Retirement Spending and Biological Age

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Abstract

We solve a retirement lifecycle model in which the consumer’s age does not move in lockstep with calendar time. Instead, biological age increases at a stochastic non-linear rate in chronological age, which one can think of as working with a clock that occasionally moves backwards in time. Our paper is inspired by the growing body of medical literature that has identified biomarkers of aging which – practically speaking – offer better estimates of expected remaining lifetime and future mortality rates. It isn’t farfetched to argue that in the not-too-distant future of wearable technology, personal age will be more closely associated with biological time vs. calendar age or time. Thus, after introducing our stochastic mortality model we derive optimal consumption rates in a classic Yaari (1965) framework adjusted to our proper clock and time. In addition to the normative implications of having access to biological age, our positive objective is to partially explain the cross-sectional heterogeneity in retirement spending rates at any given chronological age. In sum, we argue that biological age is not a sufficient statistic for making economic decisions and you need information about both your ages to behave rationally.

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1 Introduction and Motivation

In the classical Yaari (1965) model – and the thousands of lifecycle papers it has spawned over the last five decades, such as Levhari and Mirman (1977), Davies (1981), Leung (2007) or Lachance (2012) – the operating assumption is that the chronological age of the representative consumer is the only time variable that matters. In these deterministic mortality models there is a known and consistent mapping between age, time and the future hazard rate, all of which then flows into preferences via discounted utility.

In this paper we take the first steps towards solving a lifecycle model in which the consumer’s age does not move in lockstep with calendar time. Instead, biological age increases at a stochastic non-linear rate in time which is a clock that might occasionally move backwards. In addition to trying to understand the normative implications, or how one might use this (new) information in a lifecycle model, our positive empirical objective is to shed new light on the cross-sectional heterogeneity in retirement spending rates, which lately has been somewhat of a puzzle for researchers in the retirement and pensions arena.

Indeed, one of the stylized facts within the empirical lifecycle literature is the wide dispersion or heterogeneity of consumption and/or withdrawal rates during the period commonly referred to as retirement. At any given retirement age these rates vary cross-sectionally even when controlling for financial wealth, pensions and other economic variables.

To be clear, when we use the term withdrawal rate in this paper we mean the amount of dollars extracted or removed from investible net-worth in any given year, presumably for consumption purposes. When we use the term retirement spending rate we are referring to the same withdrawal amount but expressed as a percentage of financial net worth at that point in time. So, a withdrawal rate of $50,000 per year at age 65 from a (total) net-worth of $1,000,000 at age 65, is labeled a spending rate of 5% at age 65. Ten years later the same 75 year-old might continue to withdraw at the rate of $50,000 per year from his or her portfolio, but if the value of the portfolio has declined to $500,000 (for example) the spending rate at age 75 is (obviously) now 10%. Likewise, as far as terminology is concerned, a retiree with (government or corporate) pension income would add that amount to their withdrawals to arrive at total consumption.

In other words retirement consumption, withdrawal (occasionally called draw-down in the UK) and spending rate are all different quantities. And of course there is Becker’s (1965) distinction between expenditure and consumption. It might seem odd to begin with these definitions, but there is often confusion (especially among practitioners) over terminology which is why we want to get this issue out of the way. Back to our main point. As far as the data is concerned – and distinct from the focus on consumption ($) itself – retirement withdrawal rates ($) and/or spending rates (%) are quite heterogenous.
As just one example, Figure 1 displays the total (balance sheet) economic net-worth of households as a function of age in Canada. It provides a glimpse of how wealth evolves during the retirement years. The data displayed are from the Survey of Financial Security (SFS), which is collected and made available every 5 years.

Although the SFS data is a snapshot of the entire population of Canada at a point in time – and doesn’t address retirement withdrawals or spending of a given cohort per se – it sets the stage for our discussion and our terminology. In a simple linear regression with age $x_i$ as the independent variable and economic net-worth $w_i$ as the dependent variable:

$$w_i = \alpha_0 + \alpha_1 x_i + e_i,$$

the value of net-worth declines by approximately $\alpha_1 = -$18,000 per (chronological age) year. That number would correspond with our (previously explained) population average withdrawal rate. That number is for the 3,179 sampled households (whose head is) above the age of 65 in the SFS2012 dataset, which is the most recent wave of the survey.

Moreover, if we limit ourselves to only including financial assets (a.k.a. investment accounts) and scale withdrawals by the value of the assets, they decline by approximately 4% per year in a similar regression. That number would represent the population average spending rate and is quite close to the widely used and advocated (among financial planners) sustainable spending rate of no more than 4%. See for example Milevsky and Huang (2010) for more references on the topic of financial advice and retirement behavior.
Regardless of whether we focus on withdrawal rates or spending rates, the above evidence is (loosely) consistent with the Modigliani life-cycle model in which consumers accumulate wealth while working and slowly drawdown to consume during their retirement years. The question of whether or not the average or representative consumer is depleting their wealth fast enough is a fertile area of research in the academic literature, but not our focus.

Notwithstanding the trend line it is interesting to note that there is a very wide dispersion in both withdrawal and/or spending rates for retirees who are of the same chronological age, even after controlling for the level of wealth and asset holdings. For example (within the above-mentioned SFS data set) at the age of 70 over 1/4 of Canadian households have a negative portfolio withdrawal rate (i.e. they continue to save). At the age of 70 the median retirement spending rate (relative to wealth one year later at age 71) is 10.48%. Of course not all 70-year old households spend at the rate of 10.48%. According to the SFS one quarter of this group spends less than 4.35% and at the other extreme another quarter experienced an increased of 7.45% (or more) in their net worth. That group is obviously not drawing-down wealth. This is not a unique Canadian phenomenon. The dispersion in withdrawal or spending rates is not unique to Canada. In the U.S. similar evidence is provided by Poterba, Venti and Wise (2015), as well as De Nardi, French and Jones (2015), or Banks, Blundell, Levell and Smith (2015). All of this is yet another motivation for the current paper and research. Within a lifecycle framework, what might explain the reason some retirees of the exact same chronological age – Canadians, Americans or any other nationality – withdraw or spend (much) faster than others? Broadly speaking the literature offers three categories of explanations for the dispersion in these rates.

1.1 Why the Dispersion in Spending Rates?

1. Heterogeneity in Leisure, Labor & Legacy Preferences: See for example the work by Gan, Gong, Hurd and McFadden (2015), Hubener, Maurer and Mitchell (2015), as well as Farhi and Panageas (2017). Individuals with weaker bequest motives, or greater potential for future labour income or lower utility of leisure might all be inclined to spend more today relative to the average retiree. The heterogeneity of preferences would induce statistical dispersion in observed withdrawal and spending rates.

2. Heterogeneity in Portfolio Choice, Markets & Investment Views: See for example the work by Cocco, Gomes and Maenhout (2005), Horneff, Maurer, Mitchell and Stamos (2009), Cocco and Gomes (2012), Yogo (2016). Investors who perceive (or believe) that their portfolio or individual investments will earn greater returns are likely to spend or withdraw more, all else being equal. This would also tie into the behavioral finance and economics literature on irrational beliefs.
3. Heterogeneity in Longevity & Mortality Expectations. See for example Groneck, Ludwig and Zimper (2016), Kuhn, Wrzaczek, Prskawetz, Feichtinger (2015), or Spaenjers and Spira (2015). While population mortality rates are measurable and objective, individuals might have personal (perhaps incorrectly estimated) views of their survival probabilities, which would induce them to spend more or less than the average individual. This idea or explanation can also be positioned within the context of health heterogeneity, see for example the work by Rosen and Wu (2004), Berkowitz and Qiu (2006) or De Nardi, French and Jones (2010).

Our current work falls within category #3 and more specifically in the measurement of (subjective) mortality, but we frame and formulate the lifecycle problem quite differently. Instead of splitting the universe into five health categories for example – and rhetorically asking people to determine whether they are in very good health or merely in good health – we query their biological age vs. their chronological age; more on this to follow.

Generally speaking, we assume a canonical retiree with zero weight on bequest motives (eliminating explanation #1) and a portfolio that only consists of a risk-free asset (eliminating explanation #2) to focus attention on the definition of aging and its impact on portfolio withdrawal and spending rates. At the risk of placing the cart before the horse, we find that the dispersion or heterogeneity of biological ages – at any given chronological age – can (partially) explain the heterogeneity of spending rates. Our (main) contribution is to explain and illustrative how to embed this framework within a rational lifecycle model.

1.2 How Old Are You, Really?

Against this economics and financial backdrop there is a growing body of medical evidence (and mail-order kits) suggesting that an individual’s true age can be measured (more accurately) using telomere length, which is the protective ends of chromosomes and that telomeres provide incremental information above-and-beyond other biomarkers of aging. Moreover, biological age can diverge by as much as 10 to 15 years from chronological age as measured by calendar years. In other words, a 65 year-old retiree might in fact be as young as 50 or as old as 80 when measured properly in terms of forward-looking mortality and morbidity rates.

And, while the technology to accurately estimate biological age is still being refined – and other biomarkers of aging might emerge triumphant – the science does raise the possibility that individuals will soon have access to another number (perhaps on an iPhone or other wearable) quite relevant to wealth management and retirement planning. In fact, it’s not inconceivable that chronological age will take a back seat to biological age in the public discourse around retirement policy.
Biological age should not be viewed simply as an age set-back on a (deterministic) mortality table or a fixed scaling adjustment factor, both of which are quite common in actuarial practice and the health economics literature. Indeed, the joint dynamics of biological (B) age and chronological (C) age are subtle and mathematically non-trivial. For example, the divergence between B-age and C-age is highest around middle age and lowest at younger and older ages. Intuitively, a (live) centenarian’s B-age is quite close to her C-age and vice versa. In other words the dispersion of relative ages within a population is (chronologically) age dependent. For example, in one (very) widely cited socioeconomic study by Marmot and Shipley (1996), the dispersion in mortality rates – or what we would call biological age – continues well-into retirement¹.

Motivated and inspired by this new view of age, in this paper we solve a classical lifecycle model of consumption and spending in which the (rational) economic agent has two distinct and measurable ages at every point in time; we call it B-age and C-age. We assume a canonical retiree with a fixed endowment of investable wealth and then derive the optimal consumption rate as a function of the two dimensional age co-ordinates.

In the absence of any pension income, the consumption rate (in $) is the withdrawal rate (in $) and the spending rate we defined earlier is the ratio of consumption to wealth and is expressed as a percent (%). Of course, our framework collapses to the standard lifecycle model and the known Yaari (1965) results when B-age is forced to equal C-age at all times.

Probabilistically, we connect the two distinct ages via the mechanism of stochastic mortality rates. Namely, by formulating a generalized Brownian Bridge-driven model for individual mortality rates and then inverting the standard (population) Gompertz law of mortality to arrive at one’s relative B-age. All of this will be carefully explained but at this point we should note our Brownian Bridge is a novel (stochastic mortality) model and one that has not been proposed before in the actuarial arena. It certainly has not been put to work in the lifecycle economics literature. We will address why a Brownian Bridge is a suitable model for (stochastic) mortality – as well as the basic question of what is a Brownian Bridge?

1.3 Connection to the Actuarial Literature

Our paper sits at the intersection of three different fields or areas of research. On the one side is the economics literature and the data which motivated the paper belongs firmly in the lifecycle arena. At the other end, there is the field of bio-gerontology and the study of aging. In the middle of the two extremes we have the field of actuarial science and demography.

¹For example, they found that a retiree who had worked as an administrative clerk had a 50% higher mortality rate at the age of 90, compared to someone who has worked as an executive within the same governmental organization.
Regardless of the field – economics, actuarial science or gerontology – a very popular assumption regarding death and aging is that hazard rates (denoted by $\lambda_s$) obey the so-called Gompertz law of mortality, where $\lambda_{s+x} = \frac{1}{b} \exp\left(\frac{x+s-m}{b}\right)$. The way this law is expressed, $x$ is current age, the parameter $m$ is a modal value of life in years (e.g. 80) and $b$ is a dispersion parameter in years (e.g. 10). Stated briefly, mortality rates grow predictably and deterministically by approximately 9% to 10% per year. Most of the lifecycle papers in the financial and economics literature explicitly or implicitly assume this law of mortality, perhaps calibrated to discrete population mortality tables. Leung (2007) who extended Yaari (1965) used this exact law of mortality.

The Gompertz (sometimes known as Gompertz-Makeham) law is the canonical and prototypical deterministic model of mortality. It is widely taught and used in pricing insurance and annuity contracts, albeit after some adjustments for anti-selection and various discretization and smoothing techniques.

The first extension of the (deterministic) Gompertz law, to a stochastic environment was the work by Lee & Carter (1992). From that starting point the demographic and actuarial literature have proposed many models for the evolution of population mortality rates in which the hazard rate is assumed to be a diffusion process in continuous time. In this matter we refer readers to the actuarial papers by Milevsky & Promislow (2001), Dahl (2004), Biffis (2005), Renshaw & Haberman (2006), Cairns, et. al. (2006), Schrager (2006), Luciano & Vigna (2008), Plat (2009), Cairns, et. al. (2011), Huang, Milevsky and Salisbury (2012), Blackburn and Sherris (2013), Delong & Chen (2016). See also Pitacco, Denuit, Haberman & Olivieri (2008) as well as the criticism by Norberg (2010). To be clear in our positioning, this paper is not an attempt to better forecast or project future population mortality rates. Rather, our assumption is that today’s 65 year-old does not know (with certainty) what his or her mortality rate will be in 30 years. They only have a rational expectation. The question is how the uncertainty affects optimal behavior. Our language (biological vs. chronological age) is borrowed from the literature on bio-demographics. We refer the interested readers to the work by Cawthon, et. al. (2003), Dong et. al. (2016), Heidenger, et. al. (2012), Mather, et. al. (2010) and in particular Olshansky, et. al. (1990) for a general discussion of the uncertainty in future mortality rates and its relation to the limits to life.

Our work is an attempt to merge (i.) lifecycle models and (ii.) stochastic mortality on the individual level. Aside from the attempt to explain current and observed behavior, one of the things that appeals to us is that it looks ahead into the future when the measurement of biological age will presumably be much more widespread and accurate. We’re asking what the rational impact will be down the road, once everybody knows their B-age.
1.4 Overview of the Paper

The remainder of this paper is organized as follows: The next section describes and explains our stochastic process for the instantaneous mortality rate – the so called Brownian Bridge model that was briefly mentioned – which then leads to the distinct evolution of biological and chronological age. In section 3 we present the theoretical core of the paper, which is the derivation of the optimal consumption and withdrawal rate as a function of both biological and chronological age. In that section we carefully explain the difference between a model with deterministic aging, which has been part of the lifecycle literature for decades and our model of stochastic aging. Then in section 4 we provide a range of numerical results as well as the testable implications emerging from our framework. Finally, section 5 concludes the paper. All non-essential\(^2\) proofs and mathematical derivations are relegated to a technical appendix.

2 Stochastic Model of Aging

2.1 How to Think About Death

As far as the modeling is concerned we assume that one can accurately measure (e.g. using a combination of telomere length, systolic blood pressure, body-mass index, etc.) an individual’s current mortality rate and that it can be higher or lower than (average) population mortality rates at that chronological age. This then allows us to invert a (population) Gompertz law using the observed mortality rate to obtain the corresponding biological age. Stated differently, our first assumption is that the two-century old Gompertz law of mortality applies in (something we call) biological time and not in standard calendar time.

Second, although (technically) there is no maximum imposed limit to the length of human life, we do assume that at some fixed age (e.g. 110) the mortality rate of our canonical retiree will hit \(\lambda_T = 1\) with probability one and that life expectancy from that point onward is exactly one year. You stop aging.

Third and finally, we assume that the instantaneous mortality rate which follows a diffusion process wanders randomly over time but reverts back to the above mentioned \(\lambda_T\) by time \(T\) and is absorbed at that end point. Figure 2 offers a picture of the evolution of the two ages with a corresponding 90% confidence interval assuming both ages are identical at time zero (which is age 60).

\(^2\text{We concede that the word non-essential is subjective.}\)
2.2 Process for the Mortality Rate

Let $A_t$ denote biological age and $\kappa_t$ chronological age. We assume that $A_0 = \kappa_0$, $A_T = \kappa_T$, for a fixed value of $T$, and $A_t = A_T$ for $t \geq T$. We assume Gompertz mortality based on biological age, so the stochastic hazard rate is $\lambda_t = \frac{1}{b}e^{\frac{A_t - m}{b}}$. Alternatively we can write this in terms of the corresponding hazard rates $\lambda_0$ and $\lambda_T$ at times 0 and $T$ (rather than in terms of $m$ and $b$), namely:

$$\lambda_t = \frac{1}{b}e^{\frac{A_t - m}{b}} = \lambda_0e^{\frac{A_T - A_0}{b}} = \lambda_0e^{\frac{A_T - A_0}{b}} = \lambda_0\left(\frac{\lambda_T}{\lambda_0}\right)^{\frac{A_T - A_0}{\lambda_T - A_0}}$$

The two sets of parameters $(\lambda_0, \lambda_T)$ and $(m, b)$ are interchangeable, with

$$b = \frac{\kappa_T - \kappa_0}{\log(\lambda_T/\lambda_0)}, \quad m = \kappa_0 - b\log(b\lambda_0)$$

In the Partial Differential Equations (PDEs) that follow, the mortality hazard rate is expressed in terms of a biological-age variable $a$ via a function $\lambda(a)$, which is written in either of the two forms:

$$\frac{1}{b}e^{\frac{a-m}{b}} = \lambda(a) = \lambda_0\left(\frac{\lambda_T}{\lambda_0}\right)^{\frac{a - \kappa_0}{\kappa_T - \kappa_0}}$$

Just to be clear when it comes to calibration, one can assume that the end-points $(\lambda_0, \lambda_T)$ are known and then solve for the implied Gompertz parameters $(m, b)$. Or, *vice versa* one can start with a particular Gompertz parameter set and then compute the relevant end-points. For example, if we assume that mortality is pinned at $\lambda_0 = 0.005$ at the age of $x = 60$ and $\lambda_{50} = 1$ at the age of $x = 110$, then the implied Gompertz values are $m = 88.8174$ and $b = 9.4369$, according to equation (2).
We assume that the biological age can be written as \( A_t = \kappa_t + Y_t \) for \( t \leq T \), where

\[
dY_t = -\xi \frac{Y_t}{T-t} \, dt + \sigma \, dB_t.
\] (4)

Since \( d\kappa_t = dt \) this implies that, for \( t < T \),

\[
dA_t = \left(1 + \xi \frac{\kappa_t - A_t}{T-t} \right) \, dt + \sigma dB_t.
\] (5)

In other words, \( \xi \) is a type of mean reversion parameter and \( \sigma \) a volatility parameter. The singularity of the drift at \( t = T \) will force \( A_T = \kappa_T \), as we will show below. If \( \xi = \sigma = 1 \) then \( Y_t \) is in fact the well-known Brownian Bridge process.

By equation (4), we have that \( d[(T-t)^{-\xi} Y_t] = \sigma (T-t)^{-\xi} dB_t \), so:

\[
(T-t)^{-\xi} Y_t - (T-s)^{-\xi} Y_s = \sigma \int_s^t (T-q)^{-\xi} \, dB_q.
\] (6)

In other words, given the history of biological age until time \( t \), \( Y_t \) has a normal distribution with conditional mean \( (T-t)^{-\xi} \kappa_y \) and conditional variance \( \sigma^2 (T-t)^{-2\xi} \int_s^t (T-q)^{-2\xi} \, dq \). If \( \xi \neq \frac{1}{2} \) this = \( \sigma^2 \frac{T-t}{2\xi-1} [1 - (\frac{T-t}{T-s})^{2\xi-1}] \). In particular, the mean and variance of \( Y_t \) both \( \to 0 \) as \( t \uparrow T \), so \( Y_t \to 0 \) in probability as well. One can show that \( Y_t \to 0 \) a.s.

It’s important to emphasize that while the distribution of \( Y_t \) is symmetric around zero based on the construction of equation (4), one is not likely to observe a chronological \( \kappa = 75 \)-year old with a biological age of \( A = 95 \) and \( A = 55 \) with equal odds. Indeed, the path above zero for which \( Y_t > 0 \) is more hazardous and is more likely to kill the retiree along the human lifecycle relative to the path under zero for which \( Y_t < 0 \). In other words, although the odds of both events are small (and technically have measure zero), it is more likely one will observe the co-ordinates \( (B = 55, C = 75) \) vs. \( (B = 95, C = 75) \) We will return to these odds later on.

### 2.3 Intuition for the Stochastic Model

Given that our model for mortality (life and death of the retiree) is quite different from the standard models used in the lifecycle economics literature, in this subsection we provide some additional intuition and insight into the difference between stochastic and deterministic aging.

Table 1 offers some numerical values for the 95% confidence interval of biological age when the retiree is 85 years old, chronologically. It focuses on the impact of the two key parameters \( \xi \) and \( \sigma \) in our model. Notice that as the volatility of mortality \( \sigma \) increases from 0.30 to 0.90, the range in years between biological age and chronological age increases. For example when \( \xi = 1 \), which is the canonical basis for most of our numerical examples to
Table 1: 90% Confidence Interval for Biological (B) Age at Chronological (C) Age 85 (time \( t = 25 \)), under various combinations of reversion speed \( \xi \) and mortality volatility \( \sigma \).

| Lower Bound, Upper Bound & Range | \( \sigma = 0.30 \) | \( \sigma = 0.60 \) | \( \sigma = 0.90 \) |
|-----------------------------------|-----------------|-----------------|-----------------|
| \( \xi = 0.50 \)                 | [81.33, 88.17] 6.84 yrs | [79.67, 89.42] 9.75 yrs | [77.67, 90.67] 13.00 yrs |
| \( \xi = 0.75 \)                 | [81.67, 87.92] 6.25 yrs | [80.08, 89.08] 9.00 yrs | [78.25, 90.33] 12.08 yrs |
| \( \xi = 1.00 \)                 | [81.92, 87.67] 5.75 yrs | [80.50, 88.75] 8.25 yrs | [78.75, 90.00] 11.25 yrs |
| \( \xi = 2.00 \)                 | [82.58, 87.08] 4.50 yrs | [81.50, 87.92] 6.42 yrs | [80.25, 88.92] 8.67 yrs |

Note: Model assumes \( \lambda_0 = 0.005 \) at age 60 and \( \lambda_{50} = 1.00 \) at age 110.

follow, at the chronological age of 85 (which is also time \( t = 25 \)), biological age can vary by 5.75 years when \( \sigma = 0.30 \) and by as much as 11.25 years when \( \sigma = 0.90 \). This effect is quite natural and to be expected given the definition of volatility which is synonymous with dispersion. The same outcome can be observed when the speed or force of mean reversion \( \xi \) is reduced from 1.0 to 0.50. The dispersion in range increases from 5.75 years to 6.84 years. Intuitively, the underlying diffusion process is wandering more (i.e. not forced to “return quickly”) in between the fixed end-points.

These represent 90% confidence intervals, which to be clear means there is a 5% probability of observing a B-age below the lower bound and a 5% probability of observing a B-age above the upper bound, at a fixed Chronological age. Again, this is a conditional probability; conditional on survival. Obviously, using 95% or 99% values would increase the range of plausible B-ages.

On a more subtle level though it is worth examining the upper and lower bounds themselves relative to age of 85. Notice that all 12 ranges displayed are left-skewed. This is not coincidental or the result of numerical approximations. For example, in our canonical case of \( \xi = 1.0 \) and \( \sigma = 0.30 \) the lower bound is biological age 81.92 which is 3.08 years below chronological age 85. The upper bound is biological 87.67, which is only 2.67 years above 85. Indeed, the two numbers add-up to the listed range of 5.75 years. So even though \( A_0 = \kappa_0 \) at time zero, as time evolves a lower B-age is more likely to be observed (among survivors) than a higher B-age.

The asymmetry is driven by the fact that if biological age (the generalized Brownian bridge) happens to wander above chronological age over time, the higher mortality rate is more likely to kill the retiree. They are less likely to survive to the C-age of 85. On the other hand if biological age wanders and remains under chronological age as time marches on, the implied mortality rate is lower (by definition) and the retiree is more likely to survive to C-age 85. This generates the skewness that is observed in the table.
In other words although the underlying stochastic process for the (generalized) Brownian bridge $Y_t$ is perfectly symmetric around zero and the unconditional biological age process $A_t$ evolves symmetrically around $\kappa_t$, once we translate into mortality units and then condition on survival the symmetry is destroyed. For example, in the upper right corner of the table in which $\xi = 0.5$ and $\sigma = 90\%$ the lower bound is 77.67 years (7.33 years under 85) and the upper bound is 90.67 (only 5.67 years above 85.) Very loosely speaking, this is a 55\% chance of being younger than your age and a 45\% chance of being older even though you started out being exactly your age.

To provide one final perspective before we move on to moments and economics, Figure 3 displays the entire probability distribution of biological age at chronological ages 35, 60 and 85. In this particular figure we pin-down mortality rates at age 10 and at age 110, which means that they wander for 100 (chronological) years before converging at the end of the mortality table. The parameters for this figure are just one ($\xi = 1, \sigma = 0.30$) from the subset displayed in Table 1, but the main graphical insight is the same. A (live) chronological 85 year-old could conceivably have a (biologically) mortality rate that is closer to that of a 65 year old, although the probability of observing this (essentially 20 year gap in ages) is quite rare. The left tail is extremely thin at 65. But those (small) odds are relatively higher than the odds of observing a chronological 85 year-old who is 105 biologically. On the right side of that particular density curve the tail value is essentially zero. Stated differently, the large hazard rate would have surely killed the 85 year-old. This is why the three displayed (in boxes) mean values for B-age are always lower than their C-age. We pause here to stress (yet again) that this is a fundamental aspect of our model for stochastic mortality; a twisted bridge.
2.4 Expected Remaining Lifetime

Let $e(t, a)$ denote the life expectancy of an individual whose biological age at time $t$ is $a$. If $\zeta$ denotes our individual’s lifetime then on the event that $\zeta > t$ we have

$$t + e(t, A_t) = E[\zeta \mid \mathcal{F}_t].$$

(7)

where $\mathcal{F}_t$ represents the historical information set at time $t$. The latter is a martingale and has a jump of $−e$ at $t = \zeta$, so after adding back the compensated jump, we get that

$$t + e(t, A_t) - \int_0^t \lambda_s e(s, A_s) \, ds$$

(8)

is a martingale when stopped at $\zeta$. Therefore

$$1 + e_t(t, a) + \left(1 + \xi \frac{\kappa_t - a}{T - t}\right) e_a(t, a) + \frac{\sigma^2}{2} e_{aa}(t, a) - \lambda(a)e(t, a) = 0$$

(9)

for $t < T$, with boundary conditions $e(T, \cdot) = \frac{1}{\lambda_T}$ and $e(t, \infty) = 0$. A potential issue is that the drift coefficient blows up as $t \uparrow T$, which creates some (potential) stability problems, but one can obtain asymptotics of $e(t, a)$ as $t \uparrow T$, to get a sense of the impact on results. For the boundary condition at $a = -\infty$, observe from the above mean-variance calculation that if $s < t < T$ then $A_s \ll 0 \Rightarrow A_t \ll 0$. Therefore when $a \to -\infty$, we will have $\lambda \to 0$. In other words we live to time $T$ and then die exponentially: $e(t, -\infty) = T - t + \frac{1}{\lambda_T}$.

Table [2] provides some estimates of life expectancy under a variety of biological and chronological age assumptions. The bolded diagonal values which range from 24.95 years down to 3.51 years are comparable to (unisex) life expectancy values from (Canadian) population mortality tables during the ages 60 to 95.

For example, a 65 year-old retiree who is judged or estimated to be 20 years younger with a biological age of 45, has an expected remaining lifetime of 29.2 years. In contrast, the same 65 year-old with a biological age of 85 (i.e. he is 20 years older than his chronological age) will have a life expectancy of only 8.9 years. Although these values are model dependent, the main assumption being made is that the instantaneous (population) mortality rate of a 60 year-old is 0.005 and the instantaneous (population) mortality rate of a 110 year-old is exactly 1, and there is a generalized Brownian bridge that links them.

What this also means (in our model) is that one can in fact grow younger over (short) periods of time. If we think in terms of $[B, C]$ age co-ordinates in the table, a retiree at point $[B = C = 65]$ has a life expectancy of 20.70 years. Ten (calendar) years later his chronological age is 75, but his biological age (might) wander to 60, which represents a substantial improvement in health, a reduction in mortality rate and revised life expectancy of 21.44 years. In a classical actuarial (deterministic) model of aging this would be impossible. Life expectancy would (always) decline over time.
Table 2: Assuming an initial (age 60) mortality rate of $\lambda_0 = 0.005$ and terminal (age 110) rate of $\lambda_{50} = 1$. The mortality rate wanders in a Brownian Bridge, $\xi = 1$ and $\sigma = 30\%$

| Expected Remaining Lifetime (ERL) | Chronological Age $\kappa$ |
|----------------------------------|-----------------------------|
| B-Age                           | 60  | 65  | 70  | 75  | 80  | 85  | 90  | 95  |
| 45                              | 31.68| 29.18| 26.60| 23.93| 21.17| 18.29| 15.27| 12.09|
| 50                              | 29.73| 27.46| 25.12| 22.68| 20.14| 17.48| 14.67| 11.68|
| 55                              | 27.49| 25.49| 23.41| 21.23| 18.94| 16.52| 13.95| 11.19|
| 60                              | 24.95| 23.24| 21.44| 19.55| 17.55| 15.41| 13.11| 10.61|
| 65                              | 22.10| 20.70| 19.22| 17.64| 15.95| 14.12| 12.13| 9.92 |
| 70                              | 18.97| 17.90| **16.74**| 15.49| 14.13| 12.64| 10.99| 9.11 |
| 75                              | 15.67| 14.90| 14.07| **13.15**| 12.13| 10.98| 9.68 | 8.17 |
| 80                              | 12.35| 11.86| 11.31| 10.69| **9.99**| 9.18 | 8.24 | 7.10 |
| 85                              | 9.22 | 8.94 | 8.62 | 8.25 | 7.83 | **7.32**| 6.70 | 5.93 |
| 90                              | 6.48 | 6.35 | 6.19 | 6.01 | 5.79 | 5.51 | **5.17**| 4.70 |
| 95                              | 4.31 | 4.26 | 4.20 | 4.12 | 4.03 | 3.90 | 3.74 | **3.51**|

With stochastic aging this is no longer the case. Life expectancy can (occasionally) increase over time. Of course the probability that this event will occur and that your biological age will take a favorable path downward depends very much on the underlying parameters. Moreover, computing the survival probabilities requires some additional technology which is addressed in the appendix.

3 Consumption over Time and Age

With the stochastic mortality model behind us, we are ready to discuss lifecycle economics.

3.1 Optimal Spending Rate: Deterministic Aging

We begin with a (very) short review of the lifecycle model under deterministic aging along the lines initiated by Ramsey (1928), Yaari (1964, 1965), Hakansson (1969), Fischer (1973), Richard (1975), Levhari & Mirman (1977), Davies (1981), Butler (2001) or Lachance (2012). In all of those models the underlying objective is to maximize discounted utility of consumption over one’s remaining lifetime, which can be formally expressed as:

$$J = \max_{c_s} E \left[ \int_0^\infty e^{-\rho s} u(c_s) 1_{\{s \leq \zeta\}} ds \right],$$

(10)
where \( \zeta \) is the (random) remaining lifetime satisfying \( \Pr[\zeta > s] = p(s, \lambda_0) \).

When the mortality hazard rate (i.e. aging) \( \lambda_t \) is deterministic, the optimal consumption \( c^*_s \) and \( 1_{\{s \leq \zeta\}} \) are independent, so by Fubini’s theorem the objective function can be written without the expectation as:

\[
J = \max_{c_s} \int_0^\infty e^{-\rho s} u(c_s) p(s, \lambda_0) ds.
\]  

(11)

For most of the above-cited literature, and all of what follows in this paper, \( u(c) = c^{(1-\gamma)}/(1-\gamma) \), which is CRRA utility although the framework can be generalized to HARA.

The budget constraint can be written as:

\[
F'_s = (r + \lambda_s) F_s + \pi_s - c_s,
\]

(12)

where \( F_0 = W > 0 \) (initial retirement wealth); \( F_D = 0 \) (i.e. no bequest motive); \( \pi_s \) is the (possibly) time-dependent pension income, \( r \) is the risk-free rate and the added \( \lambda_s \) are the mortality credits embedded in the actuarial notes if they are purchased. Indeed, we reference the classic Yaari (1965) result that in the absence of bequest motives there is no reason to hold conventional (i.e. non actuarial) investments.

In practice actuarial notes are not available – unless and until society re-introduces tontines – people do not voluntarily annuitize wealth and Defined Benefit (DB) pensions rarely offer more than inflation protection. Here we make a simplifying assumption.

Our investable universe will not contain mortality credits other than fixed exogenous pension income and the budget constraint is therefore:

\[
F'_s = r F_s + \pi_0 - c_s.
\]

(13)

Using methods from the Calculus of Variations, by the Euler-Lagrange (EL) Theorem the optimal \( F_s \) satisfies a second-order non-homogenous differential equation in regions where \( F_s \neq 0 \) and the relevant Ordinary Differential (OD) equation is:

\[
F''_s - \left( \frac{r - \rho - \lambda_s}{\gamma} + r \right) F'_s + r \left( \frac{r - \rho - \lambda_s}{\gamma} \right) F_s = - \left( \frac{r - \rho - \lambda_s}{\gamma} \right) \pi_0.
\]

(14)

In general this equation can’t be solved explicitly unless \( \lambda \) happens to be constant (i.e. no aging). However, one can express the relevant (consumption) function analytically under some continuous mortality rate models\(^3\) In fact, when the mortality hazard rate \( \lambda_s \) is of the Gompertz type introduced earlier one can obtain an analytic expression for: \( c^*_s \) and \( F_s \).

In addition to easily computable values, using an analytic mortality law helps provide insight about the impact of longevity risk aversion \( \gamma \) on the optimal consumption and spending strategy. In particular when \( \pi_0 = 0 \) (no pension) the optimal consumption is:

\[
c^*_s = c^*_0 e^{k_s \left( p(s, \lambda_0) \right)^{1/\gamma}},
\]

(15)

\(^3\)See the original work by Leung (2007), as well as Lachance (2012), Milevsky and Huang (2010).
where the new constant \( k := (r - \rho)/\gamma \). The optimal trajectory of wealth is:

\[
F_t = \left( F_0 - c_0^* \int_0^t e^{(k-r)s}(p(s, \lambda_0))^{1/\gamma} ds \right) e^{rt}.
\]

And, since (eventually) \( F_\tau = 0 \) for some (albeit very large) \( \tau \) in a lifecycle model with no bequest motives, the initial consumption and spending rate is therefore:

\[
c_0^* = \frac{F_0}{\int_0^\tau e^{(k-r)s}(p(s, \lambda_0))^{1/\gamma} ds}, \tag{16}
\]

with some technical conditions on the relationship between \( \rho, r, \gamma \) and mortality parameters. For the sake of basic (deterministic mortality) intuition we now present a few examples to help contrast with our main (later) results. Assume a 65 year-old retiree under a Gompertz law of mortality with mode \( m = 89.3 \) (years) and dispersion: \( b = 9.5 \) (years.) This individual has a current mortality hazard rate of \( \lambda = \frac{1}{9.5} e^{(65-89.3)/10} = 0.00926 \), by construction. Under deterministic aging mortality will smoothly increase by \( 1/(9.5) = 10.5\% \) per year to reach a value of \( \lambda = \frac{1}{9.5} e^{(110-89.3)/10} = 0.83419 \) at age of 110, if he is still alive.

Furthermore, our 65 year-old starts retirement with \( F_0 = $100 \) in consumable wealth and is assumed to have preferences described by CRRA utility with longevity risk aversion \( \gamma = 8 \), subjective discount rate \( \rho = 2.5\% \) in an economy with a real (fixed) interest rate of 2.5%. Putting this all together, according to equation (16) the optimal initial consumption rate is 4.121\% of wealth, assuming no exogenous pension income. In other words, consumption will be (entirely) sourced from the investable wealth. To be consistent with language, the nest-egg’s spending rate is equal to the total consumption rate.
In contrast if the retiree is entitled to $5 of pension income (or social security) for every initial dollar of wealth, which changes the budget constraint and introduces \( \pi \), then the optimal consumption rate and the optimal spending rate from wealth is no longer 4.121\%. The optimal initial spending rate from investable wealth is 4.839\%, or $4.839 per $100 of initial wealth. The optimal (i.e. combined) consumption rate is $9.839, which includes the pension annuity. In some sense this is just a matter of terminology. Figure 4 displays the path of consumption over time.

3.2 Optimal Spending Rate: Stochastic Aging

We assume that all wealth \( W_t \) is invested at the risk-free rate denote by \( r \), thus abstracting from portfolio choice consideration, and with the usual controllable consumption rate \( c_t \). Once again utility is assumed to be CRRA(\( \gamma \)) with no utility of bequest and with a subjective discount rate \( \rho \). For now we ignore pension income (which would have required us to compute a wealth depletion time) and assume that all consumption comes from portfolio withdrawals.

Let \( v(t,a,w) \) be the value function for maximizing utility of consumption, which includes both time \( t \) and age \( a \). Namely;

\[
v(t,a,w) = \sup_{c_s} E \left[ \int_0^\infty e^{-\rho s} \frac{c^{1-\gamma}_s}{1-\gamma} 1(t+s<\zeta) \, ds \mid A_t = a, W_t = w \right],
\]

with the same budget constraint we discussed in the prior section.

\[
dW_t = (rW_t - c_t) \, dt
\]

The derivation now is trickier and we can’t use Calculus of Variation arguments due to the stochasticity of the mortality rate. We must resort to martingale methods and refer the interested reader to the derivations and proofs in the Appendix. For now, we note that \( v(t,a,w) = f(t,a)w^{1-\gamma}/(1-\gamma) \) and move on to obtain a partial differential equation (PDE) for the optimal consumption function, which can be expressed in the following way.

\[
f_t + \left(1 + \xi \frac{k_t - a}{T-t} \right) f_a + \frac{\sigma^2}{2} f_{aa} + r(1-\gamma)f - (\rho + \lambda(a)) f + \gamma f^{1-\frac{1}{\gamma}} = 0.
\]

for \( t < T \), with boundary conditions \( f(T,a) = f_T \), \( f(t,\infty) = 0 \). Optimal consumption is: \( c = w \cdot f(t,a)^{-\frac{1}{\gamma}} \). Note that for \( t \geq T \), \( f \) doesn’t depend on \( a \). Therefore the dynamics are time invariant, which means \( f \) is a constant \( f_T \). It is determined by the following equation:

\[
r(1-\gamma)f_T - (\rho + \lambda T)f_T + \gamma f_T^{1-\frac{1}{\gamma}} = 0.
\]

That is,

\[
f_T = \left( \frac{\rho + \lambda T - r(1-\gamma)}{\gamma} \right)^{-\gamma}.
\]
For the boundary condition at $a = -\infty$ we have the simpler problem where $f$ doesn’t depend on $a$, and $\lambda = 0$. In other words, we have an ODE

$$f_t + r(1 - \gamma)f - \rho f + \gamma f^{1 - \frac{1}{\gamma}} = 0$$

with the terminal condition given above.

Finally there is a stability condition that is needed when $0 < \gamma < 1$, namely that the interest rate $r$ can’t be much larger than the subjective discount rate $\rho$, which is actually a standard restriction imposed in similar lifecycle models. In particular note that if $r \gg \rho$, the rational consumer optimally defers consumption forever. For most of the numerical examples that follow (to flush out the impact of mortality) we will assume that $\rho = r$, so these stability issues will not be a concern.

### 3.3 Comparative Statics

Equation (19) contains all the information we need to establish the optimal spending (or consumption) rate at any given [C,B] age pair. So, before we move on to numerical examples and the empirical calibration of our model, in this subsection we discuss (what we know about) the impact of the mortality parameters $\lambda_0, \lambda_T$, diffusion parameters $\xi, \sigma$, preference parameters $\rho, \gamma$ and the interest rate $r$ on the optimal consumption rate. How do these seven parameters affect optimal behavior?

As far as subjective discount rate $\rho$ and relative risk aversion (or inter-temporal elasticity of substitution) $\gamma$ is concerned, it appears our stochastic mortality model doesn’t negate standard results. A higher value of $\rho$ (a.k.a. impatience) will increase current consumption at the expense of future consumption, no different from a deterministic Yaari (1965) and the thousands of extensions. With regards to $\gamma$, numerical results indicate that a higher value will lead to more cautious behavior – or greater longevity risk aversion – and current consumption is reduced. An increase in the interest rate $r$ when $\gamma > 1$ will increase the initial (current) consumption rate. The opposite occurs when $\gamma < 1$ All standard in the economics literature.

As far as the biology is concerned, if the current mortality rate $\lambda_0$ is increased then we anticipate the survival probability to any given age (or the overall curve) will be reduced. Conditional life expectancy is lower. And, when a rational consumer is subjected to a lower survival probability curve his or her optimal consumption rate will increase. This result or fact is well-known in the standard (random horizon) lifecycle literature and was originally studied and emphasized by Levhari and Mirman (1977). The same probability-based argument can be made with regards to the terminal mortality rate $\lambda_T$. Numerical results indicate that when it is held to a higher level, the conditional survival probability to any given age is reduced and consumption increases relative to a baseline scenario.
Now, with regards to the impact of volatility of mortality $\sigma$ and the speed of mean reversion $\xi$ – which are the two parameters driving the diffusion for the biological age $A_t$ in equation (5) – the situation is a bit trickier and the effects are not immediately obvious. Numerical evidence and extensive simulations suggest that when we increase $\sigma$, all else being held constant of course, the survival probability (curve) will decline. The higher volatility will kill you (faster.) This then leads to a higher consumption rate due to the lower odds of surviving to (very) old age. Consumption should be enjoyed earlier. The same mechanism or pathway is observed for $\xi$. When the speed of mean reversion is reduced (think of this as leading to more wandering) then consumption will also increase because the survival probability is lower. Note that for both $\sigma$ and $\xi$ there are offsetting impacts at work. Reducing the value $\sigma$ and/or increasing $\xi$ has the impact of moving the B-age closer to the C-age. This brings down the mortality rate when B-age is in the region that is higher than C-age, but raises it when the opposite holds. It appears that the former effect is more significant, at least for the parameters we have explored. To be clear though there are some pathological cases that might arise when B-age is (very) far from C-age.

3.4 Dispersion in Future Spending Rates

Recall that in addition to the normative objectives of this paper, we also wanted to show that even in a relatively simple life-cycle model with no bequest motives, no portfolio choice decisions and identical risk preference parameters $\rho$ and $\gamma$, one might still observe a dispersion in retirement spending rates for a homogenous group of individuals who are at the same chronological age today. This dispersion would arise solely because the retiree’s biological age – which is necessary but not sufficient for consumption decisions – will itself vary or wander over time. Recall that by time $t = T$ the wandering will cease and the spending rate (assuming our retiree is still alive) will converge to the known $f(T, \kappa_T = A_T)$, because the aging process stops.

Our plan in this section is to describe the methodology for computing the magnitude of this dispersion over time (and then offer some numerical examples.) To think about this in practical terms, we start our canonical retiree with a chronological and biological age of $\kappa = 60$ and corresponding mortality rate of $\lambda = 0.005$. We know that if-and-when he/she reaches the chronological age of $\kappa = 110$ the mortality rate will plateau at $\lambda = 1$ from there onward; these parameters are the basis for most of our numerical examples.

At time $t = 5, t = 10, t = 20, t = 30$ the chronological age will (obviously) be $\kappa = 65, \kappa = 70, \kappa = 80, \kappa = 90$ respectively, but the biological age ($\alpha$) is a random variable with a (sub) probability density function denoted by $g(t, \alpha)$. In some sense, that density is the conceptual core on which this paper is based.
The (sub-density) function $g(t,a)$ will satisfy what is known as a forward equation and the appendix contains a derivation of the PDE satisfied by $g(t,a)$, which can be written as:

$$g_t(t,a) + \frac{\partial}{\partial a} \left( \left(1 + \xi \frac{\kappa_t - a}{T-t} \right) g(t,a) \right) - \frac{\sigma^2}{2} g_{aa}(t,a) + \lambda(a) g(t,a) = 0. \quad (23)$$

This equation, like the PDE for the optimal consumption and spending rate, can be solved using numerical techniques.

Now, let $\alpha(t,q)$ denote the $q$'th percentile of the biological age at time $t$ conditional on being alive, that is $\Pr[a_t \leq \alpha(t,q) | \text{survival}] = q$. For example the expression $\alpha(15,0.95) = 75$ represents the 95% probability that biological age is less than 75 when chronological age is 70; implicitly assuming you are alive at chronological age 70 and that your current biological and chronological age is 60. Likewise, the range from $\alpha(15,0.05)$ to $\alpha(15,0.95)$ would provide a 90% confidence interval for biological age at time $t = 15$, etc., which is what we (tried to) plot in Figure 2.

So much for how $\alpha(t,q)$ is defined and what it represents, now let’s discuss how it is actually computed. By the formal definition of $q$ as a probability and the properties of the probability density function $g(t,a)$, we know that:

$$q = \frac{1}{S(t)} \int_{-\infty}^{\alpha(t,q)} g(t,a) \, da \quad (24)$$

where $S(t) = \int_{-\infty}^{\infty} g(t,a) \, da$ is a scaling factor. Then, differentiating both sides of equation (24) with respect to $q$ leads to the equality: $1 = \frac{1}{S(t)} g(t,\alpha(t,q)) \frac{\partial \alpha(t,q)}{\partial q}$. and the partial derivative term (that is the probability density function for biological age conditional on survival, can be computed as: $\frac{\partial \alpha(t,q)}{\partial q} = S(t)/g(t,\alpha(t,q))$.

Finally, since spending varies monotonically with age plugging $\alpha(t,q)$ into the optimal spending rate function, or $f(t,\alpha(t,q))^{-1/\gamma}$ will provide us with the $q$'th percentile for spending at time $t$. The range $f(t,\alpha(t,0.975))^{-1/\gamma}$ to $f(t,\alpha(t,0.025))^{-1/\gamma}$ is the 95% confidence interval for the dispersion of spending rates at time $t$ (chronological age $\kappa_t$), conditional on survival.

In other words the above enables us to map or convert the uncertainty or dispersion in observed biological ages – at any given and fixed chronological age – to a range of plausible value for withdrawal rates and spending rates.

### 4 Testable Implications

Table 3 and Table 4 provide an assortment of numerical values for spending rates under a variety of chronological and biological ages. For example let’s start with a (chronological) $\kappa = 60$ year-old whose biological age is (also) measured at $a = 60$. Assume that we are in an
Table 3: The optimal retirement spending rate (computed via $f^{-1/\gamma}$) as a function of biological and chronological age assuming $\gamma = 8$ and $\rho = r = 2.5\%$. The underlying mortality rate is $\lambda = 0.005$ at age 60 and pinned at $\lambda = 1.0$ at age 110, with $\xi = 1$ and $\sigma = 30\%$.

| B-Age | 60   | 65   | 70   | 75   | 80   | 85   | 90   | 95   |
|-------|------|------|------|------|------|------|------|------|
| 45    | 3.638%| 3.800%| 3.998%| 4.246%| 4.563%| 4.980%| 5.551%| 6.374%|
| 50    | 3.688%| 3.852%| 4.051%| 4.301%| 4.620%| 5.038%| 5.611%| 6.435%|
| 55    | 3.752%| 3.917%| 4.118%| 4.369%| 4.690%| 5.110%| 5.684%| 6.509%|
| 60    | 3.834%| 4.000%| 4.203%| 4.456%| 4.778%| 5.200%| 5.775%| 6.601%|
| 65    | 3.942%| 4.110%| 4.314%| 4.568%| 4.892%| 5.316%| 5.892%| 6.717%|
| 70    | 4.087%| 4.256%| 4.461%| 4.717%| 5.042%| 5.466%| 6.042%| 6.865%|
| 75    | 4.285%| 4.455%| 4.662%| 4.918%| 5.243%| 5.667%| 6.242%| 7.061%|
| 80    | 4.565%| 4.735%| 4.941%| 5.197%| 5.521%| 5.943%| 6.513%| 7.323%|
| 85    | 4.969%| 5.137%| 5.342%| 5.595%| 5.915%| 6.330%| 6.891%| 7.684%|
| 90    | 5.571%| 5.735%| 5.934%| 6.180%| 6.490%| 6.892%| 7.433%| 8.195%|
| 95    | 6.502%| 6.655%| 6.841%| 7.070%| 7.359%| 7.735%| 8.239%| 8.945%|

$r = 2.5\%$ interest rate environment which is identical to this individual’s subjective discount rate $\rho = 2.5\%$. The relevant mortality rate parameters are $\lambda_0 = 0.005$ and $\lambda_{110} = 1.0$.

Here is how to interpret the results: A consumer with a relatively high level of longevity risk aversion (i.e. $\gamma = 8$) will consume or spend at a rate of 3.834% of wealth under these parameter conditions. In contrast, but under the same parameter conditions, a consumer with a relatively low level of longevity risk aversion (i.e. $\gamma = 2$) will spend at a higher rate of 4.798%. There are no surprises so far and this is exactly what one might expect in the classical (deterministic aging) model which we described earlier in Section [4].

Let’s now front-forward by 10 years when our consumer (retiree) is aged 70 chronologically. Within our lifecycle model their optimal spending rate depends (also) on biological age since chronological age is no longer enough information (a.k.a. a sufficient statistic.) If their age co-ordinates happen to be: [C=70, B=70] then their spending rate is 4.461% in Table 3 or Table 4, which is the case of high risk aversion. But if they have “aged well” and their biological age is only $a = 65$, their age co-ordinates in the matrix are now [C=70,B=65] and the optimal spending rate is (a relatively lower) 4.314%. The 15 basis point difference between the old 70 year-old and the young 70-year old might not seem like much, but recall that these values depend on subjective preference parameters $\gamma, \rho$, as well as mortality pa-
Table 4: Same parameters as table 3, but with $\gamma = 2$.

| B-Age | 60   | 65   | 70   | 75   | 80   | 85   | 90   | 95   |
|-------|------|------|------|------|------|------|------|------|
| 45    | 4.242%| 4.465%| 4.743%| 5.096%| 5.560%| 6.195%| 7.116%| 8.572%|
| 50    | 4.379%| 4.607%| 4.889%| 5.247%| 5.717%| 6.359%| 7.289%| 8.757%|
| 55    | 4.559%| 4.791%| 5.078%| 5.442%| 5.918%| 6.569%| 7.509%| 8.989%|
| 60    | 4.798%| 5.035%| 5.328%| 5.698%| 6.182%| 6.841%| 7.792%| 9.286%|
| 65    | 5.123%| 5.366%| 5.665%| 6.043%| 6.535%| 7.203%| 8.166%| 9.675%|
| 70    | 5.579%| 5.827%| 6.133%| 6.518%| 7.019%| 7.697%| 8.672%| 10.196%|
| 75    | 6.233%| 6.487%| 6.800%| 7.192%| 7.701%| 8.389%| 9.374%| 10.912%|
| 80    | 7.203%| 7.462%| 7.779%| 8.177%| 8.691%| 9.386%| 10.379%| 11.923%|
| 85    | 8.691%| 8.951%| 9.268%| 9.666%| 10.180%| 10.873%| 11.863%| 13.400%|
| 90    | 11.054%| 11.306%| 11.614%| 12.000%| 12.500%| 13.175%| 14.138%| 15.635%|
| 95    | 14.937%| 15.166%| 15.447%| 15.800%| 16.258%| 16.878%| 17.768%| 19.159%|

rameters $\lambda_0, \lambda_T$ and interest rates $r$. The difference between old and young could be (much) greater depending on specific values. For example, and in the same table, at the chronological age of 80 the lowest spending rate is 5.56% and the highest spending rate is more than triple at 16.25% The odds of biological age wandering to these levels are slim but the point remains.

Figure 5 offers another (color) perspective on the impact and importance of both age co-ordinates on the optimal spending rate function. The $x,y$ plane denotes the biological and chronological age respectively and the $z$ dimension is the corresponding spending rate for those co-ordinates. Notice that in the near corner where the individual is relatively young the spending rate as a function of wealth is in the 4% range. As the individual ages in either chronological (to the right) or biological (to the left), the spending rate as a function of wealth increase. However one can see the gradient or slope is higher as one ages biologically as opposed to chronologically, all else being equal of course.

Finally, as we reach the summit or upper back corner of the graph when chronological age is 100 and biological age is 95, the spending rate function $f^{-1/\gamma}$ reaches a peak of approximately 9%, which corresponds to the numbers at the lower right corner of Table 3 or Table 4. Once again we caution the reader that these numbers and figures are quite sensitive to parameter values and therefore one shouldn’t read too much into the fact they are within the vicinity of the so-called 4% rule that has been widely advocated by the media and financial advisors. In sum, if indeed such a model were to be implemented in a
normative context – and to provide financial advice – the user would have to take great care in estimating appropriate consumer specific values of (longevity) risk aversion $\gamma$, subjective discount rates $\rho$ and a suitable long-term real-return $r$.

5 Conclusion

In this paper we took the first step towards constructing a lifecycle model of consumption and investment in which the agent’s age does not move in lockstep with calendar time. Our main break with the prior literature is that we assume a world in which (true) biological age can (i.) be measured with precision and (ii.) grows at a stochastic non-linear rate which might even decline over periods of time. We focused exclusively on the retirement stage (i.e. when human capital is exhausted) to isolate and examine the impact of aging, which is also a period when mortality rates have a noticeable impact on optimal consumer behavior and longevity uncertainty would be expected to drive economic behavior.

We re-iterate that at some level our work is (also) about relative health-status and how it affects retirement spending and consumption rates. However we frame the problem differently and instead think of everyone as having a unique biological age, a number that might soon become available to consumers. Technically speaking we modify the lifecycle optimality conditions and show how to derive the proper consumption, withdrawal or spending rate function in a Yaari (1965) model with stochastic aging, which introduces another state variable into the (partial differential) equation.

If indeed retirees soon have access to (a device with) their biological age in real time then
our model would inform them how to draw-down (spend, consume) their assets in a way that is consistent with a rational lifecycle model. As we showed, they would require both their biological and chronological age to make these economic decisions.

We believe that in addition to contribution to normative theory, our model helps explain the cross-sectional heterogeneity in retirement spending rates at any given chronological age – a phenomenon which has been documented in various countries – by suggesting that this dispersion is due to (perceived) differences in biological age. For example, under one set of parameters we show that the optimal spending rate can vary in the range of 5% to 15% at retirement – for a fixed value of longevity risk aversion (γ) and no bequest motives – when real long term investment rates are ρ = r = 2.5%. The exact number is sensitive to the usual financial and economic inputs and is not really the point of our analysis.

5.1 Next Steps

This is the beginning of a research agenda to investigate the implications of acting your age in a rational lifecycle framework. Our next step is to introduce (properly) priced fixed pensions or life annuities and thus investigate the impact of B-age vs. C-age on the demand for longevity insurance and optimal consumption. This was done by Leung (2007) in a deterministic (i.e. old) aging model. The question becomes how to solve the problem which the wealth depletion time (WDT) using Leung’s (2007) language, is prior to the end of the mortality table. That extension might also help shed light on the so-called annuity puzzle and the low demand for (voluntary) annuities. The plan is then to relax the budget constraint and allow control over the annuitization decision, in addition to the consumption vs. investment tradeoff. Likewise, our plan is to introduce a risky asset (i.e. portfolio choice) and a flexible retirement date (i.e. labour vs. leisure) to examine how those decisions – which have been thoroughly studied by financial economists in the deterministic mortality framework – would be affected by our stochastic model.

We conclude by musing whether on a societal level it might make sense to subordinate retirement policy to biological age instead of chronological age. In other words, as an alternative to extending or increasing pension eligibility ages – which has its own political problems and fears of redistribution – perhaps everyone should be allowed to retire and draw their pension at the biological age of 65. Might it be perceived as more equitable?
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6 Appendix: Derivations and Proofs

6.1 Derivation of the PDE Satisfied by Optimal Consumption

Our starting point here is the objective function in equation [17]. We know that for \( t < \zeta \), that is conditional on survival, and the optimal \( c_t \), that:

\[
\int_0^t e^{-\rho s} \frac{c_s^{1-\gamma}}{1-\gamma} ds + e^{-\rho t} v(t, A_t, W_t) = E[\int_0^\zeta e^{-\rho s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \mid F_t].
\]

The latter is a martingale and has a jump of \(-e^{-\rho t} v(t, A_t, W_t)\) at \( t = \zeta \). Adding back the compensated jump, we get that:

\[
\int_0^t e^{-\rho s} \frac{c_s^{1-\gamma}}{1-\gamma} ds + e^{-\rho t} v(t, A_t, W_t) - \int_0^t \lambda_s e^{-\rho s} v(s, A_s, W_s) ds
\]

is a martingale when stopped at \( \zeta \). Therefore for \( t < T \),

\[
\frac{c_t^{1-\gamma}}{1-\gamma} + v_t + \left( 1 + \xi \frac{\kappa_t - a}{T-t} \right) v_a + \sigma^2 v_{aa} + (r w - c) v_w - (\rho + \lambda(a)) v = 0
\]

in the optimal case. In the sub-optimal case we get a supermartingale instead, so the Hamilton-Jacobi-Belman (HJB) equation is:

\[
\sup_e \frac{c_t^{1-\gamma}}{1-\gamma} + v_t + \left( 1 + \xi \frac{\kappa_t - a}{T-t} \right) v_a + \sigma^2 v_{aa} + (r w - c) v_w - (\rho + \lambda(a)) v = 0
\]

Therefore \( c^{-\gamma} - v_w = 0 \) and as usual, this becomes

\[
v_t + \left( 1 + \xi \frac{\kappa_t - a}{T-t} \right) v_a + \sigma^2 v_{aa} + r w v_w + \frac{\gamma}{1-\gamma} v_{w}^{1-\gamma} - (\rho + \lambda(a)) v = 0.
\]

The scaling relation is that \( v(t, a, kw) = k^{1-\gamma} v(t, a, w) \).

This implies that \( v(t, a, w) = f(t, a) \frac{w^{1-\gamma}}{1-\gamma} \) for some \( f \), and we get

\[
f_t + \left( 1 + \xi \frac{\kappa_t - a}{T-t} \right) f_a + \frac{\sigma^2}{2} f_{aa} + r(1-\gamma)f - (\rho + \lambda(a)) f + \gamma f^{1-\frac{1}{\gamma}} = 0.
\]

for \( t < T \), with boundary conditions \( f(T, a) = f_T, f(t, \infty) = 0 \). Optimal consumption then is \( c = w \cdot f(t, a)^{-\frac{1}{\gamma}} \).

6.2 Special Case of Logarithmic Utility

In the special case of logarithmic, when \( \gamma = 1 \) in the CRRA utility, the same arguments used in the body of the paper (section 3) lead to:

\[
v_t + \left( 1 + \xi \frac{\kappa_t - a}{T-t} \right) v_a + \frac{\sigma^2}{2} v_{aa} + r w v_w - 1 - \log v_w - (\rho + \lambda(a)) v = 0.
\]
In this case, the well-used scaling relation is that \( v(t, a, kw) = v(t, a, w) + (\log k)E[\int_0 t^{-\rho a} ds | \mathcal{F}_t] \), which implies a solution of the form \( v(t, a, w) = f(t, a) \log w + h(t, a) \). From this we obtain \( f_t + \left(1 + \xi \frac{\kappa_t - a}{T - t}\right)f_a + \frac{\sigma^2}{2}f_{aa} + 1 - (\rho + \lambda(a))f = 0 \) \( (32) \)

\( h_t + \left(1 + \xi \frac{\kappa_t - a}{T - t}\right)h_a + \frac{\sigma^2}{2}h_{aa} - (\rho + \lambda(a))h + rf - \log f - 1 = 0 \) \( (33) \)

for \( t < T \). Boundary conditions are \( f(T, a) = f_T \) and \( h(T, a) = h_T \). For \( t \geq T \), \( f \) and \( h \) are constant, so \( f_T = \frac{1}{\rho + \xi T} \) and \( h_T = \frac{1}{\rho + \xi T}(rf_T - \log f_T - 1) \). Optimal consumption is \( c = w \cdot f(t, a) \).

Finally, in this logarithmic \( \gamma = 1 \) case, the ODE for \( f(t) \) is \( f_t + 1 - \rho f = 0 \), which has solution \( f(t) = e^{-(T-t)}(f(T) - \frac{1}{\rho}) + \frac{1}{\rho} \). There is a similar ODE for \( h(t) \), namely \( h_t - \rho h + rf - \log f - 1 = 0 \), which one would solve to obtain the maximized value of utility, necessary for welfare calculations.

### 6.3 Approximate Analytic Solution

Consider the main HJB equation (19) for consumption or the spending rate, after a dimension reduction. Note that the diffusion effect within the PDE occurs at \( \mathcal{O}(\sigma^2) \), which is smaller in magnitude than the effect in the drift, especially early on when \((T - t)\) is small. Therefore we can approximate the solution to the PDE by taking the following asymptotic expansion:

\[ f \sim f^{(0)}(t, a) + \sigma^2 f^{(1)}(t, a) + \cdots \] \( (34) \)

At the leading order, we obtain the following first order PDE for \( f^{(0)} \), which by substitution is:

\[ f_t^{(0)} + \left(1 + \xi \frac{\kappa_t - a}{T - t}\right)f_a^{(0)} + r(1 - \gamma)f^{(0)} - (\rho + \lambda(a))f^{(0)} + \gamma(f^{(0)})^{1-\frac{1}{\gamma}} = 0. \] \( (35) \)

with the same (as before) boundary condition at \( t = T \). This equation, which is obviously simpler than equation (19), can be solved by using the so-called method of characteristics, that is by by treating biological \( a \) as a deterministic function of time \( t \) given by:

\[ \frac{da}{dt} = 1 + \xi \frac{\kappa_t - a}{T - t} \] \( (36) \)

with \( a(T) = \kappa_T \), since both ages converge to each other at time \( T \). We can then solve (the modified) equation \( F(t) = f^{(0)}(t, a(t)) \) using the following ordinary differential equation (ODE) representation instead of the original PDE:

\[ \frac{dF}{dt} + r(1 - \gamma)F - (\rho + \lambda(a(t)))F + \gamma F^{1-\frac{1}{\gamma}} = 0 \] \( (37) \)

with terminal condition \( F(T) = f_T \). Of course, it remains to be seen how good this analytic approximation is relative to the (numerical) solution to the PDE we reported in the body.
of the paper. Note in fact that the equation for biological $a$ can now be explicitly solved in closed-form as:

$$a(t) = \kappa t + (a(0) - \kappa_0) \left( \frac{T - t}{T} \right)^\xi$$  \hspace{1cm} (38)

where $a(0)$ is the biological age at time 0, or chronological age $\kappa_0$. The equation for $F$ can be further simplified by introducing $G = F^{1/2}$, for which the equation for $G$ actually becomes linear, leading to:

$$\frac{dG}{dt} + 1 + \frac{1}{\gamma} [r(1 - \gamma) - (\rho + \lambda(a(t))] G = 0.$$  \hspace{1cm} (39)

In principle now, this equation can solved explicitly. Note that when both ages match and $a(0) = \kappa_0$, we have $a(t) = \kappa_t$, which is the Gompertz case and $F$ can be expressed by using the incomplete Gamma function, as reported in Milevsky and Huang (2010) for example. Another special case is when $\xi = 1$ (and $\sigma = 0$), and we have a linear relationship for the biological age:

$$a(t) = \kappa_t + (a(0) - \kappa_0) \left( \frac{T - t}{T} \right) = a(0) + kt, \quad k = \frac{\kappa_0 - a(0)}{T} + 1.$$  \hspace{1cm} (40)

In this (simplified) case, the underlying mortality hazard rate is given by:

$$\lambda = \frac{1}{b} \exp \left( \frac{a(t) - m}{b} \right) = \frac{1}{b} \exp \left( \frac{a(0) + kt - m}{b} \right),$$  \hspace{1cm} (41)

which is quite similar to the Gompertz form, thus leading to a closed-form solution for $f^{(0)}$. Then, we can improve the accuracy by going to a higher order correction $f^{(1)}$, which by the same logic as before satisfies the following PDE:

$$f^{(1)}_t + \left( 1 + \xi \frac{\kappa_t - a}{T - t} \right) f^{(1)}_a + r(1 - \gamma) f^{(1)} - (\rho + \lambda(a)) f^{(1)} + \gamma (f^{(1)})^{1-\frac{1}{\gamma}} = -\frac{f^{(0)}_{aa}}{2}.$$  \hspace{1cm} (42)

And, since (in the first approximation) we have already obtained $f^{(0)}$, the right-hand-side is known. The left-hand-side is exactly the same as that for $f^{(0)}$, which again can be solved using the (prior mentioned) method of characteristics.

### 6.4 Distribution of Future Biological Age

In this section we derive the (sub) density $g(t,a) da = P(A_t \in da, \zeta > t)$, which represents the probability that biological age is within any given ‘range’ $da$ and (of course) on being alive at time $t$, which is chronological age $\kappa_t$. The sub-density $g(t,a)$ can then be used to compute all relevant expectations and quantiles, since for any function $\phi$,

$$E[\phi(A_t) | t < \zeta] = \frac{\int \phi(a) g(t,a) da}{\int g(t,a) da}.$$  \hspace{1cm} (43)
In particular, we can then compute the conditional mean \( E[A_t \mid t < \zeta] \) for the biological age at time \( t \), second moment \( E[A_t^2 \mid t < \zeta] \), and the conditional variance (which recall is \( E^2[A_t] - E[A_t^2] \)). The conditional variance, of course, differs from the unconditional (time \( t = 0 \)) expressions worked out earlier.

Recall that by definition (i.e. aging stops at age \( T = 110 \) for example), for \( t \geq T \) the conditional mean = \( \kappa_T \) and the conditional variance is zero. So at this point we are only concerned and work with \( t < T \). Let \( \phi \) be smooth, with compact support. We know that:

\[
\phi(A_t) = \phi(\kappa_0) + \int_0^t \phi'(A_s) \left( 1 + \xi \frac{\kappa_s - A_s}{T - s} \right) ds + \int_0^t \phi'(A_s) dB_s + \int_0^t \frac{\sigma^2}{2} \phi''(A_s) ds. \tag{44}
\]

Therefore incorporating the jump,

\[
\phi(A_t)1_{\{t<\zeta\}} = \phi(\kappa_0) + \int_0^t \phi'(A_s) \left( 1 + \xi \frac{\kappa_s - A_s}{T - s} \right) 1_{\{s<\zeta\}} ds + \int_0^t \phi'(A_s) 1_{\{s<\zeta\}} dB_s + \int_0^t \frac{\sigma^2}{2} \phi''(A_s) 1_{\{s<\zeta\}} ds - \phi(A_\zeta) 1_{\{s \geq \zeta\}}. \tag{45}
\]

Adding back the compensated jump and taking expectations, we get that

\[
E[\phi(A_t)1_{\{t<\zeta\}}] = \phi(\kappa_0) + \int_0^t E[\phi'(A_s) \left( 1 + \xi \frac{\kappa_s - A_s}{T - s} \right) 1_{\{s<\zeta\}}] ds \tag{46}
\]

\[+ \int_0^t \frac{\sigma^2}{2} E[\phi''(A_s) 1_{\{s<\zeta\}}] ds - \int_0^t E[\phi(A_s) 1_{\{s<\zeta\}} \lambda_s] ds.
\]

In other words,

\[
\int \phi(a) g(t, a) da = \phi(\kappa_0) + \int_0^t \int \phi'(a) \left( 1 + \xi \frac{\kappa_s - a}{T - s} \right) g(s, a) da ds \tag{47}
\]

\[+ \int_0^t \int \frac{\sigma^2}{2} \phi''(a) g(s, a) da ds - \int_0^t \int \phi(a) \lambda(a) g(s, a) da ds.
\]

Taking \( \frac{\partial}{\partial a} \), we get that

\[
\int \phi(a) g_t(t, a) da = \int \phi'(a) \left( 1 + \xi \frac{\kappa_t - a}{T - t} \right) g(t, a) da \tag{48}
\]

\[+ \int \frac{\sigma^2}{2} \phi''(a) g(t, a) da - \int \phi(a) \lambda(a) g(t, a) da.
\]

Using integration by parts this becomes

\[
\int \phi(a) g_t(t, a) da = \int \phi(a) \left[ - \frac{\partial}{\partial a} \left( \left( 1 + \xi \frac{\kappa_t - a}{T - t} \right) g(t, a) \right) + \frac{\sigma^2}{2} g_{aa}(t, a) - \lambda(a) g(t, a) \right] da. \tag{49}
\]

And since \( \phi \) was reasonably arbitrary,

\[
g_t(t, a) = - \frac{\partial}{\partial a} \left( \left( 1 + \xi \frac{\kappa_t - a}{T - t} \right) g(t, a) \right) + \frac{\sigma^2}{2} g_{aa}(t, a) - \lambda(a) g(t, a), \tag{50}
\]
with the initial condition being a delta-function. In the forward (vs. more common backward equation) case, the boundary conditions are simply that \( g = 0 \) when \( a = \pm \infty \). Equation (50) is the PDE satisfied by the (sub) density \( g(t,a) \) we need, and is known as the so-called forward equation. This is the PDE we solve (using numerical methods) in section 4 to obtain the possible distribution (and quantiles) of spending rates at any time \( t \) or age \( \kappa_t \).

A special case gives the population (or unconditioned) survival probability

\[ tP_{\kappa_0}^{\text{pop}} = P(\zeta > t) = \int_{-\infty}^{\infty} g(t,a) \, da, \quad (51) \]

and therefore the population hazard rate

\[ \lambda_t^{\text{pop}} = -\frac{1}{tP_{\kappa_0}^{\text{pop}}} \frac{d}{dt} tP_{\kappa_0}^{\text{pop}} = -\frac{\int_{-\infty}^{\infty} g_t(t,a) \, da}{\int_{-\infty}^{\infty} g(t,a) \, da}. \quad (52) \]