THE MINIMAL REPRESENTATION OF THE CONFORMAL GROUP
AND CLASSICAL SOLUTIONS TO THE WAVE EQUATION

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Abstract. Let $n$ be an integer $\geq 2$. We consider the wave operator $\Box = -\partial_t^2 + \sum_{i=1}^n \partial_{x_i}^2$, where $(t, x) = (t, x_1, \ldots, x_n)$ are the canonical coordinates on Minkowski space $\mathbb{R}^{1,n} = \mathbb{R} \times \mathbb{R}^n$. Lie's prolongation algorithm calculates the Lie algebra of infinitesimal symmetries of $\Box$ to be isomorphic to the conformal Lie algebra $\mathfrak{g} = \mathfrak{so}(2, n+1)$ plus an infinite dimensional piece reflecting the fact that $\Box$ is linear. In particular, $\ker \Box = \{ f \in C^\infty(\mathbb{R}^{1,n}) \mid \Box f = 0 \}$ is a representation of $\mathfrak{g}$. This Lie algebra action does not exponentiate to a global action of the conformal group $G = \text{SO}(2, n+1)_0$ or any cover group. However, for $n$ odd, it is known that $\ker \Box$ contains a nice $\mathfrak{g}$-invariant subspace that carries the minimal representation of $G$. In this paper, we give a uniform realization of the minimal representation of a double cover of $G$ in $\ker \Box$ as a positive energy representation $\mathcal{H}^+$ for $n$ even and odd. Using this realization, we obtain an explicit orthonormal basis for $\mathcal{H}^+$ that is well behaved with respect to energy and angular momentum. The lowest positive energy solution is, up to normalization,

$$f(t, x) = \frac{1}{\sqrt{(1-it)^2 + \|x\|^2}^{n-1}},$$

where $\sqrt \cdot$ denotes the principal branch of the square root. Of special note, for $n$ odd, all functions in our basis are rational functions. Finally, using Fourier analysis with respect to this basis, we prove that every classical real-valued solution to the wave equation is the real part of a unique continuous element in the representation $\mathcal{H}^+$. 

1. Introduction

The minimal representations of the orthogonal groups are well known and much studied in representation theory. Consequently, some of our methods and results overlap with existing literature. We mention a few of the most relevant references here. Important early work on the minimal representation of $\text{SO}(4,4)$ was done by B. Kostant starting with the paper [4]. B. Binegar and R. Zierau in [1] then constructed the minimal representation of $\text{SO}(p, q)_0$ for $p + q$ even. Their model was based on the kernel of the ultrahyperbolic wave operator $\Box_{p,q}$ acting on the space of smooth functions on the cone $C^{p,q}_p = \{(x, y) \in \mathbb{R}^{p,q} = \mathbb{R}^p \times \mathbb{R}^q \mid \|x\| = \|y\| \neq 0\}$ of homogeneous degree $2 - \frac{p+q}{2}$. In particular, for $p = 2$ and $q = n+1$ with $n$ odd, their model was based on the kernel of the operator $\Box_{2,n+1}$ acting on homogenous function
on the cone $C^{2,n+1}$ of degree $r = \frac{1+n}{2}$. A more general study of homogenous functions on generalized light cones was given by R. Howe and E. Tan in [3] and connections to dual pairs were studied by C. Zhu and J. Huang in [12]. T. Kobayashi and B. Orsted made an exhaustive study of the minimal representation of $O(p,q)$ for $p + q$ even in [5, 6, 7]. They realized the representation as the kernel of the Yamabe operator $\Delta_{S^{p-1} \times S^{q-1}}$ acting on $C^\infty(S^{p-1} \times S^{q-1})$, as a certain subspace of the kernel of $\Box_{p-1,q-1}$ acting on $C^\infty(\mathbb{R}^{p-1,q-1})$, and, via Fourier techniques, as $L^2(C^{p-1,q-1})$. In particular, for $p = 2$ and $q = n + 1$ with $n$ odd, they realized the representation in a subspace of the kernel of the usual wave operator $\Box = \Box_{1,n}$ acting on $C^\infty(\mathbb{R}^{1,n})$. Finally, T. Kobayashi and G. Mano in [8] start with a representation of $SL(2,\mathbb{R}) \times SO(n)$ on $L^2(\mathbb{R}^n, \frac{dx}{\|x\|})$ and use a result of S. Sahi [10] to show the representation extends to a double cover of $SO(2,n+1)$ when $n$ is even.

This paper has three purposes. Firstly, we give a uniform realization of the minimal representation of a double cover $\tilde{G}$ of $SO(2,n+1)_0$ as a positive energy representation $\mathcal{H}^+$ and a negative energy representation $\mathcal{H}^-$ in the kernel of the wave operator $\Box = \Box_{1,n}$ acting on $C^\infty(\mathbb{R}^{1,n})$ for both $n$ even and odd. Our construction is related to a general construction due to T. Enright and N. Wallach who in [2] showed for the Hermitian symmetric pairs of tube type every positive energy representation that is a reduction point can be realized in the kernel of a generalized Dirac operator. In this context we are also able to answer an open question concerning the irreducibility of the kernel. Secondly, we give an explicit orthonormal basis of $\mathcal{H}^+$ relative to the Klein-Gordon inner product that is well behaved with respect to energy and angular momentum. For $n$ odd, all functions in our basis are rational smooth solutions to the wave equation. Thirdly, we use our explicit basis of $\mathcal{H}^+$ to study classical solutions to the wave equation. We prove that every classical real-valued solution to the wave equation is the real part of a unique smooth element in the representation $\mathcal{H}^+$.

For brevity, some proofs are only outlined or omitted. Complete proofs of all our results will appear in a forthcoming paper.

2. A Distinguished Copy of $SL(2,\mathbb{R})$

Throughout the paper let $n$ be an integer $\geq 2$. It is well known (c.f. [9]) that Lie’s prolongation algorithm calculates the Lie algebra of infinitesimal symmetries of the wave operator $\Box = -\partial_x^2 + \sum_{i=1}^n \partial_{x_i}^2$ on $\mathbb{R}^{1,n} = \mathbb{R} \times \mathbb{R}^n$ to be isomorphic to the conformal Lie algebra $\mathfrak{g} = so(2,n+1)$ plus an infinite dimensional piece reflecting the fact that $\Box$ is linear.
The Lie algebra \( \mathfrak{g} \) contains a distinguished copy of \( \mathfrak{sl}(2, \mathbb{R}) \) spanned by the triple
\[
\begin{align*}
    h &= 2(r - t \partial_t - \sum_i x_i \partial x_i), \\
    e^+ &= -\partial_t, \\
    e^- &= 2t(r - t \partial_t - \sum_i x_i \partial x_i) + q(t, x) \partial_t,
\end{align*}
\]
where \( r = \frac{1}{2} \) and \( q(t, x) = -t^2 + \|x\|^2 \). The centralizer of this copy of \( \mathfrak{sl}(2, \mathbb{R}) \) in \( \mathfrak{g} \) is the Lie algebra \( \mathfrak{so}(n) \) spanned by the infinitesimal rotations \(-x_j \partial x_i + x_i \partial x_j, 1 \leq i < j \leq n\). If we define \( \mathfrak{so}(2, n + 1) \) with respect to the symmetric bilinear form given by \( \text{diag}(-1, -1, \ldots, 1) \) then \( \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{so}(n) \) embeds block diagonally as the Lie subalgebra \( \mathfrak{so}(2, 1) \times \mathfrak{so}(n) \). Corresponding to this Lie subalgebra we have the block diagonally embedded subgroup \( \text{SO}(2, 1)_0 \times \text{SO}(n) \) of the conformal group \( G = \text{SO}(2, n + 1)_0 \). Also diagonally embedded in \( G \) is the maximal compact subgroup \( K = \text{SO}(2) \times \text{SO}(n + 1) \). The group \( K \) has a double cover \( \tilde{K} \cong \text{SO}(2) \times \text{SO}(n + 1) \) with covering map \( \pi : \tilde{K} \to K \) given by \( (R_\varphi, k) \mapsto (R_\varphi, k) \), where \( R_\alpha \in \text{SO}(2) \) is the rotation by \( \alpha \) and \( k \in \text{SO}(n + 1) \). It is then possible to extend \( \tilde{K} \) to a connected Lie group \( \tilde{G} \) with Lie algebra \( \mathfrak{g} \) and extend \( \pi \) to a map \( \pi : \tilde{G} \to G \) so that \( \tilde{G} \) is a double cover of \( G \) with covering map \( \pi \). The double cover \( \tilde{G} \) contains a subgroup isomorphic to \( \text{SL}(2, \mathbb{R}) \times \text{SO}(n) \) such that the restriction of the covering map \( \pi : \tilde{G} \to G \) to the first factor is the map \( \text{SL}(2, \mathbb{R}) \cong \text{Spin}(2, 1)_0 \to \text{SO}(2, 1)_0 \). The group \( \tilde{G} \) and its distinguished subgroup \( \text{SL}(2, \mathbb{R}) \times \text{SO}(n) \) will play a pivotal role in our study of solutions to the wave equation. The starting point is the formula
\[
\Omega_{\text{SL}(2)} - \Omega_{\text{SO}(n)} - r(r + 1) = \|x\|^2 \Box,
\]
where \( \Omega_{\text{SL}(2)} = \frac{1}{4} h^2 + \frac{1}{2} (e^+ e^- + e^- e^+) \) and \( \Omega_{\text{SO}(n)} = -\sum_{i < j} (-x_j \partial x_i + x_i \partial x_j)^2 \) are the Casimir elements of the Lie subalgebras \( \mathfrak{sl}(2, \mathbb{R}) \) and \( \mathfrak{so}(n) \) of \( \mathfrak{g} \) with respect to the bilinear form \( B(X, Y) = \frac{1}{2} \text{tr}(XY) \) on \( \mathfrak{g} \).

3. Degenerate Principal Series Representations

By construction via the prolongation algorithm, the Lie algebra \( \mathfrak{g} \) acts naturally on \( \ker \Box = \{ f \in C^\infty(\mathbb{R}^{1,n}) \mid \Box f = 0 \} \). This Lie algebra action does not exponentiate to an action of the group \( G \) or its double cover \( \tilde{G} \). We will now show how to obtain a group action on a subspace of \( \ker \Box \) given by restricting smooth sections in the kernel of the differential operator \( \Omega = \Omega_{\text{SL}(2)} - \Omega_{\text{SO}(n)} - r(r + 1), r = \frac{1-n}{2} \), acting on an equivariant line bundle on the conformal compactification of \( \mathbb{R}^{1,n} \).

The conformal compactification of \( \mathbb{R}^{1,n} \) arises naturally in our setting as follows. The eigenvalues of \( \text{ad}(h) \) on \( \mathfrak{g} \) are \( \{2, 0, -2\} \) which gives rise to a pair of opposite maximal
parabolic subgroups $Q^\pm$ of $G = \text{SO}(2n)$ with Langlands decompositions $Q^\pm = MAN^\pm$ for which $M \cong \text{SO}(1, n)$, $A \cong \mathbb{R}_0^\times$, and $N^\pm \cong \mathbb{R}^\times$. Similarly, we have a pair of opposite maximal parabolic subgroups $\tilde{Q}^\pm$ of $\tilde{G}$ with Langlands decompositions $\tilde{Q}^\pm = \tilde{M}\tilde{A}\tilde{N}^\pm$ for which $\tilde{M}_0 \cong \text{SO}(1, n)\circlearrowleft \tilde{A} \cong \mathbb{R}_0^\times$, and $\tilde{N}^\pm \cong \mathbb{R}^\times$. The group $\tilde{M}$ has four connected components with $\tilde{M}/\tilde{M}_0 \cong \mathbb{Z}_4$. Identifying $\mathbb{R}^{1,n}$ with $N^+ \cong \tilde{N}^+$, we obtain an embedding $\mathbb{R}^{1,n} \hookrightarrow G/Q^- \cong \tilde{G}/\tilde{Q}^-$ with open dense image. This is the conformal compactification of $\mathbb{R}^{1,n}$.

Turning to line bundles, recall that the $\tilde{G}$-equivariant line bundles on $\tilde{G}/\tilde{Q}^-$ are of the form $\mathcal{L}_\chi = \tilde{G} \times_{\tilde{Q}^-} \mathbb{C}_\chi$, where $\chi$ is a character of $\tilde{Q}^-$. The space of global sections of $\mathcal{L}_\chi$ is the \textit{degenerate principal series representation} $\text{Ind}_{\tilde{Q}^-}(\chi)$ defined by

$$\text{Ind}_{\tilde{Q}^-}(\chi) = \left\{ \phi \in C^\infty(\tilde{G}) \mid \phi(\gamma g^-) = \chi^{-1}(q^-)\phi(g) \quad \forall g \in \tilde{G}, q^- \in \tilde{Q}^- \right\}$$

with $\tilde{G}$-action given by $(g \cdot \phi)(x) = \phi(g^{-1}x)$. Restricting functions to $\tilde{N}^+ \cong \mathbb{R}^\times$, we obtain the so-called \textit{noncompact picture} $\mathcal{T}_\chi' \subset C^\infty(\mathbb{R}^{1,n})$ of $\text{Ind}_{\tilde{Q}^-}(\chi)$.

A character $\chi$ of $Q^-$ is determined by a discrete parameter $m \in \mathbb{Z}_4$ and a continuous parameter $r \in \mathbb{C}$ as follows. Let $\gamma_0 : \mathbb{Z}_4 \to S^1$ be given by $\gamma_0(j) = i^j$. Identifying $\tilde{M}/\tilde{M}_0$ with $\mathbb{Z}_4$, let $\gamma : \tilde{M} \to S^1$ be the pull back of $\gamma_0$ given by mapping $\tilde{M}$ to its component group and then applying $\gamma_0$. For $m \in \mathbb{Z}_4$ and $r \in \mathbb{C}$, we then define a character $\chi_{m,r} : \tilde{Q}^- \to \mathbb{C}^\times$ by $\chi_{m,r}(q^-) = \gamma^m(q^-) e^{sr}$, where $q^- = q^-_M q^-_A q^-_N$ is written with respect to the Langlands decomposition with $q^-_A = \exp(\mathfrak{h})$, $s \in \mathbb{R}$. We remark that every character of $\tilde{Q}^-$ is of the form $\chi_{m,r}$ and $\text{Ind}_{\tilde{Q}^-}(\chi_{m,r})$ descends to a representation of $G$ if and only if $m$ is even.

Writing $\mathcal{T}_{m,r}' \subset C^\infty(\mathbb{R}^{1,n})$ for the noncompact picture of $\text{Ind}_{\tilde{Q}^-}(\chi_{m,r})$, the Lie algebra $\mathfrak{g}$ acts on $\mathcal{T}_{m,r}'$ by first order differential operators. These differential operators only depend on the parameter $r$ and hence can be calculated explicitly by choosing $m = 0$ and viewing $\mathcal{T}_{m,r}'$ as a representation of $G$. It turns out that for the special parameter $r = 1 - \frac{n}{2}$, the $\mathfrak{g}$-action on $\mathcal{T}_{m,r}'$ is given by exactly the same differential operators as the $\mathfrak{g}$-action on $C^\infty(\mathbb{R}^{1,n})$ that is obtained via the prolongation algorithm. In particular, the subspace $\ker \Box \subset \mathcal{T}_{m,r}'$ is $\mathfrak{g}$-invariant and hence $\tilde{G}$-invariant. (Here we slightly abuse language and write $\ker \Box \subset \mathcal{T}_{m,r}'$ for $\ker \Box |_{\mathcal{T}_{m,r}'}$.) In light of formula (2.2), $\ker \Box \subset \mathcal{T}_{m,r}'$ corresponds to $\ker \Omega \subset \text{Ind}_{\tilde{Q}^-}(\chi_{m,r})$ under the isomorphism $\mathcal{T}_{m,r}' \cong \text{Ind}_{\tilde{Q}^-}(\chi_{m,r})$ for $r = 1 - \frac{n}{2}$. Before we study the $\tilde{G}$-representations $\ker \Omega \subset \text{Ind}_{\tilde{Q}^-}(\chi_{m,r})$ in detail, we will give three concrete realizations of $\text{Ind}_{\tilde{Q}^-}(\chi_{m,r})$ in the following sections.
4. Geometric Picture

Define the cone $C^{2,n+1} = \{(a, b) \in \mathbb{R}^{2,n+1} \mid \|a\| = \|b\| \neq 0\}$. The group $G$ acts on $C^{2,n+1}$ by matrix multiplication on the left. Now consider the double cover of $C^{2,n+1}$ defined by $\tilde{C}^{2,n+1} = C^{2,n+1}$ with covering map $\eta : \tilde{C}^{2,n+1} \to C^{2,n+1}$ given by $\eta(\lambda \sin \frac{\pi}{2}, \lambda \cos \frac{\pi}{2}, \lambda b) = (\lambda \sin \varphi, \lambda \cos \varphi, \lambda b)$ where $\lambda \in \mathbb{R}_{>0}$, $\varphi \in \mathbb{R}$, and $b \in S^m$. A priori, $\tilde{C}^{2,n+1}$ carries only an action of $\tilde{K} = SO(2) \times SO(n+1)$ given by matrix multiplication on the left that is compatible with $\eta$ in the sense that $\eta(k \cdot c) = \pi(k) \cdot \eta(c)$ for $k \in \tilde{K}$ and $c \in \tilde{C}^{2,n+1}$. However, using the diffeomorphisms

$$\tilde{G} / (\tilde{M}_0 \tilde{N}^-) \cong \tilde{K} / (\tilde{K} \cap \tilde{M}_0) \times \tilde{A} \cong \tilde{C}^{2,n+1},$$

it follows that the $\tilde{K}$-action on $\tilde{C}^{2,n+1}$ extends to a $\tilde{G}$-action. (This action is not the restriction of a linear action.) Furthermore, there is a commutative diagram

\[
\begin{array}{ccc}
\tilde{G} & \supset & \tilde{C}^{2,n+1} \\
\pi & \downarrow & \downarrow \eta \\
G & \supset & C^{2,n+1},
\end{array}
\]

i.e., $\eta(g \cdot c) = \pi(g) \cdot \eta(c)$ for $g \in \tilde{G}$, $c \in \tilde{C}^{2,n+1}$. As a side remark, we note that it is possible to use this diagram to realize $\tilde{G}$ as the elements $\tilde{g} \in \text{Diff}^\infty(\tilde{C}^{2,n+1})$ so that there exists a $g \in G$ satisfying $\eta(\tilde{g}(c)) = g \cdot \eta(c)$ for all $c \in \tilde{C}^{2,n+1}$.

Now let $w$ be the block-diagonally embedded element of $SO(2) \times O(n+1)$ given by $(R_{\pi}, -I_{n+1})$ and acting on $\tilde{C}^{2,n+1}$ by matrix multiplication on the left. The action of $w$ commutes with the $\tilde{G}$-action and there is a $\tilde{G}$–equivariant diffeomorphism $\tilde{G} / (\tilde{M} \tilde{N}^-) \cong \tilde{C}^{2,n+1} / \langle w \rangle$. From this we obtain a more geometric and isomorphic realization of $\text{Ind}_{\tilde{Q}^-}^{\tilde{G}}(\chi_{m,r})$ as

$$\mathcal{I}_{m,r} = \left\{ \phi \in C^\infty(\tilde{C}^{2,n+1}) \mid \phi(w \cdot c) = i^{-m} \phi(c) \text{ and } \phi(\lambda c) = \lambda^r \phi(c), \lambda \in \mathbb{R}_{>0} \right\}$$

with $\tilde{G}$ action given by $(g \cdot \phi)(c) = \phi(g^{-1} \cdot c)$, $g \in \tilde{G}$, $c \in \tilde{C}^{2,n+1}$.

5. Noncompact Picture

By letting $\tilde{N}^+ \cong \mathbb{R}^{1,n}$ act on the base point $(0, 1, -1, 0, \ldots, 0) \in \tilde{C}^{2,n+1}$, we obtain a map $\iota : \mathbb{R}^{1,n} \to \tilde{C}^{2,n+1}$. Explicitly, $\iota(t, x) = (2t, 1 + q(t, x), -1 + q(t, x), 2x)$, where $q(t, x) = -t^2 + \|x\|^2$ as before. We then define the noncompact picture of $\mathcal{I}_{m,r}$ as

$$\mathcal{I}_{m,r}' = \left\{ f \in C^\infty(\mathbb{R}^{1,n}) \mid \exists \phi \in \mathcal{I}_{m,r} \text{ so } f = \phi \circ \iota \right\}.$$

This definition of $\mathcal{I}_{m,r}'$ is equivalent to the definition that was given earlier.
Lemma 1. Let \( c = (\lambda \sin \frac{\varphi}{2}, \lambda \cos \frac{\varphi}{2}, \lambda b) \in \overset{\sim}{\mathbb{C}}^{2,n+1} \) with \( \lambda \in \mathbb{R}_{>0} \), \( \varphi \in \mathbb{R} \), and \( b = (b_0, b_1, \ldots, b_n) \in S^n \). If \( \phi \in \mathcal{I}_{m,r} \) and \( f = \phi \circ \iota \), then
\[
\phi(c) = i^{mj} \left( \frac{\lambda |\cos \varphi - b_0|}{2} \right)^r f \left( \frac{\sin \varphi}{\cos \varphi - b_0}, \frac{b_1}{\cos \varphi - b_0}, \ldots, \frac{b_n}{\cos \varphi - b_0} \right),
\]
where \( j = j(\varphi, b_0) \in \mathbb{Z}_4 \) is given by
\[
j = \begin{cases} 
0, & \text{if } \cos \varphi - b_0 > 0 \text{ and } \frac{\varphi}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2}) \text{ mod } 2\pi, \\
1, & \text{if } \cos \varphi - b_0 < 0 \text{ and } \frac{\varphi}{2} \in (0, \pi) \text{ mod } 2\pi, \\
2, & \text{if } \cos \varphi - b_0 > 0 \text{ and } \frac{\varphi}{2} \in (\frac{\pi}{2}, \frac{3\pi}{2}) \text{ mod } 2\pi, \\
3, & \text{if } \cos \varphi - b_0 < 0 \text{ and } \frac{\varphi}{2} \in (\pi, 2\pi) \text{ mod } 2\pi.
\end{cases}
\]

Proof. The formula follows from the definitions by a direct calculation.

Calculating the explicit action of \( \tilde{G} \) on \( \mathcal{I}_{m,r}' \) is rather subtle, but always possible by using the lemma. As an important example, we give (without proof) the action of the subgroup \( \text{SL}(2, \mathbb{R}) \times \text{SO}(n) \subset \tilde{G} \).

Theorem 2. Let \( g = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), k \in \text{SL}(2, \mathbb{R}) \times \text{SO}(n) \) and \( f \in \mathcal{I}_{m,r}' \). Then
\[
(g \cdot f)(t, x) = (\sqrt{\text{sgn} \delta})^m |\delta|^r f \left( \frac{(b+d)t(a-ct)+cd \|x\|^2}{\delta}, \frac{xk}{\delta} \right),
\]
where \( \delta = (a-ct)^2 - c^2 \|x\|^2 \) and \( \sqrt{\text{sgn} \delta} \) is defined as +1 if \( \delta > 0 \) and \( a-ct > c \|x\| \), as−1 if \( \delta > 0 \) and \( a-ct < c \|x\| \), and as \( i \) if \( \delta < 0 \). (See also Fig. 1.)

6. Compact Picture

The final realization of \( \text{Ind}_{\mathcal{Q}'}^{\tilde{G}'}(\chi_{m,r}) \) is essentially given by restricting the elements of \( \mathcal{I}_{m,r} \) to \( S^1 \times S^n \subset \overset{\sim}{\mathbb{C}}^{2,n+1} \), i.e., the so-called compact picture. Because of the prominent role of the group \( \text{SL}(2, \mathbb{R}) \times \text{SO}(n) \), we use spherical coordinates on \( S^n \) and push \( S^1 \) back to \( \mathbb{R} \) by the map \( \sigma : \mathbb{R} \times \mathbb{R} \times S^{n-1} \rightarrow S^1 \times S^n \) given by \( \sigma(\varphi, \theta, \tilde{x}) = (\sin \frac{\varphi}{2}, \cos \frac{\varphi}{2}, -\cos \theta, \tilde{x} \sin \theta) \). We then define
\[
\mathcal{I}''_{m,r} = \{ F \in C^\infty(\mathbb{R} \times \mathbb{R} \times S^{n-1}) | F = \phi \circ \sigma, \text{ some } \phi \in \mathcal{I}_{m,r} \}
\]
and give it a \( \tilde{G} \)-structure so that the map \( \phi \rightarrow \phi \circ \sigma \) is an isomorphism. If \( \phi \in \mathcal{I}_{m,r} \) corresponds to \( f \in \mathcal{I}_{m,r}' \) and to \( F \in \mathcal{I}''_{m,r} \), then it follows from the definitions and Lemma 1 that
\[
F(\varphi, \theta, \tilde{x}) = i^{mj} \left| \frac{\cos \varphi + \cos \theta}{2} \right|^r f \left( \frac{\sin \varphi}{\cos \varphi + \cos \theta}, \frac{\tilde{x} \sin \theta}{\cos \varphi + \cos \theta} \right) \]
Figure 1. The values of $\sqrt{\text{sgn}(\delta)}$

\( \sqrt{\text{sgn}(\delta)} = +1 \)

\( \sqrt{\text{sgn}(\delta)} = -1 \)

\( \sqrt{\text{sgn}(\delta)} = i \)

\[
(\text{6.2}) \quad f(t, x) = \lambda(t, x)^r \left( \text{sgn}(t) \cos^{-1} \left( \frac{1 + q(t, x)}{\lambda(t, x)} \right), \cos^{-1} \left( \frac{1 - q(t, x)}{\lambda(t, x)} \right), \frac{x}{\|x\|} \right),
\]

where \( j \) is as in Lemma 1 with \( b_0 = -\cos \theta \) and \( \lambda(t, x) = \left( (1 - q(t, x))^2 + 4\|x\|^2 \right)^{\frac{1}{2}} \).

**Theorem 3.** For \( r = \frac{1-n}{2} \), we have the following identity of differential operators on \( T''_{m,r} \):

\[
\Omega_{\text{SL}(2)} - \Omega_{\text{SO}(n)} - r(r + 1) = \sin^2 \theta \left( \Omega_{\text{SO}(2)} - \Omega_{\text{SO}(n+1)} - r^2 \right).
\]

**Proof.** Using the definitions and (6.2) it is easy to show that \( \Omega_{\text{SL}(2)} \) acts on \( T''_{m,r} \) by the formula \( \Omega_{\text{SL}(2)} = r(1+r) - r^2 \sin^2 \theta - 2r \cos \theta \left( \partial_{\varphi} \sin \theta \partial_{\theta} + \sin^2 \theta \left( -\partial_{\varphi}^2 + \partial_\theta^2 \right) \right) \). Moreover, \( \Omega_{\text{SO}(2)} = -\partial_{\varphi}^2 \), \( \Omega_{\text{SO}(n+1)} = -\Delta_{S^n} \), and \( \Omega_{\text{SO}(n)} = -\Delta_{S^{n-1}} \). The theorem then follows from the well known recursive formula

\[
(\text{6.3}) \quad \Delta_{S^n} = \partial_\theta^2 + (n - 1) \cot \theta \partial_\theta - \csc^2 \theta \Delta_{S^{n-1}}
\]

for the spherical Laplacian in spherical coordinates.
7. \( \tilde{K} \)-Types

For \( r = \frac{1-n}{2} \), we consider \( \Omega = \sin^2 \theta \left( \Omega_{SO(2)} - \Omega_{SO(n+1)} - r^2 \right) \) and write \( (\ker \Omega)_{\tilde{K}} \) for the space of \( \tilde{K} \)-finite vectors in \( \ker \Omega \subset I''_{m,r} \). Furthermore, we define two \( \tilde{K} \)-representations \((\mathcal{H}^+)_{\tilde{K}}\) and \((\mathcal{H}^-)_{\tilde{K}}\) by

\[
(\mathcal{H}^\pm)_{\tilde{K}} = \bigoplus_{k \geq 0} \mathbb{C} e^{\pm i(k-r)} \otimes \mathcal{H}_k(S^n),
\]

where \( \mathcal{H}_k(S^n) \) is the space of degree \( k \) harmonic polynomials on \( \mathbb{R}^{n+1} \) restricted injectively to \( S^n \).

**Theorem 4.** For \( n \) odd, as a \( \tilde{K} \)-representation,

\[
(\ker \Omega)_{\tilde{K}} \cong \begin{cases} 
(\mathcal{H}^+)_{\tilde{K}} \oplus (\mathcal{H}^-)_{\tilde{K}} & \text{if } m \equiv n - 1 \text{ mod } 4, \\
0 & \text{otherwise},
\end{cases}
\]

and for \( n \) even,

\[
(\ker \Omega)_{\tilde{K}} \cong \begin{cases} 
(\mathcal{H}^+)_{\tilde{K}} & \text{if } m \equiv -(n-1) \text{ mod } 4, \\
(\mathcal{H}^-)_{\tilde{K}} & \text{if } m \equiv +(n-1) \text{ mod } 4, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** It follows from the definition of \( I''_{m,r} \) that as a \( \tilde{K} \)-representation,

\[
I''_{m,r} \cong \{ \phi \in C^\infty(S^1 \times S^n) \mid \phi(w \cdot c) = i^{-m} \phi(c) \quad \forall c \in S^1 \times S^n \}.
\]

The space of \( \tilde{K} \)-finite vectors in \( C^\infty(S^1 \times S^n) \) decomposes as \( C^\infty(S^1 \times S^n)_{\tilde{K}} \cong \bigoplus_{p,k} \mathbb{C} e^{ip\frac{\theta}{2}} \otimes \mathcal{H}_k(S^n) \), where the sum is over all \( p \in \mathbb{Z} \) and \( k \in \mathbb{Z}_{\geq 0} \). Noting that on \( \mathbb{C} e^{ip\frac{\theta}{2}} \otimes \mathcal{H}_k(S^n) \) the operator \( \Omega_{SO(2)} - \Omega_{SO(n+1)} - r^2 \) acts as the scalar

\[
\left( \frac{p}{2} \right)^2 - k(k + n - 1) - r^2 = \left( \frac{p}{2} \right)^2 - (k - r)^2,
\]

and \( w \) acts as \( i^p(-1)^k \), the result follows immediately. \( \square \)

8. Unitarity

The *Klein-Gordon inner product* on the space of smooth solutions (satisfying appropriate decay conditions) to the wave equation on \( \mathbb{R}^{1,n} \) is defined as

\[
\langle f_1, f_2 \rangle = i \int_{\mathbb{R}^n} \left( \partial_t f_1 f_2 - \partial_1 f_1 f_2 \right) \bigg|_{t=t_0} dx.
\]

The value of the integral above is independent of the choice of \( t_0 \). In the following, we will always choose \( t_0 = 0 \).
Theorem 5. For $r = \frac{1-n}{2}$, the Klein-Gordon inner product is well defined and $\tilde{G}$-invariant on $\ker \Box \subseteq \mathcal{T}'_{m,r}$. Moreover, if $m \equiv \mp(n-1) \mod 4$, it is positive definite on the subspace of $\mathcal{T}'_{m,r}$ corresponding to $(\mathcal{H}^+)_{\tilde{K}}$ and negative definite on the subspace of $\mathcal{T}'_{m,r}$ corresponding to $(\mathcal{H}^-)_{\tilde{K}}$.

Proof. If $f_1, f_2 \in \mathcal{T}'_{m,r}$, (6.2) implies $|\partial_t f_1(0,x)f_2(0,x)| \leq C \left(1 + \|x\|^2\right)^{-n}$ and hence the Klein-Gordon inner product is well-defined on $\mathcal{T}'_{m,r}$. The fact that the Klein-Gordon inner product is $\mathfrak{g}$-invariant on $\ker \Box \subseteq \mathcal{T}'_{m,r}$ is easily checked by integration by parts.

For the last statement of the theorem it is useful to calculate the Klein-Gordon inner product in the compact picture. If $f_1, f_2 \in \mathcal{T}'_{m,r}$ and $F_1, F_2 \in \mathcal{T}''_{m,r}$ are the corresponding functions in the compact picture, then

$$\langle f_1, f_2 \rangle = i2^{-n} \int_{[0,\pi] \times S^{n-1}} \left(\frac{\partial_\varphi F_1}{\varphi} - F_1 \frac{\partial_\varphi F_2}{\varphi}\right) |_{\varphi=0} \sin^{n-1} \theta d\theta d\varphi,$$

where $d\varphi$ is the spherical measure on $S^{n-1}$. Now consider $0 \neq f \in \mathcal{T}'_{m,r}$ corresponding to a function $F \in (\mathcal{H}^+)_{\tilde{K}}$ in the $\tilde{K}$-type $\mathbb{C}e^{i(k-r)\varphi} \otimes \mathcal{H}_k(S^n)$. Then $\partial_\varphi F = i(k-r)F$ and since $k-r = k + \frac{n-1}{2} > 0$, (8.1) implies $\langle f, f \rangle > 0$. \hfill $\Box$

9. An Explicit Orthonormal Basis of Solutions

Let $r = \frac{1-n}{2}$. For $l \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{Z}_{\geq 0}$ of the form $p = 2(l + d - r)$ with $d \in \mathbb{Z}_{\geq 0}$, we define a polynomial $g_{p,l}(t,x)$ of degree $2d$ by

$$g_{p,l}(t,x) = \lambda(t,x)^d \tilde{C}_d^{l-r} \left(\frac{1 - q(t,x)}{\lambda(t,x)}\right),$$

where $q(t,x) = -t^2 + \|x\|^2$, $\lambda(t,x) = \left((1 - q(t,x))^2 + 4\|x\|^2\right)^{-\frac{1}{2}}$, and $\tilde{C}_d^{l-r}(s)$ is the normalized Gegenbauer polynomial of degree $d$ and parameter $l - r$. Let $h_{l,j}(x)$ be homogeneous harmonic polynomials on $\mathbb{R}^n$ of degree $l$ such that the functions $h_{l,j}|_{S^{n-1}}$ form an orthonormal basis for $L^2(S^{n-1})$. Without loss of generality, we may assume that the functions $h_{l,j}|_{S^{n-1}}$ are real-valued. (We will need this assumption in the last section.)

Theorem 6. For $r = \frac{1-n}{2}$ and $m = -(n-1) \mod 4$, the functions

$$f_{p,l,j}(t,x) = \frac{1}{2^{l-r}p^2} \frac{g_{p,l}(t,x)h_{l,j}(x)}{\left((1-it)^2 + \|x\|^2\right)^p}$$

form an orthonormal basis of the subspace of $\ker \Box \subseteq \mathcal{T}'_{m,r}$ corresponding to $(\mathcal{H}^+)_{\tilde{K}}$. Similarly, the complex conjugate functions $\overline{f}_{p,l,j}$ form an orthonormal basis of the subspace of $\ker \Box \subseteq \mathcal{T}'_{-m,r}$ corresponding to $(\mathcal{H}^-)_{\tilde{K}}$. 
Proof. We first work in the compact picture. Noting that $\mathcal{H}_k(S^n) \cong \bigoplus_{l=0}^{k} \mathcal{H}_l(S^{n-1})$, it is clear that the $\widetilde{K}$-type $C e^{i(k-r)\varphi} \otimes \mathcal{H}_k(S^n)$ is spanned by functions of the form $F(\varphi, \theta, \tilde{x}) = e^{i(k-r)\varphi} g(\cos \theta) \sin^l \theta h_{l,j}(\tilde{x})$, where $0 \leq l \leq k$ and $g(s)$ is a polynomial such that $g(\cos \theta) \sin^l \theta h_{l,j}(\tilde{x}) \in \mathcal{H}_k(S^n)$. Using (6.3) and substituting $s = \cos \theta$, a straightforward calculation shows that the last condition is equivalent to $g(s)$ satisfying the differential equation

$$(1 - s^2)g''(s) - (2(l - r) + 1)sg'(s) + d(2l - r + d)g(s) = 0,$$

where $d = k - l$. This equation is a Gegenbauer differential equation and has a unique (up to multiple) nonzero polynomial solution, namely the Gegenbauer polynomial $g(s) = C_d^{l-r}(s)$.

Writing $p = 2(k - r) = 2(l + d - r)$, we now define

$$F_{p,l,j}(\varphi, \theta, \tilde{x}) = p^{-\frac{1}{2}} e^{ip\varphi} C_d^{l-r}(\cos \theta) \sin^l \theta h_{l,j}(\tilde{x}).$$

Using (6.2), the functions $F_{p,l,j}$ are seen to correspond to the functions $f_{p,l,j}$ in the non-compact picture. Finally, using (8.1), it is straightforward to check that the functions $F_{p,l,j}$ form an orthonormal set and hence an orthonormal basis of the subspace of $\ker \Box \subset T_{m,r}'$ corresponding to $(\mathcal{H}^+)_{\widetilde{K}}$. 

10. Energy and Irreducibility

We define another $\mathfrak{sl}(2)$-triple $\{z, n^+, n^\}$ in $\mathfrak{g}_C$ by

$$z = i(e^+ - e^-), \quad n^+ = \frac{1}{2}(h - i(e^+ + e^-)), \quad n^- = \frac{1}{2}(h + i(e^+ + e^-)).$$

Then $z$ lies in the center of $\mathfrak{t}_C$ and $\text{ad}(z)$ acts with eigenvalues $\{-2, 0, +2\}$ on $\mathfrak{g}_C$. The corresponding eigenspace decomposition $\mathfrak{g}_C = \mathfrak{p}^- \oplus \mathfrak{t}_C \oplus \mathfrak{p}^+$ is the usual complexified Cartan decomposition. In the compact picture, the action of $\{z, n^+, n^\}$ is given by the formulas $z = -2i \partial_{\varphi}$ and $n^\pm = e^{\pm i\varphi} (r \cos \varphi \pm i \cos \varphi \partial_{\varphi} - \sin \varphi \partial_{\theta})$. In particular, this shows that the decomposition of $(\ker \Omega)_{\widetilde{K}} \subset T''_{m,r}$ into $\widetilde{K}$-types given by Theorem 4 is the decomposition into eigenspaces of $z$, which are also referred to as energy levels. Using properties of Gegenbauer polynomials, it is easy to show that $z \cdot F_{p,l,j} = pF_{p,l,j}$ and $n^\pm \cdot F_{p,l,j} \in \mathbb{C} F_{p \pm 2l,j}$ with $n^\pm \cdot F_{p,l,j} = 0$ if and only if $p = \mp 2(l - r)$, respectively. Here for $p < 0$, $F_{p,l,j} = \mathcal{T}_{-p,l,j}$. Thus, if for fixed $l \geq 0$ we define $V_{l-r} = \text{span}_C \{F_{p,l,j} \mid p \geq 2(l - r)\}$ and $V^{-(l-r)} = \text{span}_C \{F_{p,l,j} \mid p \leq -2(l - r)\}$, then $V_{l-r}$ is a lowest weight representation of $\mathfrak{sl}(2, \mathbb{C})$ with lowest weight $l - r$ and $V^{-(l-r)}$ is a highest weight representation of $\mathfrak{sl}(2, \mathbb{C})$ with highest weight $-(l - r)$. Furthermore, as an $\mathfrak{sl}(2, \mathbb{C}) \times \text{SO}(n)$-representation,

$$(\mathcal{H}^+)_{\widetilde{K}} \cong \bigoplus_{l \geq 0} V_{l-r} \otimes \mathcal{H}_l(S^{n-1})$$
and
\[
(H^-)_{\tilde{K}} \cong \bigoplus_{l \geq 0} V^{-(l-r)} \otimes \mathcal{H}_l(S^{n-1}).
\]

From these observations it follows that the \((\mathfrak{g}_C, \tilde{K})\)-module \((H^+)_{\tilde{K}}\) is an irreducible lowest weight representation with lowest weight vector \(e^{-ir\varphi}\) and \((H^-)_{\tilde{K}}\) is an irreducible highest weight representation with lowest weight vector \(e^{ir\varphi}\). To summarize we then have the following result.

**Theorem 7.** Let \(r = \frac{1-n}{2}\). Then for \(n\ odd\), \(\ker \Box \subset I'_{m,r}\ completes to a \tilde{G}\)-representation
\[
\mathbb{R} \begin{cases} 
H^+ \oplus H^- & \text{if } m \equiv n - 1 \mod 4, \\
0 & \text{otherwise,}
\end{cases}
\]
and for \(n\ even\),
\[
\mathbb{R} \begin{cases} 
H^+ & \text{if } m \equiv -(n-1) \mod 4, \\
H^- & \text{if } m \equiv +(n-1) \mod 4, \\
0 & \text{otherwise,}
\end{cases}
\]
where \(H^+\) is a unitary lowest weight representation (or positive energy representation) of lowest weight \(-r\omega_0\) and \(H^-\) is a unitary highest weight representation (or negative energy representation) of highest weight \(r\omega_0\), where \(\omega_0\) is the fundamental weight corresponding to the noncompact simple root. \(\square\)

11. Classical Solutions

It is easy to see that every element \(f \in H^+\) is a weak solution to the wave equation. Here we outline a proof that every classical real-valued solution is the real part of a unique continuous element in the representation \(H^+\).

**Theorem 8.** Suppose \(\Phi, \Psi \in S(\mathbb{R}^n)\) are real-valued Schwartz functions and let \(u \in C^\infty(\mathbb{R}^{1,n})\) be the solution to the Cauchy problem
\[
\Box u = 0, \quad u(0, x) = \Phi(x), \quad \partial_t u(0, x) = \Psi(x).
\]
Then there is a unique continuous function \(f \in H^+\) such that \(u = \text{Re} f\). Explicitly, \(f = \sum_{p,l,j} c_{p,l,j} f_{p,l,j} \) with \(c_{p,l,j} = 2 \langle f_{p,l,j}, u \rangle = 2i \int_{\mathbb{R}^n} (\partial_t f_{p,l,j}(0,x)\Phi(x) - f_{p,l,j}(0,x)\Psi(x)) dx\).

**Proof.** Let \(r = \frac{1-n}{2}\) and \(m = -(n-1) \mod 4\). The \(g\)-actions on \(I'_{m,r}\) and \(I''_{m,r}\) extend to \(g\)-actions on \(C^\infty(\mathbb{R}^{1,n})\) and \(C^\infty(\mathbb{R} \times \mathbb{R} \times S^{n-1})\), respectively. Moreover, the intertwining map \(I'_{m,r} \to I''_{m,r}\) given by (6.1) extends to an intertwining map \(T: C^\infty(\mathbb{R}^{1,n}) \to \)
\[ C^\infty \left( \{(\varphi, \theta, \widehat{x}) \in \mathbb{R} \times \mathbb{R} \times S^{n-1} \mid \cos \varphi + \cos \theta \neq 0 \} \right) \text{.} \] Let \( U = T(u) \) and view \( U \) as a function on an open dense subset of \( \mathbb{R} \times S^n \). By the assumptions of the theorem, it easily follows that \( (z^k \cdot U)|_{\varphi=0} = T(z^k \cdot u)|_{\varphi=0} \) is in \( L^2(S^n) \) with values in \( \iota^k \mathbb{R} \). Define \( G_{p,l,j} = 2^{-r} p^{-\frac{1}{2}} F_{p,l,j} \). Then the real-valued functions \( G_{p,l,j}|_{\varphi=0} \) form an orthonormal basis of \( L^2(S^n) \). Hence we can write \( (z^k \cdot U)|_{\varphi=0} = \sum_{p,l,j} a^{(k)}_{p,l,j} G_{p,l,j}|_{\varphi=0} \) with coefficients \( a^{(k)}_{p,l,j} \in \mathbb{R} \) such that \( \sum_{p,l,j} |a^{(k)}_{p,l,j}|^2 < \infty \). A direct calculation in the compact picture shows \( c_{p,l,j} = 2^{-r} p^{-\frac{1}{2}} (pa^{(0)}_{p,l,j} + a^{(1)}_{p,l,j}) \). More generally, \( p^k c_{p,l,j} = 2^k (z^k \cdot f_{p,l,j}, u) = 2^r p^{-\frac{1}{2}} (p a^{(k)}_{p,l,j} + a^{(k+1)}_{p,l,j}) \) and hence \( a_{p,l,j}^{(k)} = p^k a_{p,l,j}^{(0)} \) for \( k \) even and \( a_{p,l,j}^{(k)} = p^{k-1} a_{p,l,j}^{(1)} \) for \( k \) odd. It follows that \( \sum_{p,l,j} p^N |a^{(0)}_{p,l,j}|^2 < \infty \) and \( \sum_{p,l,j} p^N |a^{(1)}_{p,l,j}|^2 < \infty \) for every \( N \geq 0 \). Since \( a^{(0)}_{p,l,j} \in \mathbb{R} \) and \( a^{(1)}_{p,l,j} \in \mathbb{R} \), we have \( \sum_{p,l,j} p^N |c_{p,l,j}|^2 < \infty \) for every \( N \geq 0 \).

To show that \( |G_{p,l,j}| \) is independent of \( \varphi \), [11, Corollary 2.9] shows

\[
\sum_{p,l,j} |G_{p,l,j}|^2 = \frac{\dim \mathcal{H}_{n+r}(\mathbb{R}^{n+1})}{\text{Surface Area}(S^n)} \leq C p^{n-1},
\]

where \( C \) is a constant only depending on \( n \). Thus, by Hölder’s inequality,

\[
\sum_{p,l,j} \left| c_{p,l,j} p^{-\frac{1}{2}} G_{p,l,j} \right| \leq \left( \sum_{p,l,j} p^n |c_{p,l,j}|^2 \right)^{\frac{1}{2}} \left( \sum_{p} p_{-n-1} \sum_{l,j} |G_{p,l,j}|^2 \right)^{\frac{1}{2}} < \infty.
\]

This shows that \( \sum_{p,l,j} c_{p,l,j} F_{p,l,j} = 2^{-r} \sum_{p,l,j} c_{p,l,j} p^{-\frac{1}{2}} G_{p,l,j} \) converges uniformly to a continuous function \( F \) and hence \( \sum_{p,l,j} c_{p,l,j} F_{p,l,j} \) converges uniformly to a continuous function \( f \). Moreover, \( \text{Re} \, F|_{\varphi=0} = \text{Re} \sum_{p,l,j} (a^{(0)}_{p,l,j} + p^{-1} a^{(1)}_{p,l,j}) G_{p,l,j}|_{\varphi=0} = \sum_{p,l,j} a^{(0)}_{p,l,j} G_{p,l,j}|_{\varphi=0} = U|_{\varphi=0} \) and hence \( \text{Re} \, f|_{t=0} = u|_{t=0} = \Phi \).

A subtle argument using a variation of Bernstein’s inequality and again Hölder’s inequality shows

\[
\sum_{p,l,j} \left| c_{p,l,j} p^{-\frac{1}{2}} \partial_\theta G_{p,l,j} \right| < \infty.
\]

Noting that \( e^+ = -r \sin \varphi \cos \theta - (1 + \cos \varphi \cos \varphi) \partial_\varphi + \sin \varphi \sin \theta \partial_\theta \) in the compact picture, we find that \( \sum_{p,l,j} c_{p,l,j} (e^+ \cdot F_{p,l,j}) \) converges uniformly. Since \( e^+ = -\partial_\theta \) in the noncompact picture, this implies that \( \partial_\theta f = \sum_{p,l,j} c_{p,l,j} \partial_\theta F_{p,l,j} \) is continuous. Moreover, it follows as above that \( \text{Re} \partial_\theta f|_{t=0} = \partial_\theta u|_{t=0} = \Psi \).

Since \( \sum_{p,l,j} |c_{p,l,j}|^2 < \infty, f \in \mathcal{H}^+ \). Since \( \text{Re} \, f \) is a weak solution to the wave equation with \( \text{Re} \, f|_{t=0} = \Phi \) and \( \text{Re} \, \partial_\theta f|_{t=0} = \Psi \) it follows that \( u = \text{Re} \, f \). \(\square\)
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