Research Article

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Butterfly support for off diagonal coefficients and boundedness of solutions to quasilinear elliptic systems

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Abstract: We consider quasilinear elliptic systems in divergence form. In general, we cannot expect that weak solutions are locally bounded because of De Giorgi’s counterexample. Here we assume that off-diagonal coefficients have a “butterfly support”: this allows us to prove local boundedness of weak solutions.

Keywords: Quasilinear, elliptic, system, weak, solution, regularity
MSC: Primary: 35J47; Secondary: 35B65, 49N60

1 Introduction

This paper deals with quasilinear elliptic systems in divergence form

$$-\text{div}(a(x, u(x))Du(x)) = 0, \quad x \in \Omega,$$

(1.1)

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^{N \times n}$ is matrix valued with components $a_{i,j}^{\alpha,\beta}(x, y)$ where $i, j \in \{1, \ldots, n\}$ and $\alpha, \beta \in \{1, \ldots, N\}$.

On the coefficients $a_{i,j}^{\alpha,\beta}(x, y)$ we set the usual conditions, that is they are measurable with respect to $x$, continuous with respect to $y$, bounded and elliptic. When $N = 1$, that is in the case of one single equation, the celebrated De Giorgi-Nash-Moser theorem ensures that weak solutions $u \in W^{1,2}(\Omega)$ are locally bounded and even Hölder continuous, see section 2.1 in [27].

But in the vectorial case $N \geq 2$, the aforementioned result is no longer true due to the De Giorgi’s counterexample, see [6], section 3 in [27] and the recent paper [29]; see also [32] and [20].

So it arises the question of finding additional structural restrictions on the coefficients $a_{i,j}^{\alpha,\beta}$ that keep away De Giorgi’s counterexample and allow for local boundedness of weak solutions $u$, see Section 3.9 in [28].

In the present work we assume a condition on the support of off-diagonal coefficients: there exists $L_0 \in [0, +\infty)$ such that $\forall L \geq L_0$, when $\alpha \neq \beta$,

$$a_{i,j}^{\alpha,\beta}(x, y) \neq 0 \text{ and } |y^\alpha| > L \Rightarrow |y^\beta| > L,$$

(1.2)

(see Figure 1 and note that the support has the shape of a butterfly in the plane $y^\beta - y^\alpha$).

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Under such a restriction we are able to prove local boundedness of weak solutions. All the necessary assumptions and the result will be listed in section 2 while proofs will be performed in section 3.

It is worth to stress out that systems with special structure have been studied in [33], [26] and off-diagonal coefficients with a particular support have been successfully used when proving maximum principles in [21], $L^\infty$-regularity in [22], when obtaining existence for measure data problems in [23], [24] and, for the degenerate case, in [7].

Higher integrability has been studied as well in [10] when off-diagonal coefficients are small and have staircase support and in [11] when off-diagonal coefficients are proportional to diagonal ones.

Let us mention as well that when the ratio between the largest and the smallest eigenvalues of $a_{i,j}^{\alpha,\beta}$ is close to 1, then regularity of $u$ is studied at page 183 of [12]; see also [31], [18], [17], [19].

Let us also say that proving boundedness for weak solutions could be an important tool for getting fractional differentiability, see the estimate after (4.15) in [8]. In the present paper we deal with local boundedness of solutions. If the reader is interested in regularity up to a rough boundary it could be worth looking at [25].

\section{Assumptions and Result}

Assume $\Omega$ is an open bounded subset of $\mathbb{R}^n$, with $n \geq 3$. Consider the system of $N \geq 2$ equations

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \sum_{\alpha, \beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x, u) \frac{\partial}{\partial x_j} u^{\beta} \right) = 0 \text{ in } \Omega, \text{ for } \alpha = 1, \ldots, N. \quad (2.1)$$

Note that $u^{\beta}$ is the $\beta$ component of $u = (u^1, u^2, \ldots, u^N)$. We list our structural conditions.

(A) For all $i, j \in \{1, \ldots, n\}$ and all $\alpha, \beta \in \{1, \ldots, N\}$, we require that $a_{i,j}^{\alpha,\beta} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following conditions:

(A_0) $x \mapsto a_{i,j}^{\alpha,\beta}(x, y)$ is measurable and $y \mapsto a_{i,j}^{\alpha,\beta}(x, y)$ is continuous;

(A_1) (boundedness of all the coefficients) for some constant $c > 0$, we have

$$|a_{i,j}^{\alpha,\beta}(x, y)| \leq c$$

for almost all $x \in \Omega$ and for all $y \in \mathbb{R}^N$;

(A_2) (ellipticity of all the coefficients) for some constant $\nu > 0$, we have

$$\sum_{\alpha, \beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x, y) \xi_i^\alpha \xi_j^\beta \geq \nu |\xi|^2$$

for almost all $x \in \Omega$, for all $y \in \mathbb{R}^N$ and for all $\xi \in \mathbb{R}^{N+n}$;

(A_3) ("butterfly" support of off-diagonal coefficients) there exists $L_0 \in [0, +\infty)$ such that $\forall L \geq L_0$, when $\alpha \neq \beta$,

$$(a_{i,j}^{\alpha,\beta}(x, y) \neq 0 \text{ and } |y^\alpha| > L) \Rightarrow |y^\beta| > L,$$

(see Figure 1).
Fig. 1: Assumption ($A_3$): off-diagonal coefficients $a_{i,j}^{\alpha,\beta}$ vanish on the white part of the picture; they might be non zero only on the grey part.

**Remark 2.1.** Assumption ($A_3$) guarantees equality (3.2): such an equality is a basic tool for proving boundedness of solutions.

**Example 2.2.** An example of coefficients which readily satisfy the aforementioned assumptions are defined as follows:

$$a_{i,j}^{\alpha,\beta}(x, y) = a_{i,j}^{\alpha,\beta}(y) = \begin{cases} 
\delta_{ij} \frac{\max(|y| - |y^\alpha|, 0)}{1 + 2|y|} & \text{if } \alpha \neq \beta \\
\delta_{ij} & \text{if } \alpha = \beta
\end{cases}$$

where $\alpha, \beta = 1, 2$ and $i, j = 1, \ldots, n$ with $n \geq 3$ and $N = 2$. In this case we have $c = 1, \nu = 1/2$ and we can pick for instance $L_0 = 0$. 
We say that a function \( u : \Omega \rightarrow \mathbb{R}^N \) is a weak solution of the system (2.1), if \( u \in W^{1,2} \left( \Omega, \mathbb{R}^N \right) \) and

\[
\int_{\Omega} \sum_{i,j=1}^{N} \sum_{\alpha, \beta=1}^{n} a^{\alpha \beta}_{ij}(x) u(x) D_{ij} u(x) \varphi^{\alpha}(x) dx = 0, \tag{2.2}
\]

for all \( \varphi \in W_{0}^{1,2} \left( \Omega, \mathbb{R}^N \right) \).

**Theorem 2.3.** Let \( u \in W^{1,2} \left( \Omega, \mathbb{R}^N \right) \) be a weak solution of system (2.1) under the set (A) of assumptions. Then \( u \in L_{loc}^{\infty} \left( \Omega, \mathbb{R}^N \right) \) and we have the following estimate

\[
\sup_{B(x_0, r)} |u^\alpha| \leq 2 \max \left\{ L_0; \left( \frac{2(n-1)}{(n-2)} \left[ 4 + \frac{16c^2n^4k^4}{v^2} \right] \right)^{n/2} 2^{n+2} \sum_{\beta=1}^{N} \int_{B(x_0, R)} |u^{\beta}|^2 \bigg/ \bigg( \frac{2(n-1)}{(n-2)} \left[ 4 + \frac{16c^2n^4k^4}{v^2} \right] \right)^{n/2} 2^{n+2} \sum_{\beta=1}^{N} \int_{B(x_0, R)} |u^{\beta}|^2 \right\} \tag{2.3}
\]

for every \( \alpha = 1, \ldots, N \) and for every \( r, R \) with \( 0 < r < R \) and \( B(x_0, R) \subset \Omega \), where \( c \) is the constant involved in assumption (A_3), \( v \) is given in (A_2) and \( L_0 \) appears in (A_3).

**Remark 2.4.** The present local \( L^{\infty} \)–regularity result improves on [22] since assumption (A_3) allows off diagonal coefficients to have a larger support than in [22].

**Remark 2.5.** “Butterfly” support (A_3) has been used in [7] when proving the existence of at least one globally bounded solution to a (possibly) degenerate problem with zero boundary value problem. In the present work we prove local boundedness of every solution to a non degenerate system regardless of boundary values.

### 3 Proof of the result

The proof of Theorem 2.3 will be performed in several steps

#### STEP 1. Caccioppoli inequality

**Lemma 3.1.** (Caccioppoli inequality on superlevel sets) Let \( u \in W^{1,2} \left( \Omega, \mathbb{R}^N \right) \) be a weak solution of system (2.1) under assumptions (A_0), (A_1), (A_2), (A_3). For \( 0 < s < t \), let \( B(x_0, s) \) and \( B(x_0, t) \) be concentric open balls centered at \( x_0 \) with radii \( s \) and \( t \) respectively. Assume that \( B(x_0, t) \subset \Omega \) and \( L \geq L_0 \). Then

\[
\sum_{\alpha=1}^{N} \int_{\{ |u^\alpha| > L \} \cap B(x_0, s)} |D u^\alpha|^2 \, dx \leq \frac{16c^2n^4k^4}{v^2} \sum_{\alpha=1}^{N} \int_{\{ |u^\alpha| > L \} \cap B(x_0, t)} \left( \frac{|u^\alpha| - L}{t-s} \right)^2 \, dx, \tag{3.1}
\]

where \( c \) is the constant involved in assumption (A_1), \( v \) is given in (A_2) and \( L_0 \) appears in (A_3).

**Proof of Lemma 3.1** Let \( u \in W^{1,2} \left( \Omega, \mathbb{R}^N \right) \) be a weak solution of system (2.1). Let \( \eta : \mathbb{R}^n \rightarrow \mathbb{R} \) be the standard cut-off function such that \( 0 \leq \eta \leq 1 \), \( \eta \in C^0_0(B(x_0, t)) \), with \( B(x_0, t) \subset \Omega \) and \( \eta = 1 \) in \( B(x_0, s) \). Moreover, \( |D\eta| \leq 2/(t-s) \) in \( \mathbb{R}^n \). For every level \( L \geq L_0 \), consider

\[
T_L(s) = \begin{cases} 
-L & \text{if } s < -L \\
 s & \text{if } -L \leq s \leq L \\
 L & \text{if } s > L
\end{cases}
\]
and

\[ G_L(s) = s - T_L(s). \]

We define \( \varphi : \mathbb{R}^n \to \mathbb{R}^N \) with \( \varphi = (\varphi^1, \ldots, \varphi^N) \), where

\[ \varphi^a := \eta^2 G_L(u^a), \quad \text{for all } a \in \{1, \ldots, N\}. \]

Then

\[ D_i \varphi^a = \eta^2 1_{\{|u^| > L\}} D_i u^a + 2\eta (D_i \eta) 1_{\{|u^| > L\}} G_L(u^a) \quad \text{for all } i \in \{1, \ldots, n\} \text{ and } a \in \{1, \ldots, N\}. \]

Using this test function in the weak formulation (2.2) of system (2.1), we have

\[
0 = \int_{\Omega} \sum_{\alpha, \beta = 1}^{N} \sum_{i, j = 1}^{n} a_{i,j}^{\alpha, \beta} D_j \varphi^a D_i \varphi^a \, dx = \int_{\Omega} \sum_{\alpha, \beta = 1}^{N} \sum_{i, j = 1}^{n} a_{i,j}^{\alpha, \beta} D_j \varphi^a D_i u^a \, dx + \int_{\Omega} \sum_{\alpha, \beta = 1}^{N} \sum_{i, j = 1}^{n} a_{i,j}^{\alpha, \beta} D_j \eta 2\eta (D_i \eta) 1_{\{|u^| > L\}} G_L(u^a) \, dx.
\]

Now, assumption \((A_3)\) guarantees that

\[ a_{i,j}^{\alpha, \beta}(x, u(x)) = a_{i,j}^{\alpha, \beta}(x, u(x)) 1_{\{|u^| > L\}}(x) 1_{\{|u^| > L\}}(x) \quad (3.2) \]

when \( \beta \neq \alpha \) and \( L \geq L_0 \). It is worthwhile to note that (3.2) holds true when \( \alpha = \beta \) as well; then

\[
\int_{\Omega} \sum_{\alpha, \beta = 1}^{N} \sum_{i, j = 1}^{n} a_{i,j}^{\alpha, \beta} 1_{\{|u^| > L\}} D_j \varphi^a D_i u^a \, dx = -\int_{\Omega} \sum_{\alpha, \beta = 1}^{N} \sum_{i, j = 1}^{n} a_{i,j}^{\alpha, \beta} 1_{\{|u^| > L\}} D_j u^\beta 2\eta (D_i \eta) 1_{\{|u^| > L\}} G_L(u^a) \, dx. \quad (3.3)
\]

Now we can use ellipticity assumption \((A_2)\) with \( \xi^a = 1_{\{|u^| > L\}} D_i u^a \) and we get

\[
\int_{\Omega} \eta^2 \sum_{a = 1}^{N} 1_{\{|u^| > L\}} |Du^a|^2 \, dx \leq \int_{\Omega} \sum_{a, \beta = 1}^{N} \sum_{i, j = 1}^{n} a_{i,j}^{\alpha, \beta} 1_{\{|u^| > L\}} D_j u^\beta \eta^2 1_{\{|u^| > L\}} D_i u^a \, dx. \quad (3.4)
\]

Moreover

\[ |G_L(u^a)| = |u^a| - L \text{ where } |u^a| > L \quad (3.5) \]

and

\[
\int_{\Omega} \sum_{\alpha, \beta = 1}^{N} \sum_{i, j = 1}^{n} a_{i,j}^{\alpha, \beta} 1_{\{|u^| > L\}} D_j u^\beta 2\eta (D_i \eta) 1_{\{|u^| > L\}} G_L(u^a) \, dx \leq
\]

\[
\int_{\Omega} c \sum_{\beta = 1}^{N} \sum_{j = 1}^{n} 1_{\{|u^| > L\}} |D_j u^\beta| \sum_{\alpha = 1}^{N} \sum_{i = 1}^{n} 2\eta |D_i \eta| 1_{\{|u^| > L\}} |G_L(u^a)| \, dx \leq
\]

\[
\int_{\Omega} c \sum_{\beta = 1}^{N} n 1_{\{|u^| > L\}} |D_j u^\beta| \sum_{\alpha = 1}^{N} n 2\eta |D_i \eta| 1_{\{|u^| > L\}} |G_L(u^a)| \, dx \leq
\]

\[
\int_{\Omega} c n^2 \eta^2 \left( \sum_{\beta = 1}^{N} 1_{\{|u^| > L\}} |Du^\beta| \right)^2 \, dx + \int_{\Omega} \frac{c n^2 \eta^2}{e} |D\eta|^2 \left( \sum_{a = 1}^{N} 1_{\{|u^| > L\}} |G_L(u^a)| \right)^2 \, dx \leq
\]

\[
\int_{\Omega} c n^2 \eta^2 \sum_{\beta = 1}^{N} 1_{\{|u^| > L\}} |Du^\beta|^2 \, dx + \int_{\Omega} \frac{c n^2 \eta^2}{e} |D\eta|^2 \sum_{a = 1}^{N} 1_{\{|u^| > L\}} |G_L(u^a)|^2 \, dx, \quad (3.6)
\]
where we used the inequality $2ab \leq ca^2 + b^2/e$, provided $e > 0$. Merging (3.5), (3.4) and (3.6) into (3.3) we get
\[
v \int_{\Omega} \eta^2 \sum_{a=1}^{N} 1_{\{|u^a| > L\}} |Du^a|^2 \, dx \leq 
\int_{\Omega} cn^2 N^2 \eta^2 \sum_{\beta=1}^{N} 1_{\{|u^\beta| > \Omega\}} |Du^\beta| + \int_{\Omega} \frac{cn^2 N^2}{e} |D\eta|^2 \sum_{a=1}^{N} 1_{\{|u^a| > L\}} (|u^a| - L)^2 \, dx.
\]
We choose $e = v/(2cn^2 N^2)$ and we have
\[
\frac{v}{2} \int_{\Omega} \eta^2 \sum_{a=1}^{N} 1_{\{|u^a| > L\}} |Du^a|^2 \, dx \leq \int_{\Omega} \frac{2cn^2 N^2}{v} |D\eta|^2 \sum_{a=1}^{N} 1_{\{|u^a| > L\}} (|u^a| - L)^2 \, dx.
\]
Using the properties of the cut off function $\eta$ we deduce
\[
\sum_{a=1}^{N} \int_{\Omega} |Du^a|^2 \, dx \leq \frac{16cn^4 N^4}{v^2} \sum_{a=1}^{N} \int_{\Omega} (|u^a| - L)^2 \, dx.
\]  
Note that
\[
|D_\alpha u^a| = |Du^a|; 
\]
this ends the proof of Lemma 3.1.

\[\text{□}\]

**STEP 2. Sup estimate for general vectorial functions**

In the next Lemma we state and prove a general result that holds true for some general vectorial function $v \in W^{1,p}(\Omega, \mathbb{R}^N)$. Eventually, we will use such a result with $v = (|u^1|, \ldots, |u^N|)$ and $p = 2$.

**Lemma 3.2.** Assume that $\Omega$ is a bounded open subset of $\mathbb{R}^n$ and $v = (v^1, \ldots, v^N) \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $1 < p < n$.

We require the existence of constants $c_1 > 0$ and $L_0 \geq 0$ such that
\[
\sum_{a=1}^{N} \int_{\{v^a > L\} \cap B(x_0, s)} |Dv^a|^p \, dx \leq c_1 \sum_{a=1}^{N} \int_{\{v^a > L\} \cap B(x_0, t)} \left( \frac{v^a - L}{t - s} \right)^p \, dx, \tag{3.8}
\]
for every $s, t, L$, where $0 < s < t$, $B(x_0, t) \subset \Omega$ and $L \geq L_0$. Then,
\[
\sup_{B(x_0, t)} v^a \leq 2 \max \left\{ L_0; \left( \frac{[(n-1)p]^{R^p}}{[\alpha - p]^p} \frac{2^{n/p} + c_1}{(R - p)^n} \sum_{\beta=1}^{N} \int_{B(x_0, R)} \max\{v^\beta, 0\} \right)^{1/p} \right\}, \tag{3.9}
\]
for every $a = 1, \ldots, N$ and for every $r, R$ with $0 < r < R$ and $B(x_0, R) \subset \Omega$.

**Proof of Lemma 3.2** Let us consider balls $B(x_0, r_1)$ and $B(x_0, r_2)$ with $0 < r_1 < r_2$ and $B(x_0, r_2) \subset \Omega$. Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be the standard cut-off function such that $0 \leq \eta \leq 1$, $\eta \in C^1_0(B(x_0, (r_1 + r_2)/2))$, with $\eta = 1$ in $B(x_0, r_1)$. Moreover, $|D\eta| \leq 4/(r_2 - r_1)$ in $\mathbb{R}^n$. Let us set
\[
A_{r_2}^a = \{ x \in B(x_0, r) : v^a > L \}.
\]
Then, using Hölder inequality, Sobolev embedding and the properties of the cut-off function,

\[
\int_{A_{L,r_1}^a} (v^a - L)^p \leq \left( \int_{A_{L,r_1}^a} (v^a - L)^p \right)^{p/p'} |A_{L,r_1}^a|^{1-(p/p')} = \\
\left( \int_{A_{L,r_1}^a} [\eta(v^a - L)]^p \right)^{p/p'} |A_{L,r_1}^a|^{1-(p/p')} = \left( \int_{A_{L,r_1}^a} [\eta(\max\{v^a - L; 0\})]^p \right)^{p/p'} |A_{L,r_1}^a|^{1-(p/p')} \leq \\
c_2 \int_{A_{L,r_1}^a} |D[\eta(\max\{v^a - L; 0\})]|^p |A_{L,r_1}^a|^{1-(p/p')} = \\
c_2 2^p \left( \int_{A_{L,r_1}^a} |(D\eta)(v^a - L)|^p + \int_{A_{L,r_1}^a} |\eta Dv^a|^p \right) |A_{L,r_1}^a|^{1-(p/p')} \leq \\
c_2 2^p \left( \int_{A_{L,r_1}^a} \left( \frac{v^a - L}{r_2 - r_1} \right)^p + \int_{A_{L,r_1}^a} |Dv^a|^p \right) |A_{L,r_1}^a|^{1-(p/p')} \leq \\
(3.10)
\]

where \( c_2 = [(n-1)p/(n-p)]^p \). Now we sum upon \( a \) from 1 to \( N \) obtaining

\[
\sum_{a=1}^N \int_{A_{L,r_1}^a} (v^a - L)^p \leq \\
c_2 2^p \sum_{a=1}^N \left( \int_{A_{L,r_1}^a} \left( \frac{v^a - L}{r_2 - r_1} \right)^p + \int_{A_{L,r_1}^a} |Dv^a|^p \right) |A_{L,r_1}^a|^{1-(p/p')} \leq \\
c_2 2^p \sum_{a=1}^N \left( \int_{A_{L,r_1}^a} \left( \frac{v^a - L}{r_2 - r_1} \right)^p + \int_{A_{L,r_1}^a} |Dv^a|^p \right) \left( \sum_{\beta=1}^N |A_{L,r_1}^\beta| \right)^{1-(p/p')} \leq \\
(3.11)
\]

In order to control \( \sum_{a=1}^N \int |Dv^a|^p \) we use our assumption (3.8) with \( s = (r_1 + r_2)/2 \) and \( t = r_2 \): we get

\[
\sum_{a=1}^N \int_{A_{L,r_1}^a} (v^a - L)^p \leq \\
c_2 2^p \left( \int_{A_{L,r_1}^a} \left( \frac{v^a - L}{r_2 - r_1} \right)^p + c_1 2^p \sum_{a=1}^N \int_{A_{L,r_1}^a} \left( \frac{v^a - L}{r_2 - r_1} \right)^p \right) \left( \sum_{\beta=1}^N |A_{L,r_1}^\beta| \right)^{1-(p/p')} \leq 
\]
Inserting (3.16) into (3.15) we get

$$c_2 2^p [q^p + c_1 2^p] \left( \sum_{\alpha=1}^{\beta} \int_{A^\beta_{L,r_2}} (v^\alpha - L)^p \right) \left( \sum_{\beta=1}^{\beta} \int_{A^\beta_{L,r_1}} (v^\beta - L)^p \right)^{1-p/p'} .$$

(3.12)

We want to estimate \( |A^\beta_{L,r_1}| \) by means of \( \int (v^\beta - L)^p \). We are able to do that for a lower level \( \tilde{L} \). Indeed, for \( L > \tilde{L} \geq L_0 \), we have

\[
|A^\beta_{L,r_1}| = \frac{1}{(L - \tilde{L})^p} \int (L - \tilde{L})^p \leq \frac{1}{(L - \tilde{L})^p} \int (v^\beta - L)^p \leq \frac{1}{(L - \tilde{L})^p} \int (v^\beta - L)^p .
\]

(3.13)

Note that

\[ 1 - (p/p') = p/n. \]

Inserting (3.14) and (3.13) into (3.12) we deduce

\[
\sum_{\alpha=1}^{\beta} \int_{A^\alpha_{L,r_2}} (v^\alpha - L)^p \leq \frac{c_2 2^p [q^p + c_1 2^p]}{(r_2 - r_1)^p (L - \tilde{L})^p} \left( \sum_{\beta=1}^{\beta} \int_{A^\beta_{L,r_2}} (v^\beta - L)^p \right)^{p/n} .
\]

(3.15)

We want to estimate \( \int (v^\alpha - L)^p \) with \( \int (v^\alpha - \tilde{L})^p \). Since \( L > \tilde{L} \), we have

\[
\int_{A^\alpha_{L,r_2}} (v^\alpha - L)^p \leq \int_{A^\alpha_{L,r_2}} (v^\alpha - \tilde{L})^p \leq \int_{A^\alpha_{\tilde{L},r_2}} (v^\alpha - \tilde{L})^p .
\]

(3.16)

Inserting (3.16) into (3.15) we get

\[
\sum_{\alpha=1}^{\beta} \int_{A^\alpha_{L,r_2}} (v^\alpha - L)^p \leq \frac{c_2 2^p [q^p + c_1 2^p]}{(r_2 - r_1)^p (L - \tilde{L})^p} \left( \sum_{\beta=1}^{\beta} \int_{A^\beta_{\tilde{L},r_2}} (v^\beta - \tilde{L})^p \right)^{1+p/n} .
\]

(3.17)

Now we fix \( 0 < r < R \), with \( B(x_0, R) \subset \Omega \), and we take the following sequence of radii

\[ \rho_i = r + \frac{R - r}{2^i} \]

(3.18)

for \( i = 0, 1, 2, \ldots \); then \( \rho_0 = R \) and \( \rho_i - \rho_{i+1} = (R - r)/2^{i+1} > 0 \), so \( \rho_i \) strictly decreases and \( r < \rho_i \leq R \).

Let us fix a level \( d \geq L_0 \) and we take the following sequence of levels

\[ k_i = 2d \left( 1 - \frac{1}{2^{i+1}} \right) \]

(3.19)

for \( i = 0, 1, 2, \ldots \); then \( k_0 = d \) and \( k_{i-1} - k_i = d/2^{i+1} > 0 \), so \( k_i \) strictly increases and \( L_0 \leq d \leq k_i < 2d \). We can use (3.17) with levels \( L = k_{i+1} > k_i = \tilde{L} \) and radii \( r_1 = \rho_{i+1} < \rho_i = r_2 \):

\[
\sum_{\alpha=1}^{\beta} \int_{A^\alpha_{k_{i+1},r_1}} (v^\alpha - k_{i+1})^p \leq \frac{c_2 2^p [q^p + c_1 2^p]}{(R - r)/2^{i+1}} \left( d/2^{i+1} \right)^p \left( \sum_{\beta=1}^{\beta} \int_{A^\beta_{k_{i+1},r_1}} (v^\beta - k_i)^p \right)^{1+p/n} .
\]
\[
\frac{c_2 4^p [2^p + c_1] 2^{(i+1)p/n} d^{pp/n}}{(R - r)^p d^{pp/n}} \left( \sum_{B = 1}^{N} \int_{A_{v, l_i}^B} (v^\beta - k_i)^P \right)^{1+(p/n)}. \tag{3.20}
\]

Let us set
\[
J_i := \sum_{a = 1}^{N} \int_{A_{k_0, \rho_0}^a} (v^\alpha - k_i)^P; \tag{3.21}
\]
then (3.20) can be written as follows
\[
J_{i+1} \leq \frac{c_2 4^p [2^p + c_1] 2^{(i+1)p/n} d^{pp/n}}{(R - r)^p d^{pp/n}} \left( \sum_{B = 1}^{N} \int_{A_{v, l_i}^B} (v^\beta - k_i)^P \right)^{i+(p/n)}. \tag{3.22}
\]
We would like to get
\[
\lim_{i \to \infty} J_i = 0; \tag{3.23}
\]
this is true provided
\[
J_0 \leq \frac{c_2 4^p [2^p + c_1] 2^{(i+1)p/n} d^{pp/n}}{(R - r)^p d^{pp/n}} \left( \sum_{B = 1}^{N} \int_{A_{v, l_i}^B} (v^\beta - k_i)^P \right)^{1+(p/n)}; \tag{3.24}
\]
as Lemma 7.1 says at page 220 in [13]. Let us try to check (3.24): we first rewrite it as follows
\[
\sum_{a = 1}^{N} \int_{A_{k, \rho}^a} (v^\alpha - k_i)^P \leq \frac{c_2 4^p [2^p + c_1] 2^{(i+1)p/n} d^{pp/n}}{(R - r)^p d^{pp/n}} \left( \sum_{B = 1}^{N} \int_{A_{v, l_i}^B} (v^\beta - k_i)^P \right)^{nn/(pp)}; \tag{3.25}
\]
we keep in mind that \(k_0 = d\) and \(\rho_0 = R\); so, (3.25) can be written in the following way
\[
\left( \frac{c_2 4^p [2^p + c_1] 2^{(i+1)p/n} d^{pp/n}}{(R - r)^p d^{pp/n}} \right)^{n/(p)} (2^{(1+p/n)n})^{nn/(pp)} \sum_{a = 1}^{N} \int_{A_{k, \rho}^a} (v^\alpha - d)^P \leq d^P. \tag{3.26}
\]
Note that \(d \geq L_0 \geq 0\) so, when \(v^\alpha > d\), we have \(v^\alpha - d < v^\alpha = \max\{v^\alpha; 0\}\); then
\[
\int_{A_{k, \rho}^a} (v^\alpha - d)^P \leq \int_{B(x_0, R)} (\max\{v^\alpha; 0\})^P \leq \int_{B(x_0, R)} (\max\{v^\alpha; 0\})^P. \tag{3.27}
\]
Using (3.27), we get the following sufficient condition when checking (3.26):
\[
\left( \frac{c_2 4^p [2^p + c_1] 2^{(1+1)p/n}}{(R - r)^n} \right)^{n/(p)} (2^{1+p/n})^{nn/p} \sum_{a = 1}^{N} \int_{A_{k, \rho}^a} (v^\alpha - d)^P \leq d^P. \tag{3.28}
\]
Then, we fix \(d\) verifying (3.28) and \(L_0 \leq d\); then (3.24) holds true and (3.23) holds true. We keep in mind that \(r < \rho_i\) and \(k_i < 2d\), so we can use (3.16) with \(r_2 = r < \rho, L = 2d\) and \(L = \bar{L}\);
\[
\int_{\{v^\alpha > 2d\} \cap B(x_0, r)} (v^\alpha - 2d)^P \leq \int_{\{v^\alpha > k_i\} \cap B(x_0, r)} (v^\alpha - k_i)^P \leq \int_{\{v^\alpha > k_i\} \cap B(x_0, \rho_i)} (v^\alpha - k_i)^P; \tag{3.29}
\]
so that
\[
0 \leq \sum_{a = 1}^{N} \int_{\{v^\alpha > 2d\} \cap B(x_0, r)} (v^\alpha - 2d)^P \leq \sum_{a = 1}^{N} \int_{\{v^\alpha > k_i\} \cap B(x_0, \rho_i)} (v^\alpha - k_i)^P = J_i; \tag{3.30}
\]
since (3.23) holds true, we have \(\lim_i J_i = 0\), so
\[
\sum_{a = 1}^{N} \int_{\{v^\alpha > 2d\} \cap B(x_0, r)} (v^\alpha - 2d)^P = 0; \tag{3.31}
\]
this means that $|\{v^a > 2d\} \cap B(x_0, r)| = 0$, so that
\[ v^a \leq 2d \quad \text{almost everywhere in } B(x_0, r). \tag{3.32} \]

Level $d$ can be selected as follows
\[
d = \max \left\{ L_0; \left(\frac{c_2 4^p [2^p + c_1] 2^{(1+(p/n)p)n/p}}{(R - r)^n} \sum_{\beta = 1}^N \int_{B(x_0, R)} \max\{v^\beta, 0\}^p\right)^{1/p} \right\}
\]
and claim (3.9) is proved after noting that $(4^p 2^{(1+(p/n)p)n/p} 2^{(1+(p/n)n)n/p} = 2^{4n+p+nn/p}$ and $c_2 = \frac{(n - 1)p/(n - p))^p}{c_1}$. This ends the proof of Lemma 3.2.

**STEP 3. Proof of Theorem 2.3**

Caccioppoli inequality proved in Lemma 3.1 allows us to use Lemma 3.2 with $v^a = |u^a|$, $p = 2$ and $c_1 = \frac{16c^a n^N}{\nu^a}$; this gives estimate (2.3) and the proof of Theorem 2.3 ends here.

Remark 3.3. *In the present work we used a test function $\varphi$ that modifies every component of $u$; this gives the summation on the index $a$ in Caccioppoli’s inequality (3.1). In [4], [1] and [3] only one component of $u$ is modified and a Caccioppoli’s inequality without the summation on $a$ is proved.

Moreover, the Caccioppoli’s inequality proved in [4] and [1] has an exponent $p^*$ on the right–hand side in contrast with the same $p$ that we have on both sides of (3.8), see also [30], [9], [2], [5], [14], [15], [16].

Remark 3.4. *In [22] it is used $\max\{u^a - L; 0\}$ in the test function $\varphi$, see Figure 2 (left), while in the present paper we use $G_L(u^a)$ instead, see Figure 2 (right). Such a function $G_L(u^a)$ allows us to deal with support larger than in [22] for off diagonal coefficients.*

\[\text{Fig. 2: (left) graph of } u^a \rightarrow \max\{u^a - L; 0\}; \text{ (right) graph of } u^a \rightarrow G_L(u^a).\]

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