Unveiling the physics of partial differential equations with heuristics

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Abstract

Heuristic arguments and order of magnitude estimates for partial differential equations highlight essential features of the physics they describe. We present order of magnitude estimates, and their limitations, for the three classic second order PDEs of mathematical physics (wave, heat, and Laplace equations), for first order transport equations, and for two non-linear wave equations. It is beneficial to expose the beginning student to these considerations before jumping into more rigorous mathematics. Yet these simple arguments are missing from physics textbooks.

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1 Introduction

Typically, undergraduate physics students are introduced to the classic second order linear partial differential equations (PDEs) of mathematical physics through various courses in the curriculum, maybe taking a course devoted to this subject, but only in upper years. Beginning students usually struggle with PDEs, learning to make sense of them, and how to use them, only after substantial effort. After some practice with the three main classes of second order PDEs of mathematical physics, i.e., hyperbolic, parabolic, and elliptic equations exemplified by the wave, diffusion, and Laplace equations, respectively, some “sloppy math” considerations often arise, which are simple order of magnitude estimates facilitating the understanding of this subject. The early discussion of these observations when first introducing PDEs would be more beneficial. These order of magnitude estimates capture essential features of these PDEs, which are later rediscovered in specific analytical solutions (e.g., Refs.[1, 2]).

The classic wave, heat, and Laplace PDEs embody the essential features of the physical phenomena they describe, which are pillars of an education in physics, and it is best to show the student this physical content before the details. Most mathematics textbooks do not present the derivation of these PDEs from basic physical principles (conservation of energy for the heat equation, Newton’s second law for points of a string displaced from equilibrium for the wave equation, etc.). Even when such derivations are presented, many students fail to realize that the corresponding PDEs exhibit features that have a certain degree of universality. These PDEs describe phenomena that go beyond, say, the wave in a string, and they are important as prototypes for all linear transport, diffusion, or wave-like phenomena. Simple hand-waving introductions to PDEs do not find a place in mathematics or even physics textbooks but are nevertheless very useful for the physics student and should be included in physics courses in some way. Here we summarize semi-qualitative aspects of PDEs that, while certainly apparent to instructors, are probably reputed to be beneath dignified textbooks. Nevertheless, they are greatly appreciated by beginning students in introductory lectures.

2 Linear PDEs

Let us begin with the one-way transport equations in one spatial dimension

\[
\frac{\partial u}{\partial t} \mp c \frac{\partial u}{\partial x} = 0 , \tag{2.1}
\]

where \(c\) is a constant coefficient and, for definiteness, we consider infinite domains \(-\infty < x < +\infty\) in this section. Let \(T\) and \(L\) be, respectively, the time scale and spatial scale of variation of the quantity \(u(t, x)\) that is being transported. Clearly, the coefficient \(c\) describes the speed at which the quantity \(u\) is being transported. The order of magnitude estimates \(u_t \sim u/T\) and \(u_x \sim u/L\) (where \(u_t \equiv \partial u/\partial t\), \(u_x \equiv \partial u/\partial x\), etc.) give

\[
\frac{u}{T} \mp c \frac{u}{L} \sim 0 \tag{2.2}
\]
from which it follows that

\[ L = \pm cT. \]  

Thus, already at first sight, Eq. (2.1) says that some quantity \( u \) is being transported at speed \( c \), covering the distance \( L \) in the time \( T \), to the right if the upper sign is chosen or to the left if the lower sign is applied. This physical intuition motivates guessing the solutions

\[ u_1(t, x) = f(x + ct), \quad u_2(t, x) = g(x - ct) \]  

where \( f \) and \( g \) are regular functions of their arguments, say continuous with their first derivatives. It is then straightforward to verify that \( u_1(t, x) \) describes transport to the left (since \( c \sim -L/T \)) and \( u_2(t, x) \) transport to the right (since \( c \sim +L/T \)). These solutions are just the translations \( f(\xi), g(\eta) \) of static profiles \( f(x) \) and \( g(x) \) to the left or to the right, respectively, with \( x \to \xi \equiv x + ct \) or \( x \to \eta \equiv x - ct \). These Galilean translations express the motion of the whole functions along the line at constant speed to the left or to the right, and hold thanks to the constancy of the coefficient \( c \) (there are also physically meaningful situations in which \( c \) can be considered approximately constant over a certain region, or in which \( c \) varies slowly, which leads to meaningful approximations).

Let us proceed with the one-dimensional homogeneous heat or diffusion equation

\[ \frac{\partial u}{\partial t} = a \nabla^2 u \]  

describing heat conduction in the absence of sources, where \( a \) is the Fourier coefficient (here assumed to be constant) and \( u(t, x) \) is the temperature. Denote with \( T \) the time scale of variation of \( u \) and with \( L \) its spatial scale of variation. We have, in order of magnitude,

\[ \frac{\partial u}{\partial t} \sim \frac{u}{T}, \quad \frac{\partial u}{\partial x} \sim \frac{u}{L}, \quad \nabla^2 u \sim \frac{\partial^2 u}{\partial (x')^2} \sim \frac{u}{L^2}; \]  

then the heat equation (2.5) gives

\[ \frac{u}{T} \sim \frac{a}{L^2} u \]

and

\[ L = \sqrt{aT} \]  

This simple equation provides valuable physical insight: it expresses the characteristic feature of a random walk that the macroscopic distance travelled grows with the square root of time, if one thinks of \( u(t, \vec{x}) \) as something that spreads in a random way. In a random walk, the root mean square distance travelled by the average particle grows with the square root of the number \( N \) of scatterings it experiences, which is a signature of the stochastic nature of diffusion. \( N \) is proportional to the time \( t \) expired from the beginning of the walk, so \( x_{\text{rms}} \propto \sqrt{N} \sim \sqrt{Dt} \), where \( D \) is the diffusion coefficient analogous to the Fourier coefficient for heat conduction.
The simple order of magnitude estimate in the PDE reveals an essential feature of the physical process it describes. (The fact that \( x_{\text{rms}} = \sqrt{Dt} \) is rediscovered by the student in the Gaussian solution of the heat equation,\(^2\) which is the basis for the many Gaussian plume models used to describe the spreading of pollutants in environmental physics.\(^3\)).

Continuing with qualitative considerations, in the heat equation (2.5) the speed of diffusion \( \partial u/\partial t \) of the heat is proportional to the spatial “curvature” \( \nabla^2 u \) of the temperature, which is more intuitive in one dimension where \( \nabla^2 u \) reduces to \( d^2 u/dx^2 \). In first year calculus, students are instructed to regard \( d^2 y/dx^2 \) as a measure of the curvature of the graph of the function \( y(x) \) and they can easily relate to it. (See Ref. \(^4\) for a more nuanced discussion of the meaning of the Laplacian.) The larger this curvature, the faster the process. Finally, the fundamentally irreversible nature of diffusion is highlighted by the non-invariance of Eq. (2.5) under the time reversal \( t \to -t \).

Pass now to the wave equation

\[
\nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0
\]

(2.8)

with constant coefficient \( c \), for which the same order of magnitude reasoning employing time scale \( T \) and length scale \( L \) of variation of the solution \( u(t, \vec{x}) \) gives

\[
\frac{u}{L^2} \sim \frac{1}{c^2} \frac{u}{T^2}
\]

and

\[
L = cT,
\]

(2.9)

which is nothing but the relation

\[
c = \lambda \nu
\]

(2.10)

between wavelength \( \lambda \), frequency \( \nu \), and speed \( c \) of a wave, to which a student relates from algebra-based elementary physics courses (which, however, do not discuss Eq. (2.8)). The naïve order of magnitude approach tells the student that something travels with speed \( c \) and Eq. (2.10) reports the basic fact that distance covered = velocity \( \times \) time. The wave \( u(t, x) \) travels the distance \( L \) in the time \( T = \nu^{-1} \) (the period), and this is the way the relation \( c = \lambda \nu \) is introduced in elementary physics courses.

At this point in the introduction of the wave PDE, it is fruitful to focus on the one-dimensional wave equation

\[
\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0,
\]

(2.11)

and derive the general solution on the infinite line

\[
u(t, x) = f(x + ct) + g(x - ct),
\]

(2.12)
where $f$ and $g$ are regular functions of their arguments (say, continuous with their first and second order derivatives). By formally splitting the d’Alembert operator as

$$\Box \equiv \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = \left( \frac{\partial}{\partial x} + \frac{1}{c} \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial x} - \frac{1}{c} \frac{\partial}{\partial t} \right),$$

the second order one-dimensional wave equation for $u(t, x)$ is equivalent to the set of two first order PDEs

$$\frac{\partial u}{\partial x} + \frac{1}{c} \frac{\partial u}{\partial t} = 0,$$

$$\frac{\partial u}{\partial x} - \frac{1}{c} \frac{\partial u}{\partial t} = 0.$$ 

These are transport equations of the form (2.1) already discussed and help making sense of the fact that the general solution of the wave equation on the infinite line describes waves travelling to the left or to the right. In this discussion, the pulses $f(x + ct)$ and $g(x - ct)$ maintain their profiles unchanged as they propagate, hence they describe waves that do not suffer dispersion.\[5, 6, 7\]

Let us come now to the prototypical elliptic equation, the Laplace equation

$$\nabla^2 u = 0 \quad (2.16)$$

that describes static phenomena. At first sight, the order of magnitude reasoning based on the scale of variation $L$ of $u(x)$ would produce $u/L^2 \approx 0$, or $L \sim \infty$, which is not particularly enlightening and could lead one to conclude that, since nothing travels in this case, the comparison with the heat and the wave equations is fruitless. However, one can do better and note that, contrary to the case of the previous two equations, one can see the Laplace equation as a (first order) equation for the gradient of $u$, instead of $u$ itself. It is often convenient to simplify the situation and look at the special case of one spatial dimension, in which a PDE is as simple as possible. The Laplace equation becomes trivial in one dimension,

$$\frac{d^2 u}{dx^2} = 0, \quad (2.17)$$

and the solution is a straight line. The meaning is that the solutions of the Laplace equation are “as straight as possible”, which reproduces also the property that the solutions assume maxima and minima on the boundary of the domain of integration. These properties survive in higher dimension (this approach is used, e.g., in Ref.\[8\] with great benefit for the student).

In one spatial dimension, the gradient reduces to $u'(x) \equiv du/dx$ and the Laplace equation for $u'$ reads

$$\frac{du'}{dx} = 0, \quad (2.18)$$

4
expressing the fact that the slope $u'$ of the solution $u(x)$ is constant or, the graph of the solution is straight and has no curvature $u''$. With this view, the naive order of magnitude approach gives $u'/L \simeq 0$ or, the length scale of variation of $u'$ is infinite. Then $u'$ does not change: the slope is constant, which means that the graph of the solution $u$ has no curvature.

Moving to two dimensions, $u(x,y)$ can vary in two independent directions $x$ and $y$, but these variations are not independent: the Laplace equation links them. If the curvature of $u$ is positive in the $x$-direction, $u_{xx} > 0$, it must simultaneously be negative in the $y$-direction, $u_{yy} < 0$, to compensate so that the total curvature $u_{xx} + u_{yy}$ of $u$ vanishes identically. This behaviour is exemplified by the harmonic function

$$v(x, y) = x^2 - y^2,$$

for which $v_{xx} = 2$ but $v_{yy} = -2$ so that the sum $v_{xx} + v_{yy}$ vanishes. Another revealing harmonic function is

$$w(x, y) = xy$$

which is linear in each of the independent $x$- and $y$- directions: $w_x = y$ and $w_y = x$, then $w_{xx} = w_{yy} = 0$ satisfying the Laplace equation.

Another elliptic equation that the physics student is bound to encounter is the Helmoltz equation

$$\nabla^2 u + k^2 u = 0.$$ 

(2.21)

Let us refer, specifically, to the problem of solving the wave equation by separation of variables, in which the Helmoltz equation appears.[1, 2] Again, it is useful to first comment on its one-dimensional version

$$\frac{d^2 u}{dx^2} + k^2 u(x) = 0,$$

(2.22)

which is nothing but the harmonic oscillator equation describing oscillations in the variable $x$ (i.e., in space) with angular frequency $k$. By extension, when solving the wave equation by separation of variables, the higher-dimensional Helmoltz equation describes oscillations of the spatial part $u$ of the wave in space (in the direction of the wave vector $\vec{k}$). The order of magnitude estimate using the spatial scale $L$ of variation of the solution $u$ gives

$$k \sim \frac{1}{L},$$

(2.23)

which is the order of magnitude version of the relation $k = 2\pi/\lambda$ between wave vector $k$ and wavelength $\lambda$ for the propagating waves.

3 Beyond linear

The naive order of magnitude estimate still serves well the student who encounters much more complicated non-linear equations. To this regard, standard physics courses only discuss linear,
small-amplitude waves without mentioning the existence of non-linear waves. They ignore what a trip to the beach reveals, *i.e.*, large-amplitude waves in shallow water crashing on the shore. Many undergraduate physics students end their course of studies believing that all waves are linear. It is more honest to mention the existence of non-linear waves in introductory courses, and then restrict the discussion to small-amplitude waves instead of hiding this beautiful subject forever for fear of the mathematics involved. Later on, non-linear waves can still be made accessible to undergraduates.\textsuperscript{[9]}

As an example of the order of magnitude technique applied to non-linear PDEs, consider the Burgers equation

\[
\frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x} \tag{3.1}
\]

describing solitonic waves. Based on what the student now knows about transport equations, it is tempting to view Eq. (3.1) as a transport equation of the form

\[
\frac{\partial u}{\partial t} = c(u) \frac{\partial u}{\partial x} \tag{3.2}
\]

where the velocity \( c(u) = u\) depends on the wave’s amplitude according to \( c = u_t/u_x = u \). The order of magnitude estimate using temporal and spatial scales of variation \( T \) and \( L \) of the solution \( u \) still gives

\[
c = \frac{u_t}{u_x} \sim \frac{u/T}{u/L} = \frac{L}{T}, \tag{3.3}
\]

expressing the fact that something is still being transported, but now this ratio is a function of the amplitude \( u \) of the wave and is smaller for low-amplitude waves and larger for large-amplitude ones. Large-amplitude waves travel faster (*i.e.*, the dispersion depends on the wave’s amplitude), and this dependence makes the Burgers equation non-linear.

Consider now the Korteweg-de Vries (KdV) equation describing one-dimensional non-linear waves in a shallow water channel. Let \( \tau \) and \( \rho \) be the surface tension and density of water, respectively, while \( h \) is the depth of the channel, \( g \) is the acceleration of gravity, and \( u(t,x) \) is the vertical displacement of the water surface from its position of equilibrium. The third order KdV equation is

\[
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} + \gamma u \frac{\partial u}{\partial x} + \epsilon \frac{\partial^3 u}{\partial x^3} = 0, \tag{3.4}
\]

where

\[
\gamma = \frac{3c}{2h} \tag{3.5}
\]

is a parameter describing non-linearity,

\[
\epsilon = c \left( \frac{h^2}{6} - \frac{\tau}{2\rho g} \right) \tag{3.6}
\]
is a parameter describing dispersion, and $c$ is the velocity of linear (low-amplitude) waves, \textit{i.e.}, the velocity that waves would have if the terms weighted by the parameters $\gamma$ and $\epsilon$ were absent. In fact, if $\gamma = \epsilon = 0$, the KdV equation (3.4) reduces to the transport equation (2.1) already discussed, describing transport to the left.

Applying again the order of magnitude reasoning with scales of variation of the solution $u(t,x)$ in time and space $T$ and $L$, we have now

$$\frac{u}{T} \sim - \left[ (c + \gamma u) \frac{u}{L} + \epsilon \frac{u}{L^3} \right]$$

(the negative sign describing transport to the left) and the velocity of propagation is approximately

$$\frac{L}{T} \sim - \left( c + \gamma u + \frac{\epsilon}{L^2} \right) = -c \left[ 1 + \frac{3u}{2h} + \frac{h^2}{6L^2} - \frac{\tau}{2\rho g L^2} \right] \equiv -c \left[ 1 + \frac{3u}{2h} + \frac{h^2}{6L^2} - \frac{\ell_c^2}{L^2} \right],$$

where $\ell_c \equiv \sqrt{\frac{\tau}{\rho g}}$ is the capillary length used in the discussion of dispersion.\[5, 6, 7\] Equation (3.8) makes it clear that the speed of the waves depends on their amplitude as well as on the ratio between wave amplitude and depth of the channel (longer waves “feel” the depth of the channel, which is shallow for them) and the ratio between capillary length and “wavelength”. Because of the unintuitive physics, now it is not possible to justify analytical solutions as done for the linear wave equation, but they are still accessible to the undergraduate.\[9, 10\]

4 Limitations of the approach

By definition, heuristic explanations, order of magnitude estimates, and qualitative arguments are not rigorous. Their usefulness stems from the fact that they provide intuition and physical insight before going through a rigorous course and lengthy chains of arguments. Beginning students benefit from glimpsing key physical features before digesting an entire semester of theorems, proofs, corollaries, and exercises, not only in terms of motivation but also because these discoveries facilitate the understanding of rigorous results and make them more likely to be remembered. While extremely valuable, these glimpses should be tempered by an analysis of their limitations, which we address in this section.

The first issue is that the order of magnitude estimates exposed in the previous section use the notion of characteristic scale, which is extremely valuable and widely used in physics and engineering. In general, however, it is not simple to define the concept of characteristic scale for physical quantities, nor is there a unique definition. In fact, if a physical quantity is expressed by a Fourier representation with many wavenumbers, a characteristic scale is not even defined unambiguously. This is probably the reason why this concept is avoided in introductory courses, yet one is throwing out the baby with the bathwater by doing so.
A convenient (but not unique) approach to defining a characteristic length for quantities satisfying linear differential equations is based on their Fourier decomposition. It consists of introducing, say, the characteristic length scale \( L \) of a physical quantity \( u(t, x) \) at time \( t \) using the average of its wavenumber power spectrum \( \mathcal{P}(t, k) \),

\[
L \equiv \frac{2\pi}{\langle k \rangle},
\]

where

\[
\langle k \rangle = \int_0^{+\infty} dk \mathcal{P}(t, k) k.
\]

Here

\[
\mathcal{P}(t, k) = a |u(t, k)|^2,
\]

\( a > 0 \) is a normalization constant, and

\[
u(t, k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx u(t, x) e^{ikx}
\]

is the Fourier coefficient at time \( t \) of the Fourier component with wavenumber \( k \). Similarly, a characteristic time scale at the position \( x \) can be introduced as the average of the frequency power spectrum \( \mathcal{P}(x, \omega) \),

\[
T \equiv \frac{2\pi}{\langle \omega \rangle},
\]

where \( \mathcal{P}(x, \omega) = b |u(x, \omega)|^2 \). These definitions are intuitive if the spectrum is peaked around a particular wavelength (or wavenumber) and less meaningful otherwise. It is clear that, with these definitions, the concept of length scale depends on the time and that of time scale on the position, which complicates matters. Nevertheless, even though these scales are only defined locally, the concept is still useful because differential equations describe physics locally. There is merit, therefore, in referring to “characteristic scales” when introducing PDEs, invoking intuition rather than entering elaborate discussions about the Fourier representation of physical quantities. This discussion can be postponed to a time when students have already become familiar with Fourier representations and linear PDEs.

A second issue is that characteristic scales are defined in relation with initial (for evolution PDEs) and boundary conditions, which may have their own characteristic scales. Initial conditions may have length or time scales associated with them, and the same can be said of boundary conditions on a finite domain (the size of a finite domain is itself a scale and shows up in the solutions of the PDE, for example waves in a string clamped at both ends or the wavefunction for the quantum particle in a box). Initial-boundary value problems for PDEs and their well-posedness are considered in the presence of initial and boundary conditions, which is certainly mentioned in introductions to PDEs. Initial and boundary conditions, as well as possible variable source terms, may affect characteristic scales.
One can summarize as follows the conditions under which the heuristic arguments of the
previous sections are valid:

- PDEs are considered on an infinite domain (in one dimension, $-\infty < x < +\infty$). This
  assumption guarantees that no scale is associated with the size of a finite domain (or, in
  higher dimension, its shape).

- The PDEs considered have constant coefficients. This assumption may be replaced by
  assuming that these coefficients are approximately constant over a region of interest, as
  done many times in the physics curriculum. This assumption avoids introducing charac-
  teristic scales through these coefficients. For example, in the one-dimensional Schrödinger
  equation for a particle of mass $m$ described by the wave function $\psi(t,x)$
  \[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial^2 x} + V(x) \psi , \]  
  (4.6)
  the potential energy $V(x)$, in general, introduces a spatial scale.

- The PDE is homogeneous or else the source terms do not vary in time and space. When
  this is true, no scales are introduced by the sources.

- If the PDE describes time evolution, the initial conditions contain only one length or
  time scale, $i.e.$, its Fourier spectrum is peaked on a rather narrow range of frequencies or
  wavenumbers. In this case, if a physical quantity $u(t,x)$ is Fourier-decomposed as
  \[ u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \ u(t,k) , \]  
  (4.7)
  then
  \[ \frac{\partial u}{\partial x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \ ik \ u(k) , \]  
  (4.8)
  and
  \[ \frac{u}{\ell} \approx i k_0 u \]  
  (4.9)
  provided that the spectrum peaks around the wavenumber $k_0$.

Finally, if the PDE considered is linear, it is satisfied by single Fourier modes of the physical
quantity $u(t,x)$ obeying it, which has well-defined scales associated with them. Non-linearities
cause mode-mode mixing and the presence of multiple scales when two incommensurate wave
numbers $k_1$ and $k_2$ mix. This is the case, for example, of forward or inverse cascades in fluid
dynamics, where energy is transferred from large to small scales, or vice-versa, respectively.
5 Discussion

Using order of magnitude estimates for PDEs, as suggested in the previous sections, is common practice in dimensional analysis[11] in engineering and applied physics (see, e.g., Refs.[3, 12]), or in fluid mechanics and oceanography[13, 14, 15], where many terms are dropped from the relevant long PDEs, retaining only those that dominate in certain physical regimes. Therefore, there is no real reason for withdrawing this kind of analysis from the introductory discussion of PDEs when beginner students could instead benefit from them. Appealing to mathematical rigour does not justify hiding physical intuition which is very useful as an entry point into PDEs and even more for understanding rigorous results derived later in the course. Indeed, using heuristic arguments and order of magnitude estimates for PDEs with constant coefficients early on has the extra advantage that intuitive features of the physical processes described are rediscovered later in specific analytical solutions of these PDEs, reinforcing the point. As usual, heuristic arguments have limitations, which we have highlighted in Sec.[4] but their benefits outweigh these limitations.

Acknowledgments

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