ON THE PARABOLIC DONALDSON’S EQUATION OVER A COMPACT COMPLEX MANIFOLD

LIANGDI ZHANG

Abstract. We prove the uniqueness and long time existence of the smooth solution to a parabolic Donaldson’s equation on a compact complex manifold $M$. Then we show that a suitably normalized solution converges to a smooth function on $M$ in $C^\infty$ topology as $t \to \infty$.

Keywords: Parabolic Donaldson’s equation; Long time existence; Smooth convergence

2020 Mathematics Subject Classification: 53C55; 35K10; 35B45; 58J35

CONTENTS

1. Introduction
2. Short time existence and second-order estimates
3. Zero-order estimate
4. $C^\infty$ estimate and long time existence
5. Li-Yau estimate and Harnack inequality
6. Smooth convergence
Acknowledgements
References

1. Introduction

Let $M$ be a compact complex manifold of complex dimension $n$ ($n \geq 2$) with $\chi$ is a Hermitian metric and $\omega$ is another Kähler metric on $M$. In local coordinates,

$\chi = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ and $\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$.

For a given real-valued smooth function $F$ on $M$ satisfying

$\chi - \frac{n-1}{ne^F} \omega > 0$,

we study the parabolic Donaldson’s equation

$$\frac{\partial \varphi}{\partial t} = \log \frac{\chi^n_{\varphi}}{\omega \wedge \chi^{n-1}_{\varphi}} + F$$ (1.1)

with $\chi_{\varphi} := \chi + \sqrt{-1} \partial \bar{\partial} \varphi > 0$ and the initial value $\varphi(\cdot,0) = 0$.

(1.1) is a parabolic analog of the Donaldson’s equation

$$\omega \wedge \chi^{n-1}_{\varphi} = e^F \chi^n_{\varphi}.$$ (1.2)

In 2014, Y. Li [20] proved second-order, zero-order and $C^\infty$ estimates for (1.2) on an $n$-dimensional compact Hermitian manifold equipped with two Hermitian
metrics $\chi$ and $\omega$ satisfying $\chi - \frac{n-1}{n} \omega > 0$. Y. Li [20] also raised a question that whether the condition of $\chi - \frac{n-1}{n} \omega > 0$ is sufficient to produce a solution to (1.2) on a compact Hermitian manifold since it was proven to be true when both $\omega$ and $\chi$ are Kähler (see [5, 32, 33]). In this paper, we will provide a parabolic proof of this question in the special case of $\omega$ is Kähler and $F$ is modified by adding a suitable constant.

On a Kähler manifold $M$ with two Kähler classes $[\omega]$ and $[\chi]$, The J-flow is a parabolic flow in the space of Kähler potentials

$$H_\chi := \{ \varphi \in C^\infty(M, \mathbb{R}) | \chi_\varphi := \chi + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \}$$

and is defined by

$$\frac{\partial \varphi}{\partial t} = c - \frac{\omega \wedge \chi_{\varphi}^{n-1}}{\chi_{\varphi}^{n}}$$

(1.3)

with $\varphi \in H_\chi$ and the constant $c := \frac{\int_M \omega \wedge \chi^n}{\int_M \chi^n}$. The J-flow was introduced by S. K. Donaldson [10] in the setting of moment maps and independently by X. Chen [4, 5] in the study of Mabuchi energy ([22]). X. Chen [5] proved that the J-flow (1.3) always exists for all the time for any smooth initial data, and converges to a smooth critical metric $\chi_0$ satisfying

$$\omega \wedge \chi_0^{n-1} = c \chi_0^n$$

(1.4)

if the bisectional curvature of $\omega$ is semi-positive.

Historically, S. K. Donaldson [10] considered the equation (1.4) on compact Kähler manifolds and showed that a necessary condition for the existence of a solution $\chi_0$ to (1.4) in $[\chi]$ is $[n \chi] - [\omega] > 0$ and conjectured that this condition is also sufficient. When the complex dimension is 2, X. Chen [4] proved Donaldson’s conjecture by reducing (1.4) to a complex Monge-Ampère equation. B. Weinkove [32, 33] provided a parabolic proof of Donaldson’s conjecture by using the J-flow. Furthermore, J. Song and B. Weinkove [23] showed that the J-flow converges in $C^\infty$ to a smooth critical metric $\chi_0 \in [\chi]$ satisfying (1.4) if and only if there exists a metric $\chi' \in [\chi]$ with

$$(n \chi' - (n-1) \omega) \wedge \chi'^{n-2} > 0.$$ 

More recently, G. Chen [3] proved that there exists a unique Kähler metric $\chi_0$ satisfying (1.4) up to a constant if and only if $(M, [\omega], [\chi])$ is uniformly $J$-stable. Please refer to [8, 9, 12, 15, 18, 27, 31, 34, 36] for more related works on the J-flow.

The Calabi conjecture states that for any given $(1,1)$-form $\tilde{R}$ representing the first Chern class of a compact Kähler manifold $(M, \omega)$, there exists a unique Kähler metric in $[\omega]$ whose Ricci form is $\tilde{R}$. E. Calabi [1] transformed his conjecture into solving the complex Monge-Ampère equation on an $n$-dimensional compact Kähler manifold:

$$\omega + \sqrt{-1} \partial \bar{\partial} \phi = e^F \omega^n$$

(1.5)

with $\omega + \sqrt{-1} \partial \bar{\partial} \phi > 0$. By using the maximum principle, E. Calabi [1] proved the uniqueness (up to adding a constant) of the solution to (1.5). S. T. Yau [35] completely solved Calabi’s conjecture in 1970s by proving the existence of the smooth solution $\phi$ to (1.5) when $F$ satisfies the normalization of $\int_M e^F \omega^n = \int_M \omega^n$ through the continuity method.

V. Tosatti and B. Weinkove [29] obtained a uniform $C^\infty$ estimate, which depends only on $M$, $\omega$ and $F$, for a solution $\phi$ to (1.5) on the compact Hermitian manifold.
(M, ω) without additional assumptions. Based on this estimate, they [29] proved that there exists a unique smooth solution to (1.5) up to adding a constant to F and solved a Hermitian version of the Calabi conjecture that every representative of the first Bott-Chern class of M can be represented as the first Chern form of a Hermitian metric of the form ω + √−1∂∂ϕ. By adding a suitable constant to F, J. Chu, V. Tosatti and B. Weinkove [7] proved existence and uniqueness of smooth solutions to (1.5) on a compact almost complex manifold with non-integrable almost complex structure.

In 1980s, H. D. Cao [2] proved the long time existence of smooth solution to the parabolic complex Monge-Ampère equation
\[
\frac{∂ϕ}{∂t} = \log \frac{(ω + √−1∂∂ϕ)^n}{ω^n} - F \tag{1.6}
\]
on a compact Kähler manifold (M, ω) with initial condition ϕ(·, 0) = 0, and showed that the normalized solution \( v = ϕ - \frac{1}{Vol(M)} \int_M ω^n \) to (1.6) converges in \( C^∞ \) to a unique smooth solution to (1.5) up to adding a constant. On compact Hermitian manifolds, M. Gill [14] proved \( C^∞ \) convergence for the normalized solution \( \tilde{ϕ} = ϕ - \int_M ω^n \) to (1.6) and this provided a parabolic proof of the result of uniqueness and existence of smooth solutions to (1.5) by V. Tosatti and B. Weinkove [29]. J. Chu [6] obtained analogous results on compact almost Hermitian manifolds.

On a compact Kähler manifold (M, ω) of complex dimension \( n \), T. D. Tô [25] proved uniqueness and stability for the following complex Monge-Ampère flow
\[
\frac{∂ϕ}{∂t} = \log \frac{(θ_t + √−1∂∂ϕ)^n}{ω^n} - F(t, z, ϕ) \tag{1.7}
\]
where \((θ_t)_{t∈[0,T]}\) is a family of Kähler forms with \( θ_0 = ω \) and \( F \) is a smooth function. Furthermore, he [26] generalized the results to the Hermitian case. More recently, J. Zhou and Y. Chu [37] proved long-time existence and smooth convergence of the unique solution to a parabolic Hessian quotient equation on compact Hermitian manifolds by assuming some conditions on its parabolic \( C^{-}\)-subsolutions.

In [24], W. Sun considered the parabolic complex Monge-Ampère type equations on a compact Hermitian manifold (M, ω) of complex dimension \( n \geq 2 \):
\[
\frac{∂ϕ}{∂t} = \log \frac{χ_{ϕ}^n}{χ_{ϕ}^{n-a} ∧ ω^a} + F \tag{1.7}
\]
with \( χ_ϕ := χ + √−1∂∂ϕ > 0 \) and the initial value \( ϕ(·, 0) = 0 \), where \( F ∈ C^∞(M) \), \( 1 ≤ a ≤ n \) and \( χ \) is another Hermitian metric. W. Sun proved long time existence of a solution to (1.7) based on its \( C^∞ \) estimate and showed smooth convergence of the normalized solution \( \tilde{ϕ} = ϕ - \frac{1}{Vol(M)} \int_M ω^n \) under a cone condition of
\[
[χ] ∈ 𝒞_a(F) := \{[χ] : \text{ there exists } χ' ∈ [χ]^+, \ nχ' > (n-a)e^{-F}χ'^{n-a-1} ∧ ω^a \} \tag{1.8}
\]
with \([χ]^+ := \{χ' ∈ [χ] : χ' > 0\} \), and
\[
e^{-F} ≥ \frac{χ^n}{ω ∧ χ^{n-1}}. \tag{1.9}
\]
Particularly, if both ω and χ are Kähler, the condition (1.9) can be weaken to
\[
e^{-F} ≥ \frac{1}{Vol(M)} \frac{χ^n}{χ^{n-a} ∧ ω^a}. \tag{1.10}
\]
Motivated by the work of [14, 20, 24, 28], we prove uniqueness and long time existence for solutions and smooth convergence of a suitably normalized solution to (1.1). The main theorems of this paper are below.

**Theorem 1.1.** Let $M$ be an $n$-dimensional $(n \geq 2)$ compact complex manifold equipped with a Hermitian metric $\chi$ and a Kähler metric $\omega$. Then there exists a unique smooth solution $\varphi$ to the parabolic Donaldson’s equation (1.1) on $M \times [0, \infty)$.

**Theorem 1.2.** Let $M$ be an $n$-dimensional $(n \geq 2)$ compact complex manifold equipped with a Hermitian metric $\chi$ and a Kähler metric $\omega$. Let $\varphi$ be a smooth solution to the parabolic Donaldson’s equation (1.1) and

$$\tilde{\varphi} := \varphi - \int_M \varphi \omega^n.$$  \hfill (1.11)

Then $\tilde{\varphi}$ converges in $C^\infty$ topology to a smooth function $\tilde{\varphi}_\infty$ as $t \to \infty$.

Although the parabolic Donaldson’s equation (1.1) is a special case of (1.7) by taking $a = 1$, the main theorems may still be of interest since we only assuming $\omega$ is Kähler and $\chi - \frac{n-1}{n} \omega > 0$, which is easier to verify than the cone condition (1.8), without the assumption of (1.9) or (1.10).

The paper is arranged as follows. In Section 2, we prove uniqueness, short time existence and second-order estimates of a smooth solution $\varphi$ to (1.1). Based on the second-order estimates, we derive a zero-order estimate for $\varphi$ in Section 3. In Section 4, we show a uniform $C^\infty$ estimate and prove Theorem 1.1. In Section 5, we derive a Li-Yau type gradient estimate and a Harnack estimate for positive solutions to a linear parabolic PDE. We apply the Harnack estimate to finish the proof of Theorem 1.2 and prove a existence result of solutions to the Donaldson’s equation (1.2) in Section 6.

### 2. Short time existence and second-order estimates

In this section, we prove uniqueness and short time existence of a smooth solution $\varphi$ to (1.1) first. Then we derive estimates for $\text{tr}_\omega \chi_\varphi$, which serves as a connecting link between zero-order and higher-order estimates for $\varphi$.

Adopting the notation in [20], we define a Hermitian metric $h$ on $M$ with its inverse matrix is locally given by

$$h^{ij} = \chi_\varphi^{ij} \chi_\varphi^{-1} g_{kl}$$

and a second-order elliptic operator by

$$L := (\text{tr}_\chi \omega)^{-1} \Delta_h = (\text{tr}_\chi \omega)^{-1} h^{ij} \partial_i \partial_j.$$  \hfill (2.1)

Throughout this paper, $C$ denotes a positive uniform constant that depends only on $M$, $\chi$, $\omega$ and $F$.

**Theorem 2.1.** Let $M$ be an $n$-dimensional $(n \geq 2)$ compact complex manifold equipped with a Hermitian metric $\chi$ and a Kähler metric $\omega$. There exists a unique smooth solution $\varphi$ to the parabolic Donaldson’s equation (1.1) on $M \times [0, T)$, where $[0, T)$ is the maximal time interval for some $0 < T \leq \infty$.

**Proof.** Note that

$$\omega \wedge \chi_\varphi^{n-1} = \frac{1}{n} (\text{tr}_\chi \omega) \chi_\varphi^{n}.$$  \hfill (2.2)
Then \((1.1)\) is equivalent to
\[
\frac{\partial \varphi}{\partial t} = - \log(\text{tr}_\varphi \omega) + \log n + F
\]
with initial value \(\varphi(\cdot, 0) = 0\).

Denote the right-hand-side of \((2.3)\) by \(P\). It is clear that
\[
\frac{\partial P}{\partial (\partial, \partial^i \varphi)} = \frac{\partial (\log(\text{tr}_\varphi \omega))}{\partial (\partial, \partial^i \varphi)} = \frac{\chi^p_{\varphi} \chi^q_{\varphi} \delta_{ki}^j}{\text{tr}_\varphi \omega} \cdot \frac{\partial (\chi_{\varphi} \omega)}{\partial (\partial, \partial^i \varphi)} = \frac{h^i_j}{\text{tr}_\varphi \omega} > 0.
\]
By standard parabolic theory, there exists a smooth solution \(\varphi\) to \((2.3)\) on the maximal time interval \([0, T)\) for some \(0 < T \leq \infty\).

Then we estimate \(\frac{\partial \varphi}{\partial t}\) by the maximum principle for linear second order parabolic PDEs as in \([14, 6]\).

**Lemma 2.2.** Let \(\varphi\) be a solution to \((1.1)\) on \(M \times [0, T)\), then we have
\[
\sup_{M \times [0, T)} \left| \frac{\partial \varphi}{\partial t} \right| \leq C.
\]

**Proof.** It follows from \((2.2)\) that
\[
\frac{\partial}{\partial t} (\omega \wedge \chi^{n-1}_\varphi) = \frac{1}{n} \frac{\partial}{\partial t} (\text{tr}_\varphi \omega) \chi^n_{\varphi} + \frac{1}{n} \frac{\partial}{\partial t} (\text{tr}_\varphi \omega) \frac{\partial}{\partial t} \chi^n_{\varphi}
\]
\[
= \chi^n_{\varphi} \frac{\partial}{\partial t} (\chi^j_i \omega) + \frac{\omega \wedge \chi^{n-1}_\varphi}{\chi^n_{\varphi}} \cdot \frac{\partial}{\partial t} \chi^n_{\varphi}
\]
\[
= - \frac{\omega \wedge \chi^{n-1}_\varphi}{\text{tr}_\varphi \omega} \cdot \chi^k_i \chi^j_l \frac{\partial (\chi_{\varphi} \omega)}{\partial t} \delta_{ij}^k + \frac{\omega \wedge \chi^{n-1}_\varphi}{\chi^n_{\varphi}} \cdot \frac{\partial}{\partial t} \chi^n_{\varphi}
\]
\[
= - \omega \wedge \chi^{n-1}_\varphi \cdot L \frac{\partial \varphi}{\partial t} + \frac{\omega \wedge \chi^{n-1}_\varphi}{\chi^n_{\varphi}} \cdot \frac{\partial}{\partial t} \chi^n_{\varphi},
\]
i.e.,
\[
L \frac{\partial \varphi}{\partial t} = \frac{\partial}{\partial t} \frac{\omega \wedge \chi^{n-1}_\varphi}{\chi^n_{\varphi}} = \frac{\partial}{\partial t} \frac{\omega \wedge \chi^{n-1}_\varphi}{\omega \wedge \chi^{n-1}_\varphi}
\]
\[
= \frac{\partial}{\partial t} \log \chi^n_{\varphi} - \frac{\partial}{\partial t} \log (\omega \wedge \chi^{n-1}_\varphi)
\]
\[
= \frac{\partial}{\partial t} (\frac{\partial \varphi}{\partial t}),
\]
where we used \((1.1)\).

Therefore, the maximum principle implies that
\[
\sup_{M \times [0, T)} \left| \frac{\partial \varphi}{\partial t} \right| \leq \sup_M \left| \frac{\partial \varphi}{\partial t} (\cdot, 0) \right|
\]
\[
= \sup_M \left| \log \frac{\chi^n_{\varphi}}{\omega \wedge \chi^{n-1}_\varphi} + F \right| \leq C
\]
on \(M \times [0, T)\).

Choose a holomorphic normal coordinates system for the Kähler metric \(\omega\) centered at an arbitrary point \(x \in M\) so that
\[
g_{ij}(x) = \delta_{ij}\text{ and } \partial_k g_{ij}(x) = 0
\]
for all $i, j$ and $k$. In this coordinate system, $\chi_\varphi(x, t)$ (for any fixed $t \in [0, T]$) is diagonal so that
\[
(\chi_\varphi)_{ij}(x, t) = \lambda_i \delta_{ij}
\]
for some $\lambda_1, \cdots, \lambda_n > 0$. Moreover,
\[
h^{ij}(x, t) = \frac{1}{\lambda_i^2} \delta_{ij}.
\]

In this section, $E_1$ denotes a error term satisfying $|E_1|_\omega \leq C$. Without loss of generality, we can assume
\[
|E_1| \leq C tr_\omega \chi_\varphi.
\]
Otherwise the upper bounds for $tr_\omega \chi_\varphi$ follows immediately.

Define a quantity
\[
Q := \log tr_\omega \chi_\varphi + A \left( \sup_{M \times [0, T]} \varphi - \varphi \right),
\]
where $A$ is a nonnegative constant depending only on $M, \chi, \omega$ and $F$ to be determined later.

**Lemma 2.3.** There exists a positive constant $C_0$ that depends only on $M, \chi, \omega, and F$ so that following inequality holds on $M \times [0, T]$.

\[
(L - \frac{\partial}{\partial t})Q \geq \frac{C_0}{tr_\omega \chi_\varphi} - Ae^{-\frac{\omega}{\omega}} + A \frac{tr_\chi}{tr_\omega \chi_\varphi}.
\]

**Proof.** Calculating directly, we have
\[
(L - \frac{\partial}{\partial t})Q
\]
\[
= \left( \frac{\Delta h}{tr_\omega \chi_\varphi} - \frac{\partial}{\partial t} \right) \left[ \log tr_\omega \chi_\varphi + A \left( \sup_{M \times [0, T]} \varphi - \varphi \right) \right]
\]
\[
= \frac{\Delta h tr_\omega \chi_\varphi}{(tr_\omega \chi_\varphi)^2} - \frac{h^{ij} \partial_i (tr_\omega \chi_\varphi) \partial_j (tr_\omega \chi_\varphi)}{(tr_\omega \chi_\varphi)^2} - \frac{1}{tr_\omega \chi_\varphi} \Delta \frac{\partial \varphi}{\partial t} - AL \varphi + A \frac{\partial \varphi}{\partial t}.
\]

In the following, we deal with the right-hand-side of (2.8). Fixing $t \in [0, T)$, we compute in the holomorphic normal coordinate system mentioned before at an arbitrary point $(x, t) \in M \times [0, T)$.

By the same computation in [20] (see Equation (2.11), Equation (2.14) and Lines 5 to 9 at Page 874) and the fact of $d\omega = 0$, we have
\[
\Delta h tr_\omega \chi_\varphi - \frac{h^{ij} \partial_i (tr_\omega \chi_\varphi) \partial_j (tr_\omega \chi_\varphi)}{tr_\omega \chi_\varphi}
\]
\[
\geq \sum_{1 \leq i, k \leq n} h^{ik} \partial_k (\chi_\varphi)_{ii} - \sum_{1 \leq i, k \leq n} h^{ik} \partial_i g_{kk} \cdot (\chi_\varphi)_{kk}
\]
\[
+ \sum_{1 \leq i, k \leq n} h^{ik} (\partial_i \partial_k \chi_{kk} - \partial_k \partial_i \chi_{ii})
\]
\[
- \sum_{1 \leq i, j \leq n} h^{ij} \chi_\varphi^{jj} \partial_i (\chi_\varphi)_{jj} \partial_i (\chi_\varphi)_{jj}.
\]
It follows from (2.3) that
\[ tr_{\chi^\varphi} \omega = n \exp \{ F - \frac{\partial \varphi}{\partial t} \}. \quad (2.10) \]

Combining (2.4) and (2.10), we have
\[ tr_{\chi^\varphi} \omega \leq C. \quad (2.11) \]

Furthermore,
\[ \chi_{ij}^\varphi = \frac{\delta_{ij}}{\lambda_i} \leq tr_{\chi^\varphi} \omega \leq C, \quad (2.12) \]

and
\[ h_{ij} = \frac{\delta_{ij}}{\lambda_i} \leq (tr_{\chi^\varphi} \omega)^2 \leq C \quad (2.13) \]

at \((x, t)\) for all \(i, j \in \{1, 2, \ldots, n\}\).

Applying Cauchy’s inequality and (2.13) to (2.9) yields
\[
\Delta_h tr_{\omega \chi^\varphi} - \frac{h_{ij} \partial_i (tr_{\omega \chi^\varphi}) \partial_j (tr_{\omega \chi^\varphi})}{tr_{\omega \chi^\varphi}} \\
\geq \sum_{1 \leq i, k \leq n} h_{ii} \partial_k \partial_{\varphi}(\chi^\varphi)_{ii} + E_1 \\
- \sum_{1 \leq i, j \leq n} h_{ij} \chi_{ij}^\varphi \partial_i (\chi^\varphi)_{jj} \partial_i (\chi^\varphi)_{jj}.
\]

(2.14)

Note that
\[
\Delta_h \frac{\partial \varphi}{\partial t} \\
= g_{ij} \partial_i \partial_j tr_{\chi^\varphi} \omega \]
\[
= g_{ij} \partial_i (\chi^\varphi)^2 \partial_j (\chi^\varphi)_{ij} \log(\omega \wedge \chi^\varphi) + \Delta_h F \]
\[
= g_{ij} \partial_i \left( \frac{\partial_j (tr_{\chi^\varphi} \omega) \chi^\varphi + tr_{\chi^\varphi} \omega \partial_j (tr_{\chi^\varphi} \omega)}{n \omega \wedge \chi^\varphi} \right) + E_1 \\
= g_{ij} \partial_i \left( \frac{\partial_j tr_{\chi^\varphi} \omega}{tr_{\chi^\varphi} \omega} + \frac{\partial_j (tr_{\chi^\varphi} \omega)}{\chi^\varphi} \right) + E_1 \\
= - \frac{g_{ij} \partial_i \partial_j tr_{\chi^\varphi} \omega}{tr_{\chi^\varphi} \omega} + \frac{g_{ij} \partial_i (\partial_j tr_{\chi^\varphi} \omega) (\partial_j tr_{\chi^\varphi} \omega)}{(tr_{\chi^\varphi} \omega)^2} + E_1,
\]

(2.15)

where we used (1.1) in the first equality, and (2.2) in the second and third equalities.

Furthermore, we can get
\[
g_{ij} \partial_i \partial_j tr_{\chi^\varphi} \omega \\
= g_{ij} \partial_i (-\chi^\varphi_{\ell q} \chi^\varphi_{\ell q} \partial_j (\chi^\varphi)_{\ell q} g_{k\ell} + \chi^\varphi_{\ell q} \partial_j g_{k\ell}) \\
= g_{ij} \chi^\varphi_{\ell q} \chi^\varphi_{\ell q} \partial_i (\chi^\varphi)_{\ell q} g_{k\ell} + g_{ij} \chi^\varphi_{\ell q} \chi^\varphi_{\ell q} \partial_i (\chi^\varphi)_{\ell q} g_{k\ell} \\
+ g_{ij} \chi^\varphi_{\ell q} \chi^\varphi_{\ell q} \partial_i (\chi^\varphi)_{\ell q} g_{k\ell} \\
- g_{ij} \chi^\varphi_{\ell q} \chi^\varphi_{\ell q} \partial_i (\chi^\varphi)_{\ell q} g_{k\ell} \\
- g_{ij} \chi^\varphi_{\ell q} \chi^\varphi_{\ell q} \partial_i (\chi^\varphi)_{\ell q} g_{k\ell} + g_{ij} \chi^\varphi_{\ell q} \chi^\varphi_{\ell q} \partial_i (\chi^\varphi)_{\ell q} g_{k\ell} \\
= g_{ij} h_{k\ell} \chi^\varphi_{\ell q} \partial_i (\chi^\varphi)_{\ell q} g_{k\ell} + g_{ij} h_{k\ell} \chi^\varphi_{\ell q} \partial_i (\chi^\varphi)_{\ell q} g_{k\ell} \\
- g_{ij} h_{k\ell} \chi^\varphi_{\ell q} \partial_i (\chi^\varphi)_{\ell q} g_{k\ell} + g_{ij} h_{k\ell} \chi^\varphi_{\ell q} \partial_i (\chi^\varphi)_{\ell q} g_{k\ell}
\[
\begin{align*}
&= \sum_{1 \leq i,j,k \leq n} h^{ij} \chi^{ij}_{\varphi} \partial_k (\chi_{\varphi})_{ij} \partial_k (\chi_{\varphi})_{ij} \\
&+ \sum_{1 \leq i,j,k \leq n} h^{ij} \chi^{ij}_{\varphi} \partial_k (\chi_{\varphi})_{ji} \\
&- \sum_{1 \leq i,k \leq n} h^{ii} \partial_k \partial_k (\chi_{\varphi})_{ii} + E_1,
\end{align*}
\]
(2.16)

and
\[
\begin{align*}
g^{ij}(\partial_i \text{tr}_{\chi_{\varphi}} \omega)(\partial_j \text{tr}_{\chi_{\varphi}} \omega) &= g^{ij}(\partial_i \chi_{\varphi}^{kl} g_{kl})(\partial_j \chi_{\varphi}^{qp} g_{qp}) \\
&= g^{ij}(-h^{kl} \partial_i (\chi_{\varphi})_{kl})(-h^{qp} \partial_j (\chi_{\varphi})_{qp}) \\
&= \sum_{1 \leq i,j,k \leq n} h^{ij} h^{kl} \partial_k (\chi_{\varphi})_{ii} \partial_k (\chi_{\varphi})_{jj}.
\end{align*}
\]
(2.17)

Plugging (2.16) and (2.17) into (2.15) and then combining (2.14) and (2.15), we can obtain
\[
\begin{align*}
\frac{\Delta h_{\text{tr}_{\omega} \chi_{\varphi}}}{\text{tr}_{\chi_{\varphi}} \omega} &= \frac{h^{ij}\partial_i (\text{tr}_{\omega} \chi_{\varphi}) \partial_j (\text{tr}_{\omega} \chi_{\varphi})}{(\text{tr}_{\omega} \chi_{\varphi})(\text{tr}_{\omega} \chi_{\varphi})} - \Delta_{\omega} \frac{\partial \varphi}{\partial t} \\
&\geq \frac{E_1}{\text{tr}_{\chi_{\varphi}} \omega} + E_1 \\
&- \frac{1}{\text{tr}_{\chi_{\varphi}} \omega} \sum_{1 \leq i,j \leq n} h^{ij} \chi^{ij}_{\varphi} \partial_i (\chi_{\varphi})_{ij} \partial_i (\chi_{\varphi})_{ij} \\
&+ \frac{1}{\text{tr}_{\chi_{\varphi}} \omega} \sum_{1 \leq i,j,k \leq n} h^{ij} \chi^{ij}_{\varphi} \partial_i (\chi_{\varphi})_{jj} \partial_k (\chi_{\varphi})_{ij} \\
&+ \frac{1}{\text{tr}_{\chi_{\varphi}} \omega} \sum_{1 \leq i,j,k \leq n} h^{ij} \chi^{ij}_{\varphi} \partial_i (\chi_{\varphi})_{ij} \partial_k (\chi_{\varphi})_{ji} \\
&- \frac{1}{(\text{tr}_{\chi_{\varphi}} \omega)^2} \sum_{1 \leq i,j,k \leq n} h^{ij} h^{kl} \partial_k (\chi_{\varphi})_{ii} \partial_k (\chi_{\varphi})_{jj}.
\end{align*}
\]
(2.18)

It follows from Cauchy’s inequality, (2.6) and (2.11) that
\[
\begin{align*}
\frac{\Delta h_{\text{tr}_{\omega} \chi_{\varphi}}}{\text{tr}_{\chi_{\varphi}} \omega} &- \frac{\|\nabla \text{tr}_{\omega} \chi_{\varphi}\|^2}{\text{tr}_{\chi_{\varphi}} \omega} - \Delta_{\omega} \frac{\partial \varphi}{\partial t} \\
&\geq \frac{E_1}{\text{tr}_{\chi_{\varphi}} \omega} + E_1 \\
&- \frac{C \text{tr}_{\omega} \chi_{\varphi}}{\text{tr}_{\chi_{\varphi}} \omega} - C \text{tr}_{\omega} \chi_{\varphi} \\
&\geq -\frac{C \text{tr}_{\omega} \chi_{\varphi}}{\text{tr}_{\chi_{\varphi}} \omega}
\end{align*}
\]
(2.19)

for some positive constant \(C_0\) that depends only on \(M, \chi, \omega\) and \(F\).

By direct computation, we get
\[
L\varphi = \frac{1}{\text{tr}_{\chi_{\varphi}} \omega} h^{ij} \partial_i \partial_j \varphi \\
= \frac{1}{\text{tr}_{\chi_{\varphi}} \omega} h^{ij} [(\chi_{\varphi})_{ij} - \chi_{ij}]
\]
Applying (2.19) and (2.20) to (2.8), we have
\[
(L - \frac{\partial}{\partial t})Q \geq - \frac{C_0}{\text{tr}_\chi \omega} - A(1 - \frac{\text{tr}_h \chi}{\text{tr}_\chi \omega}) + A \frac{\partial \varphi}{\partial t} \geq - \frac{C_0}{\text{tr}_\chi \omega} - Ae^{-\frac{\varphi}{\alpha}} + A \frac{\text{tr}_h \chi}{\text{tr}_\chi \omega},
\]
where we used the fact of \( e^s \geq 1 + s \) for all \( s \in \mathbb{R} \).

The following elementary result presented by Y. Li [20] (see Lemma 2.6 in [20]) is necessary to derive a upper bound for \( \text{tr}_\omega \chi \varphi \).

**Proposition 2.4 ([20]).** Let \( \lambda_1, \cdots, \lambda_n \) be a sequence of positive numbers. Suppose
\[
0 \geq 1 - \alpha \sum_{1 \leq i \leq n} \frac{1}{\lambda_i} + \beta \sum_{1 \leq i \leq n} \frac{1}{\lambda_i^2}
\]
for some \( \alpha, \beta > 0 \) and \( n \geq 2 \). If
\[
\frac{4}{n} \leq \frac{\alpha^2}{\beta} < \frac{4}{n - 1}
\]
holds, then
\[
\lambda_i \leq \frac{2\beta}{\alpha - \sqrt{n\alpha^2 - 4\beta}}
\]
for each \( i \).

Now we are ready to estimate \( \text{tr}_\omega \chi \varphi \).

**Theorem 2.5.** Let \( M \) be an \( n \)-dimensional \((n \geq 2)\) compact complex manifold equipped with a Hermitian metric \( \chi \) and a Kähler metric \( \omega \). If \( \varphi \) is a smooth solution of parabolic Donaldson’s equation (1.1) on \( M \times [0, T) \), then there exist positive constants \( A_1, C_1 \) and \( C_2 \), which depend only on \( M, \chi, \omega \) and \( F \), so that
\[
C_1 \leq \text{tr}_\omega \chi \varphi \leq C_2 e^{A_1(\varphi - \inf_{M \times [0, T)} \varphi)}.
\]

**Proof.** Firstly, we estimate lower bound for \( \text{tr}_\omega \chi \varphi \).

Under the holomorphic normal coordinate system for \( \omega \) mentioned before at an arbitrary point \((x, t) \in M \times [0, T)\),
\[
\text{tr}_\omega \chi \varphi = \sum_{1 \leq i \leq n} \lambda_i \geq \frac{n}{\sum_{1 \leq i \leq n} \frac{1}{\lambda_i}} = \frac{n}{\text{tr}_\chi \omega} \geq C_1,
\]
where we used (2.11) to get the last inequality.

Then we derive the upper bound for \( \varphi \).

For any fixed \( T_0 \in [0, T) \), set \((x_0, t_0)\) be a maximum point for \( Q \) on \( M \times [0, T_0] \). By Lemma 2.3 and the fact of \( \text{tr}_\chi \omega > 0 \), we have
\[
A \text{tr}_h \chi - Ae^{-\frac{\varphi}{\alpha}} \text{tr}_\chi \omega - C_0 \leq 0
\]
at such point.
For simplicity, we define
\[ \tilde{F} = F - \frac{\partial \varphi}{\partial t}. \]

Combining (2.10) and (2.23), we obtain
\[ \operatorname{Atr} h \chi - (A e^{-\frac{\partial \varphi}{F}} + B) \sum_{1 \leq i \leq n} \frac{1}{\lambda_i} + (B n e^{\tilde{F}} - C_0) \leq 0 \]  
(2.24)

at \((x_0, t_0)\) for some constant \(B\) to be determined later.

Since \(\chi - \frac{n-1}{ne^{\tilde{F}}} > 0\), there exists a constant \(0 < \epsilon_0 \leq \frac{1}{n-1}\) so that \(\chi \geq \frac{(n-1)(1 + \epsilon_0)}{ne^{\tilde{F}}} \omega \) at \((x_0, t_0)\). Moreover, we have
\[ \operatorname{tr} h \chi \geq \left(1 + \frac{\epsilon_0}{n-1}\right) \sum_{1 \leq i \leq n} \frac{1}{\lambda_i} \]

at \((x_0, t_0)\).

It follows from (2.24) that
\[ \frac{(n-1)(1 + \epsilon_0)A}{ne^{\tilde{F}(x_0,t_0)}} \sum_{1 \leq i \leq n} \frac{1}{\lambda_i} - \left(\frac{2(n-1)(1 + \epsilon_0)}{n-\xi} - 1\right) B \sum_{1 \leq i \leq n} \frac{1}{\lambda_i} + (B n e^{\tilde{F}} - C_0) \leq 0 \]  
(2.25)

at \((x_0, t_0)\).

Suppose
\[ B n e^{\tilde{F}(x_0,t_0)} - C_0 > 0. \]  
(2.26)

Define
\[ \alpha := \frac{A e^{-\frac{\partial \varphi}{F}(x_0,t_0)} + B}{B n e^{\tilde{F}(x_0,t_0)} - C_0} \]  
(2.27)

and
\[ \beta := \frac{(n-1)(1 + \epsilon_0)A}{ne^{\tilde{F}(x_0,t_0)}(B n e^{\tilde{F}(x_0,t_0)} - C_0)}. \]  
(2.28)

We need to find positive constants \(A\) and \(B\) so that (2.26) holds and
\[ \frac{\alpha^2}{\beta} = \frac{4}{n-\xi} \]  
(2.29)

for some constant \(\xi \in [0, 1)\). An elementary computation yields that (2.29) is equivalent to
\[ \left[ A e^{-\frac{\partial \varphi}{F}(x_0,t_0)} - \left(\frac{2(n-1)(1 + \epsilon_0)}{n-\xi} - 1\right) B \right]^2 \]
\[ = \left[ \left(\frac{2(n-1)(1 + \epsilon_0)}{n-\xi} - 1\right)^2 - 1 \right] B^2 - \frac{4(n-1)(1 + \epsilon_0)AC_0}{(n-\xi)ne^{\tilde{F}(x_0,t_0)}}. \]  
(2.30)

Fix \(\xi\) such that \(0 \leq 1 - (n-1)\epsilon_0 < \xi < 1\) and define
\[ B := \frac{C_0[(1 + 2\epsilon_0)n - 2(1 + \epsilon) + \xi]}{(\epsilon_0 n - \epsilon_0 - 1 + \xi)ne^{\tilde{F}(x_0,t_0)}} \]

and
\[ A := \frac{2(n-1)(1 + \epsilon_0)}{n-\xi} B e^{-\frac{\partial \varphi}{F}(x_0,t_0)} \]
\[ = \frac{[(1 + 2\epsilon_0)n - 2\epsilon_0 - 2 + \xi][(1 + 2\epsilon_0)n - 2(1 + \epsilon) + \xi]C_0}{n(n-\xi)(\epsilon_0 n - \epsilon_0 - 1 + \xi)ne^{\tilde{F}(x_0,t_0)}}. \]
Then (2.30) holds. It is clear that $A, B > 0$ and

$$Bn\epsilon \hat{F}(x_0, t_0) - C_0 = \frac{(1 + \epsilon_0)(n - 1)}{\epsilon_0 n - \epsilon_0 - 1 + \xi} > 0.$$ 

Therefore, Proposition 2.4 and (2.4) implies that

$$\lambda_i(x_0, t_0) \leq \frac{2\beta}{\alpha - \sqrt{n\alpha^2 - 4\beta}} \leq C$$

for each $i$ with $\alpha, \beta$ are constants defined in (2.27) and (2.28). It follows that

$$tr_\omega \chi \varphi(x_0, t_0) \leq \sum_{1 \leq i \leq n} \lambda_i(x_0, t_0) \leq \frac{2n\beta}{\alpha - \sqrt{n\alpha^2 - 4\beta}} \leq C.$$ 

Hence,

$$Q = \log tr_\omega \chi \varphi + A(\sup_{M \times [0, T]} \varphi - \varphi)$$

$$\leq tr_\omega \chi \varphi(x_0, t_0) + A(\sup_{M \times [0, T]} \varphi - \varphi(x_0, t_0))$$

$$\leq C + A(\sup_{M \times [0, T]} \varphi - \inf_{M \times [0, T]} \varphi),$$

on $M \times [0, T_0]$, i.e.,

$$\log tr_\omega \chi \varphi \leq C + A(\varphi - \inf_{M \times [0, T]} \varphi)$$

(2.31)

on $M \times [0, T_0]$.

Choose

$$A_1 := \frac{[(1 + 2\epsilon_0)n - 2\epsilon_0 - 2 + \xi][(1 + 2\epsilon_0)n - 2(1 + \epsilon) + \xi]C_0}{n(n - \xi)(\epsilon_0 n - \epsilon_0 - 1 + \xi)e^{\inf_M F - \sup_{M \times [0, T]} |\partial\varphi|}}.$$ 

Then $A_1 \geq A$ and $A_1$ depends only on $M, \chi, \omega$ and $F$ by (2.4).

Since $T_0$ is arbitrary and both $A_1$ and $C$ are independent of the choice of $T_0$, we can conclude from (2.31) that

$$tr_\omega \chi \varphi \leq C_2 e^{A_1(\varphi - \inf_{M \times [0, T]} \varphi)}$$

(2.32)

on $M \times [0, T]$.

(2.21) follows by combining (2.22) and (2.32).

3. Zero-order estimate

In this section, we derive a $C^0$ estimate for a smooth solution $\varphi$ to (1.1) directly from Theorem 2.1.

It is remarked by V. Tosatti and B. Weinkov in [28] that the $C^0$ estimate for solutions to the complex Monge-Ampère equation (1.5) on a compact Hermitian manifold follows from its $C^2$ estimate $tr_\omega \omega \varphi \leq C e^{A(\varphi - \inf_M \phi)}$ and does not make use of the equation (1.5) itself. As we have obtained Theorem 2.1, we can apply the $C^0$ estimate arguments in [28] (see also [30, 33]) to the parabolic Donaldson’s equation (1.1).

Note that $0 < tr_\omega \chi \leq C$ and

$$\Delta_\omega \varphi = tr_\omega \chi \varphi - tr_\omega \chi.$$ 

Theorem 2.1 yields

$$-C \leq \Delta_\omega \varphi \leq C e^{A_1(\varphi - \inf_{M \times [0, T]} \varphi)}.$$
Fix any \( t \in [0, T) \) and substitute \( \varphi(\cdot, t) = \sup_{M \times [0,T]} \varphi \) for \( \varphi \) in Lemma 3.1, Lemma 3.2 and Lemma 3.3 in Tosatti-Weinkove [28] and following their proof step by step, we can obtain same results:

**Lemma 3.1.** Let \( a \) be a positive constant, there exists a constant \( D_a \) depending only on \( M, \chi, \omega, F \) and \( a \) so that for every \( t \in [0, T) \),

\[
\sup_{M \times [0,T]} \varphi - \inf_{M \times [0,T]} \varphi \leq D_a + \log \left( \int_{M} e^{-a(\varphi(\cdot, t) - \sup_{M \times [0,T]} \varphi)} d\mu \right)^{\frac{1}{a}} \tag{3.1}
\]

where \( d\mu = \omega^n / \int_M \omega^n \).

**Lemma 3.2.** Let \( D_1 \) be the constant corresponding to \( a = 1 \) from Lemma 3.1. Then for every fixed \( t \in [0, T) \),

\[
|\{ \varphi(\cdot, t) \leq \inf_{M \times [0,T]} \varphi + D_1 + 1 \}| \geq \frac{1}{4eD_1} \tag{3.2}
\]

where \( |\cdot| \) the measure of a set with respect to the measure \( d\mu = \omega^n / \int_M \omega^n \).

**Lemma 3.3.** For every \( t \in [0, T) \), we have

\[
\sup_{M \times [0,T]} \varphi - \int_{M} \varphi(\cdot, t) d\mu \leq C. \tag{3.3}
\]

We are ready to finish the proof of \( C^0 \) estimate for \( \varphi \).

**Theorem 3.4.** Let \( M \) be an \( n \)-dimensional \((n \geq 2)\) compact complex manifold equipped with a Hermitian metric \( \chi \) and a Kähler metric \( \omega \). If \( \varphi \) is a smooth solution of parabolic Donaldson’s equation (1.1) on \( M \times [0, T) \), then there exists a positive constant \( C_3 \) depending only on \( M, \chi, \omega, F \), so that

\[
\sup_{M \times [0,T]} \varphi - \inf_{M \times [0,T]} \varphi \leq C_3. \tag{3.4}
\]

**Proof.** Combining Lemma 3.2 and Lemma 3.3, we have

\[
C \geq \sup_{M \times [0,T]} \varphi - \int_{M} \varphi(\cdot, t) d\mu \geq \frac{1}{4eD_1}(\sup_{M \times [0,T]} \varphi - \inf_{M \times [0,T]} \varphi - D_1 - 1). \tag{3.5}
\]

Hence,

\[
\sup_{M \times [0,T]} \varphi - \inf_{M \times [0,T]} \varphi \leq C \tag{3.5}
\]

for all \( M \times [0, T) \). \( \square \)

4. \( C^\infty \) ESTIMATE AND LONG TIME EXISTENCE

In this section, we finish the proof of Theorem 1.1. We can achieve a uniform \( C^\infty \) bound on \( \varphi \) through a standard bootstrapping procedure once a \( C^\alpha \) \((0 < \alpha < 1)\) estimate is obtained as Tosatti-Weinkove [28] and M. Gill [14] did for the complex Monge-Ampère equation and the parabolic complex Monge-Ampère equation on compact Hermitian manifolds, respectively. We will appeal to a method due to L. C. Evans [11] and N. V. Krylov [16, 17] (see also in the textbooks [13, 21]) to prove the required Hölder estimate.

**Proposition 4.1.** Let \( \gamma \in \mathbb{C}^n \) be an arbitrary vector, then we have

\[
- \frac{\partial}{\partial t} \partial_{\gamma} \partial_{\gamma} \varphi + L \partial_{\gamma} \partial_{\gamma} \varphi \geq - \frac{|h_{ij} \partial_{\gamma} \partial_{\gamma} \chi_{ij}|}{t r_{\chi, \omega}} - |\partial_{\gamma} \partial_{\gamma} F|. \tag{4.1}
\]
Proof. Define \( \Phi(\chi_\varphi) := -\log(\text{tr}_{\chi_\varphi} \omega) \). Then we have
\[
\frac{\partial \Phi(\chi_\varphi)}{\partial (\chi_\varphi)_{ij}} = -\frac{\partial \log(\chi_\varphi^{kj} g_{kl})}{\partial (\chi_\varphi)_{ij}} = \frac{\chi_\varphi^{kj} \chi_\varphi^{il} g_{ab}}{\text{tr}_{\chi_\varphi} \omega} = \frac{h^{ij}}{\text{tr}_{\chi_\varphi} \omega}
\]
and
\[
\frac{\partial^2 \Phi(\chi_\varphi)}{\partial (\chi_\varphi)_{ik} \partial (\chi_\varphi)_{lj}} = -\frac{\partial^2 \log(\chi_\varphi^{kj} g_{kl})}{\partial (\chi_\varphi)_{ik} \partial (\chi_\varphi)_{lj}} = -\frac{\chi_\varphi^{kj} \chi_\varphi^{il} \chi_\varphi^{mk} \chi_\varphi^{nl} g_{ab}}{\text{tr}_{\chi_\varphi} \omega} + \frac{h^{ij} h^{kl}}{(\text{tr}_{\chi_\varphi} \omega)^2}
\]
\[
\leq - \sum_{1 \leq i,k \leq n} \frac{h^{ik} \chi_\varphi^{ij} + h^{kk} \chi_\varphi^{ij}}{\text{tr}_{\chi_\varphi} \omega} + \sum_{1 \leq i,k \leq n} \frac{h^{ii} \chi_\varphi^{ij}}{\text{tr}_{\chi_\varphi} \omega}
\]
\[
= - \sum_{1 \leq i,k \leq n} \frac{h^{kk} \chi_\varphi^{ij}}{\text{tr}_{\chi_\varphi} \omega} < 0.
\]
Differentiating (2.3) along an arbitrary vector \( \gamma \in \mathbb{C}^n \) and \( \bar{\gamma} \), and then using (4.2) and (4.3), we obtain
\[
\frac{\partial}{\partial t} \partial_\varphi \partial_\varphi = \frac{\partial \Phi(\chi_\varphi)}{\partial (\chi_\varphi)_{ik} \partial (\chi_\varphi)_{lj}} \partial_\varphi (\chi_\varphi)_{ij} \partial_\varphi (\chi_\varphi)_{kl} + \partial_\varphi, \partial_\bar{\varphi} F
\]
\[
\leq \frac{h^{ij} \partial_\varphi \partial_\varphi (\chi_\varphi)_{ij} \partial_\varphi F}{\text{tr}_{\chi_\varphi} \omega}
\]
\[
= L \partial_\varphi \partial_\varphi + \frac{h^{ij} \partial_\varphi \partial_\bar{\varphi} \partial_\varphi \chi_\varphi^{ij} \partial_\varphi F}{\text{tr}_{\chi_\varphi} \omega}
\]
\[
\leq L \partial_\varphi \partial_\varphi + \frac{|h^{ij} \partial_\varphi \partial_\bar{\varphi} \partial_\varphi \chi_\varphi^{ij}|}{\text{tr}_{\chi_\varphi} \omega} + |\partial_\varphi \partial_\bar{\varphi} F|.
\]
This proves (4.1). \( \square \)

Let \( O \subset \mathbb{C}^n \) be an open ball around the origin. A linear algebra result in M. Gill [14] (see Lemma 4.1 in [14]) is needed.

Lemma 4.2 ([14]). There exists a finite number \( N \) of unit vectors \( \gamma_\nu = (\gamma_{\nu_1}, \cdots, \gamma_{\nu_n}) \in \mathbb{C}^n \) and real-valued functions \( \beta_\nu \) on \( O \times [0, T) \), for \( \nu = 1, \cdots, N \), with
(1) \( 0 < \frac{1}{T} \leq \beta_\nu \leq C \), and
(2) \( \gamma_1, \cdots, \gamma_N \) containing an orthonormal basis of \( \mathbb{C}^n \), such that
\[
\frac{\partial \Phi(\chi_\varphi(y, t_2))}{\partial (\chi_\varphi)_{ij}} = \sum_{\nu=1}^N \beta_\nu(y, t_2) \gamma_{\nu i} \gamma_{\nu j}.
\]
As in [28, 14], we define \( \beta_0 := 1 \), \( w_0 := -\frac{\partial \varphi}{\partial t} \), and
\[
w_\nu := \partial_\nu \partial_\varphi = \sum_{i,j=1}^n \gamma_{\nu i} \gamma_{\nu j} \varphi_{ij}
\]
for \( \nu = 1, \cdots, N \).
Lemma 4.3. There exists a uniformly bounded function $H$ so that
\[-\frac{\partial w_\nu}{\partial t} + L w_\nu \geq H \tag{4.4}\]
on $O \times [0, T)$ for $\nu = 0, 1, \ldots, N$.

For all $(x, t_1)$ and $(y, t_2) \in O \times [0, T)$, we have
\[\sum_{\nu=0}^{N} \beta_\nu(y, t_2)(w_\nu(y, t_2) - w_\nu(x, t_1)) \leq C|x - y|. \tag{4.5}\]

Proof. From (2.5) and (4.1), we have
\[-\frac{\partial w_\nu}{\partial t} + L w_\nu \geq -|h^{ij}\partial_\nu \partial_i \chi_{ij}|_{\omega} - |\partial_\gamma F| := H. \]

Then we show $H$ is uniformly bounded. It follows from Theorem 2.5 and Theorem 3.4 that
\[tr_\omega \chi_\varphi \leq C_2 e^{A_1(\sup_{M \times [0, T)} \varphi - \int_{M \times [0, T)} \varphi)} \leq C. \tag{4.6}\]

At an arbitrary point $x \in M$, under the holomorphic normal coordinate system for the Kähler metric $\omega_0$ mentioned in Section 2, we have
\[tr_\omega \omega = \sum_{1 \leq i \leq n} \frac{1}{\lambda_i} \geq \frac{n}{\sum_{1 \leq i \leq n} \lambda_i} = \frac{n}{tr_\omega \chi_\varphi} \geq \tilde{C} \tag{4.7}\]
for some uniform constant $\tilde{C} > 0$ depending only on $M, \chi, \omega$ and $F$, where we used Cauchy's inequality in the first inequality and (4.6) in the second.

Moreover, (2.13) and (2.21) imply that
\[-C \leq H \leq 0. \]

This proves (4.4).

(4.3) implies that $\Phi(\chi_\varphi)$ is a concave function. Hence,
\[\Phi(\chi_\varphi(x, t_1)) - \Phi(\chi_\varphi(y, t_2)) \leq \sum \frac{\partial \Phi(\chi_\varphi(y, t_2))}{\partial(\chi_\varphi)_{ij}}(\chi_\varphi(x, t_1) - \chi_\varphi(y, t_2)) \tag{4.8}\]
for all $(x, t_1)$ and $(y, t_2) \in O \times [0, T)$.

It follows from (2.3) and (4.8) that
\[\frac{\partial \varphi}{\partial t}(x, t_1) - \frac{\partial \varphi}{\partial t}(y, t_2) + \sum \frac{\partial \Phi(\chi_\varphi(y, t_2))}{\partial(\chi_\varphi)_{ij}}(\chi_\varphi(y, t_2) - \chi_\varphi(x, t_1)) \leq F(x) - F(y) \leq C|x - y|, \tag{4.9}\]

(4.5) is obtained by applying Lemma 4.2 to (4.9). \[\square\]

Adapting the notations used in [21, 14], we define the parabolic distance between $(x, t_1)$ and $(y, t_2)$ in a domain $\Omega \subset \mathbb{C}^n \times [0, T)$ to be
\[|(x, t_1) - (y, t_2)| := \max(|x - y|, |t_1 - t_2|^{1/2}), \]
and semi-norms for a function $f$ by
\[\|f\|_{\alpha,(x_0, t_0)} = \sup_{(x, t) \in \Omega \setminus \{(x_0, t_0)\}} \frac{|f(x, t) - f(x_0, t_0)|}{|(x, t) - (x_0, t_0)|^\alpha} \]
and
\[\|f\|_{\alpha, \Omega} = \sup_{(x, t) \in \Omega} \|f\|_{\alpha,(x, t)}. \]
Fix $\epsilon \in (0, T)$ and $s \in [\epsilon, T]$. Pick $0 < R < \min \{1, \sqrt{s/10}\}$ small enough. Define a parabolic cylinders
\[ Q(jR) := \{(x, t) \in O \times [0, T] \mid x \leq jR, \ s - (jR)^2 \leq t \leq s\}, \quad j = 1, 2, \]
and
\[ \Theta(R) := \{(x, t) \in O \times [0, T] \mid x < R, \ s - 5R^2 \leq t \leq s - 4R^2\}. \]
For $\nu = 0, 1, \cdots, N$, set
\[ M_{j\nu} := \sup_{Q(jR)} w_{j\nu}, \quad m_{j\nu} := \inf_{Q(jR)} w_{j\nu} \text{ and } \Omega(jR) := \sum_{\nu=0}^{N} (M_{j\nu} - m_{j\nu}). \]
We say that $u \in W^{2,1}_{2n+1}$ if $u_x, u_{ij}, u_{ij}, u_{ij}$ and $u_t$ are all in $L^{2n+1}$. The following Harnack inequality (refer to Theorem 7.37 in [21] for its proof) is needed.

**Lemma 4.4.** If $u \in W^{2,1}_{2n+1}$ is nonnegative in $Q(4R)$ and satisfies
\[-\frac{\partial u}{\partial t} + Lu \leq f\]
in $Q(4R)$. Then there exists $p > 0$ determined by $M$, $\chi$, $\omega$ and $F$ such that
\[
\left(\frac{1}{R^{2n+2}} \int_{Q(R)} w^p\right)^{\frac{1}{p}} \leq C\left(\inf_{Q(R)} u + R^{\frac{2n}{p} + 1}\|f\|_{L^{2n+1}}\right). \tag{4.10}
\]

We give a self-contained proof for the Hölder estimate below with the arguments are almost identical to that in [14] and [28].

**Theorem 4.5.** Let $M$ be an $n$-dimensional $(n \geq 2)$ compact complex manifold equipped with a Hermitian metric $\chi$ and Kähler metric $\omega$. If $\varphi$ is a smooth solution of parabolic Donaldson’s equation (1.1) on $M \times [0, T)$. Fix $\epsilon > 0$, then there exists $\alpha \in (0, 1)$ and a positive constant $C_4$ depending only on $M$, $\chi$, $\omega$, $F$ and $\epsilon$ so that
\[ [((\chi_x)_{ij})_{\alpha, M \times [\epsilon, T]}] \leq C_4. \tag{4.11}\]

**Proof.** It is suffice to show that $\Omega(R) \leq CR^d$ for some positive constant $\delta$.

Let $l$ be an integer with $0 \leq l \leq N$. It follows from (4.4) that
\[-\frac{\partial (M_{2l} - w_l)}{\partial t} + L(M_{2l} - w_l) = \frac{\partial w_l}{\partial t} - Lw_l \leq -H.\]
By Lemma 4.4, we have
\[
\left(\frac{1}{R^{2n+2}} \int_{Q(R)} (M_{2l} - w_l)^p\right)^{\frac{1}{p}} \leq C(M_{2l} - M_l + R^{\frac{2n}{p} + 1}). \tag{4.12}\]
Similarly,
\[
\left(\frac{1}{R^{2n+2}} \int_{Q(R)} \left(\sum_{\nu \neq l} (M_{2\nu} - m_{\nu})^p\right)^{\frac{1}{p}}\right.
\leq N^{\frac{1}{p}} \sum_{\nu \neq l} \left(\frac{1}{R^{2n+2}} \int_{Q(R)} (M_{2\nu} - m_{\nu})^p\right)^{\frac{1}{p}}
\leq C\left(\sum_{\nu \neq l} (M_{2\nu} - M_\nu + R^{\frac{2n}{p} + 1})\right)
\leq C\left(\sum_{\nu \neq l} (M_{2\nu} - m_{2\nu} + m_\nu - M_\nu + R^{\frac{2n}{p} + 1})\right)
\]
\[
\leq C(\Omega(2R) - \Omega(R) + R^{\frac{2n}{2n-2}}). \tag{4.13}
\]

For some \((x, t_1) \in Q(2R)\) and every \((y, t_2) \in Q(2R)\), (4.5) yields that
\[
\beta_l(y, t_2)(w_l(y, t_2) - w_l(x, t_1)) \leq C(R + \sum_{\nu \neq l} \beta_{\nu}(y, t_2)(w_{\nu}(x, t_1) - w_{\nu}(y, t_2))).
\]

Choose \((x, t_1) \in Q(2R)\) so that \(w_l(x, t_1)\) approaches \(m_2l\) and recall that \(0 < \frac{1}{C} \leq \beta_{\nu} \leq C\) for \(\nu = 0, 1, \ldots, N\), we obtain
\[
w_l(y, t_2) - m_2l \leq C(R + \sum_{\nu \neq l} (M_{2\nu} - w_{\nu}(y, t_2))). \tag{4.14}
\]

Note that \(0 < R < 1\) is small. Integrating (4.14) and using (4.13), we can get
\[
\left(\frac{1}{R^{2n+2}} \int_{\Theta(R)} (w_l - m_2l)^p\right)^{\frac{1}{p}} \leq C(\Omega(2R) - \Omega(R) + R^{\frac{2n}{2n-2}}). \tag{4.15}
\]

Adding (4.12) and (4.15) and then summing over \(l\) shows that
\[
\Omega(2R) \leq C(\Omega(2R) - \Omega(R) + R^{\frac{2n}{2n-2}})
\]
and thus for some \(0 < \rho < 1,\)
\[
\Omega(R) \leq \rho \Omega(2R) + CR^{\frac{2n}{2n-2}}. \tag{4.16}
\]

By a standard iteration argument (see e.g. Chapter 8 in [13]), there exists a uniform constant \(\delta > 0\) so that
\[
\Omega(R) \leq CR^{\delta}.
\]

We can conclude the desired Hölder estimate on \(\partial_i \partial_j \varphi\) and hence (4.11) holds. \(\square\)

Now we finish the proof of long time existence of smooth solutions to (1.1).

**Proof of Theorem 1.1:** As we obtained Theorem 4.5, repeatedly differentiating (2.3) and applying the Schauder estimate (see e.g. Theorem 4.9 in [21]) provides that parabolic \(C^{2+\alpha}\) norms of all higher-order derivatives of \(\varphi\) are uniformly bounded by the uniform constant \(C\). This implies that the unique smooth solution \(\varphi\) to (1.1) is uniformly bounded in \(C^\infty\) on \(M \times [0, T]\).

Suppose by contradiction that \(T < \infty\). Since \(\varphi\) is uniformly bounded in \(C^\infty\), \(\varphi\) converges smoothly to a smooth function \(\bar{\varphi}(\cdot, T)\) on \(M\) as \(t \to T\). Then we can apply its short time existence (Theorem 2.1) to restart the flow (1.1) with initial value \(\bar{\varphi}(\cdot, T)\) to \([T, T + \epsilon]\) for some \(\epsilon > 0\), which contradicts that \(T\) is the maximal existence time. Hence, \(T = \infty\). \(\square\)

5. **Li-Yau estimate and Harnack inequality**

In this section, we prove a gradient estimate and associated Harnack inequality due to P. Li and S. T. Yau [19] for a positive solution \(u\) to the linear parabolic equation
\[
\frac{\partial u}{\partial t} = Lu \tag{5.1}
\]
on \(M \times [0, \infty)\) with \(L\) is the elliptic operator defined in (2.1). The Harnack inequality will be essential in the proof of Theorem 1.2.

Set
\[
|\partial v|^2_h := h^{ij} \partial_i v \partial_j v,
\]
for any real-valued smooth function \( v \) on \( M \), while
\[
\langle X, Y \rangle_h := h^{ij} X_i Y_j
\]
for any tangent vectors \( X \) and \( Y \) on \( M \).

For \( t \geq 0 \), \( f := \log u \), and some fixed constant \( \alpha > 1 \), we define
\[
G := t(\frac{|\partial f|_h^2}{\text{tr}_{\chi,\omega}} - \frac{\partial f}{\partial t}).
\]

**Proposition 5.1.** The following inequality holds on \( M \times (0, \infty) \).

\[
LG \geq -C(1 + \frac{1}{\varepsilon})t(\frac{|\partial f|_h^2}{\text{tr}_{\chi,\omega}} + \frac{(1 - Ce)t|\partial f|_h^2}{\text{tr}_{\chi,\omega}} + \frac{(1 - Ce)t|D^2 f|_h^2}{\text{tr}_{\chi,\omega}} - \frac{2\alpha(\alpha - 1)t}{\text{tr}_{\chi,\omega}} \cdot \text{Re} \langle \partial f, \partial f \rangle_h - \alpha L \frac{\partial f}{\partial t},
\]

(5.2)

where \( \varepsilon \) is a positive constant to be determined later.

**Proof.** By direct computation, we obtain
\[
\frac{\partial f}{\partial t} = \frac{1}{u} \frac{\partial u}{\partial t} = 1 \cdot \frac{h^{ij} \partial_i \partial_j f}{\text{tr}_{\chi,\omega}} = \frac{h^{ij}(\partial_i f \partial_j f + \partial_j f \partial_i f)}{\text{tr}_{\chi,\omega}} = \frac{|\partial f|_h^2}{\text{tr}_{\chi,\omega}} + Lf,
\]

(5.3)

where we used (5.1) in the second equality. Together with the definition of \( G \), we have
\[
Lf = -G + (\alpha - 1) \frac{\partial f}{\partial t}.
\]

(5.4)

Calculating directly yields
\[
LG = \frac{t}{\text{tr}_{\chi,\omega}} h^{kl} \partial_k \partial_l (\frac{|\partial f|_h^2}{\text{tr}_{\chi,\omega}} - \frac{\partial f}{\partial t})
\]
\[
= \frac{t}{\text{tr}_{\chi,\omega}} h^{kl} \partial_k \left( \frac{h^{ij} \partial_i \partial_j f \partial_j f}{\text{tr}_{\chi,\omega}} + \frac{h^{ij} \partial_i \partial_j f}{\text{tr}_{\chi,\omega}} + \frac{h^{ij} \partial_i \partial_j f}{\text{tr}_{\chi,\omega}} \right) - \frac{2(\alpha - 1)t}{\text{tr}_{\chi,\omega}} \cdot \text{Re} \langle \partial f, \partial f \rangle_h - \alpha L \frac{\partial f}{\partial t}
\]
\[
= \frac{th^{kl}}{\text{tr}_{\chi,\omega}} \left( \frac{\partial_k h^{ij} \partial_i \partial_j f}{\text{tr}_{\chi,\omega}} + \frac{\partial h^{ij} \partial_i \partial_j f}{\text{tr}_{\chi,\omega}} + \frac{\partial h^{ij} \partial_i \partial_j f}{\text{tr}_{\chi,\omega}} \right) - \frac{\partial h^{ij} \partial_i \partial_j f}{\text{tr}_{\chi,\omega}} \frac{\text{tr}_{\chi,\omega}}{h^{ij} \partial_i \partial_j f \partial_k (\text{tr}_{\chi,\omega})} + \frac{\partial h^{ij} \partial_i \partial_j f}{\text{tr}_{\chi,\omega}} \frac{\text{tr}_{\chi,\omega}}{h^{ij} \partial_i \partial_j f \partial_k (\text{tr}_{\chi,\omega})}
\]
\[
= \frac{t}{\text{tr}_{\chi,\omega}} \left( \frac{h^{ij} \partial_i \partial_j f}{\text{tr}_{\chi,\omega}} + \frac{h^{ij} \partial_i \partial_j f}{\text{tr}_{\chi,\omega}} + \frac{h^{ij} \partial_i \partial_j f}{\text{tr}_{\chi,\omega}} \right) - \frac{2(\alpha - 1)t}{\text{tr}_{\chi,\omega}} \cdot \text{Re} \langle \partial f, \partial f \rangle_h - \alpha L \frac{\partial f}{\partial t}
\]
and 

Similarly, we have 

Note that 

Moreover, 

and 

By (2.12), (2.13) and the uniform higher-order derivative bounds for \( \varphi \) from the proof of Theorem 1.1, we have 

Moreover, 

and 

Similarly, we have 

For the 1st, 4th, 13th and 16th terms of the right-hand-side of (5.5), the Cauchy’s inequality, (4.7), (5.6) and (5.7) imply that 

\[
\frac{th^{kl} \partial_k tr_{(x_\varphi, \omega)} f}{(tr_{(x_\varphi, \omega)})^2} - \frac{th^{kl} \partial_k tr_{(x_\varphi, \omega)} f}{(tr_{(x_\varphi, \omega)})^2} + \frac{th^{kl} \partial_k tr_{(x_\varphi, \omega)} f}{(tr_{(x_\varphi, \omega)})^3}
\]
Similarly, there exists some positive constant $\varepsilon$ so that the 2nd, 9th, 12th and 14th terms of the right-hand-side of (5.5) satisfy

$$\geq - \frac{Ct|\partial_t f|_h^2}{\text{tr}_{\chi,\omega}},$$

while for the 3rd, 5th, 8th and 15th terms of the right-hand-side of (5.5), we have

$$\geq - \frac{C\varepsilon t|\partial_t f|_h^2 + \frac{C}{t}|\partial f|_h^2}{\text{tr}_{\chi,\omega}}.$$
\[ -\frac{th^{ij}\partial h^{kl}\partial_{k}\partial_{l}f\partial_{j}f}{(tr_{X_{\varphi}}\omega)^{2}} - \frac{th^{ij}\partial h^{kl}\partial_{k}\partial_{l}f\partial_{j}f}{(tr_{X_{\varphi}}\omega)^{2}} + \frac{th^{kl}h^{ij}R(\omega)k_{j}p_{l}g_{p\bar{q}}\partial_{k}f\partial_{l}f}{(tr_{X_{\varphi}}\omega)^{2}} \geq -\frac{2Re(\partial G, \partial f)_{h}}{tr_{X_{\varphi}}\omega} - \frac{2(\alpha - 1)t}{tr_{X_{\varphi}}\omega} Re(\partial(\frac{\partial f}{\partial t}, \partial f))_{h} \]

\[ -\frac{C\varepsilon t|\partial\overline{f}|_{h}^{2} + \frac{C\varepsilon t|\partial f|_{h}^{2}}{tr_{X_{\varphi}}\omega}}{tr_{X_{\varphi}}\omega} \]

for some positive constant \(\varepsilon\) small enough, where we used (5.4) in the third equality. Moreover, we used Cauchy’s inequality, (4.7), (5.6) and (5.7) in the last inequality.

Applying (5.8), (5.9), (5.10) and (5.11) to (5.5), we conclude (5.2). \(\square\)

Then we deal with the last two terms of the right-hand-side of (5.2), respectively.

**Proposition 5.2.** The following inequality holds on \(M \times (0, \infty)\).

\[ 2t \cdot Re(\partial f, \partial (\frac{\partial f}{\partial t}))_{h} \leq \frac{\partial G}{\partial t} - \frac{G}{t} + \alpha \frac{\partial^{2} f}{\partial t^{2}} + \frac{C\varepsilon t|\partial f|_{h}^{2}}{tr_{X_{\varphi}}\omega}, \]

where \(\varepsilon\) is a positive constant to be determined later.

**Proof.** It follows from (2.3) that

\[
\frac{\partial}{\partial t}(\chi_{\varphi})_{ij} = \partial_{i}\partial_{j}\frac{\partial \varphi}{\partial t} = -\partial_{i}\partial_{j}\log(tr_{X_{\varphi}}\omega) + \partial_{i}\partial_{j}F = \frac{-\partial_{i}(tr_{X_{\varphi}}\omega)\partial_{j}(tr_{X_{\varphi}}\omega)}{tr_{X_{\varphi}}\omega} + \partial_{i}\partial_{j}F.
\]

Applying (4.7) and (5.7), we know that

\[ |\frac{\partial X_{\varphi}}{\partial t}| \leq C. \quad (5.13) \]

Since

\[
-\frac{\partial}{\partial t}h^{ij} = \chi_{\varphi}^{i\bar{k}}\chi_{\varphi}^{j\bar{l}}\frac{\partial}{\partial t}(\chi_{\varphi})_{\bar{k}\bar{l}} + \chi_{\varphi}^{i\bar{k}}\chi_{\varphi}^{j\bar{l}}\frac{\partial}{\partial t}(\chi_{\varphi})_{\bar{k}\bar{l}} - \chi_{\varphi}^{i\bar{k}}\chi_{\varphi}^{j\bar{l}}\frac{\partial}{\partial t}(\chi_{\varphi})_{\bar{k}\bar{l}} = \chi_{\varphi}^{i\bar{k}}\chi_{\varphi}^{j\bar{l}}\frac{\partial}{\partial t}(\chi_{\varphi})_{\bar{k}\bar{l}},
\]

and

\[
-\frac{\partial}{\partial t}(tr_{X_{\varphi}}\omega) = \chi_{\varphi}^{i\bar{k}}\chi_{\varphi}^{j\bar{l}}\frac{\partial}{\partial t}(\chi_{\varphi})_{\bar{k}\bar{l}}g_{\bar{k}\bar{l}} = h_{\bar{k}}\frac{\partial}{\partial t}(\chi_{\varphi})_{\bar{k}},
\]

we have

\[ |\frac{\partial}{\partial t}(h^{-1})| \leq C \text{ and } |\frac{\partial}{\partial t}(tr_{X_{\varphi}}\omega)| \leq C. \quad (5.14) \]

By the definition of \(G\), we have

\[
\frac{\partial G}{\partial t} = \frac{G}{t} + \frac{t(\frac{\partial f}{\partial t})_{h}\partial_{i}f\partial_{j}f}{tr_{X_{\varphi}}\omega} + \frac{th^{ij}\partial_{i}(\frac{\partial f}{\partial t})_{h}\partial_{j}f}{tr_{X_{\varphi}}\omega} + \frac{th^{ij}\partial_{i}f\partial_{j}(\frac{\partial f}{\partial t})_{h}}{tr_{X_{\varphi}}\omega},
\]
\[- \frac{t|\partial f|^2_h}{(\text{tr}_X\omega)^2} \frac{\partial}{\partial t} (\text{tr}_X\omega) - \alpha t \frac{\partial^2 f}{\partial t^2}, \]
i.e.,
\[
\frac{2t \cdot \text{Re}(\partial f, \partial (\frac{\partial f}{\partial t}))}{\text{tr}_X\omega} = \frac{\partial G}{\partial t} - \frac{G}{t} + \alpha t \frac{\partial^2 f}{\partial t^2} - \frac{t \left( \frac{\partial G}{\partial t} \right) \partial_i f \partial_j f}{\text{tr}_X\omega} + \frac{t|\partial f|^2_h}{(\text{tr}_X\omega)^2} \frac{\partial}{\partial t} (\text{tr}_X\omega)
\leq \frac{\partial G}{\partial t} - \frac{G}{t} + \alpha t \frac{\partial^2 f}{\partial t^2} + C t|\partial f|^2_h, \tag{5.15}
\]
where we used (5.13) and (5.14).
\[\square\]

**Proposition 5.3.** The following inequality holds on \(M \times (0, \infty)\).
\[
L \frac{\partial f}{\partial t} \leq \frac{G}{t^2} - \frac{1}{t} \cdot \frac{\partial G}{\partial t} - (\alpha - 1) \frac{\partial^2 f}{\partial t^2} + \frac{C \varepsilon |\partial f|^2_h}{\text{tr}_X\omega} + \frac{C}{\varepsilon}, \tag{5.16}
\]
where \(\varepsilon\) is a positive constant to be determined later.

**Proof.** Calculating directly, we have
\[
L \frac{\partial f}{\partial t} = \frac{\partial}{\partial t} (L f) - \frac{1}{\text{tr}_X\omega} \cdot \frac{\partial}{\partial t} \left( \frac{\partial G}{\partial t} \right) \partial_i f \partial_j f + \frac{h^i_j \partial_i \partial_j f}{(\text{tr}_X\omega)^2} \cdot \frac{\partial}{\partial t} (\text{tr}_X\omega)
\leq \frac{G}{t^2} - \frac{1}{t} \cdot \frac{\partial G}{\partial t} - (\alpha - 1) \frac{\partial^2 f}{\partial t^2} + C \left( \frac{1}{\text{tr}_X\omega} + \frac{1}{(\text{tr}_X\omega)^2} \right) |\partial f|^2_h
\leq \frac{G}{t^2} - \frac{1}{t} \cdot \frac{\partial G}{\partial t} - (\alpha - 1) \frac{\partial^2 f}{\partial t^2} + \frac{C \varepsilon |\partial f|^2_h}{\text{tr}_X\omega} + \frac{C}{\varepsilon},
\]
for \(t > 0\) and some positive constant \(\varepsilon\). where we used (5.4), Cauchy’s inequality, (2.13), (5.13) and (5.14) in the first inequality. Moreover, we used Cauchy’s inequality and (4.7) in the last inequality.
\[\square\]

**Lemma 5.4.** Define \(I = \frac{\partial f|^2_h}{\text{tr}_X\omega G}\). There exists a positive constant \(C_5\) depending only on \(M, \chi, \omega\) and \(F\) so that the following inequality holds on \(M \times (0, \infty)\).
\[
(L - \frac{\partial}{\partial t}) G \geq \frac{\text{tr}_X\omega}{2n} \cdot \frac{G^2[(\alpha - 1)It + 1]^2}{\alpha^2 t} - \frac{C_5(\frac{\varepsilon + 1}{\varepsilon} + \alpha - 1)IG - \frac{G}{t}}{\text{tr}_X\omega} \frac{2\text{Re}(\partial f, \partial f)_h}{\varepsilon}, \tag{5.17}
\]
where \(\varepsilon\) is a chosen constant with \(0 < \varepsilon \leq \frac{1}{2\alpha(\alpha + 1)}\).

**Proof.** By applying Proposition 5.2 and Proposition 5.3 to Proposition 5.1, we know that there exists a positive constant \(C_5\) depending only on \(M, \chi, \omega\) and \(F\) so that
\[
LG \geq -C_5 \left( 1 + \frac{1}{\varepsilon} \right) t \frac{|\partial f|^2_h}{\text{tr}_X\omega} + \frac{1 - C_5 \varepsilon |\partial f|^2_h}{\text{tr}_X\omega} + \frac{1 - C_5 \varepsilon |D^2 f|^2_h}{\text{tr}_X\omega},
\]
\[-2\text{Re}(\partial G, \partial f)_h \over \text{tr}_{Xe} \omega) - (\alpha - 1) \frac{\partial G}{\partial t} + (\alpha - 1) G \frac{t}{t} - \alpha(\alpha - 1) t \frac{\partial^2 f}{\partial t^2} \]

\[-C_5 (\alpha - 1) t \frac{\partial f}{h}_h \over \text{tr}_{Xe} \omega - \alpha G \over t + \alpha \frac{\partial G}{\partial t} \]

\[+ \alpha(\alpha - 1) t \frac{\partial^2 f}{\partial t^2} - C_5 \frac{\alpha \epsilon t}{\epsilon} \over \text{tr}_{Xe} \omega \]

\[= [1 - C_5 (\alpha + 1) \epsilon] t \frac{\overline{\partial^2 f}^2}{h}_{Xe \omega} + (1 - C_5 \epsilon) t \frac{\overline{\partial f}^2}{h}_{Xe \omega} \]

\[-C_5 \left( \frac{\epsilon + 1}{\epsilon} + \alpha - 1 \right) t \frac{\partial f}{h}_h \over \text{tr}_{Xe} \omega - \frac{G}{t} \over \text{tr}_{Xe} \omega \]

\[2\text{Re}(\partial G, \partial f)_h \over \text{tr}_{Xe} \omega + \frac{\partial G}{\partial t} - C_5 \frac{\alpha t}{\epsilon}. \tag{5.18} \]

It follows from (5.3) and the definition of $G$ and $I$ that

\[-Lf = \frac{\overline{\partial f}^2}{h}_{Xe \omega} - \alpha \frac{\partial f}{\partial t} = \frac{G}{\alpha} [(\alpha - 1) It + 1]. \tag{5.19} \]

By Cauchy’s inequality and (5.19), we have

\[\frac{\overline{\partial f}^2}{h}_{Xe \omega} \geq \frac{\text{tr}_{Xe} \omega}{\alpha^2 t} \cdot (L f)^2 = \frac{\text{tr}_{Xe} \omega}{\alpha^2 t} \cdot \frac{G^2 [(\alpha - 1) It + 1]^2}{\alpha^2 t^2}. \tag{5.20} \]

Choosing $\epsilon$ so that $1 - C_5 (\alpha + 1) \epsilon \geq \frac{1}{2}$ and applying (5.20) to (5.18), we obtain (5.17) immediately. \qed

Now we are ready to prove a Li-Yau estimate and associated Harnack estimate for $u$.

**Theorem 5.5.** Let $u$ be a positive solution to (5.1) on $M \times [0, \infty)$ and $f = \log u$, then there exist a uniform constant $\bar{C} > 0$ depending only on $M, \chi, \omega, F$ and $\alpha$ so that

\[\frac{\overline{\partial f}^2}{h}_{Xe \omega} - \alpha \frac{\partial f}{\partial t} \leq \bar{C} \tag{5.21} \]

on $M \times (0, \infty)$.

**Proof.** For any fixed $T_1 \in (0, \infty)$, let $(x_1, t_1)$ be a maximum principle of $G$ on $M \times [0, T_1]$. We may assume $t_1 > 0$ otherwise $G \leq 0$ on $M \times [0, T_1]$. By definition of $G$ and arbitrary of $T_1$, we have $\frac{\overline{\partial f}^2}{h}_{Xe \omega} - \alpha \frac{\partial f}{\partial t} \leq 0$ on $M \times (0, \infty)$ in this case.

At $(x_1, t_1)$, applying the maximum principle to Lemma 5.4 yields that

\[\frac{\text{tr}_{Xe} \omega}{2n} \cdot \frac{G^2 [(\alpha - 1) It + 1]^2}{\alpha^2 t} \leq C_5 \left( \frac{\epsilon + 1}{\epsilon} + \alpha - 1 \right) t I G + \frac{G}{t} + \frac{C_5 \alpha t}{\epsilon}. \tag{5.22} \]

Since $\alpha > 1$, we have

\[[(\alpha - 1) It + 1]^2 \geq 1 \text{ and } \frac{t I}{[(\alpha - 1) It + 1]^2} \leq \frac{1}{2(\alpha - 1)}. \tag{5.23} \]
(5.22) and (5.23) implies that
\[
\frac{\text{tr}_{\chi, \omega} \cdot G}{2n\alpha^2} \leq \frac{C_5}{2(\alpha - 1)} \left( \frac{\varepsilon + 1}{\varepsilon} + \alpha - 1 \right) \frac{G}{t} + \frac{C_5 \alpha}{\varepsilon}
\] (5.24)
at \((x_1, t_1)\).

Recall a result that if \(a_0^2 \leq a_1 + a_0 a_2\) for some \(a_0, a_1, a_2 > 0\), then
\[
a_0 \leq \frac{a_2}{2} + \sqrt{a_1 + \frac{a_2^2}{4}} \leq a_2 + \sqrt{a_1}.
\]

It follows from (5.24) that
\[
\frac{G}{t} \leq \frac{C_5}{2(\alpha - 1)} \left( \frac{\varepsilon + 1}{\varepsilon} + \alpha - 1 \right) \frac{2n\alpha^2}{\text{tr}_{\chi, \omega}} + \sqrt{\frac{2nC_5\alpha^3}{\text{tr}_{\chi, \omega}}}
\]
\[
\leq \bar{C}
\] (5.25)
at \((x_1, t_1)\), where we used (4.7).

Furthermore, we have
\[
G(x, T_1) \leq G(x, t_1) \leq \bar{C} t_1 \leq \bar{C} T_1
\]
for all \(x \in M\).

Note that \(T_1\) is arbitrary, we conclude
\[
-\alpha \frac{\partial \log u}{\partial t} = \frac{G}{t} \leq \bar{C}
\]
on \(M \times (0, \infty)\). \(\square\)

As a direct corollary of Theorem 5.5, we have the following Harnack inequality.

**Corollary 5.6.** Let \(u\) be a positive solution to (5.1) on \(M \times [0, \infty)\), then there exist a uniform constant \(\bar{C} > 0\) depending only on \(M, \chi, \omega, F\) and \(\alpha\) so that
\[
\sup_M u(\cdot, s_1) \leq \bar{C} \cdot \inf_M u(\cdot, s_2) \cdot e^{s_2 - s_1}
\] (5.26)
for any \(0 < s_1 < s_2\).

**Proof.** It follows from Theorem 5.5 that
\[
-\alpha \frac{\partial \log u}{\partial t} \leq \bar{C}.
\] (5.27)

For any \((y_1, s_1), (y_2, s_2) \in M \times (0, \infty)\) with \(0 < s_1 < s_2\), take the geodesic path \(\gamma(t)\) from \(y_1\) to \(y_2\) at time \(s_1\) parametrized proportional to arc length with parameter \(t\) starting at \(y_1\) at time \(s_1\) and ending at \(y_2\) at time \(s_2\). Now consider the path \((\gamma(t), t)\) in space-time and integrate (5.27) along \(\gamma\), we get
\[
\alpha \log \frac{u(y_1, s_1)}{u(y_2, s_2)} \leq \bar{C}(s_2 - s_1)
\] (5.28)
for any given \((y_1, s_1), (y_2, s_2) \in Q_{\frac{\varepsilon}{2} : T}\) with \(0 < s_1 < s_2\).

Exponentiate (5.28), then the arbitrary of \(y_1\) and \(y_2\) implies (5.26). \(\square\)
6. Smooth convergence

In this section we use Corollary 5.6 to prove Theorem 1.2 that \( \hat{\phi} \) converges smoothly to a smooth function \( \hat{\phi}_\infty \) on \( M \). First of all, we show a upper bound for the oscillation

\[
\theta(t) := \sup_M \frac{\partial \hat{\phi}}{\partial t}(\cdot, t) - \inf_M \frac{\partial \hat{\phi}}{\partial t}(\cdot, t)
\]

for any \( t \in (0, \infty) \).

**Lemma 6.1.** There exists positive constants \( \bar{C}_1 \) and \( \bar{C}_2 \) depending only on \( M, \chi, \omega, F \) and \( \alpha \) so that

\[
\theta(t) \leq \bar{C}_1 e^{-\bar{C}_2 t} \tag{6.1}
\]

for any \( t \in (0, \infty) \).

**Proof.** Define

\[
v_m(x, t) := \sup_{y \in M} \frac{\partial \phi}{\partial t}(y, m - 1) - \frac{\partial \phi}{\partial t}(x, m - 1 + t),
\]

and

\[
w_m(x, t) := \frac{\partial \phi}{\partial t}(x, m - 1 + t) - \inf_{y \in M} \frac{\partial \phi}{\partial t}(y, m - 1)
\]

with \( m \in \mathbb{N}_+ \).

For any fixed \( m \), applying the strong maximum principle to (2.5) on \( M \times [m - 1, \infty) \) yields that either \( v_m(x, t) = w_m(x, t) = 0 \) for \( t \geq 0 \), or \( v_m(x, t) \) and \( w_m(x, t) \) are positive for \( t > 0 \).

In the first case, the fact of \( v_1(x, t) = w_1(x, t) = 0 \) for any \( t \in (0, \infty) \) implies that \( \theta(t) \equiv 0 \) and (6.1) is true.

In the latter case, \( \{v_m\} \) and \( \{w_m\} \) are positive solutions to (5.1) on \( M \times (0, \infty) \). Take \( s_1 = \frac{1}{2} \) and \( s_2 = 1 \) in Corollary 5.6, there exists a constant \( \bar{C}_3 > 1 \) depending only on \( M, \chi, \omega, F \) and \( \alpha \) so that

\[
\sup_M v_m(\cdot, \frac{1}{2}) \leq \bar{C}_3 \inf_M v_m(\cdot, 1), \quad \text{and} \quad \sup_M w_m(\cdot, \frac{1}{2}) \leq \bar{C}_3 \inf_M w_m(\cdot, 1),
\]

i.e.,

\[
\sup_M u(\cdot, m - 1) - \inf_M u(\cdot, m - \frac{1}{2}) \leq \bar{C}_3 (\sup_M u(\cdot, m - 1) - \sup_M u(\cdot, m)), \tag{6.2}
\]

and

\[
\sup_M u(\cdot, m - \frac{1}{2}) - \inf_M u(\cdot, m - 1) \leq \bar{C}_3 (\inf_M u(\cdot, m) - \inf_M u(\cdot, m - 1)), \tag{6.3}
\]

respectively.

Adding (6.2) and (6.3), we have

\[
\theta(m - 1) \leq \theta(m - 1) + \theta(m - \frac{1}{2}) \leq \bar{C}_3 (\theta(m - 1) - \theta(m)),
\]

i.e.,

\[
\theta(m) \leq \frac{\bar{C}_3 - 1}{\bar{C}_3} \theta(m - 1). \tag{6.4}
\]

Note that \( 0 < \frac{\bar{C}_3 - 1}{\bar{C}_3} < 1 \). By induction, we conclude (6.1). \( \square \)
In the following, we finish the proof of Theorem 1.2.

**Proof of Theorem 1.2:** By the definition of \( \tilde{\varphi} \), we know that
\[
\int_M \frac{\partial \tilde{\varphi}}{\partial t} \omega^n = \int_M \tilde{\varphi} \omega^n = 0.
\]
(6.5)
Hence, for any \( t \in (0, \infty) \), there exists \( y \in M \) so that
\[
\frac{\partial \tilde{\varphi}}{\partial t}(y, t) = 0.
\]
(6.6)
For any \( x \in M \), we have
\[
|\frac{\partial \tilde{\varphi}}{\partial t}(x, t)| = |\frac{\partial \tilde{\varphi}}{\partial t}(x, t) - \frac{\partial \tilde{\varphi}}{\partial t}(y, t)|
\]
\[
= \left| \frac{\partial \varphi}{\partial t}(x, t) - \frac{\partial \varphi}{\partial t}(y, t) \right|
\]
\[
\leq \bar{C}_1 e^{-\bar{C}_2 t},
\]
(6.7)
where we used (6.6) in the first equality and Lemma 6.1 in the inequality.

For any \( (x, t) \in M \times (0, \infty) \), the continuity of \( \tilde{\varphi} \) and the fact of
\[
\int_M \tilde{\varphi} \omega^n = 0
\]
imply that there exists \( \tilde{y} \in M \) so that \( \tilde{\varphi}(\tilde{y}, t) = 0 \). Then we have
\[
|\tilde{\varphi}(x, t)| = |\tilde{\varphi}(x, t) - \tilde{\varphi}(\tilde{y}, t)|
\]
\[
= |\varphi(x, t) - \varphi(\tilde{y}, t)|
\]
\[
\leq \sup_{M \times [0, \infty)} \varphi - \inf_{M \times [0, \infty)} \varphi \leq C,
\]
(6.8)
where we used Theorem 3.4.

Define a function
\[
\Psi := \tilde{\varphi} + \frac{\bar{C}_1}{\bar{C}_2} e^{-\bar{C}_2 t}
\]
on \( M \times (0, \infty) \).

It follows from (6.8) that \(|\Psi| \leq C\) on \( M \times (0, \infty) \), while (6.7) implies that
\[
\frac{\partial \Psi}{\partial t} = \frac{\partial \tilde{\varphi}}{\partial t} - \bar{C}_1 e^{-\bar{C}_2 t} \leq 0,
\]
on \( M \times (0, \infty) \), i.e., \( \Psi \) is uniformly bounded and monotonically decreasing in \( t \).
Hence, there exists a function \( \tilde{\varphi}_\infty \) on \( M \) with
\[
\lim_{t \to \infty} \Psi(\cdot, t) = \tilde{\varphi}_\infty
\]
pointwise, moreover,
\[
\lim_{t \to \infty} \tilde{\varphi}(\cdot, t) = \lim_{t \to \infty} \Psi(\cdot, t) - \lim_{t \to \infty} \frac{\bar{C}_1}{\bar{C}_2} e^{-\bar{C}_2 t} = \tilde{\varphi}_\infty
\]
(6.9)
pointwise.

We prove that the convergence (6.9) is in \( C^\infty \) topology by contradiction. Assume there exists a time sequence \( t_m \not\to \infty \), \( x \in M \), constant \( \delta > 0 \) and integer \( k \geq 1 \), so that
\[
\|\tilde{\varphi}(x, t_m) - \tilde{\varphi}_\infty(x)\|_{C^k} \geq \delta
\]
(6.10)
for all \( m \).
Since $\varphi$ is uniformly bounded in $C^\infty$ from the proof of Theorem 1.1, we know that $\tilde{\varphi}(x, \cdot)$ is bounded in $C^\infty$. Then there exists a subsequence $t_{m_j} \to \infty$ with

$$\lim_{j \to \infty} \tilde{\varphi}(x, t_{m_j}) = \tilde{\varphi}_\infty(x)$$

in $C^k$. (6.10) yields that $\tilde{\varphi}'_\infty(x) \neq \tilde{\varphi}_\infty(x)$, which contradicts (6.9).

We conclude $\tilde{\varphi}$ converges in $C^\infty$ topology to $\tilde{\varphi}_\infty$ that is a smooth function on $M$.

As a corollary of Theorem 1.2, we show Donaldson’s equation (1.2) has a unique solution by adding a certain constant to $F$. This partially answers a question raised by Y. Li [20] (see Page 869, Lines 16 to 17).

**Corollary 6.2.** Let $M$ be an $n$-dimensional ($n \geq 2$) compact complex manifold equipped with a Hermitian metric $\chi$ and a Kähler metric $\omega$. There exists a unique real number $b$ such that $\tilde{\varphi}_\infty$ in Theorem 1.2 is a unique solution to the Donaldson’s equation

$$\omega \wedge \chi^{n-1} = e^{F+b} \chi^n \tag{6.11}$$

with

$$\chi > \frac{n-1}{neF} \omega.$$  

**Proof.** It is clear that the normalized solution (1.11) to (1.1) $\varphi$ solves

$$\frac{\partial \tilde{\varphi}}{\partial t} = \log \left( \frac{\chi^n}{\omega \wedge \chi^{n-1}} \right) + F - \int_M \frac{\partial \varphi}{\partial t} \omega^n. \tag{6.12}$$

Denote the constant $- \int_M \left( \log \frac{\chi^n}{\omega \wedge \chi^{n-1}} + F \right) \omega^n$ by $b$. By taking $t \to \infty$ in both side of (6.12) and exponentiating, we can obtain

$$\omega \wedge \chi^{n-1} = e^{F+b} \chi^n \tilde{\varphi}_\infty.$$  

\[\Box\]

**Acknowledgements**

The author would like to thank Prof. Kefeng Liu for guidance and help. The author also thanks Prof. Yi Li for his interest of this work.

**References**

[1] E. Calabi, On Kähler manifolds with vanishing canonical class, Algebraic geometry and topology, A symposium in honor of S. Lefschetz, Princeton Univ. Press, Princeton, (1957) 78-89.

[2] H. D. Cao, Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds, Invent. math. 81 (1985) 395-372.

[3] G. Chen, The J-equation and the supercritical deformed Hermitian-Yang-Mills equation. Invent. math. 225 (2021) 529-602.

[4] X. Chen, On the lower bound of the Mabuchi energy and its application, Internat. Math. Res. Notices 12 (2000) 607-623.

[5] X. Chen, A new parabolic flow in Kähler manifolds, Commun. Anal. Geom. 12(4) (2004) 837-852.

[6] J. Chu, The parabolic Monge-Ampère equation on compact almost Hermitian manifolds, J. reine angew. Math. 2020(761) (2020) 1-24.

[7] J. Chu, V. Tosatti, B. Weinkove, The Monge-Ampère equation for non-integrable almost complex structures, J. Eur. Math. Soc. 21(7) (2019) 1949-1984.

[8] T. C. Collins, G. Székelyhidi, Convergence of the J-flow on toric manifolds, J. Differential Geometry 107(1) (2017) 47-81.
[9] R. Dervan, J. Keller, A finite dimensional approach to Donaldson's J-flow, Commun. Anal. Geom. 27(5) (2019) 1025-1085.
[10] S. K. Donaldson, Moment maps and diffeomorphisms, Asian J. Math. 3(1) (1999) 1-16.
[11] L. C. Evans, Classical solutions of fully nonlinear, convex, second order elliptic equations, Comm. Pure Applied Math. 35 (1982) 333-363.
[12] H. Fang, M. Lai, J. Song, B. Weinkove, The J-flow on Kähler surfaces: a boundary case, Anal. PDE 7(1) (2014) 215-226.
[13] D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, Berlin, Heidelberg, New York, 1983.
[14] M. Gill, Convergence of parabolic complex Monge-Ampère equation on compact Hermitian manifolds, Commun. Anal. Geom. 19(2) (2011) 277-303.
[15] Y. Hashimoto, J. Keller, About J-flow, J-balanced metrics, uniform J-stability and K-stability, Asian J. Math. 22(3) (2018) 391-412.
[16] N. V. Krylov, Boundedly inhomogeneous elliptic and parabolic equations in a domain (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 47 (1983) 75-108.
[17] N. V. Krylov, Boundedly inhomogeneous elliptic and parabolic equations, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982) 487-523, 670. English translation: Math. USSR Izv. 22 (1984) 67-98.
[18] M. Lejmi, G. Székelyhidi, The J-flow and stability, Adv. Math. 274 (2015) 404-431.
[19] P. Li, S. T. Yau, On the parabolic kernel of the Schrödinger operator, Acta Math. (1986) 156(3-4) 153-201.
[20] Y. Li, A priori estimates for Donaldson’s equation over compact Hermitian manifolds, Calc. Var. 50 (2014) 867-882.
[21] G. Lieberman, Second order parabolic differential equations, World Scientific, Singapore, New Jersey, London, Hong Kong, 1996.
[22] T. Mabuchi, K-energy maps integrating Futaki invariants, Tohoku Math. J. (2) 38(4) (1986) 575-593.
[23] J. Song, B. Weinkove, On the convergence and singularities of the J-flow with applications to the Mabuchi energy, Commun. Pure Appl. Math. 61(2) (2008) 210-229.
[24] W. Sun, Parabolic complex Monge-Ampère type equations on closed Hermitian manifolds, Calc. Var. 54 (2015) 3715-3733.
[25] T. D. Tô, Regularizing properties of complex Monge-Ampère flows I: J-flow, J. Funct. Anal. 272(5) (2017) 2058-2091.
[26] T. D. Tô, Regularizing properties of complex Monge-Ampère flows II: Hermitian manifolds, Math. Ann. 372 (2018) 699-741.
[27] T. D. Tô, Degenerate J-flow on compact Kähler manifolds, arXiv: 2201.09380v2 [math.DG].
[28] V. Tosatti, B. Weinkove, Estimates for the complex Monge-Ampère equation on Hermitian and balanced manifolds, Asian J. Math. 14(1) (2010) 19-40.
[29] V. Tosatti, B. Weinkove, The complex Monge-Ampère equation on compact Hermitian manifolds, J. Am. Math. Soc. 23(4) (2010) 1187-1195.
[30] V. Tosatti, B. Weinkove, S. T. Yau, Taming symplectic forms and the Calabi-Yau equation, Proc. London Math. Soc. 97(2) (2008) 401-424.
[31] L. Vezzoni, M. Zedda, On the J-flow in Sasakian manifolds, Ann. Mat. Pur. Appl. 195(3) (2016) 757-774.
[32] B. Weinkove, Convergence of the J-flow on Kähler surfaces, Commun. Anal. Geom. 12(4) (2004) 949-965.
[33] B. Weinkove, On the J-flow in higher dimensions and the lower boundedness of the Mabuchi energy, J. Differential Geom. 73(2) (2006) 351-358.
[34] Y. Yao, The J-flow on toric manifolds, Acta Math. Sin. 31(10) (2015) 1582-1592.
[35] S. T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I, Comm. Pure Appl. Math. 31 (1978) 339-411.
[36] M. Zedda, On the convergence of the Sasaki J-flow, Ann. Glob. Anal. Geom. 49 (2016) 393-407.
[37] J. Zhou, Y. Chu, The complex Hessian quotient flow on compact Hermitian manifolds, AIMS Math. 7(5) (2022) 7441-7461.
(Liangdi Zhang)
1. **Yanqi Lake Beijing Institute of Mathematical Sciences and Applications, Beijing 101408, P. R. China**
2. **Yau Mathematical Sciences Center, Tsinghua University, Beijing 100084, P. R. China**

*Email address: ldzhang91@163.com*