Finding a unique texture for quark mass matrices

Samandeep Sharma, Priyanka Fakay, Gulsheen Ahuja∗, Manmohan Gupta
Department of Physics, Panjab University, Chandigarh, India.

∗gulsheen@pu.ac.in

March 16, 2015

Abstract

Within the Standard Model, starting with the most general mass matrices we have used the facility of making weak basis transformations and have imposed the condition of ‘naturalness’ to carry out their analysis within the texture zero approach. Interestingly, our analysis reveals that a particular set of texture 4 zero quark mass matrices can be considered to be a unique viable option for the description of quark mixing data.

One of the key challenges in the present day high energy physics is to understand vast spectrum of fermion masses and their relationships with the corresponding mixing angles as well as mass matrices. Despite impressive advances in the measurements of fermion masses and mixing parameters, we are far from having a compelling theory for flavor physics. Even for the case of quarks, where precision measurements are available, the data is understood in terms of phenomenological models having their roots in the ‘bottom up’ approach. In this context, exploring the possibility of finding a minimal set of viable quark mass matrices can perhaps be the first important step for solving the flavor riddle.

Without loss of generality, in the Standard Model (SM) and its extensions where right handed quarks are singlets, fermion mass matrices can be considered to be hermitian using ‘Polar Decomposition Theorem’, for details we refer the reader to a recent review by Gupta and Ahuja [1]. The hermitian matrices in the up and down sectors are characterized by 18 free parameters which are still large in number as compared to the 10 observables i.e. 6 quark masses, 3 mixing angles and 1 CP violating phase. In order that these matrices provide valuable clues for developing theory of flavor physics, it is desirable that following the bottom up approach the number of free parameters of these are constrained by invoking certain broad and general guidelines [1, 2].

The bottom up approach of understanding fermion masses and mixings has essentially evolved in three different directions. Firstly, on the lines of Fritzsch ansatze [3], mass matrices are formulated wherein certain elements of these are assumed to be zero, usually referred to as texture zeros, and the compatibility of the mixing
matrix so obtained from these with the low energy data ensures the viability of the formulated mass matrices. It has been shown [2, 4] that both hermiticity and texture zeros remain largely preserved while carrying out the renormalization group evolution of the texture specific mass matrices, therefore enabling one to formulate these at $M_Z$ scale. Despite showing considerable promise, in this approach the possibility to arrive at a minimal set of viable quark mass matrices emerges only by carrying out an exhaustive case by case analysis of all possible texture zero mass matrices [1].

Besides the above mentioned approach, within the framework of SM and its extensions, one has the freedom to make unitary transformations, referred to as ‘Weak Basis (WB) transformations’, which change the mass matrices without changing the quark mixing matrix. Using WB transformations, several attempts [5]-[7] have been made wherein the above freedom is exploited to introduce texture zeros in the quark mass matrices. This results in somewhat reducing the number of free parameters of general mass matrices, however, in the absence of any constraints on the elements of the mass matrices, leads to a large number of texture zero matrices which are able to explain the quark mixing data [7].

In yet another approach, advocated by Peccei and Wang [8], the concept of ‘natural mass matrices’ has been introduced to formulate viable set of mass matrices at the Grand Unified Theories (GUTs) as well as the $M_Z$ scale. In order to avoid fine tuning, the elements of the mass matrices are constrained in order to reproduce the hierarchical nature of the quark mixing angles. This results in constraining the parameter space available to the elements of the mass matrices, however without yielding a finite set of viable mass matrices at the GUTs as well as the $M_Z$ scale.

A careful perusal of the above mentioned approaches suggests that none of these lead to a finite set of viable texture specific mass matrices, therefore in order to obtain the same perhaps one needs to combine the three. The idea is to follow the texture zero approach coupled with WB transformations to reduce the number of free parameters of general hermitian mass matrices as well as to impose the condition of ‘naturalness’ for constraining the parameter space available to the elements of these. The purpose of the paper, therefore, is to start with the most general mass matrices and consequently explore the possibility of obtaining a finite set of viable texture specific mass matrices formulated by using weak basis transformations as well as the constraints imposed due to naturalness.

To begin with, we consider the following general hermitian mass matrices

$$M_q = \begin{pmatrix} E_q & A_q & F_q \\ A_q^* & D_q & B_q \\ F_q^* & B_q^* & C_q \end{pmatrix} \quad (q = U, D), \tag{1}$$

which are related to the most general mass matrices in the SM [11]. As a next step, one can introduce texture zeros in these matrices using the WB transformations [5], in particular, one can find a unitary matrix $W$ transforming $M_U \rightarrow W^\dagger M_U W$ and
$M_D \rightarrow W^\dagger M_D W$, leading to

$$
M_U = \begin{pmatrix}
E_U & A_U & 0 \\
A_U^* & D_U & B_U \\
0 & B_U^* & C_U
\end{pmatrix}, \quad M_D = \begin{pmatrix}
E_D & A_D & 0 \\
A_D^* & D_D & B_D \\
0 & B_D^* & C_D
\end{pmatrix}.
$$

(2)

The above matrices, wherein $A_q = |A_q|e^{i\alpha_q}$ and $B_q = |B_q|e^{i\beta_q}$ for $q = U, D$, can be characterized as texture 2 zero quark mass matrices. It should be noted that for $M_U$ and $M_D$ instead of zeros being in the (1,3) and (3,1) positions, these could also be in either the (1,2) and (2,1) or (2,3) and (3,2) position. These other structures are related to the above mentioned matrices as we have the facility of subjecting $M_U$ and $M_D$ to another WB transformation which can be the permutation matrix $P$. These different mass matrices, however, yield the same mixing matrix, therefore while discussing the results of the analysis, it is sufficient to discuss any one of these matrices. Therefore, the matrices given in equation (2) can now be considered as most general in the context of SM. Further, in order to incorporate the condition of ‘naturalness’ on these mass matrices, the following condition is imposed on the elements of the matrices [9]

$$(1, i) < (2, j) \lesssim (3, 3); \quad i = 1, 2, 3, \quad j = 2, 3.$$  

(3)

After obtaining these matrices, as a next step, their viability needs to be ensured by examining the compatibility of the Cabibbo-Kobayashi-Maskawa (CKM) matrix reproduced through these mass matrices with the recent quark mixing data. This also enables one to check what constraints are put on the elements of these mass matrices.

Before getting into the details of the analysis, we first present some of the essentials pertaining to the construction of the CKM matrix from these mass matrices. To facilitate diagonalization, for $q = U, D$, the mass matrix $M_q$ may be expressed as $M_q = Q_q^\dagger M_q^r Q_q$ implying $M_q^r = Q_q M_q Q_q^\dagger$ where $M_q^r$ is a symmetric matrix with real eigenvalues and $Q_q$ is the diagonal phase matrix, e.g.,

$$
M_q^r = \begin{pmatrix}
E_q & |A_q| & 0 \\
|A_q| & D_q & |B_q| \\
0 & |B_q| & C_q
\end{pmatrix}, \quad Q_q = \begin{pmatrix}
e^{-i\alpha_q} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{i\beta_q}
\end{pmatrix}.
$$

(4)

The matrix $M_q^r$ can be diagonalized using the following transformations

$$
M_q^{\text{diag}} = O_q^T M_q^r O_q = O_q^T Q_q M_q Q_q^\dagger O_q = \text{Diag}(m_1, -m_2, m_3),
$$

(5)

where the subscripts 1, 2 and 3 refer respectively to $u, c, t$ for the up sector and $d$, $s, b$ for the down sector. The exact diagonalizing transformation $O_q$ for the matrix
\[ M^r_q \] is given by

\[
O_q = \begin{pmatrix}
\sqrt{(E_u+m_2)(m_3-E_u)(m_3-m_1)} & \sqrt{(m_1-E_u)(m_3-E_u)(C_3+C_2)} & \sqrt{(m_1-E_u)(E_u+m_2)(m_3-C_2)} \\
(C_u-E_u)(m_3-m_1)(m_2+m_1) & (C_u-E_u)(m_3-m_1)(m_2+m_1) & (C_u-E_u)(m_3-m_1)(m_2+m_1) \\
(C_3-m_1)(m_1-E_u)(m_2+m_1) & (C_3-m_1)(m_1-E_u)(m_2+m_1) & (C_3-m_1)(m_1-E_u)(m_2+m_1) \\
-\sqrt{(m_1-E_u)(m_3-C_2)(C_3+m_2)} & -\sqrt{(E_u+m_2)(C_3+m_2)} & -(m_3-E_u)(m_3-C_2) \\
(C_u-E_u)(m_3-m_1)(m_2+m_1) & (C_u-E_u)(m_3-m_1)(m_2+m_1) & (C_u-E_u)(m_3-m_1)(m_2+m_1) \\
\end{pmatrix}.
\]

(6)

Further, these diagonalizing transformations are related to the mixing matrix as

\[
V_{\text{CKM}} = O^T_U Q_U Q_D^T O_D.
\]

(7)

It should be noted that for the construction of the CKM matrix, the elements \( E_U \), \( E_D \), \( D_U \) and \( D_D \) of the mass matrices have been considered as free parameters.

The inputs used for the purpose of calculations, the quark masses and the mass ratios at the \( M_Z \) scale [10], are

\[
\begin{align*}
  m_u &= 1.38^{+0.42}_{-0.41} \text{ MeV}, \quad m_d = 2.82 \pm 0.48 \text{ MeV}, \quad m_s = 57^{+18}_{-12} \text{ MeV}, \\
  m_c &= 0.638^{+0.043}_{-0.084} \text{ GeV}, \quad m_b = 2.86^{+0.16}_{-0.06} \text{ GeV}, \quad m_t = 172.1 \pm 1.2 \text{ GeV}, \\
  m_u/m_d &= 0.553 \pm 0.043, \quad m_s/m_d = 18.9 \pm 0.8.
\end{align*}
\]

(8)

The latest values [11] of precisely measured CKM parameters required for the construction of the CKM matrix pertaining to three mixing angles and one CP violating phase are

\[
\begin{align*}
  |V_{us}| &= 0.22534 \pm 0.00065, \quad |V_{ub}| = 0.00351^{+0.00015}_{-0.00014}, \quad |V_{cb}| = 0.0412^{+0.0011}_{-0.0005}, \\
  \text{Sin}2\beta &= 0.679 \pm 0.020.
\end{align*}
\]

(9)

Coming to the analysis of the mass matrices \( M_U \) and \( M_D \), the parameters \( \phi_1 \) and \( \phi_2 \), related to the phases of the mass matrices, \( \phi_1 = \alpha_U - \alpha_D \) and \( \phi_2 = \beta_U - \beta_D \), have been given full variation from 0 to \( 2\pi \). Apart from \( \phi_1 \) and \( \phi_2 \), the free parameters \( E_U \), \( E_D \), \( D_U \) and \( D_D \) have also been given wide variation in conformity with the condition of naturalness as well as to ensure that the elements of \( O_U \) and \( O_D \) should remain real. As mentioned earlier, the viability of matrices \( M_U \) and \( M_D \) can be checked by examining the compatibility of the CKM matrix so reproduced, therefore, using the relation between mass matrices and mixing matrix, given in equation (7), the resultant CKM matrix is obtained as follows

\[
V_{\text{CKM}} = \begin{pmatrix}
0.9739 - 0.9745 & 0.2246 - 0.2259 & 0.00337 - 0.00365 \\
0.2224 - 0.2259 & 0.9730 - 0.9990 & 0.0408 - 0.0422 \\
0.0076 - 0.0101 & 0.0408 - 0.0422 & 0.9990 - 0.9999
\end{pmatrix},
\]

(10)

being fully compatible with the one given by PDG [11]. Also, the CP violating Jarlskog’s rephasing invariant parameter \( J \) comes out to be \((2.494 - 3.365) \times 10^{-5}\) which again is compatible with its latest experimental range, \((2.96^{+0.20}_{-0.16}) \times 10^{-5}\).

The above mentioned compatibility leads to the viability of the general mass
matrices $M_U$ and $M_D$, however, since the number of free parameters associated with these matrices is larger than the number of observables, therefore, it becomes interesting to examine whether any of their elements is redundant. To this end, we present below the magnitudes of the elements of matrices $M_U$ and $M_D$ which reproduce the CKM matrix given in equation (10)

$$M_U = \begin{pmatrix} 0 - 0.00138 & 0.006 - 0.042 & 0 \\ 0.006 - 0.042 & 26.46 - 102.68 & 62.82 - 86.10 \\ 0 & 62.82 - 86.10 & 68.78 - 145.00 \end{pmatrix} \text{GeV}, \quad (11)$$

$$M_D = \begin{pmatrix} 0 - 0.00127 & 0.011 - 0.019 & 0 \\ 0.011 - 0.019 & 0.36 - 1.66 & 1.03 - 1.44 \\ 0 & 1.03 - 1.44 & 1.16 - 2.44 \end{pmatrix} \text{GeV}. \quad (12)$$

A closer look at the above matrices reveals that their (1,1) element is quite small in comparison with the other non-zero elements. This brings up the issue whether the elements $E_U$ and $E_D$ of the matrices $M_U$ and $M_D$ respectively can be ignored all together without loss of parameter space. To confirm this, one should examine the effect of the variation of these parameters on CKM matrix elements. To this end, in Figure (11) we have plotted the variation of the element $|V_{us}|$ and the CP asymmetry parameter $\sin 2\beta$, two of the best determined CKM parameters, with the mass matrix element $E_U$. As is evident from these plots, not only for reproducing the experimental ranges of $|V_{us}|$ and $\sin 2\beta$, mentioned in equation (9), the parameter $E_U$ assumes quite small values, < 0.0014 GeV, but also both $|V_{us}|$ and $\sin 2\beta$ seem independent of the range of $E_U$, indicating the redundancy of element $E_U$. Similar conclusions can be drawn from $E_U$ versus the other CKM matrix elements plots. In the down sector, similar plots pertaining to (1,1) element $E_D$ of the matrix $M_D$ reveal that again this parameter is also quite small and essentially redundant.

These conclusions can be understood analytically also by examining the exact transformation $O_q$, $q = U, D$, given in equation (10). Interestingly, the parameters $E_U$ and $E_D$ being of the order of the smallest mass eigenvalue $m_1$ can be ignored when these appear with $m_2$, $m_3$ and $C_q$ which are much larger than $m_1$. However, at other places $E_U$ and $E_D$ appear with $m_1$ as $(m_1 - E_q)$ which essentially implies re-scaling of $m_1$.

Keeping in mind the above discussion, ignoring the elements $E_U$ and $E_D$ of the mass matrices, one gets $M_U$ and $M_D$ as

$$M_U = \begin{pmatrix} 0 & A_U & 0 \\ A_U^* & D_U & B_U \\ 0 & B_U^* & C_U \end{pmatrix}, \quad M_D = \begin{pmatrix} 0 & A_D & 0 \\ A_D^* & D_D & B_D \\ 0 & B_D^* & C_D \end{pmatrix}, \quad (13)$$

indicating a transition from texture 2 zero mass matrices to texture 4 zero mass matrices. Carrying out a similar analysis for these matrices, the corresponding
Figure 1: Plots showing the dependence of $V_{us}$ and $\sin 2\beta$ on the parameter $E_U$.

CKM matrix comes out to be

$$V_{CKM} = \begin{pmatrix}
0.9741 - 0.9744 & 0.2246 - 0.2259 & 0.00337 - 0.00365 \\
0.2245 - 0.2258 & 0.9732 - 0.9736 & 0.0407 - 0.0422 \\
0.0071 - 0.0100 & 0.0396 - 0.0417 & 0.9990 - 0.9992
\end{pmatrix}. \quad (14)$$

This matrix is not only in agreement with the latest quark mixing matrix given by PDG [11], but is also fully compatible with the CKM matrix given in equation (10). Further, the range of the CP violating Jarlskog’s rephasing invariant parameter $J$ comes out to be $(2.50 - 3.37) \times 10^{-5}$ which again is compatible with its latest experimental range, justifying our earlier conclusion that the elements $E_U$ and $E_D$ are essentially redundant as far as reproducing the CKM parameters are concerned.

It becomes interesting to examine how the parameter space of the elements of the matrices $M_U$ and $M_D$ gets changed on going from texture 2 to texture 4 zero. To this end, reconstructing $M_U$ and $M_D$ we get

$$M_U = \begin{pmatrix}
0 & 0.031 - 0.041 & 0 \\
0.031 - 0.041 & 13.73 - 98.62 & 47.70 - 85.80 \\
0 & 47.70 - 85.80 & 72.84 - 157.73
\end{pmatrix} \text{ GeV,} \quad (15)$$

$$M_D = \begin{pmatrix}
0 & 0.012 - 0.018 & 0 \\
0.012 - 0.018 & 0.18 - 1.56 & 0.81 - 1.45 \\
0 & 0.81 - 1.45 & 1.24 - 2.61
\end{pmatrix} \text{ GeV.} \quad (16)$$

Interestingly, as expected, the above matrices appear to be quite compatible with the earlier mentioned matrices in equations (11) and (12) and the parameter space of the elements of the two also remains almost the same, again confirming the redundancy of elements $E_U$ and $E_D$.

It may be noted that apart from the form of texture 4 zero mass matrices considered above, there are several other possible texture 4 zero structures [1]. Based on whether the matrices are related through permutations or not, all possible texture 4 zero mass matrices can be classified as shown in Table (1). The matrices which are not related to each other through permutations have been put into different categories. For the matrices belonging to category 1, considering both $M_U$ and $M_D$
Table 1: Table showing all possible texture 2 zero quark mass matrices, classified into four different categories.

| Category 1 | a | b | c | d | e | f |
|------------|---|---|---|---|---|---|
|             | (0 0 0) | (0 0 0) | (D A B) | (C B 0) | (D B A) | (C 0 B) |
|             | A' D B | A C B | A' 0 0 | B' D A | B' C 0 | 0 0 A |
|             | 0 B' C | 0 B' C | 0 A' 0 | 0 A' 0 | 0 A' 0 | 0 B' A' |

| Category 2 | (D A 0) | (D 0 0) | (C 0 B) | (C 0 B) | (0 B A) | (0 B A) |
|------------|---------|---------|---------|---------|---------|---------|
|             | A' 0 B | A C B | A' 0 0 | 0 B' A' | 0 D A | 0 D A |
|             | 0 B' C | 0 B' C | 0 A' D | 0 A' D | 0 A' D | 0 A' D |

| Category 3 | (0 A D) | (0 D A) | (C 0 B) | (C 0 B) | (0 B A) | (0 B A) |
|------------|---------|---------|---------|---------|---------|---------|
|             | A' 0 B | A D C B | C B 0 | C B 0 | 0 B A | 0 B A |
|             | 0 B' C | 0 B' C | A' D C | A' D C | A' D C | A' D C |

| Category 4 | (A 0 0) | (C 0 0) | (A 0 0) | (D 0 0) | (C 0 B) | (0 B A) |
|------------|---------|---------|---------|---------|---------|---------|
|             | 0 D B | 0 A 0 | 0 0 A | 0 0 B' | 0 C B | 0 A 0 |
|             | 0 B' C | 0 B' C | 0 B' D | 0 B' D | 0 B' D | 0 B' D |

as 1a type, corresponding to the ones mentioned in equation (13), we have already shown that these are viable and explain the quark mixing data quite well. The other matrices of this category, related through permutation matrix, also yield similar results. For the matrices belonging to category 4, one finds that interestingly these are not viable as in all these matrices one of the generations gets decoupled from the other two. Further, for categories 2 and 3, again a similar numerical analysis reveals that the matrices of these classes are also not viable as can be understood from the following CKM matrices obtained for categories 2 and 3 respectively, e.g.,

\[
V_{\text{CKM}} = \begin{pmatrix}
0.9740 - 0.9744 & 0.2247 - 0.2260 & 0.0024 - 0.0099 \\
0.2205 - 0.2256 & 0.9509 - 0.9727 & 0.0596 - 0.2172 \\
0.0140 - 0.0445 & 0.0584 - 0.2127 & 0.9905 - 1.0000
\end{pmatrix}, \quad (17)
\]

\[
V_{\text{CKM}} = \begin{pmatrix}
0.9736 - 0.9744 & 0.2247 - 0.2260 & 0.0098 - 0.0331 \\
0.2226 - 0.2278 & 0.9549 - 0.9719 & 0.0659 - 0.1937 \\
0.00007 - 0.0340 & 0.0694 - 0.1928 & 0.9810 - 0.9976
\end{pmatrix}. \quad (18)
\]

A look at these matrices shows that there are certain elements which do not agree with their corresponding values given by PDG [11].

To check this rigorously, in Figure (2) we have plotted the precisely measured CKM matrix element \( |V_{cb}| \) against the phases \( \phi_1 \) and \( \phi_2 \) for the matrices belonging to categories 2 and 3. While plotting these graphs, we have constrained the value of element \( V_{us} \) by its experimental bounds given in equation (9), whereas full variation has been given to the other parameters. From these graphs, one finds that since the plotted values of element \( |V_{cb}| \) have no overlap with its experimental range, therefore, these matrices can be considered to be non viable.

The above discussion clearly brings out that only the texture 4 zero quark mass matrices belonging to category 1 of the table are found to be viable. Interestingly, the matrices given in equation (13) are quite similar to the original Fritzsch ansatze, except for their (2,2) element being non zero for both \( M_U \) and \( M_D \). In case one considers texture specific mass matrices with zeros more than 4, we find that the present data rules out all possible texture 5 and 6 zero quark mass matrices, confirming our
earlier conclusions in this regard [1].

![Graphs showing the variation of the magnitude of $V_{cb}$ with phases $\phi_1$ and $\phi_2$ for quark mass matrices belonging to categories 2 and 3 respectively.](image)

Figure 2: Plots showing the variation of the magnitude of $V_{cb}$ with phases $\phi_1$ and $\phi_2$ for quark mass matrices belonging to categories 2 and 3 respectively.

To summarize, within the context of SM, starting with the most general mass matrices, we have used the freedom of making WB transformations to reduce these to texture 2 zero quark mass matrices. Imposing the condition of ‘naturalness’ within the texture zero approach, one finds that certain elements of these matrices can be considered as essentially redundant and therefore reducing the matrices to texture 4 zero type. Numerical analysis of all possible texture 4 zero mass matrices lead to a finite set of these which can be considered as a unique viable option for the description of quark mixing data. This texture structure for quarks could be the first step towards unified textures for all fermions.

**Acknowledgements**

S.S. and G.A. respectively thank UGC and DST, Government of India (Grant No: SR/FTP/PS-017/2012) for financial support. S.S., P.F., G.A. acknowledge the Chairperson, Department of Physics, P.U., for providing facilities to work. M.G. would like to thank Walter Grimus for useful discussions.

**References**

[1] M. Gupta and G. Ahuja, Int. J. Mod. Phys. A 27, 1230033 (2012).

[2] H. Fritzsch and Z. Z. Xing, Prog. Part. Nucl. Phys. 45, 1 (2000).
[3] H. Fritzsch, Phys. Lett. B 70, 436 (1977); 73, 317 (1978).
[4] P. Ramond, R. G. Roberts and G. G. Ross, Nucl. Phys. B 406, 19 (1993).
[5] H. Fritzsch and Z. Z. Xing, Phys. Lett. B 413, 396 (1997); Nucl. Phys. B 556, 49 (1999).
[6] G. C. Branco, D. Emmanuel-Costa and R. Gonzalez Felipe, Phys. Lett. B 477, 147 (2000); G.C. Branco, D. Emmanuel-Costa, R. Gonzalez Felipe and H. Sahoo, Phys. Lett. B 670, 340 (2009).
[7] D. Emmanuel-Costa and C. Simoes, Phys. Rev. D 79, 073006 (2009).
[8] R. D. Peccei and K. Wang, Phys. Rev. D 53, 2712 (1996).
[9] S. Sharma, P. Fakay, G. Ahuja and M. Gupta, arXiv:1404.5726.
[10] Z.Z. Xing, H. Zhang and S. Zhou, Phys. Rev. D 86, 013013 (2012).
[11] J. Beringer et al., Particle Data Group, Phys. Rev. D 86, 010001 (2012).