THE AUSLANDER-REITEN COMPONENTS OF $K^b(\text{proj } \Lambda)$ FOR A CLUSTER-TILTED ALGEBRA OF TYPE $\tilde{A}$

KRISTIN KROGH ARNESEN AND YVONNE GRIMELAND

Abstract. We classify the Auslander-Reiten components of $K^b(\text{proj } \Lambda)$, where $\Lambda$ is a cluster tilted algebra of type $\tilde{A}$. The main tool is the combinatoric description of the indecomposable complexes in $K^b(\text{proj } \Lambda)$ via homotopy strings and homotopy bands.

1. Introduction

The derived category of an abelian category was introduced by Grothendieck and Verdier in the early 1960s, and in the 1980s, Happel started studying the derived category of a finite dimensional algebra [9]. The module category is embedded into the derived category of the algebra, and this expansion to larger categories provided a new tool for comparing and distinguishing the module categories of two algebras.

Let $\Lambda$ be a finite dimensional $k$-algebra, where $k$ is an algebraically closed field, and let $\text{mod } \Lambda$ denote the category of finitely generated left $\Lambda$-modules. The derived category of $\Lambda$ is denoted by $D(\text{mod } \Lambda)$, with suspension functor called shift and denoted by $[1]$. Two important subcategories is the bounded derived category, denoted by $D^b(\text{mod } \Lambda)$, and its subcategory $K^b(\text{proj } \Lambda)$, the bounded homotopy category of finitely generated projective $\Lambda$-modules. One way of describing $D(\text{mod } \Lambda)$ is to describe its Auslander-Reiten (hereafter abbreviated AR) structure. In general, the subcategory $K^b(\text{proj } \Lambda)$ has AR-triangles whenever $\Lambda$ has finite Gorenstein dimension. In particular, there is an explicit description of the AR-structure when $\Lambda$ is gentle [5].

The gentle algebras form a subclass of the special biserial algebras, introduced by Skowroński and Waschbühş in 1983 [12]. The AR-structure of the module category of a gentle algebra $\Lambda \cong kQ/I$, where $I$ is an admissible ideal, can be combinatorially described in terms of $Q$ and $I$ [13]. More recently, the AR-structure of $K^b(\text{proj } \Lambda)$ has also been given combinatorially for gentle algebras. This was done in 2011, when Bobiński gave a combinatorial algorithm for computing the AR-triangle starting in any given indecomposable object of $K^b(\text{proj } \Lambda)$, where $\Lambda$ is gentle [5].

The foundation for Bobiński’s algorithm is diverse. In 2003, Beckett and Merklen showed that the indecomposable objects of $K^b(\text{proj } \Lambda)$ are the complexes arising from so called homotopy strings and homotopy bands (in Bobiński’s renaming) [4]. Moreover, Bobiński takes advantage of the Happel functor, introduced by Happel in [7]. The Happel functor is an exact functor of triangulated categories $\Psi: D^b(\text{mod } \Lambda) \to \text{mod } \Lambda$, where the latter category is the stable module category of the repetitive algebra of $\Lambda$. When $\Lambda$ is gentle, the repetitive algebra $\Lambda$ is combinatorially described in terms of strings, see Ringel [11]. Since the repetitive algebra of a gentle algebra is special biserial, it is even possible to describe the AR-sequences of $\text{mod } \Lambda$, using the methods of Wald and Waschbühş [13].

In its original appearance, the Happel functor is rather abstract and no explicit construction is given. Nevertheless, some of its properties are quite remarkable: It
is always full and faithful, and if $\Lambda$ is of finite global dimension, it is a triangle-equivalence. It also extends the inclusion functor embedding $\text{mod} \Lambda$ into $\text{mod} \hat{\Lambda}$. By using all the known structure of $\Lambda$, $\text{mod} \Lambda$, $\text{mod} \hat{\Lambda}$ and $K_b(\text{proj} \Lambda)$ when $\Lambda$ is gentle, Bobiński constructed a formula for the Happel functor in the gentle case.

Special classes of gentle algebras include some cluster-tilted algebras. The cluster-tilted algebras were introduced in 2007 by Buan, Marsh and Reiten [6]. In 2010, Assem, Brüstle, Charbonneau-Jodoin and Plamondon showed that cluster-tilted algebras of type $A$ and $\tilde{A}$ are gentle [1]. The mutation classes of type $\tilde{A}$ and the derived equivalences between cluster-tilted algebras of type $\tilde{A}$ are described by Bastian [3].

Also worth noting is the derived invariant described by Avella-Alaminos and Geiss [2]. Using this invariant one can find an upper bound for the number of AR-components containing sequences with only one middle term.

In this paper, we classify the AR-components of $K_b(\text{proj} \Lambda)$, where $\Lambda$ is a cluster-tilted algebra of type $\tilde{A}$. Our main tool is Bobiński’s algorithm for computing AR-triangles in $K_b(\text{proj} \Lambda)$.

Parallel to this paper, a paper in progress by Fedra Babaei describes the AR-structure of $K_b(\text{proj} \Lambda)$ where $\Lambda$ is a cluster-tilted algebra of type $A$.

The paper is organized as follows: In Section 2, we state the main result of the paper. Section 3 is an overview of the background theory needed. In Section 4, we give the details needed from Bastian’s description of the mutation classes of type $\tilde{A}$. In Section 5, we introduce the combinatorial concepts of walks and reductions, which we use to restate Bobiński’s algorithm for our class of algebras in Section 6. The main result is proved in Section 7. Finally, in Section 8, we give an example. Some technical results are proved in the appendices.

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2. Main results

In this section, we state the main result. By a quiver, we mean a pair $Q = (Q_0, Q_1)$ where $Q_0$ is a set of vertices and $Q_1$ is a set of arrows, together with functions $s, t : Q_1 \to Q_0$ returning the starting and ending vertex of an arrow, respectively.

Let $Q$ be the fixed quiver given by the parameters $x, y, x'$ and $y'$, as shown in Figure 1. We require that at least one of $x, y$ and at least one of $x', y'$ are non-zero.

We then define the double quiver $Q'$ to be the quiver with vertices $Q'_0 = Q_0$ and arrows $Q'_1 = Q_1 \cup Q_1^{-1}$. Let a homotopy string denote a path in $Q'$ with no subpath of the form $\alpha \alpha^{-1}$ or $\alpha^{-1} \alpha$ for any $\alpha \in Q_1$. By a central homotopy string, we mean a homotopy string both starting and ending in a vertex marked with $\triangle$ or $\triangleleft$, excluding the homotopy strings starting with the arrows $\alpha_1$ or $\beta_1$ and the homotopy strings ending with the arrows $\alpha_1^{-1}$ or $\beta_1^{-1}$, and the trivial homotopy strings for the vertices marked with $\blacktriangle$. (Note that in case one or more of the parameters are zero, then any vertex which is adjacent to both an oriented cycle of length 3 and an arrow $\alpha$ or $\beta$ should be marked with $\blacktriangle$, and that any vertex which is adjacent to two oriented cycles of length 3 should be marked with $\triangle$.) A homotopy band is a central homotopy string starting and ending in the same vertex, with some additional constraints (see Section 5.2).

Let $\Lambda$ be the algebra $kQ/I$, where $I$ is the ideal generated by all compositions of two arrows in each directed cycle of length 3 in $Q$, and $kQ$ is the path algebra of $Q$. This is in fact a cluster-tilted algebra of type $A_n$, and all cluster-tilted algebras of type $\tilde{A}_n$ are of this form, up to derived equivalence [3]. We will now describe the AR-components of $K_b(\text{proj} \Lambda)$. First we state their types and numbers in the following theorem. Let $\tau$ denote the AR-translate in $K_b(\text{proj} \Lambda)$.
Theorem 1. Let $Q$ be a quiver as in Figure 1 and let $\Lambda = kQ/I$ where $I$ is as described above. Then the AR-quiver of $K^b(\text{proj } \Lambda)$ consists of:

(i) A class of tubes of rank one (homogeneous tubes), where up to shift, the tubes are parametrized by the set of pairs consisting of one homotopy band and one element of $k$.

(ii) A class of components given by the parameters $x$ and $y$. If $x = 0$, we get up to shift a tube of rank $y$. If $x > 0$, we get $x$ components of type $ZA_\infty$ with $\tau^x + y = [x]$.

(iii) A class of components given by the parameters $x'$ and $y'$. If $x' = 0$, we get up to shift a tube of rank $y'$. If $x' > 0$, we get $x'$ components of type $ZA_\infty$ with $\tau^{x'} + y' = [x']$.

(iv) Up to shift, one $ZA_\infty$-component containing all the stalk complexes corresponding to vertices marked by $\Box$ and $\blacklozenge$.

(v) Up to shift, a class of $ZA_\infty$-components, parametrized by the central homotopy strings.

For any quiver as in Figure 1 the edge of the components in (ii) can be described easily in terms of the quiver. Figure 2 shows the edge of a $ZA_\infty$-component where $\tau^x + y = [x]$. The edge of a $ZA_\infty$-component where $\tau^{x'} + y' = [x']$ can be found symmetrically.

Example 2.1. We now consider the quiver $Q$ given in Figure 3 and the path algebra $\Lambda = kQ/I$ where $I = \langle ih, gi, hg, ed, fe, df, ba, cb, ac, ts, ut, su, qp, rq, pr \rangle$. Figures 4 and 5 show the edges of two AR-components of type $ZA_\infty$, one given by $x$ and $y$, and one given by $x'$ and $y'$. The first component has the property $\tau^5 = [3]$ and the second component has the property $\tau^6 = [2]$. 

Figure 1. The quiver $Q$ given by $x$, $y$, $x'$ and $y'$.

Figure 2. The edge of an AR-component of type $ZA_\infty$, with the degree shown below each complex.
Figure 3. The quiver from Example 2.1.

\[ \cdots \rightarrow P_7 \rightarrow P_5 \rightarrow P_3 \rightarrow (P_9 \rightarrow P_8) \rightarrow (P_8 \rightarrow P_6) \rightarrow P_4 \rightarrow \cdots \]

Figure 4. The edge of a $\mathbb{Z}A_\infty$-component where $\tau^5 = [3]$. 

\[ \cdots \rightarrow P_{15} \rightarrow P_{16} \rightarrow (P_9 \rightarrow P_{10}) \rightarrow (P_{10} \rightarrow P_{11}) \rightarrow (P_{11} \rightarrow P_{12}) \rightarrow (P_{12} \rightarrow P_{13}) \rightarrow P_{15} \rightarrow \cdots \]

Figure 5. The edge of a $\mathbb{Z}A_\infty$-component where $\tau^6 = [2]$. 

Figure 6. The special $\mathbb{Z}A_\infty$-component.
Figures 4 and 5 show the edges of two AR-components of type $ZA_{\infty}$, one given by $x$ and $y$, and one given by $x'$ and $y'$. The first component has the property $\tau^3 = [3]$ and the second component has the property $\tau^6 = [2]$.

Figure 6 shows part of the structure in the special $ZA_{\infty}$-component. In particular, it shows the irreducible maps between the stalk complexes.

The remaining $ZA_{\infty}$-components are parametrized by the central homotopy strings, that is, each component contains exactly one central homotopy string. Note that also the trivial homotopy strings corresponding to the vertices 4, 2, 1 and 14 are central homotopy strings.

3. Background

In this section we give an overview of the theory needed to prove Theorem 1.

First we give the definition of a gentle algebra. Then we describe $k^b(\text{proj } \Lambda)$ via homotopy strings and homotopy bands, for the gentle algebra $\Lambda$. Finally, we state some results about the almost split triangles and components in the AR-quot of $k^b(\text{proj } \Lambda)$.

3.1. Gentle algebras. Let $\Lambda$ be isomorphic to $kQ/I$, for some quiver $Q$ and some admissible ideal $I$. Then $\Lambda$ is called special biserial [12] if the following are satisfied:

(a) for each vertex $x$ of $Q$ there are at most two arrows starting in $x$ and at most two arrows ending in $x$, and
(b) for any arrow $\alpha$ in $Q$ there is at most one arrow $\beta$ such that $\alpha\beta \notin I$ and at most one arrow $\gamma$ such that $\gamma\alpha \notin I$.

Furthermore, if $I$ consists of only zero-relations, then $\Lambda$ is a string algebra, and if in addition all the relations in $I$ have length 2 and for any arrow $\alpha$ in $Q$ there is at most one arrow $\beta$ such that $\alpha\beta \in I$ and at most one arrow $\gamma$ such that $\gamma\alpha \in I$, then $\Lambda$ is a gentle algebra.

We now state an equivalent definition of a gentle algebra, see [5]. This definition will be used later in the paper.

Definition 2. A finite dimensional algebra $\Lambda = kQ/I$ is gentle if there exist two functions $S,T : Q \to \{-1,1\}$ satisfying the following:

(a) if $\alpha \neq \beta$ start in the same vertex, then $S\alpha = -S\beta$, 
(b) if $\alpha \neq \beta$ end in the same vertex, then $T\alpha = -T\beta$, 
(c) if $\alpha$ starts in the vertex where $\beta$ ends, and $\alpha\beta$ is not in $I$, then $S\alpha = -T\beta$, 
(d) if $\alpha$ starts in the vertex where $\beta$ ends, and $\alpha\beta$ is in $I$, then $S\alpha = T\beta$.

3.2. Homotopy strings and the category $k^b(\text{proj } \Lambda)$. In this subsecion we first introduce the concept of homotopy strings for a gentle algebra $\Lambda$, and given a homotopy string explain how one can construct an associated string complex in $k^b(\text{proj } \Lambda)$. We also discuss some special homotopy strings called homotopy bands, which in addition to string complexes give rise to band complexes in $k^b(\text{proj } \Lambda)$. Finally, we state a result giving the connection between indecomposable objects of $k^b(\text{proj } \Lambda)$ and homotopy strings and bands.

Let $\Lambda \cong kQ/I$ be a gentle algebra. We now want to explain what the homotopy strings associated with $\Lambda$ are. First we define the double quiver $Q'$ of $Q$: Let $Q'_0 = Q_0$, and $Q'_1 = Q_1 \cup Q_1^{-1}$, where $Q_1^{-1}$ is the set of formal inverses of the arrows of $Q$; that is, for each $\alpha : x \to y$ in $Q$, we have $\alpha^{-1} : y \to x$ in $Q'$. We also add formal inverses of the trivial paths of $Q$: For a trivial path 1$_x$ in $Q$ we add the inverse $1_x^{-1}$ to $Q'_1$.

We define $(\alpha^{-1})^{-1} = \alpha$ and $(1_x^{-1})^{-1} = 1_x$ and extend the functions $s,t : Q'_1 \to Q_0$ to include $s(1_x^{-1}) = t(1_x^{-1}) = x$ for all vertices $x$ and $\varepsilon \in \{-1,1\}$. We define the homotopy strings associated with $\Lambda$ to be all paths in $Q'$ which contain no subpath
of the form $\alpha \alpha^{-1}$ or $\alpha^{-1} \alpha$ for $\alpha \in Q_1$. Note that each vertex $x$ in $Q_0$ gives rise to two trivial homotopy strings, namely the paths $1_x$ and $1_x^{-1}$ in $Q'$. We also consider the empty homotopy string, denoted by $\emptyset$.

A non-trivial, non-empty homotopy string $\omega$ can be written as $\omega = \alpha_0 \alpha_{l-1} \cdots \alpha_1$ where for each $1 \leq i \leq l$ the $i$th letter $\alpha_i(\omega) = \alpha_i$ is one arrow or the inverse of one arrow, and $l(\omega) = l$ is the number of letters, called the length of $\omega$. If $\omega$ is a trivial or empty homotopy string, then $l(\omega) = 0$. A homotopy string $\omega$ is called direct if all of the letters in $\omega$ are arrows, and inverse if all of the letters are inverse arrows.

We define $\omega^{-1} = \alpha_1^{-1} \cdots \alpha_l^{-1}$.

We now state when composition of non-trivial and non-empty homotopy strings is defined; two homotopy strings $\omega = \alpha_1 \cdots \alpha_l$ and $\omega' = \alpha'_1 \cdots \alpha'_{l'}$ where $l, l' \geq 1$, can be composed if $s(\omega) = t(\omega')$ and one of the following statements holds:

- $\alpha_1$ is direct and $\alpha'_1$ is inverse and $\alpha_1^{-1} \neq \alpha'_1$;
- $\alpha_1$ is inverse and $\alpha'_1$ is direct and $\alpha_1^{-1} \neq \alpha'_1$;
- $\alpha_1$ and $\alpha'_1$ are both direct and $\alpha_1 \alpha'_1$ is in $I$, or
- $\alpha_1$ and $\alpha'_1$ are both inverse and $\alpha'_1^{-1} \alpha_1^{-1}$ is in $I$.

The composition $\omega \cdot \omega'$ is the path $\omega \omega'$ in $kQ'$, which is also a homotopy string.

We now define composition of homotopy strings involving trivial homotopy strings. To do this we first need to extend the functions $S, T$ from Definition 2 to homotopy strings; for any arrow $\alpha$ in $Q_1$, we define $S \alpha^{-1} = T \alpha$ and $T \alpha^{-1} = S \alpha$. Furthermore, define $S \alpha^x = \varepsilon$ and $T \alpha^x = -\varepsilon$ for $\varepsilon \in \{-1, 1\}$ and $x \in Q_0$. Let $\omega$ be a non-trivial and non-empty homotopy string, then the composition $\omega \cdot 1_x^\varepsilon$ is defined (and equals $\omega$) if $x = s \omega$, and one of the following statements holds:

- $\varepsilon = S(\alpha_1(\omega))$ and $\alpha_1(\omega)$ is an arrow, or
- $\varepsilon = -S(\alpha_1(\omega))$ and $\alpha_1(\omega)$ is an inverse arrow.

Similarly, the composition $1_x^\varepsilon \cdot \omega$ is defined (and equals $\omega$) if $x = t \omega$, and one of the following statements holds:

- $\varepsilon = T(\alpha_1(\omega))$ and $\alpha_1(\omega)$ is an arrow, or
- $\varepsilon = -T(\alpha_1(\omega))$ and $\alpha_1(\omega)$ is an inverse arrow.

The composition $1_x^\varepsilon \cdot 1_y^\varepsilon$ is defined (and equals $1_x^\varepsilon$) if $x = x'$ and $\varepsilon = \varepsilon'$. Note that for any non-empty homotopy string $\omega$, we have $\emptyset \cdot \omega = \omega$ and $\omega \cdot \emptyset = \omega$.

For a homotopy string $\omega$ of positive length, there is a unique partition $\omega = \sigma_L \cdots \sigma_1$ where each $\sigma_i$ is a homotopy string of positive length; for each $1 \leq i \leq L$ the homotopy strings $\sigma_i$ and $\sigma_{i-1}$ can be composed as homotopy strings, and none of the homotopy strings $\sigma_i$ can be partitioned into non-empty non-trivial homotopy strings. We call this the homotopy partition of $\omega$, and the $\sigma_i$’s are called homotopy letters. Define $\omega^i = \sigma_L \cdots \sigma_{L-1+i}$ for $i > 0$ and $\omega^0$ to be the trivial homotopy string $1_\omega^0$ such that $1_\omega^0 \cdot \omega$ is defined as composition of homotopy strings.

Furthermore, we define the degree of $\omega$, denoted by $\deg(\omega)$, to be the number of direct homotopy letters of $\omega$ minus the number of inverse homotopy letters of $\omega$. The degree of a trivial homotopy string is defined to be 0.

A non-trivial and non-empty homotopy string $\omega = \sigma_L \cdots \sigma_1$ with $s \omega = t \omega$ is called a homotopy band if $\deg(\omega) = 0$; either $\sigma_L$ and $\sigma_L^{-1}$ are both direct homotopy letters or $\sigma_L^{-1}$ and $\sigma_1$ are both direct homotopy letters; $\sigma_1 \cdot \sigma_L$ is defined as composition of homotopy strings and there is no homotopy string $\tilde{\omega}$ with $l(\tilde{\omega}) < l(\omega)$ such that $s \tilde{\omega} = t \omega$ and $\omega = \tilde{\omega}^n$ for some positive integer $n$. If $\omega = \sigma_L \cdots \sigma_1$ is a homotopy band, then the homotopy string $\omega' = \sigma_{j-1} \cdots \sigma_1 \sigma_L \cdots \sigma_j$ is a rotation of $\omega$ if $\omega'$ is a homotopy band.

We now give an explicit description of how to construct a complex $X_{m,\omega}$ in $K^0(\text{proj} \Lambda)$ from a homotopy string $\omega$ associated to $\Lambda$ and an integer $m$. Let $P_x$ denote the indecomposable projective in vertex $x$. If $\omega = \emptyset$, then the complex $X_{m,\omega}$
is the zero complex for all integers \( m \). If \( \omega \) is trivial, that is, \( \omega = 1^e_x \) for \( e \in \{-1, 1\} \), then the complex \( X_{m, \omega} \) is the stalk complex

\[
\cdots \longrightarrow 0 \longrightarrow P_n \longrightarrow 0 \longrightarrow \cdots
\]

with \( P_i \) in degree \( m \). If \( l(\omega) > 0 \) we have the homotopy partition \( \omega = \sigma_L \cdots \sigma_1 \) with \( L \geq 1 \). Let \( \sigma_i^\ast \) be the direct homotopy string in \( \{\sigma_i, \sigma_i^{-1}\} \). Then \( \sigma_i^\ast \) gives rise to a map

\[
P_{i\sigma_i^\ast} \longrightarrow P_{i\sigma_i^\ast},
\]

sending \( e_i \sigma_i^\ast \) to \( \sigma_i^\ast e_i \sigma_i^\ast = \sigma_i^\ast \), where \( e_i \) is the primitive idempotent corresponding to the vertex \( i \) in \( Q \).

For \( m' \in \mathbb{Z} \) define an index set \( \mathcal{I}_{m'}(m, \omega) \) by

\[
\mathcal{I}_{m'}(m, \omega) = \{ i \in [0, L] | \deg(\omega[i]) + m = m' \}.
\]

The object in degree \( m' \) of \( X_{m, \omega} \) is the direct sum

\[
\bigoplus_{i \in \mathcal{I}_{m'}(m, \omega)} P_{\sigma(\omega[i])}.
\]

The differentials are defined componentwise, if \( \delta_{m'} \) is the differential from degree \( m' \) to degree \( m' + 1 \), we define

\[
(\delta_{m'})_{i,j} = \begin{cases} 
  p_{\sigma(i,-1)} & i = j - 1 \text{ and } \sigma_{L-1} \text{ is inverse} \\
  p_{\sigma(i,j)} & i = j + 1 \text{ and } \sigma_{L-j} \text{ is direct} \\
  0 & \text{otherwise}
\end{cases}
\]

for \( j \in \mathcal{I}_{m'}(m, \omega) \) and \( i \in \mathcal{I}_{m'+1}(m, \omega) \). The complexes \( X_{m, \omega} \) constructed in this way are called string complexes. Observe that \( X_{m, \omega} \cong X_{m+\deg \omega, \omega-1} \).

**Example 3.1.** Consider the algebra \( \Lambda = kQ/I \) given in Example 2.1. The homotopy string \( \omega = u^{-1}cbf \) associated with \( \Lambda \) has homotopy partition \( \omega = u^{-1}c \cdot bf \).

We compute the string complex \( X_{0, \omega} \) as follows:

We have \( \mathcal{I}_{-1}(0, \omega) = \{1\} \); \( \mathcal{I}_0(0, \omega) = \{0, 2\} \) and \( \mathcal{I}_1(0, \omega) = \{3\} \). For the differentials in the complex, we get \( (\delta_{\omega}^{-1})_{0,1} = p_c; (\delta_{\omega}^{-1})_{2,1} = p_c; (\delta_{\omega}^{-1})_{3,0} = 0 \) and \( (\delta_{\omega})_{3,2} = p_{bf} \). Hence, the complex \( X_{0, \omega} \) is

\[
\cdots \longrightarrow 0 \longrightarrow P_1 \overset{p_c}{\longrightarrow} P_6 \oplus P_3 \overset{(0 \ p_{bf})}{\longrightarrow} P_5 \longrightarrow 0 \longrightarrow \cdots
\]

with \( P_1 \) in degree \(-1\).

Since homotopy bands are homotopy strings they give rise to complexes as described above, and in addition each homotopy band \( \omega \) also gives rise to a family of band complexes \( Y_{m, \omega, \mu} \in \mathcal{K}(\text{proj } \Lambda) \), where \( m \in \mathbb{Z} \) and \( \mu \) is an indecomposable automorphism of a finite dimensional vector space.

Consider the equivalence relation on the set of all homotopy strings generated by inverting a homotopy string, and let \( \mathcal{W} \) be a complete class of representatives under this equivalence relation. Similarly, we consider the equivalence relation on the set of all homotopy bands generated by inverting a homotopy band and identifying each homotopy band with its rotations, and let \( \mathcal{B} \) be a complete set of representatives under this equivalence relation.

**Proposition 3** ([4] Theorem 3, see also [5] Proposition 3.1]). Let \( \Lambda \cong kQ/I \) be a gentle algebra. Then the indecomposable objects of \( \mathcal{K}(\text{proj } \Lambda) \) are exactly the string complexes \( X_{m, \omega} \) for \( m \in \mathbb{Z} \) and \( \omega \in \mathcal{W} \), and the band complexes \( Y_{m, \omega, \mu} \) for \( m \in \mathbb{Z} \), \( \omega \in \mathcal{B} \) and \( \mu \) an indecomposable automorphism of a finite dimensional vector space.
3.3. Almost split triangles in $\mathcal{K}(\text{proj} \Lambda)$. Before we state Bobiński’s main result, giving the connection between homotopy strings and almost split sequences, we need a result about the almost split sequences in the category $\mathcal{C}$ of indecomposable automorphisms of finite-dimensional vector spaces over $k$. Let $\mu : V \to V$ be an indecomposable object of $\mathcal{C}$ where $\dim_k V = n > 0$. Since $\mu$ is indecomposable, it is similar to a Jordan matrix $J_n(\lambda)$ consisting of one Jordan block, where $\lambda \in k$ is the eigenvalue of $\mu$. Hence the object $\mu$ of $\mathcal{C}$ can be represented by the pair $(\lambda, n)$ and we denote it by $V_n(\lambda)$, as in $[8]$. The following lemma from $[8]$ gives the AR-sequence starting in $V_n(\lambda)$ in $\mathcal{C}$.

**Lemma 4.** Let $\mu = V_n(\lambda)$ be an indecomposable object in the category $\mathcal{C}$. Then there is an AR-sequence

$$\psi : 0 \to V_n(\lambda) \to V_{n-1}(\lambda) \oplus V_{n+1}(\lambda) \to V_n(\lambda) \to 0$$

in $\mathcal{C}$, where $V_0(\lambda) = 0$.

In particular, the AR-structure of $\mathcal{C}$ consists of homogeneous tubes, parametrized by the eigenvalues $\lambda$ of the indecomposable automorphisms.

For each homotopy string $\omega$ we will give combinatorial definitions of homotopy strings $\omega^+$, $\omega^-$ and $\omega^o$ and integers $m'(\omega)$ and $m''(\omega)$, see Section $[9]$ and Appendix $[C]$ with these definitions we can state the following:

**Theorem 5** ([5], Main Theorem).

i) Let $\omega$ be a homotopy band, $m \in \mathbb{Z}$, $\lambda \in k$ and $\mu = V_n(\lambda)$ an indecomposable automorphism of a finite dimensional vector space. Then we have an almost split triangle in $\mathcal{K}(\text{proj} \Lambda)$ of the form

$$Y_{m,\omega,\mu} \to Y_{m,\omega,\mu_1} \oplus Y_{m,\omega,\mu_2} \to Y_{m,\omega,\mu} \to Y_{m,\omega,\mu}[1]$$

where $\mu_1 = V_{n-1}(\lambda)$ and $\mu_2 = V_{n+1}(\lambda)$.

ii) Let $\omega$ be a homotopy string, and $m \in \mathbb{Z}$. Then we have an almost split triangle in $\mathcal{K}(\text{proj} \Lambda)$ of the form

$$X_{m,\omega} \to X_{m+1,\omega} \oplus X_{m,\omega}^+ \to X_{m+1,\omega}^+ \to X_{m+1,\omega}$$

From now on, we will denote the string complex $X_{m,\omega}$ by $\omega[m]$. We call the integer $m$ the degree of the string complex.

3.4. **Components of the AR-quiver of $\mathcal{K}(\text{proj} \Lambda)$.** We define the number of middle terms in an AR-triangle $\chi$ to be $\alpha(\chi)$. Note that from Theorem $[4]$ we have $\alpha(\chi) \leq 2$ for all AR-triangles $\chi$ in $\mathcal{K}(\text{proj} \Lambda)$. By $[10]$ we know that any component in a stable translation quiver is of the form $\mathbb{Z}\Delta/G$ where $\Delta$ is a directed tree and $G$ is an admissible group of automorphisms on $\mathbb{Z}\Delta$. It is clear that since any component in the AR-quiver $\mathcal{K}(\text{proj} \Lambda)$ is a stable translation quiver, and $\alpha(\chi) \leq 2$ for all AR-triangles, $\Delta$ is either $A_n$, $A_{\infty}$ or $A_{\infty}$ (see also $[7]$).

4. Derived equivalence classes of $\tilde{A}_n$-quivers

In this section we describe representatives of the derived equivalence classes of the cluster-tilted algebras of type $\tilde{A}_n$. These algebras are known to be gentle by $[1]$. We now introduce some notation.

We start by fixing an embedding into the plane, to be able to make the distinction between the clockwise and the counterclockwise direction.

Let $Q_n^o$ be the class of quivers of the form as in Figure $[7]$ for some non-negative integers $r_1, r_2, s_1, s_2$ where $r = r_1 + r_2 > 0$ and $s = s_1 + s_2 > 0$, and $r + s = n$. $[3]$. For $Q$ in $Q_n^o$, we denote by $\Lambda_Q$ the path algebra of $Q$ modulo the ideal generated by all zero-relations which are compositions of two arrows in the same 3-cycle of $Q$. We have the following:
Theorem 6 \([\text{[3]}]\).

1. If \(\Lambda\) is a cluster-tilted algebra of type \(\tilde{A}_n\), then there exists \(Q\) in \(Q^*_n\) such that \(\Lambda\) and \(\Lambda_Q\) are derived equivalent.

2. If \(Q\) and \(Q'\) belong to \(Q^*_n\), then \(\Lambda_Q\) and \(\Lambda_{Q'}\) are derived equivalent if and only if they have the same parameters (up to changing the roles of \(r_i\) and \(s_i\) for \(i \in \{1, 2\}\)).

For a quiver \(Q\) in \(Q^*_n\), a clockwise oriented 3-cycle in \(Q\) will be called an \(r\)-cycle and a counter-clockwise oriented 3-cycle an \(s\)-cycle. The arrows \(\alpha_{r2+1}, \ldots, \alpha_r\) are called \(r\)-arrows, and similarly the arrows \(\beta_{s2+1}, \ldots, \beta_s\) are called \(s\)-arrows.

The vertices of a quiver \(Q\) in \(Q^*_n\) can be divided into ten disjoint sets as follows, see Figure 7:

- \(A = \{x \in Q_0 \mid x\) has valency 2 and is part of an \(r\)-cycle\}\)
- \(B = \{x \in Q_0 \mid x\) is part of two \(r\)-cycles\}\)
- \(C = \{x \in Q_0 \mid x\) has valency 2 or 3, with \(\alpha_r\) and \(\beta_s\) ending here\}\)
- \(\tilde{C} = \{x \in Q_0 \mid x\) has valency 2 and is a source\}\)
- \(D = \{x \in Q_0 \mid x\) is the starting vertex of an \(r\)-arrow\}\)
- \(F = \{x \in Q_0 \mid x\) has valency 4 \(\alpha_1\) starting here\}\)

The sets \(A', B', D'\) are defined similarly as \(A, B, D\), respectively, by replacing \(r\)-cycle with \(s\)-cycle and \(r\)-arrow with \(s\)-arrow. The set \(F'\) is defined similarly as \(F\) by replacing “\(\alpha_1\) starting here” with “\(\alpha_r\) ending here”. We will use the notation \(D_{\geq 1}\) for the subset \(D \setminus \{D_0\}\).

The types of vertices occurring in a quiver \(Q\) in \(Q^*_n\) depends on the values of the parameters \(r_1, r_2, s_1\) and \(s_2\). In particular one should note that if there is a vertex of type \(C, \tilde{C}, F\) or \(F'\) in \(Q\), then there is only one vertex of the respective type. Also, a vertex of type \(F'\) will only occur in the case when both \(r_1 = 0\) and \(s_1 = 0\). Vertices of type \(B\) will only occur when \(r_2 > 1\) and vertices of type \(D\) will only occur when \(r_1 > 0\).

Moreover, as Theorem 6 states, the roles of \(r_i\) and \(s_i\) can be interchanged while preserving the derived equivalence. Therefore, it is sufficient to consider the cases listed in Table 1. Figure 8 shows examples of quivers in the cases 2–5. Case 6 is illustrated in Figure 7. The first case is hereditary, and will not be considered in this paper since its derived category is well-known \([9]\).
Table 1. Variations of the normal form

| № | \(r_1\) | \(r_2\) | \(s_1\) | \(s_2\) | Possible vertices |
|---|---|---|---|---|---|
| 1 | \(\neq 0\) | 0 | 0 | \(\neq 0\) | \(\tilde{C}, C, D, D'\) |
| 2 | 0 | \(\neq 0\) | 0 | \(\neq 0\) | \(A, A', B, B', F, F'\) |
| 3 | 0 | \(\neq 0\) | \(\neq 0\) | 0 | \(A, B, C, D'\) |
| 4 | \(\neq 0\) | \(\neq 0\) | 0 | \(\neq 0\) | \(A, A', B, B', C, D, F\) |
| 5 | \(\neq 0\) | \(\neq 0\) | \(\neq 0\) | 0 | \(A, B, C, D, D'\) |
| 6 | \(\neq 0\) | \(\neq 0\) | \(\neq 0\) | \(\neq 0\) | \(A, A', B, B', C, D, D', F\) |

Figure 8. Some quivers in \(Q_n^*\).

5. r-walks and s-walks

In this section we define r- and s-walks, which are special ways of traversing a quiver in \(Q_n^*\). We will also define reduction of homotopy strings. The walks and reductions will later be used to describe AR-triangles in \(K^b(\text{proj} \Lambda)\).

Let \(Q\) be a quiver in \(Q_n^*\). For a vertex \(x \in Q_0\) let a walk starting in \(x\) be a possibly infinite series of homotopy strings \([w_1, w_2, \ldots]\) with \(sw_1 = x, sw_1 = tw_{i-1}\) for \(i > 1\) and such that \(w_n \cdots w_2w_1\) for any \(n > 1\) is a homotopy string. However, it is not necessary that \(w_i, w_{i-1}\) is defined as composition of homotopy strings.

We define a clockwise r-walk \(W = [w_1, w_2, \ldots]\) starting in a vertex \(x\) in \(A \cup D\):

It is recursively defined by the function \(cw_r(x)\), where if \(x\) is the vertex

- \(A_i\) for \(i > 1\): then \(cw_r(x) = \gamma_2 \gamma_1, \) going from \(A_i\) to \(A_{i-1}\)
- \(A_1\): \(\{ r_1 = 0 \) then \(cw_r(x) = \gamma_2 \beta_1 \gamma_1, \) going from \(A_1\) to \(A_{r_1}\)
- \(r_1 > 0\) then \(cw_r(x) = \alpha_1 \beta_1 \gamma_1, \) going from \(A_1\) to \(A_{r_1-1}\)
- \(D_i\) for \(i > 0\): then \(cw_r(x) = \alpha_2 \beta_1, \) going from \(D_i\) to \(D_{i-1}\)
- \(D_0\): \(\{ r_2 = 0 \) then \(cw_r(x) = \alpha_1 \beta_1 \gamma_1, \) going from \(D_0\) to \(D_{r_1-1}\)
- \(r_2 > 0\) then \(cw_r(x) = \gamma_2 \alpha_2, \) going from \(D_0\) to \(A_{r_1}\)

where the vertices \(A_i\) and \(D_i\) are as shown in Figure 8. We define \(w_1 = cw_r(x)\).

Observe that the vertex \(t(w_1)\) is always in \(A \cup D\). Further, we define the \(i\)th step of the clockwise r-walk to be \(w_i = cw_r(t(w_{i-1}))\) for \(i > 1\). Observe that \(w_i = w_{i+r}\) for \(i > 1\). Note that \(cw_r(x)\) is always the shortest homotopy string from \(x\) to \(t(w_1)\) in the clockwise direction.

A clockwise r-walk is illustrated in Figure 8a. The vertices of type \(A \cup D\) are marked with \(*\), and the paths between the \(*\)'s in the clockwise direction are the steps of the walk.

Now we extend the function \(cw_r(x)\) to vertices \(x\) of type \(B, B', C, D', F\) or \(F'\). The **clockwise r-prefix** is the shortest clockwise homotopy string \(w\) with \(s(w) = x\), such that there exists some homotopy string \(w'\) such that \(ww' = cw_r(y)\) for some...
vertex \( y \in A \cup D \). From the above, it follows that \( w \) is unique, and we denote it by \( \text{cw}\cdot r\cdot p(x) \). The extended definition of the clockwise r-walk is then

\[
\text{cw}\cdot r\cdot r(x) = \begin{cases} 
\text{cw}\cdot r\cdot r(x) & x \in A \cup D \\
\text{cw}\cdot r\cdot r(x) & x \in B \cup B' \cup C \cup D' \cup F \cup F' 
\end{cases}
\]

Note that according to Table \ref{table:properties} we always have \( r_2 > 0 \). However, we have included the case when \( r_2 = 0 \) in the definition of clockwise r-walk, as this is needed to give a complete definition of counter-clockwise s-walk which will be defined as a mirror image of the clockwise r-walk.

Next we define a counter-clockwise r-walk \( V = [v_1, v_2, \ldots] \) starting in a vertex \( x \) in \( A \cup C \cup D_{\geq 1} \). It is recursively defined by the function \( \text{ccw}\cdot r\cdot r(x) \), given by

- \( A_i \) for \( i < r_2 \): then \( \text{ccw}\cdot r\cdot r(x) = r_{i}^{-1} \), going from \( A_i \) to \( A_{i+1} \)
- \( A_{r_2} \):
  - \( r_1 = 0 \) then \( \text{ccw}\cdot r\cdot r(x) = r_{i}^{-1} \beta_1^{-1} \cdots \beta_i^{-1} \gamma_{i+1}^{-1} \), going from \( A_{r_2} \) to \( A_1 \)
  - \( r_1 > 0 \) then \( \text{ccw}\cdot r\cdot r(x) = \alpha_{r_2+1} \gamma_{i+1}^{-1} \), going from \( A_{r_2} \) to \( D_1 \), or from \( A_{r_2} \) to \( C \)
- \( C \):
  - \( r_2 = 0 \) then \( \text{ccw}\cdot r\cdot r(x) = \alpha_1 \beta_1^{-1} \cdots \beta_{r_2}^{-1} \), going from \( C \) to \( C \)
  - \( r_2 > 0 \) then, \( \text{ccw}\cdot r\cdot r(x) = \gamma_1 \beta_1^{-1} \cdots \beta_{r_2}^{-1} \), going from \( C \) to \( A_1 \)
- \( D_i \) for \( 1 \leq i \leq r_1 - 1 \): then \( \text{ccw}\cdot r\cdot r(x) = \alpha_{r_2+i+1} \), going from \( D_i \) to \( D_{i+1} \), or from \( D_i \) to \( C \)

As for the clockwise case, we define \( v_1 = \text{ccw}\cdot r\cdot r(x) \), and the \( i \)th step of the counter-clockwise r-walk is defined by \( v_i = \text{ccw}\cdot r\cdot r(t(v_{i-1})) \) for \( i > 1 \). See Figure \ref{fig:counter-clockwise-def} for an illustration of the steps in a counter-clockwise r-walk.

The \textit{counter-clockwise r-prefix} for a vertex \( x \) in \( B, B', D_0, D', F \) or \( F' \) is defined as follows: It is the shortest counter-clockwise homotopy string \( v \) with \( s(v) = x \), such that there exists some homotopy string \( v' \) such that \( vv' = \text{ccw}\cdot r\cdot p(y) \) for some vertex \( y \in A \cup C \cup D_{\geq 1} \). Again, \( v \) is unique, and we denote it by \( \text{ccw}\cdot r\cdot p(x) \). We extend the counter-clockwise r-walk as follows:

\[
\text{ccw}\cdot r\cdot r(x) = \begin{cases} 
\text{ccw}\cdot r\cdot r(x) & x \in A \cup C \cup D_{\geq 1} \\
\text{ccw}\cdot r\cdot r(x) & x \in B \cup B' \cup D_0 \cup D' \cup F \cup F' 
\end{cases}
\]

For the same reason as for the clockwise r-walk, we have included the case when \( r_2 = 0 \).

A \textit{counter-clockwise s-walk} is defined to be the mirror image of a clockwise r-walk, see Figure \ref{fig:counter-clockwise-def}. A \textit{clockwise s-walk} is defined as the mirror image of a counter-clockwise r-walk, see Figure \ref{fig:counter-clockwise-def}.

5.1. Reduction of a homotopy string. Let \( \omega \) be a non-trivial and non-empty homotopy string with one of the following properties:

(i) \( t(\omega) \in A \cup D \), and such that \( \alpha_l(\omega) \) is the last letter in the \( r \)th step of the clockwise r-walk starting in \( t(\omega) \),

(ii) \( t(\omega) \in A' \cup D' \), and such that \( \alpha_l(\omega) \) is the last letter in the \( s \)th step of the counter-clockwise s-walk starting in \( t(\omega) \),

(iii) \( t(\omega) \in A \cup C \cup D_{\geq 1} \), and such that \( \alpha_l(\omega) \) is the last letter in the \( r \)th step of the counter-clockwise r-walk starting in \( t(\omega) \),

(iv) \( t(\omega) \in A' \cup C \cup D'_{\geq 1} \), and such that \( \alpha_l(\omega) \) is the last letter in the \( s \)th step of the clockwise s-walk starting in \( t(\omega) \).

Observe that a homotopy string \( \omega \) satisfies at most one of these properties.

\textbf{Definition 7.} Let \( \omega \) be a homotopy string satisfying property (i). Let \( w_r \) be the \( r \)th step of the clockwise r-walk starting in \( t(\omega) \). We define the clockwise r-reduction of \( \omega \) to be \( \omega' \), where \( \omega' = \sigma \omega' \) for a non-trivial homotopy string \( \sigma \) satisfying...
\( w_r = \sigma \hat{w} \) for some homotopy string \( \hat{w} \), and
\( \text{there is no } \sigma' \text{ such that } \omega = \sigma' \omega'' \text{ and } w_r = \sigma' \hat{w} \text{ with } l(\sigma') > l(\sigma) \).

Similarly, we define counter-clockwise r-reduction by replacing property (i) by
property (iii), and by letting \( w_r \) be the \( r \)th step of the counter-clockwise r-walk starting in \( t(\omega) \). The clockwise s-reduction and counter-clockwise s-reduction are
defined analogously.

Note that for some homotopy strings \( \omega \), the reduction removes \( \omega \) itself – that is,
\( \omega = \sigma \omega' \) where \( \sigma = \omega \). In this case, \( \omega' \) is the trivial homotopy string \( 1_{\epsilon_{\omega}} \) such that
\( \omega \cdot 1_{\epsilon_{\omega}} \) is defined as composition of homotopy strings. To do this, we fix the string
functions \( S \) and \( T \) described in Definition 2. See Appendix A for details.

\textbf{Example 5.1.} Recall Example 2.1. Consider the homotopy string \( \omega = bfed \) associated with \( \Lambda \). It is clear that \( \omega \) satisfies property (i). Then the homotopy string
\( ed \) is the clockwise r-reduction of \( \omega \). Note that this homotopy string also satisfies
property (i), and has the clockwise r-reduction \( d \). In the last case, the homotopy
string we remove is the clockwise r-prefix for vertex 4.

Moreover, the homotopy string \( \nu = jgd \) satisfies property (ii), and has counter-clockwise r-reduction \( gd \). Here, the homotopy string we remove is the counter-clockwise r-prefix for vertex 6. Note that the homotopy string \( gd \) does not satisfy properties (i)–(iv) and hence does not have a reduction.
6. Almost split triangles for string complexes

Let $\Lambda$ be a cluster-tilted algebra of type $\tilde{A}_n$, and let $\omega[m]$ be a string complex in $K^b(\text{proj} \ \Lambda)$. In this section, we give explicit calculations of the AR-triangle starting in $\omega[m]$ and the AR-triangle ending in $\omega[m]$.

Theorem 5 states that the almost split sequence starting in $\omega[m]$ is of the form

\[
\omega[m] \rightarrow \omega^+[m + m'(\omega)] \oplus \omega^+_+[m] \rightarrow \omega^+_+[m + m''(\omega)] \rightarrow \omega[m - 1]
\]

where all the involved homotopy strings and integers can be found combinatorially by using Bobiński’s algorithm (see Appendix C). Figure 10 shows the AR-triangle ending in $\omega[m]$, and the AR-triangle starting in $\omega[m]$.

\[
\begin{align*}
\omega^-[m] & \cdots \omega^+[m + m'(\omega)] \\
\omega^-[m - m''(\omega_-)] & \cdots \omega[m] & \cdots \omega^+[m + m''(\omega)] \\
\omega_-[m - m'(\omega_-)] & \cdots \omega[m] \\
\omega_-[m - m'(\omega_-)] & \cdots \omega[m]
\end{align*}
\]

Figure 10. The AR-triangles starting and ending in $\omega[m]$.

Lemma 8 ([5]). We have that $\omega^+ = ((\omega^{-1})^{-1})^{-1}$ and $\omega^- = ((\omega^{-1})^{-1})^{-1}$.

Hence, if we have a combinatorial description of $\omega^+$ and $\omega^-$, then we also have a combinatorial description of all homotopy strings shown in Figure 10. The combinatorial descriptions of $\omega^+$ and $\omega^-$ are given in Tables 2–5. In Tables 2 and 4 we include the integer $m'(\omega)$.

Proposition 9. Let $\omega[m]$ be a string complex. The middle term $\omega^+[m + m'(\omega)]$ in the AR-triangle starting in $\omega$ is given by the entries in Tables 3 and 4. The middle term $\omega^-[m - m'(\omega_-)]$ in the AR-triangle ending in $\omega$ is given by the entries in Tables 3 and 4.

Proof. See Appendix C.

Table 2. $\omega^+$ for a non-trivial and non-empty homotopy string $\omega$

| $\alpha_i(\omega)$ | condition $\omega^+$ | $m'(\omega)$ |
|---------------------|-------------------|-------------|
| $\alpha_1$, $1 \leq i \leq r_2$ | $c_{\omega^+}(t_\omega) \cdot \omega$ | $-1$ |
| $\alpha_1$, $r_2 + 1 \leq i \leq r$ | $l(\omega) = 1$ | $\emptyset$ |
| $\alpha_1$, $r_2 + 1 \leq i \leq r$ | $l(\omega) > 1$ | ccw $r$-reduction of $\omega$ |
| $\beta_1$, $1 \leq i \leq s_2$ | $c_{\omega^-}(t_\omega) \cdot \omega$ | $-1$ |
| $\beta_1$, $s_2 + 1 \leq i \leq s$ | $l(\omega) = 1$ | $\emptyset$ |
| $\beta_1$, $s_2 + 1 \leq i \leq s$ | $l(\omega) > 1$ | cw $s$-reduction of $\omega$ |
| $\alpha_i^{-1}$, $1 \leq i \leq r$ | $c_{\omega^-}(t_\omega) \cdot \omega$ | $-1$ if $2 \leq i \leq r_2 + 1$ |
| | | $0$ if $r_2 + 2 \leq i \leq r$ |
| | | $\phi(r_1)$ if $i = 1$ |

Continued on next page
Table 2 – Continued from previous page

| $\alpha_1(\omega)$ | condition $\omega^*$ | $m'(\omega)$ |
|-------------------|---------------------|--------------|
| $\beta_i^{-1}$, $1 \leq i \leq s$ | $ccw_{\omega^*}(l\omega) \cdot \omega$ | $0$ or $-1^*$ |
| $\gamma_{2i}$, $1 \leq i \leq r_2$ | $cw_{\omega^*}(l\omega) \cdot \omega$ | $0$ if $i = 1$ and $r_1 > 0$ $-1$ otherwise |
| $\delta_{2i}$, $1 \leq i \leq s_2$ | $ccw_{\omega^*}(l\omega) \cdot \omega$ | $0$ or $-1^*$ |
| $\gamma_{2i-1}$, $1 \leq i \leq r_2$ | $ccw_{\omega^*}(l\omega) \cdot \omega$ | $\phi(s_1)$ if $i > 1$ $\phi(r_1)$ if $i = 1$ and $s_2 > 0$ $0$ otherwise |
| $\delta_{2i-1}$, $1 \leq i \leq s_2$ | $cw_{\omega^*}(l\omega) \cdot \omega$ | $0$ or $-1^*$ |
| $\gamma_{2i+1}$, $1 \leq i \leq r_2$ | $cw_{\omega^*}(l\omega) \cdot \omega$ | $\phi(s_1)$ |
| $\delta_{2i+1}$, $1 \leq i \leq s_2$ | $ccw_{\omega^*}(l\omega) \cdot \omega$ | $0$ or $-1^*$ |
| $\gamma_{2i-1}$, $1 \leq i \leq r_2$ | $ccw_{\omega^*}(l\omega) \cdot \omega$ | $cw_{r}$-reduction of $\omega$ $-1$ |
| $\delta_{2i-1}$, $1 \leq i \leq s_2$ | $cw_{s}$-reduction of $\omega$ $-1$ |

$^*$ as in above row, but interchanging $r$ and $s$.
$\phi : \mathbb{Z}_{\geq 0} \to \{-1, 0\}$ is defined by $\phi(a) = -1$ if $a = 0$, otherwise $\phi(a) = 0$.

Table 3. $\omega^*$ for a non-trivial and non-empty homotopy string $\omega$

| $\alpha_1(\omega)$ | conditions $\omega^*$ |
|-------------------|---------------------|
| $\alpha_i$, $1 \leq i \leq r$ | $ccw_{\omega^*}(l\omega) \cdot \omega$ |
| $\beta_i$, $1 \leq i \leq s$ | $cw_{\omega^*}(l\omega) \cdot \omega$ |
| $\alpha_i^{-1}$, $1 \leq i \leq r_2$ | $ccw_{\omega^*}(l\omega) \cdot \omega$ |
| $\beta_i^{-1}$, $r_2 + 1 \leq i \leq r$ | $l(\omega) = 1$ $\emptyset$ |
| $\alpha_i^{-1}$, $r_2 + 1 \leq i \leq r$ | $l(\omega) > 1$ $cw_{r}$-reduction of $\omega$ |
| $\beta_i^{-1}$, $1 \leq i \leq s_2$ | $ccw_{\omega^*}(l\omega) \cdot \omega$ |
| $\delta_{2i}$, $1 \leq i \leq r_2$ | $cw_{r}$-reduction of $\omega$ |
| $\gamma_{2i-1}$, $1 \leq i \leq r_2$ | $cw_{\omega^*}(l\omega) \cdot \omega$ |
| $\delta_{2i-1}$, $1 \leq i \leq s_2$ | $ccw_{\omega^*}(l\omega) \cdot \omega$ |
| $\gamma_{2i}$, $1 \leq i \leq r_2$ | $cw_{\omega^*}(l\omega) \cdot \omega$ |
| $\delta_{2i+1}$, $1 \leq i \leq s_2$ | $ccw_{\omega^*}(l\omega) \cdot \omega$ |
| $\gamma_{2i-1}$, $1 \leq i \leq s_2$ | $ccw_{\omega^*}(l\omega) \cdot \omega$ |
| $\delta_{2i+1}$, $1 \leq i \leq s_2$ | $ccw_{\omega^*}(l\omega) \cdot \omega$ |

Table 4. $\omega^*$ for a trivial homotopy string

| $x$ in $A$ (so $x = A_i$) | $\omega^*$ for $\omega = 1_x$ | $m'(\omega)$ | $\omega^*$ for $\omega = 1_{x^{-1}}$ | $m'(\omega)$ |
|---------------------------|-----------------------------|--------------|---------------------------------|--------------|
| $A$                        | $cw_{\omega^*}(x)$         | $\phi(r_1)$ if $i = 1$ $\emptyset$ if $i = 1$ and $r_1 > 0$ $\emptyset$ |
| $A'$                       | $ccw_{\omega^*}(x)$        | $\phi(s_1)$ if $i = 1$ $\emptyset$ if $i = 1$ and $s_2 > 0$ $\emptyset$ |
For the remaining part of this section, we will only consider properties of homotopy strings, and not complexes in $K^h(\text{proj} \Lambda)$. Note that it will never happen that $\omega$ is equal to any of $\omega^\ast$, $\omega^+$, $\omega^-$ and $\omega^\ast$.

We start by defining four diagonals for a given homotopy string $\omega$. The upper right diagonal of $\omega$ is the sequence $(\omega, \omega^+, (\omega^+)^+, \ldots)$. Furthermore, we define the lower right diagonal of $\omega$ to be the sequence $(\omega, \omega^+, (\omega_+)_+, \ldots)$. The upper and lower left diagonals of $\omega$ are defined similarly.

Next, let a $Q$-walk denote one of the following walks: clockwise r-walk, clockwise s-walk, counter-clockwise r-walk, counterclockwise s-walk. Note that it follows from Proposition 8 that the condition $l(\omega^+) > l(\omega)$ implies that $\omega^+$ is of the form $\sigma \omega$ for a step $\sigma$ in a $Q$-walk.

**Corollary 10.** Let $\omega$ be a homotopy string, let $\omega^\ast$ be either $\omega^+$ or $\omega^-$, and let $D$ be the diagonal $(\omega, \omega^+, \ldots)$. If $\omega^+$ is of the form $\sigma \omega$, where $\sigma$ is a step in a $Q$-walk $W$, then the homotopy string in $D$ succeeding $\omega^+$ is $\sigma' \omega^\ast$, where $\sigma'$ is the step succeeding $\sigma$ in $W$.

**Proof.** This follows from the definition of walks and Tables 2–5. For example, assume that $\omega^\ast = \omega^+$ and that $\omega^+ = \sigma \omega$ where $\sigma$ is a step in a clockwise r-walk. Then $t(\omega^+) \in A \cup D$, such that $\alpha(t(\omega^+))$ is an arrow $\gamma_2$, if $t(\omega^+) \in A$, and an inverse r-arrow otherwise. In all cases, $(\omega^+)^+ = \text{cw}_r(t(\omega^+)) \cdot \omega^\ast$. \(\square\)
Proposition 11. (1) Let \( \omega \) be a homotopy string. Assume that \( \omega^+ = \sigma \omega \) and \( \omega^+ = \omega \sigma' \), where \( \sigma \) is a step in a \( Q \)-walk \( W \) and \( \sigma'^{-1} \) is a step in a \( Q \)-walk \( W' \). If \( \tilde{\omega} \) is in the lower right diagonal of \( \omega \), and \( \tilde{\omega} \) is in the upper right diagonal of \( \omega \), then
\[
\tilde{\omega}^+ = \sigma \tilde{\omega} \quad \text{and} \quad \tilde{\omega}^+ = \tilde{\omega} \sigma'.
\]
(2) Let \( \omega \) be a homotopy string. Assume that \( \omega^+ = \sigma \omega \) and \( \omega^- = \omega \sigma' \), where \( \sigma \) is a step in a \( Q \)-walk \( W \) and \( \sigma'^{-1} \) is a step in a \( Q \)-walk \( W' \). If \( \tilde{\omega} \) is in the upper left diagonal of \( \omega \), and \( \tilde{\omega} \) is in the upper right diagonal of \( \omega \), then
\[
\tilde{\omega}^+ = \sigma \tilde{\omega} \quad \text{and} \quad \tilde{\omega}^- = \tilde{\omega} \sigma'.
\]
(3) Let \( \omega \) be a homotopy string. Assume that \( \omega^- = \sigma \omega \) and \( \omega^+ = \omega \sigma' \), where \( \sigma \) is a step in a \( Q \)-walk \( W \) and \( \sigma'^{-1} \) is a step in a \( Q \)-walk \( W' \). If \( \tilde{\omega} \) is in the upper left diagonal of \( \omega \), and \( \tilde{\omega} \) is in the lower left diagonal of \( \omega \), then
\[
\tilde{\omega}^- = \sigma \tilde{\omega} \quad \text{and} \quad \tilde{\omega}^- = \tilde{\omega} \sigma'.
\]
(4) Let \( \omega \) be a homotopy string. Assume that \( \omega^- = \sigma \omega \) and \( \omega^- = \omega \sigma' \), where \( \sigma \) is a step in a \( Q \)-walk \( W \) and \( \sigma'^{-1} \) is a step in a \( Q \)-walk \( W' \). If \( \tilde{\omega} \) is in the lower left diagonal of \( \omega \), and \( \tilde{\omega} \) is in the lower right diagonal of \( \omega \), then
\[
\tilde{\omega}^- = \tilde{\omega} \sigma' \quad \text{and} \quad \tilde{\omega}^- = \omega \sigma'.
\]
Proof. We prove case (1). The proofs for cases (2)–(4) are similar. If \( \omega \neq 0 \) is non-trivial, then by Corollary 10 we have that \( \alpha_1(\tilde{\omega}) = \alpha_1(\omega) \) and that \( \alpha_1(\tilde{\omega}) = \alpha_1(\omega(\tilde{\omega})) \); and \( \tilde{\omega}^+ \) is determined by \( \alpha_1(\tilde{\omega}) \), and \( \tilde{\omega}^- \) is determined by \( \alpha_1(\tilde{\omega}) \).

Assume now that \( \omega \) is trivial, that is, \( \omega = I^*_x \) for some vertex \( x \) and some \( \varepsilon \in \{-1, 1\} \). Then, by Table 2 it follows that \( x \) is in \( B \cup B' \cup D_0 \cup D'_0 \cup F \cup F' \), since these are the only vertex types where both \( I^*_x \) and \( (I^{-1}_x)^+ \) are of length greater than \( \omega \). It is easy to verify by Tables 3 and 4 that we are in this situation:

\[
\begin{array}{ccc}
\sigma & \sigma' \\
1^*_x & \sigma_1 & \sigma_1' \\
\end{array}
\]

We can now consider the upper right diagonal of \( \sigma \) and the lower right diagonal of \( \sigma' \), but then we are in the non-trivial case. \( \square \)

7. Classification of the AR-components

In this section, we give a complete classification of all the AR-components in \( K^b(\text{proj} \Lambda) \) where \( \Lambda \cong kQ/I \) is a fixed cluster-tilted algebra of type \( \tilde{A}_n \) with \( Q \) in \( Q^*_n \).

We start by defining admissible reduction for homotopy strings and we show that there are three types of homotopy strings that can not be admissibly reduced. Moreover, any other homotopy string can be admissibly reduced to a homotopy string of one of these three types. In Section 7.1 we show that one of the classes of homotopy strings that can not be admissibly reduced, gives rise to the \( \mathcal{Z}A_{\infty} \)-components and tubes containing string complexes. In Section 7.2 we parametrize the \( \mathcal{Z}A_{\infty} \)-components arising from the second class of homotopy strings that can not be admissibly reduced, and we describe the \( \mathcal{Z}A_{\infty} \)-components arising from the third class of homotopy strings that can not be admissibly reduced. In Section 7.3 we consider the AR-components containing band complexes. Finally, Section 7.4 provides a summary of the main results.
In the remainder of this section, all homotopy strings and homotopy bands are associated with the fixed algebra $\Lambda$.

**Definition 12.** Let $\omega$ be a homotopy string which is neither an $r$-arrow, nor an $s$-arrow, nor the inverse of such an arrow. If $\omega'$ is either the clockwise $r$-reduction of $\omega$, or the counter-clockwise $r$-reduction of $\omega$, or the clockwise $s$-reduction of $\omega$, or the counter-clockwise $s$-reduction of $\omega$, then we call $\omega'$ a left admissible reduction of $\omega$.

We define a right admissible reduction of $\omega$ to be $\omega''$, where $(\omega'')^{-1}$ is a left admissible reduction of $\omega^{-1}$.

If there exists a left or right admissible reduction of a homotopy string $\omega$, then we say that $\omega$ can be admissibly reduced, and the operation performed is an admissible reduction.

**Lemma 13.** Let $\omega$ be a homotopy string. If $\omega'$ is a left admissible reduction of $\omega$, then either $\omega' = \omega^+$ or $\omega' = \omega^-$. Moreover,

- if $\omega'$ is a clockwise $r$-reduction or a counter-clockwise $s$-reduction of $\omega$, then $\omega' = \omega^-$, and
- if $\omega'$ is a clockwise $s$-reduction or a counter-clockwise $r$-reduction of $\omega$, then $\omega' = \omega^+$.

Similarly, if $\omega'$ is a right admissible reduction of $\omega$, then $\omega'$ is either $\omega^+$ or $\omega^-$.

**Proof.** Let $\omega$ be a homotopy string and assume it can be left admissibly reduced and that $\omega'$ is the left admissible reduction of $\omega$. The left admissible reduction on $\omega$ is either a clockwise $r$-reduction, a counter-clockwise $r$-reduction of $\omega$, a clockwise $s$-reduction of $\omega$, or a counter-clockwise $s$-reduction of $\omega$.

If the reduction is a clockwise $r$-reduction, then $\omega$ satisfies condition $\Box$ in chapter $\Box$. In particular $\alpha_i(\omega)$ is either an arrow $\gamma_{2i}$ for some $1 \leq i \leq r_2$, or an inverse $r$-arrow. Thus we need to check that in all these cases $\omega' = \omega^-$. The entries in Table $\Box$ clearly show that this is so, except for the case when $l(\omega) = 1$ and $\omega$ is the inverse of an $r$-arrow. The clockwise $r$-reduction of the exceptions are defined, however the reductions are not admissible reductions, and therefore are not cases we need to consider, as we have assumed that $\omega$ can be admissibly reduced.

The three other cases follow by similar arguments. \hfill $\Box$

**Corollary 14.** Let $\omega$ be a homotopy string. Then $\omega'$ is a left admissible reduction of $\omega$ if and only if $\omega' \neq \emptyset$ and $\omega'$ is one of $\{\omega^+, \omega^-, \omega^+\}$ such that $l(\omega') < l(\omega)$.

**Proof.** Let $\omega$ be a homotopy string.

Assume that $\omega$ has an admissible reduction $\omega'$. By the definition of admissible reduction it is clear that $l(\omega') < l(\omega)$ and $\omega' \neq \emptyset$, and by Lemma $\Box$ either $\omega' = \omega^+$ or $\omega' = \omega^-$. Assume that $\omega' \neq \emptyset$ is either $\omega^+$ or $\omega^-$ and that $l(\omega') < l(\omega)$. By Tables $\Box$ and $\Box$ it is clear that in any of these cases $\omega^+$ and $\omega^-$ is an admissible reduction. \hfill $\Box$

It is clear from the definition of a right admissible reduction, that there is a similar result as Corollary $\Box$ for right admissible reductions. We now give the definition of central homotopy strings, which we will show form one of the classes of homotopy strings that can not be admissibly reduced.

**Definition 15.** A central homotopy string is either a trivial homotopy string corresponding to a vertex of type $B,B',F$ or $F'$, or a homotopy string $\omega$ where

- $\alpha_i(\omega) \in \{\alpha_i, \alpha_i^{-1}, \beta_j, \beta_j^{-1}, \gamma_{2i-1}, \delta_{2i-1} | 1 \leq i \leq r_2, 1 \leq j \leq s_2\}$
- $\alpha_i(\omega) \in \{\alpha_i, \alpha_i^{-1}, \beta_j, \beta_j^{-1}, \gamma_{2i-1}, \delta_{2i-1} | 1 \leq i \leq r_2, 1 \leq j \leq s_2\}$.

**Lemma 16.** If $\omega$ is a central homotopy string, then $\omega$ can not be admissibly reduced.
Proof. This follows from Tables 2 and 3. □

We have now seen that there are three classes of homotopy strings which cannot be admissibly reduced: The central homotopy strings (by the above lemma), the non-central trivial homotopy strings (because no trivial homotopy string can be reduced, by Definition 7), and the r- and s-arrows and their inverses (by Definition 12). The following lemma shows that these classes of homotopy strings are the only ones which can not be admissibly reduced.

Lemma 17. Let ω ≠ ∅ be a homotopy string. By a series of left or right admissible reductions, ω can be reduced to a homotopy string which is of one of the following types:

(i) an r- or s-arrow or an inverse of such an arrow or a trivial homotopy string corresponding to a vertex of type A or A',
(ii) a central homotopy string, or
(iii) a trivial homotopy string corresponding to a vertex of type C, D or D'.

Proof. Let ω ≠ ∅ be a homotopy string, and assume that ω is neither of type (i), nor (ii), nor (iii). Then l(ω) > 0. Denote by X the complement of \( \{ \alpha_1, \alpha_1^{-1}, \beta_j, \beta_j^{-1}, \gamma_{2i-1}, \gamma_{2i-1}^{-1}, \delta_{2j-1}, \delta_{2j-1}^{-1} | 1 \leq i \leq r_2, 1 \leq j \leq s_2 \} \) in \( Q_1 \). Let \( \hat{\omega} = \omega \) if \( \alpha_1(\omega) \in X \), otherwise let \( \hat{\omega} = \omega^{-1} \). It is clear that \( \alpha_1(\hat{\omega}) \in X \), since ω is not a central homotopy string. In any case, either \( \hat{\omega}^+ \) or \( \hat{\omega}^- \) will be an admissible reduction of \( \hat{\omega} \).

For instance, assume that \( \alpha_1(\hat{\omega}) = \gamma_{2i} \) for some \( 1 \leq i \leq r_2 \). Then by Table 3 \( \hat{\omega}^- \) is an admissible reduction of \( \hat{\omega} \).

Next, let \( \hat{\omega} \) be the admissible reduction of \( \hat{\omega} \). Repeat the above step for the homotopy string \( \hat{\omega} \). □

7.1. Characteristic components containing string complexes. By a characteristic component, we mean an AR-component containing AR-triangles with only one middle term. In this section, we will consider characteristic components containing string complexes. These components are dependent on the parameters of the quiver of the cluster-tilted algebra.

A similar result as Proposition 12 holds by exchanging the parameters \( s_1 \) and \( s_2 \) with the parameters \( r_1 \) and \( r_2 \).

Proposition 18. If \( s_2 = 0 \) then, for each \( i \in \mathbb{Z} \), there is a characteristic component with the following edge:

\[
\beta_1^{-1} \quad \ldots \quad \beta_1^{-1} \quad \beta_1^{-1} \quad \beta_1^{-1}
\]

If \( s_2 \neq 0 \), there is a class of \( s_2 \) AR-components with the following edges:

\[
1_{A_i} \quad \ldots \quad 1_{A_i} \quad 1_{A_i}^{-1} \quad 1_{A_i}^{-1} \quad \ldots \quad 1_{A_i}^{-1} \quad 1_{A_i}^{-1}
\]

Proof. From Proposition 7 it is clear that for any of the homotopy strings \( 1_{A_i} \) for \( 1 \leq i \leq s_1 \) and \( \beta_j^{-1} \) for \( s_2 + 1 \leq j \leq s \), we have \( \omega_\beta = \emptyset \). Hence the string complexes shown in the above figures are all in some characteristic component. The rest of the result follows from direct calculations. □
The characteristic components described in Proposition 13 are called s-components. Similarly, the characteristic components depending on the parameters $r_1$ and $r_2$ are called r-components.

In the next corollaries we show that the r- and s-components are tubes or $\mathbb{Z}A_\infty$-components, and that they are exactly the characteristic AR-components containing string complexes. We also describe the string complexes occurring in such components.

**Corollary 19.** Let $\omega[i]$ be a string complex occurring in an s-component. Then $\omega$ is of the following form: $\omega = w_k \cdots w_1 \omega'$ where $\omega'[j]$ is on the edge of the component (for some $j \in \mathbb{Z}$), and where $w_1, \ldots, w_k$ are the $k$ first consecutive steps of the counter-clockwise s-walk starting in $t\omega'$.

Similarly, let $\omega[i]$ be a string complex occurring in an r-component. Then $\omega$ is of the form $\omega = w_k \cdots w_1 \omega'$ where $\omega'[j]$ is on the edge of the component (for some $j \in \mathbb{Z}$), and where $w_1, \ldots, w_k$ are the $k$ first consecutive steps of the clockwise r-walk starting in $t\omega'$.

**Proof.** From Tables 2–5, we know that for a string complex $\omega[i]$ on the edge of an s-component, we have that $\omega^+ = cew_\omega(t\omega) \cdot \omega$. Then, from Corollary 10, we know that the upper right diagonal of $\omega$ will continue to grow with succeeding steps in a counter-clockwise s-walk.

The other case is proved similarly. □

**Corollary 20.** If $s_2 = 0$, then the s-components are tubes of rank $s_1$. If $s_2 > 0$, then the s-components are of type $\mathbb{Z}A_\infty$ with $\tau^s = [s_2]$.

**Corollary 21.** The r- and s-components are exactly the characteristic components containing string complexes.

**Proof.** This is clear, as in Tables 2–5 the empty homotopy string $\emptyset$ occurs only for the homotopy strings listed in Proposition 13. □

**Corollary 22.** Let $\omega[i]$ be a string complex in a characteristic component. Then $\omega$ is not a central homotopy string.

**Proof.** If $\omega[i]$ is a string complex on the edge of a characteristic component, then $\omega$ is not a central homotopy string. Let $\omega'[j]$ be a string complex in a characteristic component, which is not on the edge. Then by Corollary 10 it is clear that $\omega'$ can be admissibly reduced. By Lemma 16 no central homotopy string can be in this characteristic component. □

From Corollary 19 it is clear that no string complex $\omega[i]$ where $\omega$ is of type (iii) in Lemma 17 can be in a characteristic component.

**7.2. The $\mathbb{Z}A_\infty$-components.** From Corollary 21 Corollary 22 and Section 3.3 it is clear that if $\omega[i]$ is a string complex where $\omega$ is of type (ii) or (iii) in Lemma 17 then $\omega[i]$ is in a component of type $\mathbb{Z}A_\infty/G$ where $G$ is an admissible group of automorphisms on $\mathbb{Z}A_\infty$.

Let $\omega[i]$ be a string complex where $\omega$ is a central homotopy string. In the following lemma, we show that if $v$ is another central homotopy string, then the string complex $v[j]$ can not be in the same AR-component as $\omega[i]$.

**Lemma 23.** Let $\omega[i]$ and $v[j]$ be string complexes where $\omega$ and $v$ are central homotopy strings. If $\omega[i] \not\simeq v[j]$, then $\omega[i]$ and $v[j]$ are in different AR-components.

**Proof.** Let $\omega$ be a central homotopy string. From Tables 2–5 it is easy to verify that each of $\omega^+, \omega^-, \omega^0$ and $\omega^-$ is obtained by adding (possibly the inverse of) a step of some Q-walk to $\omega$, and that the four Q-walks involved are of four different
types. By Corollary [10] and Proposition [11] there can be no central homotopy string different from $\omega$ in the AR-component.

It follows from the proof of Lemma [23] that the string complex $\omega[i]$, where $\omega$ is a central homotopy string, never occurs twice in the same component. Hence, the only possibility for $G$ is the trivial group, and thus there is a class of $\mathcal{Z}A^\infty_{\infty}$-components which up to shift are parametrized by the central homotopy strings. Note that for each $\mathcal{Z}A^\infty_{\infty}$-component in this class, the corresponding central homotopy string $\omega$ is of strictly smaller length than any homotopy string $\nu$ where $\nu[j]$ is in the same component for some $j \in \mathbb{Z}$.

We now consider string complexes of the form $\omega[i]$ where $\omega$ is of type (iii) in Lemma [17] that is a trivial homotopy string corresponding to a vertex of type $C,D$ or $D'$.

For a string complex $\omega[i]$ in an AR-component of type $\mathcal{Z}A^\infty_{\infty}$, we define the upper right diagonal of $\omega[i]$ to be the sequence $(\omega[i], \omega^1[i], \omega^2[i], \ldots)$ where $i' = i + m(\omega)$ and $i'' = i' + m(\omega^r)$. Similarly, we define the lower right diagonal of $\omega[i]$, the upper left diagonal of $\omega[i]$ and the lower left diagonal of $\omega[i]$.

The following lemma is illustrated in Example 2.1 revisited in Section 8.

Lemma 24. All stalk complexes $\omega[i]$ where $\omega$ is a trivial homotopy string corresponding to a vertex of type $C,D$ or $D'$ and $i$ is a fixed integer, are in the same $\mathcal{Z}A^\infty_{\infty}$-component. There are irreducible maps, which depend on the values of the parameters $r_1$ and $s_1$, as described below:

- If $r_1 > 0$ the irreducible maps in the lower right diagonal of $1^{-1}_{C[i]}$ are
  \[ 1^{-1}_{C[i]} \rightarrow 1^{-1}_{D_{r_1-1}[i]} \rightarrow \cdots 1^{-1}_{D_1[i]} \rightarrow 1^{-1}_{D_0[i]]. \]

- If $s_1 > 0$ the irreducible maps in the upper right diagonal of $1^{-1}_{C[i]}$ are
  \[ 1^{-1}_{C[i]} \rightarrow 1^{-1}_{D_{s_1-1}[i]} \rightarrow \cdots 1^{-1}_{D_1[i]} \rightarrow 1^{-1}_{D_0[i]}]. \]

Furthermore, for each $j$ in $\mathbb{Z}$ with $j \neq i$ and $\omega$ as above, $\omega[j]$ and $\omega[i]$ are not in the same component.

Proof. We have seen that the string complexes arising from trivial homotopy strings corresponding to a vertex of type $C,D$ or $D'$ are in a component of the form $\mathcal{Z}A^\infty_{\infty}/G$. By a similar argument as in the proof of Lemma [23] and its subsequent comment, $G$ is trivial. The irreducible maps follow from Tables 4 and 5. □

Hence, if at least one of $r_1$ and $s_1$ is non-zero, we get a class of $\mathcal{Z}A^\infty_{\infty}$-components parametrized by $\mathbb{Z}$. If $r_1 = s_1 = 0$, then we have no vertices of type $C,D$ or $D'$, and hence no AR-component of type $\mathcal{Z}A^\infty_{\infty}$ as in Lemma 24.

7.3. Characteristic components containing band complexes. For any $Q$ in $\mathcal{Q}_n^*$ there is always a homotopy band $\omega = \beta_1^{-1} \cdots \beta_n^{-1} \alpha_1 \cdots \alpha_1$, called the central homotopy band of $Q$. Note that it follows from the definition of a homotopy band that there can be no homotopy bands starting in a vertex of type $A,A',D_{\geq 1}$ and $D'_{\geq 1}$.

Proposition 25. Let $\Lambda \cong kQ/I$ be a cluster-tiled algebra of type $\tilde{A}_n$. There are finitely many homotopy bands associated with $\Lambda$ if and only $\Lambda$ is hereditary.

Proof. It is clear that for the hereditary case, the central homotopy band is the only homotopy band. Assume now that $\Lambda$ is not hereditary, that is, we have $r_2 > 0$ (see Table 1). We can then construct one class consisting of infinitely many homotopy bands for the case when $r = r_2 = 1$, and one class consisting of infinitely many homotopy bands for the case when $r > 1$. □
First, let \( r = r_2 = 1 \). Then the homotopy string
\[
\omega = \beta_1^{-1} \cdots \beta_s^{-1} \gamma_2^{-1} \gamma_1^{-1} \alpha_1^{-1} \beta_1 \gamma_1 \gamma_2 \alpha_1
\]
is a homotopy band. Let \( \omega_c \) denote the central homotopy band. Then the homotopy string \( \omega_n = \omega \cdot (\omega_c)^n \) is a homotopy band for all positive integers \( n \).

Now let \( r > 1 \). Then the homotopy string
\[
\omega = \beta_1^{-1} \cdots \beta_s^{-1} \alpha_r \cdots \alpha_2 \gamma_2^{-1} \gamma_1^{-1} \alpha_1^{-1} \cdots \alpha_r^{-1} \beta_s \cdots \beta_1 \gamma_1 \gamma_2 \alpha_1
\]
is a homotopy band. Again, let \( \omega_c \) denote the central homotopy band. Then the homotopy string \( \omega_n = \omega \cdot (\omega_c)^n \) is a homotopy band for all positive integers \( n \). Thus, it is clear that any non-hereditary cluster-tilted algebra of type \( \tilde{A}_n \) will have infinitely many homotopy bands associated with itself.

Note that for most quivers in \( Q_n^\nu \), there are also ways of constructing classes of infinitely many homotopy bands other than in the proof.

### 7.4. Summary

In the following theorem we give a full overview of all the AR-components of \( K^b(\text{proj} \Lambda) \). Note that we always assume that \( r_2 > 0 \) (see Table 4).

#### Theorem 26

Let \( \mathcal{C} \) be the set of central homotopy strings associated with \( \Lambda \), and \( \mathcal{B} \) the set of homotopy bands associated with \( \Lambda \). The AR-quiver of \( K^b(\text{proj} \Lambda) \) consists of:

- A class of homogeneous tubes, parametrized by \( \mathcal{B} \times k \times \mathbb{Z} \).
- A class of \( s \)-components. If \( s_2 = 0 \), we get a class of tubes of rank \( s_1 \) parametrized by \( \mathbb{Z} \). If \( s_2 > 0 \), we get \( s_2 \) components of type \( \mathbb{Z} A_\infty \) with \( \tau^s = [s_2] \).
- \( r_2 \) components of type \( \mathbb{Z} A_\infty \) with \( \tau^r = [r_2] \).
- A class of \( \mathbb{Z} A_\infty \)-components containing all the stalk complexes corresponding to a vertex of type \( C \), \( D \) or \( D' \), parametrized by \( \mathbb{Z} \).
- A class of \( \mathbb{Z} A_\infty \) components parametrized by \( \mathcal{C} \times \mathbb{Z} \).

**Proof.** By Lemma 17 any homotopy string admissibly reduces to one of three types. Hence, by Corollary 21 Lemma 23 and Lemma 24 the list in the theorem gives all AR-components.

### 8. Example

In this section, we revisit Example 8. The following notation will be used: The arrows of the quivers are denoted by small letters (e.g. \( a, b \)), and for an arrow \( a \) the inverse is denoted by \( \overline{a} \).

Recall that \( \Lambda = kQ/I \) is a cluster-tilted algebra of type \( \tilde{A}_{15} \), where \( Q \) is the quiver in Figure 11 and
\[
I = \langle ih, gi, hg, ed, fe, df, ba, cb, ac, ts, ut, su, qy, rq, pr \rangle.
\]
The parameters of \( Q \) are \( r_1 = 2 \), \( r_2 = 3 \), \( s_1 = 4 \) and \( s_2 = 2 \). We will now give part of the AR-structure of \( K^b(\text{proj} \Lambda) \).

Recall from Section 4 that the 16 vertices can be divided into disjoint sets as follows: \( A = \{3, 5, 7\} \), \( A' = \{15, 16\} \), \( B = \{2, 4\} \), \( B' = \{14\} \), \( C = \{9\} \), \( D = \{6, 8\} \), \( D' = \{10, 11, 12, 13\} \) and \( F = \{1\} \).

The steps of the clockwise \( r \)-walk starting in vertex 7 are
\[
[ei, bf, \overline{edmnops}, j, h, ei, bf, \ldots].
\]
The steps of the clockwise \( r \)-walk starting in vertex 5 consists of the same steps as for 7, but deleting the first step.
By Theorem 26 we know that there are two classes of characteristic components containing string complexes, or more precisely, one class of s-components and one of r-components. Moreover, since neither of $r_2$ and $s_2$ are zero, we know that both classes consists of $\mathbb{Z}A_\infty$-components. For an r-component, the edge is given by Proposition 18. We look at the component including the stalk complex $1_{17}[0]$. For any complex on the edge of this component, the upper right diagonal of the complex is given by Corollary 19. In particular, the upper right diagonal of $1_{17}[0]$ is $(1_{17}[0], e_i[-1], bfei[-2], \ldots)$. Figure 12 shows the three lower rows of the AR-component. Note that this is the same component as in Figure 4.

Next, we give the $\mathbb{Z}A_\infty$-components, up to shift, including the trivial homotopy strings corresponding to vertices of type $C$, $D$, and $D'$. This is the special $\mathbb{Z}A_\infty$-component described in Lemma 24. The component is given in Figure 6.

Further, by Tables 2–5, we get that the upper right diagonals starting in the stalk complexes $1_{8}^{-1}[0]$, $1_{4}^{-1}[0]$ and $1_{13}[0]$ consists of subsequent steps of a counter-clockwise s-walk; the lower left diagonals starting in $1_{9}^{-1}[0]$, $1_{6}^{-1}[0]$ and $1_{15}[0]$ consists of subsequent steps of a clockwise s-walk; the lower right diagonals starting in $1_{6}^{-1}[0]$, $1_{10}[0]$, $1_{11}[0]$, $1_{12}[0]$ and $1_{13}[0]$ are inverse steps of a clockwise r-walk; and finally that the upper left diagonals starting in $1_{5}^{-1}[0]$, $1_{10}[0]$, $1_{11}[0]$, $1_{12}[0]$ and $1_{13}[0]$ are inverse steps of a counter-clockwise r-walk. Using this, we have a complete description of the AR-component. In Figure 13 a part containing the stalk complexes is shown.
Some examples of homotopy bands associated with $\Lambda$ are $\text{stuspo}n\text{mkjgda}cb\text{a}$ and $\text{stucba}$.

**Figure 13.** The special $\mathbb{Z}A_\infty^\infty$-component of $\mathcal{K}^h(\text{proj } \Lambda)$.
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Appendix A. Assignments of $S$ and $T$

Recall Definition 2 of a gentle algebra. We fix the functions $S, T : Q \to \{-1, 1\}$ for a quiver $Q$ in $Q^n$. For the arrows $\alpha_i$, where $1 < i \leq r$, we set $S\alpha_i = -1$ and $T\alpha_i = 1$. Similarly, for the arrows $\beta_j$, we set $S\beta_j = -1$ and $T\beta_j = 1$ for $1 \leq j < s$. For any 3-cycle containing neither $\alpha_1$ nor $\beta_s$, we assign values of $S$ and $T$ as shown in Figure 14a, where the arrow marked $a$ is either $\alpha_i$ or $\beta_j$ for some $1 < i \leq r$ or $1 \leq j < s$ (note that $Sa$ and $Ta$ are already taken care of by the above assignments). For the arrow $\alpha_1$, we set $S\alpha_1 = 1 = T\alpha_1$, and for the arrow $\beta_s$, we set $S\beta_s = -1 = T\beta_s$. In the case where $\alpha_1$, respectively $\beta_s$, is part of a 3-cycle, the values of $S$ and $T$ for the remaining arrows of the 3-cycle are shown in Figures 14b and 14c, respectively.

![Figure 14. Assignment of $S$ and $T$ to the arrows of a 3-cycle.](image-url)
Lemma 27. If $\Lambda \cong kQ/I$ is a cluster-tilted algebra of type $\tilde{A}_n$, then the above assignments of $S$ and $T$ to the quiver $Q$ in $Q^*_n$ satisfies the conditions given in Definition 3.

The proof is straight-forward. Note that this assignment of $S$ and $T$ is not unique. A different assignment of the functions will still give the same final result, but to give examples and technical results in an unequivocally manner, we need to fix explicitly given functions.

Appendix B. The longest antipath $\theta_{x,\varepsilon}$

An antipath in a gentle algebra is either a trivial homotopy string, or a direct homotopy string $\rho$ such that for any two consecutive arrows $\alpha$ and $\beta$ in $\rho$, we have that their composition is a relation. Bobiński defines, for each vertex $x \in Q_0$ and each $\varepsilon \in \{-1, 1\}$, the set $\Theta_{x,\varepsilon}$ consisting of all antipaths $\theta$ such that $t\theta = x$ and $T\theta = \varepsilon$. Moreover, he defines $\theta_{x,\varepsilon}$ to be the maximal element of $\Theta_{x,\varepsilon}$ if such element exists; otherwise, he defines $\theta_{x,\varepsilon} = \emptyset$.

We will now consider the possible values of $\theta_{x,\varepsilon}$ when $Q$ is a quiver in $Q^*_n$. The cases where $\theta_{x,\varepsilon}$ is not the empty string are the following:

- $\theta_{A_i, -1} = 1_{A_i}$ for $1 \leq i \leq r_2$,
- $\theta_{C_{i,1}} = \alpha_{r}$ when $r_1 > 0$,
- $\theta_{C_{i,-1}} = \beta_{s}$ when $s_1 > 0$,
- $\theta_{D_{i,1}} = \alpha_{i+r_2}$ for $1 \leq i \leq r_1 - 1$,
- $\theta_{D_{i,-1}} = 1_{D_i}$ for $0 \leq i \leq r_1 - 1$ and $r_2 > 0$,
- $\theta_{D'_{i,1}} = \beta_{j+s_2}$ for $1 \leq j \leq s_1 - 1$,
- $\theta_{D'_{i,-1}} = 1_{D'_i}$ for $0 \leq j \leq s_1 - 1$.

For the rest of the $\theta_{x,\varepsilon}$’s, the set $\Theta_{x,\varepsilon}$ is infinite with no maximal element. This is the case when there exists an arrow $\alpha$ which is part of a 3-cycle and such that $ta = x$ and $Ta = \varepsilon$.

Appendix C. Proof of Proposition 9

We now give a proof of Proposition 9 in section 6. In this proof we will use the algorithm presented by Bobiński in [5, Section 6], which we will now recall. Let $\omega = \alpha_l \cdots \alpha_1$ be a homotopy string with homotopy partition $\omega = \sigma_l \cdots \sigma_1$. Note that our convention of labeling the letters and homotopy letters is opposite of the convention used in [5].

We now state Bobiński’s algorithm for finding $\omega^+$. If $\alpha_l$ is a direct letter, let $\rho(\omega)$ denote the maximal integer $i \in [1, l]$ such that the $i$ last letters of $\omega$, that is $\alpha_l \cdots \alpha_{l-i+1}$, is an antipath. If $\alpha_l$ is an inverse letter or $\omega$ is a trivial homotopy string, we let $\rho(\omega) = 0$. Next, we define the substring $\omega'$ of $\omega$ to be

$$\omega' = \begin{cases} 
\omega & \text{if } \rho(\omega) = 0 \\
1_{S\omega} & \text{if } \rho(\omega) = l \\
\alpha_{l-\rho(\omega)} \cdots \alpha_1 & \text{if } 0 < \rho(\omega) < l
\end{cases}$$

Note that $\omega'$ is obtained by removing the longest possible antipath from the end of $\omega$, and that $\rho(\omega)$ denotes the number of letters removed.

Let $\sigma$ denote the maximal path of $Q$ with no subpath in $I$ such that $\sigma \cdot \omega$ is defined as composition of homotopy strings. Now, we have 6 cases for computing $\omega^+$, as listed below:
The calculations for Tables 2 and 4 can be done similarly. If \( \hat{\omega} \) be the homotopy string arising by performing the operation from the corresponding entry in the third column (i.e. \( \hat{\omega} = \omega_\ast \)). Next, consider all
possible $\tilde{\omega}$ such that $\tilde{\omega}^+ = \omega$ (we use Tables 2 and 4 for this). Then, for each such $\tilde{\omega}$, verify that $\tilde{\omega} = \hat{\omega}$. In this proof, we shall only do this for the entry $\alpha_l(\omega) = \gamma_{2i}$ in Table 3. The rest is left to the reader.

Assume that $\omega$ is a homotopy string with $\alpha_l(\omega) = \gamma_{2i}$ for some $1 \leq i \leq r_2$. We examine Tables 2 and 4 to find $\tilde{\omega}$ such that $\tilde{\omega}^+$ can be a homotopy string ending with such an arrow. The possibilities for $\tilde{\omega}$ are:

- $\alpha_l(\tilde{\omega}) = \alpha_i$ for $1 \leq i \leq r_2$
- $\alpha_l(\tilde{\omega}) = \alpha_i^{-1}$ for $2 \leq i \leq r_2 + 1$
- $\alpha_l(\tilde{\omega}) = \gamma_{2i}$ for $2 \leq i \leq r_2$ or $i = 1$ and $r_1 = 0$
- $\alpha_l(\tilde{\omega}) = \delta_{2i-1}$ for $1 \leq i \leq s_2$ and $r_1 = 0$
- $\alpha_l(\tilde{\omega}) = \delta_{2i}^{-1}$ for $1 \leq i \leq s_2$ and $r_1 = 0$

It is easy to see that those are all homotopy strings such that $\tilde{\omega}^+ = cw_r(t\tilde{\omega})\tilde{\omega}$. We have that $\hat{\omega}$ is the clockwise r-reduction of $\omega$, an since this operation undoes adding a step of a clockwise r-walk, we will always get $\hat{\omega} = \tilde{\omega}$.

\[\square\]