On the solution of the initial value constraints for general relativity coupled to matter in terms of Ashtekar’s variables

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Abstract

The method of solution of the initial value constraints for pure canonical gravity in terms of Ashtekar’s new canonical variables due to CDJ (see [1]) is further developed in the present paper. There are 2 new main results:

1) We extend the method of CDJ to arbitrary matter-coupling again for non-degenerate metrics: the new feature is that the 'CDJ-matrix' adopts a nontrivial antisymmetric part when solving the vector constraint and that the Klein-Gordon-field is used, instead of the symmetric part of the CDJ-matrix, in order to satisfy the scalar constraint.

2) The 2nd result is that one can solve the general initial value constraints for arbitrary matter coupling by a method which is completely independent of that of CDJ. It is shown how the Yang-Mills and gravitational Gauss constraints can be solved explicitly for the corresponding electric fields. The rest of the constraints can then be satisfied by using either scalar or spinor field momenta. This new trick might be of interest also for Yang-Mills theories on curved backgrounds.

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1 Introduction

The general solution of the initial value (iv) constraints with respect to the vector and the scalar constraint within the Ashtekar-framework (see [3]) for pure gravity has a strikingly simple structure when restricting it to non-degenerate metrics.

The gravitational action in the new canonical variables introduced by Ashtekar (see ref. [3]) has the form

\[ E = \int_R dt \int_\Sigma d^3x \left\{ P_a^i \dot{\omega}_a^i - \left[ -\omega_a^i (E_G) + N^a (E_V) - \kappa N (E_C) \right] \right\}. \]  

(1.1)

Here \( P_a^i = -i/2\kappa \sqrt{\det(q)} e_i^a \) where \( \kappa / (8\pi) \) is Newton’s coupling constant and \( e_i^a \) is the triad of the metric \( q := (q_{ab}) \) on the hypersurface \( \Sigma \) while \( \omega_a^i \) is the Ashtekar-connection. We use indices \( a, b, c, \ldots \) from the beginning of the alphabet to denote valences of tensors defined on the initial data hypersurface \( \Sigma \) while indices \( i, j, k, \ldots \) from the middle of the alphabet describe the \( O(3) \)-gauge group structure of a generalized tensor. CDJ (see [1]) have shown that the ansatz

\[ P_a^i = \Psi_{ij} B_{ji}^a, \]  

(1.2)

where \( B_i^a := \frac{1}{2} \epsilon_{abc} \Omega_{bc}^i \) is the magnetic field with respect to the Ashtekar-connection, \( \Omega_{ab}^i \) being its field strength and \( \epsilon_{abc} \) is the totally skew (metric-independent) tensor density of weight 1, solves both the gravitational part of the vector constraint

\[ E V_a := \Omega_{ab}^i P_{i}^b \frac{1}{2} = 0 \]  

(1.3)

and the gravitational part of scalar constraint

\[ E C := \Omega_{ab}^i P_{j}^a P_{k}^b \epsilon_{ijk} \frac{1}{3} = 0 \]  

(1.4)

provided the CDJ-matrix \( \Psi \) is subject to the following conditions :

\[ \Psi^T = \Psi \text{ and } (tr(\Psi))^2 - tr(\Psi^2) = 0. \]  

(1.5)

Here \( A^T \) means the transpose of the matrix \( A \). Inserting this ansatz into the gravitational part of the Gauss-constraint one obtains when using the Bianchi-identity (\( D \) is the gauge covariant differential acting on arbitrary generalized tensors)

\[ E G_i := D_a P_i^a = B_j^a D_a \Psi_{ij} \frac{1}{2} = 0 \]  

(1.6)
which is now a differential condition on $\Psi$ and cannot be solved by purely algebraic methods any more. Up to now there are no solutions known to equation (1.6) for full gravity.

Furthermore, it should be stressed that even if one had also the general solution to equation (1.6) then one would only have obtained the constraint surface of the 'non-degenerate sector' of the gravitational phase space, not its reduced phase space since one did not factor by the gauge orbits yet. Nevertheless, this method could be of some importance for obtaining the reduced phase space for full gravity as might be indicated by the fact that it is of some help in model systems (see [4]).

In this paper we are going to discuss the iv constraints in terms of Ashtekar’s new variables for arbitrary matter coupling. The analysis will be in the canonical framework as in equation (1.1), so all the fields will be subject to 4 different types of constraints, namely
1) the gravitational Gauss constraint (Lagrange multiplier : $-\omega^i_t$)
2) the Yang-Mills Gauss constraint (Lagrange multiplier : $-A^i_t$)
3) the vector constraint (Lagrange multiplier : $N^a$, the shift vector) and
4) the scalar constraint (Lagrange multiplier : $\tilde{N}$, where $\tilde{N} := \sqrt{\text{det}(q)}N$ is the lapse function).

The matter sector which we are going to discuss consists of (we will not dwell on how to derive the 3+1 form of the various actions, for details see ref. [5]; the 3+1 form of the Higgs-action is not derived there but it can be obtained by a calculation similar to that for the Klein-Gordon action so we can omit this here)

$$\text{matter } S = KG S + W S + C S + YM S + H S ,$$

i.e. a collection of (real) Klein-Gordon-fields with arbitrary potential $V(\phi)$ (we do not display the summation over the different scalar fields)

$$KG S = \int_R dt \int_{\Sigma} d^3 x \{ \pi \dot{\phi} - [N^a \pi \phi, a + \frac{1}{2} \tilde{N}(\pi^2 + \text{det}(q)(q^{ab}\phi, a\phi, b + V(\phi)))] \},$$

a (collection of) (complex valued rather than Grassmann-valued) Weyl spinor field(s) which couple only to the self-dual part of the spin-connection (this is explained in refs. [4] and [5]; we write down only one spinor field which may stand for an arbitrary number of Weyl-fields of possibly both chiralities), including an arbitrary spinor potential (e.g. the usual mass term when at
least 2 spinor fields of both chiralities are present)

\[ W S = \int_R dt \int_{\Sigma} d^3 x \{ \pi^T \dot{\psi} - [-\omega_i^i \pi^T \tau_i \psi \\
+ N^a \pi^T D_a \psi - 4N(\kappa P^a \pi^T \tau_i D_a \psi + \det(q)V(\psi^A, \pi_A))\} \] , (1.9)

where \( \tau_i = -i/2 \sigma_i \), \( \sigma_i \) the usual Pauli-matrices, \( D_a \psi := [\partial_a + \omega_i^a \tau_i] \psi \) and \( \pi = (\pi_A), \psi = (\psi^A), A = 1, 2 \) SU(2) spinor indices,

a Yang-Mills-field for a semi-simple gauge group G (such that the Cartan-Killing metric \( (d_{IJ}) \), I, J,.. internal indices of this gauge group, is non-degenerate; if \( T_I \) are the generators of the Lie algebra LG of G then \( d_{IJ} = \text{tr}(T_I T_J), [T_I, T_J] = f_{IJ}^{\phantom{IJ}K} T_K \)

\[ YM S = \int_R dt \int_{\Sigma} d^3 x \{ E_a^I \dot{A}_a^I - [-A_a^I \dot{D}_a^I E_a^I \\
+ N^a F_{ab}^I E_b^I + \frac{g^2}{2N} q_{ab} d_{IJ}^a (E_a^I E_b^J + B_a^J B_b^J)\} \] , (1.10)

where \( E_a^I \) is the YM electric field, \( B_a^J := 1/(2g^2) d_{IJ}^a \epsilon^{abc} F_{bc}^J \), \( F_{ab}^I \) being the YM-field strength, is the YM magnetic field and \( g \) the YM coupling constant,

a cosmological constant (\( \Lambda \)) term

\[ CS = \int_R dt \int_{\Sigma} d^3 x N \det(q) \Lambda \] (1.11)

and a Higgs-field for either the gravitational gauge group O(3) or the YM gauge group G

\[ EHH S = \int_R dt \int_{\Sigma} d^3 x \{ \pi_i \dot{\phi}^i - [-\omega^i_{ij} \pi_j \phi^j + N^a \pi_i D_a \phi^i \\
+ \frac{1}{2N} \n(ab \delta_{ij} (D_a \phi^i)(D_b \phi^j) + V(\phi^i))\} \] or(1.12)

\[ YMH S = \int_R dt \int_{\Sigma} d^3 x \{ \pi_I \dot{\phi}^I - [-A_I^I f_{IJ} \phi^J \pi^K + N^a \pi_I D_a \phi^I \\
+ \frac{1}{2N} (d_{IJ} \pi_I \pi_J + \det(q)(q^{ab} d_{IJ} (D_a \phi^I)(D_b \phi^J) + V(\phi^I)))\} (1.13)

After this brief introduction of our notation, we can outline the plan of the paper:
In section 2 we will show how the method of CDJ can be extended to include arbitrary physically relevant matter couplings. The deviations from the source-free case are that the CDJ-matrix $\Psi$ fails to be symmetric and that the scalar constraint is not solved for $\Psi$ but for one Klein-Gordon-momentum $\pi$. The equation for $\pi$ is purely algebraic and of 2nd or 4th order respectively depending on whether the YM field is coupled or not. This is just sufficient in order that the scalar constraint be solvable algebraically, for example, by the methods of Cardano and Ferrari (see [2]). As for the source-free case, this method does not solve the gravitational Gauss constraint.

In section 3 we show that by employing a method which is completely independent of the CDJ-framework it is possible to solve all constraints of general relativity coupled to arbitrary matter. An advantage of this method is that the solutions of the constraint equations are remarkably simpler than those obtained by the method presented in section 2. First we show how the gravitational Higgs-field can be used to solve both the gravitational Gauss constraint and the scalar constraint. The equations for the Higgs field are at most quadratic. The YM-Higgs-field can (in general) be used in order to satisfy the YM Gauss constraint. Now it is the vector constraint which is not solvable by purely algebraic methods for the gravitational field. However, by using again the gravitational Higgs field and at least 3 Klein-Gordon fields one can solve both the vector and the scalar constraint. By a similar line of approach one can also solve only the YM Gauss-constraints purely for the YM electric field. The rest of the constraints can then be satisfied by purely algebraic methods if at least 4 Weyl spinor fields are present which, of course, corresponds to the physical reality. This latter option has the advantage that one can forget about scalar fields altogether which have not yet be proved to exist at all.

The results of section 3 may be of interest also for the canonical approach to (pure) YM theory on curved or flat background metrics.

The paper concludes with an appendix in which some explicit formulas for special cases of couplings, i.e. when the YM-field is absent, are given, valid when applying the method of section 2. There we also display the formulas of Cardano and Ferrari for the reader who wants to arrive at an explicit solution of the initial value constraints including a YM-field when applying the framework of section 2.
2 Solving the vector and the scalar constraint

The full vector constraint reads
\[ V_a = \Omega_{ab}^i P_i^b + \pi \phi_a + \pi^T D_a \psi + F_{ab} E_I^b + \pi_i D_a \phi^i + \pi_f D_a \phi^f \]  
\[ (2.1) \]

As CDJ (ref. [1]) we make the ansatz
\[ P_a^i = \Psi B_a^i \]  
\[ (2.2) \]
and obtain
\[ \Omega_{ab}^i P_i^b = \epsilon_{abc} B_a^c B_j^b \Psi_{ij} = \det(B_i^a) \epsilon^{ijk} B_a^k \Psi_{ij} \],

where \( B^a_i B_b^i = \delta^a_b \), \( B^a_i B_a^j = \delta^j_i \). This ansatz poses no restrictions on \( P_a^i \) as long as the magnetic fields are non-degenerate (which however excludes flat space field configurations).

The important step is to decompose the CDJ-matrix into its symmetric and anti-symmetric part
\[ \Psi := S + A, \ S^T = S, \ A^T = -A \]  
\[ (2.3) \]
because only A enters the vector constraint which then can in fact be solved for A : let \( 2\xi_k := \epsilon_{ijk} A_{ij} \), \( B := \det(B_i^a) \), then
\[ \xi_k = \pi \left[ \frac{B_a^k}{2B} \phi_a \right] + \left[ \frac{B_a^k}{2B} \left( \pi^T D_a \psi + F_{ab} E_I^b + \pi_i D_a \phi^i + \pi_f D_a \phi^f \right) \right] =: \pi \eta_k + \theta_k \]  
\[ (2.4) \]
where the quantities \( \eta_k, \theta_k \) do not depend on \( \pi \). Contracting equation (2.4) with \( \epsilon_{ijk} \) we obtain
\[ A := \pi T + R \]  
\[ \text{where} \quad T^T = -T, \quad R^T = -R \],
\[ (2.5) \]
i.e. the matrix A is linear in \( \pi \), homogenous only if the matter different from the KG-field is absent.

We now insert this into the scalar constraint whose complete expression is given by
\[ C = -\kappa \Omega_{ab}^i P_j^a P_k^b \epsilon_{ijk} + \frac{1}{2} \left[ \pi^2 + \det(q) \left( q^{ab} \phi_a \phi_b + V(\phi) \right) \right] \]
\[-4 \left( \kappa P_i^a \pi^T D_a \psi + \det(q) V(\psi^A, \pi_A) \right) + \Lambda \det(q) + \frac{g^2}{2} q_{ab} d^{ij} (E_I^a E_J^b + B_I^a B_J^b) \]
\[ \begin{align*}
&+ \frac{1}{2} [\delta_{ij} \pi_i \pi_j + \det(q)(\delta_{ij} q^{ab}(D_a \phi^i)(D_b \phi^j) + V(\phi^j))] \\
&+ \frac{1}{2} [d^{IJ} \pi_I \pi_J + \det(q)(d_{IJ} q^{ab}(D_a \phi^I)(D_b \phi^J) + V(\phi^I))] \\
&=: E_C + KG C + W C + C + Y M C + \text{EH} C + Y MH C .
\end{align*} \] (2.6)

When inserting for \( P_{i}^a \) the various terms involve different powers of \( A \) which we now discuss in sequence:

\[ \begin{align*}
&- \frac{1}{\kappa} E C = \epsilon_{abc} B_i^a B_m^b B_n^c \epsilon^{ijk} \Psi_{jm} \Psi_{kn} = B \epsilon_{mn} \epsilon^{ij} \Psi_{jm} \Psi_{kn} = B((tr(\Psi))^2 - tr(\Psi^2)) \\
&\text{(2.7)}
\end{align*} \]

Since \( tr(A) = tr(A^T) = -tr(A) = 0, \ tr(SA) = tr(A^T S^T) = -tr(AS) = -tr(SA) = 0 \) for any symmetric matrix \( S \) and any antisymmetric matrix \( A \), it follows easily that

\[ E_C = -\kappa B((tr(S))^2 - tr(S^2) - tr(A^2)) , \] (2.8)

such that \( \pi \) enters \( E_C \) only purely quadratically without a linear term.

Let \( \phi_{,a} B_i^a := v_i \), then

\[ \det(q) q^{ab} \phi_{,a} \phi_{,b} = -4\kappa^2 P_{i}^a P_{i}^b \phi_{,a} \phi_{,b} , \] (2.9)

but \( P_{i}^a \phi_{,a} = (S_{ij} + R_{ij})v_j \) because \( T_{ij}v_j = \epsilon_{ijk}v_jv_k/(2B)\phi_{,a} = 0 \). Hence

\[ \det(q) q^{ab} \phi_{,a} \phi_{,b} = -4\kappa^2 v^T((S+R)^T(S+R))v = -4\kappa^2 tr((S^2 - R^2 + [S, R])v \otimes v) \] (2.10)

which is independent of \( \pi \).

Now

\[ \det(\det(q) q^{ab}) = (\det(q))^4(\det(q))^{-1} = (\det(q))^2 = -(2\kappa)^2 \det(P_{i}^a P_{i}^b) = -(2\kappa \det(P_{i}^a))^2 \] (2.11)

from which follows (up to a sign) that

\[ \det(q) = 2i\kappa B \det(\Psi) . \] (2.12)

Since a term cubic in \( A \) enters \( det(\Psi) \) only through \( det(A) \) (this follows from the fact that \( \Psi = A + S \) and that the determinant is a totally skew multilinear
functional) which vanishes in 3 dimensions, $det(q)$ is also only quadratic in $\pi$. The explicit formula is given by

$$\det(\Psi) = \frac{1}{3!} \epsilon^{ijk} \epsilon_{lmn} (A + S)_{il} (A + S)_{jm} (A + S)_{kn}$$

$$= \frac{1}{3!} \epsilon^{ijk} \epsilon_{lmn} (S_{il} S_{jm} S_{kn} + 3 S_{il} S_{jm} A_{kn} + 3 S_{il} A_{jm} A_{kn} + A_{il} A_{jm} A_{kn})$$

$$= \det(S) + \det(A) + \frac{1}{2} 3! \delta_i^l \delta_j^m \delta_k^n (S_{il} S_{jm} A_{kn} + S_{il} A_{jm} A_{kn})$$

$$= \det(S) + tr(SA^2) - \frac{1}{2} tr(S) tr(A^2). \quad (2.13)$$

Altogether we have therefore

$$KG = \frac{1}{2} \left[ \pi^2 - 4 \kappa^2 tr((S^2 - R^2 + [S, R])v \otimes v) \right.$$  

$$\left. + 2i \kappa B (\det(S) + tr(SA^2) - \frac{1}{2} tr(S) tr(A^2))V(\phi) \right]. \quad (2.14)$$

The Higgs-sector differs from the KG-sector as far as the appearance of $\pi$ is concerned only in that $T_{ij} B^a \phi_a = 0$, while $T_{ij} B^a D_a \phi_k T_{ij} B^a D_a \phi_k \neq 0$.

Let $v_i := B^a D_a \phi^i$, $v^j := B^a D_a \phi^j$ then

$$EH = \frac{1}{2} [\delta_i^j \pi_i \pi_j - 4 \kappa^2 \delta_i^j tr((S^2 - A^2 + [S, A])v^i \otimes v^j)$$

$$+ 2i \kappa B (\det(S) + tr(SA^2) - \frac{1}{2} tr(S) tr(A^2))V(\phi^i)]$$

$$YMH = \frac{1}{2} [d^I \pi_I \pi_I - 4 \kappa^2 d^I tr((S^2 - A^2 + [S, A])v^I \otimes v^I)$$

$$+ 2i \kappa B (\det(S) + tr(SA^2) - \frac{1}{2} tr(S) tr(A^2))V(\phi^I)]. \quad (2.15)$$

There is only a quadratic appearance of $\pi$ again.

The contribution by the YM-sector however gives rise to a quartic constraint for $\pi$.

We have

$$q_{ab} = det(q) E^i_a E^i_a = \frac{(2 \kappa)^4}{4 \det(q)} \epsilon_{acd} \epsilon_{bej} \epsilon^{ijk} \epsilon^{lmn} P^c_j P^d_k P^e_m P^f_n$$

$^{1}$This formula can also be obtained by applying the theorem of Hamilton-Cayley as was pointed out to the author by Ted Jacobson in private communication.
\[
\begin{align*}
\kappa^4 B^2 &= \frac{2}{\det(q)} \varepsilon_{rst} \varepsilon_{uvw} \partial_{[t} \partial_{m} \Psi_{jr} \Psi_{ks} \Psi_{nu} B_a^t B^u \\
\kappa^4 B^2 &= \frac{3}{2 \det(q)} \delta_{[t} \delta_{s} \delta_{lu} (\Psi^T \Psi)_{ru} (\Psi^T \Psi)_{sv} B_a^t B^u \\
\kappa^4 B^2 &= \frac{(2 \kappa)^4}{2 \det(q)} \left[ (\tau(\Psi^T \Psi))^2 - \tau((\Psi^T \Psi)^2) + 2(\Psi^T \Psi)^2 - 2\tau(\Psi^T \Psi)\Psi^T \Psi \right]_{tw} B_a^t B^u \\
\end{align*}
\]

Since \( \Psi^T \Psi = (S - A)(S + A) = S^2 - A^2 + [S, A] \), \( \tau([S, A]) = 0 \), we conclude that

\[
\begin{align*}
q_{ab} &= \frac{(2 \kappa)^4}{2 \det(q)} \left[ (\tau(S^2 - A^2))^2 - \tau((S^2 - A^2 + [S, A])^2) + 2(S^2 - A^2 + [S, A])^2 \\
&- 2\tau(S^2 - A^2)(S^2 - A^2 + [S, A])_{tw} B_a^t B^u \right].
\end{align*}
\]

As long as \( \det(q) \neq 0 \) one is allowed to multiply the scalar constraint by \( \det(q) \) so that it becomes 4th order in \( \pi \) because the contributions of the other fields are at most quadratic in \( \pi \) and \( \det(\Psi) \) is, according to formula (2.13), also only quadratic in \( \pi \).

The final step is to collect all terms and to determine the coefficients of the various powers of \( \pi \). We have

\[
\det(\Psi) = \frac{[\det(S) + \tau(S R^2) - \frac{1}{2}\tau(S) \tau(R^2)] + \pi[\tau(S(RT + TR))] - \frac{1}{2}\tau(S) \tau(TR + RT)] + \pi^2[\tau(ST^2) - \frac{1}{2}\tau(S) \tau(T^2)]}{(2i \kappa B)}
\]

and

\[
\begin{align*}
\Psi^T \Psi &= \{S^2 - R^2 - [S, R]\} - \pi \{RT + TR - [S, T]\} - \pi^2 T^2, \\
(\Psi^T \Psi)^2 &= \{[(S^2 - R^2 - [S, R])^2 \\
&- \pi[(RT + TR - [S, T])(S^2 - R^2 - [S, R]) + (S^2 - R^2 - [S, R]) (RT + TR - [S, T])] - \pi^2[(RT + TR - [S, T])T^2 \\
&+ T^2(RT + TR - [S, T]) - (S^2 - R^2 - [S, R])^2 \\
&+ \pi^3((RT + TR - [S, T])T^2 + T^2(RT + TR - [S, T])) + \pi^4 T^4 \\
&= A_4 \pi^4 + A_3 \pi^3 + A_2 \pi^2 + A_1 \pi + A_0 \}
\end{align*}
\]
such that

\[ q_{ab} \frac{2 \det(q)}{(2\kappa)^4} B^a_i B^b_j \]

\[ = \left( (tr(S^2 - R^2))^2 - 4\pi tr(S^2 - R^2)tr(RT) - 2\pi^2 (tr(S^2 - R^2)tr(T^2) \right. \]
\[ - 2(tr(RT))^2 + 4\pi^2 tr(RT)tr(T^2) + \pi^4 (tr(T^2))^2 - (tr(A_1)\pi^4 + tr(A_3)\pi^3 \]
\[ + tr(A_2)\pi^2 + tr(A_1)\pi + tr(A_0)) + 2(A_4\pi^4 + A_3\pi^3 + A_2\pi^2 + A_1\pi + A_0 \]
\[ -2(tr(S^2 - R^2)(S^2 - R^2 - [S, R]) - \pi(tr(S^2 - R^2)(RT + TR - [S, T])) \]
\[ + 2tr(RT)(S^2 - R^2 - [S, R]) \]
\[ - \pi^2 (tr(S^2 - R^2)T^2 + tr(T^2)(S^2 - R^2 - [S, R]) - 2tr(RT)(S^2 - R^2 - [S, R])) \]
\[ + \pi^3 (2tr(RT)T^2 + tr(T^2)(S^2 - R^2 - [S, R])) + \pi^4 tr(T^2T^2))_i j \].

We first collect the powers of \( \pi \) contained in the non-YM part of the scalar constraint and then multiply by \( \det(q) \).

\[ E_C + ^{KG} C + ^W C + ^C C + ^{EH} C + ^{YM} C \]

\[ = \{-\kappa B((tr(S))^2 - tr(S^2) - tr(R^2) - 2\kappa^2 tr((S^2 - R^2 + [S, R])\varphi \otimes \varphi) \]
\[ +\frac{1}{2}\varphi - 4(\kappa B_a^a (S + R)_ij \pi^T \tau_i \mathcal{D}_a \psi + cV(\psi^A, \pi_A)) + c\Lambda \]
\[ + \frac{1}{2}[\delta^{ij} \pi_i \pi_j - 4\kappa^2 \delta_{ij} tr([S^2 - R^2 + [S, R]] \varphi \otimes \varphi) \]
\[ + cV(\phi^i)] + \frac{1}{2}[d^I^J \pi_I \pi_J - 4\kappa^2 d_{IJ} tr([S^2 - R^2 + [S, R]] \varphi_I \otimes \varphi^I + cV(\phi^I))] \}
\[ + \{2\kappa B tr(RT) + \frac{b}{2} V(\phi) - 4(\kappa B^a_j T_{ij} \pi^T \tau_i \mathcal{D}_a \psi + bV(\psi^A, \pi_A)) \]
\[ + b\Lambda + \frac{1}{2}[4\kappa^2 \delta_{ij} tr([RT + TR - [S, T]] \varphi \otimes \varphi) + bV(\phi^i)] \]
\[ + \frac{1}{2}[4\kappa^2 d_{IJ} tr([RT + TR - [S, T]] \varphi^I \otimes \varphi^J + bV(\phi^I))] \\pi \]
\[ + \{\kappa B tr(T^2) + \frac{1}{2}[1 + a V(\phi)] - 4aV(\psi^A, \psi_A) + a\Lambda \]
\[ + \frac{1}{2}[4\kappa^2 \delta_{ij} tr(T^2 \varphi^I \otimes \varphi^J) \]
\[ +aV(\phi^i)] + \frac{1}{2}[4\kappa^2 d_{IJ} tr(T^2 \varphi^I \otimes \varphi^J + aV(\phi^I))] \\pi^2 \]

\[ =: d + e\pi + f\pi^2 \]
Finally we have thus, using $M^j := g^2 B^2 (2 \kappa)^4 / 4 B_a B^a_i d_{ij} (E_i E_j + B_i B_j)$ as an abbreviation

$$\det(q) C - Y^M C = af \pi^4 + (ae + fb) \pi^3 + (ad + fc + eb) \pi^2 + (bd + ec) \pi + cd.$$  

Finally we have thus, using $M^j := g^2 B^2 (2 \kappa)^4 / 4 B_a B^a_i d_{ij} (E_i E_j + B_i B_j)$ as an abbreviation

$$\det(q) C = \{ (tr(S^2 - R^2))^2 tr(M) - tr(A_0) tr(M) + 2tr(A_0 M) - 2tr(S^2 - R^2) \}$$

$$tr((S^2 - R^2 - [S, R]) M) + cd$$

$$+ \{ -4tr(S^2 - R^2) tr(RT) tr(M) - tr(A_1) tr(M) + 2tr(A_1 M) + 2(tr(S^2 - R^2) tr((RT + TR - [S, T]) M) + 2tr(RT) tr((S^2 - R^2 - [S, R]) M) + bd + ec \} \pi$$

$$+ \{ -2(tr(S^2 - R^2) tr(T^2) - 2(tr(RT))^2 tr(M) - tr(A_2) tr(M) + 2tr(A_2 M) - 2(tr(S^2 - R^2) tr(T^2 M) + tr(T^2) tr((S^2 - R^2 - [S, R]) M)$$

$$- 2tr(RT) tr((S^2 - R^2 - [S, R]) M) + ad + fc + eb \} \pi^2$$

$$+ \{ 4tr(RT) tr(T^2) tr(M) - tr(A_3) tr(M) + 2tr(A_3 M)$$

$$- 2(tr(RT) tr(T^2 M) + tr(T^2) tr((S^2 - R^2 - [S, R]) M) + ace + fb \} \pi^3$$

$$+ \{ (tr(T^2))^2 tr(M) - tr(A_4) tr(M) + 2tr(A_4 M) - 2tr(T^2) tr(T^2 M) \} \pi^4$$

$$\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\:\\
In the appendix we give the above solution for special cases i.e. when the Yang-Mills field is absent. In particular, if one only couples the Klein-Gordon field, then the scalar constraint depends only on $\pi^2$ and the above formulas simplify tremendously.

3 Solving all constraints

3.1 Solution by using the scalar fields

First of all we show how a Higgs-field can be used for any semi-simple gauge group with rank $\ell$ to satisfy the Yang-Mills gauge constraint (note that for semisimple groups $f_{IJK} := f_{IJ}^L d_{LK}$ is totally skew; see ref. [7] for the necessary Lie algebra terminology).

The YM Gauss constraint is

$$G_I := D_a E_I^a + f_{IJ}^K \phi^J \pi_K = 0. \tag{3.1}$$

Let $V$ be the complex vector space of dimension $\dim(G)$ which is the representation space of the adjoint representation of $LG$. Let further $Y := \phi^I T_I$, $(T_I)_{JK} = -f_{IJK}$ and $V^\perp(\phi^I) = \{X \in LG; \text{ad}(Y)X := [Y,X] = 0\}$ which shows that $V^\perp$ depends only on $\phi$ (even the dimension $k$ of $V^\perp$ depends, in general, on the specific element $Y$ : choose a Cartan subalgebra $H$ of $LG$ and choose the corresponding Weyl canonical form. If $Y \in H$ then $1 \leq \dim(V^\perp) = \ell$, if $Y$ is a nonzero-root vector relative to $H$ then $\dim(V^\perp)$ is not characterizable by $\ell$ only but will depend on the algebra and the root chosen. One only knows then that $\dim(V^\parallel) \geq \ell + 1$, see ref. [8], where $V^\parallel := V - V^\perp$).

We will regard the fields as Lie algebra valued by the identification $\phi^I T_I =: \phi$ etc.

Contracting equation (3.1) with $X^I \in V^\perp$ yields

$$X^I D_a E_I^a = 0 \tag{3.2}$$

because $X^I f_{IK}^J \phi^J \pi_K = tr(T_K[X,\phi]) \pi_K = tr(\pi[X,\phi]) = 0$ for any $\pi$. Hence the internal-vector density $D_a E_I^a$ is weakly 'orthogonal' to the subspace $V^\perp$. Hence, the part of the Gauss-constraint which is 'parallel' to $V^\perp$ has to be satisfied independently of the momentum $\pi_I$. We will satisfy the constraint
eqns. (3.2) which can be read as a condition on $E^a_I$ by employing the following new trick which is at the heart of the present approach:

Let

$$E^a_I := \epsilon^{abc} D_b v_{cI} \tag{3.3}$$

where the generalized tensor $v_{aI}$ is yet arbitrary. Then, using the torsion-freeness of the (purely metric-determined) Riemann-connection $\Gamma^a_{bc}$ which acts on tensor indices only (the torsion due to the spinor fields shows up in $\omega^i_a$, more precisely in the spin-connection, and is part of the reality conditions, see refs. [5],[6]) we have

$$X^I D_a E^a_I = g^{2f_{IJ}^K X^I B_{aI}^J v_{aK} = 0 \forall X \in V^\perp \tag{3.4}$$

which is a purely algebraic restriction for $v_{aI}$ and which can be satisfied identically by solving for $k$ components of $v_{aI}$ in terms of the others and of the magnetic as well as the Higgs fields. We assume that this has been done in the following. The electric fields remain, however, independent of the Higgs-momenta which is important for the sequel.

Are there any restrictions implied on the Lie-algebra valued 2-form $t_{abI} := E^c_I \epsilon_{cab}$ by representing it as a exterior differental of the Lie-algebra valued 1-form $v_{aI}$ i.e. $t_I = D \wedge v_I$ ? The purely algebraic properties of both generalized tensors are the same, however one has to worry about the 'generalized integrability conditions' obtained by taking the exterior differential of the last equation

$$D \wedge t_I = \frac{1}{2} f_{IJ}^K F_J \wedge v_K.$$  

The latter equation can now only be satisfied for an arbitrary $t_I$ if the magnetic fields $F_I$ are non-degenerate. This restriction does not directly show up in equation (3.3) because even for $A_I = 0$ the rhs of eq. (3.3) is nonvanishing for a suitable choice of $v_I$. Hence we obtain the same restriction as CDJ on the magnetic fields in order that the new trick works, at least when there is a non-trivial Yang-Mills-potential.

We decompose the dual internal vector density $\pi_I$ into a part 'parallel' and 'orthogonal' to $V^\perp$:

$$\pi_I := \pi^\perp_I + \pi^\parallel_I \tag{3.5}$$

where $\pi^\perp \in V^\perp$, $\pi^\parallel \in V^\parallel$ are dual internal vector densities. Hence it follows when inserting into the Gauss-constraint

$$D_a E^a_I + Y^J_I \pi^\parallel_J = 0 \tag{3.6}$$

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where it is understood that $E^a_I$ is replaced by equation (3.3). Equation (3.6) can be solved for $\pi^\parallel$ since $Y$ is regular on $V^\parallel$ while $\pi^\perp$ remains unspecified. Hence we can regard the gravitational and YM-Gauss law as identically satisfied in terms of 2 of the gravitational and dim(G)-k components of the YM Higgs field and 1 component of the gravitational and k components of the YM electric field. Note, however, that there is the additional spin-density $\pi^T\tau_i\psi$ contained in the gravitational Gauss constraint which upon solving the Gauss constraint becomes part of the vector $v_{ai}$.

When inserting these solutions into the vector constraint, one obtains again a genuinely differential relation between the relevant momenta $v_{ai}$, $v_{aI}$ and thus the vector constraint fails to be algebraically solvable in terms of the gravitational field. However, we can make use of the gravitational field $\pi^\perp_i$ and/or the YM field $\pi^\perp_I$ and 3 Klein-Gordon momenta $\pi_\alpha = 1, 2, 3$ in order to solve the rest of the constraints by algebraic methods (the usage of scalar fields in order to solve all constraints of GR plus scalar matter is most convenient in the old ADM variables). One solves first the 3 vector constraints in terms of $\pi_\alpha$ which will then depend linearly on the fields $\pi^\perp_i$, $\pi^\perp_I$. Hence, the scalar constraint also depends only quadratically on $\pi^\perp_i$, $\pi^\perp_I$. Accordingly, by coupling suitable matter, it turns out to be possible to solve all the constraints explicitly. So matter helps to solve the complete iv constraints with this method.

We will now give the explicit formulas. Let $\pi^\perp_i =: h_E \phi_i$ and $|\phi|^2 := \phi^i \phi^i$. The vector constraint reads

$$V_a = \begin{bmatrix} \Omega^{i}_{ab} P^b_i + F^I_{ab} E^b_i + \pi_i D_a \phi^I + \frac{1}{|\phi|^2} \epsilon_{ijk} \phi_j (D_b P^k_i)(D_a \phi^i) \\ + \pi^T D_a \psi \end{bmatrix} + h_E \phi^i D_a \phi^i + \pi_\alpha \phi^\alpha_a$$

(3.7)

where it is understood that $\phi_I$, $E^0_I$, $P^\alpha_a$ are expressed in terms of the other fields as derived above, the crucial point being that they do not depend on $h_E, \pi_\alpha$. We write this as a matrix relation

$$\bar{u}_0 + h_E \bar{u} + P(\pi_1, \pi_2, \pi_3)^T = 0$$

(3.8)

where the matrix $P$, which consists of the covariant derivatives of the fields $\phi^\alpha$, is in general non-singular so that equation (3.8) can be inverted to give

$$\pi_\alpha = -(P^{-1})^\beta_\alpha [\bar{u}_0 + h E] \beta =: (a + bh_E)_\alpha .$$

(3.9)
Here the coefficients $a_\alpha, b_\alpha$ do not depend on $h_E$.
We finally insert this into the scalar constraint (again it is to be understood that one has to insert the above solutions for $P^a_i, E^a_i, \phi^I$ and that $q_{ab}$ is expressed in terms of $P^a_i$) and obtain

$$C = \{-\kappa \Omega^i_{\alpha b} P^a_j P^b_k \epsilon_{ijk} + \frac{1}{2} a_\alpha^2 + \text{det}(q)(q^{ab} \phi^\alpha_a \phi^\alpha_b + V(\phi))\} + \Lambda \text{det}(q)$$

$$-4(\kappa P^a_i \psi^T \tau_i D_a \psi + \text{det}(q)V(A, \pi_A)) + \frac{g^2}{2} q_{ab} d^{IJ}(E^a_i E^b_j + B^a_i B^b_j)$$

$$\frac{1}{2} \left(\frac{1}{|\phi|^2} \epsilon_{ijk} \phi_j (D_b P^b_k))^2 + \text{det}(q)(\delta_{ij} q^{ab}(D_a \phi^i)(D_b \phi^j) + V(\phi^i))\right]$$

$$+ \frac{1}{2} [d^{IJ} \pi_I \pi_J + \text{det}(q)(d_{IJ} q^{ab}(D_a \phi^I)(D_b \phi^J) + V(\phi^I))]$$

$$+ \{a_\alpha b_\alpha\} h_E + \{\frac{1}{2} b_\alpha^2 + \frac{1}{2} |\phi|^2\}(h_E)^2$$

$$= \alpha (h_E)^2 + \beta h_E + \gamma \tag{3.10}$$

which yields a quadratic equation for $h_E$. Hence, the inclusion of a gravitational and YM Higgs-field together with this new method enables one to obtain the general solution of the initial value constraints for arbitrary matter coupling. Note that the new method introduced is completely independent of the CDJ-framework since the CDJ-matrix does not enter the game at any stage.

### 3.2 Solution without using the scalar fields

The method of section 3.1 (as well as of section 2) has the unattractive feature that explicit use was made of the (possibly spurious) scalar fields. We now show that one can do without them altogether by essentially the same trick if one uses Weyl fields to solve the vector and scalar constraint as well as the gravitational Gauss constraint.

Inserting the basic ansatz (3.3) into the Gauss constraint yields the purely algebraic relation

$$D_a E^a_i = g^2 f_{IJ} K B^{aJ} v_{aK} = -f_{IJ} K \phi^J \pi_K \forall X \in V^\perp \tag{3.11}$$

for $v_{aI}$. There are $3 \times \text{dim}(G)$ independent components of the electric field contained in $v_{aI}$ and $\text{dim}(G)$ constraints, so we can choose $\text{dim}(G)$ of the
components of \( v_{ai} \) to depend on the others as well as on the magnetic fields, the Higgs field and the \( V^\parallel \) part of the Higgs-momenta. We assume that this has been done from now on.

Now one inserts formula (3.11) into the rest of the constraints. We assume that at least 4 spinor fields are present and solve these remaining 7 constraints in terms of 7 components of spinor field momenta by using similar algebraic methods as in eqn. (3.8) ff. If the spinor momenta enter the scalar constraint only linearly, i.e. \( V(\psi^A, \pi_A) = V(\psi^A) \), there is not even the need to solve a quadratic equation which simplifies the relevant formulas tremendously. Hence, one could then write the vector and the scalar constraint as one matrix relation as in eqn. (3.1.8) for the 7 components of spinor momenta chosen and solve for them by methods of linear algebra. If this is not the case, then we require that the spinor potential is of at most 4th order in the spinor momenta (without spatial derivatives) in order that the scalar constraint be solvable by algebraic methods.

We refrain from giving the explicit formulas because the procedure how to get them is identically the same as the one of section (3.1).

Note: in case of the gravitational Gauss constraint one can interpreting the trick, eqn. (3.3), in the following nice way:

Provided that the magnetic fields are non-degenerate we can write \( v_{ai} = m_{ab} B^b_i \), \( m_{ab} \) being an arbitrary tensor density of weight -1. Then the Gauss-law yields

\[
D_a P^a_i = \epsilon_{ijk} B^a_j B^b_k m_{ab} = B B^c \epsilon^{abc} m_{ab} = -\epsilon_{ijk} \phi^i \pi_k
\]

such that the antisymmetric part of the tensor density \( m \) is expressible in terms of the gravitational Higgs-field

\[
\epsilon^{abc} m_{ab} = -\frac{1}{B} B^c \epsilon_{ijk} \phi^i \pi_k
\]

while its symmetric part remains unspecified. Unfortunately, the symmetric part enters the vector and scalar constraint differentiated which excludes the possibility to solve for \( m_{(ab)} \), at least by algebraic methods.
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A Appendix

A.1 The resolution formula for quartic equations

For the benefit of the reader we include here a brief recipe for solving the general quartic equation. The formulas were first found by Cardano and Ferrari (see [2]).

1) Turn the general form of a quartic equation

\[ x^4 + \alpha x^3 + \beta x^2 + \gamma x + \delta = 0 \]  

(A.1)

into its normal form by substituting \( x = y - \alpha/4 \):

\[ y^4 + py^2 + qy + r = 0 \]  

(A.2)

2) Make the ansatz

\[ y^4 + py^2 + qy + r = (y^2 + P)^2 - (Qy + R)^2 \]  

(A.3)

This factorizes into a product of two quadratic equations and can be solved by standard methods. Comparison of coefficients yields

\[ p = 2P - Q^2, \quad q = -2QR, \quad r = P^2 - R^2 \]  

(A.4)

and results in the so-called cubic resolvent

\[ P^3 - \frac{1}{2}pP^2 - rP + \frac{1}{8}q^2 + \frac{1}{2}rp =: P^3 + aP^2 + bP + c = 0 \]  

(A.5)

3) Solve the cubic equation which upon substitution \( P = t - a/3 \) adopts its normal form \( t^3 + et + f = 0 \). Let \( D := (e/3)^3 + (f/2)^2 \) (the discriminant) and

\[ u := \sqrt[3]{-\frac{f}{2} + \sqrt{D}}, \quad v := \sqrt[3]{-\frac{f}{2} - \sqrt{D}} \]  

(A.6)
then the 3 roots are given by

\[ t_1 = u + v, \ t_2 = -\frac{u + v}{2} + i\sqrt{3}\frac{u - v}{2}, \ t_3 = -\frac{u + v}{2} - i\sqrt{3}\frac{u - v}{2} \]  

(A.7)

4) Use one of the roots \( t_1, t_2, t_3 \) of the cubic resolvent to determine \( P, Q, R \) and proceed with formula (A.3).

Since already \( \alpha, \beta, \gamma, \delta \) look horrible when expressed in terms of the independent components of the fields (compare formula (2.23)) we refrain from giving \( x \) in terms of \( \alpha, \beta, \gamma, \delta \) explicitly and rather discuss a feasible example.

### A.2 The extended method of CDJ applied to a special case

First we switch off only the Yang-Mills field (and, necessarily, its associated Higgs-field). Then the scalar constraint reduces to (recall formula (2.21))

\[ f\pi^2 + e\pi + d = 0 \]  

(A.8)

where \( E^a_I = A^I_a = \pi^I = \phi^I = 0 \), also in the expressions for \( a, b, c \) - compare formula (2.18). This is now only a quadratic equation for \( \pi \).

Although the degree of the scalar constraint cannot be lowered further without switching off the Klein-Gordon field altogether, a tremendous simplification occurs when one retains only the gravitational and the KG field because then the matrix \( R \) vanishes. We obtain

\[ \det(\Psi) = [\det(S)] + \pi^2[tr(ST^2) - \frac{1}{2}tr(S)tr(T^2)] =: (a\pi^2 + c)/(2i\kappa B) \]  

(A.9)

i.e. \( b=0 \). The scalar constraint reduces to

\[ C = -\kappa B((tr(S))^2 - tr(S^2) - \pi^2 tr(T^2)) \]
\[ +\frac{1}{2}[\pi^2 - 4\kappa^2 tr(S^2 v \otimes v) + (a\pi^2 + c)V(\phi)] \]
\[ = \{-\kappa B((tr(S))^2 - tr(S^2)) + \frac{1}{2}[\pi^2 - 4\kappa^2 tr(S^2 v \otimes v) + cV(\phi)]\} \]
\[ +\{\kappa B tr(T^2) + \frac{1}{2}[1 + aV(\phi)]\} \pi^2 \]
\[ = f\pi^2 + d \]  

(A.10)

i.e. \( e=0 \), the momentum \( \pi \) enters the scalar constraint without a linear term!

The matrix \( T \) is simply \( T_{ij} = B_{k}^a/(2B)\phi_{a}\epsilon_{ijk} \). 

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