The notions of time and evolution in quantum cosmology

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Abstract
We re-examine the notions of time and evolution in the light of the mathematical properties of the solutions of the Wheeler-DeWitt equation which are revealed by an extended adiabatic treatment. The main advantage of this treatment is to organize the solutions in series that make explicit both the connections with the corresponding Schrödinger equation as well as the modifications introduced by the quantum character of gravity. When the universe is macroscopic, the ordered character of the expansion leads to connections with the Schrödinger equation so precise that the interpretation of the solutions of the Wheeler-DeWitt equation is unequivocally determined. On the contrary, when the expansion behaves quantum mechanically, i.e. in the presence of backscattering, major difficulties concerning the interpretation persist.
1 Introduction

It is now well known that in quantum cosmology, when working with the solutions of the Wheeler-DeWitt (WDW) equation, one completely looses the notion of time, i.e. the notion of an external parameter that clocks events. Indeed, the WDW constraint equation implies that the universe is in an eigenstate of zero total energy [1]-[5]. In a Schrödinger sense, this means that one works with a stationary state.

Before investigating the problem of the description of evolution in quantum cosmology, it should be noted that in classical cosmology, when working with the action in the Hamilton-Jacobi formalism, one also looses the notion of the (Newtonian) time. As in quantum cosmology, this results from the invariance of the theory under arbitrary reparametrizations of time. It should also be noticed that a similar disappearance applies to space as well, in virtue of the full reparametrization invariance of general relativity. Surprisingly, this problem has attracted much less attention that the one associated with time, see however [6][7][8]. In any case, when analyzing the solutions of the Wheeler-DeWitt equation, one must keep in mind that certain properties might possess an equivalent version in classical cosmology and, more importantly, one should sort out these classical aspects from those which are specific to quantum gravity. These concerns are rarely found in the literature, see however [9][10][11].

The absence of time in quantum cosmology leads at least to three questions:
1. Is the notion of time intrinsic to the notion of (quantum) evolution ?
2. How evolution should be described in the absence of time ?
3. When and why this new description coincides with the usual one based on a Schrödinger equation ?

That time might not be necessary in cosmology is not really a surprise. For instance, cosmological events are delivered to astronomers ordered in terms of their red shifts \( z = \frac{a_{\text{reception}}}{a_{\text{emission}}} - 1 \) which bear no direct information about lapses of proper time \( \Delta t \) between emission and reception. Indeed, the determination of the latters requires the knowledge of the expansion rate \( \dot{a}(a) = \frac{da}{dt} \) from \( a_{\text{emission}} \) until \( a_{\text{reception}} \). Thus, at first sight, since one has lost the notion of time, it seems that this rate is not physically meaningful. However, it becomes meaningful when it is compared to rates governing local processes, such as growth of perturbations. This is because these rates issue from evolution laws governed by lapses of (proper) time.

This discussion brings about an important concept that we shall use throughout in this paper: it is the comparison of the rates characterizing two different processes that leads, a posteriori, to the notion of time. This is why we shall analyze the transitions amplitudes of heavy atoms induced by absorption or emission of photons.
that occur in an expanding universe. Notice that the physical necessity of comparing two rates to re-introduce the notion of time is reminiscent of the description of planar orbits of an isolated system in Newtonian mechanics. In that case, one can describe intrinsically, i.e. without reference to an external time, the trajectory in the form \( r(\theta) \). However, to give physical sense to the “extrinsic” description \( r(t), \theta(t) \), one needs an additional system to tell what \( t \) means [10][11][12].

To further discuss the issues raised by the three questions, one must consider technical aspects more closely. Firstly, we shall pursue the discussion in mini-superspace. In this restricted space, classical cosmology is described by the history of the scale factor \( a(t) \). The main advantage of this truncation is that it leads to explicit equations whose solutions can be fully analyzed. The physical relevance of this radical truncation is greatly enlarged once it is conceived as the first order term in an expansion in local gravitational deformations [3][9]. In particular, in such an expansion, it is consistent to consider quantum matter fields carrying non-vanishing momenta evolving in homogeneous three geometries, thereby allowing the study of local phenomena [8]. When mini-superspace is conceived in this manner, the rather symmetrical roles played by \( a \) and \( \phi \) (the homogeneous part of a matter field) in a strict mini-superspace reduction become clearly distinguishable since the local perturbations of all fields (matter and gravity) are minimally coupled to \( a \) and not to \( \phi \).

Secondly, of great importance, is the choice of the formalism used to investigate the solutions of the WDW equation. Indeed, the choice of the formalism plays a double role. First, it organizes the mathematical approximations necessary to construct non-trivial solutions. Secondly, it partially determines the physical interpretation of these solutions. In fact, a careful examination of the literature reveals that authors who use very different mathematical expansions reach generally different conclusions concerning the interpretative aspects, see [1]-[5],[13]. In any case, at present, no fully consistent interpretation of the solutions of the WDW equation has been reached. At the end of this paper, we shall present the reasons that prevent a simple interpretation of these solutions. However, in spite of these difficulties, unambiguous answers to the three questions raised above can be obtained in certain cases. To this end, we shall apply, following [14], an (extended) adiabatic treatment to the WDW equation. The merits of this treatment are the following.

First, when applied both to the Schrödinger and the WDW equations describing the same quantum processes, this treatment makes the comparison of both descriptions of the transitions quite transparent. This is because, in both descriptions, the dynamical role of the expansion rate of the universe is made explicit. Indeed, non-adiabatic transitions are directly induced by the expansion. This is to be opposed to time dependent perturbation theory wherein transitions are induced
by the interaction Hamiltonian and bear therefore no direct relation to cosmology. Notice also that in the absence of adiabatic transitions, there is nothing to clock. Thus, time cannot be recovered from a universe in which matter is in an eigenstate (or a frozen superposition thereof). In fact, we shall recover the notion of time from the transition rates induced by the expansion. In this, the analysis of non-adiabatic transitions precisely formalizes the idea expressed above following which time reappears from a comparison of the rates of two physical processes. Indeed, no reference to external properties, like the trajectory of the “peak” of the wave function of the universe, will be used.

Moreover, the adiabatic treatment leads to an exact rewriting of the WDW equation in which the consequences of the dynamical character of gravity are displayed. It does not, in itself, introduce any kind of approximations nor does it require specific matter properties. Instead, it organizes the solutions of the WDW equation into series characterized by adiabatic parameters. Then, the validity of a truncation of these series requires that the parameters, say, be small. This in turn puts physical restrictions on the space of valid truncated solutions. The important fact is that this organization of the series answers by itself question 3, i.e. it both determines the precise conditions that guarantee that the classical and quantum descriptions coincide and explicitizes the mechanisms by which the two descriptions differ when the conditions are not met. In brief, the main condition that validates a truncation of the series is that the universe be macroscopic, i.e. that the matter sources driving gravity be macroscopic. (Recall that the appropriate character of the adiabatic treatment follows from the fact that one deals with dynamically coupled systems which are characterized by very different time-energy scales.) Then, the truncated adiabatic treatment shows that microscopic quantum transitions evolve according to a unitary evolution in the mean geometry parametrized by $\alpha$, in a manner similar that non-adiabatic electronic (light) transitions occurring in molecules can be parametrized in terms of the (heavy) nuclei positions. Indeed, in cosmology, the rest mass of all matter delivers through the WDW constraint a kind of (macroscopic) inertia to gravity which fixes the geometry in which the (microscopic) transitions take place. Notice that the value of the Planck mass plays no role into this division in macroscopic and microscopic energy scales. We emphasize this point: for non empty macroscopic universes, it is inappropriate to develop the solutions of the WDW equation in series of the Planck mass, contrary to what is adopted in the “standard treatment” presented for instance in [5].

Finally, the adiabatic treatment allows to question the problem of the interpretation of the solutions of the WDW equation in well defined mathematical terms. In the case of macroscopic universes, the coefficients that must be interpreted as the amplitudes to find the $n$-th state at $\alpha$ are designated by the formalism. As
pointed out to me by S. Massar, this procedure to reach the interpretation of the WDW solutions bears many similarities to the original reasoning that Max Born used to reach the probabilistic interpretation of the solutions of the Schrödinger equation \cite{16}. In both cases, it is through an examination of the mathematical properties of the solutions describing quantum transitions that the physical interpretation is, a posteriori, reached.

The properties and the inferred interpretation are the following \cite{13, 14}. As long as matter is close to equilibrium and evolving in a macroscopic universe, the evolution extracted from the WDW equation coincides with the Schrödingerian evolution of the corresponding problem. The identification of the amplitude to find matter in the n-th state is unambiguous and the physical interpretation follows from this identification. When matter is far from equilibrium but the universe still macroscopic, the evolution differs from the Schrödingerian one, even though it is still unitary. As before, the identification and the interpretation of the amplitudes evolving unitarily are unambiguous. In both cases, unitarity follows from the ordered semi-classical expansion of the macroscopic universe.

Therefore, there is a major difficulty when the cosmological expansion can no longer be described semi-classically. In this case indeed, through backscattering, one obtains non-vanishing coupling terms between quantum matter states associated with expanding and contracting universes. The present difficulties in interpreting the solutions of the WDW equation stem from the consequences of these coupling terms. Most probably, they require an extension of the usual concepts prevailing in quantum mechanics. We shall conclude this paper by presenting and commenting various approaches that have been proposed to overcome these difficulties.

2 The adiabatic treatment applied to quantum mechanics in classical cosmology

When studying quantum processes in classical cosmology, the expansion of the universe is treated at the background field approximation (BFA), i.e. \( a = a(t) \) is given from the outset and thus unmodified by the quantum transitions that one investigates. This is of course an approximation since, in general relativity, gravity is coupled to all forms of energy. To take into account this coupling is one of the jobs of quantum gravity.

In classical cosmology, the point that is crucial for us is that the expansion law, \( a = a(t) \), leads, in the general case, to time dependent hamiltonians. Indeed only degenerate cases, like purely photonic homogeneous universes, are characterized by constants of motion. Therefore, in general, one deals with the (explicitly) time
dependent Schrödinger equation

\[ i\partial_t |\psi(t)\rangle = H(t) |\psi(t)\rangle \]  \hspace{1cm} (1)

(In this equation, as everywhere in this paper, we designate by \(t\) the proper time evaluated in the classical universe one is dealing with.) The main consequence of eq. (1) is that there will be no stationary states, a very useful feature on which we shall base our investigation of the role of time in cosmology. Physically it means that the expansion rate \(\dot{a}\) induces quantum transitions. In this respect it should be noticed that the whole time dependence of \(H(t)\) comes from \(a(t)\) only, i.e.

\[ H(t) = H(a(t)) \]  \hspace{1cm} (2)

There is no external time dependence. In other words, we postulate that the universe is isolated. This possesses some flavor of General Relativity and will be automatically implemented when working in quantum cosmology.

The adiabatic treatment consists in developing the solutions of eq. (1) in terms of instantaneous (normalized) eigenstates of \(H(a)\)

\[ H(a)|\psi_n(a)\rangle = E_n(a)|\psi_n(a)\rangle \]  \hspace{1cm} (3)

\[ \langle \psi_n(a)|\psi_m(a)\rangle = \delta_{n,m} \]  \hspace{1cm} (4)

One can already see the appropriateness of this treatment. It naturally incorporates the fact that the eigenstates and their eigenvalues depend on \(t\) through \(a(t)\) only. This is exactly like the red-shift mentioned in the introduction: the energy of photons measured in proper time scales as

\[ \omega'(a) = \omega_{\text{emission}} \frac{a_{\text{emission}}}{a} \]  \hspace{1cm} (5)

which is a special case of eq. (3).

By factorizing, as in time dependent perturbation theory\[14\], the “free” kinetic factor \(\exp(-i \int^t dt' E_n(t'))\), the development in this basis reads

\[ |\psi(t)\rangle = \sum_n c_n(t) \ e^{-i \int^t dt' E_n(t')} |\psi_n(a(t))\rangle \]  \hspace{1cm} (6)

By inserting this development into eq. (1), one obtains the equation that determines the time dependence of the coefficients \(c_n(t)\):

\[ \partial_t c_n = \sum_{m \neq n} \langle \partial_t \psi_m(t)|\psi_n(t)\rangle \ e^{i \int^t dt' E_m(t')} \] \(\exp \left[ -i \int^t dt' (E_m(t') - E_n(t')) \right] c_m(t) \]  \hspace{1cm} (7)

It is instructive to compare this equation with the one obtained in time dependent perturbation theory. In that case, the matrix elements \(\langle m|V|n\rangle\) of the perturbation
V induce transitions among the “free” states $|n\rangle$ and $|m\rangle$. Here, these matrix elements are replaced by $\langle \partial_t \psi_m(t) | \psi_n(t) \rangle$. Simple algebra gives

$$\langle \psi_m | \partial_t \psi_n \rangle = \frac{\langle \psi_m | \partial_t H | \psi_n \rangle}{E_n - E_m}, \quad n \neq m$$

(8)

Thus, it is the time dependence of the instantaneous eigenstates, which is induced and determined by the time dependence of $H$, that induces, in turn, transitions among these states.

Therefore, in classical cosmology, non-adiabatic transitions are caused by the expansion law $a(t)$ since $H(t)$ depends on time through $a$ only. Moreover, one can rewrite eq. (4) directly in terms of $a$ as

$$\partial_a c_n = \sum_{m \neq n} \langle \partial_a \psi_m(a) | \psi_n(a) \rangle \exp \left[ -i \int_{a_0}^{a} \frac{1}{\dot{a}(a')} (E_m(a') - E_n(a')) \right] c_m(a)$$

(9)

One sees that the only place where time appears is in the difference of phases between neighboring states. Furthermore it appears parametrized by $a$, through the inverse rate $1/\dot{a}(a)$. Given this rate, the lapse of proper time from $a_0$ is of course given by

$$\Delta t(a) = \int_{a_0}^{a} \frac{1}{\dot{a}(a')}$$

(10)

This equation makes explicit the fact that $\dot{a}(a)$ should be known to convert Doppler shifts controlled by $a_{\text{reception}}/a_{\text{emission}}$ into proper time lapses, c.f. the Introduction.

As such eq. (4) is simply a rewriting of the Schrödinger equation in which the role of the expansion rate has been put forward. However, its real usefulness will become manifest in Section 3 since it is precisely in that guise that the evolution of the $c_n$, i.e. the weights of the instantaneous eigenstates, are delivered in quantum cosmology. This is not an accident, as we explain below. Moreover, eqs. (7, 9) prepare the analysis of the solutions both from the mathematical and the physical point of view.

We start with the physical point of view. As noticed in the Introduction, Max Born was lead to the interpretation of the wave function by the properties of first order transition amplitudes. More precisely, the three following points are such that, according to him, “only one interpretation is possible”[16]: 1. When initially $c_m = \delta(m, n)$, this means that the system is in the $n$-th state characterized by the eigen-energy $E_n$, 2. Mathematically, the final values of $c_m$ are asymptotically constant and satisfy $\sum_m |c_m|^2 = 1$ (for all hermitian hamiltonians and for normalized eigenstates) and 3. Experimentally, the asymptotic “electron” is found in one of the outcomes labeled by $n$. These three properties are also found in the adiabatic treatment of $|\psi(t)\rangle$. This is not surprising in view of the analogies with perturbation theory. Thus, the properties of $c_n(t)$, solutions of eq. (7), can also be used to
reach the interpretation of $|\psi(t)\rangle$ along Born’s lines. Moreover, similar properties will be found as well in quantum cosmology when an extended adiabatic treatment is applied to the WDW equation. We shall then base our interpretation of the wave function of the universe on the properties so obtained.

From a mathematical point of view, the first useful property of the adiabatic treatment follows from the fact that the instantaneous eigenstates form a “comoving” vector basis in Fock space. By comoving, we mean that no transition among eigenstates occurs in the limit in which the characteristic time of the expansion (i.e. $a/\dot{a}$) is much larger than the characteristic time of quantum transitions (i.e. the time for the Golden Rule to be valid). To work in the adiabatic approximation simply means that one neglects completely these transitions. Moreover, to first order in the non-adiabaticity, the probability to find a transition takes a universal form controlled by an exponentially small factor, like in a tunneling process, see [14].

The second useful property concerns the possibility of enlarging the dynamics. Up to now indeed, the adiabaticity concerned only the quantum dynamics of the (light) matter degrees of freedom since $a(t)$ was treated at the BFA. However, the adiabatic treatment is naturally enlarged so as to take into account the quantum dynamics of the (heavy) degree $a$ as well. This “extendible” property of the adiabatic treatment is precisely what we need to investigate (light) transitions in quantum cosmology. Moreover, this treatment leads to a rewriting of the WDW equation such that the comparison to the Schrödinger equation is greatly facilitated. Indeed, the description of light transitions in quantum cosmology coincides with the BFA description when a first order (light) change is applied to the heavy WKB dynamics. To establish how these mathematical properties are precisely implemented in the formalism is the first aim of next Section.

3 The extended adiabatic treatment applied to quantum cosmology

As we explained, we shall use an extended adiabatic treatment as a guide to first identify and then to interpret the coefficient $C_n(a)$, i.e. the weight of the $n$-th adiabatic state, that replaces the coefficient $c_n(a)$ of eq. (3). Before accomplishing this program, we briefly present the kinematical properties at work in quantum cosmology when the eigenstates are stationary, i.e. in the absence of transitions. It should be emphasized that this kinematical analysis is preparatory in character since it reveals the framework in which transitions take place upon considering non-degenerate cases. In those cases only, one can obtain a meaningful notion of evolution based on physical processes.
The (preparatory) notion of propagation in absence of transitions

To work with matter systems such that no transition occurs, requires that the matter states be stationary eigenstates of the Hamiltonian $H(a)$

$$H(a)|\psi_n\rangle = E_n(a)|\psi_n\rangle$$
$$\partial_a|\psi_n\rangle = 0$$  \hspace{1cm} (11)

In a Schrödinger context, this degenerate case would lead to no evolution in the sense that the coefficients $c_n$ would be constant, see eq. (9). One simply obtains a (frozen) superposition of eigenstates whose relative phases depend on time through their eigenvalues. To give a physical substance to these phases requires either the addition of internal interactions or a coupling to the external world, see [13].

In general relativity restricted to minisuperspace, when matter is characterized by an energy $E_n(a)$, the gravitational action satisfies the Hamilton-Jacobi constraint

$$H_G(a) + E_n(a) = \frac{-G^2 (\partial_a S_G(a))^2 + \kappa a^2 + \Lambda a^4}{2Ga} + E_n(a) = 0$$  \hspace{1cm} (12)

where $G$ is Newton’s constant, $\kappa$ is equal to $\pm 1$ or 0 for respectively open, closed and flat three surfaces and $\Lambda$ is the cosmological constant. The solution of this equation is simply $S_n(a) = \int a \, dp_n(a')$ where the momentum of $a$ driven by $E_n(a)$ is

$$p_n(a) = \pm G^{-1} \sqrt{\kappa a^2 + \Lambda a^4 + 2Ga E_n(a)}$$  \hspace{1cm} (13)

The sign $-$ (+) corresponds respectively to an expanding (contracting) universe.

Upon working in quantum cosmology, the Hamilton-Jacobi constraint becomes the WDW equation

$$[H_G(a) + H(a)] |\Xi(a)\rangle = 0$$  \hspace{1cm} (14)

The matter states are still the stationary eigenstates of $H(a)$ given in eq. (11). Therefore, as in Schrödinger case, the wave function $\Xi(a, \phi)$ can be decomposed as

$$\Xi(a, \phi) = \langle \phi | \Xi(a) \rangle = \sum_n C_n \Psi(a; n) \langle \phi | \psi_n \rangle$$  \hspace{1cm} (15)

where the weights $C_n$ are constant and where the gravitational waves entangled to their corresponding matter state are solutions of

$$\left[G^2 \partial_a^2 + \kappa a^2 + \Lambda a^4 + 2Ga E_n(a)\right] \Psi(a; n) = 0$$  \hspace{1cm} (16)

Being second order in $\partial_a$, each equation has two independent solutions. This has to be the case since classically we can work either with expanding or contracting universes. Indeed one verifies that the WKB waves

$$\Psi_{WKB}(a; n) = \frac{e^{i \int_a^{a'} p_n(a')da'}}{\sqrt{2|p_n(a)|}}$$  \hspace{1cm} (17)
with positive (negative) Wronskians
\[ W_n = \Psi^*(a; n) \oint_a^b \Psi(a; n) \] (18)
correspond to expanding (contracting) universes in this semiclassical limit. It is now through the sign of the Wronskian rather than at the classical level that one can still choose to work either with expanding or with contracting universes (at least far from a turning point). Thus, upon abandoning the WKB approximation, one must deal with superpositions of contracting and expanding universes. As we shall see later in this article, this mixing leads to major difficulties concerning the notion of evolution.

Before examining transitions, two aspects should be analyzed. First, we shall construct the Feynman kernel to go from \( a_0, \phi_0 \) to \( a, \phi \) in order to make contact with the notion of free evolution and with the classical theory. Secondly, we shall express the exact solutions of eq. (16) in terms of the WKB solutions eq. (17). Both aspects shall be exploited upon studying non-adiabatic transitions in quantum cosmology.

The kernel to go from \( a_0, \phi_0 \) to \( a, \phi \) can be decomposed, as usual, with the help of the quantum conserved number \( n \) according to
\[ \mathcal{K}(a, \phi; a_0, \phi_0) = \sum_n \mathcal{K}_n(a; a_0) K_n(\phi; \phi_0) \] (19)
The matter kernel \( K_n(\phi; \phi_0) \) is equal to \( \langle \phi | \psi_n \rangle \langle \psi_n | \phi_0 \rangle \) as in Schrödinger settings. The gravitational kernel \( \mathcal{K}_n(a; a_0) \) is a solution of eq. (12) and satisfies specific boundary conditions, see [7]. In the WKB approximation, for \( a > a_0 \), it is equal to
\[ \mathcal{K}_n(a; a_0) = \Psi_{WKB}(a; n) \Psi^*_{WKB}(a_0; n) \] (20)

At this point, we wish to point out that we used a dissymmetrical treatment of \( a \) and \( \phi \) when constructing the kernel \( \mathcal{K} \) or expressing the general solution of the WDW equation in eq. (15). As in Schrödinger settings, only the states of the matter field \( \phi \) were quantized, i.e. imposed to belong to a Hilbert space. This dissymmetry is reinforced upon considering many matter fields since only \( a \) will not participate to the definition of the Hilbert space.

When the field configurations are such that the dominant contribution comes from states with energy distributed around \( E_n(a) \), one can expand the gravitational kernels around that energy. For a discussion concerning the validity of this expansion, see [7]. By using eqs. (13, 17) and by developing the phase\(^1\) to first order expansion in the phase. The kernel itself is still a non-linear function of \( E_n - E_n \) since one keeps all terms of the exponential. This is how the classical behavior is recovered from quantum mechanics[12].
order in $E_n - E_{\bar{n}}$, one obtains
\begin{align*}
\mathcal{K}(a, \phi; a_0, \phi_0) &= \Psi^*(a_0; \bar{n}) \Psi(a; \bar{n}) \\
&\times \sum_n \exp \left[ -i \int_{a_0}^{a} \frac{a'}{Gp_{\bar{n}}(a')} [E_n(a') - E_{\bar{n}}(a')] \right] K_n(\phi; \phi_0) \\
&= \Psi^*(a_0; \bar{n}) \Psi(a; \bar{n}) \sum_n \exp \left[ -i \int_{0}^{t_n} dt' [E_n(t') - E_{\bar{n}}(t')] \right] K_n(\phi; \phi_0) \\
&= \Psi^*(a_0; \bar{n}) \Psi(a; \bar{n}) e^{-i \int_{0}^{t_n} dt' E_{\bar{n}}(t')} \times K(\phi; \phi_0, t_n) 
\end{align*}
(21)

In the second line, we have introduced the Hamilton-Jacobi time, the conjugate to the matter energy $E$, defined by
\begin{equation}
\Delta t_{\bar{n}}(a) = \partial_E \int_{a_0}^{a} da' p(a', E)|_{E=E_{\bar{n}}} = \int_{a_0}^{a} \frac{a'}{Gp_{\bar{n}}(a')} = \int_{a_0}^{a} \frac{1}{\dot{a}_{\bar{n}}(a')}
\end{equation}
(22)

By definition it equals the proper time evaluated in the geometry driven by $E_{\bar{n}}$ and given in eq. (10). In the third line of eq. (21), we have made use of the relation between the time dependent kernel $K(\phi; \phi_0, t)$ and its decomposition in terms of eigenstates of given energy.

Eq. (21) shows that the kernel $\mathcal{K}$ to propagate from $a_0, \phi_0$ to $a, \phi$ can be decomposed in a purely gravitational term controlled by the mean energy times the usual time dependent matter kernel $K$ evaluated in the “mean” geometry, i.e. in the geometry driven by the mean energy $E_{\bar{n}}$. Notice that this (kinematical) time dependence was obtained in two steps. Firstly, through a first order change of the gravitational action with respect to the matter energy $E_n$ and secondly, by using the “dispersion relation” $\partial_E p(a, E) = 1/\dot{a}$ (i.e. the Hamilton equation $dE/dp = \dot{a}$) stemming from the dynamical character of gravity. In view of both steps, there is a the strong analogy between this development leading to a time dependent kernel and the development that leads to the concept of a canonical partition function of a little system contained in a bigger ensemble which is microcanonically distributed, see App. B in \[7\]. In each case, it is through a first order change in the partitioning energy that the “classical” concept of time (or inverse temperature) is determined from the properties of the heavy system. Moreover, in both cases, the physical justification of considering fluctuations of the partitioning energy arises from the existence of interactions, see next subsections.

From eq. (21), it is straightforward to make contact with classical mechanics. To this end, one should suppose that the matter kernel can also be correctly approximated by its WKB expression. In this case, it is entirely governed by the matter action $S = \int_{a_0}^{a} d\phi' \pi(\phi', E_n)$ where $\pi(\phi, E_n)$ is the momentum of $\phi$ at fixed $E_n$. Then, the dominant contribution to $\mathcal{K}(a, \phi; a_0, \phi_0)$ comes from energy repartitions located near the saddle point of its phase. The location of this saddle point
is given by the solution of
\[
\partial E_n \left[ \int_{a_0}^{a} da' p(a', E_n) - \int_{\phi_0}^{\phi} d\phi' \pi(\phi', E_n) \right] = 0
\]
\[
\Delta t_n(a, a_0) - \int_{\phi_0}^{\phi} d\phi' \frac{1}{\dot{\phi}_n(\phi')} = 0
\] (23)

In the second equality, we have used the dispersion relation for the matter field to rewrite \( \partial_E \pi(\phi, E)|_{E=E_n} \) by \( 1/\dot{\phi}_n \) exactly as we just did with \( a \). Eq. (23) means that the saddle value \( E_n \) is such that the lapses of time evaluated separately for gravity and matter agree. This constructive interference condition (see Box 25.3 in [12]) can be conceived as the “dual” of a resonance condition in the following sense. In traditional time dependent settings, the dominant contribution to quantum processes arises from states such that the energy is conserved, c.f. the Golden Rule\[17\]. Here, in quantum cosmology, energy conservation is built in, thanks to the constraint equation. Thus, the phases of sub-systems interfere constructively such that their classical times agree. This is exactly like the zeroth law of thermodynamics: at equilibrium, (inverse) temperatures agree.

In physical terms, eq. (23) means that the cosmological time \( \Delta t_n(a, a_0) \) extracted from the expansion law (which has a similar status to that of the “ephemeris” time based on the solar system dynamics) is equal to the “cesium” time obtained from the (microscopic) behavior of matter. (This equality is not a tautology since it relates uncoupled dynamical systems characterized by widely separated time scales.) Notice finally that this condition of equal times can be formulated in purely classical terms. Indeed it provides the answer to the following question[11]: Given \( a_0, \phi_0 \), what is the value of \( E_n \) such that \( \phi \) is reached at \( a \) ? Thus we have established that the emergence of time in the quantum kernel \( K \), see eq. (21), makes use of classical concepts only: As in eq. (23), it follows, through a first order variation of \( E \), from the classical relationship \( \partial_{EP}(a, E) = 1/\dot{a} \).

The second point that we wish to address concerns the relationship between the exact solutions of eq. (16) and their WKB approximate expressions, eq. (17). This relationship is needed to properly investigate the consequences of quantizing the propagation of \( a \) expressed by eq. (14). Had we obtained a first order equation in \( \partial_a \), this would have meant that \( a \) had no dynamics at all, like the longitudinal part of the electric field in classical electrodynamics (or in QED) which is fully determined by the charge density (operator). Being second order, eq. (14) implies that some backscattering might, and in general will, be spontaneously generated. This quantum effect cannot be expressed in terms of matter states unlike the Coulomb-Coulomb interaction which can be represented by a composite operator of charged fields. Moreover, gravitational backscattering will modify the propagation of matter states as we shall see below.
To express the exact solutions of eq. (16) in terms of their WKB expressions, we follow the usual technique of replacing a second order equation by a set of two coupled first order ones, see [14] for more details. The exact solution is first decomposed as

$$\Psi(a; n) = C_n(a)\Psi_{WKB}(a; n) + D_n(a)\Psi^\ast_{WKB}(a; n)$$  \hspace{1cm} (24)$$

and the the coefficients $C_n(a)$ and $D_n(a)$ are fully determined by requiring that $\partial_a\Psi(a; n)$ be instantaneously decomposable into purely forward and backward waves

$$i\partial_a\Psi(a; n) = p_n(a) [C_n(a)\Psi_{WKB}(a; n) - D_n(a)\Psi^\ast_{WKB}(a; n)]$$  \hspace{1cm} (25)$$

This guarantees that $C_n(a)$ and $D_n(a)$ are constant in the adiabatic limit $\partial_a p_n / p_n^2 \to 0$. In addition, the Wronskian takes the simple form

$$W_n = \Psi^\ast(a; n) \partial_a \Psi(a; n) = |C_n(a)|^2 - |D_n(a)|^2 = \text{constant}$$  \hspace{1cm} (26)$$

Simple algebra then yields the coupled first order equations

$$\partial_a C_n(a) = \frac{1}{2} \frac{\partial_a p_n(a)}{p_n(a)} e^{-2i \int^a da' p_n(a')} D_n(a)$$

$$\partial_a D_n(a) = \frac{1}{2} \frac{\partial_a p_n(a)}{p_n(a)} e^{2i \int^a da' p_n(a')} C_n(a)$$  \hspace{1cm} (27)$$

These equations are equivalent to the original equation for $\Psi(a; n)$, eq. (16). They constitute a convenient starting point for evaluating perturbatively non-adiabatic transitions from $C_n(a)$ to $D_n(a)$ (i.e. backscattering). Moreover they resemble to the Schrödinger equation (9) that governs non-adiabatic transition in the particular case of two eigenstates with $\langle \partial_a \psi_m(a) | \psi_n(a) \rangle$ replaced by $\partial_a p_n(a)/p_n(a)$, see [14] for a more detailed comparison.

**The double adiabatic treatment**

In this subsection, we consider the non-degenerate cases in which the eigenstates of the matter hamiltonian depend on $a$. In these cases, the coefficients $C_n$ also depend on $a$, like the $c_n(a)$ in eq. (9). To obtain the equation which governs their evolution, we need to join the usual adiabatic treatment presented in eqs. (3-9) with the treatment by which eq. (16) is represented by eqs. (27).

To this end, we first carry out the instantaneous diagonalization of $H_M$, exactly like in eq. (9). We emphasize that this diagonalization does not require the “existence” of a Schrödinger equation. Using these instantaneous eigenstates, $|\Xi(a)\rangle$, solution of eq. (14), can always be decomposed as

$$|\Xi(a)\rangle = \sum_n \varphi_n(a) |\psi_n(a)\rangle$$  \hspace{1cm} (28)$$
The novelty of this decomposition with respect to eq. (15) is that the waves \( \varphi_n(a) \) are no longer decorrelated since the instantaneous eigenstates \( |\psi_n(a)\rangle \) do now depend on \( a \). The difference with the adiabatic treatment of section 2 arises from the fact that the WDW constraint is second order in \( \partial_a \). Therefore, to obtain a first order equation for the coefficients \( C_n \), we must proceed to a second adiabatic development. Thus, as in eq. (24), we express \( \varphi_n(a) \) in terms of the WKB waves

\[
\varphi_n(a) = C_n(a) \Psi_{WKB}(a; n) + D_n(a) \Psi^*_{WKB}(a; n) \quad (29)
\]

In view of the coupling among eigenstates, we must generalize eq. (25). We require now that

\[
\langle \psi_n(a) | i \partial_a | \Xi(a) \rangle = p_n(a) [C_n(a) \Psi_{WKB}(a; n) - D_n(a) \Psi^*_{WKB}(a; n)] \quad (30)
\]

This equation still expresses that \( \partial_a |\Xi(a)\rangle \) is instantaneously decomposed into a superposition of forward and backward waves. With this condition, the total current carried by \( |\Xi(a)\rangle \) contains no terms proportional to \( \partial_a C_n(a) \). Indeed, one finds

\[
\langle \Xi(a) | i \partial_a | \Xi(a) \rangle = \int d\phi \left[ \Xi^*(a, \phi) i \partial_a \Xi(a, \phi) \right] = \sum_n |C_n(a)|^2 - \sum_n |D_n(a)|^2 = C \quad (31)
\]

Notice that the absence of relative factors in the above sum follows from the usual choice of working with equally normalized eigenstates, eq. (4), as well as our choice of identical (unit) Wronskians for the WKB waves that form our gravitational basis.

Finally, by generalizing the derivation of eq. (27), one finds

\[
\partial_a C_n = \sum_{m \neq n} \langle \partial_a \psi_m | \psi_n \rangle \frac{p_n + p_m}{2\sqrt{p_n p_m}} e^{-i \int^a (p_n - p_m) da} C_m + \sum_m \langle \partial_a \psi_m | \psi_n \rangle \frac{p_n - p_m}{2\sqrt{p_n p_m}} e^{-i \int^a (p_n + p_m) da} D_m + \frac{\partial_a p_n}{2p_n} e^{-2i \int^a p_n da} \quad (32)
\]

and the same equation with \( C_n \leftrightarrow D_n \), \( i \leftrightarrow -i \). These equations are equivalent to the original WDW equation (14), exactly like eq. (9) is just a rewriting of eq. (4). Moreover they furnish a very convenient starting point for answering the three questions raised in the Introduction. We now turn to these aspects as well as those concerning the interpretation of \( |\Xi(a)\rangle \) in the light of the properties of the \( C_n(a) \).

The (physical) notion of evolution from non-adiabatic transitions

In the limit where both the adiabatic approximation for the matter states and the WKB approximation for gravity are valid, the coefficients \( C_n \) and \( D_n \) are constants.
In this case, we recover the (preparatory) situation of eq. (15) wherein there is neither correlations among the coefficients $C_n$ nor correlations with the $D_n$. One simply obtains a frozen superposition of uncorrelated eigenstates with $a$-dependent kinematical phase factors.

To given physical meaning to these phases, one must consider non-adiabatic transitions. The simplest case consists in working in the adiabatic approximation for gravity only. Then, one can neglect the coupling between the coefficients $C_n$ and $D_n$. The simplified dynamical equation is

$$\partial_a C_n = \sum_{m \neq n} \langle \partial_a \psi_m | \psi_n \rangle \frac{p_n + p_m}{2 \sqrt{p_n p_m}} e^{-i \int (p_n - p_m) da'} C_m(a)$$

which implies immediately

$$\sum_n |C_n(a)|^2 = \text{constant} \quad (34)$$

If one further assumes that the $C_n$ form a well defined wave packet in $n$ centered around the mean $\bar{n}$, one can develop eq. (33) around that mean, in power of $n - \bar{n}$. To first order in $n - \bar{n}$, one obtains

$$\partial_a C_n = \sum_{m \neq n} \langle \partial_a \psi_m | \psi_n \rangle \exp \left[ -i \int^{a} da' \frac{1}{\dot{a}_{\bar{n}}(a')} \{ E_n(a') - E_m(a') \} \right] C_m(a)$$

$$= \sum_{m \neq n} \langle \partial_a \psi_m | \psi_n \rangle \exp \left[ -i \int^{t_{\bar{n}}(a)} dt' \{ E_n(a_{\bar{n}}(t')) - E_m(a_{\bar{n}}(t')) \} \right] C_m(a)$$

This equation is identical to eqs. (7, 9). More precisely, through $dt_{\bar{n}}/da = 1/\dot{a}_{\bar{n}}(a)$, i.e. the (inverse) expansion rate of the mean universe, we recover the Schrödinger equation governing non adiabatic transitions among instantaneous matter states if one identifies the coefficients $C_n(a)$ with the probability amplitudes $c_n(a)$ to find the $n$-th state at $a$ in this mean universe. We emphasize the a posteriori character of this identification. Indeed, it is based on the comparison of eq. (33) with the resulting simplified equation governing the $C_n(a)$. Notice also that the justification of developing the expression around the mean value $\bar{n}$ follows from a closer examination of higher order terms in $n - \bar{n}$, see [7, 8]. The mathematical criterion that legitimizes a first order expansion in $n - \bar{n}$ is that the spread in energy satisfies $\langle (E_n - E_{\bar{n}})^2 \rangle \ll E_{\bar{n}}^2$. This requires that the universe be macroscopic[3][15]. Notice that the Planck mass does not appear in the mathematical criterion.

When the $C_n$ do not form a well defined wave packet, it is meaningless to expand in power of $n - \bar{n}$ around some value and to use the Hamilton-Jacobi time characterizing the corresponding geometry. Instead, one should work directly with eq. (13) and keep $a$ as the parameter to follow the evolution of the $C_n(a)$. Then,
even though each state “lives” in its geometry, i.e. is entangled to its gravitational wave, it is still mandatory to interpret the weight $C_n(a)$ as the probability amplitude to find the $n$-matter state at $a$. Indeed, the three properties used by Max Born and mentioned after eq. (10) are still found.

Once this identification is done, the interpretation of the full wave function $|\Xi(a)\rangle$ is determined. In particular, due to the second order character of the WDW constraint, the norm of $|\Xi(a)\rangle$ does not determine probabilities since

$$\langle\Xi(a)|\Xi(a)\rangle = \sum_n \langle\Xi(a)|\psi_n(a)\rangle\langle\psi_n(a)|\Xi(a)\rangle = \sum_n \frac{|C_n(a)|^2}{p_n(a)} \neq \text{constant} \quad (36)$$

Instead the current leads to the correct expression, see eqs. (31, 34). That the norm of $|\Xi(a)\rangle$ does not determine probabilities is free of physical consequences, at least in the present case where there is no coupling among $C_n(a)$ and $D_m(a)$.

Before considering the problems that arise when these couplings are taken into account, we wish to emphasize that the properties we just discussed, namely the recovery of the Schrödinger equation and the fact that the norm of $|\Xi(a)\rangle$ does not determine probabilities, are not specific to cosmology. Indeed they are also found upon considering the BFA limit in other dynamical models, such as accelerated particle detectors or mirrors [18, 19].

The difficulties induced by the $C_n$ – $D_m$ couplings

The last two terms in eq. (32) govern the couplings between the $C_n(a)$ and the $D_m(a)$. Once they are taken into account, they induce quantum transitions from $C_n|\psi_n(a)\rangle$ to $D_m|\psi_m(a)\rangle$. Mathematically, these transitions imply that the knowledge of the initial values of all $D_m$ is required in order to determine the evolution of the $C_n$. Physically, it means that matter states associated with expanding and contracting universes are quantum mechanically interacting. In other words, the matter states in our expanding universe do not form (exactly) a closed set, as if our universe were not isolated.

To our opinion, there is no satisfactory proposition which specifies how to deal with this problem. The main difficulties are:

i. how to cope with the conservation of the Wronskian, eq. (31), that is the violation of eq. (34) ?

ii. how to determine the values of the $D_m$ which are relevant for early cosmology ?

We emphasize the double aspect of the problem. There is both a conceptual side, i.e. to find the appropriate framework of interpretation, and a more pragmatic aspect which concerns the actual values of the $D_m$ and their relevance for our universe.

Various propositions have been made in order to try to cope with this double problem. All propositions somehow fall within one of the following four classes.
We refer to [4] for a detailed comparative description. In what follows, we shall briefly present what constitutes, to our opinion, their main weakness.

1. A first approach consists in rejecting the WDW equation itself since its solutions do not have a simple satisfactory interpretation. Then the main difficulty is to find a principle that selects the new equation. In particular, this principle should explain why the new equation is not the quantized version of the Hamilton-Jacobi constraint equation which is quadratic in $\partial_a S_{\text{gravity}}$. We recall that the canonical quantization of such a quadratic term inevitably generates backscattering effects that caused the abandonment of the WDW equation.

2. A second one consists in searching for a new definition of probabilities no longer based on the current carried by $|\Xi(a)|$. The so-called “conditional probabilities” constitute an example. In this case, the main difficulty is to explain why it is physically meaningful to consider probabilities which contain superpositions of expanding and contracting universes of the type given in eq. (29). What are the physical questions whose answers are sensitive either to an interfering term $C_n D_m^*$ or even to a sum like $|C_n|^2 + |D_m|^2$? Moreover, we recall that the “conditional probability” to find matter in the $n$-state depends on $a$ even in the absence of transition $[13]$.

3. Third quantization offers an alternative method to deal with $\Xi(a, \phi)$. In this perfectly consistent framework, backscattering is interpreted as pair creation of (macroscopic) universes. However, additional principles are required to define the “vacuum”, to pick the initial state as well as to be able to answer questions of the type: Given the initial state, what is the probability to find, at $a$, a specific matter state in the expanding sector? Without these additional principles, third quantization is rather useless.

4. The fourth attitude consists in admitting that probabilities and/or unitarity are intrinsically approximated concepts when restricted to the sole expanding sector of the theory. By adopting this attitude, one simply postpones confronting the problem of the $C_n - D_n$ couplings.

Conclusions

In this article, we have applied a double adiabatic treatment to the solutions of the WDW equation, eq. (14). The power of this treatment is displayed by its re-expression given in eq. (32). In itself, this rewriting does not introduce any kind of approximations. What it does instead is to deliver the precise nature of the approximations that reduces the WDW equation to the corresponding Schrödinger equation. These approximations are the following: One should neglect the coupling between expanding and contracting solutions in order to obtain a monotonic expansion and secondly, one must proceed to a first order expansion in the matter energy spread. No less no more.
Moreover, upon neglecting only the coupling between expanding and contracting solutions, one still obtains a unitary evolution even though it is not controlled by a time parameter, i.e. it cannot be obtained from a Schrödinger equation. Indeed, the evolution should be followed by $a$ itself.

Both simple results directly follow from the fact that the adiabatic treatment has been applied to the WDW equation. Indeed it re-organizes the dynamical variables so as to express the content of quantum cosmology (rather than quantum gravity in full generality), i.e. the dynamical interplay between the expansion of the universe described by the (homogeneous part of) the conformal factor $a$ and (local) matter fields. Had we used time dependent perturbation theory, no simple characterization of the approximations that should be implemented to recover the Schrödinger equation would have been obtained. The reason is that there is an interplay between higher order terms in the coupling constant and the gravitational distortion introduced by the level shifts, see [8]. The adiabatic treatment sorts out these effects by construction.

We wish to emphasize this last point: Due to the dynamical coupling between matter and gravity, the concept of the Schrödinger equation (expressable in any representation, c.f. eq. (1) and eq. (7)) no longer exists in quantum cosmology. This is particularly clear upon considering matter fields carrying non-vanishing momenta since the coupling to gravity occurs through energy. In that case indeed, the representation of Fock states in terms of local configurations, i.e. not characterized by a given energy-momentum, is a secondary approximate concept, see App. A of [7].

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