Solution for ”geodesic” motion of a Schwarzschild black hole along a magnetic field in AdS$^2 \times S^2$ space-time

George A. Alekseev

Steklov Mathematical Institute of Russian Academy of Sciences,
Gubkina 8, 119991, Moscow Russia
E-mail: G.A.Alekseev@mi.ras.ru

The exact solution of Einstein - Maxwell equations for a Schwarzschild black hole immersed in the static spatially homogeneous AdS$^2 \times S^2$ space-time of Bertotti-Robinson magnetic universe is presented. In this solution, the black hole possesses a finite initial boost in the direction of the magnetic field and performs a “geodesic” oscillating motion interacting with the background gravitational and electromagnetic fields.

Keywords: Einstein-Maxwell, Schwarzschild, Bertotti-Robinson universe, rigid frames

Introduction

The century which passed since a wonderful discovery of General Relativity, brought us a lot of developed methods for construction of exact solutions of Einstein and Einstein-Maxwell field equations and findings of numerous particular solutions. However, despite the permanent interest in the literature to different aspects of solutions which describe an interaction of various compact sources (such as black holes, particle-like singularities, matter discs, rings, etc) with the external gravitational and electromagnetic fields, very few such exact solutions are known which describe some non-trivial motion of the sources in various external fields.

In this paper, we present a construction of new solution of Einstein-Maxwell equations for a Schwarzschild black hole moving freely along a magnetic field in AdS$^2 \times S^2$ space-time of Bertotti-Robinson magnetic universe shown on Fig. 1.

Fig. 1. Schwarzschild black hole in a ”geodesic” motion along a magnetic field in Bertotti-Robinson (BR) magnetic universe. In Weyl-like coordinates, the black hole horizon is represented by segments on the axis $\rho = 0$ which constitute a black stripe on $(t, z)$-plane. $H = g_{\tau\tau}/g_{\tau\tau}^{(BR)}$ and $g_{\tau\tau}$ is the square of time-like Killing vector field $\partial/\partial T$ which becomes null on the horizon.
Among the earlier found solutions which admit some dynamical interpretation it is worth to mention the Ernst solution in Ref. [4] which describes a charged black hole immersed in Melvin electric universe and accelerated by its electric field (see also Ref. [5] for details and references) and solution derived in Ref. [6] for a Schwarzschild black hole immersed in Bertotti-Robinson magnetic universe and resting at the origin of some rigid frame without any struts or string-like singularities.

In comparison with the solution Ref. [6] the solution presented here possess one more arbitrary real parameter which determines the initial boost of a black hole along the direction of the magnetic field. This boosted black hole performs a free "geodesic" motion, oscillating around the origin of a chosen rigid reference frame. Below, we consider at first the motion of a test particles along the magnetic field in the Bertotti-Robinson background and show that any such particle performs an oscillating motion around the origin of some rigid reference frame where a geodesic particle with zero boost can be at rest. Then we show that a similar rigid frame can be associated with any other geodesic world line directed along the magnetic field. In each of these frames the metric in co-moving coordinates coincides with the same Bertotti-Robinson one. Therefore, we can use this metric as the background and apply the solution generating methods developed earlier for Einstein-Maxwell equations[4] for construction of the solution with a black hole immersed in this metric, that was done in Ref. [6]. The generated solution is static in a co-moving frame, but in the original frame it looks like a free oscillating motion of a black hole near its equilibrium position without struts and string-like singularities on the axis.

The background Bertotti-Robinson space-time with AdS$^2 \times S^2$ topology and spatially homogeneous magnetic field

The space-time of the Bertotti-Robinson magnetic universe possess AdS$^2 \times S^2$ topology and it is filled by spatially homogeneous magnetic field (see Ref. [12] for details and references). In cylindrical coordinates $\{t, \rho, z, \varphi\}$, the components of its metric, 1-form of electromagnetic potential and the only non-zero component of magnetic field measured by the local observer can be presented in the form ($\gamma = c = 1$):

$$
\begin{align*}
\text{ds}^2 &= -\cosh^2 \left(\frac{z}{b}\right) dt^2 + d\rho^2 + dz^2 + b^2 \sin^2 \left(\frac{\rho}{2b}\right) d\varphi^2 \\
A &= \left\{0, 0, 0, -2b \sin^2 \left(\frac{\rho}{2b}\right)\right\}, \\
H_\varphi &= \frac{1}{b}
\end{align*}
$$

(1)

where $b$ is a constant parameter. The sections $\{t = \text{const}, z = \text{const}\}$ possess the geometry of a 2-sphere of the radius $b$ and the space-time is closed in the $\rho$-direction.

In the limit $b \to \infty$, the magnetic field vanishes ($H_\varphi \to 0$) and the metric (1) transforms into the usual Minkowski metric in cylindrical coordinates.

\[^4\text{See, in particular, Ref. [7] for generation of Einstein-Maxwell solitons or Ref. [8,9] for application of the monodromy transform approach and the corresponding integral equation method.}\]
Geodesic motion of test particles along the $z$-axis in the metric (1)

The world lines of test particles which are at rest in the metric (1) constitute a rigid frame. These world lines possess the acceleration along $z$-axis which value is

$$W^z = \frac{1}{b} \tanh \frac{z}{b}$$

and only the world line of a test particle at rest in the position $z = 0$ is geodesic. The time-like geodesics which correspond to a motion of test particles of mass $\mu$ and the energy $E_o$ along $z$-axis in metric (1) are

$$\tanh \frac{z}{b} = \sqrt{\frac{E_o^2 - 1}{E_o}} \sin \frac{t - t^*}{b}, \quad E_o = \frac{E_o}{\mu} > 1$$

This equation shows that the motion of a freely falling neutral test particle boosted with some finite energy $E_o > 1$ along the $z$-axis in the metric (1) is periodic and it takes place in some restricted region $\cosh \frac{z}{b} < E_o$.

![Fig. 2. The left picture shows the form of $g_{tt}$ metric component which plays the role of a potential well for test particles moving with the energy $E_o$ along a magnetic field in Bertotti-Robinson magnetic universe. The right picture shows the geodesic world lines for these test particles on $(t, z)$ plane for different energies $E_o$.]

Killing vector fields in $(t, z)$-plane

The rigid frames in the metric (1) are determined by congruences of Killing vector fields in $(t, z)$-plane. In the coordinates $(t, \rho, z, \phi)$ these can be expressed as:

$$\xi^i = \{ \cosh \delta - \sinh \delta \sin \frac{t - t^*}{b} \tanh \frac{z}{b}, 0, \sinh \delta \cos \frac{t - t^*}{b}, 0 \}$$

where $\delta$ and $t^*$ are arbitrary real parameters. Thus, at every point of $(t, z)$-plane we have a family of Killing vector fields parametrized by $\delta$ which possess the norm

$$g_{\xi\xi} \equiv \xi^i \xi_i = -\cosh^2 \frac{z}{b} \left( \cosh \delta - \sinh \delta \sin \frac{t - t^*}{b} \tanh \frac{z}{b} \right)^2 + \sinh^2 \delta \cos^2 \frac{t - t^*}{b}.$$
Rigid frames associated with geodesics on \((t, z)\)-plane

The background solution \(\mathbf{I}\) possess an important symmetry: changing the reference frame with corresponding coordinate transformation \((t, \rho, z, \varphi) \rightarrow (T, \rho, Z, \varphi)\) with

\[
\begin{align*}
T &= b \arctan \left( \frac{\cosh \delta \tan \frac{t-t_s}{b} - \sinh \delta \tanh \frac{z}{b}}{\sec \frac{t-t_s}{b}} \right), \\
Z &= b \arcsinh \left[ \cos \delta \sinh \frac{z}{b} - \sin \delta \cosh \frac{z}{b} \sin \frac{t-t_s}{b} \right],
\end{align*}
\]

leaves the form of the background metric invariant:

\[
ds^2 = \frac{-c^2}{b} \left( \begin{array}{c} dt^2 + dz^2 + d\rho^2 + \sin^2 \frac{\rho}{b} \, d\varphi^2 \\ \end{array} \right) = \frac{-c^2}{b} \left( \begin{array}{c} dT^2 + dZ^2 + d\rho^2 + \sin^2 \frac{\rho}{b} \, d\varphi^2 \\ \end{array} \right)
\]

Thus, we constructed a rigid frame which motion with respect to the original rigid frame is determined by the boost parameter \(\delta\). Among the world lines \(Z = \text{const}\) only the line \(Z = 0\) is geodesic and for this we have

\[
\mathcal{E}_\alpha = \cosh \delta
\]

The other Killing lines with \(Z = \text{const} \neq 0\) and with the same \(\delta\) are not geodesics.

Schwarzschild black hole in a “geodesic” motion in metric \(\mathbf{I}\)

The solution for a Schwarzschild black hole at rest in the external Bertotti-Robinson magnetic universe was constructed in Ref. [11] in a slightly different notations it takes the form

\[
ds^2 = -H \left( \cosh \frac{Z}{b} \right)^2 \left( dT^2 + f(d\rho^2 + dZ^2) + \frac{b^2}{H} \left( \frac{\sin \frac{\rho}{b}}{b} \right)^2 \, d\varphi^2 \right)
\]

where the functions \(H, f\) and the electromagnetic potential \(A = \{0, 0, 0, A_{\varphi}\}\) are

\[
H = \frac{(x_1 + y_1)^2}{x_1^2 - 1}, \quad A_{\varphi} = -\frac{b(x_2 + 1)(1 + y_1)}{x_2 + y_1}, \quad f = \frac{(x_2 + y_1)^2}{x_2^2 - y_2^2} \left[ \frac{x_1 + x_2 - y_1 - y_2}{x_1 + x_2 + y_1 + y_2} \right]^2
\]

and the bipolar coordinates \(\{x_1, y_1\}, \{x_2, y_2\}\) are functions of \(\rho\) and \(Z\) only:

\[
\begin{align*}
&x_1 = -\frac{b}{m} \sinh \frac{Z}{b}, \quad x_2 = (R_1 + R_2)/2m, \quad y_2 = (R_1 - R_2)/2m, \\
y_1 = -\cos \frac{\rho}{b}, \quad \left. R_{\pm} = \sqrt{(b \sinh \frac{Z}{b} \cos \frac{\rho}{b} + m)^2 + b^2 \sin^2 \frac{\rho}{b}} \sin \frac{\rho}{b} \right|_{R_+ = \sqrt{(b \sinh \frac{Z}{b} \cos \frac{\rho}{b} + m)^2 + b^2 \sin^2 \frac{\rho}{b}}}
\end{align*}
\]

However, this solution will describe the boosted Schwarzschild black hole moving in Bertotti-Robinson background, if \(T\) and \(Z\) are functions of \(t\) and \(z\) defined in \(\mathbf{I}\). For small mass \(m\), this black hole produces perturbations of metric and electromagnetic field localized near \(Z = 0\) that is a background geodesic oscillating near \(z = 0\).

\[\text{It is necessary to mention here that, in contrast to such “local” (in the region near the axis } \rho = 0\text{) interpretation of this solution as a black hole in the external field, its global interpretation is more difficult because the semi-closed AdS}^2 \times S^2\text{ topology of this external space-time and focusing properties of gravitational field give rise to existence of naked singularity located on the “antipodal” axis } \rho = \pi b\text{ (see Ref. [6] for more details).}\]
Acknowledgements

This work was supported in part by the Russian Foundation for Basic Research (Grants No. 14-01-00049 and No. 14-01-00860).

References

1. Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E (2003) *Exact Solutions of Einstein’s Field Equations*, 2nd edition (Cambridge University Press).
2. Griffiths J B and Podolsky J (2009) *Exact Space-Times in Einstein’s General Relativity* (Cambridge University Press).
3. Alekseev G A, *Thirty years of studies of integrable reductions of Einstein’s field equations*, Proceedings of the Twelfth Marcel Grossmann Meeting on General Relativity, edited by Thibault Damour, Robert T Jantzen and Remo Ruffini, World Scientific, Singapore, Part A, Plenary and Review talks, p. 645 - 666, (2011); arXiv:gr-qc/1011.3846.
4. Ernst, F.J. (1976) *Removal of the nodal singularity of the C-metric* JMP **17**, 515.
5. Bicak J and Kofron D (2010) *Rotating charged black holes accelerated by an electric field* Phys.Rev. **D82**, 024006; arXiv: gr-qc/1006.4072v1.
6. Alekseev G A and Garcia A A (1996) *Schwarzschild black hole immersed in a homogeneous electromagnetic field*, Phys.Rev. **D53**, 1853-1867.
7. Alekseev G A (1980) *N-soliton solutions of Einstein - Maxwell equations*, JETP Lett. **32** (4), 277-279.
8. Alekseev G A (1985) *The method of the inverse problem of scattering and the singular integral equations for interacting massless fields*, Sov. Phys. Dokl. **30**, 565.
9. Alekseev G A (1988) *Exact solutions in General Relativity*, Proc. Steklov Inst. Maths. **3**, 215-262.