GENERALIZED HARDY TYPE AND CAFFARELLI-KOHN-NIRENBERG TYPE INEQUALITIES ON FINSLER MANIFOLDS

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ABSTRACT. In this paper we derive both local and global geometric inequalities on general Riemannian and Finsler manifolds and prove generalized Caffarelli-Kohn-Nirenberg type and Hardy type inequalities on Finsler manifolds, illuminating curvatures of both Riemannian and Finsler manifolds influence geometric inequalities.

1. INTRODUCTION

It is well-known that Hardy type inequalities have been widely used in analysis and differential equations. In [3] Caffarelli, Kohn and Nirenberg proved rather general interpolation inequalities with weights. Recently in [9], Wei and Li used comparison theorems in Riemannian geometry to prove some sharp generalized Hardy type and Caffarelli-Kohn-Nirenberg type inequalities on Riemannian manifolds. Some applications of generalized Hardy type inequalities in $p$-harmonic geometry have been studied in [4].

Finsler geometry, as the natural generalization of Riemannian geometry, has been a very active field in differential geometry and appears in a broad spectrum of contexts. (e.g., two different Finsler metrics, the Kobayashi metric and the Carathéodory metric appear very naturally in the theory of several complex variables.) The main purpose of the present paper is to, on the one hand, give a local and two $L^p$ versions of the results in [9] so that the inequalities work on every Riemannian manifold and in a wider class, and on the other hand, generalize their results from Riemannian manifolds to Finsler manifolds. We use Hessian and Laplacian comparison theorems in Finsler Geometry by constructing appropriate vector fields. It should be pointed out here that the volume form on a Riemannian manifold is uniquely determined by the given Riemannian metric, while there are different choices of volume forms for Finsler metrics. The frequently used volume
forms in Finsler geometry are the so-called Busemann-Hausdorff volume form and Holmes-Thompson volume form, and in \[10\] \[11\] we introduce the extreme volume forms \( dV_{\text{ext}} \) (include the maximal and minimal volume forms, cf. (2.3) and (2.9)) for Finsler manifolds which also play the important role in Finsler geometry. In this paper we shall mainly use the extreme volume forms.

To state our results we need some notions from Finsler geometry, for details see §2. Throughout this paper, unless otherwise stated, we let \((M, F)\) be a complete Finsler manifold with finite uniformity constant \(\mu_F\) (cf. (2.1)), \(\text{Cut}(x_0)\) be the cut locus of a fixed point \(x_0\), and \(\Omega \subset M \setminus \text{Cut}(x_0)\) be a domain in \(M\). It should be pointed out here that in general there are three completeness for Finsler manifolds: forward complete, backward complete and complete (i.e., both forward and backward complete), and they are equivalent when \(\mu_F < \infty\). In this situation, the distance function \(r = d_F(x_0, \cdot) : \Omega \to \mathbb{R}\) from \(x_0\) is smooth on \(\Omega \setminus \{x_0\}\), and thus the gradient vector field \(\nabla r\) of \(r\) (with respect to Finsler metric \(F\)) is also smooth on \(\Omega \setminus \{x_0\}\). We usually call \(\nabla r\) the radial vector field with respect to \(x_0\). We call \(x_0 \in M\) a pole, if the exponential map \(\exp_{x_0} : T_xM \to M\) is a diffeomorphism. We say that \(M\) has nonpositive (resp. nonnegative) radial flag curvature at \(x_0\) if flag curvature \(K(\nabla r; P)\) of flag \((\nabla r; P)\) whose flag pole is a radial vector is nonpositive (resp. nonnegative) for every plane \(P\) (cf. (2.5)). Similarly, we say that \(M\) has nonpositive (resp. nonnegative) radial Ricci curvature \(\text{Ric}(\nabla r)\) at \(x_0\) if \(\text{Ric}(\nabla r) \leq 0\) (resp. \(\geq 0\)) (cf. (2.6)). In this paper we first derive both local and global Geometric Inequalities 4.1 and 4.2 on every Riemannian manifold and Finsler manifold (cf. (4.1.a), (4.1.b), (4.2.a), (4.2.b)). We then prove generalized Caffarelli-Kohn-Nirenberg type and Hardy type inequalities on Finsler manifolds. The main results of this paper are the following:

**Theorem 1.1** Let \((M, F)\) be an \(n\)-dimensional complete Finsler manifold with finite uniformity constant \(\mu_F\). Let \(\text{Cut}(x_0)\) be the cut locus of a fixed point \(x_0\), and \(\Omega \subset M \setminus \text{Cut}(x_0)\) be a domain in \(M\). Suppose that the radial flag curvature \(K(\nabla r; \cdot)\) or radial Ricci curvature \(\text{Ric}(\nabla r)\) of \(M\) satisfies one of the following three conditions:

(i) \(0 \leq \text{Ric}(\nabla r)\) and \(n \leq a + b + 1\);

(ii) \(K(\nabla r; \cdot) \leq 0\) and \(a + b + 1 \leq n\);

(iii) \(K(\nabla r; \cdot) = 0\) and \(a, b \in \mathbb{R}\) are any constants.

Then for any \(u \in C_0^\infty(\Omega \setminus \{x_0\})\), the following Caffarelli-Kohn-Nirenberg type inequality holds:

\[
\int_{\Omega} |u|^2 dV_{\text{ext}} \leq \hat{\mu}_F^{\frac{n+1}{2}} \left( \int_{\Omega} |u|^p dV_{\text{ext}} \right)^{\frac{2}{p}} \left( \int_{\Omega} \left( \frac{F(\nabla u)}{r^b} \right)^q dV_{\text{ext}} \right)^{\frac{1}{q}}.
\]
In particular, if \( M \) has a pole \( x_0 \) or \( \text{Cut}(x_0) \) is empty in (i), or \( M \) is simply connected in (ii) or (iii), then for any \( u \in C_0^\infty(M \setminus \{x_0\}) \),

\[
\int_M \frac{|u|^2}{r^{a+b+1}} dV \leq \hat{\mu}_F \cdot \left( \int_M \frac{|u|^p}{r^{ap}} dV \right)^{\frac{1}{p}} \left( \int_M \frac{(F(\nabla u))^q}{r^{bq}} dV \right)^{\frac{1}{q}},
\]

where \( \hat{\mu}_F = \mu F^2 \cdot \left( \frac{2}{n-a-b-1} \right) \), \( p \in \left[ 1, \infty \right) \), \( \frac{1}{p} + \frac{1}{q} = 1 \), and if \( p = \infty \), \( \int_M \frac{|u|^p}{r^{ap}} dV \right)^{\frac{1}{p}} \) stands for the supreme of \( \frac{|u|^p}{r^{ap}} \).

This result is new, even when \( M \) is a Riemannian manifold:

**Corollary 1.1** Let \( M \) be a complete \( n \)-dimensional Riemannian manifold with the volume element \( dv \). Fix \( x_0 \in M \) let \( \Omega \subset M \setminus \text{Cut}(x_0) \) be a domain in \( M \). Suppose that radial curvature \( K_r \), or radial Ricci curvature \( \text{Ric}_{rad} \) of \( \Omega \) satisfies one of the following three conditions:

(i) \( 0 \leq \text{Ric}_{rad} \) and \( n \leq a + b + 1 \);

(ii) \( K_r \leq 0 \) and \( a + b + 1 \leq n \);

(iii) \( K_r = 0 \) and \( a, b \in \mathbb{R} \) are any constants.

Then for any \( u \in C_0^\infty(\Omega \setminus \{x_0\}) \), the following Caffarelli-Kohn-Nirenberg type inequality holds:

\[
C \cdot \int_\Omega \frac{|u|^2}{r^{a+b+1}} dv \leq \left( \int_\Omega \frac{|u|^p}{r^{ap}} dv \right)^{\frac{1}{p}} \left( \int_\Omega \frac{|\nabla u|^q}{r^{bq}} dv \right)^{\frac{1}{q}}.
\]

In particular, if in (i) \( \text{Cut}(x_0) = \emptyset \), or in (ii) or in (iii) \( \pi_1(M) = 0 \), then for any \( u \in C_0^\infty(M \setminus \{x_0\}) \),

\[
C \cdot \int_M \frac{|u|^2}{r^{a+b+1}} dv \leq \left( \int_M \frac{|u|^p}{r^{ap}} dv \right)^{\frac{1}{p}} \left( \int_M \frac{|\nabla u|^q}{r^{bq}} dv \right)^{\frac{1}{q}},
\]

where \( C = C(a,b) = \left| \frac{n-a-b-1}{2} \right| \), \( p \in \left[ 1, \infty \right) \), and \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Theorem 1.2** Let \((M, F), \Omega \) and curvature conditions as in Theorem 1.1 (i), (ii), (iii) Then for any \( u \in C_0^\infty(\Omega \setminus \{x_0\}) \), the following Caffarelli-Kohn-Nirenberg type inequality holds:

\[
\int_\Omega \frac{|u|^p}{r^{a+b+1}} dV \leq \hat{\mu}_F \cdot \left( \int_\Omega \frac{|u|^p}{r^{ap}} dV \right)^{\frac{1}{p}} \left( \int_\Omega \frac{(F(\nabla u))^q}{r^{bq}} dV \right)^{\frac{1}{q}}.
\]
In particular, if $M$ has a pole $x_0$ or $\text{Cut}(x_0)$ is empty in (i), or $M$ is simply connected in (ii) or (iii), then for any $u \in C_0^\infty(M\setminus\{x_0\})$,

$$\int_M \frac{|u|^p}{r^{a+b+1}}dV_{\text{ext}} \leq \tilde{\mu}_F^{\frac{n+1}{p}} \cdot \left( \int_M \frac{|u|^p}{r^{aq}}dV_{\text{ext}} \right)^\frac{1}{q} \left( \int_M \frac{|\nabla u|^p}{r^{bp}}dV_{\text{ext}} \right)^\frac{1}{p},$$

where $\tilde{\mu}_F^{\frac{n+1}{p}} = \mu_F^{\frac{n+1}{p}} \cdot |\frac{p}{n-a-b-1}|$, $p \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Similarly, when $M$ is a Riemannian manifold, we have the following new result:

**Corollary 1.2** Let $M, \Omega$ and curvature conditions as in Corollary 1.1 (i), (ii), (iii) Then for any $u \in C_0^\infty(\Omega\setminus\{x_0\})$, the following Caffarelli-Kohn-Nirenberg type inequality holds:

$$\tilde{\mathcal{C}} \cdot \int_\Omega \frac{|u|^p}{r^{a+b+1}}dv \leq \left( \int_\Omega \frac{|u|^p}{r^{aq}}dv \right)^\frac{1}{q} \left( \int_\Omega \frac{|\nabla u|^p}{r^{bp}}dv \right)^\frac{1}{p}.$$

In particular, if in (i) $\text{Cut}(x_0) = \emptyset$, or in (ii) or in (iii) $\pi_1(M) = 0$, then for any $u \in C_0^\infty(M\setminus\{x_0\})$,

$$\tilde{\mathcal{C}} \cdot \int_M \frac{|u|^p}{r^{a+b+1}}dv \leq \left( \int_M \frac{|u|^p}{r^{aq}}dv \right)^\frac{1}{q} \left( \int_M \frac{|\nabla u|^p}{r^{bp}}dv \right)^\frac{1}{p},$$

where $\tilde{\mathcal{C}} = \tilde{\mathcal{C}}(a, b) = \left|\frac{n-a-b-1}{p}\right|$, $p \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$.

When $p = q = 2$, the above two Theorems 1.1 and 1.2 meet and give their $L^2$ version:

**Theorem 1.3** Let $(M, F)$ be an $n$-dimensional complete Finsler manifold with finite uniformity constant $\mu_F$. Fix $x_0 \in M$ let $\Omega \subset M\setminus\text{Cut}(x_0)$ be a domain. Suppose that the radial flag curvature $K(\nabla r; \cdot)$ or radial Ricci curvature $\text{Ric}(\nabla r)$ satisfies one of the following three conditions:

(i) $0 \leq \text{Ric}(\nabla r)$ and $n \leq a + b + 1$;

(ii) $K(\nabla r; \cdot) \leq 0$ and $a + b + 1 \leq n$;

(iii) $K(\nabla r; \cdot) = 0$ and $a, b \in \mathbb{R}$ are any constants.

Then for any $u \in C_0^\infty(\Omega\setminus\{x_0\})$, the following Caffarelli-Kohn-Nirenberg type inequality holds:

$$\int_\Omega \frac{|u|^2}{r^{a+b+1}}dV_{\text{ext}} \leq \tilde{\mu}_F^{\frac{n+1}{2}} \cdot \left( \int_\Omega \frac{|u|^2}{r^{2a}}dV_{\text{ext}} \right)^\frac{1}{2} \left( \int_\Omega \frac{|F(\nabla u)|^2}{r^{2b}}dV_{\text{ext}} \right)^\frac{1}{2}.$$
In particular, if $M$ has a pole $x_0$ or $\text{Cut}(x_0)$ is empty in (i), or $M$ is simply connected in (ii) or (iii), then for any $u \in C_0^\infty(M \setminus \{x_0\})$,

$$
(1.10) \quad \int_M \frac{|u|^2}{r^{a+b+1}} dV_{\text{ext}} \leq \mu_F^{\frac{a+1}{p}} \cdot \left( \int_M \frac{|u|^2}{r^{2a}} dV_{\text{ext}} \right)^{\frac{1}{2}} \left( \int_M \frac{(F(\nabla u))^2}{r^{2b}} dV_{\text{ext}} \right)^{\frac{1}{2}}.
$$

where $\mu_F^{\frac{a+1}{p}} = \mu_F^{\frac{a+1}{p'}} \cdot \left| \frac{2}{n-a-b-1} \right|$.

**Remark** When $F$ is Riemannian, one has $\mu_F = 1$, and we recapture the corresponding results for Riemannian manifolds $[9, 14]$.

As applications, we obtain embedding theorems for weighted Sobolev spaces of functions on Finsler manifolds (cf. Theorem 7.1) and geometric differential-integral inequalities on Finsler manifolds (cf. Theorem 7.2), generalizing the work in $[9]$ in Riemannian manifolds. We then focus our study on generalized Hardy type inequalities on Finsler manifolds by using the double limiting technique in $[9]$, and extend the density argument in $[5]$. We introduced the notion of the space $W_{F,0}^{1,p}(M)$ on a Finsler manifold to be the completion of smooth compactly supported functions $u \in C_0^\infty(M)$ with respect to the “norm”

$$
(1.11) \quad \|u\|_{W_{F,0}^{1,p}(M)} := \left( \int_M (|u|^p + (F(\nabla u))^p) dV_{\text{ext}} \right)^{\frac{1}{p}}.
$$

It is easy to verify that $\| \cdot \|_{W_{F,0}^{1,p}(M)}$ satisfies the following properties:

(i) (Positive definiteness) $\|u\|_{W_{F,0}^{1,p}(M)} > 0, \forall u \in W_{F,0}^{1,p}(M)$, and $\|u\|_{W_{F,0}^{1,p}(M)} = 0$ if and only if $u = 0$ almost everywhere.

(ii) (Positive homogeneity) $\|\lambda u\|_{W_{F,0}^{1,p}(M)} = \lambda\|u\|_{W_{F,0}^{1,p}(M)}, \forall \lambda > 0$ and $u \in W_{F,0}^{1,p}(M)$.

(iii) (Triangle inequality) $\|u + v\|_{W_{F,0}^{1,p}(M)} \leq \|u\|_{W_{F,0}^{1,p}(M)} + \|v\|_{W_{F,0}^{1,p}(M)}, \forall u, v \in W_{F,0}^{1,p}(M)$.

We note here that $\|\lambda u\|_{W_{F,0}^{1,p}(M)} = |\lambda| \cdot \|u\|_{W_{F,0}^{1,p}(M)}$ does not hold for general Finsler metric, that is to say, $\| \cdot \|_{W_{F,0}^{1,p}(M)}$ is not a genuine norm for general Finsler manifold. Nevertheless we have $\|\lambda u\|_{W_{F,0}^{1,p}(M)} \leq \mu_F^{\frac{1}{p'}} |\lambda| \cdot \|u\|_{W_{F,0}^{1,p}(M)}$, and since we assume $\mu_F < \infty$, we may define Cauchy sequence $\{u_i\} \subset C_0^\infty(M)$ with respect to $\| \cdot \|_{W_{F,0}^{1,p}(M)}$ in the usual way, and thus $W_{F,0}^{1,p}(M)$ is well-defined. We say $u \in L^p(M)$ in a Finsler sense, denoted by $u \in L^p(M)$ if $\int_M \frac{|u|^p}{r^p} dV_{\text{ext}} < \infty$. In particular, we have

**Theorem 1.4** Let $(M, F)$ be an $n$-dimensional complete Finsler manifold with non-positive radial flag curvature at the pole $x_0 \in M$ and with finite uniformity constant $\mu_F$. Then for any $u \in W_{F,0}^{1,p}(M)$ and $1 < p < n$, the following Hardy type inequality
holds:
\[(1.12) \quad \int_M |u|^p r^p dV_{\text{ext}} \leq \hat{\mu}_F^{\frac{n+p}{p}} \cdot \int_M (F(\nabla u))^p dV_{\text{ext}},\]
where \(\hat{\mu}_F^{\frac{n+p}{p}} = \mu_F^{\frac{n+p}{p}} \cdot \left(\frac{p}{n-p}\right)^p\).
Furthermore, \(\frac{p}{r} \in L^p(M)\) in a Finsler sense.

This recaptures a result of Wei-Li [9, Theorem 1, Corollary 1.2] (cf. Corollary 8.2), when \(M\) is a Riemannian manifold.

**Theorem 1.5** Let \((M, F)\) be an \(n\)-dimensional complete Finsler manifold with non-negative radial Ricci curvature at the pole \(x_0 \in M\), and with finite uniformity constant \(\mu_F\). Then for any \(u \in W^{1,p}_{F,0}(M), \frac{u}{r} \in L^p_F(M)\) and \(p > n\), the following Hardy type inequality holds:
\[(1.13) \quad \int_M |u|^p r^p dV_{\text{ext}} \leq \hat{\mu}_F^{\frac{n+p}{p}} \cdot \int_M (F(\nabla u))^p dV_{\text{ext}},\]
where \(\hat{\mu}_F^{\frac{n+p}{p}} = \mu_F^{\frac{n+p}{p}} \cdot \left|\frac{p}{n-p}\right|^p\).

This recaptures a theorem of Chen-Li-Wei [5, Theorem 5] (cf. Corollary 8.4), when \(M\) is a Riemannian manifold. Furthermore, the assumption \(\frac{u}{r} \in L^p_F(M)\) cannot be dropped, or a counter-example is constructed in Section 5 in Chen-Li-Wei [5].

**Corollary 1.3** Let \((M, F)\) be an \(n\)-dimensional complete Finsler manifold with vanishing flag curvature at the pole \(x_0 \in M\), and with finite uniformity constant \(\mu_F\). Then (i) for any \(u \in W^{1,p}_{F,0}(M), \frac{u}{r} \in L^p_F(M)\) and \(1 < p < \infty\), or (ii) for any \(u \in W^{1,p}_{F,0}(M)\), and \(p < n\) the following Hardy type inequality holds:
\[(1.14) \quad \int_M |u|^p r^p dV_{\text{ext}} \leq \hat{\mu}_F^{\frac{n+p}{p}} \cdot \int_M (F(\nabla u))^p dV_{\text{ext}},\]
where \(\hat{\mu}_F^{\frac{n+p}{p}} = \mu_F^{\frac{n+p}{p}} \cdot \left|\frac{p}{n-p}\right|^p\).

Theorem 1.5 is in contrast to Theorem 1.4, in which \(\frac{u}{r} \in L^p(M)\) in a Finsler sense is a conclusion, rather than an assumption. The above theorems illuminate that curvatures of both Riemannian and Finsler manifolds influence geometric inequalities such as generalized Hardy Type and Caffarelli-Kohn-Nirenberg Type inequalities.

2. **Finsler Geometry**

In this section we shall recall some basic notations and formulas in Finsler geometry, for details we refer to [10, 12, 13]. Let \((M, F)\) be a Finsler \(n\)-manifold with Finsler metric \(F : TM \to [0, \infty)\), where \(TM\) is the tangent bundle of \(M\). Let
(x, y) = (x^i, y^j) be local coordinates on TM, and \( \pi : TM \setminus \{0\} \to M \) be the natural projection. Unlike in the Riemannian case, in general Finsler quantities are functions defined on TM rather than M. The **fundamental tensor** \( g_{ij} \) and the **Cartan tensor** \( C_{ijk} \), \( 1 \leq i, j, k \leq n \) are defined by

\[
g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}, \quad C_{ijk}(x, y) := \frac{1}{4} \frac{\partial^3 F^2(x, y)}{\partial y^i \partial y^j \partial y^k}.
\]

According to [6], the pulled-back bundle \( \pi^*TM \) admits a unique affine connection, called the **Chern connection**. Its connection forms \( \omega^i_j \) are characterized by the following two structure equations:

- **Torsion freeness:**
  \[
dx^j \wedge \omega^i_j = 0;
\]

- **Almost \( g \)-compatibility:**
  \[
dg_{ij} - g_{kj} \omega^k_i - g_{ik} \omega^k_j = 2C_{ijk}(dy^k + N^k_l dx^1),
\]
  where \( N^k_l \) are real-valued functions determined by \( N^k_l dx^1 = y^l \omega^k_l \). It is easy to know that torsion freeness is equivalent to the absence of \( dy^k \) terms in \( \omega^i_j \); namely,

\[
\omega^i_j = \Gamma^i_j dx^k,
\]
  together with the symmetry

\[
\Gamma^i_{jk} = \Gamma^i_{kj}.
\]

Let \( V = V^i \partial/\partial x^i \) be a non-vanishing vector field on an open subset \( U \subset M \). One can introduce a Riemannian metric \( \tilde{g}^V(\cdot, \cdot) = \langle \cdot, \cdot \rangle_V \) in the direction of \( V \), and a linear connection \( \nabla^V \) on the tangent bundle over \( U \) as follows:

\[
\tilde{g}^V(X, Y) = \langle X, Y \rangle_V := X^i Y^j g_{ij}(x, V), \quad \forall X = X^i \frac{\partial}{\partial x^i}, Y = Y^j \frac{\partial}{\partial x^j};
\]

\[
\nabla^V_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} := \Gamma^k_{ij}(x, V) \frac{\partial}{\partial x^k}.
\]

From the torsion freeness and almost \( g \)-compatibility of Chern connection we have

\[
(2.1) \quad \nabla^V_X Y - \nabla^V_Y X = [X, Y],
\]

\[
(2.2) \quad X \cdot \langle Y, Z \rangle_V = \langle \nabla^V_X Y, Z \rangle_V + \langle Y, \nabla^V_X Z \rangle_V + 2C_V(\nabla^V_X V, Y, Z),
\]

here \( C_V \) is defined by

\[
C_V(X, Y, Z) = X^i Y^j Z^k C_{ijk}(x, V).
\]

By definition \( C_V(X, X, Z) \) is totally symmetric with respect to \( X, Y, Z \), and by Euler’s lemma it also satisfies

\[
(2.3) \quad C_V(V, X, Y) = 0.
\]
In view of (2.1)-(2.3) we see that the Chern connection $\nabla^V$ and the Levi-Civita connection $\tilde{\nabla}^V$ of $\tilde{g}^V(\cdot, \cdot) = \langle \cdot, \cdot \rangle^V$ are related by

$$(2.4) \quad \langle \nabla^V_X Y, Z \rangle^V = \langle \tilde{\nabla}^V_X Y, Z \rangle^V - C^V(\nabla^V_X V, Y, Z) - C^V(\nabla^V_Y V, X, Z) + C^V(\nabla^V_Z V, X, Y).$$

The Chern curvature $R^V(X, Y)Z$ for vector fields $X, Y, Z$ on $\mathcal{U}$ is defined by

$$R^V(X, Y)Z := \nabla^V_X \nabla^V_Y Z - \nabla^V_Y \nabla^V_X Z - \nabla^V_{[X, Y]} Z.$$ 

In the Riemannian case this curvature does not depend on $V$ and coincides with the Riemannian curvature tensor. Let $P \subset T_xM$ be a 2-plane and $V \in P$ be a nonzero vector. We call the pair $(V; P)$ a flag with pole $V$. The flag curvature $K(V; P)$ of given flag is defined as follows:

$$(2.5) \quad K(V; P) = K(V; W) := \frac{\langle R^V(V, W)W, V \rangle^V}{\langle V, V \rangle^V - \langle V, W \rangle^V}. $$

Here $W$ is a tangent vector such that $V, W$ span the 2-plane $P$ and $V \in T_x M$ is extended to a geodesic field, i.e., $\nabla^V_v V = 0$ near $x$. The Ricci curvature $\text{Ric}(V)$ of $V \in T_xM$ is defined by

$$\text{Ric}(V) = \sum_{i=1}^n K(V; E_i),$$

here $E_1, \cdots, E_n$ is the $\tilde{g}^V$-orthonormal basis for $T_xM$.

Let $f : M \to \mathbb{R}$ be a smooth function on $M$. The gradient $\nabla f$ of $f$ is defined by

$$df(X) = \langle \nabla f, X \rangle^f \quad \forall X \in \Gamma(TM)$$

whenever $df \neq 0$, and $\nabla f = 0$ where $df = 0$. Let $\mathcal{U} = \{x \in M : \nabla f |_x \neq 0\}$. We define the Hessian $\text{Hess}(f)$ of $f$ on $\mathcal{U}$ as follows:

$$\text{Hess}(f)(X, Y) := X(Y(f)) - \nabla^V_X Y(f), \quad \forall X, Y \in \Gamma(T\mathcal{U}).$$

It is known that $\text{Hess}(f)$ is symmetric, and it can be rewritten as (see [13])

$$\text{Hess}(f)(X, Y) = \langle \nabla^V_X Y, f \rangle^f.$$

It should be noted that the notion of Hessian defined here is different from that in [8]. In that case $\text{Hess}(f)$ is in fact defined by

$$\text{Hess}(f)(X, X) = X(X(f)) - \nabla^V_X X(f),$$

and there is no definition for $\text{Hess}(f)(X, Y)$ if $X \neq Y$. The advantage of our definition is that $\text{Hess}(f)$ is a symmetric bilinear form and we can treat it by using the theory of symmetric matrix.

The following Hessian comparison theorem for distance function $r = d_F(x_0, \cdot)$ from $x_0$ first was proved in [13] with pointwise curvature bounds, and it is easy to
see from the proof that the pointwise curvature bounds can be weakened to radial curvature bounds. More precisely, we have the following:

**Theorem 2.1 (Hessian Comparison Theorem under Radial Curvature Assumptions)** Let \((M, F)\) be an \(n\)-dimensional complete Finsler manifold, and \(x_0 \in M\).

(1) Suppose that \(M\) has nonpositive radial flag curvature \(K(\nabla r; \cdot) \leq 0\), then for any tangent vector field \(X\) on \(M\) the following inequality holds whenever \(r\) is smooth:

\[
\text{Hess}(r)(X, X) \geq \frac{1}{r} \left( \langle X, X \rangle_{\nabla r} - \langle \nabla r, X \rangle^2_{\nabla r} \right);
\]

(2) Suppose that \(M\) has nonnegative radial Ricci curvature \(\text{Ric}(\nabla r) \geq 0\), then the following inequality holds whenever \(r\) is smooth:

\[
\sum_{i=1}^n \text{Hess}(r)(E_i, E_i) \leq \frac{n-1}{r},
\]

where \(E_1, \ldots, E_n\) is the local \(\langle \cdot, \cdot \rangle_{\nabla r}\)-orthonormal frame on \(M\).

Define

\[
dV_{\text{max}} = \sigma_{\text{max}}(x) dx^1 \wedge \cdots \wedge dx^n
\]

and

\[
dV_{\text{min}} = \sigma_{\text{min}}(x) dx^1 \wedge \cdots \wedge dx^n,
\]

(2.8) where

\[
\sigma_{\text{max}}(x) := \max_{y \in T_x M \setminus \{0\}} \sqrt{\det \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)_y},
\]

\[
\sigma_{\text{min}}(x) := \min_{y \in T_x M \setminus \{0\}} \sqrt{\det \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)_y}.
\]

(Since \(F(x, \lambda y) = \lambda F(x, y)\) for \(\forall \lambda > 0 \Rightarrow \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle_y := g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}\) is homogeneous of degree 0 in \(y\), both \(\max_{y \in T_x M \setminus \{0\}}\) and \(\min_{y \in T_x M \setminus \{0\}}\) in (2.8) are taken over a compact set and hence can be realized.) Then it is easy to check that the \(n\)-forms \(dV_{\text{max}}\) and \(dV_{\text{min}}\) as well as the function \(\frac{\sigma_{\text{max}}}{\sigma_{\text{min}}}\) are well-defined on \(M\). We call \(dV_{\text{max}}\) and \(dV_{\text{min}}\) the maximal volume form and the minimal volume form of \((M, F)\), respectively\[10, 12\]. Both maximal volume form and minimal volume form are called extreme volume form, and we shall denote by

\[
dV_{\text{ext}} = \text{either} \ dV_{\text{max}} \text{ or } dV_{\text{min}}
\]

(2.9)

The uniformity function \(\mu : M \to \mathbb{R}\) is defined by

\[
\mu(x) = \max_{y, z, w \in T_x M \setminus \{0\}} \frac{\langle w, w \rangle_y}{\langle w, w \rangle_z}.
\]

(2.10)
Then
\[
\mu_F = \sup_{x \in \tilde{M}} \mu(x)
\]
is called the uniformity constant. We always assume \(\mu_F < \infty\) throughout this paper. It is known that
\[
\mu_F^{-1} F^2(w) \leq \langle w, w \rangle_y \leq \mu_F F^2(w).
\]
Furthermore, we have \([10, 11]\)
\[
\frac{\sigma_{\text{max}}}{\sigma_{\text{min}}} \leq \frac{n}{\mu_F},
\]
and thus
\[
dV_{\text{max}} \leq \mu_F^n dV_{\text{min}}.
\]

3. The Induced Riemannian Metric \(\tilde{g}\) in the Radial Direction \(T\)

In this section we introduce the induced Riemannian metric \(\tilde{g}\) by the Finsler metric \(F\) in the radial direction \(T\). This will play an important role in the proof of main theorems. Let \((M, F)\) be a complete Finsler manifold, and \(x_0 \in M\). Then the distance function \(r = r(x) = d_F(x_0, x)\) is smooth on \(\tilde{M}\), where
\[
\tilde{M} := M \setminus \{x_0\} \cup \text{Cut}(x_0)
\]
and the radial vector field \(T = \nabla r\) is also smooth on \(\tilde{M}\). It is also well-known that \(T\) is the unit geodesic field, i.e., \(F(T) = 1\) and \(\nabla^2_T T = 0\). Now we define
\[
\tilde{g}(\cdot, \cdot) = \langle \cdot, \cdot \rangle_T.
\]
Then \(\tilde{g}\) is the Riemannian metric on \(\tilde{M}\) induced by \(F\) in the radial direction \(T\). It is clear from the definition of gradient that the gradient \(\tilde{\nabla} r\) of \(r\) with respect to \(\tilde{g}\) is just the radial vector field, i.e.,
\[
\tilde{\nabla} r = \nabla r = T.
\]
Thus by (2.3), (2.4) and (2.7) the Laplacian \(\tilde{\Delta} r\) of \(r\) with respect to \(\tilde{g}\) is
\[
\tilde{\Delta} r = \tilde{\text{div}}(\tilde{\nabla} r) = \sum_{i=1}^{n} \tilde{\text{Hess}}(r)(E_i, E_i) = \sum_{i=1}^{n} \langle \tilde{\nabla}_{E_i} T, E_i \rangle_T
\]
\[
= \sum_{i=1}^{n} \langle \nabla^T_{E_i} T, E_i \rangle_T = \sum_{i=1}^{n} \text{Hess}(r)(E_i, E_i),
\]
here \(\tilde{\text{div}}\) and \(\tilde{\text{Hess}}\) are the divergence and Hessian with respect to \(\tilde{g}\), and \(E_1, \ldots, E_n\) is the local \(\tilde{g}\)-orthonormal field on \(\tilde{M}\). Therefore, by Theorem 2.1 and (3.3) we clearly have the following
Theorem 3.1 (Laplacian Comparison Theorem) With the same notations as above.

1. If $M$ has nonpositive radial flag curvature at $x_0$, then $\Delta r \geq \frac{n-1}{r}$.
2. If $M$ has nonnegative radial Ricci curvature at $x_0$, then $\Delta r \leq \frac{n-1}{r}$.

Let $u : M \to \mathbb{R}$ be a smooth function on $M$. By definition, the gradient $\nabla u$ of $u$ (with respect to $F$) and the gradient $\tilde{\nabla} u$ of $u$ with respect to $\tilde{g}$ are related by

$$du(X) = \langle \nabla u, X \rangle \nabla u = \langle \tilde{\nabla} u, X \rangle T, \quad \forall X \in \Gamma(\tilde{T}M),$$

which together with (2.12) and Schwartz inequality yields

$$\langle \tilde{\nabla} u, \tilde{\nabla} u \rangle_T = \langle \tilde{\nabla} u, \nabla u \rangle \nabla u \leq F(\nabla u) F(\tilde{\nabla} u) \leq \mu F F(\nabla u) \langle \tilde{\nabla} u, \tilde{\nabla} u \rangle_T,$$

namely,

$$|\tilde{\nabla} u|_{\tilde{g}} \leq \frac{1}{\mu} F(\nabla u).$$

Here $| \cdot |_{\tilde{g}} = \langle \cdot, \cdot \rangle_T^1$ denotes the norm of vector field with respect to $\tilde{g}$. Let $dV_{\tilde{g}}$ be the Riemannian volume form of $\tilde{g}$. It is clear from (2.13) that on $\tilde{M}$ we have

$$dV_{\min} \leq dV_{\tilde{g}} \leq dV_{\max} \leq \mu \frac{n}{2} dV_{\min}. \tag{3.5}$$

A positive Radon measure $\nu$ is a linear functional on the space $C_0(\mathcal{H})$ of real-valued continuous functions on a locally compact Hausdorff space $\mathcal{H}$ with compact support, $\nu : f \mapsto \nu(f) \in \mathbb{R}$, such that $\nu(f) \geq 0$, for any $f \geq 0$. We define an upper integral for the nonnegative functions as follows. If $\xi \geq 0$ is lower semicontinuous:

$$\nu^*(\xi) = \sup \nu(f) \text{ for all nonnegative-valued } f \in C_0(\mathcal{H}) \text{ satisfying } f \leq \xi,$$

and for any function $\eta \geq 0 : \nu^*(\eta) = \inf \nu(\xi)$ for all lower semicontinuous functions $\xi$ satisfying $\eta \leq \xi$.

A function $f$ is said to be $\nu$-integrable (or integrable if without ambiguity) if there exists a sequence $\{f_n\} \subset C_0(\mathcal{H})$ such that $\nu^* (|f - f_n|) \to 0$ as $n \to \infty$. A subset $A \subset \mathcal{H}$ is measurable with finite measure $\nu(A)$, if its characteristic function $\chi_A$ is integrable. We set $\nu(A) = \int \chi_A d\nu$.

A function $f$ is said to be $\nu$-measurable (or measurable if without ambiguity) if for all compact sets $K$ and for all $\epsilon > 0$, there exists a compact set $K_\epsilon \subset K$, such that $\nu(K - K_\epsilon) < \epsilon$ and such that the restriction $f|_{K_\epsilon}$ is continuous on $K_\epsilon$ (cf. [1]).

Theorem 3.2. Let $\tilde{M}$ be as in (3.1). Then

1. The volume element $dV_{\tilde{g}}$ on Riemannian manifold $(\tilde{M}, \tilde{g})$ induces a positive Radon measure $\nu$ on a locally compact Hausdorff space $M$. 
(2) The set \( \{x_0\} \cup \text{Cut}(x_0) \) has measure \( \nu(\{x_0\} \cup \text{Cut}(x_0)) = 0 \).

Proof. (1) We first note that a Finsler manifold is a locally compact Hausdorff space. For \( f \in C_0(M) \), we define

\[
\nu(f) = \int_{M \setminus \{x_0\} \cup \text{Cut}(x_0)} f \, dV_{\tilde{g}} = \int_M f \, dV_{\tilde{g}}.
\]

Then \( \nu \) is a linear functional on \( C_0(M) \) with \( \nu(f) \geq 0 \), for any \( f \geq 0 \). Thus \( \nu \) is a positive Radon measure on \( M \).

(2) In view of the definition of the measure of a subset of \( M \) and (3.6), we have the measure

\[
\nu(\{x_0\} \cup \text{Cut}(x_0)) = \nu(\chi_{\{x_0\} \cup \text{Cut}(x_0)}) = \int_{\{x_0\} \cup \text{Cut}(x_0) \setminus \{x_0\} \cup \text{Cut}(x_0)} \chi_{\{x_0\} \cup \text{Cut}(x_0)} \, dV_{\tilde{g}} = 0.
\]

Assume \( 1 \leq p \leq \infty \) for the remaining of this section.

**Definition 3.1.** A measurable function \( u : M \to \mathbb{R} \) is said to belong to \( L^p(M) \) with respect to Riemannian metric \( \tilde{g} \), denoted by \( u \in L^p(M, \tilde{g}) \), if \( \nu(|u|^p) = \int_M |u|^p \, dV_{\tilde{g}} < \infty \), and \( u \) is said to belong to \( W^{1,p}_0(M) \) with respect to Riemannian metric \( \tilde{g} \), denoted by \( u \in W^{1,p}_0(M, \tilde{g}) \), if there exists a sequence \( \{u_i\} \) in \( C_\infty^0(M) \) such that

\[
(\int_M |u - u_i|^p + |\nabla(u - u_i)|^p \, dV_{\tilde{g}})^{\frac{1}{p}} \to 0, \quad \text{as } i \to \infty.
\]

**Theorem 3.3.** \( L^p(M, \tilde{g}) \) is complete, i.e. every Cauchy sequence \( \{u_i\} \) in \( L^p(M, \tilde{g}) \) converges (This means that if for every \( \epsilon > 0 \), there exists \( N \) such that \( (\nu^*(|u_i - u_j|^p))^{\frac{1}{p}} = (\int_M |u_i - u_j|^p \, dV_{\tilde{g}})^{\frac{1}{p}} < \epsilon \), when \( i > N \) and \( j > N \), then there exists a unique function \( u \in L^p(M, \tilde{g}) \), such that \( (\nu^*(|u_i - u|^p))^{\frac{1}{p}} = (\int_M |u_i - u|^p \, dV_{\tilde{g}})^{\frac{1}{p}} \to 0 \), as \( i \to \infty \)).

Proof. In view of Theorem 3.2.(1), \( M \) is a measure space with a Radon measure \( \nu \) as defined in (3.6). To prove that every Cauchy sequence in \( L^p(M, \tilde{g}) \) converges, it suffices to prove that for every Cauchy sequence \( \{u_i\} \) in \( L^p(M, \tilde{g}) \), there exists a subsequence \( \{u_{i_k}\} \) which converges strongly to a function \( u \) in \( L^p(M, \tilde{g}) \) as \( k \to \infty \),
by the triangle inequality. Indeed,
\[
\left( \int_M |u_i - u|^p dV_\tilde{g} \right)^{\frac{1}{p}} \leq \left( \int_M |u_i - u_{ik}|^p dV_\tilde{g} \right)^{\frac{1}{p}} + \left( \int_M |u_{ik} - u|^p dV_\tilde{g} \right)^{\frac{1}{p}}
\]
\[
< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \text{for sufficiently large } i, i_k \text{ if } \{u_{ik}\} \text{ converges to } u.
\]

This also proves that if the limit \( u \) exists, then it is unique. The subsequence can be obtained by choosing \( u_{ik} \) so that \( \left( \int_M |u_{ik} - u_n|^p dV_\tilde{g} \right)^{\frac{1}{p}} < \frac{1}{2^k} \) for all \( n > i_k \). (This is the definition of Cauchy sequence). In particular, \( \left( \int_M |u_{ik} - u_{ik+1}|^p dV_\tilde{g} \right)^{\frac{1}{p}} < \frac{1}{2^k} \) for \( k = 1, 2, \ldots \). Furthermore, this subsequence \( \{u_{ik}\} \) gives rise to a bounded monotone sequence of positive functions

\[
U_\ell = |u_{i_1}| + \sum_{k=1}^\ell |u_{ik+1} - u_{ik}|.
\]

Indeed, by the triangle inequality \( U_\ell \) is bounded in \( L^p(M, \tilde{g}) \) as

\[
\left( \int_M U_\ell^p dV_\tilde{g} \right)^{\frac{1}{p}} \leq \left( \int_M |u_{i_1}|^p dV_\tilde{g} \right)^{\frac{1}{p}} + \sum_{k=1}^\ell \frac{1}{2k} = \left( \int_M |u_{i_1}|^p dV_\tilde{g} \right)^{\frac{1}{p}} + \left( 1 - \frac{1}{2^\ell} \right).
\]

By the monotone convergence theorem, \( U_\ell \) converges pointwise a.e. to a positive function \( U \) which is in \( L^p(M, \tilde{g}) \) and hence is finite almost everywhere. The sequence

\[
u_{i_{k+1}} = u_{i_1} + \sum_{k=1}^\ell (u_{ik+1} - u_{ik})
\]

thus converges absolutely for almost every \( x \) and hence it converges for the same \( x \) to some function \( u(x) \). Since \( |u_{ik}(x)| \leq U_{k-1}(x) \leq U(x) \) a.e. and \( U \in L^p(M, \tilde{g}) \), by dominated convergence theorem (applying to the sequence \( \{|u_{ik}|^p\} \)), \( u \in L^p(M, \tilde{g}) \).

Since \( |u_{ik}(x) - u(x)| \leq U(x) + |u(x)| \in L^p(M, \tilde{g}) \), again by dominated convergence theorem (applying to the sequence \( \{|u_{ik} - u|^p\} \)), we conclude

\[
\lim_{k \to \infty} \left( \int_M |u_{ik} - u|^p dV_\tilde{g} \right)^{\frac{1}{p}} = \left( \int_M (\lim_{k \to \infty} |u_{ik} - u|)^p dV_\tilde{g} \right)^{\frac{1}{p}} = 0.
\]

That is the desired \( \left( \int_M |u_{ik} - u|^p dV_\tilde{g} \right)^{\frac{1}{p}} \to 0 \) as \( k \to \infty \), (cf. [7] for real analysis.) \( \square \)

The proof of Theorem 3.3 yields the following domination and pointwise convergence properties:

**Theorem 3.4.** If \( \{u_i\} \) is a Cauchy sequence in \( L^p(M, \tilde{g}) \), then there exists a subsequence \( \{u_{i_k}\} \) and a nonnegative function \( U \) in \( L^p(M, \tilde{g}) \), such that

1. \( |u_{i_k}| \leq U \) almost everywhere in \( M \).
(2) \( \lim_{k \to \infty} u_{i_k} = u \) almost everywhere in \( M \).

4. Geometric Inequalities on General Manifolds

In this section, we begin with the following geometric inequalities on general Riemannian manifolds and on Finsler Manifolds:

**Local and Global Geometric Inequalities 4.1** Let \( \Omega \subset M \setminus \text{Cut}(x_0) \) be a domain in a Finsler manifold \((M, F)\). For every \( u \in C^\infty_0(\Omega \setminus \{x_0\}) \), and every \( a, b \in \mathbb{R} \), the following inequality holds:

\[
(4.1.a) \quad \frac{1}{2} \left| \int \frac{|u|^2}{r^{a+b+1}} \left( r \Delta r - (a + b) \right) \, dV_\tilde{g} \right| \leq \left( \int \frac{|u|^p}{r^{aq}} \, dV_\tilde{g} \right)^{\frac{1}{p}} \left( \int \frac{\tilde{\nabla} u |_q}{r^{bq}} \, dV_\tilde{g} \right)^{\frac{1}{q}},
\]

where \( p \in [1, \infty] \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). In particular, if \( \text{Cut}(x_0) \) is empty, or \( x_0 \) is a pole, then

\[
(4.1.b) \quad \frac{1}{2} \left| \int _M \frac{|u|^2}{r^{a+b+1}} \left( r \Delta r - (a + b) \right) \, dV_\tilde{g} \right| \leq \left( \int _M \frac{|u|^p}{r^{aq}} \, dV_\tilde{g} \right)^{\frac{1}{p}} \left( \int _M \frac{\tilde{\nabla} u |_q}{r^{bq}} \, dV_\tilde{g} \right)^{\frac{1}{q}}.
\]

*Proof.* We observe that by (3.3) on any domain \( \Omega \subset M \setminus \text{Cut}(x_0) \),

\[
\tilde{\nabla} \left( \frac{|u|^2}{r^{a+b}} \right) = \frac{|u|^2}{r^{a+b}} \Delta r + 2 \tilde{g} \left( \frac{u}{r^{a+b}}, \tilde{\nabla} u \right) + \tilde{g} \left( |u|^2 T, \tilde{\nabla} r^{a-b} \right)
\]

\[
= \frac{|u|^2}{r^{a+b}} \Delta r + 2 \left\langle \frac{u T}{r^{a+b}}, \tilde{\nabla} u \right\rangle_T - (a + b) \frac{|u|^2}{r^{a+b+1}}.
\]

Hence, by the divergence theorem,

\[
\frac{1}{2} \left| \int \frac{|u|^2}{r^{a+b+1}} \left( r \Delta r - (a + b) \right) \, dV_\tilde{g} \right| = \left| \int \left\langle \frac{u T}{r^{a+b}}, \tilde{\nabla} u \right\rangle_T \, dV_\tilde{g} \right|
\]

\[
= \left| \int \left\langle \frac{u T}{r^a}, \tilde{\nabla} u \right\rangle_T \, dV_\tilde{g} \right|,
\]

(cf. [9, p.409]). Now applying the Hölder inequality to the right side of the above formula we obtain the desired inequalities. \( \Box \)

**Local and Global Geometric Inequalities 4.2** Let \( \Omega \subset M \setminus \text{Cut}(x_0) \) be a domain in a Finsler manifold \((M, F)\). For every \( u \in C^\infty_0(\Omega \setminus \{x_0\}) \), and every \( a, b \in \mathbb{R} \), the following inequality holds:

\[
(4.2.a) \quad \frac{1}{p} \left| \int \frac{|u|^p}{r^{a+b+1}} \left( r \Delta r - (a + b) \right) \, dV_\tilde{g} \right| \leq \left( \int \frac{|u|^p}{r^{aq}} \, dV_\tilde{g} \right)^{\frac{1}{p}} \left( \int \frac{\tilde{\nabla} u |_q}{r^{bq}} \, dV_\tilde{g} \right)^{\frac{1}{q}},
\]
where \( p \in (1, \infty) \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). In particular, if \( \text{Cut}(x_0) \) is empty, then

\[
\frac{1}{p} \left| \int_M \frac{|u|^p}{r^{a+b+1}} (r \Delta r - (a + b)) dV_g \right| \leq \left( \int_M \frac{|u|^p}{r^{aq}} dV_g \right)^{\frac{1}{p}} \left( \int_M \frac{\nabla u^p}{r^{bp}} dV_g \right)^{\frac{1}{p}}.
\]

**Proof.** First consider the case that \( 1 < p < 2 \). For every \( u \in C^\infty_0(\Omega \setminus \{x_0\}) \), given \( \epsilon > 0 \), consider \( I := p \int_{\Omega} \left( \frac{|u|^2 + \epsilon}{r^{a+b}} \right)^{\frac{p}{2}} \left( \frac{T_r}{r^{a+b}} \right) \frac{\nabla u^p}{r^{bp}} dV_g \). Then it follows from the Gauss lemma that

\[
I = \int_{\Omega} \nabla \left( \frac{|u|^2 + \epsilon}{r^{a+b}} \right) \frac{T_r}{r^{a+b}} dV_g - \int_{\Omega} \left( \frac{|u|^2 + \epsilon}{r^{a+b}} \right) \frac{T_r}{r^{a+b+1}} \frac{\nabla u^p}{r^{bp}} dV_g + \int_{\Omega} \frac{a + b}{r^{a+b+1}} (|u|^2 + \epsilon)^{\frac{p}{2}} dV_g.
\]

Hence by the Divergence Theorem and (3.3),

\[
I = \int_{\partial V} \left( \frac{T_r}{r^{a+b}} (|u|^2 + \epsilon)^{\frac{p}{2}} \xi \right) T_r dS - \int_{\Omega} \frac{r \Delta r - (a + b)}{r^{a+b+1}} (|u|^2 + \epsilon)^{\frac{p}{2}} dV_g
\]

\[
= \epsilon \frac{p}{2} \int_{\partial V} \left( \frac{T_r}{r^{a+b}} \xi \right) T_r dS - \int_{\Omega} \frac{r \Delta r - (a + b)}{r^{a+b+1}} (|u|^2 + \epsilon)^{\frac{p}{2}} dV_g,
\]

where \( V \) is an open set with \( \text{supp} \{u\} \subset V \subset \subset \Omega \setminus \{x_0\} \), \( \xi \) is the outward unit normal vector of \( \partial V \), \( dS \) is the area element induced from \( dV_g \).

Now the triangle inequality, (4.4), and the Hölder inequality imply that

\[
\left| \int_{\Omega} \frac{(|u|^2 + \epsilon)^{\frac{p}{2}}}{r^{a+b+1}} (r \Delta r - a - b) dV_g \right| = \left| I \right| \leq \left| I \right| = \left| \int_{\Omega} \left( \frac{|u|^2 + \epsilon}{r^{a+b+1}} \right)^{\frac{p}{2}} \left( \frac{T_r}{r^{a+b+1}} \xi \right) T_r dS \right| \leq \left( \int_{\Omega} \frac{|u|^p}{r^{aq}} u^p dV_g \right)^{\frac{1}{p}} \left( \int_{\Omega} \frac{\nabla u^p}{r^{bp}} dV_g \right)^{\frac{1}{p}}.
\]
Since $1 < p < 2$, we have $(|u|^2 + \varepsilon)^{\frac{p-2}{2}} < (|u|^2)^{\frac{p-2}{2}}$. Thus, for every $1 < p < 2$ with $\frac{1}{p} + \frac{1}{q} = 1$, one has via (4.5)

\[
\left| \int_\Omega \frac{(|u|^2 + \varepsilon)^{\frac{p}{2}}}{r^{a+b+1}} (r\Delta r - a - b) dV_{\tilde{g}} \right| \leq p \left( \int_\Omega \frac{|u|^p}{r^{a+q}} dV_{\tilde{g}} \right)^{\frac{1}{q}} \left( \int_\Omega \frac{|\nabla u|^q}{r^{bp}} dV_{\tilde{g}} \right)^{\frac{1}{p}} + \left| \varepsilon \int_{\partial V} \frac{T}{r^{a+b}} \cdot \xi \right|_{T} dS
\]

(4.6)

Since (4.6) holds for every sufficiently small $\varepsilon > 0$, we have

\[
\left| \int_\Omega \frac{(|u|^2 + \varepsilon)^{\frac{p}{2}}}{r^{a+b+1}} (r\Delta r - a - b) dV_{\tilde{g}} \right| \leq p \left( \int_\Omega \frac{|u|^p}{r^{a+q}} dV_{\tilde{g}} \right)^{\frac{1}{q}} \left( \int_\Omega \frac{|\nabla u|^q}{r^{bp}} dV_{\tilde{g}} \right)^{\frac{1}{p}}
\]

(4.7)

Let $\varepsilon \to 0$, the Monotone Convergence Theorem gives the desired result.

For the case $p \geq 2$, consider $I := p \int_\Omega \left| u |^{p-2} u \frac{T}{r^{a+b}} \cdot \nabla u \right|_{T} dV_{\tilde{g}}$. Then it follows from the Guass lemma and (3.3) that

\[
I = \int_\Omega \nabla \left( \frac{|u|^p}{r^{a+b}} T \right) dV_{\tilde{g}} - \int_\Omega \nabla \frac{\text{div} (T)}{r^{a+b}} |u|^p dV_{\tilde{g}} + \int_\Omega \frac{a + b}{r^{a+b+1}} |u|^p dV_{\tilde{g}}
\]

(4.8)

\[
= \int_\Omega \nabla \left( \frac{|u|^p}{r^{a+b}} T \right) dV_{\tilde{g}} - \int_\Omega \frac{|u|^p}{r^{a+b+1}} (r\Delta r - (a + b)) dV_{\tilde{g}}
\]

for every $u \in C_0^\infty(\Omega \setminus \{x_0\})$. Hence by the Divergence Theorem and the Hölder inequality we easily get the desired result.

5. PROOF OF THEOREMS 1.1 AND 1.2 - GENERALIZED CAFFARELLI-KOHN-NIRENBERG TYPE INEQUALITIES ON FINSLER MANIFOLDS

Proof of Theorem 1.1. Case (i) $0 \leq \text{Ric}(\nabla r)$ and $n \leq a + b + 1$:
In view of Laplacian Comparison Theorem 3.1.(2),
\[
\frac{a + b + 1 - n}{2} \int_{\Omega} \frac{|u|^2}{r^{a+b+1}} dV_g \leq \frac{1}{2} \left| \int_{\Omega} \frac{|u|^2}{r^{a+b+1}} \left( r \Delta r - (a + b) \right) dV_g \right|
\]
\[
\leq \left( \int_{\Omega} \frac{|u|^p}{r^{ap}} dV_g \right)^{\frac{1}{p}} \left( \int_{\Omega} \frac{|\tilde{\nabla} u|^2}{r^{bq}} dV_g \right)^{\frac{1}{q}}
\]
\[
\leq \mu_F \left( \int_{\Omega} \frac{|u|^p}{r^{ap}} dV_{\text{max}} \right)^{\frac{1}{p}} \left( \int_{\Omega} \frac{(\nabla u)^q}{r^{bq}} dV_{\text{max}} \right)^{\frac{1}{q}}.
\]
(5.1)

The second, third and last steps follow from Geometric Inequality 4.1.(a), (3.5), and (3.4) respectively. On the other hand, (3.5) implies that
\[
\frac{a + b + 1 - n}{2} \int_{\Omega} \frac{|u|^2}{r^{a+b+1}} dV_g \leq \frac{a + b + 1 - n}{2} \int_{\Omega} \frac{|u|^2}{r^{a+b+1}} dV_{\text{min}}
\]
\[
\leq \frac{a + b + 1 - n}{2} \int_{\Omega} \frac{|u|^2}{r^{a+b+1}} dV_g.
\]
(5.2)

Combining (5.1) and (5.2), we have proved (1.1) when the extreme volume form \( dV_{\text{ext}} \) is \( dV_{\text{max}} \). Similarly, we can prove (1.1) when \( dV_{\text{ext}} \) is the minimum volume form \( dV_{\text{min}} \) and hence (1.2) holds if in addition, \( \text{Cut}(x_0) \) is empty or \( x_0 \) is a pole. This completes the proof of Case (i). Similarly, by considering the fact that a simply connected flat Finsler manifold does not have a cut point in Case (ii), and so does a simply connected Finsler manifold with nonpositive flag curvature by a comparison theorem in Case (iii), the assertions follow.

**Proof of Theorem 1.2.** Proceeding as in the proof of Theorem 1.1 by applying Theorem 3.1.(2), (3.4), and (3.5) to Geometric Inequality 4.2.(a), the assertions follow.

**6. Proof of Theorem 1.3**

We will give two methods:

**First Method:** This follows at once from substituting \( p = q = 2 \) into Theorem 1.1 or Theorem 1.2.
Second Method: We first follow [9]. For every \( u \in W^{1,2}_0(\Omega \setminus \{x_0\},\tilde{g}) \) (cf. Definition 3.1, where \( p = 2 \)) and any \( a,b,t \in \mathbb{R} \), we have, analogous to [9] (4.1)

\[
\int_{\Omega} \left( \frac{\tilde{\nabla} u}{r^b} + t \frac{u}{r^a} T, \frac{\tilde{\nabla} u}{r^b} + t \frac{u}{r^a} T \right) dV_{\tilde{g}} \geq 0,
\]

namely,

\[
(6.1) \quad \int_{\Omega} \frac{\tilde{\nabla} u|^2_{\tilde{g}}}{r^{2b}} dV_{\tilde{g}} + t^2 \int_{\Omega} \frac{|u|^2}{r^{2a}} dV_{\tilde{g}} + 2t \int_{\Omega} \left( \frac{uT}{r^{a+b}}, \tilde{\nabla} u \right)_T dV_{\tilde{g}} \geq 0.
\]

Observing that

\[
\tilde{\nabla} \left( |u|^2_{r^a+b} T \right) = \frac{|u|^2}{r^{a+b}} \Delta r + 2\tilde{g} \left( \frac{u}{r^{a+b}}, \tilde{\nabla} u \right) + \tilde{g} \left( |u|^2 T, \tilde{\nabla} r^{-a-b} \right)
\]

becomes

\[
1
\]

\[
(6.2) \quad 2 \int_{\Omega} \left( \frac{uT}{r^{a+b}}, \tilde{\nabla} u \right)_T dV_{\tilde{g}} = -\int_{\Omega} \frac{|u|^2}{r^{a+b+1}} \left( \Delta r + (a+b) \right) dV_{\tilde{g}}.
\]

Let

\[
A = \int_{\Omega} \frac{|u|^2}{r^{2a}} dV_{\tilde{g}}, \quad B = 2 \int_{\Omega} \left( \frac{uT}{r^{a+b}}, \tilde{\nabla} u \right)_T dV_{\tilde{g}}, \quad C = \int_{\Omega} \frac{|\tilde{\nabla} u|^2_{\tilde{g}}}{r^{2b}} dV_{\tilde{g}},
\]

then (6.1) takes the form

\[
At^2 + Bt + C \geq 0, \quad A > 0
\]

for every \( t \in \mathbb{R} \) which implies that \( B^2 - 4AC \leq 0 \). Thus by (6.2) one has

\[
(6.3) \quad \frac{1}{2} \int_{\Omega} \frac{|u|^2}{r^{a+b+1}} \left( \Delta r + (a+b) \right) dV_{\tilde{g}} \leq \left( \int_{\Omega} \frac{|u|^2}{r^{2a}} dV_{\tilde{g}} \right)^{\frac{1}{2}} \left( \int_{\Omega} \frac{|\tilde{\nabla} u|^2_{\tilde{g}}}{r^{2b}} dV_{\tilde{g}} \right)^{\frac{1}{2}}.
\]

Suppose now that \( M \) has nonnegative radial Ricci curvature at \( x_0 \). Then it follows from Theorem 3.1 that \( r\Delta r \leq n - 1 \). Thus if in addition, \( n \leq a+b+1 \), then (6.3) becomes

\[
(6.4) \quad \frac{1}{2} \int_{\Omega} \frac{|u|^2}{r^{a+b+1}} \left( a + b + 1 - n \right) dV_{\tilde{g}} \leq \left( \int_{\Omega} \frac{|u|^2}{r^{2a}} dV_{\tilde{g}} \right)^{\frac{1}{2}} \left( \int_{\Omega} \frac{|\tilde{\nabla} u|^2_{\tilde{g}}}{r^{2b}} dV_{\tilde{g}} \right)^{\frac{1}{2}}.
\]

Hence, (1.9) follows directly from (3.4), (3.5) and (6.4). This proves the case (i). Analogously, we can prove the cases (ii) and (iii). In particular, if \( M \) has a pole \( x_0 \) or \( \text{Cut}(x_0) \) is empty in (i), or \( M \) is simply connected in (ii) or (iii), then we can choose \( \Omega = M \) and (1.9) becomes (1.10). This completes the proof.
Remark: The case that \( M = \mathbb{R}^n \) is due to Costa (c.f. [2]), The case that \( M \) is a Riemannian manifold is due to Wei-Li. ([9]).

7. Applications - Embedding Theorems for Weighted Sobolev Spaces and Differential-Integral Inequalities on Finsler manifolds

As in the Riemannian case [9], for giving Finsler manifold \((M, F)\) we let \( L^2_{F,a}(M) \) be the completion of \( C_0^\infty(M\setminus\{x_0\}) \) with respect to the norm
\[
\|u\|_{L^2_{F,a}(M)} := \left( \int_M \frac{|u|^2}{r^2a} dV_{\text{ext}} \right)^{\frac{1}{2}},
\]
\( D^1,2_F(M) \) be the completion of \( C_0^\infty(M\setminus\{x_0\}) \) with respect to the “norm”
\[
\|u\|_{D^1,2_F(M)} := \left( \int_M (F(\nabla u))^2 dV_{\text{ext}} \right)^{\frac{1}{2}},
\]
and \( H^1_{F,a,b}(M) \) be the completion of \( C_0^\infty(M\setminus\{x_0\}) \) with respect to the “norm”
\[
\|u\|_{H^1_{F,a,b}(M)} := \left( \int_M \left[ \frac{|u|^2}{r^2a} + \frac{(F(\nabla u))^2}{r^2b} \right] dV_{\text{ext}} \right)^{\frac{1}{2}}.
\]

It should be pointed out here that in general \( \|k \cdot u\|_{H^1_{F,a,b}(M)} = |k| \cdot \|u\|_{H^1_{F,a,b}(M)} \) holds only when \( k \geq 0 \), thus \( \| \cdot \|_{H^1_{F,a,b}(M)} \) although satisfies the triangle inequality is not a genuine norm. Nevertheless, by Theorems 1.3 we clearly have

**Theorem 7.1** Let \((M, F)\) be an \( n \)-dimensional complete Finsler manifold with nonpositive radial flag curvature or nonnegative radial Ricci curvature at the pole \( x_0 \in M \). Suppose also that \( M \) has finite uniformity constant \( \mu_F \). Then the following embeddings hold

\[
(7.1) \quad H^1_{F,a,b}(M) \subset L^2_{F,\alpha+\beta+1}(M) \quad \text{and} \quad H^1_{F,b,a}(M) \subset L^2_{F,\alpha+\beta+1}(M).
\]

As a consequence, we have differential-integral inequalities on Finsler manifolds:

**Theorem 7.2** Let \( M \) be as in Theorem 7.1. Then

i) For any \( u \in D^1,2_F(M) \),

\[
(7.2) \quad \int_M \frac{|u|^2}{r^2} dV_{\text{ext}} \leq \left( \frac{2}{n-2} \right)^2 \cdot \mu_F^{n+1} \int_M |F(\nabla u)|^2 dV_{\text{ext}};
\]
ii) For any \( u \in H^1_{F,b+1,b}(M) \),

\[
(7.3) \quad \int_M \frac{|u|^2}{r^{2(b+1)}} dV_{\text{ext}} \leq \left( \frac{n}{2} - (b + 1) \right)^{-2} \mu_F^{n+1} \int_M \frac{|F(\nabla u)|^2}{r^{2b}} dV_{\text{ext}};
\]

iii) For any \( u \in H^1_{F,a+1,a}(M) \),

\[
(7.4) \quad \left( \int_M \frac{|u|^2}{r^{2(a+1)}} dV_{\text{ext}} \right)^2 \leq \left( \frac{n}{2} - (a + 1) \right)^{-2} \mu_F^{n+1} \left( \int_M |u|^2 \right) \left( \int_M \frac{|F(\nabla u)|^2}{r^{2a}} dV_{\text{ext}} \right);
\]

iv) If \( u \in H^1_{F,-(b+1),b}(M) \) then \( u \in L^2_F(M) \) and

\[
(7.5) \quad \left( \int_M |u|^2 dV_{\text{ext}} \right)^2 \leq \frac{4}{(n-1)^2} \mu_F^{n+1} \left( \int_M |u|^2 \right) \left( \int_M \frac{|F(\nabla u)|^2}{r^{2b}} dV_{\text{ext}} \right);
\]

v) If \( u \in H^1_{F,0,1}(M) \), then \( u \in L^2_F(M) \) and

\[
(7.6) \quad \left( \int_M \frac{|u|^2}{r^2} dV_{\text{ext}} \right)^2 \leq \frac{4}{(n-1)^2} \mu_F^{n+1} \left( \int_M |u|^2 \right) \left( \int_M \frac{|F(\nabla u)|^2}{r^{2b}} dV_{\text{ext}} \right);
\]

vi) If \( u \in H^1_{F,-1,1}(M) \), then \( u \in L^2_{F,1}(M) \) and

\[
(7.7) \quad \left( \int_M \frac{|u|^2}{r} dV_{\text{ext}} \right)^2 \leq \frac{4}{(n-1)^2} \mu_F^{n+1} \left( \int_M |u|^2 \right) \left( \int_M \frac{|F(\nabla u)|^2}{r^{2b}} dV_{\text{ext}} \right);
\]

vii) If \( u \in H^1_F(M) = H^1_{F,0,0}(M) \), then \( u \in L^2_{F,1}(M) \) and

\[
(7.8) \quad \left( \int_M \frac{|u|^2}{r} dV_{\text{ext}} \right)^2 \leq \frac{4}{(n-1)^2} \mu_F^{n+1} \left( \int_M |u|^2 \right) \left( \int_M \frac{|F(\nabla u)|^2}{r^{2b}} dV_{\text{ext}} \right).
\]
Remark The case that $M = \mathbb{R}^n$ is due to [2]. The case that $M$ is a Riemannian manifold is due to [9]. Item (i) is a generalized Hardy’s inequality. In the next Section we will discuss its generalizations.

Proof. We make special choices in Theorems 1.3 as follows:

i) Let $a = 1, b = 0$;

ii) Let $a = b + 1$;

iii) Let $b = a + 1$;

iv) Let $a = -b - 1$;

v) Let $a = 0, b = 1$;

vi) Let $a = -1, b = 1$;

vii) Let $a = 0, b = 0$.

□

8. Proof of Theorems 1.4 and 1.5 - Generalized Hardy inequalities on Finsler manifolds

Employing the double limiting technique in [7], we prove the following:

Geometric Inequality 8.1 (cf. [9 (1.3)], [9 (3)]) Let $M$ be a Finsler manifold, $u \in C_0^\infty(M)$ and $\partial B_\delta(x_0)$ be the $C^1$ boundary of the geodesic ball $B_\delta(x_0)$ centered at $x_0$ with radius $\delta > 0$. Let $V$ be an open set with smooth boundary $\partial V$ such that $V \subset \subset M$, and $u = 0$ off $V$. We choose a sufficiently small $\delta > 0$ so that $\partial V \cap \partial B_\delta(x_0) = \emptyset$. Then for every $\epsilon > 0$ and $p > 1$, we have

\[
\left| - \int_{V \cap \partial B_\delta(x_0)} \frac{r}{r^p + \epsilon} |u|^p \langle T, \xi \rangle dS + \int_{\tilde{M} \setminus B_\delta(x_0)} \frac{(r^p + \epsilon)(1 + r \tilde{\Delta} r) - pr^p}{(r^p + \epsilon)^2} |u|^p dV_{\tilde{g}} \right| 
\]

(8.1) \[ \leq p \left( \int_{\tilde{M} \setminus B_\delta(x_0)} \left( \frac{|u|^{p-1} r}{r^p + \epsilon} \right)^{\frac{p}{p-1}} dV_{\tilde{g}} \right)^{\frac{p-1}{p}} \left( \int_{\tilde{M} \setminus B_\delta(x_0)} |\nabla u|^p dV_{\tilde{g}} \right)^{\frac{1}{p}}, \]

where $\tilde{M}$ is as in (3.1), $\xi$ is the outward unit normal vector field on $\partial B_\delta(x_0)$ with respect to $\tilde{g}$, and $dS$ is the volume form on $\partial B_\delta(x_0)$ induced from $dV_{\tilde{g}}$. 


Proof. Observing via Gauss Lemma

\[
\tilde{\text{div}} \left( \frac{rT}{r^p + \epsilon} |u|^p \right) = \tilde{\text{div}}(rT) \frac{|u|^p}{r^p + \epsilon} + \tilde{g} \left( rT, \tilde{\nabla} \left( \frac{|u|^p}{r^p + \epsilon} \right) \right)
\]

\[
= \frac{\text{div}(rT)}{r^p + \epsilon} |u|^p - \left\langle rT, \frac{\tilde{\nabla}r^p}{(r^p + \epsilon)^2} |u|^p \right\rangle_T + \frac{r}{r^p + \epsilon} \langle T, \tilde{\nabla} |u|^p \rangle_T
\]

we have

\[
\text{(8.2)} \quad p \int_{M \setminus B_\delta(x_0)} \left\langle |u|^{p-2} u \frac{rT}{r^p + \epsilon}, \tilde{\nabla} u \right\rangle_T dV_{\tilde{g}}
\]

\[
= \int_{M \setminus B_\delta(x_0)} \tilde{\text{div}} \left( \frac{rT}{r^p + \epsilon} |u|^p \right) dV_{\tilde{g}} - \int_{M \setminus B_\delta(x_0)} \frac{\text{div}(rT)}{r^p + \epsilon} |u|^p dV_{\tilde{g}} + \int_{M \setminus B_\delta(x_0)} \frac{pr^p}{(r^p + \epsilon)^2} |u|^p dV_{\tilde{g}}.
\]

By the divergence theorem it follows that for sufficiently small \( \delta > 0 \),

\[
\text{(8.3)} \quad \int_{M \setminus B_\delta(x_0)} \tilde{\text{div}} \left( \frac{rT}{r^p + \epsilon} |u|^p \right) dV_{\tilde{g}} = \int_{V \setminus B_\delta(x_0)} \tilde{\text{div}} \left( \frac{rT}{r^p + \epsilon} |u|^p \right) dV_{\tilde{g}}
\]

\[
= \int_{V \setminus \partial B_\delta(x_0)} \frac{r}{r^p + \epsilon} |u|^p \langle T, \xi \rangle_T dS.
\]

On the other hand,

\[
\text{(8.4)} \quad \tilde{\text{div}}(rT) = r \tilde{\text{div}}(T) + \tilde{g}(T, \tilde{\nabla} r) = 1 + r \tilde{\Delta} r.
\]

Substituting (8.3) and (8.4) into (8.2) and using the Hölder inequality one has the desired (8.1). \( \square \)

Proof of Theorem 1.4. We first assume \( u \in C_0^\infty(M) \). Since \( 1 < p < n \) and \( M \) has nonpositive radial flag curvature at \( x_0 \), by Theorem 3.1 we have \( r \tilde{\Delta} r + 1 \geq n > p \), and thus

\[
\text{(8.5)} \quad \int_{M \setminus B_\delta(x_0)} \frac{(r^p + \epsilon)(1 + r \tilde{\Delta} r) - pr^p}{(r^p + \epsilon)^2} |u|^p dV_{\tilde{g}} \geq \int_{M \setminus B_\delta(x_0)} \frac{(n-p)r^p + (n-p)\epsilon}{(r^p + \epsilon)^2} |u|^p dV_{\tilde{g}}
\]

\[
\geq (n-p) \int_{M \setminus B_\delta(x_0)} \frac{(r^p + \epsilon)^{\frac{1}{p-1}}}{(r^p + \epsilon)^{\frac{1}{p-1}}} |u|^p dV_{\tilde{g}}
\]

\[
\geq (n-p) \int_{M \setminus B_\delta(x_0)} \frac{(r^p)^{\frac{1}{p-1}}}{(r^p + \epsilon)^{\frac{1}{p-1}}} |u|^p dV_{\tilde{g}}.
\]
Substituting (8.5) into (8.1), we have for sufficiently small $\delta > 0$,

$$- \int_{V \cap \partial B_\varepsilon(x_0)} \frac{r}{r^p + \varepsilon} |u|^p \langle T, \xi \rangle_T dS + (n - p) \int_{M \setminus B_\varepsilon(x_0)} \frac{(r^p)^{\frac{1}{p-1}}}{(r^p + \varepsilon)^{\frac{1}{p-1}}} |u|^p dV_{\tilde{g}}$$

(8.6)  

\[ \leq p \left( \int_{M \setminus B_\varepsilon(x_0)} \frac{(r^p)^{\frac{1}{p-1}}}{(r^p + \varepsilon)^{\frac{1}{p-1}}} |u|^p dV_{\tilde{g}} \right)^{\frac{1}{p-1}} \left( \int_{M \setminus B_\varepsilon(x_0)} \left| \nabla u \right|^p dV_{\tilde{g}} \right)^{\frac{1}{p}}. \]

For sufficiently small $\delta > 0$, one has

(8.7)  

\[ \int_{\partial B_\varepsilon(x_0)} \frac{r}{r^p + \varepsilon} |u|^p \langle T, \xi \rangle_T dS = 0 \quad \text{if} \quad x_0 \notin V \]

and

(8.8)  

\[ \left| \int_{\partial B_\varepsilon(x_0)} \frac{r}{r^p + \varepsilon} |u|^p \langle T, \xi \rangle_T dS \right| \to 0 \quad \text{as} \quad \delta \to 0 \quad \text{if} \quad x_0 \in V, \]

Indeed, $\frac{r}{r^p + \varepsilon}$ is a continuous, nondecreasing function for $r \in [0, \delta_0]$, where $\delta_0 = (\frac{\varepsilon}{p-1})^{\frac{1}{p-1}}$ and $u$ is bounded in $M$. Hence, for $\delta < \delta_0$,

(8.9)  

\[ \left| \int_{\partial B_\varepsilon(x_0)} \frac{r}{r^p + \varepsilon} |u|^p \langle T, \xi \rangle_T dS \right| \leq \frac{\delta}{\delta^p + \varepsilon} \int_{\partial B_\varepsilon(x_0)} \max_M |u|^p dS. \]

This implies (8.8). It follows from (8.6), via (8.7) or (8.8) that for every $\varepsilon > 0$,

(8.10)  

\[ (n - p) \left( \int_{M \setminus B_\varepsilon(x_0)} \frac{(r^p)^{\frac{1}{p-1}}}{(r^p + \varepsilon)^{\frac{1}{p-1}}} |u|^p dV_{\tilde{g}} \right)^{\frac{1}{p-1}} \leq p \left( \int_{M \setminus B_\varepsilon(x_0)} \left| \nabla u \right|^p dV_{\tilde{g}} \right)^{\frac{1}{p}} \]

Monotone convergence theorem and (3.5) imply that as $\varepsilon \to 0$, for every $u \in C_0^\infty(M)$,

(8.11)  

\[ \left( \frac{n - p}{p} \right) \left( \int_{M} \frac{|u|^p}{r} dV_{\tilde{g}} \right)^{\frac{1}{p}} \leq \left( \int_{M} \left| \nabla u \right|^p dV_{\tilde{g}} \right)^{\frac{1}{p}}, \]

(8.12)  

\[ \frac{u}{r} \in L^p(M, \tilde{g}). \]

Now we extend (8.11) from $u \in C_0^\infty(M)$ to $u \in W_{F,0}^{1,p}(M)$. Let $\{u_i\}$ be a sequence of functions in $C_0^\infty(M)$ tending to $u \in W_{F,0}^{1,p}(M)$ in $\| \cdot \|_{W_{F,0}^{1,p}(M)}$ as in (1.11). Applying the inequality (8.11) to difference $u_{im} - u_{in}$, we have via (3.4) and (3.5).
(8.13) \[
\left( \int_{\tilde{M}} \left| \frac{u_{im} - u_{in}}{r} \right|^p dV_{\tilde{g}} \right)^{\frac{1}{p}} \leq \left( \frac{p}{n - p} \right) \left( \int_{\tilde{M}} \left| \tilde{\nabla}(u_{im} - u_{in}) \right|^p dV_{\tilde{g}} \right)^{\frac{1}{p}} \leq \left( \frac{p}{n - p} \right) \mu_F^\frac{1}{p} \left( \int_M \left( F(\nabla(u_{im} - u_{in})) \right)^p dV_{\max} \right)^{\frac{1}{p}}.
\]

Hence \( \{ \frac{u}{p} \} \) is a Cauchy sequence in \( L^p(M, \tilde{g}) \). By Theorem 3.3, there exists a limiting function \( f(x) \in L^p(M, \tilde{g}) \) satisfying, via (3.4) and (3.5),

(8.14) \[
\int_{\tilde{M}} |f(x)|^p dV_{\tilde{g}} = \lim_{i \to \infty} \int_{\tilde{M}} \frac{|u_i(x)|^p}{r^p} dV_{\tilde{g}} \leq \left( \frac{p}{n - p} \right) \lim_{i \to \infty} \int_{\tilde{M}} \left| \tilde{\nabla}u_i \right|^p dV_{\tilde{g}} \leq \left( \frac{p}{n - p} \right) \mu_F^\frac{1}{p} \lim_{i \to \infty} \int_M (F(\nabla u_i))^p dV_{\max} \leq \left( \frac{p}{n - p} \right) \mu_F^\frac{1}{p} \int_M (F(\nabla u))^p dV_{\max} \leq \left( \frac{p}{n - p} \right) \mu_F^\frac{1}{p} \int_M (F(\nabla u))^p dV_{\min}.
\]

On the other hand, since \( \frac{1}{r^p} \) is bounded in \( M \setminus B_\epsilon(x_0) \), where \( B_\epsilon(x_0) \) is the open geodesic ball of radius \( \epsilon > 0 \), centered at \( x_0 \), and the pointwise convergence in Theorem 3.4.(2), we have for every \( \epsilon > 0 \),

(8.15) \[
\int_{M \setminus B_\epsilon(x_0)} |f(x)|^p dV_{\tilde{g}} = \lim_{i \to \infty} \int_{M \setminus B_\epsilon(x_0)} \frac{|u_i(x)|^p}{r^p} dV_{\tilde{g}} = \int_{M \setminus B_\epsilon(x_0)} \frac{|u|^p}{r^p} dV_{\tilde{g}} = \int_{M \setminus B_\epsilon(x_0)} \frac{|u|^p}{r^p} dV_{\tilde{g}} = \int_{\tilde{M}} \chi_{\tilde{M} \setminus B_\epsilon(x_0)} \frac{|u|^p}{r^p} dV_{\tilde{g}},
\]

where \( \chi_{\tilde{M} \setminus B_\epsilon(x_0)} \) is the characteristic function on \( \tilde{M} \setminus B_\epsilon(x_0) \). As \( \epsilon \to 0 \), monotone convergence theorem and (3.5) imply that

(8.16) \[
\int_{\tilde{M}} |f(x)|^p dV_{\tilde{g}} = \lim_{i \to \infty} \int_{\tilde{M}} \frac{|u_i|^p}{r^p} dV_{\tilde{g}} = \int_{\tilde{M}} \frac{|u|^p}{r^p} dV_{\tilde{g}} \geq \int_{\tilde{M}} \frac{|u|^p}{r^p} dV_{\min} \geq \mu_F^\frac{1}{p} \int_{\tilde{M}} \frac{|u|^p}{r^p} dV_{\max}.
\]
Substituting (8.16) into (8.14) we have for every \( u \in W^{1,p}_{F,0}(M) \),
\[
\int_M \frac{|u|^p}{r^p} dV_{\text{ext}} \leq \left( \frac{p}{n-p} \right)^{\frac{n+p}{p}} \frac{\mu_F^{n+p}}{(n-p)^{n+p}} \int_M (F(\nabla u))^p dV_{\text{ext}},
\]
which implies that
\[
\int_M \chi_{M \setminus B_r(x_0)} \frac{|u|^p}{r^p} dV_{\text{ext}} = \int_{M \setminus B_r(x_0)} \frac{|u|^p}{r^p} dV_{\text{ext}} = \int_{M \setminus B_r(x_0)} \frac{|u|^p}{r^p} dV_{\text{ext}}
\leq \left( \frac{p}{n-p} \right)^{\frac{n+p}{p}} \mu_F^{n+p} \int_M (F(\nabla u))^p dV_{\text{ext}}, \quad \forall \epsilon > 0.
\]
As \( \epsilon \to 0 \), again by monotone convergence theorem we obtain the desired \( \frac{u}{r} \in L^p_F(M) \) and inequality (1.12).

The proof of Theorem 1.4 yields

**Corollary 8.1** Let \((M, F)\) be an \(n\)-dimensional complete Finsler manifold with non-positive radial flag curvature at the pole \(x_0 \in M\) and with finite uniformity constant \(\mu_F\). Then for any \(u \in W^{1,p}_{0}(M, \tilde{g})\) and \(1 < p < n\), the following Hardy type inequality holds:
\[
\left( \frac{n-p}{p} \right) \left( \int_M \frac{|u|^p}{r^p} dV_{\tilde{g}} \right)^{\frac{1}{p}} \leq \left( \int_M |\nabla u|^p dV_{\tilde{g}} \right)^{\frac{1}{p}}.
\]
Furthermore, \( \frac{u}{r} \in L^p(M, \tilde{g}) \).

Either Theorem 1.4 or Corollary 8.1 recaptures the following, when \(M\) is a Riemannian manifold.

**Corollary 8.2** ([10, Theorem 1, Corollary 1.2]) Let \(M\) be an \(n\)-dimensional complete Riemannian manifold of nonpositive radial curvature with the volume element \(dv\). Then for any \(u \in W^{1,p}_{0}(M)\) and \(1 < p < n\), the following Hardy type inequality holds:
\[
(8.17) \quad \left( \frac{n-p}{p} \right) \int_M \frac{|u|^p}{r^p} dv \leq \int_M |\nabla u|^p dv.
\]
Furthermore, \( \frac{u}{r} \in L^p(M) \).

**Proof of Theorem 1.5.** We first assume \(u \in C^\infty_0(M)\). When \(p > n\) and \(M\) has non-negative radial Ricci curvature at \(x_0\), by Theorem 3.1 we have \(r\tilde{\Delta}r + 1 \leq n < p\). In
view of this inequality and the triangle inequality, (8.18) implies (8.18)
\[
\int_{M\setminus B_\delta(x_0)} \frac{(p-n)r^p - ne}{(r^p + \epsilon)^2} |u|^p dV_{\tilde{g}} \leq p \left( \int_{M\setminus B_\delta(x_0)} \left( \frac{|u|^{p-1}r}{r^p + \epsilon} \right)^{\frac{p}{p-1}} dV_{\tilde{g}} \right) \left( \int_{M\setminus B_\delta(x_0)} |\nabla u|^p dV_{\tilde{g}} \right)^{\frac{1}{p}} \\
+ \int_{\partial B_\delta(x_0)} \frac{r}{r^p + \epsilon} |u|^p \langle T, \xi \rangle_T dS.
\]
Applying (8.7) or (8.8), and letting \( \delta \to 0 \) in (8.18), one has (8.19)
\[
\int_M \frac{(p-n)r^p - ne}{(r^p + \epsilon)^2} |u|^p dV_{\tilde{g}} \leq p \left( \int_M \left( \frac{|u|^{p-1}r}{r^p + \epsilon} \right)^{\frac{p}{p-1}} dV_{\tilde{g}} \right) \left( \int_M |\nabla u|^p dV_{\tilde{g}} \right)^{\frac{1}{p}}.
\]
We observe that the integrands in the left, and in the first factor in the right of (8.19) are monotone and uniformly bounded above by a positive constant multiple of \( |u|^p \) on \( M \). Since \( \frac{n}{p} \in L^p_F(M) \), \( \frac{n}{r} \in L^p(M, \tilde{g}) \). By the dominated convergent theorem, as \( \epsilon \to 0 \),

(8.20)
\[
(p-n) \left( \int_M \frac{|u|^p}{r^p} dV_{\tilde{g}} \right) \leq p \left( \int_M \left( \frac{|u|^{p-1}r}{r^p + \epsilon} \right)^{\frac{p}{p-1}} dV_{\tilde{g}} \right) \left( \int_M |\nabla u|^p dV_{\tilde{g}} \right)^{\frac{1}{p}}.
\]
Simplifying and raising to the \( p \)-th power,

(8.21)
\[
\left( \frac{p-n}{p} \right)^p \left( \int_M \frac{|u|^p}{r^p} dV_{\tilde{g}} \right) \leq \left( \int_M |\nabla u|^p dV_{\tilde{g}} \right)^{\frac{1}{p}}.
\]
Now analogously we extend (8.21) from \( u \in C_0^\infty(M) \) to \( u \in W^{1,p}_{F,0}(M) \). Let \( \{u_i\} \) be a sequence of functions in \( C_0^\infty(M) \) tending to \( u \in W^{1,p}_{F,0}(M) \). Applying the inequality (8.21) to difference \( u_{i_m} - u_{i_n} \), employing (3.4) and (3.5) and proceeding as in the proof of Theorem 1.4, we obtain the desired inequality (1.13) for every \( u \in W^{1,p}_{F,0}(M) \).

Similarly, the proof of Theorem 1.4 yields

Corollary 8.3 Let \( (M, F) \) be an \( n \)-dimensional complete Finsler manifold with non-negative radial Ricci curvature at the pole \( x_0 \in M \), and with finite uniformity constant \( \mu_F \). Then for any \( u \in W^{1,p}_0(M, \tilde{g}) \), \( \frac{n}{r} \in L^p(M, \tilde{g}) \) and \( p > n \), the following Hardy type inequality holds:

\[
\left( \frac{p-n}{p} \right)^p \left( \int_M \frac{|u|^p}{r^p} dV_{\tilde{g}} \right) \leq \left( \int_M |\nabla u|^p dV_{\tilde{g}} \right)^{\frac{1}{p}}.
\]

Either Theorem 1.5 or Corollary 8.3 recaptures the following, when \( M \) is a Riemannian manifold.
Corollary 8.4 (5 Theorem 5) Let $M$ be an $n$-dimensional Riemannian manifold with a pole, nonnegative radial Ricci curvature and the volume element $dv$. Then for any $u \in W^{1,p}_0(M)$, $\frac{u}{r} \in L^p(M)$ and $p > n$, the following Hardy type inequality holds:

\[
\left( \frac{p-n}{p} \right)^p \int_M \frac{|u|^p}{r^p} dv \leq \int_M |\nabla u|^p dv.
\]

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