Alternating maps on Hatcher-Thurston graphs

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Abstract

Let $S_1$ and $S_2$ be connected orientable surfaces of genus $g_1, g_2 \geq 3$, $n_1, n_2 \geq 0$ punctures, and empty boundary. Let also $\varphi : \mathcal{HT}(S_1) \to \mathcal{HT}(S_2)$ be an edge-preserving alternating map between their Hatcher-Thurston graphs. We prove that $g_1 \leq g_2$ and that there is also a multicurve of cardinality $g_2 - g_1$ contained in every element of the image. We also prove that if $n_1 = 0$ and $g_1 = g_2$, then the map $\tilde{\varphi}$ obtained by filling the punctures of $S_2$, is induced by a homeomorphism of $S_1$.

Introduction

Suppose $S_{g,n}$ is an orientable surface of finite topological type, with genus $g \geq 3$, empty boundary, and $n \geq 0$ punctures. The (extended) mapping class group is the group of isotopy classes of self-homeomorphisms of $S_{g,n}$.

In 1980 (see [4]), Hatcher and Thurston introduce the Hatcher-Thurston complex of a surface, which is the 2-dimensional CW-complex whose vertices are multicurves called cut systems, 1-cells are defined as elementary moves between cut systems, and 2-cells are defined as appropriate “triangles”, “squares” and “pentagons”. See Section 1 for the details. They used this complex to prove that the index 2 subgroup of $\text{Mod}^*(S_{g,n})$ of orientation preserving isotopy classes, is finitely presented. The 1-skeleton of this complex is called the Hatcher-Thurston graph, which we denote by $\mathcal{HT}(S_{g,n})$.

There is a natural action of $\text{Mod}^*(S_{g,n})$ on the Hatcher-Thurston complex by automorphisms, and in [9] Irmak and Korkmaz proved that the automorphism group of the Hatcher-Thurston complex is isomorphic to $\text{Mod}^*(S_{g,n})$. Inspired by the different results in combinatorial rigidity on other simplicial graphs (like the curve graph in [11] and [7], and the pants graph in [1]), we obtain analogous results concerning simplicial maps between Hatcher-Thurston graphs.

Let $S_1 = S_{g_1,n_1}$ and $S_2 = S_{g_2,n_2}$ with $g_1, g_2 \geq 2$ and $n_1, n_2 \geq 0$. A simplicial map $\varphi : \mathcal{HT}(S_1) \to \mathcal{HT}(S_2)$ is alternating if the restriction to the star of any vertex, maps cut systems that differ in exactly 2 curves to cut systems that differ in exactly 2 curves. See Section 1 for the details. In Section 2 we prove our first result concerning this type of map:

Theorem A. Let $S_1$ and $S_2$ be connected orientable surfaces, with genus $g_1, g_2 \geq 2$ respectively, with empty boundary and $n_1, n_2 \geq 0$ punctures respectively. Let $\phi : \mathcal{HT}(S_1) \to \mathcal{HT}(S_2)$ be an edge-preserving and alternating map. Then we have the following:

1. $g_1 \leq g_2$.

2. There exists a unique multicurve $M$ in $S_2$ with $g_2 - g_1$ elements such that $M \subset \phi(C)$ for all cut systems $C$ in $S_1$.

A consequence of this theorem is that whenever we have an edge-preserving alternating map $\phi : \mathcal{HT}(S_1) \to \mathcal{HT}(S_2)$ (where the conditions of Theorem A are satisfied), we can then induce an edge-preserving alternating map $\varphi : \mathcal{HT}(S_1) \to \mathcal{HT}(S_2 \setminus M)$ where $M$ is the multicurve obtained...
by Theorem A, and $S_2 \setminus M$ is connected (due to the nature of Theorem A) and has genus $g_1$. This means we can focus solely on the case where $g_1 = g_2$. However, due to the nature of $\mathcal{HT}(S_1)$ and the techniques available right now, it is quite difficult to study these maps if $n_1 > 0$, and it is possible to have edge-preserving alternating maps if $n_1 < n_2$ that are obviously not induced by homeomorphisms, e.g., creating $n_2 - n_1$ punctures in $S_1$.

A way around this particular complication is wondering if this is the only way for the edge-preserving alternating maps to be not induced by homeomorphisms, leading to the following question:

**Question B.** Let $S_1$, $S_2$ and $S_3$ be connected orientable surfaces, with genus $g \geq 3$, $n_1, n_2 \geq 0$ punctures for $S_1$ and $S_2$ respectively, and assume $S_3$ is closed. Let $\phi : \mathcal{HT}(S_1) \to \mathcal{HT}(S_2)$ be an edge-preserving alternating map. Is there a way to induce a well-defined map $\varphi : \mathcal{HT}(S_3) \to \mathcal{HT}(S_3)$ from $\phi$ by filling the punctures of $S_1$ and $S_2$? If so, is $\varphi$ induced by a homeomorphism?

In Section 3 we answer this question for a particular case. If $\pi_{\mathcal{HT}}$ is the map induced by filling the punctures of $S_2$, we have the following result:

**Theorem C.** Let $S_1$ and $S_2$ be connected orientable surfaces, with genus $g \geq 3$ and empty boundary, and assume $S_1$ is closed. Let $\phi : \mathcal{HT}(S_1) \to \mathcal{HT}(S_2)$ be an edge-preserving and alternating map. Then

$$\tilde{\phi} := \pi_{\mathcal{HT}} \circ \phi : \mathcal{HT}(S_1) \to \mathcal{HT}(S_1)$$

is induced by a homeomorphism of $S_1$.

This implies that the only way to obtain a map from $\mathcal{HT}(S_1)$ to $\mathcal{HT}(S_2)$ that is edge-preserving and alternating, is to use a homeomorphism of $S_1$ and then puncture the surface to obtain $S_2$.

Theorem C is proved by using $\phi$ to induce maps between the underlying curves of the cut systems, and eventually induce an edge-preserving self-map of the curve graph of $S_1$ (see Section 3 for the details). Then, by the Theorem A of [7] (the second article of a series of which this work is also a part) we have that said self-map is induced by a homeomorphism.

Later on, in Section 4 we prove a consequence of Theorems A and C concerning isomorphisms and automorphisms between Hatcher-Thurston graphs.

**Corollary D.** Let $S_1$ and $S_2$ be connected orientable surfaces, with genus $g_1, g_2 \geq 2$ respectively, with empty boundary and $n_1, n_2 \geq 0$ punctures respectively. If $\phi : \mathcal{HT}(S_1) \to \mathcal{HT}(S_2)$ is an isomorphism, we have that $\phi$ is an alternating map and $g_1 = g_2$. Moreover, this implies that if $S = S_{g,0}$ with $g \geq 3$, then $\text{Aut}(\mathcal{HT}(S))$ is isomorphic to $\text{Mod}^*(S)$.

We must remark that this work is the published version of the fourth chapter of the author’s Ph.D. thesis (see [5]), and the results here presented are dependent on the results found in [7], which is the published version of the third chapter. There we prove that for any edge-preserving map between the curve graphs of a priori different surfaces (with certain conditions on the complexity and genus of the surfaces) to exist, it is necessary that the surfaces be homeomorphic and that the edge-preserving map be induced by a homeomorphism between the surfaces.

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1 Preliminaries and properties

In this section we give several definitions and prove several properties of the Hatcher-Thurston graph. Here we suppose \( S = S_{g,n} \) with genus \( g \geq 1 \) and \( n \geq 0 \) punctures.

A curve \( \alpha \) is a topological embedding of the unit circle into the surface. We often abuse notation and call “curve” the embedding, its image on \( S \) or its isotopy class. The context makes clear which use we mean.

A curve is essential if it is neither null-homotopic nor homotopic to the boundary curve of a neighbourhood of a puncture.

The (geometric) intersection number of two (isotopy classes of) curves \( \alpha \) and \( \beta \) is defined as follows:

\[
i(\alpha, \beta) := \min\{|a \cap b| : a \in \alpha, b \in \beta\}.
\]

Let \( \alpha \) and \( \beta \) be two curves on \( S \). Here we use the convention that \( \alpha \) and \( \beta \) are disjoint if \( i(\alpha, \beta) = 0 \) and \( \alpha \neq \beta \).

A multicurve \( M \) is either a single curve or a set of pairwise disjoint curves. A cut system \( C \) of \( S \) is a multicurve of cardinality \( g \) such that \( S \setminus C \) is connected.

Similarly, a curve \( \alpha \) is separating if \( S \setminus \{\alpha\} \) is disconnected, and is nonseparating otherwise. Note that a cut system can only contain nonseparating curves, and also \( S \setminus C \) has genus zero, thus a cut system \( C \) can be characterized as a maximal multicurve such that \( S \setminus C \) is connected.

Two cut systems \( C_1 \) and \( C_2 \) are related by an elementary move if they have \( g - 1 \) elements in common and the remaining two curves intersect once.

The Hatcher-Thurston graph \( \mathcal{H}(S) \) is the simplicial graph whose vertices correspond to cut systems of \( S \), and where two vertices span an edge if they are related by an elementary move. We will denote by \( \mathcal{V}(\mathcal{H}(S)) \) the set of vertices of \( \mathcal{H}(S) \).

If \( M \) is a multicurve on \( S \), we will denote by \( \mathcal{H}_M(S) \) the (possibly empty) full subgraph of \( \mathcal{H}(S) \) spanned by all cut systems that contain \( M \).

**Remark 1.1.** Let \( M \) and \( M' \) be multicurves on \( S \) such that neither \( \mathcal{H}_M(S) \) nor \( \mathcal{H}_{M'}(S) \) are empty graphs. Then \( \mathcal{H}_{M'}(S) \subset \mathcal{H}_M(S) \) if and only if \( M' \subset M \). Also, if \( M \) is a multicurve such that \( \mathcal{H}_M(S) \) is nonempty, then \( \mathcal{H}_M(S) \) is naturally isomorphic to \( \mathcal{H}(S \setminus M) \).

Recalling previous work on the Hatcher-Thurston complex we have the following lemma.

**Lemma 1.2.** Let \( S \) be an orientable connected surface of genus \( g \geq 1 \), with empty boundary and \( n \geq 0 \) punctures. Then \( \mathcal{H}(S) \) is connected.

Note that this lemma and Remark 1.1 imply that if \( M \) is a multicurve on \( S \) such that \( S \setminus M \) is connected, then \( \mathcal{H}_M(S) \) is connected.

1.1 Properties of \( \mathcal{H}(S) \)

Let \( C \) be a cut system on \( S \), and denote by \( \mathcal{A}(C) \) the full subgraph spanned by the set of cut systems on \( S \) that are adjacent to \( C \) in \( \mathcal{H}(S) \) (often called the link of \( C \) in \( \mathcal{H}(S) \)). Intuitively, we want to relate the elements of \( \mathcal{A}(C) \) that are obtained by replacing the same curve of \( C \); this is done defining the relation \( \sim_C \) in \( \mathcal{A}(C) \) by

\[
C_1 \sim_C C_2 \iff C_1 \cap C = C_2 \cap C.
\]

We can easily check \( \sim_C \) is an equivalence relation, and two cut systems are related in \( \mathcal{A}(C) \) if they are obtained by replacing the same curve of \( C \) as was desired. The equivalence classes of this relation will be called colours.

This definition implies that in \( \mathcal{A}(C) \) there are \( g \) colours, each corresponding to a curve in \( C \) that was substituted; thus, we use the elements of \( C \) to index these colours.
**Remark 1.3.** We should note that if \(C_1, C_2 \in \mathcal{A}(C)\) are such that \(C_1 \sim C_2\), then \(C_1\) and \(C_2\) share exactly \(g - 2\) curves.

![Diagram of cut systems \(\{\alpha_1, \beta, \alpha_3, \alpha_4\}\) and \(\{\alpha_1, \beta', \alpha_3, \alpha_4\}\) are in the same color with respect to \(\{\alpha_1, \ldots, \alpha_4\}\), while \(\{\alpha_1, \alpha_2, \beta'', \alpha_4\}\) is not.

Let \(\gamma\) be a nonseparating curve of \(S\). Following Irmak and Korkmaz’s work on the Hatcher-Thurston complex (for which we recall \(\mathcal{HT}(S)\) is the 1-skeleton) in [9], we define the graph \(X^S_1\) as the simplicial graph whose vertices are the nonseparating curves \(\beta\) on \(S\) such that \(i(\beta, \gamma) = 1\), and two vertices \(\alpha\) and \(\beta\) span an edge if \(i(\alpha, \beta) = 1\).

In [9], we obtain the following result, modifying the statement to suit the notation used here.

**Lemma 1.4** ([9]). Let \(S = S_{g,n}\) such that \(g \geq 1\) and \(n \geq 0\), and \(\gamma\) be a nonseparating curve on \(S\). Then \(X^S_1\) is connected.

A **triangle** on \(\mathcal{HT}(S)\) is a set of three distinct cut systems on \(S\), whose elements pairwise span edges in \(\mathcal{HT}(S)\). Now we prove that for every triangle in \(\mathcal{HT}(S)\) there exists a convenient multicurve contained in each cut system.

**Lemma 1.5.** Let \(S = S_{g,n}\) such that \(g \geq 1\) and \(n \geq 0\) punctures, and \(T\) be a triangle on \(\mathcal{HT}(S)\). Then, there exists a unique multicurve \(M\), of cardinality \(g - 1\), such that \(M\) is contained in every element of \(T\).

**Proof.** Let us denote \(T = \{A, B, C\}\). Since \(A, B \in \mathcal{A}(C)\) then if \(A \sim_C B\), by Remark 1.3, \(|A \cap B| = g - 2\); but then \(A\) and \(B\) would not be able to span an edge, contradicting \(T\) being a triangle. Thus \(A \sim_C B\). Since \(A \neq B\) we have \(M = A \cap C = B \cap C\) is the desired multicurve of cardinality \(g - 1\).

**Lemma 1.6.** Let \(A, B, C\) be distinct cut systems on \(S\), such that \(A, B \in \mathcal{A}(C)\). Then \(A \sim_C B\) if and only if there exists a finite collection of triangles \(T_1, \ldots, T_m\) such that \(A, C \in T_1, B, C \in T_m\), and \(T_i\) and \(T_{i+1}\) share exactly one edge for \(i = 1, \ldots, m - 1\).

**Proof.** If \(g = 1\), then we obtain the desired result directly from Lemma 1.4, making \(C = \{\gamma\}\). So, suppose \(g > 1\).

If \(A \sim_C B\), let \(M = A \cap C = B \cap C\) be the multicurve of Lemma 1.5 with cardinality \(g - 1\). Let \(\alpha\) be the curve in \(A \setminus M\), \(\beta\) be the curve in \(B \setminus M\) and \(\gamma\) be the curve in \(C \setminus M\). Since \(A, B \in \mathcal{A}(C)\) then \(\alpha\) and \(\gamma\) intersect once, just the same as \(\beta\) and \(\gamma\); moreover, \(\alpha, \beta\) and \(\gamma\) are nonseparating curves of \(S \setminus M\) since \(A, B\) and \(C\) are cut systems. Thus \(\alpha\) and \(\beta\) are vertices in \(X^S_1\), and by Lemma 1.4 there exists a finite collection of nonseparating (in \(S \setminus M\)) curves \(c_0, \ldots, c_m\) with \(\alpha = c_0, \beta = c_m\) and \(c_i\) adjacent to \(c_{i+1}\) in \(X^S_1\). Since every \(c_i\) is a nonseparating curve of \(S_i\), then \(\{c_i\} \cup M\) is a cut system of \(S\) for each \(i\); in particular \(A = \{c_0\} \cup M\) and \(B = \{c_m\} \cup M\).
By construction, \( T_{i+1} := \{\{c_i \cup M, C, \{c_{i+1} \cup M\}\} \) is a triangle for \( i = 0, \ldots, m - 1 \), \( T_i \) and \( T_{i+1} \) share exactly one edge for \( i = 0, \ldots, m - 1 \), \( A, C \in T_1 \), and \( C, B \in T_m \).

Conversely, if \( T_1, \ldots, T_m \) is a finite collection of triangles such that \( A, C \in T_1 \), \( B, C \in T_m \) and \( T_i \) and \( T_{i+1} \) share exactly one edge for \( i = 1, \ldots, m - 1 \), we denote by \( M_i \) the multicurve corresponding to the triangle \( T_i \) obtained by Lemma 1.5. Let \( M \) be the cut systems in the triangle \( T_1 \) and \( T_2 \). Since \( D_i \cap D_j = M_i \) in \( T_i \) and \( D_i \cap D_j = M_j \) in \( T_j \), we have that \( M_i = M_j \) for \( i = 1, \ldots, m - 1 \). Thus \( M_i = M_j \) for \( i \neq j \), so \( A \cap C = B \cap C \) which by definition implies that \( A \sim C B \).

2 Proof of Theorem

In this section, let all surfaces be of genus at least 2, possibly with punctures.

An alternating square in \( HT(S) \) is a closed path with four distinct consecutive vertices \( C_1, C_2, C_3, C_4 \) such that \( C_1 \sim C_2 \sim C_3 \sim C_4 \). So, \( C_1 \) and \( C_3 \) have exactly \( g - 2 \) curves in common, and \( C_2 \) and \( C_4 \) have also exactly \( g - 2 \) curves in common. In Figure 1 the curves \( \{α_1, \α_3, \α_4\}, \{β_1, \β_3, \β_4\}, \{α_2, β_2, β''_4, α_4\} \) and \( \{α_1, β_2, β''_2, α_2\} \) form an alternating square.

**Lemma 2.1.** Let \( C_1, C_2, C_3, C_4 \) be consecutive vertices of an alternating square in \( HT(S) \). Then \( C_1 \cup C_2 \cap C_3 \cap C_4 \) has cardinality \( g - 2 \).

**Proof.** Since \( C_1, C_3 \in \mathcal{A}(C_2), C_1 \cap C_3 = C_1 \cap C_2 \cap C_3 \); analogously \( C_1 \cap C_3 = C_1 \cap C_4 \cap C_3 \). This implies that \( C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_3 \). Thus \( C_1 \cap C_2 \cap C_3 \cap C_4 = C_1 \cap C_3 \). Given that \( C_1 \sim C_2, C_3 \), we have \( |C_1 \cap C_2 \cap C_3 \cap C_4| = g - 2 \).

**Lemma 2.2.** Let \( C_1, C_2, C_3 \) be cut systems on \( S \), such that \( C_1, C_3 \in \mathcal{A}(C_2) \) and \( C_1 \sim C_2, C_3 \). There exists \( C_1', C_2', C_3' \in \mathcal{A}(C_2) \) with \( C_1' \sim C_2' \sim C_3' \), such that \( C_1, C_2, C_3' \) are consecutive vertices of an alternating square.

**Proof.** Let \( C_1, C_3 \), let \( M \) be the common multicurve of \( C_1, C_2 \) and \( C_3 \) obtained by Lemma 1.5. Let also \( α, β, α', β' \) be the curves such that \( C_1 = M \cup \{α', β\}, C_2 = M \cup \{α, β\} \) and \( C_3 = M \cup \{α, β'\} \).

Let \( T \) be a regular neighbourhood of \( \{α, α'\} \). Since \( i(α, α') = 1 \), \( T \) is homeomorphic to \( S_{1,1} \). Let \( β'' \) be a nonseparating curve of \( \mathcal{S}'(M) \) such that \( i(β', β'') = 1 \), and \( β'' \) is contained in \( \mathcal{S}'(T) \) (that is possible since \( S' \mathcal{M} \) has genus 2). By construction we have the following: \( C_1 = M \cup \{α, β''\} \) and \( C_2 = M \cup \{α', β''\} \) are cut systems such that \( C_1' = M \cup \{α, β''\} \in \mathcal{A}(C_2) \cap \mathcal{A}(C_4), C_4 \in \mathcal{A}(C_1) \cap \mathcal{A}(C_3), C_3 \sim C_2 C_3' \sim C_1 \sim C_2, C_3' \) and \( C_2 \sim C_3 C_4' \). Thus \( C_1, C_2, C_3, C_4 \) are the consecutive vertices of an alternating square.

Let \( S_1 = S_{g_1, n_1} \) and \( S_2 = S_{g_2, n_2} \) with genus \( g_1 \geq 2, g_2 \geq 1 \) and \( n_1, n_2 \geq 0 \).

A simplicial map \( φ : HT(S_1) \to HT(S_2) \) is said to be edge-preserving if whenever \( C_1 \) and \( C_2 \) are two distinct cut systems that span an edge in \( HT(S_1) \), their images under \( φ \) are distinct and span an edge in \( HT(S_2) \).

**Remark 2.3.** Note that if \( φ : HT(S_1) \to HT(S_2) \) is an edge-preserving map, then triangles are mapped to triangles.

The map \( φ \) is said to be alternating if for all cut systems \( C \) on \( S_1 \) and all \( C_1, C_2 \in \mathcal{A}(C) \) such that \( C_1 \) and \( C_2 \) differ by exactly two curves, then \( φ(C_1) \) and \( φ(C_2) \) differ by exactly two curves. Note that this condition says nothing about \( φ(C) \) and its relation with \( φ(C_1) \) and \( φ(C_2) \).
Lemma 2.4. Let $\phi : \mathcal{HT}(S_1) \to \mathcal{HT}(S_2)$ be an edge-preserving map, and $C_1$, $C_2$ and $C_3$ be cut systems on $S_1$ with $C_1, C_3 \in \mathcal{A}(C_2)$. If $C_1 \sim_{C_2} C_3$, then $\phi(C_1) \sim_{\phi(C_2)} \phi(C_3)$. If $\phi$ is also alternating, then $C_1 \sim_{C_2} C_3$ implies $\phi(C_1) \sim_{\phi(C_2)} \phi(C_3)$; in particular alternating squares go to alternating squares.

Proof. If $C_1 \sim_{C_2} C_3$, then by Lemma 1.6 there exists a finite collection of triangles $T_1, \ldots, T_m$ with $C_1, C_2 \in T_1, \ldots, C_3 \in T_m$, and $T_i, T_{i+1}$ share one edge. By Remark 2.3 $\phi(T_i)$ is a triangle for all $i = 1, \ldots, m$, with $\phi(C_1), \phi(C_2) \in \phi(T_1), \phi(C_2), \phi(C_3) \in \phi(T_m)$ and $\phi(T_i), \phi(T_{i+1})$ sharing one edge; thus, once again by Lemma 1.6 $\phi(C_1) \sim_{\phi(C_2)} \phi(C_3)$.

Let $\phi$ be also alternating, and $C_1 \sim_{C_2} C_3$. By Remark 1.3, $C_1, C_3$ differ by exactly 2 curves and since $\phi$ is an edge-preserving alternating map, we have that $\phi(C_1), \phi(C_3) \in N(\phi(C_2))$ and $\phi(C_1), \phi(C_3)$ differ by exactly 2 curves; so, $\phi(C_1) \sim_{\phi(C_2)} \phi(C_3)$.

Let $\phi$ be an edge-preserving alternating map, and $S$ be an alternating square with consecutive vertices $C_1, \ldots, C_4$. Since $C_1, C_3 \in \mathcal{A}(C_2) \cap \mathcal{A}(C_4)$ then $\phi(C_1), \phi(C_3) \in N(\phi(C_2)) \cap N(\phi(C_4))$, and as proved above $\phi(C_1) \sim_{\phi(C_2)} \phi(C_3)$ and $\phi(C_2) \sim_{\phi(C_4)} \phi(C_4)$ (since $C_1 \sim_{C_2} C_3$ and $C_2 \sim_{C_3} C_4$). Therefore $\phi(S)$ is an alternating square.

Note that this lemma allows us to see the importance of the alternating requirement for $\phi$. If $\phi$ were only edge-preserving (or locally injective), we would not have enough information to be certain that alternating squares are mapped to alternating squares, which is an important requirement if we ever want $\phi$ to be induced by a homeomorphism. Moreover, the rest of the results presented here would be much more complicated to prove if at all possible.

Now we are ready to prove Theorem [A] (which is quite similar to a result about locally injective maps for the Pants complex, that appears as Theorem C in [1], though we must note that for some of the arguments in the proof being an alternating map is a key requirement).

Proof of Theorem [A]. Let $A$ be a vertex of $\mathcal{HT}(S_1)$. Then let $\{B_1, \ldots, B_{g_1}\} \subset \mathcal{A}(A)$ be a set of representatives for the colours of $\mathcal{A}(A)$. Since $B_i \sim_A B_j$ if and only if $i = j$ then $\phi(B_i) \sim_{\phi(A)} \phi(B_j)$ if and only if $i = j$, by Lemma 2.4 thus $\mathcal{A}(\phi(A))$ has at least a many colours as $\mathcal{A}(A)$, so $g_1 \leq g_2$.

This implies that $M = \phi(B_1) \cap \cdots \cap \phi(B_{g_1})$ has cardinality $g_2 - g_1$. We must also note that $M \subset \phi(A)$. We can easily check that if $B \sim_A B_i$ for some $i$, then $M \subset \phi(B)$; by Lemma 2.4 $\phi(B) \sim_{\phi(A)} \phi(B_i)$, which means they were obtained from $\phi(A)$ by replacing the same curve, so $\phi(A) \cap \phi(B_i) \subset \phi(B)$; since $M \subset \phi(A) \cap \phi(B_i)$, then $M \subset \phi(B)$.

With this we have proved that for all $B \in A \cup \mathcal{A}(A)$, $M \subset \phi(B)$. Given that $\mathcal{HT}(S_1)$ is connected, we only need to prove that given any element $B \in \mathcal{A}(A)$, for all $C \in \mathcal{A}(B)$, we have $M \subset \phi(C)$. Let $B$ and $C$ be such cut systems.

If $C \sim_B A$, then by Lemma 2.4 $\phi(C) \sim_{\phi(B)} \phi(A)$, which means $\phi(C)$ and $\phi(A)$ were obtained by replacing the same curve of $\phi(B)$, so $\phi(C) \cap \phi(B) = \phi(A) \cap \phi(B)$. Since we have already proved that $M \subset \phi(B)$ and $M \subset \phi(A)$ then $M \subset \phi(A) \cap \phi(B) = \phi(C) \cap \phi(B)$. Thus $M \subset \phi(C)$.

If $C \sim_B A$, then by Lemma 2.2 there exists $C' \in \mathcal{A}(B)$ with $C \sim_B C'$ and $C' \sim_B A$, such that $A, B, C'$ are consecutive vertices of an alternating square $\Sigma$. By Lemma 2.4 $\phi(\Sigma)$ is also an alternating square. Let $D$ the vertex of $\Sigma$ different from $A, B$ and $C'$; since $D \in \mathcal{A}(A)$, we have proved above that $M \subset \phi(D)$, thus $M \subset \phi(A) \cap \phi(B) \cap \phi(D)$ and as we have seen in the proof of Lemma 2.1 $\phi(A) \cap \phi(B) \cap \phi(C') \cap \phi(D) = \phi(A) \cap \phi(B) \cap \phi(D)$, so $M \subset \phi(C')$. Given that $C \sim_B C'$, this leaves us in the previous case, therefore $M \subset \phi(C')$.  

\[\square\]
3 Proof of Theorem C

Hereinafter, let $S_1 = S_{g,0}$ and $S_2 = S_{g,n}$ with $g \geq 3$ and $n \geq 0$. Before giving the idea of the proof, we need the following definitions.

We define the complexity of $S_{g,n}$, denoted by $\kappa(S_{g,n})$ as $3g - 3 + n$. Note this is equal to the cardinality of a maximal multicurve.

If $S_{g,n}$ is such that $\kappa(S_{g,n}) > 1$, the curve graph $C(S_{g,n})$, introduced by Harvey in [3], is the simplicial graph whose vertices correspond to the curves of $S$, and two vertices span an edge if they are disjoint. We denote $\mathcal{V}(C(S_{g,n}))$ the set of vertices of $C(S_{g,n})$.

If $S_{g,n}$ is such that $g \geq 1$, the Schmutz graph $G(S_{g,n})$, introduced by Schmutz-Schaller in [10], is the simplicial graph whose vertices correspond to nonseparating curves of $S$, and where two vertices span an edge if they intersect once. We denote by $\mathcal{V}(G(S))$ the set of vertices of $G(S)$.

Idea of the proof: We proceed by using $\phi$ to induce a map $\psi : \mathcal{V}(G(S_1)) \to \mathcal{V}(G(S_2))$ in such a way that $\phi(\{\alpha_1, \ldots, \alpha_g\}) = \{\psi(\alpha_1), \ldots, \psi(\alpha_g)\}$. Then we induce two maps $\phi : \mathcal{H}(S_1) \to \mathcal{H}(S_2)$ and $\psi : \mathcal{V}(G(S_1)) \to \mathcal{V}(G(S_2))$ by filling the punctures of $S_2$. These maps also verify that $\phi(\{\alpha_1, \ldots, \alpha_g\}) = \{\psi(\alpha_1), \ldots, \psi(\alpha_g)\}$. Following the proofs of several properties of $\psi$ and $\tilde{\psi}$, we extend $\tilde{\psi}$ to an edge-preserving map $\tilde{\psi} : C(S_1) \to C(S_1)$ which, by Theorem A in [7], is induced by a homeomorphism of $S_1$. Therefore $\tilde{\psi}$ is induced by a homeomorphism of $S_1$.

3.1 Inducing $\psi : G(S_1) \to G(S_2)$ and $\tilde{\psi} : G(S_1) \to G(S_1)$

Let $\alpha$ be a nonseparating curve. Recall that $\mathcal{H}(\alpha)(S_1)$ is isomorphic to $\mathcal{H}(S_1\backslash\{\alpha\})$. Then, given an edge-preserving alternating map $\phi : \mathcal{H}(S_1) \to \mathcal{H}(S_2)$ we can obtain an edge-preserving alternating map $\phi_{\alpha} : \mathcal{H}(S_1\backslash\{\alpha\}) \to \mathcal{H}(S_2)$. Applying Theorem A to $\phi_{\alpha}$ we know there exists a unique multicurve on $S_2$ of cardinality 1, contained in the image under $\phi$ of every cut system containing $\alpha$; we will denote the element of this multicurve as $\psi(\alpha)$. In this way we have defined a function $\psi : \mathcal{V}(G(S_1)) \to \mathcal{V}(G(S_2))$.

Lemma 3.1. Let $\phi : \mathcal{H}(S_1) \to \mathcal{H}(S_2)$ be an edge-preserving alternating map and $\psi : \mathcal{V}(G(S_1)) \to \mathcal{V}(G(S_2))$ be the induced map on the nonseparating curves. If $\alpha$ and $\beta$ are nonseparating curves and $C$ a cut system on $S_1$, then:

1. If $\alpha \in C$, then $\psi(\alpha) \in \phi(C)$.
2. If $\alpha \neq \beta$ and $\alpha, \beta \in C$, then $\psi(\alpha) \neq \psi(\beta)$.
3. If $i(\alpha, \beta) = 1$, then $i(\psi(\alpha), \psi(\beta)) = 1$.

Proof. (1) Follows directly from the definition.

(2) Let $C = \{\alpha, \beta, \gamma_1, \ldots, \gamma_{g-2}\}$ and let $C_{\alpha}, C_{\beta}, C_{\gamma_1}, \ldots, C_{\gamma_{g-2}}$ be representatives of the colours in $\mathcal{A}(C)$ indexed by $\alpha, \beta, \gamma_1, \ldots, \gamma_{g-2}$ respectively so that $\alpha \notin C_{\alpha}, \alpha \in C_{\beta}, \gamma_1, \ldots, C_{\gamma_{g-2}}, \beta \notin C_{\beta}$ and $\beta \in C_{\alpha}, C_{\gamma_1}, \ldots, C_{\gamma_{g-2}}$. Using Lemma 2.4 we have that $\phi(\beta), \phi(\gamma_1), \ldots, \phi(\gamma_{g-2})$ are representatives of all the colours of $\mathcal{A}(\phi(C))$. By (1) we have that $\psi(\alpha) \in \phi(C) \cap \phi(\beta) \cap \phi(\gamma_1) \cap \ldots \cap \phi(\gamma_{g-2})$, so $\psi(\alpha)$ cannot be an element of $\phi(C)$ and, since $\beta \in C_{\alpha}$, by (1) again we have that $\psi(\beta) \in \phi(C)$. Therefore $\psi(\alpha) \neq \psi(\beta)$.

(3) Using a regular neighbourhood of $\{\alpha, \beta\}$, we can find a multicurve $M$ in $S_1$ such that $C' = \{\alpha\} \cup M$ and $C'' = \{\beta\} \cup M$ are cut systems; this implies that if $C'$ and $C''$ span an edge in $\mathcal{H}(S_1)$, then $\phi(C')$ and $\phi(C'')$ span an edge in $\mathcal{H}(S_2)$. By (1) and (2), $\phi(C') = \{\psi(\alpha)\} \cup \psi(M)$ and $\phi(C'') = \{\psi(\beta)\} \cup \psi(M)$, therefore $i(\psi(\alpha), \phi(\beta)) = 1$. \qed
Note that this lemma implies that if $C = \{\alpha_1, \ldots, \alpha_g\}$, we have that $\phi(C) = \{\psi(\alpha_1), \ldots, \psi(\alpha_g)\}$.

By filling the punctures of $S_2$ and identifying the resulting surface with $S_1$, we obtain a map $\pi_C : V(Y(S_2)) \to V(C(S_1))$, where $Y(S_2)$ is the subcomplex of $C(S_2)$ whose vertices correspond to curves $\gamma$ on $S_2$ such that all the connected components of $S_2 \setminus \{\gamma\}$ have positive genus. Observe that $\pi_C$ sends nonseparating curves of $S_2$ into nonseparating curves of $S_1$, and separating curves of $S_2$ that separate the surface in connected components of genus $g' > 0$ and $g'' > 0$ into separating curves of $S_1$ that separate the surface in connected components of genus $g'$ and $g''$. In particular, if $C$ is a cut system, $\pi_C(C)$ is also a cut system, thus we obtain a map $\pi_{HT} : V(HT(S_2)) \to V(HT(S_1))$.

Now, from $\phi : HT(S_1) \to HT(S_2)$ we can obtain the map

$$\tilde{\psi} : \pi_{HT} \circ \psi : V(G(S_1)) \to V(G(S_1)),$$

and the map

$$\tilde{\phi} : \pi_{HT} \circ \phi : V(HT(S_1)) \to V(HT(S_1)).$$

**Corollary 3.2.** Let $\phi : HT(S_1) \to HT(S_2)$ an edge-preserving alternating map, $\psi : V(G(S_1)) \to V(G(S_2))$ be the induced map on the nonseparating curves, and $\tilde{\phi}$ and $\tilde{\psi}$ as above. If $\alpha$ and $\beta$ are nonseparating curves and $C$ a cut system on $S_1$, then:

1. If $\alpha \in C$ then $\tilde{\psi}(\alpha) \in \tilde{\phi}(C)$.
2. If $\alpha \neq \beta$ and $\alpha, \beta \in C$ then $\tilde{\psi}(\alpha) \neq \tilde{\psi}(\beta)$.
3. If $i(\alpha, \beta) = 1$ then $i(\tilde{\psi}(\alpha), \tilde{\psi}(\beta)) = 1$.

**Proof.** (1) Follows from Lemma \ref{lemma:induced_map}

(2) If $\alpha \neq \beta$ and $\alpha, \beta \in C$ then by Lemma \ref{lemma:induced_map}, $\psi(\alpha), \psi(\beta) \in \phi(C)$ and $\psi(\alpha) \neq \psi(\beta)$. This implies that $\psi(\alpha)$ and $\psi(\beta)$ are disjoint curves that do not together separate $S_2$; these two properties together are preserved by $\pi_C$. Indeed, let $S'$ be a subsurface of $S_2$ such that $\psi(\alpha), \psi(\beta) \in C(S')$ and $S'$ is homeomorphic to $S_2$. Let $\gamma$ be the boundary curve of $S_2 \setminus \text{int}(S')$, then $\gamma$ separates $S_2$ in two connected components, each of positive genus. Thus $S'$ is unaffected by $\pi_C$, i.e. $\pi_C|_{V(C(S'))} = \text{id}|_{V(C(S'))}$. Therefore $\tilde{\psi}(\alpha) \neq \tilde{\psi}(\beta)$.

(3) Since $i(\psi(\alpha), \psi(\beta)) = 1$, let $T$ be a regular neighbourhood of $\{\psi(\alpha), \psi(\beta)\}$. Then $T$ is homeomorphic to $S_{1,1}$. Let $\gamma$ be the boundary curve in $S_2 \setminus \text{int}(T)$; then $\gamma$ is a separating curve that separates $S_2$ in two connected components, each of positive genus. Thus, as in (2), $T$ is unaffected by $\pi_C$. Therefore $i(\tilde{\psi}(\alpha), \tilde{\psi}(\beta)) = 1$. \qed

Similarly to Lemma \ref{lemma:induced_map} this implies that if $C = \{\alpha_1, \ldots, \alpha_g\}$, we have that $\tilde{\phi}(C) = \{\tilde{\psi}(\alpha_1), \ldots, \tilde{\psi}(\alpha_g)\}$.

As a consequence of Lemma \ref{lemma:induced_map} and Corollary \ref{corollary:induced_map}, we have that the maps $\psi$, $\tilde{\psi}$ and $\tilde{\phi}$ are simplicial. Moreover, we have the following result.

**Corollary 3.3.** $\psi : G(S_1) \to G(S_2)$, $\tilde{\psi} : G(S_1) \to G(S_1)$ and $\tilde{\phi} : HT(S_1) \to HT(S_1)$ are edge-preserving maps. Also, $\tilde{\phi}$ is an alternating map.

A pants decomposition of $S_i$ (for $i = 1, 2$) is a maximal multicurve of $S_i$, i.e. it is a maximal complete subgraph of $C(S_i)$. Note that any pants decomposition of $S_i$ has exactly $\kappa(S_i)$ curves.

On the other hand, we say $P$ is a punctured pants decomposition of $S_2$ if $\pi_C(P)$ is a pants decomposition of $S_1$. This implies that $S_2 \setminus P$ is the disjoint union of $3g - 3$ surfaces, with each connected component $P_i$ homeomorphic to $S_{0,3+k_i}$, such that $\sum_i k_i = n$. 

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Lemma 3.4. Let $P$ be a pants decomposition of $S_1$ such that no two curves of $P$ together separate $S_1$. Then $\psi(P)$ is a punctured pants decomposition of $S_2$ and $\tilde{\psi}(P)$ is a pants decomposition of $S_1$.

Proof. Since for any two distinct curves $\alpha, \beta \in P$ we can always find a cut system containing both of them, by Lemma 3.1 and Corollary 3.2 we know that $\psi(\alpha)$ is disjoint from $\psi(\beta)$ and $\tilde{\psi}(\alpha)$ is disjoint from $\tilde{\psi}(\beta)$. Thus, both $\psi(P)$ and $\tilde{\psi}(P)$ are multicurves of cardinality $3g - 3$, which means $\tilde{\psi}(P)$ is a pants decomposition; then, by definition, $\psi(P)$ is a punctured pants decomposition. \hfill \square

Figure 2: Pants decompositions for the closed surfaces of genus 3 (left) and genus 5 (right), such that no two curves of $P$ together separate.

The rest of this subsection consists of several technical definitions and lemmas, all of them leading to proving that both $\psi$ and $\tilde{\psi}$ preserve disjointness and intersection number 1, which we later use to extend their definitions to the respective curve complexes.

Let $\alpha$ and $\beta$ be two curve in $S_1$, and $N$ be a regular neighbourhood of $\{\alpha, \beta\}$. We say they are spherical-Farey neighbours if $N$ has genus zero and $i(\alpha, \beta) = 2$.

Let $\alpha$ and $\beta$ be two nonseparating curves in $S_1$ that are spherical-Farey neighbours, and $N(\alpha, \beta)$ be their closed regular neighbourhood. Then $N(\alpha, \beta)$ is homeomorphic to a genus zero surface with four boundary components. Let $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3$ be the boundary curves of $N(\alpha, \beta)$. We say $\varepsilon_i$ and $\varepsilon_j$ are connected outside of $\Sigma_{\alpha, \beta}$ to $\varepsilon_j$ for all $i, j \in \{0, 1, 2, 3\}$ with one endpoint in $\varepsilon_i$ and another in $\varepsilon_j$.

Remark 3.5. If $\varepsilon_i$ is a nonseparating curve, it has to be connected outside of $N(\alpha, \beta)$ to at least one other $\varepsilon_j$ (with $i \neq j$), since otherwise there would not exist any curve intersecting $\varepsilon_i$ exactly once, and thus $\varepsilon_i$ would not be nonseparating.

We say $\alpha$ and $\beta$ are of type A if $\varepsilon_i$ is a nonseparating curve for all $i$ and $\varepsilon_i$ is connected outside of $\Sigma_{\alpha, \beta}$ to $\varepsilon_j$ for all $i, j \in \{0, 1, 2, 3\}$. See Figure 3.

Remark 3.6. Remember that while $\pi_{HT}$ is an edge-preserving map it is not alternating. Also, $\pi_C$ has the property that if $\alpha$ and $\beta$ are disjoint nonseparating curves, then $i(\pi_C(\alpha), \pi_C(\beta)) = 0$, since forgetting the punctures only affects the connected components of $S\backslash\{\alpha, \beta\}$ by possibly transforming one of them into a cylinder.

Lemma 3.7. Let $\alpha$ and $\beta$ be two nonseparating curves in $S_1$ that are spherical-Farey neighbours of type A. Then $i(\psi(\alpha), \psi(\beta)) \neq 0 \neq i(\tilde{\psi}(\alpha), \tilde{\psi}(\beta))$. 

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Proof. This proof is divided in three parts: the first proves that \( \psi(\alpha) \neq \psi(\beta) \), the second proves that \( \tilde{\psi}(\alpha) \neq \tilde{\psi}(\beta) \), and finally the third proves that \( i(\psi(\alpha), \psi(\beta)) \neq 0 \neq i(\tilde{\psi}(\alpha), \tilde{\psi}(\beta)) \).

First part: Since \( \alpha \) and \( \beta \) are of type A, we can always find curves \( \gamma \) and \( \delta \) such that:

- \( i(\alpha, \gamma) = i(\beta, \delta) = 1 \).
- \( i(\alpha, \delta) = i(\beta, \gamma) = i(\gamma, \delta) = 0 \).
- There exists a multicurve \( M \) of cardinality \( g - 2 \) such that \( C_1 = \{\alpha, \delta\} \cup M \), \( C_2 = \{\beta, \gamma\} \cup M \) and \( C_0 = \{\gamma, \delta\} \cup M \) are cut systems.

See Figure 4 for a way to obtain them.

Second part: Using the cut systems \( C_1, C_2 \) and \( C_0 \) from the first part of this proof, we can then apply Corollary \( \ref{cor:cut-systems} \) thus getting that \( \tilde{\psi}(\alpha) \) is disjoint from \( \tilde{\psi}(\delta) \) while \( i(\tilde{\psi}(\beta), \tilde{\psi}(\delta)) = 1 \). Then
\( \tilde{\psi}(\alpha) \neq \tilde{\psi}(\beta) \).

**Third part:** Let \( \tilde{P} \) be a multicurve such that \( P_1 = \tilde{P} \cup \{\alpha\} \) and \( P_2 = \tilde{P} \cup \{\beta\} \) are pants decompositions such that for \( i = 1, 2 \), any two curves of \( P_i \) do not separate the surface (see Figure 5 for an example). By Lemma 3.4 then \( \tilde{\psi}(P_1) \) and \( \tilde{\psi}(P_2) \) are pants decompositions of \( S_1 \) and, by the above paragraph, will differ in exactly one curve, \( \tilde{\psi}(\alpha) \) and \( \tilde{\psi}(\beta) \), meaning that they are contained in a complexity-one subsurface of \( S_1 \); given that by the second part of the proof, these two curves are different and yet they are contained in a subsurface of complexity one, we have that \( i(\tilde{\psi}(\alpha), \tilde{\psi}(\beta)) \neq 0 \). Given that \( \tilde{\psi} = \pi_C \circ \psi \), by Remark 3.6 we have that \( i(\psi(\alpha), \psi(\beta)) \neq 0 \).

![Figure 5: \( \tilde{P} = \{\zeta_1, \ldots\} \) is a multicurve such that \( P_1 = \tilde{P} \cup \{\alpha\} \) and \( P_2 = \tilde{P} \cup \{\beta\} \) are pants decompositions such that for \( i = 1, 2 \), any two curves of \( P_i \) do not separate the surface.](image)

A **halving multicurve** of a surface \( S = S_{g,n} \) is a multicurve \( H \) whose elements are nonseparating curves on \( S \) such that: \( S \setminus H = Q_1 \sqcup Q_2 \), with \( Q_1 \) and \( Q_2 \) homeomorphic to \( S_{0,n_1} \) and \( S_{0,n_2} \) respectively, and \( n_1 + n_2 = 2(g + 1) + n \). Note that a halving multicurve has exactly \( g + 1 \) elements.

We define a **cutting halving multicurve** as a halving multicurve such that any \( g \) elements of it form a cut system. Note that there exist halving multicurves that are not cutting halving multicurves, see Figure 6 for an example.

![Figure 6: An example of a halving multicurve that is not a cutting halving multicurve.](image)

**Lemma 3.8.** If \( H \) is a cutting halving multicurve of \( S_1 \), then \( \psi(H) \) and \( \tilde{\psi}(H) \) are cutting halving multicurves of \( S_2 \) and \( S_1 \) respectively.
Proof. Since $H$ is a cutting halving multicurve of $S_1$ then, by a repeated use of Lemma \ref{lem:intermediate} and Corollary \ref{cor:intermediate}, $\psi(H)$ and $\tilde{\psi}(H)$ will contain $g + 1$ elements and any $g$ elements of $\psi(H)$ and $\tilde{\psi}(H)$ will form cut systems. Therefore $S_2 \setminus \psi(H)$ and $S_1 \setminus \tilde{\psi}(H)$ will have two connected components, each of genus zero; thus $\psi(H)$ and $\tilde{\psi}(H)$ are cutting halving multicurves of $S_2$ and $S_1$ respectively. 

**Lemma 3.9.** Let $\alpha$ and $\beta$ be two disjoint nonseparating curves such that $S_1 \setminus \{\alpha, \beta\}$ is disconnected. Then $\psi(\alpha)$ and $\psi(\beta)$ are disjoint in $S_2$ and $\tilde{\psi}(\alpha)$ and $\tilde{\psi}(\beta)$ are disjoint in $S_1$.

Proof. We claim $\psi(\alpha) \neq \psi(\beta)$ and $\tilde{\psi}(\alpha) \neq \tilde{\psi}(\beta)$.

Given the conditions, let $\gamma$ be a nonseparating curve such that $\beta$ and $\gamma$ are spherical-Farey neighbours of type A, $\alpha$ and $\gamma$ are disjoint, and $S_1 \setminus \{\alpha, \gamma\}$ is connected; then, by Lemmas \ref{lem:intermediate} and Corollary \ref{cor:intermediate}, $i(\psi(\alpha), \psi(\gamma)) = i(\tilde{\psi}(\alpha), \tilde{\psi}(\gamma)) = 0$ and $i(\psi(\beta), \psi(\gamma)) \neq 0 \neq i(\tilde{\psi}(\beta), \tilde{\psi}(\gamma))$. Therefore $\psi(\alpha) \neq \psi(\beta)$ and $\tilde{\psi}(\alpha) \neq \tilde{\psi}(\beta)$.

We claim $i(\psi(\alpha), \psi(\beta)) = i(\tilde{\psi}(\alpha), \tilde{\psi}(\beta)) = 0$.

Let $H$ be a cutting halving multicurve in $S_1$ such that $\alpha$ is contained in $S_1'$ and $\beta$ is contained in $S_1''$, where $S_1'$ and $S_1''$ are the connected components of $S_1 \setminus H$, and also such that $S_1 \setminus \{\alpha, \gamma\}$ and $S_1 \setminus \{\beta, \gamma\}$ are connected for all $\gamma \in H$. By Lemma \ref{lem:intermediate}, $\psi(H)$ is a cutting halving multicurve; let $S_2'$ and $S_2''$ be the corresponding connected components of $S_2 \setminus \psi(H)$. See Figure \ref{fig:cut} for examples. By construction $\psi(\alpha)$ and $\psi(\beta)$ are disjoint from every element in $\psi(H)$, so they are curves contained in $S_2 \setminus \psi(H)$.

If $\psi(\alpha)$ and $\psi(\beta)$ are in different connected components of $S_2 \setminus \psi(H)$ then they are disjoint. So, suppose (without loss of generality) that both representatives are in $S_2''$.

Let $M$ be a multicurve of $S_1$ with the following properties.

1. Every element of $M$ is also a curve contained in $S_1''$ and $S_1 \setminus \{\gamma, \delta\}$ is connected for all $\gamma, \delta \in M$.
2. $S_1 \setminus \{\gamma, \delta\}$ is connected for all $\gamma \in M$ and all $\delta \in H$.
3. For all $\gamma \in M$, $\beta$ and $\gamma$ are spherical-Farey neighbours of type A.
4. For all $\gamma \in M$, $S_1 \setminus \{\alpha, \gamma\}$ is connected.
5. $M$ has $g - 2$ elements.

See Figure \ref{fig:cut} for an example. By Lemma \ref{lem:intermediate}, $\psi(M)$ satisfy conditions 1, 2, 4 and 5; also, by Lemma \ref{lem:intermediate} we have that for all $\gamma \in M$, $i(\psi(\beta), \psi(\gamma)) \neq 0$. This implies that every element of $\psi(M)$ is a curve contained in $S_2''$; thus $\psi(\gamma)$ intersects $\psi(\beta)$ at least twice for all $\gamma \in M$ (since $S_2''$ has genus zero, every curve contained in it is separating in $S_2''$).

Let $U$ and $V$ be the connected components of $S_2'' \setminus \{\psi(\alpha)\}$.

Now, we prove by contradiction that the elements of $\psi(M)$ are either all in $U$ or all in $V$: Let $\gamma, \gamma' \in M$ be such that $\psi(\gamma)$ is contained in $U$ and $\psi(\gamma')$ is contained in $V$. Then we can always find a curve $\delta$ contained in $S_1''$ such that the elements of $\{\gamma, \delta\}$ and of $\{\gamma', \delta\}$ satisfy the conditions of Lemma \ref{lem:intermediate} and such that $S_1 \setminus \{\delta, \delta'\}$ is connected for all $\delta' \in H \cup \{\alpha\}$. This implies $i(\psi(\gamma), \psi(\delta)) \neq 0 \neq i(\psi(\gamma'), \psi(\delta))$, and that $\psi(\delta)$ has to be either in $U$ or $V$. These two conditions together imply that $\psi(\delta)$ is contained in both $U$ and $V$, which is a contradiction.

Therefore, $\psi(M)$ consists of $g - 2$ nonseparating curves, no two of which separate $S_2$, and (up to relabelling) all these nonseparating curves are disjointly contained in $U$. But $U$ can have at most $g - 3$ nonseparating (in $S_2$) curves that no pair of which separates $S_2$ (this number is actually the greatest possible cardinality of a punctured pants decomposition of $U$); so we have found a contradiction and thus $\psi(\alpha)$ and $\psi(\beta)$ are in different connected components and then
Figure 7: The cutting halving multicurve $H = \{\eta_1, \ldots, \eta_{g+1}\}$, the multicurve $M = \{\gamma_1, \ldots, \gamma_{g-2}\}$, and the spherical-Farey neighbours $\alpha$ and $\beta$ for the closed surfaces of genus 5 (above) and genus 7 (below).

By Remark 3.6, since $i(\psi(\alpha), \psi(\beta)) = 0$, then $i(\tilde{\psi}(\alpha), \tilde{\psi}(\beta)) = 0$.

Thus, by using Lemmas 3.1, 3.9 and Corollary 3.2 we obtain the following corollary.

**Corollary 3.10.** $\psi$ and $\tilde{\psi}$ preserve both disjointness and intersection 1.

### 3.2 Inducing $\hat{\psi} : C(S_1) \to C(S_1)$

To extend $\tilde{\psi}$, we proceed in the same way as Irmak in [8], using chains and the fact that every separating curve in $S_1$ is the boundary curve of a closed neighbourhood of a chain.

Using Lemmas 3.1, 3.9 and Corollary 3.2 we obtain the following lemma.

**Lemma 3.11.** If $X$ is a chain of length $k$, then $\psi(X)$ and $\tilde{\psi}(X)$ are chains of length $k$.

Since $S_1$ is a closed surface, then every separating curve $\alpha$ on $S_1$ can be characterized as the boundary curve of a closed regular neighbourhood of a chain $X_\alpha$. See Figure 8 for an example. We call $X_\alpha$ a defining chain of $\alpha$. Recall that every defining chain of a separating curve always has even cardinality, $2k$, and its closed regular neighbourhood will then have genus $k$.

**Lemma 3.12.** Let $\beta_1$ and $\beta_2$ be separating curves in $S_1$, and $X_1$ and $X_2$ be defining chains of $\beta_1$ and $\beta_2$ respectively. If $\beta_1 = \beta_2$, then either every element of $X_1$ is disjoint from every element of $X_2$ and viceversa, or every curve in $X_1$ intersects at least one curve in $X_2$ and viceversa.

**Proof.** Since every element in $X_1$ and $X_2$ is by definition disjoint from $\beta = \beta_1 = \beta_2$, then all the elements in $X_1$ are contained in the same connected component of $S_1 \setminus \{\beta\}$, and analogously with all the elements of $X_2$. If the elements of $X_2$ are in a different connected component from those of $X_1$ then every element of $X_1$ is disjoint from every element of $X_2$ and viceversa. If the elements of $X_2$ are in the same connected component as those of $X_1$, since $X_1$ fills its
regular neighbourhood we have that every curve in \(X_1\) intersects at least one curve in \(X_2\) and vice versa.

To extend the definition of \(\tilde{\psi}\) to \(C(S)\), we define \(\hat{\psi}\) as follows: If \(\alpha\) is a nonseparating curve, then \(\hat{\psi}(\alpha) = \tilde{\psi}(\alpha)\); if \(\alpha\) is a separating curve, let \(X_\alpha\) be a defining chain of \(\alpha\) and then we define \(\hat{\psi}(\alpha)\) as the boundary curve of a regular neighbourhood of \(\tilde{\psi}(X_\alpha)\). This makes sense given that the regular neighbourhoods of \(X_\alpha\) are all isotopic, and thus the boundary curves of any two regular neighbourhoods are isotopic.

**Lemma 3.13.** The map \(\hat{\psi}\) is well-defined.

**Proof.** Let \(\alpha\) be a separating curve and \(X_1\) and \(X_2\) be two defining chains of \(\alpha\). We divide this proof in two parts, depending on whether \(X_1\) and \(X_2\) are in the same connected component of \(S_1\setminus\{\alpha\}\) or not.

*Part 1:* If \(X_1\) and \(X_2\) are in two different connected components, then due to Corollary [3.10] we have that every element in \(\hat{\psi}(X_1) = \tilde{\psi}(X_1)\) will be disjoint from every element in \(\hat{\psi}(X_2) = \psi(X_2)\); now, if \(X_1\) (and thus also \(\tilde{\psi}(X_1)\)) has length \(2k\), then \(X_2\) (and thus also \(\tilde{\psi}(X_2)\)) has length \(2(g-k)\). If we cut \(S_1\) along the boundary curve of the regular neighbourhood of \(\tilde{\psi}(X_1)\), we obtain a surface \(S'_1\) that has two connected components, one of genus \(k\) and another of genus \(g-k\). If we cut \(S'_1\) along the boundary curve of a regular neighbourhood of \(\hat{\psi}(X_2)\) (which means we are cutting \(S'_1\) in the connected component of genus \(g-k\)), we obtain a surface with three connected components: one of genus \(k\) (since it is where the elements of \(\tilde{\psi}(X_1)\) are contained), one of genus \(g-k\) (since it is where the elements of \(\tilde{\psi}(X_2)\) are contained), and an annulus. Therefore the two boundary curves of the regular neighbourhoods are isotopic, i.e. \(\hat{\psi}(\alpha)\) is well defined for these two chains.

*Part 2:* If \(X_1\) and \(X_2\) are in the same connected component, then we can find a defining chain \(X_3\) on the other connected component such that the pairs \((X_1,X_3)\) and \((X_2,X_3)\) satisfy the conditions of the previous part, so the boundary curves of the regular neighbourhoods of the chains \((\hat{\psi}(X_1),\hat{\psi}(X_3))\) and \((\hat{\psi}(X_2),\hat{\psi}(X_3))\) are isotopic. Therefore \(\hat{\psi}(\alpha)\) is well defined.

Now we prove that \(\hat{\psi}\) is an edge-preserving map, so that we can apply Theorem A from [7].

**Lemma 3.14.** \(\hat{\psi}\) is an edge-preserving map.

**Proof.** What we must prove is that given \(\alpha\) and \(\beta\) two disjoint curves, then \(\hat{\psi}(\alpha)\) and \(\hat{\psi}(\beta)\) are disjoint. If both \(\alpha\) and \(\beta\) are nonseparating curves, then we get the result from Corollary [3.10]. If \(\alpha\) is nonseparating and \(\beta\) is separating, let \(X\) be a defining chain of \(\beta\) such that \(\alpha \in X\). Then by definition \(\hat{\psi}(\alpha)\) is disjoint from \(\hat{\psi}(\beta)\).
If $\alpha$ and $\beta$ are both separating, then we can always find two disjoint defining chains $X_\alpha$ and $X_\beta$ of $\alpha$ and $\beta$ respectively. Then by the two previous cases, every element of $\psi(X_\alpha)$ is disjoint from every element of $\hat{\psi}(X_\beta) \cup \{\hat{\psi}(\beta)\}$ and every element of $\psi(X_\beta)$ is disjoint from every element of $\hat{\psi}(X_\alpha) \cup \{\hat{\psi}(\alpha)\}$. Since by definition $\hat{\psi}(\alpha)$ is the boundary curve of a regular neighbourhood of $\psi(X_\alpha)$, if $\hat{\psi}(\beta)$ and $\hat{\psi}(\alpha)$ were to intersect each other, $\hat{\psi}(\beta)$ would have to intersect at least one element of $\psi(X_\alpha)$; thus $\hat{i}(\hat{\psi}(\beta), \hat{\psi}(\alpha)) = 0$.

To prove that $\hat{\psi}(\alpha) \neq \hat{\psi}(\beta)$, let $X_1$ and $X_2$ be chains such that $X_1$ is a defining chain of $\alpha$ and $X_1 \cup X_2$ is a defining chain of $\beta$. Thus $\psi(X_1)$ and $\hat{\psi}(X_1 \cup X_2)$ are defining chains of $\hat{\psi}(\alpha)$ and $\hat{\psi}(\beta)$ respectively. This implies that there exists (by the first case) an element in $\hat{\psi}(X_1 \cup X_2)$ that is disjoint from every element in $\hat{\psi}(X_1)$, and another element in $\hat{\psi}(X_1 \cup X_2)$ that intersects at least one element in $\hat{\psi}(X_1)$ (this happens since $\hat{\psi}|_{\hat{\psi}(S_1)} = \hat{\psi}$ and we can apply Corollary 3.10). Then by Lemma 3.12 $\hat{\psi}(\alpha) \neq \hat{\psi}(\beta)$. Therefore they are disjoint.

Now, for the sake of completeness, we first cite Theorem A from [7] and then finalize with the proof of Theorem C.

**Theorem (A in [7]).** Let $S_1 = S_{g_1,n_1}$ and $S_2 = S_{g_2,n_2}$ be two orientable surfaces of finite topological type such that $g_1 \geq 3$, and $\kappa(S_2) \leq \kappa(S_1)$; let also $\varphi : C(S_1) \to C(S_2)$ be an edge-preserving map. Then, $S_1$ is homeomorphic to $S_2$ and $\varphi$ is induced by a homeomorphism $S_1 \to S_2$.

**Proof of Theorem C** We apply Theorem A from [7] to $\hat{\psi}$, obtaining an element $h$ of $\text{Mod}^*(S_1)$ that induces it. Since $h|_{\hat{\psi}(S_1)} = \hat{\psi}|_{\hat{\psi}(S_1)} = \hat{\psi}$, we have that (by Corollary 3.2) for every cut system $C = \{\alpha_1, \ldots, \alpha_g\}$ in $S_1$, $\hat{\varphi}(C) = \{h(\alpha_1), \ldots, h(\alpha_g)\}$. Therefore $h$ induces $\varphi$.

### 4 Proof of Corollary D

To prove Corollary D we first prove a consequence of Theorem C.

**Corollary 4.1.** Let $S = S_{g,0}$ be an orientable closed surface of finite topological type of genus $g \geq 3$, and $\phi : \mathcal{HT}(S) \to \mathcal{HT}(S)$ be an edge-preserving alternating map. Then $\phi$ is induced by a homeomorphism of $S$.

**Proof.** By supposing $S = S_1 = S_2$ we have that $\pi_{\mathcal{HT}}$ is the identity, and by applying Theorem C we obtain that $\phi = \phi$ is induced by a homeomorphism.

**Proof of Corollary D** Let $\phi : \mathcal{HT}(S_1) \to \mathcal{HT}(S_2)$ be an isomorphism. We first prove that it is alternating. By Lemma 2.4, we have that for all cut systems $C$ in $S_1$, $\phi$ preserves the colours in $\mathcal{A}(C)$. Applying the same lemma to $\phi^{-1}$, we have that two cut systems in $\mathcal{A}(C)$ are in the same colour if and only if their images are in the same colour in $\mathcal{A}(\phi(C))$. This implies that $\phi$ is an alternating map.

Given that $\phi$ and $\phi^{-1}$ are isomorphisms, then they are also edge-preserving maps, and by Theorem A applied to $\phi$ and $\phi^{-1}$, we have that $g_1 = g_2$.

Now, let $S = S_{g,0}$ with $g \geq 3$. To prove that $\text{Aut}(\mathcal{HT}(S))$ is isomorphic to $\text{Mod}^*(S)$, we note there is a natural homomorphism:

$$
\Psi_{\mathcal{HT}(S)} : \text{Mod}^*(S) \to \text{Aut}(\mathcal{HT}(S)) \quad \text{[h]} \mapsto \varphi : \mathcal{HT}(S) \to \mathcal{HT}(S) \quad \{\alpha_1, \ldots, \alpha_g\} \mapsto \{h(\alpha_1), \ldots, h(\alpha_g)\}
$$
Injectivity: If $h_1$ and $h_2$ are two homeomorphisms of $S$ such that $\Psi_{HT(S)}([h_1]) = \Psi_{HT(S)}([h_2])$, then the action of $h_1$ and $h_2$ on the nonseparating curves on $S$ would be exactly the same. Recalling that Schmutz-Schaller proved in [10] that $\text{Aut}(\mathcal{G}(S))$ is isomorphic to $\text{Mod}^*(S)$, this implies that $h_1$ is isotopic to $h_2$.

Surjectivity: If $\phi \in \text{Aut}(\mathcal{HT}(S))$, then (as was proved above) it is an edge-preserving alternating map. Thus, by Corollary 4.1 we have that $\phi$ is induced by a homeomorphism. Therefore, $\text{Aut}(\mathcal{HT}(S))$ is isomorphic to $\text{Mod}^*(S)$.

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