Equivalence of black hole thermodynamics between a generalized theory of gravity and the Einstein theory

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We analyze black hole thermodynamics in a generalized theory of gravity whose Lagrangian is an arbitrary function of the metric, the Ricci tensor and a scalar field. We can convert the theory into the Einstein frame via a "Legendre" transformation or a conformal transformation. We calculate thermodynamical variables both in the original frame and in the Einstein frame, following the Iyer–Wald definition which satisfies the first law of thermodynamics. We show that all thermodynamical variables defined in the original frame are the same as those in the Einstein frame, if the spacetimes in both frames are asymptotically flat, regular and possess event horizons with non-zero temperatures. This result may be useful to study whether the second law is still valid in the generalized theory of gravity.

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I. INTRODUCTION

Black hole thermodynamics was originally formulated within Einstein’s theory of general relativity, by showing that analogous relations to the usual thermodynamical laws hold in black hole dynamics [1]. After Hawking’s discovery [2] of black hole evaporation due to black body radiation with the temperature, \( T = \hbar \kappa / 2 \pi \), where \( \kappa \) is the surface gravity of the black hole, black holes have been regarded as real thermodynamical objects. While Hawking’s discovery has elucidated the thermodynamical nature of black holes, it also left problems which have not been solved yet, such as the information loss problem, fates of black holes due to the Hawking evaporation and the origin of the black hole entropy. Those problems are expected to be explained by quantum gravity. Actually, a recent development in string theory, which is one of the most promising candidates of quantum gravity, has unveiled that the number of states of a D-brane configuration is exactly the same as the entropy of the corresponding black hole [3]. However, such a coincidence is true only for extreme black holes because of their BPS nature. While non-extreme black holes are also discussed in this context (see, e.g., Ref. [4]), further development of string theory would be required to understand the origin of the entropy of the generic black holes.

Together with string theory, investigations based on field theory including stringy effects may also be helpful to understand such microscopic black holes and their properties. One possible method to study the stringy effects is to consider effective field theories, in which corrections to Einstein gravity are expected [5], e.g., higher curvature interactions and a dilaton coupling. Higher curvature corrections arise also in the effective theory with quantum loop effects [6]. Those effective theories may not be fundamental, and the corrections should be regarded as perturbative effects. However, such corrections may modify black hole physics from that in the Einstein theory, which includes one of the most important properties of microscopic black holes. In addition, by investigating the generalized theories of gravity, as the effective theories, we may extract a universal feature of black hole physics, which cannot be understood by the analysis within the Einstein theory.

Since Hawking’s result that a black hole is regarded as a thermodynamical object does not depend on the details of the Einstein equations, we expect that a black hole in the generalized theories of gravity also shows thermal properties. In addition, when the horizon of the black hole is bifurcate, the zero-th law was proved [7] and the first law has been formulated in an arbitrary diffeomorphism invariant theory [8,9]. Therefore, we expect that the thermodynamical laws of black holes may also hold in the generalized theories. However, the entropy in dynamical processes is still to be defined, and thus, not only the third law, but also the second law has not been established so far, in general. For the theories which can be transformed into the Einstein frame via a conformal transformation, the entropy is proved to be the same as that in the Einstein frame, and the second law is proved for some specific models, by examining the null energy condition in the Einstein frame [10]. Equivalence between the entropy of a stationary black hole in the original frame and that in the Einstein frame is also shown in the theory whose Lagrangian includes a specific form of higher curvature interactions [11], based on the fact that the transformation into the Einstein frame does not alter the asymptotic structure of the spacetime, such as the mass and the angular momentum of the black hole. It is not the case, however, when the theory contains a scalar field which couples non-minimally to gravity.

In this paper, we generalize those results into a much wider class of theories. The Lagrangian we consider is an arbitrary function of the metric, the Ricci tensor, a scalar field and its derivative. This class of theories includes any combinations of the Ricci tensor of any order as well as a non-minimal coupling of a scalar field. We convert the theory into the Einstein frame via a “Legendre” transformation [12] or a conformal transformation [13], and show that all the thermodynamical variables are the same between the original frame and the Einstein frame. It is expected that the equivalence of the thermodynamical variables may provide a useful clue to the second law, as in the case of Ref. [14].

In the formulation of the first law in Ref. [8,9], the assumption that a black hole has a bifurcate Killing horizon plays a crucial role. This assumption seems to be reasonable if the event horizon is a Killing horizon, since a Killing horizon with non-vanishing temperature should be bifurcate [7], and the third law implies that a zero temperature state cannot be realized in physical processes, although its general proof is still absent. We then assume, in this paper, that the event horizon is a Killing horizon, and focus only on black holes with non-zero temperature.

As we mentioned above, we consider the effective theories, which may not be adequate to study physics at the Planck scale. Hence, we will be concerned only with black holes sufficiently larger than Planck scale. The
corrections of the higher curvature interactions in those cases will not be large enough to change drastically the geometry of a black hole spacetime from that in the Einstein theory. However, they might change black hole thermodynamics. This is the present subject.

We organize this paper as follows. We first describe the transformation of the theory into the Einstein frame in the next section. In Section III, we briefly repeat the definition of the thermodynamical variables of a black hole in the generalized theory of gravity formulated in Ref. [8,9]. By following their definition, we show, in Section IV, the equivalence between the thermodynamical variables in the original frame and those in the Einstein frame. The conclusion and discussions are described in Section V. The Appendix contains derivations of the equations used in our calculations.

We use two metrics $g^{\mu\nu}$ and $\overline{g}^{\mu\nu}$ for fundamental variables [14], and then, the inverse matrices of $g^{\mu\nu}$ and $\overline{g}^{\mu\nu}$ will be denoted as $g^{-1}_{\mu\nu}$ and $\overline{g}^{-1}_{\mu\nu}$, respectively, because two variables, $g^{-1}_{\mu\nu}$ and $g^{-1}_{\mu\alpha}g^{-1}_{\nu\beta}g_{\alpha\beta}$ (or $\overline{g}^{-1}_{\mu\nu}$ and $\overline{g}^{-1}_{\mu\alpha}\overline{g}^{-1}_{\nu\beta}g_{\alpha\beta}$), must be distinguished, although both of them are the same and are expressed as $\overline{g}_{\mu\nu}$ (or $g_{\mu\nu}$) in the standard convention. The units we adopt are $c = G = k_B = 1$.

II. TRANSFORMATION OF THEORY

The action we consider is written as

$$ I = \frac{1}{16\pi} \int d^4x \sqrt{-g} f(g^{\alpha\beta}, R_{\alpha\beta}, \phi, \nabla_{\alpha}\phi), $$

(2.1)

where $f$ is an arbitrary function of the metric, $g^{\alpha\beta}$, the Ricci tensor, $R_{\alpha\beta}$, a scalar field, $\phi$, and its derivative. $f$ of course includes an arbitrary function of a scalar curvature, $R$, because $R = g^{\alpha\beta}R_{\alpha\beta}$. We transform the action (2.1) into the Einstein frame by defining a new "metric" tensor, $\overline{g}^{\mu\nu}$, as

$$ \overline{g}^{\mu\nu} = \frac{\partial(\sqrt{-g}f)}{\partial R_{\mu\nu}} \frac{1}{\sqrt{-\det(\partial(\sqrt{-g}f)/\partial R_{\alpha\beta})}} \frac{\sqrt{-g}}{\sqrt{-\overline{g}}} \partial f \partial R_{\mu\nu}. $$

(2.2)

Since $\overline{g}^{\mu\nu}$ is a function of $g^{\alpha\beta}$, $R_{\alpha\beta}$, $\phi$, and $\nabla_\alpha \phi$, it is written as

$$ \overline{g}^{\mu\nu} = \overline{g}^{\mu\nu}(g^{\alpha\beta}, R_{\alpha\beta}, \phi, \nabla_\alpha \phi). $$

(2.3)

For an arbitrary function $f$, there are two cases, depending on whether or not $\det |\partial^2 f/\partial R_{\mu\nu}\partial R_{\rho\sigma}|$ vanishes.

If $\det |\partial^2 f/\partial R_{\mu\nu}\partial R_{\rho\sigma}| = 0$ (the degenerate case), we may use a conformal transformation to convert the theory into the Einstein frame. For example, if the action is given by

$$ I = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ f(\phi)R - \frac{\epsilon(\phi)}{2}(\nabla_\mu \phi)(\nabla^\mu \phi) - V(\phi) \right], $$

(2.4)

we can convert the action into the Einstein frame

$$ \mathcal{I} = \frac{1}{16\pi} \int d^4x \sqrt{-\overline{g}} \left[ \overline{R} - \frac{1}{2}\overline{g}^{\mu\nu}(\nabla_\mu \Phi)(\nabla_\nu \Phi) - U(\Phi) \right], $$

(2.5)

via a conformal transformation,

$$ \overline{g}^{\mu\nu} = f^{-1}(\phi)g^{\mu\nu}, $$

(2.6)

with a redefinition of the scalar field

$$ \Phi = \int d\phi \left[ \frac{\epsilon(\phi)f(\phi) + 3|df(\phi)/d\phi|^2}{f(\phi)^2} \right]^{1/2}, $$

(2.7)

where $U(\Phi) \equiv f^{-2}(\phi)V(\phi)$. 

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More difficulty appears when
\[ \text{det} \frac{\partial^2 f}{\partial R_{\mu\nu} \partial R_{\rho\sigma}} \neq 0. \]

In this case, we can solve Eq.(2.3) for \( R_{\mu\nu} \) as
\[ R_{\mu\nu} = R_{\mu\nu}(g^{\alpha\beta}, \bar{g}^{\alpha\beta}, \phi, \nabla_\alpha \phi), \]

and then, define a new action, \( \mathcal{T} \), via a kind of Legendre transformation, by
\[ \mathcal{T} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ g^{\mu\nu} R_{\mu\nu}(g^{\alpha\beta}, \partial_\gamma g^{\alpha\beta}, \partial_\gamma \partial_\lambda g^{\alpha\beta}) - \bar{g}^{\mu\nu} R_{\mu\nu}(g^{\alpha\beta}, \bar{g}^{\alpha\beta}, \phi, \nabla_\gamma \phi) \right. \]
\[ \left. + \sqrt{-g} f(g^{\alpha\beta}, R_{\alpha\beta}(g^{\alpha\beta}, \bar{g}^{\alpha\beta}, \phi, \nabla_\beta \phi), \phi, \nabla_\alpha \phi) \right], \]
where the independent variables are \( g^{\mu\nu}, \bar{g}^{\mu\nu} \) and \( \phi \). The Ricci tensor in the first term of the right hand side of Eq.(2.10) is given by the metric, \( g^{\alpha\beta} \), and its derivatives as usual, whereas those in the second and the third terms are given by the relation Eq.(2.9). We find, by varying the action(2.10), the Euler–Lagrange equation for \( \mathcal{T} \) gives the same relation as Eq.(2.2). Eq.(2.10) is rewritten into the form
\[ \mathcal{T} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R(\bar{g}^{\alpha\beta}, \partial_\gamma \bar{g}^{\alpha\beta}, \partial_\gamma \partial_\lambda \bar{g}^{\alpha\beta}) + \bar{g}^{\mu\nu} \left( \Gamma^\rho_{\mu\alpha} \Gamma^\rho_{\nu\sigma} - \Gamma^\rho_{\mu\sigma} \Gamma^\rho_{\nu\alpha} \right) \right. \]
\[ \left. - \bar{g}^{\mu\nu} R_{\mu\nu}(g^{\alpha\beta}, \bar{g}^{\alpha\beta}, \phi, \nabla_\alpha \phi) + \sqrt{-g} f(g^{\alpha\beta}, R_{\alpha\beta}(g^{\alpha\beta}, \bar{g}^{\alpha\beta}, \phi, \nabla_\beta \phi), \phi, \nabla_\alpha \phi) \right], \]
by using the relation
\[ R_{\mu\nu}(g^{\alpha\beta}, \partial_\gamma g^{\alpha\beta}, \partial_\gamma \partial_\lambda g^{\alpha\beta}) = \overline{R}_{\mu\nu}(g^{\alpha\beta}, \bar{g}^{\alpha\beta}, \partial_\gamma \bar{g}^{\alpha\beta}) \]
\[ + \Gamma^\rho_{\mu\alpha} \Gamma^\rho_{\nu\sigma} - \Gamma^\rho_{\mu\sigma} \Gamma^\rho_{\nu\alpha} + \nabla_\rho \Gamma^\rho_{\mu\nu} - \nabla_\rho \Gamma^\rho_{\nu\mu}, \]
where \( \Gamma^\rho_{\mu\nu} \equiv \frac{1}{2} g^{\rho\sigma} \left( \nabla_\mu g_{\nu\sigma}^{-1} + \nabla_\nu g_{\mu\sigma}^{-1} - \nabla_\sigma g_{\mu\nu}^{-1} \right) \). \( \nabla_\nu \) denotes the covariant derivative associated with \( g^{\mu\nu} \), and the total divergence is omitted in Eq.(2.11). Now \( \mathcal{T} \) plays the role of the metric, and the first term in the right hand side of Eq.(2.11) is the Einstein–Hilbert action of \( \bar{g}^{\mu\nu} \). The second term is interpreted as a kinetic term of an exotic spin-2 matter field, \( g^{\mu\nu} \), since it is bilinear in the first derivatives of \( g^{\mu\nu} \). The rest consists of a kinetic term of the scalar field, \( \phi \), potential and interaction of the matter fields. Hence \( \mathcal{T} \) is regarded as the action of Einstein gravity with exotic matter, which provides the equivalent equations of motion to those derived from the original action, \( I \). The equations of motion in the Einstein frame are second order differential equations, while those in the original frame are fourth order.

Although we considered the degenerate case separately, the conformal transformation(2.6) is consistent with Eq.(2.2), and the action(2.5) is formally obtained by substituting Eq.(2.6) and (2.7) into Eq.(2.11), where the Ricci tensor terms given by Eq.(2.9) cancel. Since we need only Eq.(2.2) and Eq.(2.11) to show the equivalence of the thermodynamical variables, we can treat the action(2.4) on an equal footing, as the general case, Eq.(2.1).

However, if Eq.(2.3) is not satisfied and the action is not written as Eq.(2.4), a transformation into the Einstein frame is not known. We will not discuss such cases in what follows.

### III. Definition of Thermodynamical Variables

The thermodynamical variables of black holes in the generalized theories of gravity are defined such that they satisfy the first law of black hole thermodynamics \[\text{(a)}\]. In Ref. \[\text{[8,9]}\], the first law is established for
an arbitrary perturbation from a regular background spacetime of a black hole, for which the following assumptions are made: (1) It is stationary and axisymmetric, and hence it possesses two Killing vectors, $\xi^\mu_t$ and $\xi^\mu_\varphi$, associated with the respective symmetry. (2) It is asymptotically flat, and then the mass and the angular momentum are well-defined. (3) The event horizon of the black hole is a bifurcate Killing horizon. Definition of the thermodynamical variables is based on the covariance of the Lagrangian density 4-form, $\epsilon_{\mu\nu\rho\sigma} L$, under a diffeomorphism, where $L$ is a scalar function of the field variables which we denote as $\psi_i$ collectively. The variation of $\epsilon_{\mu\nu\rho\sigma} L$ is described as

$$\delta (\epsilon_{\mu\nu\rho\sigma} L) = \epsilon_{\mu\nu\rho\sigma} E^{(i)} \delta \psi_i + \epsilon_{\mu\nu\rho\sigma} \nabla_\beta \Theta_\beta \psi_i \delta \psi_i,$$

(3.1)

where $E^{(i)} = 0$ is the equation of motion for $\psi_i$, and $\Theta_\beta \psi_i \delta \psi_i$ in the totally divergent term is a functional of the field variables, $\psi_i$, and their variations, $\delta \psi_i$. When we identify the variation, $\delta \psi_i$, with a general coordinate transformation, $L$, vanishes, because $\nabla_\beta (\zeta^\beta L) = -E^{(i)} L_{\zeta} \psi_i + \nabla_\beta \Theta_\beta \psi_i L_{\zeta} \psi_i \right.$.

Then in the integral over the bifurcation surface, $H$, i.e., the left hand side of Eq. (3.4), the term proportional to $\xi^\mu$ vanishes, because $\xi^\mu = 0$ on that surface. If there exists a vector, $B^\mu [\psi_i]$, whose variation gives the second term in the right hand side of Eq. (3.7) as

$$\delta \int \frac{1}{2} \xi_{\beta\alpha\mu} \epsilon^{\beta\alpha} [B^\alpha [\psi_i]] dx^\mu dx^\nu = \int \frac{1}{2} \xi_{\beta\alpha\mu} \epsilon^{\beta\alpha} [B^\alpha [\psi_i]] dx^\mu dx^\nu \right.$.

(3.4)

In the integral over the bifurcation surface, $H$, i.e., the left hand side of Eq. (3.4), the term proportional to $\xi^\mu$ vanishes, because $\xi^\mu = 0$ on that surface. If there exists a vector, $B^\mu [\psi_i]$, whose variation gives the second term in the right hand side of Eq. (3.7) as

$$\delta \int \frac{1}{2} \xi_{\beta\alpha\mu} \epsilon^{\beta\alpha} [B^\alpha [\psi_i]] dx^\mu dx^\nu = \int \frac{1}{2} \xi_{\beta\alpha\mu} \epsilon^{\beta\alpha} [B^\alpha [\psi_i]] dx^\mu dx^\nu \right.$.

(3.5)

Eq. (3.4) provides the relation between the variation of an integral of a geometrical quantity on the horizon and that over the 2-surface at infinity, i.e.,

$$\delta \int \frac{1}{2} \xi_{\beta\alpha\mu} \epsilon^{\beta\alpha} [B^\alpha [\psi_i]] dx^\mu dx^\nu = \delta \left( \frac{1}{2} \xi_{\beta\alpha\mu} \epsilon^{\beta\alpha} [B^\alpha [\psi_i]] dx^\mu dx^\nu \right. \right.$.

(3.6)

Then, we define the entropy, $S$, the mass, $M$, and the angular momentum, $J$, of the black hole as

$$\frac{\hbar k}{2\pi} S \equiv \int \frac{1}{2} \xi_{\beta\alpha\mu} \epsilon^{\beta\alpha} [B^\alpha [\psi_i]] m dx^\mu dx^\nu \right.$.

(3.7)

$$M \equiv \int \frac{1}{2} \xi_{\beta\alpha\mu} \epsilon^{\beta\alpha} [B^\alpha [\psi_i]] m dx^\mu dx^\nu \right.$.

(3.8)

$$J \equiv -\int \frac{1}{2} \xi_{\beta\alpha\mu} \epsilon^{\beta\alpha} [B^\alpha [\psi_i]] m dx^\mu dx^\nu \right.$.

(3.9)
respectively, where \( \kappa \) is the surface gravity of the black hole. Although the second term on the right hand side of Eq. (3.9) does not contribute, we keep this term for convenience in our discussion below. In the case of the Einstein theory, Eq. (3.7) gives \( S = A/4\bar{h} \), where \( A \) is the area of the black hole, and Eqs. (3.8) and (3.9) coincide with the expressions of the ADM mass and the Komar angular momentum, respectively. The existence of \( B^\mu \) is closely related to asymptotic flatness. In Ref. [9], it is argued that the condition on the asymptotic behavior of the field variables should be imposed to ensure the existence of \( B^\mu \), because the mass and the angular momentum are naturally defined in an asymptotically flat spacetime.

The first law of thermodynamics, for arbitrary perturbations from a stationary background solution,

\[
\frac{\hbar \kappa}{2\pi} \delta S = \delta M - \Omega H \delta J
\]

(3.10)

follows by substituting Eqs. (3.7), (3.8) and (3.9) into Eq. (3.6), and noting that \( Q^{\beta \alpha} [\psi^{(i)}, \zeta^\rho, \nabla_\lambda \zeta^\rho] \) is linear in \( \zeta^\rho \) and \( \nabla_\lambda \zeta^\rho \).

IV. EQUIVALENCE OF THERMODYNAMICAL VARIABLES

Now, following the definition in the previous section, we show the equivalence of the thermodynamical variables between the original frame and the Einstein frame, where the actions are given by Eqs. (2.1) and (2.11), respectively. First we have to mention further assumptions in addition to those in the previous section. Then, after showing the equivalence of the temperature and the angular velocity, we prove that the entropy, the mass and the angular momentum are the same in these two frames.

A. Assumptions

Since we do not specify our model explicitly, the spacetime in the Einstein frame may not be regular or asymptotically flat, in which case we have to give up discussing black hole thermodynamics. However, when we consider the effective string theory, as an example, \( g_{\mu \nu} \) is typically given as

\[
g_{\mu \nu} = e^{-\alpha \phi} \left( g_{\mu \nu} + \beta l_{PL}^2 g_{\mu \nu} + \gamma l_{PL}^2 R_{\mu \nu} + \cdots \right),
\]

(4.1)

where dots denote the higher order terms of curvature, \( \phi \) is a dilaton field, \( l_{PL} \) is the Planck length, and \( \alpha, \beta, \gamma \) are numerical constants of order unity.

In this case, \( \overline{g}^{\mu \nu} \) approaches \( g^{\mu \nu} \) in the asymptotically flat region in the original frame, if the scalar field is not singular, and hence the spacetime in the Einstein frame is also asymptotically flat. This is naturally expected to be the case in general, because the dominant contribution to the Lagrangian, \( f \), in the original action (2.1) will be proportional to the scalar curvature in the asymptotically flat region, where \( g^{\mu \nu} \rightarrow \eta^{\mu \nu} + O(r^{-1}) \). Although a non-minimal coupling of the scalar field may result in \( \overline{g}^{\mu \nu} \rightarrow e^{\omega} g^{\mu \nu} + O(r^{-1}) \), where \( \omega \) is determined by the asymptotic value of the scalar field, we can set \( \omega = 0 \) by rescaling the length unit, without loss of generality. In what follows, we assume that this treatment has been done, and so \( \overline{g}^{\mu \nu} \rightarrow g^{\mu \nu} + O(r^{-1}) \). We emphasize that this does not necessarily indicate that the mass and the angular momentum are the same between the original frame and the Einstein frame, in contrast to Ref. [11].

In addition, if the spacetime and the scalar field in the original frame are regular at least outside the horizon, \( \overline{g}^{\mu \nu} \) is also expected to be regular in the above example (4.1) if the black hole is much larger than Planck scale. This is because \( \overline{g}^{\mu \nu} \) is expressed in terms of the field variables in the original frame as Eq. (4.1), and the corrections from the higher curvature terms will be small enough so that a singularity will not appear. This seems to be true in the general case for large black holes. Therefore we will be concerned only with such cases. Since the spacetime is regular and \( \overline{g}^{\mu \nu} \) approaches \( \eta^{\mu \nu} \) in the asymptotically flat region, the signature of the metric, \( \overline{g}^{\mu \nu} \), in the Einstein frame is still Lorentzian.

As we mentioned in the Introduction, we focus only on black holes with non-zero temperature. Since physics is described in the original frame, it might not be physically reasonable to make an assumption about the temperature in the Einstein frame. However, if the temperature in the original frame is non-zero and that in the Einstein frame vanishes, we naively presume that the black hole in the original frame is...
quantum mechanically unstable due to Hawking radiation while it might be stable in the Einstein frame. This does not seem to be natural. We assume, in what follows, the temperatures of black holes in both frames are non-zero.

Our assumptions are summarized as: (1) $\mathcal{g}^{\mu \nu} \to g^{\mu \nu} + \mathcal{O}(r^{-1})$ in the asymptotically flat region, where $g^{\mu \nu} \to \eta^{\mu \nu} + \mathcal{O}(r^{-1})$. (2) The transformation into the Einstein frame is regular in the sense that the spacetime region in the Einstein frame is regular if the corresponding region in the original frame is regular. (3) The temperatures of black holes in both frames are non-zero.

B. Temperature and angular velocity

Here we show the equality of the temperature and the angular velocity in two frames [18]. For this purpose, we first show the Killing horizon in the Einstein frame appears at the same place as in the original frame.

On the Killing horizon in the original frame, we introduce a coordinate system $(\theta, \varphi)$ on the 2-dimensional hypersurface orthogonal to $\xi^\mu$, such that $\xi^\mu_{(\varphi)} (\partial / \partial x^\mu) \equiv \partial / \partial \varphi$ and denote the basis vector of the coordinate $\theta$ by $\theta^\mu (\partial / \partial x^\mu) \equiv \partial / \partial \theta$. A 2-dimensional hypersurface in the original frame is transformed into a 2-dimensional hypersurface in the Einstein frame, because the transformation is regular. We consider the vector fields in the Einstein frame, which are the same as $\xi^\mu$, $\xi^\mu_{(\varphi)}$ and $\theta^\mu$ in the original frame. Then we find that fixed points of $\xi^\mu$ in the Einstein frame appear at the same points as the bifurcation surface in the original frame, and we have, at those points,

$$
\mathcal{g}^{-1}_{\mu\nu} \xi^\mu \xi^\nu = 0, \quad \mathcal{g}^{-1}_{\mu\nu} \xi^\mu \theta^\nu = 0, \quad \mathcal{g}^{-1}_{\mu\nu} \xi^\mu \xi^\nu_{(\varphi)} = 0.
$$

(4.2)

In addition, we notice that the Killing vectors in the original frame are also Killing vectors in the Einstein frame, because the field variables in the Einstein frame are composed of those in the original frame, and hence the Lie derivatives of the field variables in the Einstein frame with respect to the Killing vectors vanish. Then we have

$$
\mathcal{L}_{\xi} \left( \mathcal{g}^{-1}_{\mu\nu} \xi^\mu \xi^\nu \right) = 0, \quad \mathcal{L}_{\xi} \left( \mathcal{g}^{-1}_{\mu\nu} \xi^\mu \theta^\nu \right) = 0, \quad \mathcal{L}_{\xi} \left( \mathcal{g}^{-1}_{\mu\nu} \xi^\mu \xi^\nu_{(\varphi)} \right) = 0,
$$

(4.3)

because $\xi^\mu$ commutes with $\theta^\mu$ and $\xi^\mu_{(\varphi)}$. Therefore, we find that Eq. (4.2) holds along the orbits of $\xi^\mu$ through the fixed points of $\xi^\mu$, and then, $\xi^\mu$ is orthogonal to the 2-dimensional hypersurface spanned by $\theta^\mu$ and $\xi^\mu_{(\varphi)}$, is null and hence hypersurface orthogonal. Thus, $\xi^\mu$ generates a Killing horizon and the fixed points of $\xi^\mu$ form a bifurcation surface also in the Einstein frame. Since $\xi^\mu$ and the location of the bifurcation surface are the same, we see the horizon in the Einstein frame appears at the same place as in the original frame.

There might exist other Killing horizons outside the above mentioned Killing horizon in the Einstein frame. However, we can show that those extra horizons cannot be bifurcate, by repeating the same argument as above. That is, if there exist bifurcate horizons in the Einstein frame, the corresponding bifurcate horizons must exist also in the original frame, whereas it is not the case. Hence all extra horizons must have vanishing temperature [6]. Since we assume the temperatures in both frames are non-zero, we exclude such an exceptional case, if any, from our consideration.

We consider the surface gravity in order to show the equivalence of the temperature, since the temperature, $T$, of a black hole is given as $T = \hbar \kappa / 2\pi$ in arbitrary theories of gravity. The surface gravity, $\kappa$, in the Einstein frame is calculated, from hypersurface orthogonality of $\xi^\mu$, by

$$
2\kappa^2 = (\nabla_{\mu} \xi^\nu) \left( \nabla_{\nu} \xi^\mu \right),
$$

(4.4)

as is the surface gravity, $\kappa$, in the original frame by

$$
2\kappa^2 = (\nabla_{\mu} \xi^\nu) \left( \nabla_{\nu} \xi^\mu \right).
$$

(4.5)

Then, by describing $\nabla_{\mu} \xi^\nu = \nabla_{\mu} \xi^\nu + \xi^\sigma \Gamma^\nu_{\mu \sigma}$, where $\Gamma^\nu_{\mu \sigma} \equiv \frac{1}{2} \mathcal{R}^{\nu\alpha\beta\gamma} \left( \nabla_{\mu} \mathcal{g}^{-1}_{\alpha\sigma} + \nabla_{\sigma} \mathcal{g}^{-1}_{\mu\alpha} - \nabla_{\alpha} \mathcal{g}^{-1}_{\mu\sigma} \right)$, and noticing $\xi^\mu = 0$ at the bifurcation surface, we find $\nabla_{\mu} \xi^\nu = \nabla_{\mu} \xi^\nu$, and hence, $\kappa = \kappa$ at the bifurcation surface. In addition, the zero-th law, i.e., constancy of $\kappa$ over the horizon, holds whenever the Killing horizon is
bifurcate \( \mathfrak{H} \). Therefore, \( \mathfrak{p} = \kappa \), not only on the bifurcation surface, but all over the horizon, and therefore the temperatures are the same in both frames.

The angular velocity, \( \mathcal{H}_H \), in the Einstein frame is given by

\[
\mathcal{H}_H^{-1} \left( \xi^\mu_{(t)} + \mathcal{H}_H \xi^\mu_{(\phi)} \right) \xi^\nu_{(\phi)} = 0 ,
\]

and we have

\[
\mathcal{H}_H^{-1} \left( \xi^\mu_{(t)} + \Omega H \xi^\mu_{(\phi)} \right) \xi^\nu_{(\phi)} = 0 ,
\]

from orthogonality of \( \xi^\mu \) to the bifurcation surface in the Einstein frame. Hence we see that the angular velocities in the two frames are the same, \( \mathcal{H}_H = \mathcal{H}_H \), from Eq.(4.6) and (4.7).

C. Entropy, mass and angular momentum

We prove the equivalence of the entropy, the mass and the angular momentum, showing that the integrand of Eq.(3.7) on the bifurcation surface, and the asymptotic forms of the integrands of Eqs.(3.8) and (3.9) at infinity are the same between the two frames.

At the bifurcation surface (\( \xi^\mu = 0 \)), the integrands of Eq.(3.7) are calculated, by setting \( \zeta^\mu = \xi^\mu \) in Eqs.(A12) and (A15) and using the Killing equation for \( \xi^\mu \),

\[
\frac{1}{2} \epsilon_{\beta\alpha\mu\nu} Q^{\beta\alpha} = - \frac{1}{16 \pi} \epsilon_{\alpha\beta\mu\nu} P^{\alpha\gamma} \nabla_{\gamma} \xi^\beta ,
\]

in the original frame, where

\[
P^{\mu\nu} \equiv \frac{\partial f}{\partial R^{\mu\nu}} ,
\]

and

\[
\frac{1}{2} \epsilon_{\beta\alpha\mu\nu} Q^{\beta\alpha} = - \frac{1}{16 \pi} \epsilon_{\alpha\beta\mu\nu} T^{\gamma\gamma} \nabla_{\gamma} \xi^\beta ,
\]

in the Einstein frame, respectively. Substituting the definition of \( \mathcal{H}^{\mu\nu} \), Eq.(2.2), into Eq.(4.10) and noticing again that \( \nabla \xi^\mu = \nabla_\mu \xi^\mu \) at the bifurcation surface, we find the tensor components of Eq.(4.10) are equal to those of Eq.(4.8) on that surface. Then, since \( \mathfrak{p} = \kappa \) which does not vanish and the bifurcation surfaces in both frames appear at the same place, the entropy is the same in these two frames.

If the integrands of Eq.(3.8) and Eq.(3.9) are the same up to the order of \( r^{-2} \) in the asymptotically flat region, the mass and the angular momentum are also the same in the two frames. Since \( \epsilon_{\alpha\beta\mu\nu} \) approaches \( \epsilon_{\alpha\beta\mu\nu} \) as \( r \to \infty \), and those integrands in both frames fall as \( r^{-2} \), it is sufficient to show that

\[
Q^{\beta\alpha} + 2 \chi^{[\beta} B^{\alpha]} = \mathcal{Q}^{\beta\alpha} + 2 \chi^{[\beta} T^{\alpha]} + \mathcal{O}(r^{-3}) ,
\]

where \( \chi^\mu \) is taken to be \( \xi^\mu_{(t)} \) and \( \xi^\mu_{(\phi)} \) to show the equivalence of the mass and the angular momentum, respectively.

The asymptotic forms of \( Q^{\beta\alpha} \) and \( \Theta^\beta \) in the original frame are calculated, from Eqs.(A12) and (A11), as

\[
Q^{\beta\alpha} \to \frac{1}{8 \pi} \left[ \nabla^{[\alpha} \chi^\beta + \chi^\nu \nabla^{[\beta} P^{\alpha]} + \chi^{[\alpha} \nabla_{\nu} P^{\beta]} + \mathcal{O}(r^{-3}) \right] ,
\]

\[
\Theta^\beta \to \frac{1}{16 \pi} \left[ g^{\mu\nu} \nabla^\beta \delta g^{\mu\nu} - \nabla_{\nu} \delta g^{\nu\beta} + v^\beta \delta \phi \right] + \mathcal{O}(r^{-3})
\]

\[
= \frac{1}{16 \pi} \left[ g^{\mu\nu} \nabla^\beta \delta g^{\mu\nu} - \partial_{\nu} \delta g^\beta + v^\beta \delta \phi \right] + \mathcal{O}(r^{-3}) ,
\]
to the order of $r^{-2}$, by replacing $\zeta^\mu$ with $\chi^\mu$ and using the facts that $\nabla_\mu \chi^\nu \sim O(r^{-2})$, $P^\mu \nu \to g^\mu \nu + O(r^{-1})$, $\delta g^\mu \nu \sim O(r^{-1})$ and hence $\nabla_\mu \delta g^\mu \nu = \partial_\mu \delta g^\mu \nu + O(r^{-3})$, where

$$\nu^\beta \equiv \frac{\partial f}{\partial (\nabla_\beta \phi)} . \tag{4.14}$$

If there exists a vector field $C^\beta$ such that $\delta C^\beta$ approaches $\nu^\beta \delta \phi$ and is of order $r^{-2}$ at most in the asymptotically flat region, then $\Theta^\beta \to \delta B^\beta + O(r^{-3})$ as $r \to \infty$, where

$$B^\beta = \frac{1}{16\pi} \left[ g_{\mu \nu}^{-1} g^{\rho \sigma} \partial_\rho \delta g^\mu \nu - \partial_\nu \delta g^\mu \nu + C^\beta \right] . \tag{4.15}$$

The vector $C^\beta$ is expected to exist in the asymptotically flat spacetime, so that $B^\beta$ exists, and thus the mass and the angular momentum are well-defined. In the usual cases, where the kinetic term of $\phi$ takes the form of $(\nabla_\mu \phi)(\nabla^\mu \phi)$, $\nu^\beta \delta \phi$ is of order $r^{-3}$ if $\phi$ approaches a constant in the asymptotically flat region, and then $C^\beta$ does not contribute to $B^\beta$ in those cases.

Since $\nabla_\mu g^\mu \nu + O(r^{-1})$, $\nabla_\mu \nu$ in the Einstein frame given by Eq.(143) with $\zeta^\mu = \chi^\mu$ is calculated as

$$\nabla_\mu \nu = H_{\mu \nu} \equiv \frac{\partial \bar{\mathcal{J}}}{\partial (\nabla_\mu g^\mu \nu)} , \tag{4.17}$$

with $\bar{\mathcal{J}}$ being the Lagrangian of the matter fields in the Einstein frame, i.e.,

$$\bar{\mathcal{J}} = \bar{\mathcal{J}}^\mu \nu \left( \Gamma^\mu_{\rho \sigma} \Gamma_{\rho \sigma \nu} - \Gamma^\mu_{\rho \nu} \Gamma^\sigma_{\rho \sigma} \right) - \bar{\mathcal{J}}^{\mu \nu} \partial_\rho \left( g^{\alpha \beta} \bar{\mathcal{J}}^{\mu \nu} \chi^\alpha \chi^\beta + g^{\alpha \beta} \nabla_\rho \chi^\alpha \chi^\beta \right) + \frac{\sqrt{-g}}{\sqrt{-\bar{g}}} f \left( g^{\sigma \beta} R_{\alpha \beta} (g^{\sigma \sigma}, \bar{\mathcal{J}}^{\mu \nu}, \phi, \nabla_\alpha) \right) \tag{4.18}$$

and the symmetry of $H_{\mu \nu}^\rho$ in the first two indices, $H_{\mu \nu}^\rho = H_{\nu \mu}^\rho$, as well as $H_{\mu \nu} \sim O(r^{-2})$, is used in Eq.(4.16). We rewrite Eq.(4.16), in terms of the tensors in the original frame, by using the relations

$$\nabla_\nu \chi^\beta = \nabla_\nu \chi^\beta - \frac{1}{2} \chi^\alpha \left( g^{\nu \rho} \nabla_\rho \chi^\beta + g^{\nu \rho} \nabla_\rho \chi^\beta - \bar{g}^{\gamma \beta} \bar{g}^{\alpha \sigma} \nabla_\gamma \bar{g}^{\alpha \sigma} \right) , \tag{4.19}$$

and

$$\bar{\mathcal{J}}^{\mu \nu} = \frac{1}{\sqrt{-g}} \frac{1}{\sqrt{-\det (P^{\alpha \beta})}} P^{\mu \nu} , \tag{4.20}$$

which follows from Eqs.(2.2) and (4.3), which leads to

$$\nabla_\nu \nu = \frac{1}{8\pi} \left[ \nabla^{\alpha} \chi^\beta + \chi^\nu \nabla^{[\beta} P^{\alpha]}_{\nu} + \frac{1}{2} \chi^{[\beta} \nabla^{\alpha]} P \right] + O(r^{-3})$$

and

$$Q^{\beta \alpha} = Q^{\beta \alpha} + \frac{1}{8\pi} \left[ \chi^{[\beta} \nabla_\nu P^{\alpha]}_{\nu} + \frac{1}{2} \chi^{[\beta} \nabla^{\alpha]} P \right] + O(r^{-3}) , \tag{4.21}$$

where $P \equiv g_{\mu \nu} P^{\mu \nu}$, and we used Eq.(4.14). By noting that $\delta g^\mu \nu \sim O(r^{-1})$ and hence $\nabla_\rho \delta g^\mu \nu = \partial_\rho \delta g^\mu \nu + O(r^{-3})$ in the asymptotically flat region, the asymptotic form of $\delta g^\mu \nu$ is given, from Eq.(4.14), by
\[ \Theta^\beta \to \frac{1}{16\pi} \left[ g_{\mu\nu}^{-1} \nabla^\beta \delta g^{\mu\nu} - \nabla_\nu \delta g^{\beta\mu} + \varpi^\beta \delta \phi \right] + \mathcal{O}(r^{-3}) \]

\[ = \frac{1}{16\pi} \left[ g_{\mu\nu}^{-1} g^{\rho\beta} \partial_\rho \delta g^{\mu\nu} - \partial_\nu \delta g^{\beta\rho} + \varpi^\beta \delta \phi \right] + \mathcal{O}(r^{-3}) . \] (4.22)

where

\[ \varpi^\beta = \frac{\partial f}{\partial (\nabla_\beta \phi)} . \] (4.23)

We see from Eqs. (4.16) and (4.22) that the contributions, \( H_{\mu\nu}^\rho \) and \( \delta g^{\mu\nu} \), of the exotic matter, \( g^{\mu\nu} \), to the mass and the angular momentum vanish. The relation between \( v^\beta \) and \( v^\beta \) is obtained, from Eq. (4.23), as

\[ v^\beta = \frac{\partial f}{\partial (\nabla_\beta \phi)} = \frac{\nabla_\mu P^{\mu\nu}}{\sqrt{-g}} \varpi^\beta + \mathcal{O}(r^{-3}) , \] (4.24)

where \( R_{\mu\nu} \) is given by Eq. (2.9), and Eq. (2.2) is used. So, \( \varpi^\beta \delta \phi \to \delta C^\beta + \mathcal{O}(r^{-3}) \), and we find \( B^\beta \) such that

\[ \Theta^\beta \to \delta B^\beta + \mathcal{O}(r^{-3}) \], (4.25)

where we have rewritten the equation in terms of the tensors in the original frame, by using Eqs. (4.15) and (4.20) with the facts \( P^{\mu\nu} \to g^{\mu\nu} + \mathcal{O}(r^{-1}) \) and thus \( \partial_\mu P^{\mu\nu} \sim \mathcal{O}(r^{-2}) \).

Then, from Eqs. (4.21) and (4.25), we obtain Eq. (4.11) with \( \chi^\mu \) being \( \xi^\mu(t) \) and \( \xi^\mu(\phi) \). Therefore, the mass and the angular momentum are the same between the original frame and the Einstein frame.

V. CONCLUSION AND DISCUSSION

In summary, we considered a generalized theory of gravity whose Lagrangian is an arbitrary function of the metric, the Ricci tensor, a scalar field and its derivative, which is converted into the Einstein frame via a “Legendre” transformation or a conformal transformation. By following the definition of the thermodynamical variables formulated in Ref. [8,9], we have shown that all the thermodynamical variables are the same between the original frame and the Einstein frame, under the assumptions that the spacetimes in both frames are asymptotically flat, regular and possess event horizons with non-zero temperatures. This conclusion is based on the equivalence of the structure of the horizon, and of the potential, Eq. (3.3), of the symplectic current 3-form, \( \omega_{\mu\nu\rho} \), both at the horizon and in the asymptotically flat region.

The above result is a generalization of the works by Jacobson, Kang and Myers [10,11]. They showed the entropy, defined in Ref. [8,9], in the theory whose Lagrangian is given by either

\[ I = \frac{1}{16\pi} \int d^4x \sqrt{-g} f(R, \phi, \nabla_\alpha \phi) \] (5.1)

or
method [10,11], not only for the entropy, but also the mass and the angular momentum. In the asymptotically flat region. Thus, the definition of Ref. [8,9] is consistent with the field redefinition the same between the two frames, by showing the equality of the potential of the symplectic current 3-form "thermodynamical" mass defined by Eq.(3.8), as well as the angular momentum defined by Eq.(3.9), are Einstein frame is different from that in the Brans–Dicke frame. We have proved, even in such cases, the asympotic structure, in general. Actually, there is an example [19] where the gravitational mass in the absence of the bifurcation surface, and hence the second law has not been established in general. One example which satisfies the second law is the case where the action is given by Eq.(5.1) [10,25]. In this case, in dynamical processes of black holes, however, the definition of the entropy is still ambiguous because of the absence of the bifurcation surface, and hence the second law has not been established in general. One example which satisfies the second law is the case where the action is given by Eq.(5.1) [10,25]. In this case, the theory is converted, via a conformal transformation, into Einstein gravity with scalar fields (one scalar field when the Lagrangian is linear in the scalar curvature, and two fields otherwise). Then, showing that the null energy condition in the Einstein frame is satisfied, we can prove that the entropy in the Einstein frame, which is proportional to the area, is a non-decreasing quantity, due to Hawking’s area theorem in Einstein gravity. Therefore, the second law holds, if we define the entropy in the original frame to be equal to that in the Einstein frame also in the dynamical processes. This definition of the dynamical entropy differs from that proposed in Ref. [8], which uses boost invariant parts of the field variables, but the second law has not been established for the latter definition even in the case of the action(5.2). In general, as the above example, the equivalence of the entropy between the original frame and the Einstein frame for stationary black holes will provide a useful clue to analyze the dynamical entropy, since useful results have been already established in Einstein gravity.

$$I = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left[ R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} \right]$$ (5.2)

It does not seem to be easily accomplished to define the dynamical entropy in the general case of the action(2.1), because a null vector in the original frame may be transformed to either a timelike, null or spacelike vector in the Einstein frame, and the energy condition of the exotic matter, $g^{\mu\nu}$, in the Einstein
frame is not trivial. Therefore, investigations of the second law in arbitrary quasi-stationary processes, which include the effects of the higher curvature interactions, seem to be important. Our result can be applied to such analyses, since the effects of the higher curvature interactions are included in the action (2.11) in the Einstein frame. We may clarify under what condition the second law is satisfied, or whether we need to consider the generalized second law, at least in the quasi-stationary processes, which might afford us a facility in the analysis of the dynamical entropy, i.e., its general definition and the second law.

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APPENDIX A: DERIVATION OF $\Theta^\beta$ AND $Q^\beta \alpha$

In this appendix, we derive the explicit forms of $\Theta^\beta$ and $Q^\beta \alpha$ both in the original frame and in the Einstein frame. We begin with an action given by

$$I = \frac{1}{16\pi} \int d^4x \sqrt{-g} L(g^{\alpha \beta}, R_{\alpha \beta \rho \lambda}, \phi, \nabla_\alpha \phi, s^{\alpha \beta}, \nabla_\rho s^{\alpha \beta}), \quad (A1)$$

where $L$ is a scalar function of the metric, $g^{\alpha \beta}$, the Riemann tensor, $R_{\alpha \beta \rho \lambda}$, a scalar field, $\phi$, a symmetric second rank tensor, $s^{\alpha \beta}$, and the derivatives of $\phi$ and $s^{\alpha \beta}$. Variation of this action gives, for the equation of motion, $E^{(i)}$ of $\psi^{(i)}$ and $\Theta^\beta$, as

$$E^{(g)}_{\mu \nu} = \frac{1}{16\pi} \left[ M_{\mu \nu} - \frac{1}{2} g_{\mu \nu} L - \nabla_\lambda X^{\alpha \beta} (\mu \rho \beta \alpha) - 2 \nabla_\rho \nabla_\lambda X^{\alpha \beta} (\mu \rho \beta \alpha) \right], \quad (A2)$$

$$E^{(s)}_{\mu \nu} = \frac{1}{16\pi} \left[ N_{\mu \nu} - \nabla_\rho K_{\rho \mu \nu} \right], \quad (A3)$$

$$E^{(\phi)} = \frac{1}{16\pi} \left[ \frac{\partial L}{\partial \phi} - \nabla_\mu \omega^\mu \right], \quad (A4)$$

$$\Theta^\beta = \frac{1}{16\pi} \left[ 2 \left( \nabla_\alpha X^{(\alpha \beta)} (\mu \rho \beta \alpha) \right) \delta g^{\mu \rho} - 2 X^{(\alpha \beta)} (\mu \rho \beta \alpha) \nabla_\alpha \delta g^{\mu \rho} \right] + K_{\mu \nu} s^{\alpha \beta} \delta g^{\mu \nu} - K_{\alpha \mu} s^{\alpha \beta} \delta g^{\mu \nu} - K_{\alpha \nu} s^{\alpha \beta} \delta g^{\mu \nu} + w^\beta \delta \phi + K_{\mu \nu} \delta s^{\mu \nu} \right], \quad (A5)$$

where

$$M_{\mu \nu} = \frac{\partial L}{\partial g_{\mu \nu}}, \quad X_{\mu \nu \rho \lambda} = \frac{\partial L}{\partial R_{\mu \nu \rho \lambda}}, \quad K_{\mu \nu} = \frac{\partial L}{\partial (\nabla_\mu s^{\mu \nu})},$$

$$N_{\mu \nu} = \frac{\partial L}{\partial s^{\mu \nu}}, \quad w^\mu = \frac{\partial L}{\partial (\nabla_\mu \phi)}.{ \quad (A6)$$

Then we regard the variation as a coordinate transformation induced by an arbitrary vector, $\zeta^\mu$, and obtain $J^\beta = \Theta^\beta - \zeta^\beta L$ of the form
\[ J^\beta = \frac{1}{8\pi} \left[ \nabla_\alpha \left\{ X^{\alpha\beta\mu\nu} \nabla_\mu \zeta_\nu - 2\zeta_\nu \nabla_\mu X^{\alpha\beta\mu\nu} + \zeta^\nu K_{\mu}^{[\alpha} s^{\beta]\mu} + \zeta^\nu K_{\mu}^{[\alpha} s^{\beta]\mu} + \zeta^\nu K_{\mu}^{[\alpha} s^{\beta]\mu} \right\} 
\]

\[ + E^{(h)} \zeta^\mu + E^{(s)} \beta^\alpha \zeta^\mu - 2\zeta^\mu \left( M^\beta - 2R_{\alpha\nu\rho\mu} X^{\alpha\nu\rho\mu} - \frac{1}{2} \eta^\beta \nabla_\mu \phi \right) \]

\[ - \frac{1}{2} K_{\alpha\nu}^{\beta} \nabla_\mu s^{\alpha\nu} + K_{\alpha\nu}^{\beta} \nabla_\mu s^{\alpha\beta} + N_{\alpha\nu} s^{\alpha\beta} \right) \right], \]  

by integrating by parts and substituting Eqs.\((A7)\) and \((A9)\). Since \(L\) is a scalar, we have

\[ 0 = L_\eta L - \eta^\mu \nabla_\mu L \]

\[ = -2 \left[ M^\beta - 2R_{\alpha\nu\rho\mu} X^{\alpha\nu\rho\mu} - \frac{1}{2} \eta^\beta \nabla_\mu \phi \right] \]

\[ - \frac{1}{2} K_{\alpha\nu}^{\beta} \nabla_\mu s^{\alpha\nu} + K_{\alpha\nu}^{\beta} \nabla_\mu s^{\alpha\beta} + N_{\alpha\nu} s^{\alpha\beta} \right] (\nabla_\beta \eta^\mu), \]  

for an arbitrary vector, \(\eta^\mu\), and thus the inside of the round bracket in Eq.\((A7)\) vanishes identically. Then the potential \(Q^{\beta\alpha}\) of \(J^\beta\) is given as

\[ Q^{\beta\alpha} = \frac{1}{8\pi} \left\{ X^{\alpha\beta\mu\nu} \nabla_\mu \zeta_\nu - 2\zeta_\nu \nabla_\mu X^{\alpha\beta\mu\nu} + \zeta^\nu K_{\mu}^{[\alpha} s^{\beta]\mu} + \zeta^\nu K_{\mu}^{[\alpha} s^{\beta]\mu} + \zeta^\nu K_{\mu}^{[\alpha} s^{\beta]\mu} \right\}, \]  

when the equations of motion, \(E^{(h)} = 0\) and \(E^{(s)} = 0\), are satisfied.

In the original frame, \(X^{\mu\nu\rho\lambda}, \eta^\mu\) and \(K^{\mu\nu}\) are given as

\[ X^{\mu\nu\rho\lambda} = \frac{1}{2} \left\{ P^{\mu[\lambda} g^{\rho\nu]} + P^{\rho[\lambda} g^{\nu\mu]} \right\}, \quad \eta^\mu = \eta^\mu, \quad K^{\mu\nu} = 0. \]  

Then, by substituting Eq.\((A10)\) into Eqs.\((A3)\) and \((A9)\), \(\Theta^\beta\) and \(Q^{\beta\alpha}\) in the original frame are calculated as

\[ \Theta^\beta = \frac{1}{32\pi} \left[ (\nabla_\nu P^{\beta}_{\mu}) \delta g^{\mu\nu} - g^{-1}_{\mu\nu} (\nabla_\alpha P^{\alpha}_{\mu}) \delta g^{\mu\nu} + (\nabla_\mu P^{\beta}_{\alpha}) \delta g^{\alpha\beta} \right. \]

\[ - (\nabla_\beta P^{\mu}_{\nu}) \delta g^{\mu\nu} + g^{-1}_{\mu\nu} P^{\alpha\beta} \nabla_\alpha \delta g^{\mu\nu} - P^{\nu}_{\alpha} \nabla_\nu \delta g^{\alpha\beta} \]

\[ - P^{\nu}_{\mu} \nabla_\nu \delta g^{\mu\nu} + P^{\mu}_{\nu} \nabla_\mu \delta g^{\alpha\beta} + 2\eta^\beta \delta \phi \right], \]  

\[ Q^{\beta\alpha} = \frac{1}{16\pi} \left\{ P^{\mu[\alpha} \nabla_\nu \zeta_{\beta]} + P^{\rho[\beta} \nabla_\alpha \zeta_{\nu]} + 2 \left( \zeta_\nu \nabla^\beta P^{\alpha\nu} + \zeta^\alpha \nabla_\nu P^{\beta\nu} \right) \right\}. \]  

On the other hand, in the Einstein frame, we have

\[ X^{\mu\nu\rho\lambda} = g^{\mu[\lambda} g^{\rho\nu]} , \quad \eta^\mu = \eta^\mu , \quad K^{\mu\nu} = H^{\mu\nu} , \]  

and hence, we obtain \(\Theta^\beta\) and \(Q^{\beta\alpha}\) in the Einstein frame, as

\[ \Theta^\beta = \frac{1}{16\pi} \left[ \nabla_{\mu\nu} \delta g^{\mu\nu} - \nabla_\mu \delta g^{\mu\nu} + \nabla_\nu \delta g^{\mu\nu} + \delta \phi^\beta \right] \]

\[ + H^{\beta}_{\mu\nu} \delta g^{\mu\nu} + H^{\beta}_{\mu\nu} \delta g^{\mu\nu} - H^{\beta}_{\mu\nu} \delta g^{\mu\nu} \]  

\[ + H^{\beta}_{\mu\nu} \delta g^{\mu\nu} + H^{\beta}_{\mu\nu} \delta g^{\mu\nu} \]  

\[ \]  

\[ \Theta^\beta = \frac{1}{8\pi} \left[ \nabla^\alpha \zeta_{\beta} + \zeta^\nu H^{\beta}_{\mu\nu} |^{\alpha} g^{\beta\mu} + \zeta^\nu H^{\beta}_{\mu\nu} |^{\alpha} g^{\beta\mu} + \zeta^\nu H^{\beta}_{\mu\nu} |^{\alpha} g^{\beta\mu} \right]. \]  

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