Complexity Analysis and Synchronization Control of Fractional-Order Jafari-Sprott Chaotic System

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ABSTRACT Aiming at the complexity problem of fractional-order Jafari-Sprott chaotic system, in this paper, Adomian decomposition method is used to study its numerical analysis and a complexity analysis method of fractional-order Jafari-Sprott chaotic system based on fuzzy entropy algorithm, sample entropy algorithm and dispersion entropy algorithm is proposed. For the synchronization and control of fractional-order Jafari-Sprott chaotic system, sliding mode control is used to achieve synchronization of fractional-order Jafari-Sprott chaotic system and a control method of fractional-order Jafari-Sprott chaotic system is proposed based on frequency distribution model of fractional-order integral operator. The main results are as follows: (1) The complexity of the fractional-order Jafari-Sprott chaotic system is greater than the integer-order Jafari-Sprott chaotic system, and fractional-order chaotic system has better application prospects. (2) Moreover, it is concluded that the effect of the dispersion entropy algorithm on detecting complexity is the best, which provides theoretical and experimental basis for the practical engineering application of the fractional-order Jafari-Sprott chaotic system. (3) Synchronization and control of fractional-order Jafari-Sprott chaotic system is accomplished by sliding model control and frequency distribution model of fractional-order integral operator respectively. In particular, the control effect of each variable is accomplished by designing a control law based on the frequency distribution model of fractional integral operator.

INDEX TERMS Fractional-order chaotic system, Adomian decomposition method, complexity analysis, sliding mode control, frequency distribution model of fractional-order integral operator.

I. INTRODUCTION Research on complexity is involved in various fields. So far there is no unified concept of complexity. Complexity refers to metric value, which has comparative significance, and different complexity algorithms characterize different aspects of complexity. Horgan [1] pointed out that there are multiple definitions of complexity, such as time complexity, space complexity, semantic complexity, Kolmogorov complexity, etc. There are multiple algorithms for calculating complexity. Lempel and Ziv [2] proposed Lempel-Ziv algorithm. Pincus [3] and Sun et al. [4] proposed an approximate entropy algorithm. Chen et al. [5] and Sun et al. [6] proposed a fuzzy entropy algorithm. Larrondo et al. [7] and Sun et al. [8] proposed a strength statistics algorithm. Azad et al. [9] proposed a symbolic entropy algorithm. At present, there are spectral entropy algorithm [10], wavelet entropy algorithm [11] and C0 algorithm [12] based on Fourier transform and wavelet transform. In addition, complexity algorithms based on entropy are often used in biological and medical research. Yang and Liao [13] applied approximate entropy to the comparison of heart rate between children with sudden infant death and normal children. Lake et al. [14] used sample entropy to analyze the change of newborn heart rate signal as the basis for diagnosis of neonatal sepsis. Liu et al. [15] used sample entropy to study the electroencephalography (EEG) signal during sleep, and distinguished the different stages of sleep through the change of sample entropy value. Yang et al. [16] analyzed depression by using sample entropy. It can be seen that entropy can be used to test the complexity of various systems. Fractional-order chaotic system is widely used because of its rich dynamic behavior, especially related to complexity. The complexity of fractional-order chaotic system analysis by fuzzy entropy algorithm, sample entropy algorithm and dispersion entropy algorithm has not been reported, so it is necessary to carry out this research.
The current synchronization control methods commonly used in chaotic system research are OGY (firstly proposed by E. Ott, C. Grebogi, J. A. Yorke) control method [17], drive response control method [18], [19], linear and nonlinear feedback control method [20], [21], real-delay feedback control method [22], linear error feedback control method [23], generalized synchronous control method [24], neural network control method [25], [26]. With the deepening research of scientists, a large number of methods have been proposed, such as self-adaptive synchronization method [27], active control synchronization method [28], [29], variable structure synchronization control method [30], [31], fuzzy control synchronization method [32], [33], synchronization method based on state observer [34], coupled synchronization method [35]–[38], etc. Scholars at home and abroad have also achieved some achievements. Mao [39] and Mao and Li [40] studied the synchronization problem of the fractional-order Duffing system with a new approach law and the synchronization control problem of the fractional-order Genesio-Tesi chaotic system. Based on the nonlinear sliding mode integral control method, the control problem of a class of systems has been studied and the proportional integral control method has been widely used in the field of cybernetics [41]. Based on proportional integral control, an accurate tracking and guidance method has been studied, and sliding mode control method has also been widely used in control field [42]. Xu et al. [43] studied on sliding mode control for a class of chaotic systems. Aghababa and Akbari [44] studied the control synchronization of two types of chaotic systems with uncertain disturbances and designed an appropriate sliding mode controller and adaptive rate to eliminate the effects of system uncertainty and external disturbances, and finally achieved system synchronization. Song et al. [45], [46] studied a class of uncertain fractional-order nonlinear systems subject to uncertainties and external disturbances and the adaptive output feedback resilient control problem, and designed a fractional adaptive backstepping neuro-fuzzy sliding mode controller with neuro-fuzzy network system and the fractional Lyapunov stability theory and the finite-time stability theory, and pointed out that observer-based adaptive output feedback control scheme for fractional-order (FO) nonlinear systems is one of our future work. Today, most three-dimensional fractional-order chaotic systems require three control laws for synchronous control [47], [48], and how to design fewer control laws needs further study, and how to optimize the control law that makes it contain only state variables is also a question worth studying.

Because of the advantages of entropy in complexity analysis, this paper combines the complexity analysis of fractional chaotic system with entropy. Firstly, for the integer-order Jafari-Sprott chaotic system, the classical Lyapunov exponent, Poincaré section, and bifurcation diagram are used to analyze dynamics. Secondly, the fractional-order Jafari-Sprott chaotic system is taken as an example to introduce the decomposition steps of the Adomian decomposition method in detail. Then, the effects of system parameters and orders on the complexity of fractional-order Jafari-Sprott chaotic system based on Adomian decomposition method and three entropy are analyzed. The combination of entropy and the complexity of fractional-order chaotic system can better reflect the characteristics of the system. Finally, synchronization control of the fractional-order Jafari-Sprott chaotic system is accomplished by sliding mode control and the frequency distribution model of the fractional-order integral operator respectively.

II. DYNAMICS ANALYSIS OF INTEGER-ORDER JAFARI-SPROTT CHAOTIC

A. INTRODUCTION TO MATHEMATICAL MODEL

The mathematical model of the integer-order Jafari-Sprott chaotic system [49] is as following:

\[
\begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= -x + yz \\
\frac{dz}{dt} &= -x - axy - bxz
\end{align*}
\]  

In this three-dimensional system, \(x, y, \) and \(z\) are system variables, \(a\) and \(b\) are system parameters. When \(a = 15, b = 1\) and the initial values is \((0, 0.5, 0.5)\), the system is in a chaotic state. By using the Runge-Kutta method, phase space diagram is shown in Figure 1.

![Phase space diagram of each plan.](image1)

Time domain waveform of each variable is shown in Figure 2.

B. B ANALYSIS OF DYNAMIC CHARACTERISTICS

1) LYAPUNOV EXPONENT

Lyapunov exponent is a quantitative criterion for describing the state evolution of dynamical systems, and used to measure the degree of attraction or separation of two adjacent trajectories with different initial conditions in phase space according to the exponential law over time. This ratio of trajectory
convergence or divergence is called the Lyapunov exponent. In a three-dimensional chaotic system, there is a Lyapunov exponent greater than 0, which means that the system is in a chaotic state.

For integer-order Jafari-Sprott chaotic system, when the system parameters $a = 15$, $b = 1$ and the initial values is $(0, 0.5, 0.5)$, the Lyapunov exponent diagram is shown in Figure 3. Fixed system parameter $b = 1$ and the system parameter $a \in [10, 20]$, its Lyapunov exponent is shown in Figure 4.

2) POINCARÉ SECTION
A section is selected in the multi-dimensional phase space. This section is not tangent to the trajectory, and convenient for observing the motion characteristics and changes of the system. This section is called the Poincaré section. When the system parameters $a = 15, b = 1, h = 0.001, h$ is the step length and the initial values is $(0, 0.5, 0.5)$, the Poincaré section of the integer-order Jafari-Sprott chaotic system is shown in Figure 5. Figure 5 shows that the Poincaré section is a dense cluster of slices, so it can be concluded that the Jafari-Sprott chaotic system is in a chaotic state.

3) BIFURCATION DIAGRAM
Bifurcation refers to the type of dynamic movement changed when a parameter changes. When the system parameters $a \in [10, 20]$, the bifurcation diagram of the Jafari-Sprott chaotic system is shown in Figure 6. It can be seen from Figure 6 with the change of the system parameter $a$, the system continuously branches between different states, and finally the system reaches a chaotic state.
the integral of the fractional order. When \( q > 0 \), \( \mathbb{D}_t^q \) represents the derivative. When \( q < 0 \), \( \mathbb{D}_t^q \) represents the integral.

2) The Riemann-Liouville fractional differential is defined as following [51]:

\[
\begin{align*}
\mathbb{D}_t^q f(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau) d\tau \quad q < 0 \\
\mathbb{D}_t^q f(t) &= \left. \frac{d^m}{dt^m} f(t) \right|_{t=t_0} \quad q = 0 \\
\mathbb{D}_t^q f(t) &= 0 \quad q > 0
\end{align*}
\]

The power series and constant of \( q \)-order differential are defined respectively as following:

\[
\begin{align*}
\mathbb{D}_t^q x(t) &= \frac{\Gamma(r+1)}{\Gamma(r+1-q)} (t-t_0)^{r-q} \\
\mathbb{D}_t^q C &= \frac{C}{\Gamma(1-q)} (t-t_0)^{-q}
\end{align*}
\]

where \( \Gamma(\cdot) \) is the Gamma function. This is the most basic function in fractional calculus, which is defined as:

\[
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt
\]

When \( x \in [-5, 5] \), the Gamma function is shown in Figure 7. Places marked in Figure 7 are 0!, 1!, 2!, 3!.

![Gamma function graph](image)

**FIGURE 7.** Gamma function graph.

3) The Caputo fractional differential is defined as following [52]:

\[
\begin{align*}
\mathbb{D}_t^q f(t) &= \frac{1}{\Gamma(m-q)} \int_0^t \frac{f(m)(\tau)d\tau}{(m-\tau)^{q+1-m}} \\
&= \left. \frac{d^m}{dt^m} f(t) \right|_{t=t_0} \quad m-1 < q < m
\end{align*}
\]

The \( q \)-order differential of constant and power function is defined respectively as following:

\[
\begin{align*}
\mathbb{D}_t^q x(t) &= \frac{\Gamma(r+1)}{\Gamma(r+1-q)} (t-t_0)^{r-q} \\
\mathbb{D}_t^q C &= 0
\end{align*}
\]

### B. THE ADOMIAN DECOMPOSITION METHOD

The Adomian decomposition method [53] is the latest proposed time-domain approximation algorithm, and suitable for numerical solution of fractional-order chaotic system. This algorithm does not require discrimination and takes up a lot of computer memory, and can provide high-precision, fast-convergent numerical analysis.

For fractional-order chaotic system

\[
\mathbb{D}_t^q x(t) = f(x(t)) + g(t)
\]

where, \( x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \) is the corresponding function variable, \( g(t) = [g_1(t), g_2(t), \ldots, g_n(t)]^T \) is constant. The system is divided into three parts as following:

\[
\begin{align*}
\mathbb{D}_t^q x(t) &= Lx(t) + Nx(t) + g(t) \\
x(k)(t_0) &= b_k, \quad k = 0, \ldots, m - 1 \\
m \in N, \quad m - m < q \leq m
\end{align*}
\]

The nonlinear term is decomposed according to the following formula:

\[
\begin{align*}
A_j &= \int_0^x \int_{t_0}^x d\lambda_1 \int_{t_0}^{\lambda_1} \cdots \int_{t_0}^{\lambda_{j-1}} N(\lambda_j) \lambda_j \lambda_{j-1} \cdots \lambda_1 dt_0 \cdots dt_{j-1} \\
&= \sum_{\lambda_j = 0}^{\infty} A_j(\lambda_0, \lambda_1, \ldots, \lambda_j)
\end{align*}
\]

where the nonlinear term can be expressed as following:

\[
Nx = \sum_{i=0}^{\infty} A_i(\lambda^0, \lambda^1, \lambda^2, \ldots, \lambda^i)
\]

The solution of the equation is as following:

\[
x = \sum_{i=0}^{\infty} x^i = J^q_{t_0} L \sum_{i=0}^{\infty} x^i + J^q_{t_0} \sum_{i=0}^{\infty} A^i + J^q_{t_0} g + \Phi
\]

where \( \Phi = \sum_{k=0}^{m-1} b_k (t-t_0)^m \) is the initial condition to satisfy the system, and its iterative relationship is as following:

\[
\begin{align*}
x^0 &= J^q_{t_0} g + \Phi \\
x^1 &= J^q_{t_0} L x^0 + J^q_{t_0} A^0 (x^0) \\
x^2 &= J^q_{t_0} L x^1 + J^q_{t_0} A^1 (x^0, x^1) \\
&\quad \vdots \\
x^{i-1} &= J^q_{t_0} L x^{i-1} + J^q_{t_0} A^{i-1} (x^0, x^1, \ldots, x^{i-1}) \\
&\quad \vdots
\end{align*}
\]

The mathematical model of the fractional-order Jafari-Sprott chaotic system is as following:

\[
\begin{align*}
D^q_{t_0} x_1 &= x_2 \\
D^q_{t_0} x_2 &= -x_1 + x_2 x_3 \\
D^q_{t_0} x_3 &= -x_1 - ax_1 x_2 - bx_1 x_3
\end{align*}
\]
The linear and nonlinear terms in this system respectively are as following:

\[
\begin{bmatrix}
Lx_1 \\
Lx_2 \\
Lx_3
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
-x_1 \\
-x_1
\end{bmatrix},
\begin{bmatrix}
N_{x_1} \\
N_{x_2} \\
N_{x_3}
\end{bmatrix} =
\begin{bmatrix}
0 \\
x_2x_3 \\
-ax_1x_2 - bx_1x_3
\end{bmatrix}
\]  

(17)

Decompose the nonlinear term according to Equation (12).

The decomposition \(N_{x_2}\) process is as following:

\[
A_2^0 = x_2^0 x_3^0
\]

\[
A_2^1 = \frac{1}{2} [(k\lambda - 1)x_2^0 (\lambda x_3^k) + (k\lambda - 1)x_3^k (\lambda x_2^k)]
\]

\[
= \frac{1}{2} [2x_2 x_3^0 + 2x_2 x_1^0 + 2x_2 x_0^2]
\]

\[
= x_2^0 x_3^0 + x_1^0 x_3^0 + x_2^0 x_3^0
\]

\[
A_2^2 = \frac{1}{2} [(k\lambda - 1)x_2^0 (\lambda x_3^k) + (k\lambda - 1)x_3^k (\lambda x_2^k)]
\]

\[
= \frac{1}{2} [4x_2 x_3^0 + 4x_2 x_1^0 + 4x_2 x_0^2 + 4x_2 x_0^3 + 4x_2 x_0^4]
\]

\[
= x_2^0 x_3^0 + x_1^0 x_3^0 + x_2^0 x_3^0 + x_2^0 x_3^0 + x_2^0 x_3^0
\]

(18)

Since each decomposition principle is the same, Equation (18) only lists the specific decomposition process of the previous three terms. The complete decomposition result is as following:

\[
\begin{bmatrix}
A_2^0 \\
A_2^1 \\
A_2^2 \\
A_2^3 \\
A_2^4 \\
A_2^5
\end{bmatrix} =
\begin{bmatrix}
0 \\
x_2^0 x_3^0 + x_1^0 x_3^0 \\
x_2^0 x_3^0 + x_1^0 x_3^0 + x_2^0 x_3^0 \\
0 \\
x_2^0 x_3^0 + x_1^0 x_3^0 + x_2^0 x_3^0 + x_2^0 x_3^0 \\
0
\end{bmatrix}
\]

(19)

Let \(c_0^0 = x_0^0, c_0^1 = x_0^1, c_0^2 = x_0^2, c_0^3 = x_0^3\), we can obtain from Equation (15):

\[
Lx_0^0 =
\begin{bmatrix}
0 \\
x_0^0 \\
x_0^0 \\
x_0^0 \\
x_0^0 \\
x_0^0
\end{bmatrix}
\]

(22)

\[
J_h^p Lx_0^0 =
\begin{bmatrix}
0 \\
x_0^0 \\
x_0^0 \\
x_0^0 \\
x_0^0 \\
x_0^0
\end{bmatrix}
\]

\[
(\tau - t_0)^\theta \Gamma(q + 1)
\]

(23)

\[
A_0^0 x_0^0 =
\begin{bmatrix}
0 \\
x_0^0 x_0^0 \\
x_0^0 x_0^0 \\
x_0^0 x_0^0 \\
x_0^0 x_0^0 \\
x_0^0 x_0^0
\end{bmatrix}
\]

(24)

\[
J_h^p A_0^0 (x_0^0) =
\begin{bmatrix}
0 \\
x_0^0 x_0^0 \\
x_0^0 x_0^0 \\
x_0^0 x_0^0 \\
x_0^0 x_0^0 \\
x_0^0 x_0^0
\end{bmatrix}
\]

\[
(\tau - t_0)^\theta \Gamma(q + 1)
\]

(25)

\[
x_1 = J_h^p Lx_0^0 + J_h^p A_0^0 (x_0^0) =
\begin{bmatrix}
0 \\
x_0^0 - c_1^0 (t - t_0)^\theta \Gamma(q + 1)
\end{bmatrix}
\]

(26)

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

(27)

Assign the coefficient to the corresponding variable, we can get:

\[
\begin{bmatrix}
c_1^0 = c_2^0 \\
c_1^1 = [-c_0^0 + c_2^0 c_3^0] \\
c_1^2 = [-c_0^0 - ac_2^0 c_3^0 - bc_4^0 c_3^0]
\end{bmatrix}
\]

(28)

The derivation method for the remaining 5 items is the same as the above formula:

\[
\begin{bmatrix}
c_2^0 = c_1^2 \\
c_2^1 = [-c_1^1 + c_2^1 c_3^1 + c_0^0 c_3^0] \\
c_2^2 = [-c_1^2 + a(-c_1^0 c_2^0 - c_0^0 c_2^0)] + b(-c_1^0 c_3^1 - c_1^1 c_3^1)
\end{bmatrix}
\]

(29)

\[
\begin{bmatrix}
c_3^0 = c_2^3 \\
c_3^1 = [-c_2^1 + c_3^1 c_3^2 + c_0^0 c_3^2] \\
c_3^2 = [-c_2^2 + a(-c_2^1 c_2^0 - c_0^0 c_2^0)] + b(-c_2^1 c_3^2 - c_2^2 c_3^2)
\end{bmatrix}
\]

(30)

The initial condition is as following:

\[
\begin{bmatrix}
x_1^0 = x_1(t_0) \\
x_2^0 = x_2(t_0) \\
x_3^0 = x_3(t_0)
\end{bmatrix}
\]

(21)
expressed as

$$\frac{\Gamma(q+1)}{\Gamma(2q+1)} - c_0^5 c_1^2 \lim_{n \to \infty} \frac{\Gamma(q+1)}{\Gamma(3q+1)}$$

$$+ b (- c_1^3 c_0^0 - c_1^3 c_1^0 + c_1^3 c_2^0) \frac{\Gamma(q+1)}{\Gamma(3q+1)}$$

(31)

(32)

(33)

At this time, the solution of the system equation can be expressed as:

$$x_j(t) = c_0^j + c_j^1 (t - t_0) + c_j^2 (t - t_0)^2 + c_j^3 (t - t_0)^3 + c_j^4 (t - t_0)^4 + c_j^5 (t - t_0)^5 + c_j^6 (t - t_0)^6$$

(34)

When \(a = 15, b = 1, q = 0.96\), and the initial values is \((0, 0.5, 0.5)\), phase space diagram is shown in Figure 8.

The Adomian decomposition method has the advantages of high precision, no occupy computer memory, no need of discretization, etc. It is suitable for cases where the maximum number of nonlinear terms is less than 3. If the highest order of nonlinear terms is greater than or equal to 3, then the calculation amount of its decomposition process is large, and the Adomian decomposition method is not recommended.

IV. COMPLEXITY ANALYSIS OF FRACTIONAL-ORDER JAFARI-SPROTT CHAOTIC SYSTEM

The complexity of chaotic characteristics is also a method for analyzing the dynamic characteristics of chaotic system. It has the same effect as the Lyapunov exponent, Poincaré section, and bifurcation diagram. In a nutshell, complexity is the degree to which a correlation algorithm is used to calculate the approximate random sequence. The greater the complexity, the closer the random sequence is, the higher the security is. The complexity of chaotic sequences is divided into behavioral complexity and structural complexity. Behavioral complexity refers to using a certain method to measure the probability of a sequence generating a new pattern in a short time window from the chaotic sequence itself. The larger the probability of generating a new pattern, the more complex the sequence is. Structural complexity refers to analyzing the complexity of a sequence by changing frequency characteristics, energy spectrum characteristics, etc. The more balanced the energy spectrum distribution of the sequence, the more complex the sequence is.

A. FUZZY ENTROPY ALGORITHM

For judging the complexity of a sequence, fuzzy entropy is an effective measurement algorithm, and it has lower sensitivity dependence on sequence length, phase space dimension, and similarity tolerance. The algorithm process [54] is as following:

Step 1: Perform phase space reconstruction of the sequence. For a given sequence \([u(1), u(2), \ldots, u(N)]\), the reconstructed phase space is:

$$X(i) = [u(i), u(i+1), \ldots, u(i+1-m)] - u_0(i)$$

(35)

where, \(u_0(i) = \frac{1}{m} \sum_{j=0}^{m-1} u(i+j)\).
Step 2: Introduce fuzzy membership function. The fuzzy membership function is defined as:

\[
A(x) = \begin{cases} 
1, & x = 0 \\
\exp[-\ln (2)(x/r)^2], & x > 0 
\end{cases}
\]  

(36)

where, \( r \) is the similar tolerance. The fuzzy membership function is given:

\[
A_{ij}^m = \exp[-\ln (2)(d_{ij}^m/r)^2], \quad j = 1, 2, \ldots, N - m + 1, \quad j \neq i
\]

(37)

where,

\[
d_{ij}^m = d[X(i), X(j)] = \max_{p=1,2,\ldots,m}|u(i+p-1) - u_0(i)| - |u(j+p-1) - u_0(j)|
\]

(38)

is the maximum absolute distance between the vectors \( X(i), X(j) \).

Step 3: Calculate the fuzzy entropy. Average value each \( i \) to get:

\[
C_i^m(r) = \frac{1}{N - m} \sum_{j=1,j\neq i}^{N-m+1} A_{ij}^m
\]

(39)

\[
\Phi^m(r) = \frac{1}{N - m + 1} \sum_{i=1}^{N-m+1} C_i^m(r)
\]

(40)

So the fuzzy entropy is:

\[
Fuzzy(m, r, N) = \ln \Phi^m(r) - \ln \Phi^{m+1}(r)
\]

(41)

where, \( m, r, N \) are the dimensions of phase space, similarity tolerance, and number of selected data, select \( m = 2, r = 0.2 \times SD, N = 15000, SD \) is the standard deviation of \( N \). When the system parameters \( a = 12, 20 \), the fuzzy entropy complexity of the fractional-order Jafari-Sprott chaotic system is shown in Figure 9 (a). When the system parameters \( a = 15, b = 1, q \in [0.2, 1] \), the fuzzy entropy complexity of the fractional-order Jafari-Sprott chaotic system is shown in Figure 9 (b). When the system parameters \( a = 15, q = 0.96, b \in [0.2, 1.2] \), the fuzzy entropy complexity of the fractional-order Jafari-Sprott chaotic system is shown in Figure 9 (c).

A single parameter change is not as complicated as a multi-parameter change. In the following, the chromatogram is used to simulate and analyze the situation under the changes of the two parameters. When \( a \in [12, 20], q \in [0.2, 1] \), the chromatogram of the change of fuzzy entropy complexity is shown in Figure 10 (a). When \( b \in [0.2, 1.2], q \in [0.2, 1] \), the chromatogram of the change of fuzzy entropy complexity is shown in Figure 10 (b). When \( a \in [12, 20], b \in [0.2, 1.2] \), the chromatogram of the change of fuzzy entropy complexity is shown in Figure 10 (c).

B. SAMPLE ENTROPY ALGORITHM

Sample entropy is a method for measuring the complexity of a sequence, and it is used in a variety of research fields, especially medical biology. The algorithmic process [55] is as following:

Step 1: Perform phase space reconstruction of the sequence. For a given sequence \([x(1), x(2), \ldots, x(N)]\), the reconstructed phase space is:

\[
\begin{align*}
X_m(1) &= \{x_1, x_2, \ldots, x_m\} - \bar{X}_1^m \\
X_m(2) &= \{x_2, x_3, \ldots, x_{m+1}\} - \bar{X}_2^m \\
&\quad \vdots \\
X_m(i) &= \{x_i, x_{i+1}, \ldots, x_{i+m}\} - \bar{X}_i^{m+1} \\
&\quad \vdots \\
X_m(N - m) &= \{x_{N-m}, x_{N-m+1}, \ldots, x_{N-1}\} - \bar{X}_{N-m}^m
\end{align*}
\]

(42)
\[
X_{m+1}(1) = \{x_1, x_2, \ldots, x_{m+1}\} - \bar{X}_1^{m+1} \\
X_{m+1}(2) = \{x_2, x_3, \ldots, x_{m+2}\} - \bar{X}_2^{m+1} \\
\vdots \\
X_{m+1}(i) = \{x_i, x_{i+1}, \ldots, x_{i+m}\} - \bar{X}_i^{m+1} \\
\vdots \\
X_{m+1}(N-m) = \{x_{N-m}, x_{N-m+1}, \ldots, x_N\} - \bar{X}_{N-m}^{m+1}
\]

where, \(\bar{X}_i^{m+1}\) is the mean of the sequences \(\{x_i, x_{i+1}, \ldots, x_{i+m}\}\) and \(\{x_i, x_{i+1}, \ldots, x_{i+m}\}\) respectively.

Step 2: Calculate the distance between the vectors. The distance between the vectors \(X_m(i), X_m(j)\) is defined as \(d\), then the distance between the two vectors can be defined as:

\[
\begin{align*}
&d[X_m(i), X_m(j)] = \max |x(i + k) - x(j + k)| \\
&d[X_m+1(i), X_m+1(j)] = \max |x(i + k) - x(j + k)|
\end{align*}
\]

Step 3: Calculate the sample entropy. Given a similar capacity \(r\), count the number that the distance between the vectors is less than the similarity capacity, and calculate the ratio to the total number \(A^m_r(i)\) and \(A^{m+1}_r(i)\). It is specifically defined as:

\[
\begin{align*}
&A^m_r(i) = \frac{d[X_m(i), X_m(j)] \leq r}{N - m - 1} \\
&A^{m+1}_r(i) = \frac{d[X_{m+1}(i), X_{m+1}(j)] \leq r}{N - m - 1}
\end{align*}
\]

The average of the reconstructed \(m\)-dimensional and \(m+1\)-dimensional sequences is recorded as \(B^m(r)\) and \(B^{m+1}(r)\), and the specifically defined is as following:

\[
\begin{align*}
&B^m(r) = \frac{1}{N - m} \sum_{i=1}^{N-m} A^m_r(i) \\
&B^{m+1}(r) = \frac{1}{N - m} \sum_{i=1}^{N-m} A^{m+1}_r(i)
\end{align*}
\]

The sample entropy calculation formula is:

\[\text{Samp}(m, r, N) = - \ln \frac{B^{m+1}(r)}{B^m(r)}\]

where, \(m, r, N\) are the dimensions of phase space, similarity tolerance, and number of selected data, select \(m = 2, r = 0.2 \ast SD, N = 15000, SD\) is the standard deviation of \(N\).

When the system parameters \(b = 1, q = 0.96, a \in [12, 20]\), the entropy complexity of the fractional-order Jafari-Sprott chaotic system is shown in Figure 11 (a). When the system parameters \(a = 15, b = 1, q \in [0.2, 1]\), the sample entropy complexity of the fractional-order Jafari-Sprott chaotic system is shown in Figure 11 (b). When the system parameters \(a = 15, q = 0.96, b \in [0.2, 1.2]\), the sample entropy complexity of the fractional-order Jafari-Sprott chaotic system is shown in Figure 11 (c).

When \(a \in [12, 20], q \in [0.2, 1]\), the chromatogram of the change of sample entropy complexity is shown in Figure 12 (a). When \(b \in [0.2, 1.2], q \in [0.2, 1]\), the chromatogram of the change of sample entropy complexity is shown in Figure 12 (b). When \(a \in [12, 20], b \in [0.2, 1.2]\),

The chromatogram of the change of sample entropy complexity is shown in Figure 12 (c).

### C. DISPERSION ENTROPY ALGORITHM

The dispersion entropy algorithm process [56] is as following:

Step 1: Map the sequence \(x_j(j = 1, 2, \ldots, N)\) to \(y_j(j = 1, 2, \ldots, N)\) by using a normal distribution, and then use a linear transformation to map \(y\) to the range of \([1, 2, \ldots, c]\)

\[z_j = \text{round} \left( cy_j + 0.5 \right)\]

where \(\text{round}\) and \(c\) respectively represent integer function and the number of categories.

Step 2: Calculate the embedded vector \(z_i^{m,c}\)

\[z_i^{m,c} = \{z_i^c, z_{i+d}^c, \ldots, z_{i+(m-1)d}^c\}\]
where \( i = 1, 2, \ldots, N - (m - 1) d \), \( m \) and \( d \) respectively represent embedding dimension and time delay.

Step 3: Create an embedding vector \( c \) with an embedding dimension \( m \) and a time delay \( d \). Each time series \( c \) is mapped to a decentralized pattern \( c \), where \( x, y = \{y_1, y_2, \ldots, y_N\} \).

Step 4: Calculate the probability \( p (\pi_{v_0, v_1, \ldots, v_{m-1}}) \) for each dispersion pattern \( \pi_{v_0, v_1, \ldots, v_{m-1}} \):

\[
p (\pi_{v_0, v_1, \ldots, v_{m-1}}) = \frac{N_b (\pi_{v_0, v_1, \ldots, v_{m-1}})}{N - (m - 1)d}
\]

where \( N_b (\pi_{v_0, v_1, \ldots, v_{m-1}}) \) represents the number of \( \xi_i^{m,c} \) mapped to \( \pi_{v_0, v_1, \ldots, v_{m-1}} \). So \( p (\pi_{v_0, v_1, \ldots, v_{m-1}}) \) can be expressed as the ratio of the number of \( \xi_i^{m,c} \) mapped to \( \pi_{v_0, v_1, \ldots, v_{m-1}} \) to the number of elements in \( \xi_i^{m,c} \).

Step 5: According to the definition of Shannon, the dispersion entropy of original time series is defined as:

\[
DE (x, m, c, d) = - \sum_{\pi=1}^{c^m} p (\pi_{v_0, v_1, \ldots, v_{m-1}}) \times \ln (p (\pi_{v_0, v_1, \ldots, v_{m-1}}))
\]

where \( x \) is the number of selected data, \( m \) is the embedding dimension, \( c \) is the category, and \( d \) is the time delay, select \( x = 15000, m = 2, c = 3, d = 1 \). When the system parameters \( b = 1, q = 0.96, a \in [12, 20] \), the dispersion entropy complexity of the fractional-order Jafari-Sprott chaotic system is shown in Figure 13 (a). When the system parameters \( a = 15, b = 1, q \in [0.2, 1] \), the dispersion entropy complexity of the fractional-order Jafari-Sprott chaotic system is shown in Figure 13 (b). When the system parameters \( a = 15, q = 0.96, b \in [0.2, 1.2] \), the dispersion entropy complexity of the fractional-order Jafari-Sprott chaotic system is shown in Figure 13 (c).

![FIGURE 13. Dispersion entropy complexity.](image)

When \( a \in [12, 20], q \in [0.2, 1] \), the chromatogram of the change of dispersion entropy complexity is shown in Figure 14 (a). When \( b \in [0.2, 1.2], q \in [0.2, 1] \), the chromatogram of the change of dispersion entropy complexity is shown in Figure 14 (b). When \( a \in [12, 20], b \in [0.2, 1.2] \), the chromatogram of the change of dispersion entropy complexity is shown in Figure 14 (c).

From Figure 9 (b), Figure 11 (b), and Figure 13 (b), the complexity is the largest at the 0.6-order, especially compared with the 1-order complexity, so we can conclude that the complexity of the fractional-order Jafari-Sprott chaotic system is greater than the integer-order Jafari-Sprott chaotic system. The maximum complexity detected by using three types of entropy when system parameter \( a, b \), and order \( q \) changed is shown in Table 1.

### TABLE 1. Comparison results of maximum complexity.

|            | Fuzzy entropy | Sample entropy | Dispersion entropy |
|------------|---------------|----------------|--------------------|
| \( a \)    | 0.009964      | 0.03686        | 1.14               |
| \( b \)    | 0.01076       | 0.03324        | 1.106              |
| \( q \)    | 0.06002       | 0.1994         | 1.249              |

According to Table 1, when using a single parameter as the independent variable, the complexity of the detection of dispersion entropy is larger than that of fuzzy entropy and sample entropy. In practical application, if the detection complexity is too small, it is not conducive to actual application.

By comparing the chromatograms in Figure 10, Figure 12, and Figure 14, we can obtain two conclusions:

1) Regardless of whether \( a, q, b, q \) or \( a, b \) are dual parameters, the color of the chromatogram of dispersion entropy is deeper than that of fuzzy entropy and sample entropy, which means that dispersion entropy can detect greater complexity not only under a single parameter, but also under the dual parameter.

2) By carefully observing the chromatograms of Figure 10, Figure 12, and Figure 14, it can be seen that dispersion entropy can detect areas where fuzzy entropy and sample entropy cannot be detected, which means that the detection area of dispersion entropy is widest.
In summary, comparing two aspects of the detection complexity value size and the detection area size, it is concluded that in the complexity detection of the fractional-order Jafari-Sprott chaotic system, the detection performance of dispersion entropy is the best.

V. SYNCHRONOUS CONTROL OF FRACTIONAL-ORDER JAFARI-SPROTT CHAOTIC SYSTEM

A. SYNCHRONIZATION OF FRACTIONAL-ORDER JAFARI-SPROTT CHAOTIC SYSTEM BASED ON SLIDING MODE CONTROL

The mathematical model of the fractional-order Jafari-Sprott chaotic system is:

\[
\begin{align*}
D_q^\theta x_1 &= x_2 \\
D_q^\theta x_2 &= -x_1 + x_2 x_3 \\
D_q^\theta x_3 &= -x_1 - ax_1 x_2 - bx_1 x_3
\end{align*}
\]  

(52)

Make (52) as the driving system, the response system is:

\[
\begin{align*}
D_q^\theta y_1 &= y_2 \\
D_q^\theta y_2 &= -y_1 + y_2 y_3 \\
D_q^\theta y_3 &= -y_1 - ay_1 y_2 - by_1 y_3
\end{align*}
\]  

(53)

The state error between the drive system and the response system is:

\[
\begin{align*}
e_1 &= y_1 - x_1 \\
e_2 &= y_2 - x_2 \\
e_3 &= y_3 - x_3
\end{align*}
\]  

(54)

Then the state error between the drive system and the response system is:

\[
\begin{align*}
D_q^\theta e_1 &= e_2 + u_1 \\
D_q^\theta e_2 &= -e_1 - e_2 e_3 + e_2 y_3 + e_3 y_2 + u_2 \\
D_q^\theta e_3 &= -e_1 + a(e_1 e_2 - e_1 y_2 - e_2 y_1) \\
&\quad + b(e_1 e_3 - e_1 y_3 - e_3 y_1) + u_3
\end{align*}
\]  

(55)

The control law design steps are as following:

Step 1: Introduce the fractional-order sliding mode surface. Three fractional-order sliding mode surfaces are:

\[
\begin{align*}
s_1(t) &= (D_q^\theta + \lambda_1) \int_{0}^{t} e_1(\tau) d\tau \\
s_2(t) &= (D_q^\theta + \lambda_2) \int_{0}^{t} e_2(\tau) d\tau \\
s_3(t) &= (D_q^\theta + \lambda_3) \int_{0}^{t} e_3(\tau) d\tau
\end{align*}
\]  

(56)

The first-order derivative of the synovial surface is:

\[
\begin{align*}
s_1'(t) &= D_q^\theta e_1(t) + \lambda_1 e_1(t) \\
s_2'(t) &= D_q^\theta e_2(t) + \lambda_2 e_2(t) \\
s_3'(t) &= D_q^\theta e_3(t) + \lambda_3 e_3(t)
\end{align*}
\]  

(57)

When the error system moves on the synovial surface, \( s_1(t) = 0 \) is satisfied. So the dynamic equation of the synovial surface is as follows:

\[
\begin{align*}
D_q^\theta e_1 &= -\lambda_1 e_1 \\
D_q^\theta e_2 &= -\lambda_2 e_2 \\
D_q^\theta e_3 &= -\lambda_3 e_3
\end{align*}
\]  

(58)

Step 2: Design the control law.

Design the first Lyapunov function as:

\[
V_1(t) = \frac{1}{2} s_1^2
\]  

(59)

Its first-order derivative can be obtained:

\[
\begin{align*}
V_1(t) &= s_1 s_1' \\
&= s_1(D_q^\theta e_1 + \lambda_1 e_1) \\
&= s_1(e_2 + u_1 + \lambda_1 e_1)
\end{align*}
\]  

(60)

We can get \( u_1 = -e_2 - \lambda_1 e_1 - k_1 s_1 \cdot \text{sign}(s_1) \).

Design the second Lyapunov function as:

\[
V_2(t) = \frac{1}{2} s_2^2
\]  

(61)

Its first-order derivative can be obtained:

\[
\begin{align*}
V_2(t) &= s_2 s_2' \\
&= s_2(D_q^\theta e_2 + \lambda_2 e_2) \\
&= s_2(-e_1 - e_2 e_3 + e_2 y_3 + e_3 y_2 + \lambda_2 e_2 + u_2)
\end{align*}
\]  

(62)

We can get \( u_2 = e_1 + e_2 e_3 - e_2 y_3 - e_3 y_2 - \lambda_2 e_2 - k_2 s_1 \cdot \text{sign}(s_2) \).

Design the third Lyapunov function as:

\[
V_3(t) = \frac{1}{2} s_3^2
\]  

(63)

Its first-order derivative can be obtained:

\[
\begin{align*}
V_3(t) &= s_3 s_3' \\
&= s_3(D_q^\theta e_3 + \lambda_3 e_3) \\
&= s_3(-e_1 + a(e_1 e_2 - e_1 y_2 - e_2 y_1) \\
&\quad + b(e_1 e_3 - e_1 y_3 - e_3 y_1) + \lambda_3 e_3 + u_3)
\end{align*}
\]  

(64)

We can get:

\[
\begin{align*}
u_3 &= e_1 - a e_1 e_2 + a e_1 y_2 + a e_2 y_1 - b e_2 e_3 + b e_2 y_3 + \\
&\quad b e_3 y_1 - \lambda_3 e_3 - k_3 s_1 \cdot \text{sign}(s_3)
\end{align*}
\]

where, \( \lambda_1, \lambda_2, \lambda_3 \) are synovial surface parameters, select \( \lambda_1 = \lambda_2 = \lambda_3 = 4 \), and \( k_1, k_2, k_3 \) are the gain of control law, select \( k_1 = k_2 = k_3 = 1 \).

The error graph between the drive and the corresponding system is shown in Figure 15.

From Figure 15, it can be seen that under the three control laws, the drive-response systems of fractional-order Jafari-Sprott chaotic system have completed synchronization, and the error tends to 0 with time, which illustrates the three control laws effectiveness and correctness.
B. FRACTIONAL-ORDER JAFARI-SPROTT CHAOTIC SYSTEM CONTROL BASED ON FREQUENCY DISTRIBUTION MODEL OF FRACTIONAL-ORDER INTEGRAL OPERATOR

The mathematical model of the fractional-order Jafari-Sprott chaotic system is:

\[
\begin{align*}
D_t^q x_1 &= x_2 \\
D_t^q x_2 &= -x_1 + x_2 x_3 \\
D_t^q x_3 &= -x_1 - \alpha_1 x_2 - bx_1 x_3
\end{align*}
\] (65)

Defining fractional-order systems \( D_t^q x(t) = y(t) \), where \( 0 < q < 1 \), it is equivalent to a linear continuous frequency distribution model [57]:

\[
\begin{align*}
\frac{\partial z(w, t)}{\partial t} &= -wz(w, t) + y(t) \\
x(t) &= \int_0^\infty \mu(w)z(w, t)dw
\end{align*}
\] (66)

where, weight function \( \mu(w) = \frac{\sin(\pi w)}{\pi w^q} \), system status \( z(w, t) \in \mathbb{R} \).

The control law design steps are as follows:

Step 1: For the first equation in the mathematical model of the fractional-order Jafari-Sprott chaotic system, define the new coordinates as:

\( w_1 = x_1 \) (67)

The dynamic equation of the first new coordinate is:

\[
D_t^q w_1 = x_2
\] (68)

According to the frequency distribution model of the fractional integral operator, Equation (68) is equivalent to the following equation:

\[
\begin{align*}
\frac{\partial z(w, t)}{\partial t} &= -wz(w, t) + x_2 \\
w_1(t) &= \int_0^\infty \mu(w)z_1(w, t)dw
\end{align*}
\] (69)

Select Lyapunov function as:

\[
V_1(t) = \frac{1}{2} \int_0^\infty \mu(w)z_1^2(w, t)dw
\] (70)

Its first-order derivative can be obtained:

\[
\dot{V}_1(t) = \int_0^\infty \mu(w)z_1(w, t) \frac{\partial z_1(w, t)}{\partial t} dw
\] (71)

Take Equation (69) into Equation (71) to get:

\[
\dot{V}_1(t) = \int_0^\infty \mu(w)z_1(w, t)(-wz_1(w, t) + x_2)dw
\]

\[
= - \int_0^\infty \mu(w)z_1^2(w, t)dw + x_2 \int_0^\infty \mu(w)z_1(w, t)dw
\]

\[
= - \int_0^\infty \mu(w)z_1^2(w, t)dw + x_2w_1
\]

\[
= - \int_0^\infty \mu(w)z_1^2(w, t)dw - w_1^2 + w_1(x_1 + x_2)
\] (72)

Step 2: Define the second new coordinate as:

\( w_2 = x_1 + x_2 \) (73)

The dynamic equation of the second new coordinate is:

\[
D_t^q w_2 = x_2 - x_1 + x_2 x_3
\] (74)

According to the frequency distribution model of the fractional integral operator, Equation (74) is equivalent to the following equation:

\[
\begin{align*}
\frac{\partial z(w, t)}{\partial t} &= -wz_2(w, t) + x_2 - x_1 + x_2 x_3 \\
w_2(t) &= \int_0^\infty \mu(w)z_2(w, t)dw
\end{align*}
\] (75)

Select Lyapunov function as:

\[
V_2(t) = V_1(t) + \frac{1}{2} \int_0^\infty \mu(w)z_2^2(w, t)dw
\] (76)

Its first-order derivative can be obtained:

\[
\dot{V}_2(t) = \dot{V}_1(t) + \int_0^\infty \mu(w)z_2(w, t) \frac{\partial z_2(w, t)}{\partial t} dw
\] (77)

By taking Equation (75) into Equation (77), we can get:

\[
\dot{V}_2(t) = \dot{V}_1(t) + \int_0^\infty \mu(w)z_2(w, t) \frac{\partial z_2(w, t)}{\partial t} dw
\]
where, $u$

\[
\frac{\partial \omega_1(t, x, y, z)}{\partial t} = -w_z^2(t, x_2 - x_1 + x_2x_3) + x_2 - x_1 + x_2x_3
\]

such that

\[
\frac{\partial \omega_2(t, x, y, z)}{\partial t} = w(t) - w_z^2(t, x_2 - x_1 + x_2x_3)
\]

\[
\frac{\partial \omega_3(t, x, y, z)}{\partial t} = 0
\]

The dynamic equation of the third new coordinate is

\[
\frac{\partial \omega_3(t, x, y, z)}{\partial t} = 0
\]

Step 3: Define the third new coordinate as:

\[
w_3 = x_1 + 2x_2 + x_2x_3
\]

The dynamic equation of the third new coordinate is obtained:

\[
D_1^\alpha w_3 = x_2 - 2x_1 + 2x_2x_3 - x_1x_3 - x_1x_2 + x_2x_3^2 - ax_1x_2^2 - bx_1x_2x_3 + u
\]

where, $u$ is the required control law. According to the frequency distribution model of the fractional integral operator, Equation (80) is equivalent to the following equation:

\[
\frac{\partial \omega_3(t, x, y, z)}{\partial t} = w(t) + \frac{\partial \omega(t, x, y, z)}{\partial t}
\]

\[
\omega(t, x, y, z) = \int_0^\infty w(t) \omega(t, x, y, z) \, dt
\]

Select Lyapunov function as:

\[
\dot{V}_3(t) = \dot{V}_2(t) + \int_0^\infty \mu(w) \omega_3(t, w) \, dt
\]

Its first-order derivative can be obtained:

\[
\dot{V}_3(t) = \dot{V}_2(t) + \int_0^\infty \mu(w) \omega_3(t, w) \, dt
\]

Take Equation (81) into Equation (83), we can get:

\[
\dot{V}_3(t) = \dot{V}_2(t) + \int_0^\infty \mu(w) \omega_3(t, w) \, dt
\]
From Equation (84) we get the law of control \( u = -4x_2 - 3x_2 x_3 + x_1 x_2 + x_1 x_3 - x_2 x_1^2 + ax_1 x_2^2 + bx_1 x_2 x_3 \), and the effect of each variable control is shown in Figure 16.

From Figure 16, we can see that under the action of the control law:
\[
u = -4x_2 - 3x_2 x_3 + x_1 x_2 + x_1 x_3 - x_2 x_1^2 + ax_1 x_2^2 + bx_1 x_2 x_3,
\]
each state variable approaches, which illustrates the effectiveness and correctness of the design of the control law. Compared with other control methods, the control method based on the frequency distribution model of the fractional integral operator has the advantage that it only needs to design a control law to complete the control of each state variable.

**VI. CONCLUSION**

Based on the Adomian decomposition method, this paper combines fuzzy entropy algorithm, sample entropy algorithm, and dispersion entropy algorithm by comparing detection complexity value size and detection area size, and the dispersion entropy algorithm in analyzing the complexity of fractional-order Jafari-Sprott chaotic system is the best. In addition, the complexity of the fractional-order Jafari-Sprott chaotic system is greater than the integer-order Jafari-Sprott chaotic system, and the complexity is the highest especially at 0.6-order. Compared with the integer-order Jafari-Sprott chaotic system, the fractional-order Jafari-Sprott system has more research significance compared with the integer-order Jafari-Sprott chaotic system, and the complexity of fractional-order Jafari-Sprott chaotic system is greater than the complexity of the integer-order Jafari-Sprott chaotic system. In addition, the complexity of fractional-order Jafari-Sprott chaotic system is accomplished by sliding model control and frequency distribution model of fractional-order integral operator respectively. In particular, the control effect of each variable is accomplished by designing a control law based on the frequency distribution model of fractional integral operator. The advantage of this method is that you only need to design a control law to complete the control of three state variables, at the same time the designed control law contains only state variables, and does not include integer or fractional order derivatives of state variables, so it is easy to implement. The system studied in the synchronization control in this paper is ideal, but in practice, many systems have external interference. How to complete the system’s synchronous control in the presence of external interference is worthy of further study.

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