On the Generalized Enveloping Algebra of a Color Lie Algebra

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Abstract

Let $G$ be an abelian group, $\epsilon$ an anti-bicharacter of $G$ and $L$ a $G$-graded $\epsilon$-Lie algebra (color Lie algebra) over $K$ a field of characteristic zero. We prove that all $G$-graded, positive filtered $A$ such that the associated graded algebra is isomorphic to the $G$-graded $\epsilon$-symmetric algebra $S(L)$, there is a $G$-graded $\epsilon$-Lie algebra $L$ and a $G$-graded scalar two cocycle $\omega \in Z^2_{gr}(L,K)$, such that $A$ is isomorphic to $U_\omega(L)$ the generalized enveloping algebra of $L$ associated with $\omega$. We also prove there is an isomorphism of graded spaces between the Hochschild cohomology of the generalized universal enveloping algebra $U(L)$ and the generalized cohomology of color Lie algebra $L$.

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Introduction

Let $G$ be an abelian group, $\epsilon$ an anti-bicharacter of $G$ and $(L,[,])$ a $G$-graded $\epsilon$-Lie algebra (color Lie algebra) over $K$ a field of characteristic zero. Let $\omega \in Z^2_{gr}(L,K)$ be a scalar graded two cocycle of degree zero in the sense of Scheunert-Zhang, \[\text{[6]}\]. The generalized enveloping algebra (or $\omega$-enveloping algebra) of $L$ is the quotient of the $G$-graded tensor algebra $T(L)$ by the $G$-graded two-sided ideal generated by the elements $v_1 \otimes v_2 - \epsilon(|v_1|,|v_2|)v_2 \otimes v_2 - [v_1,v_2] - \omega(v_1,v_2)$, where $v_1,v_2$ are homogeneous elements of $L$. The object of the present paper is to study the structure of the generalized enveloping algebra. In Section 1 we

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fix notation and provide background material concerning finite group gradings
and color Lie algebras. In Section 2 we introduce the generalized enveloping
algebra of color Lie algebra and study its properties. In particular we state
the generalized Poincaré-Birkhoff-Witt theorem for the generalized enveloping
algebra. In Section 3 we classify all $G$-graded, positive filtered $A$ such that
the associated graded algebra is isomorphic to the $G$-graded $\varepsilon$-symmetric algebra
$S(L)$ which extends the result of Sridharan for Lie algebras, \cite{7}. In Section 4
we introduce a graded generalization cohomology (or $\omega$-cohomology) of color Lie
algebras which coincides with the graded Chevalley-Eilenberg cohomology of
degree zero of $L$ introduced by Scheunert and Zhang \cite{6} in the case $\omega = 0$. We
show that there is an isomorphism of graded spaces between the Hochschild
cohomology of the generalized universal enveloping algebra and the graded $\omega$-
cohomology of color Lie algebra.

1 Preliminaries

Throughout this paper groups are assumed to be abelian and $\mathbb{K}$ is a field of
characteristic zero. We recall some notation for graded algebras and graded
modules \cite{1}, and some facts on color Lie algebras from \cite{5,6}.

1.1 Graded Hochschild cohomology

Let $G$ be a group with identity element $e$. We will write $G$ as an multiplicative
group. An associative algebra $A$ with unit $1_A$, is said to be $G$-graded, if there
is a family $\{A_g|g \in G\}$ of subspaces of $A$ such that $A = \oplus_{g \in G} A_g$ with $1_A \in A_0$
and $A_gA_h \subseteq A_{gh}$, for all $g, h \in G$. Any element $a \in A_g$ is called homogeneous
of degree $g$, and we write $|a| = g$.

A left graded $A$-module $M$ is a left $A$-module with a decomposition $M = \oplus_{g \in G} M_g$
such that $A_gM_h \subseteq M_{gh}$. Let $M$ and $N$ be graded $A$-modules. Define

$$\text{Hom}_{A-gr}(M,N) = \{f \in \text{Hom}_A(M,N)| \quad f(M_g) \subseteq N_{0g}, \forall \quad g \in G\}. \quad (1.1)$$

We obtain the category of graded left $A$-modules, denoted by $A$-gr., \cite{1}. Denote
by $\text{Ext}^n_{A-gr}(-, -)$ the $n$-th right derived functor of the functor $\text{Hom}_{A-gr}(-, -)$.

Let us recall the notion of graded Hochschild cohomology of a graded algebra $A$.
A graded $A$-bimodule is an $A$-bimodule $M = \oplus_{g \in G} M_g$ such that $A_gM_hA_k \subseteq M_{ghk}$.
Thus we obtain the category of graded $A$-bimodules, denoted by $A$-A-gr.
Let $A^e = A \otimes A^{op}$ be the enveloping algebra of $A$, where $A^{op}$ is the opposite
algebra of $A$. The algebra $A^e$ also is graded by $G$ by setting $A^e_g := \sum_{h \in G} A_h \otimes A_{h^{-1}g}$.

Now the graded $A$-bimodule $M$ becomes a graded left $A^e$-module by defining the $A^e$-action as

$$(a \otimes b)m = a.m.b, \quad (1.2)$$

and it is clear that $A^e_gM_h \subseteq M_{gh}$, i.e., $M$ is a graded $A^e$-module. Moreover,
every graded left $A^e$-module arises in this way. Precisely, the above correspondence establishes an equivalence of categories

$$A$-$A$-gr $\simeq$ $A^e$-gr. \quad (1.3)$$
In the sequel we will identify these categories. Let \( M \) be a graded \( A \)-bimodule, as above, \( M \) may be regarded as a graded left \( A^e \)-module. The \( n \)-th graded Hochschild cohomology of \( A \) with value in \( M \) is defined by
\[
\text{HH}^n_{\text{gr}}(A, M) := \text{Ext}^n_{A^e-\text{gr}}(A, M), \quad n \geq 0,
\]
where \( A \) is the graded left \( A^e \)-module induced by the multiplication of \( A \), and the algebra \( A^e = \oplus_{g \in G} A^g \) is considered as a \( G \)-graded algebra.

### 1.2 Lie color algebras

The concept of color Lie algebras is related to an abelian group \( G \) and an antisymmetric bicharacter \( \varepsilon : G \times G \rightarrow \mathbb{K}^\times \), i.e.,
\[
\varepsilon (g, h) \varepsilon (h, g) = 1, \quad (1.5)
\]
\[
\varepsilon (g, hk) = \varepsilon (g, h) \varepsilon (g, k), \quad (1.6)
\]
\[
\varepsilon (gh, k) = \varepsilon (g, k) \varepsilon (h, k), \quad (1.7)
\]
where \( g, h, k \in G \) and \( \mathbb{K}^\times \) is the multiplicative group of the units in \( \mathbb{K} \).

A \( G \)-graded space \( L = \oplus_{g \in G} L_g \) is said to be a \( G \)-graded \( \varepsilon \)-Lie algebra (or simply, color Lie algebra), if it is endowed with a bilinear bracket \( [-,-] \) satisfying the following conditions
\[
[L_g, L_h] \subseteq L_{gh}, \quad (1.8)
\]
\[
[a, b] = -\varepsilon ([a|, |b]) [b, a], \quad (1.9)
\]
\[
\varepsilon ([c|, |a]) [a, [b, c]] + \varepsilon ([a|, |b]) [b, [c, a]] + \varepsilon ([b|, |c]) [c, [a, b]] = 0, \quad (1.10)
\]
where \( g, h \in G \), and \( a, b, c \in L \) are homogeneous elements.

For example, a super Lie algebra is exactly a \( \mathbb{Z}_2 \)-graded \( \varepsilon \)-Lie algebra where
\[
\varepsilon(i, j) = (-1)^{ij}, \quad \forall \quad i, j \in \mathbb{Z}_2. \quad (1.11)
\]

Let \( L \) be a color Lie algebra as above and \( T(L) \) the tensor algebra of the \( G \)-graded vector space \( L \). It is well-known that \( T(L) \) has a natural \( \mathbb{Z} \times G \)-grading which is fixed by the condition that the degree of a tensor \( a_1 \otimes \ldots \otimes a_n \) with \( a_i \in L_{g_i}, g_i \in G \), for \( 1 \leq i \leq n \), is equal to \( (n, g_1 + \ldots + g_n) \). The subspace of \( T(L) \) spanned by homogeneous tensors of order \( \leq n \) will be denoted by \( T^n(L) \).

Let \( J(L) \) be the \( G \)-graded two-sided ideal of \( T(L) \) which is generated by
\[
a \otimes b - \varepsilon ([a|, |b]) b \otimes a - [a, b] \quad (1.12)
\]
with homogeneous \( a, b \in g \). The quotient algebra \( U(L) := T(L) / J(L) \) is called the universal enveloping algebra of the color Lie algebra \( L \). The \( \mathbb{K} \)-algebra \( U(L) \) is a \( G \)-graded algebra and has a positive filtration by putting \( U_n(L) \) equal to the canonical image of \( T_n(L) \) in \( T(L) \).

In particular, if \( L \) is \( \varepsilon \)-commutative (i.e., \( [L, L] = 0 \)), then \( U(L) = S(L) \) (the \( \varepsilon \)-symmetric algebra of the graded space \( L \)).
The canonical map $i_L : L \to U(L)$ is a $G$-graded homomorphism and satisfies
\[
i_L (a) i_L (b) - \varepsilon ([a], [b]) i_L (b) i_L (a) = i_L ([a, b]) .
\] (1.13)
The $\mathbb{Z}$-graded algebra $G(L)$ associated with the filtered algebra $U(L)$ is defined by letting $G^n (L)$ be the vector space $U_n (L) / U_{n-1} (L)$ and $G (L)$ the space $\bigoplus_{n \in \mathbb{N}} G^n (L)$ (note $U^{-1} (L) := \{0\}$). Consequently, $G (L)$ is a $\mathbb{Z} \times G$-graded algebra. The well-known generalized Poincaré-Birkhoff-Witt theorem, [4], states that the canonical homomorphism $i_L : L \to U (L)$ is an injective $G$-graded homomorphism; moreover, if $\{x_i\}_I$ is a homogeneous basis of $L$, where the index set $I$ well-ordered. Set $y_{k_i} := i (x_{k_i})$, then the set of ordered monomials $y_{k_1} \cdots y_{k_n}$ is a basis of $U (L)$, where $k_j \leq k_{j+1}$ and $k_j < k_{j+1}$ if $\epsilon (g_j, g_j) \neq 1$ with $x_{k_j} \in L_g$, for all $1 \leq j \leq n, n \in \mathbb{N}$. In case $L$ is finite-dimensional $U (L)$ is a graded two-sided Noetherian algebra (e.g., see for example [3]).

2 Generalized Enveloping Algebras

Let $L$ be a $\epsilon$-Lie algebra over $\mathbb{K}$, $U (L)$ its enveloping algebra and $S (L)$ its $\epsilon$ symmetric algebra. Let $\omega \in Z^2_{gr} (L, \mathbb{K})$ be a 2-cocycle (of degree zero) for $L$ with values in $\mathbb{K}$ considered as a $G$-graded trivial $L$-module, i.e.
\[
\epsilon ([z], [x]) \omega (x, [y, z]) + \epsilon ([x], [y]) \omega (y, [z, x]) + \epsilon ([y], [z]) \omega (z, [x, y]) = 0
\] (2.1) for all homogeneous elements $x, y, z \in L$, see [4], [2].

**Definition 2.1** Let $L$ be a $\epsilon$-Lie algebra and $\omega \in Z^2_{gr} (L, \mathbb{K})$ a scalar graded 2-cocycle. We call generalized enveloping algebra of $L$ associated with $\omega$, the algebra $U_\omega (L)$, quotient of the tensor algebra over $L$ by the $G$-graded two sided ideal generated by the elements of the form $v_1 \otimes v_2 - \epsilon (v_1, v_2) v_2 \otimes v_1 - [v_1, v_2] - \omega (v_1, v_2)$, where $v_1, v_2$ are homogeneous elements. Then the algebra $U_\omega (L)$ is $G$-graded and $\mathbb{Z}$-filtered.

Let $\omega \in Z^2 (L, \mathbb{K})$ be a scalar graded two cocycle of the color Lie algebra $L$. Let $L_\omega := L \times \mathbb{K} : x$ be a central extension of $L$ with $\omega$ such that the new bracket $[,]'$ is defined by
\[
[x_1 + ax, x_2 + bx]' := [x_1, x_2] + \omega (x_1, x_2) x
\] (2.2) where $x_1, x_2 \in L, a, b, x \in \mathbb{K}$ are homogeneous. The generalized enveloping algebra $U_\omega (L)$ is isomorphic to the $G$-graded and $\mathbb{Z}$-filtered algebra $U (L_\omega) / < y - 1 >$, with $< y - 1 >$ being the $G$-graded two-sided ideal of $U (L_\omega)$ generated by $y - 1$ and $y$ the image of $x$ in $U (L_\omega)$. Denote by
\[
\pi_\omega : U (L_\omega) \to U_\omega (L)
\] (2.3) the canonical epimorphism.

**Definition 2.2** A graded $(\omega, L)$-module over $\mathbb{K}$ is a graded $\mathbb{K}$-module $M$ endowed with a graded $\mathbb{K}$-linear map $\varphi : L \to \text{Hom}_{gr}(M, M)$ such that for all
homogeneous elements \(x, y \in L\)

\[\llbracket [\varphi(x), \varphi(y)] \rrbracket = \varphi([x, y]) + \omega(x, y)i_M\]  \hspace{1cm} (2.4)

where \(\llbracket [\varphi(x), \varphi(y)] \rrbracket = \varphi(x)\varphi(y) - \varepsilon(|x|, |y|)\varphi(y)\varphi(x)\) and \(i_M\) is the graded identity map of \(M\).

Proposition 2.1 There is a 1–1 correspondence between graded (left) \((\omega, L)\)-modules and graded (left) \(U_\omega(L)\) modules.

Proof. Let \((M, \varphi)\) be a graded left \((\omega, L)\)-module, then the graded \(\mathbb{K}\) linear map \(\varphi\) may be uniquely extended to a graded \(\mathbb{K}\) homomorphism \(\tilde{\varphi} : T(L) \to \text{Hom}_\mathbb{K}(M, M)\). It follows from the condition (2.4) that \(\tilde{\varphi}\) vanishes on the \(G\)-graded two sided ideal of \(U_\omega(L)\) generated by the elements

\[v_1 \otimes v_2 - \varepsilon(|v_1|, |v_2|)v_2 \otimes v_1 - [v_1, v_2] - \omega(v_1, v_2),\]

where \(v_1, v_2\) are homogeneous elements. The converse is trivial. \(\square\)

This proves in particular that for any \(\omega \in \mathcal{Z}_2^{gr}(L, \mathbb{K})\) there is a \((\omega, L)\)-module, we can see for example that the \(\omega\)-enveloping algebra \(U_\omega(L)\) is a graded \((\omega, L)\)-module.

Theorem 2.1 If \(L\) is a \(\mathbb{K}\)-free \(\epsilon\)-Lie algebra. Let \(\{x_i\}_{i \in I}\) be a \(G\)-homogeneous basis of \(L\), where \(I\) is a well-ordered set. For any central extension of \(L\) with \(\omega\), the set of ordered monomials \(z_{i_1} \cdots z_{i_n}\) forms a basis of \(U_\omega(L)\), where \(i_j \leq i_{j+1}\) and \(i_j < i_{j+1}\) if \(\epsilon(g_j, g_{j+1}) \neq 1\) with \(y_{j, i} \in L_{g_j}\) for all \(1 \leq j \leq n, n \in \mathbb{N}\).

Proof. Since \(\{x_i\}_{i \in I}\) is a \(G\)-homogeneous basis of the vector space \(L\), it follows that \(\{x_i, x_j\}_{i \in I}\) forms a \(G\)-homogeneous basis of the vector space \(L_\omega\). Let

\[i_\omega : L_\omega \xrightarrow{i_\omega} U(L_\omega) \xrightarrow{\pi_\omega} U_\omega(L)\]  \hspace{1cm} (2.5)

denote the composition. We set \(z_{i} := i_\omega(x_i), z := i_\omega(x), y_i := i_{L_\omega}(x_i), \) with \(i \in I\). Let \(y_{i_0}y_{i_1} \cdots y_{i_n}\) be the generators of the PBW basis of \(U(L_\omega)\) with \(i_0 \in \mathbb{N}, i_0 \leq i_1, i_0 < i_1\) if \(\varepsilon(|y_{i_0}|, |y_{i_1}|) \neq 1\). In the quotient algebra \(U_\omega(L) = U(L_\omega) / \langle y - 1 \rangle\), the element \(z_{i_0}\) is identified with 1. Then the canonical projection \(\pi_\omega\) sends \(y_{i_0}y_{i_1} \cdots y_{i_n}\) into \(z_{i_1} \cdots z_{i_n}\), and it follows that the elements \(z_{i_1} \cdots z_{i_n}\) form a basis of \(U_\omega(L)\). \(\square\)

The restriction of the canonical homomorphism \(i_\omega\) on \(L\), see (2.5), we is again denoted by \(i_\omega\), i.e., \(i_\omega : L \to U_\omega(L)\) satisfies for every \(x, y \in L\), homogeneous elements:

\[\llbracket [i_\omega(x), i_\omega(y)] \rrbracket = i_\omega([x, y]) + \omega(x, y) \cdot i_{U_\omega(L)}\]  \hspace{1cm} (2.6)

with \(\llbracket [i_\omega(x), i_\omega(y)] \rrbracket = i_\omega(x) \cdot i_\omega(y) - \varepsilon(|x|, |y|) i_\omega(y) \cdot i_\omega(x)\).

Corollary 2.1 If \(L\) is a \(\mathbb{K}\)-free \(\epsilon\)-Lie algebra, then for any central extension of \(L\) with \(\omega, i_\omega : L \to U_\omega(L)\) is an injective homomorphism.
Thus we may identify every element of $L$ with the canonical image in $U_\omega(L)$. Hence $L$ is embedded in $U_\omega(L)$ and

$$[[x, y]] = [x, y] + w(x, y) \cdot 1 \quad (2.7)$$

for all $x, y \in L$. The algebra $U_\omega(L)$ has a positive filtration defined by taking for $U_{n,\omega}(L)$ the canonical image of $U_n(L_\omega)$ by $\pi_\omega$. Denote by $G_\omega(L)$ its associated $\mathbb{Z}$-graded algebra, then $G_\omega(L)$ is a $\mathbb{Z} \times \Gamma$-graded algebra and $\epsilon$-commutative. It follows that the canonical injection

$$L \hookrightarrow U_\omega(L) \rightarrow G_\omega(L), \quad (2.8)$$

may be uniquely extended to a homomorphism $\varphi_\omega$ of the $\epsilon$-symmetric algebra $S(L)$ of $L$ into $G_\omega(L)$. If $S^n(L)$ denotes the set of elements of $S(L)$ which are homogeneous of degree $(n, g_1 + \ldots + g_n)$, then $\varphi_\omega(S^n(L)) \subset G^n_\omega(L)$.

**Proposition 2.2** The canonical homomorphism $\varphi_\omega$ of $S(L)$ into $G_\omega(L)$ is a $\mathbb{Z} \times \Gamma$-graded algebra isomorphism.

**Proof.** Let $\{x_i\}_{i \in I}$ be a $G$-homogeneous basis of $L$, with $I$ a well-ordered set. Let $y_{i_1} \cdots y_{i_n}$ be the product $x_{i_1} \cdots x_{i_n}$ calculated in $S(L)$, $z_{i_1} \cdots z_{i_n}$ the product $x_{i_1} \cdots x_{i_n}$ calculated in $U_\omega(L)$ and $z'_{i_1} \cdots z'_{i_n}$ the canonical image of $z_{i_1} \cdots z_{i_n}$ in $G_\omega(L)$. Since the set of ordered monomials $z_{i_1} \cdots z_{i_n}$ form a basis of $U_\omega(L)$, by Theorem 2.1 then the set of ordered monomials $z'_{i_1} \cdots z'_{i_n}$ is a basis of $G_\omega(L)$. Since $\varphi_\omega(y_{i_1} \cdots y_{i_n}) = z'_{i_1} \cdots z'_{i_n}$, it can be seen that $\varphi_\omega$ is bijective. \[\square\]

**Proposition 2.3** If $L$ is of finite dimensional then $U_\omega(L)$ is a graded Noetherian algebra.

**Proof.** By Proposition 2.2 the generalized enveloping algebra $U_\omega(L)$ is a positively graded filtered algebra with its associated graded algebra $gr(U_\omega(L)) \simeq S(L)$. The fact that the $\epsilon$-symmetric algebra $S(L)$ is graded Noetherian, see Lemma 2.3 and by Theorem 1.1.9 we deduce that $U_\omega(L)$ is a graded Noetherian algebra. \[\square\]

### 3 Classification of Generalized Enveloping Algebras

Fix $G$ an abelian group and $\epsilon$ an antisymmetric bicharacter on $G$. Let $V$ be a free $G$-graded vector space over $\mathbb{K}$. Let $S(V)$ denote the $\epsilon$-symmetric algebra of $V$. Consider the family of all pairs $(A, \varphi_A)$ where $A = \bigcup_{n \in \mathbb{Z}} F_n A$ is a $G$-graded, $\mathbb{Z}$-filtered algebra and $\varphi_A : S(V) \rightarrow G_F(A)$ is a $G \times \mathbb{Z}$-graded isomorphism. A map $\Psi : (A, \varphi_A) \rightarrow (B, \varphi_B)$ is a $G$-graded, $\mathbb{Z}$-filtered algebra homomorphism $\Psi : A \rightarrow B$ such that if $G(\Psi) : G(A) \rightarrow G(B)$ is the $G \times \mathbb{Z}$-graded algebra
morphism induced by $\Psi$, the diagram

\[
\begin{array}{ccc}
G(A) & \xrightarrow{G(\Psi)} & G(B) \\
\downarrow{\varphi_A} & & \downarrow{\varphi_B} \\
S(V) & & 
\end{array}
\tag{3.1}
\]

is commutative. Composition of maps is defined in the obvious way. The resulting category is denoted by $\mathcal{R}_{gr}(S(V))$. If $\Psi : (A, \varphi_A) \to (B, \varphi_B)$ is a map then $G(\Psi) : G(A) \to G(B)$ is a graded isomorphism, since $G(\Psi) = \varphi_B \circ \varphi_A^{-1}$.

**Lemma 3.1** With notation as above $\Psi : A \to B$ is a $\mathbb{Z}$-filtered, $G$-graded isomorphism.

**Proof.** Let $\Psi_p : F_p A \to F_p B$ denote the $\mathbb{K}$ linear map induced by $\Psi$. We reason by induction on the integer $p$. It is clear that $\Psi_0$ is a graded isomorphism. From the commutativity of the diagram

\[
\begin{array}{cccc}
0 & \to & F_{p-1} A & \to & F_p A & \to & G_p(A) & \to & 0 \\
\downarrow{\varphi_{p-1}} & & \downarrow{\varphi_p} & & \downarrow{G_p(\Psi)} & & & & \\
0 & \to & F_{p-1} B & \to & F_p B & \to & G_p(B) & \to & 0
\end{array}
\tag{3.2}
\]

at $\Psi_{p-1}$ and $G_p(\Psi)$ are graded isomorphisms, it is easily seen that $\Psi_p$ is also a graded isomorphism. Since $p$ is arbitrary, the assertion holds. $\square$

**Lemma 3.2** For each $(A, \varphi_A)$ pair of $\mathcal{R}(S(V))$ there is a pair $(L, [\omega])$ where $L$ is a $\epsilon$-Lie algebra and $[\omega] \in H^2_{gr}(L, \mathbb{K})$ such that $F_1 A = L_\omega = L \times \mathbb{K}$, with $\omega$ is a representative of $[\omega]$.

**Proof.** Let $a, b \in F_1 A$ be homogeneous elements, we have $[a, b] := ab - \epsilon(a, b) ba \in F_2 A$. Since $G(A)$ is $\epsilon$-commutative via $\varphi_A$, then $[a, b] \in F_1 A$. Thus $F_1 A$ acquires a structure of a $\epsilon$-Lie algebra. It is clear that $\mathbb{K} = F_0 A$ is a central $G$-graded ideal of $F_1 A$. The $G$-graded isomorphism $S_1(V) \cong F_1 A/F_0 A$ given by $\varphi_A$, induces a $\epsilon$-Lie structure on $S_1(V)$, denote it by $L$. Then the following sequence

\[
0 \to \mathbb{K} \xrightarrow{i} F_1 A \xrightarrow{\pi} L \to 0 \tag{3.3}
\]

is central $G$-graded exact and $\pi$ is induced by $\varphi_A$. Thus $i$ and $\pi$ are graded homomorphisms (of degree zero) of $\epsilon$-Lie algebras. Since $S_1(V)$ is $\mathbb{K}$-free, there exists a graded linear map $\sigma : L \to F_1 A$ (necessarily of degree zero) such that $\pi \circ \epsilon = \text{id}_{F_1 A}$. We then have

\[
\pi([[\sigma(x), \sigma(y)]] - [x, y]) = 0
\]

for all (homogeneous) $x, y \in F_1 A$. Hence, there is a unique map $\omega : L \times L \to \mathbb{K}$ such that

\[
i(\omega(x, y)) = [[\sigma(x), \sigma(y)]] - \sigma([x, y]) \tag{3.4}
\]

for all (homogeneous) $x, y \in L$, and it is easy to see that $\omega$ is a homogeneous 2-cocycle of degree zero, i.e, $\omega \in Z^2_{gr}(L, \mathbb{K})$. From [5], it follows that the cohomology class $[\omega]$ of $\omega$ is independent of the choice of $\omega$. $\square$
**Theorem 3.1** Let $G$ be an abelian group and $\epsilon$ a symmetric bicharacter on $G$. Let $V$ be $G$-graded $K$-free module. Let $S(V)$ be the $\epsilon$-symmetric algebra on $V$. The isomorphism classes of objects in $\mathfrak{R}_{gr}(S(V))$ are in a 1-1 correspondence with pairs $(L, [\omega])$ where $L$ is a $\epsilon$-Lie algebra on $V$ and $[\omega]$ is an element in $H^2_{gr}(L, K)$. If $\omega$ is a cocycle in the cohomology class $[\omega]$, then $(U_{\omega}(L), \varphi_{\omega})$ is an object in the isomorphism class determined by $(L, [\omega])$.

**Proof.** Let $L$ be a $\epsilon$-Lie algebra structure on $V$ and $\omega$ is a representative of the cohomology class $[\omega] \in H^2_{gr}(L, K)$. Using Proposition 2.2, then $(U_{\omega}(L), \varphi_{\omega})$ is an object in $\mathfrak{R}_{gr}(S)$. Consider the exact sequence of graded algebras

$$0 \to K \xrightarrow{i} F_1(U_{\omega}(L)) \xrightarrow{\pi} L \to 0$$

(3.5)

where $\pi_{\omega}$ is induced by $\varphi_{\omega}$. The map $i_{\omega} : L \to F_1(U_{\omega}(L))$ is a $K$-homogeneous linear section and the relation (2.6) shows that $(U_{\omega}(L), \varphi_{\omega})$ yields $(L, [\omega])$. Let $(A, \varphi_{\lambda}) \in \mathfrak{R}_{gr}(S(V))$ be another object. Choose $\sigma : L \to F_1A$ so that (4.1) is valid for the cocycle $\omega$. Let $\tilde{\sigma} : T(L) \to A$ be the natural homogeneous extension of $\sigma$. If $x, y \in L$ are homogeneous, then,

$$\tilde{\sigma}(x \otimes y - \epsilon([x], [y])y \otimes x - [x, y] - \omega(x, y)) = [[\sigma(x), \sigma(y)]] - \sigma([x, y]) - \omega(x, y) = 0.$$ 

Then $\tilde{\sigma}$ induces a $G$-graded, $Z$-filtered homomorphism of algebras $\overline{\varphi} : U_{\omega}(L) \to A$. We then have

$$\begin{array}{ccc}
G(U_{\omega}) & \xrightarrow{G(\overline{\varphi})} & G(A) \\
\varphi_{\omega} & \downarrow & \varphi_{\lambda} \\
S(V) & \xrightarrow{\varphi_{\lambda}} & 
\end{array}$$

(3.6)

For $x \in \mathfrak{g}$, $\sigma(x)$ is in the coset $\varphi_{\lambda}(x)$ of $F_1A$ mod $F_0A$. Thus, $G(\overline{\varphi}) \varphi_{\lambda}(x) = G(\overline{\varphi}) i_{\omega}(x) = \varphi_{\lambda}(x)$. Hence the diagram above is commutative. Thus, $\overline{\varphi} : (U_{\omega}, \varphi_{\omega}) \to (A, \varphi_{\lambda})$ is a map and then an isomorphism by Lemma 3.1.

From Theorem 3.1, we retain in particular that $(U_{\omega_1}(L), \varphi_{\omega_1})$ and $(U_{\omega_2}(L), \varphi_{\omega_2})$ are $Z$-filtered, $G$-graded isomorphic if and only if, $\omega_1$ and $\omega_2$ are (graded) cohomologous.

### 4 Homological Properties of $U_{\omega}(L)$ and Color Hopf Algebra

Let $G$ be a commutative group and $\chi : G \to K^*$ a bicharacter.

**Definition 4.1** A $(G, \chi)$-Hopf graded algebra $A$ is a 5-tuple $(A, m, \eta, \Delta, \epsilon, S)$ such that

1. $A = \bigoplus_{g \in G} A_g$ is a graded algebra with multiplication $m : A \otimes A \to A$ and the unit map $\eta : K \to A$. Moreover, $(A, \Delta, \epsilon)$ is a graded coalgebra with respect to the same grading.
2. The counit \( \epsilon : A \to K \) is an algebra map. The comultiplication
\[ \Delta : A \to (A \otimes A) \chi \]
is an algebra map, where the algebra \((A \otimes A) \chi \) is equipped with multiplication \(*\) defined by
\[ (a \otimes b) * (a' \otimes b') = \chi(|b|, |a'|)aa' \otimes bb', \] (4.1)
where \( a, a' \in A \) and \( b, b' \in B \) are homogeneous.

3. The antipode \( S : A \to A \) is a graded map such that
\[ \sum a_1 S(a_2) = \epsilon(a) = \sum S(a_1)a_2 \] (4.2)
for all homogeneous \( a \in A \), where we use Sweedler’s notation
\[ \Delta(a) = \sum a_1 \otimes a_2. \]

Definition 4.2 An algebra is said to be a color Hopf algebra if it is a \((G, \chi)\)-Hopf algebra with the antipode being an isomorphism.

Let \( M \) be a graded \( A \)-bimodule, then we define a left \( A \)-module by
\[ am = \sum \chi(|a_{(2)}|, |m|)a_{(1)}m.S(a_{(2)}), \] (4.3)
for homogeneous \( a \in A \) and \( m \in M \). It is called the adjoint \( A \)-graded module and denoted by \( ^{ad}M \).

Theorem 4.1 Let \( A = (A, m, \eta, \Delta, \epsilon, S) \) be a color Hopf algebra and let \( M \) be a graded \( A \)-bimodule. Then there exists an isomorphism of graded spaces
\[ \text{HH}^n_{gr}(A, M) \cong \text{Ext}^n_A(K, ^{ad}M), \quad n \geq 0, \]
where \( K \) is viewed as the trivial graded \( A \)-module via the counit \( \epsilon \), and \( ^{ad}M \) is the adjoint \( A \)-module associated to the graded \( A \)-bimodule \( M \).

Proof. See [2].

Proposition 4.1 Let \( L \) be a \( \epsilon \)-Lie algebra and \( \omega \in Z^{2}_{gr}(L, K) \) a scalar 2-cocycle. Then the generalized enveloping algebra \( U_{\omega}(L) \) of \( L \) is a color Hopf algebra.

Proof. It’s shown in [2] that the graded tensor algebra \( T(L) \) is a color Hopf algebra. Moreover it is easy to prove that the two-sided ideal generated by the elements, \( v_1 \otimes v_2 - \epsilon(v_1, v_2)v_2 \otimes v_1 - [v_1, v_2] - \omega(v_1, v_2) \), where \( v_1, v_2 \) are homogeneous elements, is a graded Hopf ideal. It follows that the generalized enveloping algebra becomes a color algebra Hopf by quotient.

Now we can apply the theorem for the generalized universal enveloping algebra \( U_{\omega}(L) \) of a \( G \)-graded \( \epsilon \)-Lie algebra \( L \). In fact, if \( M \) is a graded \( U_{\omega}(L) \)-bimodule, the adjoint \( U_{\omega}(L) \)-module \( ^{ad}M \) is given by
\[ ^{ad}(x)m = x.m - \epsilon(|x|, |m|)m.x = [x, m] \] (4.4)
for all homogeneous \( x \in L \) and \( m \in M \). Thus we have
Corollary 4.1 Let $L$ be a $G$-graded $\varepsilon$-Lie algebra and $U_{\omega}(L)$ its universal generalized enveloping algebra. Let $M$ be a graded $U_{\omega}(L)$-bimodule. Then there exists a graded isomorphism

$$\text{HH}_{\varepsilon}^{n}(U_{\omega}(L), M) = \text{Ext}_{U_{\omega}(L)-\text{gr}}^{n}(\mathbb{K}, ^{ad}M), \quad n \geq 0,$$

where $^{ad}M$ is the adjoint $U_{\omega}(L)$-module associated with the graded $U_{\omega}(L)$-bimodule $M$ defined by \cite{43}.

It follows from \cite{2} that the sequence

$$C : \cdots \to C_{n} \xrightarrow{d_{n}} C_{n-1} \to \cdots \xrightarrow{d_{1}} C_{1} \xrightarrow{\epsilon} C_{0} \quad (4.5)$$

is a $G$-graded $U(L_{\omega})$-free resolution of the $G$-graded trivial $U(L_{\omega})$-left module $\mathbb{K}$ via $\epsilon$ where $C_{n} = U(L_{\omega}) \otimes_{\mathbb{K}} \wedge_{n}^{\varepsilon}L_{\omega}$ and the operator $d_{n}$ is given by

$$d_{n}(u \otimes (x_{1}, \ldots, x_{n})) = \sum_{i=1}^{n}(-1)^{i+1}x_{i} \otimes \langle x_{1}, \ldots, \hat{x}_{i}, \ldots, x_{n} \rangle + \sum_{1 \leq i < j \leq n}(-1)^{i+j}x_{i}x_{j} \otimes \langle x_{1}, x_{j}, x_{1}, \ldots, \hat{x}_{i}, \ldots, \hat{x}_{j}, \ldots, x_{n} \rangle,$$

for all homogeneous elements $u \in U(L_{\omega})$ and $x_{i} \in L$, with $\varepsilon_{i} = \prod_{h=1}^{i-1} \varepsilon(|x_{h}|, |x_{i}|)$ $i \geq 2$, $\varepsilon_{1} = 1$ and the sign $\hat{\cdot}$ indicates that the element below it must be omitted. The differential operator $d$ maps $U(L_{\omega}) < y-1 > \otimes_{\mathbb{K}} \wedge_{n}^{\varepsilon}L_{\omega}$ into itself, then it passes to the quotient, i.e., $\overline{d} : \overline{C}_{n} \to \overline{C}_{n-1}$ and satisfies that $\overline{d} \circ \overline{d} = 0$ where $\overline{C}_{n} = U_{\omega}(L) \otimes_{\mathbb{K}} \wedge_{n}^{\varepsilon}L_{\omega}$.

Proposition 4.2 The sequence

$$\overline{C} : \cdots \to \overline{C}_{n} \xrightarrow{\overline{d}_{n}} \overline{C}_{n-1} \to \cdots \xrightarrow{\overline{d}_{1}} \overline{C}_{1} \xrightarrow{\overline{\epsilon}} \overline{C}_{0} \quad (4.6)$$

is a $G$-graded $U_{\omega}(L)$-free resolution of the $G$-graded trivial $U_{\omega}(L)$-left module $\mathbb{K}$ via $\epsilon$.

Proof. Let $\{x_{i}\}$ be a homogeneous basis of $L$, where $I$ is a well-ordered set. By Theorem 2.1 the elements

$$x_{k_{1}} \cdots x_{k_{m}} \otimes \langle x_{1} \cdots x_{n} \rangle \quad (4.7)$$

with

$$k_{1} \leq \cdots \leq k_{m} \quad \text{and} \quad k_{i} < k_{i+1} \quad \text{if} \quad \varepsilon(|x_{k_{i}}|, |x_{k_{i}}|) = -1 \quad (4.8)$$

and

$$l_{1} \leq \cdots \leq l_{n} \quad \text{and} \quad l_{i} < l_{i+1} \quad \text{if} \quad \varepsilon(|x_{l_{i}}|, |x_{l_{i}}|) = 1 \quad (4.9)$$

form a homogeneous basis of $\overline{C}_{n}$. The canonical filtration of $U_{\omega}(L)$, induces a filtration on the complex $\overline{C}$. The associated $Z$-graded complex $G(\overline{C})$ is $G$-graded and isomorphic to the $Z \times G$-graded complex $S(L) \otimes \wedge_{L}$. It follows from Lemma 3, \cite{3}, that the complex $G(\overline{C})$ is acyclic and consequently so is $\overline{C}$. \hfill \Box
Let $M$ be a $G$-graded left $(\omega, L)$-module, we define the $n^{th}$ graded cohomology group of $L$ with coefficients in $M$ by

$$H^n_{gr,\omega}(L, M) := \text{Ext}^n_{U_\omega(L)-gr}(K, M).$$

(4.10)

The modules on the right hand side can be computed using the left graded $U_\omega(L)$-projective resolution of $K$. If $M$ is a graded left $(\omega, L)$-module, the graded cohomology groups are the graded homology groups of the complex:

$$\text{Hom}_{U_\omega(L)-gr}(\bigwedge^n L, M) = \text{Hom}_{K-gr}(\bigwedge^n L, M).$$

The coboundary operator in this cocomplex is

$$\delta_n(f)(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} \varepsilon_i x_i f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})$$

(4.12)

$$+ \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \varepsilon_i \varepsilon_j f([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}).$$

(4.13)

**Theorem 4.2** Let $L$ be a $G$-graded $\epsilon$-Lie algebra, $\omega \in H^2_{gr}(L, K)$ and let $U_\omega(L)$ be its generalized universal enveloping algebra. Let $M$ be a graded $U_\omega(L)$-bimodule. Let $adM$ be the adjoint graded left $(L, \omega)$-module defined by

$$ad(x)m = [[x, m]] := xm - \epsilon(|x|, |m|)mx$$

for all homogeneous elements $x \in L$ and $m \in M$. There exists an isomorphism

$$H^n_{gr,\omega}(L, ad M) \simeq HH^n_{gr}(U_\omega(L), M), \quad n \geq 0.$$  

(4.14)

**Proof.** It is a direct consequence from above and Corollary 4.1. \qed

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