Fluids and vortex from constrained fluctuations around C-metric black holes

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\textbf{Abstract}

By foliating the four-dimensional C-metric black hole spacetime, we consider a kind of initial-value-like formulation of the vacuum Einstein’s equation, the holographic initial data is a double consisting of the induced metric and the Brown-York energy momentum tensor on an arbitrary initial hypersurface. Then by perturbing the initial data that generates the background spacetime, it is shown that, in an appropriate limit, the fluctuation modes are governed by the continuity equation and the compressible Navier-Stokes equation which describe the momentum transport in non-relativistic viscous fluid on a flat Newtonian space. It turns out that the flat space fluid behaves as a pure vortex and the viscosity to entropy ratio is subjected to the black hole acceleration.

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1 Introduction

In the past two decades the development of fundamental physics has been greatly promoted by the recognition of the holographic principle which emerges in the study of black holes. This bold principle was originally put forward by 't Hooft [1] and Susskind [2], and first realized concretely by Maldacena [3] in the context of string theory. In this implementation a conjectured equivalence was established between the supersymmetric Yang-Mills gauge theory and the superstring theory. Although the origin of the connection is still mysterious, this well-known AdS/CFT correspondence has been applied in many aspects as an efficient tool to perform analytical calculations in strongly coupled systems [4–6]. A canonical application of this correspondence is the analysis of shear viscosity in the strongly coupled SYM theory [4], in which the hydrodynamics arises as a classical description for the behavior of any interacting quantum field theory at long-wavelength and low-frequency. Then in the framework of AdS/CFT correspondence a connection between gravity in asymptotically AdS and relativistic hydrodynamics in one less dimension was revealed [7–12], and this is known as the Gravity/Fluid correspondence, and the relativistic gravitational field equations were reduced into the incompressible Navier-Stokes equation in an appropriate scaling limit [13].

Actually, the Gravity/Fluid correspondence can be realized independent of AdS/CFT correspondence, and it’s origin could be dated back to the late 1970’s, long before the holographic principle was proposed. Here we refer to the membrane paradigm [14–17], in which the local dynamical quantities of black holes was first studied “holographically”, and shown to be governed by the Damour-Navier-Stokes equations. Despite the technical differences, the viscosity/entropy ratio from membrane paradigm at horizon and AdS/CFT at spatial infinity was surprisingly the same, which indicates a correlation between these two approaches [18–20]. Under such background the Gravity/Fluid correspondence at arbitrary cutoff was constructed [21]. In it’s original version this was achieved by considering linearized Einstein equations while making hydrodynamic expansion and imposing appropriate boundary conditions. To be specific, the scalar and tensor modes of the fluctuations are fixed by the ingoing-wave boundary conditions on the horizon and Dirichlet boundary conditions at the cutoff, and the only dynamical modes are the vector fluctuations governed by the linearized Navier-Stokes equations. Shortly afterwards in [22] the analysis was improved to the nonlinear case, more importantly the hydrodynamic expansions and the boundary conditions are shown to be mathematically equivalent to the near horizon expansions and Petrov-like boundary conditions respectively [23].

In the recent years this reformulated Gravity/Fluid correspondence was greatly generalized [24–46], especially in [47] we studied a type of constrained perturbations around a class of black holes with curved horizons, and in the near horizon region (later this was generalized to finite cutoff [48]), we find that such kind of Petrov-type fluctuations could be mapped to a forced compressible viscous fluid in flat space of one less dimension. In these works we went beyond the framework of the bulk/boundary
duality, and the relaxation of bulk/boundary restriction may possibly be a motivator and reminder of deeper understanding about the holographic principle, also we expect that this further generalization of the Gravity/Fluid correspondence could unearth more potentiality of gravity as a powerful tool in the study of fluid dynamics. But we are far from being optimistic, since we have not yet achieved a complete construction, one of the major obstacles is the unusual body force of the dual fluid system pertaining to the surface stress. To make sense of the external force we need to apply our analysis to other type of solutions of the Einstein’s equation. So, in this paper we take that first step and study a fluid dual of a well-known four-dimensional accelerating black hole solution, i.e. the C-metric black hole. We find that in this case the form of the external force is better understood, and more intriguingly, there will be a vortex in the dual fluid system.

2 The C-metric reformulated in an appropriate coordinate system

The whole construction relies on the C-metric solution of the vacuum Einstein equation [49]. The line element reads

\[ ds^2 = \frac{1}{\mathcal{A}} \left( -Qdt^2 + \frac{dr^2}{Q} + \frac{r^2}{P} d\theta^2 + \frac{Pr^2 \sin^2 \theta}{(1 + \alpha r h)^2} d\varphi^2 \right), \tag{1} \]

where

\[ \mathcal{A}(r, \theta) = (1 + \alpha r \cos \theta)^2, \]

\[ Q(r) = (1 - \alpha^2 r^2) \left( 1 - \frac{r_h}{r} \right), \]

\[ P(\theta) = 1 + \alpha r_h \cos \theta, \]

and \( 0 \leq \alpha r_h < 1 \). This solution can be viewed as a one-parameter generalization of the Schwarzschild metric and is interpreted as an accelerating black hole solution, with the black hole event horizon located at \( r = r_h \), and the acceleration horizon at \( r = \frac{1}{\alpha} \). The positive parameter \( \alpha \) corresponds to the proper acceleration of the black hole.

To construct the hydrodynamic equations in isotropic coordinates, it is desirable to introduce a conformal isotropic coordinate system in the angular part of the line element. The new coordinates \((x^1, x^2)\) replace the old ones \((\theta, \varphi)\) via the relations

\[ x^1 = w(\theta) \cos \varphi, \]
\[ x^2 = w(\theta) \sin \varphi, \]

where

\[ w(\theta) = \left( \frac{P}{\sin \theta} \right)^{\alpha r_h} \tan \left( \frac{\theta}{2} \right)^{\frac{1}{1 - \alpha r_h}} \]
plays the role of a radial coordinate on the \((w, \varphi)\) “plane” and it is a univariate function in \(\theta\) ranging from 0 to \(+\infty\). Though seemingly weird, the construction of this new coordinate system is straightforward. Using the above transformation, the line element \((1)\) could be rewritten under the coordinates \(x^\mu = (t, r, x^1, x^2)\) in the form
\[
\begin{align*}
\mathrm{d}s^2 &= g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \frac{1}{A} \left( -Q\mathrm{d}t^2 + \frac{dr^2}{Q} + r^2 e^{\Phi(x^i)}\delta_{ij}dx^i dx^j \right), \\
\end{align*}
\]
in which
\[
\begin{align*}
A(r, x^i) &= (1 + \alpha r Z)^2, \\
e^{\Phi(x^i)} &= \frac{(1 + \alpha r_h Z)(1 - Z^2)}{(1 + \alpha r_h)^2\delta_{ij}x^i x^j}, \quad (i = 1, 2)
\end{align*}
\]
where
\[
Z(x^i) = \cos(\theta(x^i))
\]
is an implicit function of \(x^i\). For calculation purposes, it is desirable to rewrite the expression \(\delta_{ij}x^i x^j\) in terms if the implicit function \(Z(x^i)\),
\[
\begin{align*}
w^2 &= \delta_{ij}x^i x^j = \left( 1 - Z \left( \frac{1 + \alpha r_h Z}{1 - Z^2} \right)^{\alpha r_h} \right)^{\frac{1}{1-\alpha r_h}} \\
\end{align*}
\]
because of the simple relation
\[
\partial_i Z = -(1 + \alpha r_h)e^\Phi x_i,
\]
which can be verified directly. This simple relation will be very facilitating in the following calculations.

### 3 Constraints on initial hypersurface

We start by foliating the C-metric background by three-dimensional timelike hypersurfaces defined by \(r = \text{const}\). The bulk line element could be expressed as
\[
\begin{align*}
\mathrm{d}s^2 &= \frac{dr^2}{AQ} + \gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b = \frac{dr^2}{AQ} + \frac{1}{A} \left( -Q\mathrm{d}t^2 + r^2 e^{\Phi(x^i)}\delta_{ij}dx^i dx^j \right),
\end{align*}
\]
where \(x^a = (t, x^i)\), and \(\gamma_{ab}\) with \(r\) taken to be constant is the induced metric on each hypersurface. This foliation will enable us to consider the “initial value formulation” of the vacuum Einstein’s equation, taking \(r\) to be the evolution parameter\(^1\). The appropriate initial data could be chosen as a hypersurface \(\Sigma_c\) located at \(r = r_c\), together with it’s first and second fundamental forms. The first fundamental form is the projection tensor
\[
P_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu,
\]
\(^1\)Regardless of the fact that \(r\) is actually a spacelike coordinate.
where
\[ n_\mu = \frac{1}{\sqrt{AQ}}(dr)_\mu = \left(0, \frac{1}{\sqrt{AQ}}, 0, 0\right), \]
is the unit normal covector. This first fundamental form is closely related to the induced metric, in our coordinate, \( P_{\mu\nu} = \gamma_{\mu\nu} \). The second fundamental form is the extrinsic curvature of the hypersurface
\[ K_{\mu\nu} = \frac{1}{2} \mathcal{L}_n P_{\mu\nu}, \quad (4) \]
also in our choice of the coordinate system we have \( K_{\mu\tau} = K_{\tau\mu} = 0 \). After foliating the background manifold, tensors of type \((0, 2)\) could be decomposed into the following form
\[ S_{\mu\nu} = S_{\rho\sigma} P^\rho_{\mu} P^\sigma_{\nu} + n_\mu (S_{\rho\sigma} P^\rho_\nu n^\sigma) + n_\nu (S_{\rho\sigma} P^\rho_\mu n^\sigma) + n_\mu n_\nu (S_{\rho\sigma} n^\rho n^\sigma), \]
so the equivalent form of the vacuum Einstein’s equation is
\[ G_{\mu\nu} n^\mu n^\nu = 0, \]
\[ G_{\mu\nu} P^\rho_{\mu} P^\nu_{\rho} = 0, \]
\[ G_{\mu\nu} P^\mu_{\rho} P^\nu_{\sigma} = 0, \quad (5) \]
the first two lines in \((5)\) are the constraint equations of the initial data \((P_{\mu\nu}, K_{\mu\nu})\) on the initial hypersurface, and the third line is the evolution equation. According to the Gauss-Codazzi equations the constraint equations could be cast in the following form
\[ \hat{R} + K^{ab} K_{ab} - K^2 = 0, \quad (6a) \]
\[ D_a (K^a_b - \gamma^a_b K) = 0, \quad (6b) \]
where \( \hat{R} \) is the Ricci scalar of \( \Sigma_c \), \( D_a \) is the covariant derivative compatible with \( \gamma_{ab} \). The equations \((6)\) are often referred to as the Hamiltonian and momentum constraints, respectively. Equivalently we could choose \( (\gamma_{ab}, T_{ab}) \) as the initial data, here \( T_{ab} = \gamma_{ab} K - K_{ab} \) is the the Brown-York stress energy tensor, with the unit \( 8\pi G = 1 \). Then the constraint equations can be reformulated as
\[ \mathcal{H} = \hat{R} + T^a_b T_b^a - \frac{T^2}{2} = 0, \quad (7a) \]
\[ \mathcal{P}_b = D_a T^a_b = 0. \quad (7b) \]

Next we turn to the evolution equations \( G_{\mu\nu} P^\mu_{\rho} P^\nu_{\sigma} = 0 \). Rather than list their concrete forms expressed in terms of \( (\gamma_{ab}, T_{ab}) \), let us directly come to the following conclusion, i.e. if we perturb the initial data which generate the background spacetime, and demand that it evolves no singularity in the bulk, then the geometry of the perturbed spacetime should be of Petrov type I. So there are additional constraints of the initial data
\[ \mathcal{E}_{ij} = l^\mu (m_i)^\nu (m_j)^\rho C_{\mu\nu\rho\sigma} \big|_{\Sigma_c} = 0, \quad (8) \]
where $C_{\mu\nu\rho\sigma}$ is the bulk Weyl tensor, and

$$l^\mu = \frac{1}{\sqrt{2}} \left( \frac{\sqrt{A}}{\sqrt{Q}} (\partial_\nu n^\mu) - n^\mu \right),$$

$$k^\mu = -\frac{1}{\sqrt{2}} \left( \frac{\sqrt{A}}{\sqrt{Q}} (\partial_\nu n^\mu) + n^\mu \right),$$

$$(m_i)^\mu = r^{-1} e^{-\frac{\Phi}{2}} \sqrt{A} (\partial_i n^\mu),$$

are a set of Newman-Penrose basis vector fields located at the initial hypersurface, here $Q_c = Q(r_c)$, $A_c = A(r_c, x^i)$. Inserting (9) into (8), we get

$$A_c Q_c C_{titj} + \sqrt{A_c Q_c} C_{ttij} + \sqrt{A_c Q_c} C_{tjit} + C_{i(n)n} = 0,$$

and expressing these projections of the bulk Weyl tensor by $(\gamma_{ab}, T_{ab})$, the additional constraint equations will finally become

$$\mathcal{G}_{ij} = 2 \frac{Q_c}{A_c} C_{titj} + \frac{T^2}{4} \gamma_{ij} - (T^t - 2 \frac{\sqrt{A}}{\sqrt{Q}} D_t) \left( \frac{T}{2} \gamma_{ij} - T_{ij} \right)$$

$$- 2 \frac{\sqrt{Q}}{\sqrt{A}} D_i T^j - T_{ik} T^k - \hat{R}_{ij} = 0,$$

where $\hat{R}_{bca}$, $\hat{R}_{ab}$ represent the Riemann and Ricci tensors of $\Sigma_c$. Up till now we have derived all the constraint equations in our initial value formulation, and in the following sections we will see that, on highly accelerated hypersurface these equations give rise to the Navier-Stokes equation.

### 4 Non-relativistic hydrodynamic expansion and constrained fluctuations

For the background initial data, $\gamma_{ab}^{(B)}$ can be read directly from the line element of initial hypersurface $\Sigma_c$, so we can obtain the the background Brown-York tensor $T_{ab}^{(B)}$, and the only nonzero components are

$$T_{i}^{(B)} = 2 \frac{\sqrt{Q_c}}{r_c}, \quad T_{j}^{(B)} = \left( \frac{\sqrt{Q_c}}{r_c} - \alpha Z \sqrt{Q_c} + \frac{Q_c' \sqrt{A_c}}{2 \sqrt{Q_c}} \right) \delta_{ij},$$

here we have used the notations $Q_c' = Q'(r)|_{r=r_c}$, $A_c'(x^i) = \partial_\nu A(r, x^i)|_{r=r_c}$ for short, and in the rest of this paper the notations $Q_h, Q_h', A_h', A_h''$ will be similar to $Q_c, Q_c', A_c', A_c''$ with $r_c$ replaced by $r_h$, which represents the radial position of the black hole event horizon. Before imposing perturbation to the background metric, we first consider the non-relativistic limit which will be essential when constructing the non-relativistic
hydrodynamics. This could be achieved mathematically by rescaling the $t$ coordinate, $t = \frac{\tau}{\lambda\sqrt{Q_c}}$, then the background metric becomes

$$\gamma_{ab}dx^a dx^b = \frac{1}{A_c} \left( - \frac{1}{\lambda^2} (dx)^2 + r_c^2 e^\Phi dx^i dx^j \right),$$  \hspace{1cm} (13)$$

the reciprocal of the rescaling parameter $\lambda$ can viewed as the speed of light and $\lambda \rightarrow 0$ corresponds to the non-relativistic limit. By explicit calculations we find that the Brown-York tensor $T^a_{(B)}$ and the constraint equations (7) are kept invariant under this rescaling, whilst some coefficients in the additional constraint (11) is changed:

$$C_{ij} = 2\lambda^2 \gamma_{ij} D^2 + \cdot \cdot \cdot$$  \hspace{1cm} (14)$$

Then we take into account the hydrodynamic limit. As is proven in [22], there is a mathematical equivalence between the near horizon limit and the hydrodynamic limit. So let us take a particular initial hypersurface which is close to the black hole horizon at $r = r_h$. This can be realized by introducing a small parameter $\epsilon$ via $r_c - r_h = \epsilon^2$. We would like to link the two small parameters $\epsilon$ and $\lambda$ via $\epsilon = \chi \lambda$, where the constant $\chi$ is employed to balance the dimensionality. This identification makes the non-relativistic limit and the hydrodynamic limit occur simultaneously by taking $\lambda \rightarrow 0$. In this limit the background Brown-York tensor $T^a_{(B)}$ could be expanded as

$$T^\tau_{(B)} = 2\lambda \sqrt{Q'_h r_h} + \cdot \cdot \cdot, \quad T^t_{(B)} = 0,$$

$$T^i_{(B)} = \left( \frac{1}{2} \sqrt{A_h Q'_h} \right) \delta^i_j + \chi \lambda \left( \sqrt{Q'_h r_h} - \frac{1}{2} \alpha Z \sqrt{Q'_h} + \frac{1}{2} Q''_h \right) \delta^i_j + \cdot \cdot \cdot \hspace{1cm} (15)$$

Now let us consider the perturbation theory in the initial value formulation, the most general equations for the fluctuations around the background initial data can be very complicated, so we restrict ourselves to the following non-relativistic hydrodynamic $\lambda$-expansion.

$$\gamma_{ab} = \gamma_{ab}^{(B)} + \sum_{n=1}^{\infty} \gamma_{ab}^{(n)} \lambda^n, \hspace{1cm} (16a)$$

$$T^a_b = T^a_b^{(B)} + \sum_{n=1}^{\infty} \lambda^n T^a_b^{(n)}, \hspace{1cm} (16b)$$

where $(\gamma_{ab}^{(n)}, T^a_b^{(n)})$ represent the fluctuation modes. In the above double expansion the induced geometry of the initial hypersurface is perturbed, by explicit calculations we can list all the expansions of the Christoffel symbol, the Riemann tensor $\hat{R}_{abcd}$ and
of the Ricci tensor $\hat{R}_{ab}$, subjecting to the perturbed metric $\gamma_{ab}$, but for what we are considering, the only necessary expression is $\hat{R}^\tau_{irj} + \hat{R}_{ij}$

$$\hat{R}^\tau_{irj} + \hat{R}_{ij} = \left(\frac{Q_h}{r_h^2} - \frac{2\alpha Z Q_h}{r_h} + \frac{Q_h^2}{r_h}\right) \gamma_{ij}^{(B)} + \cdots. \quad (17)$$

with all of this materials, next we will derive the constraint equations for the fluctuations.

(a) Perturbed Hamiltonian constraint

The middle term in the Hamiltonian constraint (7a) can be further decomposed according to the spacial slicing, i.e.

$$H = \hat{R} + 2 T^i_{\tau} T^i_{\tau} + T^j_{\tau} T^j_{\tau} - \frac{T^2}{2} = 0, \quad (18)$$

according to (16b) and (17), we can get

$$\hat{R} = 2 \frac{Q_h^2}{r_h} + O(\lambda^1),$$

$$T^i_{\tau} T^i_{\tau} = - \frac{\gamma_{ij}^{(0)}}{A_h} T^\tau_{i} T^\tau_{j} + O(\lambda^1),$$

$$\frac{T^2}{2} = \frac{1}{2} \frac{Q_h^2}{A_h} + \frac{4}{r_h} \gamma_{ij}^{(0)} A_h - \alpha Z Q_h^2 + \chi A_h T^i_{i} + O(\lambda^1),$$

$$T^j_{j} T^i_{i} = \frac{1}{2} \frac{Q_h^2}{A_h} + 2 \frac{Q_h^2}{\sqrt{\chi A_h}} A_h - \alpha Z Q_h^2 + \chi A_h T^i_{i} + O(\lambda^1),$$

where $\gamma_{ij}^{(0)} = A_h r^{-2} e^{-\Phi} \delta_{ik}$; so, at the first nontrivial order of the perturbed Hamiltonian constraint, we have

$$\mathcal{H}^{(0)} = 0 \quad \Rightarrow \quad T^\tau_{\tau} = -2 \frac{\chi}{\sqrt{Q_h^2 A_h}} \gamma_{ij}^{(0)} T^\tau_{i} T^\tau_{j}. \quad (20)$$

(b) expansions for momentum and additional constraint

For the general case when both induced metric and and Brown-York tensor receive perturbation the covariant form of the momentum and additional constraint will not be strictly expanded as a power series in $\lambda$, so we need to express these constraint equations by ordinary derivative. Firstly inserting (16b) into (7b)

$$D_a T^a_{\tau} = D_a T^a_{\tau}^{(B)} + \chi D_a T^a_{\tau}^{(1)} + \cdots,$$

$$\Rightarrow \quad \{\begin{array}{l}
\tau\text{-component:} \quad D_a T^a_{\tau} = - \frac{1}{\lambda} \frac{\gamma_{ij}^{(0)}}{A_h} D_i T^i_{\tau}^{(1)} + \cdots, \\
\quad i\text{-component:} \quad D_a T^a_{i} = \lambda \left( D_r T^r_{i}^{(1)} + D_j T^j_{i}^{(1)} - \frac{\alpha \chi}{2 \sqrt{A_h}} e^\Phi \right) + \cdots
\end{array} \quad (21a)$$
so at the first nontrivial order of the momentum constraints, we get

\[ \mathcal{D}^{(-1)}_\gamma = 0 : \gamma^{ij(0)}(\partial_i + 3\sqrt{A_h}\frac{(1 + \alpha r_h)\alpha r_h x_i e^\Phi}{\sqrt{A_h}} )T^{(1)}_j = 0, \]

\[ \mathcal{D}^{(1)} = 0 : \partial_j T^{(1)}_j + \left( \partial_j \partial_k \Phi + 3\sqrt{A_h}\frac{(1 + \alpha r_h)\alpha r_h x_j e^\Phi}{\sqrt{A_h}} \right)T^{(1)}_j \]

\[- \frac{1}{2} T^{(1)}_j \partial_j \Phi - \frac{(1 + \alpha r_h)\alpha r_h e^\Phi}{\sqrt{A_h}} x_j T^{(1)}_j - \chi \frac{(1 + \alpha r_h)\alpha r_h x_j \sqrt{Q'_h e^\Phi}}{2\sqrt{A_h}} = 0, \]

(22b)

due to (16b) and (17)

\[ \frac{T^2}{4} = \frac{1}{4} Q'_h A_h + 2 Q'_h \sqrt{A_h} - \frac{1}{2} \alpha Z Q'_h \sqrt{A_h} + \frac{1}{2} Q''_h A_h + \frac{\sqrt{Q'_h A_h} T^{(1)}_j}{\chi} + \mathcal{O}(\lambda^1), \]

\[ T^i T^k = \left( \frac{1}{4} Q'_h A_h + 2 Q'_h \sqrt{A_h} - \frac{1}{2} \alpha Z Q'_h \sqrt{A_h} + \frac{1}{2} Q''_h A_h \right) \delta^i_j + \frac{\sqrt{Q'_h A_h} T^{(1)}_j + \mathcal{O}(\lambda^1)}{\chi}, \]

\[ T^i \delta^i_j = \lambda \left( \frac{T^{(1)}_j}{2} \delta^i_j - T^{(1)}_j + \chi \sqrt{Q'_h A_h} \right) + \mathcal{O}(\lambda^1), \]

so, the first nontrivial order of the additional constraint (14) will be \( \mathcal{O}(\lambda^0) \), and we have

\[ \mathcal{G}^{(0)} = 0 : T^{(1)}_j = \frac{\chi}{\sqrt{Q'_h A_h}}, 2 \gamma^{i(0)} \left( \frac{1}{A_h} T^{(1)}_i T^{(1)}_j - \frac{1}{\sqrt{A_h}} \zeta_{kj} \right) + \frac{T^{(1)}_j}{2} \delta^i_j, \]

(23)

where we have used the following short-hand notation

\[ \zeta_{kj} = \partial_k T^{(1)}_j - \partial_j \Phi T^{(1)}_j + \frac{1}{2} \delta^i_j \delta^{lm} \partial_i \Phi T^{(1)}_m \]

\[- \frac{(1 + \alpha r_h)\alpha r_h x_j e^\Phi (x_j T^{(1)}_j) - \delta^i_j \delta^{lm} x_l T^{(1)}_m). \]

(24)

Interestingly, from (22a) and (23) we find that

\[ \mathcal{G}^{(0)} = 0 \Rightarrow T^{(1)}_j = -2 \frac{\chi}{\sqrt{Q'_h A_h}} \left( \frac{1}{\sqrt{A_h}} T^{(1)}_i T^{(1)}_j + 3 \frac{(1 + \alpha r_h)\alpha r_h e^\Phi}{\sqrt{A_h}} x_i T^{(1)}_j \right), \]

(25)

then compare (25) with (20) we will come to the following relation

\[ \alpha \cdot \delta^i_j x_i T^{(1)}_j = 0, \]

(26)

from eqs. (20), (22) and (23) we could also establish the continuity and Navier-Stokes equation, but these equations are not expressed in a covariant form, so we would rather interpret the corresponded system as fluid lives in Newtonian spacetime, further the equation (26) implies that there is a vortex in the fluid system.
5 Vortex fluid in flat space

In this section we will study the hydrodynamic equations constructed from the constraints of the fluctuations derived in the last section, and turn our attention only to the strictly expanded constraint equations with \( \alpha \neq 0 \), because the case \( \alpha = 0 \) corresponds to fluctuations around Schwarzschild black hole solution which is somewhat well understood. Since \( \alpha \neq 0 \), eq.(26) yields

\[
\delta^{ij} x_i T^\tau_{\tau j}^{(1)} = 0,
\]

which turns out to be a critical new condition for this case. First of all, the condition (27) implies that eq.(22a) can be rearranged into

\[
\partial_j T^\tau_{\tau j}^{(1)} = 0.
\]

To analyze the “spatial” components of the perturbed momentum constraint, we need to insert (23) and (25) into (22b). The computation is rather complicated, but at the first nontrivial order the equation can be simplified into the following form

\[
\partial_{\tau} T^{\tau (1)}_i + \frac{1}{2} \partial_{j} T^{(1)} + \frac{\delta^{jk}}{r_h^2} \left[ \frac{2}{\sqrt{A_h^2}} T^{\tau (1)}_k \partial_{j} T^{\tau (1)}_i - \partial_{j} \partial_{k} T^{\tau (1)}_i - \frac{2}{\sqrt{A_h^2}} T^{\tau (1)}_j T^{\tau (1)}_k \partial_{i} \Phi \right.
\]

\[
+ \partial_{j} \Phi (\partial_{k} T^{\tau (1)}_i - \partial_{i} T^{\tau (1)}_k) + T^{\tau (1)}_i \partial_{j} \partial_{k} \Phi - \frac{(1 + \alpha r_h) \alpha r_h e^\Phi}{\sqrt{A_h}} \left( 2 x_j \partial_{i} T^{\tau (1)}_k + 2 x_j \partial_{k} T^{\tau (1)}_i \right)
\]

\[
- 2(\delta_{jk} + 2 x_j \partial_{k} \Phi + 2 \frac{(1 + \alpha r_h) \alpha r_h e^\Phi}{\sqrt{A_h}} x_j x_k) T^{\tau (1)}_i \right) + \frac{(1 + \alpha r_h) \alpha r_h e^\Phi}{2 \sqrt{A_h}} \left( T^{(1)} - \frac{Q^2}{r_h^2} \right)x_i = 0.
\]

Next we are going to interpret eqs. (28) and (29) as the continuity and the Cauchy momentum equations in flat Newtonian spacetime, so in what follows, all the indices will be raised and lowered by \( \delta^{ij} \) and its inverse \( \delta_{ij} \). As the last step we would like to introduce the following “holographic dictionary”

\[
T^{\tau (1)}_i = \rho v_i, \quad T^{(1)} = 2p,
\]

where \( \rho, v_i, p \) represent the density distribution, velocity field and the pressure of the fluid system. Under this dictionary eq.(28) becomes the continuity equation

\[
\partial^i (\rho v_i) = 0,
\]

with the density distribution \( \rho = \frac{\sqrt{A_h^2} r_h e^\Phi}{2} \), and eq.(29) becomes the Navier-Stokes equation for the viscous fluid

\[
\rho (\partial_{\tau} v_i + v_j \partial_j v_i) + \partial_{i} p - \eta \partial^j \partial_j v_i = f_i,
\]

with the dynamic viscosity \( \eta = \frac{\sqrt{A_h}}{2} \). For C-metric black hole in isotropic coordinate, if the trivial infinite factor in the black area is taken to be the volume parallel to the
event horizon, then the entropy density should be \( s = \frac{1}{4G\sqrt{A_h(1+\alpha r_h)^2}} \), therefore the viscosity to entropy ratio is

\[
\frac{\eta}{s} = \frac{1}{4\pi}(1 + \alpha r_h Z)^2(1 + \alpha r_h)^2;
\]

we see that for accelerated black hole the shear viscosity to entropy density ratio is affected by the black hole acceleration, when \( \alpha = 0 \), this ratio will be \( \frac{1}{4\pi} \). Then For the right hand side, \( f_i \) represents the body force density

\[
f_i = F(Z) \left( 2\mu x^i \partial_j v_j - 2\mu v_i + \rho v^2 x_i \right) - \frac{(1 + 2\alpha m)\alpha}{\nu r_h} \left[ \left( 2p - \frac{Q_h}{r_h} \right) x_i \frac{2}{2\mu} + \left( 4 + 2F(Z)w^2 + 3\frac{(1 + 2\alpha m)\alpha}{2\mu r_h}w^2 \right) v_i \right],
\]

in the last force term, we have used the short-hand notation

\[
F(Z) = \frac{(Z - 1)(3\alpha r_h Z + 3\alpha r_h + 2)}{w^2(1 + \alpha r_h)};
\]

it is clear that the second line in eq.\( (32) \) is a pure inertia force because of the overall factor \( \alpha \). As for the first line in eq.\( (32) \), we could recognize the first term as a kind of Coriolis force and the second term as a linear resistance force, so only the last term on the first line remains a mystery. Nevertheless one can recognize that this last force term is proportional to the kinetic energy density of the fluid component, which also sounds good. Anyway we should not forget the role of \( (27) \), which is now rewritten in terms of the holographic dictionary as

\[
x^i v_i = 0,
\]

i.e. the velocity field of the fluid is always perpendicular to its spacial displacement. This is nothing but a vortex condition. In other words, the fluid we constructed on the flat Newtonian spacetime is precisely a holographic vortex, when \( w \to \infty \), which correspond to the region near the conical singularity, the asymptotic behaviour of the velocity field is

\[
v \sim w^{-\frac{2}{1 + \alpha r_h}} \sin \left( \frac{2\sqrt{\alpha r_h}}{1 + \alpha r_h} \log w \right),
\]

it can be seen from that the velocity field does not vanish in the far zone. This effect can be interpreted by the presence of the Coriolis-like force term in the fluid equation.

### 6 Conclusion

Unlike the ordinary static black holes with maximally symmetric horizons, the C-metric black hole represents two black holes under constant proper acceleration. The acceleration of the black hole squeezes the horizon surface, leaving less symmetries than the non-accelerating black holes. Our construction reveals that Gravity/Fluid
correspondence can be realized in terms of Petrov I boundary condition even for black holes with less symmetries than the usual static black holes with maximally symmetric horizons.

To be more concrete, we have realized a fluid system from the vacuum C-metric black hole solution, which lives on a flat Newtonian spacetime, and possess non-constant but stationary density distributions and kinematic viscosities, so it is compressible viscous fluids subject to extra body forces. Compared with previous studies on Gravity/Fluid correspondences, the present work differs in two major aspects. The first difference of our present work lies in that the extra body forces arising from the C-metric black hole case can have more appropriate physical interpretations. For the flat space fluid system, the extra force are consisted of an inertial force term, a Coriolis-like term, a linear resistance term and a term proportional to the kinetic energy density of the fluid. It is remarkable that the combination of all these complicated force terms gives rise to a pure vortex behavior for this case. The second difference lies in that the shear viscosity to entropy density ratio of the corresponded system is subjected to the black hole acceleration. Besides, just as those anisotropic theories which violate the KSS bound [51–53], for the accelerated black hole with less symmetries, the ratio is also lower than the bound in the region around the conical singularity.

Before ending, let us stress that the Gravity (with curved horizon)/Flat space fluid correspondence realized in [47, 48] and the present work seems to rely on the conformal flatness of the horizon surface of the background geometry. However, going through the details of the construction, it is evident that such correspondence only requires the existence of a map from the near horizon hypersurface to the flat space, be it conformal or not. Therefore, it is tempting to consider other cases with more complicated, less symmetric black hole backgrounds. Doing so one might be able to get more general fluid systems with less constraints on the density distributions and/or kinematic viscosities. For this purpose, the black ring geometry in 5 dimensions may be a good choice as background geometry. We leave the study of such backgrounds to later works.

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