QUASICONVEXITY IN 3–MANIFOLD GROUPS

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Abstract. In this paper, we study strongly quasiconvex subgroups in a finitely generated 3–manifold group $\pi_1(M)$. We prove that if $M$ supports a Sol geometry, then $\pi_1(M)$ contains infinitely many finite height subgroups which are not strongly quasiconvex. On the other hand, if $M$ does not support the Sol geometry, then a finitely generated subgroup $H \leq \pi_1(M)$ has finite height if and only if $H$ is strongly quasiconvex. We also characterize strongly quasiconvex subgroups of graph manifold groups by using their finite height, their Morse elements, and their actions on the Bass-Serre tree of $\pi_1(M)$. This result strengthens analogous results in right-angled Artin groups and mapping class groups. Finally, we characterize strongly quasiconvex subgroups of a finitely generated 3–manifold group $\pi_1(M)$ by using their undistortedness property and their Morse elements.

1. Introduction

In geometric group theory, one method to understand the structure of a group $G$ is to investigate subgroups of $G$. Using this approach, one often investigates subgroup $H \leq G$ whose geometry reflects that of $G$. Quasiconvex subgroup of a hyperbolic group is a successful application of this approach. It is well-known that quasiconvex subgroups of hyperbolic groups are finitely generated and have finite height [GMRS98]. The height of a subgroup $H$ in a group $G$ is the smallest number $n$ such that for any $(n+1)$ distinct left cosets $g_1H, \ldots, g_{n+1}H$ the intersection $\bigcap_{i=1}^{n+1} g_iHg_i^{-1}$ is always finite. It is a long standing question asked by Swarup that whether or not the converse is true (see Question 1.8 in [Bes]). If the converse is true, then we could characterize quasiconvex subgroup $H$ of a hyperbolic group $G$ purely in terms of group theoretic notions.

Outside hyperbolic settings, quasiconvexity is not preserved under quasi-isometry. This means that we can not define quasiconvex subgroups of a non-hyperbolic group $G$ which are independent of the choice of finite generating set for $G$. Therefore, the second author [Tra19] develops a theory of strongly quasiconvex subgroups of an arbitrary finitely generated group. Strong quasiconvexity does not depend on the choice of finite generating set of the ambient group, and agrees with quasiconvexity when the ambient group is hyperbolic.

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**Definition 1.1.** Let $G$ be a finitely generated group and $H$ a subgroup of $G$. We say $H$ is strongly quasiconvex in $G$ if for every $L \geq 1$, $C \geq 0$ there is some $R = R(L, C)$ such that every $(L, C)$-quasi-geodesic in $G$ with endpoints on $H$ is contained in $R$-neighborhood of $H$.

In [Tra19], the second author shows that strongly quasiconvex subgroups of an arbitrary finitely generated group are also finitely generated and have finite height. Therefore, it is reasonable to extend the Swarup’s question to strongly quasiconvex subgroups of finitely generated groups.

**Question 1.2** (Question 1.4 [Tra17]). Let $G$ be a finitely generated group and $H$ a finitely generated subgroup. If $H$ has finite height, is $H$ strongly quasiconvex?

In this paper, we answer Question 1.2 for the case $G$ is a finitely generated 3–manifold group $\pi_1(M)$. We first prove that if $M$ has the Sol geometry, then $\pi_1(M)$ contains a finitely generated, finite height subgroup which is not strongly quasiconvex. This result gives a counter example for Question 1.2.

**Proposition 1.3.** Let $M$ be a 3–manifold with finitely generated fundamental group that supports the Sol geometry. Then, $\pi_1(M)$ contains infinitely many finitely generated, finite height subgroups which are not strongly quasiconvex.

For the proof of Proposition 1.3, by passing to a double cover, we get a manifold that is a torus bundle with Anosov monodromy $\Phi$. This implies that $\pi_1(M)$ contains an abelian-by-cyclic subgroup $\mathbb{Z}^2 \rtimes_{\Phi} \mathbb{Z}$ as a finite index subgroups. Therefore, we study all strongly quasiconvex subgroups and finite height subgroups in abelian-by-cyclic subgroups $\mathbb{Z}^k \rtimes_{\Phi} \mathbb{Z}$ (see Appendix A) and Proposition 1.3 is obtained from those results.

However, we prove that if the manifold $M$ does not support the Sol geometry, then we obtain a positive answer to Question 1.2.

**Theorem 1.4.** Let $M$ be a 3–manifold with finitely generated fundamental group that does not support the Sol geometry and let $H \leq \pi_1(M)$ be a finitely generated subgroup. Then $H$ has finite height in $\pi_1(M)$ if and only if $H$ is strongly quasiconvex in $\pi_1(M)$.

It is well know that if $F_2$ is the free group of rank 2, then the free-by-cyclic group $G = F_2 \rtimes_{\phi} \mathbb{Z}$ (for some automorphism $\phi$ from $F_2$ to $F_2$) is the fundamental group of a 3–manifold $M$, that is the mapping torus of the compact connected surface $\Sigma$ with one circle boundary and one genus. The manifold $M$ does not support Sol geometry, so Theorem 1.4 has the following corollary.

**Corollary 1.5.** Let $G = F_2 \rtimes_{\phi} \mathbb{Z}$ be a free-by-cyclic group, where $F_2$ is a free group of rank 2. Let $H$ be a finitely generated subgroup of $G$. Then $H$ has finite height in $G$ if and only if $H$ is strongly quasiconvex.
If \( \phi: F_n \to F_n \) is geometric (i.e. \( \phi \) is induced from a homeomorphism \( f: \Sigma \to \Sigma \) of some compact surface \( \Sigma \)), then the free-by-cyclic group \( F_n \rtimes_{\phi} \mathbb{Z} \) is the fundamental group of a 3–manifold \( M \) which does not support the Sol geometry. Therefore, all finitely generated finite height subgroups of \( G = F_n \rtimes_{\phi} \mathbb{Z} \) are strongly quasiconvex by Theorem 1.4. However, not all group automorphism \( \phi: F_n \to F_n \) is geometric (see [Ger94]). Therefore, the following question may be interesting.

**Question 1.6.** Let \( \phi: F_n \to F_n \) be a non-geometric automorphism. Let \( H \) be a finitely generated finite height subgroup of \( F_n \rtimes_{\phi} \mathbb{Z} \). Is \( H \) strongly quasiconvex in \( F_n \rtimes_{\phi} \mathbb{Z} \)?

We now briefly discuss the proof of Theorem 1.4. We assume that \( H \) has finite height in \( \pi_1(M) \) and we will prove that \( H \) is a strongly quasiconvex subgroup. By some standard arguments, we first reduce to the case where \( M \) is compact, connected, orientable, irreducible and \( \partial \)-irreducible. If \( M \) has empty or tori boundary, we call \( M \) is a geometric manifold if its interior admits geometric structures in the sense of Thurston, that are \( S^3, \mathbb{E}^3, \mathbb{H}^3, S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}, \widetilde{SL}(2, \mathbb{R}), \text{Nil} \) and \( \text{Sol} \). If \( M \) is not geometric, \( M \) is called nongeometric 3–manifold. When \( M \) is geometric 3–manifold, the proof is relatively easy. We refer the reader to Section 4.1 for details.

By Geometrization Theorem, a nongeometric 3–manifold can be cut into hyperbolic and Seifert fibered pieces along a JSJ decomposition. It is called a graph manifold if all the pieces are Seifert fibered spaces, otherwise it is a mixed manifold. If \( M \) is a mixed 3–manifold, then \( \pi_1(M) \) is relatively hyperbolic with respect to the fundamental groups of maximal graph manifold components, isolated Seifert components, and isolated JSJ tori (see [Dah03] or [BW13]). Therefore, we first study strongly quasiconvex subgroups and finite height subgroups in graph manifold groups and obtain Theorem 1.4 in this case. Then we use Theorem 2.18 which characterizes strongly quasiconvex subgroups of relatively hyperbolic groups to obtain Theorem 1.4 for the case of mixed manifold \( M \). We note that the undistortedness of the subgroup \( H \) must be the first step before using the above strategy. We use the recent work of Sun and the first author (see [NS19]) to prove the undistortedness of \( H \).

We note that the compact, connected, orientable, irreducible and \( \partial \)-irreducible manifold \( M \) could have boundary components that are higher genus surfaces. If we are in this situation, we use the filling argument as in [Sun] to get a mixed 3–manifold \( N \) (resp. hyperbolic 3–manifold) if \( M \) has nontrivial torus decomposition (resp. \( M \) has trivial torus decomposition) such that \( M \) is submanifold of \( N \) with some special properties. We show that \( \pi_1(M) \) is strongly quasiconvex in \( \pi_1(N) \). Then we use Proposition 2.2 to show that \( H \) also has finite height in \( \pi_1(N) \). By the previous paragraph, we have that \( H \) is strongly quasiconvex in \( \pi_1(N) \). Then we can conclude that \( H \) is strongly quasiconvex in \( \pi_1(M) \) by Proposition 4.11 in [Tra19].
The most difficult part of this paper is to study strongly quasiconvex subgroups of graph manifold groups.

**Theorem 1.7.** Let $M$ be a graph manifold. Let $H$ be a nontrivial, finitely generated subgroup of infinite index of $\pi_1(M)$. Then the following are equivalent:

1. $H$ is strongly quasiconvex;
2. $H$ has finite height in $\pi_1(M)$;
3. All nontrivial group elements in $H$ are Morse in $\pi_1(M)$;
4. The action of $H$ on the Bass-Serre tree of $M$ induces a quasi-isometric embedding of $H$ into the tree.

Moreover, $H$ is a free group if some (any) above condition holds.

Theorem 1.7 can be compared with the work in [KK14, KMT17, Tra19, Gen] which study purely loxodromic subgroups in right-angled Artin groups and the work in [BBKL, DT15, Kim] which study convex cocompact subgroups in mapping class groups. We proved Theorem 1.7 by showing that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$ and $(3) \Leftrightarrow (4)$. The heart parts of Theorem 1.7 are the implications $(3) \Rightarrow (1)$ and $(3) \Rightarrow (4)$. We note that an infinite order group element $g$ in a finitely generated group $G$ is Morse if the cyclic subgroup $\langle g \rangle$ is strongly quasiconvex in $G$. For the proof of the implication $(3) \Rightarrow (1)$ and $(3) \Rightarrow (4)$, we consider the covering space $M_H \to M$ corresponding to the subgroup $H \leq \pi_1(M)$, and then construct a Scott core $K$ of $M_H$ (i.e., a compact codim–0 submanifold such that the inclusion of the submanifold into $M_H$ is a homotopy equivalence) as in [NS19]. Take the universal cover $\tilde{M}$ of $M$ and take the preimage of $K$ in $\tilde{M}$ to get $\tilde{K} \subset \tilde{M}$. Then we show that that $\tilde{K} \subset \tilde{M}$ is contracting subset in the sense of Sisto [Sis18], and thus $\tilde{K}$ is strongly quasiconvex in $\tilde{M}$. As a consequence $H$ is strongly quasiconvex in $\pi_1(M)$. Moreover, special properties of the construction of Scott core $K$ above also allow us to get the implication $(3) \Rightarrow (4)$. We refer the reader to Section 3 for the full proof of Theorem 1.7.

We also generalize a part of Theorem 1.7 to characterize hyperbolic strongly quasiconvex subgroup in finitely generated 3–manifold groups.

**Theorem 1.8.** Let $M$ be a 3–manifold with finitely generated fundamental group. Let $H$ be an undistorted subgroup of $\pi_1(M)$ such that all infinite order group elements in $H$ are Morse in $G$. Then $H$ is hyperbolic and strongly quasiconvex.

In contrast to the case of graph manifold, a subgroup whose all infinite order elements are Morse in a finitely generated 3–manifold group can not be automatically strongly quasiconvex without the hypothesis undistortedness. For example, if $M$ is a closed hyperbolic 3–manifold and $H$ is a finitely generated geometrically infinite subgroup of $\pi_1(M)$, then all infinite order group elements in $H$ are Morse in $\pi_1(M)$. However, $H$ is not strongly quasiconvex because $H$ is exponentially distorted in $\pi_1(M)$ by the Covering
Theorem (see [Can96]) and the Subgroup Tameness Theorem (see [Ago04] and [CG06]).

We now discuss the proof of Theorem 1.8. First, we call a finitely generated subgroups \( H \) of a finitely generated group \( G \) purely Morse if all infinite order elements in \( H \) are Morse in \( G \). As in the proof of Theorem 1.4, we first reduce to the case where \( M \) is compact, connected, orientable, irreducible and \( \partial \)-irreducible. When \( M \) is a geometric 3–manifold, the proof is relatively easy. When \( M \) is a nongeometric 3–manifold, then \( M \) is either a graph manifold or a mixed manifold. In the first case, the proof is already given in Theorem 1.7. In the later case, we note that \( \pi_1(M) \) is relatively hyperbolic with respect to the fundamental groups of maximal graph manifold components, isolated Seifert components, and isolated JSJ tori (see [Dah03] or [BW13]). Thus, we first prove that if all peripheral subgroups of a relatively hyperbolic group have the property that all their undistorted purely Morse subgroups are hyperbolic and strongly quasiconvex, then so does the ambient group (see Corollary 2.21). This implies that we can prove Theorem 1.8 for the case of mixed 3–manifold by using Theorem 1.7.

If the manifold \( M \) has at least a boundary component that is higher genus surface, then this case follows from a similar filling argument as in the proof of Theorem 1.4 (although some details are different).

Overview. In Section 2, we review concepts finite height, strongly quasiconvex and stable subgroups in finitely generated groups. A proof of Theorem 1.7 is given in Section 3. In Section 4, we give a complete proof of Theorem 1.4 and Theorem 1.8. In Appendix A, we study strongly quasiconvex subgroups and finite height subgroups \( \mathbb{Z}^k \rtimes_\phi \mathbb{Z} \).

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2. Preliminaries

In this section, we review concepts finite height subgroups, strongly quasiconvex subgroups, stable subgroups, Morse elements, purely Morse subgroups, and their basic properties that will be used in this paper.

2.1. Finite height subgroups, malnormal subgroups, and their properties.

Definition 2.1. Let \( G \) be a group and \( H \) a subgroup of \( G \). Then

1. Conjugates \( g_1Hg_1^{-1}, \ldots, g_kHg_k^{-1} \) are essentially distinct if the cosets \( g_1H, \ldots, g_kH \) are distinct.
2. \( H \) has height at most \( n \) in \( G \) if the intersection of any \((n + 1)\) essentially distinct conjugates is finite. The least \( n \) for which this is satisfied is called the height of \( H \) in \( G \).
3. \( H \) is almost malnormal in \( G \) if \( H \) has the height at most 1 in \( G \).
(4) $H$ is malnormal in $G$ if for each $g \in G - H$ the subgroup $gHg^{-1} \cap H$ is trivial.

We observe that a malnormal subgroup is always almost malnormal. Moreover, if $G$ is a torsion free group, then every almost malnormal subgroup of $G$ is malnormal.

The following proposition provides some basic properties of finite height subgroups. We will use these properties many times for studying finite height subgroups of 3–manifold groups.

Proposition 2.2. Let $G$ be a group and $H$ a subgroup. Then:

1. If $H$ has finite height in $G$ and $G_1$ is a subgroup of $G$, then $H \cap G_1$ has finite height in $G_1$.
2. If $G_1$ is a finite index subgroup of $G$ and $H \cap G_1$ has finite height in $G_1$, then $H$ has finite height in $G$.
3. If $H$ is a finite height subgroup of $G$ and $K$ is a finite height subgroup of $H$, then $K$ has finite height in $G$.
4. If $H_1, H_2$ are two finite height subgroup of $G$, then $H_1 \cap H_2$ has finite height in $G$, $H_1$, and $H_2$.

Proof. We first prove Statement (1). Assume that the height of $H$ in $G$ is at most $n$. Then the intersection of any $(n+1)$ essentially distinct conjugates of $H$ in $G$ is finite. Let $H_1 = H \cap G_1$ and let $g_1 H_1, g_2 H_1, \ldots, g_n H_1, g_{n+1} H_1$ be $(n+1)$ distinct left cosets of $H_1$ in $G_1$. It is straightforward that $g_i H \neq g_j H$ for $i \neq j$. Therefore, $g_i H g_i^{-1}$ is finite. This implies that $\cap g_i H_1 g_i^{-1}$ is also finite. Therefore, the height of $H_1$ in $G_1$ is also at most $n$.

We now prove Statement (2). Assume the index of $G_1$ in $G$ is $k$. Let $H_1 = H \cap G_1$. Since $H_1$ has finite height in $G_1$, there is a number $m$ such that the intersection of any $(m+1)$ essentially distinct conjugates of $H_1$ in $G_1$ is finite. Let $n = km$ and we will prove that the height of $H$ in $G$ is at most $n$. In fact, let $A = \{g_1 H, g_2 H, \ldots, g_n H, g_{n+1} H\}$ be a collection of $(n+1)$ distinct left cosets of $H$ in $G$. Then there is $(m+1)$ left cosets in $A$ (called $g_{(1)} H, g_{(2)} H, \ldots, g_{(m)} H, g_{(m+1)} H$) such that $g_{(i)}$ lies in the same left coset $g_i H_1$. Therefore, $g_{(i)} = g_k \varepsilon_i$ for some $k_i \in G_1$. It is straightforward that $k_i H_1 \neq k_j H_1$ for $i \neq j$. Therefore, $\cap k_i H_1 k_i^{-1}$ is finite. Since $H_1$ is of finite index in $H$, the intersection $\cap k_i H k_i^{-1}$ is also finite. Therefore, $\cap g_{(i)} H g_{(i)}^{-1} = g (\cap k_i H k_i^{-1}) g^{-1}$ is finite. This implies that the height of $H$ in $G$ is at most $m$.

We now prove the Statement (3). Assume that the height of $H$ in $G$ is at most $n$ and the height of $K$ in $H$ is at most $m$. Let $k = mn+1$ We will prove that the height of $K$ in $G$ is at most $k$. Let $g_1 K, g_2 K, \ldots, g_k K, g_{k+1} K$ be $(k+1)$ distinct left cosets of $K$ in $G$. If $A = \{g_1 H, g_2 H, \ldots, g_k H, g_{k+1} H\}$ contains more than $n$ distinct left cosets of $H$ in $G$, then $g_i H g_i^{-1}$ is finite and therefore $\cap g_i K g_i^{-1}$ is also finite. Otherwise, $A = \{g_1 H, g_2 H, \ldots, g_k H, g_{k+1} H\}$ contains at most $n$ distinct left cosets of $H$ in $G$ and therefore there is a group element $g$ in $G$ and $(m+1)$ distinct elements $g_{(1)}, g_{(2)}, \ldots, g_{(m)}, g_{(m+1)}$
in \{g_1, g_2, \cdots, g_k\} such that \(g_{i(l)} = gh_i\) for some \(h_i \in H\). Since the height of \(K\) in \(H\) is at most \(m\) and \(B = \{h_1K, h_2K, \cdots, h_{m+1}K\}\) is a collection of \((m + 1)\) distinct left cosets of \(K\) in \(H\), the intersection \(\cap h_iKh_i^{-1}\) is finite. This implies that \(\cap g_{i(l)}Kg_{i(l)}^{-1} = g(\cap h_iKh_i^{-1})g^{-1}\) is also finite and therefore \(\cap g_iKg_i^{-1}\) is finite. Thus \(K\) has finite height in \(G\). Statement (4) is obtained from Statement (1) and Statement (3).

Finite subgroups and finite index subgroups always have finite height in the ambient group. On the other hand, the following proposition provides certain groups whose finite height subgroups are either finite or has finite index in the ambient groups. This proposition will help us study finite height subgroups of almost all geometric manifold groups (except hyperbolic manifold groups and Sol manifold groups) and Seifert manifold groups.

**Proposition 2.3.** Let \(G\) be a group such that the centralizer \(Z(G)\) of \(G\) is infinite. Let \(H\) be a finite height infinite subgroup of \(G\). Then \(H\) must have finite index in \(G\).

**Proof.** We first assume that \(Z(G) \cap H\) has infinite index in \(Z(G)\). Then there is an infinite sequence \((t_n)\) of elements in \(Z(G)\) such that \(t_i(Z(G) \cap H) \neq t_j(Z(G) \cap H)\) for \(i \neq j\). Therefore, it is straightforward that \(t_iH \neq t_jH\) for \(i \neq j\). Also, \(\cap t_nHt_n^{-1} = H\) is infinite. This contradicts to the fact that \(H\) has finite height. Therefore, \(Z(G) \cap H\) has finite index in \(Z(G)\). In particular, \(Z(G) \cap H\) is infinite. Assume that \(H\) has infinite index in \(G\). Then there is an infinite sequence \(\{g_nH\}\) of distinct left cosets of \(H\). However, \(\cap g_nHg_n^{-1}\) is infinite since it contains the infinite subgroup \(Z(G) \cap H\). This contradicts to the fact that \(H\) has finite height. Therefore, \(H\) must have finite index in \(G\).  

We now discuss how a finite height subgroup interacts to a normal subgroup with certain property (see Corollary 2.5). This result will be used to study finite height subgroups in abelian-by-cyclic groups \(Z^k \rtimes \Phi Z\) in Appendix A.

**Proposition 2.4** (Proposition A.1 in [Tra17]). Let \(G\) be a group and suppose there is a collection \(A\) of subgroups of \(G\) that satisfies the following conditions:

1. For each \(A\) in \(A\) and \(g \in G\) the conjugate \(g^{-1}Ag\) also belongs to \(A\) and there is a finite sequence
   \[A = A_0, A_1, \cdots, A_n = g^{-1}Ag\]
   of subgroups in \(A\) such that \(A_{j-1} \cap A_j\) is infinite for each \(j\);

2. For each \(A\) in \(A\) each finite height subgroup of \(A\) must be finite or have finite index in \(A\).

Then for each infinite index finite height subgroup \(H\) of \(G\) the intersection \(H \cap A\) must be finite for all \(A\) in \(A\).
Corollary 2.5. Let $G$ be a group and $H$ a finite height subgroup of infinite index. Let $N$ be a normal subgroup of $G$ such that each finite height subgroup of $N$ must be finite or have finite index in $N$. Then the intersection $H \cap N$ must be finite.

Proof. We use Proposition 2.4 for the case $\mathcal{A}$ consists of only element $N$. □

The following proposition studies certain property to finite height subgroups of certain graphs of groups. This proposition will be used to study finite height subgroups in graph manifold groups.

Proposition 2.6. Assume a group $G$ is decomposed as a finite graph $T$ of groups that satisfies the following.

1. For each vertex $v$ of $T$ each finite height subgroup of vertex group $G_v$ must be finite or have finite index in $G_v$.
2. Each edge group is infinite.

Then, if $H$ is a finite height subgroup of $G$ of infinite index, then $gHg^{-1} \cap G_v$ is finite for each vertex group $G_v$ and each group element $g$. In particular, if $H$ is torsion free, then $H$ is a free group.

We need the following lemma in the proof of Proposition 2.6.

Lemma 2.7. Assume a group $G$ is decomposed as a finite graph $T$ of groups such that each edge group is infinite. Let $G_v$ be a vertex subgroup. Then for each $g_1$ and $g_2$ in $G$ there is a finite sequence of conjugates of vertex subgroups $g_1G_vg_1^{-1} = Q_0, Q_1, \cdots, Q_m = g_2G_vg_2^{-1}$ such that $Q_{i-1} \cap Q_i$ is infinite for each $i \in \{1, 2, \cdots, m\}$.

Proof. Let $\tilde{T}$ be the Bass-Serre tree of the decomposition of $G$. Then conjugates of vertex groups (resp. edge groups) correspond to vertices (edges) of $\tilde{T}$. Therefore, the lemma follows the facts that $\tilde{T}$ is connected and each edge group is infinite. □

Proof of Proposition 2.6. First at all, we will show that if $H$ is a finite height subgroup of $G$ of infinite index, then $gHg^{-1} \cap G_v$ is finite for each vertex group $G_v$ and each $g \in G$. Suppose by way of contradiction that there exist a vertex group $G_v$, an element $g_0 \in G$ such that $g_0Hg_0^{-1} \cap G_v$ is infinite. Since $H$ has finite height in $G$, it follows that $g_0Hg_0^{-1}$ has finite height in $G$. By (1) in Proposition 2.2, we have $g_0Hg_0^{-1} \cap G_v$ has finite height in $G_v$. By the hypothesis, $g_0Hg_0^{-1} \cap G_v$ has finite index in $G_v$.

Claim: For any $g \in G$, the subgroup $gHg^{-1} \cap G_v$ has finite index in $G_v$. Indeed, by Lemma 2.7, there is a finite sequence of conjugates of vertex subgroups $g_0^{-1}G_vg_0 = Q_0, Q_1, \cdots, Q_{\ell} = g^{-1}Gvg$ such that $Q_{j-1} \cap Q_j$ is infinite for each $j \in \{1, \cdots, \ell\}$. Since $g_0Hg_0^{-1} \cap G_v$ has finite index in $G_v$, it follows that $H \cap Q_0 = H \cap g_0^{-1}G_vg_0$ has finite index in $Q_0$. Also, $Q_0 \cap Q_1$ is infinite, hence $H \cap Q_1$ is infinite. Using a similar argument as above, we get that $H \cap Q_1$ has finite index in $Q_1$. Repeating this process, we have
$H \cap g^{-1}G_vg = H \cap Q_{\ell}$ has finite index in $Q_{\ell} = g^{-1}G_vg$. Thus, $gHg^{-1} \cap G_v$ has finite index in $G_v$. The claim is proved.

Since $H$ has finite height in $G$, it follows that there is a number $n$ such that the intersection of any $(n + 1)$ essentially distinct conjugates of $H$ is finite. Since $H$ has infinite index in $G$, we choose $n + 1$ distinct left cosets $g_1H, \ldots, g_{n+1}H$. It follows that $\bigcap_{i=1}^{n+1} g_iHg_i^{-1}$ is finite. According to the claim above, $g_iHg_i^{-1} \cap G_v$ has finite index in $G_v$ for each $i$. Thus, $\bigcap_{i=1}^{n+1} g_iHg_i^{-1} \cap G_v$ has finite index in $G_v$. This contradicts the fact $\bigcap_{i=1}^{n+1} g_iHg_i^{-1} \cap G_v$ is finite. Therefore $gHg^{-1} \cap G_v$ is finite for each vertex group $G_v$ and each group element $g$.

Now, if we further assume that $H$ is torsion free then $gHg^{-1} \cap G_v$ is trivial for each vertex group $G_v$ and each element $g \in G$. Let $\widetilde{T}$ be the Bass-Serre tree of the decomposition of $G$. Then $G$ acts on $\widetilde{T}$ such that the stabilizer of a vertex of $T$ is a conjugate of a vertex group. To show $H$ is free, it is enough to show that $H$ acts freely on $\widetilde{T}$. To see $H$ acts freely on $\widetilde{T}$, it suffices to show that for each vertex $v \in \widetilde{T}$ then $\text{Stab}_H(v) = \{e\}$. Note that $\text{Stab}_H(v) = \text{Stab}_G(v) \cap H$. Since $gHg^{-1} \cap G_a$ is trivial for each vertex group $G_a$ and each element $g \in G$, we have that $\text{Stab}_G(v) \cap H = \{e\}$, thus $\text{Stab}_H(v) = \{e\}$.

2.2. Quasiconvex subgroups, strongly quasiconvex subgroups, and stable subgroups. We first discuss the concepts of quasiconvex subsets and strongly quasiconvex subsets in a geodesic spaces. These concepts are the foundation for the concepts of quasiconvex subgroups and strongly quasiconvex subgroups in a finitely generated group.

**Definition 2.8 ((Strongly) quasiconvex subsets).** Let $X$ be a geodesic space and let $Y$ be a subset of $X$. The subset $Y$ is *quasiconvex* in $X$ if there is a constant $D > 0$ such that every geodesic with endpoints on $Y$ is contained in the $D$–neighborhood of $Y$. The subset $Y$ is *strongly quasiconvex* if for every $K \geq 1, C \geq 0$ there is some $M = M(K,C)$ such that every $(K,C)$–quasigeodesic with endpoints on $Y$ is contained in the $M$–neighborhood of $Y$.

It follows directly from the definition that strong quasiconvexity is a quasi-isometry invariant in the following sense.

**Lemma 2.9.** Let $X$ and $Z$ be a geodesic metric spaces and $f: X \to Z$ be a quasi-isometry. If $Y$ is a strongly quasiconvex subset of $X$, then $f(Y)$ is a strongly quasiconvex subset of $Z$.

Quasiconvexity is not a quasi-isometry invariant but it is equivalent to strong quasiconvexity in the settings of hyperbolic spaces. We now define quasiconvex subgroups, strongly quasiconvex subgroups, and Morse elements in a finitely generated group.

**Definition 2.10 (Quasiconvex subgroups, strongly quasiconvex subgroups, and stable subgroups).** Let $G$ be a finitely generated group and $H$ a subgroup
of $G$. We say $H$ is quasiconvex in $G$ with respect to some finite generating set $S$ of $G$ if $H$ is a quasiconvex subset in the Cayley graph $\Gamma(G, S)$. We say $H$ is strongly quasiconvex in $G$ if $H$ is a strongly quasiconvex subset in the Cayley graph $\Gamma(G, S)$ for some (any) finite generating set $S$. We say $H$ is stable in $G$ if $H$ is strongly quasiconvex and hyperbolic.

Remark 2.11. If $H$ is a quasiconvex subgroup of a group $G$ with respect to some finite generating set $S$, then $H$ is also finitely generated and undistorted in $G$ (see Lemma 3.5 of Chapter III.Γ in [BH99]). However, we emphasize that the concept of quasiconvex subgroups depends on the choice of finite generating set of the ambient group.

The strong quasiconvexity of a subgroup does not depend on the choice of finite generating sets by Lemma 2.9. It is clear that a strongly quasiconvex subgroup is also quasiconvex with respect to some (any) finite generating set of the ambient group. In particular, each strongly quasiconvex subgroup is finitely generated and undistorted in the ambient group.

Finally, we would like to emphasis that the above definition of stable subgroup is equivalent to the definition originally given by Durham and Taylor in [DT15]. We refer the reader to the work of the second author [Tra19] to see the proof of the equivalence.

In the following theorem, we review a result proved by the second author [Tra19] that a strongly quasiconvex subgroup always has finite height.

Theorem 2.12 (Theorem 1.2 in [Tra19]). Let $G$ be a finitely generated group and $H$ a strongly quasiconvex subgroup of $G$. Then $H$ is finitely generated and has finite height in $G$.

Definition 2.13 (Morse elements and purely Morse subgroups). Let $G$ be a finitely generated group. A group element $g$ in $G$ is Morse if $g$ is of infinite order and the cyclic subgroup generated by $g$ is strongly quasiconvex. A finitely generated subgroup $H$ of $G$ is purely Morse if all infinite order elements of $H$ are Morse in $G$.

Proposition 2.14. [DT15] Let $G$ be a finitely generated subgroup and let $H$ be a stable subgroup of $G$. Then $H$ is undistorted and purely Morse.

The following proposition is an direct result of Proposition 4.10 and Proposition 4.12 in [Tra19].

Proposition 2.15. Let $G$ be a finitely generated group and let $H$ be a strongly quasiconvex subgroup of $G$. A group element $h \in H$ is Morse in $H$ if and only if it is Morse in $G$.

The following corollary is a direct result of Proposition 2.15.

Corollary 2.16. Let $G$ be a finitely generated group and let $H$ be a strongly quasiconvex subgroup of $G$. Then:

1. If $G$ has the property that all purely Morse subgroup of $G$ are undistorted, then $H$ also has the property all purely Morse subgroup of $H$ are undistorted;
(2) If \( G \) has the property that all purely Morse subgroup of \( G \) are stable, then \( H \) also has the property all purely Morse subgroup of \( H \) are stable;

(3) If \( G \) has the property that all undistorted purely Morse subgroup of \( G \) are stable, then \( H \) also has the property all undistorted purely Morse subgroup of \( H \) are stable.

We now define hyperbolic elements in relatively hyperbolic groups.

**Definition 2.17.** Let \((G,P)\) be a finitely generated relatively hyperbolic group. An infinite order element \( g \) in \( G \) is hyperbolic if \( g \) is not conjugate to an element of a subgroup in \( P \).

The following theorem provides a characterization of a strongly quasiconvex subgroup \( H \) in a relatively hyperbolic groups \((G,P)\) in terms of its interactions to peripheral subgroups. This theorem will be used to study strongly quasiconvex subgroups in mixed manifold groups and strengthen the result to finitely generated 3–manifold groups.

**Theorem 2.18** (Theorem 1.9 in [Tra19]). Let \((G,P)\) be a finitely generated relatively hyperbolic group and \( H \) a finitely generated undistorted subgroup of \( G \). Then the following are equivalent:

1. The subgroup \( H \) is strongly quasiconvex in \( G \).
2. The subgroup \( H \cap gPg^{-1} \) is strongly quasiconvex in \( gPg^{-1} \) for each conjugate \( gPg^{-1} \) of peripheral subgroup in \( P \).
3. The subgroup \( H \cap gPg^{-1} \) is strongly quasiconvex in \( G \) for each conjugate \( gPg^{-1} \) of peripheral subgroup in \( P \).

The following proposition characterizes all Morse elements in a relatively hyperbolic groups.

**Proposition 2.19.** Let \((G,P)\) be a finitely generated relatively hyperbolic group. An infinite order element \( g \) in \( G \) is Morse if and only if \( g \) is either a hyperbolic element or \( g \) is conjugate into a Morse element in some subgroup \( P \) in \( P \).

**Proof.** The “only if” direction is obtained from Proposition 2.15 and the fact that each conjugate of subgroup in \( P \) is strongly quasiconvex in \( G \). Therefore, we only need to prove the “if” direction. If \( g = hg_0h^{-1} \) for some Morse element \( g_0 \) in a subgroup \( P \in P \), then both \( g \) and \( h \) are Morse in \( G \). Otherwise, \( g \) is an hyperbolic element. Therefore, the cyclic subgroup \( \langle g \rangle \) is undistorted (see [Osi06]). We claim that \( \langle g \rangle \cap uPu^{-1} \) is trivial for each subgroup \( P \in P \) and each group element \( u \in G \). In fact, if \( \langle g \rangle \cap uPu^{-1} \) is not trivial for some \( P \in P \) and some group element \( u \in G \). Then, there is a positive integer \( n \) such that \( g^n \) in an element in \( uPu^{-1} \). Let \( g_1 = u^{-1}gu \). Then \( g_1^n \) is a group element in \( P \). Since \( g \) is hyperbolic, \( g_1 \) is not a group element in \( P \). Also, \( g_1Pg_1^{-1} \cap P \) is infinite. This contradicts to the fact that \( P \) is almost malnormal (see [Osi06]). Therefore, \( \langle g \rangle \cap uPu^{-1} \) is trivial for
each subgroup $P \in \mathcal{P}$ and each group element $u \in G$. Therefore, the cyclic subgroup $\langle g \rangle$ is strongly quasiconvex in $G$ by Theorem 2.18. Therefore, $g$ is a Morse element in $G$. □

The following theorem provides a characterization of a stable subgroup $H$ in a relatively hyperbolic groups $(G, \mathcal{P})$ in terms of its interactions to peripheral subgroups.

**Theorem 2.20** (Corollary 1.10 in [Tra19]). Let $(G, \mathcal{P})$ be a finitely generated relatively hyperbolic group and $H$ a finitely generated undistorted subgroup of $G$. Then the following are equivalent:

1. The subgroup $H$ is stable in $G$.
2. The subgroup $H \cap gPg^{-1}$ is stable in $gPg^{-1}$ for each conjugate $gPg^{-1}$ of peripheral subgroup in $\mathcal{P}$.
3. The subgroup $H \cap gPg^{-1}$ is stable in $G$ for each conjugate $gPg^{-1}$ of peripheral subgroup in $\mathcal{P}$.

**Corollary 2.21.** Let $(G, \mathcal{P})$ be a finitely generated relatively hyperbolic group. If each subgroup $P$ in $\mathcal{P}$ has the property that all undistorted purely Morse subgroups in $P$ are stable, then $G$ also has the property that all undistorted purely Morse subgroups in $G$ are stable.

**Proof.** Let $H$ be an undistorted purely Morse subgroup in $G$. We will prove that $H \cap gPg^{-1}$ is stable in $gPg^{-1}$ for each conjugate $gPg^{-1}$ of peripheral subgroup $P$ in $\mathcal{P}$. By using Proposition 2.15 and the fact that $gPg^{-1}$ is strongly quasiconvex in $G$ we can conclude that $H \cap gPg^{-1}$ is a purely Morse subgroup of $gPg^{-1}$. We now claim that $H \cap gPg^{-1}$ is an undistorted subgroup in $gPg^{-1}$. By Proposition 4.11 in [Tra19] and the fact that $gPg^{-1}$ is strongly quasiconvex in $G$, the subgroup $gPg^{-1} \cap H$ is finitely generated and undistorted in $H$. Also, the subgroup $H$ is undistorted in $G$. Then $gPg^{-1} \cap H$ is also undistorted in $G$. This implies that $gPg^{-1} \cap H$ is undistorted in $gPg^{-1}$. Therefore, $H \cap gPg^{-1}$ is stable in $gPg^{-1}$. Thus, $H$ is stable in $G$ by Theorem 2.20. □

3. **Strongly quasiconvex subgroups in graph manifold groups**

A graph manifold is a compact, irreducible, connected, orientable 3-manifold $N$ that can be decomposed along embedded incompressible tori $\mathcal{T}$ into finitely many Seifert manifolds. We specifically exclude Sol and Seifert manifolds from the class of graph manifolds. Up to isotopy, each graph manifold has a unique minimal collection of tori $\mathcal{T}$ as above [JS79, Joh79]. This minimal collection is the JSJ decomposition of $N$, and each torus of $\mathcal{T}$ is a JSJ torus.

In this section, we study strongly quasiconvex subgroups in graph manifold groups. More precisely, we prove that stable subgroups, strongly quasiconvex subgroups and finitely generated, finite height subgroups in graph manifold groups are all equivalent and we characterize these subgroups in
terms of their group elements (see Theorem 1.7). We first characterize Morse elements in graph manifold groups.

**Proposition 3.1** (Morse elements in graph manifold groups). *Let $M$ be a graph manifold group. Then a nontrivial group element $g$ in $\pi_1(M)$ is Morse if and only if $g$ is not conjugate into any Seifert subgroups.*

*Proof.* Since each Seifert subgroup is virtually product of a free group and $\mathbb{Z}$, then it can not contain an infinite cyclic subgroups which is strongly convex by Proposition 2.3 and Theorem 2.12. Therefore, if a nontrivial group element $g$ is Morse, then $g$ is not conjugate into any Seifert subgroups. On the other hand, if $g$ is not conjugate into any Seifert subgroups, then $g$ is Morse by Lemma 2.8 and Proposition 3.6 in [Sis18]. □

We now talk about the proof of Theorem 1.7. The implication $(1) \Rightarrow (2)$ is obtained from Theorem 2.12. We note that $\pi_1(M)$ is decomposed as a graph of Seifert manifold groups and this decomposition satisfies conditions (1) and (2) of Proposition 2.6. Therefore, the implication “$(2) \Rightarrow (3)$” follows from Proposition 3.1. Moreover, the fact $H$ is free when Condition (2) holds is also obtained from Proposition 2.6. Therefore, we now only need to prove the implication “$(3) \Rightarrow (1)$” (see Proposition 3.12) and the equivalence “$(3) \iff (4)$” (see Proposition 3.13).

### 3.1. Some preparations:

Firstly, by passing to a finite cover $M'$ of $M$, we can assume that each Seifert piece $M_i$ of $M$ is a product $F_i \times S^1$, and $M$ does not contains a twisted $I$–bundle over the Klein bottle (see [PW14]). We remark here that all nontrivial group elements in a finitely generated subgroup $H$ of $\pi_1(M)$ are Morse in $\pi_1(M)$ if and only if all nontrivial group elements in $H' := H \cap \pi_1(M')$ are Morse in $\pi_1(M')$. Moreover, $H$ is strongly quasiconvex in $\pi_1(M)$ if and only if $H'$ is strongly quasiconvex in $\pi_1(M')$. Therefore, it suffices to prove Proposition 3.12 for $H' \leq \pi_1(M')$, we still denote the subgroup of the 3–manifold group by $H \leq \pi_1(M)$.

**A Scott core of $M_H$:** Let $p \colon M_H \to M$ be the covering space corresponding to $H$. Since $M$ has nontrivial torus decomposition, $M_H$ has an induced graph of space structure. Each elevation (i.e. a component of the preimage) of a piece of $M$ in $M_H$ is called a *piece* of $M_H$, and each elevation of a decomposition torus of $M$ in $M_H$ is called an *edge space* of $M_H$. Since $H$ is finitely generated, there exists a finite union of pieces $M_H^c \subset M_H$, such that the inclusion $M_H^c \to M_H$ induces an isomorphism on fundamental groups, and we take $M_H^c$ to be the minimal such manifold. In [NS19], a compact Scott core $K$ of $M_H$ (and thus $\pi_1(K) = H$) has been constructed explicitly (see Preparation Step II in [NS19]) and this Scott core satisfies the following properties.

1. $K \subset M_H^c$ and for each piece $M_{H,i}$ of $M_H^c$, the intersection $K \cap M_{H,i}$ is a compact Scott core of $M_{H,i}$. Note that each $M_{H,i}$ covers a Seifert piece of $M$. 


(2) For each edge space \( E \subset M_H \) then \( K \cap E \) is either empty or a disc of \( E \) if all piece \( M_{H,i} \) of \( M_H \) are simply connected.

The following lemmas capture some geometric properties of subgroups of manifold groups whose all nontrivial elements are Morse.

**Lemma 3.2.** Let \( M \) be a graph manifold, and let \( H \) be a finitely generated purely Morse subgroup of \( \pi_1(M) \). Then each piece \( M_{H,i} \) of \( M_H \) is simply connected.

*Proof.* For each Seifert piece \( M_i \) of \( M \) and each group element \( g \in \pi_1(M) \) the subgroup \( H \cap g\pi_1(M_i)g^{-1} \) is trivial by Proposition 3.1. Since \( M_{H,i} \) is a covering space of a Seifert piece \( M_i \) of \( M \), it follows that \( M_{H,i} \) is simply connected. \( \square \)

**Lemma 3.3.** Let \( M \) be a graph manifold. Let \( H \) be a finitely generated purely Morse subgroup of \( \pi_1(M) \). Let \( K \subset M_H \) be the Scott core of \( M \) given by previous paragraphs. Let \( \tilde{K} \) be the preimage of \( K \) in the universal cover \( \tilde{M} \) of \( M \). Then there exists a positive constant \( \delta \) such that for any piece \( \tilde{M}_i \) of \( \tilde{M} \) with \( \tilde{K} \cap \tilde{M}_i \neq \emptyset \), then \( \tilde{K} \cap \tilde{M}_i \) is simply connected and \( \text{diam}(\tilde{K} \cap \tilde{M}_i) < \delta \).

*Proof.* For each piece \( M_{H,j} \) of \( M_H \), let \( K_j = K \cap M_{H,j} \). Since \( K_j \) is a Scott core of \( M_{H,j} \) and \( M_{H,j} \) is simply connected (see Lemma 3.2), it follows that \( K_j \) is simply connected. Since there are only finitely many pieces \( M_{H,j} \) of \( M_H \), it follows there are only finitely many \( K_j \).

Let \( \tilde{M}_j \) the universal cover of \( M_{H,j} \), and let \( \tilde{K}_j \) be the preimage of \( K_j \) in \( \tilde{M}_j \). Then \( \tilde{K}_j = \tilde{K} \cap \tilde{M}_j \). Since \( K_j \) is simply connected and compact, it follows that \( \tilde{K}_j \) is compact (actually \( \tilde{K}_j \) is homeomorphic to \( K_j \) since two universal covers of a common space are homeomorphic). Since there are finitely many \( K_j \) and each of them has bounded diameter. We then can find a uniform constant \( \delta \) such that the statement of the lemma holds. \( \square \)

### 3.2. \( \mathcal{PS} \)-contracting and strong quasiconvexity in graph manifold groups.

In [Sis18], Sisto constructed a certain collection of paths in the universal cover of graph manifolds which is called the path system \( \mathcal{PS} \) and the concept of \( \mathcal{PS} \)-contracting to study Morse elements in graph manifold groups. We now use these concepts to study strongly quasiconvex subgroups in graph manifold groups.

**Definition 3.4** (path system, [Sis18]). Let \( X \) be a metric space. A path system \( \mathcal{PS} \) in \( X \) is a collection of \((c,c)\)-quasi-geodesic for some \( c \) such that

1. Any subpath of a path in \( \mathcal{PS} \) is in \( \mathcal{PS} \).
2. All pairs of points in \( X \) can be connected by a path in \( \mathcal{PS} \).

**Definition 3.5** (\( \mathcal{PS} \)-contracting, [Sis18]). Let \( X \) be a metric space and let \( \mathcal{PS}(X) \) be a path system in \( X \). A subset \( A \) of \( X \) is called \( \mathcal{PS}(X) \)-contracting if there exists \( C > 0 \) and a map \( \pi : X \to A \) such that

1. For any \( x \in A \), then \( d(x, \pi(x)) \leq C \)
(2) For any \( x, y \in X \) such that \( d(\pi(x), \pi(y)) \geq C \) then for any path \( \gamma \) in \( \mathcal{PS}(X) \) connecting \( x \) to \( y \) then \( d(\pi(x), \gamma) \leq C \) and \( d(\pi(y), \gamma) \leq C \).

The map \( \pi \) will be called \( \mathcal{PS} \)-projection on \( A \) with constant \( C \).

The following lemma seems well-known to experts, but we can not find them in literature. For the benefit of the reader, we provide a proof in the Appendix.

**Lemma 3.6.** Let \( A \) be a \( \mathcal{PS} \)-contracting subset of a metric space \( X \), then \( A \) is strongly quasiconvex.

*Flip manifolds* are graph manifolds that are constructed as follows. Take a finite collection of product of \( S^1 \) with compact orientable hyperbolic surface. Glue them along boundary tori by maps which interchange the basis and fiber direction (see [KL98]). In [Sis11], Sisto constructs a path system for the universal cover of a *flip manifold*. We first review Sisto’s construction.

Let \( M \) be a flip manifold, we equip \( M \) with a nonpositively curved metric. This metric on \( M \) induces a metric on \( \tilde{M} \), which is denoted by \( d \). Lift the JSJ decomposition of the graph manifold \( M \) to the universal cover \( \tilde{M} \), and let \( T_\tilde{G} \) be the tree dual to this decomposition of \( \tilde{M} \). Each Seifert piece \( M_i \) of \( \tilde{M} \) is a product \( F_i \times S^1 \) where \( F_i \) is a compact surface with negative Euler characteristic number. Thus each piece \( \tilde{M}_i \) of \( \tilde{M} \) is the product \( \tilde{F}_i \times \mathbb{R} \). We will identify \( \tilde{F}_i \) with \( \tilde{F}_i \times \{0\} \).

**Definition 3.7** (Special paths for flip graph manifolds [Sis18]). Let \( M \) be a flip manifold. A path in \( M \) is called a *special path* if it is constructed as follows. For any \( x \) and \( y \) in \( M \). If \( x \) and \( y \) belong to the same piece \( \tilde{M}_i \) of \( \tilde{M} \) for some \( i \), the special path connecting \( x \) to \( y \) is defined to be the geodesic from \( x \) to \( y \).

We now assume that \( x \) and \( y \) belong to different pieces of \( \tilde{M} \). Let \([x, y]\) be the geodesic in \((\tilde{M}, d)\) connecting \( x \) to \( y \). The path \([x, y]\) passes through a sequence of pieces \( M_0, \ldots, M_n \) of \( \tilde{M} \) where \( x \in M_0, y \in M_n, n \geq 1 \).

For convenience, relabel \( x \) by \( x_0 \) and \( y \) by \( x_{n+1} \). For each \( i \in \{0, \ldots, n-1\} \). Let \( \tilde{T}_i = \tilde{M}_i \cap \tilde{M}_{i+1} \). The plane \( \tilde{T}_i \) covers a JSJ torus \( T_i \) obtained by identifying boundary tori \( \tilde{T}_i \) and \( \tilde{T}_i \) of Seifert pieces \( M_i \) and \( M_{i+1} \) of \( M \).

For each \( i \in \{1, \ldots, n-1\} \), let \( p_i \) and \( q_i \) be the points in the lines \( \tilde{T}_{i-1} \cap \tilde{F}_i \) and \( \tilde{T}_i \cap \tilde{F}_i \) respectively such that the geodesic \([p_i, q_i]\) is the shortest path joining two lines \( \tilde{T}_{i-1} \cap \tilde{F}_i \) and \( \tilde{T}_i \cap \tilde{F}_i \).

Let \( p_0 \) be the projection of \( x_0 \in M_0 = \tilde{F}_0 \times \mathbb{R} \) into the base surface \( \tilde{F}_0 \). Let \( q_0 \) be the point in \( \tilde{T}_0 \cap \tilde{F}_0 \) that minimizing the distance from \( p_0 \).

Let \( q_n \) be the projection of \( x_{n+1} = y \in M_n = \tilde{F}_n \times \mathbb{R} \) into the base surface \( \tilde{F}_n \). Let \( p_n \) be the point in \( \tilde{T}_{n-1} \cap \tilde{F}_n \) that minimizing the distance from \( q_n \).

So far, we have a sequence of points \( p_0, q_0, p_1, q_1, \ldots, p_n, q_n \). For each \( i \in \{0, \ldots, n-1\} \), let \( \vec{\ell}_i \) and \( \vec{\ell}_i' \) be the Euclidean geodesics in \( \tilde{T}_i \) passing through \( q_i \) and \( p_{i+1} \) such that they project to fibers \( \tilde{j}_i \subset \tilde{T}_i \) and \( \tilde{j}_i' \subset \tilde{T}_i \) respectively. Two lines \( \vec{\ell}_i \) and \( \vec{\ell}_i' \) intersects at a point in \( \tilde{T}_i \), which is denoted
by $x_{i+1}$. Hence, we have a sequence of point $x = x_0, x_1, \ldots, x_{n+1} = y$. Let $\gamma_i$ be the geodesic connecting $x_i$ to $x_{i+1}$. Let $\gamma$ be the concatenation $\gamma_0 \cdot \gamma_1 \cdots \gamma_n$. Then $\gamma$ is the special path from $x$ to $y$.

**Lemma 3.8** (Proposition 3.6 [Sis18]). Let $M$ be a flip manifold. Let $\mathcal{PS}(\tilde{M})$ be the collection of the special paths in $\tilde{M}$ then $\mathcal{PS}(\tilde{M})$ is a path system of $\tilde{M}$.

Before we get into the proofs of Proposition 3.12 and Proposition 3.13, we need several lemmas.

**Lemma 3.9.** Let $F$ be a connected compact surface with nonempty boundary and $\chi(F) < 0$. Let $M = F \times S^1$. Equip $F$ with a hyperbolic metric and equip $M$ with the product metric. Let $A$ be a subset of $\tilde{M}$ such that $\text{diam}(A) \leq \delta$ for some $\delta > 0$. Then there exists a constant $r > 0$ that depends only on $\delta$ and the metric on $F$ such that the following holds. Let $E$ and $E'$ be two planes boundaries of $\tilde{M}$ such that $A \cap E \neq \emptyset$ and $A \cap E' \neq \emptyset$. Let $\ell$ (resp. $\ell'$) be the boundary line of $\tilde{F}$ such that $\ell \subset E$ (resp. $\ell' \subset E'$). Let $p \in \ell$ and $q \in \ell'$ such that the geodesic $[p, q]$ is the shortest path joining $\ell$ to $\ell'$. For any $x \in E \cap A$ and $y \in E' \cap A$, let $u$ and $v$ be the projection of $x$ and $y$ to the lines $\ell$ and $\ell'$ respectively. Then $d(u, p) \leq r$ and $d(v, q) \leq r$.

**Proof.** The chosen hyperbolic metric on $F$ induces a metric on $\tilde{F}$ which is denoted by $d_{\tilde{F}}$. Note that $(\tilde{F}, d_{\tilde{F}})$ is Bilipschitz homeomorphic to a fattened tree (see the paragraph after Lemma 1.1 in [BN08]). Thus, there exists a constant $\epsilon > 0$ that depends only on the metric $d_{\tilde{F}}$ such that for any $s \in \ell$ and $t \in \ell'$ we have

$$d_{\tilde{F}}(s, p) + d_{\tilde{F}}(p, q) + d_{\tilde{F}}(q, t) \leq \epsilon + d_{\tilde{F}}(s, t)$$

Since $u \in \ell$, $v \in \ell'$ and $d$ is the product metric of $d_{\tilde{F}}$ with the metric on $\mathbb{R}$, we have

$$d(u, p) + d(p, q) + d(q, v) \leq \epsilon + d(u, v)$$

Since $\text{diam}(A) \leq \delta$, it follows that $d(x, y) \leq \delta$. Hence $d(u, v) \leq \delta$. Let $r = \delta + \epsilon$, it is easy to see that $d(u, p) \leq r$ and $d(v, q) \leq r$. \hfill $\square$

**Lemma 3.10.** Let $M$ be a flip manifold equipped with a nonpositively curved metric. Let $\tilde{K}$ be a subset of $\tilde{M}$ such that the following holds.

1. There is a positive constant $\delta$ such that the following holds. If $\tilde{K}$ has nonempty intersection with a piece $\tilde{M}_i$ of $\tilde{M}$, then $\tilde{K} \cap \tilde{M}_i$ is simply connected and $\text{diam}(\tilde{K} \cap \tilde{M}_i) \leq \delta$.

2. For each plane $E$ of a piece $\tilde{M}_i$, then $\tilde{K} \cap E$ is either empty or a disc. The graph $T_{\tilde{K}}$ duals to the decomposition of $\tilde{K}$ along those discs is a subtree of $T_{\tilde{M}}$.

There exists $R > 0$ such that the following holds. For any $x, y \in \tilde{K}$, let $\gamma$ be the special path in $\tilde{M}$ connecting $x$ to $y$. Let $\tilde{M}_1, \ldots, \tilde{M}_s$ be the sequence of Seifert pieces where $\gamma$ passing through. Then

$$\gamma \cap \tilde{M}_i \subset N_R(\tilde{K} \cap \tilde{M}_i).$$
Proof. Given \( \delta \), for each Seifert piece \( M_i \) of \( M \), let \( r_i \) be the constant given by Lemma 3.9. Since there are finitely many Seifert pieces of \( M \), we can choose a constant \( R_1 > 0 \) so that the conclusion of Lemma 3.9 applies to all Seifert pieces of \( M \).

Let \( R = 5R_1 + 5\delta \). We consider the following cases.

Case 1: \( x \) and \( y \) belong to the same piece \( M_i \) of \( M \). Since \( \text{diam}(K \cap M_i) \leq \delta \) and \( x, y \in K \cap M_i \), we have \( d(x, y) \leq \delta \). By definition, the special path \( \gamma \) connecting \( x \) to \( y \) is the geodesic connecting \( x \) to \( y \). Thus \( \gamma \subset N_\delta(K) \subset N_R(K) \).

Case 2: \( x \) and \( y \) belong to two distinct pieces of \( \tilde{M} \). Let \( \gamma \) be the special path connecting \( x \) to \( y \). The path \( \gamma \) is constructed explicitly in Definition 3.7.

Let \( x_0, x_1, \ldots, x_n \) be the sequence of points given by Definition 3.7. Let \( \gamma_i \) be the geodesic connecting \( x_i \) to \( x_{i+1} \). We recall that \( \gamma \) is the concatenation \( \gamma_0 \cdot \gamma_1 \cdots \gamma_n \).

Claim: \( \gamma_i \subset N_R(K) \) for each \( i \in \{0, \ldots, n\} \).

The proofs of the cases \( i = 0 \) and \( i = n \) are similar, so we only need the proof for case \( i = 0 \). The proofs of the cases \( i = 1, \ldots, n-1 \) are similar, so we only give the proof for the case \( i = 1 \).

Proof of the case \( i = 0 \):

Since \( K \cap \tilde{T}_0 \neq \emptyset \), we choose a point \( O_1 \in \tilde{K} \cap \tilde{T}_0 \). Let \( U_1 \in \tilde{T}_0 \cap \tilde{F}_0 \) be the projection of \( O_1 \) into the base surface \( \tilde{F}_0 \) of \( \tilde{M}_i = \tilde{F}_i \times \mathbb{R} \). Let \( V_1 \in \tilde{T}_0 \cap \tilde{F}_1 \) be the projection of \( O_1 \) into the base surface \( \tilde{F}_1 \) of \( M_i = \tilde{F}_i \times \mathbb{R} \).

By Lemma 3.9, we have \( d(V_1, p_1) \leq R_1 \). Since \( O_1, x_0 \in \tilde{K} \cap \tilde{M}_0 \) and \( \text{diam}(\tilde{K} \cap \tilde{M}_0) \leq \delta \), it follows that \( d(O_1, x_0) \leq \delta \). Since \( p_0 \) is the projection of \( x_0 \) to the base surface \( \tilde{F}_0 \) of \( \tilde{M}_0 \), and the metric on each piece \( \tilde{M}_i = \tilde{F}_i \times \mathbb{R} \) is the product metric, we have \( d(p_0, U_1) \leq \delta \). By the construction, \( q_0 \) is the point in \( \tilde{T}_0 \cap \tilde{F}_0 \) such that \( d(p_0, q_0) \leq d(p_0, y) \) for all \( y \in \tilde{T}_0 \cap \tilde{F}_0 \). Since \( U_1 \in \tilde{T}_0 \cap \tilde{F}_0 \), it follows that \( d(p_0, q_0) \leq d(p_0, U_1) \leq \delta \). By the triangle inequality, we have \( d(U_1, q_0) \leq d(U_1, p_0) + d(p_0, q_0) \leq \delta + \delta = 2\delta \). Using Euclidean geometry in the plane \( \tilde{T}_0 \), we have

\[
d(O_1, x_1) = \sqrt{d(U_1, q_0)^2 + d(V_1, p_1)^2} \leq d(U_1, q_0) + d(V_1, p_1) \leq 2\delta + R_1
\]

Since \( x_0, O_1 \) belong to \( \tilde{K} \cap \tilde{M}_0 \) and \( \text{diam}(\tilde{K} \cap \tilde{M}_0) \leq \delta \), it implies that \( d(O_1, x_0) \leq \delta \). Thus

\[
dx(x_0, x_1) \leq d(x_0, O_1) + d(O_1, x_1) \leq \delta + 2\delta + R_1 = 3\delta + R_1
\]

Since \( \gamma_0 \) is the geodesic connecting \( x_0 \in \tilde{K} \) to \( x_1 \in N_{2\delta + R_1}(O_1) \), it is easy to see that \( \gamma_0 \subset N_R(K) \).

Proof of the case \( i = 1 \):

Since \( \tilde{K} \cap \tilde{T}_1 \neq \emptyset \), we choose a point \( O_2 \) in \( \tilde{K} \cap \tilde{T}_1 \neq \emptyset \). Let \( U_2 \in \tilde{T}_1 \cap \tilde{F}_1 \) be the projection of \( O_2 \) to the base surface \( \tilde{F}_1 \) of \( M_i \). Let \( V_2 \in \tilde{T}_1 \cap \tilde{F}_2 \) be the projection of \( O_2 \) to the base surface \( \tilde{F}_2 \) of \( M_i \).

By Lemma 3.9, we have \( d(q_1, U_2) \leq R_1 \) and \( d(p_2, V_2) \leq R_1 \). Using Euclidean geometry in the plane \( \tilde{T}_1 \), we have

\[
d(x_2, O_2) = \sqrt{d(q_1, U_2)^2 + d(p_2, V_2)^2} \leq d(q_1, U_2) + d(p_2, V_2) \leq 2R_1
\]
Since $O_1, O_2 \in \tilde{K} \cap \tilde{M}_1$, it follows that $d(O_1, O_2) \leq \delta$. Thus $d(x_1, x_2) \leq d(x_1, O_1) + d(O_1, O_2) + d(O_2, x_2) \leq 2\delta + R_1 + \delta + 2R_1 = 3\delta + 3R_1$.

Since $\gamma_1$ is a geodesic connecting $x_1$ to $x_2$, it is easy to see that $\gamma_1 \subset N_R(\tilde{K})$.

Since $\gamma \cap \tilde{M}_i = \gamma_i$, the lemma is proved. \hfill \Box

**Lemma 3.11.** Let $M$ be a flip manifold and let $\mathcal{PS}(\tilde{M})$ be the collection of special paths in $\tilde{M}$. Let $\tilde{K}$ be the set given by Lemma 3.10. Then $\tilde{K}$ is $\mathcal{PS}$–contracting subset of $\tilde{M}$.

**Proof.** Let $\delta$ and $R$ be the constants given by Lemma 3.10. Let $C = 10\delta + R$.

First, we are going to define a $\mathcal{PS}$–projection $\pi: \tilde{M} \to \tilde{K}$ on $\tilde{K}$.

Let $\pi': T_{\tilde{M}} \to T_{\tilde{K}}$ be the projection from the tree $T_{\tilde{M}}$ to the subtree $T_{\tilde{K}}$.

For any $x \in \tilde{M}$, there exists a piece $\tilde{M}_i$ of $\tilde{M}$ such that $x \in \tilde{M}_i$. The piece $\tilde{M}_i$ is corresponding to a vertex, denoted by $v(x)$, in $T_{\tilde{M}}$. Then $\pi'(v(x))$ is a vertex of $T_{\tilde{K}} \subset T_{\tilde{M}}$. Note that $\tilde{K} \cap \tilde{M}_{\pi'(v(x))} \neq \emptyset$ and $diam(\tilde{K} \cap \tilde{M}_{\pi'(v(x))}) \leq \delta$. Choose $\pi(x)$ to be a point in $\tilde{K} \cap \tilde{M}_{\pi'(v(x))}$.

By the construction of $\pi$, if $x$ is a point in $\tilde{K}$ then $d(x, \pi(x)) \leq \delta < C$. Hence, the map $\pi$ satisfies the condition (1) of Definition 3.5.

We are going to verify (2) of Definition 3.5. Let $x$ and $y$ be two points in $\tilde{M}$ such that $d(\pi(x), \pi(y)) \geq C$. Let $\gamma$ be the special path in $\tilde{M}$ connecting $x$ to $y$. We want to show that the distance from $\pi(x)$ and $\pi(y)$ to $\gamma$ is no more than $C$. Let $\kappa$ be the number of Seifert pieces of $\tilde{M}$ where the geodesic $[\pi(x), \pi(y)]$ in $\tilde{M}$ is traveling through. Then we have $d(\pi(x), \pi(y)) \leq \kappa \delta$.

Indeed, we call these Seifert pieces by $\tilde{M}_1, \ldots, \tilde{M}_\kappa$. Note that $\tilde{K} \cap \tilde{M}_i \neq \emptyset$ with $i \in \{1, \ldots, \kappa\}$. Choose a point $s_i$ in $\tilde{M}_i \cap \tilde{M}_{i+1} \cap \tilde{K}$. We have that $d(\pi(x), s_i) \leq \delta$, $d(\pi(y), s_{i-1}) \leq \delta$ and $d(s_i, s_{i+1}) \leq \delta$ with $i \in \{1, \ldots, \kappa - 2\}$. Using triangle inequality, we have $d(\pi(x), \pi(y)) \leq \kappa \delta$. We now have

$$10\delta < C \leq d(\pi(x), \pi(y)) \leq \kappa \delta$$

Hence, $10 < \kappa$. This shows that the distance of two vertices $\pi'(v(x))$ and $\pi'(v(y))$ is at least 10.

Choose the geodesic in the tree $T_{\tilde{K}}$ connecting $\pi'(v(x))$ to $\pi'(v(y))$. Choose vertices $v_i$ and $v_j$ in this geodesic such that the distance between two vertices $v_i$ and $\pi'(v(x))$ is 3 and the distance between two vertices $v_j$ and $\pi'(v(y))$ is 3. Let $\beta$ be the geodesic in the tree $T_{\tilde{K}}$ connecting $v_i$ to $v_j$. Let $\alpha$ be the special path in $\tilde{M}$ connecting $\pi(x) \in \tilde{M}_{\pi'(v(x))}$ to $\pi(y) \in \tilde{M}_{\pi'(v(y))}$. We remark here there is a subpath of $\gamma$ (and thus this subpath is also a special path) such that this path and $\alpha$ connecting a point in $\tilde{M}_{\pi'(v(x))}$ to a point in $\tilde{M}_{\pi'(v(y))}$. By Remark 3.3 in [Sis11], we have two paths $\alpha$ and $\gamma$ coincide in pieces $\tilde{M}_v$ where $v$ is any vertex of $\beta$.

Applying Lemma 3.10 to the path $\alpha$, we have $\alpha \subset N_R(\tilde{K})$. In particular, for any vertex $v$ of $\beta$ then $\alpha \cap \tilde{M}_v \subset N_R(\tilde{K} \cap \tilde{M})$. Since $v_i$ is a vertex of $\beta$, we choose a point $u \in \alpha \cap \tilde{M}_v$ and a point $u' \in \tilde{K} \cap \tilde{M}_v$ so that $d(u, u') \leq R$. 


Since \( \pi(x) \in \tilde{K} \cap \tilde{M}_{\pi'(v(x))} \), \( u' \in \tilde{K} \cap \tilde{M}_{v_i} \) and the distance between \( v_i \) and \( \pi'(v(x)) \) in the tree is 3, we have \( d(\pi(x), u') \leq 4\delta \). Thus
\[
d(\pi(x), u) \leq d(\pi(x), u') + d(u', u) \leq 4\delta + R < C
\]
Since \( \alpha \) and \( \gamma \) coincide in \( \tilde{M}_{v_i} \), it follows that \( u \in \gamma \). Thus, \( d(\pi(x), \gamma) \leq d(\pi(x), u) < C \). Similarly, we can show that \( d(\pi(y), \gamma) < C \). Therefore, the theorem is proven. \( \square \)

**Proposition 3.12.** Let \( M \) be a graph manifold group. Let \( H \) be a finitely generated purely Morse subgroup of \( \pi_1(M) \). Then \( H \) is strongly quasiconvex.

**Proof.** We equip \( M \) with a Riemannian metric. By Theorem 2.3 in [KL98], there exists a nonpositively curved flip-manifold \( N \) and a bilipschitz homeomorphism \( \phi: M \to \tilde{N} \) such that \( \phi \) preserves their geometric decompositions.

Let \( M_H \to M \) be the covering space corresponding to the subgroup \( H \leq \pi_1(M) \). Let \( K \) be the compact Scott core of \( M_H \) given by Subsection 3.1. Let \( \tilde{K} \) be the preimage of \( K \) in the universal cover \( \tilde{M} \). Using Lemma 3.3 together with the fact \( \phi \) is bilipschitz homeomorphism, we have that the image \( \phi(\tilde{K}) \subset \tilde{N} \) satisfies (1) and (2) of Lemma 3.10. By Lemma 3.11, \( \phi(\tilde{K}) \) is \( \mathcal{PS}(\tilde{N}) \)-contracting. Thus, \( \phi(\tilde{K}) \) is strongly quasiconvex in \( \tilde{N} \) by Lemma 3.6. It follows that \( \tilde{K} \) is strongly quasiconvex in \( \tilde{M} \). As a consequence, \( H \) is strongly quasiconvex in \( \pi_1(M) \).

\( \square \)

**Proposition 3.13.** Let \( M \) be a graph manifold. Let \( H \) be a finitely generated subgroup of \( \pi_1(M) \). Then the following are equivalent:

1. \( H \) is purely Morse in \( \pi_1(M) \);
2. The action of \( H \) on the Bass-Serre tree of \( M \) induces a quasi-isometric embedding from \( H \) into the tree.

**Proof.** The implication “\( (2) \Rightarrow (1) \)” is straightforward. In fact, if some nontrivial element \( g \) in \( H \) is not Morse, then \( g \) is conjugate into a Seifert piece subgroup by Proposition 3.1. Therefore, \( g \) fixes a vertex of the Bass-Serre tree of \( M \) which contradicts to Statement (2).

Now we prove the implication “\( (2) \Rightarrow (1) \)”.

Let \( M_H \to M \) be the covering space corresponding to the subgroup \( H \leq \pi_1(M, x_0) \). Without loss of generality, we can assume that the base point \( x_0 \) belongs to the interior of some Seifert piece of \( M \). Let \( \tilde{x}_0 \) be a lift point of \( x_0 \). Let \( \tilde{K} \) be the Scott core of \( M_H \) given by Section 3.1. Let \( \tilde{M} \) be the universal cover of \( M \), and let \( \tilde{K} \) be the preimage of \( K \) in the universal cover \( \tilde{M} \).

Since all nontrivial elements in \( H \) are Morse in \( \pi_1(M) \), we recall that all pieces \( M_{H,i} \) of \( M_H \) are simply connected (see Lemma 3.2). The point \( \tilde{x}_0 \) belong to a Seifert piece \( \tilde{M}_0 \) of \( \tilde{M} \), and this Seifert piece corresponds to a vertex in the tree \( T_\tilde{M} \), denoted by \( v \). We first define a map \( \Phi: H \to T_\tilde{M} \) as the following. For any \( h \in H \), the point \( h \cdot \tilde{x}_0 \) belong to a Seifert piece of
\( \tilde{M} \), and this Seifert piece corresponds to a vertex in \( T_{\tilde{M}} \) that we denote by \( \Phi(h) \).

Since \( H \) acts on \( T_{\tilde{M}} \) by isometry and the map \( \Phi: H \to T_{\tilde{M}} \) is \( H \)-equivariant, we only need to show that there exist \( L \geq 1 \) and \( C \geq 0 \) such that for any \( h \in H \) then

\[
\frac{1}{L} d_H(e, h) - C \leq d_{T_{\tilde{M}}}(v, \Phi(h)) \leq L d_H(e, h) + C
\]

By Proposition 3.12, \( H \) is strongly quasiconvex in \( \pi_1(M) \). Hence \( H \) is undistorted in \( \pi_1(M) \). It follows that there exists a positive number \( \epsilon > 1 \) such that for any \( h, k \in H \) such that

\[
\frac{1}{\epsilon} d(h \cdot \tilde{x}_0, k \cdot \tilde{x}_0) - \epsilon \leq d_H(h, k) \leq \epsilon d(h \cdot \tilde{x}_0, k \cdot \tilde{x}_0) + \epsilon
\]

Let \( \delta \) be the constant given by Lemma 3.3. Let \( \rho \) be the minimum distance of any two distinct JSJ planes in \( \tilde{M} \). We have that the following property holds. Let \( x \neq y \) be two points in \( \tilde{K} \subset \tilde{M} \). Let \([x, y]\) be a geodesic in \( \tilde{M} \) connecting \( x \) to \( y \), and let \( n \) be the number of Seifert pieces of \( \tilde{M} \) which \([\tilde{x}_0, h \cdot \tilde{x}_0]\) passes through. Then

\[
(n - 2) \rho \leq d(x, y) \leq n \delta
\]

Let \( L = \frac{\epsilon}{\rho} + \delta \epsilon \) and \( C = \frac{\epsilon}{\delta} + 2 + \frac{\epsilon^2}{\rho} \).

For each \( h \in H \), if \( h = e \) then there is nothing to show. We consider the case that \( h \) is nontrivial element of \( H \). It follows that \( \Phi(h) \neq v \) (because each piece \( M_{H,i} \) is simply connected). Let \( n \) be the number of Seifert pieces of \( \tilde{M} \) which \([\tilde{x}_0, h \cdot \tilde{x}_0]\) passes through. We have

\[
d_{T_{\tilde{M}}}(\Phi(h), v) = n - 1
\]

Using (\( \bigcirc \)) we have

\[
(n - 2) \rho \leq d(\tilde{x}_0, h \cdot \tilde{x}_0) \leq n \delta
\]

Hence,

\[
\frac{1}{\delta} d(\tilde{x}_0, h \cdot \tilde{x}_0) - 1 \leq d_{T_{\tilde{M}}}(\Phi(h), v) \leq 1 + \frac{1}{\rho} d(\tilde{x}_0, h \cdot \tilde{x}_0)
\]

Combining with (\( \bigstar \)) we have

\[
\frac{1}{\delta \epsilon} d_H(e, h) - \frac{1}{\delta} - 1 \leq d_{T_{\tilde{M}}}(\Phi(h), v) \leq \frac{\epsilon}{\rho} d_H(e, h) + 1 + \frac{\epsilon^2}{\rho}
\]

Thus,

\[
\frac{1}{L} d_H(e, h) - C \leq d_{T_{\tilde{M}}}(v, \Phi(h)) \leq L d_H(e, h) + C
\]

for all \( h \in H \). In other words, \( \Phi: H \to T_{\tilde{M}} \) is a quasi-isometric embedding. \( \square \)
4. STRONGLY QUASICONVEX SUBGROUPS OF 3–MANIFOLD GROUPS

In this section, we complete the proof of Theorem 1.4 which states that strongly quasiconvex subgroups and finitely generated finite height subgroups are equivalent in a finitely generated group \( \pi_1(M) \), where \( M \) is a 3–manifold which does not support the Sol geometry. We remark here that we already proved this theorem when \( M \) is a graph manifold in Section 3. We now prove the theorem for the case of geometric manifold \( M \) except Sol in Section 4.1. In this section, we also show that Theorem 1.4 is not true for the case of Sol manifolds. In Section 4.2, we prove the theorem 1.4 for the case of mixed manifold \( M \) which completes the theorem for the case of nongeometric 3-manifold \( M \). Finally, we complete the theorem for the case of 3-manifold \( M \) with finitely generated fundamental group in Section 4.3.

4.1. Strongly quasiconvex subgroups of geometric 3–manifolds. We recall that a compact orientable irreducible 3–manifold \( M \) with empty or tori boundary is called geometric if its interior admits a geometric structure in the sense of Thurston which are 3–sphere, Euclidean 3–space, hyperbolic 3–space, \( S^2 \times \mathbb{R} \), \( \mathbb{H}^2 \times \mathbb{R} \), \( \widetilde{SL}(2, \mathbb{R}) \), Nil and Sol.

In the following lemma, we show that all strongly quasiconvex subgroups of Sol 3–manifold groups are either trivial or of finite index in their ambient groups.

**Lemma 4.1** (Strongly quasiconvex subgroups in Sol 3–manifold groups).

Let \( M \) be a Sol 3–manifold and let \( H \) be a strongly quasiconvex subgroup of \( \pi_1(M) \). Then \( H \) is trivial or has finite index in \( \pi_1(M) \).

**Proof.** Let \( N \) be the double cover of \( M \) that is a torus bundle with Anosov monodromy \( \Phi \). Then \( \pi_1(N) \) is an abelian-by-cyclic subgroup \( \mathbb{Z}^2 \rtimes_{\phi} \mathbb{Z} \) and it has finite index subgroup in \( \pi_1(M) \). Therefore, each strongly quasiconvex subgroups of \( \pi_1(N) \) is trivial or has finite index in \( \pi_1(N) \) by Corollary A.3. This implies that \( H \) is trivial or has finite index in \( \pi_1(M) \). \( \square \)

In the following lemma, we characterize all finite height subgroups of Sol 3–manifold groups.

**Lemma 4.2** (Finite height subgroups in Sol 3–manifolds). Let \( M \) be a Sol 3–manifold. Let \( N \) be the double cover of \( M \) that is a torus bundle with Anosov monodromy. Let \( H \) be a nontrivial finitely generated infinite index subgroup of \( \pi_1(M) \). Then \( H \) has finite height in \( \pi_1(M) \) if and only if \( H \cap \pi_1(N) \) is an infinite cyclic subgroup generated by an element in \( \pi_1(N) \) that does not belong to the fiber subgroup of \( \pi_1(N) \).

**Proof.** We note that \( \pi_1(N) \) is the semi-direct product \( \mathbb{Z}^2 \rtimes_{\phi} \mathbb{Z} \) where \( \phi \in GL_2(\mathbb{Z}) \) is the matrix corresponding to the Anosov monodromy. We note that \( \phi \) is conjugate to a matrix of the form \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) where \( ad - bc = 1 \) and \( |a + d| > 2 \). By Example A.6, we have that \( \phi^\ell \) has no nontrivial fixed point for any nonzero integer \( \ell \).
We are going to prove necessity. Assume that \( H \) has finite height in \( \pi_1(M) \). It is straightforward to see that \( H \cap \pi_1(N) \) has infinite index in \( \pi_1(N) \) and \( H \cap \pi_1(N) \) is not trivial. Since \( H \) has finite height in \( \pi_1(M) \) and \( \pi_1(N) \) is a subgroup of \( \pi_1(M) \), it follows that \( H \cap \pi_1(N) \) has finite height in \( \pi_1(N) \) (see Proposition 2.2). By Proposition A.1, the subgroup \( H \cap \pi_1(N) \) is an infinite cyclic subgroup generated by an element that does not belong to the fiber subgroup of \( \pi_1(N) \).

We are going to prove sufficiency. Suppose that \( H \cap \pi_1(N) \) is an infinite cyclic subgroup generated by an element in \( \pi_1(N) \) that does not belong to the fiber subgroup of \( \pi_1(N) \). By Proposition A.1, the subgroup \( H \cap \pi_1(N) \) has finite height in \( \pi_1(N) \). Since \( \pi_1(N) \) has finite index in \( \pi_1(M) \), it follows from (2) of Proposition 2.2 that \( H \) has finite height in \( \pi_1(M) \).

In the following lemma we study strongly quasiconvex subgroups and finite height subgroups of the fundamental group \( \pi_1(M) \), where \( M \) is a 3-manifold which has a geometric structure modeled on six of eight geometries: \( S^3, \mathbb{R}^3, S^2 \times \mathbb{R}, \text{Nil}, SL(2, \mathbb{R}), \text{or } \mathbb{H}^2 \times \mathbb{R} \).

**Lemma 4.3.** Suppose that a 3-manifold \( M \) has a geometric structure modeled on six of eight geometries: \( S^3, \mathbb{R}^3, S^2 \times \mathbb{R}, \text{Nil}, SL(2, \mathbb{R}), \text{or } \mathbb{H}^2 \times \mathbb{R} \). Let \( H \) be a nontrivial, finitely generated subgroup of \( \pi_1(M) \). Then the following statements are equivalent:

1. \( H \) has finite height in \( \pi_1(M) \).
2. \( H \) is a finite index subgroup of \( \pi_1(M) \).
3. \( H \) is strongly quasiconvex in \( \pi_1(M) \).

**Proof.** By Theorem 2.12, we have (3) implies (1). It is obvious that (2) implies (3). For the rest of the proof, we only need to show that (1) implies (2).

If the geometry of \( M \) is spherical, then it fundamental group is finite, hence it is obvious that \( H \) has finite index in \( \pi_1(M) \).

If the geometry of \( M \) is either \( S^2 \times \mathbb{R}, \mathbb{E}^3, \text{Nil}, SL(2, \mathbb{R}), \text{or } \mathbb{H}^2 \times \mathbb{R} \), then \( \pi_1(M) \) has a finite index subgroup \( K \) such that the centralizer \( Z(K) \) of \( K \) is infinite. More precisely,

1. If the geometry of \( M \) is \( S^2 \times \mathbb{R} \) then there exists a finite index subgroup \( K \leq \pi_1(M) \) such that \( K \) is isomorphic to \( \mathbb{Z} \). It is obvious that the centralizer \( Z(K) \) is infinite.
2. If the geometry of \( M \) is \( \mathbb{E}^3 \) then there exists a finite index subgroup \( K \leq \pi_1(M) \) such that \( K \) is isomorphic to \( \mathbb{Z}^3 \). Hence the centralizer \( Z(K) \) is infinite.
3. If the geometry of \( M \) is \( \text{Nil} \), then \( \pi_1(M) \) contains a discrete Heisenberg subgroup of finite index \( K \) which has infinite centralizer.
4. If the geometry of \( M \) is \( \mathbb{H}^2 \times \mathbb{R} \) then \( M \) is finitely covered by \( M' = \Sigma \times S^1 \) where \( \Sigma \) is a compact surface with negative Euler.
characteristic. Therefore, \( K = \pi_1(M') \) has finite index in \( \pi_1(M) \) and \( Z(K) \) is infinite.

(5) If \( M \) has a geometry modeled on \( SL(2, \mathbb{R}) \) then \( M \) is finitely covered by a circle bundle over surface \( M' \). Thus \( K = \pi_1(M') \) has finite index in \( \pi_1(M) \). We note that \( Z(K) \) is infinite since it contains the fundamental group of a regular fiber of \( M' \).

By Proposition 2.2, the subgroup \( H \cap K \) has finite height in the subgroup \( K \). Therefore by Proposition 2.3, the subgroup \( H \cap K \) is trivial or has finite index in the subgroup \( K \). It follows that \( H \) has finite index in \( \pi_1(M) \). □

The rest of this section is devoted to the study of strongly quasiconvex subgroups and finite height subgroups of hyperbolic 3-manifold groups.

**Remark 4.4.** If \( M \) is a closed hyperbolic 3-manifold, it is well-known that a finitely generated subgroup \( H \) has finite height in \( \pi_1(M) \) if and only if \( H \) is strongly quasiconvex (equivalently, \( H \) is geometrically finite). Indeed, by Subgroup Tameness Theorem, any finitely generated subgroup of \( \pi_1(M) \) is either geometrically finite or virtual fiber surface subgroup. If \( H \) has finite height in \( \pi_1(M) \) then \( H \) must be geometrically finite, otherwise \( H \) is a virtual fiber surface subgroup that is not a finite height subgroup. Since \( M \) is closed hyperbolic 3-manifold, it is well-known that \( \pi_1(M) \) is a hyperbolic group. Since \( H \) is geometrically finite, it follows that \( H \) is strongly quasiconvex in \( \pi_1(M) \). Conversely, if \( H \) is strongly quasiconvex in \( \pi_1(M) \) then \( H \) has finite height by a result of [GMRS98].

**Proposition 4.5.** Let \( M \) be a hyperbolic 3-manifold with tori boundary. Let \( H \) be a finitely generated subgroup of \( \pi_1(M) \). Let \( M_H \to M \) be the covering space corresponding to the subgroup \( H \leq \pi_1(M) \). Then the following statements are equivalent.

1. \( H \) has finite height in \( \pi_1(M) \).
2. \( \partial M_H \) consists only planes, tori
3. \( H \) is strongly quasiconvex in \( \pi_1(M) \).

**Proof.** We first prove that (1) implies (2). Let \( T_1, T_2, \ldots, T_n \) be the tori boundary of \( M \). We note that \( \pi_1(M) \) is hyperbolic relative to the collection \( \mathcal{P} = \{ \pi_1(T_1), \ldots, \pi_1(T_n) \} \). Since \( M_H \to M \) is a covering spaces, it follows that \( \partial M_H \) contains only tori, planes and cylinders. If \( H \) has finite height in \( \pi_1(M) \), then by Proposition 2.2 \( H \cap g\pi_1(T_i)g^{-1} \) has finite height in \( g\pi_1(T_i)g^{-1} \) for any \( g \in \pi_1(M) \) and \( i \in \{ 1, \ldots, n \} \). Since \( g\pi_1(T_i)g^{-1} \) is isomorphic to \( \mathbb{Z}^2 \), it follows from Proposition 2.3 that \( H \cap g\pi_1(T_i)g^{-1} \) is trivial or has finite index in \( g\pi_1(T_i)g^{-1} \). Thus, \( \partial M_H \) does not contain a cylinder.

Now, we are going to prove that (2) implies (3). Assume that \( \partial M_H \) consists only tori and planes. It follows that \( H \) is geometrically finite subgroup of \( \pi_1(M) \). Indeed, if not, then \( H \) is geometrically infinite. Hence \( M_H \) is homeomorphic to \( \Sigma_H \times \mathbb{R} \) or a twisted \( \mathbb{R} \)-bundle \( \Sigma_H \times \mathbb{R} \) for some compact surface \( \Sigma_H \) (with nonempty boundary). Thus, \( \partial M_H \) contains cylinders that
contradicts to our assumption. Since $H$ is geometrically finite, it follows that $H$ is undistorted in $\pi_1(M)$. Moreover, since $\partial M_H$ consists only planes and tori, it follows that the intersection of $H$ with each conjugate $gP_g^{-1}$ of peripheral subgroup in $P$ is either trivial or has finite index in $gP_g^{-1}$. Thus $H \cap gP_g^{-1}$ is strongly quasiconvex in $gP_g^{-1}$. By Theorem 2.18, it follows that $H$ is strongly quasiconvex in $\pi_1(M)$. By Theorem 2.12, we have (3) implies (1). Therefore, the theorem is proved.

\textbf{Remark 4.6} (Morse elements in geometric 3–manifold groups). Let $M$ be a geometric 3–manifold. If $M$ has a geometric structure modeled on six of eight geometries: $S^3$, $\mathbb{R}^3$, Nil, Sol, $SL(2,\mathbb{R})$ or $\mathbb{H}^2 \times \mathbb{R}$, then $\pi_1(M)$ has no Morse element by Lemma 4.1 and Lemma 4.3. If $M$ has a geometric structure modeled $S^2 \times \mathbb{R}$, then $\pi_1(M)$ is virtually an infinite cyclic subgroup. Therefore, all infinite order elements of $\pi_1(M)$ are Morse. Now we consider the case $M$ has a geometric structure modeled on $\mathbb{H}^3$. If $M$ is a closed manifold, then all infinite order elements of $\pi_1(M)$ are Morse. If $M$ is a hyperbolic 3–manifold with tori boundary, then we let $T_1, T_2, \ldots, T_n$ be the tori boundary of $M$. We note that $\pi_1(M)$ is hyperbolic relative to the collection $\mathbb{P} = \{\pi_1(T_1), \ldots, \pi_1(T_n)\}$. Therefore, an infinite order element $g$ in $\pi_1(M)$ is Morse if and if $g$ does not conjugate to an element of $\pi_1(T_i)$ (see Proposition 2.19).

4.2. \textbf{Strongly quasiconvex subgroups of mixed 3–manifold groups.} In this section, we prove Theorem 1.4 and Theorem 1.8 for the case of mixed 3–manifold $M$ which complete the proof of these theorems for the case of nongeometric 3–manifold groups.

\textbf{Proposition 4.7}. Let $M$ be a mixed 3–manifold. Let $H$ be a finitely generated subgroup of $\pi_1(M)$. Then $H$ has finite height if and only if $H$ is strongly quasiconvex.

\textit{Proof.} The sufficiency is followed from Theorem 2.12. We are going to prove the necessity. We will assume that $H$ has infinite index in $\pi_1(M)$, otherwise it is trivial. Let $M_1, \ldots, M_k$ be the maximal graph manifold components of the JSJ decomposition of $M$, let $S_1, \ldots, S_l$ be the tori in the boundary of $M$ that adjoin a hyperbolic piece, and let $T_1, \ldots, T_m$ be the tori in the JSJ decomposition of $M$ that separate two hyperbolic components of the JSJ decomposition. Then $\pi_1(M)$ is hyperbolic relative to $\mathbb{P} = \{\pi_1(M_p)\} \cup \{\pi_1(S_q)\} \cup \{\pi_1(T_r)\}$ (see [Dah03] or [BW13]).

By Theorem 2.18, to see that $H$ is strongly quasiconvex in $\pi_1(M)$, we only need to show that $H$ is undistorted in $\pi_1(M)$ and $H \cap gP_g^{-1}$ is strongly quasiconvex in $gP_g^{-1}$ for each conjugate $gP_g^{-1}$ of peripheral subgroup in $\mathbb{P}$. In the rest of the proof, we are going to verify the following claims.

**Claim 1:** $H$ is undistorted in $\pi_1(M)$.

Indeed, let $p: M_H \to M$ be the covering space corresponding to the subgroup $H \leq \pi_1(M)$. Let $M_{H,i} \subset M_H$ be the submanifold of $M_H$ given by Subsection 3.1. For each piece $M_{H,i}$ of $M_H$, let $M_i$ be a piece of $M$ such
that $M_{H,i}$ covers $M_i$. Since $H$ has finite height in $\pi_1(M)$, it follows from (1) in Proposition 2.2 that $\pi_1(M_{H,i})$ has finite height in $\pi_1(M_i)$. If $M_i$ is a Seifert piece of $M$, then $\pi_1(M_{H,i})$ is either finite or has finite index in $\pi_1(M_i)$. If $M_i$ is a hyperbolic piece of $M$, then $\pi_1(M_{H,i})$ is strongly quasiconvex in $\pi_1(M_i)$ (see Proposition 4.5), hence $\pi_1(M_{H,i})$ is geometrically finite in $\pi_1(M_i)$. Thus, the “almost fiber surface” $\Phi(H)$ of the subgroup $H \leq \pi_1(M)$ (see Section 3.1 in [Sun]) is empty. It follows from Theorem 1.4 in [NS19] that $H$ is undistorted in $\pi_1(M)$.

Claim 2: $H \cap gPg^{-1}$ is strongly quasiconvex in $gPg^{-1}$ for each conjugate $gPg^{-1}$ of peripheral subgroup in $\mathbb{P}$.

We first show that each subgroup $H \cap gPg^{-1}$ is finitely generated. In fact, since $gPg^{-1}$ is strongly quasiconvex and $H$ is undistorted, then $H \cap gPg^{-1}$ is strongly quasiconvex in $H$ by Proposition 4.11 in [Tra19]. This implies that $H \cap gPg^{-1}$ is finitely generated. We now prove that $H \cap gPg^{-1}$ is strongly quasiconvex. Since $H$ has finite height in $\pi_1(M)$, it follows that $H \cap gPg^{-1}$ has finite height in $gPg^{-1}$ (see Proposition 2.2). If $P$ is either $\pi_1(S_q)$ or $\pi_1(T_r)$ for some $\pi_1(S_q), \pi_1(T_r) \in \mathbb{P}$, then $gPg^{-1}$ is isomorphic to $\mathbb{Z}^2$. By Proposition 2.3, $H \cap gPg^{-1}$ is either finite or has finite index in $gPg^{-1}$. It implies that $H \cap gPg^{-1}$ is strongly quasiconvex in $gPg^{-1}$. If $P$ is $\pi_1(M_j)$ for some maximal graph manifold component $M_j$, then $H \cap gPg^{-1}$ is strongly quasiconvex in $gPg^{-1}$ by Theorem 1.7.

The following proposition is the direct result of Proposition 2.19 and Proposition 3.1.

Proposition 4.8 (Morse elements in mixed manifold groups). Let $M$ be a mixed 3–manifold group. Then a nontrivial group element $g$ in $\pi_1(M)$ is Morse if and only if $g$ is not conjugate into any Seifert subgroups.

The following proposition is the direct result of Theorem 1.7 and Proposition 2.21.

Proposition 4.9 (Purely Morse subgroups in mixed manifold groups). Let $M$ be a mixed 3–manifold group and let $H$ be an undistorted purely Morse subgroup of $\pi_1(M)$. Then $H$ is stable in $\pi_1(M)$.

4.3. Strongly quasiconvex subgroups of finitely generated 3–manifold groups. In Section 3, Section 4.1 and Section 4.2, we have shown that finitely generated finite height subgroups and strongly quasiconvex subgroups are equivalent in the fundamental group of a 3–manifold with empty or tori boundary (except Sol 3–manifold). In this subsection, we extend these results to arbitrary finitely generated 3–manifold groups.

Proposition 4.10. Let $M$ be a compact orientable irreducible 3–manifold that has trivial torus decomposition and has at least one higher genus boundary component. Let $H$ be a finitely generated subgroup of $\pi_1(M)$.

Then $H$ has finite height in $\pi_1(M)$ if and only if $H$ is strongly quasiconvex.
Proof. If $H$ has finite index in $\pi_1(M)$ then the result is obviously true. Hence, we will assume that $H$ has infinite index in $\pi_1(M)$. The sufficiency is followed from Theorem 2.12. We are going to prove necessity.

We paste hyperbolic 3–manifolds with totally geodesic boundaries to $M$ to get a finite volume hyperbolic 3–manifold $N$ (for example, see Section 6.3 in [Sun]).

Claim: $\pi_1(M)$ is strongly quasiconvex in $\pi_1(N)$. Indeed, by the construction of $N$, the subgroup $\pi_1(M) \leq \pi_1(N)$ is not a virtual fiber, hence it is geometrically finite. Thus, $\pi_1(M)$ is undistorted. If $N$ is closed then $\pi_1(N)$ is a hyperbolic group. Since $\pi_1(M)$ is undistorted in $\pi_1(N)$, it follows that $\pi_1(M)$ is strongly quasiconvex in $\pi_1(N)$. As a consequence, $\pi_1(M)$ has finite height in $\pi_1(N)$ (see [GMRS98]). If $N$ has nonempty boundary, then $N$ has tori boundary, we let $N_M \to N$ be the covering space corresponding to the subgroup $\pi_1(M) \leq \pi_1(N)$. We note that $\partial N_M$ does not contain any cylinder. By Proposition 4.5, we have $\pi_1(M)$ is strongly quasiconvex in $\pi_1(N)$. The claim is established.

We are now going to show that $H$ is strongly quasiconvex in $\pi_1(M)$ if $H$ has finite height in $\pi_1(M)$. Indeed, by the above claim, $\pi_1(M)$ has finite height in $\pi_1(N)$. Since $H$ has finite height in $\pi_1(M)$, it follows that $H$ has finite height in $\pi_1(N)$ (see Proposition 2.2). Applying Remark 4.4 and Proposition 4.5 to the hyperbolic manifold $N$ (in the case $\partial N = \emptyset$ and $\partial N \neq \emptyset$ respectively), we have $H$ is strongly quasiconvex in $\pi_1(N)$. Since $\pi_1(M)$ is undistorted in $\pi_1(N)$, it follows from Proposition 4.11 in [Tra19] that $H = H \cap \pi_1(M)$ is strongly quasiconvex in $\pi_1(M)$.

We are now ready for the proof of Theorem 1.4.

Proof of Theorem 1.4. By Theorem 2.12, if $H$ is strongly quasiconvex in $\pi_1(M)$ then $H$ has finite height in $\pi_1(M)$. Thus, for the rest of the proof, we only need to show that $H$ is strongly quasiconvex in $\pi_1(M)$ if $H$ has finite height. We also assume that $H$ has infinite index in $\pi_1(M)$, otherwise the result is vacuously true.

Since $\pi_1(M)$ is finitely generated group, it follows from the Scott core theorem that $M$ contains a compact codim–0 submanifold such that the inclusion map of the submanifold into $M$ is a homotopy equivalence. In particular, the inclusion induces an isomorphism on their fundamental groups. We note that $H$ has finite height (resp. strongly quasiconvex) in $\pi_1(M)$ if and only if the preimage of $H$ under the isomorphism has finite height (resp. strongly quasiconvex) in the fundamental group of the submanifold. Thus, we can assume that the manifold $M$ is compact.

We can also assume that $M$ is orientable. Indeed, let $M'$ be a double cover of $M$ that is orientable. We remark here that a finitely generated subgroup $H$ of $\pi_1(M)$ has finite height if and only if $H' := H \cap \pi_1(M')$ has finite height in $\pi_1(M')$. Moreover, $H$ is strongly quasiconvex in $\pi_1(M)$ if
and only if $H'$ is strongly quasiconvex in $\pi_1(M')$. Therefore, without losing of generality, we can assume that $M$ is orientable.

We can reduce to the case that $M$ is irreducible and $\partial$–irreducible by the following reason: Since $M$ is compact, orientable 3–manifold, it decomposes into irreducible, $\partial$–irreducible pieces $M_1, \ldots, M_k$ (by the sphere-disc decomposition). In particular, $\pi_1(M)$ is a free product $\pi_1(M_1) * \pi_1(M_2) * \cdots * \pi_1(M_k)$. Let $G_i = \pi_1(M_i)$. We remark here that $\pi_1(M)$ is hyperbolic relative to the collection $\mathbb{P} = \{G_1, \ldots, G_k\}$. By Kurosh Theorem, the subgroup $H \cong H_1 * \cdots * H_m * F_k$ where each subgroup $H_i = H \cap g_i G_{i_1} g_i^{-1}$ for some $g_i \in \pi_1(M)$, and $i_j \in \{1, \ldots, k\}$. We remark here that $H$ is strongly quasiconvex in $\pi_1(M)$ if finitely generated finite height subgroups of $G_i$ are strongly quasiconvex. Indeed, to see this, we note that $H_i$ has finite height in $g_i G_{i_1} g_i^{-1}$ since $H$ has finite height in $\pi_1(M)$. It follows that $H_i$ is strongly quasiconvex in $g_i G_{i_1} g_i^{-1}$ (by the assumption above), and hence $H_i$ is undistorted in $g_i G_{i_1} g_i^{-1}$. The argument in the second paragraph of the proof of Theorem 1.3 in [NS19] tells us that $H$ must be undistorted in $\pi_1(M)$. Using Theorem 2.18, we have that $H$ is strongly quasiconvex in $\pi_1(M)$. Therefore, for the rest of the proof we only need to show that finitely generated finite height subgroups in the fundamental group of a compact, orientable, irreducible, $\partial$–irreducible are strongly quasiconvex.

We consider the following cases.

**Case 1:** $M$ has trivial torus decomposition.

Case 1.1: $M$ has empty or tori boundary. In this case, $M$ has a geometric structure modeled on seven of eight geometries: $S^3$, $\mathbb{R}^3$, $S^2 \times \mathbb{R}$, Nil, $\text{SL}(2, \mathbb{R})$, $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{H}^3$. By Lemma 4.3, Remark 4.4, and Proposition 4.5, we have that $H$ is strongly quasiconvex in $\pi_1(M)$.

Case 1.2: $M$ has higher genus boundary. In this case, it follows from Proposition 4.10 that $H$ is strongly quasiconvex.

**Case 2:** $M$ has nontrivial torus decomposition.

Case 2.1: $M$ has empty or tori boundary. Then $M$ is a nongeometric 3–manifolds. By Theorem 1.7 and Proposition 4.7, $H$ is strongly quasiconvex.

Case 2.2: $M$ has a boundary component of genus at least 2. By Section 6.3 in [Sun], we paste hyperbolic 3–manifolds with totally geodesic boundaries to $M$ to get a 3–manifold $N$ with empty or tori boundary. The new manifold $N$ satisfies the following properties.

1. $M$ is a submanifold of $N$ with incompressible tori boundary.
2. The torus decomposition of $M$ also gives the torus decomposition of $N$.
3. Each piece of $M$ with a boundary component of genus at least 2 is contained in a hyperbolic piece of $N$.

We remark here that it has been proved in [NS19] that $\pi_1(M)$ is undistorted in $\pi_1(N)$ (see the proof of Case 1.2 in the proof of Theorem 1.3 in [NS19]).

Claim: $\pi_1(M)$ is strongly quasiconvex in $\pi_1(N)$. 

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We are going to prove the claim above. Let $M'_1, \ldots, M'_k$ be the pieces of $M$ that satisfies (3). Since $N$ is a mixed 3–manifold, we equip $N$ with a nonpositively curved metric as in [Lee95]. This metric induces a metric on the universal cover $\tilde{N}$. Let $N'_i$ be the hyperbolic piece of $N$ such that $M'_i$ is contained in $N'_i$. Note that by Proposition 4.5, the subgroup $\pi_1(M'_i)$ is strongly quasiconvex in $\pi_1(N'_i)$. Since there are only finitely many pieces $M'_1, \ldots, M'_k$, it follows that for any $K \geq 1$, $C \geq 0$, there exists $R = R(K, C)$ such that for any $(K, C)$–quasi-geodesic in $\tilde{N}'_v$ with endpoints in $\tilde{M}'_v$ (for some $\tilde{M}'_v \subset \tilde{M}$ covers a $M'_i \subset N'_i$) then this quasi-geodesic lies in the $R$–neighborhood of $\tilde{M}'_v$.

Let $\gamma$ be any $(K, C)$–quasi-geodesic in $\tilde{N}$. By Taming quasi-geodesic Lemma (see Lemma 1.11 page 403 in [BH99]), we can assume that $\gamma$ is a continuous path. By the construction of the manifold $N$, the path $\gamma$ can be decomposed into a concatenation $\gamma = \alpha_1 \cdot \beta_1 \cdot \alpha_2 \cdot \beta_2 \cdots \alpha_\ell \cdot \beta_\ell$ such that either one of the following two statements holds.

(1) For each $j$, the subpath $\beta_j$ is a subset of $\tilde{M}$, and $\alpha_j$ intersects $\tilde{M}$ only at its endpoints. Moreover, there are pieces $\tilde{M}'_j$ and $\tilde{N}'_j$ of $\tilde{M}$ and $\tilde{N}$ respectively such that $\tilde{M}'_j \subset \tilde{N}'_j$, $\alpha_j \subset \tilde{N}'_j$, and the endpoints of $\alpha_j$ in $\tilde{M}'_j$.

(2) For each $j$, the subpath $\alpha_j$ is a subset of $\tilde{M}$, and $\beta_j$ intersects $\tilde{M}$ only at its endpoints. Moreover, there are pieces $\tilde{M}'_j$ and $\tilde{N}'_j$ of $\tilde{M}$ and $\tilde{N}$ respectively such that $\tilde{M}'_j \subset \tilde{N}'_j$, $\beta_j \subset \tilde{N}'_j$, and the endpoints of $\beta_j$ in $\tilde{M}'_j$.

Without loss of generality, we assume that (1) holds. We have that $\alpha_j \subset N'_R(\tilde{M}'_j) \subset N'_R(\tilde{M})$. Thus $\gamma \subset N'_R(\tilde{M})$. Therefore, $\tilde{M}$ is strongly quasiconvex in $\tilde{N}$. It follows that $\pi_1(M)$ is strongly quasiconvex in $\pi_1(N)$.

Since $H$ has finite height in $\pi_1(M)$, it follows that $H$ has finite height in $\pi_1(N)$ by Proposition 2.2. Since $N$ is a nongeometric 3–manifold, it follows from Theorem 1.7 and Proposition 4.7 that $H$ is strongly quasiconvex in $\pi_1(N)$. Since $\pi_1(M)$ is undistorted in $\pi_1(N)$, it follows from Proposition 4.11 [Tra19] that $H = H \cap \pi_1(M)$ is strongly quasiconvex in $\pi_1(M)$. \hfill \Box

We are now ready for the proof of Theorem 1.8.

**Proof of Theorem 1.8.** Using the similar argument as in the proof of Theorem 1.4, we can reduce to the case where $M$ is compact, orientable, irreducible and $\partial$–irreducible. We leave it to the interesting reader. We consider the following cases:

Case 1: $M$ has trivial torus decomposition.

Case 1.1: $M$ has empty or torus boundary. In this case, $M$ has geometric structure modeled on $S^3$, $\mathbb{R}^3$, $S^2 \times \mathbb{R}$, $\text{Nil}$, $\text{SL}(2, \mathbb{R})$, $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{H}^3$. By Remark 4.6, if the geometry of $M$ is $S^3$, $\mathbb{R}^3$, $\text{Nil}$, $\text{SL}(2, \mathbb{R})$, $\mathbb{H}^2 \times \mathbb{R}$, then $H$ must be finite. Thus $H$ is stable. If the geometry of $M$ is $S^2 \times \mathbb{R}$, then from
the fact \( \pi_1(M) \) is virtually \( \mathbb{Z} \), it follows that \( H \) is stable. If the geometry of \( M \) is \( \mathbb{H}^3 \), then \( H \) is stable by the following reasons. If \( M \) is closed, then \( \pi_1(M) \) is hyperbolic group. Since \( H \) is undistorted in \( \pi_1(M) \), it follows that \( H \) is a hyperbolic group and \( H \) is strongly quasiconvex in \( \pi_1(M) \). Hence \( H \) is stable. If \( M \) has tori boundary, then \( H \) is stable by combination of Remark 4.6 and Theorem 2.20.

Case 1.2: \( M \) has higher genus boundary. Let \( N \) be the finite volume hyperbolic 3–manifold constructed in the proof of Proposition 4.10. It follows from Case 1.1 above that all undistorted purely Morse subgroups of \( \pi_1(N) \) are stable. By Statement (3) of Corollary 2.16, we have that all undistorted purely Morse subgroups of \( \pi_1(M) \) are stable. Thus, \( H \) is stable in \( \pi_1(M) \).

Case 2: \( M \) has nontrivial torus decomposition. If the geometry of \( M \) is \( \text{Sol} \), then \( H \) is trivial by Remark 4.6. Thus, we can assume that \( M \) does not support the \( \text{Sol} \) geometry.

Case 2.1: \( M \) has empty or tori boundary. In this case, \( M \) is either a graph manifold or a mixed manifold. If \( M \) is a graph manifold, then \( H \) is stable by Theorem 1.7 (see (2) implies (4)). If \( M \) is a mixed manifold, then it follows from Proposition 4.9 that \( H \) is stable.

Case 2.2: \( M \) has higher genus boundary. Let \( N \) be the mixed 3–manifold constructed in Case 2.2 of the proof of Theorem 1.4. We recall that we have shown in the proof of Theorem 1.4 that \( \pi_1(M) \) is strongly quasiconvex in \( \pi_1(N) \). Since \( \pi_1(N) \) has the property that all undistorted purely Morse subgroups of \( \pi_1(N) \) are stable (see Case 2.1 above), it follows that all undistorted purely Morse subgroups of \( \pi_1(M) \) are stable by Statement (3) of Corollary 2.16. Thus \( H \) is stable in \( \pi_1(M) \). \( \square \)

Appendix A

Finite height subgroups, malnormal subgroups, and strongly quasiconvex subgroups of \( \mathbb{Z}^k \rtimes \Phi \mathbb{Z} \). In this section, we study strongly quasiconvex subgroups and finite height subgroups of abelian-by-cyclic subgroups \( \mathbb{Z}^k \rtimes \Phi \mathbb{Z} \). First, we define a fixed point of a group automorphism \( \Phi : \mathbb{Z}^k \to \mathbb{Z}^k \) is a group element \( z \) in \( \mathbb{Z}^k \) such that \( \Phi(z) = z \). The main result of this section is the following proposition.

**Proposition A.1.** Let \( \Phi : \mathbb{Z}^k \to \mathbb{Z}^k \) be a group automorphism. Then the group \( G = \mathbb{Z}^k \rtimes \Phi \mathbb{Z} = \langle \mathbb{Z}^k, t|tzt^{-1} = \Phi(z) \rangle \) has a finite height subgroup which is not trivial and has infinite index if and only if for every non-zero integer \( \ell \), the group automorphism \( \Phi^\ell \) has no nontrivial fixed point. Moreover, a nontrivial, infinite index subgroup \( H \) has finite height if and only if \( H \) is a cyclic subgroup generated by a group element \( g \in G - \mathbb{Z}^k \).

The proof of Proposition A.1 is a combination of Lemma A.2, Lemma A.4, and Lemma A.5 as follows.

**Lemma A.2.** Let \( G = \mathbb{Z}^k \rtimes \Phi \mathbb{Z} = \langle \mathbb{Z}^k, t|tzt^{-1} = \Phi(z) \rangle \) and \( H \) a nontrivial subgroup of infinite index of \( G \). Assume that \( H \) is a finite height subgroup.
Then $H$ is a cyclic subgroup generated by $t^m z$ where $m$ is a positive integer and $z$ is an element in $\mathbb{Z}^k$.

**Proof.** It follows from Corollary 2.5 that that $H \cap \mathbb{Z}^k$ is trivial. Thus $H$ is a cyclic subgroup generated by $t^m z$ where $m$ is a positive integer and $z$ is an element in $\mathbb{Z}^k$. \hfill \Box

**Corollary A.3** (Strongly quasiconvex subgroups \iff are trivial or have finite index). Let $G = \mathbb{Z}^k \rtimes \Phi \mathbb{Z} = \langle \mathbb{Z}^k, t|tzt^{-1} = \Phi(z) \rangle$ and $H$ a strongly quasiconvex subgroup of $G$. Then either $H$ is trivial or $H$ has finite index in $G$.

**Proof.** We observe that the group $G$ is a solvable group. By [DS05], none of asymptotic cones of $G$ has a global cut-point. Also by [DMS10], the group $G$ does not contain any Morse element.

Assume that $H$ is not trivial and has infinite index in $G$. Then $H$ is a finite height subgroup by Theorem 2.12. By Proposition A.2, $H$ is a cyclic subgroup generated by $t^m z$ where $m$ is a positive integer and $z$ is an element in $\mathbb{Z}^k$. Therefore, $G$ contains the Morse element $t^m z$ which is a contradiction. Thus, either $H$ is trivial or $H$ has finite index in $G$. \hfill \Box

**Lemma A.4.** Let $\Phi : \mathbb{Z}^k \rightarrow \mathbb{Z}^k$ be a group automorphism such that $\Phi^\ell$ has a nontrivial fixed point $z_0$ for some non-zero integer $\ell$. Let $H$ be a finite height subgroup of $G = \mathbb{Z}^k \rtimes \Phi \mathbb{Z} = \langle \mathbb{Z}^k, t|tzt^{-1} = \Phi(z) \rangle$. Then either $H$ is trivial or $H$ has finite index in $G$.

**Proof.** Assume that $H$ is not trivial and has infinite index in $G$. By Proposition A.2, the subgroup $H$ is a cyclic subgroup generated by $t^m z$ where $m$ is a positive integer and $z$ is an element in $\mathbb{Z}^k$. Since $\Phi^\ell(z_0) = z_0$, the group element $z_0$ commutes to the group element $t^\ell$. Therefore, $z_0$ commutes to the group element $(t^m z)^\ell$ in $H$. Therefore, $\bigcap_{i=1}^{\infty} z_0^i H z_0^{-i}$ is infinite. Also $z_0^i H \neq z_0^j H$ for each $i \neq j$. Therefore, $H$ is not a finite height subgroup of $G$ which is a contradiction. Therefore, either $H$ is trivial or $H$ has finite index in $G$. \hfill \Box

**Lemma A.5.** Let $\Phi : \mathbb{Z}^k \rightarrow \mathbb{Z}^k$ be a group automorphism such that $\Phi^\ell$ has no nontrivial fixed point for every non-zero integer $\ell$. Let $G = \mathbb{Z}^k \rtimes \Phi \mathbb{Z} = \langle \mathbb{Z}^k, t|tzt^{-1} = \Phi(z) \rangle$ and $H$ be the cyclic subgroup of $G$ generated by $t^m z$ where $m$ is a positive integer and $z$ is an element in $\mathbb{Z}^k$. Then $H$ has height at most $m$. In particular, any cyclic group generated by $t z$ where $z \in \mathbb{Z}^k$ is malnormal.

**Proof.** Assume that $H$ does not have height at most $m$. Then there are $(m+1)$ distinct left cosets $g_1 H, g_2 H, g_3 H, \cdots, g_{m+1} H$ such that $\bigcap_{i=1}^{m+1} g_i H g_i^{-1}$ is infinite. We observer that there are $i \neq j$ such that $g = g_i^{-1} g_j$ can be written of the form $t^mq z_1$ for some integer $q$ and some group element $z_1 \in \mathbb{Z}$. \hfill \Box
Since \( g = t^m z_1 \) is not a group element in \( H \), we can write \( g = z_0(t^m z)^q \) for some group element \( z_0 \in \mathbb{Z} - \{e\} \).

Since the subgroup \( g_i H g_i^{-1} \cap g_j H g_j^{-1} \) is infinite, the subgroup \( H \cap gHg^{-1} \) is also infinite. Therefore, there is a non-zero integer \( p \) such that \( g(t^m z)^p g^{-1} = (t^m z)^p \). Also, \( g = z_0(t^m z)^q \) for some group element \( z_0 \in \mathbb{Z} - \{e\} \). Thus, \( z_0(t^m z)^p z_0^{-1} = (t^m z)^p \). It is straightforward that \( t^m z \) has two eigenvalues \( 1, \lambda \) and \( 1/\lambda \) which have absolute value \( 1 \). Hence \( \Phi^\ell \) has no nontrivial fixed point. Another way to see is that \( \Phi^\ell \) has the form of \((a', b', c', d')\) where \( a'd' - b'c' = 1 \). Since \( \text{Trace}(\Phi^\ell) = a' + d' = \lambda^\ell + 1/\lambda^\ell \), it follows that \( |a' + d'| > 2 \). It easy to see that the matrix \((a', b', c', d')\) has no nontrivial fixed point (otherwise \( |a' + d'| = 2 \)). By Proposition A.1 and Corollary A.3, the group \( \mathbb{Z}^2 \rtimes \Phi \mathbb{Z} \) has a finite height subgroup \( H \) which is not strongly quasiconvex.

The \( \mathcal{PS} \) system and strong quasiconvexity. In this section, we first give the proof for the statement that all special paths for a flip graph manifold are uniform quasi-geodesic. This fact seems not be proved explicitly in [Sis11] and [Sis18]. Then we give the proof for Lemma 3.6 which states that a \( \mathcal{PS} \)–contracting subset is strongly quasiconvex.

**Proposition A.7.** The special path for a flip graph manifold is uniform quasi-geodesic.

Let \( \tilde{M}_0 = \tilde{F}_0 \times \tilde{\ell}_1 \) and \( \tilde{M}_1 = \tilde{F}_1 \times \tilde{\ell}_1 \) be two pieces with common boundary \( \tilde{T}_1 = \tilde{\ell}_1 \times \tilde{\ell}_1 \). By definition of flip manifolds, the boundary line \( \tilde{\ell}_1 \) of \( \tilde{F}_0 \) is identified with the fiber \( \tilde{\ell}_1 \) of \( \tilde{M}_1 \); vice versa for \( \tilde{\ell}_1 \) of \( \tilde{F}_1 \) and \( \tilde{\ell}_1 \) of \( \tilde{M}_0 \).

By abuse of language, we denote by \( d_h \) the hyperbolic distance on \( \tilde{F}_1 \), and \( d_v \) the fiber distance of \( \tilde{M}_i \) for \( i = 0, 1 \). However, the following fact is crucial: the \( d_h \)-distance of \( \tilde{M}_0 \) on the boundary \( \tilde{\ell}_1 \) coincides with the fiber \( d_v \)-distance on \( \tilde{\ell}_1 \) of \( \tilde{M}_1 \).

Let \( \delta = [x, y] [y, z] \) be a concatenated path of geodesics \( [x, y] \) and \( [y, z] \) where \( x = (x^h, x^v) \in \tilde{M}_0 \), \( y = (y^h, y^v) = (y^v, y^h) \in \tilde{M}_0 \cap \tilde{M}_1 = \tilde{T}_1 \), and \( z = (z^h, z^v) \in \tilde{M}_1 \). Note that the coordinates of \( y \) in \( \tilde{M}_0 \) and \( \tilde{M}_1 \) are switched.
Consider a minimizing horizontal slide of \( y \) in \( \tilde{M}_0 \) which changes its \( \tilde{F}_0 \)-coordinate only so that the projection of \([x, y]\) on \( \tilde{F}_0 \) is orthogonal to \( \tilde{T}_1 \).

We need the following observation that a minimizing horizontal slide does not increase distance too much.

**Lemma A.8** (Horizontal slide). There exists a constant \( C > 0 \) depending only on \( M \) with the following property. Let \( w = (w^h, y^v) \in \tilde{T}_1 \cap \tilde{M}_0 \) be a point with same \( \tilde{F}_1 \)-coordinate as \( y \). If \( d(x^h, w^h) \) minimizes the distance of \( d(x, w, z) \), then

\[
d(x, w) + d(w, z) \leq C \ell(\delta) + C
\]

**Proof.** Since a \( L^1 \)-metric is quasi-isometric to a \( L^2 \)-metric on each piece, it follows that

\[
d(x, w) + d(w, z) \approx d_h(x^h, w^h) + d_h(y^v, z^h) + d_v(x^h, y^v) + d_v(w^h, z^h)
\]

and

\[
d(x, y) + d(y, z) \approx d_h(x^h, y^h) + d_h(y^v, z^h) + d_v(x^h, y^v) + d_v(y^h, z^v).
\]

Thus, it suffices to find a constant \( C \) such that

\[
(1) \quad d_h(x^h, w^h) + d_v(w^h, z^v) \leq d_h(x^h, y^h) + d_v(y^h, z^v) + C.
\]

By assumption, \( w_h \) is a shortest projection point of \( x_h \) to \( \tilde{F}_1 \). By hyperbolicity of \( \tilde{F}_0 \), since \( y^h \) and \( w^h \) lie on \( \tilde{F}_1 \), we have

\[
(2) \quad d_h(x^h, w^h) + d_h(w^h, y^h) \leq d_h(x^h, y^h) + C
\]

for some constant \( C > 0 \) depending only on hyperbolicity constant of \( \tilde{F}_0 \).

Since the \( d_h \)-distance on the boundary \( \tilde{F}_1 \) coincides the fiber \( d_v \)-distance of \( \tilde{M}_1 \) on \( \tilde{T}_1 \), we then obtain

\[
d_h(w^h, y^h) = d_v(w^h, y^h) = |d_v(w^h, z^v) - d_v(y^h, z^v)|,
\]

which with (2) together proves (1). The lemma is thus proved. \( \square \)

**Proof of the Proposition A.7.** We follow the notations in Definition 3.7. Let \( \gamma = \gamma_0 \cdot \gamma_1 \cdots \gamma_n \) be a special path between \( x_0 \in M_0 \) and \( x_{n+1} \in M_n \) so that

\[
\gamma_i = [x_i, x_{i+1}] \subset M_i \quad \text{where} \quad x_{i+1} \in \tilde{T}_i := \tilde{M}_i \cap \tilde{M}_{i+1} \quad \text{for} \quad 0 \leq i \leq n.
\]

Then we have

\[
\ell(\gamma) \approx \sum_{i=0}^{n} d(x_i, x_{i+1})
\]

Let \( \delta \) be the CAT(0) geodesic with same endpoints as \( \gamma \). Let \( y_i \) be the intersection point of \( \delta \) with \( \tilde{T}_i \). Then

\[
\ell(\delta) \approx \sum_{i=0}^{n} d(y_i, y_{i+1})
\]

By definition of \( \gamma_i \), its horizontal projection to \( \tilde{F}_i \) is orthogonal to the corresponding boundary lines of \( \tilde{F}_i \). Hence, if we move \( y_i \) to \( x_i \) by a minimizing horizontal slide in \( \tilde{M}_i \) from \( i = 1 \) to \( n \), we then transform the geodesic \( \delta \) to
the special path $\gamma$. By Lemma A.8, the length of $\gamma$ is linearly bounded by $\ell(\delta)$. The proposition is proved. \hfill $\square$

Now we give a proof for Lemma 3.6. Before giving the proof, we need the following fact. Recall that all paths in the $\mathcal{PS}$ system are $c$-quasi-geodesic for some uniform constant $c$.

**Lemma A.9.** [Sis18, Lemma 2.4(3)] Let $\pi$ be a $\mathcal{PS}$-projection with constant $C$ on $A \subset X$. Then for each $x \in X$ we have $\text{diam}(\pi(B_r(x))) \leq C$ for $r = d(x, A)/c - c$.

**Proof of the Lemma 3.6.** Let $\gamma$ be a $k$-quasi-geodesic with two endpoints in $A$, where $k \geq 1$. For a constant $R > 0$, we consider a connected component $\alpha$ of $\gamma - N_R(A)$. We need at most $c\ell(\alpha)/(R - c)$ balls of radius $R/c - c$ to cover $\alpha$, where $\ell(\alpha)$ denotes the length of $\alpha$.

We now set $R = c + 2Cc^2$. On the one hand, we obtain by Lemma A.9 that

$$\text{diam}(\pi(\alpha)) \leq Cc\ell(\alpha)/(R - c) \leq \ell(\alpha)/2c.$$

On the other hand, since $\alpha$ is a $(c, c)$-quasi-geodesic, we have

$$\ell(\alpha) \leq cd(\alpha, \alpha_+) + c \leq c(2R + \text{diam}(\pi(\alpha))) + c.$$

We thus obtain $\ell(\alpha) \leq 2c(2R + 1)$. As a consequence, we have that $\gamma$ is contained in a $3c(2R + 1)$-neighborhood of $A$. This proves the lemma. \hfill $\square$

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