Abstract

In this paper we deal with monogenic and $k$-hypermonogenic automorphic forms on arithmetic subgroups of the Ahlfors-Vahlen group. Monogenic automorphic forms are exactly the 0-hypermonogenic automorphic forms.

In the first part we establish an explicit relation between $k$-hypermonogenic automorphic forms and Maaß wave forms. In particular, we show how one can construct from any arbitrary non-vanishing monogenic automorphic form a Clifford algebra valued Maaß wave form.

In the second part of the paper we compute the Fourier expansion of the $k$-hypermonogenic Eisenstein series which provide us with the simplest non-vanishing examples of $k$-hypermonogenic automorphic forms.

Keywords: $k$-hypermonogenic automorphic forms, Maaß wave forms, Eisenstein series, Fourier expansion, Epstein zeta type functions, Laplace-Beltrami operator

AMS-Classification: 11F03, 11F36, 11F30, 11F37, 30G35

1 Introduction

The theory of higher dimensional Maaß wave forms has become a major topic of study in analytic number theory. Maaß wave forms are automorphic forms that are complex-valued eigensolutions to the Laplace-Beltrami operator

\[ \Delta_{LB} = x_n^2 \left( \sum_{i=0}^{n} \frac{\partial^2}{\partial x_i^2} \right) - (n-1)x_n \frac{\partial}{\partial x_n}. \]  

The classical setting is $n + 1$-dimensional upper half-space in the framework of the action of discrete arithmetic subgroups of the orthogonal group. Its study
was initiated in 1949 by H. Maaß in [27, 28]. In the late 1980s this study had a major boost by breakthrough works of J. Elstrodt, F. Grunewald and J. Mennicke, [6, 8, 9], A. Krieg [22, 23], V. Gritsenko [14] among many others.

Recently, in [18] another class of automorphic forms on these arithmetic groups has been considered. The context is again \( n + 1 \)-dimensional upper half-space. The classes of automorphic forms considered in [18], however, have different analytic and mapping properties. They are null-solutions to iterates of the Euclidean Dirac operator \( D := \sum_{i=1}^{n+1} \frac{\partial}{\partial x_i} e_i \) and, in general, they take values in real Clifford algebras. These are called \( k \)-monogenic automorphic forms. The monogenic automorphic forms on upper half-space in turn can be embedded into the general framework of \( k \)-hypermonogenic automorphic forms. The class of \( k \)-hypermonogenic functions [26, 10, 12] contains the 1-monogenic functions as the particular case \( k = 0 \). 1-monogenic functions are more known simply as monogenic functions, [5].

An important question that remained open in this context is to understand whether these classes of automorphic forms are linked with Maaß wave forms. In Section 3 we shed some light on this problem after having introduced the preliminary tools. We develop explicit relations between the classes of \( k \)-hypermonogenic automorphic forms and eigensolutions to the Laplace-Beltrami operator. In particular, we explain how we can construct Clifford algebra valued Maaß wave forms from monogenic automorphic forms.

In Section 4 we compute the Fourier expansion of \( k \)-hypermonogenic automorphic forms. The appearance of Bessel-K functions in its Fourier expansion reflects once more the close relation between the class of \( k \)-hypermonogenic automorphic forms and Maaß wave forms, see [28, 6, 22, 23]. In particular, we compute the Fourier coefficients of the \( k \)-hypermonogenic Eisenstein series explicitly.

The appearance of variants of the Epstein zeta function provides a further nice analogy to the classical theory of complex analytic automorphic forms. See for example [13, 16, 32, 33].

2 Preliminaries

2.1 Clifford algebras

We introduce the basic notions of real Clifford algebras over the Euclidean space \( \mathbb{R}^n \). For details, see for instance [5]. Throughout this paper, \( \{e_1, \ldots, e_n\} \) stands for the standard orthonormal basis in the Euclidean space \( \mathbb{R}^n \) and \( Cl_n \) denotes its associated real Clifford algebra in which the relation \( e_i e_j + e_j e_i = -2\delta_{ij} \) holds. A basis for the algebra \( Cl_n \) is given by \( 1, e_1, \ldots, e_n, e_{j_1} \ldots e_{j_r}, \ldots, e_1 \ldots e_n \) where \( 1 \leq j_1 < \ldots < j_r \leq n \). Every element \( a \) from \( Cl_n \) can hence be written in the form \( a = \sum_{A \subseteq P(\{1, \ldots, n\})} a_A e_A \) where \( a_A \) are uniquely defined real numbers. Here, \( e_\emptyset = e_0 = 1 \). Elements from \( Cl_n \) of the reduced form \( x_0 + x_1 e_1 + \cdots + x_n e_n \)
are called paravectors. \( x_0 \) is called the scalar part of \( x \) and will be denoted by \( Sc(x) \). In the case \( n = 1 \) the Clifford algebra \( Cl_1 \) only consists of paravectors of the form \( x_0 + x_1 e_1 \) which in turn can be identified with the complex numbers. In the case \( n = 2 \) we have \( Cl_2 \cong \mathbb{H} \), \( \mathbb{H} \) standing for the Hamiltonian quaternions. These in turn can be identified with the set of paravectors in \( \mathbb{R} \oplus \mathbb{R}^3 \) which have the form \( x = x_0 + x_1 e_1 + \cdots + x_3 e_3 \) when we identify the particular imaginary unit \( e_3 \) with the product \( e_1 e_2 \).

The Clifford conjugation is defined by \( \overline{ab} = \overline{b} \overline{a} \) where \( \overline{e}_j = -e_j \) for all \( j = 1, 2, \ldots, n \). Each paravector \( x \in \mathbb{R} \oplus \mathbb{R}^n \setminus \{0\} \) has an inverse element, given by \( x^{-1} = x/||x||^2 \).

The reversion anti-automorphism is defined by \( \tilde{ab} = \tilde{b} \tilde{a} \), where \( \tilde{e}_j = -e_j = e_{-j} \). The main involution is defined by \( (ab)^* = ab' \) and \( e'_i = -1 \) for \( i = 1, \ldots, n \). Furthermore, we also need the following automorphism \( * : Cl_n \to Cl_n \) defined by the relations: \( e^*_n = -e_n \), \( e^*_i = e_i \) for \( i = 0, 1, \ldots, n-1 \) and \( (ab)^* = a^*b^* \). Any element \( a \in Cl_n \) may be uniquely decomposed in the form \( a = b + ce_n \), where \( b \) and \( c \) belong to \( Cl_{n-1} \). Based on this definition one defines the mappings \( P : Cl_n \to Cl_{n-1} \) and \( Q : Cl_n \to Cl_{n-1} \) by \( Pa = b \) and \( Qa = c \).

### 2.2 Dirac type operators over \( n+1 \)-dimensional Euclidean and hyperbolic spaces

**Monogenic functions.** Let \( U \subseteq \mathbb{R} \oplus \mathbb{R}^n \) be an open set. Then a real differentiable function \( f : U \to Cl_n \) that satisfies \( Df = 0 \), resp. \( fD = 0 \), where \( D := \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} e_1 + \cdots + \frac{\partial}{\partial x_n} e_n \) is the Euclidean Cauchy-Riemann operator, is called left (resp. right) monogenic or left (resp. right) Clifford-holomorphic. Due to the non-commutativity of \( Cl_n \) for \( n > 1 \), both classes of functions do not coincide with each other. However, \( f \) is left monogenic, if and only if \( f \) is right monogenic.

The Cauchy-Riemann operator factorizes the Euclidean Laplacian \( \Delta = \sum_{j=0}^n \frac{\partial^2}{\partial x_j^2} \), viz \( D \overline{D} = \Delta \). Every real component of a monogenic function is hence harmonic.

The \( D \)-operator is left quasi-invariant under Möbius transformations. Following for example [2, 7], Möbius transformations in \( \mathbb{R} \oplus \mathbb{R}^n \) can be represented as

\[
T : \mathbb{R} \oplus \mathbb{R}^n \cup \{\infty\} \to \mathbb{R} \oplus \mathbb{R}^n \cup \{\infty\}, \quad T(x) = (ax + b)(cx + d)^{-1}
\]

with coefficients \( a, b, c, d \) from \( Cl_n \) that can all be written as products of paravectors from \( \mathbb{R} \oplus \mathbb{R}^n \) and that satisfy \( ad - bc \in \mathbb{R} \setminus \{0\} \) and \( a^{-1}b, c^{-1}d \in \mathbb{R} \oplus \mathbb{R}^n \) if \( c \neq 0 \) or \( a \neq 0 \), respectively. These conditions are often called Ahlfors-Vahlen conditions.

The set that consists of Clifford valued matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) whose coefficients satisfy the above conditions forms a group under matrix multiplication and it is called the general Ahlfors-Vahlen group, \( GAV(\mathbb{R} \oplus \mathbb{R}^n) \). The subgroup consisting of those matrices from \( GAV(\mathbb{R} \oplus \mathbb{R}^n) \) that satisfy \( ad - bc = 1 \) is called the special Ahlfors-Vahlen group. It is denoted by \( SAV(\mathbb{R} \oplus \mathbb{R}^n) \).
Now assume that $M \in GAV(\mathbb{R} \oplus \mathbb{R}^n)$. If $f$ is a left monogenic function in the variable $y = M < x > = (ax + b)(cx + d)^{-1}$, then the function $\frac{cx+d}{\|cx+d\|^{n+1}} f(M < x >)$ is again left monogenic in the variable $x$, cf. [31].

$k$-hypermonogenic functions. The class of monogenic functions belongs to the more general class of so-called $k$-hypermonogenic functions, introduced in [10]. These are defined as the null-solutions to the system

$$Df + k\left(Qf\right)' = 0$$

(2)

where $k \in \mathbb{R}$.

Under a Möbius transformation $y = M < x > = (ax + b)(cx + d)^{-1}$ such a solution $f(y)$ is transformed to the $k$-hypermonogenic function

$$F(x) := \frac{cx+d}{\|cx+d\|^{n+1-k}} f(M < x >).$$

(3)

See for instance [12]. In the case $k = 0$, we are dealing with the set of left monogenic functions introduced earlier. The particular solutions associated to the case $k = n - 1$ coincide with the null-solutions to the hyperbolic Hodge-Dirac operator with respect to the hyperbolic metric on upper half space. These are often called hyperbolic monogenic functions or simply hypermonogenic functions, see [26].

2.3 Discrete arithmetic subgroups of $GAV(\mathbb{R} \oplus \mathbb{R}^n)$

In this paper we deal with null-solutions to the above systems that are quasi-invariant under arithmetic subgroups of the general Ahlfors-Vahlen group that act totally discontinuously on upper half space

$$H^+(\mathbb{R} \oplus \mathbb{R}^n) = \{ x = x_0 + x_1e_1 + \cdots + x_ne_n \in \mathbb{R} \oplus \mathbb{R}^n : x_n > 0 \}.$$

These can be regarded as generalizations of the classical holomorphic modular forms in the context of monogenic and $k$-hypermonogenic functions.

The automorphism group of upper half-space $H^+(\mathbb{R} \oplus \mathbb{R}^n)$ is the group $SAV(\mathbb{R} \oplus \mathbb{R}^{n-1})$. Arithmetic subgroups of the special Ahlfors-Vahlen group that act totally discontinuously on upper half space are for instance considered in [23][4][9]. For convenience, we first recall the definition of the rational Ahlfors-Vahlen group on $H^+(\mathbb{R} \oplus \mathbb{R}^n)$.

**Definition 1.** The rational Ahlfors-Vahlen group $SAV(\mathbb{R} \oplus \mathbb{R}^{n-1}, \mathbb{Q})$ is the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ from $SAV(\mathbb{R} \oplus \mathbb{R}^{n-1})$ that satisfy

(i) $a\bar{\sigma}, b\bar{\tau}, c\bar{\pi}, d\bar{\varepsilon} \in \mathbb{Q}$,

(ii) $a\bar{\sigma}, b\bar{\tau} \in \mathbb{Q} \oplus \mathbb{Q}^n$,

(iii) $ax\bar{\pi} + b\bar{\sigma} \bar{x} + ax\bar{\pi} \bar{\varepsilon} + b\bar{\sigma} \bar{x} \bar{\varepsilon} \in \mathbb{Q}$ (\forall \bar{x} \in \mathbb{Q} \oplus \mathbb{Q}^n),

(iv) $ax\bar{\varepsilon} + b\bar{\pi} \bar{x} \bar{\varepsilon} \in \mathbb{Q} \oplus \mathbb{Q}^n$ (\forall \bar{x} \in \mathbb{Q} \oplus \mathbb{Q}^n).
Next we need

**Definition 2.** A \( \mathbb{Z} \)-order in a rational Clifford algebra is a subring \( R \) such that the additive group of \( R \) is finitely generated and contains a \( \mathbb{Q} \)-basis of the Clifford algebra.

The following definition, cf. [9], provides us with a whole class of arithmetic subgroups of the Ahlfors-Vahlen group which act totally discontinuously on upper half-space.

**Definition 3.** Suppose that \( p \leq n - 1 \). Let \( \mathcal{I} \) be a \( \mathbb{Z} \)-order in \( Cl_n \) which is stable under the reversion and the main involution \( \tau \) of \( Cl_n \). Then

\[
\Gamma_p(\mathcal{I}) := SAV(\mathbb{R} \oplus \mathbb{R}^p, \mathbb{Q}) \cap \text{Mat}(2, \mathcal{I}).
\]

For an \( N \in \mathbb{N} \) the principal congruence subgroup of \( \Gamma_p(\mathcal{I}) \) of level \( N \) is defined by

\[
\Gamma_p(\mathcal{I})[N] := \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_p(\mathcal{I}) \mid a - 1, b, c, d - 1 \in N\mathcal{I} \right\}.
\]

Notice that all the groups \( \Gamma_p(\mathcal{I})[N] \) have finite index in \( \Gamma_p(\mathcal{I}) \). Therefore, all of them are discrete groups and act totally discontinuously on upper half-space \( H^+(\mathbb{R} \oplus \mathbb{R}^p) \) provided \( p \leq n - 1 \). The proof of the total discontinuous action can be done in the same way as in [13].

**Special notation.** In the sequel let us denote the subgroup of translation matrices contained in \( \Gamma_p(\mathcal{I})[N] \) by \( T_p(\mathcal{I})[N] \). The associated \( p + 1 \)-dimensional lattice in \( \mathbb{R} \oplus \mathbb{R}^p \) will be denoted by \( \Lambda_p(\mathcal{I})[N] \) and its dual by \( \Lambda_p^*(\mathcal{I})[N] \). The dual lattice is contained in the subspace \( \mathbb{R} \oplus \mathbb{R}^p \), too.

The simplest concrete examples for \( \Gamma_p(\mathcal{I}) \) are obtained by taking for \( \mathcal{I} \) the standard \( \mathbb{Z} \)-order in the Clifford algebras \( Cl_p \), i.e., \( O_p := \sum_{A \subseteq P(1, \ldots, p)} \mathbb{Z} e_A \) where \( p \leq n - 1 \). In this case, the group \( \Gamma_p(\mathcal{I}) \) coincides with the special hypercomplex modular group considered in [18]. This one is generated by the matrices

\[
J := \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), T_1 := \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), T_{e_1} := \left( \begin{array}{cc} 1 & e_1 \\ 0 & 1 \end{array} \right), \ldots, T_{e_p} := \left( \begin{array}{cc} 1 & e_p \\ 0 & 1 \end{array} \right).
\]

In this particular case, we have \( T_p(O_p) =< T_1, T_{e_1}, \ldots, T_{e_p} > \) and the associated period lattice is the orthonormal \( p + 1 \)-dimensional lattice \( \mathbb{Z} + \mathbb{Z} e_1 + \mathbb{Z} e_2 + \cdots + \mathbb{Z} e_p \) which is self-dual. Its standard fundamental period cell is \([0, 1]^{p+1}\).

In the case \( n = 3 \) the set \( H^+(\mathbb{R} \oplus \mathbb{R}^3) \) can be identified with upper quaternionic half-space. In this setting further important examples of \( \mathcal{I} \) are the quaternionic orders, in particular the Hurwitz order \( \mathbb{Z} + \mathbb{Z} e_1 + \mathbb{Z} e_2 + \frac{1}{2}\mathbb{Z}(1 + e_1 + e_2 + e_3) \), considered for example in [15, 22, 23, 29]. In the quaternionic context the third unit \( e_3 \) is identified with \( e_1 e_2 \).
2.4 $k$-hypermonogenic automorphic forms

As already mentioned, any Möbius transformation $T(x) = M < x >$ induced by matrices $M$ belonging to $SAV(\mathbb{R} \oplus \mathbb{R}^{n-1})$ (as well as any of its subgroups) preserves upper half-space. Therefore, in view of (3), if $f$ is a function that is left $k$-hypermonogenic on the whole half-space then so is $\frac{cx+d}{\|cx+d\|^{n+1-k}} f(M < x >)$.

Next we introduce

**Definition 4.** Let $p \leq n - 1$. A left $k$-hypermonogenic function $f : H^+ (\mathbb{R} \oplus \mathbb{R}^n) \to Cl_n$ is called a left $k$-hypermonogenic automorphic form on $\Gamma_p(\mathcal{I})[N]$ if for all $x \in H^+ (\mathbb{R} \oplus \mathbb{R}^n)$

$$f(x) = \frac{cx+d}{\|cx+d\|^{n+1-k}} f(M < x >)$$

for all $M \in \Gamma_p(\mathcal{I})[N]$.

In the case $k = 0$ we re-obtain the class of left monogenic automorphic forms described in [18]. As the following proposition shows, there is a direct relation between $k$-hypermonogenic automorphic forms and $-k$-hypermonogenic automorphic forms.

**Proposition 1.** Suppose that $k \in \mathbb{R}$ and that $p$ is a positive integer with $p < n$. If $f : H^+ (\mathbb{R} \oplus \mathbb{R}^n) \to Cl_n$ is a $k$-hypermonogenic automorphic form on $\Gamma_p(\mathcal{I})[N]$ satisfying (4), then the function $g : H^+ (\mathbb{R} \oplus \mathbb{R}^n) \to Cl_n$ defined by $g(x) := \frac{f(x)e_n}{x_n}$ is $-k$ hypermonogenic and satisfies

$$g(M < x >) = \frac{1}{(M < x >)^k_n} f(M < x >) e_n$$

for all $M \in \Gamma_p(\mathcal{I})[N]$.

Proof. Following for example [11], a function $f$ is $k$-hypermonogenic, if and only if $\frac{f(x)e_n}{x_n}$ is $-k$ hypermonogenic. Suppose that $f$ is a $k$-hypermonogenic automorphic form satisfying (4). Then $g(x) := \frac{f(x)e_n}{x_n}$ is $-k$-hypermonogenic and satisfies the transformation law

$$g(M < x >) = \frac{1}{(M < x >)^k_n} f(M < x >) e_n$$

$$= \frac{\|cx+d\|^{2k}}{x_n^{k}} (\|cx+d\|^{n+1-k}(cx+d)^{-1} f(x)e_n$$

$$= \|cx+d\|^{n+1+k}(cx+d)^{-1} g(x).$$

Therefore, it is sufficient to restrict to non-positive values of $k$ in all that follows.

For each $\mathcal{I}$ there exists a minimal positive integer $N_0(\mathcal{I})$ such that neither the negative identity matrix $-I$ nor the other diagonal matrices of the form
\[
\begin{pmatrix}
\hat{e}_A & 0 \\
0 & \hat{e}_{1_A}
\end{pmatrix}
\] where \(A \subseteq P(1, \ldots, p)\) are no longer included in all principal congruence subgroups \(\Gamma_p(I)[N]\) with \(N \geq N_0(I)\). In the case where \(I\) is the standard \(\mathbb{Z}\)-order \(O_p\), we have \(N_0(O_p) = 3\), see [13]. For all \(N < N_0(I)\), only the zero function satisfies (4). However, for all \(N \geq N_0(I)\), one can construct non-trivial \(k\)-hypermonogenic automorphic forms that have the transformation behavior (4).

The simplest non-trivial examples of \(k\)-hypermonogenic automorphic forms on the groups \(\Gamma_p(I)[N]\) with \(N \geq N_0(I)\) are the \(k\)-hypermonogenic generalized Eisenstein series:

**Definition 5.** Let \(N \geq N_0(I)\). For \(p < n\) and real \(k\) with \(k < n - p - 2\) the \(k\)-hypermonogenic Eisenstein series on the group \(\Gamma_{n-1}(I)[N]\) acting on \(H^+(\mathbb{R} \oplus \mathbb{R}^n)\) are defined by

\[
\varepsilon_{k,p,N}(x) := \sum_{M \in \mathcal{T}_p(I)[N] \backslash \Gamma_p(I)[N]} \frac{cx + d}{\|cx + d\|^{n+1-k}}.
\]

These series converge for \(k < n - p - 2\) absolutely and uniformly on each compact subset of \(H^+(\mathbb{R} \oplus \mathbb{R}^n)\). A majorant is

\[
\sum_{M \in \mathcal{T}_p(I)[N] \backslash \Gamma_p(I)[N]} \frac{1}{\|ce_n + d\|^\alpha}
\]

whose absolute convergent absissa is \(\alpha > p + 2\), cf. for example [13, 9]. For \(p = n - 1\), this majorant converges absolutely for all \(k < -1\).

The non-vanishing behavior for \(N \geq N_0(I)\) can easily be established by considering the limit \(\lim_{x_n \to \pm \infty} \varepsilon_{k,p,M}(x_n e_n)\) which equals +1 in these cases, cf. [14].

**Remarks.** In the case \(k = 0\) and \(I = O_p\), the series (6) coincide with the monogenic Eisenstein series considered in [13, 19] in the cases \(p < n - 2\).

By adapting the Hecke trick from [13] can also introduce Eisenstein series of lower weight. This is shown in Section 4 of [3]. In particular, for \(N \geq N_0(I)\) the series

\[
\varepsilon_{0,N-1,M}(x) := \lim_{s \to 0^+} \sum_{M \in \mathcal{T}_{n-1}(I)[N] \backslash \Gamma_{n-1}(I)[N]} \left( \frac{x_n}{\|cx + d\|^2} \right)^s \frac{cx + d}{\|cx + d\|^n+1}.
\]

defines a well-defined non-vanishing left monogenic Eisenstein series on the groups \(\Gamma_{n-1}(I)[N]\) in upper half-space variable \(x\), cf. [31] Section 4. In [31] Section 4 this is done for the particular case \(I = O_p\). However, the transition to the context of more general orders \(I\) follows identically along the same lines.

In view of Proposition [14] we can directly construct non-vanishing \(j\)-hypermonogenic Eisenstein series for positive \(j\) from the \(k\)-hypermonogenic Eisenstein series of negative \(k\), simply by forming

\[
E_{-k,p,N}(x) := \frac{\varepsilon_{k,p,N}(x) e_n}{x_n^k}
\]
which then satisfy the transformation law

\[ E_{-k,p,N}(x) := \frac{cx+d}{\|cx+d\|^{n+1+k}} E_{-k,p,N}(M < x >) \]

for all \( M \in \Gamma_p(\mathcal{I})[N] \).

3 Relation to Maaß wave forms

As one directly sees, the theory of monogenic automorphic forms fits as a special case within the general framework of \( k \)-hypermonogenic automorphic forms. We can say more:

Suppose that \( f = Pf + Qf e_n \) is a \( k \)-hypermonogenic function, where \( k \) is some arbitrary fixed real number. Then, following for example [10] all the real components of \( Pf \) are contained in the set of \( k \)-hyperbolic harmonic functions. These are solutions to the system

\[ x_n \Delta u - k \frac{\partial u}{\partial x_n} = 0. \]

These solutions are also quasi-invariant under Möbius transformations that act on upper half-space: If \( f \) is a solution to (8), then following [11],

\[ F(x) = \frac{1}{\|cx+d\|^{n-1-k}} f(M < x >) \]

is \( k \)-hyperbolic harmonic, too. Notice that in the particular case \( k = n - 1 \) the correction factor disappears. This attributes a special role to the function class of \((n-1)\)-hypermonogenic functions.

The solutions to (8) in turn are directly related to the Maaß wave equation. Following for example [25], if \( u \) is a solution to (8) then \( g(x) = x_n^{(1-n+k)/2} u(x) \) is a solution to

\[ \Delta g - \frac{n-1}{x_n} \frac{\partial g}{\partial x_n} + \lambda \frac{g}{x_n^2} = 0 \]

where \( \lambda = \frac{1}{4}(n^2 - (k + 1)^2) \). The solutions to (10) have the property that they are directly preserved by all Möbius transformations that act on upper half-space. Each solution \( g \) is an eigensolution to the Laplace-Beltrami operator \( \mathcal{H} \) associated to the fixed eigenvalue \( -\frac{1}{4}(n^2 - (k + 1)^2) \).

Let \( p < n \). Now suppose that \( f : H^+(\mathbb{R} \oplus \mathbb{R}^n) \to Cl_n \) is a \( k \)-hypermonogenic automorphic form on \( \Gamma_p(\mathcal{I})[N] \) of weight \((n-k)\), satisfying the transformation law

\[ f(x) = \frac{cx+d}{\|cx+d\|^{n+1+k}} f(M < x >) \]

for all \( M \in \Gamma_p(\mathcal{I})[N] \).

Since \((M < x >)_n = \frac{x^n}{\|cx+d\|^n} \), the function \( g(x) = x_n^{-(1-n+k)/2} f(x) \) thus satisfies for all \( M \in \Gamma_p(\mathcal{I})[N] \):

\[ g(M < x >) = (M < x >)_n^{-\frac{1-n+k}{2}} f(M < x >) \]
\[
\begin{align*}
&= \left( \frac{x_n}{\| cx + d \|^2} \right)^{\frac{1-n+k}{2}} \| cx + d \|^{n+1-k} (cx + d)^{-1} f(x) \\
&= x_n^{\frac{1-n+k}{2}} \frac{1}{\| cx + d \|^{n-k-1}} \| cx + d \|^{n+1-k} (cx + d)^{-1} \frac{1-n+k}{2} g(x).
\end{align*}
\]

Hence,
\[
g(x) = \frac{cx + d}{\| cx + d \|^2} g(M < x >).
\]

Unfortunately, if \( f = Pf + Qfe \) is a \( k \)-hypermonogenic automorphic form with respect to \( \Gamma_p(\mathcal{I})[N] \), then \( Pf \) is in general not an automorphic form with respect to the full group \( \Gamma_p[N](\mathcal{I}) \). However, only the components of the \( P \)-part of \( f \) satisfy (8). The associated function \( g(x) = x_n^{-(1-n+k)/2} f(x) \) is exactly a \( \Gamma_p(\mathcal{I})[N] \)-invariant eigenfunction to the Laplace-Beltrami operator for the eigenvalue \(-\frac{1}{4}(n^2 - (k + 1)^2)\), if \( Qf = 0 \). In this case we are dealing with monogenic functions. Actually, we can construct from every left monogenic automorphic form a Maaß wave form. Following [10], the real components of the \( Q \)-part of a \( k \)-hypermonogenic function satisfies the equation
\[
x_n^2 \Delta u - kx_n \frac{\partial u}{\partial x_n} + ku = 0.
\]

In particular, if \( f \) is monogenic, then all the real components of the \( P \)-part and of the \( Q \)-part satisfy
\[
x_n^2 \Delta u = 0.
\]

Consequently, all real components of \( g(x) := x_n^{-(1-n+k)/2} f(x) \) are eigenfunctions to the Laplace-Beltrami operator for the particular eigenvalue \(-\frac{1}{4}(n^2 - 1)\).

We thus arrive at the following result:

**Theorem 1.** Suppose that \( f \) is a left monogenic automorphic form on \( \Gamma_p(\mathcal{I})[N] \) of weight \( n \), satisfying \( f(x) = \frac{cx + d}{\| cx + d \|^2} g(M < x >) \) for all \( M \in \Gamma_p(\mathcal{I})[N] \). Then \( g(x) = x_n^{-(1-n)/2} f(x) \) is a quasi-\( \Gamma_p(\mathcal{I})[N] \)-invariant Maaß wave form associated to the fixed eigenvalue \(-\frac{1}{4}(n^2 - 1)\) and has the \(-1\)-weight automorphy factor \( \frac{cx + d}{\| cx + d \|^2} \).

Notice that in the complex case \( n = 1 \), the function \( g \) is a null-solution to the Laplace-Beltrami operator.

Theorem 1 reflects a connection between the class of monogenic automorphic forms and the particular family of non-analytic automorphic forms on the Ahlfors-Vahlen group considered for instance by A. Krieg, J. Elstrodt et al. and others (see for example [23, 9, 24]). The non-analytic automorphic forms considered in [23, 9, 24] are scalar-valued eigenfunctions of the Laplace-Beltrami operator associated to a special continuous spectrum of eigenvalues described in [9]. They are all totally invariant under the group action.
Notice that the Eisenstein series considered in this paper are Clifford algebra valued in general. They are associated to a fixed eigenvalue.

Let us now analyze more into detail how the particular examples of non-analytic Maaß wave forms discussed in the above mentioned papers fit within the framework of \( k \)-hypermonogenic automorphic forms. First we observe, that evidently in all the other cases where \( f \) is a \( k \)-hypermonogenic automorphic form on \( \Gamma_p(\mathcal{I})[N] \) of weight \( (n-k) \) with \( k \neq 0 \) and \( Qf \neq 0 \), the \( P \)-part of the associated function \( g(x) = x_n^{-(n+1)/2} f(x) \) is an eigenfunction of the Laplace-Beltrami operator for the eigenvalue \(-\frac{1}{4}(n^2 - (k+1)^2)\) and satisfies

\[
P[g(x)] = P\left[ \frac{cx+d}{\|cx+d\|^2} g(M < x >) \right] \quad \forall M \in \Gamma_p(\mathcal{I})[N].
\]

However, a relation of the form

\[
P[j(M, x)g(M < x >)] = j(M, x) P[g(M < x >)]
\]

only holds if \( j(M, x) \) is a scalar-valued automorphy factor. On the other hand, as already mentioned, the real components of the \( P \)-part of a \( k \)-hypermonogenic automorphic are \( k \)-hyperbolic harmonic.

Let us now suppose that \( F \) is a \( k \)-hyperbolic harmonic automorphic form that has the following invariance behavior

\[
F(x) = \frac{1}{\|cx+d\|^{(n-k-1)/2}} F(M < x >),
\]

dealing with a scalar-valued automorphy factor. The associated function \( G(x) = x_n^{-\frac{1}{2}(1-k)} F(x) \) then has the property of being totally invariant under the group action of \( \Gamma_p(\mathcal{I})[N] \), i.e. \( G(x) = G(M < x >) \) for all \( M \in \Gamma_p(\mathcal{I})[N] \).

Non-trivial examples of \( k \)-hyperbolic harmonic functions \( F \) with this transformation behavior are the \( k \)-hyperharmonic Eisenstein series:

\[
\mathcal{E}_{k,p,N}(x) := \sum_{M: \mathcal{T}_p(\mathcal{I})[N] \setminus \Gamma_p(\mathcal{I})[N]} \frac{1}{\|cx+d\|^{n-k-1}} \quad k < n - p - 1.
\]

Indeed, when applying the transformation \( G(x) = x_n^{-(1-k)/2} \mathcal{E}_{k,p,N}(x) \), then we obtain the totally invariant Eisenstein series

\[
\mathcal{Q}_{k,p,N}(x) := \sum_{M: \mathcal{T}_p(\mathcal{I})[N] \setminus \Gamma_p(\mathcal{I})[N]} \left( \frac{x_n}{\|cx+d\|^2} \right)^{(n-k-1)/2} \quad k < n - p - 1.
\]

In the case \( p = n - 1 \), we then recover exactly the particular non-analytic Eisenstein series from \([1]\) which is associated to the fixed eigenvalue \(-\frac{1}{4}(n^2 - (k+1)^2)\).

In the particular complex case \( n = 1 \) and \( \mathcal{I} = \mathbb{Z} \), this series coincides with the complex non-analytic Eisenstein series from \([27, 33]\) that is associated to the
particular eigenvalue $-\frac{1}{4}(1 - (k + 1)^2)$. In the subcase $n = 2$ and $\mathcal{I} = \mathbb{Z}[e_1]$ we obtain the non-analytic Eisenstein series on Picard’s group introduced in [6] that is associated to the fixed eigenvalue $-\frac{1}{4}(4 - (k + 1)^2)$.

In the quaternionic setting where $n = 3$ and where $\mathcal{I}$ is the Hurwitz order, we recover one of the non-analytic Eisenstein series from [23].

4 Fourier transforms

4.1 The Fourier expansion of $k$-hypermonogenic automorphic forms

In the first part of this section we determine the general structure of the Fourier expansion of $k$-hypermonogenic automorphic forms. Afterwards we then compute the Fourier coefficients of the $k$-hypermonogenic Eisenstein series that we introduced in (6) explicitly. For technical reasons it is convenient to compute first the Fourier transform of a function that satisfies the reduced $k$-hypermonogenic equation

$$Df + \frac{k}{x_n}(Qf)' = 0$$

in reduced upper half-space $H^+(\mathbb{R}^n) := \{x_1e_1 + \cdots + x_ne_n \in \mathbb{R}^n, x_n > 0\}$, where $D := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}e_i$ is the Euclidean Dirac operator. We prove

**Theorem 2.** Suppose that $f : H^+(\mathbb{R}^n) \to \mathbb{C}$ is a solution to (13). Then its image under the Fourier transform in the directions $x_1, \ldots, x_{n-1}$ has the form

$$\alpha(\omega, x_n) = x_n^{\frac{k+1}{2}}(\langle \omega \rangle x_n - i e_n \frac{\omega}{\| \omega \|} K_{k+1}(\| \omega \| x_n))$$

if $\omega \in \mathbb{R}^{n-1} \setminus \{0\}$ and, for $\omega = 0$,

$$\alpha(\omega, x_n) = a(0) + \alpha(0)x_n^k$$

where $a(0)$ and all $\alpha(\omega)$ are well-defined Clifford numbers from the Clifford sub algebra $Cl_{n-1}$.

**Proof.** Applying the Fourier transform (in the first $(n-1)$ components) on the reduced $k$-hypermonogenic equation

$$Df + \frac{k}{x_n}(Qf)' = 0$$

leads to the following differential equation on the Fourier images

$$\alpha(\omega, x_n) = (\alpha(\omega, x_n), (\omega \in \mathbb{R}^{n-1}, x_n > 0)$$

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of its solutions
\[(i\omega + e_n \frac{\partial \alpha}{\partial x_n}) + \frac{k}{x_n} (Q\alpha)' = 0.\] (14)

Next we split \(\alpha\) into its \(P\)-part and \(Q\)-part:
\[\alpha = P\alpha + e_n(Q\alpha)'.\]

Applying this decomposition to each single term of (14) leads to
\[i\omega \alpha = i\omega(P\alpha) - e_n(i\omega Q\alpha)',\] (15)
\[e_n \frac{\partial \alpha}{\partial x_n} = -\frac{\partial}{\partial x_n}(Q\alpha)' + e_n\left(\frac{\partial}{\partial x_n}(P\alpha)\right)',\] (16)
\[\frac{k}{x_n} (Q\alpha)' = \left(\frac{k}{x_n} Q\alpha\right)'.\] (17)

Since \(\alpha\) is a solution to (14), we thus obtain the following system of two equations when collecting all terms that include or that do not include the factor \(e_n\), respectively:
\[i\omega(P\alpha) - \frac{\partial}{\partial x_n}(Q\alpha)' + \frac{k}{x_n} (Q\alpha)' = 0\] (18)
\[i\omega(Q\alpha) + \frac{\partial}{\partial x_n}(P\alpha)' = 0.\] (19)

This system is equivalent to
\[\frac{\partial}{\partial x_n}(Q\alpha)' = i\omega(P\alpha) + \frac{k}{x_n} (Q\alpha)';\] (20)
\[\frac{\partial}{\partial x_n}(P\alpha) = i\omega(Q\alpha)'.\] (21)

To proceed, we next decompose \(\alpha\) as follows
\[\alpha = A\alpha + \varpi B\alpha\] (22)
where \(A\) and \(B\) are appropriately chosen Clifford operators, projecting \(\alpha\) down to that part of the Clifford algebra which intersects the subspace generated by \(\varpi\) only at 0.

For the \(P\)-part and the \(Q\)-part we have
\[P\alpha = AP\alpha + \varpi BP\alpha\] (23)
\[(Q\alpha)' = A(Q\alpha)' + \varpi B(Q\alpha)'.\] (24)

If we apply this decomposition to (20), then we obtain
\[\frac{\partial}{\partial x_n} A(Q\alpha)' + \varpi \frac{\partial}{\partial x_n} B(Q\alpha)' = i\omega(A(P\alpha) + \varpi B(P\alpha)) + \frac{k}{x_n} (A(Q\alpha)' + \varpi B(Q\alpha)').\]
From this equation, we in turn recover the following system
\[
\frac{\partial}{\partial x_n} A(Q\alpha)' = -i\|\omega\|^2B(P\alpha) + \frac{k}{x_n}A(Q\alpha)'
\]  
(25)
\[
\frac{\partial}{\partial x_n} B(Q\alpha)' = iA(P\alpha) + \frac{k}{x_n}B(Q\alpha)'.
\]  
(26)

Similarly, when applying the decomposition (22) to (21), we obtain the equation
\[
\frac{\partial}{\partial x_n} A(P\alpha) + \omega \frac{\partial}{\partial x_n} B(P\alpha) = i\omega A(Q\alpha)' + \omega B(Q\alpha)'
\]
\[
= i\omega A(Q\alpha)' - i\|\omega\|^2B(Q\alpha)'.
\]

This finally leads to the following system
\[
\frac{\partial}{\partial x_n} A(P\alpha) = -i\|\omega\|^2B(Q\alpha)'
\]  
(27)
\[
\frac{\partial}{\partial x_n} B(P\alpha) = iA(Q\alpha)'.
\]  
(28)

For simplicity let us next use the abbreviations: \(u_1 := AQ(\alpha)', v_1 := BP(\alpha)\) and \(u_2 := AP(\alpha), v_2 := BQ(\alpha)'.\) The set of equations (25) and (28) is a system involving only the functions \(u_1\) and \(v_1\):
\[
\frac{\partial}{\partial x_n} u_1 = -i\|\omega\|^2v_1 + \frac{k}{x_n}u_1
\]  
(29)
\[
\frac{\partial}{\partial x_n} v_1 = iu_1.
\]  
(30)

The other set of equations (26) and (27) is a system involving only the other functions \(u_2\) and \(v_2\):
\[
\frac{\partial}{\partial x_n} u_2 = -i\|\omega\|^2v_2
\]  
(31)
\[
\frac{\partial}{\partial x_n} v_2 = iu_2 + \frac{k}{x_n}v_2.
\]  
(32)

In the case where \(\omega \neq 0\), the general solutions to these systems turn out to be
\[
u_1 = ix_n^{k+1}K_{\frac{k+1}{2}}(\|\omega\|x_n)\alpha_1(\omega)
\]  
(33)
\[
u_1 = \frac{1}{\|\omega\|}x_n^{k+1}K_{\frac{k+1}{2}}(\|\omega\|x_n)\alpha_1(\omega)
\]  
(34)
\[
u_2 = x_n^{k+1}K_{k-\frac{1}{2}}(\|\omega\|x_n)\alpha_2(\omega)
\]  
(35)
\[
u_2 = -x_n^{k+1}\frac{i}{\|\omega\|}K_{k-\frac{1}{2}}(\|\omega\|x_n)\alpha_2(\omega)
\]  
(36)
where $\alpha_1(\omega)$ and $\alpha_2(\omega)$ may be Clifford constants that belong to the orthogonal complement of $\text{span}_\mathbb{R}(\omega, e_n)$.

After re-substituting

\[ \alpha = P(\alpha) + e_n Q(\alpha) = u_2 + \omega v_1 + e_n u_1 + e_n \omega v_2, \]

we hence obtain for all non-vanishing frequencies $\omega \neq 0$:

\[ \alpha(\omega, x_n) = \frac{k+1}{x_n^2} \left[ K_{k+1}(||\omega||x_n) i e_n \left( -\frac{\omega}{||\omega||} \alpha_2(\omega) + \alpha_1(\omega) \right) + K_{k+1}(||\omega||x_n) \right] \]

\[ = \frac{k+1}{x_n^2} \left[ K_{k+1}(||\omega||x_n) i e_n \frac{\omega}{||\omega||} (\alpha_2(\omega) + \alpha_1(\omega)) + K_{k+1}(||\omega||x_n) \right] \]

\[ = \frac{k+1}{x_n^2} \left[ K_{k+1}(||\omega||x_n) - i e_n \frac{\omega}{||\omega||} K_{k+1}(||\omega||x_n) \right] \alpha(\omega), \]

where $\alpha(\omega)$ is a Clifford constant belonging to $\text{Cl}_{n-1}$.

It remains to treat the particular case $\omega = 0$. In this case, equation 14 simplifies to

\[ e_n \frac{\partial \alpha}{\partial x_n} + \frac{k}{x_n} (Q\alpha)' = 0. \]

In this case the system 20 and 21 simplifies to

\[ \frac{\partial}{\partial x_n} (Q\alpha)' = \frac{k}{x_n} (Q\alpha)' \]

\[ \frac{\partial}{\partial x_n} (P\alpha) = 0. \]

We obtain for $\alpha(0, x_n) = a(0) + \alpha(0) x_n^k$, where $a(0)$ is a constant belonging to the part of the Clifford algebra that has no non-trivial intersection with the vector space $\mathbb{R}e_n$.

The use of the vector formalism made this proof technically very elegant. Now we can directly translate the resulting representation formula 2 into the paravector formalism, i.e., to the context of $H^+(\mathbb{R} \oplus \mathbb{R}^n)$:

**Corollary 1.** Suppose that $f : H^+(\mathbb{R} \oplus \mathbb{R}^n) \to \text{Cl}_n$ is a solution to 2. Then its image under the Fourier transform in the directions $x_0, x_1, \ldots, x_{n-1}$ have the form

\[ \alpha(\omega, x_n) = \frac{k+1}{x_n^2} \left[ K_{k+1}(||\omega||x_n) - i e_n \frac{\omega}{||\omega||} K_{k+1}(||\omega||x_n) \right] \alpha(\omega), \]
if \( \omega \in \mathbb{R} \oplus \mathbb{R}^{n-1} \setminus \{0\} \) and, for \( \omega \neq 0 \),
\[
\alpha(\omega, x_n) a(\underline{0}) + \alpha(\underline{0}) x_n^k
\]
where \( a(\underline{0}) \) and all \( \alpha(\omega) \) are well-defined Clifford numbers from the Clifford subalgebra \( Cl_{n-1} \).

Now suppose that \( f \) is a \( k \)-hypermonogenic automorphic form on \( \Gamma_{n-1}(\mathcal{I})[N] \). These are in particular \( n \)-fold periodic with respect to the discrete period lattice \( \Lambda_{n-1}(\mathcal{I})[N] \) and we finally arrive at the main result of this subsection:

**Theorem 3.** Let \( f : H^+(\mathbb{R} \oplus \mathbb{R}^n) \to Cl_n \) be a \( k \)-hypermonogenic automorphic form on \( \Gamma_{n-1}(\mathcal{I})[N] \). Then \( f \) has a particular Fourier series representation on upper half-space of the form
\[
f(x) = a(\underline{0}) + \alpha(\underline{0}) x_n^k + \sum_{m \in \Lambda_{n-1}(\mathcal{I})[N] \setminus \{\underline{0}\}} x_n^{k+1} K_{k+1} (2\pi \|m\| x_n) - i e_n \frac{m}{\|m\|} K_k (2\pi \|m\| x_n) \alpha(m) e^{2\pi i <m, x_n>}
\]
where \( a(\underline{0}) \) and all \( \alpha(\omega) \) are well-defined Clifford numbers from the Clifford subalgebra \( Cl_{n-1} \).

**Remark.** In the monogenic case \((k = 0)\), we have
\[
K_{-\frac{1}{2}} (\|\omega\| x_n) = K_{\frac{1}{2}} (\|\omega\| x_n) = \frac{1}{\sqrt{x_n}} e^{-\|\omega\| x_n}.
\]
In this particular case, the factor simplifies to
\[
x_n^{\frac{1}{2}} \left[ K_{\frac{1}{2}} (\|\omega\| x_n) - i e_n \frac{\omega}{\|\omega\|} K_{-\frac{1}{2}} (\|\omega\| x_n) \right] = e^{-\|\omega\| x_n (1 - i e_n \frac{\omega}{\|\omega\|})}.
\]
This is a scalar multiple of an idempotent in the Clifford algebra. In the monogenic case one obtains the well-known particular Fourier series representation, involving the monogenic plane wave exponential functions (see \([5, 17]\)):
\[
f(x) = a(\underline{0}) + \sum_{m \in \Lambda_{n-1}(\mathcal{I})[N] \setminus \{\underline{0}\}} (1 - i e_n \frac{m}{\|m\|}) \alpha(m) e^{2\pi i <m, x_n>} e^{-2\pi \|m\| x_n}.
\]
In particular, one has the identity:
\[
\lim_{x_n \to +\infty} f(x_n e_n) = a(\underline{0}).
\]

**4.2 The Fourier expansion of the \( k \)-hypermonogenic Eisenstein series**

In this subsection we now apply the previous result in order to determine the Fourier coefficients of the \( k \)-hypermonogenic Eisenstein series explicitly. To proceed in this direction we first show...
Proposition 2. Let $m \in \mathbb{R} \oplus \mathbb{R}^{n-1}$ and $x_n > 0$. Then

$$\beta(m, x_n) := \int_{\mathbb{R} \oplus \mathbb{R}^{n-1}} \frac{\mathcal{F}}{\|x\|^{n+1-k}} e^{-2\pi i <\mathcal{F}, x_n>} dx_0dx_1 \ldots dx_{n-1} \quad (41)$$

$$= \frac{2^{\frac{n+k}{2}}(2\pi \|m\|)^{\frac{k+1}{2}} \pi^{\frac{n+k}{2}} x_n^{\frac{k+1}{2}}}{(1 - n + k)\Gamma(\frac{n+k}{2} - 1)} \times \left(K_{\frac{k+1}{2}}(2\pi \|m\|x_n) - i c_n \frac{m}{\|m\|} K_{\frac{k-1}{2}}(2\pi \|m\|x_n)\right)$$

Proof. We have

$$\frac{\mathcal{F}}{\|x\|^{n+1-k}} = \frac{1}{(1 - n + k)\Gamma(\frac{n-k}{2} - 1)} \frac{1}{\|x\|^{n-k-1}}.$$ 

Now we first compute

$$\int_{\mathbb{R} \oplus \mathbb{R}^{n-1}} \frac{1}{\|x + n e_n\|^{n-k-1}} e^{-2\pi i <\mathcal{F}, x_n>} dx_0dx_1 \ldots dx_{n-1}$$

$$= \int_{\mathbb{R} \oplus \mathbb{R}^{n-1}} \frac{1}{(x_0^2 + x_1^2 + \ldots + x_n^{2n-1} + x_n^2)^{\frac{n-k}{2} - \frac{1}{2}}} e^{-2\pi i <\mathcal{F}, x_n>} dx_0dx_1 \ldots dx_{n-1}$$

$$= \int_{S_{n-2}} \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \frac{r^{n-1}}{(r^2 + x_n^2)^{\frac{n-k}{2}}} e^{-2\pi i \|m\| \cos(\theta)} \sin^{n-2}(\theta) d\theta dr dS_{n-2},$$

where we made the substitution to spherical coordinates by introducing the parameters $r := x_0^2 + \ldots + x_n^2$ and $\theta$ as the angle between $m$ and $x$. $S_{n-2}$ denotes the $n-2$-dimensional unit hypersphere. This integral in turn equals

$$2^{\frac{n+k}{2}} \sqrt{2\pi} 2^{\frac{n+k}{2}} \pi^{\frac{n+k}{2}} x_n^{\frac{n+k}{2}} \frac{\Gamma(\frac{n-k}{2} - \frac{1}{2})}{\Gamma(\frac{n-k}{2})}$$

$$\times \int_{r=0}^{\infty} \frac{r^{n-1}}{(r^2 + x_n^2)^{\frac{n-k}{2}}} e^{-\frac{n+k}{2}} J_{\frac{n-k}{2}}(2\pi \|m\|r) dr$$

$$= \frac{2^{\frac{n+k}{2}}(2\pi \|m\|)^{\frac{k+1}{2}} \pi^{\frac{n+k}{2}} x_n^{\frac{k+1}{2}}}{(1 - n + k)\Gamma(\frac{n+k}{2} - 1)} K_{\frac{k+1}{2}}(2\pi \|m\|x_n).$$

Applying finally the $\overrightarrow{D}$-operator yields the above stated result. \hfill $\square$

Next we observe that we may rewrite the $k$-hypermonogenic Eisenstein series in the form

$$E_{k,n-1,N}(x) = 1 + \sum_{M: T_{n-1}(x)[N] \setminus T_{n-1}(x)[N], c \neq 0} \frac{\mathcal{F} + \tau^{-1}}{\|\mathcal{F} + \tau^{-1}\|^{n+1-k}} \frac{\mathcal{F}}{\|\mathcal{F}\|^{n+1-k}}.$$
If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ runs through a subset of representatives of $T_{n-1}(I)[N] \setminus \Gamma_{n-1}(I)[N]$, where we suppose that $N \geq N_0(I)$, then each $r \in \mathbb{Q} + \mathbb{Q}e_1 + \cdots + \mathbb{Q}e_{n-1}$ is represented in the form $c^{-1}d$ exactly once. Then $\nu(r) := \frac{\overline{r}}{r}$ becomes well-defined, so that we may re-express the Eisenstein series $E_{k,n-1,N}$ in the following form

$$E_{k,n-1,N}(x) = 1 + \sum_{r \in \mathbb{Q} \oplus \mathbb{Q}^{n-1} \setminus \{0\}} \frac{x + r}{\|x + r\|^{n+1-k} \nu(r)}.$$  

From this representation we may readily establish

**Theorem 4.** Let $k < -1$, $p = n - 1$ and $N \geq N_0(I)$. Then the Eisenstein series defined in (4) have the Fourier expansion

$$E_{k,n-1,N}(x) = 1 + \sum_{m \in \Lambda_{n-1}^{*}(I)[N] \setminus \{0\}} e^{2\pi i <m,x>} \beta(m,x) \alpha(m) \quad (42)$$

where $\beta(m,x)$ is defined as in (44) and where

$$\alpha(m) = \sum_{r \mod N, r \neq 0} \nu(r) e^{2\pi i <r,m>}$$

where $r \mod N$ indicates that $r$ runs through a set of representatives of $(\mathbb{Q} \oplus \mathbb{Q}^{n-1})/N\Lambda_{n-1}$.

In the cases where $I$ is a ring in which each non-zero element has a unique left-sided or right-sided prime factor decomposition, the Clifford group valued expressions $\alpha(m)$ can directly be related to Clifford group valued weighted variants of the Epstein zeta function.

The simplest non-trivial examples for $I$ with unique left-sided and right-sided prime factor decomposition are for instance the ring of Gaussian integers $\mathbb{Z}[e_1]$, the ring $\mathbb{Z} + \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3$ (when embedding it into the quaternions by identifying $e_3$ with the $e_1e_2$) or the Hurwitz order mentioned in Section 2.3. Notice that the left-sided prime factor decomposition differs in general from the right-sided prime factor decomposition in the non-commutative cases.

Let us write in analogy to (23) $\mathcal{R}(c)$ for a set of representatives of the right cosets $d + c(\mathbb{N} \cap \mathbb{R} \oplus \mathbb{R}^{n-1})$. For $c \neq 0$, let $\rho(c)$ be the greatest rational divisor of $c$ in $\mathbb{N}I$, i.e., $\rho(c) := \max_{l \in \mathbb{N}} \{\frac{1}{l} \in \mathbb{N}I\}$. Adapting from (23), in this case the expressions $\alpha(m)$ can be expressed explicitly in the form

$$\alpha(m) = \sum_{r \in \mathbb{Q} + \mathbb{Q}e_1 + \cdots + \mathbb{Q}e_{n-1} \setminus \{0\} \mod N} \nu(r) e^{2\pi i <r,m>}$$

Next we split the Clifford group valued expressions $\alpha(m)$ into their real-valued components, i.e. $\alpha(m) = \sum_A \alpha_A(m)e_A$. This representation is valued for all orders $I$, even for those which do not admit a unique left-sided or right-sided
prime factor decomposition. If \( I \) however has a unique left-sided or right-sided prime factor decomposition, then each non-vanishing real component \( \alpha_A(m) \) can be represented as follows:

\[
\alpha_A(m) = \left( \sum_{d \in \mathbb{N} \setminus \{0\}} \frac{d_A \operatorname{sgn}(c_A)}{\|d\|^{n-k}} \right)^{-1} \sum_{c \in \mathbb{N} \setminus \{0\}} \frac{|c_A|}{\|c\|^{n-k}} \gamma(c, m),
\]

where

\[
\gamma(c, m) = \begin{cases} 
\rho(c)\|c\|^2 & \text{if } \langle \overline{d} c^{-1}, m \rangle \in \mathbb{Z} \ \forall d \in \mathcal{R}(c) \\
0 & \text{otherwise}
\end{cases}
\]

The previous expression then involves divisor sums similarly to those described in \([23]\) for the quaternionic case.

**Remark.** Applying Proposition 1 allows us directly to set up the Fourier expansion for the \( k \)-hypermonogenic Eisenstein series with positive \( k > 1 \).

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