Supersymmetrization of the Lindblad-Franke-GKS equation.

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Abstract

We investigate departures from hamiltonian dynamics of a SUSY quantum system due to its interaction with environment that generates decoherence. The Lindblad-Franke-GKS equation is taken to approach the quantum dynamics of decoherence, the prescription for supersymmetrization of this equation is elaborated and pairs of related isospectral quantum systems are described. Some instructive examples of such a construction are built.

1. Introduction

We consider the evolution of a quantum subsystem in an environment that is not the perfect vacuum and thereby generates departures from hamiltonian dynamics in the sub-system. The dynamics of quantum systems interacting with environment (open systems) are typically described by the Lindblad-Franke-GKS master equation [1], [2], [3] for the density matrix $\rho$. This equation includes not only Hamiltonian of the system but also the so called Lindblad operators which describe the interaction with environment. We assume the large number of degrees of freedom in the environment that leads to irreversibility and to the increasing entropy. An important example of such a subsystem is provided by the neutral kaon-antikaon

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propagation which can be influenced by the vacuum quantum-gravity fluctuations \cite{4, 5, 6} and/or by the detector matter \cite{7}. There are of course much more examples in the atomic and solid state physics and optics \cite{8} (an updated list of applications see in \cite{9}). Steven Weinberg advocates the $\rho$ matrix approach as the only ingredient necessary to incorporate the measurement and the Lindblad-Franke-GKS equation to be a smooth way to describe the relevant loss of quantum coherence after the influence of device \cite{10}.

Correspondingly the consistent description is based on the formalism of the $\rho$ matrix rather than on the wave functions themselves as the decoherence converts pure states with a given wave function into mixed ones which are characterized not by a particular wave function $\psi$ but by a set of quantum pure states with normalized wave functions $\{\psi_i\}$ and classical probabilities $r_i$ to occupy them:

$$\rho = \sum_i r_i |\psi_i \rangle \langle \psi_i|, \quad \langle \psi_i | \psi_i \rangle = 1, \quad r_i \geq 0, \quad (1)$$

If the total probability is conserved then $\sum_i r_i = 1$. Notice that, in general, the wave functions involved in (1) are not necessarily orthogonal, $\langle \psi_i | \psi_j \rangle \neq 0$ for some $i \neq j$. However the $\rho$ matrix is evidently a positive operator.

In the standard (hamiltonian) Quantum Mechanics, Supersymmetrical approach \cite{11} gave a powerful impulse in construction of exactly solvable and quasi exactly solvable quantum models of different nature \cite{12}. The main elements of Supersymmetrical Quantum Mechanics (SUSY QM) are the SUSY intertwining relations which express the commutation of Superhamiltonian with Supercharges. They provide the connection between pairs of superpartner systems: spectral properties of these systems are related to each other. Thus, the SUSY method in Quantum Mechanics became one of the best tools for quantum design \cite{13}. It looks interesting to generalize the SUSY approach for quantum systems with hamiltonian dynamics to a more general case of quantum systems interacting with environment and described by the Lindblad-Franke-GKS equation. Just this problem is studied in the present paper. It is organized as follows. After the brief discussion of the Lindblad-Franke-GKS equation in Section 2, the prescription for supersymmetrization of this equation is proposed in Section 3. Finally, Section 4 contains some illustrating examples of such construction.
2. Lindblad-Franke-GKS equation

The evolution of the total system [Subsystem + (Large) Environment] is unitary, with the time evolution operator given by \( U(t) = \exp(-iH_{\text{Total}}t) \); whereas the dynamics of the subsystem alone is obtained by tracing out the environment degrees of freedom:

\[
\rho_{\text{SUB}}(t) = \text{Tr}_{\text{ENV}} \left( U(t) \rho_{\text{SUB+ENV}}(0) U^\dagger(t) \right). \tag{2}
\]

This gives a complicated evolution for \( \rho_{\text{SUB}} \), even if the initial conditions are a direct product \( \rho_{\text{SUB+ENV}}(0) = \rho_{\text{SUB}}(0) \otimes \rho_{\text{ENV}}(0) \). However, when the interaction is tiny (e.g., as in the case of a diluted environment) the dynamics of \( \rho_{\text{SUB}} \) becomes approximately free from memory effects [14] (Markovian), and it is dictated by a Lindblad master equation [1], [2], [3] of the form

\[
\dot{\rho} = -i \left( H_{\text{eff}} \rho(t) - \rho(t) H_{\text{eff}}^\dagger \right) + L[\rho], \tag{3}
\]

where

\[
L[\rho] = \sum_j \left( A_j \rho(A_j)^\dagger - \frac{1}{2} \rho(A_j)^\dagger A_j - \frac{1}{2} (A_j)^\dagger A_j \rho \right), \tag{4}
\]

with the effective hamiltonian \( H_{\text{eff}} \) being in general non-hermitian. From now on, \( \rho \) stands for \( \rho_{\text{SUB}} \).

The new piece with vanishing trace is linear in \( \rho \), and the linear superposition principle is still valid for such a quantum system. But it is quadratic\(^3\) in the unspecified operators \( A^j \): the Lindblad equation encodes a new dynamics that is not purely hamiltonian. The new term may entail decoherence by making pure states to evolve into mixed states. Since the opposite process does not occur, i.e., no mixed state evolves to a pure one, we demand that the von Neumann entropy

\[
S = \text{Tr} (-\rho \log \rho)
\]

should not decrease and it imposes the constraints on the choice of the Lindblad operators \( A^j \).

\(^3\)The form (4) is the most general hermitian expression bilinear in \( A^j \) with vanishing \( \text{Tr}(L(\rho)) \). The forms linear in operators \( A^j \) are not interesting since they can be absorbed into Schrödinger-Liouville dynamics with \( L = 0 \).
For non-Hermitean Hamiltonians one can see that during evolution there are two distinct effects that affect the density matrix, namely the loss of quantum coherence, due to entanglement with the medium, and probability depletion, due to decay of the initial states which decreases $\text{Tr}(\rho)$. It is natural to introduce [7] a new variable $\bar{\rho}$ through the relation $\rho = \bar{\rho} \text{Tr}(\rho)$, so that $\text{Tr}(\bar{\rho}) = 1$ remains normalized to unity. This conditional $\bar{\rho}$ matrix satisfies the following closed evolution equation

$$\dot{\bar{\rho}} = -i(H_{\text{eff}}\bar{\rho} - \bar{\rho}H_{\text{eff}}^\dagger) + L[\bar{\rho}] - 2\text{tr}(\bar{\rho} \text{Im}H_{\text{eff}})\bar{\rho}.$$ 

Evidently the conditional $\bar{\rho}$ matrix characterizes solely decoherence. In what follows we restrict ourselves by consideration of Hermitean Hamiltonians only.

### 3. Supersymmetrical Lindblad-Franke-GKS system

The technique of Supersymmetrical Quantum Mechanics and the related isospectral Darboux transformation [11], [15] are well designed to the $\rho$ matrix formalism.

Indeed, in the fermion number representation, the one-dimensional SUSY QM assembles a pair of isospectral Hamiltonians $h_1$ and $h_2$ into a Super-Hamiltonian,

$$H = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} = \begin{pmatrix} -\partial^2 + V_1(x) & 0 \\ 0 & -\partial^2 + V_2(x) \end{pmatrix} \equiv -\partial^2 I + V(x),$$

where $\partial \equiv d/dx$ and the potentials are taken real. We assume also the time independence of the Super-Hamiltonian and also of supercharges which are introduced below. The isospectral connection between components of the Super-Hamiltonian is provided by intertwining relations with the help of Crum-Darboux differential operators $q^+ = (q^-)^\dagger$, 

$$h_1 q^+ = q^+ h_2, \quad q^- h_1 = h_2 q^-, \quad (6)$$

which, in the framework of SUSY QM, are components of the supercharges,

$$Q = \begin{pmatrix} 0 & q^+ \\ 0 & 0 \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} 0 & 0 \\ q^- & 0 \end{pmatrix} = Q^\dagger, \quad Q^2 = \bar{Q}^2 = 0.$$ 

(7)
The isospectral transformation (6) entails the conservation of supercharges or the supersymmetry of the Super-Hamiltonian,

$$[H, Q] = [H, \bar{Q}] = 0,$$

which represents the basis of the SUSY algebra. Its algebraic closure is given, in general, by a non-linear SUSY relation,

$$\{Q, \bar{Q}\} = \mathcal{P}(H),$$

where \(\mathcal{P}(H)\) is a time-independent function of the Super-Hamiltonian.

Let us consider first the Schrödinger-Liouville dynamics (3), i.e. with vanishing Lindbladian \(L[\rho] \equiv 0\), for the system with the Hamiltonian \(h_1\) and corresponding density matrix \(\rho_1\),

$$\dot{\rho}_1 = -i (h_1 \rho_1(t) - \rho_1(t) h_1).$$

After multiplying Eq.(10) by \(q^-\) from the left side and by \(q^+\) from the right one, in the absence of decoherence effects one obtains,

$$\dot{\rho}_2 = -i (h_2 \rho_2(t) - \rho_2(t) h_2),$$

where by definition,

$$\sqrt{\mathcal{P}(h_2)} \rho_2 \sqrt{\mathcal{P}(h_2)} \equiv q^- \rho_1 q^+.$$

After multiplying of (12) by \(q^+\) from the left or by \(q^-\) from the right and using intertwining relations (6) one arrives to the equivalent set of intertwining relations for \(\rho_1\) and \(\rho_2\),

$$\sqrt{\mathcal{P}(h_1)} \rho_1 q^+ = q^+ \rho_2 \sqrt{\mathcal{P}(h_2)};$$

$$\sqrt{\mathcal{P}(h_2)} \rho_2 q^- = q^- \rho_1 \sqrt{\mathcal{P}(h_1)}.$$

Thus the supersymmetrization is perfectly compatible with the \(\rho\) matrix approach but it is realized by essentially nonlocal transformations (12). The inverse mapping also holds,

$$\sqrt{\mathcal{P}(h_1)} \rho_1 \sqrt{\mathcal{P}(h_1)} = q^+ \rho_2 q^-.$$
which is provided also by Eqs. (13). Due to intertwining relations (6) and factorization (9), one can derive from (12) and (14):

\[ \rho_1 = \sum_{j,k=1}^{M_1} \beta_{jk}^1 |\psi_+^j \rangle \langle \psi_+^k | + q^+ \frac{1}{\sqrt{\mathcal{P}(h_2)}} \rho_2 \frac{1}{\sqrt{\mathcal{P}(h_2)}} q^-, \]  

(15)

\[ \rho_2 = \sum_{j,k=1}^{M_2} \beta_{jk}^2 |\psi_-^j \rangle \langle \psi_-^k | + q^- \frac{1}{\sqrt{\mathcal{P}(h_1)}} \rho_1 \frac{1}{\sqrt{\mathcal{P}(h_1)}} q^+, \]  

(16)

where \( |\psi_\pm^i \rangle \) are zero modes of the operators \( q^\pm \), respectively, and \( \beta_{1,2}^{jk} = \langle \psi_\pm^j | \rho_{1,2} | \psi_\pm^k \rangle \).

Thus, the Super-matrix \( \hat{\rho} \) defined as

\[ \hat{\rho} \equiv \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} \]  

(17)

satisfies the following linear equation:

\[ \sqrt{\mathcal{P}(H)} \hat{\rho} \sqrt{\mathcal{P}(H)} \equiv Q \hat{\rho} \bar{Q} + \bar{Q} \hat{\rho} Q, \]  

(18)

which is an essence of the SUSY for \( \rho \) matrices. Equivalently the set (13) is reproduced by,

\[ \bar{Q} \hat{\rho} \sqrt{\mathcal{P}(H)} = \sqrt{\mathcal{P}(H)} \hat{\rho} \bar{Q}, \quad Q \hat{\rho} \sqrt{\mathcal{P}(H)} = \sqrt{\mathcal{P}(H)} \hat{\rho} Q, \]  

(19)

which are analogs of the intertwining relations (6).

At this point, one may find it useful to introduce normalized supercharges,

\[ Q_n \equiv Q \frac{1}{\sqrt{\mathcal{P}(H)}} = \frac{1}{\sqrt{\mathcal{P}(H)}} Q; \quad \bar{Q}_n \equiv \bar{Q} \frac{1}{\sqrt{\mathcal{P}(H)}} = \frac{1}{\sqrt{\mathcal{P}(H)}} \bar{Q}, \]  

(20)

which satisfy the following superalgebra:

\[ \{ Q_n, \bar{Q}_n \} = 1; \quad [Q_n, \hat{\rho}] = [\bar{Q}_n, \hat{\rho}] = 0; \quad \hat{\rho} \equiv Q_n \hat{\rho} \bar{Q}_n + \bar{Q}_n \hat{\rho} Q_n, \]  

(21)

the latter relation being actually a direct consequence of the former one (13). Thus, the \( \hat{\rho} \) matrix is supersymmetric in terms of normalized supercharges.

These results however can be defined unambiguously only on the subspace of states orthogonal to the \( L^2 \)-subspace of zero modes of \( Q, \bar{Q} \). In the absence of zero modes for
the intertwining operators $q^+, q^-$, one can deduce from (12), (14) that the total probability remains invariant,

$$\text{Tr} (\rho_2) = \text{Tr} (\rho_1),$$

(22)

Otherwise, one has to apply Eq. (22) only for $\rho_1$ and $\rho_2$ matrices which contain states orthogonal to the corresponding zero modes.

Now let us derive the SUSY covariance conditions on the Lindblad operators $A^j \rightarrow A^j_1, A^j_2$ associated with the Hamiltonians $h_1, h_2$. We anticipate the relation similar to Eq. (12):

$$q^- L[A^j_1; \rho_1] q^+ = \sqrt{P(h_2)} L[A^j_2; \rho_2] \sqrt{P(h_2)}. $$

(23)

It entails the following (similar to (13)) intertwining relations between the Lindblad operators,

$$q^- A^j_1 \sqrt{P(h_1)} = \sqrt{P(h_2)} A^j_2 q^-;$$

(24)

$$q^- \sum_j (A^j_1)^\dagger A^j_1 \sqrt{P(h_1)} = \sqrt{P(h_2)} \sum_j (A^j_2)^\dagger A^j_2 q^-.$$

(25)

Multiplying (24) on the left by $q^+$ and on the right by $1/\sqrt{P(h_2)}$ and using intertwining relations (6), one obtains explicit expressions for $A^j_2$:

$$A^j_2 = q^- \frac{1}{\sqrt{P(h_1)}} A^j_1 \frac{1}{\sqrt{P(h_1)}} q^+. $$

(26)

Simultaneously, the relations (25) also will be fulfilled.

In the case when the Lindblad operators $A^j$ are local, i.e. they are differential operators of finite order it is possible that these operators were symmetry operators for the Hamiltonians $h_{1,2},$

$$[A^j_{1,2}, h_{1,2}] = 0.$$

(27)

In one-dimensional case, such an ansatz means that the Lindblad operators are functionals of Hamiltonians:

$$A^j_{1,2} = A^j(h_{1,2}). $$

(28)

The latter functionals may be non-linear in $h_{1,2}$ if there are two supersymmetries for the same Super-hamiltonian [16].
One can also introduce the SUSY notations,
\[
\hat{A}^j \equiv \begin{pmatrix} A^j_1 & 0 \\ 0 & A^j_2 \end{pmatrix},
\]
\[
\hat{Q} \hat{A}^j \sqrt{\mathcal{P}(H)} = \sqrt{\mathcal{P}(H)} \hat{A}^j \hat{Q}, \quad \hat{Q} \sum_j (\hat{A}^j)^\dagger \hat{A}^j \sqrt{\mathcal{P}(H)} = \sqrt{\mathcal{P}(H)} \sum_j (\hat{A}^j)^\dagger \hat{A}^j \hat{Q},
\]
\[
\hat{Q}(\hat{A}^j)^\dagger \sqrt{\mathcal{P}(H)} = \sqrt{\mathcal{P}(H)} (\hat{A}^j)^\dagger \hat{Q}, \quad \hat{Q} \sum_j (\hat{A}^j)^\dagger \hat{A}^j \sqrt{\mathcal{P}(H)} = \sqrt{\mathcal{P}(H)} \sum_j (\hat{A}^j)^\dagger \hat{A}^j \hat{Q}.
\]

If one employs the normalized SUSY charges (20) then the above relations appear to be intertwining ones,
\[
\hat{Q}_n \hat{A}^j = \hat{A}^j \hat{Q}_n, \quad \hat{Q}_n \sum_j (\hat{A}^j)^\dagger \hat{A}^j = \sum_j (\hat{A}^j)^\dagger \hat{A}^j \hat{Q}_n
\]
\[
\hat{Q}_n (\hat{A}^j)^\dagger = (\hat{A}^j)^\dagger \hat{Q}_n, \quad \hat{Q}_n \sum_j (\hat{A}^j)^\dagger \hat{A}^j = \sum_j (\hat{A}^j)^\dagger \hat{A}^j \hat{Q}_n.
\]

For hermitian \( \hat{A}^j \) the relations (31), (33) are reduced to the following equations,
\[
\sqrt{\mathcal{P}(H)} \hat{A}_j \sqrt{\mathcal{P}(H)} = Q \hat{A}_j \hat{Q} + \bar{Q} \hat{A}_j Q, \quad \hat{A}_j = Q_n \hat{A}_j \hat{Q}_n + \bar{Q}_n \hat{A}_j \hat{Q}_n,
\]
\[
\sqrt{\mathcal{P}(H)} \sum_j (\hat{A}_j)^2 \sqrt{\mathcal{P}(H)} = Q \sum_j (\hat{A}_j)^2 \hat{Q} + \bar{Q} \sum_j (\hat{A}_j)^2 Q,
\]
\[
\sum_j (\hat{A}_j)^2 = Q_n \sum_j (\hat{A}_j)^2 \hat{Q}_n + \bar{Q}_n \sum_j (\hat{A}_j)^2 Q_n,
\]
which are simple consequences of the intertwining conditions (30), (31), (33). We remark that the latter relations remind the similar ones (18),(21) for the Super-matrix \( \hat{\rho} \). Thus both the \( \hat{\rho} \)-matrix and the (hermitian) matrices \( \hat{A}_j \) are solutions of the same set of equations. On the other hand, the Super-matrix \( \hat{\rho} \) is a positive operator whereas there is no reason for the Lindblad operators to be positive.

4. Examples

In this Section, we shall consider a few simple illustrative examples of SUSY construction proposed above. Let us consider the density matrix \( \rho_1 \) from (1) for one-dimensional quantum
system in the energetic basis:

\[ h_1|E_m> = E_m|E_m>, \]  

which for simplicity has only discrete spectrum states with non-degenerate mutually orthonormal eigenvectors \(|E_m>\):

\[ \rho_1(t) = \sum_{i=1}^{N} r_i(t) |\psi_i><\psi_i|, \quad |\psi_i> \equiv \sum_{m=0}^{\infty} \Lambda_{im}|E_m>. \]  

Thus,

\[ \rho_1(t) = \sum_{m,n=0}^{\infty} (\lambda^\dagger \lambda)_{nm}|E_m><E_n|; \quad \lambda_{im}(t) \equiv \sqrt{r_i(t)} \Lambda_{im}; \quad (\lambda^\dagger \lambda)_{nm} = \sum_{i=1}^{N} \lambda^*_m \lambda_{im}. \]  

The normalization condition \(<\psi_i|\psi_i> = 1\) is provided if

\[ \sum_{n=0}^{\infty} |\Lambda_{in}|^2 = (\Lambda^\dagger \Lambda)_{ii} = 1. \]  

Under the supertransformations, the eigenstates \(|E_m>\) of \(h_1\) are transformed to the eigenstates \(|\tilde{E}_m>\) of \(h_2\):

\[ |\tilde{E}_m> = \frac{1}{\sqrt{E_m}} q^-|E_m>; \quad h_2|\tilde{E}_m> = \tilde{E}_m|\tilde{E}_m>. \]  

It follows from (15) and (41) that

\[ \rho_2(t) = \sum_{m,n=0}^{\infty} (\lambda^\dagger \lambda)_{nm}|\tilde{E}_m><\tilde{E}_n|. \]  

Also, let the components \(q^\pm\) of supercharges be the differential operators of first order so that the polynomials \(\mathcal{P}(H)\) in (9) and further on are linear in the corresponding Hamiltonians with zero energy of factorization: \(\mathcal{P}(h_1) = q^+ q^- = h_1, \quad \mathcal{P}(h_2) = q^- q^+ = h_2.\) One more simplification will be supposed. Let the components \(q^\pm\) of supercharges have no normalizable zero modes, i.e. the supersymmetry is spontaneously broken. In this case the spectra of superpartners \(h_1\) and \(h_2\) coincide exactly \(\tilde{E}_n = E_n\) (see details in [17]).

1. Let’s choose the Lindblad operators \(A_i^j\) as the real functions of the Hamiltonian:

\[ A_i^j = f_j(h_1); \quad f_j = f^*_j. \]
The Lindblad-Franke-GKS equation (3), (4) for the open system with $h_1$, $A_1^j$ takes the form:

$$\sum_{n,m=0}^{\infty} \frac{d}{dt} (\lambda^\dagger \lambda)_{nm} |E_m| |E_n| =$$

$$= \sum_{n,m=0}^{\infty} (\lambda^\dagger \lambda)_{nm} \left[ -i (E_m - E_n) - \frac{1}{2} \sum_j (f_j(E_n) - f_j(E_m))^2 \right] |E_m| |E_n|,$$

providing the necessary relations for $\lambda(t)$. In particular, the asymptotically stable at $t \to \infty$ density matrix is possible if for $n \neq m$ at $t \to \infty$ the matrix $(\lambda^\dagger \lambda)_{nm} \to 0$. The diagonal elements $(\lambda^\dagger \lambda)_{nn}$ may be arbitrary up to normalization condition (40) due to vanishing expression of terms with $n = m$ in square brackets in (44). Supersymmetry between the partners $\rho_1$, $\rho_2$ leads to conclusion that the partner density matrix $\rho_2(t)$ is also asymptotically stable if its Lindblad operators are taken according to supersymmetry as in Eq.(26): $A_2^j = f_j(h_2)$.

2. In the next example, we use the only Lindblad operator $A_1 = \frac{\Delta}{h_1} q^-$ where the real constant $\Delta$ is of dimensionality $[\Delta] = [E^{1/2}]$. Then, by relation (26):

$$A_1 = A_2 = q^- \frac{\Delta}{\sqrt{h_1}} = \frac{\Delta}{\sqrt{h_2}} q^-; \quad A_1^\dagger = A_2^\dagger = q^+ \frac{\Delta}{\sqrt{h_1}} = \frac{\Delta}{\sqrt{h_2}} q^+,$$

for which the Lindbladians in the energetic basis are:

$$L[\rho_1, A_1] = \sum_{m,n=0}^{\infty} (\lambda^\dagger \lambda)_{nm} \left[ |\bar{E}_m| |E_n| - |E_m| |E_n| \right] = \rho_2 - \rho_1; \quad (46)$$

$$L[\rho_2, A_2] = \sum_{m,n=0}^{\infty} (\lambda^\dagger \lambda)_{nm} \left| E_m |E_n| - |\bar{E}_m| |\bar{E}_n| \right| = \rho_1 - \rho_2 \quad (47)$$

Taking into account, that $\bar{E}_n = E_n$ here, the Lindblad-Franke-GKS equations for superpartners $\rho_1$, $\rho_2$ can be written now in a compact form:

$$\dot{\rho}_1(t) + \dot{\rho}_2(t) = \sum_{m,n=0}^{\infty} (-i)(E_m - E_n) \left( \lambda^\dagger \lambda \right)_{nm} \left[ |E_m| |E_n| + |\bar{E}_m| |\bar{E}_n| \right]; \quad (48)$$

$$\dot{\rho}_1(t) - \dot{\rho}_2(t) = \sum_{m,n=0}^{\infty} (-i)(E_m - E_n) \left( \lambda^\dagger \lambda \right)_{nm} \left[ |E_m| |E_n| - |\bar{E}_m| |\bar{E}_n| \right] - 2\Delta^2 (\rho_1 - \rho_2). \quad (49)$$

For the considered case of Hermitean Hamiltonians, the terms with $(-i)(E_m - E_n)$ in the r.h.s. of both equations produce oscillations of $(\rho_1(t) \pm \rho_2(t))$ in time while an additional
term in (49) corresponds to asymptotical damping of \((\rho_1(t) - \rho_2(t)) \to 0 \) at \( t \to \infty \) due to negative mutiplier \((-2\Delta^2)\). Then, Eq.(48) leads to pure oscillating asymptotical behaviour of both density matrices for \( t \to \infty \):

\[
\dot{\rho}_1(t) \sim \dot{\rho}_2(t).
\] (50)

In particular, if \( \rho_1 \) is pointer, its SUSY partner \( \rho_2 \) is too. This specific example includes the case of harmonic oscillator when components of supercharges coincide with creation-annihilation operators \( q^\pm = a^\pm \) playing also the role of Lindblad operators.

3. One more example can be considered with

\[
A_1(t) = \sum_{s=1}^{2} \gamma_s |E_s><E_s|; \quad A_1^\dagger(t) = \sum_{s=1}^{2} \gamma_s^* |E_s><E_s|,
\] (51)

with complex values of \( \gamma_{1,2}(t) \). According to (26), the superpartners are:

\[
A_2(t) = \sum_{s=1}^{2} \gamma_s |\tilde{E}_s><\tilde{E}_s|; \quad A_2^\dagger(t) = \sum_{s=1}^{2} \gamma_s^* |\tilde{E}_s><\tilde{E}_s|,
\] (52)

with the same functions \( \gamma_{1,2}(t) \) and \( |\tilde{E}> \) defined by (41). Such choice for the Lindblad operators leads to the Lindbladian:

\[
L[\rho_1, A_1] = \sum_{s,u=1}^{2} \gamma_s (\lambda^\dagger \lambda)_{su} |E_u><E_u| - \frac{1}{2} \sum_{s,u=1}^{2} \sum_{n=0}^{\infty} (\lambda^\dagger \lambda)_{su} |\gamma_s|^2 |E_n><E_n| - \frac{1}{2} \sum_{s,u=1}^{2} \sum_{m=0}^{\infty} (\lambda^\dagger \lambda)_{ms} |\gamma_s|^2 |E_m><E_m|.
\] (53)

Since both the eigenenergies \( \tilde{E}_n = E_n \) and functions \( \gamma_s(t) \) for the Lindbland-Franke-GKS equations for superpartners \( \rho_1(t), \rho_2(t) \) are the same, the evolution of density matrices is also the same while the Hamiltonians \( h_1, h_2 \) are different. This fact opens an opportunity to predict evolution of \( \rho_2 \) if evolution of \( \rho_1 \) is already known.

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