Asymptotically flat gravitating spinor field solutions. Step 2 - the compatibility of Dirac equations in a curve and a flat spaces

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(Dated: January 18, 2012)

Using the fact that a spin connection is defined to an accuracy of a vector it is shown that the spin connection should be modified in such a manner that Dirac equation in a curve space would be compatible with Dirac equation in a flat space.

PACS numbers: 04.40.-b
Keywords: curve space; Dirac equation

I. DIRAC EQUATION IN A CURVE SPACE

This note is the continuation of the Ref.[1]. There it was mentioned that at the moment in general relativity asymptotically flat solutions for a spinor field are unknown although do exist asymptotically flat solutions for all known fields: scalar and gauge fields. We continue the investigations in this direction and we will consider Dirac equation in a curve space. We will show that a spin connection should be modified by the addition of a vector in order to obtain correct Dirac equation in a flat space.

A covariant derivative of a spinor \( \psi \) is
\[
\nabla_\mu \psi = \partial_\mu \psi - \Gamma_\mu \psi
\]
(1)
and the covariant derivative of a Dirac conjugated spinor \( \bar{\psi} \) is
\[
\nabla_\mu \bar{\psi} = \partial_\mu \bar{\psi} + \bar{\psi} \Gamma_\mu
\]
(2)
where a spin connection \( \Gamma_\mu \) is defined from following equation (here we follow to Ref. [2], section 1.7.2)
\[
\omega^{\bar{a}\bar{b}\mu} \gamma^{\bar{a}} \gamma^{\bar{b}} - \gamma^{\bar{a}} \Gamma_i^{\bar{a}} \gamma^{\bar{b}} + 4 \Gamma_\mu = 0.
\]
(3)
where \( \gamma^{\bar{a}} \) are Dirac matrixes in a flat (Minkowski) space, \( \gamma^\mu = e^\mu_{\bar{a}} \gamma^{\bar{a}} \) are Dirac matrixes in a curve space. For the definition of tetrad, spin connection and so on we follow to Ref. [2]. The inverse tetrad \( e^a_\mu \) satisfies
\[
e^a_\mu e^b_\nu = \delta^a_\nu, \quad e^a_\mu e^{\bar{a}}_\nu = \delta^\nu_\mu.
\]
(4)
(5)
where \( \bar{a}, \bar{b} = 0, 1, 2, 3 \) are Lorentz indices; \( \mu, \nu \) are world indices. The metric tensor \( g_{\mu\nu} \) in a curve space is related to the Minkowski metric \( \eta_{\bar{a}\bar{b}} \) through the tetrad
\[
g_{\mu\nu} = e^a_\mu e^{\bar{b}}_\nu \eta_{\bar{a}\bar{b}}.
\]
(6)
The solution of (3) is
\[
\Gamma_\mu = -\frac{1}{4} \omega^{\bar{a}\bar{b}\mu} \gamma^{\bar{a}} \gamma^{\bar{b}} - A_\mu,
\]
(7)
where \( A_\mu \) is a spinor-tensor quantity with one vector index. Substituting (7) to (3) gives us the equation for \( A_\mu \)
\[
- \gamma^{\bar{a}} A_\mu \gamma^{\bar{a}} + 4A_\mu = 0.
\]
(8)

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Here we would like to show that $A_\mu$ should be nonzero that the Dirac equation in a curve space would be consistent with the Dirac equation in a flat space. In order to find such solution we rewrite (7) in following form

$$\Gamma_\varepsilon = e_\varepsilon^\mu \Gamma_\mu = -\frac{1}{4} \sum_{a, b} \omega_{a b \varepsilon} \gamma^a \gamma^b - A_\varepsilon = -\frac{1}{4} \sum_{a, b \neq \varepsilon} \omega_{a b \varepsilon} \gamma^a \gamma^b - \frac{1}{2} \sum_{\mu \neq \varepsilon} \omega_{\mu \varepsilon} \gamma^\mu$$

(9)

here there is not summation over $\varepsilon$. In order to find $A_\varepsilon$ we calculate the term $\gamma^\varepsilon \Gamma_\varepsilon$

$$\gamma^\varepsilon \Gamma_\varepsilon = -\frac{1}{4} \sum_{a, b \neq \varepsilon} \omega_{a b \varepsilon} \gamma^a \gamma^b + \frac{1}{2} \sum_{a \neq \varepsilon} \omega_{a \varepsilon} \gamma^a - \sum_{a \neq \varepsilon} \gamma^a A_\varepsilon$$

(10)

here we calculated

$$\sum_{a \neq \varepsilon} \omega_{a b \varepsilon} \gamma^a \gamma^b = -\sum_{a \neq \varepsilon} \omega_{a \varepsilon} \gamma^a$$

(11)

We choose

$$A_\varepsilon = \frac{1}{2} \sum_{c \neq \varepsilon} \omega_{a c \varepsilon}$$

(12)

It is easy to show that (12) is the solution of (8). Let us rewrite the term

$$\sum_{a, b \neq \varepsilon} \omega_{a b \varepsilon} \gamma^a \gamma^b = \sum_{a, b \neq \varepsilon} \omega_{a b \varepsilon} \gamma^a \gamma^b$$

(13)

Finally we choose the spin connection $\Gamma_\mu$ in the form

$$\tilde{\Gamma}_\mu = -\frac{1}{4} e_\mu^\varepsilon \sum_{a, b} \omega_{a \varepsilon} \gamma^a \gamma^b - \frac{1}{2} \sum_{\varepsilon \neq \mu} \omega_{e \varepsilon}$$

(14)

here $\tilde{\Gamma}_\mu$ means that we use the Fock - Ivanenko coefficients $\omega_{a \varepsilon}$ with $\bar{a} \neq \bar{b}, \bar{b} \neq \bar{c}, \bar{a} \neq \bar{c}$. Consequently one can write Dirac equation in a curve space in following form

$$i \gamma^\mu \nabla_\mu \psi - m \psi = i \gamma^\mu \left( \partial_\mu - \tilde{\Gamma}_\mu \right) \psi - m \psi = 0$$

(15)

or

$$i \left( \nabla_\mu \tilde{\psi} \right) \gamma^\mu - m \tilde{\psi} = i \tilde{\psi} \left( \partial_\mu + \tilde{\Gamma}_\mu \right) \gamma^\mu - m \tilde{\psi} = 0$$

(16)

for the Dirac conjugated spinor $\tilde{\psi}$, here $\gamma^\mu \tilde{\psi} = \partial_\mu \tilde{\psi}$. Usually Dirac equation is written as

$$i \gamma^\mu \nabla_\mu \psi = i \gamma^\mu \left( \partial_\mu \psi - \Gamma_\mu \right) \psi - m \psi = i \gamma^\mu \left( \partial_\mu + \frac{1}{4} \omega_{a b \mu} \gamma^a \gamma^b \right) \psi - m \psi = 0$$

(17)

In order to compare equations (15) and (17) we will write Dirac operators $\mathcal{D}_1, \mathcal{D}_2$ for both equations (15) and (17)

$$\mathcal{D}_1 \psi = \left( i \gamma^\mu \partial_\mu + i \sum_{a,b,c} \omega_{a b c} \gamma^a \gamma^b + i \sum_{b \neq c} \omega_{b c} \gamma^c \right) \psi$$

(18)

$$\mathcal{D}_2 \psi = \left( i \gamma^\mu \partial_\mu + i \sum_{a,b,c} \omega_{a b c} \gamma^a \gamma^b - m \right) \psi$$

(19)

in Minkowski space for the spherical coordinate system. For the calculations of $\omega_{a b \mu}$ we use following definitions from (2) (section 1.5.4)

$$\Lambda_{\alpha \mu \nu} = \frac{1}{2} \left( \partial_\mu e^\alpha_{\nu} - \partial_\nu e^\alpha_{\mu} \right) = -\Lambda_{\nu \mu \alpha}$$

(20)

$$\omega_{\alpha \beta \gamma} = -\Lambda_{\alpha \beta \gamma} + \Lambda_{\alpha \beta \gamma} - \Lambda_{\beta \alpha \gamma} = -\omega_{\beta \alpha \gamma}$$

(21)

$$\omega_{a b \mu} = e^a_{\alpha} e^b_{\beta} \omega_{\alpha \beta \mu} = -\omega_{b a \mu}$$

(22)
that obviously is not the necessary equation.

The metric is

\[ ds^2 = dt^2 - dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \]  

(24)

The Dirac matrices for the spherical coordinate system \((24)\) are

\[
\gamma^\alpha = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\]

(25)

\[
\gamma^1 = \begin{pmatrix}
0 & 0 & \cos \theta & \sin \theta e^{-i\varphi} \\
-\cos \theta & -\sin \theta e^{-i\varphi} & 0 & 0 \\
\sin \theta & \sin \theta e^{i\varphi} & \cos \theta & 0 \\
-\cos \theta e^{i\varphi} & -\sin \theta & 0 & 0
\end{pmatrix},
\]

(26)

\[
\gamma^2 = \begin{pmatrix}
0 & 0 & -\sin \theta & \cos \theta e^{-i\varphi} \\
0 & 0 & \cos \theta e^{i\varphi} & \sin \theta \\
\sin \theta & -\cos \theta e^{-i\varphi} & \cos \theta & 0 \\
-\cos \theta e^{i\varphi} & -\sin \theta & 0 & 0
\end{pmatrix},
\]

(27)

\[
\gamma^3 = \begin{pmatrix}
0 & 0 & 0 & -i e^{-i\varphi} \\
0 & 0 & i e^{i\varphi} & 0 \\
0 & -i e^{i\varphi} & 0 & 0 \\
-i e^{i\varphi} & 0 & 0 & 0
\end{pmatrix}.
\]

(28)

The Fock - Ivanenko coefficients \((22)\) are

\[
\omega_{\tilde{1}\tilde{2}} = 1, \quad \omega_{\tilde{1}\tilde{3}} = \sin \theta, \quad \omega_{\tilde{2}\tilde{3}} = \cos \theta.
\]

(29)

For this case

\[
\omega_{\tilde{a}\tilde{b}\tilde{c}} = 0, \quad \text{with } \tilde{a}, \tilde{b} \neq \tilde{c}
\]

(30)

which we will use in \((18)\). For the first case \((18)\) we have

\[
D_1 \psi = e^{i\Omega} \begin{pmatrix}
g'(r) + \frac{2}{r} g(r) + (m + \Omega) f(r) \\
0 \\
i [-f' + (\Omega - m) g(r)] \cos \theta \\
i [-f' + (\Omega - m) g(r)] \sin \theta e^{i\varphi}
\end{pmatrix}
\]

(31)

that is agreed with Dirac equation for an electron in hydrogen atom. But for the second case \((19)\) we have

\[
D_2 \psi = e^{i\Omega} \begin{pmatrix}
-\left[ g'(r) + \frac{2}{r} g(r) + (m + \Omega) f(r) \right] \\
-\frac{i e^{i\varphi} \cot \theta}{2} g(r) \\
-i \left[ f'(r) + \frac{f(r)}{r} + (m - \Omega) g(r) \right] \cos \theta \\
i \left[ -f'(r) + \frac{-3 + \cos(2\Theta) f(r)}{4r} - (m - \Omega) g(r) \right] \sin \theta e^{i\varphi}
\end{pmatrix}
\]

(32)

that obviously is not the necessary equation.

II. ENERGY - MOMENTUM TENSOR

According to Ref. \([3]\) the energy - momentum tensor for the classical spinor field is \([4]\)

\[
T_{\mu\nu} = -\frac{i}{2} \left[ \bar{\psi} \gamma_{(\mu} \nabla_{\nu)} \psi - \nabla_{(\mu} \bar{\psi} \gamma_{\nu)} \psi \right] + g_{\mu\nu} L_{\psi},
\]

(33)

\[
L_{\psi} = \frac{i}{2} \left( \bar{\psi} \gamma^\mu \nabla_{\mu} \psi - \nabla_{\mu} \bar{\psi} \gamma^\mu \psi \right) - m \bar{\psi} \psi
\]

(34)
here $a_{(\mu\nu)} = (1/2)(a_{\mu}b_{\nu} + a_{\nu}b_{\mu})$ means the symmetrization. For our goal we change $\nabla \rightarrow \tilde{\nabla}$

$$
\tilde{T}_{\mu\nu} = \frac{i}{2} \left\{ \bar{\psi} \gamma^{\mu} (\tilde{\nabla}_{\nu}) \psi - \tilde{\nabla}_{(\mu} \bar{\psi} \gamma_{\nu)} \psi \right\} + g_{\mu\nu} \tilde{\nabla} \psi,
$$

(35)

$$
\tilde{\nabla} \psi = \frac{i}{2} \left( \bar{\psi} \gamma^{\mu} \nabla_{\mu} \psi - \nabla_{\mu} \bar{\psi} \gamma^{\mu} \psi \right) - m\bar{\psi} \psi.
$$

(36)

The calculation of the energy - momentum tensor $\tilde{T}_{a\bar{b}}$ with the spinor ansatz (23) and modified spin connection (14) gives us

$$
\tilde{T}_{\bar{0}0} = - \Omega \left[ f(r)^2 + g(r)^2 \right],
$$

(37)

$$
\tilde{T}_{\bar{1}\bar{1}} = - \left[ g(r) f'(r) - f(r) g'(r) \right],
$$

(38)

$$
\tilde{T}_{\bar{1}\bar{4}} = \tilde{T}_{\bar{4}\bar{1}} = \left[ \Omega f(r) - \frac{g(r)}{2r} \right] g(r) \sin \theta,
$$

(39)

$$
\tilde{T}_{\bar{2}\bar{2}} = \tilde{T}_{\bar{3}\bar{3}} = \frac{f(r) g(r)}{r}.
$$

(40)

The current

$$
J^\mu = \bar{\psi} \gamma^\mu \psi
$$

(41)

is

$$
J^0 = f(r)^2 + g(r)^2,
$$

(42)

$$
J^3 = \frac{2 f(r) g(r)}{r}.
$$

(43)

From (37) and (42) we see that there is the energy and charge densities. From (43) we see that there is the current $J^3$ along $\varphi$ direction leading to $\tilde{T}_{\bar{1}\bar{4}}$ component of energy - momentum (39). One can say that $\tilde{T}_{\bar{1}\bar{4}} \neq 0$ and $J^3 \neq 0$ is the consequence of a spin distinct from zero.

**III. CONCLUSIONS**

Thus we have shown that the standard definition of the spin connection in a curve space should be modified that the Dirac equation in a curve space would have correct limit in going from a curve space to a flat space. The modification of the spin connection to lie in the fact that to the standard spin connection is a vector added. So that the modified spin connection has Fock - Ivanenko coefficients with unequal indices only.

**Acknowledgements**

I am grateful to the Research Group Linkage Programme of the Alexander von Humboldt Foundation for the support of this research.

[1] V. Dzhunushaliev, “Asymptotically flat gravitating spinor field solutions. Step 1 - the statement of the problem and the comparison with confinement problem in QCD,” arXiv:0910.3352 [gr-qc].

[2] Nikodem J. Poplawski, “Spacetime and fields”, arXiv:0911.0334.

[3] Tomas Ortin, “Gravity and Strings”, Cambridge university press, 2004.

[4] It is necessary to note that the definition of Ricci tensors in Ref’s 2 and 3 have the opposite sign.