Relativistic least action principle for discontinuous hydrodynamic flows, Hamiltonian variables, helicity and Ertel invariant

A.V. KATS
Usikov Institute for Radiophysics and Electronics
National Academy of Sciences of Ukraine,
61085, 12 Ak. Proskury St., Kharkiv, Ukraine
e-mail: avkats@online.kharkiv.com

A rigorous method for introducing the variational principle describing relativistic ideal hydrodynamic flows with all possible types of breaks (including shocks) is presented in the framework of an exact Clebsch type representation of the four-velocity field as a bilinear combination of the scalar fields. The boundary conditions for these fields on the breaks are found. We also discuss the local invariants caused by the symmetries of the problem, including relabeling symmetry. In particular, the generalization of the well-known nonrelativistic Ertel invariant is presented.

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a. Introduction. In the paper we discuss some problems related to the ideal relativistic hydrodynamic (RHD) flows in the framework of the special relativity. They are pertinent to description of the flows with breaks in terms of the canonical (Hamiltonian) variables based upon the corresponding variational principle, and introducing the generalization of the Ertel invariant. These subjects are of interest both from the general point of view and are very useful in solving the nonlinear problems, specifically, for the nonlinear stability investigation, description of the turbulent flows, etc. The necessity to consider the relativistic flows is motivated by a wide area of applications to the cosmological problems.

Variational principles for the ideal relativistic hydrodynamic (RHD) flows are widely discussed in the literature, see, for instance, [1, 2, 3] and citations therein. As for the nonrelativistic flows, the least action principle is convenient to formulate in terms of the subsidiary fields and corresponding velocity representation known as the Clebsch representation, see [4, 5, 6, 7, 8, 9, 10]. These subsidiary fields can be introduced explicitly by means of the Weber transformation, [11], see also [3, 5]. Alternatively, they naturally arise from the least action principle as Lagrange multipliers for necessary constraints. Using these variables allows one to describe the dynamics in terms of canonical (Hamiltonian) variables. The nontrivial character of the Hamiltonian approach is due to the fact that the fluid dynamics corresponds to the degenerated case, see [12, 13].

In the papers [14, 15] it was shown that the hydrodynamic flows with breaks (including shocks) can be described in terms of such least action principle, which includes (as natural boundary conditions) the boundary conditions for the subsidiary fields. In the nonrelativistic case the triplet of the subsidiary fields corresponds to the Lagrange labels of the fluid particles, say, $\mu^B$, which are advected by the fluid,

$$d_t \mu^B = 0, \quad d_t \equiv \partial_t + v \cdot \nabla; \quad B = 1, 2, 3,$$

where $v$ denotes three-velocity. These equations along with the entropy advection and the fluid mass conservation are assumed as constraints. Corresponding Lagrange multipliers, $\lambda_B$, $\theta$ and $\varphi$, along with $\mu^B$ enter the Clebsch type velocity representation,

$$\rho v = -\rho \nabla \varphi - \lambda_B \nabla \mu^B - \theta \nabla s,$$

where $\rho$ and $s$ denote fluid density and the entropy per unit mass.

b. Variational principle. The relativistic least action principle can be formulated in a close analogy to the nonrelativistic one. Namely, introduce action $A$,

$$A = \int d^4 x \mathcal{L},$$

with the Lagrangian density

$$\mathcal{L} = -\epsilon(n, S) + G J^a Q_a,$$

$$G = (1, \nu_B, \Theta), \quad Q_a = (\varphi, \mu^B, S), \quad B = 1, 2, 3,$$

where $\nu_B$, $\Theta$, $\varphi$, $\mu^B$ represent subsidiary fields; $n$ and $\epsilon(n, S)$ denote the particle’s number, entropy and energy proper densities, $J^a = nu^a$ is the particle current, and $u^a$ is the four-velocity, $u^a = u^0(1, v/c)$, $u^0 = 1/\sqrt{1 - v^2/c^2}$;
comma denotes partial derivatives. Small Greek indexes run from 0 to 3, and the Latin indexes run from 1 to 3; $x^0 = ct$, $r = (x^1, x^2, x^3)$. The metric tensor, $g^{\alpha \beta}$, corresponds to the flat space-time in Cartesian coordinates, $g^{\alpha \beta} = \text{diag}\{-1, 1, 1, 1\}$. The four-velocity obeys normalization condition

$$u^\alpha u_\alpha = g_{\alpha \beta} u^\alpha u^\beta = -1. \quad (5)$$

Below we consider the four-velocity and the particle density $n$ as dependent variables expressed in terms of the particles current $J^\alpha$,

$$u^\alpha = J^\alpha / |J|, \quad n = |J| = \sqrt{-J^\alpha J_\alpha}. \quad (6)$$

The fluid energy obeys the second thermodynamic law

$$d\epsilon = n T dS + n^{-1} w dn \equiv n T dS + W dn, \quad (7)$$

where $T$ is the temperature and $w \equiv \epsilon + p$ is the proper enthalpy density, $p$ is the fluid pressure, $W = w/n$.

The action in Eq. (3) depends (for a fixed or infinite volume) on the independent variables $J^\alpha, \Theta, \text{and } Q = (\varphi, \mu^B, S, \nu_B, \Theta)$. Its variation results in the following set of equations

$$\delta J^\alpha := W u^\alpha = V^\alpha = -GQ_{,\alpha}, \quad (8)$$

$$\delta \varphi := J^\alpha = 0, \quad (9)$$

$$\delta \mu^B := \partial_\alpha (J^\alpha \nu_B) = 0, \quad \text{or } D\nu_A = 0, \quad (10)$$

$$\delta \nu_B := D\mu^B = 0, \quad (11)$$

$$\delta S := \partial_\alpha (J^\alpha \Theta), \quad \text{or } D\Theta = -T, \quad (12)$$

$$\delta \Theta := DS = 0, \quad (13)$$

where $D \equiv u^\alpha \partial_\alpha$. Eq. (8) gives us Clebsch type velocity representation, cf. Ref. [2]. Contracting it with $u^\alpha$ results in the dynamic equation for the scalar potential $\varphi$,

$$D \varphi = W. \quad (14)$$

Both triplets $\mu^B$ and $\nu_B$ represent the advected subsidiary fields and do not enter the internal energy. Therefore, it is natural to treat one of them, say, $\mu^B$ as the flow line labels.

Taking into account that the entropy and particles conservation are incorporated into the set of variational equations, it is easy to make sure that the equations of motion for the subsidiary variables along with the velocity representation reproduces the relativistic Euler equation. The latter corresponds to the orthogonal to the flow lines projection of the fluid stress-energy-momentum $T^{\alpha \beta}$ conservation, cf. Ref. [17, 18],

$$\frac{\partial T^{\alpha \beta}}{\partial x^\beta} \equiv T^{\alpha \beta}_{,\beta} = 0, \quad (15)$$

$$T^{\alpha \beta} = w u^\alpha u^\beta + pg^{\alpha \beta}. \quad (16)$$

Note that the relativistic Euler equation can be written as

$$(V_{\alpha,\beta} - V_{\beta,\alpha}) u^\beta = TS_{,\alpha}, \quad (17)$$

where the thermodynamic relation

$$dp = n dW - n T dS \quad (18)$$

is taken into account. Vector $V_\alpha$, sometimes called Taub current, [16], plays an important role in relativistic fluid dynamics, especially in the description of circulation and vorticity. Note that $W$ can be interpreted as an injection energy (or chemical potential), cf., for instance [17], i.e., the energy per particle required to inject a small amount of fluid into a fluid sample, keeping the sample volume and the entropy per particle $S$ constant. Therefore, $V_\alpha$ is identified with the four-momentum per particle of a small amount of fluid to be injected in a larger sample of fluid without changing the total fluid volume and the entropy per particle.
c. Boundary conditions. In order to complete the variational approach for the flows with breaks, it is necessary to formulate the boundary conditions for the subsidiary variables which do not imply any restrictions on the physically possible breaks (the shocks, tangential and contact breaks), are consistent with the corresponding dynamic equations and thus are equivalent to the conventional boundary conditions, i.e., to continuity of the particle and energy-momentum fluxes intersecting the break surface $R(x^\alpha) = 0$, cf. Ref. [13],

\[
\{ \tilde{J} \} = 0, \quad \tilde{J} \equiv J^{\alpha}n_\alpha,
\]  

(19)

\[
\{ T^{\alpha\beta}n_\beta \} = 0,
\]  

(20)

where $n_\alpha$ denotes the unit normal vector to the break surface,

\[
n_\alpha = N_\alpha/N, \quad N_\alpha = R_{\alpha\alpha} \quad N = \sqrt{N_\alpha N^{\alpha}},
\]  

(21)

and braces denote jump, $\{ X \} \equiv X|_{R=+0} - X|_{R=-0}$.

Our aim is to obtain boundary conditions as natural boundary conditions for the variational principle. In the process of deriving the volume equations we have applied integration by parts to the term $J^\alpha G \delta Q_{\alpha}$. Vanishing of the corresponding surface term along with that resulting from the variation of the surface itself lead to the appropriate boundary conditions after the variational principle has been specified.

Rewriting the (volume) action with the break surface being taken into account in the explicit form as

\[
A = \int d^4x \sum_{\varsigma=\pm1} \mathcal{L}^\varsigma \theta(\varsigma R),
\]  

(22)

where $\theta$ stands for the step-function, we obtain the residual part of the (volume) action in the form

\[
\delta A|_{\text{res}} = \int d^4x \sum_{\varsigma=\pm1} [\varsigma \mathcal{L} \delta \delta D(R) \delta R + \theta(\varsigma R) \partial_\alpha (J^\alpha G \delta Q)].
\]  

(23)

Here $\delta D$ denotes Dirac’s delta-function and we omit index $\varsigma$ labeling the quantities that correspond to the fluid regions divided by the interface $R = 0$; superscript $\varsigma \geq 0$ corresponds to the quantities in the regions $R \geq 0$, respectively. Integrating the second term by parts and supposing that the surface integral $\int d^4x \sum_{\varsigma=\pm1} \partial_\alpha (\theta(\varsigma R) (u^\alpha G \delta Q))$ vanishes due to vanishing of the variations $\delta Q$ at infinity, we arrive at the residual action expressed by the surface integral

\[
\delta A|_{\text{res}} = \int d^4x \sum_{\varsigma=\pm1} \varsigma \delta \delta D(R) \left[ \mathcal{L} \delta R - R_{\alpha\alpha} J^{\alpha} \bar{G} \delta Q \right].
\]  

(24)

$\bar{Q}$ here means the limit values of the volume variations, $\bar{Q}^{\pm} \equiv (\delta Q)|_{R=\pm0}$. It is convenient to express these variations in terms of variations of the boundary restrictions of the volume variables, $\bar{Q}(X_{R=\pm0}) \equiv \bar{X}^{\pm}$, and variation of the break surface. It is easy to show that

\[
\bar{Q} = \delta \bar{X} + |N|^{-1} n^\alpha X_{\alpha} \delta R - X_{\alpha} P^\alpha_{\beta} \delta f^\beta,
\]  

(25)

where $P^\alpha_{\beta} = \delta^\alpha_{\beta} - n^\alpha n_\beta$, and $\delta f^\beta$ is an arbitrary infinitesimal four-vector related to the one-to-one mapping of the surfaces $R = 0$ and $R + \delta R = 0$.

Vanishing of the action variation with respect to variations of the surface variables $\delta R$ and $\delta f^\beta$ (which are supposed to be independent) results in the following boundary conditions

\[
\delta R :\Rightarrow \{ p + (u^\alpha n_\alpha)^2 w \} = 0,
\]  

(26)

\[
\delta f^\beta :\Rightarrow P^\gamma_{\beta} \{ W J^\alpha N_{\alpha} u_\gamma \} = 0, \text{ or } P^\gamma_{\beta} \{ \tilde{J} W u_\gamma \} = 0,
\]  

(27)

which are equivalent to continuity of the momentum and energy fluxes, cf. Eq. [40]. Here we consider that the ‘on shell’ value of the volume Lagrangian density, $\mathcal{L}_{eq}$, is equal to the pressure, $\mathcal{L}_{eq} = -\epsilon + nGDQ = -\epsilon + w = p$.

Now we can complete formulation of the variational principle appropriate both for continuous and discontinuous flows. The independent volume variables are indicated above, and independent variations of the surface variables are
canonical coordinates. Then introduce the canonical (Hamiltonian) variables according to the general receipt. Namely, let the cases both with excluded and non-excluded volume constraints are discussed for ideal hydrodynamics and magnetohydrodynamics in the nonrelativistic limit. This can be done for the action the surface term respective for the surface constraints, cf. Refs. \[14, 15, 19, 20\], where such surface terms with the fluid pressure, cf. Ref. \[2\], where the continuous flows are discussed in detail. Second, we can include into four-velocity by means of representation (8). In this case the volume Lagrangian density can be chosen coinciding value of \(\Theta\). For \(\tilde{\Theta}\) specific case \(\tilde{\Theta}\) boundary values of the subsidiary variables, cf. nonrelativistic case discussed in Refs. \[14, 15\]. Note that for the variations of the subsidiary variables to be independent.

Thus, performing the Legendre transform with respect to \(Q\) dealing with the degenerated (constraint) system, cf. Refs. \[3, 6, 12, 13\]. But the constraints are of the first type. Taking into account the normalization condition for the Taub current, \(V_{\alpha}V^{\alpha} = -W^{2}\), we can transform the first term in Eq. (34) as

\[
V^{0} = \sqrt{W^{2} + V_{\alpha}V^{\alpha}}.
\]

Consequently, we arrive at the following Hamiltonian density

\[
\mathcal{H} = \mathcal{H}(P, Q, Q_{\alpha}; W) = \sqrt{W^{2} + V_{\alpha}V^{\alpha}} - p(W, S) - p(W, S).
\]
In terms of the canonical coordinates and momenta the space components of the velocity are
\[ \pi_\varphi V_a = -PQ_{\alpha}. \]

The canonical equations following from this Hamiltonian reproduce in a 3 + 1 form the above dynamic equations for the variables entering the Taub current representation. Variation of the action with respect to the chemical potential \( W \) results in
\[ n = \frac{\pi_\varphi}{\sqrt{1 + V_a V^\alpha/W^2}}. \]

Obviously, this relation is equivalent to Eq. (36), expressing the particle density \( n \) in terms of the variables entering the Hamiltonian.

Underline, that the Hamiltonian given by Eq. (57) depends not only on the generalized coordinates \( \varphi, \mu^B, S \), their spatial derivatives and conjugate momenta, but also on the chemical potential \( W \) as well. Evidently, we can consider \( W \) as the additional generalized coordinate with zero conjugate momentum, \( \pi_W = 0 \). This condition is consistent with the dynamic equations due to the fact that \( \partial_0 \pi_W = \partial H/\partial W = 0 \), cf. Eq. (39).

Bearing in mind the flows with breaks one can see that in the above discussed variant of the least action principle we do not arrive at the additional surface variables except that defining a break surface, \( R \). But it enters the action functional without derivatives. Therefore, corresponding conjugate momentum is zero-valued. Introducing the Hamiltonian variables for the flows with breaks we have to treat \( R \) as the surface function, defining some (surface) constraint. The latter is nothing else than continuity of the normal component of the fluid momentum flux, Eq. (26).

d. Poisson brackets. The Poisson brackets in the canonical variables are of a standard form. Symbolically,
\[ \{Q^A(x), P_B(y)\} = \delta_A^B \delta(x - y), \]
where \( \delta(x - y) \) is spacetime Dirac’s delta, \( x \) and \( y \) are spacetime points, \( Q^A \) is shorthand notation for \( \varphi, \mu^A \) and \( S \), and, analogously, \( P_B \) denotes corresponding conjugate momenta, \( \pi_\varphi, \pi_\mu^\alpha \) and \( \pi_S \).

e. Ertel invariant. In addition to energy, momentum, and angular momentum conservation, for the ideal hydrodynamic flows there are exist specific local conservation laws related to the dragged and frozen-in fields, and corresponding topological invariants (vorticity, helicity, Ertel invariant, etc.), cf. Refs. [3, 5, 6, 10, 21] and citations therein for the nonrelativistic case. They are caused by the relabeling symmetry. Discussion of these problems for the relativistic flows seems insufficient, see Refs. [1, 2, 3, 16] and citations therein. Exploitation of the above description permits one considering these invariants to be simplified. For example, consider here generalization of the Ertel invariant for the relativistic fluids (to my best knowledge, this item was not discussed earlier). Defining the Ertel four-current,
\[ E^\alpha = -\frac{1}{2} \epsilon^{\alpha\beta\mu\nu} \omega^\beta_{\mu} S^\nu = -\omega^\alpha_{\nu} S^\nu, \]
one can see that it is divergence-free, \( E^\alpha_{,\alpha} = 0 \). Here \( \epsilon^{\alpha\beta\mu\nu} \) is Levi-Civita tensor, \( \omega^\beta_{\mu} \) is the (Khalatnikov) vorticity tensor, \( \omega^\alpha_{\nu} = V_{\mu,\beta} - V_{\beta,\mu} \), and \( \omega^\alpha_{\nu} \) is its dual. Moreover, the Ertel four-vector \( E^\alpha \) is proportional to the particle current \( J^\alpha \),
\[ E^\alpha = E J^\alpha, \]
in view of \( E^\alpha_{,\alpha} = 0 \) resulting in \( E \equiv E^0/J^0 \) being dragged by the fluid,
\[ DE = 0, \]
i.e. \( E \) is the scalar invariant of the motion. In the nonrelativistic limit it coincides with the Ertel invariant (curl \( v \cdot \nabla s \))/\( \rho \), where \( \rho \) denotes the fluid density.

f. Helicity current. The helicity invariant in the nonrelativistic case exists for the barotropic flows and presents pseudoscalar \( v \cdot \text{curl} v \). The strict analog for the relativistic case is the pseudovector
\[ Z^\alpha = \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} \omega^\beta_{\mu} V^\nu = \omega^\alpha_{\nu} V^\nu, \]
Strict calculations show that for the isentropic flows the helicity current \( Z^\alpha \) is conserved, \( Z^\alpha_{,\alpha} = 0 \).

For the general type flows there exists generalization of the helicity current. Namely, consider reduced Taub vector,
\[ \tilde{V}_\alpha = V_\alpha + \Theta S_{,\alpha}, \]
where \( \Theta \) obeys Eq. (12), and the corresponding reduced vorticity tensor,

\[
\tilde{\omega}_{\alpha \beta} \equiv \tilde{V}_{\beta,\alpha} - \tilde{V}_{\alpha,\beta}.
\]

This tensor is orthogonal to the flow lines,

\[
\tilde{\omega}_{\alpha \beta} u^\beta = 0,
\]

and the reduced helicity current

\[
\tilde{Z}^\alpha = \ast \tilde{\omega}^{\alpha \nu} \tilde{V}_\nu
\]

is conserved for arbitrary flows,

\[
\tilde{Z}_{\alpha,\alpha} = \ast \tilde{\omega}^{\alpha \nu} \tilde{V}_{\nu,\alpha} = \frac{1}{2} \ast \tilde{\omega}^{\alpha \nu} \tilde{\omega}_{\alpha \nu} = 0.
\]

Conclusion. We have shown that it is possible to describe the relativistic ideal fluids with all physically allowable breaks in terms of the least action principle both in the Lagrangian and Hamiltonian description. The boundary conditions for the subsidiary variables entering the Clebsch type velocity representation are obtained in two different ways: one part follows from the variational principle as natural boundary conditions while the other one was obtained from the dynamic equations under assumption relating to absence of the corresponding sources and the maximal continuity compatible with the volume equations. Note that it is possible to change the variational principle in such a way that all boundary conditions will result from it, i.e., they become natural boundary conditions. For this purpose it is necessary to modify the variational principle by adding a surface term with corresponding constraints, similarly to the nonrelativistic case (compare with the papers \([14, 15]\) for the hydrodynamics and \([19, 20]\) for the magnetohydrodynamics). This variants are to be discussed in the forthcoming papers.

The approach discussed allowed us to give a simple treatment of the additional invariants of the motion, in particular, to present generalization of the Ertel invariant for the relativistic flows. This approach is suitable for the general relativity and for the relativistic magnetohydrodynamics as well. Note that for the flows without breaks the general relativity case is discussed in detail in the paper \([2]\). The discontinuous flows for the general relativity can be described in analogy to the above discussion and will be published elsewhere.

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Note that choice of the boundary conditions for the fields \( \varphi, \mu^B, \nu_B \) and \( \Theta \) is not unique due to the fact that they play roles of the generalized potentials and therefore possess the corresponding gauge freedom relating to the transformations \( \varphi, \mu^B, \nu_B, \Theta \to \varphi', \mu'^B, \nu_B', \Theta' \) such that \( u'_a = u_a \) (given by the representation (8)). For instance, it seems possible to use entropy \( S \) as one of the flow line markers. But if we are dealing with discontinuous flows then it is necessary to distinguish the Lagrange markers of the fluid lines, \( \mu^B \), and the entropy, \( S \). Namely, the label of the particle intersecting a shock surface evidently does not change, but the entropy does change. Thus, entropy can be chosen as one of the flow line markers only for the flows without entropy breaks.