The Non-Abelian Target Space Duals of Taub-NUT Space

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Abstract

We discuss the non-abelian duality procedure for groups which do not act freely. As an example we consider Taub-NUT space, which has the local isometry group $SU(2) \otimes U(1)$. We dualise over the entire symmetry group as well as the subgroups $SO(3)$ and $U(1)$, presenting unusual new solutions to low energy string theory. The solutions obtained highlight the relationship between fixed points of an isometry in one solution and singular points in another. We also find the interesting results that, in this case, the $U(1)$ and $SO(3)$ T-duality procedures commute with each other, and that the extreme points of the $O(1,1)$ duality group for the time translations have special significance under the $SO(3)$ T-duality.

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1 Introduction

The subject of target space duality, or T-duality, in string theory has generated much interest in recent years. T-duality provides a method, valid to all orders in perturbation theory, for relating seemingly inequivalent string theories.

T-duality was originally discovered for the case of a string theory defined via a $\sigma$-model with an abelian isometry (for example invariance under time translation). The isometry was then gauged, leading to another $\sigma$-model describing a different spacetime geometry. Although the resulting geometry is different from a particle point of view, the models classically describe the same string theory, due to the invariance of the generating function under such a transformation. Quantum equivalence of the two structures is achieved by an appropriate shift in the dilaton.

This procedure was then generalised to models with more than an abelian isometry group of dimension greater than one. This led to the idea of the vacuum moduli space; for example, a heterotic string theory with $d$ abelian isometries and $p$ abelian gauge fields has an $O(d,d+p)$ invariance. These sort of results and their implications have been studied in great detail for the various types of superstring. For a review see, for example, [1].

It was noticed, however, that the basic duality procedure could be generalised to the case where the original $\sigma$-model had a non-abelian group of isometries [2]. Further work in this direction was carried out in [3] [4] [5]. Due to the extra complexity of non-abelian isometry groups, the technical problems associated with T-duality in this framework have only recently been fully solved, in particular relating to the quantum equivalence of two string theories, [7]. Non-abelian duality is thus a valid symmetry of the string theory, as in the abelian case. It is therefore of great interest to study new solutions found by application of the non-abelian duality formulation. The procedure yields new and unusual solutions, thus increasing our understanding of low energy string propagation. One particularly interesting result is that singular geometries can be dual to regular geometries, from a string theory point of view. Thus duality may shed light on the role of spacetime singularities in string theory.

The non-abelian duality shares many features with the abelian counterpart, but there are also many differences; for example, in the non-abelian case, the dual model generally has fewer isometries than the original model. This means that there is no immediate way of performing the inverse duality transformation (Hence ‘duality’ is
perhaps a misnomer). The two models are equivalent as string theories however, so there should be some way of defining a duality transformation independent of the existence of isometries. This implies that for a given $\sigma$-model there should be a much larger equivalence class of solutions in the moduli space than the one given by the abelian duality transformations. A method of performing the reverse transformation and constructing the full equivalence class of solutions in the string moduli space is suggested in [8] by embedding the string solutions into an enlarged class of backgrounds, using Drinfeld doubles. For recent progress on the subject see [9][10]. Unfortunately [8] only consider classical string equivalence for the case of freely acting groups. Although interesting, this does not include cases of groups such as SO(3), which are of fundamental importance in physics.

In this paper we clarify the general method for constructing the dual models from a $\sigma$-model point of view and present an explicit calculation of the non-abelian duals of Taub-NUT space. The resulting metrics are rather unusual and have entirely non-trivial dilatons and B-fields. These solutions show that the abelian and non-abelian dualities procedures commute with each other and single out the extreme $R \to \frac{1}{R}$ points of the abelian T-duality as special solutions.

2 Low Energy String $\sigma$-Models

$T$-Duality is an exact symmetry order by order in perturbation theory, so to relate accurately solutions we need to consider low energy string propagation. This can be thought of in two ways: either we consider a string propagating on a massless background or we integrate out the massive modes of the string in the vacuum. A bosonic string propagating on a non-trivial background can be described by a general non-linear $\sigma$-model defined on a two-dimensional curved surface, because of the correspondence between the massless modes of a string and a spacetime.

In order that the quantum system is equivalent to the classical system the model is required to be conformally invariant. This is described by the non-linear $\sigma$-model in $D$ spacetime dimensions:-

$$S_{nlsm} = \frac{1}{4\pi\alpha'} \int d^2\xi \left( \sqrt{-\gamma}^{ab} g_{MN}(X) \partial_a X^M \partial_b X^N + \epsilon^{ab} B_{MN}(X) \partial_a X^M \partial_b X^N \right)$$

(2.1)

where $M, N \ldots$ run from 0...$D - 1$ and $a, b \ldots$ run over 0,1 with $\epsilon^{01} = 1$
\( \xi^a \) are the string worldsheet coordinates and \( X^M \) are coordinates in spacetime. \( \gamma \) is the induced metric on the string worldsheet, \( \Sigma \). The background spacetime metric and antisymmetric tensor fields in which the string propagates are denoted by \( g \) and \( B \) respectively.

The action has the invariance property

\[
\gamma_{ab} \to e^{\phi(\xi)} \gamma_{ab}.
\]

(2.2)

This is the conformal transformation of the metric \( \gamma_{ab} \) under which the action remains invariant, since \( \gamma \) is two dimensional.

Since we are considering bosonic string theory, there is only one more massless degree of freedom of the string, namely the dilaton \( \Phi \). This gives a contribution to the action of the form

\[
S_{\text{dil}} = -\frac{1}{8\pi} \int d^2\xi \sqrt{\gamma}R^{(2)}\Phi(X),
\]

(2.3)

where \( R^{(2)} \) is the scalar curvature of the metric \( \gamma \).

This term breaks Weyl invariance on a classical level as do the one loop corrections to \( g \) and \( B \).

Conformal invariance is restored to first order in \( \alpha' \) provided that the string \( \beta \) function equations are satisfied

\[
R_{MN} - \nabla_M \nabla_N \Phi - \frac{1}{4} H^L_{MP} H_{NLP} = 0
\]

(2.4)

\[
\nabla_L H^L_{MN} + (\nabla_L \Phi) H^L_{MN} = 0
\]

(2.5)

\[
R - \frac{D - 26 + c}{3\alpha'} - (\nabla \Phi)^2 - 2\Box \Phi - \frac{1}{12} H_{MNP} H^{MNP} = 0.
\]

(2.6)

where \( H = 3dB \) and \( c \) is a central charge.

It is interesting to note that in the field theoretic limit, the \( \beta \) function equations may be considered to be the equations of motion derived from the low energy effective action

\[
S = \frac{1}{2\kappa^2} \int d^Dx \sqrt{g} e^\phi \left( R - D_\mu \phi D^\mu \phi + \frac{1}{12} H^2 \right).
\]

(2.7)

In order to agree with the form of the standard general relativity action we may define the string metric to be related to the Einstein metric by
Note that any vacuum solution of the Einstein equations is a solution to the low energy string theory with zero dilaton and $B$-field, when tensored with an appropriate conformal field theory to provide a central charge to satisfy the third $\beta$-function equation \( 2.6 \). We thus have a good supply of well known solutions to investigate T-duality.

Finally we remark that all parts of the formalism may be made supersymmetric. Thus all the results hold in the supersymmetric case.

### 3 Gauging of the $\sigma$-Model

We now consider constructing the dual models from a given string background. From a spacetime point of view we may perform a duality procedure whenever the string background has a group of isometries. We first approach the problem for an action written in a coordinate basis, as originally considered in \([2]\), for a background with an isometry group acting without isotropy, and then look at the specific case of a symmetry group acting with isotropy. Isotropic symmetry groups are more difficult to treat due to residual gauge freedom remaining for any given gauge choice.

#### 3.1 Isometry Group Acting without Isotropy

We consider the low energy bosonic $\sigma$-model in terms of all the massless modes, and use conformal invariance and diffeomorphism invariance to put the metric $\gamma$ into flat form to obtain the action

\[
S[X] = \frac{1}{4\pi\alpha'} \int d^2z \left( Q_{\mu\nu} \partial X^\mu \partial X^\nu - \frac{\alpha'}{2} R^{(2)} \Phi \right)
\]

where $\mu, \nu$ run from $0 \ldots d - 1$, where $d$ is the dimension of the background spacetime we are considering (neglecting compactified dimensions, for example). We consider $B$ as a potential of the three form $H$

\[
Q = g + B.
\]
Suppose that $Q$ has a freely acting group of isometries $G$. This means that there exists Killing vector fields $T^i$ (the generators of $G$) such that

\[ \mathcal{L}_T g = 0 \]
\[ \mathcal{L}_T \Phi = 0 \]
\[ \mathcal{L}_T H = 0 \]
\[ \Rightarrow \mathcal{L}_T B = d\omega \text{ locally, for some one-form } \omega. \quad (3.3) \]

Note that since the commutator of two Killing vectors is also a Killing vector, the Killing vectors can be thought of as defining a Lie algebra of some group $G$.

We may introduce a gauge field into the action to describe this symmetry (or a subgroup thereof) by the minimal coupling prescription:

\[ \partial X^m \rightarrow DX^m = \partial X^m + A^a(T_a)^m_n X^n \]

The $T_a$ are the generators of the Lie algebra in the adjoint representation of the subgroup $H \subset G$ and give rise to the structure constants $f^c_{ab}$

\[
[T_a, T_b] = f^c_{ab}T_c \\
Tr(T_a T_b) = 2\eta_{ab},
\]

(3.4)

$\eta$ is the Cartan matrix of the group and $a, b$ are internal group indices running over 1..dim($H$). In order to reproduce the original action, the gauge fields must be constrained to be flat by the addition of the Lagrange multiplier term

\[ S_{\text{constraint}} = \frac{1}{4\pi\alpha'} \int d^2z \ Tr (\Lambda F). \quad (3.5) \]

The equations of motion for $\Lambda$ give a vanishing, non-dynamical, $F$.

\[ F = \partial\overline{A} - \overline{\partial}A + [A, \overline{A}] = 0 \]
\[ \Leftrightarrow A = h^{-1}\partial h \]
\[ \overline{A} = h^{-1}\overline{\partial} \overline{h} \]
\[ h \in H, \]

(3.6)
that is, $A$ is pure gauge.

As described in the model above the $X$ are the dynamical fields and the $\Lambda$ are Lagrange multipliers. However, we may also perform an integration by parts on the constraint term, which will introduce derivative terms of $\Lambda$ into the action. Thus the Lagrange multipliers will be promoted to dynamical variables and the coordinates describing the isometry become constant.

The form of the dual is given by

$$S_{\text{dual}} = \frac{1}{4\pi\alpha'} \int d^2 z \left( \bar{h}_a \left( f^{ab} \right)^{-1} h_b + Q_{ij} \partial x^i \tilde{\partial} x^j \right)$$

(3.7)

where

$$h_a = -\partial \Lambda_a + Q_{\mu n} \partial X^\mu (T_a)^n_m X^m$$

$$\bar{h}_a = \partial \Lambda_a + Q_{n\mu} \bar{\partial} X^\mu (T_a)^n_m X^m$$

$$f_{ab} = -f_{ab} c\Lambda_c + X^p (T_b)^q_p Q_{qn} (T_a)^n_m X^m.$$  

(3.8)

where $m, n, p, q : 1, 2, \ldots \text{dim} H$ and $i, j : \text{dim}(H)+1\ldots\text{dim}Q$.

We now have the dual model from which we can read off the new $Q$ field in terms of the coordinates on the dual manifold $(x^i, \Lambda^m)$. Note that if the non-abelian gauged subgroup is semisimple then the gauging procedure is well defined [7].

Classically the two solutions generated are equivalent, and in the special case where the gauged symmetry group is semisimple then in the low energy limit the new fields satisfy the beta function equations if we make the dilaton shift [12]

$$\Phi_{\text{new}} = \Phi_{\text{old}} - \log(\det f).$$

(3.9)

These equations reduce to the abelian $T$-duality equations[1] for a group of abelian isometries, in which case for the two models to be truly dual as string theories the new coordinates $\Lambda$ must have an appropriate periodicity. The periodicity is chosen to remove any potential conical singularities in the spacetime, where the dual would not satisfy the $\beta$-function equations.

Notice that the new fields will be singular at the points where $\det(f) = 0$. This certainly occurs at the fixed points of the action of the gauged symmetry group [10]. If the symmetry group gauges is isotropic then there are by definition fixed points of the action. Hence the duals to such spaces will always have curvature singularities. This is due to a bad gauge choice where the singularity occurs.
3.2 Case with Isotropy

If the gauged subgroup has an isotropy group, as is often the case for groups in physics, then there is an additional freedom in the gauge fixing. If we choose the gauge fixing \( X = \hat{X} \), then the theory is still invariant under

\[
\begin{align*}
\hat{X} & \to L\hat{X} \\
\Lambda & \to L^{-1}\Lambda L
\end{align*}
\] (3.10)

where \( L \) is in the little group of \( \hat{X} \) under the action of \( G, \text{lg}(\hat{X}) \). Thus there is a redundancy in the new coordinates \( \Lambda \), which must be fixed, by placing \( \text{dim} \left( \text{lg}(\hat{X}) \right) \) constraints on the \( \Lambda \).

Note that this extra gauge freedom implies a fixed point in the action of the group. This implies that the dual models will have curvature singularities.

4 Duality Transformation via Forms

The duality procedure becomes clearer if we choose a non-coordinate basis such that the background metric and antisymmetric tensor field are constant under the action of the symmetry group. In the case of a non-isotropic symmetry group the isometry dependent variables can be factored from \( Q \) in a unique, up to constant factors, way and the action can be rewritten in the form

\[
S = \frac{1}{4\pi\alpha'} \int d^2z \left( e^m \mathcal{E}_{mn}(x^i) e^n + e^m \mathcal{E}_{mj}(x^i) \bar{\partial} x^j + \partial x^i \mathcal{E}_{m}(x^i) e^n \\
+ \partial x^i \mathcal{E}_{ij}(x^i) \bar{\partial} x^j - \frac{\alpha'}{2} \sqrt{\gamma} R^{(2)} \Phi \right)
\] (4.1)

\[
e^m \equiv e^m_\mu dx^\mu
\]

\[
d e^m + \frac{1}{2} f^m_{pq} e^p \wedge e^q = 0
\] (4.2)

where \( m,n \) are isometry indices and \( x^i \) are the other coordinates.

\( e^m \) satisfy the Maurer-Cartan equation implying that the connection \( e^m \) is pure gauge

\[
e^m_\mu(x^\nu) \partial x^\mu = \text{Tr} \left( T^m g^{-1} \partial g \right)
\]

so we may choose \( g = 1 \iff x = 0 \).

8
Equivalently the $T_m$ act as Killing vectors on $\mathcal{E}_{\mu\nu}$.

If we gauge the $\sigma$-model with $G$ valued gauge fields then in the low energy limit the duality procedure yields

$$
\hat{S} = \frac{1}{4\pi\alpha'} \int d^2 z \left( \left( \partial \Lambda_m - \partial x^i \mathcal{E}_{im} \right) \left[ (f_{qq'})^{-1} \right]^{mn} \left( \partial \Lambda_n + \mathcal{E}_{nj} \tilde{\partial} x^j \right) 
+ \mathcal{E}_{ij} \tilde{\partial} x^i \tilde{\partial} x^j - \frac{\alpha'}{2} \sqrt{\gamma R^{(2)}} \Phi_{\text{new}} \right) 
$$

$$
\Phi_{\text{new}} = \Phi - \log(\text{det}(f)) 
$$

$$
f_{qq'} = \mathcal{E}_{qq'} + \Lambda_p f^{p}_{qq'} \tag{4.3}
$$

The original Lagrange multipliers $\Lambda$ have become new coordinates in the theory, the old isometry coordinates are now fixed. We can now read off the new $Q$ field in terms of the coordinates on the dual manifold $(x^i, \Lambda^m)$.

If the original solution was an exact conformal field theory then we may obtain an exact dual conformal field theory by an adjustment of $\alpha'$ corrections [7].

4.1 Isotropic Case

If the symmetry group is isotropic then there are extra degrees of freedom in choosing the decomposition of $Q$ into a form which has constant backgrounds with respect to the action of the isometry. For an isotropic symmetry, the dimension of the group $\dim(G)$ is larger than the dimension of the surface of transitivity $\dim(S)$.

$$
\dim(G) = \dim(S) + \dim(\text{lg}(G))
$$

In order to perform the duality transformation it is therefore necessary to consider the extension of the surface over which the isometry acts to a higher dimensional surface of dimensionality $\dim(G)$. We then choose a particular decomposition of $Q$ and perform the non-abelian duality transformation using 4.3. In the resulting solution the new coordinates are constrained to lie on some surface corresponding to the action of the isotropy group of the original gauge choice. Choosing different points on this surface merely corresponds to a change of variables in the new solution.

The action of the isotropy group, $R \subset H$, corresponds to the action

$$
f_{qq'} \rightarrow R^{-1} f_{qq'} R
$$
\[
R^{-1} \left( \mathcal{E}_{qq'} + \Lambda_p f^p_{qq'} \right) R \\
= \mathcal{E}_{qq'} + \hat{\Lambda}_p f^p_{qq'}.
\]

Thus we may choose an element \( R \) to constrain \( \Lambda \) in the above fashion.

## 5 Taub-NUT space

Taub-NUT is a vacuum solution to the Einstein equations in four dimensions.

\[
R_{\mu\nu} = 0
\]

The line element can be written as

\[
ds^2 = -f_1(dt + 2l \cos \theta d\phi)^2 + \frac{1}{f_1} dr^2 + (r^2 + l^2)(d\theta^2 + \sin \theta^2 d\phi^2),
\]

\[
f_1 = 1 - 2 \left( \frac{mr + l^2}{r^2 + l^2} \right) = \frac{(r - r_+)(r - r_-)}{(r^2 + l^2)}
\]

\[
r_\pm = m \pm \sqrt{(m^2 + l^2)}
\]

where \( l \) is the NUT charge and \( m \) is the mass.

For constant \( r \), this is the metric on a squashed three sphere, and for non-zero \( l \) is written as

\[
ds^2 = -(2l)^2 f_1(d\psi + \cos \theta d\phi)^2 + \frac{1}{f_1} dr^2 + \Omega^2(d\theta^2 + \sin \theta^2 d\phi^2)
\]

\[
\psi = \frac{t}{2l}
\]

\[
\Omega^2 = r^2 + l^2.
\]

In order to avoid conical singularities \( \psi \) must be identified with period \( 4\pi \).

For constant \( r \), each point on the manifold \( M \) defined by the line element can be thought of as representing a rotation on 3-d Cartesian vector \( \mathbf{r} \), so Taub-NUT represents a group manifold. The range of \( \psi \) determines the group as \( SU(2) \).

The group elements are

\[
G(\theta, \psi, \phi) = \exp \left( \frac{\psi \sigma_z}{2i} \right) \exp \left( \frac{\theta \sigma_x}{2i} \right) \exp \left( \frac{\phi \sigma_z}{2i} \right)
\]
which act on $r, \sigma$, where the $\sigma$ are the usual Pauli spin matrices. In this context $\psi, \phi$ and $\theta$ are the Euler angles of a general 3-d rotation. Topologically, $SU(2)$ is the three sphere $S^3$ which admits a Hopf fibration

$$S^3 \rightarrow S^2 : (\psi, \theta, \phi) \rightarrow (\theta, \phi) \text{ with fiber } S^1.$$ (5.4)

For Taub-NUT space we have the fibration

$$\pi : M \rightarrow S^2 \ (r, \psi, \theta, \phi) \rightarrow (\theta, \phi)$$ (5.5)

with the metric on the fiber given by (5.2) with constant $\theta, \phi$.

$$ds^2 = -(2l^2f_1^2d\psi^2 + \frac{1}{f_1}dr^2)$$ (5.6)

this fiber is topologically $R \times S^1$.

However, Taub-NUT space has the topology $R^1 \times S^3$ which is only locally $(R^1 \times S^1) \times S^2$. Thus Taub-NUT has a non-trivial fibration which cannot be globally written as a direct product.

### 5.1 Geometry of Taub-NUT space

Taub-NUT space is regular with no horizons, but there are still three separate regions of the spacetime which need to be considered (See figure [I]).

**Region I: $r > r_+$**

In this region the geometry of Taub-NUT space can be thought of as that of rotations of a cone in $\mathbb{R}^3$. The cone is fixed at the vertex, corresponding to $r = r_+$, and has an internal symmetry axis corresponding to the right action of $\psi$.

There are closed timelike curves in this region due to the periodicity of the $\psi$ variable.

**Region II: $r_- \leq r \leq r_+$**

This region represents a homogeneous cosmological model where $r$ acts as the timelike variable. Although this region is compact and regular there are families of incomplete timelike and null geodesics which spiral in towards $r = r_\pm$. Thus the region is *non-Hausdorff*.

**Region III**

This has a similar geometry to region I. In the massless case, $m = 0$, the two regions are isometric and may be identified with each other.
5.2 Monopole Interpretation of Taub-NUT

The non-Hausdorffness of Taub-NUT not too difficult to deal with since we do have bifurcation of geodesics. If we consider the euclidean continuation of Taub-NUT by taking \( t \to it, \ l \to i\r \), then the resulting instanton is Hausdorff. Geodesics in this metric represent asymptotic motion of monopoles [14], thus Taub-NUT space has a clearly defined physical interpretation.

5.3 Isometries of the Group Manifold

The Killing vectors for the line element (5.2) act on surfaces of constant \( r \), so we may neglect the \( dr \) terms throughout the discussion of the symmetries.

For constant \( r \), the line element becomes

\[
ds^2 = -(2l)^2 f_1 (d\psi + \cos \theta d\phi)^2 + (r^2 + l^2)(d\theta^2 + \sin \theta^2 d\phi^2). 
\]

The left invariant one forms for this line element are

\[
e^x = \sin \theta \sin \psi d\phi + \cos \psi d\theta \\
e^y = \sin \theta \cos \psi d\phi - \sin \psi d\theta \\
e^z = d\psi + \cos \theta d\phi.
\]

This enables us to write the metric in terms of the flat connection

\[
ds^2 = -(2l)^2 f_1 e^z + (r^2 + l^2)(e^x + e^y)^2, 
\]

since these are Maurer-Cartan forms obeying the relation

\[
d e^m + \frac{1}{2} f^m_{\ pq} e^p \wedge e^q = 0,
\]

where \( f^m_{\ pq} \) are the structure constants of the rotation group:

\[
f^m_{\ pq} = -e^m_{\ pq}.
\]

Equivalently, we may consider the left invariant vector fields (Killing vectors)

\[
L_x = \cos \psi \frac{\partial}{\partial \theta} - \sin \psi \left( \cot \theta \frac{\partial}{\partial \psi} - \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right) \\
L_y = -\sin \psi \frac{\partial}{\partial \theta} - \cos \psi \left( \cot \theta \frac{\partial}{\partial \psi} - \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right) \\
L_z = \frac{\partial}{\partial \psi}.
\]
The line element is invariant to first order under the action

\[ x \rightarrow x + \epsilon L \]

and the algebra of these vectors is given by

\[
\left[ L_i, L_j \right] = -\epsilon_{ijk} L_k \\
\epsilon_{123} = +1.
\]  (5.13)

This is the same as the Lie Algebra for ordinary rotations on the two sphere, because the extra \( \psi \) coordinate is abelian.

6 Duality of Taub-NUT

Although a simple solution of the Einstein equations, Taub-NUT has a rich duality structure. There are different ways in which the solution can be gauged, leading to interesting solutions with non-trivial \( B \)-fields and dilatons. The first, well known as the H-monopole (if one considers the euclidean continuation of the solution), is the abelian dual of the solution with respect to \( \frac{\partial}{\partial t} \). The H-monopole has an SO(3) symmetry over which we can take the T-dual. The second is the result of gauging out the entire symmetry group of Taub-NUT. The resulting solution has one abelian symmetry. We show that the abelian duality procedure commutes with the one for non-abelian SO(3) duality.

Note that it is also possible to consider the dual for the Killing vector \( \frac{\partial}{\partial \phi} \). The resulting solution has no spherical symmetry, and we do not consider the results obtained in this case.

6.1 SO(3) Dual of Schwarzschild Solution \((l = 0 \text{ case})\)

The Taub-NUT solution reduces to the Schwarzschild solution in the zero NUT charge limit. The group structure is discontinuous in the limit, however, since these two solutions have a different symmetry structure, in that the Schwarzschild solution has a trivial fibration which can be written globally as a direct product. The Schwarzschild solution has the isometry group \( SO(3) \), the action of which which has an isotropy subgroup.
Due to this isotropy of the SO(3) gauge group there are three Killing vectors which act on a two-dimensional surface of \((\theta, \phi)\), hence there will be an SO(2) freedom in the new coordinates if we take the dual of the Schwarzschild solution. We thus consider the solution as a surface embedded in a larger space, and perform the duality procedure over three forms \(e^1, e^2\) and \(e^3\), by choosing forms satisfying the Maurer Cartan equation [1.2] such that

\[
d\theta^2 + \sin^2 \theta d\phi^2 = (e^1)^2 + (e^2)^2 + (e^3)^2.
\] (6.1)

This choice has an ambiguity defined up to an SO(2) rotation. After taking the dual we must therefore fix the extra SO(2) gauge freedom in the new coordinates \(\Lambda\).

We write the line element in terms of the forms

\[
\begin{align*}
e^1 &= d\theta \\
e^2 &= \sin \theta d\phi \\
e^3 &= \cos \theta d\phi.
\end{align*}
\] (6.2)

Note that we have made an implicit choice of gauge. We generally may choose

\[
\begin{align*}
e^2 &= A \sin \theta + B \cos \theta \\
e^3 &= A \cos \theta - B \sin \theta \\
A^2 + B^2 &= 1.
\end{align*}
\] (6.3)

In terms of the forms [6.2], the line element reads:

\[
ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left((e^1)^2 + (e^2)^2\right)
\] (6.4)

By extending the surface of transitivity of the Killing vectors to a three-dimensional surface we can now dualise over the \(e^1, e^2\) and \(e^3\), treating the problem as for non-isotropic symmetry groups. We obtain:

\[
f_{qq'} = \mathcal{E}_{qq'} + \Lambda_p f_p^{qq'} = \begin{pmatrix} r^2 & \Lambda_3 & -\Lambda_2 \\
-\Lambda_3 & r^2 & \Lambda_1 \\
\Lambda_2 & -\Lambda_1 & 0 \end{pmatrix}
\] (6.5)

This matrix can be inverted to give the new fields. We must then constrain the new coordinates, \(\Lambda\), by making a gauge choice which leaves the original \(\mathcal{E}\) field unchanged. In this case the freedom in the dreibein is given by rotations about \(e^3\).
\[
\begin{align*}
\begin{pmatrix}
\hat{e}^1 \\
\hat{e}^2 \\
\hat{e}^3
\end{pmatrix} &= \begin{pmatrix}
\cos \Theta & \sin \Theta \\
-\sin \Theta & \cos \Theta
\end{pmatrix}
\begin{pmatrix}
e^1 \\
e^2
\end{pmatrix} \\
\Rightarrow (\hat{e}^1)^2 + (\hat{e}^2)^2 &= (e^1)^2 + (e^2)^2.
\end{align*}
\]

(6.6)

Imposing this constraint as in 4.14 by choosing, for example, \(\Lambda_2 = 0\), we obtain the extra contribution to the metric (after relabelling all the \(\Lambda_i\))

\[
\hat{g} = \begin{pmatrix}
\partial \Lambda_1 & \partial \Lambda_2 \\
\Lambda_2 & \Lambda_1 \Lambda_2 \\
\Lambda_1 \Lambda_2 & (r^4 + \Lambda_2^2)
\end{pmatrix} \frac{1}{\Delta} \quad \Delta = r^2 \Lambda_2^2.
\]

(6.7)

Of course, a different constraint choice merely corresponds to a change of variables in the new solution.

We now make the change of variables \(y = \Lambda_2\) and \(x^2 + y^2 = \Lambda_1^2\), the solution reduces to that found in [4] via coordinate methods

\[
\begin{align*}
ds^2 &= - \left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + \frac{1}{r^2(x^2 - y^2)} [r^4 dy^2 + x^2 dx^2] \\
B &= 0 \\
\Phi &= - \log[r^2(x^2 - y^2)]
\end{align*}
\]

(6.8)

### 6.1.1 Analysis of the dual space

The line element obtained by taking the \(SO(3)\) dual of Schwarzschild space is important to understand, because any space with spherical symmetry will give rise to a dual space of a similar form.

There are several points of note.

Firstly we note that the five dimensional space obtained after performing the duality procedure, but without placing a constraint on the \(\Lambda_i\), has a singular metric with zero determinant. Thus the dual solution obtained is truly four dimensional, and may not be thought of as just a slice through a larger five-space string theory solution.

Also, since the duality is performed over the spherically symmetric part of the metric, the dual solution retains the original horizon at \(r = 2m\) and a curvature...
singularity at \( r = 0 \). New curvature singularities are introduced along the lines \( x = \pm y \), corresponding to the axes of fixed points of the original symmetry, \( \sin \theta = 0 \). The \( SO(3) \) isometry is lost after the duality transformation, and it has been shown that the metric only has one continuous isometry corresponding to the abelian time shift \([3]\). The only new symmetries introduced under the duality procedure are the discrete symmetries \( x \to -x \) and \( y \to -y \), and we may choose to identify these points. To perform the inverse transformation we need to consider an enlarged background containing both the original and dual spacetimes \([8]\).

More properties of the original spacetime, other than the isometries, are also lost after dualising. The resulting spacetime is not algebraically special (Petrov type I) and, in the massless case, does not admit any Killing spinors.

Despite such problems, however, the resulting spaces may be given a qualitative analysis. Such a discussion will be given in section \([4]\).

### 6.2 H-Monopole

Standard application of the abelian duality transformation for the Killing vector \( \frac{\partial}{\partial t} \) on Taub-NUT yields the H-Monopole given by :-

\[
\begin{align*}
  ds^2 &= -\frac{1}{(2l)^2 f_1} d\Lambda_0^2 + \Omega^2 (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{1}{f_1} dr^2 \\
  B_{0\phi} &= -\cos \theta \\
  \Phi &= -\log \Delta_0 \\
  \Delta_0 &= \det(g_{00}) = -(2l)^2 f_1 .
\end{align*}
\]

This solution now has non-trivial \( H = 3dB \)-field and the line element has an \( SO(3) \) symmetry. Thus we can either perform duality over \( \Lambda_0 \) which will return us to Taub-NUT space, or we can gauge the \( SO(3) \) spherical symmetry, which will elucidate the method of calculating duals given in section \([4]\).
6.3 \textit{SO}(3) dual of H-Monopole

We rewrite the H-monopole line element in terms of the particular choice of Maurer-Cartan forms:

\begin{align*}
e^0 &= d\Lambda_0 \\
e^1 &= d\theta \\
e^2 &= \sin \theta d\phi \\
e^3 &= \cos \theta d\phi.
\end{align*}

(6.10)

yielding:

\begin{align*}
ds^2 &= -\frac{1}{(2l)^2 f_1} (e^0)^2 + \Omega^2 ((e^1)^2 + (e^2)^2) + \frac{1}{f_1} dr^2 \\
B_{0\phi} &= -\cos \theta \Rightarrow B = \cos \theta d\Lambda_0 \wedge d\phi = e^0 \wedge e^3 \\
&\Rightarrow B_{03} = 1.
\end{align*}

(6.11)

We now dualise over $e^1, e^2, e^3$ treating the choice as fixed. We obtain:

\begin{align*}
f_{qq'} &= \mathcal{E}_{qq'} + \Lambda_p f^p_{qq'} = \begin{pmatrix}
\Omega^2 & \Lambda_3 & -\Lambda_2 \\
-\Lambda_3 & \Omega^2 & \Lambda_1 \\
\Lambda_2 & -\Lambda_1 & 0
\end{pmatrix}
\end{align*}

(6.12)

\begin{align*}
\partial \Lambda_m - \partial x^i \mathcal{E}_{im} : (\partial \Lambda_1, \partial \Lambda_2, \partial \Lambda_3 - \partial \Lambda_0) \\
\bar{\partial} \Lambda_m + \mathcal{E}_{ni} \bar{\partial} x^i : (\bar{\partial} \Lambda_1, \bar{\partial} \Lambda_2, \bar{\partial} \Lambda_3 - \bar{\partial} \Lambda_0)
\end{align*}

Inverting $f$ we read off the new contributions to the metric and anti-symmetric tensor field. Since the new line element is in the five dimensional space $(\Lambda_0, r, \Lambda_1, \Lambda_2, \Lambda_3)$ we must make a constraint on the new variables $(\Lambda_1, \Lambda_2, \Lambda_3)$ by considering elements which leave the gauge choice fixed, as in 3.10. In this case we may make the particular choice $\Lambda_1 = 0$. Note that the five space metric has zero determinant, and actually describes a four dimensional surface. The extra gauge fixing merely removes the redundancy in the new coordinates $\Lambda_i$ and all such choices reduce to the same four-space.

After imposing the constraints we find that:

\begin{align*}
\hat{g} &= \begin{pmatrix}
\partial \Lambda_0 & \partial \Lambda_1 & \partial \Lambda_2 \\
\Omega^4 + \Lambda_2^2 & -\Lambda_1 \Lambda_2 & -\left(\Omega^4 + \Lambda_2^2 \right) \\
-\Lambda_1 \Lambda_2 & \Lambda_1^2 & \Lambda_1 \Lambda_2 \\
-\left(\Omega^4 + \Lambda_2^2 \right) & \Lambda_1 \Lambda_2 & \left(\Omega^4 + \Lambda_2^2 \right)
\end{pmatrix} \frac{1}{\Delta}
\end{align*}

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\[
\begin{align*}
B &= 0 \\
\Delta &= \Omega^2 \Lambda_1^2 \\
\Phi &= \Phi_{\text{monopole}} - \log \Delta \\
&= \log f_1 - \log(\Omega^2 \Lambda_1) 
\end{align*}
\]

To this metric we need to add the \((\Lambda_0, r)\) sector of the H-monopole, resulting in the solution we shall call \(H_{SO(3)}:\)

\[
ds^2 = -\frac{d\Lambda_0^2}{(2l)^2 f_1} + \frac{dr^2}{f_1} + \frac{1}{\Delta} (-\Lambda_2 d\Lambda_0 + \Lambda_1 d\Lambda_1 + \Lambda_2 d\Lambda_2)^2 + \frac{1}{\Delta} \Omega^4 (d\Lambda_0 - d\Lambda_2)^2
\]

Note we have explicitly checked that this background is a solution to \(\Sigma 4\)

Clearly this solution has the abelian isometry \(\Lambda_0 \rightarrow \Lambda_0 + \) constant. We can therefore perform the \(T\)-dual for this variable. We consider specifically the extreme case of \(\Lambda_0 \rightarrow \frac{1}{\Lambda_0}\) duality, although this could be extended to an \(O(1, 1)\) symmetry to generate an entire spectrum of solutions.

### 6.4 Abelian dual of \(H_{SO(3)}\)

Performing the standard \(T\)-duality and making the variable change

\[
\hat{\Lambda}_0 = \alpha \Lambda_1 = (2l)^2 y \Lambda_2 = (2l)^2 t
\]

we obtain the solution \(H_{SO(3), \Lambda_0}\):-

\[
ds^2 = -\frac{\Omega^4}{\Delta} dt^2 - \frac{(2l)^2}{\Delta} (tdt + ydy)^2 + \frac{f_1 \Omega^2 (2l)^2}{\Delta} (dy^2 + y^2 d\alpha^2) + \frac{dr^2}{f_1} \\
B = \left( \frac{\Omega^2 y^2 (2l)}{\Delta} - (2l) \right) d\alpha \wedge dt + \frac{(2l)^3 f_1 yt}{\Delta} d\alpha \wedge dy \\
\Phi = \log(- (2l)^2 \Delta) \\
\Delta = f_1 (\Omega^4 + (2l)^2 t^2) - \Omega^2 y^2.
\]

### 6.5 SU(2) Dual of Taub-NUT

We note that the symmetry structure of Taub-NUT has a non-trivial decomposition and we rewrite the line element \([5.11]\) using the forms \([5.8]\). We are now able to dualise over the entire SU(2), instead of just an abelian subgroup. The symmetry acts on surfaces
of constant $r$ so we may neglect the $dr^2$ term throughout the duality procedure. As we cannot decouple the $\psi$ variable from the symmetry, there is only one abelian degree of freedom in our choice of tetrad.

Equations 4.3 give:

$$f_{qq'} = \mathcal{E}_{qq'} + \Lambda_p f_p^p_{qq'} = \begin{pmatrix} -(2l)^2 f_1 & \Lambda_3 & -\Lambda_2 \\ -\Lambda_3 & \Omega^2 & \Lambda_1 \\ \Lambda_2 & -\Lambda_1 & \Omega^2 \end{pmatrix}$$

Inverting this matrix and following the procedure to read off the new fields we obtain the new solution.

Since $\mathcal{E}_{im} = 0$, $f^{-1}$ represents $\hat{g} + \hat{B}$ (For constant $r$ surfaces the $dr$ terms remain unchanged).

so

$$\hat{g} = \frac{1}{\Delta} \begin{pmatrix} \bar{\partial}\Lambda_1 & \bar{\partial}\Lambda_2 & \bar{\partial}\Lambda_3 \\ (\Omega^4 + \Lambda_1^2) & \Lambda_1 \Lambda_2 & \Lambda_1 \Lambda_3 \\ \Lambda_2^2 - (2l)^2 f_1 \Omega^2 & \Lambda_2 \Lambda_3 & \Lambda_3^2 - (2l)^2 f_1 \Omega^2 \end{pmatrix}_{sym}$$

$$\hat{B} = \frac{1}{\Delta} \begin{pmatrix} 0 & -\Omega^2 \Lambda_3 & \Omega^2 \Lambda_2 \\ 0 & (2l)^2 f_1 \Lambda_1 & 0 \\ \Lambda_2 - (2l)^2 f_1 \Lambda_1 & \Omega^4 + \Lambda_1^2 \end{pmatrix}_{asym}$$

$$\Delta = \Omega^2 (\Lambda_2^2 + \Lambda_3^2) - (2l)^2 f_1 (\Omega^4 + \Lambda_1^2)$$

$$\Phi = \log(\Delta). \quad (6.16)$$

Since SU(2) is semisimple, these massless modes are a solution of the $\beta$-function equations [24, 7]. This has been explicitly checked. Comparison with the Schwarzschild $SO(3)$ dual suggests that we make the change of coordinates:

$$\Lambda_2 = y \cos \alpha, \Lambda_3 = y \sin \alpha$$

$$\Lambda_1^2 = (2l)^2 t^2$$

yielding the solution $TN_{SU(2)}$.

$$ds^2 = -\frac{dt^2}{f_1} + \frac{dr^2}{f_1} + \frac{(2l)^2}{\Omega^2} dy^2 - \frac{(\Omega^2 y dt + (2l)^2 f_1 t dy)^2}{\Delta f_1 \Omega^2} + \frac{f_1 \Omega^2 (2l)^2}{\Delta} y^2 d\alpha^2$$
\[ B = \frac{\Omega^2 y^2(2l)}{\Delta} d\alpha \wedge dt + \frac{(2l)^3 f_1 y t}{\Delta} d\alpha \wedge dy \]
\[ \Phi = \log(-2l^2 \Delta) \]
\[ \Delta = f_1(\Omega^4 + (2l)^2 t^2) - \Omega^2 y^2 \]

(6.17)

**Periodification of coordinates on the dual manifold**

The metric defined by 6.17 is invariant under

\[ t \to -t \quad \text{and} \quad y \to -y. \]

These points may be identified to give an orbifold. However, such an identification changes the sign of the \(B\)-field. Given a coordinate basis, if we take reciprocal periodicities then the solutions are identical points in the string moduli space. In the case where we take the duals over a non-coordinate basis it is not so clear how to relate the solutions. In any case, as in Taub-NUT space, the periodicities of the new coordinates should be defined to remove any conical singularities in the new solution. The solution will then satisfy the \(\beta\)-function equations everywhere.

Finally, we see that this solution is the same as the solution 6.15, up to a constant factor in the \(B\)-field, hence we find that the abelian and non-abelian duals commute with each other:

\[ TN_{SU(2)} = TN_{t, SO(3), \Lambda_0} = H_{SO(3), \Lambda_0}. \]

(6.18)

Of course this also implies that the \(\alpha\) dual of \(TN_{SU(2)}\) gives us \(H_{SO(3)}\)

**6.6 \(O(1, 1)\) duality of Taub-NUT**

As mentioned previously, we may extend the \(t \to 1/t\) discrete duality of Taub-NUT to an \(O(1, 1)\) group of dualities. The general solution obtained is

\[ ds^2 = -\frac{f_1}{f_2}(dt + (x + 1)l \cos \theta d\phi)^2 + \frac{dr^2}{f_1} + \frac{1}{r^2 + l^2} (d\theta^2 + \sin^2 \theta d\phi^2) \]
\[ f_2 = 1 + (x - 1) \frac{mr + l^2}{r^2 + l^2} \]
\[ \Phi = \Phi(r) \]
\[ B_{\phi t} = (x - 1) \cos \theta \times (r - \text{terms}) \]
\[ x^2 \geq 1 \]

(6.19)
of which the H-monopole is the $x = -1$ case and Taub-NUT is the $x = 1$ case.

It is interesting to note that either the $dtd\phi$ metric cross terms or the $B$-field vanish only for the $x = \pm 1$ cases. In these two cases we are able to perform the duality transformation for the non-abelian spherical symmetries. In the case $|x| \geq 1$ we are unable to factor out the $SO(3)$ dependence from the matrix $Q$ to give a real $\mathcal{E}$. This creates problems with performing the duality transformation, and seems to single out the extreme cases of small and large radius compactification as being special in some way.

6.7 Equivalence of Solutions

We stress that all the solutions we have presented are completely equivalent as low energy string theory solutions since the gauged subgroups are either abelian or semisimple. There are interesting connections between the new solutions, which may be represented diagrammatically as follows:

\[
\begin{array}{c}
\text{TN} \xrightarrow{SU(2)} \text{TN}_{SU(2)} \\
\uparrow \quad \uparrow \\
\Lambda_0, \alpha \quad \Lambda_0, \alpha \\
\downarrow \quad \downarrow \\
H \xrightarrow{SO(3)} H_{SO(3)}
\end{array}
\] (6.20)

Thus the duality procedure closes, the abelian duality commutes with the non-abelian duality transformation at least for the case of Taub-NUT, and the extreme points of the abelian duality seem to have special significance with respect to the abelian duality.

7 Properties of $TN_{SU(2)}$

We now investigate the solution obtained after performing the non-abelian transformation on Taub-NUT space. Despite the complicated local behaviour of the solution, it has some global properties which may be understood from a geometrical point of view.
(see the figure). The global structure of the solution clarifies the relationship between spacetime singularities and fixed points of isometry groups.

**Signature**

The determinant of the metric is

\[
\det(g) = -y^2(2l)^4(r^2 + l^2)^2 [\Delta^{-2}]
\]

\[
\Delta = f_1(\Omega^4 + (2l)^2 t^2) - \Omega^2 y^2
\]

This is only zero for non-singular regions when \(y = 0\). The determinant never becomes positive which, coupled with an examination of the behaviour of the coordinate surfaces, shows that we still have a completely Lorentzian spacetime.

**Singularities**

The metric has curvature singularities (singular Riemann curvature terms) when:-

\[i\] \(\Omega^2 = r^2 + l^2 = 0\)

\[ii\] \(f_1 = 1 - 2(\frac{mr + l^2}{r^2 + l^2}) = 0 \iff r = r_\pm\)

\[iii\] \(\Delta = 0\)

The surface \(\Delta = 0\) has a strictly spacelike normal.

**Geometry**

There are four separate regions which need to be considered

1: \(\Delta > 0\), \(r > r_+\) \((f_1 > 0)\)

For large \(t\) and \(r\) the spacetime is non-singular and for small \(y\) approaches the metric of a 2-dimensional H-monople-type throat region where the \(y\) and \(\alpha\) dimensions are squeezed:

\[
ds^2 \rightarrow -\frac{1}{f_1} dt^2 + \frac{1}{f_1} dr^2 + O\left(\frac{1}{r^2}\right) (dy^2, dydt, d\alpha^2)
\]

\[
B \rightarrow \frac{2ly^2}{f_1r^2} d\alpha \wedge dt + O\left(\frac{1}{r^2}\right) dy \wedge dt
\]

This gives us the interpretation of \(t\) as the asymptotic time.

For a given \((t, r)\) there is a singularity, for large enough \(y = y_{\text{sing}}\), where \(\Delta\) vanishes. For an observer at fixed \(r\) and \(y\), \(y_{\text{sing}}\) increases as \(t^2\):-

\[
y_{\text{sing}}^2 = f_1 \left(\Omega^2 + \frac{(2l)^2 t^2}{\Omega^2}\right)
\]

\[
\sim \frac{t^2}{r^2}
\]
Thus the singularity accelerates away from such observers, in the manner of a particle in a constant potential. In this region when \( y^2 > f_1 \Omega^2 \), surfaces of constant \( t \) have a spacelike normal. Thus there are no timelike curves with fixed \((r, y, \alpha)\) and we have an \textit{ergoregion}. In self-dual euclidean Taub-NUT space, geodesics can be thought of as describing two monopoles interacting via a Coulomb type potential \([14]\). This seems to be a similar property to the acceleration described above, and is is possible that the dual space retains some sort of monopole interpretation.

The solution also displays some of the properties of the \( C \)-matrics, \([17]\), in this region. The general form of the solution is similar to the \( C \)-metrics, and the acceleration property of the horizon is also reminiscent of such metrics.

\[
\text{II}: \Delta < 0, \ r_- < r < r_+ \ (f_1 < 0)
\]

This region is completely non-singular. The timelike variable is given by \( r \) and all other variables are spacelike, as in region II of Taub-NUT.

\[
\text{III}: \Delta > 0, \ r < r_- \ (f_1 > 0)
\]

This region has similar properties to region I. In the special case where \( m = 0 \) the two regions I and III are isometric and may be identified.

\[
\text{IV}: \Delta < 0, \ r > r_+ \ (f_1 > 0)
\]

In this region \( \alpha \) is timelike and the normal to surfaces of constant \( \alpha \) are timelike. Therefore, since \( \alpha \) is an angular variable we have closed timelike curves along orbits of constant \( t, r, y \).

**Comparison with Taub-NUT**

The geometric interpretation of Taub-NUT space and the metric of the dual are similar: both can be thought of as two cones free to rotate on an internal axis separated by a non-singular region. The nature of the dual geometry may be intuitively understood by considering the fixed points of the Taub-NUT symmetry space.

i) The duality procedure fixes the position of the Taub-NUT cone in \( \mathbb{R}^3 \). This position corresponds to the choice of gauge.

ii) Secondly, the boundary becomes singular. This corresponds to the invariance of Taub-NUT under the right action of shifts in \( \psi \). Notice that in region II of Taub-NUT there are no boundaries (fixed points of the \( SU(2) \) action), hence region II of the dual is entirely regular.

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These are generic features of duality procedures: fixed points of the action of the isometry group on the original manifold become singular points in the dual model. This is demonstrated in the Schwarzschild dual where we have a line of singularities corresponding to the $\theta = 0$ line of fixed points of the $SO(3)$ isometry, and the abelian T-dual of two dimensional Minkowski space in polar coordinates, where a point naked singularity is introduced, corresponding to the fixed point $\theta = 0$.

8 Conclusion

We have found dual descriptions, both abelian and non-abelian, of Taub-NUT space in string theory. The dualities commute at least for the case of Taub-NUT space. The resulting spaces are rather complicated and lose their non-abelian symmetries. It seems an important task to study the nature of the dual spaces, which have properties reminiscent of some point particle spacetimes, such as those of $C$-metrics and monopole spacetimes. Although the original and dual spaces seem very different, they are equivalent as string theory vacua and in order to understand the full string theory moduli space, the relationship between these string backgrounds needs to be better understood.

The framework of non-abelian dualities may be applied to any string solution with a background which admits non-commuting Killing vectors. This has an extremely wide area of applicability: many string solutions, when considered as full ten dimensional solutions, have a pure spherical symmetry arising from extra flat dimensions, such as in plane waves. These extra dimensions may be dualised over to give complicated solutions of the form $6.8$. Perhaps in string theory, spherical symmetry is not as simple as it seems from a particle point of view.

At the very least non-abelian duality points to an equivalence of solutions in string theory beyond those given by abelian target space duality and the diffeomorphisms of general relativity.

I would like to thank Malcolm Perry for suggestions and help.
Figure 1: Taub-NUT and its dual

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