Bulk viscosity in hyperonic star and r-mode instability

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(Dated: February 23, 2010)

We consider a rotating neutron star with the presence of hyperons in its core, using an equation of state in an effective chiral model within the relativistic mean field approximation. We calculate the hyperonic bulk viscosity coefficient due to nonleptonic weak interactions. By estimating the damping timescales of the dissipative processes, we investigate its role in the suppression of gravitationally driven instabilities in the r-mode. We observe that r-mode instability remains very much significant for hyperon core temperature of around 10^8 K, resulting in a comparatively larger instability window. We find that such instability can reduce the angular velocity of the rapidly rotating star considerably up to \( \sim 0.04 \Omega_K \), with \( \Omega_K \) as the Keplerian angular velocity.

PACS numbers: 97.60.Jd, 04.30.Db, 04.40.Dg, 26.60. +c

I. INTRODUCTION

The study of neutron stars are natural testing ground for studying extremely dense matter. The densities in the interior of such stars can reach up to several times the nuclear matter saturation density. At such high densities, with higher fermi momenta being available, high mass hadrons can be accommodated leading to a hyperonic core in the neutron star interior [1, 2]. There could also be the possibility that at such densities, when the nucleons are crushed, there could be quark matter [3] which can result in a color superconducting core in the interior of the neutron star [4]. In fact, there are different possibilities of the ground state of dense matter, which could be stable strange quark matter [3], and various possibilities of color superconducting matter [5]. The reason is that the true ground state of the dense matter system for densities relevant for the densities in the interior of neutron stars is still an open problem because of the inherent nonperturbative nature of strong interaction physics. Moreover, external conditions like electrical and color charge neutrality conditions for the bulk matter in the interior of the star can also lead to various different possible phases of quark matter [7, 8]. This has given rise to various possibilities of compact stellar objects like neutron stars, strange stars, hyperonic stars or hybrid stars with a quark matter core and a crust of hadronic matter [2]. Obviously, it becomes very challenging to distinguish various compact stellar objects observationally.

One of the various signatures suggested to distinguish different compact stars has been the r-mode instability [9, 10]. Various pulsating modes exist in neutron stars classified by the nature of the restoring forces. The r-modes correspond to the pulsating modes of the rotating stars where the restoring force is the Coriolis force and these modes are axial modes [12]. As the fluid inside the star is self gravitating, these oscillations can couple to metric perturbations and lead to emission of gravitational waves [13]. Gravitational modes drive the r-modes unstable due to Chandrasekhar-Friedman-Schutz (CFS) mechanism [14]. This phenomenon will occur if the damping is sufficiently small and therefore provides a probe to study the viscosity of the matter in the interior of the star. While shear viscosity prevents differential rotation in a star, bulk viscosity dampens the density fluctuations in the star. Shear viscosity seems to be important at lower temperatures while bulk viscosity becomes the dominating dissipation mechanism at higher temperatures (\( \sim 10^8 K \)). Further, since typical r-mode frequencies are of the order of rotational frequencies of the stars (1s^{-1} < \omega < 10^3 s^{-1} ), which are of the order of weak interactions, therefore the viscosity that is of relevance for r-mode instability will be dominated by the weak processes. Bulk viscosity is produced when the mode oscillations induce perturbations in pressure and density and drives the system away from \( \beta \)-equilibrium. As a result energy is dissipated from the system as the weak interaction tries to restore the equilibrium. While for hadronic neutron star matter, modified Urca processes \((n + n \rightarrow n + p + e^- + \bar{\nu}_e)\) involving leptons are important, it has been noted that the damping of the instability is dominated by large bulk viscosity arising due to nonleptonic processes involving hyperons [15]. The corresponding viscosities are not only stronger, the temperature dependence is also different (varying as \( T^{-2} \) instead of \( T^6 \) ) which makes the hyperon bulk viscosity important at lower temperatures. Bulk viscosities of dense baryonic matter have been calculated under various assumptions as well as conditions over last few years [15, 21].

The study of the r-mode oscillations provides an avenue to study the density profile of the star along with the mass and the radius which in turn depends upon the equation of state for the matter in the interior of the star. Phenomenologically, for hadronic matter parallel to the \( \sigma - \omega \) model, popularly known as Walecka model [23, 24], the chiral effective models have been de-
density under consideration is given by \[35–37\]:

In the subsequent subsections in the section, we derive the model that we shall be using to derive the equation of state of the present investigation.

We organise the paper as follows. In the first subsection of section II, we describe very briefly the chiral model including hyperons, which has been considered earlier in the context of neutron star matter. It will be used later to calculate the bulk viscosity of the hyperonic matter and discuss their role in r-mode damping. In section III, we discuss the calculation of bulk viscosity within this model and its consequences regarding the r-mode instability problem in rotating neutron star with a hyperonic core.

We organise the paper as follows. In the first subsection of section II, we describe very briefly the chiral model that we shall be using to derive the equation of state needed to study the structure of the neutron star. In the subsequent subsections in the section, we derive the bulk viscosity of the hyperonic matter and discuss their role in r-mode damping. In section III, we discuss the results of the present calculations. Finally, in section IV, we summarise the results and give possible outlook of the present investigation.

\[\mathcal{L} = \bar{\psi}_B \left( i\gamma_\mu \partial^\mu - g_\omega B \gamma_\mu \omega^\mu - \frac{1}{2} g_\rho B \bar{\rho}_\mu \gamma^\mu \psi_B \right) - g_{\sigma B} \bar{\psi}_B \left( \sigma + i\gamma_5 \tau \cdot \bar{\tau} \right) \psi_B + \frac{1}{2} \left( \bar{\rho}_\mu \bar{\rho}^\mu + \bar{\sigma}_\mu \sigma^\mu \right) - \frac{\lambda}{4} (x^2 - x_0^2)^2 - \frac{\lambda B}{6} (x^2 - x_0^2)^3 - \frac{\lambda C}{8} (x^2 - x_0^2)^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} g_{\sigma B} x^2 \omega B \omega^\mu - \frac{1}{4} \bar{\rho}_\mu \bar{\rho}^\mu + \frac{1}{2} m_\rho^2 \bar{\rho}_\mu \bar{\rho}^\mu . \]

The first two lines of the above Lagrangian density represent the interaction of baryons $\Psi_B$ with the aforesaid mesons. In the next two lines, we have the kinetic and the non-linear terms in the pseudoscalar-isovector pion field $'\pi'$, the scalar field $'$'\sigma'$', and with $x^2 = \pi^2 + \sigma^2$. Finally in the last two lines, we have the field strength and the mass term for the vector field $'\omega'$ and the iso-vector field $'\bar{\rho}'$ meson. The terms in equation (1) with the subscript $'B'$ should be interpreted as sum over the states of lowest baryonic octet. In this paper, we shall be concerned only with the normal non-pion condensed state of matter and, so we take $<\pi>=0$.

The interaction of the scalar and the pseudoscalar mesons with the vector boson generates a dynamical mass for the vector bosons through spontaneous breaking of the chiral symmetry with scalar field getting the vacuum expectation value $x_0$. Then the masses of the baryons, the scalar and the vector mesons, are respectively given by

\[m_B = g_{\sigma B} x_0, \quad m_\sigma = \sqrt{2\lambda} x_0, \quad m_\omega = g_\omega B x_0 . \]

In the above, $x_0$ is the vacuum expectation value of the $\sigma$ field. We could have taken an interaction of the $\rho$-meson with the scalar and the pseudoscalar mesons similar to the omega meson. However, a dynamical mass generation mechanism of the $\rho$-meson in a similar manner will not generate the correct symmetry energy. Therefore, we have taken an explicit mass term for the isovector $\rho$-meson similar to what was considered in earlier works \[32, 35, 36\].

We use mean-field approximation \( (23) \) to evaluate the meson fields in our present calculations. In the mean-field treatment, one assumes the mesonic fields to be uniform, i.e., without any quantum fluctuations. The details of the model that we use in our present investigation and its attributes such as the derivation of the equation of motion of the meson fields and its equation of state \( (\varepsilon & P) \) of the many baryonic system, can be found in our preceding work \[36\]. The vector meson and the iso-vector meson field equations in the mean field framework are then given by

\[\omega_0 = \sum_B \frac{\rho_B}{g_{\omega B} x_0^2} . \]
\[ \rho_{03} = \sum_{B} \frac{g_{\rho B}}{m_{\rho}^2} I_{3B} \rho_{B}. \] (4)

In the above equations the quantity \( \rho_{B} \) is the baryon density and \( I_{3B} \) is the third component of the isospin of each baryon species.

The scalar field equation can be written in terms of the variable \( Y = x/x_0 \) with \( x = (\langle \sigma^2 + \pi^2 \rangle)^{1/2} \) as \([35, 36]\)

\[ \sum_{B} \left[ (1 - Y^2) - \frac{B}{c_{\omega B}} (1 - Y^2)^2 + \frac{C}{c_{\omega B}} (1 - Y^2)^3 \right] + \frac{2 c_{\sigma B} c_{\omega B} \rho_{B}^2}{m_B^2 Y^4} - \frac{2 c_{\sigma B} \rho_{SB}}{m_B Y} = 0, \quad (5) \]

where the effective mass of the baryonic species is \( m_B^* = Y m_B \) and \( c_{\omega B} = g_{\omega B}^2 / m_B^2 \) are the usual scalar and vector coupling constants respectively. Similarly, in the present model describing dense matter, the \( \omega \)-meson mass is generated dynamically. This vector meson mass enters in Eq. (5) through the ratio \( c_{\omega B} \equiv (g_{\omega B} / m_B^2) \equiv 1/x_0 \). Various parameters of the model for the nuclear matter case (i-th nucleon meson couplings, \( C, C_{\sigma}, C_{\omega}, C_{\rho} \) and the nonlinear couplings \( B, C \)) are fitted from nuclear matter saturation properties \([32, 33, 38]\). It was shown recently that these parameters are rather constrained by the nuclear matter saturation properties like the binding energy per nucleon, saturation density, the nuclear incompressibility, as well as the asymmetry energy, the effective mass of the nucleon and the pion decay constant \( f_\pi \). Further in Eq.(5), the quantities \( \rho_B \) and \( \rho_{SB} \) are the baryon density and the scalar density for a given baryon species given respectively as,

\[ \rho_B = \frac{\gamma}{(2\pi)^3} \int_{0}^{k_B} d^3 k, \]

\[ \rho_{SB} = \frac{\gamma}{(2\pi)^3} \int_{0}^{k_B} \frac{m_B^* d^3 k}{\sqrt{k^2 + m_B^*}}, \]

where \( k_B \) is the fermi momentum of the baryon and \( \gamma = 2 \) is the spin degeneracy factor.

For a given baryon density, the total energy density \( \varepsilon \) and the pressure \( P \) can be written in terms of the dimensionless variable \( Y = x/x_0 \) as

\[ \varepsilon = \frac{2}{\pi^2} \int_{0}^{k_B} k^2 dk \sqrt{k^2 + m_B^*} + \frac{m_B^2 (1 - Y^2)^2}{8 c_{\omega B}} \]

\[ - \frac{m_B^2 B}{12 c_{\omega B} c_{\sigma B}} (1 - Y^2)^3 + \frac{m_B^2 C}{16 c_{\omega B} c_{\sigma B}} (1 - Y^2)^4 \]

\[ + \frac{1}{2 Y^2} c_{\omega B} \rho_{B}^2 + \frac{1}{2} m_B^2 \rho_{03} \]

\[ + \frac{1}{\pi^2} \sum_{\lambda=e,\mu} \int_{0}^{k_\lambda} k^2 dk \sqrt{k^2 + m_\lambda^*}, \]

\[ P = \frac{2}{3\pi^2} \int_{0}^{k_B} k^4 dk \sqrt{k^2 + m_B^*} - \frac{m_B^2 (1 - Y^2)^2}{8 c_{\omega B}} \]

\[ + \frac{m_B^2 B}{12 c_{\omega B} c_{\sigma B}} (1 - Y^2)^3 - \frac{m_B^2 C}{16 c_{\omega B} c_{\sigma B}} (1 - Y^2)^4 \]

\[ + \frac{1}{2 Y^2} c_{\omega B} \rho_{B}^2 + \frac{1}{2} m_B^2 \rho_{03} \]

\[ + \frac{1}{\pi^2} \sum_{\lambda=e,\mu} \int_{0}^{k_\lambda} k^4 dk \sqrt{k^2 + m_\lambda^*}, \]

\[ \sum_{B} Q_B \rho_B + \sum_{l} Q_l \rho_l = 0, \quad (10) \]

where \( \rho_B \) and \( \rho_l \) are the baryon and the lepton (\( e,\mu \)) number densities with \( Q_B \) and \( Q_l \) as their respective electric charges.

B. Hyperon Bulk-viscosity

Bulk viscosity characterizes the response of the system to an externally oscillating change in the volume. The volume expansion or contraction leads to a change in density or the chemical potential of the system. This drives the system out of chemical equilibrium. The equilibrium is restored by the microscopic processes. If the equilibrium time scales are comparable to the oscillation time scales, there will be energy dissipation. In the context of neutron star, the typical oscillation frequencies are less than a kilo hertz. Therefore, the microscopic processes that will be relevant are the weak processes.

It is already known that the non-leptonic processes containing hyperons lead to high values of bulk viscosity for rotating neutron stars with temperature \( 10^9 - 10^{10} \text{K} \) \([16, 18, 39]\). The leptonic processes are suppressed by smaller phase space factors. Thus the relevant reactions which are going to give a lower limit on the rates (or upper limit on bulk viscosity) are

\[ n + n \leftrightarrow p + \Sigma^- \quad (11) \]

\[ n + p \leftrightarrow p + \Lambda^0 \quad (12) \]

\[ n + n \leftrightarrow n + \Lambda^0 \quad (13) \]

The coefficient of bulk viscosity relates difference between the perturbed pressure \( p \) and the thermodynamic pressure \( \hat{p} \) to the macroscopic expansion of the fluid as
\[ p - \dot{\rho} = -\zeta \nabla \cdot \mathbf{v} \]  \hspace{1cm} (14)

where \( \mathbf{v} \) is the velocity of the fluid element and \( \zeta \) is the coefficient of bulk viscosity, which, in general, is complex in nature \[22\].

The relativistic expression for the real part of \( \zeta \), which corresponds to the damping, can be calculated in terms of microscopic equilibrium restoring reaction rates \[18, 42\]. Within a relaxation time approximation, the real part of \( \zeta \) is given as,

\[ \zeta = \frac{p(\gamma_\infty - \gamma_0) \tau}{1 + (\omega \tau)^2} \]  \hspace{1cm} (15)

where \( \gamma_\infty \) and \( \gamma_0 \) are the “infinite” and “zero” frequency adiabatic index respectively. \( \omega \) is the angular frequency of the perturbation in co-rotating frame and \( \tau \) is the net equilibrium restoring microscopic relaxation time. The expression for \( \gamma_\infty - \gamma_0 \) is

\[ \gamma_\infty - \gamma_0 = -\frac{n_B^2}{p} \frac{\partial p}{\partial \rho_B} \frac{d\tilde{x}}{d\tilde{n}_B} \]  \hspace{1cm} (16)

Here \( n_B \) corresponds to the total baryon density and \( \tilde{x} = \frac{n_B}{\tilde{n}_B} \) is the neutron fraction. Thus the difference \( \gamma_\infty - \gamma_0 \) can be calculated from a given equation of state. The angular frequency \( \omega \) of the \( r \)-mode \( (l=2, m=2) \) in a co-rotating frame is given in terms of the Keplerian frequency \( \Omega_K \) of the rotating star as \( \omega = \frac{2\sqrt{n_B}}{l(l+1)} \Omega_K \) \[24\]).

The prominent reactions involving the lightest hyperons, \( \Sigma^- \) and \( \Lambda^0 \), which have higher population in a given star are as given by equations \[11\] and \[12\]. The rates of these reactions can be calculated from the tree-level Feynman diagrams involving the exchange of a W boson. We shall discuss more regarding this in Section III. The relaxation time \( \tau \) (at a temperature \( T \)), when both \( \Sigma^- \) and \( \Lambda^0 \) are present, is given by \[18, 19\]

\[ \frac{1}{\tau} = \left( \frac{k_B T^2}{192\pi^3} \right) \left( k_F \langle |\mathcal{M}_E^2| \rangle + k_\Lambda \langle |\mathcal{M}_A^2| \rangle \right) \frac{\delta \mu}{n_B \delta \tilde{x}_n} \]  \hspace{1cm} (17)

where \( k_B \) is the Boltzmann’s constant and \( k_F \) and \( k_\Lambda \) are the Fermi momenta of these hyperons. \( \delta \mu \equiv \delta \mu_n - \delta \mu_\Lambda \) is the chemical potential imbalance. \( \delta x_n = x_n - \tilde{x}_n \) is the small difference between the perturbed and equilibrium values of the neutron fraction. \( \langle |\mathcal{M}_E^2| \rangle \) and \( \langle |\mathcal{M}_A^2| \rangle \) are the mean squares of the reactions calculated from the Feynman diagrams. We refer \[18, 19\] for the expressions of \( \langle |\mathcal{M}_E^2| \rangle \) and \( \langle |\mathcal{M}_A^2| \rangle \). We note that the contribution from \( \Lambda \) hyperons will not be present in Equation \[17\] while considering a neutron star medium before the appearance of \( \Lambda \).

The factor \( \delta \mu / n_B \delta x_n \) is determined from the constraints imposed by the electric charge neutrality and the baryon number conservation given respectively as

\[ \delta x_p - \delta x_\Sigma = 0 \]  \hspace{1cm} (18)
\[ \delta x_n + \delta x_\Lambda + \delta x_p + \delta x_\Sigma = 0 \]  \hspace{1cm} (19)

together with the condition that the non-leptonic strong interaction reaction

\[ n + \Lambda^0 \leftrightarrow p + \Sigma^- \]  \hspace{1cm} (20)

which has a higher rate, is in equilibrium compared to weak interaction processes giving rise to the bulk viscosity. Equilibrium of this reaction implies that both the reactions \[11\] and \[12\] have equal chemical potential imbalance,

\[ \delta \mu \equiv \delta \mu_n - \delta \mu_\Lambda = 2\delta \mu_p - \delta \mu_\Sigma. \]  \hspace{1cm} (21)

Using these constraints, we can write,

\[ \frac{\delta \mu}{n_B \delta x_n} = \frac{\alpha_{nn} + (\beta_n - \beta_\Lambda)(\alpha_{np} - \alpha_{p\Sigma} + \alpha_{n\Sigma} - \alpha_{\Lambda\Sigma})}{2\beta_\Lambda - \beta_p - \beta_\Sigma} \]
\[ -\frac{\alpha_{\Lambda n} - (2\beta_n - \beta_p - \beta_\Sigma)(\alpha_{n\Sigma} - \alpha_{\Lambda\Sigma})}{2\beta_\Lambda - \beta_p - \beta_\Sigma} \]  \hspace{1cm} (22)

where \( \alpha_{ij} = \left( \frac{\partial n_i}{\partial \nu_j} \right)_{n_B, k \neq j} \) and \( \beta_i = \alpha_{ni} + \alpha_{n\Lambda} - \alpha_{pi} - \alpha_{\pi \Sigma}. \)

These expressions are for the case when both the \( \Sigma^- \) and \( \Lambda^0 \) hyperons are present. One can not use the reaction \[20\] while considering the region where we have only \( \Sigma^- \) hyperons. In that case, instead of Eq. \[22\], we have

\[ \frac{2\delta \mu}{n_B \delta x_n} = 4\alpha_{nn} - 2(\alpha_{np} + \alpha_{n\Sigma} + \alpha_{np} + \alpha_{n\Sigma}) \]
\[ + \alpha_{pp} + \alpha_{p\Sigma} + \alpha_{p\Sigma} + \alpha_{\Sigma\Sigma} \]  \hspace{1cm} (23)

Now the \( \alpha_{ij} \)’s can be found out using the expression for baryon chemical potential and the equations of motion of the mesonic fields given by equations \[34\]. In general, the form of the \( \alpha_{ij} \) is given as

\[ \alpha_{ij} = \frac{m_i^* m_j}{\sqrt{k_F^2 + m_i^*}} \frac{\partial Y}{\partial \nu_j} + \frac{\gamma_{ij}}{g_{ij}} \left( \frac{Y x_0}{(Y x_0)^3} \right) \sum_B \frac{\rho_B}{g_{z-B}} \frac{\partial Y}{\partial \nu_j} + \frac{1}{2m_p} \left( g_{\rho \rho}(I_{3i}I_{3j}) \right) \]

for \( i \neq j \) and

\[ \alpha_{ii} = \frac{m_i^* m_i}{\sqrt{k_F^2 + m_i^*}} \frac{\partial Y}{\partial \nu_i} + \frac{\gamma_i}{k_F^2 + m_i^*} \frac{1}{(Y x_0)^2} \]
\[ - \frac{2\gamma_{ii}x_0}{(Y x_0)^3} \sum_B \frac{\rho_B}{g_{z-B}} \frac{\partial Y}{\partial \nu_i} + \frac{1}{2m_p} \left( g_{\rho \rho}(I_{3i}I_{3i}) \right). \]  \hspace{1cm} (25)

As before, \( Y \) is the scalar field expectation value in the medium in units of its vacuum expectation value. Further, \( \frac{\partial Y}{\partial \nu_i} \) are calculated from the scalar field equation Eq. \[5\] and are given by

\[ \frac{\partial Y}{\partial \nu_i} = \frac{1}{D} \left( \frac{2c_{\sigma \rho} c_{\omega \rho}}{m_B^2 Y^4} - \frac{c_{\sigma i} m_i^*}{m_i Y \sqrt{k_F^2 + m_i^*}} \right) \]  \hspace{1cm} (26)
with

\[ D = \sum_B \left[ \frac{2B}{c_B} (Y^2 - 1) Y + \frac{3C}{c_B} (Y^2 - 1)^2 Y - \frac{c_B B \rho_S B}{m_B Y^2} + \frac{4c_B c_\omega B_B^2}{m_B Y^2} \right. \]
\[ \left. + \frac{c_B}{m_B Y} \frac{\gamma}{(2\pi)^3} \int_o^{k|\delta v|} d^3k \frac{k^2 m_B}{(k^2 + m_B^2)^{3/2}} \right] \]

Using equations (28)-(27), one can compute the relaxation time from Eq. (17) and hence the bulk viscosity given in Eq. (15), for a given equation of state.

C. R-mode damping

As mentioned in the introduction section, the r-modes correspond to the axial modes where the restoring force is the Coriolis force. The r-mode frequency \( \omega_c \) in the corotating frame, to first order in \( \Omega \), the rotation frequency of the star is given by

\[ \omega_c = \frac{2m\Omega}{l(l+1)} + O(\Omega^3) \]  

(28)

The r-modes correspond to \( l = m = 1 \). The r-mode frequency observed by an inertial observer is given by

\[ \omega_0 = \omega_c - m\Omega = \left( \frac{2}{l(l+1)} - 1 \right) m\Omega. \]  

(29)

Thus, \( l = m = 2 \) mode becomes most unstable to emission of gravitational waves due to Chandrasekhar Friedman Shultz (CFS) mechanism. In a rotating star emission of these waves causes the modes to grow. This instability can get damped due to the various viscosities of the stellar matter. This happens when the damping time scales associated with these viscous processes are comparable to the gravitational radiation (GR) time scale.

We need the expressions for the time scales associated with the dissipative processes and GR in order to understand the nature of the damping of the r-modes. The imaginary part of the dissipative time scale (which causes the damping) is given by

\[ \frac{1}{\tau_i} = \frac{1}{2\tilde{E}} \left( \frac{d\tilde{E}}{dt} \right) \]  

(30)

where \( i \) labels the various dissipative phenomena like hyperonic bulk viscosity (B), bulk viscosity due to Urca processes (U), shear viscosity (\( \eta \)), bulk viscosity due to Urca processes, (44)\). shear viscosity (\( \eta \)) GR etc. In the above, \( \tilde{E} \) is energy of the r-mode in the co-rotating frame. This can arise both from velocity perturbation as well as the perturbation of the gravitational potential. For a slowly rotating star, the dominant contribution is from the velocity perturbation and is given as

\[ \tilde{E} = \frac{1}{2} \int \rho|\delta\vec{v}|^2 d^3x, \]  

(31)

with, \( \rho(r) \) being the mass density of the star. Assuming the spherical symmetry, mode energy can be reduced into an one dimensional integral as

\[ \tilde{E} = \frac{1}{2} \alpha^2 \Omega^2 R^{-2l+2} \int_o^R \rho(r)r^{2l+2} dr. \]  

(32)

with \( l = m = 2 \) for the r-modes and \( R \) denotes the radius of the star. \( \alpha \) is the dimensionless amplitude coefficient of the mode, which gets cancelled out in the \( \tau \) calculation. This energy is dissipated both by gravitational radiation as well as thermodynamic transport of the fluid (12)\). The dissipation rate due to the bulk viscosity effects is given by

\[ \frac{d\tilde{E}_B}{dt} = -4\pi \int_o^R \Re \zeta(r) \left| \vec{\nabla} \cdot \delta\vec{v} \right|^2 r^2 dr. \]  

(33)

Here, in general, \( \left| \vec{\nabla} \cdot \delta\vec{v} \right|^2 \) depends upon the radial and the angular co-ordinates. In slowly rotating stars, to the lowest order, \( \zeta \) depends only on the radius. Therefore, to the lowest order in \( \Omega \), it is possible to write the bulk viscosity dissipation rate in Eq. (33) as an one dimensional integral by defining a quantity which is the angle averaged expansion squared \( \left< |\vec{\nabla} \cdot \delta\vec{v}|^2 \right> \). In terms of this quantity, Eq. (33) can be written as

\[ \frac{d\tilde{E}_B}{dt} = -4\pi \int_o^R \Re \zeta(r) \left< |\vec{\nabla} \cdot \delta\vec{v}|^2 \right> r^2 dr. \]  

(34)

where \( \left< |\vec{\nabla} \cdot \delta\vec{v}|^2 \right> \) can be determined numerically (44). However Ref.s (18)\). give an analytic expression

\[ \left< |\vec{\nabla} \cdot \delta\vec{v}|^2 \right> = \frac{\alpha^2 \Omega^2 \left( \frac{r}{R} \right)^6 \left( 1 + 0.86 \left( \frac{r}{R} \right)^2 \right)}{\pi G\rho} \frac{\rho_b}{690} \]

(35)

which fits to the numerical data. Here \( G \) is the gravitational constant and \( \rho_b \) is the mean density of the non-rotating star.

Now with the knowledge of density profile \( \rho(r) \) of the star, it is straightforward to find out the bulk viscosity damping time scales from Equations Eq. (35), Eq. (34), Eq. (32) and Eq. (30), once we know the bulk viscosity coefficient \( \zeta(r) \). In the case of bulk viscosity time scale arising due to hyperons \( (\tau_B) \), we can get \( \zeta(r) \) from Eq. (15)\). Similarly we can find out the time scale \( (\tau_l) \) associated with modified Urca processes, with the help of the expression for associated bulk viscosity \( \zeta_U \) given by

\[ \zeta_U = \frac{1.46 \rho(r)^2 \omega^{-2}}{l_M eV} \left[ k_B T \int_0^R dr \rho(r)r^{2l+2} \right] g/(\text{cm s}) \]  

(36)

The shear viscosity time scale is given by

\[ \frac{1}{\tau_\eta} = \frac{(l-1)(2l+1)}{\int_0^R dr \rho(r)r^{2l+2}} \int_0^R dr r^{2l+2}, \]  

(37)
where $\eta$ can be calculated from the prominent $nn$ scattering and is given by

$$\eta = 2 \times 10^{18} \rho_{15}^{9/4} T_{9}^{-2} \text{ g/(cm s)}. \quad (38)$$

Here $\rho_{15} = \rho/(10^{15} \text{ g/cm}^3)$ and $T_{9} = T/(10^{9} \text{ K})$ are density and temperature respectively, casted in dimensionless forms. Finally, the gravitational radiation time scale ($\tau_{GR}$) is given by

$$\frac{1}{\tau_{GR}} = \frac{32\pi G \Omega^{2l+2}}{c^{2l+3}} \frac{(l-1)^{2l}}{[(2l+1)!]^2} \left( \frac{l+2}{l+1} \right)^{2l+2}$$

$$\times \int_{0}^{R} \rho(r)r^{2l+2} dr. \quad (39)$$

The evolution of the r-mode due to dissipative viscous effects and GR can be studied by defining the overall r-mode time scale \( \tau_r \) [18, 20],

$$\frac{1}{\tau_r(\Omega, T)} = \frac{1}{\tau_{GR}(\Omega)} + \frac{1}{\tau_{B}(\Omega, T)} + \frac{1}{\tau_{U}(\Omega, T)} + \frac{1}{\tau_{\eta}(\Omega, T)}. \quad (40)$$

It appears in the decay of the mode as $e^{-t/\tau_r}$ and when $\tau_r > 0$, the mode is stable. Now from Equation (39) we can see that $\tau_{GR} < 0$, which is indicative of the fact that GR allows the modes to grow and drives them to instability, while $\tau_B$, $\tau_U$, and $\tau_\eta$ are positive and thus they try to dampen the mode. We can define a critical angular velocity $\Omega_C$ as $1/\tau_r(\Omega_C, T) = 0$; for a star at a given temperature $T$. Now if the angular velocity of the star is greater than $\Omega_C$, then the star is unstable and will be subjected to GR emission while stars with angular velocities smaller than $\Omega_C$ will be stable.

### III. RESULTS AND DISCUSSIONS

Now with the EoS discussed in subsection II.A we set out to find the coefficient of bulk viscosity as given by equation (13). The parameters in the effective chiral model that we have used are given in Table I. The parameters were so chosen that they satisfy the constraints on the equation of state from the flow data in heavy ion collisions [37]. The resulting EOS is plotted in figure 1. To calculate the bulk-viscosity, we first need to calculate $\gamma_{\infty} - \gamma_0$, the difference between fast and slow adiabatic indices, from Eq.(16). This expression can be calculated with the help of EOS alone. In figure 2 we plot $\gamma_{\infty} - \gamma_0$ as a function of the normalized baryon density ($n_b/n_0$), where $n_0 = 0.153 \text{ fm}^{-3}$ is the nuclear matter saturation density. The sudden rises in the graph can be attributed to the appearance of hyperons with increase of baryon density at the cost of neutron number density $n_n$.

Since we have $\Sigma^-$ and $\Lambda$ hyperons formed in the system with lowest threshold densities, we consider the non leptonic reactions represented by the Eq. (11) and Eq.(12), and calculate the relaxation time as given by the Eq. (17). We note that we have not considered the reaction Eq.(13) and further there are several reactions which are going to contribute to the net reaction rate [19, 21]. The reason for this is that the rate for the process given in Eq.(13) is estimated to be an order of magnitude higher than the process given by Eq.(12). This leads to subdominant contribution to the relaxation time and hence

### TABLE I: Parameter set for the model.

| $c_{\sigma N}$ | $c_{\omega N}$ | $c_{\rho N}$ | $B$ | $C$ | $K$ | $m_N/m_N$ |
|----------------|----------------|--------------|-----|-----|-----|-------------|
| (fm$^{-2}$)     | (fm$^{-2}$)    | (fm$^{-2}$)  | (fm$^{-2}$) | (fm$^{-2}$) | (MeV) |            |
| 6.79            | 1.99           | 4.66         | -4.32 | 0.165 | 300   | 0.85        |

FIG. 1: Equation of State used to calculate the stellar configurations.

FIG. 2: Thermodynamic factor $\gamma_{\infty} - \gamma_0$ that appears in the expression for hyperon bulk viscosity, is plotted against the normalised baryon density.
to the bulk viscosity [39]. Thus what we are calculating is a lower limit of the net rate which will correspond to an upper limit on the bulk viscosity. The matrix elements are calculated with the values of Fermi momenta and effective masses of various baryonic species obtained from the EoS. Here we use axial-vector coupling values $g_{np} = -1.27$, $g_{p\Lambda} = -0.72$ and $g_{n\Sigma} = 0.34$ measured in $\beta$-decay of baryons at rest and Fermi coupling constant $G_F = 1.166 \times 10^{-11}$ MeV$^{-2}$ and $\sin\theta_C = 0.222$ (where $\theta_C$ is the Cabibbo weak mixing angle) [40]. Then, we calculate $\delta\mu = \frac{\mu_s}{\mu_n}$ from Eq. (22) for the densities where both the hyperons are present ($n_B/n_0 > 2.36$), and, from Eq. (23) for lower densities where there is only $\Lambda$-hyperon present ($n_B/n_0 = 1.86 - 2.36$). Further, Eq. (24)-Eq. (27) are evaluated using the EoS under consideration. We can thus calculate the relaxation time for relevant temperatures. We show the calculated behaviour of relaxation time (in seconds) with temperature in Fig. 3. It is clear that the relaxation time increases considerably with the decrease of temperature. For a given temperature, the relaxation time is seen to decrease when both the hyperons are present, as compared to the case of presence of a single species of hyperons ($\Sigma$), since in this case $\tau$ value will be less according to Eq. (17). We then compute the coefficient of bulk viscosity responsible for the mode damping in neutron stars from the expression (15). The value of maximum frequency $\Omega_K$ is the Keplerian angular frequency of the rotating star and is set by the onset of mass shedding from the equator of the star. The bulk viscosity coefficient is calculated for the relevant temperatures and is plotted against the normalized baryon density in Fig. 4. The behaviour of the hyperon bulk viscosity is similar to that of the corresponding relaxation time as is expected from Eq. (15). The high value of the bulk viscosity coefficient at the temperature $10^8$ K is indicative of the fact that hyperon bulk viscosity plays a major role in the suppression of the r-modes. We note that our bulk viscosity values are of the order of magnitude less than the values obtained by [18]. It could be due to the fact that unlike their work we are not considering the effect of hyperon superfluidity in this calculation. It might also be noted that the non-superfluid hyperonic bulk viscosity calculated in [18] uses an EoS based on a model, where only $\Lambda$ hyperons are present at the relevant density.

We next study the effect of hyperon bulk viscosity on the r-modes. Here we need to calculate the dissipation time scale due to hyperon bulk viscosity as well as
due to other dissipative phenomena. If this time scale is greater than the gravitational radiation time scale, then the r-mode is not stable and suppressed. In order to calculate the dissipative timescales from Equations (30)-(39) we need to know the density profile \( \rho(r) \), of the neutron star under consideration. We need to know the Kepler frequency of the rotating star also. We use Tolman-Oppenheimer-Volkoff equations to construct the non-rotating stellar configurations. The maximum mass of the neutron star in this case is found to be \( 1.65 M_\odot \) with a radius of 16.7 km. We use Hartle’s slow rotation approximation to calculate the global properties of rotating neutron star \( [49] \). We get the maximum mass and radius \( (R) \) of the rotating star to be \( 1.66 M_\odot \) and 18.9 km respectively. The Kepler frequency in this case is found to be \( \Omega_K = 3998 \) Hz. A typical density profile of the rotating star is shown in the Fig. 5. This profile corresponds to a central density of \( 7.48 \times 10^{14} \text{ gm/cm}^3 \). We have also indicated the densities corresponding to the appearance of the hyperons, i.e., threshold densities of both \( \Sigma \) and \( \Lambda \) hyperons in the graph. From centre of the star upto a density of \( \rho = 6.34 \times 10^{14} \text{ gm/cm}^3 \), we have the presence of both the hyperons in the star (i.e., up to a distance 2.5 km from the centre). The presence of \( \Sigma \) alone is there upto \( \rho = 5.1 \times 10^{14} \text{ gm/cm}^3 \) (another 1.7 km) making a hyperon core of radius 4.2 km in the neutron star. Hyperon bulk viscosity time scale, and, hence its effects on r-mode is very sensitive to the hyperonic core’s constituent structure and its radius.

For the rotating neutron star with a mass of \( 1.66 M_\odot \) and \( \Omega_K = 3998 \text{ Hz} \) as considered above, we next evaluate the various dissipative time scales associated with the r-mode damping. The dissipative time scale of hyperonic bulk viscosity, denoted by \( \tau_B \), can be calculated from equations (30)-(35) with the help of density profile of the star. Here hyperonic bulk viscosity \( (\zeta) \) as a function of radius is obtained from our previous calculations of \( \zeta \) for the EoS together with the knowledge of stellar density profile i.e., \( \zeta(\rho(r)) \). The time scale associated with modified Urca processes \( \tau_U \), is calculated in the same manner as for \( \tau_B \), by using the equation (36) instead of hyperonic bulk viscosity. Next, we estimate the shear viscosity dissipative time scale \( \tau_\eta \) using the equations (37) and (38). Finally, the gravitational radiation time scale \( \tau_{GR} \) associated with the r-mode can be calculated with the help of density profile using equation (39). Figure 6 shows the calculated time scales as functions of temperature, for the star rotating with \( \Omega_K \). From figure 6 we observe that in the non-superfluid hyperonic matter, r-modes get substantially damped due to hyperonic bulk viscosity only at low temperatures \( (T < 10^8 \text{ K}) \), whereas the modified Urca bulk viscosity suppresses the r-modes rapidly only at high temperatures \( T > 5 \times 10^{10} \text{ K} \). The role of shear viscosity in suppressing the modes is not prominent in this temperature range. Consequently, the effect of r-mode instability will be prominent in the temperature window \( (10^8 - 5 \times 10^{10}) \text{ K} \). The hyperon bulk viscosity suppresses the instability for temperatures below \( 10^8 \text{ K} \) while modified Urca processes suppress the instability beyond \( 10^{10} \text{ K} \).

Now we are in a position to calculate the critical angular velocity \( \Omega_C \) of the neutron star. \( \Omega_C \) is obtained by solving equation (40) \( \frac{1}{\tau_r(\Omega_C, T)} = 0 \), for a particular value of \( T \). At this frequency, the energy fed into the r-mode per unit time by gravitational radiation is equal.
to the energy dissipated per unit time. A star rotating above this critical frequency will be subjected to r-mode instability. We have shown the $\Omega_C$ (scaled to $\Omega_K$) in Fig. 7 for the temperature regime of interest. Since $\Omega_K$ determines the maximum allowed rotation rate for the star, stable rotation at any temperature will have $\Omega_C/\Omega_K = 1$ as an upper bound. In this figure, the region above $\Omega_C$ curve is unstable and a star rotating in this region will be rapidly spun down to an angular frequency below $\Omega_C$. As expected the instability window exists in the temperature regime \((10^8 - 5 \times 10^{10})\) K, where gravitational radiation is dominant and not suppressed substantial, which shows that the neutron star with hyperonic core is unstable in this region. Low temperature regime hyperonic bulk viscosity damps the r-mode effectively whereas nucleon dominated modified Urca bulk viscosity is the cause of mode damping at high temperatures. The minima of $\Omega_C$ curve occurs at $T \approx 5 \times 10^{16}$ K with $\Omega_C \approx 0.4 \Omega_K$, which is indicative of the fact that the r-mode instability is rather strong in this hyperon core scenario. The shaded box in the Fig. 7 is where most of the observed Low Mass X-ray Binaries (LMXBs) are found. They have a core temperature in the range of \((2 \times 10^7 - 3 \times 10^8)\) K with rotation rate between 300 to 700 Hz \[10,50]\. In our case, LMXBs are placed in the stable region, unlike conventional neutron stars with npe matter \[43\].

IV. SUMMARY

Rotating equilibrium configurations of self gravitating fluids are subjected to various possibilities of instabilities at large rotation periods. In the present work, we have investigated the r-mode instability which is known to limit the angular velocities of rapidly rotating stars. The r-mode and the related instabilities are damped by various viscosities of the matter in the interior of the neutron star. Thus the microscopic models describing the matter in the interior of the star get constrained by the observations of the rapidly rotating pulsars.

In the present work, we have confined our attention to the case of neutron star with a hyperonic core. For the description of the matter in the core of the neutron star, we have used an effective chiral hadronic model generalised to include the lowest lying octet of baryons. The parameters of the model are chosen that are consistent with the flow data in heavy ion collisions, nuclear matter properties as well as observation of high masss neutron stars. In the present work, we have computed the coefficient of bulk viscosity due to the the hyperonic matter in the core of a neutron star and the resulting effects on the r-mode instability. It turns out that hyperon bulk viscosity within the model is effective in damping the instability for temperatures below $10^6$K. Beyond a temperature of about $10^{10}$K, the bulk viscosity due to modified Urca processes become effective in damping the r-mode instability. Shear viscosity of hadronic matter becomes effective in damping only at low temperatures. Within the model it turns out that the the bulk viscosity in normal hyperonic matter does play an important role in spinning down fast rotating neutron stars. However, superfluid hyperonic matter or quark matter in the core can change this conclusion.

We have not considered in the present work the phase transition to quark matter which could be most likely in a color superconducting phase. The role of quark matter in the context of r-mode characteristics has been dealt with in Ref.\[10\] which shows that the r-mode instability gets suppressed by both normal quark matter as well as gapped quark matter in the color flavor locked phase. Further, the neutron stars are also endowed with strong magnetic fields the effects of which on the r-modes have not been included in the present work. Ultra strong magnetic field seem to increase the instability window for quark matter \[11\]. It will thus be interesting to examine the scenario with a phase transition to quark matter and the effect of magnetic fields in the context of r-modes for a hybrid star with a crust of hadronic matter. Work in this direction is under progress and will be reported elsewhere \[51\].

Acknowledgments

The authors would like to thank Jitesh R Bhatt, Debadas Bandopadhyay and Debarati Chatterjee for many discussions. The authors would also like to thank Amruta Mishra for a careful reading of the manuscript.

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