Dually Degenerate Varieties and the Generalization of a Theorem of Griffiths–Harris

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Abstract. The dual variety $X^*$ for a smooth $n$-dimensional variety $X$ of the projective space $\mathbb{P}^N$ is the set of tangent hyperplanes to $X$. In the general case, the variety $X^*$ is a hypersurface in the dual space $(\mathbb{P}^N)^*$. If $\dim X^* < N - 1$, then the variety $X$ is called dually degenerate.

The authors refine these definitions for a variety $X \subset \mathbb{P}^N$ with a degenerate Gauss map of rank $r$. For such a variety, in the general case, the dimension of its dual variety $X^*$ is $N - l - 1$, where $l = n - r$, and $X$ is dually degenerate if $\dim X^* < N - l - 1$.

In 1979 Griffiths and Harris proved that a smooth variety $X \subset \mathbb{P}^N$ is dually degenerate if and only if all its second fundamental forms are singular. The authors generalize this theorem for a variety $X \subset \mathbb{P}^N$ with a degenerate Gauss map of rank $r$.

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1 Dual varieties

1.1 Varieties with Degenerate Gauss Maps. An almost everywhere smooth $n$-dimensional variety $X$ of a projective space $\mathbb{P}^N$ is called tangentially degenerate or a variety with a degenerate Gauss map if the rank of its Gauss map

$$\gamma : X \rightarrow \mathbb{G}(n, N)$$

is less than $n$, $0 \leq r = \text{rank } \gamma < n$. Here $n = \dim X$, $x \in X$, $\gamma(x) = T_x(X)$, and $T_x(X)$ is the tangent subspace to $X$ at $x$ considered as an $n$-dimensional projective space $\mathbb{P}^n$. The number $r$ is also called the rank of $X$, $r = \text{rank } X$. The case $r = 0$ is trivial one: it gives just an $n$-plane. The varieties with degenerate Gauss maps were studied in many books and papers (see, for example, [AG 02], [AG 04], [FP 01], [GH 79], [L 99], where one can find further references).

Let $X \subset \mathbb{P}^N$ be an $n$-dimensional almost everywhere smooth variety with a degenerate Gauss map. Suppose that $0 < \text{rank } \gamma = r < n$. Denote by $L$ a leaf of this map, $L = \gamma^{-1}(T_x) \subset X$; $\dim L = n - r = l$. The number $l = n - r$ is called the Gauss defect of the variety $X$. The leaves $L$ of the map $\gamma$ are $l$-dimensional subspaces of the space $\mathbb{P}^N$, $L = P^l \subset \mathbb{P}^N$ or open parts of such subspaces (see, for example, [AG 04], Theorem 3.1, p. 95, or Theorem 2.10 in [GH 79]; or §5 in [L 99]; or the Linearity Theorem in Section 2.3 of [FP 01]).

A variety with a degenerate Gauss map of rank $r$ foliates into its leaves $L$ of dimension $l$, along which the tangent subspace $T_x(X)$ is fixed.
The tangent subspace $T_x(X)$ is fixed when a point $x$ moves along regular points of $L$. This is the reason that we denote it by $T_L$, $L \subset T_L$. A pair $(L, T_L)$ on $X$ depends on $r$ parameters.

The foliation on $X$ defined as indicated above is called the Monge–Ampère foliation.

The varieties of rank $r < n$ are multidimensional analogues of developable surfaces of a three-dimensional Euclidean space.

The main results on the geometry of varieties with degenerate Gauss maps and further references can be found in Chapter 4 of the book [AG 93] and in the recently published book [AG 04].

1.2 Dual Defect and Dually Degenerate Varieties. By the duality principle, to a point $x$ of a projective space $\mathbb{P}^N$, there corresponds a hyperplane $\xi$. The set of hyperplanes of the space $\mathbb{P}^N$ forms the dual projective space $(\mathbb{P}^N)^\ast$ of the same dimension $N$. Under this correspondence, to a subspace $\mathbb{P} \subset \mathbb{P}^N$ of dimension $p$, there corresponds a subspace $(\mathbb{P}^k)^\ast \subset (\mathbb{P}^N)^\ast$ of dimension $N - p - 1$.

Under the dual map, the incidence of subspaces is reversed, that is, if $\mathbb{P}^1 \subset \mathbb{P}^2$, then $(\mathbb{P}^1)^\ast \supset (\mathbb{P}^2)^\ast$.

Let $X$ be an irreducible, almost everywhere smooth variety of dimension $n$ in the space $\mathbb{P}^N$, let $x$ be a smooth point of $X$, and let $T_xX$ be the tangent subspace to $X$ at the point $x$. A hyperplane $\xi$ is said to be tangent to $X$ at $x$ if $T_xX \subseteq \xi$. The bundle of hyperplanes $\xi$ tangent to $X$ at $x$ is of dimension $N - n - 1$.

The set of all hyperplanes $\xi$ tangent to the variety $X$ at its smooth points composes a variety

$$X^\wedge = \{ \xi \subset \mathbb{P}^N | \exists x \in X_{sm} \text{ such that } T_xX \subseteq \xi \},$$

where $X_{sm}$ is the locus of smooth points of the variety $X$. But this variety can be not closed if $X$ has singular points. The dual variety $X^\ast$ of a variety $X$ is the closure of the variety $X^\wedge$:

$$X^\ast = X^{\wedge} = \{ \xi \subset \mathbb{P}^N | \exists x \in X_{sm} \text{ such that } T_xX \subseteq \xi \}.$$ (1)

The dual variety $X^\ast$ can also be described as the envelope of the family of hyperplanes $\xi$ dual to the points $x \in X$. This gives a practical way for finding $X^\ast$, which we will use in examples.

If a variety $X$ is tangentially nondegenerate, i.e., if its rank $r = n$, then in the general case, the dimension $n^\ast$ of its dual variety $X^\ast$ is equal to

$$n^\ast = \dim X^\ast = (N - n - 1) + n = N - 1.$$ (2)

Equation (2) means that the variety $X^\ast$ is a hypersurface with a degenerate Gauss map in the space $(\mathbb{P}^N)^\ast$. The rank $r$ of $X^\ast$ equals the dimension $n$ of the variety $X$, $r = \rank X^\ast = n$, and its Gauss defect

$$\delta_r(X^\ast) = l^\ast = n^\ast - r = N - r - 1.$$
However, it may happen that $\dim X^* < N - 1$. Then the number
\[ \delta_* = N - 1 - \dim X^* \] (3)
is called the dual defect of the variety $X$, and the variety $X$ itself is said to be dually degenerate. The classification of dually degenerate smooth varieties of small dimensions $n$ with positive dual defect $\delta_*$ was found in \cite{E85, E86} for $n \leq 6$; in \cite{E85, E86} and \cite{LS87} for $n = 7$, and in \cite{BFS92} for $n \leq 10$ (see also Section 9.2.C in \cite{T01}).

The dual defect of a variety $X$ must be defined as the difference between an expected dimension of the dual variety $X^*$ and its true dimension. An expected dimension of the dual variety $X^*$ of smooth (tangentially nondegenerate) varieties equals $N - 1$. For these reasons, the above standard definitions of the dual defect and dually degenerate varieties are appropriate for smooth varieties.

However, in the books \cite{FP01} (p. 55); \cite{Ha92} (p. 199); \cite{L99} (p. 16); and \cite{T01}, the above standard definitions of the dual defect and dually degenerate varieties, which are appropriate for tangentially nondegenerate varieties, are automatically extended to varieties with degenerate Gauss maps. In our opinion, these definitions should be refined, because for these latter varieties, the expected dimension of $X^*$ is less than $N - 1$, and for them the appropriate definition of the dual defect (and dually degenerate varieties) must be different (see below).

If a variety $X$ has a degenerate Gauss map (i.e., if its rank $r < n$), then the dual variety $X^*$ is a fibration whose fiber is the bundle $\Xi = \{ \xi \subset \mathbb{P}^N | \xi \supset T_L X \}$ of hyperplanes $\xi$ containing the tangent subspace $T_L X$ and whose base is the manifold $B = X^*/\Xi$. The dimension of a fiber $\Xi$ of this fibration (as in the case $r = n$) equals $N - n - 1$, $\dim \Xi = N - n - 1$, and the dimension of the base $B$ equals $r$, $\dim B = r$, i.e., the dimension of $B$ coincides with the rank of the variety $X$. Therefore, in the general case, the dimension $n^*$ of its dual variety $X^*$ is determined by the formula
\[ \dim X^* = (N - n - 1) + r = N - l - 1, \] (4)
where $l = \dim L = \delta_\gamma = n - r$, and its Gauss defect is equal to $\delta_\gamma(X^*) = l^* = n^* - r = (N - l - 1) - r = N - n - 1 = \dim \Xi$.

Note that formula (4) for an expected dimension of the dual variety of a variety with degenerate Gauss map appeared also in the paper \cite{P02} and implicitly in the books \cite{L99} (see 7.2.1.1 and 7.3i) and \cite{FP01} (see Section 2.3.4).

However, it may happen that $\dim X^* < N - l - 1$. Then the number
\[ \delta_* = N - l - 1 - \dim X^* \] (5)
is called the refined dual defect of the variety $X$, and the variety $X$ itself is said to be dually degenerate.

We will make the following three remarks about these definitions of the refined dual defect and dually degenerate varieties with degenerate Gauss maps:

(i) The refined dual defect of tangentially nondegenerate varieties can be obtained from this definition (5) by taking $l = 0$. 

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(ii) While by the standard definition (3), all varieties with degenerate Gauss maps are dually degenerate, by the definition (5), they can be either dually degenerate or dually nondegenerate. Moreover, while by the standard definition (3), the dual defect \( \delta_* \) of a dually nondegenerate variety with degenerate Gauss map equals its Gauss defect, \( \delta_* = \delta_\gamma = n - r > 0 \), by the definition (5), the refined dual defect \( \delta_* \) of such a variety equals 0, \( \delta_* = 0 \), and this is more appropriate for a dually nondegenerate variety.

(iii) Note also that dually degenerate smooth varieties in the projective space \( \mathbb{P}^N \) are few and far between. As to dually degenerate varieties with degenerate Gauss maps, we are aware of only a few examples of dually degenerate varieties \( X \) with degenerate Gauss maps: the varieties \( X \) with degenerate Gauss maps of ranks three and four in \( \mathbb{P}^N \) were considered in [P 02]. At the end of this paper we construct another example of a dually degenerate variety with a degenerate Gauss map (see Example 9).

The following theorem follows immediately from the preceding considerations.

**Theorem 1.** Let \( X \) be a dually nondegenerate variety with a degenerate Gauss map of dimension \( n \) and rank \( r \) in the space \( \mathbb{P}^N \). Then the leaves \( L \) of the Monge–Ampère foliation of \( X \) are of dimension \( l = n - r \). The dual variety \( X^* \subset (\mathbb{P}^N)^* \) is of dimension

\[
n^* = N - l - 1
\]

and the same rank \( r \), and the leaves \( L^* \) of the Monge–Ampère foliation of \( X^* \) are of dimension

\[
l^* = N - n - 1.
\]

Under the Gauss map, the plane generator \( L^* \) corresponds to a tangent subspace \( T_x(X) \) of the variety \( X \), and the tangent subspace \( T_\xi(X^*) \) of the variety \( X^* \) corresponds to a plane generator \( L \), i.e., on \( X \) the tangent bundle \( T(X) \) and the Monge–Ampère foliation \( L(X) \) are mutually dual.

In particular, if a variety \( X \subset \mathbb{P}^N \) is tangentially nondegenerate, then we have \( n = r, l = 0 \) (i.e., \( n^* = N - 1 \)), and the dual map \((*)\) sends \( X \) to a hypersurface \( X^* \subset (\mathbb{P}^N)^* \) with a degenerate Gauss map of rank \( n \) with the leaves \( L^* \) of the Monge–Ampère foliation of dimension \( l^* = N - n - 1 \).

Conversely, if \( X \) is a hypersurface with a degenerate Gauss map of rank \( r < N - 1 \) in \( \mathbb{P}^N \), then the variety \( X^* \) dual to \( X \) is a tangentially nondegenerate variety of dimension \( r \) and rank \( r \).

In particular, the dual map \((*)\) sends a tangentially nondegenerate variety \( X \subset \mathbb{P}^N \) of dimension and rank \( r = n = N - 2 \) to a hypersurface \( X^* \subset (\mathbb{P}^{n+2})^* \) with a degenerate Gauss map of rank \( r \), and \( X^* \) bears an \( r \)-parameter family of rectilinear generators. Each of these rectilinear generators possesses \( r \) foci if each is counted as many times as its multiplicity. The hypersurface \( X^* \) is torsal and foliates into \( r \) families of toruses (for definition of the torse, see Example 5 in
The variety $X$ bears rectilinear generators corresponding to the torses of the variety $X^*$. Of course, the correspondence indicated above is mutual.

We consider an irreducible, almost everywhere smooth variety $X$ of dimension $n$ and rank $r$ in the space $\mathbb{P}^N$ in more detail. The tangent bundle $T(X)$ of $X$ is formed by the $n$-dimensional subspaces $T_x$ tangent to $X$ at points $x \in X$ and depending on $r$ parameters. The subspaces $T_x$ are tangent to $X$ along the plane generators $L$ of dimension $l = n - r$ composing on $X$ the Monge–Ampère foliation $L(X)$. The bundle $T(X)$ and the foliation $L(X)$ have a common $r$-dimensional base.

Let $(\ast)$ be the dual map of $\mathbb{P}^N$ onto $(\mathbb{P}^N)^*$. The dual map $(\ast)$ sends the variety $X$ to a variety $X^*$, which is the set of all hyperplanes $\xi \subset (\mathbb{P}^N)^*$ tangent to $X$ along the leaves $L$ of its Monge–Ampère foliation. The map $(\ast)$ sends the tangent bundle $T(X)$ and the Monge–Ampère foliation $L(X)$ of $X$ to the Monge–Ampère foliation $L(X^*)$ and the tangent bundle $T(X^*)$ of $X^*$, respectively. Thus, under the dual map $(\ast)$, we have

$$(T(X))^* = L(X^*), \quad (L(X))^* = T(X^*),$$

where $\dim T(X^*) = \dim X^* = n^* = N - l - 1$ and $\dim L(X^*) = \dim L^* = l^* = N - n - 1$.

1.3 Examples. We now consider a few examples. In particular, we consider the main types of varieties with degenerate Gauss maps (cones, multidimensional torses, and tangentially degenerate hypersurfaces) and determine their dual varieties. Most of these examples can be found in [AG 02] and [AG 04], Section 2.4. We present them here in order to illustrate the notion of the dual variety.

Example 1. First, we consider a simple example. Let $X$ be a smooth spatial curve $X$ in a three-dimensional projective space $\mathbb{P}^3$. For this curve, we have $N = 3$, $n = r = 1$, $l = 0$, and $T_x(X)$ is the tangent line to $X$ at $x$. The dual map $(\ast)$ sends a point $x \in X$ to a plane $\xi \subset X^*$, and the dual variety $X^*$ is the envelope of the one-parameter family of hyperplanes $\xi$, i.e., $X^*$ is a torse.

Using formulas (6) and (7) for $n^*$ and $l^*$, we find that $n^* = 2$, $l^* = 1$. The variety $X^*$ bears rectilinear generators $L^*$ along which the tangent planes $\xi = T(X^*)$ are constant. Hence rank $X^* = 1$. The generators $L^*$ of the torse $X^*$ are dual to the tangent lines $T(X)$ to the curve $X$.

Example 2. Let $X$ be a hypersurface with a degenerate Gauss map of rank $r < N - 1$ in $\mathbb{P}^N$. In this case, we have $l = N - 1 - r$, $n^* = (N - 1) - l = (N - 1) - (N - 1 - r) = r$, $l^* = n^* - r = 0$, i.e., the variety $X^*$ dual to $X$ is a tangentially nondegenerate variety of dimension $r$ and rank $r$.

Conversely, if a variety $X \subset \mathbb{P}^N$ is tangentially nondegenerate, then we have $n = r$, $l = 0$, and the dual map $(\ast)$ sends $X$ to a hypersurface $X^* \subset (\mathbb{P}^N)^*$ (i.e., $n^* = N - 1$) with a degenerate Gauss map of rank $n$ with the leaves $L^*$ of the Monge–Ampère foliation of dimension $l^* = N - n - 1$.  

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Example 3. In the space $\mathbb{P}^N$, $N \geq 4$, we take two arbitrary smooth space curves, $Y_1$ and $Y_2$, that do not belong to the same three-dimensional space, and the set of all straight lines intersecting these two curves. These straight lines form a three-dimensional variety $X$. Such a variety is called the join. Its dimension is three, $n = \dim X = 3$. It is easy to see that the variety $X$ has a degenerate Gauss map. In fact, the three-dimensional tangent subspace $T_x(X)$ to $X$ at a point $x$ lying on a rectilinear generator $L$ is defined by this generator $L$ and two straight lines tangent to the curves $Y_1$ and $Y_2$ at the points $y_1$ and $y_2$ of their intersection with the line $L$. Because this tangent subspace does not depend on the location of the point $x$ on the generator $L$, the variety under consideration is a variety $X = V_2^3$ with a degenerate Gauss map of rank two. Thus, in this case, we have $l = 1, r = r^* = 2, n^* = N - 2, l^* = N - 4$.

This example can be generalized by taking $k$ space curves in the space $\mathbb{P}^N$, where $N \geq 2k$ and $k > 2$, and considering a $k$-parameter family of $(k-1)$-planes intersecting all these $k$ curves.

Example 4. Suppose that $S$ is a subspace of the space $\mathbb{P}^N$, $\dim S = l - 1$, and $T$ is its complementary subspace, $\dim T = N - l$, $T \cap S = \emptyset$. Let $Y$ be a smooth tangentially nondegenerate and dually nondegenerate variety of the subspace $T$, $\dim Y = \text{rank } Y = r < N - l$. Consider an $r$-parameter family of $l$-dimensional subspaces $L_y = S \wedge y$, $y \in Y$. This variety is a cone $X$ with vertex $S$ and the director manifold $Y$. The subspace $T_x(X)$ tangent to the cone $X$ at a point $x \in L_y (x \notin S)$ is defined by its vertex $S$ and the subspace $T_y(Y), T_x(X) = S \wedge T_y(Y)$, and $T_x(X)$ remains fixed when a point $x$ moves in the subspace $L_y$. As a result, the cone $X$ is a variety with a degenerate Gauss map of dimension $n = l + r$ and rank $r$, with plane generators $L_y$ of dimension $l$. The generators $L_y$ of the cone $X$ are leaves of the Monge–Ampère foliation associated with $X$. Note that for a cone $X \subset \mathbb{P}^N$ of rank $r$ with $l$-dimensional generators, we have $n = r + l, n^* = N - l - 1, l^* = n^* - r = (N - l - 1) - r = N - n - 1$.

Example 5. Consider a smooth curve $Y$ in the space $\mathbb{P}^N$ not belonging to a subspace $\mathbb{P}^{l+1} \subset \mathbb{P}^N$ and the set of its osculating subspaces $L_y$ of order and dimension $l$. This set forms a variety $X = \cup_{y \in Y} L_y$ of dimension $l + 1$ and rank $r = 1$ in $\mathbb{P}^N$. Such a variety is called a multidimensional torse. The subspace $T_y = L_y + \frac{dy}{dy}$ is the tangent subspace to $X$ at all points of its generator $L_y$. Thus, the subspaces $L_y$ are the leaves of the Monge–Ampère foliation associated with the torse $X$.

Conversely, a variety of dimension $n$ and rank 1 is a torse formed by a family of osculating subspaces of order $n - 1$ of a curve of class $C^p, p \geq n - 1$, in the space $\mathbb{P}^N$.

Note that for the torse $X$, we have $n = l + 1, n^* = N - l - 1, l^* = N - l - 2$, i.e., the dual image $X^*$ of a torse $X$ is a torse. Note also that the torse considered in Example 1 is a particular case of a multidimensional torse corresponding to the value $l = 0$.

Example 6. Let $Y$ be an $r$-parameter family of hyperplanes $\xi$ in in the space $\mathbb{P}^{n+1}, r < n$, whose $(n - r)$-dimensional plane generators depend also on $r$.
parameters. Then the family $Y$ has an $n$-dimensional envelope $X$ which is a variety with a degenerate Gauss map rank $r$. For this hypersurface, we have $l = n - r$, $n^* = r$, $l^* = 0$.

To a cone $X$ of rank $r$ with vertex $S$ of dimension $l - 1$ (see Example 3), there corresponds a variety $X^*$ lying in the subspace $T = S^*$, $\dim T = N - l$. Because $\dim X^* = n^* = N - l - 1$, the variety $X^*$ is a hypersurface of rank $r$ in the subspace $T$. Such a hypersurface was considered in Example 5.

If a tangentially nondegenerate variety $X$, $\dim X = \text{rank } X = r$, belongs to a subspace $\mathbb{P}^{n+1} \subset \mathbb{P}^N$, then we can consider two dual maps in the spaces $\mathbb{P}^{n+1}$ and $\mathbb{P}^N$. We denote the first of these maps by $*$ and the second by $\circ$. Then under the first map, the image of $X$ is a hypersurface $X^* \subset \mathbb{P}^{n+1}$, and under the second map, the hypersurface $X$ is transferred into a cone $X^* \circ$ of rank $r$ and dimension $n^\circ = N - n + r - 1$ with an $(N - n - 2)$-dimensional vertex $S = (\mathbb{P}^{n+1})^\circ$ and $(N - n - 1)$-dimensional plane generators $L^\circ = T(X^\circ)$.

It follows that if the variety $X$ lies in a proper linear subspace of the space $\mathbb{P}^N$, then Examples 4 and 6 are mutually dual to each other.

Example 7. Now we consider the Veronese variety given as the image of the embedding $V^* : \text{Sym}(\mathbb{P}^{2*} \times \mathbb{P}^{2*}) \to \mathbb{P}^{5*}$ into the projective space $\mathbb{P}^{5*}$. This embedding is defined by the equations

$$x_{ij} = u_iu_j, \quad i, j = 0, 1, 2,$$

where $u_i$ are projective coordinates in the plane $\mathbb{P}^{2*}$, i.e., tangential coordinates in the plane $\mathbb{P}^{2}$, and $x_{ij}$ are projective coordinates in the space $\mathbb{P}^{5*}$, $x_{ij} = x_{ji}$.

It was shown in [AG 02] (see also [AG 04], p. 77) that the equation of the variety $V$ that is dual to the variety $V^* \subset \mathbb{P}^{5*}$ defined by equations (8) is

$$\det \begin{pmatrix} x^{00} & x^{01} & x^{02} \\ x^{10} & x^{11} & x^{12} \\ x^{20} & x^{21} & x^{22} \end{pmatrix} = 0, \quad x^{ij} = x^{ji}. \quad (9)$$

Equation (9) defines in the space $\mathbb{P}^5$ the cubic hypersurface dual to the Veronese variety (8) and called the cubic symmetroid.

The Veronese variety $V^*$ defined by equation (8) is a tangentially nondegenerate variety in the space $\mathbb{P}^{5*}$. Thus, by Theorem 1, its dual variety $V$ is a hypersurface with a degenerate Gauss map of rank two in the space $\mathbb{P}^5$ having two-dimensional leaves $L(V)$ of the Monge–Ampère foliation on $V$. The latter is dual to the tangent bundle $T(V^*)$ of $V^*$.

The following table shows the values of dimensions $n$ and $n^*$ of variety $X$ and $X^*$, their common rank $r$, and the dimensions of $l$ and $l^*$ of their plane generators $L$ and $L^*$ in Examples 1–7:
### 2.3 Correlative Transformations

If we have the identification $(\mathbb{P}^N)^* = \mathbb{P}^N$, the duality principle can be realized by a correlative transformation of the space $\mathbb{P}^N$.

Consider a correlative transformation $\mathcal{C}$ (a correlation) in the space $\mathbb{P}^N$ that maps a point $x \in \mathbb{P}^N$ into a hyperplane $\xi \in \mathbb{P}^N$, $\xi = \mathcal{C}(x)$, and preserves the incidence of points and hyperplanes. A correlation $\mathcal{C}$ maps a $k$-dimensional subspace $\mathbb{P}^k \subset \mathbb{P}^N$ into an $(N - k - 1)$-dimensional subspace $\mathbb{P}^{N-k-1} \subset \mathbb{P}^N$.

We assume that the correlation $\mathcal{C}$ is nondegenerate, i.e., it defines a one-to-one correspondence between points and hyperplanes of the space $\mathbb{P}^N$.

Analytically, a correlation $\mathcal{C}$ can be written in the form

$$\xi_i = c_{ij}x^j, \quad i, j = 0, 1, \ldots, N,$$

where $x^i$ are point coordinates and $\xi_i$ are tangential coordinates in the space $\mathbb{P}^N$. A correlation $\mathcal{C}$ is nondegenerate if $\det(c_{ij}) \neq 0$.

Consider a smooth curve $C$ in the space $\mathbb{P}^N$ and suppose that this curve does not belong to a hyperplane. A correlation $\mathcal{C}$ maps points of $C$ into hyperplanes forming a one-parameter family. The hyperplanes of this family envelope a hypersurface with a degenerate Gauss map of rank one with $(N - 2)$-dimensional generators.

If the curve $C$ lies in a subspace $\mathbb{P}^s \subset \mathbb{P}^N$, then a correlation $\mathcal{C}$ maps points of $C$ into hyperplanes that envelop a hypercone with an $(N - s - 1)$-dimensional vertex.

Further, let $X = V^r$ be an arbitrary tangentially nondegenerate $r$-dimensional variety in the space $\mathbb{P}^N$. A correlation $\mathcal{C}$ maps points of such $V^r$ into hyperplanes forming an $r$-parameter family. The hyperplanes of this family envelop a hypersurface $Y = V_r^{N-1}$ with a degenerate Gauss map of rank $r$. The generators of this hypersurface $X$ are of dimension $N - r - 1$ and correspond to the tangent subspaces $T_x(V^r)$.

If the tangentially nondegenerate variety $V^r$ belongs to a subspace $\mathbb{P}^s \subset \mathbb{P}^N$, $s > r$, then the hypersurface $Y = V_r^{N-1}$ corresponding to $V^r$ under a correlation $\mathcal{C}$ is a hypercone with an $(N - s - 1)$-dimensional vertex.

Now let $X = V_n^r$ be a variety with a degenerate Gauss map of rank $r$. Then we can prove the following result, which fully corresponds to Theorem 1.

|   | $X$     | $N$ | $n$ | $l$ | $r$ | $l^*$ | $n^*$ | $X^*$                  |
|---|---------|-----|-----|-----|-----|------|------|------------------------|
| 1 | Torse   | 3   | 1   | 0   | 1   | 1    | 2    | Torse                  |
| 2 | Hypersurface of rank $r$ | $N$ | $N - 1$ | $N - 1 - r$ | $r$ | 0    | $r$    | Tangentially nondegenerate variety |
| 3 | Join | $N \geq 4$ | 3   | 1   | 2   | $N - 4$ | $N - 2$ |                          |
| 4 | Cone | $N$ | $n$ | $n - r$ | $r$ | $N - n - 1$ | $N - n - 1$ |                          |
| 5 | Multidimensional torse | $N$ | $l + 1$ | $l$ | 1   | $N - l - 2$ | $N - l - 1$ |                          |
| 6 | Hypersurface of rank $r$ | $n + 1$ | $n$ | $n - r$ | $r$ | 0    | $r$    |                          |
| 7 | Cubic symmetroid | 5   | 4   | 2   | 2   | 0    | 2    | Veronese variety         |
Theorem 2. A correlation $C$ maps an $n$-dimensional dually nondegenerate variety $X = V^n_r$ with a degenerate Gauss map of rank $r$ with plane generators of dimension $l = n - r$ into an $(N - l - 1)$-dimensional variety $X^* = V^{N-l-1}_r$, with a degenerate Gauss map of the same rank $r$ with $(N - n - 1)$-dimensional plane generators.

Proof. A correlation $C$ sends an $l$-dimensional plane generator $L \subset X$ to an $(N - l - 1)$-dimensional plane $\mathbb{P}^{N-l-1}$, and a tangent subspace $T_x(X)$ to an $(N - n - 1)$-dimensional plane $\mathbb{P}^{N-n-1}$, where $\mathbb{P}^{N-n-1} \subset \mathbb{P}^{N-l-1}$. Because both of these planes depend on $r$ parameters, the planes $\mathbb{P}^{N-n-1}$ are generators of the variety $C(X)$, and the planes $\mathbb{P}^{N-l-1}$ are its tangent subspaces. Thus, the variety $C(X)$ is a variety $X^* = V^{N-l-1}_r$ of dimension $N - l - 1$ and rank $r$. \qed

2 Basic Equations of a Variety with a Degenerate Gauss Map.

In this section, we find the basic equations of a variety $X$ with a degenerate Gauss map of dimension $n$ and rank $r$ in a projective space $\mathbb{P}^N$.

In what follows, we will use the following ranges of indices:

$$a, b, c = 1, \ldots, l; \quad p, q = l + 1, \ldots, n; \quad \alpha, \beta = n + 1, \ldots, N.$$ 

A point $x \in X$ is said to be a regular point of the map $\gamma$ and the variety $X$ if $\dim T_x X = \dim X = n$, and a point $x \in X$ is called a singular point of the leaf $L \subset X$ if $\dim T_x X > \dim X = n$.

In what follows, we assume that every plane generator $L$ of a variety $X$ with a degenerate Gauss map has at least one regular point. Otherwise (i.e., if all points of $L$ are singular) the Monge–Ampère foliation is degenerate, and we will not consider this case.

We associate a family of moving frames $\{A_u\}$, $u = 0, 1, \ldots, N$, with $X$ in such a way that the point $A_0 = x$ is a regular point of $X$; the points $A_a$ belong to the leaf $L$ of the Monge–Ampère foliation passing through the point $A_0$; the points $A_p$ together with the points $A_0, A_a$ define the tangent subspace $T_L X$ to $X$; and the points $A_\alpha$ are located outside the subspace $T_L X$.

The equations of infinitesimal displacement of the moving frame $\{A_u\}$ are

$$dA_u = \omega_u^v A_v, \quad u, v = 0, 1, \ldots, N,$$  \hfill (10)

where $\omega_u^v$ are 1-forms satisfying the structure equations of the projective space $\mathbb{P}^N$:

$$d\omega_u^v = \omega_u^w \wedge \omega_w^v, \quad u, v, w = 0, 1, \ldots, N. \hfill (11)$$

As a result of the specialization of the moving frame mentioned above, we obtain the following equations of the variety $X$ (see [AG 04], Section 3.1):

$$\omega_0^\alpha = 0,$$  \hfill (12)
\[ \omega_\alpha^\alpha = 0, \quad (13) \]

\[ \omega_\alpha^\alpha = b_\alpha^\alpha \omega^\alpha, \quad b_\alpha^\alpha = b_\alpha^\alpha, \quad (14) \]

\[ \omega_\alpha^\alpha = r_\alpha^\alpha \omega^\alpha. \quad (15) \]

The 1-forms \( \omega^\alpha := \omega_\alpha^\alpha \) in these equations are basis forms of the Gauss image \( \gamma(X) \) of the variety \( X \), and the quantities \( b_\alpha^\alpha \) form the second fundamental tensor of the variety \( X \) at the point \( x \). The quantities \( b_\alpha^\alpha \) and \( r_\alpha^\alpha \) are related by the following equations:

\[ b_\alpha^\alpha r_\alpha^\alpha = b_\alpha^\alpha r_\alpha^\alpha. \quad (16) \]

Equations (14) and (15) are called the basic equations of a variety \( X \) with a degenerate Gauss map (see [AG 04], Section 3.1).

Note that under transformations of the points \( A_p \), the quantities \( r_\alpha^\alpha \) are transformed as tensors. As to the index \( a \), the quantities \( c_\alpha^\alpha \) do not form a tensor with respect to this index. Nevertheless, under transformations of the points \( A_0 \) and \( A_a \), the quantities \( r_\alpha^\alpha \) along with the unit tensor \( \delta_\alpha^\alpha \) are transformed as tensors. For this reason, the system of quantities \( c_\alpha^\alpha \) is called a quasitensor.

By (12) and (13), the equations of infinitesimal displacement of the moving frame associated with a variety \( X \) with a degenerate Gauss map have the form

\[
\begin{align*}
    dA_0 &= \omega_0^0 A_0 + \omega_0^a A_a + \omega_a^p A_p, \\
    dA_a &= \omega_a^0 A_0 + \omega_a^b A_b + \omega_a^p A_p, \\
    dA_p &= \omega_p^0 A_0 + \omega_p^a A_a + \omega_p^q A_q + \omega_p^\alpha A_\alpha, \\
    dA_\alpha &= \omega_\alpha^0 A_0 + \omega_\alpha^a A_a + \omega_\alpha^q A_q + \omega_\alpha^\beta A_\beta,
\end{align*}
\]

(17)

where here and in what follows, unless otherwise stated, the indices take the values indicated earlier.

Denote by \( B^\alpha \) and \( C_a \) the \((r \times r)\)-matrices of coefficients occurring in equations (14) and (15):

\[ B^\alpha = (b_\alpha^\alpha), \quad C_a = (c_\alpha^\alpha). \]

Sometimes we will use the identity matrix \( C_0 = (\delta_0^p) \) and the index \( i = 0, 1, \ldots, l \), i.e., \( \{i\} = \{0, a\} \). Then equations (14) and (16) can be combined and written as follows:

\[ (B^\alpha C_i)^T = (B^\alpha C_i), \]

i.e., the matrices

\[ H_i^\alpha = B^\alpha C_i = (b_\alpha^\alpha r_\alpha^\alpha) \]

are symmetric.
The second fundamental form $II$ of the variety $X$ at the point $x$ is the linear span of the system of quadratic forms

$$\Phi^{\alpha} = b_{pq}^\alpha \omega^p \omega^q$$

(see, for example, [AG 04], Section 2.2; [FP 01], Section 2.4; [GH 79]; [L 94]; and [L 99], Section 4.1.5).

Suppose now that $x = x^i A_i$, $i = 0, 1, \ldots, l$, is an arbitrary point of a generator $L$ of the variety $X$. By (17), the differential of this point has the form

$$dx = (dx^i + x^j \omega^i_j) A_i + x^i c_{iq}^p \omega^q A_p,$$

where $i, j = 0, 1, \ldots, l$ and $c_{iq}^p = \delta_{iq}^p$. The matrix

$$J_p^q = (x^i c_{iq}^p)$$

is called the *Jacobi matrix* of the variety $X$. A point $x \in X$ is *regular* if $\det(J_p^q) \neq 0$ at $x$, and a point $x \in X$ is *singular* if $\det(J_p^q) = 0$ at $x$, i.e.,

$$\det(x^i c_{iq}^p) = 0$$

(see [AG 04, Section 3.2]. The set of singular points forms the focus algebraic hypersurface of order $r$ on $L$.

Consider the bundle $\Xi$ of tangent hyperplanes $\xi$ of the variety $X$ passing through its tangent subspace $T_L$. Because $\dim T_L = n$, the dimension of the bundle $\Xi$ is $N - n - 1$, $\dim \Xi = N - n - 1$. In the space $\mathbb{P}^N$, a tangent hyperplane $\xi$ is determined by the equation $\xi = \xi^\alpha x^\alpha = 0$. In the linear system of quadratic forms defined by the forms $\Phi^{\alpha} = b_{pq}^\alpha \omega^p \omega^q$, to the tangent hyperplane $\xi = (\xi^\alpha)$, there corresponds the quadratic form

$$\Phi(\xi) = \xi^\alpha b_{pq}^\alpha \omega^p \omega^q.$$

The tangent hyperplane $\xi$ is called *singular* if the quadratic form $\Phi(\xi)$ is singular, i.e., if

$$\det(\xi^\alpha b_{pq}^\alpha) = 0.$$

3 The Generalization of a Theorem of Griffiths–Harris

3.1 A Theorem of Griffiths–Harris. Griffiths and Harris in [GH 79] (see Theorem 3.5 in [GH 79]) proved the following theorem:

**Theorem 3.** If a variety $X$ is tangentially nondegenerate, then its dual variety $X^*$ is dually degenerate if and only if at any smooth point $x \in X$ every quadratic form $\Phi = \xi^\alpha \Phi^{\alpha}$ belonging to the second fundamental form $II$ of the variety $X$ is singular.

The following example illustrates this theorem.
Example 8. The Segre variety $S(m, n)$ is the embedding of the direct product of the projective spaces $\mathbb{P}^m$ and $\mathbb{P}^n$ in the space $\mathbb{P}^{mn+m+n}$ (see [GH 79] or [T 01]):

$$S : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n},$$

defined by the equations

$$z^{ik} = x^i y^k,$$

where $i = 0, 1, \ldots, m$, $k = 0, 1, \ldots, n$, and $x^i, y^k$, and $z^{ik}$ are the coordinates of points in the spaces $\mathbb{P}^m$, $\mathbb{P}^n$, and $\mathbb{P}^{mn+m+n}$, respectively. This manifold has the dimension $m + n$, $\dim S(m, n) = m + n$.

Consider in the spaces $\mathbb{P}^m$ and $\mathbb{P}^n$ projective frames $\{A_0, A_1, \ldots, A_m\}$ and $\{B_0, B_1, \ldots, B_n\}$. Then in the space $\mathbb{P}^{mn+m+n}$ we obtain the projective frame

$$\{A_0 \otimes B_0, A_0 \otimes B_k, A_i \otimes B_0, A_i \otimes B_k\}$$

(here and in what follows $i, j = 1, \ldots, m$; $k, l = 1, \ldots, n$) consisting of $(m + 1)(n + 1)$ linearly independent points of the space $\mathbb{P}^{mn+m+n}$. The point $A_0 \otimes B_0$ is the generic point of the variety $S$.

In the spaces $\mathbb{P}^m$ and $\mathbb{P}^n$, we have the following equations of infinitesimal displacements of the moving frames $\{A_0, A_1, \ldots, A_m\}$ and $\{B_0, B_1, \ldots, B_n\}$:

$$dA_0 = \omega^0_0 A_0 + \omega^0_0 A_i, \quad dB_0 = \sigma^0_0 B_0 + \sigma^0_0 B_k.$$ 

Hence

$$d(A_0 \otimes B_0) = (\omega^0_0 + \sigma^0_0)(A_0 \otimes B_0) + \omega^0_0 (A_i \otimes B_0) + \sigma^0_0 (A_0 \otimes B_k),$$

and the subspace in $\mathbb{P}^{mn+m+n}$ spanned by the points $A_0 \otimes B_0, A_i \otimes B_0$, and $A_0 \otimes B_k$ is the tangent subspace to the Segre variety $S$ at the point $A_0 \otimes B_0$:

$$T_{A_0 \otimes B_0} = \text{Span} (A_0 \otimes B_0, A_i \otimes B_0, A_0 \otimes B_k).$$

The second differential of the point $A_0 \otimes B_0$ has the form:

$$d^2(A_0 \otimes B_0) = 2 \omega^0_0 \sigma^0_0 A_i \otimes B_k \pmod{T_{A_0 \otimes B_0}}.$$ 

Hence the osculating subspace $T^2_{A_0 \otimes B_0}(S)$ to the variety $S$ coincides with the entire space $\mathbb{P}^{mn+m+n} / T_{A_0 \otimes B_0}$, and its second fundamental form is the linear span of the system of bilinear forms

$$\Phi^{ik} = \omega^0_0 \sigma^0_0.$$ 

The total number of these forms is $mn$. The equations $\omega^0_0 = 0$ determine $n$-dimensional plane generators on $S$, and the equations $\sigma^0_0 = 0$ determine its $m$-dimensional plane generators.

Consider a tangent hyperplane to the Segre variety $S$ at the point $A_0 \otimes B_0$. Because such a hyperplane contains the tangent subspace $T_{A_0 \otimes B_0}$, its equation can be written in the form

$$\xi = \xi_{ik} z^{ik} = 0,$$ 

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where $i = 1, \ldots, m$; $k = 1, \ldots, n$, and $z^{ik}$ are coordinates of points in the space $\mathbb{P}^{mn+m+n}/T_{A_0 \otimes B_0}$. As a result, the second fundamental form of the variety $S$ with respect to the hyperplane $\xi$ is

$$\Phi(\xi) = \xi_{ik} \omega_0 \sigma_{ik}^0$$

(cf. equations (2.21) in [AG 04]). The forms $\Phi(\xi)$ is a linear combination of the linearly independent forms $\Phi^{ik}$, and the matrix of this form is

$$\Xi = \frac{1}{2} \begin{pmatrix} 0 & (\xi_{ik}) \\ (\xi_{ki}) & 0 \end{pmatrix}.$$

In this formula $(\xi_{ik})$ is a rectangular $(m \times n)$-matrix and $(\xi_{ki})$ is its transpose.

It follows that $\det \Xi = 0$ if $m \neq n$. In this case, all quadratic forms belonging to the second fundamental forms of the variety $S$ are singular, and the dual defect $\delta_* (S)$ of $S$ equals $|n - m|$: $\delta_* (S) = |n - m|$. If $m \neq n$, then the variety $S$ is dually degenerate. The variety $S$ is dually nondegenerate if and only if $m = n$.

For other proofs of the formula $\delta_* (S) = |n - m|$ see [L 99], p. 52; or [FP 01], p. 110; or [Ha 92], p. 198; or [AG 04], p. 74–76).

### 3.2 The Generalization of a Theorem of Griffiths–Harris

In Section 1.2, we defined the dual variety $X^* \subset (\mathbb{P}^N)^*$ for a variety $X \subset \mathbb{P}^N$ with a degenerate Gauss map of dimension $n$ and rank $r$ as the set of tangent hyperplanes $\xi (\xi \supset T_LX)$ to $X$. It follows that the dual variety $X^*$ is a fibration whose fiber is the bundle

$$\Xi = \{\xi | \xi \supset T_LX\}$$

of hyperplanes $\xi$ containing the tangent subspace $T_LX$ and whose base is the manifold

$$B = X^*/\Xi.$$

As we noted in Section 1.2, the dimension of a fiber $\Xi$ of this fibration equals $N - n - 1$, and the dimension of the base $B$ equals $r$, $\dim B = r$, i.e., the dimension of $B$ coincides with the rank of the variety $X$. This implies that in the general case,

$$\dim X^* = (N - n - 1) + r = N - l - 1$$

(cf. formula (6)).

As was noted in Section 2, for a dually degenerate variety $X$ with a degenerate Gauss map, we have

$$\dim X^* < N - l - 1.$$

The following theorem generalizing Theorem 3 expresses this condition in terms of the second fundamental forms of the variety $X$. 

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Theorem 4. The dual variety $X^* \subset (\mathbb{P}^N)^*$ of a variety $X$ with a degenerate Gauss map is dually degenerate if and only if at any smooth point $x \in X$ every quadratic form $\Phi = \xi_\alpha \Phi^\alpha$ belonging to the second fundamental form $II$ of the variety $X$ is singular.

Proof. Consider the bundle $\mathcal{R}(X)$ of frames associated with a variety $X$ with a degenerate Gauss map, which we constructed earlier in this section. The basis forms of the bundle $\mathcal{R}(X)$, as well as the basis forms of the tangent bundle $T(X)$ and the Monge–Ampère foliation of the variety $X$, are also called the horizontal forms, and the secondary forms of all these bundles are called the fiber or vertical forms. The horizontal forms $\omega^p$, $p = l + 1, \ldots, n$, are linearly independent, and their number equals $r$. Thus, these forms are basis forms in the bundle $\mathcal{R}(X)$.

On the bundle $\mathcal{R}(X)$ the equations of infinitesimal displacement of a frame have the form (17).

In this proof we will use the following ranges of indices:

$$
0 \leq u, v \leq N, \quad 1 \leq i, j \leq n,
1 \leq a, b \leq l, \quad l + 1 \leq p, q \leq n,
n + 1 \leq \alpha, \beta \leq N, \quad n + 1 \leq \rho, \sigma \leq N - 1.
$$

Consider now the dual coframe (or tangential frame) $\{\alpha^u\}$ in the space $(\mathbb{P}^N)^*$ to the frame $\{A_u\}$ (see Section 1.3 in [AG 04]). The hyperplanes $\alpha^u$ of the frame $\{\alpha^u\}$ are connected with the points of the frame $\{A_u\}$ by the conditions

$$
(\alpha^u, A_v) = \delta^u_v, \quad (21)
$$

Conditions (21) mean that the hyperplane $\alpha^u$ contains all points $A_v$, $v \neq u$, and that the condition of normalization $(\alpha^u, A_u) = 1$ holds.

The equations of infinitesimal displacement of the tangential frame $\{\alpha^u\}$ have the form (see Section 1.3 in [AG 04])

$$
d\alpha^u = \tilde{\omega}_v^u \alpha^v, \quad u, v = 0, 1, \ldots, N, \quad (22)
$$

where the forms $\tilde{\omega}_v^u$ are related to the forms $\omega_v^u$ by the following formulas:

$$
\tilde{\omega}_v^u = -\omega_v^u.
$$

Hence equations (22) can be written as

$$
d\alpha^u = -\omega_v^u \alpha^v. \quad (23)
$$

Recalling that

$$
\begin{align*}
\begin{cases}
    dA_0 &\equiv \omega^0 A_0 + \omega^p A_p \pmod{A_0}, \\
    dA_l &\equiv \omega^l_0 A_l \pmod{A_0, A_1, \ldots, A_l}, \\
    dA_l &\equiv \omega^l_0 A_l + \omega^N_l A_N \pmod{A_0, A_1, \ldots, A_l}
\end{cases}
\end{align*}
$$

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(cf. equations (17)) and
\[
\omega^a = 0, \quad \omega_a^a = 0, \quad \omega_p^a = b_{pq}^a \omega^p
\]
(cf. equations (12), (13), and (14)), we find from (23) that
\[
d\alpha^N \equiv -\omega_a^N \alpha^a - \omega_p^N \alpha^p - \omega_{\sigma}^N \alpha^\sigma \pmod{\alpha^N}.
\]
The \(N - n - 1\) forms \(\omega_N^\alpha\) determine the infinitesimal displacement of the hyperplane \(\xi = \alpha^N\) in the bundle \(\Xi\) of tangent hyperplanes \(\xi\) containing the tangent subspace \(T_L X\), i.e., these forms are the fiber forms on the dual variety \(X^*\). The number \(N - n - 1\) coincides with the dimension of a fiber of this bundle. Hence, forms \(\omega_N^\alpha\) are linearly independent.

A basis of the fibration \(X^*\) is the span \(S^N\) of the forms \(\omega_p^N\), i.e., the forms \(\omega_p^N\) are horizontal on \(X^*\). Because
\[
\omega_p^N = b_{pq}^N \omega^q, \quad b_{pq}^N = b_{qp}^N, \quad p, q = l + 1, \ldots, n,
\]
the dimension of \(S^N\) does not exceed the rank \(r = n - l\) of the variety \(X\).

Consider the exterior product
\[
\omega_{i+1}^N \wedge \ldots \wedge \omega_n^N = \det (b_{pq}^N) \omega_{i+1}^N \wedge \ldots \wedge \omega_n^N.
\]
It is easy to see that \(\dim S^N = \text{rank} (b_{pq}^N)\), and \(\dim S^N < r\) if and only if \(\det (b_{pq}^N) = 0\).

Because \(\alpha^N\) was any of the hyperplanes \(\alpha^\beta\), we have
\[
\det (b_{pq}^\beta) = 0.
\]
Moreover, the tangent hyperplane \(\xi\) can be chosen arbitrarily from the system \(\xi = \xi_\beta \alpha^\beta\). This system of tangent hyperplanes passing through the tangent subspace \(T_L X\) determines the system of quadratic forms
\[
\xi_\beta b_{pq}^\beta \omega^p \omega^q
\]
whose span is the second fundamental form \(II\) of the variety \(X\) and the system of second fundamental tensors
\[
\xi_\beta b_{pq}^\beta
\]
of this variety \(X\). This proves the theorem statement: the variety \(X\) is dually degenerate if and only if every quadratic form \(\Phi = \xi_\alpha \Phi^\alpha\) belonging to the second fundamental form \(II\) of the variety \(X\) is singular. \(\square\)

Note that if \(r = n\) (i.e., if a variety \(X\) is tangentially nondegenerate), then we obtain Theorem 3.

We emphasize that unlike in (24), the basis forms in Theorem 3 are the forms \(\omega^i, i = 1, \ldots, n\).
Corollary 5. A variety $X$ with a degenerate Gauss map is dually nondegenerate (i.e., the dimension of its dual variety $X^* \subset (\mathbb{P}^N)^*$ equals $N - l - 1$) if and only if at any smooth point $x \in X$ there is at least one nonsingular quadratic form $\Phi = \xi^a \Phi^a$ belonging to the second fundamental form $\mathcal{II}$ of the variety $X$.

Example 9. It is not difficult to construct an example of a dually degenerate variety $X$ with a degenerate Gauss map departing from Example 8 of a Segre variety $S(m, n) = \mathbb{P}^m \times \mathbb{P}^n \subset \mathbb{P}^{mn+m+n}$. To this end, we consider the projective space $\mathbb{P}^N$ of dimension $N = mn + m + n + l$ and its two complementary subspaces $\mathbb{P}^{mn+m+n}$ and $\mathbb{P}^{l-1}$. Consider the Segre variety $S(m, n)$ in the first of these subspaces $\mathbb{P}^{mn+m+n}$, and let $C(m, n)$ be a cone with vertex $\mathbb{P}^{l-1}$ whose director variety is the variety $S(m, n)$. The cone $C(m, n)$ has a degenerate Gauss map of rank $r = m + n = \dim S(m, n)$. The quadratic forms belonging to the second fundamental form of the cone $C(m, n)$ coincide with quadratic forms belonging to the second fundamental form of its director variety $S(m, n)$. Because the variety $S(m, n)$ is dually degenerate if $m \neq n$, the cone $C(m, n)$ is also dually degenerate.

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