No-exclaves percolation

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Abstract
Network robustness has been a pivotal issue in the study of system failure in network science since its inception. To shed light on this subject, we introduce and study a new percolation process based on a new cluster called an ‘exclave’ cluster. The entities comprising exclave clusters in a network are the sets of connected unfailed nodes that are completely surrounded by the failed (i.e., nonfunctional) nodes. The exclave clusters are thus detached from other unfailed parts of the network, thereby becoming effectively nonfunctional. This process defines a new class of clusters of nonfunctional nodes. We call it the no-exclave percolation cluster (NExP cluster), formed by the connected union of failed clusters and the exclave clusters they enclose. Here we showcase the effect of NExP cluster, suggesting a wide and disruptive collapse in two empirical infrastructure networks. We also study on two-dimensional Euclidean lattice to analyze the phase transition behavior using finite-size scaling. The NExP model considering the collective failure clusters uncovers new aspects of network collapse as a percolation process, such as quantitative change of transition point and qualitative change of transition type. Our study discloses hidden indirect damage added to the damage directly from attacks, and thus suggests a new useful way for finding nonfunctioning areas in complex systems under external perturbations as well as internal partial closures.

Keywords Complex network · Network robustness · Percolation theory · No-exclaves percolation

1 Introduction

Complex systems’ function is fulfilled by a multitude of interactions between the constituents, which can be efficiently modeled by the complex networks of links connecting the nodes [1, 2]. From this network perspective, the existence of system-wide connectivity is an elementary requirement for the proper functioning of the complex system. Failures of network connections can weaken the overall functionality of the system, and understanding the origin of system collapse is important in many disciplines. The subject of network robustness dealing with this problem has been one of the major topics of complex network theory since its inception [3]. Outstanding results include the theoretical framework for error and attack tolerance in complex networks [4–10]. The study of mutual percolation has advanced the networks robustness of multiplex beyond the existing single-network [11–14]. On the other side, percolation has also been attempted to offer basic structural insights of diverse practical problems such as the effective immunization in the spreading process [15, 16], urban transport dynamics [17, 18], power grids [19, 20], and climate system [21, 22] and the like [23].

The reduction in the largest connected component size upon the removal of nodes or links (due either to random failures or targeted attacks) is often considered a measure of the robustness of the system [3]. This rationale captures an intuitive notion for evaluating and reacting against the impact of failures. However, a system’s collective impact by the failure may also be assessed and gauged by other measures. Here we propose to consider another measure of collective impact, that is the size of the largest ‘nonfunctional’ component. In networked systems, a node can become nonfunctional not only due to its own failure but also due to isolation from other functional parts. The latter can happen when the component the node belongs to becomes completely surrounded by failed nodes. Such secondary effects of failures may drive extensive agglomerations of otherwise disconnected failed clusters, the extent...
of which we will demonstrate. Therefore, the size of the largest nonfunctional cluster can be substantially bigger than the mere size of the largest cluster of failed nodes. From a theoretical point of view, this indicates that the impact of failures gets amplified nontrivially by the network effect and the collective impact can become greater than the sum of individual impacts. From a practical network management perspective, it would be desirable not only to keep the size of functional clusters (connected components) from diminishing but also to prevent the size of nonfunctional clusters from outgrowing.

To systematically formulate the problem, we introduce a new percolation process, termed the no-exclaves percolation (NExP), that aims to find the cluster of nonfunctional nodes that comprises both the directly failed nodes and the secondarily isolated nodes. Borrowing the terminology from political geography, the cluster of unfailed nodes completely surrounded by the failed nodes forms an ‘exclave’ of unfailed nodes. The aforementioned secondary effects can be termed ‘no-exclaves’ rule, in that all the nodes inside exclaves become nonfunctional in addition to the failed nodes. The cluster of such nonfunctional nodes, formed by the aggregate of failed node clusters and the finite exclaves of unfailed nodes that they enclose, can thus be called the NExP cluster. The largest NExP cluster reveals previously-neglected aspects of network robustness beyond the conventional largest connected component measure. The information encoded by the two measures is complementary, but neither completely reciprocal nor orthogonal. Our main aim in this paper is to portray the essential understanding gained by the new measure and thereby to broaden the perspective of the study of network robustness. It is noteworthy that the NExP rule in our model can be thought of as a network generalization of the no-enclaves percolation of biofilament network studied in two-dimensional Euclidean space [24, 25].

2 No-exclaves percolation

The NExP cluster is illustrated in Fig. 1a. We use the failure probability $q$ that each node of a given network is failed as the control parameter and the probability $\Phi$ that a randomly selected node belongs to the largest cluster as the order parameter, as in a usual site percolation problem [26].

(I) Initially, all nodes are unfailed state and are functional. (II) With probability $q$, each node fails (marked blue). (III) Nodes that are surrounded by the failed clusters are identified (marked yellow). They form the exclave clusters. (IV) The exclave clusters identified in step III are conglomerated with the adjacent surrounding failed clusters to form a collectively nonfunctional component that we call NExP cluster (marked red), with which we evaluated the degree of

Fig. 1  NExP model on Chilean power grid networks. a The process of forming an NExP cluster is shown in sample networks depicted as a schematic illustration based on a site percolation process. Marked by gray and blue nodes are unfailed and failed, nodes respectively shown in (I, II). Yellow nodes are exclave nodes (or clusters) that are completely surrounded by failed nodes (III), thus red-colored largest NExP cluster is the union of exclaves and failed cluster shown in (IV). b Results of $\Phi$ vs. $q$ for two models on Chilean power grid networks. Error bars denote the standard error from $10^6$ Monte Carlo simulation runs. c Plot of cluster size distribution $n(s)$ on Chilean power grid networks at failure probability $q = 0.4$. Each Monte Carlo simulation point was obtained by averaging over $10^6$ samples.
3.1 NExP on the road network

Studies on the robustness of transportation systems such as road networks are of relevance to many industries such as transportation mechanics, industrial science, and urban engineering [17, 28]. We simulate NExP process on the large-scale London road network data [29] to estimate the size of traffic malconditions due to road closures. Thanks to its large scale, it helps us evaluate phase transition properties more reliably. Figure 2a depicts nested snapshots of the London road network reflecting the actual partial geographic locations in successive lateral scales. As shown in Fig. 2b, c, our results report two important features. First, the NExP transition point $q_\text{c}$, at which the extensive NExP cluster emerges, is reduced compared to that of random site percolation (RSP) as shown in Fig. 2b. Finite exclave clusters are absorbed by the piecemeal yet numerous failed clusters and eventually lead to the discontinuous formation of the extensive NExP cluster which connects nearly all nodes before extensive RSP cluster appears. Second, the universality class of percolation transition changes as shown in Fig. 2c. To assess the critical property more quantitatively, we examine the critical behaviors by finite-size scaling analysis of the order parameter $\Phi$, using the standard finite-size scaling ansatz in Eq. (1), with $\Psi_\Phi$ being the scaling function with $\Delta q \equiv q - q_0$ and the critical exponents $\beta$ and $\nu$ [26].

$$\Phi(q, L) = L^{-\beta/\nu} \Psi_\Phi(\Delta q L^{1/\nu}). \quad (1)$$

In Fig. 2c, we show the finite size scaling plot for three different lateral scales $L$ near critical point $q_\text{c}^{\text{NExP}} = 0.2019$. Here, $q_\text{c}$ is inferred through the best collapse curve in the scaling plot due to the finite size effect of the road network. The number of nodes $N$ contained in each unit

$$L^3$$

Fig. 2 NExP on the London road networks. a An example of the scaling samples of the London road network with lateral size $L_1$, $L_2$, and $L_3$, respectively. b The size of largest cluster, $\Phi$ as a function of failure probability $q$ of NExP and random site percolation (RSP) on urban London road network with total system size $N = 288016$ and the mean degree $z \approx 2.58$. Error bars denote the standard error from $10^5$ Monte Carlo simulation runs. c Data collapse using finite size scaling ansatz Eq. (1) of Monte Carlo simulation with different road length $L_1(\triangle), L_2(\square)$ and $L_3(\circ)$ with $q_\text{c} = 0.2019$ (where $q_\text{c} = q_\text{c}^{\text{NExP}}$). Data are well collapsed onto single line with discontinuous percolation transition critical exponents $\beta = 0$ and $\nu = 4/3$. Each Monte Carlo simulation point was obtained by averaging over $10^5$ samples.
square of $L_1 \approx 10000$ m (composed of $N_1 = 19235$), $L_2 \approx 7000$ m ($N_2 = 8923$) and $L_3 \approx 5000$ m ($N_3 = 4367$).

As a result, NExP transition behavior is characterized by exponents $\beta = 0$ and $\nu = 4/3$, different from the second-order phase transition characterizing the RSP on road network, where the order parameter $\Phi$ is a continuous function at $q = q_c^{\text{RSP}}$ and $\beta/\nu = 5/48$. The mechanism for the discontinuity of NExP is different from most existing explosive percolation models [30]. NExP cluster is formed by the merging of distributed failed clusters and exclave clusters. In that sense, it might share common mechanism with the cluster aggregation model of discontinuous percolation [31].

We also examine the results of the NExP model on road networks by applying it to other cities across the world. We apply the NExP modeling study to other cities of sufficiently large size with a publicly available dataset [29] similar to the London road network data. We selected the major cities across continents from the available dataset, applied the NExP model, and performed the finite-size scaling analysis. The critical point $q_c$ is measured in the same way as in the London road network and shown in Fig. 3a. As a result, it was confirmed that though the transition point $q_c$ varies moderately across cities the NExP model in all urban road networks displays a discontinuous percolation transition pattern ($\beta/\nu = 0$), as shown in Fig. 3b–i.

### 3.2 NExP on two-dimensional square lattice

To substantiate the conclusion of the previous empirical data analysis theoretically, we examine the NExP on two-dimensional square lattice model with periodic boundary conditions. Fig. 4a shows the size of the largest cluster size $\Phi$ as a function of failure probability $q$ for NExP and RSP on two-dimensional square lattice. It is well known that the critical point for RSP on two-dimensional square lattice is $q_c^{\text{RSP}} = 0.5927$ [26]. The appearance of NExP critical point $q_c^{\text{NExP}} = 0.4073$ is consistent with the pattern for empirical networks results in the previous section.

In Fig. 4b, we show the Monte Carlo simulation results for the average NExP cluster size as a function of $q$.

![Graphs showing NExP on road networks of different cities](image)

**Fig. 3** NExP on road networks of different other cities. a Table of the critical point for 8 cities in alphabetical order. b–i Data collapse plots obtained from scaling formula Eq. (1). Data points are well collapsed by taking critical exponent $\beta = 0$ and $\nu = 4/3$ for all cities. Each Monte Carlo simulation point was obtained by averaging over $10^5$ samples.
The size of largest failed cluster $\Phi$ as a function of failure probability $q$ of NExP and RSP on two-dimensional square lattice with lattice lateral size $L = 2000$ (NExP), $L = 5000$ (RSP) denoted by solid line with points respectively. Error bars represent the standard error of a $10^5$ Monte Carlo simulation run. b Monte Carlo simulation results of failure probability $q$ vs. average NExP cluster size $\chi$ with square lattice lateral size $L = 2000$. c The simulation results for $q_c(L) - q_c$ as the lattice lateral sizes range from $L=100$ to 3500. Data are fitted to a power law with the exponent $-0.749(7)$ and $q_c = 0.4073(1)$. Each Monte Carlo simulation point was obtained by averaging over $10^4$. d Semi-log plot of $1 - \Phi$ as a function of lattice lateral sizes range from $L=100$ to 3500 for failure probability $q_c + \epsilon$ of NExP and RSP. For convenience of numerical simulation, we set $\epsilon = 0.0027$ for NExP and $\epsilon = 0.0073$ for RSP. In case of NExP, as the lattice lateral size $L$ increases, it tends to decrease exponentially. e The simulation results of the average NExP cluster size $\chi(q_c,L)$ versus the lattice lateral sizes range from $L = 500$ to 4000 at $q_c = 0.4073$, marked red points uses a double logarithm scale. The line displays the best fit of the data points. Data are fitted to a power law with an exponent $\gamma = 2.001(2)$ at $q_c$. b, d and e Error bars denote the standard error from $10^5$ runs for this simulation, which are smaller than the point size.

Numerically, we compute the average finite NExP cluster size $\chi$ to which a randomly chosen node belongs defined as,

$$\chi(q) = \frac{\sum_{s=1}^{\text{finite}} s^2 n(s,q)}{\sum_{s=1}^{\text{finite}} s n(s,q)},$$

(2)

where $s$ is finite NExP cluster size and $n(s,q)$ is the number of NExP clusters with size $s$ at failure probability $q$.

To avoid notational excess, $q_c = q_{c}^{\text{NExP}}$ from now on. Now, we obtained the critical point $q_c$, and correlation volume exponent $\nu$ from the following ansatz [26]:

$$q_c(L) - q_c \propto L^{-\nu},$$

(3)

where $q_c(L)$ is defined as a point with the maximum value of $\chi$ with lattice lateral size $L$. We estimated $q_c$ that best fits Eq. (3) using Monte Carlo simulation results, obtain $q_c = 0.4073(1)$. In Fig. 4c, data points are well described by the power law and show the best fit denoted by a straight line with the slope $-1/\nu = -0.749(7)$, corresponding to the critical exponent $\nu = 1.33(1)$.

We show simulation results for $q = q_c + \epsilon = 0.410$ to analyze and confirm the discontinuous percolation transition of the NExP model [32]. In case of RSP, $q = q_{c}^{\text{RSP}} + \epsilon = 0.6$ it decreases as the lattice lateral size $L$ increases and then converges to a nonzero value. On the other hand, referring to Fig. 4d, when calculating $1 - \Phi$ according to lattice lateral size at $q_c + \epsilon$ for NExP, $1 - \Phi$ shows a tendency to decrease with increasing $L$ asymptotically. This supports the conjecture that $\Phi$ jumps to 1 at $q_c$.

In Fig. 4e, we show the Monte Carlo simulation results for the average NExP cluster size $\chi$ as a different lattice lateral size $L$ at $q_c = 0.4073$. The average NExP cluster size increase as a power law, $\chi(q_c;L) \propto L^{1/\nu}$ for $L \gg 1$. In the two-dimensional

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lattice model, $\gamma/\nu = 2$ should be satisfied based on the critical exponent $\beta = 0$ for the discontinuous percolation transition. Indeed the Monte Carlo simulation confirms a value of $\gamma/\nu = 2.001(2)$ at critical point $q_c$ (Fig. 4e).

Now we clarify the critical properties of NExP model on two-dimensional square lattice by finite-size scaling analysis of the order parameter $\Phi$, the average NExP cluster size $x$, and the cluster size distribution $n(s)$. In addition to Eq. (1), we have finite-size scaling equation with additional exponents $\gamma, \tau$, and $\sigma$ [26].

\begin{equation}
\chi(q, L) = L^{\gamma/\nu} \Psi_x(\Delta q L^{1/\nu}),
\end{equation}

\begin{equation}
n_s(s, q_c) = L^{-\tau/\sigma} \Psi_n(s/L^{1/\sigma}).
\end{equation}

In addition, we refer to the percolating cluster mass $M$ which is the number of nodes of the percolating cluster. The percolating cluster mass $M$ and fractal dimension $d_f$ satisfied the scaling [26].

\begin{equation}
M(L; q_c) \propto L^{d_f}.
\end{equation}

In our Monte Carlo simulation calculation, the percolating cluster corresponds to the largest NExP cluster. In Fig. 5a we show the largest NExP cluster mass $M$ as a different lattice lateral size $L$ at critical point $q_c$. Data points are well described by the power law and show a best fit denoted by a straight line, which results in a value of $d_f = 2.00025(1)$ at the critical point $q_c$. Figure 5b shows the NExP cluster size distribution $n(s)$ obtained at the critical point $q_c$. Data points are well described by the power law decay and show the best fit denoted by a straight line. The straight line has

![Fig. 5](image)

Discontinuous percolation transition of NExP model on two-dimensional square lattices. a The simulation results of the largest NExP cluster mass $M$, versus the lattice lateral sizes range from $L = 500$ to 3500 at $q_c = 0.4073$ marked red points. Data are fitted to a power law with the exponent $d_f = 2.00025(1)$ at $q_c$. Error bars denote the standard error from $10^5$ runs for this simulation, which are smaller than the point size. b Finite NExP cluster size distribution $n(s, q_c)$ with lattice lateral size $L = 2000$ and ensemble averaged over taken $10^6$ for each simulation configurations. Data are well collapsed onto a single line with critical exponents $\beta/\nu = 0$ and $\gamma/\nu = 2$. c Data collapse of order parameter $\Phi$ and the average NExP cluster size $x$ using scaling ansatz Eqs. (1, 4) near critical point $q_c$ with different lattice lateral size. Plots are obtained by using lattice lateral sizes $L = 500 (\square), L = 1000 (\bigcirc)$ and $L = 2000 (\triangle)$ and ensemble averaged over taken $10^6$ for each simulation configurations. Data are well collapsed onto a single line with critical exponents $\beta/\nu = 0$ and $\gamma/\nu = 2$. d Error bars denote the standard error from $10^5$ runs for this simulation, which are smaller than the point size.
a slope $= -2.00(1)$, corresponding to the Fisher exponent $\tau = 2.00(1)$.

Finite-size scaling plots are shown in Fig. 5c–e. Data of all simulation results are well collapsed and in good agreement with the discontinuous critical exponents, $\beta/\nu = 0$ and $\gamma/\nu = 2$. These critical exponents from simulation results are consistent with those of no-enclaves percolation (NEP) model [24]. Standard hyperscaling relation $2\beta + \gamma = d\nu$ is satisfied. It is noteworthy that, unlike NEP, NExP in square lattice is governed by the excluded hull, which might underlie the difference in the critical exponent $\tau$ [24, 25]. Finally, Fig. 5e shows the finite-size scaling results of cluster size distribution at estimated $q_c$ using scaling ansatz Eq. (5) with $1/\nu = d\nu$. From this result, we confirmed the standard scaling relation $\nu = \frac{d-1}{d}$.

### 4 Summary and outlook

In this paper, we have studied collective network collapse by introducing the NExP model. We applied NExP to two city infrastructure networks, the power grid and the road network. Through the two applied studies, we derived two rather profound but simple results that the transition point and the transition pattern of the NExP model change compared to the RSP. These effects are relevant to network system robustness and provide useful ideas for designing social infrastructure systems. They also offer warning messages. In the power grid network system, with a failure probability $q$ of less than 1/2, nearly 70% of the nodes failed according to the NExP process and the system is paralyzed. On the other hand, in the road network, even with a failure probability $q \approx 0.3$, the entire network system was paralyzed and showed an abrupt percolation transition pattern. We confirmed the abrupt transition pattern in the two-dimensional square lattice, and the obtained critical exponents indicate that it belongs to the universality of the discontinuous phase transition, which is different from the RSP. We summarized the obtained critical exponents from NExP in Table 1. Our formalism and results show the discovery of a new nontrivial critical point that the entire system collapses and demonstrate that small nonfunctional fractions can lead to the collapse of the entire system.

Amid the enduring global pandemic, it would still be timely to speculate on the implication of our results for the impact of spatial lockdown on disease spread such as COVID-19. Our modeling suggests that the impact of lockdown (node failure) can be significantly amplified due to the indirect exclave effect and permits the abrupt global breakdown due to spatial nature. In conclusion, in order to accurately and precisely understand and predict the aggregate functionality of the network, the collective characteristic of the functional failure must be explicitly emphasized. In this regard, the NExP model framework proposed in this paper gives us a promising foundation for future research and industrial applications by providing a versatile platform for more realistic modeling of percolation processes involving node functional dependencies in networked systems.

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