1$.

Since $|\corona(H)| + |\core(H)| = 2\alpha(H)$ and the assertions (i) and (ii) hold for every bipartite connected component $H$ of $G$, we may assume that every almost bipartite non-König-Egerváry graph is connected.
2 Results

Recall the following useful results.

**Lemma 2.1** [17] For every bipartite graph \( H \), a vertex \( v \in \text{core}(H) \) if and only if there exists a maximum matching that does not saturate \( v \).

Lemma 2.1 fails for non-bipartite König-Egerváry graphs; e.g., every maximum matching of the graph \( G \) from Figure 1 saturates \( c \in \text{core}(G) = \{a, b, c\} \).

**Lemma 2.2** [22] If \( G \) is an almost bipartite graph, then

1. \( n(G) - 1 \leq \alpha(G) + \mu(G) \leq n(G) \);
2. \( n(G) - 1 = \alpha(G) + \mu(G) \) if and only if each edge of its unique odd cycle is \( \alpha \)-critical.

**Theorem 2.3** [22] If \( G \) is an almost bipartite non-König-Egerváry graph, then

1. \( \text{core}(G) \cap N[V(C)] = \emptyset \);
2. \( n(G) - 1 = \alpha(G) + \mu(G) \), i.e., \( G \) is not a König-Egerváry graph.

The assertion in Theorem 2.3(ii) may fail for connected unicyclic König-Egerváry graphs. For instance,

\[
\text{core}(G_2) \neq \{u, w\} = \bigcup_{y \in V(C)} \text{core}(D_y - y),
\]

while \( \text{core}(G_1) = \bigcup_{y \in V(C)} \text{core}(D_y - y) \), where \( G_1 \) and \( G_2 \) are from Figure 3.

**Proposition 2.4** [22] Let \( G \) be an almost bipartite graph. Then the following assertions are equivalent:

1. \( y \in \text{core}(D_y) \), for every \( y \in V(C) \);
2. there exists some \( S \in \Omega(G) \), such that \( S \cap N_1(C) = \emptyset \);
3. \( n(G) - 1 = \alpha(G) + \mu(G) \), i.e., \( G \) is not a König-Egerváry graph.

**Corollary 2.5** If \( G \) is an almost bipartite non-König-Egerváry graph, then there exists some \( S \in \Omega(G) \), such that \( |S \cap V(C)| = \left\lfloor \frac{|V(C)|}{2} \right\rfloor \), where \( C \) is its unique odd cycle.
Lemma 2.6 If \( G \) is an almost bipartite non-Kö nig-Egerváry graph, then

\[
\alpha(G) = \sum_{y \in V(C)} \alpha(D_y) - \left\lfloor \frac{|V(C)|}{2} \right\rfloor - 1,
\]

where \( C \) is its unique odd cycle.

**Proof.** By Corollary 2.5, there is a maximum independent set \( S \in \Omega(G) \) such that \( |S \cap V(C)| = \left\lfloor \frac{|V(C)|}{2} \right\rfloor \). Therefore, by Proposition 2.4(i),

\[
\alpha(G) = \sum_{y \in S \cap V(C)} \alpha(D_y) + \sum_{y \in V(C) - S} (\alpha(D_y) - 1)
= \sum_{y \in V(C)} \alpha(D_y) - |V(C) - S| = \sum_{y \in V(C)} \alpha(D_y) - \left\lfloor \frac{|V(C)|}{2} \right\rfloor - 1,
\]

as required. \( \square \)

**Proposition 2.7** If \( G \) is an almost bipartite non-Kö nig-Egerváry graph, then every maximum matching of \( G \) contains at least one edge belonging to its unique odd cycle.

**Proof.** Assume, to the contrary, that there exists some maximum matching \( M \) of \( G \), such that \( M \cap E(C) = \emptyset \).

**Case 1.** There exist two consecutive vertices on \( C \), say \( y_1, y_2 \), such that \( D_{y_1} = \{y_1\} \) and \( D_{y_2} = \{y_2\} \).

Since \( G - y_1 y_2 \) is a bipartite graph, we have that

\[
\alpha(G) + \mu(G) + 1 = n(G) = n(G - y_1 y_2)
= \alpha(G - y_1 y_2) + \mu(G - y_1 y_2) = \alpha(G) + 1 + \mu(G - y_1 y_2)
\]

which leads to \( \mu(G - y_1 y_2) = \mu(G) = |M| \). Since \( M \cap E(C) = \emptyset \), we infer that \( M \cup \{y_1 y_2\} \) is a matching in \( G \), larger than \( M \), contradicting the fact that \( \mu(G) = |M| \).

**Case 2.** No two consecutive vertices on \( C \), say \( y_1, y_2 \), satisfy both \( D_{y_1} = \{y_1\} \) and \( D_{y_2} = \{y_2\} \). It follows that the number \( k \) of vertices \( y_1, y_2, \ldots, y_k \) on \( C \) with \( D_{y_i} = \{y_i\} \) satisfies \( k \leq \left\lfloor \frac{|V(C)|}{2} \right\rfloor \).

Let \( y_{k+1}, y_{k+2}, \ldots, y_{k+p} \) be all the vertices on \( C \) with \( |V(D_{y_i})| = n(D_{y_i}) \geq 2 \). Hence, \( p \geq \left\lfloor \frac{|V(C)|}{2} \right\rfloor \).

Since every \( D_{y_i} \) is bipartite, we know that \( n(D_{y_i}) = \alpha(D_{y_i}) + \mu(D_{y_i}) \). In addition,

\[
\mu(G) = \sum_{i=k+1}^{k+p} \mu(D_{y_i}), \text{ because } M \cap E(C) = \emptyset.
\]

Thus

\[
n(G) = \sum_{i=1}^{k+p} n(D_{y_i}) = \sum_{i=1}^{k} n(D_{y_i}) + \sum_{i=k+1}^{k+p} n(D_{y_i}) = k + \sum_{i=k+1}^{k+p} n(D_{y_i}).
\]
Consequently, by Proposition 2.4 (iii) and Lemma 2.6, 
\[ n(G) = \alpha(G) + \mu(G) + 1 = \sum_{y \in V(C)} \alpha(D_y) - \left\lfloor \frac{|V(C)|}{2} \right\rfloor - 1 + \mu(G) + 1 \]
\[ = \sum_{y \in V(C)} \alpha(D_y) - \left\lfloor \frac{|V(C)|}{2} \right\rfloor + \mu(G). \]

On the other hand, we have
\[ n(G) = k + \sum_{i=k+1}^{k+p} n(D_{y_i}) = k + \sum_{i=k+1}^{k+p} \alpha(D_{y_i}) + \mu(D_{y_i}) = k + \sum_{i=k+1}^{k+p} \alpha(D_{y_i}) + \mu(G). \]

Hence, we get
\[ \sum_{y \in V(C)} \alpha(D_y) - \left\lfloor \frac{|V(C)|}{2} \right\rfloor = k + \sum_{i=k+1}^{k+p} \alpha(D_{y_i}) \]
\[ \sum_{i=1}^{k} \alpha(D_{y_i}) = k + \left\lfloor \frac{|V(C)|}{2} \right\rfloor. \]

Taking into account that \( \sum_{i=1}^{k} \alpha(D_{y_i}) = k \) by definition of the sequence \( y_1, y_2, \ldots, y_k \), we arrive at a contradiction. \[\blacksquare\]

Proposition 2.7 is not true for almost bipartite König-Egerváry graphs; e.g., the graphs in Figure 4.

![Figure 4: G₁ and G₂ are almost bipartite König-Egerváry graphs](image)

**Lemma 2.8** Let \( G \) be an almost bipartite non-König-Egerváry graph with the unique odd cycle \( C \).

(i) If \( A \) is a critical independent set, then \( A \cap V(C) = \emptyset \).

(ii) \( \text{core}(G) \) is a critical set.

**Lemma 2.9** Let \( G \) be an almost bipartite graph. If there is \( x \in N_1(C) \), such that \( x \in \text{core}(D_y - y) \) for some \( y \in V(C) \), then \( G \) is a König-Egerváry graph.

**Proof.** Let \( x \in \text{core}(D_y - y) \), \( y \in N(x) \cap V(C) \), and \( z \in N(y) \cap V(C) \). Suppose, to the contrary, that \( G \) is not a König-Egerváry graph. By Lemma 2.2, the edge \( yz \) is \( \alpha \)-critical. By Lemma 2.8, \( y \notin \text{core}(G) \). Thus it follows that \( \alpha(G) = \alpha(G - y) \). By Lemma 2.1 there exists a maximum matching \( M_x \) of \( D_y - y \) not saturating \( x \). Combining \( M_x \) with a maximum matching of \( G - D_y \) we get a maximum matching \( M_y \) of \( G - y \). Hence \( M_y \cup \{xy\} \) is a matching of \( G \), which results in \( \mu(G) \geq \mu(G - y) + 1 \). Consequently,
using Lemma 2.2(ii) and having in mind that $G - y$ is a bipartite graph of order $n(G) - 1$, we get the following contradiction

$$n(G) - 1 = \alpha(G) + \mu(G) \geq \alpha(G - y) + \mu(G - y) + 1 = n(G) - 1 + 1 = n(G),$$

and this completes the proof. ■

There exist König-Egerváry and non-König-Egerváry graphs having $\text{core}(G) \neq \ker(G)$; e.g., the graphs from Figure 5: $\text{core}(G_1) = \{x, y, z\}$ and $\text{core}(G_2) = \{a, b, c\}$.

Figure 5: $\ker(G_1) = \{x, y\}$, $\ker(G_2) = \{b, c\}$ and only $G_1$ is a König-Egerváry graph

**Theorem 2.10** Let $G$ be an almost bipartite non-König-Egerváry graph with the unique odd cycle $C$. Then

$$\ker(G) = \bigcup_{y \in V(C)} \ker(D_y - y) = \bigcup_{y \in V(C)} \text{core}(D_y - y) = \text{core}(G).$$

**Proof.** By Theorem 2.3 we have that $\text{core}(G) = \bigcup_{y \in V(C)} \text{core}(D_y - y)$.

Since every $D_y - y$ is a bipartite graph, we infer that $\ker(D_y - y) = \text{core}(D_y - y)$, by Theorem 1.1(ii).

Consequently, we obtain

$$\text{core}(G) = \bigcup_{y \in V(C)} \text{core}(D_y - y) = \bigcup_{y \in V(C)} \ker(D_y - y).$$

By Lemma 2.8(ii), the set $\text{core}(G)$ is critical in $G$. Hence, we get that

$$\ker(G) \subseteq \text{core}(G) = \bigcup_{y \in V(C)} \ker(D_y - y).$$

Thus it is enough to show that

$$\bigcup_{y \in V(C)} \ker(D_y - y) \subseteq \ker(G).$$

In other words, $\ker(D_y - y) \subseteq \ker(G)|_{V(D_y - y)}$ for every $y \in V(C)$, which is equivalent to the fact that $\ker(G)|_{V(D_y - y)}$ is critical in $D_y - y$.

By Lemma 2.9 if $A \subseteq \text{core}(D_y - y)$, then $N_G(A) = N_{D_y - y}(A)$, since $G$ is a non-König-Egerváry almost bipartite graph. Hence it follows $d_G(A) = d_{D_y - y}(A)$ for every $A \subseteq \ker(D_y - y)$. Thus, in accordance with Theorem 1.1(i), if $A \subseteq \ker(D_y - y)$, then

$$d_G(A) = d_{D_y - y}(A) < d_{D_y - y}(\ker(D_y - y)) = d_G(\ker(D_y - y)). \quad (*)$$

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Since $\ker(G) \subseteq \bigcup_{y \in V(C)} \ker(D_y - y)$,

$$d_G(\ker(G)) = d_G\left(\ker(G) \cap \bigcup_{y \in V(C)} \ker(D_y - y)\right)$$

$$= d_G\left(\bigcup_{y \in V(C)} (\ker(D_y - y) \cap \ker(G))\right) = \sum_{y \in V(C)} d_G(\ker(D_y - y) \cap \ker(G)) .$$

If $\ker(G) \neq \bigcup_{y \in V(C)} \ker(D_y - y)$, then

$$\ker(G) \mid_{V(D_y - y)} \subset \ker(D_y - y)$$

for some $y \in V(C)$. Consequently, using the inequality (*) for $A = \ker(G) \mid_{V(D_y - y)}$, we obtain

$$d_G(\ker(G)) = \sum_{y \in V(C)} d_G(\ker(G) \mid_{V(D_y - y)})$$

$$< \sum_{y \in V(C)} d_G(\ker(D_y - y)) = d_G\left(\bigcup_{y \in V(C)} \ker(D_y - y)\right) = d(\core(G)) = d(G),$$

which stays in contradiction with the fact that $\ker(G)$ is critical in $G$. ■

As a consequence, we get the following.

**Corollary 2.11** [20] If $G$ is a unicyclic non-König-Egerváry graph, then $\ker(G) = \core(G)$. 

It is easy to see that for every non-negative integer $k$ there exits a graph $G$ with $|\core(G)| = k$. For instance, $|\core(K_4)| = 0$, while the graph $G$, obtained from $K_3$ by joining $k \geq 1$ leaves to one of the vertices of $K_3$, has $|\core(G)| = k$.

**Proposition 2.12** [11] If $G$ is a connected bipartite graph of order at least two, then $|\core(G)| \neq 1$.

**Corollary 2.13** If $G$ is an almost bipartite non-König-Egerváry graph, then $|\core(G)| \neq 1$.

**Proof.** Clearly, if $G = C_{2k+1}$, then $\core(G) = \emptyset$. If $G \neq C_{2k+1}$, then, by Theorem 2.10, we have that

$$\bigcup_{y \in V(C)} \core(D_y - y) = \core(G) ,$$

while by Proposition 2.12 we know that $|\core(D_y - y)| \neq 1$ for each $y \in V(C)$, since $D_y - y$ is bipartite. Hence we finally get $|\core(G)| \neq 1$. ■
Corollary 2.14 \cite{20} If $G$ is a unicyclic non-König-Egerváry graph, then $|\text{core}(G)| \neq 1$.

There exist non-bipartite König-Egerváry graphs and non-König-Egerváry graphs that have $|\text{core}(G)| = 1$; e.g., the graph $G_1$ in Figure 6 and the graphs in Figure 6.

It is worth noticing that there exists an almost bipartite König-Egerváry graph with a critical independent set meeting its unique cycle. For instance, the bull graph.

There exist non-König-Egerváry graphs satisfying $\text{corona}(G) \cup N(\text{core}(G)) \neq V(G)$; e.g., the graph in Figure 7 has $\text{corona}(G) \cup N(\text{core}(G)) = V(G) - \{a\}$.

Figure 7: $G$ is a non-König-Egerváry graph with $\text{core}(G) = \{b, c\}$

Theorem 2.15 If $G$ is an almost bipartite non-König-Egerváry graph, then

(i) $\text{corona}(G) \cup N(\text{core}(G)) = V(G)$;

(ii) $\text{corona}(G) = V(C) \cup \left( \bigcup_{y \in V(C)} \text{corona}(D_y - y) \right)$.

Proof. (i) It is enough to show that $V(G) \subseteq \text{corona}(G) \cup N(\text{core}(G))$.

Let $a \in V(G)$.

Case 1. $a \in V(C)$. If $b \in N(a) \cap V(C)$, then, by Lemma 2.2(iii), the edge $ab$ is $\alpha$-critical. Hence $a \in \text{corona}(G)$.

Case 2. $a \in V(G) - V(C)$. It follows that $a \in V(D_y - y)$, for some $y \in V(C)$.

Since $G[D_y - y]$ is bipartite, by Theorem 1.2(iii), we know that $V(D_y - y) = \text{corona}(D_y - y) \cup N(\text{core}(D_y - y))$, while by Theorem 2.3(iii), we have that $\Omega(G) |_{V(D_y - y)} = \Omega(D_y - y)$ for every $y \in V(C)$, which ensures that $\text{corona}(D_y - y) \subseteq \text{corona}(G)$.

Therefore, either $a \in \text{corona}(D_y - y) \subseteq \text{corona}(G)$, or $a \in N(\text{core}(D_y - y)) \subseteq N(\text{core}(G))$, because $\text{core}(D_y - y) \subseteq \text{core}(G)$, by Theorem 2.3(ii). Thus, $a \in \text{corona}(G) \cup N(\text{core}(G))$.

All in all, $V(G) = \text{corona}(G) \cup N(\text{core}(G))$.

(ii) In the proof of Part (i) we showed that $\text{corona}(D_y - y) \subseteq \text{corona}(G)$ for every $y \in V(C)$, and $V(C) \subseteq \text{corona}(G)$.

Hence, $V(C) \cup \left( \bigcup_{y \in V(C)} \text{corona}(D_y - y) \right) \subseteq \text{corona}(G)$. To complete the proof, it
remains to validate that \( \text{corona}(G) \subseteq V(C) \cup \left( \bigcup_{y \in V(C)} \text{corona}(D_y - y) \right) \). Let \( a \in \text{corona}(G) \). Then, \( a \in S \) for some \( S \in \Omega(G) \). Suppose \( a \notin V(C) \), then there must be \( y \in V(C) \) such that \( a \in D_y - y \). Thus, \( a \in S \cap V(D_y - y) \subseteq \text{corona}(D_y - y) \), because \( \Omega(G) \cap V(D_y - y) = \Omega(D_y - y) \), in accordance with Theorem 2.3(iii).

**Theorem 2.16** [22] *If* \( G \) *is an almost bipartite non-Kőnig-Egerváry graph, then*

\[
d(G) = \alpha(G) - \mu(G) = |\text{core}(G)| - |N(\text{core}(G))|.
\]

**Theorem 2.17** *If* \( G \) *is an almost bipartite non-Kőnig-Egerváry graph, then*

\[
|\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G) + 1.
\]

**Proof.** Let \( S \in \Omega(G) \). According to Theorem 2.15 and Lemma 2.2, we infer that

\[
|\text{corona}(G)| + |N(\text{core}(G))| = |V(G)| = \alpha(G) + \mu(G) + 1.
\]

By Theorem 2.16, we obtain

\[
|\text{corona}(G)| + |\text{core}(G)| = |\text{corona}(G)| + |N(\text{core}(G))| + \alpha(G) - \mu(G) = \alpha(G) + \mu(G) + 1 + \alpha(G) - \mu(G) = 2\alpha(G) + 1
\]

as required. \( \blacksquare \)

**Corollary 2.18** [24] *If* \( G \) *is a unicyclic non-Kőnig-Egerváry graph, then* \( |\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G) + 1 \).

### 3 Conclusions

It is known that for every graph \( \text{ker}(G) \subseteq \text{core}(G) \). In this paper we showed that an almost bipartite non-Kőnig-Egerváry graph satisfies \( \text{ker}(G) = \text{core}(G) \), like bipartite graphs and unicyclic non-Kőnig-Egerváry graphs.

**Problem 3.1** *Characterize graphs enjoying* \( \text{ker}(G) = \text{core}(G) \).

We also proved that \( \text{corona}(G) \cup N(\text{core}(G)) = V(G) \) is true for almost bipartite non-Kőnig-Egerváry graphs, like for Kőnig-Egerváry graphs.

**Problem 3.2** *Characterize graphs enjoying* \( \text{corona}(G) \cup N(\text{core}(G)) = V(G) \).

Theorem 2.17 claims that \( |\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G) + 1 \) holds for almost bipartite non-Kőnig-Egerváry graphs, like for unicyclic non-Kőnig-Egerváry graphs.

**Problem 3.3** *Characterize graphs enjoying* \( |\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G) + 1 \).

Proposition 2.7 motivates the following.

**Conjecture 3.4** *If* \( G \) *is an almost bipartite non-Kőnig-Egerváry graph, then every maximum matching of* \( G \) *contains* \( \left\lfloor \frac{|V(C)|}{2} \right\rfloor \) *edges belonging to its unique odd cycle* \( C \).
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