INTRODUCTION

Lusztig defined in [Lus91] a canonical basis of the quantum group attached to any quiver without loop. This definition was possible thanks to an isomorphism between this quantum group and the Grothendieck group of a category of perverse sheaves, generated by the so-called Lusztig sheaves. Lusztig endowed this Grothendieck group with a structure of Hopf algebra, by means of restriction and induction functors. These functors made it possible for him to perform induction proofs via a nice stratification of his category. This construction yielded a combinatorial structure on the canonical basis which would later be recognized as a Kashiwara crystal.

There are more and more evidences of the relevance of the study of quivers with loops. A particular class of such quivers are the comet-shaped quivers, which have recently been used by Hausel, Letellier and Rodriguez-Villegas in their study of the topology of character varieties, where the number of loops at the central vertex is the genus of the considered curve (see [HRV08] and [HLRV13]). We can also see quivers with loops appearing in a work of Nakajima relating quiver varieties with branching (see [Nak09]), as in the work of Okounkov and Maulik about quantum cohomology (see [MO12]).

Kang and Schiffmann generalized Lusztig constructions in the framework of generalized Kac-Moody algebra in [KS06], using quivers with loops. In this case, one has to impose a somewhat unnatural restriction on the definition of a category
of perverse sheaves, considering only those attached to complete flags on imaginary vertices.

In this article we consider the general definition of Lustig sheaves for arbitrary quivers, possibly carrying loops. We therefore follow the definition given in [Lus93], and use the results obtained in this article for quivers with one vertex and multiple loops. Note that the category hence considered is bigger than the one considered in [KS06], as one may already see in the case of the Jordan quiver. We prove a conjecture raised by Lusztig in [Lus93], asking if the more "simple" Lustig perverse sheaves are enough to span the whole Grothendieck group considered. A partial proof was given in [LL09]. Our proof is also based on induction, still with the help of restriction and induction functors, but with non trivial first steps, consisting in the study of quivers with one vertex but possible loops. We also need to consider regularity conditions on the support of our perverse sheaves to perform efficient restrictions at imaginary vertices. From our proof emerges a new combinatorial structure on our generalized canonical basis, which is more general than the usual crystals, in that there are now more operators associated to a vertex with loops, as in [Boz13] (see 1.12).

In a second part, we construct and study a Hopf algebra which generalizes the usual quantum groups. The geometric study previously made leads to a natural definition, which includes countably infinite sets of generators at imaginary roots, with higher order Serre relations and commutativity conditions imposed by the Jordan quiver case. We prove that the positive part of this algebra is isomorphic to our Grothendieck group, thanks to the study of a nondegenerate Hopf pairing.

In a final section, we try to build a bridge with the Lagrangian varieties studied in [Boz13], using our new Hopf algebra, as the classical case suggests (see [Lus91]).

Acknowledgement. I would like to thank Olivier Schiffmann for his constant support and availability during the preparation of this work.

1. Quiver Varieties

1.1. Preliminaries. Let $Q$ be a quiver, with vertex set $I$ and oriented edge set $\Omega = \{h : s(h) \to t(h)\}$. We will denote by $\Omega(i)$ the set of loops at $i$, and call $i$ imaginary if $\omega_i = |\Omega(i)| \geq 1$, real otherwise.

For every $\alpha = \sum_{i \in I} \alpha_i i \in NI$, we fix an $I$-graded vector space $V_\alpha$ of graded dimension $\alpha$. For every $I$-graded vector space $X$, we set:

$$E_X = \bigoplus_{h \in \Omega} \text{Hom}(X_{s(h)}, X_{t(h)})$$

and $E_\alpha = E_{V_\alpha}$. We also denote by $G_\alpha$ the group $\prod_{i \in I} GL(V_{\alpha_i})$, naturally acting on $E_\alpha$. Take $m > 0$ and two sequences $i = (i_1, \ldots, i_m)$ and $a = (a_1, \ldots, a_m)$ of $I$ and $\mathbb{N}_{>0}$. We write $(i, a) \vdash \alpha$ if $\sum_{1 \leq k \leq m} a_k i_k = \alpha$. We set:

$$\mathcal{F}_{i,a} = \left\{ W = (\{0\} = W_0 \subset \ldots \subset W_m = V_\alpha) \mid \forall k, \dim \frac{W_k}{W_{k-1}} = a_k i_k \right\}$$

$$\tilde{\mathcal{E}}_{i,a} = \left\{ (x, W) \mid x_h(W) \subseteq W \right\} \subseteq E_\alpha \times \mathcal{F}_{i,a}$$

so that we get a proper morphism $\pi_{i,a} : \tilde{\mathcal{E}}_{i,a} \to E_\alpha$ induced by the first projection.
Following [Lus10], we will denote by $\mathcal{M}_G(X)$ the category of $G$-equivariant perverse sheaves on an algebraic variety $X$ equipped with an action of an algebraic connected group $G$.

Thanks to the decomposition theorem of Beilinson, Bernstein and Deligne (see [BBD82]), the complex $\pi_{i,a}1$ is semisimple. Denote by $\mathcal{P}_\alpha \subseteq \mathcal{M}_G(E_\alpha)$ the additive category consisting of sums of $G_\alpha$-equivariant simple perverse sheaves appearing (possibly with a shift) in $\pi_{i,a}1$ for some $(i,a) \vdash \alpha$. Here $1$ stands for the constant perverse sheaf on $\tilde{E}_{i,a}$.

Denote by $\mathcal{Q}_\alpha$ the category of complexes isomorphic to sums of shifts of sheaves of $\mathcal{P}_\alpha$.

Let $K_\alpha$ be the Grothendieck group of $\mathcal{Q}_\alpha$, seen as a $\mathbb{Z}[v^{\pm 1}]$-module by setting $v^{\pm 1}[\mathcal{P}]=[\mathcal{P}[\pm 1]]$, $[\mathcal{P}]$ denoting the isoclass of a perverse sheaf $\mathcal{P}$. We will finally denote by $B_\alpha$ the finite set of isoclasses of simple perverse sheaves in $\mathcal{P}_\alpha$, and we set $B = \sqcup_\alpha B_\alpha$.

For every $I$-graded subspace $W \subseteq V_\alpha$ of dimension $\beta$ and codimension $\gamma$, equipped with two $I$-graded isomorphisms $p : W \xrightarrow{\sim} V_\beta$ and $q : V_\alpha/W \xrightarrow{\sim} V_\gamma$, we have the following diagram:

$$E_\beta \times E_\gamma \xrightarrow{\kappa} E_\alpha(W) \xrightarrow{\iota} E_\alpha$$

where $E_\alpha(W) = \{x \in E_\alpha \mid x(W) \subseteq W\}$, $\kappa : x \mapsto (p_*(x_{W}), q_*(x_{V_\alpha/W}))$ and $\iota$ is the inclusion. Note that $\kappa$ is a vector bundle.

We will also consider:

$$E_\beta \times E_\gamma \xrightarrow{p_1} E_{\beta,\gamma}^\dagger \xrightarrow{p_2} E_{\beta,\gamma} \xrightarrow{p_3} E_\alpha$$

where:

$$E_{\beta,\gamma}^\dagger = \left\{ (x,W,r,\bar{r}) \mid \begin{array}{ll} x \in E_\alpha \\ W \subseteq V_\alpha \text{ is } I\text{-graded and } x\text{-stable} \\ r : W \xrightarrow{\sim} V_\beta \\ \bar{r} : V_\alpha/W \xrightarrow{\sim} V_\gamma \end{array} \right\}$$

$$E_{\beta,\gamma} = \left\{ (x,W) \mid \begin{array}{ll} x \in E_\alpha \\ W \subseteq V_\alpha \text{ is } I\text{-graded and } x\text{-stable} \end{array} \right\}.$$  

These diagrams induce (cf. [Lus10] §9.2):

$$\widetilde{\text{Res}}_{\beta,\gamma} = \kappa_! \iota^* : \mathcal{Q}_\alpha \to \mathcal{Q}_\gamma \boxtimes \mathcal{Q}_\beta$$

$$\widetilde{\text{Ind}}_{\beta,\gamma} = p_3 p_2 p_1^* : \mathcal{Q}_\gamma \boxtimes \mathcal{Q}_\beta \to \mathcal{Q}_\alpha$$

and:

$$\text{Res}_{\beta,\gamma} \alpha = \widetilde{\text{Res}}_{\beta,\gamma} \alpha \left[d_1 - d_2 - 2\langle \beta, \gamma \rangle \right]$$

$$\text{Ind}_{\beta,\gamma} \alpha = \widetilde{\text{Ind}}_{\beta,\gamma} \alpha \left[d_1 - d_2 \right]$$

where $d_1$ and $d_2$ denote the dimensions of the fibers of $p_1$ and $p_2$, and $\langle \beta, \gamma \rangle = \sum_{i \in I} \beta_i \gamma_i$. These functors endow $\mathcal{K} = \oplus_\alpha K_\alpha$ with a Hopf algebra structure (see [Lus91] 10]). Setting $(\gamma,\beta) = \sum_{h \in \Omega} \gamma_h \beta_i(h)$, observe that:

$$d_1 - d_2 = (\gamma,\beta) + (\beta,\gamma)$$

$$d_1 - d_2 - 2\langle \beta, \gamma \rangle = (\gamma,\beta) - (\beta,\gamma).$$
1.2. Study of an imaginary sink. Let \( i \) be an imaginary sink, and \((i, a) \vdash \alpha\). Take \( a_i = (a_{k_1}, \ldots, a_{k_r}) \) where \( k_j < k_{j+1} \) and \( \{k_j\}_{1 \leq j \leq r} = \{ k \mid i_k = i \} \). For \( x \in E_\alpha \), we set \( x^{(i)} = (x_{h})_{h \in \Omega(i)} \) and \( x^0 = (x_{h})_{h \in \Omega(i)} \). Then, we define:

\[
\tilde{E}^{(i)}_{1,a} = \{(x, W^{(i)}) \mid x^{(i)}(W^{(i)}) \subseteq W^{(i)}\} \subseteq E_\alpha \times \mathcal{F}^{(i)}_{a_i} \\
E^0_\alpha = \{x \in E_\alpha \mid x^{(i)} = 0\}
\]

where \( \mathcal{F}^{(i)}_{a_i} \) denotes the variety of flags of \( V_{\alpha, i} \) of dimension \( a_i \). We have the following diagram:

\[
\begin{array}{ccc}
\tilde{E}^{(i)}_{1,a} & \xrightarrow{\pi^{(i)}_{1,a}} & E_\alpha \\
\psi \downarrow & & \downarrow \pi_i \\
\tilde{E}^0_{1,a} & \xrightarrow{\phi} & E^0_\alpha \times \mathcal{F}^{(i)}_{a_i}
\end{array}
\]

where \( \tilde{E}^0_{1,a} = \{(x, W) \in \tilde{E}_{1,a} \mid x^{(i)} = 0\} \). Note that \( \psi \) and \( V_{a_i} \) are vector bundles.

1.2.1. A notion of regularity. Put:

\[
E^{i, \mathrm{rss}}_\alpha = \{x \in E_\alpha \mid x_i is regular semisimple if \( h \in \Omega(i) \}\}.
\]

For any constructible subsets \( X \subseteq E_\alpha, Y \subseteq \tilde{E}_{1,a} \) and \( Z \subseteq \tilde{E}^{(i)}_{1,a} \), we put:

\[
X^{i, \mathrm{rss}} = X \cap E^{i, \mathrm{rss}}_\alpha \\
Y^{i, \mathrm{rss}} = Y \cap \pi_{1,a}^{-1}(E^{i, \mathrm{rss}}_\alpha) \\
Z^{i, \mathrm{rss}} = Z \cap \pi_{1,a}^{-1}(E^{i, \mathrm{rss}}_\alpha).
\]

We also write \( \rho_\alpha : E^{i, \mathrm{rss}}_\alpha \rightarrow E_\alpha \) for the open inclusion.

**Proposition 1.2.** Let \( P \) be any simple element of \( P_\alpha \). Then \( P = \rho_{\alpha, i} \cdot \rho'_\alpha \cdot P \), i.e. if \( P = \mathrm{IC}(X, \mathcal{L}) \) for some smooth irreducible subvariety \( Y \subseteq E_\alpha \) and some local system \( \mathcal{L} \) on \( Y \), then \( Y^{i, \mathrm{rss}} \neq \emptyset \).

**Proof.** By definition, \( P \) appears as a simple summand of \( \pi_{1,a}'' \cdot Q \) for some simple component \( Q \subseteq \pi_{1,a}' \cdot 1 \). Since in \([13]\) \( \psi \) is a vector bundle and the square is cartesian, \( Q \subseteq V_{a_i} \cdot \phi \cdot 1 \), and thus \( Q \) is of the form \( \mathrm{IC}(X, \mathcal{S}) \) where \( X = V_{a_i}^{-1}(Y) \) for an irreducible smooth subvariety \( Y \subseteq E^{\alpha}_\alpha \times \mathcal{F}^{(i)}_{a_i} \), and \( \mathcal{S} = V_{a_i} \cdot \mathcal{L} \) for an irreducible local system \( \mathcal{L} \) on \( Y \).

In the lemma below, we call **quasismall** a map of algebraic varieties \( \pi : X \rightarrow Y \) satisfying the following property: there exist stratifications \( X = \bigsqcup_{j \in I} X_j, Y = \bigsqcup_{j \in J} Y_j \) over a finite set \( J \) containing an element \( 0 \) such that:

1. \( X_0 \) and \( Y_0 \) are dense;
2. \( \pi|_{X_j} : X_j \rightarrow Y_j \) is a locally trivial fibration of fiber \( F_j \) if \( j \neq 0 \);
3. \( \pi|_{X_0} : X_0 \rightarrow Y_0 \) is a finite morphism;
4. \( 2 \dim F_j < \operatorname{codim}_Y Y_j \) if \( j \neq 0 \).

**Lemma 1.3.** Let \( S \) be a smooth irreducible subvariety of \( E^{\alpha}_\alpha \times \mathcal{F}^{(i)}_{a_i} \). Put \( \tilde{S} = V_{a_i}^{-1}(S) \) and \( \tilde{S} = \pi_{1,a}''(\tilde{S}) \). Then the map \( \pi_{1,a}'' : \tilde{S} \rightarrow \tilde{S} \) is quasismall.
Proof of the lemma. Put $S^0 = \hat{S}^{i,\mathrm{rss}}$, which is a nonempty open dense subset of $\hat{S}$. Moreover, the restriction of $\pi''_{1,a}$ to $\hat{S}^0$ is a finite morphism since a regular semisimple element $z_h$ for $h \in \Omega(i)$ stabilizes only finitely many flags of subspaces of $V_{\alpha,i}$. Put $\tilde{T} = \hat{S} \setminus \hat{S}^0$. To prove that $\pi''_{1,a}|\hat{S} : \hat{S} \to \hat{S}$ is quasismall, it now suffices to check that:

$$\dim(T \times E_\alpha \tilde{T}) < \dim \hat{S}.$$ 

Let $z = (z_h,k)$ be a $r \times r$-matrix of nonnegative integers such that $\sum_h z_h,k = a_k$, $\sum_k z_h,k = a_h$, and set:

$$(\tilde{S} \times E_\alpha \hat{S})_z = \left\{ (x, W, W') \mid \forall h, k \dim \frac{W_h \cap W'_k}{W_{h-1} \cap W'_k + W_h \cap W'_{k-1}} = z_h, k \right\}.$$ 

This yields a finite stratification $\tilde{S} \times E_\alpha \hat{S} = \sqcup z (\tilde{S} \times E_\alpha \hat{S})_z$. We use the same notations for $\tilde{S} \times E_\alpha \hat{S}$ and $\tilde{T} \times E_\alpha \tilde{T}$. The fibers of $V_{a_i}|\hat{S} : \hat{S} \to \hat{S}$ being the same as those of $\tilde{E}_{i,a_i} \to \tilde{F}_{a_i}^{(i)}$, we have for any $z$ as above:

(1.4) $$\dim(\tilde{S} \times E_\alpha \hat{S})_z - \dim(\tilde{S} \times E_\alpha \hat{S})_z = \dim(\tilde{E}_{i,a_i} \times E_\alpha \tilde{E}_{i,a_i})_z - \dim(\tilde{F}_{a_i}^{(i)} \times \tilde{F}_{a_i}^{(i)})_z$$

and:

$$\dim(\tilde{T} \times E_\alpha \tilde{T})_z - \dim(\tilde{T} \times E_\alpha \tilde{T})_z = \dim(\tilde{E}_{i,a_i} \times U_{\alpha,i} \tilde{E}_{i,a_i})_z - \dim(\tilde{F}_{a_i}^{(i)} \times \tilde{F}_{a_i}^{(i)})_z$$

where $U_{\alpha,i} = E_{\alpha,i} \setminus E_{\alpha,i}^{i,\mathrm{rss}}$. If $\omega_i = 1$, it is very well known that the map $E_{i,a_i} \to E_{\alpha,i}$ is quasismall, with $E_{\alpha,i}^{i,\mathrm{rss}}$ being the only relevant stratum. Indeed, it is true if $a_i = (1^{\alpha_i})$, and we have the following commutative diagram:

$$\begin{array}{ccc}
\tilde{E}_{i,(1^{\alpha_i})} & \xrightarrow{f} & E_{\alpha,i} \\
\downarrow g & & \downarrow h \\
\tilde{E}_{i,a_i} & \xrightarrow{h} & E_{\alpha,i}
\end{array}$$

where $g$ is projective, hence $f$ quasismall implies $h$ quasismall. It follows that:

(1.5) $$\dim(\tilde{E}_{i,a_i} \times U_{\alpha,i} \tilde{E}_{i,a_i})_z < \dim \tilde{E}_{i,a_i}.$$ 

By [Lus93], this strict inequality is also true if $\omega_i \geq 2$. Indeed, the large inequality is true for any $z$ if we replace $U_{\alpha,i}$ by $E_{\alpha,i}$, and, since $\dim U_{\alpha,i} < \dim E_{\alpha,i}$:

$$\dim(\tilde{E}_{i,a_i} \times U_{\alpha,i} \tilde{E}_{i,a_i})_z < \dim(\tilde{E}_{i,a_i} \times E_{\alpha,i} \tilde{E}_{i,a_i})_z \leq \dim \tilde{E}_{i,a_i}.$$
hence \(1.5\) is still satisfied. But then:

\[
\dim \tilde{S} - \dim (\tilde{T} \times E_{\alpha, i}\tilde{T})_z
\]

\[
= \dim \tilde{S} - \dim (S \times E_{\alpha, i}^0 S)_z + \dim (S \times E_{\alpha, i}^0 S)_z - \dim (\tilde{S} \times E_{\alpha, i}\tilde{S})_z
\]

\[
= \dim \tilde{S} - \dim (S \times E_{\alpha, i}^0 S)_z
\]

\[
- \dim (\tilde{E}_{l, a_i} \times E_{\alpha, i}, \tilde{E}_{l, a_i})_z + \dim (\mathcal{F}_{\alpha, i}^{(1)} \times \mathcal{F}_{\alpha, i}^{(1)})_z
\]

[use \(1.4\)]

\[
> \dim \tilde{S} - \dim (S \times E_{\alpha, i}^0 S)_z - \dim \tilde{E}_{l, a_i} + \dim (\mathcal{F}_{\alpha, i}^{(1)} \times \mathcal{F}_{\alpha, i}^{(1)})_z
\]

[use \(1.5\)]

\[
= \dim S - \dim (S \times E_{\alpha, i}^0 S)_z - \dim \mathcal{F}_{\alpha, i}^{(1)} + \dim (\mathcal{F}_{\alpha, i}^{(1)} \times \mathcal{F}_{\alpha, i}^{(1)})_z
\]

[use \(1.4\) with \(z\) diagonal]

\[
= \text{codim} ((E_{\alpha, i}^{(1)} \times E_{\alpha, i}^{(1)})(E_{\alpha, i}^{0}(E_{\alpha, i}^{(1)})), (S \times E_{\alpha, i}^0 S)_z) - \text{codim} E_{\alpha, i}^{(1)} S
\]

\[
\geq 0,
\]

the last inequality being true thanks to the following diagram:

\[
(S \times E_{\alpha, i}^0 S)_z \xleftarrow{\iota} X \xrightarrow{E_{\alpha, i}^0 \times (\mathcal{F}_{\alpha, i}^{(1)} \times \mathcal{F}_{\alpha, i}^{(1)})_z} \square \xrightarrow{\text{id} \times \text{pr}_1} S \xrightarrow{E_{\alpha, i}^0 \times \mathcal{F}_{\alpha, i}^{(1)}}
\]

The lemma is proved. \(\square\)

End of proof of proposition \(1.2\) For any stratum \(S \subseteq \overline{Y}\) for IC(\(Y, \xi\)), the subvariety \(\tilde{S} = V_{\alpha, i}^{-1}(S)\) is a stratum for \(\mathcal{Q}\). By \(1.3\) the restriction of \(\pi''_{\alpha, i}\) to each of these strata is quasismall. By an argument identical to that in \([KS07, 1]\), it follows that \(\pi''_{\alpha, i, !}\mathcal{Q}\) is a perverse sheaf, and that moreover any simple summand of \(\pi''_{\alpha, i, !}\mathcal{Q}\) is an intermediate extension to \(E_{\alpha, i}\) of a simple direct summand of \(\pi''_{1, \alpha, !}(V_{\alpha, i}^0(\xi_0))\) for some irreducible local system \(\xi_0\) on a stratum \(S\). In particular, it is of the form IC(\(R, \delta\)) where \(R\) is an open subset of \(\pi''_{1, \alpha, !}(\tilde{S}^0)\) for some \(S\), and \(\delta\) is an irreducible local system on \(R\). The proposition follows from the fact that, by construction, \(\pi''_{1, \alpha, !}(\tilde{S}^0) \subseteq E_{\alpha, i}^{(1)}\). \hfill \(\square\)

1.2.2. A notion of invariance. For any \(x \in E_{\alpha, i}\), put \(V_{\alpha, i}^0 = \oplus_{j \neq i} V_{\alpha, j}\) and \(\mathcal{I}_i(x) = \mathbb{C}(x).V_{\alpha, i}^0\), i.e. the smallest subspace of \(V_{\alpha, i}\) stable by \(x\) and containing \(V_{\alpha, i}^0\).

**Definition 1.6.** Let us write \(x \sim_i x'\) if the following holds:

1. \(x^0 = x'^0;\)
2. \(\mathcal{I}_i(x) \subseteq \cap_{h \in \Omega_i} \ker(x_h - x'_h);\)
3. \(\sum_{h \in \Omega_i} \text{Im}(x_h - x'_h) \subseteq \mathcal{I}_i(x).\)

**Lemma 1.7.** \(\sim_i\) is an equivalence relation.

**Proof.**

- Reflexivity is obvious.
- Symmetry: if \(x \sim_i x'\), then \(\mathcal{I}(x') = \mathcal{I}(x)\) since \(\mathbb{C}(x'^0).V_{\alpha, i}^0 = \mathbb{C}(x^0).V_{\alpha, i}^0 \subseteq \mathcal{I}_i(x)\) and since \(x^{(1)}_{\mathcal{I}_i(x)} = x'^{(1)}_{\mathcal{I}_i(x)}\). This implies \(x' \sim_i x\).
• Transitivity: if \( x \sim_i x' \) and \( x' \sim_i x'' \), we have \( \mathcal{I}_i(x) = \mathcal{I}_i(x') = \mathcal{I}_i(x'') \), \( x |\mathcal{I}_i(x) = x' |\mathcal{I}_i(x) = x'' |\mathcal{I}_i(x) \), and if \( h \in \Omega(i) \):

\[
\text{Im}(x_h - x''_h) \subseteq \text{Im}(x_h - x'_h) + \text{Im}(x'_h - x''_h) \subseteq \mathcal{I}_i(x).
\]

Hence \( x \sim_i x'' \).

Observe that equivalence classes are affine spaces. If \( x \in \mathcal{E}_\alpha \), then the equivalence class of \( x \) is of dimension equal to \( \omega_i \gamma_i (\alpha_i - \gamma_i) \) where \( \omega_i = |\Omega(i)| \) and \( \gamma_i = \text{codim}_{\mathcal{E}_\alpha} \mathcal{I}_i(x) \).

There is a stratification \( E_\alpha = \sqcup_{\gamma \geq 0} E_{\alpha,i,\gamma} \) where:

\[
E_{\alpha,i,\gamma} = \{ x \in \mathcal{E}_\alpha \mid \text{codim}_{\mathcal{E}_\alpha} \mathcal{I}_i(x) = \gamma_i \}.
\]

Note that \( E_{\alpha,i,\gamma} \) is a union of \( \sim_i \)-equivalence classes. This can be made more precise as follows. Fix \( \gamma \leq \alpha_i \) and \( W \subseteq \mathcal{V}_\alpha \) an \( I \)-graded subspace of codimension \( \gamma_i \). Let \( E_{\alpha,i,\gamma}(W) = E_{\alpha,i,\gamma} \cap E_\alpha(W) \) be the closed subvariety of \( E_\alpha \) of elements \( x \in \mathcal{E}_\alpha \) such that \( \mathcal{I}_i(x) = W \). Then, if \( P = \text{Stab}_{\mathcal{V}_\alpha}(W) \),

\[
E_{\alpha,i,\gamma} = G_\alpha \times_P E_{\alpha,i,\gamma}(W),
\]

hence the inclusion \( \iota_0 : E_{\alpha,i,\gamma}(W) \hookrightarrow E_{\alpha,i,\gamma} \) induces an equivalence of categories of perverse sheaves:

\[
\iota_0^*[-d] : \mathcal{M}_{G_\alpha}(E_{\alpha,i,\gamma}) \to \mathcal{M}_P(E_{\alpha,i,\gamma}(W))
\]

where \( d = \dim(G_\alpha/P) \). Observe also that \( E_{\alpha,i,\gamma}(W) \) is itself a union of \( \sim_i \)-equivalence classes. Here \( \iota_0 \) is a restriction of the inclusion \( \iota \) introduced in [1,1] with \( \gamma_i \) in place of \( \gamma \).

Now, as in [1,1] fix \( I \)-graded isomorphisms \( W \cong V_{\alpha - \gamma_i} \) and \( V_\alpha/W \cong V_{\gamma_i} \). We have a natural vector bundle map:

\[
\kappa_0 : E_{\alpha,i,\gamma}(W) \to E_{\alpha - \gamma_i,i,0} \times E_{\gamma_i}
\]

whose fibers are precisely the \( \sim_i \)-equivalence classes in \( E_{\alpha,i,\gamma}(W) \). Again, \( \kappa_0 \) is a restriction of the vector bundle \( \kappa \) introduced in [1,1] with \( \gamma_i \) in place of \( \gamma \). There is a fully faithful embedding:

\[
\kappa_0^*[\omega_i d] : \mathcal{M}_{G_{\alpha - \gamma_i} \times G_{\gamma_i}}(E_{\alpha - \gamma_i,i,0} \times E_{\gamma_i}) \to \mathcal{M}_P(E_{\alpha,i,\gamma}(W)).
\]

We say that a perverse sheaf \( \mathcal{P} \in \mathcal{M}_{G_\alpha}(E_{\alpha,i,\gamma}) \) is \( \sigma \)-invariant (at \( i \)) if \( \iota_0^*[-d](\mathcal{P}) \) belongs to the essential image of \( \kappa_0^*[\omega_0 d] \).

**Definition 1.8.** Let \( \mathcal{P}_{\alpha,i,\gamma} \subseteq \mathcal{P} \) be the set of perverse sheaves supported on \( E_{\alpha,i,\gamma} \). The notation \( \mathcal{P}_{\alpha,i,\gamma} \) is defined likewise, and we set \( \mathcal{P}_{\alpha,i,\gamma} = \mathcal{P}_{\alpha,i,\geq \gamma} \setminus \mathcal{P}_{\alpha,i,\gamma} \). The terms \( \mathcal{P}_{\alpha,i,\leq \gamma}, \mathcal{P}_{\alpha,i,< \gamma} \) are defined similarly.

We will need the following technical result:

**Proposition 1.9.** Let \( \mathcal{P} \) be any simple element of \( \mathcal{P}_{\alpha,i,\gamma} \). Let \( m : E_{\alpha,i,\gamma} \hookrightarrow E_{\alpha,i,\geq \gamma} \) be the open embedding. The perverse sheaf \( m^*\mathcal{P} \in \mathcal{M}_{G_\alpha}(E_{\alpha,i,\gamma}) \) is \( \sigma \)-invariant at \( i \).
Proof. The proof follows closely that of [1,2] whose notations we keep. In particular \( \mathbf{P} = \text{IC}(R, \mathfrak{A}) \) where \( R \) is an open subset of \( \pi''_{1,a}'(\tilde{S}^0) \) for some \( G_\alpha \)-invariant stratum \( S \subseteq E_\alpha \times \mathcal{F}_{a_i}^{(i)} \). Moreover \( \mathbf{P} \) appears in some complex:

\[
\mathbf{R} = j_{!*}\left( \pi''_{1,a!}( (V_a, \mathcal{L} )|_{\tilde{S}^0} ) \right)
\]

where \( j : \pi''_{1,a}(\tilde{S}^0) \to E_\alpha \) is the inclusion and where \( \mathcal{L} \) is a certain \( G_\alpha \)-equivariant local system on \( S \). It suffices to show that \( \mathbf{R} \) is \( \sigma \)-equivariant.

Consider a stratification \( \tilde{S} = \bigcup_k S(k) \) where:

\[
S(k) = \{(x^0, W) \in S \mid \text{Im}(x^0) \cap V_{\alpha,i} \subseteq W_k \text{ but } \text{Im}(x^0) \cap V_{\alpha,i} \nsubseteq W_{k-1} \}.
\]

Let \( k \) be maximal such that \( S(k) \neq \emptyset \). Then \( S(k) \) is open and dense in \( S \). Denote by \( \tilde{S} = \bigcup_l \tilde{S}(l) \) the induced stratification of \( \tilde{S} \). Then \( \tilde{S}(k) \) is also open and dense in \( \tilde{S} \). Finally, set:

\[
\tilde{S}(k)^\square = \{(x, W) \in \tilde{S}(k)^{\text{rss}} \mid \mathcal{I}_i(x) = W_k \}.
\]

It is easy to see that \( \tilde{S}(k)^\square \) is open and dense in \( \tilde{S}(k) \), hence in \( \tilde{S} \).

Put \( \gamma = \sum_{i > k} a_i \) so that \( \gamma = \text{codim}_{V_{\alpha,i}} W_k \) for any \( W \in \mathcal{F}_{a_i}^{(i)} \). Let \( W \) an \( I \)-graded subspace of \( V_a \) of codimension \( \gamma_i \) with fixed identifications \( W \simeq V_{\alpha_i - \gamma_i} \) and \( V_{\alpha}/W \simeq V_{\gamma_i} \). Consider the following diagram:

\[
\begin{array}{ccc}
S(k) & \xrightarrow{\mathcal{V}_{a_i}} & \tilde{S}(k)^\square \\
\downarrow{\mathcal{I}_0} & & \downarrow{\mathcal{I}_0} \\
S(k, W) & \xrightarrow{\mathcal{V}_{a_i}} & \tilde{S}(k, W)^\square \\
\downarrow{\mathcal{E}_0} & & \downarrow{\mathcal{E}_0} \\
\Xi & \xrightarrow{\mathcal{E}''} & E_{\alpha - \gamma_i,0} \times E_{\gamma_i}
\end{array}
\]

where:

- \( S(k, W) = \{(x^0, W) \mid W_k = W \} \cap S(k) \subseteq S(k) \);
- \( \tilde{S}(k, W)^\square = \{(x, W) \mid W_k = W \} \cap \tilde{S}(k)^\square \subseteq \tilde{S}(k)^\square \);
- \( \mathcal{I}_0, \mathcal{E}_0 \) and \( \mathcal{E}_0 \) stand for maps induced by \( \mathcal{I}_0 \) and \( \mathcal{E}_0 \);
- \( \pi''_{1,a} \) and \( \mathcal{V}_{a_i} \) (improperly) stand for maps induced by \( \pi''_{1,a} \) and \( \mathcal{V}_{a_i} \);
- \( \Xi = \kappa(\tilde{S}(k, W)^\square) \subseteq \tilde{E}'_{V_{a_i},a'i'} \times \tilde{E}'_{V_{a_i},a'i'} \) where \( (i', a') \vdash \alpha - \gamma_i \) and \( (i'', a'') \vdash \gamma_i \) are naturally induced by \( (i, a) \) and \( k \). Note the existence of an inclusion \( \theta \) making commutative the triangle appearing in the diagram.
- \( \pi'' \) is the restriction of \( \pi''_{V_{a_i},a'i'} \times \pi''_{V_{a_i},a'i'} \) to \( \Xi \).

Observe that the two rightmost squares are cartesian. This is obvious for the top square. For the bottom square, this follows from the fact that for \( x \in E_{\alpha - \gamma_i, \gamma_i} \), a flag \( W \in \mathcal{F}_{a_i}^{(i)} \) satisfying \( W_k = \mathcal{I}_i(x) \) is \( x \)-stable if and only if it is \( x' \)-stable for any \( x' \sim_i x \).

Because \( \tilde{S}(k)^\square \) is open and dense in \( \tilde{S}^0 \) and \( \pi''_{1,a}|_{\tilde{S}^0} \) is finite, we have:

\[
\mathbf{R} = j_{!*}\left( \pi''_{1,a!}( (V_a, \mathcal{L} )|_{\tilde{S}(k)^\square} ) \right)
\]
where \( j' : \pi''_{1,a}(\tilde{S}(k)) \hookrightarrow E_\alpha \) is the inclusion. Note that by construction \( R \) is a direct sum of objects in \( P_{\alpha,i,\gamma} \). We have:

\[
m^* R = j''(\pi''_{1,a}( (V_{\mathbf{a}, \mathcal{L}})_{\tilde{S}(k)} ))
\]

where now \( j'' \) and \( m \) denote the inclusions defined by the following commutative diagram:

\[
\begin{array}{ccc}
\pi''_{1,a}(\tilde{S}(k)) & \xrightarrow{j''} & E_\alpha \\
\downarrow m & & \downarrow \pi_{\alpha,i,\gamma} \\
E_{\alpha,i,\gamma} & & E_{\alpha,i,\gamma}
\end{array}
\]

Furthermore, if \( j''(W) : \pi''_{1,a}(\tilde{S}(k,W)) \hookrightarrow E_{\alpha,i,\gamma}(W) \) denotes the inclusion induced by \( j'' \),

\[
\iota_0^* m^* R = \iota_0^* j''(\pi''_{1,a}( (V_{\mathbf{a}, \mathcal{L}})_{\tilde{S}(k,W)} ))
\]

\[
= j''(W) \circ \iota_0^* \pi''_{1,a}( (V_{\mathbf{a}, \mathcal{L}})_{\tilde{S}(k,W)} )
\]

[since \( \iota_0^* \) is an equivalence of categories]

\[
= j''(W) \circ \pi''_{1,a}( (V_{\mathbf{a}, \mathcal{L}})_{\tilde{S}(k,W)} )
\]

[the highest rightmost square in (1.10) being cartesian]

\[
= j''(W) \circ \pi''_{1,a}( \kappa_0^* \theta^*( \mathcal{L}_{|S(k,W)} ) )
\]

[the triangle being commutative in (1.10)]

\[
= j''(W) \circ \kappa_0^* \pi''_{1,a}( \theta^*( \mathcal{L}_{|S(k,W)} ) )
\]

[the lowest rightmost square in (1.10) being cartesian]

\[
= \kappa_0^* \lambda \varepsilon_1 \pi''_{1,a}( \theta^*( \mathcal{L}_{|S(k,W)} ) )
\]

where \( \lambda : \pi''(\Xi) \hookrightarrow E_{\alpha-\gamma_i,0} \times E_{\gamma_i} \) is the inclusion (recall that \( \kappa_0 \) is a vector bundle). It follows that \( m^* R \) is \( \sigma \)-invariant as wanted. The proposition is proved.

\[
\square
\]

1.3. A crystal type structure on \( B \). We keep the same notations. In particular, \( i \) is an imaginary sink and \( W \) is an \( I \)-graded subspace of \( V_0 \) of codimension \( \gamma_i \), with stabilizer \( P \subseteq G_\alpha \). We also denote by \( U \) the unipotent radical of \( P \).

**Proposition 1.11.** Set \( d = \dim(G_\alpha/P) \).

(1) Consider \( A \in P_{\alpha-\gamma_i,0} \boxtimes P_{\gamma_i} \). For every \( n \) we have:

\[
\text{supp}(H^n \text{Ind}_{\alpha-\gamma_i,\gamma_i} A) \subseteq E_{\alpha,i,\gamma}.
\]

If \( n \neq 0 \), we have:

\[
\text{supp}(H^n \text{Ind}_{\alpha-\gamma_i,\gamma_i} A) \cap E_{\alpha,i,\gamma} = \emptyset.
\]

Otherwise, the sum of the simple components of \( H^0 \text{Ind}_{\alpha-\gamma_i,\gamma_i} A \) belonging to \( P_{\alpha,\gamma_i} \) is nontrivial, and we denote it by \( \xi(A) \).

(2) Consider \( B \in P_{\alpha,i,\gamma} \). If \( n \neq -2\omega_i d \), we have:

\[
\text{supp}(H^n \text{Res}_{\alpha-\gamma_i,\gamma_i} B) \cap E_{\alpha-\gamma_i,0} \times E_{\gamma_i} = \emptyset.
\]
Otherwise, the sum of the simple components of $H^{-2i}(\text{Res}_{a-i,\gamma}) B$ belonging to $\mathcal{P}_{a-i,\gamma} \subseteq \mathcal{P}_{a-i}$ is nontrivial, and we denote it by $\rho(B)$.

(3) The functors $\xi$ and $\rho$ are equivalences of categories inverse to each other.

**Proof.** We will use the following diagram:

$$
\begin{array}{ccc}
G_\alpha \times_p E_{\alpha,i,\gamma}(W) & \xrightarrow{p_0} & E_{\alpha,i,\gamma}(W) \\
\downarrow m_0 & & \downarrow m \\
G_\alpha \times_p E_\alpha(W) & \xrightarrow{p^\ast p_3} & E_\alpha(W) \\
\downarrow \kappa & & \downarrow \kappa \\
E_{\alpha-i,\gamma} \times E_{\gamma_i} & & \end{array}
$$

To prove (1), we denote by $\tilde{\mathcal{A}}$ the perverse sheaf $p_2 p_1^\ast A[\omega_i + 1]$. Therefore $\tilde{\text{Ind}}_{a-i,\gamma} A = p_1^\ast \mathcal{A}[\omega_i + 1]$, and thus the support of $\text{Ind}_{a-i,\gamma} A$ is included in the image of $p$, equal to $E_{\alpha,i,\gamma}$. The following sheaf:

$$
m^\ast \tilde{\text{Ind}}_{a-i,\gamma} A = m^\ast p_1^\ast \mathcal{A}[\omega_i + 1] = p_0 m_0^\ast \tilde{\mathcal{A}}[\omega_i + 1]
$$

is perverse since $m_0$ is an open embedding, and since $p_0$ is an isomorphism. The support of $H^n \tilde{\text{Ind}}_{a-i,\gamma} A$ being included in $E_{\alpha,i,\gamma}$ for all $n$, we get for $n \neq 0$:

$$
m^\ast H^n \tilde{\text{Ind}}_{a-i,\gamma} A = H^n m^\ast \tilde{\text{Ind}}_{a-i,\gamma} A = 0
$$

which proves (1) since $\tilde{\text{Ind}}_{a-i,\gamma} A[\omega_i + 1] = \text{Ind}_{a-i,\gamma} A$.

To prove (2), we use the fact that $m^\ast B$ is $\sigma$-equivariant, which implies that $\kappa_0 \kappa_0^\ast m^\ast B[\omega_i + 1]$ is perverse. But:

$$
\kappa_0 \kappa_0^\ast m^\ast B[\omega_i + 1] = \mu^\ast \kappa_1 \kappa_1^\ast B[\omega_i + 1] = \mu^\ast \text{Res}_{a-i,\gamma} B[\omega_i + 1],
$$

hence $\mu^\ast \text{Res}_{a-i,\gamma} B[\omega_i + 1]$ is perverse. Since $\mu$ is an open embedding, we have, for $n \neq -2\omega_i$:

$$
\mu^\ast H^n \text{Res}_{a-i,\gamma} B = H^n \mu^\ast \text{Res}_{a-i,\gamma} B = 0
$$

which ends the proof of (2).

We have the following diagram:

$$
\begin{array}{ccc}
E_{\alpha,i,\gamma}(W) & \xrightarrow{\text{pr}_{2,0}} & G_\alpha \times E_{\alpha,i,\gamma}(W) \\
\downarrow \kappa & & \downarrow \pi^p \\
E_\alpha(W) & \xrightarrow{\text{pr}_2} & G_\alpha \times E_\alpha(W) \\
\downarrow \pi^p & & \downarrow \pi^p \\
E_{\alpha-i,\gamma} \times E_{\gamma_i} & \xrightarrow{p_1} & G_\alpha \times E_\alpha(W) \\
\end{array}
$$

where $\kappa \cdot \text{pr}_2 = p_1 \pi^p$ by definition of $p_1$, hence $\text{pr}_2^\ast \kappa^* = \pi^p \ast p_1^\ast$, then $\pi^p \ast \text{pr}_2^\ast \kappa^* = p_1^\ast$, then $p_2 \pi^p \ast \text{pr}_2^\ast \kappa^* = p_2 p_1^\ast$ and thus:

$$
\pi^p_p \ast \text{pr}_2^\ast \kappa^* = p_2 p_1^\ast
$$

since $p_2 \pi^p = \pi^p_p$. 


From the proof of (2) we have \( \mu^* \rho(B) = \kappa_0 t_0^* m^* B[\omega_i + 1]d \), from which we get:
\[
m_0^\sim \rho(B) = m_0^* p_{20}^* p_1^* \rho(B)[\omega_i + 1]d \\
= m_0^* \pi_0^* P_{2,0}^* k^* \rho(B)[\omega_i + 1]d \\
= \pi_0^* P_{2,0}^* k_0^* \mu^* \rho(B)[\omega_i + 1]d \\
= \pi_0^* P_{2,0}^* k_0^* \mu_0^* \kappa_0^* m^* B \\
= \pi_0^* P_{2,0}^* k_0^* m^* B.
\]

But if we denote by \( a, b : G_\alpha \times E_{\alpha} \rightarrow E_{\alpha} \) the action of \( G_\alpha \) on \( E_{\alpha} \) and the second projection, we have:
\[
\pi_0^* P_{2,0}^* k_0^* m^* B = \pi_0^* (id_{G_\alpha} \times \iota_0)^* b^* m^* B \\
= \pi_0^* (id_{G_\alpha} \times \iota_0)^* a^* m^* B \\
[\text{by } G_\alpha\text{-equivariance of } B] \\
= \pi_0^* P_{2,0}^* \pi_0^* m^* B \\
[\text{by definition of } p_0] \\
= p_0^* m^* B.
\]

From the proof of (1), we also have \( m^* \xi(A) = p_0^* m_0^* A \), from which we get:
\[
\mu^* \rho(\xi(A)) = \kappa_0 t_0^* m^* \xi(A)[\omega_i + 1]d \\
= \kappa_0 t_0^* p_0^* m_0^* A[\omega_i + 1]d \\
= \kappa_0 t_0^* p_0^* m_0^* A
\]

but we have seen earlier that for \( G_\alpha\)-equivariant sheaves we have \( pr_{2,0}^* = p_0^* \), hence \( t_0^* p_0^* = pr_{2,0}^* \), and thus:
\[
\mu^* \rho(\xi(A)) = \kappa_0 t_0^* \mu^* A \\
= \mu^* \xi(A)
\]

but also:
\[
m^* \xi(\rho(B)) = p_0^* m_0^* \rho(B) \\
= p_0^* p_0^* m^* B \\
= m^* B.
\]

We finally get (3). \( \square \)

**Proposition 1.12.** With the same hypotheses and notations:

1. Let \( B \) be a simple object of \( \mathcal{P}_{\alpha} \). We have:
   \[
   \text{Res}_{\alpha \rightarrow \gamma} \mathcal{B} \simeq (A \boxtimes C) \oplus (\bigoplus_{j \in \mathbb{Z}} L_j[j])
   \]
   where \( A \) is a simple object of \( \mathcal{P}_{\alpha - \gamma} \), \( C \) a simple object of \( \mathcal{P}_{\gamma} \), and \( L_j \) is the tensor product of an element of \( \mathcal{P}_{\alpha - \gamma} \) and an element of \( \mathcal{P}_{\gamma} \) for all \( j \).

2. Let \( (A, C) \) be a pair of simple objects of \( \mathcal{P}_{\alpha - \gamma} \times \mathcal{P}_{\gamma} \). We have:
   \[
   \text{Ind}_{\alpha \rightarrow \gamma} (A \boxtimes C) \simeq B \oplus (\bigoplus_{j \in \mathbb{Z}} L_j'[j])
   \]
   where \( B \) is a simple object of \( \mathcal{P}_{\alpha} \) and \( L'_j \in \mathcal{P}_{\alpha, \gamma} \) for all \( j \).
(3) The maps \([B] \mapsto ([A], [C])\) and \(([A], [C]) \mapsto [B]\) induced by (1) and (2) are inverse bijections between \(B_{α,i,γ}\) and \(B_{α−γ,i,0} \times B_{γi}\).

**Proof.** As in [Lus10 10.3.2], the proof relies on [Lus93 1.11] using the Fourier-Deligne transform (the result [Lus10 10.3.1] remains true in our setting).

We are now able to answer a question asked by Lusztig in [Lus93 7]. We put \(1_{ai} = π_{ai, ai}\):  

**Proposition 1.13.** The elements \([1_{ai}]\) generate \(K, i ∈ I, a ∈ N_+\).

**Proof.** We proceed by induction on \(α\). Let \(B\) be a simple object of \(P_α\). Using the Fourier-Deligne transform, we may assume that there is a sink \(i\) such that \(B ∈ P_{α,i,γ}\) for some \(γ > 0\) (see [Lus91 7.2]). We then proceed by descending induction on \(γ\). If \(i\) is real, we can conclude as in [Lus91 7.3]. If \(i\) is imaginary, the second part of Proposition 1.12, together with the one vertex quiver case enable us to conclude. Indeed, the case of the Jordan quiver is well known (see e.g. [Sch09a], and the case of the quiver with one vertex and multiple loops is treated in [Lus93]. □

2. A GENERALIZED QUANTUM GROUP

2.1. Generators. Let \((-,-)\) denote the symmetric Euler form on \(ZI\): \((i,j)\) is equal to the opposite of the number of edges of \(Ω\) between \(i\) and \(j\) and \((i,i) = 2 - 2ω_i\). We will denote by \(I^e\) (resp. \(I^m\)) the set of real (resp. imaginary) vertices, and by \(I^o \subseteq I^m\) the set of isotropic vertices: vertices \(i\) such that \((i,i) = 0, i.e. such that \(ω_i = 1\). We also set \(I_∞ = (I^e × \{1\}) ∪ (I^m × N_+)\), and \((i, j) = l(i, j)\) if \(i = (i, l) ∈ I_∞\) and \(j ∈ I\).

**Definition 2.1.** Let \(F\) denote the \(\mathbb{Q}(v)\)-algebra generated by \((E_i)_{i ∈ I_∞}\), naturally \(NI\)-graded by \(deg(E_i,l) = li\) for \((i,l) ∈ I_∞\). We put \(F[A] = \{x ∈ F \mid |x| ∈ A\}\) for any \(A ⊆ NI\), where, for convenience, we denote by \(|x|\) the degree of an element \(x\).

For \(α = ∑ α_i i ∈ ZI\), we set:

\[ht(α) = ∑ α_i\text{ its height};\]
\[v_α = ∏ v_ι^{α_i}\text{ if } v_ι = v(ι,i)/2.\]

We endow \(F ⊗ F\) with the following multiplication:

\[(a ⊗ b)(c ⊗ d) = v((b|c|))(ac) ⊗ (bd).\]

and equip \(F\) with a comultiplication \(δ\) defined by:

\[δ(E_{i,l}) = ∑ v_{i,l'} E_{i,l} ⊗ E_{i,l'}\]

where \((i, l) ∈ I_∞\).

**Proposition 2.2.** For any family \((ν_i)_{i ∈ I_∞}\), we can endow \(F\) with a bilinear form \(\langle -, - \rangle\) such that:

\[\langle x, y \rangle = 0 \text{ if } |x| ≠ |y|;\]
\[\langle E_i, E_i \rangle = ν_i\text{ for all } i ∈ I_∞;\]
\[\langle ab, c \rangle = \langle a ⊗ b, δ(c) \rangle\text{ for all } a, b, c ∈ F.\]

**Proof.** Strictly analogous to [Lus10 Proposition 1.2.3] or [Rin96 3]. □

**Notations 2.3.** Take \(i ∈ I^m\) and \(c\) a composition (i.e. a tuple of positive integers) or a partition (i.e. a decreasing tuple of positive integers). We put \(E_{i,c} = ∏_j E_{i,c_j}\), \(ν_{i,c} = ∏_j ν_{i,c_j}\), and \(|c| = ∑ c_j\).
2.2. Relations.

**Proposition 2.4.** Consider \((i, j) \in I_\infty \times I^\text{re}\). The element:

\[
\sum_{t+t' = -(i, j)+1} (-1)^t E_j^{(t)} E_i E_j^{(t')}
\]

belongs to the radical of \(\langle -, - \rangle\).

**Proof.** Analogous to [Lus10, Proposition 1.4.3] or [Rin97]. □

**Remark 2.6.** Some higher order Serre relations are studied in [Lus10, Chapter 7], where some conditions are given to belong to the radical. However the proofs cannot be directly adapted to our setting.

The following definition is motivated by the previous proposition and our knowledge of the Jordan quiver case, which is related to the classical Hall algebra (see e.g. [Sch09]). We know that the commutators \([E_{i,l}, E_{i,k}]\) lie in the radical if \(i\) is isotropic.

**Definition 2.7.** We denote by \(\tilde{U}^+\) the quotient of \(F\) by the ideal spanned by the elements 2.5 and the commutators \([E_{i,l}, E_{i,k}]\) for every isotropic vertex \(i\), so that \(\langle -, - \rangle\) is still defined on \(\tilde{U}^+\). We denote by \(U^+\) the quotient of \(\tilde{U}^+\) by the radical of \(\langle -, - \rangle\).

**Definition 2.8.** Let \(\hat{U}\) be the quotient of the algebra generated by \(K_i^\pm, E_i, F_i \ (i \in I\) and \(\iota \in I_\infty)\ subject to the following relations:

\[
K_i K_j = K_j K_i
\]

\[
K_i K_i^\pm = 1
\]

\[
K_j E_i = v^{(j,i)} E_i K_j
\]

\[
K_j F_i = \overline{v^{(j,i)}} F_i K_j
\]

\[
\sum_{t+t' = -(i, j)+1} (-1)^t E_j^{(t)} E_i E_j^{(t')} = 0 \quad (j \in I^\text{re})
\]

\[
\sum_{t+t' = -(i, j)+1} (-1)^t F_j^{(t)} F_i F_j^{(t')} = 0 \quad (j \in I^\text{re})
\]

\[
[E_{i,l}, E_{i,k}] = 0 \quad (i \in I^\text{iso})
\]

\[
[F_{i,l}, F_{i,k}] = 0 \quad (i \in I^\text{iso}).
\]

We extend the graduation by \(|K_i| = 0\) and \(|F_i| = -|E_i|\), and we set \(K_\alpha = \prod_i K_i^{\alpha_i}\) for every \(\alpha \in \mathbb{Z} I\).

We endow \(\hat{U}\) with a comultiplication \(\Delta\) defined by:

\[
\Delta(K_i) = K_i \otimes K_i
\]

\[
\Delta(E_{i,l}) = \sum_{t+t' = l} v_t^{i,t} E_{i,l} K_{t'} \otimes E_{i,t'}
\]

\[
\Delta(F_{i,l}) = \sum_{t+t' = l} v_t^{-i,t'} F_{i,l} \otimes K_{-t} F_{i,t'}
\]

We extend \(\langle -, - \rangle\) to the subalgebra \(\hat{U}^0 \subseteq \hat{U}\) spanned by \((K_i^\pm)_{i \in I}\) and \((E_i)_{i \in I_\infty}\) by setting \(\langle x K_i, y K_j \rangle = \langle x, y \rangle v^{(i,j)}\) for \(x, y \in \hat{U}^+\).
We use the Drinfeld double process to define $\hat{U}$ as the quotient of $U$ by the relations:

\begin{equation}
\sum (a_1, b_2) \omega(b_1) a_2 = \sum (a_2, b_1) a_1 \omega(b_2)
\end{equation}

for any $a, b \in \hat{U} \geq 0$, where $\omega$ is the unique involutive automorphism of $\hat{U}$ mapping $E_i$ to $F_i$ and $K_i$ to $K_{-i}$, and where we use the Sweedler notation, for example $\Delta(a) = a_1 \otimes a_2$.

Setting $x^- = \omega(x)$ for $x \in \hat{U}$, we define $\langle -,- \rangle$ on the subalgebra $\hat{U}^- \subseteq \hat{U}$ spanned by $(F_i)_{i \in \mathbb{N}}$ by setting $\langle x, y \rangle = \langle x^-, y^- \rangle$ for any $x, y \in \hat{U}^-$. We will denote by $U^-$ (resp. $\hat{U}$) the quotient of $\hat{U}^-$ (resp. $\hat{U}$) by the radical of $\langle -,- \rangle$ restricted to $\hat{U}^- \times \hat{U}^+$.

**Proposition 2.10.** [Xia97] We can define $S, S^{\text{op}} : U \to U^{\text{op}}$ (the antipode and the skew antipode) such that:

\begin{align*}
\Delta(S \otimes 1) \Delta &= \Delta(1 \otimes S) \Delta = \epsilon 1 \\
\Delta(S^{\text{op}} \otimes 1) \Delta^{\text{op}} &= \Delta(1 \otimes S^{\text{op}}) \Delta^{\text{op}} = \epsilon 1,
\end{align*}

where $\Delta$ denotes the multiplication, $\epsilon$ denotes the counit, which is equal to 1 on $U^0$, and 0 on $U^- \times U^+$, and $\Delta^{\text{op}}$ denotes the composition of $\Delta$ and $\text{op} : U \otimes U \to U \otimes U$, $x \otimes y \mapsto y \otimes x$. We also know that $S^{\text{op}} = S^{-1}$.

2.3. The case of the quiver with one vertex and multiple loops.

**Lemma 2.11.** We have $\langle E_i | c \rangle, E_{i,c} \rangle = v_i \sum_{1 \leq j \leq l} e_k e_j \nu_{i,c}$.

**Proof.** By induction, using the definitions. \hfill \square

**Proposition 2.12.** Let $i \in I$ be a nonisotropic imaginary vertex. Assume that for every $l \geq 1$ we have:

\begin{equation}
\langle E_i | l \rangle, E_{i,l} \rangle \in 1 + v^{-1} \mathbb{N}[v^{-1}] \setminus 0.
\end{equation}

Then, for any compositions $c$ and $c'$,

\begin{equation}
\langle E_i | c \rangle, E_{i,c} \rangle \in \delta_{c,c'} + v^{-1} \mathbb{N}[v^{-1}] \setminus 0.
\end{equation}

**Proof.** For clarity, we forget the indices $i$ in this proof. Notice that by definition of $\delta$, of the multiplication on $\mathbb{F} \otimes \mathbb{F}$, and since $(i, i) < 0$, we already have:

\begin{equation}
\langle E_c | E_{c'} \rangle \in 1 + v^{-1} \mathbb{N}[v^{-1}] \setminus 0.
\end{equation}

Hence, we can work modulo $v^{-1}$, and then, setting $c = (c_1, \ldots, c_r), c' = (c'_1, \ldots, c'_s)$, $\bar{c} = (c_2, \ldots, c_r)$ and $\bar{c}' = (c'_2, \ldots, c'_s)$, we get:

\begin{align*}
\langle E_c | E_{c'} \rangle &= \left\langle E_{c_1} \otimes E_{\bar{c}}, \prod_{1 \leq j \leq s} \delta(E_{c'_j}) \right\rangle \\
&= \left\langle E_{c_1} \otimes E_{\bar{c}}, \prod_{1 \leq j \leq s} (E_{c'_j} \otimes 1 + 1 \otimes E_{c'_j}) \right\rangle \mod v^{-1} \\
&= \begin{cases} 
0 \mod v^{-1} & \text{if } c'_1 \neq c_1 \\
\langle E_{c}, E_{c'} \rangle \mod v^{-1} & \text{otherwise}
\end{cases}
\end{align*}

the second equality coming from the definition of $\delta$, and from $(i, i) < 0$; the last equality coming from the definition of the multiplication on $\mathbb{F} \otimes \mathbb{F}$, from $(i, i) < 0$...
0, from [2.11], and from the hypothesis of the proposition. We end the proof by induction. \[ \square \]

**Corollary 2.14.** Under the assumption 2.13, the restriction of \( \langle -, - \rangle \) to \( F[\mathbb{N}i] \) is nondegenerate.

**Notations 2.15.** We denote by \( C_{i,l} \) the set of compositions \( c \) (resp. partitions) such that \( |c| = l \) if \((i, i) < 0\) (resp. \((i, i) = 0\)).

### 2.4. Quasi \( \mathcal{R} \)-matrix.

**Proposition 2.16.** For any imaginary vertex \( i \) and any \( l \geq 1 \), there exists a unique element \( a_{i,l} \in F[i] \) such that, if we set \( b_{i,l} = a_{i,l}^{-1} \), we get:

1. \( \langle E_{i,l} | l \geq 1 \rangle = \langle a_{i,l} | l \geq 1 \rangle \) and \( \langle F_{i,l} | l \geq 1 \rangle = \langle b_{i,l} | l \geq 1 \rangle \) as algebras;
2. \( \langle a_{i,l}, z \rangle = \langle b_{i,l}, z^- \rangle = 0 \) for any \( z \in \langle E_{i,k} | k < l \rangle \);
3. \( a_{i,l} - E_{i,l} \in \langle E_{i,k} | k < l \rangle \) and \( b_{i,l} - F_{i,l} \in \langle F_{i,k} | k < l \rangle \);
4. \( a_{i,l} = a_{i,l} \) and \( b_{i,l} = b_{i,l} \);
5. \( \Delta(a_{i,l}) = a_{i,l} \otimes 1 + K_{ii} \otimes a_{i,l} \) and \( \Delta(b_{i,l}) = b_{i,l} \otimes K_{-li} + 1 \otimes b_{i,l} \);
6. \( S(a_{i,l}) = -K_{-li}a_{i,l} \) and \( S(b_{i,l}) = -b_{i,l}K_{li} \).

**Proof.** The properties 2 and 3 enable us to define \( a_{i,l} \) uniquely, and imply the other ones. \[ \square \]

**Notations 2.17.** Consider \( i \in \mathbb{I}^{im} \) and \( c \in C_{i,l} \). We set \( \tau_{i,l} = \langle a_{i,l}, a_{i,l} \rangle, a_{i,c} = \prod_j a_{i,c_j} \), and \( \tau_{i,c} = \prod_j \tau_{i,c_j} \). Notice that \( \{a_{i,c} \mid c \in C_{i,l}\} \) is a basis of \( F[i] \).

**Definition 2.18.** We denote by \( \delta_{i,c}, \delta^{i,c} : F \to F \) the linear maps defined by:

\[
\begin{align*}
\delta(x) &= \sum_{c \in C_{i,l}} \delta_{i,c}(x) \otimes a_{i,c} + \text{obd} \\
\delta(x) &= \sum_{c \in C_{i,l}} a_{i,c} \otimes \delta^{i,c}(x) + \text{obd}
\end{align*}
\]

where "obd" stands for terms of bidegree not in \( \mathbb{N}I \times \mathbb{N}i \) in the former equality, \( \mathbb{N}I \times \mathbb{N}I \) in the latter one.

**Proposition 2.19.** The maps \( \delta_{i,c} \) and \( \delta^{i,c} \) preserve the radical of \( \langle -, - \rangle \).

**Proof.** First consider the case where \( i \) is isotropic and \( x \) is a commutator \([E_{i,l}, E_{i,k}]\), then we have \( \delta(x) = 0 \), and thus \( \delta_{i,c}(x) = \delta^{i,c}(x) = 0 \). Thus, we can assume that \( \langle -, - \rangle \) is nondegenerate on \( F[\mathbb{N}i] \). Consider \( x \) in the radical of \( \langle -, - \rangle \). If \( |c| = l \), we have, for all \( y \in F \):

\[
0 = \langle x, ya_{i,c} \rangle = \langle \delta(x), y \otimes a_{i,c} \rangle = \sum_{|c'|=l} \langle \delta_{i,c'}(x) \otimes a_{i,c'}, y \otimes a_{i,c} \rangle = \sum_{|c'|=l} \langle \delta_{i,c'}(x), y \rangle \langle a_{i,c'}, a_{i,c} \rangle.
\]

The result comes from the nondegeneracy of the restriction of \( \langle -, - \rangle \) to \( F[\mathbb{N}i] \). \[ \square \]

**Lemma 2.20.** We have:
\begin{enumerate}
\item \langle a_{i,l}, a_{i,c} \rangle = \delta_{(i),c} \tau_{i,l};
\item \langle a_{i,l}y, z \rangle = \tau_{i,l}(y, \delta^{i,l}(z)) for any \( y, z \in \mathbb{F} \);
\item \langle ya_{i,l}, z \rangle = \tau_{i,l}(y, \delta_{i,l}(z)) for any \( y, z \in \mathbb{F} \).
\end{enumerate}

\textit{Proof}. The first point is a direct consequence of the definition of the \( a_{i,l} \), and the rest comes from it. \( \Box \)

\textbf{Definition 2.21}. Let \( U \hat{\otimes} U \) be the completion of \( U \otimes U \) with respect to the following sequence \((t \geq 1)\):

\[ F_t = \left( U^+, U^0 \sum_{|\alpha| \geq t} U^{-}[\alpha] \right) \otimes U + U \otimes \left( U^{-}, U^0 \sum_{|\alpha| \geq t} U^+[\alpha] \right). \]

\textbf{Proposition 2.22}. For any \( \alpha \in \mathbb{N}^I \), let \( B_\alpha \) be a basis of \( U^+[\alpha] = \{ x \in U^+, |x| = \alpha \} \), and \( \{ b^* | b \in B_\alpha \} \) the dual basis with respect to \( \langle -, - \rangle \). Set:

\[ \Theta_\alpha = \sum_{b \in B_\alpha} b^- \otimes b^*. \]

Then, the element \( \Theta = \sum \Theta_\alpha \in U \hat{\otimes} U \) satisfies:

\[ \Delta(u) \Theta = \Theta \Delta(u) \text{ for all } u \in U \]

where \( \Delta_u = \Delta(\pi) \) if \( u \mapsto \pi \) denotes the unique involutive \( \mathbb{Q}_2 \)-morphism of \( U \) stabilizing \( E_i \) and \( F_i \), and mapping \( K_i \) to \( K_{-i} \), and \( v \) to \( v^{-1} \).

\textit{Proof}. It’s enough to check the relation on generators. For those of real degree, the proof is identical to the one of \cite[Theorem 4.1.2]{Lus}. Consider \( i \in \mathbb{F}^m \) and \( l \geq 1 \). We have:

\[ \Delta(a_{i,l}) \Theta = \Theta \Delta(a_{i,l}) \iff \sum_{b \in B} \{ a_{i,l} b^- \otimes b^* + K_{il} b^- \otimes a_{i,l} b^* \} - b^- a_{i,l} \otimes b^* - b^- K_{-li} \otimes b^* a_{i,l} \} = 0 \]

\[ \iff \forall z \in U^+, \sum_{b \in B} \{ a_{i,l} b^- \langle b^*, z \rangle + K_{il} b^- \langle a_{i,l} b^*, z \rangle \} - b^- a_{i,l} \langle b^*, z \rangle - b^- K_{-li} \langle b^* a_{i,l}, z \rangle \} = 0 \]

\[ \iff \forall z \in U^+, \sum_{b \in B} \{ a_{i,l} b^- \langle b^*, z \rangle + K_{il} b^- \tau_{i,l} \langle b^*, \delta^{i,l}(z) \rangle \} - b^- a_{i,l} \langle b^*, z \rangle - b^- K_{-li} \tau_{i,l} \langle b^*, \delta_{i,l}(z) \rangle \} = 0 \]

\[ \iff \forall z \in U^+, a_{i,l} z^- + \tau_{i,l} K_{il} \delta^{i,l}(z) = z^- a_{i,l} + \tau_{i,l} \delta_{i,l}(z) - K_{-li} \]

which is the relation (2.9) with \( a, b = a_{i,l}, z \). The equivalence before the last one comes from 2.20. The computations are the same for \( U^{\leq 0} \):

\[ \Delta(b_{i,l}) \Theta = \Theta \Delta(b_{i,l}) \iff \sum_{b \in B} \{ b_{i,l} b^- \otimes K_{-li} b^* + b^- \otimes b_{i,l} b^* \} - b^- b_{i,l} \otimes b^* K_{il} - b^- \otimes b^* b_{i,l} \} = 0 \]
⇔ ∀ \in U^+, \sum_{b \in B} \{ \langle a, i, l \rangle \delta b, z \rangle - \langle a, i, l \rangle \delta b, z \rangle = 0

⇔ ∀ \in U^+, \sum_{b \in B} \{ \tau, i, l \delta \delta \langle b, \delta \rangle \langle b, \delta \rangle \rangle - \langle b, \delta \rangle \delta \rangle = 0

⇔ ∀ \in U^+, \tau, i, l \delta \delta \langle b, \delta \rangle \langle b, \delta \rangle \rangle + b, i, l \delta \delta \langle b, \delta \rangle \rangle = \tau, i, l \delta \delta \langle b, \delta \rangle \rangle

which matches (2.9) with \alpha, b = a, i, l, z. □

Remark 2.23. As in [Lus10, 4.1.2], one can prove that \( \Theta \) is the only element satisfying \( \Theta_0 = 1 \otimes 1 \) and \( \Delta(u)\Theta = \Theta \Delta(u) \) for all \( u \in U \).

2.5. Casimir operator.

Definition 2.24. We denote by \( C \) the category of \( U \)-modules satisfying:

1. \( M = \oplus_{\alpha \in \mathbb{Z}I} M_\alpha \) where \( M_\alpha = \{ m \in M \mid \forall i, K_i m = v(\alpha, i) m \} \);
2. For any \( m \in M \), there exists \( p \geq 0 \) such that \( x m = 0 \) as soon as \( x \in F[\alpha] \) and \( \text{ht}(\alpha) \geq p \).

Proposition 2.25. Set \( \Omega \leq p = m(S \otimes 1)(\sum_{\text{ht}(\alpha) \leq p} \Theta_\alpha) \), and \( M \in C \). Then, for every \( m \in M \), the value of \( \Omega(m) = \Omega \leq p(m) \) does not depend on \( p \) for \( p \) large enough, and we have the following identities of operators on \( M \):

\[ K_i \Omega = \Omega K_i \]
\[ K_{-i} a_i l \Omega = K_i a_i l \Omega \]
\[ b_i l K_l i \Omega K_i = b_i l \Omega \]

for any \( i \in I \) and \( l \geq 1 \).

Proof. The computations are strictly analogous to those in [Lus10, 6.1.1], thanks to the definition of \( a, i, l \) and \( b, i, l \) (see 2.16).

Definition 2.26. For any \( \alpha \in \mathbb{Z}I \), we define a Verma module:

\[ M(\alpha) = \sum_{i \in I_{\infty}} U E_i + \sum_{i \in I} U(K_i - v^{(1, i)}) \in C. \]

Proposition 2.27. Under the assumption 2.13 we have \( \tilde{U}^{-} \simeq U^{-} \).

Proof. The proof follows [Kac90], [Lus10] and more specifically [SVDB01, Proposition 2.4]. The maximal degrees of the primitive elements of the kernel of the map \( \tilde{U}^{-} \rightarrow U^{-} \) are the same as those of the primitive elements of:

\[ \ker \left( \sum_{(i, l) \in I_{\infty}} b_i l : \bigoplus_{(i, l) \in I_{\infty}} M(-i) \rightarrow M(0) \right). \]

By maximality, if \( \alpha \) is such a degree, we get \( (\alpha, i) \geq 0 \) for any vertex \( i \). Indeed, [SVDB01] §2, properties 1., 2., 3., 4.] are still satisfied in our case, in particular the second one, thanks to the higher order Serre relations.
Let $C$ denote the $\mathbb{Q}(v)$-linear map defined on $M = \bigoplus_{(i,l) \in I} M(-li)$ by:

$$Cm = v^{f(\alpha)}\Omega m$$

if $m \in M_\alpha$,

where $f(\alpha) = (\alpha, \alpha + 2\rho)$ and $\rho$ is defined by $(i, 2\rho) = (i, i)$ for every $i \in I$.

Notice that:

$$f(\alpha - li) - f(\alpha) + 2l(i, \alpha) = l(l - 1)(i, i).$$

For any $(i, l) \in I_\infty$, since $\Omega b_{i,l} = b_{i,l}\Omega K_{2li}$, we get:

$$Cb_{i,l}m = v^{f(\alpha-li)}\Omega b_{i,l}m$$

$$= v^{f(\alpha-li)}b_{i,l}\Omega K_{2li}m$$

$$= v^{f(\alpha-li)+2l(i,\alpha)}b_{i,l}\Omega m$$

$$= \begin{cases} v^{l(l-1)(i,i)}b_{i,l}Cm & \text{if } i \in I^{im} \\ b_{i,l}Cm & \text{if } i \in I^{re}. \end{cases}$$

Hence, if $m$ is a primitive vector of the kernel of the map $\bigoplus_{(i,l) \in I} M(-li) \rightarrow M(0)$ with $|m| = \alpha \in -NI$, we have:

(2.28) $$f(\alpha) = \sum_{1 \leq k \leq r} l_k(l_k - 1)(i_k, i_k)$$

where $\sum_{i \in I^{im}} \alpha_i i = \sum_{1 \leq k \leq r} l_k i_k$. Since $(\alpha, i) \geq 0$ for any real vertex $i$, we also have:

$$(\alpha, \alpha + 2\rho) = \sum_{i \in I} \alpha_i (i, \alpha + i)$$

$$= \sum_{i \in I^{re}} \alpha_i (i, \alpha) + 2 \sum_{i \in I^{re}} \alpha_i + \sum_{i \in I^{im}} \alpha_i (i, \alpha + i)$$

$$\leq 2 \sum_{i \in I^{re}} \alpha_i + \sum_{i \in I^{im}} \alpha_i (i, \alpha + i).$$

Combining with (2.28) we get:

$$\sum_{1 \leq k \leq r} l_k(l_k - 1)(i_k, i_k) \leq 2 \sum_{i \in I^{re}} \alpha_i + \sum_{i \in I^{im}} \alpha_i (i, \alpha + i)$$

$$= 2 \sum_{i \in I^{re}} \alpha_i + \sum_{i \in I^{im}} \alpha_i (\alpha_i + 1)(i, i) + \sum_{i \in I^{im}} \sum_{j \neq i} \alpha_i \alpha_j (i, j)$$

and thus:

$$0 \leq 2 \sum_{i \in I^{re}} \alpha_i + \sum_{i \in I^{im}} \alpha_i \alpha_j (i, j) + \sum_{i \in I^{im}} (i, i) \left( \alpha_i (\alpha_i + 1) - \sum_{l_k = 1}^1 l_k(l_k - 1) \right).$$

Since $\sum_{i \in I^{re}} l_k = -\alpha_i$, we have:

$$\alpha_i (\alpha_i + 1) - \sum_{l_k = 1}^1 l_k(l_k - 1) = |\alpha_i|(|\alpha_i| - 1) - \sum_{l_k = 1}^1 l_k(l_k - 1) \geq 0.$$
But we also have $\alpha_i \leq 0$, $(i, j) \leq 0$ when $i \neq j$, and $(i, i) \leq 0$ when $i$ is imaginary, hence:

$$2 \sum_{i \in I^{re}} \alpha_i + \sum_{i \in I^{im}} \alpha_i \alpha_j(i, j) + \sum_{i \in I^{im}} (i, i) \left( \alpha_i (\alpha_i + 1) - \sum_{i_k = i} l_k (l_k - 1) \right) \leq 0.$$ 

Finally every term in the sum is equal to 0, and $-\alpha$ is a sum of pairwise orthogonal imaginary vertices. Since the restriction of $\langle -, - \rangle$ to $\bar{U}^{[-N]}$ is nondegenerate for any imaginary vertex $i$, the proof is over.

\[ \tag{2.29} \]

**Theorem 2.29.** We have an isomorphism of Hopf algebras $\Psi : U_{\alpha}^{+} \cong \mathcal{K}$ defined by:

$$\begin{cases} E_{i, a} \mapsto [1_{a_i}] & \text{if } i \in I^{im} \\ E_i^{(\alpha)} \mapsto [1_{a_i}] & \text{if } i \in I^{re} \end{cases}$$

and mapping $\langle -, - \rangle$ to the geometric form $\{-, -\}$.

**Proof.** First, $\Psi$ is defined. Indeed, we know from the Jordan quiver case that the elements $(1_{a_i})_{a \geq 1}$ commute if $i$ is isotropic. Moreover the higher order Serre relations are satisfied for real vertices (see [Lus10, 7]), and, applying the Fourier transform on the imaginary vertices, we can assume that we are working with nilpotent representations. Hence we have $1_{a_i} = \mathbb{C} [\{0, 1\}]$ as if there were no loops, and the higher order Serre relations are still satisfied. For the same reason, we know that:

$$\{1_{a_i}, 1_{a_j}\} \in 1 + v^{-1} \mathbb{N} [v^{-1}].$$

Hence, setting $\langle E_{i, a}, E_{i, a} \rangle = \{1_{a_i}, 1_{a_j}\}$, $\langle -, - \rangle$ is nondegenerate (thanks to 2.12). Therefore $\Psi$ is injective, and since $\Psi$ is also surjective by [1.13], we get the result. □

3. Relation with constructible functions

We denote by $\bar{h} : \bar{t}(h) \to s(h)$ the opposite arrow of $h \in \Omega$, and $\bar{Q}$ the quiver $(I, \bar{H} = \Omega \sqcup \Omega)$, where $\bar{\Omega} = \{\bar{h} \mid h \in \Omega\}$: each arrow is replaced by a pair of arrows, one in each direction, and we set $\epsilon(h) = 1$ if $h \in \Omega$, $\epsilon(h) = -1$ if $h \in \bar{\Omega}$.

For any pair of $I$-graded $\mathbb{C}$-vector spaces $V = (V_i)_{i \in I}$ and $V' = (V'_i)_{i \in I}$, we set:

$$\bar{E}(V, V') = \bigoplus_{h \in H} \text{Hom}(V_{s(h)}, V'_{t(h)}).$$

For any dimension vector $\alpha = (\alpha_i)_{i \in I}$, we fix an $I$-graded $\mathbb{C}$-vector space $V_{\alpha}$ of dimension $\alpha$, and put $\bar{E}_{\alpha} = \bar{E}(V_{\alpha}, V_{\alpha})$. The space $\bar{E}_{\alpha} = \bar{E}(V_{\alpha}, V_{\alpha})$ is endowed with a symplectic form:

$$\omega_\alpha(x, x') = \sum_{h \in H} \text{Tr}(\epsilon(h)x_h x'_{h})$$

which is preserved by the natural action of $G_\alpha$ on $\bar{E}_\alpha$. The associated moment map $\mu_\alpha : \bar{E}_\alpha \to g_\alpha = \oplus_{i \in I} \text{End}(V_{\alpha}) i$ is given by:

$$\mu_\alpha(x) = \sum_{h \in H} \epsilon(h)x_h x_h.$$ 

Here we have identified $g_\alpha^*$ with $g_\alpha$ via the trace pairing.
Definition 3.1. An element $x \in \bar{E}_\alpha$ is said to be seminilpotent if there exists an $I$-graded flag $W = (W_0 = \{0\} \subset \ldots \subset W_r = V_\alpha)$ of $V_\alpha$ such that:

$x_h(W_\bullet) \subseteq W_{\bullet-1}$ if $h \in \Omega$,
$x_h(W_\bullet) \subseteq W_\bullet$ if $h \in \bar{\Omega}$.

We put $\Lambda(\alpha) = \{ x \in \mu_\alpha^{-1}(0) \mid x \text{ seminilpotent} \}$.

The following is proved [Boz13]:

Theorem 3.2. The subvariety $\Lambda(\alpha)$ of $\bar{E}_\alpha$ is Lagrangian.

Following [Lus00], we denote by $\mathcal{M}(\alpha)$ the $\mathbb{Q}$-vector space of constructible functions $\Lambda(\alpha) \to \mathbb{Q}$, which are constant on any $G_\alpha$-orbit. Then, we set $\mathcal{M} = \bigoplus_{\alpha \geq 0} \mathcal{M}(\alpha)$ which is a graded algebra once equipped with the product $\ast$ defined in [Lus00] 2.1.

For $Z \in \text{Irr } \Lambda(\alpha)$ and $f \in \mathcal{M}(\alpha)$, we put $\rho_Z(f) = c$ if $Z \cap f^{-1}(c)$ is an open dense subset of $Z$.

If $i \in I^{im}$ and $(l)$ denotes the trivial composition or partition of $l$, we denote by $1_{i,l}$ the characteristic function of the associated irreducible component $Z_{i,(l)} \in \text{Irr } \Lambda(l \epsilon_i)$ (the component of elements $x$ such that $x_h = 0$ for all $h \in \Omega(i)$). If $i \notin I^{im}$, we just denote by $1_l$ the function mapping to 1 the only point in $\Lambda(\epsilon_i)$.

We have $1_{i,l} \in \mathcal{M}(l \epsilon_i)$ for $i \in I^{im}$ and $1_l \in \mathcal{M}(\epsilon_i)$ for $i \notin I^{im}$. We denote by $\mathcal{M}_0 \subseteq \mathcal{M}$ the subalgebra generated by these functions.

The following was proved in [Boz13]:

Proposition 3.3. For every $Z \in \text{Irr } \Lambda(\alpha)$, there exists $f \in \mathcal{M}_0(\alpha)$ such that $\rho_Z(f) = 1$ and $\rho_{Z'}(f) = 0$ if $Z' \neq Z$.

Proposition 3.4. There exists a surjective morphism $\Phi : U_{i=1}^+ \to \mathcal{M}_0$ defined by:

$$\begin{cases} 
E_{i,a} \mapsto 1_{i,l} & \text{if } i \in I^{im} \\
E_i \mapsto 1_l & \text{if } i \in I^{re}.
\end{cases}$$

Proof. The morphism is well defined: first, the higher order Serre relations are mapped to 0. Indeed, they are for real vertices (see [Lus91] 12.11 and [Lus10] chapitre 7), and we work with semi-nilpotent representations. Hence they are still satisfied by definition of $Z_{i,(l)} \in \text{Irr } \Lambda(l \epsilon_i)$ ($x$ such that $x_h = 0$ for all $h \in \Omega(i)$). On the other hand, the commutators $[E_{i,l}, E_{i,k}]$ are also mapped to 0 if $i$ is isotropic, thanks to the following lemma:

Lemma 3.5. Let $Q$ be the Jordan quiver. We set $I = \{ \circ \}$ and $1_k = 1_{o,k}$. We have $[1_m, 1_n] = 0$ for all $m, n \in \mathbb{N}$.

Proof. Consider $(x, y) \in \Lambda(n + m)$, and set $V = \mathbb{C}^{n+m}$. We have:

$$1_m \ast 1_n(x, y) = \chi \left( \left\{ W \in \text{Grass}_n V \mid \begin{array}{c} W \text{ is } (x, y)\text{-stable} \\
x_{W}|W = 0 \\
x_{V/W}|V/W = 0 \end{array} \right\} \right).$$

This is equal to 0 except if $x \in \mathcal{O}_\lambda$, where $\lambda = (\lambda_1 \geq \lambda_2)$. Then:

$$1_m \ast 1_n(x, y) = \chi \left( \left\{ \tilde{W} \in \text{Grass}_{n-\lambda_2} \ker x \mid \tilde{W} \text{ } \tilde{y}\text{-stable} \right\} \right).$$
where $\bar{}$ stands for the quotient by $\text{Im } x$. Also:

$1_n \ast 1_m(x, y) = \chi\left(\{W \in \text{Grass}_{m-\lambda_2 \text{ker } x} \mid W \text{ y-stable}\}\right).$

Since $n - \lambda_2 + m - \lambda_2 = \lambda_1 - \lambda_2 = \dim \text{ker } x$, we get the result by duality:

$$\text{End}(\text{ker } x) \sim \rightarrow \text{End}(\text{ker } x^*)$$

$\bar{y} \mapsto \phi \mapsto \phi \circ \bar{y}.$

Finally, the surjectivity comes from the definition of $M_\phi$. □

We conjecture that $\Phi$ is an isomorphism, which should be proved by comparing the two "crystal" structures on $K$ and $M_\phi$ given by the following sets of bijections:

\[
B_{\alpha, i, \gamma} \sim \rightarrow B_{\alpha-\gamma, i, 0} \times B_{\gamma, i} \\
\text{Irr } \Lambda(\alpha)_{i, \gamma} \sim \rightarrow \text{Irr } \Lambda(\alpha-\gamma i)_{i, 0} \times \text{Irr } \Lambda(\gamma i),
\]

the latter being obtained in [Boz13]. To that end, the notion of crystal should be generalized, and results analogous to those obtained in [KS97] should be proved.

REFERENCES

[BBD82] A. A. Beilinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In Analysis and topology on singular spaces, I (Luminy, 1981), volume 100 of Astérisque, pages 5–171. Soc. Math. France, Paris, 1982.

[Boz13] Tristan Bozec. Quivers with loops and Lagrangian subvarieties. arXiv:1311.5396, 2013.

[HHRV13] Tamás Hausel, Emmanuel Letellier, and Fernando Rodriguez-Villegas. Arithmetic harmonic analysis on character and quiver varieties II. Adv. Math., 234:85–128, 2013.

[HRV08] Tamás Hausel and Fernando Rodriguez-Villegas. Mixed Hodge polynomials of character varieties. Invent. Math., 174(3):555–624, 2008. With an appendix by Nicholas M. Katz.

[Kac90] Victor G. Kac. Infinite-dimensional Lie algebras. Cambridge University Press, Cambridge, third edition, 1990.

[KS97] Masaki Kashiwara and Yoshihisa Saito. Geometric construction of crystal bases. Duke Math. J., 89(1):9–36, 1997.

[KS06] Seok-Jin Kang and Olivier Schiffmann. Canonical bases for quantum generalized Kac-Moody algebras. Adv. Math., 200(2):455–478, 2006.

[KS07] S-J Kang and Olivier Schiffmann. Addendum to “Canonical bases for quantum generalized Kac-Moody algebras”. arXiv:0711.1948, 2007.

[LL09] Yiqiang Li and Zongzhu Lin. Canonical bases of Borcherds-Cartan type. Nagoya Math. J., 194:169–193, 2009.

[Lus91] G. Lusztig. Quivers, perverse sheaves, and quantized enveloping algebras. J. Amer. Math. Soc., 4(2):365–421, 1991.

[Lus93] G. Lusztig. Tight monomials in quantized enveloping algebras. In Quantum deformations of algebras and their representations (Ramat-Gan, 1991/1992; Rehovot, 1991/1992), volume 7 of Israel Math. Conf. Proc., pages 117–132, Bar-Ilan Univ., Ramat Gan, 1993.

[Lus00] G. Lusztig. Semicanonical bases arising from enveloping algebras. Adv. Math., 151(2):129–139, 2000.

[Lus10] George Lusztig. Introduction to quantum groups. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2010. Reprint of the 1994 edition.

[MO12] Davesh Maulik and Andrei Okounkov. Quantum groups and quantum cohomology. arXiv:1211.1287, 2012.

[Nak09] Hiraku Nakajima. Quiver varieties and branching. SIGMA Symmetry Integrability Geom. Methods Appl., 5:Paper 003, 37, 2009.

[Rin96] Claus Michael Ringel. Green’s theorem on Hall algebras. In Representation theory of algebras and related topics (Mexico City, 1994), volume 19 of CMS Conf. Proc., pages 185–245. Amer. Math. Soc., Providence, RI, 1996.
[Rin97] Claus Michael Ringel. Quantum Serre relations. In Algèbre non commutative, groupes quantiques et invariants (Reims, 1995), volume 2 of Sémin. Congr., pages 137–148. Soc. Math. France, Paris, 1997.

[Sch09a] Olivier Schiffmann. Lectures on canonical and crystal bases of Hall algebras. arXiv:0910.4460, 2009.

[Sch09b] Olivier Schiffmann. Lectures on Hall algebras. arXiv:0611617v2, 2009.

[SVDB01] Bert Sevenhant and Michel Van Den Bergh. A relation between a conjecture of Kac and the structure of the Hall algebra. J. Pure Appl. Algebra, 160(2-3):319–332, 2001.

[Xia97] Jie Xiao. Drinfeld double and Ringel-Green theory of Hall algebras. J. Algebra, 190(1):100–144, 1997.

Faculté des sciences d’Orsay, Bâtiment 425, Université de Paris-Sud
F-91405 Orsay Cedex, France,
e-mail: tristan.bozec@math.u-psud.fr