The Fourier $U(2)_F$ group on square and round pixellated arrays

Kurt Bernardo Wolf$^1$ and Natig M Atakishiyev$^2$

$^1$ Instituto de Ciencias Físicas
Universidad Nacional Autónoma de México
$^2$ Instituto de Matemáticas
Universidad Nacional Autónoma de México
Av. Universidad s/n, Cuernavaca, Morelos 62251, México
E-mail: bwolf@fis.unam.mx

Abstract.

The Fourier group $U(2)_F$ is the maximal compact group of the group $Sp(4,\mathbb{R})$ of linear canonical transformations on $\mathbb{R}^2$ configuration space. When we finitely quantize the two-dimensional harmonic oscillator we turn this space into an array of $N^2$ points arranged along either Cartesian or polar coordinates, as a square or a circle, that model pixellated screens. We investigate the action of $U(2)_F$ on these two $N^2$-dimensional spaces, which become a homogeneous space for this group action.

1. Introduction: The Fourier group

Two-dimensional discrete Hamiltonian systems include finite pixellated screens, whose pixels may follow either cartesian or polar coordinates [1, 2, 3, 4]. We shall develop this model out of paraxial geometric optics with attention to the transformations that belong to its maximal compact subgroup — the unitary Fourier group. In this classical model, there are two dimensions of position $(q_x, q_y) \in \mathbb{R}^2$ and two of momentum $(p_x, p_y) \in \mathbb{R}^2$, out of which we can build the ten quadratic functions $q_i q_{i'}, q_i p_{i'}, p_i p_{i'}$, with $i, i' \in \{x, y\}$, that generate the real symplectic Lie algebra $sp(4, \mathbb{R})$ under Poisson brackets. The following four linear combinations generate the maximal compact subalgebra $u(2)_F = u(1)_F \oplus su(2)_F$, called the Fourier algebra:

\begin{align*}
\text{isotropic FT} & \quad F_0 := \frac{1}{4}(p_x^2 + p_y^2 + q_x^2 + q_y^2), \\
\text{anisotropic FT} & \quad F_1 := \frac{1}{4}(p_x^2 - p_y^2 + q_x^2 - q_y^2), \\
\text{gyration} & \quad F_2 := \frac{1}{2}(p_x p_y + q_x q_y), \\
\text{rotation} & \quad F_3 := \frac{1}{2}(q_x p_y - q_y p_x).
\end{align*}

These include the central generator $F_0$ of fractional Fourier transformations (FTs) [5, 6]. Figure 1 shows the transformations generated by $su(2)_F$ on the sphere $F_1^2 + F_2^2 + F_3^2 = F_0^2$.

In Sect. 2 we recall the finite quantization of the two-dimensional harmonic oscillator along Cartesian coordinates [1, 7], and in Sect. 3 along polar coordinates [2, 8]. Among the one-parameter subgroups of Fourier group, isotropic and anisotropic unitary Fourier transforms are domestic on Cartesian coordinates; but as shown in Sect. 4, unitary rotations can be imported.
2. Two-dimensional Cartesian \(\text{so}(4)\) algebra

The classical Fourier group \(\text{U}(2)_K\), generated by (1)–(4) will be corresponded with the ‘Fourier-Kravchuk’ group \(\text{U}(2)_K\) acting on the space of finite ‘sensor’ arrays of the finite oscillator model developed in Refs. [9, 10] in two dimensions [1]. The classical observables are corresponded with the Cartesian finite-quantum operators in \(\text{su}(2)_x \oplus \text{su}(2)_y\) as follows:

\[
\begin{align*}
\text{positions:} & \quad q_x &\mapsto Q^0_x = J_{x1}, & q_y &\mapsto Q^0_y = J_{y1}, \\
\text{momenta:} & \quad p_x &\mapsto P^0_x = -J_{x2}, & p_y &\mapsto P^0_y = -J_{y2}, \\
\text{pseudo-energies:} & \quad K^0_x := i[Q^0_x, P^0_x] = J_{x3}, & K^0_y := i[Q^0_y, P^0_y] = J_{y3},
\end{align*}
\]

where the classical oscillator energy in the \(x\)- and \(y\)-directions is related to the pseudo-energies by \(h_i \mapsto H^0_i := K^0_i + (j + \frac{1}{2})I\), with \(j = j_x = j_y\) being the representation indices of \(\text{su}(2)\) for a square pixelated screen with \(N = 2j + 1\) pixels on the side.

When we identify the \(\text{su}(2)\) commutators \([J_1, J_2] = iJ_3\) (and cyclically), the commutators satisfied by the Cartesian position and momentum operators are the two Hamilton equations, \([K^0_i, Q^0_{i'}] = -i\delta_{i,i'} P^0_i\) and \([K^0_i, P^0_{i'}] = i\delta_{i,i'} Q^0_i\) for \(i, i' \in \{x, y\}\), while \([Q^0_i, P^0_j] = -i\delta_{i,j} K^0_j\) are the nonstandard commutators of the model. The spectra of all these operators is naturally \(\{-j, -j+1, \ldots, j\} \equiv \{j\}_j\), with \(j\) integer or half-integer. If \(\kappa_i\) are the eigenvalues of \(K^0_i\), we define also the mode numbers \(n_i := \kappa_i + j\), \(n_i|_{0}^{2j}\) and the energies \(n_i + \frac{1}{2}\). The structure of the four-dimensional orthogonal Lie algebra \(\text{so}(4) = \text{su}(2)_x \oplus \text{su}(2)_y\) associated with the Cartesian-pixellated screens can be summarized by the diagram
Two-dimensional arrays of ‘sensor’ points. a) Cartesian arrangement \((q_x, q_y)\). b) Polar arrangement \((\rho, \phi_k)\) with \(\psi_{\rho} = 0\).

\[
\begin{array}{ccc}
K_x^\circ & -P_x^\circ & Q_x^\circ \\
Q_x^\circ & \phantom{0} & \phantom{0} \\
0 & \phantom{0} & \phantom{0}
\end{array}
\oplus
\begin{array}{ccc}
K_y^\circ & -P_y^\circ & Q_y^\circ \\
Q_y^\circ & \phantom{0} & \phantom{0} \\
0 & \phantom{0} & \phantom{0}
\end{array}
= \begin{array}{ccc}
\hline
K_x^\circ + K_y^\circ & -P_x^\circ - P_y^\circ & Q_x^\circ - Q_y^\circ \\
Q_x^\circ + Q_y^\circ & P_x^\circ - P_y^\circ & \phantom{0} \\
K_x^\circ - K_y^\circ & \phantom{0} & \phantom{0}
\hline
\end{array}
\tag{8}
\]

where in the \(i-i'\) box we place generators \(J_{i,i'} = -J_{i',i}\) with the \(\mathfrak{so}(4)\) commutation relations

\[
[J_{i,i'}, J_{k,k'}] = i(\delta_{i,k} J_{i,k'} + \delta_{i,k'} J_{i,k} + \delta_{k,i} J_{k,i'} + \delta_{k,i'} J_{k,i}),
\tag{9}
\]

where \(i, i', k, k' \in \{1, 2, 3, 4\}\).

The generators of isotropic and anisotropic Fourier transforms are \textit{domestic} (within the algebra): \(K_x^\circ + K_y^\circ\) and \(K_x^\circ - K_y^\circ\) respectively; the generators of rotations and gyrations are not manifestly present in the pattern (8), but will be \textit{imported} below.

Two bases for the square \((j = j_x = j_y)\) unitary irreducible representations of \(\mathfrak{so}(4)\) can be thus chosen as eigenstates of position and of mode (pseudo-energy), as

\[
\begin{align*}
Q_x^\circ |q_x, q_y\rangle_0 &= q_x |q_x, q_y\rangle_0, & Q_y^\circ |q_x, q_y\rangle_0 &= q_y |q_x, q_y\rangle_0, \\
K_y^\circ |n_x, n_y\rangle_0 &= (n_x - j) |n_x, n_y\rangle_0, & K_x^\circ |n_x, n_y\rangle_0 &= (n_y - j) |n_x, n_y\rangle_0,
\end{align*}
\tag{10, 11}
\]

for \(q_x, q_y\) and \(n_x, n_y\). In Figure 2a we show the spectrum \((q_x, q_y)\) that determines the discrete positions of the rectangular array. The Cartesian finite oscillator wavefunctions on \(N = 2j+1\) points are then defined by their overlap,

\[
\begin{align*}
\Psi_{n_x,n_y}^\circ (q_x, q_y) &= \langle q_x, q_y | n_x, n_y \rangle_0 = \Psi_{n_x}^\circ (q_x) \Psi_{n_y}^\circ (q_y), \\
\Psi_n(q) &= d_{n-j,q}^j \left(\frac{1}{2}\pi\right) = \left(-1\right)^n \frac{\sqrt{\binom{2j}{n} \binom{2j}{j+q}}}{2^n} K_n(j+q; \frac{1}{2}, 2j),
\end{align*}
\tag{12, 13}
\]

where \(d_{m,n}^j(\theta)\) is the Wigner little-d function, \(\binom{s}{r}\) are the binomial coefficients, and \(K_n(\xi; \frac{1}{2}, 2j)\) are the \(N\) symmetric Kravchuk polynomials of degrees \(n\) in \(\xi\).

### 3. Two-dimensional polar \(\mathfrak{so}(4)\) algebra

Consider the same \(\mathfrak{so}(4)\) algebra as above, but now with its generators \(J_{i,i'}\) defined by

\[
\begin{array}{ccc}
K & -P_x^\circ & -P_y^\circ \\
Q_x^\circ & \phantom{0} & \phantom{0} \\
0 & \phantom{0} & \phantom{0}
\end{array}
\oplus
\begin{array}{ccc}
Q_y^\circ & Q_x^\circ & M \\
0 & \phantom{0} & \phantom{0}
\end{array}
\rightarrow
\begin{array}{ccc}
Q_x^\circ & Q_y^\circ & M \\
0 & \phantom{0} & \phantom{0}
\end{array}
\rightarrow
\begin{array}{ccc}
M & \phantom{0} & \phantom{0} \\
0 & \phantom{0} & \phantom{0}
\end{array}
\tag{14}
\]


showing the subalgebra chain \( \mathfrak{so}(4) \supset \mathfrak{so}(3)_R \supset \mathfrak{so}(2)_M \) [2]. The commutation relations (9) now imply the oscillator Hamilton equations in the form \([K, \mathcal{Q}^j] = iP^j_\rho\), and \([K, \mathcal{P}^j] = -i\mathcal{Q}^j\), together with the nonstandard commutators \([\mathcal{Q}^j_\rho, \mathcal{P}^j_\kappa] = iK\), again for \(i \in \{x, y\}\). The \(\mathfrak{so}(4)\) algebra also implies the further commutators \([M, \mathcal{Q}^j_\rho] = i\mathcal{Q}^j_y\), \([M, \mathcal{Q}^j_y] = -i\mathcal{Q}^j_x\), \([M, \mathcal{P}^j_x] = i\mathcal{P}^j_\rho\), and \([M, \mathcal{P}^j_y] = -i\mathcal{P}^j_\rho\), clearly indicating that \(M\) generates rotation between the \(x\)- and \(y\)-axes of both positions and momenta. And now there are nonstandard commutators \([\mathcal{Q}^j_\rho, \mathcal{Q}^j_\kappa] = iM = [P^j_\rho, P^j_\kappa]\), which indicate that the two \(\mathcal{Q}^j_\rho\)’s do not commute, and neither do the \(P^j_\rho\)’s, so they cannot be simultaneously diagonalized. On the other hand, \([\mathcal{Q}^j_\rho, P^j_{i'}] = 0\) for \(i \neq i'\). Finally, \([K, M] = 0\), meaning that the total mode commutes with the operator \(M\), that we interpret as an angular momentum generator.

In the subalgebra chain (14) we use the \(\mathfrak{so}(3)_R\) Casimir operator to identify a radius operator \(R\) (which does not belong to the algebra) by

\[
R(R+1) := (\mathcal{Q}^2_\rho) + (\mathcal{Q}^2_y) + M^2. \tag{15}
\]

The reason for doing so is that the branching rules of the square representation \(\mathfrak{so}(4) \supset \mathfrak{so}(3)\) determine that the eigenvalues of \(R(R+1)\) are \(\rho(\rho+1)\), with \(\rho^2\) [2]. Finally, the eigenvalues of \(M\) in each representation \(\rho\) of \(\mathfrak{so}(3)\) are \(\rho^2, \rho\)-. Eigenbases of mode (or pseudo-energy) and angular momentum, and of radius and angular momentum, are thus defined by

\[
K |n, m\rangle = (n - 2j) |n, m\rangle, \quad M |n, m\rangle = m |n, m\rangle, \tag{16}
\]
\[
R |\rho, m\rangle = \rho |\rho, m\rangle, \quad M^2 |\rho, m\rangle = \rho^2 |\rho, m\rangle. \tag{17}
\]

For \(N = 2j+1\), the ranges are \(\rho^2\) and \(m^2\), containing a total of \(N^2\) states that have the same number as the Cartesian bases (10)–(11). Finally, using the ordinary discrete Fourier transform at \(2\rho+1\) points, we define the eigenbasis of radius and angle by

\[
|\rho, \phi_k\rangle := \frac{1}{\sqrt{2\rho+1}} \sum_{\rho \in \rho} \exp(-i\rho \phi_k) |\rho, m^\rho\rangle, \tag{18}
\]

for \(\phi_k := 2\pi k/(2\rho+1) + \psi_\rho\), \(k^\rho\), where \(\psi_\rho\) are fixed but arbitrary phases. Note that we do not claim an ‘angle’ operator, but the transformation between the bases (17) and (18) is unitary. In Figure 2 we show the values \((\rho, \phi_k)\) that represents de discrete positions of the polar array.

As in the previous section, we search for the overlap between the two bases. A glance at the placement of the diagonal operators in the pattern (14) shows that this is a recoupling of angular momenta that defines the Clebsch-Gordan coefficients—times a phase which is due to our counting of states from the ground energy up, and because the chain involves omitting the \(i = 1\) generators \(J_{i', j}\) rather than the \(i = 1\) ones. We thus find the overlap of mode \(n = \kappa+j\) and angular momentum \(m\) for radius \(\rho\) as [2]

\[
\rho_{EA}(\rho, \kappa+j, m^\rho) = \varphi(j, \rho, \kappa, m) C^j, \quad (m+n)/2, \quad (m-n)/2, \quad \rho, \tag{19}
\]

where \(\varphi(j, \rho, \kappa, m) := (-1)^{j+\rho} \exp[i\pi/2 (\kappa + |m| - m)]\) is a phase that must be computed carefully. Finally, using (18), the wavefunctions of mode and angular momentum on the coordinates of radius and angle are

\[
\Psi_{n,m}(\rho, \phi_k) := \langle \rho, \phi_k | n-j, m \rangle_{MA} \tag{20}
\]
\[
= \frac{1}{\sqrt{2\rho+1}} \sum_{\rho \in \rho} \exp(i\rho \phi_k) \varphi(j, \rho, \kappa, m) C^j, \quad (m+n-j)/2, \quad (m-n+j)/2, \quad \rho.
\]

These functions are expressible in terms of the discrete dual Hahn polynomials [2]. In the polar basis, the generators of isotropic Fourier transforms and rotations are \textit{domestic}: \(K\) and \(M\) respectively; while anisotropic Fourier transforms and gyrations [11] will have to be \textit{imported}. 
4. Importation of U(2)ₚ on Cartesian screens

The two-dimensional Schrödinger quantum oscillator is separable in Cartesian and in polar coordinates, the transformation between Hermite-Gauss and Laguerre-Gauss states is given by a Wigner little-d function and a phase,

\[ |n, m\rangle^{\text{WG}}_{\text{osc}} = \sum_{n_x+n_y=n} e^{i\pi(n_x-n_y)/4} d_{m/2,n_x-n_y/2}^{n/2} (\frac{1}{2}\pi) |n_x, n_y\rangle^{\text{LG}}_{\text{osc}}, \]

for Cartesian modes \( n_x, n_y \), total mode \( n \), and angular momenta ranging over \( m \in \{-n, -n + 2, \ldots, n\} \). This is an \((n+1) \times (n+1)\) unitary transformation which is not within \( \text{SO}(4) \), but it can be imported in the same way that we imported the Fourier transform in (18). In this way we define the Cartesian basis of mode (\( \sim \text{energy} \)) and angular momentum,

\[ |n, m\rangle^{\text{LG}}_{\text{ELA}} := \sum_{n_x+n_y=n} e^{i\pi(n_x-n_y)/4} d_{m/2,n_x-n_y/2}^{n/2} (\frac{1}{2}\pi) |n_x, n_y\rangle^{\text{LG}}_{\text{ELA}}, \]

and the overlaps are the finite equivalents of the the Laguerre-Gauss wavefunctions of quantum mechanics, but over the Cartesian screen,

\[ A_n^\Omega(q_x, q_y) := \frac{\partial}{\partial \{q_x, q_y|n, m\}}^{\text{LG}}_{\text{ELA}} \]

\[ = \sum_{n_x+n_y=n} e^{i\pi(n_x-n_y)/4} d_{m/2,n_x-n_y/2}^{n/2} (\frac{1}{2}\pi) \Psi_n^\Omega (q_x, q_y) \]

Equation (22) is the key to build the Fourier group \( U(2)_p \) on the \( N \times N \) points/pixels of the Cartesian screen. We must be careful though, to note that the classical generators (1)–(4) are one-half of the oscillator energy and angular momentum operators, and that the \( U(1)_p \subset \text{SO}(4) \) transformation generated by \( K \) is not (but close to) the discrete Fourier transform; we have called it the Fourier-Kravchuk transform (FK) [9]. The one-parameter subgroups of this group correspond with the classical generators through

\[ \text{isotropic FK } \mathcal{K}(\omega) \leftrightarrow e^{-i\omega \hat{F}_3}, \text{ anisotropic FK } \mathcal{A}(\phi) \leftrightarrow e^{-i\phi \hat{F}_1}, \text{ rotation } \mathcal{R}(\theta) \leftrightarrow e^{-i\theta \hat{F}_3}, \]

and act on the Cartesian mode states \( |n_x, n_y\rangle^\Omega_{\beta} \) as

\[ \mathcal{K}(\omega) |n_x, n_y\rangle^{\Omega}_{\beta} = \exp[-2i\omega(n_x+n_y)] |n_x, n_y\rangle^{\Omega}_{\beta}, \]

\[ \mathcal{A}(\phi) |n_x, n_y\rangle^{\Omega}_{\beta} = \exp[-2i\phi(n_x-n_y)] |n_x, n_y\rangle^{\Omega}_{\beta}, \]

\[ \mathcal{G}(\psi) |n_x, n_y\rangle^{\Omega}_{\beta} = \sum_{n_x'+n_y'=n} e^{-i\pi(n'_x-n_x-n'_y+n_y)/2} \times d_{n_x-n_y/2,n'_x-n'_y/2}^{n/2} (2\psi) |n_x', n_y'\rangle^{\Omega}_{\beta}, \]

\[ \mathcal{R}(\theta) |n_x, n_y\rangle^{\Omega}_{\beta} = \sum_{n_x'+n_y'=n} d_{n_x-n_y/2,n'_x-n'_y/2}^{n/2} (2\theta) |n_x', n_y'\rangle^{\Omega}_{\beta}, \]

where the action of gyration [11] is obtained from that of rotation through \( \mathcal{G}(\psi) = \mathcal{A}(\frac{1}{2}\pi) \mathcal{R}(\psi) \mathcal{A}(\frac{1}{2}\pi)^{-1} \).

The action of \( U(2)_p \) on images (or signals, or wavefunctions) \( f \equiv \|f(q_x, q_y)\|^2 \) on the square screen,

\[ f(q_x, q_y) := \langle q_x, q_y|f \rangle = \sum_{n_x, n_y} \langle q_x, q_y|n_x, n_y\rangle^{\Omega}_{\beta} |n_x, n_y\rangle^\Omega_{\beta}, \]

5
the Fourier group on the Cartesian screen is
\[ K(q_x, q_y; q'_x, q'_y; \omega) = \sum_{q'_x, q'_y} K(q_x, q_y; q'_x, q'_y; \omega) f(q'_x, q'_y), \]
\[ A(\phi; q_x, q_y; q'_x, q'_y; \phi) = \sum_{q'_x, q'_y} A(q_x, q_y; q'_x, q'_y; \phi) f(q'_x, q'_y), \]
\[ G(\psi; q_x, q_y; q'_x, q'_y; \psi) = \sum_{q'_x, q'_y} G(q_x, q_y; q'_x, q'_y; \psi) f(q'_x, q'_y), \]
\[ R(\theta; q_x, q_y; q'_x, q'_y; \theta) = \sum_{q'_x, q'_y} R(q_x, q_y; q'_x, q'_y; \theta) f(q'_x, q'_y), \]
with summation kernels which are \( N^2 \times N^2 \) matrices that involve the finite two-dimensional harmonic oscillator wavefunctions \( \Psi_{n_x,n_y}^{\alpha}(q_x, q_y) \) in (12). These are [4]
\[ K(q_x, q_y; q'_x, q'_y; \omega) = \sum_{n_x,n_y} \Psi_{n_x,n_y}^{\alpha}(q_x, q_y) \exp[-2i\omega(n_x+n_y)] \Psi_{n_x,n_y}^{\alpha}(q'_x, q'_y), \]
\[ A(q_x, q_y; q'_x, q'_y; \phi) = \sum_{n_x,n_y} \Psi_{n_x,n_y}^{\alpha}(q_x, q_y) \exp[-2i\phi(n_x-n_y)] \Psi_{n_x,n_y}^{\alpha}(q'_x, q'_y), \]
\[ G(q_x, q_y; q'_x, q'_y; \psi) = \sum_{\mu,\mu'} \Psi_{n_x,n_y}^{\alpha}(q_x, q_y) e^{-i\mu(\mu-\mu')/4} d_{\mu,\mu'}^{n/2}(2\psi) \Psi_{n_x,n_y}^{\alpha}(q'_x, q'_y), \]
\[ R(q_x, q_y; q'_x, q'_y; \theta) = \sum_{\mu,\mu'} \Psi_{n_x,n_y}^{\alpha}(q_x, q_y) d_{\mu,\mu'}^{n/2}(2\theta) \Psi_{n_x,n_y}^{\alpha}(q'_x, q'_y), \]
where in (36) and (37) the sums over \( \mu = \frac{1}{2}(n_x - n_y) \) and \( \mu' = \frac{1}{2}(n'_x - n'_y) \) are bound by \( n_x + n_y = n = n'_x + n'_y \) and preserve the total mode \( n \). We note that rotations (28) and their kernels (37) are real transformations that involve states with the same value of the mode number \( n \). In Figure 3 we offer an example of this rotation.

The generic element of the Fourier group \( U(2)_F = U(1)_c \otimes SU(2)_c \) in Euler angles can be found from
\[ D(\omega; \phi, \theta, \psi) = K(\frac{1}{2}\omega) R(\frac{1}{2}\phi) G(\frac{1}{2}\theta) R(\frac{1}{2}\psi). \]

Writing \( \Omega := (\phi, \theta, \psi) \), the generic action of the Fourier group on the Cartesian screen is
\[ D(\omega, \Omega; q_x, q_y; q'_x, q'_y) = \sum_{q'_x, q'_y} D(\omega; q_x, q_y; q'_x, q'_y; \omega, \Omega) f(q'_x, q'_y), \]
where the kernel is [4]
\[ D(\omega; q_x, q_y; q'_x, q'_y; \omega, \Omega) = \sum_{n_x,n_y} \Psi_{n_x,n_y}^{\alpha}(q_x, q_y) e^{-i(n-2j)\omega} D_{(n_x-n_y)/2,(n'_x-n'_y)/2}(\Omega) \Psi_{n'_x,n'_y}^{\alpha}(q'_x, q'_y), \]
with \(n_x + n_y = n = n'_x + n'_y\) and \(n_i^j\), and using the Wigner big-\(D\) functions \(D^j_{m,m'}(\phi, \theta, \psi)\). These \(N^2 \times N^2\) matrices are the unitary representations of the Fourier \(U(2)\) group acting on images over the Cartesian pixellated screen.

5. Action of \(U(2)\) on the polar screen

In Section 3 we identified in \(\mathfrak{so}(4)\) the subalgebra chain that led to bases related to the polar pixellation of a screen. In these Proceedings we shall not repeat what has been done for the pixellation in polar coordinates, but only indicate how the results of the previous section are applied to find them. In (22) we built states of mode (energy) and angular momentum, \(|n, m\rangle_\mathcal{E}\), on the Cartesian screen, while in (16) we defined similar states \(|n, m\rangle_{\mathcal{E}A}\) on the polar screen. To relate images on both screens we identify both sets of states: \(|n, m\rangle_{\mathcal{E}A} \equiv |n, m\rangle_\mathcal{E}\). In this way, generic images \(f\) on the polar screen \(f_\circ(q_x, q_y)\) can be related to images on the Cartesian screen \(f_\circ(q_x, q_y)\) by means of a unitary transformation [3],

\[
f_\circ(\rho, \phi_k) = \sum_{q_x, q_y} \langle \rho, \phi_k | f \rangle \sum_{q_x, q_y} U(\rho, \phi_k; q_x, q_y) f_\circ(q_x, q_y),
\]

where the \(N^2 \times N^2\) transform matrix is

\[
U(\rho, \phi_k; q_x, q_y) := \sum_{n,m} \langle \rho, \phi_k | n, m \rangle_{\mathcal{E}A} \langle n, m \rho, \phi_k \rangle_{\mathcal{E}A} (|n,m\rangle_{\mathcal{E}A} \otimes |\rho, \phi_k\rangle_{\mathcal{E}A}) = \sum_{n,m} \Psi_{n,m}^\circ(\rho, \phi_k) \Lambda_{n,m}^\circ(q_x, q_y)^*.
\]

In the last line this is expressed as a sum over modes and angular momenta of \(\Psi_{n,m}^\circ(\rho, \phi_k)\) on the polar screen in (20), and \(\Lambda_{n,m}^\circ(q_x, q_y)\) on the Cartesian screen in (23).

The action of the Fourier group \(U(2)\) on images on the polar screen can be found now from the action on the Cartesian one in (38) as

\[
\mathcal{D}(\omega, \Omega) : f(\rho, \phi_k) = \sum_{\rho', \phi_k'} D^\circ(\rho, \phi_k; \rho', \phi_k'; \omega, \Omega) f(\rho', \phi_k'),
\]

with the summation kernel obtained from (39) unitarily transformed by (41),

\[
D^\circ(\rho, \phi_k; \rho', \phi_k'; \omega, \Omega) = \sum_{q_x, q_y} U(\rho, \phi_k; q_x, q_y) D^\circ(\rho, \phi_k; q_x, q_y; \omega, \Omega) U(\rho', \phi_k'; q_x', q_y')^*.
\]

where we use the unitary inverse

\[
V(q_x, q_y; \rho, \phi_k) = \sum_{n,m} \Lambda_{n,m}(q_x, q_y) \Psi_{n,m}^\circ(\rho, \phi_k)^* = U(\rho, \phi_k; q_x, q_y)^*.
\]

6. Concluding remarks

The material presented here has been published before, but the order of derivation and compaction of results may provide a clearer picture of the scope of this case of separation of discrete coordinates. Certainly, the accidental equality between \(\mathfrak{su}(2) \times \mathfrak{su}(2)\) and \(\mathfrak{so}(4)\) is
unique for this matter. We have not lost hope for the search of subalgebra reductions other than $\mathfrak{so}(4) \supset \mathfrak{so}(3) \supset \mathfrak{so}(2)$ that may lead to the more general elliptic coordinates.

Whereas the discrete Fourier transform has a fast algorithm that obviates the $N^2$-growth of computation complexity, which has been extended to (finite but non-unitary) linear canonical transformations, the fractional Fourier-Kravchuk transformation that is natural in our construction is not likely to possess such convenience. Rotations of finite images are of course done through simple interpolation algorithms. The $N^4$-growth of the Fourier group transformations seems thus to be inevitable, but its unitarity ensures exact composition and inversion. Additionally, it provides an interesting application for discrete separation of variables and their special functions.

Acknowledgements
We thank Guillermo Krötzsch and Luis Edgar Vicent (deceased) for the figures as well as help and collaboration with this manuscript.

References
[1] Atakishiyev N M, Pogosyan G S, Vicent L E and Wolf K B, Finite two-dimensional oscillator. I: The Cartesian model. 2001 J. Phys. A 34 9381–9398
[2] Atakishiyev N M, Pogosyan G S, Vicent L E and Wolf K B, Finite two-dimensional oscillator. II: The radial model. 2001 J. Phys. A 34 9399–9415
[3] Vicent L E and Wolf K B, Unitary transformation between Cartesian- and polar-pixellated screens, 2008 J. Opt. Soc. Am. A 25 1875–1884.
[4] Wolf K B and Vicent L E, The Fourier U(2) group and separation of discrete variables, 2011 Sigma 7 art. 053 (18pp).
[5] Simon R and Wolf K B, Structure of the set of paraxial optical systems, 2000 J. Opt. Soc. Am. A 17 342–355.
[6] Simon R and Wolf K B, Fractional Fourier transforms in two dimensions, 2000 J. Opt. Soc. Am. A 17 2368–2381.
[7] Atakishiyev N M, Pogosyan G S and Wolf K B, Contraction of the finite one-dimensional oscillator, 2003 Int. J. Mod. Phys. A 18 317–327.
[8] Atakishiyev N M, Pogosyan G S and Wolf K B, Contraction of the finite radial oscillator, 2003 Int. J. Mod. Phys. A 18 329–341.
[9] Atakishiyev N M and Wolf K B, Fractional Fourier-Kravchuk transform, 1997 J. Opt. Soc. Am. A 14 1467–1477.
[10] Atakishiyev N M, Pogosyan G S and Wolf K B, Finite models of the oscillator, 2005 Phys. Part. Nuclei, Suppl. 3 36 521–555.
[11] Wolf K B and Alieta T, Rotation and gyration of finite two-dimensional modes, 2008 J. Opt. Soc. Am. A 25 365–370.