Variational inequalities for the commutators of rough operators with BMO functions

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Abstract Starting with the relatively simple observation that the variational estimates of the commutators of the standard Calderón-Zygmund operators with the bounded mean oscillation (BMO) functions can be obtained from their weighted variational estimates, we establish the similar variational estimates for the commutators of the BMO functions with rough singular integrals, which do not admit any weighted variational estimates. The proof involves several Littlewood-Paley-type inequalities with the commutators as well as Bony decomposition and related para-product estimates.

Keywords commutator, variational inequality, singular integral, averaging operator, rough kernel

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1 Introduction

Motivated by the modulus of continuity in Brownian motion, Lépingle [32] established the first variational inequality for general martingales (see [41] for a simple proof). Bourgain [3] was the first to exploit Lépingle’s result into obtaining corresponding variational estimates for the Birkhoff ergodic averages along the subsequences of natural numbers. He directly deduced pointwise convergence results without previous knowledge that pointwise convergence holds for a dense subclass of functions, which are not available in some ergodic models. In particular, Bourgain’s work [3] initiated a new research direction in ergodic theory and harmonic analysis. In the previous studies [6, 7, 25–27], Jones and his collaborators systematically studied variational inequalities for ergodic averages and truncated homogeneous singular integrals. Since then, many other studies enriched the literature on this subject (see [16, 19, 28, 31, 36, 40]). Recently, several works on weighted as well as vector-valued variational inequalities in ergodic theory and harmonic analysis also appeared (see [21, 24, 30, 34, 35]), and several results on ℓp(Zd)-estimates of q-variations for discrete operators of the Radon type also established (see [29, 37–39, 44]).

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Most of the operators considered in these cited papers are of the homogeneous type: it is still unknown whether variational inequalities hold for all the singular integrals of the convolution type (this statement is, however, true when the kernel is smooth enough [37]), let alone all the standard Calderón-Zygmund operators. In this paper, we establish variational estimates for the commutators of rough operators with BMO functions which are not a convolution type of integral operators.

To illustrate the observation, let us introduce some notations and recall some notions. Given a family of Lebesgue measurable functions \( F = \{ F_t : t \in \mathbb{R}_+ \} \) defined on \( \mathbb{R}^n \), the strong \( \rho \)-variation function \( V_\rho(F)(x) \) of the family \( F \) can be defined as

\[
V_\rho(F)(x) = \sup \{ \| (F_{t_k}(x) - F_{t_{k-1}}(x))_{k \geq 1} \|_{\ell^\rho} : \text{a.e. } x \in \mathbb{R}^n, \}
\]

where the supremum runs over all the increasing sequences \( \{ t_k : k \geq 1 \} \). Suppose that \( T = \{ T_t \}_{t > 0} \) is a family of operators on \( L^p(\mathbb{R}^n) \) (1 \( \leq p \leq \infty \)). The strong \( \rho \)-variation operator can be expressed as

\[
V_\rho T(f)(x) = \| \{ T_t(f(x)) \}_{t > 0} \|_{\ell^\rho}, \quad \forall f \in L^p(\mathbb{R}^n).
\]

Thus, the operator \( V_\rho T \) maps functions from \( \mathbb{R}^n \) to non-negative functions on \( \mathbb{R}^n \). From the definition of the \( \rho \)-variation norm, we can easily observe that for any \( x \), if \( V_\rho T(f)(x) < \infty \), then \( \{ T_t(f(x)) \}_{t > 0} \) converges when \( t \to 0 \) or \( t \to \infty \). In particular, if \( V_\rho T(f) \) belongs to some function spaces such as \( L^p(\mathbb{R}^n) \) or \( L^{p,\infty}(\mathbb{R}^n) \), the sequence converges almost everywhere without any additional condition. This is why the mapping property of the strong \( \rho \)-variation operator is extremely interesting in ergodic theory and harmonic analysis. Considering the influence of (1.2) for any \( f \in L^p(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \) yields the inequality expression

\[
T^*(f)(x) \leq V_\rho(Tf)(x) \quad \text{for } \rho \geq 1,
\]

where \( T^* \) is the maximal operator defined by

\[
T^*(f)(x) := \sup_{t > 0} \{ |T_t(f(x))| \}.
\]

Let \( b \in \text{BMO}(\mathbb{R}^n) \). If \( T \) is a linear operator on some measurable function space, the commutator formed by \( b \) and \( T \) can be represented by

\[
[b, T]f(x) := T((b(x) - b(\cdot))f)(x).
\]

In 1976, Coifman et al. [13, Theorem I] proved that if \( \Omega \in \text{Lip}(S^{n-1}) \), the commutator \( [b, T_\Omega] \) for \( T_\Omega \) and a BMO function \( b \) is bounded on \( L^p \) for \( 1 < p < \infty \), as expressed in the relationship

\[
[b, T_\Omega]f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x - y)}{|x - y|^n} (b(x) - b(y))f(y)dy,
\]

where

\[
T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x - y)}{|x - y|^n} f(y)dy.
\]

In the same paper [13], they also outlined a different approach that is less direct but demonstrates a close relationship between the weighted inequalities of the operator \( T \) and the weighted inequalities of the commutator \( [b, T] \). In 1993, the idea of the aforementioned study was developed by Alvarez et al.
[1, Theorem 2.13], who established a generalized boundedness criterion for the commutators of linear operators. It is well known that the commutators have played an important role in harmonic analysis and PDE, for example, in the theory of non-divergent elliptic equations with discontinuous coefficients (see [4, 11, 14, 43]).

Before presenting the main results of this paper, let us first recall the definition and some properties of $A_p$ weight on $\mathbb{R}^n$. Let $w$ be a non-negative locally integrable function defined on $\mathbb{R}^n$. We can say that $w \in A_1$ if there is a constant $C > 0$ such that $M(w)(x) \leq Cw(x)$, where $M$ is the classical Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|\leq r} |f(x-y)|dy.$$  

Equivalently, $w \in A_1$ if and only if there exists a constant $C > 0$ such that for any cube $Q$,

$$\frac{1}{|Q|} \int_Q w(x)dx \leq C \inf_{x \in Q} w(x).$$  

(1.5)

Furthermore, for $1 < p < \infty$, we can say that $w \in A_p$ if there exists a constant $C > 0$ such that

$$\sup_Q \left( \frac{1}{|Q|} \int_Q w(x)dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \right)^{p-1} \leq C.$$  

(1.6)

The smallest constant appearing in (1.5) or (1.6) is denoted by $[w]_{A_p}$. At this point, we can now state the first result of this paper, which is represented by the following theorem.

**Theorem 1.1.** Let $b \in \text{BMO}(\mathbb{R}^n)$. Let $\mathcal{T} = \{T_t : t \in \mathbb{R}_+\}$ be a family of linear operators and $\mathcal{T}_b = \{[b, T_t] : t \in \mathbb{R}_+\}$ be the family of the commutators formed by the linear operators and $b$. Moreover, let $1 < p < \infty$, $p \geq 1$ and $1 < s < \infty$. Let $w$ be a locally integrable function such that $w \in A_s$. Then $V_p \mathcal{T}_b$ is bounded on $L^p(w)$, i.e.,

$$\|V_p \mathcal{T}_b(f)\|_{L^p(w)} \leq C\|b\|_w \|f\|_{L^p(w)}$$

provided that $V_p \mathcal{T}$ is bounded on $L^p(w)$.

Now we present four consequences of Theorem 1.1. Let $K$ be a kernel on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x) : x \in \mathbb{R}^n\}$. Likewise, let us assume that $K$ satisfies the following regularity conditions. There exist two constants $\delta > 0$ and $C > 0$ such that

$$|K(x, y)| \leq \frac{C}{|x-y|^n} \text{ for } x \neq y,$$

(1.7)

$$|K(x, y) - K(z, y)| \leq \frac{C|x-z|^\delta}{|x-y|^{n+\delta}} \text{ for } |x-y| > 2|x-z|,$$

(1.8)

$$|K(y, x) - K(y, z)| \leq \frac{C|x-z|^\delta}{|x-y|^{n+\delta}} \text{ for } |x-y| > 2|x-z|.$$  

(1.9)

Let $T$ be the standard Calderón-Zygmund operator associated with the kernel $K$. For a Schwartz function $f$,

$$Tf(x) := \lim_{\varepsilon \to 0^+} T_{\varepsilon} f,$$

where $T_{\varepsilon}$ is the truncated operator

$$T_{\varepsilon} f(x) = \int_{|x-y| > \varepsilon} K(x, y)f(y)dy.$$  

For $b \in \text{BMO}(\mathbb{R}^n)$, let $[b, T_{\varepsilon}]$ be the truncated commutator

$$[b, T_{\varepsilon}]f(x) = \int_{|x-y| > \varepsilon} K(x, y)(b(x) - b(y))f(y)dy.$$
Let us define $\mathcal{T} = \{T_\varepsilon\}_{\varepsilon > 0}$ and $\mathcal{T}_b = \{[b, T_\varepsilon]\}_{\varepsilon > 0}$. Let $K$ be a kernel on $\mathbb{R}^n$ satisfying (1.7)–(1.9), and let $2 < \rho < \infty$. Ma et al. [35, Theorem 1.1] showed that if the operator $V_\rho \mathcal{T}$ is of the type $(p_0, p_0)$ for some $1 < p_0 < \infty$, then for $1 < p < \infty$ and $w \in A_p$, $V_\rho \mathcal{T}$ is bounded on $L^p(w)$; thus, we can apply Theorem 1.1 to formulate the following corollary.

**Corollary 1.2.** Let $b \in \text{BMO}(\mathbb{R}^n)$ and $K$ be the kernel on $\mathbb{R}^n$ satisfying (1.7)–(1.9). Let $2 < \rho < \infty$. If $V_\rho \mathcal{T}$ is of the type $(p_0, p_0)$ for some $1 < p_0 < \infty$, then for $1 < p < \infty$ and $w \in A_p$, there exists a constant $C$ such that

$$\|V_\rho \mathcal{T}_b(f)\|_{L^p(w)} \leq C\|b\|_* \|f\|_{L^p(w)}.$$  

In particular, if $K$ is of the convolution type satisfying the cancellation condition

$$\int_{\partial B(0, t)} K(x)dx = 0, \quad \forall t > 0$$

and either (i) $|\nabla K(x)| \leq C \frac{1}{|x|^{n+1}}$ or (ii) $K(x) = \frac{\Omega(x)}{|x|^n}$ with $\Omega \in \text{Lip}(\mathbb{S}^{n-1})$, then the associated $V_\rho \mathcal{T}_b$ is bounded on $L^p(w)$.

We refer the reader to [37, p.33] (resp. [7, Theorem A]) for the result that $V_\rho \mathcal{T}$ is $L^2$-bounded when $K$ satisfies (i) (resp. (ii)). On the other hand, Corollary 1.2 implies the main result of [33], where $T$ is the Hilbert transform.

Now, let $\varphi : \mathbb{R}^n \to [0, +\infty)$ be a radially decreasing integrable function. We have

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n} \varphi\left(\frac{x}{\varepsilon}\right), \quad \Phi_\varepsilon(f)(x) = \varphi_\varepsilon * f(x) \quad \text{and} \quad \Phi_b(f)(x) = \{\varphi_\varepsilon * f(x)\}_{\varepsilon > 0}.$$  

In [35, Corollary 1.5], it was shown that for $2 < \rho < \infty$, the operator $V_\rho \Phi$ is bounded on $L^p(w)$ for $1 < p < \infty$. Let $b \in \text{BMO}(\mathbb{R}^n)$ and $w \in A_p$. Define

$$\Phi_b(f) = \{\Phi_\varepsilon((b(x) - b(\cdot))f)\}_{\varepsilon > 0}.$$  

Thus, by applying Theorem 1.1, we have the following corollary.

**Corollary 1.3.** Let $b \in \text{BMO}(\mathbb{R}^n)$ and $\varphi : \mathbb{R}^n \to [0, +\infty)$ be a radially decreasing integrable function. Likewise, let

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n} \varphi\left(\frac{x}{\varepsilon}\right), \quad \Phi_\varepsilon(f)(x) = \varphi_\varepsilon * f(x) \quad \text{and} \quad \Phi_b(f)(x) = \{\Phi_\varepsilon((b(x) - b(\cdot))f(x))\}_{\varepsilon > 0}.$$  

Hence, for $2 < \rho < \infty$, $1 < p < \infty$ and $w \in A_p$, there exists a constant $C$ such that

$$\|V_\rho \Phi_b(f)\|_{L^p(w)} \leq C\|b\|_* \|f\|_{L^p(w)}.$$  

(1.10)

Theorem 1.1 is applicable not only to smooth singular kernels or good approximation identities but also to singular integral operators or averaging operators with some homogeneous rough kernels. Suppose that $T_{\Omega, \varepsilon}$ is the truncated singular integral operator defined by

$$T_{\Omega, \varepsilon}f(x) = \int_{|y| > \varepsilon} \frac{\Omega(y')}{|y'|^n} f(x - y)dy,$$  

(1.11)

where $\Omega \in L^1(\mathbb{S}^{n-1})$ satisfies the cancellation condition

$$\int_{\mathbb{S}^{n-1}} \Omega(y')d\sigma(y') = 0.$$  

(1.12)

Thus, for $1 < p < \infty$ and $f \in C^\infty_c(\mathbb{R}^n)$, the Calderón-Zygmund singular integral operator $T_\Omega$ with the homogeneous kernel can be expressed as

$$T_\Omega f(x) = \lim_{\varepsilon \to 0^+} T_{\Omega, \varepsilon}f(x), \quad \text{a.e.} \ x \in \mathbb{R}^n.$$  

(1.13)
Thus, we can denote the family of operators \( \{T_{\Omega,c}\}_{c>0} \) by \( \mathcal{T}_\Omega \). In [10], we showed that if \( \Omega \in L^q(S^{n-1}) \), \( q > 1 \) satisfies (1.12), then for \( \rho > 2 \), \( V_\rho \mathcal{T}_\Omega \) is bounded on \( L^p(w) \) whenever \( w \) and \( p \) satisfy one of the following conditions:

(i) \( q' \leq p < \infty \), \( p \neq 1 \) and \( w \in A_{p/q'} \);
(ii) \( 1 < p \leq q \), \( p \neq \infty \) and \( w^{-1/p} \in A_{p/q'} \).

Thus, we can invoke Theorem 1.1 to obtain the corresponding weighted estimates for the strong \( \rho \) variation of the commutators with rough kernels when \( b \in \text{BMO}(\mathbb{R}^n) \).

**Corollary 1.4.** Let \( b \in \text{BMO}(\mathbb{R}^n) \) and \( \mathcal{T}_{\Omega,b}f(x) \) be the family of the commutators

\[
\{T_{\Omega,c}((b(x) - b(\cdot))f(x))\}_{c>0}
\]

with \( \Omega \in L^q(S^{n-1}) \), \( q > 1 \) satisfying (1.12). Then for \( \rho > 2 \), there exists a constant \( C \) such that

\[
\|V_\rho \mathcal{T}_{\Omega,b}(f)\|_{L^p(w)} \leq C\|b\|_\star \|f\|_{L^p(w)},
\]

(1.14)

if \( w \) and \( p \) satisfy either (i) or (ii).

Theorem 1.1 is also applicable in the situation of averaging operators with rough kernels \( \mathcal{M}_\Omega = \{M_{\Omega,t}\}_{t>0} \), where \( M_{\Omega,t} \) is defined as

\[
M_{\Omega,t}f(x) = \frac{1}{t^n} \int_{|y|<t} \Omega(y)f(x-y)dy,
\]

(1.15)

where \( \Omega \in L^1(S^{n-1}) \). In [10], it was shown that if \( \Omega \in L^q(S^{n-1}) \), \( q > 1 \), then for \( \rho > 2 \), \( V_\rho \mathcal{M}_\Omega \) is bounded on \( L^p(w) \) if \( w \) and \( p \) satisfy either (i) or (ii) (see [18] on the maximal inequality for the family \( \mathcal{M}_\Omega \)). Now, given a BMO function \( b \), let us denote the family of operators \( \{M_{\Omega,t}((b(x) - b(\cdot))f(x))\}_{t>0} \) by \( \mathcal{M}_{\Omega,b}f(x) \).

We can invoke Theorem 1.1 to obtain the corresponding weighted estimates for the strong \( \rho \) variation of the commutators with rough kernels and \( b \in \text{BMO}(\mathbb{R}^n) \).

**Corollary 1.5.** Let \( b \in \text{BMO}(\mathbb{R}^n) \) and \( \mathcal{M}_{\Omega,b} \) be the family of the commutators of averaging operators with \( \Omega \in L^q(S^{n-1}) \), \( q > 1 \). Then for \( \rho > 2 \), there exists a constant \( C \) such that

\[
\|V_\rho \mathcal{M}_{\Omega,b}(f)\|_{L^p(w)} \leq C\|b\|_\star \|f\|_{L^p(w)},
\]

(1.16)

if \( w \) and \( p \) satisfy either (i) or (ii).

However, it is still unclear whether the operator \( V_\rho \mathcal{T}_\Omega \) or \( V_\rho \mathcal{M}_\Omega \) with \( \Omega \in L^1 \setminus \bigcup_{q>1} L^q(S^{n-1}) \) is bounded on \( L^p(w) \) (1 \( < \) \( p \) \( < \) \( \infty \)) for all \( w \in A_r \) (1 \( < \) \( r \) \( < \) \( \infty \)). Hence, if \( \Omega \in L^1 \setminus \bigcup_{q>1} L^q(S^{n-1}) \), the \( L^p \) boundedness of \( V_\rho \mathcal{T}_{\Omega,b} \) and \( V_\rho \mathcal{M}_{\Omega,b} \) cannot be deduced from Theorem 1.1. The main purpose of this paper is to give a sufficient condition that contains \( \bigcup_{q>1} L^q(S^{n-1}) \) such that the operators \( V_\rho \mathcal{T}_{\Omega,b} \) and \( V_\rho \mathcal{M}_{\Omega,b} \) are bounded on \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \).

In [5], Calderón and Zygmund proved that the rough singular integral \( T_\Omega \) is bounded on \( L^p \), 1 \( < \) \( p \) \( < \) \( \infty \) when \( \Omega \in L \log^+ L(S^{n-1}) \) satisfies (1.12). It is well known that

\[
\bigcup_{q>1} L^q(S^{n-1}) \subset L(\log^+ L)^\alpha(S^{n-1})
\]

for any \( \alpha > 0 \) and

\[
L(\log^+ L)^3(S^{n-1}) \subset L(\log^+ L)^2(S^{n-1}) \subset L\log^+ L(S^{n-1}).
\]

The second main result of our paper is represented by the next theorems.

**Theorem 1.6.** Let \( b \in \text{BMO}(\mathbb{R}^n) \) and \( \mathcal{T}_{\Omega,b} \) be the family of the commutators of truncated singular integral operators with \( \Omega \) satisfying (1.12). If \( \Omega \in L(\log^+ L)^3(S^{n-1}) \), then the following \( \rho (2 < \rho < \infty) \)-variational inequality holds for \( 1 < p < \infty \), namely,

\[
\|V_\rho \mathcal{T}_{\Omega,b}(f)\|_{L^p} \leq C\|b\|_\star \|f\|_{L^p}.
\]
The proof for this theorem is based on Fourier transform, which, although quite standard, is also technical as it involves several Littlewood-Paley-type inequalities with the commutators, as well as Bony decomposition and related para-product estimates.

Our approach to the variational estimates for singular integrals also works for the family $\mathcal{M}_{\Omega,b}$. 

**Theorem 1.7.** Let $1 < p < \infty$ and $b \in \text{BMO}(\mathbb{R}^n)$. If $\Omega \in L(\log^+L)^2(S^{n-1})$, then the $\rho(2 < \rho < \infty)$-variational inequality for the family $\mathcal{M}_{\Omega,b}$ holds, i.e.,

$$
\|V_{\rho} \mathcal{M}_{\Omega,b}(f)\|_{L^p} \leq C\|b\|_{\text{BMO}} \|f\|_{L^p}, \quad \forall f \in L^p(\mathbb{R}^n).
$$

The rest of this paper is organized as follows. Section 2 illustrates the proof of Theorem 1.1. In Section 3, we give the proof of Theorem 1.6. Section 4 is devoted to the proof of Theorem 1.7. Some notations are used throughout the paper as follows: for $p \geq 1$, $p'$ denotes the conjugate exponent of $p$, i.e., $p' = p/(p - 1)$. Likewise, the letter “C” stands for a positive constant that is independent of the essential variables and may not necessarily have the same value for each occurrence.

## 2 Proof of Theorem 1.1

For a family of Lebesgue measurable functions $\mathcal{F} = \{F_t : t \in \mathbb{R}_+\}$ defined on $\mathbb{R}^n$, the strong $\rho$-variation function $V_{\rho}(\mathcal{F})$ of the family $\mathcal{F}$ is defined as

$$
V_{\rho}(\mathcal{F})(x) = \sup \{\|F_{t_k}(x) - F_{t_{k-1}}(x)\|_{L^p} : k \geq 1\}, \quad \text{a.e. } x \in \mathbb{R}^n,
$$

where the supremum runs over all the increasing sequences $\{t_k : k \geq 1\}$. Let $\mathcal{T} = \{T_t : t \in \mathbb{R}_+\}$ be a family of linear operators and $\mathcal{T}_b = \{[b, T_t] : t \in \mathbb{R}_+\}$ be the family of the commutators formed by the linear operators and $b$. Thus, for $w \in A_p$, we obtain

$$
\|V_{\rho} \mathcal{T}_b(f)\|_{L^p(w)} = \sup \{\|[b, T_{t_k}]f - [b, T_{t_{k-1}}]f\|_{L^p(w)} : k \geq 1\},
$$

where the supremum runs over all the increasing sequences $\{t_k : k \geq 1\}$. We use an idea due to Coifman et al. [13]. Let us define $F(z) = e^{i[b(x) - b(y)]}$, $z \in \mathbb{C}$. By the analyticity of $F(z)$ on $\mathbb{C}$ and the Cauchy integration formula, we obtain that for any $\varepsilon > 0$,

$$
b(x) - b(y) = F'(0) = \frac{1}{2\pi i} \int_{|z| = \varepsilon} \frac{F(z)}{z^2} \, dz = \frac{1}{2\pi} \int_0^{2\pi} F(e^{i\theta}) \, d\theta = \frac{1}{2\pi \varepsilon} \int_0^{2\pi} e^{i\varepsilon[b(x) - b(y)]} e^{-i\varepsilon} \, d\theta.
$$

By the formula, we further obtain that for any linear operator $S$,

$$
[b, S]f(x) = S((b(x) - b(\cdot))f)(x) = \frac{1}{2\pi \varepsilon} \int_0^{2\pi} S(e^{-i\varepsilon}f(x))e^{i\varepsilon} e^{-i\varepsilon} \, d\theta.
$$

Therefore, for any sequences $\{t_k : k \geq 1\}$, we have

$$
[b, T_{t_k}]f(x) - [b, T_{t_{k-1}}]f(x) = \frac{1}{2\pi \varepsilon} \int_0^{2\pi} (T_{t_k}(h_{\theta})(x) - T_{t_{k-1}}(h_{\theta})(x))e^{i\varepsilon} e^{-i\varepsilon} \, d\theta,
$$

where $h_{\theta}(x) = f(x)e^{-b(x)e^{i\theta}}$ for $\theta \in [0, 2\pi]$. Then using (2.1) and the Minkowski inequality, we have that for $w \in A_p$,

$$
\|V_{\rho} \mathcal{T}_b(f)\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} \left(\sup \left\{\frac{1}{2\pi \varepsilon} \int_0^{2\pi} (T_{t_k}(h_{\theta})(x) - T_{t_{k-1}}(h_{\theta})(x))e^{i\varepsilon} e^{-i\varepsilon} \, d\theta \right\}_{k \geq 1}\right)^p w(x) \, dx\right)^{1/p}.
$$
\[
\begin{align*}
\leq & \frac{1}{2\pi} \int_0^{2\pi} \left( \int_{\mathbb{R}^n} \left( \sup_{k \geq 1} \| T_{k\theta}(h_{\theta})(x) - T_{k\theta-1}(h_{\theta})(x) \|_{L^p} \right)^p e^{\mu b(x) \cos \theta} w(x) dx \right)^{1/p} d\theta \\
= & \frac{1}{2\pi} \int_0^{2\pi} \| V_{\mu} T(h_{\theta}) \|_{L^p(\omega e^{\mu b(x) \cos \theta})} d\theta.
\end{align*}
\] (2.2)

Note that for \( f \in L^p(w) \), it is easy to check that for any \( \theta \in [0, 2\pi] \),
\[
\| h_{\theta} \|_{L^p(\omega e^{\mu b(x) \cos \theta})}^p = \int_{\mathbb{R}^n} | h_{\theta}(x) |^p w(x) e^{\mu b(x) \cos \theta} dx
\]
\[
= \int_{\mathbb{R}^n} | f(x) e^{-\mu b(x) \cos \theta} w(x) e^{\mu b(x) \cos \theta} dx
\]
\[
= \int_{\mathbb{R}^n} | f(x) |^p e^{-\mu b(x) \cos \theta} w(x) e^{\mu b(x) \cos \theta} dx
\]
\[
= \int_{\mathbb{R}^n} | f(x) |^p w(x) dx.
\] (2.3)

Then we get
\[
h_{\theta} \in L^p(\omega e^{\mu b(x) \cos \theta}) \quad \text{and} \quad \| h_{\theta} \|_{L^p(\omega e^{\mu b(x) \cos \theta})} = \| f \|_{L^p(w)}.
\] (2.4)

Therefore, to prove Theorem 1.1, we need to verify \( \omega e^{\mu b(x) \cos \theta} \in A_s \) for appropriate \( \varepsilon \). More precisely, we will choose \( \varepsilon \) depending on \( b \) such that
\[
[e^{\mu b(x) \cos \theta} w]_{A_s} \leq C_1 [w]_{A_s},
\] (2.5)
where \( C_1 \) is independent of \( w \) and \( b \). Hence we have to compute
\[
[e^{\mu b(x) \cos \theta} w]_{A_s} = \sup_Q \left( \frac{1}{|Q|} \int_Q e^{\mu b(x) \cos \theta} w(x) dx \right) \left( \frac{1}{|Q|} \int_Q [e^{\mu b(x) \cos \theta} w(x)]^{-\frac{s}{s-1}} dx \right)^{s-1}.\] (2.6)

Now, since \( w \in A_s \), there exists some \( r_1 > 1 \) such that
\[
\left( \frac{1}{|Q|} \int_Q w^{r_1} dx \right)^{\frac{1}{r_1}} \leq \frac{2}{|Q|} \int_Q w dx
\]
and similarly for \( w^{-\frac{1}{r'}} \in A_{s'} \), there exists some \( r_2 > 1 \) such that
\[
\left( \frac{1}{|Q|} \int_Q w^{-\frac{1}{r'}} dx \right)^{\frac{1}{r'}} \leq \frac{2}{|Q|} \int_Q w^{-1/(s-1)} dx.
\] (2.7)

By this, we know if \( r_2 < r_1 \), by Hölder’s inequality, we get
\[
\left( \frac{1}{|Q|} \int_Q w^{r_2} dx \right)^{\frac{1}{r_2}} \leq \left( \frac{1}{|Q|} \int_Q w^{r_1} dx \right)^{\frac{1}{r_1}} \leq \frac{2}{|Q|} \int_Q w dx.
\] (2.8)

If \( r_1 < r_2 \), by Hölder’s inequality, we get
\[
\left( \frac{1}{|Q|} \int_Q w^{-\frac{1}{r'}} dx \right)^{\frac{1}{r'}} \leq \left( \frac{1}{|Q|} \int_Q w^{-\frac{1}{r'}} dx \right)^{\frac{1}{r'}} \leq \frac{2}{|Q|} \int_Q w^{-\frac{1}{r'}} dx.
\]

Taking \( r = \min\{r_1, r_2\} \) and using (2.7), (2.8) and Hölder’s inequality, we have
\[
[e^{\mu b(x) \cos \theta} w]_{A_s} \leq \sup_Q \left( \frac{1}{|Q|} \int_Q e^{\mu b(x) \cos \theta} w(x) dx \right) \left( \frac{1}{|Q|} \int_Q e^{-\frac{1}{r'}} b(x) \cos \theta w(x)^{-\frac{1}{r'}} dx \right)^{s-1}
\]
\[
\leq \sup_Q \left( \frac{1}{|Q|} \int_Q w'(x) dx \right)^{\frac{1}{2}} \left( \frac{1}{|Q|} \int_Q e^{pr'[b(x) \cos \theta]} dx \right)^{\frac{1}{2}}
\] (2.8)
\[ \left( \frac{1}{|Q|} \int_Q w(x)^{-r/(s-1)} \right)^{\frac{s-1}{r}} \left( \frac{1}{|Q|} \int_Q e^{-\frac{r}{s} r' b(x) \cos \theta} \right)^{\frac{r}{r'}} \leq 4 \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(s-1)} \right)^{s-1} \times \sup_Q \left( \frac{1}{|Q|} \int_Q e^{pr' b(x) \cos \theta} dx \right)^{\frac{1}{p'}} \left( \frac{1}{|Q|} \int_Q e^{-\frac{r}{s} r' b(x) \cos \theta} dx \right)^{\frac{r}{r'}} = 4[w]_{A_s} \left[ e^{pr' b(x) \cos \theta} \right]_{A_s}^{\frac{1}{p' r' \cos \theta}}. \] (2.9)

Now, since \( b \in \text{BMO}(\mathbb{R}^n) \), we apply Lemma 2.1 to do this.

**Lemma 2.1** (See [12, Lemma 2.2]). Let \( 1 < p < \infty \) and \( b \in \text{BMO} \). There exist two-dimensional constants \( \alpha_n \) and \( \beta_n \) satisfying \( 0 < \alpha_n < 1 \) and \( 0 < \beta_n < \infty \) such that for any \( \lambda \in \mathbb{R} \) with \( |\lambda| \leq \frac{\alpha_n}{\|b\|_s} \min\{1, \frac{1}{p-1}\} \), we have

\[ e^{\lambda b} \in A_p \quad \text{and} \quad [e^{\lambda b}]_{A_p} \leq \beta_n^s. \]

Now, we choose the radius

\[ \varepsilon = \frac{\alpha_n \min\{1, \frac{1}{p'}\}}{pr' \|b\|_s}, \] (2.10)

such that

\[ |pr' \varepsilon \cos \theta| \leq \alpha_n \min\left\{1, \frac{1}{s'}\right\}, \]

and then by Lemma 2.1, we get

\[ [e^{pr' b \cos \theta}]_{A_s} \leq \beta_n^s. \] (2.11)

Combining (2.9) and (2.11), we get

\[ [e^{rb \cos \theta} w]_{A_s} \leq 4[w]_{A_s} [e^{pr' b \cos \theta}]_{A_s}^{\frac{1}{p'}} \leq 4[w]_{A_s} \beta_n^s \leq C_1 |w|_{A_s}, \] (2.12)

where \( C_1 \) is independent of \( w \) and \( b \). Now we return to (2.4). By (2.10), (2.12) and the assumption that

\[ \|V_p T(f)\|_{L_p(w)} \leq C \|f\|_{L_p(w)} \quad \text{for} \ w \in A_s, \]

we get

\[ \|V_p T_b(f)\|_{L_p(w)} \leq C \frac{1}{2\pi \varepsilon} \int_0^{2\pi} \|h\|_{L_p(e^{rb \cos \theta} w)} d\theta \]

\[ = C \frac{1}{2\pi \varepsilon} \int_0^{2\pi} \|f\|_{L_p(w)} d\theta \]

\[ \leq C \|b\| \ |f\|_{L_p(w)}. \]

Thus we complete the proof of Theorem 1.1.

## 3 Proof of Theorem 1.6

The focus of this section is to obtain the desired estimate by separately proving the long and short variational estimates, i.e., we intend to show that for \( 1 < p < \infty \),

\[ \|V_p \{[b, T_{\Omega, 2^k}] f]_k\} \|_{L^p} \leq C \|b\| \ |f\|_{L^p}, \] (3.1)

where

\[ [b, T_{\Omega, 2^k}] f(x) := \int_{|x-y| > 2^k} \frac{\Omega(x-y)}{|x-y|^n} f(y)(b(x) - b(y)) dy \]
and

\[ \|S_2(T_{\Omega,b}f)\|_{L^p} \leq C\|b\|_\ast\|f\|_{L^p}, \] (3.2)

where

\[ S_2(T_{\Omega,b}f)(x) = \left( \sum_{j \in \mathbb{Z}} |V_{2,j}(T_{\Omega,b}f)(x)|^2 \right)^{1/2}, \]

with

\[ V_{2,j}(T_{\Omega,b}f)(x) = \left( \sup_{2^j \leq l_0 < \cdots < l_N < 2^{j+1}} \sum_{l=0}^{N-1} \| [b, T_{\Omega,t_{l+1}}] f(x) - [b, T_{\Omega,t_l}] f(x) \|^2 \right)^{1/2}. \]

To deal with the long variation (3.1), we use Fourier transform and some Littlewood-Paley-type estimates involving commutators, in addition to Theorem 1.1. For handling the short variation (3.2), we exploit the Bony decomposition theorem and para-product estimates as well as the Fefferman-Stein inequality for rough maximal functions.

3.1 Proof of (3.1)

Let us begin with the definition that for \( j \in \mathbb{Z} \), let

\[ \nu_j(y) = \frac{\Omega(y)}{|y|^n} \chi_{\{2^j \leq |y| < 2^{j+1}\}}(x). \]

Then

\[ \nu_j * f(x) = \int_{2^j \leq |y| < 2^{j+1}} \frac{\Omega(y)}{|y|^n} f(x-y)dy. \]

Let \( \phi \in \mathcal{S}(\mathbb{R}^n) \) be a radial function such that \( \hat{\phi}(\xi) = 1 \) for \( |\xi| \leq 2 \) and \( \hat{\phi}(\xi) = 0 \) for \( |\xi| > 4 \). We have the following decomposition:

\[ T_{\Omega,2^k}f = \phi_k * T_{\Omega}f + \sum_{s \geq 0} (\delta_0 - \phi_k) * \nu_{k+s} * f - \phi_k * \sum_{s < 0} \nu_{k+s} * f, \]

where \( \phi_k \) satisfies \( \hat{\phi}_k(\xi) = \hat{\phi}(2^k \xi) \), \( \delta_0 \) is the Dirac measure at 0 and \( s \in \mathbb{N} \cup \{0\} \). Then we have

\[ [b, T_{\Omega,2^k}f] = [b, \phi_k * T_{\Omega}f] + \left[ b, \sum_{s \geq 0} (\delta_0 - \phi_k) * \nu_{k+s} \right] f - \left[ b, \phi_k * \sum_{s < 0} \nu_{k+s} \right] f \]

\[ =: T^1_{k,b}f + T^2_{k,b}f - T^3_{k,b}f. \]

Let \( \mathcal{T}^i_{k,b}f \) define the family \( \{T^i_{k,b}f\}_{k \in \mathbb{Z}} \) for \( i = 1, 2, 3 \). Obviously, to show (3.1) it suffices to prove the following inequalities:

\[ \|V_\rho(\mathcal{T}^i_{k,b}f)\|_{L^p} \leq C\|b\|_\ast\|f\|_{L^p}, \quad 1 < p < \infty, \quad i = 1, 2, 3. \] (3.3)

**Estimate of (3.3) for \( i = 1 \).** For this estimate, we need the following two lemmas.

**Lemma 3.1** (See [15, Lemma 2.7]). Let \( \phi_k \) be the same as that given above, and set \( \mathcal{U}f = \{\phi_k * f\}_k \). Hence, for \( 1 < p < \infty \) and \( 2 < p < \infty \),

\[ \|V_\rho(\mathcal{U}f)\|_{L^p} \leq C\|f\|_{L^p}. \]

In [10], we proved that for \( 2 < p < \infty, 1 < p < \infty \) and \( w \in A_p, V_\rho \mathcal{U} \) is bounded on \( L^p(w) \). Applying Theorem 1.1, we can formulate the following lemma.

**Lemma 3.2.** For \( k \in \mathbb{Z} \), let \( \phi_k \) be the same as above and \( b \in \text{BMO}(\mathbb{R}^n) \). Define \( \Phi_k f = \phi_k * f, k \in \mathbb{Z} \), and set \( \mathcal{U}_b f = \{[b, \Phi_k] f\}_k \). Hence, for \( 2 < p < \infty, 1 < p < \infty \) and \( w \in A_p \),

\[ \|V_\rho \mathcal{U}_b(f)\|_{L^p(w)} \leq C\|f\|_{L^p(w)}. \]
Subsequently, for \( k \in \mathbb{Z} \),
\[
T^{I}_{k,b}f = [b, \Phi_{k}][T_{\Omega}f + \Phi_{k}[b, T_{\Omega}]f].
\]
Combining Lemma 3.1 with Lemma 3.2, the \( L^{p} \)-boundedness of \( T_{\Omega} \) (see [5, Theorem 1]) and \([b, T_{\Omega}]\) (see [22, Theorem 1]), we can express the estimate
\[
\| V_{q}^{(\mathcal{P}^{1}_{b})f} \|_{L^{p}_{b}} \leq \| V_{q}^{(\{\Phi_{k}([b, T_{\Omega}f])\})} \|_{L^{p}_{b}} + \| V_{q}^{([b, \Phi_{k}][T_{\Omega}f])} \|_{L^{p}_{b}} \\
\leq C(\| b \|, \| \Phi_{k} \|, \| T_{\Omega}f \|_{L^{p}_{b}}) \\
\leq C(\| \Phi_{k} \|_{L^{1}(\log^{+} L)^{2}(S^{n-1})} \| b \|, \| f \|_{L^{p}_{b}}).
\]

**Estimate of (3.3) for \( i = 2 \).** Let
\[
E_{0} = \{ x' \in S^{n-1} : |\Omega(x')| < 2 \} \quad \text{and} \quad E_{d} = \{ x' \in S^{n-1} : 2^{d} \leq |\Omega(x')| < 2^{d+1} \}
\]
for any positive integer \( d \). For \( d \geq 0 \), let
\[
\Omega_{d}(y') = \Omega(y') \chi_{E_{d}}(y') - \frac{1}{|S^{n-1}|} \int_{E_{d}} \Omega(x')d\sigma(x').
\]
Since \( \Omega \) satisfies the cancellation condition (1.12), we have
\[
\int_{S^{n-1}} \Omega_{d}(y')d\sigma(y') = 0 \quad \text{for} \quad d \geq 0
\]
and
\[
\Omega(y') = \sum_{d \geq 0} \Omega_{d}(y').
\]
Set
\[
\nu_{j,d}(x) = \frac{\Omega_{d}(x)}{|x|^{n}} \chi(2^{d} \leq |x| < 2^{d+1})(x), \quad d \geq 0, \quad j \in \mathbb{Z}.
\]
Then by the Minkowski inequality, we get
\[
V_{q}^{C_{\eta}^{2}(f)}(x) \leq C \sum_{s \geq 0} \left( \sum_{k \in \mathbb{Z}} \| b_{s} (\delta_{0} - \phi_{k}) * \nu_{k+s,j} f(x) \|^{2} \right)^{1/2} \\
\leq C \sum_{s \geq 0} \sum_{d \geq 0} \left( \sum_{k \in \mathbb{Z}} \| b_{s} (\delta_{0} - \phi_{k}) * \nu_{k+s,d} f(x) \|^{2} \right)^{1/2}.
\]
(3.4)
Consequently, let \( \varphi \in C_{c}^{\infty}(\mathbb{R}^{n}) \) be a radial function such that \( 0 \leq \varphi \leq 1 \), \( \text{supp} \ \varphi \subset \{1/2 \leq |\xi| \leq 2\} \) and \( \sum_{l \in \mathbb{Z}} \varphi^{2}(2^{-l} \xi) = 1 \) for \( |\xi| \neq 0 \). Let the multiplier \( \Delta_{l} \) be defined as
\[
\hat{\Delta}_{l}f(\xi) = \varphi(2^{-l} \xi) \hat{f}(\xi) \quad \text{for} \quad l \in \mathbb{Z}.
\]
By \( \sum_{l \in \mathbb{Z}} \Delta_{l}^{2} = I \) (the identity operator) and the Minkowski inequality, we can express
\[
V_{p}^{C_{\eta}^{2}(f)}(x) \leq C \sum_{l \in \mathbb{Z}} \sum_{s \geq 0} \sum_{d \geq 0} \left( \sum_{k \in \mathbb{Z}} \| b_{s} (\delta_{0} - \phi_{k}) * \nu_{k+s,d} \Delta_{l-k}^{2} f(x) \|^{2} \right)^{1/2}.
\]
Define the multipliers \( F_{s,k,d}^{I} \) and \( F_{s,k,d}^{I} \), respectively by
\[
\hat{F}_{s,k,d}^{I}f(\xi) = (1 - \hat{\phi}_{k}(\xi)) \nu_{k+s,d}(\xi) \hat{f}(\xi)
\]
and
\[
\hat{F}_{s,k,d}^{I}f(\xi) = \hat{F}_{s,k,d}f(\xi) \varphi(2^{k-l} \xi) \hat{f}(\xi).
\]
Then
\[ V_\rho F_b^2(f)(x) \leq C \sum_{l \in \mathbb{Z}} \sum_{s \geq 0} \sum_{d \geq 0} \left( \sum_{k \in \mathbb{Z}} \| b, F_{s,k,d} \Delta^2_{k-l} \| f(x) \right)^{1/2}. \]

If we can prove that there exist \( \theta \in (0, 1) \) and \( \beta \in (0, 1) \) such that
\[ \left\| \left( \sum_{k \in \mathbb{Z}} \| b, F_{s,k,d} \Delta^2_{k-l} \| f(x) \right)^{1/2} \right\|_{L^2} \leq C \| \Omega_d \|_{L^\infty(S^{n-1})} 2^{-\beta} 2^{-\theta|l|} \| b \|_* \| f \|_{L^2} \]  
(3.5)
and for \( 1 < p < \infty \),
\[ \left\| \left( \sum_{k \in \mathbb{Z}} \| b, F_{s,k,d} \Delta^2_{k-l} \| f(x) \right)^{1/2} \right\|_{L^p} \leq C \| \Omega_d \|_{L^\log + L(S^{n-1})} \| b \|_* \| f \|_{L^p}, \]  
(3.6)
then we may have completed the estimate proof of (3.3) for \( i = 2 \). In fact, interpolating between (3.5) and (3.6) gives that for \( 0 < \theta_0, \beta_0 < 1 \) and \( 1 < p < \infty \),
\[ \left\| \left( \sum_{k \in \mathbb{Z}} \| b, F_{s,k,d} \Delta^2_{k-l} \| f(x) \right)^{1/2} \right\|_{L^\infty(S^{n-1})} \| f \|_{L^p}. \]  
(3.7)
Taking some ideas from [22, p. 23], we take a large positive integer \( N \) such that \( N > \max\{2\theta_0^{-1}, 2\beta_0^{-1}\} \).

For \( J_1 \), by (3.6) and (3.7), we get
\[
J_1 \leq C \| b \|_* \| f \|_{L^p} \sum_{d \geq 0} \sum_{l \in \mathbb{Z}} \left( \sum_{s \leq N \Delta} d^{2\sigma(E_d)} + \sum_{|l| \leq N \Delta} 2^{d^2 - \theta_0|l|} \right)
\leq C \| b \|_* \| f \|_{L^p} \left( \sum_{d \geq 0} d^{3\sigma(E_d)} + \sum_{d \geq 0} d^{2(1-\beta_0)N} \right)
\leq C \| \Omega \|_{L^{\log + L}(S^{n-1})} \sum_{d \geq 0} \| b \|_* \| f \|_{L^p}.
\]

For \( J_2 \), using (3.7), we get
\[
J_2 \leq C \| b \|_* \sum_{d \geq 0} \sum_{s > N \Delta} 2^{d - \beta_0} \left( \sum_{|l| \leq N \Delta} \sum_{|l| \geq N \Delta} 2^{-\theta_0|l|} \right) \| f \|_{L^p}
\leq C \| b \|_* \sum_{d \geq 0} \left( d^{2(1-\beta_0)N} + d^{2(1-\beta_0N-\theta_0N)|d|} \right) \| f \|_{L^p}
\leq C \| b \|_* \| f \|_{L^p}.
\]

Finally, combining the two estimates gives the relationship, and we get that for \( 1 < p < \infty \),
\[ \| V_\rho F_b^2(f) \|_{L^p} \leq C \left( 1 + \| \Omega \|_{L^{\log + L}(S^{n-1})} \right) \| b \|_* \| f \|_{L^p}. \]

We therefore finish the estimate of (3.3) for \( i = 2 \).

Now, let us resume proving (3.5) and (3.6). We first prove (3.5). We will begin with the statement
\[ F^t_{s,k,d} = F_{s,k,d} \Delta_{t-k}. \]
We write
\[ [b, F_{s,k,d} \Delta^2_{l-k}] f = [b, F_{s,k,d}^i] \Delta_{l-k} f + F_{s,k,d}^i [b, \Delta_{l-k}] f. \]

Therefore,
\[
\left\| \left( \sum_{k \in \mathbb{Z}} [b, F_{s,k,d} \Delta^2_{l-k}] f \right)^2 \right\|_{L^2}^{1/2} \leq C \left\| \left( \sum_{k \in \mathbb{Z}} [b, F_{s,k,d}^i] \Delta_{l-k} f \right)^2 \right\|_{L^2}^{1/2} + C \left\| \left( \sum_{k \in \mathbb{Z}} [b, \Delta_{l-k}] f \right)^2 \right\|_{L^2}^{1/2}.
\]

To proceed with (3.8), we need the following lemmas.

**Lemma 3.3** (See [22, Lemma 1]). Let \( b \in \text{BMO}(\mathbb{R}^n) \). Then for \( 1 < p < \infty \) and \( f \in L^p(\mathbb{R}^n) \), we have
(i) \( \| (\sum_{i \in \mathbb{Z}} [b, \Delta_i] f)^{1/2} \|_{L^p} \leq C(n, p) \| b \|_{\text{BMO}} \| f \|_{L^p} \);
(ii) \( \| (\sum_{i \in \mathbb{Z}} [b, \Delta_i] f)^{1/2} \|_{L^p} \leq C(n, p) \| b \|_{\text{BMO}} \| f \|_{L^p} \).

**Lemma 3.4** (See [22, Lemma 2]). Let \( m \in C_0^\infty(\mathbb{R}^n) \) and \( \text{supp} \, m \subset \{|\xi| \leq 2\sigma\} \) for some \( \sigma \in (0, \infty) \).

Suppose that \( m \) satisfies
\[
\| m \|_{L^\infty} \leq C 2^{-\gamma s} \min \{ \sigma, \sigma^{-\lambda} \}, \quad \| \nabla m \|_{L^\infty} \leq C 2^s
\]
for some constants \( C, \lambda, \gamma > 0 \) and \( s \in \mathbb{N} \). Let \( T_m \) be the multiplier operator defined by
\[
T_m f(\xi) = m(\xi) \hat{f}(\xi).
\]

Moreover, for \( b \in \text{BMO} \) and \( u \in \mathbb{N} \), denote by
\[
T_{m,b,u} f(x) = T_m ((b(x) - b(\cdot))^u f)(x)
\]
the \( u \)-th order commutator of \( T_m \). Then for any fixed \( 0 < v < 1 \), there exist positive constants \( C = C(n, u, v) \) and \( \beta \in (0, 1) \) such that
\[
\| T_{m,b,u} f \|_{L^v} \leq C 2^{-\beta s} \min \{ \sigma^v, \sigma^{-\lambda v} \} \| b \|_{\text{BMO}} \| f \|_{L^v}.
\]

Now, fix \( l \in \mathbb{Z} \). We let
\[
m_{s,k,d}^l(\xi) = (1 - \hat{\phi}_k(\xi)) \hat{\nu}_{k+s,d}(\xi) \varphi(2^{-l} \xi)
\]
and define the multiplier \( \hat{F}_{s,k,d}^l \) by
\[
\hat{F}_{s,k,d}^l(\xi) = m_{s,k,d}^l(2^{-k} \xi) \hat{f}(\xi).
\]

Since
\[
\text{supp} (1 - \hat{\phi}_k) \hat{\nu}_{k+s,d} \subset \{ \xi : |2^k \xi| > 1/2 \},
\]
by a well-known Fourier transform estimate of Duoandikoetxea and Rubio de Francia [17, pp. 551–552], it is easy to show that there is a \( \gamma \in (0, 1) \) such that
\[
\| (1 - \hat{\phi}_k(\xi)) \hat{\nu}_{k+s,d}(\xi) \|_{L^\infty(S^{n-1})} \leq C \Omega_d \| \nu_{k+s,d} \|_{L^\infty(S^{n-1})} 2^{\gamma - \gamma} \min \{|2^k \xi|, \|2^k \xi\|^{-\gamma}\}
\]
and
\[
\| \nabla [(1 - \hat{\phi}_k(\xi)) \hat{\nu}_{k+s,d}(\xi)] \|_{L^\infty(S^{n-1})} \leq C \Omega_d \| \nu_{k+s,d} \|_{L^\infty(S^{n-1})} 2^k 2^s.
\]

As a result, we have the following estimates:
\[
\text{supp} m_{s,k,d}^l(2^{-k} \xi) \subset \{ \xi : \xi \leq 2^l \}, \quad (3.9)
\]
\[
|m_{s,k,d}^l(2^{-k} \xi)| \leq C \| \Omega_d \|_{L^\infty(S^{n-1})} 2^{-\gamma s} \min \{2^l, 2^{-\gamma l}\} \quad (3.10)
\]
and
\[ |\nabla m_{s,k,d}^t(2^{-k}\xi)| \leq C\|\Omega_d\|_{L^\infty(S^{n-1})}2^s. \] (3.11)

Applying Lemma 3.4 with \( \sigma = 2^l \) to (3.9)–(3.11), there exist constants \( \beta \in (0,1) \), \( \theta \in (0,1) \) and \( C = C(n, \theta) \) such that for \( l \in \mathbb{Z} \) and \( s \geq 0 \),
\[ \| [b, F^l_{s,k,d}] f \|_{L^2} \leq C \| \Omega_d \|_{L^\infty(S^{n-1})} \| b \|_1 2^{-\beta s} 2^{-\theta l} \| f \|_{L^2}. \]

Furthermore, for \( l \in \mathbb{Z} \) and \( s \geq 0 \), the dilation-invariance implies the following:
\[ \| [b, F^l_{s,k,d}] f \|_{L^2} \leq C \| \Omega_d \|_{L^\infty(S^{n-1})} \| b \|_1 2^{-\beta s} 2^{-\theta l} \| f \|_{L^2}. \] (3.12)

Also by the Plancherel theorem to (3.10) and the dilation-invariance, we get that for \( l \in \mathbb{Z} \) and \( s \geq 0 \),
\[ \| F^l_{s,k,d} f \|_{L^2} \leq C \| \Omega_d \|_{L^\infty(S^{n-1})} 2^{-\gamma s} \min\{2^l, 2^{-\gamma l}\} \| f \|_{L^2}. \] (3.13)

Then by (3.12)–(3.13) and Lemma 3.3, we get
\[
\left\| \left( \sum_{k \in \mathbb{Z}} |[b, F_{s,k,d}\Delta^{2}_{-k} f]|^2 \right)^{1/2} \right\|_{L^p} \\
\leq C \left\| \Omega_d \|_{L^\infty(S^{n-1})} 2^{-\beta s} 2^{-\theta l} \| b \|_1 \left\| \sum_{k \in \mathbb{Z}} |\Delta_{-k} f|^2 \right\|_{L^2}^2 \\
+ C \left\| \Omega_d \|_{L^\infty(S^{n-1})} 2^{-\gamma s} 2^{-\gamma l} \left\| \sum_{k \in \mathbb{Z}} |[b, \Delta_{-k} f]|^2 \right\|_{L^2}^2 \right\|_{L^p} \\
\leq C \left\| \Omega_d \|_{L^\infty(S^{n-1})} 2^{-\beta s} 2^{-\theta l} \| b \|_1 \| f \|_{L^2}, \right. \]

which completes the proof of (3.5).

We can now prove (3.6) in a similar manner by first defining
\[ [b, F_{s,k,d}\Delta^{2}_{-k} f](x) = [b, F_{s,k,d}]\Delta^{2}_{-k} f + F_{s,k,d}[b, \Delta^{2}_{-k} f]. \]

By the Minkowski inequality, we get
\[
\left\| \left( \sum_{k \in \mathbb{Z}} |[b, F_{s,k,d}\Delta^{2}_{-k} f]|^2 \right)^{1/2} \right\|_{L^p} \\
\leq C \left\| \left( \sum_{k \in \mathbb{Z}} |[b, F_{s,k,d}]\Delta^{2}_{-k} f|^2 \right)^{1/2} \right\|_{L^p} + C \left\| \left( \sum_{k \in \mathbb{Z}} |F_{s,k,d}[b, \Delta^{2}_{-k} f]|^2 \right)^{1/2} \right\|_{L^p}. \]

Recall that \( F_{s,k,d} f(x) = (\delta_0 - \phi_k) * \nu_{k+s} * f(x) \). In fact, by
\[ \sup_{k \in \mathbb{Z}} |[b, F_{s,k,d}] f_k| \leq C \left( M_b M_{\Omega_d} + M_{M_{\Omega_d} b} \right) \left( \sup_{k \in \mathbb{Z}} |f_k| \right), \]
where \( M \) is the Hardy-Littlewood maximal operator, \( M_b \) is the commutator of the Hardy-Littlewood maximal operator,
\[ M_{\Omega_d} f(x) = \sup_{r > 0} \frac{1}{r^n} \int_{|x-y| < r} |f(y)| |\Omega_d(x-y)| dy \]
and
\[ M_{\Omega_d, b} f(x) = \sup_{r > 0} \frac{1}{r^n} \int_{|x-y| < r} |f(y)||b(x) - b(y)||\Omega_d(x-y)| dy. \]

Then we can get that for \( 1 < p < \infty \),
\[ \left\| \sup_{k \in \mathbb{Z}} |[b, F_{s,k,d}] f_k| \right\|_{L^p} \leq C \left\| \Omega_d \|_{L^{p} \log^{+} L(S^{n-1})} \| b \|_1 \sup_{k \in \mathbb{Z}} |f_k| \right\|_{L^p}^1. \]
where we have used that for $1 < p < \infty$,
\[
\|M_{\Omega_d}f\|_{L^p} \leq C\|b\|_\ast \|\Omega_d\|_{L^{\log^+ L}(S^{n-1})}\|f\|_{L^p}
\]
(see [22, Lemma 4]) and
\[
\|M_{\Omega_d}f\|_{L^p} \leq C\|\Omega_d\|_{L^1(S^{n-1})}\|f\|_{L^p}
\]
(see [22, p. 18]).

By duality, we get that for $1 < p < \infty$,
\[
\left\| \sum_{k\in\mathbb{Z}} [b, F_{s,k,d}]f_k \right\|_{L^p} \leq C\|\Omega_d\|_{L^{\log^+ L}(S^{n-1})}\|b\|_\ast \left\| \sum_{k\in\mathbb{Z}} |f_k| \right\|_{L^p}.
\]

Interpolating between the two inequalities above, we can get that for $1 < p < \infty$,
\[
\left\| \left( \sum_{k\in\mathbb{Z}} [b, F_{s,k,d}]\Delta^2_{-k}f|^2 \right)^{1/2} \right\|_{L^p} \leq C\|\Omega_d\|_{L^{\log^+ L}(S^{n-1})}\|b\|_\ast \left\| \sum_{k\in\mathbb{Z}} |f_k|^2 \right\|_{L^p}^{1/2},
\]
where $C$ is independent of $\{f_k\}$.

Thus by the Littlewood-Paley theorem, we get that for $1 < p < \infty$,
\[
\left\| \left( \sum_{k\in\mathbb{Z}} [b, F_{s,k,d}]\Delta^2_{-k}f|^2 \right)^{1/2} \right\|_{L^p} \leq C\|\Omega_d\|_{L^{\log^+ L}(S^{n-1})}\|b\|_\ast \|f\|_{L^p}.
\]  

(3.14)

Also for $1 < p < \infty$,
\[
\left\| \left( \sum_{k\in\mathbb{Z}} |F_{s,k,d}f_k|^2 \right)^{1/2} \right\|_{L^p} \leq \Omega_d\|f\|_{L^1(S^{n-1})}\left\| \sum_{k\in\mathbb{Z}} |f_k|^2 \right\|_{L^p}^{1/2},
\]
(see [22]). Then by Lemma 3.3, we get that for $1 < p < \infty$,
\[
\left\| \left( \sum_{k\in\mathbb{Z}} |F_{s,k,d}[b, \Delta^2_{-k}]f|^2 \right)^{1/2} \right\|_{L^p} \leq C\|\Omega_d\|_{L^1(S^{n-1})}\|b\|_\ast \|f\|_{L^p},
\]
which, combined with (3.14), gives (3.6).

**Estimate of (3.3) for $i = 3$.** We have the following pointwise estimate:
\[
V_p\mathcal{F}_0^4(f)(x) \leq \sum_{s<0} \left( \sum_{k\in\mathbb{Z}} |b, \phi_k \ast \nu_{k+s}|f(x)|^2 \right)^{1/2}.
\]

The proofs are essentially similar to the proof of (3.3) for $i = 2$. More precisely, the estimates on the left-hand side of (3.5)–(3.6) are obtained by replacing $(\delta_0 - \phi_k) \ast \nu_{k+s}$ by $\phi_k \ast \nu_{k+s}$. Since
\[
\text{supp}\phi_k \nu_{k+s} \subset \{\xi : |2^k \xi| < 1\}
\]
and $\Omega$ satisfies the cancellation condition (1.12), it is easy to see that
\[
|\phi_k \ast \nu_{k+s}(\xi)| \leq C2^k \|\Omega\|_{L^1(S^{n-1})} \min\{2^k |\xi|, 2^k |\xi|^{-1}\}
\]
and
\[
|\nabla \nu_{k+s}(\xi)| \leq C2^{(k+s)} \|\Omega\|_{L^1(S^{n-1})}.
\]

Set
\[
R_{s,k}(\xi) = \phi_k(\xi) \nu_{k+s}(\xi), \quad R_{s,k}(\xi) = R_{s,k}(\xi) \nu(2^{k-1} \xi).
\]
Using the two inequalities above, we can get the estimate
\[
\text{supp} R_{s,k}^i(2^{-k}\xi) \subset \{|\xi| \leq 2^i\},
\]
\[
|R_{s,k}^i(2^{-k}\xi)| \leq C 2^s \min\{2^s, 2^{-1}\} \|\Omega\|_{L^1(S^{n-1})}
\]
and
\[
|\nabla (R_{s,k}^i(2^{-k}\xi))| \leq C 2^s \|\Omega\|_{L^1(S^{n-1})},
\]
By applying Lemma 3.4 and the same arguments of the proofs of (3.3) for \(i = 2\), the right-hand side of (3.5) is then controlled by
\[
C 2^{s} 2^{-\theta \|\nu\|_{L^1(S^{n-1})}} \|b\|_p \|f\|_{L^p}
\]
for \(\theta > 0\). It is easy to get the same estimates on the right-hand side of (3.6) by using Lemma 3.3 and the results in [20, 22]. Then we get that for \(1 < p < \infty\),
\[
\|V_{\rho}^2 F_0^j(f)\|_{L^p} \leq C \|\Omega\|_{\text{Log}^+ L^p(S^{n-1})} \|b\|_p \|f\|_{L^p}.
\]
This completes the proof of (3.3) for \(i = 3\).

3.2 Proof of (3.2)

We begin the proof of (3.2). For \(t \in [1, 2)\), we define \(\nu_{0,t}\) as
\[
\nu_{0,t}(x) = \frac{\Omega(x')}{|x'|^n} \chi_{\{t \leq |x| \leq 2\}}(x) \quad \text{and} \quad \nu_{j,t}(x) = 2^{-jn} \nu_{0,t}(2^{-j}x)
\]
for \(j \in \mathbb{Z}\). Define \(T_{j,t}\) to be
\[
T_{j,t} f(x) = \nu_{j,t} * f(x).
\]
Recall that
\[
S_2(T_{\Omega,b} f)(x) = \left( \sum_{j \in \mathbb{Z}} |V_{2,j}(T_{\Omega,b} f)(x)|^2 \right)^{1/2}
\]
with
\[
V_{2,j}(T_{\Omega,b} f)(x) = \left( \sup_{2^{i<j} \leq ... \leq 2^{i+1}} \sum_{i=0}^N \|b,T_{\Omega,i+1} f(x) - b,T_{\Omega,i} f(x)|^2 \right)^{1/2}.
\]
Observe that \(V_{2,j}(T_{\Omega,b} f)(x)\) is just the strong 2-variation function of the family \(\{b,T_{j,t} f(x)\}_{t \in [1,2]}\) and \(\sum_{k \in \mathbb{Z}} \Delta_k^2 = I\) (the identity operator). Hence,
\[
S_2(T_{\Omega,b} f)(x) = \left( \sum_{j \in \mathbb{Z}} |V_{2,j}(T_{\Omega,b} f)(x)|^2 \right)^{1/2}
\]
\[
= \left( \sum_{j \in \mathbb{Z}} \|\{b,T_{j,t} f(x)\}_{t \in [1,2]}\|_{V_2}^2 \right)^{1/2}
\]
\[
\leq C \sum_{k \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}} \|\{b,T_{j,t} \Delta_{k-j}^2 f(x)\}_{t \in [1,2]}\|_{V_2}^2 \right)^{1/2}.
\]
Decompose \(\Omega = \sum_{d \geq 0} \Omega_d\) as in the estimate of (3.3) for \(i = 2\). For \(d \geq 0\) and \(j \in \mathbb{Z}\), set
\[
\nu_{j,t,d}(x) = \frac{\Omega_d(x)}{|x'|^n} \chi_{\{2^{j+t}\leq |x| < 2^{j+t+1}\}}(x) \quad \text{and} \quad T_{j,t,d} = \nu_{j,t,d} * f(x).
\]
where as those of (3.3) for and for 1

\[ S_2(\Omega;b,f)(x) \leq C \sum_{d \geq 0} \sum_{k \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}} \left\| \{ [b, T_{j,t,d} \Delta^2_{k,j}] f(x) \}_{t \in [1,2]} \right\|_{T_1}^2 \right)^{1/2} =: C \sum_{d \geq 0} \sum_{k \in \mathbb{Z}} S_{2,k,d}(\Omega;b,f)(x), \]

where

\[ S_{2,k,d}(\Omega;b,f)(x) := \left( \sum_{j \in \mathbb{Z}} \left\| \{ [b, T_{j,t,d} \Delta^2_{k,j}] f(x) \}_{t \in [1,2]} \right\|_{T_1}^2 \right)^{1/2}. \]

Now, if there exists a constant \( \theta \in (0, 1) \) such that

\[ \| S_{2,k,d}(\Omega;b,f) \|_{L^2} \leq C \| \Omega \|_{L^\infty(\mathbb{S}^{n-1})} 2^{-\theta|k|} \| b \|_{L^\infty} \| f \|_{L^2}; \]

and for \( 1 < p < \infty \),

\[ \| S_{2,k,d}(\Omega;b,f) \|_{L^p} \leq C \| \Omega \|_{L^{\log^+ L}(\mathbb{S}^{n-1})} \| b \|_{L^\infty} \| f \|_{L^p}, \]

where the constants \( C \)'s in (3.15) and (3.16) are independent of \( k \) and \( d \), then using the same arguments as those of (3.3) for \( i = 2 \), we may finish the proof of (3.2). The details of the proof are omitted.

Next, we will estimate (3.15) and (3.16), respectively. Without loss of generality, we will use \( S_{2,k} \) to replace \( S_{2,k,d} \) and \( \Omega \) to replace \( \Omega_d \).

**Proof of (3.15).** We can borrow the fact that

\[ \| a \|_{V_2} \leq \| a \|_{L^2} \| a' \|_{L^2}^{1/2}, \]

where \( a' = \{ \frac{d}{dt} a_t : t \in \mathbb{R} \} \). This is a special case of [28, p. 24, (39)]. Then

\[ |S_{2,k}(\Omega;b,f)(x)|^2 \leq \sum_{j \in \mathbb{Z}} \left( \int_1^2 \left\| b, T_{j,t} \Delta^2_{k,j} f(x) \right\|^2 dt \right)^{1/2} \left( \int_1^2 \left\| [b, \frac{d}{dt} T_{j,t} \Delta^2_{k,j}] f(x) \right\|^2 dt \right)^{1/2} =: \sum_{j \in \mathbb{Z}} I_{1,k,j}(f) \cdot I_{2,k,j}(f), \]

where

\[ I_{1,k,j}(f) := \left( \int_1^2 \left\| b, T_{j,t} \Delta^2_{k,j} f(x) \right\|^2 dt \right)^{1/2} \]

and

\[ I_{2,k,j}(f) := \left( \int_1^2 \left\| [b, \frac{d}{dt} T_{j,t} \Delta^2_{k,j}] f(x) \right\|^2 dt \right)^{1/2}. \]

By the Cauchy-Schwarz inequality, we have

\[ \| S_{2,k}(\Omega;b,f) \|_{L^2}^2 \leq \left( \sum_{j \in \mathbb{Z}} |I_{1,k,j}(f)|^2 \right)^{1/2} \left( \sum_{j \in \mathbb{Z}} |I_{2,k,j}(f)|^2 \right)^{1/2}. \]

Now, let us estimate \( \| \sum_{j \in \mathbb{Z}} |I_{1,k,j}(f)|^2 \|_{L^2} \) and \( \| \sum_{j \in \mathbb{Z}} |I_{2,k,j}(f)|^2 \|_{L^2}, \) respectively. Recall that

\[ T_{j,t} f(x) = \nu_{j,t} * f(x). \]

To estimate

\[ \left\| \sum_{j \in \mathbb{Z}} |I_{1,k,j}(f)|^2 \right\|_{L^2}^{1/2}, \]

we need the following estimates: for some \( \gamma > 0 \),

\[ |\nu_{j,t}(\xi)| \leq C \| \Omega \|_{L^\infty(\mathbb{S}^{n-1})} \min \{ |2^j \xi |^{-\gamma}, 2^j |\xi| \}. \]
and
\[
|\nabla \varphi_{1,t}(\xi)| \leq C2^j \|\Omega\|_{L^\infty(S^{n-1})}
\]
uniformly in \( t \in [1, 2) \), which have been essentially proved in [17, pp. 551–552] and [22, p. 22]. Similar to the proof of (3.5), we get that for some \( v > 0 \),
\[
\left\| \left( \sum_{j \in \mathbb{Z}} |I_{1,k,j}f|^2 \right)^{1/2} \right\|_{L^2} \leq C\|\Omega\|_{L^\infty(S^{n-1})} \|b\|_* \min\{2^{|k|}, 2^{-\gamma v}\} \|f\|_{L^2}.
\] (3.17)
Next, we estimate
\[
\left\| \left( \sum_{j \in \mathbb{Z}} |I_{2,k,j}f|^2 \right)^{1/2} \right\|_{L^2}.
\]
Write
\[
\left[ b, \frac{d}{dt} T_{j,t} \Delta^2_{k-j} \right] f = \left[ b, \Delta_{k-j} \right] \frac{d}{dt} T_{j,t} \Delta_{k-j} f + \Delta_{k-j} \left[ b, \frac{d}{dt} T_{j,t} \right] \Delta_{k-j} f + \frac{d}{dt} \Delta_{k-j} \left[ b, T_{j,t} \right] \Delta_{k-j} f.
\]
Thus, we get
\[
\left\| \left( \sum_{j \in \mathbb{Z}} |I_{2,k,j}f|^2 \right)^{1/2} \right\|_{L^2} \leq \left( \int_1^2 \left\| \left( \sum_{j \in \mathbb{Z}} |b, \Delta_{k-j} \Delta_{k-j} f|^2 \right)^{1/2} \right\|_{L^2} \left( \frac{dt}{t} \right)^{1/2}
\right)
\]
\[
+ \left( \int_1^2 \left\| \left( \sum_{j \in \mathbb{Z}} \Delta_{k-j} \left[ b, \frac{d}{dt} T_{j,t} \right] \Delta_{k-j} f \right|^2 \right\|_{L^2} \left( \frac{dt}{t} \right)^{1/2}
\right)
\]
\[
+ \left( \int_1^2 \left\| \left( \sum_{j \in \mathbb{Z}} \frac{d}{dt} \Delta_{k-j} \left[ b, T_{j,t} \right] \Delta_{k-j} f \right|^2 \right\|_{L^2} \left( \frac{dt}{t} \right)^{1/2}
\right)
\]
\[
=: I + II + III.
\]
To estimate \( I, II \) and \( III \), respectively, we need the following elementary fact:
\[
\frac{d}{dt} T_{j,t} h(x) = \frac{d}{dt} \left[ \int_{|y| \leq 2^{t+1}} \frac{\Omega(y)}{|y|^n} h(x - y) dy \right]
\]
\[
= \frac{d}{dt} \left[ \int_{S^{n-1}} \Omega(y') \int_{2^{t+1}}^{2^{t+1}} \frac{1}{r} h(x - ry') dr d\sigma(y') \right]
\]
\[
= -\frac{1}{t} \int_{S^{n-1}} \Omega(y') h(x - 2^t y') d\sigma(y')
\] (3.18)
and
\[
\|T^*_j h\|_{L^2} \leq C\|\Omega\|_{L^1(S^{n-1})} \|h\|_{L^2},
\] (3.19)
where
\[
T^*_j h(x) = \int_{S^{n-1}} |\Omega(y')| h(x - 2^t y') d\sigma(y')
\]
for \( t \in [1, 2) \). We then estimate \( I \). Indeed, by applying (3.19), Lemma 3.3 and Littlewood-Paley theory, we obtain
\[
I \leq C\|b\|_* \left( \int_1^2 \left\| \sum_{j \in \mathbb{Z}} |T^*_j \Delta_{k-j} f|^2 \right\|_{L^2} \left( \frac{dt}{t} \right)^{1/2}
\right)
\]
\[
\leq C\|b\|_* \|\Omega\|_{L^1(S^{n-1})} \left\| \left( \sum_{j \in \mathbb{Z}} |\Delta_{k-j} f|^2 \right) \right\|_{L^2}
\]
\[
\leq C\|b\|_* \|\Omega\|_{L^1(S^{n-1})} \|f\|_{L^2}.
\]
Similarly, we get
\[ III \leq C\|b\|_s\|\Omega\|_{L^1(\mathbb{S}^{n-1})}\|f\|_{L^2}. \]
For \(II\), we will apply the Bony para-product to do this. Let \(\varpi \in \mathcal{S}(\mathbb{R}^n)\) be a radial function satisfying \(0 \leq \varpi \leq 1\) with its support in the unit ball and \(\varpi(\xi) = 1\) for \(|\xi| \leq \frac{1}{2}\). The function
\[ \psi(\xi) = \varpi\left(\frac{\xi}{2}\right) - \varpi(\xi) \in \mathcal{S}(\mathbb{R}^n) \]
is supported on \(\{\frac{1}{2} \leq |\xi| \leq 2\}\) and satisfies the identity \(\sum_{j \in \mathbb{Z}} \psi(2^{-j} \xi) = 1\) for \(\xi \neq 0\). For \(j \in \mathbb{Z}\), denote by \(\Theta_j\) and \(G_j\) the convolution operators whose symbols are \(\psi(2^{-j} \xi)\) and \(\varpi(2^{-j} \xi)\), respectively, i.e., \(\Theta_j\) and \(G_j\) are defined by
\[ \Theta_j f(\xi) = \psi(2^{-j} \xi) \hat{f}(\xi) \quad \text{and} \quad G_j f(\xi) = \varpi(2^{-j} \xi) \hat{f}(\xi) \]
(see [42, p. 86] or [20]). The para-product of Bony [2] between two functions \(f\) and \(g\) is defined by
\[ \pi_j(f, g) = \sum_{j \in \mathbb{Z}} (\Theta_j f)(G_{j-3} g). \]
Our formal Bony decomposition theorem is therefore expressed as
\[ f g = \pi_0(f, g) + \pi_1(f, g) + R(f, g) \quad \text{with} \quad R(f, g) = \sum_{i \leq 2} \sum_{|k-i| \leq 2} (\Theta_i f)(\Theta_k g). \tag{3.20} \]
Letting \(f_{k,j} := \Delta_{k-j} f\) and \(T_{j,t}f := \frac{dt}{t} T_{j,t} f\) and applying these in (3.20), we have
\[
[b, T_{j,t}^\prime] f_{k,j}(x) = b(x)(T_{j,t}^\prime f_{k,j})(x) - T_{j,t}^\prime (b f_{k,j})(x)
= [\pi T_{j,t}^\prime f_{k,j}(b)(x) - T_{j,t}^\prime (\pi f_{k,j}(b))(x)]
+ [R(b, T_{j,t} f_{k,j})(x) - T_{j,t}^\prime (R(b, f_{k,j}))(x)]
+ [\pi b(T_{j,t} f_{k,j})(x) - T_{j,t}^\prime (\pi b(f_{k,j}))(x)].
\]
Thus, we now have an estimate for \(II\) as
\[
II \leq \left( \int_1^2 \left\| \sum_{j \in \mathbb{Z}} |\Delta_{k-j}| |\pi T_{j,t}^\prime f_{k,j}(b) - T_{j,t}^\prime (\pi f_{k,j}(b))| \right\|_{L^2}^2 \frac{dt}{t} \right)^{1/2}
+ \left( \int_1^2 \left\| \sum_{j \in \mathbb{Z}} |\Delta_{k-j}| |R(b, T_{j,t} f_{k,j}) - T_{j,t}^\prime (R(b, f_{k,j}))| \right\|_{L^2}^2 \frac{dt}{t} \right)^{1/2}
+ \left( \int_1^2 \left\| \sum_{j \in \mathbb{Z}} |\Delta_{k-j}| |\pi b(T_{j,t} f_{k,j}) - T_{j,t}^\prime (\pi b(f_{k,j}))| \right\|_{L^2}^2 \frac{dt}{t} \right)^{1/2}
=: II_1 + II_2 + II_3.
\]
We will estimate \(II_i, i = 1, 2, 3\), respectively as will be discussed below. For \(II_1\), noting that
\[ \Theta_i \Delta_{k-j} g = 0 \]
for \(g \in \mathcal{S}''(\mathbb{R}^n)\) when \(|i - (k - j)| \geq 3\), by (3.18), we get
\[
|\pi T_{j,t}^\prime f_{k,j}(b)(x) - T_{j,t}^\prime (\pi f_{k,j}(b))(x)|
= \sum_{i \in \mathbb{Z}} \left\{ [T_{j,t}^\prime \Theta_i \Delta_{k-j} f](G_{i-3} b)(x) - T_{j,t}^\prime (\Theta_i \Delta_{k-j} f)(G_{i-3} b)(x)] \right\}
= \sum_{|i-(k-j)| \leq 2} \frac{1}{7} \int_{\mathbb{S}^{n-1}} \Omega(y)'(G_{i-3} b(x) - G_{i-3} b(x - 2^i t y'))(\Theta_i \Delta_{k-j} f)(x - 2^i t y') d\sigma(y'). \tag{3.21}
\]
To estimate the above inequality, we need the following lemma.
Lemma 3.5 (See [9, Lemma 3.2]). For any fixed $0 < \tau < 1/2$, we have

$$|G_k b(x) - G_k b(y)| \leq C \frac{q^{k \tau}}{\tau} |x - y|^\tau \|b\|_*,$$

where $C$ is independent of $k$ and $\tau$.

By Lemma 3.5, note that for $t \in [1, 2)$, we get

$$\left| \frac{1}{t} \int_{S^{n-1}} \Omega(y'(G_{i-3} b(x) - G_{i-3} b(x - 2^1 t y'))(\Theta_i \Delta_{k-j} f)(x - 2^1 t y') d\sigma(y') \right| \leq C \frac{q^{(i+j)\tau}}{\tau} \|b\|_* \int_{S^{n-1}} |\Omega(y')||(\Theta_i \Delta_{k-j} f)(x - 2^1 t y')| d\sigma(y')$$

$$ \leq C \frac{q^{(i+j)\tau}}{\tau} \|b\|_* \|T_{j,t}^* (\Theta_i \Delta_{k-j} f)(x).$$

Then by (3.21)–(3.22), (3.19) and Littlewood-Paley theory, we have

$$II_1 \leq C \|b\|_* \left[ \frac{q^{k \tau}}{\tau} \sum_{|l| \leq 2} \left( \left( \sum_{j \in \mathbb{Z}} |T_{j,l}^* (\Theta_{k-j+l} \Delta_{k-j} f)|^2 \right)^{1/2} \right)^2 \right]^{1/2} \left( \sum_{|l| \leq 2} dt \right)^{1/2}$$

$$\leq C \frac{q^{k \tau}}{\tau} \|b\|_* \|\Omega\|_{L^1(S^{n-1})} \|f\|_{L^2},$$

where $C$ is independent of $k$ and $\tau$.

Next, we estimate $II_2$. Clearly, $\Theta_{i+l} \Delta_{k-j} g = 0$ for $g \in \mathcal{S}'(\mathbb{R}^n)$ when $|l| \leq 2$ and $|i - (k-j)| \geq 8$. Thus, by (3.18),

$$[R(b, T_{j,t}^* f_{k,j})(x) - T_{j,t}^* (R(b, f_{k,j}))(x)]$$

$$= \sum_{i \in \mathbb{Z}} \sum_{|l| \leq 2} \left( \Theta_{i+l} (T_{j,t} \Theta_{i+l} \Delta_{k-j} f)(x) - T_{j,t} \left( \sum_{i \in \mathbb{Z}} \sum_{|l| \leq 2} \Theta_{i+l} \Delta_{k-j} f \right) \right)(x)$$

$$= \sum_{l=-2}^{2} \sum_{|i - (k-j)| \leq 7} \left( \Theta_{i+l} \Delta_{k-j} f \right)(x) - T_{j,t} \left( \sum_{i \in \mathbb{Z}} \sum_{|l| \leq 2} \Theta_{i+l} \Delta_{k-j} f \right)(x)$$

$$= \sum_{l=-2}^{2} \sum_{|i - (k-j)| \leq 7} \left[ \frac{1}{t} \int_{S^{n-1}} \Omega(y') \Theta_{i+l} \Delta_{k-j} f(x) \right] \left[ \frac{1}{t} \int_{S^{n-1}} |\Omega(y')| \|b(x - 2^1 t y')\| d\sigma(y') \right]$$

$$\leq C \sup_{i \in \mathbb{Z}} \|\Theta_{i+l} \Delta_{k-j} f\|_{L^\infty} \int_{S^{n-1}} |\Omega(y')| \|b(x - 2^1 t y')\| d\sigma(y')$$

$$\leq C \|b\|_* \|T_{j,t}^* f(x).$$

Using (3.23)–(3.24), (3.19) and Littlewood-Paley theory, we have

$$II_2 \leq C \|b\|_* \|\Omega\|_{L^1(S^{n-1})} \sum_{|l| \leq 7} \left( \int_{1}^{2} \left( \sum_{j \in \mathbb{Z}} |T_{j,t}^* (\Theta_{k-j+l} \Delta_{k-j} f)|^2 \right)^{1/2} \right)^{1/2} \left( \sum_{|l| \leq 2} dt \right)^{1/2}$$

$$\leq C \|b\|_* \|\Omega\|_{L^1(S^{n-1})} \|f\|_{L^2}.$$

Finally, we estimate $II_3$. Note that $\Delta_{k-j} ((\Theta_{i} g)(G_{i-3} h)) = 0$ for $g, h \in \mathcal{S}'(\mathbb{R}^n)$ if $|i - (k-j)| \geq 5$. Thus, we get

$$\Delta_{k-j} [\pi_b(T_{j,t}^* f_{k,j})(x) - T_{j,t}^* (\pi_b(f_{k,j}))](x)$$
\[
= \Delta_{k-j} \left( \sum_{i \in \mathbb{Z}} (\Theta_i b)(G_{i-3} T^i_{j,t} \Delta_{k-j} f) - T^i_{j,t} \left( \sum_{i \in \mathbb{Z}} (\Theta_i b)(G_{i-3} \Delta_{k-j} f) \right) \right)(x)
\]

\[
= \sum_{|i-(k-j)| \leq 4} \Delta_{k-j} \left( (\Theta_i b T^i_{j,t} (G_{i-3} \Delta_{k-j} f)) - T^i_{j,t} ((\Theta_i b)(G_{i-3} \Delta_{k-j} f)) \right)(x).
\]

Since

\[
(\Theta_i b T^i_{j,t} (G_{i-3} \Delta_{k-j} f)) - T^i_{j,t} ((\Theta_i b)(G_{i-3} \Delta_{k-j} f))(x)
\]

\[
= \frac{1}{2} \int_{S^{n-1}} \Theta_i b(x) - \Theta_i b(x - 2^{i} t y') G_{i-3} \Delta_{k-j} f(x - 2^{i} t y') d\sigma(y'),
\]

by (3.24), (3.19) and Littlewood-Paley theory, we get

\[
II_3 \leq C ||b||_{L^{1}(S^{n-1})} \sum_{|l| \leq 4} \left( \int_{1}^{2} \left( \sum_{j \in \mathbb{Z}} |T^i_{j,t} G_{k-j+l-3} \Delta_{k-j} f|^2 \right)^{1/2} \frac{dt}{t} \right)^{1/2}
\]

\[
\leq C ||b||_{L^{1}(S^{n-1})} ||f||_{L^{2}}.
\]

Together with the estimates of \(II_1\), \(II_2\) and \(II_3\), we get

\[
II \leq C \max \left\{ 2^{\tau k}, 2^{\gamma k} \right\} ||b||_{L^{1}(S^{n-1})} ||f||_{L^{2}} \quad \text{for} \, k \in \mathbb{Z}, \tag{3.25}
\]

where \(C\) is independent of \(k\) and \(\tau\). Taking \(\tau = \frac{1}{4}\) in (3.25), we get

\[
II \leq C (|k| + 1)||b||_{L^{1}(S^{n-1})} ||f||_{L^{2}} \quad \text{for} \, k \in \mathbb{Z}.
\]

Combining this with the estimates of \(I\) and \(II_1\), we get

\[
\left\| \left( \sum_{j \in \mathbb{Z}} |I_{2,k,j} f|^2 \right)^{1/2} \right\|_{L^{2}} \leq C (|k| + 1)||b||_{L^{1}(S^{n-1})} ||f||_{L^{2}}. \tag{3.26}
\]

Combining the estimates of (3.17) and (3.26), we get that for some constant \(\theta \in (0, 1)\) and \(k \in \mathbb{Z},\)

\[
\left\| S_{2,k}(T_{\Omega, b} f) \right\|_{L^{2}} \leq C \min \{2^{\nu k}, 2^{-\gamma k}\} (1 + |k|)||b||_{L^{2}} ||f||_{L^{2}} \leq C 2^{-\theta |k|} ||b||_{L^{2}} ||f||_{L^{2}}^2.
\]

This finishes the proof of (3.15).

**Proof of (3.16).** Let

\[
B = \left\{ (a_{j,t})_{j \in \mathbb{Z}, t \in [1, 2]) : \|a_{j,t}\|_{B} = \left( \sum_{j \in \mathbb{Z}} ||a_{j,t}||^{2}_{L^{2}} \right)^{1/2} < \infty \right\}.
\]

Clearly, \((B, \| \cdot \|_{B})\) is a Banach space. Then

\[
S_{2,k}(T_{\Omega, b} f)(x) = \left( \sum_{j \in \mathbb{Z}} \sup_{t_1 < \cdots < t_N} \sum_{l=1}^{N-1} \left| b, T_{j,t_l} \Delta_{k-j}^{2} f(x) - [b, T_{j,t_{l+1}} \Delta_{k-j}^{2} f(x)] \right|^{2/3} \right)^{1/2}
\]

\[
= \left( \sum_{j \in \mathbb{Z}} \sup_{t_1 < \cdots < t_N} \sum_{l=1}^{N-1} \left| b, T_{j,t_l,t_{l+1}} \Delta_{k-j}^{2} f(x) \right|^{2} \right)^{1/2},
\]

where

\[
T_{j,t_l,t_{l+1}} f(x) := \int_{2^{l} t_l < |y| \leq 2^{l} t_{l+1}} f(x - y) \frac{\Omega(y)}{|y|^n} dy \quad \text{and} \quad [t_l, t_{l+1}] \subset [1, 2].
\]
Then we get
\[
S_{2,k}(T_{\Omega,b,f})(x) \leq C \left( \sum_{j \in \mathbb{Z}} \sup_{t_{l} < \cdots < t_{N}} \left( \sum_{l=1}^{N-1} |[b, T_{j,t_{l},t_{l+1}}] \Delta_{k}^{2-j} f(x)|^{2} \right)^{\frac{1}{2}} \right)
\]
\[
+ C \left( \sum_{j \in \mathbb{Z}} \sup_{t_{l} < \cdots < t_{N}} \left( \sum_{l=1}^{N-1} |T_{j,t_{l},t_{l+1}} [b, \Delta_{k}^{2-j}] f(x)|^{2} \right)^{\frac{1}{2}} \right).
\]
For \([t_{l}, t_{l+1}] \subset [1,2] \), let
\[
\tilde{T}_{j,t_{l},t_{l+1}} f(x) := \int_{2^{-j} t_{l} < |x-y| < 2^{-j} t_{l+1}} |f(y)| \Omega(x-y) |dy|
\]
and
\[
[b, \tilde{T}_{j,t_{l},t_{l+1}}] f(x) := \int_{2^{-j} t_{l} < |x-y| < 2^{-j} t_{l+1}} |f(y)||b(x) - b(y)| \Omega(x-y) |dy|
\]
Then
\[
S_{2,k}(T_{\Omega,b,f})(x) \leq C \left( \sum_{j \in \mathbb{Z}} \sup_{t_{l} < \cdots < t_{N}} \left( \sum_{l=1}^{N-1} |[b, \tilde{T}_{j,t_{l},t_{l+1}}] \Delta_{k}^{2-j} f(x)|^{2} \right)^{\frac{1}{2}} \right)
\]
\[
+ C \left( \sum_{j \in \mathbb{Z}} \sup_{t_{l} < \cdots < t_{N}} \left( \sum_{l=1}^{N-1} |\tilde{T}_{j,t_{l},t_{l+1}} [b, \Delta_{k}^{2-j}] f(x)|^{2} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}
\]
\[
= C \left( \sum_{j \in \mathbb{Z}} \sup_{t_{l} < t_{N}} |[b, \tilde{T}_{j,t_{l},t_{N}}] (\Delta_{k}^{2-j} f)(x)|^{2} \right)^{\frac{1}{2}}
\]
\[
+ C \left( \sum_{j \in \mathbb{Z}} \sup_{t_{l} < t_{N}} |\tilde{T}_{j,t_{l},t_{N}} ([b, \Delta_{k}^{2-j}] f)(x)|^{2} \right)^{\frac{1}{2}}
\]
Therefore, we get
\[
S_{2,k}(T_{\Omega,b,f})(x) \leq C \left( \sum_{j \in \mathbb{Z}} |[b, T_{\Omega}^{*}] (\Delta_{k}^{2-j} f)(x)|^{2} \right)^{\frac{1}{2}} + C \left( \sum_{j \in \mathbb{Z}} |T_{\Omega}^{*} [b, \Delta_{k}^{2-j}] f)(x)|^{2} \right)^{\frac{1}{2}},
\]
where
\[
[b, T_{\Omega}^{*}] f(x) = \sup_{r > 0} \frac{1}{r^{n}} \int_{|x-y| < r} |f(y)||b(x) - b(y)| \Omega(x-y) |dy|
\]
and
\[
T_{\Omega}^{*} f(x) = \sup_{r > 0} \frac{1}{r^{n}} \int_{|x-y| < r} |f(y)| \Omega(x-y) |dy|
\]
Since for \(1 < p < \infty\),
\[
\left\| \left( \sum_{j \in \mathbb{Z}} |[b, T_{\Omega}^{*}] f|^{2} \right)^{1/2} \right\|_{L^{p}} \leq C \|b\|_{L^{\infty}} \|\Omega\|_{L^{\log^{+}}L(S^{n-1})} \left\| \left( \sum_{j \in \mathbb{Z}} |f|^{2} \right)^{1/2} \right\|_{L^{p}}
\]
and
\[
\left\| \left( \sum_{j \in \mathbb{Z}} |T_{\Omega}^{*} f|^{2} \right)^{1/2} \right\|_{L^{p}} \leq C \|\Omega\|_{L^{1}(S^{n-1})} \left\| \left( \sum_{j \in \mathbb{Z}} |f|^{2} \right)^{1/2} \right\|_{L^{p}},
\]
which were established in [22, Lemma 4] and [8, Lemma 2.3] and the Littlewood-Paley theorem, we get a reasonable expression below: for \(1 < p < \infty\),
\[
\| S_{2,k}(T_{\Omega,b,f}) \|_{L^{p}} \leq C \|b\|_{L^{\infty}} \|\Omega\|_{L^{\log^{+}}L(S^{n-1})} \|f\|_{L^{p}},
\]
which gives (3.16).
4 Proof of Theorem 1.7

For definition, we have
\[
\Omega(x') = \left[ \Omega(x') - \frac{1}{\omega_{n-1}} \int_{\mathbf{S}^{n-1}} \Omega(y')d\sigma(y') \right] + \frac{1}{\omega_{n-1}} \int_{\mathbf{S}^{n-1}} \Omega(y')d\sigma(y')
\]
\[
=: \Omega_0(x') + C(\Omega, n),
\]
where \(\omega_{n-1}\) denotes the area of \(\mathbf{S}^{n-1}\). Thus,
\[
[b, M_{\Omega, t}]f(x) = \frac{1}{t^n} \int_{|x-y|<t} \Omega_0(x-y)(b(x) - b(y))f(y)dy
\]
\[
+ C(\Omega, n) \frac{1}{t^n} \int_{|x-y|<t} (b(x) - b(y))f(y)dy
\]
\[
= [b, M_{\Omega, t}]f(x) + C(\Omega, n)[b, M_t]f(x),
\]
where \(\Omega_0\) satisfies the cancellation condition (1.12). Denote the operator family \([a, M_{\Omega, t}]\)_{t>0} by \(M_{\Omega, b}\) and \([a, M_t]\)_{t>0} by \(M_b\). By Corollary 1.5, we get that for \(1 < p < \infty\),
\[
\|V_pM_b(f)\|_{L^p} \leq C\|b\|_*\|f\|_{L^p}.
\]

Since
\[
\|\Omega_0\|_{L^{\log^+ L(\mathbf{S}^{n-1})}} \leq C\|\Omega_0\|_{L^{\log^+ L(\mathbf{S}^{n-1})}},
\]
to prove Theorem 1.7, we can begin with observing
\[
\|V_pM_{\Omega_0, b}(f)\|_{L^p} \leq C\|b\|_*\|\Omega_0\|_{L^{\log^+ L(\mathbf{S}^{n-1})}}\|f\|_{L^p}. \tag{4.1}
\]

Similarly, the proof of (4.1) is reduced to prove
\[
\|V_p([a, M_{\Omega_0, 2^k}]f)_{k\in\mathbb{Z}}\|_{L^p} \leq C\|b\|_*\|\Omega_0\|_{L^{\log^+ L(\mathbf{S}^{n-1})}}\|f\|_{L^p} \tag{4.2}
\]
and
\[
\|S_2(M_{\Omega_0, b}f)\|_{L^p} \leq C\|b\|_*\|\Omega_0\|_{L^{(\log^+ L)^2(\mathbf{S}^{n-1})}}\|f\|_{L^p}. \tag{4.3}
\]

First, (4.2) can be obtained from the pointwise domination
\[
V_p([a, M_{\Omega_0, 2^k}]f)_{k\in\mathbb{Z}} \leq \left( \sum_{k\in\mathbb{Z}} \|a, M_{\Omega_0, 2^k}f\|^2 \right)^{1/2}
\]
and
\[
\left\| \left( \sum_{k\in\mathbb{Z}} \|a, M_{\Omega_0, 2^k}f\|^2 \right)^{1/2} \right\|_{L^p} \leq C\|\Omega_0\|_{L^{(\log^+ L)^2(\mathbf{S}^{n-1})}}\|f\|_{L^p}, \quad 1 < p < \infty,
\]
which is a known result in a previous study [23, Theorem 2].

For (4.3), observe that \(V_{2^j}(M_{\Omega_0, b}f)\) is just the strong 2-variation function of the family,
\[
\{[a, M_{\Omega_0, 2^j}f]_{l\in\{1, 2\}}\}
\]
hence,
\[
S_2(M_{\Omega_0, b}f)(x) = \left( \sum_{j\in\mathbb{Z}} |V_{2^j}(M_{\Omega_0, b}f)(x)|^2 \right)^{1/2}.
\]

Similar to the proof of (3.2), we can obtain that for \(1 < p < \infty\),
\[
\|S_2(M_{\Omega_0, b}f)\|_{L^p} \leq C\|b\|_*\|\Omega_0\|_{L^{(\log^+ L)^2(\mathbf{S}^{n-1})}}\|f\|_{L^p}.
\]
This completes our proof for (4.3).

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