COMPARISON OF JUMP AND BRIDGE RESETING IN DIFFUSIVE SEARCH FOR A RANDOM TARGET ON THE LINE AND IN SPACE

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Abstract. Fix $D > 0$. For a parameter $r > 0$, and for $d \geq 1$, let $X^{(d,r)}(\cdot)$ be a $d$-dimensional Brownian motion with diffusion coefficient $D$, equipped with an exponential clock with rate $r$, so that when the clock rings the process jumps to the origin, where it resets and begins anew according to the same rule. Denote expectations with respect to this process by $E_0^{(d,r)}$. This process, by now rather well-studied, is called Brownian motion with resetting. For a parameter $T > 0$, consider also a process $X^{bb,d,T}(\cdot)$ that performs a $d$-dimensional Brownian bridge with diffusion coefficient $D$ and bridge interval $T$, and then at time $T$ starts anew from the origin according to the same rule. Denote expectations with respect to this process by $E_0^{bb,d,T}$. The two resetting processes, one with jumps and the other continuous, search for a random target $a \in \mathbb{R}^d$ that has a known distribution $\mu$ on $\mathbb{R}^d$. Fix $\epsilon_0 > 0$ and define

\begin{equation}
\tau_a = \begin{cases} 
\inf\{t \geq 0 : Y(t) = a\}, & d = 1; \\
\inf\{t \geq 0 : |Y(t) - a| \leq \epsilon_0\}, & d \geq 2,
\end{cases}
\end{equation}

where $Y(\cdot) = X^{(d,r)}(\cdot)$ or $Y = X^{bb,1/T}(\cdot)$; $\tau_a$ is the time until a target at $a \in \mathbb{R}^d$ is “located”. The expected time to locate the target is $\int_{\mathbb{R}^d} (E_0^{(d,r)} \tau_a) \mu(da)$ for the first process and $\int_{\mathbb{R}^d} (E_0^{bb,d,T} \tau_a) \mu(da)$ for the second process. An explicit formula is known for $E_0^{(d,r)} \tau_a$. We calculate explicitly $E_0^{bb,d,T} \tau_a$ in dimensions $d = 1$ and $d = 3$. Let $\mu^{Gauss, d}$ denote a centered Gaussian target distribution with variance $\sigma^2$, which is the mean-squared distance of the target from the origin. For $d = 1$, we calculate $\int_{\mathbb{R}} (E_0^{(1,r)} \tau_a) \mu^{Gauss,1}(da)$ and $\int_{\mathbb{R}} (E_0^{bb,1,T} \tau_a) \mu^{Gauss,1}(da)$, and then compare $\inf_{r > 0} \int_{\mathbb{R}} (E_0^{(1,r)} \tau_a) \mu^{Gauss,1}(da)$ to $\inf_{T > 0} \int_{\mathbb{R}} (E_0^{bb,1,T} \tau_a) \mu^{Gauss,1}(da)$. For $d = 3$, we calculate $\lim_{\epsilon_0 \to 0} \int_{\mathbb{R}^2} (E_0^{(3,r)} \tau_a) \mu_{Gauss,2}(da)$ and $\lim_{\epsilon_0 \to 0} \int_{\mathbb{R}^3} (E_0^{bb,3,T} \tau_a) \mu_{Gauss,2}(da)$, and then compare the infimum over $r > 0$ in the first expression to the infimum over $T > 0$ in the second one. In the one-dimensional case, for each of the search processes, the optimal expected time to locate the target is an appropriate universal constant times $\sigma^2$, whereas in the three-dimensional case it is an appropriate universal constant times $\frac{\sigma^2}{2}$.

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1. Introduction and Statement of Results

The use of resetting in search problems is a common phenomenon in various contexts. For example, in everyday life, one might be searching for some target, such as a face in a crowd or a misplaced object. After having searched unsuccessfully for a while, there is a tendency to return to the starting point and begin the search anew. Other contexts where search problems frequently involve resetting include animal foraging \[1, 18\] and internet search algorithms.

Over the past decade or so, a variety of stochastic processes with resetting have attracted much attention. See \[9\] for a rather comprehensive, recent overview. Prominent among such processes is the diffusive search process with resetting, which we now describe. Consider a random stationary target \(a \in \mathbb{R}^d, d \geq 1\), with known distribution \(\mu\) on \(\mathbb{R}^d\), and consider a search process that sets off from the origin and performs \(d\)-dimensional Brownian motion with diffusion coefficient \(D > 0\), which is fixed once and for all. The search process is also equipped with an exponential clock with rate \(r\), so that if it has failed to locate the target by the time the clock rings, then its position is reset to the origin and it continues its search anew independently with the same rule. We consider \(r\) as a parameter that can be varied. In dimension one, the target is considered “located” when the process hits the point \(a\), while in dimensions two and higher, one chooses an \(\epsilon_0 > 0\) and the target is considered “located” when the process hits the \(\epsilon_0\)-ball centered at \(a\).

Denote the search process by \(X^{(d;r)}(\cdot)\) and let \(P_0^{(d;r)}\) and \(E_0^{(d;r)}\) denote probabilities and expectations for the process starting from 0. Let

\[
\tau_a = \begin{cases} 
\inf\{t \geq 0 : X^{(1;r)}(t) = a\}, & d = 1; \\
\inf\{t \geq 0 : |X^{(d;r)}(t) - a| \leq \epsilon_0\}, & d \geq 2
\end{cases}
\]

denote the time at which a target at \(a \in \mathbb{R}^d\) is located. Throughout the paper, we suppress the dependence on \(\epsilon_0\) in the notation. One may be interested in several statistics, the most important one probably being the expected time to locate the target, \(\int_{\mathbb{R}} \left(E_0^{(d;r)}\tau_a\right)\mu(da)\). See, for example, \[5, 6, 7, 8, 13, 12, 4, 16\] for a sampling of articles on this model and related ones. See \[15\] for an analysis of the large time behavior of \(\int_{\mathbb{R}} P_0^{(d;r)}(\tau_a > t)\mu(da)\).
The resetting in the above model is of course discontinuous—at the ring of the exponential clock, the search process instantaneously jumps back to its initial position at the origin. In certain applications, this is a reasonable assumption, but in others, it is more reasonable to consider a type of resetting for which the search process remains continuous. For example, while the instantaneous jump model might be reasonable for an internet search, perhaps a continuous type of resetting would be more reasonable for animal foraging. In this paper, we introduce a continuous search process with resetting via the Brownian bridge, and for dimensions \( d = 1 \) and \( d = 3 \), we compare its efficiency to the above search process. Of course, this continuous search will be less efficient since it diffuses back rather than jumps back to its initial point.

As above, fix once and for all the diffusion coefficient \( D > 0 \). Recall that the one-dimensional Brownian bridge with bridge time interval \( T \) is the one-dimensional Brownian motion conditioned to be at the origin at time \( T \). As is well-known [11, 17], a one-dimensional Brownian bridge with bridge time interval \( T \) and diffusion coefficient \( D \) can be represented as

\[
\sqrt{D} \left( W(t) - \frac{t}{T} W(T) \right), \quad 0 \leq t \leq T,
\]

where \( W(\cdot) \) is a standard one-dimensional Brownian motion. The \( d \)-dimensional Brownian bridge with bridge time interval \( T \) and diffusion coefficient \( D \) is the process in \( \mathbb{R}^d \) whose components are independent one-dimensional Brownian bridges with bridge time interval \( T \) and diffusion coefficient \( D \).

For \( T > 0 \), let \( \{B_n^{bb,d,T}(t), 0 \leq t \leq T\}_{n=1}^\infty \) be a sequence of independent \( d \)-dimensional Brownian bridges with bridge time interval \( T \) and diffusion coefficient \( D \). Define the search process \( X^{bb,d,T}(\cdot) \) by

\[
X^{bb,d,T}(t) = B_n^{bb,d,T}(t-nT), \quad t \in [nT, (n+1)T], \quad n = 0, 1, 2, \ldots
\]

We consider \( T \), the time interval between resets, to be a parameter that can be varied.

Let \( P_0^{bb,d,T} \) and \( E_0^{bb,d,T} \) denote probabilities and expectations for the search process \( X^{bb,d,T}(\cdot) \) starting from 0. Let

\[
\tau_a = \begin{cases} 
\inf\{t \geq 0 : X^{bb,1,T}(t) = a\}, & d = 1; \\
\inf\{t \geq 0 : |X^{bb,d,T}(t) - a| \leq \epsilon_0\}, & d \geq 2
\end{cases}
\]

denote the expected time at which a target at \( a \in \mathbb{R}^d \) is located. We wish to compare the expected time to locate a target for this Brownian bridge search process \( X^{bb,d,T}(\cdot) \) to that of the jump resetting Brownian search process \( X^{(d;r)}(\cdot) \).
defined above. (We use the same notation $\tau_a$ for the hitting time of both processes; the different notation for the expectations will distinguish them.) From now on we will refer to $X^{(d,r)}(\cdot)$ as the Brownian jump reset process and to $X^{bb,d,T}(\cdot)$ as the Brownian bridge reset process.

It is known that for the Brownian jump reset process, one has in dimension one,

\begin{equation}
E_0^{(1;r)}\tau_a = \frac{1}{r}(e^{\sqrt{2\pi}|a|} - 1), \ a \in \mathbb{R},
\end{equation}

and in dimension $d \geq 2,$

\begin{equation}
E_0^{(d;r)}\tau_a = \frac{1}{r} \left[ (\frac{\epsilon_0}{|a|})^{1-\frac{d}{2}} \frac{K_{1-\frac{d}{2}}(\sqrt{2\pi}\epsilon_0)}{K_{1-\frac{d}{2}}(\sqrt{2\pi}|a|)} - 1 \right],
\end{equation}

where $K_\nu$ denotes the modified Bessel function of the second kind of order $\nu$ \cite{5,7}.

From now on we consider only dimensions $d = 1$ and $d = 3.$ It is known that

\begin{equation}
K_{-\frac{d}{2}}(y) = \left(\frac{\pi}{2y}\right)^{\frac{d}{2}}e^{-y}.
\end{equation}

Thus, from (1.5) we have

\begin{equation}
E_0^{(3;r)}\tau_a = \frac{1}{r} \left[ \frac{|a|}{\epsilon_0} e^{\sqrt{2\pi}(|a| - \epsilon_0)} - 1 \right].
\end{equation}

From (1.4) and (1.6), it follows that for $d = 1$ and $d = 3,$ a necessary and sufficient condition for the finiteness of $\inf_{r>0} \int_{\mathbb{R}^d} (E_0^{(d;r)}\tau_a) \mu(da),$ the infimum over resetting rates of the expected time to locate the target, is that the target distribution $\mu$ possess some absolute exponential moment; that is, $\int_{\mathbb{R}^d} e^{\rho|x|} \mu(dx) < \infty,$ for some $\rho > 0.$ (We note that one can also consider spatially dependent resetting rates $r = r(x).$ It turns out that in dimension $d = 1,$ for any $l > 2,$ if $\mu$ possesses its absolute $l$th moment, then one can choose a rate $r(x)$ for which the expected time to locate the target is finite \cite{14}.

We now turn to the Brownian bridge reset process. For the one dimensional case, we have the following theorem, which is for the Brownian bridge reset process the counterpart of (1.4) for the Brownian jump reset process.

**Theorem 1.**

\begin{equation}
E_0^{bb,1:T}\tau_a = T(e^{2a^2\frac{\epsilon_0}{D}} - 1) + |a|e^{2a^2\frac{\epsilon_0}{D}} \int_0^T \frac{e^{-2Dt(1-\frac{x^2}{4})}}{\sqrt{2\pi D t(1-\frac{x^2}{4})}} dt, \ a \in \mathbb{R}.
\end{equation}
For the three-dimensional case, we have the following theorem, a counterpart to (1.6), giving upper and lower bounds in terms of $\epsilon_0$ for the expecting hitting time $\tau_a$.

**Theorem 2.**

\begin{align*}
E^0_{bb,3;\tau_a} & \leq T\left[\frac{|a| + \epsilon_0}{|a| - \epsilon_0} \frac{2(|a| + \epsilon_0)^2}{|a|^2} - 1\right] + T\frac{|a| + \epsilon_0}{2(|a| - \epsilon_0)} e^{\frac{8a_0|x|}{|a|^2}}, |a| > \epsilon_0; \\
E^0_{bb,3;\tau_a} & \geq T\left[\frac{|a|}{2\epsilon_0} e^{\frac{2(|a| - \epsilon_0)^2}{|a|^2}} - 1\right] + T\frac{|a| - \epsilon_0}{2(|a| + \epsilon_0)} e^{-\frac{8a_0|x|}{|a|^2}}, |a| > \epsilon_0.
\end{align*}

Also, for any $A > 0$,

\begin{equation}
\sup_{\epsilon_0,a:0<\epsilon_0<|a|\leq A} \epsilon_0 E^0_{bb,3;\tau_a} < \infty.
\end{equation}

It follows from Theorems 1 and 2 that for $d = 1$ and $d = 3$, a necessary and sufficient condition for the finiteness of $\inf_{T>0} \int_{\mathbb{R}^d} (E^0_{bb;\tau_a}) \mu(da)$, the infimum over bridge time intervals, of the expected time to locate the target, is that $\int_{\mathbb{R}^d} e^{\rho|x|^2} \mu(dx) < \infty$, for some $\rho > 0$. In particular, if the target distribution $\mu$ does not satisfy this condition, but does possess some absolute exponential moment, then the expected time to locate the target in the case of the Brownian jump reset process will be finite for appropriate resetting rates $r$, but will be infinite in the case of the Brownian bridge reset process for every bridge time interval $T$.

We now come to the main point of this paper, which is to compare the expected time to hit a centered Gaussian target for the two search processes. Let $\mu_{\sigma^2}^{Gauss,d}$ denote the centered Gaussian distribution on $\mathbb{R}^d$ with variance $\sigma^2$. Of course, $\sigma^2$ is then the mean-squared displacement of the target from the search origin.

We begin with the one-dimensional case. Here is the result for the Brownian jump reset process.

**Proposition 1.**

\begin{equation}
\int_{\mathbb{R}} (E^{(1;r)}_0;\tau_a) \mu_{\sigma^2}^{Gauss,1}(da) = \frac{1}{r}(2e^{\frac{r^2}{D}} \int_{-\sqrt{\frac{D}{2\pi}}}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx - 1).
\end{equation}

Equivalently, writing $r = \frac{D}{\sigma^2}s$, with $s > 0$,

\begin{equation}
\int_{\mathbb{R}} (E^{(1;\frac{D}{\sigma^2}s)}_0;\tau_a) \mu_{\sigma^2}^{Gauss,1}(da) = \frac{\sigma^2}{D} \left(2e^{\frac{s}{\sqrt{2\pi}}} \int_{-\sqrt{\frac{D}{2\pi}}}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx - 1\right).
\end{equation}
One has
\[
\inf_{r>0} \int_{\mathbb{R}} (E_r(1;r)a_0) \mu_{\sigma^2} \, (da) \approx 3.548 \frac{\sigma^2}{D} \quad \text{with the infimum attained at } r \approx 0.491 \frac{D}{\sigma^2}.
\]

Here is the corresponding result for the one-dimensional Brownian bridge search process.

**Theorem 3.**

\[
\int_{\mathbb{R}} (E_0^{bb;1;T}a_0) \mu_{\sigma^2} \, (da) = \begin{cases} 
T\left(\frac{DT}{DT - 4\sigma^2}\right)^{\frac{1}{2}} - T + \frac{T\sigma}{\sqrt{DT + 2\sigma^2}}, & T > \frac{4\sigma^2}{D}; \\
\infty, & T \leq \frac{4\sigma^2}{D}.
\end{cases}
\]

Equivalently, writing \( T = \frac{\sigma^2}{D} \mathcal{T} \), with \( \mathcal{T} > 0 \),
\[
\int_{\mathbb{R}} (E_0^{bb;1;\frac{\sigma^2}{D}\mathcal{T}}a_0) \mu_{\sigma^2} \, (da) = \begin{cases} 
\frac{\sigma^2}{D}\left(\mathcal{T}\left(\frac{T}{(T - 4)}\right)^{\frac{1}{2}} - \mathcal{T} + \frac{T}{2\sqrt{T}}\right), & \mathcal{T} > 4; \\
\infty, & \mathcal{T} \leq 4.
\end{cases}
\]

One has
\[
\inf_{T>0} \int_{\mathbb{R}} (E_0^{bb;1;T}a_0) \mu_{\sigma^2} \, (da) \approx 4.847 \frac{\sigma^2}{D} \quad \text{with the infimum attained at } T \approx 10.136 \frac{\sigma^2}{D}.
\]

**Conclusion for the one-dimensional case.** From Proposition 1 and Theorem 3, it follows that the appropriate scaling unit for the resetting rate \( r \) in the case of the Brownian jump reset process is \( \frac{D}{\sigma^2} \), and the appropriate scaling unit for the bridge time interval \( T \) in the case of the Brownian bridge reset process is \( \frac{\sigma^2}{D} \). In both of these cases, the corresponding expected time to locate the target is an appropriate constant times \( \frac{\sigma^2}{D} \). Furthermore, the expected time to locate the target is about 36.6 percent longer under the Brownian bridge reset process with optimal bridge time interval than it is under the Brownian jump reset process with optimal jump rate.

We now turn to the corresponding results in three dimensions. The expected hitting time of a point \( a \in \mathbb{R}^3 \) with \( |a| > \epsilon_0 \) depends on \( \epsilon_0 \), and from (1.6) and (1.8), it diverges on the order \( \frac{1}{\epsilon_0} \) as \( \epsilon_0 \to 0 \), for both the Brownian jump reset process and the Brownian bridge reset process. Thus, we consider the quantity
\[
\lim_{\epsilon_0 \to 0} \epsilon_0 \int_{\mathbb{R}^3} (E_0^{(3;r)}a_0) \mu_{\sigma^2} \, (da).
\]

Here is the result for the Brownian jump reset process.
Proposition 2.

\[
\lim_{\epsilon_0 \to 0} \epsilon_0 \int_{\mathbb{R}^3} (E_{0}^{(3;r)} \tau_a) \mu_{\sigma^2}^{Gauss,3} (da) = \frac{1}{r} \int_{\mathbb{R}^3} |a| e^{\frac{-|a|^2}{2}} \frac{e^{-\frac{|a|^2}{2\sigma^2}}}{(2\pi\sigma^2)^{\frac{3}{2}}} da = \frac{2\sigma}{\sqrt{2\pi r}} \int_{0}^{\infty} x^3 e^{\frac{-x^2}{2\sigma^2}} e^{-\frac{x^2}{2} dx}.
\]

Equivalently, writing \( r = \frac{D}{\sigma^2} s \), with \( s > 0 \),

\[
\lim_{\epsilon_0 \to 0} \epsilon_0 \int_{\mathbb{R}^3} (E_{0}^{(3;r)} \tau_a) \mu_{\sigma^2}^{Gauss,3} (da) = \frac{\sigma^3}{D} \left( \frac{2}{\sqrt{2\pi s}} \int_{0}^{\infty} x^3 e^{\frac{-x^2}{2\sigma^2}} e^{-\frac{x^2}{2} dx} \right).
\]

One has

\[
\inf_{r > 0} \lim_{\epsilon_0 \to 0} \epsilon_0 \int_{\mathbb{R}^3} (E_{0}^{(3;r)} \tau_a) \mu_{\sigma^2}^{Gauss,3} (da) \approx 13.09 \frac{\sigma^3}{D},
\]

with the infimum attained at \( r \approx 0.738 \frac{D}{\sigma^2} \).

Here is the corresponding result for the three-dimensional Brownian bridge search process.

Proposition 3.

\[
\lim_{\epsilon_0 \to 0} \epsilon_0 \int_{\mathbb{R}^3} (E_{0}^{bb;3:T} \tau_a) \mu_{\sigma^2}^{Gauss,3} (da) = \begin{cases} 
\frac{2T^3 D^2 \sigma}{\sqrt{2\pi(DT - 4\sigma^2)^2}}, & T > 4\frac{\sigma^2}{D}; \\
\infty, & T \leq 4\frac{\sigma^2}{D}.
\end{cases}
\]

Equivalently, writing \( T = \frac{\sigma^2}{D} T \),

\[
\lim_{\epsilon_0 \to 0} \epsilon_0 \int_{\mathbb{R}^3} (E_{0}^{bb;3:T} \tau_a) \mu_{\sigma^2}^{Gauss,3} (da) = \begin{cases} 
\frac{\sigma^3}{D} \left( \frac{2T^3}{\sqrt{2\pi(T - 4)^2}} \right), & T > 4; \\
\infty, & T \leq 4.
\end{cases}
\]

One has

\[
\inf_{T > 0} \lim_{\epsilon_0 \to 0} \epsilon_0 \int_{\mathbb{R}^3} (E_{0}^{bb;3:T} \tau_a) \mu_{\sigma^2}^{Gauss,3} (da) \approx 21.52 \frac{\sigma^3}{D},
\]

with the infimum attained at \( T = 12 \frac{\sigma^2}{D} \).

Conclusion for the three-dimensional case. From Proposition 2 and Theorem 3, it follows that the appropriate scaling unit for the resetting rate \( r \) in the case of the Brownian jump reset process is \( \frac{D}{\sigma^2} \), and the appropriate scaling unit for the bridge time interval \( T \) in the case of the Brownian bridge reset process is \( \frac{\sigma^2}{D} \); these are the same as in the one dimensional case. However, for both of these processes, the corresponding expected time to locate the target is an appropriate constant times \( \frac{\sigma^3}{D} \), as opposed to \( \frac{\sigma^2}{D} \) in
the one-dimensional case. The expected time to locate the target is about 64.4 percent longer under the Brownian bridge reset process with optimal bridge time interval than it is under the Brownian jump reset process with optimal jump rate.

We note that a recent paper [3] introduced a hybrid version of the two one-dimensional search processes considered in this paper. The process in that paper is the classical search process with instantaneous resetting, conditioned to return to the origin at time $T$. The authors study various properties of this process on the time interval $[0, T]$. One can construct a search process on the entire time line $[0, \infty)$ by repeating independent copies of this process on $[0, T]$, just as was done to construct the process $X^{bb,1:T}(\cdot)$ from independent copies of the Brownian bridge. It would be interesting to investigate how this search process performs with respect to a centered Gaussian target. Such an analysis seems difficult because the expression for the expectation of $\tau_a$, analogous to (1.7), is quite unwieldy.

We first prove the results for the one-dimensional case and then prove the results for the three-dimensional case. For the one-dimensional case, we prove Theorem 1, Proposition 1 and Theorem 3 in sections 2, 3 and 4 respectively. For the three-dimensional case, we prove Theorem 2, Proposition 2 and Proposition 3 in sections 5, 6 and 7 respectively.

2. Proof of Theorem 1

By symmetry, it suffices to consider $a > 0$. Let $W(\cdot)$ be a Brownian motion with diffusion coefficient $D$ starting from the origin (the generator of the process is $\frac{D}{2} \frac{d^2}{dx^2}$), and denote probabilities and expectations for this process by $P_0$ and $E_0$. It is well-known [11] that $\tau_a$, the hitting time of $a \in \mathbb{R}$, satisfies

$$P_0(\tau_a < T|W(T) = 0) = e^{-\frac{2a^2}{D}}.$$

It follows from the definition of the Brownian bridge and the definition of $X^{bb,1:T}(\cdot)$ that the above equation is equivalent to

$$P_0^{bb,1:T}(\tau_a < T) = e^{-\frac{2a^2}{D}}.$$  

It is very well-known from the reflection principle [11] [17] that the hitting time $\tau_a$ for $W(\cdot)$ has a density $f(t)$ given by

$$f(t) = \frac{1}{\sqrt{2\pi D t^2}} \frac{a}{t} e^{-\frac{a^2}{2Dt}}, \ t > 0.$$
Of course, the density of $W(t)$ is $\frac{e^{-\frac{|a|^2}{2\pi Dt}}}{\sqrt{2\pi Dt}}$, $a \in \mathbb{R}$. Thus, $\tau_a$, the hitting time for $W(t)$, $0 \leq t \leq T$, conditioned on $W(T) = 0$, or equivalently, the hitting time under $P_{0,\mu}^{b,1,T}$, has sub-density

$$f(t) = \frac{1}{\sqrt{2\pi D(T-t)}} e^{-\frac{a^2}{2D(T-t)}}, \quad 0 < t < T.$$  

(2.2)

Consequently, from the definition of $X^{b,1,T}()$,

$$E_0^{b,1,T}(\tau_a 1_{\tau_a < T}) = E_0(\tau_a 1_{\tau_a < T} | W(T) = 0) = a \int_0^T \frac{e^{-\frac{a^2}{2D(T-t)}}}{\sqrt{2\pi Dt}} dt.$$  

From (2.1) and the definition of $X^{b,1,T}()$, it follows that

$$P_{0,\mu}^{b,1,T}(\tau_a \in (nT, (n + 1)T)) = (1 - e^{-\frac{2a^2}{D\tau}})^n e^{-\frac{a^2}{D\tau}}.$$  

(2.4)

Also, from (2.1), (2.3) and the definition of $X^{b,1,T}()$ it follows that

$$E_0^{b,1,T}(\tau_a | \tau_a \in (nT, (n + 1)T)) = nT + E_0^{b,1,T}(\tau_a | \tau_a < T) =$$

$$nT + \frac{E_0^{b,1,T}(\tau_a 1_{\tau_a < T})}{P_{0,\mu}^{b,1,T}(\tau_a < T)} = nT + a e^{\frac{2a^2}{Dt}} \int_0^T \frac{e^{-\frac{a^2}{2D(T-t)}}}{\sqrt{2\pi Dt}} dt.$$  

(2.5)

Now (2.4) and (2.5) yield

$$E_0^{b,1,T}(\tau_a) = \sum_{n=0}^{\infty} nT + \frac{E_0^{b,1,T}(\tau_a | \tau_a \in (nT, (n + 1)T))}{P_{0,\mu}^{b,1,T}(\tau_a \in (nT, (n + 1)T))} P_{0,\mu}^{b,1,T}(\tau_a \in (nT, (n + 1)T)) =$$

$$\sum_{n=0}^{\infty} \left( nT + a e^{\frac{2a^2}{Dt}} \int_0^T \frac{e^{-\frac{a^2}{2D(T-t)}}}{\sqrt{2\pi Dt}} dt \right) \left( 1 - e^{-\frac{2a^2}{D\tau}} \right)^n e^{-\frac{a^2}{D\tau}} =$$

$$T(e^{\frac{2a^2}{D\tau}} - 1) + a e^{\frac{2a^2}{Dt}} \int_0^T \frac{e^{-\frac{a^2}{2D(T-t)}}}{\sqrt{2\pi Dt}} dt,$$

which completes the proof of the theorem. \hfill \Box

3. Proof of Proposition 1

From (1.4), we have

$$\int_{\mathbb{R}} \left( E_0^{(1,r)} \tau_a \right) \mu_{\sigma^2}^{Gauss,1}(da) = 2 \int_0^\infty e^{\frac{\sigma^2}{r} a} - 1 \frac{e^{-\frac{a^2}{2\pi \sigma}}}{\sqrt{2\pi \sigma}} da.$$  

(3.1)
Also,

\begin{equation}
\int_0^\infty \frac{e^{\frac{a^2}{2\sigma^2} - \frac{a^2}{2\sigma^2}}}{\sqrt{2\pi \sigma}} \, da = e^{\frac{1}{2\sigma^2}} \int_0^\infty \frac{e^{\frac{(a-\sqrt{2}D\sigma^2)^2}{2\sigma^2}}}{\sqrt{2\pi \sigma}} \, da = e^{\frac{1}{2\sigma^2}} \int_{-\infty}^\infty e^{-\frac{x^2}{2\pi \sigma}} \, dx.
\end{equation}

We obtain (1.10) from (3.1) and (3.2). A change of variables in (1.10) yields (1.11). Finally, (1.12) was obtained from (1.11) using the Desmos graphing calculator.

\[\Box\]

4. Proof of Theorem 3

From (1.7), we have

\begin{equation}
\int_{\mathbb{R}} (E_{0}^{bb,1,\tau}(T_0)\mu_{\sigma^2}^{Gauss,1}(da) = 2T \int_0^\infty (e^{\frac{a^2}{2\sigma^2} - \frac{a^2}{2\sigma^2}} - 1) e^{-\frac{a^2}{2\sigma^2}} \sqrt{2\pi \sigma} \, da + \int_0^\infty e^{\frac{a^2}{2\sigma^2} - \frac{a^2}{2\sigma^2}} \sqrt{2\pi \sigma} \, da.
\end{equation}

The first integral on the right hand side of (4.1) is infinite if \( T \leq \frac{4\sigma^2}{D} \). From now on, we assume that \( T > \frac{4\sigma^2}{D} \). We have

\[2 \int_0^\infty e^{\frac{a^2}{2\sigma^2} - \frac{a^2}{2\sigma^2}} \sqrt{2\pi \sigma} \, da = 2 \int_0^\infty e^{\frac{1}{2\sigma^2} \frac{DT - 4\sigma^2}{2\pi \sigma} - \frac{a^2}{2\sigma^2}} \sqrt{2\pi \sigma} \, da = \sqrt{\frac{DT \sigma^2}{DT - 4\sigma^2}}.
\]

Thus, the first term on the right hand side of (4.1) satisfies

\[2T \int_0^\infty (e^{\frac{a^2}{2\sigma^2} - \frac{a^2}{2\sigma^2}} - 1) e^{-\frac{a^2}{2\sigma^2}} \sqrt{2\pi \sigma} \, da + \int_0^\infty e^{\frac{a^2}{2\sigma^2} - \frac{a^2}{2\sigma^2}} \sqrt{2\pi \sigma} \, da = T \left( \frac{DT}{DT - 4\sigma^2} \right)^{\frac{1}{2}} - T.
\]

We now turn to the second term on the right hand side of (4.1). We have

\begin{equation}
\int_0^\infty ae^{\frac{a^2}{2\sigma^2} - \frac{a^2}{2\sigma^2}} \sqrt{2\pi \sigma} \, da = \int_0^\infty ae^{-\frac{1}{2\sigma^2} \frac{DT - 4\sigma^2}{2\pi \sigma} a^2} \sqrt{2\pi \sigma} \, da = \frac{DtT(T-t)\sigma^2}{T^2\sigma^2 + t(T-t)(DT - 4\sigma^2)}.
\end{equation}

Using (4.3), we can write the second term on the right hand side of (4.1) as

\[2 \int_0^\infty ae^{\frac{a^2}{2\sigma^2} - \frac{a^2}{2\sigma^2}} \sqrt{2\pi \sigma} \, da = \frac{T^2 \sigma}{\pi} \int_0^T \frac{\sqrt{Dt(T-t)}}{T^2\sigma^2 + t(T-t)(DT - 4\sigma^2)} \, dt = \frac{2T^2 \sigma}{\pi} \int_0^T \frac{\sqrt{Dt(T-t)}}{T^2\sigma^2 + t(T-t)(DT - 4\sigma^2)} \, dt.
\]
Make the substitution \( x = \sqrt{t(T-t)} \). Then \( t = \frac{1}{2}(T - (T^2 - 4x^2)^{\frac{1}{2}}) \) and \( dt = 2x(T^2 - 4x^2)^{-\frac{1}{2}}dx \). We obtain

\[
(4.5) \int_0^T \frac{\sqrt{Dt(T-t)}}{T^2\sigma^2 + t(T-t)(DT - 4\sigma^2)} \, dt = 2\sqrt{D} \int_0^T \frac{1}{(T^2 - 4x^2)^\frac{1}{2}} \left( \frac{x^2}{T^2\sigma^2 + x^2(DT - 4\sigma^2)} \right) \, dx.
\]

Now make the substitution \( x = \frac{T}{2} \sin \theta \). Then \( dx = \frac{T}{2} \cos \theta \, d\theta \). We obtain

\[
(4.6) \int_0^\frac{\pi}{2} \frac{1}{(T^2 - 4x^2)^\frac{1}{2}} \left( \frac{x^2}{T^2\sigma^2 + x^2(DT - 4\sigma^2)} \right) \, dx = \frac{1}{8} \int_0^\frac{\pi}{2} \frac{\sin^2 \theta}{\sigma^2 + \frac{4\sigma^2}{DT - 4\sigma^2}} \, d\theta.
\]

We write

\[
\frac{\sin^2 \theta}{\sigma^2 + \frac{4\sigma^2}{DT - 4\sigma^2} \sin^2 \theta} = \frac{4}{DT - 4\sigma^2} \frac{\sin^2 \theta}{\sin^2 \theta + \frac{4\sigma^2}{DT - 4\sigma^2}} = \frac{16}{(DT - 4\sigma^2)^2} \frac{1}{\sin^2 \theta + \frac{4\sigma^2}{DT - 4\sigma^2}}.
\]

Thus,

\[
(4.7) \int_0^\frac{\pi}{2} \frac{\sin^2 \theta}{\sigma^2 + \frac{4\sigma^2}{DT - 4\sigma^2} \sin^2 \theta} \, d\theta = \frac{2\pi}{2\sqrt{A}} \frac{16\sigma^2}{(DT - 4\sigma^2)^2} \int_0^\frac{\pi}{2} \frac{1}{\sin^2 \theta + \frac{4\sigma^2}{DT - 4\sigma^2}} \, d\theta.
\]

Making the substitution \( \tan \theta = s \), in which case \( \sin \theta = \frac{s}{\sqrt{1+s^2}} \) and \( d\theta = \frac{1}{\sqrt{1+s^2}} \, ds \), we obtain for any \( A > 0 \),

\[
(4.8) \int_0^\frac{\pi}{2} \frac{1}{\sin^2 \theta + A} \, d\theta = \int_0^\infty \frac{1}{\frac{A}{A+1} + A} \frac{1}{\sqrt{A+1}} \frac{1}{A\sqrt{\frac{A+1}{A} - 8}} \, ds = \frac{\pi}{2\sqrt{A+1}} \left( \frac{A+1}{A} \right)^{-\frac{1}{2}} \left( \frac{4\sigma^2}{DT - 4\sigma^2} \right)^{\frac{1}{2}} \left( \frac{4\sigma^2}{DT - 4\sigma^2} + 1 \right)^{-\frac{1}{2}} \frac{\pi}{4\sigma\sqrt{DT}}.
\]

From (4.8), we have

\[
(4.9) \int_0^\frac{\pi}{2} \frac{1}{\sin^2 \theta + \frac{4\sigma^2}{DT - 4\sigma^2} \sin^2 \theta} \, d\theta = \frac{\pi}{2} \left( \frac{4\sigma^2}{DT - 4\sigma^2} \left( \frac{4\sigma^2}{DT - 4\sigma^2} + 1 \right)^{-\frac{1}{2}} \right) \frac{\pi}{4\sigma\sqrt{DT}}.
\]
From (4.4)- (4.7) and (4.9), we obtain

\[(4.10)\]

\[
2 \int_0^\infty ae^{\frac{a^2}{2Dt}} \left( \int_0^T \frac{e^{-\frac{a^2}{2Dt}}}{\sqrt{2\pi Dt(1 - \frac{t}{T})}} \, dt \right) e^{-\frac{a^2}{2\sigma^2}} \, da =
\]

\[
\left(\frac{2T^2\sigma}{\pi}\right) (2\sqrt{D})(\frac{1}{8}) \left( \frac{2\pi}{DT - 4\sigma^2} - \frac{16\sigma^2}{(DT - 4\sigma^2)^2} \right) \pi(DT - 4\sigma^2) =
\]

\[
\frac{T^2\sigma \sqrt{D}}{2\pi} \frac{2\pi \sqrt{DT - 4\sigma^2}}{(DT - 4\sigma^2)\sqrt{DT}} = \frac{T^2\sigma \sqrt{D}}{2\pi} \frac{2\pi \sqrt{DT - 2\sigma}}{(DT - 4\sigma^2)\sqrt{DT}} = \frac{T\sigma}{\sqrt{DT + 2\sigma}}.
\]

Now (4.13) follows from (4.1), (4.2) and (4.10). A change of variables in (4.13) yields (1.14). Finally, (1.15) was obtained from (1.14) using the Desmos graphing calculator.

5. PROOF OF THEOREM 2

Let \( W(t) \) be a three-dimensional Brownian motion with diffusion coefficient \( D \) and denote probabilities for the process starting from \( b \in \mathbb{R}^3 \) by \( P_b \).

We use the same notation for hitting times for all the processes; so

\[ \tau_b = \inf\{t \geq 0 : |W(t) - b| \leq \epsilon_0\}, \quad b \in \mathbb{R}^3. \]

Abusing notation, let \( P_0(\tau_a = t) \) denote the density of the distribution of \( \tau_a \) under \( P_0 \); we will make similar abuses of notation for other densities in the sequel. Of course by isotropy, \( P_0(\tau_a = t) = P_a(\tau_0 = t) \), and this latter density can be found in [10] in the case that the diffusion coefficient \( D \) is equal to one. After appropriate scaling to take into account \( D \), this gives

\[(5.1)\]

\[ P_0(\tau_a = t) = \frac{\epsilon_0}{|a|} \frac{|a| - \epsilon_0}{\sqrt{2\pi Dt}} e^{-\frac{(|a| - \epsilon_0)^2}{2Dt}}, \quad |a| > \epsilon_0. \]

In the sequel, in all formulas involving \( a \) and \( \epsilon_0 \), it will be tacitly assumed that \( |a| > \epsilon_0 \). From (5.1), the strong Markov property and the definition of \( X^{b,b,3,T}(\cdot) \), we have

\[(5.2)\]

\[ P_0^{b,3,T}(\tau_a = t) = \frac{P_0(\tau_a = t) \int_{\nu \in \mathbb{R}^3 : \nu = 1} P_a + \epsilon_0 \nu (W(T - t) = 0) \mu_{a,t}(d\nu)}{P_0(W(T) = 0)}, \]

where \( \mu_{a,t} \) is the distribution under \( P_0 \) of \( W(\tau_a) \), conditioned on \( \{\tau_a = t\} \).

Of course,

\[(5.3)\]

\[ P_b(W(t) = a) = \frac{e^{-\frac{|b - a|^2}{2Dt}}}{(2\pi Dt)^{3/2}}, \quad a, b \in \mathbb{R}^3. \]
Thus, $P_{a+\epsilon_0\nu}(W(T-t) = 0)$ attains its maximum and minimum over $\{\nu \in \mathbb{R}^3 : |\nu| = 1\}$ at $\nu = -\frac{a}{|a|}$ and $\nu = \frac{a}{|a|}$ respectively, giving
\begin{equation}
(5.4) \quad \frac{e^{-\frac{(|a|+\epsilon)^2}{2D(T-t)}}}{(2\pi D(T-t))^\frac{3}{2}} \leq P_{a+\epsilon_0\nu}(W(T-t) = 0) \leq \frac{e^{-\frac{(|a|-\epsilon)^2}{2D(T-t)}}}{(2\pi D(T-t))^\frac{3}{2}}, \quad |\nu| = 1.
\end{equation}

From (5.1)-(5.4), we conclude that
\begin{equation}
(5.5) \quad \frac{\epsilon_0 |a| - \epsilon_0 e^{-\frac{(|a|+\epsilon)^2}{2D(1-T)}} e^{-\frac{(|a|-\epsilon)^2}{2Dt}}}{|a| \sqrt{2\pi D}} \leq P_{0}^{\nu_{b,b},b;T}(\tau_a = t) \leq \frac{\epsilon_0 |a| - \epsilon_0 e^{-\frac{(|a|-\epsilon)^2}{2D(1-T)}}}{|a| \sqrt{2\pi D}} (t(1 - \frac{t}{T}))^\frac{3}{2}.
\end{equation}

We will use the lower bound in (5.5) to prove (1.9), however for the proof of the lower bound in (1.8), we will use the following lower bound, which is obtained by replacing the term $|a| - \epsilon$ on the left hand side of (5.5) by $|a| + \epsilon$:
\begin{equation}
(5.6) \quad \frac{\epsilon_0 |a| - \epsilon_0 e^{-\frac{(|a|+\epsilon)^2}{2D(1-T)}}}{|a| \sqrt{2\pi D}} \leq P_{0}^{\nu_{b,b},b;T}(\tau_a = t).
\end{equation}

From (2.1) and (2.2), it follows that
\begin{equation}
(5.7) \quad \frac{1}{\sqrt{2\pi D}} \int_0^T \frac{b e^{-\frac{\nu^2}{2D(1-T)}}}{(1 - \frac{t}{T})^\frac{3}{2}t^2} dt = e^{\frac{2D^2}{b}}, \quad b > 0.
\end{equation}

Making the change of variables $s = T - t$ in (5.7) gives
\begin{equation}
(5.8) \quad \frac{1}{\sqrt{2\pi D}} \int_0^T \frac{b e^{-\frac{\nu^2}{2D(1-T)}}}{t^2(1 - \frac{t}{T})^\frac{3}{2}} dt = Te^{\frac{2D^2}{b}}.
\end{equation}

Now (5.5) and (5.8) give
\begin{equation}
(5.9) \quad E_{0}^{\nu_{b,b},b;T} \tau_a \leq T e^{-\frac{2(|a|-\epsilon)^2}{D(1-T)}},
\end{equation}

\begin{equation}
(5.9) \quad E_{0}^{\nu_{b,b},b;T} \tau_a \geq \frac{\epsilon_0 |a| - \epsilon_0 e^{-\frac{(|a|+\epsilon)^2}{2D(1-T)}}}{|a| \sqrt{2\pi D}} \leq \frac{\epsilon_0 |a| - \epsilon_0 e^{-\frac{(|a|-\epsilon)^2}{2D(1-T)}}}{|a| |a| + \epsilon_0 e^{-\frac{2(|a|-\epsilon)^2}{D(1-T)}}}.
\end{equation}
From (5.5), we have
\begin{equation}
\frac{\epsilon_0 |a| - \epsilon_0}{|a| \sqrt{2\pi D}} \int_0^T e^{-\frac{(s-a+\epsilon_0)^2}{2D}} dt \leq P_0^{bb,3;T} (\tau_a \leq T) \leq \frac{\epsilon_0 |a| - \epsilon_0}{|a| \sqrt{2\pi D}} \int_0^T e^{-\frac{(s-a-\epsilon_0)^2}{2D}} dt.
\end{equation}

We now show that
\begin{equation}
\int_0^T e^{-\frac{\xi^2}{2D(t - \frac{1}{T})}} dt = \frac{2\sqrt{2\pi D}}{b} e^{-\frac{\xi^2}{2D}}, \ b > 0.
\end{equation}

Indeed, (5.11) follows from the following calculation, in which we use (5.1) for the first equality and (5.8) for the last inequality.
\begin{align*}
\frac{\sqrt{2\pi D}}{b} e^{-\frac{\xi^2}{2D}} &= \int_0^T e^{-\frac{\xi^2}{2D(t - \frac{1}{T})}} dt = \int_0^T e^{-\frac{\xi^2}{2D(t - \frac{1}{T})}} \left(1 - \frac{t}{T}\right) dt = \\
\int_0^T e^{-\frac{\xi^2}{2D(t - \frac{1}{T})}} \left(1 - \frac{t}{T}\right) dt - &\frac{1}{T} \int_0^T e^{-\frac{\xi^2}{2D(t - \frac{1}{T})}} \frac{1}{T} dt = \int_0^T e^{-\frac{\xi^2}{2D(t - \frac{1}{T})}} \left(1 - \frac{t}{T}\right) dt - \frac{\sqrt{2\pi D}}{b} e^{-\frac{\xi^2}{2D}}.
\end{align*}

From (5.10) and (5.11), we have
\begin{equation}
\frac{2\epsilon_0 |a| - \epsilon_0}{|a| |a| + \epsilon_0} e^{-\frac{2(a+\epsilon_0)^2}{D}} \leq P_0^{bb,3;T} (\tau_a \leq T) \leq \frac{2\epsilon_0 |a| - \epsilon_0}{|a|} e^{-\frac{2(a-\epsilon_0)^2}{D}}.
\end{equation}

We now write
\begin{equation}
E_0^{bb,3;T} (\tau_a = \sum_{n=0}^\infty E_0^{bb,3;T} (\tau_a | \tau_a \in (nT, (n + 1)T]) P_0^{bb,3;T} (\tau_a \in (nT, (n + 1)T]).
\end{equation}

From the definition of $X^{bb,3;T}$, we have
\begin{equation}
E_0^{bb,3;T} (\tau_a \in (nT, (n+1)T]) = nT + E_0^{bb,3;T} (\tau_a \leq T) = nT + \frac{E_0^{bb,3;T} (\tau_a \leq T)}{P_0^{bb,3;T} (\tau_a \leq T)}
\end{equation}
and
\begin{equation}
P_0^{bb,3;T} (\tau_a \in (nT, (n + 1)T]) = \left(P_0^{bb,3;T} (\tau_a > T)\right)^n P_0^{bb,3;T} (\tau \leq T).
\end{equation}
Since for $q \in (0, 1)$, one has $\sum_{n=0}^{\infty} n(1 - q)^n q = \frac{1 - q}{q}$, which is decreasing in $q$, we have from (5.5)

(5.16)\[
\sum_{n=0}^{\infty} n (P_{0}^{b,3:T}(\tau_a > T)) P_{0}^{b,3:T}(\tau \leq T) \leq \frac{|a|(|a| + \epsilon_0)}{2c_0(\sigma_0 - \epsilon_0)} e^{\frac{2(|a|+\epsilon_0)^2}{T D}} - 1;
\]

(5.17)\[
\sum_{n=0}^{\infty} n (P_{0}^{b,3:T}(\tau_a > T)) P_{0}^{b,3:T}(\tau \leq T) \geq \frac{|a|}{2c_0} e^{\frac{2(|a|+\epsilon_0)^2}{T D}} - 1.
\]

Now (1.8) follows from (5.13)-(5.16) along with (5.5) and (5.9).

We now prove (1.9). From the upper bound in (1.8), it suffices to consider the case that $|a| - \epsilon_0$ is small. We will assume that $(|a| - \epsilon_0)^2 \leq \frac{T}{4}$. From the lower bound in (5.5), we have

(6.1)\[
\epsilon_0 \int_{\mathbb{R}^3} \left( E_0^{(3;r)}(\tau_a) \right) \mu_{\sigma^2}^{Gauss,3} (da) = \frac{1}{r} \int_{\mathbb{R}^3} \left( |a| e^{\frac{|a|}{\sqrt{T}}(\sigma_0 - \epsilon_0)} e^{\frac{|a|^2}{2\sigma^2}} \right) da.
\]

Thus,

(6.2)\[
\lim_{\epsilon_0 \to 0} \epsilon_0 \int_{\mathbb{R}^3} \left( E_0^{(3;r)}(\tau_a) \right) \mu_{\sigma^2}^{Gauss,3} (da) = \frac{1}{r} \int_{\mathbb{R}^3} \left( |a| e^{\frac{|a|}{\sqrt{T}}(\sigma_0 - \epsilon_0)} e^{\frac{|a|^2}{2\sigma^2}} \right) da.
\]
Letting \( R = |a| \) and then letting \( x = \frac{R}{\sigma} \), we have

\[
\frac{1}{r} \int_{\mathbb{R}^3} |a|e^{\frac{|a|^2}{r^2}} e^{\frac{-|a|^2}{2\tau^2}} \, da = \frac{1}{r} \int_0^\infty Re\sqrt{\pi} e^{\frac{-r^2}{2\tau^2}} \frac{4\pi R^2 \, dR}{(2\pi\sigma^2)^{\frac{3}{2}}} = \frac{2\sigma}{\sqrt{2\pi r}} \int_0^\infty x^3 e^{\frac{\sigma^2 x^2}{2}} e^{-\frac{x^2}{2}} \, dx.
\]

(6.3)

Letting \( s = \sigma^2 \frac{r}{D} \), we obtain

\[
\frac{2\sigma}{\sqrt{2\pi r}} \int_0^\infty x^3 e^{\frac{\sigma^2 x^2}{2}} e^{-\frac{x^2}{2}} \, dx = \frac{\sigma^3}{D} \left( \frac{2}{\sqrt{2\pi s}} \int_0^\infty x^3 e^{\frac{\sigma^2 x^2}{2}} e^{-\frac{x^2}{2}} \, dx \right),
\]

(6.4)

Therefore, (1.16) follows from (6.2) and (6.3), and (1.17) follows from (6.2)-(6.4). Finally, (1.18) follows from (1.17) using the Desmos graphing calculator.

7. Proof of Proposition 3

From (1.8) and (1.9), it follows that

\[
\lim_{\epsilon_0 \to 0} \epsilon_0 \int_{\mathbb{R}^3} (F_{\epsilon_0}^{bb,3;T})_\sigma \mu_{Gauss,3}(da) = \frac{T}{2} \int_{\mathbb{R}^3} |a|e^{\frac{2|a|^2}{r^2}} e^{\frac{-|a|^2}{2\tau^2}} \, da.
\]

(7.1)

We have \( \frac{2|a|^2}{T^2} - \frac{|a|^2}{2\sigma^2} = -\left( \frac{T^2 - 4\sigma^2}{T^2 \sigma^2} \right) \frac{|a|^2}{2} \). Thus,

\[
\int_{\mathbb{R}^3} |a|e^{\frac{2|a|^2}{T^2}} e^{-\frac{|a|^2}{2\sigma^2}} \, da = \int_{\mathbb{R}^3} |a|e^{-\frac{|a|^2}{2\sigma^2}} \left( \frac{T^2 - 4\sigma^2}{T^2 \sigma^2} \right) \, da = \int_0^\infty re^{-\frac{r^2}{2} \left( \frac{T^2 - 4\sigma^2}{T^2 \sigma^2} \right)} 4\pi r^2 \, dr.
\]

(7.2)

Integrating by parts yields

\[
\int_0^\infty re^{-\frac{r^2}{2} \left( \frac{T^2 - 4\sigma^2}{T^2 \sigma^2} \right)} 4\pi r^2 \, dr = \begin{cases} 8\pi \left( \frac{T^2 \sigma^2}{T^2 - 4\sigma^2} \right)^2, & T > \frac{4\sigma^2}{D}; \\ \infty, & T \leq \frac{4\sigma^2}{D}. \end{cases}
\]

(7.3)

Now (1.19) follows from (7.1)-(7.3), (1.20) follows immediately from (1.19) and (1.21) is obtained using the Desmos graphing calculator.

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