Coupled post-glitch response of the crust and interior of neutron stars

C. A. van Eysden
NORDITA, Roslagstullsbacken 23, SE-10691 Stockholm, Sweden

Abstract.
The spin-up of magnetized plasma and its role in the post-glitch response of neutron stars was first studied over thirty years ago, where it was demonstrated that co-rotation between crust and plasma is established rapidly (within a few seconds) by a process analogous to Ekman pumping in a viscous fluid. However, if the magnetized plasma is considered to be ideal, conservation of energy implies that the final state cannot be co-rotation. Using an exact analytical solution for the coupled motion of the crust and plasma, we demonstrate that the system oscillates persistently and explore the consequences for neutron star observations.

Introduction
Pulsars are rapidly rotating neutron stars that emit radio waves along their magnetic dipole axis, which is misaligned with the rotation axis, creating a lighthouse effect that is observed at earth as pulsations. The pulse arrival time of these objects can be measured with extremely high precision, however, about 5% of the known pulsar population is occasionally observed to ‘glitch’, where the pulse frequency undergoes a sudden increase, followed by a recovery. The glitch event itself occurs over a timescale of less than a minute [2], while the recovery typically takes days to weeks [3]. The increase in pulse frequency during a glitch ranges from one part in $10^{-4}$ to $10^{-9}$, and is typically $10^{-6}$ and $10^{-8}$ in the Crab and Vela pulsars respectively [4, 3].

The mechanism for driving a pulsar glitch is still not completely understood, but it is generally attributed to an avalanche unpinning event of pinned superfluidity in the neutron star crust [5, 6, 9]. The core of the star, which consists of a neutron superfluid and a plasma of electrons and superconducting protons, responds to the glitch via magnetic, viscous and mutual friction forces. Ref. [10] showed that, for typical pulsars, the magnetic tension couples the plasma and the crust on a timescale of $\sim 2$ s, bringing the two components into co-rotation. As such, the crust and the plasma are usually assumed to be rigidly locked together in models of pulsar glitch recovery.

In this article, we re-visit the problem of the response of a magnetised plasma to an impulsive acceleration of the container and the analysis of [10]. We are interested in particular in two aspects: (i) what is the coupled response of container, and (ii) why is the final state of the plasma and container that of co-rotation? The first is motivated by recent studies of glitch recovery that calculate the motion of the neutron star crust in response to the interior fluid for comparison to observational data, which can potentially extract information about the transport coefficients of the interior fluid of the neutron star [4]. The second is motivated by the fact that the magnetised plasma is assumed to be ideal, making this problem qualitatively different from
To investigate the coupled response of a magnetised plasma and its container, we consider cylindrical geometry, with a uniform magnetic field aligned with the rotation axis. Classical spin up where there is dissipation. Conservation of energy and momentum arguments suggest that this system should be executing torsional oscillations, similar to those identified in magnetar quasi-periodic oscillations Ref. [11].

The model  
To study the coupled response of a magnetised plasma and its container, we consider the cylindrical geometry illustrated in figure 1. For $t < 0$, the container and fluid are rotating uniformly with angular velocity $\Omega$ about the vertical axis, denoted $\hat{k}$. The fluid and container also possess a uniform magnetic field of strength $B_0$, aligned with the rotation axis. At $t = 0$, the angular velocity of the container is impulsively increased to $\Omega + \delta \Omega(0)$ and the container and fluid are left to respond self-consistently. In contrast to previous studies, we also consider the motion of the container, so that $\delta \Omega$ is a function of $t$.

The equations describing the system are those of magnetohydrodynamics. In general, $\delta \Omega(0)/\Omega$ is much smaller than unity, so the equations can be linearised. Also, since the magnetic diffusivity is extremely small in neutron stars it is neglected. The scaled, linearised equations are

$$
\frac{\partial \delta \mathbf{v}}{\partial t} + 2\hat{k} \times \delta \mathbf{v} = -\nabla \delta P - \xi^2 \hat{k} \times (\nabla \times \delta \mathbf{B}) + E \nabla^2 \delta \mathbf{v},
$$

$$
\nabla \cdot \delta \mathbf{v} = 0,
$$

$$
\frac{\partial \delta \mathbf{B}}{\partial t} = \nabla \times (\delta \mathbf{v} \times \hat{k}).
$$

where we define the dimensionless coefficients

$$
E = \frac{\nu}{L^2 \Omega}, \quad \xi = \frac{1}{L \Omega} \sqrt{\frac{B_0^2}{\mu_0 \rho}}.
$$

**Figure 1.** To investigate the coupled response of a magnetised plasma and its container, we consider cylindrical geometry, with a uniform magnetic field aligned with the rotation axis.
Figure 2. Motion of the container for slow rotation, i.e., $\xi = 10$, $E = 0$ and $K = 1$.

The first is the usual Ekman number [12] and denotes the relative strength of viscous forces to inertial forces. The second, defined by Ref. [10] is the ratio of magnetic forces to inertial forces. We consider the infinite parallel plate geometry (containers of infinite aspect ratio) as in the classic studies of spin up [12, 10]. The usual no-slip boundary conditions apply to the fluid at the upper and lower plates

$$\delta \mathbf{v} = x \times \delta \Omega(t) \hat{k}.$$  \hfill (5)

As there is no magnetic diffusivity the magnetic field is flux-frozen, and Eq. (5) also satisfies the continuity requirements for the magnetic field at the boundary. The evolution of $\delta \Omega(t)$ is determined by considering a torque balance on the container, and evaluating the hydrodynamic torque using the stress tensor of the fluid.

The solution to this system of equations is obtained using Laplace transform techniques, the details of which will be presented elsewhere. However, in contrast to previous studies, we make no approximations regarding the magnitude of $E$ and $\xi$, and study the behaviour of the system for a range of these parameters.

Results

First, let us consider the limit $\xi \gg 1$, $E = 0$, i.e., when the rotational period is much longer than the Alfvén crossing time and the fluid is inviscid. We also take $K = 1$, where $K$ is the ratio of the moments of inertia of the fluid (rotating as a rigid body) to that of the container. The motion of the container is given in Fig. 2. We find that the solution oscillates, as expected, where the restoring force for the oscillations is provided by the magnetic field line tension. Unexpectedly, however, we find that these oscillations are not smooth, but incoherent. There are two reasons for this. First, the container is coupled to a spectrum of Alfvén waves in the fluid, and as a result behaves like a mass coupled to many smaller masses via springs. Importantly, the system is not chaotic (the equations are linear) or random, and the complicated motion arises from the many degrees of freedom in the system. Second, the eigenvalues of the system are solutions to a complicated transcendental equation and are unevenly spaced. This means that the nodes of different modes do not necessarily coincide, therefore the modes do not add to form a coherent, periodic result. This effect becomes more pronounced at larger times, when small misalignments of nodes have accumulated into larger phase differences between the modes.
Figure 3. Motion of the container for fast rotation, i.e., $\xi = 0.1$, $E = 0$, $K = 10$.

In Fig. 3, we consider a faster rotating container, with numbers closer to those corresponding to neutron stars, $\xi = 0.1$, $E = 0$ and $K = 10$. The rotation period is now much faster than the Alfvén crossing time. We find that the behavior identified by Ref. [10] is observed initially, namely, the container overshoots, oscillates briefly before settling to a state of co-rotation with the interior fluid. This is achieved by a mechanism closely analogous to Ekman pumping in a viscous fluid, in which a secondary flow circulates fluid from boundary layers at the container walls into the interior, spinning up the fluid [12]. However, co-rotation only persists until a dimensionless time of 20, at which point oscillations of the nature of those observed in Fig. 2 are present. The onset of these oscillations correspond exactly with the Alfvén crossing time of the container, when the magneto-inertial wave excited by the impulsive acceleration of the container reaches the opposing boundary and is reflected. At short times, when this wave has only propagated a short distance from the boundary, it facilitates the Ekman-like circulation identified by Ref. [10] in an analogous manner to the viscous boundary layer in classic Ekman pumping. After the Alfvén crossing time, however, it induces oscillations like in a slowly rotating container. These oscillations are not identified by Ref. [10], who studies the limit $\xi \to \infty$, because the magneto-inertial oscillations take an infinitely long time to traverse the container.

In Fig. 4, we present the solution for numbers expected in a neutron star, namely, $\xi = 0.001$, and $K = 50$. We also consider the inviscid limit, $E = 0$. The top-left panel shows the response for five Alfvén crossing times, while the remaining panels show short sections of the top-left panel in greater detail. We find that the qualitative features in Fig. 4 are present and exaggerated further. Oscillations onset after the Alfvén crossing time, and become messier at later times. The amplitude of oscillation is approximately one tenth of the glitch amplitude, and will onset approximately 20 s after the glitch for a pulsar with $\Omega = 100 \, \text{rad s}^{-1}$. Note, however, that in spherical geometry the Alfvén crossing time varies with latitude and is much shorter near the equator, resulting in quantitatively different behavior. Also note that because we have neglected other components of the star, i.e., the neutron superfluid, that glitch recovery is almost immediate, namely, 2 s for a pulsar with $\Omega = 100 \, \text{rad s}^{-1}$.

In Fig. 5, we consider the same parameters as those in Fig. 4, but include viscosity, taking $E = 10^{-7}$. We find similar qualitative behavior to that seen in Fig. 4, however the oscillations are significantly damped and have a maximum amplitude of approximately 1/200 of the glitch size. The time taken for the oscillations to become messy is also much longer. In Fig. 6, we plot the response to the event shown in Fig. 5 at four later times. Each panel has the same
scale on the time and amplitude axes for comparison. Clearly, as time progresses, the oscillation amplitude decreases while the period becomes longer. This is a result of viscosity damping the higher wavelength modes faster, so that at later times only the long wavelength (corresponding to long period) persist. Examining the top-right panel, the oscillation has an amplitude roughly a factor of $1/2000$ of the glitch amplitude and a frequency of $20$ s at a time of $2.78$ hrs after the glitch, assuming $\Omega = 100$ rad s$^{-1}$.

**Conclusions**

We have shown that in response to a glitch, the crust and interior plasma are expected to execute torsional oscillations. The state of co-rotation between the crust and plasma only persists until the Alfvén crossing time, when magneto-inertial waves excited by the glitch have traversed the
star and are reflected from the opposing boundary. The period of oscillation for a typical neutron star is predicted to be of the order of seconds. The oscillations are damped by viscosity, which establishes co-rotation on the viscous diffusion time. The oscillations are incoherent, and may be a contributing factor in timing noise. The model considered here neglects the important dynamics of the neutron superfluid in both the crust and the core, which is expected to result in the longer time-scales observed in glitch recovery, and which may affect the predictions presented here. This will be considered in future work.

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