Hermite-Hadamard’s Inequalities for Preinvex Function via Fractional Integrals and Related Fractional Inequalities

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Received August 12, 2013; Revised August 24, 2013; Accepted August 28, 2013

Abstract In this paper, the author has established Hermite-Hadamard’s inequalities for preinvex functions and has extended some estimates of the right side of a Hermite-Hadamard type inequalities for preinvex functions via fractional integrals.

Keywords: Hermite-Hadamard’s inequalities, invex set, preinvex function, fractional integrals

Cite This Article: İmdat İşcan, “Hermite-Hadamard’s Inequalities for Preinvex Function via Fractional Integrals and Related Fractional Inequalities.” American Journal of Mathematical Analysis 1, no. 3 (2013): 33-38. doi: 10.12691/ajma-1-3-2.

1. Introduction

Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be a convex mapping defined on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \), then

\[
\left( \frac{a+b}{2} \right) f'(x) \leq \frac{b}{b-a} \int_a^b f(x) dx \leq \frac{b}{2} f(a) + \frac{b}{2} f(b).
\]

This doubly inequality is known in the literature as Hermite-Hadamard integral inequality for convex mapping. We note that Hadamard’s inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen’s inequality. For several recent results concerning the inequality (1) we refer the interested reader to [3,5,6,8,9,11,18,21,22] and the references cited therein.

Definition 1.1 The function \( f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R} \) is said to be convex if the following inequality holds:

\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\]

for all \( x, y \in [a, b] \) and \( t \in [0,1] \). We say that \( f \) is concave if \((-f)\) is convex.

In [18] Pearce and Pečarić established the following result connected with the right part of (1).

Theorem 1.2 Let \( f : I' \subset \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( I' \), \( a, b \in I' \) with \( a < b \), and let \( q \geq 1 \). If the mapping \( |f|^q \) convex on \( [a, b] \), then

\[
\frac{1}{2} \int_a^b f(x) dx \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{1}{4} \left[ \frac{1}{2} f(a) + \frac{1}{2} f(b) \right]^2.
\]

The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function \( f : [a, b] \rightarrow \mathbb{R} \).

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1.3 Let \( f \in L[a, b] \). The Riemann-Liouville integrals \( J^\alpha_a f \) and \( J^\alpha_b f \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[
J^\alpha_a f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, x > a
\]

and

\[
J^\alpha_b f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, x < b
\]

respectively, where \( \Gamma(\alpha) \) is the Gamma function and \( J^\alpha_a f(x) = J^\alpha_b f(x) = f(x) \).

In the case of \( \alpha = 1 \), the fractional integral reduces to the classical integral. Properties concerning this operator can be found ([7,12,17]).

For some recent result connected with fractional integral see ([4,19,20,22]).

In [19] Sarıkaya et al. proved the following Hadamard type inequalities for fractional integrals as follows.
Theorem 1.4 Let \( f: [a, b] \rightarrow \mathbb{R} \) be a positive function with \( 0 \leq a < b \) and \( f \in L[a, b] \). If \( f \) is a convex function on \([a, b]\), then the following inequalities for fractional integrals hold:

\[
\frac{f(a) + f(b)}{2} \leq \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_a^\alpha f(x) + J_b^\alpha f(x) \right] \leq \frac{f(a) + f(b)}{2} \tag{3}
\]

with \( \alpha > 0 \).

Using the following identity Sankarya et al. in [17] established the following result which hold for differentiable functions.

Lemma 1.5 Let \( f: [a, b] \rightarrow \mathbb{R} \) be a differentiable mapping on \((a, b)\) with \( a < b \). If \( f' \in L[a, b] \), then the following equality for fractional integrals holds:

\[
\frac{f(a) + f(b)}{2} \leq \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_a^\alpha f(x) + J_b^\alpha f(x) \right] \leq \frac{f(a) + f(b)}{2} \tag{4}
\]

Theorem 1.6 Let \( f: [a, b] \rightarrow \mathbb{R} \) be a differentiable mapping on \((a, b)\) with \( a < b \). If \( f' \) is a convex function on \([a, b]\), then the following inequalities for fractional integrals hold:

\[
\frac{f(a) + f(b)}{2} \leq \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_a^\alpha f(x) + J_b^\alpha f(x) \right] \leq \frac{b-a}{2} \left( 1 - \frac{1}{\alpha^2} \right) \left[ f'(a) + f'(b) \right] \tag{5}
\]

In recent years several extentions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson in [8]. Weir and Mond [23] introduced the concept of preinvex functions and generalization of convex functions. Later, Mohan and Neogy [24] introduced condition C defined as follows

Condition C: Let \( A \subseteq \mathbb{R}^n \) be an invex subset with respect to \( \eta: A \times A \rightarrow \mathbb{R}^n \). Then, for any distinct points \( x, y \in \mathbb{R}^n \) and any \( t \in [0, 1] \),

\[
\eta(y, x + t\eta(y, x)) \leq (1-t)\eta(y, x) + t\eta(x, y) \quad \eta(x, y + t\eta(x, y)) = (1-t)\eta(x, y). \tag{6}
\]

Note that for every \( x, y \in \mathbb{R}^n \) and every \( t_1, t_2 \in [0, 1] \),

\[
\eta(y + t_2\eta(x, y), y + t_1\eta(x, y)) = (t_2 - t_1)\eta(x, y).
\]

we will use the condition in our main results.

In [16], Noor proved the Hermite-Hadamard inequality for the preinvex functions as follows:

Theorem 1.9 Let \( f: K = [a, a + \eta(b, a)] \rightarrow (0, \infty) \) be a preinvex function on the interval of real numbers \( K^* \) (the interior of \( K \)) and \( a, b \in K^* \) with \( a < a + \eta(b, a) \). Then the following inequality holds:

\[
f\left( \frac{2a + \eta(b, a)}{2} \right) \leq \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(x)dx \leq \frac{f(a) + f(b)}{2} \tag{7}
\]

In [2] Barani, Gahazanfari, and Dragomir proved the following theorems:

Theorem 1.10 Let \( A \subseteq \mathbb{R}^n \) be an open invex subset with respect to \( \eta: A \times A \rightarrow \mathbb{R}^n \). Suppose that \( f: A \rightarrow \mathbb{R} \) is a differentiable function. If \( f' \) is preinvex on \( A \) then, for every \( a, b \in A \) with \( \eta(b, a) \neq 0 \) the following inequalities hold:

\[
\left| f(a) + f'(a + \eta(b, a)) \right| \leq \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f'(x)dx \leq \frac{f(a) + f(b)}{2} \tag{8}
\]

Theorem 1.11 Let \( A \subseteq \mathbb{R}^n \) be an open invex subset with respect to \( \eta: A \times A \rightarrow \mathbb{R}^n \). Suppose that \( f: A \rightarrow \mathbb{R} \) is a differentiable function. Assume that \( p \in \mathbb{R} \) with \( p > 1 \). If

\[
f\left( \frac{p}{p-1} \right) \text{ is preinvex on } A \text{ then, for every } a, b \in A \text{ with } \eta(b, a) \neq 0 \text{ the following inequalities hold}
\]
2. Main Results

Throughout this section, Let $A \subseteq \mathbb{R}$ be an open invex subset. In this section, firstly we will establish Hermite-Hadamard’s inequalities for preinvex functions via fractional integrals. Secondly we will introduce some generalizations of the right side of a Hermite-Hadamard type inequalities for functions whose first derivatives absolute values are preinvex via fractional integrals.

**Theorem 2.1** Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b, a)$. If $f : [a, a + \eta(b, a)] \to (0, \infty)$ is a preinvex function, $f \in L[a, a + \eta(b, a)]$ and $\eta$ satisfies condition C then, the following inequalities for fractional integrals holds:

$$
\frac{f(a) + f(a + \eta(b, a))}{2} \leq \frac{\Gamma(\alpha + 1)}{2\eta^\alpha(b, a)} \left[ f_0^\alpha f(a) + f_0^\alpha f(a + \eta(b, a)) - f(a) \right]
$$

(10)

$$
\leq f(a) + f(a + \eta(b, a)) \leq \frac{f(a) + f(b)}{2}
$$

with $\alpha > 0$.

**Proof.** Since $a, b \in A$ and $A$ is an invex set with respect to $\eta$, for every $t \in [0, 1]$, we have $a + \eta(b, a) \in A$. By preinvexity of $f$, we have for every $x, y \in [a, a + \eta(b, a)]$ with $t = \frac{x + y}{2}$

$$
f(x + \eta(y, x)) \leq f(x) + f(y)
$$

i.e. with $x = a + (1 - t)\eta(b, a)$, $y = a + t\eta(b, a)$ from equality (6) we get

$$
2f\left(a + (1 - t)\eta(b, a) + \frac{\eta(a + t\eta(b, a), a + (1 - t)\eta(b, a))}{2}\right)
$$

$$
= 2f\left(a + (1 - t)\eta(b, a) + (\frac{(1-t)\eta(b, a))}{2}\right)
$$

$$
= 2f\left(\frac{2a + \eta(b, a)}{2}\right)
$$

$$
\leq f\left(a + (1 - t)\eta(b, a)\right) + f\left(a + t\eta(b, a)\right)
$$

Multiplying both sides (11) by $t^{\alpha - 1}$, then integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$
\frac{2}{\alpha} f\left(\frac{2a + \eta(b, a)}{2}\right)
$$

$$
\leq \frac{1}{0} \int_0^t f\left(a + (1 - t)\eta(b, a)\right) dt + \frac{1}{0} \int_0^t f\left(a + t\eta(b, a)\right) dt
$$

$$
= \frac{1}{\eta^\alpha(b, a)} \left[ \int_0^a (a + \eta(b, a) - u)^{\alpha - 1} f(u) du + \int_a^1 (u - a)^{\alpha - 1} f(u) du \right]
$$

$$
= \Gamma(\alpha) \left[ \int_0^a f(a + \eta(b, a)) + f(a + \eta(b, a))\right]
$$

i.e.

$$
\frac{f(a) + f(a + \eta(b, a))}{2}
$$

and the fist inequality is proved.

For the proof of the second inequality in (11) we first note that if $f$ is a preinvex function on $[a, a + \eta(b, a)]$ and the mapping $\eta$ satisfies condition C then for every $t \in [0,1]$, from inequality (6) it yields

$$
f\left(a + t\eta(b, a)\right) = f\left(a + \eta(b, a) + (1 - t)\eta(a, a + \eta(b, a))\right)
$$

(12)

$$
\leq tf\left(a + \eta(b, a)\right) + (1 - t)f(a)
$$

and similarly

$$
f\left(a + (1 - t)\eta(b, a)\right) = f\left(a + \eta(b, a) + t\eta(a, a + \eta(b, a))\right)
$$

(13)

$$
\leq (1 - t)f\left(a + \eta(b, a)\right) + tf(a)
$$

By adding these inequalities we have

$$
f\left(a + t\eta(b, a)\right) + f\left(a + (1 - t)\eta(b, a)\right) \leq f(a) + f\left(a + \eta(b, a)\right)
$$

Then multiplying both (13) by $t^{\alpha - 1}$ and integrating the resulting inequality with respect to $t$ over $[0,1]$, we obtain

$$
\frac{1}{\alpha} \int_0^t f\left(a + t\eta(b, a)\right) dt + \frac{1}{\alpha} \int_0^t f\left(a + (1 - t)\eta(b, a)\right) dt
$$

$$
\leq \left[ f(a) + f\left(a + \eta(b, a)\right)\right] \frac{1}{\alpha} \int_0^t f\left(a + \eta(b, a)\right) dt
$$

i.e.

$$
\frac{f(a) + f\left(a + \eta(b, a)\right)}{\alpha}
$$

Using the mapping $\eta$ satisfies condition C the proof is completed.

**Remark 2.2** a) If in Theorem 2.1, we let $\eta(b, a) = b - a$, then inequality (10) become inequality (3) of Theorem 1.4.
b) If in Theorem 2.1, we let \( \alpha = 1 \), then inequality (10) become inequality (7) of Theorem 1.9.

Now we give the following lemma which is a generalization of Lemma 1.5 to invex setting.

**Lemma 2.3** Let \( A \subset \mathbb{R} \) be an open invex subset with respect to \( \eta : A \times A \to \mathbb{R} \) and \( a, b, \in A \) with \( a < a + \eta(b, a) \). If \( f : A \to \mathbb{R} \) is a differentiable function such that \( f' \in L[a, a + \eta(b, a)] \) then, the following equality holds:

\[
\frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2\eta^a(b, a)} \left[ \int_a^a f(a + \eta(b, a)) + \int_{a+\eta(b, a)} f(a) \right] = \frac{\eta(b, a)}{2} \int_0^a (a - 1)^{\alpha - 1} f'(a + \eta(b, a)) \, dt \tag{14}
\]

Proof: It suffices to note that

\[
I = \int_0^a (a - 1)^{\alpha - 1} f'(a + \eta(b, a)) \, dt \]

integrating by parts

\[
I_1 = \int_0^a f(a + \eta(b, a)) \left\{ \frac{\eta(b, a)}{a} \right\}^{a+\eta(b, a)} f(x) \, dx \]

\[
I_2 = \int_0^a (a - 1)^{\alpha - 1} f'(a + \eta(b, a)) \, dt \tag{15}
\]

and similarly we get

\[
I_2 = \int_0^a (a - 1)^{\alpha - 1} f'(a + \eta(b, a)) \, dt \tag{16}
\]

Using (16) and (17) in (15), it follows that

\[
I = f(a) + f(a + \eta(b, a)) - \frac{\Gamma(\alpha + 1)}{2\eta^a(b, a)} \left[ \int_a^a f(a + \eta(b, a)) + \int_{a+\eta(b, a)} f(a) \right] \tag{17}
\]

Thus, by multiplying both sides by \( \frac{\eta(b, a)}{2} \), we have conclusion (14).

**Remark 2.4** If in Lemma 2.3, we let \( \eta(b, a) = b - a \), then inequality (14) become inequality (4) of Lemma 1.5.

**Theorem 2.5** Let \( A \subset \mathbb{R} \) be an open invex subset with respect to \( \eta : A \times A \to \mathbb{R} \) and \( a, b, \in A \) with \( a < a + \eta(b, a) \). Suppose that \( f : A \to \mathbb{R} \) is a differentiable function such that \( f' \in L[a, a + \eta(b, a)] \). If \( f' \) is preinvex function on \( [a,a + \eta(b, a)] \) then the following inequality for fractional integrals with \( \alpha > 0 \) holds:

\[
\frac{f(a) + f(a + \eta(b, a))}{2} - \frac{\Gamma(\alpha + 1)}{2\eta^a(b, a)} \left[ \int_a^a f(a + \eta(b, a)) + \int_{a+\eta(b, a)} f(a) \right] \leq \frac{\eta(b, a)}{2(\alpha + 1)} \left[ \int f'(a) + \int f'(b) \right] \tag{18}
\]

Proof: Using lemma 2.3 and the preinvexity of \( f' \) we get

\[
f(a) + f(a + \eta(b, a)) - \frac{\Gamma(\alpha + 1)}{2\eta^a(b, a)} \left[ \int_a^a f(a + \eta(b, a)) + \int_{a+\eta(b, a)} f(a) \right] \]

\[
\leq \frac{\eta(b, a)}{2(\alpha + 1)} \left[ \int f'(a) + \int f'(b) \right] \]

which completes the proof.

**Remark 2.6** a) If in Theorem 2.5, we let \( \eta(b, a) = b - a \), then inequality (18) become inequality (5) of Theorem 1.6.

b) If in Theorem 2.5, we let \( \alpha = 1 \), then inequality (18) become inequality (8) of Theorem 1.10.

c) In Theorem 2.5, assume that \( f' \) satisfies condition C and using inequality (12) for \( f' \) we get
we have is a differentiable function. If is preinvex on for some fixed , for some $(a,b)$. Suppose , and then inequality (19) become inequality (9) of Theorem1.11. Then the following inequality holds:

\[
\frac{f(a) + f(a + \eta(b,a))}{2} - \eta(b,a) \left( \frac{1}{2} \frac{f'(a) + f'(a + \eta(b,a))}{2} \right) \leq 0
\]

Theorem 2.7 Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a,b \in A$ with $a < a + \eta(b,a)$ such that $f' \in L[a,a + \eta(b,a)]$. Suppose that $f_a$ is preinvex function on $[a,a + \eta(b,a)]$ for some fixed $q > 1$ then the following inequality holds:

\[
\frac{f(a) + f(a + \eta(b,a))}{2} - \frac{\Gamma(a+1)}{2\eta^q(b,a)} \left[ J^a_{a^+} f(a + \eta(b,a)) + J^a_{(a + \eta(b,a))^-} f(a) \right] \leq \frac{\eta(b,a)}{2(\alpha p+1)} \left( \left| f'(a) \right|^q + \left| f'(a + \eta(b,a)) \right|^q \right)^{\frac{1}{q}}
\]

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha \in [0,1]$. Proof. From lemma 2.3 and using Hölder inequality with properties of modulus, we have

\[
\frac{f(a) + f(a + \eta(b,a))}{2} - \frac{\Gamma(a+1)}{2\eta^q(b,a)} \left[ J^a_{a^+} f(a + \eta(b,a)) + J^a_{(a + \eta(b,a))^-} f(a) \right] \leq \frac{\eta(b,a)}{2} \left[ \left| f'(a + \eta(t\eta(b,a))) \right| dt \right]^{\frac{1}{q}}
\]

We know that for $\alpha \in [0,1]$ and $\forall t_1,t_2 \in [0,1]$, $\left| t_1^p - t_2^p \right| = \left| t_1 - t_2 \right|^p$, therefore

\[
\int_0^{1} \left| f'(a + \eta(t\eta(b,a))) \right|^q dt \leq \int_0^{1} \left| t_1 - t_2 \right|^p dt
\]

On the other hand, we have

Remark 2.8 a) If in Theorem 2.7, we let $\eta(b,a) = b - a$ and $\alpha = 1$ then inequality (19) become inequality (9) of Theorem1.11. b) In Theorem 2.7, assume that $\eta$ satisfies condition C and using inequality (12) we get

\[
\frac{f(a) + f(a + \eta(b,a))}{2} - \frac{\Gamma(a+1)}{2\eta^q(b,a)} \left[ J^a_{a^+} f(a + \eta(b,a)) + J^a_{(a + \eta(b,a))^-} f(a) \right] \leq \frac{\eta(b,a)}{2(\alpha p+1)} \left( \left| f'(a) \right|^q + \left| f'(a + \eta(b,a)) \right|^q \right)^{\frac{1}{q}}
\]

Theorem 2.9 Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \rightarrow \mathbb{R}$ and $a,b \in A$ with $a < a + \eta(b,a)$. Suppose that $f : A \rightarrow \mathbb{R}$ is a differentiable function such that $f' \in L[a,a + \eta(b,a)]$. If $f_a$ is preinvex function on $[a,a + \eta(b,a)]$ for some fixed $q > 1$ then the following inequality holds:

\[
\frac{f(a) + f(a + \eta(b,a))}{2} - \frac{\Gamma(a+1)}{2\eta^q(b,a)} \left[ J^a_{a^+} f(a + \eta(b,a)) + J^a_{(a + \eta(b,a))^-} f(a) \right] \leq \frac{\eta(b,a)}{2(\alpha p+1)} \left( \left| f'(a) \right|^q + \left| f'(a + \eta(b,a)) \right|^q \right)^{\frac{1}{q}}
\]

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha > 0$. Proof. From lemma 2.3 and using Hölder inequality with properties of modulus, we have

\[
\frac{f(a) + f(a + \eta(b,a))}{2} - \frac{\Gamma(a+1)}{2\eta^q(b,a)} \left[ J^a_{a^+} f(a + \eta(b,a)) + J^a_{(a + \eta(b,a))^-} f(a) \right] \leq \frac{\eta(b,a)}{2} \left[ \left| f'(a + \eta(t\eta(b,a))) \right| dt \right]^{\frac{1}{q}}
\]

Since $f_a$ is preinvex on $[a,a + \eta(b,a)]$, we have inequality (19), which completes the proof.
\[
\int_0^1 \left[ f^\alpha - (1-t)^\alpha \right] \, dt = \frac{1}{2} \left[ (1-t)^\alpha - t^\alpha \right] \, dt + \frac{1}{2} \left[ f^\alpha - (1-t)^\alpha \right] \, dt
\]
\[
= \frac{2}{\alpha + 1} \left( 1 - \frac{1}{2\alpha} \right)
\]

Since \( f^\alpha \) is preinvex function on \( A \), we obtain
\[
\left[ f'(a + \eta(b,a)) \right]^\alpha \leq (1-t)\left[ f'(a) \right]^\alpha + t\left[ f'(b) \right]^\alpha , \quad t \in [0,1]
\]
and
\[
\int_0^1 \left[ f^\alpha - (1-t)^\alpha \right] \left[ f'(a + \eta(b,a)) \right]^\alpha \, dt
\]
\[
\leq \int_0^1 \left[ f^\alpha - (1-t)^\alpha \right] \left[ (1-t)\left[ f'(a) \right]^\alpha + t\left[ f'(b) \right]^\alpha \right] \, dt
\]
\[
= \frac{1}{2} \left[ (1-t)^\alpha - t^\alpha \right] \left[ (1-t)\left[ f'(a) \right]^\alpha + t\left[ f'(b) \right]^\alpha \right] \, dt
\]
\[
+ \frac{1}{2} \left[ t^\alpha - (1-t)^\alpha \right] \left[ (1-t)\left[ f'(a) \right]^\alpha + t\left[ f'(b) \right]^\alpha \right] \, dt
\]
\[
= \frac{1}{\alpha + 1} \left( 1 - \frac{1}{2\alpha} \right) \left[ f'(a) \right]^\alpha + \left[ f'(b) \right]^\alpha \right] \]

From here we obtain inequality (20) which completes the proof.

**Remark 2.10**

a) If in Theorem 2.9, we let \( \eta(b,a) = b - a \) and \( \alpha = 1 \) then inequality (20) becomes inequality (2) Theorem 1.2.

b) In Theorem 2.9, assume that \( \eta \) satisfies condition C, using inequality (12) we get
\[
\left[ f(a) + f(a + \eta(b,a)) \right] \leq \frac{\Gamma(\alpha + 1)}{2\eta^\alpha (b,a)} \left[ f^\alpha (a + \eta(b,a)) + f^\alpha (a) \right] \leq \frac{\eta(b,a)}{(\alpha + 1)} \left( 1 - \frac{1}{2\alpha} \right) \left[ f'(a) \right]^\alpha + \left[ f'(b) \right]^\alpha \right] \]

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