Method for constructing elliptic curves using complex multiplication
and its optimizations

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Abstract
Elliptic curves over finite fields with predefined conditions in the order are practically constructed using
the theory of complex multiplication. The stage with longest calculations in this method reconstructs some
polynomial with integer coefficients. We will prove theoretical results and give a detailed account of the
method itself and how one can use a divisor of the mentioned polynomial with coefficients in some extension
of the field of rational numbers.

1 Introduction
Elliptic curves play an important role in a variety of different applications. For example, elliptic curves form a
basis for some public-key cryptosystems [1, 2], primality tests [3] and factorization [4] of rational integers. The
applications use elliptic curves over finite fields with the order satisfying several restrictions. For instance, for
cryptographical applications the order should be a prime number or, at least, should have a large prime divisor.

One of methods for constructing elliptic curves with predefined restrictions on the order is the following.
First, we generate an equation of an elliptic curve with random coefficients. Next, we calculate the order of
the generated curve and check whether the order satisfies the predefined conditions. If so, the construction is
done; otherwise, we repeat the process from the beginning. Possible values for arising orders are distributed
approximately uniformly (the precise statement for prime fields with characteristic greater than 3 can be found
in [3]). The calculation of order of an elliptic curve has a polynomial complexity. However, in practice the
complexity grows quite fast, so this method is quite slow.

The complex multiplication gives another, more practical method for constructing elliptic curves with pre-
defined restrictions on the order. This article is devoted to the complex multiplication method. Here we start
with calculating an order satisfying the predefined conditions and then construct an elliptic curve with this
order. Section 2 describes the details and some known optimizations.

We suggest a new optimization for calculations. Further theoretical results, used by this optimization, are
proved in Sections 3 and 4. We describe the overall approach in Section 5 and the details in next sections.

2 The CM method
2.1 Theoretical basis
Hereafter we always assume that $D \in \mathbb{Z}$ satisfies the following condition:

$$ D < 0 \text{ and either } D \equiv 0 \pmod{4} \text{ or } D \equiv 1 \pmod{4}. $$

(1)

Consider the field $K = \mathbb{Q}(\sqrt{D})$. Let $d$ be the discriminant of $K$. Then $d < 0$ and

- either
  $$ d \equiv 1 \pmod{4} \text{ and } d \text{ is square-free}, $$
  (2)

- or
  $$ d \equiv 0 \pmod{4}, \quad \frac{d}{4} \text{ is square-free}, \quad \frac{d}{4} \not\equiv 1 \pmod{4}. $$

(3)

In addition, $D = f^2d$, where $f \in \mathbb{N}$. Let $\mathcal{O} = \mathbb{Z}\left[\frac{d+\sqrt{d}}{2}\right]$ be the ring of algebraic integers in the field $K$. Let $\mathcal{O}_D = \mathbb{Z}\left[\frac{d+\sqrt{d}}{2}\right]$ be the order in $\mathcal{O}$ with conductor $f$. For any number field $M$ we denote the ring of integers
for $M$ by $\mathcal{O}_M$; e.g. $\mathcal{O}_K = \mathcal{O}$.
We define a fractional ideal of the order $\mathcal{O}_D$ as a subset of $K$ which is a finitely generated $\mathcal{O}_D$-module and contains a nonzero element. We define an ideal of $\mathcal{O}_D$ as a fractional ideal which is a subset of $\mathcal{O}_D$ (this is the same as the standard definition of ring ideal except for $(0)$, which is not considered). We define a proper (fractional) ideal of the order $\mathcal{O}_D$ as a (fractional) ideal $a$ such that \( (\beta \in K : \beta a \subset a) = \mathcal{O}_D \). All proper fractional ideals of an order form an abelian group under the multiplication of ideals (\cite{5} §7). We denote this group by $I(\mathcal{O}_D)$. It is easy to see that principal fractional ideals, i.e. $\alpha \mathcal{O}_D$ with $\alpha \in K^*$, form a subgroup in $I(\mathcal{O}_D)$. We denote this subgroup by $P(\mathcal{O}_D)$. Two ideals $a$ and $b$ are equivalent when they differ by multiplication by a principal ideal. It is easy to see that this relation is an equivalence relation; we denote it by $a \sim b$. For the sake of brevity we call a class of equivalent proper fractional ideals an ideal class. Since $P(\mathcal{O}_D)$ is a subgroup, all ideal classes form a factorgroup $\mathcal{H}_D = I(\mathcal{O}_D)/P(\mathcal{O}_D)$. It is called the ideal class group of $\mathcal{O}_D$. The ideal class group is a finite abelian group (\cite{5} §7). Since $\mathcal{O}_D$ and $K$ are invariant under complex conjugation, the conjugation of a fractional ideal as a set is itself a fractional ideal; the complex conjugation induces a well-defined operation on $\mathcal{H}_D$.

We define a quadratic form as an expression of the form $Ax^2 + Bxy + Cy^2$, where $A, B, C \in \mathbb{Z}$. We also use $(A, B, C)$ as another notation for the same quadratic form. We define a discriminant of the quadratic form as $B^2 - 4AC$. Two forms are equivalent if one can be transformed to another using change of variables $x' = ax + by, y' = cx + dy$ with $a, b, c, d \in \mathbb{Z}, ad - bc = 1$; it is easy to see that this is indeed an equivalence relation. The quadratic form $(A, B, C)$ is positive definite if $A > 0$ and $B^2 - 4AC < 0$; it is primitive if $\gcd(A, B, C) = 1$. Hereafter we consider only primitive positive definite forms of the discriminant $D$, calling them just forms for the sake of brevity. We define the root of the form as the (only) root $\tau$ of the equation $Ax^2 + Bx + C = 0$ from the upper half-plane $\mathbb{H} = \{ z \in \mathbb{C} : \text{Im } z > 0 \}$, i.e. $\tau = \frac{-B + \sqrt{D}}{2A} \in K \cap \mathbb{H}$. The form $(A, B, C)$ is reduced when $|B| \leq A \leq C$ and if $B < 0$, then $|B| < A < C$. Every form is equivalent to exactly one reduced form (\cite{5} Theorem 2.8).

There is one-to-one correspondence between elements of the group $\mathcal{H}_D$ and reduced forms. We denote this correspondence by $\mathfrak{i}$. Namely (\cite{5} Theorem 7.7), a form $\xi = (A, B, C)$ with the root $\tau$ corresponds to a class $\mathfrak{i}(\xi) = \mathfrak{i}(A, B, C)$ of $\mathcal{O}_D$-ideals containing $(1, \tau)\mathbb{Z}$ (which is a proper fractional $\mathcal{O}_D$-ideal), and two equivalent forms correspond to the same ideal class.

It is easy to enumerate all reduced forms: obviously, such a form has $|B| \leq A \leq \sqrt{4AC}$ and for fixed $A, B$ there exists at most one $C$. So reduced forms give a convenient way to organize elements of $\mathcal{H}_D$.

The classical $j$-invariant is the function from the upper half-plane $\mathbb{H}$ to $\mathbb{C}$ (\cite{6} §46). It can be also defined on lattices in $\mathbb{C}$ (\cite{5} §10) so that it does not change when a lattice is multiplied by any nonzero complex number and $j(\tau) = j((1, \tau)\mathbb{Z})$ for $\tau \in \mathbb{H}$. Any proper fractional $\mathcal{O}_D$-ideal $a$ is also a lattice in $\mathbb{C}$. Obviously, $j(a)$ depends only on the ideal class of $a$. From the computational point of view, if the fractional ideal $a$ belongs to the ideal class corresponding to the form $Ax^2 + Bxy + Cy^2$ with the root $\tau$ (i.e. $a \sim (1, \tau)\mathbb{Z}$), then $j(a) = j(\tau)$.

For any elliptic curve and $n \in \mathbb{Z}$ we define the map $[n]$ which maps a point $P$ to $nP$. In particular, $[1]$ is the identity map. We define a isogeny of two elliptic curves as a morphism (in the sense of algebraic geometry) which maps the infinite point of the first curve to the infinite point of the second curve. We define an endomorphism of an elliptic curve as an isogeny of the curve to itself. For any $n \in \mathbb{Z}$ and any elliptic curve the map $[n]$ is an endomorphism (\cite{7} Example III.4.1) and commutes with any other endomorphism (because any isogeny is a homomorphism of groups of points due to \cite{7} Example III.4.8); the ring of endomorphisms of any elliptic curve is a $\mathbb{Z}$-module with an action $nP = [n] \circ \varphi$, where $n \in \mathbb{Z}, \varphi$ is an endomorphism. Endomorphisms $\{[n] : n \in \mathbb{Z}\}$ form the ring isomorphic to $\mathbb{Z}$ (\cite{7} Proposition III.4.2).

The ring of endomorphisms of an elliptic curve over $\mathbb{C}$ is either equal to $\{[n] : n \in \mathbb{Z}\}$ or isomorphic to an order in some imaginary quadratic field (\cite{7} Corollary III.9.4 and Exercise 3.18b)). In the last case the curve is said to have complex multiplication by this order. There exists exactly $|\mathcal{H}_D|$ nonisomorphic elliptic curves with complex multiplication by $\mathcal{O}_D$ (\cite{7} Proposition C.11.1)). These curves can be characterized as follows: the $j$-invariant of a curve equals one of values of modular $j$-invariant in an ideal representing an ideal class for $\mathcal{O}_D$. These values are called singular values (of the function $j$). Any singular value generates over $\mathbb{K}$ the same field $L = L(D) = K(j(\alpha))$, which is called the ring class field for $\mathcal{O}_D$ (\cite{5} Theorem 11.1). The Galois group of the extension $L/K$ is isomorphic to $\mathcal{H}_D$ (\cite{5} §9). We denote the canonical isomorphism by $\Omega$; $\Omega$ maps an ideal class containing $b$ to the automorphism mapping $j(a)$ to $j(ab^{-1})$ (\cite{5} Corollary 11.37). The complex conjugation acts as follows: $j(\overline{a}) = j(\overline{b})$ by the definition of $j$ (\cite{5} §10), therefore $\Omega(\mathfrak{a}) = j(\mathfrak{a}^{-1})$.

Let us consider the polynomial $H_D(j)(x) = \prod_{i=1}^{\alpha} (x - j(\alpha_i))$, where $h = |\mathcal{H}_D|$ and $\alpha_i$ represent all ideal classes of $\mathcal{O}_D$. The coefficients of $H_D$ are elements of $L$, are invariant under the action of $\text{Gal}(L/K)$ and complex conjugation, therefore, they lie in $\mathbb{Q}$. Moreover, the values $j(\alpha_i)$ are algebraic integers (\cite{2} Theorem 11.1)), so $H_D(j)(x) \in \mathbb{Z}[x]$.

Let $p$ be a prime number, $n$ a natural number, $q = p^n$. Let $E$ be an elliptic curve defined over the finite field $\mathbb{F}_q$. Unless explicitly specified, we consider $\mathbb{F}_q$-points and $\mathbb{F}_q$-endomorphisms of the curve $E$. The order of the curve is the number of $\mathbb{F}_q$-points. The ring of endomorphisms $\text{End}(E)$ is isomorphic either to an order in some imaginary quadratic field or to an order in some quaternion algebra over $\mathbb{Q}$ (\cite{7} Corollary III.9.4 and Theorem
V.3.1]). In the last case the curve $E$ is called supersingular, and we are not interested in these. In the first case $\text{End}(E) \cong \mathcal{O}_D$ for some $D$ satisfying \((\text{I})\): $\text{End}(E) = \langle [1], \alpha \rangle \mathbb{Z}$, where $\alpha$ is some endomorphism of $E$. It appears (\cite{S} Theorem 13.14) that the curve and one endomorphism $\alpha$ can be “lifted” to $\mathbb{C}$ in the following sense: there exists a number field $L$, an elliptic curve $E'$ defined over $L$, an endomorphism $\alpha'$ of $E'$, an ideal $\mathfrak{B}' \subseteq \mathcal{O}_L$, lying above $p$ (i.e. $\mathfrak{B}' \cap \mathbb{Z} = p\mathbb{Z}$) and a reduction of $E'$ modulo $\mathfrak{B}'$ so that the reduced curve is isomorphic to $E$ and $\alpha'$ corresponds to $\alpha$ under the reduction. Since $\alpha \notin \{[n] : n \in \mathbb{Z}\}$, we have $\alpha' \notin \{[n] : n \in \mathbb{Z}\}$, so $\text{End}(E') \neq \mathbb{Z}$ and $E'$ has a complex multiplication by some order in some imaginary quadratic field. Due to properties of reduction (\cite{S} Theorem 13.12), it induces an isomorphism of $\text{End}(E')$ to a subring in $\text{End}(E)$. Since $\alpha'$ reduces to $\alpha$, we have $\text{End}(E') \cong \text{End}(E) \cong \mathcal{O}_D$.

The ring $\text{End}(E)$ contains a Frobenius isogeny $Fr : (x, y) \mapsto (x^q, y^q)$ and the dual isogeny $\overline{Fr}$. We have $Fr \circ \overline{Fr} = [q]$ (\cite{Th} Theorem III.6.2 and Proposition 2.11)) and $|\text{End}(\mathbb{F}_q)| = |\text{Ker}([1] - Fr)| = ([1] - Fr) \circ ([1] - \overline{Fr})$ (the first equality follows from the fact that $Fr$ fixes $\mathbb{F}_q$-points and only them; the second equality follows from \cite{Th} Theorem III.4.10, Corollary III.5.5, Theorem III.6.2]). Let $\pi \in \mathcal{O}_D \cong \text{End}(E)$ be the element corresponding to $Fr$ and $\overline{\pi}$ be the element corresponding to $\overline{Fr}$. Then $\pi \overline{\pi} = q$ and $(1 - \pi)(1 - \overline{\pi}) = |\text{End}(\mathbb{F}_q)|$. In particular, if $\pi \not\in \mathbb{R}$, these equations imply that $\overline{\pi}$ is indeed a complex conjugate to $\pi$; otherwise, $\pi \in \mathbb{R} \cap \mathcal{O}_D = \mathbb{Z}$, so $Fr = [\pi]$ and in this case $\overline{Fr} = Fr$ (\cite{Th} Theorem III.6.2)), hence $\overline{\pi}$ is equal to complex conjugate to $\pi$ too.

Since $\pi \in \mathcal{O}_D$, there exist $u, v \in \mathbb{Z}$ such that $\pi = u + \sqrt{D}/2$. Then $\pi = u - \sqrt{D}/2$ and $q = \pi \overline{\pi} = u^2 - D/v^2$ or
\[
4q = u^2 + |D|v^2.
\]
The order of $E$ is $1 - \pi - \overline{\pi} + \pi \overline{\pi} = q + 1 - u$. Due to \cite{Th} Exercise 5.10 the non-supersingularity of $E$ implies $\gcd(q, u) = 1$.

Let us sum up the above. If $E$ is a non-supersingular elliptic curve over $\mathbb{F}_q$, then there exist an integer $D$, a number field $L'$ and an elliptic curve $E'$ over $L'$ such that

- $E'$ has a complex multiplication by $\mathcal{O}_D$,
- there exists a reduction of $E'$ isomorphic to $E$,
- the order of $E$ is $q + 1 - u$, where $u \in \mathbb{Z}$ is such that for some $v \in \mathbb{Z}$ the equality $4q = u^2 + |D|v^2$ holds.

### 2.2 Basic algorithm

We want to go in the other direction and construct such curves $E$. In order to do this, we implement the following scheme.

1. Select the numbers $q = p^n$, $p$ is a prime, and $\hat{u}, \hat{v}, D \in \mathbb{Z}$ such that $D$ satisfies \((\text{I})\),
\[
4q = \hat{u}^2 + |D|\hat{v}^2, \quad (4)
\]
and the field size $q$ and the order of a future curve $q + 1 - \hat{u}$ satisfy the predefined restrictions required by concrete applications.

2. Calculate the polynomial $H_D[j](x)$.

3. Reduce the polynomial $H_D[j](x)$ modulo $p$. Obtain the polynomial over $\mathbb{F}_p \subset \mathbb{F}_q$; this polynomial (as we will show) splits into linear factors in $\mathbb{F}_q$. Calculate any root of this polynomial. Generate an elliptic curve $E''$ over $\mathbb{F}_q$ such that its $j$-invariant equals the found root.

4. The curve $E''$ has complex multiplication by $\mathcal{O}_D$. An isomorphism does not change the ring of complex multiplication, but can change the number of $\mathbb{F}_q$-points. Construct the curve isomorphic to $E''$ with the order $q + 1 - \hat{u}$.

We define the Kronecker symbol $\left(\frac{a}{b}\right)$, where $a \in \mathbb{Z}$, $b \in \mathbb{N}$, as follows. If $b$ is an odd prime, the Kronecker symbol equals the Legendre symbol. If $b = 2$, the Kronecker symbol is defined only for $a \equiv 0, 1 \pmod{4}$ and equals
\[
\left(\frac{a}{2}\right) = \begin{cases} 
1, & \text{if } a \equiv 1 \pmod{8}, \\
-1, & \text{if } a \equiv 5 \pmod{8}, \\
0, & \text{if } a \equiv 0 \pmod{4}.
\end{cases}
\]

In the general case the Kronecker symbol is defined as being multiplicative in $b$.

The conditions \((\text{I})\) and \((\text{II})\) impose quite strong restrictions on $q$ and $p$. In particular, the following lemma holds.
Lemma 1. Let $d < 0$ satisfy one of conditions (2) or (3), $D = df^2$. Assume that $q = p^v$ and some integers $u, v$ satisfy the conditions

$$4q = u^2 + |D|v^2, \quad \gcd(q, u) = 1.$$ 

Then

1. $\left(\frac{D}{p}\right) = (\frac{d}{p}) = 1;$

2. $p \nmid f$;

3. $p\mathcal{O} = p\mathfrak{p}^n$, where $p \neq \mathfrak{p}$ are prime ideals of $\mathcal{O}$;

4. $\frac{u^2 + \sqrt{D}}{2} \in \mathcal{O}_D$;

5. $\frac{u + v\sqrt{D}}{2} \mathcal{O} = p^n$ or $\frac{u + v\sqrt{D}}{2} \mathcal{O} = \mathfrak{p}^n$.

Proof. The equality $4q = u^2 + |D|v^2$ and the condition $p \nmid u$ imply that $p \nmid D$ and $p \nmid v$. Reducing modulo $p$, we obtain $u^2 - Du^2 \equiv 0 \pmod{p}$, so $D \equiv (uv^{-1})^2 \pmod{p}$. To conclude the proof of the first assertion, it remains to note that $D = df^2$.

The second assertion follows obviously from $p \nmid D$.

The third assertion follows from the first one due to the well-known fact from the theory of quadratic fields (e.g. [9] Propositions 13.1.3 and 13.1.4).

To prove the fourth assertion, we reduce the equality $4q = u^2 + |D|v^2$ modulo $2$. If $D$ is even, then $u$ is even, $\mathcal{O}_D = Z\left[\frac{D + \sqrt{D}}{2}\right] = Z\left[\sqrt{2}\right]$ and $\frac{u^2 + \sqrt{D}}{2} = \frac{u}{2} + v\sqrt{2} \in \mathcal{O}_D$. If $D$ is odd, then $u \equiv v \pmod{2}$, $\mathcal{O}_D = Z\left[\frac{D + \sqrt{D}}{2}\right] = Z\left[\frac{1 + \sqrt{D}}{2}\right]$ and $\frac{u^2 + \sqrt{D}}{2} = \frac{u}{2} + \frac{1}{2}\sqrt{2} \in \mathcal{O}_D$.

Finally, note that

$$\frac{u + v\sqrt{D}}{2} \mathcal{O} = \frac{u - v\sqrt{D}}{2} \mathcal{O} = q\mathcal{O} = p^n\mathcal{O} = p^n\mathfrak{p}^n.$$

Since $p \nmid u$, we have $p\mathfrak{p} = p\mathcal{O} \supseteq \frac{u + v\sqrt{D}}{2} \mathcal{O}$. The last assertion follows from the uniqueness of the factorization to prime ideals in $\mathcal{O}$.

Now we discuss some implementation details of the generic scheme.

The implementation of the first stage depends on restrictions for the field size $q$ and the curve order.

If $q$ is fixed, scan over integers $D$ satisfying (1). For every $D$, first check the necessary condition (3); if it does not hold, continue to the next $D$. Assume that $D$ satisfies (3). Apply the Cornacchia algorithm (10) that solves the equation $x^2 + |D|y^2 = m$, to $m = 4q$. If there is no solution, continue to the next $D$. If a solution is found, check whether $q + 1 \pm x$ satisfies the restrictions for the order.

If $q$ is not fixed, it is more efficiently to fix $D$ instead of the previous method. First, fix $D$ satisfying (1). Next, generate $\hat{u}, \hat{v}$ at random and calculate from (\ref{1}) and $q + 1 \pm \hat{u}$; repeat until the required restrictions are met. Some improvements of this method are suggested in \cite{11} and \cite{12}. In essence, these improvements implement the following idea: one can select parameters $\hat{u}, \hat{v}$ less randomly and guarantee the absence of small prime divisors of $q = \frac{\hat{u}^2 + |D|\hat{v}^2}{4}$ and $q + 1 \pm \hat{u}$ (or, at least, decrease the probability of such divisors). As an example, assume the following restrictions: $p$ is odd and one of $q + 1 \pm u$ is an odd prime (that is the case in \cite{12}). It is easy to see that $D \equiv 5 \pmod{8}$ and $\hat{u}, \hat{v}$ must be odd. \cite{12} suggests starting from $\hat{u} = 210\hat{u}_0 + 1, \hat{v} = 210\hat{u}_0 + 105, \hat{u}_0, \hat{v}_0$ are random integers; if the initial values are bad, continue with adding to $\hat{u}$ numbers 106 and 104 = 210 - 106 in turn. Note that $210 = 105 \cdot 2 = 2 \cdot 3 \cdot 5 \cdot 7$. This choice guarantees that $\frac{\hat{u}^2 + |D|\hat{v}^2}{4}$ and one of $q + 1 \pm \hat{u}$ do not divide by 2, 3, 5, 7. The method from \cite{11} uses more small divisors and is cumbersome, so we do not quote it here. The performance of different methods is compared in \cite{12}.

The second stage consists of calculating the polynomial $H_D[j](x)$. Enumerate all reduced forms (there are $h = |\mathcal{H}_D|$ of them). Calculate their roots $\tau_1, \ldots, \tau_h$. Calculate the values $j(\tau_1), \ldots, j(\tau_h)$ as complex numbers with sufficiently large precision. Calculate the coefficients of the polynomial $H_D[j](x)$ approximately. If the precision is large enough, then possible error in coefficients is less than $\frac{1}{2}$ and the exact values (which are integer numbers) can be calculated by rounding.

For a number field $M$, a prime ideal $\mathfrak{C} \subseteq \mathcal{O}_M$ and $z \in \mathcal{O}_M$ we denote by $R_{\mathfrak{C}}(z)$ the reduction of $z$ modulo $\mathfrak{C}$. So $R_{\mathfrak{C}}$ is a map from $\mathcal{O}_M$ to a finite field. The map $R_{\mathfrak{C}}$ also acts on polynomials from $\mathcal{O}_M[x]$, reducing every coefficient.

For the third stage we must show that the polynomial

$$R_{\mathfrak{C}}(H_D[j](x))$$

\hspace{1cm} 4
splits into linear factors in \( F_q \). Also we must construct an elliptic curve by its \( j \)-invariant.

Lemma 1 implies that \( p\mathcal{O} = p\mathbb{F} \). Let \( \mathfrak{B} \subset \mathcal{O}_L \) be a prime ideal lying above \( p \). Since \( \mathfrak{B} \cap \mathbb{Z} = p \cap \mathbb{Z} = p\mathbb{Z} \), we have

\[
R_{p\mathfrak{B}}(H_D[j](x)) = R_\mathfrak{B}(H_D[j](x)) = \prod_{i=1}^k (x - R_\mathfrak{B}(j(\alpha_i)))
\]

so \( R_{p\mathfrak{B}}(H_D[j](x)) \) splits into linear factors in \( \mathcal{O}_L/\mathfrak{B} \). Therefore, it remains to prove the following theorem.

**Theorem 1.**

\( \mathcal{O}_L/\mathfrak{B} \subset F_q. \)

**Proof.** Let \( \mathfrak{c} \subset \mathcal{O} \) be a prime ideal unramified in \( L \). Let \( \mathfrak{C} \subset \mathcal{O}_L \) be a prime ideal lying above \( \mathfrak{c} \). Let \( \left( \frac{L/K}{\mathfrak{c}} \right) \) denote the Artin symbol \([5, \S 5]\) (it is defined for any Galois extension \( K \subset L \), but we use it only for the fields \( K \) and \( L \) defined above), namely, the unique \((5, \text{Lemma 5.19})\) element \( \sigma \in \text{Gal}(L/K) \) such that \( \sigma(\alpha) \equiv \alpha^{\text{Norm}(\mathfrak{c})} \) (mod \( \mathfrak{C} \)) for any \( \alpha \in \mathcal{O}_L \). Since \( \text{Gal}(L/K) \equiv \mathcal{H}_D \) is Abelian, the Artin symbol depends only on \( \mathfrak{c} \) \((5, \text{Corollary 5.21})\) and can be denoted as \( \left( \frac{L/K}{\mathfrak{c}} \right) \). For a fractional \( \mathcal{O} \)-ideal \( \mathfrak{b} = \mathfrak{c}_1^a \ldots \mathfrak{c}_s^b \) we define \( \left( \frac{L/K}{\mathfrak{b}} \right) = \left( \frac{L/K}{\mathfrak{c}_1} \right)^{a_1} \ldots \left( \frac{L/K}{\mathfrak{c}_s} \right)^{a_s} \). The map \( \left( \frac{L/K}{\mathfrak{b}} \right) \) is a homomorphism from the group of those fractional \( \mathcal{O} \)-ideals whose factorization does not contain prime ideals ramified in \( L \), to the group \( \text{Gal}(L/K) \). This homomorphism is called the Artin map.

Let \( P_{K,Z}(f) \) \((5, \S 9)\) denote the subgroup of fractional \( \mathcal{O} \)-ideals generated by principal ideals of the form \( \alpha \mathcal{O}, \alpha \equiv a \pmod{\mathfrak{c} \mathcal{O}} \), for some \( a \in \mathbb{Z} \) with \( \text{gcd}(a, f) = 1 \). According to \((5, \S 9)\), the ring class field \( L \) for \( \mathcal{O}_D \) is the unique abelian extension of \( K \) such that

- all prime ideals \( \mathfrak{O} \) ramified in \( L \) (consequently, all ideals from \( P_{K,Z}(f) \) are unramified in \( L \): if \( \alpha \equiv a \pmod{\mathfrak{c} \mathcal{O}} \), then \( \text{gcd}(\alpha \mathcal{O}, f) = \text{gcd}(a \mathcal{O}, f) = \text{gcd}(a, f) \mathcal{O} = \mathcal{O} \), so \( \alpha \mathcal{O} \) is prime to \( f \mathcal{O} \),

- the kernel of the Artin map is \( P_{K,Z}(f) \).

Let \( \hat{\pi} = \frac{a + \sqrt{-3}}{2} \); Lemma 1 implies that either \( p^n = \hat{\pi} \mathcal{O} \) or \( p^n = \hat{\pi} \mathcal{O} \). In both cases the ideal \( p^n \) is principal and lies in \( P_{K,Z}(f) \) (this is easy to see from \( \hat{\pi} = \frac{a + \sqrt{-3}}{2} \). \( \mathcal{O} = \frac{a + \sqrt{-3}}{2} \in \mathcal{O} \), \( \frac{a + \sqrt{-3}}{2} \in \mathcal{O} \), from the definition of \( P_{K,Z}(f) \), Lemma 1 and \((5, \S 9)\)). Therefore, \( p^n \) lies in the kernel of the Artin map. Equivalently, \( \left( \frac{L/K}{p} \right)^n = 1 \mathcal{O} \). The automorphism \( \left( \frac{L/K}{p} \right) \) acts on \( \mathcal{O}_L/\mathfrak{B} \) as \( x \mapsto x^{\text{Norm}(p)} = x^p \), so its \( n \)-th power acts as \( x \mapsto x^{n^p} = x^n \). This means that the operation \( x \mapsto x^n \) acts trivially on \( \mathcal{O}_L/\mathfrak{B} \), which is possible only if \( \mathcal{O}_L/\mathfrak{B} \subset F_q \). \( \square \)

Using formulas from \([7, \text{Proposition A.1.1}]\), it is easy to check that for \( j \in F_q \) or \( j \in C \) the following curves, defined over \( F_q \) or \( C \) respectively, have the \( j \)-invariant equal to \( j \):

- if the field characteristic is 0 or greater than 3, when \( j \neq 0, j \neq 1728 \): \( y^2 = x^3 + 3cx + 2c \), where \( c = \frac{x}{1728 - x} \);

- if the field characteristic is 0 or greater than 3, when \( j = 0 \): \( y^2 = x^3 + 1 \);

- if the field characteristic is 0 or greater than 3, when \( j = 1728 \): \( y^2 = x^3 + x \);

- over the field \( F_q \) of the characteristic 2, when \( j \in F_q^* \): \( y^2 + xy = x^3 + j^{-1} \);

- over the field \( F_q \) of the characteristic 3, when \( j \in F_q^* \): \( y^2 = x^3 + 3x^2 - j^{-1} \).

The missing cases with \( j = 0 \) in characteristics 2 and 3 correspond to supersingular curves \((7, \text{Exercise 5.7, Theorem 4.1})\), so they cannot arise.

For the fourth stage we must show that the curve \( E'' \) (which is constructed in the third stage) has complex multiplication by \( \mathcal{O}_D \) (in particular, it is non-supersingular).

It follows from the construction of the curve \( E'' \) and from \([7, \text{Lemma 5.1.2}]\) that the \( j \)-invariant of \( E'' \) equals \( R_{\mathfrak{B}}(j(\alpha)) \), where \( \alpha \) is some proper fractional \( \mathcal{O}_D \)-ideal. Since \( j(\alpha) \in \mathcal{O}_L \), there exists \((12, \S 4.3)\) a finite extension \( L' \) of \( L \), a curve \( E' \) defined over \( L' \) and a prime ideal \( \mathfrak{B}' \subset \mathcal{O}_{L'} \) lying above \( \mathfrak{B} \) such that \( j(E') = j(\alpha) \) and the equation of \( E' \) reduced modulo \( \mathfrak{B}' \) gives a non-singular curve (over a finite field). Since \( j(E') \) equals a singular value, \( E' \) has complex multiplication by \( \mathfrak{B} \). Since \( j(E') \) equals the reduced \( j \)-invariant (because \( j \)-invariant is a rational function in coefficients) and \( \mathfrak{B}' \cap \mathcal{O}_L = \mathfrak{B} \), \( j \)-invariant of the curve \( E' \) reduced modulo \( \mathfrak{B}' \) equals \( R_{\mathfrak{B}}(j(\alpha)) = R_{\mathfrak{B}}(j(\alpha)) = j(E'') \). Two elliptic curves are isomorphic if and only if their \( j \)-invariants are equal \((7, \text{Proposition III.1.4})\), therefore \( E'' \) is isomorphic to the reduction of \( E' \). Finally, Lemma 1 and the properties of the reduction \((8, \text{Theorem 13.12})\) imply that \( E'' \) is non-supersingular and \( \text{End}(E'') \cong \text{End}(E') \cong \mathcal{O}_D \).
Therefore, the fourth stage starts with the curve $E''$ defined over $\mathbb{F}_q$, which has complex multiplication by $O_D$. As shown above, the order of the curve $E''$ equals $q + 1 - u$, where $4q = u^2 + |D|v^2$ and gcd$(q, u) = 1$, but $u, v$ are not necessarily equal to $\hat{u}, \hat{v}$. Let $\pi = \frac{-\sqrt{4q}}{q} \in O_D \subset O$. Due to Lemma 1 there is either $\pi O = \mathfrak{p}^n$ or $\pi O = \mathfrak{p}$. The same holds for $\hat{\pi}$. Thus, there is either $\pi O = \hat{\pi} O$ or $\pi O = \pi O$. Since the norm of the $O_D$-ideal $\pi O_D$ equals $|\text{Norm}(\pi)| = q$ and is prime to the conductor $f$ of the order $O_D$, we have $\pi O_D = \pi O \cap O_D$. Similarly, we have $\hat{\pi} O_D = \hat{\pi} O \cap O_D$. Thus, there is either $\pi O_D = \hat{\pi} O_D$ or $\pi O_D = \pi O_D$. Equivalently, the number $x$ is associated either with $\hat{\pi}$ or with $\pi$ in the ring $O_D$.

It is well known (e.g. [3, Proposition 13.1.5]) that the group of units in $O_D$ is $\{\pm 1\}$ if $D \not\equiv \{-3, -4\}$, $(\pm 1, \pm \Omega, \pm \Theta)$ if $D = -3$, $\{\pm 1, \pm \Omega\}$ if $D = -4$. Here $\zeta = e^{2\pi i/3} = \frac{-1 + \sqrt{-3}}{2}$.

If $D \not\equiv \{-3, -4\}$, we have $x = \pm \zeta$ or $x = \pm \bar{\zeta}$. This corresponds to $u = \pm \hat{u}$. Therefore, in this case $|E''(\mathbb{F}_q)| = q + 1 - \hat{u}$. If $|E''(\mathbb{F}_q)| = q + 1 - \hat{u}$, the curve $E''$ is the one we search for. Otherwise, we construct the quadratic twist of $E''$ as follows. If $p \not\equiv 2$, the normal Weierstrass form of the curve equation is $y^2 = f(x)$, where $f$ is a polynomial of degree 3 with the highest-order coefficient equal to 1 (in particular, the formulas above give the equation in this form), and the curve $y^2 = c^3 f(x/c)$, where $c$ is any quadratic non-residue in $\mathbb{F}_q$, has the required order ($|\mathcal{E}|$). If $p = 2$, the normal form is $y^2 + xy = x^3 + a_2x^2 + a_6$ and the curve $y^2 + xy = x^3 + (a_2 + \gamma)x^2 + a_6$, where $T_{\mathcal{R}_2, \beta_2} \gamma = 1$, has the required order ($|\mathcal{E}|$).

If $D = -3$, the procedure for calculating $H_3[j]$ yields the polynomial $H_{-3}[j](x) = x$, it has the only root $j = 0$. The formula (10) in this case is $\left(\frac{D}{p}\right) = 1$ and implies $p \equiv 1$ (mod 3), in particular, $p > 3$. Any curve of the form $y^2 = x^3 + b, b \neq 0$, has the $j$-invariant equal to 0 (Theorem A.1.1), and all such curves are $\mathbb{F}_q$-isomorphic (because they have the same j-invariant).

Let $\chi$ be the unique multiplicative character on $\mathbb{F}_q$ of order 2. Let $S_2(b) = \sum_{x \in E_q(x)} \chi(x^3 + b)$. It is easy to see that the order of the curve $y^2 = x^3 + b$ is equal to $q + 1 - S_2(b)$. The equality $p \equiv 1$ (mod 3) implies $q \equiv 1$ (mod 3). According to [16] (the article [16] considers only the case $q = p$, where $\chi$ is the Legendre symbol, but the arguments can be trivially generalized), there exist $k, l \in \mathbb{Z}$ such that for any cubic non-residue $c \in \mathbb{F}_q^*$, the equalities $S_2(c) = -2k + 3l, S_2(c^2) = -k + 3l$ and $q = k^2 + 3l^2$ hold. Moreover, $S_2(b) = \chi(t)b^2$ for any $t \in \mathbb{F}_q^*$. The curve $E''$ generated in the third stage is $y^2 = x^3 + 1$; therefore, $b = -S_2(1) = -2k, \quad q = k^2 + 3l^2, \quad l = \frac{S_2(c) - S_2(c^2)}{6}.$ For any cubic non-residue $c \in \mathbb{F}_q^*$, the $j$-invariant equal to 0 to $\mathbb{F}_q$-isomorphic (because they have the same j-invariant).

Let $S_1(b) = \sum_{x \in \mathbb{F}_q} \chi(x^3 + b)$, where $\chi$ is the unique multiplicative character on $\mathbb{F}_q$ of order 2, as above. It is easy to see that the order of the curve $y^2 = x^3 + bx$ equals $q + 1 + S_1(b)$. The equality $p \equiv 1$ (mod 4) implies $q = 1$ (mod 4). According to [16], there exist $k, l \in \mathbb{Z}$ such that $k$ is odd, $S_1(1) = 2k$, $S_1(b) = \pm 2l$ for any quadratic non-residue $b$ and $S_1(b) = \chi(t)b^2$ for any $t \in \mathbb{F}_q^*$. The curve $E''$ generated in the third stage is $y^2 = x^3 + x$; therefore, $u = -S_1(1) = -2k, \quad q = k^2 + l^2$. Since $|\mathcal{E}|^2 = q$, it follows that $\pi = -k \pm li$.

Similarly to the previous case, there are 4 possible variants for $b$: $\pm 2 \Re \pi = \pm 2k$ and $\pm 2 \Re (\pi i) = \pm 2l$. If $\hat{u} = \pm 2k$, one of curves $y^2 = x^3 + x$ and $y^2 = x^3 + g^2x$, where $g$ is any quadratic non-residue in $\mathbb{F}_q$, has the required order. If $\hat{u} = \pm 2l$, one of curves $y^2 = x^3 + gx$ and $y^2 = x^3 + g^3x$ has the required order.

### 2.3 Some known optimizations

The coefficients of the polynomial $H_D[j]$ grow quite fast with $|D|$. For example, $H_{-40}[j](x) = x^2 - 425692800x + 9103145472000$. Consequently, it is useful to search for another functions with singular values in $\mathbb{L}$, which have a smaller height of the characteristic polynomial.
Let $z \in \mathbb{H}, q = e^{2\pi i z}$. Let us introduce some functions following [13]:

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)^{a_n} = q^{\frac{1}{24}} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{2n^2 + n}{12}},$$

$$f(z) = e^{\frac{1}{24} \eta(z)}, \quad f_1(z) = \frac{\eta(z)}{\eta(z)}, \quad f_2(z) = \sqrt{2} \frac{\eta(2z)}{\eta(z)},$$

$$\gamma_2(z) = \frac{f_2 - 16}{f_8} = \frac{f_2 + 16}{f_8} = \frac{f_1^2 + 16}{f_3^2}, \quad j(z) = \gamma_2(z)^3. \quad (9)$$

Let $N$ be a natural number. We define a $N$-system (following [17]) as a set of forms $(A_1, B_1, C_1), \ldots, (A_h, B_h, C_h)$ such that

- the set $\{\eta(A_i, B_i, C_i) : 1 \leq i \leq h\}$ is the complete system of representatives of the group $\mathcal{H}_D$,
- the relations

$$\gcd(A_i, N) = 1; \quad B_i \equiv B_j \pmod{2N}$$

hold.

Note that for any form $(A_i, B_i, C_i)$ the congruence $B_i \equiv D \pmod{2}$ is true, so the first condition implies that $B_i \equiv B_j \pmod{2}$ for any $i, j$.

If a set of forms satisfying the first condition is known, it is easy to construct a $N$-system. For example, the complete set of reduced forms can be used as a starting point. The corresponding algorithm can be found in [17] proof of Proposition 3. (We assume that the prime factorization of $N$ is known.)

1. First, achieve the condition $\gcd(A_i, N) = 1$ for all $i$.

   Obviously, it is sufficient to solve the next task: achieve $\gcd(A_i, N_0) = 1$ assuming that $\gcd(A_i, N_0) = 1$ where $l$ is the next prime divisor of $N$ not dividing $N_0$.

   The number $l$ can not divide all of numbers $A_i, A_i + N_0 B_i + N_0^2 C_i, l^2 A_i + l N_0 B_i + N_0^2 C_i$, because otherwise the numbers $A_i, B_i, C_i$ would have the common divisor $l$ and the form $(A_i, B_i, C_i)$ would not be primitive. 

   - Assume $l \nmid A_i$. Then the condition $\gcd(A_i, N_0)$ already holds.
   - Assume $l \mid A_i + N_0 B_i + N_0^2 C_i$. Change the variables $x = x', y = N_0 x' + y'$ and replace the current form with the new form (obviously, it is equivalent).
   - Assume $l \mid l^2 A_i + l N_0 B_i + N_0^2 C_i$. Find $a, b \in \mathbb{Z}$ such that $al - bN_0 = 1$, change the variables $x = ax' + by', y = N_0 x' + ay'$ and replace the current form with the new form (obviously, it is equivalent).

2. Next, achieve the condition $B_i \equiv B_1 \pmod{2N}$ for all $i$. The change of the variables $x = x' + ay', y = y'$ transforms the form $(A_i, B_i, C_i)$ to the equivalent form $(A_i, B_i + 2aA_i, C_i + aB_i + a^2A_i)$. Since $\gcd(A_i, N) = 1$, it is sufficient to apply this transformation with $a = A_i^{-1} \frac{B_i - B_1}{2}$ mod $N$.

**Theorem 2.** (17 Theorem 4) Let $\alpha \in \mathbb{H}$ be the root of the form

$$(A, B, C), \quad 2 \nmid A, \quad 32 \mid B,$$

with the discriminant $B^2 - 4AC = D = -4m, m \in \mathbb{N}$. Let $g(\alpha)$ be defined by the following formulæ:

$$\left(\left(\frac{2}{A}\right)\frac{1}{\sqrt{2}} \eta(\alpha)^2\right)^3, \quad \text{if } m \equiv 1 \pmod{8},$$

$$f(\alpha)^3, \quad \text{if } m \equiv 3 \pmod{8},$$

$$\left(\frac{1}{2} \eta(\alpha)^4\right)^3, \quad \text{if } m \equiv 5 \pmod{8},$$

$$\left(\frac{2}{A}\right)\frac{1}{\sqrt{2}} \eta(\alpha), \quad \text{if } m \equiv 7 \pmod{8},$$

$$\left(\frac{2}{A}\right)\frac{1}{\sqrt{2}} f_1(\alpha)^2, \quad \text{if } m \equiv 2 \pmod{4},$$

$$\left(\frac{2}{A}\right)\frac{1}{2\sqrt{2}} f_1(\alpha)^4, \quad \text{if } m \equiv 4 \pmod{8}.$$

Then $g(\alpha) \in \mathcal{O}_L$.

If $\alpha_1, \alpha_2, \ldots, \alpha_6$ are roots of the elements of 16-system, the singular values $g(\alpha_i)$ form the complete set of different conjugates over $\mathbb{Q}$. 

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Theorem 3. ([17] Theorem 2]) Let $\alpha \in \mathbb{H}$ be the root of the form

$$(A, B, C), \ 3 \nmid A, \ 3 \mid B,$$

with the discriminant $B^2 - 4AC = D$. Then

$$Q(\gamma_2(\alpha)) = \begin{cases} Q(j(\alpha)), & 3 \nmid D, \\ Q(j(3\alpha)), & 3 \mid D. \end{cases}$$

Moreover, if $3 \nmid D$ and $\alpha_1 = \alpha, \ldots, \alpha_8$ are roots of the elements of $3$-system, the singular values $\gamma_2(\alpha_i)$ form the complete set of different conjugates over $\mathbb{Q}$. In addition, $\gamma_2(\alpha_i)$ are algebraic integers.

Let $p_1, p_2$ be prime numbers. Following [18], we introduce the function

$$m_{p_1,p_2}(z) = \frac{\eta\left(\frac{z}{p_1}\right) \eta\left(\frac{z}{p_2}\right)}{\eta(z) \eta\left(\frac{z}{p_1p_2}\right)}$$

and define $s = \frac{24}{\gcd(24, (p_1-1)(p_2-1))}$.

Theorem 4. ([18] Theorems 3.2, 3.3, Corollary 3.1]) Let $D$ satisfy (1), $N = p_1p_2$, $p_1$ and $p_2$ are primes such that

1) $(\frac{D}{p_1}), (\frac{D}{p_2}) \neq -1$ if $p_1 \neq p_2$;

• either 2a) $(\frac{D}{p_1}) = 1$ if $p_1 = p_2 = p$, or 2b) $p \mid f$ if $p_1 = p_2 = p$.

Then there exists a form $(A_1, B_1, C_1)$ such that $\gcd(A_1, N) = 1$ and $N \mid C_1$. Let $\alpha_1 \in \mathbb{H}$ be the root of this form. The singular value $m_{p_1,p_2}^*(\alpha_1)$ lies in $L$. All conjugates over $K$ to $m_{p_1,p_2}^*(\alpha_1)$ are $m_{p_1,p_2}^*(\alpha_1)$, where $\alpha_1$ are roots of elements of $N$-system. The numbers $m_{p_1,p_2}^*(\alpha_1)$ are algebraic integers.

If one of conditions 1) and 2a) holds, the numbers $m_{p_1,p_2}^*(\alpha_1)$ are units (i.e. the numbers $m_{p_1,p_2}^{-1}(\alpha_1)$ are algebraic integers too).

If primes $p_1$ and $p_2$ satisfy the stronger condition:

• $(\frac{D}{p_1}), (\frac{D}{p_2}) \neq -1$ and $p_1, p_2 \nmid f$ when $p_1 \neq p_2$;

• $(\frac{D}{p_1}) = 1$ or $p \mid f$ when $p_1 = p_2 = p \neq 2$;

• either $(\frac{D}{p_2}) = 1$, or $2 \mid f$ and $D \neq 4 \pmod{32}$ when $p_1 = p_2 = 2$,

then the complex conjugation rearranges $m_{p_1,p_2}^*(\alpha_i)$.

We need more precise statements for the following. The formulations of theorems [2, 3, 4] do not give the full information regarding the action of $\text{Gal}(L/K)$ on singular values. However, the proofs from the articles [17] and [18] contain this information.

Statement 1. ([17] Theorem 7]) Let $\theta$ be the function from one of theorems [2, 3, 4] (in the last one the weak condition on $p_1, p_2$ is sufficient). Let $a, b$ be two elements of the $N$-system from the same theorem. Then $\alpha$ be the root of $a$, $\beta$ be the root of $b$, $\Omega : \mathcal{H}_D \to \text{Gal}(L/K)$ be the canonical isomorphism. Then

$$\theta(\alpha)^{\Omega(h(a)\beta^{-1})} = \theta(\beta).$$

This formula holds for $\theta = j$ too, as mentioned above.

It is more convenient to use Statement [1] in the form of a formula which specifies the action of a given automorphism from $\text{Gal}(L/K)$ on a given singular value. We remind that $h$ is surjective and $N$-system contains representatives of all classes in $\mathcal{H}_D$.

Corollary 1. Let $\theta, N$-system and $a$ be the same as in Statement [1]. Let $c \in \mathcal{H}_L$. Then there exists a form $b$ from the $N$-system such that

$$h(b) = h(a)c^{-1}.$$

If $\beta$ is the root of $b$, then

$$\theta(\alpha)^{\Omega(c)} = \theta(\beta). \quad (10)$$
Further we assume that $p_1$ and $p_2$ satisfy the strong condition of the theorem; it is easy to see that such primes can be found for any $D$.

Theorems 2 and 3 can be joined: if the discriminant $D$ and the form $(A, B, C)$ satisfy the assumptions of Theorem 2 and also $3 \mid D$, $3 \mid A$, $3 \mid B$, then the function $g(\alpha)$ can be defined without the exponent 3 and the consequence of Theorem 2 still holds. For example, let us consider the case $m \equiv 3 \pmod 8$. According to (3),

$$f(\alpha) = \frac{(f(\alpha))^{3/2} \gamma_2(\alpha)}{f(\alpha)^3 - 16}$$  

(11)

Since $f(\alpha)^3 \in L$ and $\gamma_2(\alpha) \in L$, we have $f(\alpha) \in L$. Statement 1 now implies that any automorphism from $\text{Gal}(L/K)$ maps $f(\alpha)^3$ to $f(\alpha')^3$ and $\gamma_2(\alpha)$ to $\gamma_2(\alpha')$, where $\alpha'$ depends only on the automorphism; (11) implies that $f(\alpha)$ is mapped to $f(\alpha')$. Finally, $f(\alpha)$ is an algebraic integer e.g. as a cubic root from $(f(\alpha)^3)$ which is an algebraic integer due to Theorem 2. In other cases formulas are slightly more complicated, but the reasoning is the same.

Let $\theta$ and $\alpha_n = \{\alpha_1, \ldots, \alpha_h\}$ be the polynomial and the set of roots from one of theorems 2–4. Let us consider the polynomial in one variable

$$H_D[\theta, \alpha_n](x) = \prod_{i=1}^{h}(x - \theta(\alpha_i)).$$

This polynomial has integer coefficients. For functions from Theorems 2 and 3 this follows directly from the consequence of theorem. For $\theta = m_{p_1, p_2}$ it is easy to see from Statement 1 that $H_D[\theta, \alpha_n]$ is invariant under $\text{Gal}(L/K)$ and therefore is in $K[x]$ and it remains to apply Theorem 4.

For example, $H_{-40}[\gamma_2, \alpha_n](x) = x^2 - 780x^2 + 20880, H_{-40}[\gamma_2, \alpha_n](x) = x^2 - x - 1$ with $g(\alpha) = \frac{1}{3} \gamma_1(\alpha)^2$, $H_{-40}[m_{1, 7}, \alpha_n](x) = x^2 - x - 1, H_{-40}[m_{11, 13}, \alpha_n](x) = x^2 + 2x + 1$. (The choice of $\alpha_n$ does not affect the polynomial in first three cases; there are two variants for the polynomial depending on $\alpha_n$ in the last case.) The last example shows that the values of $m_{p_1, p_2}(\alpha_n)$ can coincide, so in the general case $H_D[m_{p_1, p_2}, \alpha_n]$ is some power of the minimal polynomial.

Since the polynomial $H_D[\theta, \alpha_n]$ has integer coefficients similar to $H_D[j]$, it also can be calculated by calculating sufficiently accurate approximations to the singular values $\theta(\alpha_i)$, multiplying factors $x - \theta(\alpha_i)$ and rounding coefficients to integers. Since $\theta(\alpha_i) \in \mathbb{O}_L$, it has a representative in $\mathbb{O}_L/\mathfrak{B} \subset \mathbb{F}_q$ (Theorem 4), so the reduction of $H_D[\theta, \alpha_n](x)$ modulo $p$ splits into linear factors in $\mathbb{F}_q$. It remains to calculate the $j$-invariant by the reduction of $\theta(\alpha_i)$ in $\mathbb{F}_q$. The formulas (9) give the answer for $\theta = \gamma_2$ and $\theta$ being a power of $j$ from Theorem 2. The situation for $\theta = m_{p_1, p_2}$ is more complicated. There exists the polynomial $\Phi_{p_1, p_2}(x, y) \in \mathbb{Z}[x, y]$ such that the identity $\Phi_{p_1, p_2}(m_{p_1, p_2}(z), j(z)) = 0$ holds (13). Substituting $z = \alpha_n$ and reducing modulo $\mathfrak{B}$ (since $\mathfrak{B} \cap \mathbb{Z} = p\mathbb{Z}$, it is sufficient to reduce $\Phi_{p_1, p_2}$ modulo $p$), we obtain a polynomial equation for the required $j$-invariant. Solving this equation gives several variants for the $j$-invariant. The correct one can be selected e.g. as follows: construct an elliptic curve (and its quadratic twist) for every variant and check whether its order equals $q + 1 - \hat{u}$. For example, cryptographic applications require that $q + 1 - \hat{u}$ has a large prime divisor; in this case a simple test $(q + 1 - \hat{u})P = 0$ for a random point $P$ is good for eliminating wrong candidates. Note that the right order does not guarantee that the endomorphism ring is precisely $\mathbb{O}_D$, but such a subtle difference is usually not important; more detailed discussion can be found in (15).

3 Properties of the isomorphism $\Omega$

We recall that the group $\mathcal{H}_D$ is the factorgroup of the group $\mathcal{I}(\mathcal{O}_D)$ of proper fractional $\mathcal{O}_D$-ideals by the subgroup $P(\mathcal{O}_D)$ of principal ideals.

An $\mathcal{O}_D$-ideal $a$ is prime to $f$ when $a + f\mathcal{O}_D = \mathcal{O}_D$. This is equivalent to $\gcd(Norm(a), f) = 1$, and every ideal prime to the conductor is proper (5 Lemma 7.18). Let $I(\mathcal{O}_D, f)$ denote the subgroup in $I(\mathcal{O}_D)$ generated by ideals prime to $f$. Let $P(\mathcal{O}_D, f)$ denote the subgroup in $P(\mathcal{O}_D)$ generated by principal ideals $\alpha\mathcal{O}_D$ with $\gcd(Norm(\alpha), f) = 1$. The inclusion $I(\mathcal{O}_D, f) \subset I(\mathcal{O}_D)$ induces an isomorphism $I(\mathcal{O}_D, f)/P(\mathcal{O}_D, f) \cong \mathcal{H}_D$ (5 Proposition 7.19).

An $\mathcal{O}$-ideal $a$ is prime to $f$ if and only if $\gcd(Norm(a), f) = 1$ (5 Lemma 7.18). Let $I(\mathcal{O}, f)$ denote the subgroup of fractional $\mathcal{O}$-ideals generated by ideals prime to $f$. We recall that $P_{K, Z}(f)$ denotes the subgroup of $\mathcal{O}$-ideals generated by principal ideals of the form $\alpha\mathcal{O}$ with $\alpha \in \mathcal{O}, \alpha \equiv a \pmod f\mathcal{O}$ for some $a \in \mathbb{Z},$ $\gcd(a, f) = 1$. The map $\Omega_1: a \mapsto a\mathcal{O}$ gives a group isomorphism $I(\mathcal{O}_D, f) \to I(\mathcal{O}, f)$ which preserves the norm (5 Proposition 7.20). In addition (5 Proposition 7.22), $\Omega_1$ induces an isomorphism $I(\mathcal{O}_D, f)/P(\mathcal{O}_D, f) \cong I(\mathcal{O}, f)/P_{K, Z}(f)$.

Thus, we have an isomorphism $\Omega_2: H_D \to I(\mathcal{O}, f)/P_{K, Z}(f)$. The Artin map $I(\mathcal{O}, f) \to Gal(L/K)$ (denoted as $(\mathcal{O}_p(f)) \to I(\mathcal{O}, f)/P_{K, Z}(f) \to Gal(L/K)$. The composition of the last isomorphism with $\Omega_2$ is the canonical isomorphism $\Omega$ referenced in Statement 1 (5 §9).
Let us sum up the above maps. There exists a commutative diagram

\[
\begin{array}{ccc}
I(\mathcal{O}_D) & \supset & I(\mathcal{O}_D, f) \\
\mathcal{H}_D & \to & I(\mathcal{O}_D, f)/P(\mathcal{O}_D, f) \\
& & \Omega_f \\
& & I(\mathcal{O}, f)/P_{K,\mathbb{Z}}(f) \\
\end{array}
\]

where vertical arrows denote projections of a group to its factor group and horizontal arrows in the second line are isomorphisms.

**Theorem 5.** Let \( (A, B, C) \) be a form with \( \gcd(A, D) = 1 \). Let \( q \mid D \) be an integer satisfying one of the conditions:

- \(|q| \) is an odd prime, \( q \equiv 1 \pmod{4} \); or
- \( q \in \{-4, \pm 8\} \), \( \frac{Q}{q} \equiv 0 \pmod{4} \) or \( \frac{Q}{q} \equiv 1 \pmod{4} \).

Then

1. \( \sqrt{q} \in L \).
2. \( a = (A, \frac{-B+\sqrt{D}}{2})_{\mathbb{Z}} \in I(\mathcal{O}_D, f) \), \( \text{Norm}(a) = A \).
3. \( \left( \frac{L/K}{\Omega_1(a)} \right) (\sqrt{q}) = \left( \frac{q}{A} \right) \sqrt{q} \).

**Proof.** The first assertion follows from [19, Theorem 2.2.23 and (2.2.8)].

[3] Theorem 7.7 implies that \( a \) is a proper \( \mathcal{O}_D \)-ideal. Its norm is \( \text{Norm}(a) \) by definition; it is easy to see that every coset in \( \mathcal{O}_D/a \) contains exactly one integer from \( 0, \ldots, A \), so \( \text{Norm}(a) = A \). Since \( \gcd(A, f) = 1 \), the ideal \( a \) is prime to \( f \). The second assertion is proved.

Let \( \Omega_1(a) = p_1 \ldots p_s \), where \( p_i \) are prime \( \mathcal{O} \)-ideals (not necessarily different). Since

\[ A = \text{Norm}(a) = \text{Norm}(p_1) \ldots \text{Norm}(p_s) \]

and the Kronecker symbol is multiplicative, it is sufficient to prove that for every prime ideal \( p \) dividing \( \Omega_1(a) \) the equality with the Artin symbol

\[ \left( \frac{L/K}{p} \right) (\sqrt{q}) = \left( \frac{q}{\text{Norm}(p)} \right) \sqrt{q}. \]  

(13)

holds. The left-hand side is an image of \( \sqrt{q} \) under an automorphism, so it must be one of \( \pm \sqrt{q} \).

Assume first that \( p \mid \Omega_1(a) \), \( p \cap \mathbb{Z} = p\mathbb{Z} \), \( p \) is odd. Let \( \mathfrak{B} \) be a prime \( \mathcal{O}_L \)-ideal lying above \( p \). Since \( \gcd(A, D) = 1 \) and \( q \mid D \), we have \( 2\sqrt{q} \notin \mathfrak{B} \) and therefore \( \sqrt{q} \neq -\sqrt{q} \pmod{\mathfrak{B}} \). By definition

\[ \left( \frac{L/K}{p} \right) (\sqrt{q}) \equiv \sqrt{q}^{\text{Norm}(p)} = q^{\text{Norm}(p)-1} \sqrt{q} \pmod{\mathfrak{B}}. \]

If the ideal \( p\mathcal{O} \) is prime (i.e. \( p = p\mathcal{O} \)), then \( \text{Norm}(p) = p^2 \) and the right-hand side of (13) equals \( \sqrt{q} \). On the other part, \( q^{\text{Norm}(p)-1} = (q^{p-1})^{\frac{1}{p-1}} \equiv 1 \pmod{p} \), so the left-hand side of (13) is congruent to \( \sqrt{q} \) modulo \( \mathfrak{B} \) and therefore is equal to \( \sqrt{q} \). Thus, (13) is proved in this case.

If the ideal \( p\mathcal{O} \) is not prime, then \( \text{Norm}(p) = p \) and the right-hand side of (13) equals \( \left( \frac{q}{p} \right) \sqrt{q} \). On the other part, \( q^{\text{Norm}(p)-1} = q^{\frac{p-1}{p}} \equiv \left( \frac{q}{p} \right) \pmod{p} \), so the left-hand side of (13) is congruent to \( \left( \frac{q}{p} \right) \sqrt{q} \) modulo \( \mathfrak{B} \) and therefore is equal to \( \left( \frac{q}{p} \right) \sqrt{q} \). Thus, (13) is proved in this case too.

Assume now that \( p \mid \Omega_1(a) \), \( p \cap \mathbb{Z} = 2\mathbb{Z} \), a prime \( \mathcal{O}_L \)-ideal \( \mathfrak{B} \) lies above \( p \). In this case 2 \( \mid \mathcal{O} \), the assumption of theorem implies that 2 \( \mid D \) and \( q \) is odd. Since \( B^2 - 4AC = D \), we have \( D = 2B^2 \equiv 1 \pmod{8} \). Thus \( d \equiv 1 \pmod{8} \) and the ideal 2\( \mathcal{O} \) is not prime ([9, Proposition 13.1.4]), so \( \text{Norm}(p) = 2 \). Therefore, the right-hand side of (13) equals \( \left( \frac{q}{2} \right) \sqrt{q} \). To calculate the left-hand side of (13), consider

\[ \left( \frac{L/K}{p} \right) \left( \frac{1+\sqrt{q}}{2} \right). \]
This expression must be equal to one of \( \frac{1 + \sqrt{q}}{2} \), two possible values are different modulo \( \mathfrak{B} \). By definition

\[
\left( \frac{L/K}{p} \right) \left( 1 + \frac{\sqrt{q}}{2} \right) \equiv \left( 1 + \frac{\sqrt{q}}{2} \right)^2 = \frac{q - 1}{4} + \frac{1 + \sqrt{q}}{2} \quad (\text{mod } \mathfrak{B}).
\]

If \( \left( \frac{q}{2} \right) = 1 \), then \( q \equiv 1 \pmod{8} \), \( \frac{q - 1}{4} \) is even and hence lies in \( \mathfrak{B} \). If \( \left( \frac{q}{2} \right) = -1 \), then \( q \equiv 5 \pmod{8} \), \( \frac{q - 1}{4} \) is odd and therefore is congruent to \(-1 \equiv 1 \pmod{2} \). In both cases there is

\[
\left( \frac{L/K}{p} \right) \left( 1 + \frac{\sqrt{q}}{2} \right) \equiv \frac{1 + \left( \frac{q}{2} \right) \sqrt{q}}{2} \quad (\text{mod } \mathfrak{B}),
\]

which implies \((13)\).

**Lemma 2.** Let \( d < 0 \) satisfy one of conditions \((2)\) and \((3)\). There exists the unique (up to the order of factors) representation of \( d \) as the product

\[ d = q_1^* \cdots q_t^*, \]

where all \( q_i^* \) are pairwise relatively prime,

\[ q^* = (-1)^\frac{d+1}{2} q, \]

if \( q > 0 \) is an odd prime, and \( q^* \in \{ -4, \pm 8 \} \) if \( q = 2 \).

**Proof.** The uniqueness is obvious, we need to prove the existence.

If \( d \) satisfies \((3)\), the prime factorization of \( d \) has the form \( d = -q_1 \cdots q_t \), where \( q_i \) are different odd primes; since \( q_i^* = \pm q_i \), it follows that \( d = \pm q_1^* \cdots q_t^* \); finally, the sign is correct due to \( d \equiv 1 \pmod{4} \) and \( q_i^* \equiv 1 \pmod{4} \) for all \( t \).

Assume that \( d \) satisfies \((2)\). The prime factorization of \( \frac{d}{4} \) has one of the forms \( \frac{d}{4} = -q_1 \cdots q_{t-1} \) or \( \frac{d}{4} = -2q_1 \cdots q_{t-1} \), where \( q_i \) are different odd primes in both forms. If \( \frac{d}{4} \) is odd, similarly to the previous case we obtain \( \frac{d}{4} = \pm q_1^* \cdots q_{t-1}^* \), but this time \((2)\) implies \( \frac{d}{4} \not\equiv 1 \pmod{4} \), so the sign is “-”. Multiplying by 4, we obtain the assertion of the lemma. Finally, if \( \frac{d}{4} \) is even, we have \( \frac{d}{4} = \pm 2q_1^* \cdots q_{t-1}^* \). Selecting the correct sign in \( q_t^* = \pm 8 \), we obtain the assertion of the lemma.

It is easy to see that the numbers \( q_i^* \) from Lemma \((2)\) satisfy the assumptions of Theorem \((3)\). Therefore, \( K(\sqrt{q_1^*}, \ldots, \sqrt{q_t^*}) \subset L \). The field \( K(\sqrt{q_1^*}, \ldots, \sqrt{q_t^*}) \) depends only on the field \( K \) (which defines \( d \) but not \( f \)) and is called the genus field for \( K \). Hereafter we use the notation

\[ K(\sqrt{q_1^*}, \ldots, \sqrt{q_t^*}) = K_G. \]

### 4 Ring of algebraic integers in the genus field

Let \( q_i^* \) be as in Lemma \((2)\). There are three cases.

1. All \(|q_i|\) are odd primes.
2. \( q_t^* = \pm 8 \).
3. \( q_t^* = -4 \).

We need to know a basis of algebraic integers in the field \( K_G \) over \( \mathbb{Z} \). Since \( d = q_1^* \cdots q_t^* \), we have \( \sqrt{d} \in \mathbb{Q}(\sqrt{q_1^*}, \ldots, \sqrt{q_t^*}) \) and therefore \( K_G = \mathbb{Q}(\sqrt{q_1^*}, \ldots, \sqrt{q_t^*}) \). The formulas are slightly different in different cases, so we consider each case separately.

**Lemma 3.** Let \( M \) be a number field. Let \( p \in \mathbb{Z} \) be a prime such that the ideal \( p\mathbb{Z} \) is unramified in \( M \). Let \( c \in M \) satisfy the condition \( pc^2 \in \mathcal{O}_M \). Then \( c \in \mathcal{O}_M \).

**Proof.** Assume that \( c \not\in \mathcal{O}_M \). The fractional ideal \( c\mathcal{O}_M \) has the factorization to the prime ideals \( c\mathcal{O}_M = q_1 \cdots q_n \), where \( q_i \) are pairwise different and \( s_i < 0 \). The degree of \( q_1 \) in the prime factorization of \( p\mathcal{O}_M \) is at most 1 because \( p\mathcal{O}_M \) is unramified. The degree of \( q_1 \) in the prime factorization of \( c^2\mathcal{O}_M \) is at most \(-2\). Therefore, the degree of \( q_1 \) in the prime factorization of \( pc^2\mathcal{O}_M \) is negative. The contradiction with \( pc^2 \in \mathcal{O}_M \) proves the lemma.

**Theorem 6.** Let \( \tilde{q}_1, \ldots, \tilde{q}_m \) be pairwise different integers such that \(|\tilde{q}_i|\) are odd primes and \( \tilde{q}_i \equiv 1 \pmod{4} \). Let \( \alpha_i = \frac{1 + \sqrt{q_i}}{2} \) and \( \tilde{\alpha}_1 = \frac{1 - \sqrt{q_i}}{2} \). Then:

1. The set \( \{ \alpha_1^{s_1} \cdots \alpha_m^{s_m} : (s_1, \ldots, s_t) \in \{0,1\}^t \} \) is a basis of integers in the field \( \mathbb{Q}(\sqrt{q_1}, \ldots, \sqrt{q_t}) \) over \( \mathbb{Z} \).
2. The set \( \{ \alpha_1^{s_1} \alpha_1^{1-s_1} \ldots \alpha_r^{s_r} \alpha_r^{1-s_r} : (s_1, \ldots, s_r) \in \{0,1\}^r \} \) is a basis of integers in the field \( \mathbb{Q}(\sqrt{q_1}, \ldots, \sqrt{q_r}) \) over \( \mathbb{Z} \).

Proof. We prove the theorem by induction on \( r \). For \( r = 0 \) the theorem is trivial. Assume that the theorem is proved for all fields \( M_i = \mathbb{Q}(\sqrt{q_1}, \ldots, \sqrt{q_i}) \) with \( i = 1, \ldots, r - 1 \).

**Lemma 4.** Let \( p \in \mathbb{Z} \) be a prime not dividing any of numbers \( q_1, \ldots, q_{r-1} \). Then the ideal \( p\mathbb{Z} \) is unramified in \( M_{r-1} \).

Proof. It is sufficient to check that any prime ideal of the field \( M_{r-1} \) dividing \( p\mathcal{O}_{M_{r-1}} \) is unramified in \( M_i = M_{r-1}(\sqrt{q_i}) \) for all \( 1 \leq i \leq r - 1 \).

Let \( p \) be a prime ideal of the field \( M_{r-1} \) such that \( p \cap \mathbb{Z} = p\mathbb{Z} \). The extension \( M_{r-1} \subset M_i \) is generated by \( \alpha_i \); the inductive hypothesis implies that \( (1, \alpha_i) \) is a basis of \( \mathcal{O}_M / \mathcal{O}_{M_{r-1}} \). The only nontrivial automorphism in \( \text{Gal}(M_i/M_{r-1}) \) transforms this basis to \( (1, \tilde{\alpha}_i) \). According to [20] Propositions III.8 and III.14, \( p \) is unramified if \( p \) does not divide \( \det (1 - \tilde{\alpha}_i/\alpha_i) \). This is true, because \( p \) does not divide \( q_i \).

Apply Lemma [4] to \( p = |q_r| \). The factorization of \( \sqrt{q_r} \mathcal{O}_{M_{r-1}} \) in the prime ideals does not contain squares. In particular, \( \sqrt{q_r} \not\in M_{r-1} \) because otherwise \( \sqrt{q_r} \mathcal{O}_{M_{r-1}} = (\sqrt{q_i} \mathcal{O}_{M_{r-1}})^2 \). Therefore, \( (1, \alpha_r) \) is a \( M_{r-1} \)-basis of \( M_r \).

Let \( a + b\alpha_r \) be an algebraic integer and \( a, b \in M_{r-1} \). The number \( b(1 - \alpha_r) \) is conjugate to \( a + b\alpha_r \) and hence is also an algebraic integer. Thus, their sum \( x = 2a + b \) and product \( y = a^2 + b^2 + 2ab \) are also algebraic integers and lie in \( \mathcal{O}_{M_{r-1}} \). Furthermore, \( x^2 - 4y = q_r \beta^2 \in \mathcal{O}_{M_{r-1}} \), Lemma [3] implies that \( b \in \mathcal{O}_{M_{r-1}} \), and \( \mathcal{O}_{M_r} = (1 + \sqrt{q_r} \mathcal{O}_{M_{r-1}}) \). Applying Lemmas [4] and [3] to \( p = 2 \), we obtain a \( \in \mathcal{O}_{M_{r-1}} \). Thus, \( (1, \alpha_r) \) is a \( M_{r-1} \)-basis of \( \mathcal{O}_{M_r} \). Therefore, \( (1, \alpha_r) \) is a \( M_{r-1} \)-basis of \( \mathcal{O}_{M_r} \). This proves the inductive step for the set \( \{ \alpha_1^{s_1} \ldots \alpha_r^{s_r} \} \). Therefore, the product of any set of \( \alpha_i \)’s is an algebraic integer and \( a, b \in M_{r-1} \), then \( a, b \in \mathcal{O}_{M_{r-1}} \). The converse assertion is obvious, so \( (1, \alpha_r) \) is a \( \mathcal{O}_{M_{r-1}} \)-basis of \( \mathcal{O}_{M_r} \). This proves the theorem.

**Theorem 7.** Let \( q_1, \ldots, q_{r-1} \) be the same as in Theorem [2] and \( \tilde{q}_r = \pm 8 \). Let \( \alpha_r = \sqrt{q_r} \). Then:

1. The set \( \{ \alpha_1^{s_1} \ldots \alpha_r^{s_r} : (s_1, \ldots, s_r) \in \{0,1\}^r \} \) is a basis of integers in the field \( \mathbb{Q}(\sqrt{q_1}, \ldots, \sqrt{q_r}) \) over \( \mathbb{Z} \).

2. The set \( \{ \tilde{\alpha}_1^{s_1} \alpha_1^{1-s_1} \ldots \tilde{\alpha}_{r-1}^{s_{r-1}} \alpha_{r-1}^{1-s_{r-1}} \alpha_r^{s_r} : (s_1, \ldots, s_r) \in \{0,1\}^r \} \) is a basis of integers in the field \( \mathbb{Q}(\sqrt{q_1}, \ldots, \sqrt{q_r}) \) over \( \mathbb{Z} \).

Proof. Let \( M = \mathbb{Q}(\sqrt{q_1}, \ldots, \sqrt{q_{r-1}}) \). Apply Lemma [4] with \( p = 2 \) and Theorem [3]. The ideal \( 2\mathbb{Z} \) is unramified in \( M \). As shown above, this implies that \( \sqrt{q_r} \not\in M \) and \( 1, \sqrt{q_r} \) is a \( M \)-basis of \( \mathbb{Q}(\sqrt{q_r}) \).

Let \( a + b\alpha_r \) be an algebraic integer and \( a, b \in M \). The number \( a - b\alpha_r \) is conjugate to \( a + b\alpha_r \) and therefore is also an algebraic integer. Thus, their sum \( 2a \) and product \( a^2 + b^2 \) are algebraic integers and lie in \( \mathcal{O}_M \). Furthermore, \( (2a)^2 - 4(a^2 + b^2) = 2(2b)^2 \in \mathcal{O}_M \), with /Lemma [2] this implies \( 2b \in \mathcal{O}_M \). Now \( (2a + \sqrt{q_r} b)^2 \) is in \( \mathcal{O}_M \), and with /Lemma [3] this implies \( b \in \mathcal{O}_M \). Therefore, \( (1, \alpha_r) \) is an \( M \)-basis of the ring of integers in \( M(\sqrt{q_r}) \). Use of Theorem [3] concludes the proof.

**Theorem 8.** Let \( q_1, \ldots, q_{r-1} \) be the same as in Theorem [2] and \( \tilde{q}_r = -4 \). Let \( \alpha_r = \sqrt{q_r} = i \). Then:

1. The set \( \{ \alpha_1^{s_1} \ldots \alpha_r^{s_r} : (s_1, \ldots, s_r) \in \{0,1\}^r \} \) is a basis of integers in the field \( \mathbb{Q}(\sqrt{q_1}, \ldots, \sqrt{q_r}) \) over \( \mathbb{Z} \).

2. The set \( \{ \tilde{\alpha}_1^{s_1} \alpha_1^{1-s_1} \ldots \tilde{\alpha}_{r-1}^{s_{r-1}} \alpha_{r-1}^{1-s_{r-1}} \alpha_r^{s_r} : (s_1, \ldots, s_r) \in \{0,1\}^r \} \) is a basis of integers in the field \( \mathbb{Q}(\sqrt{q_1}, \ldots, \sqrt{q_r}) \) over \( \mathbb{Z} \).

Proof. Let \( M = \mathbb{Q}(\sqrt{q_1}, \ldots, \sqrt{q_{r-1}}) \). The identity \( 2 = -i(1 + i)^2 \) shows that the ideal \( 2\mathbb{Z} \) is ramified in any field containing \( i \). Lemma [2] and Theorem [3] imply that \( 2\mathbb{Z} \) is not ramified in \( M \). Therefore, \( i \not\in M \).

Let \( a + bi \) be an algebraic integer and \( a, b \in M \). The number \( a - bi \) is conjugate to \( a + bi \) and therefore is also an algebraic integer. Thus, their sum \( 2a \) and product \( a^2 + b^2 \) are also algebraic integers and lie in \( \mathcal{O}_M \). Furthermore, \( 2(a^2 + b^2) = 2a(2a + b)^2 \in \mathcal{O}_M \), with /Lemma [2] and [3] with \( p = 2 \) and Theorem [4] imply that \( a + b \in \mathcal{O}_M \). Now \( 2a - (a + b) = a - b \in \mathcal{O}_M \), \( (a + b)(a - b) = a^2 - b^2 \in \mathcal{O}_M \), \( 2a^2 \in \mathcal{O}_M \), \( 2b^2 \in \mathcal{O}_M \). Applying Lemmas [4] and Theorem [3] again, we obtain \( a, b \in \mathcal{O}_M \). Thus, \( (1, \alpha_1) \) is a \( M \)-basis of the ring of integers in \( M(\sqrt{q_r}) \). Use of Theorem [3] concludes the proof.
Let $\oplus$ denote the addition of integer numbers modulo 2.

In each case we have $[K_G : \mathbb{Q}] = 2^t$. Thus, \( \sqrt{q_i} \notin \mathbb{Q}(\ldots, \sqrt{q_{j-1}}^2, \sqrt{q_{j+1}}^2, \ldots) \) for any $1 \leq j \leq t$. Therefore, $\text{Gal}(K_G/\mathbb{Q})$ has $t$ elements $\tau_j$ with the following action:

\[
\tau_j \left( \sqrt{q_i} \right) = -\sqrt{q_i}, \quad \tau_j \left( \sqrt{q_i} \right) = \sqrt{q_i} \text{ for } i \neq j.
\]  

(14)

Let 
\[
\tau^\mu = \tau_1^\mu \cdots \tau_t^\mu \in \text{Gal}(K_G/\mathbb{Q})
\]
for $\mu \in \{0, 1\}^t$. Comparing the action of $\tau^\mu$ on $\sqrt{q_i}$, it is easy to see that $\tau^\mu$ are pairwise different. We obtain $2^t = |\text{Gal}(K_G/\mathbb{Q})|$ different elements of $\text{Gal}(K_G/\mathbb{Q})$, so this group does not contain other elements.

The theorems above give a $\mathbb{Z}$-basis of $\mathcal{O}_{K_G}$. We also need the intersection $\mathcal{O}_{K_G} \cap \mathbb{R}$ (obviously, it is the ring of integers in $K_G \cap \mathbb{R}$) and the intersection $\mathcal{O}_{K_G} \cap i\mathbb{R}$ (obviously, it is a $\mathbb{Z}$-module). There is at least one negative $q^*_i$. Let $u$ be the number of positive $q^*_i$, $0 \leq u < t$. We assume without loss of generality that $q^*_1 > 0$, 
\[
\ldots, q^*_u > 0, q^*_{u+1} < 0, \ldots, q^*_t < 0.
\]

The complex conjugation acts on $\sqrt{q_i}$ same as the composition $\tau_{u+1} \cdots \tau_t$. Since $K_G \cap \mathbb{R}$ is the fixed field and the complex conjugation restricted to $K_G$, the group $\text{Gal}(K_G \cap \mathbb{R}/\mathbb{Q})$ is isomorphic to the factorgroup of $\text{Gal}(K_G/\mathbb{Q})$ by the subgroup generated by the complex conjugation. We select an element with $\mu_t = 0$ as a representative in each coset and obtain that $\text{Gal}(K_G \cap \mathbb{R}/\mathbb{Q})$ consists of the automorphisms

\[
\tau^\lambda = \tau^\lambda_1 \cdots \tau^\lambda_t = \tau^\lambda_1 \cdots \tau^\lambda_{t-1}
\]

(15)

for $\lambda \in \{0, 1\}^{t-1}$, $\tau^\lambda$ are pairwise different for different $\lambda$.

Note that $\sqrt{d}$ has two possible values. Further we select the value that equals the product $\sqrt{q_1} \cdots \sqrt{q_t}$, where the values of individual square roots are the same as in definition of $\alpha_i$ and $\tilde{\alpha}_i$.

**Theorem 9.** Let $q^*_1, \ldots, q^*_t$ be as in Lemma \([2]\) odd and numbered so that $q^*_i > 0$ for $1 \leq i \leq u$, $q^*_i < 0$ for $u < i \leq t$, where $0 \leq u \leq t - 1$. Let $K_G = \mathbb{Q}(\sqrt{q^*_1}, \ldots, \sqrt{q^*_t})$. Let $\tau^\lambda$ be defined by (15).

1. Define

\[
\beta_{s_1, \ldots, s_{t-1}} = \beta_{s_1, \ldots, s_{t-1}}(q^*_1, \ldots, q^*_t) = \left( \prod_{i=1}^{u} (\tilde{\alpha}_i^{s_i} - \alpha_i^t) \right) \left( \prod_{i=u+1}^{t-1} \tilde{\alpha}_i^{s_i} \right) \left( \alpha_i^{t-1-s_i} \right) \left( \prod_{i=1}^{t-1} \tilde{\alpha}_i^{s_i} \right) - \left( \prod_{i=1}^{t} \tilde{\alpha}_i^{s_i} \right) - \left( \prod_{i=1}^{t-1} \tilde{\alpha}_i^{s_i} \right).
\]

The set $\{\beta_{s_1, \ldots, s_{t-1}} : (s_1, \ldots, s_{t-1}) \in \{0, 1\}^{t-1}\}$ is a $\mathbb{Z}$-basis of the ring of integers in $K_G \cap \mathbb{R}$.

2. Define

\[
\beta^*_{s_1, \ldots, s_{t-1}} = \beta^*_1 \cdots \beta^*_t = \beta^*_1 \cdots \beta^*_t(q^*_1, \ldots, q^*_t) = \left( \prod_{i=1}^{u} (-\tilde{\alpha}_i)^{s_i} \right) \left( \prod_{i=u+1}^{t-1} (-\tilde{\alpha}_i)^{s_i} \right) - \left( \prod_{i=1}^{t-1} \tilde{\alpha}_i^{s_i} \right) - \left( \prod_{i=1}^{t-1} \tilde{\alpha}_i^{s_i} \right).
\]

The set $\{\beta^*_{s_1, \ldots, s_{t-1}} : (s_1, \ldots, s_{t-1}) \in \{0, 1\}^{t-1}\}$ is a $\mathbb{Z}$-basis of the $\mathbb{Z}$-module $\mathcal{O}_{K_G} \cap i\mathbb{R}$.

3. For any $\eta, \nu \in \{0, 1\}^{t-1}$

\[
\sum_{\mu \in \{0, 1\}^{t-1}} (-1)^{\mu_1 + \cdots + \mu_t} \tau^\mu \left( \beta_{\eta_1, \ldots, \eta_{t-1}} \beta^*_{\nu_1, \ldots, \nu_{t-1}} \right) = \begin{cases} \sqrt{d}, & \text{if } \eta = \nu, \\ 0, & \text{otherwise}. \end{cases}
\]

**Proof.** Let $\beta^*_{s_1, \ldots, s_t}$ be the element of the basis from second assertion of Theorem 8 corresponding to the set $(s_1, \ldots, s_t)$.

A number from $\mathcal{O}_{K_G}$ is in $K_G \cap \mathbb{R}$ if and only if it is invariant under the complex conjugation. It is easy to see that the complex conjugation maps $\beta^*_{s_1, \ldots, s_t}$ to $\beta^*_{s_1, \ldots, s_t} \gamma_{1, \ldots, s_t}$. Thus, a $\mathbb{Z}$-linear combination of $\beta^*_{s_1, \ldots, s_t}$ is invariant if and only if coefficients of $\beta^*_{s_1, \ldots, s_t}$ and $\beta^*_{s_1, \ldots, s_t}$ are equal for any set $(s_1)$. Now Theorem 8 implies that $\{\beta^*_{s_1, \ldots, s_t}, 0 + \beta^*_{s_1, \ldots, s_t}, 1 - \beta^*_{s_1, \ldots, s_t}, 1-1 \}$ is a required basis. From the definition of $\beta^*$ it is easy to see that this sum is equal to $\beta^*_{s_1, \ldots, s_t}$. This concludes the proof of the first assertion.

A number from $\mathcal{O}_{K_G}$ is in $K_G \cap i\mathbb{R}$ if and only if it changes the sign under the complex conjugation. Similarly to the first assertion, we obtain that $\{\beta^*_{s_1, \ldots, s_t}, 0 - \beta^*_{s_1, \ldots, s_t}, 1 - \beta^*_{s_1, \ldots, s_t}, 1-1 \}$ is a required basis. From the definition of $\beta^*$ it is easy to see that this difference is equal to $\pm \beta^*_{s_1, \ldots, s_t}$. This concludes the proof of the second assertion.
The last assertion is checked by a direct calculation. It is easy to see that

$$\tau_{\mu} (\beta_{\eta_1,\ldots,\eta_{u-1}}) = \left( \prod_{i=1}^{u} \tilde{\alpha}_i^{\mu_i \oplus \eta_i} \alpha_i^{1-(\mu_i \oplus \eta_i)} \right) \times \left( \prod_{i=u+1}^{t-1} \tilde{\alpha}_i^{\mu_i \oplus \eta_i} \alpha_i^{1-(\mu_i \oplus \eta_i)} \right) \alpha_t + \left( \prod_{i=u+1}^{t-1} \tilde{\alpha}_i^{1-(\mu_i \oplus \eta_i)} \alpha_i^{\mu_i \oplus \eta_i} \right) \tilde{\alpha}_t,$$

$$\tau_{\mu} (\beta^*_\nu_1,\ldots,\nu_{v-1}) = \left( \prod_{i=1}^{v} (-1)^{\nu_i} \tilde{\alpha}_i^{\nu_i \oplus \nu_i} \alpha_i^{1-(\mu_i \oplus \eta_i)} \right) \times \left( \prod_{i=v+1}^{t-1} (-1)^{\nu_i} \tilde{\alpha}_i^{\nu_i \oplus \nu_i} \alpha_i^{1-(\mu_i \oplus \eta_i)} \right) \alpha_t - \left( \prod_{i=v+1}^{t-1} (-1)^{\nu_i} \tilde{\alpha}_i^{1-(\mu_i \oplus \eta_i)} \alpha_i^{\mu_i \oplus \eta_i} \right) \tilde{\alpha}_t.$$

Substitute these formulas to the product \(\tau_{\mu} (\beta_{\eta_1,\ldots,\eta_{u-1}}) \tau_{\mu} (\beta^*_{\nu_1,\ldots,\nu_{v-1}})\), obtain the formula of the form \((a+b)(c-d)\). Expand it and obtain four operands \(ac + bc - ad - bd\). Let \(\delta_{ij}\) be the Kronecker delta: \(\delta_{ii} = 1, \delta_{ij} = 0\) if \(i \neq j\).

Note that

$$\sum_{\mu_i=0}^{1} (-1)^{\mu_i} (-1)^{\nu_i} \tilde{\alpha}_i^{(\mu_i \oplus \eta_i) + (\mu_i \oplus \eta_i)} \alpha_i^{1-(\mu_i \oplus \eta_i) + 1-(\mu_i \oplus \eta_i)}$$

$$= (-1)^{\nu_i} \left( \tilde{\alpha}_i^{\eta_i + \nu_i} \alpha_i^{2-(\eta_i + \nu_i)} - \tilde{\alpha}_i^{2-(\eta_i + \nu_i)} \alpha_i^{\eta_i + \nu_i} \right) = \delta_{\eta_i \nu_i} (\alpha_i^2 - \tilde{\alpha}_i^2) = \delta_{\eta_i \nu_i} \sqrt{q_i^*},$$

and transposing of \(\alpha_i\) with \(\tilde{\alpha}_i\) gives two more products with values multiplied by \((-1)\).

Therefore,

$$\sum_{\mu \in \{0,1\}^{t-1}} (-1)^{\mu_1 + \ldots + \mu_{t-1}} \tau_{\mu} (\beta_{\eta_1,\ldots,\eta_{u-1}} \beta^*_{\nu_1,\ldots,\nu_{v-1}})$$

$$= \left( \prod_{i=1}^{u} \delta_{\eta_i \nu_i} \sqrt{q_i^*} \right) \left( \prod_{i=u+1}^{t-1} \delta_{\eta_i \nu_i} \sqrt{q_i^*} + \tilde{\alpha}_t \alpha_t \prod_{i=u+1}^{t-1} \delta_{\eta_i \nu_i} \sqrt{q_i^*} \right)$$

$$- \tilde{\alpha}_t \alpha_t \prod_{i=u+1}^{t-1} (-\delta_{\eta_i \nu_i} \sqrt{q_i^*}) = \sqrt{q_t^*} \prod_{i=1}^{t-1} \delta_{\eta_i \nu_i} \sqrt{q_i^*}.$$
1. Define
\[
\beta_{s_1, \ldots, s_{t-1}} = \beta_{s_1, \ldots, s_{t-1}}(q_1^\ast, \ldots, q_t^\ast) = \sqrt{2^{s_1}} \left( \prod_{i=2}^{u} \tilde{\alpha}_i^{s_1} \alpha_i^{1-s_1} \right) 
\times \left( \prod_{i=u+1}^{t-1} \tilde{\alpha}_i \alpha_i^{-s_1} \right) \alpha_t + \left( \prod_{i=u+1}^{t-1} \tilde{\alpha}_i \alpha_i^{-s_1} \right) \tilde{\alpha}_t.
\]

The set \( \{ \beta_{s_1, \ldots, s_{t-1}} : (s_1, \ldots, s_{t-1}) \in \{0,1\}^{t-1} \} \) is a \( \mathbb{Z} \)-basis of the ring of integers in \( K_G \cap \mathbb{R} \).

2. Define
\[
\beta_{s_1, \ldots, s_{t-1}}^\ast = \beta_{s_1, \ldots, s_{t-1}}^\ast(q_1^\ast, \ldots, q_t^\ast) = \sqrt{2^{s_1}} \left( \prod_{i=2}^{u} (-\tilde{\alpha}_i)^{s_1} \alpha_i^{1-s_1} \right)
\times \left( \prod_{i=u+1}^{t-1} (-\tilde{\alpha}_i)^{s_1} \alpha_i^{1-s_1} \right) \alpha_t - \left( \prod_{i=u+1}^{t-1} \tilde{\alpha}_i \alpha_i^{-s_1} \right) (-\alpha_t)^{s_1} \alpha_t.
\]

The set \( \{ \beta_{s_1, \ldots, s_{t-1}}^\ast : (s_1, \ldots, s_{t-1}) \in \{0,1\}^{t-1} \} \) is a \( \mathbb{Z} \)-basis of the \( \mathbb{Z} \)-module \( \mathcal{O}_{K_G} \cap i\mathbb{R} \).

3. For any \( \eta, \nu \in \{0,1\}^{t-1} \)
\[
\sum_{\mu \in \{0,1\}^{t-1}} (-1)^{\mu_1 + \ldots + \mu_{t-1}} \tau_{\mu} \left( \beta_{\eta_1, \ldots, \eta_{t-1}}^\ast \beta_{\nu_1, \ldots, \nu_{t-1}}^\ast \right) = \begin{cases} \sqrt{d}, & \text{if } \eta = \nu, \\ 0, & \text{otherwise.} \end{cases}
\]

**Proof.** The arguments are similar to Theorem 7. Calculating the expression from the third assertion yields an additional factor
\[
\sum_{\mu_1=0}^{1} (-1)^{\mu_1} \left( -1 \right)^{\mu_1} \sqrt{2}^{\eta_1} \left( -1 \right)^{\mu_1} \sqrt{2}^{1-\nu_1} = \sqrt{2}^{1+\eta_1-\nu_1} \left( 1 + (-1)^{\eta_1+\nu_1} \right) = 2\sqrt{2} \delta_{\eta_1 \nu_1}.
\]

**Theorem 11.** Let \( q_1^\ast, \ldots, q_{t-1}^\ast \) be the same as in Theorem 4 and \( q_i^\ast \in \{-4,-8\} \). Let \( q_i^\ast \) be numbered so that \( q_i^\ast > 0 \) for \( 1 \leq i \leq u \) and \( q_i^\ast < 0 \) for \( u < i \leq t \), where \( 0 \leq u \leq t-2 \). Let \( K_G = \mathbb{Q} (\sqrt{q_1^\ast}, \ldots, \sqrt{q_t^\ast}) \). Let \( \tau_\lambda \) be defined by (13).

1. Define
\[
\beta_{s_1, \ldots, s_{t-1}} = \beta_{s_1, \ldots, s_{t-1}}(\sqrt{q_1^\ast}, \ldots, \sqrt{q_t^\ast}) = \left( \prod_{i=1}^{u} \tilde{\alpha}_i s_1 \alpha_i^{1-s_1} \right)
\times \left( \prod_{i=u+1}^{t-1} \tilde{\alpha}_i s_1 \alpha_i^{1-s_1} \right) \alpha_t - \left( \prod_{i=u+1}^{t-1} \tilde{\alpha}_i s_1 \alpha_i^{1-s_1} \right) (-\alpha_t)^{s_1} \alpha_t.
\]

The set \( \{ \beta_{s_1, \ldots, s_{t-1}} : (s_1, \ldots, s_{t-1}) \in \{0,1\}^{t-1} \} \) is a \( \mathbb{Z} \)-basis of the ring of integers in \( K_G \cap \mathbb{R} \).

2. Define
\[
\beta_{s_1, \ldots, s_{t-1}}^\ast = \beta_{s_1, \ldots, s_{t-1}}(\sqrt{q_1^\ast}, \ldots, \sqrt{q_t^\ast}) = \left( \prod_{i=1}^{u} (-\tilde{\alpha}_i)^s \alpha_i^{1-s} \right)
\times \left( \prod_{i=u+1}^{t-1} (-\tilde{\alpha}_i)^s \alpha_i^{1-s} \right) \alpha_t - \left( \prod_{i=u+1}^{t-1} (-\tilde{\alpha}_i)^s \alpha_i^{1-s} \right) (-\alpha_t)^{s} \alpha_t.
\]

The set \( \{ \beta_{s_1, \ldots, s_{t-1}}^\ast : (s_1, \ldots, s_{t-1}) \in \{0,1\}^{t-1} \} \) is a \( \mathbb{Z} \)-basis of the \( \mathbb{Z} \)-module \( \mathcal{O}_{K_G} \cap i\mathbb{R} \).

3. For any \( \eta, \nu \in \{0,1\}^{t-1} \)
\[
\sum_{\mu \in \{0,1\}^{t-1}} (-1)^{\mu_1 + \ldots + \mu_{t-1}} \tau_{\mu} \left( \beta_{\eta_1, \ldots, \eta_{t-1}}^\ast \beta_{\nu_1, \ldots, \nu_{t-1}}^\ast \right) = \begin{cases} \sqrt{d}, & \text{if } \eta = \nu, \\ 0, & \text{otherwise.} \end{cases}
\]
Proof. Let \( \beta'_{s_1,...,s_t} \) be the element of the basis from second assertion of Theorem 8 corresponding to the set \((s_1,...,s_t)\). 

A number from \( \mathcal{O}_{K_G} \) is in \( K_G \cap \mathbb{R} \) if and only if it is invariant under the complex conjugation. It is easy to see that the complex conjugation maps \( \beta'_{s_1,...,s_t} \) to \((-1)^{s_1} \beta'_{s_1,...,s_t,1-s_{t+1},...,1-s_{t+2},s_{t+1}} \). Thus, a \( \mathbb{Z} \)-linear combination of \( \beta'_{s_1,...,s_t} \) is invariant if and only if coefficients of \( \beta'_{s_1,...,s_t} \) and \( \beta'_{s_1,...,s_t,1-s_{t+1},...,1-s_{t+2},s_{t+1}} \) are the same for \( s_t = 0 \) and differ in the sign for \( s_t = 1 \). Now Theorem 8 implies that \( \{ \beta'_{s_1,...,s_t,0,s_t} + (-1)^{s_t} \beta'_{s_1,...,s_t,1-s_{t+1},...,1-s_{t+2},s_{t+1}} \} \) is a required basis. From the definition of \( \beta' \) it is easy to see that this sum is equal to \( \beta'_{s_1,...,s_t-1,s_t} \). This concludes the proof of the first assertion.

The second assertion is proved similarly to the first one.

The third assertion is checked by a direct calculation. Similar to the proof of Theorem 8 we obtain

\[
\sum_{\mu \in \{0,1\}^{t-1}} (-1)^{\mu_1+...+\mu_{t-1}} \tau_\mu \left( \beta_{\eta_1,...,\eta_{t-1}} \beta_{\eta_1,...,\eta_{t-1}}^* \right) = \left( \prod_{\alpha_{i=1}^u} \delta_{\eta_1,...,\eta_{t-1}} \sqrt{q_t^*} \right) \alpha_{t}^{\eta_1+...+\eta_{t-1}} + (-1)^{\eta_1+...+\eta_{t-1}} \left( \prod_{\alpha_{i=1}^u} \delta_{\eta_1,...,\eta_{t-1}} \sqrt{q_t^*} \right) \alpha_{t}^{\eta_1+...+\eta_{t-1}} \]  

Since \( q_1^* \ldots q_t^* < 0 \), the number of negative \( q_t^* \) (i.e. \( t - u \)) is odd. Therefore, \( \prod_{\alpha_{i=1}^u} (-1) = (-1)^{t-u-2} = -1. \)

\[
\sum_{\mu \in \{0,1\}^{t-1}} (-1)^{\mu_1+...+\mu_{t-1}} \tau_\mu \left( \beta_{\eta_1,...,\eta_{t-1}} \beta_{\eta_1,...,\eta_{t-1}}^* \right) = \left( \prod_{\alpha_{i=1}^u} \delta_{\eta_1,...,\eta_{t-1}} \sqrt{q_t^*} \right) \alpha_{t}^{\eta_1+...+\eta_{t-1}} \]  

\[= \delta_{\eta_1,...,\eta_{t-1}}^* \left( \prod_{\alpha_{i=1}^u} \delta_{\eta_1,...,\eta_{t-1}} \sqrt{q_t^*} \right) 2\alpha_{t}. \]

**Theorem 12.** Let \( q_1^*, \ldots, q_{t-1}^* \) are positive odd, \( q^*_t = -4 \) or \( q^*_t = -8 \).

1. Define \( \beta_{s_1,...,s_{t-1}} = \beta_{s_1,...,s_{t-1}} (q_1^*, \ldots, q_t^*) = \prod_{\alpha_{i=1}^u} \alpha_{i}^{s_1+...+s_{t-1}}. \)

The set \( \{ \beta_{s_1,...,s_{t-1}} : (s_1, \ldots, s_{t-1}) \in \{0,1\}^{t-1} \} \) is a \( \mathbb{Z} \)-basis of the ring of integers in \( K_G \cap \mathbb{R}. \)

2. Define \( \beta_{s_1,...,s_{t-1}}^* = \beta_{s_1,...,s_{t-1}}^* (q_1^*, \ldots, q_t^*) = \prod_{\alpha_{i=1}^u} (-\alpha_{i})^{s_1+...+s_{t-1}} \sqrt{q_t^*}. \)

The set \( \{ \beta_{s_1,...,s_{t-1}}^* : (s_1, \ldots, s_{t-1}) \in \{0,1\}^{t-1} \} \) is a \( \mathbb{Z} \)-basis of the \( \mathbb{Z} \)-module \( \mathcal{O}_{K_G} \cap i\mathbb{R}. \)

3. For any \( \eta, \nu \in \{0,1\}^{t-1} \)

\[
\sum_{\mu \in \{0,1\}^{t-1}} (-1)^{\mu_1+...+\mu_{t-1}} \tau_\mu \left( \beta_{\eta_1,...,\eta_{t-1}} \beta_{\eta_1,...,\eta_{t-1}}^* \right) = \begin{cases} \sqrt{d}, & \text{if } \eta = \nu, \\ 0, & \text{otherwise.} \end{cases} \]

**Proof.** Obviously, here \( K_G = M(\sqrt{q_t^*}) \) with \( M \subset \mathbb{R}. \) Thus \( K_G \cap \mathbb{R} = M, K_G \cap i\mathbb{R} = \sqrt{q_t^*} \cdot M. \) First two assertions follow from Theorem 8 The last assertion is checked by a direct calculation similar to the one from the proof of Theorem 8.

For convenience, we denote \( \beta_\mu = \beta_{\mu_1,...,\mu_{t-1}} \) for \( \mu \in \{0,1\}^{t-1}. \) Let \( M \) denote the field \( K_G \cap \mathbb{R}. \) The set \( \{ \beta_\mu \} \) is a \( \mathbb{Z} \)-basis of \( \mathcal{O}_M. \)

Let \( z \) be any element of \( \mathcal{O}_{K_G}. \) Since \( z \in K_G, \) also \( \overline{z} \in K_G \) and \( z + \overline{z} = 2 \text{Re} \ z \in K_G \cap \mathbb{R} \) and \( z - \overline{z} = 2i \text{Im} \ z \in K_G \cap i\mathbb{R}. \) Moreover, \( z \) and \( \overline{z} \) are algebraic integers, so \( 2 \text{Re} \ z \) and \( 2i \text{Im} \ z \) are algebraic integers too. Thus, \( 2 \text{Re} \ z = \sum_\mu b_\mu \beta_\mu \) and \( 2i \text{Im} \ z = \sum_\mu b_\mu^* \beta^*_\mu. \) Hereafter sums with parameter given by a Greek letter without an
Theorem 5. Then \((\frac{q^i}{A})\) implies that the image of \(\Omega(1)\) to \(\Omega(1)\) equals \(\Omega(\tilde{\theta})\) for \(z\) by an approximate expression for \(z\) by an approximate value, it is sufficient to solve the next task: restore the coefficients of the decomposition of a number given by a sufficiently accurate approximation, by a real basis.

The scheme of next sections is following.

- Consider a divisor of the polynomial \(H_D[\theta, \alpha_*]\) over the field \(K_D\). The degree of this divisor is \(\frac{N}{2}\). Section 3 deals with this task. The ultimate goal is to use this divisor instead of the full polynomial, thus decreasing the number and the magnitude of coefficients to be calculated.

- Calculate an apriori upper bound for all conjugates to coefficients of the divisor. This is done in Section 6.

- The main idea for calculating exact values is to use simultaneous rational approximations to the elements of a basis. Section 7 shows how to construct such approximations for \(\beta_*\) and \(\beta^*\) with any predefined precision. The actual precision depends on the bound from Section 6.

- Finally, Section 8 shows how to calculate exact values by approximations. Also Section 8 sums up all the steps used in our optimization.

5 Divisor of \(H_D[\theta, \alpha_*](x)\)

Let \(\tilde{a} \in H_D\). Select a form \((A, B, C)\) such that \(\eta(A, B, C) = \tilde{a}\) and \(\gcd(A, D) = 1\); this is possible because \(\eta\) depends only on the equivalence class of a form and each class contains a form \((A, B, C)\) with \(\gcd(A, D) = 1\) due to \(\eta\) Lemmas 2.25 and 2.3]. Let \(\varphi : H_D \rightarrow \{\pm 1\}^t\) be the map defined by the formula

\[\varphi(\tilde{a}) = \left(\left(\frac{q^i}{A}\right)\right)\in \Omega(1)\] .

This definition is correct because the Artin map depends only on an ideal class in \(I(O, f)/P_K(\tau(f))\) and Theorem 5 implies that \(\left(\frac{q^i}{A}\right)\) does not change when a form \((A, B, C)\) is replaced to an equivalent form.

**Theorem 13.** The image of the map \(\varphi\) is the group \(\{(\varepsilon_1, \ldots, \varepsilon_t) : \varepsilon_i = 1\}\). The map \(\varphi\) is a group homomorphism. The fixed field \(L^{\Omega(\Ker \phi)}\) is \(\{x \in L : \tau(x) = x\} \text{ for all } \tau \in \Omega(\Ker \phi)\). \(K(\sqrt{q^i_1}, \ldots, \sqrt{q^i_t})\).

**Proof.** The assertion 3 of Theorem 5 and the fact that the Artin map is a homomorphism imply that \(\varphi\) is a homomorphism.

Let \(a\) be the ideal for the form \((A, B, C)\) defined in Theorem 5. We have

\[\left(\frac{q^i}{A}\right) = \frac{1}{\sqrt{q^i}} \left(\frac{L/K}{\Omega(1)}\right)(\sqrt{q^i}).\]

Multiplying over all \(i\) and using Lemma 2 we obtain

\[\left(\frac{q^i}{A}\right) \ldots \left(\frac{q^i}{A}\right) = \frac{1}{\sqrt{d}} \left(\frac{L/K}{\Omega(1)}\right)(\sqrt{d}).\]

Since \(\sqrt{d} \in K\) and \(\left(\frac{L/K}{\Omega(1)}\right)\) is an element of Gal\((L/K)\), the right-hand side equals 1. This proves the inclusion of image of \(\varphi\) to \(\{(\varepsilon_i) : \varepsilon_i \in \{\pm 1\}\} \text{ with } \varepsilon_i = 1\).

Let \(\tilde{a}\) lie in the kernel of \(\varphi\) (i.e. \(\varphi(\tilde{a}) = (1, \ldots, 1)\)). Let \(a\) be the representative of \(\tilde{a}\) from the assertion 2 of Theorem 5. Then

\[\left(\frac{L/K}{\Omega(1)}\right)(\sqrt{q^i}) = \sqrt{q^i}.\]

Equivalently, the image of \(\Omega(1)\) under the Artin map acts trivially on all \(\sqrt{q^i}\). Due to the commutativity of the diagram (12) this image equals \(\Omega(\tilde{a})\). This proves the inclusion \(\left(\sqrt{q^i_1}, \ldots, \sqrt{q^i_t}\right) \subset L^{\Omega(\Ker \phi)}\).

According to Galois theory, \(\text{Gal}\left(L^{\Omega(\Ker \phi)}/K\right) \cong \text{Gal}\left(L/K\right)/\Omega(\Ker \phi) \cong H_D/Ker \phi \cong \text{Im} \varphi\). In particular, \([L^{\Omega(\Ker \phi)}/K] = |\text{Im} \varphi| \leq 2^{t-1}$. We proved in Section 4 that \([K(\sqrt{q^i_1}, \ldots, \sqrt{q^i_t}) : Q] = 2^t\), so \([K(\sqrt{q^i_1}, \ldots, \sqrt{q^i_t}) : K] = 2^{t-1}\). Thus, the chain of inequalities \([K(\sqrt{q^i_1}, \ldots, \sqrt{q^i_t}) : K] \leq [L^{\Omega(\Ker \phi)} : K] = |\text{Im} \varphi| \leq 2^{t-1}\) is possible only if \(|\text{Im} \varphi| = 2^{t-1}\) and \((\sqrt{q^i_1}, \ldots, \sqrt{q^i_t}) = L^{\Omega(\Ker \phi)}\). □
We suggest to calculate the polynomial
\[
\hat{H}_D[\theta, \alpha_*](x) = \prod_{i: \varphi(b(A_i, B_i, C_i))=(1,\ldots,1)} (x - \theta(\alpha_i)),
\]
which obviously divides \(H_D[\theta, \alpha_*]\), instead of the entire polynomial \(H_D[\theta, \alpha_*]\). Here the function \(\theta\) and the \(N\)-
-system \(\{(A_i, B_i, C_i)\}\) satisfy the assumption of one of Theorems 2-4, and \(\alpha_*\) is the root of the form \((A_i, B_i, C_i)\).

The main obstacle is that \(H_D[\theta, \alpha_*]\) is not invariant under \(\text{Gal}(L/K)\) and therefore does not lie in \(\mathbb{Q}[x]\).

Note that \(\varphi\) is a homomorphism. Using the formula (10), it is easy to see that \(\Omega(\text{Ker}\varphi)\) fixes \(\hat{H}_D[\theta, \alpha_*](x)\), therefore, this polynomial has coefficients in \(K_G\). All numbers \(\theta(\alpha)\) are algebraic integers (Theorems 2-4), so the coefficients of \(\hat{H}_D[\theta, \alpha_*]\) are also algebraic integers. Therefore, to use the polynomial \(\hat{H}_D[\theta, \alpha_*]\) in the complex multiplication method, one must know how to recover an algebraic integer from \(K_G\) by its sufficiently accurate approximation. Assuming that such a procedure is implemented, the other actions to generate an elliptic curve are the same as in the original method.

An idea to use the genus field in the CM method was already considered in [4] (1993). There the main obstacle for an algebraic integer \(z\) is solved in the following way. All conjugates of \(z\) are calculated. One looks for the exact value of \(z\) in the form of linear combination of some generators with unknown coefficients. Any conjugate of \(z\) is a linear combination of conjugates to generators with the same unknown coefficients. The known approximations for all conjugates give a system of linear equations for these coefficients, it allows to calculate them (approximately and then round to integer). We refer to [4] for the details. Note that this solution requires to calculate values \(\theta(\alpha)\) for roots of all elements of a \(N\)-system and all conjugate polynomials to \(\hat{H}_D[\theta, \alpha_*]\). Thus the optimization is only in the magnitude of the coefficients.

Our approach requires to calculate only the polynomial \(\hat{H}_D[\theta, \alpha_*]\) itself (although with a greater precision); in particular, it is sufficient to know only values \(\theta(\alpha)\) for roots of forms \(a\) with \(\varphi(\mathfrak{a}(a)) = (1, \ldots, 1)\). Theorem 13 obviously implies that the number of these forms is \(2^{r-1}\) times less than size of the \(N\)-system.

6 Bound for coefficients of \(\hat{H}_D[\theta, \alpha_*]\)

According to Theorems 2-4 each coefficient of the polynomial \(\hat{H}_D[\theta, \alpha_*]\) can be represented with a formula
\[
\frac{1}{2} \left( \sum_{\mu} b_\mu \beta_\mu + \sum_{\mu} b'_\mu \beta'_\mu \right),
\]
where \(b_\mu, b'_\mu \in \mathbb{Z}, \beta_\mu \in \mathbb{R}, \beta'_\mu \in i\mathbb{R}\). We need a bound for all conjugates,
\[
\frac{1}{2} \left| \sum_{\mu} b_\mu \beta_\mu + \sum_{\mu} b'_\mu \beta'_\mu \right| \leq T_0.
\]

Note that the polynomial \(\hat{H}_D[j, \alpha_*]\) does not depend on the set \(\alpha_*\), so the short notation \(\hat{H}_D[j] = \hat{H}_D[j, \alpha_*]\) is correct.

For theoretical bounds we apply the method from [21].

Let us consider along with \(\hat{H}_D[j]\) also polynomials
\[
\hat{H}_{D, \varphi_0}[j](x) = \prod_{i: \varphi(b(A_i, B_i, C_i))=\varphi_0} (x - j(\alpha_i)),
\]
where \(\varphi_0 \in \{0, 1\}^t\), \((A_i, B_i, C_i)\) runs over representatives of all form classes, \(\alpha_i\) is the root of \((A_i, B_i, C_i)\).

By definition, \(\hat{H}_D[j] = \hat{H}_{D,(1,\ldots,1)}[j]\). Similarly to \(\hat{H}_D[j]\), the polynomial \(\hat{H}_{D, \varphi_0}[j]\) is in \(O_{K_G}[x]\) for each \(\varphi_0\).

Moreover, if \(\sigma \in \text{Gal}(L/\mathbb{Q})\) is the automorphism corresponding to an ideal class \(b \in \mathcal{H}_G\), then \(\hat{H}_{D, \varphi_0}[j]_\sigma = \hat{H}_{D, \varphi_0[\varphi_0(\sigma)]}[j]\) due to Corollary 1.

Since any automorphism of the field \(K_G\) can be extended to an element of \(\text{Gal}(L/Q)\), for each \(\tau \in \text{Gal}(K_G/Q)\) there exists \(\varphi_0 = \varphi_0(\tau)\) such that \(\hat{H}_{D, \varphi_0}[j]_\tau = \hat{H}_{D, \varphi_0[\varphi_0(\tau)]}[j]\) for any \(\varphi_1 \in \{1, \pm 1\}^t\).

**Theorem 14.** The absolute value of each coefficient of the polynomial \(\hat{H}_{D, \varphi_0}[j]\) does not exceed
\[
\exp \left( c_5 h + c_1 N \left( \ln^2 N + 4\gamma \ln N + c_6 + \frac{\ln N + \gamma + 1}{N} \right) \right) \leq \exp \left( c_1 N \ln^2 N + c_2 N \ln N + c_3 N + c_1 \ln N + c_4 \right) = T_0,
\]
where \(N = \sqrt{|D|}, \gamma = 0.577... \) is the Euler constant, \(c_1 = \sqrt{3}\pi = 5.441..., c_2 = 18.587..., c_3 = 17.442..., c_4 = 11.594..., c_5 = 3.011... c_6 = 2.566...\) The asymptotic upper bound
\[
T_0 = \exp O \left( \sqrt{|D|} \ln^2 |D| \right)
\]
holds for other functions \(\theta\) too.
Proof. We follow [21] Section 4.

We can assume that \((A_i, B_i, C_i)\) in (17) are reduced forms, because a change of a form to an equivalent form corresponds to some \(SL_2(\mathbb{Z})\)-transformation of the form root and the function \(j\) is invariant under these. Let \((A, B, C)\) be a reduced form; we need an upper bound for \(|j\left(\frac{-B + \sqrt{D}}{2A}\right)|\). The argument of \(j\) lies in the area \(\{z \in \mathbb{H} : |z| \geq 1, \text{Re} \ z \leq \frac{1}{2}\}\). Therefore, \(\text{Im} \ z \geq \frac{\sqrt{3}}{2}\) and \(|q| = |e^{2\pi iz}| \leq e^{-\pi \sqrt{3}}\). Furthermore,

\[ j(z) = \frac{1}{q} + 744 + \sum_{m=1}^{\infty} c_m q^m, \]

where \(|c_m| \leq \frac{4 \pi m}{\sqrt[4]{m}}\) due to [22]. Thus,

\[ |j\left(\frac{-B + \sqrt{D}}{2A}\right) - \frac{1}{q}| \leq 744 + \sum_{m=1}^{\infty} \frac{e^{4 \pi m}}{\sqrt{2m}^{3/4}} e^{-\pi \sqrt{3m}} = k_1 = 2114.566... \]

and \(|j\left(\frac{-B + \sqrt{D}}{2A}\right)| \leq \frac{1}{|q|} + k_1 \leq k_2\) with \(k_2 = 1 + k_1 e^{-\pi \sqrt{3}} = 10.163....\)

Assume that all reduced forms are numbered so that \(\{(A_i, B_i, C_i) : 1 \leq i \leq \deg \hat{H}_D[j]\}\) are all reduced forms from the product (17) ordered by increasing \(|q| = e^{\pi \sqrt{|D|}/A_i}\). The absolute value of the coefficient of \(x^i\) in \(\hat{H}_{D,\varphi_0}[j]\) does not exceed

\[ C_{\deg \hat{H}_D[j]}^{k_2} \prod_{i=k+1}^{\deg \hat{H}_D[j]} \frac{k_2}{|q_i|} \leq (2k_2)^{h/2^{i-1}} \prod_{i=1}^{h/2^{i-1}} e^{\pi \sqrt{|D|}/A_i}. \]

Therefore, the logarithm of any coefficient of \(\hat{H}_{D,\varphi_0}[j]\) does not exceed

\[ \frac{h}{2^{i-1}} \ln(2k_2) + \pi \sqrt{|D|} \sum_{i=1}^{h/2^{i-1}} \frac{1}{A_i} \leq h \ln(2k_2) + \pi \sqrt{|D|} \sum_{i=1}^{h} \frac{1}{A_i}. \]

The bound for the last sum proved in [21] Theorem 1.2 concludes the proof for \(j\).

The bound for other functions \(\theta\) follows from the proved one and [23] Proposition 3.

In practice it is better to use heuristic, but more accurate bounds.

The article [23] suggests the following upper bound for logarithms of absolute values of coefficients of the polynomial \(H_D[j]\):

\[ \pi \sqrt{|D|} \sum_{(A, B, C)} \frac{1}{A}, \]

where the sum is over all reduced forms. This bound is heuristic, but sufficiently close to the exact value. The same article suggests multiplying this sum by some constant depending on \(\theta\) to obtain the analogous bound for \(H_D[\theta]\). The constant is the ratio \(\frac{\deg \Phi}{\deg \Phi}\), where a polynomial \(\Phi\) in two variables links functions \(\theta\) and \(j\) so that \(\Phi(\theta(z), j(z)) = 0\).

Trivial changes of the arguments from [23] with respect to \(\hat{H}_D[j]\) give the heuristic bound

\[ \ln T_0 \sim \pi \sqrt{|D|} \max_{\varepsilon \in \{\pm 1\}^3} \text{max}_{(A, B, C), \varepsilon(A, B, C) = \varepsilon} \sum_{A, B, C} \frac{1}{A} \]  

for the invariant \(j\). Again, for other invariants this bound should be multiplied by \(\frac{\deg \Phi}{\deg \Phi}\).

Let

\[ z = \frac{1}{2} \left( \sum_{\mu} b_{\mu} \beta_{\mu} + \sum_{\mu} b_{\mu}' \beta_{\mu}' \right) \]

be a coefficient of the polynomial \(\hat{H}_{D,\varphi_0}[j]\),

\(b_{\mu}, b_{\mu}' \in \mathbb{Z}\). As mentioned above, the action of \(\text{Gal}(K_G/Q)\) maps the polynomial \(\hat{H}_{D,\varphi_1}[j]\) to the polynomial of the same type. So for any \(\lambda \in \{0, 1\}^3\) the following inequation holds:

\[ |\tau_{\lambda}(z)| \leq T_0. \]
7 Construction of rational approximations to a basis of ring of algebraic integers

There is a number of different algorithms for constructing simultaneous rational approximations to a given set of real numbers. The book [24] covers many of them. Properties of approximations differ significantly for different algorithms. For practical purposes the inner product algorithm from [24, Chapter 6A] seems to be the best in the general case. Unfortunately, it is quite difficult to prove good theoretical bounds for universal algorithms. Therefore we suggest another algorithm which allows to obtain theoretical bounds, but works only for very specific sets.

In essence, the main part of the following theorem is contained in the article [25]. Main differences between the following theorem and [25] are following: the explicit formulation, including explicit constants; the function $\mathfrak{M}$ (25) deals with dual bases which is equivalent to $\mathfrak{M} = 1$; specialization for our case (25) does not require for $M/Q$ to be Galois and also contains a converse theorem.

**Theorem 15.** Let $M \subset \mathbb{R}$ be a field such that $M/Q$ is a Galois extension of degree $m$. Let $W_1, \ldots, W_m$ and $W_1^*, \ldots, W_m^*$ be two bases of $M$. Let $\mathfrak{M} : \text{Gal}(M/Q) \to \mathbb{R}$ be a function (not necessarily a homomorphism) such that for each $1 \leq l, l' \leq m$ the following equality holds:

$$\sum_{\tau \in \text{Gal}(M/Q)} \mathfrak{M}(\tau)\tau(W_l W_{l'}^*) = \begin{cases} 1, & \text{if } l = l', \\ 0, & \text{if } l \neq l'. \end{cases}$$

Let

$$C = \sum_{\tau \in \text{Gal}(M/Q), \tau \neq \text{Id}} |\mathfrak{M}(\tau)(W_1)|$$

and

$$C_i = \sum_{\tau \in \text{Gal}(M/Q), \tau \neq \text{Id}} |\mathfrak{M}(\tau)(\tau(W_i) - W_i \tau(W_i))|$$

for $i = 2, \ldots, m$. Let a positive number $\Delta$ and integers $\Lambda_1, \ldots, \Lambda_m$ satisfy the inequalities

$$\sum_{i=1}^m \Lambda_i W_i^* = Z \geq 1,$$

$$\left| \tau \left( \sum_{i=1}^m \Lambda_i W_i^* \right) \right| \leq \frac{\Delta}{Z^{m-1}} \quad \text{for each } \tau \in \text{Gal}(M/Q), \tau \neq \text{Id}.$$

Then:

- $|\Lambda_1| \geq |\mathfrak{M}(\text{Id})W_1|Z - C\Delta$.
- If $|\Lambda_1| > C\Delta$, then $\mathfrak{M}(\text{Id}) \neq 0$ and the following bound holds for each $i = 2, \ldots, m$:

$$\left| \frac{\Lambda_i}{\Lambda_1} - \frac{W_i}{W_1} \right| \leq C_i \frac{\Delta}{|\Lambda_1| \left( \frac{|\Lambda_1| - C\Delta}{|\mathfrak{M}(\text{Id})W_1|} \right)^{m-1}}.$$

**Proof.** For each $l = 1, \ldots, m$

$$\Lambda_l = \sum_{\nu=1}^m \Lambda_{\nu} \left( \sum_{\tau \in \text{Gal}(M/Q)} \mathfrak{M}(\tau)(W_l W_{\nu}^*) \right)$$

$$= \sum_{\tau \in \text{Gal}(M/Q)} \mathfrak{M}(\tau)(W_l) \left( \sum_{\nu=1}^m \Lambda_{\nu} \tau(W_{\nu}^*) \right)$$

$$= \mathfrak{M}(\text{Id})W_l Z + \sum_{\tau \in \text{Gal}(M/Q), \tau \neq \text{Id}} \mathfrak{M}(\tau)\tau(W_l)\tau(Z). \quad (19)$$

Substitute $l = 1$:

$$\Lambda_1 = \mathfrak{M}(\text{Id})W_1 Z + \sum_{\tau \in \text{Gal}(M/Q), \tau \neq \text{Id}} \mathfrak{M}(\tau)(W_1)\tau(Z). \quad (20)$$
Using the definition of $C$ and the bound for $\tau(Z)$, we obtain
\[ |A_1 - \mathfrak{M}(Id)W_1 Z| \leq \frac{C\Delta}{Z^{-\varepsilon}} \leq C\Delta. \]

This proves the first assertion.
Assume that $|A_1| > C\Delta$. Then
\[ |\mathfrak{M}(Id)W_1 Z| \geq |A_1| - C\Delta. \]

Therefore, $\mathfrak{M}(Id) \neq 0$ and
\[ Z \geq \frac{|A_1| - C\Delta}{|\mathfrak{M}(Id)W_1|}. \] (21)

Multiply the equality (20) by $\frac{W_1}{|A_1|}$ and subtract from (19). Then use the definition of $C_l$ and the bound for $\tau(Z)$:
\[ \left| \frac{A_1 - W_1}{W_1} \right| \leq C_l \frac{\Delta}{|A_1|Z^{-\varepsilon}}. \]

Divide the last inequality by $|A_1|$:
\[ \left| \frac{A_1}{A_1} - \frac{W_1}{W_1} \right| \leq C_l \frac{\Delta}{|A_1|Z^{-\varepsilon}}. \]

Now it is sufficient to use (21) to conclude the proof. □

The article uses a knowledge of group of units in $O_M$ (Dirichlet theorem) and looks for $\sum_{i=1}^m A_iW_i^{*}$ as a unit of a special form. It allows to prove interesting theoretical results, but it is quite inconvenient from the practical point of view. We use another approach.

We want to construct simultaneous approximations to elements of the field $M = K_G \cap \mathbb{R}$. In order to do this, we apply Theorem 15 to the field $M = M$. Thus, $m = [M : \mathbb{Q}] = 2^{t-1}$, $t \geq 2$, and $\text{Gal}(M/\mathbb{Q})$ consists of automorphisms $\tau_\lambda$ defined by (15), $\lambda \in \{0, 1\}^{t-1}$.

It is convenient to enumerate sets related to the field $M$ by vectors from $\{0, 1\}^{t-1}$. Hereafter we assume that two basises $\omega_{\mu}$ and $\omega_{\mu}^{*}$ of $M$ over $\mathbb{Q}$ and a function $\mathfrak{M} : \text{Gal}(M/\mathbb{Q}) \to \mathbb{R}$ are given and satisfy the following conditions:

1. $\omega_{0, \ldots, 0} = 1$.
2. Any element of $O_M$ is a linear combination of $\{\omega_{\mu}^{*}\}$ with integer coefficients.
3. For any $\lambda, \lambda' \in \{0, 1\}^{t-1}$,
\[ \sum_{\mu \in \{0, 1\}^{t-1}} \mathfrak{M}(\tau_\mu) (\omega_\lambda \omega_{\lambda'}) = \begin{cases} 1, & \text{if } \lambda = \lambda', \\ 0, & \text{if } \lambda \neq \lambda'. \end{cases} \] (22)

We call such a pair an $\mathfrak{M}$-pair. It is easy to see that these conditions imply conditions on basises from Theorem 15 applied to the numbers
\[ W_1 + \mu_1 + 2\mu_2 + 2^2\mu_3 + \ldots + 2^{t-2}\mu_{t-1} = \omega_{\mu}, \]
\[ W_1^{*} + \mu_1 + 2\mu_2 + 2^2\mu_3 + \ldots + 2^{t-2}\mu_{t-1} = \omega_{\mu}^{*}. \]

Note that if $x \in O_M$, then $x\beta_{0, \ldots, 0}^{*} \in O_{K_G} \cap \mathbb{R}$. Two following corollaries follow easily from Theorems 9 12. As in these theorems, the value of $\sqrt{d}$ is chosen as the product $\sqrt{q_1} \ldots \sqrt{q_t}$.

**Corollary 2.** Conditions 1–3 hold for
\[ \omega_{\mu_1, \ldots, \mu_{t-1}}^{*} = \frac{\beta_{\mu_1, \ldots, \mu_{t-1}}}{\sqrt{d}}, \]
\[ \omega_{\mu_1, \ldots, \mu_{t-1}} = \frac{\beta_{\mu_1, \ldots, \mu_{t-1}}}{\sqrt{d}}. \] (23)

**Corollary 3.** Conditions 1–3 hold for
\[ \mathfrak{M}(\tau_{\mu_1, \ldots, \mu_{t-1}}) = (-1)^{\mu_1 + \ldots + \mu_{t-1}} (\beta_{\mu_1, \ldots, \mu_{t-1}} \beta_{0, \ldots, 0}^{*})/\sqrt{d}. \]

\[ \mathfrak{M}(\tau_{\mu_1, \ldots, \mu_{t-1}}) = (-1)^{\mu_1 + \ldots + \mu_{t-1}} (\beta_{\mu_1, \ldots, \mu_{t-1}} \beta_{0, \ldots, 0}^{*})/\sqrt{d}. \] (24)
Theorem 13 also uses integer numbers $\Lambda_i$ and a constant $\Delta$. The rest of this section deals with construction of a set $A_\mu$ such that the numbers

$$A_{1+\mu_1+2\mu_2+2^2\mu_3+\ldots+2^{t-1}\mu_{t-1}} = A_{\mu_1,\ldots,\mu_{t-1}}$$

satisfy the assumption of Theorem 13 with some $\Delta$.

We need the following quantities to describe the algorithm. Let $\lambda \in \{0, 1\}^{t-1}$, $\lambda \neq (0, \ldots, 0)$. Define $\delta_\lambda = (q_1^\lambda)^{\lambda_1} \ldots (q_{t-1}^\lambda)^{\lambda_{t-1}} (q_t^\lambda)^{\lambda_{t+1} \oplus \ldots \oplus \lambda_1 - 1}$. If $\delta_\lambda$ is even, set $g_\lambda = \sqrt{\delta_\lambda}/2$; otherwise set $g_\lambda = 1 + \sqrt{\delta_\lambda}/2$.

Then $g_\lambda \in O_M$.

We use continued fractions. We remind that for any number $X \in \mathbb{R}$ two sequences are defined: complete quotients $X_0, X_1, X_2, \ldots$, and partial quotients $a_0, a_1, a_2, \ldots$, where $X_0 = X$, $a_n = \lfloor X_n \rfloor$, $X_{n+1} = X - a_n$. These sequences are finite (i.e. $X_n$ is indefinite for some $n$) if and only if $X \in \mathbb{Q}$. In addition, the sequence of convergents $P_0/Q_0, P_1/Q_1, P_2/Q_2, \ldots$ is defined as follows: $P_0 = 0, Q_0 = 0, P_0 = a_0, Q_0 = 1, P_{n+1} = a_{n+1}P_n + P_{n-1}, Q_{n+1} = a_{n+1}Q_n + Q_{n-1}$. It is well known (e.g. §11.10), that for any $n \geq 0$

$$\left| X - \frac{P_n}{Q_n} \right| < \frac{1}{Q_nQ_{n+1}}, \text{ if } X_{n+2} \text{ is defined;}$$

$$Q_n \geq 2^\frac{n}{x}. \quad (25)$$

In the case of quadratic irrationals these sequences have an additional structure. We use some results from §11.10 collected in the next statement.

**Statement 2.** Let $a, b, c$ be integer numbers with $\gcd(a, b, c) = 1$. Let $\delta = b^2 - 4ac > 0$ be not an exact square. We call the roots of the equation $ax^2 + bx+c = 0$ as irrationals of determinant $\delta$.

Let $X = \frac{-b+\sqrt{\delta}}{a}$ be an irrational of determinant $\delta$. Then all complete quotients $X_n$ are also irrationals of determinant $\delta$ and have a form $X_n = \frac{x_n + \sqrt{\delta}}{y_n}$, where $x_n, y_n \in \mathbb{Z}$ are uniquely determined. Let $a_n = \lfloor X_n \rfloor$ be partial quotients for $X$. Define $y_{-1} = -c = \frac{\delta - b^2}{a} \in \mathbb{Z}$. The following recurrent formulas hold:

$$x_n = y_{n-1}a_n - x_{n-1}, \quad n \geq 1;$$

$$\delta = x_n^2 + y_ny_{n-1}, \quad n \geq 0;$$

$$y_n = y_{n-2} - a_{n-1}(x_n - x_{n-1}), \quad n \geq 1. \quad (27)$$

Moreover, for $n \geq 0$

$$X_1 \ldots X_n = \frac{(-1)^n}{P_{n-1} - Q_{n-1}X};$$

$$aP_{n-1}^2 + 2bP_{n-1}Q_{n-1} + cQ_{n-1}^2 = (-1)^n y_n. \quad (28)$$

A number $\frac{x+y\sqrt{\delta}}{y}$ with $x, y \in \mathbb{Z}$ is reduced if $\frac{x+y\sqrt{\delta}}{y} > 1$ and $-1 < \frac{x+y\sqrt{\delta}}{y} < 0$. A number $\frac{x+y\sqrt{\delta}}{y}$ is reduced if and only if $0 < \sqrt{\delta} - x < y < \sqrt{\delta} + x$. If $X$ is reduced, then all complete quotients for $X$ are also reduced.

We calculate continued fractions for all numbers $g_\lambda$ in parallel, $\lambda \in \{0, 1\}^{t-1}$, $\lambda \neq 0$. Let $X_{\lambda, n}$ be complete quotients for $g_\lambda$, $a_{\lambda, n}$ be partial quotients for $g_\lambda$. Let $P_{\lambda, n}$ and $Q_{\lambda, n}$ be numerators and denominators of convergents of $g_\lambda$ respectively. Let $x_{\lambda, n}, y_{\lambda, n}$ be the quantities $x_n, y_n$ from Statement 2 calculated for $X = g_\lambda$. Let $\sigma_\lambda$ denote the only nontrivial automorphism of the field $\mathbb{Q}(g_\lambda)$.

If $\delta_\lambda$ is odd, then $g_\lambda$ is an irrational of determinant $\delta$, $x_{\lambda, 0} = 1, y_{\lambda, 0} = 2, y_{\lambda, 1} = \frac{\delta_\lambda + 1}{2}$. It is easy to see from §11.10 by induction that $x_{\lambda, n}$ is odd and $y_{\lambda, n}$ is even for all $n$. Let $x'_{\lambda, n} = \frac{y_{\lambda, n} - 1}{2} \in \mathbb{Z}$ and $y'_{\lambda, n} = \frac{y_{\lambda, n} + 1}{2} \in \mathbb{Z}$. The quadratic polynomial $ax^2 + 2bx+c$, where $a, b, c$ are defined in Statement 2 has the first coefficient 2 and roots $g_\lambda, \sigma_\lambda(g_\lambda)$. Thus, (28) is equivalent to $2(P_{\lambda, n-1} - Q_{\lambda, n-1}y_{\lambda, n})\sigma_\lambda(P_{\lambda, n-1} - Q_{\lambda, n-1}y_{\lambda, n}) = (-1)^ny_{\lambda, n} = (-1)^{n+1}y'_{\lambda, n}$.

If $\delta_\lambda$ is even, then $g_\lambda$ is an irrational of determinant $\frac{\delta_\lambda}{4}$, $x_{\lambda, 0} = 0, y_{\lambda, 0} = 1, y_{\lambda, 1} = \frac{\delta_\lambda}{2}$. Let $x'_{\lambda, n} = x_{\lambda, n}$ and $y'_{\lambda, n} = y_{\lambda, n}$. The quadratic polynomial $ax^2 + 2bx+c$, where $a, b, c$ are defined in Statement 2 has the first coefficient 1 and roots $g_\lambda, \sigma_\lambda(g_\lambda)$. Thus, (28) is equivalent to $(P_{\lambda, n-1} - Q_{\lambda, n-1}y_{\lambda, n})\sigma_\lambda(P_{\lambda, n-1} - Q_{\lambda, n-1}y_{\lambda, n}) = (-1)^ny_{\lambda, n} = (-1)^ny'_{\lambda, n}$.\n
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In both cases
\[ X_{\lambda,n} = \frac{g_{\lambda} + x'_{\lambda,n}}{y'_{\lambda,n}}; \]

\[ (P_{\lambda,n-1} - Q_{\lambda,n-1}g_{\lambda})\sigma_{\lambda}(P_{\lambda,n-1} - Q_{\lambda,n-1}g_{\lambda}) = (-1)^n y'_{\lambda,n}. \quad (29) \]

Statement 2 gives an efficient method to calculate numbers \( x'_{\lambda,n}, y'_{\lambda,n} \), \( \sigma_{\lambda} = [X_{\lambda,n}] \) in sequence and then \( P_{\lambda,n} \) and \( Q_{\lambda,n} \). The algorithm uses numbers \( x'_{\lambda,n}, y'_{\lambda,n} \) and
\[ z_{\lambda,n} = \frac{1}{X_{\lambda,1} \ldots X_{\lambda,n}} = (-1)^n (P_{\lambda,n-1} - Q_{\lambda,n-1}g_{\lambda}) \in \mathcal{O}_M. \quad (30) \]

This definition and the equality (20) imply that for any \( n \geq 0 \)
\[ z_{\lambda,n} \sigma_{\lambda}(z_{\lambda,n}) = (-1)^n y'_{\lambda,n}. \quad (31) \]

Numbers \( A_\mu \) are taken from the equality
\[ \prod_{\lambda \neq 0} (-1)^n \sigma_{\lambda}(z_{\lambda,n}) = \sum_\mu A_\mu \omega^*_\mu. \]

The left-hand side is the product of algebraic integers due to (20), so the condition 2 on \( \mathcal{M} \)-pair guarantees that \( A_\mu \) are integers.

Each step of the algorithm increments exactly one of numbers \( n_\lambda \). This multiplies \( \sum_\mu A_\mu \omega^*_\mu \) by
\[ \frac{(-1)^{n+1} \sigma_{\lambda}(z_{\lambda,n+1})}{(-1)^n \sigma_{\lambda}(z_{\lambda,n})} = \frac{y'_{\lambda,n+1}/z_{\lambda,n+1}}{y'_{\lambda,n}/z_{\lambda,n}} = \frac{X_{\lambda,n+1}y'_{\lambda,n+1}}{y'_{\lambda,n}} = \frac{g_{\lambda} + x'_{\lambda,n+1}}{y'_{\lambda,n}}. \]

Thus, we need to switch from the set \( A_\mu \) to the set \( A'_\mu \) such that
\[ \left( \sum_\mu A'_\mu \omega^*_\mu \right) = \left( \sum_\xi A_\xi \omega^*_\xi \right) \frac{g_{\lambda} + x_{\lambda}}{y_{\lambda}} \]
(where \( x_{\lambda} = x'_{\lambda,n+1} \) and \( y_{\lambda} = y'_{\lambda,n} \)). Since \( \{ \omega^*_\mu \} \) is a \( \mathbb{Q} \)-basis of \( \mathcal{M} \) and \( g_\mu \in \mathcal{M} \), we can precompute numbers \( c_{\mu \xi \eta} \in \mathbb{Q} \) such that
\[ \omega^*_\xi g_\eta = \sum_\mu c_{\mu \xi \eta} \omega^*_\mu. \]

On each step we calculate
\[ \left( \sum_\xi A_\xi \omega^*_\xi \right) \frac{g_{\lambda} + x_{\lambda}}{y_{\lambda}} = \frac{1}{y_{\lambda}} \left( \sum_\xi A_\xi \sum_\mu c_{\mu \xi \eta} \omega^*_\mu + \sum_\xi A_\xi \omega^*_\xi x_{\lambda} \right) = \sum_\mu \sum_\xi A_\xi c_{\mu \xi \eta} + A_\mu x_{\lambda} \omega^*_\mu. \]

Now we are ready to show the algorithm.

**Algorithm for construction of simultaneous approximations.** Input data: the sets \( \delta_\lambda, g_\lambda, c_{\mu \xi \eta} \) as above, the threshold \( N_0 > 0 \). Output data: the set of integer numbers \( A_\mu \) such that \( |A_0, \ldots, A| \geq N_0 \) and \( A_\mu \) is an approximation to \( \frac{\omega^*_\mu}{\omega^*_{\lambda,n}} \) for each \( \mu \in \{0,1\}^{t-1} \).

The algorithm keeps a set of \( 2^{t-1} \) integer numbers \( A_\mu \) and auxiliary sets of non-negative integers \( x_{\lambda}, \) positive integers \( (y_{\lambda}, \tilde{y}_{\lambda}) \) and positive reals \( (z_{\lambda}, \tilde{z}_{\lambda}) \) for \( \lambda \in \{0,1\}^{t-1}, \lambda \neq (0, \ldots, 0) \). These sets have the following sense: if each vector \( \lambda \) was selected \( n_\lambda \) times during the step 3 below, then
\[ x_{\lambda} = x'_{\lambda,n_\lambda}, \]
\[ (y_{\lambda}, \tilde{y}_{\lambda}) = (y'_{\lambda,n_\lambda}, y'_{\lambda,n_\lambda-1}), \]
\[ (z_{\lambda}, \tilde{z}_{\lambda}) = (z_{\lambda,n_\lambda}, z_{\lambda,n_\lambda-1}), \]
\[ \sum_\mu A_\mu \omega^*_\mu = \prod_{\lambda \neq 0} (-1)^n \sigma_{\lambda}(z_{\lambda,n_\lambda}). \]

The algorithm consists of the following steps.
1. **Initialization.** For each \( \lambda \in \{0, 1\}^{t-1}, \lambda \neq (0, \ldots, 0) \) set
\[
A_{0,\ldots,0} := 1, \\
A_{\lambda} := 0, \\
x_{\lambda} := 0, \\
(y_{\lambda}, \tilde{y}_{\lambda}) := (1, \lceil \frac{1}{2} \rceil) \\
(z_{\lambda}, \tilde{z}_{\lambda}) := (1, g_{\lambda}).
\]

2. **Iterations.** Repeat the following steps while \(|A_{0,\ldots,0}| < N_0\).

3. Select any \( \lambda \) such that \( z_{\lambda} = \max_{\mu \neq (0, \ldots, 0)} z_{\mu} \).

4. Calculate \( a = \left\lfloor \frac{g_{\lambda} + x_{\lambda}}{y_{\lambda}} \right\rfloor \).

5. Set \( (z_{\lambda}, \tilde{z}_{\lambda}) := (\tilde{z}_{\lambda} - az_{\lambda}, z_{\lambda}) \).

6. Save \( x = x_{\lambda} \). Set \( x_{\lambda} := ay_{\lambda} - x_{\lambda} - 4 \left\{ \frac{1}{2} \right\} \). Set \( (y_{\lambda}, \tilde{y}_{\lambda}) := (\tilde{y}_{\lambda} - a(x_{\lambda} - x), y_{\lambda}) \). (As shown below, the new value of \( x_{\lambda} \) is always a non-negative integer, the new value of \( y_{\lambda} \) is always a positive integer.)

7. For each \( \mu \) calculate
\[
A'_{\mu} = \sum_{\xi} A_{\xi} c_{\mu\xi} + A_{\mu} x_{\lambda}.
\]

(As shown above, \( A'_{\mu} \in \mathbb{Z} \) for all \( \mu \).) Set \( A_{\mu} := A'_{\mu} \).

**Theorem 16.** The algorithm completes in \( O(\ln N_0) \) steps. The following inequalities hold in every step of the algorithm:
\[
0 \leq x_{\lambda} < \sqrt{\delta_{\lambda}} - g_{\lambda}, \\
0 < y_{\lambda} < \sqrt{\delta_{\lambda}}; \\
Z = \sum_{\mu} A_{\mu} \omega_{\mu} \geq 1, \\
\left| \tau_{\lambda} \left( \sum_{\mu} A_{\mu} \omega_{\mu}^{\star} \right) \right| \leq \frac{\sqrt{d_{\lambda}^{\star}}}{Z} \quad \text{for} \quad \lambda \neq (0, \ldots, 0).
\]

**Proof.** We start from the bounds for \( x'_{\lambda,n}, y'_{\lambda,n} \).

**Lemma 5.** Let \( \lambda \in \{0, 1\}^{t-1}, \lambda \neq 0 \). Let \( n \geq 1 \) be an integer. Then
\[
0 \leq x'_{\lambda,n} < \sqrt{\delta_{\lambda}} - g_{\lambda}, \\
0 < y'_{\lambda,n} < \sqrt{\delta_{\lambda}}, \\
-1 < \sigma_{\lambda}(X_{\lambda,n}) < 0.
\]

**Proof.** Assume first that \( \delta_{\lambda} \) is odd. By definition, \( X_{\lambda,1} = \frac{1}{g_{\lambda} - |x_{\lambda}|} \). Obviously, \( X_{\lambda,1} > 1 \). In addition, \( \sigma_{\lambda}(X_{\lambda,1}) = \frac{1}{1-g_{\lambda} - |x_{\lambda}|} \) and \( g_{\lambda} > 1 \) imply that \( -1 < \sigma_{\lambda}(X_{\lambda,1}) < 0 \). Therefore, due to Statement 2 all complete quotients of \( g_{\lambda} \) starting from \( X_{\lambda,1} \) are reduced irrationals of determinant \( \delta_{\lambda} \). That is, \( 0 < \sqrt{\delta_{\lambda}} - x_{\lambda,n} < y_{\lambda,n} < \sqrt{\delta_{\lambda}} + x_{\lambda,n} \) for \( n \geq 1 \). Since \( x'_{\lambda,n} = \frac{x_{\lambda,n} - 1}{2} \) and \( y'_{\lambda,n} = \frac{y_{\lambda,n} - 1}{2} \) in this case, we obtain the required bounds.

Assume now that \( \delta_{\lambda} \) is even. As in the first case, \( X_{\lambda,1} = \frac{1}{g_{\lambda} - |x_{\lambda}|} > 1 \). In addition, \( \sigma_{\lambda}(X_{\lambda,1}) = \frac{1}{g_{\lambda} + |x_{\lambda}|} \) and \( g_{\lambda} > 1 \) imply that \( -1 < \sigma_{\lambda}(X_{\lambda,1}) < 0 \). Therefore, due to Statement 2 all complete quotients of \( g_{\lambda} \) starting from \( X_{\lambda,1} \) are reduced irrationals of determinant \( \frac{1}{2} \). That is, \( 0 < \sqrt{\delta_{\lambda}} - x_{\lambda,n} < y_{\lambda,n} < \sqrt{\delta_{\lambda}} + x_{\lambda,n} \) for \( n \geq 1 \). Since \( x'_{\lambda,n} = x_{\lambda,n} \) and \( y'_{\lambda,n} = y_{\lambda,n} \) in this case, we obtain the required bounds.

Since \( X_{\lambda,n} = \frac{g_{\lambda} + x'_{\lambda,n}}{y'_{\lambda,n}} \), Lemma 5 immediately implies

**Corollary 4.** For \( n \geq 1 \)
\[
X_{\lambda,n} < \sqrt{\delta_{\lambda}}.
\] (32)

Let \( n_{\lambda} \) denote the number of times when \( \lambda \) was selected in the step 3 of the algorithm, \( \lambda \neq 0 \).

The inequality \( Z = \prod_{\lambda \neq 0} ((-1)^{n_{\lambda}} \sigma_{\lambda}(z_{\lambda,n_{\lambda}})) \geq 1 \) follows immediately from the last inequality of Lemma 5 and the definition \( z_{\lambda,n_{\lambda}} = \frac{1}{X_{\lambda,1} \cdots X_{\lambda,n_{\lambda}}} \).
Lemma 6.

\[
\max_{\mu \neq (0, \ldots, 0)} z_{\mu, n_\mu} = \max_{\mu \neq (0, \ldots, 0)} z_{\mu, n_\mu},
\]

\[
\min_{\mu \neq (0, \ldots, 0)} z_{\mu, n_\mu} \leq \sqrt{|d|}.
\]

Proof. Before iterations the left-hand side equals 1, so the inequality holds. Assume that the inequality holds after some number of iterations. Assume that the step 3 of the next iteration selects the value \( \lambda \), i.e.

\[
z_{\lambda, n_\lambda} = \max_{\mu \neq (0, \ldots, 0)} z_{\mu, n_\mu}.
\]

Let \( n'_\lambda = n_\lambda + 1 \) and \( n'_\mu = n_\mu \) for \( \mu \neq \lambda, \mu \neq (0, \ldots, 0) \). Obviously, \( X_{\lambda, n'_\lambda} > 1 \), so \( z_{\lambda, n'_\lambda} < z_{\lambda, n_\lambda} \). There are two possible cases:

- \( z_{\lambda, n'_\lambda} \geq \min_{\mu \neq (0, \ldots, 0)} z_{\mu, n_\mu} \). In this case

\[
\min_{\mu \neq (0, \ldots, 0)} z_{\mu, n'_\mu} = \min_{\mu \neq (0, \ldots, 0)} z_{\mu, n_\mu},
\]

therefore,

\[
\max_{\mu \neq (0, \ldots, 0)} z_{\mu, n'_\mu} \leq \max_{\mu \neq (0, \ldots, 0)} z_{\mu, n_\mu} \leq \sqrt{|d|}.
\]

- \( z_{\lambda, n'_\lambda} < \min_{\mu \neq (0, \ldots, 0)} z_{\mu, n_\mu} \). In this case \( \min_{\mu \neq (0, \ldots, 0)} z_{\mu, n'_\mu} = z_{\lambda, n'_\lambda} \); using \([32]\), we obtain

\[
\max_{\mu \neq (0, \ldots, 0)} z_{\mu, n'_\mu} \leq \frac{z_{\lambda, n_\lambda}}{z_{\lambda, n'_\lambda}} = X_{\lambda, n_\lambda + 1} < \sqrt{\delta} \leq \sqrt{|d|}.
\]

We recall that \( \sigma_\lambda \) is an automorphism of the field \( \mathbb{Q}(g_\lambda) \subset M \). Note that for any \( \lambda \) and \( \mu \) the automorphism \( \tau_\mu \) can be restricted to the field \( \mathbb{Q}(g_\lambda) \). Since

\[
\tau_\mu \left( \sqrt{(q_1^*)^{\lambda_1} \cdots (q_t^*)^{\lambda_t-1}} \right) = ((-1)^{\mu_1} \sqrt{q_1^*})^{\lambda_1} \cdots ((-1)^{\mu_t-1} \sqrt{q_t^*})^{\lambda_t-1} = (-1)^{\sum_{i=1}^{t-1} \lambda_i \mu_i} \sqrt{(q_1^*)^{\lambda_1} \cdots (q_t^*)^{\lambda_t-1}},
\]

the restriction \( \tau_\mu |_{\mathbb{Q}(g_\lambda)} \) acts trivially if \( \sum_{i=1}^{t-1} \lambda_i \mu_i \equiv 0 \) (mod 2) and coincides with \( \sigma_\lambda \) otherwise.

Let \( \max_{\mu \neq (0, \ldots, 0)} z_{\mu, n_\mu} = \varepsilon \). Lemma 6 implies that

\[
\frac{\varepsilon}{\sqrt{|d|}} \leq z_{\lambda, n_\lambda} \leq \varepsilon
\]

for each \( \lambda \neq (0, \ldots, 0) \). Equalities \([30]\), \([31]\) and Lemma \([5]\) imply that \( 1 \leq z_{\lambda, n_\lambda} |\sigma_\lambda (z_{\lambda, n_\lambda})| < \sqrt{\delta} \leq \sqrt{|d|} \). Thus,

\[
\frac{1}{\varepsilon} \leq |\sigma_\lambda (z_{\lambda, n_\lambda})| \leq \frac{|d|}{\varepsilon}.
\]

By construction,

\[
Z = \prod_{\lambda \neq 0} |\sigma_\lambda(z_{\lambda, n_\lambda})| \leq \left( \frac{|d|}{\varepsilon} \right)^{m-1},
\]

so

\[
\varepsilon \leq \frac{|d|}{Z^{1/m}}.
\]

Let \( \lambda \neq 0 \). The condition \( \sum_{i=1}^{t-1} \lambda_i \mu_i \equiv 0 \) (mod 2) as an equation for \( \mu \in \{0, 1\}^{t-1} \) has exactly \( \frac{|d|}{2} \) solutions, including zero.

\[
\left| \tau_\lambda \left( \sum_{\mu} A_\mu \omega_\mu \right) \right| = \prod_{2[\sum, \lambda_i \mu_i, \mu \neq 0]} |\sigma_\mu (z_{\mu, n_\mu})| \cdot \prod_{2[\sum, \lambda_i \mu_i]} |z_{\mu, n_\mu}|
\]

\[
\leq \left( \frac{|d|}{\varepsilon} \right)^{t-1} \varepsilon^{\frac{m}{2}} = \frac{|d|^{m-1} \varepsilon^{\frac{m}{2}}}{Z^{1/m}} \leq \frac{|d|^{m}}{Z^{1/m}}.
\]
It remains to show that the algorithm completes in $O(\ln N_0)$ iterations. A part of theorem which is already proved allows to apply Theorem 15. Thus, the following inequality holds in any step of the algorithm:

$$|A_{0,\ldots,0}| \geq \Re(\langle d \rangle) \omega_{0,\ldots,0} |Z - C \sqrt{|d|}^m|,$$

where constants $\Re(\langle d \rangle) \omega_{0,\ldots,0} \neq 0$ and $C \sqrt{|d|}^m$ depend only on basises.

Now (31) implies

$$Z = \prod_{\lambda \neq 0} y^\lambda_{n,\lambda} \geq \left( \prod_{\lambda \neq 0} z_{\lambda, n, \lambda} \right)^{-1},$$

with (30), (25) and (26) this yields

$$Z \geq \left( \prod_{\lambda \neq 0} |P_{\lambda, n, \lambda} - Q_{\lambda, n, \lambda} \beta| \right)^{-1} \geq \prod_{\lambda \neq 0} Q_{\lambda, n, \lambda} \geq \prod_{\lambda \neq 0} 2^{\frac{\alpha - 1}{2}} = \prod_{\lambda \neq 0} n_{\lambda}^{-(m-1)}.$$

The sum $\sum_{\lambda \neq 0} n_{\lambda}$ is the number of algorithm iterations. Thus, after $O(\ln N_0)$ iterations the following inequality is reached:

$$Z \geq N_0 + C \sqrt{|d|}^m \Re(\langle d \rangle) \omega_{0,\ldots,0}|.$$  

This implies $|A_{0,\ldots,0}| \geq N_0$ and concludes the proof. □

8 Calculation of an algebraic integer by its approximation

We want to calculate numbers $b_\mu \in \mathbb{Z}$ by an approximate value of $\sum_\mu b_\mu \beta_\mu$, and also numbers $b'_\mu \in \mathbb{Z}$ by an approximate value of $\sum_\mu b'_\mu \beta'_\mu$. Section 6 gives apriori bounds of the form

$$\left| \tau_\lambda \left( \sum_\mu b_\mu \beta_\mu \right) \right| \leq T_0, \quad \left| \tau_\lambda \left( \sum_\mu b'_\mu \beta'_\mu \right) \right| \leq T_0,$$

where $T_0$ depends only on $D$. Section 2 gives a set of simultaneous approximations to the numbers $\frac{\beta_\mu}{\eta_{0,\ldots,0}}$ and another set for the numbers $\frac{\beta'_\mu}{\eta_{0,\ldots,0}}$. The precision of these approximations depends on a parameter $N_0$.

Approximations constructed in Section 7 satisfy Theorem 16 which will be used. (One can prove that any simultaneous approximations $\Lambda_i$ to a basis $W_i$ with a bound of the form $\left| \frac{\lambda_i}{\Lambda_i} - \frac{w_i}{W_i} \right| \leq \frac{C_i}{|\Lambda_i|^{1+\alpha}}$ satisfy the last bound from Theorem 16 with an exponent $\alpha$ instead of $\frac{1}{m-1}$. Thus, actually any sufficiently good approximations can be used.)

We continue to use the basises $\omega_\mu, \omega_\mu^*$ and the function $\Re$ defined in (23) (for $b_\mu$) or (24) (for $b'_\mu$). It is easy to see that they satisfy the following property additionally to properties 1–3 of $\Re$-pairs:

2'. If $x \in \mathcal{O}_M$, then $\omega_\xi x$ is a linear combination of $\{\omega_\mu\}$ with integer coefficients.

For definiteness, we show how to find $b_\mu$; the method for $b'_\mu$ is analogous.

Let $X_\eta \in \mathcal{O}_M$ be a set of $m = 2^{t-1}$ numbers linearly independent over $\mathbb{Q}$. For example, one possible choice is $X_\eta = \beta_\eta$; another possible choice is $X_{0,\ldots,0} = 1$ and $X_\eta = g_\eta$ for $\eta \neq (0,\ldots,0)$. The property 2' implies that

$$\omega_\xi X_\eta = \sum_\mu x_\mu \xi \omega_\mu,$$

with $x_\mu \xi \in \mathbb{Z}$. (The choice $X_\eta = g_\eta$ is convenient in that $x_\mu \xi \eta$ are the same as $c_\mu \xi \eta$ with transposed $\beta_\mu$ and $\beta_\mu^*$. The choice $X_\eta = \beta_\eta$ results in numbers $x_\mu \xi \eta$ which are slightly less in the absolute value.)

Assume that the precision $\varepsilon$ is selected. We know the value of the sum $\sum_\xi b_\xi \beta_\xi$ with the precision $\varepsilon$; in other words, we know a number $\gamma$ such that $\left| \sum_\xi b_\xi \beta_\xi - \gamma \right| < \varepsilon$. Divide this inequality by $\beta_{0,\ldots,0}$ and multiply by $X_\eta$,

$$\left| \sum_\xi b_\xi \omega_\xi X_\eta - \frac{\gamma X_\eta}{\beta_{0,\ldots,0}} \right| \leq \frac{\varepsilon |X_\eta|}{|\beta_{0,\ldots,0}|},$$

$$\left| \sum_\mu \left( \sum_\xi b_\xi x_\mu \xi \eta \right) \omega_\mu - \frac{\gamma X_\eta}{\beta_{0,\ldots,0}} \right| \leq \frac{\varepsilon |X_\eta|}{|\beta_{0,\ldots,0}|}.$$  

(35)
Therefore, \((36)\), \((35)\) and Theorem 16 imply that
\[
|\lambda| \text{ holds, and then the second term in the right-hand side of (37) is less than}
\]
holds. Then the first term in the right-hand side of (37) is also less than

The matrix
Assume that this matrix is singular. Equivalently, there exist numbers \(s\) such that not all of them
are zero and

Summarizing, the equality \((22)\) can be interpreted as matrix equality \(M_1^T M_3 M_2 = E\), where \(E\) is the identity matrix. In particular, \(M_1, M_2\) and \(M_3\)

\[
\sum_{\mu} A_{\mu} B_{\mu} \leq |\mathfrak{M}(\text{Id})| Z \frac{\gamma X_0}{\beta_0,...,0} \leq |\mathfrak{M}(\text{Id})| Z \frac{\varepsilon |X_0|}{|\beta_0,...,0|} + \sum_{\lambda \neq 0} |\mathfrak{M}(\tau_{\lambda})| \frac{\sqrt{|d|}^m}{Z^{m-1}} T_0 |\tau_{\lambda}(X_0)|,
\]

where \(Z = \sum_{\mu} A_{\mu} \omega^*_{\mu}\) as above.

Assume that such a threshold \(N_0\) is selected. Calculate simultaneous approximations \(A_{\mu}\), then compute \(Z\). Select \(\varepsilon\) so that for each \(\eta\) the inequality
\[
\varepsilon < \frac{1}{4} \frac{|\beta_0,...,0|}{|\mathfrak{M}(\text{Id})X_0|Z}.
\]
holds. Then the first term in the right-hand side of (37) is also less than \(\frac{1}{4}\). Thus, the left-hand side of (37) is less than \(\frac{1}{2}\). Since \(\sum_{\mu,\eta} A_{\mu} B_{\mu,\eta} \in \mathbb{Z}\), we can recover the exact value of this sum by rounding \(\mathfrak{M}(\text{Id})Z \frac{\gamma X_0}{\beta_0,...,0}\) to an integer.

Now we obtain a system of linear equations for \(b_\xi\) with the left-hand side

\[
\sum_{\mu} A_{\mu} B_{\mu,\eta} = \sum_{\xi} \left( \sum_{\mu} A_{\mu} x_{\mu,\xi,\eta} \right) b_\xi.
\]
are invertible. The set of all equalities obtained from (34) under the action of all \( \tau^\mu \), can be interpreted as matrix equality

\[ M_4 M_1 = M_1 X \]

Thus

\[ X^T = M_1^T M_4(M_4^T)^{-1} = M_1^{-1} M_4 M_3 M_2. \]

Since any two diagonal matrices commute, \( M_4 M_3 = M_3 M_4 \), so \( M_2 X^T = M_4 M_2 \).

Comparing the element in the line 1 and the column \( \mu \), we obtain

\[ \sum_\xi \omega_1^\mu x_{\mu \xi \eta} = X_{\eta \omega_1^\mu}. \]

Now let \( \eta \) vary. Multiply (41) by \( \omega_1^\mu \) and sum over all \( \xi \in \{0,1\}^{t-1} \):

\[ \sum_\eta \sum_\mu A_\mu X_{\eta \omega_1^\mu} y_\eta = 0, \]

\[ \left( \sum_\eta X_{\eta \eta} \right) \left( \sum_\mu A_\mu \omega_1^\mu \right) = 0. \]

But the first factor is nonzero because \( X_{\eta \eta} \) are linearly independent over \( \mathbb{Q} \) and not all of \( y_\eta \in \mathbb{Q} \) are zero. The second factor is nonzero due to Theorem 16. The contradiction proves the lemma.

So it is sufficient to solve a linear system \( m \times m \) with nonsingular matrix to find \( \{b_\mu\} \). For example, one can use the standard Gaussian elimination.

Finally, we give an overall scheme for our optimization of the CM method.

1. Select numbers \( q = p^n \), \( \hat{u}, \hat{v}, D \in \mathbb{Z} \) as in the stage 1 of the basic algorithm from Subsection 2.2. The future curve will be defined over \( \mathbb{F}_q \) and have the order \( q + 1 - \hat{u} \).

2. Enumerate all reduced forms. Calculate \( T_0 \) from (18), \( N_0 \) from (38), using (21). Apply the algorithm from Section 7.

3. Calculate the required precision \( \varepsilon \) from (39). Calculate the polynomial \( \hat{H}_D[j] \) by the definition (16) approximately with the precision \( \varepsilon \).

4. For each coefficient of the polynomial calculate the decomposition of doubled real part as a \( \mathbb{Z} \)-linear combination of \( \beta_\mu \). In order to do this, obtain a system of linear equations with the left-hand side (40) using (37) and solve this system. Similarly calculate the decomposition of doubled imaginary part as a \( \mathbb{Z} \)-linear combination of \( \beta_\mu^* \). (If the coefficient is known to be real, the stage for imaginary part is not necessary and one can avoid doubling the real part.)

5. Reduce the polynomial modulo any prime ideal of \( \mathcal{O}_{K_G} \) lying above \( p \), obtain a polynomial over \( \mathbb{F}_q \). Calculate any root in \( \mathbb{F}_q \) (there always is one). Construct an elliptic curve \( E'' \) over \( \mathbb{F}_q \) with \( j \)-invariant equal to the found root.

6. If the order \( E'' \) is not the same as required, apply an isomorphism from Subsection 2.3 (quadratic twist if \( D < -4 \)).

As in the original method, one can use another functions \( \theta \) (described in Subsection 2.3) instead of \( j \). This requires correcting the bound \( T_0 \) as described in Section 6 using \( \hat{H}_D[\theta, \alpha] \) instead of \( \hat{H}_D[j] \), and calculating \( j \)-invariant by the found value of \( \theta \) as described in Subsection 2.3.

References

[1] Koblitz N. Elliptic curve cryptosystems // Mathematics of Computation. 1987. Vol. 48. Pp. 203–209.

[2] Miller V. S. Uses of elliptic curves in cryptography // Advances in Cryptology — CRYPTO ’85. Vol. 218 of Lecture Notes in Computer Science, Springer-Verlag, 1986. Pp. 417–426.

[3] Lenstra H. W. Factoring integers with elliptic curves // Annals of Mathematics. 1987. Vol. 126. Pp. 649–673.

[4] Atkin A. O. L., Morain F. Elliptic curves and primality proving // Mathematics of Computation. 1993. Vol. 61. N 203. Pp. 29–68.

[5] Cox D. A. Primes of the form \( x^2 + ny^2 \). New York: Wiley, 1989.

[6] Weber H. Lehrbuch der Algebra. 3rd edition. New York: Chelsea Publishing Company, 1908. Vol. 3.

[7] Silverman J. H. The Arithmetic of Elliptic Curves. Springer, 1986.
[8] Lang S. Elliptic functions. Addison-Wesley, 1973.

[9] Ireland K., Rosen M. A classical introduction to modern number theory. 2nd edition. Vol. 84 of Graduate Texts in Mathematics, Springer, 1990.

[10] Cornacchia G. Su di un metodo per la risoluzione in numeri interi dell’equazione $\sum_{h=0}^{n} C_h x^{n-h} y^h = P$ // Giornale di Matematiche di Battaglini. 1908. N 46. Pp. 33–90.

[11] Baier H. Efficient algorithms for generating elliptic curves over finite fields suitable for use in cryptography. Department of Computer Science, Technical University of Darmstadt, 2002.

[12] Konstantinou E., Kontogeorgis A., Zaroliagis C. D. On the Efficient Generation of Prime-Order Elliptic Curves // J. Cryptology. 2010. Vol. 23. N 3. Pp. 477–503.

[13] Deuring M. Die Typen der Multiplikatorenringe elliptischer Funktionenkörper // Abh. Math. Sem. Han-nschen Univ. 1941. Vol. 14. Pp. 197–272.

[14] Lay G.-J., Zimmer H. G. Constructing elliptic curves with given group order over large finite fields // Algorithmic Number Theory, First International Symposium, Vol. 877 of Lecture Notes in Computer Science, Springer, 1994. Pp. 250–263.

[15] von Schrutka L. Ein Beweis für die Zerlegbarkeit der Primzahlen von der Form $6n + 1$ in ein einfaches und ein dreifaches Quadrat // J. reine und ang. Math. 1911. Vol. 140. Pp. 252–265.

[16] Jacobstahl E. Über die Darstellung der Primzahlen der Form $4n + 1$ als Summe zweier Quadrate // J. reine und ang. Math. 1907. Vol. 132. Pp. 238–245.

[17] Schertz R. Weber’s class invariants revisited // Journal de Théorie des Nombres de Bordeaux. 2002. Vol. 14. N 1. Pp. 325–343.

[18] Enge A., Schertz R. Constructing elliptic curves over finite fields using double eta-quotients // Journal de Théorie des Nombres de Bordeaux. 2004. Vol. 16. N 3. Pp. 555–568.

[19] Cohn H. Introduction to the construction of class fields. Cambridge University Press, 1985.

[20] Lang S. Algebraic number theory. 2nd edition. Vol. 110 of Graduate Texts in Mathematics, Springer, 1994.

[21] Enge A. The complexity of class polynomial computation via floating point approximations // Mathematics of Computation. 2009. Vol. 78. N 266. Pp. 1089–1107.

[22] Brisebarre N., Philibert G. Effective lower and upper bounds for the Fourier coefficients of powers of the modular invariant $j$ // Journal of the Ramanujan Mathematical Society. 2005. Vol. 20. Pp. 255–282.

[23] Enge A., Morain F. Comparing invariants for class fields of imaginary quadratic fields // Algorithmic Number Theory — ANTS-V (Berlin), Vol. 2369 of Lecture Notes in Computer Science, Springer-Verlag, 2002. Pp. 252–266.

[24] Brentjes A. J. Multi-dimensional continued fraction algorithms. Amsterdam: Mathematisch Centrum, 1981.

[25] Peck L. G. Simultaneous rational approximations to algebraic numbers // Bull. Amer. Math. Soc. 1961. Vol. 67. Pp. 197–201.

[26] Khinchin A. Ya. Continued fractions. University of Chicago Press, 1961.

[27] Venkov B. A. Elementary number theory. Wolters-Noordhoff Publishing Groningen, 1970.