Extraction of a Weak Phase from $B \to D^{(*)}\pi$

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To observe CP-violating asymmetries through the interference of a weaker amplitude with a stronger one in $B^0 \to D^{(*)}\pi$ and $\bar{B}^0 \to D^{(*)}\pi$ decays, one must collect enough events that the intensity associated with the weaker amplitude would be statistically significant. We show that provided the weaker amplitude is measured separately in $B^\pm \to D^{(*)}\pm\pi^0$ decays, the time-integrated approach requires around $2.5 \cdot 10^8 B\bar{B}$ pairs for measurements of the weak phase $\sin(2\beta + \gamma)$ with an uncertainty of 0.05 or better. We also determine the optimal conditions for precise $2\beta + \gamma$ measurements and discuss the possibilities for resolving a discrete ambiguity.

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I Introduction

The phases of elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix describing the weak charge-changing interactions of quarks are of fundamental importance. Together with magnitudes of the matrix elements and masses of the six quarks $(u,c,t)$ and $(d,s,b)$, these phases must be explained by any theory which extends our knowledge beyond the Standard Model (SM) of electroweak and strong interactions.

Indirect information on CKM phases is now being supplemented by measurements of CP-violating asymmetries in $B$ decays which provide direct phase measurements. The weak phase $\beta \equiv \text{Arg}(-V_{cb}^* V_{cd}/V_{tb}^* V_{td})$ is determined by measurements of the rate asymmetry in decays such as $B^0 \to J/\psi K_S$, while $\alpha \equiv \text{Arg}(-V_{tb}^* V_{td}/V_{ub}^* V_{ud})$ will be determined by measurements in decays such as $B \to \pi\pi$ and $B \to \rho\pi$. Information on all charge modes will be needed to separate contributing amplitudes from one another.

Information on $\gamma \equiv \text{Arg}(-V_{ub}^* V_{ud}/V_{cb}^* V_{cd})$ is more difficult to obtain. The decays $B^\pm \to D^0 K^\pm$, $B^\pm \to D^0 K^\mp$, and $B^\pm \to D_{CP} K^\pm$, where $D_{CP}$ is a CP eigenstate, permit one to perform a triangle construction to extract the weak phase $\gamma$. The
interference of the Cabibbo-favored decay $D^0 \to K^-\pi^+$ and the doubly-Cabibbo-suppressed decay $D^0 \to K^+\pi^-$ introduces an important subtlety in this method [8]. Numerous determinations of $\gamma$ using nonstrange and strange $B$ decays to $\pi\pi$ and $K\pi$ are subject to questions associated with SU(3) flavor violation, electroweak penguin contributions, and rescattering [2].

The Cabibbo-favored decays $B_0^+ \to D^{(*)-}\pi^+$ and $B^{(*)+}\pi^-$ and the corresponding doubly-Cabibbo-suppressed modes $B_0^\to D^{(*)-}\pi^+$ and $B^0 \to D^{(*)+}K\pi^-$ can provide information on the weak phase $2\beta+\gamma$ [9, 10, 11, 12]. (One can substitute $\rho^\pm$ or $a_1^\pm$ for the charged pion.) These methods typically require measuring either a very small rate asymmetry (for the Cabibbo-favored modes) or a very small rate (for the Cabibbo-suppressed modes). It was therefore suggested recently [13] that one instead measure $2\beta+\gamma$ via the interference of a small amplitude with a larger one in decays of the form $B \to V_1 V_2$, where, for example, $V_1 = D^*$ and $V_2 = \rho$. The interference is to be detected through characteristic angular distributions in decay products of the vector mesons, and through time-dependent measurements. Refs. [12] and [14] contain some useful results regarding these distributions.

In the present paper we analyze the possibilities of precise measurements of $2\beta+\gamma$ for the simplest case of $B \to D^{(*)}\pi$ decays. We find the optimal conditions for measuring $2\beta+\gamma$. We also estimate the number of $B\bar{B}$ pairs needed for such measurements that will reduce the allowed range of $2\beta+\gamma$ values to the currently achieved indirect bounds coming from measurements of other CKM parameters.

A general feature of CP-violating asymmetries detected through the interference of a weaker amplitude with a stronger one is that one must be able to detect processes at the level of the absolute square of the weaker amplitude [14]. We find that this situation holds for $B \to D^{(*)}\pi$ decays. One still has to be able to collect enough events such that the absolute square of the Cabibbo-suppressed amplitude would be detectable with good statistical significance. This translates to the need for several times $10^8$ produced $B\bar{B}$ pairs. (Ref. [13] cites a figure of $10^8$ pairs for a useful measurement of $\sin(2\beta+\gamma)$ using $B \to V_1 V_2$ decays.) In fact, our best determination makes use of a direct measurement of the weaker amplitude through a factorization relation between $B^0 \to D^{(*)-}\pi^-$ and $B^+ \to D^{(*)+}\pi^0$ [11]. For both pseudoscalar and vector $D$ mesons in the final state, we employ different models to anticipate the size of the weaker amplitude. However, direct measurements of the rates for $B^+ \to D^{+}\pi^0$ and $B^+ \to D^{(*)+}\pi^0$ will eventually give us these amplitudes directly.

In Section II we introduce our notation and predictions for decay rates of neutral $B$ mesons in the framework of the time-integrated approach. We shall quote results for $B \to D^*\pi$ decays because of advantages in $D^*$ detection, recognizing that many are also valid for $B \to D\pi$. Decay rates as functions of a minimum vertex separation (expressed in terms of proper time) are of particular interest to us in Section III as we try to find the optimal conditions for measuring the weak phase $2\beta+\gamma$ with high precision. In Section IV we circumvent the problem of measuring the small weaker-to-stronger amplitude ratio $R$ by making a foray into charged $B$ meson decays, using the process $B^+ \to D^{(*)+}\pi^0$. Estimates of the minimum number of $B\bar{B}$ pairs required for precise measurements of $2\beta+\gamma$ are obtained in Section V. These are convoluted
II Notation and predictions

The “right-sign” decays $B^0 \rightarrow D^{*-}\pi^+$ and $\overline{B}^0 \rightarrow D^{*+}\pi^-$ are governed by the Cabibbo-favored combination of CKM matrix elements $V^*_{ub}V_{ud}$ or charge-conjugate, while the “wrong-sign” decays $\overline{B}^0 \rightarrow D^{-}\pi^+$ and $B^0 \rightarrow D^{+}\pi^-$ are governed by the doubly-Cabibbo-suppressed combination $V^*_{cd}V_{ub}$ or charge-conjugate. We denote $f \equiv D^{*-}\pi^+$ and $\bar{f} \equiv D^{*+}\pi^-$. Then from

$$\langle f|B^0\rangle = A_1 e^{i\phi_1 e^{i\delta_1}} , \quad \langle f|\overline{B}^0\rangle = A_2 e^{i\phi_2 e^{i\delta_2}}$$

it follows that

$$\langle \bar{f}|\overline{B}^0\rangle = A_1 e^{-i\phi_1 e^{i\delta_1}} , \quad \langle \bar{f}|B^0\rangle = A_2 e^{-i\phi_2 e^{i\delta_2}} ,$$

where the weak phases $\phi_i$ change sign under CP conjugation, while the strong phases $\delta_i$ do not. The amplitudes are in the ratio

$$R \equiv \frac{A_2}{A_1} = \left| \frac{V^*_{ub}V_{ud}}{V^*_{cd}V_{ub}} \right| r = | - \lambda^2 (\rho - i\eta) | r \simeq 0.02 r ,$$

where $\lambda \simeq 0.22$, $\rho$, and $\eta$ are parameters [16] which describe CKM matrix elements, and $r = O(1)$ describes a ratio of decay constants and form factors. The weak phase difference is

$$\phi_1 - \phi_2 = \text{Arg} \left( \frac{V^*_{ub}V_{ud}}{V^*_{cd}V_{ub}} \right) = \pi + \gamma .$$

We write the time-dependent decay amplitudes in terms of the functions [9]

$$f_+(t) \equiv e^{-imt} e^{-i\Gamma t/2} \cos(\Delta m t/2) , \quad f_-(t) \equiv e^{-imt} e^{-i\Gamma t/2} i \sin(\Delta m t/2) ,$$

where $m = (m_L + m_H)/2$ is the average of the two mass eigenvalues, $\Delta m = m_H - m_L$ is their difference, $\Gamma = (\Gamma_1 + \Gamma_2)/2$ is the average decay rate of the eigenstates, and we neglect $\Delta \Gamma = \Gamma_H - \Gamma_L$. Then

$$\langle f|B^0(t)\rangle = f_+(t)\langle f|B^0\rangle + \frac{q}{p} f_-(t)\langle f|\overline{B}^0\rangle$$

$$= e^{-imt} e^{-i\Gamma t/2} \left( A_1 e^{i\phi_1 e^{i\delta_1}} \cos \frac{\Delta m t}{2} + i \frac{q}{p} A_2 e^{i\phi_2 e^{i\delta_2}} \sin \frac{\Delta m t}{2} \right) ,$$

$$\langle \bar{f}|\overline{B}^0(t)\rangle = f_+(t)\langle \bar{f}|\overline{B}^0\rangle + \frac{p}{q} f_-(t)\langle \bar{f}|B^0\rangle$$

$$= e^{-imt} e^{-i\Gamma t/2} \left( A_1 e^{-i\phi_1 e^{i\delta_1}} \cos \frac{\Delta m t}{2} + i \frac{p}{q} A_2 e^{-i\phi_2 e^{i\delta_2}} \sin \frac{\Delta m t}{2} \right) .$$
If $B^0\rightarrow \bar{B}^0$ mixing is described primarily by standard model loop contributions dominated by intermediate $t\bar{t}$ pairs, we have $q/p = e^{-2i\beta}$, and

$$
|\langle f | B^0(t) \rangle|^2 = \frac{A_1^2}{2} e^{-\Gamma t} \left[ 1 + R^2 - (1 - R^2) \cos \Delta mt \right] - 2R \sin(2\beta + \gamma - \delta) \sin \Delta mt \tag{8}
$$

$$
|\langle f | \bar{B}^0(t) \rangle|^2 = \frac{A_1^2}{2} e^{-\Gamma t} \left[ 1 + R^2 - (1 - R^2) \cos \Delta mt \right] + 2R \sin(2\beta + \gamma + \delta) \sin \Delta mt \tag{9}
$$

where $\delta \equiv \delta_2 - \delta_1$.

Retracing the above steps for the “wrong-sign” decays $B^0 \rightarrow D^{*+}\pi^-$ and $\bar{B}^0 \rightarrow D^{*-}\pi^+$, we find

$$
|\langle f | B^0(t) \rangle|^2 = \frac{A_1^2}{2} e^{-\Gamma t} \left[ 1 + R^2 - (1 - R^2) \cos \Delta mt \right] - 2R \sin(2\beta + \gamma + \delta) \sin \Delta mt \tag{10}
$$

$$
|\langle f | \bar{B}^0(t) \rangle|^2 = \frac{A_1^2}{2} e^{-\Gamma t} \left[ 1 + R^2 - (1 - R^2) \cos \Delta mt \right] + 2R \sin(2\beta + \gamma - \delta) \sin \Delta mt \tag{11}
$$

Let us now consider the production of a $B^0\bar{B}^0$ pair in $e^+e^- \rightarrow \Upsilon(4S) \rightarrow B^0\bar{B}^0$, so that the pair is in a state $\Psi_-$ of negative charge-conjugation eigenvalue. Assume that we “tag” the initial production of a $\bar{B}^0(\hat{p})$ with a $B^0(\hat{\bar{p}})$, and the initial production of a $B^0(\hat{p})$ with a $\bar{B}^0(\hat{\bar{p}})$. Then if we define the proper decay time of the state $f$ with center-of-mass direction $\hat{p}$ as $t_f$, that of the tagging state with direction $\hat{\bar{p}}$ as $t_t$, and $t' \equiv t_f - t_t$, $T \equiv t_f + t_t$, we find [3, 12, 17]

$$
|\langle B^0(-\hat{p}), D^{*+}\pi^\pm(\hat{\bar{p}})|\Psi_- \rangle|^2 = e^{-\Gamma T} |A_1|^2 \left[ 1 + R^2 \pm (1 - R^2) \cos \Delta mt' \right] - 2R \sin(2\beta + \gamma \mp \delta) \sin \Delta mt' \tag{12}
$$

$$
|\langle \bar{B}^0(-\hat{p}), D^{*+}\pi^\mp(\hat{\bar{p}})|\Psi_- \rangle|^2 = e^{-\Gamma T} |A_1|^2 \left[ 1 + R^2 \pm (1 - R^2) \cos \Delta mt' \right] + 2R \sin(2\beta + \gamma \pm \delta) \sin \Delta mt' \tag{12}
$$

One can express the time-integrated decay rates as

$$
\int_0^\infty dt_f \int_0^\infty dt_t \langle B^0(-\hat{p}), D^{*+}\pi^\pm(\hat{\bar{p}})|\Psi_- \rangle|^2 \propto \int_{-\infty}^\infty dt' e^{-\Gamma |t'|} [A_\pm(t') + B_\mp(t')] \tag{13}
$$

$$
\int_0^\infty dt_f \int_0^\infty dt_t \langle \bar{B}^0(-\hat{p}), D^{*+}\pi^\mp(\hat{\bar{p}})|\Psi_- \rangle|^2 \propto \int_{-\infty}^\infty dt' e^{-\Gamma |t'|} [A_\pm(t') - B_\pm(t')] \tag{13}
$$

where

$$
A_\pm(t') \equiv (1 + R^2) \pm (1 - R^2) \cos \Delta mt' \tag{14}
$$

$$
B_\pm(t') \equiv -2R \sin(2\beta + \gamma \pm \delta) \sin \Delta mt' \tag{15}
$$

are even and odd functions of $t'$, respectively.
Now we introduce notation for measurable decay numbers. The number of $B^0 \rightarrow D^{*+}\pi^-$ decays with vertex separation $t' > 0$ is
\[ N^r_+ \propto \int_0^\infty dt' e^{-\Gamma|t'|}[A_+(t') + B_-(t')], \] (16)
while those with $t' < 0$ is
\[ N^r_- \propto \int_{-\infty}^0 dt' e^{-\Gamma|t'|}[A_+(t') + B_-(-t')]. \] (17)
Here the superscript “\( r \)” denotes right-sign decays. The corresponding expressions \( N^w_\pm \) for the wrong-sign (superscript “\( w \)” decays $B^0 \rightarrow D^{*+}\pi^-$ with $t' > 0$ and \( N^w_- \) for $B^0 \rightarrow D^{*+}\pi^-$ with $t' < 0$ are
\[ N^w_\pm \propto \int_0^\infty dt' e^{-\Gamma|t'|}[A_-(-t') \pm B_+(t')]. \] (18)
Similar expressions for $\overline{B}^0$ decays are
\[ \overline{N}^r_\pm \propto \int_0^\infty dt' e^{-\Gamma|t'|}[A_+(t') \mp B_-(t')], \] (19)
\[ \overline{N}^w_\pm \propto \int_0^\infty dt' e^{-\Gamma|t'|}[A_-(-t') \mp B_+(t')]. \] (20)
Note that the following 4 linear relations among the 8 decay numbers
\[ N^r_+ + N^r_- = N^w_+ + N^w_-, \] (21)
\[ N^r_+ - N^r_- = N^w_+ - N^w_-, \] (22)
limit the number of independent quantities to 4. In principle, that allows one to forgo measurements of $\overline{B}^0$ decay numbers. However, that method would lead to larger uncertainties in determination of $2\beta + \gamma$ and we shall not use it.

We shall investigate the dependence of the time-integrated rates on a minimum vertex separation $t_0$. The aim of the calculation is to find the optimal conditions for measuring $\sin(2\beta + \gamma)$. Fig. 1 shows that indirect bounds on that weak phase coming from measurements of other CKM parameters [2, 18] limit the expected value of $\sin(2\beta + \gamma)$ to the region between 0.89 and 1. To get in the same ballpark with the indirect bounds we will calculate the number of $\overline{B}B$ pairs required to determine $\sin(2\beta + \gamma)$ with an uncertainty of 0.05. This is the main goal of the paper.

### III Decays with vertex separation greater than $t_0$

If one only takes into account decays with vertex separation greater than $t_0$, Eqs. (16–20) become
\[ N^r_+(t_0) \propto \int_{t_0}^\infty dt' e^{-\Gamma|t'|}[A_+(t') \pm B_-(t')], \] (23)
\[ N^w_+(t_0) \propto \int_{t_0}^\infty dt' e^{-\Gamma|t'|}[A_-(t') \pm B_+(t')], \] (24)
Figure 1: Contours of $\sin(2\beta + \gamma)$ (thin curves with values to the right) in $(\rho, \eta)$ plane. Thick lines denote current limits on CKM matrix parameters \cite{2, 18}. Solid circles denote limits on $|V_{ub}/V_{cb}|$ from charmless $b$ decays, dashed circles denote limits on $V_{td}$ from $B^0 - \bar{B}^0$ mixing, and the dotted circle denotes the lower limit on $|V_{ts}/V_{td}|$ from the lower limit on $B_s - \bar{B}_s$ mixing. Dot-dashed hyperbolae come from limits on CP-violating $K^0 - \bar{K}^0$ mixing (the parameter $\epsilon$). Two solid rays denote the recent world average $\pm 1\sigma$ limits $\sin(2\beta) = 0.79 \pm 0.10$ from neutral $B$ meson decays. The allowed range is shaded gray.
\begin{align}
\mathcal{N}_\pm(t_0) & \propto \int_{t_0}^\infty dt\ e^{-R|t'|}[A_+(t') \mp B_+(t')], \\
\mathcal{N}_\mp(t_0) & \propto \int_{t_0}^\infty dt\ e^{-R|t'|}[A_-(t') \mp B_-(t')].
\end{align}

There are several ways to combine these decay numbers together into algebraic sums. Some of the resulting combinations may include one of the following expressions: \(A_+(t') + A_-(t')\), \(A_+(t') - A_-(t')\), \(B_+(t') + B_-(t')\), or \(B_+(t') - B_-(t')\). Composing ratios of these algebraic sums (see \(f_1(t_0)\), \(f_2(t_0)\) and \(f_3(t_0)\) below), we can extract the parameters \(R\), \(\sin(2\beta + \gamma)\) \(\cos \delta\) and \(\cos(2\beta + \gamma)\) \(\sin \delta\):

\[
R = \sqrt{\frac{a(t_0) - f_1(t_0)}{a(t_0) + f_1(t_0)}},
\]

\[
SC \equiv \sin(2\beta + \gamma) \cos \delta = \frac{1 + R^2}{2b(t_0) R} f_2(t_0),
\]

\[
CS \equiv \cos(2\beta + \gamma) \sin \delta = \frac{1 + R^2}{2b(t_0) R} f_3(t_0),
\]

where

\[
a(t_0) \equiv \Gamma e^{\Gamma t_0} \int_{t_0}^\infty dt\ e^{-\Gamma t'} \cos(\Delta mt') = \frac{1}{\sqrt{1 + x_d^2}} \cos(x_d \Gamma t_0 + \Delta),
\]

\[
b(t_0) \equiv \Gamma e^{\Gamma t_0} \int_{t_0}^\infty dt\ e^{-\Gamma t'} \sin(\Delta mt') = \frac{1}{\sqrt{1 + x_d^2}} \sin(x_d \Gamma t_0 + \Delta),
\]

\[
\Delta = \arctan x_d, \quad x_d \equiv \frac{\Delta m}{\Gamma}.
\]

and

\[
f_1(t_0) \equiv \frac{(N_+^r + N_+^w + N_+^r + N_+^w) - (N_+^w + N_+^w + N_+^w + N_+^w)}{N},
\]

\[
f_2(t_0) \equiv \frac{(N_+^r + N_+^w + N_+^r + N_+^w) - (N_+^r + N_+^r + N_+^r + N_+^r)}{N},
\]

\[
f_3(t_0) \equiv \frac{(N_+^r + N_+^w + N_+^r + N_+^w) - (N_+^r + N_+^w + N_+^r + N_+^r)}{N},
\]

with

\[
N \equiv N_+^r + N_+^r + N_+^r + N_+^r + N_+^r + N_+^r + N_+^r + N_+^r.
\]

We have suppressed \((t_0)\) after the decay numbers in the last four formulae.

It has been noted in [11, 13] that \(R\) is too small to be determined by this method. Indeed, calculations show that the smallest uncertainty in \(R\) is achieved at \(t_0 = 0\) and is equal to

\[
\sigma(R) = \sqrt{\frac{x_d^2 (2 + x_d^2)}{16 R^2} \frac{1}{\epsilon (B^r + B^w) N_B}} \approx 0.03,
\]

with \(\epsilon\) being the tagging efficiency. We take \(\epsilon\) to be \(0.684 \pm 0.007\) [1]. \(B^r\), the branching ratio of the “right-sign” decays \(B^0 \to D^*+\pi^+\), equals \((2.76 \pm 0.21) \times 10^{-3}\) [19]. One
can show that for $x_d \approx 0.756 \pm 0.012$ [20] the branching ratio of “wrong-sign” decays is $\mathcal{B}^w = k\mathcal{B}^r \approx 0.61 \cdot 10^{-3}$, with $k \approx x_d^2/(2 + x_d^2) \approx 0.22$.

The error $\sigma(R) = 0.03$ is bigger than the approximate $R$ value itself [Eq. (3)]. Thus, one has to search for another method of measuring $R$.

### IV Ratio of amplitudes

The main reason one cannot get $R$ directly from the ratio of $B^0 \to D^{(*)+} \pi^-$ and $B^0 \to D^{(*)-} \pi^+$ decay rates is that the large $B^0 \to \bar{B}^0$ mixing amplitude in the former overwhelms the smaller direct tree contribution. One can circumvent this obstacle by considering decays of charged $B$ mesons, e.g. $B^\pm \to D^{(*)\pm} \pi^0$, as suggested in [11]. The tree amplitude is dominant in these decays and is proportional to $A_2^2/2$. Thus, the ratio of $B^\pm \to D^{(*)\pm} \pi^0$ and $B^0 \to D^{(*)-} \pi^+$ decay rates can be used to provide a simple way to estimate $R$.

The $B^+ \to D^{(*)+} \pi^0$ decay rate can be estimated by assuming factorization:

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} V_{ub}^* V_{cd} (\pi(p - q)) \langle \bar{b} \gamma_\mu u | B(p) \rangle \langle D^{(*)} | V_\mu | 0 \rangle.$$ \hfill (38)

Using the standard parameterization [21], one obtains the ratio $r$ defined in Eq. (3):

$$r(D^*) = \frac{f_D^* \cdot F_{B\pi}^D(m_D^2)}{f_\pi A_B^{BD}(m_\pi^2)};$$

$$r(D) = \frac{f_D (m_B^2 - m_\pi^2) F_0^{B\pi}(m_B^2)}{f_\pi (m_B^2 - m_\pi^2) F_0^{BD}(m_\pi^2)}. \hfill (39)$$

In Table I, we give the values of $r$ for $\bar{B}^0 \to D^{(*)} \pi$ decays in several models. In all cases, the models predict that $r$ is close to unity, i.e. $R \sim 0.02$.

The error on $R$ can be estimated using the method described in the beginning of this Section. Suppose that the number of detected $B^+ \to D^{(*)+} \pi^0$ decays is $N_2$ out of $N$ tagged $B^+$'s, while the number of detected $B^0 \to D^{(*)-} \pi^+$ decays is $N_1$ out of the same number $N$ of tagged $B^0$'s. Then, assuming equal charged and neutral $B$ production, the value

$$R = \sqrt{2 \frac{\tau_B}{\tau_{B^+}} \frac{N_2}{N_1}}.$$ \hfill (40)
has an uncertainty
\[
\sigma(R) = \sqrt{\frac{2\tau^0_B}{\tau^0_B + 2\sqrt{N_1}}} \sqrt{1 + \frac{N_2}{N_1}} \approx \frac{1}{\sqrt{2N_1}}. \tag{41}
\]

Taking into account \(B^- \to D^{(*)-}\pi^0\) and \(\bar{B}^0 \to D^{(*)+}\pi^-\) decays increases statistics by a factor of 2, leading to \(\sigma(R) = 1/\sqrt{2N_r}\), where \(N_r = 2N_1\) is the number of \(B^0 \to D^{(*)-}\pi^+\) decays plus the number \(\bar{B}^0 \to D^{(*)+}\pi^-\) decays. To make connection with the total number \(N_B\) of produced \(\bar{B}B\) pairs, note that the number of tagged events is \(N_r = \epsilon B^r N_B\). Thus,
\[
\sigma(R) = \frac{1}{\sqrt{2\epsilon B^r N_B}}. \tag{42}
\]

For \(10^8\) produced \(\bar{B}B\) pairs \(\sigma(R) = 0.17 \cdot 10^{-2}\), i.e. less than 10\% of its value. Thus, measurements of \(B^+ \to D^{(*)+}\pi^0\) decay rates provide the ratio of amplitudes with a high precision. This information may be used in the time-integrated approach discussed in the previous Section. Now we can go a step further and estimate the uncertainty in determination of \(\sin(2\beta + \gamma)\cos \delta\) and \(\cos(2\beta + \gamma)\sin \delta\).

In the following analysis, we will take \(r = 1\) (corresponding to \(R \approx 0.02\)) and use Eq. (42) to estimate the error on the ratio \(R\).

V Uncertainties in \(\sin(2\beta + \gamma)\cos \delta\) and \(\cos(2\beta + \gamma)\sin \delta\) with perfect time resolution and no mistagging

The uncertainties in the ratios \(f_2\) and \(f_3\) [see Eqs. (34) and (33)] are
\[
\sigma(f_2) \approx \sigma(f_3) \approx \frac{1}{\sqrt{N(t_0)}} \left[ \frac{e^{\Gamma t_0}}{\epsilon(\Gamma^r + B^u)N_B} \right] \left[ \frac{e^{\Gamma t_0}}{\epsilon(1 + k) B^r N_B} \right]. \tag{43}
\]

Eqs. (28) and (29) allow an estimate of the values of \(f_2\) and \(f_3\): \(f_2 \approx 2R b(t_0) SC\), \(f_3 \approx 2R b(t_0) CS\). Now that Eq. (42) provides the error in \(R\), we can calculate the uncertainties in \(SC\) and \(CS\):
\[
\sigma(SC) \approx \frac{1}{2b(t_0)} \sqrt{\frac{f_2^2(t_0)}{R^4} \sigma^2(R) + \frac{\sigma^2(f_2)}{R^2}} \leq \frac{1}{2b(t_0) R} \sqrt{\frac{2(1 + k) b^2(t_0) + e^{\Gamma t_0}}{\epsilon(1 + k) B^r N_B}}, \tag{44}
\]
\[
\sigma(CS) \approx \frac{1}{2b(t_0)} \sqrt{\frac{f_3^2(t_0)}{R^4} \sigma^2(R) + \frac{\sigma^2(f_3)}{R^2}} \approx \frac{\sigma(f_3)}{2b(t_0) R} \approx \frac{1}{2b(t_0) R} \sqrt{\frac{e^{\Gamma t_0}}{\epsilon(1 + k) B^r N_B}}. \tag{45}
\]

Finally, one can calculate the number of \(\bar{B}B\) pairs needed to get any particular precision \(\sigma_0(SC)\):
\[
N_B \approx \frac{2(1 + k) b^2(t_0) + e^{\Gamma t_0}}{4\epsilon(1 + k) b^2(t_0) R^2 B^r \sigma_0^2(SC)}, \tag{46}
\]
or \( \sigma_0(CS) \):

\[
N_B \approx \frac{e^{Pt_0}}{4\epsilon(1+k)B^2(t_0)R^2B^r\sigma_0^2(CS)}.
\] (47)

As seen from the figure, these two quantities have the same minimum location because they only differ by a constant independent of \( t_0 \). Here we have assumed that \( f_3(t_0) \) is proportional to \( \cos(2\beta + \gamma)\sin\delta \), which is expected to be small, and that \( SC \) is close to 1. However, the neglected \( SC \) and \( CS \) dependence can be readily put back if necessary, and one finds that the position of the minima would remain the same, independent of the values of \( SC \) and \( CS \), for both curves.

Fig. 2 shows the \( N_B \) dependence on \( t_0 \) according to the above two equations. The curves were calculated under the assumption that one needs to get \( \sigma_0 = 0.1 \). We found out that this precision level is sufficient to determine \( \sin(2\beta + \gamma) \) with an uncertainty of 0.05 (Section VII). The optimal conditions for both measurements are achieved if one only takes into account decays with vertex separation greater than \( \sim 0.45/\Gamma \). That one needs fewer \( B\bar{B} \) pairs to reach the same precision for \( \cos(2\beta + \gamma)\sin\delta \) as indicated in Fig. 2 reflects our previous assumption of small \( \cos(2\beta + \gamma)\sin\delta \). Thus, the minimum uncertainties one can obtain if \( N_B \) \( B\bar{B} \) pairs are available are

\[
\sigma_{\min}(SC) \approx 0.1 \sqrt{\frac{1.62 \times 10^8}{N_B}},
\] (48)
Now we shall check how these formulae change if we take into account finite time resolution and realistic mistagging probabilities.

VI Finite time resolution; mistagging

Measurements of the decay numbers are smeared by finite resolution of vertex separation. For simplicity we shall assume a single Gaussian resolution function. The observed decay numbers are given by Eqs. (23) − (26) convoluted with the resolution function

\[ R(t_0) \equiv \int_{-\infty}^{+\infty} d\mu \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{\mu - t_0}{\sigma}}. \]  

For example,

\[ N_{r+}(t_0) = N_0 \cdot R(t_0) \otimes N_{r+}^\prime(\mu), \]  

and similar convoluted relations for \( N_{r-}^w(t_0), N_{w+}^r(t_0), \) and \( N_{w-}^r(t_0). \) Here \( N_0 \) is a normalization factor and \( \sigma \) is the resolution of time separation between vertices. For the BaBar detector the average resolution of space separation between vertices is 180 \( \mu \) \( m \) while the average separation is \( \beta\gamma c\tau_{B_0} = 260 \mu m, \) implying \( \sigma\Gamma = 180/260 \approx 0.69. \)

The algebraic sums of decay numbers that enter Eqs. (34) and (35) have to be modified correspondingly. For example, Eq. (36) becomes

\[ N(t_0) = 2 N_0 \int_{-\infty}^{+\infty} d\mu \Phi\left(\frac{\mu - t_0}{\sqrt{2\sigma}}\right) \int_{\mu}^{+\infty} dt' e^{-|t'|} [A_+(t') + A_-(t')] \]

\[ = 2 N_0 \int_{-\infty}^{+\infty} d\tilde{t} e^{-|\tilde{t}|} [A_+ (\tilde{t}/\Gamma) + A_- (\tilde{t}/\Gamma)] \]

\[ + 2 N_0 \int_{-\infty}^{+\infty} d\tilde{\mu} \Phi\left(\frac{\tilde{\mu} - \Gamma t_0}{\sqrt{2\sigma \Gamma}}\right) e^{-|\tilde{\mu}|} [A_+ (\tilde{\mu}/\Gamma) + A_- (\tilde{\mu}/\Gamma)] \]

\[ = 4 N_0 \frac{A_1^2}{A_1^2} (J_1 + J_2), \]  

Similarly, one obtains

\[ (N_{r+} + N_{w+} + N_{r+}^w + N_{w+}^r) - (N_{r+} + N_{w+} + N_{r+}^w + N_{w+}^r) = 8 \frac{N_0}{\Gamma} R J_3 \text{SC}, \]  

\[ (N_{r+} + N_{w+} + N_{r+}^w + N_{w+}^r) - (N_{r+} + N_{w+} + N_{r+}^w + N_{w+}^r) = 8 \frac{N_0}{\Gamma} R J_3 \text{CS}. \]  

In the above equations, \( \Phi(x) \equiv (2/\sqrt{\pi}) \int_0^x e^{-z^2} dz \) is the error function and

\[ J_1 \equiv \int_{-\infty}^{+\infty} e^{-|\tilde{t}|} d\tilde{t} = 2, \]

\[ J_2 \equiv \int_{-\infty}^{+\infty} d\tilde{\mu} \Phi\left(\frac{\tilde{\mu} - \Gamma t_0}{\sqrt{2\sigma \Gamma}}\right) e^{-|\tilde{\mu}|}, \]

\[ J_3 \equiv \int_{-\infty}^{+\infty} d\tilde{\mu} \Phi\left(\frac{\tilde{\mu} - \Gamma t_0}{\sqrt{2\sigma \Gamma}}\right) e^{-|\tilde{\mu}|} \sin x d\tilde{\mu}, \]
The last two integrals have been numerically evaluated for different values of $t_0$ in the range from 0 to $1.5/\Gamma$. Now $SC$ and $CS$ can be rewritten in terms of the ratios $f_2$ and $f_3$ as

$$SC, CS = \frac{J_1 + J_2}{1 - 2w} \frac{1 + R^2}{R \sqrt{J_3}} f'_{2,3}(t_0).$$

Next, we will take into account the mistagging factor. Mistagging refers to the cases where a decay ($B^0 \rightarrow \text{tag}, \bar{B}^0 \rightarrow D^{*-\pi^+}$) was incorrectly identified as ($\bar{B}^0 \rightarrow \text{tag}, B^0 \rightarrow D^{*-\pi^+}$), and vice versa. Thus, one sees that decays labelled as $B^0 \rightarrow D^{*-\pi^+}$ (“right-sign” decays) actually contain some $\bar{B}^0 \rightarrow D^{*-\pi^+}$ (“wrong-sign” decays) events. As a result, experimental measurements only provide decay numbers smeared by the mistagging effect. For instance, the numbers of apparent right-sign events are

$$N'_\pm(t_0) = (1 - w)N'_\pm(t_0) + wN''_\pm(t_0),$$

where $w$ is the mistagging probability. For the BaBar detector the tagging efficiency is $\epsilon = \sum \epsilon_i = 0.684 \pm 0.007$ while the effective tagging efficiency is $Q = \sum \epsilon_i(1 - 2w)^2 = 0.261 \pm 0.012$. For our purposes we will simplify calculations by assuming the single tagging option with $\epsilon = 0.684$ and $Q = \epsilon(1 - 2w)^2 = 0.261$. Thus, the effective mistagging probability is $w = 0.191$.

Note that the sum of all smeared decay numbers is still equal to $N$, the sum of all physical decay numbers. One can show that the ratios $f'_2$ and $f'_3$ composed of smeared decay numbers are related to $f_2$ and $f_3$ by $f'_2 = (1 - 2w)f_2$, $f'_3 = (1 - 2w)f_3$. Thus, experimental measurements of smeared decay numbers allow the direct calculations of $SC$ and $CS$:

$$SC, CS = \frac{J_1 + J_2}{1 - 2w} \frac{1 + R^2}{R \sqrt{J_3}} f'_{2,3}(t_0).$$

Assuming that experimental uncertainties are $\sigma[N'_\pm(t_0)] = \sqrt{N'_\pm(t_0)}$, $\sigma[N''_\pm(t_0)] = \sqrt{N''_\pm(t_0)}$, etc., we can estimate the uncertainties in $f'_2$ and $f'_3$ measurements to be

$$\sigma(f'_2) \approx \sigma(f'_3) \approx \frac{1}{\sqrt{\sigma(N(t_0))}} = \frac{e^{\Gamma t_0}}{\sqrt{\epsilon(1 + k) B^{\ast} N_B}}.$$

The uncertainties in $SC$ and $CS$ measurements are

$$\sigma(SC) \leq \frac{1}{1 - 2w} \frac{J_1 + J_2}{2J_3} \frac{1}{R} \sqrt{\frac{2(1 + k)[J_3/(J_1 + J_2)]^2(1 - 2w)^2 + e^{\Gamma t_0}}{\epsilon(1 + k) B^{\ast} N_B}},$$

$$\sigma(CS) \approx \frac{1}{1 - 2w} \frac{J_1 + J_2}{2J_3} \frac{1}{R} \sqrt{\frac{e^{\Gamma t_0}}{\epsilon(1 + k) B^{\ast} N_B}}.$$

We assumed a small $CS$ in deriving the second equation. Finally, one can calculate the number of $B\bar{B}$ pairs needed to get any particular precision $\sigma_0(SC)$:

$$N_B \approx \frac{2(1 - 2w)^2(1 + k) J_3^2 + e^{\Gamma t_0} (J_1 + J_2)^2}{4\epsilon(1 - 2w)^2(1 + k) J_3^2 R^2 B^{\ast} \sigma_0^2(SC)},$$
Figure 3: Number of produced $B\bar{B}$ pairs needed to achieve the uncertainty of 0.1 in measurements of $\sin(2\beta + \gamma) \cos \delta$ (solid line) and $\cos(2\beta + \gamma) \sin \delta$ (dashed line) vs. minimum vertex separation $t_0$.

or $\sigma_0(CS)$:

$$N_B \approx \frac{e^{\Gamma t_0} (J_1 + J_2)^2}{4\epsilon(1 - 2w)^2(1 + k) J_3^2 R^2 B^{-r} \sigma_0^2(CS)}.$$  \hfill (65)

As in the previous Section, the position of the minima is the same for both curves and is independent of the values of $SC$ and $CS$.

Fig. 3 shows the $N_B$ dependence on $t_0$. The curves were calculated under the assumption that one needs to get $\sigma_0 = 0.1$. The optimal conditions for measurements are achieved if one only takes into account decays with vertex separation greater than $0.44/\Gamma$. Then

$$\sigma_{min}(SC) \simeq 0.1 \sqrt{\frac{5.06 \cdot 10^8}{N_B}},$$  \hfill (66)

$$\sigma_{min}(CS) \simeq 0.1 \sqrt{\frac{4.40 \cdot 10^8}{N_B}}.$$  \hfill (67)

If BaBar is able to improve its performance to the level quoted in [27], i.e. $\sigma(\Delta z) = 110 \, \mu m$, $\epsilon = 0.767$ and $Q = 0.279$, then the required minimum number of $B\bar{B}$ pairs reduces by a factor of $\sim 1.4$ for both $SC$ and $CS$ measurements. Besides, the position of the minima is shifted to a slightly larger value ($t_0 \sim 0.53/\Gamma$) of vertex separation.
Figure 4: Contours of $s_-$, $s_+$, and their uncertainties $\sigma(s_-)$ and $\sigma(s_+)$ in the $(SC,CS)$ plane. Only the first quadrant of the plane is shown. The plots in other quadrants are symmetric to those in the first one since the plotted quantities only depend on the absolute values of $SC$ and $CS$. The blank triangle above the line $SC + CS = 1$ denotes the forbidden region on the plane: $SC + CS = \sin(2\beta + \gamma + \delta)$ should always be smaller or equal to 1. When $CS = 0$, $\sigma(s_-)$ achieves its smallest values: 0.1 and 0.07 for plots (c) and (e) respectively.
VII Extraction of $\sin(2\beta + \gamma)$ and $\cos \delta$

If one measures $\sin(2\beta + \gamma) \cos \delta$ and $\cos(2\beta + \gamma) \sin \delta$ values to be $SC$ and $CS$, then trigonometry dictates the following values for $\sin^2(2\beta + \gamma)$ and $\cos^2 \delta$:

$$
\sin^2(2\beta + \gamma), \cos^2 \delta = s^2_+ \equiv \frac{1}{2} \left( 1 + SC^2 - CS^2 \pm \sqrt{\lambda(1,SC^2,CS^2)} \right),
$$

where $\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$. When one root corresponds to $\sin^2(2\beta + \gamma)$, the other corresponds to $\cos^2 \delta$. There is an ambiguity: Which is which? One cannot resolve it without making additional assumptions.

If the value of $\sin(2\beta + \gamma)$ is assumed to be in agreement with the indirect bounds (Fig. 1) then it should be larger then 0.89. However, indications that $\cos \delta$ is large, too [28], do not allow an easy distinction between the two quantities. Figs. 4(a,b) show the contours of $s_-$ and $s_+$ values in the $(SC,CS)$ plane. There is a big region on the plane where $s_- < 0.89$ while $s_+ > 0.89$. If the measured values of $SC$ and $CS$ fall inside this region then $\sin(2\beta + \gamma)$ could only be associated with $s_+$ and the ambiguity might be resolved. The possibility of resolution also depends on the uncertainties $\sigma(s_-)$ and $\sigma(s_+)$. Those are calculated from Eq. (68) with the help of Eqs. (63)–(67). The contours of these uncertainties are shown in Figs. 4(c,d,e,f) for two different numbers of $B\bar{B}$ pairs. For example, if the number of produced $B\bar{B}$ pairs is $5.06 \cdot 10^8$ and $(SC,CS) = (0.75, 0.15)$, then we can calculate $s_+ = 0.97 \pm 0.04$ and $s_- = 0.77 \pm 0.12$. In this case, $s_-$ does not take values that are larger than 0.89 within the $1\sigma$ level, and the solution favored for consistency with Fig. 1 is $\sin^2(2\beta + \gamma) = s^2_+$, $\cos^2 \delta = s^2_-$. The 4-fold ambiguity in $2\beta + \gamma$ remains but reduces to a 2-fold one when we take into account that only positive values of $\sin(2\beta + \gamma)$ are consistent with indirect bounds. One can see from Fig. 1 that if $2\beta + \gamma < \pi/2$ then $\sin(2\beta + \gamma)$ should be larger than 0.97. This fact might completely resolve the ambiguity in favor of $\pi/2 < 2\beta + \gamma < \pi$ if values larger than 0.97 are measured to be inconsistent with $\sin(2\beta + \gamma)$ within the $1\sigma$ level.

Of course, one cannot exclude the possibility that $\sin(2\beta + \gamma)$ is inconsistent with indirect bounds and is substantially smaller than 0.89 while $|\cos \delta|$ is close to unity. In that case, one would make a wrong assignment of $s_+$ and $s_-$ to $\sin(2\beta + \gamma)$ and $\cos \delta$, respectively. Therefore, it is preferable to make other measurements of $\sin(2\beta + \gamma)$ in decays like $B \to D^{(*)} \rho$ or $B \to D^{(*)} a_1$ where strong phase might differ from $\delta$ in $B \to D^{(*)} \pi$ decays.

It is also worth noting that for the overwhelming part of the region where $s_+ > 0.89$, the uncertainty in $\sin(2\beta + \gamma)$ is at most 0.05 [cf. Figs. 4(b) and 4(d)]. Thus, for many values of $SC$ and $CS$ this method allows a very precise determination of $\sin(2\beta + \gamma)$ and a good measurement of the strong phase $\delta$.

Besides, the method can be used to detect deviations from the Standard Model. If the measured values of $\sin(2\beta + \gamma) \cos \delta$ and $\cos(2\beta + \gamma) \sin \delta$ fall into the upper left corner of the $(SC,CS)$ plane, then both $s_-$ and $s_+$ would be inconsistent with the $0.89 - 1.0$ range expected from the unitarity of the CKM matrix.
VIII Conclusions

This paper has explored the optimal conditions for measurements of weak phase angle $2\beta + \gamma$ and strong phase $\delta$ between Cabibbo-allowed and doubly-Cabibbo-suppressed amplitudes in $B \to D^{(*)}\pi$ decays. We have found that in the time-integrated approach it is advantageous to only consider events with vertex separation greater than $t_0$ which is equal to $0.44/\Gamma$ for the BaBar detection parameters. The loss in statistics is outweighed by an increase in the integrated asymmetry.

Fig. 3 shows that production of approximately $5 \cdot 10^8 B\bar{B}$ pairs is needed to reduce the uncertainty in determination of $\sin(2\beta + \gamma) \cos \delta$ to 0.1 in $B \to D^*\pi$ decays. A smaller error on $\cos(2\beta + \gamma) \sin \delta$ will be achieved at the same time if its value is small. $B \to D\pi$ decays have the advantage of a slightly higher branching ratio but a setback in $D$ meson detection. The combination of both types of decays might reduce the number of needed $B\bar{B}$ pairs to $2.5 \cdot 10^8$, an amount within the reach of both BaBar and BELLE in the next few years. A time-dependent analysis [17] does not lead to any improvement with respect to this figure.

If the strong phase $\delta$ is not very close to 0 or $\pi$, the ambiguity between $\sin(2\beta + \gamma)$ and $\cos \delta$ can be resolved. This method allows $\sin(2\beta + \gamma)$ to be determined with a precision of 0.05 or better.

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